ANGLED CRESTED TYPE WATER WAVES WITH SURFACE TENSION II:
ZERO SURFACE TENSION LIMIT

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Abstract. This is the second paper in a series of papers analyzing angled crested type water waves with surface tension. We consider the 2D capillary gravity water wave equation and assume that the fluid is inviscid, incompressible, irrotational and the air density is zero. In the first paper [1] we constructed a weighted energy which generalizes the energy of Kinsey and Wu [22] to the case of non-zero surface tension, and proved a local wellposedness result. In this paper we prove that under a suitable scaling regime, the zero surface tension limit of these solutions with surface tension are solutions to the gravity water wave equation which includes waves with angled crests.

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1. INTRODUCTION

This is the second paper in a series of papers analyzing angled crested type water waves with surface tension. As in the first paper we will identify 2D vectors with complex numbers. We consider a 2D fluid which we assume to be inviscid, incompressible and irrotational and the fluid is subject to a uniform gravitational field \(-i\) acting in the downward direction. The fluid domain \(\Omega(t)\) and the air region are separated by an interface \(\Sigma(t)\) which we assume to be homeomorphic to \(\mathbb{R}\) and which tends to the real line at infinity. We do not assume that the interface is a graph. The air and the fluid are assumed to have constant densities of 0 and 1 respectively. The fluid is below the air region and its motion is governed by the Euler equation

\[
\begin{align*}
    v_t + (v \cdot \nabla)v &= -i - \nabla P & \text{on } \Omega(t) \\
    \text{div } v &= 0, \quad \text{curl } v &= 0 & \text{on } \Omega(t)
\end{align*}
\]

along with the boundary conditions

\[
\begin{align*}
    P &= -\sigma \partial_s \theta & \text{on } \Sigma(t) \\
    (1, v) &= \text{tangent to the free surface } (t, \Sigma(t)) \\
    v &\to 0, \nabla P \to -i & \text{as } |(x, y)| \to \infty
\end{align*}
\]

Here \(\theta = \text{angle the interface makes with the } x-\text{axis}, \partial_s = \text{arc length derivative}, \sigma = \text{coefficient of surface tension} \geq 0\).

The earliest results on local well-posedness for the Cauchy problem are for small data in 2D and were obtained by Nalimov [26], Yoshihara [37, 38] and Craig [17]. In the case of zero surface tension, Wu [33, 34] obtained the proof of local well-posedness for arbitrary data in Sobolev spaces. This result has been extended in various directions, see the works in [15, 24, 23, 39, 14, 7, 20, 8, 19, 18, 3, 5, 4]. In the case of non-zero surface tension, the local well-posedness of the equation in Sobolev spaces was established by Beyer and Gunther in [12]. See also related works in [21, 9, 29, 16, 32, 6, 13, 27]. The zero surface tension limit of the water wave equation in Sobolev spaces was proved by Ambrose and Masmoudi [10, 11]. See also the works in [28, 31, 25, 13, 30].

All of the above results are for interfaces with regularity at least \(C^{1,\alpha}\) for some \(\alpha > 0\). In an important work Kinsey and Wu [22] proved an apriori estimate for angled crested water waves in the case of zero surface tension. Using this Wu [36] proved a local wellposedness result which allows for angled crested interfaces as initial data. In [2] the author proved that these singular solutions are rigid and in particular the angle of the corner does not change with time.

In the first part of this series of papers [1], we took the first step in extending this theory of angled crested water waves to the case of non-zero surface tension. We constructed an energy functional \(E_\sigma(t)\) which generalizes the energy of [22] to the case of \(\sigma \geq 0\), and proved a local wellposedness result based on this energy (see Theorem 2.1). For initial data in appropriate Sobolev spaces, this existence result gives us a uniform time of existence of solutions for \(0 \leq \sigma \leq \sigma_0\) for arbitrary \(\sigma_0 > 0\), thereby recovering the uniform time of existence result of Ambrose and Masmoudi [10] in this case. In addition to this, the energy \(E_\sigma(t)\) has several interesting properties: for example if \(\sigma = 0\) then it reduces to a lower order version of the energy of [22] and allows angled crested interfaces, however
for $\sigma > 0$ the energy $E_\sigma(t)$ does not allow any singularities of the interface. On the other hand the energy does allow large curvature when $\sigma > 0$ and in particular the $L^\infty$ norm of the initial curvature can be as large as $\sigma^{-\frac{1}{2} + \epsilon}$ for any $\epsilon > 0$ (see Corollary 2.2). This growth rate of $\sigma^{-\frac{1}{2}}$ is explained by the fact that the quantity $||\sigma^{\frac{1}{2}} \kappa||_\infty$ where $\kappa$ is the curvature, is a scaling invariant quantity for the problem for zero surface tension limit (see the introduction and Remark 3.4 of [1] for more details).

In this paper we continue the study of angled crested water waves and consider the zero surface tension limit. We construct a weighted energy $E_\Delta(t)$ for the difference of two solutions of the water wave equation, one with zero surface tension and one with surface tension $\sigma$, and prove in our main result Theorem 3.1 that $E_\Delta(t) \to 0$ as the surface tension $\sigma \to 0$. This is a generalization of the convergence result of [10] where convergence is proven in Sobolev spaces. The advantage of using this energy $E_\Delta(t)$ for the difference of the solutions, instead of the usual Sobolev norms, is that the rate of growth of the energy $E_\Delta(t)$ does not depend on the Sobolev norms of the initial data but only on weighted norms such as $E_\sigma(0)$. Hence this result allows us to control the difference of the solutions independent of how close the initial interface is to an angled crested interface.

As an application of our main result Theorem 3.1 we show that under a suitable scaling regime, smooth solutions of the water wave equation with surface tension converge to the singular solution of the water wave equation with zero surface tension. More precisely we consider an initial data $(Z, Z_t)(0)$ with angled crested interface and consider the singular solution $(Z, Z_t)(t)$ to the water wave equation with zero surface tension as obtained from [36]. We then consider smooth solutions $(Z^{\epsilon, \sigma}, Z^{\epsilon, \sigma}_t)(t)$ to the water wave equation with surface tension $\sigma$ with initial data $(Z^{\epsilon, \sigma}, Z^{\epsilon, \sigma}_t)(0) = (Z_t*P_\epsilon, Z_t*P_\epsilon)(0)$ where $P_\epsilon$ is the Poisson kernel. We show in Corollary 3.2 that if $\max\{\sigma/\epsilon^2, \epsilon\} \to 0$, then the solutions $(Z^{\epsilon, \sigma}, Z^{\epsilon, \sigma}_t)(t) \to (Z, Z_t)(t)$ in a suitable norm. The existence of these solutions $(Z^{\epsilon, \sigma}, Z^{\epsilon, \sigma}_t)(t)$ on a uniform time interval was already shown in [11] (see Corollary 2.2 and this factor of $\sigma/\epsilon^2$ already appeared there (see Section 3.1 of [11] for more details)). The novelty of Corollary 3.2 is the convergence aspect.

The proof of Theorem 3.1 is based on energy estimates. The usual strategy of directly subtracting terms in the energy $E_\sigma(t)$ and using that as the energy for the difference of the solutions fails in our case because of the weighted nature of the energy $E_\sigma(t)$. To remedy this, we introduce a coupling term $\sigma(E_{aux})_b(t)$ which couples the carefully constructed weighted energy $(E_{aux})_b(t)$ of the zero surface tension solution, with the surface tension $\sigma$ coming from the solution with surface tension. This coupling term $\sigma(E_{aux})_b(t)$ is a part of the energy $E_\Delta(t)$ and is crucial to closing the energy estimate of $E_\Delta(t)$ at the highest order. We believe that using such coupling terms could be useful in other convergence problems more generally. We discuss the necessity and usefulness of this coupling term in more detail in §3.2.

The paper is organized as follows: In §2 we introduce the notation, the main system of equations we solve and recall the results from [1]. In §3 we state our main results and discuss the main ideas behind the proof. Then in §4 we collect all the identities and estimates from [1] that we frequently use in our calculations. In §5 we prove an apriori estimate for the energy $E_{high}(t)$ which is an energy for the zero surface tension solutions, and this is used in the apriori estimate for $E_{aux}(t)$. In §6 we prove an apriori energy estimate for the energy $E_{aux}(t)$ which is also a weighted energy for the zero surface tension solution. As explained before this energy is crucial in proving Theorem 3.1. In §7 we prove an apriori estimate for our main energy $E_\Delta(t)$. Finally in §8 we complete the proofs of our results Theorem 3.1 and Corollary 3.2. In Appendix A §9 we collect all the identities and estimates from the appendix of [1] which are needed for this paper and in Appendix B §10 we prove some new estimates that we use throughout the paper.
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2. Notation and previous work from part I

2.1. Notation

In this section we recall the notation used in [1] and explain the main results proved there. The Fourier transform is defined as

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int e^{-ix\xi} f(x) \, dx$$

We will denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space of rapidly decreasing functions and $\mathcal{S}'(\mathbb{R})$ is the space of tempered distributions. A Fourier multiplier with symbol $a(\xi)$ is the operator $T_a$ defined formally by the relation $\hat{T_a} f = a(\xi) \hat{f}(\xi)$. The operators $|\partial_x|^s$ for $s \in \mathbb{R}$ are defined as the Fourier multipliers with symbol $|\xi|^s$. The Sobolev space $H^s(\mathbb{R})$ for $s \geq 0$ is the space of functions with $\|f\|_{H^s} = \|(1 + |\xi|^2)^{s/2} \hat{f}(\xi)\|_{L^2(\mathbb{R})} < \infty$. The homogenous Sobolev space $\dot{H}^s(\mathbb{R})$ is the space of functions modulo constants with $\|f\|_{\dot{H}^s} = \|\xi^{s/2} \hat{f}(\xi)\|_{L^2(\mathbb{R})} < \infty$. The Poisson kernel is given by

$$K_\epsilon(x) = \frac{\epsilon}{\pi(x^2 + \epsilon^2)} \quad \text{for } \epsilon > 0 \quad (3)$$

From now on compositions of functions will always be in the spatial variables. We write $f = f(\cdot,t), g = g(\cdot,t), f \circ g(\cdot,t) := f(g(\cdot,t),t)$. Define the operator $U_g$ as given by $U_g f = f \circ g$. Observe that $U_f U_g = U_{g \circ f}$. Let $[A,B] := AB - BA$ be the commutator of the operators $A$ and $B$. If $A$ is an operator and $f$ is a function, then $(A + f)$ will represent the addition of the operators $A$ and the multiplication operator $T_f$ where $T_f(g) = fg$. We denote the convolution of $f$ and $g$ by $f * g$. We will denote the spacial coordinates in $\Omega(t)$ with $z = x + iy$, whereas $z' = x' + iy'$. Let $z \rightarrow \{x, y\} \in \mathbb{R}^2$ when $y < 0$. As we will frequently work with holomorphic functions, we will use the holomorphic derivatives $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{z'} = \frac{1}{2}(\partial_{x'} - i\partial_{y'})$. In this paper all norms will be taken in the spacial coordinates unless otherwise specified. For example for a function $f : \mathbb{R} \times [0, T) \rightarrow \mathbb{C}$ we write $\|f\|_2 = \|f(\cdot,t)\|_2 = \|f(\cdot,t)\|_{L^2(\mathbb{R}, dt')}$. Also for a function $f : P_- \rightarrow \mathbb{C}$ we write $\sup_{y' < 0} \|f\|_{L^2(\mathbb{R}, dt')} = \sup_{y' < 0} \|f(\cdot,y')\|_{L^2(\mathbb{R}, dt')}$. We write $a \lesssim b$ if there exists a universal constant $C > 0$ so that $a \leq C b$. This notation may be changed in different sections to simplify calculations in those sections. For functions $f_1, f_2, f_3 : \mathbb{R} \rightarrow \mathbb{C}$ we define the function $[f_1, f_2, f_3] : \mathbb{R} \rightarrow \mathbb{C}$ as

$$[f_1, f_2, f_3](\alpha') = \frac{1}{i\pi} \int \left( \frac{f_1(\alpha') - f_1(\beta')}{\alpha' - \beta'} \right) \left( \frac{f_2(\alpha') - f_2(\beta')}{\alpha' - \beta'} \right) f_3(\beta') \, d\beta' \quad (4)$$

and if $g : \mathbb{R} \rightarrow \mathbb{R}$ is a homeomorphism then we define $[f_1, f_2, f_3]_g : \mathbb{R} \rightarrow \mathbb{C}$ as

$$[f_1, f_2, f_3]_g(\alpha') = \frac{1}{i\pi} \int \left( \frac{f_1(\alpha') - f_1(\beta')}{g(\alpha') - g(\beta')} \right) \left( \frac{f_2(\alpha') - f_2(\beta')}{g(\alpha') - g(\beta')} \right) f_3(\beta') \, d\beta' \quad (5)$$

Let the interface be parametrized in Lagrangian coordinates by $z(\cdot,t) : \mathbb{R} \rightarrow \Sigma(t)$ satisfying $z_\alpha(z(\alpha,t),t) \neq 0$ for all $\alpha \in \mathbb{R}$. Hence $z_t(\alpha,t) = v(z(\alpha,t),t)$ is the velocity of the fluid on the interface and $z_{tt}(\alpha,t) = (v_t + (v \nabla)v)(z(\alpha,t),t)$ is the acceleration.
Let $\Psi(\cdot, t) : P_\rightarrow \rightarrow \Omega(t)$ be conformal maps satisfying $\lim_{z \rightarrow \infty} \Psi(z, t) = 1$ and $\lim_{z \rightarrow \infty} \Psi_t(z, t) = 0$. With this, the only ambiguity left in the definition of $\Psi$ is that of the choice of translation of the conformal map at $t = 0$, which does not play any role in the analysis. Let $\Phi(\cdot, t) : \Omega(t) \rightarrow P_\rightarrow$ be the inverse of the map $\Psi(\cdot, t)$ and define $h(\cdot, t) : \mathbb{R} \rightarrow \mathbb{R}$ as

$$h(\alpha, t) = \Phi(z(\alpha, t), t)$$

hence $h(\cdot, t)$ is a homeomorphism. As we use both Lagrangian and conformal parameterizations, we will denote the Lagrangian parameter by $\alpha$ and the conformal parameter by $\alpha'$. Let $h^{-1}(\cdot, t)$ be its spacial inverse i.e.

$$h(h^{-1}(\alpha', t), t) = \alpha'$$

From now on, we will fix our Lagrangian parametrization at $t = 0$ by imposing

$$h(\alpha, 0) = \alpha \text{ for all } \alpha \in \mathbb{R}$$

Hence the Lagrangian parametrization is the same as conformal parametrization at $t = 0$. Define the variables

$$Z(\alpha', t) = z \circ h^{-1}(\alpha', t), \quad Z_{\alpha'}(\alpha', t) = \partial_{\alpha'} Z(\alpha', t), \quad \text{Hence } \left( \frac{\partial}{\partial \alpha} \right) h^{-1} = Z_{\alpha'}$$

$$Z_t(\alpha', t) = z_t \circ h^{-1}(\alpha', t), \quad Z_{t, \alpha'}(\alpha', t) = \partial_{\alpha'} Z_t(\alpha', t), \quad \text{Hence } \left( \frac{\partial}{\partial \alpha} \right) h^{-1} = Z_{t, \alpha'}$$

$$Z_{tt}(\alpha', t) = z_{tt} \circ h^{-1}(\alpha', t), \quad Z_{tt, \alpha'}(\alpha', t) = \partial_{\alpha'} Z_{tt}(\alpha', t), \quad \text{Hence } \left( \frac{\partial}{\partial \alpha} \right) h^{-1} = Z_{tt, \alpha'}$$

Hence $Z(\alpha', t), Z_t(\alpha', t)$ and $Z_{tt}(\alpha', t)$ are the parameterizations of the boundary, the velocity and the acceleration in conformal coordinates and in particular $Z(\cdot, t)$ is the boundary value of the conformal map $\Psi(\cdot, t)$. Note that as $Z(\alpha', t) = z(h^{-1}(\alpha', t), t)$ we see that $\partial_t Z \neq Z_t$. Similarly $\partial_t Z_t \neq Z_{tt}$. The substitute for the time derivative is the material derivative. Define the operators

$$D_t = \text{material derivative} = \partial_t + b \partial_{\alpha'}$$

where

$$b = h_t \circ h^{-1}$$

$$D_{\alpha'} = \frac{1}{Z_{\alpha'}} \partial_{\alpha'}$$

$$D_{\alpha''} = \frac{1}{Z_{\alpha''}} \partial_{\alpha''}$$

$$|D_{\alpha'}| = \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'}$$

$$\mathbb{H} = \text{Hilbert transform} = \text{Fourier multiplier with symbol } - \text{sgn}(\xi)$$

$$\mathbb{H} f(\alpha') = \frac{1}{i \pi} \text{p.v.} \int \frac{1}{\alpha' - \beta'} f(\beta') d\beta'$$

$$\mathbb{P}_H = \text{Holomorphic projection} = \frac{1 + \mathbb{H}}{2}$$

$$\mathbb{P}_A = \text{Antiholomorphic projection} = \frac{1 - \mathbb{H}}{2}$$

$$|\partial_{\alpha'}| = i \mathbb{P}_{A, \alpha'} = \sqrt{-\Delta} = \text{Fourier multiplier with symbol } |\xi|$$

$$|\partial_{\alpha'}|^{1/2} = \text{Fourier multiplier with symbol } |\xi|^{1/2}$$

Now we have $D_t Z = Z_t$ and $D_t Z_t = Z_{tt}$ and more generally $D_t (f(\cdot, t) \circ h^{-1}) = (\partial_t f(\cdot, t) \circ h^{-1}$ or equivalently $\partial_t (F(\cdot, t) \circ h) = (D_t F(\cdot, t)) \circ h$. This means that $D_t = U_h^{-1} \partial_t U_h$ i.e. $D_t$ is the material
derivative in conformal coordinates. We also define a few more variables related to the interface:

\[ g = \text{Im}(\log(Z,_{\alpha'})) \quad \text{Hence} \quad \left| D_{\alpha'} \right| g = -iD_{\alpha'} \frac{Z,_{\alpha'}}{|Z,_{\alpha'}|} \]

\[ \Theta = (I + H)|D_{\alpha'}|g = -i(I + H)D_{\alpha'} \frac{Z,_{\alpha'}}{|Z,_{\alpha'}|} \] \hspace{1cm} (7)

\[ \omega = e^{ig} = \frac{Z,_{\alpha'}}{|Z,_{\alpha'}|} \quad \text{Hence} \quad \left| D_{\alpha'} \right| \omega = i\omega \text{Re}\Theta \]

Observe that \( g \) is the angle the interface makes with the x-axis in conformal coordinates, i.e. \( g = \theta \circ h^{-1} \) where \( \frac{i}{|z|} = e^{ig} \). Hence \( \text{Re}\Theta = \kappa \circ h^{-1} \) where \( \kappa \) is the curvature of the interface.

### 2.2. The system

To solve the system (1), (2) in [1] we obtained a system for the variables \((Z, Z_t)\) which we then solve. The system is as follows:

\[ b = \text{Re}(I - H) \left( \frac{Z_t}{Z,_{\alpha'}} \right) \]

\[ A_1 = 1 - \text{Im}[Z_t, H]Z,_{t,\alpha'} \]

\[ (\partial_t + b\partial_{\alpha'})Z,_{\alpha'} = Z,_{t,\alpha'} - b_{\alpha'}Z,_{\alpha'} \] \hspace{1cm} (8)

\[ (\partial_t + b\partial_{\alpha'})\vec{Z} = i - \frac{A_1}{Z,_{\alpha'}} + \frac{\sigma}{Z,_{\alpha'}} \partial_{\alpha'}(I + H)\left\{ \text{Im} \left( \frac{1}{Z,_{\alpha'}} \partial_{\alpha'}Z,_{\alpha'} \right) \right\} \]

along with the condition that their harmonic extensions, namely \( \Psi_{z'}(\cdot + iy) = K_{-y} * Z,_{\alpha'} \) and \( U(\cdot + iy) = K_{-y} * \vec{Z}_t \) for all \( y < 0 \), are holomorphic functions on \( P_- \) and satisfy \(^2\) \( \lim_{c \to \infty} \sup_{|z'| \geq c} \{|\Psi_{z'}(z')| - 1| + |U(z')|\} = 0 \) and \( \Psi_{z'}(z') \neq 0 \) for all \( z' \in P_- \).

After solving the above system one can obtain \( Z(\cdot, t) \) by the formula

\[ Z(\alpha', t) = Z(\alpha', 0) + \int_0^t \{ Z_t(\alpha', s) - b(\alpha', s)Z,_{\alpha'}(\alpha', s) \} \, ds \]

and hence \( (\partial_t + b\partial_{\alpha'})Z = D_tZ = Z_t \). Hence one can view the system being in variables \((Z, Z_t)\) instead of the variables \((Z,_{\alpha'}, Z_t)\).

We observe that the above system allows self intersecting interfaces. However if the interface is self-intersecting then it becomes nonphysical and so its relation to the Euler equation (1), (2) is lost. See [1] for more details.

From the calculation in [22] we have \( A_1 \geq 1 \). Now to get the function \( h(\alpha, t) \), we solve the ODE

\[ \frac{dh}{dt} = b(h, t) \]

\[ h(\alpha, 0) = \alpha \] \hspace{1cm} (9)

Observe that as long as \( \sup_{[0,T]} \|b(\alpha, t)\|_{\infty} < \infty \) we can solve this ODE uniquely and for any \( t \in [0, T] \) we have that \( h(\cdot, t) \) is a homeomorphism. Hence it makes sense to talk about the functions

\(^1\)Here \( K_{-y} \) is the Poisson kernel \([3]\)

\(^2\)We observe that for such a \( \Psi_{z'} \) we can uniquely define \( \log(\Psi_{z'}) : P_- \to \mathbb{C} \) such that \( \log(\Psi_{z'}) \) is a continuous function with \( \Psi_{z'} = \exp(\log(\Psi_{z'})) \) and \( (\log(\Psi_{z'}))(z') \to 0 \) as \( z' \to \infty \).
\[ Z_t - i = -\frac{A_1}{Z_{\alpha'}} + \sigma D_{\alpha'} \Theta \quad (10) \]

2.3. Previous result

Let us now describe the main result of [1]. For \( \sigma \geq 0 \) define the energy

\[ \mathcal{E}_{\sigma,1} = \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2^2 + \left\| \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{H^{\frac{3}{2}}}^2 + \left\| \sigma \partial_{\alpha'} \Theta \right\|_{H^{\frac{3}{2}}}^2 + \left\| \sigma Z_{\alpha'}^{\frac{5}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{2}^2 + \left\| \sigma Z_{\alpha'}^{\frac{5}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{\infty}^2 \\
= \left\| \frac{Z_{\alpha'}}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{2}^2 + \left\| \frac{Z_{\alpha'}}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{H^{\frac{3}{2}}}^2 + \left\| \sigma \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{2}^2 + \left\| \sigma \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{H^{\frac{3}{2}}}^2 \\
\mathcal{E}_{\sigma,2} = \left\| Z_{l,\alpha'} \right\|_2^2 + \left\| \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{2}^2 + \left\| \frac{Z_{\alpha'}}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{2}^2 + \left\| \frac{Z_{\alpha'}}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{2}^2 \\
\mathcal{E}_{\sigma} = \mathcal{E}_{\sigma,1} + \mathcal{E}_{\sigma,2} \]

**Theorem 2.1** ([1]). Let \( \sigma > 0 \) and assume the initial data \((Z, Z_t)(0)\) satisfies \((Z_{\alpha'} - 1, \frac{1}{Z_{\alpha'}} - 1, Z_t)(0) \in H^{3.5}(\mathbb{R}) \times H^{3.5}(\mathbb{R}) \times H^3(\mathbb{R})\). Then \( \mathcal{E}_\sigma(0) < \infty \) and there exists \( T, C_1 > 0 \) depending only on \( \mathcal{E}_\sigma(0) \) such that the initial value problem to (8) has a unique solution \((Z, Z_t)(t)\) in the time interval \([0, T]\) satisfying \((Z_{\alpha'} - 1, \frac{1}{Z_{\alpha'}} - 1, Z_t) \in C^l([0, T], H^{3.5-\frac{5}{2}}(\mathbb{R}) \times H^{3.5-\frac{5}{2}}(\mathbb{R}) \times H^{3-\frac{5}{2}}(\mathbb{R}))\) for \( l = 0, 1 \) and \( \sup_{t \in [0, T]} \mathcal{E}_\sigma(t) \leq C_1 < \infty \)

The main feature of this existence result is that the time of existence depends only on \( \mathcal{E}_\sigma(0) \) and not on the \( H^s \) norms of the initial data. The energy \( \mathcal{E}_\sigma(t) \) can be viewed as a weighted Sobolev norms of \((Z, Z_t)\) with the weight being powers of \( \frac{1}{Z_{\alpha'}} \) along with appropriate powers of \( \sigma \). The energy \( \mathcal{E}_\sigma(t) \) has several interesting properties such as:

1. For \( \sigma = 0 \) the energy \( \mathcal{E}_\sigma(t) \) reduces to a lower order version of the energy of Kinsey and Wu [22]. In particular it allows singular interfaces such as interfaces with angled crests and cusps (see [2]).
2. For \( \sigma > 0 \) the energy \( \mathcal{E}_\sigma(t) \) does not allow any singularities in the interface. In particular it does not allow angled crested interfaces.
3. For \( \sigma > 0 \) even though the energy \( \mathcal{E}_\sigma(t) \) does not allow singularities in the interface, it does allow interfaces with large curvature. It allows the \( L^\infty \) norm of the curvature of the initial interface to be as large as \( \sigma^{-\frac{5}{2} + \epsilon} \) for any \( \epsilon > 0 \). In particular for \( \sigma \) small, the energy allows interfaces with very large curvature.
4. Note that in the statement of the theorem, there are no assumptions on the Taylor sign condition. Also the energy \( \mathcal{E}_\sigma(t) \) is an increasing function of \( \sigma \) and hence for initial data in appropriate Sobolev spaces, this result implies a uniform time of existence of solutions for \( 0 \leq \sigma \leq \sigma_0 \) for arbitrary \( \sigma_0 > 0 \), thereby recovering the uniform time of existence result of Ambrose and Masmoudi [10] in this case.

These properties are discussed in more detail in [1]. One of the more interesting consequences of this theorem is that it can be used to prove the existence of solutions to (8) with the initial interface close to being angled crested. Let us now explain this result.
As before let \((\Psi, U)(\cdot, 0) : P_\sim \to \C\) be holomorphic maps with \(\Psi \not\equiv 0\) for all \(z \in P_\sim\) and with their boundary values being the initial data namely \((Z, Z)(\alpha', 0) = (\Psi, U)(\alpha', 0)\) for all \(\alpha' \in \R\). Recall the notation namely that for \(z' \in P_\sim\) we write \(z' = x' + iy'\). At \(t = 0\) define the quantity

\[
M = \sup_{y' < 0} \left| \Psi_z \frac{\partial}{\partial x'} \left( \frac{1}{\Psi_{z'}} \right) \right|_{L^2} \left( R, dx' \right) + \sup_{y' < 0} \left| \Psi_z \frac{\partial}{\partial y'} \left( \frac{1}{\Psi_{z'}} \right) \right|_{L^2} \left( R, dx' \right) + \sup_{y' < 0} \left| \frac{\partial}{\partial y'} \left( \frac{1}{\Psi_{z'}} \right) \right|_{L^2} \left( R, dx' \right)
\]

It is easy to check that if \(M < \infty\), then the initial data satisfies the hypothesis of Theorem 3.9 of Wu [36] and we get a unique solution \((Z, Z_\sim)(t)\) to [8] for \(\sigma = 0\). Also by exactly the same argument as in section 5 of [2], \(M < \infty\) allows interfaces with angled crests of angles \(\nu \pi\) with \(0 < \nu < \frac{1}{2}\) and also allows certain cusps (see [2] for more details). With this we can now state the main corollary of this theorem.

**Corollary 2.2** (1). Consider an initial data \((Z, Z_\sim)(0)\) with \(M < \infty\). Let \((Z, Z_\sim)(t)\) be the unique solution of equation (8) for \(\sigma = 0\) with initial data \((Z, Z_\sim)(0)\) as obtained in [36]. For \(0 < c \leq 1\) and \(\sigma \geq 0\) denote by \((Z^{c, \sigma}, Z^{c, \sigma}_\sim)(t)\) the unique solution to the equation (8) with surface tension \(\sigma\) and with initial data \((Z^{c, \sigma}, Z^{c, \sigma}_\sim)(0) = (Z \ast P_\sim, Z \ast P_\sim)(0)\) where \(P_\sim\) is the Poisson kernel. Then we have the following

1. For any \(c > 0\), \(c_1 > 0\) depending only on \(c\) and \(M\) such that for all \(\sigma \geq 0\) and \(0 < \epsilon \leq 1\) satisfying \(\frac{\sigma}{\epsilon^2} \leq c\), the solutions \((Z^{c, \sigma}, Z^{c, \sigma}_\sim)(t)\) exist in the time interval \([0, T]\)

\[
\text{with } \sup_{t \in [0, T]} \mathcal{E}_\sigma(Z^{c, \sigma}, Z^{c, \sigma}_\sim)(t) \leq c_1 < \infty.
\]

2. If the initial interface \(Z(\cdot, 0)\) has only one angled crest of angle \(\nu \pi\) with \(0 < \nu < \frac{1}{2}\), then the \(L^\infty\) norm of the curvature \(\kappa^{c, \sigma}\) of the initial interface \(Z^{c, \sigma}(\cdot, 0)\) satisfies \(\|\kappa^{c, \sigma}\|_{\infty} \sim \epsilon^{-\nu}\) as \(\epsilon \to 0\). In particular for any \(0 < c_1 < 1\) arbitrarily small, choosing \(\nu = 1 - \frac{1}{2} \delta^2\) and \(\sigma = \epsilon^{\frac{2}{\nu}}\), we obtain \(\|\kappa^{c, \sigma}\|_{\infty} \sim \sigma^{-\frac{1}{2} + \delta}\) as \(\sigma \to 0\). Hence Theorem 2.1 allows initial interfaces with large curvature when \(\sigma\) is small.

See Figure [1] for a comparison between the interfaces \(Z^{c, \sigma}(\cdot, t)\) and \(Z(\cdot, t)\). The interesting thing about this result is that under the assumption of \(\sigma \leq \epsilon^{\frac{2}{\nu}}\), it proves the existence of the solutions \((Z^{c, \sigma}, Z^{c, \sigma}_\sim)(t)\) on a uniform time interval \([0, T]\). Generally if one uses standard energy estimates (as compared with the energy \(\mathcal{E}_\sigma(t)\)), one gets an upper bound on the time of existence as \(T \sim \|\kappa\|_{-1}^{-2}\) where \(\kappa\) is the initial curvature. As one can see, the initial interface of \(Z^{c, \sigma}(\cdot, 0)\) has a very large curvature for \(\epsilon\) small, and so one cannot obtain a result such as the above corollary by standard energy estimates and one has to use weighted energy estimates as done in Theorem 2.1.

The scaling factor \(\sigma/\epsilon^{\frac{2}{\nu}}\) comes from the scaling of the equation and we refer to the introduction and Remark 3.4 of [1] for more details.
3. Main results and discussion

3.1. Results

We now explain our results about convergence. Let \((Z, Z_t)_a\) and \((Z, Z_t)_b\) be two solutions of the water wave equation \((8)\) with surface tensions \(\sigma_a\) and \(\sigma_b\) respectively. We denote the two solutions as \(A\) and \(B\) respectively for simplicity. We will denote the terms and operators for each solution by their subscript \(a\) or \(b\). For example \((\partial_t Z_{t,\alpha'})_a\) and \((\partial_t Z_{t,\alpha'})_b\) denotes the spacial derivative of the velocity for the respective solutions. Similarly we also have the operators

\[
(\|D_{\alpha'}\|)_a = \frac{1}{|Z_{t,\alpha'}|} \partial_{\alpha'} \quad (\|D_{\alpha'}\|)_b = \frac{1}{|Z_{t,\alpha'}|} \partial_{\alpha'} \quad \text{etc.}
\]

Let \(h_a, h_b\) be the homeomorphisms from \((\Sigma)\) for the respective solutions and let the material derivatives by given by \((D_t)_a = U_{h_a}^{-1} \partial_t U_{h_a}\) and \((D_t)_b = U_{h_b}^{-1} \partial_t U_{h_b}\). We define

\[
\tilde{h} = h_b \circ h_a^{-1} \quad \text{and} \quad \tilde{U} = U_b^{-1} U_{h_a}
\]

While taking the difference of the two solutions, we will subtract in Lagrangian coordinates and then bring it to the Riemmanian coordinate system of \(A\). The reason we want to subtract in the Lagrangian coordinate system is that in our proof of the energy estimate we mainly use the material derivative, and in the Lagrangian coordinate system the material derivative for both the solutions is given by the same operator \(\partial_t\) and subtracting in Lagrangian coordinates helps us avoid a loss of derivatives. The operator \(\tilde{U}\) takes a function in the Riemmanian coordinate system of \(B\) to the Riemmanian coordinate system of \(A\). We define

\[
\Delta(f) = f_a - \tilde{U}(f_b)
\]

For example \(\Delta(\tilde{Z}_{t,\alpha'}) = (\tilde{Z}_{t,\alpha'})_a - \tilde{U}(\tilde{Z}_{t,\alpha'})_b\), where we have written \(\tilde{U}(f)_b\) instead of \(U(f)_b\) for easier readability for the term \(\tilde{U}(\tilde{Z}_{t,\alpha'})_b\). This notation allows us subtract the corresponding quantities of the two solutions in the correct manner, while still using conformal coordinates.

To describe our main result Theorem \((T)\) on the zero surface tension limit, we let \((Z, Z_t)_a\) be the solution with surface tension \(\sigma\) and let \((Z, Z_t)_b\) denote the solution with zero surface tension. We want to show that

\[
(Z, Z_t)_a \to (Z, Z_t)_b \quad \text{as} \quad \sigma \to 0
\]

in a suitable norm. Note that with this notation equation \((\Theta)\) becomes

\[
(\tilde{Z}_{tt})_a - i = -i \left( \frac{A_1}{Z_{t,\alpha'}} \right)_a + \sigma(D_{\alpha'}\Theta)_a \quad \text{where as} \quad (\tilde{Z}_{tt})_b - i = -i \left( \frac{A_1}{Z_{t,\alpha'}} \right)_b
\]

and we have

\[
\Delta(\tilde{Z}_{tt}) = -i \left\{ \left( \frac{A_1}{Z_{t,\alpha'}} \right)_a - \tilde{U} \left( \frac{A_1}{Z_{t,\alpha'}} \right)_b \right\} + \sigma(D_{\alpha'}\Theta)_a = -i \Delta \left( \frac{A_1}{Z_{t,\alpha'}} \right)_a + \sigma(D_{\alpha'}\Theta)_a
\]

To state our convergence result, we first need to define a few more norms. We define the higher order energy for zero surface tension solutions as

\[
\mathcal{E}_{\text{high}} = \left\| \partial_{\alpha'} \frac{1}{Z_{t,\alpha'}} \right\|_2^2 + \left\| \frac{1}{Z_{t,\alpha'}} \partial_{\alpha'}^2 \frac{1}{Z_{t,\alpha'}} \right\|_2^2 + \left\| \frac{1}{Z_{t,\alpha'}} \partial_{\alpha'} Z_{t,\alpha'} \right\|_2^2 + \left\| \frac{1}{Z_{t,\alpha'}} \partial_{\alpha'} Z_{t,\alpha'} \right\|_2^2 + \left\| \frac{1}{Z_{t,\alpha'}} \partial_{\alpha'} Z_{t,\alpha'} \right\|_2^2
\]

\[\tag{12}\]
The energy $\mathcal{E}_{\text{high}}$ is used in Kinsey-Wu [22] and Wu [30] to prove local wellposedness for angled crested water waves. Note that this energy is half spacial derivative higher order than the energy $\mathcal{E}_\sigma|_{\sigma=0}$. We also define an auxiliary energy for the zero surface tension solution

$$\mathcal{E}_{\text{aux}} = \left\| Z_{\alpha'} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_\infty^2 + \left\| \frac{1}{Z_{\alpha'}} \partial_{\alpha'} Z_{\alpha'} \right\|_2^2 + \left\| \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2^2$$

Let us now understand this norm. Observe that the terms of $E$ used in [22] and [36] also have the lower order term

$$\mathcal{E}_{\text{aux}} = \frac{1}{Z_{\alpha'}} \partial_{\alpha'} Z_{\alpha'}$$

This energy will be key in proving the convergence result. We can now define the norm in which we prove convergence. Define the energy $\mathcal{E}_\Delta$ as

$$\mathcal{E}_{\Delta,1} = \left\| \Delta \left( \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\|_2^2 + \left\| \Delta \left( \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\|_H^2 + \left\| \left( \sigma^2 \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\|_2^6$$

$$\mathcal{E}_{\Delta,2} = \left\| \Delta \left( Z_{t,\alpha'} \right) \right\|_2^2 + \left\| \Delta \left( \frac{1}{Z_{t,\alpha'}} \partial_{\alpha'} Z_{t,\alpha'} \right) \right\|_2^2 + \left\| \left( \sigma^2 \partial_{\alpha'} \frac{1}{Z_{t,\alpha'}} \right) \right\|_2^2$$

$$\mathcal{E}_\Delta = \mathcal{E}_{\Delta,1} + \mathcal{E}_{\Delta,2} + \sigma(\mathcal{E}_{\text{aux}})_b$$

Let us now understand this norm. Observe that the terms of $\mathcal{E}_{\Delta,1}$ and $\mathcal{E}_{\Delta,2}$ are obtained by taking the difference of the terms of the energy $\mathcal{E}_\sigma$ defined in [22,30] for the solutions $A$ and $B$. The term $\sigma(\mathcal{E}_{\text{aux}})_b$ is a coupling term as it couples the energy $(\mathcal{E}_{\text{aux}})_b$ of the zero surface tension solution $B$ with the surface tension coefficient $\sigma$ of the solution $A$. This term is crucial to close the energy estimate for $\mathcal{E}_\Delta$ at the highest order. The reason for the need to include this term and why it works is explained in more detail in [33].

We now state our main result on convergence. We will state the theorem here only for the special case of the two solutions having the same initial data and a more general result is stated in [47]. The existence part of our result follows from earlier results: For $\sigma = 0$, one can use the existence result Theorem 3.9 of [30], where a wellposedness result is proved in terms of the energy $\mathcal{E}_{\text{high}}$ $^3$. For $\sigma > 0$ we can use Theorem 2.1 for an existence result in terms of the energy $\mathcal{E}_\sigma(t)$. The main result of this paper is as follows:

**Theorem 3.1.** Let $\sigma > 0$ and let $(Z, Z_t)(0)$ be an initial data such that $(Z_{\alpha'} - 1, \frac{1}{Z_{\alpha'}} - 1, Z_t)(0) \in H^{3.5}(\mathbb{R}) \times H^{3.5}(\mathbb{R}) \times H^{3.5}(\mathbb{R})$. Let $L > 0$ be such that

$$\mathcal{E}_{\text{high}}(0), \mathcal{E}_\sigma(0) \leq L$$

---

$^3$The energy used in [22] and [30] also has the lower order term $\left\| \frac{1}{Z_{\alpha'}} \right\|_\infty^2$ or $\left\| \frac{1}{Z_{\alpha'}} \right\|_2^2$ which we don’t have.

$^4$In Theorem 3.9 of [30], the time of existence also depends on the lower order term $\left\| \frac{1}{Z_{\alpha'}} \right\|_\infty^2$. In our case this can be avoided as we show in Theorem 3.1 and the proof of Theorem 3.1.
Then there exists \( T, C_0, C_1, C_2 > 0 \) depending only on \( L \) so that we have a unique solution \((Z, Z_t)(t)\) to (8) with zero surface tension with initial data \((Z, Z_t)(0)\) in time interval \([0, T]\) satisfying \((Z_{\alpha'} - 1, \frac{1}{Z_{\alpha'}}, 1, Z_t) \in C^l([0, T], H^{3-\frac{3}{2}}(\mathbb{R}) \times H^{3-\frac{3}{2}}(\mathbb{R}) \times H^{3.5-\frac{3}{2}}(\mathbb{R}))\) for \( l = 0, 1 \) along with the estimate sup\(_{t \in [0, T]} \| E(high)(Z, Z_t)(t) \| \leq C_2 \), and we have a unique solution \((Z^\sigma, Z^\sigma_t)(t)\) to (8) with surface tension \( \sigma \) with the same initial data \((Z, Z_t)(0)\) in the time interval \([0, T]\) satisfying \((Z^\sigma_{\alpha'} - 1, \frac{1}{Z^\sigma_{\alpha'}}, 1, Z^\sigma_t) \in C^l([0, T], H^{3.5-\frac{3}{2}}(\mathbb{R}) \times H^{3-\frac{3}{2}}(\mathbb{R}) \times H^{3.5-\frac{3}{2}}(\mathbb{R}))\) for \( l = 0, 1 \) and sup\(_{t \in [0, T]} \| E(\sigma)(Z^\sigma, Z^\sigma_t)(t) \| \leq C_2 \), and we have the estimate

\[
\sup_{t \in [0, T]} E_\Delta(Z^\sigma, Z)(t) \leq C_1 e^{C_0 T} E_\Delta(Z^\sigma, Z)(0)
\]

In the above result \( E_\Delta(Z^\sigma, Z)(t) \) is the energy \( E_\Delta(t) \) where \((Z^\sigma, Z^\sigma_t)(t)\) is the solution \( A \) and \((Z, Z_t)(t)\) is the solution \( B \). From the energy \( E_\Delta \) defined above we observe that if the initial data for the two solutions is then same, then \( E_\Delta(Z^\sigma, Z)(0) \leq \sigma \to 0 \) as \( \sigma \to 0 \), and hence sup\(_{t \in [0, T]} \| E_\Delta(Z^\sigma, Z)(t) \| \to 0 \) as \( \sigma \to 0 \). This result should be contrasted with the result of Ambrose-Masmoudi [10] where the convergence is proved in Sobolev spaces. Note that the theorem above implies convergence in Sobolev spaces so we recover the result of [10]. The novelty of the above convergence result is that the rate of growth of the energy \( E_\Delta \) does not depend on the Sobolev norms of the initial data (as is the case in [10]) but only on the energies \( E_{\text{high}} \) and \( E_\sigma \), as the constant \( C_0 \) appearing in the estimate for \( E_\Delta \), namely

\[
\sup_{t \in [0, T]} E_\Delta(Z^\sigma, Z)(t) \leq C_1 e^{C_0 T} E_\Delta(Z^\sigma, Z)(0)
\]

depends only on \( L \) which in turn depends only on \( E_{\text{high}}(0), E_\sigma(0) \). If the initial interface is close to being an angled crest, then \( E_{\text{high}} \) and \( E_\sigma \) remain bounded \(^5\) where as the \( C^{1,\alpha} \) norm (for any \( 0 < \alpha \leq 1 \)) of the interface \( Z \) blows up as the interface gets closer to being angled crested. Hence this result allows us to control the difference of the solutions independent of how close the initial interface is to an angled crest interface. It is also worthwhile to note that in the proof we show that the energy \( E_\Delta \) is quite strong, and in particular it directly controls \( \| \theta^\sigma - \theta \|_2^\infty \) and hence we have the angle of the interface \( \theta^\sigma \to \theta \) in \( L^\infty \) as \( \sigma \to 0 \). Hence the approximation between the solutions with non-zero surface tension and zero surface tension is quite strong.

We prove this theorem in [8]. To prove this theorem, we first prove an apriori energy estimate for the energy \( E_{\text{aux}} \), by proving that as long as \( E_{\text{high}} \) is controlled, then \( E_{\text{aux}} \) is also controlled. This is proved in [8]. Then using this energy estimate we prove an apriori energy estimate for \( E_\Delta \) in [7]. We then use both of these energy estimates to finish the proof in [8].

Let us now apply this result explicitly to angled crested interfaces. For two solutions \( A \) and \( B \) of water waves, we define the following norm

\[
\mathcal{F}_\Delta = \| (z_t)_a - (z_t)_b \|_{H^\frac{3}{2}} + \| (ztt)_a - (ztt)_b \|_{H^\frac{3}{2}} + \left\| \frac{h_a}{z_a} \right\|_{H^\frac{3}{2}} + \left\| \frac{h_b}{z_b} \right\|_{H^\frac{3}{2}} + \left\| (h_a)_a - (h_a)_b \right\|_2 + \left\| (h_b)_a - (h_b)_b \right\|_2
\]

\[\tag{14}\]

\(^5\) \( E_\sigma(t) \) remains bounded provided the surface tension is small enough depending on how close it is an angled crest interface. See Corollary [7,2].
This norm was introduced by Wu in Theorem 3.7 of [36] to establish uniqueness of angled crested water waves without surface tension. Let us now consider Corollary 2.2 and let solution $A$ be $(Z_{t}^{\epsilon,\sigma}, Z_{t}^{\epsilon,\sigma})(t)$ and solution $B$ be $(Z, Z_{t})(t)$. We will use the above norm $\mathcal{F}_{\Delta}$ to establish convergence for $(Z_{t}^{\epsilon,\sigma}, Z_{t}^{\epsilon,\sigma})(t) \rightarrow (Z, Z_{t})(t)$ as $\epsilon, \sigma \rightarrow 0$ under suitable conditions.

\textbf{Corollary 3.2.} Consider an initial data $(Z, Z_{t})(0)$ with $M < \infty$ where $M$ is defined in \footnote{See \cite{AG} for an equivalent form of $\mathcal{F}_{\Delta}$ written in the Riemannian coordinate system of solution $A$, as was written in \cite{CM}. Also instead of the term $\|(h_{\alpha})_{u} - (h_{\alpha})_{u}\|_{2}$ above, in \cite{AG} the term which shows up is $\left\| \frac{(h_{\alpha})_{u}}{(h_{\alpha})_{u} - 1} \right\|_{2}$. Both of these are equivalent as $h_{\alpha}$ has a lower and upper bound as long as $\mathcal{E}_{\sigma}(t)$ remains bounded. See the paragraph following \cite{CM}.} and fix $c > 0$. Let $(Z, Z_{t})(t)$ be the unique solution of equation \footnote{See \cite{AG} for an equivalent form of $\mathcal{F}_{\Delta}$ written in the Riemannian coordinate system of solution $A$, as was written in \cite{CM}. Also instead of the term $\|(h_{\alpha})_{u} - (h_{\alpha})_{u}\|_{2}$ above, in \cite{AG} the term which shows up is $\left\| \frac{(h_{\alpha})_{u}}{(h_{\alpha})_{u} - 1} \right\|_{2}$. Both of these are equivalent as $h_{\alpha}$ has a lower and upper bound as long as $\mathcal{E}_{\sigma}(t)$ remains bounded. See the paragraph following \cite{CM}.}. Let us now consider Corollary 2.2 and let solution $A$ be $(Z_{t}^{\epsilon,\sigma}, Z_{t}^{\epsilon,\sigma})(t)$ and solution $B$ be $(Z, Z_{t})(t)$. We will use the above norm $\mathcal{F}_{\Delta}$ to establish convergence for $(Z_{t}^{\epsilon,\sigma}, Z_{t}^{\epsilon,\sigma})(t) \rightarrow (Z, Z_{t})(t)$ as $\epsilon, \sigma \rightarrow 0$ under suitable conditions.

\begin{enumerate}
\item[(1)] $\sup_{t \in [0,T_{2}]} \mathcal{E}_{\Delta}(Z_{t}^{\epsilon,\sigma}, Z_{t}^{\epsilon,\sigma})(t) \leq \frac{\sigma^{2}}{\epsilon^{2}} t_{\epsilon}^{2} C_{1} e^{C_{2} T_{2}}$
\item[(2)] $\sup_{t \in [0,T]} \mathcal{F}_{\Delta}(Z_{t}^{\epsilon,\sigma}, Z_{t})(t) \rightarrow 0$ as $\max\left\{ \frac{\sigma^{2}}{\epsilon^{2}}, \epsilon \right\} \rightarrow 0$, where $T = \min\{T_{1}, T_{2}\}$.
\end{enumerate}

In the above result $\mathcal{E}_{\Delta}(Z_{t}^{\epsilon,\sigma}, Z_{t}^{\epsilon,\sigma})(t)$ is the energy $\mathcal{E}_{\Delta}(t)$ for the two solutions $(Z_{t}^{\epsilon,\sigma}, Z_{t}^{\epsilon,\sigma})(t)$ and $(Z_{t}, Z_{t})(t)$ and similarly $\mathcal{F}_{\Delta}(Z_{t}^{\epsilon,\sigma}, Z_{t})(t)$ is the energy $\mathcal{F}_{\Delta}(t)$ for the two solutions $(Z_{t}^{\epsilon,\sigma}, Z_{t}^{\epsilon,\sigma})(t)$ and $(Z, Z_{t})(t)$. As explained in \cite{CM} $M < \infty$ allows interfaces with angled crests of angles $\nu \pi$ with $0 < \nu < \frac{1}{2}$ and also allows certain cusps, and so $(Z, Z_{t})(0)$ is allowed to be singular. As explained previously the existence part of this result has already been proved in Corollary 2.2 and the novelty here is the convergence aspect. The first part of the convergence result says that as long as $\sigma \lesssim \epsilon^{2}$, then the difference of the solution with surface tension $(Z_{t}^{\epsilon,\sigma}, Z_{t}^{\epsilon,\sigma})(t)$ to the solution without surface tension $(Z_{t}, Z_{t})(t)$ does not depend on how close the initial interface is to being angled crested (i.e. independent of $\epsilon$). The second part shows that the smooth solutions $(Z_{t}^{\epsilon,\sigma}, Z_{t}^{\epsilon,\sigma})(t)$ to \footnote{See \cite{AG} for an equivalent form of $\mathcal{F}_{\Delta}$ written in the Riemannian coordinate system of solution $A$, as was written in \cite{CM}. Also instead of the term $\|(h_{\alpha})_{u} - (h_{\alpha})_{u}\|_{2}$ above, in \cite{AG} the term which shows up is $\left\| \frac{(h_{\alpha})_{u}}{(h_{\alpha})_{u} - 1} \right\|_{2}$. Both of these are equivalent as $h_{\alpha}$ has a lower and upper bound as long as $\mathcal{E}_{\sigma}(t)$ remains bounded. See the paragraph following \cite{CM}.} with surface tension $\sigma$ converge to the singular solution $(Z, Z_{t})(t)$ of \footnote{See \cite{AG} for an equivalent form of $\mathcal{F}_{\Delta}$ written in the Riemannian coordinate system of solution $A$, as was written in \cite{CM}. Also instead of the term $\|(h_{\alpha})_{u} - (h_{\alpha})_{u}\|_{2}$ above, in \cite{AG} the term which shows up is $\left\| \frac{(h_{\alpha})_{u}}{(h_{\alpha})_{u} - 1} \right\|_{2}$. Both of these are equivalent as $h_{\alpha}$ has a lower and upper bound as long as $\mathcal{E}_{\sigma}(t)$ remains bounded. See the paragraph following \cite{CM}.} with zero surface tension, and the convergence happens in the norm $\mathcal{F}_{\Delta}$.

![Figure 1. Waves with and without surface tension](image-url)
This corollary is proved in §8. The first part of the convergence result follows directly from Theorem 3.1. The second part follows from the observation that the norm $F_\Delta$ is weaker than the norm $E_\Delta$ and hence we essentially have
\[
F_\Delta(Z^{e,\sigma}, Z)(t) \leq F_\Delta(Z^{e,\sigma}, Z')(t) + F_\Delta(Z', Z)(t) \\
\lesssim \{E_\Delta(Z^{e,\sigma}, Z')(t)\}^\alpha + F_\Delta(Z', Z)(t)
\]
for some $\alpha > 0$. Now the proof follows from using the fact that $E_\Delta(Z^{e,\sigma}, Z')(t) \to 0$ as $\max \left\{ \frac{\sigma}{\epsilon^2}, \epsilon \right\} \to 0$ from part (1) of the corollary, and $F_\Delta(Z', Z)(t) \to 0$ by Theorem 3.7 of [36]. See the proof in §8 for more details.

The norm $F_\Delta$ being weaker than the norm $E_\Delta$ has one consequence being that it can only show $\theta^{e,\sigma} \to \theta$ in $L^2$ instead of the convergence in $L^\infty$, which is what you would obtain if one could use the stronger norm $E_\Delta$. We do not know whether the convergence of $\theta^{e,\sigma} \to \theta$ can be proved in a stronger space and this is a problem we leave for future work.

### 3.2 Discussion

Let us first discuss how the standard energy estimate for convergence in Sobolev spaces generally works. Let $u_a$ and $u_b$ be solutions to the equations
\[
(\partial_t^2 + |\partial_\alpha'|^3 + \sigma |\partial_\alpha'|^3)u_a = N_a \quad (\partial_t^2 + |\partial_\alpha'|)u_b = N_b
\]
These are highly simplified versions of the water wave equation with surface tension $\sigma$ and with zero surface tension respectively. Now the corresponding energies for the solutions $u_a$ and $u_b$ are of the form
\[
(E_s)_a(t) = \|\partial_\alpha u_a\|_{H^\frac{1}{2}}^2 + \|\partial_\alpha^{s+1} u_a\|_2^2 + \|\alpha^{s+2} \partial_\alpha^{s+2} u_a\|_2^2 \quad (E_s)_b(t) = \|\partial_\alpha u_b\|_{H^\frac{1}{2}}^2 + \|\partial_\alpha^{s+1} u_b\|_2^2
\]
Hence to prove convergence at the level of $E_s$, we first subtract the equations to get
\[
(\partial_t^2 + |\partial_\alpha'|)(u_a - u_b) + \sigma |\partial_\alpha'|^3 u_a = N_a - N_b
\]
From which we get the energy
\[
E_{\Delta, s}(t) = \|\partial_\alpha u_a\|_{H^\frac{1}{2}}^2 + \|\partial_\alpha^{s+1} (u_a - u_b)\|_2^2 + \|\alpha^{s+2} \partial_\alpha^{s+2} u_a\|_2^2
\]
Differentiating this we see that
\[
\frac{d}{dt} E_{\Delta, s}(t) = 2\text{Re} \left\{ \int (|\partial_\alpha'| |\partial_\alpha u_a| (u_a - u_b) \partial_\alpha u_a) \right\} + 2\text{Re} \left\{ \int (\partial_t \partial_\alpha^{s+1} (u_a - u_b) \partial_\alpha^{s+1} u_a) \right\} + 2\text{Re} \left\{ \int (\partial_t \partial_\alpha^{s+2} u_a) (\sigma \partial_\alpha^{s+2} u_a) \right\}
\]
Now in the last integral we rewrite $\partial_t \partial_\alpha^{s+2} u_a = \partial_t \partial_\alpha^{s+2} (u_a - u_b) + \partial_t \partial_\alpha^{s+2} u_b$ and use the identities $|\partial_\alpha'| = i \mathbb{H} \partial_\alpha'$ and $\mathbb{H}^2 = 1$ to get
\[
\frac{d}{dt} E_{\Delta, s}(t) = 2\text{Re} \left\{ \int (|\partial_\alpha'| |\partial_\alpha u_a| (u_a - u_b) \partial_\alpha u_a) \right\} + 2\text{Re} \left\{ \int (\sigma^{s+2} \partial_\alpha^{s+2} u_b) (\sigma^{s+2} \partial_\alpha^{s+2} u_a) \right\}
\]
Now the first integral can be controlled by using the equation for the difference. To control the second integral, one can assume control of the energy \((E_{s+3/2})_b(t)\) to get control of \(\|\partial_t \partial_{s+2}^\ast u_b\|_2\), and so the second integral can be estimated via

\[
\text{Re} \left\{ \int (\sigma^{\frac{2}{s+2}} \partial_t \partial_{s+2}^\ast u_b)(\sigma^{\frac{2}{s+2}} \partial_{s+2}^\ast u_a) \right\} \lesssim \sigma^{\frac{2}{s+2}} \|\partial_t \partial_{s+2}^\ast u_b\|_2 \|\sigma^{\frac{2}{s+2}} \partial_{s+2}^\ast u_a\|_2 \to 0 \quad \text{as } \sigma \to 0 \tag{15}
\]

We follow essentially the same strategy as above except that at the last step this strategy fails. To see this, first note that the integral we are left to estimate by trying to control \(E_{\Delta,4}\) in Theorem 7.1 is an integral of the form

\[
2\text{Re} \int \hat{U} \left\{ \frac{\sigma^{\frac{2}{s+2}}}{|Z_{\alpha'}|^{\frac{2}{s+2}}} \partial_{\alpha'}|D_{\alpha'}|(D_t D_{\alpha'} Z_t) \right\}_a \left\{ \frac{\sigma^{\frac{2}{s+2}}}{|Z_{\alpha'}|^{\frac{2}{s+2}}} \partial_{\alpha'}|D_{\alpha'}|(D_t D_{\alpha'} Z_t) \right\}_b
\]

See (7.2.4) for the details. Now \(\left\{ \frac{1}{|Z_{\alpha'}|^{\frac{2}{s+2}}} \partial_{\alpha'}|D_{\alpha'}|(D_t D_{\alpha'} Z_t) \right\}_b \in L^2\) as it is part of the energy \(E_{\Delta,4}\) (This is analogous to the control of \(\|\sigma^{\frac{2}{s+2}} \partial_{s+2}^\ast u_a\|_2\) one has from \(E_{\Delta,4}\) in (15)). However we cannot assume control of \(\left\{ \frac{1}{|Z_{\alpha'}|^{\frac{2}{s+2}}} \partial_{\alpha'}|D_{\alpha'}|(D_t D_{\alpha'} Z_t) \right\}_b \in L^2\) in an analogous manner as was done in (15) (by assuming control of \(\|\partial_t \partial_{s+2}^\ast u_b\|_2\)), as heuristically if the zero surface tension solution \(B\) has an angle crest of angle \(\nu \pi\) then \(Z(\alpha') \sim (\alpha')^{\nu - \frac{1}{2}}\) near \(\alpha' = 0\), and hence using (10) we heuristically have

\[
\frac{1}{|Z_{\alpha'}|^{\frac{2}{s+2}}} \partial_{\alpha'}|D_{\alpha'}|(D_t D_{\alpha'} Z_t) \sim \frac{1}{Z_{\alpha'}^{\frac{3}{2}}} \partial_{\alpha'}^3 \sim (\alpha')^{-\frac{5}{2} + \frac{1}{2}}
\]

which does not belong to \(L^2\) for \(\frac{3}{2} \leq \nu < \frac{1}{2}\). Hence assuming control of this term will force us to severely restrict the initial data allowed in our results by imposing the restriction \(0 < \nu < \frac{3}{2}\). To overcome this difficulty, we observe that we do not really need to control

\[
\left\{ \frac{1}{|Z_{\alpha'}|^{\frac{2}{s+2}}} \partial_{\alpha'}|D_{\alpha'}|(D_t D_{\alpha'} Z_t) \right\}_b \in L^2,
\]

but what we really need to control is

\[
\left\{ \frac{1}{|Z_{\alpha'}|^{\frac{2}{s+2}}} \partial_{\alpha'}|D_{\alpha'}|(D_t D_{\alpha'} Z_t) \right\}_b \in L^2\] and ensure that this goes to zero as \(\sigma \to 0\). This is achieved by the term \(\sigma(\mathcal{E}_{\text{aux}})_b(t)\) which controls precisely terms of this type and ensures that it goes to zero as \(\sigma \to 0\). The energy \(\sigma(\mathcal{E}_{\text{aux}})_b(t)\) not only allows us to prove the apriori estimate for \(\mathcal{E}_{\Delta}\) in Theorem 3.1 but moreover ensures that the scaling \(\sigma^{\frac{2}{s+2}}\) obtained originally in Corollary 2.2 as part of (1) remains the same in Corollary 3.2. This shows that the energy \(\sigma(\mathcal{E}_{\text{aux}})_b(t)\) scales in the correct manner and does not weaken the statement of our results in Theorem 3.1 or Corollary 3.2.

4. Identities and equations from previous work

In this section we collect the identities and estimates proved in (1) which we use in this paper. We are collecting them essentially all in this section as this is easier than to constantly refer to the paper (1) when we prove our energy estimates in later sections.

4.1. Main identities

Here we first collect the main identities commonly used in this paper. See Section 4 of (1) for a proof of these identities.
a) We have
\[ \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \frac{1}{Z_{\alpha'}} = \frac{1}{Z_{\alpha'}} + \omega|D_{\alpha'}| \mathbb{D} \]
Observe that \( \frac{1}{Z_{\alpha'}} \) is real valued and \( \omega|D_{\alpha'}| \mathbb{D} \) is purely imaginary. From this we obtain
\[ \text{Re} \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \frac{1}{Z_{\alpha'}} \right) = \frac{1}{Z_{\alpha'}} \quad \text{and} \quad \text{Im} \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \frac{1}{Z_{\alpha'}} \right) = i(\mathbb{D}D_{\alpha'}|\omega|) = -\text{Re}\Theta \quad (16) \]

b) We have
\[ D_{t}g = -\text{Im}(\overline{D}_{\alpha'}Z_{t}) \quad (17) \]

c) For any complex valued function \( f \), we have \( \mathbb{H}(\text{Re}) = i\text{Im}(\mathbb{H}f) \) and \( \mathbb{H}(i\text{Im}) = \text{Re}(\mathbb{H}f) \). Hence we get the following identities
\[ (\mathbb{I} + \mathbb{H})(\text{Re}) = f - i\text{Im}(\mathbb{I} - \mathbb{H})f \quad (18) \]
\[ (\mathbb{I} + \mathbb{H})(i\text{Im}) = f - \text{Re}(\mathbb{I} - \mathbb{H})f \quad (19) \]

d) We have
\[ A_{1} = 1 + iZ_{t}Z_{t,\alpha'} - i(\mathbb{I} + \mathbb{H})\{\text{Re}(Z_{t}\overline{Z}_{t,\alpha'})\} \quad (20) \]

e) We have
\[ b = \frac{Z_{t}}{Z_{\alpha'}} - i(\mathbb{I} + \mathbb{H})\left\{\text{Im}\left(\frac{Z_{t}}{Z_{\alpha'}}\right)\right\} \]
and hence
\[ b_{\alpha'} = D_{\alpha'}Z_{t} + Z_{t}\partial_{\alpha'} - i\partial_{\alpha'}(\mathbb{I} + \mathbb{H})\left\{\text{Im}\left(\frac{Z_{t}}{Z_{\alpha'}}\right)\right\} \quad (21) \]

f) We now record some frequently used commutator identities.
\[ [\partial_{\alpha'}, D_{t}] = b_{\alpha'}\partial_{\alpha'} \quad \text{and} \quad \text{[}D_{\alpha'}, D_{t}\text{]} = \text{Re}(D_{\alpha'}Z_{t})|D_{\alpha'}| = \text{Re}(\overline{D}_{\alpha'}Z_{t})|D_{\alpha'}| \quad (22) \]
\[ [D_{\alpha'}, D_{t}] = (D_{\alpha'}Z_{t})D_{\alpha'} \quad \text{and} \quad [\overline{D}_{\alpha'}, D_{t}] = (\overline{D}_{\alpha'}Z_{t})\overline{D}_{\alpha'} \quad (23) \]

Using these we also obtain the following formulae
\[ D_{t}|Z_{\alpha'}| = D_{t}e^{	ext{Re} \log Z_{\alpha'}} = |Z_{\alpha'}|\{\text{Re}(D_{\alpha'}Z_{t}) - b_{\alpha'}\} \quad (24) \]
\[ D_{t}\frac{1}{Z_{\alpha'}} = \frac{-1}{Z_{\alpha'}}(D_{\alpha'}Z_{t} - b_{\alpha'}) = \frac{1}{Z_{\alpha'}}\{b_{\alpha'} - D_{\alpha'}Z_{t} - \overline{D}_{\alpha'}Z_{t} + \overline{D}_{\alpha'}Z_{t}\} \quad (25) \]

Observe that \( (b_{\alpha'} - D_{\alpha'}Z_{t} - \overline{D}_{\alpha'}Z_{t}) \) is real valued and this fact will be useful later on.

g) We have the formula
\[ \Theta = i \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} - i\text{Re}(\mathbb{I} - \mathbb{H})\left(\frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right) \quad (26) \]

h) We have the formula
\[ \text{Re}(D_{t}\Theta) = -\text{Im}\{(|D_{\alpha'}| + i\text{Re}\Theta)\overline{D}_{\alpha'}Z_{t}\} \quad (27) \]

We also have the formula
\[ D_{t}\Theta = i(|D_{\alpha'}| + i\text{Re}\Theta)\overline{D}_{\alpha'}Z_{t} - i\text{Re}(\mathbb{I} - \mathbb{H})\{(|D_{\alpha'}| + i\text{Re}\Theta)\overline{D}_{\alpha'}Z_{t}\} + i\text{Im}(\mathbb{I} - \mathbb{H})D_{t}\Theta \quad (28) \]
along with

\[(\mathbb{I} - \mathbb{H}) D_t \Theta = [D_t, \mathbb{H}] \Theta = [b, \mathbb{H}] \partial_\alpha \Theta \]  

(29)

4.2. Quasilinear equations

We now write down the quasilinear equations obtained in [1] that we will need to prove our energy estimates. See Section 4 of [1] for a derivation of these equations.

(1) Define the real valued variable \( J_1 \) as

\[ J_1 = D_t A_1 + A_1 (b_\alpha' - D_\alpha' Z_t - \overline{D_\alpha'} Z_t) + \sigma \partial_\alpha \text{Re}(\mathbb{I} - \mathbb{H}) \left\{ (|D_\alpha'|^2 + i \text{Re}\Theta) \overline{D_\alpha'} Z_t \right\} - \sigma \partial_\alpha \text{Im}(\mathbb{I} - \mathbb{H}) D_t \Theta \]

Using this we get

\[
\overline{Z}_{1t} + i A_1 \overline{D_\alpha'} Z_t \overline{Z}_{1t} - i \sigma D_\alpha' (|D_\alpha'| + i \text{Re}\Theta) \overline{D_\alpha'} Z_t = -\sigma (D_\alpha' Z_t) D_\alpha' \Theta - i \frac{J_1}{Z_{1t}} \overline{Z}_{1t}, \alpha' \]

We also have

\[
\overline{Z}_{1t} Z_{1t} + i A_1 \overline{D_\alpha'} Z_t - i \sigma \partial_\alpha' \left( \frac{1}{Z_{\alpha'}} |D_\alpha'| \overline{Z}_{1t}, \alpha' \right) = i \sigma \partial_\alpha' \left\{ (|D_\alpha'|^2 + i \text{Re}\Theta) \overline{D_\alpha'} Z_t \right\} - \sigma (D_\alpha' Z_t) \partial_\alpha' \Theta - i \frac{J_1}{Z_{1t}, \alpha'} \overline{Z}_{1t} \overline{Z}_{1t}, \alpha' \]

This equation gives rise to the energy \( E_{\sigma,1} \) in the energy estimate Theorem \( 4.1 \)

(2) \[
\left( \frac{D_t^2 + i A_1}{|Z_{\alpha'}|^2} \partial_\alpha' - i \sigma |D_\alpha'|^3 \right) \overline{D_\alpha'} Z_t = R_1 - i \left( \frac{1}{Z_{\alpha'}} \overline{Z}_{1t}, \alpha' \right) J_1 - i \frac{1}{|Z_{\alpha'}|^2} \partial_\alpha' J_1 \quad (33)
\]

where

\[
R_1 = -2(\overline{D_\alpha'} Z_t)(D_t \overline{D_\alpha'} Z_t) - 2 \sigma \text{Re}(D_\alpha' Z_t) \overline{D_\alpha'} D_\alpha' \Theta - \sigma (\overline{D_\alpha'} D_\alpha' Z_t) D_\alpha' \Theta + i \sigma \left( 2i \text{Re}(|D_\alpha'| \Theta) + (\text{Re}\Theta)^2 \right) |D_\alpha'| \partial_\alpha' \overline{Z}_t - \sigma \text{Re} \left( |D_\alpha'|^2 \Theta \right) \overline{D_\alpha'} Z_t \]

(34)

and \( J_1 \) was defined in [30]. This equation gives rise to the energy \( E_{\sigma,4} \) in the energy estimate Theorem \( 4.1 \)

(3) Multiply the equation for \( \overline{D_\alpha'} Z_t \) in (33) by \( Z_{\alpha'} \) to get the equation

\[
\left( \frac{D_t^2 + i A_1}{|Z_{\alpha'}|^2} \partial_\alpha' - i \sigma |D_\alpha'|^3 \right) Z_{1\alpha'} = R_1 Z_{1\alpha'} - i \left( \partial_\alpha' \frac{1}{Z_{\alpha'}} \right) J_1 - i D_\alpha' J_1 - Z_{1\alpha'} \left[ \frac{D_t^2 + i A_1}{|Z_{\alpha'}|^2} \partial_\alpha' - i \sigma |D_\alpha'|^3 \right] \overline{Z}_{1\alpha'} \quad (35)
\]

This equation gives rise to the energy \( E_{\sigma,2} \) in the energy estimate Theorem \( 4.1 \)

(4) We have

\[
\left( \frac{D_t^2 + i A_1}{|Z_{\alpha'}|^2} \partial_\alpha' - i \sigma |D_\alpha'|^3 \right) \Theta = R_2 + i J_2 \quad (36)
\]
where

\[ R_2 = -2i(D_{\alpha'}Z_t)\langle |D_{\alpha'}|D_{\alpha'}Z_t \rangle + (\text{Re}\Theta)\left\{ (D_{\alpha'}Z_t)^2 + iD_{\alpha'}\left( \frac{A_1}{Z_{\alpha'}} \right) + i\sigma(\text{Re}\Theta)|D_{\alpha'}|\Theta \right\} \]

\[ + \sigma\text{Re}(|D_{\alpha'}|\Theta)|D_{\alpha'}|\Theta + \left( |D_{\alpha'}|\frac{A_1}{Z_{\alpha'}} \right) \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) + |D_{\alpha'}| \left( \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'}A_1 \right) \]

\[ + (\mathbb{I} + \mathbb{H})\text{Im}\{\text{Re}(\overline{D}_{\alpha'}Z_t)|D_{\alpha'}|D_{\alpha'}Z_t - i\text{Re}((D_t\Theta)\overline{D}_{\alpha'}Z_t)\} \]

\[ + \frac{A_1}{|Z_{\alpha'}|^2}\partial_{\alpha'}\text{Re}(\mathbb{I} - \mathbb{H})\left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \]

\[ J_2 = \text{Im}(\mathbb{I} - \mathbb{H})(D_t^2\Theta) - \text{Re}(\mathbb{I} - \mathbb{H})\{(|D_{\alpha'}| + i\text{Re}\Theta)D_t\overline{D}_{\alpha'}Z_t\} \]

Note that the variable \( J_2 \) is real valued. This equation gives rise to the energy \( E_{\sigma,3} \) in the energy estimate Theorem 4.1

4.3. Previous apriori estimate

We now describe the main apriori estimate proved in Section 5 of [1]. We will need a modification of this energy when we prove our main energy estimate Theorem 7.1. Define

\[ E_{\sigma,0} = \left\| \sigma^{\frac{1}{2}}Z_{\sigma'} |\frac{1}{2}\partial_{\alpha'} \frac{1}{Z_{\sigma'}} \right\|_2^2 + \left\| \sigma^{\frac{1}{2}}Z_{\sigma'} |\frac{1}{2}\partial_{\alpha'} \frac{1}{Z_{\sigma'}} \right\|_2^6 + \left\| \partial_{\alpha'} \frac{1}{Z_{\sigma'}} \right\|_2^2 + \left\| \sigma^{\frac{1}{2}}Z_{\sigma'} \partial_{\alpha'}^2 \frac{1}{Z_{\sigma'}} \right\|_2^2 \]

\[ E_{\sigma,1} = \left\| (\overline{Z}_{tt} - i)Z_{\sigma'} \right\|_2 + \left\| \overline{A_1}Z_{t,\alpha'} \right\|_2^2 \]

\[ E_{\sigma,2} = \left\| D_tZ_{t,\alpha'} \right\|_2^2 + \left\| \overline{A_1}Z_{t,\alpha'} \right\|_2^2 \]

\[ E_{\sigma,3} = \left\| D_t\Theta \right\|_2^2 + \left\| \overline{A_1} \Theta \right\|_2^2 \]

\[ E_{\sigma,4} = \left\| D_t\overline{D}_{\alpha'}Z_t \right\|_2^2 + \left\| \overline{A_1}|D_{\alpha'}|D_{\alpha'}Z_t \right\|_2^2 \]

\[ E_{\sigma} = E_{\sigma,0} + E_{\sigma,1} + E_{\sigma,2} + E_{\sigma,3} + E_{\sigma,4} \]

**Theorem 4.1 ([1])**. Let \( \sigma \geq 0 \) and let \( (Z, Z_t)(t) \) be a smooth solution to (8) in \([0, T]\) for \( T > 0 \), such that for all \( s \geq 3 \) we have \( (Z_{\sigma'} - 1, \frac{1}{Z_{\sigma'}} - 1, Z_t) \in L^\infty([0, T], H^{s+\frac{1}{2}}(\mathbb{R}) \times H^{s+\frac{1}{2}}(\mathbb{R}) \times H^s(\mathbb{R})) \).

Then \( \sup_{t \in [0, T]} E_{\sigma}(t) < \infty \) and there exists a polynomial \( P \) with universal non-negative coefficients such that for all \( t \in [0, T] \) we have

\[ \frac{dE_{\sigma}(t)}{dt} \leq P(E_{\sigma}(t)) \]

**Remark 4.2**. Note that the energy \( E_{\sigma,0} \) contains a term which is the \( L^\infty \) norm of a function and hence may not in general be differentiable in time even for smooth solutions. Hence for this term the time derivative is replaced by the upper Dini derivative \( \limsup_{t \to t^+} \frac{\|f(t+s)\|_{L^\infty} - \|f(t)\|_{L^\infty}}{s} \).
In this section when we write $f \in L^2$, what we mean is that there exists a universal polynomial $P$ with non-negative coefficients such that $\|f\|_2 \leq P(E_\sigma)$. Similar definitions for $f \in \dot{H}^\frac{1}{4}$, $f \in L^\infty$ etc. This notation was used in [1] and we use it here as well as it simplifies the presentation. In order to prove this theorem, two new weighted spaces were also introduced to simplify the calculations. Define the spaces $\mathcal{C}$ and $\mathcal{W}$:

(1) If $w \in L^\infty$ and $|D_\alpha| w \in L^2$, then we say $w \in \mathcal{W}$. Define
\[
\|w\|_\mathcal{W} = \|w\|_{L^\infty} + \|D_\alpha| w\|_2
\]  
(2) If $f \in \dot{H}^\frac{1}{4}$ and $f|Z_\alpha| \in L^2$, then we say $f \in \mathcal{C}$. Define
\[
\|f\|_\mathcal{C} = \|f\|_{\dot{H}^\frac{1}{4}} + \left(1 + \left\|\frac{1}{|Z_\alpha|}\right\|_{L^2}\right)\|f|Z_\alpha|\|_2
\]

Also define the norm $\|f\|_{\mathcal{W}\cap\mathcal{C}} = \|f\|_\mathcal{W} + \|f\|_\mathcal{C}$. The main lemma governing the behavior of these spaces is given as follows:

**Lemma 4.3.** The following properties hold for the spaces $\mathcal{W}$ and $\mathcal{C}$

(1) If $w_1, w_2 \in \mathcal{W}$, then $w_1 w_2 \in \mathcal{W}$. Moreover $\|w_1 w_2\|_\mathcal{W} \leq \|w_1\|_\mathcal{W} \|w_2\|_\mathcal{W}$

(2) If $f \in \mathcal{C}$ and $w \in \mathcal{W}$, then $fw \in \mathcal{C}$. Moreover $\|fw\|_\mathcal{C} \leq \|f\|_\mathcal{C} \|w\|_\mathcal{W}$

(3) If $f, g \in \mathcal{C}$, then $fg|Z_\alpha| \in L^2$. Moreover $\|fg\|_\mathcal{C} \|Z_\alpha\|_2 \leq \|f\|_\mathcal{C} \|g\|_\mathcal{C}$

This lemma was presented in [1] and its proof follows directly from Proposition 9.9. We will continue to use these spaces as they are quite useful in our energy estimates.

In the proof of Theorem 4.1 several quantities were shown to be controlled by the energy $E_\sigma$. We now list some of the quantities controlled by $E_\sigma$ which we will use frequently in our energy estimates in [1] and [10]. By Proposition 6.1 of [1], the energy $E_\sigma$ is equivalent to the energy $E_\sigma$ and so the following terms are also controlled by $E_\sigma$. It is important to note that as the estimates are true for all $\sigma \geq 0$, they are in particular true for $\sigma = 0$. We will now list all the terms controlled in Section 5.1 in [1] which remain non-zero when we specialize to $\sigma = 0$ case.

1) $\bar{Z}_t, Z_\alpha \in L^2, |D_\alpha| \bar{Z}_t, Z_\alpha \in L^2$

2) $A_1 \in L^\infty \cap \dot{H}^\frac{1}{4}$

3) $\partial_\alpha \frac{1}{Z_\alpha} \in L^2, \partial_\alpha \frac{1}{Z_\alpha} |Z_\alpha| \in L^2, |D_\alpha| \omega \in L^2$ and hence $\omega \in \mathcal{W}$

4) $\bar{D}_\alpha Z_t \in L^\infty, |D_\alpha| \bar{Z}_t \in L^\infty, D_\alpha \bar{Z}_t \in L^\infty$

5) $\bar{D}_\alpha Z_t \in L^2, |D_\alpha| \bar{Z}_t \in L^2, D_\alpha \bar{Z}_t \in L^2$

6) $\bar{D}_\alpha Z_t \in \mathcal{W}\cap\mathcal{C}, |D_\alpha| \bar{Z}_t \in \mathcal{W}\cap\mathcal{C}, D_\alpha \bar{Z}_t \in \mathcal{W}\cap\mathcal{C}$

7) $\partial_\alpha \varphi_\lambda \left(\frac{Z_t}{Z_\alpha}\right) \in L^\infty$

8) $|D_\alpha| A_1 \in L^2$ and hence $A_1 \in \mathcal{W}, \sqrt{A_1} \in \mathcal{W}, \frac{1}{A_1} \in \mathcal{W}, \frac{1}{\sqrt{A_1}} \in \mathcal{W}$

9) $\Theta \in L^2, D_\theta \Theta \in L^2$

10) $\frac{1}{|Z_\alpha|} \in \mathcal{C}$

11) $D_\alpha \frac{1}{Z_\alpha} \in \mathcal{C}, |D_\alpha| \frac{1}{Z_\alpha} \in \mathcal{C}, |D_\alpha| \frac{1}{|Z_\alpha|} \in \mathcal{C}, \frac{1}{|Z_\alpha|} \partial_\alpha \omega \in \mathcal{C}$

12) $\frac{1}{|Z_\alpha|^2} \partial_\alpha A_1 \in L^\infty \cap \dot{H}^\frac{1}{4}$ and hence $\frac{1}{|Z_\alpha|^2} \partial_\alpha A_1 \in \mathcal{C}$
13) \( \frac{1}{|Z_{a'}|^2} \partial_{a'} A_1 \in L^2, |D_{a'}| \left( \frac{1}{|Z_{a'}|^2} \partial_{a'} A_1 \right) \in L^2 \) and hence \( \frac{1}{|Z_{a'}|^2} \partial_{a'} A_1 \in W \)

14) \( b_{a'} \in L^\infty \cap \dot{H}^\frac{1}{2} \) and \( \mathbb{H}(b_{a'}) \in L^\infty \cap \dot{H}^\frac{1}{2} \)

15) \( |D_{a'}|b_{a'} \in L^2 \) and hence \( b_{a'} \in W \)

16) \( \partial_{a'} D_t \frac{1}{|Z_{a'}|^2} \partial_{a'} \in L^2 \)

17) \( Z_{t,a'} \in L^2 \)

18) \( D_{a'} Z_{tt} \in C, |D_{a'}| Z_{tt} \in C, D_{a'} \bar{D}_{a'} \bar{Z}_t \in C \) and \( D_{a'} |D_{a'}| Z_{t} \in C \)

19) \( D_t A_1 \in L^\infty \cap \dot{H}^\frac{1}{2} \)

20) \( D_t (b_{a'} - D_{a'} Z_t - \bar{D}_{a'} \bar{Z}_t) \in L^\infty \cap \dot{H}^\frac{1}{2} \) and hence \( D_t b_{a'} \in \dot{H}^\frac{1}{2}, \partial_{a'} D_t b \in \dot{H}^\frac{1}{2} \)

21) \( (I - \mathbb{H}) D_t^2 \Theta \in L^2, (I - \mathbb{H}) D_t^2 Z_{t,a'} \in L^2, (I - \mathbb{H}) D_t^2 D_{a'} Z_t \in \dot{H}^\frac{1}{2} \)

22) \( \left[ D_t^2, \frac{1}{|Z_{a'}|^2} \partial_{a'}, \frac{1}{|Z_{a'}|^2} \right] Z_{t,a'} \in C \)

23) \( i \left[ A_1 \right] \left[ \frac{1}{|Z_{a'}|^2} \partial_{a'}, \frac{1}{|Z_{a'}|^2} \right] Z_{t,a'} \in C \)

24) \( j_1 \in C \)

25) \( j_1 \in L^\infty \cap \dot{H}^\frac{1}{2} \)

26) \( |D_{a'}| j_1 \in L^2 \) and hence \( j_1 \in W \)

27) \( j_2 \in L^2 \)

28) \( j_2 \in L^2 \)

29) \( (I - \mathbb{H}) D_t^2 \bar{D}_{a'} Z_t \in \dot{H}^\frac{1}{2} \)

30) \( \frac{1}{|Z_{a'}|} \partial_{a'} j_1 \in \dot{H}^\frac{1}{2} \) and hence \( \frac{1}{|Z_{a'}|} \partial_{a'} j_1 \in C \)

In addition to controlling these terms, some other estimates were also proved to the energy estimate Theorem 4.1. Now we give a useful lemma proved in Section 5.2 in [1].

Lemma 4.4. Let \( T > 0 \) and let \( f, b \in C^2([0,T], H^2(\mathbb{R})) \) with \( b \) being real valued. Let \( D_t = \partial_t + b \partial_a \). Then

\[
\begin{align*}
(1) \quad & \frac{d}{dt} \int f \, da' = \int D_t f \, da' + \int b_{a'} f \, da' \\
(2) \quad & \frac{d}{dt} \int |f|^2 \, da' - 2 \Re \int \bar{f} (D_t f) \, da' \leq \|f\|_2^2 \|b_{a'}\|_\infty \\
(3) \quad & \frac{d}{dt} \int (|\partial_{a'} f| f) \, da' - 2 \Re \int \left\{ (|\partial_{a'} f| f) D_t f \, da' \right\} \leq \|f\|_2^2 \|b_{a'}\|_\infty
\end{align*}
\]

In proving Theorem 4.1 several estimates were proven in Section 5.2.3 of [1] in order to close the energy estimate. We now list some of them below. In the following \( P \) represents a universal polynomial with non-negative coefficients and \( (Z, Z_t)(t) \) is a solution to (5) with surface tension \( \sigma \).

(1) We have the estimate

\[
\left\| D_t \left( \frac{\sqrt{A_1}}{|Z_{a'}|} \bar{f} \right) - \frac{\sqrt{A_1}}{|Z_{a'}|} D_t \bar{f} \right\|_{\dot{H}^\frac{1}{2}} \lesssim P(E_\sigma) \left\| \frac{f}{|Z_{a'}|} \right\|_C
\]

(2) If \( \mathbb{F} \bar{H} f = f \) then we have the estimate

\[
\left\| \frac{\sqrt{A_1}}{|Z_{a'}|} \partial_{a'} \left( \frac{\sqrt{A_1}}{|Z_{a'}|} \bar{f} \right) - i \frac{A_1}{|Z_{a'}|^2} \partial_{a'} \bar{f} \right\|_{\dot{H}^\frac{1}{2}} \lesssim P(E_\sigma) \left\{ \left\| \frac{\sqrt{A_1}}{|Z_{a'}|} \bar{f} \right\|_{\dot{H}^\frac{1}{2}} + \left\| \frac{f}{|Z_{a'}|} \right\|_C \right\}
\]
Finally in addition to the above estimates it was shown in Section 6 of [1] that the energies $E$ and $\mathcal{E}_\sigma$ are equivalent.

**Proposition 4.5.** There exists universal polynomials $P_1, P_2$ with non-negative coefficients so that if $(Z, Z_t)(t)$ is a smooth solution to the water wave equation (8) for $\sigma \geq 0$ in the time interval $[0, T]$ satisfying $(Z, \alpha - 1, \frac{1}{Z_{\alpha'}} - 1, Z_t) \in L^\infty([0, T], H^{s+\frac{1}{2}}(\mathbb{R}) \times H^{s+\frac{1}{2}}(\mathbb{R}) \times H^s(\mathbb{R}))$ for all $s \geq 3$, then for all $t \in [0, T]$ we have

$$E_\sigma(t) \leq P_1(\mathcal{E}_\sigma(t)) \quad \text{and} \quad \mathcal{E}_\sigma(t) \leq P_2(E_\sigma(t))$$

5. **Higher order energy $\mathcal{E}_{high}$**

In this section we prove an apriori energy estimate for solutions of (8) for $\sigma = 0$ for the energy $\mathcal{E}_{high}$ defined in [22, 36]. We will first construct an energy $E_{high}$, prove an apriori estimate for it in Theorem 5.1 and then show in Proposition 5.2 that $E_{high}$ and $\mathcal{E}_{high}$ are equivalent energies. Note that we already have an energy estimate for $\sigma = 0$ by simply taking the special case of $\sigma = 0$ in the energy $E$ in Theorem 4.1. The energy $E_{high}$ is higher order than $E_{\sigma = 0}$ by half spatial derivative. We need an estimate for a higher order norm as we will need it to control $\mathcal{E}_{aux}$ in [36] which in turn needed to prove Theorem 4.1.

This higher order energy $E_{high}$ is essentially equivalent to the energy used in [22, 36], (more precisely we do not have the dependence on the lower order term $\|\tfrac{1}{Z_{\alpha'}}\|_\infty(t)$ or $\|\tfrac{1}{Z_{\alpha'}}(0, t)$ in the energy which they have) and in those papers an apriori estimate is already established. The main reason we are proving an apriori estimate here is because in the proof of the energy estimate of $E_{high}$, we prove several additional things such as control of several more terms (for example $|D_{\alpha'}|\tfrac{1}{Z_{\alpha'}} \in \mathcal{W} \cap \mathcal{C}, \tfrac{1}{Z_{\alpha'}}\partial_{\alpha'} \omega \in \mathcal{W} \cap \mathcal{C}, \tfrac{1}{|Z_{\alpha'}|} \partial_{\alpha'} Z_{t} \in \mathcal{C}$ etc.) and estimates regarding the time derivative of the energy, which are not available in [22, 36]. We need these additional estimates in [8] to prove Theorem 6.1. An additional benefit of proving the apriori estimate is that it also allows us to remove the dependence on lower order terms such as $\|\tfrac{1}{Z_{\alpha'}}\|_\infty(t)$ or $\|\tfrac{1}{Z_{\alpha'}}(0, t)$.

Let us now define the energy $E_{high}$ and state the apriori estimate. Define

$$E_{high} = E_0|_{\sigma = 0} + \|D_1D_{\alpha'}^2 Z_t\|_2^2 + \|\sqrt{\frac{1}{|Z_{\alpha'}|}} D_1^2 D_{\alpha'} Z_t\|_{H^{s+\frac{1}{2}}}^2 \quad (43)$$

**Theorem 5.1.** Let $T > 0$ and let $(Z, Z_t)(t)$ be a smooth solution to (8) with $\sigma = 0$ in the time interval $[0, T]$, such that for all $s \geq 2$ we have $(Z, \alpha' - 1, \frac{1}{Z_{\alpha'}} - 1, Z_t) \in L^\infty([0, T], H^s(\mathbb{R}) \times H^{s+\frac{1}{2}}(\mathbb{R}))$. Then $\sup_{t \in [0, T]} E_{high}(t) < \infty$ and there exists a universal polynomial $P$ with non-negative coefficients such that for all $t \in [0, T]$ we have

$$\frac{dE_{high}(t)}{dt} \leq P(E_{high}(t))$$

To prove this theorem, we first obtain the quasilinear equation in [5.1] then control the quantities controlled by $E_{high}$ in [5.2] and then finish the proof in [5.3]. Finally we will show in [5.4] that $E_{high}$ and $\mathcal{E}_{high}$ are equivalent energies.
5.1. Quasilinear equation

Let us now derive the quasilinear equation relevant for the energy $E_{\text{high}}$. Plugging in $\sigma = 0$ in the equation for $Z_t$ from (31) we obtain

$$
\left( D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \right) Z_t = -i \frac{J_1}{Z_{\alpha'}}
$$

with $J_1$ defined by (30) with $\sigma = 0$, i.e. $J_1 = D_t A_1 + A_1 \left( b_{\alpha'} - D_{\alpha'} Z_t - \mathcal{D}_{\alpha', Z_t} \right)$. Applying $D_{\alpha'}^2$ to the above equation we obtain

$$
\left( D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \right) D_{\alpha'}^2 Z_t = -i D_{\alpha'}^2 \left( \frac{J_1}{Z_{\alpha'}} \right) + \left[ D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'}, D_{\alpha'}^2 \right] Z_t
$$

Let us try to simplify the terms above. We will heavily use the commutator estimates from (22) and also use the equation (10) for $\sigma = 0$.

a) We see that

$$
\left[ D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'}, D_{\alpha'} \right] = D_t[D_t, D_{\alpha'}] + [D_t, D_{\alpha'}] D_t + [(Z_t + i) D_{\alpha'}, D_{\alpha'}]
$$

$$
= -D_t \{(D_{\alpha'} Z_t) D_{\alpha'} \} - (D_{\alpha'} Z_t) D_{\alpha'} D_t - (D_{\alpha'} Z_t) D_{\alpha'}
$$

$$
= \{ -2(D_{\alpha'} Z_t) + 2(D_{\alpha'} Z_t)^2 \} D_{\alpha'} - 2(D_{\alpha'} Z_t) D_{\alpha'} D_t
$$

b) We have the relation

$$
\left[ D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'}, D_{\alpha'} \right] D_{\alpha'}
$$

$$
= \{ -2(D_{\alpha'} Z_t) + 2(D_{\alpha'} Z_t)^2 \} D_{\alpha'}^2 - 2(D_{\alpha'} Z_t) D_{\alpha'} D_t D_{\alpha'}
$$

$$
= \{ -2(D_{\alpha'} Z_t) + 2(D_{\alpha'} Z_t)^2 \} D_{\alpha'}^2 - 2(D_{\alpha'} Z_t) \{ -2(D_{\alpha'} Z_t) D_{\alpha'} + D_{\alpha'} D_t \}
$$

$$
= 2(D_{\alpha'} Z_t)(D_{\alpha'}^2 Z_t) D_{\alpha'} + \{ -2(D_{\alpha'} Z_t) + 4(D_{\alpha'} Z_t)^2 \} D_{\alpha'}^2 - 2(D_{\alpha'} Z_t) D_{\alpha'}^2 D_t
$$

c) We similarly have

$$
D_{\alpha'} \left[ D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'}, D_{\alpha'} \right]
$$

$$
= \{ -2(D_{\alpha'} Z_t) + 4(D_{\alpha'} Z_t)(D_{\alpha'}^2 Z_t) \} D_{\alpha'} + \{ -2(D_{\alpha'} Z_t) + 2(D_{\alpha'} Z_t)^2 \} D_{\alpha'}^2
$$

$$
- 2(D_{\alpha'}^2 Z_t) D_{\alpha'} D_t - 2(D_{\alpha'} Z_t) D_{\alpha'}^2 D_t
$$
d) Hence we have
\[
\left[ D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'}, D_{\alpha'}^2 \right]
\]
\[
= \left[ D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'}, D_{\alpha'} \right] D_{\alpha'} + D_{\alpha'} \left[ D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'}, D_{\alpha'} \right]
\]
\[
= \{-2(D_{\alpha'}^2 Z_t) + 6(D_{\alpha'} Z_t)(D_{\alpha'}^2 Z_t)\} D_{\alpha'} + \{-4(D_{\alpha'} Z_t) + 6(D_{\alpha'} Z_t)^2\} D_{\alpha'}^2
\]
\[
- 2(D_{\alpha'}^2 Z_t) D_{\alpha'} D_t - 4(D_{\alpha'} Z_t) D_{\alpha'}^2 D_t
\]
e) We see that
\[
-i D_{\alpha'}^2 \left( \frac{J_1}{Z_{\alpha'}} \right) = -i D_{\alpha'} \left\{ \frac{\omega^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1 + J_1 \left( D_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\}
\]
\[
= -i \omega^3 |D_{\alpha'}| \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1 \right) - 2i \omega(D_{\alpha'} \omega) \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1 \right)
\]
\[
- i(D_{\alpha'} J_1) \left( D_{\alpha'} \frac{1}{Z_{\alpha'}} \right) - i J_1 \left( D_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right)
\]
Combining the above identities we get the equation for $D_{\alpha'}^2 \bar{Z}_t$
\[
\left( D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \right) D_{\alpha'}^2 \bar{Z}_t = -i \omega^3 |D_{\alpha'}| \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1 \right) + R_3 \tag{44}
\]
where
\[
R_3 = \{-2(D_{\alpha'}^2 Z_t) + 6(D_{\alpha'} Z_t)(D_{\alpha'}^2 Z_t)\} (D_{\alpha'} \bar{Z}_t) + \{-4(D_{\alpha'} Z_t) + 6(D_{\alpha'} Z_t)^2\} (D_{\alpha'}^2 \bar{Z}_t)
\]
\[
- 2(D_{\alpha'}^2 Z_t)(D_{\alpha'} \bar{Z}_t) - (D_{\alpha'} Z_t)(D_{\alpha'}^2 \bar{Z}_t) - 2i \omega(D_{\alpha'} \omega) \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1 \right)
\]
\[
- i(D_{\alpha'} J_1) \left( D_{\alpha'} \frac{1}{Z_{\alpha'}} \right) - i J_1 \left( D_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right)
\]

5.2. Quantities controlled by the energy $E_{\text{high}}$

In this section whenever we write $f \in L^2$, what we mean is that there exists a universal polynomial $P$ with nonnegative coefficients such that $\|f\|_2 \leq P(E_{\text{high}})$. Similar definitions for $f \in H^\infty$, $f \in \mathcal{C}$ or $f \in W$ where the definitions for the spaces $\mathcal{C}$ and $W$ are as in \[43\], \[44\]. Note that $E_{\text{high}}$ controls the energy $E_{\sigma}|_{\sigma=0}$ and hence we already have control of a lot of quantities as listed in \[43\]. We will freely use the quantities controlled by $E_{\sigma}|_{\sigma=0}$ to prove the above theorem. In particular we will also be making use of Lemma \[44\]. Let us now establish the quantities controlled by $E_{\text{high}}$ which are not controlled by $E_{\sigma}|_{\sigma=0}$. 
1) \( D_t D_{\alpha'}^2 Z_t \in L^2 \), \( D_{\alpha'}^2 Z_{tt} \in L^2 \), \( |D_{\alpha'}|^2 Z_{tt} \in L^2 \) and \( D_{\alpha'}^2 Z_{tt} \in L^2 \). Hence from Lemma 4.3 we have

\[
D_t D_{\alpha'}^2 Z_t = [D_t, D_{\alpha'}^2] Z_t + D_{\alpha'}^2 Z_{tt}
\]

\[
= D_{\alpha'} \{- (D_{\alpha'} Z_t) D_{\alpha'} Z_t \} - (D_{\alpha'} Z_t) D_{\alpha'}^2 Z_t + D_{\alpha'}^2 Z_{tt}
\]

\[
= -(D_{\alpha'}^2 Z_t) D_{\alpha'} Z_t - 2(D_{\alpha'} Z_t) D_{\alpha'}^2 Z_t + D_{\alpha'}^2 Z_{tt}
\]

Now \( D_t D_{\alpha'}^2 Z_t \in L^2 \) as it part of the energy and hence we have

\[
\| D_{\alpha'}^2 Z_{tt} \|_2 \lesssim \| D_t D_{\alpha'}^2 Z_t \|_2 + \| D_{\alpha'}^2 Z_{tt} \|_2 \| D_{\alpha'} Z_t \|_\infty
\]

Now we observe that

\[
D_{\alpha'}^2 Z_{tt} = D_{\alpha'} (\overline{\omega} D_{\alpha'} \langle Z_{tt} \rangle) = \overline{\omega} (|D_{\alpha'} \langle Z_{tt} \rangle| D_{\alpha'} Z_{tt} + \overline{\omega}^2 |D_{\alpha'}|^2 Z_{tt})
\]

Hence from Lemma 4.3 we have

\[
\| D_{\alpha'}^2 Z_{tt} \|_2 \lesssim \| D_{\alpha'}^2 Z_{tt} \|_2 + \left\| \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \omega \right\|_C \| D_{\alpha'} Z_t \|_C
\]

The terms \( D_{\alpha'}^2 Z_{tt} \in L^2 \), \( |D_{\alpha'}| D_t D_{\alpha'} Z_t \in L^2 \) are proven similarly.

2) \( D_{\alpha'} Z_{tt} \in W \cap C \), \( |D_{\alpha'}| Z_{tt} \in W \cap C \) and \( D_{\alpha'} Z_{tt} \in W \cap C \)

Proof: We already know that \( D_{\alpha'} Z_{tt} \in C \) and \( D_{\alpha'} Z_{tt} \in L^2 \). Hence using \( f = D_{\alpha'} Z_{tt} \) and \( w = \frac{1}{Z_{\alpha'}} \) in Proposition 4.8 we obtain

\[
\| D_{\alpha'} Z_{tt} \|_2^2 \lesssim \| Z_{tt,\alpha'} \|_2 \| D_{\alpha'}^2 Z_{tt} \|_2
\]

Hence \( D_{\alpha'} Z_{tt} \in W \). We also have from Lemma 4.3 that \( \| D_{\alpha'} Z_{tt} \|_W \lesssim \| \omega \|_W \| D_{\alpha'} Z_{tt} \|_W \). The proof of \( D_{\alpha'} Z_{tt} \in W \cap C \) is done similarly.

3) \( |D_{\alpha'}| \frac{1}{Z_{\alpha'}} \in L^\infty \), \( |D_{\alpha'}| \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \omega \in L^\infty \)

Proof: We observe from (10) that \( Z_{tt} - i = -\frac{A_1}{Z_{\alpha'}} \) and hence we have

\[
|D_{\alpha'}| Z_{tt} = -i \frac{\overline{\omega}}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 - i A_1 |D_{\alpha'}| \frac{1}{Z_{\alpha'}}
\]

As \( A_1 \geq 1 \) we have

\[
\left\| \left| D_{\alpha'} \right| \frac{1}{Z_{\alpha'}} \right\|_\infty \lesssim \left\| \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right\|_\infty + \left\| D_{\alpha'} Z_{tt} \right\|_\infty
\]

Hence \( |D_{\alpha'}| \frac{1}{Z_{\alpha'}} \in L^\infty \). Now using (16) we see that

\[
\text{Re} \left( D_{\alpha'} \frac{1}{Z_{\alpha'}} \right) = |D_{\alpha'}| \frac{1}{|Z_{\alpha'}|} \quad \text{Im} \left( D_{\alpha'} \frac{1}{Z_{\alpha'}} \right) = i \left( \frac{\overline{\omega}}{|Z_{\alpha'}|^2} \partial_{\alpha'} \omega \right)
\]

Hence we obtain \( |D_{\alpha'}| \frac{1}{Z_{\alpha'}} \in L^\infty \) and \( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \omega \in L^\infty \).
4) \(|D_{\alpha'}|^2 \frac{1}{Z_{\alpha'}} \in L^2, D_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \in L^2\) and similarly \(|D_{\alpha'}|^2 \frac{1}{|Z_{\alpha'}|} \in L^2, \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'}^3 \omega \in L^2\). We also have \(\frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \in L^2, \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \Theta \in L^2\).

Proof: From (10) we have \(Z_{tt} - i = -i \frac{A_1}{Z_{\alpha'}}\) and hence

\[
|D_{\alpha'}|^2 Z_{tt} = |D_{\alpha'}|^2 \left( -i \frac{\bar{\omega}}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 - i A_1 |D_{\alpha'}|^2 \frac{1}{Z_{\alpha'}} \right)
\]

\[-\bar{\omega} D_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right) - i (|D_{\alpha'}|^2 \bar{\omega}) \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right) - i (|D_{\alpha'}|^2 A_1) \left( \frac{1}{Z_{\alpha'}} \right)
\]

\[-i A_1 |D_{\alpha'}|^2 \frac{1}{Z_{\alpha'}}
\]

As \(A_1 \geq 1\) we see that

\[
\left\| |D_{\alpha'}|^2 \frac{1}{Z_{\alpha'}} \right\|_2 \lesssim \left\| |D_{\alpha'}|^2 Z_{tt} \right\|_2 + \left\| \frac{1}{Z_{\alpha'}} \partial_{\alpha'} A_1 \right\|_2 + \left\| \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right\|_\infty
\]

Now we see that

\[
D_{\alpha'}^2 \frac{1}{Z_{\alpha'}} = D_{\alpha'} \left( \bar{\omega} |D_{\alpha'}|^2 \frac{1}{Z_{\alpha'}} \right) = \bar{\omega} (|D_{\alpha'}|^2 \bar{\omega}) |D_{\alpha'}|^2 \frac{1}{Z_{\alpha'}} + \bar{\omega}^2 |D_{\alpha'}|^2 \frac{1}{Z_{\alpha'}}
\]

Hence we have

\[
\left\| D_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right\|_2 \lesssim \left\| D_{\alpha'} \bar{\omega} \right\|_2 \left\| D_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_\infty + \left\| D_{\alpha'} \right\|_2 \left\| \frac{1}{Z_{\alpha'}} \right\|_2
\]

Now using the formula (16) we have

\[
\text{Re} \left( D_{\alpha'} \frac{1}{Z_{\alpha'}} \right) = |D_{\alpha'}| \frac{1}{|Z_{\alpha'}|}, \quad \text{Im} \left( D_{\alpha'} \frac{1}{Z_{\alpha'}} \right) = i \left( \frac{\bar{\omega}}{|Z_{\alpha'}|^2} \partial_{\alpha'} \omega \right)
\]

Hence we have

\[
\left\| |D_{\alpha'}|^2 \frac{1}{Z_{\alpha'}} \right\|_2 \lesssim \left\| |D_{\alpha'}|^2 \omega \right\|_2 \left\| \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \omega \right\|_\infty + \left\| |D_{\alpha'}|^2 \frac{1}{Z_{\alpha'}} \right\|_2
\]

\[
\left\| \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'}^2 \omega \right\|_2 \lesssim \left\| |D_{\alpha'}|^2 \omega \right\|_2 \left\| \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \omega \right\|_\infty + \left\| |D_{\alpha'}|^2 \frac{1}{Z_{\alpha'}} \right\|_2 + \left\| D_{\alpha'} \bar{\omega} \right\|_2 \left\| \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \omega \right\|_\infty
\]

\[
+ \left\| |D_{\alpha'}|^2 \frac{1}{|Z_{\alpha'}|^2} \right\|_\infty \left\| |D_{\alpha'}|^2 \omega \right\|_2
\]

We also see that

\[
|D_{\alpha'}|^2 \frac{1}{Z_{\alpha'}} = |D_{\alpha'}| \left( \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) = \left( |D_{\alpha'}| \frac{1}{|Z_{\alpha'}|} \right) \partial_{\alpha'} \frac{1}{Z_{\alpha'}} + \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}}
\]
We have
\[ \left\| \frac{1}{|Z_{\sigma'}|} \partial_{\alpha'} \frac{1}{Z_{\sigma'}} \right\|_2 \leq \left\| D_{\alpha'} \right\|_\infty \left\| \partial_{\alpha'} \frac{1}{Z_{\sigma'}} \right\|_2 + \left\| D_{\alpha'} \right\|^2 \left\| \frac{1}{Z_{\sigma'}} \right\|_2 \]
Now recall the formula of Θ from (26)
\[ \Theta = i \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\sigma'}} - i \operatorname{Re}(I - \mathbb{H}) \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\sigma'}} \right) \]
We have
\[ \left\| \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\sigma'}} \right) \right\|_2 \leq \left\| \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \omega \right\|_\infty \left\| \partial_{\alpha'} \frac{1}{Z_{\sigma'}} \right\|_2 + \left\| \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'}^2 \frac{1}{Z_{\sigma'}} \right\|_2 \]
and hence from Proposition 9.5 we obtain
\[ \left\| \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \Theta \right\|_2 \leq \left\| D_{\alpha'} \right\|_\infty \left\| \partial_{\alpha'} \frac{1}{Z_{\sigma'}} \right\|_2 + \left\| \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\sigma'}} \right) \right\|_2 \]
5) \[ |D_{\alpha'}| \frac{1}{|Z_{\sigma'}|} \in \mathcal{W} \cap \mathcal{C}, |D_{\alpha'}| \frac{1}{|Z_{\sigma'}|} \in \mathcal{W} \cap \mathcal{C} \text{ and } \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \omega \in \mathcal{W} \cap \mathcal{C} \]
Proof: The inclusion in \( \mathcal{C} \) is known as it is part of energy estimate for \( E_\sigma \) for \( \sigma = 0 \). Now we have already shown that all the quantities are in \( L^\infty \) and using the above estimates like \( |D_{\alpha'}|^2 \frac{1}{|Z_{\sigma'}|} \in L^2 \) we are done.
6) \[ D_t b_{\alpha'} \in L^\infty, \partial_{\alpha'} D_t b \in L^\infty \]
Proof: We already know that \( E_\sigma |_{\sigma = 0} \) controls \( D_t (b_{\alpha'} - D_{\alpha'} Z_t - \overline{D}_{\alpha'} Z_t) \in L^\infty \). Now
\[ D_t D_{\alpha'} Z_t = -(D_{\alpha'} Z_t)^2 + D_{\alpha'} Z_{tt} \]
and hence as \( D_{\alpha'} Z_t \in L^\infty, D_{\alpha'} Z_{tt} \in L^\infty \) we have \( D_t D_{\alpha'} Z_t \in L^\infty \). Hence we have \( D_t b_{\alpha'} \in L^\infty \).
We now have \( \partial_{\alpha'} D_t b = b_{\alpha'}^2 + D_t b_{\alpha'} \) and as \( b_{\alpha'} \in L^\infty \) we see that \( \partial_{\alpha'} D_t b \in L^\infty \).
7) \[ \frac{1}{|Z_{\alpha'}|} D_{\alpha'}^2 Z_t \in \mathcal{C}, \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} D_{\alpha'} Z_t \in \mathcal{C}, \frac{1}{|Z_{\alpha'}|^3} \partial_{\alpha'} Z_{t,\alpha'} \in \mathcal{C} \text{ and } \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \overline{D}_{\alpha'} Z_t \in \mathcal{C} \text{ and similarly } D_{\alpha'}^2 \overline{D}_{\alpha'} Z_t \in \mathcal{C} \]
Proof: From the energy we know that \( \frac{\sqrt{A}}{|Z_{\alpha'}|} D_{\alpha'}^2 Z_t \in \dot{H}^{\frac{5}{2}} \). Hence as \( \sqrt{A} D_{\alpha'}^2 Z_t \in L^2 \) we see that \( \sqrt{A} D_{\alpha'}^2 Z_t \in \mathcal{C} \). Hence from Lemma 4.3 we see that
\[ \left\| \frac{1}{|Z_{\alpha'}|} D_{\alpha'}^2 Z_t \right\|_C \leq \left\| \frac{\sqrt{A}}{|Z_{\alpha'}|} D_{\alpha'}^2 Z_t \right\|_C \left\| \frac{1}{\sqrt{A}} \right\| \]
As \( \omega \in \mathcal{W} \) we again get from Lemma 4.3 that \( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} D_{\alpha'} Z_t \in \mathcal{C} \).
Now
\[ \frac{1}{|Z_{\alpha'}|} D_{\alpha'}^2 Z_t = \left( D_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right) D_{\alpha'} |Z_t| + \frac{\sqrt{A}}{|Z_{\alpha'}|^3} \partial_{\alpha'} Z_{t,\alpha'} \]
Hence from Lemma 4.3 we have
\[ \left\| \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} Z_{t,\alpha'} \right\|_C \lesssim \| \omega \|_W^2 \left\| \frac{1}{|Z_{\alpha'}|} D^2_{\alpha'} Z_t \right\|_C + \| \omega \|_W^2 \left\| D_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right\|_C \| D_{\alpha'} Z_t \|_W \]

Now we see that
\[ \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} D_{\alpha'} Z_t = \left( |D_{\alpha'}| \frac{1}{Z_{\alpha'}} \right) |D_{\alpha'}| Z_t + \frac{\omega}{|Z_{\alpha'}|^2} \partial_{\alpha'} Z_{t,\alpha'} \]

Hence by using Lemma 4.3 we obtain
\[ \left\| \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} D_{\alpha'} Z_t \right\|_C \lesssim \left\| D_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_C \| D_{\alpha'} Z_t \|_W + \| \omega \|_W \left\| \frac{1}{|Z_{\alpha'}|^3} \partial_{\alpha'} Z_{t,\alpha'} \right\|_C \]

Now we recall the equation for \( D_{\alpha'} Z_t \) for \( \sigma = 0 \) from (33) and (34)
\[ \left( D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \right) D_{\alpha'} Z_t = -2(D_{\alpha'} Z_t)(D_t D_{\alpha'} Z_t) - i \left( D_{\alpha'} \frac{1}{Z_{\alpha'}} \right) J_1 - i \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1 \]

Hence from Lemma 4.3 we have
\[ \| D_t^2 D_{\alpha'} Z_t \|_C \lesssim \| A_1 \|_W \left\| \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} D_{\alpha'} Z_t \right\|_C + \| D_{\alpha'} Z_t \|_W \left\| D_t D_{\alpha'} Z_t \right\|_C \]
\[ \quad + \| D_{\alpha'} \frac{1}{Z_{\alpha'}} \|_C \| J_1 \|_W + \left\| \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1 \right\|_C \]

8) (I - \( \mathbb{H} \)) \( D_t^2 D_{\alpha'}^2 Z_t \in L^2 \)

Proof: For a function \( f \) satisfying \( \mathbb{P}_A f = 0 \) we have from Proposition 9.1
\[ (I - \mathbb{H}) D_t^2 f = [D_t, \mathbb{H}] D_t f + D_t[D_t, \mathbb{H}] f \]
\[ \quad = [b, \mathbb{H}] \partial_{\alpha'} D_t f + D_t[b, \mathbb{H}] \partial_{\alpha'} f \]
\[ \quad = 2[b, \mathbb{H}] \partial_{\alpha'} D_t f + [D_t b, \mathbb{H}] \partial_{\alpha'} f - [b, b_\alpha] \partial_{\alpha'} f \]

As \( \mathbb{P}_A D_{\alpha'}^2 Z_t = 0 \) we obtain from Proposition 9.6 and Proposition 9.7
\[ \| (I - \mathbb{H}) D_t^2 D_{\alpha'}^2 Z_t \|_2 \lesssim \| b_{\alpha'} \|_\infty \| D_t D_{\alpha'}^2 Z_t \|_2 + \| \partial_{\alpha'} D_t b \|_{H^2} \| D_{\alpha'}^2 Z_t \|_2 + \| b_{\alpha'} \|_\infty^2 \| D_{\alpha'}^2 Z_t \|_2 \]

9) (I - \( \mathbb{H} \)) \( i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} D_{\alpha'}^2 Z_t \) \( \in L^2 \)

Proof: We see that
\[ (I - \mathbb{H}) \left( i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} D_{\alpha'}^2 Z_t \right) = i \left[ \frac{A_1}{|Z_{\alpha'}|^2}, \mathbb{H} \right] \partial_{\alpha'} D_{\alpha'}^2 Z_t \]
and hence we have from Proposition 9.5
\[ \| (I - \mathbb{H}) \left( i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} D_{\alpha'}^2 Z_t \right) \|_2 \lesssim \| D_{\alpha'}^2 Z_t \|_2 \left\{ \| \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} A_1 \|_\infty + \| A_1 \|_\infty \| D_{\alpha'} \frac{1}{|Z_{\alpha'}|} \|_\infty \right\} \]
10) $R_3 \in L^2$

Proof: We recall from (45) the formula of $R_3$

$$R_3 = \{-2(D_{\alpha}^2 Z_{tt}) + 6(D_{\alpha} Z_t)(D_{\alpha}^2 Z_t)\} \{D_{\alpha} Z_t\} + \{-4(D_{\alpha} Z_{tt}) + 6(D_{\alpha} Z_t)^2\} \{D_{\alpha}^2 Z_t\}$$

$$- 2(D_{\alpha}^2 Z_t)(D_{\alpha} Z_{tt}) - 4(D_{\alpha} Z_t)(D_{\alpha}^2 Z_{tt}) - 2i\overline{\omega}(D_{\omega}\overline{\omega}) \left( \frac{1}{|Z_{\alpha}|^2} \partial_{\alpha} J_1 \right)$$

$$- i(D_{\alpha} J_1) \left( D_{\alpha} \frac{1}{Z_{\alpha}} \right) - i J_1 \left( D_{\alpha}^2 \frac{1}{Z_{\alpha}} \right)$$

Hence using Lemma 4.3 we easily have the estimate

$$\|R_3\|_2 \lesssim \left\{ \|D_{\alpha}^2 Z_{tt}\|_2 + \|D_{\alpha} Z_t\|_\infty \|D_{\alpha}^2 Z_t\|_\infty \right\} \|D_{\alpha} Z_t\|_\infty + \left\{ \|D_{\alpha} Z_{tt}\|_\infty + \|D_{\alpha} Z_t\|_\infty^2 \right\} \|D_{\alpha}^2 Z_t\|_2$$

$$+ \|D_{\alpha} Z_t\|_2 \|D_{\alpha} Z_{tt}\|_\infty + \|D_{\alpha} Z_t\|_\infty \|D_{\alpha}^2 Z_{tt}\|_2 + \left\| \frac{1}{|Z_{\alpha}|^2} \partial_{\alpha} \overline{\omega} \right\|_c \left\| \frac{1}{|Z_{\alpha}|^2} \partial_{\alpha} J_1 \right\|_c$$

$$+ \|D_{\alpha} J_1\|_2 \left\| D_{\alpha} \frac{1}{Z_{\alpha}} \right\|_\infty + \|J_1\|_\infty \left\| D_{\alpha}^2 \frac{1}{Z_{\alpha}} \right\|_2$$

11) $|D_{\alpha}| \left( \frac{1}{|Z_{\alpha}|^3} \partial_{\alpha} J_1 \right) \in L^2$ and hence $\frac{1}{|Z_{\alpha}|^2} \partial_{\alpha} J_1 \in L^\infty$

Proof: As $J_1$ is real valued we have

$$|D_{\alpha}| \left( \frac{1}{|Z_{\alpha}|^3} \partial_{\alpha} J_1 \right)$$

$$= \text{Re}(\mathbb{I} - \mathbb{H}) \left\{ \frac{1}{|Z_{\alpha}|^3} \partial_{\alpha} \left( \frac{1}{|Z_{\alpha}|^2} \partial_{\alpha} J_1 \right) \right\}$$

$$= \text{Re} \left\{ \left[ \frac{1}{|Z_{\alpha}|^3}, \mathbb{H} \right] \partial_{\alpha} \left( \frac{1}{|Z_{\alpha}|^2} \partial_{\alpha} J_1 \right) - \omega^3 \left[ \overline{\mathbb{W}}^3 \right] \partial_{\alpha} \left( \frac{1}{|Z_{\alpha}|^2} \partial_{\alpha} J_1 \right) \right\}$$

$$+ \text{Re} \left\{ \omega^3 (\mathbb{I} - \mathbb{H}) \left[ \overline{\mathbb{W}}^3 \right] \partial_{\alpha} \left( \frac{1}{|Z_{\alpha}|^2} \partial_{\alpha} J_1 \right) \right\}$$

Now applying $(\mathbb{I} - \mathbb{H})$ on the equation for $D_{\alpha}^2 Z_t$ from (44) we obtain

$$\left\| (\mathbb{I} - \mathbb{H}) \left\{ \omega^3 |D_{\alpha}| \left( \frac{1}{|Z_{\alpha}|^3} \partial_{\alpha} J_1 \right) \right\} \right\|_2 \lesssim \left\| (\mathbb{I} - \mathbb{H}) D_{\alpha}^2 D_{\alpha}^2 Z_t \right\|_2 + \|R_3\|_2$$

$$+ \left\| (\mathbb{I} - \mathbb{H}) \left( i \frac{A_1}{|Z_{\alpha}|^2} \partial_{\alpha} D_{\alpha}^2 Z_t \right) \right\|_2$$
Hence we have from Proposition 9.5
\[
\left\| D_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1 \right) \right\|_2 \lesssim \left\{ \left\| \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \right) \right\|_2 + \left\| D_{\alpha'} (p) \right\|_2 \right\} \left\| \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1 \right\|_2 \\
+ \left\| (I - H) \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1 \right) \right\|_2
\]

Now we just use Proposition 9.8 with the functions \( f = \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1 \) and \( w = \frac{1}{|Z_{\alpha'}|^2} \) and we easily get that \( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1 \in L^\infty \).

5.3. Closing the energy estimate for \( E_{\text{high}} \)

We now complete the proof of Theorem 5.1. To simplify the calculations, we will continue to use the notation used in 5.2 and introduce another notation: If \( a(t), b(t) \) are functions of time we write \( a \approx b \) if there exists a universal polynomial \( P \) with non-negative coefficients such that \( |a(t) - b(t)| \leq P(E_{\text{high}}(t)) \). Observe that \( \approx \) is an equivalence relation. With this notation, proving Theorem 5.1 is equivalent to showing \( \frac{dE_{\text{high}}(t)}{dt} \approx 0 \).

Now we know from Theorem 4.4 that
\[
\frac{dE_\sigma(t)}{dt} \leq P(E_\sigma(t))
\]
and hence this is true for \( \sigma = 0 \) with the same polynomial \( P \). Hence we have
\[
\frac{d(E_\sigma|_{\sigma=0})(t)}{dt} \leq P((E_\sigma|_{\sigma=0})(t)) \leq P(E_{\text{high}}(t))
\]

Hence only need to control the time derivative of \( E_{\text{high}} - E_\sigma|_{\sigma=0} \). Hence
\[
\frac{dE_{\text{high}}(t)}{dt} \approx \frac{d}{dt} \left\{ \int |D_t D_{\alpha'}^2 Z_t|^2 \, d\alpha' + \int |\partial_{\alpha'} |^2 \left( \frac{\sqrt{A_1}}{|Z_{\alpha'}|} D_{\alpha'}^2 Z_t \right) |^2 \, d\alpha' \right\}
\]
The right hand side is the time derivative of
\[
\int |D_t f|^2 \, d\alpha' + \int |\partial_{\alpha'} |^2 \left( \frac{\sqrt{A_1}}{|Z_{\alpha'}|} f \right) |^2 \, d\alpha'
\]
where \( f = D_{\alpha'} Z_t \) and we have \( \mathbb{P}_H f = f \). Now by using Lemma 4.4 we see that
\[
\frac{dE_{\text{high}}(t)}{dt} \approx 2 \text{Re} \left\{ \int (D_t \bar{f})(\bar{D}_t f) \, d\alpha' + \int \left| \partial_{\alpha'} | \left( \frac{\sqrt{A_1}}{|Z_{\alpha'}|} f \right) \right| D_t \left( \frac{\sqrt{A_1}}{|Z_{\alpha'}|} \bar{f} \right) \, d\alpha' \right\}
\]
Now using \([11]\) we obtain
\[
\frac{dE_{\text{high}}(t)}{dt} \approx 2 \text{Re} \left\{ \int (D_t \bar{f})(\bar{D}_t f) \, d\alpha' + \int \left( \frac{\sqrt{A_1}}{|Z_{\alpha'}|} |\partial_{\alpha'} | \left( \frac{\sqrt{A_1}}{|Z_{\alpha'}|} f \right) \right) (D_t \bar{f}) \, d\alpha' \right\}
\]
Now using \([12]\) and combing terms we get
\[
\frac{dE_{\text{high}}(t)}{dt} \approx 2 \text{Re} \int \left( D_t^2 f + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} f \right) (D_t \bar{f}) \, d\alpha'
\]
As $D_t f = D_i D_{x_i} Z_i \in L^2$ we only need to show that the other term in in $L^2$. Now the equation for $D_{\alpha'} Z_t$ from (44) implies

$$
\left( D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \right) D_{\alpha'}^2 Z_t = -i \omega^3 |D_{\alpha'}| \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J \right) + R_3
$$

As we have already shown that $|D_{\alpha'}| \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J \right) \in L^2$ and $R_3 \in L^2$, this implies that the right hand side of the above equation is in $L^2$ and the proof of Theorem 5.1 is complete.

5.4. **Equivalence of $E_{\text{high}}$ and $E_{\text{high}}$**

We now give a simpler description of the energy $E_{\text{high}}$. Recall the definition of $E_{\text{high}}$ from (3.1)

$$
E_{\text{high}}(t) = \left\{ \begin{array}{c}
\left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2^2 + \left\| \frac{1}{Z_{\alpha'}} \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right\|_2^2 + \left\| Z_{t,\alpha'} \right\|_2^2 + \left\| \frac{1}{Z_{\alpha'}} \partial_{\alpha'} Z_{t,\alpha'} \right\|_2^2 + \left\| \frac{1}{Z_{\alpha'}} \partial_{\alpha'} Z_{t,\alpha'} \right\|_2^2 \end{array} \right. \quad \in H^\perp$

**Proposition 5.2.** There exists universal polynomials $P_1, P_2$ with non-negative coefficients so that if $(Z, Z_t)(t)$ is a smooth solution to the water wave equation (5) with $\sigma = 0$ in the time interval $[0, T]$ satisfying $(Z_{\alpha'} - 1, \frac{1}{Z_{\alpha'}}, -1, Z_t) \in L^\infty([0, T], H^*(\mathbb{R}) \times H^*(\mathbb{R}) \times H^*+\mathbb{R})$ for all $s \geq 2$, then for all $t \in [0, T]$ we have

$$
E_{\text{high}}(t) \leq P_1(E_{\text{high}}(t)) \quad \text{and} \quad E_{\text{high}}(t) \leq P_2(E_{\text{high}}(t))
$$

**Proof.** Let $E_{\text{high}}(t) < \infty$ and recall the definition of $E_{\text{high}}$, $E_{\text{high}}$ from (3.1), (12) respectively. As $E_{\text{high}}$ controls $E_{\|\sigma\| = 0}$, we see from Proposition 4.4 that we already have control over $E_{\|\sigma\| = 0}$. Now from 5.2 we have that $\left\| \frac{1}{Z_{\alpha'}} \partial_{\alpha'} Z_{t,\alpha'} \right\|_2^2 \lesssim P_2(E_{\text{high}}(t))$. The last term to control is $\frac{1}{Z_{\alpha'}} \partial_{\alpha'} Z_{t,\alpha'}$, which can be easily controlled in $H^\perp$ by using Lemma 4.3

$$
\left\| \frac{1}{Z_{\alpha'}} \partial_{\alpha'} Z_{t,\alpha'} \right\|_c^2 \lesssim \left\| \omega \right\|_{W}^3 \left\| \frac{1}{Z_{\alpha'}} \partial_{\alpha'} Z_{t,\alpha'} \right\|_c \lesssim P_2(E_{\text{high}}(t))
$$

Now we assume that $E_{\text{high}}(t) < \infty$. We use Proposition 9.8 with $f = \frac{1}{Z_{\alpha'}}$, and $w = \frac{1}{Z_{\alpha'}}$ to obtain

$$
\left\| \frac{1}{Z_{\alpha'}} \right\|_{L^\infty(\mathbb{R})}^2 \lesssim \left\| \frac{1}{Z_{\alpha'}} \right\|_2^2 \left\| \frac{1}{Z_{\alpha'}} \right\|_2 \left\| \frac{1}{Z_{\alpha'}} \right\|_2^4 \lesssim \left\| \frac{1}{Z_{\alpha'}} \right\|_2^2 \left\| \frac{1}{Z_{\alpha'}} \right\|_2 \left\| \frac{1}{Z_{\alpha'}} \right\|_2^2 \left\| \frac{1}{Z_{\alpha'}} \right\|_2^2 \left\| \frac{1}{Z_{\alpha'}} \right\|_2^4 \lesssim P_1(E_{\text{high}}(t)).
$$

Hence we see that $E_{\|\sigma\| = 0}$ is controlled by $E_{\text{high}}$ and hence by Proposition 4.4 we know that $E_{\|\sigma\| = 0}$ is controlled. Hence we only need to control the last two terms of $E_{\text{high}}$ from (43). Now following
the proof of $\frac{1}{|Z_{\alpha'}|} \partial^2_{\alpha} \frac{1}{Z_{\alpha'}} \in L^2$ in 6.2 we see that $\|D_{\alpha'}^2 \mathcal{Z}_t\|_2 \lesssim P_1(\mathcal{E}_{\text{high}}(t))$. Following the proof of $|D_{\alpha'}^2 \mathcal{Z}_t| \in L^2$ in 6.2 we see that $\|D_{\alpha'}^2 \mathcal{Z}_t\|_2 \lesssim P_1(\mathcal{E}_{\text{high}}(t))$.

We now observe from Lemma 4.3 that

$$\|\frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \mathcal{Z}_t, t, \alpha\|_c \lesssim \|\omega\|^3 \|\frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \mathcal{Z}_t, t, \alpha\|_c \lesssim P_1(\mathcal{E}_{\text{high}}(t))$$

Now by following the proof of $\frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \mathcal{Z}_t, t, \alpha\|_c \in C$ in 6.2 we see that $\|\frac{1}{|Z_{\alpha'}|} D_{\alpha'}^2 \mathcal{Z}_t\|_c \lesssim P_1(\mathcal{E}_{\text{high}}(t))$. Hence we have from Lemma 4.3

$$\|\nabla A_1 \|_c \lesssim \|\nabla A_1\|_W \|\frac{1}{|Z_{\alpha'}|} D_{\alpha'}^2 \mathcal{Z}_t\|_c \lesssim P_1(\mathcal{E}_{\text{high}}(t))$$

This proves the proposition. □

6. Auxiliary Energy $\mathcal{E}_{\text{aux}}$

In this section we again consider a solution to the water wave equation with zero surface tension and prove an apriori energy estimate for the energy $\mathcal{E}_{\text{aux}}$ defined in 3.3. More precisely we show that as long as $\mathcal{E}_{\text{high}}$ is controlled, the energy $\mathcal{E}_{\text{aux}}$ is also controlled. To prove this, we first define an energy $E_{\text{aux}}$, prove an apriori estimate for it in Theorem 6.1 and then in Proposition 6.4 show that the energies $\mathcal{E}_{\text{aux}}$ and $E_{\text{aux}}$ are equivalent.

The energy $\mathcal{E}_{\text{aux}}$ is used in the definition of the energy $\mathcal{E}_\Delta$ in 3.1 and is used crucially in the proof of Theorem 3.1. We refer to 3.2 to the discussion regarding the necessity of having the energy $\mathcal{E}_{\text{aux}}$ as part of the energy $\mathcal{E}_\Delta$ and how it is used.

Let us now define the energy $E_{\text{aux}}$ and state out main result for this section. Define

$$E_{\text{aux}} = \|Z_{\alpha'}\|_\infty \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \|_\infty^2 + \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \mathcal{Z}_t, t, \alpha\|_c^2 + \|\frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \mathcal{Z}_t, t, \alpha\|_c^2 \|\frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} D_{\alpha'}^2 \mathcal{Z}_t\|_2^2 \|D_{\alpha'} \left( \frac{1}{Z_{\alpha'}} \partial_{\alpha'} D_{\alpha'}^2 \mathcal{Z}_t \right)\|_2^2 + \|\frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} D_{\alpha'}^2 \mathcal{Z}_t\|_2^2 \|D_{\alpha'} \left( \frac{1}{Z_{\alpha'}} \partial_{\alpha'} D_{\alpha'}^2 \mathcal{Z}_t \right)\|_2^2 (46)$$

**Theorem 6.1.** Let $T, \lambda > 0$ and let $(Z, \mathcal{Z}_t)(t)$ be a smooth solution to 3.1 with $\sigma = 0$ in the time interval $[0, T]$, such that for all $s \geq 2$ we have $(Z_{\alpha'} - 1, \frac{1}{Z_{\alpha'}}, -1, Z_t) \in L^\infty([0, T], H^s(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s+\frac{1}{2}}(\mathbb{R}))$. Then $\sup_{t \in [0, T]} \mathcal{E}_{\text{high}}(t) < \infty$, $\sup_{t \in [0, T]} E_{\text{aux}}(t) < \infty$ and there exists a universal polynomial $P$ with non-negative coefficients such that for all $t \in [0, T)$ we have

$$\frac{d}{dt}(\lambda E_{\text{aux}}(t)) \leq P(\mathcal{E}_{\text{high}}(t))(\lambda E_{\text{aux}}(t)) (47)$$

**Remark 6.2.** Similar to the case of energy $E_\sigma$ as mentioned in Remark 4.2 the energy $E_{\text{aux}}$ contains a term which is the $L^\infty$ norm of a function and hence for this term we replace the time derivative by the upper Dini derivative.
Note that in the above result, the constant \( \lambda > 0 \) does not actually play any role and one can simply rewrite (47) as

\[
\frac{d}{dt} E_{\text{aux}}(t) \leq P(\mathcal{E}_{\text{high}}(t)) E_{\text{aux}}(t)
\]

The reason we add a \( \lambda \) to the energy estimate (47) is that later on we will need to replace this \( \lambda \) by \( \sigma \) which denotes the surface tension of solution \( A \) in Theorem 3.1. Another important reason is that having a \( \lambda \) makes the proof of the Theorem 6.1 more readable, as we will have lots of terms of both \( \mathcal{E}_{\text{high}} \) and \( E_{\text{aux}} \) showing up in the proof of Theorem 6.1 and the constant \( \lambda \) will help us distinguish whether the term corresponds to \( \mathcal{E}_{\text{high}} \) or \( E_{\text{aux}} \).

We employ a similar strategy to prove this theorem as we used to prove Theorem 5.1. To prove this theorem, we first obtain the quasilinear equation in §6.1, then control the quantities controlled in §6.2 and then finish the proof in §6.3. Finally we show in §6.4 that \( E_{\text{aux}} \) and \( \mathcal{E}_{\text{aux}} \) are equivalent energies.

6.1. Quasilinear equation

Let us recall the equation of \( D^2_{\alpha'} Z_t \) from (44)

\[
\left( D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \right) D^2_{\alpha'} Z_t = -i \omega^3 |D_{\alpha'}| \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1 \right) + R_3
\]

with \( R_3 \) as given in (45) along with the identities \( J_1 = D_t A_1 + A_1 (b_{\alpha'} - D_{\alpha'} Z_t - \bar{D}_{\alpha'} Z_t) \) and \( Z_{tt} - i = -i \frac{A_1}{Z_{\alpha'}} \) from (90) and (10) respectively. Applying \( \frac{\lambda^+}{Z_{\alpha'}^{1/2}} \partial_{\alpha'} \) to the above equation we obtain

\[
\left( D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \right) \frac{\lambda^+}{Z_{\alpha'}^{1/2}} \partial_{\alpha'} D^2_{\alpha'} Z_t = -i \omega^3 \frac{\lambda^+}{Z_{\alpha'}^{1/2}} \partial_{\alpha'} |D_{\alpha'}| \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1 \right) + R_4
\]

where

\[
R_4 = -3 \omega^3 \left( \frac{\lambda^+}{Z_{\alpha'}^{1/2}} \partial_{\alpha'} \omega \right) |D_{\alpha'}| \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1 \right) + \frac{\lambda^+}{Z_{\alpha'}^{1/2}} \partial_{\alpha'} R_3
\]

\[
+ \left[ D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} , \frac{\lambda^+}{Z_{\alpha'}^{1/2}} \partial_{\alpha'} \right] D^2_{\alpha'} Z_t
\]

Let us try to simplify the terms above

a) \( D_t \frac{1}{Z_{\alpha'}^{1/2}} = -\frac{1}{2Z_{\alpha'}^{3/2}} D_t Z_{\alpha'} = -\frac{1}{2Z_{\alpha'}^{3/2}} (D_{\alpha'} Z_t - b_{\alpha'}) \)

b) \( \partial_{\alpha'} \frac{1}{Z_{\alpha'}^{1/2}} = \frac{1}{2} Z_{\alpha'}^{1/2} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} = \left( \frac{1}{2} Z_{\alpha'} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \frac{1}{Z_{\alpha'}^{1/2}} \)

c) \( \frac{\lambda^+}{Z_{\alpha'}^{1/2}} \partial_{\alpha'} , D_t = \frac{\lambda^+}{Z_{\alpha'}^{1/2}} |\partial_{\alpha'} , D_t| + \left[ \frac{\lambda^+}{Z_{\alpha'}^{1/2}} , D_t \right] \partial_{\alpha'} = \left( \frac{b_{\alpha'}}{2} + \frac{D_{\alpha'} Z_t}{2} \right) \frac{\lambda^+}{Z_{\alpha'}^{1/2}} \partial_{\alpha'} \)
Combining the above formulæ we have

\[ D_t \left[ \frac{\lambda^2}{Z_{\alpha'}} \partial_{\alpha'} D_t \right] = D_t \left[ \frac{\lambda^2}{Z_{\alpha'}} \partial_{\alpha'} D_t \right] + \left[ \frac{\lambda^2}{Z_{\alpha'}} \partial_{\alpha'} D_t \right] D_t \]

\[ = \left\{ \frac{D_t b_{\alpha'}}{2} + \frac{D_t D_{\alpha'} Z_t}{2} - \left( \frac{b_{\alpha'}}{2} + \frac{D_{\alpha'} Z_t}{2} \right) \right\} \frac{\lambda^2}{Z_{\alpha'}^2} \partial_{\alpha'} + \left( b_{\alpha'} + D_{\alpha'} Z_t \right) \frac{\lambda^2}{Z_{\alpha'}^2} \partial_{\alpha'} D_t \]

\[ e) \left[ \frac{\lambda^2}{Z_{\alpha'}^2} \partial_{\alpha'}, \frac{i}{\alpha} \frac{A_1}{\alpha} \partial_{\alpha'} \right] \]

\[ = \left\{ 2i A_1 \left( \frac{1}{Z_{\alpha'}} \right) - \frac{i}{\alpha} A_1 \left( \frac{1}{Z_{\alpha'}} \right) \right\} \frac{\lambda^2}{Z_{\alpha'}^2} \partial_{\alpha'} \]

Combining the above formulæ we have

\[ \left( D_t^2 + i \frac{A_1}{\alpha} \right) \frac{\lambda^2}{Z_{\alpha'}^2} \partial_{\alpha'} D_t^2 \frac{\lambda^2}{Z_{\alpha'}^2} \partial_{\alpha'} - 2i A_1 \left( \frac{1}{Z_{\alpha'}} \right) + \frac{i}{\alpha} A_1 \left( \frac{1}{Z_{\alpha'}} \right) \frac{\lambda^2}{Z_{\alpha'}^2} \partial_{\alpha'} D_t^2 \frac{\lambda^2}{Z_{\alpha'}^2} \partial_{\alpha'} \]

where

\[ R_4 = \left\{ \frac{D_t b_{\alpha'}}{2} + \frac{D_t D_{\alpha'} Z_t}{2} - \left( \frac{b_{\alpha'}}{2} + \frac{D_{\alpha'} Z_t}{2} \right) \right\} \frac{\lambda^2}{Z_{\alpha'}^2} \partial_{\alpha'} D_t^2 \frac{\lambda^2}{Z_{\alpha'}^2} \partial_{\alpha'} \]

\[ = \left\{ 2i A_1 \left( \frac{1}{Z_{\alpha'}} \right) + \frac{i}{\alpha} A_1 \left( \frac{1}{Z_{\alpha'}} \right) \right\} \frac{\lambda^2}{Z_{\alpha'}^2} \partial_{\alpha'} D_t^2 \frac{\lambda^2}{Z_{\alpha'}^2} \partial_{\alpha'} \]

\[ - 3i \omega^2 \left( \frac{\lambda^2}{Z_{\alpha'}^2} \partial_{\alpha'} \right) \right| D_{\alpha'} \right| \left( \frac{1}{Z_{\alpha'}} \right) \frac{\lambda^2}{Z_{\alpha'}^2} \partial_{\alpha'} D_t^2 \frac{\lambda^2}{Z_{\alpha'}^2} \partial_{\alpha'} \]

and \( R_3 \) is as defined in \( (49) \).

6.2. Quantities controlled by the energy \( \lambda E_{aux} \)

In this section whenever we write \( f \in L^2_{\alpha} \), what we mean is that there exists a universal polynomial \( P \) with nonnegative coefficients such that \( \| f \|_2 \leq (\lambda E_{aux})^\alpha P(\epsilon_{high}) \). Similar definitions for \( f \in \dot{H}^\alpha_{\epsilon} \) and \( f \in L^2_{\alpha} \). We define the spaces \( C_{\lambda} \) and \( W_{\lambda} \) as follows

1. If \( w \in L^\infty_{\alpha} \) and \( |D_{\alpha'}|w \in L^2_{\alpha} \), then we say \( f \in W_{\lambda} \). Define

\[ \| w \|_{W_{\lambda}} = \| w \|_{\infty} + \| |D_{\alpha'}|w \|_2 \]

2. If \( f \in \dot{H}^\frac{\alpha}{2} \) and \( f|Z_{\alpha'}| \in L^2_{\alpha} \), then we say \( f \in C_{\lambda} \). Define

\[ \| f \|_{C_{\lambda}} = \| f \|_C = \| f \|_{\dot{H}^\frac{\alpha}{2}} + \left( 1 + \| \partial_{\alpha'} \| |Z_{\alpha'}| \|_2 \right) \| f \| |Z_{\alpha'}| \|_2 \]

Analogous to Lemma \( 4.3 \) we have the following lemma

**Lemma 6.3.** Let \( \alpha_1, \alpha_2, \alpha_3 \geq 0 \) with \( \alpha_1 + \alpha_2 = \alpha_3 \). Then the following properties hold for the spaces \( W_{\lambda} \) and \( C_{\lambda} \)
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(1) If \( w_1 \in \mathcal{W}_{\lambda_1}, \ w_2 \in \mathcal{W}_{\lambda_2}, \) then \( w_1 w_2 \in \mathcal{W}_{\lambda_3}. \) Moreover we have the estimate \( \|w_1 w_2\|_{\mathcal{W}_{\lambda_3}} \lesssim \|w_1\|_{\mathcal{W}_{\lambda_1}} \|w_2\|_{\mathcal{W}_{\lambda_2}}. \)

(2) If \( f \in \mathcal{C}_{\lambda_1} \) and \( w \in \mathcal{W}_{\lambda_2}, \) then \( f \circ w \in \mathcal{C}_{\lambda_3}. \) Moreover \( \|f \|_{\mathcal{C}_{\lambda_3}} \lesssim \|f\|_{\mathcal{C}_{\lambda_1}} \|w\|_{\mathcal{W}_{\lambda_2}}. \)

(3) If \( f \in \mathcal{C}_{\lambda_1}, \ g \in \mathcal{C}_{\lambda_2}, \) then \( f g \|_{\mathcal{W}_{\lambda_3}} \lesssim \|f\|_{\mathcal{C}_{\lambda_1}} \|g\|_{\mathcal{C}_{\lambda_2}}. \)

When we write \( f \in L^2 \) we mean \( f \in L^2_{\lambda_3} \) with \( \alpha = 0. \) Similar notation for \( \hat{H}^\perp, L^\infty, C \) and \( \mathcal{W}. \) This notation is now consistent with the notation used in \( \{1,2,3\}. \) Let us now control the important terms controlled by the energy \( \lambda E_{\text{aux}}. \)

1) \( \lambda^\frac{1}{2}|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \in L^\infty \mathcal{X}, \) \( \lambda^\frac{1}{2}|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \in L^\infty \mathcal{X} \) and \( \lambda^\frac{1}{2}|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \omega \in L^\infty \mathcal{X}. \)

Proof: From the energy we already know that \( \lambda^\frac{1}{2}|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \in L^\infty \mathcal{X}. \) Hence we easily have

\( \lambda^\frac{1}{2} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \in L^\infty \mathcal{X}. \) Recall from \( \{10\} \) that

\[
\text{Re} \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right) = \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \quad \text{and} \quad \text{Im} \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right) = i(\mathcal{Z}|D_{\alpha'}|\omega)
\]

Hence we obtain \( \lambda^\frac{1}{2}|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \in L^\infty \mathcal{X} \) and \( \lambda^\frac{1}{2}|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \omega \in L^\infty \mathcal{X}. \)

2) \( \lambda^\frac{1}{2}|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \in L^2 \mathcal{X}, \) \( \lambda^\frac{1}{2}|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \in L^2 \mathcal{X} \) and \( \lambda^\frac{1}{2}|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \omega \in L^2 \mathcal{X} \) and hence we have that

\( \lambda^\frac{1}{2}|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \in \mathcal{W}_{\mathcal{X}}, \) \( \lambda^\frac{1}{2}|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \in \mathcal{W}_{\mathcal{X}} \) and \( \lambda^\frac{1}{2}|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \omega \in \mathcal{W}_{\mathcal{X}}. \)

Proof: Observe that \( \lambda^\frac{1}{2}|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \in L^2 \mathcal{X} \) as it is part of the energy. Recall from \( \{10\} \) that

\[
\text{Re} \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right) = \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \quad \text{and} \quad \text{Im} \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right) = i(\mathcal{Z}|D_{\alpha'}|\omega)
\]

Hence taking derivatives we obtain

\[
\frac{\lambda^\frac{1}{2}|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|}}{|Z_{\alpha'}|^{\frac{1}{2}} |Z_{\alpha'}|} \leq \left( \lambda^\frac{1}{2}|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \omega \right) \| \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \|_2 + \left( \lambda^\frac{1}{2}|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right) \| D_{\alpha'} |\omega | \|_2
\]

We also see that

\[
\| D_{\alpha'} | \left( \lambda^\frac{1}{2}|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right) \|_2 \leq \left( \lambda^\frac{1}{2}|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right) \| D_{\alpha'} | \|_2 + \left( \lambda^\frac{1}{2}|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right) \| D_{\alpha'} | \|_2
\]

Hence \( \lambda^\frac{1}{2}|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \in \mathcal{W}_{\mathcal{X}}. \) The rest are proven similarly.
3) \( \frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D^2_{\alpha'} Z_t \in L^2_{\sqrt{X}} \) and similarly we have \( \frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D^2_{\alpha'} Z_t \in L^2_{\sqrt{X}} \) and \( \frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D_{\alpha'} Z_t \in L^2_{\sqrt{X}} \) and \( \frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} |D_{\alpha'}| D_{\alpha'} Z_t \in L^2_{\sqrt{X}} \)

Proof: We already have \( \frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D^2_{\alpha'} Z_t \in L^2_{\sqrt{X}} \) as it is part of the energy. We now have

\[
\frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D^2_{\alpha'} Z_t = \left( \frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) (D_{\alpha'} |D_{\alpha'} Z_t|) + \frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D^2_{\alpha'} Z_t
\]

and hence

\[
\left\| \frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D^2_{\alpha'} Z_t \right\| \lesssim \left\| \frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\| \left\| D_{\alpha'} |D_{\alpha'} Z_t| \right\|_2 + \left\| \frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D^2_{\alpha'} Z_t \right\|_2
\]

Now we have

\[
\frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D^2_{\alpha'} Z_t = \left( \frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) |D_{\alpha'}| Z_t + 2 \left( \frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \left( \frac{1}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} Z_{t, \alpha'} \right)
\]

From this we obtain

\[
\left\| \frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} Z_{t, \alpha'} \right\|_2 \lesssim \left\| \frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2 \left\| D_{\alpha'} |Z_t| \right\|_\infty + \left\| \frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2 \left\| \frac{1}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} Z_{t, \alpha'} \right\|_2
\]

Hence we have \( \frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} Z_{t, \alpha'} \in L^2_{\sqrt{X}} \) By taking conjugation and retracing the steps backwards we easily obtain the other estimates \( \frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D^2_{\alpha'} Z_t \in L^2_{\sqrt{X}} \) \( \frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D_{\alpha'} Z_t \in L^2_{\sqrt{X}} \) and \( \frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} |D_{\alpha'}| D_{\alpha'} Z_t \in L^2_{\sqrt{X}} \)

4) \( \frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} Z_{t, \alpha'} \in W_{\sqrt{X}} \cap C_{\sqrt{X}} \) \( \frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D^2_{\alpha'} Z_t \in W_{\sqrt{X}} \cap C_{\sqrt{X}} \) \( \frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D_{\alpha'} Z_t \in W_{\sqrt{X}} \cap C_{\sqrt{X}} \) \( \frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} |D_{\alpha'}| D_{\alpha'} Z_t \in W_{\sqrt{X}} \cap C_{\sqrt{X}} \) and similarly \( \frac{\lambda^\beta}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} |D_{\alpha'}| Z_t \in W_{\sqrt{X}} \cap C_{\sqrt{X}} \)
Proof: We first see that
\[ \left\| \partial_{\alpha'} \left( \frac{\lambda^\frac{3}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} Z_{t,\alpha'} \right) \right\|_2 \lesssim \left\| \lambda^\frac{3}{2} |Z_{\alpha'}|^{\frac{3}{2}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right\|_\infty \left\| \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} Z_{t,\alpha'} \right\|_2 + \left\| \lambda^\frac{3}{2} |Z_{\alpha'}|^{\frac{3}{2}} \partial^2_{\alpha'} Z_{t,\alpha'} \right\|_2 \]

Now using Proposition 9.8 with \( f = \frac{\lambda^\frac{3}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} Z_{t,\alpha'} \) and \( w = \frac{1}{|Z_{\alpha'}|} \) we see that
\[ \left\| \frac{\lambda^\frac{3}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} Z_{t,\alpha'} \right\|^2_{L^\infty \cap H^\frac{1}{2}} \lesssim \left\| \frac{\lambda^\frac{3}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} Z_{t,\alpha'} \right\|_2 \left\| \partial_{\alpha'} \left( \frac{\lambda^\frac{3}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} Z_{t,\alpha'} \right) \right\|_2 + \left\| \frac{\lambda^\frac{3}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} Z_{t,\alpha'} \right\|^2_2 \left\| \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right\|^2_2 \]

As \( |Z_{\alpha'}|f \in L^2_{W} \) we have \( \frac{\lambda^\frac{3}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} Z_{t,\alpha'} \in W_{\mathcal{X}} \cap C_{\mathcal{X}} \). Now we have
\[ \frac{\lambda^\frac{3}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} D_{\alpha'} Z_t = \left( \frac{\lambda^\frac{3}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) |D_{\alpha'}| \hat{Z}_t + \frac{\lambda^\frac{3}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} Z_{t,\alpha'} \]

Hence from Lemma 6.3 we have
\[ \left\| \frac{\lambda^\frac{3}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} D_{\alpha'} Z_t \right\|_{W_{\mathcal{X}} \cap C_{\mathcal{X}}} \lesssim \left\| \lambda^\frac{3}{2} |Z_{\alpha'}|^{\frac{3}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{W_{\mathcal{X}}} \|D_{\alpha'}| \hat{Z}_t\|_{W_{\mathcal{X}}} + \|\hat{Z}_t\|_{W_{\mathcal{X}}} \left\| \frac{\lambda^\frac{3}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} Z_{t,\alpha'} \right\|_{W_{\mathcal{X}} \cap C_{\mathcal{X}}} \]

The estimates \( \frac{\lambda^\frac{3}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} D_{\alpha'} Z_t \in W_{\mathcal{X}} \cap C_{\mathcal{X}} \), \( \frac{\lambda^\frac{3}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} |D_{\alpha'}| Z_t \in W_{\mathcal{X}} \cap C_{\mathcal{X}} \) are proven similarly.

5) \( \frac{\lambda^\frac{3}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} A_1 \in W_{\mathcal{X}} \)

Proof: The proofs for \( \frac{\lambda^\frac{3}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} A_1 \in L^\infty_{\mathcal{X}} \) and \( \frac{\lambda^\frac{3}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial^2_{\alpha'} A_1 \in L^2_{\mathcal{X}} \) follow in exactly the same way as the proofs of \( \frac{\sigma^\frac{3}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} A_1 \in L^\infty \) and \( \frac{\sigma^\frac{3}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial^2_{\alpha'} A_1 \in L^2 \) as done in Section 5.1 in [II].
Hence we have the estimates

\[
\left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} A_1 \right\|_{\infty} \lesssim \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} \left( \frac{1}{Z_{\alpha'}} \right) \right\|_{\infty} \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} Z_{t,\alpha'} \right\| + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} Z_{t,\alpha'} \right\| \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} Z_{t,\alpha'} \right\|_{2}
\]

\[
\left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'}^2 A_1 \right\|_{2} \leq \left\{ \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'}^2 \right\|_{2} + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} \right\|_{2} \right\} \left\| A_1 \right\|_{\infty}
\]

+ \left\| Z_{t,\alpha'} \right\|_{2} \left\{ \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} Z_{t,\alpha'} \right\|_{\infty} + \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{2} \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} Z_{t,\alpha'} \right\|_{2} \right\}.
\]

From this we get

\[
\left\| D_{\alpha'} \left( \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} A_1 \right) \right\|_{2} \lesssim \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{2} \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} A_1 \right\|_{2} + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'}^2 A_1 \right\|_{2}
\]

\[
6) \quad \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'}^2 A_1 \in L^2_{\mathcal{V} X}, \quad \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} \left( \frac{1}{Z_{\alpha'}} \partial_{\alpha'} A_1 \right) \in L^2_{\mathcal{V} X} \text{ and hence } \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'}^2 A_1 \in \mathcal{W}_{\mathcal{V} X} \cap C_{\mathcal{V} X}
\]

Proof: We will first show that \((\mathbb{I} - \mathbb{H}) \left\{ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'}^3 A_1 \right\} \in L^2_{\mathcal{V} X} \). Now from \((20)\) we see that

\[
A_1 = 1 + iZ_t \bar{Z}_{t,\alpha'} - i(\mathbb{I} + \mathbb{H}) \left\{ \text{Re}(Z_t \bar{Z}_{t,\alpha'}) \right\}. \quad \text{Hence we have}
\]

\[
(\mathbb{I} - \mathbb{H}) \left\{ \frac{\lambda^{\frac{1}{2}}}{Z_{\alpha'}^{\frac{1}{2}}} \partial_{\alpha'}^3 A_1 \right\}
\]

\[
= i(\mathbb{I} - \mathbb{H}) \left\{ \left( \frac{Z_t}{Z_{\alpha'}} \right) \frac{\lambda^{\frac{1}{2}}}{Z_{\alpha'}^{\frac{1}{2}}} \partial_{\alpha'}^2 \bar{Z}_{t,\alpha'} \right\}
\]

\[
+ i(\mathbb{I} - \mathbb{H}) \left\{ \left( \frac{\lambda^{\frac{1}{2}}}{Z_{\alpha'}^{\frac{1}{2}}} \partial_{\alpha'}^2 Z_{t,\alpha'} \right) (D_{\alpha'} \bar{Z}_t) + 3 \left( \frac{1}{Z_{\alpha'}^{\frac{1}{2}}} \partial_{\alpha'} Z_{t,\alpha'} \right) \left( \frac{\lambda^{\frac{1}{2}}}{Z_{\alpha'}^{\frac{1}{2}}} \partial_{\alpha'} Z_{t,\alpha'} \right)
\]

\[
+ 3(D_{\alpha'} Z_t) \left( \frac{\lambda^{\frac{1}{2}}}{Z_{\alpha'}^{\frac{1}{2}}} \partial_{\alpha'}^2 Z_{t,\alpha'} \right) \right\}
\]

Now we see that

\[
i(\mathbb{I} - \mathbb{H}) \left\{ \left( \frac{Z_t}{Z_{\alpha'}^{\frac{1}{2}}} \right) \frac{\lambda^{\frac{1}{2}}}{Z_{\alpha'}^{\frac{1}{2}}} \partial_{\alpha'}^3 \bar{Z}_{t,\alpha'} \right\} = \frac{-5i}{2} [Z_t, \mathbb{H}] \left\{ \left( \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \left( \frac{\lambda^{\frac{1}{2}}}{Z_{\alpha'}^{\frac{1}{2}}} \partial_{\alpha'}^2 Z_{t,\alpha'} \right) \right\}
\]

\[
+ i \left[ \mathbb{P}_A \left( \frac{Z_t}{Z_{\alpha'}^{\frac{1}{2}}} \right), \mathbb{H} \right] \partial_{\alpha'} \left( \frac{\lambda^{\frac{1}{2}}}{Z_{\alpha'}^{\frac{1}{2}}} \partial_{\alpha'}^2 Z_{t,\alpha'} \right)
\]
Hence using Proposition 9.3 we have the estimate

\[
\| (I - \mathbb{H}) \left\{ \frac{\lambda^\frac{1}{2}}{Z_{\alpha'}^{\frac{1}{2}}} \partial^2_{\alpha'} \partial_{\alpha'} A_1 \right\} \|_2 \\
\leq \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial^2_{\alpha'} Z_{t,\alpha'} \right\|_2 \left\| D_{\alpha'} |Z_t| \right\|_\infty + \left\| \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} Z_{t,\alpha'} \right\|_2 \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} Z_{t,\alpha'} \right\|_\infty \\
+ \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial^2_{\alpha'} Z_{t,\alpha'} \right\|_2 \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2 + \left\| \partial_{\alpha'} \partial_{\alpha'} A_1 \right\|_\infty \right\} 
\]

Now let’s come back to proving the main estimate \( \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial^2_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right) \in L^2_{\mathbb{V}_X} \). Now as \( A_1 \) is real valued we see that

\[
\frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial^2_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right) = \lambda^\frac{1}{2} \text{Re} \left\{ \frac{\omega^\frac{1}{2} \tau^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{1}{2}}} (I - \mathbb{H}) \partial^2_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right) \right\} 
\]

Hence it is enough to show that we have \( \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{1}{2}}} (I - \mathbb{H}) \partial^2_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right) \in L^2_{\mathbb{V}_X} \). Now observe that

\[
\frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{1}{2}}} (I - \mathbb{H}) \partial^2_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right) \\
= -\left[ \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial^2_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right) \right] (I - \mathbb{H}) \left\{ \lambda^\frac{1}{2} \partial^2_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right) \right\} 
\]

Hence by expanding the second term we see that

\[
\frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{1}{2}}} (I - \mathbb{H}) \partial^2_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right) \\
= -\left[ \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial^2_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right) \right] (I - \mathbb{H}) \left\{ \lambda^\frac{1}{2} \partial^2_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right) \right\} \\
+ (I - \mathbb{H}) \left\{ \frac{2\lambda^\frac{1}{2} \tau^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right) \right\} + (I - \mathbb{H}) \left\{ \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial^3_{\alpha'} A_1 \right\} \\
+ (I - \mathbb{H}) \left\{ \frac{4\lambda^\frac{1}{2} \tau^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right) \right\} 
\]
Hence from Proposition 9.5 we have the estimate

\[
\left\| \frac{\lambda^\frac{7}{2}}{|Z_{\alpha'}|^\frac{7}{2}} \partial_{\alpha'}^2 \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right) \right\|_2 \\
\lesssim \left\| (I - \mathbb{H}) \left\{ \frac{\lambda^\frac{7}{2}}{|Z_{\alpha'}|^\frac{7}{2}} \partial_{\alpha'} A_1 \right\} \right\|_2 + \left\| \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right) \right\|_\infty \\
+ \left\| \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right\|_\infty \left\{ \left\| \frac{\lambda^\frac{7}{2}}{|Z_{\alpha'}|^\frac{7}{2}} \partial_{\alpha'}^2 \omega \right\|_2 + \left\| \frac{\lambda^\frac{7}{2}}{|Z_{\alpha'}|^\frac{7}{2}} \partial_{\alpha'} \omega \right\|_\infty \left\| D_{\alpha'} \omega \right\|_2 \right\} \\
+ \left\| \frac{\lambda^\frac{7}{2}}{|Z_{\alpha'}|^\frac{7}{2}} \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right) \right\|_\infty \\
+ \left\| \frac{1}{|Z_{\alpha'}|^3} \partial_{\alpha'} A_1 \right\|_2 \right\}
\]

From this we easily have the estimate

\[
\left\| \frac{\lambda^\frac{7}{2}}{|Z_{\alpha'}|^\frac{7}{2}} \partial_{\alpha'} A_1 \right\|_2 \\
\lesssim \left\| \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right\|_\infty \left\{ \left\| \frac{\lambda^\frac{7}{2}}{|Z_{\alpha'}|^\frac{7}{2}} \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right) \right\|_2 + \left\| \frac{\lambda^\frac{7}{2}}{|Z_{\alpha'}|^\frac{7}{2}} \partial_{\alpha'}^2 \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right) \right\|_2 \\
+ \left\| \frac{\lambda^\frac{7}{2}}{|Z_{\alpha'}|^\frac{7}{2}} \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right) \right\|_2 \right\}
\]

and

\[
\left\| \partial_{\alpha'} \left( \frac{\lambda^\frac{7}{2}}{|Z_{\alpha'}|^\frac{7}{2}} \partial_{\alpha'}^2 A_1 \right) \right\|_2 \lesssim \left\| \frac{\lambda^\frac{7}{2}}{|Z_{\alpha'}|^\frac{7}{2}} \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right) \right\|_\infty \left\| \frac{1}{|Z_{\alpha'}|^3} \partial_{\alpha'} A_1 \right\|_2 + \left\| \frac{\lambda^\frac{7}{2}}{|Z_{\alpha'}|^\frac{7}{2}} \partial_{\alpha'}^3 A_1 \right\|_2
\]

We also get the estimate \( |D_{\alpha'}| \left( \frac{\lambda^\frac{7}{2}}{|Z_{\alpha'}|^\frac{7}{2}} \partial_{\alpha'}^2 A_1 \right) \in L^2_{\sqrt{\chi}} \) similarly. Hence now using Proposition 9.8 with \( f = \frac{\lambda^\frac{7}{2}}{|Z_{\alpha'}|^\frac{7}{2}} \partial_{\alpha'}^2 A_1 \) and \( w = \frac{1}{|Z_{\alpha'}|^2} \) we obtain

\[
\left\| \frac{\lambda^\frac{7}{2}}{|Z_{\alpha'}|^\frac{7}{2}} \partial_{\alpha'} A_1 \right\|_{L^\infty \cap H^{}_{\frac{7}{2}}} \\
\lesssim \left\| \frac{\lambda^\frac{7}{2}}{|Z_{\alpha'}|^\frac{7}{2}} \partial_{\alpha'} A_1 \right\|_2 \left\| \partial_{\alpha'} \left( \frac{\lambda^\frac{7}{2}}{|Z_{\alpha'}|^\frac{7}{2}} \partial_{\alpha'}^2 A_1 \right) \right\|_2 + \left\| \frac{\lambda^\frac{7}{2}}{|Z_{\alpha'}|^\frac{7}{2}} \partial_{\alpha'}^2 A_1 \right\|_2 \left\| \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right) \right\|_2
\]

7) \( \frac{\lambda^\frac{7}{2}}{|Z_{\alpha'}|^\frac{7}{2}} \partial_{\alpha'} b_{\alpha'} \in L^\infty_{\sqrt{\chi}} \)
Proof: The proof of this estimate is the same as the proof of \( \frac{\sigma}{|Z_{\alpha'}|^2} \partial_\alpha b_{\alpha'} \in L^\infty \) as done in Section 5.1 in [1]. Hence we have the estimate
\[
\left\| \frac{\lambda}{|Z_{\alpha'}|^2} \partial_\alpha b_{\alpha'} \right\|_\infty \lesssim \left\| \frac{\lambda}{|Z_{\alpha'}|^2} |D_{\alpha'}|b_{\alpha'} \right\|_2 + \left\| \frac{\lambda}{|Z_{\alpha'}|^2} \partial_\alpha Z_{t,\alpha'} \right\|_2
\]
and using this we have
\[
\left\| \frac{\lambda}{|Z_{\alpha'}|^2} \partial_\alpha b_{\alpha'} \right\|_2 \lesssim \left\| b_{\alpha'} \right\|_\infty \left\{ \left\| \frac{\lambda}{|Z_{\alpha'}|^2} |D_{\alpha'}|b_{\alpha'} \right\|_2 + \left\| \frac{\lambda}{|Z_{\alpha'}|^2} \partial_\alpha Z_{t,\alpha'} \right\|_2 \right\}
\]
and using this we have
\[
\left\| \frac{\lambda}{|Z_{\alpha'}|^2} \partial_\alpha b_{\alpha'} \right\|_2 \lesssim \left\| b_{\alpha'} \right\|_\infty \left\{ \left\| \frac{\lambda}{|Z_{\alpha'}|^2} |D_{\alpha'}|b_{\alpha'} \right\|_2 + \left\| \frac{\lambda}{|Z_{\alpha'}|^2} \partial_\alpha Z_{t,\alpha'} \right\|_2 \right\}
\]
The estimates for the other terms can now be shown easily.

9) \( \frac{\lambda}{|Z_{\alpha'}|^2} \partial_\alpha Z_{t,\alpha'} \in L^2 \)

Proof: From (10) we know that \( Z_{tt} - i = -\frac{A_1}{Z_{\alpha'}} \). Hence we have
\[
\left\| \frac{\lambda}{|Z_{\alpha'}|^2} \partial_\alpha Z_{t,\alpha'} \right\|_2 \lesssim \left\| \frac{\lambda}{|Z_{\alpha'}|^2} |D_{\alpha'}|A_1 \right\|_2 + \left\| |D_{\alpha'}|A_1 \right\|_2 + \left\| \frac{\lambda}{|Z_{\alpha'}|^2} \partial_\alpha Z_{t,\alpha'} \right\|_2
\]
10) \( \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}} \partial_{\alpha'}^2 Z_{tt,\alpha'} \in L^2_{\sqrt{\lambda}} \) \( \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}} \partial_{\alpha'} D_{\alpha'}^2 Z_{tt} \in L^2_{\sqrt{\lambda}} \), \( \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}} \partial_{\alpha'} D_{\alpha'}^2 Z_{tt} \in L^2_{\sqrt{\lambda}} \) and similarly we also have \( \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}} \partial_{\alpha'} D_{\alpha'}^2 Z_{\alpha} \in L^2_{\sqrt{\lambda}} \), \( \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}} \partial_{\alpha'} D_{\alpha'}^2 Z_{\alpha} \in L^2_{\sqrt{\lambda}} \).

Proof: From the energy we have \( D_t \left( \frac{\lambda^\frac{1}{2}}{Z^{1/2}_{\alpha'}} \partial_{\alpha'} D_{\alpha'}^2 Z_t \right) \in L^2_{\sqrt{\lambda}}. \) Hence we have

\[
\left\| \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}} \partial_{\alpha'} D_{\alpha'}^2 Z_t \right\|_2 \leq \left\| \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}} \partial_{\alpha'} D_{\alpha'}^2 Z_t \right\|_2 \left\{ \| b_{\alpha'} \|_\infty + \| D_{\alpha'} Z_t \|_\infty \right\} + \left\| D_t \left( \frac{\lambda^\frac{1}{2}}{Z^{1/2}_{\alpha'}} \partial_{\alpha'} D_{\alpha'}^2 Z_t \right) \right\|_2
\]

Using the commutator \( [D_{\alpha'}, D_t] = (D_{\alpha'} Z_t) D_{\alpha'} - (D_{\alpha'} Z_t) D_{\alpha'} \) we see that

\[
D_t D_{\alpha'}^2 Z_t = -2(D_{\alpha'} Z_t) D_{\alpha'}^2 Z_t - (D_{\alpha'} Z_t) D_{\alpha'}^2 Z_t + D_{\alpha'}^2 Z_{tt}
\]

Hence we have

\[
\left\| \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}} \partial_{\alpha'} D_{\alpha'}^2 Z_t \right\|_2 \leq \left\{ \left\| \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}} \partial_{\alpha'} D_{\alpha'}^2 Z_t \right\|_2 \right\} \left\{ \| b_{\alpha'} \|_\infty + \| D_{\alpha'} Z_t \|_\infty \right\} + \left\| D_t \left( \frac{\lambda^\frac{1}{2}}{Z^{1/2}_{\alpha'}} \partial_{\alpha'} D_{\alpha'}^2 Z_t \right) \right\|_2
\]

Now we have the identity

\[
D_{\alpha'}^2 Z_{tt} = \left( D_{\alpha'} \left( \frac{1}{Z_{\alpha'}} \right) \right) Z_{tt,\alpha'} + \frac{1}{Z_{\alpha'}} \partial_{\alpha'} Z_{tt,\alpha'}
\]

Hence we have

\[
\left\| \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}} \partial_{\alpha'}^2 Z_{tt,\alpha'} \right\|_2 \leq \left\| D_{\alpha'} |Z_{tt,\alpha'}| \right\| \left\{ \left\| \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}} \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right\|_\infty \right\} \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2 + \left\| \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}} \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right\|_2
\]

We can prove \( \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}} \partial_{\alpha'} D_{\alpha'}^2 Z_t \in L^2_{\sqrt{\lambda}} \) similarly. Now observe that

\[
|D_{\alpha'}| D_t D_{\alpha'} Z_t = |D_{\alpha'}| (D_{\alpha'} Z_{tt} - (D_{\alpha'} Z_t)^2) = \omega D_{\alpha'}^2 Z_{tt} - 2(D_{\alpha'}^2 Z_t)
\]
Hence we have
\[
\left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |D_{\alpha^1} D_{t} D_{\alpha^1} Z_t| \right\|_2 \leq \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |D_{\alpha^1} Z_t| \right\|_2 + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |D_{\alpha^1}^2 Z_{tt}| \right\|_2 + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |D_{\alpha^1}^3 Z_t| \right\|_2
\]
\[
\leq \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |Z_{t}\alpha^1| \right\|_2 + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |D_{\alpha^1} Z_t| \right\|_2 + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |D_{\alpha^1}^2 Z_{tt}| \right\|_2 + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |D_{\alpha^1}^3 Z_t| \right\|_2
\]

11) \[ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |Z_{t\alpha^1}| \in W_{\sqrt{\lambda}} \cap C_{\sqrt{\lambda}}, \quad \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |D_{t} Z_{t\alpha^1}| \in W_{\sqrt{\lambda}} \cap C_{\sqrt{\lambda}}, \quad \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |D_{\alpha^1} Z_{tt}| \in W_{\sqrt{\lambda}} \cap C_{\sqrt{\lambda}} \]

and \[ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |D_{t} Z_{tt}| \in W_{\sqrt{\lambda}} \cap C_{\sqrt{\lambda}} \]

Proof: Let \( f = \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |Z_{t\alpha^1}| \) and \( w = \frac{1}{|Z_{\alpha^1}|} \). Then we see that \( \frac{f}{w} \in L^2_{\sqrt{\lambda}} \). Now we have
\[
\left\| \partial_{\alpha^1} \left( \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |Z_{t\alpha^1}| \right) \right\|_2 \leq \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |Z_{t\alpha^1}| \right\|_2 + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |D_{\alpha^1} Z_{tt}| \right\|_2
\]
\[
\leq \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |Z_{t\alpha^1}| \right\|_2 + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |D_{\alpha^1} Z_{tt}| \right\|_2
\]

As \( w' \in L^2 \), using Proposition 9.3 we obtain
\[
\left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |Z_{t\alpha^1}| \right\|_2 \leq \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |Z_{t\alpha^1}| \right\|_2 + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |D_{\alpha^1} Z_{tt}| \right\|_2
\]

Hence we have \( \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |Z_{t\alpha^1}| \in W_{\sqrt{\lambda}} \cap C_{\sqrt{\lambda}} \). Now
\[
\frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |D_{t} Z_{t\alpha^1}| = - \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} (b_{\alpha^1} |Z_{t\alpha^1}|) + \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |Z_{t\alpha^1}| \]

Hence from Lemma 6.3 we obtain
\[
\left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |D_{t} Z_{t\alpha^1}| \right\|_{W_{\sqrt{\lambda}} \cap C_{\sqrt{\lambda}}}
\]
\[
\leq \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} b_{\alpha^1} \right\|_{W_{\sqrt{\lambda}}} + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |Z_{t\alpha^1}| \right\|_{W_{\sqrt{\lambda}} \cap C_{\sqrt{\lambda}}}
\]
\[
+ \left\| b_{\alpha^1} \right\|_{W_{\sqrt{\lambda}}} \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha^1}|^{\frac{1}{2}}} \partial_{\alpha^1} |Z_{t\alpha^1}| \right\|_{W_{\sqrt{\lambda}} \cap C_{\sqrt{\lambda}}}
\]
We also have from Lemma 6.3
\[
\left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|} \partial_{\alpha'} D_{\alpha'} Z_{tt} \right\|_{W_{\nu} \cap C_{\nu}} \lesssim \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|} |\partial_{\alpha'}| \frac{1}{Z_{\alpha'}} \right\|_{W_{\nu} \cap C_{\nu}} \| D_{\alpha'} |Z_{tt}| \|_{W_{\nu} \cap C_{\nu}} + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|} \partial_{\alpha'} Z_{tt,a'} \right\|_{W_{\nu} \cap C_{\nu}}
\]
We prove \( \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|} \partial_{\alpha'} D_{\alpha'} Z_{tt} \in W_{\nu} \cap C_{\nu} \) similarly.

12) \( \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|} \partial_{\alpha'} D_{\alpha'} \frac{1}{Z_{\alpha'}} \in L^2_{\nu} \) and similarly we also have \( \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|} \partial_{\alpha'} D^2_{\alpha'} \frac{1}{Z_{\alpha'}} \in L^2_{\nu} \)

Proof: We know from (10) that \( Z_{tt} - i = -i \frac{A_1}{Z_{\alpha'}} \) and as \( A_1 \geq 1 \) we have
\[
\left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2 \lesssim \left\| \frac{1}{Z_{\alpha'}} \partial_{\alpha'} A_1 \right\|_2 + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2 + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2
\]
Recall from (10) that
\[
\text{Re} \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) = \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \quad \text{Im} \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) = i \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}
\]
Hence by taking derivatives we obtain
\[
\left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2 \lesssim \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2 + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2 + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2
\]
and also
\[
\left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2 \lesssim \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2 + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2 + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2
\]
The estimate \( \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{3}{2}} \partial_\alpha^2 D_{\alpha}^2 \frac{1}{Z,\alpha'} \in L^2_{\sqrt{\lambda}} \) can be proved similarly.

13) \( \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{3}{2}} \partial_\alpha^2 \frac{1}{Z,\alpha'} \in W_{\sqrt{\lambda}} \cap C_{\sqrt{\lambda}}, \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{3}{2}} \partial_\alpha^2 \frac{1}{Z,\alpha'} \in W_{\sqrt{\lambda}} \cap C_{\sqrt{\lambda}} \) and \( \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{3}{2}} \partial_\alpha^2 \omega \in W_{\sqrt{\lambda}} \cap C_{\sqrt{\lambda}} \)

Proof: We first note that

\[
\left\| \partial_\alpha \left( \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{3}{2}} \partial_\alpha^2 \frac{1}{Z,\alpha'} \right) \right\|_2 \leq \left\| D_{\alpha} \left( \frac{1}{|Z,\alpha'|} \right) \right\|_\infty \left\| \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{3}{2}} \partial_\alpha^2 \frac{1}{Z,\alpha'} \right\|_2 + \left\| \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{3}{2}} \partial_\alpha^3 \frac{1}{Z,\alpha'} \right\|_2
\]

Now taking \( f = \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{3}{2}} \partial_\alpha^2 \frac{1}{Z,\alpha'} \) and \( w = \frac{1}{|Z,\alpha'|} \) in Proposition 9.8 we obtain

\[
\left\| \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{3}{2}} \partial_\alpha^2 \frac{1}{Z,\alpha'} \right\|_{L^\infty \cap H^\frac{1}{2}}^2 \lesssim \left( \left\| \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{3}{2}} \partial_\alpha^2 \frac{1}{Z,\alpha'} \right\|_2 + \left\| \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{3}{2}} \partial_\alpha^3 \frac{1}{Z,\alpha'} \right\|_2 \right)^2
\]

As \( \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{3}{2}} \partial_\alpha^2 \frac{1}{Z,\alpha'} \in L^2_{\sqrt{\lambda}} \) this shows that \( \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^\frac{3}{2}} \partial_\alpha^2 \frac{1}{Z,\alpha'} \in W_{\sqrt{\lambda}} \cap C_{\sqrt{\lambda}} \). We can prove the other estimates similarly.

14) \( \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^{1/2}} \partial_\alpha D_{\alpha}^2 \frac{Z_t}{|Z,\alpha'|^{1/2}} \in C_{\sqrt{\lambda}}, \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^{1/2}} \partial_\alpha D_{\alpha} \frac{Z_t}{|Z,\alpha'|^{1/2}} \in C_{\sqrt{\lambda}}, \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^{1/2}} \partial_\alpha^2 \frac{Z_{t,\alpha'}}{|Z,\alpha'|^{1/2}} \in C_{\sqrt{\lambda}} \) and similarly \( \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^{1/2}} \partial_\alpha D_{\alpha} \frac{Z_t}{|Z,\alpha'|^{1/2}} \in C_{\sqrt{\lambda}}, \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^{1/2}} \partial_\alpha |D_{\alpha}| \frac{Z_t}{|Z,\alpha'|^{1/2}} \in C_{\sqrt{\lambda}} \)

Proof: We first see that \( \frac{\lambda^\frac{1}{2}}{Z,\alpha'|^{1/2}} \partial_\alpha D_{\alpha} \frac{Z_t}{|Z,\alpha'|^{1/2}} \in L^2_{\sqrt{\lambda}} \) as \( A_1 \in L^\infty \) and \( \frac{\lambda^\frac{1}{2}}{Z,\alpha'|^{1/2}} \partial_\alpha D_{\alpha} \frac{Z_t}{|Z,\alpha'|^{1/2}} \in L^2_{\sqrt{\lambda}} \) as it is part of the energy. Hence from the energy we see that \( \frac{\lambda^\frac{1}{2}}{Z,\alpha'|^{1/2}} \partial_\alpha D_{\alpha}^2 \frac{Z_t}{|Z,\alpha'|^{1/2}} \in C_{\sqrt{\lambda}} \). Now using Lemma 10.6 we get

\[
\left\| \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^{1/2}} \partial_\alpha D_{\alpha}^2 \frac{Z_t}{|Z,\alpha'|^{1/2}} \right\|_{C_{\sqrt{\lambda}}} \lesssim \left\| \frac{\lambda^\frac{1}{2}}{|Z,\alpha'|^{1/2}} \partial_\alpha D_{\alpha}^2 \frac{Z_t}{|Z,\alpha'|^{1/2}} \right\|_{C_{\sqrt{\lambda}}} \left\| \frac{1}{\sqrt{\lambda}} \right\|_W
\]
As \( \omega^\bot \in \mathcal{W} \) we see that \( \frac{\lambda^\bot}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D_{\alpha'}^2 Z_t \in C_\sqrt{\lambda} \). Now again using Lemma 6.3 we have

\[
\left\| \frac{\lambda^\bot}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'}^2 D_{\alpha'} Z_t \right\|_{C_\sqrt{\lambda}} \lesssim \|\omega\|_W \left( \frac{\lambda^\bot}{|Z_{\alpha'}|^\frac{3}{2}} \left\| \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \right\|_{W_{\sqrt{\lambda}}} \right) \left\| \frac{1}{Z_{\alpha'}} \partial_{\alpha'} D_{\alpha'} Z_t \right\|_C + \|\omega\|_W \left( \frac{\lambda^\bot}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D_{\alpha'}^2 Z_t \right) \right.
\]

Now observe that

\[
\frac{\lambda^\bot}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'}^2 D_{\alpha'} Z_t = \frac{\lambda^\bot}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} \left\{ \left( \frac{\partial_{\alpha'} - \frac{1}{Z_{\alpha'}}} \right) Z_t, \frac{1}{Z_{\alpha'}} \partial_{\alpha'} Z_t \right\}
\]

Hence from Lemma 6.3 we have

\[
\left\| \frac{\lambda^\bot}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'}^2 D_{\alpha'} Z_t \right\|_{C_\sqrt{\lambda}} \lesssim \|\omega\|_W \left( \frac{\lambda^\bot}{|Z_{\alpha'}|^\frac{3}{2}} \left\| \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \right\|_{W_{\sqrt{\lambda}}} \right) \left\| \frac{1}{Z_{\alpha'}} \partial_{\alpha'} D_{\alpha'} Z_t \right\|_C + \|\omega\|_W \left| D_{\alpha'} \right|_{W_{\sqrt{\lambda}}} \left\| \frac{\lambda^\bot}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D_{\alpha'}^2 Z_t \right\|_{C_\sqrt{\lambda}} + \|\omega\|_W \left| D_{\alpha'} \right|_{W_{\sqrt{\lambda}}} \left\| \frac{\lambda^\bot}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D_{\alpha'}^2 Z_t \right\|_{C_\sqrt{\lambda}}
\]

Now the estimates \( \frac{\lambda^\bot}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D_{\alpha'} Z_t \in C_\sqrt{\lambda} \), \( \frac{\lambda^\bot}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} |D_{\alpha'} D_{\alpha'} Z_t \in C_\sqrt{\lambda} \) are proven similarly.

15) \( \frac{\lambda^\bot}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} \Theta \in L^2_{\sqrt{\lambda}} \)

Proof: Using formula (18) we see that

\[
\frac{\lambda^\bot}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} \Theta = (I + \mathbb{H}) \text{Re} \left\{ \frac{\lambda^\bot}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} \Theta \right\} + i \text{Im} (I - \mathbb{H}) \left\{ \frac{\lambda^\bot}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} \Theta \right\}
\]

Now using the formula (16) we see that

\[
\text{Re} \left\{ \frac{\lambda^\bot}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} \Theta \right\} = -i \frac{\lambda^\bot}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} (D_{\alpha'} \omega)
\]

Hence we have

\[
\left\| \text{Re} \left\{ \frac{\lambda^\bot}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} \Theta \right\} \right\|_2 \lesssim \left\| \frac{\lambda^\bot}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{W_{\sqrt{\lambda}}} \left\| D_{\alpha'} |\omega| \right\|_2 + \left\| \frac{\lambda^\bot}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} \omega \right\|_2
\]

We also have

\[
(I - \mathbb{H}) \left\{ \frac{\lambda^\bot}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} \Theta \right\} = \left[ \frac{\lambda^\bot}{|Z_{\alpha'}|^\frac{3}{2}}, \mathbb{H} \right] \partial_{\alpha'} \Theta
\]
and hence we have from Proposition 16.5
\[ \left\| (I - \mathbb{H}) \left\{ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} \Theta \right\} \right\|_2 \lesssim \left\| \frac{\lambda^{\frac{1}{2}} |Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} }{\infty} \right\|_2 \| \Theta \|_2 \]

16) \( \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} \Theta \in C_{\sqrt{X}}, \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} \Theta \in C_{\sqrt{X}} \)

Proof: As \( \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} \Theta \in L^2_{\sqrt{X}} \), we only need to prove \( \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} \Theta \in \dot{H}^{\frac{1}{2}}_{\sqrt{X}} \). Now using formula (18) we see that
\[ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} \Theta = (I + \mathbb{H}) \text{Re}\left\{ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} \Theta \right\} + i \text{Im}(I - \mathbb{H}) \left\{ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} \Theta \right\} \]
Using the formula (16) we see that
\[ \text{Re}\left\{ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} \Theta \right\} = -i \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} (D_{\alpha'} \omega) \]
Hence we have from Lemma 6.3
\[ \left\| \text{Re}\left\{ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} \Theta \right\} \right\|_{\dot{H}^{\frac{1}{2}}} \]
\[ \lesssim \left\| \frac{\lambda^{\frac{1}{2}} |Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} }{\infty} \right\|_{L^2} \| \Theta \|_{c_{\sqrt{X}}} \]
Now observe that
\[ (I - \mathbb{H}) \left\{ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} \Theta \right\} = \lambda^{\frac{1}{2}} \left[ \frac{\omega^{\frac{1}{2}}}{|Z_{\alpha'}|} \right]^{\frac{1}{2}} \frac{1}{Z_{\alpha'}^{1/2}} \partial_{\alpha'} \Theta \]
Hence we have from Proposition 9.5
\[ \left\| (I - \mathbb{H}) \left\{ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} \Theta \right\} \right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \left( \| D_{\alpha'} \omega \|_2 + \| \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \|_2 \right) \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} \Theta \right\|_2 \]
Hence \( \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} \Theta \in C_{\sqrt{X}} \). Now as \( \mathbb{W}^{\frac{1}{2}} \in \mathbb{W} \) we obtain the other estimate easily by multiplying and using Lemma 6.3

17) \( \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} D_t \Theta \in L^2_{\sqrt{X}} \)

Proof: Using formula (18) we see that
\[ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} D_t \Theta = (I + \mathbb{H}) \text{Re}\left\{ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} D_t \Theta \right\} + i \text{Im}(I - \mathbb{H}) \left\{ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} D_t \Theta \right\} \]
We control the terms individually.
(a) Using the formula \[\{27\}\] we see that

\[
\text{Re}\left\{ \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} D_t \Theta \right\} = - \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \text{Im}\{ |D_{\alpha'}| \overline{Z_t} + i(\text{Re}\Theta) \overline{D_{\alpha'} Z_t} \}
\]

Hence we have

\[
\left\| \text{Re}\left\{ \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} D_t \Theta \right\} \right\|_2 \lesssim \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} |D_{\alpha'}| \overline{Z_t} \right\|_2 + \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \Theta \right\|_2 \left\| \overline{D_{\alpha'} Z_t} \right\|_\infty
\]

\[
+ \left\| \Theta \right\|_2 \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \overline{D_{\alpha'} Z_t} \right\|_\infty
\]

(b) We note that

\[
(\mathbb{I} - \mathbb{H}) \left\{ \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} D_t \Theta \right\} = \left[ \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \mathbb{H} \right] \partial_{\alpha'} D_t \Theta - \omega \left\{ \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \mathbb{H} \right\} \partial_{\alpha'} D_t \Theta
\]

\[
+ \omega (\mathbb{I} - \mathbb{H}) \left\{ \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} D_t \Theta \right\}
\]

Now observe that using the identities from \[\{6.1\}\] we have

\[
D_t \left( \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \Theta \right) = \left[ D_t, \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \right] \partial_{\alpha'} \Theta + \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} D_t \Theta
\]

\[
= - \left( \frac{b_{\alpha'}}{2} + \frac{D_{\alpha'} Z_t}{2} \right) \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \Theta + \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} D_t \Theta
\]

Now as \( \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \Theta \) is holomorphic and \( D_t = \partial_t + b \partial_{\alpha'} \) we have

\[
(\mathbb{I} - \mathbb{H}) \left\{ \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} D_t \Theta \right\}
\]

\[
= (\mathbb{I} - \mathbb{H}) \left\{ \left( \frac{b_{\alpha'}}{2} + \frac{D_{\alpha'} Z_t}{2} \right) \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \Theta \right\} + [b, \mathbb{H}] \partial_{\alpha'} \left\{ \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \Theta \right\}
\]

Hence using Proposition \[\{3.5\}\] we obtain

\[
\left\| (\mathbb{I} - \mathbb{H}) \left\{ \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} D_t \Theta \right\} \right\|_2 \lesssim \left( \|b_{\alpha'}\|_\infty + \|D_{\alpha'} Z_t\|_\infty \right) \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \Theta \right\|_2
\]

\[
+ \left( \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right\|_\infty + \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \omega \right\|_2 \right) \|D_t \Theta\|_2
\]
18) \( \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D_t \Theta \in C_{\sqrt{\lambda}} \)

Proof: Note that we only need to prove the \( \dot{H}^{\frac{1}{2}} \) estimate. Using formula (18) we see that
\[
\frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D_t \Theta = (I + \mathbb{H}) \text{Re} \left\{ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D_t \Theta \right\} + i \text{Im}(I - \mathbb{H}) \left\{ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D_t \Theta \right\}
\]

We control the term individually.
(a) Using the formula (27) we see that
\[
\text{Re} \left\{ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D_t \Theta \right\} = -\frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} \text{Im} \{ |D_{\alpha'}| D_t Z_t + i \text{Re} \Theta \}
\]

Hence we have from Lemma (28)
\[
\left\| \text{Re} \left\{ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D_t \Theta \right\} \right\|_{C_{\sqrt{\lambda}}} \leq \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} |D_{\alpha'}| D_t Z_t \right\|_{C_{\sqrt{\lambda}}} + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} \Theta \right\|_{C_{\sqrt{\lambda}}} \left\| D_{\alpha'} Z_t \right\|_W
\]

(b) We note that
\[
(I - \mathbb{H}) \left\{ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D_t \Theta \right\} = \left[ \frac{1}{|Z_{\alpha'}|^\frac{3}{2}} \right] \left( I - \mathbb{H} \right) \left\{ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} \right\} + \omega \frac{\sqrt{\lambda}}{|Z_{\alpha'}|^\frac{1}{2}} (I - \mathbb{H}) \left\{ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D_t \Theta \right\}
\]

as we have from Proposition (35)
\[
\left\| \left[ \frac{1}{|Z_{\alpha'}|^\frac{3}{2}} \right] \left( I - \mathbb{H} \right) \left\{ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D_t \Theta \right\} \right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \left\| \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^\frac{3}{2}} \right) \right\|_{L^2} \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D_t \Theta \right\|_2
\]

we only need to show the second term is in \( \dot{H}^{\frac{1}{2}} \). Now as \( \omega \frac{\sqrt{\lambda}}{|Z_{\alpha'}|^\frac{1}{2}} \in W \), it is enough to show that
\[
\left[ \frac{\sqrt{\lambda}}{|Z_{\alpha'}|^\frac{1}{2}} (I - \mathbb{H}) \right] \left\{ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D_t \Theta \right\} \in C_{\sqrt{\lambda}}. \quad \text{As } \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D_t \Theta \in L^2_{\sqrt{\lambda}}, \quad \text{we only need to show the } \dot{H}^{\frac{1}{2}} \text{ estimate. Now}
\]
\[
\left[ \frac{\sqrt{\lambda}}{|Z_{\alpha'}|^\frac{1}{2}} (I - \mathbb{H}) \right] \left\{ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D_t \Theta \right\} = -\left[ \frac{\sqrt{\lambda}}{|Z_{\alpha'}|^\frac{1}{2}} , \mathbb{H} \right] \left\{ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D_t \Theta \right\} + (I - \mathbb{H}) \left\{ \frac{\lambda^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D_t \Theta \right\}
\]
Now by a similar computation as done in (50), we see that

\[
(\mathbb{1} - \mathcal{H}) \left\{ \frac{\lambda^\frac{1}{2}}{Z_{\alpha',\alpha'}^\frac{3}{2}} \partial_{\alpha'} D_t \Theta \right\} = (\mathbb{1} - \mathcal{H}) \left\{ \left( \frac{b_{\alpha'}^2}{2} + \frac{3D_{\alpha'}Z_t}{2} \right) \frac{\lambda^\frac{1}{2}}{Z_{\alpha',\alpha'}^\frac{3}{2}} \partial_{\alpha'} \Theta \right\} + \left\{ b, \mathcal{H} \right\} \partial_{\alpha'} \left\{ \frac{\lambda^\frac{1}{2}}{Z_{\alpha',\alpha'}^\frac{3}{2}} \partial_{\alpha'} \Theta \right\}
\]

From this we obtain by Proposition 9.3 and Lemma 8.3

\[
\left\| \frac{\omega}{|Z_{\alpha'}^\frac{3}{2}|} (I - \mathcal{H}) \left\{ \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}^\frac{3}{2}|} \partial_{\alpha'} D_t \Theta \right\} \right\|_{H^{-\frac{1}{2}}}
\]

\[
\lesssim \left\{ \left\| \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}^\frac{3}{2}|} \right) \right\|_2 + \left\| |D_{\alpha'}| \omega \right\|_2 \right\} \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}^\frac{3}{2}|} \partial_{\alpha'} D_t \Theta \right\|_2
\]

\[
+ \left( \left\| b_{\alpha'} \right\|_W + \left\| D_{\alpha'} Z_t \right\|_W \right) \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}^\frac{3}{2}|} \partial_{\alpha'} \Theta \right\|_c, \Theta + \left\| b_{\alpha'} \right\|_{H^{-\frac{1}{2}}} \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}^\frac{3}{2}|} \partial_{\alpha'} \Theta \right\|_{H^{-\frac{1}{2}}} \right.
\]

19) \[
\frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}^\frac{3}{2}|} \partial_{\alpha'} R_3 \in L^2 \sqrt{X}
\]

Proof: Recall from (10) the formula of \( R_3 \)

\[
R_3 = \left\{ -2(D_{\alpha'}^2 Z_{tt}) + 6(D_{\alpha'} Z_t)(D_{\alpha'}^2 Z_t) \right\} (D_{\alpha'} Z_t) + \left\{ -4(D_{\alpha'}^2 Z_{tt}) + 6(D_{\alpha'} Z_t)^2 \right\} (D_{\alpha'} Z_t)
\]

\[
- 2(D_{\alpha'}^2 Z_t)(D_{\alpha'} Z_{tt}) - 4(D_{\alpha'} Z_t)(D_{\alpha'}^2 Z_{tt}) - 2i\omega(D_{\alpha'} \bar{\omega}) \left( \frac{1}{|Z_{\alpha'}^\frac{3}{2}|^2} \partial_{\alpha'} J_1 \right)
\]

\[
- i(D_{\alpha'} J_1) \left( D_{\alpha'} \frac{1}{Z_{\alpha'}^\frac{3}{2}} \right) - iJ_1 \left( D_{\alpha'}^2 \frac{1}{Z_{\alpha'}^\frac{3}{2}} \right)
\]

Let us control the terms individually

(a) We see that

\[
\left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}^\frac{3}{2}|} \partial_{\alpha'} \left\{ -2(D_{\alpha'}^2 Z_{tt}) + 6(D_{\alpha'} Z_t)(D_{\alpha'}^2 Z_t) \right\} (D_{\alpha'} Z_t) \right\|_2
\]

\[
\lesssim \left\{ \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}^\frac{3}{2}|} \partial_{\alpha'} D_{\alpha'}^2 Z_{tt} \right\|_2 \left\| D_{\alpha'} Z_t \right\|_\infty + \left\| D_{\alpha'}^2 Z_{tt} \right\|_2 \right\} \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}^\frac{3}{2}|} \partial_{\alpha'} D_{\alpha'} Z_t \right\|_\infty
\]

\[
+ \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}^\frac{3}{2}|} \partial_{\alpha'} D_{\alpha'} Z_t \right\|_\infty \left\| D_{\alpha'}^2 Z_t \right\|_2 \left\| D_{\alpha'} Z_t \right\|_\infty + \left\| D_{\alpha'} Z_t \right\|_\infty \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}^\frac{3}{2}|} \partial_{\alpha'} D_{\alpha'}^2 Z_t \right\|_2
\]

\[
+ \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha'}^\frac{3}{2}|} \partial_{\alpha'} D_{\alpha'} Z_t \right\|_\infty \left\| D_{\alpha'}^2 Z_t \right\|_2 \left\| D_{\alpha'} Z_t \right\|_\infty
\]
(b) We have

\[ \left\| \frac{\lambda^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} \left( \left\{ -4(D_{\alpha'} Z_{tt}) + 6(D_{\alpha'} Z_t)^2 \right\} (D_{\alpha'} Z_t) \right) \right\|_2 \]

\[ \lesssim \left\| \frac{\lambda^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} D_{\alpha'} Z_{tt} \right\|_2 \left\| D_{\alpha'} Z_t \right\|_2 + \left\| D_{\alpha'} Z_{tt} \right\|_\infty \left\| \frac{\lambda^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} D_{\alpha'} Z_t \right\|_2 \]

\[ + \left\| \frac{\lambda^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} D_{\alpha'} Z_t \right\|_\infty \left\| D_{\alpha'} Z_t \right\|_\infty \left\| \frac{\lambda^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} D_{\alpha'} Z_t \right\|_2 \]

(c) We have

\[ \left\| \frac{\lambda^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} \left( -2(D_{\alpha'} Z_t) (D_{\alpha'} Z_{tt}) - 4(D_{\alpha'} Z_t) (D_{\alpha'} Z_{tt}) \right) \right\|_2 \]

\[ \lesssim \left\| \frac{\lambda^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} D_{\alpha'} Z_t \right\|_2 \left\| D_{\alpha'} Z_{tt} \right\|_\infty + \left\| D_{\alpha'} Z_t \right\|_2 \left\| \frac{\lambda^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} D_{\alpha'} Z_{tt} \right\|_\infty \]

\[ + \left\| \frac{\lambda^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} D_{\alpha'} Z_t \right\|_\infty \left\| D_{\alpha'} Z_{tt} \right\|_2 + \left\| D_{\alpha'} Z_t \right\|_\infty \left\| \frac{\lambda^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} D_{\alpha'} Z_{tt} \right\|_2 \]

(d) We have

\[ \left\| \frac{\lambda^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} \left( -2i\varpi(D_{\alpha'} \varpi) \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1 \right) \right) \right\|_2 \]

\[ \lesssim \left\| \frac{\lambda^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} \varpi \right\|_2 \left\| D_{\alpha'} \varpi \right\|_2 \left\| \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1 \right\|_\infty + \left\| \frac{\lambda^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} \varpi \right\|_2 \left\| \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1 \right\|_\infty \]

\[ + \left\| \frac{\lambda^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} \varpi \right\|_\infty \left\| D_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1 \right) \right\|_2 \]

(e) We have

\[ \left\| \frac{\lambda^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} \left( -i(D_{\alpha'} J_1) \left( D_{\alpha'} \frac{1}{Z_{\alpha'}} \right) - i J_1 \left( D_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right) \right) \right\|_2 \]

\[ \lesssim \left( \left\| \frac{\lambda^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} \varpi \right\|_\infty + \left\| \frac{\lambda^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_\infty \right) \left\| \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1 \right\|_\infty \left\| \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2 \]

\[ + \left\| D_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1 \right) \right\|_2 \left\| \frac{\lambda^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_\infty + \left\| \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1 \right\|_\infty \left\| \frac{\lambda^2}{|Z_{\alpha'}|^2} \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right\|_2 \]

\[ + \left\| J_1 \right\|_\infty \left\| \frac{\lambda^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} D_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2 \]
20) $R_4 \in L^2_{\mathcal{A}}$

Proof: Recall from (49) the formula of $R_4$

$$R_4 = -\left\{ \frac{D_t b_{\alpha'}}{2} + \frac{D_t D_{\alpha'} Z_t}{2} - \left( \frac{b_{\alpha'}}{2} + \frac{D_{\alpha'} Z_t}{2} \right)^2 \right\} \frac{\lambda_2^\alpha}{Z_{\alpha'}^{1/2}} \partial_{\alpha'} D_{\alpha'}^2 Z_t + \frac{\lambda_2^\alpha}{Z_{\alpha'}^{1/2}} \partial_{\alpha'} R_3$$

$$- \left\{ 2i A_1 \left( |D_{\alpha'}| \frac{1}{|Z_{\alpha'}|} \right) + \frac{i}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 - \frac{i A_1}{2} \left( \overline{D_{\alpha'}} \frac{1}{Z_{\alpha'}} \right) \right\} \frac{\lambda_2^\alpha}{Z_{\alpha'}^{1/2}} \partial_{\alpha'} D_{\alpha'}^2 Z_t$$

$$- \left\{ 3i \omega^2 \left( \frac{\lambda_2^\alpha}{Z_{\alpha'}^{1/2}} \partial_{\alpha'} \omega \right) |D_{\alpha'}| \left( \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} J_1 \right) \right\} - (b_{\alpha'} + D_{\alpha'} Z_t) \frac{\lambda_2^\alpha}{Z_{\alpha'}^{1/2}} \partial_{\alpha'} D_{\alpha'}^2 Z_t$$

Hence we have the estimate

$$\| R_4 \|_2 \leq \left\{ \| D_t b_{\alpha'} \|_\infty + \| D_t D_{\alpha'} Z_t \|_\infty + (\| b_{\alpha'} \|_\infty + \| D_{\alpha'} Z_t \|_\infty)^2 \right\} \frac{\lambda_2^\alpha}{Z_{\alpha'}^{1/2}} \partial_{\alpha'} D_{\alpha'}^2 Z_t$$

$$+ \left\{ \| \lambda_2^\alpha \|_\infty \| R_3 \|_2 + \left\{ \| A_1 \|_\infty \| |D_{\alpha'}| \frac{1}{|Z_{\alpha'}|} \|_\infty + \| \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1 \|_\infty \right\} \| D_{\alpha'} \|_\infty \| \lambda_2^\alpha \|_\infty \| \partial_{\alpha'} D_{\alpha'}^2 Z_t \|_2$$

$$+ \{ \| b_{\alpha'} \|_\infty + \| D_{\alpha'} Z_t \|_\infty \} \| \lambda_2^\alpha \|_\infty \| \partial_{\alpha'} D_{\alpha'}^2 Z_t \|_2$$

21) $(I - \mathcal{H}) D_{t}^2 \left( \frac{\lambda_2^\alpha}{Z_{\alpha'}^{1/2}} \partial_{\alpha'} D_{\alpha'}^2 Z_t \right) \in L^2_{\mathcal{A}}$

Proof: For a function $f$ satisfying $\mathbb{P}_A f = 0$ we have from Proposition 9.1

$$(I - \mathcal{H}) D_{t}^2 f = [D_t, \mathcal{H}] D_{t} f + D_t [D_t, \mathcal{H}] f$$

$$= [b, \mathcal{H}] \partial_{\alpha'} D_{t} f + D_t [b, \mathcal{H}] \partial_{\alpha'} f$$

$$= 2[b, \mathcal{H}] \partial_{\alpha'} D_{t} f + [D_t b, \mathcal{H}] \partial_{\alpha'} f - [b, b; \partial_{\alpha'} f]$$

As $\mathbb{P}_A \left( \frac{\lambda_2^\alpha}{Z_{\alpha'}^{1/2}} \partial_{\alpha'} D_{\alpha'}^2 Z_t \right) = 0$ we obtain from Proposition 9.5

$$\left\| (I - \mathcal{H}) D_{t}^2 \left( \frac{\lambda_2^\alpha}{Z_{\alpha'}^{1/2}} \partial_{\alpha'} D_{\alpha'}^2 Z_t \right) \right\|_2 \leq \| b_{\alpha'} \|_\infty \left\| D_t \left( \frac{\lambda_2^\alpha}{Z_{\alpha'}^{1/2}} \partial_{\alpha'} D_{\alpha'}^2 Z_t \right) \right\|_2 + \| \partial_{\alpha'} D_t b \|_\mathcal{H} \left\| \frac{\lambda_2^\alpha}{Z_{\alpha'}^{1/2}} \partial_{\alpha'} D_{\alpha'}^2 Z_t \right\|_2$$

$$+ \| b_{\alpha'} \|_\infty \left\| \frac{\lambda_2^\alpha}{Z_{\alpha'}^{1/2}} \partial_{\alpha'} D_{\alpha'}^2 Z_t \right\|_2$$
22) \((I - \mathbb{H}) \left\{ i \frac{A_1}{|Z_{\alpha}|} \partial_{\alpha'} \left( \frac{\lambda_{\alpha}}{Z_{\alpha}^{1/2}} \partial_{\alpha'} Z_t \right) \right\} \in L^2_{\sqrt{\lambda}} \)

Proof: We see that

\[
(I - \mathbb{H}) \left\{ i \frac{A_1}{|Z_{\alpha}|} \partial_{\alpha'} \left( \frac{\lambda_{\alpha}}{Z_{\alpha}^{1/2}} \partial_{\alpha'} Z_t \right) \right\} = i \left[ \frac{A_1}{|Z_{\alpha}|} \mathbb{H} \right] \partial_{\alpha'} \left( \frac{\lambda_{\alpha}}{Z_{\alpha}^{1/2}} \partial_{\alpha'} Z_t \right)
\]

and hence we have from Proposition 9.5

\[
\left\| (I - \mathbb{H}) \left\{ i \frac{A_1}{|Z_{\alpha}|} \partial_{\alpha'} \left( \frac{\lambda_{\alpha}}{Z_{\alpha}^{1/2}} \partial_{\alpha'} Z_t \right) \right\} \right\|_2 \lesssim \left\| \frac{\lambda_{\alpha}}{Z_{\alpha}^{1/2}} \partial_{\alpha'} Z_t \right\|_2 \left\{ \left\| \frac{1}{|Z_{\alpha}|} \partial_{\alpha'} A_1 \right\|_{\infty} + \|A_1\|_{\infty} \left\| \frac{1}{|Z_{\alpha}|} \right\|_{\infty} \right\}
\]

23) \(\frac{\lambda_{\alpha}}{|Z_{\alpha}|} \partial_{\alpha'} |D_{\alpha'}| \left( \frac{1}{|Z_{\alpha}|^2} \partial_{\alpha'} J_1 \right) \in L^2_{\sqrt{\lambda}} \)

Proof: Recall the equation of \(\frac{\lambda_{\alpha}}{Z_{\alpha}^{1/2}} \partial_{\alpha'} Z_t \) from (18)

\[
(D_t^2 + i \frac{A_1}{|Z_{\alpha}|} \partial_{\alpha'}) \frac{\lambda_{\alpha}}{Z_{\alpha}^{1/2}} \partial_{\alpha'} Z_t = -i \omega \frac{\lambda_{\alpha}}{Z_{\alpha}^{1/2}} \partial_{\alpha'} |D_{\alpha'}| \left( \frac{1}{|Z_{\alpha}|^2} \partial_{\alpha'} J_1 \right) + R_4
\]

Applying \((I - \mathbb{H})\) to the above equation we obtain the estimate

\[
\left\| (I - \mathbb{H}) \left\{ \frac{\lambda_{\alpha}^{\sqrt{3}}}{Z_{\alpha}^{1/2}} \partial_{\alpha'} |D_{\alpha'}| \left( \frac{1}{|Z_{\alpha}|^2} \partial_{\alpha'} J_1 \right) \right\} \right\|_2 \lesssim \|R_4\|_2 + \left\| (I - \mathbb{H}) D_t^2 \left( \frac{\lambda_{\alpha}}{Z_{\alpha}^{1/2}} \partial_{\alpha'} Z_t \right) \right\|_2 + \left\| (I - \mathbb{H}) \left\{ i \frac{A_1}{|Z_{\alpha}|} \partial_{\alpha'} \left( \frac{\lambda_{\alpha}^{\sqrt{3}}}{Z_{\alpha}^{1/2}} \partial_{\alpha'} Z_t \right) \right\} \right\|_2
\]

Now as \(J_1\) is real valued we see that

\[
\frac{\lambda_{\alpha}}{|Z_{\alpha}|^{\frac{1}{2}}} \partial_{\alpha'} |D_{\alpha'}| \left( \frac{1}{|Z_{\alpha}|^2} \partial_{\alpha'} J_1 \right) = \text{Re} \left\{ \frac{\lambda_{\alpha}^{\sqrt{3}}}{Z_{\alpha}^{1/2}} \partial_{\alpha'} (I - \mathbb{H}) |D_{\alpha'}| \left( \frac{1}{|Z_{\alpha}|^2} \partial_{\alpha'} J_1 \right) \right\}
\]

and we see that

\[
\frac{\lambda_{\alpha}^{\sqrt{3}}}{Z_{\alpha}^{1/2}} \partial_{\alpha'} (I - \mathbb{H}) |D_{\alpha'}| \left( \frac{1}{|Z_{\alpha}|^2} \partial_{\alpha'} J_1 \right) = -\left[ \frac{\lambda_{\alpha}^{\sqrt{3}}}{Z_{\alpha}^{1/2}} \partial_{\alpha'} (I - \mathbb{H}) |D_{\alpha'}| \left( \frac{1}{|Z_{\alpha}|^2} \partial_{\alpha'} J_1 \right) \right] \]

\[
+ (I - \mathbb{H}) \left\{ \frac{\lambda_{\alpha}^{\sqrt{3}}}{Z_{\alpha}^{1/2}} \partial_{\alpha'} |D_{\alpha'}| \left( \frac{1}{|Z_{\alpha}|^2} \partial_{\alpha'} J_1 \right) \right\}
\]
Hence from Proposition 0.5 we obtain
\[
\left\| \frac{\lambda^\frac{p}{2}}{|Z,\alpha'|^{\frac{p}{2}}} \partial_{\alpha'} |D,\alpha'| \left( \frac{1}{|Z,\alpha'|^2} \partial_{\alpha'} J_1 \right) \right\|_2 \leq \left\{ \left\| \frac{\lambda^\frac{p}{2}}{|Z,\alpha'|^{\frac{p}{2}}} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_\infty + \left\| \frac{\lambda^\frac{p}{2}}{|Z,\alpha'|^{\frac{p}{2}}} \partial_{\alpha'} \partial_t \lambda E \left( \frac{1}{Z,\alpha'}^2 \partial_{\alpha'} J_1 \right) \right\|_2 \\
+ \left\{ (\mathbb{I} - \mathbb{I}) \left\{ \frac{\lambda^\frac{p}{2}}{|Z,\alpha'|^{\frac{p}{2}}} \partial_{\alpha'} |D,\alpha'| \left( \frac{1}{|Z,\alpha'|^2} \partial_{\alpha'} J_1 \right) \right\} \right\|_2 \right\}. 
\]

6.3. Closing the energy estimate for \( \lambda E_{aux} \)

We now complete the proof of Theorem 6.1. To simplify the calculations, we will continue to use the notation used in 6.2 and introduce another notation: If \( a(t), b(t) \) are functions of time we write \( a \approx b \) if there exists a universal polynomial \( P \) with non-negative coefficients so that \( |a(t) - b(t)| \leq P(\epsilon_{high}(t))(\lambda E_{aux}(t)) \). Observe that \( \approx \) is an equivalence relation. With this notation, proving Theorem 6.1 is equivalent to showing \( \frac{d}{dt}(\lambda E_{aux}(t)) \approx 0 \). We control the first four terms of the energy \( \lambda E \) directly and for the last two terms we use the equation \( \lambda E \).

(1) Controlling the time derivative of \( \left\| \frac{\lambda^\frac{p}{2}}{|Z,\alpha'|^{\frac{p}{2}}} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_\infty^2 \) proceeds in exactly the same as that

of controlling \( \left\| \sigma^\frac{p}{2} |Z,\alpha'|^{\frac{p}{2}} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_\infty^2 \) for the energy \( \lambda E_{aux} \) which was done in Sec 5.2.1 in [1]. First observe that

\[
\left\| D_t \left( \frac{\lambda^\frac{p}{2}}{|Z,\alpha'|^{\frac{p}{2}}} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right) \right\|_\infty \leq \left( \left\| D_t |Z,\alpha'| \right\|_\infty + \left\| b_{\alpha'} \right\|_\infty \right) \left\| \frac{\lambda^\frac{p}{2}}{|Z,\alpha'|^{\frac{p}{2}}} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_\infty \left\| \lambda^\frac{p}{2} \partial_{\alpha'} b_{\alpha'} \right\|_\infty \left\| \lambda^\frac{p}{2} \partial_{\alpha'} D_t |Z,\alpha'| \right\|_\infty \right. 
\]

Now using the computation from Sec 5.2.1 in [1] we obtain
\[
\limsup_{s \to 0^+} \frac{\left\| \frac{\lambda^\frac{p}{2}}{|Z,\alpha'|^{\frac{p}{2}}} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_\infty^2 (t + s) - \left\| \frac{\lambda^\frac{p}{2}}{|Z,\alpha'|^{\frac{p}{2}}} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_\infty^2 (t)}{s} \leq \left\| \frac{\lambda^\frac{p}{2}}{|Z,\alpha'|^{\frac{p}{2}}} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_\infty(t) \right. 
\]

We first observe that
\[
D_t \left( \frac{\lambda^\frac{p}{2}}{|Z,\alpha'|^{\frac{p}{2}}} \partial_{\alpha'} Z,\alpha' \right) = - \left( \frac{b_{\alpha'}}{2} + \frac{D_t Z,\alpha'}{2} \right) \frac{\lambda^\frac{p}{2}}{|Z,\alpha'|^{\frac{p}{2}}} \partial_{\alpha'} Z,\alpha' + \frac{\lambda^\frac{p}{2}}{|Z,\alpha'|^{\frac{p}{2}}} \partial_{\alpha'} \left( -b_{\alpha'} Z,\alpha' + Z,tt,\alpha' \right) \right. 
\]
Hence
\[
\left\| D_t \left( \frac{\lambda^\frac{1}{2}}{Z_{\alpha}} \partial_{\alpha'} Z_{\alpha} \right) \right\|_2 \lesssim \left\{ \left\| b_{\alpha'} \right\|_\infty + \left\| D_{\alpha'} Z_t \right\|_\infty \right\} \left\| \frac{\lambda^\frac{1}{2}}{Z_{\alpha}^2} \partial_{\alpha'} Z \right\|_2 + \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha}|^\frac{1}{2}} \partial_{\alpha'} b_{\alpha'} \right\|_\infty \left\| Z_{\alpha} \right\|_2 + \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha}|^\frac{1}{2}} \partial_{\alpha'} Z_{\alpha} \right\|_2
\]

Now by using Lemma 4.4 we obtain
\[
\frac{d}{dt} \int \left| \frac{\lambda^\frac{1}{2}}{Z_{\alpha}^\frac{1}{2}} \partial_{\alpha'} Z_{\alpha} \right|^2 d\alpha'
\lesssim \left\| b_{\alpha'} \right\|_\infty \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha}|^\frac{1}{2}} \partial_{\alpha'} Z_{\alpha} \right\|_2^2 + \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha}|^\frac{1}{2}} \partial_{\alpha'} Z_{\alpha} \right\|_2 \left\| D_t \left( \frac{\lambda^\frac{1}{2}}{Z_{\alpha}^\frac{1}{2}} \partial_{\alpha'} Z_{\alpha} \right) \right\|_2
\lesssim P(E_{\text{high}}) (\lambda E_{\text{aux}})
\]

(3) We observe that from (25)
\[
D_t \left( \frac{\lambda^\frac{1}{2}}{Z_{\alpha}^\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha}} \right)
= - \left( \frac{b_{\alpha'}}{2} + \frac{D_{\alpha'} Z_t}{2} \right) \frac{\lambda^\frac{1}{2}}{Z_{\alpha}^\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha}} + \frac{\lambda^\frac{1}{2}}{Z_{\alpha}^\frac{1}{2}} \partial_{\alpha'} \left( D_t \partial_{\alpha'} \frac{1}{Z_{\alpha}} \right)
= - \left( \frac{b_{\alpha'}}{2} + \frac{D_{\alpha'} Z_t}{2} \right) \frac{\lambda^\frac{1}{2}}{Z_{\alpha}^\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha}} + \frac{\lambda^\frac{1}{2}}{Z_{\alpha}^\frac{1}{2}} \partial_{\alpha'} \left( D_{\alpha'} b_{\alpha'} - D_{\alpha'}^2 Z_t - \left( \partial_{\alpha'} \frac{1}{Z_{\alpha}} \right) D_{\alpha'} Z_t \right)
\]

Hence we have
\[
\left\| D_t \left( \frac{\lambda^\frac{1}{2}}{Z_{\alpha}^\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha}} \right) \right\|_2 \lesssim \left\{ \left\| b_{\alpha'} \right\|_\infty + \left\| D_{\alpha'} Z_t \right\|_\infty \right\} \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha}|^\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha}} \right\|_2 + \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha}|^\frac{1}{2}} \partial_{\alpha'} D_{\alpha'}^2 Z_t \right\|_2
\]

Now by using Lemma 4.4 we obtain
\[
\frac{d}{dt} \int \left| \frac{\lambda^\frac{1}{2}}{Z_{\alpha}^\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha}} \right|^2 d\alpha'
\lesssim \left\| b_{\alpha'} \right\|_\infty \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha}|^\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha}} \right\|_2^2 + \left\| \frac{\lambda^\frac{1}{2}}{|Z_{\alpha}|^\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha}} \right\|_2 \left\| D_t \left( \frac{\lambda^\frac{1}{2}}{Z_{\alpha}^\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha}} \right) \right\|_2
\lesssim P(E_{\text{high}}) (\lambda E_{\text{aux}})
\]
(4) Using Lemma 4.4 we see that
\[ \frac{d}{dt} \int \left| \frac{\lambda^\frac{1}{2}}{Z_\alpha^1} \partial_\alpha D_\alpha^2 \overline{Z}_t \right|^2 \, d\alpha' \]
\[ \lesssim \| b_{\alpha'} \|_{\infty} \left( \frac{\lambda^\frac{1}{2}}{|Z_\alpha^1|} \right) \| \partial_\alpha D_\alpha^2 \overline{Z}_t \|_2^2 + \left( \frac{\lambda^\frac{1}{2}}{|Z_\alpha^1|} \right) \| \partial_\alpha D_\alpha^2 \overline{Z}_t \|_2 \| D_t \left( \frac{\lambda^\frac{1}{2}}{Z_\alpha^1} \partial_\alpha D_\alpha^2 \overline{Z}_t \right) \|_2 \]
\[ \lesssim P(\mathcal{E}_{\text{high}})(\lambda E_{\text{aux}}) \]

(5) The quantity left to control is the time derivative of
\[ \int |D_t f|^2 \, d\alpha' + \int \left| \partial_\alpha \left( \frac{\sqrt{A_1}}{|Z_\alpha^1|} f' \right) \right|^2 \, d\alpha' \]
where \( f = \frac{\lambda^\frac{1}{2}}{Z_\alpha^1} \partial_\alpha D_\alpha^2 \overline{Z}_t \) and we have \( \mathcal{P}_H f = f \). Following the same computation as done in [49,3] for the energy \( E_{\text{high}} \) we see that
\[ \frac{d}{dt} (\lambda E_{\text{aux}}(t)) \approx 2 \text{Re} \int \left( D_t^2 f + i \frac{A_1}{|Z_\alpha^1|^2} \partial_\alpha f \right) (D_t f) \, d\alpha' \]
As \( D_t \left( \frac{\lambda^\frac{1}{2}}{Z_\alpha^1} \partial_\alpha D_\alpha^2 \overline{Z}_t \right) \in L^2_{\sqrt{\lambda}} \) we only need to show that the other term in in \( L^2_{\sqrt{\lambda}} \). Now the equation for \( \frac{\lambda^\frac{1}{2}}{Z_\alpha^1} \partial_\alpha D_\alpha^2 \overline{Z}_t \) from [45] implies
\[ \left( D_t^2 + i \frac{A_1}{|Z_\alpha^1|^2} \partial_\alpha \right) \frac{\lambda^\frac{1}{2}}{Z_\alpha^1} \partial_\alpha D_\alpha^2 \overline{Z}_t = -i \omega^2 \frac{\lambda^\frac{1}{2}}{Z_\alpha^1} \partial_\alpha |D_\alpha| \left( \frac{1}{|Z_\alpha^1|^2} \partial_\alpha J_1 \right) + R_4 \]
As we have shown \( \frac{\lambda^\frac{1}{2}}{|Z_\alpha^1|^2} \partial_\alpha |D_\alpha| \left( \frac{1}{|Z_\alpha^1|^2} \partial_\alpha J_1 \right) \in L^2_{\sqrt{\lambda}} \) and \( R_4 \in L^2_{\sqrt{\lambda}} \), this implies that the right hand side is in \( L^2_{\sqrt{\lambda}} \) and so the proof of Theorem 6.1 is complete.

6.4. Equivalence of \( \lambda E_{\text{aux}} \) and \( \lambda \mathcal{E}_{\text{aux}} \)

We now give a simpler description of the energy \( E_{\text{aux}} \) defined by (49). Recall the definition of \( \mathcal{E}_{\text{aux}} \) from (13)
\[ \mathcal{E}_{\text{aux}} = \left( \frac{1}{Z_\alpha^1} \partial_\alpha \overline{Z}_t \right)^2 + \left( \frac{1}{Z_\alpha^1} \partial_\alpha \overline{Z}_t \right)^2 + \left( \frac{1}{Z_\alpha^1} \partial_\alpha \overline{Z}_t \right)^2 \]

Proposition 6.4. There exists universal polynomials \( P_1, P_2 \) with non-negative coefficients so that if \( (Z, Z_t)(t) \) is a smooth solution to the water wave equation (8) with \( \sigma = 0 \) in the time interval \( [0, T] \) satisfying \( (Z_\alpha^1, Z_t^1) \in L^\infty([0, T], H^s(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s+\frac{1}{2}}(\mathbb{R})) \) for all \( s \geq 2 \) and if \( \lambda > 0 \) is some given constant, then for all \( t \in [0, T] \) we have
\[ \lambda E_{\text{aux}} \leq P_1(\mathcal{E}_{\text{high}})(\lambda \mathcal{E}_{\text{aux}}) \quad \text{and} \quad \lambda \mathcal{E}_{\text{aux}} \leq P_2(\mathcal{E}_{\text{high}})(\lambda E_{\text{aux}}) \]
Proof. Let $\lambda E_{aux} < \infty$. We have already pretty much controlled all the terms of $\lambda E_{aux}$ and the only term not directly controlled is $\frac{\lambda^\frac{1}{2}}{|Z_{t,\alpha'}|^\frac{3}{2}} \partial_\alpha^2 Z_{t,\alpha'}$. This term can be easily controlled by using Lemma $\ref{lem:control}$

$$
\left\| \frac{\lambda^\frac{1}{2}}{|Z_{t,\alpha'}|^\frac{3}{2}} \partial_\alpha^2 Z_{t,\alpha'} \right\|_{C_{\sqrt{\lambda}}} \lesssim \left\| \rho \right\|_{W} \left\| \frac{\lambda^\frac{1}{2}}{|Z_{t,\alpha'}|^\frac{3}{2}} \partial_\alpha^2 Z_{t,\alpha'} \right\|_{C_{\sqrt{\lambda}}}
$$

Now assume that $\lambda E_{aux} < \infty$. We see that the first three terms of $\lambda E_{aux}$ are controlled and following the proofs of $\lambda^\frac{1}{2}|Z_{t,\alpha'}|^\frac{1}{2} \partial_\alpha^{1} \frac{1}{|Z_{t,\alpha'}|} \in L^\infty_{\sqrt{\lambda}}$ and $\frac{\lambda^\frac{1}{2}}{|Z_{t,\alpha'}|^\frac{3}{2}} \partial_\alpha^2 \omega \in L^2_{\sqrt{\lambda}}$ in $\ref{lem:control}$ we immediately get

$$
\left\| \lambda^\frac{1}{2}|Z_{t,\alpha'}|^\frac{1}{2} \partial_\alpha^{1} \frac{1}{|Z_{t,\alpha'}|} \right\|_{\infty} + \left\| \lambda^\frac{1}{2}|Z_{t,\alpha'}|^\frac{1}{2} \partial_\alpha^{1} \omega \right\|_{\infty} + \left\| \lambda^\frac{1}{2}|Z_{t,\alpha'}|^\frac{1}{2} \partial_\alpha^{2} \omega \right\|_{2} + \left\| \frac{\lambda^\frac{1}{2}}{|Z_{t,\alpha'}|^\frac{3}{2}} \partial_\alpha^2 \omega \right\|_{2} \lesssim P_1(\mathcal{E}_{high})(\lambda E_{aux})^\frac{1}{2}
$$

Now following the proof of $\frac{\lambda^\frac{1}{2}}{|Z_{t,\alpha'}|^\frac{3}{2}} \partial_\alpha D_{\alpha'} Z_t \in W_{\sqrt{\lambda}} \cap C_{\sqrt{\lambda}}$ from $\ref{lem:control}$ we see that

$$
\left\| \frac{\lambda^\frac{1}{2}}{|Z_{t,\alpha'}|^\frac{3}{2}} \partial_\alpha D_{\alpha'} Z_t \right\|_{W_{\sqrt{\lambda}} \cap C_{\sqrt{\lambda}}} \lesssim P_1(\mathcal{E}_{high})(\lambda E_{aux})^\frac{1}{2}
$$

Similarly following the proof of $\frac{\lambda^\frac{1}{2}}{|Z_{t,\alpha'}|^\frac{3}{2}} \partial_\alpha^2 Z_t \in W_{\sqrt{\lambda}} \cap C_{\sqrt{\lambda}}$ from $\ref{lem:control}$ we see that

$$
\left\| \frac{\lambda^\frac{1}{2}}{|Z_{t,\alpha'}|^\frac{3}{2}} \partial_\alpha^2 Z_t \right\|_{W_{\sqrt{\lambda}} \cap C_{\sqrt{\lambda}}} \lesssim P_1(\mathcal{E}_{high})(\lambda E_{aux})^\frac{1}{2}
$$

Now following the proofs of $\frac{\lambda^\frac{1}{2}}{|Z_{t,\alpha'}|^\frac{3}{2}} \partial_\alpha^3 A_t \in L^2_{\sqrt{\lambda}}$ and $\frac{\lambda^\frac{1}{2}}{|Z_{t,\alpha'}|^\frac{3}{2}} \partial_\alpha^2 \int Z_t \in L^2_{\sqrt{\lambda}}$ in $\ref{lem:control}$ we see that

$$
\left\| \frac{\lambda^\frac{1}{2}}{|Z_{t,\alpha'}|^\frac{3}{2}} \partial_\alpha^3 A_t \right\|_{2} + \left\| \frac{\lambda^\frac{1}{2}}{|Z_{t,\alpha'}|^\frac{3}{2}} \partial_\alpha^2 \int Z_t \right\|_{2} \lesssim P_1(\mathcal{E}_{high})(\lambda E_{aux})^\frac{1}{2}
$$

Now following the proof of $\frac{\lambda^\frac{1}{2}}{|Z_{t,\alpha'}|^\frac{3}{2}} \partial_\alpha^3 Z_t \in L^2_{\sqrt{\lambda}}$ from $\ref{lem:control}$ we obtain

$$
\left\| \frac{\lambda^\frac{1}{2}}{|Z_{t,\alpha'}|^\frac{3}{2}} \partial_\alpha^3 Z_t \right\|_{2} \lesssim P_1(\mathcal{E}_{high})(\lambda E_{aux})^\frac{1}{2}
$$
Now we follow the proof of $\frac{\lambda^\frac{1}{2}}{|Z_{a\alpha}|^2} \partial^3_{a\alpha} Z_{tt} \in L^2_\lambda$ from [6.2] to obtain

$$\left\| D_t \left( \frac{\lambda^\frac{1}{2}}{|Z_{a\alpha}|^2} \partial_{a\alpha}^2 D_{a\alpha}^2 Z_t \right) \right\|_2 \lesssim P_1(\mathcal{E}_{\text{high}})(\lambda \mathcal{E}_{\text{aux}})^\frac{1}{2}$$

Now we use Lemma 6.3 to obtain the estimate

$$\left\| \frac{\lambda^\frac{1}{2}}{|Z_{a\alpha}|^2} \partial_{a\alpha}^2 Z_{t,\alpha} \right\|_{C_\lambda} \lesssim \| \omega \|_W \left\| \frac{\lambda^\frac{1}{2}}{|Z_{a\alpha}|^2} \partial_{a\alpha}^2 Z_{t,\alpha} \right\|_{C_\lambda} \lesssim P_1(\mathcal{E}_{\text{high}})(\lambda \mathcal{E}_{\text{aux}})^\frac{1}{2}$$

Hence by following the proof of $\frac{\lambda^\frac{1}{2}}{|Z_{a\alpha}|^2} \partial_{a\alpha}^2 Z_{t,\alpha} \in C_\lambda$ from [6.2] we see that

$$\left\| \sqrt{A_1} \left( \frac{\lambda^\frac{1}{2}}{|Z_{a\alpha}|^2} \partial_{a\alpha}^2 D_{a\alpha}^2 Z_t \right) \right\|_{C_\lambda} \lesssim P_1(\mathcal{E}_{\text{high}})(\lambda \mathcal{E}_{\text{aux}})^\frac{1}{2}$$

Hence proved. □

7. The energy $\mathcal{E}_\Delta$

In this section we consider two solutions of the water wave equation (8), one with and one without surface tension and prove an apriori estimate for the difference of the two solutions. Let $(Z, Z_t)_a$ and $(Z, Z_t)_b$ be two solutions of the water wave equation (8) with surface tensions $\sigma_a = \sigma$ and $\sigma_b = 0$ respectively, i.e. $(Z, Z_t)_a$ is the solution with surface tension $\sigma$ and $(Z, Z_t)_b$ is the solution with zero surface tension. We denote the two solutions as $A$ and $B$ respectively for simplicity. In this section we consider an energy $\mathcal{E}_\Delta$ defined below, which is a weighted Sobolev norm for the difference of the two solutions $A$ and $B$, and prove our main energy estimate for this energy in Theorem 7.1. To prove this theorem, we first control the quantities controlled by $\mathcal{E}_\Delta$ in (7.1) and then close the energy estimate in (7.2). Finally in (7.3) we show that the energy $\mathcal{E}_\Delta$ is equivalent to the energy $\mathcal{E}_\Delta$ defined in (7.1).

Theorem 7.1 allows different initial data for the solutions $A$ and $B$ and hence the energies $\mathcal{E}_\Delta$ and $\mathcal{E}_\Delta$ take this into account. The energy $\mathcal{E}_\Delta$ used in (7.1) is a simplified version of the actual energy defined in (7.1). In this section whenever we talk about the energy $\mathcal{E}_\Delta$, we will mean the one given by (7.1).

Let us recall some of the notation used in (3.1) We will denote the terms and operators for each solution by their subscript $a$ or $b$. Let $h_a, h_b$ be the homeomorphisms from (9) for the respective solutions and let the material derivatives by given by $(D_t)_a = U_{h_a}^{-1} \partial_t U_{h_a}$ and $(D_t)_b = U_{h_b}^{-1} \partial_t U_{h_b}$. We define

$$\tilde{h} = h_b \circ h_a^{-1} \quad \text{and} \quad \tilde{U} = U_{\tilde{h}} = U_{h_a}^{-1} U_{h_b}$$

We also define

$$\Delta(f) = f_a - \tilde{U}(f_b)$$

(51)

(52)
See [3.3] for more details about the notation $\Delta(f)$. Define the operators
\begin{align*}
(\mathcal{H}f)(\alpha') &= \frac{1}{i\pi} \text{p.v.} \int \frac{\tilde{h}(\beta')}{h(\alpha') - h(\beta')} f(\beta') \, d\beta' \\
(\overline{\mathcal{H}}f)(\alpha') &= \frac{1}{i\pi} \text{p.v.} \int \frac{1}{h(\alpha') - \tilde{h}(\beta')} f(\beta') \, d\beta'
\end{align*}
(53)

See Proposition [10.3] for some properties of these operators. In this section in addition to these above operators, we will also heavily use the notations $[f_1, f_2; f_3]$ and $[f_1, f_2; f_3]_\sigma$ defined in [4] and [3] respectively.

We are now ready to define the energy for the difference of the solutions. Define
\begin{align*}
E_{\Delta,0} &= \left\| \left( \frac{\sigma^\frac{1}{2}}{|Z_{\alpha'}|} \frac{1}{Z_{\alpha'}} \right) \right\|_{L^\infty}^2 + \left\| \left( \frac{\sigma^\frac{1}{2}}{|Z_{\alpha'}|} \frac{1}{Z_{\alpha'}} \right) \right\|_{H^\frac{1}{2}}^2 + \left\| \left( \frac{\sigma^\frac{1}{2}}{|Z_{\alpha'}|} \frac{1}{Z_{\alpha'}} \right) \right\|_{H^\frac{1}{2}}^2 + \left\| \Delta(\omega) \right\|_{L^\infty}^2 + \left\| \Delta \left( \frac{2}{Z_{\alpha'}} \right) \right\|_{L^\infty}^2 + \left\| \Delta \left( \frac{2}{Z_{\alpha'}} \right) \right\|_{L^\infty}^2 + \left\| \Delta \left( \frac{2}{Z_{\alpha'}} \right) \right\|_{L^\infty}^2 + \left\| \Delta \left( \frac{2}{Z_{\alpha'}} \right) \right\|_{L^\infty}^2 \\
E_{\Delta,1} &= \left\| \Delta\left( (\mathcal{Z}_{tt,i})Z_{\alpha'} \right) \right\|_{H^\frac{1}{2}}^2 + \left\| \left( \sqrt{\mathcal{A}} \right)_{\alpha'} \Delta (\mathcal{Z}_{t,\alpha'}) \right\|_{H^\frac{1}{2}}^2 + \left\| \left( \frac{\sigma^\frac{1}{2}}{|Z_{\alpha'}|} \frac{1}{Z_{\alpha'}} \right) \right\|_{H^\frac{1}{2}}^2 \\
E_{\Delta,2} &= \left\| \Delta(D_t \mathcal{Z}_{t,\alpha'}) \right\|_{H^\frac{1}{2}}^2 + \left\| \left( \sqrt{\mathcal{A}} \right)_{\alpha'} \Delta (\mathcal{Z}_{t,\alpha'}) \right\|_{H^\frac{1}{2}}^2 + \left\| \left( \frac{\sigma^\frac{1}{2}}{|Z_{\alpha'}|} \frac{1}{Z_{\alpha'}} \right) \right\|_{H^\frac{1}{2}}^2 \\
E_{\Delta,3} &= \left\| \Delta(D_t \mathcal{Z}_{t,\alpha'}) \right\|_{H^\frac{1}{2}}^2 + \left\| \left( \sqrt{\mathcal{A}} \right)_{\alpha'} \Delta (\mathcal{Z}_{t,\alpha'}) \right\|_{H^\frac{1}{2}}^2 + \left\| \left( \frac{\sigma^\frac{1}{2}}{|Z_{\alpha'}|} \frac{1}{Z_{\alpha'}} \right) \right\|_{H^\frac{1}{2}}^2 \\
E_{\Delta,4} &= \left\| \Delta(D_t \mathcal{Z}_{t,\alpha'}) \right\|_{H^\frac{1}{2}}^2 + \left\| \left( \sqrt{\mathcal{A}} \right)_{\alpha'} \Delta (\mathcal{Z}_{t,\alpha'}) \right\|_{H^\frac{1}{2}}^2 + \left\| \left( \frac{\sigma^\frac{1}{2}}{|Z_{\alpha'}|} \frac{1}{Z_{\alpha'}} \right) \right\|_{H^\frac{1}{2}}^2 \\
E_\Delta &= \sigma(E_{\text{aux}})_{b} + E_{\Delta,0} + E_{\Delta,1} + E_{\Delta,2} + E_{\Delta,3} + E_{\Delta,4}
\end{align*}

Note that here $(E_{\text{aux}})_{b}$ is the energy $E_{\text{aux}}$ defined in [10] for the solution $B$. Hence the term $\sigma(E_{\text{aux}})_{b}$ couples the zero surface tension solution $B$ with the value of surface tension $\sigma$ of the solution $A$. This coupling term is crucial to closing the energy estimate and this is discussed in more detail in [3.2]. The other terms in the energy $E_\Delta$ come from taking a difference in the energy $E_\sigma$ defined in [4.3]. We can now state our main apriori energy estimate about the difference of the two solutions.

**Theorem 7.1.** Let $T > 0$ and let $(Z, Z_t)_{\alpha}(t), (Z, Z_t)_{\alpha}(t)$ be two smooth solutions in $[0, T]$ to (8) with surface tension $\sigma$ and zero surface tension respectively, such that for all $s \geq 2$ we have $(Z_{\alpha'} - 1, \frac{1}{Z_{\alpha'}}, 1, Z_t)_{i} \in L^\infty([0,T], H^s(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s+\frac{1}{2}}(\mathbb{R}))$ for both $i = a, b$. Let $L_1 > 0$ be such that
\begin{align*}
\sup_{t \in [0,T]} (E_{\text{high}})_{b}(t), \sup_{t \in [0,T]} (E_{\sigma})_{a}(t), \left\| Z_{\alpha'} \right\|_{H^\frac{1}{2}}(0), \left\| \frac{1}{Z_{\alpha'}} \right\|_{H^\frac{1}{2}}(0), \left\| \frac{1}{Z_{\alpha'}} \right\|_{H^\frac{1}{2}}(0) \leq L_1
\end{align*}
Then there exists a constant $C(L_1)$ depending only on $L_1$ so that for all $t \in [0,T)$ we have

\[
\frac{d}{dt} E(\Delta)(t) \leq C(L_1) E(\Delta)(t)
\]

Note that the above theorem allows the initial data of the two solutions $A$ and $B$ to be different. The theorem simplifies a little bit if we work with the same initial data for the two solutions $A$ and $B$ which was how Theorem 3.1 was stated. The energy $E(\Delta)$ is a very strong norm which compares the two solutions $A$ and $B$. In particular observe that the energy $E_{\Delta,0}$ controls $\|\Delta(\omega)\|_{\infty}$ and hence the above theorem implies that if the initial data of $A$ and $B$ are the same, then the angle of the interface $\theta_a \to \theta_b$ in $L^\infty$ as $\sigma \to 0$ (as stated in [3.1]).

The rest of this section is devoted to the proof of Theorem [7.1]. We will first need some basic identities regarding the operators $\bar{U}, \mathcal{H}, \bar{\mathcal{H}}$ and $\Delta$.

**Lemma 7.2.** Let $\bar{U}$ be defined by (31) and let $\mathcal{H}, \bar{\mathcal{H}}$ be defined by (32). Then

1. $(D_t)_a \bar{U} = \bar{U}(D_t)_b$
2. $\partial_{\alpha'} \bar{U} = \tilde{h}_{\alpha'} \bar{U} \partial_{\alpha'} \partial_{\alpha'} \bar{U}^{-1} = \frac{1}{h_{\alpha'} \circ h^{-1}} \bar{U}^{-1} \partial_{\alpha'} \partial_{\alpha'} \bar{U}^{-1} = \tilde{h}_{\alpha'} U_{h_{\alpha'}} \left( \frac{(h_{\alpha'})_b}{(h_{\alpha'})_a} \right)$
3. $\mathcal{H} \bar{U} = \bar{U} \mathcal{H}$
4. $\bar{U} [f, \mathcal{H}] \partial_{\alpha'} g = [\bar{U} f, \bar{\mathcal{H}}] \partial_{\alpha'} (\bar{U} g)$
5. $\bar{U} [f_1, f_2; \partial_{\alpha'} f_3] = [\bar{U} f_1, \bar{U} f_2; \partial_{\alpha'} (\bar{U} f_3)] \bar{h}$

**Proof.** The proofs are quite straightforward.

1. We see that $(D_t)_a \bar{U} = (U_{h^{-1}_a} \partial_{U_{h_a}})(U_{h^{-1}_a} U_{h_b}) = U_{h^{-1}_a} \partial_{U_{h_b}}$ and similarly we have $\bar{U}(D_t)_b = (U_{h^{-1}_b} \partial_{U_{h_b}})(U_{h^{-1}_b} \partial_{U_{h_b}}) = U_{h^{-1}_b} \partial_{U_{h_b}}$.
2. As $\bar{U}(f)(\alpha') = \bar{h}(\alpha')$ we see that $\partial_{\alpha'} \bar{U}(f)(\alpha') = \tilde{h}_{\alpha'} \alpha' f_{\alpha'} (\tilde{h}(\alpha'))$ and hence $\partial_{\alpha'} \bar{U} = \tilde{h}_{\alpha'} \bar{U} \partial_{\alpha'}$. Similarly we have $\partial_{\alpha'} \bar{U}^{-1} = \frac{1}{h_{\alpha'} \circ h^{-1}} \bar{U}^{-1} \partial_{\alpha'}$. Now as $\bar{h} = h_b \circ h_a^{-1}$ we see that

$$\tilde{h}_{\alpha'} = \frac{(h_{\alpha'})_b \circ h_a^{-1}}{(h_{\alpha'})_a \circ h_a^{-1}}$$

3. We see that

$$\mathcal{H} \bar{U} f(\alpha') = \frac{1}{i\pi} \int \frac{\tilde{h}_{\beta'}(\beta')}{h(\alpha') - \bar{h}(\beta')} f(\tilde{h}(\beta')) d\beta' = \frac{1}{i\pi} \int \frac{1}{\tilde{h}(\alpha') - \beta'} f(\beta') d\beta' = \bar{U} \mathcal{H} f(\alpha')$$

4. We observe that

$$\bar{U} [f, \mathcal{H}] \partial_{\alpha'} g = \bar{U} [f \mathcal{H}(\partial_{\alpha'} g) - \mathcal{H}(f \partial_{\alpha'} g)]$$

$$= \bar{U} f \mathcal{H}(\partial_{\alpha'} g) - \mathcal{H}(\bar{U} f \partial_{\alpha'} g)$$

$$= \bar{U} f \mathcal{H}(\tilde{h}_{\alpha'} \bar{U} g) - \mathcal{H}(\bar{U} f \partial_{\alpha'} \bar{U} g)$$

$$= [(\bar{U} f), \mathcal{H}] \partial_{\alpha'} (\bar{U} g)$$
estimates for all we write Lemma 7.4. For the third estimate we see that
\[ \Delta \left( U_1, U_2 ; \partial_{\alpha'} U_3 \right) \]
\[ = \frac{1}{i\pi} \int \left( f_1 \left( \tilde{h} (\alpha') - f_1 (\beta') \right) \right) \left( f_2 \left( \tilde{h} (\alpha') - f_2 (\beta') \right) \right) (f_3 \circ \tilde{h}) (\beta') \, d\beta' \]
\[ = \frac{1}{i\pi} \int \left( f_1 \left( \tilde{h} (\alpha') - f_1 \tilde{h} (\beta') \right) \right) \left( f_2 \left( \tilde{h} (\alpha') - f_2 \tilde{h} (\beta') \right) \right) (f_3 \circ \tilde{h}) (\beta') \, d\beta' \]
\[ = \left[ (U_1), (U_2) ; \partial_{\alpha'} (U_3) \right] \]

Lemma 7.3. Let \( \Delta \) be defined by (52). Then

1. \( \Delta (f_1 f_2 \cdots f_n) = \sum_{i=1}^{n} \{ U(f_1) b \cdots U(f_{i-1}) b \} \Delta (f_i) \{ (f_{i+1}) a \cdots (f_n) a \} \)
2. \( \Delta \left( f, \mathbb{H} \right) \partial_{\alpha'} g = [\Delta f, \mathbb{H}] \partial_{\alpha'} g + (U(f), \mathbb{H} - \tilde{h}) \partial_{\alpha'} (U_1 \Delta (\tilde{h} g)) \)
3. \( \Delta (f_1, f_2 ; \partial_{\alpha'} f_3) = \Delta (f_1, (f_2) a ; \partial_{\alpha'} (f_3) a) + \{ U(f_1), \Delta f_2 ; \partial_{\alpha'} (f_3) a \] 
\[ + \left[ U(f_1), U(f_2) ; \partial_{\alpha'} (U_3) \right] \]
\[ + \left[ (U(f_1), U(f_2)) ; \partial_{\alpha'} \Delta (f_3) \right] \]
\[ + \left[ U(f_1), U(f_2) ; \partial_{\alpha'} \Delta (f_3) \right] \]
\[ + \left[ (U(f_1), U(f_2)) ; \partial_{\alpha'} \Delta (f_3) \right] \]

Proof. The first identity follows immediately from the definition. For the second, we see that
\[ \Delta (f_1, f_2 \cdots f_n) = \sum_{i=1}^{n} \{ U(f_1) b \cdots U(f_{i-1}) b \} \Delta (f_i) \{ (f_{i+1}) a \cdots (f_n) a \} \]
\[ = \Delta \left( f, \mathbb{H} \right) \partial_{\alpha'} g - \tilde{U} \left( f, \mathbb{H} \right) \partial_{\alpha'} \tilde{U} \]
\[ = \left\{ U(f_1), U(f_2) \right\} ; \partial_{\alpha'} \Delta (f_3) \]
\[ + \left\{ U(f_1), U(f_2) \right\} ; \partial_{\alpha'} \Delta (f_3) \]
\[ + \left\{ U(f_1), U(f_2) \right\} ; \partial_{\alpha'} \Delta (f_3) \]

For the third estimate we see that
\[ \Delta (f_1, f_2 ; \partial_{\alpha'} f_3) = \{ (f_1) a, (f_2) a ; \partial_{\alpha'} (f_3) a \} - \tilde{U} \{ (f_1) b, (f_2) b ; \partial_{\alpha'} (f_3) b \} \]
\[ = \{ (f_1) a, (f_2) a ; \partial_{\alpha'} (f_3) a \} - \tilde{U} \{ (f_1) b, \tilde{U}(f_2) b ; \partial_{\alpha'} \tilde{U}(f_3) b \} \]
\[ + \left\{ U(f_1), U(f_2) ; \partial_{\alpha'} \Delta (f_3) \right\] 
\[ + \left\{ U(f_1), U(f_2) ; \partial_{\alpha'} \Delta (f_3) \right\] 
\[ + \left\{ U(f_1), U(f_2) ; \partial_{\alpha'} \Delta (f_3) \right\] 

The following lemma gives us control of some basic quantities required for the proof of Theorem 7.1.

Lemma 7.4. Assume the hypothesis of Theorem 7.1. We will suppress the dependence of \( L_1 \) i.e. we write \( a \lesssim b \) instead of \( a \leq C(L_1) b \). Let \( f \in S(\mathbb{R}) \). With this notation we have the following estimates for all \( t \in [0, T) \)
We will prove each of the estimates individually.

(1) \[ \|\bar{h}_{\alpha'}\|_{L^\infty}(t), \left\| \frac{1}{h_{\alpha'}} \right\|_{L^\infty}(t) \leq 1 \]

(2) \[ \left\| \frac{\bar{h}(\alpha', t) - \bar{h}(\beta', t)}{\alpha' - \beta'} \right\| \leq 1 \text{ for all } \alpha' \neq \beta' \]

(3) \[ \|Uf\|_2 \leq \|f\|_2 \text{ and } \|\bar{U}f\|_{H^\frac{1}{2}} \leq \|f\|_{H^\frac{1}{2}}, \text{ These estimates are also true for the operator } \bar{U}^{-1} \text{ instead of } \bar{U}. \]

(4) \[ \|\bar{H}(f)\|_2 \leq \|f\|_2, \|\bar{H}(f)\|_{H^\frac{1}{2}} \leq \|f\|_{H^\frac{1}{2}} \text{ and } \|\bar{H}(f)\|_2 \leq \|f\|_2 \]

(5) \[ \left\| \bar{h}_{\alpha'} \right\|_{H^\frac{1}{2}}(t), \left\| \frac{1}{h_{\alpha'}} \right\|_{H^\frac{1}{2}}(t) \leq 1 \]

(6) \[ \left\| Z_{\alpha'} \right\|_a \bar{U} \left( \frac{1}{Z_{\alpha'}} \right) b_\infty(t), \left\| \frac{1}{Z_{\alpha'}} \bar{U} \left( |Z_{\alpha'}| \right) b_\infty(t) \right\| \leq 1 \]

(7) \[ \left\| (D_{\alpha'}) a \bar{h}_{\alpha'} \right\|_2(t) \leq 1 \]

(8) \[ \left\| (D_{\alpha'}) a \left\{ Z_{\alpha'} \right\} \bar{U} \left( \frac{1}{Z_{\alpha'}} \right) b_2(t), \left\{ (D_{\alpha'}) a \left\{ \frac{1}{Z_{\alpha'}} \bar{U} \left( |Z_{\alpha'}| \right) b_2(t) \right\} \right\}_2 \leq 1 \]

Proof. We will prove each of the estimates individually.

(1) From (13) we know that \( h_a(\alpha', 0) = \bar{h}_b(\alpha', 0) = \alpha' \) at time \( t = 0 \). Now observe that \( U_{h}^{-1} \left( \frac{h_{\alpha}}{h_{\alpha}} \right) = b_{\alpha'} \) and \( \partial_t h_{\alpha} = \left( \frac{h_{\alpha}}{h_{\alpha}} \right) h_{\alpha} \) and \( \partial_{\alpha'} \left( \frac{1}{h_{\alpha}} \right) = -\left( \frac{h_{\alpha}}{h_{\alpha}} \right) \frac{1}{h_{\alpha}} \)

and as \( \| (b_{\alpha'}) b_\infty \) is controlled by \( (\mathcal{E}_{a} h_b)(t) \) and \( \| (b_{\alpha'}) a_\infty \) is controlled by \( (\mathcal{E}_{a})_a(t) \), we see that \( (h_{\alpha})_a \) and \( \left( \frac{1}{h_{\alpha}} \right)_a \) remain bounded for both \( i = a, b \). Now

\[ (D_{\alpha'})_a \bar{h}_{\alpha'} = U_{h}^{-1} \left( \partial_t \left( \frac{h_{\alpha}}{h_{\alpha}} \right) \right) = \bar{h}_{\alpha'} U_{h}^{-1} \left\{ \left( \frac{h_{\alpha}}{h_{\alpha}} \right) - \left( \frac{h_{\alpha}}{h_{\alpha}} \right) \right\} = \bar{h}_{\alpha'} (\bar{U} (b_{\alpha'})_a - (b_{\alpha'})_a) \]

Hence as \( \bar{h}_{\alpha'} = 1 \) at time \( t = 0 \), we see that \( \| \bar{h}_{\alpha'} \|_\infty(t) \leq 1 \). Similarly for \( \frac{1}{h_{\alpha'}} \).

(2) This is an easy consequence of \( \| \bar{h}_{\alpha'} \|_L^\infty(t), \left\| \frac{1}{h_{\alpha'}} \right\|_L^\infty(t) \leq 1 \) and the fact that \( \bar{h}(\cdot, t) \) is a homeomorphism.

(3) We see that

\[ \|\bar{U}f\|_2^2 = \int f(\bar{h}(\alpha'))^2 d\alpha' = \int \frac{|f(s)|^2}{|\bar{h}(\alpha' h^{-1})(s)|} ds \lesssim \|f\|_2^2 \]

Similarly we have that

\[ \|\bar{U}f\|_{H^\frac{1}{2}}^2 = \frac{1}{2\pi} \int \int \frac{|f(\bar{h}(\alpha')) - f(\bar{h}(\beta'))|^2}{(\alpha' - \beta')^2} d\alpha' d\beta' \]

\[ = \frac{1}{2\pi} \int \int \left( \frac{|f(x) - f(y)|^2}{(h^{-1}(x) - h^{-1}(y))^2} \right) \frac{1}{|\bar{h}(\alpha' h^{-1})(x)(\bar{h}(\alpha' h^{-1})(y)|} dx dy \]

\[ \lesssim \frac{1}{2\pi} \int \int \frac{|f(x) - f(y)|^2}{(\alpha' - \beta')^2} dx dy \]

\[ \lesssim \|f\|_{H^\frac{1}{2}}^2 \]
In the same way we can prove the estimates for $\tilde{U}^{-1}$.

(4) This follows directly from Proposition [10,3].

(5) As $\|\langle b_\alpha \rangle \|_{L^\infty \cap L^\frac{2}{3}}$ and is controlled by $C(L_1)$ for $i = a, b$, using Lemma [4,4] and Proposition [9,6] we see that

\[
\frac{d}{dt} \| \tilde{h}_\alpha \|^2_{H^\frac{2}{3}} \lesssim \| \tilde{h}_\alpha \|^2_{H^\frac{2}{3}} + \| \tilde{h}_\alpha \|_{H^\frac{2}{3}} \| (D_t)_a \tilde{h}_\alpha \|_{H^\frac{2}{3}} \\
\lesssim \| \tilde{h}_\alpha \|^2_{H^\frac{2}{3}} + \| \tilde{h}_\alpha \|_{H^\frac{2}{3}} \| \tilde{h}_\alpha (\tilde{U}(b_\alpha)_b - \langle b_\alpha \rangle_b) \|_{H^\frac{2}{3}} \\
\lesssim \| \tilde{h}_\alpha \|^2_{H^\frac{2}{3}} + \| \tilde{h}_\alpha \|_{H^\frac{2}{3}} (\| \tilde{h}_\alpha \|_\infty + \| \tilde{h}_\alpha \|_{H^0}) \\
\lesssim (\| \tilde{h}_\alpha \|^2_{H^\frac{2}{3}} + 1)
\]

Now as $\| \tilde{h}_\alpha \|_{H^\frac{2}{3}}(0) = 0$, we obtain that $\| \tilde{h}_\alpha \|_{H^\frac{2}{3}}(t) \lesssim 1$ for $t \in [0, T)$.

(6) Using (24) we observe that

\[
(D_t)_a \left\{ |Z_\alpha|_a \tilde{U} \left( \frac{1}{|Z_\alpha|} \right)_b \right\} \\
= \left\{ |Z_\alpha|_a \tilde{U} \left( \frac{1}{|Z_\alpha|} \right)_b \right\} \text{Re} \left\{ (D_\alpha Z_t)_a - \tilde{U}(D_\alpha Z_t)_b - \langle b_\alpha \rangle_a + \tilde{U}(b_\alpha)_b \right\}
\]

Hence we have that $\left\| |Z_\alpha|_a \tilde{U} \left( \frac{1}{|Z_\alpha|} \right)_b \right\|_\infty(t) \lesssim 1$ for all $t \in [0, T)$. The other estimate is proven similarly.

(7) We first observe that

\[
(D_t)_a |D_\alpha|_a \tilde{h}_\alpha = \text{Re}(D_\alpha Z_t)_a |D_\alpha|_a \tilde{h}_\alpha + |D_\alpha|_a (D_t)_a \tilde{h}_\alpha \\
= \text{Re}(D_\alpha Z_t)_a |D_\alpha|_a \tilde{h}_\alpha + |D_\alpha|_a \left\{ \tilde{h}_\alpha (\tilde{U}(b_\alpha)_b - \langle b_\alpha \rangle_a) \right\} \\
= \text{Re}(D_\alpha Z_t)_a |D_\alpha|_a \tilde{h}_\alpha + (|D_\alpha|_a \tilde{h}_\alpha)(\tilde{U}(b_\alpha)_b - \langle b_\alpha \rangle_a) \\
+ \tilde{h}_\alpha \left\{ \tilde{h}_\alpha \left( \frac{1}{|Z_\alpha|_a} \right) \tilde{U}(|Z_\alpha|_b) \tilde{U}(|D_\alpha| b - |D_\alpha| b) \right\}
\]

Hence using Lemma [4,4] we have

\[
\frac{d}{dt} \left\| |D_\alpha|_a \tilde{h}_\alpha \right\|^2_2 \lesssim \left\| |D_\alpha|_a \tilde{h}_\alpha \right\|^2_2 + \left\| |D_\alpha|_a \tilde{h}_\alpha \right\|^2_2 + \left\| (D_t)_a |D_\alpha|_a \tilde{h}_\alpha \right\|^2_2 \\
\lesssim \left\{ \left\| |D_\alpha|_a \tilde{h}_\alpha \right\|^2_2 + 1 \right\}
\]

As $|D_\alpha|_a \tilde{h}_\alpha = 0$ at time $t = 0$, we are done.

(8) Using (24) we observe that

\[
(|D_\alpha|_a) \left\{ |Z_\alpha|_a \tilde{U} \left( \frac{1}{|Z_\alpha|} \right)_b \right\} \\
= \left\{ |Z_\alpha|_a \tilde{U} \left( \frac{1}{|Z_\alpha|} \right)_b \right\} \left\{ - \partial_\alpha \left( \frac{1}{|Z_\alpha|} \right)_a + \tilde{h}_\alpha \left( \frac{1}{|Z_\alpha|_a} \right) \tilde{U} \left( \partial_\alpha \frac{1}{|Z_\alpha|} \right) \right\}
\]

Hence we see that $\left\| (|D_\alpha|_a) \left\{ |Z_\alpha|_a \tilde{U} \left( \frac{1}{|Z_\alpha|} \right)_b \right\} \right\|_2(t) \lesssim 1$ for all $t \in [0, T)$. The other estimate is proven similarly.
7.1. Quantities controlled by $E_\Delta$

In this section whenever we write $f \in L^2_{\Delta, \alpha}$, what we mean is that there exists a constant $C(L_1)$ depending only on $L_1$ such that $\|f\|_2 \leq C(L_1)(E_\Delta)^\alpha$. Similar definitions for $f \in L^1_{\Delta, \alpha}$, $f \in \tilde{H}^\frac{1}{2}_{\Delta, \alpha}$ and $f \in L^\infty_{\Delta, \alpha}$. We define the spaces $C_{\Delta, \alpha}$ and $W_{\Delta, \alpha}$ as follows

1. If $w \in L^\infty_{\Delta, \alpha}$ and $|D_{\alpha'}|w \in L^2_{\Delta, \alpha}$, then we say $f \in W_{\Delta, \alpha}$. Define

$$\|w\|_{W_{\Delta, \alpha}} = \|w\|_W = \|w\|_{L^\infty} + \|D_{\alpha'}|w\|_2$$

2. If $f \in \tilde{H}^\frac{1}{2}_{\Delta, \alpha}$ and $f[Z_{\alpha'}]_a \in L^2_{\Delta, \alpha}$, then we say $f \in C_{\Delta, \alpha}$. Define

$$\|f\|_{C_{\Delta, \alpha}} = \|f\|_C = \|f\|_{\tilde{H}^\frac{1}{2}} + \left(1 + \\left\|\left(\frac{1}{|Z_{\alpha'}|}\right)_a\right\|_2\right)\|f[Z_{\alpha'}]_a\|_2$$

Now analogous to Lemma 4.3 we have the following lemma

**Lemma 7.5.** Let $\alpha_1, \alpha_2, \alpha_3 \geq 0$ with $\alpha_1 + \alpha_2 = \alpha_3$. Then the following properties hold for the spaces $W_{\Delta, \alpha}$ and $C_{\Delta, \alpha}$

1. If $w_1 \in W_{\Delta, \alpha_1}$, $w_2 \in W_{\Delta, \alpha_2}$, then $w_1w_2 \in W_{\Delta, \alpha_3}$. Moreover we have the estimate $\|w_1w_2\|_{W_{\Delta, \alpha_3}} \lesssim \|w_1\|_{W_{\Delta, \alpha_1}}\|w_2\|_{W_{\Delta, \alpha_2}}$

2. If $f \in C_{\Delta, \alpha_1}$ and $w \in W_{\Delta, \alpha_2}$, then $fw \in C_{\Delta, \alpha_3}$. Moreover $\|fw\|_{C_{\Delta, \alpha_3}} \lesssim \|f\|_{C_{\Delta, \alpha_1}}\|w\|_{W_{\Delta, \alpha_2}}$

3. If $f \in C_{\Delta, \alpha_1}$, $g \in C_{\Delta, \alpha_2}$, then $fg[Z_{\alpha'}]_a \in L^2_{\Delta, \alpha_3}$. Moreover $\|fg[Z_{\alpha'}]_a\|_2 \lesssim \|f\|_{C_{\Delta, \alpha_1}}\|g\|_{C_{\Delta, \alpha_2}}$

When we write $f \in L^2$ we mean $f \in L^2_{\Delta, \alpha}$ with $\alpha = 0$. Similar notation for $\tilde{H}^\frac{1}{2}, L^\infty, C$ and $W$. It is important to note that if $f \in L^2$ and $f \in L^2_{\Delta, \alpha}$, then we have $f \in L^2_{\Delta, \beta}$ for all $0 < \beta < \alpha$. We say that $a \approx_{L^2_{\Delta, \alpha}} b$ if $a - b \in L^2_{\Delta, \alpha}$. It should be noted that $\approx_{L^2_{\Delta, \alpha}}$ is an equivalence relation. Similar definitions for $\approx_{L^\infty_{\Delta, \alpha}}, \approx_{\tilde{H}^\frac{1}{2}_{\Delta, \alpha}}, \approx_{W_{\Delta, \alpha}}$ and $\approx_{C_{\Delta, \alpha}}$.

In this section, we will need to commute weights and derivatives with the operators $\tilde{U}$ and $\Delta$ quite frequently and hence the following lemma will be very frequently used.

**Lemma 7.6.** Let $f, g \in S(\mathbb{R})$ and let $\alpha \in \mathbb{R}$. Then

1. If $g_\alpha\tilde{U}(\partial_{\alpha'}f)_b \in L^2$ then
   (a) $g_\alpha\partial_{\alpha'}\tilde{U}(f)_b \in L^2$
   (b) $g_\alpha\tilde{U}(\partial_{\alpha'}f)_b \approx_{L^2_{\Delta, \alpha}} g_\alpha\partial_{\alpha'}\tilde{U}(f)_b$
   (c) $g_\alpha\Delta(\partial_{\alpha'}f) \approx_{L^2_{\Delta, \alpha}} g_\alpha\partial_{\alpha'}\Delta(f)$

   These estimates are also true if we replace $(L^2, L^2_{\sqrt{\Delta}})$ with $(L^\infty, L^{\infty}_{\sqrt{\Delta}})$, $(L^\infty \cap \tilde{H}^\frac{1}{2}, L^{\infty}_{\sqrt{\Delta}} \cap \tilde{H}^\frac{1}{2}_{\sqrt{\Delta}})$, $(W, W_{\sqrt{\Delta}})$ or $(C, C_{\sqrt{\Delta}})$.

2. If $g_\alpha\tilde{U}(f)_b \in L^2$ then
   (a) $(g[Z_{\alpha'}]_a)_b \tilde{U}\left([Z_{\alpha'}]^{-\alpha}_a f\right)_b \in L^2$
   (b) $g_\alpha\tilde{U}(f)_a \approx_{L^2_{\Delta, \alpha}} (g[Z_{\alpha'}]_a)_a \tilde{U}\left([Z_{\alpha'}]^{-\alpha}_a f\right)_b$
   (c) $g_\alpha\Delta(f) \approx_{L^2_{\Delta, \alpha}} (g[Z_{\alpha'}]_a)_a \Delta([Z_{\alpha'}]^{-\alpha}_a f)$

   These estimates are also true if we replace $(L^2, L^2_{\sqrt{\Delta}})$ with $(L^\infty, L^{\infty}_{\sqrt{\Delta}})$, $(W, W_{\sqrt{\Delta}})$ or $(C, C_{\sqrt{\Delta}})$.\]

\[ \square \]
(3) If \( g_a \tilde{U}(f)_b \in L^2 \) then
   
   (a) \( (g_\alpha^a)_a \tilde{U}(\omega^{-\alpha}f)_b \in L^2 \)
   
   (b) \( g_a \tilde{U}(f)_b \approx_{L^2_{\sqrt{\Delta}}} \) \( (g_\alpha^a)_a \tilde{U}(\omega^{-\alpha}f)_b \)
   
   (c) \( g_a \Delta(f) \approx_{L^2_{\sqrt{\Delta}}} \) \( (g_\alpha^a)_a \Delta(\omega^{-\alpha}f) \)

These estimates are also true if we replace \( L^2_{\sqrt{\Delta}} \) with \( (L^\infty, L^{\infty}_{\sqrt{\Delta}}) \), \( (W, W_{\sqrt{\Delta}}) \) or \( (C, C_{\sqrt{\Delta}}) \).

Proof. We prove each of the statements individually.

(1) We first observe that the energy \( E_\Delta \) controls \( (\tilde{h}_a - 1) \in L^\infty_{\sqrt{\Delta}} \cap H^\frac{1}{2}_{\sqrt{\Delta}} \cap W_{\sqrt{\Delta}} \). Now notice that

   \[ g_a \partial_\alpha \tilde{U}(f)_b = \tilde{h}_a g_a \tilde{U}(\partial_\alpha f)_b. \]

   As \( \tilde{h}_a \in L^\infty \) we see that \( g_a \partial_\alpha \tilde{U}(f)_b \in L^2 \). Now we have

   \[ g_a \partial_\alpha \tilde{U}(f)_b - g_a \tilde{U}(\partial_\alpha f)_b = (\tilde{h}_a - 1) g_a \tilde{U}(\partial_\alpha f)_b \]

   and hence \( \|g_a \partial_\alpha \tilde{U}(f)_b - g_a \tilde{U}(\partial_\alpha f)_b\|_2 \leq \|\tilde{h}_a - 1\|_{\infty} \|g_a \tilde{U}(\partial_\alpha f)_b\|_2 \leq C(L_1)(E_\Delta)^\frac{1}{2}. \) The other estimates are shown similarly using the fact that \( \tilde{h}_a \in L^\infty \cap H^\frac{1}{2} \cap \mathcal{W} \) and \( (\tilde{h}_a - 1) \in L^\infty_{\sqrt{\Delta}} \cap H^\frac{1}{2}_{\sqrt{\Delta}} \cap W_{\sqrt{\Delta}} \).

(2) Observe that the energy \( E_\Delta \) controls \( \Delta w \in L^\infty_{\sqrt{\Delta}} \) and \( \Delta \left( \partial_\alpha \frac{1}{|Z_\alpha|} \right) \in L^2_{\sqrt{\Delta}} \). Hence by using \([10]\)

   \[ \Delta \left( \partial_\alpha \frac{1}{|Z_\alpha|} \right) \in L^2_{\sqrt{\Delta}} \] and \( \Delta(\partial_\alpha |\omega|) \in L^2_{\sqrt{\Delta}} \).

   Now observe that for \( \alpha \in \mathbb{R} \), we have \( |\alpha - 1| \leq C(\alpha) |z_1 - 1| \max(|\alpha|, 1) \) for \( z \in \mathbb{C} \). Hence using the fact that \( |Z_\alpha|_a \tilde{U} \left( \frac{1}{|Z_\alpha|} \right)_b - 1 \in L^\infty_{\sqrt{\Delta}} \) we see that \( \left( |Z_\alpha|_a \tilde{U} \left( \frac{1}{|Z_\alpha|} \right)_b \right)^\alpha - 1 \in L^\infty_{\sqrt{\Delta}} \).

   In particular we have \( \frac{1}{|Z_\alpha|} \tilde{U} \left( \frac{1}{|Z_\alpha|} \right)_b - 1 \in L^\infty_{\sqrt{\Delta}} \). Now we have

   \[ \langle |\partial_\alpha |\omega| \rangle \left\{ |Z_\alpha|_a \tilde{U} \left( \frac{1}{|Z_\alpha|} \right)_b \right\} \]

   \[ = \left\{ |Z_\alpha|_a \tilde{U} \left( \frac{1}{|Z_\alpha|} \right)_b \right\} - \left( \partial_\alpha \frac{1}{|Z_\alpha|} \right)_a \cdot \tilde{h}_a \left( \frac{1}{|Z_\alpha|} \tilde{U} \left( \frac{1}{|Z_\alpha|} \right)_b \right) \left( \partial_\alpha \frac{1}{|Z_\alpha|} \right)_b \]

   Hence we see that \( |Z_\alpha|_a \tilde{U} \left( \frac{1}{|Z_\alpha|} \right)_b - 1 \in W_{\sqrt{\Delta}} \) or more generally \( \left( |Z_\alpha|_a \tilde{U} \left( \frac{1}{|Z_\alpha|} \right)_b \right)^\alpha - 1 \in W_{\sqrt{\Delta}} \). Now coming back we see that

   \[ \langle |\partial_\alpha |\omega| \rangle \left( |Z_\alpha|_a \tilde{U} \left( \frac{1}{|Z_\alpha|} \right)_b \right)^\alpha = \left( |Z_\alpha|_a \tilde{U} \left( \frac{1}{|Z_\alpha|} \right)_b \right)^\alpha g_a \tilde{U}(f)_b \]

   Now as \( |Z_\alpha|_a \tilde{U} \left( \frac{1}{|Z_\alpha|} \right)_b \in L^\infty \), we see that \( \langle |\partial_\alpha |\omega| \rangle \left( |Z_\alpha|_a \tilde{U} \left( \frac{1}{|Z_\alpha|} \right)_b \right)^\alpha \in L^2 \). Now

   \[ \langle |\partial_\alpha |\omega| \rangle \left( |Z_\alpha|_a \tilde{U} \left( \frac{1}{|Z_\alpha|} \right)_b \right)^\alpha - g_a \tilde{U}(f)_b = \left\{ \left( |Z_\alpha|_a \tilde{U} \left( \frac{1}{|Z_\alpha|} \right)_b \right)^\alpha - 1 \right\} g_a \tilde{U}(f)_b \]

   Hence we have \( \| \langle |\partial_\alpha |\omega| \rangle \left( |Z_\alpha|_a \tilde{U} \left( \frac{1}{|Z_\alpha|} \right)_b \right)^\alpha - g_a \tilde{U}(f)_b \|_2 \leq C(L_1)(E_\Delta)^\frac{1}{2}. \) The other estimates are proven similarly using \( |Z_\alpha|_a \tilde{U} \left( \frac{1}{|Z_\alpha|} \right)_b \in L^\infty \cap \mathcal{W} \) and \( \left\{ \left( |Z_\alpha|_a \tilde{U} \left( \frac{1}{|Z_\alpha|} \right)_b \right)^\alpha - 1 \right\} \in L^\infty_{\sqrt{\Delta}} \cap W_{\sqrt{\Delta}} \).
(3) This is proved exactly the same as above. Here we use the estimate $\Delta(w) \in L^\infty_{\sqrt{\Delta}}$ and $|w| = 1$ to see that $\omega_a \tilde{U}(\omega^{-1})_b - 1 \in L^\infty_{\sqrt{\Delta}}$ or more generally $\omega^a \tilde{U}(\omega^{-a})_b - 1 \in L^\infty_{\sqrt{\Delta}}$. Now we observe that

$$\begin{align*}
|D\alpha_a| \left(\omega_a \tilde{U}(\omega^{-1})_b\right) \\
= \left(\omega_a \tilde{U}(\omega^{-1})_b\right) \left(\sum_a |D\alpha_a^t| a_a - \frac{\vec{h}_a}{|Z,\alpha_a^t| a} \tilde{U}(|Z,\alpha_a^t|) \tilde{U}(|D\alpha^t|)\right)
\end{align*}$$

Now as the energy $E_\Delta$ controls $\Delta(|D\alpha^t|) \in L^2_{\sqrt{\Delta}}$, we see that $\omega_a \tilde{U}(\omega^{-1})_b - 1 \in W\sqrt{\Delta}$ or more generally $\omega^a \tilde{U}(\omega^{-a})_b - 1 \in W\sqrt{\Delta}$. Now using

$$(g\omega^a)_a \tilde{U}(\omega^{-a} f)_b - g_a \tilde{U}(f)_b = g_a \tilde{U}(f)_b \left\{\omega^a \tilde{U}(\omega^{-a})_b - 1\right\}$$

we easily reach the desired conclusion.

One important conclusion from the proof of the previous lemma is that for any $\alpha \in \mathbb{R}$ we have

$$\left\{\begin{align*}
|Z,\alpha_a^t| \tilde{U} \left(\frac{1}{|Z,\alpha_a^t|}\right) \\
\end{align*}\right\}_a - 1 \in W\sqrt{\Delta}$$

This estimate along with the fact that $(\vec{h}_a - 1) \in W\sqrt{\Delta} \cap \dot{H}^\frac{1}{2}\sqrt{\Delta}$ will be heavily used in the proof of the energy estimate.

Let us now control the main terms controlled by $E_\Delta$. We will heavily use the lemmas proven earlier namely Lemma 7.2, Lemma 7.3, Lemma 7.4, Lemma 7.5, Lemma 7.6 along with Proposition 9.5 and Proposition 10.3 from the appendix. As we will be using Lemma 7.2, Lemma 7.3 and Lemma 7.4 in almost every step, we will skip referencing them. The proof of the estimates in this section follow exactly analogous to the proof of the control of the quantities controlled by $E_\alpha$ from Sec 5.1 of [1], and in particular we use the identities established there and then subtract the terms of the two solutions. As the proofs are straightforward modifications of the proofs in Sec 5.1 of [1], we will just control a few terms and show how it is done and the rest are proved analogously. A few terms require a bit more work and we give more details for those terms. The numbering system employed here for the quantities controlled is the same as used in Sec 5.1 of [1].

1. $\Delta(Z_{t,\alpha_a}) \in L^2_{\sqrt{\Delta}}$, $\partial_{\alpha_a} \Delta(Z_{t}) \in L^2_{\sqrt{\Delta}}$ and $|D\alpha_a| \Delta(D\alpha_a) \Delta(Z_{t}) \in L^2_{\sqrt{\Delta}}$

Proof: As $(A_1)_t \geq 1$, we see that $E_\Delta$ controls $\Delta(Z_{t,\alpha_a}) \in L^2_{\sqrt{\Delta}}$ and $|D\alpha_a| \Delta(D\alpha_a) \Delta(Z_{t}) \in L^2_{\sqrt{\Delta}}$

We obtain the other two estimates by using Lemma 7.6

2. $\Delta(A_1) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$

Proof: Recall from [8] that $A_1 = 1 - \text{Im} |Z_t, \mathbb{H}| Z_{t,\alpha_a}$ and hence we have $\Delta(A_1)$

$$\begin{align*}
\Delta(A_1) \\
= -\text{Im} \left\{\Delta |Z_t, \mathbb{H}| \partial_{\alpha_a} Z_{t}\right\} \\
= -\text{Im} \left\{\Delta |Z_t, \mathbb{H}| \partial_{\alpha_a} (Z_{t}) + \left[\tilde{U}(Z_{t})_b, H - \mathbb{H}\right] \partial_{\alpha_a} (Z_{t})_b + \tilde{U}(\{(Z_{t})_b, \mathbb{H}\} | \partial_{\alpha_a} (\tilde{U}^{-1}(\Delta Z_{t}))\right\}
\end{align*}$$

Hence using Proposition 9.3 Proposition 10.3 and Lemma 7.6 we get $\|\Delta(A_1)\|_{L^\infty_{\sqrt{\Delta}} \cap \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}} \leq C(L_1)(E_\Delta)\frac{1}{2}$. 
(3) \( \Delta \left( \frac{1}{Z_{,\alpha'}} \right) \in L^2_{\sqrt{\lambda}} \), \( \Delta \left( \frac{1}{Z_{,\alpha'}} \right) \in L^2_{\sqrt{\lambda}} \), \( \Delta(|D_{\alpha'}|\omega) \in L^2_{\sqrt{\lambda}} \) and hence \( \Delta(\omega) \in W_{\sqrt{\lambda}} \).

Proof: Observe that \( \Delta \left( \frac{1}{Z_{,\alpha'}} \right) \in L^2_{\sqrt{\lambda}} \) as it is part of the energy \( E_{\Delta,0} \). Recall from (16) that

\[
\text{Re} \left( \frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) = \frac{1}{|Z_{,\alpha'}|} \partial_{\alpha'} \text{Im} \left( \frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) = i(\overline{\omega} |D_{\alpha'}| \omega)
\]

Using \( \Delta(\omega) \in L^\infty_{\sqrt{\lambda}} \) we obtain \( \Delta \left( \frac{1}{Z_{,\alpha'}} \right) \in L^2_{\sqrt{\lambda}} \) and \( \Delta(|D_{\alpha'}|\omega) \in L^2_{\sqrt{\lambda}} \). Now using Lemma 7.6 we obtain \( |D_{\alpha'}|_{a} \Delta(\omega) \in L^2_{\sqrt{\lambda}} \) and hence \( \Delta(\omega) \in W_{\sqrt{\lambda}} \).

(4) \( \Delta(\overline{D}_{\alpha'} Z_{t}) \in L^\infty_{\sqrt{\lambda}}, \Delta(|D_{\alpha'}|Z_{t}) \in L^\infty_{\sqrt{\lambda}} \) and \( \Delta(D_{\alpha'} Z_{t}) \in L^\infty_{\sqrt{\lambda}} \).

Proof: First observe that from Lemma 7.6 we have \( |Z_{,\alpha'}|_{a} \Delta(\overline{D}_{\alpha'} Z_{t}) \in L^2_{\sqrt{\lambda}} \). \Delta(\overline{D}_{\alpha'} Z_{t}) \in L^2_{\sqrt{\lambda}} \). The other ones are obtained easily by using Lemma 7.6.

(5) \( \Delta(\overline{D}_{\alpha'} Z_{t}) \in L^2_{\sqrt{\lambda}}, \Delta(|D_{\alpha'}|^2 Z_{t}) \in L^2_{\sqrt{\lambda}} \) and \( \Delta \left( \frac{1}{Z_{,\alpha'}} \partial_{\alpha'} Z_{t,a'} \right) \in L^2_{\sqrt{\lambda}} \).

Proof: We already know that \( \Delta(|D_{\alpha'}|^2 Z_{t}) \in L^2_{\sqrt{\lambda}} \) and hence using Lemma 7.6 we have \( \Delta(\overline{D}_{\alpha'} Z_{t}) \in L^2_{\sqrt{\lambda}} \) and \( \Delta \left( \frac{1}{Z_{,\alpha'}} \partial_{\alpha'} Z_{t,a'} \right) \in L^2_{\sqrt{\lambda}} \) is proven similarly.

(6) \( \Delta(\overline{D}_{\alpha'} Z_{t}) \in W_{\sqrt{\lambda}} \cap C_{\sqrt{\lambda}}, \Delta(|D_{\alpha'}|^2 Z_{t}) \in W_{\sqrt{\lambda}} \cap C_{\sqrt{\lambda}}, \Delta(D_{\alpha'} Z_{t}) \in W_{\sqrt{\lambda}} \cap C_{\sqrt{\lambda}} \).

Proof: As \( \Delta(\overline{D}_{\alpha'} Z_{t}) \in L^\infty_{\sqrt{\lambda}} \) and \( |Z_{,\alpha'}|_{a} \Delta(\overline{D}_{\alpha'} Z_{t}) \in L^2_{\sqrt{\lambda}} \) we see that \( \Delta(\overline{D}_{\alpha'} Z_{t}) \in W_{\sqrt{\lambda}} \). Now \( |Z_{,\alpha'}|_{a} \Delta(\overline{D}_{\alpha'} Z_{t}) \in L^2_{\sqrt{\lambda}} \) and using Proposition 0.8 with \( f = \Delta(\overline{D}_{\alpha'} Z_{t}) \) and \( w = \frac{1}{|Z_{,\alpha'}|_{a}} \) we see that

\[
\left\| \Delta(\overline{D}_{\alpha'} Z_{t}) \right\|_{2}^{2} \leq \left\| \frac{1}{|Z_{,\alpha'}|_{a}} \Delta(\overline{D}_{\alpha'} Z_{t}) \right\|_{2} \left\| \partial_{\alpha'} \left( \frac{1}{|Z_{,\alpha'}|_{a}} \Delta(\overline{D}_{\alpha'} Z_{t}) \right) \right\|_{2}
\]

From this we obtain \( \Delta(\overline{D}_{\alpha'} Z_{t}) \in W_{\sqrt{\lambda}} \cap C_{\sqrt{\lambda}} \). As \( \omega \in W \), using Lemma 7.6 and Lemma 7.5 we see that \( \Delta(|D_{\alpha'}|^2 Z_{t}) \in W_{\sqrt{\lambda}} \cap C_{\sqrt{\lambda}} \). The other one is proven similarly.
(7) \( \Delta \left\{ \partial_{\alpha'} P_A \left( \frac{Z_t}{Z_{\alpha'}} \right) \right\} \in L^\infty_{\sqrt{\Delta}} \)

Proof: In Sec 5.1 of \[1\] we have shown the formula
\[
2\partial_{\alpha'} P_A \left( \frac{Z_t}{Z_{\alpha'}} \right) = 2D_{\alpha'} Z_t + \left[ \frac{1}{Z_{\alpha'}}, \mathbb{H} \right] Z_t + [Z_t, \mathbb{H}] \partial_{\alpha'} \frac{1}{Z_{\alpha'}}
\]
Now applying \( \Delta \) to the formula above, the estimate follows easily from Proposition 9.5, Proposition 10.4 and Lemma 7.6.

(8) \( \Delta(|D_{\alpha'}| A_1) \in L^2_{\sqrt{\Delta}} \) and hence \( \Delta(A_1) \in W_{\sqrt{\Delta}}, \Delta(\sqrt{A_1}) \in W_{\sqrt{\Delta}} \) and also \( \Delta \left( \frac{1}{\sqrt{A_1}} \right) \in W_{\sqrt{\Delta}} \)

Proof: In Sec 5.1 of \[1\] we established the formula
\[
(1 - \mathbb{H}) D_{\alpha'} A_1 = i(1 - \mathbb{H})((D_{\alpha'} Z_t) Z_{t, \alpha'}) + i \left[ P_A \left( \frac{Z_t}{Z_{\alpha'}} \right), \mathbb{H} \right] \partial_{\alpha'} Z_{t, \alpha'}
\]
Now by applying \( \Delta \) to the formula above and using Proposition 9.5, Proposition 10.4 and Lemma 7.6, we see that \((1 - \mathbb{H}) D_{\alpha'} A_1 \in L^2_{\sqrt{\Delta}} \). Now in Sec 5.1 of \[1\] we also established the formula
\[
|D_{\alpha'}| A_1 = \text{Re}\{\omega(1 - \mathbb{H}) D_{\alpha'} A_1\} - \text{Re}\left\{ \omega \left[ \frac{1}{Z_{\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} A_1 \right\}
\]
Hence by applying \( \Delta \) to the formula above and using Proposition 9.5, Proposition 10.4 and Lemma 7.6, we easily get \( \Delta(|D_{\alpha'}| A_1) \in L^2_{\sqrt{\Delta}} \). Hence using Lemma 7.6, we see that \( \Delta(A_1) \in W_{\sqrt{\Delta}} \). Now as \((A_1)_a \geq 1\) and \((A_1)_a \in W\), we get from Lemma 7.6 that \( \frac{1}{(A_1)_a} \tilde{U}(A_1)_b - 1 \in W_{\sqrt{\Delta}} \). The inequality \(|z^\alpha - 1| \leq C(\alpha)|z - 1| \max(|z|^\alpha, 1)\) for \( z \in \mathbb{C}, \alpha \in \mathbb{R} \) implies that we have \( \left( \frac{1}{(A_1)_a} \tilde{U}(A_1)_b \right)^\alpha - 1 \in W_{\sqrt{\Delta}} \). Choosing suitable values of \( \alpha \) imply all the other estimates.

(9) \( \Delta(\Theta) \in L^2_{\sqrt{\Delta}} \) and \( \Delta(D, \Theta) \in L^2_{\sqrt{\Delta}} \)

Proof: The proof of \( \Delta(\Theta) \in L^2_{\sqrt{\Delta}} \) follows easily by applying \( \Delta \) to the formula (26) and using Proposition 10.4. Also \( \Delta(D, \Theta) \in L^2_{\sqrt{\Delta}} \) as it part of \( E_\Delta \).

(10) \( \frac{1}{|Z_{\alpha'}|_a} \Delta(\Theta) \in C_{\sqrt{\Delta}}, \Delta \left( \frac{\Theta}{|Z_{\alpha'}|} \right) \in C_{\sqrt{\Delta}} \)

Proof: The energy \( E_\Delta \) controls \( \sqrt{(A_1)_a} \Delta(\Theta) \in H^2_{\sqrt{\Delta}} \). Now as \( \sqrt{(A_1)_a} \Delta(\Theta) \in L^2_{\sqrt{\Delta}} \) this implies that \( \frac{\sqrt{(A_1)_a}}{|Z_{\alpha'}|_a} \Delta(\Theta) \in C_{\sqrt{\Delta}} \). Hence by using Lemma 7.6 and observing that \( \frac{1}{\sqrt{(A_1)_a}} \in W \) we obtain \( \frac{1}{|Z_{\alpha'}|_a} \Delta(\Theta) \in C_{\sqrt{\Delta}} \). The other estimate is now obtained by using Lemma 7.6.

From now on we will just state the estimates and the proof follows exactly as in Sec 5.1 of \[1\] and can be easily obtained by applying \( \Delta \) to the formula there and using Lemma 7.6.
Lemma \textbf{[7,6]} Proposition \textbf{[9,5]} and Proposition \textbf{[10,4]} as shown in the above examples. For estimates which do not follow this pattern we give more details.

(11) $\Delta \left( D_{\alpha'} \left| \frac{1}{Z_{\alpha'}} \right| \right) \in C_{\sqrt{\Delta}} \quad \Delta \left( |D_{\alpha'}| \left| \frac{1}{Z_{\alpha'}} \right| \right) \in C_{\sqrt{\Delta}} \quad \Delta \left( \left| D_{\alpha'} \right| \left| \frac{1}{Z_{\alpha'}} \right| \right) \in C_{\sqrt{\Delta}}$ and similarly we have $\Delta \left( \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \omega \right) \in C_{\sqrt{\Delta}}$

(12) $\Delta \left( \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} A_1 \right) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^\frac{1}{2}_{\sqrt{\Delta}}$ and hence $\Delta \left( \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} A_1 \right) \in C_{\sqrt{\Delta}}$

(13) $\Delta \left( \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} A_1 \right) \in L^2_{\sqrt{\Delta}} \quad |D_{\alpha'}| \Delta \left( \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} A_1 \right) \in L^2_{\sqrt{\Delta}}$ and $\Delta \left( \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} A_1 \right) \in W_{\sqrt{\Delta}}$

(14) $\Delta(b_{\alpha'}) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^\frac{1}{2}_{\sqrt{\Delta}}$ and $\Delta(\mathbb{H}(b_{\alpha'})) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^\frac{1}{2}_{\sqrt{\Delta}}$

(15) $\Delta(|D_{\alpha'}|b_{\alpha'}) \in L^2_{\sqrt{\Delta}}$ and hence $\Delta(b_{\alpha'}) \in W_{\sqrt{\Delta}}$

(16) $\Delta \left( \partial_{\alpha'} D_t \left| \frac{1}{Z_{\alpha'}} \right| \right) \in L^2_{\sqrt{\Delta}} \quad \Delta \left( D_t \partial_{\alpha'} \left| \frac{1}{Z_{\alpha'}} \right| \right) \in L^2_{\sqrt{\Delta}}$

(17) $\Delta(\overline{Z_{\alpha'}}) \in L^2_{\sqrt{\Delta}}$

(18) $\Delta(\overline{D_{\alpha'}Z_t}) \in C_{\sqrt{\Delta}} \quad \Delta(|D_{\alpha'}|\overline{Z_t}) \in C_{\sqrt{\Delta}} \quad \Delta(D_t\overline{D_{\alpha'}Z_t}) \in C_{\sqrt{\Delta}} \quad \Delta(D_t|D_{\alpha'}|\overline{Z_t}) \in C_{\sqrt{\Delta}}$

(19) $\Delta(D_tA_1) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^\frac{1}{2}_{\sqrt{\Delta}}$

(20) $\Delta(D_t(b_{\alpha'} - D_{\alpha'}Z_t - D_{\alpha'}Z_t)) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^\frac{1}{2}_{\sqrt{\Delta}}$ and hence $\Delta(D_t b_{\alpha'}) \in \dot{H}^\frac{1}{2}_{\sqrt{\Delta}} \Delta(\partial_{\alpha'} D_t b) \in \dot{H}^\frac{1}{2}_{\sqrt{\Delta}}$

Now we start controlling terms with surface tension. Note that these estimates are only for the solution $A$ and hence the estimates have already been shown in Sec 5.1 in $\mathbb{H}$ and no new work has to be done at all. For most of the estimates we will have that the power of $\sigma$ will be the same as that of the power of $\Delta$. For e.g. we have $\left( \sigma^\frac{1}{2} |Z_{\alpha'}| \frac{1}{Z_{\alpha'}} \right) \in L^2_\Delta$ and both $\sigma$ and $\Delta$ are raised to the same power $1/6$. However the estimates derived from $\sigma \partial_{\alpha'} \Theta, \sigma \partial_{\alpha'} D_{\alpha'} \Theta$ and $\sigma D_{\alpha'}^2 \Theta$ will not follow this pattern. For e.g. we will have $\left( \sigma^\frac{1}{2} \Theta \right) \in L^\infty_\Delta$ and not $\left( \sigma^\frac{1}{4} \Theta \right) \in L^\infty_\Delta$. The reason is that $E_\Delta$ controls $\Delta((\overline{Z_{\alpha'} - i})Z_{\alpha'}) \in \dot{H}^\frac{1}{2}_{\sqrt{\Delta}}$ not $\dot{H}^\frac{1}{2}_{\Delta}$ and hence we have $\left( \sigma \partial_{\alpha'} \Theta \right) \in \dot{H}^\frac{1}{2}_{\sqrt{\Delta}}$. Similarly for $\sigma \partial_{\alpha'} D_{\alpha'} \Theta$ and $\sigma D_{\alpha'}^2 \Theta$.

(21) $\left( \frac{1}{Z_{\alpha'}} \right) \in L^\infty_{\sqrt{\Delta}} \quad \left( \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \in L^\infty_{\sqrt{\Delta}} \quad \left( \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \omega \right) \in L^\infty_{\sqrt{\Delta}}$ and $\left( \sigma^\frac{1}{2} |Z_{\alpha'}| \frac{1}{Z_{\alpha'}} \right) \in L^\infty_{\sqrt{\Delta}}$

(22) $\left( \frac{1}{Z_{\alpha'}} \right) \in L^2_{\Delta^\frac{1}{2}} \quad \left( \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \in L^2_{\Delta^\frac{1}{2}} \quad \left( \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \omega \right) \in L^2_{\Delta^\frac{1}{2}}$

(23) $\left( \partial_{\alpha'} \Theta \right) \in \dot{H}^\frac{1}{2}_{\sqrt{\Delta}}$
Proof: Note carefully that we claim the estimate $(\sigma \partial_{\alpha'} \Theta)_a \in \dot{H}^{\frac{1}{2}}_{\Delta}$ and not $(\sigma \partial_{\alpha'} \Theta)_a \in \dot{H}^{\frac{1}{2}}_{\Delta}$. From the fundamental equation (10) we have

$$\Delta((Z_{tt} - i)Z_{\alpha'}) = -i \Delta(A_1) + (\sigma \partial_{\alpha'} \Theta)_a$$

and we know that $E_\Delta$ controls $\Delta((Z_{tt} - i)Z_{\alpha'}) \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$. As $\Delta(A_1) \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$, we obtain the above estimate.

(24) \((\sigma^{\frac{1}{2}} \partial_{\alpha} \Theta)_a \in \dot{L}^{2}_{\Delta}$$

(25) \((\sigma^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}})_a \in \dot{L}^{2}_{\Delta}$$ (\sigma^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|})_a \in \dot{L}^{2}_{\Delta}, \left(\sigma^{\frac{1}{2}} \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \omega\right)_a \in \dot{L}^{2}_{\Delta}$$ and similarly we have \((\sigma^{\frac{1}{2}} \partial_{\alpha'} |D_{\alpha'}| |\omega|)_a \in \dot{L}^{2}_{\Delta}$$

Proof: Note carefully that we have \(\dot{L}^{2}_{\Delta}$$ and not \(\dot{L}^{2}_{\Delta}$$ in the above estimate. First see that \((\sigma^{\frac{1}{2}} |Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}})_a \in \dot{L}^{2}$$ as \((E_\alpha)_a(t) \leq C(L_1)$$ and \((E_\alpha)_a$$ controls it. But we have already shown above that \((\sigma^{\frac{1}{2}} |Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}})_a \in \dot{L}^{2}_{\Delta}$$ Hence we have \((\sigma^{\frac{1}{2}} |Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}})_a \in \dot{L}^{2}_{\Delta}$$ for all \(0 \leq \beta \leq 1/2$$ In a similar way we can show that \((\sigma^{\frac{1}{2}} |Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}})_a \in \dot{L}^{2}_{\Delta}$$ etc. for all \(0 \leq \beta \leq 1/2$$. Due to this argument, the proof of the above estimates follow in the same way as is shown in Sect 5.1 in [1].

(26) \((\sigma^{\frac{1}{2}} \Theta)_a \in \dot{L}^{\infty}_{\Delta}$$ and \(\dot{H}^{\frac{1}{2}}_{\Delta}$$

(27) \((\sigma^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}})_a \in \dot{L}^{\infty}_{\Delta}$$ and \(\dot{H}^{\frac{1}{2}}_{\Delta}$$, \left(\sigma^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|}\right)_a \in \dot{L}^{\infty}_{\Delta}$$ and \(\dot{H}^{\frac{1}{2}}_{\Delta}$$, \left(\sigma^{\frac{1}{2}} \partial_{\alpha'} \omega\right)_a \in \dot{L}^{\infty}_{\Delta}$$ and \(\dot{H}^{\frac{1}{2}}_{\Delta}$$

(28) \((\partial_{\alpha'} D_{\alpha'} \Theta)_a \in \dot{L}^{2}_{\sqrt{\Delta}}$$ and \(\dot{H}^{\frac{1}{2}}_{\Delta}$$, \left(\partial_{\alpha'} D_{\alpha'} \partial_{\alpha'} \Theta\right)_a \in \dot{L}^{2}_{\sqrt{\Delta}}$$ etc. for all \(0 \leq \beta \leq 1/2$$

(29) \left(\sigma \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right)_a \in \dot{L}^{2}_{\sqrt{\Delta}}$$ \left(\sigma \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|}\right)_a \in \dot{L}^{2}_{\sqrt{\Delta}}$$ \left(\sigma \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \omega\right)_a \in \dot{L}^{2}_{\sqrt{\Delta}}$$ and also \(\sigma \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \Theta\)_a \in \dot{L}^{2}_{\sqrt{\Delta}}$$

(30) \left(\sigma \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right)_a \in \dot{L}^{2}_{\sqrt{\Delta}}$$ \left(\sigma \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|}\right)_a \in \dot{L}^{2}_{\sqrt{\Delta}}$$ \left(\sigma \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \omega\right)_a \in \dot{L}^{2}_{\sqrt{\Delta}}$$ and also \(\sigma \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \Theta\)_a \in \dot{L}^{2}_{\sqrt{\Delta}}$$

(31) \left(\sigma \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right)_a \in \dot{W}_{\sqrt{\Delta}}$$ \left(\sigma \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|}\right)_a \in \dot{W}_{\sqrt{\Delta}}$$ \left(\sigma \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \omega\right)_a \in \dot{W}_{\sqrt{\Delta}}$$ and \(\sigma \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \Theta\)_a \in \dot{W}_{\sqrt{\Delta}}$$

(32) \left(\sigma \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \Theta\right)_a \in \dot{L}^{\infty}_{\Delta}$$ and \(\dot{H}^{\frac{1}{2}}_{\Delta}$$
\( (33) \left( \frac{\sigma^\frac{1}{2}}{|Z_{a'}|^{\frac{1}{2}}} \partial_{a'}^2 \frac{1}{|Z_{a'}|} \right)_a \in L^\infty_{\Delta^\frac{1}{2}} \cap \dot{H}^\frac{1}{2}_{\Delta^\frac{1}{2}}, \quad \left( \frac{\sigma^\frac{1}{2}}{|Z_{a'}|^{\frac{1}{2}}} \partial_{a'} \frac{1}{|Z_{a'}|} \right)_a \in L^\infty_{\Delta^\frac{1}{2}} \cap \dot{H}^\frac{1}{2}_{\Delta^\frac{1}{2}} \), and similarly
\( \left( \frac{\sigma^\frac{1}{2}}{|Z_{a'}|^{\frac{1}{2}}} \partial_{a'} \omega \right)_a \in L^\infty_{\Delta^\frac{1}{2}} \cap \dot{H}^\frac{1}{2}_{\Delta^\frac{1}{2}} \)

\( (34) \left( \frac{\sigma^\frac{1}{2}}{|Z_{a'}|^{\frac{1}{2}}} \partial_{a'} \Theta \right)_a \in C_{\sqrt{\Delta}} \)

\( (35) \left( \frac{\sigma^\frac{1}{2}}{|Z_{a'}|^{\frac{1}{2}}} \partial_{a'}^2 \frac{1}{|Z_{a'}|} \right)_a \in C_{\sqrt{\Delta}}, \quad \left( \frac{\sigma^\frac{1}{2}}{|Z_{a'}|^{\frac{1}{2}}} \partial_{a'}^3 \frac{1}{|Z_{a'}|} \right)_a \in C_{\sqrt{\Delta}}, \quad \left( \frac{\sigma^\frac{1}{2}}{|Z_{a'}|^{\frac{1}{2}}} \partial_{a'} \omega \right)_a \in C_{\sqrt{\Delta}} \)

\( (36) \left( \sigma \partial_{a'} \Theta \right)_a \in C_{\sqrt{\Delta}}, \quad \left( \sigma \partial_{a'}^2 \Theta \right)_a \in C_{\sqrt{\Delta}}, \quad \left( \sigma \partial_{a'}^3 \Theta \right)_a \in C_{\sqrt{\Delta}} \)

\( (37) \left( \frac{\sigma^\frac{1}{2}}{|Z_{a'}|^3} \partial_{a'}^1 \frac{1}{|Z_{a'}|} \right)_a \in C_{\sqrt{\Delta}}, \quad \left( \frac{\sigma^\frac{1}{2}}{|Z_{a'}|^2} \partial_{a'} \frac{1}{|Z_{a'}|} \right)_a \in C_{\sqrt{\Delta}}, \quad \left( \sigma \partial_{a'}^2 \frac{1}{|Z_{a'}|} \right)_a \in C_{\sqrt{\Delta}} \), and similarly
\( \left( \frac{\sigma^\frac{1}{2}}{|Z_{a'}|^3} \partial_{a'} \omega \right)_a \in C_{\sqrt{\Delta}} \)

\( (38) \left( \frac{\sigma^\frac{1}{2}}{|Z_{a'}|^{\frac{1}{2}}} \partial_{a'} \partial_{t,a'} Z_{a'} \right)_a \in L^2_{\sqrt{\Delta}}, \quad \left( \sigma \frac{1}{2} |Z_{a'}|^{\frac{1}{2}} \partial_{a'} D_{a'} |Z_{a'}| \right)_a \in L^2_{\sqrt{\Delta}} \) and \( \left( \sigma \frac{1}{2} |Z_{a'}|^{\frac{1}{2}} \partial_{a'} D_{a'} \partial_{t} Z_{a'} \right)_a \in L^2_{\sqrt{\Delta}} \)

\( (39) \left( \frac{\sigma^\frac{1}{2}}{|Z_{a'}|^{\frac{1}{2}}} \partial_{a'}^2 \partial_{t} Z_{a'} \right)_a \in L^2_{\sqrt{\Delta}}, \quad \left( \frac{\sigma^\frac{1}{2}}{|Z_{a'}|^2} \partial_{a'} D_{a'} Z_{a'} \right)_a \in L^2_{\sqrt{\Delta}}, \quad \left( \sigma \frac{1}{2} |Z_{a'}|^{\frac{1}{2}} \partial_{a'} D_{a'} |Z_{a'}| \right)_a \in L^2_{\sqrt{\Delta}} \)

\( \left( \sigma \frac{1}{2} |Z_{a'}|^{\frac{1}{2}} \partial_{a'} D_{a'} \partial_{t} Z_{a'} \right)_a \in L^2_{\sqrt{\Delta}} \)

\( (40) \left( \frac{\sigma^\frac{1}{2}}{|Z_{a'}|^{\frac{1}{2}}} \partial_{a'} \partial_{t} Z_{a'} \right)_a \in W_{\sqrt{\Delta}} \cap C_{\sqrt{\Delta}}, \quad \left( \frac{\sigma^\frac{1}{2}}{|Z_{a'}|^2} \partial_{a'} |D_{a'}| Z_{a'} \right)_a \in W_{\sqrt{\Delta}} \cap C_{\sqrt{\Delta}} \), and similarly we have
\( \left( \frac{\sigma^\frac{1}{2}}{|Z_{a'}|^{\frac{1}{2}}} \partial_{a'} D_{a'} Z_{a'} \right)_a \in W_{\sqrt{\Delta}} \cap C_{\sqrt{\Delta}}, \quad \left( \frac{\sigma^\frac{1}{2}}{|Z_{a'}|^2} \partial_{a'} D_{a'} \partial_{t} Z_{a'} \right)_a \in W_{\sqrt{\Delta}} \cap C_{\sqrt{\Delta}} \)

\( (41) \left( \frac{\sigma^\frac{1}{2}}{|Z_{a'}|^{\frac{1}{2}}} \partial_{a'} D_{a'} \partial_{t} Z_{a'} \right)_a \in L^2_{\Delta^{\frac{1}{2}}}, \quad \left( \frac{\sigma^\frac{1}{2}}{|Z_{a'}|^2} \partial_{a'} |D_{a'}| \partial_{t} Z_{a'} \right)_a \in L^2_{\Delta^{\frac{1}{2}}}, \quad \left( \sigma \frac{1}{2} |Z_{a'}|^{\frac{1}{2}} \partial_{a'} D_{a'} Z_{a'} \right)_a \in L^2_{\Delta^{\frac{1}{2}}} \)

\( (42) \left( \frac{\sigma^\frac{1}{2}}{|Z_{a'}|^{\frac{1}{2}}} \partial_{a'} \partial_{t} Z_{a'} \right)_a \in L^2_{\Delta^{\frac{1}{2}}}, \quad \left( \sigma \frac{1}{2} |Z_{a'}|^{\frac{1}{2}} \partial_{a'} D_{a'} \partial_{t} Z_{a'} \right)_a \in L^2_{\Delta^{\frac{1}{2}}} \)

\( (43) \left( \frac{\sigma^\frac{1}{2}}{|Z_{a'}|^{\frac{1}{2}}} \partial_{a'} \partial_{t} Z_{a'} \right)_a \in W_{\Delta^{\frac{1}{2}}} \)

\( (44) \left\{ \frac{\sigma^\frac{1}{2}}{|Z_{a'}|^{\frac{1}{2}}} \partial_{a'} \frac{1}{Z_{a'}^{\frac{1}{2}}} \right\} \in L^\infty_{\Delta^{\frac{1}{2}}} \)

\( (45) \left( \frac{\sigma^\frac{1}{2}}{|Z_{a'}|^{\frac{1}{2}}} \partial_{a'} Z_{a'} \right)_a \in L^\infty_{\Delta^{\frac{1}{2}}} \cap \dot{H}^\frac{1}{2}_{\Delta^{\frac{1}{2}}}. \)
Now we apply $\Delta$ to the above equation and handle each term individually.

**Proof:** Observe that the terms are of the form $\Delta((\mathbb{I} - \mathbb{H})D_t^2\mathbf{f})$ with $\mathbf{f}$ satisfying $P_A\mathbf{f} = 0$. Hence we can use Proposition 10.1 to get

$$(\mathbb{I} - \mathbb{H})D_t^2\mathbf{f} = 2\left[P_A\left(\frac{Z_t}{Z_{t,\alpha'}}\right), \mathbb{H}\right] \partial_{\alpha'}(D_t\mathbf{f}) - \left[P_A\left(\frac{Z_t}{Z_{t,\alpha'}}\right), P_A\left(\frac{Z_t}{Z_{t,\alpha'}}\right)\right] \partial_{\alpha'}\mathbf{f}$$

$$+ \frac{1}{4}((\mathbb{I} - \mathbb{H})\left\{\left[\frac{1}{Z_{t,\alpha'}}, \mathbb{H}\right] Z_{t,\alpha'} \right\} [Z_t, \mathbb{H}] D_t\mathbf{f} - \frac{1}{4}((\mathbb{I} - \mathbb{H})\left\{[Z_t, \mathbb{H}] \partial_{\alpha'} \frac{1}{Z_{t,\alpha'}}\right\}^2$$

$$+ \frac{1}{2}\left[\left[Z_t, Z_{t,\alpha'}\mathbb{H}\right]\partial_{\alpha'} \frac{1}{Z_{t,\alpha'}} \mathbb{H}\right] \partial_{\alpha'}(\mathbf{f} Z_{t,\alpha'}) + [Z_{tt}, \mathbb{H}] D_t\mathbf{f}$$

Now we apply $\Delta$ to the above equation and handle each term individually.

(a) Using Proposition 9.9 and Proposition 10.4 the term $\Delta\left\{P_A\left(\frac{Z_t}{Z_{t,\alpha'}}\right), \mathbb{H}\right\] \partial_{\alpha'}(D_t\mathbf{f})$ is easily shown to be in $L^2_{\Delta^\frac{1}{2}}$ for $\mathbf{f} = \Theta, Z_{t,\alpha'}$ and in $\dot{H}^\frac{1}{2}_{\Delta^\frac{1}{2}}$ for $\mathbf{f} = D_{\alpha'} Z_t$.

(b) Using Proposition 9.7 and Proposition 10.4 we see that $\Delta\left[P_A\left(\frac{Z_t}{Z_{t,\alpha'}}\right), P_A\left(\frac{Z_t}{Z_{t,\alpha'}}\right)\right] \partial_{\alpha'}\mathbf{f}$ is in $L^2_{\Delta^\frac{1}{2}}$ for $\mathbf{f} = \Theta, Z_{t,\alpha'}$ and in $\dot{H}^\frac{1}{2}_{\Delta^\frac{1}{2}}$ for $\mathbf{f} = D_{\alpha'} Z_t$.

(c) Using Proposition 9.9 and Proposition 10.4 we see that $\Delta\left[\frac{1}{Z_{t,\alpha'}}, \mathbb{H}\right] Z_{t,\alpha'} \in L^2_{\Delta^\frac{1}{2}} \cap \dot{H}^\frac{1}{2}_{\Delta^\frac{1}{2}}$

(d) For $f = D_{\alpha'} Z_t$ we see by using Proposition 9.9 and Proposition 10.4 that $\Delta([Z_t, \mathbb{H}] D_t\mathbf{f}) \in L^2_{\Delta^\frac{1}{2}} \cap \dot{H}^\frac{1}{2}_{\Delta^\frac{1}{2}}$ for $\mathbf{f} = \Theta, Z_{t,\alpha'}$. Hence combining these estimates with the previous estimate we see

$$[Z_t, \mathbb{H}] D_t\mathbf{f} = \left[P_A\left(\frac{Z_t}{Z_{t,\alpha'}}\right), \mathbb{H}\right] \partial_{\alpha'}\mathbf{f}$$

Hence by using Proposition 9.9 and Proposition 10.4 we see that $\Delta([Z_t, \mathbb{H}] D_t\mathbf{f}) \in L^2_{\Delta^\frac{1}{2}}$ for $\mathbf{f} = \Theta, Z_{t,\alpha'}$. Hence combining these estimates with the previous estimate we see
that \( \Delta(1 - \mathbb{H})\left\{ \left[ \frac{1}{Z_{t,\alpha'}}, \mathbb{H} \right] Z_{t,\alpha'} \right\} \) is in \( L^2_{\sqrt{\mathbb{A}}} \) for \( f = \Theta, Z_{t,\alpha'} \) and in \( \dot{H}^\frac{3}{2}_{\sqrt{\mathbb{A}}} \) for \( f = D_{\alpha'} Z_t \).

(e) Using Proposition 9.5 and Proposition 10.4 we see that \( \Delta \left( \left[ Z_t, \mathbb{H} \right] \partial_{\alpha'} \frac{1}{Z_{t,\alpha'}} \right) = \Theta_{t,\alpha'} \in L^2_{\sqrt{\mathbb{A}}} \cap \dot{H}^\frac{3}{2}_{\sqrt{\mathbb{A}}} \)

Hence using this we see that \( \Delta(1 - \mathbb{H})\left\{ \left[ Z_t, \mathbb{H} \right] \partial_{\alpha'} \frac{1}{Z_{t,\alpha'}} \right\} \) is in \( L^2_{\sqrt{\mathbb{A}}} \) for \( f = \Theta, Z_{t,\alpha'} \) and in \( \dot{H}^\frac{3}{2}_{\sqrt{\mathbb{A}}} \) for \( f = D_{\alpha'} Z_t \).

(f) Using Proposition 9.5 and Proposition 10.4 we see that \( \partial_{\alpha'} \Delta \left( \left[ Z_t, \mathbb{H} \right] \partial_{\alpha'} \frac{1}{Z_{t,\alpha'}} \right) = \Theta_{t,\alpha'} \in L^2_{\sqrt{\mathbb{A}}} \)

(g) For \( f = D_{\alpha'} Z_t \) we observe that \( \partial_{\alpha'} \Delta \left( \frac{f}{Z_{t,\alpha'}} \right) = \Theta_{t,\alpha'} \in L^2_{\sqrt{\mathbb{A}}} \) for \( f = \Theta, Z_{t,\alpha'} \) we see that \( \Delta \left( \frac{f}{Z_{t,\alpha'}} \right) = \Theta_{t,\alpha'} \in C_{\sqrt{\mathbb{A}}} \), Hence combining this with the previous estimate and using Proposition 9.5 and Proposition 10.4 we see that \( \left[ Z_t, \mathbb{H} \right] \partial_{\alpha'} \frac{1}{Z_{t,\alpha'}} \in L^2_{\sqrt{\mathbb{A}}} \) for \( f = \Theta, Z_{t,\alpha'} \) and in \( \dot{H}^\frac{3}{2}_{\sqrt{\mathbb{A}}} \) for \( f = D_{\alpha'} Z_t \).

(h) For \( f = D_{\alpha'} Z_t \) we observe that by using Proposition 9.5 and Proposition 10.4 we obtain \( \Delta(1 - \mathbb{H})D_{\alpha'} f = \Theta_{t,\alpha'} \in \dot{H}^\frac{3}{2}_{\sqrt{\mathbb{A}}} \) for \( f = \Theta, Z_{t,\alpha'} \) we see that

\[
[Z_{tt}, \mathbb{H}] D_{\alpha'} f = -[Z_{tt}, \mathbb{H}] \left\{ \left( \partial_{\alpha'} \frac{1}{Z_{t,\alpha'}} \right) f \right\} + [Z_{tt}, \mathbb{H}] \partial_{\alpha'} \left( \frac{f}{Z_{t,\alpha'}} \right)
\]

Now observe that for \( f = \Theta, Z_{t,\alpha'} \) we have \( \Delta \left\{ \left( \partial_{\alpha'} \frac{1}{Z_{t,\alpha'}} \right) f \right\} = \Theta_{t,\alpha'} \in L^1_{\sqrt{\mathbb{A}}} \) and \( \Delta \left( \frac{f}{Z_{t,\alpha'}} \right) = \Theta_{t,\alpha'} \in C_{\sqrt{\mathbb{A}}} \), Hence using Proposition 9.5 and Proposition 10.4 we see that \( \Delta(1 - \mathbb{H})D_{\alpha'} f = \Theta_{t,\alpha'} \in L^2_{\sqrt{\mathbb{A}}} \) for \( f = \Theta, Z_{t,\alpha'} \). Hence we have the required estimate.

(53) \( \left( \sigma(1 - \mathbb{H}) \right) D_{\alpha'}^3 \Theta \right\}_a \in L^2_{\sqrt{\mathbb{A}}} \), \( \left( \sigma(1 - \mathbb{H}) \right) D_{\alpha'}^3 Z_{t,\alpha'} \right\}_a \in L^2_{\sqrt{\mathbb{A}}} \left( \sigma(1 - \mathbb{H}) \right) D_{\alpha'}^3 Z_t \right\}_a \in \dot{H}^\frac{3}{2}_{\sqrt{\mathbb{A}}}
\]

(54) \( \Delta \left\{ \left[ D_{t,\alpha'}, \frac{1}{Z_{t,\alpha'}} \right] \right\} \in C_{\sqrt{\mathbb{A}}} \), \( \Delta \left\{ \left[ D_{t,\alpha'}, \frac{1}{Z_{t,\alpha'}} \right] \right\} \in C_{\sqrt{\mathbb{A}}} \)

(55) \( \Delta \left\{ \left[ i \frac{A_1}{Z_{t,\alpha'}} \partial_{\alpha'}, \frac{1}{Z_{t,\alpha'}} \right] \right\} \in C_{\sqrt{\mathbb{A}}} \), \( \Delta \left\{ \left[ i \frac{A_1}{Z_{t,\alpha'}} \partial_{\alpha'}, \frac{1}{Z_{t,\alpha'}} \right] \right\} \in C_{\sqrt{\mathbb{A}}} \)

(56) \( \left\{ \left( 1 - \mathbb{H} \right) \left| i \sigma D_{\alpha'} \right|^3, \frac{1}{Z_{t,\alpha'}} \right\}_a \in \dot{H}^\frac{3}{2}_{\sqrt{\mathbb{A}}} \), \( \left\{ \left( 1 - \mathbb{H} \right) \left| i \sigma D_{\alpha'} \right|^3, \frac{1}{Z_{t,\alpha'}} \right\}_a \in \dot{H}^\frac{3}{2}_{\sqrt{\mathbb{A}}} \) and we also have \( \left\{ \left| Z_{t,\alpha'} \right| \right\}_a \in L^2_{\sqrt{\mathbb{A}}} \), \( \left\{ \left| Z_{t,\alpha'} \right| \right\}_a \in L^2_{\sqrt{\mathbb{A}}} \)

(57) \( \Delta(\mathbb{R}_t) \in C_{\sqrt{\mathbb{A}}} \)

(58) \( \Delta(\mathbb{J}_t) \in L^2_{\sqrt{\mathbb{A}}} \cap \dot{H}^\frac{3}{2}_{\sqrt{\mathbb{A}}} \)

(59) \( \Delta(D_{\alpha'} J_1) \in L^2_{\sqrt{\mathbb{A}}} \) and hence \( \Delta(J_1) \in W_{\sqrt{\mathbb{A}}} \)
(60) \( \Delta(R_2) \in L^2_{\sqrt{\Delta}} \)
(61) \( \Delta(J_2) \in L^2_{\sqrt{\Delta}} \)

(62) \( \left( \sigma \frac{1}{|Z_{\alpha'}|} |D_{\alpha'}|^3 \mathcal{Z}_{t,\alpha'} \right)_a \in \dot{H}^\frac{3}{2}_{\sqrt{\Delta}} \), \( \left( \sigma \frac{1}{|Z_{\alpha'}|} |D_{\alpha'}|^3 \mathcal{Z}_{t,\alpha'} \right)_a \in \dot{H}^\frac{3}{2}_{\sqrt{\Delta}} \)

(63) \( \Delta \{ (I - H) D^2_{I} \mathcal{D}_{\alpha'} \mathcal{Z}_I \} \in \dot{H}^\frac{3}{2}_{\sqrt{\Delta}} \)

(64) \( \left( \sigma (I - H) |D_{\alpha'}|^3 \mathcal{D}_{\alpha} \mathcal{Z}_I \right)_a \in \dot{H}^\frac{3}{2}_{\sqrt{\Delta}} \)

(65) \( \Delta \left( \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} J_1 \right) \in \dot{H}^\frac{3}{2}_{\sqrt{\Delta}} \) and hence \( \Delta \left( \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} J_1 \right) \in \mathcal{C}_{\sqrt{\Delta}} \)

7.2. Closing the energy estimate for \( E_{\Delta} \)

We now complete the proof of Theorem 7.1. To simplify the calculations, we will continue to use the notation used in 6.1 and introduce another notation: If \( a(t), b(t) \) are functions of time we write \( a \approx b \) if there exists a constant \( C(L_1) \) depending only on \( L_1 \) (where \( L_1 \) was defined in Theorem 7.1) with \( |a(t) - b(t)| \leq C(L_1)E_\Delta(t) \). Observe that \( \approx \) is an equivalence relation. With this notation, proving Theorem 7.1 is equivalent to showing \( \frac{dE_\Delta(t)}{dt} \approx 0 \).

First observe that by replacing \( \lambda \) by \( \sigma \) we get from Theorem 6.1

\[
\frac{d}{dt}(\sigma(E_{aux})_b(t)) \leq P((E_{high})_b(t))(\sigma(E_{aux})_b(t))
\]

Now from the assumptions of Theorem 6.1 we have that \( \sup_{t \in [0,T]}(E_{high})_b(t) \leq L_1 \) and so we get

\[
\frac{d}{dt}(\sigma(E_{aux})_b(t)) \leq C(L_1)E_\Delta(t)
\]

So we now need to control the other components of \( E_\Delta \).

7.2.1. Controlling \( E_{\Delta,0} \)

We recall that

\[
E_{\Delta,0} = \left\| \left( \sigma \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right)_a \right\|^2_\infty + \left\| \left( \sigma \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right)_a \right\|^6_2 + \left\| \left( \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right)_a \right\|^2_2 + \left\| \Delta(\partial_{\alpha'} \frac{1}{|Z_{\alpha'}|}) \right\|^2_2 + \left\| (\tilde{h}_{\alpha'} - 1) \right\|^2_2 L_{\infty \cap \dot{H}^\frac{3}{2}} + \left\| (D_{\alpha'} |_a (\tilde{h}_{\alpha'} - 1)) \right\|^2_2 \\
+ \left\| Z_{\alpha'} |_a \tilde{U} \left( \frac{1}{|Z_{\alpha'}|} \right) - 1 \right\|^2_\infty
\]

The control of the time derivatives of the first three quantities is the same as the one done in Sec 5.2.1 in \( H \) while controlling the time derivative of \( E_{\sigma,0} \) there. Now we control the other quantities.

(1) We observe from 17.2 that

\[
(D_t)_a \Delta(\omega) = \Delta(D_t \omega) = -\Delta(i\omega \text{Im}(D_{\alpha'} \mathcal{Z}_I)) \in L^\infty_{\sqrt{\Delta}}
\]
Now using the computation from Sec 5.2.1 in [1] we obtain
\[
\frac{d}{dt} \| \Delta(\omega) \|_\infty^2 \leq \| \Delta \omega \|_\infty \| (D_t) \Delta(\omega) \|_\infty \leq C(L_1) E_\Delta
\]

(2) By using Lemma 4.4 we obtain
\[
\frac{d}{dt} \left\| \Delta \left( \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\|_2 \leq \| (\omega_{\alpha'})_a \|_\infty \| \Delta \left( \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \|_2 + \left\| \Delta \left( \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\|_2 \| (D_t) \Delta \left( \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \|_2 \leq C(L_1) E_\Delta
\]

(3) By the calculation of Lemma 4.4 we have
\[
(D_t)_a \tilde{h}_{\alpha'} = \tilde{h}_{\alpha'}(\tilde{U}(b_{\alpha'})_b - (b_{\alpha'})_a) = -\tilde{h}_{\alpha'} \Delta(b_{\alpha'})
\]

As \( \Delta(b_{\alpha'}) \in L^\infty_{\sqrt{\Delta}} \cap H^1_{\sqrt{\Delta}} \) we have that \((D_t)_a \tilde{h}_{\alpha'} \in L^\infty_{\sqrt{\Delta}} \cap H^1_{\sqrt{\Delta}} \). Hence by Lemma 4.4 and by using the computation from Sec 5.2.1 in [1] we have
\[
\frac{d}{dt} \left\| \tilde{h}_{\alpha'} - 1 \right\|_{L^\infty_{\sqrt{\Delta}} H^1_{\sqrt{\Delta}}} \leq C(L_1) \left\| \tilde{h}_{\alpha'} - 1 \right\|_{L^\infty_{\sqrt{\Delta}} H^1_{\sqrt{\Delta}}} \| (D_t)_a \tilde{h}_{\alpha'} \|_{L^\infty_{\sqrt{\Delta}} H^1_{\sqrt{\Delta}}} \leq C(L_1) E_\Delta
\]

(4) By the calculation of Lemma 4.4 we have
\[
(D_t)_a \left| D_{\alpha'} \right|_a \tilde{h}_{\alpha'} = -\text{Re}(D_{\alpha'} Z_t)_a D_{\alpha'} \tilde{h}_{\alpha'} + \left| D_{\alpha'} \right|_a \tilde{h}_{\alpha'} (\tilde{U}(b_{\alpha'})_b - (b_{\alpha'})_a) + \tilde{h}_{\alpha'} \left\{ \frac{1}{Z_{\alpha'}} \tilde{U}(|Z_{\alpha'}|)_b \right\} \tilde{U}(|D_{\alpha'}|_a)_b - (|D_{\alpha'}|_a)_b
\]

Now as \( \left| D_{\alpha'} \right|_a \tilde{h}_{\alpha'} \in L^2_{\sqrt{\Delta}} \tilde{h}_{\alpha'} - 1 \in L^\infty_{\sqrt{\Delta}} \left| Z_{\alpha'} \right|_a \tilde{U}(|Z_{\alpha'}|)_b - 1 \in L^\infty_{\sqrt{\Delta}} \) and \( \Delta(|D_{\alpha'}|_a)_b \in L^2_{\sqrt{\Delta}} \) we see that \((D_t)_a \left| D_{\alpha'} \right|_a \tilde{h}_{\alpha'} \in L^2_{\sqrt{\Delta}} \). Hence by Lemma 4.4
\[
\frac{d}{dt} \left\| \left| D_{\alpha'} \right|_a \tilde{h}_{\alpha'} - 1 \right\|_2 \leq C(L_1) \left\| \left| D_{\alpha'} \right|_a \tilde{h}_{\alpha'} - 1 \right\|_2 \left\| (D_t)_a \left| D_{\alpha'} \right|_a \tilde{h}_{\alpha'} \right\|_2 \leq C(L_1) E_\Delta
\]

(5) By the calculation of Lemma 4.4 we have
\[
(D_t)_a \left\{ \left| Z_{\alpha'} \right|_a \tilde{U} \left( \frac{1}{|Z_{\alpha'}|} \right)_b \right\} = \left\{ \left| Z_{\alpha'} \right|_a \tilde{U} \left( \frac{1}{|Z_{\alpha'}|} \right)_b \right\} \text{Re} \left\{ (D_{\alpha'} Z_t)_a - \tilde{U}(D_{\alpha'} Z_t)_b - (b_{\alpha'})_a + \tilde{U}(b_{\alpha'})_b \right\}
\]

Now as \( \Delta(D_{\alpha'} Z_t) \in L^\infty_{\sqrt{\Delta}} \) and \( \Delta(b_{\alpha'}) \in L^\infty_{\sqrt{\Delta}} \) we see that \( (D_t)_a \left\{ \left| Z_{\alpha'} \right|_a \tilde{U} \left( \frac{1}{|Z_{\alpha'}|} \right)_b \right\} \in L^\infty_{\sqrt{\Delta}} \)

Hence by using the computation from Sec 5.2.1 in [1] we have
\[
\frac{d}{dt} \left\| \left| Z_{\alpha'} \right|_a \tilde{U} \left( \frac{1}{|Z_{\alpha'}|} \right)_b - 1 \right\|_\infty \leq C(L_1) \left\| \left| Z_{\alpha'} \right|_a \tilde{U} \left( \frac{1}{|Z_{\alpha'}|} \right)_b - 1 \right\|_\infty \left\| (D_t)_a \left\{ \left| Z_{\alpha'} \right|_a \tilde{U} \left( \frac{1}{|Z_{\alpha'}|} \right)_b \right\} \right\|_\infty \leq C(L_1) E_\Delta
\]
7.2.2. Controlling $E_{\Delta,1}$

Recall that

$$E_{\Delta,1} = \left\| \Delta \{(\overline{Z}_{tt} - i)Z_{,\alpha'}\} \right\|_{\mathcal{H}^+}^2 + \left\| (\sqrt{A})_{\alpha} \Delta (\overline{Z}_{t,\alpha'}) \right\|_{L^2}^2 + \left\| \left( \frac{\sigma^{1/2}}{|Z_{,\alpha'}|^{1/2}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right) \right\|_{L^2}^2$$

We will first simplify the time derivative of each of the individual terms before combining them.

(1) By using Lemma 4.4 we get

$$\frac{d}{dt} \int \left| \partial_{\alpha'} \right| \Delta \{(\overline{Z}_{tt} - i)Z_{,\alpha'}\} \, d\alpha' \approx 2 \text{Re} \int \{ \alpha_{\alpha'} | \Delta (\overline{Z}_{tt} + i)Z_{,\alpha'}) \} \right| (\overline{Z}_{tt} - i)Z_{,\alpha'}) \, d\alpha'$$

Now \((D_t)_{\alpha} \Delta (\overline{Z}_{tt} - i)Z_{,\alpha'}) = \Delta (D_t(\overline{Z}_{tt} - i)Z_{,\alpha'}) \) and we have

$$D_t(\overline{Z}_{tt} - i)Z_{,\alpha'}) = \overline{Z}_{tt}Z_{,\alpha'} + \beta_{\alpha'}(\overline{Z}_{tt} - i)Z_{,\alpha'}$$

Now by following the argument in Sec 5.2.2 of [1] we see that \(\Delta ((D_{\alpha'} Z_t - b_{\alpha'})(-iA_1 + \sigma \partial_{\alpha'} \Theta)) \in \mathcal{H}^+\). Hence we have

$$\frac{d}{dt} \int \left| \partial_{\alpha'} \right| \Delta \{(\overline{Z}_{tt} - i)Z_{,\alpha'}\} \, d\alpha' \approx 2 \text{Re} \int \{ \partial_{\alpha'} | \Delta (\overline{Z}_{tt} + i)Z_{,\alpha'}) \} \Delta (\overline{Z}_{tt}Z_{,\alpha'}) \, d\alpha'$$

(2) We see from Lemma 4.4 that

$$\frac{d}{dt} \int (A_{1})_{\alpha} | \Delta (Z_{,\alpha'}) \|^2 \, d\alpha'$$

$$= \int (b_{\alpha'} A_t + D_t A_{1})_{\alpha} | \Delta (Z_{,\alpha'}) \|^2 \, d\alpha' + 2 \text{Re} \int (A_{1})_{\alpha} \Delta (Z_{,\alpha'}) \Delta (b_{\alpha'} Z_{t,\alpha'} + Z_{tt,\alpha'}) \, d\alpha'$$

$$\leq C(L_1)E_{\Delta}$$

(3) By following the proof of time derivative of $E_{\sigma,1}$ in Sec 5.2.2 in [1] we get

$$\sigma \frac{d}{dt} \int \left| \left( \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right) \right|_{a}^2 \, d\alpha'$$

$$\approx 2 \text{Re} \int \left\{ -i \sigma \partial_{\alpha'} \left( \frac{1}{Z_{,\alpha'}} \right) D_{\alpha'} \left( \overline{Z}_{t,\alpha'} \right) \right\} \, d\alpha'$$

$$= 2 \text{Re} \int \left\{ -i \sigma \partial_{\alpha'} \left( \frac{1}{Z_{,\alpha'}} \right) D_{\alpha'} \left( \overline{Z}_{t,\alpha'} \right) \right\} \, d\alpha'$$

$$+ 2 \text{Re} \int \left\{ -i \sigma \partial_{\alpha'} \left( \frac{1}{Z_{,\alpha'}} \right) D_{\alpha'} \left( \overline{Z}_{t,\alpha'} \right) \right\} \, d\alpha'$$

We now show that the second term is controlled. Observe that \((Z_{tt} + i)Z_{,\alpha'}) = i(A_{1})_{\alpha} \) and that

$$\left( \frac{\sigma^{1/2}}{|Z_{,\alpha'}|^{1/2}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right) \in L^2_{\mathcal{H}_{\Delta}} \). Hence we only need to show that \( \frac{\sigma^{1/2}}{|Z_{,\alpha'}|^1} \partial_{\alpha'} | \overline{U} (A_{1})_{\alpha} \| \in L^2_{\mathcal{H}_{\Delta}} \).
Now
\[ \frac{\sigma}{|Z_{\alpha'}|^2} \partial_{\alpha'} \bar{U}(A_1)_b = \frac{\sigma}{|Z_{\alpha'}|^2} \left( \frac{\sigma}{|Z_{\alpha'}|^2} \frac{1}{\sqrt{2}} \nabla_2 \bar{U}(A_1)_b \right) \]
The first term is easily shown to be in \( L^2_{\sqrt{\Delta}} \) as from Proposition 9.5 we have
\[ \left\| \frac{\sigma}{|Z_{\alpha'}|^2} \frac{1}{\sqrt{2}} \nabla_2 \bar{U}(A_1)_b \right\|_2 \lesssim \left\| \frac{\sigma}{|Z_{\alpha'}|^2} \partial_{\alpha'} \left( \frac{1}{\sqrt{2}} \nabla_2 \bar{U}(A_1)_b \right) \right\| \]
Hence it is enough to show that \( \frac{\sigma}{|Z_{\alpha'}|^2} \partial_{\alpha'} \bar{U}(A_1)_b \in L^2_{\sqrt{\Delta}} \). We see that
\[ \frac{\sigma}{|Z_{\alpha'}|^2} \partial_{\alpha'} \bar{U}(A_1)_b = \frac{\sigma}{|Z_{\alpha'}|^2} \partial_{\alpha'} \left( \bar{h}_{\alpha'} \bar{U}(\partial_{\alpha'} A_1)_b \right) \]
and hence we have the estimate
\[ \left\| \frac{\sigma}{|Z_{\alpha'}|^2} \partial_{\alpha'} \bar{U}(A_1)_b \right\| \leq C(L_1) \left\| D_{\alpha'} \bar{h}_{\alpha'} \right\|_2 \left\| \frac{\sigma}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right\|_\infty + C(L_1) \left\| \frac{\sigma}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right\|_2 \]
Now \( \frac{\sigma}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \in L^2_{\sqrt{\Delta}} \) and \( \frac{\sigma}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \in L^2_{\sqrt{\Delta}} \) as they are controlled by \( \sigma(E_{aux})_b \).
Hence we have shown
\[ \sigma \frac{d}{dt} \int \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} Z_{t,\alpha'} \right)_b \left| \partial_{\alpha'} \Delta (Z_{it} + i) \right| \, d\alpha' \]
\[ \equiv 2 \Re \int \left\{ -i \sigma \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \right) D_{\alpha'} |Z_{t,\alpha'} \rangle \left| \partial_{\alpha'} \Delta ((Z_{it} + i) Z_{\alpha'}) \right| \, d\alpha' \right\} \]
(4) Now combining the terms we have
\[ \frac{d}{dt} \Delta_{t,1} \approx 2 \Re \int \left\{ \Delta (Z_{it} Z_{\alpha'}) - i \sigma \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \right) D_{\alpha'} |Z_{t,\alpha'} \rangle \right\} \left| \partial_{\alpha'} \Delta ((Z_{it} + i) Z_{\alpha'}) \right| \, d\alpha' \]
Recall from \((32)\) that
\[ Z_{it} Z_{\alpha'} + i A_1 D_{\alpha'} Z_{t} - i \sigma \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \right) D_{\alpha'} |Z_{t,\alpha'} \rangle \]
\[ = i \sigma \partial_{\alpha'} \left( \left| D_{\alpha'} \right|^2 \right) |Z_{t,\alpha'} \rangle - \sigma (D_{\alpha'} Z_{t}) \partial_{\alpha'} \Theta - \sigma \partial_{\alpha'} \left( \left| \Re \Theta \right| D_{\alpha'} \right) |Z_{t} \rangle - i J_1 \]
Now we just apply \( \Delta \) to the above equation and control the quantities. We see that \( \Delta (J_1) \in \dot{H}^\frac{1}{2}_{\sqrt{\Delta}}, \Delta (A_1 D_{\alpha'} Z_{t}) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^\frac{1}{2}_{\sqrt{\Delta}} \) and the other terms with \( \sigma \) are controlled as in the proof of
the time derivative of $E_{\sigma,1}$ in Sec 5.2.2 in \[1\]. Hence

$$\Delta(\mathcal{Z}_{tt} Z_{t,\alpha'}) - i \sigma \delta_{\alpha'}(\frac{1}{\mathcal{Z}_{t,\alpha'}}|D_{t,\alpha'}|\mathcal{Z}_{t,\alpha'})_a \in \dot{H}^{\frac{1}{2}}\sqrt{\Delta}$$

and hence we have shown that $\frac{d}{dt}E_{\Delta,1} \leq C(L_1)E_{\Delta}$.

### 7.2.3. Controlling $E_{\Delta,2}$ and $E_{\Delta,3}$

Note that both $E_{\Delta,2}$ and $E_{\Delta,3}$ are of the form

$$E_{\Delta,i} = \|\Delta(D_t f)\|^2 + \left\| \left( \frac{\sqrt{A_1}}{|Z_{t,\alpha'}|}_a \right) \Delta(f) \right\|^2 + \left\| \left( \frac{\sigma^{\frac{1}{2}}}{|Z_{t,\alpha'}|}_a \partial_{\alpha'} f \right) \right\|^2$$

Where $f = \mathcal{Z}_{t,\alpha'}$ for $i = 2$ and $f = \Theta$ for $i = 3$. Also note that $\mathbb{F}_H f = f$ for these choices of $f$. We will simplify the time derivative of each of the terms individually before combining them.

1. From Lemma \[23\] we have

$$\frac{d}{dt} \int |\Delta(D_t f)|^2 d\alpha' \approx 2 \text{Re} \int (\Delta(D_t^2 f))|\Delta(D_t f)| d\alpha'$$

2. By following the proof of time derivative of $E_{\sigma,2}, E_{\sigma,3}$ in Sec 5.2.3 in \[1\] we have

$$\frac{d}{dt} \int \left| \partial_{\alpha'} \left( \frac{\sqrt{A_1}}{|Z_{t,\alpha'}|}_a \Delta(f) \right) \right|^2 d\alpha' 
\approx 2 \text{Re} \int \left\{ \left( \frac{\sqrt{A_1}}{|Z_{t,\alpha'}|}_a \right) |\partial_{\alpha'}| \left( \left( \frac{\sqrt{A_1}}{|Z_{t,\alpha'}|}_a \Delta(f) \right) \right) \Delta(D_t f) d\alpha'$$

Now using Proposition \[9.5\] and Lemma \[7.5\] we see that

$$\left( \frac{\sqrt{A_1}}{|Z_{t,\alpha'}|}_a \right) |\partial_{\alpha'}| \left( \left( \frac{\sqrt{A_1}}{|Z_{t,\alpha'}|}_a \Delta(f) \right) \right) \approx \mathcal{L}_2 \mathbb{H} \left\{ \left( \frac{\sqrt{A_1}}{|Z_{t,\alpha'}|}_a \right) \partial_{\alpha'} \left( \left( \frac{\sqrt{A_1}}{|Z_{t,\alpha'}|}_a \Delta(f) \right) \right) \right\}$$

Now using Lemma \[8.6\] we have

$$\mathbb{H} \left\{ \left( \frac{\sqrt{A_1}}{|Z_{t,\alpha'}|}_a \right) \partial_{\alpha'} \Delta(f) \right\} \approx \mathcal{L}_2 \mathbb{H} \left\{ \Delta \left( \frac{A_1}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} f \right) \right\}$$

Now as $\left\{ \frac{A_1}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} f \right\}_b \in L^2$ we can replace $\mathbb{H}$ in the second term with $\mathcal{H}$. Hence we have

$$\left( \frac{\sqrt{A_1}}{|Z_{t,\alpha'}|}_a \right) |\partial_{\alpha'}| \left( \left( \frac{\sqrt{A_1}}{|Z_{t,\alpha'}|}_a \Delta(f) \right) \right) \approx \mathcal{L}_2 \mathbb{H} \left\{ \partial_{\alpha'} \left( \left( \frac{\sqrt{A_1}}{|Z_{t,\alpha'}|}_a \Delta(f) \right) \right) \right\}$$
We can simplify the above term by using \( \mathbb{H} f = f \). We see that
\[
\text{i} \mathbb{H} \left( \frac{A_1}{|Z,\alpha|^{2}} \partial_{\alpha'} f \right) = -i \left[ \frac{A_1}{|Z,\alpha|^{2}}, \mathbb{H} \right] \partial_{\alpha'} f + i \frac{A_1}{|Z,\alpha|^{2}} \partial_{\alpha'} f
\]

Now apply \( \Delta \) to the above equation. We can easily control the first term in \( L^2_{\sqrt{\Delta}} \) by using Proposition 7.3, Proposition 11.3 and Lemma 7.5 and hence we have
\[
\left( \frac{\sqrt{A_1}}{|Z,\alpha'|} \right) |\partial_{\alpha'}| \left( \left( \frac{\sqrt{A_1}}{|Z,\alpha'|} \right) \Delta(f) \right) \approx L^2_{\sqrt{\Delta}} \Delta \left( i \frac{A_1}{|Z,\alpha'|^2} \partial_{\alpha'} f \right)
\]

Finally using this we obtain
\[
\frac{d}{dt} \int \left| |\partial_{\alpha'}| \right| \frac{1}{|Z,\alpha'|^2} \partial_{\alpha'} f \right)_{a} \left( |\partial_{\alpha'}| \right) \Delta(D_i \bar{f}) \text{d} \alpha' \approx 2 \text{Re} \left( \frac{\sqrt{A_1}}{|Z,\alpha'|^2} \partial_{\alpha'} f \right) \Delta(D_i \bar{f}) \text{d} \alpha'
\]

(3) By using the argument in controlling the time derivative of \( E_{\sigma,2}, E_{\sigma,3} \) in Sec 5.2.3 in [1] we have
\[
\frac{d}{dt} \sigma \int \left| |\partial_{\alpha'}| \right| \frac{1}{|Z,\alpha'|^2} \partial_{\alpha'} f \right)_{a} \left( |\partial_{\alpha'}| \right) \Delta(D_i \bar{f}) \text{d} \alpha' \approx -2 \sigma \text{Re} \left( \frac{1}{|Z,\alpha'|^2} \partial_{\alpha'} \right) \left( \frac{1}{|Z,\alpha'|^2} \partial_{\alpha'} f \right) \left( D_i \bar{f} \right) \text{d} \alpha'
\]

We now show that the second term is controlled. We see that
\[
-2 \sigma \text{Re} \left( \frac{1}{|Z,\alpha'|^2} \partial_{\alpha'} \right) \left( \frac{1}{|Z,\alpha'|^2} \partial_{\alpha'} f \right) \left( D_i \bar{f} \right) \text{d} \alpha'
\]
\[
= 2 \sigma \text{Re} \left( \frac{1}{|Z,\alpha'|^2} \partial_{\alpha'} \right) \left( \frac{1}{|Z,\alpha'|^2} \partial_{\alpha'} f \right) \left( D_i \bar{f} \right) \text{d} \alpha'
\]

Now we know that \( \left( \frac{\sigma^{-\frac{1}{2}}}{|Z,\alpha'|^2} \partial_{\alpha'} D_t \Theta \right) \in C_{\sqrt{\Delta}} \) and \( \left( \frac{\sigma^{-\frac{1}{2}}}{|Z,\alpha'|^2} \partial_{\alpha'} D_t \bar{Z}_{t,\alpha'} \right) \in C_{\sqrt{\Delta}} \) as they are both controlled by \( \sigma(E_{aux}) \). Hence we also have that \( \frac{\sigma^{-\frac{1}{2}}}{|Z,\alpha'|^2} \partial_{\alpha'} \bar{U} (D_t \bar{Z}_{t,\alpha'}) \in C_{\sqrt{\Delta}} \) and
\[
\sigma^\frac{1}{2} \frac{\partial_{\alpha'}}{|Z_{\alpha'}|^2} \bar{U}(D_t \Theta)_b \in C_{\sqrt{\Delta}}
\]
by using Lemma 7.6 Therefore we now have
\[
\frac{d}{dt} \sigma \int \left| \partial_{\alpha'} \right|^2 \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} f \right) \nonumber\]
\[
\approx -2 \text{Re} \int \partial_{\alpha'} \left\{ \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} f \right) \right\} \Delta(D_t \bar{f}) d\alpha'
\]
Now from the proof from Sec 5.2.3 in [1] we obtain
\[
\sigma \partial_{\alpha'} \left\{ \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} f \right) \right\} \approx |t|^{-\frac{3}{2}} \left( i \sigma |D_{\alpha'}|^3 f \right)
\]
So we finally have
\[
\frac{d}{dt} \sigma \int |\partial_{\alpha'}|^2 \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} f \right) \nonumber\]
\[
\approx -2 \text{Re} \int \left( \text{Re} |D_{\alpha'}|^3 f \right) \Delta(D_t \bar{f}) d\alpha'
\]
(4) Now combining all three terms we have for \( i = 2, 3 \)
\[
\frac{d}{dt} E_{\Delta,i} \approx 2 \text{Re} \int \left\{ \Delta(D_t^2 f) + \Delta \left( i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} f \right) - i \sigma(|D_{\alpha'}|^3 f)_a \right\} \Delta(D_t \bar{f}) d\alpha'
\]
For \( f = \bar{Z}_{t,\alpha'} \) we obtain from [34]
\[
\left( D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} - i \sigma |D_{\alpha'}|^3 \right) \bar{Z}_{t,\alpha'}
\]
\[
= R_1 \bar{Z}_{t,\alpha'} - i \left( \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|^2} \right) J_1 - i D_{\alpha'} J_1 - \bar{Z}_{t,\alpha'} \left\{ D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} - i \sigma |D_{\alpha'}|^3 \right\} \bar{Z}_{t,\alpha'}
\]
Hence applying \( \Delta \) on both sides, we easily see that the terms on the right hand side are in \( L^2_{\sqrt{\Delta}} \).
Similarly for \( f = \Theta \) we have from [36]
\[
\left( D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} - i \sigma |D_{\alpha'}|^3 \right) \Theta = R_2 + i J_2
\]
In this case also we apply \( \Delta \) on both sides and see that the terms on the right are controlled.
Hence we have shown that for \( i = 2, 3 \) we have
\[
\frac{d}{dt} E_{\Delta,i} \leq C(L_1) E_{\Delta}
\]
7.2.4. **Controlling \( E_{\Delta,4} \)**
Recall that
\[
E_{\Delta,4} = \left\| \Delta(D_t \bar{D}_{\alpha'} Z_t)_a \right\|_{H^2}^2 + \left\| \left( \sqrt{A_1} \right)_a |D_{\alpha'}|_a \Delta(D_{\alpha'} Z_t)_a \right\|_{L^2}^2 + \left\| \left( \frac{\sigma^\frac{1}{2}}{|Z_{\alpha'}|^2} \partial_{\alpha'} |D_{\alpha'}|_a \bar{D}_{\alpha'} Z_t \right)_a \right\|_{L^2}^2
\]
We again simplify the terms individually before combining them.
(1) By Lemma [4.4] we have

\[ \frac{d}{dt} \int |\partial_{\alpha'}|^2 \Delta(D_{t, \alpha'} Z_t)|^2 \, d\alpha' \approx 2 \Re \int \Delta(D_{t, \alpha'} Z_t)|\partial_{\alpha'}|\Delta(D_{t, D_{\alpha'} Z_t}) \, d\alpha' \]

Now as \( \Delta((I - H)D_{t, \alpha'} Z_t) \in \dot{H}^\frac{1}{2} \) we see that

\[ \Delta(D_{t, \alpha'} Z_t) \approx \dot{H}^\frac{1}{2} \Delta(I - H)D_{t, \alpha'} Z_t \approx \dot{H}^\frac{1}{2}(D_{t, \alpha'} Z_t)_a - H\bar{U}(D_{t, \alpha'} Z_t)_b \]

But we know that \( (D_{t, \alpha'} Z_t)_b \in \dot{H}^\frac{1}{2} \) as it is controlled by \( E_{high}\)_b. Hence we now have \( (I - H)\bar{U}(D_{t, \alpha'} Z_t)_b \in \dot{H}^\frac{1}{2} \). From this we get

\[ \Delta(D_{t, \alpha'} Z_t) \approx \frac{1}{\dot{H}^\frac{1}{2}} \Delta(D_{t, \alpha'} Z_t) \]

Now we use the fact that |\partial_{\alpha'}| = i\hat{\partial}_{\alpha'} to obtain

\[ \frac{d}{dt} \int |\partial_{\alpha'}|^2 \Delta(D_{t, \alpha'} Z_t)|^2 \, d\alpha' \approx 2 \Re \int \Delta(D_{t, \alpha'} Z_t)|-i\partial_{\alpha'} \Delta(D_{t, D_{\alpha'} Z_t})| \, d\alpha' \]

(2) By following the proof of control of the time derivative of \( E_{\alpha,4} \) from Sec 5.2.4 in [1] we see that

\[ \frac{d}{dt} \int (A_1)_{a} |D_{\alpha'}|_a \Delta(D_{\alpha'} Z_t)|^2 \, d\alpha' \approx 2 \Re \int \left( i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \Delta(D_{\alpha'} Z_t) \right) \{ -i \partial_{\alpha'} \Delta(D_{t, D_{\alpha'} Z_t}) \} \, d\alpha' \]

Now we know that \( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} D_{\alpha'} Z_t \in \mathcal{C} \) as it is controlled by \( E_{high}\)_b. Hence as \( (A_1)_b \in \mathcal{W} \),

we have \( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} D_{\alpha'} Z_t \in \mathcal{C} \). Hence we see that

\[ i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \Delta(D_{\alpha'} Z_t) \approx \dot{H}^\frac{1}{2} \Delta \left( i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} D_{\alpha'} Z_t \right) \]

From this we get

\[ \frac{d}{dt} \int (A_1)_{a} |D_{\alpha'}|_a \Delta(D_{\alpha'} Z_t)|^2 \, d\alpha' \approx 2 \Re \int \Delta \left( i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} D_{\alpha'} Z_t \right) \{ -i \partial_{\alpha'} \Delta(D_{t, D_{\alpha'} Z_t}) \} \, d\alpha' \]
(3) By following the proof of control of the time derivative of $E_{\sigma,4}$ from Sec 5.2.4 in [1] we see that

$$\frac{d}{dt} \sigma \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} |D_{\alpha'}| \mathcal{D}_{\alpha'} Z_t \right) \left| \mathcal{D}_{\alpha'} Z_t \right| \approx 2 \text{Re} \left\{ \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} |D_{\alpha'}| \mathcal{D}_{\alpha'} Z_t \right\} \left\{ \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} |D_{\alpha'}| \Delta(D_t D_{\alpha'} Z_t) \right\}$$

$$\approx 2 \text{Re} \left\{ \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} |D_{\alpha'}| \mathcal{D}_{\alpha'} Z_t \right\} \left\{ \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} |D_{\alpha'}| \Delta(D_t D_{\alpha'} Z_t) \right\}$$

$$+ 2 \text{Re} \left\{ \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} |D_{\alpha'}| \mathcal{D}_{\alpha'} Z_t \right\} \left\{ \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} |D_{\alpha'}| \tilde{U}(D_t D_{\alpha'} Z_t) \right\}$$

We now show that the second term is controlled. We first observe that

$$\frac{\sigma^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} |D_{\alpha'}| \tilde{U}(D_t D_{\alpha'} Z_t)$$

$$= \frac{\sigma^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} \left( \frac{\tilde{h}_{\alpha'}}{|Z_{\alpha'}|^2} \tilde{U}(|Z_{\alpha'}|) b \tilde{U}(|D_{\alpha'}| D_t D_{\alpha'} Z_t) \right)$$

$$= |Z_{\alpha'}|^2 \partial_{\alpha'} \left( \frac{\sigma^2}{|Z_{\alpha'}|^2} \tilde{h}_{\alpha'} \tilde{U}(|Z_{\alpha'}|) b \tilde{U}(|D_{\alpha'}| D_t D_{\alpha'} Z_t) \right)$$

$$+ \frac{\sigma^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} \tilde{U}(|D_{\alpha'}| D_t D_{\alpha'} Z_t)$$

Now we know that

$$\left( \frac{\sigma^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} |D_{\alpha'}| \tilde{h}_{\alpha'} \tilde{U}(|Z_{\alpha'}|) b \right) \in L^\infty_{s,\mathcal{X}}$$

and

$$\left( \frac{\sigma^2}{|Z_{\alpha'}|^2} \partial_{\alpha'} |D_{\alpha'}| \tilde{U}(|D_{\alpha'}| D_t D_{\alpha'} Z_t) \right) \in L^2_{s,\mathcal{X}}$$

as they are controlled by $\sigma(E_{aux})$. We also know that $\tilde{h}_{\alpha'} \in \mathcal{W}$, $\frac{1}{|Z_{\alpha'}|^2} \tilde{U}(|Z_{\alpha'}|) b \in \mathcal{W}$ and hence the above terms are controlled. Hence we have

$$\frac{d}{dt} \sigma \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} |D_{\alpha'}| \mathcal{D}_{\alpha'} Z_t \right) \left| \mathcal{D}_{\alpha'} Z_t \right| \approx 2 \text{Re} \left\{ \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} |D_{\alpha'}| \mathcal{D}_{\alpha'} Z_t \right\} \left\{ \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} |D_{\alpha'}| \Delta(D_t D_{\alpha'} Z_t) \right\}$$

$$\approx 2 \text{Re} \left\{ -i \sigma |D_{\alpha'}|^3 \mathcal{D}_{\alpha'} Z_t \right\} \left\{ -i \partial_{\alpha'} \Delta(D_t D_{\alpha'} Z_t) \right\}$$

(4) Combining the three terms we obtain

$$\frac{d}{dt} E_{\Delta,4} \approx 2 \text{Re} \left\{ -i \partial_{\alpha'} \Delta(D_t D_{\alpha'} Z_t) \right\} \left\{ \Delta(D_t^2 \mathcal{D}_{\alpha'} Z_t) + \Delta \left( i \frac{A_1^{(1)}}{|Z_{\alpha'}|^2} \partial_{\alpha'} \mathcal{D}_{\alpha'} Z_t \right) \right.$$
From equation [33] we see that
\[
(D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} - i \sigma |D_{\alpha'}|^3) \mathbb{M}_{\alpha'} Z_t = R_1 - i \left( \mathbb{M}_{\alpha'} \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \right) J_1
\]
Now we apply \( \Delta \) to the above equation and see that the terms on the right hand side terms are controlled in \( \dot{H}^{\frac{1}{2}} \). Hence we have
\[
\frac{d}{dt} E_{\Delta, t} \leq C(L_1) E_\Delta
\]
This concludes the proof of Theorem 7.1.

7.3. Equivalence of \( E_\Delta \) and \( \mathcal{E} \)

We now give a simpler description of the energy \( E_\Delta \). Define
\[
\mathcal{E}_{\Delta, 0} = \| \Delta (\omega) \|_{\infty}^2 + \| \tilde{h}_{\alpha'} - 1 \|_{L^\infty \cap \dot{H}^{\frac{1}{2}}}^2 + \| D_{\alpha'} |_a (\tilde{h}_{\alpha'} - 1) \|_\infty^2 + \| Z_{\alpha'} |_a U \left( \frac{1}{|Z_{\alpha'}|_b} \right) \|_\infty^2
\]
\[
\mathcal{E}_{\Delta, 1} = \Delta \left( \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\|_{\dot{H}^{\frac{1}{2}}}^2 + \| \Delta \left( \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \|_{\dot{H}^{\frac{1}{2}}}^2 + \left( \frac{\sigma^2}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\|_{\dot{H}^{\frac{1}{2}}}^2
\]
\[
\mathcal{E}_{\Delta, 2} = \| \Delta (Z_{t, \alpha'}) \|_{\dot{H}^{\frac{1}{2}}}^2 + \| \Delta \left( \frac{1}{Z_{t, \alpha'}} \partial_{\alpha'} Z_{t, \alpha'} \right) \|_{\dot{H}^{\frac{1}{2}}}^2 + \left( \frac{\sigma^2}{Z_{t, \alpha'}} \partial_{\alpha'} Z_{t, \alpha'} \right) \right\|_{\dot{H}^{\frac{1}{2}}}^2
\]
\[
\mathcal{E}_\Delta = \mathcal{E} = (\mathcal{E}_{aux})_b + \mathcal{E}_{\Delta, 0} + \mathcal{E}_{\Delta, 1} + \mathcal{E}_{\Delta, 2}
\]
Note that if the two solutions have the same initial data, then \( \mathcal{E}_{\Delta, 0}(0) = 0 \) and hence we obtain the representation of the energy as stated in [85].

**Proposition 7.7.** Let \( T > 0 \) and let \( (Z, Z_t)_a(t) \), \( (Z, Z_t)_b(t) \) be two smooth solutions in \([0, T]\) to \( \mathbb{S} \) with surface tension \( \sigma \) and zero surface tension respectively, such that for all \( s \geq 2 \) we have \( (Z_{t, \alpha'} - 1, \frac{1}{Z_{t, \alpha'}} - 1, Z_t)_i \in L^\infty ([0, T], H^s(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s+\frac{1}{2}}(\mathbb{R})) \) for both \( i = a, b \). Let \( L_1 > 0 \) be such that
\[
\sup_{t \in [0, T]} (\mathcal{E}_{\mathcal{E}_{high}})_b(t), \sup_{t \in [0, T]} (\mathcal{E}_S)_a(t) \left( \frac{1}{|Z_{\alpha'}|_a} U \left( \frac{1}{|Z_{\alpha'}|_b} \right) \right) \|_\infty \leq L_1
\]
Then there exists constants \( C_1(L_1), C_2(L_1) > 0 \) depending only on \( L_1 \) so that for all \( t \in [0, T] \) we have
\[
E_\Delta \leq C_1(L_1) \mathcal{E}_\Delta \quad \text{and} \quad \mathcal{E}_\Delta \leq C_2(L_1) E_\Delta
\]
**Proof.** Let us first show \( \mathcal{E}_\Delta \leq C_2(L_1) E_\Delta \). From Proposition [6.4] we see that \( \sigma (\mathcal{E}_{aux})_b \) is controlled by \( \sigma (E_{aux})_b \). Also observe that the energy \( E_\Delta \) directly controls \( \mathcal{E}_{\Delta, 0} \) and most of the terms of \( \mathcal{E}_{\Delta, 1} \).
and $E_{\Delta,2}$. The terms which are not directly controlled are $\left(\frac{\sigma_1^2}{Z_{a',\alpha}^2} \frac{1}{Z_{a',\alpha'}}\right)_a$ and $\left(\frac{\sigma_2^2}{Z_{a',\alpha}^2} \frac{1}{Z_{a',\alpha'}}\right)_a$
both in $H^\frac{1}{2}_\Delta$. To control these, we use Lemma 7.5 to see that
\[
\left\| \left(\frac{\sigma_1^2}{Z_{a',\alpha}^2} \frac{1}{Z_{a',\alpha'}}\right)_a \right\|_{H^\frac{1}{2}} \lesssim \left\| \left(\frac{\sigma_2^2}{Z_{a',\alpha}^2} \frac{1}{Z_{a',\alpha'}}\right)_a \right\|_{C^\infty_\Delta} \leq C_2(L_1) E_{\Delta}^\frac{1}{2}
\]

Similarly we have
\[
\left\| \left(\frac{\sigma_1^2}{Z_{a',\alpha}^2} \frac{1}{Z_{a',\alpha'}}\right)_a \right\|_{H^\frac{1}{2}} \lesssim \left\| \left(\frac{\sigma_2^2}{Z_{a',\alpha}^2} \frac{1}{Z_{a',\alpha'}}\right)_a \right\|_{C^\infty_\Delta} \leq C_2(L_1) E_{\Delta}^\frac{1}{2}
\]

This proves that $E_{\Delta} \leq C_2(L_1) E_{\Delta}$.

Let us first show $E_{\Delta} \leq C_1(L_1) E_{\Delta}$. We again observe from Proposition 6.4 that $\sigma(E_{aux})_b$ is controlled by $\sigma(E_{aux})_b$. It is also clear that $E_{\Delta}$ controls $E_{\Delta,0}$. Hence we now need to control $E_{\Delta,1}, E_{\Delta,2}, E_{\Delta,3}$ and $E_{\Delta,4}$. As the proof of this is a little more involved, to simplify the presentation we will continue to use the same notation as in 6.4 except for a few minor modifications. In the definitions, instead of using the energy $E_{\Delta}$ we will use the energy $E_{\Delta}$. So now whenever we write $f \in L^2_{\Delta,\alpha}$, what we mean is that there exists a constant $C_1(L_1)$ depending only on $L_1$ such that $\|f\|_2 \leq C_1(L_1)(E_{\Delta})^\alpha$. Similar modifications for $f \in L^1_{\Delta,\alpha}$, $f \in H^2_{\Delta,\alpha}$ and $f \in L^\infty_{\Delta,\alpha}$. The definitions of the spaces $C_{\Delta,\alpha}$ and $W_{\Delta,\alpha}$ remain the same except for the fact that we have now changed the underlying definition of the spaces $L^1_{\Delta,\alpha}, H^2_{\Delta,\alpha}$ and $L^\infty_{\Delta,\alpha}$. We say that $f \in L^2$ if $f \in L^2_{\Delta,\alpha}$ for $\alpha = 0$. Similar definitions for $f \in H^2, f \in L^\infty, f \in C$ and $f \in W$. The definitions of $\approx_{L^1_{\Delta,\alpha}}, \approx_{L^2_{\Delta,\alpha}}, \approx_{L^\infty_{\Delta,\alpha}}, \approx_{H^1_{\Delta,\alpha}}, \approx_{W_{\Delta,\alpha}}$ and $\approx_{C_{\Delta,\alpha}}$ remain the same except the changes to the underlying spaces. Observe that there is no change to Lemma 7.5.

We now make the important observation that Lemma 7.6 still remains true with the new definitions. This is because in the proof of Lemma 7.6 the only properties of $E_{\Delta}$ used were the control of $(\delta_3 - 1) \in L^\infty_{\sqrt{\Delta}} \cap H^2_{\sqrt{\Delta}}$ and $W_{\sqrt{\Delta}}$. $\Delta w \in L^\infty_{\sqrt{\Delta}} \Delta\left(\alpha_{a',\alpha}\right) \in L^2_{\sqrt{\Delta}}$ and the term $|Z_{\alpha'}|_a \left(\frac{1}{|Z_{\alpha'}|}\right)_a \in L^\infty_{\sqrt{\Delta}}$. All of the these quantities are also controlled by $E_{\Delta}$ and hence the lemma still holds. Let us now control $E_{\Delta,1}$ for $1 \leq i \leq 4$.

1. Controlling $E_{\Delta,1}$: From $E_{\Delta,2}$ we have $\Delta(Z_{\Delta,\alpha'}) \in L^2_{\sqrt{\Delta}}$. Hence we have $(\sqrt{A_1})_a \Delta(Z_{\Delta,\alpha'}) \in L^2_{\sqrt{\Delta}}$

Hence now via [111] we have $\Delta(A_1) \in L^\infty_{\sqrt{\Delta}} \cap H^2_{\sqrt{\Delta}}$. Now we know from (111) that $(Z_{\Delta} - i)Z_{\alpha'} = -iA_1 + \sigma\partial_{\alpha'} \Theta$ and hence

$\Delta\left\{(Z_{\Delta} - i)Z_{\alpha'}\right\} = -i\Delta(A_1) + (\sigma\partial_{\alpha'} \Theta)_a$

As $(\sigma\partial_{\alpha'} \Theta)_a \in H^\frac{1}{2}_{\sqrt{\Delta}}$ we see that $\Delta\left\{(Z_{\Delta} - i)Z_{\alpha'}\right\} \in H^\frac{1}{2}_{\sqrt{\Delta}}$. From $E_{\Delta}$ we also clearly see that

$\left(\frac{\sigma_1^2}{|Z_{\alpha'}|_a^2} \frac{1}{Z_{\alpha'}}\right)_a \in L^2_{\sqrt{\Delta}}$ and hence $E_{\Delta,1}$ is controlled.

2. Controlling $E_{\Delta,2}$: We prove this step by step.
(a) As $E_\Delta$ controls $\Delta\left(\partial_{a'}\frac{1}{Z_{a'}}\right) \in L^2_{\sqrt{\Delta}}$, from (7.1) we easily obtain control of $\Delta\left(\partial_{a'}\frac{1}{Z_{a'}}\right) \in L^2_{\sqrt{\Delta}}$, $\Delta(|D_{a'}|\omega) \in L^2_{\sqrt{\Delta}}$ and $\Delta(\omega) \in W_{\sqrt{\Delta}}$.

(b) As $E_\Delta$ controls $\Delta\left(\frac{1}{Z_{a'}}\partial_{a'}Z_{t,a'}\right) \in L^2_{\sqrt{\Delta}}$, by using Lemma 7.6 repeatedly we also have $\frac{1}{(Z_{a'})^2_a}\partial_{a'}\Delta(Z_{t,a'}) \in L^2_{\sqrt{\Delta}}$. Hence we see that

$$\partial_{a'}\left(\frac{1}{(Z_{a'})_a}\Delta(Z_{t,a'})\right) = 2\left(\frac{1}{(Z_{a'})_a}\Delta(Z_{t,a'})\right) \partial_{a'}\left(\frac{1}{(Z_{a'})_a}\Delta(Z_{t,a'})\right)$$

$$= 2\left(\frac{1}{(Z_{a'})_a}\Delta(Z_{t,a'})\right) \left(\partial_{a'}\frac{1}{Z_{a'}}\right) \Delta(Z_{t,a'})$$

$$+ 2\Delta(Z_{t,a'}) \left(\frac{1}{(Z_{a'})^2_a}\partial_{a'}\Delta(Z_{t,a'})\right)$$

From this we obtain

$$\left\|\frac{1}{(Z_{a'})_a}\Delta(Z_{t,a'})\right\|_2 \leq C_1(L_1)\left\|\frac{1}{(Z_{a'})_a}\Delta(Z_{t,a'})\right\|_2$$

$$+ C_1(L_1)\left\|\Delta(Z_{t,a'})\right\|_2\left\|\frac{1}{(Z_{a'})^2_a}\partial_{a'}\Delta(Z_{t,a'})\right\|_2$$

Now using the inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, we see that $\frac{1}{(Z_{a'})_a}\Delta(Z_{t,a'}) \in L^\infty_{\sqrt{\Delta}}$. Now by using Lemma 7.6 we see that $\Delta(D_{a'}Z_t) \in L^\infty_{\sqrt{\Delta}}$, $\Delta(|D_{a'}|Z_t) \in L^\infty_{\sqrt{\Delta}}$ and $\Delta(D_{a'}Z_t) \in L^\infty_{\sqrt{\Delta}}$.

(c) Observe that

$$\Delta(D^2_{a'}Z_t) = \Delta\left\{\left(\partial_{a'}\frac{1}{Z_{a'}}\right)D_{a'}Z_t\right\} + \Delta\left(\frac{1}{Z_{a'}}\partial_{a'}Z_{t,a'}\right)$$

Hence we have $\Delta(D^2_{a'}Z_t) \in L^2_{\sqrt{\Delta}}$. Similarly we can also show $\Delta\left(|D_{a'}|^2Z_t\right) \in L^2_{\sqrt{\Delta}}$ and $\Delta(D^2_{a'}Z_t) \in L^2_{\sqrt{\Delta}}$. Now using Lemma 7.6 we see that $\Delta(|D_{a'}|D_{a'}Z_t) \in L^2_{\sqrt{\Delta}}$ and $|D_{a'}|^2\Delta(D_{a'}Z_t) \in L^2_{\sqrt{\Delta}}$. This in particular implies $(\sqrt{A_1})_aD_{a'}\Delta(D_{a'}Z_t) \in L^2_{\sqrt{\Delta}}$ which is part of $E_{\Delta,A}$.

(d) Following the proof in (7.1) we see that $\Delta(D_{a'}Z_t) \in W_{\sqrt{\Delta}} \cap C_{\sqrt{\Delta}}$, $\Delta(|D_{a'}|Z_t) \in W_{\sqrt{\Delta}} \cap C_{\sqrt{\Delta}}$ and $\Delta(D_{a'}Z_t) \in W_{\sqrt{\Delta}} \cap C_{\sqrt{\Delta}}$. Hence using Lemma 7.6 we have $\frac{1}{(Z_{a'})_a}\Delta(Z_{t,a'}) \in C_{\sqrt{\Delta}}$.

As $(\sqrt{A_1})_a \in W$ we now obtain $\left(\frac{\sqrt{A_1}}{(Z_{a'})_a}\right)\Delta(Z_{t,a'}) \in C_{\sqrt{\Delta}}$ and hence we have controlled the second term of $E_{\Delta,2}$.

(e) Following the proof in (7.1) we see that $\Delta(|D_{a'}|A_1) \in L^2_{\sqrt{\Delta}}$ and hence we have $\Delta(A_1) \in W_{\sqrt{\Delta}}$ and $\Delta(\sqrt{A_1}) \in W_{\sqrt{\Delta}}$.

(f) From (7.1) we see that $\Delta(b_{a'}) \in W^\infty_{\sqrt{\Delta}} \cap H^4_{\sqrt{\Delta}}$, $\Delta(|D_{a'}|b_{a'}) \in L^2_{\sqrt{\Delta}}$ and $\Delta(b_{a'}) \in W_{\sqrt{\Delta}}$. 

(g) As \((\sigma \partial_{\alpha'} \Theta)_{a} \in \dot{H}_{\Delta}^{\frac{1}{2}}\) and \(\Theta_{a} \in L^{2}\), by interpolation we see that \((\sigma^{\frac{1}{2}} \partial_{\alpha'} \Theta)_{a} \in L^{2}_{\Delta}H_{\Delta}^{\frac{1}{2}}\) and 

(h) Following the proof in Sec 5.1 of [1] we see that \(\left(\frac{\sigma^{\frac{1}{2}} \partial_{\alpha'}^{2}}{2} \frac{1}{|Z_{\alpha'}|} \right)_{a} \in L^{2}_{\Delta}H_{\Delta}^{\frac{1}{2}}\) and similarly \(\left(\frac{\sigma^{\frac{1}{2}} \partial_{\alpha'}^{2}}{2} \frac{1}{|Z_{\alpha'}|} \frac{1}{|D_{\alpha'}|} \right)_{a} \in L^{2}_{\Delta}H_{\Delta}^{\frac{1}{2}}\). In the same way we have 

\(\left(\sigma^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right)_{a} \in L^{2}_{\Delta}H_{\Delta}^{\frac{1}{2}}\) and \(\left(\sigma^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right)_{a} \in L^{2}_{\Delta}H_{\Delta}^{\frac{1}{2}}\). Thus 

(i) Following the proof in Sec 5.1 of [1] we see that \(\left(\frac{\sigma}{|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \Theta} \right)_{a} \in L^{2}_{\Delta}H_{\Delta}^{\frac{1}{2}}\) and from this we easily get \((\sigma \partial_{\alpha'} D_{\alpha'} \Theta)_{a} \in L^{2}_{\Delta}H_{\Delta}^{\frac{1}{2}}\). Similarly we also get \(\left(\frac{\sigma}{|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|}} \right)_{a} \in L^{2}_{\Delta}H_{\Delta}^{\frac{1}{2}}\) and 

(j) Following the proof in Sec 5.1 of [1] we see that we have \(\left(\frac{\sigma^{\frac{1}{2}} \partial_{\alpha'}^{2}}{2} \frac{1}{|Z_{\alpha'}|} \right)_{a} \in L^{2}_{\Delta}H_{\Delta}^{\frac{1}{2}}\). 

\(\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}} \right)_{a} \in L^{2}_{\Delta}H_{\Delta}^{\frac{1}{2}}\) and 

\(\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \Theta} \right)_{a} \in L^{2}_{\Delta}H_{\Delta}^{\frac{1}{2}}\). Similarly we also obtain control 

of \(\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}} \right)_{a} \in W_{\Delta}^{\frac{1}{2}}\). 

\(\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|}} \right)_{a} \in W_{\Delta}^{\frac{1}{2}}\) and 

\(\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \omega} \right)_{a} \in W_{\Delta}^{\frac{1}{2}}\). 

(k) We now recall from [10]

\[Z_{tt} - i = -i \frac{A_{1}}{Z_{\alpha'}} + \sigma D_{\alpha'} \Theta\]

Taking derivatives on both sides and applying \(\Delta\) we get

\[\Delta(Z_{tt,\alpha'}) = -i \Delta \left(\frac{A_{1} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}}{Z_{\alpha'}}\right) - i \Delta(D_{\alpha'} A_{1}) + (\sigma \partial_{\alpha'} D_{\alpha'} \Theta)_{a}\]

Hence we see that \(\Delta(Z_{tt,\alpha'}) \in L^{2}_{\Delta}\). As \(D_{t} Z_{t,\alpha'} = -b_{\alpha'} Z_{t,\alpha'} + Z_{tt,\alpha'}\) and \(\Delta(b_{\alpha'}) \in L^{\infty}_{\Delta}\) we obtain \(\Delta(D_{t} Z_{t,\alpha'}) \in L^{2}_{\Delta}\) which is the first term of \(E_{\Delta,2}\).

(l) By exactly following the proof of \(\frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} Z_{t,\alpha'}} \in L^{2}\) in Sec 5.1 of [1], we easily get that 

\(\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} D_{\alpha'} |D_{\alpha'} Z_{t}|} \right)_{a} \in L^{2}_{\Delta}\). Note that this is the last term of \(E_{\Delta,\alpha}\).
(m) Now we use can Proposition 9.8 with \( \omega = \frac{1}{|Z_{\alpha'}|_a} \) and \( f = \left( \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|_a^2} \partial_{\alpha'} Z_{\alpha'} \right)_a \) and we obtain \( \left( \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|_a^2} \partial_{\alpha'} Z_{\alpha'} \right)_a \in L_{2,\infty} \cap H_{2,\infty}^{\frac{1}{2}} \). Hence \( E_{\Delta,2} \) is controlled.

(3) Controlling \( E_{\Delta,3} \): We prove this step by step.

(a) By (26) we see that \( \Delta(\Theta) \in L_{\infty}^{2,\infty} \). Similarly from (28) and (29) we obtain \( \Delta(D_t \Theta) \in L_{\infty}^{2,\infty} \). Hence the first term of \( E_{\Delta,3} \) is controlled.

(b) As we have \( \Delta \left( D_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \in C_{\infty} \), using Lemma 7.6 and Lemma 7.5 we see that we have \( \Delta \left( \frac{D_{\alpha'}}{Z_{\alpha'}} \right) \in C_{\infty} \). Now following the proof of \( \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} A_1 \in C \) in Sec 5.1 of [1], we easily get \( \Delta \left( \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} A_1 \right) \in C_{\infty} \).

(c) Following the proof of \( D_{\alpha'} \frac{1}{Z_{\alpha'}} \in C \) in Sec 5.1 of [1], we see that \( \Delta \left( \frac{\Theta}{|Z_{\alpha'}|} \right) \in C_{\infty} \). Hence by Lemma 7.6 and Lemma 7.5 we see that \( \frac{1}{|Z_{\alpha'}|} \Delta(\Theta) \in C_{\infty} \) and \( \left( \frac{\sqrt{A}}{Z_{\alpha'}} \right) \Delta(\Theta) \in C_{\infty} \). Hence the second term of \( E_{\Delta,3} \) is controlled.

(d) As \( \left( \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|_a^2} \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right)_a \in C_{\infty} \), by using Lemma 7.6 we see that \( \left( \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|_a^2} \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right)_a \in C_{\infty} \). Now by following the proof of \( \frac{1}{|Z_{\alpha'}|_a^2} \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \in C \) in Sec 5.1 of [1], we easily obtain \( \left( \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|_a^2} \partial_{\alpha'} \Theta \right)_a \in C_{\infty} \), \( \left( \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|_a^2} \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right)_a \in C_{\infty} \) and \( \left( \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|_a^2} \partial_{\alpha'}^2 \omega \right)_a \in C_{\infty} \). Hence \( E_{\Delta,3} \) is controlled.

(4) Controlling \( E_{\Delta,4} \): Observe that we have already controlled the second and third term of \( E_{\Delta,4} \). Now by following the proof of \( \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|_a^2} \partial_{\alpha'}^3 \frac{1}{Z_{\alpha'}} \in C \) and \( \frac{\sigma}{|Z_{\alpha'}|_a^2} \partial_{\alpha'}^2 \Theta \in C \) in Sec 5.1 of [1], we see that \( (\sigma \bar{D}_{\alpha'} D_{\alpha'} \Theta)_a \in C_{\infty} \). Hence by applying \( \bar{D}_{\alpha'} \) in the formula (10) we obtain
\[
\Delta(D_{\alpha'} \bar{Z}_{tt}) = -i \Delta \left( A_1 \bar{D}_{\alpha'} \frac{1}{Z_{\alpha'}} \right) - i \Delta \left( \frac{1}{|Z_{\alpha'}|^2} A_1 \right) + (\sigma \bar{D}_{\alpha'} D_{\alpha'} \Theta)_a
\]
Hence \( \Delta(\bar{D}_{\alpha'} \bar{Z}_{tt}) \in C_{\infty} \). Now by applying \( \Delta \) to the equation \( D_t \bar{D}_{\alpha'} \bar{Z}_t = -(\bar{D}_{\alpha'} \bar{Z}_t)^2 + \bar{D}_{\alpha'} \bar{Z}_{tt} \) we see that \( \Delta(D_t \bar{D}_{\alpha'} \bar{Z}_t) \in H_{\infty}^{\frac{1}{2}} \) which shows that \( E_{\Delta,4} \) is controlled. Hence proved.

\[ \square \]

8. Proof of Theorem 3.1 and Corollary 3.2

We now prove our main results stated in [3.1]. We first prove a basic lemma.
Lemma 8.1. Let \((Z, Z_t)(t)\) be a smooth solution to the water wave equation \(^8\) for \(\sigma \geq 0\) in the time interval \([0, T]\) for \(T > 0\), satisfying \((Z_{\alpha'}, -1, \frac{1}{Z_{\alpha'}}, -1, Z_t) \in L^\infty([0, T], H^{s + \frac{3}{2}}(\mathbb{R}) \times H^{s + \frac{1}{2}}(\mathbb{R}) \times H^s(\mathbb{R}))\) for all \(s \geq 3\). Let

\[ R_0 = \left\| \frac{1}{Z_{\alpha'}} - 1 \right\|_2(0) + \|Z_t\|_2(0) \quad \text{and} \quad M_0 = \|Z_{\alpha'}\|_\infty(0) + \left\| \frac{1}{Z_{\alpha'}} - 1 \right\|_2(0) + \|Z_t\|_2(0) \]

Then there exists a universal increasing function \(F : [0, \infty) \to [0, \infty)\) so that

(1) If \(\sigma > 0\), then

\[ \sup_{t \in [0, T]} \left\{ \|Z_{\alpha'} - 1\|_{H^{3.5}}(t) + \left\| \frac{1}{Z_{\alpha'}} - 1 \right\|_{H^{3.5}}(t) + \|Z_t\|_{H^3}(t) \right\} \leq F \left( M_0 + \sup_{t \in [0, T]} \mathcal{E}_\sigma(t) + T + \sigma + \frac{1}{\sigma} \right) \]

(2) If \(\sigma = 0\), then

\[ \sup_{t \in [0, T]} \left\{ \|Z_{\alpha'} - 1\|_{H^3}(t) + \left\| \frac{1}{Z_{\alpha'}} - 1 \right\|_{H^3}(t) + \|Z_t\|_{H^{3.5}}(t) \right\} \leq F \left( M_0 + \sup_{t \in [0, T]} \mathcal{E}_{\text{high}}(t) + \sup_{t \in [0, T]} \mathcal{E}_{\text{aux}}(t) + T + 1 \right) \]

(3) For \(\sigma \geq 0\) we define

\[ S(t) = \left\| \frac{1}{Z_{\alpha'}} - 1 \right\|_2(t) + \left\| \frac{1}{Z_{\alpha'}} \right\|_{L^\infty \cap H^{\frac{3}{2}}}(t) + \|Z_t\|_2(t) + \|Z_t\|_{H^{\frac{3}{2}}}(t) + \|Z_t\|_2(t) + \|Z_t\|_{L^\infty \cap H^{\frac{1}{2}}}(t) + \|b\|_2(t) + \|b\|_{L^\infty \cap H^{\frac{1}{2}}}(t) + \|b_{\alpha'}\|_2(t) + \|Z_{\alpha'}\|_2(t) + \|Z_{\alpha'}\|_{L^\infty \cap H^{\frac{1}{2}}}(t) \]

Then we have the estimate

\[ \sup_{t \in [0, T]} S(t) \leq F \left( R_0 + \sup_{t \in [0, T]} \mathcal{E}_\sigma(t) + \sigma + T + 1 \right) \]

Proof. The first estimate was already proved in Lemma 6.2 of \([1]\). Let us now prove the second estimate. Observe from the definition of \(\mathcal{E}_{\text{high}}\) in \([13]\), Proposition 5.2 and Proposition 4.3 that \(\mathcal{E}_{\text{high}}\) controls \(\mathcal{E}_\sigma|_{\sigma = 0}\). Hence from Lemma 6.2 of \([1]\) we get

\[ \sup_{t \in [0, T]} \left\{ \|Z_{\alpha'} - 1\|_{H^{1.5}}(t) + \left\| \frac{1}{Z_{\alpha'}} - 1 \right\|_{H^{1.5}}(t) + \|Z_t\|_{H^2}(t) \right\} \leq F \left( M_0 + \sup_{t \in [0, T]} \mathcal{E}_{\text{high}}(t) + T + 1 \right) \]

Hence we see that for each \(t \in [0, T]\), we have \(Z_{\alpha'} \in L^\infty, |D_{\alpha'}|Z_{\alpha'} \in L^2\) and hence \(Z_{\alpha'} \in \mathcal{W}\). Now using the energy \(\mathcal{E}_{\text{aux}}(t)\) we see that

\[ \left\| \frac{1}{Z_{\alpha'}} - 1 \right\|_2 \lesssim \|Z_{\alpha'}\|_{L^\infty} \left\| \frac{1}{Z_{\alpha'}} - 1 \right\|_2 \left\| \frac{1}{Z_{\alpha'}} \right\|_{L^\infty} \]
and by Lemma [4.3]
\[ \| \partial_{\alpha'}^2 Z_{t,\alpha'} \|_c \lesssim \| Z_{t,\alpha'} \|_{H^{\frac{3}{2}}_V} \| \frac{1}{Z_{\alpha'}} \partial_{\alpha'} Z_{t,\alpha'} \|_c \]

We control \( Z_{\alpha'} - 1 \in H^3 \) similarly. This proves the second estimate.

Let us now prove the third estimate. The proof follows in a similar fashion as the proof of Lemma 6.2 of [1]. To simplify the calculation we define
\[ M = R_0 + \sup_{t \in [0,T]} E_\sigma(t) + \sigma + T + 1 \]
and we write \( a \lesssim b \) if \( a \leq C(M)b \) where \( C(M) \) is a constant which depends only on \( M \). Hence we need to prove that \( \sup_{t \in [0,T]} S(t) \lesssim 1 \). First observe that
\[ \| \frac{1}{Z_{\alpha'}} \|_\infty (0) \lesssim 1 + \| \frac{1}{Z_{\alpha'}} - 1 \|_2 (0) \| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \|_2 (0) \lesssim 1 \]
Now the evolution equation (8) gives us
\[ (\partial_t + b \partial_{\alpha'}) Z_{\alpha'} = Z_{t,\alpha'} - b_{\alpha'} Z_{\alpha'} = (D_{\alpha'} Z_t - b_{\alpha'}) Z_{\alpha'} \]
Hence for all \( 0 \leq t \leq T \) we have the estimate
\[ \| \frac{1}{Z_{\alpha'}} \|_\infty (t) \leq \| \frac{1}{Z_{\alpha'}} \|_\infty (0) \exp \left\{ \int_0^t \| D_{\alpha'} Z_t \|_\infty (s) + \| b_{\alpha'} \|_\infty (s) \| ds \right\} \lesssim 1 \]
Define
\[ f(t) = \left\| \frac{1}{Z_{\alpha'}} - 1 \right\|_2^2 (t) + \| Z_t \|_2^2 (t) + 1 \]
Observe that \( f(0) \lesssim 1 \). Now it was proved as part of the proof of Lemma 6.2 in [1] that
\[ \| b \|_2 + \| b \|_\infty + \| b_{\alpha'} \|_2 + \| A_1 - 1 \|_2 + \| Z_{tt} \|_2 \lesssim f^\frac{1}{2} \]
and that \( \partial_t f \lesssim f \). Hence we have
\[ \sup_{t \in [0,T]} \left\{ \left\| \frac{1}{Z_{\alpha'}} - 1 \right\|_2 (t) + \| Z_t \|_2^2 (t) + \| b \|_2 (t) + \| b \|_\infty (t) + \| b_{\alpha'} \|_2 (t) + \| A_1 - 1 \|_2 (t) + \| Z_{tt} \|_2 (t) \right\} \lesssim 1 \]
The other terms are easily controlled. Observe that
\[ \left\| \frac{1}{Z_{\alpha'}} \right\|_{H^{\frac{3}{2}}} \lesssim \left\| \frac{1}{Z_{\alpha'}} - 1 \right\|_2 \| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \|_2 \lesssim 1 \]
We also see that \( \| Z_t \|_{L^\infty \cap H^{\frac{3}{2}}} \lesssim \| Z_t \|_2^\frac{1}{2} \| Z_{t,\alpha'} \|_2^\frac{3}{2} \lesssim 1 \). Similarly we get \( \| b \|_{L^\infty \cap H^{\frac{3}{2}}} \lesssim 1 \) and \( \| Z_{tt} \|_{L^\infty \cap H^{\frac{3}{2}}} \lesssim 1 \). This finishes the proof of the lemma.

We can now prove our main theorem.
Proof of Theorem 3.1. In the proof we will frequently denote a constant which depends only on $L$ by $C(L)$. Let $M_0, N_0$ be defined as

$$M_0 = \|Z_{\alpha'}\|_\infty(0) + \|\frac{1}{Z_{\alpha'}} - 1\|_2(0) + \|Z_t\|_2(0)$$

and

$$N_0 = E_{aux}(Z, Z_t)(0)$$

Let $\epsilon \geq 0$ and consider the mollified initial data given by $(Z^\epsilon, Z_t^\epsilon)(0) = (P_\epsilon * Z, P_\epsilon * Z_t)(0)$ where $P_\epsilon$ is the Poisson kernel. Observe that there exists an $0 < \epsilon_0 \leq 1$ small enough so that for all $0 < \epsilon \leq \epsilon_0$ we have

$$E_{high}(Z^\epsilon, Z_t^\epsilon)(0), E_\sigma(Z^\epsilon, Z_t^\epsilon)(0) \leq 2L$$

and

$$E_\Delta(Z^\epsilon, Z_t^\epsilon)(0) \leq 2E_\Delta(Z^\sigma, Z)(0)$$

and we also have

$$\|Z_{\alpha'\alpha'}\|_\infty(0) + \|\frac{1}{Z_{\alpha'}} - 1\|_2(0) + \|Z_t\|_2(0) \leq M_0$$

and

$$E_{aux}(Z^\epsilon, Z_t^\epsilon)(0) \leq N_0$$

Now let $0 \leq \epsilon < \epsilon_0$. As $(Z_{\alpha'}, 1, \frac{1}{Z_{\alpha'}} - 1, Z_t)(0) \in H^{3.5}(\mathbb{R}) \times H^{3.5}(\mathbb{R}) \times H^3(\mathbb{R})$, by Theorem 2.1 there exists a $T_1 > 0$ depending only on $L$ so that we have a unique solution $(Z^\epsilon, Z_t^\epsilon)(t)$ in $[0, T_1]$ to (5) with surface tension $\sigma$ and initial data $(Z^\epsilon, Z_t^\epsilon)(0)$, satisfying $(Z_{\alpha'}^\epsilon - 1, \frac{1}{Z_{\alpha'}} - 1, Z_t^\epsilon)^\epsilon) \in C^l([0, T_1] \times H^{3.5-\frac{1}{2}}(\mathbb{R}) \times H^{3-\frac{1}{2}}(\mathbb{R}))$ for $l = 0, 1$ and we also have

$$\sup_{t \in [0, T_1]} E_{aux}(Z^\epsilon, Z_t^\epsilon)(t) \leq C(L)$$

We denote the solution for $\epsilon = 0$ in this case simply by $(Z^\sigma, Z_t^\sigma)(t)$. Using Lemma 8.1 we see that

$$\sup_{t \in [0, T_1]} \left\{ \left\|Z_{\alpha'\alpha'}^\sigma - 1\right\|_{H^{3.5}}(t) + \left\|\frac{1}{Z_{\alpha'}}^\sigma - 1\right\|_{H^{3.5}}(t) + \|Z_t^\sigma\|_{H^3}(t) \right\} \leq C(L, M_0, T_1, \sigma)$$

Also from Corollary 7.9 of [1] we obtain

$$\sup_{t \in [0, T_1]} \left\{ \left\|Z_{\alpha'}^\sigma - Z_{\alpha'}^\epsilon\right\|_{H^2}(t) + \left\|\frac{1}{Z_{\alpha'}}^\sigma - \frac{1}{Z_{\alpha'}}^\epsilon\right\|_{H^2}(t) + \|Z_t^\sigma - Z_t^\epsilon\|_{H^1}(t) \right\} \to 0 \quad \text{as} \quad \epsilon \to 0$$

Now let $0 < \epsilon \leq \epsilon_0$. Using Theorem 2.3 of [36] we see that there exists a $T_2 > 0$ so that we have a unique smooth solution $(Z^\epsilon, Z_t^\epsilon)(t)$ in $[0, T_2]$ to (5) with zero surface tension and initial data $(Z^\epsilon, Z_t^\epsilon)(0)$, satisfying $(Z_{\alpha'}^\epsilon - 1, \frac{1}{Z_{\alpha'}} - 1, Z_t^\epsilon) \in L^\infty([0, T], H^s(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s+\frac{1}{2}}(\mathbb{R}))$ for all $s \geq 4$.

Therefore by Theorem 5.1 we see that for all $t \in [0, T_2)$ we have

$$\frac{dE_{high}(Z^\epsilon, Z_t^\epsilon)(t)}{dt} \leq P(E_{high}(Z^\epsilon, Z_t^\epsilon)(t))$$

Also from Theorem 6.1 we have

$$\frac{dE_{aux}(Z^\epsilon, Z_t^\epsilon)(t)}{dt} \leq P(E_{high}(Z^\epsilon, Z_t^\epsilon)(t))E_{aux}(Z^\epsilon, Z_t^\epsilon)(t)$$

Hence using Proposition 5.2, Proposition 6.4, Lemma 8.1 and the blow up criterion of Theorem 2.3 of [36], we see that there exists a $T_2 > 0$ depending only on $L$ so that the solution $(Z^\epsilon, Z_t^\epsilon)(t)$ in fact exists in the time interval $[0, T_2]$ and we have the estimates

$$\sup_{t \in [0, T_2]} E_{high}(Z^\epsilon, Z_t^\epsilon)(t) \leq C(L)$$

and

$$\sup_{t \in [0, T_2]} E_{aux}(Z^\epsilon, Z_t^\epsilon)(t) \leq C(L, N_0)$$
proof that the solutions \((Z^\epsilon, Z^\epsilon_t)(t)\) and \((Z^{\epsilon,\sigma}, Z^{\epsilon,\sigma}_t)(t)\) are smooth for \(0 < \epsilon \leq \epsilon_0\), we can now use Theorem \(\frac{\sigma}{\epsilon^{3/2}}\) to see that for all \(t \in [0, T)\) we have

\[
\frac{d}{dt} E_\Delta(Z^{\epsilon,\sigma}, Z^\epsilon)(t) \leq C(L) E_\Delta(Z^{\epsilon,\sigma}, Z^\epsilon)(t)
\]

Hence using Proposition \(\frac{\sigma}{\epsilon^{3/2}}\) and the fact that \(E_\Delta(Z^{\epsilon,\sigma}, Z^\epsilon)(0) \leq 2 E_\Delta(Z^\sigma, Z)(0)\), we see that there are constants \(C_0, C_1\) depending only on \(L\) so that

\[
\sup_{t \in [0, T]} E_\Delta(Z^{\epsilon,\sigma}, Z^\epsilon)(t) \leq C_1 e^{C_0 T} E_\Delta(Z^\sigma, Z)(0)
\]

Hence proved. \(\square\)

**Proof of Corollary** \[1\]. Without loss of generality we assume that \(c = 1\) and define \(\tau = \frac{\sigma}{\epsilon^{3/2}}\) which implies that \(\tau \leq 1\). To simplify the proof we will suppress the dependence of \(M\) in the inequalities i.e. when we write \(a \lesssim b\), what we mean is that there exists a constant \(C(M)\) depending only on \(M\) such that \(a \leq C(M)b\).

If \(\sigma \geq 0, 0 < \epsilon \leq 1\) and \(\tau \leq 1\), then it was already shown in the proof of Corollary \[2\] that we have \(E_\sigma(Z^{\epsilon,\sigma}, Z^{\epsilon,\sigma}_t)(0) \lesssim 1\). It is also clear from the definition of \(M\) in \[1\] that we have \(E_{\text{high}}(Z^\epsilon, Z^\epsilon_t)(0) \lesssim 1\). Hence by Theorem \[3\] there exists \(T_2, C_0 > 0\) depending only on \(M\) so that the solutions \((Z^{\epsilon,\sigma}, Z^{\epsilon,\sigma}_t)(t)\) exist in the time interval \([0, T_2]\), we have \(\sup_{t \in [0, T_2]} E_{\text{high}}(Z^\epsilon, Z^\epsilon_t)(t) \lesssim 1\) and \(\sup_{t \in [0, T_2]} E_{\sigma}(Z^{\epsilon,\sigma}, Z^{\epsilon,\sigma}_t)(t) \lesssim 1\), and we have

\[
\sup_{t \in [0, T_2]} E_\Delta(Z^{\epsilon,\sigma}, Z^\epsilon)(t) \lesssim e^{C_0 T_2} E_\Delta(Z^{\epsilon,\sigma}, Z^\epsilon)(0)
\]

Let us now prove each of the statements in the corollary.

**Part 1:** From the above equation it is clear that we only need to prove \(E_\Delta(Z^{\epsilon,\sigma}, Z^\epsilon)(0) \lesssim \tau\). As we only need to suppress the time dependence of the solutions e.g. we will write \((Z \ast P_\epsilon, Z_t \ast P_\epsilon)_{t=0}\) by \((Z, Z_t)\), for simplicity.

Recall that \(E_\Delta(Z^{\epsilon,\sigma}, Z^\epsilon)(0) = E_{\Delta,1}(Z^{\epsilon,\sigma}, Z^\epsilon)(0) + E_{\Delta,2}(Z^{\epsilon,\sigma}, Z^\epsilon)(0) + \sigma(E_{\text{aux}})(0)\) where the term \(\sigma(E_{\text{aux}})(0)\) is given by

\[
\sigma(E_{\text{aux}})(0) = \left( \left\| \left( \frac{\sigma^2}{Z^\epsilon_{\alpha'}} \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right) \right\|_{\infty}^2 + \left\| \left( \frac{\sigma^2}{Z^\epsilon_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\|_2^2 + \left\| \left( \frac{\sigma^2}{Z^\epsilon_{\alpha'}} \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right) \right\|_2^2 \right)^{1/2}
\]

\[
+ \left( \left\| \left( \frac{\sigma^2}{Z^\epsilon_{\alpha'}} \partial_{\alpha'} Z_{\alpha'} \right) \right\|_2^2 + \left\| \left( \frac{\sigma^2}{Z^\epsilon_{\alpha'}} \partial_{\alpha'}^2 Z_{\alpha'} \right) \right\|_2^2 + \left\| \left( \frac{\sigma^2}{Z^\epsilon_{\alpha'}} \partial_{\alpha'}^2 Z_{\alpha'} \right) \right\|_{H^{\frac{3}{2}}}^2 \right)^{1/2}
\]
Now looking at the proof of Corollary 2.2 in [1], it was shown there all the terms involving \( \sigma \) in \( \mathcal{E}_\tau(Z^{r,\sigma}, Z_i^{r,\sigma}) (0) \) are bounded above by \( \tau \). Hence this directly implies that \( \mathcal{E}_{\Delta,1}(Z^{r,\sigma}, Z_i^{r,\sigma}) (0) + \mathcal{E}_{\Delta,2}(Z^{r,\sigma}, Z_i^{r,\sigma}) (0) \lesssim \tau \). Hence we only need to prove that \( \sigma(\mathcal{E}_{\text{aux}})(0) \lesssim \tau \). Now observe that the only terms of \( \sigma(\mathcal{E}_{\text{aux}})(0) \) we really need to control are \( \left( \frac{\sigma^2}{Z_{\alpha'}^r} \partial^3_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \in L^2 \) and \( \left( \frac{\sigma^2}{Z_{\alpha'}^r} \partial^2_{\alpha'} Z_{1,\alpha'} \right) \in H^\frac{1}{2} \), as all the other terms are controlled as part of the terms involving \( \sigma \) in \( \mathcal{E}_\sigma(Z^{r,\sigma}, Z_i^{r,\sigma})(0) \). Let us now control these two terms.

1. We use \( \sup_{\gamma < 0} \left\| \frac{1}{\Psi_2} \partial_{\alpha'}^3 \left( \frac{1}{\Psi_2} \right) \right\|_{L^1(\mathbb{R}, dx)} \lesssim 1 \) and Lemma 9.10 to get

\[
\left\| \sigma \partial_{\alpha'} \left( \frac{1}{Z_{\alpha'}^r} \partial^3_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\|_2 \lesssim \tau
\]

From this we also obtain

\[
\left\| \left( \frac{\sigma}{Z_{\alpha'}^r} \partial^3_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\|_2 \lesssim \left\| \left( \frac{1}{Z_{\alpha'}^r} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\|_2 \left\| \left( \sigma^2 \partial^3_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\|_2 \lesssim \tau
\]

Now we have

\[
\left\| \left( \sigma |Z_{\alpha'}|^2 \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial^3_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right) \right\|_2 \lesssim \left\| \left( \sigma \partial^3_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\|_2 \left\| \left( \sigma |Z_{\alpha'}|^2 \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial^3_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right) \right\|_2 \lesssim \tau
\]

where the last two terms in the product above were controlled in the proof of Corollary 2.2 in [1]. Hence by using integration by parts we see that

\[
\left\| \left( \frac{\sigma^2}{Z_{\alpha'}^r} \partial^3_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\|_2 \lesssim \left\| \left( \frac{1}{|Z_{\alpha'}|^2} \partial^2_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\|_2 \left\| \left( \sigma |Z_{\alpha'}|^2 \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial^3_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right) \right\|_2 \lesssim \tau
\]

(2) Using Proposition 9.3 we see that

\[
\left\| \left( \frac{\sigma^2}{Z_{\alpha'}^r} \partial^3_{\alpha'} Z_{1,\alpha'} \right) \right\|_2 \lesssim \sigma \left\| \partial_{\alpha'} \left( \frac{1}{Z_{\alpha'}} \right) \right\|_2 + \sigma \left\| \partial_{\alpha'} \left( \frac{1}{Z_{\alpha'}} \right) \right\|_2 \lesssim \tau
\]

This finishes the proof of \( \mathcal{E}_{\Delta}(Z^{r,\sigma}, Z^r)(0) \lesssim \tau \) and hence we have proved the first part.

**Part 2:** From the definition of \( \mathcal{F}_{\Delta} \) in [14] we observe that

\[
\mathcal{F}_{\Delta}(Z^{r,\sigma}, Z)(t) \leq \mathcal{F}_{\Delta}(Z^{r,\sigma}, Z^r)(t) + \mathcal{F}_{\Delta}(Z^r, Z)(t)
\]
Now from Theorem 3.7 of [36] we see that \( \sup_{t \in [0,T]} \mathcal{F}_\Delta(Z^\epsilon, Z)(t) \to 0 \) as \( \epsilon \to 0 \). Hence it is enough to show that \( \sup_{t \in [0,T]} \mathcal{F}_\Delta(Z^\epsilon, Z^\epsilon)(t) \to 0 \) as \( \tau \to 0 \). Let \((Z^\epsilon, Z_t^\epsilon)(t)\) be the solution \( A \) and let \((Z_t^\epsilon, Z_t^\epsilon)(t)\) be solution \( B \). Hence from Lemma [7.3] we see that \( \tilde{U} = U_h^t = U_{h_a}^t U_{h_b} \) is bounded on \( L^2 \) and \( H^2 \), and the same is true for \( \tilde{U}^{-1} \). Hence we see that

\[
\mathcal{F}_\Delta(Z^\epsilon, Z^\epsilon)(t) \lesssim \tilde{\mathcal{F}}_\Delta(Z^\epsilon, Z^\epsilon)(t)
\]

where

\[
\tilde{\mathcal{F}}_\Delta(Z^\epsilon, Z^\epsilon)(t) = \|\Delta(Z_t^\epsilon)\|_{H^2} + \|\Delta(Z_t)\|_{H^2} + \left\| \Delta \left( \frac{1}{Z_{\alpha'\tau}} \right) \right\|_{H^2} + \|\Delta(h_{\alpha} \circ h^{-1})\|_2
\]

(55)

We now control each of these terms. Hence using Lemma [8.1] we see that

\[
\|\Delta(Z_t^\epsilon)\|_{L^\infty \cap H^2} \lesssim \|\Delta(Z_t)\|_2 \|\partial_{\alpha'} \Delta(Z_t)\|_2 \lesssim (\|Z_t\|_a + \|Z_t\|_b) \|\partial_{\alpha'} \Delta(Z_t)\|_2 \lesssim \mathcal{E}_\Delta(Z^\epsilon, Z^\epsilon)(t)^{1/2}
\]

This implies that \( \|\Delta(Z_t^\epsilon)\|_{L^\infty \cap H^2} \to 0 \) as \( \tau \to 0 \). By the same argument we also see that

\[
\|\Delta(Z_t)\|_{L^\infty \cap H^2} \to 0 \) as \( \tau \to 0 \). Now using Lemma [8.1] we observe that

\[
\|\Delta(D_{\alpha'}Z_t^\epsilon)\|_2 \lesssim \left\| \Delta \left( \frac{1}{Z_{\alpha'}} \right) \right\|_\infty + \|\Delta(Z_t)\|_2
\]

Therefore \( \|\Delta(D_{\alpha'}Z_t)\|_2 \to 0 \) as \( \tau \to 0 \). Now we recall the formula of \( A_1 \) from [8] which is \( A_1 = 1 - \text{Im}[Z_t, \mathbb{I}] \mathbb{Z}_{t,\alpha'} \). Hence by using Lemma [7.3] Lemma [8.1] Proposition [9.5] and Proposition [10.4] we see that

\[
\|\Delta(A_1)\|_2 \lesssim \|\Delta(Z_t)\|_{H^2} + \|\tilde{h}_{\alpha'} - 1\|_\infty + \|\partial_{\alpha'} \Delta(Z_t)\|_2
\]

This shows that \( \|\Delta(A_1)\|_2 \to 0 \) as \( \tau \to 0 \). Now applying \( \text{Re}(\mathbb{I} - \mathbb{I}) \) to the formula of \( b_{\alpha'} \) from [24] we see that

\[
b_{\alpha'} = \text{Re} \left\{ \left[ \frac{1}{Z_{\alpha'}}, \mathbb{I} \right] Z_{t,\alpha'} + 2D_{\alpha'}Z_t + [Z_t, \mathbb{I}] \left( \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\}
\]

Hence by using Lemma [7.3] Lemma [8.1] Proposition [9.5] and Proposition [10.4] we see that

\[
\|\Delta(b_{\alpha'})\|_2 \lesssim \left\| \Delta \left( \frac{1}{Z_{\alpha'}} \right) \right\|_{H^2} + \|\tilde{h}_{\alpha'} - 1\|_\infty + \|\partial_{\alpha'} \Delta(Z_t)\|_2 + \|\Delta(D_{\alpha'}Z_t)\|_2 + \|\Delta(Z_t)\|_{H^2}
\]

This shows that \( \|\Delta(b_{\alpha'})\|_2 \to 0 \) as \( \tau \to 0 \). Finally using Lemma [11.3] we see that

\[
\frac{d}{dt}\left( \|\Delta(h_{\alpha} \circ h^{-1})\|_2^2 \right) \lesssim \left( \|\Delta(h_{\alpha} \circ h^{-1})\|_2^2 + \|\Delta(h_{\alpha} \circ h^{-1})\|_2^2 \|\Delta(D_{\alpha})\|_2 \right) \|\Delta(h_{\alpha} \circ h^{-1})\|_2
\]

\[
\lesssim \left( \|\Delta(h_{\alpha} \circ h^{-1})\|_2^2 + \|\Delta(h_{\alpha} \circ h^{-1})\|_2^2 \|\Delta(h_{\alpha} \circ h^{-1})\|_2 \right) \|\Delta(b_{\alpha'})\|_2
\]

\[
\lesssim \left( \|\Delta(h_{\alpha} \circ h^{-1})\|_2^2 + \|\Delta(b_{\alpha'})\|_2^2 \right)
\]

Now by integrating the above inequality and using the fact that \( \|\Delta(b_{\alpha'})\|_2 \to 0 \) as \( \tau \to 0 \), we see that \( \|\Delta(h_{\alpha} \circ h^{-1})\|_2 \to 0 \) as \( \tau \to 0 \). Hence proved. \[\square\]
9. Appendix A

Here we collect the main identities and estimates proved in the appendix of [1] that we need in this paper. For the proof of the estimates we refer to the appendix of [1]. Let \( D_t = \partial_t + b\partial_\alpha \), where \( b \) is given by (8) and recall that \([f; g; h]\) is defined as

\[
[f_1; f_2; f_3](\alpha') = \frac{1}{i\pi} \int \left( \frac{f_1(\alpha') - f_1(\beta')}{\alpha' - \beta'} \right) \left( \frac{f_2(\alpha') - f_2(\beta')}{\alpha' - \beta'} \right) f_3(\beta') \, d\beta'.
\]

**Proposition 9.1.** Let \( f, g, h \in S(\mathbb{R}) \). Then we have the following identities

1. \( h\partial_\alpha [f, H]\partial_\alpha g = [h\partial_\alpha f, H]\partial_\alpha g + [f, H]\partial_\alpha (h\partial_\alpha g) - [h, f; \partial_\alpha g] \)
2. \( D_t[f, H]\partial_\alpha g = [D_t f, H]\partial_\alpha g + [f, H]\partial_\alpha (D_t g) - [b, f; \partial_\alpha g] \)

**Proof.** See [1] for a proof. \( \square \)

**Proposition 9.2.** Let \( H \in C^1(\mathbb{R}), A_i \in C^1(\mathbb{R}) \) for \( i = 1, \ldots, m \) and \( F \in C^\infty(\mathbb{R}) \). Define

\[
C_1(H, A, f)(x) = p.v. \int \frac{F(H(x) - H(y))}{x - y} \prod_{i=1}^m (A_i(x) - A_i(y)) \, f(y) \, dy
\]

\[
C_2(H, A, f)(x) = p.v. \int \frac{F(H(x) - H(y))}{x - y} \prod_{i=1}^m (A_i(x) - A_i(y)) \, \partial_y f(y) \, dy
\]

then there exists constants \( c_1, c_2, c_3, c_4 \) depending only on \( F \) and \( \|H'\|_\infty \) so that

1. \( \|C_1(H, A, f)\|_2 \lesssim c_1 \|A'_1\|_\infty \cdots \|A'_m\|_\infty \|f\|_2 \)
2. \( \|C_2(H, A, f)\|_2 \lesssim c_2 \|A'_1\|_2 \|A'_2\|_\infty \cdots \|A'_m\|_\infty \|f\|_2 \)
3. \( \|C_2(H, A, f)\|_2 \lesssim c_3 \|A'_1\|_\infty \cdots \|A'_m\|_\infty \|f\|_2 \)
4. \( \|C_2(H, A, f)\|_2 \lesssim c_4 \|A'_1\|_2 \|A'_2\|_\infty \cdots \|A'_m\|_\infty \|f\|_2 \)

**Proof.** See [1] for a proof. \( \square \)

**Proposition 9.3.** Let \( T : D(\mathbb{R}) \to D'(\mathbb{R}) \) be a linear operator with kernel \( K(x, y) \) such that on the open set \( \{(x, y) : x \neq y\} \subset \mathbb{R} \times \mathbb{R} \), \( K(x, y) \) is a function satisfying

\[
|K(x, y)| \leq \frac{C_0}{|x - y|} \quad \text{and} \quad |\nabla_x K(x, y)| \leq \frac{C_0}{|x - y|^2}
\]

where \( C_0 \) is a constant. If \( T \) is continuous on \( L^2(\mathbb{R}) \) with \( \|T\|_{L^2 \to L^2} \leq C_0 \) and if \( T(1) = 0 \), then \( T \) is bounded on \( H^s \) for \( 0 < s < 1 \) with \( \|T\|_{H^s \to H^s} \lesssim C_0 \)

**Proof.** See [1] for a proof. \( \square \)

**Proposition 9.4.** Let \( f \in S(\mathbb{R}) \). Then we have

1. \( \|f\| \lesssim \|f\|_{H^{\frac{1}{2}}} \) if \( s > \frac{1}{2} \) and for \( s = \frac{1}{2} \) we have \( \|f\|_{BMO} \lesssim \|f\|_{H^{\frac{1}{2}}} \)
2. \( \int \left| \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} \right|^2 \, d\beta' \lesssim \|f\|_2^2 \)
3. \( \sup_{\beta'} \left| \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} \right| \lesssim \|f\|_2 \)
4. \( \|f\|_{H^\frac{1}{2}}^2 = \frac{1}{2\pi} \int \int \left| \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} \right|^2 \, d\beta' \, d\alpha' \)
5. \( \left| \partial_\beta' \left( \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} \right) \right|_{L^2(\mathbb{R}^2, d\alpha' d\beta')} \lesssim \|f\|_{H^\frac{1}{2}} \)

**Proof.** See [1] for a proof. \( \square \)
Proof. See [1] for a proof.

Proposition 9.5. Let $f, g \in \mathcal{S}(\mathbb{R})$ with $s, a \in \mathbb{R}$ and $m, n \in \mathbb{Z}$. Then we have the following estimates

1. $\left\| \partial_{\alpha}^{|s|} f, H \right\|_{2} \lesssim \left\| \partial_{\alpha}^{|s+a|} g, H \right\|_{2}$ for $s \geq 0$
2. $\left\| \partial_{\alpha}^{|s+1|} f, H \right\|_{2} \lesssim \left\| \partial_{\alpha}^{|s+a|} g, H \right\|_{2}$ for $s \geq 0$ and $a > 0$
3. $\left\| \partial_{\alpha} \frac{1}{2} g \right\|_{2} \lesssim \left\| \partial_{\alpha} \frac{1}{2} f \right\|_{2}$
4. $\left\| f, \partial_{\alpha} \frac{1}{2} \right\|_{2} \lesssim \left\| \partial_{\alpha} \frac{1}{2} f \right\|_{2}$
5. $\left\| f, H \partial_{\alpha}^{|s|} g \right\|_{2} \lesssim \left\| \partial_{\alpha}^{|s+m|} f \right\|_{2}$ for $m, n \geq 0$
6. $\left\| \partial_{\alpha}^{|s+1|} f, H \partial_{\alpha}^{|s|} g \right\|_{2} \lesssim \left\| \partial_{\alpha}^{|s+m+n|} f \right\|_{2}$ for $m, n \geq 0$
7. $\left\| \partial_{\alpha}^{|s+1|} f, H \partial_{\alpha}^{|s|} g \right\|_{2} \lesssim \left\| \partial_{\alpha}^{|s+m+n|} f \right\|_{2}$ for $m \geq 0$ and $n \geq 1$
8. $\left\| f, H \partial_{\alpha}^{|s|} g \right\|_{2} \lesssim \left\| f \right\|_{2} \left\| g \right\|_{2}

Proof. See [1] for a proof.

Proposition 9.6. Let $f, g, h \in \mathcal{S}(\mathbb{R})$ with $s, a \in \mathbb{R}$ and $m, n \in \mathbb{Z}$. Then we have the following estimates

1. $\left\| \partial_{\alpha}^{|s|} (fg) \right\|_{2} \lesssim \left\| \partial_{\alpha}^{|s|} f \right\|_{2} \left\| g \right\|_{2} \left\| f \right\|_{\infty} \left\| \partial_{\alpha}^{|s|} g \right\|_{2}$ for $s \geq 0$
2. $\left\| \partial_{\alpha} \frac{1}{2} g \right\|_{2} \lesssim \left\| \partial_{\alpha} \frac{1}{2} f \right\|_{2} \left\| g \right\|_{2} \left\| f \right\|_{\infty} \left\| \partial_{\alpha} \frac{1}{2} g \right\|_{2}$
3. $\left\| f, H \right\|_{2} \lesssim \left\| f \right\|_{2} \left\| g \right\|_{2} \left\| f \right\|_{\infty} \left\| g \right\|_{2} \left\| H \right\|_{2}$

Proof. See [1] for a proof.

Proposition 9.7. Let $f, g, h \in \mathcal{S}(\mathbb{R})$. Then we have the following estimates

1. $\left\| f, g, h \right\|_{2} \lesssim \left\| f \right\|_{2} \left\| g \right\|_{2} \left\| h \right\|_{2}$
2. $\left\| \partial_{\alpha} \frac{1}{2} (fg) \right\|_{2} \lesssim \left\| \partial_{\alpha} \frac{1}{2} f \right\|_{2} \left\| g \right\|_{2} \left\| f \right\|_{\infty} \left\| \partial_{\alpha} \frac{1}{2} g \right\|_{2}$
3. $\left\| f, g, h \right\|_{2} \lesssim \left\| f \right\|_{2} \left\| g \right\|_{2} \left\| h \right\|_{2}$
4. $\left\| f, g, h \right\|_{2} \lesssim \left\| f \right\|_{2} \left\| g \right\|_{2} \left\| h \right\|_{2}$
5. $\left\| f, g, h \right\|_{2} \lesssim \left\| f \right\|_{2} \left\| g \right\|_{2} \left\| h \right\|_{2}$

Proof. See [1] for a proof.

Proposition 9.8. Let $f \in \mathcal{S}(\mathbb{R})$ and let $w$ be a smooth non-zero weight with $w, \frac{1}{w} \in L^{\infty}(\mathbb{R})$ and $w' \in L^{2}(\mathbb{R})$. Then

1. $\left\| f \right\|_{2} \lesssim \left\| \frac{1}{w} \right\|_{2} \left\| w(f') \right\|_{2}$
2. $\left\| f \right\|_{2} \lesssim \left\| \frac{1}{w} \right\|_{2} \left\| (w') \right\|_{2} + \left\| \frac{1}{w} \right\|_{2} \left\| w' \right\|_{2}^{2}$

Proof. See [1] for a proof.

Proposition 9.9. Let $f, g \in \mathcal{S}(\mathbb{R})$ and let $w, h \in L^{\infty}(\mathbb{R})$ be smooth functions with $w', h' \in L^{2}(\mathbb{R})$. Then

$$\left\| fwh \right\|_{2} \lesssim \left\| f \right\|_{2} \left\| w \right\|_{\infty} \left\| h \right\|_{2} + \left\| f \right\|_{2} \left\| (wh)' \right\|_{2} + \left\| f \right\|_{2} \left\| w' \right\|_{2} \left\| h \right\|_{\infty}$$

If in addition we assume that $w$ is real valued then

$$\left\| fgw \right\|_{2} \lesssim \left\| f \right\|_{2} \left\| w \right\|_{2} \left\| g \right\|_{2} + \left\| gw \right\|_{2} \left\| f \right\|_{2} + \left\| f \right\|_{2} \left\| g \right\|_{2} \left\| w' \right\|_{2} \left\| h \right\|_{\infty}$$

Proof. See [1] for a proof.
Lemma 9.10. Let $K_\epsilon$ be the Poisson kernel from [3]. If $f \in L^q(\mathbb{R})$, then for $s \geq 0$ an integer we have

$$
\| (\partial_{x,s}^\alpha f) * P_\epsilon \|_p \lesssim \| f \|_q e^{-s(\frac{1}{p} - \frac{1}{q})} \quad \text{for} \quad 1 \leq q \leq p \leq \infty
$$

Similarly for $s \in \mathbb{R}, s \geq 0$ we have

$$
\| (\partial_{x,s}^\alpha f) * P_\epsilon \|_p \lesssim \| f \|_q e^{-s(\frac{1}{p} - \frac{1}{q})} \quad \text{for} \quad 1 \leq q \leq p \leq \infty
$$

Proof. See [1] for a proof.

10. Appendix B

We now prove some estimates which are new. We first need some identities for holomorphic functions.

Proposition 10.1. Let $f \in \mathcal{S}(\mathbb{R})$ with $P_A f = 0$. Then we have the following identities

1. $(\mathbb{I} - H)D_{t,f} = (\mathbb{I} - H)(Z_t D_{\alpha^t} f)$
2. $(\mathbb{I} - H)D_{t,f}^2$

Proof. These identities essentially says that the material derivative of a holomorphic function remain essentially holomorphic. These identities are proved in [35], see Appendix B for the first identity and section 4 for the second identity.

Corollary 10.2. Let $H \in C^1(\mathbb{R}), A_i \in C^1(\mathbb{R})$ for $i = 1, \cdots m$ and let $\delta > 0$ be such that

$$
\delta \leq \left| \frac{H(x) - H(y)}{x - y} \right| \leq \frac{1}{\delta} \quad \text{for all} \quad x \neq y
$$

Let $0 \leq k \leq m + 1$ and define

$$
T(A, f)(x) = \text{p.v.} \int \frac{\prod_{i=1}^{m} (A_i(x) - A_i(y))}{(x - y)^{m+1-k}(H(x) - H(y))} f(y) dy
$$

then we have the estimates

1. $\|T(A, f)\|_2 \leq C(\|H'\|_\infty, \delta) \|A_1\|_\infty \cdots \|A_m\|_\infty \|f\|_2$
2. $\|T(A, f)\|_2 \leq C(\|H'\|_\infty, \delta) \|A_1\|_2 \cdots \|A_m\|_\infty \|f\|_\infty$

Proof. If $k = 0$, then the result follows directly from Proposition 0.2. If $k \geq 1$, we choose a smooth function $F$ with compact support such that $F(x) = 0$ if $|x| \leq \frac{\delta}{2}, F(x) = 0$ if $|x| \geq \frac{\delta}{2}$ and $F(x) = x^{-k}$ if $\delta \leq |x| \leq \frac{1}{\delta}$. The result now follows from Proposition 0.2.
We now prove some basic estimates for the operators defined in \[\mathcal{H}\]. Recall from (53) that

\[
(\mathcal{H}f)(\alpha') = \frac{1}{i\pi} p.v. \int \frac{\tilde{h}_{\beta'}(\beta')}{h(\alpha') - \tilde{h}(\beta')} f(\beta') d\beta'
\]

\[
(\tilde{\mathcal{H}}f)(\alpha') = \frac{1}{i\pi} p.v. \int \frac{1}{h(\alpha') - \tilde{h}(\beta')} f(\beta') d\beta'
\]

**Proposition 10.3.** Let \(\mathcal{H}, \tilde{\mathcal{H}}\) be defined as in (53) and let \(f \in \mathcal{S}(\mathbb{R})\). Assume that there is a constant \(L > 0\) so that

\[
\frac{1}{L} \leq \left| \frac{\tilde{h}(x) - \tilde{h}(y)}{x - y} \right| \leq L \quad \text{for all } x \neq y
\]

We will suppress the dependence of \(L\) i.e. we write \(a \lesssim b\) instead of \(a \leq C(L)b\). With this notation we have the following estimates

1. \(\|\mathcal{H}(f)\|_2 \lesssim \|f\|_2\) and \(\|\tilde{\mathcal{H}}(f)\|_2 \lesssim \|f\|_2\).
2. \(\|\mathcal{H}(f)\|_{\dot{H}^{1/2}} \lesssim \|f\|_{\dot{H}^{1/2}}\).

**Proof.** The proofs are quite straightforward

1. The estimate \(\|\tilde{\mathcal{H}}(f)\|_2 \lesssim \|f\|_2\) follows directly from Corollary 10.2. Hence we see that \(\|\mathcal{H}(f)\|_2 \lesssim \|\tilde{\mathcal{H}}(f)\|_2 \lesssim \|f\|_2\).
2. We see that \(\mathcal{H}(1) = 0\) and \(\|\mathcal{H}\|_{L^2 \to L^2} \lesssim 1\). The kernel of \(\mathcal{H}\) is

\[
K(\alpha', \beta') = \frac{\tilde{h}_{\beta'}(\beta')}{h(\alpha') - \tilde{h}(\beta')}
\]

From this we see that

\[
|K(\alpha', \beta')| \lesssim \frac{1}{|\alpha' - \beta'|} \quad \text{and} \quad |\nabla_{\alpha'} K(\alpha', \beta')| \lesssim \frac{1}{|\alpha' - \beta'|^2}
\]

Hence from Proposition 3.3 we obtain \(\|\mathcal{H}\|_{\dot{H}^{1/2} \to \dot{H}^{1/2}} \lesssim 1\).

We now prove the main estimate used to handle the difference of terms in the energy estimate of \(E_\Delta\). Recall that \([f_1, f_2; f_3]_{\tilde{h}}\) is defined as

\[
[f_1, f_2; f_3]_{\tilde{h}}(\alpha') = \frac{1}{i\pi} \int \left( \frac{f_1(\alpha') - f_1(\beta')}{h(\alpha') - h(\beta')} \right) \left( \frac{f_2(\alpha') - f_2(\beta')}{h(\alpha') - h(\beta')} \right) f_3(\beta') d\beta'
\]

**Proposition 10.4.** Let \(\mathbb{H}\) be the Hilbert transform and let \(\mathcal{H}, \tilde{\mathcal{H}}\) be defined as in (53) and let \(f, f_1, f_2, f_3, g \in \mathcal{S}(\mathbb{R})\). Assume that there is a constant \(L > 0\) so that

\[
\frac{1}{L} \leq \left| \frac{\tilde{h}(x) - \tilde{h}(y)}{x - y} \right| \leq L \quad \text{for all } x \neq y
\]

We will suppress the dependence of \(L\) i.e. we write \(a \lesssim b\) instead of \(a \leq C(L)b\). With this notation we have the following estimates

1. \(\|(\mathbb{H} - \mathcal{H})f\|_2 \lesssim \|\tilde{h}_{\alpha'} - 1\|_\infty \|f\|_2\)
2. \(\|(\mathbb{H} - \mathcal{H})f\|_{\dot{H}^{1/2}} \lesssim \|\tilde{h}_{\alpha'} - 1\|_\infty \|f\|_{\dot{H}^{1/2}}\)
3. \(\|[f, \mathbb{H} - \tilde{\mathcal{H}}]g\|_2 \lesssim \|\tilde{h}_{\alpha'} - 1\|_\infty \|f'\|_2 \|g\|_1\)

□
(4) \[ \|f, \mathbb{H} - \tilde{H} \|_2 \lesssim \|\tilde{h}_{\alpha'} - 1\|_\infty \|f\|_{\tilde{H}^\perp} \|g\|_2 \]
\[ (5) \|f, \mathbb{H} - \tilde{H} \|_{\partial\alpha' g} \lesssim \|\tilde{h}_{\alpha'} - 1\|_\infty \|f'\|_\infty \|g\|_2 \]
\[ (6) \|f, \mathbb{H} - \tilde{H} \|_{\partial\alpha' g} \lesssim \|\tilde{h}_{\alpha'} - 1\|_\infty \|f'\|_2 \|g\|_\infty \]
\[ (7) \|f, \mathbb{H} - \tilde{H} \|_{\partial\alpha' g} \lesssim \|\tilde{h}_{\alpha'} - 1\|_\infty \|f'\|_2 \|g\|_{\tilde{H}^\perp} \]
\[ (8) \|f, \mathbb{H} - \tilde{H} \|_{\partial\alpha' g} \lesssim \|\tilde{h}_{\alpha'} - 1\|_\infty \{\|f'\|_{\tilde{H}^\perp} \|g\|_2 + \|f'g\|_2\} \]
\[ (9) \|\partial_{\alpha'} [f_1, [f_2, \mathbb{H} - \tilde{H}]] \|_2 \lesssim \|\tilde{h}_{\alpha'} - 1\|_\infty \|f'\|_2 \|f'\|_2 \|f'\|_2 \]
\[ (10) \|f, \mathbb{H} - \tilde{H} \|_{L^\infty \cap \tilde{H}^\perp} \lesssim \|\tilde{h}_{\alpha'} - 1\|_\infty \|f'\|_2 \|g\|_2 \]
\[ (11) \|f, \mathbb{H} - \tilde{H} \|_{\tilde{H}^\perp} \lesssim \|\tilde{h}_{\alpha'} - 1\|_\infty \|f'\|_\infty \|g\|_{\tilde{H}^\perp} \]
\[ (12) \|f_1, f_2, f_3 \|_{L^\infty \cap \tilde{H}^\perp} \lesssim \|\tilde{h}_{\alpha'} - 1\|_\infty \|f_1\|_\infty \|f_2\|_\infty \|f_3\|_2 \]
\[ (13) \|f_1, f_2, f_3, \partial_{\alpha'} f_3 \|_2 \lesssim \|\tilde{h}_{\alpha'} - 1\|_\infty \|f_1\|_\infty \|f_2\|_\infty \|f_3\|_2 \]
\[ (14) \|f_1, f_2, \partial_{\alpha'} f_3 \|_{\tilde{H}^\perp} \lesssim \|\tilde{h}_{\alpha'} - 1\|_\infty \|f_1\|_\infty \|f_2\|_\infty \|f_3\|_2 \]

Proof. We first observe that from Proposition \[10.3\] we get \(\|\mathcal{H}(f)\|_2 \lesssim \|f\|_2\), \(\|\mathcal{H}(f)\|_{\tilde{H}^\perp} \lesssim \|f\|_{\tilde{H}^\perp}\) and \(\|\tilde{H}(f)\|_2 \lesssim \|f\|_2\). To simplify the calculations we define

\[ F(a, b) = \frac{f(a) - f(b)}{a - b} \quad \quad F_h(a, b) = \frac{f(a) - f(b)}{h(a) - h(b)} \]
\[ G(a, b) = \frac{g(a) - g(b)}{a - b} \quad \quad G_h(a, b) = \frac{g(a) - g(b)}{h(a) - h(b)} \]
\[ F_1(a, b) = \frac{f_1(a) - f_1(b)}{a - b} \quad \quad F_{h1}(a, b) = \frac{f_1(a) - f_1(b)}{h(a) - h(b)} \]
\[ H(a, b) = \frac{(h(a) - a) - (h(b) - b)}{a - b} \quad \quad H_{h}(a, b) = \frac{(h(a) - a) - (h(b) - b)}{h(a) - h(b)} \]

We have the identities

\[ \frac{F(\alpha', s) - F(\beta', s)}{\alpha' - \beta'} = \frac{F(\alpha', \beta') - F(\beta', s)}{\alpha' - s} \]
\[ \frac{H_h(\alpha', s) - H_h(\beta', s)}{\alpha' - \beta'} = \frac{1}{h(\alpha') - h(s)} \left\{ H(\alpha', \beta') - H_h(\beta', s) \left( \frac{\tilde{h}(\alpha') - \tilde{h}(\beta')}{\alpha' - \beta'} \right) \right\} \]

(1) We write \((\mathbb{H} - \tilde{H}) = (\mathbb{H} - \tilde{H}) + (\tilde{H} - \mathcal{H})\). Observe that

\( (\tilde{H} - \mathcal{H}) f = \tilde{H}((1 - \tilde{h}_{\alpha'}) f) \)

Hence as \(\tilde{H}\) is bounded on \(L^2\) we have \(\|(\tilde{H} - \mathcal{H}) f\|_2 \lesssim \|\tilde{h}_{\alpha'} - 1\|_\infty \|f\|_2\). Now we have

\[ ((\mathbb{H} - \tilde{H}) f)(\alpha') = \frac{1}{i\pi} \int \left( \frac{1}{\alpha' - \beta'} - \frac{1}{h(\alpha') - h(\beta')} \right) f(\beta') d\beta' \]
\[ = \frac{1}{i\pi} \int \frac{H_h(\alpha', \beta')}{\alpha' - \beta'} f(\beta') d\beta' \]

Now using Corollary \[10.2\] we see that \(\|(\mathbb{H} - \tilde{H}) f\|_2 \lesssim \|\tilde{h}' - 1\|_\infty \|f\|_2\). Hence the required estimate follows.

(2) Observe that \((\mathbb{H} - \tilde{H})(1) = 0\) and that the kernel of this operator is

\[ K(\alpha', \beta') = \frac{1}{\alpha' - \beta'} - \frac{\tilde{h}_{\beta'}(\beta')}{h(\alpha') - h(\beta')} \]

\]
Hence this kernel satisfies
\[ |K(\alpha', \beta')| \lesssim \frac{||\hat{h}_{\alpha'} - 1||_\infty}{|\alpha' - \beta'|} \quad |\nabla_{\alpha'} K(\alpha', \beta')| \lesssim \frac{||\hat{h}_{\alpha'} - 1||_\infty}{|\alpha' - \beta'|^2} \]
and by the first estimate of this proposition we also have \(|\|H - \mathcal{H}\|_{L^2 \rightarrow L^2} \lesssim ||\hat{h}_{\alpha'} - 1||_\infty|\). Hence by Proposition 10.3 we have boundedness on \(\dot{H}^{s \frac{1}{2}}\) with \(\|H - \mathcal{H}\|_{\dot{H}^{s \frac{1}{2}} \rightarrow \dot{H}^{s \frac{1}{2}}} \lesssim ||\hat{h}_{\alpha'} - 1||_\infty|\).

(3) Note that
\[ ([f, \mathbb{H} - \mathcal{H}]g)(\alpha') = \frac{1}{i \pi} \int F(\alpha', \beta')H_h(\alpha', \beta')g(\beta') \, d\beta' \]
and hence by Cauchy Schwartz we have
\[ ||[f, \mathbb{H} - \mathcal{H}]g||_2(\alpha') \lesssim ||\hat{h}_{\alpha'} - 1||_\infty \|g\|_1 \left( \int |F(\alpha', \beta')|^2 |g(\beta')| \, d\beta' \right)^{\frac{1}{2}} \]
The estimate now follows from Hardy’s inequality in Proposition 9.4.

(4) From the earlier computation we see that
\[ ([f, \mathbb{H} - \mathcal{H}]_\partial g)(\alpha') \lesssim ||\hat{h}_{\alpha'} - 1||_\infty \|g\|_2 \left( \int |F(\alpha', \beta')|^2 \, d\beta' \right)^{\frac{1}{2}} \]
We now obtain the estimate easily as \(\int \int |F(\alpha', \beta')|^2 \, d\beta' \, d\alpha' \lesssim ||f||^{\frac{1}{2}}_{\dot{H}^{s \frac{1}{2}}}\) from Proposition 9.4.

(5) We observe
\[
([f, \mathbb{H} - \mathcal{H}]_\partial g)(\alpha') \\
= \frac{1}{i \pi} \int F(\alpha', \beta')H_h(\alpha', \beta')g(\beta') \, d\beta' \\
= \frac{1}{i \pi} \int \frac{H_h(\alpha', \beta')}{\alpha' - \beta'} f_{\beta'}(\beta')g(\beta') \, d\beta' + \frac{1}{i \pi} \int \frac{F(\alpha', \beta')}{\hat{h}(\alpha') - \hat{h}(\beta')} (\hat{h}_{\beta'}(\beta') - 1)g(\beta') \, d\beta' \\
- \frac{1}{i \pi} \int \frac{F(\alpha', \beta')}{\alpha' - \beta'} H_h(\alpha', \beta')g(\beta') \, d\beta' - \frac{1}{i \pi} \int \frac{F(\alpha', \beta')H_h(\alpha', \beta')}{\hat{h}(\alpha') - \hat{h}(\beta')} \hat{h}_{\beta'}(\beta')g(\beta') \, d\beta'
\]
The estimate now follows from Corollary 10.2.

(6) This also follows from the computation above and Corollary 10.2.

(7) We see that
\[
([f, \mathbb{H} - \mathcal{H}]_\partial g)(\alpha') = \frac{1}{i \pi} \int F(\alpha', \beta')H_h(\alpha', \beta')g(\beta') \, d\beta' \\
= \frac{1}{i \pi} \int \partial_{\beta'}(F(\alpha', \beta')H_h(\alpha', \beta'))(g(\alpha') - g(\beta')) \, d\beta'
\]
Now as \(F(\alpha', \beta') = \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'}\), if the derivative falls on \(f\) then we can use estimate 4 of this proposition. All other terms are bounded point-wise by
\[ ||\hat{h}_{\alpha'} - 1||_\infty \int \left| \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} \right| \left| \frac{g(\alpha') - g(\beta')}{\alpha' - \beta'} \right| \, d\beta' \]
Now use Cauchy Schwartz and Hardy’s inequality from Proposition 9.4.
(8) We see that
\[
[f, H - \tilde{H}]\partial_{\alpha'} g
= \frac{1}{i\pi} \int F(\alpha', \beta') H_h(\alpha', \beta') g(\beta') d\beta'
- \frac{1}{i\pi} \int (\partial_\beta F(\alpha', \beta')) H_h(\alpha', \beta') g(\beta') d\beta'
- \frac{1}{i\pi} \int \{\partial_\beta H(\alpha', \beta')\} f_{3\beta'}(\beta') g(\beta') d\beta'
- \frac{1}{i\pi} \int \left(\frac{F(\alpha', \beta') - f_{3\beta'}(\beta')}{\alpha' - \beta'}\right) ((\alpha' - \beta')\partial_\beta h(\alpha', \beta')) g(\beta') d\beta'
\]

For the first term we use Cauchy Schwartz inequality along with Proposition 9.4. The second term is easily handled by Corollary 10.2. For the last term we observe that \( \partial_\beta F(\alpha', \beta') = \frac{F(\alpha', \beta') - f_{3\beta'}(\beta')}{\alpha' - \beta'} \) and then use Cauchy Schwartz inequality along with Proposition 9.4.

(9) We have
\[
\partial_{\alpha'} [f_1, [f_2, H - \tilde{H}]]\partial_{\alpha'} f_3
= \frac{1}{i\pi} \partial_{\alpha'} \int (f_1(\alpha') - f_1(\beta')) F_2(\alpha', \beta') H_h(\alpha', \beta') f_{3\beta'}(\beta') d\beta'
= f_1'(\alpha') \left( \frac{1}{i\pi} \int F_2(\alpha', \beta') H_h(\alpha', \beta') f_{3\beta'}(\beta') d\beta' \right)
+ f_2'(\alpha') \left( \frac{1}{i\pi} \int F_1(\alpha', \beta') H_h(\alpha', \beta') f_{3\beta'}(\beta') d\beta' \right)
+ \frac{1}{i\pi} \int F_1(\alpha', \beta') F_2(\alpha', \beta') \{ (\alpha' - \beta') \partial_\beta h(\alpha', \beta') - H_h(\alpha', \beta') \} f_{3\beta'}(\beta') d\beta'
\]

Each of the terms are now easily controlled by using Cauchy Schwartz inequality and using Proposition 9.4 and Proposition 9.7.

(10) Note that
\[
[f, H - \tilde{H}] g = \frac{1}{i\pi} \int F(\alpha', s) H_h(\alpha', s) g(s) ds
\]

Hence the \( L^\infty \) estimate follows immediately from Cauchy Schwartz inequality and Proposition 9.4. We now show the \( H^2 \) estimate by using identity 4 in Proposition 9.4. We see that
\[
\frac{(|f, H - \tilde{H}|g)(\alpha') - (|f, H - \tilde{H}|g)(\beta')}{\alpha' - \beta'}
= \frac{1}{i\pi} \int \frac{F(\alpha', s) H_h(\alpha', s) - F(\beta', s) H_h(\beta', s)}{\alpha' - \beta'} g(s) ds
= \frac{1}{i\pi} \int \frac{F(\alpha', s) - F(\beta', s)}{\alpha' - \beta'} H_h(\alpha', s) g(s) ds + \frac{1}{i\pi} \int \frac{H_h(\alpha', s) - H_h(\beta', s)}{\alpha' - \beta'} F(\beta', s) g(s) ds
\]
Now using the identities mentioned at the start of the proof of this proposition we get

\[
\frac{([f, \mathbb{H} - \widetilde{H}]g)(\alpha') - ([f, \mathbb{H} - \widetilde{H}]g)(\beta')}{\alpha' - \beta'}
\]

\[
= \frac{F(\alpha', \beta')}{i\pi} \int \frac{H_h(\alpha', s)}{\alpha' - s} g(s) \, ds - \frac{1}{i\pi} \int \frac{H_h(\alpha', s)}{\alpha' - s} F(\beta', s) g(s) \, ds
\]

\[+
\frac{H(\alpha', \beta')}{i\pi} \int \frac{F(\beta', s)}{h(\alpha') - h(s)} g(s) \, ds
\]

\[= \frac{1}{i\pi} \left( \frac{\tilde{h}(\alpha') - \tilde{h}(\beta')}{\alpha' - \beta'} \right) \int \frac{F(\beta', s)}{h(\alpha') - h(s)} H_h(\beta', s) g(s) \, ds
\]

\[= I + II + III + IV
\]

We can control each of the terms. The first term is controlled as

\[
\left\| \frac{F(\alpha', \beta')}{i\pi} \int \frac{H_h(\alpha', s)}{\alpha' - s} g(s) \, ds \right\|_{L^2(\mathbb{R} \times \mathbb{R}, d\alpha' \, d\beta')}
\]

\[
\lesssim \left\| \frac{F(\alpha', \beta')}{L^\infty(\mathbb{R})} \right\| \left\| \int \frac{H_h(\alpha', s)}{\alpha' - s} g(s) \, ds \right\|_{L^2(d\alpha')} \left\| \frac{L^2(d\beta')}{L^2(d\beta')} \right\|
\]

\[
\lesssim \| \tilde{h}_{\alpha'} - 1 \|_\infty \| \tilde{f}' \|_2 \| g \|_2
\]

where we used Corollary 10.2 and Proposition 9.4. For the second term we have

\[
\left\| \frac{1}{i\pi} \int \frac{H_h(\alpha', s)}{\alpha' - s} F(\beta', s) g(s) \, ds \right\|_{L^2(\mathbb{R} \times \mathbb{R}, d\alpha' \, d\beta')}
\]

\[
\lesssim \left\| \int \frac{H_h(\alpha', s)}{\alpha' - s} F(\beta', s) g(s) \, ds \right\|_{L^2(d\alpha')} \left\| \frac{L^2(d\beta')}{L^2(d\beta')} \right\|
\]

\[
\lesssim \| \tilde{h}_{\alpha'} - 1 \|_\infty \| g \|_2 \| F(\beta', \cdot) \|_{L^\infty} \left\| \frac{L^2(d\beta')}{L^2(d\beta')} \right\|
\]

\[
\lesssim \| \tilde{h}_{\alpha'} - 1 \|_\infty \| \tilde{f}' \|_2 \| g \|_2
\]

Similarly the third term is controlled as

\[
\left\| \frac{H(\alpha', \beta')}{i\pi} \int \frac{F(\beta', s)}{h(\alpha') - h(s)} g(s) \, ds \right\|_{L^2(\mathbb{R} \times \mathbb{R}, d\alpha' \, d\beta')}
\]

\[
\lesssim \| \tilde{h}_{\alpha'} - 1 \|_\infty \left\| \int \frac{F(\beta', s)}{h(\alpha') - h(s)} g(s) \, ds \right\|_{L^2(d\alpha')} \left\| \frac{L^2(d\beta')}{L^2(d\beta')} \right\|
\]

\[
\lesssim \| \tilde{h}_{\alpha'} - 1 \|_\infty \| g \|_2 \| F(\beta', \cdot) \|_{L^\infty} \left\| \frac{L^2(d\beta')}{L^2(d\beta')} \right\|
\]

\[
\lesssim \| \tilde{h}_{\alpha'} - 1 \|_\infty \| \tilde{f}' \|_2 \| g \|_2
\]

and the last term is controlled similarly to the second term. Hence we have the required estimate. (11) From estimate 5 of this proposition we see that the operator \( T : g \mapsto [f, \mathbb{H} - \widetilde{H}]\partial_{\alpha'}(g) \) is bounded on \( L^2 \) with \( \|T\|_{L^2 \to L^2} \lesssim \| \tilde{h}_{\alpha'} - 1 \|_\infty \| f' \|_\infty \) and that \( T(1) = 0 \). It is also easy to see that its kernel satisfies the conditions of Proposition 9.3 and hence the estimate follows.
This is proved in exactly the same way as we proved estimate 10 of this proposition. We leave the details to the reader.

Consider the operator $T : f_3 \mapsto [f_1, f_2; \partial_{\alpha'} f_3] - [f_1, f_2; \partial_{\alpha'} f_3]$. We observe that

$$T(f_3)(\alpha') = \frac{1}{\pi} \int \partial_{\beta'} \left\{ \frac{f_1(\alpha') - f_1(\beta')}{\alpha' - \beta'} \frac{f_2(\alpha') - f_2(\beta')}{\alpha' - \beta'} - \frac{f_1(\alpha') - f_1(\beta')}{h(\alpha') - h(\beta')} \frac{f_2(\alpha') - f_2(\beta')}{h(\alpha') - h(\beta')} \right\} f_3(\beta') d\beta'.$$

Now by repeated use of Corollary 11.2 we obtain $\|T\|_{L^2 \to L^2} \lesssim \|h_{\alpha'} - 1\|_{\infty} \|f_1\|_{\infty} \|f_2\|_{\infty}$.

We again consider the operator $T : f_3 \mapsto [f_1, f_2; \partial_{\alpha'} f_3] - [f_1, f_2; \partial_{\alpha'} f_3]$ and from the previous estimate we have $\|T\|_{L^2 \to L^2} \lesssim \|h_{\alpha'} - 1\|_{\infty} \|f_1\|_{\infty} \|f_2\|_{\infty}$. We observe that $T(1) = 0$ and it is also easy to see that its kernel satisfies the conditions of Proposition 1.3. Hence the required estimate follows.

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