Representation Theory for Default Logic

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Abstract

Default logic can be regarded as a mechanism to represent families of belief sets of a reasoning agent. As such, it is inherently second-order. In this paper, we study the problem of representability of a family of theories as the set of extensions of a default theory. We give a complete solution to the representability by means of normal default theories. We obtain partial results on representability by arbitrary default theories. We construct examples of denumerable families of non-including theories that are not representable. We also study the concept of equivalence between default theories.

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1 Introduction

In this paper we investigate the issues related to the expressibility of default logic, a knowledge representation formalism introduced by Reiter \cite{Rei80} and extensively investigated by the researchers of logical foundations of Artificial Intelligence \cite{Eth88, Bes89, Bre91}. A default theory $\Delta$ describes a family (possibly empty) of belief sets of an agent reasoning with $\Delta$. In that, default logic is inherently second-order, but in a sense different from that used by logicians. Whereas a logical theory $S$ describes a subset of the set of all formulas (specifically, the set $Cn(S)$ of logical consequences of $S$), a default theory $\Delta$ describes a collection of subsets of the set of all formulas, namely the family of all extensions of $\Delta$, $\text{ext}(\Delta)$. Hence, default theories can be viewed as encodings of families of subsets of some universe described by a propositional language. Examples of encodings of the problems of existence of colorings, kernels, and hamilton cycles in graphs are given in \cite{CMMT95}.

This second-order flavor of default logic makes it especially useful in knowledge representation. An important question, then, is to characterize those families of sets that can be represented as the set of extensions of a certain default theory. This is the topic of our paper.

There is a constraint on the family $\mathcal{T}$ of extensions of a default theory $\Delta$. Namely such family must be non-including \cite{Rei80}. In this paper we exhibit several classes of families of non-including theories that can be represented by default theories. We also show that there are non-representable families of non-including theories. The existential proof follows easily from a cardinality argument. There are continuum-many default theories in a given (denumerable) language, while there is more than continuum-many families of non-including theories. In the paper, we actually construct a family of non-including theories that is not represented by a default theory. Moreover, our family is denumerable (the cardinality argument mentioned above does not guarantee the existence of a non-representable denumerable family of non-including theories).

The family of extensions of a normal default theory is not only non-including, but all its members are pairwise inconsistent \cite{Rei80}. In this paper, we fully characterize these families of theories which are of the form $\text{ext}(\Delta)$, for a normal default theory $\Delta$. In addition, we construct examples of denumerable families of pairwise inconsistent theories which are not
Let \( T = \text{ext}(\Delta) \), for some default theory \( \Delta \). Clearly, there are other default theories \( \Delta' \) such that \( \text{ext}(\Delta') = T \). In other words, \( \Delta \) is not uniquely determined by \( T \). Thus, it is natural to search for alternative default theories \( \Delta' \) with the same set of extensions as \( \Delta \). Let us call \( \Delta' \) equivalent to \( \Delta \) if \( \text{ext}(\Delta) = \text{ext}(\Delta') \). We show that for every \( \Delta \) we can effectively (without constructing extensions of \( \Delta \)) find an equivalent theory \( \Delta' \) with all defaults in \( D' \) prerequisite-free (this result was obtained independently by Schaub [Sch92], and Bonatti and Eiter [BE95]). An important feature of our approach is that it shows that when \( \Delta \) is normal, we can construct a normal prerequisite-free default theory \( \Delta' \) equivalent to \( \Delta \).

We also present results that allow us to replace some normal theories \( \Delta \) with equivalent normal default theories of the form \((D', \emptyset)\). At present, it is an open problem to decide whether such replacement is possible for every normal default theory \( \Delta \) with \( W \) consistent.

We discuss yet another (weaker) form of equivalence and prove that every normal default theory is equivalent to a theory closely related to the closed world assumption over a certain set of atoms.

This paper sheds some light on the issue of expressibility of default logic and, in particular, on expressibility of normal default logic. We firmly believe that the success of default logic as a knowledge representation mechanism depends on a deeper understanding of expressibility issues.

## 2 Preliminaries

In this paper, by \( L \) we denote a language of propositional logic with a denumerable set of atoms \( A \). By a **theory** we always mean a subset of \( L \) *closed under propositional provability*. Let \( B \) be a set of standard monotone inference rules. The formal system obtained by extending propositional calculus with the rules from \( B \) will be denoted by \( PC + B \). The corresponding provability operator will be denoted by \( \vdash_B \) and the consequence operator by \( Cn^B(\cdot) \) [MT93].

A **default** is an expression \( d \) of the form \( \frac{\alpha}{\beta} \), where \( \alpha \) and \( \beta \) are formulas from \( L \) and \( \Gamma \) is a **finite** subset of \( L \). The formula \( \alpha \) is called the **prerequisite**, formulas in \( \Gamma \) — the
justifications, and β — the consequent of d. The prerequisite, the set of justifications and the consequent of a default d are denoted by p(d), j(d) and c(d), respectively. If p(d) is a tautology, d is called prerequisite-free (p(d) is then usually omitted from the notation of d). This terminology is naturally extended to a set of defaults D.

By a default theory we mean a pair ∆ = (D, W), where D is a set of defaults and W is a set of formulas, is called a default theory. A default theory ∆ = (D, W) is called finite if both D and W are finite. For a set of defaults D, define

\[ \text{Mon}(D) = \left\{ \frac{p(d)}{c(d)} : d \in D \right\}. \]

A default d (a set of defaults D) is applicable with respect to a theory S (is S-applicable) if S \( \not\vdash \neg \gamma \) for every \( \gamma \in j(d) \) (j(D), respectively). Let D be a set of defaults. By the reduct \( D_S \) of D with respect to S we mean the set of monotone inference rules:

\[ D_S = \text{Mon}(\{d \in D : d \text{ is S-applicable}\}). \]

A theory S is an extension\(^4\) of a default theory (D, W) if and only if

\[ S = \text{Cn}^{D_S}(W). \]

The family of all extensions of (D, W) is denoted by \( \text{ext}(D, W) \).

A family \( T \) of subsets of \( \mathcal{L} \) is non-including if:

1. each \( T \in T \) is closed under propositional consequence, and
2. \( T \) is an antichain, that is, for every \( T, T' \in T \), if \( T \subseteq T' \) then \( T = T' \).

Let S be a theory. A default d is generating for S if d is S-applicable and \( p(d) \in S \). The set of all defaults in D generating for S is denoted by GD(D, S). It is well-known\(^5\) that

(P1) If S is an extension of (D, W) then \[ S = \text{Cn}(W \cup c(GD(D, S))). \]

(P2) If all defaults in D are prerequisite-free then S is an extension of (D, W) if and only if \[ S = \text{Cn}(W \cup c(GD(D, S))). \]

\(^4\)Our definition of extension is different from but equivalent to the original definition by Reiter. See\(^6\) for details.
We will define now the key concepts of the paper.

**Definition 2.1** Default theories $\Delta$ and $\Delta'$ are *equivalent* if $\text{ext}(\Delta) = \text{ext}(\Delta')$.

**Definition 2.2** Let $\Delta$ be a default theory over a language $\mathcal{L}$ and let $\Delta'$ be a default theory over a language $\mathcal{L}'$ such that $\mathcal{L} \subseteq \mathcal{L}'$. Theory $\Delta$ is *semi-equivalent* to $\Delta'$ if $\text{ext}(\Delta) = \{T \cap \mathcal{L} : T \in \text{ext}(\Delta')\}$.

**Definition 2.3** Let $\mathcal{T}$ be a family of theories contained in $\mathcal{L}$. The family $\mathcal{T}$ is *representable* by a default theory $\Delta$ if $\text{ext}(\Delta) = \mathcal{T}$.

### 3 Default theories without normality restriction

We start with the result that allows us to replace any default theory with an equivalent default theory in which all defaults are prerequisite-free. As mentioned, this result was known before. However, our argument shows that if we start with a normal default theory, its prerequisite-free equivalent replacement can also be chosen to be normal.

**Theorem 3.1** For every default theory $\Delta$ there is a prerequisite-free default theory $\Delta'$ equivalent to $\Delta$. Moreover, if $\Delta$ is normal then $\Delta'$ can be chosen to be normal, too.

Proof: Let $\Delta = (D,W)$. By a *quasi-proof* from $D$ and $W$ we mean any proof from $W$ in the system $PC + \text{Mon}(D)$. For every quasi-proof $\epsilon$ from $D$ and $W$ let $D_\epsilon$ be the set of all defaults used in $\epsilon$. For each such proof $\epsilon$, define

$$d_\epsilon = \frac{j(D_\epsilon)}{\land \text{cons}(D_\epsilon)}.$$

Next, define

$$E = \{d_\epsilon : \epsilon \text{ is a quasi-proof from } W\}.$$

Each default in $E$ is prerequisite-free. Put $\Delta' = (E,W)$. We will show that $\Delta'$ has exactly the same extensions as $(D,W)$. To this end, we will show that for every theory $S$ and for every formula $\varphi$,

$$W \vdash_{D_S} \varphi \iff W \vdash_{E_S} \varphi.$$
Assume first that $W \vdash_{DS} \varphi$. Then, there is a quasi-proof $\epsilon$ of $\varphi$ such that all defaults in $D_\epsilon$ are applicable with respect to $S$. Moreover, $W \cup c(D_\epsilon) \vdash \varphi$. Observe that $c(d_\epsilon) \vdash c(D_\epsilon)$. Since $d_\epsilon$ is prerequisite-free and $S$-applicable, $W \vdash_{ES} W \cup c(D_\epsilon)$. Hence, $W \vdash_{ES} \varphi$.

To prove the converse implication, observe that since all defaults in $E$ are prerequisite-free,

$$\{ \varphi: W \vdash_{ES} \varphi \} = \text{Cn}(W \cup c(E_S)).$$

Hence, it is enough to show that

$$W \vdash_{DS} W \cup c(E_S).$$

Clearly, for every $\varphi \in W$, $W \vdash_{DS} \varphi$. Consider then $\varphi \in c(E_S)$. It follows that there is a quasi-proof $\epsilon$ such that $d_\epsilon$ is $S$-applicable and $c(d_\epsilon) = \varphi$. Consequently, all defaults occurring in $\epsilon$ are $S$-applicable. Thus, for every default $d \in D_\epsilon$,

$$W \vdash_{DS} c(d).$$

Since $\varphi = \bigwedge c(D_\epsilon)$,

$$W \vdash_{DS} \varphi.$$

To prove the claim for normal default theories, observe that if each default in $D$ is normal, then each default in $E$ is of the form

$$\frac{\Gamma}{\text{A} \Gamma}.$$

Let $\hat{E}$ be a set of defaults obtained from $E$ by replacing each default $\frac{\Gamma}{\text{A} \Gamma}$ by the normal default $\frac{\text{A} \Gamma}{\Gamma}$. It is easy to see that $S$ is an extension of $(E,W)$ if and only if $S$ is an extension of $(\hat{E},W)$. \qed

The next result fully characterizes families of theories representable by default theories with a finite set of defaults.

**Theorem 3.2** The following statements are equivalent:

(i) $T$ is representable by a default theory $(D,W)$ with finite $D$
(ii) $T$ is a finite set of non-including theories, finitely generated over the intersection of $T$

Proof: Assume (i). Since every extension of $(D, W)$ is of the form $Cn(W \cup c(D'))$, for some $D' \subseteq D$, it follows that $\text{ext}(D, W)$ is finite. It is also well-known ([Rei80, MT93]) that $\text{ext}(D, W)$ is non-including. Let $U$ be the intersection of all theories in $\text{ext}(D, W)$. Then $W \subseteq U$. Consequently, each extension in $\text{ext}(D, W)$ is of the form $Cn(U \cup c(D'))$. Hence, each extension is finitely generated over the intersection of $\text{ext}(D, W)$.

Now, assume (ii). Let $U$ be the intersection of all theories in $T$. It follows that there is a positive integer $k$ and formulas $\varphi_1, \ldots, \varphi_k$ such that $T = \{T_1, \ldots, T_k\}$ and each $T_i = Cn(U \cup \{\varphi_i\})$.

Assume first that $k = 1$. Then, it is evident that $T$ is the family of extensions of the default theory $(\emptyset, T_1)$. Hence, assume that $k \geq 2$. Since the theories in $T$ are non-including, for every $j \neq i$ we have

$$U \cup \{\varphi_i\} \not\models \varphi_j.$$  \hspace{1cm} (1)

In particular, each theory in $T$ is consistent and so is $U$. Moreover, it follows from (1) that for every $j = 1, \ldots, k$,

$$U \not\models \varphi_j.$$ \hspace{1cm} (2)

Define

$$d_i = \{\neg \neg \varphi_1, \ldots, \neg \neg \varphi_{i-1}, \neg \neg \varphi_{i+1}, \ldots, \neg \neg \varphi_k\} \cup \varphi_i,$$  

$i = 1, \ldots, k$. Next, define $D = \{d_1, \ldots, d_k\}$. We will show that $\text{ext}(D, U) = T$.

Let $T$ be an extension of $(D, U)$. Then, there is a subset $\Phi$ of $\{\varphi_1, \ldots, \varphi_k\}$ such that $T = Cn(U \cup \Phi)$. Assume that $|\Phi| = 0$. Then, by (2), $D_T = \{\varphi_i : i = 1, \ldots, k\}$. Consequently, $U = T = Cn^{Dr}(U) = Cn(U \cup \{\varphi_1, \ldots, \varphi_k\})$. Hence, for every $i$, $U \not\models \varphi_i$, a contradiction (with (2)). Hence, $|\Phi| > 0$. Assume that $|\Phi| > 1$. By the definition of $D$, $D_T = \emptyset$. Consequently, $T = Cn(U \cup \Phi) = Cn^{Dr}(U) = Cn(U)$. Let $\varphi \in \Phi$ (recall that $\Phi \neq \emptyset$). Then, $U \not\models \varphi$, a contradiction. Hence, every extension $T$ of $(D, W)$ is of the form $Cn(U \cup \{\varphi_i\})$ for some $i$, $1 \leq i \leq k$.

To complete the proof, consider an arbitrary $i$, $1 \leq i \leq k$. We will show that $T_i$ is an extension of $(D, W)$. First, observe that, by (1), $D_{T_i} = \{\varphi_i\}$. Consequently,
\[ \text{Cn}^{D_i}(U) = \text{Cn}(U \cup \{ \varphi_i \}) = T_i. \] Hence, \( T_i \) is an extension of \((D, U)\). \( \square \)

This result and its argument provide the following corollary which gives a complete characterization of families of theories representable by finite default theories, that is, theories \((D, W)\) with both \( D \) and \( W \) finite.

**Corollary 3.3** The following statements are equivalent:

1. \( T \) is representable by a finite default theory
2. \( T \) is a finite set of finitely generated non-including theories

As pointed out in the introduction, the cardinality argument implies the existence of non-representable families of non-including theories. However, it does not imply the existence of denumerable non-representable families. We will now show two examples of such families. The first family consists of non-including finitely generated theories. The second one consists of mutually inconsistent theories.

**Theorem 3.4** There exists a countable family of finitely generated non-including theories \( T \) such that \( T \) is not representable by a default theory.

Proof: Let \( \{ p_0, p_1, \ldots \} \) be a set of propositional atoms. Define \( T_i = \text{Cn}(\{ p_i \}), i = 0, 1, \ldots \), and \( T = \{ T_i; i = 0, 1, \ldots \} \). It is clear that \( T \) is countable and consists of non-including theories. We will show that \( T \) is not representable by a default theory.

Assume that \( T \) is represented by a default theory \((D, W)\). By Theorem 3.1, we may assume that all defaults in \( D \) are prerequisite-free. We can also assume that no default in \( D \) contains a justification which is contradictory (such defaults are never used to construct extensions).

Consider a default \( d \in D \). Since \( j(d) \) is finite, there is \( k \) such that for all \( m > k \), all formulas in \( j(d) \) are consistent with \( T_m \). Since \( T_m \) is an extension of \((D, W)\), \( c(d) \in T_m \), for \( m > k \). Since

\[ \bigcap_{m > k} T_m = \text{Cn}(\emptyset), \]

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c(d) is a tautology. Since d was arbitrary, it follows that (D, W) possesses only one extension, namely Cn(W), a contradiction.

\[ \square \]

**Theorem 3.5** There exists a countable family of mutually inconsistent theories \( T \) such that \( T \) is not representable by a default theory. In particular \( T \) is not representable by a normal default theory.

**Proof:** Let \( \{p_0, p_1, \ldots\} \) be a set of propositional atoms. Define

\[ T_i = Cn(\{\neg p_i, p_{i+1}, \ldots\}), \]

for \( i = 0, 1, \ldots \), and \( T = \{T_i: i = 0, 1, \ldots\} \). It is clear that \( T \) is countable and consists of pairwise inconsistent theories. Now, we apply precisely the same argument as in the proof of Theorem 3.4. \[ \square \]

Our counterexamples have an additional property that their infinite subsets and all supersets are also counterexamples.

Finally, we show that if infinite sets of justifications are allowed in defaults, every theory of non-including theories can be represented as the family of extensions.

4 Eliminating extensions

In this section, we consider the problem of representability of subfamilies of a representable family. We present a technique for constructing default theories representing some subfamilies of a family of extensions of a given default theory \( \Delta \). Such techniques are important when we have to redesign the default theory to exclude extensions containing a specific formula and preserve all the remaining extensions unchanged.

Let \( \varphi \in \mathcal{L} \). Define

\[ d_\varphi = \varphi : \bot. \]

**Theorem 4.1** Let \( E \subseteq \mathcal{L} \) be consistent and let \( (D, W) \) be a default theory. Then, \( E \) is an extension of \( (D \cup \{d_\varphi\}, W) \) if and only if \( \varphi \notin E \) and \( E \) is an extension of \( (D, W) \).
Proof: Since $E$ is consistent,

$$(D \cup \{d_\varphi\})_E = D_E \cup \{\varphi : \perp\}.$$  

Assume that $\varphi \notin E$ and that $E$ is an extension of $(D, W)$. Then

$$E = Cn^{D_E}(W)$$

and $\varphi \notin Cn^{D_E}(W)$. Consequently,

$$E = Cn^{D_E}(W) = Cn^{D_E \cup \{\varphi\}}(W) = Cn^{(D \cup \{d_\varphi\})_E}(W).$$

Hence, $E$ is an extension of $(D \cup \{d_\varphi\}, W)$.

Conversely, assume that $E$ is an extension of $(D \cup \{d_\varphi\}, W)$. Then,

$$E = Cn^{(D \cup \{d_\varphi\})_E}(W) = Cn^{D_E \cup \{\varphi\}}(W).$$

Since $E$ is consistent, it follows that $\varphi \notin Cn^{D_E}(W)$. Consequently,

$$Cn^{D_E}(W) = Cn^{D_E \cup \{\varphi\}}(W) = E.$$  

Hence, $\varphi \notin E$ and $E$ is an extension of $(D, W)$. \hfill \square

We say that a family $\mathcal{F}$ of theories closed under propositional consequence has a \textit{strong system of distinct representatives} (SSDR, for short) if for every $F \in \mathcal{F}$ there is a formula $\varphi_F \in F$ which does not belong to any other theory in $\mathcal{F}$.

\textbf{Theorem 4.2} If $\mathcal{F}$ is representable by a default theory and has an SSDR, then every family $\mathcal{G} \subseteq \mathcal{F}$ is representable by a default theory.

Proof: The claim is obvious if $\mathcal{F} = \{\mathcal{L}\}$. So, assume that all members of $\mathcal{F}$ are consistent (since $\mathcal{F}$ is an antichain, there are no other possibilities). Let $(D, W)$ be a default theory such that $Ext(D, W) = \mathcal{F}$. Define

$$\overline{D} = D' \cup \{d_{\varphi_F} : F \in \mathcal{F} \setminus \mathcal{G}\}.$$  

Since all theories in $\mathcal{F}$ are consistent, the assertion follows from the definition of an SSDR and from Theorem \[4.1\]. \hfill \square
Let us conclude this section with two observations. First, there are families of theories closed under propositional consequence which possess SSDRs but which are not representable by a default theory (the examples presented in the paper have this property). Second, not every subfamily of a representable family is representable. It follows by the cardinality argument from the the fact that there are representable families of cardinality continuum.

5 Normal default theories

Our first result in this section describes the family of extensions of an arbitrary prerequisite-free normal default theory.

**Theorem 5.1** Let \( W, \Psi \subseteq L \). Let \( D = \{ \frac{\varphi}{\neg \varphi} \mid \varphi \in \Psi \} \). If \( W \) is inconsistent then \( \text{ext}(D, W) = \{ L \} \). Otherwise, \( \text{ext}(D, W) \) is exactly the family of all theories of the form \( Cn(W \cup \Phi) \), where \( \Phi \) is a maximal subset of \( \Psi \) such that \( W \cup \Phi \) is consistent.

Proof: The case of inconsistent \( W \) is evident. Hence, let us assume that \( W \) is consistent. Let \( T \) be an extension of \( (D, W) \). Since \( W \) is consistent, \( T \) is consistent, too. Let \( \Phi = \{ \varphi \in \Psi : T \not\vdash \neg \varphi \} \). Clearly, \( \Phi = c(GD(D, T)) \). By (P2), \( T = Cn(W \cup \Phi) \). Moreover, since \( T \) is consistent, \( W \cup \Phi \) is consistent. We will show that \( \Phi \) is a maximal subset of \( \Psi \) with this property. Let \( \Phi' \) be such that \( \Phi \subseteq \Phi' \subseteq \Psi \). Assume that \( W \cup \Phi' \) is consistent. Then, \( T \cup \Phi' \) is consistent. Hence, \( \Phi' \subseteq \Phi \) and, consequently, \( \Phi = \Phi' \).

Assume next that \( T = Cn(W \cup \Phi) \), where \( \Phi \) is a maximal subset of \( \Psi \) such that \( W \cup \Phi \) is consistent. Then, it is easy to see that

\[
\frac{\varphi}{\neg \varphi} ; \varphi \in \Phi
\]

Hence, \( \Phi = c(GD(D, T)) \) and \( T = Cn(W \cup c(GD(D, T))) \). Since all defaults in \( D \) are prerequisite-free, it follows by the property (P2) that \( T \) is an extension of \( (D, W) \).

As a corollary, we obtain a full characterization of families of theories that are representable by normal default theories.
Corollary 5.2 A family \( T \) of theories in \( \mathcal{L} \) is representable by a normal default theory if and only if \( T = \{ \mathcal{L} \} \) or there is a consistent set of formulas \( W \) and a set of formulas \( \Psi \) such that \( T = \{ \text{Cn}(W \cup \Phi) : \Phi \subseteq \Psi \text{ and } \Phi \text{ is maximal so that } W \cup \Phi \text{ is consistent} \} \).

Proof: By Theorem 3.1, \( T \) is representable by a normal default theory if and only if it is representable by a normal default theory with all defaults prerequisite-free. Hence, the assertion follows from Theorem 5.1. \( \square \)

Corollary 5.3 A family of theories \( T \) is representable by a normal default theory with empty objective part if and only if there is a set of formulas \( \Psi \) such that \( T = \{ \text{Cn}(\Phi) : \Phi \text{ is maximal consistent subset of } \Psi \} \).

We will next study the issue of equivalence between normal default theories. We have already seen that we can replace any normal default theory with an equivalent normal prerequisite-free one (Theorem 3.1). The problem of interest now will be to establish when a normal default theory can be replaced by an equivalent normal default theory with empty objective part. We have only partial answers to this problem.

First, consider a normal default theory \((D,W)\) such that \( W \) is inconsistent. Then \( \text{ext}(D,W) = \{ \mathcal{L} \} \). On the other hand, for every set of normal defaults \( D' \), \( \text{ext}(D',\emptyset) \) contains only consistent extensions. Hence, \((D,W)\) is not equivalent to any normal default theory with empty objective part. From now on we will focus on normal default theories \((D,W)\) for which \( W \) is consistent.

Theorem 5.4 For every normal default theory \((D,W)\) with \( W \) consistent and finite there exists a prerequisite-free normal default theory \((D',\emptyset)\) equivalent to \((D,W)\).

Proof: By Theorem 3.1, without loss of generality we can assume that each default in \( D \) is prerequisite-free. Define \( \omega = \bigwedge W \).

First, assume that the justification of every default in \( D \) is inconsistent with \( \omega \). Then, \( \text{ext}(D,W) = \{ \text{Cn}(W) \} \). Let \( D' = \{ \frac{\omega}{\omega} \} \). Clearly, \( \text{ext}(D',\emptyset) = \text{ext}(D,W) \).

Hence, assume that there are defaults in \( D \) whose justifications are consistent with \( \omega \). For every default \( d = \frac{\beta}{\beta} \) in \( D \), define \( d_\omega = \frac{\beta \wedge \omega}{\beta \wedge \omega} \). Finally, define \( D' = \{ d_\omega : d \in D \} \). The
statement now follows from Theorem 5.1.

Next, we will study normal default theories of a special form. Let $P$ be a subset of the set of atoms of the language $L$. By a $P$-literal we mean an element of $P$ or the negation of an element of $P$. Define

$$D^{\text{COMP}} = \left\{ \frac{\neg q}{q} : \text{where } q \text{ is a } P\text{-literal} \right\}.$$  

A theory $T$ is $P$-complete if it contains $p$ or $\neg p$ for every $p \in P$. We will now describe theories representable by default theories of the form $(D^{\text{COMP}}, W)$.

**Proposition 5.5** For every consistent theory $W$, $T$ is an extension of $(D^{\text{COMP}}, W)$ if and only if a theory $T$ is inclusion-minimal among theories which are $P$-complete and contain $W$.

Proof: If $T$ is an extension of $(D^{\text{COMP}}, W)$ then $W \subseteq T$. In addition, since exactly one of every pair $\frac{q}{q}$ and $\frac{\neg q}{q}$ of defaults from $D^{\text{COMP}}$ is used when constructing $T$, $T$ is $P$-complete. Let $T' \subseteq T$ be $P$-complete and contain $W$. It follows that $T'$ has exactly the same $P$-literals as $T$. Hence, $T = Cn_{D^{\text{COMP}}}^T (W) \subseteq T'$ and, consequently, $T = T'$.

Conversely, let $T$ be inclusion-minimal $P$-complete theory such that $W \subseteq T$. Let $P$ be the set of all $P$-literals in $T$. Since $W$ is consistent, there exist $P$-complete, consistent theories containing $W$. Therefore all inclusion-minimal $P$-complete theories containing $W$ are consistent. Hence, for every atom $p \in P$, exactly one of $p$ and $\neg p$ belongs to $T_P$. It is now clear that $D^{\text{COMP}}_T = \left\{ \frac{\neg q}{q} : q \in T_P \right\}$. Hence, $T \supseteq Cn_{D^{\text{COMP}}}^T (W)$. Since $Cn_{D^{\text{COMP}}}^T (W)$ is $P$-complete and contains $W$, by minimality of $T$, $T = Cn_{D^{\text{COMP}}}^T (W)$. Consequently, $T$ is an extension of $(D^{\text{COMP}}, W)$. □

**Proposition 5.6** Let $P$ be a subset of the set of atoms of $L$. For every consistent $W$, the family $T_{W,P} = \{ T : T $ is inclusion-minimal among all $P$-complete theories containing $W \}$ is representable by a normal prerequisite-free default theory with empty-objective part.

Proof: Proposition 5.3 implies that we can represent $T_W$ by a default theory $(D^{\text{COMP}}, W)$. We will now construct another set of defaults $D$ such that $(D, \emptyset)$ represents $T_W$.  

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If the language $\mathcal{L}$ has only a finite number of atoms, we can assume that $W$ is finite. Hence, the assertion follows from Theorem 5.4. So assume that $\mathcal{L}$ has infinite number of atoms. Let $p_0, p_1, \ldots$ be an enumeration of atoms in $\mathcal{L}$. For the purpose of this proof we define, for an atom $p$, $0p = p$, $1p = \neg p$.

Consider a set $\text{tree}^W$ of all finite sequences $\langle \epsilon_0 p_0, \ldots, \epsilon_n p_n \rangle$ such that $W \cup \{ \epsilon_0 p_0, \ldots, \epsilon_n p_n \}$ is consistent.

The set $\text{tree}^W$ forms a tree. That is, if $\langle \epsilon_0 p_0, \ldots, \epsilon_n p_n \rangle$ belongs to $\text{tree}^W$ and $m < n$ then also $\langle \epsilon_0 p_0, \ldots, \epsilon_m p_m \rangle$ belongs to $\text{tree}^W$.

An infinite branch in $\text{tree}^W$ determines the infinite sequence $\langle \epsilon_0 p_0, \ldots \rangle$. Since for all $n$, $W \cup \{ \epsilon_0 p_0, \ldots, \epsilon_n p_n \}$ is consistent, $\text{Cn}(\{ \epsilon_0 p_0, \ldots, \epsilon_n p_n \})$ is also consistent. It is also complete and therefore, as $W$ is consistent with it, $W \subseteq \text{Cn}(\{ \epsilon_0 p_0, \ldots \}).$ Conversely, if $T$ is consistent and complete then there is a sequence $\langle \epsilon_0 p_0, \ldots \rangle$ such that $T = \text{Cn}(\{ \epsilon_0 p_0, \ldots \}).$ If $W \subseteq T$ then for all $n$, $W \cup \{ \epsilon_0 p_0, \ldots, \epsilon_n p_n \}$ is consistent. Thus we proved that there is a one-to-one correspondence between the branches in $\text{tree}^W$ and complete, consistent theories containing $W$.

Now define:

$$D = \left\{ \frac{\epsilon_0 p_0 \land \ldots \land \epsilon_n p_n}{\epsilon_0 p_0 \land \ldots \land \epsilon_n p_n} : \langle \epsilon_0 p_0, \ldots, \epsilon_n p_n \rangle \in \text{tree}^W \right\}$$

We show that the extensions of $(D, \emptyset)$ are precisely the theories of the form $\text{Cn}(\{ \epsilon_0 p_0, \ldots, \epsilon_n p_n \})$, where $\langle \epsilon_0 p_0, \ldots, \epsilon_n p_n, \ldots \rangle$ is an infinite branch through $\text{tree}^W$.

Indeed, if $\langle \epsilon_0 p_0, \ldots, \epsilon_n p_n, \ldots \rangle$ is an infinite branch through $\text{tree}^W$ then $T = \text{Cn}(\{ \epsilon_0 p_0, \ldots, \epsilon_n p_n, \ldots \})$ is a complete theory. The only default rules in $D$ that have conclusions in $T$ are the rules $\frac{\epsilon_0 p_0 \land \ldots \land \epsilon_n p_n}{\epsilon_0 p_0 \land \ldots \land \epsilon_n p_n}$ for $n \in N$. This implies that $\text{Cn}(\{ \epsilon_0 p_0, \ldots, \epsilon_n p_n, \ldots \})$ is an extension of $(D, \emptyset)$.

Conversely, if $T$ is an extension of $(D, \emptyset)$ then if $\frac{\epsilon_0 p_0 \land \ldots \land \epsilon_n p_n}{\epsilon_0 p_0 \land \ldots \land \epsilon_n p_n}$ is a generating rule for $T$ then for all $m < n$, $\frac{\epsilon_0 p_0 \land \ldots \land \epsilon_m p_m}{\epsilon_0 p_0 \land \ldots \land \epsilon_m p_m}$ is also a generating rule for $T$. Next, since $W = \emptyset$, $T$ must be consistent. This means that if two rules $\frac{\epsilon_0 p_0 \land \ldots \land \epsilon_m p_m}{\epsilon_0 p_0 \land \ldots \land \epsilon_m p_m}$ and $\frac{\delta_0 p_0 \land \ldots \land \delta_m p_m}{\delta_0 p_0 \land \ldots \land \delta_m p_m}$ are both generating for $T$ then $m \leq n$ and $\delta_0 = \epsilon_0, \ldots, \delta_m = \epsilon_m$ or $n \leq m$ and $\delta_0 = \epsilon_0, \ldots, \delta_n = \epsilon_n$. Thus, in order to complete our argument it is enough to show that the set of generating rules for $T$ is infinite. Assume otherwise. Then, there exists a sequence $\langle \epsilon_0 p_0, \ldots, \epsilon_n p_n \rangle$ such that $T = \text{Cn}(\{ \epsilon_0 p_0, \ldots, \epsilon_n p_n \})$. But recall that $\langle \epsilon_0 p_0, \ldots, \epsilon_n p_n \rangle \in \text{tree}^W$. Therefore
$W \cup \{\epsilon_0p_0, \ldots, \epsilon_np_n\}$ is consistent and so there is a complete theory $T'$ containing $W \cup \{\epsilon_0p_0, \ldots, \epsilon_np_n\}$. In particular $p_{n+1} \in T'$ or $\neg p_{n+1} \in T'$. Without loss of generality assume that $p_{n+1}$ (i.e. $0p_{n+1}$) belongs to $T$. Setting $\epsilon_{n+1} = 0$ we have $$\langle \epsilon_0p_0, \ldots, \epsilon_{n+1}p_{n+1} \rangle \in \text{tree}^W.$$ Hence, the default rule $\frac{\epsilon_0p_0 \land \cdots \land \epsilon_{n+1}p_{n+1}}{\neg p_{n+1}}$ belongs to $D$. Since $T \subseteq T'$, $p_{n+1}$ is consistent with $T$. Therefore $\epsilon_0p_0 \land \cdots \land \epsilon_{n+1}p_{n+1}$ is consistent with $T$ and thus $\epsilon_0p_0 \land \cdots \land \epsilon_{n+1}p_{n+1}$ belongs to $T$. In particular $p_{n+1} \in T$, a contradiction. Therefore the set of generating default rules for $T$ is infinite and determines a branch through $\text{tree}^W$. Thus $T$ is a complete theory containing $W$. This completes the proof. 

\section{6 Representability with Closed World Assumption}

Next, we explore the connections of normal default logic with the Closed World Assumption. Consider a set of atoms $P$. Define the set of defaults $D_{CWA}^P = \{ \neg p : p \in P \}$. Informally, a default $\frac{\neg p}{p}$ allows us to derive $\neg p$ if $p$ is not derivable. This has the flavor of the Closed World Assumption. The exact connection with CWA is given by the following result [MT93]: If $P = At$ then $W$ is CWA-consistent if and only if $(D_{CWA}^P, W)$ possesses a unique consistent extension.

**Theorem 6.1** For every normal default theory $(D, W)$ in $\mathcal{L}$ there exists a language $\mathcal{L}' \supseteq \mathcal{L}$, a set of atoms $P$ in $\mathcal{L}'$, and $W' \subseteq \mathcal{L}'$ such that $(D, W)$ is semi-equivalent to a default theory $(D_{CWA}^P, W')$.

Proof: By Theorem 3.1 we can assume that all defaults in $D$ are prerequisite-free. Let $\Psi$ be the set of consequences of defaults in $D$. For each $\psi \in \Psi$ select a new atom not belonging to $At$ (recall that $At$ is the set of atoms in $\mathcal{L}$). This atom is denoted by $p_\psi$ and the set $P$ is defined as $\{ p_\psi : \psi \in \Psi \}$. Define now $\mathcal{L}'$ to be the language generated by the set of atoms $At' = At \cup P$. Next, define $V$ as this set of formulas:

$$\{ \neg p_\psi : \psi : \psi \in \Psi \}.$$
We notice the following fact:

**(F1)** Let \(\Phi \subseteq \Psi\). Then \(W \cup \Phi\) is consistent if and only if \(W \cup V \cup \{\neg p_\psi : \psi \in \Phi\}\) is consistent.

Indeed, for a model \(v\) of \(W \cup \Phi\), define \(v'\) as follows:

\[
v'(p) = \begin{cases} 
v(p) & \text{if } p \in At \\
1 - v(\psi) & \text{if } p = p_\psi
\end{cases}
\]

It is clear that \(v'\) is a model of \(W \cup V \cup \{\neg p_\psi : \psi \in \Phi\}\). Conversely, when \(v'\), a valuation of \(At'\) is a model of \(W \cup V \cup \{\neg p_\psi : \psi \in \Phi\}\) then \(v = v'|_{At}\) is a model of \(W \cup \{\psi : \psi \in \Phi\}\).

Hence, (F1) follows.

Observation (F1) implies that \(\Phi\) is a maximal subset of \(\Psi\) consistent with \(W\) if and only if \(\{\neg p_\psi : \psi \in \Phi\}\) is a maximal subset of \(\{\neg p_\psi : \psi \in \Phi\}\) which is consistent with \(W \cup V\).

Next, observe that if \(\Phi\) is a maximal set of formulas contained in \(\Psi\) and consistent with \(W\) then for all \(\theta \in \Psi \setminus \Phi\)

\[W \cup V \cup \{\neg p_\psi : \psi \in \Phi\} \vdash p_\theta.\]

We are now ready to construct the desired default theory. We put \(W' = W \cup V\) and \(D' = \{\neg p_\psi : \psi \in \Psi\}\). Clearly, \(D' = CWA^P\). Using the observations listed above, we will now show that the theory \((D', W')\) semi-represents \(ext(D, W)\).

We will apply Theorem 4.21 of [MT93]. In order to apply this result, we need to show that whenever \(\Phi \subseteq \Psi\) is a maximal subset of \(\Psi\) consistent with \(W\) then \(T = Cn(W \cup V \cup \{\neg p_\psi : \psi \in \Phi\})\) is inclusion-minimal and \(\subseteq_P\)-minimal among theories that are complete with respect to \(P\). In addition, we need to show that, \(Cn(W \cup \Phi) = T \cap L\).

To this end, we proceed as follows. When \(\Phi\) is a maximal subset of \(\Psi\) consistent with \(W\) then, by the above fact, \(\{\neg p_\psi : \psi \in \Phi\}\) is a maximal subset of \(\{\neg p_\psi : \psi \in \Phi\}\) consistent with \(W \cup V\). Thus \(\{p_\phi : \phi \in \Psi \setminus \Phi\}\) is a minimal set of atoms in a theory \(T\) which contains \(W \cup V\).

Moreover, among such theories \(Cn(W \cup V \cup \{\neg p_\psi : \psi \in \Phi\})\) is clearly inclusion-minimal. Moreover, by the argument used in the proof of (F1) we have:

\[Cn(W \cup V \cup \{\neg p_\psi : \psi \in \Phi\}) \cap L = Cn(W \cup \Phi)\]

Thus the extensions of \((D, W)\) are restrictions to \(L\) of extensions of \((D', W')\). It is easy to see that such extension of \((D', W')\) is unique.
In order to apply Theorem 4.21 we also need to prove the converse statement, that is, that if \( T \) is complete with respect to \( P \), \( T \) contains \( W \cup V \), \( T \) is \( \sqsubseteq \)-minimal with these properties, and \( \Phi_T = \{ \psi : \psi \in \Psi \} \cap T \), then \( \Phi_T \) is consistent with \( W \) and is maximal among the subsets of \( \Phi \) with this property, \( T = Cn(W \cup V = \{ \neg p_\psi : \psi \in \Phi_T \}) \) and \( Cn(W \cup \Phi_T) = T \cap \mathcal{L} \). Since the argument is entirely analogous to the one used above, we omit the details.

\[ \square \]

7 Conclusions

The concepts studied in this paper, representability and equivalence, are of key importance for default logic and its applications. Representability provides insights into the expressive power of default logic, while equivalence provides normal form results for default logic, allowing the user to find simpler representations for his/her default theories.

In this paper we characterized those families of theories that can be represented by default theories with a finite set of defaults (Theorem 3.2 and Corollary 3.3). However, we have not found a characterization of families of theories that are representable by default theories with an infinite set of defaults. This problem seems to be much more difficult and remains open. In the paper, we present two countable families that are not representable and completely solved the representability problem if infinitary defaults are allowed.

We also studied representability by means of normal default theories. Here, our results are complete. Corollary 5.2 provides a full description of families of theories that are collections of extensions of normal default theories.

Another notion studied in the paper was equivalence of default theories. We showed (Theorem 3.1) that for every normal default theory there exists a normal default theory consisting of prerequisite-free defaults and having exactly the same extensions as the original one. We also exhibited some cases when, for a given normal default theory, an equivalent normal default theory can be found with empty objective part. Finding a complete description of normal default theories for which it is possible remains an open problem.
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