ON A NON-LINEAR SIZE-STRUCTURED POPULATION MODEL

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(Communicated by Yuan Lou)

ABSTRACT. This paper deals with a size-structured population model consisting of a quasi-linear first-order partial differential equation with nonlinear boundary condition. The existence and uniqueness of solutions are firstly obtained by transforming the system into an equivalent integral equation such that the corresponding integral operator forms a contraction. Furthermore, the existence of global attractor is established by proving the asymptotic smoothness and eventual compactness of the nonlinear semigroup associated with the solutions. Finally, we discuss the uniform persistence and existence of compact attractor contained inside the uniformly persistent set.

1. Introduction. Mathematical modeling for population growth has a long history dated back to the eighteenth century when the Malthus exponential growth model was proposed in [28]. The simplest model of population dynamics is based on the Malthusian law

\[
N'(t) = rN(t) \quad (r = \text{constant}),
\]

where \(N(t)\) is the population density at time \(t\) and \(r\) is the growth rate. This model (1.1) is inapplicable to situations where the population has a competition on resources (e.g., food and space), in these situations \(r\) should depend on the size of the population, i.e., the larger the population, the slower should be its rate of growth. In order to overcome the deficiency in the Malthusian law, Verhulst [37] considered the Logistic law

\[
N'(t) = rN(t) \left( 1 - \frac{N(t)}{K} \right) \quad (r, K = \text{constant}).
\]

The solution of (1.2) has been applied successfully to fit the growth curves of various types of populations. However, the above model yields no information whatsoever concerning the age distribution of the population. In fact, the birth and death
rates should be age-dependent. A model applicable to age-dependent population
dynamics was firstly proposed by McKendrick and Von Förster [30,38] by using a
standard linear first-order partial differential equation with boundary condition

\[
\frac{\partial n(t, a)}{\partial t} + \frac{\partial n(t, a)}{\partial a} = -\delta(a)n(t, a), \quad t > 0,
\]

\[
n(t, 0) = b \left(t, \int_0^\infty r(a)n(t, a)da\right), \quad t > 0,
\]

(1.3)

where \(n(t, a)\) is the population density of individuals of age \(a\) at time \(t\), \(\delta(a)\) is the
death rate of the death-modules and \(r(a)\) is the birth rate of the birth-modulus.
The model is often called age-structured model. It is a known result that age-
structured model can be reduced to delay differential equation (DDE) and can be
studied intensively by using the theory of DDE [3,7,11]. Naturally the delay is the
developmental time from birth to maturation and is a constant since the characteristics of (1.3) are family of straight lines, i.e., \(da/dt = 1\).

However, in reality, the maturation level of an individual is dependent on individ-
ual size, not age. For some species, the maturation occurs when an immature
individual accumulates enough of some quantity, such as length or weight. For ex-
ample, the metamorphic molt is actually triggered by the size of the larva and not
by chronological age in insects [8,10]. In this work, we consider size structure in a
population. Let \(n(t, s)\) is the population density of individuals of size \(s\) at time \(t\).

For some species, the rate of change of the size of an individual depends on
a number of factors such as temperature, food, space and intra- or inter-speci-
cific competition [12,15,31]. Competition for food is known to occur in most populations.
The immediate effect of competition for food among individuals is to slow down their
growth, i.e., the larger the population, the slower should be its rate of growth or
the longer should be its maturation period. In such sense, the growth rate of the
size changes with the quantity of the population

\[
\frac{ds}{dt} = k(N(t))
\]

where \(N(t) = \int_0^{+\infty} n(t, s)ds\) is the total population density at \(t\). It is natural to
assume that \(k(\cdot)\) is a decreasing function of \(N\) because an increase in their numbers
will slow down their growth. With these assumptions, a nonlinear size-structured
model is described by a quasi-linear first-order partial differential equation with
nonlinear boundary condition

\[
\frac{\partial n(t, s)}{\partial t} \left[k(N(t))\right] \frac{\partial n(t, s)}{\partial s} = -\delta(t, s, \int_0^{+\infty} \mu(t, s)n(t, s)ds)n(t, s),
\]

\[
k(N(t))n(t, 0) = b \left(\int_0^{+\infty} r(t, s), n(t, s)ds\right),
\]

\[
\lim_{s \to \infty} n(t, s) = 0.
\]

(1.4)

We also assume that \(k(\cdot)\) has a saturating effect in the limitation of food due to
density since the quantity of food available is shared in equal parts by all individuals
occupying the same habitat. Thus, the characteristics of (1.4) are family of curves.
How to study the system of a hyperbolic partial differential equation. Smith [33,34,35] gave an important method that the system can be reduced to a state-dependent delay differential equation (SDDE) and finally to a constant delay differential equation by considering different life states and a change of variables in terms of \( k(\cdot) \). By means of this method, various nonlinear size-structured population models arising in infection diseases [18,32], population growth [4,5,23] and cell production [2,27] can be transformed into differential equations with state-dependent delay or constant delay. Lv et al. in [19,20,21,22,23,24] considered some properties of solutions of some SDDEs models. However, there is lack of works which relate directly to the theory of size structured model. We will provide a rigorous mathematical framework to study the general model (1.4).

With the help of some existing approaches for investigating age structured models, we will analyze the size structured model by studying the nonlinear semigroup generated by the family of solutions. An important method is to use the theory of integrated semigroups [26,36], another method is to employ the integrating solutions along characteristics to obtain an equivalent integral equation [39]. For the first approach, the equation should be written as an abstract Cauchy problem with non-dense domain by enlarging the state space and defining a linear operator. However, the linear operator for the size structured model (1.4) can not be defined for us because the characteristics are family of curves, and thereby the integrated semigroup theory fails. This paper will develop the second method. The system (1.4) will be rewritten as an integral equation to establish the existence and uniqueness of solutions. In contrast to the age structured model, the integral form solution of (1.4) becomes far more complex and is quite difficult to study in light of the contraction mapping theorem. We will extend the work [39] to the size structured model and take a slightly different method combined with the properties of the characteristics \( k(\cdot) \). Furthermore, we will use fundamental principles, results from Hale on asymptotic smoothness [13], and compactness condition for \( L^p \) spaces to rigorously prove the existence and uniqueness of solutions, and eventual compactness of the nonlinear semigroup associated with the solution of (1.4). The global behavior is discussed by following a similar method to [9,26], and the uniform persistence is also investigated by using results from Hale and Waltman [14], and the existence of a compact global attractor contained inside the uniformly persistent set is obtained by using results from Magal and Zhao [25], and the stability of equilibrium is considered by constructing Lyapunov function.

The organization of this paper is follows. The model is formulated in the next Section through arguments. Then the existence, uniqueness and boundedness of an equivalent integral formulation are discussed in Section 3. The existence of global attractor is established in Section 4. The uniform persistence is investigated and the existence of compact attractor is obtained in Section 5. A brief discussion Section ends the paper.

2. Model derivation. We consider a variable \( s \) that could denote the size of an individual. Rather than specifying that the developmental level of an individual reaches a certain age, we specify that it reaches a certain size. Authors in [6] showed that the developmental level for any individual is subject to the amount food which has eaten in the past time. The growth rate has a density-dependence effect since all individuals compete for food resources, thereby slowing the growth of each individual. Based on this, we assume that the growth rate of the size of an individual is affected by the total number \( N(t) \) of the population, and is described
by
\[ s'(t) = k(N(t)). \] (2.1)

In the context of fish modelling in [6], it gave the form of the function \( k(N_i(t)) = K_1/(N_i(t) + C_1) \) where \( K_1 \) is the quantity of food entering into the species habitat per unit of place, per unit of time, \( C_1 \) stands for the food consumed by individuals of other species. Based on laboratory experiments in [29], they measured the development rate of the zooplankton \( Daphnia \) as a function \( F \), the amount of available food, and found that the development rate is proportional to \( F/(F + F_{half}) \). Similarly, we consider the general case of \( k(N_i) \) and make the following assumption.

\( (A1) \) Function \( k : R^+ \to (0, +\infty) \) is Lipschitz continuous and continuously differentiable with \( k(N) > 0 \) and \( k'(N) \leq 0 \) for \( N \in R^+ \), and \( \lim_{N \to +\infty} k(N) = 0, k(0) = k_0 > 0 \).

The decreasing function in \( N \) means that (2.1) models the competition for food among individuals. The saturating effect is considered in the case of low density, \( k(0) = k_0 > 0 \), i.e. the quantity of food flowing into the species habitat per unit of volume is limited.

The monotone increasing function \( s \) (the positivity of \( k \) shows this) can be viewed as a “physiological age” function, since the development level of an individual is dependent on its size. The state variable \( n(t, s) \) is the population density of individuals at time \( t \) in the “physiological age”-interval \((s, s + ds)\). If \( t \) is increased by \( h \) units, the individual size by \( h \) units; thus
\[ s(t + h) - s(t) = s'(t + \theta h)h = k(N(t + \theta h))h \]
for \( \theta \in (0, 1) \), and
\[ Dn(t, s) = \lim_{h \to 0} \frac{n(t + h, s(t + h)) - n(t, s(t + h)) + n(t, s(t + h)) - n(t, s)}{h} \]
is the rate at which the population is changing in time. This rate plus the death rate
\[ \delta \left( t, s, \int_0^{+\infty} \mu(t, s)n(t, s)ds \right) \]
with weight function \( \mu(t, s) \)
must equal to zero, i.e.,
\[ Dn(t, s) + \delta \left( t, s, \int_0^{+\infty} \mu(t, s)n(t, s)ds \right) n(t, s) = 0. \]
If \( n \) is continuously differentiable, then
\[ Dn(t, s) = \lim_{h \to 0} \frac{n(t + h, s(t + h)) - n(t, s(t + h)) + n(t, s + s'(t + \theta h)h) - n(t, s)}{h} \]
\[ = \frac{\partial n}{\partial t} + \frac{\partial n}{\partial s} \lim_{h \to 0} \frac{s'(t + \theta h)h}{h} \]
\[ = \frac{\partial n}{\partial t} + \frac{ds}{dt} \frac{\partial n}{\partial s} = \frac{\partial n}{\partial t} + k(N(t)) \frac{\partial n}{\partial s} \]
and
\[ \frac{\partial n}{\partial t} + k(N(t)) \frac{\partial n}{\partial s} = -\delta \left( t, s, \int_0^{+\infty} \mu(t, s)n(t, s)ds \right) n(t, s). \] (2.2)
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The function \( n \) is not necessarily continuously differentiable (and sometimes not even continuous), and so this identification is not always possible. It is always possible however, to interpret \( Dn(t, s) \) at the (classical) directional derivative alone the direction vector \((1, k(N))\).

The following assumption is natural given by

\[
\lim_{s \to +\infty} n(t, s) = 0, \text{ for } t \geq 0. \tag{2.3}
\]

Here, the birth law

\[
k(N(t))n(t, 0) = b \left( \int_0^{+\infty} r(t, s)n(t, s) ds \right) \tag{2.4}
\]

is dependent on “physiological age” \( s \) and population density with a weight function \( r(t, s) \). Furthermore, initial conditions are given by nonnegative \( L^1[0, +\infty) \) function \( n(s, 0) \) such that

\[
n(0, s) = n_0(s), \quad s \geq 0. \tag{2.5}
\]

The condition \( n(0, s) = 0 \) corresponds to the assumption that no individuals is present at the beginning of time. The boundary and initial conditions imply that

\[
n_0(0) = \frac{b \left( \int_0^{+\infty} r(0, s)n_0(s)ds \right)}{k(N(0))}.
\]

The following assumptions are natural given.

\textbf{A2} Both the death rate \( \delta(t, s, x) \) and birth rate \( b(x) \) are non-negative and Lipschitzian functions with respect of \( x \) variable with Lipschitzian constants \( L_\delta \) and \( L_b \) respectively, and there exists a positive constant \( \delta_m \) such that \( \delta(t, s, x) > \delta_m > 0 \) for all \( t, x \geq 0 \) and \( s > 0 \). Naturally, \( b(0) = 0 \), and \( b(x) \leq L_b x \).

\textbf{A3} Both weight functions \( \mu(t, \cdot) \) and \( r(t, \cdot) \) are non-negative in \( L^1[0, +\infty) \).

3. Existence and non-negativeness of solutions. In this section, the uniqueness and existence of solutions are studied by using the contraction mapping theorem for the integral form solution of (2.2-2.5).

Firstly, we give the integral form solution through the method of integration along characteristics. Let \( \Gamma(t) = \int_0^t k(N(\sigma)) d\sigma \). Then \( \Gamma'(t) = k(N(t)) > 0 \) and \( \Gamma^{-1} \) is the inverse of \( \Gamma \). Fix \( h \) and let \( w(h) = n(\Gamma^{-1}(\Gamma(t) + h), s + h) \) along almost every characteristic curve \( h = s - \Gamma(t) \). It is easy to verify that

\[
\frac{d}{dh} \Gamma^{-1}(\Gamma(t) + h) = \frac{1}{k(N(\Gamma^{-1}(\Gamma(t) + h)))}.
\]

According to the method of characteristic, the solution \( w(h) \) is given by

\[
\frac{d}{dh} w(h) = \frac{d}{dh} n(\Gamma^{-1}(\Gamma(t) + h), s + h) \nonumber = \frac{1}{k(N(\Gamma^{-1}(\Gamma(t) + h)))} \left( \frac{\partial}{\partial t} n(\Gamma^{-1}(\Gamma(t) + h), s + h) + \frac{\partial}{\partial s} n(\Gamma^{-1}(\Gamma(t) + h), s + h) \right) - \delta \left( \int_0^{+\infty} \mu(\Gamma^{-1}(\Gamma(t) + h), r) n(\Gamma^{-1}(\Gamma(t) + h), r) dr \right) \frac{1}{k(N(\Gamma^{-1}(\Gamma(t) + h)))} w(h).
\]
Solves the above equation and yields
\[
w(0) = w(h) \exp \left( \int_0^h \frac{1}{k(N(\Gamma^{-1}(\Gamma(t) + \tau)))} \delta \left( \Gamma^{-1}(\Gamma(t) + \tau), s + \tau \right) \, d\tau \right)
\]

\[
= w(h) \exp \left( \int_0^h \delta \left( \Gamma^{-1}(\Gamma(t) + \tau), s + \tau \right) \, d\tau \right)
\]

\[
= w(h) \exp \left( \int_0^h \mu(\Gamma^{-1}(\Gamma(t) + \tau), p) n(\Gamma^{-1}(\Gamma(t) + \tau), p) \, dp \, d\tau \right)
\]

\[
= w(h) \exp \left( \int_0^h \int_0^{+\infty} \mu(\Gamma^{-1}(\Gamma(t) + \tau), p) n(\Gamma^{-1}(\Gamma(t) + \tau), p) \, d\Gamma^{-1}(\Gamma(t) + \tau) \right)
\]

\[
= w(h) \exp \left( \int_0^h \Gamma^{-1}(\Gamma(t) + \tau) \, d\delta \left( \xi, s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(\xi, p) n(\xi, p) \, dp \right) \right).
\]

If \( s > \Gamma(t) \), setting \( h = -\Gamma(t) \), and if \( s < \Gamma(t) \), setting \( h = -s \), respectively, then the solution of (2.2-2.5) is given by

\[
n(t, s) = \frac{b}{k} \left( \int_0^{+\infty} r(\Gamma^{-1}(\Gamma(t) - s), p) n(\Gamma^{-1}(\Gamma(t) - s), p) \, dp \right) 1_{s < \Gamma(t)}
\]

\[
\exp \left( - \int_0^t \int_{\Gamma^{-1}(\Gamma(t) - s)}^{+\infty} \delta \left( \xi, s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(\xi, p) n(\xi, p) \, dp \right) \, d\xi \right) + 1_{s \geq \Gamma(t)}
\]

\[
n_0 (s - \Gamma(t)) \exp \left( - \int_0^t \delta \left( \xi, s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(\xi, p) n(\xi, p) \, dp \right) \, ds \right) \quad (3.1)
\]

where

\[
1_{s < \Gamma(t)} = 1, \quad 0 < s < \Gamma(t) \quad 1_{s \geq \Gamma(t)} = 0, \quad 0 < s < \Gamma(t) \quad 1_{s < \Gamma(t)} = 0, \quad s > \Gamma(t) \quad 1_{s \geq \Gamma(t)} = 1, \quad s > \Gamma(t).
\]

Based on the classical approach in [9,16], we now discuss the existence and uniqueness of the solution of (3.1), and hence to the system (2.2-2.5).

**Theorem 3.1.** For \( n_0(.) \in L^1([0, +\infty]) \), there exists \( \varepsilon > 0 \) and an open neighborhood \( B_0 \in L^1([0, +\infty]) \) with \( n_0 \in B_0 \) such that there exists a unique continuous function \( \psi : [0, \varepsilon] \times B_0 \to L^1([0, +\infty]) \) where \( \psi(t, s) \) is the solution of (3.1) with \( \psi(0, l) = l \) for \( l \in B_0 \).

**Proof.** Let \( Y := C([0, \varepsilon] \times B_0, L^1([0, +\infty])) \) be the set of all continuous functions from \([0, \varepsilon] \times B_0 \) to \( L^1([0, +\infty]) \) with the norm \( \| \cdot \|_Y \) defined by

\[
\| \psi \|_Y = \sup_{t \in [0, \varepsilon]} \int_0^{+\infty} |\psi(t, l)(s)| \, ds
\]

where \( \varepsilon > 0 \) and \( B_0 \subset L^1([0, +\infty]) \) is a neighborhood containing \( n_0 \), which are to be determined. Let \( B = \overline{B(n_0(\cdot), r)} \) be the closed ball of radius \( r \) centered around the initial function where the value of \( r \) will be determined later. Define \( B^* \subset Y \) by
functions whose ranges lies in $B \subset L^1[0, +\infty)$. Then, $B^*$ is a closed subset of the complete metric space $Y$. Consider the following operator $\Lambda$ on $B^*$ for any $l(\cdot) \in B_0$ and $\eta(t, l(\cdot)) \in B^*$,
\[
\Lambda(\eta)(t, l) = \frac{b \left( \int_0^{+\infty} r(\Gamma^{-1}(\Gamma(t) - s), p)\eta(\Gamma^{-1}(\Gamma(t) - s), l(p))dp \right)}{k \left( \int_0^{+\infty} \eta(\Gamma^{-1}(\Gamma(t) - s), l(p))dp \right)} 1_{s \in \Gamma(t)}
\exp \left( -\int_t^s \delta \left( \xi, s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(\xi, p)\eta(\xi, l(p))dp \right) d\xi \right) 1_{s \geq \Gamma(t)}
+ l(s - \Gamma(t)) \exp \left( -\int_0^s \delta \left( \xi, s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(\xi, p)\eta(\xi, l(p))dp \right) d\xi \right).
\]

We now prove the operator $\Lambda$ has a fixed point by the following three steps.

For any $\eta \in B^*$, we firstly show $\Lambda(\eta) \in Y$. It follows that
\[
\int_0^1 |\Lambda(\eta)(t, l)(s)| ds
= \int_0^1 \left| \frac{b \left( \int_0^{+\infty} r(\Gamma^{-1}(\Gamma(t) - s), p)\eta(\Gamma^{-1}(\Gamma(t) - s), l(p))dp \right)}{k \left( \int_0^{+\infty} \eta(\Gamma^{-1}(\Gamma(t) - s), l(p))dp \right)} 1_{s \in \Gamma(t)}
\exp \left( -\int_t^s \delta \left( \xi, s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(\xi, p)\eta(\xi, l(p))dp \right) d\xi \right) 1_{s \geq \Gamma(t)}
+ l(s - \Gamma(t)) \exp \left( -\int_0^s \delta \left( \xi, s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(\xi, p)\eta(\xi, l(p))dp \right) d\xi \right) \right| ds
\leq \int_0^{\Gamma(t)} \frac{b_M e^{-\delta_m(t - \Gamma^{-1}(\Gamma(t) - s))}}{k \left( \int_0^{+\infty} \eta(\Gamma^{-1}(\Gamma(t) - s), l(p))dp \right)} ds + \int_0^{+\infty} e^{-\delta_m t} |l(s - \Gamma(t))| ds
\leq \int_0^{\Gamma(t)} -b_M e^{-\delta_m(t - \Gamma^{-1}(\Gamma(t) - s))} d\Gamma^{-1}(\Gamma(t) - s) + \int_0^{+\infty} |l(s - \Gamma(t))| ds
\leq \int_t^0 -b_M e^{-\delta_m(t-s)} ds + \int_0^{+\infty} |l(s - \Gamma(t))| ds
\leq b_M \frac{1 - e^{-\delta_m t}}{\delta_m} + \|l\| < +\infty
\]
for all $t \in [0, \varepsilon]$, where $\varepsilon$ is sufficiently small number, $\| \cdot \|$ is defined by $\int_0^{+\infty} |x(s)| ds$, $b_M$ is the maximal number of $b \left( \int_0^{+\infty} r(t, s)\eta(t, l(s))ds \right)$ on $B_0$ since the birth function is continuous in the closed region. Thus, $\Lambda(\eta) \in Y$ for any $\eta \in B^*$.

For any $\eta \in B^*$, we secondly prove $\Lambda(\eta) \in B^*$, i.e., $\Lambda : B^* \to B^*$. Set $B_0 = B(l^*, r/2)$. Then
\[
\left\| \Lambda(\eta)(t, l) - l \right\| = \int_0^{+\infty} \left| \frac{b \left( \int_0^{+\infty} r(\Gamma^{-1}(\Gamma(t) - s), p)\eta(\Gamma^{-1}(\Gamma(t) - s), l(p))dp \right)}{k \left( \int_0^{+\infty} \eta(\Gamma^{-1}(\Gamma(t) - s), l(p))dp \right)}
\exp \left( -\int_t^s \delta \left( \xi, s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(\xi, p)\eta(\xi, l(p))dp \right) d\xi \right) 1_{s \in \Gamma(t)} \right| ds
\leq \int_0^{\Gamma(t)} \frac{b_M e^{-\delta_m(t - \Gamma^{-1}(\Gamma(t) - s))}}{k \left( \int_0^{+\infty} \eta(\Gamma^{-1}(\Gamma(t) - s), l(p))dp \right)} ds + \int_0^{+\infty} e^{-\delta_m t} |l(s - \Gamma(t))| ds
\leq \int_0^{\Gamma(t)} -b_M e^{-\delta_m(t - \Gamma^{-1}(\Gamma(t) - s))} d\Gamma^{-1}(\Gamma(t) - s) + \int_0^{+\infty} |l(s - \Gamma(t))| ds
\leq \int_t^0 -b_M e^{-\delta_m(t-s)} ds + \int_0^{+\infty} |l(s - \Gamma(t))| ds
\leq b_M \frac{1 - e^{-\delta_m t}}{\delta_m} + \|l\| < +\infty
\]
For all 

\[ + l (s - \Gamma(t)) 1_{s \geq \Gamma(t)} \]

\[ \exp \left( - \int_0^t \delta \left( \xi, s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(\xi, p) \eta(\xi, l)(p) \, dp \right) \, d\xi \right) - l' (s) \right| ds \]

\[ \leq b_M \frac{1 - e^{-\delta_m t}}{\delta_m} + \int_0^{+\infty} \left| l (s - \Gamma(t)) 1_{s \geq \Gamma(t)} \right| ds \]

\[ \exp \left( - \int_0^t \delta \left( \xi, s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(\xi, p) \eta(\xi, l)(p) \, dp \right) \, d\xi \right) - l' (s) \right| ds \]

\[ \leq b_M \frac{1 - e^{-\delta_m t}}{\delta_m} + \int_0^{+\infty} 1_{s \geq \Gamma(t)} \left| l (s - \Gamma(t)) - l' (s - \Gamma(t)) \right| \]

\[ \exp \left( - \int_0^t \delta \left( \xi, s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(\xi, p) \eta(\xi, l)(p) \, dp \right) \, d\xi \right) - l' (s) \right| ds \]

\[ + \int_0^{+\infty} \left| l' (s - \Gamma(t)) 1_{s \geq \Gamma(t)} \right| ds \]

\[ \exp \left( - \int_0^t \delta \left( \xi, s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(\xi, p) \eta(\xi, l)(p) \, dp \right) \, d\xi \right) - l' (s) \right| ds \]

For all \( t \in [0, \varepsilon] \), we can choose \( \varepsilon \) sufficiently small that \( b_M \left( 1 - e^{-\delta_m t} \right) \delta_m^{-1} \leq \frac{\varepsilon}{4} \). Notice that

\[ \int_0^{+\infty} \exp \left( - \int_0^t \delta \left( \xi, s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(\xi, p) \eta(\xi, l)(p) \, dp \right) \, d\xi \right) \]

\[ 1_{s \geq \Gamma(t)} \left| l (s - \Gamma(t)) - l' (s - \Gamma(t)) \right| ds \]

\[ \leq \int_0^{+\infty} \left| l (s - \Gamma(t)) - l' (s - \Gamma(t)) \right| ds = \left\| l - l' \right\| < \frac{\varepsilon}{2} \]

Furthermore,

\[ \int_0^{+\infty} \left| 1_{s \geq \Gamma(t)} l' (s - \Gamma(t)) \right| ds \]

\[ \exp \left( - \int_0^t \delta \left( \xi, s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(\xi, p) \eta(\xi, l)(p) \, dp \right) \, d\xi \right) - l' (s) \right| ds \]

\[ \leq \int_0^{+\infty} \left| \exp \left( - \int_0^t \delta \left( \xi, s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(\xi, p) \eta(\xi, l)(p) \, dp \right) \, d\xi \right) - 1 \right| \]

\[ 1_{s \geq \Gamma(t)} l' (s - \Gamma(t)) ds + \int_0^{+\infty} \left| l' (s - \Gamma(t)) \right| 1_{s \geq \Gamma(t)} - l' (s) \right| ds. \]

It follows from Dominated Convergence Theorem that

\[ \lim_{t \rightarrow 0} \int_0^{+\infty} 1_{s \geq \Gamma(t)} l' (s - \Gamma(t)) \]

\[ \left| \exp \left( - \int_0^t \delta \left( \xi, s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(\xi, p) \eta(\xi, l)(p) \, dp \right) \, d\xi \right) - 1 \right| ds = 0. \]

Therefore, if \( \varepsilon \) is so small that for all \( t \in [0, \varepsilon] \)

\[ \int_0^{+\infty} 1_{s \geq \Gamma(t)} l' (s - \Gamma(t)) \]

\[ \left| \exp \left( - \int_0^t \delta \left( \xi, s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(\xi, p) \eta(\xi, l)(p) \, dp \right) \, d\xi \right) - 1 \right| ds < \frac{\varepsilon}{16} \]
Based on the fact that the set of all continuous functions with compact support is dense in $L^1[0, +\infty)$, there is a continuous function $\xi$ with compact support in $[0, +\infty)$ such that $\|\xi'(s) - \xi\| < \frac{\varepsilon}{15}$. Furthermore, there exists a bounded and closed interval $I \subset [0, +\infty)$ such that $\xi(y) = 0$ for $\forall y \notin I$ since the function with compact support vanishes at the boundary. Then

$$
\int_0^{+\infty} \left| ' \left( s - \Gamma(t) \right) 1_{s \geq \Gamma(t)} - ' \left( s \right) \right| ds
\leq \int_0^{\Gamma(t)} ' \left( s \right) ds + \int_{\Gamma(t)}^{+\infty} \left| ' \left( s - \Gamma(t) \right) - ' \left( s \right) \right| ds
\leq \int_0^{\Gamma(t)} ' \left( s \right) ds + \int_{\Gamma(t)}^{+\infty} \left| ' \left( s - \Gamma(t) \right) - \xi \left( s - \Gamma(t) \right) \right| ds
\quad + \int_{\Gamma(t)}^{+\infty} \left| \xi \left( s - \Gamma(t) \right) - \xi \left( s \right) \right| ds + \int_{\Gamma(t)}^{+\infty} \left| \xi \left( s \right) - ' \left( s \right) \right| ds
\leq \int_0^{\Gamma(t)} ' \left( s \right) ds + 2 \int_0^{+\infty} \left| ' \left( s \right) - \xi \left( s \right) \right| ds + \int_{\Gamma(t)}^{+\infty} \left| \xi \left( s - \Gamma(t) \right) - \xi \left( s \right) \right| ds
\leq \frac{r}{32} + \frac{r}{8} + \frac{r}{32} = \frac{3r}{16}
\text{ for } \varepsilon \text{ sufficiently small.}
$$

Thus,

$$
\int_0^{+\infty} 1_{s \geq \Gamma(t)} ' \left( s - \Gamma(t) \right) \exp \left( - \int_0^t \delta \left( \xi, s - \Gamma(t) + \Gamma(s), \int_0^{+\infty} \mu(s, \xi) l(s, \xi) ds \right) d\xi \right) - ' \left( s \right) \right| ds
\leq \int_0^{+\infty} \left| \exp \left( - \int_0^t \delta \left( \xi, s - \Gamma(t) + \Gamma(s), \int_0^{+\infty} \mu(s, \xi) l(s, \xi) ds \right) d\xi \right) - 1 \right|
\quad 1_{s \geq \Gamma(t)} ' \left( s - \Gamma(t) \right) ds + \int_0^{+\infty} \left| ' \left( s - \Gamma(t) \right) 1_{s \geq \Gamma(t)} - ' \left( s \right) \right| ds
\leq \frac{r}{16} + \frac{3r}{16} = \frac{r}{4}
$$

In summary, we have

$$
\left\| \Lambda(\eta)(t, l) - l \right\| \leq b_M \delta^{-1} \left( 1 - e^{-\delta = t} \right) + \left\| l - l' \right\| + \frac{r}{4} \leq \frac{r}{2} + \frac{r}{4} + \frac{r}{4} = r
$$

for all $t \in [0, \varepsilon]$, where $\varepsilon > 0$ is small enough. Thus, for any $\eta \in B^*$, it follows that $\Lambda(\eta) \in B^*$, i.e., $\Lambda : B^* \rightarrow B^*$.

Finally, we show $\Lambda$ is a contraction mapping on $B^*$ for $\varepsilon$ sufficiently small. As $k$ is locally Lipschitz, we can assume $L_1$ and $L_2$ are Lipschitzian constants for $k(\cdot)$ and $1/k(\cdot)$, respectively, restricted to a compact subset $B^*$. For any $\eta_1, \eta_2 \in B^*$, we have

$$
\left\| \Lambda(\eta_1)(t, l) - \Lambda(\eta_2)(t, l) \right\|
\leq \int_0^{+\infty} 1_{t < \Gamma(t)} \left| b \left( \int_0^{+\infty} r(\Gamma^{-1}(t) - s), p \eta_1(\Gamma^{-1}(t) - s), l(s) ds \right) k \left( \int_0^{+\infty} \eta_2(\Gamma^{-1}(t) - s), l(s) ds \right) \right|
\exp \left( - \int_{\Gamma^{-1}(t) - s}^t \delta \left( \xi, s - \Gamma(t) + \Gamma(s), \int_0^{+\infty} \mu(s, \xi) l(s, \xi) ds \right) d\xi \right)
$$
\[
\begin{align*}
&\quad - b \left( \int_0^{\Gamma(t)} r(\Gamma^{-1}(\Gamma(t) - s), p) \eta_2(\Gamma^{-1}(\Gamma(t) - s), l)(p) \, dp \right) \\
&\quad \quad + \frac{\exp \left( - \int_0^{\Gamma(t)} \delta \left( \xi, s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(\xi, p) \eta_2(\xi, l)(p) \, dp \right) \, d\xi \right)}{k \left( \int_0^{+\infty} \eta_2(\Gamma^{-1}(\Gamma(t) - s), l)(p) \, dp \right)} \\
&\quad \quad + \int_0^{+\infty} 1_{s \geq \Gamma(t)} \left| \exp \left( - \int_0^{\Gamma(t)} \delta \left( \xi, s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(\xi, p) \eta_2(\xi, l)(p) \, dp \right) \, d\xi \right) \right| \, ds \\
&\quad \quad - \exp \left( - \int_0^{\Gamma(t)} \delta \left( \xi, s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(\xi, p) \eta_2(\xi, l)(p) \, dp \right) \, d\xi \right) \, ds \\
&\quad \quad + \int_0^{\Gamma(t)} \left| \exp \left( - \int_0^{\Gamma(t)} \delta \left( \xi, s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(\xi, p) \eta_2(\xi, l)(p) \, dp \right) \, d\xi \right) \right| \, ds \\
&\quad \quad - \exp \left( - \int_0^{\Gamma(t)} \delta \left( \xi, s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(\xi, p) \eta_2(\xi, l)(p) \, dp \right) \, d\xi \right) \, ds \\
&\quad \quad + \int_0^{\Gamma(t)} \left| \frac{1}{k \left( \int_0^{+\infty} \eta_2(\Gamma^{-1}(\Gamma(t) - s), l)(p) \, dp \right)} - \frac{1}{k \left( \int_0^{+\infty} \eta_2(\Gamma^{-1}(\Gamma(t) - s), l)(p) \, dp \right)} \right| \, ds \\
&\quad \quad + \int_0^{\Gamma(t)} b_M \, ds + \int_0^{\Gamma(t)} \frac{1}{k \left( \int_0^{+\infty} \eta_2(\Gamma^{-1}(\Gamma(t) - s), l)(p) \, dp \right)} e^{-\delta_\infty(t - \Gamma^{-1}(\Gamma(t) - s))} \\
&\quad \quad \quad \quad - b \left( \int_0^{+\infty} r(\Gamma^{-1}(\Gamma(t) - s), p) \eta_1(\Gamma^{-1}(\Gamma(t) - s), l)(p) \, dp \right) \\
&\quad \quad \quad \quad - b \left( \int_0^{+\infty} r(\Gamma^{-1}(\Gamma(t) - s), p) \eta_2(\Gamma^{-1}(\Gamma(t) - s), l)(p) \, dp \right) \, ds \\
&\quad \quad \quad \quad + \int_0^{\Gamma(t)} \frac{b_M}{k \left( \int_0^{+\infty} \eta_2(\Gamma^{-1}(\Gamma(t) - s), l)(p) \, dp \right)} \, ds \\
\end{align*}
\]
Proof. The smoothing property of convolution shows that 
\[ \int_{\Gamma^{-1}(\Gamma(t) - s)}^t \left[ \delta \left( \xi, s - \Gamma(t) + \Gamma(t), \int_0^\infty \mu(\xi, p)\eta_1(\xi, t)(p)dp \right) - \delta \left( \xi, s - \Gamma(t) + \Gamma(t), \int_0^\infty \mu(\xi, p)\eta_2(\xi, l)(p)dp \right) \right] d\xi ds \]
\[ + \int_{\Gamma(t)}^{t-1} \int_0^t \left[ \delta \left( \xi, s - \Gamma(t) + \Gamma(t), \int_0^\infty \mu(\xi, p)\eta_1(\xi, t)(p)dp \right) - \delta \left( \xi, s - \Gamma(t) + \Gamma(t), \int_0^\infty \mu(\xi, p)\eta_2(\xi, l)(p)dp \right) \right] d\xi ds \]
\[ \leq b_M L_2 \| \eta_1 - \eta_2 \| ds + \int_0^{\Gamma(t)} e^{-\delta_m \tau(\Gamma(t) - s)} \int_0^{t-1} r(\Gamma(t) - s, p) L_b \| \eta_1(\Gamma^{-1}(\Gamma(t) - s), l)(p) - \eta_2(\Gamma^{-1}(\Gamma(t) - s), l)(p) \| dp d\xi d\Gamma^{-1}(\Gamma(t) - s) \]
\[ + \int_0^{\Gamma(t)} b_M L_\delta \int_{\Gamma(t)}^{t-1} \int_0^{\Gamma(t) - s} \| \mu(s, p) \| \eta_1(l)(p) - \eta_2(l)(p) \| dp d\xi d\Gamma^{-1}(\Gamma(t) - s) \]
\[ + L_\delta \int_0^{\Gamma(t)} \int_0^{t-1} \int_0^{\Gamma(t) - s} \| \mu(s, p) \| \| \eta_1(l)(p) - \eta_2(l)(p) \| dp d\xi d\Gamma^{-1}(\Gamma(t) - s) \]
\[ \leq b_M L_2 \Gamma(t) \| \eta_1 - \eta_2 \| + \int_0^t e^{-\delta_m \tau} L_b \| \eta_1(t, s)(p) - \eta_2(t, s)(p) \| dp ds \]
\[ + \int_0^{t-1} b_M L_\delta \int_{t-\delta_m}^t \| \eta_1 - \eta_2 \| \| d\xi \| + L_\delta L_\theta d\Gamma(t) \int_0^{\Gamma(t)} \int_0^{\Gamma(t) - s} \| \eta_1 - \eta_2 \| \| d\xi \| d\Gamma^{-1}(\Gamma(t) - s) \]
\[ \leq \left( b_M L_2 \Gamma(t) + \frac{1 - e^{-\delta_m \tau}}{\delta_m} L_b \| \eta_1 - \eta_2 \| \right) \| \eta_1 - \eta_2 \| \]
\[ \leq \left( b_M L_2 \Gamma(\varepsilon) + \frac{1 - e^{-\delta_m \varepsilon}}{\delta_m} L_b \| \eta_1 - \eta_2 \| \right) \| \eta_1 - \eta_2 \| \]
\[ = M \| \eta_1 - \eta_2 \| , \]
with some constants \( M > 0, \overline{\tau} = \sup_{t \geq 0, s \geq 0} \tau(t, s) \) and \( \overline{\eta} = \sup_{t \geq 0, s \geq 0} \mu(t, s) \). In the above proof, we have used \( |e^{-x} - e^{-y}| \leq |x - y| \), for all \( x, y > 0 \).

Thus, \( \Lambda \) is a contraction mapping on \( B^* \) for \( \varepsilon \) sufficiently small. The contraction mapping theorem guarantees the existence of a unique fixed point of \( \Lambda \) in \( B^* \), denoted by \( \psi \). In summary, \( \psi(t, l) \) is the continuous solution to (3.1) on \([0, \varepsilon] \times B_0 \) with \( \psi(0, l) = l(s) \) for any \( l \in B_0 \).

It is easy to verify from the integral form (3.1) that the solution of (2.2.2-2.5) is non-negative whenever it exists for any non-negative initial value. Furthermore, we have the following result.

**Theorem 3.2.** Solution given by (3.1) is bounded in forward time if the birth function is bounded, i.e., \( b(x) \leq \overline{B} \).

**Proof.** The smoothing property of convolution shows that \( \int_0^{\infty} n(t, s)ds \) is differentiable in \( t \). Thus
\[ \frac{d}{dt} \int_0^{\infty} n(t, s)ds = -k(N(t) \frac{\partial}{\partial s} n(t, s) - \delta \left( t, s, \int_0^{\infty} \mu(t, s)n(t, s)ds \right) n(t, s)ds \]
\[ b \left( \int_{0}^{+\infty} r(t, s)n(t, s)ds \right) = b \left( \int_{0}^{+\infty} \delta(t, s) \mu(t, s)n(t, s)ds \right) \]
\[ \leq B - \delta_{m} \int_{0}^{+\infty} n(t, s)ds. \]

Then
\[ \limsup_{t \to \infty} \int_{0}^{+\infty} n(t, s)ds \leq \limsup_{t \to \infty} e^{-\delta_{m}t} \left( \int_{0}^{+\infty} n(0, s)ds - \frac{B}{\delta_{m}} - \frac{\bar{B}}{\delta_{m}} \right). \]

This implies the non-negative solutions are bounded.

From the standpoint of biology, the boundedness of the function \( b(\cdot) \) in Theorem 3.2 may be viewed as a natural restriction to birth as a result of limited resources.

4. Global attractor. In this section, we mainly discuss the existence of global attractor for the autonomous model (2.2-2.5) where the functions \( \delta, \mu \) and \( r \) are independent on the time \( t \). For the convenience of the reader we remind some basic concepts and results for dissipative dynamical system.

For the complete metric space \( X = L^{1}_{+}(0, +\infty) \), we can define \( \psi(t, l) \) by the solution of (2.2-2.5) with the initial condition \( l = n(0, s) \in X \). As in Theorems 3.1 and 3.2, the solution, which is bounded and forward complete, defines the flow, \( S(t) : X \to X \) as \( S(t)l = \psi(t, l) \) for \( t \geq 0 \). The standard arguments imply that
\[ S(0)l = \psi(0, l) = l, \]
\[ S(t+s)l = \psi(t+s, l) = \psi(t, \psi(s, l)) = \psi(t, S(s)l) = S(t)S(s)l. \]

In fact, if \( \eta(t) = \psi(t+s, l) \) then \( \eta(t) \) is a solution of (2.2-2.5) with initial condition \( \psi(s, l) \) and then invoke forward uniqueness from Theorem 3.1. The continuity of the semigroup also follows from Theorem 3.1. Thus the family of functions \( \{S(t)\}_{t \geq 0} \) is a \( C^{0} \) semigroup on \( X \).

The set \( B \subset X \) is forward invariant if \( S(t)B \subset B \) for all \( t \geq 0 \); and set \( B \subset X \) is invariant if \( S(t)B = B \) for all \( t \geq 0 \). As discussed as in [25], a set \( A \subset X \) attracts a set \( B \subset X \) if \( \text{dist}(S(t)B, A) \to 0 \) as \( t \to +\infty \) where \( \text{dist}(B, A) \) is the distance from set \( B \) to set \( A \), i.e.,
\[ \text{dist}(B, A) := \sup_{y \in B} \inf_{x \in A} \|y - x\|. \]

A set \( A \subset X \) is defined to be an attractor if \( A \) is non-empty, compact and invariant. There is an open neighborhood \( U \) of \( A \) in \( X \) such that \( A \) attracts \( U \). A global attractor is defined to be an attractor which attracts every point in \( X \).

The semigroup \( S(t) \) is point dissipative, i.e., there is a bounded set \( B \subset X \) which attracts all points of \( X \). Let \( U \subset X \) be bounded, and
\[ \gamma^{+}(U) = \{ y \in X : y = \psi(t, x), t \geq 0, x \in U \}. \]

Theorem 3.2 implies that positive orbits of compact sets are bounded in \( X \). In order to obtain the existence of global attractor, we apply Theorem 2.6 in [25] and state it as follows.

**Lemma 4.1.** (Theorem 2.6 [25]) Let \( S(t) \) be a continuous map on a complete metric space \( (M, D) \). Assume that
(a) $S$ is point dissipative and asymptotically smooth;
(b) Positive orbits of compact subsets of $M$ for $S$ are bounded.

Then $S(t)$ has a global attractor $A \subset M$. Moreover, for each subset $B$ of $M$, if there exists $k \geq 0$ such that $\gamma^+(S^k(B))$ is bounded, then $A$ attracts $B$ for $S$.

To show the global dynamical properties of the flow, it needs to prove that the semigroup is asymptotically smooth. A $C^0$ semigroup $S(t) : X \to X$ is asymptotically smooth if, for any nonempty, closed set $B \subset X$ for which $S(t)B \subset B$, there exists a compact set $J \subset B$ such that $J$ attracts $B$ [13]. A semigroup $S(t)$ is completely continuous if for $t > 0$ and bounded set $B \subset X$ then $\{S(q)B, 0 \leq q \leq t\}$ is bounded and $S(t)B$ is precompact. The following lemma will be used as below.

**Lemma 4.2.** ([13]) For $t \geq 0$, we assume $S(t) = U(t) + C(t) : X \to X$ has the property that $C(t)$ is completely continuous and there exists a continuous function $k(t, r) : R^+ \times R^+ \to R^+$ such that $k(t, r) \to 0$ as $t \to \infty$ and $\|U(t)l\| \leq k(t, l)$ if $\|l\| \leq r$. Then $S(t)$, $t \geq 0$, is asymptotically smooth.

Besides, we need a notion of compactness in $L^1(0, +\infty)$. Being an infinite dimensional space, boundedness does not imply precompactness.

**Lemma 4.3.** ([1]) Set $K \subset L^p(0, +\infty)$ be closed and bounded where $p \geq 1$. Then $K$ is compact if and only if the following hold:
(i) $\lim_{h \to 0} \int_0^{+\infty} |u(z + h) - u(z)|^p dz = 0$ uniformly for $u \in K$. $(u(z + h) = 0$ if $z + h < 0$).
(ii) $\lim_{h \to +\infty} \int_h^{+\infty} |u(z)|^p dz = 0$ uniformly for $u \in K$.

Based the above two lemmas, we have the following result.

**Theorem 4.1.** The semigroup $S(t)$ generated by (3.1) is asymptotically smooth.

**Proof.** In our case, we project $S(t)$ on to $L^1(0, +\infty)$, $\pi S(t)$, can be written as $\pi S(t) = U(t) + C(t)$, where

1) there is $k(t, r) \to 0$ as $t \to +\infty$ such that $\|U(t)l\| \leq k(t, r)$ if $\|l\| \leq r$,
2) for any bounded $B \subset X$ which is closed and bounded, we have $C(t)B$ is compact.

In order to follow this plan of action, we consider $\pi S(t) = U(t) + C(t)$ where

\[
(U(t)) (s) = l(s - \Gamma(t)) 1_{s \geq \Gamma(t)} \\
\quad \exp \left( - \int_0^t \delta \left( s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(p)/\mu(p)(\xi, p) dp \right) d\xi \right),
\]

\[
(C(t)) (s) = b \left( \int_0^{+\infty} \mu(p)/\mu(p)(\Gamma(t), p) dp \right) 1_{s < \Gamma(t)} \\
\quad \exp \left( - \int_0^t \delta \left( s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(p)/\mu(p)(\xi, p) dp \right) d\xi \right),
\]

for $l = \psi(s)$ and $n(s) = \psi(t, l)(s)$. Thus, we have

\[
\|U(t)l\|_{L^1} = \int_{\Gamma(t)} \left| l(s - \Gamma(t)) \exp \left( - \int_0^t \delta \left( s - \Gamma(t) + \Gamma(\xi), \int_0^{+\infty} \mu(p)/\mu(p)(\xi, p) dp \right) d\xi \right) \right| ds \\
\leq e^{-\delta n t} \int_{\Gamma(t)} \left| l(s - \Gamma(t)) \right| ds \leq e^{-\delta m t} \|l\|.
\]
Assumptions A2 and A3 show that the solution with initial condition holds for the set $C$.

Dominated Convergence Theorem implies that

$$\lim_{h \to 0} \int_0^t \left\| \exp \left( - \int_0^t \delta \left( \Gamma(\xi) - \Gamma(\theta), \int_0^{+\infty} \mu(p) n(\xi, p) \, dp \right) \, d\xi \right) - \exp \left( - \int_{\Gamma^{-1}(\Gamma(t) - s - h)} \delta \left( \Gamma(\xi) - \Gamma(\theta) + h, \int_0^{+\infty} \mu(p) n(\xi, p) \, dp \right) \, d\xi \right) \right\| d\theta$$



For the set $C(t)B \subset L^1(0, +\infty)$, it follows that

$$\lim_{h \to 0} \int_0^t \left\| \exp \left( - \int_0^t \delta \left( \Gamma(\xi) - \Gamma(\theta), \int_0^{+\infty} \mu(p) n(\xi, p) \, dp \right) \, d\xi \right) - \exp \left( - \int_{\Gamma^{-1}(\Gamma(t) - s - h)} \delta \left( \Gamma(\xi) - \Gamma(\theta) + h, \int_0^{+\infty} \mu(p) n(\xi, p) \, dp \right) \, d\xi \right) \right\| d\theta$$



Therefore, condition (ii) of Lemma 4.3 holds for the set $C(t)B \subset L^1(0, +\infty)$.

By the boundedness of solutions, there exists a constant $M_1 > 0$ such that the solution with initial condition $l \in B$ satisfies $\|\psi(t, l)\|_{L^1} \leq M_1$ for all $t \geq 0$. Assumptions A2 and A3 show that

$$b \left( \int_0^{+\infty} r(s) n(t, s) \, ds \right) \leq L_0 \pi M_1,$$
By integral formation, it follows that

\[ \int_0^t \lim_{h \to 0} \left| \exp \left( - \int_0^t \delta \left( \Gamma(\xi) - \Gamma(\theta), \int_0^{+\infty} \mu(p)n(\xi, p)dp \right) d\xi \right) \right| d\theta = 0. \]

Thus

\[ \lim_{h \to 0} \sup_{\theta \in [0, t]} \left( \theta - \Gamma^{-1}(\Gamma(\theta) - h) \right) \int_0^t e^{-\delta_m(t-\Gamma^{-1}(\Gamma(\theta)-h))} d\theta = 0, \]

\[ \lim_{h \to 0} \sup_{\theta \in [0, t]} \left( \epsilon^{-1}(\theta) - \Gamma^{-1}(\theta - h) \right) \int_0^{t} e^{-\delta_m(t-\Gamma^{-1}(\theta-h))} d\theta = 0. \]

Therefore (4.1) converges uniformly to 0 as \( h \to 0 \) and condition (i) of Lemma 4.3 is obtained for \( C(t)B \). Hence, Lemma 4.3 implies that \( C(t)B \) is compact. It follows from the above argument and Lemma 4.2 that \( S(t) \) is asymptotically smooth.

Besides, Theorem 3.2 implies that positive orbits of compact sets are bounded in \( X \). Lemma 4.1 implies the following result.

**Theorem 4.2.** The semigroup \( S(t) \) has a global attractor \( A \) contained in \( X \).
5. Uniform persistence. In this section, we mainly prove the uniform persistence and the existence of compact attractor. The long-term dynamics of the system (2.2-2.5) in the case where the death and birth rates are simplified, $\delta(s)$ and $b(N(t))$, as follows

$$\frac{\partial n}{\partial t} + k(N(t)) \frac{\partial n}{\partial s} = -\delta(s)n(t, s),$$

$$k(N(t))n(t, 0) = b(N(t)),$$

$$\lim_{s \to \infty} n(t, s) = 0,$$

$$n(0, s) = n_0(s) \in L^1_{+}[0, \infty).$$

(5.1)

It is easy to verify that the system (5.1) has two equilibria 0 and $n^*(s)$, where

$$n^*(s) = \frac{b(N^*)}{k(N^*)} \exp \left( -\int_0^s \frac{\delta(\theta)d\theta}{k(N^*)} \right), \quad \text{and} \quad N^* = \int_0^{+\infty} n^*(s)ds.$$

The linearized system of the zero equilibrium 0 can be given as

$$\frac{\partial n}{\partial t} + k(0) \frac{\partial n}{\partial s} = -\delta(s)n(t, s),$$

$$k(0)n(t, 0) = b'(0) \int_0^{+\infty} n(t, s)ds,$$

$$\lim_{s \to \infty} n(t, s) = 0,$$

$$n(0, s) = n_0(s) \in L^1_{+}[0, \infty).$$

(5.2)

Here, a threshold $R_0$ can be introduced into this linearized system. In such sense, $\int_0^{+\infty} l(s)ds$ is the initial distribution of all individuals. Then $b'(0) \int_0^{+\infty} l(s)ds$ is the density distribution of newly population which is produced by all individuals. Thus

$$\frac{b'(0)}{k(0)} \int_0^{+\infty} l(s)ds \exp \left( -\int_0^s \frac{\delta(\theta)d\theta}{k(0)} \right)$$

is the distribution of those individuals who were newly population with size 0 and still survive in the environment with size $s$ for $s \geq 0$. Hence the integral

$$\frac{b'(0)}{k(0)} \int_0^{+\infty} l(s)ds \int_0^{+\infty} \exp \left( -\int_0^{s} \frac{\delta(\theta)d\theta}{k(0)} \right) ds$$

is the distribution of accumulative new individuals produced by all individuals $\int_0^{+\infty} l(s)ds$. Let

$$R_0 = \frac{b'(0)}{k(0)} \int_0^{+\infty} \varphi(s)ds \quad \text{where} \quad \varphi(s) = \exp \left( -\int_0^{s} \frac{\delta(\theta)d\theta}{k(0)} \right).$$

Thus, $R_0$ is a measure of how many new individuals will be produced by an old individual. It becomes one of the most important key parameters in dynamics of system (5.1).
Substituting $n(t, s) = l(s)e^{M}$ into (5.2) yields that
\[ l(s) = l(0) \exp \left( - \int_0^s \frac{\delta(\theta) + \lambda}{k(0)} \, d\theta \right), \]
\[ k(0)l(0) = b'(0) \int_0^{+\infty} l(s) \, ds. \quad (5.3) \]

Solving the equation of (5.3) shows that
\[ 1 = \frac{b'(0)}{k(0)} \int_0^{+\infty} \varphi(s) \exp \left( - \frac{\lambda s}{k(0)} \right) \, ds. \]

Substituting the complex number $\lambda = x + yi$ into the above equation yields
\[ 1 = \frac{b'(0)}{k(0)} \int_0^{+\infty} \varphi(s) \exp \left( - \frac{x s}{k(0)} \right) \cos \left( \frac{y s}{k(0)} \right) \, ds. \quad (5.4) \]

If $R_0 < 1$, then the above equation has always a negative real parts. Otherwise, there is a $x_0 > 0$ such that (5.4) holds, and
\[ b'(0) \int_0^{+\infty} \varphi(s) \exp \left( - \frac{x_0 s}{k(0)} \right) \cos \left( \frac{y s}{k(0)} \right) \, ds - 1 \leq 0 \]
\[ \text{this contradicts (5.4). Let} \]
\[ H(x) = \frac{b'(0)}{k(0)} \int_0^{+\infty} \varphi(s) \exp \left( - \frac{x s}{k(0)} \right) \, ds - 1. \]

If $R_0 > 1$, then (5.4) has a positive root with real parts. It is easy to verify that for $\lambda = x > 0$,
\[ H(0) = R_0 - 1 > 0, \quad \text{and} \quad H(x) \to -1 < 0 \quad \text{as} \quad x \to +\infty. \]

Thus, the zero steady state $0$ of the system is locally asymptotically stable if $R_0 < 1$ and is unstable if $R_0 > 1$. Now, we discuss the global asymptotic stability of the zero steady state by constructing Lyapunov functions.

**Theorem 5.1.** If $R_0 < 1$ and $b(x) \leq b'(0)x$, then the zero steady state is globally asymptotically stable.

**Proof.** We firstly give a positive function as
\[ \alpha(s) = \left( 1 - \int_0^s \frac{b'(0)}{k(0)} \exp \left( - \int_0^\omega \frac{\delta(\theta)}{k(0)} \, d\omega \right) \exp \left( \int_0^s \frac{\delta(\theta)}{k(0)} \, d\theta \right) \right). \]

It follows that
\[ \alpha(s) \geq \left( 1 - R_0 \right) \exp \left( \int_0^s \frac{\delta(\theta)}{k(0)} \, d\theta \right) > 0, \quad \text{for} \quad s \geq 0. \]
Here, we used the conditions

\[ R_0 < 1. \]

For any solution of (5.1) with non-negative initial value, we define a function \( V_1(t) \) as follows

\[ V_1(t) = \int_0^{\infty} \alpha(s)n(t, s)ds. \]

Differentiating \( V_1(t) \) along a solution of (5.1) yields

\[
\frac{dV_1(t)}{dt} = \int_0^{\infty} \alpha(s) \frac{\partial n(t, s)}{\partial t} ds = - \int_0^{\infty} \alpha(s) \left( k(N(t)) \frac{\partial n(t, s)}{\partial s} + \delta(s)n(t, s) \right) ds
\]

\[
= - k(N(t)) \alpha(s)n(t, s) \bigg|_0^{\infty} + \int_0^{\infty} k(N(t)) \alpha'(s)n(t, s)ds
\]

\[
- \int_0^{\infty} \alpha(s)\delta(s)n(t, s)ds
\]

\[
\leq b(N(t))\alpha(0) + \int_0^{\infty} k(0)\alpha'(s)n(t, s)ds - \int_0^{\infty} \alpha(s)\delta(s)n(t, s)ds
\]

\[
\leq b'(0)\alpha(0)N(t) + \int_0^{\infty} \left( \alpha(s)\delta(s) - b'(0)\alpha(0) \right) n(t, s)ds
\]

\[
- \int_0^{\infty} \alpha(s)\delta(s)n(t, s)ds = 0.
\]

Here, we used the conditions

\[
\lim_{t \to \infty} n(t, s) = 0, \quad \alpha'(s) = \alpha(s) \frac{\delta(s)}{k(0)} \geq 0, \quad k(N(t)) < k(0), \quad b(N(t)) \leq b'(0)N(t).
\]

The zero steady state is locally asymptotically stable if \( R_0 < 1 \). Thus, the zero steady state is globally asymptotically stable by Lyapunov-LaSalle asymptotic stability Theorem for semiflows.

Define

\[ X^0 = \left\{ l(s) \in X : \int_0^{\infty} l(s)ds > 0 \right\} \text{ and } \partial X^0 = X \setminus X^0. \]

**Theorem 5.2.** Both sets \( X^0 \) and \( \partial X^0 \) are forward invariant under semiflow \( \{ S(t) \} \) for \( t \geq 0 \) generated by system (5.1) on \( X \).

**Proof.** We firstly show \( S(t) : \partial X^0 \to \partial X^0 \). Otherwise, there exists \( \tau = \inf \{ t > 0 : S(t)x \in X^0 \} \) for \( x \in \partial X^0 \). The continuity of \( S(t) \) implies that \( S(\tau)x \in \partial X^0 \), and \( \Gamma^{-1}(\Gamma(\tau) - s) < \tau \) for \( s < \Gamma(\tau) \),

\[
n(t, s) = l(s - \Gamma(\tau)) \exp \left( - \int_0^\tau \delta(s - \Gamma(\tau) + \Gamma(s)) ds \right) 1_{s \geq \Gamma(\tau)} + 1_{s < \Gamma(\tau)}
\]

\[
\frac{b \left( \int_0^{\infty} n(\Gamma^{-1}(\Gamma(\tau) - s), p)dp \right)}{k \left( \int_0^{\infty} n(\Gamma^{-1}(\Gamma(\tau) - s), p)dp \right)} \exp \left( - \int_0^\tau \delta(s - \Gamma(\tau) + \Gamma(s)) ds \right)
\]

\[
= l(s - \Gamma(\tau)) \exp \left( - \int_0^\tau \delta(s - \Gamma(\tau) + \Gamma(s)) ds \right) 1_{s \geq \Gamma(\tau)}.
\]
Define
\[ \eta(t) = l(s - \Gamma(t + t)) \exp \left( -\int_0^{T+t} \delta(s - \Gamma(t + t) + \Gamma(s)) \, ds \right) \mathbf{1}_{s \geq \Gamma(t + t)} \]
for \( \eta(0) \in \partial X^0 \) and \( t \geq 0 \). Then, \( \eta(t) \) is a solution of (2.2-2.5) with initial condition \( S(\tau)x \). It follows from the forward uniqueness of solutions and Theorem 3.1 that \( S(t)x \in \partial X^0 \) for \( t \geq 0 \). In fact,
\[ \|\eta\|_{L^1} \leq \int_{\Gamma(t + t)}^{\infty} l(s - \Gamma(t + t)) \, ds = 0. \]

For the initial point \( x(s) = n(0, s) \in X^0 \) of \( n(t, s) \), we have
\[ \frac{d}{dt} \int_0^\infty n(t, s) \, ds = b \left( \int_0^\infty n(t, s) \, ds \right) - \int_0^\infty \delta(s) n(t, s) \, ds \geq -\delta_M \int_0^\infty n(t, s) \, ds \]
where \( \delta_M = \text{ess sup}_{s \in (0, +\infty)} \delta(s) \). Then
\[ \int_0^\infty n(t, s) \, ds \geq \int_0^{+\infty} x(s) \, ds \exp(-\delta_M t) > 0, \text{ for } x \in X^0. \]

This implies that \( X^0 \) is forward invariant.

In order to show the uniform persistence of \( S(t) \) and the existence of global attractor in \( X^0 \), we cite the following Lemmas in Hale and Waltman [14]. Define \( \hat{A}_0 = \bigcup_{x \in A_0} \omega(x) \) where \( A_0 \) is global attractor in \( \partial X^0 \), and
\[ \omega(x) = \{ y \in X : \exists t_n \uparrow +\infty \text{ such that } S(t_n)x \to y \} \]
is omega limit set of \( x \). The stable or attracting set of a compact invariant set \( A \) is denoted by
\[ W^s(A) = \{ x : x \in X, \omega(x) \neq \emptyset, \omega(x) \subset A \}. \]
The unstable or repelling set of a compact invariant set \( A \) is denoted by
\[ W^u(A) = \{ x : x \in X, \exists \text{ a backward orbit } \gamma^- (x) \text{ such that } \alpha_\gamma (x) \neq \emptyset, \alpha_\gamma (x) \subset A \} \]
where
\[ \alpha_\gamma (x) = \{ y \in X : \exists t_n \uparrow -\infty \text{ such that } S(t_n)x \to y \} \]
is alpha limit set of \( x \).

**Lemma 5.1.** (Theorem 4.2 [14]) Suppose metric space \( X \) is a closure of an open set \( X^0 \), i.e., \( X = X^0 \cup \partial X^0 \), and \( S(t) \) satisfies conditions
\[ S(t) : X^0 \to X^0, \quad S(t) : \partial X^0 \to \partial X^0, \quad (5.5) \]
and
(i) \( S(t) \) is asymptotically smooth,
(ii) \( S(t) \) is point dissipative in \( X \),
(iii) \( \gamma^-(U) \) is bounded in \( X \) if \( U \) is bounded in \( X \),
(iv) \( A_0 \) is isolated and has an acyclic covering \( M \).
then $S(t)$ is uniformly persistent if and only if for each $M_i \in M$, $W^s(M_i) \cap X^0 = \emptyset$.

**Lemma 5.2.** (Theorem 3.3 [14]) Suppose $S(t)$ satisfies conditions (5.5) and (i) – (iii) in Lemma 5.1, and we have:
(i) $S(t)$ is uniformly persistent,
(ii) $\gamma^+(M)$ is strongly bounded in $X^0$ if $M$ is strongly bounded in $X^0$.

Then, there are global attractors $A$ in $X$ and $A_0$ in $\partial X^0$ and a global attractor $A_0$ in $X^0$ relative to strongly bounded sets. Furthermore, $A = A_0 \cup W^u(A_0)$.

Now, we are ready to state and prove the uniform persistence of system (5.1).

**Theorem 5.3.** If $R_0 > 1$ and $b(x) \geq b'(0)x$ in a neighbourhood of 0, then the semiflow $\{S(t)\}_{t \geq 0}$ is uniformly persistent with respect to both sets $X^0$ and $\partial X^0$. That is, there is $\epsilon > 0$ such that for each $x \in X^0$,
\[ \liminf_{t \to +\infty} d(S(t)x, \partial X^0) \geq \epsilon. \]

There is a compact subset $A_0 \subset X^0$ which is a global attractor for $\{S(t)\}_{t \geq 0}$ in $X^0$.

**Proof.** Lemmas 5.1 and 5.2 will be used to establish the uniform persistence and the existence of compact attractor.

Let $U \subset X$ be bounded, and
\[ \gamma^+(U) = \{ y \in X : y = \psi(t, x), t \geq 0, x \in U \}. \]

It follows from Theorems 3.2 and 4.1 that the semigroup $\{S(t)\}_{t \geq 0}$ is asymptotically smooth and point dissipative. Then $\gamma^+(U)$ is bounded. Let $A_0$ be the global attractor in $\partial X^0$. Theorem 5.2 shows that $A_0$ is the fixed point 0 and $A_0 = \bigcup_{x \in A_0} \omega(x) = A_0$.

In order to show the condition (iv) of Lemma 5.1, we will prove $\{0\}$ is acyclic. It is obvious that $\partial X^0 \subset W^s(\{0\})$ and $\partial X^0 \setminus \{0\} \cap W^s(\{0\}) = \emptyset$. Since $\partial X^0$ and $X^0$ are forward invariant, it follows that any backward orbit of $x \in \partial X^0 \setminus \{0\}$ stays in $\partial X^0$. If $x = l(s) = 0$ then it is a unique equilibrium 0. Thus, all points in $\partial X^0$ approach 0. In order to prove that $\{0\}$ to be acyclic, it needs to prove $W^s(\{0\}) \cap X^0 = \emptyset$. Otherwise, there exists $x \in X^0 \cap W^s(\{0\})$ such that $S(t)x \to 0$ as $t \to +\infty$. In fact, if $S(t)x \to 0$, then it has a $\epsilon > 0$ such that $\|S(t)x - 0\| \geq \epsilon$ as $t \to +\infty$. It follows from Theorem 4.1 that the semigroup $S(t) = C(t) + U(t)$ has the property: $C(t)x$ is pre-compact. There has a convergent subsequence $C(t_n)x \to n^* \neq 0$. Thus $S(t_n)x \to n^*$ since $\|U(t_n)\| \to 0$. This contradicts the assumption that $x \in W^s(\{0\})$. Hence $S(t)x \to 0$ as $t \to +\infty$. A sequence $\{x_n\} \subset X^0$ can be found to show $\|S(t)x_n - 0\| < 1/n$, for all $t \geq 0$. Let $S(t)x_n = n_n(t, s)$ and $x_n = l_n(s)$. Then, we have $\|n_n(t, s)\| < 1/n$. Let $N_n(t) = \int_0^t n_n(t, s)ds$. The assumption $R_0 > 1$ implies that there is a $\epsilon > 0$ such that $b'(0) - \epsilon \geq k(0)$. By differentiating the formula $N_n(t)$ and (5.1) yield that
\[
\frac{dN_n(t)}{dt} = b(N_n(t)) - \int_0^t \delta(s)n_n(t, s)ds \\
\geq (b'(0) - \epsilon)N_n(t) - \int_0^t \delta(s)n_n(t, s)ds \\
\geq k(0)N_n(t) - \frac{\delta M}{n},
\]

$N_n(0) = \int_0^t l_n(s)ds$. 

\[ \cdot \]
where $\delta_M = \max_{s \geq 0} \delta(s)$. For $n$ sufficiently large, the solution of the above equation can be given as

$$
N(t) \geq \exp \left( k(0)t \right) \left( \int_0^{+\infty} l_n(s) ds - \frac{1}{n} \frac{\delta_M}{k(0)} \right) + \frac{1}{n} \frac{\delta_M}{k(0)}.
$$

There is a $K$ such that $N_n(t) = S(t)l_n$ is unbounded if $n > K$, hence, which is a contradiction. Thus, $W^s(\{0\}) \cap X^0 = \emptyset$. It follows from Lemma 5.1 that $S(t)$ is uniformly persistent, and from Lemma 5.2 that there exists a compact set $A_0 \subset X^0$ which is global attractor for $\{S(t)\}_{t \geq 0}$ in $X^0$.

Here, we assume that the birth rate satisfies $b(N(t)) \geq b'(0)N(t)$ in a neighborhood of 0. This implies that the species which can exhibit an accelerated population specific growth rate for small values of population as a strategy to avoid extinction.

6. Discussion. In this paper, a size-structured population model, a quasi-linear first-order partial differential equation with nonlinear boundary condition and initial condition, is established to describe the growth of a single species where the development level of an individual is dependent on its size (such as weight, length). Since all individuals occupying at the same place tend to share the food inside the place equally, the growth rate along the size is assumed to be a function of total number of the population, thereby modelling intra-specific competitive effect for food among individuals. That is the larger the population, the slower should be its rate of growth or the longer should be its developmental time.

Since the characteristics of the equation are family of curves, it is key and important to give explicitly an equivalent integral equation obtained by the method along characteristics. By extending the work [39] to size structured model and taking a slightly different method combined with properties of characteristics, we establish the existence and uniqueness of solutions in light of the contraction mapping theorem. Besides, we prove rigorously the existence, uniqueness and eventual compactness of the nonlinear semigroup associated with the solution of (2.2-2.5) by using results from Hale on asymptotic smoothness [13] and compactness condition for $L^p$ spaces. Finally, we discuss the uniform persistence by using results from Hale and Waltman [14], and establish the existence of a compact global attractor contained inside the uniformly persistent set by using results from Magal and Zhao [25].

The threshold $R_0$ describes the ability of newly developed individuals. Our results show that $R_0$ is a threshold parameter for the extinction and uniform persistence of the population. That is the species will be extinct if $R_0 < 1$, and the species persists if $R_0 > 1$. The global attractor for the semigroup in the suitable space is also obtained if $R_0 > 1$. The global asymptotic stability of the zero steady state is discussed by constructing Lyapunov function. However, the global asymptotic stability of the positive steady state still remains open. In fact, this problem for age-structured models has been discussed, such as SEIR epidemic model for a disease with age-structure in [17]. In our model, the size-structure makes the problem more difficult to solve. We will consider this interesting question via Lyapunov function which is defined on the compact global attractor in the future.

In order to show that the impact of the growth rate of individual size on the extinction of the species, we compute the derivative of the threshold $R_0$ with respect to $k$

$$
\frac{dR_0}{dk} = b'(0) \int_0^{+\infty} \exp \left( - \int_0^s \delta(\theta) d\theta \right) \left( 1 - \frac{1}{k^2(0)} \int_0^s \delta(\theta) d\theta \right) ds.
$$

(5.4)
Thus, if the integral death rate is less than the square of the growth rate of individual size at the population 0, \( \int_0^{+\infty} \delta(t) d\theta < k^2(0) \), then the derivative of the threshold is above zero. Hence, the threshold \( R_0 \) is far from zero as the growth rate of individual size increases. We can improve the growth rate of individual size by increasing the supply of the food, and decreases the death rate by limiting the category and quantity of the species’ natural enemy so as to promote their growth and thus avoid their extinction.

Acknowledgments. We would like to thank the referees very much for their valuable comments and suggestions.

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Received April 2019; revised October 2019.

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