Sensitivity investigation for unequal-arm LISA and TAIJI: the first-generation time-delay interferometry optimal channels

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Three spacecraft of LISA/TAIJI mission follow their respective geodesic trajectories, and the arm lengths formed by the pairs of spacecraft are unequal due to solar system dynamics. Time delay interferometry is proposed to suppress the laser frequency noise raised by the unequal-arm-ness. By employing a set of numerical mission obit achieved from an ephemeris framework, we investigate the averaged sensitivity of the first-generation time-delay interferometry Michelon configuration and the corresponding optimal A, E, and T channels. We find that the sensitivity of the T channel is differing from the equal-arm case including the response function and the noise level, and its performance is sensitive to the inequality of the arm lengths. We also examine the laser frequency noise due to the mismatch of laser beam paths, and show that these channels are significantly subject to the laser frequency noise at their characteristic frequencies.

I. INTRODUCTION

Gravitational wave (GW) started to become a new method to observe the universe since the first detection of advanced LIGO – GW150914 [1]. During the advanced LIGO and advanced Virgo O1 – O3 run, GW signals from stellar mass compact binary coalescences were frequently detected/identified [2–10, and references therein], and were used to explore the fundamental physics and astronomy [11–16, and references therein]. With the joining of KAGRA [17, 18], the detectability and sky localization for the GW signals will be improved by the LIGO-Virgo-KAGRA network [19].

Besides the high frequency GW (10–2000 Hz) searching by the ground-based interferometers, the researches and developments are also thriving in the other frequency bands. In the middle frequency band (0.1–10 Hz), various ground-based and space-borne GW detectors/concepts have been newly proposed. The ground-based approaches include AI (Atom Interferometer) (including MIGA [20, 21], MAGIS-100 [22], ZAIGA [23], ELGAR [24], and AION [25]), MI (Michelson Interferometer) [20], SOGRO (Superconducting Omni-directional Gravitational Radiation Observatory) [27, 28], and TOBA (Torsion-Bar Antenna) [29]. In space, BBO [30] and DECIGO [31, 32] are the first mission proposals for the mid-frequency GW detection. By employing different approaches, various missions were proposed subsequently including AEDGE [33], AIGSO [34, 35], AMIGO [36, 37], B-DECIGO [32, DO [38], and INO [39].

In the milli-Hz frequency band (0.1 mHz–1 Hz), besides the LISA [40, 41], two Chinese space missions were proposed–TAIJI [42] and TianQin [43]. The TAIJI mission is considered to be a LISA-like configuration in a heliocentric orbit leading the Earth by 20°, and TianQin uses triangular interferometry in a geocentric orbit configuration. The studies about TAIJI and TianQin are actively ongoing [44–51, and references therein]. Beyond the LISA, there are multiple detectors proposed with the detectability extended to the micro-Hz frequency band (0.1 µHz–100 µHz) including ASTROD-GW [52], µ-Aries [53], Folkner’s mission [54], and Super-ASTROD [55].

For laser interferometry in space, there are two classes of flight formations, the first class employs constant arm configuration which keeps distance equal among spacecraft (S/C) by using the thrusters, and the second class employs the (quasi-)geodesic configuration and keeps the proof mass drag-free. The B-DECIGO and DECIGO are in the first class, and LISA, TAIJI, TianQin, ASTROD-GW, DO, µ-Aries, and Super-ASTROD are in the second class. AMIGO can use both methods and can be in either class [37]. For the constant arm configuration, we estimated the thruster requirement for DECIGO and AMIGO missions, as well as for the assumed constant arm LISA and TAIJI [56]. For the (quasi-)geodesic class, the arm lengths vary with time due to solar-system dynamics. Drag-free control is demanded to achieve targeting sensitivity. LISA Pathfinder launched on December 3, 2015, has successfully demonstrated and satisfied the LISA’s drag-free requirement [57, 58].

For the geodesic mission, with the present/state-of-the-art technology, the laser frequency noise is overwhelming for traditional Michelson configuration to satisfy the GW detection requirement. Time-delay interferometry (TDI) is proposed for LISA-like missions to suppress the laser...
frequency noise. The previous studies showed that the TDI could effectively suppress the laser noise [59–63, and references therein]. In the process of TDI, the secondary noise and the GW signals also are canceled or accumulated. And the final response of a TDI combination to a GW signal is formed by combining the measurements from time shifted laser links.

To investigate the response function and noise level in TDI, multiple simulators were developed for the LISA mission. The *LISA Simulator* was developed by the group at Montana State University to calculate the response function and noise [70, 71]. The *Synthetic LISA* was developed by Vallisneri to simulate the LISA measurement process considering the level of scientific and technical requirements [65]. *LISACode* was built and targeting to pave the road between the basic principles of LISA and sophisticated simulator [68], and its successor *LISANode* is developed to adapt to the new LISA design [69].

In our previous paper [43], by using recipes in these simulators and the numerical orbit we achieved, we investigated average sensitivities of each first-generation TDI channels, and the angular resolution of LISA-TAIJI network to the supermassive black hole binaries and the monochromatic sources. In this process, we realized that the performance of the optimal T channel is rather different from the previous results by assuming the equal-arm case [72, 73]. In this paper, we focus on the sensitivity investigations of the optimal channels (A, E, and T) combined from the first-generation Michelson TDI channels (X, Y, and Z). By considering the time-varying unequal-arm numerical orbits, the laser frequency noise, acceleration noise, and the optical path noise, we evaluate the response function and the noise level in the X channel and optimal channels and their changes with time dependent arms. We find that the performance of the T channel is divergent from the equal-arm case for both response function and the noise level, and is sensitive to the inequality of the arm lengths. We examine the laser frequency noise due to the mismatch of laser beam paths, and show that these channels are significantly subject to the laser frequency noise at their characteristic frequencies.

This paper is organized as follows. In Sec. II, we introduce the recipe in our investigation, review the numerical mission orbits and TDI channels. In Sec. III, we calculate both location dependent and sky-averaged responses to GW sources for the X, A, E, and T channels, and analyze the irregular response of the T channel. In Sec. IV, we examine laser frequency noise due to the mismatch of laser beam paths in the TDI, together with the acceleration noise and optical path noise in TDI channels. In Sec. V, we synthesize sensitivities of TDI channels from the time and frequency varying response and noise level. And we recapitulate our conclusions in Sec. VI. (We set \( G = c = 1 \) in this work.)

**II. RECIPE OF THE INVESTIGATIONS**

**A. Numerical mission orbits**

The current LISA configuration is proposed to have \( 2.5 \times 10^6 \) km arms and trailing the Earth by 20\(^\circ\), and the plane formed by 3 S/C has about 60\(^\circ\) inclination angle with respect to the ecliptic plane [41]. TAIJI mission is planned to be LISA-like formation with \( 3 \times 10^6 \) km arms and leading the Earth by 20\(^\circ\) [42, 51].

In our previous works [56, 74–80], we developed a workflow to design and optimize the mission orbits for GW space missions by using an ephemeris framework, as well as to calculate the path differences of the TDI laser beams. Based on the orbital requirements for new LISA configuration [41], we achieved the numerical LISA orbit for 6 years satisfying the criteria: 1) the relative velocities between S/C are smaller than 5 m/s; 2) the changes of breathing angles are less than 1 deg, and 3) the trailing angle is in the range [19\(^\circ\), 23\(^\circ\)] [80]. For larger arm length of the TAIJI mission, the relative velocities are loosened up to be less than 6 m/s. The optimized orbits achieved for LISA and TAIJI missions are shown in Fig. 1. The orbits are calculated in the Solar System barycentric (SSB) coordinates, and start on March 22nd, 2028 (JD2461853.0). As the plots shown, the orbits can maintain in required status for 2200 days (6 years), and we select the first 400 days to investigate the performances of the TDI channels in the unequal-arm case as the shadow areas shown in Fig. 1.

**B. TDI channels**

In previous works [59–61, 66, 69, 72, 81, and references therein], the sensitivity of LISA was studied by assuming equal-arms or the Keplerian orbits. By employing the numerical orbits, we focus on the investigations of the response functions, noise levels, and sensitivities on the optimal (A, E, and T) channels combined from the first-generation TDI Michelson channels (X, Y, and Z), as well as the Michelson X channel. We adopt the original LISA optical design used in [60, 62, 66] and references therein as shown in the schematic diagram Fig. 2.

There are two optical benches on each S/C as shown in Fig. 2 and the symbols are explained as follow.

- \( y_{ij} \) denotes the laser link Doppler measurement from S/C\(_i\) to \( j \), which contains the GW signal and the noises \( y_{ij} = y_{ij}^G + y_{ij}^N \), where \( y_{ij}^G \) is the response function to the GW signal, \( h \) is the GW signal, and \( y_{ij}^N \) is the noise contained in the measurement.
- \( z_{ij} \) denotes intra-spacecraft measurement on the optical bench in S/C\(_i\) pointing to S/C\(_j\) which measures the laser from another optical bench in the
FIG. 1. The numerical orbits for LISA (upper row) and TAIJI (lower row) mission used in the investigations. The arm length $L_{ij}$ changes with time are shown in the left plot, and the relative velocities $\dot{L}_{ij}$ between S/C are shown in the right panel. In the following calculation, the first 400 days shown by the shadow areas are used to simulate the performances of the TDI channel in the unequal-arm situation.

same S/C by optical fibers. And this measurement is assumed to have no response to the GW signal since the short path.

- $C_{ij}$ denotes the laser frequency noise on the optical bench in S/C$i$ pointing to S/C$j$.

- $L_{ij}$ denotes the arm length from S/C$i$ to S/C$j$. In the static assumption, the arm from S/C$i$ to S/C$j$ is equal to the arm from S/C$j$ to S/C$i$, $L_{ij} = L_{ji}$, while in the dynamical case, the these arms could be not equal, $L_{ij} \neq L_{ji}$ and $L_{12} \neq L_{13} \neq L_{23}$ as shown in Fig. 1.

In this work, we assume the triangle configuration is static during a TDI laser beam propagation time (for instance, propagation time is $4L \simeq 33$ s for LISA Michelson channels) when we calculate the GW response and secondary noise, and adopt the dynamical case when we calculate the mismatch of laser beam paths.

The first-generation Michelson TDI configuration (X channel) is shown in Fig. 2. The expression of measurements in the X channel is shown in Eq. (1) \[81, 82\]. The Y and Z channels could be obtained by cyclical permutation of the spacecraft indexes.

\[
X(t) = \left[ y_{31}(t) + D_{31}y_{13}(t) + D_{13}D_{31}y_{21}(t) + D_{21}D_{13}D_{31}y_{12}(t) \right] \\
- \left[ y_{21}(t) + D_{21}y_{12}(t) + D_{12}D_{21}y_{31}(t) + D_{31}D_{12}D_{21}y_{13}(t) \right] \\
+ \left[ -z_{13}(t) + D_{13}D_{31}z_{13}(t) + D_{12}D_{21}z_{13}(t) - D_{13}D_{31}D_{12}D_{21}z_{13}(t) \right]/2 \\
+ \left[ z_{12}(t) - D_{13}D_{31}z_{12}(t) - D_{12}D_{21}z_{12}(t) + D_{12}D_{21}D_{13}D_{31}z_{12}(t) \right]/2
\]

(1)

where $D_{ij}$ is time-delay operators and act on a measurement $y(t)$ by

\[
D_{ij}y(t) = y(t - L_{ij}(t)), \\
D_{mn}D_{ij}y(t) = y(t - L_{ij}(t) - L_{mn}(t - L_{ij}(t))),
\]

(2)
A group of optimal TDI channels, (A, E, and T), can be formed from linear combinations of the three Michelson channels (X, Y, and Z) \[72, 73\],

\[ A = \frac{Z - X}{\sqrt{2}}, \quad E = \frac{X - 2Y + Z}{\sqrt{6}}, \quad T = \frac{X + Y + Z}{\sqrt{3}}. \] (3)

To understand the impacts from different factors, we split the investigations into three steps, 1) the response of TDI channels to the GW signal, 2) the noises (including laser frequency noise, optical noise, and acceleration noise) in TDI measurements, and 3) the average sensitivities synthesized from the response and noise levels.

\[ \mathcal{O}_1 = \begin{pmatrix}
\sin \lambda \cos \psi - \cos \lambda \sin \beta \sin \psi \\
\cos \lambda \cos \psi - \sin \lambda \sin \beta \sin \psi \\
- \cos \lambda \cos \psi - \cos \lambda \sin \beta \sin \psi
\end{pmatrix} \begin{pmatrix}
- \sin \lambda \sin \psi - \cos \lambda \sin \beta \cos \psi - \cos \lambda \cos \beta \\
\cos \lambda \sin \psi - \sin \lambda \sin \beta \cos \psi - \sin \lambda \cos \beta \\
\cos \beta \cos \psi - \sin \beta
\end{pmatrix}, \] (6)

where \( \psi \) is the polarization angle. The response to the GW in the link from S/C \( i \) to \( j \) is

\[ y_{ij}^h(f) = \frac{(1 + \cos^2 \iota)\hat{n}_{ij} \cdot e_+ \cdot \hat{n}_{ij} + i(-2 \cos \iota)\hat{n}_{ij} \cdot e_\times \cdot \hat{n}_{ij}}{4(1 - \hat{n}_{ij} \cdot \hat{k})} \times \exp(2\pi i f (L_{ij} + \hat{k} \cdot p_i)) - \exp(2\pi i f \cdot p_j), \] (7)

where \( \hat{n}_{ij} \) is the unit vector from SC\( i \) to \( j \), \( L_{ij} \) is the arm length between SC\( i \) and \( j \), \( p_i \) is the position of the S/C in the SSB coordinates, and \( \iota \) is the inclination of the GW source from the line of sight.

When the TDI is implemented, the GW signal are only incorporated in the measurements between two S/C \( y_{ij} \). Therefore, the GW signal in Michelson-X channel are only contained in the first two rows in Eq. (1). And its response in the frequency domain could be described by

\[ F_X^h(f) = (-\Delta_{21} + \Delta_{21}\Delta_{13}\Delta_{31})y_{12}^h + (-1 + \Delta_{13}\Delta_{31})y_{21}^h + (\Delta_{31} - \Delta_{31}\Delta_{12}\Delta_{21})y_{13}^h + (1 - \Delta_{12}\Delta_{21})y_{31}^h, \] (8)

where \( \Delta_{ij} = \exp(2\pi i f L_{ij}) \). The responses of A, E, and T channels in frequency-domain could be obtained by using the Eq. (3).
B. The response based on source locations

The previous studies have shown that the optimal channels A and E have the equivalent averaged sensitivity [15, 72, 73]. Nevertheless, as Eqs. (4)-(8) shown, the response of each link depends on the four geometric angles (source location \( \lambda \) and \( \beta \), polarization \( \psi \), and inclination \( \iota \)). We can expect that the instantaneous responses of A and E channels to the different directions would also be different. As the plots shown in Appendix Fig. 10 and Fig. 11 the responses of the TDI channels X, A and E to the monochromatic GW at 20 mHz (\( \sim 1/2\pi L \)) from various directions. The most sensitive directions is around the normal direction of the plane formed by three S/C, and the three channels are slightly different. The responses of the A and E channels are stronger than the X channel.

In one orbital period, the response regions in Fig. 10 and Fig. 11 will change with the constellation’s motion and rotation. And the response will modulate for a source from fixed ecliptic latitude. By selecting four latitudes \( (0^\circ, 30^\circ, 60^\circ, \text{and } 90^\circ) \) on a longitude, the sum of response in the A, E and T channels at 20 mHz, \( \sum_{AET} F^2(f = 20 \text{ mHz}, \psi = 0, \iota = 0) \), are shown in Fig. 3 upper panel. The responses at different latitudes are rather different and change with the orbit motion periodically. For a monochromatic GW signal in a TDI channel, the area between \( y = 0 \) axis and a curve is proportional to the square of SNR, \( \rho^2 \) in the selected time period.

To compare the detectability of the detector to the sources at different latitude, the response, \( \sum_{AET} F^2(f = 20 \text{ mHz}, \psi = 0, \iota = 0) \), is integrated over one year for various latitude and normalized by its maximum value as shown in Fig. 3 lower panel. The most sensitive latitude for LISA-like orbit mission should be at \( \sim \pm 18^\circ \). And the source from polar direction is 0.72 of the most sensitive direction, corresponding to the \( \sqrt{0.72} \approx 0.85 \) of the SNR (In principle, the response should be weighted by the PSD of each channels to be proportional to the corresponding SNR. In practice, the noise PSD of A and E channels are identical, and the weighted sensitivity of T channel is lower than A and E channels by one order and could be negligible at frequency 20 mHz as shown in Sec. V). This result indicates the efficiency of the LISA/TAIJI sky coverage.

C. The average response for TDI channels

To evaluate the detectability of a TDI channel, a widely used method is to calculate the averaged response over sky and polarization at each frequency

\[
R_{TDI}^2(f) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{\pi} |F_{TDI}(f, \iota = 0)|^2 \cos \beta d\psi d\beta d\lambda.
\]

(9)

The averaged responses at different frequencies for X, A, E and T channels are shown in Fig. 4. The A and E channels curves are identical and slightly higher than X channel. The T channel are rather different from others, the dark grey area shows the best 50% percentile of the response in first 400 days of the mission orbit, and the dark with light grey area shows the best 90% percentile in 400 days. The right panel in each plots shows the histograms of the response in T channel at the frequency 0.01 mHz, 0.1 mHz and 1 mHz.

Our further investigations show that, in the low frequency band \( (f \ll 1/2\pi L) \), the response of X, A and E channel should be proportional to \( L^2 \), and the response variation with the unequal-arm is negligible since the arm length variances is less than 1% as shown in Fig. 4. On the other side, from the low frequency limit approximation, we found the response of T channel is sensitive to the variance of the arm lengths and proportional to the difference of arm lengths \( \sum (L_{ij} - L_{ik})(L_{ij} - L_{jk}) \), and the mainly proportional to the defined inequality factor...
that the response of T channel, analytical approximation and tentative fitting, we deduce
tice their linear relation in the log-log plot. Based on the
inequality shown in Fig. 5’s lower panel, we can no-

period and the first 400 days are shown in Fig. 5’s

upper panel, and its value are mainly in the range of

FIG. 4. The average responses of TDI X, A, E and T channels
in frequency-domain (LISA results are shown in upper panel,
and TAIJI results are shown in lower panel). The T channel
is sensitive to the variances of arm lengths, and the dark grey
region shows the best 50% percentile in 400 days, and the dark
and light grey area together show the best 90% percentile.
The right panel in each plots shows the histogram of the T
channel’s response at frequencies 0.01 mHz, 0.1 mHz and 1
mHz.

η in Eq. (10),

\[ \eta(t) \equiv \frac{\sqrt{(L_{12} - L_{23})^2 + (L_{12} - L_{13})^2 + (L_{13} - L_{23})^2}}}{L}. \]

The inequality factor of the numerical orbits for full
period and the first 400 days are shown in Fig. 5’s
upper panel, and its value are mainly in the range of
\([4 \times 10^{-4}, 5 \times 10^{-3}]\). As the response of T channel versus
the inequality shown in Fig. 5’s lower panel, we can notice
their linear relation in the log-log plot. Based on the
analytical approximation and tentative fitting, we deduce
that the response of T channel, \( R_T^2 \), should be (approx-
imately) proportional to \( \eta^2 \) and \( (2\pi fL)^4 \). Consequently,
we fitted the parameters in this relation by assuming

\[ R_T^2(f) \approx a(2\pi fL)^4\eta^2, \] (11)

where \( a \) is the parameter to be determined and approx-

limately equal to \( a \approx 8.87 \) from the fitting results at fre-

quencies 0.01 mHz, 0.1 mHz and 1 mHz.

FIG. 5. The histogram of inequality \( \eta \) in full period and first
400 days of the LISA and TAIJI mission orbits (upper panel),
and the relation between the inequality parameter \( \eta \) and the
response of T channel at selected frequency (0.01 mHz, 0.1
mHz and 1 mHz).

The relation achieved from Eq. (11) is universal for
low frequency limit \( f << 5 \text{ mHz} \). The inequality of
arm lengths, \( \eta(t) \), is straightforward to obtain from the
numerical orbit, and the response of T channel in the
low frequency band could be inferred from the inequality
value.

IV. NOISES IN TIME-DELAY
INTERFEROMETRY

TDI is essential for LISA-like missions to suppress the
laser frequency noise and to achieve sensitivity goals. The
first-generation TDI combinations could cancel out the
laser frequency noise in a static unequal-arm configuration,
and the second-generation TDI combinations could
further cancel the frequency noise in a configuration with
relative movement. In this section, we estimate the resid-
ual laser noise and secondary (core) noise (here only ac-
celeration noise and optical path noise are considered) in
the X, A, E, and T channels.

The noises in the measurement \( y_{ij} \) in Fig. 2 could be described by

\[ y_{ij}^n(t) = C_{ij}(t - L_{ij}) - C_{ji}(t) + y_{ij}^{op}(t) - 2y_{ji}^{acc}(t), \] (12)
and the noise in \( z_{ij} \) could be

\[
z_{ij}^n(t) = C_{ik}(t) - C_{kj}(t) + 2y_{ij}^{op}(t),
\]

where \( C_{ij} \) is the laser frequency noise on the optical bench \( S/C_i \) facing to \( S/C_j \), \( y_{ij}^{op} \) is the optical-path noise in the measurement of beam from \( S/C_i \) to \( S/C_j \), and \( y_{ij}^{acc} \) is the acceleration noise raised by proof mass on the optical bench of \( S/C_j \) pointing to \( S/C_i \). To understand the impacts of the noise sources, we investigate the noise levels of the laser frequency noise and secondary noise separately in this section.

### A. Laser frequency noise

Laser frequency noise is dominant for LISA-like missions and needs to be suppressed by TDI technology. As we can see the Eqs. (11), (12) and (13), the laser frequency noise widely exist in the measurements of X channel. By substituting the laser frequency noise parts in Eqs. (12) and (13) into Eq. (11), we can attain the laser noise level in the Michelson X channel \( 8^2 \)

\[
X_{\text{laser}} \simeq \frac{1}{2} (\dot{C}_{12} + \dot{C}_{13}) \Delta t,
\]

where \( \Delta t \) is the time difference between the path lengths of two laser beams pass by. From the Fourier derivative theorem, the corresponding PSD (power spectral density) of laser frequency noise is

\[
|\delta \tilde{C}(f)|^2 = 2(\pi f \Delta t)^2 |\tilde{C}(f)|^2,
\]

where the \( \tilde{C}(f) \) is the one-side square-root spectrum density of laser source stability. The laser noise level is proportional to the square of time difference \( \Delta t^2 \). And we implement two approaches to calculate the path differences in the TDI channels as follow.

The first approach is the numerical method which calculates the light propagation time along each arm in time sequential order in the TDI channel. The position of a receiver \( S/C \) is determined by iterative interpolation. The time delay from other effects, for example, Shapiro time delay, could be incorporated in the calculation as we implemented in the previous works \([71, 73, 70]\). Our numerical method was initially developed for the TDI calculation in ASTROD-GW concept which has the 1.73 AU arm length and surrounding the Sun \([71, 73, 74, 78]\). And it was applied to the LISA and TAIJI mission subsequently \([70, 73, 80]\). The numerical method can provide high accuracy results especially for the mission orbit perturbed by planets etc. The cumulative histogram of time difference amplitude \( |\Delta t| \) for X, A, E and T channels by using numerical algorithm are shown in Fig. 6 upper panel, and the values for the A, E and T channels are inferred from the Eq. (10). In 2200 days, the time differences are mostly smaller than 0.8 \( \mu s \) for LISA, and the time differences are smaller than 1.0 \( \mu s \) for TAIJI mission.

\[
|\delta \tilde{C}_A|^2 = \frac{|\delta \tilde{C}_Z|^2 + |\delta \tilde{C}_X|^2}{2}
\]

\[
|\delta \tilde{C}_E|^2 = \frac{4|\delta \tilde{C}_X|^2 + |\delta \tilde{C}_Z|^2}{6}
\]

\[
|\delta \tilde{C}_T|^2 = \frac{|\delta \tilde{C}_X|^2 + |\delta \tilde{C}_Z|^2 + |\delta \tilde{C}_T|^2}{3}.
\]

The second approach is the approximate method which can simplify the light propagation time calculation to the arm length and its first derivative with respect to time as shown in Eq. (17). We have verified that this approach can achieve a good precision for the first-generation TDI calculation for LISA-like orbit since the \( 3L \) \((\simeq 25 \, s) \) for LISA and \( \simeq 30 \, s \) for TAIJI time delay is short compared to the relative motions between \( S/C \). As the figure shown in Fig. 6 lower panel, the difference of results from two approaches is within 8 \( \mu s \) for the X channel. There is a caveat that the precision of approximation could be declined with the increase of arm length and/or total propagation time, and it has been reflected in the plot comparing the longer arm length of TAIJI \((3 \times 10^6 \, km)\) and relatively short arm LISA \((2.5 \times 10^6 \, km)\). Also for the second-generation TDI configurations, with the increase of propagation time, the accuracy of the approximate algorithm could further degenerate.

\[
\Delta L_X(t) = [L_{31}(t) + D_{31}L_{31}(t) + D_{13}D_{31}L_{21}(t) + D_{21}D_{13}D_{31}L_{12}(t) - [L_{21}(t) + D_{21}L_{12}(t)] + D_{12}D_{21}L_{31}(t) + D_{31}D_{12}D_{21}L_{13}(t)]
\]

\[
\simeq -L_{31}L_{13} - (L_{13} + L_{31})L_{21}
\]

\[
-(L_{21} + L_{13} + L_{31})\dot{L}_{12} + L_{21}\dot{L}_{12}
\]

\[
+(L_{12} + L_{21})\dot{L}_{31} + (L_{31} + L_{12} + L_{21})\dot{L}_{13}
\]

\[
\simeq 4L_{12}\dot{L}_{13} - 4L_{13}\dot{L}_{12}.
\]

After the time differences have been obtained from either the numerical or approximate method, we can estimate the laser frequency noise in each TDI channels. The laser frequency noise will be treated as the white noise widely exist in the measurements of X channel. By comparing the longer arm length of TAIJI \((3 \times 10^6 \, km)\) and relatively short arm LISA \((2.5 \times 10^6 \, km)\). And it was applied to the LISA and TAIJI mission subsequently \([70, 73, 80]\). The numerical method can provide high accuracy results especially for the mission orbit perturbed by planets etc. The cumulative histogram of time difference amplitude \( |\Delta t| \) for X, A, E and T channels by using numerical algorithm are shown in Fig. 6 upper panel, and the values for the A, E and T channels are inferred from the Eq. (10). In 2200 days, the time differences are mostly smaller than 0.8 \( \mu s \) for LISA, and the time differences are smaller than 1.0 \( \mu s \) for TAIJI mission.
FIG. 6. The cumulative histogram of the time difference for X, A, E, and T channels interferometry paths in the 2200 days (upper panel), and the time difference between the numerical method and approximate method (lower panel).

B. Secondary noise

In this work, the secondary noise is considered to incorporate acceleration noise and optical path noise. And the noise level could be estimated by substituting the corresponding terms in Eqs. (12) and (13) into Eq. (1).

By assuming there is no correlation between the different test masses and optical benches, the PSD functions of X, A, and E channels could be approximated by

$$S_X(f) = 16S_{op}(f) \sin^2 x + 16S_{acc}(f)(3 + \cos 2x) \sin^2 x,$$

$$S_A = S_E \approx 8S_{op}(2 + \cos x) \sin^2 x + 16S_{acc}(3 + 2\cos x + \cos 2x) \sin^2 x,$$

where $x = 2\pi f L$, where $S_{op}$ is the PSD of the optical path noise level, and $S_{acc}$ is the PSD level from the test mass acceleration noise.

The current requirements of acceleration noise $S_{acc}$ for LISA and TAIJI missions are supposed to be the same which are [41, 51],

$$S_{acc}^{1/2} \leq 3 \times 10^{-15} \frac{m/s^2}{\sqrt{Hz}} \left[ \sqrt{1 + \left( \frac{0.4 mHz}{f} \right)^4} + \sqrt{1 + \left( \frac{f}{8 mHz} \right)^4} \right].$$

The optical path noise $S_{op}$ requirement for LISA and TAIJI missions are slightly different:

$$S_{op, LISA}^{1/2} \leq 10 \times 10^{-12} \frac{m}{\sqrt{Hz}} \left[ \sqrt{1 + \left( \frac{2 mHz}{f} \right)^4} \right],$$

$$S_{op, TAJI}^{1/2} \leq 8 \times 10^{-12} \frac{m}{\sqrt{Hz}} \left[ \sqrt{1 + \left( \frac{2 mHz}{f} \right)^4} \right].$$

The noise PSD of the X, A, and E channels could be obtained by applying the requirements into the Eq. (18),

$$S_X(f), S_A, S_E \approx 8S_{op}(2 + \cos x) \sin^2 x + 16S_{acc}(3 + 2\cos x + \cos 2x) \sin^2 x.$$
and their PSD curves for LISA and TAIJI are shown in Fig. 7.

For the T channel, in the previous studies, its PSD is approximated by Eq. (21) by assuming the equal-arm configuration [73, 81].

\[
S_T = 16S_{op}(1 - \cos x)\sin^2 x + 128S_{acc}\sin^2 x\sin^4(x/2).
\]  

(21)

By using the unequal-arm configuration, we found the noise PSD of T channel diverges from the equal arm case and varies with the time (actually with the inequality of the arm lengths), especially for the frequency lower than 1 mHz. The numerical substitutions are required to calculate the PSD of T channel, because the PSD is affected by the arm length differences and Eq. (21) cannot properly describe it at the lower frequency. The PSD of the T channel in the unequal-arm case is shown in Fig. 7. The dark blue area shows the highest 50% percentile of noise PSD in first 400 days, and the dark blue together with the light blue areas show the noise level in 90% of the 400 days.

The PSD of these four channels for LISA and TAIJI are shown in upper and lower panels in Fig. 7, respectively. As we can see that the secondary noises in the X, A and E channels are higher than that in the T channel, and generally higher than the laser frequency noise except at the characteristic frequencies \( f = \frac{n}{2L} \) where \( n = 1, 2, 3 \ldots \). For TAIJI mission, the laser frequency noise is slightly higher than or comparable to the secondary noise in X, A, and E channels at frequency range \([2, 10]\) mHz. These results will be reflected in the sensitivity achieved in the next section.

V. SENSITIVITIES

Based on the average response and noise levels, the average sensitivity of a TDI channel could be obtained by weighting the noise PSD by the averaged response, \( S_{avg} = S_n/R^2 \). To understand the impacts of laser frequency noise and secondary noise, we examine the sensitivities with secondary noise only first, and the sensitivity from secondary noise together with laser frequency noise thereafter.

1. Average sensitivities without laser frequency noise

When only secondary noise is considered, the averaged sensitivities of LISA and TAIJI in X, A, E, and T channels are shown in Fig. 8. On account of lower optical path noise requirement and longer arm length, the sensitivity of TAIJI is slightly better than LISA at the frequency around 10 mHz. The sensitivity of the X channel is usually considered as the fiducial sensitivities for a mission. The A/E channel is slightly better than the X channel in the frequency band \([10, 50]\) mHz. Due to the time-varying response and noise level, the T channel significantly diverges from other channels in the frequency range \([0.2, 50]\) mHz. Comparing to the sensitivity of the T channel in the equal-arm case (grey dashed line), the T channel’s sensitivity from unequal-arm configuration is enhanced for the frequency lower than 10 mHz. And its sensitivity becomes equivalent to other channels at the frequency lower than 0.2 mHz. The dark grey area shows the best sensitivity of the T channel in the 50% time of the first 400 days, and the light grey area shows the additional 40% of the time.
2. Average sensitivity with laser frequency noise

If the laser frequency noise is incorporated together with the secondary noise, the averaged sensitivities of the channels are shown in Fig. 9. The dark and light colors show the 50% and 90% percentiles of the sensitivity in the first 400 days. The sensitivity of the T channel deteriorates significantly because of the laser frequency noise. However it still better than equal-arm sensitivity most of the time. The histograms of sensitivities on the T channel at 0.1 mHz, 1 mHz, and 5 mHz are shown in the right panel of each plot. At the higher frequency band, all the four channels subject to the laser frequency noise significantly at their characteristic frequency \( f = \frac{\pi}{2L} \) where \( n = 1, 2, 3 \ldots \). As we can expect in Fig. 4, the sensitivity of TAIJI also subject to the laser noise in the frequency band [2, 20] mHz.

\[ S(f) = \frac{1}{L^2} \]  

The dark grey area shows the best sensitivity in the 90% of 400 days for X, A, and T channels, and the dark together with the light grey area show the best sensitivity in 90% of 400 days. The sensitivity curves of the T channel for TDI path mismatch calculation, we realized it would be \( \sim 2 \times 10^3 \frac{M}{\sqrt{\text{Hz}}} \) would merge below the \( f = 1/2L \approx 0.06 \text{ Hz} \) for LISA, the first-generation TDI may be used for the lower frequency GW detection and data analysis. Another approach is to improve the laser frequency stability by 1–2 orders to \( 0.3–3 \text{ Hz/√Hz} \) to mitigate the impact of laser frequency noise.

The second-generation TDI also could be employed to overcome the laser noise and achieve the sensitivity goal. However, the second-generation TDI configuration would require more than twice links combination than the first-generation combinations, in other words, the travel time of laser beams will be at least twice of the first-generation cases. In previous works, lots of second-generation configurations have been proposed [63, 64, 67, 82, 84 and references therein], as well as some new TDI combinations are recently found in [65]. And the residual laser frequency noises in these TDI configurations should be trivial and not affecting the sensitivity significantly. This has been verified in our previous studies [7, 64, 65, 80]. In these works, we numerically calculated the time difference for selected second-generation TDI configurations, and the laser frequency noise inferred by Eq. (15) is (mostly) negligible.

Based on our experiences in the orbits optimization and TDI path mismatch calculation, we realized it would be harder to minimize the relative velocities between the S/C for a LISA-like mission with longer arm length [80]. As the Eq. (17) shown, the mismatch of the TDI beams increases with the production of arm lengths and relative velocities. And it indicates that the laser frequency noise in the first-generation will decrease with the shorten of arm lengths based on a power-law with the index \( \gtrsim 2 \).
This is reflected by comparing the LISA and TAIJI in this work, the arm length of TAIJI (nominal $3 \times 10^6$ km) is longer than LISA ($2.5 \times 10^6$ km) by 20%, the relative velocities between S/C of TAIJI ($|\dot{L}| \leq 6$ m/s) is also larger than LISA’s ($|\dot{L}| \leq 5$ m/s) by 20%, as a result, the laser frequency noise in TDI channel of TAIJI is higher than LISA’s by $\sim 40\%$. And the sensitivity of TAIJI mission at $[2, 20]$ mHz is affected by the laser noise more than that of LISA. For the shorter arm LISA-like mission concept—AMIGO (nominal arm length $\sim 10^4$ km) [37], our preliminary results showed that the first generation TDI could be enough to suppress the laser frequency noise and detect the GW in the middle frequency band.

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Appendix A: Appendix
FIG. 10. The response of LISA in X, A and E channels at frequency 20 mHz at 70th day for the directions in ecliptic coordinates. The contours show the values of the $|F_{TDI}(f = 20 \text{ mHz}, \psi = 0, \iota = 0)|^2$. 

FIG. 11. The response of TAIJI in X, A and E channels at frequency 20 mHz at 70th day for the directions in ecliptic coordinates. The contours show the values of the $|F_{TDI}(f = 20 \text{ mHz}, \psi = 0, \iota = 0)|^2$. 

[19] B. Abbott et al. (KAGRA, LIGO Scientific, VIRGO), Prospects for Observing and Localizing Gravitational-Wave Transients with Advanced LIGO, Advanced Virgo and KAGRA, Living Rev. Rel. 21, 3 (2018), arXiv:1304.0670 [gr-qc].

[20] R. Geiger et al., Matter-wave laser Interferometric Gravitation Antenna (MIGA): New perspectives for fundamental physics and geosciences, in 50th Rencontres de Moriond on Gravitation: 100 years after GR (2015) pp. 163-172, arXiv:1505.07137 [physics.atom-ph].

[21] J. Junca et al. (MIGA consortium), Characterizing Earth gravity field fluctuations with the MIGA antenna for future Gravitational Wave detectors, Phys. Rev. D 99, 104026 (2019), arXiv:1902.05337 [physics.atom-ph].

[22] J. Coleman, MAGIS-100 at Fermilab, arXiv e-prints, arXiv:1812.00482 (2018), arXiv:1812.00482 [physics.ins-det].
