Some non-Gorenstein Hecke algebras attached to spaces of modular forms.

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1 Introduction

In this paper we exhibit some examples of non-Gorenstein Hecke algebras, and hence some modular forms for which mod 2 multiplicity one does not hold.

Define $S_2(\Gamma_0(N))$ to be the space of classical cuspidal modular forms of weight 2, level $N$, and trivial character. The Hecke algebra $T_N$ is defined to be the subring of End($S_2(\Gamma_0(N))$ generated by the Hecke operators $\{T_p : p \nmid N\}$ and $\{U_q : q|N\}$. Let $\mathfrak{m}$ be a maximal ideal of $T_N$, and let $\ell$ denote the characteristic of the finite field $T_N/\mathfrak{m}$. By work of Shimura, one can associate to $\mathfrak{m}$ a semi-simple Galois representation $\rho_{\mathfrak{m}} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(T_N/\mathfrak{m})$ satisfying $\text{tr}(\rho(\text{Frob}_p)) \equiv T_p \mod \mathfrak{m}$ for all primes $p \nmid N\ell$. We say that $\mathfrak{m}$ is non-Eisenstein if $\rho_{\mathfrak{m}}$ is absolutely irreducible.

As an example, if $E$ is a (modular) elliptic curve over $\mathbb{Q}$ of conductor $N$, let $f := \sum_{n \geq 1} a_n q^n$ be the modular form in $S_2(\Gamma_0(N))$ associated to $E$. Associated to $f$ is a minimal prime ideal of $T_N$; we say that $\mathfrak{m}$ is associated to $f$, or to $E$, if $\mathfrak{m}$ contains this minimal prime ideal. In this case, the representation associated to $\mathfrak{m}$ is isomorphic to the semisimplification of $E[\ell]$, where $\ell$ is the characteristic of $T_N/\mathfrak{m}$.

The localisation $(T_N)_\mathfrak{m}$ of $T_N$ at a maximal ideal $\mathfrak{m}$ is frequently a Gorenstein ring, and such a phenomenon is related to the study of the $\mathfrak{m}$-torsion in the Jacobian $J_0(N)$ of $X_0(N)$. For example, if $\mathfrak{m}$ is non-Eisenstein then by the main result of [4], the $\mathfrak{m}$-torsion in $J_0(N)$ is isomorphic to a direct sum of $d \geq 1$ copies of $\rho_{\mathfrak{m}}$. If $d = 1$ then one says that the ideal $\mathfrak{m}$ satisfies “mod $\ell$ multiplicity one”, or just “multiplicity one”. In this case, the localisation $(T_N)_\mathfrak{m}$ is known to be Gorenstein.

Multiplicity one is a common phenomenon for maximal ideals $\mathfrak{m}$ of $T_N$. Let us restrict for the rest of this paper to the case of non-Eisenstein maximal ideals $\mathfrak{m}$. The first serious study of these mod $\ell$ multiplicity one questions is that of Mazur [10], who proves that if $N = q$ is prime and the characteristic of the

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finite field $\mathbb{T}_q/m$ is not 2 then $m$ satisfies multiplicity one and hence $(\mathbb{T}_q)_m$ is Gorenstein. He also showed that, if $\mathbb{T}_q/m$ has characteristic 2 and $T_2 \in m$ then $(\mathbb{T}_q)_m$ is Gorenstein. This work has been generalised by several authors, to higher weight cases and non-prime level. Rather than explaining these generalisations in complete generality, we summarise what their implications are in the case left open by Mazur.

We fix notation first. From now on, the level $N = q$ is prime, $m$ is a non-Eisenstein maximal ideal of $\mathbb{T}_q$, and the characteristic of $\mathbb{T}_q/m$ is 2. Let $f = \sum a_n q^n$ be the mod 2 modular form associated to $f$.

As we said already, if $a_2 = 0$ then Mazur proved that $m$ satisfies multiplicity one. Gross proves in [8] that if $a_2 \neq 1$ then $m$ satisfies multiplicity one, and in Chapter 12 of [8] (see the top of page 494) he states as an open problem whether multiplicity one holds in the remaining case $a_2 = 1$.

Edixhoven [6] proves that if $\rho_m$ is ramified at 2, then $m$ satisfies multiplicity 1 (Theorem 9.2, part 3). Buzzard [3] and [4] shows that if $\rho_m$ is unramified at 2 but if the image of Frobenius at 2 is not contained in the scalars of $\text{GL}_2(\mathbb{T}_q/m)$ then $m$ satisfies multiplicity one; see Proposition 2.4 of [3].

The main theorem of this paper is that, for $q \in \{431, 503, 2089\}$ (note that all of these are prime), there is a non-Eisenstein maximal ideal $m$ of $\mathbb{T}_q$ lying above 2, such that $(\mathbb{T}_q)_m$ is not Gorenstein. As a consequence, these $m$ do not satisfy mod 2 multiplicity one. In fact, the theorems above served as a very useful guide to where to search for such $m$.

It is proved in Matsumura [9], Theorem 21.3, that if a $\mathbb{Z}_2$-algebra is finitely generated and a complete intersection, then it is Gorenstein. Hence the $\mathbb{T}_q$ above are not complete intersections. The work of Wiles [15] and Taylor-Wiles [14] on Fermat’s Last Theorem proves as a byproduct that certain Hecke algebras are complete intersections. Hence this paper gives a bound on how effective Wiles and Taylor’s methods can be in characteristic 2.

Each of the maximal ideals $m \subset \mathbb{T}_q$ that we construct are non-Eisenstein and have the property that $\mathbb{T}_q/m = \mathbb{F}_2$, so let us first consider a general surjective representation $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{F}_2)$. For such a representation, the trace of $\rho$ at any unramified prime $p$ can be computed if one knows the splitting of $p$ in the extension of $\mathbb{Q}$ of degree 6 cut out by $\rho$. Let $K$ denote this extension. If $p$ splits completely then Frob$_p$ has order 1 and if $p$ splits into 3 primes in $K$, then Frob$_p$ has order 2. In both cases the trace of $\rho(\text{Frob}_p)$ is 0. The other possibility is that $p$ splits into two primes, and then Frob$_p$ has order 3 and the trace of $\rho(\text{Frob}_p)$ is 1.

By the theorems of Mazur, Gross, Edixhoven and Buzzard above, if we wish to find examples of maximal ideals where multiplicity one fails, we could adopt the following approach: we firstly search for modular elliptic curves $E$ of prime conductor $q$, such that 2 is unramified in $K = \mathbb{Q}(E[2])$, the field generated over $\mathbb{Q}$ by the coordinates of the 2-torsion of $E$. We then require that 2 splits completely in $K$ and that $K$ has degree 6 over $\mathbb{Q}$. Note that these conditions imply that $E$ has good ordinary reduction at 2. For any such elliptic curves that one may find, the maximal ideals of $\mathbb{T}_q$ associated to $E[2]$ will be maximal ideals not covered by any of the multiplicity one results above.
We search for curves like this by using a computer to search pre-compiled
tables, such as Cremona [5], and compute the associated maximal ideals of $T_q$.
For each such maximal ideal, we then explicitly construct the completion of
$(T_q)_m$ as a subring of a direct sum of finite extensions of $\mathbb{Q}_2$ and check to see
whether it is Gorenstein. Note that a Noetherian local ring is Gorenstein if and
only if its completion is. Hence this procedure will test the Gorenstein-ness of
localisations of $T_q$ at non-Eisenstein maximal ideals not covered by the above
theorems.

It is slightly surprising to note that after trying only one or two examples,
one finds maximal ideals where multiplicity one fails. Our discoveries are sum-
marrised in the following theorem.

**Theorem 1** Assume that $q \in \{431, 503, 2089\}$. Then there is a maximal ideal $m$
of $T_q$ such that $(T_q)_m$ is not Gorenstein, and hence for which mod 2 multiplicity
one does not hold.

Note that as a consequence, the ring $T_q$ is in these cases, by definition, not
Gorenstein.

The calculations behind this theorem were made possible with the HECKE
package, running on the MAGMA computer algebra system [1]. MAGMA includes
an environment for specialised number-theoretic calculations, and also incorpo-
rates Cremona’s database of elliptic curves, and the HECKE package, written by
William Stein, implements efficient algorithms to generate spaces of modular
forms of arbitrary level, weight ($\geq 2$) and character over global and finite fields.
Without this computing package this paper could not have been written.

An application of this paper’s results can be found in Section 6 of Emer-
ton [7]. Let $X$ be the free $\mathbb{Z}$-module of divisors supported on the set of singular
points of the curve $X_0(q)$ in characteristic $q$. Theorem 0.5 of [7] shows that
$T_q$ is Gorenstein if and only if $X$ is an invertible $T_q$-module. The module $X$
can be explicitly calculated quickly using the Mestre-Oesterle method of graphs,
from [11], as implemented in, e.g., MAGMA.

The exact sequence of $\text{Gal}(\overline{K}/K)$-modules on page 488 of Gross [8], is split
as an exact sequence of Hecke modules, in the Gorenstein case. Emerton proves
that the analogue of this short exact sequence in the non-Gorenstein case is
never split. He uses this to prove results about the $m$-adic Tate module of
$J_0(q)$.

In Ribet-Stein [12] the existence of non-Gorenstein Hecke algebras is dis-
cussed in the context of the level optimisation procedure associated with Serre’s
conjecture (see section 3.7.1).

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or her suggestions, which have improved the paper and its exposition markedly.

2 $T_{431}$
Theorem 2 There is a maximal ideal $\mathfrak{m}$ of $T_{431}$ such that $(T_{431})_\mathfrak{m}$ is not Gorenstein.

Set $q = 431$. There are two non-isogenous modular elliptic curves $E_1$ and $E_2$ of conductor $q$, defined over $\mathbb{Q}$, with corresponding minimal Weierstrass equations

$$
E_1 : y^2 + xy = x^3 - 1
$$
$$
E_2 : y^2 + xy + y = x^3 - x^2 - 9x - 8.
$$

The corresponding modular forms $d_1, d_2 \in S_2(\Gamma_0(431))$ have Fourier coefficients in $\mathbb{Z}$. One can easily check that the fields $\mathbb{Q}(E_i[2])$ are isomorphic, and of degree 6 over $\mathbb{Q}$, which shows that the two elliptic curves have isomorphic and irreducible mod 2 Galois representations. One checks that 2 splits completely in $K = \mathbb{Q}(E_i[2])$.

Let

$$
\rho : \text{Gal}(K/\mathbb{Q}) \rightarrow \text{Aut}(E_i[2])
$$

denote the associated mod 2 Galois representation. We claim that there are precisely four eigenforms in $S_2(\Gamma_0(431))$ giving rise to $\rho$. To see this, note firstly that $\rho(\text{Frob}_q)$ has order 3 and hence has trace 1. We compute the characteristic polynomial of $T_3$ acting on $S_2(\Gamma_0(431))$ and reduce it modulo 2. We find that the resulting polynomial is of the form $(X - 1)^2 q(X)$, where $q(1) \neq 0$. This shows already that there are at most four eigenforms which could give rise to $\rho$. We now compute the $q$-expansions of these four candidate eigenforms, as elements of $\mathbb{Q}_2[[q]]$. Two of the eigenforms, say $d_1$ and $d_2$, corresponding to the elliptic curves $E_1$ and $E_2$ above, of course have coefficients in $\mathbb{Z}$. The other two, $d_3$ and $d_4$, are conjugate and are defined over $\mathbb{Q}_2(\sqrt{10})$. We now check that the reductions to $\mathbb{F}_2$ of the first 72 $q$-expansion coefficients (the Sturm bound) of all four of these eigenforms are equal. By [13], this implies that the four forms themselves are congruent, and hence all give rise to isomorphic mod 2 Galois representations. By construction, these representations must all be isomorphic to $\rho$.

Let $\mathfrak{m}$ denote the corresponding maximal ideal of $T_q$. Our goal now is to compute the completion $\mathbf{T}$ of $(T_q)_{\mathfrak{m}}$ explicitly. Let $V$ be the 4-dimensional space over $\mathbb{Q}_2$ spanned by the four eigenforms giving rise to $\rho$. The ring $\mathbf{T}$ is the $\mathbb{Z}_2$-subalgebra of $\text{End}(V)$ generated by the Hecke operators $T_p$ for $p \neq 431$, and $U_{431}$. By the theory of the Sturm bound, this algebra equals the $\mathbb{Z}_2$-algebra generated by $T_n$ for $n \leq 72$. One readily computes this algebra. Let $\alpha$ be the coefficient of $q^3$ in $d_3$. Then $d_3$ and $d_4$ are defined over $\mathbb{Q}_2(\alpha) = \mathbb{Q}_2(\sqrt{10})$. It turns out that $\mathbf{T}$ is isomorphic to the subring of $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2[\alpha]$ generated as a $\mathbb{Z}_2$-module by $\{(1, 1, 1), (0, 2, 0), (0, 0, \alpha + 1), (0, 0, 2)\}$.

This is a local ring, with unique maximal ideal $\mathfrak{m}$ generated as a $\mathbb{Z}_2$-module by $\{(2, 0, 0), (0, 2, 0), (0, 0, \alpha + 1), (0, 0, 2)\}$.

Now we claim that there is a reducible parameter ideal (recall that a parameter ideal is an ideal that contains a power of the maximal ideal). This will show that $\mathbf{T}$ is not Gorenstein (see [8], Theorem 18.1). (Note that $\mathbf{T}$ is
Cohen-Macaulay, as it has Krull dimension one and has no nonzero nilpotent elements (see [9], Section 17, page 139).

The ideal \( i \) generated by \((2, 2, \alpha + 1)\) is a parameter ideal, since it contains \( m^2 \).

We observe that the ideals

\[ i_1 = ((0, 2, 0), (0, 2, \alpha + 1)) \quad \text{and} \quad i_2 = ((0, 2, 0), (2, 0, \alpha + 1)) \]

have intersection exactly \( i \). Hence \( i \) is a reducible parameter ideal, so \( T \) is not Gorenstein, and as \( T \) is the completion of the Noetherian local ring \((T_{431})_m\) we deduce that this ring is also not Gorenstein. Hence mod 2 multiplicity one fails for \( m \).

3 \( T_{503} \)

**Theorem 3** There is a maximal ideal \( m \) of \( T_{503} \) such that \((T_{503})_m\) is not Gorenstein.

The argument is similar to that of the previous section, and we merely sketch it. In fact, the case \( q = 503 \) is slightly technically simpler than the case of \( q = 431 \) because all four relevant newforms are defined over \( \mathbb{Z}_2 \).

There are three isogeny classes of modular elliptic curves of conductor 503. Let \( F_1, F_2 \) and \( F_3 \) denote representatives in each class, and let \( f_1, f_2 \) and \( f_3 \) denote the corresponding modular forms. One checks that the fields \( \mathbb{Q}(F_i[2]) \) generated by the 2-torsion of the three elliptic curves are all isomorphic. If \( K \) denotes this extension, then one checks that \( K \) has degree 6 over \( \mathbb{Q} \), and that 2 is unramified and splits completely in the integers of \( K \). Let \( \rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{F}_2) \) denote the corresponding Galois representation, and let \( m \) denote the maximal ideal of \( T_{503} \) corresponding to the 2-torsion in any of these curves.

An explicit computation of \( S_2(\Gamma_0(503)) \) and the eigenvalues mod 2 of the Hecke operator \( T_{11} \) shows that there can be at most one other eigenform \( f_4 \) of level 503 giving rise to \( \rho \), and indeed one can check that such a form \( f_4 \) exists, defined over \( \mathbb{Q}_2 \) but not \( \mathbb{Q} \) (one need only check congruence for the first 84 coefficients). The completion of \((T_{503})_m\) now can be checked to be isomorphic to the subring of \((\mathbb{Z}_2)^4\) generated as a \( \mathbb{Z}_2 \)-module by the elements

\[ \{(1, 1, 1, 1), (0, 2, 0, 0), (0, 0, 2, 2), (0, 0, 0, 4)\} \subset (\mathbb{Z}_2)^4. \]

The unique maximal ideal of this ring is generated as a \( \mathbb{Z}_2 \)-module by the elements

\[ \{(2, 0, 0, 0), (0, 2, 0, 0), (0, 0, 2, 2), (0, 0, 0, 4)\}. \]

We see easily that \((T_{503})_m\) is Cohen-Macaulay, as it has Krull dimension 1 and no nilpotent elements. The ideal \( i \) generated by \((2, 2, 2, 2)\) is a parameter ideal, as \( m^2 \subseteq i \). Finally, the ideals

\[ i_1 := ((0, 2, 2, 2), (2, 0, 0, 0)) \quad \text{and} \quad i_2 := ((2, 0, 2, 2), (0, 2, 0, 0)) \]
have i as their intersection, hence i is reducible, and therefore the localisation 
\((T_{503})_m\) is not Gorenstein and hence \(T_{503}\) is not Gorenstein. Hence as before, 
mod 2 multiplicity one fails for \(m\).

4 Other examples

After these initial examples were discovered, William Stein suggested another 
way to check directly that multiplicity one fails in these and other cases, by an 
explicit computation in the Jacobian of the relevant modular curve.

Let \(q\) be a prime and let \(E_1\) and \(E_2\) denote two non-isogenous new optimal 
modular elliptic curves (also called strong Weil curves) of conductor \(q\), viewed as 
abelian subvarieties of the abelian variety \(J_0(q)\). Suppose that the two modular 
forms corresponding to \(E_1\) and \(E_2\) are congruent modulo 2. Let \(m\) be the 
corresponding maximal ideal of \(T_q\) over 2. If mod 2 multiplicity one holds 
for \(m\), then \(J_0(q)[m] = E_1[2] = E_2[2]\), viewed as subsets of \(J_0(q)\). In particular, 
\(E_1(C) \cap E_2(C)\) is non-zero. But this intersection can be explicitly computed 
using the \texttt{IntersectionGroup} command in Magma, and if it is zero then we 
have verified that mod 2 multiplicity one has failed without having to compute 
the completions of the relevant Hecke algebra explicitly.

In the case \(q = 431\), we find that \(E_1(C) \cap E_2(C) = \{0\}\), so we have another 
proof that mod 2 multiplicity one and the Gorenstein property fail for the Hecke 
algebra of level 431. A similar argument verifies failure of mod 2 multiplicity 
one at level 503.

This method can also be used to check our third example. There are five 
isogeny classes of elliptic curves with conductor 2089, of which four, the ones 
labelled 2089A, 2089C, 2089D, 2089E in Cremona’s tables, have isomorphic mod 2 
representations. (Note that 2089B, has rational 2-torsion, so its associated 
Galois representation is reducible.) Using the results of the previous section, 
we find that the intersection in \(J_0(2089)\) of the elliptic curves labeled 2089A 
and 2089B is \(\{0\}\). (Incidentally, the intersection of the curves labeled 2089A 
and 2089E is \((\mathbb{Z}/2\mathbb{Z})^2\).) Hence we have another example of failure of mod 2 
multiplicity one and Gorenstein-ness.

This paper answers the question raised in §3 and in §, in that it exhibits 
specific examples of non-Gorenstein Hecke algebras. This raises the natural 
question of deciding exactly which Hecke algebras are not Gorenstein, without 
explicitly computing them, and proving theorems like those of Buzzard, Edix-
hoven, Gross and Mazur for these algebras. A natural next step is to ask the 
following:

**Question 4** Are there infinitely many prime integers \(q\) such that \(T_q\) is not 
Gorenstein, and hence where mod 2 multiplicity one fails?

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