Cover and Conquer: Augmenting Decompositions for Connectivity Problems

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Abstract

We explore and introduce tools to design algorithms with improved approximation guarantees for edge connectivity problems such as the traveling salesman problem (TSP) and the 2-edge-connected spanning multigraph problem (2EC) on (restricted classes of) weighted graphs. In particular, we investigate decompositions of graphs that cover small-cardinality cuts an even number of times. Such decompositions allow us to design approximation algorithms that consist of augmenting an initial subgraph (e.g. cycle cover or spanning tree) with a cheap addition to form a feasible solution. Boyd, Iwata and Takazawa recently presented an algorithm to find a cycle cover that covers all 3- and 4-edge cuts in a bridgeless, cubic graph with applications to 2EC on 3-edge-connected, cubic graphs [BIT13]. We present several applications of their algorithm to connectivity problems on node-weighted graphs, which has been suggested as an intermediate step between graph and general metrics. Specifically, on 3-edge-connected, cubic, node-weighted graphs, we present a $\frac{7}{5}$-approximation algorithm for TSP and a $\frac{13}{10}$-approximation algorithm for 2EC. Moreover, we use a cycle cover that covers all 3- and 4-edge cuts to show that the everywhere $\frac{19}{18}$ vector is a convex combination of incidence vectors of tours, answering a question of Sebő [SBS15, BS17].

Extending this approach to input graphs that are 2-edge connected necessitates finding methods for covering 2-edge cuts. We therefore present a procedure to decompose an optimal solution for the subtour linear program into spanning, connected subgraphs that cover each 2-edge cut an even number of times. We use this decomposition to design a simple $\frac{4}{3}$-approximation algorithm for the 2-edge connected problem on subcubic, node-weighted graphs. For TSP on node-weighted, cubic graphs, we use this decomposition to show that an algorithm similar to that of Christofides has an approximation factor better than $\frac{3}{2}$ when the weight of an optimal subtour solution is strictly larger than twice the sum of the node weights.
1 Introduction

The **Traveling Salesperson problem** (TSP) is a canonical problem in combinatorial optimization that has spawned by itself a whole new set of techniques in the quest for solving it to optimality [ABCC06]; however, the basic LP relaxation for the metric version of problem (the so-called subtour relaxation) has a conjectured integrality gap of $\frac{4}{3}$ while the best known upper bound has stood firmly at the ratio of $\frac{3}{2}$ proven by Christofides [Chr76]. A recent spate of work has focused on the special case of graph-TSP when the underlying weights arise from hop distances in an undirected graph [OSS11, MS16, AGG11, BSvdSS14, Muc14, SV14]. However the current best ratio of $\frac{7}{5}$ [SV14] for the graph-TSP case has not been extended to more general metric versions. A problem closely related to TSP is the minimum-weight 2-edge-connected multigraph problem (2EC) for which the lower bound on the integrality gap for the natural LP relaxation is even lower than that for TSP [ABE06]. A parallel line of work has improved the ratio for this problem in the unweighted case as well [SV14, BFS14], which is commonly referred to as the 2-edge connected spanning subgraph problem or 2ECSS for short.

In this paper, we develop new tools for improving the approximation factors for both TSP and 2EC for a larger class of instances than graph-TSP. Since proving approximation factors for such covering problems is implied by showing convex decomposition results [CV02], we follow this approach. Moreover, we focus on cubic graphs; the most interesting class of TSP extreme points that have proven important in the study of metric TSP are cubic [BC11, CR98, SWvZ13]. Finally, we study an intermediate class of weight functions between graph-induced metrics and general metrics, namely weights induced by nodes on a skeleton graph.

In particular, we consider the **node-weight metric**, in which we are given an undirected graph and each vertex is assigned a nonnegative integer; the weight of an edge in the graph is the sum of the weights of its endpoints, and the weight of non-edges are completed as the shortest paths between them using these edge weights. The node-weight metric has been suggested as a bridge for TSP between weighted and unweighted graphs and has been previously studied by Frank [Fra92], and recently by Svensson, who gave a constant factor approximation algorithm in the asymmetric setting [Swe15].

1.1 Related Work

Currently, the best approximation factors for the graph-TSP and 2ECSS are $\frac{7}{5}$ and $\frac{4}{3}$, respectively [SVL]. We refer the reader to the comprehensive survey of Vygen for a summary of recent developments for TSP [Vyg12]. For general metric edge weight functions, the best-known approximation factor for both problems is $\frac{3}{2}$ via Christofides’ famous algorithm [Chr76]. Moreover, the only case beyond the unit edge weight function for which the barrier of $\frac{3}{2}$ has been successfully breached is the very restricted case where all edges in a complete graph
have weights 1 or 2 \cite{QSWvZ15}. While the TSP contains a connectivity and Eulerian (even degree at nodes) requirement, the 2EC does not have the degree parity requirement but an enhanced connectivity requirement; this should seemingly make 2EC easier than TSP as reflected in the best-known lower bounds for the integrality gaps for these problems \( \frac{6}{5} \) and \( \frac{4}{3} \) respectively. However, even the \( \frac{3}{2} \) upper bound result for metric weights for 2EC \cite{FJ81} uses Christofides’ algorithm that returns a tour. Similarly, even for the best known upper bound of \( \frac{4}{3} \) for the unweighted case of 2ECSS \cite{SV14}, some of the cases involve returning a solution for graph-TSP. One of our goals is to develop methods that more clearly separate the approaches in solving these problems.

In a related problem, the approximation factor for the path-TSP with metric edge weights almost meets \cite{SvZ16} its conjectured bound of \( \frac{3}{2} \) that has been achieved in the graph metric \cite{SV14, Gao13}.

### 1.2 Current Techniques

We consider the Symmetric Traveling Salesperson Problem (STSP) with metric weights: given an undirected complete graph \( G = (V, E) \) with nonnegative weights \( w \) on the edges that obey the triangle inequality (\( w_{ij} \leq w_{ik} + w_{kj} \) for all distinct \( i, j, k \in V \)), find the minimum-weights Hamilton cycle. The problem has been extensively studied in overlapping areas of theoretical computer science, operations research and discrete mathematics: two among several survey books \cite{LLRK85, ABCC06} on the subject offer a glimpse of the wide-ranging interest in the problem.

A natural integer programming formulation for the problem \cite{DFJ54} uses a binary variable \( x \) for each edge indicating its inclusion in the Hamilton cycle. As usual, we use \( \delta(v) \) to denote the set of edges incident on \( v \).

\[
\begin{align*}
\sum_{e \in V \times V} w_e x_e & \quad \forall S \subset V \quad \sum_{e \in S \times S} x_e \leq |S| - 1 \\
\sum_{e \in \delta(v)} x_e & \quad \forall v \in V \quad x_e \in \{0, 1\}
\end{align*}
\]

Figure 1: The subtour elimination integer program

Let us denote by \( z_{SER} \) the optimal value of the LP relaxation obtained by replacing the last set of integrality constraints with their continuous version: \( 0 \leq x_e \leq 1 \ \forall e \in V \times V \). The first set of constraints lead to the popular name for this bound as the “subtour elimination relaxation” (\( z_{SER} \) is also known as the Held-Karp bound \cite{HK70} in the literature). A folklore conjecture sometimes tersely called the “four-thirds conjecture” (see, e.g. \cite{CV04, Goe95}) is
that the optimal (integral) solution for the metric TSP is no more than \( \frac{4}{3} \) times the subtour elimination linear programming relaxation, or that \( z_{TSP} \leq \frac{4}{3} \cdot z_{SER} \).

**Structural Methods:** The Christofides’s heuristic [Chr76] for the metric TSP uses an augmentation approach. It begins by finding a minimum weight spanning tree that satisfies the connectivity requirement in the solution. By doubling every edge in the MST, we obtain a graph with all degrees even, and hence an Eulerian connected graph. Taking an Euler tour visiting each edge exactly once in this graph, and short-cutting over non-first visits to every node, we obtain a Hamilton cycle. Since the shortcuts have weights no more than that of the paths they replace by the triangle inequality (and induction), we get a tour of weight at most twice that of the MST. Since any TSP minus a single edge gives a feasible spanning tree, the minimum weight spanning tree has lower weight than \( z_{TSP} \) and hence this doubled short-cutting of the MST gives a simple 2-approximation algorithm. Christofides’ approach replaces the doubling with a minimum weight matching on the odd degree nodes of the MST. This matching along with the original MST also has even degree at every node and is connected. Proceeding as before this Eulerian connected graph can also be short-cut to a tour of no higher weight. By considering an optimal TSP tour induced only on the odd degree nodes of the MST, we can partition the tour into exactly two perfect matchings between these nodes, and thus one of them has weight at most half that of the TSP tour. These observations show that the method gives a \( \frac{3}{2} \)-approximation with respect to the optimal integral TSP tour.

(A similar method by Hoogeveen for minimum-weight \( s-t \) Hamilton path [Hoo91] also consists of a single augmentation of an arbitrary spanning tree.)

To show that it has the same ratio even with respect to \( z_{SER} \) requires a bit more work: a solution to \( z_{SER} \) can be decomposed into a convex combination of 1-trees [HK70]. Similarly to bound the matching over the odd-degree nodes w.r.t \( z_{SER} \), the set of such matching solutions can be characterized by weighted T-joins [EJ73] with \( T \) being the set of odd-degree nodes in the 1-tree. By showing that the LP solution achieving value \( z_{SER} \) scaled down by 2 satisfies all the constraints for this T-join problem, we obtain a proof that Christofides’ solution is at most \( \frac{3}{2} \) the value of the subtour elimination LP bound \( z_{SER} \) [Wo80, SW90]. Surprisingly, this bound on the integrality gap has not yet been improved in nearly 40 years. Moreover, this analysis and the factor of \( \frac{3}{2} \) is also the best approximation guarantee known for the 2EC problem.

**1.3 Graph-TSP and 2ECSS**

Most recent progress on STSP has been in the case where the weights arise as distances in a given unweighted undirected graph, also termed graph-TSP. The first improvement to slightly below \( \frac{3}{2} \) [OSS11] showed that the support of the subtour elimination LP relaxation either has a large number of nearly integral edges or a large number of edges which do not occur in any
near-minimal odd cuts; the latter fact was exploited in the matching phase of Christofides’ algorithm. A second stream of results [BSvdSS14, AGG11] proved the conjectured \( \frac{4}{3} \) bound for cubic, and cubic 3-edge-connected graphs respectively but again only for the graph-TSP. These papers take the approach of finding appropriate cycle covers and repairing them until every cycle is large (say of average size at least six). This allows them to connect the cycles using a doubled spanning tree on the graph obtained after contracting these cycles and thus obtain an Eulerian connected subgraph of size at most \( \frac{4}{3} \) the number of nodes.

A third breakthrough idea of Mömke and Svensson [MS16] uses matching theory to get a 1.461-approximation for the graph-TSP problem with respect to the Held-Karp lower bound, by using another structural idea of removable pairs of edges; its analysis was further refined to give a \( \frac{13}{9} \) approximation [Muc14], again for the graph-TSP problem. Continuing on this trend, an alternate view of the structure of removable pairs using ear decompositions was proposed by Sebő and Vygen [SV14] who designed an algorithm that achieves the currently best-known integrality gap of \( \frac{7}{5} \) for graph-TSP. It is useful to point out that all these improvement have used structural methods of bounding the LP value.

There is a parallel line of work on 2ECSS. As previously noted, Sebő and Vygen gave a \( \frac{4}{3} \)-approximation algorithm for the 2ECSS problem [SV14], and Boyd, Fu and Sun gave a \( \frac{5}{4} \)-approximation for the 2ECSS problem on bridgeless, cubic graphs [BFS14]. Note that a TSP tour is a 2-edge connected spanning multigraph that is in addition Eulerian, i.e., each vertex has even degree.

Much of this recent progress for graph-TSP and 2ECSS has stemmed from tools that seems difficult to apply to weighted metrics, e.g. the aforementioned removable pairings used by Mömke and Svensson [MS16]. We now look more closely at some approaches that are closely related to the theorems we present in this paper.

**Augmenting a DFS tree.** As an intermediate step in their algorithm, Mömke and Svensson augment a DFS tree to find a 2-vertex-connected subgraph [MS16, New14]. These subgraphs are constructed by using a circulation to find a small subset of back edges to augment a carefully chosen DFS tree. Boyd, Fu and Sun [BFS14] follow this approach, essentially based on DFS tree augmentation, to design a \( \frac{5}{4} \)-approximation algorithm for 2ECSS on bridgeless, subcubic (unweighted) graphs. This approach would extend to a weighted metric if one could find a DFS tree with weight at most that of an optimal 2EC solution. However, there is no guarantee that such a DFS tree exists. Moreover, the problem of finding a minimum weight DFS tree is NP-hard to approximate within any multiplicative factor. (See the Appendix A for more discussion). It is therefore not straightforward to generalize the techniques in [MS16] to weighted graphs.

**Cycle Covers in Cubic Graphs Covering Certain Cuts.** The first improvement for TSP over Christofides’ algorithm was due to Gamarnik et. al [GLS05] for graph-TSP on 3-edge-connected, cubic graphs. In their algorithm, they find a cycle cover (which always
exists via Petersen’s Theorem) containing strictly fewer (i.e. by a constant factor) than \( \frac{n}{4} \) cycles. They then contract the cycles in the cycle cover and augment the cycle cover with a doubled spanning tree of the contracted graph, thereby obtaining a tour of weight no more than \((\frac{3}{2} - \frac{5}{389})n\). Boyd et. al greatly expanded this technique to improve the approximation ratio to \( \frac{4}{3} \) \([\text{BSvdSS14}]\). This technique has also been used in approximation algorithms for 2ECSS. Extending a nonconstructive result of Kaiser and Škréklovský \([\text{KS08}]\), Boyd, Iwata and Takazawa, presented a polynomial time algorithm to find a cycle cover that covers every 3-edge and 4-edge cut in a cubic graph \([\text{BIT13}]\). In a 3-edge-connected cubic graph, contracting the cycles in such a cycle cover results in a 5-edge-connected graph, which they use to give a \( \frac{6}{5} \)-approximation for 2ECSS.

1.4 Our Contributions

We present applications of the algorithm of Boyd, Iwata and Takazawa for finding a cycle cover that covers all 3-edge and 4-edge cuts in an unweighted, bridgeless, cubic graph \([\text{BIT13}]\). In particular, we show the following.

- For 3-edge-connected, cubic, and node-weighted graphs, we give a \( \frac{7}{5} \)-approximation algorithm for TSP and a \( \frac{13}{10} \)-approximation algorithm for 2EC (Theorems 2 and 3 in Section 3.1). This improves the current best \( \frac{3}{2} \)-approximation for these problems that follows from Christofides.

In an attempt to extend these results to bridgeless (but not necessarily 3-edge-connected) graphs, we next explore constructing and augmenting spanning, multigraphs that cover all 2-edge cuts. In particular, we show the following.

- We show that a feasible solution for the subtour elimination linear program on any bridgeless graph \( G \) can be decomposed into a convex combination of connected, spanning multigraphs such that each multigraph in this decomposition intersects every 2-edge cut in \( G \) an even number of times. (Section 4.)

- We give a \( \frac{4}{3} \)-approximation algorithm for 2EC on bridgeless, subcubic, node-weighted graphs. This algorithm uses an algorithm for rounding a half-integral solution for the tree augmentation problem as a subroutine. (Theorem 6 in Section 5.1.) Again, this is an improvement over the factor of \( \frac{3}{2} \) due to Christofides.

- We give a \( (1 + \frac{\beta}{3}) \)-approximation algorithm for TSP when the total edge weight \( \sum_{e \in E} w(e) \) is at most \( \beta \cdot z_{SER} \). (Theorem 8 in Section 5.2.) This implies a \( \left( \frac{3}{2} - \frac{3\epsilon}{4} \right) \)-approximation algorithm for TSP on node-weighted, subcubic graphs when \( z_{SER} \) is strictly larger than twice the sum of the node weights by a multiplicative factor of \((1 + \epsilon)\). (Theorem 9 in Section 5.2.)
Finally, we use a combination of these tools to obtain improved constructive results for uniform covers of 3-edge-connected cubic graphs.

- We show that for all 3-edge-connected, cubic graphs, the everywhere \( \frac{18}{19} \) vector (i.e. \( \frac{18}{19} \) on all edges) can be written as a convex combination of tours. Our proof is constructive. This answers a question posed by Sebő [BS15, BS17]. Note that the best known factor for this problem so far is 1 [BS15].

- We show that there is an efficient algorithm for finding a decomposition of the everywhere \( \frac{15}{19} \) vector into 2-edge-connected spanning multigraphs. (Theorems 10 and 12 in Section 6.) Note that the best known constructive factors for this problem so far is 1, while the best known nonconstructive factor is \( \frac{7}{9} \) [Leg17].

# 2 Notation and Preliminaries

In this paper, \( G = (V, E) \) will denote a weighted graph and \( w(e) \) denotes the weight of edge \( e \in E \). Graph \( G \) is node-weighted if there is a function \( f: V \to \mathbb{R}^+ \) such that for each \( e = uv \in E \) we have \( w(e) = f(v) + f(u) \). In this case, we say \( G \) is node-weighted with node-weight function \( f \). Denote by \( w(E) \) the total edge weight: \( \sum_{e \in E} w(e) \). For ease of notation let \( n = |V| \). For vectors \( a, b \in \mathbb{R}^m \) we say \( a \) dominates \( b \) if \( a \geq b \).

We will work with multisets of edges of \( G \). For a multiset \( F \) of \( E \), the submultigraph induced by \( F \) (henceforth referred to simply as a multigraph) is a graph that has the same number of copies of each edge as in \( F \). For a positive integer \( t \), the multiset \( t \cdot F \) is the multiset that contains \( t \) copies of each element in \( F \). For multisets \( F \) and \( F' \), we denote by \( F \cup F' \) the multiset that contains as many copies of each edge as those in \( F \) plus those in \( F' \). For a multiset of edges (or a multigraph) \( F \) the summand \( \sum_{e \in F} w(e) \) counts each edge \( e \in F \) as many times as the number of copies it has in \( F \).

A multigraph \( F \) of \( G \) is a tour if the vertex set of \( F \) spans \( G \), \( F \) is connected, and every vertex in \( F \) has even degree. We henceforth use the term 2-edge-connected multigraph of \( G \) to refer to a 2-edge-connected spanning multigraph, i.e. a multigraph that spans all the vertices of \( G \). Subgraph \( T \) of \( G \) is a 1-tree of \( G \) if it consists of a spanning tree plus an extra edge. For a subset of edges \( S \subseteq E \), the graph \( G/S \) is the graph obtained from \( G \) by contracting the edges in \( S \).

## 2.1 Subtour Elimination Linear Program

Consider a (not necessarily complete) weighted graph \( G = (V, E) \) with edge weights \( w(e) \) for \( e \in E \). The output of TSP and 2EC on input graph \( G \) is a minimum weight tour and a minimum weight 2-edge connected multigraph of \( G \), respectively. The following relaxation
provides a lower bound for the weight of the optimal solution to both problems.

\[ z_G = \min \sum_{e \in E} w(e) x_e \]

\[ x(\delta(S)) \geq 2 \quad \text{for } \emptyset \subset S \subset V \quad \text{(SUBTOUR(G))} \]

\[ x_e \geq 0 \quad \text{for } e \in E. \]

The metric completion of \( G \) is the complete graph \( G^{\text{met}} \) on the vertex set of \( G \) such that for \( u, v \in V \) the weight of the \( uv \) edge in \( G^{\text{met}} \) is the weight of the shortest path between \( u \) and \( v \) in \( G \). Clearly, these weights obey the triangle inequality. TSP on \( G \) is equivalent to finding a minimum weight Hamilton cycle in \( G^{\text{met}} \). Also, Cunningham showed that the bound \( z_G \) is equal to the bound \( z_{\text{SER}} \) computed on the metric completion of \( G \) (see [MMP90, GB93]).

### 3 Applications of BIT Cycle Covers

Given a graph \( G = (V, E) \), a cycle cover (also known as a 2-factor) of \( G \) is a collection of vertex disjoint cycles whose vertex sets partition \( V \). Cycle covers have been extensively studied in the area of matching theory and have been also used to obtain approximation algorithms for TSP.

Kaiser and Škrekovski [KŠ08] proved that every bridgeless, cubic graph has a cycle cover that covers all 3-edge and 4-edge cuts of the graph. Their proof is not algorithmic and a constructive version was given by Boyd, Iwata and Takazawa [BIT13].

**Theorem 1** (Boyd, Iwata and Takazawa [BIT13]). Let \( G \) be a bridgeless, cubic graph. Then there is an algorithm whose running time is polynomial in the size of \( G \) that finds a cycle cover of \( G \) covering every 3-edge and 4-edge cut of \( G \).

We will show that this powerful theorem can be used to obtain approximation algorithms for TSP and 2EC. A straightforward corollary of Theorem 1 is the following.

**Corollary 1.1.** Let \( G \) be a 3-edge-connected, cubic graph. Let \( C \) be a cycle cover that covers 3-edge cuts and 4-edge cuts in the graph. Then \( G/C \) is a 5-edge-connected graph.

In this section we use Theorem 1 to design approximation algorithm for special cases of TSP and 2EC. We will use Theorem 1 again in Section 6 to answer a question about uniform covers posed by Sebő [SBS15, BS17].

### 3.1 Augmenting BIT Cycle Covers

Suppose that \( G \) is node-weighted with function \( f : V \to \mathbb{R}^+ \). The following lemmas are specific to node-weighted, 3-edge-connected, cubic graphs.
Lemma 1. We have $z_G = 2 \cdot \sum_{v \in V} f_v$.

Proof. For any $x \in \text{Subtour}(G)$ we have $x(\delta(v)) \geq 2$. So,

$$\sum_{e \in E} w(e)x_e = \sum_{v \in V} x(\delta(v)) \cdot f_v \geq 2 \cdot \sum_{v \in V} f_v.$$ 

Thus, $z_G \geq 2 \cdot \sum_{v \in V} f_v$. On the other hand, let $x'_e = \frac{2}{5}$ for all $e \in E$. Notice $x' \in \text{Subtour}(G)$ and $\sum_{e \in E} w(e)x'_e = 2 \cdot \sum_{v \in V} f_v$. Hence $z_G \leq 2 \cdot \sum_{v \in V} f_v$. □

Also notice that for a cycle cover $C$ of $G$, we have $\sum_{e \in C} w(e) = 2 \cdot \sum_{v \in V} f_v = z_G$. Similarly, for a matching $M$ of $G$ we have $\sum_{e \in M} w(e) = \sum_{v \in V} f_v = \frac{5z_G}{2}$.

Theorem 2. There is a $\frac{7}{5}$-approximation algorithm for TSP on node-weighted, 3-edge-connected, cubic graphs.

Proof. Let $C$ be a cycle cover of $G$ that covers all 3-edge and 4-edge cuts of $G$. By Corollary 1.1 the graph $G/C$ is 5-edge-connected. For $e \in E(G/C)$, let $y_e = \frac{2}{5}$. Notice, $y \in \text{Subtour}(G/C)$, since for every cut $\delta(S)$ of $G/C$ we have $y(\delta(S)) \geq \frac{2}{5} \cdot 5 \geq 2$. Let $S$ be the minimum spanning tree of $G/C$.

$$\sum_{e \in S} w(e) \leq \sum_{e \in E(G/C)} w(e)y_e \leq \frac{1}{5} z_G$$

Finally, notice $C \cup 2S$ is a tour of $G$ and

$$\sum_{e \in C \cup 2S} w(e) \leq \sum_{e \in C} w(e) + 2 \cdot \sum_{e \in S} w(e) \leq z_G + \frac{2}{5} z_G = \frac{7}{5} z_G.$$ □

Next we show that we can use a very similar approach to 2EC on node-weighted, 3-edge-connected, cubic graphs.

Theorem 3. There is a $\frac{13}{10}$-approximation algorithm for 2EC on node-weighted, 3-edge-connected, cubic graphs.

Proof. Let $C$ be a cycle cover of $G$ that covers all 3-edge and 4-edge cuts of $G$. By Corollary 1.1 graph $G/C$ is 5-edge-connected. For $e \in E(G/C)$ let $y_e = \frac{2}{5}$. Notice that $y \in \text{Subtour}(G/C)$. By Christofides' algorithm, one can find a 2-edge-connected multigraph $S$ on $G/C$, such that
\[ \sum_{e \in S} w(e) \leq \frac{3}{2} \sum_{e \in E(G/C)} w(e) y_e. \] In particular,

\[
\sum_{e \in S} w(e) \leq \frac{3}{2} \sum_{e \in E(G/C)} w(e) y_e \\
\leq \frac{3}{2} \sum_{e \in E(G) \setminus C} w(e) y_e (E(G/C) \subseteq E(G) \setminus C) \\
\leq \frac{3}{5} \sum_{e \in E(G) \setminus C} w(e) (y_e \leq \frac{2}{5} \text{ for } e \in E(G) \setminus C) \\
\leq \frac{3}{10} z_G (E(G) \setminus C \text{ is a matching of } G).
\]

Note that \( C \cup S \) is a 2-edge-connected multigraph of \( G \) and

\[
\sum_{e \in C \cup S} w(e) \leq \sum_{e \in C} w(e) + \sum_{e \in S} w(e) \leq z_G + \frac{3}{10} z_G = \frac{13}{10} z_G. \tag{4}
\]

We note that Legault recently showed that \( \frac{7}{6} x^* \) is a convex combination of 2-edge-connected subgraphs for a 3-edge-connected, cubic graph \( G \) \cite{Leg17}. This result is not constructive, but it does give a stronger bound on the integrality gap than that implied by Theorem 3.

We will see more applications of a BIT cycle cover in Section 6.

## 4 A Connector Decomposition Theorem

The result in Theorems 2 and 3 do not apply to bridgeless, cubic graphs. In this section, we give an alternative tool to the BIT cycle cover for graphs that are not 3-edge-connected, i.e. graphs that contain 2-edge cuts. In particular, we find a decomposition of an optimal solution \( x^* \) for \( \text{Subtour}(G) \) that has certain properties. Many approaches for TSP decompose \( x^* \) into a convex combination of spanning trees, whose average weight does not exceed \( z_G \). In this section, we propose an alternate way of decomposing \( x^* \) into connectors.

**Definition 1.** A connector \( F \) of graph \( G \) is a (multi) subset of edges of \( G \) such that \( F \) is connected and spanning and contains at most two copies of each edge in \( G \).

It is known that a vector \( x^* \in \text{Subtour}(G) \) dominates a convex combination of spanning trees (and hence connectors) of \( G \). We now show that \( x^* \) can be decomposed into connectors with the additional property that every 2-edge cut is covered an even number of times. These connectors can be augmented to obtain a tour or a 2-edge-connected multigraph of \( G \), and under certain conditions, this property can be exploited to bound the weight of an augmentation.
Theorem 4. Let $x^* \in \text{SUBTOUR}(G)$ Then $x^*$ can be decomposed into a family of connectors, $\mathcal{F} = \{F_1, \ldots, F_\ell\}$, such that

(a) $x^* \geq \sum_{i=1}^\ell \lambda_i F_i$, where $\sum \lambda_i = 1$ and $\lambda_i > 0$, and

(b) every $F_i$ has an even number of edges crossing each 2-edge cut in $G$.

We note that $G$ can be assumed to be the support of $x^*$, so every $F_i$ will actually have an even number of edges crossing each 2-edge cut in the support of $G$ on $x^*$. Moreover, note that if $x^*_e \leq 1$ for all $e \in E$, then a decomposition of $x^*$ into 1-trees respects property (b). (In fact, for an unweighted graph $G$, there always exists an optimal solution $x^*$ for $\text{SUBTOUR}(G)$ such that $x^*_e \leq 1$ for all edges $e$ [Vyg12].)

4.1 Proof of Theorem 4

To prove Theorem 4 we need to understand the structure of 2-edge cuts in a 2-edge connected graph. Assume $G = (V, E)$ is a 2-edge-connected graph. For $S \subseteq V$, let $G[S]$ denote the subgraph induced by vertex set $S$. Let $\delta(S) \subseteq E$ denote the edges crossing the cut $(S, V \setminus S)$.

Lemma 2. If $S \subseteq V$ and $|\delta(S)| = 2$, then $G[S]$ is connected.

Proof. Suppose not, then $S$ can be partitioned into $S_1$ and $S_2$, such that there is no edge in $G$ between $S_1$ and $S_2$. Hence, $|\delta(S_1)| + |\delta(S_2)| = 2$. However, since $G$ is 2-edge-connected we have $|\delta(S_1)| + |\delta(S_2)| \geq 4$, which is a contradiction. \hfill \square

Lemma 3. Let $e, f$ and $g$ be distinct edges of $G$. If $\{e, f\}$ and $\{f, g\}$ are each 2-edge cuts in $G$, then $\{e, g\}$ is also a 2-edge cut in $G$.

Proof. Let $S, T \subseteq V$ be such that $\delta(S) = \{e, f\}$ and $\delta(T) = \{f, g\}$. Without loss of generality, we can assume that neither endpoint of $e$ belongs to $T$. (If both endpoints of $e$ belong to $T$, we set $T$ equal to its complement.) Moreover, we can assume that $S \cap T \neq \emptyset$ (since otherwise we can set $S$ equal to its complement). We can also assume that $S \setminus T \neq \emptyset$ (since at least one endpoint of $e$ belongs to $S$ but not to $T$). Suppose $T \setminus S$ is not empty. By Lemma 2 $G[T]$ is connected. Hence there exists an edge $h$ from $S \cap T$ to $T \setminus S$. Notice $h \in \delta(S)$, and $h \notin \delta(T)$. Therefore, $h = e$. However, since both endpoints of $h$ are in $T$, this is a contradiction. So we can assume that $T \setminus S = \emptyset$. In other words, $T \subseteq S$.

Now we show that $\delta(S \setminus T) = \{e, g\}$. Since $T \subseteq S$ and neither endpoint of $e$ belongs to $T$, it follows that $e \in \delta(S \setminus T)$. Moreover, since only one endpoint of $g$ belongs to $T$ (and therefore to $S$) and $g \notin \delta(S)$, it follows that $g \in \delta(S \setminus T)$. So we have $\{e, g\} \subseteq \delta(S \setminus T)$. Suppose there is another edge $h \in \delta(S \setminus T)$ with endpoints $v \in S \setminus T$ and $u \notin S \setminus T$. Note that $h \neq f$, because neither endpoint of $f$ belongs to $S \setminus T$. If $u \in T$, then $h \in \delta(T)$ which is a contradiction to $T$ being a 2-edge cut. Otherwise if $u \in V \setminus S$, then $h \in \delta(S)$ which is again a contradiction to $S$ being a 2-edge cut. \hfill \square
We will later use these properties when building a family of connectors to delete and replace edges along the 2-edge cuts of the graph. Next, we need a decomposition lemma for $x^*$.

**Lemma 4.** A vector $x^* \in \text{SUBTOUR}(G)$ can be represented as a convex combination of polynomially many connectors of $G$.

*Proof.* By Theorem 50.8b in [Sch03] the following polytope is the convex hull of connectors of $G$.

$$
\begin{align*}
  x(\delta(\mathcal{P})) & \geq |\mathcal{P}| - 1 & \text{for } \mathcal{P} \in \Pi_n \\
  0 & \leq x \leq 2 & \text{for } e \in E
\end{align*}
$$

Here, $\Pi_n$ is the collection of partitions of $V$. For $\mathcal{P} \in \Pi_n$, we denote by $\delta(\mathcal{P})$ the set of edges with endpoint in different parts of partition $\mathcal{P}$, and $|\mathcal{P}|$ is the number of parts in partition $\mathcal{P}$.

Notice that for any partition $\mathcal{P}$ of $V$ with parts $P_1, \ldots, P_{|\mathcal{P}|}$ we have

$$
  x^*(\delta(\mathcal{P})) = \frac{1}{2} \sum_{i=1}^{|\mathcal{P}|} x^*(\delta(P_i)) \geq |\mathcal{P}|.
$$

Therefore, $x^*$ can be written as a convex combination of connectors of $G$. The fact that the number of connectors in the convex combination is polynomial follows from the fact that the polytope above is separable, and hence we can apply the constructive version of Carathéodory’s theorem to get the result [GLSSS, Sch03].

By Lemma 4 there exists positive reals $\lambda_1, \ldots, \lambda_\ell$, such that $\sum_{i=1}^\ell \lambda_i = 1$, and connectors $F_1, \ldots, F_\ell$ such that

$$
  x^* = \sum_{i=1}^\ell \lambda_i \chi^{F_i},
$$

where $\chi^{F_i}$ is the characteristic vector of $F_i$ for $i \in \{1, \ldots, \ell\}$. Furthermore, we can find this decomposition in time polynomial in the size of $G$. Notice $F_1, \ldots, F_\ell$ satisfy (a) in the statement of Theorem 4. We will now show that given $F_1, \ldots, F_\ell$, we can obtain a new family of connectors satisfying both (a) and (b) from Theorem 4.

**Lemma 5.** Given a family of connectors $F_1, \ldots, F_\ell$ of $G$ such that $x^* = \sum_{i=1}^\ell \lambda_i \chi^{F_i}$, $\lambda_i > 0$ for $i \in \{1, \ldots, \ell\}$, and $\sum_{i=1}^\ell \lambda_i = 1$, there is a polynomial-time algorithm that outputs connectors $F'_1, \ldots, F'_\ell$ such that

1. $x^* = \sum_{i=1}^\ell \lambda_i \chi^{F'_i}$.

2. If $x^*_e \geq 1$, then $\chi^{F'_i}(e) \geq 1$ for all $i \in \{1, \ldots, \ell\}$. 

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(3) If $x_e^* < 1$, then there is no $i \in \{1, \ldots, \ell\}$ such that $\chi^{F_i}(e) = 2$.

Proof. Call a tuple $(e, i, j)$ where $e \in E$, $i, j \in \{1, \ldots, \ell\}$ bad if

$$\chi^{F_i}(e) = 2 \text{ and } \chi^{F_j}(e) = 0.$$ 

Let $m$ be the number of bad tuples and let $(e, i, j)$ be a bad tuple. Then

$$F'_i = F_i - e, \quad F'_j = F_j + e, \quad \text{and } F'_p = F_p \text{ for } p \in \{1, \ldots, \ell\} \setminus \{i, j\},$$

satisfies properties (1) and (2). Notice that now $F'_1, \ldots, F'_\ell$ has at most $m - 1$ bad tuples; no new bad tuples are created by the above procedure. Thus, after at most $m$ iterations, we have that for each $e \in E$, there is no $i, j \in \{1, \ldots, \ell\}$ such that $\chi^{F'_i}(e) = 2$ and $\chi^{F'_j}(e) = 0$. This implies properties (2) and (3) in the statement of the lemma. Finally, it is also easy to see that fixing each tuple can be done in polynomial time, and that the number of tuples is polynomial in the size of $G$. 

We now proceed to the proof of Theorem 4. By Lemma 3 all edges that belong to a 2-edge cut in $G$ can be partitioned into a collection $D$ of disjoint subsets of edges of $G$ such that for all $D \in D$: (i) $|D| \geq 2$, and (ii) for each pair of edges $\{e, f\} \subseteq D$, edges $e$ and $f$ form a 2-edge cut of $G$. Note that for $D \in D$ and any distinct edges $e, f \in D$, it cannot be the case that both $x_e^* < 1$ and $x_f^* < 1$, since $\{e, f\}$ is a 2-edge cut and $x^* \in \text{SUBTOUR}(G)$. We classify the subsets in $D$ into two types:

$$D_1 = \{D \in D : \text{ for all } e \in D, \ x_e^* \geq 1\},$$

$$D_2 = \{D \in D : D \text{ there is exactly one edge } e \in D \text{ such that } x_e^* < 1\}.$$ 

Let $F_1, \ldots, F_\ell$ be a family of connectors satisfying properties (1), (2) and (3) in Lemma 5. We propose a procedure to modify these connectors and output $F'_1, \ldots, F'_\ell$ such that for each $D \in D$, property (b) in Theorem 4 is satisfied while property (a) is preserved. In particular, by property (1) from Lemma 5, we have

$$\sum_{i=1}^\ell \chi^{F_i}(e) = x_e^* \text{ for } e \in E.$$ 

Our specific procedure depends on whether $D \in D_1$ or $D \in D_2$.

**Case 1 ($D \in D_1$):** In this case, we have $\chi^{F_i}(e) \geq 1$ for all $e \in D$ and $i \in \{1, \ldots, \ell\}$, by property (2) in Lemma 5. For $i \in \{1, \ldots, \ell\}$ let $F'_i$ be such that

$$\chi^{F'_i}(e) = 1 \text{ for } e \in D \text{ and } \chi^{F'_i}(e) = \chi^{F_i}(e) \text{ for } e \in E \setminus D.$$
Now we reset $F_1, \ldots, F_\ell := F'_1, \ldots, F'_\ell$, and proceed to the next $D \in D_1$.

It is easy to see that we can apply this procedure iteratively for $D \in D_1$. This is because after this applying this operation on $D \in D_1$, properties (2) and (3) in Lemma 5 are preserved. Moreover, property (1) in Lemma 5 is also preserved for every edge not in $D$, i.e.

$$
\sum_{i=1}^{\ell} \lambda_i \chi_{F'_i}(e) = x^*_e \text{ for all } e \in E \setminus D \quad \text{and} \quad \sum_{i=1}^{\ell} \lambda_i \chi_{F'_i}(e) \leq x^*_e \text{ for all } e \in D.
$$

In addition, given any 2-edge cut $\{e, f\}$ such that $\{e, f\} \subseteq D$ for $D \in D_1$, we have $\chi_{F'_i}(e) + \chi_{F'_i}(f) = 1 + 1 = 2$ for all $i \in \{1, \ldots, \ell\}$.

**Case 2 ($D \in D_2$):** Let $e$ be the unique edge in $D$ with $x^*_e < 1$. By property (3) in Lemma 5 we have $\chi_{F'_i}(e) \leq 1$ for all $i \in \{1, \ldots, \ell\}$. Without loss of generality, assume for $\chi_{F'_i}(e) = 1$ for $i = 1, \ldots, p$ and $\chi_{F'_i}(e) = 0$ for $i = p + 1, \ldots, k$. For $i = 1, \ldots, p$, let $F'_i$ be such that

$$
\chi_{F'_i}(f) = 1 \text{ for } f \in D \text{ and } \chi_{F'_i}(f) = \chi_{F_i}(f) \text{ for } f \in E \setminus D.
$$

For $i = p + 1, \ldots, \ell$, let $F'_i$ be such that

$$
\chi_{F'_i}(e) = 0, \chi_{F'_i}(f) = 2 \text{ for } f \in D \setminus \{e\} \text{ and } \chi_{F'_i}(f) = \chi_{F_i}(f) \text{ for } f \in E \setminus D.
$$

Now we reset $F_1, \ldots, F_\ell := F'_1, \ldots, F'_\ell$, and proceed to the next $D \in D_2$. After each iteration, we observe that

$$
\sum_{i=1}^{\ell} \lambda_i \chi_{F'_i}(e) = \sum_{i=1}^{p} \lambda_i \chi_{F'_i}(e) + \sum_{i=p+1}^{\ell} \lambda_i \chi_{F'_i}(e)
= \sum_{i=1}^{\ell} \lambda_i = x^*_e.
$$

(6)

For $f \in D \setminus \{e\}$, we have

$$
\sum_{i=1}^{\ell} \lambda_i \chi_{F'_i}(f) = \sum_{i=1}^{p} \lambda_i \chi_{F'_i}(f) + \sum_{i=p+1}^{\ell} \lambda_i \chi_{F'_i}(f)
= \sum_{i=1}^{p} \lambda_i + 2 \sum_{i=p+1}^{\ell} \lambda_i
= 2 - x^*_e \quad \text{(From (6).)}
\leq x^*_f. \quad \text{(Since } x^* \in \text{ Subtour}(G).)
This also clearly holds for any $f \in E \setminus D$ as we do not touch these edges. Note that after the final iteration, $F_1, \ldots, F_\ell$ are connected, spanning multigraphs of $G$, because we began with connected, spanning multigraphs and we only remove an edge $f$ from $F_i$ if it contained at least two copies of $f$.

Finally, note that given any 2-edge cut $\{e, f\} \in D$ for $D \in \mathcal{D}_2$, we have $\chi^{F_i}(e) + \chi^{F_i}(f) = 1 + 1 = 2$, $\chi^{F_i}(e) + \chi^{F_i}(f) = 0 + 2 = 2$ or $\chi^{F_i}(e) + \chi^{F_i}(f) = 2 + 2 = 4$ for all $i \in \{1, \ldots, \ell\}$. This concludes the proof of Theorem 4.

5 Augmenting Connector Decompositions

We now present two applications of Theorem 4. In the first application, we show that there is a set of edges that can be added to a connector to yield a 2-edge connected graph, and that this set of edges is cheap for a node-weighted, subcubic graph. This addition can be found via a reduction to the 1-cover problem. The 1-cover problem generalizes the tree augmentation problem: given a connected subgraph, the goal is to find an additional subset of edges (from the complement) to make the original structure 2-edge connected. While the best-known approximation factor for this problem is 2 [HJ81], when the solution is half-integral, there is a $\frac{4}{3}$-approximation [CR98]. In the second application, we show that for a node-weighted, subcubic graph, Christofides’ algorithm has an approximation factor better than $\frac{3}{2}$ when the weight of an optimal subtour solution is strictly larger than twice the sum of the node weights.

5.1 An Approximation Algorithm for 2EC

Let $G = (V, E)$ be a weighted graph. The goal of this section is to prove the following theorem.

**Theorem 5.** If $w(E) \leq \beta \cdot z_G$, then there is a $\frac{1+2\beta}{3}$-approximation algorithm for 2EC on $G$.

From Theorem 5, we can obtain the following corollary for node-weighted graphs.

**Theorem 6.** If $G$ is a node-weighted, subcubic graph, then there exists a $\frac{4}{3}$-approximation for 2EC on $G$.

**Proof.** In this case $w(E) \leq 3 \sum_{v \in V} f(v)$, where $f : V \rightarrow \mathbb{R}^+$ is the node-weight function. Also, notice $z_G \geq 2 \sum_{v \in V} f(v)$. Thus, $\beta$ in statement of Theorem 5 is $\frac{3}{2}$. \qed

Before proving Theorem 5 we need to state the 1-cover problem on a laminar family of sets. A family of sets $\mathcal{S}$ is called laminar if for any $S$ and $S'$ in $\mathcal{S}$, the set $S \cap S'$ is equal to either $S$, $S'$ or $\emptyset$. In this problem, we are given a laminar family of sets, $\mathcal{S}$, where each set in $\mathcal{S}$ consists of a subset of vertices. Additionally, we are given a set of edges $E$ with nonnegative edge weights $w(e)$ for $e \in E$. The 1-cover problem on family $\mathcal{S}$ asks for a 1-cover of $\mathcal{S}$: a minimum weight subset of edges $C \subseteq E$ such that $|C \cap \delta(S)| \geq 1$ for all $S \in \mathcal{S}$. Indeed, we
are interested in a special case of the 1-cover problem on a laminar family of sets. Let $F$ be a spanning, connected multigraph of a given graph $G$, and let $S$ be the family of 1-edge cuts of $F$: $S_F = \{ S : |\delta(S) \cap F| = 1 \}$. In this case, we refer to a 1-cover of $S_F$ as a 1-cover of $F$. Observe that $S_F$ is a laminar family. A natural linear programming relaxation for this problem for a fixed $G = (V, E)$ and $F \subset E$ is:

$$\min \sum_{e \in E} w(e) y_e$$

subject to:

$$\sum_{e \in E \cap \delta(S)} y_e \geq 1, \quad S \text{ is a 1-edge cut of } F$$

$$y_e \geq 0, \quad \text{for all } e \in E.$$ 

Let us denote the feasible region of the above linear program by $\text{Cover}(G, F)$. By contracting the 2-edge-connected components of $F$, we get a tree on these contracted nodes and the 1-cover problem equivalent to the tree augmentation problem [FJ81]. Its integrality gap is known to be between $\frac{3}{2}$ and 2 [FJ81, Jai01, CKKK08]. However, in the special case of half-integral points, the integrality gap is much smaller.

**Theorem 7** (Cheriyan, Jordán and Ravi [CJR99]). If $y \in \text{Cover}(G, F)$ and $y_e \in \{0, \frac{1}{2}, 1\}$ for all $e \in E$, then there is an algorithm, whose running time is polynomial in the size of $G$, that writes the vector $\frac{4}{3} \cdot y$ as a convex combination of 1-covers $C_1, \ldots, C_h$ of $F$.

Recall the set up for 2EC. We are given a graph $G = (V, E)$ with nonnegative weights $w(e)$ for $e \in E$. Our goal is to find a minimum weight 2-edge-connected multigraph of $G$.

**Proof of Theorem 5.** Let $x^*$ be an optimal solution for $\text{Subtour}(G)$. By Lemma 4, there is an algorithm, whose running time is polynomial in $G$, that finds a family of connectors $\mathcal{F} = \{F_1, \ldots, F_\ell\}$ such that $\sum_{i=1}^{\ell} \lambda_i F_i \leq x^*$ and for each $F_i \in \mathcal{F}$, $F_i$ has an even number of edges in every 2-edge cut of $G$. For $i \in \{1, \ldots, \ell\}$, let $S_i$ be the family of 1-edge cuts of $F_i$. As noted previously, each $S_i$ forms a laminar family. Define vector $y^i \in \mathbb{R}^E$ as follows: $y^i_e = 0$ for $e \in F_i$ and $y^i_e = \frac{1}{2}$ for $e \in E \setminus F_i$.

**Claim 1.** For $i \in \{i, \ldots, \ell\}$, we have $y^i \in \text{Cover}(G, F_i)$.

**Proof.** Let $S$ be a 1-edge cut of $F_i$. Then $\delta(S) \cap F_i$ contains exactly one edge $e$. Note that it cannot be the case that $|\delta(S)| = 2$. This is because if $\delta(S)$ were a 2-edge cut of $G$, then by property (b) in Theorem 4 there would be an even number of edges in $F_i$ that are also in $\delta(S)$. Hence, $|\delta(S)| \geq 3$. So we have

$$\sum_{e \in \delta(S)} y_e = \sum_{e \in \delta(S) \setminus F_i} \frac{1}{2} = \sum_{e \in \delta(S) \setminus \{e\}} \frac{1}{2} = \frac{|\delta(S) \setminus \{e\}|}{2} \geq 1.$$
This concludes the proof of the claim.

For $i \in \{1, \ldots, \ell\}$, define vector $r^i$ as follows: $r^i_e = 0$ for $e \in F_i$ and $r^i_e = \frac{2}{3}$ for $e \in E \setminus F_i$.

**Claim 2.** For $i \in \{1, \ldots, \ell\}$, the vector $r^i$ can be written as a convex combination of 1-covers of $F_i$.

**Proof.** By Claim 1 and Theorem 7, vector $\frac{4}{3}y^i$ can be written as a convex combination of 1-covers of $F_i$, and $\frac{4}{3}y^i = r^i$.

By Claim 2, we can write $r^i$ as $\sum_{j=1}^{\ell_i} \lambda_j^i C_j^i$ for $i \in \{1, \ldots, \ell\}$. Let $R_j^i = F_i \cup C_j^i$. Notice, for all choices of $i$ and $j$, $R_j^i$ is a 2-edge-connected multigraph of $G$. To argue that there exists a low-weight, 2-edge-connected multigraph, we show the following claim.

**Claim 3.** There exists $i \in \{1, \ldots, \ell\}$ and $j \in \{1, \ldots, \ell_i\}$ such that $R_j^i \leq \frac{1+2\beta}{3} z_G$.

**Proof.** Pick $i \in \{1, \ldots, \ell\}$ at random according to the probability distribution defined by $\lambda_1, \ldots, \lambda_\ell$. Now, pick $j \in \{1, \ldots, \ell_i\}$ at random according to the probability distribution defined by $\lambda_1^i, \ldots, \lambda_{\ell_i}^i$.

$$
E[w(R_j^i)] = E[w(F_i)] + E[w(C_j^i)]
$$

$$
= \sum_{e \in E} \left(2 \cdot w(e) \cdot \Pr[\chi^{F_i}(e) = 2] + w(e) \cdot \Pr[\chi^{F_i}(e) = 1]\right) + \sum_{e \in E} w(e) \cdot \Pr[e \in C_j^i]
$$

$$
= \sum_{e \in E} \left(2 \cdot w(e) \cdot \Pr[\chi^{F_i}(e) = 2] + w(e) \cdot \Pr[\chi^{F_i}(e) = 1]\right) + \sum_{e \in E} \frac{2}{3} w(e) \cdot \Pr[\chi^{F_i}(e) = 0]
$$

$$
= \sum_{e \in E: x_e^i > 1} \left(2 \cdot w(e) \cdot \Pr[\chi^{F_i}(e) = 2] + w(e) \cdot \Pr[\chi^{F_i}(e) = 1]\right) + \frac{2}{3} \sum_{e \in E: x_e^i = 0} w(e) \cdot \Pr[\chi^{F_i}(e) = 0]
$$

$$
+ \sum_{e \in E: x_e^i \leq 1} \left(2 \cdot w(e) \cdot \Pr[\chi^{F_i}(e) = 2] + w(e) \cdot \Pr[\chi^{F_i}(e) = 1]\right) + \frac{2}{3} \sum_{e \in E: x_e^i = 1} w(e) \cdot \Pr[\chi^{F_i}(e) = 0]
$$

$$
\leq \sum_{e \in E: x_e^i > 1} 2x_e^i w(e) - 2w(e) + 2w(e) - x_e^i w(e) + \sum_{e \in E: x_e^i \leq 1} x_e^i w(e) + \frac{2}{3} w(e) - \frac{2}{3} x_e^i w(e)
$$

$$
\leq \sum_{e \in E: x_e^i > 1} x_e^i w(e) + \sum_{e \in E: x_e^i \leq 1} \frac{1}{3} x_e^i w(e) + \frac{2}{3} w(e).
$$

Observe

$$
\frac{E[w(R_j^i)]}{z_G} \leq \frac{\sum_{e \in E: x_e^i > 1} x_e^i w(e) + \sum_{e \in E: x_e^i \leq 1} \frac{1}{3} x_e^i w(e) + \frac{2}{3} w(e)}{\sum_{e \in E} x_e^i w(e) + \sum_{e \in E} \frac{1}{3} x_e^i w(e) + \frac{2}{3} w(e)} ,
$$

and the maximum value that the RHS of Inequality (7) can take is when $x_e^i \leq 1$ for all $e \in E$. In this case the RHS becomes
Figure 2: The graph in 2b has a total of 10t (here t = 5) vertices: each circular vertex corresponds to the gadget in 2a. The weight of each diamond vertex is 1, and all other vertices have weight zero. A minimum spanning tree (denoted by the thick, blue edges) has weight $5t - 2$ while sum of the node weights is $2t$.

$$\sum_{e \in E} \frac{1}{3} x^*_e w(e) + \frac{2}{3} w(e) = \frac{1}{3} z_G + \frac{2}{3} w(E) \leq \frac{z_G + 2\beta z_G}{3z_G} \leq \frac{1 + 2\beta}{3}.$$ 

This concludes the proof of Theorem 5.

We extend this approach to the k-EC problem. Details can be found in Appendix B.

5.2 An Algorithm for TSP à la Christofides

In the graph metric, every spanning tree has weight at most $n$. It follows that in the case where $z_G \geq (1 + \epsilon)n$, Christofides’ algorithm has an approximation guarantee strictly better than $\frac{3}{2}$ (in fact, at most $(\frac{3}{2} - \frac{\epsilon}{1+\epsilon})$). This implies that, in some sense, the most difficult case for graph-TSP is when $z_G = n$. It seems that it should also be the case for node-weighted graphs: the most difficult case should be when $z_G = 2 \cdot \sum_{v \in V} f_v$ and when $z_G \geq (1 + \epsilon) \cdot 2 \cdot \sum_{v \in V} f_v$, then Christofides’ algorithm should give an approximation guarantee strictly better than $\frac{3}{2}$.

However, in the case of node-weighted graphs (even for subcubic graphs), a minimum spanning tree of $G$ may have weight exceeding $2 \cdot \sum_{v \in V} f_v$ when $z_G > 2 \cdot \sum_{v \in V} f_v$. See Figure 2 for an example. Thus, proving an approximation factor strictly better than $\frac{3}{2}$ for node-weighted graphs in this scenario does not follow the same argument as in the graph metric. Nevertheless, we can prove the following theorem using connectors.

**Theorem 8.** If $w(E) \leq \beta \cdot z_G$, then there is a $(1 + \frac{\beta}{2})$-approximation algorithm for TSP on $G$. 

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Along the same lines, we can also prove the following theorem on node-weighted, subcubic graphs.

**Theorem 9.** Let $G$ be a node-weighted, subcubic graph. If $z_G \geq 2 \cdot (1 + \epsilon) \cdot \sum_{v \in V} f_v$, then there is an $(\frac{3}{2} - \frac{3\epsilon}{4})$-approximation algorithm for TSP on $G$.

In Christofides’ algorithm, one can write an optimal solution $x^*$ for $\text{SUBTOUR}(G)$ as a convex combination of 1-trees [HK70]. Each of these spanning trees is then augmented with a $T$-join, where $T \subseteq V$ is the set of odd-degree nodes in the spanning tree. In particular, for a 1-tree $F$ of $G$, let $T$ be the set of odd-degree vertices of $F$. Then, $\frac{x^*}{2}$ dominates a convex combination of $T$-joins in $G$. In other words, $\frac{x^*}{2}$ is a feasible solution for the following system of inequalities.

\[
x(\delta(S)) \geq 1 \quad \text{for } S \subseteq V \text{ and } |S \cap T| \text{ is odd,}
\]
\[
x_e \geq 0 \quad \text{for } x \in E.
\]

If we decompose the optimal solution for $\text{SUBTOUR}(G)$ to a family of connectors according to Theorem 4, then we can augment each connector by a $T$-join that is obtained from a convex combination of $T$-joins dominated by the vector $\{\frac{1}{3}\}^E$. For this, it is sufficient to show that $\{\frac{1}{3}\}^E$ belongs to $T$-$\text{JOIN}^\dagger(G)$.

**Lemma 6.** Let $F$ be a family of connectors for $G = (V, E)$ satisfying properties (a) and (b) from Theorem 4. For an $F_i \in F$, let $T$ denote the odd-degree vertices in $F_i$. Then the vector $\{\frac{1}{3}\}^E$ belongs to $T$-$\text{JOIN}^\dagger(G)$.

**Proof.** For $S \subseteq V$ such that $|\delta(S)| \geq 3$, we have $x(\delta(S)) \geq 3 \cdot \frac{1}{3} \geq 1$. Hence, we only need to show that if $|\delta(S)| = 2$, then $|S \cap T|$ is even.

Let $\delta(S) = \{e, f\}$. Consider graph $G[S]$. Connector $F$ induces a multigraph on $G[S]$, namely $H$. Note that the number of odd-degree vertices in $H$ has the same parity as $|S \cap T|$, since $\chi^F(e) + \chi^F(f)$ is even. Hence, $|S \cap T|$ is even. \( \square \)

**Proof of Theorem 8.** By Theorem 4, there is a connector $F$, such that $\sum_{e \in F} w(e) \leq z_G$, and $\chi^F(e) + \chi^F(f)$ is even for all $\{e, f\}$ that form a 2-edge cut of $G$. Let $T$ be the odd-degree vertices of $F$. By Lemma 6, there is a $T$-join $J$ in $G$ such that $\sum_{e \in J} w(e) \leq \frac{w(E)}{3}$. Note that $F \cup J$ is a tour and
\[
\sum_{e \in F \cup J} w(e) \leq z_G + \frac{w(E)}{3} \leq (1 + \frac{\beta}{3}) \cdot z_G.
\]

**Proof of Theorem 9.** We will apply Theorem 8. For a node-weighted, subcubic graph, we have
\[
w(E) \leq 3 \cdot \sum_{v \in V} f_v.
\]
By the assumption of the Theorem, we have
\[ \beta = \frac{3}{2} \left(1 + \epsilon\right) \]
\[ = \frac{3}{2} - \frac{3\epsilon}{2 + 2\epsilon} \]
\[ \leq \frac{3}{2} - \frac{3\epsilon}{4}. \]

\[ \square \]

6 Uniform Covers

The four-thirds conjecture is equivalent to the following conjecture [CV04].

Conjecture 1. If \( x \in \text{Subtour}(G) \) the vector \( \frac{4}{3} \cdot x \) dominates a convex combination of tours in \( G \).

The conjecture has been open for decades and the only result known is for \( \frac{3}{2} \cdot x \). For a graph \( G = (V, E) \) (following the terminology of Boyd and Sebő [BS17]), let the everywhere \( r \) vector for \( G \), be the vector in \( \mathbb{R}^E \) that is \( r \) in all coordinates. The polyhedral proof of Christofides’ algorithm by Wolsey [Wol80] shows that the everywhere 1 vector for \( G \) (or a uniform cover of \( G \)) dominates a convex combination of tours of such a graph \( G \). Since the everywhere \( \frac{2}{3} \) vector for a 3-edge-connected cubic graph \( G \) is in \( \text{Subtour}(G) \), the following is a relaxed version of Conjecture 1 [SBS15, BS17].

Conjecture 2. Let \( G = (V, E) \) be a 3-edge-connected, cubic graph. The everywhere \( \frac{8}{9} \) vector for \( G \) dominates a convex combination of tours of \( G \).

For the special class of 3-edge-connected, cubic graphs that are also Hamiltonian, Boyd and Sebő show that the everywhere \( \frac{4}{5} \) vector for \( G \) dominates a convex combination of tours [BS17]. With techniques similar to those used in the proof of Theorem 2 we are able to improve the result from 1 to \( \frac{18}{19} \) for all 3-edge-connected, cubic graphs.

Theorem 10. Let \( G = (V, E) \) be a 3-edge-connected, cubic graph. The everywhere \( \frac{18}{19} \) vector for \( G \) dominates a convex combination of tours of \( G \) and this convex combination can be found in polynomial time.

Proof. By Theorem 1, graph \( G \) has a cycle cover \( C \) such that \( C \) covers every 3-edge and 4-edge cut of \( G \). Let \( G/C \) be the graph obtained by contracting each cycle of \( C \) in \( G \). By Corollary 1.1 \( G/C \) is 5-edge-connected. Define vector \( y \in \mathbb{R}^{E(G/C)} \) as follows: \( y_e = \frac{2}{5} \) for \( e \in E(G/C) \). Observe that \( y \in \text{Subtour}(G/C) \). Thus, \( y \) can be decomposed into a convex combination of 1-trees of \( G/C \) [HK70]. More precisely, we can write \( y = \sum_{i=1}^{\ell} \lambda_i x_i \), where
$T_i$ is a 1-tree of $G/C$, $\sum_{i=1}^\ell \lambda_i = 1$, and $\lambda_i > 0$ for $i \in \{1, \ldots, \ell\}$. Consequently, we have $2y = \sum_{i=1}^\ell \lambda_i\chi^{2T_i}$, i.e., the vector $2y$ can be written as convex combination of doubled 1-trees of $G/C$. Let $M$ be the set of edges in $E \setminus C$ that are not in $G/C$; these are the edges that connect two vertices of the same cycle in $C$. Define vector $v \in \mathbb{R}^E$ as follows: $v_e = 1$ for $e \in C$, $v_e = 0$ for $e \in M$ and $v_e = \frac{1}{2}$ for $e \in E \setminus (M \cup C)$. Note that $v = \sum_{i=1}^\ell \lambda_i\chi^{C \cup 2T_i}$. For $i \in \{1, \ldots, \ell\}$, the graph induced by $C \cup 2T_i$ is a tour.

Now we define $u \in \mathbb{R}^E$ as follows: $u_e = \frac{1}{2}$ for $e \in C$ and $u_e = 1$ for $e \in E \setminus C$. We have $u \in \text{SUBTOUR}(G)$: for each cut $D$ of $G$, if $|D| \geq 4$, clearly $\sum_{e \in D} u_e \geq 2$. If $|D| = 3$, then cycle cover $|C \cap D| = 2$, so $\sum_{e \in D} u_e = 2 \cdot \frac{1}{2} + 1 \geq 2$. We can write $\frac{3}{2}u$ as a convex combination of tours $\text{WOL80}$.

Now vector $\frac{15}{19}v + \frac{4}{19}(\frac{3}{2}u)$ can be written as convex combination of tours of $G$. For edge $e \in C$ we have $\frac{15}{19}v_e + \frac{4}{19}(\frac{3}{2}u_e) = \frac{15}{19} + \frac{4}{19}(\frac{3}{2} \cdot \frac{1}{2}) = \frac{18}{19}$. For $e \in E(G/C)$ we have $\frac{15}{19}v_e + \frac{4}{19}(\frac{3}{2}u_e) = \frac{15}{19} \cdot \frac{1}{2} + \frac{4}{19}(\frac{3}{2}) = \frac{18}{19}$. For $e \in M$, we have $\frac{15}{19}v_e + \frac{4}{19}(\frac{3}{2}u_e) = 0 + \frac{4}{19}(\frac{3}{2}) = \frac{6}{19}$. Therefore $\frac{15}{19}v + \frac{4}{19}(\frac{3}{2}u)$ is dominated by the everywhere $\frac{8}{19}$ vector for $G$.

We can further relax Conjecture [2] and ask whether or not the everywhere $\frac{8}{9}$ vector for a 3-edge-connected, cubic graph $G$ can be written as a convex combination of 2-edge-connected multigraphs of $G$. In fact, Boyd and Legault have proved an even stronger result: the everywhere $\frac{6}{7}$ vector for $G$ can be written as a convex combination of 2-edge-connected multigraphs $\text{BL17}$. More recently, Legault improved the factor of $\frac{8}{9}$ to $\frac{7}{9}$ $\text{Leg17}$. The method used in both these proofs is to disprove the existence of a minimal counterexample. Hence, they do not lead to a polynomial-time algorithm. We next present a different proof that is also nonconstructive, since it uses the following nonconstructive theorem due to Carr and Ravi $\text{CR98}$.

**Theorem 11** (Carr and Ravi $\text{CR98}$). If $x \in \text{SUBTOUR}(G)$, $x \in \{0, \frac{1}{2}, 1\}$, and $x(\delta(v)) = 2$ for all $v \in V$, then the vector $\frac{4}{3}x$ dominates a convex combination of 2-edge-connected multigraphs of $G$.

**Lemma 7.** Let $G = (V, E)$ be a 3-edge-connected, cubic graph. The everywhere $\frac{8}{9}$ vector for $G$ dominates a convex combination of 2-edge-connected multigraphs of $G$.

**Proof.** Let $y$ be the everywhere $\frac{2}{3}$ vector for $G$. By Corollary 30.8a in $\text{Sch03}$, $y$ is in the convex hull of cycle covers of $G$. Thus, there are cycle covers $C_1, \ldots, C_\ell$ and positive multipliers $\lambda_1, \ldots, \lambda_\ell$ such that $\sum_{i=1}^\ell \lambda_i = 1$ and $y = \sum_{i=1}^\ell \lambda_i C_i$. For $i \in \{1, \ldots, \ell\}$, define vector $y^i \in \mathbb{R}^E$ as follows: $y^i_e = \frac{1}{2}$ for $e \in C_i$ and $y^i_e = 1$ for $e \in E \setminus C_i$. Observe that $y^i \in \text{SUBTOUR}(G)$ for $i \in \{1, \ldots, \ell\}$. Furthermore, we have $y = \sum_{i=1}^\ell \lambda_i y^i$: for $e \in E$ we have $\sum_{i \in E \setminus C_i} \lambda_i = \frac{2}{3}$ and $\sum_{i \in E \setminus C_i} \lambda_i = \frac{1}{3}$. Hence $\sum_{i=1}^\ell \lambda_i y^i = \sum_{i \in E \setminus C_i} \lambda_i \cdot \frac{2}{3} + \sum_{i \in E \setminus C_i} \lambda_i = ye$.

Since the vector $y^i$ is half-integral, by Theorem [11] the vector $\frac{4}{3}y^i$ dominates a convex combination of 2-edge-connected multigraphs of $G$ for $i \in \{1, \ldots, \ell\}$. Therefore, $\frac{4}{3}y$ dominates
a convex combination of 2-edge-connected multigraphs of $G$. Vector $\frac{2}{3}y$ is the everywhere $\frac{8}{9}$ vector for $G$.

Applying ideas from our algorithm for 2EC on 2-edge-connected graphs (Theorem 5), we obtain a different, constructive proof of Lemma 7.

**Lemma 8.** Let $G = (V, E)$ be a 3-edge-connected, cubic graph. The everywhere $\frac{8}{9}$ vector for $G$ dominates a convex combination of 2-edge-connected multigraphs of $G$ and this convex combination can be found in polynomial time.

Proof. Let $y$ be the everywhere $\frac{2}{3}$ vector for $G$. Since $y \in \mathbf{Subtour}(G)$, we can write $y$ as a convex combination of $1$-trees $T_1, \ldots, T_\ell$ and positive multipliers $\lambda_1, \ldots, \lambda_\ell$ such that $\sum_{i=1}^\ell \lambda_i = 1$ and $y = \sum_{i=1}^\ell \lambda_i \chi_{T_i}$. For $i \in \{1, \ldots, \ell\}$ define vector $y^i \in \mathbb{R}^E$ as follows: $y^i_e = 0$ for $e \in T_i$ and $y^i_e = \frac{1}{2}$ for $e \notin T_i$. Since $G$ is 3-edge-connected, we have $y^i \in \mathbf{Cover}(G, T_i)$ for $i \in \{1, \ldots, \ell\}$. By Theorem 7, there is a polynomial-time algorithm that finds 1-covers $C^i_1, \ldots, C^i_{\ell_i}$ of $T_i$ for $i \in \{1, \ldots, \ell\}$ and positive multipliers $\lambda^i_1, \ldots, \lambda^i_{\ell_i}$ such that $\sum_{j=1}^{\ell_i} \lambda^i_j = 1$ and $\frac{2}{3}y^i = \sum_{j=1}^{\ell_i} \lambda^i_j \chi_{C^i_j}$ for $i \in \{1, \ldots, \ell\}$. Note that $T_i + C^i_j$ is a 2-edge-connected subgraph of $G$ for $i \in \{1, \ldots, \ell\}$ and $j \in \{1, \ldots, \ell_i\}$. Hence,

$$u = \sum_{i \in \{1, \ldots, \ell\}} \sum_{j \in \{1, \ldots, \ell_i\}} \lambda_i \lambda^i_j \mathbf{T}(U_{C^i_j}), \quad \text{where} \quad \sum_{i \in \{1, \ldots, \ell\}} \sum_{j \in \{1, \ldots, \ell_i\}} \lambda_i \lambda^i_j = 1,$$

is a convex combination of 2-edge-connected multigraphs of $G$. By construction, an edge cannot belong both to a tree $T_i$ and to a 1-cover $C^i_j$. Thus, there are no doubled edges in any solution. Vector $u$ is the everywhere $\frac{8}{9}$ vector for $G$: for $e \in E$, we have

$$u_e = \sum_{i \in T_i} \sum_{j=1}^{\ell_i} \lambda^i_j + \sum_{i \notin T_i} \sum_{j \in C^i_j} \lambda^i_j = \frac{2}{3} + \frac{1}{3} \cdot \frac{2}{3} = \frac{8}{9}.$$

Combining the ideas from our algorithm for 2EC on 2-edge-connected graphs (Theorem 5) with the ideas from our algorithm for 2EC on 3-edge-connected, cubic graphs (Theorem 3), we can improve the bound in Lemma 8 from $\frac{8}{9}$ to $\frac{15}{17}$.

**Theorem 12.** Let $G = (V, E)$ be a 3-edge-connected, cubic graph. The everywhere $\frac{15}{17}$ vector for $G$ dominates a convex combination of 2-edge-connected multigraphs of $G$ and this convex combination can be found in polynomial time.

Proof. Let $C$ be a cycle cover of $G$ that covers every 3-edge and 4-edge cut of $G$. By Corollary 1.1, the graph $G/C$ is 5-edge-connected. Let $M = E \setminus (C \cup E(G/C))$. Define $r \in \mathbb{R}^{E(G/C)}$ as follows: $r_e = \frac{2}{5}$ for $e \in E(G/C)$. We have $r \in \mathbf{Subtour}(G/C)$, so $\frac{3}{2}r$ dominates a convex
combination of tours of $G/C$: namely $R_1, \ldots, R_\ell$. Observe that the graph induced by $C \cup R_i$ is a 2-edge-connected multigraph of $G$ for $i \in \{1, \ldots, \ell\}$. So, the vector $v \in \mathbb{R}^E$ where $v_e = 1$ for $e \in C$, $v_e = \frac{1}{2}$ for $e \in E(G/C)$, and $v_e = 0$ for $e \in M$ dominates a convex combination of 2-edge-connected multigraphs of $G$.

Now define $y \in \mathbb{R}^E$ as follows: $y_e = \frac{1}{2}$ for $e \in C$ and $y_e = 1$ for $e \in E \setminus C$. Since $y \in \text{SUBTOUR}(G)$, we can efficiently find 1-trees $T_1, \ldots, T_\ell$ of $G$ and positive multipliers $\lambda_1, \ldots, \lambda_\ell$ such that $y = \sum_{i=1}^\ell \lambda_i x^{T_i}$. For $i \in \{1, \ldots, \ell\}$ define $y^i \in \mathbb{R}^E$ as follows: $y^i_e = \frac{1}{2}$ for $e \notin T_i$ and $y^i_e = 0$ otherwise. By Theorem 7, there is a polynomial-time algorithm that finds 1-covers $C_1^i, \ldots, C_\ell^i$ of $T_i$ for $i \in \{1, \ldots, \ell\}$ and positive multipliers $\lambda^i_1, \ldots, \lambda^i_\ell$ such that $\sum_{j=1}^\ell \lambda^i_j = 1$ and $\frac{1}{3} y^i = \sum_{j=1}^\ell \lambda^i_j x^{C_j^i}$ for $i \in \{1, \ldots, \ell\}$. Note that $T_i + C^j_i$ is a 2-edge-connected subgraph of $G$ for $i \in \{1, \ldots, \ell\}$ and $j \in \{1, \ldots, \ell_i\}$. Hence,

$$u = \sum_{i \in \{1, \ldots, \ell\}} \sum_{j \in \{1, \ldots, \ell_i\}} \lambda^i_j x^{T_i \cup C^j_i},$$

where $\sum_{i \in \{1, \ldots, \ell\}} \sum_{j \in \{1, \ldots, \ell_i\}} \lambda^i_j = 1$,

is a convex combination of 2-edge-connected multigraphs of $G$. For $e \in C$, we have

$$u_e = \sum_{i \in \ell_i} \sum_{j \in T_i} \lambda^i_j + \sum_{i \notin T_i, j \in C^j_i} \lambda^i_j = \frac{1}{2} + \frac{1}{2} = \frac{5}{6}.$$ 

For $e \notin C$, we have

$$u_e = \sum_{i \in \ell_i} \sum_{j \in T_i} \lambda^i_j + \sum_{i \notin T_i, j \in C^j_i} \lambda^i_j = 1 + 0 = 1.$$

Finally we conclude that the vector $\frac{5}{17} v + \frac{12}{17} u$ can be efficiently written as convex combination of 2-edge-connected multigraphs of $G$. For $e \in C$ we have $\frac{5}{17} v_e + \frac{12}{17} u_e = \frac{5}{17} + \frac{12}{17} \cdot \frac{5}{6} = \frac{15}{17}$. For $e \in G/C$ we have $\frac{5}{17} v_e + \frac{12}{17} u_e = \frac{5}{17} \cdot \frac{1}{2} + \frac{12}{17} = \frac{12}{17}$. Therefore $\frac{5}{17} v + \frac{12}{17} u$ is dominated by the everywhere $\frac{15}{17}$ vector for $G$.

We note that in the proof of Theorem 12 since the vector $y$ is half-integral, we can apply Theorem 11 to conclude that $\frac{7}{8} y$ dominates a convex combination of 2-edge-connected multigraphs of $G$. This shows that the everywhere $\frac{7}{8}$ vector for $G$ dominates a convex combination of 2-edge-connected multigraphs, but this approach is nonconstructive. (Specifically, $\frac{3}{8} (\frac{4}{3} y) + \frac{7}{8} v$ is dominated by the everywhere $\frac{7}{8}$ vector for $G$.) Thus an interesting open problem is to find a constructive version of Theorem 11 since it would immediately lead to a constructive version of Theorem 12 with the factor of $\frac{7}{8}$ being substituted for $\frac{15}{17}$. A more ambitious goal would be to design efficient algorithms to match the factor of $\frac{6}{7}$ given by Boyd and Legault [BL17] and the factor of $\frac{5}{6}$ given by Legault [Leg17].
7 Concluding remarks

In Theorem 2 we provided an approximation algorithm that improves over Christofides’ algorithm for node-weighted, 3-edge-connected, cubic graphs. We also observed that there is an algorithm (Theorem 9) that takes advantage of the gap between the node-weight lower bound for TSP and the subtour lower bound—if it exists—when dealing with node-weighted, subcubic graphs. In fact, 3-edge-connected, cubic graphs are a special class of cubic graph where the subtour lower bound and the node-weight lower bound are always equal. A broader class of these graphs are node-weighted, cubic graphs for which the subtour lower bound meets the node-weight lower bound even after adding the cycle cover constraints, which are valid for TSP. (For a description of the cycle-cover polytope, see Corollary 30.8a, page 530 [Sch03].) This class includes 3-edge-connected, cubic graphs. We discuss this in more detail in Appendix C. A nice property of this class is that a vector in the subtour elimination polytope of a graph in this class can be decomposed into a convex combination of cycle covers each of which covers all 2-edge and 3-edge cuts of the graph, providing a tool similar to a BIT cycle cover to work with. Is there a $(\frac{3}{2} - \epsilon)$-approximation for TSP for graphs in this class?

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References

[ABCC06] David L. Applegate, Robert E. Bixby, Vašek Chvátal, and William J. Cook, editors. The Traveling Salesman Problem: A Computational Study. Princeton University Press, 2006.

[ABE06] Anthony Alexander, Sylvia Boyd, and Paul Elliott-Magwood. On the integrality gap of the 2-edge connected subgraph problem. Technical report, TR-2006-04, SITE, University of Ottawa, Ottawa, Canada, 2006.
Nishita Aggarwal, Naveen Garg, and Swati Gupta. A 4/3-approximation for TSP on cubic 3-edge-connected graphs. *CoRR*, abs/1101.5586, 2011.

Sylvia Boyd and Robert Carr. Finding low cost TSP and 2-matching solutions using certain half-integer subtour vertices. *Discrete Optimization*, 8(4):525 – 539, 2011.

Sylvia Boyd, Yao Fu, and Yu Sun. A $\frac{5}{4}$-approximation for subcubic 2EC using circulations. In *Proceedings of 17th International Conference on Integer Programming and Combinatorial Optimization*, pages 186–197. Springer, 2014.

Sylvia Boyd, Satoru Iwata, and Kenjiro Takazawa. Finding 2-factors closer to TSP tours in cubic graphs. *SIAM Journal on Discrete Mathematics*, 27(2):918–939, 2013.

Sylvia Boyd and Philippe Legault. Toward a 6/5 bound for the minimum cost 2-edge connected spanning subgraph. *SIAM Journal on Discrete Mathematics*, 31(1):632–644, 2017.

Sylvia Boyd and András Sebő. The salesman’s improved tours for fundamental classes. In *Proceedings of 19th International Conference on Integer Programming and Combinatorial Optimization*, pages 111–122, 2017.

Sylvia Boyd, René Sitters, Suzanne van der Ster, and Leen Stougie. The traveling salesman problem on cubic and subcubic graphs. *Mathematical Programming*, 144(1-2):227–245, 2014.

Nicos Christofides. Worst-case analysis of a new heuristic for the travelling salesman problem. Technical Report 388, Graduate School of Industrial Administration, Carnegie Mellon University, 1976.

Joseph Cheriyan, Tibor Jordán, and R. Ravi. On 2-coverings and 2-packings of laminar families. In *European Symposium on Algorithms*, pages 510–520. Springer, 1999.

Joseph Cheriyan, Howard Karloff, Rohit Khandekar, and Jochen Könemann. On the integrality ratio for tree augmentation. *Operations Research Letters*, 36(4):399–401, 2008.

Robert Carr and R. Ravi. A new bound for the 2-edge connected subgraph problem. In *Proceedings of 6th International Conference on Integer Programming and Combinatorial Optimization*, pages 112–125. Springer, 1998.
[CV02] Robert Carr and Santosh Vempala. Randomized metarounding. *Random Structures and Algorithms*, 20(3):343–352, 2002.

[CV04] Robert Carr and Santosh Vempala. On the Held-Karp relaxation for the asymmetric and symmetric traveling salesman problems. *Mathematical Programming*, 100(3):569–587, 2004.

[DFJ54] George Dantzig, Ray Fulkerson, and Selmer Johnson. Solution of a large-scale traveling-salesman problem. *Journal of the Operations Research Society of America*, 2(4):393–410, 1954.

[EJ73] Jack Edmonds and Ellis L. Johnson. Matching, Euler tours and the Chinese postman. *Mathematical Programming*, 5(1):88–124, 1973.

[FJ81] Greg N. Frederickson and Joseph Ja’Ja’. Approximation algorithms for several graph augmentation problems. *SIAM Journal on Computing*, 10(2):270–283, 1981.

[Fra92] András Frank. Augmenting graphs to meet edge-connectivity requirements. *SIAM Journal on Discrete Mathematics*, 5(1):25–53, 1992.

[Gao13] Zhihan Gao. An LP-based-approximation algorithm for the s-t path graph traveling salesman problem. *Operations Research Letters*, 41(6):615–617, 2013.

[GB93] Michel X. Goemans and Dimitris J. Bertsimas. Survivable networks, linear programming relaxations and the parsimonious property. *Mathematical Programming*, 60(1):145–166, 1993.

[GLS88] Martin Grötschel, László Lovász, and Alexander Schrijver. *Geometric Algorithms and Combinatorial Optimization*. Springer, 1988.

[GLS05] David Gamarnik, Moshe Lewenstein, and Maxim Sviridenko. An improved upper bound for the TSP in cubic 3-edge-connected graphs. *Operations Research Letters*, 33(5):467–474, 2005.

[Goe95] Michel X. Goemans. Worst-case comparison of valid inequalities for the TSP. *Mathematical Programming*, 69(1):335–349, 1995.

[HK70] Michael Held and Richard M. Karp. The traveling-salesman problem and minimum spanning trees. *Operations Research*, 18(6):1138–1162, 1970.

[Hoo91] J. A. Hoogeveen. Analysis of Christofides’ heuristic: Some paths are more difficult than cycles. *Operations Research Letters*, 10(5):291–295, 1991.
[Jai01] Kamal Jain. A factor 2 approximation algorithm for the generalized Steiner network problem. *Combinatorica*, 21(1):39–60, Jan 2001.

[KŠ08] Tomáš Kaiser and Riste Škrekovski. Cycles intersecting edge-cuts of prescribed sizes. *SIAM Journal on Discrete Mathematics*, 22(3):861–874, 2008.

[Leg17] Philippe Legault. Towards New Bounds for the 2-Edge Connected Spanning Subgraph Problem. Master’s thesis, University of Ottawa, 2017.

[LLRK85] Eugene L. Lawler, Jan Karel Lenstra, and Alexander H.G. Rinnooy Kan. *The Traveling Salesman Problem*. John Wiley & Sons, 1985.

[MMP90] Clyde L. Monma, Beth Spellman Munson, and William R. Pulleyblank. Minimum-weight two-connected spanning networks. *Mathematical Programming*, 46(1):153–171, 1990.

[MS16] Tobias Mömke and Ola Svensson. Removing and adding edges for the traveling salesman problem. *Journal of the ACM*, 63(1):2, 2016.

[Muc14] Marcin Mucha. 13/9-approximation for graphic TSP. *Theory of Computing Systems*, 55(4):640–657, 2014.

[New14] Alantha Newman. An improved analysis of the Mömke-Svensson algorithm for graph-TSP on subquartic graphs. In *European Symposium on Algorithms*, pages 737–749. Springer, 2014.

[OSS11] Shayan Oveis Gharan, Amin Saberi, and Mohit Singh. A randomized rounding approach to the traveling salesman problem. In *Proceedings of the 52nd Annual Symposium on Foundations of Computer Science*, pages 550–559. IEEE, 2011.

[QSWvZ15] Jiawei Qian, Frans Schalekamp, David P. Williamson, and Anke van Zuylen. On the integrality gap of the subtour LP for the 1,2-TSP. *Mathematical Programming*, 150(1):131–151, 2015.

[SBS15] András Sebő, Yohann Benchetrit, and Matej Stehlik. Problems about uniform covers, with tours and detours. *Mathematisches Forschungsinstitut Oberwolfach Report*, 2014(51):2912—2915, 2015.

[Sch03] Alexander Schrijver. *Combinatorial Optimization: Polyhedra and Efficiency*. Springer, 2003.

[SV14] András Sebő and Jens Vygen. Shorter tours by nicer ears: 7/5-approximation for the graph-TSP, 3/2 for the path version, and 4/3 for two-edge-connected subgraphs. *Combinatorica*, 34(5):597–629, 2014.
The problem of finding a minimum weight DFS tree is not only NP-hard, but is it NP-hard to approximate to within any constant factor. To observe this, assume for contradiction that there is a polynomial time algorithm that given a weighted graph, finds a DFS tree of weight $c \cdot \text{OPT}$, where $c \geq 1$ is a constant and $\text{OPT}$ is the weight of the minimum weight DFS tree. We use this algorithm to decide if an arbitrary graph $G$ contains a Hamilton path. Consider the complete graph on the vertex set of $G$ and let the edges of $G$ have weight zero and all other edges have weight 1. Now, find the minimum weight DFS tree on the new instance. Notice that since the graph is complete the DFS tree is a path. The optimal DFS tree to this instance has value zero if and only if $G$ contains a Hamilton path. Thus, the algorithm outputs zero if and only if $G$ contains a Hamilton path. This is a contradiction assuming $P \neq NP$.

B An Approximation Algorithm for $k$-EC

We show how to extend the approach from Section 5.1 to design an approximation algorithm for the $k$-EC problem. Recall that in the $k$-EC problem, we are given a 2-edge connected
graph and the goal is to find a minimum weight spanning (sub)multigraph such that each cut is covered by at least \(k\) edges. For general weighted graphs, the best-known approximation factor for this problem is \(\frac{3}{2}\), which can be achieved by taking \(\frac{k}{2}\) copies of traveling salesman tour.

The following linear programming relaxation is a generalization of \([\text{SUBTOUR}(G)]\) and provides a lower bound for the \(k\)-EC problem.

\[
\min \sum_{e \in E} w(e)x_e \\
x(\delta(S)) \geq k \quad \text{for } \emptyset \subset S \subset V \quad (8) \\
x_e \geq 0 \quad \text{for } e \in E.
\]

Let \(x^*\) be an optimal solution to the above LP. Note that \(\frac{2}{k} : x^* \in [\text{SUBTOUR}(G)]\). Hence, by Theorem [4], we can find connectors \(F_1, \ldots, F_\ell\) each having an even number of edges on every 2-edge cut of \(G\) such that \(\frac{2}{k} : x^* \geq \sum_{i=1}^\ell \lambda_i F_i\), where \(\sum_{i=1}^\ell \lambda_i = 1\) and \(\lambda_i > 0\) for \(i = 1, \ldots, \ell\). Consider the multigraph \(\mathbb{F}_i\) which consists of \(\left\lceil \frac{k}{2} \right\rceil\) copies of the connector \(F_i\). Then we have

\[
\left\lceil \frac{k}{2} \right\rceil : \frac{2}{k} : x^* \geq \sum_{i=1}^\ell \lambda_i \left(\left\lceil \frac{k}{2} \right\rceil \cdot F_i\right) = \sum_{i=1}^\ell \lambda_i \mathbb{F}_i. 
\]

Consider \(\mathbb{F}_i\) for some \(i \in \{1, \ldots, \ell\}\). Note that each 2-edge cut in \(G\) is covered by at least \(k\) edges in \(\mathbb{F}_i\). However, a cut of cardinality of three or more edges in \(G\) may not be covered by at least \(k\) edges in \(\mathbb{F}_i\). Thus, we may need to augment \(\mathbb{F}_i\) to obtain a \(k\)-edge-connected graph.

In order to be able to apply Claim [2] (which we use in the third and fourth cases below), we define vector \(r^i \in \mathbb{R}^E\) such that \(r^i_e = \frac{2}{k}\) for \(e \notin F_i\) and \(r^i_e = 0\) if \(e \in F_i\). By Claim [2], there are 1-covers of the 1-edge cuts of \(F_i\), namely \(C_1^i, \ldots, C_\ell^i\), such that \(r^i = \sum_{j=1}^\ell \lambda_j C_j^i\), and \(\sum_{j=1}^\ell \lambda_j^i = 1\) and \(\lambda_j^i > 0\) for \(j \in \{1, \ldots, \ell_i\}\). We consider the following cases for augmentation.

- **Case 1** \((k = 0 \mod 4)\). Define \(\lambda^i\) as follows: \(\chi^{\lambda^i}(e) = \frac{k}{4}\) for \(e \notin F_i\) and \(\chi^{\lambda^i}(e) = 0\) for \(e \in F_i\).
- **Case 2** \((k = 1 \mod 4)\). Define \(\lambda^i\) as follows: \(\chi^{\lambda^i}(e) = \frac{k-1}{4}\) for \(e \notin F_i\) and \(\chi^{\lambda^i}(e) = 0\) for \(e \in F_i\).
- **Case 3** \((k = 2 \mod 4)\). Define \(\lambda^i\) as follows: \(\chi^{\lambda^i}(e) = \frac{k-2}{4}\) for \(e \notin F_i\) and \(\chi^{\lambda^i}(e) = 0\) for \(e \in F_i\). For \(j \in \{1, \ldots, \ell_i\}\), define \(\lambda_j^i = \lambda^i \cup C_j^i\).
- **Case 4** \((k = 3 \mod 4)\). Define \(\lambda^i\) as follows: \(\chi^{\lambda^i}(e) = \frac{k-3}{4}\) for \(e \notin F_i\) and \(\chi^{\lambda^i}(e) = 0\) for \(e \in F_i\). For \(j \in \{1, \ldots, \ell_i\}\) define \(\lambda_j^i = \lambda^i \cup C_j^i\).
Lemma 9. If \( k = 0 \mod 4 \) or \( k = 1 \mod 4 \), then for \( i \in \{1, \ldots, \ell\} \), the multigraph \( F_i \cup A^i \) is \( k \)-edge-connected.

Proof. For some \( S \subseteq V \), let \( D = \delta(S) \subseteq E \) be a cut in \( G \). We want to show that \( \chi^{F_i \cup A^i}(D) \geq k \). Note that if \( \chi^{F_i}(D) \geq 2 \), then \( \chi^{F_i}(D) = \chi^{A^i}(D) \geq k \). In particular, this is the case when \( |D| = 2 \); by property (b) in Theorem 4, if \( |D| = 2 \), then we have \( \chi^{F_i}(D) \geq 2 \). Thus, we may assume \( |D| \geq 3 \) and \( \chi^{F_i}(D) = 1 \). We now consider two cases.

- **Case 1** \( (k = 0 \mod 4) \). We have \( \chi^{A^i}(D) \geq 2 \cdot \frac{k}{4} \), and \( \chi^{F_i}(D) = \frac{k}{2} \).

- **Case 2** \( (k = 1 \mod 4) \). We have \( \chi^{A^i}(D) \geq \frac{k-1}{4} \cdot 2 = \frac{k-1}{2} \), and \( \chi^{F_i}(D) = \frac{k}{2} = \frac{k+1}{2} \).

In each case, \( \chi^{F_i}(D) + \chi^{A^i}(D) \geq k \).

Lemma 10. If \( k = 2 \mod 4 \) or \( k = 3 \mod 4 \), then for \( i \in \{1, \ldots, \ell\} \) and \( j \in \{1, \ldots, \ell_i\} \), the multigraph \( F_i \cup A^i_j \) is \( k \)-edge-connected.

Proof. For some \( S \subseteq V \), let \( D = \delta(S) \subseteq E \) be a cut in \( G \). We want to show that \( \chi^{F_i \cup A^i_j}(D) \geq k \). As in Lemma 9, we only need to consider the case where \( |D| \geq 3 \) and \( \chi^{F_i}(D) = 1 \). We have the following two cases.

- **Case 1** \( (k = 2 \mod 4) \). We have \( \chi^{A^i_j}(D) = \chi^{A^i}(D) + \chi^{C^i_j}(D) \geq 2 \cdot \frac{k-2}{4} + 1 \), and \( \chi^{F_i}(D) = \frac{k}{2} \).

- **Case 2** \( (k = 3 \mod 4) \). We have \( \chi^{A^i_j}(D) = \chi^{A^i}(D) + \chi^{C^i_j}(D) \geq 2 \cdot \frac{k-3}{4} + 1 \), and \( \chi^{F_i}(D) = \frac{k+1}{2} \).

In each case, \( \chi^{F_i}(D) + \chi^{A^i_j}(D) \geq k \).

The proof of the next theorem is analogous to that of Theorem 5.

Theorem 13. If \( w(E) \leq \beta \cdot z_G \), then there is an \( \alpha \)-approximation algorithm for the \( k \)-EC algorithm, where

- \( \alpha \leq \frac{1+\beta}{2} \), if \( k = 0 \mod 4 \);
- \( \alpha \leq \frac{k+3}{2k} + \frac{k-1}{2k} \cdot \beta \), if \( k = 1 \mod 4 \);
- \( \alpha \leq \frac{1}{2} - \frac{1}{3k} + (\frac{1}{2} + \frac{1}{3k}) \cdot \beta \), if \( k = 2 \mod 4 \);
- \( \alpha \leq \frac{k+5}{2k} - \frac{1}{3k} + (\frac{k-3}{2k} + \frac{1}{3k}) \cdot \beta \), if \( k = 3 \mod 4 \).

Proof. We consider the four cases for \( k \mod 4 \).
• **Case 1** ($k = 0 \mod 4$). Choose $i \in \{1, \ldots, \ell\}$ at random according to the probability distribution $\lambda_1, \ldots, \lambda_\ell$.

\[
\mathbb{E}[w(F_i + A^i)] = \sum_{e \in E: x^*_e > 1} k \cdot w(e) \cdot (x^*_e - 1) + \frac{k}{2} \cdot w(e) \cdot (2 - x^*_e) \\
+ \sum_{e \in E: x^*_e \leq 1} \frac{k}{2} \cdot w(e) \cdot x^*_e + \frac{k}{4} \cdot w(e) \cdot (1 - x^*_e) \\
= \sum_{e \in E: x^*_e > 1} \frac{k}{2} \cdot w(e) \cdot x^*_e + \sum_{e \in E: x^*_e \leq 1} \frac{k}{4} \cdot w(e) \cdot x^*_e + w(e) \cdot \frac{k}{4}.
\]

Recall that $\frac{k}{2} \cdot z_G$ is a lower bound for an optimal solution to the $k$-EC problem. So,

\[
\mathbb{E}[w(F_i + A^i)] \leq \frac{\sum_{e \in E: x^*_e > 1} w(e) \cdot x^*_e + \sum_{e \in E: x^*_e \leq 1} \frac{1}{2} \cdot w(e) \cdot x^*_e + \frac{1}{2} \cdot w(e)}{\sum_{e \in E: x^*_e > 1} w(e) \cdot x^*_e + \sum_{e \in E: x^*_e \leq 1} w(e) \cdot x^*_e}.
\]

Note that RHS of (10) is maximized if there is no edge $e$ with $x^*_e > 1$. Hence,

\[
\alpha \leq \frac{1}{2} \cdot z_G + \frac{1}{2} \cdot w(E) \leq \frac{1 + \beta}{2}.
\]

• **Case 2** ($k = 1 \mod 4$). Similar to Case 1, choose a random $i \in \{1, \ldots, \ell\}$.

\[
\mathbb{E}[w(F_i + A^i)] = \sum_{e \in E: x^*_e > 1} \frac{k+1}{2} \cdot w(e) \cdot x^*_e + \sum_{e \in E: x^*_e \leq 1} \frac{k+3}{4} \cdot w(e) \cdot x^*_e + w(e) \cdot \frac{k-1}{4}.
\]

Since $\frac{k}{2} \cdot z_G$ is a lower bound for the optimal solution to the $k$-EC problem, we have

\[
\mathbb{E}[w(F_i + A^i)] \leq \frac{\sum_{e \in E: x^*_e > 1} \frac{k+1}{k} \cdot w(e) \cdot x^*_e + \sum_{e \in E: x^*_e \leq 1} \frac{k+3}{2k} \cdot w(e) \cdot x^*_e + \frac{k-1}{2k} \cdot w(e)}{\sum_{e \in E: x^*_e > 1} w(e) \cdot x^*_e + \sum_{e \in E: x^*_e \leq 1} w(e) \cdot x^*_e}.
\]

Again, the RHS of (12) is maximized if for all $e \in E$, we have $x^*_e \leq 1$.

\[
\alpha \leq \frac{\frac{k+3}{2k} \cdot z_G + \frac{k-1}{2k} \cdot w(E)}{z_G} \leq \frac{1}{2k} \cdot \frac{k+3}{2k} + \frac{1}{2k} \cdot \beta.
\]

• **Case 3** ($k = 2 \mod 4$). Choose $i \in \{1, \ldots, \ell\}$ according to the probability distribution
defined by $\lambda_1, \ldots, \lambda_\ell$ and $j \in \{1, \ldots, \ell_i\}$ according to probabilities $\lambda_i^1, \ldots, \lambda_i^\ell_i$.

$$
\mathbb{E}[w(F^i + A^i_j)] = \sum_{e \in E: x_e^* > 1} k \cdot w(e) \cdot (x_e^* - 1) + \frac{k}{2} \cdot w(e) \cdot (2 - x_e^*)
$$

$$
+ \sum_{e \in E: x_e^* \leq 1} \frac{k}{2} \cdot w(e) \cdot x_e^* + \left(\frac{k - 2}{4} + \frac{2}{3}\right) \cdot w(e) \cdot (1 - x_e^*)
$$

$$
= \sum_{e \in E: x_e^* > 1} \frac{k}{2} \cdot w(e) \cdot x_e^* + \sum_{e \in E: x_e^* \leq 1} \left(\frac{k}{4} - \frac{1}{6}\right) \cdot w(e) \cdot x_e^* + \left(\frac{k}{4} + \frac{1}{6}\right) \cdot w(e)
$$

Similar to other case we can conclude that

$$
\alpha \leq \frac{\left(\frac{1}{2} - \frac{1}{3k}\right) \cdot z_G + \left(\frac{1}{2} + \frac{1}{3k}\right) \cdot w(E)}{z_G} \leq \frac{1}{2} - \frac{1}{3k} + \left(\frac{1}{2} + \frac{1}{3k}\right) \cdot \beta. \quad (14)
$$

- **Case 4 ($k = 3 \mod 4$).** Similar to the previous cases we have

$$
\alpha \leq \frac{\left(\frac{k + 5}{2k} - \frac{1}{3k}\right) \cdot z_G + \left(\frac{k - 3}{2k} + \frac{1}{3k}\right) \cdot w(E)}{z_G} \leq \frac{k + 5}{2k} - \frac{1}{3k} + \left(\frac{k - 3}{2k} + \frac{1}{3k}\right) \cdot \beta. \quad (15)
$$

This concludes the proof. \qed

Theorem 13 has the following implications.

**Theorem 14.** If $G$ is a node-weighted graph with maximum degree $d$, then there exists a $(\frac{1}{2} + \frac{d}{4})$-approximation for $k$-EC for sufficiently large $k$.

*Proof.* In this case we have $\beta \leq \frac{d}{2}$. The result follows from Theorem 13 when we take the limit when $k \to \infty$. \qed

**Corollary 14.1.** If $G$ is a node-weighted subcubic graph, then there exists a $\frac{5}{4}$-approximation for $k$-EC on $G$ for sufficiently large $k$.

C **A Special Class of Node-Weighted Cubic Graphs**

Inspired by Theorem 9, we would like to focus on node-weighted, cubic graphs for which $z_G = 2 \cdot \sum_{v \in V} f_v$. Suppose $x^* \in \text{SUBTOUR}(G)$ is a solution whose objective value is $z_G$. Since in this case, $z_G = 2 \sum_{v \in V} f_v$, we have $x^*(\delta(v)) = 2$ for $v \in V$.

Also, if we extend $x^* \in \mathbb{R}^E$ to the vector $x^*_{\text{met}} \in \mathbb{R}^{V \times V}$ by adding zeros for all $uv \notin E$, it would be the case that $z_G = z_{\text{SER}}$, where $z_{\text{SER}}$ is the objective value for $x^*_{\text{met}}$ obtained on the relaxation of the integer program given in Figure 1. Notice that the feasible integer solutions to the subtour elimination integer program in Figure 1 are Hamilton cycles of $G^{\text{met}}$, hence cycle covers of $G^{\text{met}}$. Thus, one can add the cycle cover constraints to this relaxation
Let $z'_G$ be the optimal solution for this relaxation after adding the cycle cover constraints. If $z'_G > (1 + \epsilon)z_G$, then Christofides’ algorithm is a $(\frac{3}{2} - \frac{\epsilon}{2})$-approximation algorithm for TSP. (This follows directly from the fact that there exists a 1-tree of weight at most $z_G$ and a $T$-join for the odd-degree vertices of this 1-tree of weight at most $(1 + \epsilon)\frac{z_G}{2}$.) This further draws our attention to the case where

$$z'_G = z_G = 2 \cdot \sum_{v \in V} f_v. \quad (16)$$

An example of node-weighted graphs that satisfy Equation (16) are 3-edge-connected, cubic graphs: let $x'_e = \frac{2}{3}$ for $e \in E$ and $x'_e = 0$ otherwise. It is easy to check that all constraints are satisfied. In fact, there are also cubic graphs with 2-edge cuts that fall into this class of graphs. Hence, we believe this is a nice intermediate class between node-weighted 3-edge-connected cubic and node-weighted cubic graphs for solving TSP.

From now on assume $z'_G = z_G = 2 \cdot \sum_{v \in V} f_v$. Let $x'$ be vector in the dimension of $G^{met}$ (i.e. $\mathbb{R}^{V \times V}$) that has objective value $z'_G$. Since $z'_G = 2 \cdot \sum_{v \in V} f_v$, we have $x'_e = 0$ for $e \notin E$. Thus, $x'$ limited to the dimension of $G$ is in $\text{Subtour}(G)$ and also it satisfies the cycle cover constraints on $G$. By Corollary 30.8a in [Sch03], $x'$ can be decomposed into convex combination of cycle covers of $G$. The following lemma shows that these cycle covers have some cut covering properties that might be useful in improved approximation algorithms for this class of graphs.

**Lemma 11.** Let $G = (V, E)$ be a node-weighted, bridgeless graph that satisfies Equation (16). Then the optimal solution $x^*$ can be decomposed into a convex combination of cycle covers each of which covers all 2-edge and 3-edge cuts of $G$.

**Proof.** Let $x^* \in \text{Subtour}(G)$ be the solution in $\mathbb{R}^E$ with objective value $z_G$. Decompose $x^*$ into a convex combination of cycle covers of $G$: $y = \sum_{i=1}^\ell \lambda_i C_i$, where $C_i$ is a cycle cover of $G$, $\sum_{i=1}^\ell \lambda_i = 1$, and $\lambda_i > 0$ for $i \in \{1, \ldots, \ell\}$.

Let $\{e, f\}$ be a 2-edge cut of $G$. Assume $|C_i \cap \{e, f\}| < 2$ for some $i \in \{1, \ldots, \ell\}$, we have $x^*(\{e, f\}) = \sum_{i=1}^\ell \lambda_i |C_i \cap \{e, f\}| < \sum_{i=1}^\ell 2\lambda_i = 2$, a contradiction to the feasibility of $x^*$. On the other hand, for each 3-edge cut $\{e, f, g\}$ of $G$, we have $|C_i \cap \{e, f, g\}| = 0$ or 2 for $i \in \{1, \ldots, \ell\}$. However, if $|C_i \cap \{e, f, g\}| = 0$ for some $i \in \{1, \ldots, \ell\}$, then $x^*(\{e, f, g\}) < 2$ which is a contradiction to feasibility of $x^*$. \qed