Quantum computational path summation for relativistic quantum mechanics and a time dilation relation for a Dirac Hamiltonian generator on a qubit array

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(Dated: September 26, 2015, Revised August 4, 2017)

Dirac particle dynamics is encoded as a unitary path summation rule and implemented on a qubit array, where the qubit array represents both spacetime and the fermions contained therein. The unitary path summation rule gives a quantum algorithm to model a many-body system of Dirac particles in a gauge field with Lorentz invariance down to the grid scale (Planck scale)—the lattice-based model neither suffers the Fermi-sign problem nor breaks Lorentz invariance. Yet, for the Dirac Hamiltonian to generate the unitary evolution of the 4-spinor field at the Planck scale, there is time dilation between the shortest observable time near a single space point and that time measured at long-wavelength scales. We find gravitational time dilation where the model space around each point (with an even number of qubits) is curved like the space around a Schwarzschild black hole.

Keywords: quantum information dynamics, unitary path summation, qubit array, quantum lattice gas algorithm, Dirac Hamiltonian, gravitational time dilation, Schwarzschild black hole

Introduction.—Quantum computation used for quantum simulation offers a new way to explore relativistic quantum mechanics and demonstrate relativistic effects in the Dirac equation [1–4]. Quantum lattice gas models have been explored for relativistic quantum mechanics [5–7] and tested in one-body simulations in 1+1 dimensions [8–12]. Quantum lattice gas and quantum Boltzmann models for relativistic quantum systems have been investigated for quantum simulation applications [13–16].

Presented here is a novel quantum information dynamics model of relativistic quantum mechanics expressed as a simple rule to calculate a Feynman path integral in 3+1 dimensions. The probability amplitude, say $K$, for a relativistic Dirac particle of mass $m$ to go from point $x_a^\mu = (ct_a, x_a)$ to point $x_b^\mu = (ct_b, x_b)$ is calculated by summing over all paths that connect the points $a$ to $b$:

$$K_{ab}^{\text{Feynman}} = \sum_{R \geq 0} \Phi_{ab}(R)(\sqrt{1 - \epsilon^2})^R(-i\epsilon)^R, \quad (1)$$

where $\epsilon \equiv mct/\hbar$, $R$ is the number of bends (in space) in a path, $R$ is the number of unbends, and $\Phi_{ab}(R)$ is the number of paths with $R$ bends. Once the start and end points are selected, the only free parameter is $m$, as the reduced Planck constant $\hbar$, Planck length $\ell$, Planck time $\tau$, and speed of light $c = \ell/\tau$ are constants. A spacetime path can only be resolved up to the Planck scales, so with integer $N = (t_b - t_a)/\tau$ time steps, the number of unbends is $R = N - R$. The dimensionless parameter $\epsilon \equiv mct/\hbar \leq 1$ is a small parameter when the Planck length $\ell$ is smaller than the reduced Compton wavelength $\hbar/mc$ of the Dirac particle. Yet, the path summation rule (1) models the particle physics all the way up to $\epsilon = 1$, called the highest energy (HE) limit.

Kernel (1) appears simple, yet it is rich in particle physics. It applies to antiparticles as well. By writing it in spin variables, (1) leads to an efficient quantum computing algorithm for modeling Dirac particles. In this sense, it models relativistic quantum mechanics, including how the dynamical behavior of the fermionic 4-spinor field $\psi(x)$ is generated by the Dirac Hamiltonian. It also explains the particle motion in a background 4-potential field $A^\mu(x)$. Even particle interactions mediated by exchange of quanta of the gauge field $A^\mu(x)$ is represented by (1), and this quantum field theory application is given an another communication [17]. In this Letter, (1) is evaluated on a 3+1 spacetime lattice for relativistic particle dynamics in a background 4-potential field.

Expanding in $\ell$, $K_{ab}^{\text{HI}} = K_{ab}^{(1)} + K_{ab}^{(2)} + \cdots$. The lowest-order term $K_{ab}^{(1)} = \sum_{R \geq 0} \Phi_{ab}(R)(-i\epsilon)^R$ is not unitary on a finite spacetime lattice with 4-volume $(T\tau)/(L\ell)^3$. Yet, $K_{ab}^{(1)}$ becomes unitary in the continuum limit $\mathcal{L} \to \infty$ of Minkowski space so long as the number of time steps is much smaller that the system size $N \ll \mathcal{L}$.

$$K_{ab}^{(1)}_{\text{Feynman}} = \lim_{L \to \infty} \sum_{R \geq 0} \Phi_{ab}(R)(-i\epsilon)^R. \quad (2)$$

Analytically evaluating (2) on a 1+1 dimensional spacetime lattice is known as the Feynman chessboard algorithm [18–20]. The associated quantum lattice gas algorithm for kernel (2) on 3+1 dimensional spacetime lattice was previously found [7], providing the first efficient quantum computational algorithm for Dirac particle dynamics [15, 16]. Yet, the unitarity of (1) requires no limiting process, and it becomes a relativistic path integral where Lorentz invariance is retained down to the Planck scale. Therefore, it can be used to explore particle physics at the Planck scale. For example, (1) predicts that Schwarzschild time-dilation occurs near this scale. In short, this Letter presents a novel quantum information dynamics algorithm for quantum simulation of relativistic quantum mechanics and presents the discovery of gravitational time-dilation that occurs at the highest-energy scales when the particle physics is given (1).

Spin-chain encoded path.—The particle physics of a
massless fermion of energy $E$ moving at the speed of light $c$ is described by the $(\sqrt{1-c^2})_\mu$ term in (1); the chiral fermion’s 4-momentum $p^\mu = (p_0, \mathbf{p})$ is light like, $p_\mu p^\mu = 0$. The $(-i\epsilon)^R$ term in (1) encodes chiral symmetry breaking: the fermion’s 4-momentum becomes $p^\mu = (E/c, mcu)$ and $p_\mu p^\mu = mc^2$. The proper velocity $\mathbf{u}$ can be parametrized by the spherical Euler angles $\theta$ and $\varphi$, viz. $\mathbf{u} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$.

Casting (1) in spin variables provides a way to derive a quantum algorithm for relativistic quantum mechanics. The spin space is a tensor product space over a discrete field of qubits. This discrete field is called a qubit array. So let a qubit, say $|q_\mu\rangle$, encode the direction of $\mathbf{u}$. The path summation rule becomes a quantum information dynamics model. The conventions of particle physics are inverted using a qubit to encode the spin state of a spin-1/2 particle: the qubit is the fundamental object and a spin-1/2 particle is a bit contained in the qubit.

All the translational degrees of freedom of a particle on the spacetime lattice are encoded in spin variables; the fermion’s path from point $a$ to $b$ is specified by a chain of spin 4-vectors, where each 4-vector at a point is denoted $s_\mu = (s_0, s_1, s_2, s_3)$. The quartic $s^2 s_\mu = s_0^2 - s_1^2 - s_2^2 - s_3^2 = 0$ is light like [7] [22]. Since qubits contain the fermionic bits, each bit can simultaneously occupy all points in the qubit array in quantum superposition representing how a fermion can occupy all points of space in superposition. Using the Pauli matrices

$$\sigma_x = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_y = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

as the fundamental representation of the special unitary group $SU(2)$ and spin-space 3-vector $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ and 4-vector $\sigma^\mu = (1, \sigma)$, Using the spin-1/2 operator $\frac{1}{\sqrt{2}}\sigma$, the proper velocity $\mathbf{u}$ may be cast in matrix form as a spin operator: $\mathbf{u} \mapsto \hat{S} = \frac{1}{\sqrt{2}} \sigma \cdot \mathbf{u}$.

$$\hat{S} = \frac{\hbar}{2} \begin{pmatrix} u_z & u_x - i u_y \\ u_x + i u_y & -u_z \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} \cos \theta & \sin \theta e^{-i\varphi} \\ \sin \theta e^{i\varphi} & -\cos \theta \end{pmatrix}$$

The spin eigenvalue $\hat{S}|\pm_s\rangle = \pm \frac{1}{2}|\pm_s\rangle$ has spin-1/2 eigenstates

$$| +_s \rangle = \cos \frac{\theta}{2} e^{-i\varphi} |0\rangle + \sin \frac{\theta}{2} e^{i\varphi} |1\rangle \quad (5a)$$

$$| -_s \rangle = -\sin \frac{\theta}{2} e^{-i\varphi} |0\rangle + \cos \frac{\theta}{2} e^{i\varphi} |1\rangle \quad (5b)$$

Therefore, the fermion’s 3-momentum $\mathbf{p} = (q_\mu, \frac{mc}{\hbar} \hat{S} |q_\mu\rangle)$ is encoded on the Bloch sphere of a 2-level qubit as

$$|q_\mu\rangle = e^{-i\frac{\varphi}{2}} \left[ \cos \left( \frac{\theta}{2} \right) |0\rangle + e^{i\varphi} \sin \left( \frac{\theta}{2} \right) |1\rangle \right]$$

where the fermion’s energy determines the overall phase. The fermion’s wave 4-vector is $k_\mu = \frac{2\pi}{T} \left( \frac{n_x}{T}, \frac{n_y}{T}, \frac{n_z}{T}, \frac{n_t}{T} \right)$, for integers $n_t$, $n_x$, $n_y$ and $n_z$ and for system size $T$ in time and $L$ in space. For a massive fermion, $k_\mu = \sqrt{E_m^2 - k_\mu^2} \neq 0$. The fermion’s 4-momentum is $p_\mu = \hbar k_\mu$.

The motion of a fermion/antifermion moving say from point $(ct_a, \mathbf{x}_a)$ to $(ct_b, \mathbf{x}_b)$ is constrained such that $s_0 k_0 = \pm |k_0|$. The particle dynamics in (1) never changes a fermion to an antifermion (nor vice versa), so there is no loss of generality by restricting our attention to the dynamics of a positive-energy fermion. The treatment for negative-energy fermions is similar.

Although a fermion’s direction in time does not change while it exists, its 3-momentum changes direction upon photon absorption/emission. The path summation (1) takes this process into account. At the $w$th time step, the fermion’s local outgoing 4-momentum is

$$p_\mu(A) = p_\mu - \frac{e}{c} A_\mu$$

which is the incoming 4-momentum $p_\mu$ at the $w$th time step of the path in the spacetime lattice minus the photon’s 4-momentum [23].

During the $w$th time step $s_{nw}$, and the fermion’s momentum is $p_\mu = (E/csw_0, mcsw_0) = (|p_0| s_0, |p_0| s_0)$ in spin variables. Moreover, a quanta of the background Maxwell field $A_\mu$ that interacts with this fermion during its $w$th time step is expressed in spin variables as $A_\mu = A_{nw} (0, A_{nw} s_0)$. The fermion’s interaction with a photon does not cause the fermion to reverse direction in time. The outgoing 4-momentum (7) may be expressed in spin variables as

$$s_\mu(A) = s_\mu - \frac{eA_{nw}}{|p_0| c} \frac{\epsilon A_{nw}}{|p_0| c}$$

Contracting (8) with $p_\mu$ gives an identity needed later

$$s_\mu(A) p_\mu = eA_{nw} p_0 - \frac{eA_{nw}}{|p_0| c} \frac{\epsilon A_{nw}}{|p_0| c}$$

$$= s_\mu p_\mu - eA_{nw} s_\mu$$

The displacement $x_\mu - x_\mu = (N, M_x, M_y, M_z) \ell$ of the fermion on the spacetime lattice is specified by the positive integers $N$, $M_x$, $M_y$, $M_z$. Let $s_\mu$ denote the 4-vector spin state of a Dirac particle at the initial point $x_a$ and $s_\mu$ the spin state at the end point $x_b$. Moreover, let us consider the case when $s_\mu = s_\mu$. The path summation (1) is equivalent to a sum over a set of spin chains $\{s_\mu, \ldots, s_\mu\}$ with 4-magnetization $M^\mu = \ell \sum_{w=0}^{N-1} s_\mu$.

$$K_{ab}^{\text{RE}} = \sum_{\{s_\mu, \ldots, s_\mu\}} \left( \ell - \epsilon^2 \right) R(A) (-i\epsilon)^{R(A)}$$

Since $\ell \sum_{w=0}^{N-1} s_\mu = x_\mu - x_\mu = s_\mu - s_\mu$, the constant magnetization spin chains are equivalent to fixed length paths in spacetime. Also, with $s_\mu = s_\mu$, each spin chain is a closed loop (viz. periodic boundary conditions in spin variables).
\[ R(A) = \frac{1}{2} \sum_{w=0}^{N-1} s^\mu_w s_{\mu,w+1}, \quad \overline{R}(A) = \sum_{w=0}^{N-1} \left( 1 - \frac{1}{2} s^\mu_w s_{\mu,w+1} \right). \]

The Kronecker delta is

\[ \delta^{(4)}(M^\mu, \sum_{w=0}^{N-1} s^\mu_w) = \frac{1}{T L^3} \sum_n e^{-i s^\mu_k n^\mu / \hbar} e^{i t \sum s^\mu_n k^\mu_n} \delta^{(4)}(s^\mu_n) \]

in 3+1 dimensions \cite{24} \cite{25}. Using \( g \equiv -\frac{1}{2} \log (-i) \) and \( g' \equiv -\frac{1}{2} \log (\sqrt{1 - \epsilon^2}) \), the kernel (10) becomes

\[ K^{\text{HE}}_{ab} = \frac{1}{2} \sum_{\{s^\mu_w, \ldots, s^\mu_{N-1}\}} \delta^{(4)}(M^\mu, \sum_{w=0}^{N-1} s^\mu_w(A)) \times e^{\sum_{w=0}^{N-1} (-g s^\mu_w s_{\mu,w+1} - g' (21 - s^\mu_w s_{\mu,w+1}))}. \]

With \( s^\mu_w = s^\mu_k \) the spin chains are periodic, so
\[ \sum_{w=0}^{N-1} s^\mu_w = \frac{1}{2} \sum_{w=0}^{N-1} (s^\mu_w + s^\mu_{w+1}). \] Inserting (12b) into (13) the kernel becomes a partition function of an ensemble of spin chains

\[ K^{\text{HE}}_{ab} = \frac{1}{T L^3} \sum_n e^{-i s^\mu_k n^\mu / \hbar} \prod_{w=0}^{N-1} \left[ e^{i \frac{1}{2} (s^\mu_w + s^\mu_{w+1}) k^\mu_{n'} - g s^\mu_w s_{\mu,w+1} - g' (21 - s^\mu_w s_{\mu,w+1})} \right] \times e^{-i \frac{1}{2} (s^\mu_w + s^\mu_{w+1}) k^\mu_{n'}}. \]

Quantum algorithm.—At the \( w \)th time step in (14), spacetime transfer operator \( U(n) \) depends on the two 4-vectors \( s^\mu_w \) and \( s^\mu_{w+1} \), or \( 2^{4+4} \) combinations of spin variables. Yet, by evaluating \( U(n) \) with respect to the straight path connecting \( x^\mu_w \) to \( x^\mu_{n'} \), the number of spin variables reduces to just \( s^\mu_w, s_{0,w+1}, s_w \cdot k_n \) and \( s_{w+1} \cdot k'_{n'} \). In this frame, the \( 2^4 \) combinations of \( \pm 1 \) variables allows \( U(n) \) to be cast as a \( 4 \times 4 \) matrix and the \( w \)-product in (14) as matrix multiplication. In turn, the transfer operator is \( U(n) = e^{-i (E \tau / \hbar) \cdot A} U(n) \), where

\[ U(n) = e^{-i \frac{1}{2} (s^\mu_w + s_{w+1}) k^\mu_{n'}} \times e^{-i (s^\mu_w + s_{w+1}) \cdot k'_{n'}} \times e^{-i (s^\mu_w + s_{w+1}) \cdot k'^{\mu}_{n'}} \times e^{-i (s^\mu_w + s_{w+1}) \cdot k^{\mu}_{n'}} \]

\[ = \left( e^{i \sigma \cdot k_n - 2 \mu} 1, \quad (s_w \cdot k_n, s_{w+1} \cdot k'_{n'}) = (-1, -1) \right) \]

\[ e^{-2 \nu} 1, \quad (s_w \cdot k_n, s_{w+1} \cdot k'_{n'}) = (-1, 1) \]

\[ e^{-2 \nu} 1, \quad (s_w \cdot k_n, s_{w+1} \cdot k'_{n'}) = (1, -1) \]

\[ e^{-i \sigma \cdot k_n - 2 \mu} 1, \quad (s_w \cdot k_n, s_{w+1} \cdot k'_{n'}) = (1, 1). \]

So at the \( w \)th step for either unbent or bent path segment pairs \( (s_w \cdot k_n, s_{w+1} \cdot k'_{n'}) \), in matrix form (15) is

\[ U(n) = \begin{pmatrix} U_{-1,-1} & U_{-1,1} \\ U_{1,-1} & U_{1,1} \end{pmatrix} = \begin{pmatrix} e^{i \sigma \cdot k_n - 2 \mu} 1 & e^{-2 \nu} 1 \\ e^{-2 \nu} 1 & e^{-i \sigma \cdot k_n - 2 \mu} 1 \end{pmatrix} \]

\[ = e^{i \sigma \cdot k_n - 2 \mu} 1, \quad \sqrt{1 - \epsilon^2} 1 = \begin{pmatrix} -i e^{-i \sigma \cdot k_n'} \sqrt{1 - \epsilon^2} 1, \quad -i e^{-i \sigma \cdot k_n'} \sqrt{1 - \epsilon^2} 1 \end{pmatrix}, \]

using the convention \( \sigma \cdot \sigma \equiv \sigma \otimes \sigma \) for tensor products. This transfer operator constitutes an accurate quantum lattice gas algorithm, the product of stream and collide operators \( U(n) = S(n) \cdot C(n) \). Since \( S(n) \equiv e^{i \sigma \cdot \sigma \cdot k_n'} \) is manifestly unitary, one can show the unitarity of \( U(n) \) by showing the unitarity of the collide operator

\[ C(n) = \sqrt{1 - \epsilon^2} 1 = -i e^{-i \sigma \cdot k_n'} \sqrt{1 - \epsilon^2} 1 \]

\[ = \exp \left[ -i \cos^{-1} \left( \sqrt{1 - \epsilon^2} \right) \sigma_z 1 \cdot e^{i \sigma \cdot \sigma \cdot k_n'} \right], \]

using the identity \( \sin(\cos^{-1} \sqrt{1 - \epsilon^2}) = \epsilon \) and realizing \( \sigma_z 1 \cdot e^{i \sigma \cdot \sigma \cdot k_n'} \) is an idempotent operator. Multiplying (17a) by \( S(n) \) gives an accurate quantum algorithm useful for simulating Dirac particle dynamics

\[ U(n) = \sqrt{1 - \epsilon^2} S(n) - i \sigma_z 1. \]

This operator captures the quantum information dynamics intrinsic to relativistic quantum mechanics.

\[ \text{Dirac Hamiltonian}.—\text{One is free to write the relativistic energy relation} \ E^2 = (\hbar k'_n)^2 + (mc)^2 \ (\text{and using} \ \epsilon = mc^2 / \hbar) \text{in spin space as} \]

\[ (1 - \epsilon^2) 1_4 = \begin{pmatrix} 1 - \left( \frac{E^2}{\hbar^2} \right) \end{pmatrix} 1_4 + \begin{pmatrix} \epsilon \sigma \cdot \sigma \cdot k'_n \end{pmatrix}^2. \]

Its square root with respect to complex conjugation is

\[ \sqrt{1 - \epsilon^2} S(n) = \sqrt{1 - \left( \frac{E^2}{\hbar^2} \right)} + i \sigma_z 1 \cdot k'_{n}. \]

Using \( p'_n = \hbar k'_n \) and inserting (20) into (18), the spacetime transfer operator is unitary

\[ U(n) = \begin{pmatrix} e^{-i \hbar n_0 / \hbar} \sqrt{1 - \left( \frac{E^2}{\hbar^2} \right)} + i \sigma_z 1 \cdot k'_{n} - i \sigma_z 1 \end{pmatrix} \]

\[ = e^{-i \hbar n_0 / \hbar} \exp \left[ -i \cos^{-1} \left( \sqrt{1 - \left( \frac{E^2}{\hbar^2} \right)} \right) \frac{\hbar n_0}{E^2} \right]. \]

This demonstrates that the unitary evolution from (1) is generated without approximation by the Dirac Hamiltonian \( h_D(n) = -\sigma_z 1 \cdot (p_n c - e A) + \sigma_1 mc^2 \).

\[ \text{Time dilation}.—\text{The transfer operator (21b) is} \]

\[ U(n) = e^{-i (E \tau / \hbar) A} e^{-\frac{\tau}{n} h_D(n)}, \]
where $\zeta = \cos^{-1}\left(\sqrt{1 - E^2/\ell^2}/\ell^2\right)/E'$ is a dimensionless time scale factor, which can be written as
\[ E' \tau / \hbar = \sin E' \zeta \tau / \hbar. \]  
(23)

At the grid-scale $E = h/\tau$, (23) reduces to $1 = \sin \zeta$, so here $\zeta = \pi/2$. Yet, at QFT scales $E \ll h/\tau$, (23) implies $\zeta \rightarrow 1$. So the scale range is $1 \leq \zeta \leq \pi/2$. This implies that the smallest observable intervals are the radial distance $r = \ell \zeta$ and elapsed time $t_r = \tau/c = \zeta t$. 

At immediate scales between the Planck scale and Fermi gas (e.g. Fermi condensate), the fermion's body Fermi gas is described in the mean-field limit, a low-energy expansion (23) takes the form
\[ \frac{1}{\zeta} = 1 - \frac{\zeta^2 \tau^2}{3h^2 c^4/(p'/c)^2} + \cdots, \]  
(24)

where relativistic relation $vE' = p'/c^3$ is used and $v < c$ is the fermion's velocity in the medium [26]. In a many-body Fermi gas (e.g. Fermi condensate), the fermion's velocity ('acoustic' velocity) is less than the speed of light by a factor of $1/\sqrt{3}$ due to fermion-fermion interactions. Upon rearranging, taking the square root and setting $v = c/\sqrt{3}$, (23) becomes a time dilation equation
\[ \frac{\zeta \tau}{h/(p'/c)} = \sqrt{1 - \frac{1}{\zeta}} \rightarrow \frac{t_r}{t} + \cdots = \sqrt{1 - \frac{r_s}{\ell}} \]  
(25)

where the Schwarzschild radius is $r_s = \ell$ and in the long-wavelength limit the time is $t = h/(p'c)$. 

**Path integral.**—At QFT scales ($\zeta \approx 1$), (14) becomes
\[ K_{ab}^{\text{HE}} (21b) = \frac{1}{T L^3} \sum_n \sum_{\text{paths}} e^{-ix_n/a_{\mu}} e^{-\pi \sum_{w=1}^{N-1} \delta t (E - eA_n)} / |\psi(t)|. \]  
(26)

First, in the flat-space $L \rightarrow \infty$ continuum limit (with $\ell \sim dx \sim dy \sim dz$ and $2\pi/(L\ell) \rightarrow dk = dp/\hbar$), the summations over $n = (n_1, n_2, n_3, n_4)$ and paths connecting $x_n$ to $x'_n$ in (26) map to momentum-space and path integrals, respectively,
\[ \sum_n \frac{1}{(2\pi)^4} \left( \frac{2\pi}{L\ell} \right)^4 \sum_{\text{paths}} e^{i \delta t (E - eA_n)} / |\psi(t)|. \]  
(27)

Second, regarding stream operator $S(n) = e^{i\sigma \cdot p_n}$, where $p_n = 2\pi n h/(L\ell)$, if the continuum limit $L \rightarrow \infty$ in a many-fermion system is invoked, then one replaces qubit-array operators $p_n$ and $E_n$ by operators acting on a Dirac spinor field $\psi(x)$ in Minkowski spacetime $-ih\nabla$ and $E_n \rightarrow ih\nabla$. The transfer operator maps to
\[ \mathcal{U}(n) (18) e^{-iE_n t} \left[ \sqrt{1 - e^{2} e^{i\sigma \cdot \nabla + iA_n/mc^2}} - i\epsilon \sigma \tau \right] \]  
(28)

$A_0(x)$ causes an overall phase rotation and $A(x)$ causes a phase rotation during streaming by $S(n)$. In the continuum $L \rightarrow \infty$ limit of Minkowski space, the kernel (26) becomes a path integral
\[ K_{ab}^{\text{HE}} (27) = \int \frac{dp^4}{(2\pi)^4} \int _n \mathcal{D}\{ x \} e^{-i\frac{\tau}{\hbar} p' \cdot c} e^{-\pi \delta t L(x)}, \]  
(29)

where the Lagrangian operator $L$ is given by
\[ L(x) = \left( E' - n - (c^2 \cdot p' + mc^2) \right), \]  
(30)

and where the chiral representation of the Dirac matrices [21] $\gamma_0 = \sigma_x 1_2$ and $\gamma = \sigma_y \sigma_x$ is recovered. Since $p'_n \rightarrow -ih\nabla - eA/c$ and $E_n \rightarrow ih\nabla - eA_0 = ihc\partial_0 + eA_0$, the Lagrangian operator (30) maps to $L \rightarrow \gamma_0 \left( [ihe\partial_0 - eA_0 - (c^2 \cdot eA/c) - mc^2] \right) \rightarrow \gamma_0 \left( ihe\gamma_0 D^\mu - mc^2 \right) \psi$ as the effective quantum field theory for the quantum lattice gas. This is the covariant Lagrangian density for a Dirac 4-spinor field in the chiral representation, where chiral symmetry is broken by the mass term, and where $D^\mu \equiv \partial^\mu + ieA_\mu(x)/(hc)$ is the generalized 4-derivative with a background 4-potential $A_\mu(x)$. The matrix element of (26) takes the form of a Feynman path integral $\langle \mathcal{K}_{ab} \rangle \rightarrow \int \mathcal{D}\{ x \} \int \frac{dp^4}{(2\pi)^4} e^{i\frac{\tau}{\hbar} p' \cdot c} e^{-\pi \delta t L_0} \mathcal{D}\{ x \}$. 

**Acknowledgements.**—This work was supported by the grant “Quantum Computational Mathematics for Efficient Computational Physics” from the Air Force Office of Scientific Research.
As shorthand, the summation over wave vector modes is
\[ \sum_n \equiv \sum_{n_1=-l/2}^{(L/2)-1} \sum_{n_2=-l/2}^{(L/2)-1} \sum_{n_3=-l/2}^{(L/2)-1} \sum_{n_4=-l/2}^{(L/2)-1} \]
Cancellation of the path summation occurs outside of the light cone by the Kronecker delta (12).

In the small-\( E \) expansion, rescaling \( \tau \rightarrow \sqrt{2}\tau \) is applied in the equation for \( \zeta \). So (23) becomes \( \sqrt{2}\tau = \frac{k}{\hbar} \sin \frac{\sqrt{2}\tau}{\hbar} = \sqrt{2}\tau \left( 1 - \frac{2\tau^2}{3\hbar^2} + \cdots \right) \), and then both sides are divided by \( \sqrt{2}\tau \).