SPECTRAL GAP OF SCL IN FREE PRODUCTS

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Abstract. Let $G = \ast \lambda G_\lambda$ be a free product of torsion-free groups, and let $g \in [G, G]$ be any element not conjugate into a $G_\lambda$. Then scl$_G(g) \geq 1/2$. This generalizes, and gives a new proof of a theorem of Duncan–Howie [10].

1. Introduction

For any group $G$, let $[G, G]$ denote its commutator subgroup. For any $g \in [G, G]$, the commutator length $\text{cl}(g)$ is the minimal number $n$ such that $g = [a_1, b_1][a_2, b_2] \cdots [a_n, b_n]$ for some $a_i, b_i \in G$, and the stable commutator length $\text{scl}(g)$ is the limit

$$\lim_{n \to \infty} \frac{\text{cl}(g^n)}{n}.$$

The spectrum of $\text{scl}$ is the set of values of $\text{scl}(g)$ as $g$ runs over elements of $[G, G]$.

1.1. Main results. In this paper, our main result is

**Theorem A.** Let $G = \ast \lambda G_\lambda$ be a free product of torsion-free groups, and suppose $g \in [G, G]$ is not conjugate into any $G_\lambda$, then

$$\text{scl}_G(g) \geq 1/2.$$

This statement must be modified when the factors have torsions. For instance, we have a lower bound $1/2 - 1/N$ if every nontrivial element in each factor group has order at least $N \geq 2$; for details, see Theorem 3. The same statement and proof are still valid if the assumption $g \in [G, G]$ is weakened to $g^n \in [G, G]$ for some $n \geq 1$ since we use the geometric interpretation (See Section 2).

A special case is when each $G_\lambda = \mathbb{Z}$. In this case $G$ is free, and no $g \in [G, G]\{id\}$ is conjugate into a factor. Thus we obtain a new proof of

**Corollary B.** If $F$ is free, and $g \in [F, F]\{id\}$, then

$$\text{scl}(g) \geq 1/2.$$

Duncan–Howie [10] proved Theorem A when $G_\lambda$ are locally indicable. Our proof is new even in that case.

A group $G$ with the property that either $\text{scl}(g) = 0$ or $\text{scl}(g) \geq C$ for some $C = C(G) > 0$ for all $g \in [G, G]$ is said to have a spectral gap $C$ for $\text{scl}$. Residually free groups have spectral gap $1/2$ [4] Corollary 4.113 using Duncan–Howie’s result); $\delta$-hyperbolic groups have a spectral gap that can be estimated by the number of generators and $\delta$ (Calegari–Fujiwara [5]); finite index subgroups of mapping
class groups also have a spectral gap (Bestvina–Bromberg–Fujiwara [1]); Baumslag–Solitar groups have a spectral gap $1/12$ (Clay–Forester–Louwsma [8]); Right angled Artin groups have a spectral gap $1/24$ (Fernós–Forester–Tao [11]).

Our results imply that $*G_i$ has a spectral gap $\min\{C, 1/2 - 1/N\}$ ($N \geq 3$) if all $G_i$ have spectral gap $C$ and contains no $(N-1)$-torsion. Without the assumption on torsions, the spectral gap $\min\{C, 1/12\}$ has been obtained in [8], Theorem 6.9.

In fact we give two logically independent proofs of Corollary B.

Ivanov–Klyachko [12] recently independently obtained Theorem A together with other estimates related to Theorem 3.1 in terms of commutator length. Their argument uses a different language (diagrams) but the idea behind is similar, especially in the case of free groups. Corollary 3.7 provides the connection.

1.2. Contents of paper. Section 2 introduces the geometric language of fatgraphs, used to study scl in free groups. We give a new proof of Corollary B and discuss potential generalizations to integral chains.

Section 3 introduces some techniques to study surface maps into a wedge of spaces. We use these techniques to prove Theorem 3.1 which allows the factors to have torsion, then we deduce Theorem A.

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2. GEOMETRIC DEFINITION OF SCL

Our arguments are geometric, and depend on an interpretation of scl in terms of maps of surfaces.

Let $G$ be a group. Let $X$ be a $K(G, 1)$. Each conjugacy class $g \in [G, G]$ corresponds to a free homotopy class of loop $\gamma : S^1 \to X$.

An admissible surface for $g$ is a compact, oriented surface $R$ without disk or sphere components, together with a free homotopy class of map $f : R \to X$ for which the following diagram commutes

\[
\begin{array}{ccc}
\partial R & \overset{\partial f}{\longrightarrow} & S^1 \\
\downarrow & & \downarrow \gamma \\
R & \overset{f}{\longrightarrow} & X
\end{array}
\]

and $\partial f$ is a positively oriented (possibly disconnected) covering (of degree $n(R)$).

These are sometimes called monotone admissible surfaces ([4], Definition 2.12).

Lemma 2.1 ([4], Proposition 2.10 and 2.13).

\[
scl(g) = \inf \frac{-\chi(R)}{2n(R)}
\]

over all admissible surfaces for $g$.

This also defines $scl(g)$ for $g \in G$ with $g^n \in [G, G]$ for some $n \geq 1$.

The following corollary is well-known to people studying scl, we include it for readers interested in commutator length.
Corollary 2.2. Let $g_i$ be conjugates of $g$. Unless $m = 1$ and $g_1^{n_1} = id$, we have

$$\text{cl}(g_1^{n_1} \cdots g_m^{n_m}) \geq \text{scl}(g) \left| \sum_{i=1}^{m} n_i \right| - \frac{m}{2} + 1.$$ 

Proof. If $g_1^{n_1} \cdots g_m^{n_m} = id$, then the inequality does not hold only when $m = 1$. From now on, assume $g_1^{n_1} \cdots g_m^{n_m} \neq id$ and can be written as a product of $k$ commutators, then we obtain a surface $R$ of genus $k$ with $m$ boundary components and a map $f : R \to X$ such that the boundary components wrap $n_i$ times respectively around a loop $\gamma$ representing the conjugacy class $g$. Then $-\chi(R) = 2k + m - 2$ and $R$ is not a disk by assumption. If all $n_i$ have the same sign, the inequality follows from Lemma 2.1; for the general case, apply Proposition 2.10 in [4] instead. \hfill \Box

If $G$ is free, we can take $X$ to be a wedge of circles. In this case, any admissible $S$ can be compressed (reducing $-\chi(S)$ without changing $n(R)$) until it is represented by a fatgraph (see [9]). See [12] or [13] for an introduction to fatgraphs.

Informally, a fatgraph is a graph $Y$ with a cyclic ordering of edges at each vertex, which lets $Y$ embed canonically as the spine of a compact oriented surface $S(Y)$ (the fattening) which deformation retracts to $Y$. $\partial S(Y)$ has an induced simplicial structure.

If $Y$ comes with a simplicial map $f : Y \to X$, then we get a surface map $\bar{f} : S(Y) \to X$, which is simplicial on $\partial S(Y)$, by pre-composing with the deformation retraction $S(Y) \to Y$. We decorate the oriented edges of $\partial S(Y)$ by generators of $F$ to indicate where they are mapped in $X$ by $\bar{f}$. See Figure 1 for an example.

Any cyclically reduced $g$ in a free group $F$ is represented by a simplicial immersion $\phi : C_{|g|} \to X$, where $|g|$ is the word length, and $C_{|g|}$ is the simplicial oriented $S^1$ with $|g|$ vertices. For an admissible fatgraph $Y$ (with a simplicial map $f : Y \to X$), the oriented covering map $\partial f : \partial S(Y) \to C_{|g|}$ can be taken to be simplicial. Every admissible surface (up to compression and homotopy) can be put in this form.

Now we prove

Corollary B. If $F$ is free, and $g \in [F,F]\setminus\{id\}$, then

$$\text{scl}(g) \geq 1/2.$$
Proof. Take any fatgraph $Y$ with fattening $S = S(Y)$ admissible of degree $n$ for $g$. Label the vertices of $C_{|g|}$ cyclically as $1, 2, \ldots, |g|$. Pull back the labels to $\partial S$ via the covering map $\partial \bar{f}$, then each edge on $\partial S$ also gets labeled as $(i, i + 1)$ for $i < |g|$ or $(|g|, 1)$.

Two (distinct) edges of $\partial S$ are paired if they are mapped to the same edge of $Y$ under the deformation retraction. Two (distinct) vertices of $\partial S$ are paired if they are end points of a pair of paired edges that correspond to some edge $e$ in $Y$ and these two vertices correspond to the same end point of $e$. In Figure 1, $v_4$ and $v_5$ are paired vertices; $v_1, v_2$ and $v_3$ are mutually paired; $v_6$ is paired with $v_7$ and $v_9$ but not with $v_8$.

Claim 2.3. Paired vertices have distinct labels.

Proof. Suppose not, then we will have two paired edges mapped to two consecutive edges of $C_{|g|}$ under $\partial f$. But paired edges are decorated by inverse letters, so the cyclic word decorating $\partial S$ is not cyclically reduced, contrary to the assumption. □

Now construct a directed graph $G$ (possibly with multi-edge) as follows. The vertex set is $\{1, 2, \ldots, |g|\}$. Whenever we have a pair of paired edges on $\partial S$ labeled as $(i, i + 1)$ and $(j, j + 1)$ respectively, add a directed edge from $i + 1$ to $j$ and another from $j + 1$ to $i$. We say a directed edge from $i$ to $j$ is descending if $i > j$. See Figure 2 for an example. The graph $G$ resembles the turn graph introduced by Brady–Clay–Forester [2] to compute scl in free groups.

For each vertex $v$ of $Y$, let $d(v)$ denote the valence.

Claim 2.4.

$$\sum d(v) = n|g|,$$

where the sum is over all vertices of $Y$.

Proof. For each vertex $v$ of $Y$ with valence $d(v)$, there are exactly $d(v)$ vertices on $\partial S$ that deformation retract to $v$. Thus $\sum d(v)$ is the number of vertices of $\partial S$, which equals the number of edges on $\partial S$, which is $n|g|$. □

Claim 2.5.

$$\#(\text{vertices in } Y) \leq n \left(\frac{1}{2} |g| - 1\right).$$
Proof. In the proof above, we see that there are exactly $d(v)$ vertices on $\partial S$ that deformation retract to $v$. These vertices contribute to exactly $d(v)$ directed edges in $G$ which form a directed cycle. This gives a decomposition of $G$ into cycles as $v$ ranges over all vertices of $Y$. Note that each directed cycle in $G$ must contain a descending edge since there is no self loop according to Claim 2.3. Therefore, the number of vertices in $Y$ is no more than the number of descending edges in $G$.

On the other hand, for each pair of paired edges on $\partial S$ labeled as $(i,i+1)$ and $(j,j+1)$ respectively, if neither of $i, j$ is $|g|$, then exactly one of the two directed edges in $G$ contributed by this pair is descending; if $i = |g|$ or $j = |g|$, then neither of the two directed edges is descending. Thus the number of descending edges in $G$ is $n(|g|/2 - 1)$.

Combining the results above, we have

$$-\chi(S) = -\chi(Y) = \sum \frac{d(v) - 2}{2} = \frac{1}{2} \sum d(v) - \#(\text{vertices of } Y) \geq \frac{1}{2} n|g| - n \left( \frac{1}{2} |g| - 1 \right) = n$$

for all admissible fatgraphs, which implies $\text{scl}(g) \geq 1/2$. □

$scl$ is extended to integral chains (formal sums of elements) in [4].

Conjecture 2.6 (Calegari).

$$\text{scl}(c) \geq \frac{1}{2}$$

for any integral chain $c$ in a free group, unless an admissible surface of $c$ is annuli (in which case $\text{scl}(c) = 0$).

Remark 2.7. Using the argument above, any ordering of the vertices on $C_{|g|}$ gives a lower bound on $\text{scl}(g)$, which also works for chains. However, few orderings provide the correct lower bound 1/2, and it seems difficult to show such good orderings exist for general chains. Computer experiments give evidence for Conjecture 2.6.

Remark 2.8. Duncan–Howie’s proof depends on the existence of a left-ordering on torsion-free one-relator quotients of the free groups. There is no analogy of their argument for integral chains. Thus one motivation of our work is to find a new proof of their result which does not depend on orderability.

Remark 2.9. Spectral gap $1/2$ is often useful to certify extremal surfaces (those admissible surfaces realizing the infimum in the geometric interpretation). For example, Corollary 2.B implies that the once-punctured torus bounding $[x,y]$ is extremal when $x, y \in F$ do not commute. Similarly, the special case $c = x^{-1} + y^{-1} + xy$ in Conjecture 2.6 is asking whether the thrice-punctured sphere bounding $c$ is extremal when $x, y$ do not commute, which is still open.

3. Free product case

In this section we prove Theorem A. This will follow by induction and a finiteness argument from:
Theorem 3.1. Let $G = A * B$ and $g = a_1b_1 \cdots a_Lb_L$ with $a_i \in A \setminus \{id\}$, $b_i \in B \setminus \{id\}$ and $L \geq 1$ such that $g \in [G,G]$. Let $N \geq 2$ be the minimal order of $a_i$ and $b_i$, then
\[
scl_G(g) \geq 1/2 - 1/N.
\]

The formalism of fatgraphs is inadequate when factors are not free; thus we introduce a new formalism following \[3\].

If $G = A * B$, then we can build a $K(G,1)$ by taking $X := K(A,1) \vee K(B,1)$ to be a wedge. Then an admissible surface $R \rightarrow X$ decomposes into subsurfaces $R_A \rightarrow K(A,1)$ and $R_B \rightarrow K(B,1)$. $R_A$ and $R_B$ are surfaces with corners, which each contributes $1/4$ to $-\chi$.

Calegari \[3\] shows how to compute scl in certain free products by a pair of linear programming problems, one for each of $A$ and $B$.

When $A,B$ are abelian (the case Calegari considers), the contribution of $R_A$ to scl comes from a linear term, together with a nonlinear contribution from disk components, i.e. components of $R_A$ which are homeomorphic to $D^2$.

Example 3.2. Let $R$ be the fatgraph admissible for $g = abaBafaB \in \mathbb{Z} * \mathbb{Z}$ in Figure 2. It decomposes into $R_A$ and $R_B$ as in Figure 3. $R_A$ contributes
\[
1/4 \#\text{corners} - \#\text{disks} = 2 - 1 = 1
\]
to $-\chi(R)$, and similarly the contribution of $R_B$ is 0. In general, to minimize $-\chi(R)/2n$ is to maximize the number of disk components — a nonlinear problem.

For arbitrary $A,B$ we obtain a lower bound on scl by ignoring the (positive) contribution to $-\chi$ of non-disk components of $R_A$ and $R_B$. Equality holds (by a covering argument, see \[3\] or \[7\]) when scl vanishes on both $A$ and $B$. Formally, fix $G$ and $g$ as in Theorem 3.1.

Definition 3.3. Let $W$ be a vector space formally spanned by the set of ordered pairs $(i,j)$, $1 \leq i,j \leq L$. Let
\[
V := \left\{ \sum_{1 \leq i,j \leq L} x_{ij}(i,j) \mid x_{ij} \geq 0, \sum_i x_{ij} = 1, \sum_j x_{ij} = 1 \right\} \subset W,
\]
\[ D_A := \left\{ \sum_{j=1}^{k} (i_j, i_{j+1}) \mid i_{k+1} = i_1, \prod_{j=1}^{k} a_{i_j} = 1 \in A, k > 0 \right\} \subset W. \]

Each element in \( D_A \) is called a disk vector in \( A \). For any \( v \in V \), define

\[ \kappa_A(v) := \sup \left\{ \sum t_i \mid v = \sum t_i d_i + \sum x_{ij}'(i, j), t_i \geq 0, d_i \in D_A, x_{ij}' \geq 0 \right\}. \]

Define \( D_B \) and \( \kappa_B \) similarly. Finally define \( \phi : V \to V \) to be the affine map given by \( \phi(i, j) = (j - 1, i) \) (replace \( j - 1 \) by \( L \) if \( j = 1 \)).

Up to compression of \( R \), the boundary \( \partial R_A \) alternates between arcs mapped to one of \( a_i \) and inessential arcs (mapped to the wedge point). Encode \( R_A \) as \( v_A = \sum x_{ij}(i, j) \in V \), where \( x_{ij} \) is the number, divided by \( n(R) \), of inessential arcs on \( \partial R_A \) that go from \( a_i \) to \( a_j \). Disk components contribute to disk vectors. Such an encoding cannot reconstruct \( R_A \) but is enough to bound from below the contribution of \( R_A \) to \(-\chi(R)\), and \( \kappa_A(v_A) \) is the (normalized) maximal number of disk components \( R_A \) can have.

Note that \( R_A \) and \( R_B \) can be glued up along inessential arcs, such that \( a_i \) should be followed by \( b_i \) and \( b_{j-1} \) should be followed by \( a_j \). Thus the vectors \( v_A \) and \( v_B \) corresponding to \( R_A \) and \( R_B \) satisfy \( v_B = \phi(v_A) \). Based on these, \( \text{scl}_G \) can be estimated using the following lemma.

**Lemma 3.4.** Under the notation above,

\[ 2 \cdot \text{scl}_G(g) \geq L - \sup_{v_A \in V} \{ \kappa_A(v_A) + \kappa_B(\phi(v_A)) \}. \]

Equality holds if \( \text{scl}_A \) and \( \text{scl}_B \) are identically zero.

This is Corollary 4.17 in [17] in the case \( G_1 = A, G_2 = B \) and \( z = g \) since \((v_A, v_B) \in Y \) means \( v_B = \phi(v_A) \) in our notation and we have \( |v_A| = |v_B| = L \).

Now we are ready to prove Theorem 3.1

**Proof of Theorem 3.1.** By Lemma 3.4 it suffices to show

\[ \kappa_A(v_A) + \kappa_B(\phi(v_A)) \leq L - 1 + 2/N \]

for any \( v_A = \sum x_{ij}(i, j) \in V \).

**Claim 3.5.** For all \( v = \sum x_{ij}(i, j) \in V \) we have

\[ \kappa_A(v) \leq \frac{1}{N} \sum_{i \geq j} x_{ij} + \left( 1 - \frac{1}{N} \right) \sum_{i < j} x_{ij}, \text{ and similarly for } \kappa_B(v). \]

**Proof.** Suppose \( v = \sum t_i d_i + \sum x_{ij}'(i, j) \) with \( t_i \geq 0, d_i \in D_A \) and \( x_{ij}' \geq 0 \). It suffices to show that each \( d_i \) contributes at least 1 to the right hand side of the inequality.

For any disk vector \( d = \sum_{j=1}^{k} (i_j, i_{j+1}) \), if all \( i_j \) are equal, then \( k \geq N \), and thus the contribution of \( d \) to the right hand side is \( k/N \geq 1 \).

Suppose there are at least two distinct \( i_j \)'s, then there exist \( j_1 \) and \( j_2 \) such that \( i_{j_1} > i_{j_1 + 1} \) and \( i_{j_2} < i_{j_2 + 1} \), thus the contribution of \( d \) to the right hand side is at least \( 1/N + (1 - 1/N) = 1 \).

Geometrically, each \( v \in V \) can be thought as an \( L \times L \) doubly stochastic matrix. Since each row sums to 1, the estimate in the claim above is equivalent to saying that \( \kappa_A(v) - L/N \) and \( \kappa_B(v) - L/N \) are at most \( (1 - 2/N) \) times the sum of entries.
in the strictly upper triangular region $U$. The pull back $\phi^{-1}(U)$ and $U$ together form $L-1$ columns (Figure 4 illustrates the case $L=5$), in which the entries sum to $L-1$. Combining these, we get the desired inequality.

Formally, by the claim above, for any $v_A = \sum x_{ij} (i,j) \in V$, we have

$$\kappa_A(v_A) + \kappa_B(\phi(v_A)) \leq \frac{2L}{N} + \left(1 - \frac{2}{N}\right) \sum_{i<j} x_{ij} + \sum_{1\leq j-1<i} x_{ij}$$

$$= \frac{2L}{N} + \left(1 - \frac{2}{N}\right) \left[ \sum_{i<j} x_{ij} + \sum_{2\leq j \leq i} x_{ij} \right]$$

$$= \frac{2L}{N} + \left(1 - \frac{2}{N}\right) \sum_{2\leq j \leq L} x_{ij}$$

$$= \frac{2L}{N} + \left(1 - \frac{2}{N}\right) (L-1)$$

$$= L - 1 + \frac{2}{N}$$

as desired. \qed

Remark 3.6. The estimate in Theorem 3.1 is sharp, since we have ([7], Proposition 5.6)

$$scl_{G_1 \ast G_2}(a,b) = \frac{1}{2} - \frac{1}{\min(n_a, n_b)},$$

where $a \in G_1$, $b \in G_2$ and $n_a, n_b$ are the orders of $a, b$ respectively.

Now we apply Theorem 3.1 with $N = +\infty$ to prove

**Theorem A.** Let $G = \ast_{\lambda=1}^n G_{\lambda}$ be a free product of torsion-free groups, and suppose $g \in [G,G]$ is not conjugate into any $G_{\lambda}$, then

$$scl_G(g) \geq 1/2.$$

**Proof.** First notice that for each $g \in [G,G]$, there are finitely many $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $g \in [H,H]$ where $H = \ast_{i=1}^n G_{\lambda_i} \leq G$. Moreover, we have $scl_H(g) = scl_G(g)$ since $H$ is a retract of $G$ and $scl$ is monotone under homomorphism. Thus it suffices to show $scl_H(g) \geq 1/2$.

Now we induct on $n$. The case $n = 2$ directly follows from Theorem 3.1. Now suppose $n > 2$, then $G$ is the free product of two torsion free groups $\ast_{i=1}^n G_{\lambda_i}$ and
If $g$ is not conjugate into either of them, then the result follows from Theorem 3.1; otherwise, by assumption, $g$ is conjugate into $*_i G_{\lambda_i}$ but not any $G_{\lambda}$, then the result follows from the inductive assumption.

Finally, Corollary 2.2 and Theorem 3.1 together imply the following result about commutator length similar to the results obtained by Ivanov–Klyachko [12]:

**Corollary 3.7.** Let $G = A * B$ and $g = a_1 b_1 \cdots a_L b_L$ with $a_i \in A \setminus \{id\}$, $b_i \in B \setminus \{id\}$ and $L \geq 1$ such that $g \in [G, G]$. Let $N \geq 2$ be the minimal order of $a_i$ and $b_i$, and $g_i$ be conjugates of $g$, then

$$2 \cdot \text{cl}(g_1^n \cdots g_m^m) - 2 \geq \sum_{i=1}^{m} (n_i - 1) - \left\lfloor \frac{2}{N} \sum_{i=1}^{m} n_i \right\rfloor,$$

where $[x]$ denotes the largest integer no greater than $x$.

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