ABSTRACT. Bauer, Di Francesco, Itzykson and Zuber proposed recently an algorithm to construct all singular vectors of the Virasoro algebra. It is based on the decoupling of (already known) singular fields in the fusion process. We show how to extend their algorithm to the Neveu-Schwarz superalgebra.

RESUME. Bauer, Di Francesco, Itzykson and Zuber proposaient récemment un algorithme pour construire tous les vecteurs singuliers de l’algèbre de Virasoro. Cet algorithme repose sur le découplage de champs singuliers (déjà connus) lors du processus de fusion. Nous montrons comment étendre leur algorithme à l’algèbre de Neveu-Schwarz.
I. INTRODUCTION.

Explicit expressions for various objects have great importance in theoretical physics. One might think, for example, to the discovery of the one-instanton solution by Belavin, Polyakov, Schwartz and Tyupkin and its impact on non-perturbative effects in QCD. A similar role was played by $N$-solitons solutions of the Korteweg-de Vries equation for the field of integrable models. In representation theory such examples are numerous: from Racah formula for $SU(2)$ 6-\textit{j} coefficients to Weyl formula for characters of finite-dimensional irreducible representations of simple Lie algebras, these expressions are now common tools in modern physics. Besides their computational usefulness, these expressions brought with their discovery a wealth of information about their properties and their relationships with other often remote concepts of the theory.

Singular vectors of the Virasoro algebra and of its supersymmetric extensions, the Neveu-Schwarz and the Ramond superalgebras, play a central role in conformal quantum field theories. For this reason efforts were made to discover their expressions. According to early works by Kac [1] and Feigin and Fuchs [2], we know that, for the Virasoro algebra, singular vectors arise at level $pq$ in Verma modules $V_{(c,h)}$ whose highest weight $h$ lies on one of the parametrized curves $h_{p,q}(t) = \frac{1}{4}(1 - p^2)t^{-1} + \frac{1}{2}(1 - pq) + \frac{1}{4}(1 - q^2)t$, where $p$ and $q$ are two positive integers. The continuous parameter $t$ is related to the central charge $c = 13 + 6(t + t^{-1})$. The authors [3] were able to give an explicit expression for the singular vectors whenever $p$ or $q$ equals to 1. Though this expression is particularly simple, it does not provide any clue on a possible generalization for general $p$ and $q$ nor does it establish any relationship between the algebraic concept and the more physical quantities of conformal quantum field theory (CQFT). Exploiting the operator product expansion of CQFT, Bauer, di Francesco, Itzykson and Zuber [4] were able to provide such a relationship.
Their algorithm reproduces the singular vectors given in [3] though it does not lead to any new general expressions for $p$ and $q$ different than 1. Hence this new result is not important because it answers the problem of explicit expressions for singular vectors but because it links in a deep way the very existence of these vectors (an algebraic concept) with the consistency of the operator product expansion (a physical and analytic concept).

The Neveu-Schwarz and Ramond singular vectors are as intriguing as Virasoro ones. In an earlier communication [5] we gave an explicit expression for the Neveu-Schwarz singular vectors with one index equal to one. Due to the importance of the results by Bauer $et$ $al$, we feel that it is worthwhile to extend them to the Neveu-Schwarz supersymmetric extension. This goal is accomplished in the present paper and, as will be seen, the superfield formulation plays here a significant role. Consequently the Ramond case remains an open problem. We recognize however that the Neveu-Schwarz and the Ramond cases might be intimately connected.

The overall organization of the paper is as follows: notations and basic results are gathered in the next section. Section 3 develops the core of the algorithm and gives a non-trivial example. Section 4 is devoted to the proof of one of the main identities (eq. (3.10)). Concluding remarks follow.

**II. SOME PARTICULAR SINGULAR VECTORS.**

The purpose of this section is to gather the basic ingredients to extend to the Neveu-Schwarz algebra the algorithm proposed by Bauer, Di Francesco, Itzykson and Zuber (hereafter BdFIZ). Though nothing is new here, some of the results are presented in an unusual way that is more in line with the current goal. The three important points to be discussed are: ($i$) the Neveu-Schwarz superalgebra and its singular vectors, ($ii$) the particular case of the singular
vectors $|\psi_{1,q}\rangle$ and (iii) the differential operators associated with the generators $L_{-n}$ and $G_{-r}$, $n, r > 0$.

The energy-momentum tensor $T(Z)$ is the generator of superconformal transformations in a superconformal quantum field theory. Throughout we are using the superfield formalism in which $Z$ stands for the pair $(z, \theta)$, $z$ being the usual "chiral" coordinate $z = x + iy$ ($x, y \in \mathbb{R}$) and $\theta$ a Grassman variable that allows expansion of superfields in bosonic and fermionic parts. The superfield $T(Z)$ is fermionic and has conformal weight $\frac{3}{2}$: $T(Z) = T_F(z) + \theta T_B(z)$. Its fermionic and bosonic modes

$$T_F = \sum_{n \in \mathbb{Z}} \frac{1}{2} z^{-n-2} G_{n+\frac{1}{2}}$$

$$T_B = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n$$

span, together with the central element, the Neveu-Schwarz algebra:

$$[L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} (n^2 - 1) n \delta_{n,-m}$$

$$[L_m, G_r] = \left( \frac{m}{2} - r \right) G_{m+r}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{3} (r^2 - \frac{1}{4}) \delta_{r,-s},$$

where $n, m \in \mathbb{Z}$ and $r, s \in \mathbb{Z} + \frac{1}{2}$. The complex number $c$ is the central charge of the theory.

Verma modules $V_{(c, h)}$ are constructed from a highest weight vector $|h\rangle$ with the following properties

$$L_n|h\rangle = G_r|h\rangle = 0 \quad \text{for} \quad n \geq 1, r \geq \frac{1}{2}$$

$$L_0|h\rangle = h|h\rangle$$

where $h$ is a complex number. The vectors

$$L_{-m_1} \cdots L_{-m_k} G_{-r_1} \cdots G_{-r_l}|h\rangle$$

with

$$m_1 \geq m_2 \geq \cdots \geq m_k \geq 1 \quad \text{and} \quad r_1 > r_2 > \cdots > \frac{1}{2}$$
form a basis for the vector space. The vector (2.3) has weight $h + \sum_{i=1}^{k} n_i + \sum_{j=1}^{l} r_j$. We also say that it has level $\sum_{i=1}^{k} n_i + \sum_{j=1}^{l} r_j$. A singular vector $|v\rangle \in V_{(c,h)}$ is a vector that satisfies the defining properties of the highest weight vector:

$$L_n |v\rangle = G_r |v\rangle = 0 \quad \text{for } n \geq 1, r \geq \frac{1}{2}$$

$$L_0 |v\rangle = (h + n) |v\rangle$$

but has a weight strictly larger than $h$: $n \geq 1$. It is easily seen that, by acting with the $L_{-n}$ and $G_{-r}$, $n, r > 0$, one can construct from $|v\rangle$ an invariant submodule of $V_{(c,h)}$. The physically natural inner product on $V_{(c,h)}$ is defined by $\langle h|h\rangle = 1$ and $L^1_n = L_{-n}$ and $G^1_r = G_{-r}$. Hence a singular vector (and any vector in the submodule generated from it) has zero length and should be removed in order to get a well-defined Hilbert space. Not all Verma modules $V_{(c,h)}$ have singular vectors. They arise in the Verma modules whose pair $(c, h)$ lies on at least one of the following curves $(c(t), h_{p,q}(t))$ in the $(c, h)$-plane:

$$c(t) = \frac{15}{2} + 3t^{-1} + 3t$$

$$h_{p,q}(t) = \frac{1 - p^2}{8} t^{-1} + \frac{1 - pq}{4} + \frac{1 - q^2}{8} t,$$  \hspace{1cm} (2.4)

where $t \in \mathbb{C}$ and $p, q \in \mathbb{N}$ with $p = q \mod 2$. If $(c, h) = (c(t), h_{p,q}(t))$ for some $t$, $V_{(c,h)}$ will have a singular vector of weight $h + \frac{pq}{2}$. (See Kac [1].) Though not conventional in the physics literature, the parametrization (2.4) is appropriate to understand that singular vectors are not restricted to unitary modules (the very special case $t = -\frac{m}{m+2}$ with $m = 2, 3, \ldots$) and that the formulae for singular vectors that follow apply to all cases.

Explicit expressions for singular vectors are known only in the special cases when either $p$ or $q$ is equal to 1.[5] If $p = 1$, the singular vector $|\psi_{1,q}\rangle \in$
\( V_{(c(t),h_1,q(t))} \) is of weight \( h_1, q(t) + q/2 \) and takes the following form:

\[
|\psi_{1,q}\rangle = \sum_{\text{partitions}\{k_1,k_2,\ldots,k_N\}} \sum_{\text{permutations}\ \sigma} (\frac{q}{2})^{\frac{q-N}{2}} c[k_{\sigma(1)},k_{\sigma(2)},\ldots,k_{\sigma(n)}] \times |k_{\sigma(1)}k_{\sigma(2)}\ldots k_{\sigma(N)}\rangle \]

(2.5)

where

\[
|k_1,k_2,\ldots,k_N\rangle = G_{-\frac{k_1}{2}}G_{-\frac{k_2}{2}}\ldots G_{-\frac{k_N}{2}}|h\rangle,
\]

\[
c[k_1,k_2,\ldots,k_N] = \prod_{i=1}^{N} \left( \frac{k_i - 1}{(k_i - 1)/2} \right)^{(N-1)/2} \prod_{j=1}^{q} \left( \frac{2}{\sigma_{2j}} \cdot \frac{2}{\rho_{2j}} \right),
\]

\[
\sigma_j = \sum_{\ell=1}^{j} k_{\ell} \quad \text{and} \quad \rho_j = \sum_{\ell=j}^{q} k_{\ell}.
\]

The singular vectors are normalized so that \( c[1,1,\ldots,1] = \left(\frac{q-1}{2}\right)!^{-2} \). This requirement insures that the above expression never vanishes. Let us stress that this form depends only on the \( G_{-r} \) (some \( G_{-r} \)’s being repeated) and consequently that this form is not in the usual basis (2.3).

In the Virasoro case, singular vectors had their importance revealed in the original paper by Belavin, Polyakov and Zamolodchikov [6]. The secondary field \( \psi(z) = \sum c[k_1,\ldots,k_M] \mathcal{L}_{-k_1}(z)\ldots \mathcal{L}_{-k_M}(z) \phi_h(z) \) which creates out of the vacuum the singular vector \( |\psi\rangle = \sum c[k_1,\ldots,k_M] L_{-k_1} \ldots L_{-k_M} |h\rangle \)

1 belonging to the Verma modula \( V_{(c,h)} \) should have zero correlation functions with any field in the theory. Even though 2- and 3-point correlation functions are determined by conformal invariance up to a constant, \( N \)-point ones, \( N \geq 4 \), are not and the vanishing of the correlation functions of the singular field \( \phi(z) \) gives differential equations for them. These differential equations are obtained through

\footnote{As usual, we define \( |h\rangle \equiv \phi_h(0)|0\rangle \) and the operators \( \mathcal{L}_k(z) \) are related to the \( L_k \) by \( \mathcal{L}_k(0) = L_k \). The primary field \( \phi_h(z) \), the secondary fields and their linear combinations constitute the conformal family \( [\phi_{(c,h)}(z)] \) which is in itself a highest weight representation of the Virasoro algebra.}
the isomorphism between the subalgebra of the Virasoro algebra spanned by
the $L_{-n}$'s with $n \geq 1$ and the subalgebra generated by the following differential
operators $L_{-n}$. If one is computing the $(N + 1)$-point correlation function
$\langle \psi(z_0)\phi_1(z_1)\ldots\phi_N(z_N) \rangle$ where the $\phi_i(z_i)$ are primary, the differential equation reads

$$
\sum c[k_1, \ldots, k_N] \hat{L}_{-k_1} \ldots \hat{L}_{-k_N} \langle \phi(z_0)\phi_1(z_1)\ldots\phi_N(z_N) \rangle = 0
$$

where the $\hat{L}_{-n}$ are the differential operators

$$
\hat{L}_{-n} = \sum_{i=1}^{N} \left\{ \frac{(k - 1)h_i}{z_{i0}^n} - \frac{1}{z_{i0}^{n-1}} \frac{\partial}{\partial z_{i0}} \right\}
$$

with $h_i$ the conformal weight of the field $\phi_i(z_i)$ and $z_{i0} = z_i - z_0$. It is easy
to check that $[\hat{L}_m, \hat{L}_n] = (m - n)\hat{L}_{m+n}$. The analogues of these differential
operators for both the $L_{-n}$'s and the $G_{-r}$'s where already given in one of the
earliest works on superconformal field theories [7]:

$$
\hat{L}_{-n} = \sum_{i=1}^{N} \left\{ \frac{n - 1}{z_i^n} (h_i + \frac{1}{2} \theta_i \partial \theta_i) - \frac{1}{z_i^{n-1}} \partial z_i \right\}
$$

$$
\hat{G}_{-r} = \sum_{i=1}^{N} \left\{ \frac{1}{z_i^{-r}} (\theta_i \partial z_i - \partial \theta_i) + \frac{(1 - 2r)h_i}{z_i^{r+\frac{1}{2}}} \theta_i \right\}.
$$

In these expressions, both $z_0$ and $\theta_0$ have been set to zero. We shall need however
the expression for general $z_0$ and $\theta_0$. Since correlation functions are invariant
under supertranslations both the correlation functions and the differential
operators $\hat{L}_{-n}$'s and $\hat{G}_{-r}$'s should be expressible in terms of supertranslational
invariants. For the $(N + 1)$-tuplet $(Z_0, Z_1, \ldots, Z_N)$, a possible set is known to be:

$$
\begin{align*}
\hat{z}_{i0} &= z_i - z_0 - \theta_i \theta_0 \\
\hat{\theta}_{i0} &= \theta_i - \theta_0
\end{align*}
$$

$$
1 \geq i \geq N. \quad (2.6)
$$
It is then easy to write down the general differential operators \( \hat{\mathcal{L}}_{-n} \) and \( \hat{\mathcal{G}}_{-r} \):

\[
\hat{\mathcal{L}}_{-n}(Z_0; Z_1, \ldots, Z_N) = \sum_{i=1}^{N} \left\{ \frac{(n-1)}{\tilde{z}_{i0}^n} \left( h_i + \frac{1}{2} \theta_{i0} \partial \theta_{i0} \right) - \frac{1}{\tilde{z}_{i0}^{n-1}} \partial \tilde{z}_{i0} \right\}, \quad n \geq 1
\]

\[
\hat{\mathcal{G}}_{-r}(Z_0; Z_1, \ldots, Z_N) = \sum_{i=1}^{N} \left\{ \frac{(1-2r)h_i}{\tilde{z}_{i0}^{r+\frac{1}{2}}} \theta_{i0} - \frac{1}{\tilde{z}_{i0}^{r+\frac{1}{2}}} (\partial \theta_{i0} - \theta_{i0} \partial \tilde{z}_{i0}) \right\}, \quad r \geq \frac{1}{2}
\]

Finally, it is worth noticing that throughout the rest of the paper we will use the same notation \( (L \text{ and } G) \) for two essentially different but related operators; those acting on a field of the conformal family \( \phi(c, h)(Z) \), i.e. the \( \mathcal{L}_{-n}(Z) \)'s and the \( \mathcal{G}_{-r}(Z) \)'s, and those acting on the vectors of its associated Verma module \( V(c, h) \), i.e. the \( L_{-n} \)'s and the \( G_{-r} \)'s. The context will unambiguously indicates which one we should use.

III. FUSION AND SINGULAR VECTORS.

One of the key arguments of the BdFIZ algorithm is that, since singular fields decouple completely from the theory, fusion between primary fields whose descendants include singular fields should carry this information to the expanded product. Among other things the fusion of a singular field with a primary one should be identically zero. This idea is rather straightforward though it does not indicate how to extract new singular vectors out of known ones. In fact it does so in a very subtle way and it is the goal of this section to present the BdFIZ algorithm and to extend it to supersymmetric conformal field theory.

The fusion of two primary superfields \( \phi_0(Z_0) \) and \( \phi_1(Z_1) \) of conformal weights \( h_0 \) and \( h_1 \) is usually written as:

\[
\phi_0(Z_0) \times \phi_1(Z_1) = \sum_{h} \tilde{z}_{01}^{-h_0-h_1} g(h; h_0, h_1) \sum_{\alpha=0,1} \tilde{z}_{01}^n \theta_{01}^\alpha \phi_h^{(n+\frac{\alpha}{2})}(Z_1).
\]

The first sum runs over all conformal families \( h \) that is contained in the fusion of \( \phi_0 \) and \( \phi_1 \). Since we will be dealing with only one of these families at a time,
we will select one of these and note by $F_h(\phi_0(Z_0) \times \phi_1(Z_1))$ the restriction of the rhs to the family $h$. The constant $g(h; h_0, h_1)$ is an overall factor that depends, besides the conformal weight of the fields involved, on the sectors it is interpolating from and to. While these constants $g$ are important in the normalization of the quantum group action coefficients, they are irrelevant for the present purpose. They will be absorbed in the fields $\phi_h^{(n+\frac{\alpha}{2})}(Z_1)$ which are descendants of weight $h + n + \frac{\alpha}{2}$ of the primary field $\phi_h(Z_1) = \phi_h^{(0)}(Z_1)$.

Let us recall, once for all, that the equality (3.1) is to be understood inside a correlation function.

As said earlier, the decoupling of a singular field $\psi$ is equivalent to the vanishing of their fusion with primary fields. In the notation just introduced this is simply

$$F_h(\psi(Z_0) \times \phi_1(Z_1)) = 0 \quad (3.2)$$

for all $h$. (This condition seems extremely stringent as it sets all $\phi_h^{(n+\frac{\alpha}{2})}(Z)$ in the expansion to zero.) Recall that, with $\psi = \psi_{1,q}$ for example, this condition has the form

$$0 = F_h(\psi_{1,q}(Z_0) \times \phi_1(Z_1))$$

$$= \sum \left(\frac{t}{2}\right)^{(q-N)/2} c[k_{\sigma(1)} \ldots k_{\sigma(N)}]$$

$$\times F_h(G_{-k_1/2}(Z_0) \ldots G_{-k_N/2}(Z_0) \phi_0(Z_0) \times \phi_1(Z_1))$$

where we have used (2.5) and abbreviated the sums over partitions and permutations by a single sum sign. Can we express $F_h((G_{-k_1/2} \ldots G_{-k_N/2}\phi_0)(Z_0) \times \phi_1(Z_1))$ in terms of $F_h(\phi_0(Z_0) \times \phi_1(Z_1))$? This is surely possible since the fusion takes place inside a correlation function and we know how to take the generators $G_{-k/2}$ out of it by using the differential operators $\hat{G}_{-k/2}$ introduced
in (2.7). Hence we can write

\[ F_h \left( \left( G_{-k_1/2} \ldots G_{-k_N/2} \phi_0 \right) (Z_0) \times \phi_1(Z_1) \right) = F_h (G_{-k_1/2} \times 1) \ldots F_h (G_{-k_N/2} \times 1) F_h (\phi_0 (Z_0) \times \phi_1(Z_1)) \]

for some operators \( F_h (G_{-k_i/2} \times 1) \) that we now construct. Since the correlation function \( \langle \phi_0(Z_0) \phi_1(Z_1) \phi_2(Z_2) \ldots \phi_N(Z_N) \rangle \) after fusion will look like

\[ \sum_{n, \alpha} \hat{z}_{01}^{h_0 - h_1 + n} \theta_{01}^{\alpha} \langle \phi_h^{(n+\frac{\alpha}{2})} (Z_1) \phi_2(Z_2) \ldots \phi_N(Z_n) \rangle, \]

we want to write \( F_h (G_{-k_i/2} \times 1) \) as a sum of differential terms acting on \( \hat{z}_{01} \) and \( \theta_{01} \) and of operators \( \hat{G}_{-r}(Z_1; Z_2, \ldots, Z_N) \) and \( \hat{L}_{-n}(Z_1; Z_2, \ldots, Z_N) \) that could be brought back inside the correlation function in the form of \( G_{-r} \) and \( L_{-n} \) acting on \( \phi_h^{(n+\frac{\alpha}{2})} \). Notice that the expression for the operator \( \hat{G}_{-k/2}(Z_0; Z_1, \ldots, Z_N) \) is for a \( G_{-k/2}(Z_0) \) acting on the field \( \phi_0(Z_0) \) as the operators \( \hat{G}_{-k/2}(Z_1; Z_2, \ldots, Z_N) \) will be for a \( G_{-k/2}(Z_1) \) acting on the field \( \phi_h^{(n+\frac{\alpha}{2})}(Z_1) \). The independent variables used to describe the functions \( \langle \phi_0(Z_0) \ldots \phi_N(Z_N) \rangle \) were \( z_{0i} \) and \( \theta_{0i} \), \( 1 \leq i \leq N \), and the ones used for \( \langle \phi_h^{(n+\frac{\alpha}{2})}(Z_1) \ldots \phi_N(Z_N) \rangle \) should be \( \hat{z} = \hat{z}_{01} \), \( \theta = \theta_{01} \) and \( \hat{z}_{i1} \) and \( \theta_{i1} \), \( 2 \leq i \leq N \). (In the latter set of variables, we have labelled the variables \( \hat{z}_{01} \) and \( \theta_{01} \) as \( \hat{z} \) and \( \theta \) respectively, to avoid any confusion.) From here on, it is a simple exercise in many variable calculus. The new variables are related to the old ones by

\[ \begin{align*} 
\theta &= -\theta_{10}, \\
\hat{z} &= -\hat{z}_{10}, \\
\theta_{i1} &= \theta_{i0} - \theta_{10}, \\
\hat{z}_{i1} &= \hat{z}_{i0} - \hat{z}_{10} - \theta_{i0} \theta_{10}, \quad i \geq 2 
\end{align*} \]
and the partial derivatives (old ones in terms of new ones):

\[
\partial_{\hat{z}_{10}} = -\partial_{\hat{z}} - \sum_{i \geq 2} \partial_{\hat{z}_{i1}}, \\
\partial_{\hat{z}_{10}} = -\partial_{\hat{z}} - \sum_{i \geq 2} \partial_{\hat{z}_{i1}}, \\
\partial_{\theta_{10}} = \partial_{\theta_{11}} + \theta \partial_{\hat{z}_{11}}, \\
\partial_{\hat{z}_{10}} = \partial_{\hat{z}_{11}}, \quad i \geq 2.
\]  

(3.4)

After this preamble, the rest of the calculation is straightforward:

\[
\mathcal{F}(G_{-r}(Z_0) \times 1) = \frac{1}{(-\hat{z})^{r+\frac{1}{2}}} \left\{ (2r-1)h_1 \theta - \hat{z}(\partial_{\theta} + \theta \partial_{\hat{z}}) + \hat{z}(G_{-\frac{1}{2}}(Z_1) + 2\theta L_{-1}(Z_1)) \right\} \\
+ \sum_{m=0}^{\infty} \left( \frac{r + m - \frac{3}{2}}{m} \right) \hat{z}^m (G_{-m-r}(Z_1) + 2\theta L_{-m-r-\frac{1}{2}}(Z_1)).
\]

(3.5a)

The operator \( \mathcal{F}(L_{-n}(Z_0) \times 1) \) is obtained in the same way:

\[
\mathcal{F}(L_{-n}(Z_0) \times 1) = \frac{1}{(-\hat{z})^n} \left\{ (n-1)h_1 + \frac{1}{2}(n-1)\theta (\partial_{\theta} - G_{\frac{1}{2}}(Z_1)) + \hat{z}(L_{-1}(Z_1) - \partial_{\hat{z}}) \right\} \\
+ \sum_{m=0}^{\infty} \left( \frac{n + m - 2}{m} \right) \hat{z}^m (L_{-m-n}(Z_1) - \frac{1}{2}(n + m - 1)\theta G_{-m-n-\frac{1}{2}}(Z_1)).
\]

(3.5b)

The \( r = \frac{1}{2} \) and \( n = 1 \) cases are the limits of these expressions; they are:

\[
\mathcal{F}(G_{-\frac{1}{2}}(Z_0) \times 1) = \theta \partial_{\hat{z}} + \partial_{\theta} \quad \text{and} \quad \mathcal{F}(L_{-1}(Z_0) \times 1) = \partial_{\hat{z}}.
\]

(3.5c)

One could check that the operators \( \mathcal{F}(G_{-r}(Z_0) \times 1) \) and \( \mathcal{F}(L_{-n}(Z_0) \times 1) \) satisfy the commutation rules of the subalgebra \( \{G_{-r}, L_{-n}, n, r \geq 0\} \) of the Neveu-Schwarz algebra. Since these expressions do not depend on \( h \), we have dropped the index on \( \mathcal{F}_h \).

We are now ready to examine eq. (3.2), i.e. what is the content of the fusion of singular fields with primary ones. To be more specific, we concentrate our
efforts on the singular vectors $|\psi_{1,q}\rangle$. Let $M_{1,q}$ be the generators that act on the highest weight vector to create the singular vector $|\psi_{1,q}\rangle$:

$$M_{1,q} = \sum \left( \frac{t}{2} \right)^{(q-N)/2} c[k_{\sigma(1)}, \ldots, k_{\sigma(N)}] G_{-k_{\sigma(1)}/2} \cdots G_{-k_{\sigma(N)}/2}.$$ 

Then the decoupling condition is

$$\mathcal{F}(M_{1,q} \times 1) \mathcal{F}_h(\phi_0(Z_0) \times \phi_1(Z_1)) = 0. \quad (3.6)$$

It is easy to compute the action of one term $\mathcal{F}(G_{-r} \times 1)$ on the fusion of $\phi_0$ and $\phi_1$ to understand the form the above equation will take. Defining

$$H = h - h_0 - h_1,$$

we get

$$\mathcal{F}(G_{-r}(Z_0) \times 1) (\hat{z}^H \sum_{n \geq 0} \hat{z}^n (\phi_h^{(n)} + \theta \phi_h^{(n+\frac{1}{2})}))$$

$$= \hat{z}^{H-r-\frac{1}{2}} \sum_{n \geq 0} \hat{z}^n \left\{ \left( -1 \right)^{r+\frac{1}{2}} (-\phi_h^{(n-\frac{1}{2})} + G_{-\frac{1}{2}} \phi_h^{(n-1)})

+ \sum_{m=0}^{n-r-\frac{1}{2}} \binom{r + m - \frac{3}{2}}{m} G_{-m-r} \phi_h^{(n-m-r-\frac{1}{2})} \right\}

+ \theta \left\{ \left( -1 \right)^{r+\frac{1}{2}} ((2r - 1) h_1 - (H + n)) \phi_h^{(n)} - G_{-\frac{1}{2}} \phi_h^{(n+\frac{1}{2})} + 2L_{-1} \phi_h^{(n-1)}

+ \sum_{m=0}^{n-r-\frac{1}{2}} \binom{r + m - \frac{3}{2}}{m} (-G_{-m-r} \phi_h^{(n-r-m)} + 2L_{-m-r-\frac{1}{2}} \phi_h^{(n-r-m-\frac{1}{2})}) \right\} \right\}

= \hat{z}^{H-r-\frac{1}{2}} \sum_{n \geq 0} \hat{z}^n (\lambda_h^{(n)} + \theta \lambda_h^{(n+\frac{1}{2})})$$

where

$$\lambda_h^{(n)} = \bar{N}_{n-\frac{1}{2}} \phi_h^{(n-\frac{1}{2})} + \sum_{s=1}^{2(n-\frac{1}{2})} A_s^1 \phi_h^{(n-\frac{s}{2}-\frac{1}{2})}$$

$$\lambda_h^{(n+\frac{1}{2})} = \bar{N}_n \phi_h^{(n)} + \sum_{s=1}^{2n} A_s^0 \phi_h^{(n-\frac{s}{2})}.$$
The coefficients $A^0_\frac{s}{2}$ and $A^1_\frac{s}{2}$ are functions of level $\frac{s}{2}$ of the operators $G_{-r}$ and $L_{-m}$. The coefficient $\bar{N}_{n-\frac{1}{2}}$ and $\bar{N}_n$ of the highest level field, which will turn out to be crucial in the next lines, are generated from the part of $\mathcal{F}(G_{-r}(Z_0) \times 1)$ that does not depend on the operators $G_{-r}(Z_1)$ and $L_{-m}(Z_1)$:

$$\bar{N}_{n-\frac{1}{2}} = \hat{z}^{-(H+n-r-\frac{j}{2})} \left( \frac{1}{(-\hat{z})^{r+\frac{1}{2}}} \{ (2r-1)h_1\theta - \hat{z}(\partial_\theta + \theta \partial_{\hat{z}}) \} \right) \hat{z}^{H+n}\theta^\alpha$$

$$\equiv \hat{z}^{-(H+n-r-\frac{j}{2})} \hat{g}_{-r}(\hat{z}, \theta) \hat{z}^{H+n}\theta^\alpha, \quad \alpha = 0, 1. \quad (3.7)$$

To get the decoupling equation (3.6), we have to apply several other generators $\mathcal{F}(G_{-r} \times 1)$ and gather them into $\mathcal{F}(M_1,q \times 1)$. The explicit equations might be long to write down but they will have the form

$$N_{n,\alpha}(H, h_1)\phi_h^{(n+\frac{q}{2})} + \sum_{s=1}^{2n+\alpha} B_{\frac{s}{2}}\phi_h^{(n+\frac{q}{2}-\frac{s}{2})} = 0 \quad (3.8)$$

for $n \geq 0$ and $\alpha = 0, 1$. Again $B_{\frac{s}{2}}$ is a function of the $G_{-r}$ and $L_{-m}$ with level $\frac{s}{2}$. Comparing with (3.7), one can see that the coefficients $N_{n,\alpha}$ of the highest level field may be put in a compact form as:

$$\theta^{1-\alpha}N_{n,\alpha}(H, h_1) = \hat{z}^{-(H+n+\alpha-q-\frac{j}{2})}M_{1,q}\left(G_{-r} \rightarrow \hat{g}_{-r}(\hat{z}, \theta)\right)\hat{z}^{H+n}\theta^\alpha \quad (3.9)$$

where the arrow means that all generators $G_{-r}$ in $M_{1,q}$ have to be replaced by the expression of $\hat{g}_{-r}(\hat{z}, \theta)$ defined implicitly in (3.7). The next section will be devoted to the proof of the following identity for $N_{n,\alpha}(H, h_1)$:

$$N_{n,\alpha}(H, h_1) = \prod_{-j+\frac{q}{2} \leq M \leq j-\frac{q}{2}} (h_{p',q'} + n + \frac{\alpha}{2} - h_{P,Q+4M}) \quad (3.10)$$

where:

$$h = h_{p',q'}, \quad h_0 = h_{1,q}, \quad h_1 = h_{P,Q} \quad \text{and} \quad j = (q-1)/4.$$

The notation (2.4) is being used.
To sum up, the decoupling equations of the singular vector $\phi_{1,q}$ in the conformal family $[\phi_{h_{1,q}}]$ are recursive equations for the fields $\phi_h^{(n)}$ and $\phi_h^{(n+\frac{1}{2})}$ of the fusion $\mathcal{F}_h(\phi_0(Z_0) \times \phi_1(Z_1))$, with $h = h_{1,q}$. This is of course if the numerical constant $N_{n,\alpha}$ is not zero. In fact, the very first $N_{n=0,\alpha=0}$ must be zero since otherwise by (3.8) $\phi_h^{(0)} = 0$ and the fusion $\mathcal{F}_h(\phi_0(Z_0) \times \phi_1(Z_1)) = 0$:

$$\prod (h_{p',q'} - h_{P,Q+4M}) = 0. \quad (3.11)$$

In other words, if $N_{n=0,\alpha=0} \neq 0$, the equation (3.8)

$$N_{0,0}\phi_h^{(0)} = 0$$

simply says that the conformal family $[\phi_h]$ does not arise in the fusion of $\phi_0$ and $\phi_1$. Hence the condition $N_{n=0,\alpha=0} = 0$ is a fusion criterion. Using (3.10), one concludes that the family $[\phi_{h_{p',q'}}]$ appears in the fusion of $[\phi_{h_{1,q}}]$ and $[\phi_{h_{P,Q}}]$ if and only if there exists an integer $M \in [-j,j]$ such that $h_{p',q'} = h_{P,Q+4M}$.

A sufficient condition is obviously $p' = P$ and $q' = Q + 4M$ for some integer $M \in [-j,j]$. (For specific values of $t$, the condition $N_{0,0} = 0$ does not exclude more general fusion rules than the ones predicted in [6].) This is the first consequence of the decoupling equations.

The second consequence is precisely our goal: the construction of new singular vectors. What happens if, besides $N_{0,0}$, another $N_{n,\alpha}$ vanishes? The recursive process breaks down then and the decoupling equations (3.8) for this particular $n$ and $\alpha$ is:

$$\sum_{s \geq 1} B_{\frac{s}{2}} \phi_h^{(n+\frac{s}{2} - \frac{3}{2})} = 0.$$ 

Since the previous fields $\phi_h^{(m+\frac{\beta}{2})}$, $m + \frac{\beta}{2} < n + \frac{\alpha}{2}$, have been determined recursively in terms of primary field $\phi_h^{(0)} = \phi_h$, this is a linear combination of various homogeneous products of generators $G_{-r}$ and $L_{-m}$ (of level $n + \frac{\alpha}{2}$) acting on the primary field $\phi_h$. This expression must then be a new singular vector of
level \( n + \frac{\alpha}{2} \) unless the lhs is identically zero. Although this is not ruled out, in all examples calculated the result has only a finite number of zeros and singularities in the variable \( t \). Since the singular vector is always defined up to an overall factor that can be chosen to depend on \( t \), this zeros and singularities can be eliminated and the resulting vector is regular and nonzero for all \( t \).

If \( h = h_{p',q'} \), the singular vector is expected at the level \( \frac{p'q'}{2} \). The condition \( N_{n,\alpha} = 0 \) is then

\[
\prod (h_{p',q'} + p'q'/2 - h_{P,Q+4M}) = \prod (h_{p',-q'} - h_{P,Q+4M}) = 0. \tag{3.12}
\]

The index \( M \) runs over the set \( \{-j, -j + 1, \ldots, j - 1, j\} \) if the product \( p'q' \) is even and over \( \{-j + \frac{1}{2}, -j + \frac{3}{2}, \ldots, j - \frac{3}{2}, j - \frac{1}{2}\} \) if it is odd. If one sticks to generic values of \( t \), the conditions (3.11) and (3.12) can be summed up in:

\[
q' \leq q - 1, \quad P = p', \quad Q \in \{q' - q + 1, q' - q + 5, \ldots, -q' + q - 5, -q' + q - 1\} \\
\text{for } p' \text{ and } q' \text{ even,}
\]

\[
q' \leq q - 2, \quad P = p', \quad Q \in \{q' - q + 3, q' - q + 7, \ldots, -q' + q - 7, -q' + q - 3\} \\
\text{for } p' \text{ and } q' \text{ odd.} \tag{3.13}
\]

Hence, this algorithm allows for the construction of the singular vector at level \( \frac{p'q'}{2} \) of the Verma module \( V_{(c(t),h_{p',q'}(t))} \), for all \( p', q' \in \mathbb{N}, p' = q' \mod 2 \), of the Neveu-Schwarz algebra.

To close this section, we outline the construction of the singular vector \( |\psi_{2,2}\rangle \) of level 2 in the Verma module \( V_{(c(t),h_{2,2}(t))} \). Here \( h = h_{p',q'} = h_{2,2} \). The smallest value of \( q \) allowed by (3.13) is \( q = 3 \) and the \( Q \in \{0\} \). Hence, if \( q \) is chosen to be 3, the only possible value of \( h_1 \) (for generic \( t \)) is \( h_{2,0} \). The
conformal weights are then:

\[ h = h_{2,2} = -\frac{3}{8} t^{-1} - \frac{3}{4} t - \frac{3}{8} t \]
\[ h_0 = h_{1,3} = -\frac{1}{2} - t \]
\[ h_1 = h_{2,0} = -\frac{3}{8} t^{-1} + \frac{1}{4} t + \frac{1}{8} t \]

with

\[ H = h - h_0 - h_1 = -\frac{1}{2} + \frac{1}{2} t. \]

The singular vector \( |\psi_{1,3}\rangle \in V_{c(t),h_{1,3}(t)} \) is known by (2.5):

\[ |\psi_{1,3}\rangle = (tG_{-\frac{3}{2}} + G_{-\frac{3}{2}}^3)|h_{1,3}\rangle. \]

The algebraic generators \( G_{-\frac{3}{2}} \) and \( G_{-\frac{1}{2}} \) correspond to the following generators related to the fusion process (eq. (3.5)):

\[ \mathcal{F}(G_{-\frac{3}{2}} \times 1) = \partial_\theta + \theta \partial_{\hat{z}} \]
\[ \mathcal{F}(G_{-\frac{1}{2}} \times 1) = \frac{1}{\hat{z}^2} \left\{ 2h_1 \theta - \hat{z}(\partial_\theta + \theta \partial_{\hat{z}}) + \sum_{k \geq 1} \hat{z}^k (G_{-k+\frac{1}{2}} + 2\theta L_{-k}) \right\}. \]

Computing \((t\mathcal{F}(G_{-\frac{3}{2}} \times 1) + \mathcal{F}(G_{-\frac{1}{2}}^3 \times 1))\mathcal{F}_h(\phi_0(Z_0) \times \phi_1(Z_1))\), one gets the following recursion equations (eq. (3.8)):

\[ \left( \frac{t}{2} + \frac{1}{2} - n \right) \phi_h^{(n + \frac{1}{2})} = \sum_{k=1}^{n+1} tG_{-k+\frac{1}{2}} \phi_h^{(n-k+1)} \]
\[ n(n - 2) \phi_h^{(n)} = t \sum_{k=1}^{n} (G_{-k+\frac{1}{2}} \phi_h^{(n-k+\frac{1}{2})} - 2L_{-k} \phi_h^{(n-k)}). \]

Notice that the coefficient \( N_{n,\alpha=0} = n(n - 2) \) plainly exhibits the required properties \( N_{n=0,\alpha=0} = N_{n=2,\alpha=0} = 0. \) The three intermediate fields \( \phi_h^{(\frac{1}{2})}, \phi_h^{(1)} \) and \( \phi_h^{(\frac{3}{2})} \) are obtained easily:

\[ \phi_h^{(\frac{1}{2})} = \frac{2t}{t+1} G_{-\frac{3}{2}} \phi_h^{(0)} \]
\[ \phi_h^{(1)} = \frac{2t}{t+1} G_{-\frac{1}{2}} \phi_h^{(0)} \]
\[ \phi_h^{(\frac{3}{2})} = \frac{2t}{t-1} G_{-\frac{3}{2}} \phi_h^{(0)} + \frac{4t^2}{t^2-1} G_{-\frac{1}{2}} \phi_h^{(0)}. \]
The recursion breaks down for $\phi_h^{(2)}$; we get instead the expression for the singular field of level 2:

$$0 = tG_{-\frac{1}{2}}\phi_h^{(2)} + tG_{-\frac{3}{2}}\phi_h^{(2)} - 2tL_{-2}\phi_h^{(0)} - 2tL_{-1}\phi_h^{(1)}$$

which yields:

$$|\psi_{2,2}\rangle = \left(\frac{4t}{t^2 - 1}G_{-\frac{1}{2}} + \frac{t + 1}{t - 1}G_{-\frac{3}{2}}G_{-\frac{1}{2}} + \frac{t - 1}{t + 1}G_{-\frac{3}{2}}G_{-\frac{1}{2}}\right)|h_{2,2}\rangle. \quad (3.15)$$

(The relation (2.1) was used to get rid of the $L_{-n}$.) The singularities at $t = \pm 1$ can be removed by simply multiplying the whole expression by $t^2 - 1$. A direct check ($G_{-\frac{1}{2}}|\psi_{2,2}\rangle = G_{-\frac{3}{2}}|\psi_{2,2}\rangle = 0$) shows that this expression is a singular vector for all values of $t$.

**IV. THE COEFFICIENT $N_{n,\alpha}(\lambda, \mu)$.**

The purpose of the present section is to prove the expression for $N_{n,\alpha}(\lambda, \mu)$ given in eq. (3.10):

$$N_{n,\alpha}(\lambda, \mu) = \prod_{-j + \frac{\alpha}{2} \leq M \leq j - \frac{\alpha}{2}} (h_{p',q'} + n + \frac{\alpha}{2} - h_{P,Q} - 4M) \quad (4.1)$$

where

$$\lambda = -h_{P,Q}$$

$$\mu = h_{1,q} + h_{P,Q} - h_{p',q'} - n$$

$$j = \frac{q - 1}{4}. \quad (4.2)$$

Since the proof is rather technical, we spell out here the various steps:

1. the problem of calculating $N_{n,\alpha}(\lambda, \mu)$ is recast in the algebraic problem of calculating the determinant of a $q \times q$ matrix $C(\lambda)$ whose elements depend on $\lambda$, i.e.: $N_{n,\alpha}(\lambda, \mu) = \det C(\lambda)$;
2. the determinant $\det C(\lambda)$ is seen to be a polynom of degree at most $j$ (resp. $j + \frac{1}{2}$) if $j$ is an integer (resp. an integer+$\frac{1}{2}$);
3. the constant term $\det C(\lambda = 0)$ is evaluated;
4. the determinant is evaluated for any $\lambda$. 
The coefficient $N_{n,\alpha}(\lambda, \mu)$ has been defined in the previous section (eq. (3.9)) by:

$$
\theta^{1-\alpha} N_{n,\alpha}(\lambda, \mu) = \hat{z}^{\frac{q}{2} + \frac{1}{2} + \mu - \alpha} \times M_{1,q} \left( G_{-r} \rightarrow \frac{1}{(-\hat{z})^{r+\frac{1}{2}}} [(1 - 2r)\lambda \theta - \hat{z}(\partial_{\theta} + \theta \partial_{\hat{z}})] \right) \hat{z}^{-\mu} \theta^\alpha
$$

(4.3)

where we have already used the coefficients $\lambda$ and $\mu$ defined above. Though it is not necessary for the present argument, it is interesting to note that the replacement of the odd generators $G_{-r}$ in the expression for the singular vector $M_{1,q}$ is done with the differential operators extending the Witt algebra to a supersymmetric algebra:

$$
l_{-n}(\lambda) = -z^{-n}(z\partial_z - \frac{n-1}{2}\theta \partial_{\theta} + \lambda(n - 1)), \quad n \in \mathbb{Z}
$$

$$
g_{-r}(\lambda) = z^{-r-\frac{1}{2}}(z\partial_{\theta} - \theta z \partial_z - \theta \lambda(2r - 1)), \quad r \in \mathbb{Z} + \frac{1}{2}.
$$

(4.4)

The differential generators in (4.3) coincide with $g_{-r}$ if $\hat{z} \rightarrow -z$. Hence, the coefficient $N_{n,\alpha}(\lambda, \mu)$ can be rewritten as

$$
\theta^{1-\alpha} N_{n,\alpha}(\lambda, \mu) = (-1)^{\frac{q+1}{2} + \alpha} z^{\frac{q+1}{2} + \mu - \alpha} M_{1,q}(G_{-r} \rightarrow g_{-r}(\lambda)) z^{-\mu} \theta^\alpha.
$$

(4.5)

Since the sign $(-1)^{\frac{q+1}{2} + \alpha}$ is not relevant for the algorithm, we shall omit it in the rest of the section. Accordingly, the equalities for $N_{n,\alpha}$ are to be understood up to a sign.

The following formulation [8] for the singular vectors $M_{1,q}$ will be used instead of eq. (2.5) to transform the problem of calculating $N_{n,\alpha}$ into an algebraic problem. Let $\Phi$ and $\Psi$ be two $q$-component vectors with the following forms:

$$
\Phi = \begin{pmatrix}
\phi_{q-1} \\
\phi_{q-2} \\
\vdots \\
\phi_1 \\
\phi_0
\end{pmatrix} \quad \text{and} \quad \Psi = \begin{pmatrix}
\phi_q \\
0 \\
\vdots \\
0
\end{pmatrix}.
$$

(4.6)

Each of the components is itself a vector in the Verma module $V_{(c,h_{p,q})}$, with $\phi_0$ being the highest weight vector $|h_{1,q}\rangle$ and $\phi_q = M_{1,q}\phi_0$ the singular vector
at level $q/2$. Then, the singular vector $\phi_q$ can be obtained recursively through the algebraic equation:

$$
\Psi = B\Phi = \left( -J_+ + \sum_{k=0}^{(q-1)/2} \binom{2k}{k} G_{-k-\frac{1}{2}} (\nu J_+)^{2k} \right) \Phi, \quad \text{with} \quad \nu = \sqrt{-\frac{t}{2}}
$$

(4.7)

where:

$$
J_- = \begin{pmatrix}
0 & 0 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix}, \quad
J_+ = \begin{pmatrix}
0 & q-1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & -2 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & q-3 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & -q+1 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0
\end{pmatrix}.
$$

The matrix elements of $J_+$ are $(J_+)^{ij} = \frac{1}{4}((q-i)(1-(-1)^i)-i(1+(-1)^i))\delta_{i+1,j})$.

Together with $J_0 = \frac{1}{4} \text{diag}(q-1, q-3, \ldots, -q+3, -q+1)$ and $J_+^2, J_-^2$, these $q \times q$ matrices close an $osp(1|2)$ superalgebra:

$$
[J_0, J_\pm] = \pm \frac{1}{2} J_\pm, \quad \{J_+, J_-\} = 2J_0.
$$

(4.8)

Using this formulation, the definition of $N_{n,\alpha}$ becomes:

$$
\theta^{1-\alpha} z^{-\alpha-\mu-\frac{q+1}{2}} \left( \begin{array}{c} N_{n,\alpha} \\
0 \\
0 \\
\vdots \\
0 \end{array} \right) = \left( -J_- + \sum_{k=0}^{(q-1)/2} \binom{2k}{k} (\nu J_+)^{2k} g_{-k-\frac{1}{2}} (\lambda) \right) \left( \begin{array}{c} \phi_{q-1} \\
\phi_{q-2} \\
\vdots \\
\phi_1 \\
\theta^\alpha z^{-\mu} \end{array} \right).
$$

(4.9)

From the realization of the $g_{-k-\frac{1}{2}} (\lambda)$ and the structure of the matrix $B$, one can conclude that the components of $\Phi$ will have the following dependancy on $\theta$ and $z$:

$$
\phi_0 = z^{-\mu}, \quad \phi_1 = c_1 z^{-\mu-1} \theta, \quad \phi_2 = c_2 z^{-\mu-1},
$$

$$
\phi_3 = c_3 z^{-\mu-2} \theta, \quad \ldots, \quad \phi_{q-1} = c_{q-1} z^{-\mu-\frac{2q-1}{2}} \quad \text{if} \quad \alpha = 0
$$
\[
\phi_0 = z^{-\mu} \theta, \quad \phi_1 = c_1 z^{-\mu}, \quad \phi_2 = c_2 z^{-\mu} \theta, \\
\phi_3 = c_3 z^{-\mu-1}, \quad \ldots, \quad \phi_{q-1} = c_{q-1} z^{-\mu - \frac{q-1}{2}} \theta \quad \text{if} \quad \alpha = 1
\]

where the \(c_i\)'s may depend on \(\alpha, \mu\) and \(\lambda\) but on neither \(\theta\) nor \(z\). Because the parity of the components \(\phi_i, 0 \leq i \leq q - 1\), alternate between even and odd, it is natural to introduce the two matrices

\[
A = \text{diag} \left(1, 0, 1, \ldots, 0, 1\right) \quad \text{and} \quad A' = 1 - A = \text{diag} \left(0, 1, 0, \ldots, 1, 0\right)
\]

To decompose the action of the \(g_{-k^{-1/2}}\) on \(\Phi\) as

\[
g_{-k^{-1/2}}(A + A')\Phi = -\theta z^{-k-1}(z \partial_z + 2\lambda k)A\Phi + z^{-k} \partial_\theta A'\Phi.
\]

if \(\alpha = 0\). For \(\alpha = 1\), the roles of \(A\) and \(A'\) are simply interchanged. If \(\alpha = 0\), the first term \(-\theta z^{-k-1}(z \partial_z + 2\lambda k)A\Phi = -\theta z^{-k-1}(-\mu - j - J_0 + 2\lambda k)A\Phi\) where we have used \(j = (q - 1)/4\). In the case \(\alpha = 1\), the relation holds with again \(A \leftrightarrow A'\). With these observations, one sees that for each side of the linear system (4.9) the line \(i\) is a monomial of the form \(z^{-\mu -(q-i)/2-\beta} \theta^\beta\), where \(\beta = (i + \alpha) \mod 2\). Hence, the \(z\) and \(\theta\) dependence can be factored out leaving

\[
\begin{pmatrix}
N_{n,\alpha} \\
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
= \tilde{C}
\begin{pmatrix}
c_{q-1} \\
c_{q-2} \\
\vdots \\
c_1 \\
1
\end{pmatrix}
\]

whose solution is simply

\[
N_{n,\alpha}(\lambda, \mu) = \det \tilde{C},
\]

because \(\tilde{C}\) is the sum of an upper triangular matrix and of \(-J_-\):

\[
\tilde{C} = -J_- + \sum_{k=0}^{(q-1)/2} \binom{2k}{k} (\nu J_+) 2^k [(J_0 + \mu + j - 2\lambda k)A + A']
\]
for $\alpha = 0$, and with $A$ and $A'$ interchanged for $\alpha = 1$. From now on, we will concentrate on the case $\alpha = 0$ only giving the answer for $\alpha = 1$. Using the nilpotency of $J_+$, one can transform $\bar{C}$ into

$$\bar{C} = -J_- + (1 + 2tJ_+^2)^{-\frac{3}{2}}[(\mu + j + J_0)A + A'] + 2\lambda t(1 + 2tJ_+^2)^{-\frac{3}{2}}J_+^2.$$

Since $\det (1 + 2tJ_+^2)^{\frac{3}{2}} = 1$, we define a new matrix $C$ as

$$C = (1 + 2tJ_+^2)^{\frac{3}{2}} \bar{C}$$

and

$$N_{n,0}(\lambda, \mu) = \det C$$

$$= \det \left\{ -(1 + 2tJ_+^2)^{\frac{3}{2}}J_- + (1 + 2tJ_+^2)[(\mu + j + J_0)A + A'] + 2\lambda tJ_+^2 A \right\}$$

(4.10)

and $N_{n,1} = N_{n,0}(A \leftrightarrow A')$. The calculation of $N_{n,\alpha}(\lambda, \mu)$ is now a purely algebraic problem.

The second steps consists in showing that $N_{n,0}(\lambda, \mu)$ is a polynomial in $\lambda$ of degree at most $j$ (resp. $(j + \frac{1}{2})$) if $j$ is an integer (resp. an integer+$\frac{1}{2}$). To prove this, observe first that the variable $\lambda$ appears in $C$ only in the matrix elements of the form $C_{n,n+2}$ with $1 \leq n \leq q - 2$ and $n$ odd. (For the present argument, the lines and columns are numbered by indices running from 1 to $q$.) Moreover these matrix elements are linear in $\lambda$. The determinant is a sum of products $\prod_{i=1}^{q} C_{i,k_i}$ where $(k_1, k_2, \ldots, k_q)$ is a permutation of the first $q$ integers. The second crucial observation is that, if a product contains a pair $C_{n,n+2}C_{n+2,n+4}$, it must vanish. To see this, let us try to match the remaining values for the lines:

$$\{1, 2, \ldots, n - 1, n + 1, n + 3, n + 4, n + 5, \ldots, q\}$$

with the remaining ones for the columns:

$$\{1, 2, \ldots, n - 1, n, n + 1, n + 3, n + 5, \ldots, q\}.$$
Because \( C \) has non-vanishing entries only on and over the non-zero diagonal of \( J_- \), i.e. \( C_{ij} = 0 \) if \( i \geq j + 2 \), then we have to find a one-to-one match between the line numbers \( \{n + 3, n + 4, n + 5, \ldots, q\} \) with the column numbers \( \{n + 3, n + 5, \ldots, q\} \), which is clearly impossible. Hence, at best, the contributing products \( \prod_{i=1}^{q} C_{i,k} \) will skip every other \( \lambda \). That ends the proof. The proof proceeds in the same way for the \( \alpha = 1 \) case where one finds that \( N_{n,1}(\lambda, \mu) \) is a polynomial of degree at most \( j \) (resp. \( j - \frac{1}{2} \)) if \( j \) is an integer (resp. an integer + \( \frac{1}{2} \)).

As third step, we obtain the constant term in the polynomial \( N_{n,\alpha}(\lambda, \mu) \). To do so, we multiply the matrix \( \bar{C} \) evaluated at \( \lambda = 0 \) on the left by

\[
B = [1 + t(J_-J_+^2 + J_+^2J_-)A'][A' + A(1 + 2tJ_+^2)^{\frac{1}{2}}]
\]

and on the right by

\[
B' = [A + A'(1 + 2tJ_+^2)^{\frac{1}{2}}].
\]

These two factors have determinant one and do not change the determinant of \( \bar{C} \). Using the following properties

\[
AJ_{\pm} = J_{\pm}A', \quad A'J_{\pm} = J_{\pm}A, \quad (4.11a)
\]

\[
[J_-, (1 + 2tJ_+^2)^{\pm\frac{1}{2}}] = \pm tJ_+(1 + 2tJ_+^2)^{-1\pm\frac{1}{2}}, \quad (4.11b)
\]

the determinant takes the simple form:

\[
\det \bar{C} = \det \{ - J_- + A' + A(\mu + j + J_0) - t(J_-J_+^2 + J_+^2J_-)J_-A \} \quad (4.12)
\]

which is lower triangular. Using

\[
A = \frac{[J_+, J_-] + 2j + 1}{4j + 1}, \quad (4.13)
\]

one finds:

\[
N_{n,\alpha=0}(\lambda = 0, \mu) = \prod_{-j\leq M\leq j} [\mu + j + M + t(j + M)(2j - 2M + 1)].
\]
Since the properties (4.11) are invariant under the interchange $A \leftrightarrow A'$, equation (4.12) holds also for $\alpha = 1$, provided we interchange $A$ and $A'$ in the matrices $B$ and $B'$. Only the last step of the calculation differs for $\alpha = 1$ and the results can be gathered in a single relation:

$$N_{n,\alpha}(\lambda = 0, \mu) = \prod_{-j + \frac{\alpha}{2} \leq M \leq j - \frac{\alpha}{2}} [\mu + j + M + t(j + M)(2j - 2M + 1)], \quad \alpha = 0, 1. \quad (4.14)$$

In the fourth and last step, we show that

$$N_{n,\alpha}(\lambda, \mu)^2 = \prod_{-j + \frac{\alpha}{2} \leq M \leq j - \frac{\alpha}{2}} \left[ (\mu + (j + M)(1 + t(2j - 2M + 1))) \times (\mu + (j - M)(1 + t(2j + 2M + 1))) - 8\lambda M^2 t \right]. \quad (4.15)$$

Let $n_M$ be the factor in the product corresponding to the value $M$ of the index. Then one notes that $n_M = n_{-M}$ and that the square root can be taken easily. It is also clear that (4.15) (at $\lambda = 0$) agrees with (4.14). Since the relation (4.15) predicts precisely the upper limit for the number of zeros of the polynomial $N_{n,\alpha}(\lambda, \mu)$ (see the second step above), the proof of the above relation is reduced to verifying that

$$\lambda_M = \frac{1}{8tM^2} (\mu + (j + M)(1 + t(2j - 2M + 1))) (\mu + (j - M)(1 + t(2j + 2M + 1)))$$

for $-j + \frac{\alpha}{2} \leq M \leq j - \frac{\alpha}{2}$, $M \neq 0$, are indeed zeros of the polynomial.

To do that, we go back to the matrix $C$ (eq. (4.10)) which we evaluate at $\lambda = \lambda_M$ and write in terms of a new variable $\bar{\mu}$:

$$\mu = -2M\bar{\mu} - (j + M)(1 + (2j + 2M + 1)t) \quad \text{for } M \neq 0,$$

as:

$$C(\lambda = \lambda_M) = -(1 + 2tJ_+^2)^{\frac{3}{2}}J_- + (1 + 2tJ_+^2)((J_0 + j(2\bar{\mu} + 1))A + A')$$

$$+ \bar{\mu}(\bar{\mu} + 1 + t)AJ_+^2 - (j + M)(1 + 2\bar{\mu} + (2j + 2M + 1)t)A.$$
The goal is now to replace $C$ by a lower triangular matrix whose determinant is trivial to calculate. As we shall see, the following matrix $D$ has this property:

$$D = (1 - t(2J_0 + 2j + 1)J_+ A')(1 - \bar{\mu}J_+ A')e^{-X} C(\lambda_M)e^X$$

$$\times (1 - A' \bar{\mu}J_+)(A + A'(1 - 2tJ_+^2))$$

for

$$X = -J_+^2(\bar{\mu} + 2t(J_0 + j))$$

and its determinant is obviously identical to the one of $C$. To actually compute $D$, one needs the following identities:

$$J_0 e^X = e^X (J_0 + X)$$

$$J_+ e^X = e^{X + tJ_+^2} J_+$$

$$J_- e^X = e^{X - tJ_+^2} (J_- - J_+ (\bar{\mu} + 2t(J_0 + j)))$$

$$e^{-X} e^{X - tJ_+^2} = \sqrt{1 - 2tJ_+^2}$$

$$e^{-X} e^{X + tJ_+^2} = \frac{1}{\sqrt{1 - 2tJ_+^2}}.$$ (4.17)

By first conjugating $C(\lambda_M)$ by $e^{-X}$, then multiplying by the first matrices on each side of the result in (4.16) and then by the two last ones, we get with the use of (4.13):

$$D = -J_- + A' + [(J_0 + j)(1 + 2\bar{\mu} + t(2J_0 + 2j + 1))$$

$$- (M + j)(1 + 2\bar{\mu} + t(2M + 2j + 1))] A.$$ (4.18)

The determinant of this lower triangular matrix is 0 since one of the diagonal entries of $J_0$ is precisely $M$. If $\alpha = 1$ then we interchange $A$ and $A'$ in $C(\lambda = \lambda_M)$ and the matrix $D(A \leftrightarrow A')$ is obtain through

$$(A' + A(1 - 2tJ_+^2))(1 - \bar{\mu}J_+ A)e^{-X} C(A \leftrightarrow A')e^X$$

$$\times (1 - A\bar{\mu}J_+)(1 - tAJ_+(2J_0 + 2j)).$$
This ends the proof of (4.15). It is then simply a matter of replacing, in (4.15),
the variables $\lambda$ and $\mu$ by their original values (4.2) to get the desired expression
(4.1).

V. CONCLUDING REMARKS.

The extension of the BdFIZ algorithm to the Neveu-Schwarz superalgebra
represents some improvement on the actual computation of singular vectors.
The defining equations $(G_{-\frac{1}{2}}|\psi_{p,q}\rangle = G_{-\frac{3}{2}}|\psi_{p,q}\rangle = 0)$ represent a linear system
of $P'(pq - 1) + P'(pq - 3)$ equations into $P'(pq)$ variables. (Here $P'(n)$ represent
the number of partitions of the integer $n$ into integers, the odd ones being all
distinct.) The present technique amounts to solve a $q \times q$ linear system or,
equivalently, to solve $q - 1$ recursive equations. Both algorithms are difficult to
use for, say, $pq \geq 6$ without the use of a computer.

In their work, Bauer et al reproduced the subseries of singular vectors $|\psi_{1,q}\rangle$
from the single vector $|\psi_{2,1}\rangle$. We have not been able to reach a similar result for
the Neveu-Schwarz subseries. That might be due to the absence of a singular
vector $|\psi_{2,1}\rangle$ for the Neveu-Schwarz algebra. There is however such a singular
vector in the representation theory of the Ramond algebra. It would be in-
structive to understand the interplay of these two sectors with respect to the
singular vector problem.

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