On the Properties and Applications of Transmuted Odd Generalized Exponential-Exponential Distribution

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This work was carried out in collaboration between all authors. Author TGI designed the study and performed the statistical analysis, Author JA wrote the protocol and the first draft of the manuscript. Authors UKA and DAK managed the analyses of the study. Author AAU managed the literature searches. All authors read and approved the final manuscript.

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Abstract

This article proposed a new distribution referred to as the transmuted odd generalized exponential-exponential distribution (TOGEEED) as an extension of the popular odd generalized exponential-exponential distribution by using the Quadratic rank transmutation map (QRTM) proposed and studied by [1]. Using the transmutation map, we defined the probability density function (pdf) and cumulative distribution function (cdf) of the transmuted odd generalized Exponential- Exponential distribution. Some properties of the new distribution were extensively studied after derivation. The estimation of the distribution’s parameters was also done using the method of maximum likelihood estimation. The performance of the proposed probability distribution was checked in comparison with some other generalizations of Exponential distribution using a real life dataset.
1. Introduction

An Exponential distribution which can be used in Poisson processes gives a description of the time between events. The distribution has been applied widely in life testing experiments. The distribution exhibits memoryless property with a constant failure rate which makes the distribution unsuitable for real life problems and hence creating a vital problem in statistical modeling and applications.

There are several ways of adding one or more parameters to a distribution function which makes the resulting distribution richer and more flexible for modeling data. A recap of some of these ways or methods include the beta generalized family (Beta-G) by Eugene et al. [2], the Kumaraswamy-G family by Cordeiro and de Castro [3], Transmuted family of distributions by Shaw and Shaw [1], Gamma-G (type 1) by Zografos and Balakrishnan [4], McDonald-G by Alexander et al. [5], Gamma-G (type 2) by Ristic and Balakrishnan [6], Gamma-G (type 3) by Torabi and Montazari [7], Log-gamma-G by Amini et al. [8], Exponentiated T-X by Alzaghal et al. [9], Exponentiated-G (EG) by Cordeiro et al. [10], Logistic-G by Torabi and Montazari [11], Gamma-X by Alzaatreh et al. [12], Logistic-X by Tahir et al. [13], Weibull-X by Alzaatreh et al. [14], Weibull-G by Bourguignon et al. [15], a new Weibull-G family by Tahir et al. [16], a Lomax-G family by Cordeiro et al. [17], a new generalized Weibull family by Cordeiro et al. [18] and Beta Marshall-Olkin family of distributions by Alizadeh et al. [19] and several other studies. As a result of the above mentioned methods, some studies have been done to extend the exponential distribution and some of the recent studies on the generalization of exponential distribution are; a quasi exponential distribution by Shanker et al. [20], the Gompertz inverse exponential distribution by Oguntunde et al. [21], the Marshall-Olkin Logistic-inverse distribution by Mansoor et al. [22], the transmuted exponential distribution by [23], transmuted inverse exponential distribution by Oguntunde and Adejumo [24], the odd generalized exponential-inverse distribution by Mai and Pramanik [25] and the Weibull-Exponential distribution by Oguntunde et al. [26]. Of interest to us in this article is the odd generalised exponential-inverse distribution (OGEED) which can be use in various fields to model variables whose chances of success decreases with time whereas those of failure increases as time increases. It was also discovered that the OGEED is positively skewed and performed better than some existing distributions like Gamma, exponentiated exponential, Weibull and Pareto distributions.

According to Mai and Pramanik [25], a random variable X is said to follow a odd generalized exponential-inverse distribution with parameters $\alpha$ and $\theta$ if its probability density function (pdf) is given as

$$f(x) = \alpha \theta e^{\theta x} e^{-\alpha (e^{\theta x} - 1)}$$  \hfill (1.1)$$
and the corresponding cumulative distribution function (cdf) is given as

$$F(x) = 1 - e^{-\alpha (e^{\theta x} - 1)}$$  \hfill (1.2)$$
The pdf and cdf of the transmuted OGEED are obtained using the steps proposed by Shaw and Buckley [1]. According to Shaw and Buckley [1], a random variable $X$ is said to have a transmuted distribution function if its pdf and cdf are respectively given by;

$$f(x) = g(x)[1 + \lambda - 2\lambda G(x)]$$  \hfill (1.3)$$
and

$$F(x) = (1 + \lambda)G(x) - \lambda[G(x)]^2$$  \hfill (1.4)$$
respectively.

where; $x > 0$, and $-1 \leq \lambda \leq 1$ is the transmuted parameter, $G(x)$ is the cdf of any continuous distribution while $f(x)$ and $g(x)$ are the associated pdf of $F(x)$ and $G(x)$, respectively.
The aim of this paper is to introduce a new continuous distribution called the Transmuted Odd Generalized Exponential-Exponential distribution (TOGEED) with a decreasing failure rate useful for analysing real life data using the proposed quadratic rank transmutation map by Shaw and Buckley [1]. The remaining parts of this paper are presented in sections as follows: We defined the new distribution and give its plots in section 2. Section 3 derived some properties of the new distribution. The estimation of parameters using maximum likelihood estimation (MLE) is provided in section 4. An application of the new model with other two existing one to a real life data is done in section 5 and some useful conclusions are made in section 5.

2. The Transmuted Odd Generalized Exponential-Exponential Distribution (TOGEED)

Substituting equation (1.1) and (1.2) in (1.3) and (1.4) and simplifying, we obtain the cdf and pdf of the TOGEED as follows:

$$F(x) = 1 - e^{-\alpha(e^{\theta x} - 1)} \left[ 1 - \lambda + \lambda e^{-\alpha(e^{\theta x} - 1)} \right]$$

and

$$f(x) = \alpha \theta e^{\theta x} e^{-\alpha(e^{\theta x} - 1)} \left[ 1 - \lambda + 2\lambda e^{-\alpha(e^{\theta x} - 1)} \right]$$

respectively. Where, $x > 0$, $\alpha > 0$, $\theta > 0$, $-1 \leq \lambda \leq 1$, $\alpha$ is the shape parameters respectively and $\theta$ is the exponential parameter while $\lambda$ is called the transmuted parameter.

The pdf and cdf of the TOGEED using some parameter values are displayed in Figs. 2.1 and 2.2 as follows.

![PDF of the TOGEED](image)

**Fig. 2.1. PDF of the TOGEED for different values of the parameters $\alpha$, $\theta$ and $\lambda$ as displayed in the key on the graph.**

Fig. 2.1 indicates that the TOGEED distribution could be positively skewed and otherwise depending on the parameter values. This means that distribution can be very useful for datasets with right-tailed shapes.
Fig. 2.2. CDF of the TOGEED for different values of the parameters $\alpha, \theta$ and $-\lambda$ as displayed in the key on the graph.

From the above cdf plot, the cdf increases when $X$ increases, and approaches 1 when $X$ becomes large, as expected.

3. Properties

In this section, we defined and discuss some properties of the TOGEED distribution.

3.1 Moments

Let $X$ denote a continuous random variable, the $n^{th}$ moment of $X$ is given by:

$$
\mu_n = E(X^n) = \int_0^\infty x^n f(x) \, dx
$$

(3.1.1)

where $f(x)$ the pdf of the TOGEED is as given in equation (3.1.1).

$$
f(x) = \alpha \theta e^{\theta x} e^{-\alpha(e^{\theta x} - 1)} \left[1 - \lambda + 2\lambda e^{-\alpha(e^{\theta x} - 1)}\right]
$$

Expansion and simplification of the pdf gives:

$$
f'(x) = (1 - \lambda) \alpha \theta e^{\theta x} e^{-\alpha(e^{\theta x} - 1)} + 2\lambda \alpha \theta e^{\theta x} e^{-2\alpha(e^{\theta x} - 1)}
$$

(3.1.2)

Hence,
\[ \mu_n = E \left( X^n \right) = \int_0^\infty x^n f(x) \, dx = \int_0^\infty x^n \left\{ (1-\lambda)\alpha \theta e^{\theta x} e^{-\alpha \left( e^{\theta x} - 1 \right)} + 2\lambda \alpha \theta e^{\theta x} e^{-2\alpha \left( e^{\theta x} - 1 \right)} \right\} \, dx \]

\[ \mu_n = \int_0^\infty x^n (1-\lambda)\alpha \theta e^{\theta x} e^{-\alpha \left( e^{\theta x} - 1 \right)} \, dx + \int_0^\infty x^n 2\lambda \alpha \theta e^{\theta x} e^{-2\alpha \left( e^{\theta x} - 1 \right)} \, dx \]

\[ \mu_n = (1-\lambda)\alpha \theta \int_0^\infty x^n e^{\theta x} e^{-\alpha \left( e^{\theta x} - 1 \right)} \, dx + 2\lambda \alpha \theta \int_0^\infty x^n e^{\theta x} e^{-2\alpha \left( e^{\theta x} - 1 \right)} \, dx \]

(3.1.3)

Using integration by substitution in (3.1.3) above we have,

Let

\[ w = e^{\theta x} - 1 \Rightarrow x = \frac{\ln(w)}{\theta} \]

\[ \frac{dw}{dx} = \theta e^{\theta x} \Rightarrow dx = \frac{dw}{\theta e^{\theta x}} \]

Therefore, substituting for \( w \) and \( dx \) in equation (3.1.3) above and simplifying, we have:

\[ \mu_n = \frac{(1-\lambda)\alpha}{\theta^n} \int_0^\infty \left( \ln(w) \right)^n e^{-\alpha w} \, dw + \frac{2\lambda \alpha}{\theta^n} \int_0^\infty \left( \ln(w) \right)^n e^{-2\alpha w} \, dw \]

(3.1.4)

Using a result by Nadarajah [27], Bourguignon et al. [28] and Yahaya and Ieren [29],

\[ \int_0^\infty \left( \ln(w) \right)^n e^{-\alpha w} \, dw = I(n, \alpha) \]

\[ I(n, \alpha) = \left( \frac{\partial}{\partial \alpha} \right)^n \left[ (\alpha)^{-\alpha} \Gamma(\alpha) \right] |_{\alpha = 1} \]

Reduces to

Hence, substituting and simplifying in (3.1.4), we have:

\[ \mu_n = \frac{(1-\lambda)\alpha}{\theta^n} \left( \frac{\partial}{\partial \alpha} \right)^n \left[ (\alpha)^{-\alpha} \Gamma(\alpha) \right] |_{\alpha = 1} + \frac{2\lambda \alpha}{\theta^n} \left( \frac{\partial}{\partial \alpha} \right)^n \left[ (2\alpha)^{-\alpha} \Gamma(\alpha) \right] |_{\alpha = 1} \]

(3.1.5)

The Mean

The mean of the \( \text{TOGEED} \) can be obtained from the \( n^{th} \) moment of the distribution when \( n=1 \) as follows:

\[ \mu_1 = \frac{(1-\lambda)\alpha}{\theta} \left( \frac{\partial}{\partial \alpha} \right) \left[ (\alpha)^{-\alpha} \Gamma(\alpha) \right] |_{\alpha = 1} + \frac{2\lambda \alpha}{\theta} \left( \frac{\partial}{\partial \alpha} \right) \left[ (2\alpha)^{-\alpha} \Gamma(\alpha) \right] |_{\alpha = 1} \]

(3.1.6)

Also the second moment of the \( \text{TOGEED} \) is obtained from the \( n^{th} \) moment of the distribution when \( n=2 \) as

\[ \mu_2 = \frac{(1-\lambda)\alpha}{\theta^2} \left( \frac{\partial}{\partial \alpha} \right)^2 \left[ (\alpha)^{-\alpha} \Gamma(\alpha) \right] |_{\alpha = 1} + \frac{2\lambda \alpha}{\theta^2} \left( \frac{\partial}{\partial \alpha} \right)^2 \left[ (2\alpha)^{-\alpha} \Gamma(\alpha) \right] |_{\alpha = 1} \]

(3.1.7)
The Variance

The $n^{th}$ central moment or moment about the mean of $X$, say $\mu_n$, can be obtained as

$$\mu_n = E \left( X - \mu \right)^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \mu_i^* \mu_{n-i}$$  \hspace{1cm} (3.1.8)

By obtaining the central moment when $n=2$, the variance of $X$ for $\text{TOGEDD}$ is obtained as follows:

$$\text{Var}(X) = E \left( X^2 \right) - \left( E \left( X \right) \right)^2$$  \hspace{1cm} (3.1.9)

$$\text{Var}(X) = \mu_2^* - \left( \mu_1^* \right)^2$$  \hspace{1cm} (3.1.10)

Where $\mu_1^*$ and $\mu_2^*$ are the mean and second moment of the $\text{TOGEDD}$ all obtainable from equation (3.1.5).

3.2 Moment Generating Function

The mgf of a random variable $X$ can be defined and obtained as given below:

$$M_x(t) = E \left( e^{tX} \right) = \int_0^\infty e^{tx} f(x) dx$$ \hspace{1cm} (3.2.1)

$$M_x(t) = \sum_{n=0}^{\infty} \frac{\mu_n^*}{n!} t^n = \sum_{n=0}^{\infty} \frac{\mu_n^*}{n!} \frac{(1-\lambda)\alpha}{\sigma^2} \Gamma\left(\frac{\alpha}{\lambda}\right) \Gamma\left(\alpha\right) \left[ \frac{2\lambda a}{\sigma^2} \right]^{2\lambda a} \left[ (2\lambda a)^2 \right]^{\alpha/2}$$ \hspace{1cm} (3.2.2)

3.3 Characteristics Function

The characteristics function of a random variable $X$ is defined and given as follows;

$$\varphi_x(t) = E \left( e^{ix} \right) = E \left[ \cos(tx) + i \sin(tx) \right] = E \left[ \cos(tx) \right] + E \left[ i \sin(tx) \right]$$ \hspace{1cm} (3.3.1)

Simple algebra and power series expansion proves that

$$\phi_x(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \mu_{2n}^* + i \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \mu_{2n+1}$$  \hspace{1cm} (3.3.2)

Where $\mu_{2n}^*$ and $\mu_{2n+1}^*$ are the moments of $X$ for $n=2n$ and $n=2n+1$ respectively and can be obtained from $\mu_n^*$ in equation (3.1.5).

3.4 Reliability Analysis of the \textit{TOGEDD}.

The reliability function in engineering is the characteristic of a variable that maps a set of events, usually associated with mortality or failure of some system onto time. It is used to study the efficiency or life span of products or machines during the production process in engineering. The Survival function describes the likelihood that a system or an individual or component will not fail after a given time. Mathematically, the survival function is given by:
Applying the cdf of the TOGEED in (2.1), the survival function for the TOGEED is obtained as:

\[ S(t) = 1 - F(t) \]  

(3.4.1)

The following is a plot for the survival function of the TOGEED using different parameter values as shown in Fig. 3.4.1.

The graph in Fig. 3.4.1 shows that the probability of the survival equals one (1) at initial time or early age and it decreases as \( t \) increases and equals zero (0) as \( t \) becomes larger.

Hazard function is the probability that a component will fail or die for an interval of time. The hazard function is defined as;

\[ h(t) = \frac{f(t)}{S(t)} = \frac{f(t)}{1 - F(t)} \]  

(3.4.3)

Meanwhile, the expression for the hazard rate of the TOGEED is given by

\[ h(t) = \frac{\alpha \theta e^{\theta t} \left[ 1 - \lambda + 2 \lambda e^{-\lambda t} \right]}{1 - \lambda + \lambda e^{-\lambda t}} \]  

(3.4.4)

where \( \alpha, \theta > 0 \) and \(-1 \leq \lambda \leq 1\).

The following is a plot of the hazard function at chosen parameter values in Fig. 3.4.2

Interpretation: the figure above revealed that the TOGEED has a decreasing failure rate which implies that the probability of failure for any random variable following a TOGEED decreases as time increases, that is, as time goes on, the probability of death decreases.

![Survival Function of the TOGEED](image-url)
Fig. 3.4.2. The hazard function of the TOGEED for different values of the parameters as displayed in the key on the graph

3.5 Quantile Function

Hyndman and Fan [30] defined the quantile function for any distribution in the form

\[ Q(u) = F^{-1}(u) \]

where \( Q(u) \) is the quantile function of \( F(x) \) for \( 0 < u < 1 \)

Taking \( F(x) \) to be the cdf of the TOGEED and inverting it as above will give us the quantile function as follows:

\[
F(x) = 1 - e^{-\alpha e^{\alpha x}} \left[ 1 - \lambda e^{-\lambda e^{\alpha x}} \right] = u
\]  

(3.5.1)

Simplifying equation (3.5.1) above, we obtain:

\[
Q(u) = X_{y} = \left\{ -\frac{\ln(1-u)}{2\theta} \right\}^{\frac{1}{\alpha}} - 1
\]  

(3.5.2)

This function as derived above is used for obtaining some moments like skewness and kurtosis as well as the median and for generation of random variables from the distribution in question.

3.6 Skewness and Kurtosis

This paper presents the quantile based measures of skewness and kurtosis due to non-existence of the classical measures in some cases.

The Bowley’s measure of skewness by [31] based on quartiles is given by:

\[
SK = \frac{Q_{\frac{3}{4}} - Q_{\frac{1}{2}} + Q_{\frac{1}{4}}}{Q_{\frac{3}{4}} - Q_{\frac{1}{4}}}
\]  

(3.6.1)
And the [32] kurtosis is on octiles and is given by;

$$KT = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) - Q(\frac{3}{8}) + Q(\frac{1}{8})}{Q(\frac{6}{8}) - Q(\frac{1}{8})}$$  \hspace{1cm} (3.6.2)

Where \(Q(.)\) is obtainable with the help of equation (3.5.2).

### 3.7 Order Statistics

Let \(X_{(1)}\) denote the smallest of \(X_1, X_2, \ldots, X_n\), \(X_{(2)}\) denote the second smallest of \(X_1, X_2, \ldots, X_n\), and similarly \(X_{(i)}\) denote the \(i^{th}\) smallest of \(X_1, X_2, \ldots, X_n\). Then the random variables \(X_{(1)}, X_{(2)}, \ldots, X_{(n)}\), called the order statistics of the sample \(X_1, X_2, \ldots, X_n\), has probability density function of the \(i^{th}\) order statistic, \(X_{(i)}\), as:

$$f_{x_{(i)}}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) F(x)^{i-1} \left[1 - F(x)\right]^{n-i}$$  \hspace{1cm} (3.7.1)

Where \(f(x)\) and \(F(x)\) are the pdf and cdf of the TOGEDE respectively.

Using (2.1) and (2.2), the pdf of the \(i^{th}\) order statistics \(X_{(i)}\), can be expressed from (3.7.1) as:

$$f_{x_{(i)}}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} [\alpha k e^{-\theta x_{(i-1)}}]\left[1 - \lambda + 2\lambda e^{-\theta x_{(i-1)}}\right]^{i-1}$$  \hspace{1cm} (3.7.2)

Hence, the pdf of the minimum order statistic \(X_{(1)}\) and maximum order statistic \(X_{(n)}\) of the TOGED are given by:

$$f_{x_{(1)}}(x) = n \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} [\alpha^n e^{-\lambda x_{(1)}}]\left[1 - \lambda + 2\lambda e^{-\lambda x_{(1)}}\right]$$  \hspace{1cm} (3.7.3)

$$f_{x_{(n)}}(x) = n [\alpha e^{-\lambda} e^{-\lambda x_{(n-1)}}]\left[1 - \lambda + 2\lambda e^{-\lambda x_{(n-1)}}\right]$$  \hspace{1cm} (3.7.4)

and

respectively.
4. Estimation of Parameters

Let \( X_1, \ldots, X_n \) be a sample of size ‘n’ independently and identically distributed random variables from the TOGEED with unknown parameters \( \alpha, \Theta, \) and \( \lambda \) defined previously. The pdf of the TOGEED is given as

\[
f(x) = \alpha \Theta \lambda x e^{-\alpha d^{\Theta} - \lambda d^{\Theta}} \left[ 1 - \lambda + 2 \lambda e^{-\alpha d^{\Theta}} \right]
\]

The likelihood function is given by;

\[
L(X_1, X_2, \ldots, X_n / \alpha, \Theta, \lambda) = (\alpha \Theta \lambda)^n e^{\sum_{i=1}^{n} \left( x_i - \alpha \sum_{i=1}^{n} e^{\Theta - \lambda e^{-\alpha d^{\Theta}}} \right)} \sum_{i=1}^{n} \left( 1 - \lambda + 2 \lambda e^{-\alpha d^{\Theta}} \right)
\]

(4.1)

Let the log-likelihood function, \( l = \log L(X_1, X_2, \ldots, X_n / \alpha, \Theta, \lambda) \), therefore

\[
l = n \log \alpha + n \log \Theta + \sum_{i=1}^{n} x_i - \alpha \sum_{i=1}^{n} (e^{\Theta} - 1) + \sum_{i=1}^{n} \log \left( 1 - \lambda + 2 \lambda e^{-\alpha d^{\Theta}} \right)
\]

(4.2)

Differentiating \( l \) partially with respect to \( \alpha \), \( \Theta \), and \( \lambda \) respectively gives;

\[
\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} (e^{\Theta} - 1) - 2 \lambda \sum_{i=1}^{n} \left( e^{-\alpha d^{\Theta}} \left( e^{\Theta} - 1 \right) \right) \left( 1 - \lambda + 2 \lambda e^{-\alpha d^{\Theta}} \right)
\]

(4.3)

\[
\frac{\partial l}{\partial \Theta} = \frac{n}{\Theta} - \sum_{i=1}^{n} x_i - \alpha \sum_{i=1}^{n} \left( x_i e^{-\alpha d^{\Theta}} - \lambda e^{-\alpha d^{\Theta}} \right) \left( 1 - \lambda + 2 \lambda e^{-\alpha d^{\Theta}} \right)
\]

(4.4)

\[
\frac{\partial l}{\partial \lambda} = \sum_{i=1}^{n} \left( \frac{2 e^{-\alpha d^{\Theta}} - 1}{1 - \lambda + 2 \lambda e^{-\alpha d^{\Theta}}} \right)
\]

(4.5)

Equating equations (4.3), (4.4) and (4.5) to zero and solving for the solution of the non-linear system of equations will give us the maximum likelihood estimates of parameters \( \alpha, \Theta, \) and \( \lambda \) respectively. However, the solution cannot be obtained analytically except numerically with the aid of suitable statistical software like Python, R, SAS, e.t.c when data sets are given.

5. Applications

In this section, we have applied and compared the performance of the Transmuted odd generalised exponential exponential distribution (TOGED) to that of odd generalised exponential exponential distribution (OGEED) and Weibull-exponential distribution (WED) using the following dataset.

Data set: This data represents the remission times (in months) of a random sample of 128 bladder cancer patients. It has previously been used by [33] and [34]. It’s summarised as follows.
From the descriptive statistics in table 5.1, we observed that the data set is positively skewed with a very high coefficient of kurtosis and therefore suitable for flexible and skewed distributions.

To compare the distributions listed above, we have used several measures of model fit such as $AIC$ (Akaike Information Criterion), Cram’er-Von Mises ($W^*$), Anderson Darling ($A^*$) Kolmogorov smirnov ($K-S$) statistics.

| Parameters | n | Minimum | $Q_1$ | Median | $Q_3$ | Mean | Maximum | Variance | Skewness | Kurtosis |
|------------|---|---------|-------|--------|-------|------|---------|----------|----------|----------|
| Values     | 128 | 0.0800  | 3.348 | 6.395  | 11.840 | 9.366 | 79.05   | 110.425  | 3.3257   | 19.1537  |

Table 5.2. The statistics $ll$, $AIC$, $A^*$, $W^*$ and $K-S$ for the fitted models to the dataset

| Distributions | Parameter estimates | $ll=(log$-likelihood value) | $AIC$ | $A^*$ | $W^*$ | $K-S$ | $P$-Value | Ranks |
|---------------|---------------------|-----------------------------|-------|-------|-------|-------|-----------|-------|
| $TOGED$       | $\theta=0.0182$; $\alpha=2.7822$; $\lambda=0.7591$ | 416.5186 | 839.0372 | 1.0381 | 0.1747 | 0.1079 | 0.1014 | 1 |
| $WED$         | $\theta=0.0293$; $\alpha=1.9653$; $\beta=0.6759$ | 546.5931 | 1099.186 | 1.4123 | 0.2357 | 0.1476 | 0.0076 | 2 |
| $OGEED$       | $\theta=0.0346$; $\alpha=1.6066$ | 439.5273 | 883.0546 | 3.2153 | 0.5463 | 0.2341 | 1.6e-06 | 3 |

Note that the model with the lowest values of these statistics shall be chosen as the best model to fit the data.

It is shown from Table 5.2 above that the Transmuted odd generalized exponential exponential distribution ($TOGED$) corresponds to the lower values of $ll$, $AIC$, $A^*$, $W^*$ and $K-S$ compared to those of the odd generalized exponential-exponential distribution ($OGEED$) and the Weibull-exponential distribution ($WED$) and therefore we chose the TOGED as the best model that fits the data.
The following figure displayed the histogram and estimated densities as well as empirical cumulative distribution function (ECDF) and estimated CDFs of the fitted models for the bladder cancer patients’ data set under study.

From the estimated density plots in figures 5.1 it is very clear that the performance of the TOGEED is better than the other two Models.

5. Conclusion

In this article, we proposed a new distribution, derived and study some of its properties with graphical analysis and discussion on its usefulness and applications. Hence, having demonstrated earlier in the previous section, we have a conclusion based on our applications of the model to a real life data that the new distribution (TOGEED) has a better fit compared to the other two already existing models based on the data set considered in this study.

Competing Interests

Authors have declared that no competing interests exist.

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