COMPLETE 4-MANIFOLDS WITH UNIFORMLY POSITIVE ISOTROPIC CURVATURE

HONG HUANG

Abstract. In this note we prove the following result: Let \((X, g_0)\) be a complete, connected 4-manifold with uniformly positive isotropic curvature and with bounded geometry. Then there is a finite collection \(\mathcal{F}\) of manifolds of the form \(S^3 \times \mathbb{R}/G\), where \(G\) is a fixed point free discrete subgroup of the isometry group of the standard metric on \(S^3 \times \mathbb{R}\), such that \(X\) is diffeomorphic to a (possibly infinite) connected sum of copies of \(S^4\), \(RP^4\) and/or members of \(\mathcal{F}\). This extends recent work of Chen-Tang-Zhu and Huang. The proof uses Ricci flow with surgery on complete orbifolds.

1. Introduction

This is a continuation of our previous work [Hu1] which was inspired by the recent work [BBM] of Bessières, Besson and Maillot. We will try to remove the condition of no essential incompressible space form in [Hu1] and obtain the following

Theorem 1.1. Let \((X, g_0)\) be a complete, connected 4-manifold with uniformly positive isotropic curvature and with bounded geometry. Then there is a finite collection \(\mathcal{F}\) of manifolds of the form \(S^3 \times \mathbb{R}/G\), where \(G\) is a fixed point free discrete subgroup of the isometry group of the standard metric on \(S^3 \times \mathbb{R}\), such that \(X\) is diffeomorphic to a (possibly infinite) connected sum of copies of \(S^4\), \(RP^4\) and/or members of \(\mathcal{F}\).

(By [MW] it is easy to see that the converse is also true: Any 4-manifold as in the conclusion of the theorem admits a complete metric with uniformly positive isotropic curvature and with bounded geometry. The notion of a (possibly infinite) connected sum will be given later in this section; cf. [BBM].)

This also extends recent work of Chen-Tang-Zhu [CTZ] to the noncompact case.

Recall ([MM], [MW]) that a Riemannian manifold \(M\) is said to have uniformly positive isotropic curvature if there exists a constant \(c > 0\) such that for all points \(p \in M\) and all orthonormal 4-frames \(\{e_1, e_2, e_3, e_4\} \subset T_p M\) the curvature tensor satisfies

\[
R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \geq c.
\]

This notion can be easily adapted to the case of Riemannian orbifolds.

Now we consider in particular a 4-dimensional manifold (or orbifold) \(X\). If we decompose the bundle \(\Lambda^2 TX\) into the direct sum of its self-dual and anti-self-dual

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\[ \Lambda^2 TX = \Lambda_+^2 TX \oplus \Lambda_-^2 TX, \]

then the curvature operator can be decomposed as

\[ \mathcal{R} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}. \]

Denote the eigenvalues of the matrices \( A, C \) and \( \sqrt{BB^T} \) by \( a_1 \leq a_2 \leq a_3, c_1 \leq c_2 \leq c_3 \) and \( b_1 \leq b_2 \leq b_3 \) respectively. It is easy to see (cf. Hamilton [H4]) that for a Riemannian 4-manifold/ orbifold the condition of uniformly positive isotropic curvature is equivalent to that there is a positive constant \( c \) such that \( a_1 + a_2 \geq c, c_1 + c_2 \geq c \) everywhere.

Also recall that a complete Riemannian manifold/ orbifold \( M \) is said to have bounded geometry if the sectional curvature is bounded (in both sides) and the volumes of all unit balls in \( M \) are uniformly bounded below away from zero.

Now we explain the notion of (possibly infinite) connected sum which slightly generalizes that from [BBM]. Let \( \mathcal{X} \) be a class of 4-manifolds. A 4-manifold \( X \) is said to be a connected sum of members of \( \mathcal{X} \) if there exists a countable graph \( G \) and a map \( v \mapsto X_v \) which associates to each vertex of \( G \) a copy of some manifold in \( \mathcal{X} \), such that by removing from each \( X_v \) as many 4-balls as vertices incident to \( v \) and gluing the thus punctured \( X_v \)'s to each other along the edges of \( G \) using diffeomorphisms of the boundary 3-spheres, one obtains a 4-manifold diffeomorphic to \( X \). Note that we do not assume that the elements in \( \mathcal{X} \) are closed or the graph is locally finite; compare [BBM].

Our proof of the theorem uses a version of Hamilton-Perelman’s Ricci flow with surgery on complete orbifolds, extending the work [BBM] and [Hu1] which treat the manifold case. It is somewhat different from the one adopted in Chen-Tang-Zhu [CTZ]. (See also the introduction in [Hu1].)

To consider Ricci flow with surgery on complete orbifolds will encounter some difficulties which do not occur in the compact manifold case. In [Hu1] we have established a weak openness (w.r.t. time) property of the canonical neighborhood condition for the noncompact manifold case, which can be easily extended to the noncompact orbifold case. However, there is an additional difficulty in the orbifold case, that is, the canonical neighborhoods in the orbifold case may a-priori be very collapsed. To overcome it we will pull back the solutions locally via Hamilton’s canonical parametrization, an idea already exploited in [CTZ]. We will give more details in some places. In particular we establish bounded curvature at bounded distance and persistence of almost standard caps in the orbifold case, which are crucial in the process of constructing \((r, \delta, \kappa)\)-surgical solutions.

In Section 2 we give some definitions and preliminary results, in Section 3 we prove the existence of \((r, \delta, \kappa)\)-surgical solution with initial data a complete 4-orbifold with isolated singularities, with uniformly positive isotropic curvature and with bounded geometry. In Section 4 we prove Theorem 1.1. In Appendix A we prove some technical results on gluing \( \varepsilon \)-necks. In most cases we will follow the notations and conventions in [BBB⁺], [BBM] and [Hu1].
2. Surgical solutions on 4-orbifolds

Let \((X, g_0)\) be a complete 4-orbifold with \(|Rm| \leq K\). Consider the Ricci flow ([H1])

\[
\frac{\partial g}{\partial t} = -2\text{Ric}, \quad g|_{t=0} = g_0.
\]

Clearly Shi’s short time existence for Ricci flow with initial data a complete manifold with bounded sectional curvature ([S]) extends to the orbifold case, so (2.1) has a short time solution with bounded curvature. By extending Chen-Zhu ([CZ1]) to the orbifold case this solution is unique (in the category of complete orbifolds with bounded curvature).

Now we assume that the orbifold \((X, g_0)\) has uniformly positive isotropic curvature (see Section 1). Then we can easily generalize Hamilton’s pinching result in [H4] to our situation.

**Lemma 2.1.** (Hamilton [H4]) Let \((X, g_0)\) be a complete 4-orbifold with uniformly positive isotropic curvature \((a_1 + a_2) \geq c, (c_1 + c_2) \geq c\) and with bounded curvature \((|Rm| \leq K)\). Then there exist positive constants \(\rho, \Psi, L, P, S < +\infty\) depending only on the initial metric (through \(c, K\)), such that the complete solution to the Ricci flow (2.1) with bounded curvature satisfies

\[
a_1 + \rho > 0, \quad c_1 + \rho > 0,
\]

\[
\max\{a_3, b_3, c_3\} \leq \Psi(a_1 + \rho), \quad \max\{a_3, b_3, c_3\} \leq \Psi(c_1 + \rho),
\]

\[
\frac{b_3}{\sqrt{(a_1 + \rho)(c_1 + \rho)}} \leq 1 + \frac{L e^{P t}}{\max\{\ln (a_1 + \rho)(c_1 + \rho), S\}}
\]

at all points and times.

Since the 4-orbifolds we consider have uniformly positive isotropic curvature, and in particular, have uniformly positive scalar curvature, the Ricci flow (2.1) will blow up in finite time. By Lemma 2.1, any blow-up limit (if it exists) satisfies the following restricted isotropic curvature condition

\[
a_3 \leq \Psi a_1, \quad c_3 \leq \Psi c_1, \quad b_3^2 \leq a_1 c_1,
\]

and in particular, has nonnegative curvature operator.

Following Hamilton’s idea [H4], we will do surgery before the curvature blows up.

To describe the surgery procedure, we first define necks and caps, following [CTZ]. A (topological) neck is defined to be diffeomorphic to \(S^3/\Gamma \times \mathbb{R}\), here \(\Gamma\) is a finite, fixed point free subgroup of isometries of \(S^3\). If \(S^3/\Gamma\) admits a fixed point free isometric involution \(\sigma\) (i.e. \(\sigma^2 = 1\), then the quotient \((S^3/\Gamma \times \mathbb{R})/\{1, \hat{\sigma}\}\), denoted by \(C^e\), is a smooth manifold with neck like end \(S^3/\Gamma \times \mathbb{R}\), where \(\hat{\sigma}\) is the reflection on the manifold \(S^3/\Gamma \times \mathbb{R}\) defined by \(\hat{\sigma}(x, s) = (\sigma(x), -s)\) for \(x \in S^3/\Gamma\) and \(s \in \mathbb{R}\). As noted in [CTZ], \(\hat{\sigma}\) may also be seen as an isometry of the round \(S^4\) in a natural way, and \(C^e\) is diffeomorphic to the smooth manifold obtained by removing the unique singularity from \(S^4/\{\Gamma, \hat{\sigma}\}\). We define smooth caps to be \(C^e\) and \(\mathbb{B}^4\). The orbifold caps have two types. The orbifold cap of type I, denoted by \(C^I\), is obtained from \(S^3/\Gamma \times [0, 1)\) (with \(\Gamma\) as above) by shrinking the boundary \(S^3/\Gamma \times \{0\}\) to a point. The orbifold cap of type II, denoted by \(S^4/(x, \pm x') \setminus \mathbb{B}^4\),
is obtained by removing a smooth point from some spherical orbifold \( S^4/(x, \pm x') \). Here the orbifold \( S^4/(x, \pm x') \) is a quotient of \( S^4 \) by the involution \((x_1, x_2, \cdots, x_5) \mapsto (x_1, -x_2, \cdots, -x_5)\), so it has two isolated singularities.

To give the next theorem we recall the notion of orbifold connected sum, cf. [CTZ]. Let \( O_i \) \((i = 1, 2)\) be two \( n \)-orbifolds, and let \( D_i \subset O_i \) be two suborbifolds-with boundary, both diffeomorphic to some quotient orbifold \( \mathbb{B}^n / \Gamma \), where \( \mathbb{B}^n \) is the closed unit \( n \)-ball, and \( \Gamma \) is a finite subgroup of \( O(n) \). Choose a diffeomorphism \( f : \partial D_1 \to \partial D_2 \), and use it to glue together \( O_1 \setminus \text{int}(D_1) \) and \( O_2 \setminus \text{int}(D_2) \). The result is called the orbifold connected sum of \( O_1 \) and \( O_2 \) by gluing map \( f \), and is denoted by \( O_1 \#_f O_2 \). If \( D_i \) \((i = 1, 2)\) are disjoint suborbifolds-with boundary (both diffeomorphic to some quotient orbifold \( \mathbb{B}^n / \Gamma \)) in the same \( n \)-orbifold \( O \), the result of similar process as above is called the orbifold connected sum on (the single orbifold) \( O \), and is denoted by \( O \#_f \). Sometimes, by abuse of notation, we will omit the \( f \) in \( O_1 \#_f O_2 \) and \( O \#_f \).

Now we extend the notion of infinite connected sum in the Introduction to the orbifold case. An \( n \)-orbifold \( O \) is a (possibly infinite) orbifold connected sum of members of a collection \( \mathcal{F} \) of \( n \)-orbifolds if there exist a countable graph \( G \) (in which we allow an edge to connect some vertex to itself), a map \( v \mapsto F_v \) which associates to each vertex of \( G \) a copy of some orbifold in \( \mathcal{F} \), and a map \( e \mapsto f_e \) which associates to each edge of \( G \) a diffeomorphism of some \((n-1)\)-dimensional spherical orbifold, such that if we do an orbifold connected sum (as defined above) along each edge \( e \) using the gluing map \( f_e \), we obtain an \( n \)-orbifold diffeomorphic to \( O \). This also extends the notion of (finite) orbifold connected sum in [CTZ].

For example, let \( \Gamma \) be a finite, fixed point free subgroup of isometries of \( S^3 \). Note that \( \Gamma \) acts on \( S^4 \) by suspension. Using our ambiguous orbifold connected sum notation, \( \mathbb{R}^4 / \Gamma \approx S^4 / \Gamma \times S^1 / \Gamma \times \cdots \cdot \), the cylinder \( S^3 / \Gamma \times \mathbb{R} \approx \cdots \cdot S^3 / \Gamma \times \mathbb{R} \times S^1 / \Gamma \times \cdots \cdot \). Also note that for a diffeomorphism \( f : S^3 / \Gamma \to \) the mapping torus \( S^3 / \Gamma \times_f S^1 \approx S^4 / \Gamma \times_f \). By the work [Mc], the mapping class group of \( S^3 / \Gamma \) is finite. So given \( \Gamma \), there are only a finite number of manifolds of the forms \( S^3 / \Gamma \times_f S^1 \) up to diffeomorphism. Finally note that the smooth cap \( C^n_\Gamma \approx S^4 / \Gamma \times_f S^4 / \Gamma \times \cdots \cdot \), the orbifold cap of type II, \( S^4/(x, \pm x') \setminus \mathbb{B}^4 \approx S^4/(x, \pm x') \setminus S^4 / \Gamma \times \cdots \cdot \).

We will use Ricci flow with surgery on orbifolds to prove the following theorem which is more general than Theorem 1.1, which extends [CTZ, Theorem 2.1] to the noncompact case, and whose proof will be postponed to Section 4.

**Theorem 2.2.** Let \( X \) be a complete, connected Riemannian 4-orbifold with at most isolated singularities, with uniformly positive isotropic curvature and with bounded geometry. Then there is a finite collection \( \mathcal{F} \) of spherical 4-orbifolds with at most isolated singularities such that \( X \) is diffeomorphic to a (possibly infinite) orbifold connected sum of members of \( \mathcal{F} \).

Note that the graph \( G \) which describes the (possibly infinite) orbifold connected sum appeared in the above theorem is locally finite. So in fact the conclusion of Theorem 2.2 is equivalent to say that \( X \) contains a locally finite collection \( S \) of pairwise disjoint, embedded \( S^3 / \Gamma \)'s (where \( \Gamma \)'s are finite fixed point free subgroups.
of isometries of $S^3$), such that by cutting off $X$ along $S$ and gluing back $\tilde{\mathbb{R}}^4/G$'s one gets a disjoint union of spherical 4-orbifolds diffeomorphic to members of $\mathcal{F}$.

Recently I [Hu2] have been able to remove the restriction condition in Theorem 2.2 that the singularities should be isolated.

Now we will adapt some definitions from [BBM] and [Hu1].

**Definition** (cf. [BBM] and [Hu1]) Given an interval $I \subset \mathbb{R}$, an evolving Riemannian orbifold is a pair $(X(t), g(t))$ $(t \in I)$, where $X(t)$ is a (possibly empty or disconnected) orbifold and $g(t)$ is a Riemannian metric on $X(t)$. We say that it is piecewise $C^1$-smooth if there exists a discrete subset $J$ of $I$, such that the following conditions are satisfied:

i. On each connected component of $I \setminus J$, $t \mapsto X(t)$ is constant (in topology), and $t \mapsto g(t)$ is $C^1$-smooth;

ii. For each $t_0 \in J$, $X(t_0) = X(t)$ for any $t < t_0$ sufficiently close to $t_0$, and $t \mapsto g(t)$ is left continuous at $t_0$;

iii. For each $t_0 \in J \setminus \{\sup I\}$, $t \mapsto (X(t), g(t))$ has a right limit at $t_0$, denoted by $(X_+(t_0), g_+(t_0))$.

As in [BBM] and [Hu1], a time $t \in I$ is regular if $t$ has a neighborhood in $I$ where $X(\cdot)$ is constant and $g(\cdot)$ is $C^1$-smooth. Otherwise it is singular. We also denote by $f_{\text{max}}$ and $f_{\text{min}}$ the supremum and infimum of a function $f$, respectively, as in [BBM].

**Definition** (Compare [BBM] and [Hu1]) A piecewise $C^1$-smooth evolving Riemannian 4-orbifold $\{(X(t), g(t))\}_{t \in I}$ with isolated singularities, with bounded curvature and with uniformly positive isotropic curvature is said to be a surgical solution to the Ricci flow if it has the following properties:

i. The equation $\frac{d}{dt} g = -2 \text{Ric}$ is satisfied at all regular times;

ii. For each singular time $t$ one has $(a_1 + a_2)_{\text{min}}(g_+(t)) \geq (a_1 + a_2)_{\text{min}}(g(t))$, $(c_1 + c_2)_{\text{min}}(g_+(t)) \geq (c_1 + c_2)_{\text{min}}(g(t))$, and $R_{\text{min}}(g_+(t)) \geq R_{\text{min}}(g(t))$;

iii. For each singular time $t$ there is a locally finite collection $S$ of disjoint embedded $S^3/\Gamma$'s in $X(t)$ (where $\Gamma$'s are finite, fixed point free subgroups of isometries of $S^3$), and an orbifold $X'$ such that

(a) $X'$ is obtained from $X(t) \setminus S$ by gluing back $\tilde{\mathbb{R}}^4/\Gamma$'s,

(b) $X_+(t)$ is a union of connected components of $X'$ and $g_+(t) = g(t)$ on $X_+(t) \cap X(t)$, and

(c) Each component of $X' \setminus X_+(t)$ is diffeomorphic to a closed spherical orbifold, or $S^3 \times \mathbb{R}/G$ (where $G$ is a fixed point free discrete subgroup of the isometry group of the standard metric on $S^3 \times \mathbb{R}$), or a smooth cap, or an orbifold cap (of type I or II), or $S^4/(x, \pm x')g\mathbb{RP}^4$, or $S^4/(x, \pm x')g\mathbb{S}^4/(x, \pm x')$.

**Lemma 2.3.** (cf. [Hu1]) Any complete surgical solution with $a_1 + a_2 \geq c$, $c_1 + c_2 \geq c$ must become extinct at some time $T < \frac{1}{c^2}$.

Let $\{(X(t), g(t))\}_{t \in I}$ be a surgical solution and $t_0 \in I$. As in [BBM], if $t_0$ is singular, we set $X_{\text{reg}}(t_0) := X(t_0) \cap X_+(t_0)$, and $X_{\text{sing}}(t_0) := X(t_0) \setminus X_{\text{reg}}(t_0)$. If $t_0$ is regular, $X_{\text{reg}}(t_0) = X(t_0)$ and $X_{\text{sing}}(t_0) = \emptyset$. Let $t_0 \in [a, b] \subset I$ be a time, and
Y be a subset of $X(t_0)$ such that for every $t \in [a, b)$, we have $Y \subset X_{reg}(t)$. Then as in [BBM], we say the set $Y \times [a, b]$ is unscathed.

In [H4] Hamilton devised a quantitative metric surgery procedure, and Perelman [P2] gave a somewhat different version, and in particular, he had the crucial notion of “canonical neighborhood”. To describe it we need some more notions such as $\varepsilon$-neck, $\varepsilon$-cap and strong $\varepsilon$-neck as given in [P2], [CTZ].

An open subset $U$ of a Riemannian orbifold $(X, g)$ is an $\varepsilon$-neck if there is a diffeomorphism $\varphi : (S^3/\Gamma) \times \mathbb{I} \to U$ such that the pulled back metric $\varphi^* g$, scaling with some factor, is $\varepsilon$-close (in $C^{[\varepsilon^{-1}]}$ topology) to the standard metric $(S^3/\Gamma) \times \mathbb{I}$ with scalar curvature 1 and $\mathbb{I} = (-\varepsilon^{-1}, \varepsilon^{-1})$. (Here $\Gamma$ is a finite, fixed point free subgroup of isometries of $S^3$.) An open subset $U$ is an $\varepsilon$-cap if $U$ is diffeomorphic to a smooth cap ($S^3$ or $\mathbb{B}^4$), or an orbifold cap of Type I or II ($C_1$ or $S^4/(x, x') \setminus \mathbb{B}^4$), and some region $N$ around the end is an $\varepsilon$-neck. (Any point in $U \setminus N$ is called a center of the $\varepsilon$-cap.) A strong $\varepsilon$-neck $U$ at $(x, t)$ in a surgical solution of the Ricci flow is the time $t$ slice of the parabolic region $\{(x', t') | x' \in U, t' \in [t - R(x, t)^{-1}, t]\}$ where the solution is well-defined and has the property that there is a diffeomorphism $\varphi : (S^3/\Gamma) \times \mathbb{I} \to U$ such that, the pulled back solution $\varphi^* g(\cdot, \cdot)$ scaling with factor $R(x, t)$ and shifting the time $t$ to 0, is $\varepsilon$-close (in $C^{[\varepsilon^{-1}]}$ topology) to the subset $(S^3/\Gamma \times \mathbb{I}) \times [-1, 0]$ of the evolving round cylinder $S^3/\Gamma \times \mathbb{R}$, with scalar curvature one and length $2\varepsilon^{-1}$ to $\mathbb{I}$ at time zero.

Motivated by the structure theorems of 4-dimensional ancient $\kappa$-orbifold solution ([CTZ, Theorem 3.10]) and the standard solution ([CZ, Corollary A.2], which can be easily adapted to the case of orbifold standard solution (which will be defined later via lifting), following [P2] (compare [BBM], [CaZ], [KL] and [MT]), [CZ] and [CTZ], we introduce the notion of canonical neighborhood.

**Definition** Let $\varepsilon$ and $C$ be positive constants. A point $(x, t)$ in a surgical solution to the Ricci flow is said to have an $(\varepsilon, C)$-canonical neighborhood if it has an open neighborhood $U$, $B_{\varepsilon}(x, \sigma) \subset U \subset B_{\varepsilon}(x, 2\sigma)$ with $C^{-1} R(x, t)^{-\frac{3}{2}} < \sigma < C R(x, t)^{-\frac{3}{2}}$, which falls into one of the following four types:

(a) $U$ is a strong $\varepsilon$-neck with center $(x, t)$,
(b) $U$ is an $\varepsilon$-cap with center $x$ for $g(t)$,
(c) at time $t$, $U$ is diffeomorphic to a closed spherical orbifold $S^4/\Gamma$ with at most isolated singularities,

and if moreover, the scalar curvature in $U$ at time $t$ satisfies the derivative estimates

$$|\nabla R| < CR^\frac{3}{2} \quad \text{and} \quad |\frac{\partial R}{\partial t}| < CR^2,$$

and, for cases (a) and (b), the scalar curvature in $U$ at time $t$ is between $C^{-1} R(x, t)$ and $CR(x, t)$, and for case (c), the curvature operator of $U$ is positive, and the infimal sectional curvature of $U$ is greater than $C^{-1} R(x, t)$.

**Remark 1** Our definition of canonical neighborhood is slightly different from that in [CTZ]. We include the derivative estimates for the scalar curvature in case (c) also (while [CTZ] does not) for convenience. Note that by pulling back to orbifold covering (an argument similar to that used in the proof of [CTZ, Proposition 3.5]) and using [CZ2, Proposition 3.6], these derivative estimates hold uniformly for all ancient $\kappa$-orbifold solutions satisfying (2.3). We also impose a sectional curvature
condition in case (c). Note that by using orbifold coverings and arguing as in the
proof of [KL, Lemma 59.7], it is easy to see that this condition is reasonable.

**Remark 2** Note that by [CTZ, Proposition 3.5, Theorem 3.8]) and [CZ, Corol-
ary A.2] (as adapted to the case of orbifold standard solution), for every \( \varepsilon > 0 \),
there exists a positive constant \( C(\varepsilon) \) such that each point in any ancient \( \kappa \)-orbifold
solution or the orbifold standard solution has an \( (\varepsilon, C(\varepsilon)) \)-canonical neighborhood,
except that for the orbifold standard solution, an \( \varepsilon \)-neck may not be strong.

We choose \( \varepsilon_0 > 0 \) such that \( \varepsilon_0 < 10^{-4} \) and such that when \( \varepsilon \leq 2\varepsilon_0 \), Lemma
A.1 in Appendix A and the results in the paragraph following its proof hold true.
Let \( \beta := \beta(\varepsilon_0) \) be the constant given by Lemma A.2 in Appendix A. Define \( C_0 := 
\max\{100\varepsilon_0^{-1}, 2C(\beta\varepsilon_0/2)\} \}, \) where \( C(\cdot) \) is given in the Remark 2 above. Fix \( c_0 > 0 \).
Let \( \varrho_0, \Psi_0, L_0, P_0, S_0 \) be the constants given in Lemma 2.1 by setting \( \varepsilon = c_0 \) and
\( K = 1 \).

Now we consider some a priori assumptions, which consist of the pinching as-
sumption and the canonical neighborhood assumption.

**Pinching assumption:** Let \( \varrho_0, \Psi_0, L_0, P_0, S_0 \) be positive constants as given
above. A surgical solution to the Ricci flow satisfies the pinching assumption (with
pinching constants \( \varrho_0, \Psi_0, L_0, P_0, S_0 \)) if there hold
\[
\begin{align*}
& a_1 + \varrho_0 > 0, \quad c_1 + \varrho_0 > 0, \\
& \max\{a_3, b_3, c_3\} \leq \Psi_0(a_1 + \varrho_0), \quad \max\{a_3, b_3, c_3\} \leq \Psi_0(c_1 + \varrho_0),
\end{align*}
\]
(2.4)

and
\[
\frac{b_3}{\sqrt{(a_1 + \varrho_0)(c_1 + \varrho_0)}} \leq 1 + \frac{L_0e^{P_0t}}{\max\{\ln \sqrt{(a_1 + \varrho_0)(c_1 + \varrho_0)}, S_0\}}
\]
at all points and times.

**Canonical neighborhood assumption:** Let \( \varepsilon_0 \) and \( C_0 \) be given as above.
Let \( r : [0, +\infty) \to (0, +\infty) \) be a non-increasing function. An evolving Riemanian
4-orbifold \( \{X(t), g(t)\}_{t \in I} \) satisfies the canonical neighborhood assumption (CN),
if any space-time point \((x, t)\) with \( R(x, t) \geq r^{-2}(t) \) has an \((\varepsilon_0, C_0)\)-canonical neigh-
borhood.

Bounded curvature at bounded distance is one of the key ideas in Perelman [P1],
[P2]; compare [MT, Theorem 10.2], [BBB+, Theorem 6.1.1] and [BBM, Theorem
6.7]. 4-dimensional versions have appeared in [CZ] and [Hu1]. The following is a
extension of the version in [Hu1, Theorem B.1].

**Proposition 2.4** For each \( c, \varrho, \Psi, L, P, S, A, C > 0 \) and each \( \varepsilon \in (0, 2\varepsilon_0) \),
there exists \( Q = Q(c, \varrho, \Psi, L, P, S, A, \varepsilon, C) > 0 \) and \( \Lambda = \Lambda(c, \varrho, \Psi, L, P, S, A, \varepsilon, C) > 0 \)
with the following property. Let \( I = [a, b]\) \( (0 \leq a < b < \frac{1}{2}) \) and \( \{X(t), g(t)\}_{t \in I} \)
be a surgical solution with uniformly positive isotropic curvature \( (a_1 + a_2 > c, \ c_1 + c_2 > c) \),
with bounded curvature, and satisfying the pinching condition (2.2)
(with constants \( \varrho, \Psi, L, P, S \)). Let \((x_0, t_0)\) be a space-time point such that:

1. \( R(x_0, t_0) \geq Q \);
2. For each point \( y \in B(x_0,t_0,AR(x_0,t_0)^{-1/2}) \), if \( R(y,t_0) \geq 4R(x_0,t_0) \), then \( (y,t_0) \) has an \((\varepsilon,C)\)-canonical neighborhood. Then for any \( y \in B(x_0,t_0,AR(x_0,t_0)^{-1/2}) \), we have

\[
\frac{R(y,t_0)}{R(x_0,t_0)} \leq \Lambda.
\]

**Proof** We will adapt the proof of [BBB^+, Theorem 6.1.1] and [BBM, Theorem 6.4] to our situation, incorporating an idea from [CTZ]. We argue by contradiction. Suppose the result is not true. Then there exist constants \( c, \rho, \Psi, L, P, S, A, C > 0 \) and \( \varepsilon \in (0,2\varepsilon_0] \), sequences \( Q_k \to +\infty \), \( \Lambda_k \to +\infty \), and a sequence of pointed surgical solutions \((X(t), g(t), (x_k, t_k)) (0 \leq a \leq b < \frac{1}{2\varepsilon})\) with uniformly positive isotropic curvature \((a_1 + a_2 \geq c, c_1 + c_2 \geq c)\), with bounded curvature and satisfying the pinching condition (2.2) (with constants \( \rho, \Psi, L, P, S \)), such that:

1. \( R(x_k, t_k) \geq Q_k \);
2. for each point \( y \in B(x_k, t_k, AR(x_k, t_k)^{-1/2}) \), if \( R(y, t_k) \geq 4R(x_k, t_k) \), then \( (y, t_k) \) has an \((\varepsilon,C)\)-canonical neighborhood;
3. for each \( k \), there exists \( z_k \in B(x_k, t_k, AR(x_k, t_k)^{-1/2}) \) with

\[
\frac{R(z_k, t_k)}{R(x_k, t_k)} > \Lambda_k.
\]

For each \( k \), consider the parabolic rescaling

\[
\tilde{g}_k(\cdot) := \frac{R(x_k, t_k)g_k(t_k + \frac{\cdot}{R(x_k, t_k)})}{R(x_k, t_k)}
\]

We will adopt the convention in [BBB^+] and [BBM] to put a bar on the points when the relevant geometric quantities are computed w.r.t. the metric \( \tilde{g}_k \).

Define

\[
\rho := \sup\{ s > 0 | \exists C(s) > 0, \forall k \in \mathbb{N}, \forall \tilde{y} \in B(\tilde{x}_k, 0, s), R(\tilde{y}, 0) \leq C(s) \}.
\]

It is easy to see that there exists, up to extracting a subsequence, \( \tilde{y}_k \in B(\tilde{x}_k, 0, \rho) \) such that

\[
R(\tilde{y}_k, 0) \to +\infty \quad \text{and} \quad d_0(\tilde{x}_k, \tilde{y}_k) \to \rho \quad \text{as} \quad k \to \infty.
\]

We choose points \( \tilde{x}_k' \) and \( \tilde{y}_k' \) for large \( k \) such that \( R(\tilde{x}_k', 0) = 2C \), \( R(\tilde{y}_k, 0) = R(\tilde{y}_k, 0)/(2C) \), and \([\tilde{x}_k' \tilde{y}_k'] \subset [\tilde{x}_k \tilde{y}_k] \) is a maximal subsegment on which

\[
2C \leq R(\cdot, 0) \leq \frac{R(\tilde{y}_k, 0)}{2C},
\]

with \( \tilde{x}_k' \) closest to \( \tilde{x}_k \).

As in [BBB^+] we can show that each point \( \tilde{z} \in [\tilde{x}_k' \tilde{y}_k'] \) has a \((\varepsilon,C)\) canonical neighborhood which is a strong \( \varepsilon \)-neck, say \( U(\tilde{z}) \), centered at \((\tilde{z}, 0)\). Let \( U_k \) be the union of these \( U(\tilde{z}) \)'s. The most part of \( U_k \) (that is, except for the part near the two ends), denoted by \( T_k \), admits Hamilton’s canonical parametrization, \( \Phi_k : S^3 \times [A_k, B_k] \to T_k \), (cf. Appendix A). Then similarly as in the proof of [CTZ, Proposition 4.4], we pull back the rescaled solution \( \tilde{g}_k(\cdot) \) via \( \Phi_k \). Then the pulled-back solutions (with the appropriate base points) sub-converge smoothly to a partial Ricci flow (cf. [BBB^+]). Now the rest of the arguments is almost identical to that in [BBB^+, Theorem 6.1.1]. For some of the details one can also consult Step 2 of proof of [CZ2, Theorem 4.1] (for the smooth (without surgery) case) and Step 3 of proof of [CZ2, Proposition 5.4] (for the surgical case).
**Remark 3** For the estimate above, under a parabolic rescaling of the metrics, $c, q, P, R,$ etc. will change in general, and $Q$ will change with the same scaling factor as $R$ does, but $\Lambda$ is scaling invariant.

The following proposition extends [Hu1, Proposition 2.3]; compare [BBM, Theorem 6.8] and [BBB, Theorem 6.2.1].

**Proposition 2.5.** Fix $c_0 > 0$. For any $r, \delta > 0$, there exist $h \in (0, \delta r)$ and $D > 10$, such that if $(X(\cdot), g(\cdot))$ is a complete surgical solution with uniformly positive isotropic curvature $(a_1 + a_2 \geq c_0, a_1 + a_2 \geq c_0)$, with bounded curvature, defined on a time interval $[a, b]$ ($0 \leq a < b < \frac{1}{2c_0}$) and satisfying the pinching assumption and the canonical neighborhood assumption $(CN)_r$, then the following holds:

Let $t \in [a, b]$ and $x, y, z \in X(t)$ such that $R(x, t) \leq 2/r^2, R(y, t) = h^{-2}$ and $R(z, t) \geq D/h^2$. Assume there is a curve $\gamma$ in $X(t)$ connecting $x$ to $z$ via $y$, such that each point of $\gamma$ with scalar curvature in $[2C_0r^{-2}, C_0^{-1}Dh^{-2}]$ is the center of an $\varepsilon_0$-neck. Then $(y, t)$ is the center of a strong $\delta$-neck.

**Proof** We essentially follow the proof of [BBM, Theorem 6.8] and [BBB, Theorem 6.2.1]. (Compare [P2, Lemma 4.3], [CTZ, Proposition 4.4].) We argue by contradiction. Otherwise, there exist $r, \delta > 0$, sequences $h_k \to 0, D_k \to +\infty$, a sequence of complete surgical solutions $(X_k(\cdot), g_k(\cdot))$ with bounded curvature and with uniformly positive isotropic curvature $(a_1 + a_2 \geq c_0, a_1 + a_2 \geq c_0)$ satisfying the the pinching assumption (with constants $\delta_0, \Psi_0, L_0, P_0, S_0$) and $(CN)_r$, and sequences $0 < t_k < \frac{1}{2c_0}, z_k \in X_k(t_k)$ with $R(x_k, t_k) \leq 2r^{-2}$ and $R(z_k, t_k) \geq D_k/h_k^{-2}$, and finally a sequence of curves $\gamma_k$ in $X_k(t_k)$ connecting $x_k$ with $z_k$, whose points of scalar curvature in $[2C_0r^{-2}, C_0^{-1}D_kh_k^{-2}]$ are centers of $\varepsilon$-necks, but none of the points $y_k \in \gamma_k$ with $R(y_k, t_k) = h_k^{-2}$ is the center of a strong $\delta$-neck.

For each $k$, first we choose a point $x_k' \in \gamma_k$ with $R(x_k', t_k) = 2C_0 r^{-2}$ such that it is the furthest point to $x_k$ (measured w.r.t. the arc length of $\gamma_k$) among all points with such properties. Then we choose a point $z_k' \in \gamma_k$ lying between $x_k'$ and $z_k$ with $R(z_k', t_k) = C_0^{-1}D_kh_k^{-2}$ such that it is the furthest point to $z_k$ (measured w.r.t. the arc length of $\gamma_k$) among all points with such properties. Finally we choose a point $y_k \in \gamma_k$ lying between $x_k'$ and $z_k'$ with $R(y_k, t_k) = h_k^{-2}$. Then $y_k$ is not the center of a strong $\delta$-neck. Now all points in $\gamma_k$ lying between $x_k'$ and $z_k'$ have scalar curvature in $[2C_0r^{-2}, C_0^{-1}D_kh_k^{-2}]$, and it follows from the assumptions that all these points are centers of $\varepsilon$-necks, therefore are centers of strong $\varepsilon$-necks by the a priori assumptions. As in the proof of Proposition 2.4, the most part of the union of these necks (that is, except for the part near the two ends), denoted by $\mathcal{T}_k$, admits Hamilton’s canonical parametrization, $\Phi_k : S^3 \times [A_k, B_k] \to \mathcal{T}_k$. Then we pull back the rescaled solution $h_k^{-2}g_k(t_k + h_k^2 t)$ to $S^3 \times [A_k, B_k]$. The rest of the proof is almost the same as in that of [BBM, Theorem 6.8] and [BBB, Theorem 6.2.1].

Now we describe more precisely Hamilton’s surgery procedure [H4] as adapted to the orbifold case in [CTZ]. We will follow [CaZ], [CZ2], [CTZ] closely. First we describe the model surgery on the standard cylinder, and define the orbifold.
standard solution. Consider the semi-infinite cylinder $N_0 = (S^3/\Gamma) \times (-\infty, 4)$ with the standard metric $\hat{g}_0$ of scalar curvature 1, where $\Gamma$ is a finite, fixed point free subgroup of isometries of $S^3$. Let $f$ be a smooth nondecreasing convex function on $(-\infty, 4)$ defined by

$$
\begin{cases}
  f(z) = 0, & z \leq 0; \\
  f(z) = w_0 e^{-\frac{w_0}{z}}, & z \in (0, 3]; \\
  f(z) \text{ is strictly convex}, & z \in [3, 3.9]; \\
  f(z) = -\frac{1}{2} \ln(16 - z^2), & z \in [3.9, 4),
\end{cases}
$$

(where $w_0$ is a small positive constant and $W_0$ is a large positive constant to be determined in Lemma 2.6 below). Replace the standard metric $\hat{g}_0$ on the subspace $(S^3/\Gamma) \times [0, 4)$ in $N_0$ by $e^{-2f} \hat{g}_0$. The resulting metric will induce a complete metric (denoted by $\tilde{g}$) on the orbifold cap $C_\Gamma$. We call the complete Ricci flow $(C_\Gamma, \tilde{g}(\cdot))$ with initial data $(C_\Gamma, \tilde{g})$ with bounded curvature in any compact subinterval of $[0, \frac{2}{3})$ the orbifold standard solution, which exists on the time interval $[0, \frac{2}{3})$. Note that when $\Gamma = \{1\}$, $(C_\Gamma, \tilde{g}(\cdot))$ is actually the smooth standard solution of $g(\cdot)$ with the initial metric $(\mathbb{R}^4, \tilde{g}(\cdot))$ constructed in [CZ2, Appendix]. There is a natural orbifold covering $\pi_\Gamma : (\mathbb{R}^4, \tilde{g}) \to (C_\Gamma, \tilde{g})$. Denote by $O$ the tip of the smooth standard solution, (which is the fixed point of the $SO(4)$-action on the initial metric $(\mathbb{R}^4, \tilde{g})$), and by $p_\Gamma = \pi_\Gamma(O) \in C_\Gamma$ the corresponding tip of the orbifold standard solution. We refer the reader to [CZ2, Appendix] for some of the main properties of 4-dimensional smooth standard solution.

Then we describe a similar surgery procedure for the general case. Suppose we have a $\delta$-neck $N$ centered at $x_0$ in a Riemannian 4-orbifold $(X, g)$ with at most isolated singularities. Sometimes we will call $R^{-\frac{2}{3}}(x_0)$ the radius of this neck. Let $\Phi : S^3 \times [-l, l] \to V \subset N$ be Hamilton’s canonical parametrization; see Appendix A. Assume the center of the $\delta$-neck $N$ has $\mathbb{R}$ coordinate $z = 0$. The surgery is to cut off the $\delta$-neck along the center and glue back two orbifold caps (of Type I) $C_\Gamma$ separately. We construct a new metric on the glued back orbifold cap (of Type I) $C_\Gamma$ (say on the left hand side) as follows,

$$
\tilde{g} = \begin{cases}
g, & z = 0; \\
e^{-2f}g, & z \in [0, 2]; \\
\varphi e^{-2f}g + (1 - \varphi)e^{-2f}h^2\hat{g}_0, & z \in [2, 3]; \\
e^{-2f}h^2\hat{g}_0, & z \in [3, 4],
\end{cases}
$$

where $\varphi$ is a smooth bump function with $\varphi = 1$ for $z \leq 2$, and $\varphi = 0$ for $z \geq 3$, $h = R^{-\frac{2}{3}}(x_0)$, and $\hat{g}_0$ is as above. We also perform the same surgery procedure on the right hand side with parameters $\tilde{z} \in [0, 4]$ ($\tilde{z} = 8 - z$).

The following lemma of Hamilton justifies the pinching assumption of surgical solution.

**Lemma 2.6** (Hamilton [H4,D3.1]; compare [CZ2, Lemma 5.3], [CTZ, Lemma 4.3]) There exist universal positive constants $\delta_0, w_0, W_0$, and a constant $h_0$ depends only on $c_0$, such that given any surgical solution with uniformly positive isotropic curvature $(a_1 + a_2 \geq c_0, c_1 + c_2 \geq c_0)$, satisfying the pinching assumption, defined on $[a, t_0]$ $(0 \leq a < t_0 < \frac{1}{2c_0})$, if we perform Hamilton’s surgery as described above at a $\delta$-neck (if it exists) of radius $h$ at time $t_0$ with $\delta < \delta_0$ and $h \leq h_0$, then after
the surgery, the pinching assumption still holds at all points at time \( t_0 \). Moreover, after the surgery, any metric ball of radius \( \delta^{-\frac{1}{2}}h \) with center near the tip (i.e. the origin of the attached cap) is, after scaling with the factor \( h^{-2} \), \( \delta^2 \)-close to the corresponding ball of \((C_\Gamma, \hat{g})\) for some \( \Gamma \).

Usually we will be given two non-increasing step functions \( r, \delta : [0, +\infty) \to (0, +\infty) \) as surgery parameters. Let \( h(r, \delta), D(r, \delta) \) be the associated parameter as determined in Proposition 2.5, \( (h\) is also called the surgery scale,) and let \( \Theta := 2Dh^{-2} \) be the curvature threshold for the surgery process (as in [BBM] and [Hu1]), that is, we will do surgery when \( R_{\max} \) reaches \( \Theta \). Now we adapt two more definitions from [BBM] and [Hu1].

**Definition** (compare [BBM] and [Hu1]) Given an interval \( I \subset [0, +\infty) \), fix surgery parameter \( r, \delta : I \to (0, +\infty) \) and let \( h, D, \Theta = 2Dh^{-2} \) be the associated cutoff parameters. Let \( (X(t), g(t)) \ (t \in I) \) be an evolving Riemannian 4-orbifold with bounded curvature and with uniformly positive isotropic curvature. Let \( t_0 \in I \) and \((X_+, g_+)\) be a (possibly empty) Riemannian 4-orbifold. We say that \((X_+, g_+)\) is obtained from \((X(\cdot), g(\cdot))\) by \((r, \delta)\)-surgery at time \( t_0 \) if the following conditions are satisfied:

i. \( R_{\max}(g(t_0)) = \Theta(t_0) \), and there is a locally finite collection \( S \) of disjoint embedded \( S^3/\Gamma \)'s in \( X(t_0) \) which are in the middle of strong \( \delta \)-necks with radius equal to the surgery scale \( h(t_0) \), such that \( X_+ \) is obtained from \( X(t_0) \) by doing Hamilton's surgery along these necks as described above (where \( \Gamma \)'s are finite, fixed point free subgroups of isometries of \( S^3 \), and removing the components that are diffeomorphic to

a. spherical 4-orbifolds, and either have sectional curvature bounded below by \( C_0^{-1}/100 \) or are covered by \( \varepsilon_0 \)-necks and \( \varepsilon_0 \)-caps, or
b. \( S^3 \times \mathbb{R}/G \) (where \( G \) is a fixed point free discrete subgroup of the isometry group of the standard metric on \( S^3 \times \mathbb{R} \)), and are covered by \( \varepsilon_0 \)-necks, or
c. \( \varepsilon_0 \)-caps, and each of which is covered by \( \varepsilon_0 \)-necks and an \( \varepsilon_0 \)-cap, or
d. \( S^4/(x, \pm x') \mathbb{RP}^4 \), or
e. \( S^4/(x, \pm x') \mathbb{S}^3/(x, \pm x'); \)

ii. If \( X_+ \neq \emptyset \), then \( R_{\max}(g_+) \leq \Theta(t_0)/2 \).

**Definition** (cf. [BBM] and [Hu1]) A surgical solution \((X(\cdot), g(\cdot))\) defined on some time interval \( I \subset [0, +\infty) \) is an \((r, \delta)\)-surgical solution if it has the following properties:

i. It satisfies the pinching assumption, and \( R(x, t) \leq \Theta(t) \) for all \((x, t)\);
ii. At each singular time \( t_0 \in I \), \((X_+(t_0), g_+(t_0))\) is obtained from \((X(\cdot), g(\cdot))\) by \((r, \delta)\)-surgery at time \( t_0 \); and
iii. Condition \((CN)_r\) holds.

Recall that in our 4-dimensional case, \( g(\cdot) \) is \( \kappa \)-noncollapsed (for some \( \kappa > 0 \)) on the scale \( r \) at time \( t \) if at any point \( x \), whenever \( |Rm| \leq r^{-2} \) on \( P(x, t, r, -r^2) \) and \( P(x, t, r, -r^2) \) is unscathed we have \( \text{vol}B(x, t, r) \geq \kappa r^4 \). Let \( \kappa : I \to (0, +\infty) \) be a function. We say \( \{ (X(t), g(t))_{t \in I} \} \) has property \((NC)_\kappa\) if it is \( \kappa(t) \)-noncollapsed on all scales \( \leq 1 \) at any time \( t \in I \). An \((r, \delta)\)-surgical solution which also satisfies condition \((NC)_\kappa\) is called an \((r, \delta, \kappa)\)-surgical solution.
Lemma 2.7 (cf. [BBM, Lemma 5.4] and [Hu1, Lemma 2.5]) Suppose we have fixed two constants \( r, \delta > 0 \) as surgery parameters on an interval \( [a, b] \). Let \((X(t), g(t))\) be an \((r, \delta)\)-surgical solution on \([a, b]\). Let \( a \leq t_1 < t_2 < b \) be two singular times (if they exist). Then \( t_2 - t_1 \) is bounded from below by a positive number depending only on \( r, \delta \).

The following proposition extends [Hu1, Proposition 2.6] and a result in [CTZ], and is similar to [BBM, Theorem 7.4].

Proposition 2.8 Let \( \varepsilon \in (0, 2\varepsilon_0] \). Let \((X, g)\) be a complete, connected 4-orbifold with at most isolated singularities. If each point of \( X \) is the center of an \( \varepsilon \)-neck or an \( \varepsilon \)-cap, then \( X \) is diffeomorphic to \( S^3/\Gamma, S^3 \times \mathbb{R}/G \) (where \( G \) is a fixed point free discrete subgroup of the isometry group of the standard metric on \( S^3 \times \mathbb{R} \)), \( \mathbb{R}^4, C_1, S^3/(\pm x') \setminus S^3, S^3/(\pm x') \mathbb{H} \mathbb{P}^4 \), or \( S^3/(\pm x') \mathbb{H} \mathbb{P}^4 \).

Proof. The result in the compact case has been shown in [CTZ], using a theorem in [Mc] which says that any diffeomorphism of a 3-dimensional spherical space form is isotopic to an isometry. So below we will assume that \( X \) is not compact.

Claim Let \( \varepsilon \in (0, 2\varepsilon_0] \). Let \((X, g)\) be a complete, noncompact, connected 4-orbifold with at most isolated singularities. If each point of \( X \) is the center of an \( \varepsilon \)-neck, then \( X \) is diffeomorphic to \( S^3/\Gamma \times \mathbb{R} \) (where \( \Gamma \) is a finite, fixed point free subgroup of isometries of \( S^3 \)).

Proof of Claim. Let \( x_1 \) be a point of \( X \), and let \( N_1 \) be a \( \varepsilon \)-neck centered at \( x_1 \), given by some diffeomorphism \( \psi_1 : S^3/\Gamma_1 \times (-\varepsilon^{-1}, \varepsilon^{-1}) \to N_1 \), where \( \Gamma_1 \) is a finite, fixed point free subgroup of \( SO(4) \). Consider Hamilton’s canonical parametrization \( \Phi_1 : S^3 \times [-L_1, L_1] \to V_1 \subset N_1 \) such that \( V_1 \) contains the portion \( \psi_1(S^3/\Gamma_1 \times (-0.98\varepsilon^{-1}, 0.98\varepsilon^{-1})) \) in \( N_1 \). Now choose a point \( x_2 \) in \( \Phi_1(S^3 \times \{0.9L_1\}) \), and let \( N_2 \) be a \( \varepsilon \)-neck centered at \( x_2 \), given by some diffeomorphism \( \psi_2 : S^3/\Gamma_2 \times (-\varepsilon^{-1}, \varepsilon^{-1}) \to N_2 \). Again consider Hamilton’s canonical parametrization \( \Phi_2 : S^3 \times [-L_2, L_2] \to V_2 \subset N_2 \) such that \( V_2 \) contains the portion \( \psi_2(S^3/\Gamma_2 \times (-0.98\varepsilon^{-1}, 0.98\varepsilon^{-1})) \) in \( N_2 \). Then as Appendix A we know that \( \Gamma_1 \) is conjugate to \( \Gamma_2 \) in \( SO(4) \), and the embedded \( S^3/\Gamma_1 \) through \( x_1 \) (of constant mean curvature) in \( V_1 \) is isotopic to the embedded \( S^3/\Gamma_2 \) through \( x_2 \) (of constant mean curvature) in \( V_2 \). Then we go on, choose \( x_3, N_3, \Phi_3, \cdots \) This way the desired result follows.

Now consider the case that \( X \) contains at least one \( \varepsilon \)-cap. In this case, since we are assuming \( X \) is noncompact, arguing as above, one see that \( X \) is diffeomorphic to a cap union a neck, where they glue nicely. So in this case \( X \) is diffeomorphic to a cap.

The following proposition extends [Hu1, Proposition 2.7], and is analogous to [BBM, Proposition A].

Proposition 2.9 Fix \( \epsilon_0 > 0 \). There exists a positive constant \( \delta \) (depending only on \( \epsilon_0 > 0 \)) with the following property: Let \( r, \delta \) be surgery parameters, let
\{(X(t), g(t))\}_{t \in [a,b]} (0 < a < b < \frac{1}{2q_0}) be an \((r, \delta)\)-surgical solution with uniformly positive isotropic curvature \((a_1 + a_2 \geq c_0, c_1 + c_2 \geq c_0)\). Suppose that \(\delta \leq \tilde{\delta}\), and \(R_{\max}(b) = \Theta = \Theta(b)\). Then there exists a Riemannian orbifold \((X_+, g_+)\) which is obtained from \((X(t), g(t))\) by \((r, \delta)\)-surgery at time \(b\), such that

i. \(g_+\) satisfies the pinching assumption at time \(b\);

ii. \((a_1 + a_2)_{\min}(g_+(b)) \geq (a_1 + a_2)_{\min}(g(b)), (c_1 + c_2)_{\min}(g_+(b)) \geq (c_1 + c_2)_{\min}(g(b))\), and \(R_{\min}(g_+(b)) \geq R_{\min}(g(b))\).

**Proof** It follows from Proposition 2.5, Lemma 2.6, and Proposition 2.8, as in the proof of [BBM, Proposition A] and [Hu1, Proposition 2.7]. Let \(\delta_0\) and \(h_0\) be as given in Lemma 2.4. Set \(\tilde{\delta} = \frac{1}{4}\min\{c_0 h_0, \delta_0\}\). The idea is to consider a maximal collection \(\{N_i\}\) of pairwise disjoint cutoff necks in \(X(b)\), whose existence is guaranteed by Zorn’s Lemma. (Here, following [BBM], we call cutoff neck a strong \(\delta\)-neck centered at some point \((x, t)\) of scalar curvature \(h^{-2}(t)\).) We want to show that such a collection is locally finite. Note that in the orbifold case we need a new argument to guarantee this. We argue by contradiction. Otherwise there is a sequence of cutoff necks (still denoted by \(\{N_i\}\) ) with center \(y_i\) (with \(R(y_i, b) = h^{-2}(b)\)), where \(N_i\) is diffeomorphic to \((S^2/\Gamma_i) \times \mathbb{I} \) with \(|\Gamma_i| \to \infty\) as \(i \to \infty\), and all \(N_i\)’s are contained in a compact subset \(K\) of \(M(b)\). Then there is a subsequence of \(\{y_i\}\) (still denoted by \(\{y_i\}\)) which converges to a point \(y\) in \(K\). We have \(R(y, b) = h^{-2}(b)\), so \((y, b)\) has a canonical neighborhood \(U\), which is impossible, as can be seen as follows.

i. If \(U\) is in case (a) in the definition of canonical neighborhood, we get a contradiction by using the assumptions \(|\Gamma_i| \to \infty\) and \(y_i \to y\) as \(i \to \infty\) and Appendix A.

ii. If \(U\) is in case (b), we may assume that \((y, b)\) is not the center of an \(\epsilon_0\)-neck, otherwise we can argue as in case i. above and get a contradiction. Then, for \(y_i\) sufficiently close to \(y\), \(y_i\) cannot be the center of any strong \(\delta\)-neck. Again a contradiction.

iii. If \(U\) is in case (c), we get a contradiction by comparing sectional curvature. Now as in [BBM], let \(\mathcal{G}\) (resp. \(\mathcal{O}\), resp. \(\mathcal{R}\)) be the set of points of \(X(b)\) of scalar curvature less than \(2r^{-2}(b)\) (resp. \(\in [2r^{-2}(b), \Theta(b)/2)\), resp. \(\geq \Theta(b)/2\)).

**Claim 1** Any connected component of \(X(b) \setminus \bigcup_i N_i\) is contained either in \(\mathcal{G} \cup \mathcal{O}\) or in \(\mathcal{R} \cup \mathcal{O}\).

**Proof of Claim 1.** We argue by contradiction. Otherwise there is some component \(W\) of \(X(b) \setminus \bigcup_i N_i\) containing at least one point \(x \in \mathcal{G}\) and one point \(z \in \mathcal{R}\). Choose a minimizing geodesic path \(\gamma\) in \(W\) connecting \(x\) with \(z\). In the following Claim 2, we will show each point of \(\gamma\) with scalar curvature in \([2C_0 r^{-2}(b), C_0^{-1} D(b) h^{-2}(b)]\) is the center of an \(\epsilon\)-neck. Then we can apply Proposition 2.5 to conclude that there exists some point \(y \in \gamma\) with \(R(y, b) = h^{-2}(b)\) which is the center of a strong \(\delta\)-neck. This will contradict the maximality of \(\{N_i\}\).

**Claim 2** Each point of such \(\gamma\) with scalar curvature in \([2C_0 r^{-2}(b), C_0^{-1} D(b) h^{-2}(b)]\) is center of an \(\epsilon_0\)-neck.
Proof of Claim 2. The proof is a minor modification of that of the second claim in Lemma 7.6 of [BBM] and Claim 2 in [Hu1, Proposition 2.7]. Let \( y \in \gamma \) be such a point. Then \( y \) is center of a \((\varepsilon_0, C_0)\)-canonical neighborhood. Clearly \( U \) cannot be a closed manifold by the curvature assumptions. We will show \( U \) cannot be an \((\varepsilon_0, C_0)\)-cap either. Otherwise \( U = N \cup C \), where \( N \) is an \( \varepsilon_0 \)-neck, \( N \cap C = \emptyset \), \( \nabla \cap C = \partial C \) and \( y \in \text{Int}(C) \). Let \( \psi : S^3 / \Gamma \times (-\varepsilon_0^{-1}, \varepsilon_0^{-1}) \to N \) be the diffeomorphism which defines the neck \( N \). We use Hamilton’s method to give a canonical parametrization \( \Phi : S^3 \times [-L, L] \to V \subset N \) such that \( V \) contains the portion \( \psi(S^3 / \Gamma \times (-0.98\varepsilon_0^{-1}, 0.98\varepsilon_0^{-1})) \) (cf. Appendix A). Let \( S = \Phi(S^3 \times \{0\}) \). We rescale the metric such that the scalar curvature of \( N \) is close to 1. Clearly \( \gamma \) is not minimizing in \( U \), since if \( x' \) (resp. \( z' \)) is an intersection of \( \gamma \) with \( S \) between \( x \) and \( y \) (resp. \( y \) and \( z \)), then \( d(x', z') \ll d(x', y) + d(y, z') \). The geodesic segment (in \( U \)) \([x' z']\) is not contained in \( W \) by the minimality of \( \gamma \) in \( W \). So \([x' z'] \cap \partial W \neq \emptyset \). By definition of \( W \), the corresponding component of \( \partial W \) is a boundary component, denoted by \( S^+_i \), of some cutoff neck \( N_i \). Then \( d(S^+_i, S) < \text{diam}(S) \) since \([x' z'] \cap S^+_i \neq \emptyset \). We use Hamilton’s method to give a canonical parametrization \( \Phi' : S^3 \times [-L', L'] \to V' \subset N_i \) such that one of the ends of \( V' \) is at the rescaled distance 0.02\( \varepsilon_0^{-1} \) from the end \( S^+_i \) of \( N_i \). Pick a point \( p' \) in \( V' \) which is at rescaled distance 0.2\( \varepsilon_0^{-1} \) from \( \partial_i V' \). Then \( d(p', S) \leq d(p', \partial_i V') + d(\partial_i V', S^+_i) + d(S^+_i, S) < 0.2\varepsilon_0^{-1} + 0.02\varepsilon_0^{-1} + \text{diam}(S) < 0.3\varepsilon_0^{-1} \). Then it follows from Appendix A that the embedded \( S^3 / \Gamma' \) (with constant mean curvature) in \( V' \) which contains \( p' \) is isotopic to \( S \) in \( N \). It follows that \( \gamma \cap N_i \neq \emptyset \), which is impossible by the definition of \( W \).

Then we do Hamilton’s surgery along these \( N_i \)’s, and obtain an orbifold \((X', g_+)\). By Lemma 2.6 the pinching assumption is satisfied. Moreover we throw away those components of \( X' \) which are covered by canonical neighborhoods, and hence whose diffeomorphism types are identified with the help of Proposition 2.8. The resulting orbifold is the desired \((X_+, g_+)\).

3. Existence of \((r, \delta, \kappa)\)-surgical solutions

As in [BBM], if \((X(\cdot), g(\cdot))\) is a piecewise \(C^1\) evolving orbifold defined on some interval \( I \subset \mathbb{R} \) and \([a, b] \subset I\), the restriction of \( g \) to \([a, b]\), still denoted by \( g(\cdot) \), is the evolving orbifold

\[
 t \mapsto \begin{cases} 
 (X_+(a), g_+(a)), & t = a, \\
 (X(t), g(t)), & t \in (a, b]. 
\end{cases}
\]

The following proposition extends [Hu1, Proposition 3.1]. Compare [P2, Lemma 4.5], [BBB, Theorem 8.1.2], [BBM, Theorem 8.1], [CaZ, Lemma 7.3.6], [KL, Lemma 74.1], [MT, Proposition 16.5] and [Z, Lemma 9.1.1], see also the formulation in the proof of [CZ2, Lemma 5.5].

**Proposition 3.1** Fix \( c_0 > 0 \). For all \( A > 0, \theta \in (0, \frac{2}{7}) \) and \( \hat{r} > 0 \), there exists \( \delta = \hat{\delta}(A, \theta, \hat{r}) > 0 \) with the following property. Let \( r(\cdot) \geq \hat{r}, \delta(\cdot) \leq \hat{\delta} \) be two positive, non-increasing step functions on \([a, b]\) \((0 \leq a < b < \frac{1}{2c_0})\), and let \((X(\cdot), g(\cdot))\) be a surgical solution with uniformly positive isotropic curvature \((a_1 + a_2 \geq c_0, c_1 + c_2 \geq c_0)\), defined on \([a, b]\), such that it satisfies the pinching assumption on \([a, b]\), that \( R(x, t) \leq \Theta(r(t), \delta(t)) \) for all space-time points with \( t \in [a, b] \), that at any singular time \( t_0 \in [a, b) \), \((X_+(t_0), g_+(t_0))\) is obtained from \((X(\cdot), g(\cdot))\) by
\[(r, \delta)-\text{surgery, and that any point } (x, t) \ (t \in [a, b]) \text{ with } R(x, t) \geq (\frac{r(t)}{\delta})^{-2} \text{ has a } (2\varepsilon_0, 2C_0)\text{-canonical neighborhood.} \]

\[\text{Let } t_0 \in [a, b] \text{ be a singular time. Consider the restriction of } (X(\cdot), g(\cdot)) \text{ to } [t_0, b]. \text{ Let } p \in X_+(t_0) \text{ be the tip of some surgery cap of scale } h(t_0), \text{ and let } t_1 \leq \min \{b, t_0 + \theta h^2(t_0)\} \text{ be maximal (subject to this inequality) such that } P(p, t_0, Ah(t_0), t_1 - t_0) \text{ is unscathed. Then the following holds:} \]

\[\text{i. The parabolic neighborhood } P(p, t_0, Ah(t_0), t_1 - t_0) \text{ is, after scaling with factor } h^{-2}(t_0) \text{ and shifting time } t_0 \text{ to zero, } A^{-1}\text{-close to } P(p_T, 0, A, (t_1 - t_0)h^{-2}(t_0)) \text{ (where } p_T \text{ is the tip of the cap of the orbifold standard solution } C_T \text{ for some } \Gamma; \]

\[\text{ii. If } t_1 < \min \{b, t_0 + \theta h^2(t_0)\}, \text{ then } B(p, t_0, Ah(t_0)) \subset X_{sing}(t_1) \text{ disappear at time } t_1. \]

\[\text{We will follow the proof of [BBB}^+, \text{ Theorem 8.1.2] and [BBM, Theorem 8.1].} \]

\[\text{Let } M_0 = (\mathbb{R}^4, \hat{g}(\cdot)) \text{ be the smooth standard solution, and } 0 < T_0 < \frac{4}{T}. \]

**Lemma 3.2** (Compare [BBB}^+, \text{ Theorem 8.1.3]) \[\text{For all } A, \Lambda > 0, \text{ there exists } \rho = \rho(M_0, A, \Lambda) > A \text{ with the following properties. Let } \Gamma \text{ be a finite, fixed point free subgroup of isometries of } S^3. \text{ Let } U \text{ be an open subset of } C_T \text{ and } T \in (0, T_0]. \]

\[\text{Let } g(\cdot) \text{ be a Ricci flow defined on } U \times [0, T], \text{ such that the ball } B(p_T, 0, \rho) \subset U \text{ is relatively compact. Assume that} \]

\[(i) \ |RM(g(\cdot))|_{0, U \times [0, T], g(\cdot)} \leq \Lambda, \]

\[(ii) \ g(0) \text{ is } \rho^{-1}\text{-close to } \hat{g}(0) \text{ on } B(p_T, 0, \rho). \]

\[\text{Then } g(\cdot) \text{ is } A^{-1}\text{-close to } \hat{g}(\cdot) \text{ on } B(p_T, 0, A) \times [0, T]. \]

Here, \[|RM(g(\cdot))|_{0, U \times [0, T], g(\cdot)} := \sup_{U \times [0, T]} \{ |RM(g(t))|_{g(t)} \}. \]

**Proof** We argue by contradiction. Otherwise there exist } A, \Lambda > 0, \text{ and a sequence of Ricci flows } g_k(\cdot) \text{ defined on } U_k \times [0, T_k] \text{ (where } U_k \subset C_{\Gamma_k} \text{ with } \Gamma_k \text{ a finite, fixed point free subgroup of isometries of } S^3, T_k \leq T_0), \text{ a sequence } p_k = p_{T_k}, \text{ and} \]

\[(i) \ |RM(g_k(\cdot))|_{g_k(t)} \leq \Lambda \text{ on } U_k \times [0, T_k], \]

\[(ii) \ g_k(0) \text{ is } \rho^{-1}_k\text{-close to } \hat{g}(0) \text{ on } B(p_k, 0, \rho_k), \]

\text{but for some } t_k \in [0, T_k], \text{ } g_k(t_k) \text{ is not } A^{-1}\text{-closed to } \hat{g}(t_k) \text{ on } B(p_k, 0, A). \]

\[\text{We pull back the solutions } g_k(\cdot) \text{ (and } \hat{g}(\cdot)) \text{ to } \mathbb{R}^4 \text{ via } \pi_k = \pi_{\Gamma_k} : \mathbb{R}^4 \to C_{\Gamma_k}. \]

\[\text{Then we see that} \]

\[|RM(\pi_k \circ g_k(\cdot))|_{\pi_k \circ g_k(\cdot)} \leq \Lambda \text{ on } \pi_k^{-1}(U_k) \times [0, T_k], \]

\[\pi_k^{-1}g_k(0) \text{ is } \rho^{-1}_k\text{-close to } \hat{g}(0) \text{ on } B(O, 0, \rho_k), \]

\[\pi_k^{-1}g_k(t_k) \text{ is not } A^{-1}\text{-close to } \hat{g}(t_k) \text{ on } B(O, 0, A). \]

\[\text{Now we can argue as in [BBB}^+, \text{ using a stronger version of Shi’s derivative estimates ([LT,Theorem 11], see also [MT, Theorem 3.29]), Hamilton’s compactness} \]

\[\text{theorem for Ricci flow ([H3]) and Chen-Zhu’s uniqueness theorem for complete Ricci flow (CZ1)], to get a contradiction.} \]

**Corollary 3.3** (Compare [BBM, Corollary 8.2]) \[\text{Let } A > 0. \text{ There exists } \rho = \rho(M_0, A) > A \text{ with the following properties. Let } \{(X(t), g(t))\}_{t \in [0, T]} \ (T \leq T_0) \text{ be} \]

\[\text{a surgical solution. Assume that} \]

\[(i) \ (X(\cdot), g(\cdot)) \text{ is a parabolic rescaling of some surgical solution which satisfies} \]

\[\text{the pinching assumption,} \]

\[(ii) \ |\frac{d}{dt}R| < 2C_0R^2 \text{ at any space-time point } (x, t) \text{ with } R(x, t) \geq 1. \]
Let \( p \in X_+(0) \) and \( t \in (0, T] \) be such that

(iii) \( B(p, 0, \rho) \) is \( \rho^{-1} \)-close to \( B(p_\Gamma, 0, \rho) \subset C_\Gamma \) for some \( \Gamma \),

(iv) \( P(p, 0, \rho, t) \) is unscathed.

Then \( P(p, 0, A, t) \) is \( A^{-1} \)-close to \( P(p_\Gamma, 0, A, t) \).

**Proof** The proof is similar to that of Corollaries 8.2.2 and 8.2.4 in [BBB+].

Then to finish the proof of Proposition 3.1, we can proceed as in the proof of [BBM, Theorem 8.1], using Corollary 3.3.

The following theorem extends [Hu1, Theorem 3.4]. Compare [P2, Proposition 5.1], [BBM, Theorems 5.5 and 5.6] and [MT, Theorem 15.9].

**Theorem 3.4** Given \( c_0, v_0 > 0 \), there are surgery parameter sequences

\[
K = \{ \kappa_i \}_{i=1}^\infty, \quad \Delta = \{ \delta_i \}_{i=1}^\infty, \quad \nu = \{ \nu_i \}_{i=1}^\infty
\]

such that the following holds. Let \( r(t) = r_i \) and \( \bar{\delta}(t) = \delta_i \) on \( [(i-1)2^{-5}, i \cdot 2^{-5}) \), \( i = 1, 2, \cdots \). Suppose that \( \delta : [0, \infty) \to (0, \infty) \) is a non-increasing step function with \( \delta(t) \leq \bar{\delta}(t) \). Then the following holds: Suppose that we have a surgical solution \((X(\cdot), g(\cdot))\) with uniformly positive isotropic curvature, defined on \([0, T] \) (for some \( T < \infty \)), which satisfies the following conditions:

1. the initial data \((X(0), g(0))\) is a complete 4-orbifold with at most isolated singularities, with uniformly positive isotropic curvature \((a_1 + a_2 \geq c_0, c_1 + c_2 \geq c_0)\), with \( |Rm| \leq 1 \), and with vol \(B(x, 1) \geq v_0\) at any point \(x\),
2. the solution satisfies the pinching assumption, and \( R(x, t) \leq \Theta(r(t), \delta(t)) \) for all space-time points,
3. at each singular time \( t_0 \in (0, T) \), \((X_+(t), g_+(t))\) is obtained from \((X(\cdot), g(\cdot))\) by \((r, \delta)\)-surgery at time \(t_0\), and
4. on each time interval \([(i-1)2^{-5}, i \cdot 2^{-5}] \cap [0, T] \) the solution satisfies \((CN)_{r_i} \) and \((NC)_{\nu_i} \).

Then there is an extension of \((X(\cdot), g(\cdot))\) to a surgical solution defined for \( 0 \leq t \leq T' \) (where \( T' < \frac{1}{2c_0} \) is the extinction time) and satisfying the above four conditions with \( T \) replaced by \( T' \).

We can prove Theorem 3.4 similarly as that in [Hu1] with the help of Lemma 2.1, Lemma 2.6, Lemma 2.7, Proposition 2.9, Lemma 3.5 and Proposition 3.6 below.

The following lemma extends [Hu1, Lemma 3.5], and guarantees the non-collapsing under the canonical neighborhood assumption. Compare [P2, Lemma 5.2], [CTZ, Lemma 4.5], [KL, Lemma 79.12] and [BBM, Proposition C].

**Lemma 3.5** Fix \( c_0 > 0 \). Suppose \( 0 < r_- \leq \varepsilon_0, \kappa_- > 0 \), and \( 0 < E_- < E < \frac{1}{2c_0} \).

Then there exists \( \kappa_+ = \kappa_+(r_-, \kappa_-, E_-, E) > 0 \), such that for any \( r_+ \), \( 0 < r_+ \leq r_- \), one can find \( \delta' = \delta'(r_-, r_+, \kappa_-, E_-, E) > 0 \), with the following property.

Suppose that \( 0 \leq a < b < d < \frac{1}{2c_0}, \ b - a \geq E_-, \ d - a \leq E \). Let \( r \) and \( \delta \) be two positive, non-increasing step functions on \([a, d]\) with \( \varepsilon_0 \geq r \geq r_- \) on \([a, b]\), \( \varepsilon_0 \geq r \geq r_+ \) on \([b, d]\) and \( \delta \leq \delta' \) on \([a, d]\). Let \((X(\cdot), g(\cdot))\) be a surgical solution
with uniformly positive isotropic curvature \((a_1 + a_2 \geq c_0, c_1 + c_2 \geq c_0)\), defined on the time interval \([a, d]\), such that it satisfies the pinching assumption on \([a, d]\), that \(R(x, t) \leq \Theta(r(t), \delta(t))\) for all space-time points with \(t \in [a, d]\), that at any singular time \(t_0 \in [a, d]\), \((X_+(t_0), g_+(t_0))\) is obtained from \((X(\cdot), g(\cdot))\) by \((r, \delta)\)-surgery, that the conditions \((CN)_r\) and \((NC)_{\kappa_+}\) hold on \([a, b]\), and that any point \((x, t)\) \((t \in [b, d])\) with \(R(x, t) \geq (\frac{\delta}{\delta + 1})^{-2}\) has a \((2\varepsilon_0, 2C_0)\)-canonical neighborhood. Then \((X(\cdot), g(\cdot))\) satisfies \((NC)_{\kappa_+}\) on \([b, d]\).

**Proof** Using Proposition 3.1, the proof of [CTZ, Lemma 4.5] can be adapted to our case without essential changes.

The following proposition extends [Hu1, Proposition 3.6], and justifies the canonical neighborhood assumption needed. Compare [P2, Section 5], [MT, Proposition 17.1], [BBM, Proposition B] and [CZ2, Proposition 5.4].

**Proposition 3.6** Given \(c_0 > 0\). Suppose that for some \(i \geq 1\) we have surgery parameter sequences \(\delta \geq \delta_1 \geq \delta_2 \geq \cdots \geq \delta_i > 0\) and \(\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_i > 0\), where \(\delta\) is the constant given in Proposition 2.7. Then there are positive constants \(r_{i+1} \leq r_i\) and \(\delta_{i+1} \leq \min\{\delta_i, \delta'\}\), where \(\delta' = \delta'(r_i, r_{i+1}, \kappa_i)\) is the constant given in Lemma 3.5 by setting \(r_- = r_i, \kappa_- = \kappa_i, r_+ = r_{i+1}, E_- = 2^{-5}\) and \(E = 2^{-4}\), such that the following holds. Let \(r(t) = r_j\) and \(\delta(t) = \delta_j\) on \([j \cdot 2^{-5}, (j+1)2^{-5})\), \(j = 1, 2, \cdots, i+1\). Suppose that \(\delta : [0, (i+1)2^{-5}) \to (0, \infty)\) is a non-increasing step function with \(\delta(t) \leq \delta(t)\). Let \((X(\cdot), g(\cdot))\) be any surgical solution to Ricci flow with uniformly positive isotropic curvature \((a_1 + a_2 \geq c_0, c_1 + c_2 \geq c_0)\), defined in \([0, T]\) for some \(T \in (i \cdot 2^{-5}, (i+1)2^{-5}]\), such that \(R(x, t) \leq \Theta(r(t), \delta(t))\) for all space-time points with \(t \in [0, T]\), that at each singular time \(t_0 \in (0, T)\), \((X_+(t_0), g_+(t_0))\) is obtained from \((X(\cdot), g(\cdot))\) by \((r, \delta)\)-surgery at time \(t_0\). Suppose that the restriction of the surgical solution to \([0, i \cdot 2^{-5})\) satisfies the four conditions given in Theorem 3.4. Suppose also that \(\delta(t) \leq \delta_{i+1}\) for all \(t \in [(i-1)2^{-5}, T]\), and that the pinching assumption is satisfied up to time \(T\). Then \((X(\cdot), g(\cdot))\) satisfies the condition \((CN)_{r_{i+1}}\) in \([i \cdot 2^{-5}, T]\).

We can prove Proposition 3.6 similarly as that in [Hu1], using a weak openness (w.r.t. time) property of the canonical neighborhood condition in the noncompact orbifold case (extending the noncompact manifold case in [Hu1]), Proposition 2.4, Proposition 3.1, Lemma 3.5, Hamilton’s Harnack estimate [H2], and the compactness theorem for Ricci flow ([H3], [Lu]).

4. PROOF OF THEOREM 1.1

The following proposition extends [Hu1, Lemma 3.7]; compare [BBM, Proposition 2.3].

**Proposition 4.1** Let \(X'\) be a class of closed 4-orbifolds with isolated singularities. Let \(X\) be a 4-orbifold with isolated singularities. Suppose there exists a finite sequence of 4-orbifolds \(X_0, X_1, \cdots, X_k\), all with isolated singularities, such that \(X_0 = X, X_k = \emptyset\), and for each \(i (1 \leq i \leq k)\), \(X_i\) is obtained from \(X_{i-1}\) by cutting off along a locally finite collection of pairwise disjoint, embedded spherical 3-manifolds
$S^3/\Gamma$’s, gluing back $D^4/\Gamma$’s, and removing some components that are orbifold connected sums of members of $\mathcal{X}$. Then each component of $X$ is an orbifold connected sum of members of $\mathcal{X}$.

**Proof** The proof is elementary, and is almost identical to that of [BBM, Proposition 2.3], so we will omit it.

Note that each orbifold appeared in the list of Proposition 2.8 is a (possibly infinite) orbifold connected sum of spherical 4-orbifolds. (See Section 2.) Combining with Theorem 3.4 (in particular the finite time extinction result) and Proposition 4.1, it implies the following: Let $X$ be a complete, connected Riemannian 4-orbifold with at most isolated singularities, with uniformly positive isotropic curvature and with bounded geometry, then $X$ is diffeomorphic to a (possibly infinite) orbifold connected sum of spherical 4-orbifolds with isolated singularities. Now we argue that the diffeomorphism types of the spherical 4-orbifolds that appear in the orbifold connected sum of $X$ is finite, which will finish the proof of Theorem 2.2. We divide the analysis into two cases.

i. Consider those components which are removed in our surgery procedure and each of which contains at least an $\varepsilon_0$-neck. Our assumption on uniformly positive isotropic curvature imply that the $S^3/\Gamma$ cross section of these $\varepsilon_0$-necks must have Ricci curvature uniformly bounded below away from zero, which, combined with the non-collapsing property and the boundedness of the sectional curvature, implies that the isomorphism classes of $\Gamma$ are finite. As already noted in Section 2, by the work [Mc], given a finite, fixed point free subgroup $\Gamma$ of isometries of $S^3$, the diffeomorphism types of the mapping tori $S^3/\Gamma \times_f S^1$ are finite, where $f$ runs over all diffeomorphisms of $S^3/\Gamma$.

ii. By definition of our surgery procedure, any spherical orbifold component (say diffeomorphic to $S^4/\Gamma$, where $\Gamma$ is a finite subgroup of $O(5)$ such that $S^4/\Gamma$ has at most isolated singularities,) which is removed in our surgery process but is not covered by $\varepsilon_0$-necks and $\varepsilon_0$-caps must have sectional curvature bounded from below by some positive constant. By the orbifold Myers theorem (cf. for example [Lu]) the diameters of these components are uniformly bounded above, which combined with the non-collapsing property and the boundedness of sectional curvature gives the desired finiteness of isomorphism classes of $\Gamma$ (cf. [B]).

**Proof of Theorem 1.1.** Let $Y_1, Y_2, \cdots$, be the summands appearing in the orbifold connected sum decomposition of $X$ given in Theorem 2.2. Similarly as in the proof of Main Theorem in [CTZ], we can divide the orbifold connected sum procedure (which recovers $X$ from $Y_1, Y_2, \cdots$) into two steps. The first step is to resolve (by orbifold connected sums) all singularities of $Y_1, Y_2, \cdots$ which are introduced pairwise during the surgery process of the Ricci flow, and get smooth manifolds, denoted by $Z_1, Z_2, \cdots$. Similarly as in [CTZ] one can show that each of $Z_i$ is diffeomorphic to $S^4$, $\mathbb{RP}^4$, or $S^3 \times \mathbb{R}/G$, where $G$ is a fixed point free discrete subgroup of the isometry group of the round cylinder $S^3 \times \mathbb{R}$. The second step is to perform ordinary connected sums on $Z_1, Z_2, \cdots$. Finally the finiteness result in Theorem 1.1 follows from that in Theorem 2.2. Thus the proof of Theorem 1.1 is complete.
The following lemma is essentially due to Hamilton [H4], and was used implicitly in [CTZ]. As in [CTZ] we will call the normal neck defined in [H4, Section C.2] (which is a local diffeomorphism satisfying certain fine geometric properties) Hamilton’s canonical parametrization.

Lemma A.1 Let \( \varepsilon > 0 \) be sufficiently small. Suppose that \( N \) is an \( \varepsilon \)-neck centered at \( x \), with a diffeomorphism \( \psi : \mathbb{S}^3/\Gamma \times (-\varepsilon^{-1}, \varepsilon^{-1}) \to N \), in a Riemannian 4-orbifold \((X, g)\) with at most isolated singularities. Then we have Hamilton’s canonical parametrization \( \Phi : \mathbb{S}^3 \times [-l, l] \to N \), such that \( V \) contains the portion \( \psi(\mathbb{S}^3/\Gamma \times (-0.98\varepsilon^{-1}, 0.98\varepsilon^{-1})) \) in \( N \).

Proof The proof is similar as that of [Hu1, Lemma A.1] (cf. the proof of [H4, Theorem C2.2]).

Now suppose that \( N_i \) is an \( \varepsilon \)-neck centered at \( x_i \), with a diffeomorphism \( \psi_i : \mathbb{S}^3/\Gamma_i \times (-\varepsilon^{-1}, \varepsilon^{-1}) \to N_{i\varepsilon} \), \( i = 1, 2 \), in a Riemannian 4-orbifold \((X, g)\) with at most isolated singularities. Let \( \pi_i : N_i \to (\mathbb{S}^3/\Gamma_i \times (-\varepsilon^{-1}, \varepsilon^{-1}) \to N_{i\varepsilon} \), \( i = 1, 2 \) be the composition of \( \psi_i^{-1} \) with the projection of \( \mathbb{S}^3/\Gamma_i \times (-\varepsilon^{-1}, \varepsilon^{-1}) \) onto its second factor. Assume that \( N_{i\varepsilon} \) contains a point \( y \) with \(-0.9\varepsilon^{-1} \leq \pi_i(y) \leq 0.9\varepsilon^{-1} \) \( (i = 1, 2) \). Then by the above lemma we have Hamilton’s canonical parametrization \( \Phi_i : \mathbb{S}^3 \times [-l_i, l_i] \to V_i \subset N_i \), such that \( V_i \) contains the portion \( \psi_i(\mathbb{S}^3/\Gamma_i \times (-0.98\varepsilon^{-1}, 0.98\varepsilon^{-1})) \) in \( N_i \). If \( \varepsilon \) is sufficiently small, we can use [H4, Theorem C2.4] to combine the parametrizations \( \Phi_1 \) and \( \Phi_2 \), that is, we can get Hamilton’s canonical parametrization \( \Phi : \mathbb{S}^3 \times [-l, l] \to V_1 \cup V_2 \), and diffeomorphisms \( F_1 \) and \( F_2 \) of the cylinders, such that \( \Phi_1 = \Phi \circ F_1 \) and \( \Phi_2 = \Phi \circ F_2 \). \( F_1 \) and \( F_2 \) are in fact isometries in the standard metrics on the cylinders by [H4, Lemma C2.1]. Then the groups \( \Gamma_1 \) and \( \Gamma_2 \) are conjugate in \( SO(4) \). Moreover we know that for all \( x \in [-l, l] \) and all \( \beta \in [-l_2, l_2] \), \( \Phi_1(\mathbb{S}^3 \times \{ \alpha \}) \) is isotopic to \( \Phi_2(\mathbb{S}^3 \times \{ \beta \}) \).

Let \( K_{st} \) be the supremum of the sectional curvatures of the (4-dimensional) smooth standard solution on \([0, 4/3] \). The following lemma extends [Hu1, Lemma A.2]; compare [BBB⁺, Lemma 4.3.5] and [BBM, Lemma 4.5].

Lemma A.2 For any \( \varepsilon \in (0, 10^{-4}) \) there exists \( \beta = \beta(\varepsilon) \in (0, 1) \) with the following property.

Let \( a, b \) be real numbers satisfying \( a < b < 0 \) and \( |b| \leq \frac{1}{2} \), let \((X(\cdot), g(\cdot))\) be a surgical solution defined on \((a, 0] \), and \( x \in X \) be a point such that:
(i) \( R(x, b) = 1 \);
(ii) \( (x, b) \) is the center of a strong \( \beta \varepsilon \)-neck;
(iii) \( P(x, b, (\beta \varepsilon)^{-1}, |b|) \) is unscathed and satisfies \( |Rm| \leq 2K_{st} \).
Then \( (x, 0) \) is the center of a strong \( \varepsilon \)-neck.

Proof We argue by contradiction. Otherwise there exist \( \varepsilon \in (0, 10^{-4}) \), a sequence \( \beta_k \to 0 \), sequences \( a_k < b_k \), \( b_k \in \mathbb{S}^3/\Gamma_k \), and a sequence of surgical solution \((X_k(t), g_k(t)) \) \( (t \in [a_k, 0]) \) with a point \( x_k \in X_k \) such that:
(i) \( R(x_k, b_k) = 1 \);
(ii) \( (x_k, b_k) \) is center of a strong \( \beta_k \varepsilon \)-neck \( N_k \).
(iii) $P(x_k, b_k, (\beta_k \varepsilon)^{-1}, b_k)$ is unscathed and satisfies $|Rm| \leq 2K_{st}$, but
(iv) $(x_k, 0)$ is not the center of any strong $\varepsilon$-neck.

For each $k$, let $\Phi_k : S^3 \times [-L_k, L_k] \to V_k \subset N_k$ be Hamilton’s canonical parametrization such that $V_k$ contains the portion $\psi_k(S^3/\Gamma_k \times (-0.98(\beta_k \varepsilon)^{-1}, 0.98(\beta_k \varepsilon)^{-1}))$, where $\psi_k$ is the diffeomorphism which defines $N_k$. Then we pull back $(X_k(t), g_k(t))$ ($t \in [a_k, 0]$) to $S^3 \times [-L_k, L_k]$ via $\Phi_k$. Now we can proceed as in [BBB],Lemma 4.3.5.

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School of Mathematical Sciences, Key Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing 100875, P. R. China

E-mail address: hhuang@bnu.edu.cn