Lusternik-Schnirelmann category of non-simply connected compact simple Lie groups

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Abstract

Let $F \hookrightarrow X \rightarrow B$ be a fibre bundle with structure group $G$, where $B$ is $(d-1)$-connected and of finite dimension, $d \geq 1$. We prove that the strong L-S category of $X$ is less than or equal to $m + \dim B$, if $F$ has a cone decomposition of length $m$ under a compatibility condition with the action of $G$ on $F$. This gives a consistent prospect to determine the L-S category of non-simply connected Lie groups. For example, we obtain $\text{cat} (\text{PU}(n)) \leq 3(n-1)$ for all $n \geq 1$, which might be best possible, since we have $\text{cat} (\text{PU}(p^r)) = 3(p^r-1)$ for any prime $p$ and $r \geq 1$. Similarly, we obtain the L-S category of $\text{SO}(n)$ for $n \leq 9$ and $\text{PO}(8)$. We remark that all the above Lie groups satisfy the Ganea conjecture on L-S category.

Key words: Lusternik-Schnirelmann category; cone decomposition; Lie group; Ganea conjecture

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1 Introduction

The Lusternik-Schnirelmann category $\text{cat}(X)$, L-S category for short, is the least integer $m$ such that there is a covering of $X$ by $(m+1)$ open subsets each of which is contractible in $X$.

Ganea [5] introduced a stronger notion of L-S category, $\text{Cat}(X)$, which is equal to the cone-length, that is, the least integer $m$ such that there is a set of cofibre sequences $\{A_i \to X_{i-1} \hookrightarrow X_i\}_{1 \leq i \leq m}$ with $X_0 = \{\ast\}$ and $X_m$ homotopy equivalent to $X$.

The weak L-S category $\text{wcat}(X)$ is the least integer $m$ such that the reduced diagonal map $\Delta^{m+1} : X \to \wedge^{m+1}X$ is trivial where $\wedge^{m+1}X$ is the smash product. The stabilised version of the invariant $\text{wcat}(X)$ is given as the least integer $m$ such that the reduced diagonal map $\Delta^{m+1} : X \to \wedge^{m+1}X$ is stably trivial. Let us denote it by $\text{cup}(X)$, the cup-length of $X$.

In 1971, Ganea [6] posed 15 problems on L-S category and its related topics: Computation of L-S category for various manifolds is given as the first problem and the second problem is known as the Ganea conjecture on L-S category. These problems especially the first two problems have attracted many authors such as James and Singhof [15], [28], [25], [26], [27], [16], Gómez-Larrañaga and González-Acuna [7], Montejano [18], Oprea and Rudyak [20], [21], [19] and the authors [10], [11], [12], [13], [14]. In [11,12], the first author gave a counter example as a manifold to the Ganea conjecture on L-S category.

Especially for L-S category of compact connected simple Lie groups, the followings have already been known:

\[
\begin{align*}
\text{cat}(\text{Sp}(1)) &= \text{cat}(\text{SU}(2)) = \text{cat}(\text{Spin}(3)) = 1, \\
\text{cat}(\text{SU}(3)) &= 2, \\
\text{cat}(\text{SO}(3)) &= 3,
\end{align*}
\]

since $\text{Sp}(1) = \text{SU}(2) = \text{Spin}(3) = S^3$, $\text{SU}(3) = \Sigma \mathbb{C}P^2 \cup e^8$ and $\text{SO}(3) = \mathbb{R}P^3$. Schweitzer [24] showed

\[
\text{cat}(\text{Sp}(2)) = 3
\]

using functional cohomology operations. Singhof [25,27] showed

\[
\begin{align*}
\text{cat}(\text{SU}(n)) &= n-1, \\
\text{cat}(\text{Sp}(n)) &\geq n + 1, \quad \text{if} \ n \geq 2.
\end{align*}
\]

Also we know

\[
\text{cat}(G_2) = 4
\]

by [15] (see [13]). James and Singhof [16] showed

\[
\text{cat}(\text{SO}(5)) = 8.
\]
The first and second authors [13] and Fernández-Suárez, Gómez-Tato, Strom and Tanré [4] proved
\[
\cat(\text{Sp}(3)) = 5, \\
\cat(\text{Sp}(n)) \geq n + 2 \quad \text{if } n \geq 3,
\]
by showing the reduced diagonal \(\bar{\Delta}^5\) is given by the Toda bracket \(\{\eta, \nu, \eta\} = \nu^2\). The authors [14] showed
\[
\cat(\text{Spin}(7)) = 5, \quad \cat(\text{Spin}(8)) = 6
\]
using explicit cone decompositions of \(\text{Spin}(7)\) and \(\text{SU}(4)\). Then the Ganea conjecture on L-S category holds for all these Lie groups, since the L-S and the strong L-S categories are equal to the cup-length:

**Fact 1.1** If \(\cat(X) = \cupcat X\), then the Ganea conjecture on L-S category holds for \(X\), i.e., \(\cat(X \times S^n) = \cat(X) + 1\) for all \(n \geq 1\).

In fact, we have \(\cupcat(X \times S^n) = \cupcat(X) + 1\) in general.

For any multiplicative cohomology theory \(h\), we define \(\cupcat(X; h)\), the cup-length with respect to \(h\), by the least integer \(m\) such that \(u_0 \cdots u_m = 0\) for any \(m+1\) elements \(u_i \in \tilde{h}^*(X)\). When \(h\) is the ordinary cohomology theory with coefficient ring \(R\), \(\cupcat(X; h)\) is often denoted as \(\cupcat(X; R)\).

**Theorem 1.2** For any CW-complex \(X\) we have
\[
\cupcat(X) = \max\{\cupcat(X; h) \mid h \text{ is any multiplicative cohomology theory}\}.
\]

*Proof.* It is easy to see that \(\cupcat(X) \geq \cupcat(X; h)\), and hence we have \(\cupcat(X) \geq \max\{\cupcat(X; h) \mid h \text{ is any multiplicative cohomology theory}\}\). Thus we must show
\[
\cupcat(X) \leq \max\{\cupcat(X; h) \mid h \text{ is any multiplicative cohomology theory}\}.
\]

Let \(m = \max\{\cupcat(X; h) \mid h \text{ is any multiplicative cohomology theory}\}\) and \(h_X\) be the multiplicative cohomology theory represented by the following wedge sum of iterated smash products of suspension spectrum \(\Sigma^\infty X\):
\[
S^0 \lor \Sigma^\infty X \lor \Sigma^\infty \Lambda^2 X \lor \cdots \lor \Sigma^\infty \Lambda^i X \lor \cdots.
\]
Let \(\iota \in \tilde{h}^*_X(X)\) be the element which is represented by the inclusion map into the second factor \(\Sigma^\infty X\) of the above wedge sum. Then by the definition of the cup-length, we have \(\iota^{m+1} = 0\) which is represented by the reduced diagonal map \(\bar{\Delta}^{m+1} : X \to \Lambda^{m+1} X\) in the \((m+2)\)-nd factor \(\Sigma^\infty \Lambda^{m+1} X\) of the above wedge sum. Hence we have \(\cupcat(X) \leq m\) the desired inequality. Thus we obtain the result. \(\square\)
Let $P^m(\Omega X)$ be the $m$-th projective space, in the sense of Stasheff [29], such that there is a homotopy equivalence $P^\infty(\Omega X) \simeq X$. The following theorem is obtained by Ganea (see also [10] and Sakai [23]).

**Theorem 1.3 (Ganea [5])**\[\text{cat} (X) \leq m\text{ if and only if there is a map } \sigma : X \to P^m(\Omega X)\text{ such that } e^X_m \circ \sigma \sim 1_X, \text{ where } e^X_m : P^m(\Omega X) \hookrightarrow P^\infty(\Omega X) \simeq X.\]

Using this, Rudyak [21,22] introduced a stable L-S category, $r \text{cat} (X)$, which is the least integer $m$ such that there is a stable map $\sigma : X \to P^m(\Omega X)$ satisfying $e^X_m \circ \sigma \sim 1_X$, another stabilised version of L-S category.

Rudyak [20] [21] and Strom [30] introduced the following invariant to calculate $r \text{cat} (X)$: Let $\text{wgt}(X; h)$ be the least integer $m$ such that the homomorphism $(e^X_m)^* : h^*(X) \to h^*(P^m(\Omega X))$ is injective for any cohomology theory $h$. When $h$ is the ordinary cohomology theory with coefficient ring $R$, $\text{wgt}(X; h)$ is often denoted as $\text{wgt}(X; R)$.

Since a product of any $m+1$ elements of $h^*(P^m(\Omega X))$ is trivial, we have $\text{cup}(X; h) \leq \text{wgt}(X; h)$ for any multiplicative cohomology theory $h$. Hence we have $\text{cup}(X) \leq \text{wgt}(X)$, where we denote $\text{wgt}(X) = \max\{\text{wgt}(X; h) \mid h \text{ is any cohomology theory}\}$.

**Remark 1.4** For any ring $R$, we know $\text{cup}(\text{Sp}(2); R) = \text{wgt}(\text{Sp}(2); R) = 2 < 3 = \text{cat} (\text{Sp}(2))$. But an easy calculation of algebra structure of $KO^*(\text{Sp}(2))$ yields $\text{cup}(\text{Sp}(2); KO) = \text{wgt}(\text{Sp}(2); KO) = 3 = \text{cat} (\text{Sp}(2))$.

The following theorem is due to Rudyak [21,22], although we do not know the precise relation between $w \text{cat} (X)$ and $r \text{cat} (X)$.

**Theorem 1.5** For any CW complex $X$, we have

$$r \text{cat} (X) = \text{wgt } X$$

and hence we have the following relations among categories:

$$\text{cup}(X) \leq w \text{cat} (X), r \text{cat} (X) \leq \text{cat} (X) \leq \text{Cat} (X).$$

Using this stabilised version of L-S category, we have the following theorem.

**Theorem 1.6 (Rudyak [21,22])** If $\text{cat} (X) = r \text{cat} (X)$, then the Ganea conjecture on L-S category holds for $X$.

In fact, we have $r \text{cat} (X \times S^n) = r \text{cat} (X)+1$ in general ([21,22]).
2 Main results

From now on, we work in the category of connected CW-complexes and continuous maps. We denote by $Z^{(k)}$ the $k$-skeleton of a CW complex $Z$.

**Theorem 2.1 (James [15], Ganea [5])** Let $X$ be a $(d-1)$-connected space of finite dimension. Then $cat(X) \leq \text{Cat}(X) \leq \left\lfloor \frac{\dim(X)}{d} \right\rfloor$, where $[a]$ denotes the biggest integer $\leq a$.

In this paper, we extend this for a total space of a fibre bundle, to determine L-S categories of $SO(n)$ for $n \leq 9$, $PO(8)$ and $PU(p^r)$ (and the other quotient groups of $SU(p^r)$), which also gives an alternative proof of a result due to James and Singhof [16] on $SO(5)$.

We assume that $B$ is a $(d-1)$-connected finite dimensional CW complex ($d \geq 1$), whose cells are concentrated in dimensions $0, 1, \cdots, s \mod d$ for some $s$, $(0 \leq s \leq d-1)$. Let $F \hookrightarrow X \to B$ be a fibre bundle with structure group $G$, a compact Lie group. Then we have the associated principal bundle $G \hookrightarrow E \xrightarrow{\pi} B$ with $G$-action $\psi : G \times F \to F$ on $F$ and hence $X = E \times_G F$.

Let $K_i \xrightarrow{\partial} F_{i-1} \hookrightarrow F_i$, $(1 \leq i \leq m)$ be $m$ cofibre sequences with $F_0 = \{\star\}$ and $F_m$ homotopy equivalent to $F$. We consider the following compatibility condition of the above cone decomposition of $F$ and the action of $G$ on $F$.

**Assumption 1** $\psi|_{G^{(d(i+1)+s-1)} \times F_j} : G^{(d(i+1)+s-1)} \times F_j \to F$ is compressible into $F_{i+j}$, $0 \leq i, j \leq i+j \leq m$.

**Remark 2.2** (1) Let $F = G$ and $X = E$ be the total space of a principal bundle over a path-connected space $B$ and $d = 1$. Then any cone decomposition of $F$ such that $F_i = F^{(n_i)}$ with $0 < n_1 < n_2 < \cdots < n_m = \dim(F)$ satisfies Assumption 1 with $s = 0$.

(2) Let $F \hookrightarrow X \to B$ be a trivial bundle. Then any cone decomposition of $F$ satisfies the compatibility Assumption 1 with $s = d-1$.

Our main result is stated as follows:

**Theorem 2.3** Let $B$ be a $(d-1)$-connected finite dimensional CW complex ($d \geq 1$), whose cells are concentrated in dimensions $0, 1, \cdots, s \mod d$ for some $s$, $0 \leq s \leq d-1$. Let $F \hookrightarrow X \to B$ be a fibre bundle with fibre $F$ whose structure group is a compact Lie group $G$. If $F$ has a cone decomposition with the compatibility Assumption 1 for $d$, then $cat(X) \leq m + \left\lceil \frac{\dim B}{d} \right\rceil$.

**Corollary 2.4** If $F$ has a cone decomposition with the compatibility Assumption 1 for $s = d-1$ and also $m = \text{Cat}(F)$, then $cat(X) \leq \text{Cat}(F) + \left\lceil \frac{\dim B}{d} \right\rceil$.  


Remark 2.5 Without Assumption 1, we only have

$$\text{Cat}(X)+1 \leq (\text{Cat}(F)+1)(\text{Cat}(B)+1)$$

which is obtained immediately from the definition of Cat by Ganea [5] and the corresponding results of Varadarajan [31] and Hardie [8] for cat. For example, the principal bundle $\text{Sp}(1) \hookrightarrow \text{Sp}(2) \to S^7$ does satisfy Assumption 1 for $d \leq 3$, but not if $d \geq 4$, and we have $\text{Cat}(\text{Sp}(2)) \leq \text{Cat}(\text{Sp}(1)) + [\frac{7}{4}] = 3 > 2 = \text{Cat}(\text{Sp}(1)) + [\frac{7}{4}]$. In fact by Schweitzer [24], we know $\text{Cat}(\text{Sp}(2)) = 3$.

Remark 2.6 By Remark 2.2 (2), Theorem 2.3 generalises Theorem 2.1.

By applying this, we first obtain the following general result:

Theorem 2.7 Let $C_m < SU(n)$ be a central (cyclic) subgroup of order $m$. Then we have $\text{Cat}(SU(n)/C_m) \leq 3(n-1)$ for all $n \geq 1$.

This might be best possible, because we also obtain the following result.

Theorem 2.8 We have

$$\text{Cat}(SU(p^r)/C_{p^s}) = \text{cat}(SU(p^r)/C_{p^s}) = r\text{cat}(SU(p^r)/C_{p^s}) = 3(p^r-1)$$

where $p$ is a prime and $1 \leq s \leq r$.

Similarly we obtain the following result.

Theorem 2.9 We have

$$\text{Cat}(SO(6)) = \text{cat}(SO(6)) = \text{cup}(SO(6)) = 9,$$
$$\text{Cat}(SO(7)) = \text{cat}(SO(7)) = \text{cup}(SO(7)) = 11,$$
$$\text{Cat}(SO(8)) = \text{cat}(SO(8)) = \text{cup}(SO(8)) = 12,$$
$$\text{Cat}(SO(9)) = \text{cat}(SO(9)) = \text{cup}(SO(9)) = 20,$$
$$\text{Cat}(PO(8)) = \text{cat}(PO(8)) = \text{cup}(PO(8)) = 18.$$

Remark 2.10 Theorem 2.3 also provides an alternative proof for a result of James-Singhof [16], that is, $\text{Cat}(SO(5)) = \text{cat}(SO(5)) = \text{cup}(SO(5)) = 8$ (see Section 4).

We summarise all the known cases in the following table, where each number given in the right hand side of a connected, compact, simple Lie group indicates
its L-S category.

| rank | 1     | 2     | 3     | 4     | \( n (\geq 5) \) |
|------|-------|-------|-------|-------|-----------------|
| \( A_n \) | SU(2) | SU(3) | SU(4) | SU(5) | SU(n+1) |
|       | PU(2) | 3     | 9     | PU(5) | \( n \) |
| \( B_n \) | Spin(3) | 1     | Spin(7) | 5     | Sp(n) |
|       | SO(3) | 8     | 11 | SO(9) | SO(2n+1) |
| \( C_n \) | Sp(1) | 3     | Sp(3) | 5     | Sp(4) |
|       | PSp(1) | 8     | PSp(3) | – | PSp(n) |
| \( D_n \) | 3     | Spin(6) | 3     | Spin(8) | Spin(2n) |
|       | SO(6) | 9     | SO(8) | 12 | SO(2n) |
|       | PO(6) | 9     | PO(8) | 18 | PO(2n) |
|       | Ss(2n) | – | – | – | – |
| Except. types | G2 | 4 | F4 | – | E_{6}, E_{7}, E_{8} |

where "-" indicates the unknown case.

Remark 2.11 We recall that \( A_1 = B_1 = C_1 \), \( B_2 = C_2 \) and \( A_3 = D_3 \), and that the semi-spinor group \( Ss(2n) \) is defined only for \( n \) even.

Taking into account the above table, we get the following by Theorem 1.6:

Corollary 2.12 The Ganea conjecture on L-S category holds for every connected, compact, simple Lie group \( G \) when L-S category is known as above.

The paper is organised as follows: In Section 3 we prove Theorem 2.3. In Section 4 we determine \( \text{cat}(SO(n)) \) for \( n = 5, 6, 7, 8, 9 \) and \( \text{cat}(PO(8)) \). In Section 5 we prove Theorem 2.7 and determine \( \text{cat}(SU(p^r)/C_{p^r}) \).

3 Proof of Theorem 2.3

Let \( B_i \) be the \((d-i+s)\)-skeleton of \( B \) and \( n = \left[ \frac{\dim B}{d} \right] \) the biggest integer not exceeding \( \frac{\dim B}{d} \). Then by Ganea [5], Theorem 2.1 implies that there are \( n \) cofibre sequences \( A_i \xrightarrow{\lambda} B_{i-1} \hookrightarrow B_i, 1 \leq i \leq n \) with \( B_0 = \{\ast\}, B_n = B \). Note
that \( A_i \) is \((d\cdot i-2)\)-connected and of dimension \((d\cdot i+s-1)\). Hence we obtain

\[
B_i = B_{i-1} \cup_{\lambda_i} C(A_i), \quad \lambda_i : A_i \to B_{i-1}
\]

\[
A_i = A_i^{(d\cdot i+s-1)} = \bigcup_{a=0}^{s} A_i^{(d\cdot i+a-1)}, \quad 1 \leq i \leq n,
\]

\[
B_0 = \{\ast\}, \quad B_n \simeq B.
\]

Then there is a filtration of \( E \) by \( E|_{B_i}, 0 \leq i \leq n \), as follows (see Whitehead [32], for example):

\[
E|_{B_i} = E|_{B_{i-1} \cup A_i} C(A_i) \times G, \quad \Lambda_i : A_i \times G \to E|_{B_{i-1}}, \quad 1 \leq i \leq n,
\]

\[
E|_{B_0} = \{\ast\} \times G, \quad E|_{B_n} \simeq E,
\]

and \( \bar{\lambda}_i = \Lambda_i|_{A_i} : A_i \to E|_{B_{i-1}} \) gives a lift of \( \lambda_i : A_i \to B_{i-1} \). Then by induction on \( i \), we have

\[
E|_{B_i} = \{\ast\} \times G \cup_{\Lambda_1} C(A_1) \times G \cup_{\Lambda_2} \cdots \cup_{\Lambda_i} C(A_i) \times G,
\]

\[
\Lambda_i : A_i \times G \xrightarrow{\bar{\lambda}_i \times 1_G} E|_{B_{i-1}} \times G
\]

\[
= \left( \{\ast\} \times G \cup_{\Lambda_1} C(A_1) \times G \cdots \cup_{\Lambda_{i-1}} C(A_{i-1}) \times G \right) \times G
\]

\[
\xrightarrow{1 \times \mu} \left( \{\ast\} \times G \cup_{\Lambda_1} C(A_1) \times G \cdots \cup_{\Lambda_{i-1}} C(A_{i-1}) \times G \right) \times G = E|_{B_{i-1}},
\]

where \( \mu \) is the multiplication of \( G \). For dimensional reasons, we may regard

\[
\bar{\lambda}_i : (A_i, A_i^{(d\cdot i+a-1)}) \to (E^{(d\cdot i+s-1)}|_{B_{i-1}}, E^{(d\cdot i+a-1)}|_{B_{i-1}}), \quad 0 \leq a \leq s,
\]

and \( \mu(G^{(i)} \times G^{(j)}) \subseteq G^{(i+j)} \) up to homotopy. Then we have the following descriptions for all \( k \geq d\cdot i-1 \) and \( j \geq d-1 \):

\[
E^{(k)}|_{B_i} = \left( \{\ast\} \times G \cup_{\Lambda_1} C(A_1) \times G \cup_{\Lambda_2} \cdots \cup_{\Lambda_i} C(A_i) \times G \right)^{(k)},
\]

\[
= \left( \{\ast\} \times G^{(k)} \cup _{\Lambda_1} \bigcup_{l=0}^{s} (C(A_1^{(d\cdot i+l-1)}) \times G^{(k-d\cdot l)}) \right.
\]

\[
\left. \cdots \cup _{\Lambda_i} \bigcup_{l=0}^{s} (C(A_i^{(d\cdot i+l-1)}) \times G^{(k-d\cdot l)}) \right),
\]

\[
\Lambda_i : A_i^{(d\cdot i+l-1)} \times G^{(j-l)} \xrightarrow{\bar{\lambda}_i \times 1_G^{(j)}} E^{(d\cdot i+l-1)}|_{B_{i-1}} \times G^{(j-l)}
\]

\[
= \left( \{\ast\} \times G^{(d\cdot i+l-1)} \cup _{\Lambda_1} \bigcup_{a=0}^{s} (C(A_1^{(d\cdot i+a-1)}) \times G^{(d\cdot i-1)+\ell-a-1)}) \right)
\]

\[
\left. \cdots \cup _{\Lambda_i} \bigcup_{a=0}^{s} (C(A_i^{(d\cdot i+a-1)}) \times G^{(d\cdot i-1)+a-1)}) \right) \times G^{(j-l)},
\]

\[
\xrightarrow{1 \times \mu} \left( \{\ast\} \times G^{(d\cdot i+j-1)} \cup _{\Lambda_1} \bigcup_{a=0}^{s} (C(A_1^{(d\cdot i+a-1)}) \times G^{(d\cdot i-1)+j-a-1)}) \right)
\]

\[
\left. \cdots \cup _{\Lambda_i} \bigcup_{a=0}^{s} (C(A_i^{(d\cdot i+a-1)}) \times G^{(d\cdot i-1)+a-1)}) \right).
\]
\[
= \left( \{ \ast \} \times G \cup_{A_i} C(A_1) \times G \cup_{A_2} \cdots \cup_{A_{i-1}} C(A_{i-1}) \times G \right)^{(d+i+j-1)} \\
= E^{(d+i+j-1)}|_{B_{i-1}}.
\]

Similarly, we obtain the following filtration \( \{ E'_k \}_{0 \leq k \leq n+m} \) of \( E \times_G F \).

\[
E'_k = \begin{cases} 
F_k \cup \Lambda'_1 C(A_1) \times F_k-1 \cup \Lambda'_2 \cdots \cup \Lambda'_{k} C(A_k) \times F_0, & k \leq n, \\
F_k \cup \Lambda'_1 C(A_1) \times F_k-1 \cup \Lambda'_2 \cdots \cup \Lambda'_n C(A_n) \times F_k-n, & n \leq k,
\end{cases}
\]

\[
\Lambda'_1 : A_i \times F_j \xrightarrow{\lambda_i \times 1_{F_j}} E^{(d+i+s-1)}|_{B_{i-1}} \times F_j
\]

\[
= \left( G^{(d+i+s-1)} \cup_{A_i} \cup_{a=0}^s (C(A_1^{(d+a-1)}) \times G^{(d+i+s-1)} \cup_{a=0}^s (C(A_i^{(d+i+s-1)}) \times G^{(d+i+s-1)} \right) \times F_j \\
\xrightarrow{1 \times \psi} \left( F_{i+j-1} \cup_{A'_1} \cup_{a=0}^s (C(A_1^{(d+a-1)}) \times F_{i+j-2}) \right) \\
\cdots \cup_{A'_{i-1}} \cup_{a=0}^s (C(A_i^{(d+i+s-1)}) \times F_j) \\
= F_{i+j-1} \cup_{A'_1} C(A_1) \times F_{i+j-2} \cdots \cup_{A'_{i-1}} C(A_{i-1}) \times F_j \\
= E'_{i+j-1}|_{B_{i-1}},
\]

since \( \psi(G^{(d+i+s-1)} \times F_j) \subseteq \psi(G^{(d+i+s-1)} \times F_j) \subseteq F_{i+j} \) by Assumption 1.

The above definition of \( \Lambda'_1 \) also determines a map

\[
\psi_{i,j} : E^{(d+i+s-1)}|_{B_i} \times F_j \rightarrow E'_{i+j}|_{B_i}
\]

so that \( \Lambda'_1 = \psi_{i-1,j} \circ (\lambda_i \times 1) \). Let us recall that \( F_j = F_{j-1} \cup_{\rho_j} C(K_j) \) for \( 1 \leq j \leq m \). Then the definition of \( E'_k \) implies

\[
E'_k = \begin{cases} 
E'_{k-1} \cup C(K_k) \cup C(A_1) \times C(K_{k-1}) \cup \cdots \cup_{a=0}^s (C(A_k) \times \{ \ast \} \\
\cdots \cup_{A_{k-1}} C(K_{k-1}) \cup C(A_k) \times \{ \ast \} \\
E'_{k-1} \cup C(K_k) \cup C(A_1) \times C(K_{k-1}) \cup \cdots \cup_{A_{n-1}} C(K_{k-n+1}) \cup C(A_n) \times C(K_{k-n}) \\
\cdots \cup_{A_{k-1}} C(K_{k-1}) \cup C(A_k) \times \{ \ast \} \\
\end{cases}
\]

for \( k \leq n \),

\[
E'_k = \begin{cases} 
E'_{k-1} \cup C(K_k) \cup C(A_1) \times C(K_{k-1}) \cup \cdots \cup_{a=0}^s (C(A_k) \times \{ \ast \} \\
\cdots \cup_{A_{n-1}} C(K_{k-n+1}) \cup C(A_n) \times C(K_{k-n}) \\
\end{cases}
\]

for \( k > n \).

To observe the relation between \( \text{Cat} \left( E'_{k-1} \right) \) and \( \text{Cat} \left( E'_k \right) \), we introduce the following two relative homeomorphisms:

\[
\chi(\rho_j) : (C(K_j), K_j) \rightarrow (F_{j-1} \cup C(K_j), F_{j-1}) \ (= (F_j, F_{j-1}))
\]
\[ \chi(\tilde{\lambda}_i) : (C(A_i), A_i) \to (E^{d+i+s-1}|_{B_i} \cup C(A_i), E^{d+i+s-1}|_{B_{i-1}}) \]

\[
(\subset (E^{d+i+s}|_{B_i} , E^{d+i+s-1}|_{B_{i-1}})).
\]

Then the attaching map of \( C(A_i) \times C(K_j) \) is given by the Whitehead product 
\[ [\chi(\tilde{\lambda}_i), \chi(\rho_j)] : A_i \ast K_j = (C(A_i) \times K_j) \cup (A_i \times C(K_j)) \to E'_{i+j-1} \] defined as follows:

\[
[\chi(\tilde{\lambda}_i), \chi(\rho_j)]|_{C(A_i) \times K_j} : C(A_i) \times K_j \xrightarrow{\chi(\tilde{\lambda}_i) \times 1} E^{d+i+s}|_{B_i} \times F_{j-1}
\]

\[
\subseteq E^{d+(i+1)+s-1}|_{B_i} \times F_{j-1} \xrightarrow{\psi_{i,j-1}} E'_{i+j-1} |_{B_i} \subseteq E'_{i+j-1},
\]

\[
[\chi(\tilde{\lambda}_i), \chi(\rho_j)]|_{A_i \times C(K_j)} : A_i \times C(K_j) \xrightarrow{\tilde{\lambda}_i \times \chi(\rho_j)} E^{d+i+s-1}|_{B_{i-1}} \times F_j
\]

\[
\xrightarrow{\psi_{i-1,j}} E'_{i+j-1} |_{B_{i-1}} \subseteq E'_{i+j-1}.
\]

This implies immediately that \( \text{Cat} (E'_k) \leq \text{Cat} (E'_{k-1}) + 1 \). Then by induction on \( k \), we obtain that \( \text{Cat} (E'_k) \leq k \). Thus we have \( \text{Cat} (X) = \text{Cat} (E \times_G F) = \text{Cat} (E'_{m+n}) \leq m+n \leq m+\frac{\dim B}{a} \). This completes the proof of Theorem 2.3.

### 4 Proof of Theorem 2.9

As is well known, we have the following principal bundles (see for example [2], [34] and [9] in particular for the last fibration):

- \( \text{Sp}(1) \to \text{Sp}(2) \to S^7 \),
- \( \text{SU}(3) \to \text{SU}(4) \to S^7 \),
- \( G_2 \to \text{Spin}(7) \to S^7 \),
- \( \text{Spin}(7) \to \text{Spin}(9) \to S^{15} \),
- \( G_2 \to \text{Spin}(8) \to S^7 \times S^7 \).

Each scalar matrix \((-1) \in \text{Sp}(2)\) and \((-1) \in \text{SU}(4)\) acts on \( S^7 \) as the antipodal map, and so does the center of \( \text{Spin}(7)\). Similarly the center of \( \text{Spin}(9)\) acts on \( S^{15} \) as the antipodal map. Recall that the center of \( \text{Spin}(8) \) is isomorphic to \( \mathbb{Z}/2 \times \mathbb{Z}/2 \), each generator of which acts on \( S^7 \) as the antipodal map respectively. Since there are isomorphisms \( \text{Sp}(2) \cong \text{Spin}(5) \) and \( \text{SU}(4) \cong \text{Spin}(6) \), we obtain principal bundles:

- \( \text{Sp}(1) \to \text{SO}(5) \to \mathbb{R}P^7 \),
- \( \text{SU}(3) \to \text{SO}(6) \to \mathbb{R}P^7 \),
- \( G_2 \to \text{SO}(7) \to \mathbb{R}P^7 \),
- \( \text{Spin}(7) \to \text{SO}(9) \to \mathbb{R}P^{15} \),
- \( G_2 \to \text{PO}(8) \to \mathbb{R}P^7 \times \mathbb{R}P^7 \).
Cone decompositions of the fibres except Spin(7) are given as follows (see Theorem 2.1 of [13] for $G_2$):

\[
\begin{align*}
\ast \subset & \text{Sp}(1) = S^3, \\
\ast \subset & SU(3)^{(5)} \subset SU(3), \\
\ast \subset & G_2^{(5)} \subset G_2^{(8)} \subset G_2^{(11)} \subset G_2,
\end{align*}
\]

where $SU(3)^{(5)} = G_2^{(5)} = \Sigma \mathbb{CP}^2$, $SU(3) = SU(3)^{(5)} \cup CS^7$, $G_2^{(8)} \simeq G_2^{(5)} \cup C(S^5 \cup e^7)$, $G_2^{(11)} \simeq G_2^{(8)} \cup C(S^8 \cup e^{10})$ and $G_2 = G_2^{(11)} \cup CS^{13}$. Since these fibres satisfy the conditions in Remark 2.2 (1), we obtain $\text{Cat} (SO(5)) \leq 8$, $\text{Cat} (SO(6)) \leq 9$, $\text{Cat} (SO(7)) \leq 11$ and $\text{Cat} (PO(8)) \leq 18$ using Theorem 2.3. By virtue of the mod 2 cup-lengths we have that $\text{cup}(SO(5)) \geq 8$, $\text{cup}(SO(6)) \geq 9$, $\text{cup}(SO(7)) \geq 11$ and $\text{cup}(PO(8)) \geq 18$ respectively. Thus we obtain the results for $SO(5)$, $SO(6)$, $SO(7)$ and $PO(8)$.

A cone decomposition of Spin(7) is given as follows in [14]:

\[
\ast = F_0 \subset F_1 \subset F_2 \subset F_3 \subset F_4 \subset F_5 = \text{Spin(7)},
\]

where $F_1 = SU(4)^{(7)}$, $F_2 = SU(4)^{(12)} \cup e^6$, $F_3 = SU(4) \cup e^6 \cup e^9 \cup e^{11} \cup e^{13}$ and $F_4 = \text{Spin(7)}^{(18)}$. We need here to check if the filtration satisfies Assumption 1; the only problem is to determine whether $\psi_{|\text{Spin(7)}^{(3)} \times F_1} : \text{Spin(7)}^{(3)} \times F_1 \to F$ is compressible into $F_4$ or not. Since Spin(7) and $F_1$ are included in $SU(4) \subset F_4$, we have $\text{Im} (\psi_{|\text{Spin(7)}^{(3)} \times F_1}) \subset F_4$. Then we obtain $\text{Cat} (SO(9)) \leq 20$ using Theorem 2.3. The mod 2 cup-length implies that $\text{cup}(SO(9)) \geq 20$. Thus we obtain the result for $SO(9)$.

Since $SO(8)$ is homeomorphic to $SO(7) \times S^7$, we easily see that

\[\text{Cat} (SO(8)) \leq \text{Cat} (SO(7)) + \text{Cat} (S^7) = 12\]

by Takens [15]. The mod 2 cup-length implies that $\text{cup}(SO(8)) \geq 12$. Thus we obtain the result for $SO(8)$. This completes the proof of Theorem 2.9.

### 5 Proof of Theorems 2.7 and 2.8

Firstly, we show Theorem 2.7. The following principal bundle is well-known:

\[
SU(n-1) \longrightarrow SU(n) \longrightarrow S^{2n-1}.
\]

The central (cyclic) subgroup $C_m$ of $SU(n)$ acts on $S^{2n-1}$ freely and hence we obtain a principal bundle:

\[
SU(n-1) \longrightarrow SU(n)/C_m \longrightarrow L^{2n-1}(m),
\]
where $L^{2n-1}(m)$ is a lens space of dimension $2n-1$.

A cone decomposition of $SU(n-1)$ is constructed by Kadsza [17]:

$$\star \subset V \subset V^2 \subset \cdots \subset V^{n-2} = SU(n-1),$$

where $V^k \subseteq SU(n-1)$ is a representing subspace of the quotient module $H^*(SU(n-1))/D^{k+1}$ and $D^{k+1}$ is the submodule generated by products of $k+1$ elements in positive degrees, which satisfies $V^i \cdot V^j \subseteq V^{i+j}$ for any $i$ and $j$. Thus $V$ is the subcomplex $S^3 \cup e^5 \cup e^7 \cup \cdots \cup e^{2n-3}$ of $SU(n-1)$ which is homeomorphic to $\Sigma \mathbb{C}P^{n-2}$ (see [33], for example). Then Assumption 1 is automatically satisfied, and hence using $SU(n-1)^{(k)} \subset V^k$, we obtain

$$\text{Cat} (SU(n)/C_m) \leq 3(n-1)$$

by Theorem 2.3. This completes the proof of Theorem 2.7.

Secondly, we show Theorem 2.8. By Rudyak [20] [21] and Strom [30], we know the following proposition.

**Proposition 5.1** (Rudyak [20] [21], Strom [30]) Let $h$ be a cohomology theory. For an element $u \in \check{h}^*(X)$, let $\text{wgt}(u; h)$ be the minimal number $k$ such that $(e^h_k)^*(u) \neq 0$ where $e^h_k : P^k \Omega X \to P^\infty \Omega X \simeq X$, which satisfies

1. We have $\text{wgt}(0; h) = \infty$ and $\infty > \text{wgt}(u; h) \geq 1$ for any $u \neq 0$ in $\check{h}^*(X)$.
2. For any cohomology theory $h$, we have
   $$\min \{\text{wgt}(u; h), \text{wgt}(v; h)\} \leq \text{wgt}(u + v; h).$$
3. For any multiplicative cohomology theory $h$, we have
   $$\text{wgt}(u; h) + \text{wgt}(v; h) \leq \text{wgt}(u \cdot v; h).$$
4. $\text{wgt}(X; h) = \max\{\text{wgt}(u; h) \mid u \in \check{h}^*(X), u \neq 0\}$.

Let us recall that, for any compact Lie group $G$, the ordinary cohomology of $\Omega G$ is concentrated in even degrees. Then, for any element $u$ of even degree in $\check{h}^*(G; \mathbb{Z}/p)$, we have $\text{wgt}(u; H\mathbb{Z}/p) \geq 2$, since $P^1(\Omega G) = \Sigma \Omega(G)$.

The cohomology rings of $SU(p^r)/C_{p^s}$ for a prime $p$ and $1 \leq s \leq r$ are given as follows (see [3]):

$$H^*(SU(p^r)/C_{p^s}; \mathbb{Z}/p) = \mathbb{Z}/p[x_2]/(x_2^{p^r}) \otimes \Lambda(x_1, x_3, \ldots, x_{2p^r-3}).$$

Note that $x_1^2 = x_2$ if $p = 2$ and $s = 1$. Then, using Proposition 5.1, we obtain

$$\text{wgt}(SU(p^r)/C_{p^s}; H\mathbb{Z}/p) \geq \text{wgt}(x_1 \cdot x_2^{p^r-1} \cdot x_3 \cdots x_{2p^r-3}; H\mathbb{Z}/p) \geq 3(p^r - 1),$$

since $\text{wgt}(x_2; H\mathbb{Z}/p) \geq 2$. Thus we have the following lemma.
Lemma 5.2 \( r\text{cat}(\text{SU}(p^r)/C_{p^s}) \geq 3(p^r - 1) \) for any prime \( p \) and \( 1 \leq s \leq r \).

By using Theorem 2.7, we obtain Theorem 2.8.

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