UNIVERSAL ADAPTIVE ESTIMATIONS AND CONFIDENCE INTERVALS IN THE NONPARAMERIC STATISTICS.

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\textbf{Abstract.} The paper considers so-called adaptive estimations of regression, distribution density and spectral density of a Gaussian stationary sequence, asymptotically optimal in order at a growing number of observation on any regular subspace compactly embedded in space $L_2$, and confidence intervals, also adaptive, are constructed on their basis for the estimated functions in an integral norm.

\textit{Key words:} Adaptive estimations, regression, spectrum, Fourier series, entropy, Banach spaces of random variables, confidence interval

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\section{Introduction.}

Among the latest fashions in nonparametric statistics are the so-called adaptive estimations (AE), i.e. estimations that use no a priori information about the estimated function. Many publications have recently appeared where AE are constructed which are optimal in order at a growing number of current observations on a continuum of various functional classes (cf. References for a list of works on AE, which does not, however, claim to be exhaustive).

In (Polyak B., at al., 1990), (Polyak B. at al., 1992), (Golubev G. at al., 1992) for instance, AE were constructed for the problem of estimating regression (R) which are optimal in order on many subspaces of space $L_2$, and non-adaptive confidence intervals were elaborated on the basis of the obtained estimations for the estimated regression function also in norm $L_2$, which later were somewhat improved in (Golubev at al., 1992).

In (Efroimovich S., 1985) AE were constructed for problem (D) of estimating distribution density, which are optimal on ellipsoids in $L_2$, while in (Golubev, 1994)
AE were constructed for problem (S) of estimating spectral (smooth) density, and so on.

In numerous publications by D. Donoho et al. (Donoho D at al., 1993(1), 1993(2), 1996, 1999(1), 1999(2) ) and in some others AE are constructed (and implemented) which are optimal in order on a number of Besov spaces. In those papers as well as in (Golybev G. at al., 1994), (Nussbaum M., 1985), (Tony Cai at al., 1999), (Lee G., 2003) diverse orthonormalized systems of functions are used to construct AE, such as wavelets, wedgelets, unconditional bases, splines, Demmler - Reinsch bases, Ridgelets (Candes E.J., 2003), (Dette H., 2003) etc.

The recent results about kernel estimations in the considered problems see, for example, (AAD W Van Der Vaart at al., 2003), Allal J., at al., 2003), (Corinne Berzin at al., 2003).

In our present work, as in the previous ones (Ostrovsky E.I., 1996, 1997(1); 1997(2), 1999) we construct and analyze AE on the basis of the classical apparatus of the well known trigonometric approximation theory (Nikolsky S., 1951), (Timan A., 1960), (Bernstein S, 1952, 1954).

The AE proposed herein feature a speed of convergence which is optimal in order on any regular subspace compactly embedded in space $L_2$, the estimations are universal and very simple in form, which significantly facilitates their implementation; finally, we construct exponential adaptive confidence intervals (ACI), i.e. such that the tail of the confidence probability decreases with exponential speed.

To the best of our knowledge, adaptive confidence intervals first appeared in our publications (Ostrovsky E. at al., 1996, 1997, 1999). The precursor for the present paper is perhaps (Ostrovsky E. at al., 1997); in comparison with it we now improve the confidence interval and strengthen the convergence type of random values - instead of convergence by probability we establish convergence with unit probability; (we stipulate here that all convergences of a random values sequence are understood with probability 1 only.)

### 2 Problem statement. Denotations. Conditions.

The following three problems are classical in nonparametric statistics.

**R. The regression problem.** Let $f(x), \ x \in [0,1]$ be an unknown function, Riemann-integrable with a square and measured at points of net $x_i = x_{i,n} = i/n, \ i = 1,2,\ldots,n$ with random independent centered identically distributed errors $\{\xi_i\} : y_i = f(x_i) + \xi_i$. It is required to estimate the function $f(x)$ with the best possible precision from the values $\{y_i\}$.

**D. Estimation of distribution density.** On the basis of a set of independent identically distributed values $\{\xi_i\}, \ \xi_i \in [0,1], \ i = 1,2,\ldots,n$ it is required to estimate their common density $f(x)$ (assumed to exist).
S. Spectral statistics. Let \( \{\xi_i\} \) be a Gaussian stationary centered sequence with spectral density \( f(x) \). The estimation object is \( f(x) \). We assume for convenience that \( x \in [0, 1] \).

It is supposed that all the estimated functions \( f(\cdot) \in L_2[0, 1] \), therefore they are expanded in the norm of this space into a Fourier series in the complete orthonormalized trigonometric system \( \{\varphi_j(\cdot)\} \) on set \([0, 1]\): \( \varphi_1(x) = 1; \)

\[
l > 1 \Rightarrow \varphi_2l(x) = \sqrt{2} \cos(2\pi lx); \quad \varphi_{2l+1}(x) = \sqrt{2} \sin(2\pi lx);\]

\[
f(x) = \sum_{j=1}^{\infty} c_j \varphi_j(x); \quad c_j = \int_0^1 \varphi_j(x) f(x) dx.
\]

Let us set \( \rho(N) = \rho(f, N) = \sum_{j=N+1}^{\infty} c_j^2 \). Evidently \( \lim_{N \to \infty} \rho(N) = 0 \). Let us also assume that only the non-trivial infinite-dimensional case will be considered, when an infinite multitude of Fourier coefficients \( f \) differs from zero, i.e. \( \forall N \geq 1 \Rightarrow \rho(N) > 0 \). Otherwise our estimations will converge with speed \( 1/\sqrt{n} \).

Now let us formulate the exact definition of an asymptotically optimal adaptive estimation (ADE), or, more precisely, a sequence of estimations. Let \( K(\theta), \theta \in \Theta \) be some set of Banach subspaces of space \( L_2[0, 1] \); ( only the case when \( K(\theta) \) are compactly embedded in \( L_2[0, 1] \) is non-trivial.) Set

\[
V(n, \theta) = \inf_{g(\cdot)} \sup_{f \in K(\theta)} E||g(n) - f||^2,
\]

where \( \{g(n)\} \) is any sequence of estimations of \( f \). A sequence of adaptive estimations \( f(n) \) is called asymptotically optimal on the set of classes \( K(\theta) \) if \( \forall \theta \in \Theta \)

\[
\sup_n \sup_{f \in K(\theta)} E||f(n) - f||^2/V(n, \theta) < \infty.
\]

Of course, the quadratic function of losses \( l = l(g(n), f) = ||g(n) - f||^2 \) can be replaced by another loss function, non-negative, monotonically depending on the norm \( ||g(n) - f|| \), so that

\[
\forall \epsilon > 0 \ \exists C(\epsilon) < \infty, \Rightarrow l(z) \leq C(\epsilon) (\exp(z^\epsilon) - 1), \ z \geq 0.
\]

Here is an important example of class \( K(\theta) \). Let \( \theta = \theta(N) \) be an arbitrary monotonically non-increasing numerical sequence such that \( \lim \theta(N) = 0; \ N \to \infty \). Denote

\[
K(\theta) = \{f, \ ||f||^2(\theta) \overset{d}{=} \sup_{N \geq 1} \rho(f, N)/\theta(N) < \infty\}.
\]

Relative to the norm \( ||\cdot||(\theta) \) class \( K(\theta) \) is a Banach space compactly embedded in \( L_2[0, 1] \), while the inverse is also true: any subspace compactly embedded in \( L_2[0, 1] \) is a subspace of some \( K(\theta) \).

The value \( \rho(f, N) \) is known and well studied in the approximation theory. Namely, \( \rho(f, N) = E_{N,2}^2(f) \), where \( E_{N,p}(f) \) is the error of the best approximation of
f by trigonometrical polynomials of power not exceeding \( N \) in \( L_p \) metrics: for 
\[
g : [0, 1] \to \mathbb{R}^1 \text{ we shall denote} \]
\[
\|g\|_p = \left( \int_0^1 |g(x)| \, dx \right)^{1/p}, \quad p \in [1, \infty); \quad \|g\|_\infty = \sup_{x \in [0, 1]} |g(x)|,
\]
and closely connected with module of continuity of the form
\[
\omega_{p, 2}(f^{(k)}, \delta) = \sup_{h: |h| \leq \delta} \|f^{(k)}(x + h) - 2f^{(k)}(x) + f^{(k)}(x - h)\|_p,
\]
(Timan A., 1960, p. 275); arithmetical operations on the arguments of function \( f \) and their derivatives are understood modulo 1 (periodicity).

Everywhere below condition (\( \gamma 1 \)) will be considered fulfilled:
\[
(\gamma 1) : \lim_{N \to \infty} \rho(2N)/\rho(N) \overset{def}{=} \gamma < 1,
\]
sometimes stronger conditions (\( \gamma \)) as well:
\[
(\gamma) : \exists \lim_{N \to \infty} \rho(2N)/\rho(N) \overset{def}{=} \gamma < 1; \quad (2)
\]
\[
(\gamma 0) : \gamma = 0. \quad (3)
\]
It is easy to show that from condition (1) follows
\[
\rho(N) \leq CN^{-2\beta}; \quad 2\beta \overset{def}{=} \log_2(1/\gamma) > 0. \quad (4)
\]
In problem (R) it will be assumed that \( \beta > 1/2 \). There are some grounds to suppose that at \( \beta < 1/2 \) asymptotically optimal AE do not exist in the regression problem; for a similarly stated problem this was proved by Lepsky (Lepsky O., 1990).

Also denote \( \kappa = \max(1, 2\beta) \). Here and below the symbols \( C, C_r \) will denote positive finite constructive constants inessential in this context, \( \asymp \) is the usually symbol, in detail:
\[
A(n) \asymp B(n) \iff C_1 \leq \liminf_{n \to \infty} A(n)/B(n) \leq C_2, \quad \exists C_1, C_2 \in (0, \infty).
\]
the symbol \( A \sim B \) means that in the given concrete passage to the limit \( \lim A/B = 1 \).

**Example 1.** Denote by \( W(C, \alpha, \beta) \) a class of functions \( \{f\} \) such that
\[
\rho(f, N) \sim CN^{-2\beta}(\log N)^\alpha, \quad \exists C, \beta > 0; \alpha = \text{const}.
\]
Also denote \( W(\alpha, \beta) = \cup_{C>0} W(C, \alpha, \beta) \);
\[
W(\beta) = W(0, \beta); \quad W = \cup_{\beta>0} W(\beta).
\]
For the class of functions $W$ condition $(\gamma)$ is fulfilled. It is known from [19, pp. 275, 353] that $f \in W(\alpha, \beta)$ if and only if at $\delta \to 0^+$, $\delta \in (0, 1/2]$

$$\omega_{2,2}(f^{[\beta]}, \delta) \asymp \delta^{[\beta]}|\log \delta|^\alpha/2,$$

$$\forall j \leq [\beta] \Rightarrow f^{(j)}(1 - 0) = f^{(j)}(0 + 0),$$

where $[\beta]$ denotes the integer part of $\beta$ and $\{\beta\} = \beta - [\beta]$. At $\{\beta\} = 0$ function $f^{[\beta]}(x)$ is assumed to be continuous.

Example 2. Let us denote

$$Z(\alpha, \beta) = \{f : \rho(f, N) \sim \alpha \beta^N\}, \alpha > 0, \beta \in (0, 1);$$

and $Z = \bigcup_{\alpha > 0, \beta \in (0, 1)} Z(\alpha, \beta)$. For functions of class $Z$ condition $(\gamma 0)$ is fulfilled. Besides, functions of class $Z$ are analytical [20, p. 129].

Denote for the problems $R, D, S$ respectively at $j < n$ \(\hat{c}_j = (1/n)\times\)

$$(1/n) \sum_{i=1}^{n} y_j \varphi_j(x_i); \quad \hat{c}_j = (1/n) \sum_{i=1}^{n} \varphi_j(\xi_i); \quad \hat{c}_j = \sum_{i=1}^{n-j} \xi_{i+j}/(n - j),$$

$j = 0, 2, 4, \ldots$ and $\hat{c}_j = 0$ other case; and for the regression problem

$$c_j(n) = n^{-1} \sum_{i=1}^{n} \varphi_j(x_i), \quad B(n, N) = \sum_{k=N+1}^{2N} c_k(n)^2 + \Delta_1 N/n;$$

$$\Delta_1 = \sigma^2 = D\xi_i; \text{ for problem D}$$

$$B(n, N) = \sum_{k=N+1}^{2N} c_k^2 + \Delta_2 N/n, \quad \Delta_2 = 1;$$

for the spectral problem

$$B(n, N) = \sum_{k=N+1}^{2N} c_k^2 + \Delta_3 N/n, \quad \Delta_3 = \|f\|^2;$$

and again for all the problems set $B(n) =$

$$\min_{N=1,2,\ldots,[n/3]} B(n, N), \quad N^0 = N^0(n) = \arg\min_{N=1,2,\ldots,[n/3]} B(n, N);$$

$$A(n, N) = \rho(N) + \Delta_s N/n, \quad A(n) = \min_{N=1,2,\ldots,[n/3]} A(n, N),$$

where $s$ is the problem number.

For instance, suppose that $f \in W(C, \alpha, \beta)$, then $A(n) \asymp n^{-2\beta/(2\beta+1)}(\log n)^{\alpha/(2\beta+1)}$, and in case $f \in Z(\alpha, \beta) \Rightarrow A(n) \asymp \log n/n.$
Our notation should not be surprising, as it follows from the Bernstein theorem [20, p.242] and from condition (γ1) that all the introduced functionals \( \{B(n, N)\}, \{B(n)\} \) arising from different problems are mutually \( \asymp \) equivalent. Besides, for the same reasons

\[
A(n, N) \asymp B(n, N); \quad A(n) \asymp B(n).
\]

Let us make another additional assumption with regard to the class of estimated functions \( \{f\} : (v) : \forall v \geq 1, \forall N \in [1, N^0/v] \cup [N^0, v, [n/3]] \Rightarrow
\]

\[
B(n, N) - B(n) \geq C_1(v - 1)^2(1 + C_2|v - 1|)^{-1}B(n).
\]

(At \( v \geq N^0 \) the left interval of (6) is absent, at \( v \geq n/(3N^0) \) right interval of (6) is absent.)

The classes of functions satisfying conditions (γ1) and (v) will be called regular. Classes \( W \) and \( Z \) are regular.

Apart from that it is clear that in the regression problem conditions must be imposed not only on the estimated function, but on the measurement errors \( \xi_i \) too. Two kinds of such conditions will be considered:

\[
(Rk) : \exists k = 2, 3, \ldots, \mu_{2k} \overset{d}{=} E\xi_i^{2k} < \infty;
\]

(the power level) and the exponential level:

\[
(Rq) : \exists q, Q \in (0, \infty), \Rightarrow P(|\xi_i| > x) \leq \exp\left(-\frac{x}{Q^q}\right), x > 0.
\]

The classical projective estimates by N. N. Tchentsov (Tchentsov N.N., 1972, p. 286) will be considered as estimates of the function \( f \):

\[
f(n, N, x) = \sum_{j=1}^{N} \hat{c}_j \varphi_j(x).
\]

Since, as shown by Tchentsov, \( E\|f(n, N, \cdot) - f(\cdot)\|^2 \asymp B(n, N) \), the selection of the number of harmonics \( N \) optimal by order in the sense of \( L_2(\Omega) \times L_2[0,1] \) is given by the expression \( N = N^0(n) \) with the speed of convergence \( f(n, N^0, \cdot) \rightarrow f(\cdot) \) in the above-mentioned sense \( \sqrt{A(n)} \). I. A. Ibragimov and R. Z. Khasminsky (Ibragimov I., Khasminsky R., 1982) proved that no faster convergence exists on regular classes of functions given by the value \( \sqrt{A(n)} \).

However, the value \( \rho(f, N) \) or at least its order at \( N \rightarrow \infty \) are practically unknown as a rule. Below the adaptive estimation of \( f \) will be studied based only on observations \( \{\xi_i\} \) and using no apriory information regarding \( f \), and yet possessing the optimal speed of convergence at apparently weak restrictions. Set
\[ \tau(N) = \tau(n, N) \overset{\text{def}}{=} \sum_{k=N+1}^{2N} \hat{c}_k^2, \quad N(n) \overset{\text{def}}{=} \arg\min_{N \in \{1, \ldots, [n/3]\}} \tau(n, N), \] (7)

\[ \tau^*(n) = \min_{N \in \{1, \ldots, [n/3]\}} \tau(n, N), \]

Our adaptive estimations \( \hat{f} \) in all considering problems have a universal view:

\[ \hat{f} = f(n, N(n), x) = \sum_{j=1}^{N(n)} \hat{c}_j \varphi_j(x). \] (8)

In case of a non-unique number of harmonics \( N(n) \) in (7) we choose the largest. Below the value \( N \) will always be in the set of integers numbers of segment \( 1, 2, \ldots, [n/3] \).

Before proceeding to formulations and proofs let us clarify informally our idea for choosing \( N(n) \). It is easy to find by direct calculation that

\[ \mathbb{E} \tau(n, N) \approx B(n, N), \quad \mathbb{D} \tau(n, N) \approx B(n, N)/n, \] (9)

and therefore

\[ N \to \infty, N/n \to 0 \Rightarrow \sqrt{\mathbb{D} \tau(n, N)}/\mathbb{E} \tau(n, N) \to 0. \]

(In the case of the regression problem the condition \( \beta > 1/2 \) is essential which is common in statistical research (Polyak B. at al., 1990, 1992), (Lepsky O., 1990). It follows from (9) that there are some grounds to assume

\[ \tau(n, N) \overset{a.s.}{\approx} \mathbb{E} \tau(n, N) \approx A(n, N) \]

and therefore

\[ N(n) = \arg\min_{N \leq n/3} \tau(n, N) \sim \arg\min_{N \leq n/3} \mathbb{E} \tau(n, N) = N^0(n). \]

Also note that the number of harmonics \( N(n) \) proposed by us is a random value (!) and that estimation (8) is non-linear by the totality of empirical Fourier coefficients \( \{\hat{c}_j\} \).

In the case of problem S our estimation \( \hat{f} \) is homogeneos of degree 2 as a function of a initial data \( \{\xi_i\} \) but also non-linear.

### 3 Formulation of the main results.

Let us denote

\[ P_f(u) = P \left( B^{-1}(n)||\hat{f} - f||^2 > u \right), \quad u > C, \quad C \in (0, \infty). \]
Theorem R.1(k). If the conditions \((\gamma 1)\) and \(\mu_4 < \infty\) is fulfilled in the regression problem, then
\[
P_f(u) \leq C_1 \mu_4 u^{-1} \log^2(C_2 u), \quad u > c/C_2,
\]
\[
C_1 = \min_{X \in (0, 1/2)} (X(0.5 - X)^2 + X^{-1}) \approx 7.221039 \ldots,
\]
\[
C_2 = \arg\min_{X \in (0, 1/2)} (X(0.5 - X)^2 + X^{-1}) \approx 0.198340 \ldots.
\]
This result was proved in (Bobrov P. at al., 1997), but here the values of the constants have been corrected.

Theorem R.2(k). If the conditions \((\gamma 1)\) and \((Rk)\) for some \(k = 3, 4, \ldots\) is fulfilled in the regression problem, then
\[
P_f(u) \leq 2^2 k^k \mu_2 u^{-k/2}, \quad u > 0.
\]

Theorem R.3(q). In the conditions \((Rq), (\gamma 1), (v)\) in the same problem at \(u > C = 2(1 - \gamma)^{-1}Q\) the following inequality is true:
\[
P_f(u) \leq 5 \exp \left[ -C_1 \frac{N^0(n) (u - C)/Q^{q/(2q+4)}}{\log B(n)} \right].
\]

Theorem D. If in addition to the formulated conditions the boundedness of \(f\) is presumed, then in problem \((D)\) at \(u \geq C = (1 - \gamma)^{-1}\)
\[
P_f(u) \leq 5 \exp \left[ -C_2 \frac{(u - C)N^0(n)}{\log B(n)} \right].
\]

Theorem S. If spectral density \(f(x)\) is bounded and conditions \((\gamma 1), (v)\) are fulfilled, then at \(u \geq C = (1 - \gamma)^{-1}\)
\[
P_f(u) \leq 5 \exp \left[ -C_3 \frac{(u - C)N^0(n)}{\log B(n)} \right].
\]

Theorem (R.k) a.s. If in problem \(R\) condition \((Rk)\) is fulfilled and the series
\[
\sum_{n=1}^{\infty} n^{-k/2} A^{-k/2+2\kappa}(n) < \infty,
\]
converges, then
\[
\lim_{n \to \infty} \frac{\tau^*(n)}{B(n)} = 1, \quad (10)
\]
and if condition \((v)\) is also fulfilled, then
lim_{n \to \infty} N^0(n)/N(n) = 1. \quad (11)

(Recall that the convergence of a r.v. is understood in this paper only with probability 1).

**Theorem** (Rq) a.s. If in the same problem under condition (Rq) for any \( \varepsilon > 0 \) the series

\[
\sum_{\{n: 3A(n) < 1\}} \exp \left( -\varepsilon \frac{(nA(n))^{q/(2q+4)}}{|\log A(n)|} \right) < \infty,
\]

(12) converges, then propositions (10) and (11) hold as well.

**Theorem (D)(S) a.s.** Let for problems (D), (S), besides the above-formulated assumptions, condition (12) also be fulfilled with \( q/(2q+4) \) replaced by 1. Then the factual convergences of (10) and (11) are asserted here as well.

(In comparison with (Bobrov P. at al., 1997) the exponent indices are significantly decreased.)

4 Auxiliary results.

The technical apparatus for the proofs is the theory of so-called \( G(\psi) \)-spaces, i.e. Banach spaces of random values with rapidly diminishing tails of the distributions [16, 23]. For the reader's convenience the necessary information from that theory will be provided here without proof.

A random value \( \eta \) determined, like all the other values in the present paper, on a fixed probability space, belongs to the space \( G(\psi) \), where \( \psi = \psi(m) \) is a function monotonically increasing on the set \( m \in (1, \infty) \) and finite at at least one value \( m > 1 \), if the norm

\[
||\eta||(G(\psi)) \overset{def}{=} \sup_{k \geq 1} |\eta|_k/\psi(k) < \infty;
\]

is finite. If \( \psi(m) = m^{1/q}, q = \text{const} > 0; \) then the corresponding space will be denoted \( G_p; \) \( p = q/(q-1); q = 1 \) \( \Rightarrow p = +\infty \) and the norm in it \( |||\eta|||_p; \) while \( \eta \in G_p \) then and only then, if

\[
\exists C \in (0, \infty), \forall x > 0 \quad P(|\eta| > x) \leq \exp (-Cx^p).
\]

(13)

Now let \( \eta(t), t \in T, - \) be a separable random field, \( T \) an arbitrary set, and \( \sup_{t \in T} |||\eta(t)|||_p \leq 1. \) Introduce a so-called natural metric (more exactly semi-metric) \( d_p(t, s) = |||\eta(t) - \eta(s)|||_p \) and denote by \( N(d_p, \varepsilon) \) the least number \( d_p \) of spheres with radius \( \varepsilon > 0 \) covering the entire set \( T. \) If the so-called entropic integral

\[
J = \int_0^1 (\log N(d_p, \varepsilon))^{1/q} d\varepsilon < \infty
\]
converges, then
\[ ||| \sup_{t \in T} |\eta(t)| ||_p \leq C_1 + C_2 J. \] (14)

A similar result for spaces \( L_k(\Omega) \) was obtained by G. Pizier (Pizier G., 1979 - 1980.) It is asserted that if
1) \( \exists k > 1 \Rightarrow \sup_{t \in T} |\eta(t)|^k \leq 1; \)
2) \( I \overset{df}{=} \int_0^1 N^{1/k}(r_k, \varepsilon) d\varepsilon < \infty, \)

where \( r_k(t, s) = |\eta(t) - \eta(s)|_k \), then
\[ \sup_{t \in T} |\eta(t)|_k \leq C_1 + C_2 I. \] (15)

5 Proofs

The proofs of the theorems referring to different problems are similar. The assertions referring to problem \( R \), which is the most complicated, will be proved below in detail, and after that the changes will be indicated that arise in considering problems \( D \) and \( S \). Some additional notations have to be introduced: for \( f : [0,1] \rightarrow R \) and \( p \geq 2 \) we shall denote
\[ ||f||_{p,d} = \left( \frac{1}{n} \sum_{i=1}^{n} |f(x_i)|^p \right)^{1/p}, \]
while in the case of \( p = 2 \) the index \( p \) of the norm sign will be omitted. Further,
\[ \Phi(N, f, x) = \Phi(N, x) = \Phi(N) = \sum_{j=1}^{N} c_j \varphi_j(x) \]
are partial Fourier sums for the function \( f(x) \),
\[ T(N) = T(N, x) = \Phi(2N, f, x) - \Phi(N, x), \ N \leq n/3. \]

Lemma 1. For all \( p \geq 2 \)
\[ ||T(N)||_p \asymp ||T(N)||_{p,d} \leq C N^{1/2-1/p} \sqrt[3]{\rho(f, N)}. \]

Proof. The first assertion follows from the fact that \( N \leq n/3 \) and from the Bernstein inequality [19, p. 245]. The other uses the Nikolsky inequality (Timan A., 1960, p. 245):
\[ n^{-1} \sum_{i=1}^{n} |T(N, x_i)|^p = ||\Phi(2N) - \Phi(N)||_{p,d} \leq 2^p ||\Phi(2N) - \Phi(N)||_p \leq \]

\[ 6^p N^{p/2-1} \| \Phi(2N) - \Phi(N) \|_p^p \leq 6^p N^{p/2-1} (\rho(N) - \rho(2N))^{p/2} < 6^p N^{p/2-1} \rho^{p/2}(N). \]

**Lemma 2.** Let us consider on the set \( S = \{1, 2, \ldots, n\} \) the metric 
\[ d(N_1, N_2) = |\rho(N_1) - \rho(N_2)| + n^{-1} |N_1 - N_2|. \]
It is asserted that the entropy of the set \( S \) in the metric \( d \), i.e. \( H(S, d, \varepsilon) = \log N(S, d, \varepsilon), \varepsilon \in (0, 1] \) satisfies the inequality 
\[ H(S, d, \varepsilon) \leq C + \kappa |\log \varepsilon|, \kappa = \max(1, 2\beta). \]

**Proof.** Set \( K = C \left[ \varepsilon^{-1/\beta} \right] \) and consider \( S(\varepsilon) - \) the net \( S \) in the metric \( d \) of the form 
\[ S(\varepsilon) = [\{(1, 2, \ldots, K\}] \cup (\cup_j \{[nj\varepsilon/2]\})] \cap S. \]
Calculation of the number of elements in \( S(\varepsilon) \) convinces us of the correctness of the lemma.

The central moment in all the further considerations is the so-called *expansion of the basic functional* \( \tau(n, N) \). In all the three problems under consideration \( \tau(n, N) \) is of the form 
\[ \tau(n, N) = E_\tau(n, N) + 2\Psi_1(N) + \Psi_2(N), \quad E_\tau(n, N) \sim B(n, N), \]
\[ E\Psi(N) = E\Psi_2(N) = 0; \quad \Psi_s(N) = \Psi_s(n, N), \]
where in the case of the regression problem \( \Psi_1(N) = n^{-1} \times \)
\[ \times \sum_{i=1}^n \xi_i T(N, x_i), \quad \Psi_2(N) = n^{-2} \sum_{i,j=1}^n a_{i,j}(n, N)(\xi_i \xi_j - E\xi_i \xi_j), \]
\[ a_{i,j}(n, N) = D_{2N}(x_i, x_j) - D_N(x_i, x_j), \]
\[ D_N(x, y) = \sum_{j=1}^N \varphi_j(x)\varphi_j(y) \] – is the Dirichlet kernel.

It is easy to obtain by direct calculation for problem R (and then for the remaining problems) that 
\[ D\Psi_1(N) \asymp \rho(N)/n, \quad D\Psi_2(N) \asymp N/n. \]

**Lemma 3.** In the regression problem under conditions \((Rk)\) the following inequality holds: 
\[ |\Psi_1(N)|_{2k} \leq C k \mu_{2k}^{1/2k} \sqrt{\rho(N)/n}, \quad k = 2, 3, \ldots. \]  

**Proof.** We shall apply the moments inequalities for the sums of centered independent variables \( \{\varepsilon_i\}, \quad i = 1, 2, \ldots, n \) at (Rosental H., 1970), (Johnson W.B., Shechtman G, Zinn J. at al., 1985): \( p \geq 2 \\Rightarrow \)
\[ | \sum_{i=1}^n \varepsilon_i |_p \leq 3(p/ \log p) \max \left( | \sum_{i=1}^n \varepsilon_i |_2, (\sum_{i=1}^n |\varepsilon_i |_p)^{1/p} \right). \]
Here $\varepsilon_i = \xi_i T(N, x_i)$, $\sum_i = \sum_{i=1}^n$, $p = 2k$. As long as, on the basis lemma 1,

$$\sum_i |T|^p(N, x_i) = n|T(N)||_{p,d}^p \leq C n|T(N)||_{p}^p \leq C n N^{p/2-1} |T(N)||_2^p = C n N^{p/2-1} \rho^{p/2}(N),$$

we obtain the conclusion of lemma 3.

**Lemma 4.** In the same problem and in the same assumptions

$$|\Psi_2(N)|_k \leq C k \sqrt[2k]{\mu_{2k}^{1/k}} \sqrt{N}/n. \quad (17)$$

**Proof.** It is sufficient to prove (17) for even $k$, while for the odd ones it is necessary to consider the moment of order $k+1$ and make use of the Lyapunov inequality. The functional $\Psi_2(N)$ is the quadratic centered form from the random values $\{\xi_i\}, i = 1, 2, \ldots, n$. In order to estimate its $k$-th moment we will estimate its cumulant of the same order. According to [29, p. 101],

$$\frac{\Gamma_k(\Psi_2(N))}{D^{k/2} \Psi_2(N)} \leq C k^k \mu_{2k} \cdot \left( \frac{W_n}{D^{(k-2)/2} \Psi_2(N)} \right)^{k-2}, \quad k \geq 4,$$

where $\Gamma_k(\xi)$ denotes the $k$-th semi-invariant of the value $\xi$, $W_n = \max_i \sum_{j=1}^n |a_{i,j}(n, N)| \leq \max_{y \in [0,1]} \sum_{j=1}^n \sum_{l=N+1}^{2N} \varphi_l(y) \varphi_l(x_i)$.

Analogously to the estimations of the Lebesgue constants in the theory of trigonometrical series we can estimate $W_n \leq C \log N/n$, and consequently at $k \geq 4$

$$\Gamma_k(\Psi_2(N)) \leq C k^k \mu_{2k} \cdot \log N \cdot D^{k/2}(\Psi_2(N)).$$

Proceeding by the well-known Leonov - Shiryaev formulas (Shiryaev A.N., 1989, p.311) from semi-invariants to moments, we obtain the proposition of lemma 4.

**Lemma 5.** Under the conditions of Lemmas 3 and 4 the following inequalities hold respectively:

$$|\tau(n, N) - E\tau(n, N)|_k \leq C k \sqrt[2k]{\mu_{2k}^{1/k}} \sqrt{A(n, N)/n},$$

and if condition $(Rq)$ is fulfilled, then on the basis of the properties of spaces $G(\psi)$

$$|||\tau(n, N) - E\tau(n, N)|_r \leq C \sqrt{A(n, N)/n}, \quad r = q/(q + 2).$$

The assertion of the lemma 5 it follows from the inequality of the triangle for the used norms.

Let us consider the centered and normalized random field

$$\zeta(N) = \zeta(n, N) = \sqrt{n/(A(n, N)} \ [\tau(n, N) - E\tau(n, N)],$$

where
so that \( E\zeta(N) = 0, \sup_{N \leq [n/3]} ||\zeta(N)||_r \leq C. \)

**Lemma 6.**

\[
(R_k) \Rightarrow |\max_N |\zeta(N)|_k \leq CA^{-2k/k}(n), \ k > 3\kappa;
\]

\[
(R_q) \Rightarrow |||\max_N |\zeta(N)||_r \leq C|\log A(n)|^{1/r}.
\]

The **proof** will be given for the second assertion alone, as the first is simpler because the spaces \( L_k(\Omega) \) are more customary. We obtain on the basis of lemma 5, put

\[
\Psi(N) = 2\Psi_1(N) + \Psi_2(N) = \tau(n, N) - E\tau(n, N):
\]

\[
n^{-1/2} (\zeta(N_1) - \zeta(N_2)) = \frac{(\sqrt{A(n, N_1)} - \sqrt{A(n, N_2)}) \Psi(N_1)}{\sqrt{A(n, N_1)A(n, N_2)}} +
\]

\[
[\Psi(N_1) - \Psi(N_2)]/\sqrt{A(n, N_2)} \overset{def}{=} \zeta_1 + \zeta_2;
\]

\[
|||\zeta_2|||_r \leq C\sqrt{|A(n, N_1) - A(n, N_2)|/(A(n)\sqrt{n})} \leq 
\]

\[
C\sqrt{|\rho(N_1) - \rho(N_2)| + n^{-1}|N_1 - N_2|/(A(n)\sqrt{n})},
\]

since \( A(n, N) \geq A(n), a \geq b \geq 0 \Rightarrow \sqrt{a} - \sqrt{b} \leq \sqrt{a - b}, \)

\[
|||\Psi(N_1) - \Psi(N_2)|||_r \leq C\sqrt{|A(n, N_1) - A(n, N_2)|}/n.
\]

Further, \( |||\zeta_1|||_r \leq |||\Psi(N_1)|||_r \times 
\]

\[
\left[\frac{1}{\sqrt{A(n, N_1) - \sqrt{A(n, N_2)}}}\right] \cdot \left[\sqrt{A(n, N_1)A(n, N_2)}\right]^{-1/2} \leq 
\]

\[
C\sqrt{|\rho(N_1) - \rho(N_2)| + n^{-1}|N_1 - N_2|/(A(n)\sqrt{n})},
\]

since \( |||\Psi(N_1)|||_r \leq C/\sqrt{n}. \) The random field \( \zeta(N) \) is thus bounded in the norm \( ||| \cdot |||_r \)- and

\[
d_1(N_1, N_2) \overset{def}{=} |||\zeta(N_1) - \zeta(N_2)|||_r \leq C\sqrt{d(N_1, N_2)/A(n)}.
\]

Since \( H(S, d_1, \varepsilon) \leq H(S, \sqrt{d}/(CA(n)), \varepsilon) = 
\]

\[
H(S, d, (C\varepsilon A(n))^2) \leq C_1 + 2\kappa|\log \varepsilon| + 2\kappa|\log A(n)|.
\]

the assertion of the lemma follows from the properties of the spaces \( G(\psi) \) (13, 14, 15).
Inequalities (16) and (17) can be reformulated as follows in forms more convenient for further application. Under conditions (16) and (17) respectively the sequence \( \{\tau(n, N)\} \) can be expended into
\[
\tau(n, N) = E\tau(n, N) + \sqrt{E\tau(n, N)/n} \cdot \mu_{2k}^{1/k} (A(n))^{-2\kappa/k} \nu(n, N),
\]
where \( E\nu(n, N) = 0 \),
\[
\sup_n E \max_N |\nu(n, N)|^k = C < \infty; \quad (18)
\]
and in the other case \((Rq)\)
\[
\tau(n, N) = E\tau(n, N) + \sqrt{E\tau(n, N)/n} \cdot |\log A(n)|^{1/r} \cdot \nu(n, N),
\]
\( E\nu(n, N) = 0; \quad \sup_n \|\max_N |\nu(n, N)|\|_r = C < \infty. \)

**Lemma 7.** Let \( M \) be some subset of an integral segment \( S = [1, 2, \ldots, n], \overline{M} = S \setminus M, \pi(M) = P(N(n) \in M), \)
\[
v = v(n, M) \overset{\text{def}}{=} \inf_{N \in M} B(n, N)/B(n) \geq 2.
\]
Then under conditions \((Rq)\)
\[
\pi(M) \leq 2 \exp \left( -C (vnB(n))^{r/2}/|\log B(n)| \right)
\]
and under conditions \((Rk)\)
\[
\pi(M) \leq \frac{C^k k^k \mu_{2k}^k}{v^{k/2} n^{k/2} B^{k/2 - \kappa k}}.
\]

**Proof.** We obtain for the case of \((Rq)\), denoting \( \overline{\nu} = \max_{N \in S} |\nu(n, N)| \): \(\pi(M) = P(N(n) \in M) = P(\min_{N \in M} \tau(n, N) > \min_{N \in M} \tau(n, N)) = \)
\[
P \left( \min_{N \in M} (B(n, N) + \sqrt{B(n, N)/n} |\log B(n)|^{1/r} \nu(n, N)) \right) > \min_{N \in M} \left( B(n, N) + \sqrt{B(n, N)/n} |\log B(n)|^{1/r} \nu(n, N) \right) \leq \)
\[
P(B(n) + \sqrt{B(n)/n} |\log B(n)|^{1/r} \overline{\nu}) > vB(n) - \sqrt{vB(n)/n (|\log B(n)|)^{1/r} \overline{\nu}}.
\]
We find solving the inequality under the probability symbol relative to \( \overline{\nu} \) (the case of \( v \geq C/B(n) \) is trivial): \(\pi(M) \leq \)
\[
P(C \overline{\nu} (1 + \sqrt{\overline{\nu}}) \sqrt{B(n)/n} |\log B(n)|^{1/r} \geq (v - 1)B(n)) \leq \]
\[
\frac{C^k k^k \mu_{2k}^k}{v^{k/2} n^{k/2} B^{k/2 - \kappa k}}.
\]
Using the estimations of lemma 6, we arrive at inequalities (19) and (20). The case of \((RK)\) is considered analogously.

**Proof of Theorem (R.k) a.s.** It follows from expansion (18) that

\[
\tau^*(n) \leq B(n) + \sqrt{B(n)/n} C(k) B^{-2k/k(n)} \mathcal{V},
\]

therefore

\[
\left| \frac{\tau^*(n)}{B(n)} - 1 \right| \leq \frac{C(k)}{\sqrt{\mathcal{V}B^{1/2-2k/k(n)}}}.
\]

We receive in accordance to the Chebyshev inequality:

\[
P_{n}(\varepsilon) \overset{df}{=} \mathbb{P} \left( \left| \frac{\tau^*(n)}{B(n)} - 1 \right| > \varepsilon \right) \leq \frac{C(k)}{\varepsilon^{k}n^{k/2}B^{k/2-2k/n}}.
\]

Since for any \(\varepsilon > 0\) the series \(\sum_{n} P_{n}(\varepsilon)\) converges, the first assertion to be proved follows from the Borel - Cantelli lemma. The other is proved analogously if it is taken into account that \(N \geq N^0(n)(1 + \varepsilon), \varepsilon \in (0, 1]\) and condition \((v)\) lead to the inequality \(B(n, N) \geq (1 + C\varepsilon^2)B(n), \varepsilon \in (0, 1)\) and lemma 7 is applied.

Analogously we can prove the theorem \((Rq)\) 'a.s', on the basis of inequality:

\[
P_{n}(\varepsilon) \leq \exp \left( -C\varepsilon^r \frac{(nA(n))\log A(n))}{|\log A(n)|} \right).
\]

**Remark 1.** Let us note, and use it below, a slight difference in the behaviors of the values \(\tau(n, N)\) and \(N(n)\) which consists in the peculiarity of condition \((v)\). At \(v \geq 2\) we have (under the same conditions \((Rq), (v)\) :

\[
\max \left( \mathbb{P} \left( \frac{N(n)}{N^0(n)} \leq \frac{1}{v} \right), \mathbb{P} \left( \frac{N(n)}{N^0(n)} > v \right) \right) \leq \exp \left( -Cu^r \frac{(nA(n))^{r/2}}{|\log A(n)|} \right).
\]

An analogous estimation for the probability \(\mathbb{P}(\tau^*(n)/B(n) > v)\) holds even without condition \((v)\).

**Remark 2.** The consistency of the proposed estimations in the above-mentioned sense follows from the assertions already proved. Indeed, since

\[
A(n) = A(n, \lceil \sqrt{n} \rceil) \leq Cn^{-1/2} + \rho(\lceil \sqrt{n} \rceil) \to 0,
\]

then \(N^0(n) \to \infty, N^0(n)/n \to 0\), because otherwise the value

\[
A(n) = A(n, N_0^0(n)) \approx N^0(n)/n + \rho(N^0)
\]

would not tend to zero.
Using triangular inequality for the G

The other probabilities $P(C) = 1$

We obtain therefore, based on the properties of spaces $A$ and $C$

We find analogously to lemma 4:

\[
|||\hat{\Psi}_3(N) = \frac{1}{n} \sum_{l,s=1, l \neq s} V(\xi_l, \xi_s), V(x, y) = \sum_{j=1}^{N} (\varphi_j(x) - c_j)(\varphi_j(y) - c_j),
\]

and has the same form and the same estimation as $\Psi_2(N)$.

Then we will use the elementary inequality $P(A) \leq P(ABC) + P(B) + P(C)$, in which $A, B, C$ are events. Setting $A = \{||\hat{f} - f||^2/B(n) > u\}$, $B = \{1/v \leq \tau^*(n)/B(n) \leq v\}$, $C = \{N^0/v \leq N(n) \leq vN^0(n)\}$ we have at $v \in (2, u - C)$ :

\[
P_0 \overset{def}{=} P(ABC) \leq P(v + \max_{N \leq vN^0} \Psi_3(N))/B(n) > u).
\]

We find analogously to lemma 4: $|||\Psi_3(N)|||_r \leq C\sqrt{n}/n$, $|||\Psi_3(N) - \Psi_3(N_2)|||_r \leq \sqrt{|N_1 - N_2|}/n$

and since the entropy integral converges, then (see (15))

\[
|||\max_{N \leq vN^0(n)} \Psi_3(N)\| \||_r \leq C\sqrt{vN^0(n)/n} \leq C\sqrt{v}/\sqrt{N^0(n)}.
\]

Using triangular inequality for the $G(\psi)$-norms we obtain:

\[
P \left( \left| \frac{\tau^*(n)}{B(n)} - 1 \right| + \left| \frac{\Psi_3(N)}{B(n)} \right| > v \right) \leq \exp \left( -C_5 \frac{v^r (nA(n))^{r/2}}{|\log A(n)|} \right).
\]

We obtain therefore, based on the properties of spaces $G(\psi)$ :

\[
P_0 \leq \exp \left( -C_6 \left( u^{-1/2} (u - C - v) \sqrt{N^0(n)} \right) \right).
\]

The other probabilities $P(B), P(C)$ were estimated above, and we find by summing ($C = 1/(1 - \gamma)$):

\[
P_f(u) = P(A) \leq \exp \left( - \left( C_1(u - C - v) \sqrt{N^0(n)/v} \right) \right).
\]
+ 4 \exp \left( - C_2 v^{r/2} (nA(n))^{r/2} \right).}

Taking into account that \( nB(n) > N^0 \) and choosing \( v = C_4(u - C), \ C_4 \in (0, 1) \), we arrive at the assertion of the theorem.

We proceed now to the problem of estimating density \( (D) \). The functional \( \Psi(n, N) = \Psi(n, N) \) has in it the following form:

\[
\Psi(N) = n^{-1} \sum_{i=1}^{n} \sum_{j=N+1}^{2N} (c_j \varphi_j(\xi_i) - c_j^2).
\]

Using the Rosenthal inequality once more, we obtain

\[
E(\Psi_1(N))^{2k} \leq 2C(2k) n^{-k} E \left( \sum_{j=N+1}^{2N} c_j \varphi_j(\xi_1) \right)^{2k} \leq
\]

\[
2 n^{-k} C(2k) \int_0^1 \left( \sum_{j=N+1}^{2N} c_j \varphi_j(x) \right)^{2k} f(x) \, dx \leq
\]

\[
C \cdot C(2k)n^{-k} \int_0^1 \left( \sum_{j=1}^{2N} c_j \varphi_j(x) \right)^{2k} \, dx,
\]

since \( f(x) \) is presumed to be bounded. Then, since

\[
\left\| \sum_{j=N+1}^{2N} c_j \varphi_j(x) \right\| = \left\| \Phi(2N, x) - \Phi(N, x) \right\| \to 0, \ N \to \infty,
\]

we have in according to the Riesz theorem (Timan A., 1960, p. 305)

\[
||\Phi(2N) - \Phi(N)||^{2k} \leq C^k k^k ||\Phi(2N) - \Phi(N)||^{2k} =
\]

\[
C^k k^k (\rho(N) - \rho(2N))^k < C^k k^k \rho^k(N),
\]

so that

\[
E\Psi_1^{2k}(N) \leq C^k k^k n^{-k} \rho^k(N). \quad (21)
\]

In the language of \( G(\psi) \) - spaces inequality (21) means that

\[
|||\Psi_1(N)|||_r \leq C\rho(N)/n.
\]

It is proved analogously that

\[
|||\Psi_1(N_1) - \Psi_1(N_2)|||_r \leq C|\rho(N_1) - \rho(N_2)|/n.
\]
The functional $\Psi_2(N) = \Psi_2(n, N)$ has the form

$$\Psi_2(N) = \sum_{l=1}^{n} \sum_{s=1}^{n} U(\xi_l, \xi_s),$$

where

$$U(x, y) = U(N, x, y) = \sum_{j=N+1}^{2N} (\varphi_j(x) - c_j)(\varphi_j(y) - c_j),$$

and is consequently a so-called $U-$ statistic with the kernel $U = U(N, x, y)$. At the same time our $U-$ statistic is singular. The asymptotics of the moments of this kind of statistics and the limiting distribution for them are to be found e.g. in (Korolyuk V.S. et al., 1989), (Ronzin A., 1982). However, here we need non-asymptotic estimations from above, and therefore additional reasoning will be required. Note first of all that

$$E |U(\xi_1, \xi_2)|^m \leq C^m N^{m-1}, \; m = 3, 4, \ldots$$

Let us prove (22).

$$E |U(\xi_1, \xi_2)|^m \leq 4^m C^m \int_0^1 \int_0^1 |D_{2N}(x, y)|^m f(x)f(y)dxdy \leq$$

$$\leq C^m \int_0^1 \int_0^1 |D_{2N}(x, y)|^m dxdy,$$

where, let us recall, $f$ is bounded and $D_N$ is the Dirichlet kernel. The last integral is easily estimated and we arrive at (22). Then on account of the singularity of the statistics we have in the case of even $k$:}

$$n^k E\Psi_2^k(N) \leq \sum_{i_1=1}^{n} \sum_{j_1=1}^{n} \ldots \sum_{i_k=1}^{n} \sum_{j_k=1}^{n} E U(\xi_{i_1}, \xi_{j_1}) \ldots U(\xi_{i_k}, \xi_{j_k}) \leq$$

$$C^k k^k \sum_{i_1=1}^{n} \sum_{j_1=1}^{n} \ldots \sum_{i_{k/2}=1}^{n} \sum_{j_{k/2}=1}^{n} E U^2(\xi_{i_1}, \xi_{j_1}) \ldots E U^2(\xi_{i_{k/2}}, \xi_{j_{k/2}}) \leq$$

$$C^k k^k |U(\xi_1, \xi_2)|^k \leq C^k k^k N^{k/2}.$$

In the case of odd $k$ we consider the moment of order $k + 1$; in [16, p. 42] the equivalence is proved of the norms $G(\psi)$, constructed by even moments alone, to the initial norm.

Thus $|||\Psi_2(N)|||_r \leq C \sqrt{n}/n$, and the further course of reasoning is fully analogous to the ground for estimation of regression.

**Consider now the problem of spectral statistics (S).** It turns out unexpectedly that the reasoning here is even simpler than in problem (D). The fact is that the initial sequence $\{\xi_i\}$ is assumed to be Gaussian, the empirical Fourier coefficients $\hat{c}_k$, i.e. empirical correlation coefficients, are quadratic functionals from
the trajectory \(\{\xi_i\}, \ i = 1, 2, \ldots, n\), while the functional \(\tau(n, N)\) is a polynomial functional of the 4th power and therefore has the expansion

\[
\tau(n, N) = E\tau(n, N) + \sum_{m=1}^{4} \Psi_m(n, N),
\]

where the expansion components are not correlated between themselves and \(\Psi_m\) can be written as an \(m\) - dimensional stochastic Ito-Wiener integral according to the orthogonal Gaussian measure. At the same time

\[
CN/n \geq D\tau(n, N) = \sum_{m=1}^{4} \Psi_m(n, N),
\]

therefore \(D\Psi_m(n, N) \leq CN/n\). The Plikusas theorem (Plikusas A., 1981) asserts that the distribution \(\Psi_m(n, N)\) is estimated only through dispersion:

\[
|||\Psi_m(n, N)|||_{2/m} \leq C(m)D^{1/2}\Psi_m(n, N) \leq C\sqrt{N/n},
\]

consequently \(|||\tau(n, N) - E\tau(n, N)|||_{1/2} \leq C\sqrt{N/n}\). Analogously considering the dispersion of the value

\[
\zeta(N) = \zeta(n, N) = \sqrt{n/A(n, N)} [\tau(n, N) - E\tau(n, N)],
\]

we find that \(D\zeta(N) \leq C/B(n)\) and therefore \(|||\zeta(N)|||_{1/2} \leq C/B(n)\), and the difference \(\zeta(N_1) - \zeta(N_2)\) is estimated likewise:

\[
|||\zeta(N_1) - \zeta(N_2)|||_{1/2} \leq C\sqrt{d(N_1, N_2)/B(n)}.
\]

As a result we obtain for the functional \(\tau(n, N)\) expansion (18), which is of key importance for us:

\[
\tau(n, N) = E\tau(n, N) + \sqrt{E\tau(n, N)/n \cdot \log^2 B(n) \cdot \tau},
\]

\[
\sup_{n} \left\|\max_{N \leq n/3} |\tau| \right\|_{1/2} = C < \infty.
\]

The other details of the proof are analogous to the case of regression and ought to be omitted.

### 6 Adaptive confidence intervals.

Let us now describe the use of our results for the construction of ACI. Note first of all that the probability \(P_f(u)\) with rather weak conditions (except \((R_k)\) ) in all the considered problems permits estimation of the form

\[
P_f(u) \leq 5 \exp \left( -\varphi(C, N_0^0(n), B(n))u^{r/2} \right), \ u > C.
\]

(23)
As proved above, the values $N^0, B(n)$ have respective consistent estimates

$$N^0(n) \approx \arg\min_{N \leq n/3} \tau(n, N), \quad B(n) \approx \min_{N \leq n/3} \tau(n, N) = \tau^*(n).$$

The value $C$ also depends on $\gamma$ and on the constants $C_j$ appearing in the definition of condition $(v)$. With very weak conditions they can also be estimated consistently by the sampling in the following way. Set $M = M(n) = \left[ \exp(\sqrt{\log n}) \right]$; then, if conditions $(\gamma), (v)$ are fulfilled, a system of asymptotic equalities can be written:

$$\tau(M) - \Delta_s M/n \sim (1 - \gamma)\rho(M);$$
$$\tau(2M) - 2\Delta_s M/n \sim \gamma(1 - \gamma)\rho(M);$$
$$\tau(4M) - 4\Delta_s M/n \sim \gamma^2(1 - \gamma)\rho(M).$$

Solving this system, we find the consistent ($\text{mod } P$) estimate of $\gamma$:

$$\hat{\gamma} = \frac{\tau(4M) - 2\tau(2M)}{\tau(2M) - 2\tau(M)}.$$

(The parameter $\Delta_s$ can also be estimated consistently, but that is not necessary for us). Further, since

$$\frac{\tau(n, N(n)(1 + v))}{\tau^*(n)} \sim \frac{B(n, N(n)(1 + v))}{B(n)} \sim \frac{C_1 v^2}{1 + C_2 v}, \quad (24)$$

the constants $C_1, C_2$ can be determined from (24), for instance by the least-squares method. Substituting the obtained estimates of all the parameters into (23), we get the estimate of the confidence probability

$$P_f(u) \leq 5 \exp \left( -\phi(C(\hat{\gamma}, \hat{C_1}, \hat{C_2}), N(n), \tau^*(n)) u^{r/2} \right) \overset{\text{def}}{=} \hat{P}_f(u). \quad (25)$$

then, equating the right-hand part (25) of the unreliability of the confidence interval $\delta$ to, say, the magnitude 0.05 or 0.01, we calculate $u = u(\delta)$ from the relation

$$\hat{P}_f(u(\delta)) = \delta$$

and obtain approximately the adaptive confidence interval for $f$ reliability $1 - \delta$ of the form

$$||\hat{f} - f||^2 \leq u(\delta) \min_{N \leq n/3} \tau(n, N). \quad (26)$$

But for a rough estimate of the error from replacing $f$ by $\hat{f}$ the following quite simple method can be recommended. Since

$$\frac{||\hat{f} - f||^2}{B(n)} = \frac{A(n, N(n))}{B(n)} + \frac{\Psi_3(N(n))}{B(n)}, \quad (27)$$
and the second term in the right-hand part of (27) a.s. tends to zero, while the first term, if conditions \((\gamma), (v)\) are fulfilled, has \(1/(1 - \gamma)\) as its limit, we thus prove the following assertion apparently well known to specialists in nonparametric statistics for non-adaptive estimation:

**Theorem c.i.** If the following conditions are fulfilled in our problems: in problem \(R (Rq), (\gamma), (v)\) or \((\gamma), (v)\) in problems \(D, S\), then

\[
\lim_{n \to \infty} \|\hat{f} - f\|^2 / B(n) < 1/(1 - \gamma).
\]  

(28)

In order to construct an adaptive confidence interval assertion (28) can be reformulated as follows. With probability tending to 1 at \(n \to \infty\)

\[
\|\hat{f} - f\|^2 \leq B(n)/(1 - \gamma),
\]

and ACI is constructed by replacing the values \(B(n), \gamma\) by their consistent estimates:

\[
\|\hat{f} - f\|^2 \leq \tau^*(n) \frac{\tau(2M) - 2\tau(M)}{3\tau(2M) - 2\tau(M) - \tau(4M)}.
\]

A more exact result will be obtained by taking into account the following term of the expansion of the value \(\|\hat{f} - f\|^2\):

\[
\frac{\|\hat{f} - f\|^2}{B(n)} \leq \frac{1}{1 - \gamma} + \frac{\zeta}{\sqrt{N^0(n)}} (1 + \epsilon_n),
\]

where \(\epsilon_n \to 0\). \(\mathbf{P}(|\zeta| > u) \leq 2\exp(-Cu^{r/2})\) and \(C\) no longer depends on \(n\). Equating the probability \(\mathbf{P}(|\zeta| > u)\), more exactly its estimate \(2\exp(-Cu^{r/2})\) to the value \(\delta, \delta \approx 0+\), we will easily find \(u = u(\delta)\) and construct an approximate ACI with reliability \(\approx 1 - \delta\) of the form

\[
\|\hat{f} - f\|^2 \leq \frac{\tau^*(n)}{1 - \gamma} + \tau^*(n) \frac{u(\delta)}{\sqrt{N(n)}}.
\]

Closer consideration reveals an effect that somewhat reduces the exactness of ACI. Let (as is true in all the three considered problems under the formulated assumptions)

\[
\mathbf{P} \left(\|\hat{f} - f\|^2 / B(n) > u\right) \leq \exp(-\phi(C_1 u)),
\]

\[
\mathbf{P} \left(\tau^*(n) / B(n) < 1/u\right) \leq \exp(-\phi(C_2 u)), \quad u > C,
\]

where at \(u \to \infty\) \(\Rightarrow \phi(u) \to 0\). We denote

\[
\mathbf{Q}(u) = \mathbf{P} \left(\|\hat{f} - f\|^2 / \tau^*(n) > u\right).
\]

**Theorem \(\tau\).** At \(u \leq C/B(n)\) the following inequality holds:

\[
\mathbf{Q}(u) \leq 2\exp(-\phi(C\sqrt{u})).
\]

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**Proof.** We have by the full probability formula we (we shall understood $P(A/B)$ as the conditional probabilities, if, of course, $A$ and $B$ are events):

$$Q(u) \leq P\left(\frac{||\hat{f} - f||^2}{\tau^*(n)} > u/\tau^*(n) \right) \cdot P\left(\frac{\tau^*(n)}{B(n)} > \frac{1}{v}\right) +$$

$$+ P\left(\frac{||\hat{f} - f||^2}{\tau^*(n)} > u/\tau^*(n) \right) \cdot P\left(\frac{\tau^*(n)}{B(n)} \leq \frac{1}{v}\right) \overset{def}{=} Q_1 + Q_2;$$

$$Q_1 \leq P\left(||\hat{f} - f||^2/B(n) > u/v\right) \leq \exp(-\phi(C_1 u/v));$$

$$Q_2 \leq P\left(\tau^*(n)/B(n) \leq 1/v\right) \leq \exp(-\phi(C_2 v)) .$$

Summing up and put $v = C_3 \sqrt{u}$, we obtain the assertion of the theorem.

The increase in the probability $Q$ compared to $P_f$ is apparently explained by the ability of the denominator, i.e. $\tau^*(n)$ to take values close to zero.

Note in conclusion that the estimates proposed by us have successfully passed experimental tests on problems $R$, $D$, $S$ by simulate modeled with the use of pseudo-random numbers as well as on real data (of seismic signals etc.) for which our estimations of the spectrum were compared with classical estimates obtained by the spectral window method.

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