Joint universality of the Riemann zeta-function and Lerch zeta-functions

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Received: 7 September 2012 / Revised: 27 March 2013 / Published online: 18 June 2013

Abstract. In the paper, we prove a joint universality theorem for the Riemann zeta-function and a collection of Lerch zeta-functions with parameters algebraically independent over the field of rational numbers.

Keywords: Lerch zeta-function, Riemann zeta-function, limit theorem, universality.

1 Introduction

Let \( \lambda \in \mathbb{R} \) and \( 0 < \alpha \leq 1 \), be fixed parameters. The Lerch zeta-function \( L(\lambda, \alpha, s) \), \( s = \sigma + it \), is defined, for \( \sigma > 1 \), by

\[
L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m + \alpha)^s}.
\]

For \( \lambda \in \mathbb{Z} \), the function \( L(\lambda, \alpha, s) \) reduces to the Hurwitz zeta-function \( \zeta(s, \alpha) \) which is a meromorphic function with a unique simple pole at the point \( s = 1 \) with residue 1. If \( \lambda \notin \mathbb{Z} \), then the Lerch zeta-function has analytic continuation to an entire function. In view of the periodicity of \( e^{2\pi i \lambda m} \), we can suppose that \( 0 < \lambda \leq 1 \).

It is well known that the Lerch zeta-function \( L(\lambda, \alpha, s) \) with transcendental parameter \( \alpha \) is universal (see [1], also [2]). Let \( D = \{ s \in \mathbb{C} : 1/2 < \sigma < 1 \} \). Denote by \( K \) the class of compact subsets of the strip \( D \) with connected complements, and, for \( K \in K \), denote by \( H(K) \) the set of continuous functions on \( K \) which are analytic in the interior of \( K \).

\textsuperscript{1}The author is supported by the European Community’s Seventh Framework Programme FP7/2007-2013, project INTEGER (grant No. 266638).
Moreover, we use the notation $\text{meas}\{A\}$ for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then the universality of $L(\lambda, \alpha, s)$ is contained in the following theorem.

**Theorem 1.** Suppose that $\alpha$ is transcendental. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\epsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\{\tau \in [0, T]: \sup_{s \in K} |L(\lambda, \alpha, s + i\tau) - f(s)| < \epsilon\} > 0.$$ 

Thus, the universality of $L(\lambda, \alpha, s)$ means that the shifts $L(\lambda, \alpha, s + i\tau)$ approximate with a given accuracy a wide class of analytic functions.

The functions $\zeta(s, \alpha)$, $\alpha \neq 1, 1/2$, and $L(\lambda, \alpha, s)$ with rational $\lambda$ are also universal in the above sense with rational parameter $\alpha$. The case of $\zeta(s, \alpha)$ has been examined in [3]. The universality of $L(\lambda, \alpha, s)$ follows from its expression by a linear combination of Hurwitz zeta-functions.

Also, in [4–6] and [7], the joint universality of Lerch zeta-functions has been considered. We state a general result from [7].

**Theorem 2.** Suppose that the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over the field of rational numbers $\mathbb{Q}$. For $j = 1, \ldots, r$, let $\lambda_j \in (0, 1)$, $K_j \in \mathcal{K}$, and $f_j(s) \in H(K_j)$. Then, for every $\epsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\{\tau \in [0, T]: \sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(\lambda_j, \alpha_j, s + i\tau) - f_j(s)| < \epsilon\} > 0.$$ 

We note that the algebraic independence of the numbers $\alpha_1, \ldots, \alpha_r$ can be replaced by a more general hypothesis that the set

$$L(\alpha_1, \ldots, \alpha_r) = \{\log(m + \alpha_j): m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, j = 1, \ldots, r\}$$

is linearly independent over $\mathbb{Q}$. In the case $\lambda_j \in \mathbb{Z}$, $j = 1, \ldots, r$, this was done in [8].

In [9], a joint universality theorem for the Riemann zeta-function $\zeta(s)$ and periodic Hurwitz zeta-functions has been obtained. Let $\mathfrak{A} = \{a_m: m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period $k \in \mathbb{N}$. We remind that the periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{A})$ with parameter $\alpha$, $0 < \alpha \leq 1$, is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha; \mathfrak{A}) = \sum_{m=0}^{\infty} \frac{a_m}{(m + \alpha)^s},$$

and is meromorphically continued to the whole complex plane with a unique possible pole at the point $s = 1$ with residue

$$\alpha \overset{\text{def}}{=} \frac{1}{k} \sum_{m=0}^{k-1} a_m.$$ 

If $\alpha = 0$, then $\zeta(s, \alpha; \mathfrak{A})$ is an entire function.
For \( j = 1, \ldots, r \), let \( l_j \in \mathbb{N} \). In [9], the joint universality for the functions
\[
\zeta(s), \zeta(s, \alpha_1; \mathfrak{A}_{1}), \ldots, \zeta(s, \alpha_1; \mathfrak{A}_{l_1}), \ldots, \zeta(s, \alpha_r; \mathfrak{A}_{1}), \ldots, \zeta(s, \alpha_r; \mathfrak{A}_{l_r})
\]  
(1)
has been proved. Here a collection of periodic sequences \( \mathfrak{A}_{jl} \), \( \mathfrak{A}_{jl} = \{a_{mjl}: m \in \mathbb{N}_0\} \), with minimal period \( k_{jl} \in \mathbb{N} \), \( l = 1, \ldots, l_j \), corresponds the parameter \( \alpha_j \), \( 0 < \alpha_j \leq 1 \), \( j = 1, \ldots, r \). For \( K \in \mathcal{K} \), denote by \( H_0(K) \) the class of continuous non-vanishing functions on \( K \) which are analytic in the interior of \( K \). Let \( k_j \) be the least common multiple of the periods \( k_{j1}, \ldots, k_{jl} \), and
\[
A_j = \begin{pmatrix}
a_{1j1} & a_{1j2} & \ldots & a_{1jl_j} \\
a_{2j1} & a_{2j2} & \ldots & a_{2jl_j} \\
\vdots & \vdots & \ddots & \vdots \\
a_{kj1} & a_{kj2} & \ldots & a_{kj_lj}
\end{pmatrix}, \quad j = 1, \ldots, r.
\]

Then the main result of [9] is of the form.

**Theorem 3.** Suppose that the numbers \( \alpha_1, \ldots, \alpha_r \) are algebraically independent over \( \mathbb{Q} \), and that \( \text{rank}(A_j) = l_j \), \( j = 1, \ldots, r \). For \( j = 1, \ldots, r \) and \( l = 1, \ldots, l_j \), let \( k_{jl} \in \mathcal{K} \) and \( f_{jl} \in H(K_{jl}) \). Moreover, let \( K \in \mathcal{K} \) and \( f(s) \in H_0(K) \). Then, for every \( \epsilon > 0 \),
\[
\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{ r \in [0; T]: \sup_{1 \leq j \leq r} \sup_{1 \leq l \leq l_j} \sup_{s \in K_{jl}} \left| \zeta(s + \imath r, \alpha_j; \mathfrak{A}_{jl}) - f_{jl}(s) \right| < \epsilon, \right. \sup_{s \in K} \left| \zeta(s + \imath r) - f(s) \right| < \epsilon \right\} > 0.
\]

We call the approximation property of the functions (1) in Theorem 3 a mixed joint universality because the function \( \zeta(s) \) and the functions \( \zeta(s, \alpha_j; \mathfrak{A}_{jl}) \) are of different types: the function \( \zeta(s) \) has Euler product, while the functions \( \zeta(s, \alpha_j; \mathfrak{A}_{jl}) \) with transcendental \( \alpha_j \) do not have Euler product over primes. This is reflected in the approximated functions: the function \( f(s) \) must be non-vanishing on \( K \), while the functions \( f_{jl} \) are arbitrary continuous functions on \( K_{jl} \).

The first mixed joint universality theorem has been obtained by Mishou [10] for the Riemann zeta-function and Hurwitz zeta-function \( \zeta(s, \alpha) \) with transcendental parameter \( \alpha \). This result in [11] has been generalized for a periodic zeta-function and a periodic Hurwitz zeta-function. In [12], the latter mixed joint universality theorem has been extended for several periodic zeta-functions and periodic Hurwitz zeta-functions.

Universality theorems for zeta-functions have a series of interesting applications. From them, for example, various denseness results of Bohr’s type for values of zeta-functions follow. The universality implies the functional independence of zeta-functions. This property of zeta-functions is applied to the zero-distribution of those zeta-functions. In [13], the universality has been applied to the famous class number problem. Universality theorems find applications even in solving some problems of physics [14]. For the above mentioned and other facts related to universality and references, we refer to [2, 15–20].
Thus, the universality of zeta-functions is a very interesting and useful property which motivates to continue investigations in the field.

The aim of this paper is to replace the zeta-functions $\zeta(s,\alpha_j;A_{jl})$ with periodic coefficients in Theorem 3 by Lerch zeta-functions $L(\lambda_j,\alpha_j,s)$ with arbitrary $\lambda_j \in (0,1]$ whose coefficients, in general, are not periodic. This is the novelty of the paper.

**Theorem 4.** Suppose that the numbers $\alpha_1,\ldots,\alpha_r$ are algebraically independent over $\mathbb{Q}$. For $j = 1,\ldots,r$, let $\lambda_j \in (0,1]$, $K_j \in K$ and $f_j \in H(K_j)$. Moreover, let $K \in K$ and $f(s) \in H_0(K)$. Then, for every $\epsilon > 0$,

$$\lim inf_{T \to \infty} \frac{1}{T} \text{meas}\left\{ \tau \in [0;T]; \sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(\lambda_j,\alpha_j,s+i\tau) - f_j(s)| < \epsilon, \sup_{s \in K} |\zeta(s+i\tau) - f(s)| < \epsilon \right\} > 0.$$

We note that the linear independence of the set $L(\alpha_1,\ldots,\alpha_r)$ is not sufficient for the proof of Theorem 4 because we need the linear independence of the set $L := \{(\log p; p \in \mathcal{P}), L(\alpha_1,\ldots,\alpha_r)\}$, where $\mathcal{P}$ is the set of all prime numbers. This set consists of logarithms of all prime numbers and of all logarithms $\log(m+\alpha_j)$, $m \in \mathbb{N}$, $j = 1,\ldots,r$. Really, $L$ is a multiset. For example, if $L$ has two identical elements, then it is linearly dependent over $\mathbb{Q}$. The proof of Theorem 4 is based on a joint limit theorem on weakly convergent probability measures in the space of analytic functions.

### 2 Joint limit theorem

Denote by $\mathcal{B}(S)$ the $\sigma$-field of Borel sets of the space $S$, and by $\gamma$ the unit circle on the complex plane. Define

$$\hat{\Omega} = \prod_p \gamma_p \quad \text{and} \quad \Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_p = \gamma$ for all $p \in \mathcal{P}$, and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. By the Tikhonov theorem, with the product topology and pointwise multiplication the tori $\hat{\Omega}$ and $\Omega$ are compact topological Abelian groups. Moreover, let

$$\Omega = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for all $j = 1,\ldots,r$. Then $\Omega$ again is a compact topological Abelian group. This gives the probability spaces $(\hat{\Omega},\mathcal{B}(\hat{\Omega}),\hat{m_H})$, $(\Omega_j,\mathcal{B}(\Omega_j),m_{jH})$ and $(\Omega,\mathcal{B}(\Omega),m_H)$, where $\hat{m_H}$, $m_{jH}$ and $m_H$ are the probability Haar measures on $(\hat{\Omega},\mathcal{B}(\hat{\Omega}))$, $(\Omega_j,\mathcal{B}(\Omega_j))$ and $(\Omega,\mathcal{B}(\Omega))$, respectively, $j = 1,\ldots,r$. We note that the
measure $m_H$ is the product of the measures $m_{H_1}, m_{H_2}, \ldots, m_{H_r}$. Denote by $\hat{\omega}(p)$ the projection of $\hat{\omega}$ to $\gamma_p$, $p \in P$, and by $\omega_j(m)$ the projection of $\omega_j$ to $\gamma_m$, $m \in N_0$. For brevity, we set $\alpha = (\alpha_1, \ldots, \alpha_r)$, $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\omega = (\omega, \omega_1, \ldots, \omega_r) \in \Omega$.

Let $H(D)$ be the space of analytic functions on $D$ endowed with the topology of uniform convergence on compacta, and let $r_1 = r + 1$. On the probability space $(\Omega, B(\Omega), m_H)$, define the $H^{r_1}(D)$-valued random element $\zeta(s, \alpha, \lambda, \omega)$ by the formula

$$\zeta(s, \alpha, \lambda, \omega) = \left( \zeta(s, \hat{\omega}), L(\lambda_1, \alpha_1, s, \omega_1), \ldots, L(\lambda_r, \alpha_r, s, \omega_r) \right),$$

where

$$\zeta(s, \hat{\omega}) = \prod_p \left( 1 - \frac{\hat{\omega}(p)}{p^s} \right)^{-1}$$

and

$$L(\lambda_j, \alpha_j, s, \omega_j) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda_j m} \omega_j(m)}{(m + \alpha_j)^s}, \quad j = 1, \ldots, r.$$

Let $P_{\zeta}$ stand for the distribution of the random element $\zeta(s, \alpha, \lambda, \omega)$, i.e., $P_{\zeta}$ is the probability measure on $(H^{r_1}(D), B(H^{r_1}(D)))$ given by

$$P_{\zeta}(A) = m_H(\omega \in \Omega : \zeta(s, \alpha, \lambda, \omega) \in A).$$

We set

$$\zeta(s, \alpha, \lambda) = \left( \zeta(s), L(\lambda_1, \alpha_1, s), \ldots, L(\lambda_r, \alpha_r, s) \right).$$

Now we state a limit theorem on the space $(H^{r_1}(D), B(H^{r_1}(D)))$.

**Theorem 5.** Suppose that the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and $\lambda_j \in (0, 1]$, $j = 1, \ldots, r$. Then

$$P_{\zeta}(A) \overset{\text{def}}{=} \frac{1}{T} \text{meas}\{ \tau \in [0, T] : \zeta(s + i\tau, \alpha, \lambda) \in A \}, \quad A \in B(H^{r_1}(D)),$$

converges weakly to the measure $P_{\zeta}$ as $T \to \infty$.

We divide the proof of Theorem 5 into lemmas. The first lemma is a limit theorem on the torus $\Omega$. For $A \in B(\Omega)$, define

$$Q(A) = \frac{1}{T} \text{meas}\{ ((p^{-i\tau} : p \in P), (m + \alpha_j)^{-i\tau} : m \in N_0, j = 1, \ldots, r) \in A \}. $$

**Lemma 1.** Suppose that the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then $Q_T$ converges weakly to the Haar measure $m_H$ as $T \to \infty$.

**Proof.** The proof of the lemma is given in [9, Lemma 1].
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Let \( \sigma_1 > 1/2 \) be a fixed number, and

\[
\nu_n(m) = \exp \left\{ -\left( \frac{m}{n} \right)^{\sigma_1} \right\}, \quad m, n \in \mathbb{N},
\]

\[
\nu_n(m, \alpha) = \exp \left\{ -\left( \frac{m + \alpha}{n + \alpha} \right)^{\sigma_1} \right\}, \quad m \in \mathbb{N}_0, \quad n \in \mathbb{N}.
\]

Define the series

\[
\zeta_n(s) = \sum_{m=1}^{\infty} \frac{\nu_n(m)}{m^s},
\]

and

\[
L_n(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{\nu_n(m, \alpha)}{(m + \alpha)^s}, \quad j = 1, \ldots, r,
\]

and, for \( \omega \in \Omega \),

\[
\zeta_n(s, \hat{\omega}) = \sum_{m=1}^{\infty} \frac{\hat{\omega}(m)\nu_n(m)}{m^s},
\]

\[
L_n(\lambda, \alpha, \omega, s) = \sum_{m=0}^{\infty} \frac{\nu_n(m, \alpha)}{(m + \alpha)^s}, \quad j = 1, \ldots, r.
\]

It is known, see, for example, [2, 16], that all above series converge absolutely for \( \sigma > 1/2 \). Let

\[
\zeta_n(s, \alpha, \lambda) = \left( \zeta_n(s), L_n(\lambda_1, \alpha_1, s), \ldots, L_n(\lambda_r, \alpha_r, s) \right)
\]

and

\[
\zeta_n(s, \alpha, \lambda, \omega) = \left( \zeta_n(s, \hat{\omega}), L_n(\lambda_1, \alpha_1, \omega_1, s), \ldots, L_n(\lambda_r, \alpha_r, \omega_r, s) \right).
\]

**Lemma 2.** Suppose that the numbers \( \alpha_1, \ldots, \alpha_r \) are algebraically independent over \( \mathbb{Q} \), and \( \omega \in \Omega \). Then

\[
\frac{1}{T} \operatorname{meas}\left\{ \tau \in [0, T]: \zeta_n(s + i\tau, \alpha, \lambda) \in A \right\}, \quad A \in \mathcal{B}(H^1(D)),
\]

and

\[
\frac{1}{T} \operatorname{meas}\left\{ \tau \in [0, T]: \zeta_n(s + i\tau, \alpha, \lambda, \omega) \in A \right\}, \quad A \in \mathcal{B}(H^1(D))
\]

converges weakly to the same probability measure \( P_n \) on \((H^1(D), \mathcal{B}(H^1(D)))\) as \( T \to \infty \).

**Proof.** The proof uses Lemma 1 and does not depend on the coefficients of the functions \( L_n(\lambda_j, \alpha_j, s), j = 1, \ldots, r \). Therefore, it coincides with the proof of [9, Lemma 2]. \( \square \)
Now we define a metric on $H^r_1(D)$ which induces the topology of uniform convergence on compacta. For $g_1, g_2 \in H(D)$, we define
\[
\rho(g_1, g_2) = \sum_{m=1}^{\infty} 2^{-m} \sup_{s \in K_m} |g_1(s) - g_2(s)| / \left(1 + \sup_{s \in K_m} |g_1(s) - g_2(s)|\right),
\]
where $\{K_m: m \in \mathbb{N}\}$ is a sequence of compact subsets of the strip $D$ such that
\[
D = \bigcup_{m=1}^{\infty} K_m,
\]
$K_m \subset K_{m+1}$ for all $m \in \mathbb{N}$, and, if $K \subset D$ is a compact set, then $K \subset K_m$ for some $m \in \mathbb{N}$. The existence of the sequence $\{K_m\}$ follows from a general theorem, see, for example, [21], however, in the case of the region $D$, it is easily seen that we can take closed rectangles. Clearly, $\rho$ is a metric on $H(D)$ inducing its topology. For $g_j = (g_{j1}, g_{j2}, \ldots, g_{jr}) \in H^r_1(D)$, $j = 1, 2$, we put
\[
\rho(g_1, g_2) = \max\left(\rho(g_1, g_2), \max_{1 \leq j \leq r} \rho(g_{1j}, g_{2j})\right).
\]
Then we have that $\rho$ is a desired metric on $H^r_1(D)$. Using this metric, we approximate $\zeta(s, \alpha, \lambda)$ and $\zeta(s, \alpha, \lambda, \omega)$ by $\zeta_n(s, \alpha, \lambda)$ and $\zeta_n(s, \alpha, \lambda, \omega)$, respectively.

Lemma 3. We have
\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho(\zeta(s + it, \alpha, \lambda), \zeta_n(s + it, \alpha, \lambda)) \, dt = 0.
\]
Moreover, suppose that the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then, for almost all $\omega \in \Omega$,
\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho(\zeta(s + it, \alpha, \lambda, \omega), \zeta_n(s + it, \alpha, \lambda, \omega)) \, dt = 0.
\]
Proof. In [16], it is proved that
\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho(\zeta(s + it), \zeta_n(s + it)) \, dt = 0,
\]
and, for almost all $\omega \in \hat{\Omega}$
\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho(\zeta(s + it, \omega), \zeta_n(s + it, \omega)) \, dt = 0.
\]
Since the numbers \( \alpha_1, \ldots, \alpha_r \) are algebraically independent over \( \mathbb{Q} \), each number \( \alpha_j \) is transcendental. Therefore, in [2], it was obtained that, for \( j = 1, \ldots, r \),

\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho(L(\lambda_j, \alpha_j, s + i\tau), L_n(\lambda_j, \alpha_j, s + i\tau)) \, d\tau = 0,
\]

and, for almost all \( \omega_j \in \Omega_j \),

\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho(L(\lambda_j, \alpha_j, \omega_j, s + i\tau), L_n(\lambda_j, \alpha_j, \omega_j, s + i\tau)) \, d\tau = 0.
\]

All these equalities together with the definition of the metric \( \rho \) prove the lemma.

**Lemma 4.** Suppose that the numbers \( \alpha_1, \ldots, \alpha_r \) are algebraically independent over \( \mathbb{Q} \). Then \( P_T \) and \( \hat{P}_T \) both converge weakly for almost all \( \omega \in \Omega \) to the same probability measure \( P \) on \((H^\infty(D), \mathcal{B}(H^\infty(D)))\) as \( T \to \infty \).

**Proof.** We give a shortened proof because we apply similar arguments as in [9]. Let \( \theta \) be a random variable defined on a certain probability space \((\Omega_0, \mathcal{A}, P)\) and uniformly distributed on \([0, 1]\). Let

\[
X_{T,n}(s) = \zeta_n(s + \theta T, \alpha, \lambda).
\]

Then Lemma 2 implies that, for every \( \epsilon > 0 \),

\[
\lim_{n \to \infty} \limsup_{T \to \infty} P_n(\rho(X_T(s), X_{T,n}(s)) \geq \epsilon) = 0.
\]

This, (2), (3) and Theorem 4.2 of [22] show that

\[
X_T \xrightarrow{D} P,
\]

and this is equivalent to the weak convergence of \( P_T \) to \( P \) as \( T \to \infty \).
Repeating the above arguments for the random elements
\[
\hat{X}_{T,n}(s) = \zeta_n(s + i\theta T, \alpha, \lambda, \omega)
\]
and
\[
\hat{X}_T(s) = \zeta(s + i\theta T, \alpha, \lambda, \omega),
\]
and using Lemmas 2 and 3, we find that the measure \(\hat{P}_T\) also converges weakly to \(P\) as \(T \to \infty\) for almost all \(\omega \in \Omega\).

**Proof of Theorem 5.** In virtue of Lemma 4, it suffices to check that the measure \(P\) in Lemma 4 coincides with \(P_{\zeta}\).

Let, for \(\tau \in \mathbb{R}\),
\[
a_\tau = (\{p^{-i\tau}: p \in \mathcal{P}\}, \{(m + \alpha_j)^{-i\tau}: m \in \mathbb{N}_0, j = 1, \ldots, r\}),
\]
and
\[
\Phi_\tau(\omega) = a_\tau \omega, \quad \omega \in \Omega.
\]
Then \(\{\Phi_\tau: \tau \in \mathbb{R}\}\) is an ergodic group of measurable measure preserving transformations on \(\Omega\) (see [12]).

Let \(\xi\) be a random variable on \((\Omega, \mathcal{B}(\Omega), m_H)\) given by
\[
\xi(\omega) = \begin{cases} 
1 & \text{if } \zeta(s, \alpha, \lambda, \omega) \in A, \\
0 & \text{if } \zeta(s, \alpha, \lambda, \omega) \notin A,
\end{cases}
\]
where \(A\) is a fixed continuity set of the measure \(P\).

By Lemma 4, for almost all \(\omega \in \Omega\),
\[
\lim_{T \to \infty} \frac{1}{T} \text{meas}\{\tau \in [0, T]: \zeta(s + i\tau, \alpha, \lambda, \omega) \in A\} = P(A). \tag{4}
\]
The ergodicity of the group \(\{\Phi_\tau: \tau \in \mathbb{R}\}\) implies that of the process \(\xi(\Phi_\tau(\omega))\). Therefore, the classical Birkhoff–Khintchine theorem shows that, for almost all \(\omega \in \Omega\),
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \xi(\Phi_\tau(\omega)) \, d\tau = E\xi, \tag{5}
\]
where \(E\xi\) denotes the expectation of \(\xi\). The definitions of \(\xi\) and of \(\Phi_\tau\) give the equalities
\[
E\xi = \int_{\Omega} \xi \, dm_H = m_H(\omega \in \Omega: \zeta(s, \alpha, \lambda, \omega) \in A) = P_{\zeta}(A), \tag{6}
\]
\[
\frac{1}{T} \int_0^T \xi(\Phi_\tau(\omega)) \, d\tau = \frac{1}{T} \text{meas}\{\tau \in [0, T]: \zeta(s + i\tau, \alpha, \lambda, \omega) \in A\}.
\]
Thus, by (5) and (6),
\[
\lim_{T \to \infty} \frac{1}{T} \text{meas } \{ \tau \in [0, T] : \zeta(s + i\tau, \alpha, \lambda, \omega) \in A \} = P_\zeta(A).
\]
This and (4) show that \( P(A) = P_\zeta(A) \) for all continuity sets of \( P \). Hence, \( P = P_\zeta \). The theorem is proved.

3 Support

A proof of Theorem 4 is based on Theorem 5 and the support of the limit measure \( P_\zeta \) in it. We remind that the support of \( P_\zeta \) is a minimal closed set \( S_{P_\zeta} \subset H^r(D) \) such that \( P_\zeta(S_{P_\zeta}) = 1 \). The set \( S_{P_\zeta} \) consists of all elements \( g \in H^r(D) \) such that, for every open neighbourhood \( G \) of \( g \), the inequality \( P_\zeta(G) > 0 \) is satisfied.

Define
\[
S = \{ g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0 \}.
\]

Theorem 6. The support of the measure \( P_\zeta \) is the set \( S = S \times H^r(D) \).

Proof. We write
\[
H^r(D) = H(D) \times H(D) \times \cdots \times H(D).
\]
The space \( H(D) \) is separable, therefore, it follows from [22] that
\[
B(H^r(D)) = B(H(D)) \times B(H(D)) \times \cdots \times B(H(D)).
\]
Thus, it suffices to consider the measure \( P_\zeta \) on the sets of the form
\[
B = A \times A_1 \times \cdots \times A_r, \quad A, A_j \in B(H(D)), \quad j = 1, \ldots, r.
\]
Since the measure \( m_H \) is the product of the measures \( \hat{m}_H, m_{1H}, \ldots, m_{rH} \), the definition of \( P_\zeta \) gives the equality
\[
P_\zeta(B) = m_H(A \times A_1 \times \cdots \times A_r) = \hat{m}_H(A)m_{1H}(A_1) \cdots m_{rH}(A_r).
\] (7)

In [16], it is proved that the support of the random element \( \zeta(s, \hat{\omega}) \) is the set \( S \). The algebraic independence of the numbers \( \alpha_1, \ldots, \alpha_r \) implies their transcendence. Therefore, by [2] the support the random element \( L(\lambda_j, \alpha_j, s, \omega_j) \) is the space \( H(D) \), \( j = 1, \ldots, r \).

On the other hand, the distribution \( P_\zeta \) of \( \zeta(s, \hat{\omega}) \) is
\[
P_\zeta(A) = \hat{m}_H(\omega \in \hat{\Omega} : \zeta(s, \omega) \in A), \quad A \in B(H(D)),
\]
and the distribution \( P_{L_j} \) of \( L(\lambda_j, \alpha_j, s, \omega_j) \), \( j = 1, \ldots, r \), is
\[
P_{L_j}(A_j) = m_{jH}(\omega_j \in \Omega_j : L(\lambda_j, \alpha_j, s, \omega_j) \in A_j), \quad A_j \in B(H(D)).
\]
In view of (7),

\[ P_\zeta(B) = P_\zeta(A) P_{L_1}(A_1) \cdots P_{L_r}(A_r). \]

Hence, obviously, \( P_\zeta(S) = 1 \). Moreover, if \( A \in \mathcal{B}(H(D)) \) with \( A \not\subseteq S \), or \( A_j \in \mathcal{B}(H(D)) \), for some \( j \), then, in view of the minimality of \( S \) and \( H(D) \) for \( P_\zeta(A) \) and \( P_{L_j}(A_j) \), respectively, we have that \( P_\zeta(A) < 1 \) or \( P_{L_j}(A_j) < 1 \). Thus, then \( P_\zeta(B) < 1 \). Hence, the minimality of \( S \) follows. \( \Box \)

4 Universality theorem

In this section, we will prove Theorem 4. Its proof is based on Theorems 5 and 6 as well as on the Mergelyan theorem on the approximation of analytic functions by polynomials. We state this theorem as the next lemma.

**Lemma 5.** Let \( K \subset \mathbb{C} \) be a compact set with connected complement, and \( f(s) \) be a continuous function on \( K \) which is analytic in the interior of \( K \). Then, for every \( \epsilon > 0 \), there exists a polynomial \( p(s) \) such that

\[ \sup_{s \in K} |f(s) - p(s)| < \epsilon. \]

**Proof.** The proof of the lemma can be found in [23], see also [24]. \( \Box \)

**Proof of Theorem 4.** By Lemma 5, there exists a polynomial \( p(s) \) such that

\[ \sup_{s \in K} |f(s) - p(s)| < \frac{\epsilon}{4}. \] (8)

Since \( f(s) \neq 0 \) on \( K \), \( p(s) \neq 0 \) on \( K \) as well provided \( \epsilon \) is small enough. Thus, we can define on \( K \) a continuous branch of \( \log p(s) \) which will be analytic in the interior of \( K \). Applying Lemma 5 once more, we obtain that there exists a polynomial \( q(s) \) such that

\[ \sup_{s \in K} |p(s) - e^{q(s)}| < \frac{\epsilon}{4}. \]

This together with (8) shows that

\[ \sup_{s \in K} |f(s) - e^{q(s)}| < \frac{\epsilon}{2}. \] (9)

Again, by Lemma 5, there exist polynomials \( p_j(s) \) such that

\[ \sup_{1 \leq j \leq r} \sup_{s \in K_j} |f_j(s) - p_j(s)| < \frac{\epsilon}{2}. \] (10)

Define

\[ G = \left\{ (g, g_1, \ldots, g_r) \in H^r(D) : \sup_{s \in K} |g(s) - e^{q(s)}| < \frac{\epsilon}{2}, \right. \]

\[ \left. \sup_{1 \leq j \leq r} \sup_{s \in K_j} |g_j(s) - p_j(s)| < \frac{\epsilon}{2} \right\}. \]
Then $G$ is an open set, and, in view of Theorem 6, \( e^{q(s)} \) is an element of the support of the measure \( \mathcal{P}_{\zeta} \). Therefore, an equivalent of the weak convergence of probability measures in terms of open sets, see Theorem 2.1 of [22], together with Theorem 5 and properties of the support give the inequality
\[
\liminf_{T \to \infty} \frac{1}{T} \text{meas}\{ \tau \in [0, T] : \zeta(s + i\tau, \alpha, \lambda) \in G \} \geq \mathcal{P}_{\zeta}(G) > 0.
\]

Hence, by the definition of $G$, we find that
\[
\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - e^{q(s)}| < \frac{\epsilon}{2}, \sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(\lambda_j, \alpha_j, s + i\tau) - p_j(s)| < \frac{\epsilon}{2} \right\} > 0. \tag{11}
\]

Inequalities (9) and (10) show that
\[
\left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - e^{q(s)}| < \frac{\epsilon}{2}, \sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(\lambda_j, \alpha_j, s + i\tau) - p_j(s)| < \frac{\epsilon}{2} \right\} \subset \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \epsilon, \sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(\lambda_j, \alpha_j, s + i\tau) - f_j(s)| < \epsilon \right\}.
\]

Combining this with (11) gives the assertion of the theorem.

**Acknowledgment.** The authors thank the anonymous referees for remarks and suggestions.

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