Measurable Iso-Functions

Svetlin G. Georgiev

Department of Mathematics, Sorbonne University, Paris, France

Email address: svetlingeorgiev1@gmail.com

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Abstract: In this article are given definitions definition for measurable is-functions of the first, second, third, fourth and fifth kind. They are given examples when the original function is not measurable and the corresponding iso-function is measurable and the inverse. They are given conditions for the isotopic element under which the corresponding is-functions are measurable. It is introduced a definition for equivalent iso-functions. They are given examples when the iso-functions are equivalent and the corresponding real functions are not equivalent. They are deducted some criterions for measurability of the iso-functions of the first, second, third, fourth and fifth kind. They are investigated for measurability the addition, multiplication of two iso-functions, multiplication of iso-function with an iso-number and the powers of measurable iso-functions. They are given definitions for step iso-functions, iso-step iso-functions, characteristic iso-functions, iso-characteristic iso-functions. It is investigated for measurability the limit function of sequence of measurable iso-functions. As application they are formulated the iso-Lebesgue’s theorems for iso-functions of the first, second, third, fourth and fifth kind. These iso-Lebesgue’s theorems give some information for the structure of the iso-functions of the first, second, third, fourth and fifth kind.

Keywords: Measurable Iso-Sets, Measurable Is-Functions, Is-Lebesgue Theorems

1. Introduction

Genious idea is the Santilli’s generalization of the basic unit of quantum mechanics into an integro-differential operator \( I \) which is as positive-definite as +1 and it depends of local variables and it is assumed to be the inverse of the isotopic element \( \tilde{T} \)

\[
+1 > 0 \rightarrow I(t,r,p,a,E,\ldots) = \frac{1}{\tilde{T}} > 0
\]

and it is called Santilli isounit. Santilli introduced a generalization called lifting of the conventional associative product \( ab \) into the form

\[
ab \rightarrow a \tilde{x} b = a \tilde{T} b
\]

Called isoproduct for which

\[
\tilde{I} \tilde{x} a = \frac{1}{\tilde{T}} \tilde{T} a = a \tilde{x} \tilde{I} = a \tilde{T} \frac{1}{\tilde{T}} = a
\]

For every element \( a \) of the field of real numbers, complex numbers and quaternions. The Santilli isonumbers are defined as follows: for given real number or complex number or quaternion \( a \),

\[
\tilde{a} = a I
\]

With isoproduct

\[
\tilde{a} \tilde{x} \tilde{b} = \tilde{a} \tilde{T} b = a \frac{1}{\tilde{T}} \tilde{T} b \frac{1}{\tilde{T}} = ab \frac{1}{\tilde{T}} = \tilde{a} b.
\]

If \( a \neq 0 \), the corresponding isoélement of \( \frac{1}{a} \) will be denoted with \( \tilde{a}^{-1} \) or \( \tilde{I} \times \tilde{a} \).

With \( \tilde{P}_a \) we will denote the field of the is-numbers \( \tilde{a} \) for which \( a \in \mathbb{R} \) and basic isounit \( \tilde{I}_1 \).

In [1], [3]-[12] are defined isocontinuous isofunctions and isoderivative of isofunction and in [1] are proved some of their properties. If \( D_1 \) is a isoset in \( \tilde{P}_{\mathbb{R}} \), the class of isofunctions is denoted by \( \tilde{F}_C\tilde{D}_1 \) and the class of isodifferentiable isofunctions is denoted by \( \tilde{F}_C\tilde{D}_1^* \), with the same basic isounit \( \tilde{I} = \frac{1}{\tilde{T}} \) it is supposed

\[
\tilde{P} \in C^1(D_1), \tilde{P} > 0 \text{ in } D_1.
\]

Here \( D_1 \) is the corresponding real set of \( \tilde{D}_1 \). If \( x \) is an independent variable, then the corresponding lift is \( \frac{x}{\tilde{T}(x)} \), if \( f \) is real-valued function on \( D_1 \), then the corresponding lift of first kind is defined as follows
We suppose that \( A \) is a given point set, \( \hat{T} : A \to \mathbb{R} \), \( \hat{T}(x) > 0 \) for every \( x \in A \), \( \hat{T}_i > 0 \) be a given constant, \( f : A \to \mathbb{R} \) be a given real-valued function. With \( \hat{f} \) we will denote the corresponding is-function of the first, second, third, fourth and fifth kind. More precisely,

1. \( \hat{f}(x) \equiv \hat{f}^\wedge(x) = \frac{\hat{f}(x)}{\hat{T}(x)} \) when \( \hat{f} \) is an is-function of the first kind.
2. \( \hat{f}(x) = \hat{f}^\wedge(x) = \frac{\hat{f}(x)(x)}{\hat{T}(x)} \) when \( \hat{f}(x) \in A \) for \( x \in A \), when \( \hat{f} \) is an is-function of the second kind.
3. \( \hat{f}(x) = \hat{f}^\wedge(x) = \frac{\hat{f}(x)}{\hat{T}(x)} \) when \( \hat{f}(x) \in A \) for \( x \in A \), when \( \hat{f} \) is an is-function of the third kind.
4. \( \hat{f}(x) \equiv \hat{f}^\wedge(x) = \hat{f}(x) \hat{T}(x) \), when \( x \neq \hat{T}(x) \) for \( x \in A \), when \( \hat{f} \) is an is-function of the fourth kind.
5. \( \hat{f}(x) = \hat{f}^\wedge(x) = \hat{f}(x) \frac{\hat{T}(x)}{\hat{T}(x)} \) when \( x \neq \hat{T}(x) \) for \( x \in A \), when \( \hat{f} \) is an is-function of the fifth kind.

For \( a \in A \) with \( A^\wedge > a \) we will denote the set

\[
A^\wedge > a = \{ x \in A : \hat{f}(x) > a \}.
\]

We define the symbols \( A(\hat{f} \geq a), A(\hat{f} = a), A(\hat{f} < a) \), \( A(a < \hat{f} < b) \) and etc., in the same way.

If the set on which the is-function \( \hat{f} \) is defined is designated by a letter C or D, we shall write \( C(\hat{f} > a) \) or \( D(\hat{f} > a) \).

Definition 2.1. The is-function \( \hat{f} \) is said to be measurable if
1. The set \( A \) is measurable.
2. The set \( A^\wedge > a \) is measurable for all \( a \in A \).

Theorem 2.3. Let \( \hat{f} \) be a measurable is-function defined on the set \( A \). If \( B \) is a measurable subset of \( A \), then the is-function \( \hat{f}(x) \), considered only for \( x \in B \), is measurable.

Proof. Let \( a \in \mathbb{R} \) be arbitrarily chosen and fixed. We will prove that

\[
B(\hat{f} > a) = B \cap A(\hat{f} > a).
\]  

(1)

Really, let \( x \in B(\hat{f} > a) \) be arbitrarily chosen. Then \( x \in B \) and \( \hat{f}(x) > a \). Since \( B \subseteq A \), we have that \( x \in A \). From \( x \in A \) and \( \hat{f}(x) > a \) it follows that \( x \in A(\hat{f} > a) \). Therefore \( x \in B(\hat{f} > a) \). Because \( x \in B(\hat{f} > a) \) was arbitrarily chosen and for it we get that it is an element of the set \( B(\hat{f} > a) \), we conclude that

\[
B(\hat{f} > a) \subseteq B \cap A(\hat{f} > a).
\]  

(2)

Let now \( x \in B \cap A(\hat{f} > a) \) be arbitrarily chosen. Then \( x \in B \) and \( x \in A(\hat{f} > a) \). Hence \( x \in B \) and \( \hat{f}(x) > a \). Therefore \( x \in B(\hat{f} > a) \). Because \( x \in B \cap A(\hat{f} > a) \) was arbitrarily chosen and we get that it is an element of \( B(\hat{f} > a) \), we conclude that

\[
B(\hat{f} > a) \subseteq B \cap A(\hat{f} > a).
\]

From the last relation and from (2) we prove the relation (1).

Since the is-function \( \hat{f} \) is a measurable function on the set \( A \), we have that \( A(\hat{f} > a) \) is a measurable set. As the intersection of two measurable sets is a measurable set, we have that \( B \cap A(\hat{f} > a) \) is a measurable set. Consequently, using (1), the set \( B(\hat{f} > a) \) is measurable set. In this way we have

1. \( B \) is a measurable set.
2. \( B(\hat{f} > a) \) is a measurable set for all \( a \in \mathbb{R} \).

Therefore the is-function \( \hat{f} \), considered only for \( x \in B \), is a measurable is-function.

Theorem 2.4. Let \( \hat{f} \) be defined on the set \( A \), which is the union of a finite or denumerable number of measurable sets \( A_k, A = \bigcup_k A_k \). If \( \hat{f} \) is measurable on each of the sets \( A_k \), then it is also measurable on \( A \).

Proof. Let \( a \in \mathbb{R} \) be arbitrarily chosen. We will prove that
\[ A(\hat{f} > a) = \bigcup_k A_k(\hat{f} > a). \]  

Let \( x \in A(\hat{f} > a) \) be arbitrarily chosen. Then \( x \in A \) and \( f(x) > a \). Since \( x \in A \) and \( A = \bigcup_k A_k \), there exists \( k \) such that \( x \in A_k \). Therefore \( x \in A_k \) and \( \hat{f}(x) > a \). Hence, \( x \in A_k(\hat{f} > a) \) and \( x \in \bigcup_k A_k(\hat{f} > a) \). Because \( x \in A(\hat{f} > a) \) was arbitrarily chosen and for it we get that it is an element of \( \bigcup_k A_k(\hat{f} > a) \), we conclude that

\[ A(\hat{f} > a) \subset \bigcup_k A_k(\hat{f} > a). \]  

From the last relation and from (4) we prove the relation (3).

Since the union of finite or denumerable number of measurable sets is a measurable set, using that the sets \( A(\hat{f} > a) \) are measurable, we obtain that \( A \) and \( A(\hat{f} > a) \) are measurable sets. Therefore \( \hat{f} \) is a measurable is-function.

Definition 2.5. Two is-functions \( \hat{f} \) and \( \hat{g} \), defined on the same set \( A \), are said to be equivalent if

\[ \mu \left( A(\hat{f} \neq \hat{g}) \right) = 0. \]

We will write

\[ \hat{f} \sim \hat{g}. \]

Remark 2.6. There is a possibility \( f \sim g \) and in the same time \( \hat{f} \sim \hat{g} \).

Let \( A = [1, 2], f(x) = x, g(x) = x + 1, \) \( \hat{f}(x) = \frac{-1 + \sqrt{1 + 4x^2}}{2x}, x \in A. \)

Then

\[ f \sim g. \]

On the other hand,

\[ f^{\wedge}(x) = \frac{x}{\hat{T}(x)} = \frac{-1 + \sqrt{1 + 4x^2}}{2x}, x \in A. \]

Then

\[ f^{\sim} \sim g. \]

Proposition 2.8. The functions \( f \) and \( g \) are equivalent if and only if the functions \( \hat{f}^{\wedge \wedge} \) and \( \hat{g}^{\wedge \wedge} \) are equivalent.

Proof. We have

\[ \mu \left( A(\hat{f}^{\wedge \wedge} \neq \hat{g}^{\wedge \wedge}) \right) = 0 \iff \mu \left( A(\hat{f} \neq \hat{g}) \right) = 0 \]

Definition 2.9. Let some property \( P \) holds for all the points of the set \( A \), except for the points of a subset \( B \) of the set \( A \). If \( = 0 \), we say that the property \( P \) holds almost everywhere on the set \( A \), or for almost all points of \( A \).

Definition 2.10. We say that two is-functions defined on
the set A are equivalent if they are equal almost everywhere on the set A.

Theorem 2.11. If \( \hat{f}(x) \) is a measurable is-function defined on the set A, and if \( \hat{f} \sim \hat{g} \), then the is-function \( \hat{g}(x) \) is also measurable.

Proof. Let

\[
B := A(\hat{f} \neq \hat{g}), \quad D := A \setminus B.
\]

Because \( \hat{f} \sim \hat{g} \) we have that

\[
\mu(A(\hat{f} \neq \hat{g})) = 0
\]
or \( \mu B = 0 \).

Since every function, definite on a set with measure zero is measurable on it, we have that the is-function \( \hat{g} \) is measurable on the set B.

We note that the is-functions \( \hat{f}(x) \) and \( \hat{g}(x) \) are identical on D and since the is0-function \( \hat{f} \) is measurable on D, we get that the is-function \( \hat{g} \) is measurable on D.

Consequently the is-function \( \hat{g} \) is measurable on

\[
B \cup D = A.
\]

Theorem 2.12. If the is-function \( \hat{f}(x) \), defined on the set A, is measurable, then the sets

\[
A(\hat{f} \geq a), A(\hat{f} = a), A(\hat{f} \leq a), A(\hat{f} < a)
\]

Are measurable for all \( a \in \mathbb{R} \).

Proof. We will prove that

\[
A(\hat{f} \geq a) = \prod_{n=1}^{\infty} A(\hat{f} > a - \frac{1}{n}).
\] (5)

Really, let \( x \in A(\hat{f} \geq a) \) be arbitrarily chosen. Then \( x \in A \) and \( \hat{f}(x) \geq a \). Hence, for every \( n \in \mathbb{N} \) we have \( \hat{f}(x) > a - \frac{1}{n} \). Therefore

\[
x \in \prod_{n=1}^{\infty} A(\hat{f} > a - \frac{1}{n}).
\]

Because \( x \in A(\hat{f} \geq a) \) was arbitrarily chosen and for it we obtain \( x \in \prod_{n=1}^{\infty} A(\hat{f} > a - \frac{1}{n}) \).

We conclude that

\[
A(\hat{f} \geq a) \subset \prod_{n=1}^{\infty} A(\hat{f} > a - \frac{1}{n}).
\] (6)

Let now \( x \in \prod_{n=1}^{\infty} A(\hat{f} > a - \frac{1}{n}) \) be arbitrarily chosen. Then \( x \in A(\hat{f} > a - \frac{1}{n}) \) for every natural number n. From here \( x \in A \) and

\[
\hat{f}(x) > a - \frac{1}{n}
\]

For all natural number n. Consequently

\[
\lim_{n \to \infty} \hat{f}(x) \geq \lim_{n \to \infty} \left(a - \frac{1}{n}\right)
\]
or

\[
f(x) \geq a
\]

and \( x \in A(f \geq a) \). Since \( x \in \prod_{n=1}^{\infty} A(\hat{f} > a - \frac{1}{n}) \) was arbitrarily chosen and we get that \( x \in A(f \geq a) \), we conclude

\[
\prod_{n=1}^{\infty} A(\hat{f} > a - \frac{1}{n}) \subset A(f \geq a).
\]

From the last relation and from (6) we obtain the relation (5).

Because the intersection of denumerable measurable sets is a measurable set, using the relation (5) and the fact that all sets \( A(\hat{f} > a - \frac{1}{n}) \) are measurable for all natural numbers n, we conclude that the set \( A(\hat{f} \geq a) \) is a measurable set.

The set \( A(\hat{f} = a) \) is a measurable set because

\[
A(f = a) = A(f \geq a) \setminus A(f > a).
\]

The set \( A(\hat{f} \leq a) \) is measurable since

\[
A(\hat{f} = a) = A \setminus A(\hat{f} > a).
\]

The set \( A(\hat{f} < a) \) is measurable since

\[
A(\hat{f} = a) = A \setminus A(\hat{f} \geq a).
\]

Remark 2.13. We note that if at least one of the sets

\[
A(\hat{f} \geq a), A(\hat{f} = a), A(\hat{f} \leq a), A(\hat{f} < a)
\]

Is measurable for all \( a \in \mathbb{R} \), then the iso-function \( \hat{f} \) is measurable on the set A.

Really, let \( A(\hat{f} \geq a) \) is measurable for all \( a \in \mathbb{R} \). Then, using the relation

\[
A(\hat{f} > a) = \prod_{n=1}^{\infty} A(\hat{f} \geq a - \frac{1}{n}),
\] (7)

we obtain that the set \( A(\hat{f} > a) \) is measurable for all \( a \in \mathbb{R} \).

If \( A(\hat{f} \leq a) \) is measurable for all \( a \in \mathbb{R} \), then using the relation

\[
A(\hat{f} > a) = A \setminus A(\hat{f} \leq a),
\]

we get that the set \( A(\hat{f} > a) \) is measurable for all \( a \in \mathbb{R} \).

If \( A(\hat{f} < a) \) is measurable for all \( a \in \mathbb{R} \), then using the relation

\[
A(\hat{f} > a) = A \setminus A(\hat{f} \leq a),
\]

we conclude that the set \( A(\hat{f} > a) \) is measurable for all \( a \in \mathbb{R} \).

Theorem 2.14. If \( \hat{f}(x) = c = \text{const} \) for all points of a measurable set A, then the is-function \( \hat{f}(x) \) is measurable.

Proof. For all \( a \in \mathbb{R} \) we have that

\[
A(\hat{f} > a) = A \text{ if } c > a \text{ and } A(\hat{f} > a) = \emptyset \text{ if } c \leq a.
\]

Since the sets \( A \) and \( \emptyset \) are measurable sets, then \( A(\hat{f} > a) \)
is measurable for all \( a \in \mathbb{R} \). Therefore the is-function \( \hat{f}(x) \) is measurable.

Definition 2.15. An is-function \( f(x) \) defined on the closed interval \([a, b]\) is said to be a step is-function if there is a finite number of points

\[
a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b
\]

Such that \( \hat{f}(x) \) is a constant on \((a_i, a_{i+1})\), \( i = 0, 1, 2, \ldots, n - 1 \). From the previous theorem we have that \( f(x) \) is measurable on \((a_i, a_{i+1})\), \( i = 0, 1, 2, \ldots, n \). We note that the sets \( \{a_i\}, i = 0, 1, 2, \ldots, n - 1 \), are sets with measure zero. Therefore the is-function \( \hat{f}(x) \) is measurable on \([a_i, a_{i+1})\), \( i = 0, 1, 2, \ldots, n \). From here, using that

\[
[a, b] = \bigcup_{i=0}^{n} (a_i, a_{i+1}) \bigcup_{i=0}^{n} [a_i],
\]

We conclude that the is-function \( \hat{f}(x) \) is measurable on \([a, b]\).

Theorem 2.17. If the is-function \( \hat{f}(x) \), defined on the set \( A \) is measurable and \( c \in \mathbb{R}, c \neq 0 \), then the is-functions

1. \( \hat{f}(x) + c \),  
2. \( c \hat{f}(x) \),  
3. \( |\hat{f}(x)| \),  
4. \( \hat{f}^2(x) \),  
5. \( \frac{1}{\hat{f}(x)} \)

are also measurable.

Proof. Let \( a \in \mathbb{R} \) be arbitrarily chosen. The assertion follows from the following relations.

1. \( A(\hat{f} + c > a) = A(\hat{f} > c - a) \).
2. \( A(cf > a) = A(\hat{f} > \frac{a}{c}) \) if \( c > 0 \), \( A(cf > a) = A(\hat{f} < \frac{a}{c}) \) if \( c < 0 \).
3. \( A(|\hat{f}| > a) = A(\hat{f} > a) \cup A(\hat{f} < -a) \) if \( a \geq 0 \).
4. \( A(\hat{f}^2 > a) = A(\hat{f} > \sqrt{a}) \) if \( a \geq 0 \).
5. \( A(\frac{1}{\hat{f}} > a) = A(\hat{f} > 0) \cap A(\hat{f} < \frac{1}{a}) \) if \( a > 0 \), \( A(\frac{1}{\hat{f}} > a) = A(\hat{f} > 0) \cup A(\hat{f} < 0) \cap A(f < 0) \cap A(f > 0) \) if \( a < 0 \).

Definition 2.18. An is-function \( \hat{f} \), defined on the closed interval \([p, b]\), is said to be is-step is-function, if there is a finite number of points

\[
a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b,
\]

such that

\[
\hat{f}(x) = \frac{c_i}{T(x)}, x \in [a_i, a_{i+1}), c_i = \text{const}, i = 0, 1, \ldots, n - 1.
\]

Theorem 2.19. Let \( \hat{T}(x) > 0 \) for every \( x \in [a, b] \) and \( \hat{T}(x) \) is measurable on \([a, b] \). Let also, \( \hat{T}(x) \) is an is-step is-function on \([a, b] \). Then \( \hat{f}(x) \) is measurable on \([a, b] \).

Proof. Let

\[
a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b,
\]

be such that

\[
\hat{f}(x) = \frac{c_i}{\hat{T}(x)}, x \in [a_i, a_{i+1}), c_i = \text{const}, i = 0, 1, \ldots, n - 1.
\]

From the last theorem it follows that \( \frac{c_i}{\hat{T}(x)} \) is a measurable is-function on \([a_i, a_{i+1})\), \( i = 0, 1, 2, \ldots, n - 1 \). From-1 here and from

\[
[a, b] = \bigcup_{i=0}^{n} [a_i, a_{i+1}) \cup \{b\}.
\]

Since \( \{b\} \) is a set with measure zero, we conclude that the is-step is-function \( \hat{f} \) is measurable on \([a, b]\).

Definition 2.20. Let \( M \) be a subset of the closed interval \([a, b]\). The function \( \varphi_M(x) = 0 \) for \( x \in [a, b] \) \( \setminus M \) and \( \varphi_M = 1 \) for \( x \in M \), is called the characteristic function of the set \( M \).

Theorem 2.21. If the set \( M \) is a measurable subset of the closed interval \( A = [a, b] \), then the characteristic function \( \varphi_M(x) \) is measurable on \([a, b]\).

Proof. The assertion follows from the following relations. \( \varphi_M > a \) if \( a \geq 1 \), \( \varphi_M > a \) if \( 1 > a \geq 0 \), \( \varphi_M > a \) if \( a = 0 \).

Definition 2.22. Let \( M \) be a subset of the set \( A = [a, b] \). The iso- function \( \hat{\varphi}_M(x) = 0 \) if \( x \in A \) \( \setminus M \) and \( \hat{\varphi}_M = \frac{1}{\hat{T}(x)} \) if \( x \in M \), will be called characteristic is-function of the set \( M \).

Theorem 2.23. \( \hat{T}(x) \) is a measurable function on \( A = [a, b] \), \( M \) be a measurable subset of \( A \). Then the characteristic is-function \( \hat{\varphi}_M(x) \) of the set \( M \) is measurable.

Proof. Let \( a \in \mathbb{R} \) be arbitrarily chosen. Then

\[
A(\hat{\varphi}_M > a) = (A \setminus M)(0 > a) \cup M(\frac{1}{\hat{T}(x)} > a).
\]

From here, using that the sets \( (A \setminus M)(0 > a) \) and \( M(\frac{1}{\hat{T}(x)} > a) \) are measurable sets, we conclude that \( A(\hat{\varphi}_M > a) \) is a measurable set. Because the constant \( a \) was arbitrarily chosen, we have that the characteristic function \( \hat{\varphi}_M \) is a measurable is-function.

Theorem 2.24. Let \( f \) and \( \hat{T} \) are continuous functions on the closed set \( A \). Then the is-function \( \hat{f}(\hat{T}(x)) \) is measurable.

Proof. Let \( a \in \mathbb{R} \) be arbitrarily chosen. Since every closed set is a measurable set, we conclude that the set \( A \) is a measurable set.
We will prove that the set \( A(\hat{f} \land \leq a) \) is a closed set.

Let \( \{x_n\}_{n=1}^{\infty} \) be a sequence of elements of the set \( A(\hat{f} \land \leq a) \) such that

\[
\lim_{n \to \infty} x_n = x_0.
\]

Since \( A(\hat{f} \land \leq a) \) is a subset of the set \( A \) we have that \( \{x_n\}_{n=1}^{\infty} \subset A \). Because the set \( A \) is a closed set, we obtain that \( x_0 \in A \). From the definition of the set \( A(\hat{f} \land \leq a) \) we have that

\[
\hat{f}^\land(\hat{x}_n) = \frac{f(x_n)}{\mathcal{T}(x_n)} \leq a,
\]

Hence, when \( n \to \infty \), using that \( f \) and \( \hat{T} \) are continuous functions on the set \( A \), we get

\[
\lim_{n \to \infty} \hat{f}^\land(\hat{x}_n) = \lim_{n \to \infty} \frac{f(x_n)}{\mathcal{T}(x_n)} = \frac{f(x_0)}{\mathcal{T}(x_0)} = \hat{f}^\land(\hat{x}_0) \leq a,
\]

i.e., \( x_0 \in A(\hat{f} \land \leq a) \). Therefore the set \( A(\hat{f} \land \leq a) \) is a closed set. From here, the set \( A(\hat{f}^\land \leq a) \) is a measurable set.

Because the difference of two measurable sets is a measurable set, we have that the set

\[
A(\hat{f}^\land > a) = A \setminus A(\hat{f}^\land \leq a)
\]

is a measurable set.

Since \( a \in \mathbb{R} \) was arbitrarily chosen, we obtain that the is-function of the first kind \( \hat{f}^\land \) is measurable.

Theorem 2.25. Let \( f \) and \( \hat{T} \) are continuous functions on the closed set \( A \). The the is-functions

\[
\hat{f}^\land(x), \hat{f}(\hat{x}), \hat{f}^\land(x), f^\land(x)
\]

are measurable on \( A \).

Theorem 2.26. If two measurable is-functions \( \hat{f} \) and \( \hat{g} \) are defined on the set \( A \), then the set \( A(\hat{f} > \hat{g}) \) is measurable.

Proof. We enumerate all rational numbers \( r_1, r_2, r_3, \ldots \).

We will prove that

\[
A(\hat{f} > \hat{g}) = \bigcup_{k=1}^{\infty} \left( A(\hat{f} > r_k) \cap A(\hat{g} < r_k) \right).
\]

Let \( x \in A(\hat{f} > \hat{g}) \)

Be arbitrarily chosen. Then

\[
x \in A, \hat{f}(x) > \hat{g}(x).
\]

There exists a rational number \( r_k \) such that

\[
\hat{f}(x) > r_k > \hat{g}(x).
\]

Therefore

\[
x \in A and \hat{f}(x) > r_k; x \in A and r_k > \hat{g}(x),
\]

i.e.,

\[
x \in A(\hat{f} > r_k), x \in A(\hat{g} < r_k).
\]

Consequently

\[
x \in A(\hat{f} > r_k) \cap A(\hat{g} < r_k)
\]

And

\[
x \in \bigcup_{k=1}^{\infty} \left( A(\hat{f} > r_k) \cap A(\hat{g} < r_k) \right).
\]

Because \( x \in A(\hat{f} > \hat{g}) \) was arbitrarily chosen and for it we get

\[
x \in \bigcup_{k=1}^{\infty} \left( A(\hat{f} > r_k) \cap A(\hat{g} < r_k) \right),
\]

we conclude that

\[
A(\hat{f} > \hat{g}) \subset \bigcup_{k=1}^{\infty} \left( A(\hat{f} > r_k) \cap A(\hat{g} < r_k) \right).
\]

Let no \( x \in \bigcup_{k=1}^{\infty} \left( A(\hat{f} > r_k) \cap A(\hat{g} < r_k) \right) \) be arbitrarily chosen. Then there exists a natural \( k \) so that

\[
x \in A(\hat{f} > r_k) \cap A(\hat{g} < r_k).
\]

Hence,

\[
x \in A(\hat{f} > r_k), x \in A(\hat{g} < r_k).
\]

Then

\[
x \in A, \hat{f}(x) > r_k, r_k < \hat{g}(x)
\]

or

\[
x \in A, \hat{f}(x) > r_k > \hat{g}(x).
\]

Therefore

\[
x \in A(\hat{f} > \hat{g}).
\]

Because

\[
x \in \bigcup_{k=1}^{\infty} \left( A(\hat{f} > r_k) \cap A(\hat{g} < r_k) \right)
\]

Was arbitrarily chosen and for it we get that \( x \in A(\hat{f} > \hat{g}) \), we conclude that

\[
\bigcup_{k=1}^{\infty} \left( A(\hat{f} > r_k) \cap A(\hat{g} < r_k) \right) \subset A(\hat{f} > \hat{g}).
\]

From the last relation and from the relation (9) we get the relation (8).

Since \( \hat{f} \) and \( \hat{g} \) are measurable iso-functions on \( A \), we have that the sets

\[
A(\hat{f} > r_k), A(\hat{g} < r_k)
\]
are measurable sets for every natural k, whereupon the sets
\[ A(f > r_k) \cap A(\bar{g} < r_k) \]
are measurable sets for every natural k.

Therefore, using the relation (8), we obtain that the set
\[ A(f > \bar{g}) \]
is a measurable set.

**Theorem 2.27.** Let \( f(x) \) and \( \bar{g}(x) \) be finite measurable is-functions on the set \( A \). Then each of the is-functions
1. \( f(x) - \bar{g}(x) \),
2. \( f(x) + \bar{g}(x) \),
3. \( f(x)\bar{g}(x) \),
4. \( \frac{f(x)}{\bar{g}(x)} \) if \( \bar{g}(x) \neq 0 \) on \( A \),
is measurable.

**Proof.**
1. Let \( a \in \mathbb{R} \) be arbitrarily chosen. Since \( \bar{g}(x) \) is measurable, then \( a + \bar{g}(x) \) is measurable. From here and from the last theorem it follows that the set
\[ A(f(x) - \bar{g}(x) > a) = A(f(x) > a + \bar{g}(x)) \]
is measurable. Because \( a \in \mathbb{R} \) was arbitrarily chosen, we conclude that the function \( f(x) - \bar{g}(x) \) is measurable.

2. Since \( \bar{g} \) is a measurable is-function, we have that the function \( -\bar{g} \) is a measurable is-function. From here and from 1) we conclude that the is-function
\[ \hat{f} + \hat{g} = \hat{f} - (-\hat{g}) \]
is measurable.

3. We note that
\[ f(x)\bar{g}(x) = \frac{1}{2}(f(x) + \bar{g}(x))^2 - \frac{1}{2}(f(x) - \bar{g}(x))^2. \]
Since \( f(x) \) and \( \bar{g}(x) \) are measurable is-functions, using 1) and 2) we have that
\[ \hat{f}(x) + \hat{g}(x) and \hat{f}(x) - \hat{g}(x) \]
are measurable is-functions. Hence the is-functions
\[ (f(x) + \bar{g}(x))^2, (f(x) - \bar{g}(x))^2 \]
are measurable, whereupon
\[ \frac{1}{2}(f(x) + \bar{g}(x))^2 and \frac{1}{2}(f(x) - \bar{g}(x))^2 \]
are measurable. From here, using 1) and 10), we conclude that \( \hat{f}(x)\bar{g}(x) \) is measurable.

4. Since \( \bar{g}(x) \) is measurable and \( \bar{g}(x) \neq 0 \) on \( A \), we have that the is-function \( \frac{1}{\bar{g}(x)} \) is measurable. From here and from 3) the is-function
\[ \frac{f(x)}{\bar{g}(x)} = \hat{f}(x) \frac{1}{\bar{g}(x)} \]
is measurable.

**Theorem 2.28.** Let \( \{\hat{f}_n(x)\}_{n=1}^{\infty} \) be a sequence of measurable is-functions defined on the set \( A \). If
\[ \lim_{n \to \infty} \hat{f}_n(x) = \hat{f}(x) \]
exists for every \( x \in A \), then the is-function \( \hat{f}(x) \) is measurable.

**Proof.** Let \( a \in \mathbb{R} \) be arbitrarily chosen. For \( n, k, m \in \mathbb{N} \) we define the sets
\[ A_{m,k} := A(\hat{f}_k > a + \frac{1}{m}), B_{m,n} := \bigcap_{k=n}^{\infty} A_{m,k}. \]
We will prove that
\[ A(\hat{f} > a) = \bigcup_{n,m} B_{m,n}. \]
Let
\[ x \in A(\hat{f} > a) \]
be arbitrarily chosen. Then
\[ x \in A and \hat{f}(x) > a. \]
Hence, there is enough large natural number \( m_1 \) such that
\[ \hat{f}(x) > a + \frac{1}{m_1}. \]
Using (11), there are enough large natural numbers \( k \) and \( m \) such that
\[ \hat{f}_k(x) > a + \frac{1}{m}, \]
i.e., \( x \in A_{m,k} \).
From here, it follows that there is enough large \( n \) so that \( x \in A_{m,k} \) for every \( k \geq n \), i.e., \( x \in B_{m,n} \) and then \( x \in \bigcup_{m,n} B_{m,n} \).
Since \( x \in A(\hat{f} > a) \) was arbitrarily chosen and we get that it is an element of the set \( \bigcup_{m,n} B_{m,n} \), we conclude that
\[ A(\hat{f} > a) \subset \bigcup_{m,n} B_{m,n}. \]
Let now \( x \in \bigcup_{m,n} B_{m,n} \) be arbitrarily chosen. Then, there are \( m_2, n \in \mathbb{N} \) so that
\[ x \in B_{m_2,n_1} = \bigcap_{k=n_1}^{\infty} A_{m_2,k_1} \]
or
\[ \hat{f}_{k_1}(x) > a + \frac{1}{m_2} for \forall k \geq n_1. \]
Hence,
\[ \lim_{k_1 \to \infty} \hat{f}_{k_1}(x) \geq \lim_{k_1 \to \infty} \left(a + \frac{1}{m_2}\right) \]
or
\[ f'(x) \geq a + \frac{1}{m_2} > a. \]

Therefore
\[ x \in A(\hat{f} > a). \]

Since \( x \in \bigcup_{m,n} B_{m,n} \) was arbitrarily chosen and for it we obtain \( x \in A(\hat{f} > a) \), we conclude that
\[ \bigcup_{m,n} B_{m,n} \subset A(\hat{f} > a). \]

From the last relation and from (13) it follows the relation (12).

Since \( f_k(x) \) are measurable, we have that the sets \( A_{m,k} \) are measurable for every \( m,k \in \mathbb{N} \), hence \( B_{m,n} \) are measurable for every \( m,n \in \mathbb{N} \) and then, using (12), the set \( A(\hat{f} > a) \) is measurable. Consequently the is-function \( \hat{f} \) is measurable.

Theorem 2.29. be a sequence of measurable is-functions defined on the set \( A \). If
\[ \lim_{n \to \infty} f_n(x) = \hat{f}(x) \quad (14) \]

Exists for almost everywhere \( x \in A \), then the is-function \( \hat{f}(x) \) is measurable.

Proof. Let \( B \) be the subset of \( A \) so that the relation (14) holds for every \( x \in B \). From the previous theorem it follows that the is-function \( \hat{f}(x) \) is measurable on the set \( B \).

We note that
\[ \mu(A \setminus B) = 0. \]

Therefore the is-function \( \hat{f}(x) \) is measurable on \( A \setminus B \). Hence, the is-function \( \hat{f}(x) \) is measurable on \( A \).

Let
\[ \hat{f}_n, \hat{T}: A \to (0, \infty), f_n, f: A \to \mathbb{R}, \]

\[ 0 < q_1 \leq \hat{T}_n(x), \hat{T}(x) \leq q_2 \text{ for } x \in A, n \in \mathbb{N}. \]

Then
1. \( f_k^+(x) = f_n(x), \hat{T}_n(x), f^+(x) = \frac{f(x)}{\hat{T}(x)} \)
2. \( f_k^+(x) = f_n(x\hat{T}_n(x)), \hat{T}_n(x), f^+(x) = \frac{f(x\hat{T}(x))}{\hat{T}(x)} \)
3. \( f_k^+(x) = f_n(x\hat{T}_n(x)), \hat{T}(x), f^+(x) = \frac{f(x\hat{T}(x))}{\hat{T}(x)} \)
4. \( f_k^+(x) = f_n(x\hat{T}_n(x)), f^+(x) = f(x\hat{T}(x)) \)
5. \( f_k^+(x) = f_n(x\hat{T}_n(x)), f^+(x) = f(x\hat{T}(x)) \)

\[ f_k^+(x) = f_n(x\hat{T}_n(x)), f^+(x) = f(x\hat{T}(x)) \]

If
\[ \frac{x}{\hat{T}_n(x)} \cdot \frac{x}{\hat{T}(x)}, x \in A. \]

3. The Structure of the Measurable Is-Functions

Theorem 3.1. (is-Lebesgue theorem for is-functions of the first kind) Let there be given a sequence \( \{f_n(x)\}_{n=1}^\infty \) of measurable functions on a set \( A \), all of which are finite almost everywhere. Let also, \( \{\hat{T}_n(x)\}_{n=1}^\infty \) be a sequence of measurable functions on the set \( A \),
\[ 0 < q_1, q_2, \lim_{n \to \infty} f_n(x) = f(x), \]

\[ \lim_{n \to \infty} \hat{T}_n(x) = \hat{T}(x) \]

Almost everywhere on the set \( A \), and \( f(x) \) is finite almost everywhere on \( A \),
\[ q_1 \leq \hat{T}(x) \leq q_2 \]

For all \( x \in A \). Then
\[ \lim_{n \to \infty} \mu A \left( \int f_n(x) - f^+(x) \right) \geq \sigma = 0 \]

For all \( \sigma \geq 0 \).

Proof. We will note that the limit functions \( f(x) \) and \( \hat{T}(x) \) are measurable and the sets under considerations are measurable.

Let
\[ A := A(f) = \infty, \]
\[ B_n := A(f_n) = \infty, \]
\[ C := A(f_n \to f), \]
\[ D := B \cup \bigcup_{n=1}^\infty B_n \cup C. \]

Since
\[ \mu B = 0, \mu C = 0, \mu B_n = 0, \]

using the properties of the measurable sets, we have that
\[ \mu Q = 0. \]

Let
\[ A_k(\sigma) = A \left( \frac{f_k - f}{\hat{T}} \geq \sigma \right). \]
We have that
\[ R_1(\sigma) \supset R_2(\sigma) \supset \cdots. \]
Hence,
\[ \lim_{n \to \infty} \mu R_n(\sigma) = \mu M. \]

Let us assume that \( x_0 \not\in \mathbb{Q} \). Then, using the definition of the set \( \mathbb{Q} \), we have
\[ \lim_{n \to \infty} \frac{f_k(x_0)}{\hat{T}_k(x_0)} = \frac{f(x_0)}{\hat{T}(x_0)} \]
Since
\[ 0 < q_1 \leq \hat{T}_n(x), \hat{T}(x) \leq q_2, k=1,2,\ldots,n, \]
we have that
\[ f_1(x_0), f_2(x_0), \ldots, f_k(x_0), \ldots \]
and their limit
\[ \frac{f(x_0)}{\hat{T}(x_0)} \]
are finite.
Therefore there is an enough large natural \( n \) such that
\[ \left| \frac{f_k(x_0)}{\hat{T}_k(x_0)} - \frac{f(x_0)}{\hat{T}(x_0)} \right| < \sigma \]
for every \( k \geq n \). Then \( x_0 \not\in A_k(\sigma), k \geq n \), where \( x_0 \not\in R_n(\sigma) \) and from here \( x_0 \not\in M \).

Consequently \( M \subset \mathbb{Q} \).

Because \( \mu \mathbb{Q} = 0 \), from the last relation, we have that
\[ \mu M = 0, \text{ i.e.,} \]
\[ \lim_{n \to \infty} R_n(\sigma) = 0, \]
and since
\[ A_n(\sigma) \subset R_n(\sigma), \]
\[ \lim_{n \to \infty} R_n(\sigma) = 0 \]
or
\[ \lim_{n \to \infty} \mu A(\left| f_n^\ast (\hat{\varphi}) - f^\ast (\hat{\varphi}) \right| \geq \sigma) = 0. \]

As in above one can prove the following results for the other kinds of is-functions.

Theorem 3.2. (is-Lebesgue theorem for is-functions of the second kind) Let there be given a sequence \( \{f_n(x)\}_{n=1}^\infty \) of measurable functions on a set \( A \), all of which are finite almost everywhere. Let also, \( \{\hat{T}_n(x)\}_{n=1}^\infty \) be a sequence of measurable functions on the set \( A \),
\[ 0 < q_1 \leq \hat{T}_n(x) \leq q_2 \]
For all natural numbers \( n \) and for all \( x \in A \), where \( q_1 \) and \( q_2 \) are positive constants. Suppose that
\[ \lim_{n \to \infty} f_n(x) = f(x), \]
\[ \lim_{n \to \infty} \hat{T}_n(x) = \hat{T}(x) \]
Almost everywhere on the set \( A \), and \( f(x) \) is finite almost everywhere on \( A \),
\[ q_1 \leq \hat{T}(x) \leq q_2 \]
For all \( x \in A \). Then
\[ \lim_{n \to \infty} \mu A(\left| f_n^\ast (\hat{\varphi}) - f^\ast (\hat{\varphi}) \right| \geq \sigma) = 0. \]

Theorem 3.3. (is-Lebesgue theorem for is-functions of the third kind) Let there be given a sequence \( \{f_n(x)\}_{n=1}^\infty \) of measurable functions on a set \( A \), all of which are finite almost everywhere. Let also, \( \{\hat{T}_n(x)\}_{n=1}^\infty \) be a sequence of measurable functions on the set \( A \),
\[ 0 < q_1 \leq \hat{T}_n(x) \leq q_2 \]
For all natural numbers \( n \) and for all \( x \in A \), where \( q_1 \) and \( q_2 \) are positive constants. Suppose that
\[ \lim_{n \to \infty} f_n(x) = f(x), \]
\[ \lim_{n \to \infty} \hat{T}_n(x) = \hat{T}(x) \]
Almost everywhere on the set \( A \), and \( f(x) \) is finite almost everywhere on \( A \),
\[ q_1 \leq \hat{T}(x) \leq q_2 \]
For all \( x \in A \). Then
\[ \lim_{n \to \infty} \mu A(\left| f_n^\ast (\hat{\varphi}) - f^\ast (\hat{\varphi}) \right| \geq \sigma) = 0. \]

Theorem 3.4. (is-Lebesgue theorem for is-functions of the fourth kind) Let there be given a sequence \( \{f_n(x)\}_{n=1}^\infty \) of measurable functions on a set \( A \), all of which are finite almost everywhere. Let also, \( \{\hat{T}_n(x)\}_{n=1}^\infty \) be a sequence of measurable functions on the set \( A \),
\[ 0 < q_1 \leq \hat{T}_n(x) \leq q_2 \]
For all natural numbers \( n \) and for all \( x \in A \), where \( q_1 \) and \( q_2 \) are positive constants. Suppose that
\[ \lim_{n \to \infty} f_n(x) = f(x), \]
Almost everywhere on the set $A$, and $f(x)$ is finite almost everywhere on $A$,

$$q_1 \leq \tilde{f}(x) \leq q_2$$

For all $x \in A$. Then

$$\lim_{n \to \infty} \mu A(\{f_n(x) - f(x)\} \geq \sigma) = 0$$

for all $\sigma \geq 0$.

References

[1] S. Georgiev, Foundations of Iso-Differential Calculus, Vol. 1. Nova Science Publishers, Inc., 2014.

[2] P. Roman and R. M. Santilli, "A Lie-admissible model for dissipative plasma," Lettere Nuovo Cimento 2, 449-455 (1969).

[3] R. M. Santilli, "Embedding of Lie-algebras into Lie-admissible algebras," Nuovo Cimento 51, 570 (1967), 33http://www.santillifoundation.org/docs/Santilli-54.pdf

[4] R. M. Santilli, "An introduction to Lie-admissible algebras," Suppl. Nuovo Cimento, 6, 1225 (1968).

[5] R. M. Santilli, "Lie-admissible mechanics for irreversible systems." Meccanica, 1, 3 (1969).

[6] R. M. Santilli, "On a possible Lie-admissible covering of Galilei's relativity in Newtonian mechanics for nonconservative and Galilei form-noninvariant systems.," 1, 223-423(1978), available in free pdf download from http://www.santillifoundation.org/docs/Santilli-58.pdf

[7] R. M. Santilli, "Need of subjecting to an Experimental verication the validity within a hadron of Einstein special relativity and Pauli exclusion principle," Hadronic J. 1, 574-901 (1978), available in free pdf download from http://www.santillifoundation.org/docs/Santilli-73.pdf

[8] R. M. Santilli, Lie-admissible Approach to the Hadronic Structure, Vols. I and II, Hadronic Press (1978) http://www.santillifoundation.org/docs/Santilli-71.pdf http://www.santillifoundation.org/docs/Santilli-72.pdf

[9] R. M. Santilli, Foundation of Theoretical Mechanics, Springer Verlag. Heidelberg, Germany, Volume I (1978), The Inverse Problem in newtonian mechanics, http://www.santillifoundation.org/docs/Santilli-209.pdf Volume II, Birkhoan generalization of hamiltonian mechanics, (1982), http://www.santillifoundation.org/docs/Santilli-69.pdf

[10] R. M. Santilli, "A possible Lie-admissible time-asymmetric model of open nuclear reactions," Lettere Nuovo Cimento 37, 337-344 (1983) http://www.santillifoundation.org/docs/Santilli-53.pdf

[11] R. M. Santilli, "Invariant Lie-admissible formulation of quantum deformations," Found. Phys. 27, 1159-1177 (1997) http://www.santillifoundation.org/docs/Santilli-06.pdf

[12] R. M. Santilli, "Lie-admissible invariant representation of irreversibility for matter and antimatter at the classical and operator levels," Nuovo Cimento B 121, 443 (2006), http://www.santillifoundation.org/docs/Lie-admiss-NCB-1.pdf

[13] R. M. Santilli and T. Vougiouklis. "Lie-admissible hyperalgebras," Italian Journal of Pure and Applied Mathematics, in press (2013) http://www.santillifoundation.org/Lie-admhyprstr.pdf

[14] R. M. Santilli, Elements of Hadronic Mechanics, Volumes I and II Ukraine Academy of Sciences, Kiev, second edition 1995, http://www.santillifoundation.org/docs/Santilli-300.pdf http://www.santillifoundation.org/docs/Santilli-301.pdf

[15] R. M. Santilli, Hadronic Mathematics, Mechanics and Chemistry, Vols. I [18a], II [18b], III [18c], IV [18d] and [18e], International Academic Press, (2008), available as free downloads from http://www.i-b-r.org/HadronicMechanics.htm

[16] R. M. Santilli, "Lie-isotopic Lifting of Special Relativity for Extended Deformable Particles."

[17] R. M. Santilli, Isotopic Generalizations of Galilei and Einstein Relativities, Volumes I and II, International Academic Press (1991) , http://www.santillifoundation.org/docs/Santilli-01.pdf 34 http://www.santillifoundation.org/docs/Santilli-61.pdf

[18] R. M. Santilli, "Origin, problematic aspects and invariant formulation of q-, kand other deformations," Intern. J. Modern Phys. 14, 3157 (1999), available as free download from http://www.santillifoundation.org/docs/Santilli-104.pdf

[19] R. M. Santilli, "Isonumbers and Genonumbers of Dimensions 1, 2, 4, 8, their Isoduals and Pseudoduals, and "Hidden Numbers" of Dimension 3, 5, 6, 7," Algebras, Groups and Geometries Vol. 10, 273 (1993), http://www.santillifoundation.org/docs/Santilli-34.pdf

[20] R. M. Santilli, "Nonlocal-Integral Isotopies of Dierential Calculus, Mechanics and Geometries," in Isotopies of Contemporary Mathematical Structures, P. Vetro Editor, Rendiconti Circolo Matematico Palermo, Suppl. Vol. 42, 7-82
