Strong Coupling Perturbation Theory in Quantum Mechanics

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Abstract

We present a full introduction to the recent devised perturbation theory for strong coupling in quantum mechanics. In order to put the theory in a proper historical perspective, the approach devised in quantum field theory is rapidly presented, showing how it implies a kind of duality in perturbation theory, from the start. The approach of renormalization group in perturbation theory is then presented. This method permits to resum secularities in perturbation theory and makes fully algorithmical the resummation, transforming the perturbation calculations in a step by step computational procedure. The general theorem on which is founded a proper application of the strong coupling expansion, based on a result in the quantum adiabatic theory, is then exposed. This theorem gives the leading order of a strong coupling expansion. Then, after the introduction of the principle of duality in perturbation theory that puts in a proper context the quantum field theory method, the resulting theory of the strong coupling expansion and the free picture are presented. An algorithm for the computation of the perturbation series is finally given. This approach has a lot of applications in fields as quantum optics, condensed matter and so on, extending the original expectations of the quantum field theory method. So, we give some examples of application for a class of two-level systems that, in recent years, proved to be extremely important. One of the most interesting concepts that can be obtained in
this way is that of a Quantum Amplifier (QAMP) that permits to obtain an amplification to the classical level of the quantum fluctuations of the ground state of a single radiation mode.
1 Introduction

After the discovery of the quantum chromodynamics (QCD) and asymptotic freedom [1], it became increasingly important to treat in some way a theory having a non-perturbative behavior. Since then, different approaches have been devised to recover the spectrum of QCD at low energies but here we focus on a perturbative method that, although did not prove to be useful to treat QCD problems, paved the way toward a strong coupling expansion with a possible wider scope. This approach, the strong coupling expansion, reached its best formulation in a paper by Bender and coworkers [2, 3, 4] where, being applied to a $\lambda\phi^4$ quantum field theory, it was proved that a lattice formulation could give a manageable formulation but that the problem is moved on taking the limit of zero lattice spacing on some very singular series for the relevant quantities of the theory. Some resummation techniques were devised without much success.

In quantum field theory it is customary to start with a path integral in the Euclidean space like for a $\lambda\phi^4$ theory (here and in the following $\hbar = c = 1$) [2]

$$Z[J] = \int D\phi \exp \left\{ -\int dx \left[ \frac{1}{2} (\partial\phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \phi^4 + J\phi \right] \right\} (1)$$

and to obtain a weak coupling expansion in $\lambda$ as

$$Z[J] = \exp \left[ -\frac{1}{4} \lambda \int dx \frac{\delta^4}{\delta J(x)} \right] \int D\phi \exp \left\{ -\int dx \left[ \frac{1}{2} (\partial\phi)^2 + \frac{1}{2} m^2 \phi^2 + J\phi \right] \right\} (2)$$

where we have extract formally the quartic term from the path integral and put a functional derivative for each power of the field. Then, we are left with a gaussian integral that can be easily computed and we have the sought expansion that can be cast in the form

$$Z[J] = N \left\{ 1 + \sum_{k=1}^{\infty} A_k [J] \lambda^k \right\}. (3)$$

At this stage one may ask what happens if we do the opposite operation, that is, if we consider the kinetic term $\frac{1}{2} (\partial\phi)^2 + \frac{1}{2} m^2 \phi^2$ as a perturbation. This can be realized rewriting the path integral as

$$Z[J] = \exp \left[ -\frac{1}{2} \int dx \int dy \frac{\delta}{\delta J(x)} G^{-1}(x-y) \frac{\delta}{\delta J(y)} \right] \int D\phi \exp \left\{ -\int dx \left[ \frac{1}{4} \lambda \phi^4 + J\phi \right] \right\} (4)$$
being \( G^{-1}(x-y) = (−\partial^2 + m^2)\delta(x-y) \) the inverse of the free Euclidean Green function. It can be proven that, on a lattice being the theory highly singular, the expansion takes the form [2]

\[
Z[J] = N \left\{ 1 + \sum_{k=1}^{\infty} B_k[J] \lambda^{-k/2} \right\},
\]

where one can see a dependence on the inverse of the coupling constant \( \lambda \). We see that we have arbitrarily chosen different part of the Euclidean action as a perturbation, a possibility offered by the freedom proper to this choice, and, by doing that, we get two perturbation series having as development parameter one the inverse of the other. This kind of “duality” can be found e.g. in fluid mechanics with the Navier-Stokes equation where we can have different perturbative regimes by taking as unperturbed part the Eulerian term or the Navier-Stokes term [3]. This regimes are characterized by large Reynolds number and small Reynolds number respectively. This kind of duality in perturbation theory can be indeed seen rather widely and appear an ubiquitous property due to the freedom in the choice of what a perturbation is. The first formulation in this sense in quantum mechanics appeared in Ref. [6] following a series of works where a strong perturbation theory in quantum mechanics has been formulated [7, 8, 9, 10]. In this work it was shown that the adiabatic approximation, formulated as in [11, 12], is the leading order approximation of what can be called a dual Dyson series, as the weak coupling expansion is the well-known Dyson series.

Being a dual series to the standard Dyson expansion it shares the same problems. Particularly, one of the most relevant questions one has to face is that of secularities [13, 14, 15]. A secularity is a polynomial contributions to the series that increases without bound making the series itself useless. The name “secularity” is taken from celestial mechanics where firstly these terms appeared in perturbation series with a secular timescale. Then, in order to have an useful tool to make computations one has to devise a way to remove such singular terms. In the course of time several approaches have been proposed to this aim [13, 14, 15]. The difficulties with these methods are essentially linked to the impossibility to make them algorithmic in some way. But, recently a new approach has been proposed [16, 17, 18, 19] that solved this problem making perturbation computations straightforward to realize. This method relies on the renormalization group techniques that
aim to find the envelope of the computed perturbation series to the desired order. This approach has been successfully applied in quantum mechanics and to the dual Dyson series [20, 21, 22, 23, 24, 25].

The existence of a strong coupling expansion can prove to be very important as it gives the opportunity to study the solution of a differential equation in different regimes in the parameter space. In turn, this means that new physics can be uncovered. Although the strong coupling expansion has found several applications in different fields as strong atom-laser interaction [26, 27], quantum chaos [28, 29, 30] and quantum Zeno effect [31, 32], the workhorse for a lot of studies in fields as quantum optics, quantum computation and condensed matter is the two-level system [33, 34]. Then, to present a lot of examples of application of the strong coupling expansion, we analyze some examples of two-level Hamiltonians showing how relevant physics can be exploited in this regime giving finally rise to the concept of a quantum amplifier (QAMP), able to amplify the quantum fluctuations of a radiation mode in the ground state to the classical level [35, 36, 37, 38].

This review is structured as follow. In Sec.2 we introduce the duality principle in perturbation theory for the Schrödinger equation. In Sec.3 the fundamental theorem of the strong coupling expansion to the leading order is proved. In Sec.4 the problem of secularities is put forward as a general problem in perturbation theory that is inherited by the strong coupling expansion. In Sec.5 the renormalization group method to solve the secularity problem is presented. In Sec.6 the approaches discussed above are merged to formulate an algorithm for doing perturbation theory. In Sec.7 we apply the strong coupling expansion to a two-level model well-known in quantum optics and finally, in Sec.8 we show how the two-level model discussed in Sec.7 can amplify the quantum fluctuation to the classical level producing a classical field (QAMP). In Sec.9 the conclusions are given.

2 Duality in Perturbation Theory

The starting point of our analysis is given by a quantum system with a Hamiltonian

\[ H = H_0 + V. \]  

(6)

It is usually assumed that the dynamics for th $H_0$ part is known. Adding the $V$ part can make the problem unmanageable unless one uses perturbation theory and the $V$ part is smaller in some sense
with respect to $H_0$. To account for this, we introduce an arbitrary parameter $\lambda$ that we now consider small but we take to be unity at the end of the computation. So, we write $H = H_0 + \lambda V$ and aim to solve the Schrödinger equation

$$(H_0 + \lambda V)|\psi\rangle = i\frac{\partial|\psi\rangle}{\partial t}. \quad (7)$$

When $\lambda$ is a small one does the transformation (interaction picture)

$$|\psi\rangle = e^{-iH_0t}|\psi_I\rangle \quad (8)$$

obtaining the equation to solve

$$e^{iH_0t}\lambda Ve^{-iH_0t}|\psi_I\rangle = i\frac{\partial|\psi_I\rangle}{\partial t}. \quad (9)$$

The solution of this equation, generally know as Dyson series, can be written as

$$|\psi_I\rangle = T \exp \left[ -i \int_0^t dt' e^{iH_0t'} \lambda Ve^{-iH_0t'} \right] |\psi(0)\rangle \quad (10)$$

being $T$ the so called time ordering operator and $|\psi(0)\rangle$ the initial wave function. This is just a formal writing for the small perturbation series

$$|\psi_I\rangle = \left( I - i \int_0^t dt' e^{iH_0t'} \lambda Ve^{-iH_0t'} - \int_0^t dt' e^{iH_0t'} \lambda V e^{-iH_0t'} \int_0^{t'} dt'' e^{iH_0t''} \lambda V e^{-iH_0t''} + \cdots \right) |\psi(0)\rangle \quad (11)$$

that we recognize as a power series, $|\psi_I\rangle = \sum_{n=0}^{\infty} \lambda^n |\phi_n(t)\rangle$, that can have a meaning only for small values of $\lambda$.

We have done some assumptions to derive the Dyson series. We have assumed that $H_0$ is the unperturbed part of the Hamiltonian whose dynamics is known and that $\lambda$ was a small parameter. Then, we have obtained a result that could be meaningful, at least asymptotically. But we can relax both these assumptions in view of the fact that the choice of an unperturbed part and a perturbation is totally arbitrary and one may ask what would be the face of a series where the role of $H_0$ and $V$ is interchanged. We can easily work out this exchange into the Dyson series obtaining

$$|\psi_F\rangle = T \exp \left[ -i \int_0^t dt' e^{i\lambda Vt'} H_0 e^{-i\lambda Vt'} \right] |\psi(0)\rangle. \quad (12)$$
To understand what we have done we also transform the integration variable as $\tau = \lambda t'$ and we have

$$|\psi_F\rangle = T \exp \left[ -i \frac{1}{\lambda} \int_0^{\lambda t} d\tau e^{iV \tau} H_0 e^{-iV \tau} \right] |\psi(0)\rangle. \quad (13)$$

that means the series

$$|\psi_F\rangle = \left( I - i \frac{1}{\lambda} \int_0^{\lambda t} d\tau e^{iV \tau} H_0 e^{-iV \tau} - \frac{1}{\lambda} \int_0^{\lambda t} d\tau e^{iV \tau} H_0 e^{-iV \tau} \frac{1}{\lambda} \int_0^{\lambda t} d\tau' e^{iV \tau'} H_0 e^{-iV \tau'} \right) |\psi(0)\rangle. \quad (14)$$

A meaning can be attached to this series only if we take the limit $\lambda \to \infty$ and we have obtained a dual series with respect to the Dyson series having a development parameter inverse to the latter. So, using a symmetry in the choice of what a perturbation is we were able to uncover a new perturbation series, dual to the small perturbation series, useful in problems with a strong perturbation. Finally, we can set $\lambda = 1$ and we will have the Dyson series as

$$|\psi_I\rangle = T \exp \left[ -i \int_0^t dt' e^{iH_0 t'} V e^{-iH_0 t'} \right] |\psi(0)\rangle \quad (15)$$

after passing to the interaction picture with the unitary transformation $\exp(-iH_0 t)$ and a dual Dyson series

$$|\psi_F\rangle = T \exp \left[ -i \int_0^t dt' e^{iV t'} H_0 e^{-iV t'} \right] |\psi(0)\rangle. \quad (16)$$

passing to the free picture with the unitary transformation $\exp(-iV t)$. It is easily realized that the two series will share the same kind of problems as secularities, asymptotic convergence or divergence and so on. But, we are aware that this does not diminish the usefulness of the perturbative approach.

3 The Leading Order of the Strong Coupling Perturbation Theory

The leading order of the dual Dyson series is quite straightforward to define if the perturbation $V$ does not depends on time. Things are more involved otherwise. Indeed, the dual Dyson series must be redefined if $V$ depends on time and we will prove that a series, derived
formally from the adiabatic approximation, due to Mostafazadeh \cite{11}, is recovered in this case. Again we consider the problem

$$ (H_0 + \lambda V(t))|\psi(t)\rangle = i \frac{\partial |\psi(t)\rangle}{\partial t} $$

(17)

with $\lambda \to \infty$. To recover the dual Dyson series, we have to determine the unitary operator $U_F(t)$ such that

$$ \lambda V(t)U_F(t) = i \frac{\partial U_F(t)}{\partial t} $$

(18)

with the same limit for $\lambda$. It is very easy to recognize here the starting point of the proof of the adiabatic theorem in quantum mechanics\cite{38,12} and we are recovering a series, in the same framework, due to Mostafazadeh \cite{11,12}. This is due to the slowing down implied by the parameter $\lambda$ going to infinity. Then, without recurring to any adiabatic hypothesis, we can write \cite{6}

$$ U_F(t) = \sum_n e^{i\gamma_n(t)}e^{-i\int_0^t v_n(t')dt'}|n;t\rangle\langle n;0| $$

(19)

being $\gamma_n(t)$ the geometric part of the phase and $v_n(t)$ the dynamical part such that $V(t)|n;t\rangle = v_n(t)|n;t\rangle$. We will show that this gives the proper dual Dyson series. Higher order corrections can be written down and are given in Refs.\cite{38,11,12}. It is interesting to note that this theorem has found an application in the studies of Zeno effect in quantum systems \cite{31}.

Once the unitary operator $U_F(t)$ is known, we are able to pursue the computation to the end. Indeed, we have to solve the Schrödinger equation

$$ H_F|\psi_F(t)\rangle = i \frac{\partial |\psi_F(t)\rangle}{\partial t} $$

(20)

being

$$ H_F = \sum_n \sum_m e^{i[\gamma_n(t) - \gamma_m(t)]}e^{-i\int_0^t[v_n(t') - v_m(t')]dt'}\langle m,t|H_0|n,t\rangle|n,m\rangle\langle n,0| $$

(21)

the transformed Hamiltonian. We easily realize that the Hamiltonian $H_F$ can be split in two parts as

$$ H_F = \sum_n \langle n,t|H_0|n,t\rangle|n,0\rangle\langle n,0| $$

(22)

$$ + \sum_{m \neq n} e^{i[\gamma_n(t) - \gamma_m(t)]}e^{-i\int_0^t[v_n(t') - v_m(t')]dt'}\langle m,t|H_0|n,t\rangle|n,m\rangle\langle n,0| $$
and we are able to obtain the analogous equations of the interaction picture for the probability amplitudes that apply in the case of the dual Dyson series. In order to obtain this result we write the solution of eq. (20) as

$$|ψ_F(t)⟩ = \sum_n c_n(t)e^{-i \int_0^t h_{0n}(t')dt'}|n,0⟩$$

being

$$h_{0n}(t) = ⟨n,t|H_0|n,t⟩.$$  \hspace{1cm} (24)

This gives the equations for the amplitudes

$$i\dot{c}_n(t) = \sum_k e^{i(\gamma_k(n)−\gamma_n(t))}e^{-i \int_0^t [ε_k(t')−ε_n(t')]dt'}⟨n, t|H_0|k, t⟩c_k(t)$$

with $ε_n(t) = h_{0n}(t) + v_n(t)$. These equations are similar to the amplitude equations in interaction picture, normally found in textbooks.

Some considerations are in order at this point. By the adiabatic theorem we can only apply the above approach for a discrete spectrum on the perturbation. This in turn implies that, if the spectrum of the perturbation is continuous as in the coordinate space, then a lattice regularization is needed. This takes us back to Bender et al. approach \cite{2,3,4} with all the difficulties this means by taking the limit of the lattice spacing going to zero.

\section{The Secularity Problem in Perturbation Theory}

A perturbation series is plagued by secularities when polynomial terms in time appear in it. These terms have the property of being not bounded for large time making the series generally useless.

In order to have an idea of what really happens in these situations, let us consider the well known quantum mechanical problem

$$H = \frac{\Delta}{2} σ_3 + gσ_1 \cos(ωt)$$

representing a two level atom driven by an oscillating field with frequency ω. $σ_1, σ_3$ are Pauli matrices, $g$ is the coupling constant and $Δ$ is the separation between the atom levels. This problem has a
large body of literature due to its vast field of applications (see e.g. Ref. [23, 33]).

We can apply to this problem all the machinery devised in the preceding section for the strong coupling expansion. So, one has

$$U_F(t) = e^{-i\frac{\Delta}{2} \sin(\omega t)} |+\rangle\langle +| + e^{i\frac{\Delta}{2} \sin(\omega t)} |-\rangle\langle -|$$  \hspace{1cm} (27)

being $$\sigma_1|\pm\rangle = \pm|\pm\rangle$$. By using the explicit expression for the states $$|\pm\rangle$$ one can prove that $$U_F(t) = e^{-i\sigma_1 \frac{\Delta}{2} \sin(\omega t)}$$ as it should be for the free picture. It is very easy to obtain

$$H_F = \frac{\Delta}{2} \left[ e^{-i\frac{2\Delta}{\omega} \sin(\omega t)} |-\rangle\langle +| + e^{i\frac{2\Delta}{\omega} \sin(\omega t)} |+\rangle\langle -| \right].$$ \hspace{1cm} (28)

Looking for a solution in the form $$|\psi_F(t)\rangle = c_+(t)|+\rangle + c_-(t)|-\rangle$$ we obtain the equations for the amplitudes

$$i\dot{c}_+(t) = \frac{\Delta}{2} e^{i\frac{2\Delta}{\omega} \sin(\omega t)} c_-(t)$$  \hspace{1cm} (29)

$$i\dot{c}_-(t) = \frac{\Delta}{2} e^{-i\frac{2\Delta}{\omega} \sin(\omega t)} c_+(t)$$

that give rise to the perturbation series till first order

$$c_+(t) = c_+(0) - \frac{i\Delta}{2} J_0(z)c_-(0)t - \frac{\Delta}{2} \sum_{n \neq 0} J_n(z) \frac{e^{i\omega n} - 1}{n\omega} c_-(0) + \cdots$$  \hspace{1cm} (30)

$$c_-(t) = c_-(0) + \frac{i\Delta}{2} J_0(z)c_+(0)t + \frac{\Delta}{2} \sum_{n \neq 0} J_n(z) \frac{e^{-i\omega n} - 1}{n\omega} c_+(0) + \cdots$$

where use has been made of the relation $$e^{iz \sin(\omega t)} = \sum_{n=-\infty}^{\infty} J_n(z) e^{i\omega nt}$$ with $$J_n(z)$$ the n-th Bessel function of integer order and $$z = \frac{2\Delta}{\omega}$$ in our case. We see immediately that the perturbation series is plagued with secularities and so is useless at this stage. We have to understand what is going on here by properly resum such terms. In this way, we will discover here a physical effect, i.e. Rabi oscillations between the states $$|\pm\rangle$$. The resummation technique to accomplish our task is described in the next section where we will complete our computation.

5 Renormalization Group Method for the Resummation of Secular Terms

The method of renormalization group to resum secularities in a perturbation series was firstly proposed in [16,17]. Here we present an el-
egant reformulation obtained by the mathematical theory of envelopes by Kunihiro [18, 19].

Kunihiro approach can be described as follows. Let us consider the following equation

\[
\dot{x}(t) = f(x(t), t) \tag{31}
\]

being \(x(t)\) a vector in \(\mathbb{R}^n\). The initial condition is given by \(x(t_0) = X(t_0)\). At this stage we assume \(X(t_0)\) not yet specified. We write the solution of this equation as \(x(t; t_0, X(t_0))\) that is exact. If we change \(t_0\) to \(t'_0\) we are able to determine \(X(t_0)\) by assuming that the solution should not change

\[
x(t; t_0, X(t_0)) = x(t; t'_0, X(t'_0)) \tag{32}
\]

that in the limit \(t_0 \to t'_0\) becomes

\[
\frac{dx}{dt_0} \bigg|_{t_0=t} = \frac{\partial x}{\partial t_0} \bigg|_{t_0=t} + \frac{\partial x}{\partial X} \frac{\partial X}{\partial t_0} \bigg|_{t_0=t} = 0 \tag{33}
\]

giving the evolution equation or flow equation of the initial value \(X(t_0)\). We recognize here a renormalization group equation and this gives the name to the method.

Till now, all our equations are exact and no perturbation theory entered in any part of our argument. But, except for a few cases, the solution \(x(t; t_0, X(t_0))\) is only known perturbatively and such a solution are generally valid only locally, i.e. for \(t \sim t_0\) and \(t \sim t'_0\) and a more restrictive request should be demanded to our renormalization group equation

\[
\frac{dx}{dt_0} \bigg|_{t_0=t_0} = \frac{\partial x}{\partial t_0} \bigg|_{t_0=t_0} + \frac{\partial x}{\partial X} \frac{\partial X}{\partial t_0} \bigg|_{t_0=t_0} = 0. \tag{34}
\]

But this equation can be interpreted by the mathematical theory of envelopes [18]. Indeed, varying \(t_0\) we have that \(x(t; t_0, X(t_0))\) is a family of curves with \(t_0\) being a characterizing parameter. Then, eq. (34) becomes an equation to compute the envelope of such a family of curves. Such an envelope is given by \(x(t; t_0 = t) = X(t)\), the initial condition. It can be proven that \(X(t)\) satisfies the equation (31) in a global domain up to the order with which \(x(t; t_0)\) satisfies it locally for \(t \sim t_0\). This gives the condition for the computation of the envelope

\[
\frac{dx}{dt_0} \bigg|_{t_0=t} = 0 \tag{35}
\]
The Kunihiro method is very effective to build resummed perturbation series, eliminating the secularities that appear to plague them. Besides, it permits to transform a perturbation computation in an algorithm straightforward to apply as we are going to see in the next section.

6 An Algorithm for Doing Perturbation Theory in Quantum Mechanics

In order to exploit what we mean by an algorithmic computation of a perturbation series, we come back to the example given in sec 4. All we have to do, as our first step, is to recompute the perturbation series at a generic initial time \( t_0 \) and to assume generic initial conditions. This yields

\[
\begin{align*}
c_+(t) &= \tilde{c}_+(t_0) - \frac{\Delta}{2} J_0(z) \tilde{c}_-(t_0) (t - t_0) - \frac{\Delta}{2} \sum_{n \neq 0} J_n(z) \frac{e^{i\omega t} - e^{i\omega t_0}}{n\omega} \tilde{c}_-(t_0) + \cdots \tag{36} \\
c_-(t) &= \tilde{c}_-(t_0) - \frac{\Delta}{2} J_0(z) \tilde{c}_+(t_0) (t - t_0) + \frac{\Delta}{2} \sum_{n \neq 0} J_n(z) \frac{e^{-i\omega t} - e^{-i\omega t_0}}{n\omega} \tilde{c}_+(t_0) + \cdots.
\end{align*}
\]

We realize easily that the envelope could not be computed with this series as is. What we need here is to dress all the phases in the exponentials. This gives

\[
\begin{align*}
c_+(t) &= \tilde{c}_+(t_0) - \frac{\Delta}{2} J_0(z) \tilde{c}_-(t_0) (t - t_0) - \frac{\Delta}{2} \sum_{n \neq 0} J_n(z) \frac{e^{i\omega t} - e^{-i\omega \phi(t_0)}}{n\omega} \tilde{c}_-(t_0) + \cdots \tag{37} \\
c_-(t) &= \tilde{c}_-(t_0) - \frac{\Delta}{2} J_0(z) \tilde{c}_+(t_0) (t - t_0) + \frac{\Delta}{2} \sum_{n \neq 0} J_n(z) \frac{e^{-i\omega t} - e^{-i\omega \phi(t_0)}}{n\omega} \tilde{c}_+(t_0) + \cdots.
\end{align*}
\]

where we have introduced a renormalizable phase \( \phi(t_0) = -t_0 \). This is strictly linked to the property of quantum systems to have a freedom in the choice of the initial phase. At this point we use the renormalization group or envelope equation giving

\[
\begin{align*}
\frac{\partial \tilde{c}_+(t)}{\partial t} + i \frac{\Delta}{2} J_0(z) \tilde{c}_-(t) + \cdots &= 0 \quad (38) \\
\frac{\partial \tilde{c}_-(t)}{\partial t} + i \frac{\Delta}{2} J_0(z) \tilde{c}_+(t) + \cdots &= 0 \\
\frac{\partial \phi(t)}{\partial t} + \cdots &= 0
\end{align*}
\]
and the perturbation solution is then given by
\begin{align}
    c_+(t) &= \tilde{c}_+ - \frac{\Delta}{2} \sum_{n \neq 0} J_n(z) \frac{e^{in\omega t} - 1}{n\omega} \tilde{c}_-(t) + \cdots \quad (39) \\
    c_-(t) &= \tilde{c}_- + \frac{\Delta}{2} \sum_{n \neq 0} J_n(z) \frac{e^{-in\omega t} - 1}{n\omega} \tilde{c}_+(t) + \cdots.
\end{align}
completely solving our problem, till first order, and having all secular terms properly removed in an algorithmic and simple way.
We can finally exploit fully our algorithm for doing perturbation theory in presence of secular terms in the strong coupling regime for quantum mechanics. The rules for computing the unitary evolution operator, at any desired order, are the following [23]:

1. Consider the following unitary transformation on the equation \[ U_F(t) = \sum_n e^{i\gamma_n(t)} e^{-i \int_0^t \gamma_n(t') dt'} |n; t\rangle \langle n; 0| \quad (40) \]
with the eigenstates of the perturbation \( |n; t\rangle \). This gives the transformed Hamiltonian
\[ H_F(t) = U_F^\dagger(t) H_0 U_F(t). \quad (41) \]
The dual Dyson series is computed by [7, 8, 9, 10]
\[ S_D(t, t_0) = T \exp \left[ -ie \int_{t_0}^t H_F(t') dt' \right] \quad (42) \]
being as usual \( T \) the time ordering operator and an ordering parameter \( \epsilon \) has been introduced that will be taken unity at the end of computation. It is fundamental for our argument that the computation of this series is performed at a different starting point \( t_0 \).

2. Assume, at the start, that the time evolution operator has the form
\[ U(t, t_0) = U_F(t) S_D(t, t_0) U_R(t_0) \quad (43) \]
where \( U_R(t_0) \) is a “renormalizable” part of the unitary evolution.
3. At the given order one gets \( S_D(t, t_0) \) as
\[
S_D(t, t_0) = I - ie f_1(t, t_0) - \epsilon^2 f_2(t, t_0) + \ldots \tag{44}
\]
and, at this stage, if some oscillating functions in \( t_0 \) appear like \( e^{-i\omega t_0} \) then introduce the phase \( \phi(t_0) = -t_0 \) as a "renormalizable" parameter rewriting it as \( e^{i\omega \phi(t_0)} \). The secularities must be left untouched.

4. Eliminate the dependence on \( t_0 \) by requiring\[18, 19\]
\[
\frac{dU(t, t_0)}{dt_0} \bigg|_{t_0=t} = 0 \tag{45}
\]
and one obtains the renormalization group equations
\[
\frac{dU_R(t)}{dt} = \epsilon g_1 U_R(t) + \epsilon^2 g_2 U_R(t) + O(\epsilon^3) \tag{46}
\]
\[
\frac{d\phi(t)}{dt} = \epsilon \phi_1(t) + \epsilon^2 \phi_2(t) + O(\epsilon^3)
\]
where, at some stage, to obtain such equations at the second order, we have to use their expressions at the first order as, to compute their form at order \( n \)-th one have to use these equations at the order \((n-1)\)-th, into the condition \[15\]. This is a step toward the computation of the envelope of the perturbation series as said in sec.5

5. Finally, the renormalization equations should be solved and substituted into the equation
\[
U(t, t_0) \big|_{t_0=t} \tag{47}
\]
giving the solution, i.e. the envelope, we were looking for without secularities at the order we made the computation.

Once the unitary evolution is known, we can easily compute the wave function, given the initial condition, recovering the case we have shown of the driven two-level system.

We are going to see this approach at work in the next sections.

7 Two-Level Systems and the Strong Coupling Perturbation Theory

In quantum optics the interaction between a single radiation mode and a two-level atom proves to be a paradigm for most of applications
The Hamiltonian is
\[ H = \frac{\Delta}{2} \sigma_z + \omega a^\dagger a + g \sigma_x (a^\dagger + a) \] (48)
that differs from Hamiltonian (26) by having a fully quantized radiation field of frequency \( \omega \), rather than a classical field, whose creation and annihilation operator are \( a^\dagger \) and \( a \).

The standard approach [33] to this problem is given by doing the unitary transformation
\[ U_I = e^{-i \frac{\Delta}{2} \sigma_z t} e^{-i \omega a^\dagger a t} \] (49)
and then the rotating wave approximation is applied keeping only the near resonant terms (\( \Delta \approx \omega \)). This reduces our model to the well-known Jaynes-Cummings Hamiltonian [39, 40]
\[ H_{JC} = \frac{\delta}{2} \sigma_z + g (a \sigma_x + a^\dagger \sigma_x) \] (50)
being now \( \delta \) the detuning between the frequency of the radiation field \( \omega \) and the separation between the levels of the atom \( \Delta \). This aspect is well known having a large body of literature since its inception and being a foundational matter for quantum optics.

Instead, here our aim is to realize a complete study of the Hamiltonian (48) from the point of view of our strong coupling approach [42, 43, 44, 45]. For our aims we have to compute \( U_F(t) \) and this is done by solving the eigenvalues problem
\[ [\omega a^\dagger a + g \sigma_x (a^\dagger + a)] |n; \lambda\rangle = E_{n,\lambda} |n; \lambda\rangle \] (51)
whose solution is given by
\[ |n; \lambda\rangle = e^{\frac{\Delta}{2} \lambda (a - a^\dagger)} |n\rangle |\lambda\rangle \] (52)
with \( \sigma_x |\lambda\rangle = \lambda |\lambda\rangle \), \( \lambda = \pm 1 \) and the eigenvalues, independent on \( \lambda \) being \( E_n = n \omega - \frac{g^2}{\omega} \). Then, it is straightforward to write the unitary evolution operator as
\[ U_{F0}(t) = \sum_{n,\lambda} e^{-i E_n t} \langle n; \lambda | [n; \lambda] \rangle \] (53)
that gives rise to the Hamiltonian
\[ H_F = U_{F0}(t) \frac{\Delta}{2} \sigma_z U_{F0}(t) \] (54)
It is easily realized that it can be rewritten in the form

\[ H_F = H'_0 + H_1 \]  

being

\[ H'_0 = \Delta \sum_n e^{-\frac{2g^2}{\omega^2}} L_n \left( \left\langle [n; 1] \langle [n; -1]|1 - 1 \rangle + [n; -1]\right\rangle [n; 1]|1 \right\rangle \]  

(56)

being \( L_n \) the n-th Laguerre polynomial \[46\] and

\[ H_1 = \Delta \sum_{m,n,m \neq n} e^{-i(n-m)t} \left[ \langle n| e^{-\frac{2g^2}{\omega^2} (a-a^\dagger)} |m\rangle \langle [n; 1] [m; 1]|1 \rangle \right] \]  

At this point we can iterate the procedure by diagonalizing the Hamiltonian \[56\]. The eigenstates are

\[ |\psi_n; \sigma\rangle = \frac{1}{\sqrt{2}} [\sigma [n; 1]|1 \rangle + [n; -1]|-1 \rangle] \]  

(58)

and the eigenvalues are

\[ \tilde{E}_{n,\sigma} = \sigma \Delta e^{-\frac{2g^2}{\omega^2}} L_n \left( \frac{4g^2}{\omega^2} \right) \]  

(59)

being \( \sigma = \pm 1 \). So, we can write the unitary transformation

\[ U_{F1}(t) = \sum_{n,\sigma} e^{-i\tilde{E}_{n,\sigma}t} |\psi_n; \sigma\rangle \langle \psi_n; \sigma| \]  

(60)

and get the transformed Hamiltonian

\[ H'_1 = U_{F1}(t)^\dagger H_1 U_{F1}(t) \]  

(61)

that is

\[ H'_1 = \Delta \sum_{m,n,m \neq n} \sum_{\sigma_1,\sigma_2} \mathcal{R}_{mn,\sigma_1\sigma_2} e^{-i(n-m)\omega - (\tilde{E}_{n,\sigma_1} - \tilde{E}_{m,\sigma_2})t} |\psi_n; \sigma_1\rangle \langle \psi_m; \sigma_2| \]  

(62)

being

\[ \mathcal{R}_{mn,\sigma_1\sigma_2} = \frac{1}{2} \left[ \langle n| e^{-\frac{2g^2}{\omega^2} (a-a^\dagger)} |m\rangle \sigma_1 + \langle n| e^{\frac{2g^2}{\omega^2} (a-a^\dagger)} |m\rangle \sigma_2 \right] . \]  

(63)
So, we have accomplished the unitary transformation

\[ U_F(t) = U_{F0}(t)U_{F1}(t) \]  \hspace{1cm} (64)

and we are left with the Schrödinger equation for our aims

\[ H'_{1}S_D(t,t_0) = i\frac{\partial S_D(t,t_0)}{\partial t} \]  \hspace{1cm} (65)

that we solve by the perturbation theory obtaining the strong coupling expansion for this problem. As said in the formulation of the algorithm in Sec 6, we assume a solution in the form

\[ U(t,t_0) = U_{F}(t)S_D(t,t_0)U_R(t_0) \]  \hspace{1cm} (66)

being \( U_R(t_0) \) a renormalizable part of the unitary evolution.

In order to obtain the sought series we need to understand where resonances occur, that is where the condition \((n - m)\omega - (\tilde{E}_{n,\sigma_1} - \tilde{E}_{m,\sigma_2}) = 0\) is met. This happens for \(n \neq m\) and for the two other conditions \(\sigma_1 = \sigma_2\) (intraband resonance) or \(\sigma_1 \neq \sigma_2\) (interband resonance). Then, as required by our algorithm, we compute the dual Dyson series at an initial time \(t_0\) obtaining

\[ S_D(t,t_0) = I - i\frac{\Delta}{2} \left[ \sum_{\text{intraband}} R_{mn,\sigma_1,\sigma_2} |\psi_n;\sigma_1\rangle \langle \psi_m;\sigma_2| (t-t_0) \\
+ \sum_{\text{interband}} R_{mn,\sigma_1,\sigma_2} |\psi_n;\sigma_1\rangle \langle \psi_m;\sigma_2| (t-t_0) \\
+ \sum_{m,n,\sigma_1,\sigma_2 \text{ out of resonance}} R_{mn,\sigma_1,\sigma_2} \times \\
e^{-i[(n-m)\omega - (\tilde{E}_{n,\sigma_1} - \tilde{E}_{m,\sigma_2})]t} - e^{-i[(n-m)\omega - (\tilde{E}_{n,\sigma_1} - \tilde{E}_{m,\sigma_2})]t_0} - i[(n-m)\omega - (\tilde{E}_{n,\sigma_1} - \tilde{E}_{m,\sigma_2})] |\psi_n;\sigma_1\rangle \langle \psi_m;\sigma_2| + \ldots \right] \]  \hspace{1cm} (67)

The next step is to introduce the phase \(\phi(t_0) = -t_0\) into the exponentials changing the series into

\[ S_D(t,t_0) = I - i\frac{\Delta}{2} \left[ \sum_{\text{intraband}} R_{mn,\sigma_1,\sigma_2} |\psi_n;\sigma_1\rangle \langle \psi_m;\sigma_2| (t-t_0) \\
+ \sum_{\text{interband}} R_{mn,\sigma_1,\sigma_2} |\psi_n;\sigma_1\rangle \langle \psi_m;\sigma_2| (t-t_0) \\
+ \sum_{m,n,\sigma_1,\sigma_2 \text{ out of resonance}} R_{mn,\sigma_1,\sigma_2} \times \\
e^{-i[(n-m)\omega - (\tilde{E}_{n,\sigma_1} - \tilde{E}_{m,\sigma_2})]t} - e^{-i[(n-m)\omega - (\tilde{E}_{n,\sigma_1} - \tilde{E}_{m,\sigma_2})]t_0} - i[(n-m)\omega - (\tilde{E}_{n,\sigma_1} - \tilde{E}_{m,\sigma_2})] |\psi_n;\sigma_1\rangle \langle \psi_m;\sigma_2| + \ldots \right] \]  \hspace{1cm} (68)
Finally, we can compute the envelope of $U(t, t_0)$ obtaining the renormalization group equations

$$\frac{dU_R(t)}{dt} = -i\Delta \left[ \sum_{\text{intra\text{band}}} R_{mn,\sigma_1\sigma_2} |\psi_n;\sigma_1\rangle\langle\psi_m;\sigma_2| + \sum_{\text{inter\text{band}}} R_{mn,\sigma_1\sigma_2} |\psi_n;\sigma_1\rangle\langle\psi_m;\sigma_2| U_R(t) + \ldots \right]$$

$$\frac{d\phi(t)}{dt} + \ldots = 0.$$

Then, computing $U(t, t_0)|_{t_0=t}$, we get the series

$$U(t) = U_F(t) \left[ I + \frac{\Delta}{2} \sum_{m,n,\sigma_1,\sigma_2} R_{mn,\sigma_1\sigma_2} \times \right.$$

$$\left. \frac{e^{-i[(n-m)\omega-(\tilde{E}_{n,\sigma_1}-\tilde{E}_{m,\sigma_2})]t - 1}}{(n-m)\omega - (\tilde{E}_{n,\sigma_1} - \tilde{E}_{m,\sigma_2})} |\psi_n;\sigma_1\rangle\langle\psi_m;\sigma_2| + \ldots \right] \times U_R(t).$$

The relevant result is that, by the renormalization group method, we have resummed the perturbation series obtaining the unitary evolution, $U_R(t)$, proper to Rabi oscillations as it should be [42, 43, 44, 45] plus a first order correction. Rabi oscillations in the strong coupling regime, as described here, have been recently observed in Josephson junctions [47].

### 8 An Application: The Quantum Amplifier (QAMP)

The next step is to generalize the model of sec.7 to $N$ two-level atoms. We will find a new physical effect that can be seen as a quantum amplification of the vacuum fluctuations, that is, we realize a quantum amplifier (QAMP). For our aims, it is very easy to generalize the
Hamiltonian (48) as
\[ H_N = \frac{\Delta}{2} \sum_{i=1}^{N} \sigma_{zi} + \omega a^\dagger a + g \sum_{i=1}^{N} \sigma_{xi}(a^\dagger + a). \] (71)

Now, we introduce the analogous of angular momentum operators as
\[
S_x = \frac{1}{2} \sum_{i=1}^{N} \sigma_{xi}
\]
\[
S_y = \frac{1}{2} \sum_{i=1}^{N} \sigma_{yi}
\]
\[
S_z = \frac{1}{2} \sum_{i=1}^{N} \sigma_{zi}
\]
\[
S^2 = S_x^2 + S_y^2 + S_z^2
\]
with the well-known commutation relations \([S, S_i] = 0\) and \([S_i, S_j] = i\epsilon_{ijk}S_k\), with the index \(i, j, k\) that can take the values \(x, y, z\). Now, we have that, depending on \(N\) being even we can have a zero momentum state, otherwise
\[
|S_x| \leq S \leq \frac{N}{2}
\]
\[-\frac{N}{2} \leq S_x \leq \frac{N}{2}
\]
giving the Dicke states \(|S, S_x\rangle\).

At this stage, we can iterate the procedure in sec 7 by diagonalizing the Hamiltonian
\[ H_F = \omega a^\dagger a + 2gS_x(a^\dagger + a) \] (74)
with the eigenstates
\[
|n; S, S_x\rangle = e^{\frac{2g}{\omega}S_x(a-a^\dagger)}|n\rangle|S, S_x\rangle
\]
(75)
and eigenvalues
\[
E_{n,S_x} = \left[ n - 4g^2S_x^2 \right] \omega
\]
(76)
and this time we have no degeneracy with respect to the Dicke states as happened for the single two-level atom. Then, it is straightforward to write down the unitary transformation as
\[
U_F(t) = \sum_n \sum_{S, S_x} e^{-iE_{n,S_x}t}|n; S, S_x\rangle\langle n; S, S_x|.
\]
(77)
Already at this stage we can have quantum amplification. Indeed, let us take as initial state $|\psi(0)\rangle = |0\rangle |\frac{\sqrt{N}}{2}, \frac{\sqrt{N}}{2}\rangle$, that is, we have the radiation field in the ground state and the maximal Dicke state. In the ground state, the radiation field has vacuum fluctuations as it is well-known. This gives

$$|\psi(t)\rangle = U_F(t)|\psi(0)\rangle = \sum_n e^{-iE_n t/2} |n; \frac{N}{2}\rangle e^{-\frac{\alpha_n^2}{2} \frac{\alpha_n^2}{\sqrt{n!}}}$$

(78)

that is nothing else that the solution of Ref. [35, 36, 37, 33], that is, a coherent state with a parameter increasing for large $N$. At this stage, we can take two different thermodynamic limits. The first one is given by statistical mechanics, i.e. $N \to \infty$, $V \to \infty$ and $\frac{N}{V} = \text{const}$, being $V$ the volume that contains the radiation mode (e.g. a cavity). The second one is given just by the limit $N \to \infty$ keeping fixed the volume. In the former case, we observe that $g \propto \frac{1}{\sqrt{V}}$ and so, the thermodynamic limit gives a classical radiation state in the thermodynamic limit. In the latter case, the result is similar but we have the parameter of the coherent state increasing as $N$, i.e. faster. Again, we get a classical radiation field due to the fact that the vacuum fluctuations are washed out in both limits. These have been amplified to the classical level and we have produced an intense radiation field. We have a QAMP.

Higher order corrections can now be computed by the dual Dyson series as usual by the Hamiltonian

$$H_F = U_F(t) \Delta S_x U_F(t)$$

(79)

that gives us

$$H_F = \sum_n \sum_{S,S',S_x} \langle [n; S', S_x] | \Delta S_x | [n; S, S_x] \rangle \langle [n; S, S_x] | [n; S, S_x] \rangle$$

$$+ \sum_{m,n} \sum_{S,S',S_x} e^{-i(E_n - E_{m,S_x}) t} \langle [m; S', S_x] | \Delta S_x | [n; S, S_x] \rangle \langle [m; S', S_x] | [n; S, S_x] \rangle \langle [n; S, S_x] | [n; S, S_x] \rangle.$$

(80)

The situation is more involved than the model of a single two-level atom given in sec? but the approach is identical. Here, the main result is that, in the limit $N \to \infty$, one gets a classical radiation field. This tends to become an exact result [37], so, even if we started with the strong coupling expansion, we arrived to a non-perturbative result.
9 Conclusions

We have reviewed the strong coupling expansion as can be applied to time dependent problems in quantum mechanics. This approach proved to be very fruitful for the study of a quantum system in different regime, that is, in different regions of the parameter space of the Hamiltonian.

Having introduced the renormalization group method for removing secularities in the perturbation series, in a formulation due to Kunihiro, we have built an algorithm for doing perturbation theory, making very simple the computation of higher order terms in the series, without any unbounded term in time.

We have seen the method in action by the analysis of a two-level atom in a single radiation mode. We have obtained the Rabi oscillation in the strong coupling regime that have been recently observed in Josephson junctions.

The generalization of this model to $N$ two-level atoms gives a possible description of a new effect that can be seen as a quantum amplifier (QAMP) of vacuum fluctuations of the radiation field. The effect appears in the thermodynamic limit $N \to \infty$.

We can conclude that a fruitful approach for doing perturbation theory is now available to analyze quantum systems, in the time domain, in different regions of the parameter space of the Hamiltonian.

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