On the $d$-dimensional Quasi-Equally Spaced Sampling

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Abstract

We study a class of random matrices that appear in several communication and signal processing applications, and whose asymptotic eigenvalue distribution is closely related to the reconstruction error of an irregularly sampled bandlimited signal. We focus on the case where the random variables characterizing these matrices are $d$-dimensional vectors, independent, and quasi-equally spaced, i.e., they have an arbitrary distribution and their averages are vertices of a $d$-dimensional grid. Although a closed form expression of the eigenvalue distribution is still unknown, under these conditions we are able (i) to derive the distribution moments as the matrix size grows to infinity, while its aspect ratio is kept constant, and (ii) to show that the eigenvalue distribution tends to the Marčenko-Pastur law as $d \to \infty$. These results can find application in several fields, as an example we show how they can be used for the estimation of the mean square error provided by linear reconstruction techniques.

EDICS: DSP-RECO Signal reconstruction, DSP-SAMP Sampling, SPC-PERF Performance analysis and bounds.
I. INTRODUCTION

Consider the class of random matrices of size \((2M + 1) \times r\), with entries given by

\[
G = \frac{1}{\sqrt{2M + 1}} \begin{bmatrix}
e^{j2\pi M x_1} & \cdots & e^{j2\pi M x_r} \\
\vdots & \ddots & \vdots \\
e^{+j2\pi M x_1} & \cdots & e^{+j2\pi M x_r}
\end{bmatrix}
\]  

(1)

The generic element of \(G\) can be written as: \(G_{\ell,q} = \frac{1}{\sqrt{2M + 1}} e^{j2\pi \ell x_q}\), \(\ell = -M, \ldots, M\), \(q = 0, \ldots, r - 1\), where \(x_q\) are independent random variables characterized by a probability density function (pdf) \(f_{x_q}(z)\), with \(0 \leq z \leq 1\). These matrices are Vandermonde matrices with complex exponential entries; they appear in many signal/image processing applications and have been studied in a number of recent works, (see e.g., [1]–[8]). More specifically, in the field of signal processing for sensor networks, [1], [2] studied the performance of linear reconstruction techniques for physical fields irregularly sampled by sensors. In such scenario, the random variables \(x_q\) in (1) represent the coordinates of the sensor nodes. The work in [3] addressed the case where these coordinates are uniformly distributed and subject to an unknown jitter. In the field of communications, the study in [8] presented a number of applications where these matrices appear, which range from multiuser MIMO systems to multifold scattering.

In spite of their numerous applications, few results are known for the Vandermonde matrices in (1). In particular, a closed form expression for the eigenvalue distribution of the Hermitian Toeplitz matrix \(GG^\dagger\), as well as its asymptotic behavior, would be of great interest. As an example, in [1], [2], [6], it has been observed that the performance of linear techniques for reconstructing a signal from a set of irregularly-spaced samples with known coordinates is a function of the asymptotic eigenvalue distribution of \(GG^\dagger\). The asymptotic eigenvalue distribution of \(GG^\dagger\) is defined as the distribution of its eigenvalues, in the limit of \(M\) and \(r\) growing to infinity while their ratio is kept constant. Unfortunately, such distribution is still unknown.

In this work, we consider a general formulation which extends the model in (1) to the \(d\)-dimensional domain. We study the properties of random matrices of size \((2M + 1)^d \times r\) and entries given by

\[
(G_d)_{\nu(\ell),q} = \frac{1}{\sqrt{(2M + 1)^d}} e^{j2\pi \ell^T x_q}
\]  

(2)

where the vectors \(x_q = [x_{q1}, \ldots, x_{qd}]^T\) have independent entries, characterized by the pdf \(f_{x_qm}(z)\),
\( q = 0, \ldots, r - 1, m = 1, \ldots, d, \) and \( d \) is the number of dimensions. The invertible function

\[
\nu(\ell) = \sum_{m=1}^{d} (2M + 1)^{m-1} \ell_m
\]

maps the vector of integers \( \ell = [\ell_1, \ldots, \ell_d]^T \), \( \ell_m = -M, \ldots, M \) onto a scalar index, i.e., the row index of the matrix \( G_d \). Notice that, when \( d = 1 \), \( G_d \) reduces to (1).

For the matrix model in (2), we study the interesting case where \( x_q \) are independent, quasi-equally spaced random variables in the \( d \)-dimensional hypercube \([0,1)^d\). In other words, we assume that the averages of \( x_q \) are the vertices of a \( d \)-dimensional grid in \([0,1)^d\). This is often the case arising in measurement systems affected by jitter, or in sensor network deployments where the sensors sampling the physical field can only be roughly placed at equally spaced positions, due to terrain conditions and deployment practicality [9]. Note that the distribution of the random variables \( x \) can be of any kind, the only assumption we make is on their averages being equally spaced. Since an analytic expression of the eigenvalue distribution of \( G_d G_d^\dagger \) is unknown, we derive a closed form expression for its moments. This enables us to show that, as \( d \to \infty \), the eigenvalue distribution tends to the Marčenko-Pastur law [12]. At the end of the paper, we present some numerical results and applications where the moments and the asymptotic approximation to the eigenvalue distribution of \( G_d G_d^\dagger \) can be of great use.

II. PREVIOUS RESULTS AND PROBLEM FORMULATION

As a first step, we briefly review previous results on the \( G_d \) matrices. In a one-dimensional domain \((d = 1)\), the work in [1] considered an irregularly sampled bandlimited signal, which is reconstructed using linear techniques and assuming the samples coordinates to be known. The performance of the reconstruction system was derived as a function of the eigenvalue distribution \( f_\lambda(1, \beta, z) \) of the matrix \( T_1 = \beta G_1 G_1^\dagger \), where \( \beta \) is the aspect ratio\(^1\) of \( G_1 \) [1], [2]. An explicit expression of the moments

\[
\mathbb{E}[\lambda_{1,\beta}^p] = \int_0^\infty z^p f_\lambda(1, \beta, z) \, dz
\]

was attained in [4], [5], for the specific case where \( x_q \) are uniformly distributed in \([0,1)\). Also, in the case where \( x_q \) are independent, quasi-equally spaced random variables, the analytic expression of the second moment of the eigenvalue distribution of \( T \), i.e., \( \mathbb{E}[\lambda_{1,\beta}^2] \), was obtained in [3]. Then, in [7] the moments \( f_\lambda(1, \beta, z) \) were derived for an arbitrary distribution \( f_{x_q}(z) \).

In [4], [5], the \( d \)-dimensional model (2) was also investigated. There, the properties of the random matrices \( G_d \) were studied in the case where the vectors \( x_q = [x_{q1}, \ldots, x_{qd}]^T \) have independent entries.

\(^1\)The aspect ratio of \( G \) is the ratio between the number of rows and the number of columns of the matrix
uniformly distributed in the hypercube \([0, 1]^d\). Under such assumptions, and for given \(d\) and aspect ratio \(\beta\), an analytic expression of the moments of \(f_{\lambda}(d, \beta, z)\) was derived and it was shown that, as \(d \to \infty\), \(f_{\lambda}(d, \beta, z)\) tends to the Marčenko-Pastur law \([12]\), i.e.,

\[
\lim_{d \to \infty} f_{\lambda}(d, \beta, z) = f_{\text{MP}}(\beta, z) = \frac{\sqrt{(c_1 - z)(z - c_2)}}{2\pi z\beta}
\]

where \(c_1, c_2 = (1 \pm \sqrt{\beta})^2, 0 < \beta \leq 1, c_2 \leq x \leq c_1\).

The following sections detail the problem addressed in this work and introduce some useful notations.

A. The quasi-equally spaced multidimensional model

We consider the matrix class in (2) and assume that the vectors \(\mathbf{x}\) are independent, quasi-equally spaced random variables in the \(d\)-dimensional hypercube \([0, 1]^d\), i.e., the averages of \(\mathbf{x}\) are the vertices of a \(d\)-dimensional grid in \([0, 1]^d\).

We define \(\rho\) as the number of vertices per dimension, thus the total number of vertices is \(r = \rho^d\). We denote the coordinate of a generic vertex of the grid by the vector \(q/\rho \in [0, 1]^d\), where \(q = [q_1, \ldots, q_d]^T\), is an integer vector and \(q_m = 0, \ldots, \rho - 1\). For notation simplicity and in analogy with (3), we identify the vertex with coordinate \(q/\rho\) by the scalar index

\[
\mu(q) = \sum_{m=1}^{d} \rho^{m-1} q_m
\]

Notice that \(0 \leq \mu(q) \leq r - 1\) is an invertible function and allows us to write

\[
\mathbf{x}_{\mu(q)} = \mathbf{q}/\rho + \hat{x}_{\mu(q)}/\rho
\]

where the average

\[
\mathbb{E}[\mathbf{x}_{\mu(q)}] = \mathbf{q}/\rho + 1/2\rho
\]

is the coordinate of the sample identified by the scalar label \(\mu(q)\) and \(\mathbf{1}\) is the all ones vector. Furthermore, we assume that the entries of the vectors \(\hat{x}_{\mu(q)}\) are i.i.d. with pdf \(f_{\hat{x}}(z)\) which does not depend on \(r, M,\) or \(q\). By using this notation, the entries of \(\mathbf{G}_d\) are then given by

\[
(G_d)_{\nu(\ell), \mu(q)} = \frac{1}{\sqrt{(2M + 1)^d}} e^{-j2\pi \ell^T \mathbf{x}_{\mu(q)}}
\]

while the aspect ratio is

\[
\beta = \frac{(2M + 1)^d}{r} = \left(\frac{2M + 1}{\rho}\right)^d
\]

The Hermitian Toeplitz matrix \(\mathbf{T}_d = \beta \mathbf{G}_d \mathbf{G}_d^\dagger\) is defined as

\[
(T_d)_{\nu(\ell), \nu(\ell')} = \frac{1}{\rho^d} \sum_{q} e^{-j2\pi \mathbf{x}_{\mu(q)}^T(\ell - \ell')}
\]
where $\sum_{\mathbf{q}}$ represents a $d$-dimensional sum over all vectors $\mathbf{q}$ such that $q_m = 0, \ldots, \rho - 1$, $m = 1, \ldots, d$.

Our goals are (i) to derive the analytic expression of the moments of $f_\lambda(d, \beta, z)$ with quasi-equally spaced vectors $\mathbf{x}_{\mu(q)}$ (Section III), and (ii) to show that as $d \to \infty$, $f_\lambda(d, \beta, z)$ tends to the Marchenko-Pastur law (Section IV).

III. CLOSED FORM EXPRESSION OF THE MOMENTS OF THE ASYMPTOTIC EIGENVALUE PDF

Following the approach adopted in [13], [14], in the limit for $M$ and $r$ growing to infinity with constant aspect ratio $\beta$ and dimension $d$, we compute the closed form expression of $\mathbb{E}[X_{d,\beta}]$, which can be obtained from the powers of $T_d$ as [15],

$$
\mathbb{E}[X_{d,\beta}] = \lim_{M, r \to +\infty} \frac{\operatorname{Tr}\{ \mathbb{E}[\mathbf{X}^T] \}}{(2M + 1)^d} 
$$

In (8) the symbol $\operatorname{Tr}$ identifies the matrix trace operator, and the average $\mathbb{E}[\cdot]$ is computed over the set of random variables $\mathcal{X} = \{x_0, \ldots, x_{r-1}\}$. An efficient method to compute (8) exploits set partitioning. Indeed, note that the power $T_d^p$ is the matrix product of $p$ copies of $T_d$. This operation yields exponential terms, whose exponents are given by a sum of $p$ terms of the form $x_{\mu(q_i)}^T (\ell_i - \ell_{i+1})$ (see also (22) in Appendix A). The average of this sum depends on the number of distinct vectors $\mathbf{q}_i$, and all possible cases can be described as partitions of the set $\mathcal{P} = \{1, \ldots, p\}$. In particular, the case where in the set $\{\mathbf{q}_1, \ldots, \mathbf{q}_p\}$ there are $1 \leq k \leq p$ distinct vectors, corresponds to a partition of $\mathcal{P}$ in $k$ subsets. It follows that a fundamental step to calculate (8) is the computation of all possible partitions of set $\mathcal{P}$.

Before proceeding further in our analysis, we therefore introduce some useful definitions related to set partitioning.

A. Definitions

Let the integer $p$ denote the moment order and let the vector $\mu = [\mu_1, \ldots, \mu_p]$ be a possible combination of $p$ integers. In our specific case, each entry of the vector $\mu$ is given by the expression in (4), i.e., $\mu_i = \mu(q_i)$ and, thus, can range between 0 and $r - 1$.

We define:

- the scalar integer $1 \leq k(\mu) \leq p$ as the number of distinct entries of the vector $\mu$;
- $\gamma(\mu)$ as the vector of integers, of length $k(\mu)$, whose entries $\gamma_j(\mu)$, $j = 1, \ldots, k(\mu)$, are the entries of $\mu$ without repetitions, in order of appearance within $\mu$;
- $\mathcal{P}_j(\mu)$ as the set of indices of the entries of $\mu$ with value $\gamma_j(\mu)$, $j = 1, \ldots, k(\mu)$.
the vector \( \omega(\mu) = [\omega_1(\mu), \ldots, \omega_p(\mu)] \) such that, for any given \( j = 1, \ldots, k(\mu) \), we have \( \omega_i(\mu) = j \) if \( i \in P_j(\mu), \ i = 1, \ldots, p \).

**Example 1:** Let \( \mu = [1, 5, 2, 8, 5, 3, 2] \), then \( k(\mu) = 5 \) since the entries of \( \mu \) take 5 distinct values (i.e., \{1, 5, 2, 8, 3\}). Such values, taken in order of appearance in \( \mu \) form the vector \( \gamma(\mu) = [1, 5, 2, 8, 3] \). The value \( \gamma_1 = 1 \) appears at position 1 in \( \mu \), therefore \( P_1(\mu) = \{1\} \). The value \( \gamma_2 = 5 \) appears at positions 2 and 5 in \( \mu \), therefore \( P_2(\mu) = \{2, 5\} \). Similarly \( P_3(\mu) = \{3, 7\} \), \( P_4(\mu) = \{4\} \), and \( P_5(\mu) = \{6\} \). By using the sets \( P_j \) we build the vector \( \omega(\mu) \). For each \( j = 1, \ldots, k(\mu) \) we assign the value \( j \) to every \( \omega_i \) such that \( i \in P_j(\mu) \). For example, \( \omega_2 = \omega_5 = 2 \) since the integers 2 and 5 are in \( P_2 \). In conclusion \( \omega(\mu) = [1, 2, 3, 4, 2, 5, 3] \).

Furthermore, we define:

- \( \Omega_p \) as the set of partitions of \( P \);
- \( \Omega_{p,k} \) as the set of partitions of \( P \) in \( k \) subsets, \( 1 \leq k \leq p \), with \( \bigcup_{k=1}^{p} \Omega_{p,k} = \Omega_p \).

Note that: (i) the cardinality of \( \Omega_p \), denoted by \( B(p) = |\Omega_p| \), is the \( p \)-th Bell number [16] and (ii) the cardinality of \( \Omega_{p,k} \), denoted by \( S(p, k) = |\Omega_{p,k}| \), is a Stirling number of the second kind [17].

From the above definitions, it follows that:

1) the vector \( \mu \) induces a partition of the set \( P \) which is identified by the subsets \( P_j(\mu) \). These subsets have the following properties

\[
\bigcup_{j=1}^{k(\mu)} P_j(\mu) = P, \quad P_j(\mu) \cap P_{j'}(\mu) = \emptyset \quad \text{for} \ j \neq j'
\]

Even though the partition identified by \( \mu \) is often represented as \( \{P_1, \ldots, P_{k(\mu)}\} \), by its definition, an equivalent representation of such partition is given by the vector \( \omega(\mu) \). Therefore, from now on we will refer to \( \omega(\mu) \) as a partition of the \( p \) element set \( P \) induced by \( \mu \) (for simplicity, however, often we will not explicit the dependency of \( \omega \) on \( \mu \));

2) \( k(\omega) = k(\mu) \), since the entries of \( \omega \) take all possible values in the set \( \{1, \ldots, k(\mu)\} \);

3) \( P_j(\omega) = P_j(\mu) \), for \( j = 1, \ldots, k(\mu) \).

At last, we define \( M(\omega) \) as the set of \( \mu \) inducing the same partition \( \omega \) of \( P \).
Example 2: Let \( r = 3 \) and \( p = 3 \). Since \( \mu = [\mu_1, \ldots, \mu_p] \) and \( \mu_i = 0, \ldots, r - 1, \ i = 1, \ldots, p \), we have \( r^p = 27 \) possible vectors \( \mu \), namely, \{[0, 0, 0], [0, 0, 1], \ldots, [2, 2, 1], [2, 2, 2]\}. Each \( \mu \) identifies a partition \( \omega \in \Omega_{3,k} \), with \( k = 1, \ldots, 3 \), as described in Example 1. The sets of partitions \( \Omega_{3,k} \) are given by \( \Omega_{3,1} = \{[1, 1, 1]\}, \Omega_{3,2} = \{[1, 1, 2], [1, 2, 1], [1, 2, 2]\}, \) and \( \Omega_{3,3} = \{[1, 2, 3]\} \), and have cardinality \( S(3, 1) = 1, S(3, 2) = 3 \) and \( S(3, 3) = 1 \), respectively.

The set of vectors \( \mu \) identifying the partition \( \omega = [1, 1, 1] \), i.e., \( \mathcal{M}([1, 1, 1]) \), is given by: 
\[
\mathcal{M}([1, 1, 1]) = \{[0, 0, 0], [1, 1, 1], [2, 2, 2]\}. 
\]
Similarly,
\[
\begin{align*}
\mathcal{M}([1, 1, 2]) & = \{[0, 0, 1], [0, 0, 2], [1, 1, 0], [1, 1, 2], [2, 2, 0], [2, 2, 1]\} \\
\mathcal{M}([1, 2, 1]) & = \{[0, 1, 0], [0, 2, 0], [1, 0, 1], [1, 2, 1], [2, 0, 2], [2, 1, 2]\} \\
\mathcal{M}([1, 2, 2]) & = \{[0, 1, 1], [0, 2, 2], [1, 0, 0], [1, 2, 2], [2, 0, 0], [2, 1, 1]\} \\
\mathcal{M}([1, 2, 3]) & = \{[0, 1, 2], [0, 2, 1], [1, 0, 2], [1, 2, 0], [2, 0, 1], [2, 1, 0]\} 
\end{align*}
\]

B. Closed form expression of \( \mathbb{E}[\lambda_{d,\beta}^p] \)

By using the definitions in Section III-A and by applying set partitioning to (8), we can state the first main result of this work:

**Theorem 3.1:** Let \( T_d \) be a \((2M + 1)^d \times (2M + 1)^d\) Hermitian random matrix as defined in (7), where the properties of the random vectors \( x_{j'/(q)} \) are described in Section II-A. Then, for any given \( \beta \) and \( d \), the \( p \)-th moment of the asymptotic eigenvalue distribution of \( T_d \) is given by:

\[
\mathbb{E}[\lambda_{d,\beta}^p] = \sum_{k=1}^{p} \sum_{h=1}^{k} \beta^{p-h} \sum_{\omega \in \Omega_{p,k}} \sum_{\omega' \in \Omega_{k,h}} u(\omega') v(\omega, \omega')^d \tag{9}
\]

where

\[
u(\omega, \omega') = (-1)^{k-h} \prod_{j'=1}^{h} (|\mathcal{P}_{j'}(\omega')| - 1)! \tag{10}\]

\[
v(\omega, \omega') = \begin{cases} 
\int_{\mathcal{H}_p} \prod_{j=1}^{k} C \left( -j2\pi \beta^{1/d} w_j(\omega) \right) d\gamma & h = 1 \\
\int_{\mathcal{H}_p} \prod_{j=1}^{k} C \left( -j2\pi \beta^{1/d} w_j(\omega) \right) \prod_{j'=1}^{h} \delta_D \left( \sum_{j' \in \mathcal{P}_{j'}(\omega')} w_{j'}(\omega) \right) d\gamma & 1 < h < k \\
\int_{\mathcal{H}_p} \prod_{j=1}^{k} \delta_D \left( w_j(\omega) \right) d\gamma & h = k 
\end{cases} \tag{11}\]

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TABLE I

Partition sets $\Omega_{n,m}$ for $n = 1, 2, 3,$ and $1 \leq m \leq n$. Each partition is represented through its associated vector $\omega$ and the value of $u(\omega)$

| $\omega, u(\omega)$ | $m = 1$ | $m = 2$ | $m = 3$ |
|---------------------|---------|---------|---------|
| $n = 1$             | [1], 1  |         |         |
| $n = 2$             | [1,1], -1 | [1,2], 1 |         |
| $n = 3$             | [1,1,1], 2 | [1,1,2], -1 | [1,2,3], 1 |

and $v(\omega, \omega') = 1$ for $k = 1$. In (11), we defined $\mathcal{H}_p$ as the $p$-dimensional hypercube $[-1/2, 1/2]^p$, $C(s) = \mathbb{E}[e^{sz}]$ as the characteristic function of $\tilde{x}$, $\delta_D(\cdot)$ as the Dirac’s delta, and

$$w_j(\omega) = \sum_{i \in P_j(\omega)} y_i - y_{i+1}$$

$y_i \in \mathbb{R}$, $i = 1, \ldots, p$, and $j = 1, \ldots, k(\omega)$.

Proof: The proof can be found in Appendix A.

With the aim to give an intuitive explanation of the above expressions, note that the right hand side of (9) counts all possible partitions of the set $\mathcal{P} = \{1, \ldots, p\}$, $C(s)$ in (11) accounts for the generic distribution of the variables $\tilde{x}$, and the quantity $w_j(\omega)$ represents the indices pairing that appears in the exponent of the generic entry of the power $T_d^p$.

To further clarify the moments computation, Table I reports an example of partition sets $\Omega_{n,m}$ for $n = 1, \ldots, 3$ and $1 \leq m \leq n$, while Example 3 shows the computation of the second moment of the eigenvalue distribution.
Example 3: We compute the analytic expression of $E[\lambda_{d,\beta}^2]$. Using (9), we get:

$$E[\lambda_{d,\beta}^2] = \sum_{k=1}^{2} \sum_{h=1}^{k} \beta^{2-h} \sum_{\omega \in \Omega_{2,k}} \sum_{\omega' \in \Omega_{d,h}} u(\omega') v(\omega, \omega')^d$$

By expanding this expression and using Table I, we obtain

$$E[\lambda_{d,\beta}^2] = \beta v([1,1],[1])^d - \beta v([1,2],[1,1])^d + v([1,2],[1,2])^d$$

We notice that, for $k = 1$, $v([1,1],[1]) = 1$. The term $v([1,2],[1,2])$ refers instead to the case $k = h = 2$, and it is given by

$$v([1,2],[1,2]) = \int_{\mathcal{H}_2} \prod_{j=1}^{2} \delta_D(w_j([1,2])) \, dy$$

with $w_1([1,2]) = y_1 - y_2$ and $w_2([1,2]) = y_2 - y_1$. It follows that

$$v([1,2],[1,2]) = \int_{\mathcal{H}_2} \delta_D(y_1 - y_2) \delta_D(y_2 - y_1) \, dy = 1$$

Finally,

$$v([1,2],[1,1]) = \int_{\mathcal{H}_2} \prod_{j=1}^{2} C \left( -j2\pi\beta^{1/d} w_j([1,2]) \right) \, dy$$

$$= \int_{\mathcal{H}_2} \left| C \left( -j2\pi\beta^{1/d}(y_1 - y_2) \right) \right|^2 \, dy$$

Thus, we write

$$E[\lambda_{d,\beta}^2] = 1 + \beta - \beta \left[ \int_{\mathcal{H}_2} \left| C \left( -j2\pi\beta^{1/d}(y_1 - y_2) \right) \right|^2 \, dy \right]^d$$

IV. CONVERGENCE TO THE MARČENKO-PASTUR DISTRIBUTION

In this section we show that the asymptotic eigenvalue distribution of the matrix $T_d$ tends to the Marčenko-Pastur law [12], as $d \to \infty$. This is equivalent to prove that, as $d \to \infty$, the $p$-th moment of $\lambda_{d,\beta}$ tends to the $p$-th moment of the Marčenko-Pastur distribution with parameter $\beta$, for every $p \geq 1$.

Theorem 4.1: Let $T_d$ be a $(2M+1)^d \times (2M+1)^d$ Hermitian random matrix as defined in (7), where the properties of the random vectors $x_{\mu(q)}$ are described in Section II-A. Let $E[\lambda_{d,\beta}^p]$ be the $p$-th moment of the asymptotic eigenvalue distribution of $T_d$, given by Theorem 3.1. Then, for any given $\beta$,

$$\lim_{d \to \infty} E[\lambda_{d,\beta}^p] = E[\lambda_{\infty,\beta}^p] = \sum_{k=1}^{p} \beta^{p-k} N(p, k)$$

(12)
where \(N(p, k)\) are the Narayana numbers [18], [19] and \(E_{\infty, \beta}^\lambda\) are the Narayana polynomials, i.e., the moments of the Marčenko-Pastur distribution [12].

**Proof:**

We first look at the expression of the \(p\)-th asymptotic moment and observe that, for \(h = k\), the contribution of the term in the right hand side of (9) reduces to

\[
\sum_{k=1}^{p} \beta^{p-k} \sum_{\omega \in \Omega_{p,k}} \sum_{\omega' \in \Omega_{k}} u(\omega') v(\omega, \omega')
\]

(13)

The cardinality of \(\Omega_{k,k}\) is \(S(k, k) = 1\) and \(\Omega_{k,k} = \{1, \ldots, k\}\). Thus, we only consider \(\omega' = [1, \ldots, k]\). Moreover, using (10) we have \(u([1, \ldots, k]) = 1\) since each subset \(\mathcal{P}_j([1, \ldots, k])\) has cardinality 1, \(j = 1, \ldots, k\). Therefore, the term in (13) becomes

\[
\sum_{k=1}^{p} \sum_{\omega \in \Omega_{p,k}} \beta^{p-k} v(\omega, [1, \ldots, k])
\]

Using (11) with \(h = k\), we have:

\[
v(\omega, [1, \ldots, k]) = \int_{\mathcal{D}} \prod_{j=1}^{k} \delta_D(w_j(\omega)) \, dy \triangleq v(\omega)
\]

(14)

Hence, the contribution to the \(p\)-th moment reduces to

\[
\sum_{k=1}^{p} \beta^{p-k} \sum_{\omega \in \Omega_{p,k}} v(\omega)
\]

(15)

In [4], [5] it is shown that, as \(d \rightarrow \infty\), (15) tends to the Narayana polynomial of order \(p\). It follows that, in order to prove the theorem, it is enough to show that for \(h < k\) the contribution of the term in the right hand side of (9), to the expression of the \(p\)-th asymptotic moment, vanishes as \(d \rightarrow \infty\). In practice we have to show that, for each \(\omega \in \Omega_{p,k}\) and \(\omega' \in \Omega_{k,h}\) with \(h < k\),

\[
\lim_{d \rightarrow \infty} v(\omega, \omega')^d = 0
\]

or, equivalently, that \(|v(\omega, \omega')| < 1\).

We first notice that for \(1 < h < k\)

\[
|v(\omega, \omega')| = \left| \int_{\mathcal{D}} \prod_{j=1}^{k} C\left(-j2\pi\beta^{1/d}w_j(\omega)\right) \prod_{j'=1}^{h} \delta_D\left(\sum_{i' \in \mathcal{P}_j(\omega')} w_{i'}(\omega)\right) \, dy \right|
\]

\[
\leq \int_{\mathcal{D}} \prod_{j=1}^{k} C\left(-j2\pi\beta^{1/d}w_j(\omega)\right) \prod_{j'=1}^{h} \delta_D\left(\sum_{i' \in \mathcal{P}_j(\omega')} w_{i'}(\omega)\right) \, dy
\]

\[
= \int_{\mathcal{D}} \prod_{j=1}^{k} C\left(-j2\pi\beta^{1/d}w_j(\omega)\right) \prod_{j'=1}^{h} \delta_D\left(\sum_{i' \in \mathcal{P}_j(\omega')} w_{i'}(\omega)\right) \, dy
\]

(16)
Moreover, we have:

\[ |C (-j2\pi \beta^{1/d} w_j(\omega))| = \left| \int_{-\infty}^{+\infty} \exp(-j2\pi \beta^{1/d} w_j(\omega)z) \, f_\tilde{z}(z) \, dz \right| \]

\[ \leq \int_{-\infty}^{+\infty} \left| \exp(-j2\pi \beta^{1/d} w_j(\omega)z) \, f_\tilde{z}(z) \right| \, dx \]

\[ = \int_{-\infty}^{+\infty} f_\tilde{z}(z) \, dz = 1 \]  

(17)

The equality (a) arises if the condition \( w_j(\omega) = 0 \) is always verified, otherwise, if \( w_j(\omega) \neq 0 \), \( |C (-j2\pi \beta^{1/d} w_j(\omega))| < 1 \).

Next, we make the following observations: (i) since we consider partitions \( \omega' \) of the form \( \{1, \ldots, k\} \) in \( h \) subsets with \( h < k \), then at least one of the sets \( \mathcal{P}_{j'}(\omega') \) has cardinality \( |\mathcal{P}_{j'}(\omega')| > 1 \); (ii) the term

\[ \prod_{j'=1}^{h} \delta_D \left( \sum_{\nu \in \mathcal{P}_{j'}(\omega')} w_\nu(\omega) \right) \]

gives a non-zero contribution to the integral in (16) only when \( \sum_{\nu \in \mathcal{P}_{j'}(\omega')} w_\nu(\omega) = 0 \). Hence, if \( |\mathcal{P}_{j'}(\omega')| > 1 \) for some \( j' \), then some \( w_\nu(\omega') \neq 0 \) will provide a non-zero contribution to the integral in (16). In this case, we can write

\[ |v(\omega, \omega')| \leq \int_{\mathcal{H}_{j'}} \prod_{j'=1}^{h} \left| C (-j2\pi \beta^{1/d} w_j(\omega)) \right| \prod_{j'=1}^{h} \delta_D \left( \sum_{\nu \in \mathcal{P}_{j'}(\omega')} w_\nu(\omega) \right) \, dy \]

\[ < \int_{\mathcal{H}_{j'}} \prod_{j'=1}^{h} \delta \left( \sum_{\nu \in \mathcal{P}_{j'}(\omega')} w_\nu(\omega) \right) \, dy \leq 1 \]  

(18)

which proves the claim.

When \( h = 1 \), again, there is a measurable subset of \( \mathcal{H}_p \) for which \( w_j(\omega) \neq 0 \), hence,

\[ |v(\omega, \omega')| \leq \int_{\mathcal{H}_{j'}} \prod_{j=1}^{k} \left| C (-j2\pi \beta^{1/d} w_j(\omega)) \right| \, dy < 1 \]

i.e., the strict inequality holds.

In Figure 11 we show the empirical eigenvalue distribution of the matrix \( T_d \) for \( \beta = 0.55 \), \( d = 1, 2, 3 \), and \( \tilde{x} \) uniformly distributed in \([0, 1] \). The empirical distribution is compared to the Marčenko-Pastur distribution (solid line). We observe that as, \( d \) increases, the Marčenko-Pastur distribution law becomes a good approximation of \( f_\lambda(d, \beta, z) \). In particular, the two curves are relatively close for small \( z \), already for \( d = 3 \).
Fig. 1. Comparison between the Marčenko-Pastur distribution and the empirical distribution obtained for $\beta = 0.55$ and $d = 1, 2, 3$ in the quasi equally space case, and uniform $f_{\tilde{x}}(z)$

V. APPLICATIONS

Here we present some applications where the results derived in this work can be used.

The closed form expression of the moments of $f_\lambda(d, \beta, z)$, given by (34), can be a useful basis for performing deconvolution operations, as proposed in [8]. As for the asymptotic approximation, we show below how to exploit our results for the estimation of the MSE provided by linear reconstruction techniques of irregularly sampled signals.

Let us assume a general linear system model affected by additive noise. For simplicity, consider a one-dimensional signal, $s(x)$. When observed over a finite interval, it admits an infinite Fourier series expansion [1], [2]. We can think of the largest index $M$ of the non-negligible Fourier coefficients of the expansion as the approximate one-sided bandwidth of the signal. We therefore represent $s(x)$ by using $2M + 1$ complex harmonics as

$$s(x) = \frac{1}{\sqrt{2M+1}} \sum_{k=-M}^{M} a_k e^{j2\pi kx}$$

(19)
Now, consider that the signal is observed within one period interval \([0, 1]\) and sampled in \(r\) points placed at positions \(x = [x_0, \ldots, x_{r-1}]^T\), \(x_q \in [0, 1]\), \(q = 0, \ldots, r - 1\). The complex numbers \(a_\ell\) represent amplitudes and phases of the harmonics in \(s(x)\). The signal samples \(s = [s(x_0), \ldots, s(x_{r-1})]^T\) can be written as \(s = G^\dagger a\), where the matrix \(G\) is given in (1). The signal discrete spectrum is given by the \(2M + 1\) complex vector \(a = [a_{-M}, \ldots, a_0, \ldots, a_M]^T\). We can now write the linear model for a measurement sample vector \(p = [p(x_0), \ldots, p(x_{r-1})]^T\) taken at the sampling points \(x_q\)

\[
p = s + n = G^\dagger a + n
\]

(20)

where \(n\) is a random vector representing measurement noise. The general problem is to reconstruct \(s\) or \(a\) given the noisy measurements \(p\) [4], [5]. A commonly used parameter to measure the quality of the estimate of the reconstructed signal is the mean square error (MSE). In [1]–[3] it has been shown that, when linear reconstruction techniques are used and the sample coordinates are known, the asymptotic MSE (i.e., as the number of harmonics and the number of samples tend to infinity while their ratio is kept constant) is a function of the asymptotic eigenvalue distribution of the matrix \(T = \beta GG^\dagger\), i.e.,

\[
\text{MSE} = E \left[ \frac{\beta}{\lambda \text{SNR}_m + \beta} \right]
\]

(21)

where the random variable \(\lambda\) has distribution \(f_\lambda(d, \beta, z)\) and \(\text{SNR}_m\) is the signal-to-noise ratio on the measure. We therefore exploit our asymptotic approximation to \(f_\lambda(d, \beta, z)\) to compute (21).

Figure 2 shows the MSE obtained as a function of the signal-to-noise ratio \(\text{SNR}_m\). The curves with markers labeled by “\(d = 1, 2, 3\)” refer to the cases where the signal has dimension \(d\) and the sampling points are quasi-equally spaced with jitter \(\tilde{x}\), uniformly distributed over \([0, 1]\), and \(\beta = 0.729\). The curve labeled by “MP” (thick line) reports the results derived through our asymptotic \((d \to \infty)\) approximation to the eigenvalue distribution, while the curve labeled by “Equally spaced” (dashed line) represents the MSE achieved under a perfect equally spaced sample placement, i.e., when the eigenvalue distribution is given by \(f_\lambda(d, \beta, z) = \delta_D(z - 1)\). Notice that the MSE grows as \(d\) increases and tends to the MSE obtained by a Marčenko-Pastur eigenvalue distribution. Instead, as expected, the “Equally spaced” curve represents a lower bound to the system performance.

Figure 3 presents similar results but obtained for \(d = 2\) and different values of \(\beta\). We observe that the MSE obtained through our asymptotic approximation (the curve labeled by “MP”) gives excellent results for values of \(\beta\) as small as 0.2, even when compared against the numerical results derived by fixing \(d = 2\). For \(\beta = 0.6\) (i.e., when the ratio of the number of signal harmonics to the number of samples increases), the approximation becomes slightly looser, and the MSE computed by using the Marčenko-Pastur distribution gives an upper limit to the quality of the reconstructed signal. Note that the smaller
the β, the higher the oversampling rate relative to the equally spaced minimal sampling rate $\beta = 1$. We thus observe how our bound becomes tighter as the oversampling rate increases.

To conclude, we describe some areas in signal processing where the above system model and results find application.

i) Spectral estimation with noise. Spectral estimation from high precision sampling and quantization of bandlimited signals uses measurement systems which are usually affected by jitter [20]. In such applications the quantization noise corresponds to the measurement noise and the jitter is caused by the limited accuracy of the timing circuits. In this case the sampling points are mismatched with respect to the nominal values, thus for $d = 1$ we have: $x_q = \frac{q}{r} + \frac{\tilde{x}}{r}$ with some sampling rate $1/r$. Note that the exact positions of the samples are not known and the case studied in this paper (i.e., MSE with exact positions) gives a lower bound to the reconstruction error.

ii) Signal reconstruction in sensor networks. Sensor networks, whose nodes sample a physical field, like air temperature, light intensity, pollution levels or rain falls, typically represent an example of quasi-
equally spaced sampling [3], [9], [21], [22]. Indeed, often sensors are not regularly deployed in the area of interest due to terrain conditions and deployment practicality and, thus, the physical field is not regularly sampled in the space domain. Sensors report the data to a common processing unit (or sink node), which is in charge of reconstructing the sensed field, based on the received samples and on the knowledge of their coordinates. If the field can be approximated as bandlimited in the space domain, then an estimate of the discrete spectrum can be obtained by using linear reconstruction techniques [3], [23], even in presence of additive noise. In this case, our approximation allows to compute the MSE on the reconstructed field.

iii) Stochastic sampling in computer graphics and image processing. Jittered sampling was first examined by Balakrishnan in [24], who analyzed it as an undesirable effect in sampling continuous time functions. More than twenty years later, Cook [25] realized that the effect of stochastic sampling can be advantageous in computer graphics to reduce aliasing artifacts, and considered jittering a regular grid as an effective sampling technique. Another example of sampling with jitter was recently
proposed in [26], for robust authentication of images.

VI. CONCLUSIONS

We studied the behavior of the eigenvalue distribution of a class of random matrices, which find large application in signal and image processing. In particular, by using asymptotic analysis, we derived a closed-form expression for the moments of the eigenvalue distribution. Using these moments, we showed that, as the signal dimension goes to infinity, the asymptotic eigenvalue distribution tends to the Marčenko-Pastur law. This result allowed us to obtain a simple and accurate bound to the signal reconstruction error, which can find application in several fields, such as jittered sampling, sensor networks, computer graphics and image processing.

APPENDIX A

PROOF OF THEOREM 3.1

Using (7), the term \( \text{Tr} \mathbb{E}_X \left[ T^p_d \right] \) in (8) can be written as:

\[
\text{Tr} \mathbb{E}_X \left[ T^p_d \right] = \mathbb{E}_X \left[ \sum_{\ell_1} (T^p_d)_{\nu(\ell_1),\nu(\ell_1)} \right] \\
= \mathbb{E}_X \left[ \sum_{\ell_1} \cdots \sum_{\ell_p} (T^p_d)_{\nu(\ell_1),\nu(\ell_2)} \cdots (T^p_d)_{\nu(\ell_p),\nu(\ell_1)} \right] \\
= \frac{1}{r^p} \sum_{\ell_1} \cdots \sum_{\ell_p} \sum_{q_1} \cdots \sum_{q_p} \mathbb{E}_X \left[ \exp \left( -j2\pi \sum_{i=1}^p x^T_{\mu(q_i)} (\ell_i - \ell_{[i+1]}) \right) \right] \\
= \frac{1}{r^p} \sum_{L \in \mathcal{L}_d} \sum_{Q \in \mathcal{Q}_d} \mathbb{E}_X \left[ \exp \left( -j2\pi \sum_{i=1}^p x^T_{\mu(q_i)} (\ell_i - \ell_{[i+1]}) \right) \right] \\
\tag{22}
\]

where \( \mathcal{Q}_d \) and \( \mathcal{L}_d \) are sets of integer matrices such that

\[
\mathcal{Q}_d = \{ Q | Q = [q_1, \ldots, q_p], \ q_i = [q_{i,1}, \ldots, q_{i,d}]^T, q_{i,m} = 0, \ldots, \rho - 1 \} \\
\mathcal{L}_d = \{ L | L = [\ell_1, \ldots, \ell_p], \ \ell_i = [\ell_{i,1}, \ldots, \ell_{i,d}]^T, \ell_{i,m} = -M, \ldots M \}
\]

and

\[
[i+1] = \begin{cases} 
  i + 1 & 1 \leq i < p \\
  1 & i = p
\end{cases}
\]
A. Set partitioning

We now apply the definitions in Section III-A in order to rewrite (22) using set partitioning. In particular by considering the vector \( \mu = \mu(Q) \triangleq [\mu_1, \ldots, \mu_p]^T \) where \( \mu_i = \mu(q_i) \) and \( q_i \) is the \( i \)-th column of \( Q \), we observe that:

- the vector \( \mu \) is uniquely defined by \( Q \), and a given \( \mu \) uniquely defines a matrix \( Q \in Q_d \) since \( \mu(\cdot) \) is an invertible function;
- a given \( \mu \) induces a partition \( \omega(\mu) \);
- since \( r \) is the number of values that the entries \( \mu_i \) can take, there exist \( r!/(r - k(\mu))! \) matrices \( Q \in Q_d \) generating a given partition of \( P \) made of \( k(\mu) \) subsets. In other words \( r!/(r - k(\mu))! \) distinct \( \mu \)'s yield the same partition \( \omega(\mu) \).

Since the random vectors \( x_{\mu(q')} \) and \( x_{\mu(q'')} \) are independent for \( q' \neq q'' \), for any given \( Q \) the average operator in (22) factorizes into \( k(\mu) \) terms, i.e.,

\[
\mathbb{E}_{x} \left[ \exp \left( -j2\pi \sum_{i=1}^{p} x_{\mu(q_i)}^T (\ell_i - \ell_{[i+1]}) \right) \right] = \mathbb{E}_{x} \left[ \exp \left( -j2\pi \sum_{i=1}^{p} x_{\mu_i}^T (\ell_i - \ell_{[i+1]}) \right) \right] = \prod_{j=1}^{k(\mu)} \mathbb{E}_{x_{\gamma_j}} \left[ \exp \left( -j2\pi x_{\gamma_j}^T \sum_{i \in P_j(\mu)} \ell_i - \ell_{[i+1]} \right) \right] = \prod_{j=1}^{k(\mu)} \zeta^{\rho x_{\gamma_j}^T \hat{w}_j(\mu)} \tag{23}
\]

indeed, for every \( i \in P_j(\mu) \), we have \( \mu_i = \gamma_j \). In the last line of (23), we exploited the following two definitions

\[
\zeta = \exp(-j2\pi/\rho)
\]

and

\[
\hat{w}_j(\mu) = \sum_{i \in P_j(\mu)} \ell_i - \ell_{[i+1]} \tag{24}
\]

Also, note that, in the product in (23), each factor depends on a single random vector, \( x_{\gamma_j} \). Since \( x_{\mu(q)} = q/\rho + \bar{x}_{\mu(q)}/\rho \) and \( \mu(\cdot) \) is invertible then, by defining \( x_{\gamma_j} = \mu^{-1}(\gamma_j) \) we have

\[
x_{\gamma_j} = \bar{x}_{\gamma_j}/\rho + \bar{x}_{\gamma_j}/\rho
\]

and

\[
\mathbb{E}_{x_{\gamma_j}} \left[ \zeta^{\rho x_{\gamma_j}^T \hat{w}_j(\mu)} \right] = \zeta^{\bar{x}_{\gamma_j}^T \hat{w}_j(\mu)} \mathbb{E}_{x_{\gamma_j}} \left[ \zeta^{\bar{x}_{\gamma_j}^T \hat{w}_j(\mu)} \right] = \zeta^{\bar{x}_{\gamma_j}^T \hat{w}_j(\mu)} \mathbb{E}_{\bar{x}} \left[ \zeta^{\bar{x}^T \hat{w}_j(\mu)} \right] \tag{25}
\]
In the last term of (25) we removed the subscript $\gamma_j$ from the argument of the average operator, since the distribution of $\tilde{x}_{\gamma_j}$ does not depend on $\gamma_j$. Summarizing, the term $\text{Tr} \mathbb{E}_\mathcal{X} \left[ T_d^p \right]$ in (8) can be written as

$$
\text{Tr} \mathbb{E}_\mathcal{X} \left[ T_d^p \right] = \frac{1}{r^p} \sum_{Q \in \mathcal{Q}_d} \sum_{L \in \mathcal{L}_d} \sum_{j=1}^{k(\mu)} \prod_{\gamma_j} \mathbb{E}_\mathcal{X} \left[ \zeta^{T} \tilde{w}_j(\mu) \right]
$$

(26)

Since each $Q$ is uniquely identified by a vector $\mu$, we can observe that

$$
\sum_{Q \in \mathcal{Q}_d} f(\mu) = \sum_{\omega \in \Omega_p} \sum_{\mu \in \mathcal{M}(\omega)} f(\mu) = \sum_{k=1}^{p} \sum_{\omega \in \Omega_p} \sum_{\mu \in \mathcal{M}(\omega)} f(\mu)
$$

for every function $f(\mu)$. Recall that, in (27), $\mathcal{M}(\omega)$ represents the set of $\mu$ inducing a given partition $\omega$.

From the definitions in Section [III-A] it follows that, if $\mu$ induces $\omega$, then $k(\mu) = k(\omega)$, $P_j(\mu) = P_j(\omega)$, and $\tilde{w}_j(\mu) = \tilde{w}_j(\omega)$, $j = 1, \ldots, k(\omega)$. Therefore,

$$
\text{Tr} \mathbb{E}_\mathcal{X} \left[ T_d^p \right] = \frac{1}{r^p} \sum_{k=1}^{p} \sum_{\omega \in \Omega_p} \sum_{\mu \in \mathcal{M}(\omega)} \sum_{L \in \mathcal{L}_d} \prod_{j=1}^{k(\mu)} \mathbb{E}_\mathcal{X} \left[ \zeta^{T} \tilde{w}_j(\omega) \right]
$$

$$
= \frac{1}{r^p} \sum_{k=1}^{p} \sum_{\omega \in \Omega_p} \sum_{\mu \in \mathcal{M}(\omega)} \sum_{L \in \mathcal{L}_d} \prod_{j=1}^{k(\omega)} \mathbb{E}_\mathcal{X} \left[ \zeta^{T} \tilde{w}_j(\omega) \right]
$$

$$
= \frac{1}{r^p} \sum_{k=1}^{p} \sum_{\omega \in \Omega_p} \sum_{\mu \in \mathcal{M}(\omega)} \sum_{L \in \mathcal{L}_d} \prod_{j=1}^{k(\omega)} \mathbb{E}_\mathcal{X} \left[ \zeta^{T} \tilde{w}_j(\omega) \right]
$$

(28)

In (28) we defined

$$
\eta(\omega, L) = \prod_{j=1}^{k} \mathbb{E}_\mathcal{X} \left[ \zeta^{T} \tilde{w}_j(\omega) \right] = \prod_{j=1}^{k} \prod_{m=1}^{d} \mathbb{E}_\mathcal{X} \left[ \zeta^{T} \tilde{w}_j(\omega) \right]
$$

(29)

where $\tilde{x}_m$ and $\tilde{w}_{jm}$ are the $m$-th entries of $\tilde{x}$ and $\tilde{w}_j$, respectively. In the equality “(a)” we exploited the fact that the term $\zeta^{T} \tilde{w}_j(\omega)$ does not depend on $\mu$ and can be factored from the sum over $\mu$. As for the term $\sum_{\mu \in \mathcal{M}(\omega)} \prod_{j=1}^{k} \zeta^{T} \tilde{w}_j$, we have the following lemma.

**Lemma A.1:** Let $\omega \in \Omega_p$, let $\tilde{w}_1, \ldots, \tilde{w}_k$ be vectors of size $d$ with integer entries, defined as in (24).

Let $\mathcal{M}(\omega)$ be the set of vectors $\mu$ inducing $\omega$. Then

$$
\sum_{\mu \in \mathcal{M}(\omega)} \prod_{j=1}^{k} \zeta^{T} \tilde{w}_j = \sum_{h=1}^{k-h} \sum_{\omega' \in \Omega_k} u(\omega') \prod_{j=1}^{h} \delta_j \left( \sum_{j'=1}^{h} \tilde{w}_{j'}(\omega') \right)
$$

(30)

where $u(\omega') = (-1)^{k-h} \prod_{j=1}^{h} (|P_j(\omega')| - 1)!$, $\gamma_j = \gamma_j(\mu)$, and where $\Omega_k$ is the set of vectors $\omega'$ of size $k$, representing the partitions of the set $P' = \{1, \ldots, k\}$ in $h$ subsets, namely, $P'_1(\omega'), \ldots, P'_h(\omega')$. 

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DRAFT
Proof: The proof can be found in Appendix B.

By applying the result of Lemma A.1 to (28), we get

\[
\text{Tr}_X \mathbb{E}[\mathbf{T}^p_d] = \sum_{h=1}^{p} \sum_{k=1}^{k} \sum_{\omega' \in \Omega_{k,h}} \frac{r^h u(\omega')}{r_p} \sum_{L \in \mathcal{L}_d} \eta(\omega, L) \prod_{j'=1}^{h} \delta \left( \sum_{i' \in \mathcal{P}_{j'}(\omega')} \hat{\omega}_{i'}(\omega) \right)
\]  

(31)

Considering that

\[
\prod_{j'=1}^{h} \delta \left( \sum_{i' \in \mathcal{P}_{j'}(\omega')} \hat{\omega}_{i'}(\omega) \right) = \prod_{j'=1}^{h} \prod_{m=1}^{d} \delta \left( \sum_{i' \in \mathcal{P}_{j'}(\omega')} \hat{\omega}_{i'm}(\omega) \right)
\]

and by using (29) and (31), we have

\[
\sum_{L \in \mathcal{L}_d} \eta(\omega, L) \prod_{j'=1}^{h} \delta \left( \sum_{i' \in \mathcal{P}_{j'}(\omega')} \hat{\omega}_{i'}(\omega) \right)
\]

\[
= \sum_{\ell_1 \in \mathcal{L}_1} \cdots \sum_{\ell_h \in \mathcal{L}_h} \prod_{j=1}^{h} \sum_{w_{j} \in \mathcal{L}_h} \prod_{m=1}^{d} \mathbb{E}[\hat{x}_m \hat{\omega}_{i'm}(\omega)] \prod_{j'=1}^{h} \prod_{m=1}^{d} \delta \left( \sum_{i' \in \mathcal{P}_{j'}(\omega')} \hat{\omega}_{i'm}(\omega) \right)
\]

\[
= \left[ \sum_{\ell \in \mathcal{L}} \prod_{j=1}^{h} \mathbb{E}[\hat{x}_j \omega_{j}(\omega)] \prod_{j'=1}^{h} \delta \left( \sum_{i' \in \mathcal{P}_{j'}(\omega')} \hat{\omega}_{i'}(\omega) \right) \right]^{d} = \psi_{M}(\omega, \omega')^{d}
\]  

(32)

where the subscript \(M\) highlights the dependency of \(\ell\) on \(M\).

In conclusion,

\[
\text{Tr}_X \mathbb{E}[\mathbf{T}^p_d] = \sum_{h=1}^{p} \sum_{k=1}^{k} \sum_{\omega' \in \Omega_{k,h}} \frac{r^h}{r_p} u(\omega') \psi_{M}(\omega, \omega')^{d}
\]  

(33)

To compute \(\mathbb{E}[\lambda_{d,\beta}^p]\), we consider the limit in (8). By using the definition (5), we first notice that

\[
\frac{r^h}{r_p(2M + 1)^{d}} = \frac{\beta^{p-h}}{(2M + 1)^{d(p-h+1)}}
\]

Then, by using (33) in (8), we obtain

\[
\mathbb{E}[\lambda_{d,\beta}^p] = \lim_{M, \beta \rightarrow \infty} \sum_{k=1}^{p} \sum_{h=1}^{k} \frac{\beta^{p-h}}{(2M + 1)^{d(p-h+1)}} \sum_{\omega \in \Omega_{p,k}} \sum_{\omega' \in \Omega_{k,h}} u(\omega') \psi_{M}(\omega, \omega')^{d}
\]

\[
= \sum_{k=1}^{p} \sum_{h=1}^{k} \frac{\beta^{p-h}}{(2M + 1)^{d(p-h+1)}} \sum_{\omega \in \Omega_{p,k}} \sum_{\omega' \in \Omega_{k,h}} u(\omega') \left[ \lim_{M \rightarrow \infty} \frac{\psi_{M}(\omega, \omega')}{(2M + 1)^{p-h+1}} \right]^{d}
\]

\[
= \sum_{k=1}^{p} \sum_{h=1}^{k} \frac{\beta^{p-h}}{(2M + 1)^{d(p-h+1)}} \sum_{\omega \in \Omega_{p,k}} \sum_{\omega' \in \Omega_{k,h}} u(\omega') v(\omega, \omega')^{d}
\]

(34)

The second equality in (34) holds since, for any given \(p\), the sums \(\sum_{\omega \in \Omega_{p,k}}\) and \(\sum_{\omega' \in \Omega_{k,h}}\) are over a finite number of terms, and the coefficients \(u(\omega')\) are finite and do not depend on \(M\). Therefore, the
The limit operator can be swapped with the summations. The coefficient \( v(\omega, \omega') \) is defined as

\[
v(\omega, \omega') = \lim_{M \to \infty} \frac{\psi_M(\omega, \omega')}{(2M + 1)^{p-h+1}} \]

\[
= \lim_{M \to \infty} \frac{1}{(2M + 1)^{p-h+1}} \sum_{\ell \in \mathcal{L}} \prod_{j=1}^{k} \mathbb{E}^{x}[\bar{\psi}_M(\omega)] \prod_{j'=1}^{h} \delta \left( \sum_{i' \in \mathcal{P}_{j'}(\omega)} \bar{w}_{i'}(\omega) \right)
\]

\[
\overset{(a)}{=} \lim_{M \to \infty} \frac{1}{(2M + 1)^{p-h+1}} \sum_{\ell \in \mathcal{L}} \prod_{j=1}^{k} C(\frac{-j2\pi \beta_1}{\rho}) \prod_{j'=1}^{h} \delta \left( \sum_{i' \in \mathcal{P}_{j'}(\omega)} \bar{w}_{i'}(\omega) \right)
\]

(35)

where, in the equality \( (a) \), we introduced the characteristic function of \( \bar{x} \), defined as \( C(s) = \mathbb{E}[e^{sx}]. \) We now consider three possible cases:

- If \( h = 1 \), then \( \Omega_{k,1} = \{1, \ldots, 1\} \), thus we only consider \( \omega' = [1, \ldots, 1] \). Then, \( \mathcal{P}_1(\omega') = \{1, \ldots, k\} \) and

\[
\sum_{i' \in \mathcal{P}_1(\omega')} \bar{w}_{i'}(\omega) = \sum_{i'=1}^{k} \bar{w}_{i'}(\omega)
\]

\[
= \sum_{i'=1}^{k} \bar{w}_{i'}(\omega)
\]

\[
= \sum_{i'=1}^{k} \sum_{i \in \mathcal{P}_{i'}(\omega)} \ell_i - \ell_{[i+1]}
\]

\[
= \sum_{i=1}^{p} \ell_i - \ell_{[i+1]} = 0
\]

(36)

and by consequence \( \delta \left( \sum_{i' \in \mathcal{P}_{j'}(\omega')} \bar{w}_{i'}(\omega) \right) = 1 \). Hence,

\[
v(\omega, \omega') = \int_{\mathcal{H}_{p}} \prod_{j=1}^{k} C \left( -j2\pi \beta_{1/d} w_j(\omega) \right) \, dy
\]

(37)

where, in analogy with (24), we defined

\[
w_j = \sum_{i \in \mathcal{P}_{j}(\omega)} y_i - y_{[i+1]}
\]

\( y_i \in \mathbb{R}, i = 1, \ldots, p \). We denote by \( y \) the vector \( y = [y_1, \ldots, y_p]^T \);

- If \( 1 < h < k \), the argument of the \( \delta(\cdot) \) function in (35) is always a function of the indices \( \ell_i \). Thus

\[
\int_{\mathcal{H}_{p}} \prod_{j=1}^{k} C \left( -j2\pi \beta_{1/d} w_j(\omega) \right) \prod_{j'=1}^{h} \delta_D \left( \sum_{i' \in \mathcal{P}_{j'}(\omega')} w_{i'}(\omega) \right) \, dy
\]

where \( \delta_D(\cdot) \) denotes the Dirac’s delta;
\* If \( h = k \), the cardinality of \( \Omega_{k,h} = \Omega_{k,k} \) is \( S(k,k) = 1 \) and \( \Omega_{k,k} = \{1, \ldots, k\} \). Thus, we only consider \( \omega' = [1, \ldots, k] \). It follows that:

\[
v(\omega, \omega') = \int_{\mathcal{H}_p} \prod_{j=1}^{k} C \left( -j2\pi \beta I/d \cdot w_j(\omega) \right) \prod_{j=1}^{k} \delta_D \left( \sum_{\mu \in \mathcal{P}_j([1, \ldots, k])} \right) \ dy
\]

\[
= \int_{\mathcal{H}_p} \prod_{j=1}^{k} C \left( -j2\pi \beta I/d \cdot w_j(\omega) \right) \delta_D \left( \sum_{\mu \in \mathcal{P}_j([1, \ldots, k])} \right) \ dy
\]

Since \( \mathcal{P}_j([1, \ldots, k]) = \{j\} \) and \( C(0) = 1 \), we have

\[
v(\omega, [1, \ldots, k]) = \int_{\mathcal{H}_p} \prod_{j=1}^{k} C \left( -j2\pi \beta I/d \cdot w_j(\omega) \right) \delta_D (w_j(\omega)) \ dy
\]

\[
= \int_{\mathcal{H}_p} \prod_{j=1}^{k} C(0) \delta_D (w_j(\omega)) \ dy
\]

\[
= \int_{\mathcal{H}_p} \prod_{j=1}^{k} \delta_D (w_j(\omega)) \ dy \tag{38}
\]

As a last remark, if \( k = 1 \), we have \( h = 1 \) and \( \Omega_{p,k} = \Omega_{p,1} = \{1, \ldots, 1\} \). Then \( w_j(\omega) = \sum_{i=1}^{p} w_i = 0 \). Using (37), we obtain

\[
v(\omega, \omega') = \int_{\mathcal{H}_p} \prod_{j=1}^{k} C \left( -j2\pi \beta I/d \cdot w_j(\omega) \right) \ dy = \int_{\mathcal{H}_p} \prod_{j=1}^{k} C(0) \ dy = 1
\]

**Appendix B**

**Proof of Lemma A.1**

Recall that \( \mathcal{M}(\omega) \) denotes the set of vectors \( \mu = [\mu_1, \ldots, \mu_p] \) inducing the same partition \( \omega \). As defined in Section III-A if \( \omega \in \Omega_{p,k} \), then each \( \mu \in \mathcal{M}(\omega) \) contains \( k \) distinct values, namely, \( \gamma = [\gamma_1, \ldots, \gamma_k] \) where \( 0 \leq \gamma_j < r, j = 1, \ldots, k \) and \( \gamma_j \neq \gamma_{j'} \) for each \( j, j' = 1, \ldots, k \) and \( j \neq j' \). Therefore, from (A.1) we can write

\[
\sum_{\mu \in \mathcal{M}(\omega)} \prod_{j=1}^{k} \tilde{x}_{\gamma_j}^{T} w_j = \sum_{\mu \in \mathcal{M}(\omega)} \prod_{j \neq j'}^{k} \tilde{x}_{\gamma_j}^{T} w_j
\]

where the symbol \( \sum_{\gamma_1, \ldots, \gamma_k} \) indicates a sum over the variables \( \gamma_1, \ldots, \gamma_k \) with the constraint that \( \gamma_j \neq \gamma_{j'} \) for every \( j, j' = 1, \ldots, k \) and \( j \neq j' \). Notice that the values \( \gamma_j \) \( (j = 1, \ldots, k) \) are the scalar counterparts of the integer vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_k \), \( \mathbf{v}_j = [v_{j1}, \ldots, v_{jd}]^T \), \( 0 \leq v_{jm} < \rho, m = 1, \ldots, d \), through the invertible function \( \mu(\cdot) \), i.e., \( \gamma_j = \mu(\mathbf{v}_j), j = 1, \ldots, k \). Hence, by definition of \( \bar{x} \), we have \( \bar{x}_{\gamma_j} = \bar{x}_{\mu(\mathbf{v}_j)} = \mathbf{v}_j \) and
in conclusion
\[
\sum_{\mu \in M(\omega)} \prod_{j=1}^{k} \zeta^{v_j^T \hat{w}_j} = \sum_{\nu_1, \ldots, \nu_k} \prod_{j=1}^{k} \zeta^{v_j^T \hat{w}_j} = \sum_{\nu_1, \ldots, \nu_k} \zeta^{v_1^T \hat{w}_1 + \cdots + v_k^T \hat{w}_k}
\] (40)

We now compute the last term of (40) by summing over one variable at a time. We first notice that, for every set \( v_1, \ldots, v_n \) of distinct vectors
\[
\sum_{v \neq v_1, \ldots, v_n} \zeta^{v^T \hat{w}} = \begin{cases} r - n & \hat{w} = 0 \\ - \sum_{j=1}^{n} \zeta^{v_j^T \hat{w}} & \hat{w} \neq 0 \end{cases}
\]

In particular when \( \hat{w} \neq 0 \), \( \sum_v \zeta^{v^T \hat{w}} = 0 \).

Let us arbitrarily choose the variable \( v_k \). If by hypothesis \( \hat{w}_k \neq 0 \), then by summing (40) over \( v_k \) we get
\[
\sum_{v_1, \ldots, v_{k-1} \neq v_k} \zeta^{v_1^T \hat{w}_1 + \cdots + v_{k-1}^T \hat{w}_{k-1}} \zeta^{v_k^T \hat{w}_k} = - \sum_{j=1}^{k-1} \sum_{v_1, \ldots, v_{k-1}} \zeta^{v_j^T \hat{w}_1 + \cdots + v_{k-1}^T \hat{w}_{k-1}} \zeta^{v_k^T \hat{w}_k}
\] (41)

We compute separately each of the \( k-1 \) contributions in (41). In particular, the generic \( j' \)-th term \((j = j')\) is given by
\[
- \sum_{v_1, \ldots, v_{k-1} \neq v_k} \zeta^{v_1^T \hat{w}_1 + \cdots + v_{k-1}^T \hat{w}_{k-1}} \zeta^{v_k^T \hat{w}_k} = - \sum_{v_1, \ldots, v_{k-1} \neq v_k} \zeta^{v_1^T \hat{w}_1 + \cdots + \hat{w}_{j'} + \hat{w}_k + \hat{w}_{k-1}^T \hat{w}_{k-1}}
\]

We now proceed by summing over the variable \( v_{j'} \). If by hypothesis \( \hat{w}_{j'} + \hat{w}_k \neq 0 \), this summation produces \( k-2 \) terms. Again, we consider each term separately. This procedure repeats until a subset \( S \) of \( \{1, \ldots, k\} \) is found, such that \( s = \sum_{i \in S} \hat{w}_i = 0 \).

In this case, the contribution of the \( n \)-th sum is given by \( r - (k - n) \) where \( n = |S| \) is the cardinality of \( S \). Overall, after \( n \) sums the total contribution is
\[
(-1)^{n-1}(n-1)!(r - (k - n)) \sum_{v, j \in \{1, \ldots, k\} - S \; j \in \{1, \ldots, k\} - S} \prod_{j \neq k} \zeta^{v_j^T \hat{w}_j}
\]

The factor \( (n-1)! \) accounts for the number of permutations of the elements in \( S \), once the first element is fixed (remember that we arbitrarily chose the first variable of the summation). The factor \( (-1)^{n-1} \) takes into account that we summed \( n - 1 \) times with the condition \( \hat{w} \neq 0 \), which implies \( n - 1 \) sign changes. Eventually, the term \( \sum_{v, j \in \{1, \ldots, k\} - S} \prod_{j \in \{1, \ldots, k\} - S} \zeta^{v_j^T \hat{w}_j} \) is similar to the last term in (40) where only \( k - n \) variables \( v \) are involved.

This procedure repeats until we sum over all variables \( v \). This is equivalent to check if for all possible partitions of \( \{1, \ldots, k\} \) in \( h \) subsets \( P_1, \ldots, P_h \), \( h = 1, \ldots, k \) the condition \( s_1 = s_2 = \cdots = s_h = 0 \).
holds, with $s_j = \sum_{i \in P_j} \hat{w}_i$, $n_j = |P_j|$, and $\sum_j n_j = k$. In this case, the contribution is given by

$$\prod_{j=1}^h (-1)^{n_j-1} (n_j - 1)! p_r(n_1, \ldots, n_h)$$

and it is 0 otherwise. Here $p_r(n_1, \ldots, n_h) = (r - (k - n_1))(r - (k - n_1 - n_2)) \cdots (r - (k - n_1 - n_2 - \cdots - n_{h-1}))$.

In conclusion, we can write

$$\sum_{\nu_1, \ldots, \nu_k \neq 0} \zeta^{\nu_1^T \hat{w}_1 + \cdots + \nu_k^T \hat{w}_k} = \sum_{h=1}^k \sum_{\omega' \in \Omega_{k,h}} u(\omega') p_r(\omega') \prod_{j'=1}^h \delta \left( \sum_{i' \in P_j(\omega')} \hat{w}_{i'}(\omega) \right)$$

where $u(\omega') = (-1)^{k-h} \prod_{j=1}^h (|P_j(\omega')| - 1)!$ and $p_r(\omega')$ is a polynomial in $r$ of degree $h$. For large $r$, $p_r(\omega') \simeq r^h$, thus proving the lemma.

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