RIGIDITY OF DETERMINANTAL POINT PROCESSES ON THE UNIT DISC WITH SUB-BERGMAN KERNELS

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ABSTRACT

We give natural constructions of number rigid determinantal point processes on the unit disc $\mathbb{D}$ with sub-Bergman kernels of the form

$$K_{\Lambda}(z, w) = \sum_{n \in \Lambda} (n + 1)z^n\bar{w}^n, \quad z, w \in \mathbb{D},$$

with $\Lambda$ an infinite subset of non-negative integers. Our constructions are given in both deterministic and probabilistic methods. In the deterministic method, our proofs involve the classical Bloch functions.

1. Introduction

In the present paper, we construct some non-trivial and natural number rigid determinantal point processes over bounded domains in the complex plane.

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1.1. Determinantal point processes. We start by recalling some basic materials on point processes. Let $M$ be a locally compact and complete separable metric space equipped with a $\sigma$-finite non-negative Radon measure $\mu$. We assume that the metric on $M$ is such that all bounded subsets are relatively compact. Let $\text{Conf}(M)$ denote the space of (locally finite) configurations over $M$ consisting of non-negative integer-valued Radon measures on $M$. Consider the topology of vague convergence on the set of Radon measures on $M$, then the induced topology on $\text{Conf}(M)$ makes it a Polish space; see, e.g., Daley and Vere-Jones [9, Theorem 9.1. IV] for more details. The Borel $\sigma$-algebra on $\text{Conf}(M)$ coincides with the $\sigma$-algebra generated by all the functions $\#_B : \text{Conf}(M) \to \mathbb{N} = \{0, 1, 2, \ldots\}$ (with $B$ ranges over all Borel subsets of $M$) defined by

$$\#_B(X) := X(B) = \int_B dX \quad \text{for any } X \in \text{Conf}(M).$$

By definition, any Borel probability measure $P$ on the configuration space $\text{Conf}(M)$ is called a point process on $M$.

Let $K$ be a locally trace class positive contractive operator on the complex Hilbert space $L^2(M, \mu)$. Then $K$ is an integral operator and by slightly abusing the notation, we denote its kernel by $K(x, y)$. By a theorem obtained by Macchi [17] and Soshnikov [22, 23], as well as Shirai and Takahashi [21], the kernel $K(\cdot, \cdot)$ induces a unique point process on $M$, denoted by $P_K$ and called the determinantal point process induced by $K$, such that the equalities

$$\mathbb{E}_{P_K} \left[ \prod_{i=1}^{n} \frac{(\#_{B_i})!}{(\#_{B_i} - n_i)!} \right] = \int_{B_1^{n_1} \times \cdots \times B_m^{n_m}} \det[K(x_i, x_j)]_{i,j=1}^{n} d\mu(x_1) \cdots d\mu(x_n)$$

hold for any disjoint bounded Borel subsets $B_1, \ldots, B_m \subset M$ with $m \geq 1, n_i \geq 1$ and $n_1 + \cdots + n_m = n$. Here by convention, we set

$$\frac{(\#_{B_i})!}{(\#_{B_i} - n_i)!} = 0 \quad \text{whenever } \#_{B_i} - n_i < 0.$$ 

We refer the reader to [3, 16, 19, 21, 22, 23] for further background and details of determinantal point processes.

We now recall the definition of the number rigidity property of point processes on $M$. For a given Borel subset $C \subset M$, let $\mathcal{F}_C$ denote the $\sigma$-algebra on $\text{Conf}(M)$ generated by all functions $X \mapsto \#_B(X)$ with all Borel subsets $B \subset C$. For any point process $P$ on $M$, we denote by $\mathcal{F}_C^P$ the completion of the $\sigma$-algebra $\mathcal{F}_C$ with respect to $P$. A point process $P$ on $M$ is called number rigid
if for any bounded Borel subset $B \subset M$, the random variable $\#_B$ is $\mathcal{F}_{M \setminus B}$-measurable. This definition of number rigidity is due to Ghosh [10] where he shows that the sine-process is number rigid and Ghosh–Peres [14] where they show that the Ginibre process and the zero set of Gaussian analytic functions on the plane are number rigid. Bufetov [4] shows that determinantal point processes with the Airy, the Bessel and the Gamma kernels are number rigid. For more results on the number rigidity of point processes, we refer the reader to [5, 7, 11, 12, 13, 18, 20].

However, Holroyd and Soo show that the determinantal point process on the unit disc $\mathbb{D}$ with the standard Bergman kernel (with respect to the normalized Lebesgue measure on the unit disc $\mathbb{D}$):

$$K_{\mathbb{D}}(z, w) = \frac{1}{(1 - z\bar{w})^2} = \sum_{n=0}^{\infty} (n + 1)z^n\bar{w}^n$$

is not number rigid [15]. The proof of Holroyd and Soo relies on the Peres–Virág’s explicit model [19] of the determinantal point process $\mathbb{P}_{K_{\mathbb{D}}}$: it is given by the set of zeros of a particular Gaussian analytic functions on $\mathbb{D}$. See also [8] for an alternative proof of the non-rigid property of the determinantal point process $\mathbb{P}_{K_{\mathbb{D}}}$. More generally, among other things, Bufetov, Fan and Qiu [6] show that for any domain $U$ in the $d$-dimensional complex Euclidean space $\mathbb{C}^d$ without Liouville property (that is, there exists a non-constant bounded holomorphic function $f : U \to \mathbb{C}$) and any weight $\omega : U \to \mathbb{R}^+$ locally uniformly away from zero, the determinantal point process associated with the reproducing kernel of the weighted Bergman space $L^2_a(U; \omega)$ is not number rigid.

These results lead us to ask whether there exist natural number rigid determinantal point processes on a bounded domain of the complex plane (of course, any finite rank orthogonal projection yields a number rigid determinantal point process, so here we are only interested in infinite rank orthogonal projections). In this paper, we answer affirmatively this question in both deterministic and probabilistic ways.

1.2. Main results. From now on, we focus on the case of the unit disc $\mathbb{D}$ equipped with the normalized Lebesgue measure $dm$. Recall that the **Bergman space** $L^2_a(\mathbb{D})$ is defined by

$$L^2_a(\mathbb{D}) := \left\{ f : \mathbb{D} \to \mathbb{C} \mid f \text{ is holomorphic and } \int_{\mathbb{D}} |f(z)|^2dm(z) < \infty \right\}.$$
The Bergman space $L^2_a(D)$ is a closed subspace of $L^2(D) = L^2(D, dm)$ and is a reproducing kernel Hilbert space whose reproducing kernel is given by the formula (1). To any infinite subset $\Lambda \subset \mathbb{N}$ of non-negative integers is associated an orthogonal projection kernel (which we call a sub-Bergman kernel) defined by

$$K_{\Lambda}(z, w) = \sum_{n \in \Lambda} (n + 1)z^n\bar{w}^n. \tag{2}$$

Our main results, Theorem 1.1 and Theorem 1.3, provide constructions of infinite subsets $\Lambda \subset \mathbb{N}$, in a deterministic way and in a probabilistic way respectively, such that the associated sub-Bergman kernels $K_{\Lambda}$ induce number rigid determinantal point processes on $D$.

To indicate the idea of our proofs, in what follows, given any holomorphic function on $D$:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

we write

$$K^f(z, w) := \sum_{n=0}^{\infty} a_n (n + 1)z^n\bar{w}^n, \quad z, w \in D.$$

For a subset $\Lambda \subset \mathbb{N}$, we denote

$$f_{\Lambda}(z) = \sum_{n \in \Lambda} z^n. \tag{3}$$

Then we have

$$K_{\Lambda}(z, w) = K^{f_{\Lambda}}(z, w).$$

Recall the definition of the Bloch space $\mathcal{B}$ on the unit disc $D$:

$$\mathcal{B} := \left\{ f : D \to \mathbb{C} \mid f \text{ is holomorphic and } \|f\|_{B} := \sup_{z \in D} (1 - |z|)|f'(z)| < \infty \right\}. \tag{4}$$

**Theorem 1.1:** Let $\Lambda \subset \mathbb{N}$ be an infinite subset. Suppose that the function $f_{\Lambda}$ defined in (3) satisfies

$$f_{\Lambda} \in \mathcal{B}.$$

Then the determinantal point process on $D$ induced by the sub-Bergman kernel

$$K_{\Lambda}(z, w) = K^{f_{\Lambda}}(z, w)$$

is number rigid.
We have an explicit criterion for when $f_\Lambda$ is included in the Bloch space. Let \( \Lambda = \{\lambda_1, \lambda_2, \ldots\} \) be a subset of \( \mathbb{N} \) with \( 1 \leq \lambda_1 < \lambda_2 < \cdots \). We say that \( \Lambda \) is a \textit{lacunary sequence} (in the sense of Hadamard) if it satisfies the gap condition
\[
\rho_\Lambda := \liminf_{k \in \mathbb{N}} \frac{\lambda_{k+1}}{\lambda_k} > 1.
\]
The following characterization was already hinted in the proof of [1, Lemma 10] by Anderson and Shields. We note a short proof for completeness.

\textbf{Proposition 1.2:} Let \( \Lambda \) be a subset of \( \mathbb{N} \). Then \( f_\Lambda \in \mathcal{B} \) if and only if \( \Lambda \) is a union of finitely many lacunary subsets of \( \mathbb{N} \).

Now we turn to the probabilistic construction. Throughout the paper, suppose that \( \xi = (\xi_n)_{n=0}^\infty \) is a sequence of independent Bernoulli random variables with
\[
\xi_n = \begin{cases} 
1 & \text{with probability } \frac{1}{n+1}, \\
0 & \text{with probability } 1 - \frac{1}{n+1}.
\end{cases}
\]
We shall consider the random holomorphic function on the unit disc \( \mathbb{D} \):
\[
f_\xi = \sum_{n=0}^\infty \xi_n z^n.
\]
By the Kolmogorov Three Series Theorem, almost surely, we have
\[
\sum_{n=0}^\infty \xi_n = \infty.
\]
Therefore, for almost every realization \( (\xi_n)_{n=0}^\infty \), the kernel
\[
K^{f_\xi}(z, w) = \sum_{n=0}^\infty \xi_n (n+1) z^n \bar{w}^n
\]
is an orthogonal projection onto the following infinite-dimensional subspace:
\[
\overline{\text{span}} L^2(\mathbb{D}) \{z^n \mid n \in \mathbb{N} \text{ such that } \xi_n = 1\} \subset L^2_a(\mathbb{D}).
\]

\textbf{Theorem 1.3:} Consider the sequence \( \xi \) of independent Bernoulli random variables defined by (6). For almost every realization \( \xi \), the determinantal point process induced by the kernel \( K^{f_\xi}(z, w) \) is number rigid.

We also show that our probabilistic method indeed yields different constructions of number rigid determinantal point processes on \( \mathbb{D} \) by the following
Proposition 1.4: Consider the sequence $\xi$ of independent Bernoulli random variables defined by (6). Almost surely, the function $f_\xi$ defined by (7) is not included in the Bloch space $\mathcal{B}$. Or equivalently, almost surely, the subset 
\[ \Lambda_\xi := \{ n \in \mathbb{N} | \xi_n = 1 \} \]

is not a union of finitely many lacunary subsets of $\mathbb{N}$.

1.3. Comments. It is worthwhile to compare our main results, Theorem 1.1 and Theorem 1.3, with the result in [6] mentioned in subsection §1.1. Let us state an immediate corollary of [6, Theorem 1.2] as follows. Denote by $H^\infty(\mathbb{D})$ the algebra consisting of all bounded holomorphic functions on $\mathbb{D}$. Clearly, the Bergman space $L_a^2(\mathbb{D})$ is an $H^\infty(\mathbb{D})$-module. Let $H \subset L_a^2(\mathbb{D})$ be a non-zero closed subspace which is an $H^\infty(\mathbb{D})$-submodule of $L_a^2(\mathbb{D})$. If the orthogonal projection $K_H : L_a^2(\mathbb{D}) \to H$ is locally of trace class, then the associated determinantal point process $\mathbb{P}_{K_H}$ is not number rigid. In fact, in the above statement, the algebra $H^\infty(\mathbb{D})$ can be replaced by any closed non-trivial sub-algebra of $H^\infty(\mathbb{D})$ (here by trivial sub-algebras of $H^\infty(\mathbb{D})$, we mean the subalgebra consisting only of the zero function or the subalgebra consisting of constant functions on $\mathbb{D}$). This implies in particular that for any integer $\ell \geq 1$, if $\Lambda_{\ell} = \ell \mathbb{N} = \{0, \ell, 2\ell, 3\ell, \ldots\}$, then the corresponding determinantal point process induced by the kernel $K_{\Lambda_{\ell}}$ is not number rigid.

The key difference between our setting in this paper and the setting in [6] is, in both Theorem 1.1 and Theorem 1.3, that the subset $\Lambda$ and the random subset $\Lambda_\xi$ are sparse and the associated subspace of $L_a^2(\mathbb{D})$ is not a module over a non-trivial sub-algebra of $H^\infty(\mathbb{D})$.

From the above discussions, it seems natural to ask the following question.

Question 1: Does there exist a subset $\Lambda \subset \mathbb{N}$ having arbitrarily long arithmetic progressions such that the determinantal point process induced by $K_{\Lambda}$ is number rigid?

2. Rigidity of DPP with sub-Bergman kernels

For a bounded measurable compactly supported function $\phi$ on $\mathbb{D}$, we denote by $S_{\phi}$ the additive functional on the configuration space $\text{Conf}(\mathbb{D})$ defined by the formula
\[ S_{\phi}(X) = \int_{\mathbb{D}} \phi dX. \]
The following sufficient condition for number rigidity of a point process is showed by Ghosh [10] and Ghosh, Peres [14].

**Proposition 2.1** (Ghosh and Peres): Let $\mathbb{P}$ be a Borel probability measure on $\text{Conf}(M)$. Assume that for any $\epsilon > 0$, and any bounded subset $B \subseteq M$, there exists a bounded measurable function $\phi : M \to \mathbb{C}$ of compact support such that $\phi \equiv 1$ on $B$, and $\text{Var}_{\mathbb{P}}(S_{\phi}) \leq \epsilon$. Then $\mathbb{P}$ is number rigid.

Recall that for the determinantal point process $\mathbb{P}_K$ with an orthogonal projection kernel $K(x, y)$ on the unit disc $\mathbb{D}$, we have

$$\text{Var}_{\mathbb{P}_K}(S_{\phi}) = \frac{1}{2} \iint_{\mathbb{D}} |\phi(x) - \phi(y)|^2 \cdot |K(x, y)|^2 dm(x)dm(y).$$

(9)

The following lemma is our key estimation.

**Lemma 2.2:** For any $\epsilon > 0$ and $0 < r_0 < 1$, there exists a bounded measurable function $h : [0, 1) \to \mathbb{R}$ of compact support on $[0, 1)$ such that $h \equiv 1$ on $[0, r_0]$ and

$$\int_0^1 \int_0^1 |h(t) - h(s)|^2 \frac{1}{(1 - st)^2} ds dt < \epsilon.$$

We postpone the proof of Lemma 2.2 to the last section.

### 2.1. Rigid Kernel via the Bloch Functions.

Recall that, by the definition of the Bloch space $\mathcal{B}$, any function $f \in \mathcal{B}$ satisfies the inequality

$$\quad (1 - |z|)|f'(z)| \leq \|f\|_{\mathcal{B}}, \quad z \in \mathbb{D}.\quad (10)$$

We shall also use the following classical estimate on functions belonging to the Bloch space; see, e.g., [24, Theorem 5.4] for the details.

**Theorem 2.3:** If $f : \mathbb{D} \to \mathbb{C}$ is a holomorphic function, then $f \in \mathcal{B}$ if and only if the function $(1 - |z|)^2 f''(z)$ is bounded in $\mathbb{D}$.

By the Closed Graph Theorem, there exists a numerical constant $C > 0$ such that for any $f \in \mathcal{B}$, we have

$$\sup_{z \in \mathbb{D}} (1 - |z|)^2 |f''(z)| \leq C\|f\|_{\mathcal{B}}.\quad (11)$$
Before passing to the proof of Theorem 1.1, let us briefly explain how the estimates (10) and (11) come to play a role. In fact, the estimates (10) and (11) will be used through an identity of the variance of linear statistics. More precisely, let \( \Lambda \subset \mathbb{N} \) and

\[
f_\Lambda(z) = \sum_{n \in \Lambda} z^n.
\]

Consider the determinantal point process \( \mathbb{P}_{K_\Lambda} \) on \( \mathbb{D} \) induced by the orthogonal projection kernel

\[
K^{f_\Lambda}(z, w) = K_\Lambda(z, w) = \sum_{n \in \Lambda} (n + 1)z^n \bar{w}^n.
\]

Then for any compactly supported bounded radial function \( \phi : \mathbb{D} \to \mathbb{R}_+ \), that is \( \phi(z) = \phi(|z|) \) for all \( z \in \mathbb{D} \), we have

\[
\text{Var}_{\mathbb{P}_{K_\Lambda}}(S_\phi) = 2 \int_0^1 \int_0^1 |\phi(t) - \phi(s)|^2 M(s, t) \, dt \, ds
\]

where

\[
M(s, t) = (t^4 s^4 f_\Lambda''(t^2 s^2) + 3t^2 s^2 f_\Lambda'(t^2 s^2) + f_\Lambda(t^2 s^2)) ts.
\]

The estimates (10) and (11) together with the elementary inequality

\[
f_\Lambda(r) \leq \sum_{n=0}^\infty r^n = \frac{1}{1 - r}, \quad r \in [0, 1)
\]

imply the following useful upper-estimate of \( M(s, t) \): there exists a constant \( C > 0 \) such that

\[
M(s, t) \leq \frac{C}{(1 - st)^2}, \quad s, t \in [0, 1).
\]

**Proof of Theorem 1.1.** Note that \( K_\Lambda \) is the orthogonal projection onto the following subspace of \( L^2_a(\mathbb{D}) \):

\[
\text{span}L^2(\mathbb{D}) \{ z^n \mid n \in \Lambda \} \subset L^2_a(\mathbb{D}).
\]

Assume that

\[
f_\Lambda = \sum_{n \in \Lambda} z^n \in \mathcal{B}
\]

and consider the determinantal point process \( \mathbb{P}_{K_\Lambda} \) on \( \mathbb{D} \) induced by the orthogonal projection kernel

\[
K^{f_\Lambda}(z, w) = K_\Lambda(z, w).
\]
Since $f_\Lambda \in \mathcal{B}$, by (10) and (11), there exists $C > 0$ such that

\[(1 - |z|)|f'_\Lambda(z)| \leq \|f_\Lambda\|_{\mathcal{B}} \quad \text{and} \quad (1 - |z|)^2|f''_\Lambda(z)| \leq C\|f_\Lambda\|_{\mathcal{B}}, \quad z \in \mathbb{D}.\]

Let $\phi : \mathbb{D} \to \mathbb{R}_+$ be any compactly supported bounded radial function. Then by (9), we have

\[2\text{Var}_{\mathbb{P}_{K_\Lambda}}(S_{\phi}) = \int_0^1 \int_0^1 |\phi(t) - \phi(s)|^2 |K_\Lambda(t, s)|^2 \, dt \, ds \leq \frac{C}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} |K_\Lambda(te^{i\alpha}, se^{i\beta})|^2 \, d\alpha \, d\beta \quad t, s \in \mathbb{D}.\]

Note that for any fixed $\beta \in [0, 2\pi)$, we have

\[\int_0^{2\pi} |K_\Lambda(te^{i\alpha}, se^{i\beta})|^2 \, d\alpha = \int_0^{2\pi} \sum_{n \in \Lambda} (n + 1)^2 (ts)^{2n} \, d\alpha \leq \sum_{n \in \Lambda} (n + 1)^2 (ts)^{2n}.\]

Moreover, we have

\[\sum_{n \in \Lambda} (n + 1)^2 (ts)^{2n} = \sum_{n \in \Lambda} [n(n - 1)(ts)^{2n-4}t^4s^4 + 3n(ts)^{2n-2}t^2s^2 + (ts)^{2n}] = t^4s^4f''_\Lambda(t^2s^2) + 3t^2s^2f'_\Lambda(t^2s^2) + f_\Lambda(t^2s^2).\]

Therefore, by (12), there exists $C' > 0$ such that

\[2\text{Var}_{\mathbb{P}_{K_\Lambda}}(S_{\phi}) = 4 \int_0^1 \int_0^1 |\phi(t) - \phi(s)|^2 \sum_{n \in \Lambda} (n + 1)^2 (ts)^{2n} \, dt \, ds \leq 4C' \int_0^1 \int_0^1 |\phi(t) - \phi(s)|^2 \frac{1}{(1 - t^2s^2)^2} \, dt \, ds \leq 4C' \int_0^1 \int_0^1 |\phi(t) - \phi(s)|^2 \frac{1}{(1 - ts)^2} \, dt \, ds.\]

It follows that, for any $\epsilon > 0$ and any $0 < r_0 < 1$, if we take $h_{r_0, \epsilon}$ to be the function appearing in Lemma 2.2 and set

\[\phi_{r_0, \epsilon}(z) = h_{r_0, \epsilon}(|z|),\]
then \( \phi_{r_0, \epsilon} : \mathbb{D} \rightarrow \mathbb{R} \) is a compactly supported bounded measurable function such that \( \phi_{r_0, \epsilon} \equiv 1 \) on \( \{ z \in \mathbb{D} : |z| \leq r_0 \} \) and

\[
\text{Var}_{\mathcal{K}_\Lambda} (S_{\phi_{r_0, \epsilon}}) \leq \epsilon.
\]

Since any compact subset \( B \subset \mathbb{D} \) is included in \( \{ z \in \mathbb{D} : |z| \leq r_0 \} \) for some \( r_0 \in (0, 1) \), by Proposition 2.1, we complete the proof of Theorem 1.1 with the use of Lemma 2.2.

**Proof of Proposition 1.2.** Suppose that

\[
f_{\Lambda}(z) = \sum_{n \in \Lambda} z^n \in \mathcal{B}.
\]

Then by the definition (4) of the Bloch space \( \mathcal{B} \), we have that

\[
\sum_{k \in \Lambda} k r_k^k = r f'(r) \leq \frac{\|f\|_{\mathcal{B}}}{1-r}, \quad r \in [0, 1).
\]

For any fixed integer \( M \geq 2 \), set \( r = 1 - \frac{1}{M} \). Note that there exists a positive number \( c > 0 \) depending on \( M \) such that \( r^k > c \) for any \( k \leq M \). It follows that for the constant \( c' = \frac{1}{c} > 0 \),

\[
\sum_{1 \leq k < M} k \frac{r^k}{c} \leq c' \sum_{k \in \Lambda} k r^k \leq c' \|f\|_{\mathcal{B}} \frac{1}{1-r} = c' \|f\|_{\mathcal{B}M}.
\]

This implies that

\[
\sum_{2^n \leq k < 2^{n+1}} \frac{k}{2^n} \leq \frac{c' \|f\|_{\mathcal{B}} 2^{n+1}}{2^n} \leq 2c' \|f\|_{\mathcal{B}}.
\]

To ease the notations, write

\[
q := [2c' \|f\|_{\mathcal{B}}] + 1
\]

and

\[
\Lambda^{\text{even}} := \Lambda \cap \bigcup_{m \in \mathbb{N}} I_{2m}, \quad \Lambda^{\text{odd}} := \Lambda \cap \bigcup_{m \in \mathbb{N}+1} I_{2m+1},
\]

where

\[
I_n = \{ k \in \mathbb{N} : 2^n \leq k < 2^{n+1} \}.
\]

Then the inequalities (13) imply that there exist subsets \( \{ \Lambda^{\text{even}}_i \}_{i=1}^q \) such that

\[
\Lambda^{\text{even}} = \bigcup_{i=1}^q \Lambda^{\text{even}}_i,
\]
and each $\Lambda_i^{\text{even}}$ has at most one element inside $I_{2m}$ and has no element inside $I_{2m+1}$ for any $m \in \mathbb{N}$. That is, each $\Lambda_i^{\text{even}}$ is either a finite subset or a subset which satisfies the gap condition (5) with the gap ratio not less than 2. This implies $\Lambda^{\text{even}}$ is the union of at most $q$ many lacunary subsets. The same argument also holds for $\Lambda^{\text{odd}}$. Therefore, $\Lambda$ is the union of at most $2q$ many lacunary subsets.

On the other hand, without loss of generality, suppose that $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$ is a lacunary subset with $1 \leq \lambda_1 < \lambda_2 < \cdots$ and

$$\inf_{k \in \mathbb{N}} \frac{\lambda_{k+1}}{\lambda_k} \geq 2.$$ 

This implies that

$$\sum_{2^n \leq k < 2^{n+1}, k \in \Lambda} 1 \leq 1, \quad \forall n \in \mathbb{N}$$

and hence for any $r \in (0, 1)$ and any integer $n \geq 1$, we have

$$\sum_{2^n \leq k < 2^{n+1}, k \in \Lambda} kr^k \leq \sup_{2^n \leq k < 2^{n+1}} kr^k \leq 2^{n+1} r 2^n.$$ 

Therefore, by noting

$$\sum_{2^{n-1} \leq k < 2^n} 4r^k \geq 4r^{2^n} (2^n - 2^{n-1}) = 2^{n+1} r 2^n,$$

for any $r \in (0, 1)$, we have

$$\sum_{k \in \Lambda, k \geq 2} kr^k = \sum_{n=1}^{\infty} \sum_{2^n \leq k < 2^{n+1}, k \in \Lambda} kr^k \leq \sum_{n=1}^{\infty} 2^{n+1} r 2^n$$

$$\leq \sum_{n=1}^{\infty} \sum_{2^{n-1} \leq k < 2^n} 4r^k \leq \frac{4}{1-r}.$$ 

It follows that

$$\sup_{z \in \mathbb{D}} (1 - |z|) |f_\Lambda'(z)| = \sup_{z \in \mathbb{D}} (1 - |z|) \sum_{k \in \Lambda} k |z|^{k-1} = \sup_{0 < r < 1} (1 - r) \sum_{k \in \Lambda} kr^{k-1} < \infty.$$ 

By the definition (4) of the Bloch space $\mathcal{B}$, this means that $f_\Lambda \in \mathcal{B}$. 

\[\blacksquare\]
2.2. Rigid kernel via probabilistic methods.

Proof of Theorem 1.3. By the definition (8) of the kernel $K_{f\xi}(z,w)$, we have that

$$\int_0^{2\pi} |K_{f\xi}(te^{i\alpha}, se^{i\beta})|^2 \frac{d\alpha}{2\pi} = \int_0^{2\pi} \left| \sum_{n=0}^{\infty} \xi_n (n+1) t^n s^n e^{in(\alpha-\beta)} \right|^2 \frac{d\alpha}{2\pi}$$

(14)

$$= \sum_{n=0}^{\infty} \xi_n (n+1)^2 t^{2n} s^{2n}.$$  

For any compact subset $B \subset \mathbb{D}$, there exists $r_0 \in (0,1)$ such that

$$B \subset \{ z \in \mathbb{C} : |z| \leq r_0 \}.$$  

For such a real number $r_0 \in (0,1)$ and any $\epsilon > 0$, let $h_{r_0,\epsilon}$ be the function appearing in Lemma 2.2 and set

$$\phi_{B,\epsilon}(z) = h_{r_0,\epsilon}(|z|), \quad z \in \mathbb{D}.$$  

By (14), the definition (6) of the random variables $\xi_n$ and the following elementary identity

$$\sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}, \quad |x| < 1,$$

we obtain

$$\mathbb{E} \left[ \int_{\mathbb{D}} \int_{\mathbb{D}} |\phi_{B,\epsilon}(z) - \phi_{B,\epsilon}(w)|^2 |K_{f\xi}(z,w)|^2 dm(z) dm(w) \right]$$

$$= \mathbb{E} \left[ \int_{[0,1]} \int_{[0,1]} ts|h_{r_0,\epsilon}(t) - h_{r_0,\epsilon}(s)|^2 dtds \int_0^{2\pi} \int_0^{2\pi} |K_{f\xi}(te^{i\alpha}, se^{i\beta})|^2 \frac{d\alpha}{\pi} \frac{d\beta}{\pi} \right]$$

$$= 4 \int_0^1 \int_0^1 ts|h_{r_0,\epsilon}(t) - h_{r_0,\epsilon}(s)|^2 \sum_{n=0}^{\infty} (n+1)^2 (ts)^2n \mathbb{E} \xi_n dtds$$

$$= 4 \int_0^1 \int_0^1 |h_{r_0,\epsilon}(t) - h_{r_0,\epsilon}(s)|^2 \sum_{n=0}^{\infty} (n+1)(ts)^2n tdsdtds$$

$$= 4 \int_0^1 \int_0^1 |h_{r_0,\epsilon}(t) - h_{r_0,\epsilon}(s)|^2 \frac{1}{(1-st)^2} tdsdtds$$

$$\leq 4 \int_0^1 \int_0^1 |h_{r_0,\epsilon}(t) - h_{r_0,\epsilon}(s)|^2 \frac{1}{(1-st)^2} dtds$$

$$\leq 4\epsilon.$$
Now for any integer $n \geq 1$, set
\begin{equation}
\phi_n(z) := \phi_{B,n^{-2}}(z).
\end{equation}
By the above computation, for each integer $n \geq 1$, we have
\begin{equation}
\mathbb{E} \left[ \int_D \int_D |\phi_n(z) - \phi_n(w)|^2 |K^{f_\xi}(z,w)|^2 dm(z) dm(w) \right] \leq \frac{4}{n^2}
\end{equation}
and hence
\begin{equation}
\sum_{n=1}^{\infty} \mathbb{E} \left[ \int_D \int_D |\phi_n(z) - \phi_n(w)|^2 |K^{f_\xi}(z,w)|^2 dm(z) dm(w) \right] < \infty.
\end{equation}
The Levi lemma implies that
\begin{equation}
\sum_{n=1}^{\infty} \int_D \int_D |\phi_n(z) - \phi_n(w)|^2 |K^{f_\xi}(z,w)|^2 dm(z) dm(w) < \infty, \quad \text{a.s.}
\end{equation}
It follows that
\begin{equation}
\lim_{n \to \infty} \int_D \int_D |\phi_n(z) - \phi_n(w)|^2 |K^{f_\xi}(z,w)|^2 dm(z) dm(w) = 0, \quad \text{a.s.}
\end{equation}
Note that by (15), (16) and the property of $h_{r_0,\epsilon}$, we know that for each $n \geq 1$, the function $\phi_n : D \to \mathbb{R}$ is a compactly supported bounded measurable function such that $\phi_n \equiv 1$ on $B$. Therefore, by Proposition 2.1 and the equality (9), the limit relation (17) implies that, for almost every realization of $\xi$, the determinantal point process induced by the orthogonal projection kernel $K^{f_\xi}(z,w)$ is number rigid. \hfill \Box

**Proof of Proposition 1.4.** Let
\begin{equation}
I_n = [2^n, 2^{n+1}] \cap \mathbb{N} \quad \text{and} \quad \mathcal{N}_n = \sum_{k \in I_n} \xi_k.
\end{equation}
We claim that for any integer $C \geq 1$,
\begin{equation}
\limsup_n \mathcal{N}_n \geq C, \quad \text{a.s.}
\end{equation}
Note that for a lacunary set $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$ with the gap ratio
\begin{equation}
\rho = \liminf_k \frac{\lambda_{k+1}}{\lambda_k} > 1,
\end{equation}
the subset $\Lambda \cap I_n$ has at most
\begin{equation}
\left\lfloor \frac{\log 2}{\log \left(\frac{\rho+1}{2}\right)} \right\rfloor + 1
\end{equation}
elements when $n$ is sufficiently large. Here $[x]$ denotes the largest integer not larger than $x$. More generally, for an integer set

$$\Lambda = \bigcup_{i=1}^{p} \Lambda_i$$

with each lacunary set $\Lambda_i$ having the gap ratio $\rho_i$, one has that the set $\Lambda \cap I_n$ contains at most

$$\sum_{i=1}^{p} \left\lfloor \frac{\log 2}{\log(\rho_i + 1)} \right\rfloor + p$$

elements when $n$ is sufficiently large. Combining this with the claimed inequality (18), we conclude that almost surely, the subset

$$\Lambda_\xi = \{k \in \mathbb{N} | \xi_k = 1\}$$
is not a union of finitely many lacunary sets.

We next prove the inequality (18). Noting that for $k \in I_n = (2^n, 2^{n+1}] \cap \mathbb{N}$,

$$\text{Prob}[\xi_k = 1] = \frac{1}{k+1} \geq \frac{1}{2^n+1+1}$$

and

$$\text{Prob}[\xi_k = 0] = 1 - \frac{1}{k+1} \geq 1 - \frac{1}{2^n},$$

we have that for a fixed integer $C \geq 1$,

$$\text{Prob}[N_n = C] \geq \sum_{A \subseteq I_n, |A| = C} \left( \frac{1}{1 + 2^{n+1}} \right)^C \left( 1 - \frac{1}{2^n} \right)^{2^n-C}$$

$$= \binom{2^n}{C} \frac{1}{(1 + 2^{n+1})^C} \left( 1 - \frac{1}{2^n} \right)^{2^n-C}.$$

Note that

$$\lim_{n \to \infty} \binom{2^n}{C} \frac{1}{(1 + 2^{n+1})^C} = \lim_{n \to \infty} \frac{2^n(2^n-1) \cdots (2^n-C+1)}{C!(1 + 2^{n+1})^C} = \frac{1}{2^C C!}$$

and

$$\lim_{n \to \infty} \left( 1 - \frac{1}{2^n} \right)^{2^n-C} = \lim_{n \to \infty} \left\{ \left[ 1 - \frac{1}{2^n} \right]^{2^n} \right\}^{2^n-C} = e.$$

Therefore, there exists an integer $M$ and $\beta > 0$ such that for any integer $n > M$,

$$\binom{2^n}{C} \frac{1}{(1 + 2^{n+1})^C} \left( 1 - \frac{1}{2^n} \right)^{2^n-C} > \beta.$$

This implies that, for any integer $n > M$,

$$\text{Prob}[N_n \geq C] \geq \text{Prob}[N_n = C] > \beta.$$
and hence
\[ \sum_{n=0}^{\infty} \text{Prob}[\mathcal{N}_n \geq C] = \infty. \]

Noting that the random variables \( \mathcal{N}_n \) are independent, by the Borel–Cantelli lemma, we have
\[ \limsup_{n \to \infty} \mathcal{N}_n \geq C \quad \text{a.s.} \]
This completes the proof. \( \blacksquare \)

3. Proof of Lemma 2.2

In this section we will find a suitable function \( h \) such that the integral in Lemma 2.2 is small enough.

3.1. Some preliminary discussions. Our proof of Lemma 2.2 is inspired by some trace formulas in the classical Hankel operator theory on weighted Bergman spaces (see, e.g., [24, Chapter 8] for a brief introduction).

Let us recall some basic definitions of Hankel operators on weighted Bergman spaces. For simplicity, here we only consider the Hankel operators with bounded symbols. For any \( \alpha > -1 \), let \( dm_\alpha \) denote the classical weight on the unit disc:
\[ dm_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dm(z). \]

Let \( L^2_{a}(\mathbb{D}, dm_\alpha) \subset L^2(\mathbb{D}, dm_\alpha) \) denote the associated weighted Bergman space consisting of all holomorphic functions \( f : \mathbb{D} \to \mathbb{C} \) inside \( L^2(\mathbb{D}, dm_\alpha) \). Then the orthogonal projection (called the Bergman projection)
\[ P_\alpha : L^2(\mathbb{D}, dm_\alpha) \to L^2_{a}(\mathbb{D}, dm_\alpha) \]
is an integral operator with kernel
\[ K_\alpha(z, w) = \frac{1}{(1 - z\bar{w})^{2+\alpha}}. \]

For any given bounded symbol \( g \) on \( \mathbb{D} \), that is \( g \in L^\infty(\mathbb{D}) \), we define the Hankel operator
\[ H_g : L^2_{a}(\mathbb{D}, dm_\alpha) \to L^2_{a}(\mathbb{D}, dm_\alpha) = L^2_{a}(\mathbb{D}, dm_\alpha) \oplus L^2_{a}(\mathbb{D}, dm_\alpha) \]
by the formula
\[ H_g(f) := (I - P_\alpha)(gf) = gf - P_\alpha(gf), \quad f \in L^2_{a}(\mathbb{D}, dm_\alpha). \]
In other words,
\[ H_g(f)(z) = \int_{\mathbb{D}} (g(z) - g(w))K_\alpha(z,w)f(w)dm_\alpha(w), \quad f \in L^2_a(\mathbb{D}, dm_\alpha). \]

Recall the definition (4) for the Bloch space \( \mathcal{B} \). By [24, Lemma 8.22], one sees that for any \( g \in \mathcal{B} \),
\[
\int_{\mathbb{D}} \int_{\mathbb{D}} \left| g(z) - g(w) \right|^2 \left| 1 - z\overline{w} \right|^{2(z+\alpha)} dm_\alpha(z)dm_\alpha(z) = \text{tr}(H^*_gH_g) = \|H_g\|_{HS}^2,
\]
where \( \| \cdot \|_{HS} \) denotes the Hilbert–Schmidt norm. For general cases, it is shown in [2, Theorem 1.1] that the formula
\[
\int_{\Omega} \int_{\Omega} \left| g(z) - g(w) \right|^2 K_\mu(z,w)^2d\mu(z)d\mu(w) = \int_{\Omega} |g'(z)|^2 \frac{dxdy}{\pi}
\]
holds under some mild conditions, where \( K_\mu(z,w) \) is the reproducing kernel of the Bergman space \( L^2_a(\Omega,d\mu) \) over the planar domain \( \Omega \) and \( g \) is a holomorphic function on \( \Omega \).

Although the trace formula (19) only deals with holomorphic functions \( g \), it hints that the integral on the left-hand side of the equality (19) is related to the size of the Hankel operator \( H_g \) for general symbols. Since the integral in Lemma 2.2 has a certain similarity to the integral on the left-hand side of the equality (19), it is natural to guess that it is small when the size of the Hankel operator \( H_g \) with the radial symbol \( z \mapsto g(z) = h(|z|) \) is small.

For a general symbol, the size of the related Hankel operator is usually given in terms of its operator norm, which is described as follows. For any \( g \in L^\infty(\mathbb{D}) \), we define its \( BMO_\partial \)-norm by
\[
\|g\|_{BMO_\partial}^2 := \sup_{z \in \mathbb{D}} \int_{\mathbb{D}} \left| g \circ \varphi_z(w) - \int_{\mathbb{D}} g \circ \varphi_z dm_\alpha \right|^2 dm_\alpha(w),
\]
where
\[
\varphi_{z_1}(z_2) = \frac{z_2 - z_1}{1 - \overline{z}_1 z_2}
\]
is the Möbius transformation on the unit disc \( \mathbb{D} \). It is well known that the operator norm of the Hankel operators \( H_g, H_g \) are controlled by \( \|g\|_{BMO_\partial} \); see, e.g., [24, Theorem 8.20]. Although the definition of the \( BMO_\partial \)-norm seems to depend on the parameter \( \alpha \), it is actually independent of \( \alpha \). Namely, it follows from [24, Theorem 8.14] that
\[
\|g\|_{BMO_\partial} = \inf \{ \|g_1\|_{BO} + \|g_2\|_{BA} : g = g_1 + g_2 \},
\]
where the $BA$-norm is defined by
\[ \|g\|_{BA}^2 := \sup_{z \in \mathbb{D}} \int_{\mathbb{D}} |g \circ \varphi_z(w)|^2 dm(w) \]
and the $BO$-norm is defined by
\[ \|g\|_{BO} := \sup \{ |g(z) - g(w)| \colon \rho(z, w) \leq 1 \} \]
with $\rho(\cdot, \cdot)$ being the Poincaré metric:
\begin{equation}
\rho(z_1, z_2) = \log \frac{1 + \varphi_{z_1}(z_2)}{1 - \varphi_{z_1}(z_2)} = \log \frac{|z_1 - z_2| + |1 - \bar{z}_1 z_2|}{|z_1 - z_2| - |1 - \bar{z}_1 z_2|}, \quad z_1, z_2 \in \mathbb{D}.
\end{equation}
Therefore, it is natural to guess that a function $h$ on $[0, 1]$ would satisfy the requirement in Lemma 2.2 if $g(z) = h(|z|)$ has a small $BO$ norm.

### 3.2. The Proof of Lemma 2.2

Recall the definitions (20) and (21). For any $r_0, r$ with $0 < r_0 < r < 1$, we write
\[ h^{(r_0, r)}(t) = \begin{cases} 
1 & t \leq r_0, \\
\frac{\rho(t, r)}{\rho(r_0, r)} & r_0 \leq t \leq r, \\
0 & t \geq r.
\end{cases} \]
Note that $h^{(r_0, r)}$ is a bounded measurable function of compact support on $[0, 1)$.

The following theorem implies our technical Lemma 2.2.

**Theorem 3.1:** For any fixed $r_0 \in (0, 1)$, the following integral
\[ \int_0^1 \int_0^1 |h^{(r_0, r)}(t) - h^{(r_0, r)}(s)|^2 \frac{1}{(1 - st)^2} ds dt \]
tends to 0 when $\varphi_{r_0}(r) \to 1$.

**Proof.** When $0 < t, s < r_0$ or $r < t, s < 1$, then the expression of the integral is equal to zero. So we shall estimate our integral over the following three domains:
\[ 0 < t < r_0 < s; \quad r_0 < t < s < r; \quad r_0 < t < r < s. \]
The first case: The integral over the domain $0 < t < r_0 < r < s$ can be calculated explicitly as follows:

$$(I) := \int_0^{r_0} dt \int_r^1 ds \frac{1}{(1-st)^2} = \int_r^1 ds \left[ \frac{1}{s(1-st)} \right]_0^{r_0} = \int_r^1 \frac{r_0}{1-r_0 s} ds = \log \frac{1-r_0 r}{1-r_0} = \log \frac{1+r_0}{1+r_0 \varphi_{r_0}(r)}.$$ 

Therefore, for any fixed $r_0 \in (0, 1)$, the integral $(I)$ tends to 0 when $\varphi_{r_0}(r) \to 1$.

The second case: We estimate the integral over the domain $r_0 < t < s < r$:

$$(II) := \int_{r_0}^r dt \int_t^r ds \rho^2(s,t) \frac{1}{\rho^2(r_0,r) (1-st)^2}.$$ 

By the substitutions $t \to \varphi_{r_0}(t), s \to \varphi_{r_0}(s)$, we obtain

$$\int_{r_0}^r dt \int_t^r ds \frac{\rho^2(s,t)}{\rho^2(r_0,r)} \frac{1}{(1-st)^2} = \int_0^{\varphi_{r_0}(r)} dt \int_t^{\varphi_{r_0}(r)} ds \frac{\rho^2(s,t)}{\rho^2(0,\varphi_{r_0}(r)) (1-st)^2} \frac{1-r_0^2}{(1+r_0 t)^2}.$$ 

Now making the substitution $s \to \varphi_{t}(s)$, we get

$$(II) = \int_0^{\varphi_{r_0}(r)} dt \int_0^{\varphi_{r_0}(r)} ds \frac{\rho^2(s,r)}{\rho^2(0,\varphi_{r_0}(r)) (1-st)^2} = \int_0^{\varphi_{r_0}(r)} dt \int_0^{\varphi_{t}(\varphi_{r_0}(r))} ds \frac{\rho^2(s,0)}{\rho^2(0,\varphi_{r_0}(r))} \frac{1}{1-t^2} \leq \frac{1}{\rho^2(0,\varphi_{r_0}(r))} \int_0^{\varphi_{r_0}(r)} \frac{1}{1-t^2} dt \int_0^1 \rho^2(s,0) ds = \frac{C_2}{2\rho(0,\varphi_{r_0}(r))},$$

where

$$C_2 := \int_0^1 \rho^2(s,0) ds < \infty.$$ 

Therefore, the integral $(II)$ tends to 0 when $\varphi_{r_0}(r) \to 1$. 
The third case: We now estimate the integral over the domain $r_0 < t < r < s$:

\[(III) := \int_{r_0}^{r} dt \int_{r}^{r_1} ds \frac{\rho^2(r, t)}{\rho^2(r_0, r)} \frac{1}{(1 - st)^2}.\]

With the substitutions $t \to \varphi_r(t), s \to \varphi_r(s)$, the change of variables formula yields that

\[(III) = \int_{r_0}^{r} dt \int_{r}^{r_1} ds \frac{\rho^2(r, t)}{\rho^2(r_0, r)} \frac{1}{(1 - st)^2}
\]

\[= \int_{-\varphi_{r_0}(r)}^{0} dt \int_{0}^{1} ds \frac{\rho^2(0, t)}{\rho^2(0, \varphi_{r_0}(r))} \frac{1}{(1 - st)^2}
\]

\[\leq \frac{1}{\rho^2(0, \varphi_{r_0}(r))} \int_{-1}^{0} \rho^2(t, 0) dt \int_{0}^{1} ds
\]

\[= \frac{C_2}{\rho^2(0, \varphi_{r_0}(r))}.\]

Therefore, the integral $(III)$ tends to 0 when $\varphi_{r_0}(r) \to 1$, which completes the proof.

\[\blacksquare\]

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