BRST Structure of Polynomial Poisson Algebras

Alain Dresse

Marc Henneaux*

Faculté des Sciences, Université Libre de Bruxelles,
Campus Plaine C.P. 231, B-1050 Bruxelles (Belgium)

March 28, 2022

Abstract

The BRST structure of polynomial Poisson algebras is investigated. It is shown that Poisson algebras provide non trivial models where the full BRST recursive procedure is needed. Quadratic Poisson

*Also at Centro de Estudios Científicos de Santiago, Casilla 16443, Santiago 9, Chile
algebras may already be of arbitrarily high rank. Explicit examples are provided, for which the first terms of the BRST generator are given. The calculations are cumbersome but purely algorithmic, and have been treated by means of the computer algebra system REDUCE. Our analysis is classical (= non quantum) throughout.

1 Introduction

Polynomial algebras with a Lie bracket fulfilling the derivation property

\[ [fg, h] = f[g, h] + [f, h]g \]  

are called polynomial Poisson algebras and play an increasingly important role in various areas of theoretical physics [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. In terms of a set of independent generators \( G_a, a = 1, \ldots, n, \) the brackets are given by

\[ [G_a, G_b] = C_{ab}(G) \]
where $C_{ab} = -C_{ba}$ are polynomials in the $G$'s. If the polynomials $C_{ab}(G)$ vanish when the $G$'s are set equal to zero, i.e. if they have no constant part, the polynomial algebra is said to be first class, in analogy with the terminology for constrained Hamiltonian systems (see e.g. [1]). An important class of first class Poisson algebras are symmetric algebras over a finite dimensional Lie algebra. In that case, the bracket (2) belongs to the linear span of the $G_a$'s, i.e. the $C_{ab}(G)$ are homogeneous of degree one in the $G$'s, $[G_a, G_b] = C_{abc} G_c$. We shall call this situation the “Lie algebra case”, and refer to the non Lie algebra case as the “open algebra case” using again terminology from the theory of first class constrained systems [1].

The purpose of this paper is to investigate the BRST structure of first class Poisson algebras. The BRST formalism has turned out recently to be the arena of a fruitful interplay between physics and mathematics (see e.g. [1] and references therein). A crucial ingre-

---

1 We shall restrict here the analysis to ordinary polynomial algebras with commuting generators, but one can easily extend the study to the graded case with both commuting and anticommuting generators.

2 It should be stressed that the polynomial algebra generated by the $G$'s, equipped with the bracket (2) is always an infinite-dimensional Lie algebra, even in the “open algebra” case.
dient of BRST theory is the recursive pattern of homological perturbation theory [12] which allows one to construct the BRST generator step by step. In most applications, however, this recursive construction collapses almost immediately, and, to our knowledge, no example has been given so far for which the full BRST machinery is required (apart from the field-theoretical membrane models [13, 14]). We show in this paper that Poisson algebras—actually, already quadratic Poisson algebras—offer splendid examples illustrating the complexity of the BRST construction. While Lie algebras yield a BRST generator of rank 1 (see e.g. [11]), the BRST charge for quadratic Poisson algebras can be of arbitrarily high rank. We also point out that BRST concepts provide intrinsic characterizations of Poisson algebras.

In the next section, we briefly review the BRST construction. We then discuss how it applies to Poisson algebras, even when the generators $G_a$ are not realized as phase space functions of some dynamical system. We analyze the BRST cohomology and introduce the concepts of covariant and minimal ranks, for which an elementary theorem is proven. Quadratic algebras are then shown to provide models with arbitrarily high rank. These contain “self-reproducing” algebras for which the bracket of $G_a$ with $G_b$ is proportional to the product $G_aG_b$. 

4
The first few terms in the BRST generator are also computed for more general algebras by means of a program written in REDUCE. The paper ends with some concluding remarks on the quantum case.

2 A Brief Survey of the BRST Formalism

We follow the presentation of [11], to which we refer for details and proofs. Given a set of independent functions $G_a(q, p)$ defined in some phase space $P$ with local coordinates $(q^i, p_i)$ and fulfilling the first class property $[G_a, G_b] \approx 0$, where $\approx$ denotes equality on the surface $G_a(q, p) = 0$, one can introduce an odd generator $\Omega$ (“the BRST generator”) in an extended phase space containing further fermionic conjugate pairs $(\eta^a, \mathcal{P}_a)$ (the “ghost pairs”) which has the following properties:

\[
[\Omega, \Omega] = 0 \quad (3)
\]
\[
\Omega = G_a \eta^a + \text{“more”}. \quad (4)
\]

Here, “more” stands for terms containing at least one ghost momentum $\mathcal{P}_a$. We take the ghosts $\eta^a$ to be real and their momenta imaginary.
nary, with graded Poisson bracket

\[ [P_a, \eta^b] = -\delta_a^b \]  \hspace{1cm} (5)

The BRST derivation \( s \) in the extended phase space is generated by \( \Omega \),

\[ s \bullet = [\bullet, \Omega] \]  \hspace{1cm} (6)

and is a differential \((s^2 = 0)\) because of (3). One also introduces a grading, the “ghost number” by setting

\[ \text{gh} \eta^a = \text{gh} P_a = 1, \quad \text{gh} q^i = \text{gh} p_i = 0. \]  \hspace{1cm} (7)

The ghost number of the BRST generator is equal to 1.

The BRST generator \( \Omega \) is constructed recursively as follows. One sets

\[ \begin{aligned}
\Omega &= (0) \Omega + (1) \Omega + \cdots \\
(0) \Omega &= G_a \eta^a \\
(1) \Omega &= \cdots \\
(2) \Omega &= \cdots \\
& \vdots \\
(k) \Omega &= \cdots \\
\end{aligned} \]  \hspace{1cm} (8)

where \( \Omega \) contains \( k \) ghost momenta. One has \( \Omega = G_a \eta^a \). The nilpotency condition becomes, in terms of \( \Omega \),

\[ \delta^{(p+1)} \Omega + D = 0 \]  \hspace{1cm} (9)
where $D$ involves only the lower order $\Omega$ with $s \leq p$ and is defined by

$$D = \frac{1}{2} \left[ \sum_{k=0}^{p} [\Omega, \Omega]_{\text{orig}} + \sum_{k=0}^{p-1} [\Omega, \Omega]_{\mathcal{P},\eta} \right]. \quad (10)$$

Here, the bracket $[\ , \ ]_{\text{orig}}$ refers to the Poisson bracket in the original phase space, which only acts on the $q^i$ and $p_i$, and not on the ghosts, whereas $[\ , \ ]_{\mathcal{P},\eta}$ refers to the Poisson bracket acting only on the ghost and ghost momenta arguments and not on the original phase space variables. The “Koszul” differential $\delta$ in (9) is defined by

$$\delta q^i = \delta p_i = 0, \quad \delta \eta^a = 0, \quad \delta \mathcal{P}_a = -G_a \quad (11)$$

and is extended to arbitrary functions on the extended phase space as a derivation. One easily verifies that $\delta^2 = 0$.

Given $\Omega$ with $s \leq p$, one solves (9) for $\Omega$. This can always be done because $\delta D = 0$, and because $\delta$ is acyclic in positive degree. One then goes on to $\Omega$ etc... until one reaches the complete expression for $\Omega$. The last function $\Omega$ that can be non zero is $\Omega$ where $n$ is the number of constraints. Indeed, the product $\eta^{a_1} \cdots \eta^{a_n} \eta^{a_{n+1}}$ of
\( n + 1 \) anticommuting ghost variables in \( \Omega \) is zero. The function \( \Omega \) is determined by \( [\square] \) up to a \( \delta \)-exact term. This amounts to making a canonical transformation in the extended phase space.

### 3 First Class Polynomial Poisson algebras

The standard BRST construction recalled in the previous section assumes that the \( G_a \)'s are realized as functions on some phase space, and allows the \( C^c_{ab} \) in

\[
[G_a, G_b] = C^c_{ab} G_c
\]

(12)
to be functions of \( q^i \) and \( p_i \). However, when the \( C^c_{ab} \)'s depend on the \( q \)'s and \( p \)'s only through the \( G_a \)'s themselves, as is the case when the \( G_a \)'s form a first class polynomial Poisson algebra, one can define the BRST generator directly in the algebra \( \mathbb{C} (\mathcal{P}_\Omega) \otimes \mathbb{C} (\mathcal{G}_\Omega) \otimes \mathbb{C} (\eta^\Omega) \) of polynomials in the \( G \)'s, the \( \eta \)'s and the \( \mathcal{P} \)'s without any reference to the explicit realization of the \( G \)'s as phase space functions\(^3\). That is,

\(^3\) In agreement with the notations of [\textit{[1]}], we denote the algebra of polynomials in the anticommuting variables \( \mathcal{P}_a \) with complex coefficients by \( \mathbb{C} (\mathcal{P}_\Omega) \), and not by the more
the BRST generator can be associated with the Poisson algebra itself.

The reason for which this can be done is that both the Koszul differential $\delta$ defined by (11) and the $D$ in (10) involve only $G_a$ and not $q^i$ or $p_i$ individually. Thus, $\Omega$ can be taken to depend only on $G_a$. The BRST generator is defined accordingly in the algebra $C(P) \otimes C(G) \otimes C(\eta)$. One can give an explicit solution of (9) in terms of the homotopy $\sigma$ defined on the generators by

$$\sigma G_a = -P_a, \quad \sigma P_a = \sigma G_a = 0 \quad (13)$$

and extended to the algebra $C(P) \otimes C(G) \otimes C(\eta)$ as a derivation,

$$\sigma = -P_a \frac{\partial}{\partial G_a}. \quad (14)$$

One has

$$\sigma \delta + \delta \sigma = N \quad (15)$$

where $N$ counts the degree in the $G$’s and the $P$’s. Hence, if $D_m$ is familiar notation $\Lambda(P_a)$. A typical element of $C(P)$ is $a + bP_1$ with $a, b \in \mathbb{C}$ since $(P_1)^2 = 0$. 

the term of degree $m$ in $(G, P)$ of $D$, a solution of (14) is given by

$$\Omega^{(p)} = - \sum_{m} \frac{1}{m} \left( \sigma D_{m} \right)^{(p)}$$

(16)

since $\delta D = 0 \ [14]$ and $m > 0$ (one has $m \geq p$ and for $p = 0$, $m \geq 1$ because $[\Omega, \Omega]$ contains $G_a$ by the first class property).

As mentioned earlier, the solution (16) of the equation (9) is not well defined because the functions on which they act depend only on $G_a$. For an arbitrary function of $q^i, p_i, \partial F/\partial G_a$ would not be well defined even if the constraints $G_a$ are independent (i.e. irreducible) as here. One must specify what is kept fixed. For example, if there is one constraint $p_1 = 0$ on the four-dimensional phase space $(q^1, p_1), (q^2, p_2)$, then $\partial p_2/\partial p_1 = 0$ if one keeps $q^1, q^2$ and $p_2$ fixed, but $\partial p_2/\partial p_1 = 1$ if one keeps $q^1, q^2$ and $p_2 - p_1$ fixed. Note that the subsequent developments require only that the $C_{abc}^a$ be functions of the $G_a$, but not that these functions be polynomials. We consider here the polynomial case for the sole sake of simplicity.
unique. We call it the “covariant solution” because the homotopy \( \sigma \) defined by (14) is invariant under linear redefinitions of the generators.

**Example:** for a Lie algebra

\[
[G_a, G_b] = C^c_{ab} G_c
\]  

(17)

the covariant BRST generator is given by

\[
\Omega = G_a \eta^a - \frac{1}{2} P_a C^a_{bc} \eta^c \eta^b.
\]  

(18)

Its nilpotency expresses the Jacobi identity for the structure constants \( C^a_{bc} \). One has \( \Omega = 0 \) for \( p \geq 2 \).

In general, the BRST generator \( \Omega \) for a generic Poisson algebra contains higher order terms whose calculation may be quite cumbersome. However, because the procedure is purely algorithmic, it can be performed by means of an algebraic program like REDUCE.

The cohomology of the Poisson algebra may be defined to be the cohomology of the BRST differential \( s \) in the algebra \( \mathbb{C} (\mathcal{P}_\mathcal{D}) \otimes \mathbb{C} (\mathcal{G}_\mathcal{D}) \otimes \mathbb{C} (\eta_\mathcal{D}) \). Because \( s \) contains \( \delta \) as its piece of lowest antighost number (with \( \text{antigh}(P_a) = 1, \text{antigh}(\text{anything else}) = 0 \) ), and because \( \delta \) provides a resolution of the zero-dimensional point \( G_a = 0 \), standard arguments show that the cohomology of \( s \) is isomorphic to the coho-
mology of the differential $s'$ in $\mathbb{C}(\eta^2)$,

$$s'\eta^a = 1/2C_{bc}^a \eta^b \eta^c$$  \hspace{1cm} (19)

where $C_{bc}^a$ is defined by

$$C_{bc}^a = \left. \frac{\partial C_{bc}}{\partial G^a} \right|_{G=0}$$  \hspace{1cm} (20)

The $C_{bc}^a$ fulfill the Jacobi identity so that $s'^2 = 0$. Hence, they are the structure constants of a Lie algebra, which is called the Lie algebra underlying the given Poisson algebra.

Because of (19), the BRST cohomology of a Poisson algebra is isomorphic to the cohomology of the underlying Lie algebra. For a different and more thorough treatment of Poisson cohomology, see [15].

4 Rank

Again in analogy with the terminology used in the theory of constrained systems, we shall call “covariant rank” of a first class polynomial Poisson algebra the degree in $P_a$ of the covariant BRST generator. This concept is invariant under linear redefinitions of the generators because the covariant BRST generator is itself invariant if one trans-
forms the ghosts and their momenta as

$$G_a \rightarrow \eta^a = A^{-1}_a \eta^b$$
(23)

$$P_a \rightarrow \bar{P}_a = A^{-1}_a P_b$$
(22)

We shall call "minimal rank" the degree in $P_a$ of the solution of

$$[\Omega, \Omega] = 0$$
(24)
in $P$ (i.e., one chooses at each stage $\Omega$ in such a way that $\Omega$ has lowest possible degree in $P$). It is easy to see that for a Lie algebra, the concepts of covariant and minimal ranks coincide. As we shall see on an explicit example below, they do not in the general case.

Now, for a Lie algebra, the rank is not particularly interesting in the sense that it does not tell much about the structure of the algebra: the rank of a Lie algebra is equal to zero if and only if the algebra is abelian. It is equal to one otherwise. For non linear Poisson algebras, the rank is more useful. All values of the rank compatible with the trivial inequality

$$rank \leq n - 1$$
(24)

may occur. Thus, the rank of the BRST generator provides a non trivial characterization of Poisson algebras. Conversely, non linear Poisson
algebras yield an interesting illustration of the full BRST machinery where higher order terms besides $\Omega$ are required in $\Omega$ to achieve nilpotency.

5 Upper bound on the rank

One can understand the fact that the rank of a Lie algebra is at most equal to one by introducing a degree in $\mathbb{C}(P_{\Omega}) \otimes \mathbb{C}(G_{\Omega}) \otimes \mathbb{C}(\eta_{\Omega})$ different from the ghost degree as follows.

Theorem 1 Assume that one can assign a “degree” $n_a \geq 1$ to the generators $G_a$ in such a way that the bracket decreases the degree by at least one,

$$\deg G_a = n_a, \quad \deg([G_a, G_b]) \leq n_a + n_b - 1.$$  \hfill (25)

Then, one can bound the covariant and minimal ranks of the algebra by $\sum_a (n_a - 1) + 1$,

$$r \leq \sum_a (n_a - 1) + 1$$ \hfill (26)

In the case of a Lie algebra, one can take $n_a = 1$ for all the generators since $\deg([G_a, G_b])$ is then equal to one and fulfills (25). The
Theorem then states that the rank is bounded by one, in agreement with (18).

**Proof** Assign the following degrees to $\eta^a$ and $P_a$,

$$\text{deg } \eta^a = -n_a + 1, \text{deg } P_a = n_a - 1$$  \hspace{1cm} (27)

If $\delta A = B$ and $\text{deg } B = b$, then $\text{deg } A = b - 1$ since $\delta$ increases the degree by one. Now $\Omega = G_a \eta^a$ is of degree one. It follows that $\text{deg } (0)^{(0)} \Omega = [\Omega, \Omega]\text{orig}$ is of degree $\leq 1$ and hence, by (9) and (10),

$$\text{deg } (0)^{(0)} \Omega \leq 0.$$  

More generally, one has $\text{deg } (k)^{(k)} \Omega \leq -k + 1$. Indeed, if this relation is true up to order $k - 1$, then it is also true at order $k$ because in

$$\delta (k)^{(k)} \Omega \sim [\Omega, \Omega]\text{orig} + [\Omega, \Omega] P_a, \eta$$  \hspace{1cm} (28)

$(r + s = k - 1, r' + s' = k)$, the right hand side is of degree $\leq -k + 2$. Thus $\text{deg } (k)^{(k)} \Omega \leq -k + 2 - 1 = -k + 1$.

But the element with most negative degree in the algebra is given by the product of all the $\eta$’s, which has degree $-\sum_a (n_a - 1)$. Accordingly, $\Omega$ is zero whenever $-k + 1 \geq -\sum_a (n_a - 1)$, which implies $r \leq \sum_a (n_a - 1) + 1$ as stated in the theorem.

$\blacksquare$
Remarks:

1. One can improve greatly the bound by observing that the \( \eta \)'s do not come alone in \( \Omega \). There are also \( k \) momenta \( P_a \) which carry positive degree. This remark will, however, not be pursued further here.

2. One can actually assign degrees smaller than one to the generators \( G_a \). For instance, in the case of an Abelian Lie algebra, one may take \( \deg G_a = 1/2 \), \( \deg \eta^a = 1/2 \), \( \deg P_a = -1/2 \). Because the degree of a ghost number one object is necessarily greater than or equal to 1/2, the condition \( \deg(\Omega) \leq -k + 1 \) (if \( \Omega \neq 0 \)) implies \( \Omega = 0 \) for \( k > 0 \).

6 Self-reproducing algebras

While Lie algebras are characterized by the existence of a degree that is decreased by the bracket, one may easily construct examples of Poisson algebras for which such a degree does not exist. The simplest ones are quadratic algebras for which \([G_a, G_b]\) is proportional to \( G_a, G_b \)

\[ [G_a, G_b] = M_{ab} G_a G_b \quad \text{no summation on } a, b \quad (29) \]
with $M_{ab} = -M_{ba}$. The Jacobi identity is fulfilled for arbitrary $M$’s. Since $\deg(G_a G_b) = n_a + n_b$, the inequality (25) is violated for any choice of $n_a$. Because $[G_a, G_b]$ is proportional to $G_a G_b$, we shall call these algebras “self-reproducing algebras”.

The most general self-reproducing algebra with three generators is given by

\[
\begin{align*}
[G_1, G_2] &= \alpha G_1 G_2, \\
[G_2, G_3] &= \beta G_2 G_3, \\
[G_3, G_1] &= \gamma G_1 G_3.
\end{align*}
\] (30) (31) (32)

This Poisson algebra can be realized on a six-dimensional phase space by setting

\[
G_1 = \exp(p_2 + \alpha q_3), G_2 = \exp(p_3 + \beta q_1), G_3 = \exp(p_1 + \gamma q_2).
\] (33)

The covariant BRST charge for this model is equal to

\[
\Omega = \eta^1 G_1 + \eta^2 G_2 + \eta^3 G_3 + \\
\frac{1}{2} (\alpha \eta^2 \eta^1 P_2 G_1 - \alpha \eta^2 \eta^1 P_1 G_2 - \beta \eta^3 \eta^2 P_3 G_2 \\
- \beta \eta^3 \eta^2 P_2 G_3 + \gamma \eta^3 \eta^1 P_3 G_1 + \gamma \eta^3 \eta^1 P_1 G_3) + \\
\frac{1}{12} ((-\alpha \beta + 2 \alpha \gamma - \beta \gamma) \eta^3 \eta^2 \eta^1 P_3 P_2 G_1 + \\
(-2 \alpha \beta + \alpha \gamma + \beta \gamma) \eta^3 \eta^2 \eta^1 P_3 P_1 G_2 +
\]
\((-\alpha \beta - \alpha \gamma + 2 \beta \gamma) \eta^3 \eta^2 \eta^1 \mathcal{P}_2 \mathcal{P}_1 G_3\) 

and is of rank 2 (the maximum possible rank) unless \(\alpha = \beta = \gamma\), or \(\alpha = \beta = 0, \gamma \neq 0\), in which case it is of rank 1.

7 Examples

We now give the BRST charge (or the first terms of the BRST charge) for some particular Poisson algebras. The examples have been treated using REDUCE, using the treatment of summation over dummy indices developed in [16, 17]. Details of the implementation of the BRST algorithm can be found in [18]. All dummy variables are noted as \(d_i\) where \(i\) is an integer. Unless stated otherwise, there is an implicit summation on all dummy variables. For the examples in which the Jacobi identity is not trivially satisfied, the expressions have been normalized so that no combinations of terms in a polynomial belongs to the polynomial ideal generated by the left hand side of the Jacobi identity. In particular, polynomials in this ideal are represented by identically null expressions.
7.1 Self-Reproducing Algebras

As we have just defined, the basic Poisson brackets for the generators $G_d$ of the self-reproducing algebra are given by

$$[G_{d_1}, G_{d_2}] = M_{d_1 d_2} G_{d_1} G_{d_2}$$  \hspace{1cm} (35)

without summation over the dummy variables $d_1$ and $d_2$. The matrix $M$ is antisymmetric, but otherwise arbitrary.

The seven first orders of the covariant BRST charge are given by

$$(^{(0)} \Omega = G_{d_1} \eta^{d_1})$$

$$(^{(1)} \Omega = \frac{G_{d_1} M_{d_1 d_2} \eta^{d_1} \eta^{d_2} \mathcal{P}_{d_2}}{2})$$

$$(^{(2)} \Omega = \frac{-(G_{d_1} M_{d_1 d_2} \eta^{d_1} \eta^{d_2} \eta^{d_3} \mathcal{P}_{d_2} \mathcal{P}_{d_3} (M_{d_1 d_3} + M_{d_2 d_3}))}{12})$$

$$(^{(3)} \Omega = \frac{-(G_{d_1} M_{d_1 d_2} M_{d_1 d_3} M_{d_2 d_3} \eta^{d_1} \eta^{d_2} \eta^{d_3} \eta^{d_4} \mathcal{P}_{d_2} \mathcal{P}_{d_3} \mathcal{P}_{d_4})}{24})$$

$$(^{(4)} \Omega = \left(G_{d_1} \eta^{d_1} \eta^{d_2} \eta^{d_3} \eta^{d_4} \mathcal{P}_{d_2} \mathcal{P}_{d_3} \mathcal{P}_{d_4} \mathcal{P}_{d_5}\right)$$
\[\Omega = \left( - (M_{d1d2} M_{d1d3} M_{d1d4} M_{d1d5}) + 4 M_{d1d2} M_{d1d4} M_{d1d5} M_{d2d3} \\
+ 2 M_{d1d2} M_{d1d4} M_{d2d3} M_{d4d5} + M_{d1d2} M_{d1d5} M_{d2d3} \\
- M_{d1d4} + 2 M_{d1d4} M_{d2d3} M_{d2d5} - M_{d1d4} M_{d1d5} \\
M_{d2d3} M_{d2d4} - 2 M_{d1d4} M_{d2d5} M_{d2d4} M_{d4d5} + M_{d1d5} \\
M_{d2d3} M_{d2d5} - M_{d1d5} M_{d2d3} M_{d2d4} M_{d4d5} \right) / 720 \]

(5)

\[\Omega = \left( G_{d1} M_{d1d2} M_{d3d4} \eta^d_1 \eta^{d_2} \eta^{d_3} \eta^{d_4} \eta^{d_5} \eta^{d_6} P_{d1} P_{d3} P_{d4} P_{d5} P_{d6} P_{d7} \\
- (M_{d2d3} M_{d2d5} M_{d2d6}) + 2 M_{d2d3} M_{d2d5} M_{d5d6} + M_{d2d3} \\
M_{d2d6} M_{d3d5} - M_{d2d3} M_{d3d5} M_{d5d6} - M_{d2d5} \\
M_{d2d6} M_{d3d5} - 2 M_{d2d5} M_{d3d6} M_{d5d6} \\
+ M_{d2d6} M_{d3d6} M_{d3d5} - M_{d2d6} M_{d3d5} M_{d5d6} \right) / 1440 \]

(6)

\[\Omega = \left( G_{d1} \eta^{d_1} \eta^{d_2} \eta^{d_3} \eta^{d_4} \eta^{d_5} \eta^{d_6} P_{d1} P_{d2} P_{d3} P_{d4} P_{d5} P_{d6} P_{d7} \\
- (M_{d1d2} M_{d1d3} M_{d1d4} M_{d2d4} M_{d4d5} M_{d6d7}) - M_{d1d2} M_{d1d3} \\
M_{d2d4} M_{d2d6} M_{d4d5} M_{d6d7} + 2 M_{d1d2} M_{d1d3} M_{d2d4} \\
M_{d4d5} M_{d4d6} M_{d4d7} + M_{d1d2} M_{d1d3} M_{d2d6} M_{d4d5} M_{d4d6} \\
M_{d6d7} - 2 M_{d1d2} M_{d1d3} M_{d2d4} M_{d4d5} M_{d4d6} M_{d4d7} \\
+ 2 M_{d1d2} M_{d1d3} M_{d2d5} M_{d4d6} M_{d6d7} \\
- 2 M_{d1d2} M_{d2d3} M_{d2d4} M_{d3d6} M_{d4d5} M_{d6d7} \right) \]
\[ -13 M_{d_1 d_3} M_{d_2 d_4} M_{d_3 d_6} M_{d_4 d_5} M_{d_6 d_7} \\
+ 4 M_{d_1 d_2} M_{d_2 d_3} M_{d_4 d_5} M_{d_4 d_6} M_{d_4 d_7} \\
+ 2 M_{d_1 d_2} M_{d_2 d_3} M_{d_4 d_5} M_{d_4 d_6} M_{d_6 d_7} \\
- 4 M_{d_1 d_2} M_{d_2 d_3} M_{d_4 d_5} M_{d_4 d_6} M_{d_4 d_7} \\
+ 4 M_{d_1 d_2} M_{d_2 d_3} M_{d_4 d_5} M_{d_4 d_6} M_{d_6 d_7} \\
- 2 M_{d_1 d_2} M_{d_2 d_3} M_{d_3 d_4} M_{d_4 d_5} M_{d_6 d_7} \\
+ M_{d_1 d_3} M_{d_1 d_4} M_{d_1 d_6} M_{d_2 d_6} M_{d_4 d_5} M_{d_6 d_7} - 2 M_{d_1 d_3} \\
M_{d_1 d_4} M_{d_2 d_4} M_{d_4 d_5} M_{d_4 d_6} M_{d_4 d_7} + 5 M_{d_1 d_3} M_{d_1 d_4} \\
M_{d_2 d_7} M_{d_3 d_7} M_{d_4 d_5} M_{d_4 d_6} + M_{d_1 d_3} M_{d_1 d_5} M_{d_2 d_3} M_{d_2 d_4} \\
M_{d_4 d_5} M_{d_6 d_7} - 2 M_{d_1 d_3} M_{d_1 d_6} M_{d_2 d_6} M_{d_4 d_5} M_{d_4 d_6} \\
M_{d_4 d_5} M_{d_6 d_7} - 5 M_{d_1 d_3} M_{d_1 d_6} M_{d_2 d_7} M_{d_3 d_7} M_{d_4 d_5} M_{d_4 d_6} \\
+ M_{d_1 d_3} M_{d_1 d_7} M_{d_2 d_7} M_{d_4 d_6} M_{d_4 d_7} - M_{d_1 d_3} M_{d_1 d_7} \\
M_{d_2 d_7} M_{d_3 d_5} M_{d_4 d_6} M_{d_6 d_7} - M_{d_1 d_3} M_{d_2 d_3} M_{d_2 d_4} M_{d_2 d_6} \\
M_{d_4 d_5} M_{d_6 d_7} - 8 M_{d_1 d_3} M_{d_2 d_5} M_{d_2 d_4} M_{d_3 d_6} M_{d_4 d_5} M_{d_6 d_7} \\
+ 2 M_{d_1 d_3} M_{d_2 d_3} M_{d_2 d_4} M_{d_4 d_5} M_{d_4 d_6} M_{d_4 d_7} \\
+ M_{d_1 d_3} M_{d_2 d_3} M_{d_2 d_6} M_{d_4 d_5} M_{d_4 d_6} M_{d_6 d_7} - 2 M_{d_1 d_3} \\
M_{d_2 d_3} M_{d_3 d_7} M_{d_4 d_5} M_{d_4 d_6} M_{d_4 d_7} + 2 M_{d_1 d_3} M_{d_2 d_3} \\
M_{d_2 d_7} M_{d_3 d_5} M_{d_4 d_6} M_{d_6 d_7} - 2 M_{d_1 d_3} M_{d_2 d_3} M_{d_3 d_4} \]
\[ M_{d_3d_5} M_{d_3d_6} M_{d_3d_7} + 12 M_{d_1d_3} M_{d_2d_3} M_{d_3d_4} M_{d_3d_6} \\
M_{d_3d_7} M_{d_4d_5} - 6 M_{d_1d_3} M_{d_2d_3} M_{d_3d_4} M_{d_3d_5} M_{d_4d_6} \\
M_{d_6d_7} - 5 M_{d_1d_3} M_{d_2d_3} M_{d_3d_4} M_{d_3d_7} M_{d_4d_5} M_{d_4d_6} \\
+ 5 M_{d_1d_3} M_{d_2d_3} M_{d_3d_4} M_{d_4d_5} M_{d_4d_6} M_{d_4d_7} \\
+ 5 M_{d_1d_3} M_{d_2d_3} M_{d_3d_6} M_{d_3d_7} M_{d_4d_5} M_{d_4d_6} \\
+ 13 M_{d_1d_3} M_{d_2d_3} M_{d_3d_6} M_{d_4d_5} M_{d_4d_6} M_{d_4d_7} \\
- 5 M_{d_1d_3} M_{d_2d_3} M_{d_3d_7} M_{d_4d_5} M_{d_4d_6} M_{d_4d_7} \\
+ 5 M_{d_1d_3} M_{d_2d_3} M_{d_3d_7} M_{d_4d_5} M_{d_4d_6} M_{d_4d_7} \\
+ 2 M_{d_1d_3} M_{d_2d_4} M_{d_3d_4} M_{d_4d_5} M_{d_4d_6} M_{d_4d_7} \\
- 8 M_{d_1d_3} M_{d_2d_6} M_{d_3d_4} M_{d_4d_6} M_{d_4d_5} M_{d_4d_7} \\
+ 2 M_{d_1d_3} M_{d_2d_6} M_{d_3d_6} M_{d_4d_5} M_{d_4d_6} M_{d_4d_7} \\
- M_{d_1d_3} M_{d_3d_7} M_{d_4d_5} M_{d_4d_6} M_{d_4d_7} \\
+ M_{d_1d_3} M_{d_3d_7} M_{d_4d_5} M_{d_4d_6} M_{d_4d_7} - 2 M_{d_1d_4} \\
M_{d_2d_4} M_{d_3d_4} M_{d_4d_5} M_{d_4d_6} M_{d_4d_7} + M_{d_1d_4} M_{d_2d_6} M_{d_3d_4} \\
M_{d_3d_6} M_{d_4d_5} M_{d_4d_6} + 5 M_{d_1d_6} M_{d_2d_6} M_{d_3d_4} M_{d_3d_6} \\
M_{d_4d_5} M_{d_4d_7} - 8 M_{d_1d_6} M_{d_2d_6} M_{d_3d_4} M_{d_4d_5} M_{d_4d_6} \\
M_{d_4d_7} - 2 M_{d_1d_7} M_{d_2d_7} M_{d_3d_5} M_{d_4d_5} M_{d_4d_6} \\
+ 6 M_{d_1d_7} M_{d_2d_7} M_{d_3d_7} M_{d_4d_5} M_{d_4d_6} M_{d_4d_7} - 6 \]
\[ M_{d_1 d_7} M_{d_2 d_7} M_{d_3 d_7} M_{d_4 d_5} M_{d_4 d_6} M_{d_6 d_7}) / 60480 \]

These expressions are not particularly illuminating but are of interest because they generically do not vanish and hence, define higher order BRST charges. This can be seen by means of the following example, in which only the brackets of the first generator with the other ones are non vanishing, and taken equal to

\[ [G_1, G_\alpha] = G_1 G_\alpha = -[G_\alpha, G_1] \quad (\alpha = 2, 3, \ldots, n), \quad (36) \]

\[ [G_\alpha, G_\beta] = 0. \quad (37) \]

For this particular self-reproducing algebra, all orders of the covariant BRST charge can be explicitly computed. One finds

\begin{align*}
^{(0)} \Omega &= \eta^a G_a \\
^{(k)} \Omega &= \alpha_k (T_1 + (-)^{k+1} T_2) \quad (39)
\end{align*}

where

\begin{align*}
^{(k)} T_1 &= G_1 \eta^{\alpha_1} \cdots \eta^{\alpha_k} \eta^1 \mathcal{P}_{\alpha_1} \cdots \mathcal{P}_{\alpha_k} \\
^{(k)} T_2 &= G_{\alpha_k} \eta^{\alpha_1} \cdots \eta^{\alpha_k} \eta^1 \mathcal{P}_1 \mathcal{P}_{\alpha_1} \cdots \mathcal{P}_{\alpha_{k-1}} \quad (41)
\end{align*}
and

\begin{align}
\alpha_1 &= -1/2, \quad \alpha_2 = -1/12, \quad \alpha_3 = 0 \quad \text{(42)} \\
\alpha_k &= -\frac{1}{k+1} \sum_{l=1}^{k-3} \alpha_{l+1} \alpha_{k-l-1} \quad \text{for } k > 3. \quad \text{(43)}
\end{align}

This can be seen as the only non zero brackets involved in the construction of the BRST charge are

\begin{align}
[T_1, \Omega]_{\text{orig}} &= (-)^{k+1} S \quad \text{(44)} \\
[T_1, T_2]_{\eta P} &= (-)^{l(k+1)} S, \quad \text{(45)}
\end{align}

where

\begin{equation}
S = G_1 G_\alpha \eta^{\alpha_1} \cdots \eta^{\alpha_k} \eta^{\eta_1} \cdots \eta^{\eta_{k-1}} \mathcal{P} \alpha_1 \cdots \mathcal{P} \alpha_k. \quad \text{(46)}
\end{equation}

We further have

\begin{equation}
\sigma S = (-)^{k+1} T_2 - T_1 \quad \text{(47)}
\end{equation}

Given these relations, it is straightforward to verify (39). First, one easily checks that (39) is correct for \( k = 1 \) with \( \alpha_1 \) equal to 1/2. Let us then assume that (39) is true for \( k = 0,1 \ldots \) up to \( p \). One then obtains

\begin{equation}
D = \beta_p S \quad \text{(48)}
\end{equation}
with $\beta_p$ given by
\[
\beta_p = (-)^{p+1} \alpha_p - \sum_{k=0}^{p-1} (-)^{p(k+1)} \alpha_{k+1} \alpha_{p-k} = - \sum_{k=1}^{p-2} (-)^{p(k+1)} \alpha_{k+1} \alpha_{p-k}
\]
\[(p+1)\]
from which one gets, using (47), that $\Omega$ is indeed given by (39) with $\alpha_{p+1}$ equal to
\[
\alpha_{p+1} = \frac{\beta_p}{p+2}
\]
\[(50)\]
Observe now that $\alpha_k = 0$ for $k$ odd, $k \neq 1$. This can again be shown by recurrence. First note that $\alpha_3 = 0$. Now let $p$ be even, $p > 3$. Suppose $\alpha_k = 0$ for $k$ odd, $1 < k < p$. All terms in the relation defining $\beta_p$ are proportional to an $\alpha_m$ with $m$ odd, $1 < m < p$, since $k + 1$ and $p - k$ have opposite parities. Therefore, $\beta_p = 0 = \alpha_{p+1}$ and thus $\alpha_k = 0$ for $k$ odd, $k > 1$. Accordingly only $\alpha_k$ with $k$ even can be different from zero. The expression for $\alpha_k$ reduces then to (43) since $k + 1$ must be even in (49).

Although $\alpha_k = 0$ for $k$ odd, $k > 1$, one easily sees that $\alpha_k < 0$ for $k$ even. This is true for $k = 2$ as $\alpha_2 = -1/12$. Let $p$ be even, and suppose $\alpha_m < 0$ for $1 < m < p$, $m$ even. Then, all terms in the sum in the recurrence relation (43) are strictly positive, so that $\alpha_p < 0$. Since $\alpha_p \neq 0$, the quadratic algebra (36) provides examples of systems with
arbitrarily high covariant rank.

Note also that the minimal rank is equal to one: indeed, the non-covariant BRST charge given by

\[ \tilde{\Omega} = \Omega + (T_1 - T_2)/2 = \eta^a G_a - G_a \eta^a \eta^1 \mathcal{P}_1 \]  

is nilpotent and

\[ \delta(T_1 - T_2) = 0 \]  

so \( \tilde{\Omega} \) is indeed a valid BRST charge. This shows that the minimal and covariant ranks are in general different.

Finally, it is easy to modify slightly the basic brackets so as to induce non-zero covariant \( \Omega \) with \( k \) odd. One simply replaces (36) by

\[ [G_{n-1}, G_n] = G_{n-1} G_n = -[G_n, G_{n-1}] \]  

\[ [G_1, G_n] = -G_1 G_n, \]  

\[ [G_1, G_\alpha] = G_1 G_\alpha \ (\alpha \neq n). \]  

7.2 Purely Quadratic Algebras

A generalization of the above is the pure quadratic algebra. The basic Poisson brackets are then given by

\[ [G_{d_1}, G_{d_2}] = D_{d_1 d_2}^{d_3 d_4} G_{d_3} G_{d_4} \]  

26
where $D_{d_1d_2}^{d_3d_4}$ is antisymmetric in $d_1, d_2$ and symmetric in $d_3, d_4$. The Jacobi identity implies that

$$D_{d_3d_4}^{d_5d_6} D_{d_2d_3}^{d_1d_7} + \text{symm}(d_5, d_6, d_7) + \text{cyclic}(d_1, d_2, d_3) = 0 \quad (57)$$

The first orders of the covariant BRST charge are given by

$$\Omega^{(0)} = G_{d_1} \eta d_1$$

$$\Omega^{(1)} = \frac{D_{d_1d_2}^{d_3d_4} G_{d_4} \eta d_1 \eta d_2 P_{d_3}}{2}$$

$$\Omega^{(2)} = \frac{D_{d_3d_4}^{d_5d_6} D_{d_6d_1}^{d_7d_4} G_{d_7} \eta d_1 \eta d_2 \eta d_3 \eta d_4 P_{d_4} P_{d_5}}{6}$$

$$\Omega^{(3)} = \frac{\left( \eta d_1 \eta d_2 \eta d_3 \eta d_4 \eta d_5 \mathcal{P}_{d_6} \mathcal{P}_{d_7} \mathcal{P}_{d_8} \mathcal{P}_{d_9} \right) - \left( 3 D_{d_1d_2}^{d_3d_6} D_{d_4d_5}^{d_6d_9} D_{d_8d_9}^{d_10d_6} G_{d_{10}} \eta d_1 \eta d_2 \eta d_3 \eta d_4 \mathcal{P}_{d_5} \mathcal{P}_{d_6} \mathcal{P}_{d_7} \mathcal{P}_{d_8} \mathcal{P}_{d_9} \right)}{24}$$

$$\Omega^{(4)} = \left( \eta d_1 \eta d_2 \eta d_3 \eta d_4 \eta d_5 \mathcal{P}_{d_6} \mathcal{P}_{d_7} \mathcal{P}_{d_8} \mathcal{P}_{d_9} \right) \left( 3 D_{d_1d_2}^{d_3d_6} D_{d_4d_5}^{d_6d_9} D_{d_8d_9}^{d_10d_6} G_{d_{10}} + 4 D_{d_3d_4}^{d_10d_6} D_{d_5d_2}^{d_13d_6} D_{d_10d_11}^{d_12d_8} \right)$$

$$\left( D_{d_3d_4}^{d_5d_6} G_{d_{12}} + 4 D_{d_4d_5}^{d_10d_6} D_{d_11d_13}^{d_12d_7} \right) / 360$$
\[ \Omega = \left( \frac{D_{d_1 d_2 d_3}^{d_4 d_5} D_{d_4 d_5 d_6}^{d_7 d_8} D_{d_5 d_6 d_7}^{d_8 d_9} D_{d_6 d_7 d_8}^{d_9 d_10} D_{d_7 d_8 d_9}^{d_10 d_11} D_{d_8 d_9 d_{10}}^{d_{10} d_{11} d_{12}}}{720} + 3 D_{d_1 d_2 d_3}^{d_4 d_5 d_6} \right) / 15120 \]
Again, these expressions are not particularly illuminating. The point emphasized here is that the calculation of the BRST charge is purely algorithmic and follows a general, well-established pattern.

Since homogeneous quadratic algebras contain the self-reproducing algebras as special case, they are generically of maximal covariant rank. More on this in [19].

7.3 L-T algebras

We now consider adding a linear term to the quadratic algebra above. The basic Poisson brackets for the generators $G_d$ are given by

$$[G_{d_1}, G_{d_2}] = C_{d_1 d_2}^{d_3} G_{d_3} + D_{d_1 d_2}^{d_3 d_4} G_{d_3} G_{d_4}$$  \hspace{1cm} (58)

where $C_{d_1 d_2}^{d_3}$ and $D_{d_1 d_2}^{d_3 d_4}$ are antisymmetric in $d_1, d_2$, and $D_{d_1 d_2}^{d_3 d_4}$ is symmetric in $d_3, d_4$. A particular instance of such an algebra is given by Zamolodchikov algebras [4]. We will start with a specific example, and consider general quadratically nonlinear Poisson algebras next.

The generators in the example are assumed to split into $L_a$ and $T_b$, $a = 1, \ldots, n_1$, $b = n_1 + 1, \ldots, n$, with the brackets

$$[L_{a_1}, L_{a_2}] = C_{a_1 a_2}^{a_3} L_{a_3}$$

$$[L_{a_1}, T_{b_1}] = C_{a_1 b_1}^{a_2} L_{a_2} + C_{a_1 b_1}^{b_2} T_{b_2}$$  \hspace{1cm} (59)
\[
[T_{b_1}, T_{b_2}] = \tilde{C}^{a_1}_{b_1 b_2} L_{a_1} + \tilde{C}^{b_2}_{b_1 b_2} T_{b_3} + \tilde{D}^{a_1 a_2}_{b_1 b_2} L_{a_1} L_{a_2}
\]

so that contractions of \( \tilde{D} \) are impossible.

Going back to the notations \( G_{d_i} = \{L_a, T_b\} \), \( d = 1, \ldots, n \) the Jacobi identity imply

\[
C^{d_4}_{d_1 d_2} C^{d_5}_{d_3 d_4} + \text{cyclic}(d_1, d_2, d_3) = 0 \quad (60)
\]

\[
\{D^{d_1 d_5}_{d_1 d_2} C^{d_6}_{d_3 d_4} + \text{symm}(d_5, d_6)\} + C^{d_4}_{d_1 d_2} D^{d_5 d_6}_{d_3 d_4} + \text{cyclic}(d_1, d_2, d_3) = 0 \quad (61)
\]

and contractions of \( D \) vanish.

For instance, the conditions (59) are fulfilled if one takes for the \( L \)'s the generators of a semi-simple Lie algebra and take the \( T \)'s to commute with the \( L \)'s and to close on the Casimir element:

\[
[L_a, T_b] = 0 \quad (62)
\]

\[
[T_{b_1}, T_{b_2}] = \delta_{b_1 b_2} k^{a_1 a_2} L_{a_1} L_{a_2} \quad (63)
\]

where \( k^{a_1 a_2} \) is the Killing bilinear form. The Jacobi identity is verified because the Casimir element commutes with the \( L \)'s.

The previous theorem on the rank yields, by taking \( n(l) = 1 \) and \( n(T) = 3/2 \), that the rank is bounded by \( 1/2m + 1 \), where \( m \) is the number of \( T \)-generators. Actually, the rank is much lower, since the
covariant BRST charge is computed to be

\[ \Omega = \frac{1}{2} C_{d_1 d_2}^{d_3} \eta^{d_1} \eta^{d_2} \mathcal{P}_{d_3} + \frac{1}{24} C_{d_8 d_9}^{d_6} D_{d_1 d_2}^{d_3 d_4} D_{d_3 d_4}^{d_5 d_6} \eta^{d_1} \eta^{d_2} \eta^{d_3} \eta^{d_4} \mathcal{P}_{d_5} \mathcal{P}_{d_6} \]

\[ + \frac{1}{2} D_{d_1 d_2}^{d_3 d_4} G_{d_4} \eta^{d_1} \eta^{d_2} \mathcal{P}_{d_3} + G_{d_4} \eta^{d_1} \]

which is identical to the result in [20].

### 7.4 Generalizations

The previous L-T algebras can be generalized in various directions. One may consider the general quadratic non homogeneous Poisson structure

\[ [G_{d_1}, G_{d_2}] = C_{d_1 d_2}^{d_3} G_{d_3} + D_{d_1 d_2}^{d_3 d_4} G_{d_3} G_{d_4} \]

\[ C_{d_1 d_2}^{d_3} C_{d_3 d_4}^{d_5} + \text{cyclic}(d_1, d_2, d_3) = 0 \]

\[ \{ D_{d_1 d_2}^{d_3 d_4} C_{d_3 d_4}^{d_5} + \text{symm}(d_5, d_6) \} + C_{d_1 d_2}^{d_4} D_{d_3 d_4}^{d_5 d_6} + \text{cyclic}(d_1, d_2, d_3) = 0 \]

\[ D_{d_4 d_1}^{d_5 d_6} D_{d_3 d_4}^{d_1 d_2} + \text{symm}(d_5, d_6, d_7) + \text{cyclic}(d_1, d_2, d_3) = 0 \]

with \( C_{d_1 d_2}^{d_3} \) and \( D_{d_1 d_2}^{d_3 d_4} \) antisymmetric in \( d_1, d_2 \), and \( D_{d_1 d_2}^{d_3 d_4} \) symmetric in \( d_3, d_4 \). One may also include higher order terms in the bracket while preserving the existence of a degree decreased by the bracket, as in
the so called spin 4 algebra:

\[
\begin{align*}
[L_{a1}, L_{a2}] &= C_{a1a2}^a L_a \\ (68) \\
[L_{a1}, T_{b1}] &= C_{a1b1}^b T_b \\ (69) \\
[L_{a1}, W_{c1}] &= C_{a1c1}^c W_c \\ (70) \\
[T_{b1}, T_{b2}] &= C_{b1b2}^{a1} L_{a1} + C_{b1b2}^{a2} T_{a2} + D_{b1b2}^{a1a2} L_{a1} L_{a2} \\ (71) \\
[T_{b1}, W_{c1}] &= C_{b1c1}^{b1} T_{b1} + D_{b1c1}^{a1} L_{a1} T_{b2} \\ (72) \\
[W_{c1}, W_{c2}] &= C_{c1c2}^{a1} L_{a1} + D_{c1c2}^{a1a2} L_{a1} L_{a2} + E_{c1c2}^{a1a2a3} L_{a1} L_{a2} L_{a3}. (73)
\end{align*}
\]

If one sets \( n(L) = 1, n(T) = 3/2, L(W) = 2 \), one gets \( n([A, B]) \leq n(A) + n(B) \).

We have checked, using REDUCE, that in both cases the first seven terms in \( \Omega \) are generically non zero.

### 8 Conclusion

We have shown in this paper that polynomial Poisson algebras provide a rich arena in which the perturbative features of the BRST construction are perfectly illustrated. We believe this to be of interest because models of higher rank are rather rare and are usually thought not to arise in practice. Non polynomial Poisson algebras(e.g. of the type
arising in the study of quantum groups) can also be analyzed along the same BRST lines and should provide further models of higher rank.

We have not discussed the quantum realization of Poisson algebras, and whether the nilpotency condition for the BRST generator is maintained quantum-mechanically. This is a difficult question, which is model-dependent. Indeed, while the ghost contribution to $\Omega^2$ can be evaluated independently of the specific form of the $G_a$'s in terms of the canonical variables $(q^i, p_i)$ (once a representation of the ghost anticommutation relations is chosen), the “matter” contribution to $\Omega^2$ depends on the “anomaly” terms in $[G_a, G_b]$, which, in turn, depend on the specific form of the $G_a$'s. It would be interesting to pursue this question further.

9 Acknowledgments

We are grateful to Jim Stasheff and Claudio Teitelboim for fruitful discussions at the early stages of this research. This work has been supported in part by research funds from FNRS (Belgium) and by a research contract with the Commission of the European Communities.
References

[1] M. Nakamura. *Prog. Theor. Phys.*, 37:195, 1967.

[2] S.B. Priddy. *Trans. Amer. Math. Soc.*, 152:39, 1970.

[3] E.K. Sklyanin. *Funct. Anal. Appl.*, 16:263, 1982.

[4] A. B. Zamolodchikov. *Theor. Math. Phys.*, 65:1205, 1985.

[5] V.A. Fateev and A. B. Zamolodchikov. *Nucl. Phys.*, B280:644, 1987.

[6] Y.G. Oh. *Lett. Math. Phys.*, 12:87, 1986.

[7] V.O. Tarasov, L.A. Takhatadzlupan, and L.D. Faddeev. *Theor. Math. Phys.*, 57:1059, 1983.

[8] I Bakas and P. Mathieu. *Phys. Lett.*, 208B:101, 1988.

[9] K. H. Bhaskara and K. Rama. *J. Math. Phys.*, 32:2319, 1991.

[10] Ya. I. Granovskii, A. S. Zhedanov, and I. M. Lutsenko. *Theor. Math. Phys.*, 91:474, 1992.

[11] M. Henneaux and C. Teitelboim. *Quantization of Gauge Systems*. Princeton University Press, 1992.
[12] V.K.A.M. Gugenheim and J.D. Stasheff. *Bull. Soc. Math. Bel-
gique*, 38:237, 1986; J.D. Stasheff. *Bull. Amer. Soc.*, 19:287, 1988
and references therein.

[13] M. Henneaux. *Phys. Lett.*,, 120B:75, 1983.

[14] K. Fujikawa and J. Kubo. *Phys. Lett.*, 199B:75, 1987.

[15] J. Huebschmann. Extensions of Lie algebras. Heidelberg preprint,
1989.

[16] A. Dresse. Canonical form of expressions involving dummy vari-
ables. Submitted to J. Symb. Comp.

[17] A. Dresse. Treatment of dummy variables and BRST theory in
computer algebra. In *Proceedings of the IMACS Symposium SC–
1993*, 1993.

[18] A. Burnel, H. Caprasse, and A. Dresse. in preparation.

[19] A. Dresse. Ph.D. Thesis, U.L.B. (in preparation).

[20] K. Schoutens, A. Sevrin, and P. van Nieuwenhuizen. *Commun. 
Math. Phys.*, 124:87, 1989.