Central and medial quasigroups of small order

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Abstract. We enumerate central and medial quasigroups of order less than 128 up to isomorphism, with the exception of those quasigroups that are isotopic to $C_4 \times C_2^4$, $C_2^6$, $C_4^3$ or $C_3^5$. We give an explicit formula for the number of quasigroups that are affine over a finite cyclic group.

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This paper was written on the occasion of the 90th anniversary of Valentin Danilovich Belousov’s birthday. Prof. Belousov pioneered enumerative results for quasigroups in his book “Fundamentals of the theory of quasigroups and loops” and his work has been a frequent source of inspiration for the Prague algebraic school.

1 Introduction

Given an abelian group $(G, +)$, automorphisms $\varphi$, $\psi$ of $(G, +)$, and an element $c \in G$, define a new operation $*$ on $G$ by

$$x \ast y = \varphi(x) + \psi(y) + c.$$  

The resulting quasigroup $(G, \ast)$ is said to be affine over $(G, +)$, and it will be denoted by $Q(G, +, \varphi, \psi, c)$. Quasigroups that are affine over an abelian group are called central quasigroups or T-quasigroups. We will use the terms “quasigroup affine over an abelian group” and “central quasigroup” interchangeably. Central quasigroups are precisely the abelian quasigroups in the sense of universal algebra [15].

A quasigroup $(Q, \cdot)$ is called medial if it satisfies the medial law

$$(x \cdot y) \cdot (u \cdot v) = (x \cdot u) \cdot (y \cdot v).$$

Medial quasigroups are also known as entropic quasigroups. The fundamental Toyoda-Bruck theorem [13, Theorem 3.1] states that, up to isomorphism, medial quasigroups are precisely central quasigroups $Q(G, +, \varphi, \psi, c)$ with commuting automorphisms $\varphi$, $\psi$.

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The classification of central (or medial) quasigroups up to isotopy is trivial in the sense that it coincides with the classification of abelian groups up to isomorphism. Indeed:

- If \((G, \ast) = Q(G, +, \varphi, \psi, c)\) is a central quasigroup then \((G, \ast)\) is isotopic to \((G, +)\) via the isotopism \(x \mapsto \varphi(x), x \mapsto \psi(x) + c, x \mapsto x\).

- If two central quasigroups \(Q_i = Q(G_i, +_i, \varphi_i, \psi_i, c_i)\) are isotopic then the underlying groups \((G_i, +_i)\) are isotopic. But isotopic groups are necessarily isomorphic, cf. [10, Proposition 1.4].

Classifying and enumerating central and medial quasigroups up to isomorphism is nontrivial, however, and that is the topic of the present paper.

There are not many results in the literature concerning enumeration and classification of central and medial quasigroups.

Simple idempotent medial quasigroups were classified by Smith in [9, Theorem 6.1]. Sokhatsky and Syvakivskij [12] classified \(n\)-ary quasigroups affine over cyclic groups and obtained a formula for the number of those of prime order. Kirnasovsky [5] carried out a computer enumeration of central quasigroups up to order 15, and obtained more classification results in his PhD thesis [6]. Idempotent medial quasigroups of order \(p^k, k \leq 4\), were classified by Hou [4, Table 1].

At the time of writing this paper, the On-line Encyclopedia of Integer Sequences [8] gives the number of medial quasigroups of order \(\leq 8\) up to isomorphism as the sequence A226193, and there appears to be no entry for the number of central quasigroups up to isomorphism.

Drápal [1] and Sokhatsky [11] obtained a general isomorphism theorem for quasigroups isotopic to groups, cf. [1, Theorem 2.10] and [11, Corollary 28], and for central quasigroups in particular, cf. [1, Theorem 3.2], or its restatement, Theorem 2.5. Drápal applied the machinery to calculate isomorphism classes of quasigroups of order 4 (by hand), and Kirnasovsky used Sokhatsky’s theory for the calculations mentioned above. In the present paper, we use a similar approach to obtain stronger enumeration results, taking advantages of the computer system GAP [2].

We refer the reader to [10] for general theory of quasigroups, to [1] for a more extensive list of references on central quasigroups, to [11] for results on quasigroups isotopic to groups, to [13] for results on quasigroups affine over various kinds of loops, and to [14, 15] for a broader context on affine representation of general algebraic structures. The article [3] gives a gentle introduction into automorphism groups of finite abelian groups and points to original sources on that topic.

The paper is organized as follows.

In Section 2, we formulate an isomorphism theorem for central quasigroups, Theorem 2.4, which is less general than [1, Theorem 2.10] or [11, Corollary 28], and equivalent to but less technical than [1, Theorem 3.2]. We also present the enumeration algorithm in detail.
In Section 3, we establish our own version of [12, Theorem 2] and [1, Theorem 3.5] for cyclic $p$-groups, Theorem 3.1, providing an explicit formula for the number of isomorphism classes. We were informed that the same result was obtained by Kirnasovsky in his unpublished PhD thesis [6]. Since the automorphism groups of cyclic groups are commutative, Theorem 3.1 also yields the number of medial quasigroups up to isomorphism over finite cyclic groups, and of prime order in particular.

Finally, the results of the enumeration are presented in the Appendix.

2 Isomorphism theorem and enumeration algorithm

2.1 Elementary properties of the counting functions $cq$ and $mq$

For an abelian group $G$, let $cq(G)$ (resp. $mq(G)$) denote the number of all central (resp. medial) quasigroups over $G$ up to isomorphism. For $n \geq 1$, let $cq(n)$ (resp. $mq(n)$) denote the number of all central (resp. medial) quasigroups of order $n$ up to isomorphism.

Let us establish two fundamental properties of the counting functions.

First, by the remarks in the introduction,

$$cq(n) = \sum_{|G|=n} cq(G) \quad \text{and} \quad mq(n) = \sum_{|G|=n} mq(G),$$

where the summations run over all abelian groups of order $n$ up to isomorphism.

Second, Proposition 2.1 shows that the classification of central and medial quasigroups can be reduced to prime power orders. As far as enumeration is concerned, Proposition 2.1 implies that the functions $cq, mq : \mathbb{N}^+ \to \mathbb{N}^+$ are multiplicative in the number-theoretic sense.

**Proposition 2.1.** Let $G = H \times K$ be an abelian group such that $\gcd(|H|, |K|) = 1$. Up to isomorphism, any quasigroup affine over $G$ can be expressed in a unique way as a direct product of a quasigroup affine over $H$ and a quasigroup affine over $K$. In particular,

$$cq(G) = cq(H) \cdot cq(K) \quad \text{and} \quad mq(G) = mq(H) \cdot mq(K).$$

**Proof.** Any automorphism of $G$ decomposes uniquely as a direct product of an automorphism of $H$ and an automorphism of $K$, cf. [3, Lemma 2.1]. The rest is easy. \( \square \)

2.2 The isomorphism problem for central quasigroups

Let us now consider the isomorphism problem for quasigroups affine over a fixed abelian group $(G, +)$.

Consider any group $A$. (Later we will take $A = \text{Aut}(G, +)$.) Then $A$ acts on itself by conjugation, and $A$ also acts on $A \times A$ by a simultaneous conjugation in both coordinates, i.e., $(\alpha, \beta)^\gamma = (\alpha^\gamma, \beta^\gamma)$. 
Lemma 2.2. Let $A$ be a group. Let $X$ be a complete set of orbit representatives of the conjugation action of $A$ on itself. For $\xi \in X$, let $Y_\xi$ be a complete set of orbit representatives of the conjugation action of the centralizer $C_A(\xi)$ on $A$. Then
\[ \{(\xi, v) : \xi \in X, v \in Y_\xi\} \]
is a complete set of orbit representatives of the conjugation action of $A$ on $A \times A$.

Proof. For every $(\alpha, \beta) \in A \times A$ there is a unique $\xi \in X$ and some $\gamma \in A$ such that $(\alpha, \beta)$ and $(\xi, \gamma)$ are in the same orbit. For a fixed $\xi \in X$ and some $\beta, \gamma \in A$, we have $(\xi, \beta)$ in the same orbit as $(\xi, \gamma)$ if and only if there is $\delta \in C_A(\xi)$ such that $\beta^\delta = \gamma$.

Lemma 2.3. Let $(G, +)$ be an abelian group, $A = \text{Aut}(G, +)$ and $\alpha, \beta \in A$. Then $C_A(\alpha) \cap C_A(\beta)$ acts naturally on $G / \text{Im}(1 - \alpha - \beta)$.

Proof. Let $U = \text{Im}(1 - \alpha - \beta)$. It suffices to show that for every $\gamma \in C_A(\alpha) \cap C_A(\beta)$ the mapping $u + U \mapsto \gamma(u) + U$ is well-defined. Now, if $u + U = v + U$ then $u = v + w - \alpha(w) - \beta(w)$ for some $w \in G$ and we have $\gamma(u) = \gamma(v) + \gamma(w) - \gamma\alpha(w) - \gamma\beta(w) = \gamma(v) + \gamma(w) - \alpha\gamma(w) - \beta\gamma(w) = \gamma(v) + (1 - \alpha - \beta)(\gamma(w)) \in \gamma(v) + U$.

We will now state a theorem that solves the isomorphism problem for central and medial quasigroups over $(G, +)$. Instead of showing how it follows from the more general [1, Theorem 2.10], we show that it is equivalent to [1, Theorem 3.2], which we restate as Theorem 2.5 here.

Theorem 2.4 (Isomorphism problem for central quasigroups). Let $(G, +)$ be an abelian group, let $\varphi_1, \psi_1, \varphi_2, \psi_2 \in \text{Aut}(G, +)$, and let $c_1, c_2 \in G$. Then the following statements are equivalent:

(i) the central quasigroups $Q(G, +, \varphi_1, \psi_1, c_1)$ and $Q(G, +, \varphi_2, \psi_2, c_2)$ are isomorphic;

(ii) there is an automorphism $\gamma$ of $(G, +)$ and an element $u \in \text{Im}(1 - \varphi_1 - \psi_1)$ such that

$\varphi_2 = \gamma\varphi_1\gamma^{-1}, \quad \psi_2 = \gamma\psi_1\gamma^{-1}, \quad c_2 = \gamma(c_1 + u)$.

Theorem 2.5 ([1, Theorem 3.2]). Let $(G, +)$ be an abelian group and denote $A = \text{Aut}(G, +)$. The isomorphism classes of central quasigroups (resp. medial quasigroups) over $(G, +)$ are in one-to-one correspondence with the elements of the set

$\{(\varphi, \psi, c) : \varphi \in X, \psi \in Y_\varphi, c \in G_{\varphi, \psi}\}$,

where
• \( X \) is a complete set of orbit representatives of the conjugation action of \( A \) on itself;

• \( Y_\varphi \) is a complete set of orbit representatives of the conjugation action of \( C_A(\varphi) \) on \( A \) (resp. on \( C_A(\varphi) \)), for every \( \varphi \in X \);

• \( G_{\varphi,\psi} \) is a complete set of orbit representatives of the natural action of \( C_A(\varphi) \cap C_A(\psi) \) on \( G/\text{Im}(1-\varphi-\psi) \).

Here is a proof of the equivalence of Theorems 2.4 and 2.5: By Lemma 2.2, we can assume that we are investigating the equivalence of two triples \((\varphi, \psi, c_1)\) and \((\varphi, \psi, c_2)\) for some \( \varphi \in X \), \( \psi \in Y_\varphi \) and \( c_1, c_2 \in G \). Let \( U = \text{Im}(1-\varphi-\psi) \). The following conditions are then equivalent for any \( \gamma \in \text{Aut}(G,+) \), using Lemma 2.3: \( c_2 = \gamma(c_1 + u) \) for some \( u \in U \), \( c_2 \in \gamma(c_1 + U) = \gamma(c_1) + U \), \( c_2 + U = \gamma(c_1) + U = \gamma(c_1 + U) \). This finishes the proof.

2.3 The algorithm

Theorem 2.5 together with the results of Subsection 2.1 gives rise to the following algorithm that enumerates central and medial quasigroups of order \( n \). In the algorithm we denote by \( R(H,X) \) a complete set of representatives of the (clear from context) action of \( H \) on \( X \).

Algorithm 2.6.
Input: positive integer \( n \)
Output: \( cq(n) \) and \( mq(n) \)

\[
\begin{align*}
cqn &:= 0; \quad mqn := 0; \\
\text{for } G \text{ in the set of abelian groups of order } n \text{ up to isomorphism do} \\
\quad &\text{cqG} := 0; \quad \text{mqG} := 0; \\
\quad &A := \text{automorphism group of } G; \\
\quad &\text{for } f \text{ in } R(A,A) \text{ do} \\
\quad &\quad \text{for } g \text{ in } R(C_A(f),A) \text{ do} \\
\quad &\quad &\text{for } c \text{ in } R(\text{Intersection}(C_A(f),C_A(g)), G/\text{Im}(1-f-g) ) \text{ do} \\
\quad &\quad &\quad \text{cqG} := \text{cqG} + 1; \\
\quad &\quad &\quad \text{if } f*g=g*f \text{ then } \text{mqG} := \text{mqG} + 1; \text{ fi}; \\
\quad &\quad &\quad \text{od}; \\
\quad &\quad &\text{od}; \\
\quad &\text{cqn} := \text{cqn} + \text{cqG}; \quad \text{mqn} := \text{mqn} + \text{mqG}; \\
\quad &\text{od}; \\
\text{return } \text{cqn}, \text{mqn};
\end{align*}
\]

The algorithm was implemented in the GAP system [2] in a straightforward fashion, taking advantage of some functionality of the LOOPS [7] package. The code is available from the second author at \texttt{www.math.du.edu/~petr}. 

In small situations it is possible to directly calculate the orbits of the conjugation action of $A = \text{Aut}(G,+)$ on $A \times A$. For larger groups, it is safer (due to memory constraints) to work with one conjugacy class of $A$ at a time, as in Algorithm 2.5.

Among the cases we managed to calculate, the elementary abelian group $C_2^5$ took the most effort, about 4 hours on a standard personal computer. It might not be difficult to calculate some of the missing entries for $mq(G)$. However, $cq(C_2^6)$, for instance, appears out of reach without further theoretical advances or more substantial computational resources.

The outcome of the calculation can be found in the Appendix.

3 Quasigroups affine over cyclic groups

Let $G$ be a cyclic group. Since $\text{Aut}(G)$ is commutative, every quasigroup affine over $G$ is medial.

**Theorem 3.1 ([6, p. 70]).** Let $p$ be a prime and $k$ a positive integer. Then

$$cq(C_p^k) = mq(C_p^k) = p^{2k} + p^{2k-2} - p^{k-1} - \sum_{i=k-1}^{2k-1} p^i.$$  

**Proof.** Let $G = C_p^k$ and $A = \text{Aut}(G)$. We will identify $A$ with the $p^k - p^{k-1}$ elements of $G^* = \{a \in G : p \nmid a\}$. We will follow Algorithm 2.6. Since $A$ is commutative, the conjugation action is trivial and we have to consider every $(\varphi, \psi) \in A \times A$. For a fixed $(\varphi, \psi) \in A \times A$, we must consider a complete set of orbit representatives $G_{\varphi,\psi}$ of the action of $A = C_A(\varphi) \cap C_A(\psi)$ on $G/\text{Im}(1 - \varphi - \psi)$. Now, $\text{Im}(1 - \varphi - \psi)$ is equal to $p^i G$ if and only if $p^i \mid 1 - \varphi - \psi$ and $p^{i+1} \nmid 1 - \varphi - \psi$.

**Case** $i = 0$, i. e.,

$$\varphi + \psi \not\equiv 1 \pmod{p}.$$  

In this case, we can take $G_{\varphi,\psi} = \{0\}$. How many such pairs $(\varphi, \psi)$ exist? First, let us count those with $\varphi \equiv 1 \pmod{p}$. Then $\psi \in G^*$ can be chosen arbitrarily, hence we have $p^{k-1}(p^k - p^{k-1})$ such pairs. Next, let us count those with $\varphi \not\equiv 1 \pmod{p}$. Then $\psi \in G^*$ must satisfy $\psi \not\equiv 1 - \varphi \pmod{p}$, hence we have $(p^k - 2p^{k-1})(p^k - 2p^{k-1})$ such pairs. Since $|G_{\varphi,\psi}| = 1$, this case contributes to $cq(G)$ by

$$p^{k-1}(p^k - p^{k-1}) + (p^k - 2p^{k-1})^2.$$  

**Cases** $i = 1, \ldots, k - 1$, i. e.,

$$\varphi + \psi \equiv 1 \pmod{p^i} \quad \text{and} \quad \varphi + \psi \not\equiv 1 \pmod{p^{i+1}}.$$  

In this case, we can take $G_{\varphi,\psi} = \{0, p^0, \ldots, p^{i-1}\}$. How many such pairs $(\varphi, \psi)$ exist? For $\varphi \equiv 1 \pmod{p}$, any solution $\psi$ to the congruence above is divisible by $p$, hence there is no such solution $\psi \in G^*$. For $\varphi \not\equiv 1 \pmod{p}$, we have precisely $p^{k-i} - p^{k-i-1}$
solutions to the conditions in $G^*$. Since $|G_{\varphi,\psi}| = i + 1$, this case contributes to $cq(G)$ by
\[(p^k - 2p^{k-1})(p^{k-i} - p^{k-i-1})(i + 1).\]

Case $i = k$, i.e., $\varphi + \psi = 1$.

In this case, we can take $G_{\varphi,\psi} = \{0, p^0, \ldots, p^{k-1}\}$. How many such pairs $(\varphi, \psi)$ exist? Since $\psi$ is uniquely determined by $\varphi$ and neither of $\varphi, \psi$ shall be divisible by $p$, we have precisely $p^k - 2p^{k-1}$ such pairs. Since $|G_{\varphi,\psi}| = k + 1$, this case contributes to $cq(G)$ by
\[(p^k - 2p^{k-1})(k + 1).\]

Summarized, the cases $i = 1, \ldots, k$ contribute to $cq(G)$ the total of
\[(p^k - 2p^{k-1}) \left( \sum_{i=1}^{k-1} (p^{k-i} - p^{k-i-1}) \cdot (i + 1) \right) + (k + 1),\]
which, after rearrangement, gives
\[(p^k - 2p^{k-1})(2p^{k-1} + p^{k-2} + p^{k-3} + \cdots + p + 1).\]

The total sum is then
\[
cq(G) = p^{2k-1} - p^{2k-2} + (p^k - 2p^{k-1})((p^k - 2p^{k-1})
+ (2p^{k-1} + p^{k-2} + p^{k-3} + \cdots + p + 1))
= p^{2k-1} - p^{2k-2} + (p^k - 2p^{k-1})(p^k + p^{k-2} + p^{k-3} + \cdots + p + 1)
= p^{2k} - p^{2k-1} - p^{2k-3} - \cdots - p^k - 2p^{k-1},
\]
which can be expressed as in the statement of the theorem. \hfill \Box

**Corollary 3.2.** For any $k \geq 1$ we have $cq(C_{2^k}) = mq(C_{2^k}) = 2^{2k-2}$.

**Corollary 3.3.** For any prime $p$ we have $cq(p) = mq(p) = p^2 - p - 1$.

Corollary 3.3 is a special case of [12, Corollary 2] for binary quasigroups. As a counterpart to Theorem 3.1, we ask:

**Problem 3.4.** For a prime $p$ and $k > 1$, find explicit formulas for $cq(C_p^k)$ and $mq(C_p^k)$. 
Appendix: Central and medial quasigroups of order less than 128

The following table contains the results of our enumeration of central and medial quasigroups of order less than 128.

If a row in the table starts with \( n/k \) then: column “\( G \)” gives the catalog number \( n/k \) corresponding to the abelian group \( \text{SmallGroup}(n,k) \) of GAP; column “structure” gives a structural description of the group \( G \) from which a decomposition of \( G \) into \( p \)-primary components is readily seen and hence Proposition 2.1 can be routinely applied; column “[\( A \])” gives the cardinality of the group \( A = \text{Aut}(G) \); column “[\( |X| \])” gives the number of conjugacy classes of \( A \); column “[\( |O| \])” gives the number of orbits of the conjugation action of \( A \) on \( A \times A \) (with action \( (f,g)^h = (f^h, g^h) \)), which is a lower bound on the number of quasigroups affine over \( G \); column “\( cq \)” gives the number of quasigroups affine over \( G \) up to isomorphism; column “[\( |O_c| \])” gives the number of orbits in \( O \) with a representative \( (f,g) \) such that \( fg = gf \), which is a lower bound on the number of medial quasigroups over \( G \) up to isomorphism; and column “ref” gives a reference to a numbered result within this paper if the entries in the row follow from the cited result and possibly also from previously listed table entries.

If a row in the table starts with \( n \) then: column “\( G \)” gives the order \( n \); column “\( cq \)” gives the number of central quasigroups of order \( n \) up to isomorphism; and column “\( mq \)” gives the number of medial quasigroups of order \( n \) up to isomorphism.

Entries that we were not able to establish are denoted by “?" or “?".

All entries corresponding to prime-power orders were explicitly calculated by Algorithm 2.6 although the cyclic cases follow from Theorem 3.1. Many of the entries corresponding to the remaining orders were also initially obtained by Algorithm 2.6 (to test the algorithm) but in the final version they were calculated directly from earlier entries using Proposition 2.1.

To reduce the number of transcription and arithmetical errors, the entries and the \( \LaTeX \) source of the table were computer generated.

| \( G \) structure | \( |A| \) | \( |X| \) | \( |O| \) | \( cq \) | \( |O_c| \) | \( mq \) | ref |
|-----------------|-------|-------|-------|-------|-------|-------|-----|
| 1/1 \( C_1 \)  | 1     | 1     | 1     | 1     | 1     | 1     |     |
| 2/1 \( C_2 \)  | 1     | 1     | 1     | 1     | 1     | 1     | 3.1 |
| 3/1 \( C_3 \)  | 2     | 2     | 4     | 5     | 4     | 5     | 3.1 |
| 4/1 \( C_4 \)  | 2     | 2     | 4     | 4     | 4     | 4     | 3.1 |
| 4/2 \( C_2^2 \) | 6     | 3     | 11    | 15    | 8     | 9     |     |
| 5/1 \( C_5 \)  | 4     | 4     | 16    | 19    | 16    | 19    | 3.1 |
| 6/2 \( C_2 \times C_3 \) | 2     | 2     | 4     | 5     | 4     | 5     | 2.1 |
| 7/1 \( C_7 \)  | 6     | 6     | 36    | 41    | 36    | 41    | 3.1 |
| $G$ | structure | $|A|\times|X|$ | $|O|$ | $cq$ | $|O_c|$ | $mq$ | ref |
|-----|-----------|-----------------|---------|-------|--------|--------|-----|
| 8/1 | $C_8$     | 4 4             | 16 16   | 16 16 | 16 16  | 3.1    |
| 8/2 | $C_4 \times C_2$ | 8 5            | 28 28   | 22 22 | 22 22  | 3.1    |
| 8/5 | $C_2^3$   | 168 6          | 197 341 | 32 35 | 32 35  | 3.1    |
| 8   |           |                 | 385 73  |       |        |        |
| 9/1 | $C_9$     | 6 6             | 36 48   | 36 48 | 36 48  | 3.1    |
| 9/2 | $C_3^2$   | 48 8           | 136 183 | 56 68 | 56 68  | 3.1    |
| 9   |           |                 | 231 116 |       |        |        |
| 10/2| $C_2 \times C_5$ | 4 4          | 16 19   | 16 19 | 16 19  | 2.1    |
| 10  |           |                 | 19 19   |       |        |        |
| 11/1| $C_{11}$  | 10 10          | 100 109 | 100 109 | 100 109 | 3.1 |
| 11  |           |                 | 109 109 |       |        |        |
| 12/2| $C_4 \times C_3$ | 4 4         | 16 20   | 16 20 | 16 20  | 2.1    |
| 12/5| $C_2^2 \times C_3$ | 12 6   | 44 75   | 32 45 | 32 45  | 2.1    |
| 12  |           |                 | 95 65   |       |        |        |
| 13/1| $C_{13}$  | 12 12          | 144 155 | 144 155 | 144 155 | 3.1 |
| 13  |           |                 | 155 155 |       |        |        |
| 14/2| $C_2 \times C_7$ | 6 6       | 36 41   | 36 41 | 36 41  | 2.1    |
| 14  |           |                 | 41 41   |       |        |        |
| 15/1| $C_3 \times C_5$ | 8 8       | 64 95   | 64 95 | 64 95  | 2.1    |
| 15  |           |                 | 95 95   |       |        |        |
| 16/1| $C_{16}$  | 8 8             | 64 64   | 64 64 | 64 64  | 3.1    |
| 16/2| $C_2^4$   | 96 14          | 400 624 | 168 188 | 168 188 | 3.1 |
| 16/5| $C_8 \times C_2$ | 16 10   | 112 112 | 88 88 | 88 88  | 3.1    |
| 16/10| $C_4 \times C_2^2$ | 192 13 | 564 820 | 146 150 | 146 150 | 3.1 |
| 16/14| $C_2^3$  | 20160 14      | 20747 39767 | 160 179 | 160 179 | 3.1 |
| 16  |           |                 | 41387 669 |       |        |        |
| 17/1| $C_{17}$  | 16 16          | 256 271 | 256 271 | 256 271 | 3.1 |
| 17  |           |                 | 271 271 |       |        |        |
| 18/2| $C_2 \times C_9$ | 6 6       | 36 48   | 36 48 | 36 48  | 2.1    |
| 18/5| $C_2 \times C_3^2$ | 48 8   | 136 183 | 56 68 | 56 68  | 2.1    |
| 18  |           |                 | 231 116 |       |        |        |
| 19/1| $C_{19}$  | 18 18          | 324 341 | 324 341 | 324 341 | 3.1 |
| 19  |           |                 | 341 341 |       |        |        |
| 20/2| $C_4 \times C_5$ | 8 8       | 64 76   | 64 76 | 64 76  | 2.1    |
| 20/5| $C_2^2 \times C_5$ | 24 12 | 176 285 | 128 171 | 128 171 | 2.1    |
| 20  |           |                 | 361 247 |       |        |        |
| 21/2| $C_3 \times C_7$ | 12 12   | 144 205 | 144 205 | 144 205 | 2.1 |
| 21  |           |                 | 205 205 |       |        |        |
| 22/2| $C_2 \times C_{11}$ | 10 10 | 100 109 | 100 109 | 100 109 | 2.1    |
| 22  |           |                 | 109 109 |       |        |        |
| 23/1| $C_{23}$  | 22 22          | 484 505 | 484 505 | 484 505 | 3.1    |
| 23  |           |                 | 505 505 |       |        |        |
| $G$ | structure | $|A|$ | $|X|$ | $|O|$ | $cq$ | $O_c$ | $mq$ | ref |
|-----|-----------|------|------|------|-----|-----|-----|-----|
| 24/2 | $C_8 \times C_3$ | 8    | 8    | 64   | 80  | 64  | 80  | 2.1 |
| 24/9 | $C_4 \times C_2 \times C_3$ | 16   | 10   | 112  | 140 | 88  | 110 | 2.1 |
| 24/15 | $C_2^3 \times C_3$ | 336  | 12   | 788  | 1705| 128 | 175 | 2.1 |
| 24   |          |      |      |      | 1925|     | 365 |     |
| 25/1 | $C_{25}$  | 20   | 20   | 400  | 490 | 400 | 490 | 3.1 |
| 25/2 | $C_5^2$   | 480  | 24   | 2336 | 2847| 512 | 594 |     |
| 25   |          |      |      |      | 3337|     | 1084|     |
| 26/2 | $C_2 \times C_{13}$ | 12   | 12   | 144  | 155 | 144 | 155 | 2.1 |
| 26   |          |      |      |      |     | 155 |     |     |
| 27/1 | $C_{27}$  | 18   | 18   | 324  | 441 | 324 | 441 | 3.1 |
| 27/2 | $C_9 \times C_3$ | 108  | 20   | 864  | 1356| 336 | 528 |     |
| 27/5 | $C_3^3$   | 11232| 24   | 23236| 34321|484 | 605 |     |
| 27   |          |      |      |      | 36118|    | 1574|     |
| 28/2 | $C_4 \times C_7$ | 12   | 12   | 144  | 164 | 144 | 164 | 2.1 |
| 28/4 | $C_2^2 \times C_7$ | 36   | 18   | 396  | 615 | 288 | 369 | 2.1 |
| 28   |          |      |      |      | 779 |     | 533 |     |
| 29/1 | $C_{29}$  | 28   | 28   | 784  | 811 | 784 | 811 | 3.1 |
| 29   |          |      |      |      | 811 |     | 811 |     |
| 30/4 | $C_2 \times C_3 \times C_5$ | 8    | 8    | 64   | 95  | 64  | 95  | 2.1 |
| 30   |          |      |      |      | 95  |     | 95  |     |
| 31/1 | $C_{31}$  | 30   | 30   | 900  | 929 | 900 | 929 | 3.1 |
| 31   |          |      |      |      | 929 |     | 929 |     |
| 32/1 | $C_{32}$  | 16   | 16   | 256  | 256 | 256 | 256 | 3.1 |
| 32/3 | $C_8 \times C_4$ | 128  | 26   | 1216 | 1216| 592 | 592 |     |
| 32/16| $C_{16} \times C_2$ | 32   | 20   | 448  | 448 | 352 | 352 |     |
| 32/21| $C_2^2 \times C_3$ | 1536 | 30   | 6224 | 9808| 884 | 904 |     |
| 32/36| $C_6 \times C_2^2$ | 384  | 26   | 2256 | 3280| 584 | 600 |     |
| 32/45| $C_4 \times C_3^2$ | 21504| 30   | 48412| 87580|804 | 834 |     |
| 32/51| $C_2^5$   | 9999360| 27  | 10024077| 19721077|590 | 655 |     |
| 32   |          |      |      |      | 19823665|   | 4193|     |
| 33/1 | $C_3 \times C_{13}$ | 20   | 20   | 400  | 545 | 400 | 545 | 2.1 |
| 33   |          |      |      |      | 545 |     | 545 |     |
| 34/2 | $C_2 \times C_{17}$ | 16   | 16   | 256  | 271 | 256 | 271 | 2.1 |
| 34   |          |      |      |      | 271 |     | 271 |     |
| 35/1 | $C_5 \times C_7$ | 24   | 24   | 576  | 779 | 576 | 779 | 2.1 |
| 35   |          |      |      |      | 779 |     | 779 |     |
| 36/2 | $C_4 \times C_9$ | 12   | 12   | 144  | 192 | 144 | 192 | 2.1 |
| 36/5 | $C_2^2 \times C_9$ | 36   | 18   | 396  | 720 | 288 | 432 | 2.1 |
| 36/8 | $C_4 \times C_3^2$ | 96   | 16   | 544  | 732 | 224 | 272 | 2.1 |
| 36/14| $C_2^3 \times C_3^2$ | 288  | 24   | 1496 | 2745| 448 | 612 | 2.1 |
| 36   |          |      |      |      | 4389|     | 1508|     |
| 37/1 | $C_{37}$  | 36   | 36   | 1296 | 1331| 1296| 1331| 3.1 |
| 37   |          |      |      |      | 1331|     | 1331|     |
| \( G \) | structure | \( |A| \) | \( |X| \) | \( |O| \) | \( cq \) | \( |O_c| \) | \( mq \) | ref |
|---|---|---|---|---|---|---|---|---|
| 38/2 | \( C_2 \times C_{19} \) | 18 | 18 | 324 | 341 | 324 | 341 | 2.1 |
| 39 | \( C_3 \times C_{13} \) | 24 | 24 | 576 | 775 | 576 | 775 | 2.1 |
| 40/2 | \( C_5 \times C_5 \) | 16 | 16 | 256 | 304 | 256 | 304 | 2.1 |
| 40/9 | \( C_4 \times C_2 \times C_5 \) | 32 | 20 | 448 | 532 | 352 | 418 | 2.1 |
| 40/14 | \( C_2^2 \times C_5 \) | 672 | 24 | 3152 | 6479 | 512 | 665 | 2.1 |
| 41 | \( C_{41} \) | 40 | 40 | 1600 | 1639 | 1600 | 1639 | 3.1 |
| 42/6 | \( C_2 \times C_3 \times C_7 \) | 12 | 12 | 144 | 205 | 144 | 205 | 2.1 |
| 43/1 | \( C_{43} \) | 42 | 42 | 1764 | 1805 | 1764 | 1805 | 3.1 |
| 44/2 | \( C_4 \times C_{11} \) | 20 | 20 | 400 | 436 | 400 | 436 | 2.1 |
| 44/4 | \( C_2^2 \times C_{11} \) | 60 | 30 | 1100 | 1635 | 800 | 981 | 2.1 |
| 45/1 | \( C_5 \times C_5 \) | 24 | 24 | 576 | 912 | 576 | 912 | 2.1 |
| 45/2 | \( C_2^2 \times C_5 \) | 192 | 32 | 2176 | 3477 | 896 | 1292 | 2.1 |
| 46/2 | \( C_2 \times C_{23} \) | 22 | 22 | 484 | 505 | 484 | 505 | 2.1 |
| 47/1 | \( C_{47} \) | 46 | 46 | 2116 | 2161 | 2116 | 2161 | 3.1 |
| 48/2 | \( C_{16} \times C_3 \) | 16 | 16 | 256 | 320 | 256 | 320 | 2.1 |
| 48/20 | \( C_4^2 \times C_3 \) | 192 | 28 | 1600 | 3120 | 672 | 940 | 2.1 |
| 48/23 | \( C_5 \times C_2 \times C_3 \) | 32 | 20 | 448 | 560 | 352 | 440 | 2.1 |
| 48/44 | \( C_4 \times C_2^2 \times C_3 \) | 384 | 26 | 2256 | 4100 | 584 | 750 | 2.1 |
| 48/52 | \( C_2^2 \times C_5 \) | 40320 | 28 | 82988 | 198835 | 640 | 895 | 2.1 |
| 48 | \( \) | \( \) | \( \) | \( \) | \( \) | \( \) | \( \) | \( \) |
| 49/1 | \( C_{49} \) | 42 | 42 | 1764 | 2044 | 1764 | 2044 | 3.1 |
| 49/2 | \( C_7^3 \) | 2016 | 48 | 13896 | 16055 | 2088 | 2344 | 2.1 |
| 49 | \( \) | \( \) | \( \) | \( \) | \( \) | \( \) | \( \) | \( \) |
| 50/2 | \( C_2 \times C_{25} \) | 20 | 20 | 400 | 490 | 400 | 490 | 2.1 |
| 50/5 | \( C_2 \times C_5^2 \) | 480 | 24 | 2336 | 2847 | 512 | 594 | 2.1 |
| 50 | \( \) | \( \) | \( \) | \( \) | \( \) | \( \) | \( \) | \( \) |
| 51/1 | \( C_3 \times C_{17} \) | 32 | 32 | 1024 | 1355 | 1024 | 1355 | 2.1 |
| 51 | \( \) | \( \) | \( \) | \( \) | \( \) | \( \) | \( \) | \( \) |
| 52/2 | \( C_4 \times C_{13} \) | 24 | 24 | 576 | 620 | 576 | 620 | 2.1 |
| 52/5 | \( C_2^2 \times C_{13} \) | 72 | 36 | 1584 | 2325 | 1152 | 1395 | 2.1 |
| 52 | \( \) | \( \) | \( \) | \( \) | \( \) | \( \) | \( \) | \( \) |
| 53/1 | \( C_{53} \) | 52 | 52 | 2704 | 2755 | 2704 | 2755 | 3.1 |
| 53 | \( \) | \( \) | \( \) | \( \) | \( \) | \( \) | \( \) | \( \) |
| $G$ | structure | $|A|$ | $|X|$ | $|O|$ | $cq$ | $|O_2|$ | $mq$ | ref |
|-----|-----------|------|------|------|------|------|------|-----|
| 54/2 | $C_2 \times C_{27}$ | 18 | 18 | 324 | 441 | 324 | 441 | 2.1 |
| 54/9 | $C_2 \times C_9 \times C_3$ | 108 | 20 | 864 | 1356 | 336 | 528 | 2.1 |
| 54/15 | $C_2 \times C_3^3$ | 11232 | 24 | 23236 | 34321 | 484 | 605 | 2.1 |
| 54 | $C_2 \times C_3^3$ | 36118 | | 1574 | | | | |
| 55/2 | $C_5 \times C_{11}$ | 40 | 40 | 1600 | 2071 | 1600 | 2071 | 2.1 |
| 55 | $C_5 \times C_{11}$ | 2071 | | 2071 | | | | |
| 56/2 | $C_8 \times C_7$ | 24 | 24 | 576 | 656 | 576 | 656 | 2.1 |
| 56/8 | $C_4 \times C_2 \times C_7$ | 48 | 30 | 1008 | 1148 | 792 | 902 | 2.1 |
| 56/13 | $C_3^2 \times C_7$ | 1008 | 36 | 7092 | 13981 | 1152 | 1435 | 2.1 |
| 56 | $C_3^2 \times C_7$ | 15785 | | 2993 | | | | |
| 57/2 | $C_3 \times C_{19}$ | 36 | 36 | 1296 | 1705 | 1296 | 1705 | 2.1 |
| 57 | $C_3 \times C_{19}$ | 1705 | | 1705 | | | | |
| 58/2 | $C_2 \times C_{29}$ | 28 | 28 | 784 | 811 | 784 | 811 | 2.1 |
| 58 | $C_2 \times C_{29}$ | 811 | | 811 | | | | |
| 59/1 | $C_{59}$ | 58 | 58 | 3364 | 3421 | 3364 | 3421 | 3.1 |
| 59 | $C_{59}$ | 3421 | | 3421 | | | | |
| 60/4 | $C_2 \times C_3^3 \times C_5$ | 16 | 16 | 256 | 380 | 256 | 380 | 2.1 |
| 60/13 | $C_2^4 \times C_3 \times C_5$ | 48 | 24 | 704 | 1425 | 512 | 855 | 2.1 |
| 60 | $C_2^4 \times C_3 \times C_5$ | 1805 | 1235 | | | | |
| 61/1 | $C_{61}$ | 60 | 60 | 3600 | 3659 | 3600 | 3659 | 3.1 |
| 61 | $C_{61}$ | 3659 | 3659 | | | | |
| 62/2 | $C_2 \times C_{31}$ | 30 | 30 | 900 | 929 | 900 | 929 | 2.1 |
| 62 | $C_2 \times C_{31}$ | 929 | 929 | | | | |
| 63/2 | $C_5 \times C_7$ | 36 | 36 | 1296 | 1968 | 1296 | 1968 | 2.1 |
| 63/4 | $C_3^2 \times C_7$ | 288 | 48 | 4896 | 7503 | 2016 | 2788 | 2.1 |
| 63 | $C_3^2 \times C_7$ | 9471 | 4756 | | | | |
| 64/1 | $C_{64}$ | 32 | 32 | 1024 | 1024 | 1024 | 1024 | 3.1 |
| 64/2 | $C_8^2$ | 1536 | 60 | 13568 | 22784 | 3072 | 3408 | |
| 64/26 | $C_{16} \times C_4$ | 256 | 52 | 4864 | 4864 | 2368 | 2368 | |
| 64/50 | $C_{32} \times C_2$ | 64 | 40 | 1792 | 1792 | 1408 | 1408 | |
| 64/55 | $C_4^3$ | 86016 | 60 | 206144 | 441664 | 4448 | 4672 | |
| 64/83 | $C_8 \times C_4^2 \times C_2$ | 2048 | 104 | 31168 | 31168 | 7240 | 7240 | |
| 64/183 | $C_8^3 \times C_2$ | 768 | 52 | 9024 | 13120 | 2336 | 2400 | |
| 64/192 | $C_8^2 \times C_2^3$ | 147456 | 100 | 550480 | 1239472 | 9108 | 9656 | |
| 64/246 | $C_8 \times C_4^3$ | 43008 | 60 | 193648 | 350320 | 3216 | 3336 | |
| 64/260 | $C_4 \times C_2^4$ | 10321920 | 69 | | | | | |
| 64/267 | $C_6^6$ | 20158709760 | 60 | | | | | |
| 65/1 | $C_5 \times C_{13}$ | 48 | 48 | 2304 | 2945 | 2304 | 2945 | 2.1 |
| 65 | $C_5 \times C_{13}$ | 2945 | 2945 | | | | |
| 66/4 | $C_2 \times C_3 \times C_{11}$ | 20 | 20 | 400 | 545 | 400 | 545 | 2.1 |
| 66 | $C_2 \times C_3 \times C_{11}$ | 545 | 545 | | | | |
| 67/1 | $C_{67}$ | 66 | 66 | 4356 | 4421 | 4356 | 4421 | 3.1 |
| 67 | $C_{67}$ | 4421 | 4421 | | | | |
| G     | structure            | |A| |X| |O| |cq| |Oc| |mq| ref |
|-------|----------------------|:-:|:-:|:-:|:-:|:-:|:-:|:-:|:-:|:-:|:-:|
| 68/2  | $\mathcal{C}_4 \times \mathcal{C}_{17}$ | 32 | 32 | 1024 | 1084 | 1024 | 1084 | 2.1 |
| 68/5  | $\mathcal{C}_2^2 \times \mathcal{C}_{17}$ | 96 | 48 | 2816 | 4065 | 2048 | 2439 | 2.1 |
| 68    |                     | | | | | 5149 | | 3523 | 2.1 |
| 69/1  | $\mathcal{C}_3 \times \mathcal{C}_{23}$ | 44 | 44 | 1936 | 2525 | 1936 | 2525 | 2.1 |
| 69    |                     | | | | | 2525 | | 2525 | 2.1 |
| 70/4  | $\mathcal{C}_2 \times \mathcal{C}_5 \times \mathcal{C}_7$ | 24 | 24 | 576  | 779  | 576  | 779  | 2.1 |
| 70    |                     | | | | | 779  | | 779  | 2.1 |
| 71/1  | $\mathcal{C}_{71}$  | 70 | 70 | 4900 | 4969 | 4900 | 4969 | 3.1 |
| 71    |                     | | | | | 4969 | | 4969 | 2.1 |
| 72/2  | $\mathcal{C}_8 \times \mathcal{C}_9$   | 24 | 24 | 576  | 768  | 576  | 768  | 2.1 |
| 72/9  | $\mathcal{C}_4 \times \mathcal{C}_2 \times \mathcal{C}_9$ | 48 | 30 | 1008 | 1344 | 792  | 1056 | 2.1 |
| 72/14 | $\mathcal{C}_8 \times \mathcal{C}_3^2$ | 192 | 32 | 2176 | 2928 | 896  | 1088 | 2.1 |
| 72/18 | $\mathcal{C}_2^2 \times \mathcal{C}_9$   | 1008 | 36 | 7092 | 16368 | 1152 | 1680 | 2.1 |
| 72/36 | $\mathcal{C}_4 \times \mathcal{C}_2 \times \mathcal{C}_3^2$ | 384 | 40 | 3808 | 5124 | 1232 | 1496 | 2.1 |
| 72/50 | $\mathcal{C}_2^2 \times \mathcal{C}_3^2$ | 8064 | 48 | 26792 | 62403 | 1792 | 2380 | 2.1 |
| 72    |                     | | | | | 88935 | | 8468 | 2.1 |
| 73/1  | $\mathcal{C}_{73}$  | 72 | 72 | 5184 | 5255 | 5184 | 5255 | 3.1 |
| 73    |                     | | | | | 5255 | | 5255 | 2.1 |
| 74/2  | $\mathcal{C}_2 \times \mathcal{C}_{37}$ | 36 | 36 | 1296 | 1331 | 1296 | 1331 | 2.1 |
| 74    |                     | | | | | 1331 | | 1331 | 2.1 |
| 75/1  | $\mathcal{C}_3 \times \mathcal{C}_{25}$ | 40 | 40 | 1600 | 2450 | 1600 | 2450 | 2.1 |
| 75/3  | $\mathcal{C}_3 \times \mathcal{C}_5^2$  | 960 | 48 | 9344 | 14235 | 2048 | 2970 | 2.1 |
| 75    |                     | | | | | 16685 | | 5420 | 2.1 |
| 76/2  | $\mathcal{C}_4 \times \mathcal{C}_{19}$ | 36 | 36 | 1296 | 1364 | 1296 | 1364 | 2.1 |
| 76/4  | $\mathcal{C}_2^2 \times \mathcal{C}_{19}$ | 108 | 54 | 3564 | 5115 | 2592 | 3069 | 2.1 |
| 76    |                     | | | | | 6479 | | 4433 | 2.1 |
| 77/1  | $\mathcal{C}_7 \times \mathcal{C}_{11}$ | 60 | 60 | 3600 | 4469 | 3600 | 4469 | 2.1 |
| 77    |                     | | | | | 4469 | | 4469 | 2.1 |
| 78/6  | $\mathcal{C}_2 \times \mathcal{C}_3 \times \mathcal{C}_{13}$ | 24 | 24 | 576  | 775  | 576  | 775  | 2.1 |
| 78    |                     | | | | | 775  | | 775  | 2.1 |
| 79/1  | $\mathcal{C}_{79}$  | 78 | 78 | 6084 | 6161 | 6084 | 6161 | 3.1 |
| 79    |                     | | | | | 6161 | | 6161 | 3.1 |
| 80/2  | $\mathcal{C}_{18} \times \mathcal{C}_5$  | 32 | 32 | 1024 | 1216 | 1024 | 1216 | 2.1 |
| 80/20 | $\mathcal{C}_4^2 \times \mathcal{C}_5$   | 384 | 56 | 6400 | 11856 | 2688 | 3572 | 2.1 |
| 80/23 | $\mathcal{C}_8 \times \mathcal{C}_2 \times \mathcal{C}_5$ | 64 | 40 | 1792 | 2128 | 1408 | 1672 | 2.1 |
| 80/45 | $\mathcal{C}_4 \times \mathcal{C}_2^2 \times \mathcal{C}_5$ | 768 | 52 | 9024 | 15580 | 2336 | 2850 | 2.1 |
| 80/52 | $\mathcal{C}_2 \times \mathcal{C}_5$     | 80640 | 56 | 331952 | 755573 | 2560 | 3401 | 2.1 |
| 80    |                     | | | | | 786353 | | 12711 | 2.1 |
| 81/1  | $\mathcal{C}_{81}$  | 54 | 54 | 2916 | 3996 | 2916 | 3996 | 3.1 |
| 81/2  | $\mathcal{C}_9^2$   | 3888 | 78 | 35316 | 54405 | 5616 | 8055 | 2.1 |
| 81/5  | $\mathcal{C}_{27} \times \mathcal{C}_3$ | 324 | 60 | 7776 | 12897 | 3024 | 5157 | 2.1 |
| 81/11 | $\mathcal{C}_6 \times \mathcal{C}_3^2$ | 23328 | 74 | 152892 | 270441 | 4176 | 7167 | 2.1 |
| 81/15 | $\mathcal{C}_3^4$   | 24261120 | 78 | ? | ? | ? | ? | 2.1 |
| $G$   | structure          | $|A|$ | $|X|$ | $|O|$ | $cq$ | $|O_c|$ | $mq$ | ref |
|-------|--------------------|------|------|------|------|--------|------|-----|
| 82/2  | $C_2 \times C_{41}$ | 40   | 40   | 1600 | 1639 | 1600   | 1639 | 2.1 |
| 82    |                    |      |      |      |      |        |      |     |
| 83/1  | $C_{83}$          | 82   | 82   | 6724 | 6805 | 6724   | 6805 | 3.1 |
| 83    |                    |      |      |      |      |        |      |     |
| 84/6  | $C_4 \times C_3 \times C_7$ | 24 | 24 | 576 | 820 | 576 | 820 | 2.1 |
| 84/15 | $C_2^2 \times C_3 \times C_7$ | 72 | 36 | 1584 | 3075 | 1152 | 1845 | 2.1 |
| 84    |                    |      |      |      |      |        |      |     |
| 85/1  | $C_5 \times C_{17}$ | 64   | 64   | 4096 | 5149 | 4096   | 5149 | 2.1 |
| 85    |                    |      |      |      |      |        |      |     |
| 86/2  | $C_2 \times C_{43}$ | 42   | 42   | 1764 | 1805 | 1764   | 1805 | 2.1 |
| 86    |                    |      |      |      |      |        |      |     |
| 87/1  | $C_3 \times C_{29}$ | 56   | 56   | 3136 | 4055 | 3136   | 4055 | 2.1 |
| 87    |                    |      |      |      |      |        |      |     |
| 88/2  | $C_8 \times C_{11}$ | 40   | 40   | 1600 | 1744 | 1600   | 1744 | 2.1 |
| 88/8  | $C_4 \times C_2 \times C_{11}$ | 80 | 50 | 2800 | 3052 | 2200 | 2398 | 2.1 |
| 88/12 | $C_2^2 \times C_{11}$ | 1680 | 60 | 19700 | 37169 | 3200 | 3815 | 2.1 |
| 88    |                    |      |      |      |      |        |      |     |
| 89/1  | $C_{89}$          | 88   | 88   | 7744 | 7831 | 7744   | 7831 | 3.1 |
| 89    |                    |      |      |      |      |        |      |     |
| 90/4  | $C_2 \times C_5 \times C_5$ | 24 | 24 | 576 | 912 | 576 | 912 | 2.1 |
| 90/10 | $C_2 \times C_3^2 \times C_5$ | 192 | 32 | 2176 | 3477 | 896 | 1292 | 2.1 |
| 90    |                    |      |      |      |      |        |      |     |
| 91/1  | $C_7 \times C_{13}$ | 72   | 72   | 5184 | 6355 | 5184   | 6355 | 2.1 |
| 91    |                    |      |      |      |      |        |      |     |
| 92/2  | $C_2 \times C_{23}$ | 44   | 44   | 1936 | 2020 | 1936   | 2020 | 2.1 |
| 92/4  | $C_2^2 \times C_{23}$ | 132 | 66 | 5324 | 7575 | 3872 | 4545 | 2.1 |
| 92    |                    |      |      |      |      |        |      |     |
| 93/2  | $C_3 \times C_{31}$ | 60   | 60   | 3600 | 4645 | 3600   | 4645 | 2.1 |
| 93    |                    |      |      |      |      |        |      |     |
| 94/2  | $C_2 \times C_{47}$ | 46   | 46   | 2116 | 2161 | 2116   | 2161 | 2.1 |
| 94    |                    |      |      |      |      |        |      |     |
| 95/1  | $C_5 \times C_{19}$ | 72   | 72   | 5184 | 6479 | 5184   | 6479 | 2.1 |
| 95    |                    |      |      |      |      |        |      |     |
| 96/2  | $C_{32} \times C_3$ | 32   | 32   | 1024 | 1280 | 1024   | 1280 | 2.1 |
| 96/46 | $C_8 \times C_4 \times C_3$ | 256 | 52 | 4864 | 6080 | 2368 | 2960 | 2.1 |
| 96/59 | $C_{16} \times C_2 \times C_3$ | 64 | 40 | 1792 | 2240 | 1408 | 1760 | 2.1 |
| 96/161| $C_8^2 \times C_2 \times C_3$ | 3072 | 60 | 24896 | 49040 | 3536 | 4520 | 2.1 |
| 96/176| $C_8 \times C_3^2 \times C_3$ | 768 | 52 | 9024 | 16400 | 2336 | 3000 | 2.1 |
| 96/220| $C_4 \times C_2^3 \times C_3$ | 43008 | 60 | 193648 | 437900 | 3216 | 4170 | 2.1 |
| 96/231| $C_2^2 \times C_3^2$ | 19998720 | 54 | 40096308 | 98605385 | 2360 | 3275 | 2.1 |
| 96    |                    |      |      |      |      |        |      |     |
| 97/1  | $C_{97}$          | 96   | 96   | 9216 | 9311 | 9216   | 9311 | 3.1 |
| 97    |                    |      |      |      |      |        |      |     |

**Central and Medial Quasigroups of Small Order**
| $G$  | structure          | $|A|$ | $|X|$ | $|O|$ | $e_q$ | $|O_{C}$ | $mq$ | ref |
|------|--------------------|------|------|------|------|--------|------|-----|
| 98   | $C_2 \times C_{19}$ | 42   | 42   | 1764 | 2044 | 1764   | 2044 | 2.1 |
| 98   | $C_2 \times C_7^2$  | 2016 | 48   | 13896| 16055| 2088   | 2344 | 2.1 |
|      |                    |      |      | 18099|      |        |      |     |
| 99   | $C_6 \times C_{11}$ | 60   | 60   | 3600 | 5232 | 3600   | 5232 | 2.1 |
| 99   | $C_3^2 \times C_{11}$ | 480  | 80   | 13600| 19947| 5600   | 7412 | 2.1 |
|      |                    |      |      | 25179|      |        |      |     |
| 100  | $C_4 \times C_{25}$ | 40   | 40   | 1600 | 1960 | 1600   | 1960 | 2.1 |
| 100  | $C_2^2 \times C_{25}$ | 120  | 60   | 4400 | 7350 | 3200   | 4410 | 2.1 |
| 100  | $C_4 \times C_5^2$  | 960  | 48   | 9344 | 11388| 2048   | 2376 | 2.1 |
| 100  | $C_2^2 \times C_5^2$ | 2880 | 72   | 25696| 42705| 4096   | 5346 | 2.1 |
|      |                    |      |      | 63403|      |        |      | 14092|
| 101  | $C_{101}$          | 100  | 100  | 10000| 10099| 10000  | 10099| 3.1 |
|      |                    |      |      | 10099|      |        |      | 10099|
| 102  | $C_2 \times C_3 \times C_{17}$ | 32   | 32   | 1024 | 1355 | 1024   | 1355 | 2.1 |
|      |                    |      |      | 1355 |      |        |      | 1355 |
| 103  | $C_{103}$          | 102  | 102  | 10404| 10505| 10404  | 10505| 3.1 |
|      |                    |      |      | 10505|      |        |      | 10505|
| 104  | $C_8 \times C_{13}$ | 48   | 48   | 2304 | 2480 | 2304   | 2480 | 2.1 |
| 104  | $C_4 \times C_2 \times C_{13}$ | 96   | 60   | 4032 | 4340 | 3168   | 3410 | 2.1 |
| 104  | $C_2 \times C_{13}$ | 2016 | 72   | 28368| 52855| 4608   | 5425 | 2.1 |
|      |                    |      |      | 59675|      |        |      | 11315|
| 105  | $C_3 \times C_5 \times C_7$ | 48   | 48   | 2304 | 3895 | 2304   | 3895 | 2.1 |
|      |                    |      |      | 3895 |      |        |      | 3895 |
| 106  | $C_2 \times C_{53}$ | 52   | 52   | 2704 | 2755 | 2704   | 2755 | 2.1 |
|      |                    |      |      | 2755 |      |        |      | 2755 |
| 107  | $C_{107}$          | 106  | 106  | 11236| 11341| 11236  | 11341| 3.1 |
|      |                    |      |      | 11341|      |        |      | 11341|
| 108  | $C_4 \times C_{27}$ | 36   | 36   | 1296 | 1764 | 1296   | 1764 | 2.1 |
| 108  | $C_2 \times C_{27}$ | 108  | 54   | 3564 | 6615 | 2592   | 3969 | 2.1 |
| 108  | $C_4 \times C_9 \times C_3$ | 216  | 40   | 3456 | 5424 | 1344   | 2112 | 2.1 |
| 108  | $C_2 \times C_9 \times C_3$ | 648  | 60   | 9504 | 20340| 2688   | 4752 | 2.1 |
| 108  | $C_4 \times C_3^3$  | 22464| 48   | 92944| 137284| 1936   | 2420 | 2.1 |
| 108  | $C_2^2 \times C_3^3$ | 67392| 72   | 255596| 514815| 3872   | 5445 | 2.1 |
|      |                    |      |      | 686242| 20462|        |      |     |
| 109  | $C_{109}$          | 108  | 108  | 11664| 11771| 11664  | 11771| 3.1 |
|      |                    |      |      | 11771|      |        |      | 11771|
| 110  | $C_2 \times C_5 \times C_{11}$ | 40   | 40   | 1600 | 2071 | 1600   | 2071 | 2.1 |
|      |                    |      |      | 2071 |      |        |      | 2071 |
| 111  | $C_3 \times C_{37}$ | 72   | 72   | 5154 | 6655 | 5154   | 6655 | 2.1 |
|      |                    |      |      | 6655 |      |        |      | 6655 |
| $G$ | structure | $|A|$ | $|X|$ | $|Q|$ | $cq$ | $|Q_c|$ | $mq$ | ref |
|-----|-----------|------|------|------|------|------|------|-----|
| 112/2 | $C_{16} \times C_7$ | 48 | 48 | 2304 | 2624 | 2304 | 2624 | 2.1 |
| 112/19 | $C_4^2 \times C_7$ | 576 | 84 | 14400 | 25584 | 6048 | 7708 | 2.1 |
| 112/22 | $C_8 \times C_2 \times C_7$ | 96 | 60 | 4032 | 4592 | 3168 | 3608 | 2.1 |
| 112/37 | $C_4 \times C_2^2 \times C_7$ | 1152 | 78 | 20304 | 33620 | 5256 | 6150 | 2.1 |
| 112/43 | $C_2^4 \times C_7$ | 120960 | 84 | 746892 | 1630447 | 5760 | 7339 | 2.1 |
| 112 | | | | $1696867$ | | $27429$ | |
| 113/1 | $C_{113}$ | 112 | 112 | 12544 | 12655 | 12544 | 12655 | 3.1 |
| 113 | | | | $12655$ | | $12655$ | |
| 114/6 | $C_2 \times C_3 \times C_{19}$ | 36 | 36 | 1296 | 1705 | 1296 | 1705 | 2.1 |
| 115 | | | | $1705$ | | $1705$ | |
| 115/1 | $C_5 \times C_{23}$ | 88 | 88 | 7744 | 9595 | 7744 | 9595 | 2.1 |
| 116 | | | | $9595$ | | $9595$ | |
| 116/2 | $C_4 \times C_{29}$ | 56 | 56 | 3136 | 3244 | 3136 | 3244 | 2.1 |
| 116/5 | $C_2^2 \times C_{29}$ | 168 | 84 | 8624 | 12165 | 6272 | 7299 | 2.1 |
| 117 | | | | $15409$ | | $10543$ | |
| 117/2 | $C_9 \times C_{13}$ | 72 | 72 | 5184 | 7440 | 5184 | 7440 | 2.1 |
| 117/4 | $C_3^2 \times C_{13}$ | 576 | 96 | 19584 | 28365 | 8064 | 10540 | 2.1 |
| 118/2 | $C_2 \times C_{59}$ | 58 | 58 | 3364 | 3421 | 3364 | 3421 | 2.1 |
| 118 | | | | $3421$ | | $3421$ | |
| 119/1 | $C_7 \times C_{17}$ | 96 | 96 | 9216 | 11111 | 9216 | 11111 | 2.1 |
| 119 | | | | $11111$ | | $11111$ | |
| 120/4 | $C_8 \times C_3 \times C_5$ | 32 | 32 | 1024 | 1520 | 1024 | 1520 | 2.1 |
| 120/31 | $C_4 \times C_2 \times C_3 \times C_5$ | 64 | 40 | 1792 | 2660 | 1408 | 2090 | 2.1 |
| 120/47 | $C_2^2 \times C_3 \times C_5$ | 1344 | 48 | 12608 | 32395 | 2048 | 3325 | 2.1 |
| 120 | | | | $36575$ | | $6935$ | |
| 121/1 | $C_{121}$ | 110 | 110 | 12100 | 13288 | 12100 | 13288 | 3.1 |
| 121/2 | $C_1^2$ | 13200 | 120 | 144200 | 158199 | 13400 | 14508 | |
| 121 | | | | $171487$ | | $27796$ | |
| 122/2 | $C_2 \times C_{61}$ | 60 | 60 | 3600 | 3659 | 3600 | 3659 | 2.1 |
| 122 | | | | $3659$ | | $3659$ | |
| 123/1 | $C_3 \times C_{41}$ | 80 | 80 | 6400 | 8195 | 6400 | 8195 | 2.1 |
| 123 | | | | $8195$ | | $8195$ | |
| 124/2 | $C_4 \times C_{31}$ | 60 | 60 | 3600 | 3716 | 3600 | 3716 | 2.1 |
| 124/4 | $C_2^2 \times C_{31}$ | 180 | 90 | 9900 | 13935 | 7200 | 8361 | 2.1 |
| 124 | | | | $17651$ | | $12077$ | |
| 125/1 | $C_{125}$ | 100 | 100 | 10000 | 12325 | 10000 | 12325 | 3.1 |
| 125/2 | $C_{25} \times C_5$ | 2000 | 104 | 47200 | 66580 | 9280 | 13270 | |
| 125/5 | $C_5^3$ | 148800 | 120 | ? | ? | ? | ? | |
| 125 | | | | ? | | ? | |
| 126/6 | $C_2 \times C_9 \times C_7$ | 36 | 36 | 1296 | 1968 | 1296 | 1968 | 2.1 |
| 126/16 | $C_2 \times C_3^2 \times C_7$ | 288 | 48 | 4896 | 7503 | 2016 | 2788 | 2.1 |
| 126 | | | | $9471$ | | $4756$ | |
| 127/1 | $C_{127}$ | 126 | 126 | 15876 | 16001 | 15876 | 16001 | 3.1 |
| 127 | | | | $16001$ | | $16001$ | |
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