THE BREATHER WAVE SOLUTIONS, M-LUMP SOLUTIONS AND SEMI-RATIONAL SOLUTIONS TO A (2+1)-DIMENSIONAL GENERALIZED KORTEweg-de VRIES EQUATION

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Abstract Under investigation in this work is a (2+1)-dimensional generalized Korteweg-de Vries equation, which can be used to describe many nonlinear phenomena in plasma physics. By using the properties of Bell’s polynomial, we obtain the bilinear formalism of this equation. The expression of \( N \)-soliton solution is established in terms of the Hirota’s bilinear method. Based on the resulting \( N \)-soliton solutions, we succinctly show its breather wave solutions. Furthermore, with the aid of the corresponding soliton solutions, the \( M \)-lump solutions are well presented by taking a long wave limit. Two types of hybrid solutions are also represented in detail. Finally, some graphic analysis are provided in order to better understand the propagation characteristics of the obtained solutions.

Keywords A (2+1)-dimensional generalized Korteweg-de Vries equation, bilinear form, breather wave solutions, lump solutions, hybrid solutions.

MSC(2010) 35Q51, 35Q53, 35C99.

1. Introduction

It is widely acknowledged that the research of studying integrable properties and constructing exact solutions for the nonlinear evolution equations (NLEEs) is one of the most meaningful and interesting works in the field of mathematical physics. Additionally, finding exact solutions of NLEEs is also a hot topic for research workers all the time. More recently, the breather wave solutions and lump solutions have gradually occupied the eyes of researchers. The breather wave solutions possess periodicity in one direction which could be transformed into rogue wave solutions in view of certain circumstances. The lump solutions, as special localized waves, are a kind of rational solutions in all space directions. It was first discovered in 1977 by Manakov et al. in [21]. In the past few years, it has been reported in many nonlinear fields, such as optic media, the plasma, and shallow
water wave [34, 35, 45, 46]. Meanwhile, various skills, such as taking a long wave limit of the corresponding $N$-soliton solutions [2, 33], the inverse scattering transformation [1], Darboux transformation method [9, 22], Lie symmetry method [3], Bäcklund transformation [14], and the Hirota’s bilinear method [12, 13], etc, have been used to solve exact lump solutions of NLEEs. By utilizing the Hirota’s bilinear method and symbolic computation [5, 11, 15, 28, 32, 36–38, 43, 44, 47–49, 51, 54, 56], lots of works about breather wave solutions and lump solutions have been done in the past time [6–8, 10, 16, 18, 23–26, 29, 50, 58–61]. More importantly, taking a long wave limit for the corresponding $N$-soliton solutions has great significance in the research of lump solutions for NLEEs. Hence, we will mainly focus on this topic.

In this work, we would like to consider a $(2+1)$-dimensional generalized Korteweg-de Vries (gKdV) equation of the following form

$$\begin{align*}
&{u_t + u_x + \alpha (6 uu_x + u_{xxx}) + \beta v_y = 0,} \\
&u_y = v_x,
\end{align*}$$

(1.1)

where $u = u(x, y, t), v = v(x, y, t)$, and $\alpha, \beta$ are both arbitrary constants. It’s Bäcklund transformation, infinite conservation laws, soliton solutions and periodic wave solutions have been detailedly reported in [55]. Although some works have been represented for the gKdV equation (1.1), high-order breather solutions, $M$-lump solutions and relevant hybrid solutions have not been investigated before. Moreover, Eq.(1.1) has widespread application in terms of shallow water waves with weakly nonlinear restoring forces, thus we will focus on its above properties that have not been studied before.

The outline of present paper is given as follows. In section 2, we first obtain $N$-soliton solutions of Eq.(1.1), then we further derive the $n$th-order breather wave solutions by taking suitable parameters. Subsequently, we choose the first-order breather wave solution as an example, and carry out the detailed analysis. In section 3, by virtue of the corresponding $N$-soliton solutions, we systematically construct $M$-lump solutions of the gKdV equation (1.1) by taking a long wave limit. In section 4, we consider the behavior characteristics of hybrid solutions, which are hybrid of lump solution and soliton solution, and hybrid of lump solution and breather wave solution. Finally, some conclusions and discussions of this work are revealed in the last section.

2. The breather wave solutions

By employing the results provided in [30, 39–42, 52, 53, 57], we first know that Eq.(1.1) can be mapped into

$$\begin{align*}
(D_x D_t + D_x^2 + \alpha D_x^4 + \beta D_y^2)f \cdot f = 0,
\end{align*}$$

(2.1)

with a variable transformation

$$u = 2(ln f)_{xx},$$

(2.2)

where $f = f(x, y, t)$ is a real function, and the derivatives $D_x D_t, D_x^2, D_x^4, D_y^2$ are all the bilinear derivative operators defined by

$$D^m_x D^n_y D^l_t (f \cdot g) =$$
\[
\left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^n \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^l f(x, y, t) \cdot g(x', y', t') \bigg|_{x=x', y=y', z=z'}.
\]

(2.3)

Then, we further get the N-soliton solution of the gKdV equation (1.1)

\[
f = f_N = \sum_{\sigma=0,1} \exp \left( \sum_{1 \leq i < j \leq N} \sigma_i \sigma_j A_{ij} + \sum_{i=1}^N \sigma_i \xi_i \right),
\]

(2.4)

with

\[
\xi_i = k_i(x + p_i y - (\alpha k_i^2 + \beta p_i^2 + 1)t) + \xi_i^{(0)},
\]

\[
e^A_{ij} = \frac{3\alpha(k_i - k_j)^2 - \beta(p_i - p_j)^2}{3\alpha(k_i + k_j)^2 - \beta(p_i - p_j)^2},
\]

(2.5)

where \(k_i, p_i, \xi_i^{(0)}\) are all arbitrary parameters, and the notation \(\sum_{\sigma=0,1}\) is a summation that takes over all possible combinations \(\sigma_i = 0, 1\) \((i = 1, 2, \cdots, N)\).

On the basis of previous works [4,27,31], one can obtain the nth-order breather wave solutions of the (2+1)-dimensional gKdV equation (1.1) by taking those parameters in above expression (2.4)

\[
N = 2n, \quad k_j^* = k_{j+1}, \quad p_j^* = p_{j+1}, \quad \xi_j^* = \xi_{j+1}.
\]

(2.6)

Without the loss of generality, we focus on the first-order breather wave solution of Eq.(1.1). Therefore, we have

\[
N = 2, \quad k_1^* = k_2, \quad p_1^* = p_2, \quad \xi_1^* = \xi_2.
\]

(2.7)

If we take parameters

\[
k_1 = k_2^* = i, \quad p_1 = p_2^* = 2 + i, \quad \xi_1^{(0)} = \xi_2^{(0)} = 0, \quad \alpha = 1, \quad \beta = -1,
\]

(2.8)

it is not difficult to find that the expression \(f\) in (2.4) can be rewritten as

\[
f = 1 + 2 \cos(x + 2y + 3t) \sinh(-y - 4t) + 2 \cos(x + 2y + 3t) \cosh(-y - 4t)
\]

\[
+ 4 \sinh(-2y - 8t) + 4 \cosh(-2y - 8t).
\]

(2.9)

Next, by inserting Eq.(2.9) into Eq.(2.2), the obtained result is called the first-order breather wave solution of Eq.(1.1). In order to understand propagation characteristics of the first-order breather wave solution (2.2) with (2.9) intuitively, we plot the following Figure 1. It is necessary to point out that these three pictures are plotted at \(t = 0\). Moreover, it is worth noting that the first-order breather wave solution is periodic in \(x\) axis and localized in \(y\) axis, as described in Figure 1(a).
3. \(M\)-lump solutions

In this part, our main goal is to construct lump solutions of Eq.(1.1) in detail by taking a long wave limit for the corresponding \(N\)-soliton solutions. From the expression (2.4), we easily find that the first three soliton solutions as follows

\[
\begin{align*}
    f_1 &= 1 + \exp \xi_1, \\
    f_2 &= 1 + \exp \xi_1 + \exp \xi_2 + \exp(\xi_1 + \xi_2 + A_{12}), \\
    f_3 &= 1 + \exp \xi_1 + \exp \xi_2 + \exp \xi_3 + \exp(\xi_1 + \xi_2 + A_{12}) + \exp(\xi_1 + \xi_3 + A_{13}) \\
    &\quad + \exp(\xi_2 + \xi_3 + A_{23}) + \exp(\xi_1 + \xi_2 + \xi_3 + A_{12} + A_{13} + A_{23}).
\end{align*}
\]

(3.1)

Subsequently, we want to get \(M\)-lump solutions of Eq.(1.1), so we take

\[
\exp(\xi_1^{(0)}) = -1, \quad 1 \leq i \leq N,
\]

(3.2)

and let a limit \(k_i \to 0\) in (2.4). Then, the following theorem can be presented.

**Theorem 3.1.** Eq.(1.1) has the \(M\)-lump solutions in the following form

\[
u = 2(\ln f_N)_{xx},
\]

(3.3)

with

\[
\begin{align*}
    f_N &= \prod_{i=1}^{N} \eta_i + \frac{1}{2} \sum_{i,j}^{N} B_{ij} \prod_{l \neq i,j}^{N} \eta_l + \frac{1}{2!} \sum_{i,j,s,r}^{N} B_{ij} B_{sr} \prod_{l \neq i,j,s,r}^{N} \eta_l + \cdots \\
    &\quad + \frac{1}{M!2^M} \sum_{i,j,\ldots,m,n}^{N} B_{ij} B_{el} \cdots B_{mn} \prod_{p \neq i,j,e,l,\ldots,m,n}^{N} \eta_p + \cdots,
\end{align*}
\]

(3.4)

and

\[
\begin{align*}
    \eta_i &= x + p_i y - (\beta p_i^2 + 1)t, \\
    B_{ij} &= \frac{12\alpha}{\beta(p_i - p_j)^2},
\end{align*}
\]

(3.5)
where \( \sum_{i,j,\cdots,m,n} \) stands for the summation roundly feasible combinations of \( i,j, \cdots,m,n \), which are chosen from \( 1,2,\cdots,N \) and they are all distinct. A class of nonsingular lump solutions which were confirmed by Satsuma and Ablowitz (see Ref. [33]) can be derived, if we choose the parameters \( p_{n+i} = p_i^* (i = 1,2,\cdots,n) \) with \( N = 2n \).

### 3.1. 1-lump solution

In this subsection, 1-lump solution of Eq.(1.1) can be obtained from 2-soliton solution when we take \( n = 1, N = 2 \). Meanwhile, Eq.(3.4) can be represented as

\[
f_2 = \eta_1 \eta_2 + B_{12},
\]

with

\[
\eta_i = x + p_i y - (\beta p_i^2 + 1)t, \quad i = 1, 2,
\]

\[
B_{12} = \frac{12\alpha}{\beta(p_1 - p_2)^2}.
\]

By taking \( p_2 = p_1^* \), we have a nonsingular solution

\[
f_2 = \eta_1 \eta_1^* + \frac{12\alpha}{\beta(p_1 - p_1^*)^2}.
\]

Inserting Eq.(3.8) into \( u = 2(\ln f_2)_{xx} \) and setting \( p_1 = a + bi \), 1-lump solution of Eq.(1.1) is obtained by

\[
u = 2 \frac{\partial^2}{\partial x^2} \ln[(x' + ay')^2 + (by')^2 - \frac{3\alpha}{\beta b^2}] = 4 \frac{-(x' + ay')^2 + (by')^2 - \frac{3\alpha}{\beta b^2}}{[(x' + ay')^2 + (by')^2 - \frac{3\alpha}{\beta b^2}]^2},
\]

with

\[
x' = x + (a^2 \beta + b^2 \beta - 1)t, \quad y' = y - 2a\beta t.
\]

It is necessary to point out that the rational solution (3.9) is a permanent 1-lump solution, this solution decaying as \( O(1/x^2, 1/y^2) \) for \( |x|, |y| \to \infty \) and moving with the velocity \( v_x = 1 - a^2 \beta - b^2 \beta \) and \( v_y = 2a\beta \). As shown in Figure 2, the evolution of the solution (3.9) is plotted with an appropriate choice of the parameters \( a, b, \alpha \) and \( \beta \). Moreover, it is easily find that \( f_2 \) is a positive quadratic function, which is consistent with the results in [17,19,20].

### 3.2. Multiple-lump solutions

Here, we will derive multiple-lump solutions of Eq.(1.1). We first take \( n = 2, N = 4 \), then Eq.(3.4) can be expressed as \( f_4 \), given by

\[
f_4 = \eta_1 \eta_2 \eta_3 \eta_4 + B_{12} \eta_3 \eta_4 + B_{13} \eta_2 \eta_4 + B_{14} \eta_2 \eta_3 + B_{23} \eta_1 \eta_4 + B_{24} \eta_1 \eta_3 + B_{34} \eta_1 \eta_2 + B_{12} B_{34} + B_{13} B_{24} + B_{14} B_{23},
\]

with

\[
\eta_i = x + p_i y - (\beta p_i^2 + 1)t, \quad i = 1, 2, 3, 4,
\]

By taking \( p_2 = p_1^* \), we have a nonsingular solution

\[
f_2 = \eta_1 \eta_1^* + \frac{12\alpha}{\beta(p_1 - p_1^*)^2}.
\]

Inserting Eq.(3.8) into \( u = 2(\ln f_2)_{xx} \) and setting \( p_1 = a + bi \), 1-lump solution of Eq.(1.1) is obtained by

\[
u = 2 \frac{\partial^2}{\partial x^2} \ln[(x' + ay')^2 + (by')^2 - \frac{3\alpha}{\beta b^2}] = 4 \frac{-(x' + ay')^2 + (by')^2 - \frac{3\alpha}{\beta b^2}}{[(x' + ay')^2 + (by')^2 - \frac{3\alpha}{\beta b^2}]^2},
\]

with

\[
x' = x + (a^2 \beta + b^2 \beta - 1)t, \quad y' = y - 2a\beta t.
\]

It is necessary to point out that the rational solution (3.9) is a permanent 1-lump solution, this solution decaying as \( O(1/x^2, 1/y^2) \) for \( |x|, |y| \to \infty \) and moving with the velocity \( v_x = 1 - a^2 \beta - b^2 \beta \) and \( v_y = 2a\beta \). As shown in Figure 2, the evolution of the solution (3.9) is plotted with an appropriate choice of the parameters \( a, b, \alpha \) and \( \beta \). Moreover, it is easily find that \( f_2 \) is a positive quadratic function, which is consistent with the results in [17,19,20].

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\[
f_4 = \eta_1 \eta_2 \eta_3 \eta_4 + B_{12} \eta_3 \eta_4 + B_{13} \eta_2 \eta_4 + B_{14} \eta_2 \eta_3 + B_{23} \eta_1 \eta_4 + B_{24} \eta_1 \eta_3 + B_{34} \eta_1 \eta_2 + B_{12} B_{34} + B_{13} B_{24} + B_{14} B_{23},
\]

with

\[
\eta_i = x + p_i y - (\beta p_i^2 + 1)t, \quad i = 1, 2, 3, 4,
\]
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Figure 2. (Color online) Evolution plots of 1-lump solution for Eq.(1.1) by choosing suitable parameters \( a = 1, b = 1, \alpha = 1, \beta = 1 \). (a) \( t = -10 \). (b) \( t = 0 \). (c) \( t = 10 \).

\[
B_{ij} = \frac{12\alpha}{\beta(p_i - p_j)^2}, \quad i < j, j = 2, 3, 4. \quad (3.12)
\]

Putting \( p_1 = p_R + p_Ii, p_2 = \lambda_R + \lambda_Ii \), we have \( p_3 = p_1^* = p_R - p_Ii, p_4 = p_2^* = \lambda_R - \lambda_Ii. \) It is necessary to illustrate that the parameters \( p_R, p_I, \lambda_R, \lambda_I \) are all arbitrary real constants. Noted that the expression \( f_4 \) is a positive function consisted of quartic and quadratic perfect square functions. Next, substituting Eq.(3.11) into \( u = 2(\ln f_4)_{xx} \), the obtained result is 2-lump solution of Eq.(1.1).

Similarly, Eq.(3.4) can be transformed into \( f_6 \) by choosing \( n = 3, N = 6 \). It is not difficult to find that the expression \( f_6 \) has 76 terms according to Eq.(3.4). Substituting \( f_6 \) into the transformation \( u = 2(\ln f_6)_{xx} \), and putting \( p_4 = p_1^* = p_R - p_Ii, p_5 = p_2^* = h_R - h_Ii, p_6 = p_3^* = q_R - q_Ii \), we get a nonsingular rational solution which is called 3-lump solution of Eq.(1.1).

In what follows, the evolution plots of 2-lump solution and 3-lump solution of Eq.(1.1) are drawn by taking suitable parameters in appropriate time, respectively. Figure 3 displays the propagation feature of 2-lump solution at time \( t = -10, t = 0, t = 10 \). Additionally, we can see the propagation characteristic of 3-lump solution from Figure 4.

Figure 3. (Color online) Evolution plots of 2-lump solution for Eq.(1.1) by choosing suitable parameters \( p_R = 1, p_I = 1, \lambda_R = 1.4, \lambda_I = 0.5, \alpha = -1, \beta = 1 \). (a) \( t = -10 \). (b) \( t = 0 \). (c) \( t = 10 \).
4. Semi-rational solutions

Now, we are in a position to investigate the semi-rational solutions of Eq.(1.1). By taking a long wave limit for the partial exponential functions in Eq.(2.4), a combination of polynomial and exponential functions can be derived, which also be called as semi-rational solutions or hybrid solutions. In order to illustrate the solution systematically, we will consider the following two types of hybrid solutions.

4.1. Hybrid of lump solution and soliton solution

In this subsection, we first consider the situation of $N = 3$. Let

$$N = 3, \quad \xi_1^{(0)} = \xi_2^{(0)} = i\pi,$$

and take $k_1, k_2 \to 0$ in Eq.(2.4). One can find that

$$f = (\eta_1 \eta_2 + B_{12}) + (\eta_1 \eta_2 + B_{12} + B_{13} \eta_2 + B_{23} \eta_1 + B_{12} B_{23}) e^{\xi_3},$$

with

$$B_{s3} = -\frac{12\alpha k_3}{3\alpha k_3^2 - \beta(p_s - p_3)^2}, \quad s = 1, 2,$$

where $\eta_1, \eta_2, B_{12}$ are given by Eq.(3.5), and $\xi_3$ is defined as Eq.(2.5). Then, we let $p_2 = p_1^* = a - bi$, where $a, b, k_3, p_3, \alpha$ and $\beta$ are all arbitrary real constants. The corresponding hybrid solution $u$ defined by Eq.(2.2) with Eq.(4.2) is derived.

In order to better analysis dynamical behavior of hybrid of 1-lump solution and 1-soliton solution, we give its three dimensional plots in different time. As shown in Figure 5, it is easily to see that the lump moves and passes the soliton and in the interaction domain of the two waveforms the amplitude increases considerably.
4.2. Hybrid of lump solution and breather wave solution

Higher-order semi-rational solutions consisting of lump solution and breather wave solution can also be generated in a similar way. For instance, we set

\[ N = 4, \quad p_2 = p_1^*, \quad p_4 = p_3^*, \quad k_4 = k_3^*, \quad \xi_1^{(0)} = \xi_2^{(0)} = i\pi, \]  

and take \( k_1, k_2 \to 0 \) in Eq. (2.4), one can derive

\[ f = e^{A_{34}}(B_{13}B_{23} + B_{13}B_{24} + B_{13}\eta_2 + B_{14}B_{23} + B_{14}B_{24} + B_{14}\eta_1 + B_{23}\eta_1 + B_{24}\eta_1 + \eta_1\eta_2 + B_{12})e^{\xi_1^*\xi_4^* + (B_{13}B_{23} + B_{13}\eta_2 + B_{23}\eta_1 + \eta_1\eta_2 + B_{12})e^{\xi_4}} + (B_{14}B_{24} + B_{14}\eta_2 + B_{24}\eta_1 + \eta_1\eta_2 + B_{12})e^{\xi_4^* + \eta_1\eta_2 + B_{12}}, \]

with

\[ B_{sj} = \frac{12\alpha k_j}{3\alpha k_j^2 - \beta(p_s + p_j)^2}, \quad s = 1, 2, j = 3, 4, \]

where \( \eta_1, \eta_2, B_{12} \) are given by Eq. (3.5), and \( \xi_3, \xi_4, e^{A_{34}} \) are defined as Eq. (2.5). By the substitution of Eq. (4.5) into Eq. (2.2), the corresponding hybrid solution between lump solution and breather wave solution is deduced.

In order to better illustrate the hybrid of lump solution and breather wave solution, we give a special example with the free parameters selection being taken as follows

\[ \alpha = -1, \quad \beta = 1, \quad p_1 = 0.5 + i, \quad p_2 = 0.5 - i, \quad p_3 = 1 + 0.5i, \]
\[ p_4 = 1 - 0.5i, \quad k_3 = 1.5i, \quad k_4 = -1.5i, \quad \xi_3^{(0)} = \xi_4^{(0)} = 4\pi. \]

To observe the dynamic characteristics of the hybrid between lump solution and breather wave solution more intuitively, the corresponding figure is revealed in Figure 6. It is obvious to see that the hybrid solution (2.2) with (4.5) is a mixture of a lump solution and a breather wave solution.


5. Conclusions and discussions

In this paper, we have researched a (2+1)-dimensional generalized Korteweg-de Vries equation. Its $N$-soliton solutions have been constructed by employing the Hirota’s bilinear method. We further derived the breather wave solutions of the equation based on the obtained $N$-soliton solutions. Moreover, $M$-lump solutions and semi-rational solutions have also been established in detail by taking a long wave limit. Most importantly, the figures of breather wave solution, 1-lump solution, 2-lump solution, 3-lump solution and two types of hybrid solutions have been presented in Figures 1-6 in order to better understand their dynamical behavior characteristics.

The paper shows an effect and powerful method to seek exact solutions of N-LEEs, which is worthy of further exploration to other models in mathematical physics and engineering. Finally, we hope that our results provided in this work are helpful to understand the breather wave solutions, the lump solutions and hybrid solutions for more models.

Acknowledgements

The authors would like to thank the editor and the referees for their valuable comments and suggestions.

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