SCHRÖDINGER OPERATORS WITH SINGULAR GORDON POTENTIALS

R. O. HRYNIV AND YA. V. MYKYTYUK

Abstract. Singular Gordon potentials are defined to be distributions from the space $W_{2,\text{unif}}^{-1}(\mathbb{R})$ that are sufficiently fast approximated by periodic ones. We prove that Schrödinger operators with singular Gordon potentials have no point spectrum and show that a rich class of quasiperiodic distributions consists of singular Gordon potentials.

1. Introduction

In the Hilbert space $L_2(\mathbb{R})$, we consider a Schrödinger operator

$$ S = -\frac{d^2}{dt^2} + q $$

with potential $q$ that is a real-valued distribution from the space $W_{2,\text{unif}}^{-1}(\mathbb{R})$. The exact definition of this space is given in Section 2 below, and at this point we remark that any $q \in W_{2,\text{unif}}^{-1}(\mathbb{R})$ can be represented (not uniquely) in the form $q = \sigma' + \tau$, where $\sigma$ and $\tau$ are real-valued functions from $L_{2,\text{unif}}(\mathbb{R})$ and $L_{1,\text{unif}}(\mathbb{R})$, respectively, i.e.,

$$ \|\sigma\|_{2,\text{unif}} := \sup_{t \in \mathbb{R}} \int_t^{t+1} |\sigma(s)|^2 ds < \infty, $$

$$ \|\tau\|_{1,\text{unif}} := \sup_{t \in \mathbb{R}} \int_t^{t+1} |\tau(s)| ds < \infty, $$

and the derivative is understood in the sense of distributions. For singular potentials of this type the domain and the action of the operator $S$ are given by

$$ \mathcal{D}(S) = \{ f \in W_2^1(\mathbb{R}) \mid f^{[1]} := f' - \sigma f \in W_1^{1,\text{loc}}(\mathbb{R}), \ l(f) \in L_2(\mathbb{R}) \} $$

and

$$ Sf = l(f), $$

where

$$ l(f) := -(f' - \sigma f)' - \sigma f' + \tau f. $$

It is easily seen that $l(f) = -f'' + qf$ in the sense of distributions, which implies, firstly, that the operator $S$ so defined does not depend on the particular choice of $\sigma \in L_{2,\text{unif}}(\mathbb{R})$ and $\tau \in L_{1,\text{unif}}(\mathbb{R})$ in the decomposition $q = \sigma' + \tau$ and, secondly, that for regular potentials $q \in L_{1,\text{loc}}(\mathbb{R})$ the above definition coincides with the classical one.

The above regularisation by quasi-derivative procedure was first suggested in [1] for the potential $1/x$ on $[-1,1]$ and then systematically developed in [2] for a general class of singular (i.e., not necessarily locally integrable) potentials from $W_{2,\text{loc}}^{-1}(\mathbb{R})$. We observe that this class includes, among others, the classical examples of Dirac and

\textit{Date:} November 20, 2018.

1991 Mathematics Subject Classification. Primary 34L05; Secondary 34L40, 47A75.

\textit{Key words and phrases.} Schrödinger operators, singular potentials, eigenvalues.

\textit{†}The work was partially supported by Ukrainian State Foundation for Basic Research.
Coulomb potentials that are used to model zero- and short-range interactions in quantum mechanics. On the other hand, this approach is not applicable to locally more singular interactions like $\delta'$ or $1/|x|^{\gamma}$ with $\gamma \geq 3/2$; see, e.g., [3] and the references therein and in [2, 4] for some of the related literature on Schrödinger operators with singular potentials.

Potentials $q$ from the space $W_{2,\text{unif}}^{-1}(\mathbb{R})$ were considered in detail in [4]. In particular, it was shown there that the corresponding operator $S$ is selfadjoint, bounded below, and coincides with the form-sum of the operator $-\frac{d^2}{dx^2}$ and the multiplication operator by $q$. In the present paper we shall be interested in spectral properties of the operator $S$; more precisely, we would like to find conditions on the potential $q \in W_{2,\text{unif}}^{-1}(\mathbb{R})$ under which the operator $S$ does not have point spectrum.

It is a classical result, for instance, that the spectrum of $S$ is purely absolutely continuous if the potential $q$ belongs to $L_{1,\text{loc}}(\mathbb{R})$ and is periodic. Gordon [5] introduced a class of bounded below $L_{2,\text{loc}}(\mathbb{R})$-potentials that are well enough approximated by periodic ones and for which the corresponding Schrödinger operators do not possess eigenvalues. Damanik and Stolz [6] enlarged recently this class to $L_{1,\text{loc}}(\mathbb{R})$-valued functions. More exactly, they defined $q$ to be a generalized Gordon potential if $q \in L_{1,\text{unif}}(\mathbb{R})$ and there exist $T_m$-periodic functions $q_m \in L_{1,\text{loc}}(\mathbb{R})$ such that $T_m \to \infty$ and for an arbitrary $C > 0$ it holds

$$\lim_{m \to \infty} \exp(C T_m) \int_{-T_m}^{T_m} |q(t) - q_m(t)| dt = 0.$$  

It is proved in [6] that the Schrödinger operators with generalized Gordon potentials have empty point spectrum. In particular, the authors studied potentials of one-dimensional quasicrystal theory [3] of the form

$$q(t) = q_1(t) + q_2(\alpha t + \theta), \quad (1.2)$$

where the functions $q_1, q_2 \in L_{1,\text{loc}}(\mathbb{R})$ are 1-periodic and $\alpha, \theta \in [0, 1)$, and showed that (1.2) is a generalized Gordon potential if $\alpha$ is a Liouville number [7] and $q_2$ is either (i) Hölder continuous or (ii) a step function or (iii) $|x|^{-\gamma}$ on $[-1/2, 1/2]$ with $\gamma \in (0, 1)$ or (iv) a linear combination of such functions.

The class of generalized Gordon potentials, even though being large enough, does not contain, for example, potentials modelling $\delta$-interactions that arise in one-dimensional crystal and quasicrystal theory (cf. Kronig-Penney models and their various generalizations in [3, Ch.III.2], [8], [9]). The main aim of the present article is to describe singular potentials in $W_{2,\text{unif}}^{-1}(\mathbb{R})$ (with say $\delta$-like singularities) for which the above-defined operator $S$ has empty point spectrum. As it is shown in [4], so are all periodic potentials in $W_{2,\text{unif}}^{-1}(\mathbb{R})$; here we introduce singular analogues of Gordon potentials that possess the property mentioned.

Namely, we say that $q = \sigma' + \tau \in W_{2,\text{unif}}^{-1}(\mathbb{R})$ with $\sigma \in L_{2,\text{unif}}(\mathbb{R})$ and $\tau \in L_{1,\text{unif}}(\mathbb{R})$ is a singular Gordon potential if there exist $T_m$-periodic functions $\sigma_m \in L_{2,\text{unif}}(\mathbb{R})$ and $\tau_m \in L_{1,\text{unif}}(\mathbb{R})$ such that $T_m \to \infty$ and

$$\lim_{m \to \infty} \exp(C T_m) \left\{ \|\sigma - \sigma_m\|_{L_2[-T_m, 2T_m]} + \|\tau - \tau_m\|_{L_1[-T_m, 2T_m]} \right\} = 0$$

for every $C < \infty$. (In fact, Theorem 1.1 below remains true if we only require this relation to hold for all $C < C_q$, where $C_q$ is some constant dependent on $q$; see Remark 3.3.) Observe that every generalized Gordon potential is evidently a singular
Gordon one and that \( q = \sigma' + \tau \) is a singular Gordon potential if and only if
\[
\liminf_{T \to \infty} \frac{1}{T} \log \left\{ \| \sigma - \sigma(T) \|_{L_2[-T;T]} + \| \tau - \tau(T) \|_{L_1[-T;T]} \right\} = -\infty;
\]
here \( \sigma(T)(t) := \sigma(t + T) \) and \( \tau(T)(t) := \tau(t + T) \).

The main results of the present paper are as follows.

**Theorem 1.1.** Suppose that \( q \) is a singular Gordon potential. Then the operator \( S \) in (1.1) does not have any eigenvalues.

A rich class of singular Gordon potentials (in addition to those of the form (1.2) pointed out in the work [6]) is formed by functions
\[
q(t) = \sigma'_1(t) + \sigma'_2(\alpha t + \theta) + \tau_1(t) + \tau_2(\alpha t + \theta),
\]
(1.3)
where \( \sigma_1, \sigma_2 \in L_{2,\text{loc}}(\mathbb{R}) \) and \( \tau_1, \tau_2 \in L_{1,\text{loc}}(\mathbb{R}) \) are 1-periodic and \( \alpha, \theta \in [0, 1) \). Namely, we prove the following statement.

**Theorem 1.2.** Under the above assumptions the function \( q \) of (1.3) is a singular Gordon potential if \( \alpha \) is a Liouville number and \( \sigma_2 \) belongs to the space \( W^{s}_{2,\text{loc}}(\mathbb{R}) \) with some \( s > 0 \).

In particular, Theorem 1.2 shows that the requirement of [6] that the function \( q_2 \) of (1.2) be of the above form (i)–(iv) for \( S \) to have empty point spectrum is superfluous; in addition, it may have power-like singularities \( |x|^{-\gamma} \) with \( \gamma \in (0, 3/2) \).

Absence of the point spectrum for the operator \( S \) is proved by the scheme analogous to the one used in [5, 6]. Since \( q = \sigma' + \tau \) is well approximated by periodic functions \( q_m = \sigma'_m + \tau_m \), a Gronwall-type inequality in conjunction with some growth estimate gives proximity of solutions of the equations
\[
-(u' - \sigma u)' - \sigma u' + \tau u = \lambda u, \tag{1.4}
\]
\[
-(u'_m - \sigma_m u_m)' - \sigma_m u_m + \tau_m u_m = \lambda u_m, \tag{1.5}
\]
satisfying the same initial data. If \( u \) is an eigenfunction of the operator \( S \), then we prove that
\[
|u(t)|^2 + |u^{[1]}(t)|^2 \to 0 \quad \text{as} \quad |t| \to \infty; \tag{1.6}
\]
on the other hand, (1.6) does not hold for \( u_m \) by the so-called lemma on three periods. This contradicts the closeness of \( u \) and \( u_m \); therefore \( u \) cannot be an eigenfunction and \( \lambda \) cannot be an eigenvalue of \( S \).

The paper is organized as follows. Some auxiliary results (a structure theorem for the space \( W^{-1}_{2,\text{anif}}(\mathbb{R}) \), relation (1.6) for eigenfunctions, the lemma on three periods and some estimates for solutions of equation (1.4)) are established in Section 2. The Gronwall-type estimates for closeness of solutions \( u \) and \( u_m \) of equations (1.4) and (1.5) and the proof of Theorem 1.1 are given in Section 3. Finally, Theorem 1.2 is proved in Section 4.

2. Preliminaries and some auxiliary results

We collect in this section some auxiliary results to be exploited later on. In the following, \( W^s_2(\mathbb{R}), s \in \mathbb{R} \), will denote the Sobolev spaces \([10], \| \cdot \| \) without any subscript will always stand for the \( L_2(\mathbb{R}) \)-norm and \( f^{[1]} \) for the quasi-derivative \( f' - \sigma f \) of a function \( f \).
2.1. The structure of the space $W_{2,\text{unif}}^{-1}(\mathbb{R})$. We recall that $W_{2,\text{unif}}^{-1}(\mathbb{R})$ is the dual space to the Sobolev space $W_2^1(\mathbb{R})$, i.e., it consists of those distributions $[11]$, which define continuous functionals on $W_2^1(\mathbb{R})$. With $\langle \cdot, \cdot \rangle$ denoting the duality, we have for $f \in W_{2,\text{unif}}^{-1}(\mathbb{R})$

$$\|f\|_{W_{2,\text{unif}}^{-1}(\mathbb{R})} := \sup_{0 \neq \psi \in W_2^1(\mathbb{R})} \frac{|\langle \psi, f \rangle|}{\|\psi\|_{W_2^1(\mathbb{R})}}.$$ 

The local uniform analogue of this space is constructed as follows. Put

$$\phi(t) := \begin{cases} 2(t + 1)^2 & \text{if } t \in [-1, -1/2), \\ 1 - 2t^2 & \text{if } t \in [-1/2, 1/2), \\ 2(t - 1)^2 & \text{if } t \in [1/2, 1], \\ 0 & \text{otherwise,} \end{cases} \quad (2.1)$$

and $\phi_n(t) := \phi(t - n)$ for $n \in \mathbb{Z}$. We say that $f$ belongs to $W_{2,\text{unif}}^{-1}(\mathbb{R})$ if $f \phi_n$ is in $W_{2,\text{unif}}^{-1}(\mathbb{R})$ for all $n \in \mathbb{Z}$ and

$$\|f\|_{W_{2,\text{unif}}^{-1}(\mathbb{R})} := \sup_{n \in \mathbb{Z}} \|f \phi_n\|_{W_{2,\text{unif}}^{-1}(\mathbb{R})} < \infty.$$ 

The space $W_{2,\text{unif}}^{-1}(\mathbb{R})$ has the following structure (see details in [4]).

**Theorem 2.1 ([4]).** For any $f \in W_{2,\text{unif}}^{-1}(\mathbb{R})$ there exist functions $\sigma \in L_{2,\text{unif}}(\mathbb{R})$ and $\tau \in L_{1,\text{unif}}(\mathbb{R})$ such that $f = \sigma' + \tau$ and

$$C^{-1}(\|\sigma\|_{2,\text{unif}} + \|\tau\|_{1,\text{unif}}) \leq \|f\|_{W_{2,\text{unif}}^{-1}(\mathbb{R})} \leq C(\|\sigma\|_{2,\text{unif}} + \|\tau\|_{1,\text{unif}}) \quad (2.2)$$

with some constant $C$ independent of $f$. Moreover, the function $\tau$ can be chosen uniformly bounded.

We say that a distribution $f$ is $T$-periodic if $\langle f, \psi(t) \rangle = \langle f, \psi(t + T) \rangle$ for any test function $\psi$. It follows from the proof of Theorem 2.1 that for a $T$-periodic potential $f \in W_{2,\text{unif}}^{-1}(\mathbb{R})$ the functions $\sigma$ and $\tau$ may be taken $T$-periodic and constant, respectively (see [4, Remark 2.3]).

2.2. Inequalities, quasi-derivatives etc. The aim of the next three lemmata is to prove that any eigenfunction vanishes at infinity together with its quasi-derivative. For Schrödinger operators with regular potentials this result is usually derived from Harnack’s inequality (see, e.g., [12]).

**Lemma 2.2.** Suppose that $f \in W_2^1(\mathbb{R})$; then

$$\max_{t \in \mathbb{R}} |f(t)| \leq \|f\|_{W_2^1(\mathbb{R})}.$$ 

**Proof.** Since $f\overline{f} \in L_1(\mathbb{R})$, the relation

$$|f(t)|^2 - |f(s)|^2 = 2 \int_s^t \text{Re } f\overline{f}$$

implies that $\lim_{s \to -\infty} |f(s)| = 0$. Therefore

$$|f(t)|^2 = 2 \int_{-\infty}^t \text{Re } f\overline{f} \leq \int_{-\infty}^t (|f'|^2 + |f|^2) \leq \|f\|^2_{W_2^1(\mathbb{R})}$$

and the proof is complete. \qed
Lemma 2.3. The quasi-derivative \( f^{[1]} := f' - \sigma f \) of any function \( f \in \mathcal{D}(S) \) belongs to \( L_2(\mathbb{R}) \).

Proof. It suffices to show that \( \sigma f \in L_2(\mathbb{R}) \) for any \( \sigma \in L_{2, \text{unif}}(\mathbb{R}) \) and any \( f \in W_2^1(\mathbb{R}) \). Let \( \phi \) be defined through (2.1); then \( 0 \leq \phi \leq 1, |\phi'| \leq 2, \) and \( \sum_{n=-\infty}^{\infty} \phi(t-n) \equiv 1 \). Denote \( \phi_n(t) := \phi(t-n) \) and \( f_n(t) := f(t)\phi_n(t) \); then \( \sum_{n \in \mathbb{Z}} f_n = f \),

\[
\|f_n\|_{W^2_2(\mathbb{R})}^2 \leq 2\|f'\phi_n\|^2 + 2\|f\phi_n'\|^2 + \|f\phi_n\|^2 \leq 9\|f\|_{W^2_2[-1,n+1]}^2,
\]

and hence

\[
\sum_{n \in \mathbb{Z}} \|f_n\|_{W^2_2(\mathbb{R})}^2 \leq 18\|f\|_{W^2_2(\mathbb{R})}^2 < \infty.
\]

Recalling now Lemma 2.2, we derive the inequalities

\[
\|\sigma f\|^2 \leq 2 \left( \sum_{n \in \mathbb{Z}} \|\sigma f_{2n}\|^2 \right) + 2 \left( \sum_{n \in \mathbb{Z}} \|\sigma f_{2n-1}\|^2 \right) \leq 2 \sum_{n \in \mathbb{Z}} \|\sigma f_n\|^2 \leq 2 \sum_{n \in \mathbb{Z}} \|\sigma\|_{L_2(n-1,n+1)}^2 \max |f_n(t)|^2 \leq 2 \sum_{n \in \mathbb{Z}} \|\sigma\|_{L_2(n-1,n+1)}^2 \|f_n\|_{W^2_2(\mathbb{R})}^2 \leq 4 \|\sigma\|_{L_{2, \text{unif}}}^2 \sum_{n \in \mathbb{Z}} \|f_n\|_{W^2_2(\mathbb{R})}^2 < \infty,
\]

whence \( \sigma f \in L_2(\mathbb{R}) \). The lemma is proved. \( \square \)

Lemma 2.4. Suppose that \( u \) is an eigenfunction of the operator \( S \) corresponding to an eigenvalue \( \lambda \). Then

\[
|u(t)|^2 + |u^{[1]}(t)|^2 \to 0 \quad \text{as} \quad |t| \to \infty.
\]

Proof. Since \( u \in W_2^1(\mathbb{R}) \), we have \( \lim_{|t| \to \infty} u(t) = 0 \) and

\[
\int_t^{t+1} |\sigma u'| \leq \left( \int_t^{t+1} |\sigma|^2 \right)^{1/2} \left( \int_t^{t+1} |u'|^2 \right)^{1/2} \leq \|\sigma\|_{L_{2, \text{unif}}} \left( \int_t^{t+1} |u'|^2 \right)^{1/2} \to 0
\]

as \( |t| \to \infty \). Hence

\[
\sup_{\alpha, \beta \in [t,t+1]} |u^{[1]}(\beta) - u^{[1]}(\alpha)| \leq \int_t^{t+1} |(u^{[1]})'|
\]

\[
\leq \int_t^{t+1} |\sigma u'| + \int_t^{t+1} |\tau u| + \int_t^{t+1} |\lambda u| \to 0
\]

as \( |t| \to \infty \), which implies existence of the limits \( u_\pm := \lim_{t \to \pm \infty} u^{[1]}(t) \). By Lemma 2.3 \( u^{[1]} \in L_2(\mathbb{R}) \), whence \( u_\pm = 0 \) and the proof is complete. \( \square \)

2.3. Some results on first order differential systems. It is convenient to rewrite the eigenvalue equation (1.4) as a first order system

\[
\frac{d}{dt} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \sigma & 1 \\ -\sigma^2 + \tau - \lambda & -\sigma \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.
\]

(2.3)

A function \( u \) is easily seen to be a solution of (1.4) if and only if the vector-function \( U := (u, u^{[1]})^T \) solves (2.3). Lemma 2.4 states that if \( u \) is in addition an eigenfunction of \( S \), then \( |U(t)| \to 0 \) as \( |t| \to \infty \), where for a vector \( X \in \mathbb{C}^2 \) we denote by \( |X| \) its Euclidean norm. The following statement (known as the “lemma on three periods”, cf. [5, 6]) shows that for periodic potentials \( q = \sigma' + \tau \in W_2^{-1, \text{unif}}(\mathbb{R}) \) no nontrivial solution of (2.3)
vanishes at infinity (and hence by Lemma 2.4 the corresponding Schrödinger operators have empty point spectrum; in fact, the spectrum is then absolutely continuous [4]).

**Lemma 2.5.** Suppose that \( A \) is a \( T \)-periodic \( 2 \times 2 \) matrix with locally integrable entries and zero trace and let \( U \) be any solution of the equation \( \dot{U} = AU \). Then
\[
\max\{|U(-T)|, |U(T)|, |U(2T)|\} \geq \frac{1}{2}|U(0)|.
\]

**Proof.** Denote by \( M(t) \) the monodromy matrix of the equation \( \dot{U} = AU \); then we conclude by the Liouville and Cayley-Hamilton theorems that \( \det M \equiv 1 \) and
\[ M^2 + (\tr M)M + I = 0. \]
Since \( (MU)(t) = U(t + T) \), this yields
\[ U(t + 2T) + (\tr M)U(t + T) + U(t) \equiv 0. \]
If \( |\tr M| \leq 1 \), we take \( t = 0 \) in the above equality to derive that
\[ \max\{|U(2T)|, |U(T)|\} \geq \frac{1}{2}|U(0)|, \]
while otherwise \( t = -T \) produces the estimate
\[ \max\{|U(T)|, |U(-T)|\} \geq \frac{1}{2}|U(0)|. \]
The lemma is proved. \( \square \)

Next we adopt the arguments from [13] to system (2.3) to derive some a priori estimates for its solutions.

**Lemma 2.6.** Let \( a \) and \( b \) be some locally summable functions and \( X(t) \) be an arbitrary nonzero solution of the equation
\[
\frac{dX}{dt} = \begin{pmatrix} a & 1 \\ b & -a \end{pmatrix} X.
\]
Then for any constant \( c \geq 1 \) and any \( t \in \mathbb{R} \) the following inequality holds:
\[
|X(t)| \leq c \exp\left\{ \frac{1}{2} \int_{\min\{0,t\}}^{\max\{0,t\}} \sqrt{4a(s)^2 + (c + b(s)/c)^2} \, ds \right\} |X(0)|. \tag{2.4}
\]

**Proof.** It suffices to consider the case \( t \geq 0 \). Denote
\[
A := \begin{pmatrix} a & 1 \\ b & -a \end{pmatrix}, \quad G := \begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix}, \quad F := GA + A^*G = \begin{pmatrix} 2ca & c + b/c \\ c + b/c & -2a/c \end{pmatrix}.
\]
Then \( \xi(t) := (GX(t), X(t)) \) satisfies the relation
\[
\dot{\xi}(t) = (GX(t), X(t)) + (GX(t), \dot{X}(t)) = (FX(t), X(t)).
\]
If \( \lambda^+ \) is larger of two eigenvalues of the pencil \( F - \lambda G \), then \( F \leq \lambda^+ G \), whence
\[
\xi(t) \leq \lambda^+(t)\xi(t) \text{ and } \log \xi(t) - \log \xi(0) \leq \int_0^t \lambda^+(s) \, ds.
\]
Taking the exponents of both sides, we arrive at the inequality
\[
|X(t)|^2 \leq c \xi(t) \leq c \exp\left\{ \int_0^t \lambda^+(s) \, ds \right\} \xi(0) \leq c^2 \exp\left\{ \int_0^t \lambda^+(s) \, ds \right\} |X(0)|.
\]
To calculate \( \lambda^+ \), we observe that it is larger of two eigenvalues of the matrix
\[
F_1 := G^{-1/2}FG^{-1/2} = \begin{pmatrix} 2a & 0 \\ c + b/c & -2a/c \end{pmatrix}.
\]
Since $F_1$ has a zero trace we have
\[ \lambda^+ = \sqrt{-\det F_1} = \sqrt{4a^2 + (c + b/c)^2}, \]
and the proof is complete. \( \square \)

For equation (2.3), Lemma 2.6 yields the following result.

**Lemma 2.7.** Suppose that the functions $\sigma$ and $\tau$ belong to $L_{2, loc}(\mathbb{R})$ and $L_{1, loc}(\mathbb{R})$ respectively. Then any solution $X(t)$ of equation (2.3) satisfies the inequality
\[ |X(t)| \leq C_1 \exp \left\{ \frac{1}{2} \int_{\min\{0,t\}}^{\max\{0,t\}} (2 - \varepsilon + 2\sqrt{(-\lambda)_+} + \sigma^2 + |\tau|) \, ds \right\} |X(0)|, \]
where $(-\lambda)_+ = \max\{-\lambda, 0\}$ and the constants $C_1 > 0$ and $\varepsilon > 0$ only depend on $\lambda$.

**Proof.** We shall show that for any $\lambda \in \mathbb{R}$ there exist constants $C_1 \geq 1$ and $\varepsilon > 0$ such that
\[ 4\sigma^2 + (C_1^2 - \sigma^2 + \tau - \lambda)^2 / C_1^2 < (2 - \varepsilon + 2\sqrt{(-\lambda)_+} + \sigma^2 + |\tau|)^2; \]
then the statement will follow from Lemma 2.6.

For $\lambda \geq 0$ we put $C_1 = \sqrt{\lambda + 1/4} + 1/2$; then $C_1^2 - \lambda = C_1$, whence
\[ 4\sigma^2 + (C_1^2 - \sigma^2 + \tau - \lambda)^2 / C_1^2 = 4\sigma^2 + (C_1 - \sigma^2 + \tau)^2 / C_1^2 \leq (\sigma^2 + |\tau|)^2 + 2(\sigma^2 + |\tau|)(2 - 1/C_1) + (2 - 1/C_1)^2 \]
\[ = (\sigma^2 + |\tau| + 2 - 1/C_1)^2. \]

For $\lambda \in (-1, 0)$ we take $C_1 = 1$; then
\[ 4\sigma^2 + (1 - \sigma^2 + \tau - \lambda)^2 = (\sigma^2 - \tau)^2 - 2(\sigma^2 - \tau)(1 - \lambda) + 4\sigma^2 + (1 - \lambda)^2 \]
\[ \leq (\sigma^2 + |\tau| + 1 - \lambda)^2. \]

Finally, for $\lambda = -\nu^2 \leq -1$ we put $C_1 = \nu = \sqrt{(-\lambda)_+}$; then
\[ 4\sigma^2 + (C_1^2 - \sigma^2 + \tau - \lambda)^2 / C_1^2 = 4\sigma^2 + (2\nu^2 - \sigma^2 + \tau)^2 / \nu^2 \]
\[ = (\sigma^2 - \tau)^2 / \nu^2 - 4(\sigma^2 - \tau) + 4\sigma^2 + 4\nu^2 \]
\[ \leq (\sigma^2 + |\tau| + 2\nu)^2. \]

The proof is complete. \( \square \)

**Corollary 2.8.** If the functions $\sigma$ and $\tau$ belong to $L_{2, unif}(\mathbb{R})$ and $L_{1, unif}(\mathbb{R})$ respectively, then any solution $X(t)$ of equation (2.3) satisfies the inequality
\[ |X(t)| \leq C_1 \exp \left\{ (|t| + 1)(1 - \varepsilon + \sqrt{(-\lambda)_+} + \frac{1}{2} \|\sigma\|_{2, unif}^2 + \frac{1}{2} \|\tau\|_{1, unif}) \right\} |X(0)| \]
with suitable $C_1 > 0$ and $\varepsilon > 0$ independent of $\sigma$, $\tau$, and $t$.

3. **Proof of Theorem 1.1**

Denote by $\gamma_0$ the lower bound of the operator $S$ and by $\gamma$ any number less than $\gamma_0$. We fix now any $\lambda > \gamma$ and shall prove that $\lambda$ is not an eigenvalue of $S$. The eigenvalue equation $l(u) = \lambda u$ can be recast as a first order system
\[ \frac{d}{dt} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \sigma & 1 \\ -\sigma^2 + \tau - \lambda & -\sigma \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \]
with \( u_1 = u \) and \( u_2 = u^{[1]} \). Set
\[
U(t) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ \gamma - \lambda & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} \sigma & 0 \\ -\sigma^2 + \tau - \gamma & -\sigma \end{pmatrix};
\]
then
\[
\frac{dU}{dt} = AU + BU.
\]
Let \( \tilde{U} \) be a solution of the equation
\[
\frac{d\tilde{U}}{dt} = A\tilde{U} + \tilde{B}\tilde{U}
\]
with the initial condition \( \tilde{U}(0) = U(0) \); here
\[
\tilde{U}(t) = \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix}, \quad \tilde{B}(t) = \begin{pmatrix} \tilde{\sigma} & 0 \\ -\tilde{\sigma}^2 + \tilde{\tau} - \gamma & -\tilde{\sigma} \end{pmatrix}.
\]
In the following \( |U(t)|, |A|, |B(t)| \) etc. will denote the Euclidean norms of the vector \( U(t) \) and matrices \( A \) and \( B(t) \) respectively.

**Lemma 3.1.** There exists a constant \( C_2 = C_2(\lambda) > 0 \) such that for all \( t \in \mathbb{R} \) the following inequality holds:
\[
|U(t) - \tilde{U}(t)| \leq C_2 \int_{\min(0,t)}^{\max(0,t)} |B(s) - \tilde{B}(s)||\tilde{U}(s)|\ ds \exp\left\{\int_{\min(0,t)}^{\max(0,t)} |B(s)|\ ds\right\}.
\]

**Proof.** To be definite, we consider the case \( t > 0 \). The function \( V := U - \tilde{U} \) solves the equation
\[
\frac{dV}{dt} = AV + BV + (B - \tilde{B})\tilde{U}
\]
and satisfies the initial condition \( V(0) = 0 \), whence
\[
V(t) = \int_0^t e^{(t-s)A}B(s)V(s)\ ds + \int_0^t e^{(t-s)A}(B(s) - \tilde{B}(s))\tilde{U}(s)\ ds.
\]
Since \( \lambda - \gamma > 0 \), the group \( e^{tA} \) is uniformly bounded by some constant \( C_3 \), whence
\[
|V(t)| \leq C_3 \int_0^t |B(s)||V(s)|\ ds + C_3 \int_0^t |B(s) - \tilde{B}(s)||\tilde{U}(s)|\ ds.
\]
The Gronwall inequality now yields
\[
|V(t)| \leq C_2 \int_0^t |B(s) - \tilde{B}(s)||\tilde{U}(s)|\ ds \exp\left\{\int_0^t |B(s)|\ ds\right\}
\]
with \( C_2 = C_3 e^{C_3} \), and the lemma is proved. \( \square \)

**Lemma 3.2.** For any \( a, b \in \mathbb{C} \),
\[
\left| \begin{pmatrix} a & 0 \\ b & -a \end{pmatrix} \right| \leq |b|/2 + (|b|^2/4 + |a|^2)^{1/2}.
\]
Proof. We have to maximize the expression
\[ \left| \begin{pmatrix} a & 0 \\ b & -a \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right|^2 = |az_1|^2 + |bz_1|^2 - 2 \Re ab \bar{z}_1 \bar{z}_2 + |az_2|^2 \]
over \((z_1, z_2)^T \in \mathbb{C}^2\) of unit length. Introducing \(\theta \in [0, \pi/2]\) via \(|z_1| = \cos \theta, |z_2| = \sin \theta\) and using the relations \(2|z_1z_2| = \sin 2\theta, |z_1|^2 = (1 + \cos 2\theta)/2\), we bound the above expression by
\[ |a|^2 + |b|^2(1 + \cos 2\theta)/2 + |a||b| \sin 2\theta \leq |a|^2 + |b|^2/2 + |b|(|b|^2/4 + |a|^2)^{1/2} \]
\[ = \left( |b|/2 + (|b|^2/4 + |a|^2)^{1/2} \right)^2, \]
and the proof is complete. \(\square\)

Proof of Theorem 1.1. Assume that the assumptions of the theorem hold and \(\lambda > \gamma\) is an eigenvalue of the operator \(S\) with the corresponding eigenfunction \(u\) normalized by the condition \(|u(0)|^2 + |u'|^2(0)|^2 = 1\). Fix a sequence \(\sigma_m = \sigma'_m + \tau_m\) of \(T_m\)-periodic potentials from the class \(W^{1}_{2, \text{uni}}(\mathbb{R})\) such that \(T_m \to \infty\) and
\[ \lim_{m \to \infty} \exp\left( C T_m \right) \left\{ \left( \int_{-T_m}^{2T_m} |\sigma - \sigma_m|^2 \right)^{1/2} + \int_{-T_m}^{2T_m} |\tau - \tau_m| \right\} = 0 \quad (3.2) \]
for any \(C < \infty\).

Put \(U = (u, u')^T\) and denote by \(U_m\) a solution of equation
\[ \frac{dU_m}{dt} = (A + B_m)U_m, \quad B_m(t) := \begin{pmatrix} \sigma_m & 0 \\ -\sigma_m' + \tau_m - \gamma & -\sigma_m \end{pmatrix}, \]
with the initial condition \(U_m(0) = U(0)\); then by Lemma 2.5
\[ \max\{|U_m(-T_m)|, |U_m(T_m)|, |U_m(2T_m)|\} \geq 1/2. \quad (3.3) \]
According to Corollary 2.8
\[ |U_m(t)| \leq C_1 \exp\left\{ (|t| + 1)(1 + |\gamma|^{1/2} + \frac{1}{2}\|\sigma_m\|^2_{2, \text{uni}} + \frac{1}{2}\|\tau_m\|_{1, \text{uni}}) \right\} \]
and therefore (3.1) gives
\[ |U(t) - U_m(t)| \leq C_1 C_2 \int_{\min\{0,t\}}^{\max\{0,t\}} |B(s) - B_m(s)| ds \exp\left\{ \int_{\min\{0,t\}}^{\max\{0,t\}} |B(s)| ds \right\} \times \]
\[ \times \exp\left\{ (|t| + 1)(1 + |\gamma|^{1/2} + \frac{1}{2}\|\sigma_m\|^2_{2, \text{uni}} + \frac{1}{2}\|\tau_m\|_{1, \text{uni}}) \right\}. \quad (3.4) \]
The integrals in (3.4) can be estimated by means of Lemma 3.2 as follows. First, we see that
\[ |B - B_m| \leq |\tau - \tau_m| + |\sigma - \sigma_m|(|\sigma| + |\sigma_m| + 1), \]
whence
\[ \int_{\min\{0,t\}}^{\max\{0,t\}} |B(s) - B_m(s)| ds \leq \int_{\min\{0,t\}}^{\max\{0,t\}} |\tau - \tau_m| \]
\[ + \left( \int_{\min\{0,t\}}^{\max\{0,t\}} (|\sigma|^2 + |\sigma_m|^2 + 1) \right)^{1/2} \left( \int_{\min\{0,t\}}^{\max\{0,t\}} |\sigma - \sigma_m|^2 \right)^{1/2}. \quad (3.5) \]
In the same manner we conclude that
\[ |B| \leq 1 + |\gamma| + |\sigma|^2 + |\tau|, \]
and using the relations 2|z_1z_2| = sin 2\theta, |z_1|^2 = (1 + cos 2\theta)/2, we bound the above expression by
\[ |a|^2 + |b|^2(1 + cos 2\theta)/2 + |a||b| \sin 2\theta \leq |a|^2 + |b|^2/2 + |b|(|b|^2/4 + |a|^2)^{1/2} \]
\[ = \left( |b|/2 + (|b|^2/4 + |a|^2)^{1/2} \right)^2, \]
which implies
\[
\int_{\min \{0,t\}}^{\max \{0,t\}} |B(s)| \, ds \leq (|t| + 1)(1 + |\gamma| + \|\sigma\|_{2,\text{uni}f}^2 + \|\tau\|_{1,\text{uni}f}). \tag{3.6}
\]

Since \(\|\sigma_m\|_{2,\text{uni}f}^2 \leq 2\|\sigma\|_{2,\text{uni}f}^2\) and \(\|\tau_m\|_{1,\text{uni}f} \leq 2\|\tau\|_{1,\text{uni}f}\) for all \(m\) large enough, combination of relations (3.4)–(3.6) yields the inequality
\[
|U(t) - U_m(t)| \leq C_3 \exp\left\{ |t|(2 + |\gamma|1/2 + |\gamma| + 2\|\sigma\|_{2,\text{uni}f}^2 + 2\|\tau\|_{1,\text{uni}f}) \right\} \times
\left\{ \left( \int_{\min \{0,t\}}^{\max \{0,t\}} |\sigma - \sigma_m|^2 \right)^{1/2} + \int_{\min \{0,t\}}^{\max \{0,t\}} |\tau - \tau_m| \right\}
\]
with some \(C_3\) independent of \(m\). Substituting now \(t = \pm T_m, 2T_m\) and recalling (3.2), we find that
\[
\lim_{m \to \infty} \max_{t=\pm T_m,2T_m} |U(t) - U_m(t)| = 0.
\]
By virtue of (3.3) this implies that \(U(t)\) does not tend to 0 as \(|t| \to \infty\). Therefore \(u\) is not an eigenfunction of \(S\) and hence the point spectrum of \(S\) is empty. The theorem is proved.

**Remark 3.3.** Observe that we only need (3.2) to hold for
\[
C \leq C_q := 4 + 2|\gamma|1/2 + 2|\gamma| + 4\|\sigma\|_{2,\text{uni}f}^2 + 4\|\tau\|_{1,\text{uni}f}.
\]
Recall [4] that \(\gamma_0 \geq -(a\|q\|_{W^{-1,2,\text{uni}f}(\mathbb{R})} + b)^4\) with certain \(a, b > 0\) independent of \(q\) and that \(\|\sigma\|_{2,\text{uni}f}^2 \leq 64\|q\|_{W^{-1,2,\text{uni}f}(\mathbb{R})}^2\) and \(\|\tau\|_{1,\text{uni}f} \leq 3\|q\|_{W^{-1,2,\text{uni}f}(\mathbb{R})}\); therefore for some \(a', b' > 0\) independent of \(q\) we have
\[
C_q \leq (a'\|q\|_{W^{-1,2,\text{uni}f}(\mathbb{R})} + b')^4.
\]

4. APPLICATION TO QUASIPERIODIC POTENTIALS

In this section, we shall establish Theorem 1.2 showing which of the functions (1.3) are singular Gordon potentials. We shall prove first some auxiliary results.

**Lemma 4.1.** Suppose that \(f \in W^1_2(a, b)\) with \(b - a \geq 1\); then for any \(c \in (a, b)\) and any \(\varepsilon \in (0, b - c)\) the following inequality holds:
\[
\int_a^c |f(t + \varepsilon) - f(t)|^2 \, dt \leq 7\varepsilon^2 \|f\|_{W^1_2(a, b)}^2.
\]

**Proof.** We put
\[
\psi(t) = \begin{cases} 
  f(t) & \text{if } t \in [a, b], \\
  f(2a - t) \frac{t + b - 2a}{b - a} & \text{if } t \in (2a - b, a), \\
  f(2b - t) \frac{2b - a - t}{b - a} & \text{if } t \in (b, 2b - a), \\
  0 & \text{otherwise.}
\end{cases}
\]

Then \(\psi \in W^1_2(\mathbb{R})\) and
\[
\|\psi\|_{W^1_2(\mathbb{R})}^2 = \|\psi\|_{L^2(\mathbb{R})}^2 + \|\psi'\|_{L^2(\mathbb{R})}^2 \leq 7 \left( \|f\|_{L^2(a, b)}^2 + \|f'\|_{L^2(a, b)}^2 \right) = 7\|f\|_{W^1_2(a, b)}^2.
\]
Denote by \( \hat{\psi} \) the Fourier transform of \( \psi \). Then using Plancherel’s theorem and recalling the equivalent definition of the norm in \( W^1_2(\mathbb{R}) \), we get
\[
\int_a^c |f(t + \varepsilon) - f(t)|^2 \, dt \leq \int_{\mathbb{R}} |\psi(t + \varepsilon) - \psi(t)|^2 \, dt = \int_{\mathbb{R}} |\hat{\psi}(u)|^2 |e^{i\varepsilon u} - 1|^2 \, du \\
\leq \varepsilon^2 \max_{u \in \mathbb{R}} \frac{|e^{i\varepsilon u} - 1|^2}{|\varepsilon u|^2} \int_{\mathbb{R}} (1 + u^2)|\hat{\psi}(u)|^2 \, du \leq \varepsilon^2 \|\psi\|_{W^1_2(\mathbb{R})}^2,
\]
and the lemma follows. \( \square \)

**Lemma 4.2.** Suppose that \( f \in W^1_2(\mathbb{R}) \) and \( a > 1 \); then for any \( b \geq c \)
\[
\int_1^b |f(t) - f(at)|^2 \, dt \leq 7ab^2(a - 1)^2 \|f\|_{W^1_2(\mathbb{R})}^2.
\]

**Proof.** Put \( c = \log ab \) and \( g(t) := f(e^t) \); then \( g \in W^1_2(0, c) \) and by Lemma 4.1 we have
\[
\int_1^b |f(t) - f(at)|^2 \, dt = \int_0^{\log b} |g(u) - g(u + \log a)|^2 e^u \, du \\
\leq b \int_0^{\log b} |g(u) - g(u + \log a)|^2 \, du \leq 7b(\log a)^2 \|g\|_{W^1_2(0, c)}^2.
\]
Observe that
\[
\int_0^c |g(u)|^2 \, du = \int_0^{\log ab} |f(e^u)|^2 \, du = \int_1^b |f(t)|^2 \frac{dt}{t} \leq \|f\|_{L^2(\mathbb{R})}^2
\]
and
\[
\int_0^c |g'(u)|^2 \, du = \int_0^{\log ab} |f'(e^u)|^2 e^{2u} \, du = \int_1^b |f'(t)|^2 t \, dt \leq ab \|f'\|_{L^2(\mathbb{R})}^2.
\]
Therefore
\[
\|g\|_{W^1_2(0, c)}^2 \leq ab \|f\|_{W^1_2(\mathbb{R})}^2
\]
and the result follows. \( \square \)

**Lemma 4.3.** For any \( s \in [0, 1] \) there exists a constant \( C_s > 0 \) such that the inequality
\[
\int_{-T}^{2T} |f(\alpha t + \theta) - f(\beta t + \theta)|^2 \, dt \leq C_s T^{2s} \alpha^{-1}(\beta - \alpha)^2s \|f\|_{W^s_2(\mathbb{R})}^2
\]
holds for all \( f \in W^s_2(\mathbb{R}) \), all \( \alpha, \beta > 0 \), \( \alpha \leq \beta \leq 2\alpha \), all \( \theta \in \mathbb{R} \), and all \( T \geq 1/\alpha \).

**Proof.** Denote by \( A_s \) an operator from \( W^s_2(\mathbb{R}) \) into \( L^2(-T, 2T) \) defined by
\[
(A_s f)(t) := f(\alpha t + \theta) - f(\beta t + \theta);
\]
then the statement of the lemma asserts that
\[
\|A_s\|^2 \leq C_s T^{2s} \alpha^{-1}(\beta - \alpha)^{2s} \tag{4.1}
\]
with some constant \( C_s > 0 \) independent of \( \alpha, \beta, \theta, \) and \( T \). We shall prove inequality (4.1) for \( s = 0 \) and \( s = 1 \) and then interpolate (see [14, Ch.1] for details) to cover all \( s \) in between.

It is easily seen that
\[
\|A_0 f\|_{L^2(-T, 2T)}^2 \leq 2 \frac{\alpha + \beta}{\alpha \beta} \|f(\theta)\|_{L^2(\mathbb{R})}^2 \leq 4\alpha^{-1} \|f(t)\|_{L^2(\mathbb{R})}^2,
\]
whence (4.1) holds for \( s = 0 \) with \( C_0 = 4 \).
Moreover, respectively. Combining the above inequalities we finally get
These two summands are estimated by Lemmata 4.1 and 4.2 as
and (4.1) holds for

Theorem 4.4. Suppose now that
We denote by
We recall that an irrational number

where

and (4.1) holds for

respectively. Combining the above inequalities we finally get

and (4.1) holds for

We denote by
the set of all functions

for any

where

and put

Theorem 4.4. Suppose that
are such that

for any

We have

and the proof is complete.

Theorem 4.4. Suppose that
are such that

for any

We have

and the proof is complete.

Theorem 4.4. Suppose that
are such that

for any

We have

and the proof is complete.

Theorem 4.4. Suppose that
are such that

for any

We have

and the proof is complete.

Theorem 4.4. Suppose that
are such that

for any

We have

and the proof is complete.
with a suitable constant $C$. Combining the above results, we can now prove Theorem 1.2.

**Proof of Theorem 1.2.** Suppose that 

$$q(t) = \sigma'_1(t) + \sigma'_2(\alpha t + \theta) + \tau_1(t) + \tau_2(\alpha t + \theta),$$

where $\sigma_1, \sigma_2 \in L_{2,\text{loc}}(\mathbb{R})$ and $\tau_1, \tau_2 \in L_{1,\text{loc}}(\mathbb{R})$ are 1-periodic, $\alpha, \theta \in [0, 1)$ and $\alpha$ is a Liouville number.

First of all we represent the function $\tau_2$ as $\sigma'_3 + c$ where $c \equiv \int_0^1 \tau_2$ and $\sigma_3$ is the 1-periodic primitive of $\tau_2 - c$. Then $\sigma_3 \in W^{1,\text{loc}}(\mathbb{R})$ and hence $\sigma_3 \in W^{s,\text{loc}}(\mathbb{R})$ with any $s < 1/2$ by the Sobolev embedding theorem [10]. Therefore

$$q(t) = \sigma'_1(t) + \tilde{\sigma}'_2(\alpha t + \theta) + \tilde{\tau}_1(t),$$

where $\tilde{\sigma}_2 = \sigma_2 + \sigma_3$ and $\tilde{\tau}_1 = \tau_1 + c$. We approximate $q$ by a sequence of $T_m$-periodic potentials

$$q_m(t) := \sigma'_1(t) + \tilde{\sigma}'_2(\alpha_m t + \theta) + \tilde{\tau}_1(t)$$

where $\alpha_m = R_m/T_m$ is the $m$-th approximate of $\alpha$ satisfying (4.2). Combination of Theorem 4.4 and relation (4.2) shows that $q$ is a singular Gordon potential, and the proof is complete. \hfill $\Box$

**References**

[1] F. V. Atkinson, W. N. Everitt, and A. Zettl, Regularization of a Sturm-Liouville problem with an interior singularity using quasi-derivatives, *Diff. Integr. Equat.* 1(1988), no. 2, 213–221.

[2] A. M. Savchuk and A. A. Shkalikov, Sturm-Liouville operators with singular potentials, *Matem. Zametki (Math. Notes)*, 66(1999), No. 6, p. 897–912.

[3] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics*, Springer-Verlag, New York-Berlin-Heidelberg-London-Paris-Tokyo, 1988.

[4] R. O. Hryniv and Ya. V. Mykytyuk, Schrödinger operators with periodic singular potentials, preprint (2001).

[5] A. Ya. Gordon, On the point spectrum of the one-dimensional Schrödinger operator, *Usp. Mat. Nauk* 31(1976), 257–258.

[6] D. Damanik and G. Stolz, A generalization of Gordon’s theorem and applications to quasiperiodic Schrödinger operators, *Electron. J. Diff. Equat.* (2000), no. 55, 8pp.

[7] A. Ya. Khinchin, *Continued Fractions*, Dover Publ., Mineola, 1997.

[8] F. Gesztesy and W. Kirsch, One-dimensional Schrödinger operators with interactions singular on a discrete set, *J. Reine Angew. Math.* 362(1985), 28–50.

[9] D. Buschmann, G. Stolz, and J. Weidmann, One-dimensional Schrödinger operators with local point interactions, *J. Reine Angew. Math.* 467(1995), 169–186.

[10] V. G. Mazya, *Sobolev Spaces*, Springer-Verlag, Berlin, 1985.

[11] G. V. Shilov, *Mathematical Analysis. The Second Special Course*, Moscow Univ. Publ., Moscow, 1984.

[12] B. Simon, Schrödinger semigroups, *Bull. Amer. Math. Soc.* 7(1982), 447–526.

[13] V. A. Yakubovich and V. M. Starzhinsky, *Linear Differential Equations with Periodic Coefficients*, Nauka Publ., Moscow, 1972.

[14] J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications, I*, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

Institute for Applied Problems of Mechanics and Mathematics, 3b Naukova str., 79601 LVIV, UKRAINE

E-mail address: hryinv@mebm.lviv.ua

Department of Mechanics and Mathematics, LVIV National University, 1 Universytrytska str., 79602 LVIV, UKRAINE

E-mail address: mykytyuk@email.lviv.ua