Analytic Continuation of Thermal $N$-Point Functions
from Imaginary- to Real-Energies

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Abstract

We consider thermal $n$-point Green functions in the framework of quantum field theory at finite temperature. We show how analytic continuations from imaginary to real energies relate these functions originally defined in the imaginary-time formalism to retarded and advanced real-time ones. The described method is valid to all orders of perturbation theory. It has the further advantage that it is independent of approximations often applied in actual finite-order calculations.

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I. INTRODUCTION

In quantum field theory at finite temperature and density two convenient formalisms that enable the use of conventional Feynman rules in momentum space are applied; the imaginary-time formalism (ITF) and the real-time formalism (RTF). (For books and reviews on thermal field theory, see, e.g. [4–7]). The former one is particularly suited for the evaluation of the static or thermodynamic quantities of finite-temperature systems, while the latter is preferred for the evaluation of time-dependent quantities.

Studies of the two-point functions have a long history. The relationship between their representations — the one obtained from ITF and the other from RTF — is well known. On the other hand the interest in the three- and \(n\)\((\geq 4)\)-point functions started rather recently.

An apparent difference between the results for three-point functions obtained from ITF and from RTF has posed the question: through analytic continuations of the Green functions evaluated in ITF, what kind of (combinations of) thermal Green functions in RTF are obtained?

Several papers have been devoted to this issue. It has been shown, either to one-loop order or in all orders of perturbation theory, that the three-point function in ITF, when analytically continued to real energies, becomes the retarded or advanced three-point Green function. Recently, the \(n\)\((\geq 4)\)-point functions have been investigated and it is shown that different analytic continuations of the ITF result yield different RTF Green functions, including the retarded and advanced Green functions.

The purpose of this paper is to show that the most straightforward analytic continuation of the non-amputated \(n\)-point Green function in ITF leads to the retarded or advanced \(n\)-point Green function.

In Sec. II, we introduce the \(n\)-point thermal Green functions with time arguments on a — to a large extent arbitrary — contour in the complex time plane. We then formulate the problem of analytic continuation of the Green function in ITF from imaginary- to real-
energies. For the purpose of illustrating our procedure, we discuss in Sec. III the analytic continuation of the two-point function. In Sec. IV, we carry out the analytic continuation of the \( n \)-point Green functions evaluated in ITF, and obtain the retarded- and advanced-Green functions. Sec. V is devoted to conclusions.

II. PRELIMINARIES

Throughout this paper we consider a real scalar field \( \phi(x) \). Generalizations to other kind of fields are straightforward. The thermal Green functions are defined as the statistical average of a product of Heisenberg fields,

\[
G(\{t\}) = G(t_1, t_2, \cdots, t_n) \\
\equiv Tr \left[ e^{-\beta H} T_c (\phi(t_1)\phi(t_2)\cdots\phi(t_n)) \right] / Tr e^{-\beta H} \\
\equiv \langle T_c (\phi(t_1)\phi(t_2)\cdots\phi(t_n)) \rangle ,
\]

where \( \beta = T^{-1} \) is the inverse temperature and \( H \) is the Hamiltonian of the system such that \( \phi(t) = e^{iHt} \phi(0) e^{-iHt} \). In Eq. (1) and in the following we suppress explicit reference to the space variables. The arguments \( t_1, t_2, \cdots, t_n \) lie on the contour \( C \) running from \( t_0 \) to \( t_0 - i\beta \) in the complex time plane. The symbol \( T_c \) in (1) is the time-ordering operator along this path \( C \). That is, it prescribes that the operators it is applied to be arranged in the order in which their arguments lie along \( C \), with those nearest to the beginning at \( t_0 \) to the right, and those nearest to the end \( t_0 - i\beta \) to the left.

To perform the thermal trace (1), we insert a complete set of states by choosing the Hamiltonian eigenstate basis:

\[
G(\{t\}) Tr e^{-\beta H} = \sum_{p_n} \left[ \prod_{j=1}^{n-1} \theta_c(t_j - t_{j+1}) \right] \sum_{l_1, l_2, \cdots, l_n} \exp(-iE_{l_1}(t_\pi - t_\tau - i\beta)) \\
\times \langle l_1 | \phi(0) | l_2 \rangle \prod_{j=2}^{n} \left[ \exp \left( iE_{l_j} (t_j - t_{j-1}) \right) \langle l_j | \phi(0) | l_{j+1} \rangle \right] , \quad (l_{n+1} \equiv l_1),
\]

with \( \theta_c \) the contour step function. The summation here is carried out over all permutations of \( n \) numbers \( p_n \).
\[
\begin{pmatrix}
1 & 2 & \cdots & n \\
\Phi & \Xi & \cdots & \Pi
\end{pmatrix}.
\]

Following common practice, we assume that the convergence of the above trace sum (2) is controlled by the exponential factors, so that it converges if, for every pair of \( t_i \) and \( t_j \) such that \( \theta_c(t_i - t_j) = 1 \), \( \Im(t_i - t_j) < 0 \), and \( \Im(t_s - t_i) < \beta \) with \( t_s \) (the “smallest”) (“largest”) time. This condition guarantees the existence of \( G(\{t\}) \) (Eq. (2)) as an analytic function of \( \{t\} = \{t_1, t_2, \cdots, t_n\} \). The limit of an analytic function on the boundaries of its domain of definition, where it is still continuous, is a generalized function. This implies that the thermal Green function \( G(\{t\}) \) is well defined for \( \Im(t_i - t_j) \leq 0 \) when \( \theta_c(t_i - t_j) = 1 \), and \( \Im(t_s - t_i) \leq \beta \). This imposes the restriction on \( C \) that as a point moves along \( C \) from \( t_0 \) to \( t_0 - i\beta \) its imaginary part is nonincreasing. Then, an analytic continuation of \( G(\{t\}) \) can be done by deforming the contour \( C \) with the end points \( t_0 \) and \( t_0 - i\beta \) held fixed, keeping in mind the “nonincreasing” condition for the imaginary part of the points on \( C \).

Among the above class of paths, we consider first a special path \( C_I \) depicted in Fig. 1, which defines the ITF. We note that we can choose any value for \( \Re t_0 \) because of the property of time-translation invariance of (1). Next we evaluate the Fourier component of (1),

\[
G(\{\omega\}) = G(\omega_1, \omega_2, \cdots, \omega_n) = \prod_{j=1}^{n} \left( \int_{C_j} dt_j e^{-\omega_j t_j} \right) G(t_1, t_2, \cdots, t_n), \quad \omega_j = 2\pi l_j / \beta, \quad l_j = 0, \pm 1, \pm 2, \cdots, \pm \infty. \tag{4}
\]

It is to be noted that real (discrete) \( \omega_j \)'s here are what we call imaginary energies. By using once more the time translation invariance we rewrite (4) as

\[
G(\{\omega\}) = -i\beta \delta(\sum_{j=1}^{n} \omega_j ; 0) \tilde{G}(\omega_2, \omega_3, \cdots, \omega_n), \tag{5a}
\]

\[
\tilde{G}(\omega_2, \omega_3, \cdots, \omega_n) = \prod_{j=2}^{n} \left( \int_{t_0-t_1}^{t_0-t_1-i\beta} dt_j e^{-\omega_j t_j} \right) G(0, t_2, \cdots, t_n). \tag{5b}
\]
Here the integrations are performed along the imaginary-time axis. In (5a) $\delta(\cdots;\cdots)$ denotes the Kronecker $\delta$ symbol. In order to obtain (5) the KMS condition \cite{15}, which represents the invariance of the trace under the following cyclic permutation (cf. (1)),

$$\langle \phi(t_1) \cdots \phi(t_{n-1}) \phi(t_n) \rangle = \langle \phi(t_n - i\beta) \phi(t_1) \cdots \phi(t_{n-1}) \rangle$$

(6)

is used. In ITF one calculates $G$ or $\hat{G}$ as defined in (5). In the following we focus our attention on how to continue (5) from imaginary to real energies.

III. TWO-POINT THERMAL GREEN FUNCTION

Although the relation between ITF and RTF is well understood \cite{8} for two-point thermal Green functions, we start with the thermal two-point Green function in ITF for the purpose of illustrating our procedure:

$$G(\omega_1, \omega_2) = \int_{C_I} dt_1 \int_{C_I} dt_2 e^{-(\omega_1 t_1 + \omega_2 t_2)} \langle T_{C_I}(\phi(t_1) \phi(t_2)) \rangle .$$  (7)

As explained in Sec. II, the integrand of (7) is an analytic function of $t_1$ and $t_2$ in the strip, $\Im t_0 - \beta < \Im t_j < \Im t_0$ ($j = 1, 2$) with $t_1 \neq t_2$, and we may deform the contour $C_I$ keeping the property as mentioned above after (3). In this way we may obtain the contour $C_R = C_1 \oplus C_2 \oplus C_3 \oplus C_4$ as depicted in Fig. 2: the path goes down from $t_0$ to $t_I \equiv \Re t_0$, continues from $t_I$ to $t_F$ along the real axis, returns back to $t_I$ along the real axis, and ends at $t_0 - i\beta$.

Because of time translation invariance $G(\omega_1, \omega_2)$ is nonvanishing only for $\omega_1 + \omega_2 = 0$ (cf.\,(5a)), and the integrand of (7) — on the path $C_R$ — is a function of $t_2 - t_1$. This, together with the above mentioned analyticity property of $G(t_1, t_2)$, allows us to evaluate $G(\omega_1, \omega_2)$ by fixing $t_1$ on any point on the contour $C_R = C_1 \oplus C_2 \oplus C_3 \oplus C_4$. We first fix $t_1$ on $C_1$, to obtain

$$G(\omega_1, \omega_2) = -i\beta \delta(\omega_1; -\omega_2) \int_{C_3} dt_2 e^{-\omega_2(t_2 - t_1)} \langle \phi(t_1) \phi(t_2) \rangle .$$
\[ + \int_{C_1} dt_2 e^{-\omega_2(t_2-t_1)} \langle T C_1(\phi(t_1)\phi(t_2)) \rangle + \int_{C_2 \oplus C_4} dt_2 e^{-\omega_2(t_2-t_1)} \langle \phi(t_2)\phi(t_1) \rangle \] (8a)

\[ = -i\beta \delta(\omega_1; -\omega_2) \int_{t_I}^{t_F} dt_2 e^{-\omega_2(t_2-t_1)} \left[ \langle T(\phi(t_1)\phi(t_2)) \rangle - \langle \phi(t_2)\phi(t_1) \rangle \right] \]

\[ + \text{(contributions from } C_3 \text{ and } C_4 \text{)} . \] (8b)

The symbol \( T \) is the ordinary time-ordering symbol. Eq. (8) is now well suited for the purpose of an analytic continuation of \( G(\omega_1, \omega_2) \) to real energies. In order to arrive at RTF with real energy \( p_{20}(= -p_{10}) \) we take the limit \( t_I (= \Re t_0) \rightarrow -\infty \) and \( t_F \rightarrow +\infty \), taking into account that \( t_I (= \Re t_0) \) may be chosen arbitrarily.

We realize (cf. [6]) that the term in the square bracket in (8b) may be written as

\[ \langle T(\phi(t_1)\phi(t_2)) \rangle - \langle \phi(t_2)\phi(t_1) \rangle = \theta(t_1 - t_2) \langle [\phi(t_1), \phi(t_2)] \rangle . \] (9)

Then, from (8) and (9), we learn that the above limit, \( t_I \rightarrow -\infty \) and \( t_F \rightarrow +\infty \), can be taken if we continue to the real energy as follows,

\[ \omega_2 \rightarrow -i(p_{20} - i\epsilon) , \] (10a)

or equivalently,

\[ \omega_1 \rightarrow -i(p_{10} + i\epsilon) , \] (10b)

with \( \epsilon \) an infinitesimal positive number.

The time path for \( G \) consists of two real-time segments, namely \( C_1 \) from \(-\infty \rightarrow +\infty \) and \( C_2 \) from \(+\infty \rightarrow -\infty \), and of \( C_3 \oplus C_4 \) (\( C_3 \) from \(-\infty + i\Im t_0 \rightarrow -\infty \) and \( C_4 \) from \(-\infty \rightarrow -\infty + i(\Im t_0 - \beta) \)). Then, in evaluating (8) with (10) by perturbative methods, there are four kind of bare thermal propagators, \( G^{(0)}_{ij}(t_i, t_j) \) \((i, j = 1 - 4)\), present with \( t_i \in C_i \). Following [10], one can show that as far as the thermal Green functions (like (8) with (10)) with their time arguments lying on \( C_1 \) and/or \( C_2 \) are concerned, the net effects of the contributions from \( C_3 \) and \( C_4 \) are to modify \( G^{(0)}_{ij} \) \((i, j = 1, 2)\) to the familiar forms [17] in two-component
real-time thermal field theory constructed on the basis of the closed-time path $C_1 \oplus C_2$.  

The analytic continuation of the energy conservation $\delta$-function in (8) is given by the prescription [7]:  

$$\delta(\omega_1; -\omega_2) \rightarrow 2\pi i\beta^{-1}\delta(p_{10} + p_{20}).$$

Thus we obtain the analytically continued $G$ with real energies $p_{10}$ and $p_{20}$,  

$$\lim_{\epsilon \rightarrow +0} G(p_{10} + i\epsilon, p_{20} - i\epsilon) = G_{++} - G_{+-} \quad (11a)$$

$$= 2\pi \delta(p_{10} + p_{20})$$

$$\times \lim_{\epsilon \rightarrow +0} \int_{-\infty}^{\infty} dt \ e^{i(p_{20} - i\epsilon)t} \theta(-t) \langle [\phi(0), \phi(t)] \rangle, \quad (11b)$$

where we follow e.g. [6] and introduce

$$G_{\alpha\beta}(p_{10}, p_{20})$$

$$= \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \ e^{i(p_{10}t_1 + p_{20}t_2)} \langle A_{\alpha\beta}(t_1, t_2) \rangle, \quad (12)$$

with the definitions

$$A_{++}(t_1, t_2) = \theta(t_1 - t_2)\phi(t_1)\phi(t_2) + \theta(t_2 - t_1)\phi(t_2)\phi(t_1)$$

$$A_{+-}(t_1, t_2) = \phi(t_2)\phi(t_1)$$

$$A_{-+}(t_1, t_2) = \theta(t_1 - t_2)\phi(t_2)\phi(t_1) + \theta(t_2 - t_1)\phi(t_1)\phi(t_2)$$

$$A_{--}(t_1, t_2) = \phi(t_1)\phi(t_2). \quad (13)$$

The functions $G_{\alpha\beta}$ ($\alpha, \beta = +, -$) in (12) denote Green functions in the two-component real-time thermal field theory, applying the closed time-path formalism introduced by Schwinger and Keldysh [3]. It is to be noted, in passing, that the $A_{\alpha\beta}$’s are not independent since they satisfy the relation [3],

$$A_{++}(t_1, t_2) + A_{--}(t_1, t_2) - A_{+-}(t_1, t_2) - A_{-+}(t_1, t_2) = 0. \quad (14)$$
It is important to note that the continued function $G$ in (11b) is identical with the retarded Green function. In this way the “physical” representation as discussed in [6] is established, where this function is denoted by $G = iG_{21}(p_{10}, p_{20})$, however, the primary quantities that are calculated directly in real-time thermal field theory are $G_{\alpha\beta}$ in (12).

When we fix $t_2$, instead of $t_1$, on $C_1$, and repeating similar steps as above we obtain the retarded Green function with respect to $\phi(t_2)$, i.e. the advanced Green function with respect to $\phi(t_1)$. Fixing the time variable as $t_1 \in C_2$ leads to the same result as above, (11), and likewise for the choice $t_2 \in C_2$. Fixing $t_1$ on $C_3$ or $C_4$, $t_1 \in C_3 \oplus C_4$, is not suitable for analytic continuations under consideration and we recover the original formula (7); this holds likewise for the choice $t_2 \in C_3 \oplus C_4$.

Finally we may deform the contour $C_I$ in (7) to the one that is mirror symmetric to the one of Fig. 2. Changing the time variable $t_j$ as $t_j \equiv \Re t_j + i \Im t_j \rightarrow -\Re t_j + i \Im t_j$, we get back the time path $C_1 \oplus C_2 \oplus C_3 \oplus C_4$ of Fig. 2. Then, fixing $t_1$ on the upper path on the real axis in the complex time-plane, we deduce

$$\lim_{\epsilon \rightarrow +0} G(p_{10} - i\epsilon, p_{20} + i\epsilon) = -G_{++} + G_{-+}$$

$$(15a)$$

$$= -2\pi \delta(p_{10} + p_{20})$$

$$\times \lim_{\epsilon \rightarrow +0} \int_{-\infty}^{\infty} dt e^{i(p_{20} + i\epsilon)t} \theta(t) \langle [\phi(0), \phi(t)] \rangle,$$  

$$(15b)$$
i.e. the advanced Green function.

IV. THERMAL $N$-POINT GREEN FUNCTION

We proceed in a similar manner as in Sec. III. Starting with the thermal $n$-point function in ITF, Eq. (4), we deform the contour $C_I$ to $C_R$ as depicted in Fig. 2. Under the constraint $\sum_j \omega_j = 0$ (cf. (5a)), we evaluate (4) with $C_R$ for $C_I$ by fixing $t_1$ on $C_1$, $t_1 \in C_1$, which is suitable for continuation to real energies. In place of Eq. (8) we now have

$$G(\{\omega\}) = -i\beta \delta(\sum_j \omega_j; 0) \prod_{j=2}^{n} \left( \int_{C_1 \oplus C_2} dt_j e^{-\omega_j(t_j - t_1)} \right)$$

8
\[
\times \langle T_{C_1 \oplus C_2} (\phi(t_1) \cdots \phi(t_{n-1}) \phi(t_n)) \rangle + \cdots \quad (t_1 \in C_1), \tag{16a}
\]
\[
= -i \beta \delta(\sum_j \omega_j; 0) \prod_{j=2}^{n} \left( \int_{t_j}^{t_F} dt_j \ e^{-\omega_j(t_j-t_1)} \right) \times \left[ \sum_{\alpha_2,\alpha_3,\cdots,\alpha_n=+,\cdots,\alpha_n=-} (-)^s \langle T_{C_1 \oplus C_2} (\phi_+(t_1) \phi_{\alpha_2}(t_2) \cdots \phi_{\alpha_n}(t_n)) \rangle \right] + \cdots , \tag{16b}
\]
where
\[
s = \frac{1}{2} \sum_{j=2}^{n} (1 - \alpha_j) , \tag{17}
\]
and the notation of [6] is used: for \( n > 2\) it becomes more transparent to make the subscripts +, − explicit, which indicate that the times \( t_i \) assume values on either the “positive” time path \( C_1 \) or on the “negative” one \( C_2 \); e.g. this implies for all + subscripts time ordering, whereas for all − subscripts anti-time ordering. This generalizes the expressions in (13) to the \( n \geq 3\) cases (for \( n=3\) explicit expressions are given in [6]). In Eq. (16) the dots indicate the contributions when some of the \( t_i \)'s among \( t_2, \cdots, t_n \) are on \( C_3 \oplus C_4 \); this portion is treated according to the arguments given in Sec.III [16].

Next we transform to the “physical” representation. By applying the procedure described in [6], we express the term in the square bracket in (16b) as,
\[
\sum_{p_{n-1}} \theta(1, 2, \cdots, n) \langle [\cdots [\phi(t_1), \phi(t_2)], \phi(t_3)], \cdots, \phi(t_n) \rangle , \tag{18}
\]
where \( \theta \) is the multi-step function defined by \( \theta(1, 2, \cdots, n) = \theta(1, 2) \theta(2, 3) \cdots \theta(n-1, n) \) with \( \theta(1, 2) = \theta(t_1 - t_2) \). Eq. (18) is the generalization of the two-point function case Eq. (9). The summation here is carried out over all permutations of \( n-1\) numbers \( p_{n-1} \)
\[
\begin{pmatrix}
  2 & 3 & \cdots & n \\
  2 & 3 & \cdots & n
\end{pmatrix} . \tag{19}
\]
Eq. (18) tells us that \( t_1 \) is the largest time, therefore we are allowed to take the limit \( t_f (= \Re t_0) \rightarrow -\infty\) and \( t_F \rightarrow +\infty\) if we continue \( \omega_j \) in (16b) to real energies as
\[
\omega_j \rightarrow -i(p_{j0} - i\epsilon) , \quad j = 2, 3, \cdots, n , \tag{20a}
\]
\[ \omega_1 \rightarrow -i(p_{10} + i(n - 1)\epsilon) . \]  

(20b)

Then we arrive at the Green function in RTF,

\[
\lim_{\epsilon \rightarrow +0} G(p_{10} + i(n - 1)\epsilon, \{p_{j0} - i\epsilon; j = 2, 3, \ldots, n\}) = \sum_{\alpha_2, \ldots, \alpha_n = +, -} (-)^s G_{+\alpha_2, \ldots, \alpha_n} \]  

(21a)

\[
= 2\pi \delta(\sum_{j=1}^{n} p_{j0}) \lim_{\epsilon \rightarrow +0} \prod_{j=2}^{n} \left( \int_{-\infty}^{\infty} dt_j e^{i(p_{j0} - i\epsilon)(t_j - t_1)} \right) \times \sum_{\alpha_{n-1}} \theta(1, 2, \ldots, n) \{ \cdots [\phi(t_1), \phi(t_2)], \phi(t_3), \ldots, \phi(t_n) \} \]  

(21b)

\[
\equiv i^{n-1} \tilde{G}_{211\ldots 1}(p_{10}, \ldots, p_{n0}) . \]  

(21c)

where \( s \) is given in (17). Thus we have derived the Green function (21c) in the “physical” representation, denoted by \( \tilde{G}_{211\ldots 1} \), which is the thermal \( n \)-point retarded function as seen in (21b). In (21a), \( G_{+\alpha_2, \ldots, \alpha_n} \) is a Green function in RTF and it is defined analogously to (12) and (13),

\[
G_{\alpha_1, \alpha_2, \ldots, \alpha_n} = \prod_{j=1}^{n} \left( \int_{-\infty}^{\infty} dt_j e^{i(p_{j0} - i\epsilon)t_j} \right) \times \langle T_{C_1 \oplus C_2} (\phi_{\alpha_1}(t_1) \phi_{\alpha_2}(t_2) \cdots \phi_{\alpha_n}(t_n)) \rangle . \]  

(22)

As in the two-point function case, (14), there is one identity;

\[
\sum_{\alpha_1, \ldots, \alpha_n = +, -} (-)^{s'} G_{\alpha_1, \alpha_2, \ldots, \alpha_n} = 0 \]  

(23)

where

\[
s' = \frac{1}{2} \sum_{j=1}^{n} (1 - \alpha_j) . \]  

(24)

It is worth mentioning that the primary quantities that are evaluated in real-time thermal field theory are \( G_{\alpha_1, \alpha_2, \ldots, \alpha_n} \) defined in (22), through which the retarded Green function \( G \) in (21b) is obtained.
We have developed the derivation by fixing $t_1$ on $C_1$. Of course, we may proceed in a similar manner by fixing other $t_j$ ($2 \leq j \leq n$) on $C_1$: $n-1$ different retarded Green functions are the result.

In case Eq.(4) is evaluated with $C_R$ for $C_I$ by fixing $t_1$ on $C_2$, we obtain the same result as above, (21); for $t_1 \in C_3 \oplus C_4$ we recover the original formula (4).

When we deform the contour $C_I$ in (4) to the one that is mirror symmetric to the one of Fig. 2, and fixing $t_1$ on the upper path on the real axis in the complex-time plane, i.e. the counterpart of $C_1$ in Fig. 2, we derive the advanced Green function,

$$\lim_{\epsilon \to +0} G(p_{10} - i(n-1)\epsilon, \{p_{j0} + i\epsilon; j = 2, 3, \cdots, n\})$$

$$= -2\pi \delta(\sum_{j=1}^{n} p_{j0}) \lim_{\epsilon \to +0} \prod_{j=2}^{n} \left( \int_{-\infty}^{\infty} dt_j e^{i(p_{j0}+i\epsilon)(t_j-t_1)} \right) \times \sum_{p_{n-1}} \theta(\pi, \cdots, \pi, 1) \langle [\cdots [\phi(t_1), \phi(t_2)], \phi(t_3)], \cdots, \phi(t_n)] \rangle. \quad (25)$$

V. CONCLUSIONS

In this paper we have addressed the question: what kind of thermal functions in RTF emerge by analytic continuations of the $n$-point thermal Green functions defined in ITF. This amounts to perform analytic continuations in the energies of the external legs from the discrete imaginary values to real continuous ones.

The thermal Green functions are defined on a path in the complex time plane, a path which is to a large extent arbitrary. On the basis of this observation, we have carried out the above mentioned continuations in the most straightforward and familiar manner by deforming the contour, starting from the one that defines ITF to the one defining RTF. In this way, we show that ITF $n$-point Green functions become retarded or advanced thermal Green functions.

The results obtained in this paper, being exact and valid independent of the approximation used in actual mainly perturbative calculations, are partly discussed in [13], and we
expect that our approach gives additional information on the underlying relations between ITF and RTF.
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17 The modifications are as follows: $G_{11}^{(0)} \rightarrow G_{++}$, $G_{12}^{(0)} \rightarrow G_{+-}$, $G_{22}^{(0)} \rightarrow G_{--}$ and $G_{21}^{(0)} \rightarrow G_{-+}$. Here $G$’s are given in (12) and (13) below.
FIGURE CAPTIONS

FIG.1. The contour $C_I$ in the complex time plane, which defines ITF.

FIG.2. The contour $C_R = C_3 \oplus C_1 \oplus C_2 \oplus C_4$ in the complex time plane; the segments $C_1$ and $C_2$ lie on the real axis, and $t_I = \Re t_0$. 