BILINEAR RIESZ MEANS ON THE HEISENBERG GROUP

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Abstract. In this article, we investigate the bilinear Riesz means $S_{\alpha}$ associated to the sublaplacian on the Heisenberg group. We prove that the operator $S_{\alpha}$ is bounded from $L^{p_1} \times L^{p_2}$ into $L^{p}$ for $1 \leq p_1, p_2 \leq \infty$ and $1/p = 1/p_1 + 1/p_2$ when $\alpha$ is large than a suitable smoothness index $\alpha(p_1, p_2)$. There are some essential differences between the Euclidean space and the Heisenberg group for studying the bilinear Riesz means problem. We make use of some special techniques to obtain a lower index $\alpha(p_1, p_2)$.

1. Introduction

The bilinear Bochner-Riesz means problem originates from the study of the summability of the product of two $n$-dimensional Fourier series. This leads to the study of the $L^{p_1} \times L^{p_2} \rightarrow L^{p}$ boundedness of the bilinear Bochner-Riesz multiplier

$$B_{\alpha}(f, g)(x) = \int_{|\xi|^2 + |\eta|^2 \leq 1} (1 - |\xi|^2 - |\eta|^2)\alpha \hat{f}(\xi)\hat{g}(\eta)e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta.$$ 

Here $x \in \mathbb{R}^n$, $f$, $g$ are functions on $\mathbb{R}^n$ and $\hat{f}$, $\hat{g}$ are their Fourier transforms. Bernicot et al. [1] gave a comprehensive study on the $L^{p_1} \times L^{p_2} \rightarrow L^{p}$ boundedness of the operator $B_{\alpha}$. Inspired by their work, we investigate the corresponding problem on the Heisenberg group.

Strichartz [8, 9] developed the harmonic analysis on the Heisenberg group as the spectral theory of the sublaplacian. We can define the bilinear Riesz means in terms of the spectral decomposition of the sublaplacian. Let

$$\mathcal{L} f = \int_0^{\infty} \lambda P_{\lambda} f d\mu(\lambda).$$

be the spectral decomposition of the sublaplacian $\mathcal{L}$. The bilinear Riesz means associated to the sublaplacian $\mathcal{L}$ is defined by

$$S_{\alpha}(f, g) = \int_0^{\infty} \int_0^{\infty} (1 - \lambda_1 - \lambda_2)\alpha P_{\lambda_1} f P_{\lambda_2} g d\mu(\lambda_1)d\mu(\lambda_2).$$

As same as the Euclidean case, we hope to obtain a lower smoothness index $\alpha(p_1, p_2)$ such that the operator $S_{\alpha}$ is bounded from $L^{p_1} \times L^{p_2}$ into $L^{p}$ for $1 \leq p_1, p_2 \leq \infty$ and $1/p = 1/p_1 + 1/p_2$ when $\alpha > \alpha(p_1, p_2)$.

There are some essential differences between the Euclidean space and the Heisenberg group for studying the bilinear Riesz means problem. Firstly, the kernel of the bilinear Bochner-Riesz operator $B_{\alpha}$ on $\mathbb{R}^n$ coincides with the kernel of the Bochner-Riesz operator on $\mathbb{R}^{2n}$. The pointwise estimate of this kernel is well known and gives a basic result: $B_{\alpha}$ is bounded from $L^{p_1} \times L^{p_2}$ into $L^{p}$ when $\alpha > n - \frac{1}{2}$, which is optimal in case of $(p_1, p_2, p) = (1, 1, \frac{1}{2})$. It is not the case for the Heisenberg group because the product of two Heisenberg groups is not a Heisenberg...
group. We will give a pointwise estimate for the kernel of the bilinear Riesz means $S^\alpha$, which is similar to the known estimate for the kernel of the Riesz means on the Heisenberg group but very worse than the corresponding estimate for bilinear Bochner-Riesz operator $B^\alpha$. Such a pointwise estimate only gives a very rough result for smoothness index $\alpha(p_1,p_2)$. We don’t know if there exist a better pointwise estimate even for the Riesz means on the Heisenberg group (cf. [4] or [10]). We have to develop a new technique to obtain a better result about index $\alpha(p_1,p_2)$, for example, in case of $(p_1,p_2,p) = (\infty, \infty, \infty)$. Secondly, on Euclidean space, the Fourier transform of the product of two functions is the convolution of Fourier transforms of two functions because the dual of the Euclidean space is itself. As a consequence, the $L^2 \times L^2 \rightarrow L^2$ boundedness holds for suitable bilinear Fourier multiplier, which play an important role for giving the estimate in case of $(p_1,p_2,p) = (2, \infty, 2)$. But this convenience false on the Heisenberg group. Finally, the restriction estimate for the sublaplacian is very different from that for the Laplacian on the Euclidean space because the Heisenberg group has the center of dimension one. Fefferman [2] pointed out that the restriction theorem apply to the study of the boundedness of Bochner-Riesz means. Stein’s earlier result in [7] was improved by using the restriction theorem. Mauceri [4] investigated the $L^p$ boundedness of Riesz means on the Heisenberg group. Mauceri’s result corresponds with Stein’s earlier result. M¨uller [5] construct a counter-example to show that the usual norm estimate of restriction operators holds only in the trivial case $p = 1$. M¨uller [6] gave a new proof of Mauceri’s result by using a revised restriction estimate but didn’t improve Mauceri’s result. As a result of above reasons, our techniques are very different from that in [4].

This article is organized as follows. In the next section, we state some basic facts about the Heisenberg group, and summarize our full results in the main theorem. In Section 3, we give the pointwise estimate for the kernel of $S^\alpha$. We prove the $L^p_1 \times L^p_2 \rightarrow L^p$ boundedness of $S^\alpha$ in the case of $1 \leq p_1, p_2 \leq 2$ in Section 4 and for some particular triples of points $(p_1,p_2,p)$ in Section 5. The boundedness in other cases are derived from the results in Section 4 and Section 5 by using of bilinear interpolation method. We outline this argument in Appendix for reader’s convenience.

2. Preliminaries

First we recall some basic facts about the Heisenberg group. These facts are familiar and easy to find in many references. Let $\mathbb{H}^n$ denote the Heisenberg group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$ and the group law is given by

$$ (z,t)(w,s) = (z+w,t+s+\frac{1}{2}\text{Im}(z \cdot \overline{w})). $$

The Haar measure on $\mathbb{H}^n$ coincides with the Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$. A homogeneous structure on $\mathbb{H}^n$ is given by the non-isotropic dilations $\delta_r(z,t) = (rz, r^2t)$. We define a homogeneous norm on $\mathbb{H}^n$ by

$$ |x| = \left( \frac{1}{16} |z|^4 + t^2 \right)^\frac{1}{4}, \quad x = (z,t) \in \mathbb{H}^n. $$

This norm satisfies the triangle inequality and leads to a left-invariant distance $d(x,y) = |x^{-1}y|$. The ball of radius $r$ centered at $x$ is

$$ B(x,r) = \{ y \in \mathbb{H}^n : |x^{-1}y| < r \}. $$

The Haar measure $dx$ satisfies $d\delta_r(x) = r^Q dx$ where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$. If $f$ and $g$ are functions on $\mathbb{H}^n$, their convolution is defined by

$$ (f \ast g)(x) = \int_{\mathbb{H}^n} f(xy^{-1}) g(y) dy, \quad x,y \in \mathbb{H}^n. $$
For each $\lambda \in \mathbb{R}^*$ and $f \in \mathcal{S}(\mathbb{H}^n)$, the inverse Fourier transform of $f$ in variable $t$ is defined by

$$f^\lambda(z) = \int_{-\infty}^{\infty} e^{i\lambda t} f(z, t) \, dt.$$ 

An easy calculation shows that

$$(f * g)^\lambda(z) = \int_{\mathbb{C}^n} f^\lambda(z - \omega) g^\lambda(\omega) e^{\frac{i}{2} \lambda \text{Im}(z \cdot \omega)} \, d\omega, \quad z, \omega \in \mathbb{C}^n.$$ 

Thus, we are led to the convolution of the form

$$f * \lambda g = \int_{\mathbb{C}^n} f(z - \omega) g(\omega) e^{\frac{i}{2} \lambda \text{Im}(z \cdot \omega)} \, d\omega,$$

which are called the $\lambda$-twisted convolution.

The sublaplacian $L$ is defined by

$$L = -\sum_{j=1}^{n} (X_j^2 + Y_j^2)$$

where

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t}, \quad j = 1, 2, \cdots, n,$$

$$Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial t}, \quad j = 1, 2, \cdots, n,$$

are left invariant vector fields on $\mathbb{H}^n$. Up to a constant multiple, $L$ is the unique left invariant, rotation invariant differential operator that is homogeneous of degree two. Therefore, it is regarded as the counterpart of the Laplacian on $\mathbb{R}^n$. The sublaplacian $L$ is a positive and essentially self-adjoint operator. In the following, we state the spectral decomposition of $L$ (cf. [10]).

Let $\varphi_k$ be the Laguerre functions on $\mathbb{C}^n$ given by

$$\varphi_k(z) = L_k^{n-1} \left( \frac{1}{2} |z|^2 \right) e^{-\frac{1}{4} |z|^2},$$

where $L_k^{n-1}$ are the Laguerre polynomials of type $n - 1$ defined on $\mathbb{R}$ by

$$L_k^{n-1}(t) e^{-t} t^{n-1} = \frac{1}{k!} \left( \frac{d}{dt} \right)^k (e^{-t} t^{k+n-1}).$$

Define functions

$$e_\lambda^k(z, t) = e^{-i\lambda t} \varphi_k^\lambda(z) = e^{-i\lambda t} \varphi_k(\sqrt{\lambda} |z|), \quad \lambda \in \mathbb{R}^*.$$ 

For $f \in L^2(\mathbb{H}^n)$, we have the expansion

$$f(z, t) = \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} f * e_\lambda^k(z, t) \, d\mu(\lambda)$$

where $d\mu(\lambda) = (2\pi)^{-n-1} |\lambda|^n \, d\lambda$ is the Plancherel measure for $\mathbb{H}^n$. Each $f * e_\lambda^k$ is the eigenfunction of $L$ with eigenvalue $(2k + n) |\lambda|$. We also have the Plancherel formula

$$||f||_2^2 = (2\pi)^{-2n-1} \sum_{k=0}^{\infty} \int_{\mathbb{C}^n} \left| f^\lambda * \lambda \varphi_k^\lambda(z) \right|^2 \, d^2 z \, d\lambda.$$ 

Defining

$$\tilde{e}_\lambda^k(z, t) = e_\lambda^k(z, t),$$
we can rewrite the decomposition (2.1) as
\[ f(z, t) = \int_0^{\lambda} \sum_{k=0}^{\infty} (2k + n)^{-\alpha} f \ast \tilde{c}_k \, d\lambda. \]

Let
\[ P_\lambda f(z, t) = \sum_{k=0}^{\infty} (2k + n)^{-\alpha} f \ast (\tilde{c}_k^\lambda + \tilde{c}_k^{-\lambda})(z, t). \]

Then (2.1) can be written as
\[ f(z, t) = \int_0^{\lambda} P_\lambda f(z, t) \, d\lambda. \]

It is clear that \( P_\lambda f \) is an eigenfunction of the \( \mathcal{L} \) with eigenvalue \( \lambda \) and we have the spectral decomposition
\[ \mathcal{L} f = \int_0^{\lambda} \lambda P_\lambda f \, d\lambda. \]

Now we define the bilinear Riesz means associated to the sublaplacian \( \mathcal{L} \) for \( f, g \in \mathcal{S}(\mathbb{H}^n) \) by
\[ S_\alpha^R(f, g) = \int_0^{\lambda} \int_0^{\lambda} \left( 1 - \frac{\lambda^1 + \lambda^2}{R} \right) \left( 1 - \frac{(2k + n) |\lambda^1| + (2l + n) |\lambda^2|}{R} \right) \]
\[ \times |\tilde{c}_k\lambda^1(z_1, t_1)|^\alpha \times |\tilde{c}_k\lambda^2(z_2, t_2)|^\alpha \, d\lambda^1 d\lambda^2, \]

where the kernel is given by
\[ S_\alpha^R((z_1, t_1), (z_2, t_2)) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \int_0^{\lambda} \int_0^{\lambda} \left( 1 - \frac{(2k + n) |\lambda^1| + (2l + n) |\lambda^2|}{R} \right) \]
\[ \times |\tilde{c}_k\lambda^1(z_1, t_1)|^\alpha |\tilde{c}_k\lambda^2(z_2, t_2)|^\alpha \, d\lambda^1 d\lambda^2. \]

Because
\[ S_\alpha^R((z_1, t_1), (z_2, t_2)) = R^Q S_\alpha^0(\sqrt{R}z_1, \sqrt{R}t_1, \sqrt{R}z_2, \sqrt{R}t_2). \]

By a dilation argument, the \( L^{p^1} \times L^{p^2} \rightarrow L^p \) boundedness of \( S_\alpha^R \) is deduced from the \( L^{p^1} \times L^{p^2} \rightarrow L^p \) boundedness of \( S_\alpha^0 \) as \( 1/p = 1/p^1 + 1/p^2 \). We will concentrate on the operator \( S_\alpha^0 \) and write \( S^\alpha \) instead of \( S_\alpha^0 \).

Our full results are summarized in the following theorem.

**Main Theorem.** Let \( 1 \leq p_1, p_2 \leq \infty \) and \( 1/p = 1/p^1 + 1/p^2 \).

1. **(region I)** For \( 2 \leq p_1, p_2 \leq \infty \) and \( p \geq 2 \), if \( \alpha > Q \left( 1 - \frac{1}{p^1} \right) - \frac{1}{2} \), then \( S^\alpha \) is bounded from \( L^{p^1}(\mathbb{H}^n) \times L^{p^2}(\mathbb{H}^n) \) to \( L^p(\mathbb{H}^n) \).

2. **(region II)** For \( 2 \leq p_1, p_2 \leq \infty \) and \( 1 \leq p \leq 2 \), if \( \alpha > (Q - 1) \left( 1 - \frac{1}{p} \right) \), then \( S^\alpha \) is bounded from \( L^{p^1}(\mathbb{H}^n) \times L^{p^2}(\mathbb{H}^n) \) to \( L^p(\mathbb{H}^n) \).

3. **(region III)** For \( 1 \leq p_1 \leq 2 \leq p_2 \leq \infty \) and \( p \geq 1 \), if \( \alpha > Q \left( \frac{1}{2} - \frac{1}{p^2} \right) - \left( 1 - \frac{1}{p} \right) \), then \( S^\alpha \) is bounded from \( L^{p^1}(\mathbb{H}^n) \times L^{p^2}(\mathbb{H}^n) \) to \( L^p(\mathbb{H}^n) \); For \( 1 \leq p_2 \leq 2 \leq p_1 \leq \infty \) and \( p \geq 1 \), if \( \alpha > Q \left( \frac{1}{2} - \frac{1}{p^1} \right) - \left( 1 - \frac{1}{p} \right) \), then \( S^\alpha \) is bounded from \( L^{p^1}(\mathbb{H}^n) \times L^{p^2}(\mathbb{H}^n) \) to \( L^p(\mathbb{H}^n) \).
(4) (region IV) For $1 \leq p_1 \leq 2 \leq p_2 \leq \infty$ and $p \leq 1$, if $\alpha > Q \left( \frac{1}{p_1} - \frac{1}{2} \right)$, then $S^\alpha$ is bounded from $L^{p_1}(\mathbb{H}^n) \times L^{p_2}(\mathbb{H}^n)$ to $L^p(\mathbb{H}^n)$; For $1 \leq p_2 \leq 2 \leq p_1 \leq \infty$ and $p \leq 1$, if $\alpha > Q \left( \frac{1}{p_2} - \frac{1}{2} \right)$, then $S^\alpha$ is bounded from $L^{p_1}(\mathbb{H}^n) \times L^{p_2}(\mathbb{H}^n)$ to $L^p(\mathbb{H}^n)$.

(5) (region V) For $1 \leq p_1, p_2 \leq 2$, if $\alpha > Q \left( \frac{1}{p} - 1 \right)$, then $S^\alpha$ is bounded from $L^{p_1}(\mathbb{H}^n) \times L^{p_2}(\mathbb{H}^n)$ to $L^p(\mathbb{H}^n)$.

3. Pointwise estimate for the kernel

Note that $S^\alpha((z_1, t_1), (z_2, t_2))$ is bi-radial with respect to $z_1$ and $z_2$, which means $S^\alpha((z_1, t_1), (z_2, t_2))$ depends only on $|z_1|, |z_2|, t_1, t_2$. If $F((z_1, t_1), (z_2, t_2))$ is bi-radial with respect to $z_1$ and $z_2$, we also write it as $F((r_1, t_1), (r_2, t_2))$ for convenience where $r_1 = |z_1|, r_2 = |z_2|$. Suppose $F((z_1, t_1), (z_2, t_2))$ is bi-radial with respect to $z_1$ and $z_2$, we define

$$R_{k,l}(\lambda_1, \lambda_2, F) = \frac{2^{(1-n)k!}}{(k+n-1)!} \frac{2^{(1-n)l!}}{(l+n-1)!} \int_0^\infty \int_0^\infty F^{\lambda_1, \lambda_2}(r_1, r_2) \varphi_k^{\lambda_1}(r_1) \varphi_l^{\lambda_2}(r_2) r_1^{2n-1} r_2^{2n-1} dr_1 dr_2,$$

where

$$F^{\lambda_1, \lambda_2}(r_1, r_2) = \int_{-\infty}^\infty \int_{-\infty}^\infty e^{i \lambda_1 t_1} e^{i \lambda_2 t_2} F((r_1, t_1), (r_2, t_2)) dt_1 dt_2.$$

Taking use of the Laguerre transform, we have

$$F((r_1, t_1), (r_2, t_2)) = \sum_{k=0}^\infty \sum_{l=0}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-i(\lambda_1 t_1 + \lambda_2 t_2)} R_{k,l}(\lambda_1, \lambda_2, F) \varphi_k^{\lambda_1}(r_1) \varphi_l^{\lambda_2}(r_2) d\mu(\lambda_1) d\mu(\lambda_2).$$

Especially, if

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \left( \sum_{k=0}^\infty \sum_{l=0}^\infty \left| R_{k,l}(\lambda_1, \lambda_2, F) \right| \frac{(k+n-1)! (l+n-1)!}{k! l!} \right) d\mu(\lambda_1) d\mu(\lambda_2) < \infty,$$
then $F$ is bounded, because
\[ \|\varphi_k\|_\infty = c_n \frac{(k + n - 1)!}{k!}. \]

**Theorem 1.** If $\alpha > 4m - 1$ where $m$ is a positive integer, then for any $\omega_1 = (z_1, t_1), \omega_2 = (z_2, t_2) \in \mathbb{H}^n$.
\[ |S^\alpha(\omega_1, \omega_2)| \leq C(1 + |\omega_1|)^{-2m}(1 + |\omega_2|)^{-2m}. \]

**Proof.** Set $F((z_1, t_1), (z_2, t_2)) = S^\alpha((z_1, t_1), (z_2, t_2))$. If we can show that
\[ (3.2) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} |R_{k,l}(\lambda_1, \lambda_2, (it_1 - \frac{1}{4} |z_1|^2)^m (it_2 - \frac{1}{4} |z_2|^2)^m F)| \times \frac{(k + n - 1)! (l + n - 1)!}{k! l!} \right) d\mu(\lambda_1)d\mu(\lambda_2) < \infty, \]
then $(it_1 - \frac{1}{4} |z_1|^2)^m (it_2 - \frac{1}{4} |z_2|^2)^m S^\alpha((z_1, t_1), (z_2, t_2))$ is bounded. It follows that
\[ |S^\alpha(\omega_1, \omega_2)| \leq C(1 + |\omega_1|)^{-2m}(1 + |\omega_2|)^{-2m}, \]
and Theorem 1 is proved.

It is clear that $R_{k,l}(\lambda_1, \lambda_2, F) = \{(1 - (2k + n) |\lambda_1| - (2l + n) |\lambda_2|)^\alpha \}$. We first calculate $R_{k,l}(\lambda_1, \lambda_2, (it_1 - \frac{1}{4} |z_1|^2)^m (it_2 - \frac{1}{4} |z_2|^2)^m F)$ and assume that $\lambda_1, \lambda_2 > 0$. From (3.1), we have
\[ R_{k,l}(\lambda_1, \lambda_2, it_1 \cdot it_2 F) = \frac{2^{(1-n)k}!}{(k + n - 1)!} \frac{2^{(1-n)l}!}{(l + n - 1)!} \int_0^\infty \int_0^\infty (it_1 \cdot it_2 F)^{\lambda_1, \lambda_2} (r_1, r_2) \varphi_k^{1} (r_1) \varphi_k^{2} (r_2) r_1^{2n-1} r_2^{2n-1} dr_1 dr_2. \]

Since that
\[ (it_1 \cdot it_2 F)^{\lambda_1, \lambda_2} (r_1, r_2) = \frac{\partial}{\partial \lambda_1} \frac{\partial}{\partial \lambda_2} F^{\lambda_1, \lambda_2} (r_1, r_2), \]
we get
\[ R_{k,l}(\lambda_1, \lambda_2, it_1 \cdot it_2 F) = \frac{\partial}{\partial \lambda_1} \frac{\partial}{\partial \lambda_2} R_{k,l}(\lambda_1, \lambda_2, F) \]
\[ \div \frac{2^{(1-n)k}!}{(k + n - 1)!} \frac{2^{(1-n)l}!}{(l + n - 1)!} \int_0^\infty \int_0^\infty \frac{\partial}{\partial \lambda_1} F^{\lambda_1, \lambda_2} (r_1, r_2) \varphi_k^{1} (r_1) \left( \frac{\partial}{\partial \lambda_2} \varphi_k^{2} (r_2) \right) r_1^{2n-1} r_2^{2n-1} dr_1 dr_2 \]
\[ \div \frac{2^{(1-n)k}!}{(k + n - 1)!} \frac{2^{(1-n)l}!}{(l + n - 1)!} \int_0^\infty \int_0^\infty \frac{\partial}{\partial \lambda_2} F^{\lambda_1, \lambda_2} (r_1, r_2) \left( \frac{\partial}{\partial \lambda_1} \varphi_k^{1} (r_1) \right) \varphi_k^{2} (r_2) r_1^{2n-1} r_2^{2n-1} dr_1 dr_2 \]
\[ \div \frac{2^{(1-n)k}!}{(k + n - 1)!} \frac{2^{(1-n)l}!}{(l + n - 1)!} \int_0^\infty \int_0^\infty F^{\lambda_1, \lambda_2} (r_1, r_2) \left( \frac{\partial}{\partial \lambda_1} \varphi_k^{1} (r_1) \right) \left( \frac{\partial}{\partial \lambda_2} \varphi_k^{2} (r_2) \right) r_1^{2n-1} r_2^{2n-1} dr_1 dr_2. \]

Noticing the fact (see [10], p. 92)
\[ \frac{\partial}{\partial \lambda} \varphi_k^{1} (r) = \frac{k}{\lambda} \varphi_k^{1} (r) - \frac{(k + n - 1)}{\lambda} \varphi_{k-1}^{1} (r) - \frac{1}{4} r^2 \varphi_k^{1} (r), \]
it then follows that
\[ R_{k,l}(\lambda_1, \lambda_2, it_1 \cdot it_2 F) = \frac{\partial}{\partial \lambda_1} \frac{\partial}{\partial \lambda_2} R_{k,l}(\lambda_1, \lambda_2, F) - R_{k,l} \left( \lambda_1, \lambda_2, \frac{1}{4} |z_1|^2 |z_2|^2 \right) \]
\begin{align*}
+ \frac{l}{\lambda_2} \frac{\partial}{\partial \lambda_1} & \left( R_{k,l-1}(\lambda_1, \lambda_2, F) - R_{k,l}(\lambda_1, \lambda_2, F) \right) \\
+ \frac{k}{\lambda_1} \frac{\partial}{\partial \lambda_2} & \left( R_{k-1,l}(\lambda_1, \lambda_2, F) - R_{k,l}(\lambda_1, \lambda_2, F) \right) \\
+ \frac{k}{\lambda_1} \frac{1}{\lambda_2} & \left( R_{k-1,l}(\lambda_1, \lambda_2, F) + R_{k,l-1}(\lambda_1, \lambda_2, F) - R_{k,l}(\lambda_1, \lambda_2, F) - R_{k-1,l-1}(\lambda_1, \lambda_2, F) \right) \\
+ \frac{\partial}{\partial \lambda_1} & R_{k,l}(\lambda_1, \lambda_2, \frac{1}{4} \, |z_2|^2 F) + \frac{\partial}{\partial \lambda_2} R_{k,l}(\lambda_1, \lambda_2, \frac{1}{4} \, |z_1|^2 F) \\
+ \frac{k}{\lambda_1} & \left( R_{k,l}(\lambda_1, \lambda_2, \frac{1}{4} \, |z_2|^2 F) - R_{k-1,l}(\lambda_1, \lambda_2, \frac{1}{4} \, |z_2|^2 F) \right) \\
+ \frac{l}{\lambda_2} & \left( R_{k,l}(\lambda_1, \lambda_2, \frac{1}{4} \, |z_1|^2 F) - R_{k,l-1}(\lambda_1, \lambda_2, \frac{1}{4} \, |z_1|^2 F) \right).
\end{align*}

Similarly, since that
\begin{align*}
\left(-it \frac{1}{4} \, |z_2|^2 F\right)^{\lambda_1, \lambda_2} (r_1, r_2) &= -\frac{\partial}{\partial \lambda_1} \left( \frac{1}{4} \, |z_2|^2 F \right)^{\lambda_1, \lambda_2} (r_1, r_2), \\
\left(-it \frac{1}{4} \, |z_1|^2 F\right)^{\lambda_1, \lambda_2} (r_1, r_2) &= -\frac{\partial}{\partial \lambda_2} \left( \frac{1}{4} \, |z_1|^2 F \right)^{\lambda_1, \lambda_2} (r_1, r_2),
\end{align*}

we have
\begin{align*}
R_{k,l}(\lambda_1, \lambda_2, -it \frac{1}{4} \, |z_2|^2 F) &= \frac{\partial}{\partial \lambda_1} R_{k,l}(\lambda_1, \lambda_2, \frac{1}{4} \, |z_2|^2 F) - R_{k,l} \left( \lambda_1, \lambda_2, \frac{1}{4} \, |z_1|^2 |z_2|^2 \right) \\
&+ \frac{k}{\lambda_1} \left( R_{k,l}(\lambda_1, \lambda_2, \frac{1}{4} \, |z_2|^2 F) - R_{k-1,l}(\lambda_1, \lambda_2, \frac{1}{4} \, |z_2|^2 F) \right),
\end{align*}

and
\begin{align*}
R_{k,l}(\lambda_1, \lambda_2, -it \frac{1}{4} \, |z_1|^2 F) &= \frac{\partial}{\partial \lambda_2} R_{k,l}(\lambda_1, \lambda_2, \frac{1}{4} \, |z_1|^2 F) - R_{k,l} \left( \lambda_1, \lambda_2, \frac{1}{4} \, |z_1|^2 |z_2|^2 \right) \\
&+ \frac{l}{\lambda_2} \left( R_{k,l}(\lambda_1, \lambda_2, \frac{1}{4} \, |z_1|^2 F) - R_{k,l-1}(\lambda_1, \lambda_2, \frac{1}{4} \, |z_1|^2 F) \right).
\end{align*}

Therefore,
\begin{align*}
R_{k,l}(\lambda_1, \lambda_2, \left(it - \frac{1}{4} \, |z_1|^2 \right) \left(it - \frac{1}{4} \, |z_2|^2 \right) F) &= \frac{\partial}{\partial \lambda_1} \frac{\partial}{\partial \lambda_2} R_{k,l}(\lambda_1, \lambda_2, F) - 2R_{k,l}(\lambda_1, \lambda_2, \frac{1}{4} \, |z_1|^2 \frac{1}{4} \, |z_2|^2) \\
&+ \frac{l}{\lambda_2} \frac{\partial}{\partial \lambda_1} \left( R_{k,l-1}(\lambda_1, \lambda_2, F) - R_{k,l}(\lambda_1, \lambda_2, F) \right) \\
&+ \frac{k}{\lambda_1} \frac{\partial}{\partial \lambda_2} \left( R_{k-1,l}(\lambda_1, \lambda_2, F) - R_{k,l}(\lambda_1, \lambda_2, F) \right) \\
&+ \frac{k}{\lambda_1} \frac{1}{\lambda_2} \left( R_{k-1,l}(\lambda_1, \lambda_2, F) + R_{k,l-1}(\lambda_1, \lambda_2, F) - R_{k,l}(\lambda_1, \lambda_2, F) - R_{k-1,l-1}(\lambda_1, \lambda_2, F) \right) \\
&+ 2 \frac{k}{\lambda_1} \left( R_{k,l}(\lambda_1, \lambda_2, \frac{1}{4} \, |z_2|^2 F) - R_{k,l}(\lambda_1, \lambda_2, \frac{1}{4} \, |z_2|^2 F) \right) \\
&+ 2 \frac{1}{\lambda_2} \left( R_{k,l}(\lambda_1, \lambda_2, \frac{1}{4} \, |z_1|^2 F) - R_{k,l}(\lambda_1, \lambda_2, \frac{1}{4} \, |z_1|^2 F) \right).
\end{align*}
Let \( \sigma = (2k + n)\lambda_1 + (2l + n)\lambda_2 \) so that \( R_{k,l}(\lambda_1, \lambda_2, F) = \psi(\sigma) \) where
\[
\psi(\sigma) = (1 - \sigma)^n.
\]

Define
\[
\psi_1(\sigma) = \left( it_1 - \frac{1}{4} |z_1|^2 \right) \left( it_2 - \frac{1}{4} |z_2|^2 \right) F.
\]

It can be written as
\[
\psi_1(\sigma) = \frac{\partial}{\partial \lambda_1} \frac{\partial}{\partial \lambda_2} \psi(\sigma) - \frac{l}{\lambda_2} \frac{\partial}{\partial \lambda_1} (\psi(\sigma) - \psi(\sigma - 2\lambda_2)) - k \frac{\partial}{\lambda_1} \frac{\partial}{\lambda_2} (\psi(\sigma) - \psi(\sigma - 2\lambda_1))
\]
\[
- \frac{k l}{\lambda_1 \lambda_2} (\psi(\sigma) - \psi(\sigma - 2\lambda_2) - \psi(\sigma - 2\lambda_1) + \psi(\sigma - 2\lambda_1 - 2\lambda_2))
\]
\[
+ C_1 \frac{k}{\lambda_1} (\psi(\sigma) - \psi(\sigma - 2\lambda_1)) + C_2 \frac{l}{\lambda_2} (\psi(\sigma) - \psi(\sigma - 2\lambda_2)) - C_3 \psi(\sigma).
\]

The function \( \psi(\sigma) \) has two properties: the first one is that \( \psi(\sigma) \) is supported in a set of the form 0 < \( (2k + n)\lambda_1 + (2l + n)\lambda_2 \leq c \) for large \( k \) and \( l \), and the second is
\[
\int_0^\infty \int_0^\infty |\psi ((2k + n)\lambda_1 + (2l + n)\lambda_2)| \, d\mu(\lambda_1) d\mu(\lambda_2) \leq C(2k + n)^{-n-1}(2l + n)^{-n-1}
\]

We claim that the function \( \psi_1(\sigma) \) also satisfies the same conditions. The first property is obvious. To verify that \( \psi_1(\sigma) \) satisfies the second property (3.3), we rewrite \( \psi_1(\sigma) \) as
\[
\psi_1(\sigma) = \frac{nk}{2\lambda_1 \lambda_2} \frac{\partial}{\partial t} \frac{\partial}{\partial k} \psi((2k + n)\lambda_1 + (2l + n)\lambda_2)
\]
\[
- \frac{nk}{2\lambda_1 \lambda_2} \frac{\partial}{\partial t} \psi((2k + n)\lambda_1 + (2l + n)\lambda_2) - \psi((2k - 2 + n)\lambda_1 + (2l + n)\lambda_2))
\]
\[
+ \frac{nl}{2\lambda_1 \lambda_2} \frac{\partial}{\partial k} \psi((2k + n)\lambda_1 + (2l + n)\lambda_2)
\]
\[
- \frac{nl}{2\lambda_1 \lambda_2} \frac{\partial}{\partial k} \psi((2k + n)\lambda_1 + (2l + n)\lambda_2) - \psi((2k + n)\lambda_1 + (2l - 2 + n)\lambda_2))
\]
\[
- \frac{kl}{\lambda_1 \lambda_2} \frac{\partial}{\partial k} \psi((2k + n)\lambda_1 + (2l + n)\lambda_2)
\]
\[
+ \frac{kl}{\lambda_1 \lambda_2} \frac{\partial}{\partial k} \psi((2k + n)\lambda_1 + (2l + n)\lambda_2) - \psi((2k + n)\lambda_1 + (2l - 2 + n)\lambda_2))
\]
\[
+ \frac{kl}{\lambda_1 \lambda_2} \frac{\partial}{\partial k} \psi((2k + n)\lambda_1 + (2l + n)\lambda_2)
\]
\[
- \frac{kl}{\lambda_1 \lambda_2} \psi((2k + n)\lambda_1 + (2l + n)\lambda_2) + \psi((2k + n)\lambda_1 + (2l - 2 + n)\lambda_2))
\]
\[
- \frac{kl}{\lambda_1 \lambda_2} \psi((2k + n)\lambda_1 + (2l + n)\lambda_2)
\]
\[
+ \frac{kl}{\lambda_1 \lambda_2} \psi((2k - 2 + n)\lambda_1 + (2l + n)\lambda_2) - \psi((2k - 2 + n)\lambda_1 + (2l - 2 + n)\lambda_2))
\]
\[
+ 2 \frac{kl}{\lambda_1 \lambda_2} \frac{\partial}{\partial t} \frac{\partial}{\partial k} \psi((2k + n)\lambda_1 + (2l + n)\lambda_2)
\]
\[
- 2 \frac{kl}{\lambda_1 \lambda_2} \frac{\partial}{\partial t} \frac{\partial}{\partial k} \psi((2k - 2 + n)\lambda_1 + (2l + n)\lambda_2) - \psi((2k - 2 + n)\lambda_1 + (2l + n)\lambda_2))
\]
Using Taylor expansion, we have that
\[
+2 \frac{kl}{\lambda_1 \lambda_2} \frac{\partial}{\partial k} \frac{\partial}{\partial l} \psi((2k + n)\lambda_1 + (2l + n)\lambda_2)
\]
\[-2 \frac{kl}{\lambda_1 \lambda_2} \frac{\partial}{\partial k} \psi((2k + n)\lambda_1 + (2l + n)\lambda_2) - \psi((2k + n)\lambda_1 + (2l + 2 + n)\lambda_2))
\]
\[-C_1 \frac{k}{\lambda_1} \frac{\partial}{\partial k} \psi((2k + n)\lambda_1 + (2l + n)\lambda_2)
\]
\[+C_1 \frac{k}{\lambda_1} (\psi((2k + n)\lambda_1 + (2l + n)\lambda_2) - \psi((2k - 2 + n)\lambda_1 + (2l + n)\lambda_2))
\]
\[-C_2 \frac{l}{\lambda_2} \frac{\partial}{\partial l} \psi((2k + n)\lambda_1 + (2l + n)\lambda_2)
\]
\[+C_1 \frac{k}{\lambda_1} \frac{\partial}{\partial k} \psi((2k + n)\lambda_1 + (2l + n)\lambda_2) + C_2 \frac{l}{\lambda_2} \frac{\partial}{\partial l} \psi((2k + n)\lambda_1 + (2l + n)\lambda_2)
\]
\[\quad - C_0 \psi((2k + n)\lambda_1 + (2l + n)\lambda_2).
\]

Using Taylor expansion, we have that
\[
\psi_1(\sigma) = 4nk\lambda_1 \int_{k-1}^k (t - 1 - k)\psi^{(3)}((2t + n)\lambda_1 + (2l + n)\lambda_2)dt
\]
\[+4nl\lambda_2 \int_{l-1}^l (s + 1 - l)\psi^{(3)}((2k + n)\lambda_1 + (2s + n)\lambda_2)ds
\]
\[-16kl\lambda_1 \lambda_2 \int_{k-1}^k \int_{k-1}^k (s + 1 - l)(t + 1 - k)\psi^{(4)}((2t + n)\lambda_1 + (2s + n)\lambda_2)dtds
\]
\[+16kl\lambda_1 \int_{k-1}^k (t - 1 - k)\psi^{(3)}((2t + n)\lambda_1 + (2l + n)\lambda_2)dt
\]
\[+16kl\lambda_2 \int_{l-1}^l (s + 1 - l)\psi^{(3)}((2k + n)\lambda_1 + (2s + n)\lambda_2)ds
\]
\[+C_1k\lambda_1 \int_{k-1}^k (t - 1 - k)\psi^{(2)}((2t + n)\lambda_1 + (2l + n)\lambda_2)dt
\]
\[+C_2l\lambda_2 \int_{l-1}^l (s + 1 - l)\psi^{(2)}((2k + n)\lambda_1 + (2s + n)\lambda_2)ds
\]
\[+(n^2 - 8kl)\psi^{(2)}((2k + n)\lambda_1 + (2l + n)\lambda_2)
\]
\[+C_1k\psi^{(1)}((2k + n)\lambda_1 + (2l + n)\lambda_2) + C_2l\psi^{(1)}((2k + n)\lambda_1 + (2l + n)\lambda_2)
\]
\[\quad - C_0 \psi((2k + n)\lambda_1 + (2l + n)\lambda_2).
\]

It follows that
\[
\int_0^\infty \int_0^\infty |\psi_1(\sigma)| \, d\mu(\lambda_1) \, d\mu(\lambda_2)
\]
\[\leq c_0 \int_0^\infty \int_0^\infty \left(1 - (2k + n)\lambda_1 - (2l + n)\lambda_2\right)^\alpha d\mu(\lambda_1) \, d\mu(\lambda_2)
\]
Proof. Then (3.2) holds and the proof of Theorem 1 is completed. We can get the same result by a similar calculation when

\[ \lambda (1 - (2t + n)\lambda_1 - (2l + n)\lambda_2)_{+}^{\alpha - 2} d\mu(\lambda_1)d\mu(\lambda_2) \]

\[ \lambda_1 (1 - (2t + n)\lambda_1 - (2l + n)\lambda_2)_{+}^{\alpha - 3} d\mu(\lambda_1)d\mu(\lambda_2)dt \]

\[ \lambda_2 (1 - (2k + n)\lambda_1 - (2s + n)\lambda_2)_{+}^{\alpha - 3} d\mu(\lambda_1)d\mu(\lambda_2)ds \]

\[ \lambda_1_2 (1 - (2t + n)\lambda_1 - (2s + n)\lambda_2)_{+}^{\alpha - 4} d\mu(\lambda_1)d\mu(\lambda_2)dtds \]

\[ \leq C'(2k + n)^{-n-1}(2l + n)^{-n-1}. \]

This proves that \( \psi_1(\sigma) \) also has the second property (3.3). An iteration of the process shows that

\[ R_{k,l}(\lambda_1, \lambda_2, (it_1 - \frac{1}{4} r_1^2)^j (it_2 - \frac{1}{4} r_2^2)^j F) = \psi_j(\sigma) \]

satisfies the condition

\[ \int_0^{\infty} \int_0^{\infty} |\psi_j((2k + n)\lambda_1 + (2l + n)\lambda_2)| d\mu(\lambda_1)d\mu(\lambda_2) \leq C_j(2k + n)^{-n-1}(2l + n)^{-n-1} \]

provided \((1 - \lambda_1 - \lambda_2)_{+}^{\alpha - 1}\) is integrable. Thus, when \( j = m \) and \( \alpha > 4m - 1 \), we have

\[ \int_0^{\infty} \int_0^{\infty} R_{k,l}(\lambda_1, \lambda_2, (it_1 - \frac{1}{4} |z_1|^2)^m (it_2 - \frac{1}{4} |z_2|^2)^m F) d\mu(\lambda_1)d\mu(\lambda_2) \]

\[ \leq C_m(2k + n)^{-n-1}(2l + n)^{-n-1}, \]

which implies

\[ \int_0^{\infty} \int_0^{\infty} \left( \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} R_{k,l}(\lambda_1, \lambda_2, (it_1 - \frac{1}{4} |z_1|^2)^m (it_2 - \frac{1}{4} |z_2|^2)^m F) \right) ^{1/k!} \frac{(k + n - 1)! (l + n - 1)!}{l!} d\mu(\lambda_1)d\mu(\lambda_2) < \infty. \]

We can get the same result by a similar calculation when \((\lambda_1, \lambda_2)\) belongs to other quadrants. Then (3.2) holds and the proof of Theorem 1 is completed.

As a consequence of Theorem 1, we have

**Corollary 1.** Let \( 1 \leq p_1, p_2 \leq \infty \) and \( 1/p = 1/p_1 + 1/p_2 \). If \( \alpha > 2Q + 3 \), then \( S^\alpha \) is bounded from \( L^{p_1}(\mathbb{H}^n) \times L^{p_2}(\mathbb{H}^n) \) into \( L^p(\mathbb{H}^n) \).

**Proof.** We take \( m = \frac{Q}{2} + 1 \). If \( \alpha > 2Q + 3 \), we have \( \alpha > 4m - 1 \). Then, Theorem 1 is available. By Hölder’s inequality and Young’s inequality, we conclude that

\[ \| S^\alpha(f, g) \|_p \leq C \| f \|_{p_1} \| g \|_{p_2} \int_{\mathbb{H}^n} (1 + |\omega_1|)^{-2m} d\omega_1 \int_{\mathbb{H}^n} (1 + |\omega_2|)^{-2m} d\omega_2 \]
Then, for any

**Lemma 1.**

\[ d \text{istinguish various cases and manage to give lower indices.} \]

Applying the Plancherel theorem in the variable \( t \), we get that

\[ \|T_m f\|_2 \leq C \|m\|_{\infty} ((b-a)b^n)^{\left(\frac{1}{2} - \frac{1}{p}\right)} \|f\|_p. \]

**Proof.** Since that

\[ P_\lambda f = \sum_{k=0}^{\infty} (2k + n)^{-n-1} f * \left( \hat{e}_k^\lambda + \hat{e}_k^{-\lambda} \right)(z, t), \]

the operator \( T_m \) can be written as

\[ T_m f = \int_a^b m(\lambda)P_\lambda f d\mu(\lambda) = f * G_m, \]

where the kernel is given by

\[ G_m(z, t) = \int_a^b m(\lambda) \sum_{k=0}^{\infty} (2k + n)^{-n-1} \left( \hat{e}_k^\lambda + \hat{e}_k^{-\lambda} \right) d\mu(\lambda) \]

we get that

\[ \|G_m\|_2^2 \leq C \int_{\mathbb{R}^n} \left( \sum_{k=0}^{\infty} \int_a^b (e^{-iM} + e^{iM}) m((2k + n)\lambda) |\varphi_k^\lambda(z)| |\lambda|^n d\lambda \right)^2 dt dz \]

\[ = C \int_{\mathbb{R}^n} \left( \sum_{k=0}^{\infty} \chi_{\frac{a}{2k+n}} \frac{b}{2k+n} \left( |\lambda| \right) m((2k + n)|\lambda|) |\varphi_k^\lambda(z)| |\lambda|^n d\lambda \right)^2 d\lambda \]

\[ \leq C \left( \sum_{k=0}^{\infty} (2k + n)^{-n-1} k^{n-1} \right) \int_a^b |m(\lambda)|^2 |\lambda|^n d\lambda \]

\[ \leq C \|m\|_{\infty}^2 (b-a)b^n. \]

The proof is completed.

We note that the index in Corollary 1 is very high. In the rest part of this paper, we will distinguish various cases and manage to give lower indices.

4. BOUNDEDNESS OF \( S^\alpha \) FOR \( 1 \leq p_1, p_2 \leq 2 \)

In this Section, we investigate the \( L^{p_1} \times L^{p_2} \rightarrow L^p \) boundedness of the bilinear Riesz means \( S^\alpha \) for \( 1 \leq p_1, p_2 \leq 2 \).

**Lemma 1.** Suppose \( m \in L^\infty(\mathbb{R}) \). Define operator \( T_m f = \int_0^b m(\lambda) P_\lambda f d\mu(\lambda) \) for \( 0 \leq a < b \). Then, for any \( 1 \leq p \leq 2 \), we have

\[ \|T_m f\|_p \leq C \|m\|_{\infty} ((b-a)b^n)^{\left(\frac{1}{2} - \frac{1}{p}\right)} \|f\|_p. \]

**Proof.** Since that

\[ P_\lambda f = \sum_{k=0}^{\infty} (2k + n)^{-n-1} f * \left( \hat{e}_k^\lambda + \hat{e}_k^{-\lambda} \right)(z, t), \]

the operator \( T_m \) can be written as

\[ T_m f = \int_a^b m(\lambda)P_\lambda f d\mu(\lambda) = f * G_m, \]

where the kernel is given by

\[ G_m(z, t) = \int_a^b m(\lambda) \sum_{k=0}^{\infty} (2k + n)^{-n-1} \left( \hat{e}_k^\lambda + \hat{e}_k^{-\lambda} \right) d\mu(\lambda) \]

we get that

\[ \|G_m\|_2^2 \leq C \int_{\mathbb{R}^n} \left( \sum_{k=0}^{\infty} \int_a^b (e^{-iM} + e^{iM}) m((2k + n)\lambda) |\varphi_k^\lambda(z)| |\lambda|^n d\lambda \right)^2 dt dz \]

\[ = C \int_{\mathbb{R}^n} \left( \sum_{k=0}^{\infty} \chi_{\frac{a}{2k+n}} \frac{b}{2k+n} \left( |\lambda| \right) m((2k + n)|\lambda|) |\varphi_k^\lambda(z)| |\lambda|^n d\lambda \right)^2 d\lambda \]

\[ \leq C \left( \sum_{k=0}^{\infty} (2k + n)^{-n-1} k^{n-1} \right) \int_a^b |m(\lambda)|^2 |\lambda|^n d\lambda \]

\[ \leq C \|m\|_{\infty}^2 (b-a)b^n. \]
This implies that
\[ \|T_m f\|_2 \leq C ((b-a) b^a)^{\frac{1}{2}} \|m\|_\infty \|f\|_1. \]
By interpolation with the trivial estimate
\[ \|T_m f\|_2 \leq \|m\|_\infty \|f\|_2, \]
we conclude that
\[ \|T_m f\|_2 \leq C ((b-a) b^a)^{\frac{1}{p} - \frac{1}{2}} \|m\|_\infty \|f\|_p. \]
The proof is completed. \(\square\)

**Theorem 2.** Suppose that \(1 \leq p_1, p_2 \leq 2\) and \(1/p = 1/p_1 + 1/p_2\). If \(\alpha > Q \left( \frac{1}{p} - 1 \right)\), then \(S^\alpha\) is bounded from \(L^{p_1}(\mathbb{H}^n) \times L^{p_2}(\mathbb{H}^n)\) into \(L^p(\mathbb{H}^n)\).

**Proof.** We choose a nonnegative function \(\varphi \in C_0^\infty(\frac{1}{2}, 2)\) satisfying \(\sum_{j=0}^{\infty} \varphi(2^j s) = 1, s > 0\).

For each \(j \geq 0\), we set function
\[ \varphi_j^\alpha(s, t) = (1 - s - t)^\alpha \varphi(2^j (1 - s - t)), \]
and define bilinear operator
\[ T_j^\alpha(f, g) = \int_0^\infty \int_0^\infty \varphi_j^\alpha(\lambda_1, \lambda_2) P_{\lambda_1} f P_{\lambda_2} g \, d\mu(\lambda_1) d\mu(\lambda_2). \]
It is obvious that
\[ S^\alpha = \sum_{j=0}^{\infty} T_j^\alpha, \]
and our result would follow if we can show that when \(\alpha > Q \left( \frac{1}{p} - 1 \right)\), there exists an \(\varepsilon > 0\) such that for each \(j \geq 0\),
\begin{equation}
\|T_j^\alpha\|_{L^1 \times L^{p_2} \to L^p} \leq C 2^{-\varepsilon j}.
\end{equation}
Fixing \(j \geq 0\). In order to prove (4.2), we define \(B_j = \{ \omega : |\omega| \leq 2^{j(1+\gamma)} \} \subseteq \mathbb{H}^n\) and split the kernel \(K_j^\alpha\) of \(T_j^\alpha\) into four parts:
\[ K_j^\alpha = K_j^1 + K_j^2 + K_j^3 + K_j^4, \]
where
\begin{align*}
K_j^1(\omega_1, \omega_2) &= K_j^0(\omega_1, \omega_2) \chi_{B_j}(\omega_1) \chi_{B_j}(\omega_2), \\
K_j^2(\omega_1, \omega_2) &= K_j^0(\omega_1, \omega_2) \chi_{B_j}(\omega_1) \chi_{B_j^c}(\omega_2), \\
K_j^3(\omega_1, \omega_2) &= K_j^0(\omega_1, \omega_2) \chi_{B_j^c}(\omega_1) \chi_{B_j}(\omega_2), \\
K_j^4(\omega_1, \omega_2) &= K_j^0(\omega_1, \omega_2) \chi_{B_j^c}(\omega_1) \chi_{B_j^c}(\omega_2).
\end{align*}
Here \(\chi_A\) stands for the characteristic function of set \(A\) and \(\gamma > 0\) is to be fixed. Let \(T_j^l\) be the bilinear operator with kernel \(K_j^l, l = 1, 2, 3, 4\). Then, (4.2) would be the consequence of the estimates
\[ \|T_j^l\|_{L^{p_1} \times L^{p_2} \to L^p} \leq C 2^{-\varepsilon j}, \quad l = 1, 2, 3, 4.\]

We first consider \(T_j^4\). Set \(R_j^l(\omega)\) to be the kernel of the Riesz means \(\int_0^\infty (1 - \frac{t}{j})^l P_{\lambda} d\mu(\lambda)\). Then,
\[ t \to R_j^0(\omega) \]
is a function of bounded variation and the kernel of $T_j^\alpha$ can be written as

$$K_j^\alpha(\omega_1, \omega_2) = \int \varphi_j^\alpha(s, t) \frac{\partial}{\partial s} R_s^0(\omega_1) \frac{\partial}{\partial t} R_t^0(\omega_2) \, ds \, dt.$$ 

Intergrating by parts and using the identity

$$\frac{\partial}{\partial t}(t^m R_t^m(\omega)) = mt^{m-1} R_t^{m-1}(\omega),$$

where $m$ is a positive integer, we get the relation

$$K_j^\alpha(\omega_1, \omega_2) = c_m \int \left( (\partial_s \partial_t)^{2m+2} \varphi_j^\alpha(s, t) \right) s^{2m+1} R_s^{2m+1}(\omega_1) t^{2m+1} R_t^{2m+1}(\omega_2) \, ds \, dt.$$

Note that (see Theorem 2.5.3 in [10])

$$|R_t^{2m+1}(\omega)| \leq C t^{n+1}(1 + t^2 |\omega|)^{-2m},$$

This estimate, together with the bound

$$|q_j^{2m+2} \varphi_j^\alpha| \leq C 2^{j(4m+4)}$$

imply that

$$|K_j^\alpha(\omega_1, \omega_2)| \leq C 2^{j(4m+4)} (1 + |\omega_1|)^{-2m} (1 + |\omega_2|)^{-2m}.$$ 

So, we can use Hölder’s inequality and Young’s inequality to get that

$$\|T_j^4(f, g)\|_p \leq C \|f\|_{p_1} \|g\|_{p_2} 2^{j(4m+4)} \int_{|\omega_1| \geq 2^{j(1+\gamma)}} (1 + |\omega_1|)^{-2m} d\omega_1$$

$$\times \int_{|\omega_2| \geq 2^{j(1+\gamma)}} (1 + |\omega_2|)^{-2m} d\omega_2$$

$$\leq C 2^{j(4m+4)} 2^{j(1+\gamma)(-4m+2Q)} \|f\|_{p_1} \|g\|_{p_2}.$$ 

Choosing $m$ large enough such that

$$4m\gamma \geq 2Q(1 + \gamma) + 4,$$

we have

$$\|T_j^4\|_{L^p \times L^p \rightarrow L^p} \leq C 2^{-\epsilon j}$$

for some $\epsilon > 0$.

Next, we consider the estimate of $T_j^3$. Note that $K_j^\alpha$ also can be written as

$$K_j^\alpha(\omega_1, \omega_2) = c_m \int_0^1 \int_0^1 \left( \partial_{\lambda_1}^{2m+2} \varphi_j^\alpha(\lambda_1, \lambda_2) \right) \lambda_1^{2m+1} R_{\lambda_1}^{2m+1}(\omega_1) G_{\lambda_2}(\omega_2) \, d\lambda_1 \, d\mu(\lambda_2)$$

where

$$G_{\lambda_2}(\omega_2) = G_{\lambda_2}(z, t) = \sum_{k=0}^{\infty} (2k + n)^{-n-1} (\tilde{c}_{k+2}^{\lambda_2} + \tilde{c}_{k}^{\lambda_2})(z, t)$$

is the kernel of the projection operator $P_{\lambda_2}$. Then, it follows that

$$|K_j^3(\omega_1, \omega_2)|$$

$$= |K_j^\alpha(\omega_1, \omega_2) \chi_{B_j^\alpha}(\omega_1) \chi_{B_j^\alpha}(\omega_2)|$$

$$\leq C \int_0^1 |\lambda_1^{2m+1} R_{\lambda_1}^{2m+1}(\omega_1) \chi_{B_j^\alpha}(\omega_1)| \left| \int_0^1 \left( \partial_{\lambda_2}^{2m+2} \varphi_j^\alpha(\lambda_1, \lambda_2) \right) G_{\lambda_2}(\omega_2) \chi_{B_j^\alpha}(\omega_2) \, d\mu(\lambda_2) \right| \, d\lambda_1$$

$$\leq C \int_0^1 |\lambda_1^{2m+1} R_{\lambda_1}^{2m+1}(\omega_1) \chi_{B_j^\alpha}(\omega_1)| \left| \int_0^1 \left( \partial_{\lambda_2}^{2m+2} \varphi_j^\alpha(\lambda_1, \lambda_2) \right) G_{\lambda_2}(\omega_2) \chi_{B_j^\alpha}(\omega_2) \, d\mu(\lambda_2) \right| \, d\lambda_1.$$
Thus, we get

$$
\| T^3_j(f,g) \|_p \leq C \| f \|_{p_1} \| g \|_{p_2} \int_{|\omega_1| \geq 2j^{(1+\gamma)}} \sup_{\lambda_1 \in [0,1]} \left| \lambda_1^{2m+1} R_{\lambda_1}^{2m+1}(\omega_1) \right| \left| \int_0^1 \int_0^1 \left( \partial_{\lambda_2}^{2m+2} \varphi_j^\alpha(\lambda_1, \lambda_2) \right) G_{\lambda_2}(\omega_2) \chi_{B_j}(\omega_2) \, d\mu(\lambda_2) \right| \, d\lambda_1 \, d\omega_1
$$

Applying the Plancherel theorem in the variable \( t \), the orthogonality of \( \varphi_k^\alpha \) and (4.1), we have that

$$
\int_0^1 \int_{|\omega_2| \leq 2j^{(1+\gamma)}} \left| \int_0^1 \left( \partial_{\lambda_2}^{2m+2} \varphi_j^\alpha(\lambda_1, \lambda_2) \right) G_{\lambda_2}(\omega_2) \, d\mu(\lambda_2) \right| \, d\omega_2 \, d\lambda_1
$$

$$
\leq C 2^{j^{(1+\gamma)}Q} \frac{1}{2} \int_0^1 \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{k=0}^{\infty} \int_{0}^{\frac{2\pi}{2\pi+k}} e^{-i\lambda_2 t} \left( \partial_{\lambda_2}^{2m+2} \varphi_j^\alpha(\lambda_1, (2k+n)|\lambda_2|) \right) \varphi_k^\lambda(z) \, d\mu(\lambda_2) \right| \, dtd\lambda_1
$$

$$
\leq C 2^{j^{(1+\gamma)}Q} \frac{1}{2} \int_0^1 \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{k=0}^{\infty} \chi_{[0,\frac{2\pi}{2\pi+k}]}(\lambda_2) \left( \partial_{\lambda_2}^{2m+2} \varphi_j^\alpha(\lambda_1, (2k+n)|\lambda_2|) \right) \varphi_k^\lambda(z) \right| \, d\mu(\lambda_2) \right| \, d\omega_2 \, d\lambda_1
$$

$$
= C 2^{j^{(1+\gamma)}Q} \frac{1}{2} \int_0^1 \left( \sum_{k=0}^{\infty} \int_{0}^{\frac{2\pi}{2\pi+k}} \left| \partial_{\lambda_2}^{2m+2} \varphi_j^\alpha(\lambda_1, (2k+n)|\lambda_2|) \right| ^2 \, |\lambda_2|^{2n} \, d\lambda_2 \right)^{\frac{1}{2}} \, d\lambda_1
$$

$$
\leq C 2^{j^{(1+\gamma)}Q} \frac{1}{2} \int_0^1 \left( \sum_{k=0}^{\infty} (2k+n)^{-n-1} k^{-n-1} \int_0^1 \left| \partial_{\lambda_2}^{2m+2} \varphi_j^\alpha(\lambda_1, |\lambda_2|) \right| ^2 \, |\lambda_2|^{2n} \, d\lambda_2 \right)^{\frac{1}{2}} \, d\lambda_1
$$

$$
\leq C 2^{j^{(1+\gamma)}Q} \frac{1}{2} \int_0^1 \left( \int_0^1 \left| \partial_{\lambda_2}^{2m+2} \varphi_j^\alpha(\lambda_1, |\lambda_2|) \right| ^2 \, |\lambda_2|^{2n} \, d\lambda_2 \right)^{\frac{1}{2}} \, d\lambda_1
$$

On the other hand, from (4.3), we have

$$
\sup_{\lambda_1 \in [0,1]} \left| \lambda_1^{2m+1} R_{\lambda_1}^{2m+1}(\omega_1) \right| \leq C (1 + |\omega_1|)^{-2m},
$$

which implies that

$$
\int_{|\omega_1| \geq 2j^{(1+\gamma)}} \sup_{\lambda_1 \in [0,1]} \left| \lambda_1^{2m+1} R_{\lambda_1}^{2m+1}(\omega_1) \right| \, d\omega_1 \leq C \int_{|\omega_1| \geq 2j^{(1+\gamma)}} (1 + |\omega_1|)^{-2m} \, d\omega_1 \leq C 2^{j^{(1+\gamma)}(-2m+Q)}.
$$

Thus,

$$
\| T^3_j(f,g) \|_p \leq C \| f \|_{p_1} \| g \|_{p_2} 2^{j^{(1+\gamma)}Q} 2^{j(2m+2)} 2^{j(1+\gamma)(-2m+Q)},
$$

$$
\| T^3_j(f,g) \|_p \leq C \| f \|_{p_1} \| g \|_{p_2} 2^{j^{(1+\gamma)}Q} 2^{j(2m+2)} 2^{j(1+\gamma)(-2m+Q)},
$$
and we can choose \( m \) large enough such that
\[
2m\gamma \geq \frac{3}{2}Q(1 + \gamma) + 2,
\]
which yields that
\[
(4.5) \quad \|T_j^3\|_{L^p_1 \times L^{p_2} \to L^p} \leq C 2^{-\varepsilon j}
\]
for some \( \varepsilon > 0 \). Obviously, (4.5) also holds for \( T_j^2 \).

Now, it remains to estimate \( T_j^1 \). For \( \xi \in \mathbb{H}^n \), we set \( B_j(\xi, R) = \{ \omega : |\xi^{-1}\omega| \leq R 2^{j(1+\gamma)} \} \) with \( R > 0 \), and split the functions \( f \) and \( g \) into three parts respectively: \( f = f_1 + f_2 + f_3 \), \( g = g_1 + g_2 + g_3 \), where
\[
\begin{align*}
 f_1 &= f \chi_{B_j(\xi, \frac{3}{4})}, \\
 f_2 &= f \chi_{B_j(\xi, \frac{3}{4}) \setminus B_j(\xi, \frac{5}{4})}, \\
 f_3 &= f \chi_{\mathbb{H}^n \setminus B_j(\xi, \frac{3}{4})}.
\end{align*}
\]
Assume that \( |\xi^{-1}\omega| \leq \frac{1}{2} 2^{j(1+\gamma)} \). Since that \( f_3 \) is supported on \( \mathbb{H}^n \setminus B_j(\xi, \frac{5}{4}) \), then \( f_3 \neq 0 \) leads to
\[
|\xi^{-1}\omega| \geq \frac{5}{4} 2^{j(1+\gamma)}.
\]
It follows that
\[
|\omega^{-1}\omega| \geq 2^{j(1+\gamma)}.
\]
Note that the kernel \( K_j^1 \) is supported on \( B_j \times B_j \). Hence, \( T_j^1(f_3, g) = 0 \). In the same way, \( T_j^1(f_2, g_3) = 0 \). Since that \( f_2 \) and \( g_2 \) are supported on \( B_j(\xi, \frac{5}{4}) \setminus B_j(\xi, \frac{3}{4}) \), then \( f_2, g_2 \neq 0 \) yields that
\[
|\omega^{-1}\omega| \geq \frac{1}{2} 2^{j(1+\gamma)} \quad \text{and} \quad |\omega^{-1}\omega| \geq \frac{1}{2} 2^{j(1+\gamma)}.
\]
Repeating the proof of (4.4), we get
\[
(4.6) \quad \|T_j^1(f_2, g_2)\|_{L^p(B_j(\xi, \frac{3}{4}))} \leq C 2^{-\varepsilon j} \|f_2\|_{L^p} \|g_2\|_{L^{p_2}} \leq C 2^{-\varepsilon j} \|f\|_{L^p(B_j(\xi, \frac{3}{4}))} \|g\|_{L^{p_2}(B_j(\xi, \frac{3}{4}))}.
\]
Taking the \( L^p \) norm with respect to \( \xi \) on the both side of (4.6) and using Hölder’s inequality, we have that
\[
\left( \int_{\mathbb{H}^n} \int_{B_j(\xi, \frac{3}{4})} |T_j^1(f_2, g_2)(\omega)|^p \, d\omega d\xi \right)^{\frac{1}{p}} \leq C 2^{-\varepsilon j} \left( \int_{\mathbb{H}^n} \int_{B_j(\xi, \frac{3}{4})} |f(\omega)|^{p_1} \, d\omega d\xi \right)^{\frac{1}{p_1}} \left( \int_{\mathbb{H}^n} \int_{B_j(\xi, \frac{3}{4})} |g(\omega)|^{p_2} \, d\omega d\xi \right)^{\frac{1}{p_2}}.
\]
Changing variable and exchanging the order of integration, the left side equals to
\[
\left( \int_{\mathbb{H}^n} \int_{|\omega| \leq \frac{1}{4} 2^{j(1+\gamma)}} |T_j^1(f_2, g_2)(\xi\omega)|^p \, d\omega d\xi \right)^{\frac{1}{p}} = \left( \int_{|\omega| \leq \frac{1}{4} 2^{j(1+\gamma)}} \int_{\mathbb{H}^n} |T_j^1(f_2, g_2)(\xi\omega)|^p \, d\xi d\omega \right)^{\frac{1}{p}} = \left( \frac{1}{4} 2^{j(1+\gamma)} \right)^{\frac{Q}{p}} \|T_j^1(f_2, g_2)\|_{L^p}.
\]
and the right side equals to
\[ C 2^{-\epsilon j} \left( \frac{5}{4} 2^{j(1+\gamma)} \right)^{\frac{q}{2}} \| f \|_{p_1} \left( \frac{5}{4} 2^{j(1+\gamma)} \right)^{\frac{q}{2}} \| g \|_{p_2} = C 2^{-\epsilon j} \left( \frac{5}{4} 2^{j(1+\gamma)} \right)^{\frac{q}{2}} \| f \|_{p_1} \| g \|_{p_2}. \]
This yields that
\[ (4.7) \quad \| T_j^1(f_2, g_2) \|_p \leq C 2^{-\epsilon j} \| f \|_{p_1} \| g \|_{p_2}. \]
Since \( f_1 \) is supported on \( B_j(\xi, \frac{3}{4}) \), then \( f_1, g_2 \neq 0 \) implies that
\[ |\omega_1^{-1} \omega| \leq 2^{j(1+\gamma)} \quad \text{and} \quad |\omega_2^{-1} \omega| \geq \frac{1}{2} 2^{j(1+\gamma)}. \]
We can repeat the proof of (4.5) to get that
\[ \| T_j^1(f_1, g_2) \|_{L^p(B_j(\xi, \frac{3}{4}))} \leq C 2^{-\epsilon j} \| f_1 \|_{p_1} \| g_2 \|_{p_2} \leq C 2^{-\epsilon j} \| f \|_{L^p(B_j(\xi, \frac{3}{4}))} \| g \|_{L^p(B_j(\xi, \frac{3}{4}))}. \]
Taking the \( L^p \) norm with respect to \( \xi \) as above, we have
\[ (4.8) \quad \| T_j^1(f_1, g_2) \|_p \leq C 2^{-\epsilon j} \| f \|_{p_1} \| g \|_{p_2}. \]
Obviously, (4.8) also holds for \( T_j^1(f_2, g_1) \). Finally, we consider \( T_j^1(f_1, g_1) \). Because \( f_1, g_1 \neq 0 \) implies that
\[ |\omega_1^{-1} \omega| \leq 2^{j(1+\gamma)} \quad \text{and} \quad |\omega_2^{-1} \omega| \leq 2^{j(1+\gamma)}, \]
we have
\[ (4.9) \quad T_j^1(f_1, g_1)(\omega) = T_j^\alpha(f_1, g_1)(\omega). \]
Note that \( T_j^\alpha \) can be written as
\[
T_j^\alpha(f, g) = \int_{-1}^{1} \int_{-1}^{1} \varphi_j^\alpha(\lambda_1, \lambda_2) P_{\lambda_1} f P_{\lambda_2} g d\mu(\lambda_1) d\mu(\lambda_2) = C \int_{[-1,1]^2} \varphi_j^\alpha(\{\lambda_1, \lambda_2\}) P_{\lambda_1} f P_{\lambda_2} g d\mu(\lambda_1) d\mu(\lambda_2). 
\]
Because for any fixed \( s \in [-1, 1] \), the function
\[ t \to \varphi_j^\alpha(|s|, |t|) \]
is supported in \([-1, 1]\) and vanishes at endpoints \( \pm 1 \), so we can expand this function in Fourier series by considering a periodic extension on \( \mathbb{R} \) of period 2. Then, we have
\[ \varphi_j^\alpha(|s|, |t|) = \sum_{k \in \mathbb{Z}} \gamma_j^{\alpha,k}(s) e^{i\pi k t} \]
with Fourier coefficients
\[ \gamma_j^{\alpha,k}(s) = \frac{1}{2} \int_{-1}^{1} \varphi_j^\alpha(|s|, |t|) e^{-i\pi k t} dt. \]
It is easy to see that for any \( 0 < \delta < \alpha \),
\[ \sup_{s \in [-1,1]} |\gamma_j^{\alpha,k}(s)| (1 + |k|)^{1+\delta} \leq C 2^{-j(\alpha-\delta)}. \]
\( T_j^\alpha \) can be expressed by
\[ T_j^\alpha(f, g) = C \int_{[-1,1]^2} \varphi_j^\alpha(\{\lambda_1, \lambda_2\}) P_{\lambda_1} f P_{\lambda_2} g d\mu(\lambda_1) d\mu(\lambda_2). \]
Combining (4.7), (4.8) and (4.11), we conclude that

$$= C \sum_{k \in \mathbb{Z}} \int_{[-1,1]^2} \gamma_{j,k}^\alpha(\lambda_1) e^{i\pi_k \lambda_2} P_{|\lambda_1|} f P_{|\lambda_2|} g \, d\mu(\lambda_1) d\mu(\lambda_2)$$

$$= C \sum_{k \in \mathbb{Z}} \int_{-1}^1 \gamma_{j,k}^\alpha(\lambda_1) P_{|\lambda_1|} f \, d\mu(\lambda_1) \int_{-1}^1 e^{i\pi_k \lambda_2} P_{|\lambda_2|} g \, d\mu(\lambda_2).$$

Applying Cauchy-Schwartz’s inequality and Lemma 1, we have

$$\|T_j^\alpha(f, g)\|_1 \leq C \sum_{k \in \mathbb{Z}} \left\| \int_{-1}^1 \gamma_{j,k}^\alpha(\lambda_1) P_{|\lambda_1|} f \, d\mu(\lambda_1) \right\|_2 \left\| \int_{-1}^1 e^{i\pi_k \lambda_2} P_{|\lambda_2|} g \, d\mu(\lambda_2) \right\|_2$$

$$\leq C \sum_{k \in \mathbb{Z}} (1 + |k|)^{-1/\delta} \left( \sup_{s \in [-1,1]} |\gamma_{j,k}^\alpha(s)| (1 + |k|)^{1+\delta} \right) \|f\|_{p_1} \|g\|_{p_2}$$

$$\leq C 2^{-j(\alpha - \delta)} \|f\|_{p_1} \|g\|_{p_2}.$$

Then, using Hölder’s inequality and (4.9), it follows that

$$\|T_j^1(f_1, g_1)\|_{L^p(B_j(e^\frac{i}{4}))} \leq 2^{j(1+\gamma)Q\left(\frac{1}{p} - 1\right)} \|T_j^1(f_1, g_1)\|_{L^1(B_j(e^\frac{i}{4}))}$$

$$= 2^{j(1+\gamma)Q\left(\frac{1}{p} - 1\right)} \|T_j^\alpha(f_1, g_1)\|_{L^1(B_j(e^\frac{i}{4}))}$$

$$\leq C 2^{-j(\alpha - \delta)} 2^{j(1+\gamma)Q\left(\frac{1}{p} - 1\right)} \|f_1\|_{p_1} \|g_1\|_{p_2}$$

$$\leq C 2^{-j(\alpha - \delta)} 2^{j(1+\gamma)Q\left(\frac{1}{p} - 1\right)} \|f\|_{L^{p_1}(B_j(e^\frac{i}{4}))} \|g\|_{L^{p_2}(B_j(e^\frac{i}{4}))}.$$

Taking the $L^p$ norm with respect to $\xi$ yields that

$$\|T_j^1(f_1, g_1)\|_p \leq C 2^{-j(\alpha - \delta)} 2^{j(1+\gamma)Q\left(\frac{1}{p} - 1\right)} \|f\|_{p_1} \|g\|_{p_2}.$$

Combining (4.7), (4.8) and (4.11), we conclude that

$$\|T_j^1(f, g)\|_p \leq C 2^{-j(\alpha - \delta)} 2^{j(1+\gamma)Q\left(\frac{1}{p} - 1\right)} \|f\|_{p_1} \|g\|_{p_2}.$$

Thus, whenever $\alpha > Q\left(\frac{1}{p} - 1\right)$, we can choose $\gamma, \delta > 0$ such that

$$\alpha > Q(1 + \gamma)\left(\frac{1}{p} - 1\right) + \delta,$$

which implies that there exists an $\varepsilon > 0$ such that

$$\|T_j^1\|_{L^1 \times L^p \rightarrow L^p} \leq 2^{-\varepsilon j}.$$

The proof of Theorem 2 is completed. \(\square\)

5. Boundedness of $S^\alpha$ for particular points

In this section, we investigate the boundedness of $S^\alpha$ for some specific triples of points $(p_1, p_2, p)$. 
5.1. **The point** \((1, \infty, 1)\).

**Theorem 3.** If \(\alpha > \frac{Q}{2}\), then \(S^\alpha\) is bounded from \(L^1(\mathbb{H}^n) \times L^\infty(\mathbb{H}^n)\) to \(L^1(\mathbb{H}^n)\).

**Proof.** We keep the notations in Section 4. Note that \((4.4), (4.5), (4.7)\) and \((1.8)\) hold for any \(\alpha > 0\), the proof of Theorem 2 is valid apart from the estimate of \(T_j^1(f_1, g_1)\). According to \((4.10)\), for any \(0 < \delta < \alpha\),

\[
\|T_j^\alpha(f, g)\|_1 \leq C2^{-j(\alpha-\delta)\|f\|_1\|g\|_2},
\]

we have

\[
\|T_j^1(f_1, g_1)\|_{L^1(B_j(\xi, \frac{1}{2}))} = \|T_j^\alpha(f_1, g_1)\|_{L^1(B_j(\xi, \frac{1}{2}))} \\
\leq C2^{-j(\alpha-\delta)}\|f\|_{L^1(B_j(\xi, \frac{3}{2}))}\|g\|_{L^2(B_j(\xi, \frac{3}{2}))} \\
\leq C2^{-j(\alpha-\delta)}2^{j(1+\gamma)\frac{Q}{2}}\|f\|_{L^1(B_j(\xi, \frac{3}{2}))}\|g\|_{L^\infty(B_j(\xi, \frac{1}{2}))}.
\]

It follows that

\[
\|T_j^1(f_1, g_1)\|_1 \leq C2^{-j(\alpha-\delta)}2^{j(1+\gamma)\frac{Q}{2}}\|f\|_1\|g\|_\infty.
\]

Thus, when \(\alpha > \frac{Q}{2}\), we can choose \(\gamma, \delta > 0\) such that \(\alpha > \frac{(1+\gamma)Q}{2} + \delta\), which yields that there exists \(\varepsilon > 0\) such that

\[
\|T_j^1(f_1, g_1)\|_1 \leq C2^{-\varepsilon j}\|f\|_1\|g\|_\infty.
\]

The proof is completed. \(\square\)

5.2. **The point** \((\infty, \infty, \infty)\).

**Theorem 4.** If \(\alpha > Q - \frac{1}{2}\), then \(S^\alpha\) is bounded from \(L^\infty(\mathbb{H}^n) \times L^\infty(\mathbb{H}^n)\) into \(L^\infty(\mathbb{H}^n)\).

**Proof.** We still keep the notation in Section 4. To prove this theorem, it suffices to estimate \(T_j^1(f_1, g_1)\). Since that

\[
P_{\lambda}f(z, t) = \sum_{k=0}^{\infty}(2k + n)^{-n-1}f^* (\tilde{c}_k^\lambda + \tilde{c}_k^{-\lambda})(z, t), \quad \lambda > 0,
\]

and

\[
f \ast e_k^\lambda = e^{-i\lambda t}f^\lambda \ast_{\lambda} \varphi_k^\lambda,
\]

we can write \(T_j^\alpha(f, g)\) as

\[
T_j^\alpha(f, g)(z, t) = \int_0^\infty \int_0^\infty \varphi_j^\alpha(\lambda_1, \lambda_2) \sum_{k=0}^{\infty}(2k + n)^{-n-1}f^* (\tilde{c}_k^\lambda + \tilde{c}_k^{-\lambda})(z, t) \\
\times \sum_{l=0}^{\infty}(2l + n)^{-n-1}g^* (\tilde{c}_l^\lambda + \tilde{c}_l^{-\lambda})(z, t) \, d\mu(\lambda_1)d\mu(\lambda_2)
\]

\[
= C \int_\mathbb{R} \int_\mathbb{R} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} e^{-i(\lambda_1 + \lambda_2)t} \varphi_j^\alpha ((2k + n)|\lambda_1|, (2l + n)|\lambda_2|) \\
\times (f^\lambda \ast_{\lambda_1} \varphi_k^\lambda)(z) (g^{\lambda_2} \ast_{\lambda_2} \varphi_l^\lambda)(z) |\lambda_1|^n |\lambda_2|^n \, d\lambda_1d\lambda_2.
\]

Using again \((4.1)\), for \(0 < \delta < 1\), we get

\[
\|T_j^\alpha(f, g)\|_\infty
\]
\[
\leq C \int \int_R \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} |\varphi_j^o ((2k + n) |\lambda_1|, (2l + n) |\lambda_2|)| \\
\times \|f^{\lambda_1} * \varphi_{k}^{\lambda_1}\|_{\infty} \|g^{\lambda_2} * \varphi_{l}^{\lambda_2}\|_{\infty} |\lambda_1|^{n} |\lambda_2|^{n} d\lambda_1 d\lambda_2
\]
\[
\leq C \int \int_R \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} |\varphi_j^o ((2k + n) |\lambda_1|, (2l + n) |\lambda_2|)| \\
\times \|f^{\lambda_1}\|_{2} \|\varphi_{k}^{\lambda_1}\|_{2} \|g^{\lambda_2}\|_{2} \|\varphi_{l}^{\lambda_2}\|_{2} |\lambda_1|^{n} |\lambda_2|^{n} d\lambda_1 d\lambda_2
\]
\[
\leq C \left( \int \int_R \left( \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} |\varphi_j^o ((2k + n) |\lambda_1|, (2l + n) |\lambda_2|)| |\lambda_1|^{\frac{n+4}{2}} |\lambda_2|^{\frac{n+4}{2}} \right)^{\frac{1}{2}} d\lambda_1 d\lambda_2 \right)
\]
\[
\leq C \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} k^{n+1} l^{n+1} \left( \int \int_R |\varphi_j^o ((2k + n) |\lambda_1|, (2l + n) |\lambda_2|)|^{2} |\lambda_1|^{n+\delta} |\lambda_2|^{n+\delta} d\lambda_1 d\lambda_2 \right)^{\frac{1}{2}}
\]
\[
\leq C \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} k^{-\frac{1}{2}} l^{-\frac{1}{2}} \left( \int \int_R |\varphi_j^o ((|\lambda_1|, |\lambda_2|)|^{2} |\lambda_1|^{n+\delta} |\lambda_2|^{n+\delta} d\lambda_1 d\lambda_2 \right)^{\frac{1}{2}}
\]
\[
\leq C 2^{j(\alpha + \frac{3}{2})} \left( \int_{|\lambda_1| \leq 1} |f^{\lambda_1}|_{2}^{2} |\lambda_1|^{-\delta} d\lambda_1 \right)^{\frac{1}{2}} \left( \int_{|\lambda_2| \leq 1} |g^{\lambda_2}|_{2}^{2} |\lambda_2|^{-\delta} d\lambda_2 \right)^{\frac{1}{2}}.
\]

Because
\[
T_j^f(f_1, g_1)(\omega) = T_j^g(f_1, g_1)(\omega), \quad \omega \in B_j(\xi, \frac{1}{4}),
\]
we have
\[
(5.1) \quad \|T_j^f(f_1, g_1)\|_{L^\infty(B_j(\xi, \frac{1}{4}))} = \|T_j^g(f_1, g_1)\|_{L^\infty(B_j(\xi, \frac{1}{4}))}
\]
\[
\leq C 2^{j(\alpha + \frac{3}{2})} \left( \int_{|\lambda_1| \leq 1} |f_1^{\lambda_1}|_{2}^{2} |\lambda_1|^{-\delta} d\lambda_1 \right)^{\frac{1}{2}} \left( \int_{|\lambda_2| \leq 1} |g_1^{\lambda_2}|_{2}^{2} |\lambda_2|^{-\delta} d\lambda_2 \right)^{\frac{1}{2}}.
\]
Let us consider the integral about $\lambda_1$. Note that
\[
\| f_{1}^{\lambda_1} \|_2 \leq C2^{\alpha j(1+\gamma)} \| f_{1}^{\lambda_1} \|_\infty
\]
\[
\leq C 2^{j(1+\gamma)}(\frac{\alpha}{2}+1) \| f \|_{L^\infty(B_j(\xi,\frac{\alpha}{4}))},
\]
we obtain
\[
\int_{|\lambda_1| \leq 1} \| f_{1}^{\lambda_1} \|_2^2 |\lambda_1|^{-\delta} \, d\lambda_1
\]
\[
= \int_{2^{-2j(1+\gamma)} \leq |\lambda_1| \leq 1} \| f_{1}^{\lambda_1} \|_2^2 |\lambda_1|^{-\delta} \, d\lambda_1 + \int_{|\lambda_1| \leq 2^{-2j(1+\gamma)}} \| f_{1}^{\lambda_1} \|_2^2 |\lambda_1|^{-\delta} \, d\lambda_1
\]
\[
\leq 2^{2dj(1+\gamma)} \| f_{1} \|_2^2 + C 2^{j(1+\gamma)}(Q+2) \| f \|_{L^\infty(B_j(\xi,\frac{\alpha}{4}))} \int_{|\lambda_1| \leq 2^{-2j(1+\gamma)}} |\lambda_1|^{-\delta} \, d\lambda_1
\]
\[
\leq C 2^{j(1+\gamma)(Q+2\delta)} \| f \|_{L^\infty(B_j(\xi,\frac{\alpha}{4}))}^2.
\]
In the same way, we have
\[
\int_{|\lambda_1| \leq 1} \| g_{1}^{\lambda_2} \|_2^2 |\lambda_2|^{-\delta} \, d\lambda_2 \leq C 2^{j(1+\gamma)(Q+2\delta)} \| g \|_{L^\infty(B_j(\xi,\frac{\alpha}{4}))}^2.
\]
From (5.1) and above estimates, we get
\[
\| T_j^f(f_1, g_1) \|_{L^\infty(B_j(\xi,\frac{\alpha}{4}))} \leq C 2^{-j(\alpha+\frac{1}{2})} 2^{j(1+\gamma)(Q+2\delta)} \| f \|_{L^\infty(B_j(\xi,\frac{\alpha}{4}))} \| g \|_{L^\infty(B_j(\xi,\frac{\alpha}{4}))}.
\]
It follows that
\[
\| T_j^f(f_1, g_1) \|_{L^\infty} \leq C 2^{-j(\alpha+\frac{1}{2})} 2^{j(1+\gamma)(Q+2\delta)} \| f \|_{L^\infty} \| g \|_{L^\infty}.
\]
Thus, whenever $\alpha > Q - \frac{1}{2}$, we can choose $\gamma, \delta > 0$ such that $\alpha > (1+\gamma)(Q+2\delta) - \frac{1}{2}$, which implies there exists an $\varepsilon > 0$ such that
\[
\| T_j^f(f_1, g_1) \|_{L^\infty} \leq C 2^{-\varepsilon j} \| f \|_{L^\infty} \| g \|_{L^\infty}.
\]
The proof of Theorem 4 is completed. \qed

5.3. The point $(2, \infty, 2)$.

**Theorem 5.** If $\alpha > \frac{Q-1}{2}$, then $S^\alpha$ is bounded from $L^2(\mathbb{H}^n) \times L^\infty(\mathbb{H}^n)$ to $L^2(\mathbb{H}^n)$.

**Proof.** As above, it suffices to estimate $T_j^a(f_1, g_1)$. We write $T_j^a(f, g)$ as
\[
T_j^a(f, g)(z, t) = \int_{0}^{\infty} \int_{0}^{\infty} \varphi_j^\alpha(\lambda_1, \lambda_2) \sum_{k=0}^{\infty} (2k+n)^{-n-1} f(\tilde{e}_k^{\alpha \lambda_1} + \tilde{e}_k^{\alpha \lambda_2})(z, t)
\]
\[
\times \sum_{l=0}^{\infty} (2l+n)^{-n-1} g(\tilde{e}_l^{\alpha \lambda_2} + \tilde{e}_l^{\alpha \lambda_2})(z, t) \, d\mu(\lambda_1) \, d\mu(\lambda_2)
\]
\[
= C \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\lambda_1+\lambda_2)t} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \varphi_j^\alpha ((2k+n) |\lambda_1|, (2l+n) |\lambda_2|)
\]
\[
\times \left( f_{\lambda_1}^{\lambda_1} \varphi_k^{\lambda_1} \right)(z) \left( g_{\lambda_2}^{\lambda_2} \varphi_l^{\lambda_2} \right)(z) |\lambda_1|^n |\lambda_2|^n \, d\lambda_1 \, d\lambda_2
\]
\[
= C \int_{-\infty}^{\infty} e^{-i\lambda_1 t} \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \varphi_j^\alpha ((2k+n) |\lambda_1 - \lambda_2|, (2l+n) |\lambda_2|)
\]
\[
\times \left( f_{\lambda_1}^{\lambda_1} \varphi_k^{\lambda_1} \right)(z) \left( g_{\lambda_2}^{\lambda_2} \varphi_l^{\lambda_2} \right)(z) \, d\lambda_1 \, d\lambda_2.
\]
Then, applying the Plancherel theorem in variable \( t \), Mikowski’s inequality, the orthogonality of the special Hermite projections, \([4.1]\) and Plancherel formula \([2.2]\), we get that

\[
\int_{\mathbb{C}^n} \int_{\mathbb{R}} |T^t(f, g)(z, t)|^2 \, dtdz = C \int_{\mathbb{C}^n} \int_{\mathbb{R}} e^{-i\lambda_1 t} \int_{\mathbb{R}} \sum_{k=0}^{\infty} \phi_j^s((2k + n) |\lambda_1 - \lambda_2|, (2l + n) |\lambda_2|) \\
\times \left( f^{\lambda_1 - \lambda_2} \ast_{\lambda_1 - \lambda_2} \varphi_k^{\lambda_1 - \lambda_2} \right) (z) \left( g^{\lambda_2} \ast_{\lambda_2} \varphi_l^{\lambda_2} \right) (z) |\lambda_1 - \lambda_2|^n |\lambda_2|^n \, d\lambda_1 \, d\lambda_2 \right)^2 \\
= C \int_{\mathbb{C}^n} \int_{\mathbb{R}} \sum_{k=0}^{\infty} \phi_j^s((2k + n) |\lambda_1 - \lambda_2|, (2l + n) |\lambda_2|) \\
\times \left( f^{\lambda_1 - \lambda_2} \ast_{\lambda_1 - \lambda_2} \varphi_k^{\lambda_1 - \lambda_2} \right) (z) \left( g^{\lambda_2} \ast_{\lambda_2} \varphi_l^{\lambda_2} \right) (z) |\lambda_1 - \lambda_2|^n |\lambda_2|^n \, d\lambda_1 \, d\lambda_2 \\
\leq C \int_{\mathbb{R} \times \mathbb{C}^n} \left( \int_{\mathbb{R}} \sum_{l=0}^{\infty} \phi_j^s((2k + n) |\lambda_1 - \lambda_2|, (2l + n) |\lambda_2|) \left( f^{\lambda_1 - \lambda_2} \ast_{\lambda_1 - \lambda_2} \varphi_k^{\lambda_1 - \lambda_2} \right) (z) \right) \\
\times \left( g^{\lambda_2} \ast_{\lambda_2} \varphi_l^{\lambda_2} \right) (z) |\lambda_1 - \lambda_2|^n |\lambda_2|^n \, d\lambda_2 \\
\leq C \int_{\mathbb{R} \times \mathbb{C}^n} \left( \int_{\mathbb{R}} \sum_{l=0}^{\infty} \phi_j^s((2k + n) |\lambda_1 - \lambda_2|, (2l + n) |\lambda_2|) \right)^2 \left( f^{\lambda_1 - \lambda_2} \ast_{\lambda_1 - \lambda_2} \varphi_k^{\lambda_1 - \lambda_2} \right)^2 \\
\times \|g^{\lambda_2}\|_2 \|\varphi_l^{\lambda_2}\|_2 |\lambda_1 - \lambda_2|^n |\lambda_2|^n \, d\lambda_2 \\
\leq C \left( \int_{|\lambda_2| \leq 1} \sum_{l \leq \frac{1}{|\lambda_2|}} \|g^{\lambda_2}\|_2 \|\varphi_l^{\lambda_2}\|_2 |\lambda_2|^{2n - \delta} \, d\lambda_2 \right) \\
\times \left( \int_{\mathbb{R}} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \phi_j^s((2k + n) |\lambda_1 - \lambda_2|, (2l + n) |\lambda_2|) \right)^2 \\
\times \left( f^{\lambda_1 - \lambda_2} \ast_{\lambda_1 - \lambda_2} \varphi_k^{\lambda_1 - \lambda_2} \right)^2 |\lambda_1 - \lambda_2|^{2n} |\lambda_2|^{\delta} \, d\lambda_2 \, d\lambda_1 \\
= C \left( \int_{|\lambda_2| \leq 1} \sum_{l \leq \frac{1}{|\lambda_2|}} \|g^{\lambda_2}\|_2 \|\varphi_l^{\lambda_2}\|_2 |\lambda_2|^{2n - \delta} \, d\lambda_2 \right)
The proof of Theorem 5 is completed. □

It follows from (5.2) that

Thus, whenever

Because

Taking the $L^2$ norm with respect to $\xi$ yields that

Thus, whenever $\alpha > \frac{Q-1}{2}$, we can choose $\gamma, \delta > 0$ such that $\alpha > (1 + \gamma)(\frac{Q}{2} + \delta) - \frac{1}{2}$, which yields that there exists $\varepsilon > 0$ such that

The proof of Theorem 5 is completed. □

**Appendix: Bilinear Interpolation**

Because $p_1$ and $p_2$ are symmetric, we have obtained, in two sections above, the boundedness of the bilinear Riesz means $S^\alpha$ at some specific triples of points $(p_1, p_2, p)$ like

$(1, 1, \frac{1}{2}), (2, 2, 1), (\infty, \infty, \infty), (1, 2, \frac{2}{3}), (2, 1, \frac{2}{3}), (1, \infty, 1), (\infty, 1, 1), (2, \infty, 2), (\infty, 2, 2)$.

We can obtain the intermediate boundedness of $S^\alpha$ by using of the bilinear interpolation via complex method adapted to the setting of analytic families or real method in [3]. Bernicot et al. [11] described how to make use of real method. The full results in Main Theorem are proved in this way. We outline this argument for reader’s convenience.

Consider a spherical decomposition of $S^\alpha$ as

$$S^\alpha = \sum_{j=0}^{\infty} 2^{-ja} T_{j,\alpha},$$
where
\[ T_{j,\alpha}(f, g) = \int_0^\infty \int_0^\infty \varphi_{j,\alpha}(\lambda_1, \lambda_2) P_{\lambda_1} f P_{\lambda_2} g d\mu(\lambda_1) d\mu(\lambda_2) \]
and
\[ \varphi_{j,\alpha}(s, t) = 2^{j\alpha}(1 - s - t)_{+}^\alpha \varphi(2^j (1 - s - t)). \]
In the preceding sections, we have actually obtained the estimates of the form
\[(5.3) \quad \|T_{j,\alpha}\|_{L^{p_1} \times L^{p_2} \to L^p} \leq C 2^{j\alpha(p_1, p_2)} \]
at some triples of points \((p_1, p_2, p)\). Since \(\alpha(p_1 p_2)\) only depends on the point \((p_1, p_2, p)\), (5.3) also holds for any other \(T_{j,\alpha'}\), i.e.,
\[ \|T_{j,\alpha'}\|_{L^{p_1} \times L^{p_2} \to L^p} \leq C 2^{j\alpha(p_1, p_2)}. \]
So, fixing \(j\) and \(\alpha'\) and applying bilinear real interpolation theorem on \(T_{j,\alpha'}\), we can conclude that if the point \((p_1, p_2, p)\) satisfies
\[ \left( \frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p} \right) = (1 - \theta) \left( \frac{1}{p_1^0}, \frac{1}{p_2^0}, \frac{1}{p} \right) + \theta \left( \frac{1}{p_1^1}, \frac{1}{p_2^1}, \frac{1}{p} \right) \]
for some \(\theta \in (0, 1)\) and \((p_1^0, p_2^0, p^0), (p_1^1, p_2^1, p^1)\), we have that
\[ \|T_{j,\alpha'}\|_{L^{p_1} \times L^{p_2} \to L^p} \leq C 2^{j((1-\theta)\alpha(p_1^0, p_2^0)+\theta\alpha(p_1^1, p_2^1))}. \]
Define \(\alpha(p_1, p_2) = (1 - \theta)\alpha(p_1^0, p_2^0) + \theta\alpha(p_1^1, p_2^1)\) and let \(\alpha' = \alpha\). It follows that
\[ \|S^\alpha\|_{L^{p_1} \times L^{p_2} \to L^p} \leq \sum_{j=0}^\infty 2^{-j\alpha} \|T_{j,\alpha}\|_{L^{p_1} \times L^{p_2} \to L^p} \leq C \sum_{j=0}^\infty 2^{-j\alpha} 2^{j\alpha(p_1, p_2)}. \]
Thus, when \(\alpha > \alpha(p_1, p_2)\), we have \(\|S^\alpha\|_{L^{p_1} \times L^{p_2} \to L^p} \leq C\), i.e., \(S^\alpha\) is bounded from \(L^{p_1}(\mathbb{H}^n) \times L^{p_2}(\mathbb{H}^n)\) to \(L^p(\mathbb{H}^n)\).

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