A representation of the exchange relation for affine Toda field theory

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Vertex operators are constructed providing representations of the exchange relations containing either the S-matrix of a real coupling (simply-laced) affine Toda field theory, or its minimal counterpart. One feature of the construction is that the bootstrap relations for the S-matrices follow automatically from those for the conserved quantities, via an algebraic interpretation of the fusing of two particles to form a single bound state.

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1. Introduction

Stimulated by a study of perturbed conformal field theory \[1,2\], there has been something of a revival of interest in two-dimensional affine Toda field theories, whose study was begun long ago \[3\] but is nowhere near complete. One of the striking features of two-dimensional integrable theories is the possibility of making plausible guesses for their exact S-matrices on the basis of the bootstrap and Yang-Baxter equations \[4,5\]. For the real coupling affine Toda field theories, based on the ADE Lie algebras, the Yang-Baxter equation itself plays no rôle because the S-matrices are entirely diagonal. Nevertheless, the proposed S-matrices enjoy an interesting analytic structure as a consequence of the bootstrap alone \[6–13\]. At least some of this interesting structure can be seen in perturbation theory but proofs of the conjectures are not yet available. The same bootstrap structure appears in perturbations of certain conformal field theories, namely the coset models \(g^{(1)} \times g^{(1)}/g^{(2)}\). However, there the S-matrices are slightly different; while the affine Toda S-matrices have a factor with coupling constant dependent zeroes in the physical strip, this is absent from the proposals for perturbed conformal field theory. In fact, it is these ‘minimal’ S-matrices, of interest in their own right, that will be discussed first below.

In \[14,15\], the bootstrap and its accompanying fusing rules were found to be intimately related to the geometry of root systems. Moreover, formulae for the conjectured S-matrices were discovered which made clearer their structural relationship with the roots and with the bootstrap. The ideas might well have more general significance given the marked similarities between the affine Toda theory S-matrices and those conjectured, for example, for the principal chiral models \[16\]. Very similar mathematical structures have already been observed in the context of certain \(N = 2\) Landau-Ginzburg models \[17\].

The S-matrices for ADE affine Toda field theory can be written in a number of equivalent ways. The expressions most suitable for the present discussion will be summarised in section two, alongside some useful facts concerning the action of the Coxeter element of the Weyl group on the roots and weights. For more details concerning the latter, see for example \[18\].

Section three returns to an old idea in which the S-matrix appears in a ‘braiding’, or ‘exchange’ relation \[5,19\]:

\[
V_a(\theta_a)V_b(\theta_b) = S_{ab}(\theta_a - \theta_b)V_b(\theta_b)V_a(\theta_a),
\]

where each of the operators \(V(\theta)\) is formally associated with a particle of the theory, \(\theta\) denoting its rapidity.
According to the present understanding of real coupling affine Toda theory, each operator is a singlet since the particles are distinguished by conserved charges of non-zero spin. The assumption of associativity for the exchange relation then has no consequence for the S-matrix—which is merely a set of numbers, one for each pair of particles. In more general situations, at least some particles will be degenerate and the associativity of (1.1) implies the Yang-Baxter or factorisation equation for the S-matrix.

The expressions for the S-matrices given in section two are very suggestive and in section three a vertex operator representation of the exchange relation will be presented, thus providing a set of generating relations for the S-matrix in a fairly natural way. The vertex operators to be used in this context are reminiscent of those used by Lepowsky and Wilson \[20,21\] to obtain (twisted) representations of Kac-Moody algebras. It appears at first sight that the twisted vertex operators are more appropriate in the context of affine Toda field theory than those used by Frenkel, Kac and Segal \[22\] to provide the level one representations of simply-laced affine algebras. Despite the similarities, there are some crucial differences too, the most important being the absence of an obvious action of the conformal, or Virasoro, generators. Since the field theories under discussion are massive, and therefore not conformal, this is hardly surprising.

Given the exchange relation (1.1), it is natural to ask about the bootstrap itself. Suppose there is an ‘operator product’ expansion of \(V_a(\theta_a)V_b(\theta_b)\), not in the sense of a short distance expansion but rather in the sense of a bound-state fusing relation. In other words, suppose that, for certain (imaginary) rapidity differences, the two-particle state containing particles \(a\) and \(b\) is indistinguishable in terms of its quantum numbers from another (on-shell) single particle state \(c\). From the point of view of the operators, it might then be expected that there should be a relation of the form,

\[
V_a(\theta_a)V_b(\theta_b) \sim \frac{c^{abc}V_c(\theta_c)}{(\theta_a - \theta_b - iU_{ab}^c)^{n_{abc}}},
\]

where

\[
\theta_a \sim \theta_c - iU_{ac}^b,
\]

\[
\theta_b \sim \theta_c + iU_{bc}^a,
\]

and \(U = \pi - U\). Momentum conservation (corresponding to the first conserved charge), together with the fact that the particles created by the operators \(V(\theta)\) are always on-shell, requires that the relative rapidity of \(a\) and \(b\) is just \(i\) times \(U_{ab}^c\), the fusing angle for a bound state in \(ab \rightarrow ab\) scattering. It is a feature of the construction presented in section
three that the numerical factor in (1.2) is meromorphic, the quantities \( n_{abc} \) being certain integers whose properties are described briefly at the end of that section.

Multiplying (1.2) by \( V_d(\theta_d) \) and using (1.1) leads to the bootstrap relation between different S-matrix elements:

\[
S_{cd}(\theta_c - \theta_d) = S_{ad}(\theta_c - \theta_d - iU_{ac}^b)S_{bd}(\theta_c - \theta_d + iU_{bc}^a).
\]  

(1.3)

The affine Toda field theories have infinitely many conserved charges \( P_r, P_{-r} \) where, modulo the Coxeter number \( h \), the (positive) spin-label \( r \) runs over the exponents of the algebra defining the theory. It would be expected that

\[
[P_r, V_a(\theta_a)] = p_r^a e^{r\theta_a} V_a(\theta_a),
\]  

(1.4)

where \( p_r^a \) is the eigenvalue of the charge with spin \( r \) on the single type-\( a \) particle state

\[
P_{s+kh} |p_a \rangle = p_{s+kh}^a e^{(s+kh)\theta_a} |p_a \rangle.
\]  

(1.5)

Given (1.4), (1.3) implies the bootstrap equation for the charges, namely

\[
p_r^a e^{-irU_{ac}^b} + p_r^b e^{irU_{bc}^a} = p_r^c.
\]  

(1.6)

In this article, a formalism will be developed far enough to provide a representation of (1.1) and (1.2) appropriate to the known minimal solutions of (1.3) and, after a simple modification, to real coupling affine Toda S-matrices. Despite the absence of any derivation of these operators from a quantisation of the original affine Toda Lagrangian, their form is suggestive, and they seem to provide a natural setting in which to place the fusing rule and S-matrix formulae described in [14,15].

2. Preliminaries

The discussion will be restricted to the theories associated with the simply-laced (ADE) series of Lie algebras. For a given theory, each particle is unambiguously associated to one of the spots of the relevant Dynkin diagram. This follows from the observation [7,23] (now proved Lie algebraically [24]) that the set of classical masses of the Toda particles can be arranged to be the components of the smallest eigenvalue eigenvector of the corresponding (non-affine) Cartan matrix. Besides picking a basis of simple roots for the algebra, it also appears to be useful to divide the particles of the field theory into two sets.
This division reflects a special property of root systems which allows the simple roots to be split into two sets (black and white) so that within each set, all the roots are orthogonal to one another. The sets, the roots belonging to them, the particles themselves, and the fundamental weights associated with the simple roots in each set will be distinguished wherever necessary by the symbols • or ◦. (In ref [14] a slightly different notation was used, the black roots being referred to as type \( \alpha \) the white roots as type \( \beta \).)

Corresponding to the colouring of the simple roots, it is natural to choose a particular Coxeter element of the Weyl group. Let \( w_i \) be the Weyl reflection corresponding to the simple root \( \alpha_i \) \((i = 1, \ldots, r)\) and define
\[
w_{\bullet} = \prod_{i \in \bullet} w_i \quad w_{\circ} = \prod_{i \in \circ} w_i
\]
where the black set is labelled \( i = 1, \ldots, b \) and the white set \( i = b + 1, \ldots, r \). Then \( w \), defined by
\[
w = w_{\bullet}w_{\circ},
\]
is a Coxeter element. Another subset of linearly independent roots is defined in terms of the set of simple roots \( \alpha_i \) in the following way. Set
\[
\phi_i = w_r w_{r-1} \cdots w_{i+1} \alpha_i
\]
so that with the labelling introduced above
\[
\phi_{\circ} = \alpha_{\circ} \quad \text{and} \quad \phi_{\bullet} = w_{\circ} \alpha_{\bullet}.
\]
Moreover, if \( \lambda_i \) denotes the fundamental weights, satisfying (for simply-laced algebras),
\[
\lambda_i \cdot \alpha_j = \delta_{ij},
\]
then the roots defined by (2.1) satisfy
\[
\phi_i = (1 - w^{-1}) \lambda_i. \tag{2.2}
\]
This formula has also been used to relate the fusing rule given in [14] to the previously observed Clebsch-Gordan property of the affine Toda couplings [25].
It will be useful to have a (complex) basis of eigenvectors of the Coxeter element \( w \). The elements of this basis are conveniently labelled by the exponents of the algebra. Thus, for each exponent \( s \), there is an eigenvector \( e_s \) satisfying

\[
we_s = e^{2\pi i s/h} e_s
\]  

and normalised so that

\[
e_s \cdot e_{s'} = \delta_{s+s',h}
\]

where \( h \) is the Coxeter number (ie order of the Coxeter element). Occasionally, exponents \( h/2 \) are repeated (in the \( D_{\text{even}} \) series). However, even in these cases the same notation will suffice without confusion.

The exponents of the algebra also label the eigenvectors of the Cartan matrix: for each exponent there is an eigenvector

\[
C_{ij} q_j^{(s)} = (2 - 2 \cos \pi s/h) q_i^{(s)}
\]

and the eigenvectors are orthogonal for the ADE series.

For computational purposes, it is often useful to have an expression for the basis (2.3) which makes its relationship with the eigenvectors of the Cartan matrix explicit. For each exponent \( s \), define

\[
a_{\bullet}^{(s)} = \sum_{\bullet} q_i^{(s)} \alpha_i, \quad a_{\circ}^{(s)} = \sum_{\circ} q_i^{(s)} \alpha_i
\]

\[
l_{\bullet}^{(s)} = \sum_{\bullet} q_i^{(s)} \lambda_i, \quad l_{\circ}^{(s)} = \sum_{\circ} q_i^{(s)} \lambda_i.
\]

Then, provided the eigenvectors of the Cartan matrix have unit length, the vectors \( a_{\bullet}, a_{\circ} \) are unit vectors while \( |l_{\bullet}| = |l_{\circ}| = 1/2 \sin \theta_s \), where \( \theta_s = s\pi/h \), and they enjoy a number of other properties, including

\[
a_{\bullet}^{(s)} \cdot a_{\circ}^{(s)} = -\cos \theta_s, \quad l_{\bullet}^{(s)} : l_{\circ}^{(s)} = \frac{\cos \theta_s}{4 \sin^2 \theta_s}
\]

\[
a_{\bullet}^{(s)} \cdot l_{\circ}^{(s)} = 0 = a_{\circ}^{(s)} \cdot l_{\bullet}^{(s)}, \quad a_{\bullet}^{(s)} \cdot l_{\bullet}^{(s)} = a_{\circ}^{(s)} \cdot l_{\circ}^{(s)} = \frac{1}{2}.
\]

The eigenvectors of the Coxeter element can then be written in terms of these; for example, a convenient choice is

\[
e_s = \rho_s (a_{\bullet}^{(s)} + e^{i\pi s/h} a_{\circ}^{(s)}),
\]
where \( \rho_s = 1/\sqrt{2} \sin \theta_s \). The choice of normalisation and the condition

\[
e_{h-s} = e_s^*,
\]

require

\[
a^{(h-s)}_\bullet = a^{(s)}_\bullet \quad a^{(h-s)}_\circ = -a^{(s)}_\circ
\]

and

\[
q^{(s)}_\bullet = q^{(h-s)}_\bullet \quad q^{(s)}_\circ = -q^{(h-s)}_\circ.
\]

Armed with these facts, it is straightforward to obtain a representation of roots or weights in this basis; for example, the fundamental weights have components given by

\[
\lambda^{(s)}_k = \begin{cases} 
\rho_s q^{(s)}_k, & \text{if } k \in \bullet \\
\rho_s q^{(s)}_k e^{i\pi s \over h}, & \text{if } k \in \circ.
\end{cases}
\]

The masses of the affine Toda theory are proportional to the components of \( q^{(1)} \). Moreover, assuming the other classically conserved quantities are preserved in the quantum theory and are compatible with the bootstrap (1.6) leads to the conclusion [13,14] that the single-particle states in the quantum theory are eigenstates of the quantum operators \( P_{s+kh} \), with eigenvalues related to the eigenvectors \( q^{(s)} \) of the Cartan matrix. Invariance under parity requires \( p^a_r = p^a_{-r} \) and the fact that all fusing angles appearing in (1.6) are multiples of \( \pi / h \) requires \( p^a_s = p^a_{s+2kh} \) where \( k \) is any integer. Together, these two requirements imply:

\[
p^a_{s+2kh} \propto q^{(s)}_a \quad p^a_{s+(2k+1)h} \propto q^{(h-s)}_a.
\]

Hence, using (2.11) above,

\[
p^{\bullet}_{s+kh} \propto q^{(s)}_\bullet \quad p^{\circ}_{s+kh} \propto (-)^k q^{(s)}_\circ.
\]

(Note, \( s \) will always be taken to lie in the range \( 1, \ldots, h \).)

For spin \( \pm 1 \), the conserved charges are the light-cone momentum components and the eigenvalues are just the masses. The particles are distinguished from each other using the conserved quantities but the single particle states will be labelled by their momenta only, all other labels being suppressed for convenience of notation.

For later use, a couple of alternative expressions for the S-matrices for the various affine Toda theories will be given, each of them equivalent (with the proviso noted below)
to the expressions provided in \[14\]. Here, as there and in earlier works \[7,8\], the block notation will be adopted in which the basic element of any of the conjectured S-matrices is constructed from the element

\[
(x, \Theta)_+ = \sinh\left(\frac{\Theta}{2} + \frac{i\pi x}{2h}\right) \tag{2.15}
\]

where \(\Theta\) is the rapidity difference for the process and \(x\) is an integer. The \(\Theta\) dependence is made explicit here for reasons which will become apparent in the next section. The basic building block itself is then defined to be

\[
\{x, \Theta\}_+ = \frac{(x - 1, \Theta)_+(x + 1, \Theta)_+}{(x - 1 + B, \Theta)_+(x + 1 - B, \Theta)_+} \tag{2.16}
\]

where the function

\[
B(\beta) = \frac{1}{2\pi} \frac{\beta^2}{1 + \beta^2 / 4\pi}
\]

contains the conjectured coupling constant dependence. The first expression, given in \[13\], can be summarised as follows:

\[
S_{ab}(\Theta) = \prod_{p=1}^{h} \{2p + 1, \Theta\}^{\lambda_a \cdot w^{-p} \phi_b} \quad \Theta = \theta_a - \theta_b \tag{2.17}
\]

at least provided the two particles \(a\) and \(b\) share the same colour. If the two particles correspond to different colours then the appropriate expression is \(2.17\) but the particle labelled \(\circ\) has its rapidity effectively incremented by \(i\pi/h\). In other words, the appropriate S-matrix elements are obtained by replacing the rapidity \(\Theta\) by

\[
\Theta_{\circ \bullet} = \Theta + i\pi/h \quad \text{or} \quad \Theta_{\bullet \circ} = \Theta - i\pi/h. \tag{2.18}
\]

There are other, equivalent expressions in terms of the ‘unitary’ block

\[
\{x, \Theta\}_+ / \{-x, \Theta\}_+,
\]

but they are not so useful here. The corresponding minimal S-matrix is obtained from \(2.17\) by simply deleting any \(\beta\) dependent term in \(2.16\).

These S-matrices are unitary, satisfy crossing requirements and fulfil the bootstrap conditions on the odd order poles. They are analytic in the rapidity difference \(\Theta\) with \(\beta\)-independent poles on the physical strip (\(\text{Im} \Theta \in [0, \pi]\)). The poles may be of quite a high order but appear to be compatible with perturbation theory as far as has been checked \[3\].
It will be convenient in the next section to consider a two (complex) dimensional space of rapidity variables, $\theta$ and $\bar{\theta}$, corresponding to a complexification of the light-cone momentum variables $p_{\pm}$. A representation of the exchange relation will be constructed in this complex space in the first instance and subsequently restricted to a ‘physical’ submanifold. In part, this is motivated by an analogy with conformal field theory where the two variables $z$ and $\bar{z}$, on which all conformal fields depend, are often treated as independent complex coordinates with a restriction to the euclidean section $z^* = \bar{z}$ left to a late stage of a calculation. In the present case, it is the mass-shell condition $p\bar{p} = p_+ p_- = m^2$, or $\bar{\theta} = -\theta$ which selects the physical submanifold. Note, the full vertex is automatically an analytic function of $\theta$ on this submanifold, in contrast to the conformal field theory situation in which a typical (non-chiral) vertex operator is the product of a function of $z$ and a function of $z^*$ on the euclidean section. A feature of this kind is clearly needed—the S-matrix is itself analytic in $\theta$, while typically, correlation functions in conformal field theory are not.

Consider the minimal S-matrix, in which the factors containing the coupling constant dependence are omitted. There are a number of continuations of (2.17) off the $\Theta = -\Theta$ sub-manifold. Here, just one will be given:

$$S_{\text{min}}^{ab}(\Theta_{ab}, \Theta_{ab}) = \frac{F_{ab}(\Theta_{ab}, \Theta_{ab})}{F_{ba}(\Theta_{ba}, \Theta_{ba})}$$ (2.19)

where,

$$F_{ab}(\Theta_{ab}, \Theta_{ab}) = \frac{\prod_{p=1}^{h}(-2p, \Theta_{ab})_{+}^{\lambda_{a} \cdot w_{-}^{p} \lambda_{b}}}{\prod_{p=1}^{h}(-2p, \Theta_{ab})_{+}^{\lambda_{a} \cdot w_{-}^{p-1} \lambda_{b}}}$$ (2.20)

at least when the two particles share the same colour. When the colours are different, both $\theta_{c}$ and $\bar{\theta}_{c}$ are shifted by $-i\pi/h$. Note, for $\theta_{c}$ this is the opposite sign to the shift appearing in the previous formula (2.18). To check agreement between (2.19) and (2.17) (when $\Theta_{ab} = -\Theta_{ab}$), the following inner product identities are useful:

$$\lambda_{c} \cdot w_{-}^{p} \lambda'_{c} = \lambda'_{c} \cdot w_{-}^{p} \lambda_{c} \quad \lambda_{c} \cdot w_{-}^{p} \lambda'_{c} = \lambda'_{c} \cdot w_{-}^{p} \lambda_{c}$$

(2.21)

It should be noted that whereas (2.17) is a meromorphic function of $\Theta_{ab}$ this is not generally true for the expressions occuring in (2.20); since the inner products of weights are not usually integers, these functions will have individually a complicated cut structure.
3. Representing the exchange relation

The basic ingredients of the construction will be described first and then elaborated to provide a representation of the exchange relation for the minimal S-matrix.

For each fundamental weight \( \lambda \) define a string-like, rapidity dependent field as follows:

\[
X^\lambda(\theta) = \sum_{r=s+kh} \frac{\hbar}{r} e^{-r\theta} \lambda^{(h-s)} c_r.
\] (3.1)

In (3.1), the sum extends over all integers \( k \) and exponents \( s \) and the Fock space annihilation and creation operators satisfy the commutation relations

\[
[c_r, c_{r'}] = (r/h) \delta_{r+r',0}.
\] (3.2)

The field \( X^\lambda \) is periodic in \( \theta \) with period \( 2\pi i \) and a Coxeter rotation of the weight is equivalent to a shift \( 2\pi i/h \) in \( \theta \). In that sense, the field is ‘twisted’. There is a ground state satisfying

\[
c_r |0 \rangle > 0 \quad r > 0.
\]

The commutation relation between the annihilation part of a field \( X^\lambda_+(\theta) \) and the creation part of another, similar, field \( X^{\lambda'}_- (\theta') \) is given by:

\[
\left[ X^\lambda_+(\theta), X^{\lambda'}_- (\theta') \right] = \sum_{p=1}^h (\lambda \cdot W^{-p} \lambda') \ln \left( 1 - W^p e^{\theta' - \theta} \right),
\] (3.3)

at least provided \( \text{Re} \theta > \text{Re} \theta' \). Define a vertex operator to be the normal-ordered exponential of such a field

\[
V^\lambda(\theta) = : \exp X^\lambda(\theta) : \equiv e^{X^\lambda_-(\theta)} e^{X^\lambda_+(\theta)}.
\] (3.4)

Then, the product of two vertex operators can be normal-ordered producing an extra factor depending upon the rapidity difference:

\[
V^\lambda(\theta) V^{\lambda'}(\theta') = \prod_{p=1}^h (1 - \exp(\theta' - \theta + 2\pi ip/h))^{\lambda \cdot W^{-p} \lambda'} : V^\lambda(\theta) V^{\lambda'}(\theta') :,
\] (3.5)

provided the rapidities bear the above relation to each other. The factor on the right hand side of (3.5) is strikingly similar to the factors appearing in the S-matrix formulae (2.19) and (2.20), but not the same. A similar calculation with the ordering of the two operators reversed produces the same factor (via manipulations valid in the complementary
region \( \text{Re}\theta < \text{Re}\theta' \) and up to a possible phase factor independent of rapidity). Hence, a comparison of the two orderings after analytic continuation in rapidity provides a trivial exchange relation.

The vertex operator introduced above is a conformal operator with respect to the Virasoro generators built from the \( c \)-Fock space operators:

\[
L_n = \frac{1}{2} \sum_r c_r c_{n-r}.
\]

This observation explains the feature just described. To obtain an exchange relation corresponding to the massive affine Toda field theories the vertex operator will need to be ‘delocalised’ and its conformal nature destroyed.

A second remark concerns the commutation relation of a Fock space operator with the vertex:

\[
\left[ c_{s+k\hbar}, V^\lambda(\theta) \right] = \lambda(s) e^{(s+k\hbar)\theta} V^\lambda(\theta).
\]

Recalling the earlier observations (2.12) and (2.14), it is tempting to regard the annihilation operators in the Fock space as the conserved charges of the affine Toda theory, and take the vertex operators (suitably modified) to describe the single particle states. Indeed, the \(-i\pi/\hbar\) shift in rapidity for a type \( \circ \) particle, bearing in mind (2.14) and (2.12), renders (3.6) compatible with this assumption for either of the two colours. This rather natural point of view is the one adopted here, at least tentatively, bearing in mind that the relationship between this representation of the conserved quantities and the usual one in terms of the fundamental fields of the theory is missing, and that there is certainly a subtlety to be understood with regard to the hermiticity of the operators concerned; the conserved quantities are given classically as real functionals of the fields and their derivatives.

Since, as remarked earlier, there are two mutually commuting sets of conserved quantities of opposite spin there will have to be at least one extra set of Fock space operators. These are quite naturally associated with the variable \( \bar{\theta} \) which will, as mentioned above, be regarded as independent of \( \theta \) for most computational purposes. The conserved quantities with the opposite spin nevertheless share the same eigenvalues on the particle states apart from their rapidity dependence. Thus, if an extra set of Fock space operators \( \bar{c}_r \) is introduced they and their associated vertex operator \( \bar{V}^\lambda(\bar{\theta}) \) will be expected to satisfy

\[
\left[ \bar{c}_{s+k\hbar}, \bar{V}^\lambda(\bar{\theta}) \right] = \lambda(s) e^{(s+k\hbar)\bar{\theta}} \bar{V}^\lambda(\bar{\theta}),
\]

(3.7)
at least for \( r > 0 \). Note, this expression is also compatible with the shift in \( \bar{\theta} \) for a type \( \circ \) particle.

Introducing a second set of operators also provides the opportunity for some ‘delocalisation’ in the following sense. The conformal nature of the vertex operator can be destroyed by shifting the rapidity dependence in the annihilation part of the vertex (ie without upsetting (3.7)) relative to that in the creation part. Indeed, a shift of \( 2\pi i/h \) seems to be exactly what is required to produce the minimal S-matrix (2.17). Thus, introducing the field \( \bar{X}^\lambda(\bar{\theta}) \), it is convenient to set
\[
\bar{V}^\lambda(\bar{\theta}) = e^{\bar{X}_\lambda^-(\bar{\theta})} e^{\bar{X}_+^\lambda(\bar{\theta})}
\] (3.8)
and straightforward to repeat the normal-ordering calculation, giving
\[
\bar{V}^\lambda(\bar{\theta}) \bar{V}^\lambda'(\bar{\theta}') = \prod_{p=1}^{h} \left( 1 - \exp(\bar{\theta}' - \bar{\theta} + 2\pi ip/h) \right)^{\lambda - \lambda'} \bar{V}^\lambda(\bar{\theta}) \bar{V}^\lambda'(\bar{\theta}'),
\] (3.9)
this time valid in the region \( \text{Re}\bar{\theta} > \text{Re}\bar{\theta}' \). Unfortunately, this is not quite what is required for (2.20), the exponent in (3.9) has the wrong sign. To put that right, the signature of the barred Fock-space operators should be reversed:
\[
[\bar{c}_r, \bar{c}_{r'}] = -(r/h)\delta_{r+r',0}.
\] (3.10)

The two calculations (3.5) and (3.9) indicate that a vertex operator associated with a particle of type \( \bullet \) could be
\[
V^a(\theta, \bar{\theta}) = V^{\lambda_a}(\theta) \bar{V}^\lambda(\bar{\theta})
\] (3.11)
while that associated with a particle of type \( \circ \) is a similar expression but with \( \theta \) and \( \bar{\theta} \) each shifted by \(-i\pi/h\). The latter is required to match the slight difference in the formulae for the S-matrices for the two types, mentioned earlier. Adopting the composite vertex operator and putting together the reordering effects gives
\[
V^a(\theta_a, \bar{\theta}_a) V^b(\theta_b, \bar{\theta}_b) = F_{ab}(\Theta_{ab}, \bar{\Theta}_{ab}) : V^a(\theta_a, \bar{\theta}_a) V^b(\theta_b, \bar{\theta}_b) :,
\] (3.12)
where \( F_{ab} \) is defined in (2.20). Repeating the calculation in the opposite order gives a similar expression to (3.12), but with labels \( a, b \) reversed. Assuming an analytic continuation into a common region of complex rapidity and comparing the two reordering expressions yields the exchange relation, (1.1) for the minimal part of the S-matrix.
Having achieved a representation of the minimal S-matrix, the same ideas may be adapted to introduce the coupling constant dependence. One way to do so is to introduce another pair of string-like fields $Y^\lambda(\theta)$ and $\bar{Y}^\lambda(\bar{\theta})$, with corresponding sets of annihilation and creation operators $d_r, \bar{d}_r$, together with corresponding vertex operators $W^\lambda(\theta)$ and $\bar{W}^\lambda(\bar{\theta})$ defined by

$$W^\lambda(\theta) =: e^{Y^\lambda(\theta)} e^{-i\pi(2-\beta)/\hbar} :\quad \bar{W}^\lambda(\bar{\theta}) =: e^{\bar{Y}_+^\lambda(\bar{\theta})} e^{-i\pi(2+\beta)/\hbar} :. \quad (3.13)$$

Note, both constituent vertex operators are ‘delocalised’ this time by an amount dependent upon $B(\beta)$. Note also, the commutation relations of the $d, \bar{d}$ Fock space operators have the same signature as the $c, \bar{c}$ operators, respectively.

The operator representing a particle of type • is now taken to be

$$V^a(\theta_a, \bar{\theta}_a) = V_+^{\lambda_a}(\theta_a) \bar{V}^{\lambda_a}(\bar{\theta}_a) W^\lambda(\theta_a) \bar{W}^\lambda(\bar{\theta}_a). \quad (3.14)$$

Effectively, the four sets of annihilation and creation operators can be combined into a four-dimensional vector in a space with a metric of signature (+ + --). This fact is reminiscent of a comment by Ward [26] concerning the embedding of an affine Toda field theory in a self-dual gauge theory. This construction is certainly ad hoc and there may be other, more subtle, ways of achieving the same goal. Nevertheless, the example given establishes that the exchange relation can be represented in principle.

Returning to the vertex operators it is worth emphasising that the coefficients of the Fock space creation operators in (3.1) are proportional to the eigenvalues of the conserved quantities. This follows, irrespective of the colour of the particles, from the relations (2.12) and (2.13), taking into account the relative rapidity shift $-i\pi/\hbar$ for the type • particles.

In other words, the coefficients in (3.1) may be taken to be the conserved charges, up to a factor that may depend upon $\beta$ and the spin, but not on the particle type. Thus for any particle $a$, regardless of type, the string-like field may be written

$$X^a(\theta) = \sum_{r=s+kh} \frac{\hbar}{r} e^{-r\theta} \pi^a_{s+kh} c_r, \quad (3.15)$$

where

$$\pi^a_{s+kh} \propto p^a_{s+kh}.$$ 

This remark is very interesting in the context of the fusing relation (1.2), since it implies that the fusing relation for the operators follows from the bootstrap for the eigenvalues of
the conserved charges. To see this, consider a normal-ordered product of vertex operators for a pair of particles \( a \) and \( b \), such as that appearing on the right hand side of (3.3), and evaluate it for the values of rapidity appropriate to a fusing process (1.2). Thus, it is required to evaluate

\[
: V^a(\theta_c - i\mathcal{U}^b_{ac}) V^b(\theta_c + i\mathcal{U}^a_{bc}) :
\]

\[
= : \exp \left[ \sum_{r=s+kh} \frac{\hbar e^{-r\theta_c}}{r} \left( \pi^a_{-r} e^{i r \mathcal{U}^b_{ac}} + \pi^b_{-r} e^{-i r \mathcal{U}^a_{bc}} \right) c_r \right] :. \tag{3.16}
\]

However, using the conserved quantity bootstrap (1.6), valid for any spin \( r \), and remembering the \( \pi \)'s and \( p \)'s are proportional with a factor independent of the species of particle, the right hand side of equation (3.16) is precisely \( V^c(\theta_c) \), as desired. The same calculation works for all the other pieces of any vertex operator, since the ‘delocalisation’ is independent of the particle type, and hence for the complete operators \( V^a(\theta_a, \bar{\theta}_a) \) defined in (3.11) or (3.14).

It remains to study the singularity structure of the normal-ordered operator product (3.12), at least when the particles are on-shell so that \( \bar{\theta} = -\theta \). Only brief comments will be made here, restricted to the minimal case, a fuller analysis will be given elsewhere.

Firstly, although the numerator and denominator on the right hand side of (2.20) have branch cuts (since the inner products of weights are not usually integers), when \( \bar{\theta} = -\theta \) these combine nicely to yield poles and zeroes. Moreover, the poles and zeroes also appear in the S-matrix, unless they cancel in the exchange relation. There are just two types of cancelling poles and zeroes; zeroes at \( \Theta_{ab} = 0 \) and poles at \( \Theta_{ab} = i\pi \), the latter occurring only when \( b = \bar{a} \), the former for \( b = a \).** The coefficient of the pole at \( \Theta_{a\bar{a}} = i\pi \) in the exchange relation and ought not to participate in the bootstrap; if

** Note, the vertex operator \( V^{\ast a}(\theta_a, \bar{\theta}_a) \), obtained from (3.14) by reversing the sign of the weight \( \lambda_a \) on the right hand side, is in a sense, the operator conjugate to \( V^a(\theta_a, \bar{\theta}_a) \); it satisfies the same exchange relation, it has a commutator with the conserved quantities with the opposite sign to (3.6), and there is a simple pole at \( \theta_a = \theta'_a \) in the operator product of \( V^a(\theta_a, \bar{\theta}_a) V^{\ast a}(\theta'_a, \bar{\theta'}_a) \).
\( m = n + 1 \), the pole indicates the presence of a forward-channel bound state and participates in the bootstrap; if \( m = n - 1 \) it does not. In other words, it is the algebraic sum of poles and zeroes that appears to be relevant for the fusing relation, not the mere existence of poles. Except by examining the S-matrix, it is not clear why this should be so. For a deeper analysis of the S-matrix pole structure, including an explanation of some of the above remarks, see ref. [15].

4. Discussion

Despite the lack of any construction in terms of the elementary Toda fields, the vertex operators presented here do at least provide a succinct summary of the affine Toda S-matrices and elucidate further the algebraic structure of these theories. Moreover, given (1.2), the S-matrix expressions (2.17) automatically satisfy (1.3), a fact which otherwise needed to be checked separately. A particularly nice feature of the full vertex operator, expressed in terms of fields (3.13), is its dependence (without distinction of colour), on the eigenvalues of the conserved charges. These statements apply equally well to the full S-matrix or its minimal part, the vertex representation of the former being admittedly the more ad hoc. An important ingredient in the construction presented here is the delocalisation of the vertex operators as a means to obtain the necessary breaking of conformal symmetry. This is fairly arbitrary, though in fact reminiscent of a vertex operator construction of quantum affine algebras, due to Frenkel and Jing [27].

At a technical level, the problem of performing the analytic continuations to make the comparison between the two halves of the operator reordering calculations in (3.12) is quite delicate because of the branch points in the terms in (2.21); there may be rapidity-independent phases to be taken care of by the introduction of matrix factors in the vertex operators (see, for example, [21]). In fact, the construction presented here is ‘heterotic’, being asymmetrical between the terms depending upon \( \Theta \) and \( \overline{\Theta} \). Indeed, as remarked earlier, the part of the exchange relation arising from (3.5) is merely a phase independent of \( \Theta \). However, it plays an important rôle at intermediate stages, ensuring that the poles in the fusing relation (1.2) have integer powers. This and the desire to have quantities to represent all (positive and negative) spin conserved charges motivated the inclusion of an apparently redundant set of Fock space operators. It is also possible they serve to eliminate the need for ‘cocycle’ factors of the type mentioned above.
Finally, it is worth remarking that the construction of classical soliton solutions in terms of $\tau$-functions \cite{28} involved vertex operators of a similar type, although in the context of affine Toda theory a complete discussion of solitons is not yet available \cite{29}. It will be interesting to see how the classical information is related to the real $\beta$ scattering theory.
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