Ground-state fidelity at first-order quantum transitions

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(Dated: July 5, 2018)

We propose a finite-size scaling (FSS) theory for the fidelity, and the corresponding susceptibility, that holds whenever a given control parameter $\lambda$ is varied across a quantum phase transition. Our working hypothesis is based on a scaling assumption of the fidelity in terms of the FSS variables associated to $\lambda$ and to its variation $\delta \lambda$. This framework entails the FSS predictions for continuous transitions, and meanwhile enables to extend them to first-order transitions, where the FSS becomes qualitatively different. The latter is supported by analytical and numerical analyses of the quantum Ising chain along its first-order quantum transition line, driven by an external longitudinal field.

Quantum transitions (QTs) in many-body systems are related to significant changes of the ground state and low-excitation properties, induced by small variations of a driving parameter [1]. They are continuous when the ground state of the system changes continuously at the transition point, and correlation functions develop a divergent length scale. They are instead of first order when the ground-state properties are discontinuous across the transition point, generally arising from level crossings in the infinite-volume limit. In view of their key role played in several contexts of modern statistical mechanics, quantum information and condensed matter physics, it is of crucial importance to devise suitable tools for a proper characterization of their main features. To this purpose, different quantum-information based concepts have been recently put forward, in order to spotlight ground-state variations at QTs, such as the entanglement, as well as its fidelity and its susceptibility [2–4]. The net advantage of these approaches is that they do not rely on the identification of an order parameter with the corresponding symmetry breaking pattern.

The usefulness of the fidelity as a tool to distinguish quantum states can be traced back to Anderson’s orthogonality catastrophe [5]: the overlap of two many-body ground states corresponding to Hamiltonians differing by a small perturbation vanishes in the thermodynamic limit. It is thus tempting to quantify how this paradigm reflects in a system of finite size, where the forerunners of QTs may significantly emerge. Besides that, the fidelity susceptibility covers a central role in quantum estimation theory [6] [7], being proportional to the Fisher information. The latter indeed quantifies the inverse of the smallest variance in the estimation of the varying parameter, such that, in proximity of QTs, metrological performances are believed to drastically improve [8] [9].

The last decade has seen the birth of an intense theoretical activity focusing on the behavior of the fidelity and of the corresponding susceptibility (more generally, of the geometric tensor) [10] [12] at continuous QTs (CQTs). The establishment of a non-analytic behavior at large size has been exploited to evidence CQTs in several different contexts, which have been deeply scrutinized both analytically and numerically. We quote, for example, free-fermion models [13] [16], interacting spin [17] [22] and particle models [23] [29], as well as systems presenting peculiar topological [30] [32] and non-equilibrium steady-state transitions [33] [34]. However a characterization of first-order QTs (FOQTs) in this context is still missing, despite the fact that they are of great phenomenological interest. Indeed they occur in a large variety of many-body systems, including quantum Hall samples [35], itinerant ferromagnets [36], heavy fermion metals [37] [39], disordered systems [40] [41] and infinite-range models [42] [43].

The aim of this paper is to define a general finite-size scaling (FSS) framework where to discuss the fidelity and the corresponding susceptibility at arbitrary QTs. Assuming that the fidelity of finite systems is an analytic function of the relevant scaling variables associated to the driving parameter and to its variation, we put forward a FSS behavior that entails the expected power-law divergences associated with CQTs, and also enables to extend the analysis to FOQTs. In the latter, the type of divergence is controlled by the closure of the gap between the two lowest energy levels, being exponential in most of the cases. A scaling theory for the fidelity provides a simple and intuitive route towards a complete understanding of the behaviors of finite-size many-body systems at CQTs and FOQTs, which is mandatory to distinguish them, and obtain correct interpretations of experimental and numerical results at QTs.

We define our setting by considering a $d$-dimensional quantum many-body system of size $L^d$, with Hamiltonian

$$H(\lambda) = H_0 + \lambda H_1,$$  \hspace{1cm} (1)

where $[H_0, H_1] \neq 0$ and the parameter $\lambda$ drives the QT located at $\lambda = 0$. The fidelity

$$F(\lambda, \delta \lambda, L) \equiv \langle |\Psi_0(\lambda + \delta \lambda, L)\rangle |\Psi_0(\lambda, L)\rangle |$$  \hspace{1cm} (2)

is a geometrical object that can be used to monitor the changes of the ground-state wave function $|\Psi_0(\lambda, L)\rangle$ when varying the control parameter $\lambda$ by a small amount $\delta \lambda$ around its transition value. Assuming $\delta \lambda$ sufficiently small, one can expand Eq. (2) in powers of $\delta \lambda$:

$$F(\lambda, \delta \lambda, L) = 1 - \frac{1}{2}(\delta \lambda^2) \chi_F(\lambda, L) + O(\delta \lambda^3),$$  \hspace{1cm} (3)

where $\chi_F$ defines the fidelity susceptibility [3].
The interplay between $\lambda$ and $L$ at QTs can be described within FSS frameworks at both CQTs and FOQTs. The FSS limit is generally obtained at large $L$, keeping an appropriate combination $\kappa$ of $\lambda$ and $L$ fixed. At CQTs, this is generally given by

$$\kappa = \lambda L y_\lambda,$$  \hfill (4)

$y_\lambda$ being the renormalization-group (RG) dimension of the parameter $\lambda$. The power-law dependence is universal within the QT universality class, in particular it does not depend on the boundary conditions. Generic observables $O$ behave as $O(\lambda, L) \approx L^{y_O} f_O(\kappa)$, where $y_O$ is the RG dimension associated with $O$, and $f_O(\kappa)$ a scaling function depending on the boundaries. FSS behaviors also develop at FOQTs, although with significant differences. In particular, they turn out to be more sensitive to the boundary conditions, which may give rise to different functional dependencies of the corresponding scaling variable $\kappa$, leading to both exponential and power laws.

FOQTs generally arise from level crossings. However, as we develop the asymptotic FSS behavior (6) with $\kappa$ fixed. The fidelity susceptibility is obtained using the correlation function (2) arising from changes of the longitudinal field $H\sigma$, $\sigma = \sigma_1 + \sigma_2 + \sigma_3$. Its Hamiltonian reads $H_{\text{FSS}} = -J \sum_{x,y} \sigma_x^z(3) \sigma_y^z(3) - \delta h \sum_x \sigma_x^z(3) - \hbar \sum_x \sigma_x^z(3)$, where $\sigma(3)$ are the Pauli matrices, the first sum is over all bonds connecting nearest-neighbor sites $(x,y)$, while the other sums are over the $L$ sites. We assume $h = k_B T = 1$, $J = 1$ and $g > 0$. At $g = 1$ and $h = 0$, the model undergoes a CQT belonging to the two-dimensional Ising universality class, separating a disordered phase ($g < 1$) from an ordered ($g > 1$) one. For any $g < 1$, the field $h$ drives FOQTs along the $h = 0$ line. Relevant observables at the FOQT line are the energy difference $\Delta(h, L)$ of the lowest levels and the magnetization $m = L^{-1} \sum_x \sigma_x^z(3)$. We are interested in the behavior of the ground-state fidelity (2) arising from changes of the longitudinal field $h = \lambda$, keeping $\delta \lambda$ fixed. The fidelity susceptibility is obtained by expanding $F$ to second order in powers of $\delta h$.

At the CQT point $g = 1$, the system is expected to develop the asymptotic FSS behavior (6) with $\kappa = h L y_h$ and $y_h = 15/8$. Correspondingly, in the large-$L$ limit the fidelity susceptibility diverges as $\chi_F(h, L) \sim L^{2y_h} F_2(\kappa)$. We remark that, in the usual setting at $h = 0$, where the transverse field $g$ is varied, $H_{\text{FSS}} = -J \sigma_x^z(3) \sigma_y^z(3) - \hbar \sum_x \sigma_x^z(3)$, an analogous FSS follows, with $\kappa_g = (g - 1) L y_g$ and $y_g = 1/\nu = 1$, thus $\chi_F(g, L) \sim L^{2y_g} F_2^{(g)}(\kappa_g)$. The FOQTs, occurring at $g < 1$ along the line $h = 0$, can be related to the level crossing of the two lowest magnetized states $|+\rangle$ and $|-\rangle$ for $h = 0$, such that $\langle \sigma_x^z(3) \rangle = \pm m_0$, with $m_0 = (1 - g^2)^{1/8}$. In

Note that these FSS behaviors are supposed to hold at both CQTs and FOQTs, in terms of the corresponding scaling variable $\kappa$. At CQTs they give the expected power-law behavior: $\chi_F(\lambda, L) \approx L^{2y_h} F_2(\kappa)$ [3] [52]. At FOQTs instead we obtain:

$$\chi_F(\lambda, L) \approx \Delta_0(0)^{-2} (\partial E_\lambda/\partial \lambda)^2 F_2(\kappa).$$  \hfill (9)

It is possible to generalize the above FSS framework to a finite temperature $T$ [3]. In such case, the quantum system is described by the density matrix $\rho_n \equiv \rho(\lambda, T, L) = Z^{-1} \sum_n e^{-E_n/k_B T} |\Psi_n\rangle \langle \Psi_n|$, where $Z$ is the partition function. The fidelity between two mixed states can be defined as [53]: $F(\lambda, \delta \lambda, T, L) = \text{Tr} \sqrt{\rho_n} \rho_{\lambda+\delta \lambda} \sqrt{\rho_n}$, which reduces to Eq. (2) for $T \to 0$. The corresponding fidelity susceptibility can be extracted analogously to Eq. (3). At a QT, the $T = 0$ scaling (6) can be straightforwardly extended to keep into account the temperature, by adding a further scaling variable $\tau = T/T_0(0)$, so that

$$F(\lambda, \delta \lambda, T, L) \approx F(\kappa, \delta \kappa, \tau).$$  \hfill (10)

At QTs, $\tau \sim TL^z$, where $z$ is the dynamic exponent.

We now verify the above FSS predictions by presenting analytical and numerical evidence for the paradigmatic one-dimensional quantum Ising model in the presence of transverse and longitudinal fields. Its Hamiltonian reads

$$H_{\text{Is}} = -J \sum_{x,y} \sigma_x^z(3) \sigma_y^z(3) - \delta h \sum_x \sigma_x^z(3) - \hbar \sum_x \sigma_x^z(3),$$  \hfill (11)

where $\sigma(3)$ are the Pauli matrices, the first sum is over all bonds connecting nearest-neighbor sites $(x,y)$, while the other sums are over the $L$ sites. We assume $h = k_B T = 1$, $J = 1$ and $g > 0$. At $g = 1$ and $h = 0$, the model undergoes a CQT belonging to the two-dimensional Ising universality class, separating a disordered phase ($g < 1$) from an ordered ($g > 1$) one [1]. For any $g < 1$, the field $h$ drives FOQTs along the $h = 0$ line. Relevant observables at the FOQT line are the energy difference $\Delta(h, L)$ of the lowest levels and the magnetization $m = L^{-1} \sum_x \sigma_x^z(3)$. We are interested in the behavior of the ground-state fidelity (2) arising from changes of the longitudinal field $h = \lambda$, keeping $\delta \lambda$ fixed. The fidelity susceptibility is obtained by expanding $F$ to second order in powers of $\delta h$.

At the CQT point $g = 1$, the system is expected to develop the asymptotic FSS behavior (6) with $\kappa = h L y_h$ and $y_h = 15/8$. Correspondingly, in the large-$L$ limit the fidelity susceptibility diverges as $\chi_F(h, L) \sim L^{2y_h} F_2(\kappa)$. We remark that, in the usual setting at $h = 0$, where the transverse field $g$ is varied, $H_{\text{FSS}} = -J \sigma_x^z(3) \sigma_y^z(3) - \hbar \sum_x \sigma_x^z(3)$, an analogous FSS follows, with $\kappa_g = (g - 1) L y_g$ and $y_g = 1/\nu = 1$, thus $\chi_F(g, L) \sim L^{2y_g} F_2^{(g)}(\kappa_g)$. The FOQTs, occurring at $g < 1$ along the line $h = 0$, can be related to the level crossing of the two lowest magnetized states $|+\rangle$ and $|-\rangle$ for $h = 0$, such that $\langle \sigma_x^z(3) \rangle = \pm m_0$, with $m_0 = (1 - g^2)^{1/8}$. In
a finite system of size $L$ with periodic or open boundary conditions (PBC and OBC, respectively), the lowest states are superpositions of $|+\rangle$ and $|-\rangle$, due to tunneling effects. Their energy difference $\Delta_0(L) \sim g^L$ vanishes exponentially with $L$. Conversely, the difference $\Delta_{0,i} \equiv E_i - E_0$ for higher excited states ($i > 1$) remains finite for $L \to \infty$. The interplay of the size $L$ and the field $h$ gives rise to the FSS of the low-energy properties \cite{45}. Its scaling variable is obtained from Eq. (5), i.e. $\kappa = 2m_0hL/\Delta_0(L)$, using the fact that $E_h = 2m_0hL$ is the energy variation associated with $h$. The FSS limit corresponds to $L \to \infty$ and $h \to 0$, keeping $\kappa$ fixed. Correspondingly, the energy difference of the lowest states and the magnetization behave as \cite{15} $\Delta(h,L) \approx \Delta_0(L)D(\kappa)$ and $m(h) \approx m_0M(\kappa)$, where $D(\kappa)$ and $M(\kappa)$ are scaling functions independent of $g$. The FSS of the fidelity and its susceptibility is given by Eqs. (9) and (15). We obtain

$$\chi_F(h,L) \approx \left[ \frac{2m_0L}{\Delta_0(L)} \right]^2 \mathcal{F}_2(\kappa), \quad \kappa = \frac{2m_0hL}{\Delta_0(L)}$$

implying that it exponentially diverges with $L$. This is confirmed by the numerical results \cite{50} of Fig. 1 where the curves of $\chi_F$ for PBC display sharp, and exponentially increasing, peaks around $h=0$, while $\chi_F = O(L)$ for larger $|h|$. Since the low-energy spectrum for PBC and OBC across the FOQT is characterized by the level crossing of the two lowest states, while the energy differences with the other ones remain $O(1)$, the asymptotic FSS can be exactly obtained by performing a two-level truncation of the spectrum \cite{45,57,58}, keeping only the lowest energy levels $|\pm\rangle$. Using the corresponding effective Hamiltonian, we obtain $\mathcal{F}_2^{(a)}(\kappa, \delta \kappa) = \cos(\delta \alpha/2)$, where $\delta \alpha = |h|\Delta_0(L)/L \approx 2m_0hL/\Delta_0(L)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig1}
\caption{Fidelity susceptibility $\chi_F(h,L)$ for the Ising model \cite{11} with $g=0.9$ and PBC, associated with changes of the longitudinal parameter $h$, for some values of $L$, up to $L=16$. The inset displays curves for $\chi_F/[2m_0L/\Delta_0(L)]^2$, as a function of $\kappa = 2m_0hL/\Delta_0(L)$ [see Eq. (12)], which exponentially approach the scaling function $\mathcal{F}_2^{(a)}(\kappa)$ (thick black line), cf. Eq. (13). Analogous results are obtained for other $g<1$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig2}
\caption{Same as in Fig. 1 but for $g=0.5$ and ABC. The inset shows the rescaled fidelity susceptibility according to Eq. (14), for $\kappa = hL^2$. The curves for $\chi_F/L^6$ clearly approach a scaling function of $\kappa$, with $O(L^{-2})$ corrections.}
\end{figure}
The inset shows the FSS of $\chi_F$ around $h = h_{tr}(L)$, with $\kappa = [h - h_{tr}(L)]L/\Delta_m(L)$. With increasing $L$, the curves of $(\Delta_m/L)^2 \chi_F(h, L)$ rapidly (exponentially) approach the two-level scaling function, cf. Eq. (15), with $a \approx 0.67$, $b \approx 1.64$.

to that of neutral boundary conditions, such as PBC and ABC [51]. When $h = 0$, the system is in the negatively magnetized phase, and $\Delta_0(L) = 4(1 - g) + O(L^{-2})$. For sufficiently small $h$, the observables depend smoothly on it. Then the system undergoes a sharp transition to the other phase at $h \approx h_{tr}(L) > 0$, which tends to zero with increasing $L$, asymptotically as $h_{tr}(L) \approx \eta(g)/L$, where $\eta(g)$ is a $g$-dependent constant [51]. This sharp transition corresponds to the minimum $\Delta_m(L)$ of the energy difference $\Delta(h, L)$ of the lowest levels, which vanishes exponentially with increasing $L$, as $\Delta_m(L) \sim e^{-b(g)L}$. Around $h_{tr}$, the suitable scaling variable turns out to be $\kappa = [h - h_{tr}(L)]L/\Delta_m(L)$, analogously to that of PBC and OBC, apart from the $1/L$ shift of the transition point. The corresponding scaling behaviors, $\Delta(h, L) \approx \Delta_m(L) D(\kappa)$ and $m(h, L) \approx m_0 M(\kappa)$, turn out to be those emerging from an avoided two-level crossing, similarly to the case of PBC.

Figure 3 shows the $h$-dependence of the fidelity susceptibility $\chi_F(h, L)$ for several values of $L$ [56]. Its behavior reflects that of other observables. In particular it is smooth around $h = 0$, since we checked that the ratio $\chi_F(h, L)/\chi_F(h = 0, L)$ rapidly approaches a function of $h$ only, with $\chi_F(0, L) = O(L)$ (not shown). Then, with increasing $h$, the curves show a sharp peak around $h_{tr}(L)$, whose maximum rapidly increases with $L$, and becomes narrower and narrower. For even larger $h$, $\chi_F(h, L)$ tends to rapidly become independent of $L$; this is related to the fact that the ground state is essentially given by spatially separated kink and antikink structures, whose position depends smoothly on $h$ [51]. The scaling behavior around $h_{tr}(L)$ can be inferred from the general ansatz [8]:

$$\frac{\Delta_m^2}{L^2} \chi_F(h, L) \approx a F_2^{(2)}(b\kappa), \quad \kappa = \frac{[h - h_{tr}(L)]L}{\Delta_m(L)}, \quad (15)$$

where $F_2^{(2)}(x)$ is the two-level scaling function [13], while $a$ and $b$ are appropriate normalizations. This is confirmed by numerical data in the inset of Fig. 3.

The case of fixed but opposite boundary conditions, i.e. $|\downarrow\rangle$ and $|\uparrow\rangle$ at the ends of the chain, is supposed to be similar to that with ABC [49, 50], because the low-energy states are again one-kink states. Thus, $\Delta_0(L) \sim L^{-2}$ as well, and a power-law behavior such as $L^{-1/2}$ is expected.

In conclusion we have shown that the ground-state fidelity and the corresponding susceptibility develop scaling behaviors at both CQTs and FOQTs, arising from the interplay between the driving parameter and the system size. This can be achieved within a general FSS framework. CQTs are characterized by power laws, where the scaling variable behaves as $\kappa = \lambda L^{\alpha_1}$, and boundary conditions only affect the scaling functions of observables. This contrasts with FOQTs, whose distinctive feature is a remarkable qualitative dependence of $\kappa$ on the boundary conditions, ranging from exponential to power-law FSS. Our FSS framework is quite general: we expect it to hold even in higher dimensions and for FOQTs of other models, where it would be tempting to have a direct numerical validation. Moreover, the possibility to generalize it to finite temperature makes it relevant also to quantum thermometry close to criticality, where estimation performances depend on the scaling behavior [8].

As suggested from the present study, the FSS of the fidelity is amenable to a direct experimental verification by means of small-size quantum simulators (i.e., of the order of ten spins), which can thus serve as a probe of the nature of the transition itself. To be more specific, the best viable strategy would be to measure the Loschmidt echo after a sudden quench [50, 51], a quantity strictly related to the fidelity susceptibility [51], which might shed light on the mutual interplay between QTs, entanglement and decoherence [55, 57].

We thank R. Fazio and A. Pelissetto for fruitful discussions.

Appendix A: Two-level reduction of the spectrum across the FOQT line

In the thermodynamic limit, the low-energy spectrum for PBC and for OBC across the FOQT is characterized by the level crossing of the two lowest states, while the energy differences with the other ones remain finite. The asymptotic FSS behavior for the fidelity and for its susceptibility can be thus exactly obtained by performing a two-level truncation of the spectrum, following Refs. [44, 57, 58], keeping only the lowest energy levels. For the sake of completeness, here we sketch this derivation.

The effective Hamiltonian, written in the Hilbert space spanned by the two lowest magnetized states $|+\rangle$ and $|-\rangle$ for $h = 0$, i.e. such that $\langle \pm | \sigma_x^{(3)} | \pm \rangle = \pm m_0 \langle \pm | | \pm \rangle$
\[ m_0 = (1 - g^2)^{1/8} \]
reads:

\[ H_2(h) = -\beta \sigma^{(3)} + \delta \sigma^{(1)}. \tag{A1} \]

The parameters \( \beta \) and \( \delta \) correspond to \( \beta = m_0 h L \) and \( \delta = \Delta_0 / 2 \), such that \( \kappa(h) = \beta / \delta \). The eigenstates are

\[
\begin{align*}
|0\rangle &= \sin(\alpha/2)|-\rangle + \cos(\alpha/2)|+\rangle, \\
|1\rangle &= \cos(\alpha/2)|-\rangle - \sin(\alpha/2)|+\rangle,
\end{align*}
\]
(A2)
where \( \tan \alpha = \kappa^{-1} \) with \( \alpha \in (0, \pi) \), and \( E_1 - E_0 = \Delta_0 \sqrt{1 + \kappa^2} \).

1. Zero temperature

Straightforward calculations confirm the FSS behavior of the fidelity at zero temperature, \( F(\lambda, \delta \lambda, L) \approx \mathcal{F}(\kappa, \delta \kappa) \). Indeed, we obtain

\[ F(h, \delta h, L) \approx \mathcal{F}^{(2)}(\kappa, \delta \kappa) = \cos(\delta \alpha / 2), \tag{A4} \]
where \( \tan(\alpha + \delta \alpha) = (\kappa + \delta \kappa)^{-1} \). Fig. 4 shows some plots of \( \mathcal{F}^{(2)}(\kappa, \delta \kappa) \) for two values of \( \kappa \). Finally, the corresponding scaling function of the fidelity susceptibility is easily obtained:

\[ \chi_{F}^{(2)}(\kappa) = \frac{1}{4(1 + \kappa^2)^2}. \tag{A5} \]

2. Finite temperature

The definition of fidelity can be extended to finite temperature, as well. In this case, the quantum system is described by the density matrix

\[ \rho_\lambda = \rho(\lambda, T, L) = Z^{-1} \sum_n e^{-E_n / k_b T} |\Psi_n\rangle\langle \Psi_n|, \tag{A6} \]

where \( |\Psi_n\rangle \) is the Hamiltonian eigenstate corresponding to the eigenvalue \( E_n \), while \( Z = \sum_n |\Psi_n\rangle e^{-E_n / k_b T} |\Psi_n\rangle \) is the partition function. The quantum fidelity between two mixed states can be defined as:

\[ F(\lambda, \delta \lambda, T, L) = \text{Tr} \sqrt{\rho_\lambda^{1/2} \rho_{\lambda+\delta \lambda} \rho_\lambda^{1/2}}. \tag{A7} \]

At a quantum transition, the \( T = 0 \) scaling can be straightforwardly extended to keep into account the finite temperature: \( F(\lambda, \delta \lambda, T, L) \approx \mathcal{F}(\kappa, \delta \kappa, \tau) \), where \( \tau = T / \Delta_0 (L) \) [see Eq. (10)]. The computation based on the two-level truncation confirms this scaling behavior. In Fig. 5 we report some plots of the scaling function \( \mathcal{F}^{(2)}(\kappa, \delta \kappa, \tau) \), for different values of \( \kappa \) and \( \tau \). Note that, for \( \kappa = 0 \), the zero-temperature fidelity at large \( \delta \kappa \) approaches the asymptotic value \( |\langle + |0\rangle| = 2^{-1/2} \approx 0.707 \).

On the other hand, for \( \kappa \to -\infty \), it approaches zero, since it corresponds to abruptly sweeping from one side of the transition, to the other one. The effect of the temperature is to progressively smoothen the behavior of the various curves with \( \delta \kappa \).

FIG. 4: Scaling function of the fidelity susceptibility for two different values of \( \kappa \), at finite temperature \( \tau \), as obtained in a two-level truncation scheme. The continuous black curves correspond to the zero-temperature case, for which the analytic curve of Eq. (A4) holds.

FIG. 5: Convergence of the finite-size fidelity susceptibility to the asymptotic scaling function \( \mathcal{F}_\lambda(\kappa) \). Data are for \( \kappa = 0 \), corresponding to a longitudinal field \( h = 0 \) with the choice of boundary conditions adopted here. The upper panel is for PBC, with \( g = 0.9 \). We plot the rescaled susceptibility \( \chi_F / (2 m_0 L / \Delta_0 (L))^2 \) as a function of \( L \), subtracting the asymptotic value given by \( \mathcal{F}_\lambda^{(2)}(0) = 1/4 \). The red line is an exponential fit of the data for \( 8 \leq L \leq 18 \). The lower panel is for ABC, with \( g = 0.5 \). We plot the rescaled susceptibility \( \chi_F / L^6 \) as a function of \( L^{-2} \). The red line is a power-law fit of the data for \( 14 \leq L \leq 24 \).
Appendix B: Finite-size corrections to the asymptotic behavior

We now provide details on the approach to the scaling function for the fidelity susceptibility. As claimed in the main text, for PBC and for EFBC, we expect exponential finite-size corrections to the asymptotic two-level scaling function obtained above. On the other hand, for ABC, a slower power-law convergence with $L$ to the asymptotic behavior is expected.

The two paradigmatic cases of PBC and ABC are shown in Fig. 5. Here we report numerical data for behavior is expected.

As evidenced in Eq. (12), the rescaled susceptibility $\chi_F/[2\mu_0L/\Delta_0(L)]^2$ with PBC is expected to converge to the two-level scaling function in Eq. (A5). The upper panel highlights a clear exponential convergence with $L$ to $F_{2(1)}^\beta(0) = 1/4$. Such convergence becomes even faster if one chooses a smaller transverse field $|g|$. Conversely for ABC, Eq. (14) tells us that, in order to observe convergence, one has first to divide the susceptibility $\chi_F(h, L)$ by $L^6$. After such rescaling, a power-law convergence with $O(L^{-2})$ to the asymptotic value can be observed (see the bottom panel).

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