Pauli Blocking for a Relativistic Fermi Gas in Quasielastic Lepton Nucleus Scattering

Arie Bodek

Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627-0171 USA

Received: date / version 1. November 5, 2021

Abstract. The expressions for the overall effect of Pauli blocking in quasielastic (QE) lepton scattering from nuclear targets within the framework of the Relativistic Fermi Gas (RFG) given in several publications are incorrect. For example, the expressions published by Bell and Llewelyn Smith in 1972 are incorrect (probably a typographical error). The expressions for Pauli blocking presented in several subsequent publications including the paper of C.H. Llewelyn Smith in 1972 and the paper of Paschos and Yu in 2002 have the same error. Another example is a 1992 paper by Singh and Oset which has a different (probably also a typographical) error. Other papers such as the papers of Tsai in 1974 and Bosted and Mamyan in 2012 use the correct expressions. In this short preprint we review the geometrical derivation of the overall reduction of the QE cross sections due to Pauli blocking for electron and neutrino scattering for targets with equal numbers of neutrons and protons and for targets with unequal numbers of neutrons and protons and derive the correct expressions. However, we note that the effects of Pauli blocking within the framework of the RFG model are much larger than in other more realistic models of QE scattering such as $\psi'$ superscaling or a spectral function approach.

PACS. 13.15.+g Neutrino interactions – 25.30.Pt Neutrino scattering – 25.30.Dh, 25.30.Fj electron scattering inelastic

1 Introduction

The expressions for the overall effect of Pauli blocking in quasielastic (QE) lepton scattering from nuclear targets within the framework of the Relativistic Fermi Gas (RFG) given in several published papers are incorrect (probably due to typographical errors). In this short preprint we review the geometrical derivation of the overall reduction of the QE electron and neutrino cross sections due to Pauli blocking for the RFG model for targets with equal numbers of neutrons and protons and for targets with unequal numbers of neutrons and protons and derive the correct expressions. However, we note that the effects of Pauli blocking within the framework of the RFG model are much larger than in other more realistic models of QE scattering such as $\psi'$ superscaling or a spectral function approach.

1.1 Simple Geometrical Derivation of RFG Pauli Blocking

The first calculations of Pauli blocking in quasielastic (QE) electron and neutrino scattering within the framework a Relativistic Fermi Gas (RFG) were published by Gatto in 1953. A more complete expression for the case of neutrino scattering from nuclei with an unequal number of neutrons and protons was reported in a CERN conference proceeding by Berman in 1961.

Fig. 1 shows the momentum transfer to a nucleon in a nuclear target in QE lepton scattering within the framework of the RFG model for the case in which the Fermi momentum $K_F$ for the ensemble of nucleons in the initial state is the same as $K_F$ of the ensemble of spectator nucleons in the final state. This is the case of QE electron scattering from neutrons or protons in any nucleus. It is also the case for QE scattering of neutrinos and antineutrinos from nuclear targets in which the number of protons and neutrons is the same. The QE scattering process transfers a 3-momentum $q$ to any of the nucleons in the initial state.
Pauli blocking factor

The Fermi momentum $K_F$ for the ensemble of nucleons in the initial state is the same as $K_F$ for the ensemble of spectator nucleons in the final state. This is the case for QE electron scattering from neutrons or protons in any nucleus. It is also the case for QE scattering of neutrinos and antineutrinos from nuclear targets in which the number of protons and neutrons is the same. In the QE scattering process a 3-momentum $|q|$ is transferred to one of the nucleons in the final state.

The Pauli suppression factor is equal to 1.0 minus fraction of the overlap volume between the left sphere of the spectator nucleons and the right sphere of the interacting nucleons which were given a 3-momentum transfer $|q|$. The volume of a dome of radius $R$ and height $h$ is $\frac{3}{4}\pi h^2(3R-h)$. The overlap volume is twice that or $\frac{3}{2}\pi h^2(3R-h)$ and the total volume of the final state nucleon sphere is $V_{FS} = \frac{4}{3}\pi R^3$.

In Fig. 1, $R = K_F$ and $h = K_F - \frac{|q|}{2}$. For electron scattering an for neutrino scattering on nuclear targets with equal number of neutrons and protons this leads to an overall Pauli blocking factor $P_{RF} = \frac{P_{electrons}}{P_{pauli}} = \frac{P_{neutrinos}}{P_{antineutrinos}}$ where:

$$ P_{RF} = \begin{cases} 3 |q| & (for \ |q| < 2K_F) \\ \frac{1}{16} \left(\frac{|q|}{K_F}\right)^3 & (for \ |q| > 2K_F) \end{cases} $$

If we define $x = \frac{|q|}{2K_F}$ we obtain:

$$ P_{RF} = \begin{cases} 3x & x < 1 \\ \frac{3}{4} - x^3 & (for \ |q| < 2K_F) \end{cases} $$

The above equation is identical to the expression published by Gatto in 1953. It is also indentical to the expression published in a CERN conference proceeding by Berman in 1961 for the case in which the number of protons $Z$ equal to the number of neutrons $N$. This expression is also used in the paper by Tsai in 1974, and in recent analyses of Jefferson Lab electron scattering data.

However, the expression published in by Bell and Llewelyn Smith in 1972 is incorrect (probably due to a typographical error). For the case of a nuclear target with equal numbers of protons it gives a Pauli blocking factor $P_{incor} = \frac{3}{2}2x - \frac{3}{2}x^3$. The expressions for Pauli blocking presented in several subsequent publications including C.H. Llewelyn Smith in 1972 and Paschos and Yu in 2002 have the same (probably typographical) error. A 1992 paper by Singh and Oset has a different (probably typographical) error.

Fig. 2 shows a comparison of the RFG overall Pauli blocking factor for $^{12}$C (as a function of the square of the four-momentum transfer $Q^2$) for $K_F = 0.221$ GeV (dashed red line) and for $K_F = 0.228$ GeV (dotted blue line). Also shown is the overall Pauli blocking factor for the Bodek-Ritchie Fermi Gas with high momentum tail as implemented in genie version 2 (solid black line) and GENIE V2. There is reasonable agreement with the genie calculation (exact agreement is not expected since slightly different momentum distributions are used).
2 Scattering of neutrinos and antineutrinos from nuclei with N protons and Z neutrons

Fig. 3 illustrates QE scattering within the framework of the RFG model for QE scattering from nuclei with different numbers of neutrons (N) and protons (Z) (with Fermi momenta $K_F^N$ and $K_F^P$, respectively). The top panel shows the case for neutrino QE scattering on the neutrons in the nucleus and the bottom panel shows the case for antineutrino QE scattering on the protons in the nucleus.

Fig. 4 shows the expression (12) for the overlap volume $V$ of two spheres of radii $R$ and $r$ and distance $d$.

The effects of the Pauli exclusion principle is then given by the overlap volume of the displaced neutron sphere with the spectator proton sphere. In this case the overlap volume $V$ is calculated for $d = |q|$, $R = K_F^N$ and $r = K_F^P$.

Therefore, for neutrino QE scattering from the neutrons in the nucleus ($N > Z$) we obtain:

$$P_N = \left[1 - \left(\frac{K_F^N)^3}{(K_F^P)^3}\right)\right] \quad |q| < K_F^N - K_F^P$$

$$P_N = 1 - \frac{3 - V}{4\pi (K_F^P)^3} \quad K_F^N - K_F^P < |q| < K_F^N + K_F^P$$

$$P_N = 1 \quad |q| > (K_F^N + K_F^P) \quad (3)$$

For the case of antineutrinos, the QE scattering occurs on the protons in the nucleus. The effects of the Pauli exclusion principle is then given by the overlap volume of the displaced proton sphere with the spectator neutron sphere. He also the overlap volume $V$ is calculated for $d = |q|$, $R = K_F^P$ and $r = K_F^P$.

$$P_Z = 0 \quad |q| < K_F^N - K_F^P$$

$$P_Z = 1 - \frac{3 - V}{4\pi (K_F^P)^3} \quad K_F^N - K_F^P < |q| < K_F^N + K_F^P$$

$$P_Z = 1 \quad |q| > (K_F^N + K_F^P) \quad (4)$$

Numerically, the above exact expressions give similar results to earlier expression by Berman [2], and the (corrected) expression of J.S. Bell, and C. H. Llewellyn Smith [6]. Those two calculations make additional assumptions about the relationship between the Fermi momenta of neutrons and protons.

Table 1 shows the Fermi momenta $K_F^P$ for protons and neutrons for various nuclei. The relativistic Fermi gas values are from Ref. 13 [13]. The results from a $\psi'$ superscaling fit are from reference 15.

The Pauli blocking factors for several nuclei are given in Table 2. Here, for $^{12}_6C$ we compare GENIE v2 (for a Bodek-Ritchie momentum distribution) to RFG calculation with two values of $K_F$. (In this case the factors are the same for electrons and neutrinos). Also shown are the RFG Pauli blocking factors for neutrino QE scattering on neutrons and antineutrino QE scattering on protons for $^{40}_{18}Ar$, $^{56}_{26}Fe$ and $^{208}_{82}Pb$.

3 Conclusions

In conclusion, we present the correct expressions for the overall Pauli blocking within the framework of the RFG model for electron scattering, for neutrino scattering on targets with the same number of neutrons and protons, and for neutrino and antineutrino scattering on nuclei with unequal numbers of neutrons and protons. The Pauli blocking factors for $^{12}_{6}C$ in GENIE v2 are consistent with the geometrical calculations for the Relativistic Fermi Gas. However, we note that the effects of Pauli blocking within the framework of the RFG model are much larger than in other more realistic models of QE scattering, such as $\psi'$ superscaling [16] or a spectral function approach. In addition to Pauli blocking, there are other sources of suppression of QE scattering at low values of $q$. Our experimental
investigation of the suppression of low $q$ QE electron scattering cross sections within the framework of superscaling is presented in another publication [17].

4 Acknowledgements

Research supported by the U.S. Department of Energy under University of Rochester grant number DE-SC0008475.

References

1. R. Gatto, Nuovo Cimento 10 (1953) 1559; 2 (1955) 670.
2. M. Berman, Proc. high energy Theor. Conf., CERN 61-22, 9 (1961).
3. Y. S. Tsai. Rev. Mod. Phys. 46, 815 (1974).
4. P.E. Bosted and V. Mamyan, arXiv:1203.2262 (2012).
5. A. Bodek, H. S. Budd and M. E. Christy, Eur. Phys. J. C 71, 1726 (2011).
6. J.S. Bell, and C. H. Llewellyn Smith, Nucl. Phys. B28 (1971) 317.
7. C. H. Llewellyn Smith, Physics Reports 3 (1972) 261.
8. E. A. Paschos and J. Y. Yu, Phys.Rev. D65, 033002 (2002).
9. S.K. Singh and E. Oset, Nucl. Phy. A542 (1992) 587. (This paper as a different typographical error: Here $x = |q|/K_F$ is used.
10. C. Andreopoulos [GENIE Collaboration], Acta Phys. Polon. B 40, 2461(2009). C.Andreopoulos (GENIE), Nucl. Instrum. Meth.A614, 87,2010; Steven J. Gardiner, private communication.
11. A. Bodek, and J. L. Ritchie, Phys. Rev. D23, 1070 (1980).
   A. Bodek and J. L. Ritchie, Phys. Rev. D24, 1400 (1981).
12. https://mathworld.wolfram.com/Sphere-SphereIntersection.html
13. Arie Bodek and Tejin Cai, Eur. Phys. J. C. (2019) 79
14. E. J. Moniz, et al., Phys. Rev. Lett. 26, 445 (1971); E. J. Moniz, Phys. Rev. 184, 1154 (1969); R. R. Whitney et al. Phys, Rev. C9, 2230 (1974), R. A. Smith and E. J. Moniz, Nucl. Phys B43, 605 (1972)
15. C. Maieron, T.W. Donnelly, I. Sick, Phys.Rev. C65 (2002) 025502; J.E. Amaro, M.B. Barbaro, J.A. Caballero, T.W. Donnelly, A. Molinari, and I. Sick, Phys. Rev. C 71, 015501 (2005).
16. Megías Vazquez, G.D. (2017). Charged-current neutrino interactions with nucleons and nuclei at intermediate energies. (Tesis Doctoral). Universidad de Sevilla, Sevilla. https://idus.us.es/handle//11441/74826 ???
17. A. Bodek and M. E. Christy, in preparation (2021).
| Q2 | \(q^2\) | GENIE v2 | RFG | RFG | \(\nu - N\) | \(\bar{\nu} - P\) | \(\nu - N\) | \(\bar{\nu} - P\) |
|----|-------|---------|-----|-----|--------|--------|--------|--------|
| 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.131 | 0.00 | 0.149 | 0.00 |
| 0.01 | 0.10 | 0.375 | 0.334 | 0.324 | 0.339 | 0.239 | 0.342 | 0.227 |
| 0.02 | 0.14 | 0.494 | 0.465 | 0.451 | 0.445 | 0.361 | 0.445 | 0.349 |
| 0.03 | 0.17 | 0.582 | 0.560 | 0.544 | 0.523 | 0.452 | 0.522 | 0.439 |
| 0.04 | 0.20 | 0.652 | 0.635 | 0.619 | 0.587 | 0.525 | 0.583 | 0.512 |
| 0.05 | 0.22 | 0.710 | 0.698 | 0.681 | 0.641 | 0.578 | 0.627 | 0.460 |
| 0.06 | 0.24 | 0.759 | 0.751 | 0.733 | 0.688 | 0.641 | 0.684 | 0.628 |
| 0.07 | 0.27 | 0.802 | 0.796 | 0.778 | 0.729 | 0.689 | 0.724 | 0.676 |
| 0.08 | 0.28 | 0.838 | 0.835 | 0.818 | 0.766 | 0.730 | 0.760 | 0.718 |
| 0.09 | 0.30 | 0.870 | 0.869 | 0.852 | 0.798 | 0.768 | 0.792 | 0.756 |
| 0.10 | 0.32 | 0.897 | 0.897 | 0.881 | 0.827 | 0.801 | 0.821 | 0.789 |
| 0.11 | 0.33 | 0.920 | 0.922 | 0.906 | 0.853 | 0.831 | 0.846 | 0.820 |
| 0.12 | 0.35 | 0.959 | 0.959 | 0.947 | 0.897 | 0.881 | 0.890 | 0.871 |
| 0.13 | 0.36 | 0.969 | 0.973 | 0.962 | 0.915 | 0.902 | 0.909 | 0.893 |
| 0.14 | 0.38 | 0.980 | 0.984 | 0.975 | 0.931 | 0.921 | 0.925 | 0.912 |
| 0.15 | 0.40 | 0.988 | 0.992 | 0.985 | 0.946 | 0.937 | 0.940 | 0.929 |
| 0.16 | 0.42 | 0.993 | 0.997 | 0.992 | 0.958 | 0.952 | 0.953 | 0.944 |
| 0.17 | 0.43 | 0.996 | 1.000 | 0.997 | 0.969 | 0.964 | 0.964 | 0.957 |
| 0.18 | 0.44 | 0.997 | 1.000 | 0.999 | 0.978 | 0.974 | 0.973 | 0.968 |
| 0.19 | 0.46 | 1.000 | 1.000 | 1.000 | 0.985 | 0.983 | 0.981 | 0.978 |
| 0.20 | 0.47 | 0.998 | 0.998 | 0.998 | 0.991 | 0.989 | 0.988 | 0.985 |
| 0.21 | 0.48 | 0.998 | 0.998 | 0.998 | 0.995 | 0.994 | 0.993 | 0.991 |
| 0.22 | 0.49 | 0.998 | 0.998 | 0.998 | 0.998 | 0.998 | 0.996 | 0.996 |
| 0.23 | 0.50 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0.24 | 0.51 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0.25 | 0.52 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0.26 | 0.53 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0.27 | 0.54 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0.28 | 0.55 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0.29 | 0.56 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0.30 | 0.57 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0.31 | 0.58 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

Table 2. Pauli blocking factors: For \(^{12}\)C, we compare GENIE v2 (for a Bodek-Ritchie momentum distribution) to RFG with two values of \(K_F\) (for \(^{12}\)C the Pauli blocking factors are the same for electrons and neutrinos). Also shown are the RFG Pauli blocking factors for neutrino QE scattering on neutrons and antineutrino QE scattering on protons for \(^{40}\)Ar, \(^{56}\)Fe and \(^{208}\)Pb.