DYNAMICS OF SIEGEL RATIONAL MAPS WITH PRESCRIBED COMBINATORICS

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ABSTRACT. We extend Thurston’s combinatorial criterion for postcritically finite rational maps to a class of rational maps with bounded type Siegel disks. The combinatorial characterization of this class of Siegel rational maps plays a special role in the study of general Siegel rational maps. As one of the applications, we prove that for any quadratic rational map with a bounded type Siegel disk, the boundary of the Siegel disk is a quasi-circle which passes through one or both of the critical points.

1. INTRODUCTION

Let $f : S^2 \to S^2$ be an orientation-preserving branched covering map. We call

$$\Omega_f = \{ x \mid \deg_x f > 1 \}$$

the critical set of $f$, and

$$P_f = \bigcup_{1 \leq k < \infty} f^k(\Omega_f)$$

the postcritical set. A branched covering map of the topological two sphere is called postcritically finite if its postcritical set is a finite set. Let $f, g : S^2 \to S^2$ be two orientation-preserving branched covering maps. We say $f$ and $g$ are combinatorially equivalent if there exist two homeomorphisms $\phi, \phi' : (S^2, P_f) \to (S^2, P_g)$, such that the diagram

$$(S^2, P_f) \xrightarrow{\phi'} (S^2, P_g)$$

$$f \downarrow \quad \downarrow g$$

$$(S^2, P_f) \xrightarrow{\phi} (S^2, P_g)$$

commutes, and $\phi$ is isotopic to $\phi'$ rel $P_f$. Thurston proved that an orientation-preserving and postcritically finite branched covering map with hyperbolic orbifold is combinatorially equivalent to a rational map if and only if it has no Thurston obstructions [26]. A detailed proof of this theorem was presented in Douady and Hubbard’s paper [9]. Since then, it has been a tantalizing problem to see to what extent such a combinatorial characterization is possible beyond

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the *postcritically finite* setting. Some progress has been made towards this direction. For instance, McMullen proved that for any rational map, there exist no Thurston obstructions outside all the possible rotation domains (Siegel disks or Herman rings) [16]. On the other hand, it was illustrated by Cui, Jiang, and Sullivan that there are *geometrically finite* branched covering maps which have no Thurston obstructions, but are not *combinatorially equivalent* to rational maps [5]. Here we say a branched covering map is *geometrically finite* if its *postcritical set* is an infinite set but has finitely many accumulation points. The example implies that, to make a *postcritically infinite* branched covering map *combinatorially equivalent* to a rational map, besides the non-existence of Thurston obstructions, some additional conditions have to be imposed on its local combinatorial structure around the accumulation points of the *postcritical set*. For a *geometrically finite* branched covering map, such a local condition was found by Cui, Jiang, and Sullivan, which they called *locally linearizable*. According to [5], a *geometrically finite* branched covering map is called *locally linearizable* if it can be *combinatorially equivalent* to some "normalized" one such that the later map is either holomorphically attracting or super-attracting in a neighborhood of each accumulation point of the *postcritical set*. They proved that a *geometrically finite* branched covering map is *combinatorially equivalent* to a sub-hyperbolic rational map if and only if it is *locally linearizable* and has no Thurston obstructions. Cui also studied under what condition, a *geometrically finite* branched covering map is *combinatorially equivalent* to a rational map with parabolic cycles. The situation in this case becomes more subtle where a new type of obstructions, called *invariant connecting arcs*, have to be considered as well as Thurston obstructions. We refer the reader to [3] for the details.

The main purpose of this work is to extend Thurston’s combinatorial criterion for *postcritically finite* rational maps to a class of Siegel rational maps, and then applies it to quadratic rational maps with *bounded type* Siegel disks. Here we call an irrational number $0 < \theta < 1$ of *bounded type* if $\sup \{a_i\} < \infty$ where $[a_1, \cdots, a_n, \cdots]$ is its continued fraction. We shall assume throughout this paper that $0 < \theta < 1$ is an irrational number of *bounded type*.

**Definition 1.1.** We use $R_{g}^{geom}$ to denote the class of all the rational maps $g$ such that

1. $g$ has a Siegel disk $D_g$ with rotation number $\theta$, and
2. $\partial D_g$ is a quasi-circle, and
3. $P_g - \overline{D_g}$ is a finite set.

**Remark 1.1.** Assume that $f$ has a bounded type Siegel disk $D$ such that $\overline{D} \subset U$ where $U$ is a domain on which $f$ is holomorphic. Then $\partial D$ must contain at least one critical point of $f$ [11]. It follows that for any $g \in R_{\theta}^{geom}$, $\partial D_g \cap \Omega_g \neq \emptyset$.

**Definition 1.2.** We use $R_{\theta}^{top}$ to denote the class of all the orientation-preserving branched covering maps $f : S^2 \to S^2$ such that
1. $f|\Delta : z \to e^{2\pi i \theta}z$ is a rigid rotation where $\Delta = \{z| |z| < 1\}$ is the unit disk, and
2. $\partial \Delta \cap \Omega_f \neq \emptyset$, and
3. $P_f - \Delta$ is a finite set.

We call the unit disk $\Delta$ the rotation disk of $f$.

For a branched covering map $f \in R^{top}_\theta$, we say $f$ is realized by a Siegel rational map $g \in R^{geom}_\theta$, if $f$ and $g$ are combinatorially equivalent to each other, and furthermore, when restricted to the Siegel disk, the combinatorial equivalence is a holomorphic conjugation. More precisely,

**Definition 1.3.** Let $f \in R^{top}_\theta$ and $g \in R^{geom}_\theta$. Let $\Delta$ be the unit disk, and $D_g$ be the Siegel disk of $g$. We say $f$ is realized by $g$ if
1. $f = \phi_1^{-1} \circ g \circ \phi_2$, and
2. $\phi_1$ is isotopic to $\phi_2$ relative to $P_f$, and
3. $\phi_1|_\Delta = \phi_2|_\Delta : \Delta \to D_g$ is holomorphic.

We now present a quick summary of our results. The first theorem extends Thurston’s combinatorial criterion for postcritically finite rational maps to the class $R^{geom}_\theta$. The proof is given in Section 2.

**Theorem A.** Let $0 < \theta < 1$ be an irrational number of bounded type. Then a branched covering map $f \in R^{top}_\theta$ can be realized by a Siegel rational map $g \in R^{geom}_\theta$ if and only if $f$ has no Thurston obstructions on the outside of the rotation disk.

The necessary part is a direct consequence of a theorem of McMullen. For a proof, see Appendix B of [16]. We need only to prove the sufficient part. The idea of the proof is as follows. First we construct a symmetric branched covering map $F$ such that when restricted on the outside of the unit disk, $F$ has the same combinatorial structure as that of $f$. Based on the branched covering map $F$, we construct a sequence of symmetric and postcritically finite branched covering maps $\{F_n\}$ such that $F_n \to F$ uniformly, and $|P_{F_n} - \partial \Delta| = |P_F - \partial \Delta|$ (Proposition 2.1). Then we show that for $n$ large enough, $F_n$ has no Thurston obstructions, and hence by Thurston’s theorem, it is combinatorially equivalent to some rational map $G_n$ (Lemma 2.4). Since $F_n$ is symmetric about the unit circle, it follows that $G_n$ is an Blaschke product. We then prove that the sequence $\{G_n\}$ is contained in some compact set of $R_{2d-1}$, the space of all the rational maps of degree $2d - 1$ (Lemma 2.16). By passing to a convergent subsequence, we may assume that $G_n \to G$ where $G$ is an Blaschke product of degree $2d - 1$. Then we show that $F$ and $G$ are combinatorially equivalent to each other (§2.4). The proof of Theorem A is then completed by a standard quasiconformal surgery on $G$ (§2.5).

The second theorem shows that the Julia set of any $f \in R^{geom}_\theta$ has zero Lebesgue measure. In particular, it implies the combinatorial rigidity of the maps in $R^{geom}_\theta$. The proof is given in Section 3.
Theorem B. Let \( f \in R^\text{geom}_\theta \). Then the Julia set of \( f \) has zero Lebesgue measure. In particular, if \( f \in R^\text{top}_\theta \) has no Thurston obstructions outside the rotation disk \( \Delta \), then up to a Möbius conjugation, there is a unique Siegel rational map \( g \in R^\text{geom}_\theta \) to realize \( f \).

The main part of the proof is to show that the Julia set of any Siegel rational map in \( R^\text{geom}_\theta \) has zero Lebesgue measure. The assertion of the rigidity then follows easily. For a quadratic polynomial with a bounded type Siegel disk, the zero measure statement was already proved by Petersen [19]. Petersen’s proof is based on a delicate geometric object, the so-called Petersen’s puzzle. Since for a map in \( R^\text{geom}_\theta \), the boundary of the Siegel disk may contain several critical points, each of which may have a different degree, there seems no easy way to construct the puzzles which is suitable for all the cases. To avoid this difficulty, we will introduce a new method, the minimal neighborhood method, which allows us to treat all these cases in a uniform way. One advantage of this method is that it may also be applied in the study of the Julia sets of entire functions with bounded type Siegel disks where Petersen puzzles are not available [24].

Let us briefly sketch the proof of the zero measure statement of Theorem B. Let \( g \in R^\text{geom}_\theta \). We first show that there is a Blaschke product \( \mathcal{G} \) which models \( g \). That is to say, the dynamics of \( g \) on the outside of the Siegel disk is quasiconformally conjugate to the dynamics of \( \mathcal{G} \) on the outside of the unit disk. Therefore it suffices to show that the set

\[
J_{\mathcal{G}} = J_G - \bigcup_{k=0}^{\infty} G^{-k}(\Delta)
\]

has zero Lebesgue measure. Assume that it is not true. It follows that there is a Lebesgue point of \( J_{\mathcal{G}} - \bigcup_{k=0}^{\infty} G^{-k}(\partial \Delta) \), say \( z_0 \), such that \( G^k(z_0) \to \partial \Delta \) as \( k \to \infty \) (Lemma 3.2). Now we define a sequence \( \{m(k)\} \) such that for each \( m(k) \), the point \( z_{m(k)} \) is the nearest one to \( \partial \Delta \) among all the points \( z_0, z_1, \ldots, z_{m(k)} \). Here by nearest we mean that \( z_{m(k)} \) is contained in some minimal neighborhood which is attached to the unit circle (see Definition 3.1). The importance of the sequence \( \{m(n)\} \) is that for each \( m(n) \), there is a number \( \tau(n) < m(n) \), such that the inverse branch of \( G \) which maps \( z_{\tau(n)+1} \) to \( z_{\tau(n)} \) strictly contracts the hyperbolic metric in some hyperbolic Riemann surface, and moreover, \( \tau(n) \to \infty \) as \( n \to \infty \) (Lemma 3.12). This allows us to construct a sequence of nested neighborhoods of \( z_0 \) such that the pre-images of \( \Delta \) count a definite part in each of these neighborhoods (§3.5). It follows that \( z_0 \) is not a Lebesgue point of \( J_{\mathcal{G}} - \bigcup_{k=0}^{\infty} G^{-k}(\partial \Delta) \). But this is a contradiction with our assumption.

As an application of Theorem A and Theorem B, in §4, we prove

Theorem C. For any bounded type irrational number \( \theta \), there is a constant \( 1 < K < \infty \) dependent only on \( \theta \), such that for any quadratic rational map with a Siegel disk of rotation number \( \theta \), the boundary of the Siegel disk is a \( K \)-quasi-circle which passes through one or both of the critical points of \( f \).
It was conjectured by Douady and Sullivan that the boundary of a Siegel disk for a rational map is a Jordan curve. The conjecture is still open and is far from being solved. For Siegel disks of polynomial maps, however, there has been some progress towards this conjecture [6], [24] and [30]. We especially refer the readers to [31] for a survey of all the relative results in this aspect.

Let us sketch the proof of Theorem C as follows. In §4.1, we consider a quadratic rational map $g$ with a Siegel disk of rotation number $\theta$. By a Möbius conjugation, we may normalize $g$ such that the Siegel disk is centered at the origin, and $g'(1) = 0$, $g(\infty) = \infty$. Let $\Sigma$ denote the space of all such maps. Each map in $\Sigma$ has exactly two critical points 1 and some $c \neq 1$. Under this parameterization, the space $\Sigma$ is homeomorphic to $\hat{\mathbb{C}} - \{0, 1, -1\}$.

In §4.2, we consider a family of degree-2 topological branched covering maps $f_t \in R_\theta^{top}$, $0 < t < 2\pi$ such that both the critical points of $f_t$ are on the unit circle and span an angle $t$ (see Figure 15). Clearly such $f_t$ has no Thurston obstructions outside the rotation disk. It follows that for each $0 < t < 2\pi$, there is a unique $c(t) \in \mathbb{C} - \{0, 1, -1\}$, such that $g_{c(t)}$ realizes $f_t$ (in the sense of Lemma 4.2). Similarly, we consider the family of topological branched covering maps $\tilde{f_t} \in R_\theta^{top}$ (see Figure 16), and for each $0 < t < 2\pi$, we get a unique $\tilde{c}(t) \in \mathbb{C} - \{0, 1, -1\}$ such that $g_{\tilde{c}(t)}$ realizes $\tilde{f_t}$ (in the sense of Lemma 4.2).

In §4.3, we prove that there is a uniform $1 < K < \infty$, which is independent of $t$, such that the boundary of the Siegel disk for any map $g_{c(t)}$ is a $K$—quasicircle (Lemma 4.3).

In §4.4, we prove that $\gamma = \{c(t) \mid 0 < t < 2\pi\}$ is a continuous curve segment which connects 1 and $-1$. By the same way, we get that $\tilde{\gamma} = \{\tilde{c}(t) \mid 0 < t < 2\pi\}$ is also a continuous segment which connects 1 and $-1$. We then show that $\xi = \gamma \cup \tilde{\gamma} \cup \{1, -1\}$ is a simple closed curve, which separates 0 and the infinity, and moreover, $\xi$ is invariant under $c \to 1/c$ (Lemma 4.7).

Let $\Omega_\infty$ be the unbounded component of $\hat{\mathbb{C}} - \xi$. In §4.5, §4.6, and §4.7, we show that for any four distinct integers $0 \leq k < l < m < n$, the cross-ratios of $g_k^c(1)$, $g_l^c(1)$, $g_m^c(1)$, and $g_n^c(1)$ are holomorphic functions on $\Omega_\infty$ and have no zeros. Moreover, each cross-ratio function can be continuously extended to $\partial \Omega_\infty = \xi$. This implies that the modulus of each cross-ratio function obtains its maximum and minimum on the boundary $\xi$ (Proposition 4.1). This is the key idea of the proof.

In §4.8, for each $c \in \Omega_\infty$, we define a map $T_c : \{e^{2k\pi i\theta} \mid k \geq 0\} \to \hat{\mathbb{C}}$ by $T(e^{2k\pi i\theta}) = g_k^c(1)$. We show that $T_c$ can be continuously extended to a homeomorphism $T_c : \partial \Delta \to \{g_k^c\}_{k \geq 0}(1)$. It follows that $\gamma_c = \{g_k^c\}_{k \geq 0}(1)$ is a Jordan curve (Lemma 4.25). We then show that for every four ordered points $z_1$, $z_2$, $z_3$, $z_4$ on $\gamma_c$,

$$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)} \geq \delta$$
for some \( \delta > 0 \). This, together with Lemma 9.8[23] (see also Lemma 4.27), implies that \( \gamma_c \) is actually a quasi-circle. The same cross ratio argument also implies that \( \gamma_c \) moves continuously as \( c \) varies on \( \Omega_\infty \) (Lemma 4.26). It follows that \( \gamma_c \) is the boundary of the Siegel disk of \( g_c \) which is centered at the origin. This proves Theorem C.

For reader’s convenience, in §5, we give a brief introduction of Thurston’s characterization theory on postcritically finite rational maps. The version we present here is slightly different from the one in [9]: the postcritical set \( P_f \) is replaced by a \( f \)-invariant set which contains \( P_f \) as its subset. We also present several results on short simple closed geodesics on hyperbolic Riemann surfaces, which will be used in several places in this paper. We numbered them by Theorem A.1, Theorem A.2, Theorem A.3, and Theorem A.4. The reader may refer to [9] for the details of the proofs.

This work is based on my Ph.D. thesis at CUNY [33]. I would like to express my gratitude to my advisor, Prof. Yunping Jiang for suggesting this problem, and also for his constant encouragement. Further thanks are due to Prof. Linda Keen and Prof. Frederick Gardiner for many useful conversations during the writing of the paper.

2. Realize a Siegel Disk with Prescribed Combinatorics

2.1. Constructing Symmetric Branched Covering Maps.

2.1.1. Notations. Let \( S^2 \) denote the topological two sphere. Let \( \Delta \) and \( \mathbb{T} \) denote the unit disk and unit circle, respectively. For a set \( P \subset S^2 \), let \( |P| \) denote the cardinality of the set \( P \). Let \( \mathbb{P}^1 \) denote the Riemann sphere with the standard complex structure. Given a point \( w \in \mathbb{P}^1 \), let \( w^* \) denote the symmetric image of \( w \) about the unit circle, i.e., \( w^* = 1/\bar{w} \). For a set \( W \subset \mathbb{P}^1 \), let \( W^* = \{ w^* \mid w \in W \} \). For \( x \in S^2 \), and \( \delta > 0 \), let \( B_\delta(x) \) denote the open disk with center at \( x \) and radius \( \delta \) with respect to the spherical metric. For \( x, y \in S^2 \), we use \( d_{S^2}(x, y) \) to denote the spherical distance between \( x \) and \( y \). For two maps \( f, g : S^2 \to S^2 \), the distance between \( f \) and \( g \) is defined to be \( d(f, g) = \sup_{x \in S^2} d_{S^2}(f(x), g(x)) \). For two subsets \( A, B \subset S^2 \), define \( d_{S^2}(A, B) = \inf_{x \in A, y \in B} d_{S^2}(x, y) \).

2.1.2. The Choice of the Infinity. Suppose \( f \in R_0^{top} \) has no Thurston obstructions outside the rotation disk \( \Delta \). Let \( d \geq 2 \) be the degree of \( f \). By a standard topological argument, it follows that \( f \) has at least one fixed point in the outside of the unit disk. There are two cases. In the first case, \( P_f \) contains a fixed point of \( f \). In this case, up to a combinatorial equivalence, we may assume that \( \infty \in P_f \) and \( f(\infty) = \infty \). In the second case, \( P_f \) does not contain any fixed point of \( f \). In this case, up to a combinatorial equivalence, we assume that the infinity is one of the fixed points of \( f \).

2.1.3. Construction of \( F \). Since \( \Omega_f \cap \partial \Delta \neq \emptyset \), we may also assume that \( 1 \in \Omega_f \). It follows that there is a curve segment, say \( \gamma_f \), which is attached to 1 from
the outside of the unit disk, such that $f(\gamma_f) \subset \partial \Delta$. Let

$$X = \{ z \in \Omega_f - \Delta \mid f^i(z) \in \Delta - \{0\} \text{ for some } i > 0 \}.$$ 

For each $z \in X$, let $i_z > 0$ be the smallest integer such that $f^{i_z}(z) \in \Delta$. Let $\tilde{X} = \{ f^{i_z}(z) \mid z \in X \}$ and $\sigma : S^2 \to S^2$ be a homeomorphism such that $\sigma|(S^2 - \Delta) = id$ and $\sigma(\tilde{X}) \subset \gamma_f^*$. Note that by our notation, $\gamma_f^*$ is the symmetric image of $\gamma_f$ about the unit circle. Let $\tilde{f} = \sigma \circ f$. Define a symmetric branched covering map of the sphere by

$$F(z) = \begin{cases} \tilde{f}(z) & \text{if } |z| \geq 1, \\ (\tilde{f}(z^*))^* & \text{for otherwise.} \end{cases}$$

From the construction of $F$, it follows that $P_F - \partial \Delta$ is a finite set, and moreover, for $z \in X$, $F^{i_z}(z) \in \gamma_f^*$, and hence $F^{i_z+1}(z) \in \partial \Delta$ (see Figure 1).

2.1.4. Construction of $F_n$. Let $\theta_n = p_n/q_n$ be a sequence of rational numbers such that $\theta_n \to \theta$ as $n$ goes to $\infty$. Let $O_n = \{ e^{2\pi ik\theta_n} \mid 0 \leq k < q_n \}$. Let $A(a,b)$ be the annulus with outer radius $a$ and inner radius $b$. Since $P_F - \partial \Delta$ is a finite set, there are $0 < r < 1 < R$ such that $(A(R,r) - \partial \Delta) \cap (\Omega_F \cup P_F) = \emptyset$. Set

$$Y = \{ z \in (\Omega_F \cup P_F) - \partial \Delta \mid F(z) \in \partial \Delta \},$$

and

$$Z = (\Omega_F \cap \partial \Delta) \cup F(Y).$$

Clearly, $Z$ is a finite set. It follows that for every $n$ large enough, there is a homeomorphism $\sigma_n : \partial \Delta \to \partial \Delta$ such that

1. $\sigma_n(1) = 1$,
2. $\sigma_n^{-1}(Z) \subset O_n$,
3. \( \sigma_n \) preserves the orbit relations among the points in the set \( Z \) in the following sense: If there is an \( m > 0 \) and \( x, y \in Z \) such that \( F^m(x) = y \) then \( e^{2\pi i m \theta_n} \sigma_n^{-1}(x) = \sigma_n^{-1}(y) \).

4. \( \sigma_n \to id \) uniformly as \( n \to \infty \).

We then extend \( \sigma_n \) to be a homeomorphism of the sphere to itself, which is still denoted by \( \sigma_n \), such that

1. \( \sigma_n = id \) outside \( A(R, r) \),
2. \( \sigma_n(z)^* = \sigma_n(z^*) \),
3. as \( n \to \infty \), \( \sigma_n \to id \) uniformly with respect to the spherical metric.

Now for every \( n \) large enough, let us define a homeomorphism \( h_n : \partial \Delta \to \partial \Delta \) by

\[
h_n(z) = e^{2\pi i \theta_n} \sigma_n^{-1}(e^{-2\pi i \theta_n} z).
\]

We then extend \( h_n \) to be a homeomorphism of the sphere to itself, which is still denoted by \( h_n \), such that

1. \( h_n = id \) outside \( A(R, r) \),
2. \( h_n(z)^* = h_n(z^*) \),
3. as \( n \to \infty \), \( h_n(z) \to id \) uniformly with respect to the spherical metric.

Let \( \tilde{F}_n = h_n \circ F \circ \sigma_n \). It follows that

1. \( (\tilde{F}_n|\partial \Delta)(z) = e^{2\pi i \theta_n} z \),
2. \( P_{\tilde{F}_n} - \partial \Delta = P_{\tilde{F}_n} - \partial \Delta \),
3. \( \Omega_{\tilde{F}_n} - \partial \Delta = \Omega_{\tilde{F}_n} - \partial \Delta \).

For each \( \xi \in Y \), take a small closed topological disk \( U_\xi \) containing \( \xi \) in its interior such that

1. all \( U_\xi, \xi \in Y \) are disjoint with each other, and \( U_\xi \cap \partial \Delta = \emptyset \),
2. \( \tilde{F}_n(U_\xi) \subset A(R, r) \),
3. \( \tilde{U}_\xi = U_\xi \),
4. \( \tilde{F}_n(U_\xi) \) is a closed topological disk and \( \tilde{F}_n(\partial U_\xi) = \partial \tilde{F}_n(U_\xi) \).

For each \( \xi \in Y \), let us define a homeomorphism \( g_{n, \xi} : \tilde{F}_n(U_\xi) \to \tilde{F}_n(U_\xi) \) such that

1. \( g_{n, \xi} = id \) on \( \partial \tilde{F}_n(U_\xi) \),
2. \( g_{n, \xi}(\tilde{F}_n(\xi)) = \sigma_n^{-1}(F(\xi)) \),
3. \( g_{n, \xi}(z)^* = g_{n, \xi}(z^*) \),
4. as \( n \to \infty \), \( g_{n, \xi} \to id \) uniformly with respect to the spherical metric.

Now let us define

\[
F_n(z) = \begin{cases} 
  g_{n, \xi} \circ \tilde{F}_n(z) & \text{for } z \in \bigcup_{\xi \in Y} U_\xi, \\
  \tilde{F}_n(z) & \text{for otherwise.}
\end{cases}
\]

Let \( Z_n = (\Omega_{F_n} \cap \partial \Delta) \cup F_n(Y) \). It follows from the construction that \( |Z_n| = |Z| \) for all \( n \) large enough. Moreover, for each \( x \in Z \), there is an \( x_n \in Z_n \), such that \( x_n \to x \) as \( n \to \infty \). It follows that for all \( n \) large enough, the map \( x \to x_n \) is a one-to-one correspondence between \( Z \) and \( Z_n \). By the
construction of $F_n$, the reader shall easily supply a proof of the following proposition:

**Proposition 2.1.** The sequence $\{F_n\}$ satisfy the following properties,

1. $F_n \to F$ uniformly with respect to the spherical metric,
2. $F_n$ is an orientation-preserving and postcritically finite branched covering map such that $F_n(z)^* = F_n(z)^*$,
3. $|P_{F_n} - \Delta| = |P_F - \Delta|$ for every $n$ large enough,
4. $(F_n|\partial \Delta)(z) = e^{2\pi i \theta_n}z$,
5. $P_{F_n} \cap \partial \Delta = O_n$.
6. For every $n$ large enough, $F_n$ preserves the orbit relations among the points in the set $Z$ in the following sense: if for $x, y \in Z$ and some integer $m \geq 0$, $F^m(z) = y$, then for the correspondent points $x_n$ and $y_n$, $F^m_{\infty}(x_n) = y_n$.
7. For every $n$ large enough, there is a curve segment $\gamma_n$ attached to $1$ from the outside of the unit disk such that $F_n(\gamma_n) \subset \partial \Delta$, and moreover, if for some $z \in (\Omega_{F_n} \cup P_{F_n}) - \Delta$, $F_n(z) \in \Delta - \{0\}$, then $F_n(z) \in \gamma_n$.

**Remark 2.1.** Note that the combinatorial structure of $f$ in the inside of the rotation disk is not reflected by $F$. We will use an additional argument to take care of this in §2.5.

2.2. No Thurston Obstructions of $F_n$ for Large $n$. Let $P'_{F_n}$ and $P_P$ denote the set $P_{F_n} \cup \{0, \infty\}$ and the set $P_F \cup \{0, \infty\}$, respectively. For a finite subset $P \subset S^2$ with $|P| \geq 4$, we say a simple closed curve $\gamma \subset S^2 - P$ is non-peripheral if each component of $S^2 - \gamma$ contains at least two points of $P$. Let $\phi : S^2 \to \mathbb{P}^1$ be a homeomorphism. For each non-peripheral curve $\gamma \subset S^2 - P$, there is a unique simple closed geodesic $\eta \subset \mathbb{P}^1 - \phi(P)$ in the homotopy class of $\phi(\gamma)$. We use $\|\gamma\|_{\phi, P}$ to denote the hyperbolic length of $\eta$. We say $\gamma$ is a $(\phi, P)$-geodesic if $\eta = \phi(\gamma)$.

2.2.1. Thurston's pull back. Now let $n \geq 1$ be fixed. Let $\phi_0 = Id$. For $m = 1, 2, \cdots$, let $\tau_m$ be the complex structures on $S^2$ which is obtained by pulling back the standard complex structure $\tau_0$ by $F^m_n$. Associated to each $\tau_m$ is a quasiconformal homeomorphism $\phi_m : S^2 \to \mathbb{P}^1$ which fixes 0, 1 and $\infty$. Let $G_m = \phi_m \circ F_n \circ \phi_m^{-1}$, then the following diagram

$$
\begin{array}{ccc}
(S^2, P'_{F_n}) & \stackrel{\phi_m^{-1}}{\longrightarrow} & (\mathbb{P}^1, \phi_{m+1}(P'_{F_n})) \\
F_n & \downarrow & G_m \\
(S^2, P'_{F_n}) & \stackrel{\phi_m}{\longrightarrow} & (\mathbb{P}^1, \phi_m(P'_{F_n}))
\end{array}
$$

commutes and $G_m$ is a rational map of the Riemann sphere $\mathbb{P}^1$.

Since $F_n(z^*) = F_n(z)^*$, by induction we have $\phi_m(z^*) = \phi_m(z)^*$ and hence $G_m(z^*) = G_m(z)^*$ for all $m = 0, 1, \cdots$. Therefore, $G_m$ is a Blaschke product.
on \( \mathbb{P}^1 \). By the assumption that \( f(\infty) = \infty \), it follows that \( F(\infty) = \infty \), and therefore, \( G_m(\infty) = \infty \). We write

\[
G_m(z) = \lambda_m \prod_{1 \leq k \leq d-1} \frac{z - p_{k,m}}{1 - \bar{p}_{k,m}z} \prod_{1 \leq k \leq d-1} \frac{z - q_{k,m}}{1 - \bar{q}_{k,m}z}
\]

where \( d \geq 2 \) is the degree of \( f \), and \( p_{k,m}, q_{k,m} \in \mathbb{C} - \Delta \), \( 1 \leq k \leq d-1 \), and \( \lambda_m = e^{2\pi i \alpha_m} \) for some real constant \( 0 \leq \alpha_m < 1 \).

### 2.2.2. Analysis of short simple closed geodesics.

Let \( \gamma \) be a short simple closed \( (\phi_m, P_{F_n}^') \) - geodesic. If \( \gamma \) intersects the unit circle, we use \( D(\gamma) \) to denote the component of \( S^2 - \gamma \) which does not contain the origin. Otherwise, we use \( D(\gamma) \) to denote the component of \( S^2 - \gamma \) which does not contain the unit circle.

**Lemma 2.1.** Let \( \gamma \) be a simple closed \( (\phi_m, P_{F_n}^') \) - geodesic which intersects the unit circle such that \( \|\gamma\|_{D(\gamma), P_{F_n}'} < \log(\sqrt{2} + 1) \). Then \( \gamma \) is symmetric about the unit circle. In particular, \( \gamma \cap \partial \Delta \) contains exactly two points.

*Proof.* Let \( \gamma^* \) be the symmetric image of \( \gamma \) about the unit circle. Clearly, \( \gamma^* \) is also a simple closed \( (\phi_m, P_{F_n}^') \) - geodesic and \( \|\gamma^*\|_{\phi_m, P_{F_n}'} = \|\gamma\|_{\phi_m, P_{F_n}'} < \log(\sqrt{2} + 1) \). Since \( \gamma \cap \gamma^* \neq \emptyset \), by Theorem A.1, we get that \( \gamma = \gamma^* \).

*□*

**Lemma 2.2.** For every \( n \) large enough, there is a \( \delta > 0 \) independent of \( m \) such that for every simple closed \( (\phi_m, P_{F_n}^') \) - geodesic \( \gamma \) which intersects the unit circle, we have \( \|\gamma\|_{\phi_m, P_{F_n}'} \geq \delta \).

The idea behind the proof is as follows. Let \( \gamma \) be a simple closed geodesic which intersects the unit circle. If \( \gamma \) is short enough, its images under the forward iterations of \( F_n \) generate a set of short simple closed geodesics which intersect the unit circle. The number of the short simple closed geodesics in this set can be very large if \( \gamma \) is short enough. But on the other hand, there cannot be too many such short simple closed geodesics, for otherwise, there would be two of them which intersect with each other, and this is a contradiction with Theorem A.1.

*Proof.* We prove it by contradiction. We claim that for every \( n \) large enough, there exist \( \delta' > 0 \) and \( 1 < C < \infty \) independent of \( m \), such that if \( \gamma \subset S^2 - P_{F_n}' \) is a simple closed \( (\phi_m, P_{F_n}^') \) - geodesic with \( \|\gamma\|_{\phi_m, P_{F_n}'} < \delta' \), there is a simple closed \( (\phi_m, P_{F_n}^') \) - geodesic \( \xi \) which is symmetric about the unit circle such that \( \|\xi\|_{\phi_m, P_{F_n}'} < C \delta' \) and \( D(\xi) \cap \partial \Delta \cap P_{F_n}' \) contains at least two points. Let us prove the claim. Suppose it is not true. Then \( D(\gamma) \cap \partial \Delta \cap P_{F_n}' \) contains at most one point. Now take \( \delta' > 0 \) small, so that the simple closed geodesics generated in the following are all short enough. Let \( N = |P_{F_n}' - \Delta| \), and hence \( |P_{F_n}' - \partial \Delta| = N \) by (3) of Proposition 2.1. For each \( k = 1, 2, \cdots, N + 2 \), Let
Let $\eta_k \subset S^2 - F_{n}^{-k}(P^r_{F_n})$ be the shortest simple closed $(\phi_m, F_{n}^{-k}(P^r_{F_n}))$ -- geodesic which is homotopic to $\gamma$ in $S^2 - P^r_{F_n}$. By Theorem A.3, we have
\[ \|\eta_k\|_{\phi_m,F_n^{-k}(P^r_{F_n})} < C_1 \|\gamma\|_{\phi_m,P^r_{F_n}} \] where $1 < C_1 < \infty$ depends only on $k$ and $|P^r_{F_n}|$. From Theorem A.2, we conclude that $F^r_k(\eta_k)$ covers a simple closed $(\phi_{m-k}, P^r_{F_n})$ -- geodesic $\xi'_k$. Hence
\[ \|\xi'_k\|_{\phi_{m-k},P^r_{F_n}} \leq \|\eta_k\|_{\phi_m,F_n^{-k}(P^r_{F_n})} \] Let $\xi_k \subset S^2 - P^r_{F_n}$ be the simple closed $(\phi_m, P^r_{F_n})$ -- geodesic which is homotopic to $\xi'_k$ in $S^2 - P^r_{F_n}$. By Theorem A.4 and the fact that Thurston’s pull back does not increase the Teichmüller distance (see Proposition 3.3, [9]), it follows that there is a constant $1 < C_2 < \infty$ independent of $m$, such that
\[ \|\xi_k\|_{\phi_m,P^r_{F_n}} < C_2 \|\xi'_k\|_{\phi_{m-k},P^r_{F_n}} \] Now by taking $\delta'$ small, we conclude that $\xi_1, \cdots, \xi_{N+2}$ are all short simple closed $(\phi_m, P^r_{F_n})$ -- geodesics which intersect the unit circle. By Lemma 2.1 they are all symmetric about the unit circle. Now let us show that the domains $D(\xi_1), \cdots, D(\xi_{N+2})$ are disjoint with each other. Suppose this is not true. Then by Theorem A.1, we have $D(\xi_i) \subset D(\xi_j)$ for some $1 \leq i \neq j \leq N+2$. We may assume that $|D(\xi_i) \cap \partial \Delta \cap P^r_{F_n}| \leq 1$, for otherwise the claim is proved. It then follows that $\xi_i$ intersects either exactly one of the connected components of $\partial \Delta - P^r_{F_n}$ or two of them which are adjacent to each other. Let $I$ be a component of $\partial \Delta - P^r_{F_n}$ which intersects both $\xi_i$ and $\xi_j$. Let $l = |j - i| \leq N + 1$. Then $I$ is either periodic under $F_n$ or is mapped by $F_n^l$ to one of its adjacent component of $\partial \Delta - P^r_{F_n}$. Since $(F_n|\partial \Delta)(z) = e^{2\pi i \theta_n}z$ and $\theta_n \to \theta$ as $n \to \infty$, both of the two cases are impossible when $n$ is large enough.

If none of $D(\xi_i), 1 \leq i \leq N + 2$ contains at least two points in $P^r_{F_n} - \partial \Delta$, we have for every $1 \leq i \leq N + 2$, $|D(\xi_i) \cap (P^r_{F_n} - \partial \Delta)| \geq 2$ and hence $|P^r_{F_n} - \partial \Delta| \geq 2N + 2$. This is a contradiction with that $|P^r_{F_n} - \Delta| = N$. This proves the claim.

Now we may assume that $D(\gamma) \cap \partial \Delta \cap P^r_{F_n}$ contains at least two points. There are two cases. In the first case, $(\partial \Delta \cap P^r_{F_n}) - D(\gamma) = \emptyset$. It follows that $\gamma$ intersects exactly one of the connected components of $\partial \Delta - P^r_{F_n}$. When $n$ is large enough, by the same argument as before, we get $N+2$ short simple closed $(\phi_m, P^r_{F_n})$ -- geodesics $\xi_1, \cdots, \xi_{N+2}$. It follows from (5) of Proposition 2.4 that every $\xi_i$ also intersects exactly one of the connected components of $\partial \Delta - P^r_{F_n}$, for $1 \leq i \leq N + 2$. It follows that each $D(\xi_i)$ either contains all the points in $\partial \Delta \cap P^r_{F_n}$, or contains none of them. We claim that there are $\xi_i, \xi_j$ such that $D(\xi_i) \subset D(\xi_j)$ for some $1 \leq i \neq j \leq N + 2$. In fact, if this is not true, by Theorem A.1, the domains $D(\xi_1), \cdots, D(\xi_{N+2})$ are disjoint with each other. It follows that there are at least $N + 1$ domains of $D(\xi_i), 1 \leq i \leq N + 2$ which contain none of the points in $\partial \Delta \cap P^r_{F_n}$. Therefore, each of these domains must contain at least two points in $P^r_{F_n} - \partial \Delta$, and this implies that
\[ |P^*_{F_n} - \partial \Delta| \geq 2(N + 1), \] which is a contradiction with that \(|P^*_{F_n} - \Delta| = N\). The claim follows. Now assume that \(D(\xi_i) \subset D(\xi_j)\) for some \(1 \leq i \neq j \leq N + 2\). We claim that each component of \(\partial \Delta - P^*_{F_n}\) intersects at most one of the curves in \(\xi_1, \ldots, \xi_{N+2}\). In fact, if some component, say \(I\), of \(\partial \Delta - P^*_{F_n}\) interests both \(\xi_i\) and \(\xi_m\) for some \(1 \leq i < m \leq N + 2\), then \(I\) is periodic under \(F^m_{n+i}\), which is impossible when \(n\) is large enough. It follows that \(D(\xi_j)\) must contain all the other \(D(\xi_k), 1 \leq k \leq N + 2, k \neq j, \) and hence the \(N + 1\) domains \(D(\xi_k), 1 \leq k \leq N + 2, k \neq j, \) must be disjoint with each other, and moreover, each of them contains none of the points in \(\partial \Delta \cap P^*_{F_n}\). By counting the number of the points in \(P^*_{F_n} - \partial \Delta\), we get a contradiction again.

In the second case, \((\partial \Delta \cap P^*_{F_n}) - D(\gamma) \neq \emptyset\). Let \(I = \partial \Delta \cap D(\gamma)\). Since \(O_n = P^*_{F_n} \cap \partial \Delta\) is a periodic cycle of \(F_n\) with period \(q_n\), it follows that there is an integer \(0 < k < q_n\) such that (1) \(F^k_n(I) \cap I \cap P^*_{F_n} \neq \emptyset\), (2) \((I - F^k_n(I)) \cap P^*_{F_n} \neq \emptyset\) and (3) \(F^k_n(I) - I \cap P^*_{F_n} \neq \emptyset\). Let \(\eta_k \subset S^2 - F^k_n(P^*_{F_n})\) be a simple closed \((\phi_m, F^k_n(P^*_{F_n})) - \text{geodesic}\) which is homotopic to \(\gamma\) in \(S^2 - P^*_{F_n}\). By Theorem A.2, \(F^k_n(\eta_k)\) covers a short simple closed \((\phi_{m-k}, P^*_{F_n}) - \text{geodesic}\) \(\xi^*_k\). By Theorem A.4, there is a short simple closed \((\phi_m, P^*_{F_n}) - \text{geodesic}\) \(\xi_k \subset S^2 - P^*_{F_n}\) which is homotopic to \(\xi^*_k\) in \(S^2 - P^*_{F_n}\). It follows that \(D(\gamma) \cap D(\xi_k) \neq \emptyset\) and neither of them is contained in the other one. This implies that \(\gamma \cap \xi_k \neq \emptyset\). This is a contradiction with Theorem A.1.

\[ \square \]

**Lemma 2.3.** Let \(\gamma\) be a simple closed \((\phi_m, P^*_{F_n}) - \text{geodesic}\) which is contained in the inside of the unit disk. If \(\|\gamma\|_{\phi_m, P^*_{F_n}}\) is small enough, then each non-peripheral component of \(F^{-1}_n(\gamma)\) is totally contained in the inside of the unit disk also.

**Proof.** Suppose \(\|\gamma\|_{\phi_m, P^*_{F_n}}\) is small enough. Let \(\eta\) be a non-peripheral component of \(F^{-1}_n(\gamma)\). Clearly, \(\eta\) is a short simple closed \((\phi_{m+1}, F^{-1}_n(P^*_{F_n})) - \text{geodesic}\). Since \(F_n: \partial \Delta \to \partial \Delta\) is a homeomorphism, it follows that \(\eta\) does not intersect the unit circle. Suppose \(\eta\) is in the outside of the unit disk. Take a point, say \(x \in D(\gamma)\). Let us contract \(\gamma\) continuously to \(x\). There are two cases. In the first case, \(D(\gamma)\) does not contain any critical value \(v = F_n(c)\) for some \(c \in D(\gamma) \cap \Omega_{F_n}\). Then we can lift the contraction by \(F_n\). It follows that \(D(\eta)\) will contract to some point in \(D(\eta)\). Let \(z \in D(\eta) \cap P^*_{F_n}\). In this case, \(F_n(z) \in D(\gamma)\). By (7) of proposition 2.1, it follows that \(F_n(z) \in \gamma_n\), where \(\gamma_n\) is a curve segment which is attached to the point 1 and lies in the outside of the unit disk such that \(F_n(\gamma_n) \subset \partial \Delta\). Now let \(\xi\) be one of the shortest simple closed \((\phi_m, F^{-1}_n(P^*_{F_n})) - \text{geodesics}\) which are homotopic to \(\gamma\) in \(S^2 - P^*_{F_n}\). It follows that \(\xi \cap (\partial \Delta \cup \gamma_n^* \neq \emptyset\). This implies that \(F_n(\xi) \cap \partial \Delta \neq \emptyset\). But on the other hand, \(\|\xi\|_{\phi_m, F^{-1}_n(P^*_{F_n})}\) goes to 0 as \(\|\gamma\|_{\phi_m, P^*_{F_n}}\) goes to 0 by Theorem A.3, and so does \(\|F_n(\xi)\|_{\phi_m - 1, P^*_{F_n}}\). This is a contradiction with Lemma 2.2.

In the second case, the contraction of \(D(\gamma)\) can not be lifted to a contraction of \(D(\gamma)\). This implies that there is a point \(w \in \Omega_{F_n} \cap D(\eta) \subset \Omega_{F_n} - \Delta\) such
Lemma 2.4. $F_n$ has no Thurston obstructions in $S^2 - P_{F_n}$ for every $n$ large enough.

Proof. First let us prove that $F_n$ has no Thurston obstructions in $S^2 - P_{F_n}$. Suppose $\Gamma$ is a $F_n$-stable family which consists of all the short simple closed geodesics. By Lemma 2.2, if $\gamma$ is disjoint from the unit circle, then $\gamma$ must also belong to $\Gamma$. We order the curves in $\Gamma$ as $\gamma_1, \gamma_2, \cdots, \gamma_l$, where $\gamma_i \subset \Delta$, and $\gamma_i^*$ is the symmetric image of $\gamma_i$ about the unit circle, $1 \leq i \leq l$. Now let $A$ be the associated Thurston linear transformation matrix of $\Gamma$ (see [9] or §5 for the definition). By Lemma 2.2, any non-peripheral component of $F_{n-1}^{-1}(\gamma_i)$ must be homotopic to one of the curves in $\gamma_i, 1 \leq i \leq l$, and by the same reason, any non-peripheral component of $F_{n-1}^{-1}(\gamma_i^*)$ must be homotopic to one of the curves in $\gamma_i^*, 1 \leq i \leq l$. It follows that $\{\gamma_1, \cdots, \gamma_l\}$ is a $f$-stable family. Let $B$ be its associated Thurston linear transformation matrix. Then we have

$$A = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$$

Since $f$ has no Thurston obstructions outside the unit disk, so $\|B\| < 1$. Therefore, $\|A\| < 1$.

Now let us prove that $F_n$ has no Thurston obstructions in $S^2 - P_{F_n}$. By the choice of the infinity, let us assume that $P_{F_n}^l \neq P_{F_n}$, for otherwise, the lemma has been proved. Let us suppose that $F_n$ has Thurston obstructions in $S^2 - P_{F_n}$. Since $F_n$ has no Thurston obstructions in $S^2 - P_{F_n}$, it follows that any short simple closed ($\varphi_m, P_{F_n}^l$-geodesics) $\gamma \subset S^2 - P_{F_n}$ must be homotopic to a point in $S^2 - P_{F_n}$. This implies that there are exactly two short simple closed geodesics in $S^2 - P_{F_n}$, say $\gamma$ and $\gamma^*$, such that $\gamma$ is contained in the outside of the unit disk, and $D(\gamma)$ contains exactly two distinct points in $P_{F_n}$, one is the infinity, and the other one, say $x$, is a point in $P_{F_n}$. By the same argument as before, we can show that any non-peripheral component of $F_{n-1}^{-1}(\gamma)$ is contained in the outside of the unit disk, and hence homotopic to $\gamma$ in $S^2 - P_{F_n}$. Similarly, any non-peripheral component of $F_{n-1}^{-1}(\gamma^*)$ is contained in the inside of the unit disk, and hence homotopic to $\gamma^*$ in $S^2 - P_{F_n}$. It follows that the associated Thurston linear transformation matrix is a $2 \times 2$ diagonal matrix, and hence equal to the identity matrix. This implies that there is a simple closed curve $\gamma'$ which is homotopic to $\gamma$ in $S^2 - P_{F_n}$ and $F_n : \gamma' \rightarrow \gamma$ is a homeomorphism. Now continuously contract $\gamma$ to $x$. Since $D(\gamma) - \{x\}$ contains no critical values of $F_n$, it follows that the contraction can be lifted to a contraction of $\gamma'$, which then must contract to $x$ too. This implies that $F_n(x) = x$. But this is a contradiction with our choice of the infinity. The proof of the lemma is completed. □
2.3. The Compactness of \( \{ G_n \} \) and Bounded Geometry of \( P_{G_n} \).

2.3.1. The sequence of Blaschke products \( \{ G_n \} \). For \( n \) large enough, by Lemma 2.4, \( F_n \) has no Thurston obstructions in \( S^2 - P_{F_n}' \). By Thurston’s characterization theorem on postcritically finite rational maps, it follows that there is a Blaschke product \( G_n \) which is combinatorially equivalent to \( F_n \) rel \( P_{F_n}' \) (see [2], or §5). That is to say, there is a pair of homeomorphisms \( \phi_n, \psi_n \) of the sphere which are isotopic to each other rel \( P_{F_n}' \), such that \( G_n = \phi_n \circ F_n \circ \psi_n^{-1} \).

In this section, we will show that the sequence \( \{ G_n \} \) is contained in a compact set of the space of all the rational maps of degree \( 2d - 1 \), and moreover, the geometry of \( P_{G_n} \) is uniformly bounded.

2.3.2. Analysis of short simple closed geodesics in \( P^1 - (X^g_n \cup P_n) \). We would like to mention that all the proofs in this subsection does not rely on the condition that \( \theta \) is of bounded type. The only arithmetic condition of \( \theta \) we used is that it is an irrational number.

Let \( L \geq 1 \) be an integer. Define
\[
X^g_L = \{ G_n^k(x) \mid x \in \Omega_{G_n}, -L \leq k \leq L \} \cap \partial \Delta
\]
and
\[
P_n = (P_{G_n} - \partial \Delta) \cup \{0, \infty\}.
\]

Let \( I \subset \partial \Delta \) be an arc segment (it may be open, closed, or half open and half closed). Define
\[
\sigma_n(I) = \frac{|I \cap P_{G_n}|}{|\partial \Delta \cap P_{G_n}|}.
\]

Since \( P_{G_n} \cap \partial \Delta \) consists of a periodic orbit and since \( G_n|_\partial \Delta : \partial \Delta \to \partial \Delta \) is a homeomorphism, it follows that \( \sigma_n \) is \( G_n \)–invariant, i.e., for any \( I \subset \partial \Delta \),
\[
\sigma_n(I) = \sigma_n(G_n(I)).
\]

Let \( x, y \in \partial \Delta \) be two distinct points. They separate \( \partial \Delta \) into two arc segments \( I \) and \( J \). Let \( \overline{I} \) and \( \overline{J} \) denote the closure of \( I \) and \( J \), respectively. Define
\[
d_{\sigma_n}(x, y) = \min\{\sigma_n(I), \sigma_n(J)\}.
\]

It is clear that
\[
d_{\sigma_n}(x, z) \leq d_{\sigma_n}(x, y) + d_{\sigma_n}(y, z).
\]

**Lemma 2.5.** For any \( k \geq 1 \), there is an \( \epsilon > 0 \) such that for any \( x \in \partial \Delta \), the following inequality holds for all \( n \) large enough
\[
d_{\sigma_n}(x, G_n^k(x)) \geq \epsilon.
\]

**Proof.** Assume that \( n \) is large enough. Then \( x \) and \( G_n^k(x) \) separate \( \partial \Delta \) into two arc intervals \( I \) and \( J \). Since \( \theta_n \) converges to \( \theta \), there is an \( m \geq 1 \) dependent only on \( \theta \) and \( k \) such that for all \( n \) large enough,
\[
\partial \Delta \subset \bigcup_{i=0}^{m} G_n^{ik}(I) \text{ and } \partial \Delta \subset \bigcup_{i=0}^{m} G_n^{ik}(J).
\]
Since $\sigma_n$ is $G_n$-invariant, it follows that
$$\min\{\sigma_n(I), \sigma_n(J)\} \geq 1/(m+1).$$
This implies Lemma 2.5. □

As before, let $N = |P'_n - \Delta|$. It follows that for every $n$ large enough,
\begin{equation}
|P_n| = 2N.
\end{equation}

**Lemma 2.6.** Let $L \geq N + 2$ and $M \geq 1$ be some integers. Then for any $1 \leq k \leq M$, and every $n$ large enough, $X^p_{L} \cup P_n \subset G^{-k}_{L+M}(X^p_{L+M} \cup P_n)$. Moreover, the map
$$G^k : \mathbb{P}^1 - G^{-k}_{L+M}(X^p_{L+M} \cup P_n) \to \mathbb{P}^1 - (X^p_{L+M} \cup P_n)$$
is a holomorphic covering map.

**Proof.** Let $z \in X^p_{L} \cup P_n$ and $1 \leq k \leq M$. We have two cases. In the first case, $z \in X^p_{L}$. It follows from (6) that $G^k_{L+M}(z) \in X^p_{L+M}$. In the second case, $z \in P_n$. Then from (7) of Proposition 2.1, there is some critical point $c \in \Omega_{G_n}$ and some integer $0 \leq i \leq N + 1$ such that
$$z = G^i_n(c).$$
Since $L \geq N + 2$, it follows from (6) that $G^k_{L+M}(z) \in X^p_{L+M}$. This proves the first assertion.

The second assertion follows since $L \geq N + 2$, and therefore, for any $c \in \Omega_{G_n}$, the forward orbit segment
$$\{G^i_n(c) \mid 1 \leq i \leq M\}$$
is contained in $X^p_{L+M} \cup P_n$. □

For $L > 0$ and $n$ large enough, set
$$r^n_L = \max\{\sigma_n(I) \mid I \text{ is an interval component of } \partial \Delta - X^p_{L}\}.$$

**Lemma 2.7.** Let $\epsilon > 0$ be an arbitrary number. Then there exist $L'$ and $N'$ such that $r^n_L < \epsilon$ provided that $L > L'$ and $n > N'$.

**Proof.** Let us consider the combinatorial model $F_n$ instead of $G_n$. That is, replace $G_n$ by $F_n$ in the definitions of $X^p_{L}$, $\sigma_n$, and $r^n_L$. Let us still keep the same notations.

For $\epsilon > 0$ given, let $K$ be the least integer such that $K \geq 1/\epsilon$. Since $0 < \theta < 1$ is irrational, there is a $0 < \delta < 1$ which depends only on $\theta$ such that for any closed arc segment $I \subset \partial \Delta$ with $|I| < \delta$, the $K + 1$ arc segments $e^{2\pi ik\theta}I, 0 \leq k \leq K$ are disjoint. For such $\delta$, there is an integer $L'$ which depends only on $\delta$ and $\theta$ such that for all $L > L'$, every component of
$$\Xi = \partial \Delta - \{e^{2\pi ik\theta} \mid -L \leq k \leq L\}$$
has Euclidean length less than $\delta/2$. It follows that for every component $I$ of $\Xi$, the closure of the arc segments $e^{2\pi ik\theta}I, 0 \leq k \leq K$, are disjoint.
\(\theta_n \rightarrow \theta\) as \(n \rightarrow \infty\), it follows that there is an \(N' > 0\) such that for all \(n > N'\), and any component \(I\) of

\[
\Xi_n = \partial \Delta - \{e^{2\pi i k \theta_n} \mid -L \leq k \leq L\},
\]

the closure of the \(K + 1\) arc segments \(e^{2\pi i k \theta_n} I, 0 \leq k \leq K\), are disjoint. Since \(\sigma_n\) is \(F_n\)–invariant, we have

\[
\sigma_n(T) = \sigma_n(e^{2\pi i \theta_n} T) = \cdots = \sigma_n(e^{2\pi i K \theta_n} T).
\]

From the disjointness, we have

\[
\sigma_n(T) + \sigma_n(e^{2\pi i \theta_n} T) + \cdots + \sigma_n(e^{2\pi i K \theta_n} T) \leq 1.
\]

It follows that \(\sigma_n(T) < 1/K \leq \epsilon\). The lemma then follows since \(I\) is an arbitrary component of \(\Xi_n\) and \(\{e^{2\pi i k \theta_n} \mid -L \leq k \leq L\}\) is contained in \(X^n_\Delta\).

\[\square\]

Recall that \(N = |P' - \Delta|\). As a consequence of Lemma 2.8 and 2.9 we have

**Corollary 2.1.** There exist integers \(L_0\) and \(N_0\) such that when \(L > L_0\) and \(n > N_0\), the inequality

\[d_{\sigma_n}(x, G_n(x)) > 3r_L^n\]

holds for any \(x \in \partial \Delta\) and every \(1 \leq k \leq N + 1\).

For a simple closed geodesic \(\xi\) which intersects the unit circle, we use \(D(\xi)\) to denote the bounded component of \(S^2 - \gamma\).

**Lemma 2.8.** Let \(L_0\) be the number in Corollary 2.1. Let \(L > \max\{N + 2, L_0\}\). Then there is a \(\delta > 0\) and \(1 < C < \infty\) such that for every \(n\) large enough and any simple closed geodesic \(\gamma \subset \mathbb{P}^1 - (X^n_\Delta \cup P_n)\) with \(\|\gamma\|_{\mathbb{P}^1 - (X^n_\Delta \cup P_n)} \leq \delta\) and \(\gamma \cap \partial \Delta \neq \emptyset\), one of the following two cases must be true:

1. \(|(\partial \Delta - D(\gamma)) \cap X^n_\Delta| \geq 2\) and \(|D(\gamma) \cap X^n_\Delta| \geq 2\),
2. there is a short simple closed geodesic \(\eta \subset \mathbb{P}^1 - (X^n_{\Delta + N + 2} \cup P_n)\) with \(\|\eta\|_{\mathbb{P}^1 - (X^n_{\Delta + N + 2} \cup P_n)} \leq C\|\gamma\|_{\mathbb{P}^1 - (X^n_\Delta \cup P_n)}\)

such that \(|(\partial \Delta - D(\eta)) \cap X^n_{\Delta + N + 2}| \geq 2\) and \(|D(\eta) \cap X^n_{\Delta + N + 2}| \geq 2\).

**Proof.** Let \(\gamma \subset \mathbb{P}^1 - (X^n_\Delta \cup P_n)\) be a short simple closed geodesic with \(\gamma \cap \partial \Delta \neq \emptyset\) and \(\|\gamma\|_{\mathbb{P}^1 - (X^n_\Delta \cup P_n)} \leq \delta\) for some \(\delta > 0\). Assume that the first case does not hold, that is, either \(|(\partial \Delta - D(\gamma)) \cap X^n_\Delta| < 2\) or \(|D(\gamma) \cap X^n_\Delta| < 2\). Let us prove the second case must hold. Let \(\gamma \cap \partial \Delta = \{x, y\}\). It follows that

\[
d_{\sigma_n}(x, y) \leq 2r_L^n.
\]

See Figure 2 for an illustration (Here \(|D(\gamma) \cap X^n_\Delta| = 1\)).
Let us assume that \( \delta \) is so small that the simple closed geodesics generated in the following are all short enough. In Lemma 2.6, taking \( M = N + 2 \), then for any \( 1 \leq k \leq N + 2 \), we have

\[
P^1 - G^{-k}_n(X_{L+N+2}^n \cup P_n) \subset P^1 - (X_L^n \cup P_n).
\]

Since \( |G^{-k}_n(X_{L+N+2}^n \cup P_n) - (X_L^n \cup P_n)| \) depends only on \( L, N, \) and \( d \), by Theorem A.3, there is a constant \( C \) dependent only on \( L, N, \) and \( d \), such that for each \( 1 \leq k \leq N + 2 \), there is a simple closed geodesic \( \xi_k' \subset P^1 - G^{-k}_n(X_{L+N+2}^n \cup P_n) \) which is homotopic to \( \gamma \) in \( P^1 - (X_L^n \cup P_n) \) such that

\[
\|\xi_k\|_{P^1 - G^{-k}_n(X_{L+N+2}^n \cup P_n)} \leq C\|\gamma\|_{P^1 - (X_L^n \cup P_n)}.
\]

When \( \delta \) is small, by Lemma 2.6 and Theorem A.2, \( G^n_k(\xi_k') \) covers a simple closed geodesic \( \xi_k \) in \( P^1 - (X_L^n \cup P_n) \), and hence

\[
\|\xi_k\|_{P^1 - (X_{L+N+2}^n \cup P_n)} \leq \|\xi_k'\|_{P^1 - G^{-k}_n(X_{L+N+2}^n \cup P_n)}.
\]

Now it suffices to prove that there is some \( \xi_k, 1 \leq k \leq N + 2 \), such that

\[
|(\partial \Delta - D(\xi_k)) \cap X_{L+N+2}^n| \geq 2
\]

and

\[
|D(\xi_k) \cap X_{L+N+2}^n| \geq 2.
\]

Assume at least one of the above two inequalities were not true. We will get a contradiction as follows.

We first claim that for every \( n \) large enough, each component of \( \partial \Delta - X_{L+N+2}^n \) intersects at most one of \( \xi_k, 1 \leq k \leq N + 2 \), and in particular,

\[
(11) \quad \xi_i \neq \xi_j
\]

for \( 1 \leq i \neq j \leq N + 2 \). Suppose this is not true. Then there exist \( 1 \leq i < j \leq N + 2 \) and a component of \( \partial \Delta - X_{L+N+2}^n \), say \( I \), such that \( \xi_i \cap I \neq \emptyset \) and \( \xi_j \cap I \neq \emptyset \).
Recall that $\xi_i'$ and $\xi_j'$ cover $\xi_i$ and $\xi_j$, respectively. Let $x' \in \xi_i' \cap \partial \Delta$ and $y' \in \xi_j' \cap \partial \Delta$ such that $G_n^i(x') \in \xi_i \cap I$ and $G_n^j(y') \in \xi_j \cap I$. Since both $\xi_i'$ and $\xi_j'$ are homotopic to $\gamma$ in $\mathbb{P}^1 - (X^+_n \cup P_n)$, it follows from (10) that

$$d_{\sigma_n}(x', y') \leq 2r_n^L.$$ 

See Figure 2 for an illustration (Since $x'$ and $y'$ belong to the arc interval $(a, c)$ whose $\sigma_n$-length is not more than $2r_n^L$).

Since $\sigma_n$ is $G_n$-invariant, we have

$$(12) \quad d_{\sigma_n}(G_n^i(x'), G_n^i(y')) \leq 2r_n^L.$$ 

On the other hand, since $G_n^j(y')$, $G_n^i(x') \in I$, we get

$$(13) \quad d_{\sigma_n}(G_n^j(y'), G_n^i(x')) = d_{\sigma_n}(G_n^{j-i}(G_n^i(y')), G_n^i(x')) \leq \sigma_n(I) \leq r_n^L.$$ 

It follows from (12), (12), and (13), that

$$d_{\sigma_n}(G_n^i(y'), G_n^i(y')) \leq 3r_n^L.$$ 

This is a contradiction with the definition of $L_0$ in Corollary 2.1 and the choice of $L$. The claim follows.

Now there are two cases. In the first case, all the domains $D(\xi_i), 1 \leq i \leq N + 2$ are disjoint. Since each component of $\partial \Delta - X^+_{n + N + 2}$ intersects at most one of $\xi_i, 1 \leq i \leq N + 2$, it follows from the claim that for every $1 \leq i \leq N + 2$,

$$(14) \quad |(\partial \Delta - D(\xi_i)) \cap X^+_{n + N + 2}| \geq 2.$$ 

This is because otherwise, there would be two domains $D(\xi_i)$ and $D(\xi_j)$ with $1 \leq i \neq j \leq N + 2$ such that one is contained in the other one, and this is impossible since we have assumed that all the domains $D(\xi_i), 1 \leq i \leq N + 2$, are disjoint in this case. From (14), it follows that

$$|D(\xi_i) \cap X^+_{n + N + 2}| \leq 1.$$
for every $1 \leq i \leq N+2$. For otherwise, the lemma has been proved. Since $\xi_i$ is non-peripheral, it follows that $D(\xi_i) \cap P_n$ is non-empty, and by the symmetric property of $\xi_i$ and $P_n$, we have

$$|D(\xi_i) \cap P_n| \geq 2.$$  

We thus get

$$|P_n| \geq \sum_{1 \leq i \leq N+2} |D(\xi_i) \cap P_n| \geq 2(N + 2).$$

This is a contradiction with (9).

In the second case, there are two domains $D(\xi_i)$ and $D(\xi_j)$ such that $D(\xi_i) \subset D(\xi_j)$ where $1 \leq i \neq j \leq N+2$. By the claim which we proved previously, it follows that none of the components of $\partial \Delta - X^n_{L+N+2}$ intersect both $\xi_i$ and $\xi_j$. This implies that

$$|D(\xi_j) \cap X^n_{L+N+2}| \geq 2.$$  

See Figure 3 for an illustration. We thus get by assumption that

$$(15) \quad |(\partial \Delta - D(\xi_j)) \cap X^n_{L+N+2}| \leq 1.$$  

It follows that all the domains $D(\xi_k), k \neq j$, are contained in $D(\xi_j)$. This is because if some $D(\xi_k), k \neq j$, is not contained in $D(\xi_j)$, from (15), it follows that one component of $\partial \Delta - X^n_{L+N+2}$ would intersect both $\xi_j$ and $\xi_k$, but this again contradicts with the claim we previously proved. See Figure 4 for an illustration. In this figure, we assume that $(\partial \Delta - D(\xi_j)) \cap X^n_{L+N+2}$ contains a single point $a$.

Now we claim that all the domains $D(\xi_k), k \neq j$, are disjoint with each other. In fact, if $D(\xi_{i'}) \subset D(\xi_{j'})$ for some $i'$ and $j'$ such that $i' \neq j', i' \neq
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if \( j, j' \neq j \), then by the same argument as above, we get that \( D(\xi_{j'}) \) contains all the other domains \( D(\xi_k), 1 \leq k \leq N + 2, k \neq j' \). In particular,

\[
D(\xi_j) \subset D(\xi_{j'})
\]

and hence \( \xi_j = \xi_{j'} \). This is a contradiction with (11). The claim follows.

Since for any \( 1 \leq k \leq N + 2, k \neq j \), \( D(\xi_k) \subset D(\xi_j) \) and each component of \( \partial \Delta - X_{L+N+2}^n \) can not intersect both \( \xi_j \) and \( \xi_k \), it follows that for every \( 1 \leq k \leq N + 2 \) and \( k \neq j \),

\[
|\partial \Delta - D(\xi_k)) \cap X_{L+N+2}^n| \geq 2.
\]

See Figure 4 for an illustration.

By assumption, we have

\[
|D(\xi_k) \cap X_{L+N+2}^n| \leq 1.
\]

As before, it follows that

\[
|D(\xi_k) \cap P_n| \geq 2
\]

for every \( 1 \leq k \leq N + 2, k \neq j \). This implies that

\[
|P_n| \geq \sum_{1 \leq k \leq N+2, k \neq j} |D(\xi_k) \cap P_n| \geq 2(N + 1).
\]

This is a contradiction with (11). The proof of Lemma 2.8 is completed.

\[\square\]

**Lemma 2.9.** For any \( L > 0 \) there is an \( \epsilon > 0 \) such that for every \( n \) large enough, we have

\[
d_{\sigma_n}(x, y) > \epsilon
\]

for any two distinct points \( x, y \in X_L^n \).

**Proof.** As in the proof of Lemma 2.7 we may consider the combinatorial model \( F_n \) instead of \( G_n \). That is,

\[
X_L^n = \{ F_n^k(x) \mid x \in \Omega_{F_n}, -L \leq k \leq L \} \cap \partial \Delta.
\]

Let us also define

\[
X_L = \{ F_n^k(x) \mid x \in \Omega_{F}, -L \leq k \leq L \} \cap \partial \Delta.
\]

Now for \( L > 0 \) given, \( X_L^n \to X_L \) as \( n \to \infty \). Let \( I \) be the smallest component of \( \partial \Delta - X_L \). Since \( \theta \) is an irrational number, there is a least integer \( m > 0 \) such that

\[
\partial \Delta \subset \bigcup_{0 \leq l \leq m} e^{2\pi i l \theta} I.
\]

Since each \( e^{2\pi i l \theta} I \) is open, \( 0 \leq l \leq m \), it follows that there is an \( N_1 > 0 \), such that for all \( n > N_1 \), and any component \( I \) of \( \partial \Delta - X_L^n \), we have

\[
\partial \Delta \subset \bigcup_{0 \leq l \leq m} e^{2\pi i l \theta} I.
\]

Since \( \sigma_n \) is \( F_n \)-invariant, it follows that for any \( x \) and \( y \) in \( X_L^n \),

\[
d_{\sigma_n}(x, y) > 1/(m + 1)
\]
for all $n > N_1$. \qed

**Lemma 2.10.** For any $0 < \epsilon < 1$, there exist $0 < \mu < 1/2$ and an integer $L(\epsilon) \geq 1$ dependent only on $\epsilon$ and $\theta$ such that for all $n$ large enough and any arc segment $I$ with $\epsilon \leq \sigma_n(I) \leq 1 - \epsilon$, there is an integer $1 \leq l \leq L(\epsilon)$ such that the following inequalities hold:

1. $\sigma_n(I \cap G_n^l(I)) > \mu \epsilon$,
2. $\sigma_n(I - G_n^l(I)) > \mu \epsilon$,
3. $\sigma_n(G_n^l(I) - I) > \mu \epsilon$.

**Proof.** As in the proofs of Lemma 2.7 and 2.9, let us consider the combinatorial model $F_n$ instead of $G_n$. In particular, in the definition of $\sigma_n$, $G_n$ is replaced by $F_n$, and $\sigma_n$ is thus $F_n$-invariant.

**Claim 1:** For any $0 < \delta < 1$, there exist $0 < \nu < 1/2$ and an integer $K(\delta) \geq 1$ dependent only on $\delta$ and $\theta$ such that for any arc segment $I$ with $\delta \leq |I| \leq 2\pi - \delta$, there is an integer $1 \leq l \leq K(\delta)$ such that the following inequalities hold:

1. $|I \cap e^{2\pi i l \theta_n} I| > \nu \delta$,
2. $|I - e^{2\pi i l \theta_n} I| > \nu \delta$,
3. $|e^{2\pi i l \theta_n} I - I| > \nu \delta$.

By using the fact that $\theta$ is an irrational number, the claim can be proved by a compacting argument. We leave the details to the reader.

**Claim 2:** For any $0 < \delta < 1$, there exist $0 < \nu < 1/2$ and an integer $K(\delta) \geq 1$ dependent only on $\delta$ and $\theta$ such that for all $n$ large enough and any arc segment $I$ with $\delta \leq |I| \leq 2\pi - \delta$, there is an integer $1 \leq l \leq K(\delta)$ such that the following inequalities hold:

1. $|I \cap e^{2\pi i l \theta_n} I| > \nu \delta$,
2. $|I - e^{2\pi i l \theta_n} I| > \nu \delta$,
3. $|e^{2\pi i l \theta_n} I - I| > \nu \delta$.

Claim 2 follows from Claim 1 and the fact that $\theta_n \to \theta$ as $n \to \infty$.

**Claim 3:** For any $\epsilon > 0$, there is a $\delta > 0$ dependently only on $\epsilon$ and $\theta$ such that for all $n$ large enough and any arc segment $I$, $\sigma_n(I) > \delta$ provided that $|I| > \epsilon$, and moreover, the converse is also true: $|I| > \delta$, provided that $\sigma_n(I) > \epsilon$.

Let us prove the first assertion. Suppose $|I| > \epsilon$. Then there is an integer $K \geq 1$ dependent only on $\theta$ and $\epsilon$ such that for all $n$ large enough, the following holds:

$$\partial \Delta \subset \bigcup_{0 \leq l \leq K} e^{2\pi i l \theta_n} I.$$  

Since $\sigma_n$ is $F_n$-invariant, it follows that $\sigma_n(I) > 1/(K + 1)$. This proves the first assertion.

Let us prove the second assertion now. It is sufficient to prove that $\sigma_n(I) < \epsilon$ provided that $|I| < \delta$. For $\epsilon > 0$ given, let $K$ be the least integer such that $K > 1/\epsilon$. Since $\theta$ is an irrational number, it follows that there is a $\delta > 0$ such
that for any arc segment $I$ with $|I| < \delta$, the closure of the arc segments
$e^{2\pi il\theta}I, 0 \leq l \leq K$
are disjoint with each other. Since $\theta_0 \to \theta$ as $n \to \infty$, it follows that for all $n$ large enough, and any arc segment $I$ with $|I| < \delta$, the closure of the arc segments
$e^{2\pi il\theta}I, 0 \leq l \leq K$
are disjoint with each other. Since $\sigma_n$ is $F_n$-invariant, it follows that
$\sigma_n(I) < 1/(1 + K) < \epsilon$.
This completes the proof of Claim 3. Now the lemma follows directly as a consequence of Claim 2 and 3.

**Lemma 2.11.** Let $\tau > 0$. Then there exist $K > 0$ and $N_0 > 0$ dependent only on $\tau$ and $\theta$, such that for all $L \geq K$ and $n \geq N_0$ and any arc segment $I$ with $\sigma_n(I) > \tau$, the following inequality
$|I \cap X^n_L| \geq 2$
holds.

**Proof.** As in the proofs of Lemma 2.7 and 2.9 let us consider the combinatorial model $F_n$ instead of $G_n$. Assume that $\sigma_n(I) > \tau$ for some $\tau > 0$. From Claim 3 in the proof of Lemma 2.10, there exist $\epsilon > 0$ and $N_1 > 0$ which depend only on $\theta$ and $\tau$ such that for all $n \geq N_1$, the following inequality
$|I| > \epsilon$
holds provided that $\sigma_n(I) > \tau$. For such $\epsilon > 0$, since $\theta$ is irrational, it follows that there exists a $K > 0$ which depends only on $\theta$ and $\epsilon$ such that for any $I \subset \partial \Delta$ with $|I| > \epsilon/2$,
$|I \cap \{e^{2\pi il\theta} \mid -K \leq l \leq K\}| \geq 2$.
Since $\theta_0 \to \theta$, it follows that there exists an $N_2 > 0$ which depends only on $K$ and $\epsilon$, such that for all $n \geq N_2$ and any arc segment $I \subset \partial \Delta$ with $|I| > \epsilon$,
$|I \cap \{e^{2\pi il\theta_0} \mid -K \leq l \leq K\}| \geq 2$.
Let $N_0 = \max\{N_1, N_2\}$. Then for all $L \geq K$ and $n \geq N_0$, we have
$|I \cap \{e^{2\pi il\theta_0} \mid -L \leq l \leq L\}| \geq |I \cap \{e^{2\pi il\theta_0} \mid -K \leq l \leq K\}| \geq 2$.
The lemma follows.

**Lemma 2.12.** For any $L$ large enough there is a $\delta > 0$ such that for all $n$ large enough, the hyperbolic length of every simple closed geodesic in $\mathbb{P}^1 - (X^n_L \cup P_n)$, which intersects the unit circle, is greater than $\delta$.

The proof is by contradiction. Assuming that the Lemma were not true. The basic idea is to construct two short simple closed geodesics so that they intersect with each other. This is realized by first constructing two short simple closed geodesics $\eta'$ and $\eta''$ which intersect with each other, but which
belong to different hyperbolic Riemann surfaces. The next step is to find a common hyperbolic Riemann surface in which there exist two simple closed geodesics $\xi'$ and $\xi''$ which are, respectively, homotopic to $\eta'$ and $\eta''$, and most importantly, separate some set $Z \subset \partial \Delta$ in the same way as $\eta'$ and $\eta''$. This implies that $\xi'$ and $\xi''$ must intersect with each other. This is a contradiction with Theorem A.1 (see Figure 5 for an illustration).

**Proof.** Let $L_0$ be the number defined in Corollary 2.1. Suppose that $L > \max\{N + 2, L_0\}$ and that $\gamma$ is a simple closed geodesic in $\mathbb{P}^1 - (X_L^n \cup P_n)$, which intersects the unit circle, and has length less than $\delta$. By Lemma 2.8 and replacing $L$ by $L + N + 2$ if necessary, we will have a short simple closed geodesic $\eta$ in $\mathbb{P}^1 - (P_n \cup X_L^n)$ such that

$$|D(\eta) \cap X_L^n| \geq 2$$

and

$$|(\partial \Delta - D(\eta)) \cap X_L^n| \geq 2.$$

Let

$$I \subset \partial \Delta \cap D(\eta)$$

be the maximal closed arc segment such that $\partial I \subset X_L^n$. Here $\partial I$ denote the set of the two end points of $I$. Similarly, let

$$J \subset \partial \Delta - D(\eta)$$

be the maximal closed arc segment such that $\partial J \subset X_L^n$. From Lemma 2.9 and the above two inequalities, it follows that there is an $\epsilon > 0$ which depends only on $L$ and $\theta$ such that

$$\min\{\sigma_n(I), \sigma_n(J)\} \geq \epsilon$$

for all $n$ large enough. For such $\epsilon$, let $0 < \mu < 1/2$ and $L(\epsilon) \geq 1$ be the numbers given in Lemma 2.10. Now take $\tau = \mu \epsilon$ in Lemma 2.11 and let $K > 0$ be the value there. Let

$$S = K + L + L(\epsilon).$$

By Lemma 2.10 and (16), there is an $0 < l < L(\epsilon)$ such that the following inequalities hold for all $n$ large enough:

1. $\sigma_n(I \cap G_n^l(I)) > \tau$,
2. $\sigma_n(I - G_n^l(I)) > \tau$,
3. $\sigma_n(G_n^l(I) - I) > \tau$.

Let $Z = X_K^n \cap G_n^{-l}(X_K^n)$. It follows that $X_K^n \subset Z$. From Lemma 2.11 and the above three inequalities, we have

i. $|I \cap G_n^l(I) \cap Z| \geq 2$,
ii. $|(I - G_n^l(I)) \cap Z| \geq 2$,
iii. $|(G_n^l(I) - I) \cap Z| \geq 2$. 
Now let us assume that $\delta$, and hence $\|\gamma\|_{P^1-(P_n\cup X^0_S)}$ and $\|\eta\|_{P^1-(P_n\cup X^0_S)}$ are small enough so that Theorem A.2 and Theorem A.3 can be applied in the following discussion.

From Lemma 2.6 by taking $M = K + L(\epsilon)$, we have

$$(P_n \cup X^n_L) \subset G^{-1}(P_n \cup X^n_S)$$

and

$$|G^{-1}(P_n \cup X^n_S) - (P_n \cup X^n_L)| \leq C,$$

where $C$ only depends on $L, N, S$, and the degree of $F$. By Theorem A.3, there exists a simple closed geodesic $\eta'$ in $P^1 - G^{-1}(P_n \cup X^n_S)$, which is homotopic to $\eta$ in $P^1 - (P_n \cup X^n_L)$ such that

$$\|\eta'\|_{P^1-G^{-1}(P_n\cup X^n_S)} \leq C' \|\eta\|_{P^1-(P_n\cup X^n_L)}$$

where $C'$ depends only on $L, N, S$, and the degree of $F$. Since $\eta'$ is homotopic to $\eta$ in $P^1 - (P_n \cup X^n_L)$, we have

$$I \subset \partial \Delta \cap D(\eta') \text{ and } J \subset \partial \Delta - D(\eta').$$

Let

$$\partial \Delta \cap \eta' = \{a, b\}.$$

By theorem A.2, $G^i_n(\eta')$ covers a short simple closed geodesic $\eta''$ in $P^1 - (P_n \cup X^n_S)$, and therefore,

$$\|\eta''\|_{P^1-(P_n\cup X^n_S)} \leq \|\eta'\|_{P^1-G^{-1}(P_n\cup X^n_S)}.$$

Most importantly, $\eta''$ separates $G^i_n(I)$ and $G^i_n(J)$, that is, one of them is contained in $D(\eta'')$ and the other one is contained in the outside of $D(\eta'')$. 

Figure 5. The geodesics $\eta', \eta'', \xi', \xi''$ and the points in $Z$
Let
\[ \partial\Delta \cap \eta'' = \{ G_n^a(a), G_n^b(b) \} = \{ c, d \}. \]

From the inequalities (i), (ii), and (iii), it follows that one can label the four intersection points \(a, b, c,\) and \(d\) appropriately, such that they are distributed in the order of \(a, c, b,\) and \(d,\) and moreover, each of the three segments \([a, c],[c, b],\) and \([b, d]\) contains at least two points in \(Z.\) See Figure 5 for an illustration.

From above it follows that both \(\eta'\) and \(\eta''\) are non-peripheral curves in \(\mathbb{P}^1 - (P_n \cup Z).\) Let \(\xi'\) be the simple closed geodesic in \(\mathbb{P}^1 - (P_n \cup Z)\) which is homotopic to \(\eta'\) in \(\mathbb{P}^1 - (P_n \cup Z).\) Since
\[ \mathbb{P}^1 - (P_n \cup Z) \supset \mathbb{P}^1 - G^{-l}(P_n \cup X_n^S) \]
by the definition of \(Z,\) it follows that
\[ \| \xi' \|_{\mathbb{P}^1 - (P_n \cup Z)} \leq \| \eta' \|_{\mathbb{P}^1 - G^{-l}(P_n \cup X_n^S)}. \]
Suppose \(\xi'\) intersects with \(\partial\Delta\) at the two points \(x\) and \(y.\) Similarly, let \(\xi''\) be the simple closed geodesic in \(\mathbb{P}^1 - (P_n \cup Z)\) which is homotopic to \(\eta''\) in \(\mathbb{P}^1 - (P_n \cup Z).\) Since
\[ \mathbb{P}^1 - (P_n \cup Z) \supset \mathbb{P}^1 - (P_n \cup X_n^S) \]
by the definition of \(Z,\) it follows that
\[ \| \xi'' \|_{\mathbb{P}^1 - (P_n \cup Z)} \leq \| \eta'' \|_{\mathbb{P}^1 - (P_n \cup X_n^S)}. \]
Suppose \(\xi''\) intersects with \(\partial\Delta\) at the two points \(z\) and \(w.\) One can label \(x, z, y,\) and \(w\) so that they are in the same order as \(a, c, b,\) and \(d,\) This implies that \(\xi'\) and \(\xi''\) separates the points in \(Z\) in the same way as \(\eta'\) and \(\eta''.\) It follows that
\[ \xi' \cap \xi'' \neq \emptyset. \]
See Figure 5 for an illustration. But (17), (18), (19) and (20) imply that both \(\xi'\) and \(\xi''\) can be short to any extent provided that \(\delta\) is small. This is a contradiction with Theorem A.1.

\[ \square \]

**Lemma 2.13.** There exist \(L \geq N + 2\) and a \(\delta > 0\) such that for every \(n\) large enough, any simple closed geodesic \(\gamma\) in \(\mathbb{P}^1 - (P_n \cup X_n^S)\) has length greater than \(\delta.\)

Our argument is an adapted version of the one used in §8 of [9]. In fact, all the short simple closed geodesics which do not intersect the unit circle, consist of a \(G_n\) stable family \(\Gamma.\) Since \(|P_{G_n} - \partial\Delta| = |P_{F'\Gamma} - \partial\Delta|\) does not depend on \(n,\) there is an \(m > 0\) independent of \(n\) such that \(\| A^m \| < 1/2\) where \(A\) is the associated linear transformation matrix. Then by using a similar argument with the one in [9], it follows that the simple closed geodesics in \(\Gamma\) can not be too short. In the following proof, we will present the details at the place where the situation here is different from that in §8 of [9], and only give a sketch if they are the same.
Proof. By Lemma 2.12, we can take \( N + 2 \leq L < \infty \) and \( \epsilon > 0 \) such that for all \( n \) large enough, any simple closed geodesic in \( \mathbb{P}^1 - (P_n \cup X^p_L) \), which intersects the unit circle, has length greater than \( \epsilon \).

Let \( 0 < \delta < \epsilon \) and consider the family \( \Gamma_\delta \) of all the simple closed geodesics in \( \mathbb{P}^1 - (P_n \cup X^p_L) \) which has length \( \leq \delta \). By using the same argument as in the proof of Proposition 8.1 in [9], it follows that if \( \Gamma_\delta \neq \emptyset \) for \( \delta \) small enough, then there is a \( G_n \)-stable family in \( \mathbb{P}^1 - P_{G_n} \cup \{0, \infty\} \), say \( \Gamma \), which consists of short simple closed geodesics in \( \mathbb{P}^1 - (P_n \cup X^p_L) \) and which satisfies certain gap property. Roughly speaking, the gap property means that there is a uniform \( \tau > 0 \), such that every simple closed geodesic in \( \mathbb{P}^1 - (P_n \cup X^p_L) \) either belongs to \( \Gamma \) or has hyperbolic length greater than \( \tau \). We refer the reader to §8 of [9] for more details about this property.

Now let \( A \) be the associated linear transformation matrix of \( \Gamma \). According to Thurston’s characterization theorem (see §5), we have \( \|A\| < 1 \). Since the curves in \( \Gamma \) do not intersect the unit circle, and \( \|P_n\| = 2N \), it follows that the number of the curves in \( \Gamma \) has an upper bound which is independent of \( n \). This implies that the number of all the possible linear transformation matrices also has an upper bound independent of \( n \). Therefore, there is an \( 0 < m < \infty \), which is independent of \( n \), such that \( \|A^m\| \leq 1/2 \) where \( A \) is the Thurston linear transformation matrix for any such \( G_n \)-stable family \( \Gamma \). Note that \( m \) does not depend on \( L \). In the following we may assume that \( L > m \) by increasing \( L \) if necessary.

Now assume that \( \gamma \in \Gamma \) is a short simple closed geodesic in \( \mathbb{P}^1 - (P_n \cup X^p_L) \). Recall that for a hyperbolic Riemann surface \( X \), we use \( \|\gamma\|_X \) to denote the hyperbolic length of the simple closed geodesic which is homotopic to \( \gamma \) in \( X \). Let us consider the set of all the simple closed geodesics in \( \mathbb{P}^1 - (P_n \cup X^p_{L+2m}) \) which are homotopic to \( \gamma \) in \( \mathbb{P}^1 - (P_n \cup X^p_L) \) and with length less than \( \log(\sqrt{2} + 1) \). The number of the curves in this set is not more than

\[
|X^p_{L+2m} - X^p_L|
\]

which is independent of \( n \). By Lemma 2.12 it follows that among all these curves, only the one, which does not intersect the unit circle (therefore, is homotopic to \( \gamma \) in \( \mathbb{P}^1 - P_n \cup X^p_{L+2m} \), can be short. The length of all the other curves has a positive lower bound which is dependent of \( n \). By Theorem A.3 we get

\[
\|\gamma\|_{\mathbb{P}^1 - (P_n \cup X^p_L)}^{-1} \leq \|\gamma\|_{\mathbb{P}^1 - (P_n \cup X^p_{L+2m})}^{-1} + C_1,
\]

where \( C_1 \) is some constant independent of \( n \).

Let

\[
Y = \partial \Delta \cap G_n^{-m}(X^p_L).
\]

It follows that

\[
\mathbb{P}^1 - (P_n \cup X^p_{L+2m}) \subset \mathbb{P}^1 - (P_n \cup Y).
\]

So

\[
\|\gamma\|_{\mathbb{P}^1 - (P_n \cup X^p_{L+2m})}^{-1} < \|\gamma\|_{\mathbb{P}^1 - (P_n \cup Y)}^{-1}.
\]

\[\text{(21)}\]
From (21) and (22), we have
\[ \|\gamma\|_{p_1-(P_n \cup X_L^n)}^{-1} \leq \|\gamma\|_{p_1-(P_n \cup Y)}^{-1} + C_1. \]

Note that
\[ \mathbb{P}^1 - G_n^{-m}(P_n \cup X_L^n) \subset \mathbb{P}^1 - (P_n \cup Y). \]
For each \( \gamma_i \in \Gamma \), let
\[ \gamma_{i,j,\alpha} \subset \mathbb{P}^1 - G_n^{-m}(P_n \cup X_L^n), \ \alpha \in \Lambda_{i,j}, \]
be all the components of \( G_n^{-m}(\gamma_j) \), which are homotopic to \( \gamma_i \) in \( \mathbb{P}^1 - (P_n \cup Y) \), and whose length is less than \( \log(\sqrt{2} + 1) \). Here \( \Lambda_{i,j} \) is a finite set.

By the gap property of \( \Gamma \), there is a uniform positive lower bound \( B > 0 \) independent of \( n \) such that every simple closed geodesic in \( \mathbb{P}^1 - G_n^{-m}(P_n \cup X_L^n) \), which is homotopic to some \( \gamma_i \) in \( \mathbb{P}^1 - (P_n \cup Y) \), but does not belong to \( \{ \gamma_{i,j,\alpha} \} \), must have length greater than \( B \) (This is the place where the gap property is required). This, together with Theorem A.3, and the fact that
\[ |G_n^{-m}(P_n \cup X_L^n) - (P_n \cup Y)| \]
depends only on \( L, N, m \), and the degree of \( F \), implies that there is a \( 0 < C_2 < \infty \) independent of \( n \) such that
\[ \|\gamma_i\|_{p_1-(P_n \cup Y)}^{-1} \leq \sum_j \sum_{\alpha} \|\gamma_{i,j,\alpha}\|_{p_1-G_n^{-m}(P_n \cup X_L^n)}^{-1} + C_2. \]

Since \( L > m \), it follows that
\[ G^n : \mathbb{P}^1 - G_n^{-m}(P_n \cup X_L^n) \to \mathbb{P}^1 - (P_n \cup X_L^n) \]
is a holomorphic covering map. This, together with the inequality \( \|A\|^m \leq \frac{1}{2} \)
implies
\[ \sum_i \sum_j \sum_{\alpha} \|\gamma_{i,j,\alpha}\|_{p_1-G_n^{-m}(P_n \cup X_L^n)}^{-1} \leq \frac{1}{2} \sum_i \|\gamma_i\|_{p_1-(P_n \cup X_L^n)}^{-1}. \]
From (23), (24), and (25), we have
\[ \sum_i \|\gamma_i\|_{p_1-(P_n \cup X_L^n)}^{-1} \leq \frac{1}{2} \sum_i \|\gamma_i\|_{p_1-(P_n \cup X_L^n)}^{-1} + C, \]
and hence
\[ \sum_i \|\gamma_i\|_{p_1-(P_n \cup X_L^n)}^{-1} \leq 2C. \]
where \( 0 < C < \infty \) depends only on \( L, m, N \) and the degree of \( F \). Lemma 2.13 follows.

Let \( Z_n \) and \( P_n \) denote the set of the zeros and poles of \( G_n \), respectively. The following two lemmas imply the bounded geometry of \( P_{G_n} \) and the compactness of the sequence \( \{ G_n \} \).

**Lemma 2.14.** There is a \( \delta > 0 \) independent of \( n \) such that for any two points in \( P_n \cup Z_n \cup P_n \), say \( x \) and \( y \), we have \( d_{S^2}(x, y) > \delta \).
Proof. Let $L \geq N + 2$ be the number in Lemma $[2.13]$ It is clear that
\[ P_n \cup Z_n \cup P_n \subset G^{-1}_n(P_n \cup X^n_L). \]
Consider the space
\[ Y_n = \mathbb{P}^1 - G^{-1}_n(P_n \cup X^n_L). \]
Assume that Lemma $[2.14]$ were not true. Then we would have a sequence of integers, say \( \{n_k\} \), such that \( n_k \to \infty \) as \( k \to \infty \), and a sequence of short simple closed geodesics, say \( \gamma_{n_k} \subset Y_{n_k} \), such that \( \|\gamma_{n_k}\|_{Y_{n_k}} \to 0 \). Then every \( G_{n_k}(\gamma_{n_k}) \) covers a short simple closed geodesic \( \xi_{n_k} \subset \mathbb{P}^1 - (P_{n_k} \cup X_{n_k}^L) \) whose length goes to 0 as \( k \to \infty \). This is a contradiction with Lemma $[2.13]$.

Lemma 2.15. There is a \( \delta > 0 \) independent of \( n \) such that for any point in \( P_n \cup Z_n \cup P_n \), say \( x \), we have \( d_{S^2}(x, \partial \Delta) > \delta \).

Proof. Let \( x^* \) be the symmetric image of \( x \) about the unit circle. It follows that \( x^* \in P_n \cup Z_n \cup P_n \). By Lemma $[2.14]$ \( d_{S^2}(x, \partial \Delta) = d_{S^2}(x, x^*)/2 \), and therefore has a positive lower bound independent of \( n \).

2.3.3. Bounded geometry of \( P_{G_n} \) on \( \partial \Delta \). By passing to a convergent subsequence, we may now assume that \( G_n \to G \). It follows that \( G|\partial \Delta \) is an analytic critical circle homeomorphism with rotation number \( \theta \). It was proved by Herman and Swiatek that such a critical circle homeomorphism is quasisymmetrically conjugate to the rigid rotation \( R_\theta \) if \( \theta \) is of bounded type (see [20] for a detailed proof). Let \( h : \partial \Delta \to \partial \Delta \) be the quasi-symmetric homeomorphism such that
\[ h(1) = 1 \quad \text{and} \quad G|\partial \Delta = h \circ R_\theta \circ h^{-1}. \]
Since \( G_n \) and \( F_n \) are combinatorially equivalent to each other rel \( P'_{F_n} \), there exist a pair of homeomorphisms \( \phi_n, \psi_n : S^2 \to \mathbb{P}^1 \) such that
\[
\begin{array}{ccc}
(S^2, P'_{F_n}) & \psi_n \downarrow & (\mathbb{P}^1, P'_{G_n}) \\
F_n & \downarrow & G_n \\
(S^2, P'_{F_n}) & \phi_n \downarrow & (\mathbb{P}^1, P'_{G_n})
\end{array}
\]
and \( \phi_n \) is isotopic to \( \psi_n \) rel \( P'_{F_n} \).

Lemma 2.17. \( \psi_n|\partial \Delta \to h \), \( \phi_n|\partial \Delta \to h \) uniformly as \( n \to \infty \).

Proof. We need only to prove that \( \psi_n \to h \) uniformly as \( n \to \infty \). The other one can be proved by the same argument. Let \( N \) be an integer such that the length of each interval component of \( \partial \Delta - \{G^k(1)\}_{0 \leq k \leq N} \) is less than one-sixth of the whole circle. Since \( G_n \to G \) uniformly as \( n \to \infty \), it follows that...
when $n$ is large enough, the length of each component of $\partial \Delta - \{G_n^k(1)\}_{0 \leq k \leq N}$ is less than one-fifth of the whole circle. Let

$$\delta_1 = \min\{|I|/6 \mid I \text{ is a component of } \partial \Delta - \{e^{2k\pi i \theta}\}_{0 \leq k \leq N}\}.$$  

It follows that for every $n$ large enough, the image of an arc segment with length less than $6\delta_1$ will be mapped by $\psi_n$ to some arc segment less than one half of the whole circle. In fact, if $\psi_n(I)$ contains a half of the circle, then it contains at least two components of $\partial \Delta - \{G_n^k(1)\}_{0 \leq k \leq N}$. This implies that $I$ contains at least two components of $\partial \Delta - \{e^{2k\pi i \theta}\}_{0 \leq k \leq N}$. But this is a contradiction with the definition of $\delta_1$.

Now for any given $\epsilon > 0$, since $h$ is uniformly continuous, we have a $\delta_2 > 0$ such that for any $x, x' \in \partial \Delta$ and $|x - x'| < 4\delta_2$,

$$|h(x) - h(x')| < \epsilon/5.$$  

Take $\delta = \min\{\delta_1, \delta_2\}$. For such $\delta$, there is an integer $M > 0$ such that for any $x$ in the unit circle, there are two integers $0 < k_1, k_2 < M$ such that

$$e^{2\pi i k_1 \theta} \in (x + \delta, x + 2\delta) \text{ and } e^{2\pi i k_2 \theta} \in (x - 2\delta, x - \delta).$$  

For such $\epsilon, \delta$, and $M$, take $N$ large enough such that when $n > N$,

i. $|\theta_n - \theta| < \delta/2\pi M$;

ii. $|G_n^k(x) - G^k(x)| < \epsilon/5$ for all $1 \leq k \leq M$ and all $x \in \partial \Delta$.

From (28) and Property (i) above, it follows that we have

$$e^{2\pi i k_1 \theta_n} \in (x, x + 3\delta) \text{ and } e^{2\pi i k_2 \theta_n} \in (x - 3\delta, x).$$  

This implies that $e^{2\pi i k_1 \theta_n}, x$, and $e^{2\pi i k_2 \theta_n}$ are contained in an arc segment with length less than $6\delta$, which is mapped by $\psi_n$ to some arc segment less than one half of the circle. It follows that

$$|\psi_n(x) - \psi_n(e^{2\pi i k_1 \theta_n})| \leq |\psi_n(e^{2\pi i k_1 \theta_n}) - \psi_n(e^{2\pi i k_2 \theta_n})|.$$  

We thus have the following,

$$|\psi_n(x) - h(x)| \leq |\psi_n(x) - \psi_n(e^{2\pi i k_1 \theta_n})| + |\psi_n(e^{2\pi i k_1 \theta_n}) - h(e^{2\pi i k_1 \theta})| + |h(e^{2\pi i k_1 \theta}) - h(x)|$$

$$\leq |\psi_n(e^{2\pi i k_1 \theta_n}) - \psi_n(e^{2\pi i k_2 \theta_n})| + |\psi_n(e^{2\pi i k_1 \theta_n}) - h(e^{2\pi i k_1 \theta})| + |h(e^{2\pi i k_1 \theta}) - h(x)|$$

$$= |G_n^{k_1}(1) - G_n^{k_1}(1)| + |G_n^{k_1}(1) - G^{k_1}(1)| + |h(e^{2\pi i k_1 \theta}) - h(x)|$$

$$\leq |G_n^{k_1}(1) - G^{k_1}(1)| + |G^{k_1}(1) - G^{k_2}(1)| + |G^{k_2}(1) - G_n^{k_2}(1)|$$

$$+ |G_n^{k_2}(1) - G^{k_2}(1)| + |h(e^{2\pi i k_1 \theta}) - h(x)| \leq \epsilon.$$  

Let us explain how the last inequality comes. The inequalities $|G_n^{k_1}(1) - G^{k_1}(1)| < \epsilon/5$, $|G^{k_2}(1) - G_n^{k_2}(1)| < \epsilon/5$, and $|G^{k_1}(1) - G^{k_1}(1)| < \epsilon/5$ come from the property (ii) above. The inequality $|G^{k_1}(1) - G^{k_2}(1)| < \epsilon/5$ comes from (27) and (28). The inequality $|h(e^{2\pi i k_1 \theta}) - h(x)| < \epsilon/5$ comes from (27) and (29).
2.4. The Candidate Blaschke Product. In this section, we will show that
\( G \) is the desired Blaschke product by showing that there are homeomorphisms 
\( \phi, \psi : S^2 \to S^2 \) which fix 0, 1, and \( \infty \), such that 
\( G = \phi \circ F \circ \psi^{-1} \) and \( \phi, \psi \) are
isotopic to each other rel \( P'_F \). Recall that for every \( n \) large enough, there is
a pair of homeomorphisms \( \phi_n \) and \( \psi_n \) such that 
\( G_n = \phi_n \circ F_n \circ \psi_n^{-1} \) and \( \phi_n \) and \( \psi_n \) are isotopic to each other rel \( P'_{F_n} \). The aim of this section is to show
that the homotopy classes of \( \phi_n \) and \( \psi_n \) converge to the same one as \( n \to \infty \).

First we will show that for every \( n \) large enough, by deforming \( \phi_n \) and \( \psi_n \)
in their isotopy class, we can make \( \phi_n \) and \( \psi_n \) satisfy some local properties
around each point in \( \Omega_{F_n} \cup \Delta \) (Lemma 2.18). Secondly we will prove that
for every \( \rho > 0 \), provided that \( n \) is large enough, the map \( \phi_n \) and \( \psi_n \) can be
perturbed within their \( \rho \)-neighborhood into a pair of homeomorphisms \( \hat{\phi}_n \) and
\( \hat{\psi}_n \) such that 
\( G = \hat{\phi}_n \circ F \circ \hat{\psi}_n^{-1} \) (Lemma 2.19). Finally we will prove
that when \( \rho \) is small, the maps \( \hat{\phi}_n \) and \( \hat{\psi}_n \) are isotopic to each other rel \( P'_F \)(Lemma 2.20).

2.4.1. Deforming \( \phi_n \) and \( \psi_n \) in their isotopy class. Let \( r > 0 \) be a number
such that
\[ d_{S^2}(x, y) > r \]
for any two distinct points \( x \) and \( y \) in \( \Omega_F \cup (P'_F - \Delta) \). Since \( F_n \to F \)
uniformly, it follows that for any \( x \in \Omega_F \cup (P'_F - \Delta) \), and every large \( n \),
\( B_{r/3}(x) \) contains exactly one point in \( \Omega_{F_n} \cup (P'_{F_n} - \Delta) \). Let us denote this
point by \( \tau_n(x) \). It is easy to see that \( \tau_n(x) \to x \) as \( n \to \infty \). By passing to
a convergent subsequence, and by Lemma 2.18, we may assume that for any
\( x \in \Omega_F \cup (P'_F - \Delta) \), \( \psi_n(\tau_n(x)) \) and \( \phi_n(F_n(\tau_n(x))) \) converge as \( n \to \infty \).

Lemma 2.18. For any \( r, \delta > 0 \) there exist \( 0 < r_0 < r \) and \( 0 < \delta_0 < \delta \), such
that for any \( 0 < r' < r_0 \) and \( 0 < \delta' < \delta_0 \), there exist \( 0 < r_1 < r_2 < r_3 < r' \),
and \( 0 < \delta_1 < \delta_2 < \delta_3 < \delta' \) such that for every \( n \) large enough, there exist
homeomorphisms \( \phi_n \) and \( \psi_n : S^2 \to S^2 \) such that

1. \( \psi_n \) and \( \phi_n \) are isotopic to each other rel \( P'_{F_n} \),
2. \( G_n = \phi_n \circ F_n \circ \psi_n^{-1} \),
3. for any \( x \in \Omega_F \cup (P'_F - \Delta) \), by taking a convergent subsequence,
\( \psi_n(\tau_n(x)) \) and \( \phi_n(F_n(\tau_n(x))) \) converge as \( n \to \infty \),
4. for every \( x \in P'_{F_n} \cup \Omega_{F_n} \), \( B_{\delta_1}(\phi_n(x)) \subset \phi_n(B_{r_1}(x)) \subset \phi_n(B_{r_2}(x)) \subset \phi_n(B_{r_3}(x)) \subset \phi_n(B_{\delta_2}(\phi_n(x))) \subset \phi_n(B_{\delta_3}(\phi_n(x))) \subset \phi_n(B_{\delta_4}(\phi_n(x))) \subset \phi_n(B_{\delta_5}(\phi_n(x))) \subset \phi_n(B_{r_0}(x)) \)
and this inclusion relation also holds if we replace \( \phi_n \) by \( \psi_n \).

Proof. From the previous sections, it follows that for every \( n \) large enough,
there exist a pair of homeomorphisms \( \phi_n \) and \( \psi_n \) such that (1), (2) and
(3) hold. For any \( r, \delta > 0 \), since \( P_{G_n} \) has uniform bounded geometry(see
Lemma 2.14, 2.15 and 2.17), we can take \( r' \ll r \) and \( \delta' \ll \delta \) such that
for every \( n \) large enough, \( \phi_n \) can be deformed in its isotopic class so that it satisfies
\[ B_{\delta_n}(\phi_n(x)) \subset \phi_n(B_{r_0}(x)) \]
for every $x \in P'_F \cup \Omega_{F_n}$. Then we lift $\phi_n$ by the equation
\[ \phi_n \circ F_n = G_n \circ \psi_n \]
and get $\psi_n$. Since $F_n \to F$ and $G_n \to G$ uniformly in the spherical metric, it follows that there exist $r_0''$ and $\delta_0''$, which can be taken arbitrarily small provided that $r_0'$ and $\delta_0'$ are small, such that
\[ B_{\delta_0''}(\psi_n(x)) \subset \psi_n(B_{r_0''}(x)) \]
for every $x \in P'_F \cup \Omega_{F_n}$. Now take $r_0 = \max\{r_0', r_0''\}$ and $\delta_0 = \min\{\delta_0', \delta_0''\}$. By taking $r_0'$, $\delta_0'$ small enough, we can assure $r_0 < r$ and $\delta_0 < \delta$. In particular,
\[ B_{\delta_0}(\phi_n(x)) \subset \phi_n(B_{r_0}(x)), \text{ and } B_{\delta_0}(\psi_n(x)) \subset \psi_n(B_{r_0}(x)) \]
for every $x \in P'_F \cup \Omega_{F_n}$. Now let $r' < r_0$ and $\delta' < \delta_0$ be given. We may use the same process to get $r_3, \delta_3$ as follows.

Deform $\phi_n$ in a smaller disk around each point $x \in P'_F \cup \Omega_{F_n}$ so that
\[ \phi_n(B_{r_3'}(x)) \subset B_{\delta_3'}(\phi_n(x)) \subset B_{\delta_0}(\phi_n(x)) \]
for some $r_3' \ll r', \delta_3' \ll \delta'$, and then get $\psi_n$ by lifting $\phi_n$. As in the last step, By choosing $r_3', \delta_3'$ small, we can get $0 < r_3 < r'$ and $0 < \delta_3 < \delta'$, such that
\[ \phi_n(B_{r_3}(x)) \subset B_{\delta_3}(\phi_n(x)) \subset B_{\delta_0}(\phi_n(x)) \]
and
\[ \psi_n(B_{r_3}(x)) \subset B_{\delta_3}(\psi_n(x)) \subset B_{\delta_0}(\psi_n(x)) \]
for all $x \in P'_F \cup \Omega_{F_n}$. Since we deform $\phi_n$ only in a smaller disk, this step will not affect the relations obtained in the last step. We may repeat this procedure and get $r_1, r_2, \delta_1, \delta_2$ so that the corresponding relations are also satisfied. The proof of the lemma is completed. \qed

2.4.2. Perturbing $\phi_n$ and $\psi_n$.

**Lemma 2.19.** Let $\rho > 0$ be an arbitrary number. Then there exists an $N > 0$ such that for every $n > N$, there exist homeomorphisms $\widehat{\phi}_n$, $\phi_n$, $\widehat{\psi}_n$, and $\psi_n$ of the sphere such that

1. $G_n = \phi_n \circ F_n \circ \psi_n^{-1}$, and $G = \widehat{\phi}_n \circ F \circ \widehat{\psi}_n^{-1}$,
2. $\phi_n$ and $\psi_n$ are isotopic to each other rel $P'_F$,
3. $\max_{z \in S^2} d_{S^2}(\phi_n(z), \phi_n(z)) < \rho$ and $\max_{z \in S^2} d_{S^2}(\psi_n(z), \psi_n(z)) < \rho$,
4. $\widehat{\phi}_n(\Omega_F) = \widehat{\psi}_n(\Omega_F) = \Omega_G$, and $\widehat{\phi}_n(P'_F) = \widehat{\psi}_n(P'_F) = P'_G$,
5. $\phi_n|\partial \Delta = \psi_n|\partial \Delta = h$ where $h : \partial \Delta \to \partial \Delta$ is the quasi-symmetric homeomorphism in Lemma 2.17.

**Proof.** Let
\[ 0 < r_1 < r_2 < r_3 < r' < r_0 < r \]
and
\[ 0 < \delta_1 < \delta_2 < \delta_3 < \delta' < \delta_0 < \delta, \]
be a group of constants as in Lemma 2.18 such that
\[ d_{S^2}(x, y) > \delta \]
for any two distinct points \( x \) and \( y \) in \( \Omega_G \cup (P_G' - \partial \Delta) \). Let \( \phi_n \) and \( \psi_n \) be homeomorphisms which satisfy the conditions in Lemma 2.18 with the constants given above. We will adjust these constants appropriately as the proof proceeds.

By taking \( r' \) small, and hence \( r_2 \) small, we may assume that for each \( c \in \Omega_F \), there is an open topological disk \( B_c \) containing \( c \) such that

1. \( F : B_c \to B_{r_2}(F(c)) \) is a \( d_c \)-to-1 branched covering map, where \( d_c \geq 2 \) is the local degree of \( F \) at \( c \),
2. for \( c \in \Omega_F - \partial \Delta \), \( (B_c - \{c\}) \cap P_F' = \emptyset \),
3. all \( B_c, c \in \Omega_F \), are disjoint.

Since \( F_n \to F \) uniformly as \( n \to \infty \), it follows that for any \( c \in \Omega_F \) and every \( n \) large enough,

\[
B_{r_1}(F_n(\tau_n(c))) \subset F_n(B_c) \subset B_{r_2}(F_n(\tau_n(c))).
\]

This, together with (4) of Lemma 2.18 implies that

\[
B_{\delta_1}(\phi_n(F_n(\tau_n(c)))) \subset \phi_n(B_{r_1}(F_n(\tau_n(c)))) \subset \phi_n(F_n(B_c)) = G_n \circ \psi_n(B_c)
\]

and

\[
G_n \circ \psi_n(B_c) = \phi_n(F_n(B_c)) \subset \phi_n(B_{r_2}(F_n(\tau_n(c)))) \subset B_{\delta_3}(\phi_n(F_n(\tau_n(c)))).
\]

From

\[
B_{\delta_1}(\phi_n(F_n(\tau_n(c)))) \subset G_n \circ \psi_n(B_c)
\]

and

\[
G_n \circ \psi_n(B_c) \subset B_{\delta_3}(\phi_n(F_n(\tau_n(c)))),
\]

it follows that there exist \( \nu, \mu > 0 \) such that for every \( n \) large enough, and any \( c \in \Omega_F \),

\[
B_{\mu}(\psi_n(\tau_n(c))) \subset \psi_n(B_c) \subset B_{\nu}(\psi_n(\tau_n(c))).
\]

Since as \( \delta' \to 0, \delta_3 \to 0 \), one can take \( \mu \) and \( \nu \) such that \( \mu, \nu \to 0 \) as \( \delta' \to 0 \).

In particular, by taking \( \delta' \) small, we may assume that \( \nu \leq \delta_0/40 \).

Set

\[
U = \bigcup_{c \in \Omega_F} B_c.
\]

From the first inclusion of (30) and the fact that \( \tau_n(c) \to c \) for any \( c \in \Omega_F \), it follows that for all \( n \) large enough,

\[
B_{\mu/2}(\psi_n(z)) \cap \Omega_G = \emptyset
\]

holds for any \( z \in S^2 - U \). For such \( \mu \), there is an \( 0 < \epsilon < \mu/10 \) such that for any \( z \in S^2 - \bigcup_{c \in \Omega_G} B_{\nu/2}(c) \), \( G \) is injective on the disk \( B_{\epsilon}(z) \).

For any \( \eta > 0 \), from the bounded geometry of \( P_G' \), and Lemma 2.17, there is an \( N \) large enough, such that for every \( n > N \), there exists a homeomorphism \( \hat{\phi}_n : S^2 \to S^2 \) such that

1. \( d(\hat{\phi}_n, \phi_n) = \max_{z \in S^2} d_{S^2}(\phi_n(z), \hat{\phi}_n(z)) \leq \eta \).
2. \( \hat{\phi}_n(P_G') = P_G \).
3. \( \hat{\phi}_n(\Omega_F) = \Omega_G \), and
iv. \( \hat{\psi}_n | \partial \Delta = h \), where \( h : \partial \Delta \to \partial \Delta \) is the quasi-symmetric homeomorphism in Lemma 2.17.

We now claim that by taking \( \eta > 0 \) small enough, we can make sure that for every \( n \) large enough, and any \( x \in S^2 - U \), there is a unique point \( y \in B_c(\psi_n(x)) \), such that

\[
(32) \quad G(y) = \hat{\psi}_n(F(x))
\]

where \( \hat{\psi}_n \) is the map defined previously so that (i) - (iv) are satisfied.

Let us prove the claim. Since \( F_n \to F \) and \( G_n \to G \) uniformly as \( n \to \infty \), when \( \eta \) is small and \( n \) is large enough, \( \hat{\psi}_n(F(x)) \in G(B_c(\psi_n(x)) \). This implies the existence of the point \( y \). Since \( B_{\mu/2}(\psi_n(x)) \cap \Omega_G = \emptyset \) by (31), from the choice of \( \epsilon \) above, it follows that \( G \) is injective on \( B_c(\psi_n(x)) \). Therefore, such \( y \) must be unique and does not belong to \( \Omega_G \). This proves the claim. We define \( \hat{\psi}_n(x) = y \) for \( x \in S^2 - U \). It follows that \( \psi_n \) is continuous and locally injective in \( S^2 - U \), and

\[
(33) \quad \hat{\psi}_n(S^2 - U) \cap \Omega_G = \emptyset.
\]

Since \( \hat{\psi}_n(x) = y \in B_c(\psi_n(x)) \), it follows that \( |\hat{\psi}_n(x) - \psi_n(x)| < \epsilon \).

We now claim that for all \( n \) large enough, and every \( c \in \Omega_F \), \( \hat{\psi}_n | \partial B_c \) is injective and hence \( \hat{\psi}_n(\partial B_c) \) is a Jordan curve. In fact, since for each \( c \in \Omega_F \), by the definition of \( B_c \), \( F : B_c \to B_{r_2}(F(c)) \) is a \( d_c \)-to-1 branched covering map, where \( d_c \geq 2 \) is the local degree of \( F \) at \( c \), it follows that \( F(\partial B_c) = \partial B_{r_2}(F(c)) \) is a Jordan curve, and hence \( \hat{\psi}_n(\partial B_c) \) is a Jordan curve. From the construction of \( \hat{\psi}_n \), we have

\[
(34) \quad G(\hat{\psi}_n(\partial B_c)) = \hat{\psi}_n(F(\partial B_c)).
\]

Since \( \hat{\psi}_n(\partial B_c) \cap \Omega_G = \emptyset \) by (31), it follows that \( \hat{\psi}_n(\partial B_c) \) does not intersect with itself, and is therefore a Jordan curve. Note that by the construction, we have

1. \( B_{\mu}(\psi_n(\tau_n(c))) \subset \psi_n(\partial B_c) \) by (30),
2. \( \epsilon < \mu/10 \) by the choice of \( \epsilon \),
3. \( \psi_n | \partial B_c : \partial B_c \to \psi_n(\partial B_c) \) is a homeomorphism,
4. \( |\hat{\psi}_n(z) - \psi_n(z)| < \epsilon \) for \( z \in \partial B_c \).

All the above implies that the topological degree of \( \hat{\psi}_n : \partial B_c \to \hat{\psi}_n(\partial B_c) \) must be 1. Since \( \hat{\psi}_n \) is locally injective in \( S^2 - U \), in particular, it is locally injective on \( \partial B_c \). It follows that \( \hat{\psi}_n \) is injective on \( \partial B_c \). The claim follows.

For any \( c \in \Omega_F \), by (3) of Lemma 2.18 \( \psi_n(\tau_n(c)) \) converges. Let us denote it by \( c' \). From (30), and the fact that \( 0 < \epsilon < \mu/10 \), and that \( |\hat{\psi}_n(z) - \psi_n(z)| < \epsilon \) for \( z \in \partial B_c \), it follows that for every \( n \) large enough, \( \hat{\psi}_n(\partial B_c) \cap \Omega_G = \{\} \).

From (34), \( G : \hat{\psi}_n(\partial B_c) \to \hat{\psi}_n(F(\partial B_c)) \) is a \( d_c = 1 \) branched covering map, where \( d_c \) is the local degree of \( F \) at \( c \), which is equal to the local degree
Let $D_c$ be the component of $S^2 - \hat{\phi}_n(F(\partial B_c))$ which contains $G(c')$. It follows that $G : \hat{\psi}_n(\partial B_c) \to \partial D_c$ is a $d_c : 1$ branched covering map. This allows us to continuously extend $\hat{\psi}_n$ to the inside of every $B_c$, such that $\hat{\psi}_n(c') = c'$ and moreover,

$$G(z) = \hat{\phi}_n \circ F \circ \hat{\psi}_n^{-1}(z)$$

holds on the whole sphere. It is also easy to see that $\hat{\psi}_n : S^2 \to S^2$ is a homeomorphism.

From the construction of $\hat{\phi}_n$ and $\hat{\psi}_n$, it follows that $d(\hat{\phi}_n, \phi_n) < \eta$ and $d(\hat{\psi}_n, \psi_n) < \epsilon + 2\nu < \delta_0/15$. This completes the proof since $\eta$ and $\delta_0$ can be taken arbitrarily small.

\[ \Box \]

**Lemma 2.20.** There is a $\rho_0 > 0$ small such that for all $0 < \rho < \rho_0$, the maps $\hat{\phi}_n$ and $\hat{\psi}_n$ obtained in Lemma 2.19 are isotopic to each other rel $P'_F$.

**Proof.** Since $\hat{\phi}_n|\partial \Delta = \hat{\psi}_n|\partial \Delta = h$, it is sufficient to prove that the restrictions of $\hat{\phi}_n$ and $\hat{\psi}_n$ on the unit disk are isotopic to each other rel $P'_F \cap \Delta$. This is then equivalent to show that for any curve segment $\gamma \subseteq \Delta$ which connects two distinct points $a$ and $b$ in $P'_F$, the image curve segments $\hat{\phi}_n(\gamma)$ and $\hat{\psi}_n(\gamma)$ are homotopic to each other rel $\{a', b'\}$ where $a' = \hat{\phi}_n(a) = \hat{\psi}_n(a)$ and $b' = \hat{\phi}_n(b) = \hat{\psi}_n(b)$.

It is sufficient to consider two cases. In the first case, neither $a$ nor $b$ is on the unit circle. In the second case, $a$ is on the unit circle, but $b$ is not on the unit circle. The proofs are quite direct. Let us explain the idea only and the reader shall have no difficulty to supply the details.

Suppose that we are in the first case. Let $\delta > 0$ be such that for any $z \in P'_F - \partial \Delta$,

$$B_{3\delta}(z) \cap P'_G = \{z\}.$$
Let \( a_n \) and \( b_n \) be the two points in \( P'_{G_n} \) which are correspond to \( a \) and \( b \), respectively. Let \( \gamma_n \) be a curve segment which connects \( a_n \) and \( b_n \) and is close to \( \gamma \). Let \( a'_n = \phi_n(a_n) = \psi_n(a_n) \) and \( b'_n = \phi_n(b_n) = \psi_n(b_n) \). By deforming \( \gamma \) in its homotopy class rel \( \{ a, b \} \) and changing \( \gamma_n \) correspondingly, we may assume that each of \( \phi_n(\gamma_n) \) and \( \psi_n(\gamma_n) \) is the union of three curve segments described as follows. The first piece is a straight segment which connects \( a'_n \) and \( \partial B_\delta(a'_n) \). The second piece is some curve segment which does not intersect the \( \delta \)-neighborhood of \( P'_G \) and connects \( \partial B_\delta(a'_n) \) and \( \partial B_\delta(b'_n) \). The third piece is a straight segment connects \( \partial B_\delta(b'_n) \) and \( b'_n \).

Now in Lemma 2.19, by taking \( 0 < \rho \ll \delta \) small and thus \( n \) large, we may assume that each of \( \hat{\phi}_n(\gamma) \) and \( \hat{\psi}_n(\gamma) \) is the union of three curve segments described as follows. The first piece is a curve segment which connects \( a' \) and \( \partial B_\delta(a'_n) \) and is contained in \( B_{2\delta}(a'_n) \). The second piece is some curve segment which connects \( \partial B_\delta(a'_n) \) and \( \partial B_\delta(b'_n) \), and is contained in the \( 2\rho \)-neighborhood of the corresponding second piece described as above. The third piece is a curve segment which connects \( \partial B_\delta(b'_n) \) and \( b' \) and is contained in \( B_{2\delta}(b'_n) \).

For an illustration of these curves, see Figure 6.

Now the homotopy is realized by two steps. In the first step, we can deform \( \hat{\phi}_n(\gamma) \) in \( \Delta - P'_G \) so that the first and the third pieces are still contained in \( B_{2\delta}(a'_n) \) and \( B_{2\delta}(b'_n) \), respectively, but the second piece coincide with the second piece of \( \hat{\phi}_n \). Then do the same thing for \( \hat{\psi}_n(\gamma) \). For an illustration of the resulted curves, see Figure 7.

Let us use \([A, B]\) and \([C, D]\) to denote the second pieces of \( \hat{\phi}_n(\gamma) \) and \( \hat{\psi}_n(\gamma) \), respectively. Since \( \hat{\phi}_n(\gamma) \) is homotopic to \( \hat{\psi}_n(\gamma) \), the curve segment \([C, D]\) can be deformed to \([A, B]\) in \( \Delta - P'_{G_n} \) so that \( C \) moves to \( A \) along \( \partial B_\delta(a'_n) \) and \( D \) moves to \( B \) along \( \partial B_\delta(b'_n) \), and moreover, the deformation can be taken such that it does not intersect the \( \delta/2 \)-neighborhood of \( P'_{G_n} \). Since \( P'_{G_n} \) is close to \( P_G \) as \( n \) is large, this deformation does not intersect \( P'_G \). Since

\[
B_{2\delta}(a'_n) \cap (P'_G - \{ a' \}) = B_{2\delta}(b'_n) \cap (P'_G - \{ b' \}) = \emptyset,
\]

Figure 7. The resulted curves after the first step.
the first piece and the third piece of $\tilde{\phi}_n(\gamma)$ can be deformed to the corresponding piece of $\tilde{\psi}_n(\gamma)$ in $B_{2\delta}(a'_n)$ and $B_{2\delta}(b'_n)$, respectively. It is not difficult to see that these deformations can be taken carefully so that they can be glued into a homotopy between $\tilde{\phi}_n(\gamma)$ and $\tilde{\psi}_n(\gamma)$ in $\Delta - P'_G$.

The second case can be treated in a similar way. We leave it to the reader. $\square$

2.5. Proof of Theorem A.

2.5.1. Realizing the combinatorics in the rotation disk. Let $G$ be the Blaschke product obtained in the last section. Since $G|\partial \Delta$ is an analytic critical circle homeomorphism with bounded type rotation number, by Herman-Swiatek’s theorem, $G|\partial \Delta = h \circ R_\theta \circ h^{-1}$ where $h : \partial \Delta \to \Delta$ is a quasi-symmetric homeomorphism with $h(1) = 1$. All we need to do now is to follow the standard procedure to do the quasiconformal surgery on $G$. There are many places where a detailed description of this surgery can be found (see for example, [19], [29] and [30]).

The first thing we need to take care of is the combinatorial structure of $f$ in the inside of the rotation disk, which is not reflected by the Blaschke product $G$ (see Remark 2.1). Recall that

$$X = \{ z \in \Omega_f \mid f^i(z) \in \Delta - \{0\} \text{ for some } i > 0 \}.$$  

We may assume that $X \neq \emptyset$, for otherwise, we just skip this step. For $z \in X$, let $i_z > 0$ be the least integer such that $f^{i_z}(z) \in \Delta$. Now we can extend $h : \partial \Delta \to \partial \Delta$ to a quasiconformal homeomorphism $H : \Delta \to \Delta$ by using Douady-Earle’s extension theorem [7]. By composing $H$ with an appropriate quasiconformal homeomorphism $\tau : \Delta \to \Delta$ with $\tau|\partial \Delta = id$, which is still
denoted by $H$, we may assume that $H(0) = 0$ and

$$H(f^iz(z)) = G^i((\psi(z))$$

for each $z \in X$ (see Figure 8).

2.5.2. Quasiconformal surgery. Define a modified Blaschke product as follows.

$$(35) \quad \hat{G}(z) = \begin{cases} G(z) & \text{for } |z| \geq 1, \\ H \circ R_\theta \circ H^{-1}(z) & \text{for } z \in \Delta. \end{cases}$$

**Lemma 2.21.** $\hat{G}$ is combinatorially equivalent to $f$ rel $P_f \cap \{\infty\}$.

**Proof.** Let $\hat{\phi}_n$ and $\hat{\psi}_n$ be the homeomorphisms obtained in §2.4. Let $\phi = \hat{\phi}_n$ and $\psi = \hat{\psi}_n$. By Lemma 2.19 and 2.20, $\phi$ and $\psi$ are isotopic to each other rel $P_f'$, and $G = \phi \circ F \circ \psi^{-1}$. Define

$$(36) \quad \omega_0(z) = \begin{cases} \phi(z) & \text{for } |z| \geq 1, \\ H(z) & \text{for } z \in \Delta. \end{cases}$$

Since $\hat{G}$ and $f$ have the same combinatorial structure on the outside of the unit disk, for $k = 1, 2, \cdots$, we can lift $\omega_{k-1}$ by the equation

$$\hat{G} \circ \omega_k = \omega_{k-1} \circ f$$

and get a sequence of quasiconformal homeomorphisms $\omega_n$. Note that $\omega_1 = \psi$ on the outside of $f^{-1}(\Delta)$. It follows that up to a homotopy, the only possible places where $\omega_{k-1}$ and $\omega_k$ are different are the components of $\bigcup_{i=1}^{\infty} f^{-i}(\Delta)$ which intersect $P_f$. Let $N = |P_f - \partial \Delta|$. It follows that for each $x \in P_f$, either the forward orbit of $x$ under $f$ is eventually finite, or $f^{N+1}(x) \in \Delta$. This implies if a component of $\bigcup_{i=1}^{\infty} f^{-i}(\Delta)$ intersects $P_f$, it must be one of the components of $f^{-N-1}(\Delta)$. On the other hand, it is easy to see that

$$\omega_{N+1} = \omega_{N+2}$$

on all the components of $f^{-N-1}(\Delta)$. It follows that $\omega_{N+1}$ and $\omega_{N+2}$ are combinatorially equivalent to each other rel $P_f \cup \{\infty\}$. The lemma follows.

Now let us define a $\hat{G}$-invariant complex structure $\mu$ as follows. Let

$$\mu(z) = \frac{(H^{-1})_z}{(H^{-1})_z}$$

for $z \in \Delta$. For $z \notin \Delta$, there are two cases. In the first case, the forward orbit of $z$ falls into the inside of the unit disk. Let $k > 0$ be the least integer such that $G^k(z) \in \Delta$. We define $\mu(z) = (G^k)^*(\mu(G^k(z)))$, that is, we pull back by $G^k$ the complex structure of $H^{-1}$ at $G^k(z)$ to $z$. In the second case, the forward orbit of $z$ is contained in the outside of the unit disk. In this case, we define $\mu(z) = 0$. By this way we get a $\hat{G}$-invariant complex structure $\mu(z)$.
on the whole Riemann sphere. Since \( \hat{G} \) is holomorphic outside the unit disk, it follows that
\[
\|\mu\|_\infty = \sup_{z \in \Delta} \left| \frac{(H^{-1})_z \bar{z}}{(H^{-1})_z} \right| < 1.
\]
By Ahlfors-Bers theorem, there is a quasiconformal homeomorphism \( \Phi : S^2 \to S^2 \) such that \( \mu_k = \mu \) and \( \Phi \) fixes 0, 1, and the infinity. Now let
\[
g = \Phi \circ \hat{G} \circ \Phi^{-1}.
\]
It follows that \( g \) is a rational map which has a Siegel disk centered at the origin. Let us denote the Siegel disk by \( D_g \). It follows that \( \partial D_g \) is the image of the unit circle under \( \Phi \), hence is a quasi-circle which passes through the critical point 1 of \( g \). This implies that \( g \in R_{\theta_1}^{geom} \). By Lemma 2.21 we have
\[
g = \Phi \circ h_{N+1} \circ f \circ h_{N+2}^{-1} \circ \Phi^{-1}.
\]
Note that \( h_{N+2}^{-1} \circ \Phi^{-1} | D_g : D_g \to \Delta \) is a holomorphic homeomorphism. Therefore, \( g \) realizes the topological branched covering map \( f \) in the sense of Definition 1.3. This completes the proof of Theorem A.

3. Combinatorial Rigidity of the Maps in \( R_{\theta}^{geom} \)

3.1. Blaschke Models for Maps in \( R_{\theta}^{geom} \). Let \( G \) be a Blaschke product such that \( G|\partial \Delta = h \circ R_{\theta} \circ h^{-1} \) where \( h : \partial \Delta \to \partial \Delta \) is a quasi-symmetric homeomorphism with \( h(1) = 1 \). Let \( H : \Delta \to \Delta \) be a quasiconformal extension of \( h \) to the unit disk. Let \( \hat{G} \) be the modified Blaschke product defined by (35). We say a Siegel rational map \( g \) is modeled by the Blaschke product \( G \) if there is a quasiconformal homeomorphism \( \phi : S^2 \to S^2 \) such that \( g = \phi^{-1} \circ \hat{G} \circ \phi \).

We have

**Lemma 3.1.** For each \( g \in R_{\theta}^{geom} \), there is a Blaschke product \( G \) which models \( g \).

**Proof.** Let \( D_g \) be the Siegel disk of \( g \) and \( \psi : D_g \to \Delta \) be the holomorphic homeomorphism which conjugates \( g|D_g \) to the rigid rotation \( R_{\theta} : \Delta \to \Delta \). Since \( \partial D_g \) is a quasi-circle, we can extend \( \psi \) to be a quasiconformal homeomorphism of the Riemann sphere: \( S^2 \to S^2 \). Then \( f = \psi \circ g \circ \psi^{-1} \in R_{\theta}^{top} \) is realized by \( g \). Since \( g \) has no Thurston obstructions outside the Siegel disk, \( f \) has no Thurston obstructions outside its rotation disk. By Theorem A, there is a \( \hat{g} \in R_{\theta}^{geom} \) which realizes \( f \), and which can be modeled by some Blaschke product \( G \). That is to say, \( \hat{g} = \phi_1 \circ \hat{G} \circ \phi_1^{-1} \) where \( \phi_1 : S^2 \to S^2 \) is a quasiconformal homeomorphism and \( \hat{G} \) is the modified Blaschke product.

On the other hand, since \( g \) and \( \hat{g} \) both realize the topological branched covering map \( f \), it follows that \( g \) and \( \hat{g} \) are combinatorially equivalent. Since the boundaries of the Siegel disks \( D_g \) and \( D_{\hat{g}} \) are both quasi-circles, and since \( P_g - D_g \) and \( P_{\hat{g}} - D_{\hat{g}} \) are both finite sets, it follows that \( g \) and \( \hat{g} \) are quasiconformally equivalent. By a theorem of McMullen (see also [4]), \( g \) and \( \hat{g} \) are quasiconformally conjugate to each other. Let \( \phi_2 : S^2 \to S^2 \) be a
quasiconformal homeomorphism such that \( g = \phi_2 \circ \hat{g} \circ \phi_2^{-1} \). We thus get 
\( g = \phi_2 \circ \phi_1 \circ \hat{G} \circ \phi_1^{-1} \circ \phi_2^{-1} \). The lemma follows.

Let \( g \in \mathcal{R}_g^{geom} \) and \( D_g \) be the Siegel disk. Assume that \( J_g \) has positive measure. Since \( \partial D_g \) is a quasi-circle, it follows that \( \bigcup_{k=0}^{\infty} g^{-k}(\partial D_g) \) is a zero measure set. Let \( x_0 \) be a Lebesgue point of \( J_g \). By Proposition 1.14[15], \( w(x_0) \subset P_g' \) where \( w(x_0) \) is the \( w \)-limit set of \( x_0 \) and \( P_g' \) is the derived set of \( P_g \). Since \( P_g \setminus \partial D_g \) is a finite set, it follows that \( g^k(x_0) \to \partial D_g \) as \( k \to \infty \). By Lemma 5.1, \( g \) is modeled by a Blaschke product \( G \). That is to say, there is a quasiconformal homeomorphism \( \phi: \mathbb{S}^2 \to \mathbb{S}^2 \) such that \( g = \phi \circ \hat{G} \circ \phi^{-1} \). Let \( z_0 = \phi^{-1}(x_0) \), and

\[
J_{\hat{G}} = J_G - \bigcup_{k=0}^{\infty} G^{-k}(\Delta).
\]

It follows that \( J_{\hat{G}} = \phi^{-1}(J_g) \) is a set of positive measure. Since quasiconformal maps preserve zero-measure sets, we have

**Lemma 3.2.** \( z_0 \) is Lebesgue point of \( J_{\hat{G}} = \bigcup_{k=0}^{\infty} G^{-k}(\partial \Delta) \), and \( G^k(z_0) \to \partial \Delta \) as \( k \to \infty \).

### 3.2. Contraction Regions of \( G^{-1} \)

Let \( c \in \partial \Delta \cap \Omega_G \) and \( v = G(c) \). Suppose that the local degree of \( G \) at \( c \) is \( 2m + 1 \) for some integer \( m \geq 1 \). For \( \delta > 0 \) small, denote \( U_\delta(v) = B_\delta(v) \cap \{ |z| > 1 \} \). Then there are exactly \( m + 1 \) inverse branches of \( G \) which map \( U_\delta(c) \) to \( m + 1 \) domains which are attached to \( c \) from the outside of the unit disk. In this section, we will show that for each \( c \in \Omega_G \cap \partial \Delta \), there exists a region \( W_v \subset U_\delta(v) \) which is attached to the critical value \( v \), such that when restricted on \( W_v \), all these \( m + 1 \) inverse branches of \( G \) strictly contract the hyperbolic metric on some appropriate Riemann surface.

Let \( \Omega_* = \mathbb{P}^1 \setminus (\Delta \cup P_G) \) and \( \Omega^* = \mathbb{P}^1 \setminus (\Delta \cup (G^{-1}(\Delta \cup P_G))) \). Note that \( \Omega^* \) may not be connected, and in that case, each component of \( \Omega^* \) is a hyperbolic Riemann surface. We use \( d\rho_* = \lambda_{\Omega_*} |dz| \) to denote the hyperbolic metric of \( \Omega_* \). To save the symbols, we use the same notation \( \Omega^* \) to denote the component with which we are concerned, and \( d\rho^* = \lambda_{\Omega^*} |dz| \) to denote the hyperbolic metric on that component. It follows that \( G: \Omega^* \to \Omega_* \) is a holomorphic covering map.

Let \( r > 0 \) be small enough and \( B_r(c) \) be the disk centered at \( c \) with radius \( r \). Then there are exactly \( m + 1 \) domains which are contained in

\[
B_r(c) \cap \{ |z| > 1 \},
\]

and which are mapped to the outside of the unit disk. Each of these domains is attached to \( c \). Moreover, for each of such domains, the boundary of the domain has an inner angle \( \pi/(2m + 1) \) at \( c \). Take \( 0 < \epsilon < 1/(4m + 2) \). Let \( R \) and \( L \) be the two rays starting from \( c \) such that the angles between \( \partial \Delta \) and \( R \), \( \partial \Delta \) and \( L \), are both equal to \( \epsilon \pi \). Let \( S^*_c \) be the cone spanned by \( R \) and \( L \)
which is attached to $c$ from the outside of the unit disk (see Figure 9, where $m = 2$). Set

$$
\Omega^c_{\epsilon, r} = S^c_{\epsilon} \cap \Omega^* \cap B_r(c).
$$

The following lemma says that on $\Omega^c_{\epsilon, r}$, $G$ strictly increases the hyperbolic metric in $\Omega_*$. The lemma is a general version of Lemma 1.11 in [Pe], Lemma 3.3.

**Lemma 3.3.** There is a $\delta > 0$ which depends only on $\epsilon$ such that for all $r > 0$ small enough and any $c \in \partial \Delta \cap \Omega_G$, we have

$$
\lambda_{\Omega_*}(G(x)) |G'(x)| \geq (1 + \delta) \lambda_{\Omega_*}(x)
$$

where $x$ is an arbitrary point in $\Omega^c_{\epsilon, r}$.

**Proof.** Assume that $r > 0$ is small. Take any point $x \in \Omega^c_{\epsilon, r}$. Note that $\Omega^c_{\epsilon, r}$ may not be connected. We need only to consider the case that $x$ lies in a component which has part of its boundary on $R$ or $L$, for in the other cases, $G^{-1}(\partial \Delta)$ does much more contributions to the hyperbolic density function $\lambda_{\Omega_*}$, and therefore the value $\delta$ can actually be made bigger. This will be clear from the following proof.

Since $G : \Omega^* \to \Omega_*$ is a holomorphic covering map, we have

$$
\lambda_{\Omega_*}(G(x)) |G'(x)| = \lambda_{\Omega_*}(x).
$$

So it is sufficient to prove that

$$
\lambda_{\Omega_*}(x) / \lambda_{\Omega_*}(x) \geq 1 + \delta.
$$
Since $r$ is small, when viewed from the point $x$, $\Omega^*$ is approximately an angle domain near the vertex $c$, with angle $\alpha \pi$, where $\alpha = 1/(2m + 1)$. By taking an appropriate coordinate system, we may write $x = c + \eta e^{i\lambda \pi}$ where $\epsilon < \lambda < \alpha < 1$, and $0 < \eta < r$. Thus we get

$$\lambda_{\Omega}(x) \approx \frac{1}{\alpha} \frac{1}{\eta \sin \frac{1}{\alpha} \lambda \pi} = \frac{1}{\eta \sin \frac{1}{\alpha} \lambda \pi}. $$

On the other hand, when viewed from $x$, $\Omega^*$ is approximately the half plane, therefore,

$$\lambda_{\Omega^*}(x) \approx \frac{1}{\eta \sin \lambda \pi}. $$

This gives us

$$\lambda_{\Omega^*}(x)/\lambda_{\Omega}(x) \approx \frac{\sin \alpha \pi}{\alpha \sin \frac{1}{\alpha} \lambda \pi} > \frac{\sin \epsilon \pi}{\alpha \sin \frac{1}{\alpha} \epsilon \pi} > 1. $$

### 3.3. Closed Half Hyperbolic Neighborhood

Let $I = [a, b] \subset \mathbb{R}$ be an interval segment. Denote $\mathbb{C}_I = \mathbb{C} - (\mathbb{R} - I)$. For a given $d > 0$, the hyperbolic neighborhood of the interval $I$ in the slit plane $\mathbb{C}_I$ is defined to be the set which consists of all the points $x$ such that $d_{\mathbb{C}_I}(x, I) < d$, where $d_{\mathbb{C}_I}$ denotes the hyperbolic distance in $\mathbb{C}_I$. Let us use $U_d(I)$ to denote this hyperbolic neighborhood. It is known that the set $U_d(I)$ is a domain bounded by two Euclidean arcs which are symmetric about the real line. The exterior angle between the Euclidean arc and the interval $I$ is uniquely determined by $d$, and let us denote this angle by $\alpha(d)$ (for an explicit formula of $\alpha(d)$, see [28]). Such an object was first introduced by Sullivan to complex dynamics and now becomes a popular tool in this area.

Now let us adapt this object so that it is suitable for our situation. For each arc segment $I \subset \partial \Delta$, Let

$$\Omega_I = \mathbb{P}^1 - (P_G - I).$$

For any two points $x, y \in \Omega_I$, let $d_{\Omega_I}(x, y)$ denote the distance between $x$ and $y$ with respect to the hyperbolic metric on $\Omega_I$. Let

$$H_d(I) = \{ z \in \Omega_I | d_{\Omega_I}(z, I) \leq d, \text{ and } |z| \geq 1 \}. $$

where $d_{\Omega_I}(x, y)$ is the hyperbolic distance between $x$ and $y$ in $\Omega_I$.

Let $A_\alpha(I) \subset \{ z : |z| > 1 \}$ denote the arc segment of some Euclidean circle such that it has the same ending points as $I$ and such that the exterior angle between $A_\alpha(I)$ and $I$ is equal to $\alpha$. Let

$$\gamma_d(I) = \partial H_d(I) - I. $$

Note that $\gamma_d(I)$ may not be an arc segment of some Euclidean circle. But since $P_G - \partial \Delta$ is a finite set, it follows that when $|I|$ is small enough, the set $P_G - \partial \Delta$ will do very little contribution to the hyperbolic density of the points near the arc $I$, and thus $\gamma_d(I)$ is like the Euclidean arc $A_{\omega_d}(I)$. Let us formulate this as the next lemma and leave the proof to the reader.
Lemma 3.4. For any $\delta > 0$, there is an $\epsilon > 0$, such that when $|I| < \epsilon$, $\gamma_d(I)$ lies in between the two Euclidean arcs $A_{\alpha(d)+\delta}(I)$ and $A_{\alpha(d)-\delta}(I)$.

Let us fix $d > 0$ throughout the following discussions. Let
\[ \Lambda_G = \{ c \in \Omega_G - \partial \Delta \mid \exists k \geq 1 \text{ such that } G^k(c) \in \partial \Delta \}. \]
For any $c \in \Lambda_G$, let $k(c) \geq 1$ be the least integer such that $G^{k(c)}(c) \in \partial \Delta$. Let
\[ X_G = \{ G^{k(c)}(c) \mid c \in \Lambda_G \}. \]

Lemma 3.5. Let $J \subset \partial \Delta$ with $J \cap \Omega_G = \emptyset$. Let $I = G(J)$. Suppose that $I \cap X_G = \emptyset$. Then $V \subset H_d(J)$ where $V$ is the connected component of $G^{-1}(H_d(I))$ which is attached to $J$ from the outside of the unit disk.

Proof. Let $\tilde{\Omega}_J = \mathbb{P}^1 - G^{-1}(P_G - I)$. By the assumption, it follows that $G : \tilde{\Omega}_J \to \Omega_I$ is a holomorphic covering map. For any two points $x, y \in \tilde{\Omega}_J$, let $d_{\tilde{\Omega}_J}(x, y)$ be the distance between $x$ and $y$ with respect to the hyperbolic metric on $\tilde{\Omega}_J$. It follows that
\[ V \subset \{ z \mid d_{\tilde{\Omega}_J}(z, J) \leq d, |z| \geq 1 \}. \]
Since $I \cap X_G = \emptyset$, we have $\tilde{\Omega}_J \subset \Omega_J$, and therefore $d_{\Omega_J}(x, y) < d_{\tilde{\Omega}_J}(x, y)$ for any two points in $\tilde{\Omega}_J$. This implies that
\[ \{ z \mid d_{\tilde{\Omega}_J}(z, J) \leq d, |z| \geq 1 \} \subset H_d(J). \]
The lemma follows. \qed

Lemma 3.6. Let $d' > d$. Then there is a $\ell > 0$ such that for every $J \subset \partial \Delta$ satisfying
1. $|J| < \ell$,
2. \( J \cap \Omega_G = \emptyset \),
3. \( I \cap X_G \neq \emptyset \) where \( I = G(J) \),
we have \( V \subset H_d(J) \) where \( V \) is the connected component of \( G^{-1}(H_d(I)) \) which is attached to \( J \) from the outside of the unit disk.

**Proof.** Since \( P_G - \partial \Delta \) is a finite set, there is a \( \delta > 0 \) such that 
\[
(B_\delta(x) - \partial \Delta) \cap P_G = \emptyset.
\]
where \( x \) is the mid-point of \( I \). Let 
\[
\tilde{\Sigma} = B_\delta(x) - (\partial \Delta - I).
\]
It is clear that \( \tilde{\Sigma} \) is a simply connected domain. Since \( J \cap \Omega_G = \emptyset \) and \( \tilde{\Sigma} - I \) contains no critical value of \( G \), there is an inverse branch of \( G \), say, \( \Phi \), defined on \( \tilde{\Sigma} \) which maps \( \tilde{\Sigma} \) to some domain containing \( J \). Let 
\[
\Sigma^* = \Phi(\tilde{\Sigma}).
\]
See Figure 10 for an illustration. Let \( d_\Sigma(\cdot) \) and \( d_{\Sigma^*}(\cdot) \) denote the hyperbolic distance in \( \tilde{\Sigma} \) and \( \Sigma^* \), respectively. Define 
\[
\tilde{H}_d(I) = \{ z \in \tilde{\Sigma} \mid d_\Sigma(z, I) \leq d' \text{ and } |z| \geq 1 \}
\]
and 
\[
H^*_d(I) = \{ z \in \Sigma^* \mid d_{\Sigma^*}(z, I) \leq d \text{ and } |z| \geq 1 \}.
\]
Since for \( I \) small, when viewed from the points near \( I \), the difference between \( \Omega_I \) and \( \tilde{\Sigma} \) is small, it follows that 
\[
H_d(I) \subset \tilde{H}_d(I)
\]
provided that \( |I| \) is small enough. Since \( \Phi : \tilde{\Sigma} \to \Sigma^* \) is a holomorphic isomorphism and \( \Sigma^* \subset \Omega_J \), we have 
\[
V = \Phi(H_d(I)) \subset \Phi(\tilde{H}_d(I)) = H^*_d(J) \subset H_d(J).
\]
The lemma follows. \( \Box \)

### 3.4. Minimal Neighborhoods

Let \( z_0 \) be the point in Lemma 3.2. Let 
\[
z_k = G^k(z_0) \text{ for } k \geq 1.
\]
In \( \S 3.2 \), we show that there exist regions which are attached to the critical values on the unit circle, such that in these regions, \( G^{-1} \) strictly contracts the hyperbolic metric in \( \Omega_* \). Our next step is to show that, there will be some infinite subsequence of \( \{ z_k \} \) which passes through these contraction regions. To prove the existence of such infinite subsequence, we will introduce an object, called *minimal neighborhood*.

Recall that \( G|\partial \Delta = h \circ R_0 \circ h^{-1} \), where \( h : \partial \Delta \to \partial \Delta \) is a quasi-symmetric homeomorphism with \( h(1) = 1 \). Now for each arc \( I \subset \partial \Delta \), we define \( \sigma(I) = |(h^{-1}(I))| \). It follows from the definition that \( \sigma \) is \( G \)-invariant.

**Lemma 3.7.** Let \( \delta > 0 \) be small. Then there exists a \( \tau > 0 \) such that for any two arcs \( I, J \subset \partial \Delta \) with \( I \cap J \neq \emptyset \) and \( |J| < \tau |I| \), we have \( \sigma(J) < \delta \sigma(I) \).
The proof is easy and we shall leave the details to the reader.

**Lemma 3.8.** Let $\delta > 0$ be small. Then there is a $\rho > 0$ and an $\epsilon > 0$ such that for any $I \subset \partial \Delta$ with $|I| < \epsilon$ and any $x \in H_d(I)$ and $y \in I$, if the angle between the segment $[x, y]$ and $\partial \Delta$ is less than $\rho$, then there is an arc $J \subset \partial \Delta$ such that

1. $x \in H_d(J)$, and
2. $\sigma(J) < \delta \sigma(I)$.

**Proof.** We may consider the worst case, that is, $x \in \gamma_d(I)$. See Figure 11 for an illustration. If $\epsilon$ is small, by Lemma 3.4, $\gamma_d(I)$ lies in between two Euclidean arcs which have the same ending points as the arc $I$. So if $\rho$ is small enough, $x$ must be close to one of the end points of $I$. On the other hand, by Lemma 3.4 again, if $x \in \gamma_d(I)$ is close to one of the end points of $I$, say $a$, it must be contained in $H_d(J)$ for some $J$ with

$$|J| = O(d(x, a))$$

and $a$ being the middle point of $J$. Clearly, as $\rho \to 0$,

$$d(x, a)/|I| \to 0.$$

It follows that by taking $\rho$ small, $|J|/|I|$ can be as small as wanted, and hence by Lemma 3.7, $\sigma(J) < \delta \sigma(I)$. Lemma 3.8 follows.

Let $k \geq 0$ be an integer. Define

$$\Phi_k = \{ I \subset \partial \Delta \mid z_k \in H_d(I) \}.$$ 

and

$$l_k = \inf \{ \sigma(I) \mid I \in \Phi_k \}.$$

**Remark 3.1.** Note that by taking a limit of a convergent subsequence of the intervals, the value $l_k$ in (40) can be obtained by some interval $I \in \Phi_k$. 
Since $z_k \to \partial \Delta$, we have

**Lemma 3.9.** $l_k \to 0$ as $k \to \infty$.

**Definition 3.1.** For each $n$, we define $0 \leq m(n) \leq n$ to be the least integer such that

$$l_{m(n)} = \min\{l_k \mid 0 \leq k \leq n\}. \quad (41)$$

The following two lemmas follow directly from the definition of $m(n)$ and Lemma 3.9.

**Lemma 3.10.** $m(n) \leq m(n + 1)$, and $m(n) \to \infty$ as $n \to \infty$.

**Lemma 3.11.** For each $m(n)$, there is an open arc, say $I_{m(n)} \subset \partial \Delta$, which may not be unique, such that $\sigma(I_{m(n)}) = l_{m(n)}$ and $z_{m(n)} \in H_d(I_{m(n)})$.

**Proof.** As mentioned in Remark 3.1, by taking a convergent subsequence of the intervals, we can get an interval $I \subset \partial \Delta$ such that $\sigma(I) = l_{m(n)}$ and $I \in \Phi_{m(n)}$. Let us denote this interval by $I_{m(n)}$. By the minimal property of $I_{m(n)}$, it follows that $d_{\Omega_{m(n)}}(z_{m(n)}, I_{m(n)}) = d$ and therefore, $z_{m(n)} \in \gamma_d(I_{m(n)}) \subset H_d(I_{m(n)})$. \hfill $\square$

We call the region $H_d(I_{m(n)})$ in Lemma 3.11 a minimal neighborhood associated to the number $m(n)$. From the proof of Lemma 3.11, $z_{m(n)} \in \gamma_d(I_{m(n)})$.

**Lemma 3.12.** There exist $\epsilon > 0$ and $r > 0$ and an increasing sequence of integers $\{\kappa(n)\}$ such that for all $n$ large enough, $z_{\kappa(n)} \in \Omega_{\epsilon, r}$ for some $c \in \Omega_G \cap \partial \Delta$.

The idea of the proof is based on the following fact: for $r > 0$ small and $c \in \Omega_G \cap \partial \Delta$, if $y, y' \in B_r(c)$ such that $G(y) = G(y')$, then $\alpha$ and $\alpha'$ cannot both be small, where $\alpha$ is the angle between $\partial \Delta$ and the straight segment $[c, y]$, and $\alpha'$ is the angle between $\partial \Delta$ and the straight segment $[c, y']$.

**Proof.** To fixed the ideas, let $\epsilon > 0$ and $r > 0$ be two small numbers and $N$ be a large integer. By taking $N$ large, we may assume that when $n \geq N$, the interior of the minimal neighborhood $H_d(I_{m(n)})$ does not intersect $P_G$. We may also assume that $I_{m(n)}$ does not contain any critical point of $G$. Since otherwise, by Lemma 3.11 and the minimal property of $m(n)$, it follows that the angle between the straight segment $[c, z_{m(n)}]$ and $\partial \Delta$ has a uniform positive lower bound, and this will imply the lemma. Take $n \geq N$. Let

$$M = \min\{k \geq 1 \mid \exists c \in \Omega_G \cap \partial \Delta \text{ such that } G^k(c) \in I_{m(n)}\}.$$

For $0 \leq l \leq M$, let $J_l \subset \partial \Delta$ be the arc segment such that

$$G^l(J_l) = I_{m(n)}.$$

Since $\theta$ is of bounded type, there exists a number $K$ which depends only on $\theta$ and $|X_G|$ such that the number of the intervals $J_l$, $1 \leq l \leq M$, which intersect $X_G$ is not more than $K$ (Note that $M$ can be arbitrarily large if $I_{m(n)}$ is small). Moreover, by taking $n$ large enough and thus $I_{m(n)}$ is small, we may
also assume that each $J_l$, $0 \leq l \leq M$, contains at most one of the points in $X_G$. For $0 \leq l \leq M - 1$, let $V_l$ denote the pull back of $H_d(I_{m(n)})$ by $G^l$ along the orbit $\{J_t\}_{0 \leq t \leq M - 1}$. It follows that for any $d' > d$, there is an $\eta > 0$, such that

$$V_l \subset H_{d'}(J_l)$$

for all $0 \leq l \leq M - 1$ provided that $|I_{m(n)}| < \eta$. This is because the number of $J_l$, $0 \leq l \leq M$, which intersect $X_G$, is not more than $K$, and thus we need only apply Lemma 3.6 at most $K$ times, and for all other $J_l$, we can apply Lemma 3.5 in which case $d$ is not increased. From now on, let us fix a $d' > d$ and suppose that $n$ is large enough such that $|I_{m(n)}| < \eta$.

Now there are two cases. In the first case, there is some $1 \leq k(n) \leq M - 1$ such that

$$z_{m(n)} - l \in H_{d'}(J_l)$$

holds for all $0 \leq l \leq k(n) - 1$, but

$$z_{m(n) - k(n)} \notin H_{d'}(J_{k(n)}).$$

In the second case,

$$z_{m(n) - 1} \in H_{d'}(J_l)$$

for all $0 \leq l \leq M - 1$.

In the first case, let $J_{k(n)} = [a, b]$ where $b$ is such that $|a - c| < |b - c|$. Let $z' \in H_{d'}(J_{k(n)})$ be such that

$$G(z') = G(z_{m(n) - k(n)}) = z_{m(n) - k(n) + 1} \in H_{d'}(J_{k(n) - 1}).$$

Since

$$\sigma(J_{k(n) - 1}) = \sigma(I_{m(n)}) \text{ and } z_{m(n) - k(n) + 1} \in H_{d'}(J_{k(n) - 1}),$$

Figure 12. $|a - b| \ll |c - b|$
Figure 13. \( \sigma(J') \ll \sigma(J) \)

it follows that when \( n \) is large, \( z_{m(n)-k(n)+1} \) is near \( \partial \Delta \) and thus \( m(n)-k(n)+1 \) is large. In particular, \( z_{m(n)-k(n)} \) is near \( \partial \Delta \). Since the restriction of \( G \) on \( \partial \Delta \) is a homeomorphism, it follows from (42) that there is some \( c \in \Omega_G \cap \partial \Delta \) such that both \( z' \) and \( z_{m(n)-k(n)} \) belong to a small neighborhood of \( c \). Let \( \tau(n) = m(n) - k(n) \). The first case is now separated into two subcases (i) and (ii).

In subcase (i), \( |a - b| \) is small compared with \( |b - c| \). Then the angle between the straight segment \([c, z']\) and the unit circle is small. It follows that the angle between the straight segment \([c, z_{\tau(n)}]\) and the unit circle can not be small (in this case, it is at least about \( \pi/(2m+1) \) where \( 2m+1 \geq 3 \) is the degree of \( G \) at \( c \), see Figure 12).

In subcase (ii), there is a uniform \( 0 < k < 1 \) such that \( |a - b| > k|c - b| \). See Figure 13 for an illustration. Since \( G(z) \) is like \( G(c) + \mu(z - c)^{2m+1} \) in \( B_r(c) \), where \( \mu \neq 0 \) is some constant, it follows that when \( r \) is small,

\[
|c - z_{\tau(n)}| \asymp |c - z'|.
\]

Note that if the angle between the straight segment \([c, z_{\tau(n)}]\) and the unit circle were small, there would be an arc segment \( J' \subset \partial \Delta \) such that \( z_{\tau(n)} \in H_d(J') \), and

\[
|J'| \ll |c - z_{\tau(n)}| \asymp |c - z'| \leq |c - b| \leq |J|.
\]

From (43) and Lemma 3.7, we have

\[
\sigma(J') \ll \sigma(J) = \sigma(I_{m(n)})
\]

provided that \( |J'| \) is small enough. But this contradicts with the minimal property of \( m(n) \).
In the second case, $J_{M-1}$ contains a critical value $v = G(c)$ with $c \in \Omega_G \cap \partial \Delta$, and $z_{m(n)-M+1} \in H_d(J_{M-1})$. As in the first case, as $n$ is large, $\sigma(J_{M-1}) = \sigma(I_{m(n)})$ is small, and therefore, $z_{m(n)-M+1}$ is close to $\partial \Delta$. It follows that $m(n) - M + 1$ is large provided $n$ is large. Let $\tau(n) = m(n) - M$.

Now we claim that the angle between the segment of $[z_{\tau(n)+1}, v]$ and the unit circle $\partial \Delta$ must have a positive lower bound, which is independent of $n$. In fact, if this were not true, by Lemma 3.8, we would have a $J \subset \partial \Delta$ small such that

$$z_{\tau(n)+1} \in H_d(J)$$

but

$$\sigma(J) < \sigma(J_{M-1}) = \sigma(I_{m(n)}).$$

But this contradicts with the minimal property of $m(n)$ and the claim is proved. From the claim, it follows that $[z_{\tau(n)}, c]$ and the unit circle has a uniform lower bound $\epsilon > 0$, which is independent of $n$.

Now we get a sequence of integers $\tau(n)$ such that the angle between $\partial \Delta$ and $[c, z_{\tau(n)}]$ has a positive lower bound independent of $n$. Since $\{\tau(n)\}$ is unbounded by the proof above, we may assume that $\tau(n)$ is an increasing sequence by taking a subsequence. This completes the proof of the lemma.

3.5. Pull back argument. For a subset $X \subset \Omega$, we use $\text{Diam}_h(X)$ to denote the diameter of $X$ with respect to the hyperbolic metric $d_\ast$ of $\Omega$.

For any subset $E$ of the complex plane, we use $\text{area}(E)$, and $\text{Diam}(E)$ to denote the area and the diameter of $E$ with respect to the Euclidean metric respectively. Let $\{\tau(n)\}$ be the sequence obtained in Lemma 3.12.

**Lemma 3.13.** There exist $K_1, K_2, K_3 > 0$ independent of $n$, such that for every $n$ large enough, there are open simply connected domains $C^n \subset B^n \subset A^n \subset \Omega$, satisfying

1. $z_{\tau(n)} \in B^n$,
2. $G(C^n) \subset \Delta$,
3. $\text{mod}(A^n - B^n) \geq K_1$,
4. $\text{area}(C^n)/\text{Diam}(B^n)^2 \geq K_2$,
5. $d_\ast(A^n) \leq K_3$.

**Proof.** Assume $d(z_{\tau(n)}, \partial \Delta)$ is small enough. Let $2m + 1$ be the local degree of $G$ at $c$ where $m \geq 1$ is some integer. As in the proof of Lemma 3.12 for $r > 0$ small, there are $m + 1$ domains which are attached to $c$ and contained in $B_r(c)$ and which are mapped into the outside of the unit disk. There are two of such domains which are tangent with the unit disk at $c$. To fix the discussions, let us assume that $z_{\tau(n)}$ lies in one of these two domains, say $U$. All the other cases can be treated in the same way. We also know that there are $m$ domains, which are contained in $B_r(c)$ and attached to $c$ from the outside of the unit disk, and which are mapped into the inside of the unit disk.

Let $V$ be one of these domains such that $V$ is adjacent to $U$. Let $L$ and $R$ be
the two half rays which are tangent with $U$ at $c$. In a small neighborhood of $z_{\tau(n)}$, $\partial U$ is approximately the union of two straight segments starting from $c$ and which lie on $R$ and $L$, respectively. To simplify the notation, we still use $R$ and $L$ to denote them. Suppose that the angle between $R$ and $L$ is $\alpha \pi$. Let $T$ be the straight segment between $R$ and $L$ and which is on the boundary of $\Omega^c_{\epsilon,r}$ (see Figure 14). By assumption, the angle between $T$ and $L$ is $\epsilon \pi$ where $0 < \epsilon < \alpha < \frac{1}{2}$. For convenience, we use the polar coordinate system formed by $(c, L)$. by Lemma 3.12, $z_{\tau(n)} \in \Omega^c_{\epsilon,r}$, therefore, we have

$$z_{\tau(n)} = r_0 e^{\lambda \pi},$$

for some $\epsilon < \lambda < \alpha$ and $0 < r_0 < r$. Now let $A^n$ be the region bounded by

$$\frac{1}{4} \epsilon \pi \leq \theta \leq (\alpha + \epsilon) \pi,$$

and

$$r_0/2 \leq r \leq 3r_0/2.$$

Let $B^n$ be the region bounded by

$$\frac{1}{2} \epsilon \pi \leq \theta \leq (\alpha + \epsilon) \pi,$$

and

$$3r_0/4 \leq r \leq 5r_0/4.$$

Let $C^n = B \cap V$. It is not difficult to check that for the domains defined above, there are constants $K_i > 0, 1 \leq i \leq 3$ such that the conditions in the Lemma are all satisfied. We leave the details to the reader. □

Let us prove Theorem B now. By taking $n$ large enough, we may assume that $A^n \cap P_G = \emptyset$. Now let us consider the pull back of $(A^n, B^n, C^n, z_{\tau(n)})$
From Lemma 3.3 and (45), it follows that there is a \( K \) such that for every \( \{z_k\} \) for some \( c \), there exist two quasiconformal homeomorphisms of the sphere \( f \) such that for all \( n \)

\[
A_{\tau(n)}(A^n) \subseteq K_3
\]

where \( K_3 \) is the constant in (5) of Lemma 3.12.

By Lemma 3.12 there is an \( N_0 > 0 \) such that when \( k > N_0 \), \( z_{\tau(k)} \in \Omega_{c,r} \) for some \( c \in \Omega_{c,r} \). Since \( z_k \to \partial \Omega \) and \( \tau(k) \to \infty \), by (44), there is an \( N_1 \) and an \( 0 < \eta < 1 \) such that for all \( k > N_1 \),

\[
A^n_k \subseteq \Omega_{\eta r,r}.
\]

From Lemma 3.3 and (45), it follows that there is a \( \delta > 0 \) such that for every \( k \) with \( \max\{N_0, N_1\} \leq k \leq n \),

\[
Diam_{\Omega_{\tau(k)}}(A^n_{\tau(k)}) \leq (1 - \delta)Diam_{\Omega_{\tau(k+1)}}(A^n_{\tau(k+1)}).
\]

Since \( \{\tau(k)\} \) is an infinite sequence, by (44), it follows that as \( n \to \infty \), \( Diam(A^n_k) \to 0 \) and hence \( Diam(B^n_k) \to 0 \) as \( n \to \infty \). On the other hand, by (3), (4) of Lemma 3.13 and Koebe’s distortion theorem, we get a constant \( 0 < C < \infty \) such that for all \( n \) large enough, the following conditions hold:

1. \( z_0 \in B^n_0 \), and
2. \( C^n_0 \subseteq B^n_0 \), and
3. \( areaC^n_0 \geq C diam(B^n_0)^2 \).

By (2) of Lemma 3.13 \( C^n_0 \subseteq \bigcup_{k=0}^{\infty} G^{-k}(\Omega) \). This implies that \( z_0 \) is not a Lebesgue point of \( J_{\Omega} \), which is a contradiction. The proof of the zero measure statement of Theorem B is completed.

Now let us prove the rigidity statement of Theorem B.

**Lemma 3.14.** Let \( f \in R^{top}_g \) and suppose that \( f \) has no Thurston obstructions outside the rotation disk, and is realized by two maps \( g, h \in R^{com}_g \). Then there exist two quasiconformal homeomorphisms of the sphere \( \phi_1 \) and \( \phi_2 \) such that

1. \( \phi_1 \) and \( \phi_2 \) are combinatorially equivalent to each other rel \( P_g \), and
2. \( \phi_1[D_g] = \phi_2[D_g] \) are holomorphic on the Siegel disk, and
3. For each super-attracting periodic point \( x \) of \( g \), there is a neighborhood of \( x \), say \( U_x \), such that \( \phi_1[U_x] = \phi_2[U_x] \) are holomorphic, and
4. \( g = \phi_1^{-1} \circ h \circ \phi_2 \).

The proof is easy and we leave the details to the reader.

Now for \( k \geq 2 \), since \( g \) and \( h \) are combinatorially equivalent, we can lift \( \phi_k \) by the equation

\[
g = \phi_k^{-1} \circ h \circ \phi_{k+1}.
\]
and get $\phi_{k+1}$. In this way we get a sequence of quasiconformal homeomorphisms $\{\phi_k\}$ of the sphere such that $\phi_k|P_g = \phi_{k+1}|P_g$ for all $k \geq 1$. Let $\mu_k$ be the dilation of $\phi_k$. Since both $g$ and $h$ are rational maps, it follows that $\|\mu_k\|_\infty < K < 1$ where $K$ is some constant independent of $k$. Since any periodic Fatou component of $g$ must either be the Siegel disk, or a super-attracting periodic Fatou component, it follows that $\mu_k \to 0$ on the Fatou set of $g$. Since the Julia set of $g$ has zero Lebesgue measure, and $\phi_k|P_g = \phi_{k+1}|P_g$ for all $k \geq 1$, it follows that $\phi_k$ converges to the same Möbius map. We complete the proof of Theorem B.

4. Quadratic Rational Maps with Bounded Type Siegel Disks

4.1. Quadratic Siegel Rational Maps. Let $g$ be a quadratic rational map which has a bounded type Siegel disk. Up to a Möbius conjugation, we may assume that the center of the Siegel disk is at the origin and $g(\infty) = \infty$. Then $g$ has the following normalized form,

$$g(z) = \frac{az^2 + e^{2\pi i \theta}z}{bz + 1}$$

From Riemann-Huiwitz formula, it follows that any quadratic rational map has exactly two distinct critical points. Through a Möbius conjugation, we may further assume that 1 is one of the critical points of $g$, that is, $g'(1) = 0$. By a simple calculation, this is equivalent to

$$a(b + 2) + e^{2\pi i \theta} = 0$$

Let us denote the other critical point of $g$, which is different from 1, by $c_g$.

Lemma 4.1. Let $\Sigma$ be the space of all the normalized quadratic Siegel rational maps $g$ such that $g(0) = 0$, $g(\infty) = \infty$, and $g'(1) = 0$. Then the map $\rho : g \to c_g$ is a homeomorphism between $\Sigma$ and $\hat{\mathbb{C}} - \{0, 1, -1\}$.

Proof. Since $g'(0) = e^{2\pi i \theta}$ and the two critical points of $g$ must be distinct from each other, it follows that $c_g \neq 0, 1$. By a simple calculation, we get

$$g'(z) = \frac{abz^2 + 2az + e^{2\pi i \theta}}{(bz + 1)^2}.$$  

From (49) and $g'(1) = g'(c_g) = 0$, it follows that 1 and $c_g$ are the two roots of the quadratic polynomial equation

$$abz^2 + 2az + e^{2\pi i \theta} = 0.$$ 

This implies that $c_g \neq -1$. In fact, if $c_g = -1$, we have $2/b = -(1 + c_g) = 0$, and hence $b = \infty$. This is a contradiction.

Now for $c_g \neq 0, 1$, and $-1$, we can solve $a = -e^{2\pi i \theta}(1 + c_g)/2c_g$ and $b = -2/(1 + c_g)$. Therefore, $g$ is uniquely determined by $c_g$, and we have

$$g(z) = \frac{-e^{2\pi i \theta}(1 + c_g)^2z^2 + 2e^{2\pi i \theta}c_g(1 + c_g)z}{-4c_gz + 2c_g(1 + c_g)}.$$
In particular, as $c_\theta \to \infty$, $a \to -e^{2\pi i \theta}/2$, $b \to 0$ and hence $g(z) \to -e^{2\pi i \theta}z^2/2 + e^{2\pi i \theta}z = g_\infty(z) \in \Sigma$. The lemma follows. \hfill \Box

Now for each $c \in \hat{\mathbb{C}} - \{0,1,-1\}$, we use $g_c$ to denote the normalized quadratic Siegel rational map which has 1 and $c$ as its critical points.

4.2. Branched Covering Maps $f_t \in R^{\text{top}}_{\theta}$.

4.2.1. Branched covering maps $f_t$ and Siegel rational maps $g_c(t)$. In this section, we will construct a family of topological branched covering maps $f_t \in R^{\text{top}}_{\theta}, 0 < t < 2\pi$. This family of topological branched covering maps will provide models of a continuous family of quadratic Siegel rational maps $g_c(t) \in R^{\text{geom}}_{\theta}, 0 < t < 2\pi$, where $c(t), 0 < t < 2\pi$ is a continuous curve in the critical parameter plane. Later we will see that this curve plays a fundamental role in the proof of Theorem C.

**Definition of $f_t$.** For each $0 < t < 2\pi$, let $c \in \partial \Delta$ such that the angle spanned by 1 and $c$ is $t$. Let $\eta_1, \eta_2$ be two curve segments connecting 1 and $c$ as indicated in Figure 15. Let $D_1$ be the domain bounded by $\eta_1$ and the arc from 1 to $c$, anticlockwise, and $D_2$ denote the domain bounded by $\eta_1$ and $\eta_2$. Let $D_3$ denote the domain which contains the infinity and which is bounded by $\eta_2$ and the arc from $c$ to 1, anticlockwise. Let $f_t \in R^{\text{top}}_{\theta}$ be a topological branched covering map defined as follows: $(f_t|\Delta)(z) = e^{2\pi i \theta}z$, and $f_t : D_1 \to S^2 - \Delta, D_2 \to \Delta, D_3 \to S^2 - \Delta$ are all homeomorphisms. Moreover, $f_t(\infty) = \infty$. 
Definition of $\tilde{f}_{2\pi-t}$. For each $0 < t < 2\pi$, let $c \in \partial \Delta$ such that the angle spanned by $1$ and $c$ is $t$. Let $\eta_1, \eta_2$ be two curve segments connecting $1$ and $c$ as indicated in Figure 16. Let $D_1$ denote the domain bounded by $\eta_1$ and the arc from $1$ to $c$ anticlockwise. Let $D_2$ denote the domain bounded by $\eta_1$ and $\eta_2$. Let $D_3$ denote the domain bounded by $\eta_2$ and the arc from $c$ to $1$ anticlockwise. Define $\tilde{f}_{2\pi-t}$ as follows: $(\tilde{f}_{2\pi-t}|_{\Delta}\ z) = e^{2\pi i \theta}z$, and $\tilde{f}_{2\pi-t} : D_2 \to \Delta, D_1 \to S^2 - \Delta, D_3 \to S^2 - \Delta$ are all homeomorphisms. Moreover, $\tilde{f}_{2\pi-t}(\infty) = \infty$.

Since any simple closed curve $\gamma \subset S^2 - \Delta$ is peripheral, it follows that $f_t(\tilde{f}_t)$ has no Thurston obstructions outside the rotation disk $\Delta$ for all $0 < t < 2\pi$. By Theorem A and Theorem B, we have

Lemma 4.2. For each $0 < t < 2\pi$, there is a unique $c(t)(\tilde{c}(t)) \in \mathbb{C} - \{0, 1, -1\}$ such that $g_{c(t)}(g_{\tilde{c}(t)})$ realizes $f_t(\tilde{f}_t)$ in the sense that

$$f_t = \phi^{-1} \circ g_{c(t)} \circ \psi(\tilde{f}_t = \phi^{-1} \circ g_{\tilde{c}(t)} \circ \psi)$$

where $\phi$ and $\psi : S^2 \to S^2$ are homeomorphisms which fix $0, 1$, and the infinity, and are isotopic to each other rel $P_f$.

Inner angle between the two critical points. Let $g_c \in R^\text{geom}_\theta$ and $D$ be the Siegel disk of $g_c$ centered at the origin such that $\partial D$ passes through both the two critical points $1$ and $c$. Let $\phi : D \to \Delta$ be the holomorphic map which conjugates $g_c|D$ to the rigid rotation $R_\theta$ on $\Delta$. Since $\partial D$ is a quasi-circle, it follows that $\phi$ can be homeomorphically extended to $\partial D \to \partial \Delta$. We use $A_c$
to denote the angle from \( \phi(1) \) to \( \phi(c) \) anticlockwise. We call it the *inner angle* between 1 and \( c \).

**Remark 4.1.** Let \( g_c \in R^\text{geom}_g \) such that both of the critical points of \( g_c \) are on the boundary of the Siegel disk. Suppose that the inner angle between 1 and \( c \) is \( t \). Let \( D_c \) be the Siegel disk of \( g_c \). Then \( g_c \) realizes \( f_t \) for some \( 0 < t < 2\pi \) in the sense of Lemma 4.2, if and only if the boundary of the bounded component of \( g_c^{-1}(S^2 - \overline{D_c}) \) contains the part of the boundary of \( D_c \), which connects 1 to \( c \) anticlockwise. By contrary, \( g_c \) realizes \( \tilde{f}_t \) for some \( 0 < t < 2\pi \) in the sense of Lemma 4.2, if and only if the boundary of the bounded component of \( g_c^{-1}(S^2 - \overline{D_c}) \) contains the boundary arc of the Siegel disk, which connects 1 to \( c \) clockwise.

### 4.2.2. Some basic facts about \( f_t \)

It is useful to find the Möbius transformations which conjugate a normalized quadratic Siegel rational map to another normalized one. Let \( g_c \) be a normalized quadratic Siegel rational map given by (47). There are two cases.

In the first case, \( g_c \) has exactly two fixed points 0 and \( \infty \). By a simple calculation, this is equivalent to that \( c \) is one of the two roots of the following equation,

\[
    c^2 + (4e^{-2\pi i \theta} + 2)c + 1 = 0.
\]

It follows that in this case, there are exactly two normalized quadratic Siegel rational maps which have exactly two fixed points 0 and \( \infty \), and which are conjugate to each other by \( z \rightarrow z/c \).

In the second case, \( g_c \) has exactly three distinct fixed points 0, \( \infty \), and some complex value \( p \). Let \( \phi \) be a Möbius transformation such that \( \phi \circ g_c \circ \phi^{-1} \) has the normalized form. Then \( \phi \) is determined by one of the following four conditions,

1. \( \phi = \text{id} \).
2. \( \phi(0) = 0, \phi(1) = 1, \text{ and } \phi(p) = \infty \).
3. \( \phi(z) = z/c_g \).
4. \( \phi(0) = 0, \phi(c_g) = 1, \text{ and } \phi(p) = \infty \).

Let us collect some basic facts about the topological branched covering maps \( f_t \), \( 0 < t < 2\pi \), which can be easily seen from Figure 15 and 16. The rigorous proofs of these facts are not difficult and shall be left to the reader.

**Fact 1.** Let \( 0 < t < 2\pi \). Suppose that \( g_{c(t)} \in R^\text{geom}_g \) realizes \( f_t \). Then \( g_{c(t)} \) has exactly three distinct fixed points, 0, \( \infty \), and \( p \).

**Fact 2.** Let \( g_{\tilde{c}(t)} \in R^\text{geom}_g \) realizes the topological branched covering map \( \tilde{f}_t \) for \( 0 < t < 2\pi \). Then \( g_{\tilde{c}(t)} \) is conjugate to \( g_{c(t)} \) by the Möbius map determined by (2) above.

**Fact 3.** By just exchanging the positions of 1 and \( c \) in Figure 15, with all the other topological data being fixed, we will get an another new topological branched covering map in \( R^\text{top}_g \) indicated by Figure 16. This new topological branched covering map models the Siegel rational map \( g_{\tilde{c}(2\pi - t)} \in R^\text{geom}_g \). It
is clear that the maps \( g_{c(2\pi-t)} \) and \( g_{c(t)} \) are conjugate to each other by the Möbius map determined by the condition (3).

Fact 4. If we compose the last two conjugations in either order (that is, we may first change \( f_t \) to \( \tilde{f}_t \), and then exchange the positions of 1 and \( c \) in \( \tilde{f}_t \) and finally get \( f_{2\pi-t} \), or we first exchange the positions of 1 and \( c \) in \( f_t \) and get \( \tilde{f}_{2\pi-t} \) and then change it to \( f_{2\pi-t} \)), we will get the same topological branched covering map \( f_{2\pi-t} \in R_{\theta}^{\text{top}} \) which models the Siegel rational map \( g_{c(2\pi-t)} \in R_{\theta}^{\text{geom}} \). It follows that \( g_{c(2\pi-t)} \) is conjugate to \( g_{c(t)} \) by the Möbius map determined by the condition (4).

4.3. A Distortion Lemma. For each \( 0 < t < 2\pi \), suppose that \( f_t \) is realized by a Siegel rational map \( g_{c(t)} \in R_{\theta}^{\text{geom}} \). By Lemma 5.1, there is a Blaschke product, say \( G_t \), which models \( g_{c(t)} \). The main purpose of this section is to prove

**Lemma 4.3.** There is a constant \( 1 < K < \infty \) which depends only on \( \theta \) such that for every Siegel rational map in \( R_{\theta}^{\text{geom}} \) which is modeled by \( f_t \) for some \( 0 < t < 2\pi \), the boundary of the Siegel disk is a \( K \)-quasi-circle.

In the procedure of the quasiconformal surgery in §2.5.2, if we just take \( H \) to be the Douady-Earle extension of \( h \) and do not require that \( H(0) = 0 \), then by the conformal natural property of Douady-Earle extension, we can reduce Lemma 4.3 to the following lemma. For \( 0 < t < 2\pi \), let \( h_t : \partial \Delta \to \partial \Delta \) be the quasisymmetric homeomorphism such that \( h_t(1) = 1 \) and

\[ G_t|\partial \Delta = h_t \circ R_\theta \circ h_t^{-1}. \]

**Lemma 4.4.** There is a uniform \( 1 < K < \infty \), such that for every \( 0 < t < 2\pi \), there is a Möbius map \( \sigma_t \) which fixes 1 and maps the unit circle to itself with orientation preserved, such that the map \( \sigma_t \circ h_t \) is a \( K \)-quasisymmetric homeomorphism.

**Remark 4.2.** Let \( d \geq 3 \) be an integer and \( 0 < \theta < 1 \) be a bounded type irrational number. Let \( B^0_d \) denote the family of all the Blaschke products such that the restriction of every \( B \in B^0_d \) to the unit circle is a critical circle homeomorphism of rotation number \( \theta \). By using Buff-Cheritat’s Relative Schwartz lemma, it was recently proved that the above bound \( K \) actually exists for all the maps in \( B^0_d \) and \( K \) depends only on \( \theta \) and \( d \).

**Sublemma 1.** There exist \( 0 < \delta_0 < 2\pi \) and \( 0 < \epsilon_0 < 2\pi \) such that for any \( 0 < t < 2\pi \), there exist four distinct points \( x_1, x_2, x_3, x_4 \in \partial \Delta \) and a Möbius map \( \sigma_t \) which maps the unit circle to itself and preserves the orientation, such that the arc length of each component of \( \partial \Delta - \{x_1, x_2, x_3, x_4\} \) is \( \geq \delta_0 \), and the arc length of each component of \( \partial \Delta - \{\tau^{-1}_t(x_1), \tau^{-1}_t(x_2), \tau^{-1}_t(x_3), \tau^{-1}_t(x_4)\} \) is \( \geq \epsilon_0 \), where \( \tau_t = \sigma_t \circ h_t \).

**Proof.** Take \( I \) large enough such that \( \{I\theta\} < \pi / 2 \), where \( \{\cdot\} \) is used to denote the fraction part of a number. Let \( I \) be an arc segment with minimal arc
length such that \(|h_t^{-1}(I)| = \{t\theta\}\). Let \(L\) and \(R\) be the two adjacent arc segments of \(I\) on \(\mathbb{T}\) such that

\[
|h_t^{-1}(L)| = |h_t^{-1}(R)| = \{t\theta\}.
\]

We now claim that there exists an \(1 < M < \infty\) which does not depend on \(t\) such that one of the following two inequalities hold:

\[
|L| \leq M|I| \quad \text{or} \quad |R| \leq M|I|.
\]

Let us prove the claim now. Assume that it is not true. Then there is a sequence \(t_n \in (0, 2\pi)\) such that for each \(n\), there exist three adjacent intervals \(L_n, I_n\) and \(R_n\) in \(\mathbb{T}\) so that

\[
|h_t^{-1}(L_n)| = |h_t^{-1}(I_n)| = |h_t^{-1}(R_n)| = \{t\theta\},
\]

but both of the above two inequalities do not hold. By passing to a subsequence, we may assume that \(|R_n|/|I_n| \to \infty\) and \(|L_n|/|I_n| \to \infty\). Take \(n\) large enough. Let \(\Pi_{t_n}\) be the set of the critical points of \(G_{t_n}\). Let

\[
X_n = \tilde{\mathbb{C}} - (\partial \Delta - (R_n \cup L_n)) \cup \bigcup_{1 \leq i \leq l} G_{t_n}^i(\Pi_{t_n}),
\]

and

\[
Y_n = G_{t_n}^{−l}(X_n).
\]

It follows that

\[
G_{t_n}^l : Y_n \to X_n
\]

is a holomorphic covering map.

Since \(I_n\) has a large space around it in \(L_n \cup I_n \cup R_n\), it follows that there is a short simple closed geodesic \(\gamma_n \subset X_n\) which separates \(I_n\) and \(\partial \Delta - L_n \cup I_n \cup R_n\). We thus get that \(\|\gamma_n\|_{X_n} \to 0\) as \(n \to \infty\). Let \(\xi_n\) denote the component of \(G_{t_n}^{-1}(\gamma_n)\) which intersects the unit circle. It follows that \(\xi_n\) is also a short simple closed geodesic, which is symmetric about the unit circle. Moreover, \(\|\xi_n\|_{Y_n} \to 0\) as \(n \to \infty\). Most importantly, by \((52)\), it follows that

\[
G_{t_n}^l(I_n) = L_n
\]

and therefore the geodesic \(\xi_n\) separates \(L_n\) and \(R_n\). But since \(|R_n|/|I_n| \to \infty\) and \(|L_n|/|I_n| \to \infty\), it follows that the length of any simple closed geodesic which separates \(L_n\) and \(R_n\) has a positive lower bound. This is a contradiction and the claim has been proved.

Now we may assume that \(|L| \leq M|I|\) (the case that \(|R| < M|I|\) can be treated in the same way). Let

\[
S = \partial \Delta - L \cup I \cup R.
\]

By the choice of \(l\) and \(I\), it follows that \(|h_t^{-1}(S)| > \{t\theta\}\) and hence \(|S| > |I|\). Let \(z \in \Delta\) be the point which lies in the straight line which passes through the origin and the middle point of \(I\) such that \(d(z, I) = |I|\). Define the Möbius map \(\sigma_t\) such that \(\sigma_t(1) = 1, \sigma_t(z) = 0\) and \(\sigma_t(\mathbb{T}) = \mathbb{T}\). Let \(t_1, t_2, t_3\) and \(t_4\) be the end points of the interval of \(L, I\) and \(R\). Let \(x_1, x_2, x_3\) and \(x_4\) be the images of \(t_1, t_2, t_3\) and \(t_4\) under the map \(\sigma_t\). It follows that there is
a uniform $\delta_0 > 0$ such that each component of $\partial \Delta - \{x_1, x_2, x_3, x_4\}$ has arc length $\geq \delta_0$. (To get this, one can consider the cross ratio of the four end points of the intervals $L, I, R,$ and $S$. Use the fact that $|I| < |L| < |M|, |I| < |R|,$ and $|I| < |S|$ and that Möbius maps preserve cross ratios). Let $\tau_t = \sigma_t \circ h_t$. Then the arc length of each component of

$$\partial \Delta - \{\tau_t^{-1}(x_1), \tau_t^{-1}(x_2), \tau_t^{-1}(x_3), \tau_t^{-1}(x_4)\}$$

is $\geq \epsilon_0 = \{|l\theta\}$. The proof of Sublemma 1 is completed.

To simplify the notations, in the following we use $G_t$ and $h_t$ instead of $\sigma_t \circ G_t \circ \sigma_t^{-1}$ and $\sigma_t \circ h_t$, and assume that there exist $0 < \delta_0 < 2\pi$ and $0 < \epsilon_0 < 2\pi$ such that for any $0 < t < 2\pi$, there exist four distinct points $x_1, x_2, x_3, x_4 \in \partial \Delta$ such that the arc length of each component of $\partial \Delta - \{x_1, x_2, x_3, x_4\}$ is $\geq \delta_0$, and the arc length of each component of $\partial \Delta - \{h_t^{-1}(x_1), h_t^{-1}(x_2), h_t^{-1}(x_3), h_t^{-1}(x_4)\}$ is $\geq \epsilon_0$, where $h_t : \partial \Delta \to \partial \Delta$ is the quasisymmetric homeomorphism such that $h_t(1) = 1$ and $G_t \partial \Delta = h_t \circ R_\theta \circ h_t^{-1}$.

Let $J \subset I \subset \mathbb{T}$ such that both the components of $I - J$, say $R$ and $L$, are non-trivial arc segments. Define

$$C(I, J) = \frac{|I||J|}{|R||L|}.$$ 

The value $C(I, J)$ measures the space around $J$ in $I$. Let

$$X = \hat{\mathbb{C}} - (\partial \Delta - R \cup L).$$

Let $\gamma \subset X$ be the simple closed geodesic which separates $J$ and $\partial \Delta - I$. The proof of the following lemma is direct, and we shall leave the details to the reader:

**Sublemma 2.** Let $\delta, C > 0$. Then there exists a $\lambda(\delta, C) > 0$ dependent only on $\delta$ and $C$ such that if $|\partial \Delta - I| > \delta$ and $\|\gamma\|_X \leq C$, then $C(I, J) \leq \lambda(\delta, C)$. Moreover, if $|\partial \Delta - I| > \delta$ and $C(I, J) \leq C$, then $\|\gamma\|_X \leq \lambda(\delta, C)$.

**Remark 4.3.** Sublemma 2 implies that the existence of the upper bound of the length of the simple closed geodesic which separates $J$ and $\partial \Delta - I$ is equivalent to the existence of some definite space around $J$ inside $I$ provided that $\partial \Delta - I$ is not too small.

Given a collection of arc segments

$$\mathcal{I} = \{I^k \subset \partial \Delta, k \in \Lambda\},$$

the intersection multiplicity of $\mathcal{I}$ is defined to be the largest integer $n \geq 0$ such that there exist $n$ distinct arc segments in $\mathcal{I}$ whose intersection is not empty.

For an arc segment $I \subset \partial \Delta$, we use $I^k_t \subset \partial \Delta$ to denote the component of $G_t^{-k}(I)$ which lies in the unit circle. In particular, $I^0_t = I$. 
Lemma 4.5. For each $K > 0$, $l \geq 1$ and $\rho > 0$, there is a constant $\lambda(K, l, \rho) > 0$, which is independent of $t$, such that for any arc segments $M \subset T \subset \partial \Delta$, if the following three conditions are satisfied,

1. $C(T, M) < K$,
2. the intersection multiplicity of $\{T_i, i = 0, 1, \cdots, N\}$ is less than $l$,
3. $|\partial \Delta - T_i| > \rho$ for $0 \leq i \leq N$,
then $C(T^N, M^N) < \lambda(K, l, \rho)$.

Proof. Let $M \subset T \subset \partial \Delta$. Let $\xi_i, i = 1, 2$ be the two critical points and $\nu_i, i = 1, 2$ the two critical values of $G_t$. For a given $0 \leq k \leq n$, there are two cases.

In the first case, $T^k_t$ contains some critical value of $G_t$. Set
\[ A_k = (\partial \Delta - T^k_t) \cup M^k_t \cup (T^k_t \cap \{\nu_1, \nu_2\}) , \]
and
\[ B_k = (\partial \Delta - T^k_t) \cup M^k_t . \]

Now let us consider the following three hyperbolic Riemann surfaces,
\[ X_k = \mathbb{P}^1 - A_k, \tag{53} \]
\[ Y_k = \mathbb{P}^1 - B_k, \tag{54} \]
and
\[ Z_k = \mathbb{P}^1 - G_t^{-1}(A_k) . \tag{55} \]

By the assumption that $C(T, M) < K$ and $|\partial \Delta - T| > \rho$, it follows from Sublemma 2 that there is a simple closed geodesic in $Y_0$ which separates $M$ and $\partial \Delta - T$ whose hyperbolic length has an upper bound which depends only on $K$.

Since $Y_k - X_k \subset \{\nu_1, \nu_2\}$ is a finite set, it follows that there is a uniform constant $1 < C < \infty$ such that for the simple closed geodesic $\xi \subset Y_k$, there is a simple closed geodesic $\xi' \subset X_k$ which is homotopy to $\xi$ in $Y_k$, such that $l_{X_k}(\xi') < Cl_{Y_k}(\xi)$.

Let $\eta \subset Y_k$ be the simple closed geodesic which separates $M^k_t$ and $\partial \Delta - T^k_t$. Take a simple closed geodesic $\eta' \subset X_k$ such that $\eta'$ is homotopy to $\eta$ in $Y_k$ and such that $l_{X_k}(\eta') < Cl_{Y_k}(\eta)$ where $C > 0$ is the uniform constant above. Let $\eta''$ be the simple closed geodesic in $Z_k$ which separates $\partial \Delta - T^k_t$ and $M^{k+1}_t$ such that the image of $\eta''$ under $G_t$ covers $\eta'$. Since
\[ G_t : Z_k \rightarrow X_k \]
is a holomorphic covering map of degree 3, it follows that $l_{Z_k}(\eta'') \leq 3l_{X_k}(\eta')$. Therefore, we have
\[ l_{Y_{k+1}}(\eta'') < l_{Z_k}(\eta'') < 3l_{X_k}(\eta') < 3Cl_{Y_k}(\eta) . \tag{56} \]

In the second case, $T^k_t$ does not contain any critical value of $G_t$. Let $\eta \subset Y_k$ be a simple closed geodesic which separates $M^k_t$ and $\partial \Delta - T^k_t$. It follows that
there is a simple closed geodesic \( \eta' \subset Z_k \) which separates \( \partial \Delta - T^k_{i+1} \) and \( M^{k+1} \) such that the image of \( \eta' \) under \( G_t \) covers \( \eta \) exactly one time. It follows that

\[
l_{Y_{i+1}}(\eta') < l_{Z_n}(\eta') = l_{Y_n}(\eta).
\]

Since the intersection multiplicity of \( \{ T^k \} \) is \( l \), and \( G_t \) has only two critical values, it follows that, when \( k \) runs through \( 0, 1, \cdots, N-1 \), case 1 can happen at most \( 2l \) times. Therefore, there is a simple closed geodesic which separates \( \partial \Delta - T^N_i \) and \( M^N_i \) whose length has an upper bound dependent only on \( K \) and \( l \). Note that \( |\partial \Delta - T^N_i| > \rho \). The lemma then follows from Sublemma 2 and Remark 4.3.

Let \( I = [a, b] \subset \partial \Delta \). We use \( |a - b| \) or \( |I| \) to denote the Euclidean length of the arc \( I \). For \( K > 1 \), we say two intervals \( I, J \subset \partial \Delta \) are \( K \)-comparable if \( K^{-1} < |I|/|J| < K \). Let \( p_n/q_n, n = 1, 2, \cdots \) be the convergents of \( \theta \).

**Lemma 4.6.** There is a constant \( K > 1 \) which is only dependent on \( \theta \) such that for all \( 0 < t < 2\pi, z \in \partial \Delta \) and \( n \geq 1 \), the following two inequalities hold,

\[
1/K \leq \frac{|G_t^{-q_n}(z) - z|}{|G_t^{q_n}(z) - z|} \leq K
\]

and

\[
1/K \leq \frac{|G_t^{q_n+1}(z) - z|}{|G_t^{q_n}(z) - z|} \leq K.
\]

The idea of the proof is taken from §3 of [8].

**Proof.** Let \( M \) be an integer such that

\[
|h_t^{-1}(x) - h_t^{-1}(G_t^{q_n}(x))| < \varepsilon_0/3
\]

holds for all \( n \geq M \) and \( 0 < t < 2\pi \) where \( \varepsilon_0 \) is the number in Sublemma 1. It is sufficient to prove that there is a \( K > 1 \) such that the above two inequality hold for all \( n \geq M \) and \( 0 < t < 2\pi \). The case for \( n < M \) then follows by noting the fact that \( \theta \) is of bounded type.

Take \( x \in \partial \Delta \) such that it attains the minimum of \( |G_t^{q_n}(y) - y| \). Then \( [x, G_t^{q_n}(x)] \) has a definite space around it inside \( [G_t^{-q_n}(x), G_t^{2q_n}(x)] \). Let \( M = [x, G_t^{q_n}(x)] \) and \( T = [G_t^{-q_n}(x), G_t^{2q_n}(x)] \). Since \( \theta \) is of bounded type, the intersection multiplicity for \( \{ T^k, 0 \leq k \leq 5q_n \} \) has a uniform upper bound dependent only on \( \theta \). Applying Lemma 4.5 to the intervals \( M \subset T \) and \( N = q_n, 2q_n, 3q_n, 4q_n, \) and \( 5q_n \), respectively. Note that the multiplicity of the corresponding collection of intervals is bounded above by some constant dependent only on \( \theta \). It follows that the six intervals \( [G_t^{-5q_n}(x), G_t^{-4q_n}(x)], [G_t^{-4q_n}(x), G_t^{-3q_n}(x)], [G_t^{-3q_n}(x), G_t^{-2q_n}(x)], [G_t^{-2q_n}(x), G_t^{-q_n}(x)], [G_t^{-q_n}(x), G_t^{q_n}(x)], [x, G_t^{q_n}(x)] \) are \( L \)-comparable with each other, where \( L \) is a constant dependent only on \( \theta \). Let \( l \) be the minimum of the length of these six intervals.

For any \( z \in \partial \Delta \), it follows from the property of the closed returns that there is an \( 0 \leq i < 2q_{n+1} \) such that \( G_t^i(z) \in [G_t^{-5q_n}(x), G_t^{-4q_n}(x)]\).
Let us prove (58) first. There are two cases. In the first case, there is some $1 \leq j \leq 3$ such that $[G_t^{i+jq_n}(z), G_t^{i+(j+1)q_n}(z)]$ has length less than $l/2$. Then

$$[G_t^{i+jq_n}(z), G_t^{i+(j+1)q_n}(z)]$$

has a definite space around it inside $[G_t^{i+(j-1)q_n}(z), G_t^{i+(j+1)q_n}(z)]$. Let

$$M = [G_t^{i+jq_n}(z), G_t^{i+(j+1)q_n}(z)]$$

and $T = [G_t^{i+(j-1)q_n}(z), G_t^{i+(j+2)q_n}(z)]$.

Apply Lemma 4.5 to the intervals $M \subset T$ and $N = i + jq_n$. Again note that the multiplicity of the corresponding collection of intervals is bounded above by some constant dependent only on $\theta$. We thus get a definite space around $[z, G_t^{q_n}(z)]$ inside $[G_t^{-q_n}(z), G_t^{2q_n}(z)]$. This proves (58) in the first case.

In the second case, for each $j = 1, 2, 3$, $[G_t^{i+jq_n}(z), G_t^{i+(j+1)q_n}(z)]$ has length not less than $l/2$. It follows that the interval $[G_t^{i+2q_n}(z), G_t^{i+3q_n}(z)]$ has definite space around it inside the interval $[G_t^{i+q_n}(z), G_t^{i+4q_n}(z)]$. As before, by applying Lemma 4.5 we get a definite space around $[z, G_t^{q_n}(z)]$ inside $[G_t^{-q_n}(z), G_t^{2q_n}(z)]$. This proves (58) in the second case.

Now let us prove (59). Let $b = \sup\{a_k\} < \infty$ where $[a_1, \cdots, a_n]$ is the continued fraction of $\theta$. Note that $[G_t^{-q_n+i}(z), z] \subset [G_t^{q_n}(z), z]$, so from (58), we have

$$|G_t^{q_n+i}(z) - z| \leq K|G_t^{-q_n+i}(z) - z| < K|G_t^{q_n}(z) - z|,$$

and this implies the right hand of (59). To prove the left hand, Note that

$$[G_t^{q_n}(z), z] \subset \bigcup_{0 \leq i \leq b} [G_t^{i-q_n+i}(z), G_t^{-(i+1)q_n+i}(z)],$$

This implies that

$$|G_t^{q_n}(z) - z| < \sum_{0 \leq i \leq b} |G_t^{i-q_n+i}(z) - G_t^{-(i+1)q_n+i}(z)|.$$

Applying (58) again, we have

$$|G_t^{i-q_n+i}(z) - G_t^{-(i+1)q_n+i}(z)| \leq K^{i+1}|G_t^{q_n+i}(z) - z|$$

for each $0 \leq i \leq b$. Therefore, we get

$$|G_t^{q_n}(z) - z| < \sum_{0 \leq i \leq b} K^{i+1}|G_t^{q_n+i}(z) - z|.$$

By modifying the value $K$, (59) follows.

□

It is the time to prove Lemma 4.4.

Proof. We need only to prove that there is an $M > 1$ dependent only on $\theta$ such that for any $x \in \partial \Delta$ and $0 < \delta < 2\pi$, the following inequality hold for all $0 < t < 2\pi$,

$$\frac{1}{M} < \frac{|h_t(x + \delta) - h_t(x)|}{|h_t(x - \delta) - h_t(x)|} < M.$$
Now for given $\delta$ and $x$, let us take $k \geq 1$ to be the least integer such that one of the intervals $[x - \delta, x]$ and $[x, x + \delta]$ contains $[G_t^{-qk} (x), x]$ or $[x, G_t^{qk} (x)]$. Without loss of generality, Let us suppose $[G_t^{-qk} (x), x] \subset [x - \delta, x]$. From the definition of $k$, $[x - \delta, x] \subset [G_t^{-qk-2} (x), x]$. Since $\theta$ is of bounded type, by Lemma 4.6, it follows that $[G_t^{-qk} (x), x]$ and $[x - \delta, x]$ are $L$–comparable where $1 < L < \infty$ is some constant dependent only on $\theta$. On the other hand, by the definition of $k$, we have $[x, x + \delta] \subset [x, G_t^{qk-1} (x)]$. By Lemma 4.6, $[x, G_t^{qk-1} (x)]$ and $[G_t^{-qk} (x), x]$ are $K$–comparable for some $1 < K < \infty$ dependent only on $\theta$. Therefore, $[x, x + \delta]$ and $[x, G_t^{qk-1} (x)]$ are $KL$–comparable. So we have

$$|G_t^{qk-1} (x) - x| < KL\delta.$$ 

By Lemma 4.3 again, there is an $\epsilon > 0$ dependent only on $\theta$ such that

$$|G_t^{qk^*} (x) - x| > (1 + \epsilon)|G_t^{qk+2} (x) - x|,$$

holds for all $x \in \partial \Delta$. Take $l > 1$ to be the least integer such that $KL < (1+\epsilon)^l$. It follows that $l$ depends only on $\theta$ and

$$|G_t^{qk-1} (x) - x| > (1 + \epsilon)^l |x - G_t^{qk+2l-1} (x)|.$$ 

It follows that $[x, G_t^{qk+2l-1} (x)] \subset [x, x + \delta]$. We then get

$$(60) \quad [x, G_t^{qk+2l-1} (x)] \subset [x, x + \delta] \subset [x, G_t^{qk+1} (x)],$$

and

$$(61) \quad [G_t^{-qk} (x), x] \subset [x - \delta, x] \subset [G_t^{-qk-2} (x), x].$$

Now for $x \in \mathbb{R}$, let $\{x\} \in (-1/2, 1/2)$ be the number such that $x - \{x\} \in \mathbb{Z}$. From (60) and (61), we have

$$|\{qk+2l-1\theta\}| \leq |h_l(x + \delta) - h(x)| < |\{qk-1\theta\}|,$$

and

$$|\{qk\theta\}| \leq |h_l(x - \delta) - h(x)| < |\{qk-2\theta\}|.$$

Now the lemma follows from the assumption that $\theta$ is of bounded type.

\[\square\]

4.4. Quadratic Siegel Rational Maps Modeled by $f_\alpha$. In this section we will determine all the critical parameters $c$ such that $g_c \in R^\text{geom}_\theta$ and the boundary of the Siegel disk of $g_c$ passes through both of the critical points.

**Lemma 4.7.** For $0 < t < 2\pi$, let $g_{c(t)}$ be the Siegel rational map which realizes $f_t$ in the sense of Lemma 4.2 Then $c(t)$ is continuous in $(0, 2\pi)$.

**Proof.** Let $t_k \to t$ for some $0 < t < 2\pi$. We first claim that the sequence $\{c(t_k)\}$ is contained in some compact set of $\mathbb{C} - \{0, 1, -1\}$. Let us prove the claim now. 

Note that for each $0 < t_k < 2\pi$, $P_{f_{t_k}} = \partial\Delta$ does not contain the infinity, and that the infinity is fixed by $f_{t_k}$. Following the same steps in the proof
of Theorem A, we can construct a Blaschke product, say $G_k$, to model $f_{t_k}$. Write
\begin{equation}
G_k(z) = \lambda_k \frac{z - p_k}{1 - \overline{p_k} z} \frac{z - q_k}{1 - \overline{q_k} z},
\end{equation}
where $|\lambda_k| = 1$ is some constant and $|p_k| > 1, |q_k| < 1$. In particular, by the construction, $G'_k(1) = 0$ for all $k \geq 0$.

Let $h_k : \partial \Delta \to \partial \Delta$ be the quasi-symmetric homeomorphism such that $h_k(1) = 1$ and $G_k|_{\partial \Delta} = h_k \circ R_\theta \circ h_k$. Let $H_k : \Delta \to \Delta$ be the Douady-Earle extension of $h_k$. By Lemma 4.4 and the conformal natural property of Douady-Earle extension, there exists a uniform $0 < \delta < 1$ which depends only on $M$ such that
$$\sup_{z \in \Delta} \left| \frac{(H_k^{-1})_{\bar{z}}}{(H_k^{-1})_{z}} \right| \leq \delta.$$Define
$$\widehat{G}_k(z) = \begin{cases} G_k(z) & \text{for } |z| \geq 1, \\ H_k \circ R_\theta \circ H_k^{-1}(z) & \text{for } z \in \Delta. \end{cases}$$

Now as in the proof of Theorem A, we can pull back the complex structure of $H_k^{-1}$ by $G_k$ and get a $\widehat{G}_k$–invariant complex structure $\mu_k$ on the whole sphere. Let $\phi_k$ be the quasiconformal homeomorphism of the sphere which solves the Beltrami equation given by $\mu_k$ such that $\phi_k(1) = 1, \phi_k(\infty) = \infty$, and $\phi_k(0) = H(0)$. Then $\phi_k^{-1} \circ \widehat{G}_k \circ \phi_k$ is a Siegel rational map in $R_{geom}^{\theta}$ which realizes $f_{t_k}$ in the sense of Lemma 4.2. We thus have
$$g_{c(t_k)} = \phi_k^{-1} \circ \widehat{G}_k \circ \phi_k.$$Since when $|c|$ is large enough, $g_c$ has an attracting fixed point at the infinity, and when $|c|$ is small enough, $g_c$ has an attracting fixed point at the origin, by passing to a convergent subsequence, we may assume that either $c(t_k) \to 1$ or $c(t_k) \to -1$.

First let us assume that $c(t_k) \to 1$. From (47) and a direct calculation, it follows that $g_{c(t_k)} \to e^{2\pi i \theta} z$ uniformly in any compact set of the complex plane which does not contain 1. Let $D_k$ denote the Siegel disk of $g_{c(t_k)}$. By Lemma 4.3, $\partial D_k$ is a $K$–quasi-circle for some uniform $K$. Therefore, $D_k \to \Delta$ in the Carathéodory sense. This implies that as $k \to \infty$, the inner angle between 1 and $c(t_k)$ either converges to 0 or converges to $2\pi$. This contradicts with the assumption that $t_k \to t$ for some $0 < t < 2\pi$.

Now let us assume that $c(t_k) \to -1$. Let
$$g_{c(t_k)} = \frac{a_k z^2 + e^{2\pi i \theta} z}{b_k z + 1}.$$From (50), we get
$$a_k \to 0 \text{ and } b_k \to \infty.$$
as \( c(t_k) \to -1 \). Let \( p_k \) be the fixed point of \( g_{c(t_k)} \) which is distinct from 0 and the infinity. By a direct calculation, we have
\[
p_k = \frac{1 - e^{2\pi i \theta}}{a_k - b_k}.
\]
We thus have \( p_k \to 0 \) as \( c(t_k) \to -1 \).

Let \( \psi_k \) be the Möbius transformation which maps 0 to 0, 1 to 1, and \( p_k \) to the infinity. It follows that
\[
\psi_k(z) = \frac{z(1 - p_k)}{z - p_k}.
\]
Consider the map
\[
g_{\tilde{c}(t_k)} = \psi_k \circ g_{c(t_k)} \circ \psi_k^{-1}
\]
where
\[
\tilde{c}(t_k) = \psi_k(c(t_k)) = \frac{c(t_k)(1 - p_k)}{c(t_k) - p_k}.
\]
Since \( p_k \to 0 \) as \( c(t_k) \to -1 \), it follows that \( \tilde{c}(t_k) \to 1 \).

Now let us prove that the sequence \( \{c(t_k)\} \) is convergent. By passing to a subsequence, we may assume that \( c(t_k) \to c \in \mathbb{C} - \{0, 1, -1\} \). Since \( \partial D_k \) is a \( K \)-quasi-circle passing through 1 and \( c(t_k) \to c \), it follows that there is a \( \delta > 0 \) such that \( B_\delta(0) \subset D_{c(t_k)} \).

**Lemma 4.8.** \( \lim_{k \to \infty} c(t_k) = 1 \) or \(-1 \).

**Proof.** Let us prove it by contradiction. Since when \( |c| \) is large enough, \( g_c \) has an attracting fixed point at the infinity, and when \( |c| \) is small enough, \( g_c \) has an attracting fixed point at the origin, we may assume that there is a sequence \( t_k \to 0 \) such that \( c(t_k) \to c \) for some \( c \in \mathbb{C} - \{0, 1, -1\} \). Let \( D_c \) and \( D_{c(t_k)} \) denote respectively the Siegel disks of \( g_c \) and \( g_{c(t_k)} \), which are centered at the origin. Since every \( \partial D_{c(t_k)} \) is a \( K \)-quasi-circle passing through 1 and \( c(t_k) \) and \( c(t_k) \to c \), it follows that there is a \( \delta > 0 \) such that
\[
B_\delta(0) \subset D_{c(t_k)}
\]
for all \( t_k \). Let \( p \neq 0 \) be such that \( g_c(p) = 0 \). Then there is a \( r > 0 \) such that
\[
B_r(p) \cap D_c = B_r(p) \cap D_{c(t_k)} = \emptyset
\]
for all \( t_k \). Let \( \phi(z) = 1/(z-p) \). Set
\[
T_k(z) = \phi \circ g_{c(t_k)} \circ \phi^{-1} \text{ and } T(z) = \phi \circ g_c \circ \phi^{-1}.
\]
Denote the corresponding Siegel disks of \( T_k \) and \( T \) by \( D_{T_k} \) and \( D_T \), respectively. Clearly, as \( k \to \infty \), \( T_k \to T \) uniformly with respect to the spherical metric, and moreover, there is a compact set \( E \) of the complex plane such that \( D_T \subset E \) and \( D_{T_k} \subset E \) for all \( k \geq 1 \). Since every \( \partial D_{c(t_k)} \) is a \( K \)-quasi-circle for some uniform \( 1 < K < \infty \) by Lemma 4.3, it follows that every \( \partial D_{T_k} \) is a \( K \)-quasi-circle. Let \( h_k : \Delta \to D_{T_k} \) be the univalent map such that \( h_k'(0) > 0 \) and \( h_k^{-1} \circ T_k \circ h_k = R_0 \). Since \( \partial D_{T_k} \) is a uniform \( K \)-quasi-circle, by passing to a convergent subsequence, we may assume that \( h_k \) uniformly converges to \( h \) on \( \Delta \) such that \( h^{-1} \circ T \circ h = R_0 \). This implies that \( \partial D_T \) is a quasi-circle also and passes through both the two critical points of \( T \). In particular, the inner angle of the two critical points of \( T \) must be 0 or \( 2\pi \), and therefore, the two critical points of \( T \) coincide. It follows that \( 1 = c \). This is a contradiction and the lemma follows.

**Lemma 4.9.** \( \{\lim_{t \to 0} c(t), \lim_{t \to 2\pi} c(t)\} = \{1, -1\} \).

**Proof.** For \( 0 < t < 2\pi \), let \( \tilde{c}(t) \) be the critical parameter such that \( g_{\tilde{c}(t)} \) realizes \( \tilde{f}_t \) in the sense of Lemma 4.2. Let \( p_t \) be the fixed point of \( g_{c(t)} \) which is distinct from 0 and the infinity. From Fact 2 in \( \S 4.2.2 \), it follows that the Möbius transformation
\[
\phi_t(z) = \frac{(1-p_t)z}{z-p_t}
\]
conjugates \( g_t \) to \( g_{\tilde{c}(t)} \). By a direct calculation, we get
\[
\tilde{c}(t) = \phi_t(c(t)) = \frac{e^{2\pi i\theta} - 2c(t) + e^{2\pi i\theta}}{e^{2\pi i\theta}c(t) + 2 - e^{2\pi i\theta}}.
\]
From Fact 3 in \( \S 4.2.2 \), it follows that the Möbius transformation
\[
\psi_t(z) = z/\tilde{c}(t)
\]
conjugates \( g_{\tilde{c}(t)} \) to \( g_{c(2\pi-t)} \). In particular,
\[
(63) \quad c(2\pi - t) = 1/\tilde{c}(t).
\]
By Lemma 4.3 either \( \lim_{t \to 0} c(t) = 1 \), or \( \lim_{t \to 0} c(t) = -1 \). If \( \lim_{t \to 0} c(t) = 1 \), then
\[
\lim_{t \to 2\pi} c(t) = \lim_{t \to 0} c(2\pi - t) = \lim_{t \to 0} 1/\tilde{c}(t) = \lim_{t \to 0} \frac{-e^{2\pi i\theta}c(t) + 2 - e^{2\pi i\theta}}{e^{2\pi i\theta}c(t) + 2 - e^{2\pi i\theta}} = -1.
\]
If \( \lim_{t \to 0} c(t) = -1 \), then
\[
\lim_{t \to 2\pi} c(t) = \lim_{t \to 0} c(2\pi - t) = \lim_{t \to 0} 1/\tilde{c}(t) = \lim_{t \to 0} \frac{-e^{2\pi i\theta}c(t) + 2 - e^{2\pi i\theta}}{e^{2\pi i\theta}c(t) + 2 - e^{2\pi i\theta}} = 1.
\]
Figure 17. Critical parameters determined by $f_t$ and $\tilde{f}_t$ for $0 < t < 2\pi$

Lemma 4.9 follows. □

From Lemma 4.7 and Lemma 4.9, it follows that $c(t)$, $0 < t < 2\pi$ is a continuous curve segment which does not intersect with itself and which connects 1 and $-1$. By using the same argument, the same conclusion can be derived for the curve $\tilde{c}(t)$, $0 < t < 2\pi$. Let $\gamma = \{c(t) | 0 < t < 2\pi\}$ and $\gamma' = \{\tilde{c}(t) | 0 < t < 2\pi\} = \{1/c(t) | 0 < t < 2\pi\}$. It is clear that except the two end points, $\gamma$ does not intersect $\gamma'$. (This is simply because for $0 < t, t' < 2\pi$, $g_c(t)$ and $g_{\tilde{c}(t)}$ realize different topological models, which are indicated by Figure 15 and Figure 16, respectively). It follows that

$$\xi = \gamma \cup \gamma' \cup \{1, -1\}$$

is a simple closed curve. From (63), the map $c \rightarrow 1/c$ preserves the curve $\xi$ but reverses its orientation. It follows that $\xi$ separates 0 and the infinity. We summarize these as follows:

**Lemma 4.10.** The curve $\xi = \gamma \cup \gamma' \cup \{1, -1\}$ is a simple closed curve which separates 0 and the infinity. Moreover, $\xi$ is invariant under the map $z \rightarrow 1/z$.

4.5. Quadratic Siegel Rational Maps with One Finite Critical Orbit.

In this section, we consider all those quadratic rational maps which have a fixed Siegel disk of rotation number $\theta$ and a critical point with finite forward orbit. The aim of this section is to show that such Siegel rational maps belong to $R_{g}^{\text{geom}}$. That is, for any such map, the another critical point must lie in the boundary of the Siegel disk which is a quasi-circle. Before we state the result, let us introduce some notations first.

Let $0 \leq m < n$ be integers and $t \in \mathbb{C}$. Let us define $\mathcal{Z}_{m,n}^\ast$ to be the set of all the quadratic rational maps $g$ such that

1. $g'(1) = g'(c) = 0$,
2. $g^m(c) = g^n(c)$,
3. $g$ fixes 0 and the infinity,
4. \( g'(0) = s \).

Recall that \( \lambda = e^{2\pi i \theta} \). Define \( R_{m,n}^\theta = Z_{m,n}^\lambda \cap R_{\theta}^{\text{geom}} \). The main result of this section is as follows.

**Lemma 4.11.** \( Z_{m,n}^\lambda = R_{m,n}^\theta \).

Before the proof of Lemma 4.11 let us prove a few lemmas.

**Lemma 4.12.** Let \( 0 \leq m < n \) be two integers. Then for any \( \epsilon > 0 \), there is some \( 0 < |s| < 1 \), such that for any quadratic rational map \( g_c \in Z_{m,n}^\lambda \), there is a quadratic rational map \( g \in Z_{m,n}^s \) such that \( d(g, g_c) < \epsilon \).

**Proof.** For \( s \neq 0 \), and \( t \neq 0, 1, -1 \), consider the function

\[
F_{s,t}(z) = \frac{a(s,t)z^2 + sz}{b(s,t)z + 1}
\]

where \( a(s,t) = -s(1 + t)/2t \) and \( b(s,t) = -2/(1 + t) \). It follows that \( F_{s,t}'(0) = s \), and \( F_{s,t}'(1) = F_{s,t}(t) = 0 \). There are three cases.

Case 1. \( c \neq \infty \), and \( g_{c}^{m}(c) = g_{c}^{n}(c) \neq \infty \). It is clear that \( F_{\lambda,c}(z) = g_{c}(z) \). It follows that there is an open neighborhood of \( \lambda \), say \( U \), and an open neighborhood of \( c \), say \( V \), such that both the functions \( F_{s,t}(t) \) and \( F_{s,c}(t) \) are holomorphic for \( (s,t) \in U \times V \). In particular, by taking \( V \) smaller, we can assume that as \( s \to \lambda, F_{s,t}'(t) \to F_{\lambda,t}'(t) \) and \( F_{s,c}'(t) \to F_{\lambda,c}'(t) \) uniformly for \( t \in V \). Since \( F_{\lambda,t}'(t) - F_{\lambda,c}'(t) \) has a zero at \( c \in V \), it follows from Rouché theorem that for every possible \( r > 0 \), there is a \( \delta > 0 \), such that for every \( s \in B_{\delta}(\lambda) \), there is a point \( c_{s} \in B_{r}(c) \) such that \( F_{s,c_{s}}^{m}(c_{s}) - F_{s,c_{s}}^{n}(c_{s}) = 0 \).

Case 2. \( c \neq \infty \), \( g_{c}^{l}(c) = \infty \) for some \( 1 \leq l \leq m \). We may assume that \( l \) is the least positive integer such that \( g_{c}^{l}(c) = \infty \). From (64), it follows that

\[
b(\lambda, c)g_{c}^{l-1}(c) + 1 = 0.
\]

Then instead of considering the function \( F_{s,t}'(t) - F_{s,c}'(t) \), this time we consider the function \( b(s,t)F_{s,t}'(t) + 1 \). The lemma in this case then follows by using the same argument as in the proof of the first case. The reader shall have no difficulty to supply the details.

Case 3. \( c = \infty \). In this case, just take \( g = s g_{c} \) where \( s \) is any number close enough to \( \lambda \) with \( |s| < 1 \).

\( \square \)

**Lemma 4.13.** Let \( 0 < |s| < 1 \). Then every \( f \in Z_{m,n}^{s} \) has exactly three distinct fixed points \( 0, \infty \) and some complex value \( p \).

**Proof.** In fact, if this were not true, then the infinity would be a double root of \( f(z) - z \) and hence a parabolic fixed point of \( f \). Therefore, one of the forward critical orbit approaches to the infinity and the other one approaches to the
origin. This is a contradiction with the assumption that \( f^m(c) = f^n(c) \) for some \( 0 \leq m < n \). The lemma follows. \( \square \)

**Lemma 4.14.** Let \( 0 < |s| < 1 \). Then for every \( f \in Z_{m,n}^s \), the conformal equivalent class \([f]\) of \( f \) contains exactly two elements in \( Z_{m,n}^s \).

**Proof.** Let \( g \in [f] \) such that \( g \neq f \). Assume that \( g = \phi \circ f \circ \phi^{-1} \) for some Möbius map \( \phi \). Let \( p \neq 0, \infty \) be the fixed point of \( f \). Since \( g^m(c) = g^n(c) \), it follows that \( g \) has exactly one non-repelling fixed point which is the origin. This implies that the forward orbit \( \{g^k(1)\} \) is the only infinite critical orbit of \( g \). Therefore, we get \( \phi(0) = 0 \) and \( \phi(1) = 1 \). Now let \( 0, \infty, q \) be the three fixed points of \( g \). It follows that \( \{\phi(\infty), \phi(p)\} = \{\infty, q\} \). Note that \( \phi(\infty) \neq \infty \), for otherwise \( \phi = id \) and hence \( g = f \), which contradicts with the assumption that \( g \neq f \). It follows that \( \phi(p) = \infty \). This implies that \( \phi \) is uniquely determined by \( f \) and the lemma follows. \( \square \)

By using the same argument as in the proof of Lemma 4.13, one can show that every \( f \in R^\theta_{m,n} \) also has three distinct fixed points. Then using the same argument as in the proof of Lemma 4.14 one has

**Lemma 4.15.** For every \( f \in R^\theta_{m,n} \), the conformal equivalent class \([f]\) of \( f \) contains exactly two elements in \( R^\theta_{m,n} \).

For \( |s| < 1 \), let \( R_s \) be the set which consists of all the quadratic rational maps \( f \) such that \( f(0) = 0 \) and \( f'(0) = s \). For each \( f \in R_s \), the map \( f \) restricted to a suitable neighborhood of its Julia set is polynomial-like of quadratic with connected Julia set and hence is hybrid equivalent to a unique quadratic polynomial \( z^2 + c \) for some \( c \in M \) where \( M \) is the Mandelbrot set. This induces a homeomorphism between the set of the conformal equivalent classes of \( R_s \), say \( M_s \), and the Mandelbrot set \( M \) (see [10], or the proof of Lemma 8.5, [13]). Let \( Q_{m,n} \) be the set of all the quadratics \( q_c(z) = z^2 + c \) such that \( q^m_c(0) = q^n_c(0) \). It follows from Lemma 4.14 that

**Lemma 4.16.** For \( 0 < |s| < 1 \) and any integers \( 0 \leq m < n \), \( |Z_{m,n}^s| = 2|Q_{m,n}| \).

Now let us prove Lemma 4.11

**Proof.** It suffices to show that \( |Z_{m,n}^\lambda| \leq |R^\theta_{m,n}| \). By Lemma 4.12, we have \( |Z_{m,n}^\lambda| \leq |Z_{m,n}^s| \) for some \( 0 < |s| < 1 \). Note that each element \( f \) in \( Q_{m,n} \) induces a topological branched covering map \( \tilde{f} \) in \( R^\theta \) by topologically mating itself with \( z^2 + \lambda z \) (for one way of the construction of such mating, see §7 of [13]). Clearly, the resulted map \( \tilde{f} \) has no Thurston obstructions outside the rotation disk. Moreover, if \( f_1, f_2 \) are two different elements in \( Q_{m,n} \), then the two maps \( \tilde{f}_1, \tilde{f}_2 \) in \( R^\theta \) induced by \( f_1 \) and \( f_2 \) belong to different combinatorial classes. This, together with Theorem A and Lemma 4.15 implies that \( 2|Q_{m,n}| \leq |R^\theta_{m,n}| \). It follows from Lemma 4.12 and 4.16 that \( |Z_{m,n}^\lambda| \leq |Z_{m,n}^s| = 2|Q_{m,n}| \leq |R^\theta_{m,n}| \). The lemma follows. \( \square \)
4.6. Critical Parameterization. In this section, we will give a critical parameterization of the space of all the Blaschke products in the following form,

\[ B_{p,q}(z) = \frac{z - p}{1 - \bar{p}z} \frac{z - q}{1 - \bar{q}z} \]

where \(|p| > 1, |q| < 1\).

**Lemma 4.17.** For any compact set \(K \subset \mathbb{C} - \partial \Delta\), there is a \(\delta > 0\), such that in either of the following two cases,

1. \(p \in K, |p| > 1, \text{ and dist}(q, \partial \Delta) < \delta\), or
2. \(q \in K, |q| < 1, \text{ and dist}(p, \partial \Delta) < \delta\).

\(B_{p,q}\) has at least two distinct critical points in \(\partial \Delta\).

**Proof.** Let \(T_{p,q}(\alpha) = -i \log B_{p,q}(e^{i\alpha})\) for \(0 \leq \alpha \leq 2\pi\). Then

\[ T'_{p,q}(\alpha) = 1 + \frac{1 - |q|^2}{|1 - qe^{i\alpha}|^2} + \frac{1 - |p|^2}{|1 - pe^{i\alpha}|^2}. \]

Let us assume that we are in the first case and the second case can be proved in the same way. Suppose that the lemma were not true. By passing to a convergent subsequence, we may assume that there exist a sequence \(p_k \rightarrow p \in K\) and a sequence \(q_k \rightarrow e^{i\alpha} \in \partial \Delta\) such that \(B_{p_k,q_k}\) has at most one critical point in the unit circle. Since

\[ \int_0^{2\pi} \frac{|p|^2 - 1}{|1 - pe^{i\alpha}|^2} d\alpha = 2\pi, \]

it follows that there exist \(\beta_1 < \alpha < \beta_2\) such that

\[ \frac{|p|^2 - 1}{|1 - pe^{i\beta_1}|^2} > 1, \text{ and } \frac{|p|^2 - 1}{|1 - pe^{i\beta_2}|^2} < 1. \]

Note that as \(q_k \rightarrow e^{i\alpha}\), \(\alpha_k \rightarrow \alpha\) where \(a_k = \text{arg}(q_k)\). By a simple calculation, it is easy to see that

\[ \frac{1 - |q_k|^2}{|1 - q_ke^{i\alpha}|^2} \rightarrow 0 \]

uniformly on any closed sub-interval of \([0, 2\pi]\) which does not contain \(\alpha\), and

\[ \frac{1 - |q_k|^2}{|1 - q_ke^{i\alpha_k}|^2} \rightarrow \infty. \]

It follows from (66) and (67) that for all \(k\) large enough, we have

\[ T'_{p_k,q_k}(\alpha_k) > 0, T'_{p_k,q_k}(\beta_1) < 0, \text{ and } T'_{p_k,q_k}(\beta_2) < 0. \]

Since \(\beta_1 < \alpha_k < \beta_2\) for all \(k\) large enough, the proof of the first case can thus be completed by using the Immediate Value Theorem. \(\Box\)

**Lemma 4.18.** For any compact set \(K \subset \mathbb{C} - \partial \Delta\), there is a \(\delta > 0\) such that if \(B_{p,q}\) has a critical point in \(K\), then \(d(p, \partial \Delta)L\delta\) and \(d(q, \partial \Delta) > \delta\).
Proof. This is because as \( p \) and \( q \) approach \( \partial \Delta \), by passing to a subsequence, \( B_{p,q} \) converges to a rigid rotation uniformly in any compact set \( K \subset \mathbb{C} - \partial \Delta \).

**Definition 4.1.** Let \( \mathcal{B} \) be the set which consists of all the Blaschke products \( B_{p,q} \) satisfying the following three properties:

1. \( B_{p,q} \) has a double critical point at \( 1 \),
2. the other two critical points \( c \) and \( \frac{1}{c} \) are symmetric about the unit circle such that \( c \in \hat{\mathbb{C}} - \Delta \),
3. \( |p| > 1 \) and \( |q| < 1 \).

**Lemma 4.19.** Let \( B_{p,q} \in \mathcal{B} \). Then \( B_{p,q}|\partial \Delta : \partial \Delta \to \partial \Delta \) is a homeomorphism which preserves the orientation.

Proof. Since
\[
\int_0^{2\pi} T'_{p,q}(\alpha) d\alpha = 2\pi,
\]

it follows that the topological degree of \( B|\partial \Delta : \partial \Delta \to \partial \Delta \) is 1. If \( B|\partial \Delta \) is not a homeomorphism, then \( B|\partial \Delta \) would have two distinct critical points. This is a contradiction with the definition of \( \mathcal{B} \). The fact that \( B|\partial \Delta \) preserves the orientation also follows. The proof of the lemma is complete.

**Critical Parameterization of \( \mathcal{B} \).** Let \( B_{p,q} \in \mathcal{B} \) and let \( w = p + q, v = pq \). Assume that \( c \neq \infty \). Therefore, \( q \neq 0 \) and hence \( v \neq 0 \). By a direct calculation, we get
\[
B'_{p,q}(z) = \frac{\overline{v}z^4 - 2\overline{w}z^3 + (3 + |w|^2 - |v|^2)z^2 - 2wz + v}{(\overline{v}z^2 - \overline{w}z + 1)^2}
\]

The numerator of \( B'_{p,q}(z) \) can be written into
\[
\overline{v}(z^4 - 2\overline{w}z^3 + (3 + |w|^2 - |v|^2)z^2 - 2wz + v) = \overline{v}(z - 1)^2(z - c)(z - \frac{1}{c}).
\]

It follows that \( v/\bar{v} = c/\bar{c} \) and hence \( v/c \) is a real number. So we have either \( v = c|v|/|c| \) or \( v = -c|v|/|c| \). Set \( t = (2 + c + \frac{1}{c})/2 \) and \( s = 1 + \frac{c}{2} + 2(c + \frac{1}{c}) \).

By comparing the coefficients of the two polynomials in the above equation, it follows that \( \frac{v}{\bar{v}} = t \) and hence \( |w|^2 = |t|^2|v|^2 \). It also follows that \( \frac{3}{\overline{v}} + |w|^2 - v = s \). This gives us

(68) \[
\frac{3}{\overline{v}} + |t|^2v - v = s.
\]

Note that \( \overline{v}s = 2(1 + |c|^2) + c + \overline{c} = 1 + |c|^2 + 1 + c^2 > 0 \), so if \( v = c|v|/|c| \), from (68) we get

(69) \[
(|t|^2 - 1)|v|^2 - |s||v| + 3 = 0,
\]

and if \( v = -c|v|/|c| \), we get

(70) \[
(|t|^2 - 1)|v|^2 + |s||v| + 3 = 0.
\]
Lemma 4.20. The map $\Phi : B \to \hat{\mathbb{C}} - \mathbb{X}$

by $\Phi(B_{p,q}) = c$.

Remark 4.4. For any $c$ with $|c| > 1$, let $(p_c, q_c)$ be one of the solutions obtained above such that $|p_c| > 1$ and $|q_c| < 1$. Then $B_{p_c, q_c}$ has exactly a double critical point at 1 and two distinct critical points at $c$ and $1/c$.

Recall that $t = (2 + c + \frac{1}{c})/2$. By a direct calculation, it follows that the curve

$$\gamma = \{ (c, t) \mid |c| > 1, |t| = 1 \} = \{ (r \cos t, r \sin t) \mid r > 0 \}$$

separates $\hat{\mathbb{C}} - \mathbb{X}$ into two components. Let us denote them by $U$ and $V$ respectively (see Figure 18). Define

$$\Phi : B \to \hat{\mathbb{C}} - \mathbb{X}$$

Example 1. Let $c = 2$. Then $t = \frac{3}{4}$ and $s = 7$. Since $|t|^2 - 1 > 0$, it follows that $v = c|v|/|c|$. By (71) and (72), we have $|v| = \frac{4}{3}$, or $|v| = \frac{12}{13}$. Then we have two cases:

Case 1. $v = \frac{4}{3}$, and $w = 9/5$. $(p, q) = \{1, \frac{4}{3}\}$.

Case 2. $v = \frac{12}{13}$, and $w = \frac{936491}{13}$. $(p, q) = \{1, \frac{12}{13}\}$.

Example 2. Let $c = -2$. Then $t = -\frac{1}{4}$ and $s = -3$. Since $|t|^2 - 1 < 0$, we have again two cases. In the first case, $v = c|v|/|c|$ and in the second case, $v = -c|v|/|c|$. By (73) and (74), we get $v = -4/5$ or $v = 4$.

Case 1. $v = -\frac{4}{5}$, and $w = \frac{5}{3}$. $(p, q) = \{1, -\frac{4}{5}\}$.

Case 2. $v = 4$ and $w = -1$. $(p, q) = \{-0.5 + 1.936491i, -0.5 - 1.936491i\}$. 

Since $|s|^2 - 12(|t|^2 - 1) = |c + \frac{1}{c} - 2|^2 > 0$ for all $c \neq 1$, it follows that for $|t|^2 - 1 > 0$, (70) has no positive solutions and $|v|$ must satisfy (69). Therefore, $v = c|v|/|c|$, and

$$|v| = \frac{|s| - |c + \frac{1}{c} - 2|}{2(|t|^2 - 1)},$$

or

$$|v| = \frac{|s| + |c + \frac{1}{c} - 2|}{2(|t|^2 - 1)}.$$
In the four cases of the two examples above, we see that only Case 2 of Example 1 produces the desired Blaschke product $B_{p,q}$ which satisfies $|p| > 1$ and $|q| < 1$. In the following, we will use continuation method to show that along this branch, all the other critical parameters in $U$ can produce a unique desired Blaschke product $B_{p,q}$, and that along all the other three branches, the solution pairs $\{p,q\}$ obtained do not satisfy the condition $|p| > 1$ and $|q| < 1$, that is, either one of them lies in the unit circle, or both of them belong to the outside of the unit disk.

**Proof.** It is clear that $\Phi$ is continuous. First let us prove that for any $B_{p,q} \in \mathcal{B}$, $\Phi(B_{p,q}) \in U$. Assume that this is not true. Let $\Phi(B_{p,q}) = c_0$. There are two cases.

In the first case, $c_0 \in \gamma$ where $\gamma$ is the open curve segment defined in (75). That is to say,

$$|v|^2 - 1 = |(2 + c_0 + \frac{1}{c_0})/2|^2 - 1 = 0.$$ 

It follows that $|v|$ must satisfy (69), which is degenerated to a linear equation in this case. So $|v|$ can be computed as the limit of (71) or (72) by letting $c \to c_0$ from the inside of $U$. It is easy to see that in (72), $|v|$ approaches to the infinity as $c$ approach to $c_0$ (the numerator has a positive lower bound but the denominator goes to zero). It thus follows that in this case, $|v|$ must be equal to the limit of (71) as $c$ approaches to $c_0$ from the inside of $U$. Take a curve segment $\eta \subset U$ which connects $c_0$ and the point 2 such that $d(\eta, \partial \Delta) > 0$. 

![Figure 18. The critical parameter space $U$](image-url)
For each $c \in \eta$, denote the corresponding values $s, t, v, w, p, q$ by $s_c, t_c, v_c, w_c, p_c, q_c$, respectively. Since $|t_c|^2 - 1 > 0$, $v_c$ satisfies (69). We thus have $v_c = c|v_c|/|c|$. For $c \in \eta$, we solve $|v_c|$ by (71) and get $w_c$ by the relation 

$$\bar{w}_c/\bar{v}_c = t_c = (2 + c + 1/\bar{c})/2.$$ 

Now we solve the pair $p_c, q_c$ which are the two solutions of the quadratic equation 

$$x^2 - w_c + v_c = 0.$$ 

Clearly, $p_c$ and $q_c$ depend continuously on $c$. From Case 1 of Example 1, it follows that $\{p_2, q_2\} = \{1, 4/5\}$. We now claim that there is a $\delta > 0$, such that for each $c \in \eta$, either $d(p_c, \partial\Delta) > \delta$, or $d(q_c, \partial\Delta) > \delta$. In fact, if this were not true, then we would have a sequence $\{c_k\} \subset \eta$ such that 

$$p_{c_k} \to \partial\Delta \text{ and } q_{c_k} \to \partial\Delta.$$ 

By passing to a convergent subsequence, it follows from (65) that there is some real constant $\alpha$ such that 

$$B_{p_{c_k}, q_{c_k}} \to e^{i\alpha} z$$ 

uniformly in any compact subset of $\mathbb{C} - \partial\Delta$. In particular, 

$$d(c_k, \partial\Delta) \to 0$$ 

as $k \to \infty$. But this is a contradiction with $d(\eta, \partial\Delta) > 0$. The claim has been proved.
Since $|p_{c_0}| > 1$ and $|q_{c_0}| < 1$, and $\{p_2, q_2\} = \{1, 4/5\}$, it follows that there is a sequence $\{c_k\} \subset \eta$ such that either $p_{c_k}$ is contained in some compact set in the outside of the unit disk and $q_{c_k}$ lies in the inside of the unit disk and approaches $\partial \Delta$, or $q_{c_k}$ is contained in some compact set in the inside of the unit disk and $p_{c_k}$ lies in the outside of the unit disk and approaches $\partial \Delta$. But by Lemma 4.17, both of the two possibilities imply that $B_{p_{c_k}, q_{c_k}}$ has two distinct critical points on $\partial \Delta$ for all $k$ large enough. This is a contradiction with Remark 4.4.

In the second case, $c_0 \in V$. Then we take a curve segment $\eta' \subset V$ which connects $-2$ and $c$. See Figure 20 for an illustration. There are two curves of $\{p_{c}, q_{c}\}$ which are determined by the two choices of $\{p_{-2}, q_{-2}\}$ in Example 2 respectively.

For the first choice, $\{p_{-2}, q_{-2}\} = \{1, 4/5\}$. We can get a contradiction by using the same argument as in the proof of the first case.

For the second choice, $\{p_{-2}, q_{-2}\} = \{-0.5 + 1.936491i, -0.5 - 1.936491i\}$. So $|p_{-2}| = |q_{-2}| > 1$. Since $|p_{c_0}| > 1$ and $|q_{c_0}| < 1$, and since $p_c$ and $q_c$ can not be both close to $\partial \Delta$ with $|p_c| > 1$ and $|q_c| < 1$(otherwise we get a contradiction by Lemma 4.18 and Remark 4.4), there would be a sequence $\{c_k\} \subset \eta'$ such that either $p_{c_k}$ is contained in some compact set in the outside of the unit disk and $q_{c_k}$ lies in the inside of the unit disk and approaches $\partial \Delta$, or $q_{c_k}$ is contained in some compact set in the inside of the unit disk and $p_{c_k}$ lies in the outside of the unit disk and approaches $\partial \Delta$. Again by Lemma 4.17, both of the two possibilities imply that $B_{p_{c_k}, q_{c_k}}$ has two distinct critical points on $\partial \Delta$ for all $k$ large enough. This is a contradiction with Remark 4.4.
The above argument implies that $\Phi(B) \subset U$. Next we need to prove that for each $c_0 \in U$, there is a $B_{p,q} \in \mathcal{B}$ such that $\Phi(B_{p,q}) = c_0$. In fact, since $U$ is simply connected, we can take a curve segment, say $\eta'' \subset U$ to connect the point 2 and $c_0$. For each $c \in \eta''$, we solve $|v_c|$ by (72) and then get $v_c = c|v_c|/|c|$, and a continuous curve of $\{p_c, q_c\}$. From Case (2) of Example 1, we have that $|p_2| > 1$ and $|q_2| < 1$. We claim that $|p_{c_0}| > 1$ and $|q_{c_0}| < 1$. Suppose this were not true. Then the same argument as above will induce a contradiction again. This implies that $\Phi(B) = U$. Finally let us show that $\Phi$ is injective. Assume that for some $c \in U$, we have two different pairs $\{p, q\}$ and $\{p', q'\}$ such that $|p| > 1, |q| < 1, |p'| > 1, |q'| < 1$ and $\Phi(B_{p,q}) = \Phi(B_{p',q'}) = c$. Take a curve segment $\eta \subset U$ which connects $c$ and the point 2. Then we have two curves of pairs $\{p_c, q_c\}, c \in \eta$. It follows that one of them is determined by (71), along which we get $\{p_2, q_2\} = \{1, 4/5\}$, which is Case (1) of Example 1. Now the same argument above will induce a contradiction again. This proves that $\Phi$ is injective.

Finally let us show that $\Phi^{-1}$ is continuous also. In fact, for each $c \in U$, compute $|v_c|$ by (72) and get $v_c$ by

$$v_c = c|v_c|/|c|.$$ 

Then we get $w_c$ by the relation

$$\tilde{w}_c/\tilde{v}_c = t_c = (2 + c + 1/\bar{c})/2.$$ 

Now the pair $p_c, q_c$ is determined by the solutions of the quadratic equation

$$x^2 - w_c + v_c = 0.$$
By Case (2) of Example 1, and the same argument as before, it follows that one of the two solutions lies in the outside of the unit disk, and the other one lies in the inside of the unit disk. Let $p_c$ be the one such that $|p_c| > 1$ and $q_c$ be the other one such that $|q_c| < 1$. It is clear that $p_c$ and $q_c$ depend continuously on $c$.

Note that as $c \to \infty$, by solving (72) we get $p_c \to 3$ and $q_c \to 0$. We can thus define $\Phi(B_{3,0}) = \infty$. This completes the proof of the lemma. □

By Proposition 11.7 of [14] (and see also §9 of [30]), we have

**Lemma 4.21.** For each $B_{p,q} \in B$, there is a unique $t \in [0,2\pi)$ such that the rotation number of $e^{it}B_{p,q}|_{\partial \Delta}$ is $\theta$. Moreover, $t$ depends continuously on $B_{p,q}$.

Let us denote $e^{it}B_{p,q}$ by $G_{p,q}$.

### 4.7. The Cross Ratio Function $\lambda(k,l,m,n)$

For any four distinct points $z_1, z_2, z_3, z_4$, their cross ratios have several definitions. Since the properties established in this sections are true for any of them, let us simply use the same notation $C(z_1, z_2, z_3, z_4)$ to denote them. For $0 \leq k < l < m < n$, set

$$
\lambda_{k,l,m,n}(c) = C(g_k^c(1), g_l^c(1), g_m^c(1), g_n^c(1))
$$

and

$$
\alpha(k,l,m,n) = C(e^{2\pi ik\theta}, e^{2\pi il\theta}, e^{2\pi im\theta}, e^{2\pi in\theta}).
$$

Let $\xi \subset \hat{C}$ be the simple closed curve in Lemma 4.10. Let $\Omega_0$ and $\Omega_{\infty}$ denote the bounded and unbounded components of $\hat{C} - \xi$, respectively. For $R > 0$ large enough such that $\xi \subset \{z||z| < R\}$, let $U_R = \{||z| > R\}$ and $\Omega_R = \Omega_{\infty} - U_R$.

**Lemma 4.22.** Let $c \in \Omega_{\infty}$. Then the forward orbit of 1 under $g_c$ is not finite.

**Proof.** Let us prove it by contradiction. Assume that $g_k^c(1) = g_l^c(1)$ for some integers $0 \leq k < l$. Let $u = 1/c$. It follows that $u \in \Omega_0$ and $g_k^u(u) = g_l^u(u)$.

By Lemma 4.11, $g_u \in R_{g_{\text{geom}}}$ and hence is modeled by some Blaschke product $G_{p,q} = e^{it}B_{p,q}$ where $t \in [0,2\pi)$ and $B_{p,q} \in B$ (see Lemma 4.21). Let $c_0 = \Phi(B_{p,q})$.

Let $U$ be the parameter space in Lemma 4.20. Take a continuous curve $\gamma : [0,1] \to U$ such that $\gamma(0) = c_0$ and $\gamma(1) = \infty$. For each $s \in [0,1]$, let $B_s = \Phi^{-1}(\gamma(s))$ and $G_s$ be the corresponding Blaschke product determined by $B_s$ (see Lemma 4.21). We thus get a continuous family of Blaschke products $G_s$, $0 \leq s \leq 1$. Now for each $0 \leq s \leq 1$, we may perform a quasiconformal surgery on $G_s$ as described in the proof of Lemma 4.7 and get a Siegel rational map $g_{c(s)}$. This surgery induces a surgery map

$$
S : [0,1] \to \hat{C} - \{0,1,-1\}.
$$
by $S(s) = c(s)$. It is clear that

$$S(0) = u \text{ and } S(1) = \infty.$$  

By using Zakeri’s argument (see §12 of [30]), one can show that the map $S$ is continuous in $[0, 1]$. That is, $S(s), 0 \leq s \leq 1$ is a continuous curve connecting $u$ and the infinity. Since $u \in \Omega_0$, by Lemma 4.10 the curve $S(s), 0 \leq s \leq 1$ intersects $\xi$ at some point. But on the other hand, since $G_s$ has exactly one double critical point at $1$ and the other two critical points do not lie in the unit circle for every $0 \leq s \leq 1$, it follows that $S(s)$ does not lies in the boundary of the Siegel disk. In particular, the curve $S(s), 0 \leq s \leq 1$ does not intersect $\xi$. This is a contradiction. The lemma follows.

From Lemma 4.22, it follows that $\lambda_{k,l,m,n}$ is holomorphic and has no zeros in $\Omega_R$.

**Lemma 4.23.** $\lambda_{k,l,m,n}(c)$ can be continuously extended to $\partial\Omega_R$.

**Proof.** It suffices to prove that both $\lim_{c \to 1} \lambda_{k,l,m,n}(c)$ and $\lim_{c \to -1} \lambda_{k,l,m,n}$ exist and are finite. In fact, from (51) in §4.1, it follows that

$$\lim_{c \to 1} g_c(1) = \lambda,$$

and for $z \neq 1$,

$$\lim_{c \to -1} g_c(z) = \lambda z,$$

where $\lambda = e^{2\pi i \theta}$. This implies that for given $0 \leq k < l < m < n$,

$$\lim_{c \to 1} \lambda_{k,l,m,n}(c) = \alpha(k, l, m, n).$$

This proves that $\lambda_{k,l,m,n}(c)$ can be continuously extended to the point $1$.

Now let us consider the case that $c \to -1$. By solving $g_c(z) = z$, it follows that as $c$ is close to $-1$, $g_c$ has three distinct fixed points $0, p_c, \text{ and } \infty$ where

$$p_c = \frac{(1 - \lambda)2c(1 + c)}{4c - \lambda(1 + c)^2} \to 0$$

as $c$ approaches $-1$. Let $\phi_c$ be the M"{o}bius map such that $\phi_c(0) = 0, \phi_c(1) = 1$ and $\phi_c(p_c) = \infty$. Then $\phi_c \circ g_c \circ \phi_c^{-1} = g'_{c'}$ where $c' = \phi_c(c)$. By a direct calculation, we have

$$\phi_c(z) = (1 - p_c)z/(z - p_c).$$

It follows that

$$c' = \phi_c(c) = (1 - p_c)c/(c - p_c).$$

Since $p_c \to 0$ as $c \to -1$, it follows that $c' \to 1$. Since the cross ratio $\lambda_{k,l,m,n}(c)$ is preserved by M"{o}bius transformations, it follows that

$$\lambda_{k,l,m,n}(c) = \lambda_{k,l,m,n}(c').$$

We thus get that

$$\lim_{c \to -1} \lambda_{k,l,m,n}(c) = \lim_{c' \to 1} \lambda_{k,l,m,n}(c') = \alpha(k, l, m, n).$$
Lemma 4.24. $\lambda_{k,l,m,n}$ has a removable singularity at $\infty$, and moreover, $\lim_{c \to \infty} \lambda_{k,l,m,n}(c) \neq 0$.

Proof. It suffices to prove that $\lim_{c \to \infty} \lambda_{k,l,m,n}(c)$ exists. From (51) in §4.1, it follows that for any compact set $K$ of the complex plane,
$$\lim_{c \to \infty} g_c(z) = \lambda z - \frac{\lambda^2}{2} = g_\infty(z)$$
uniformly on $K$. Therefore, for any integer $k \geq 0$, $g^k_c(1) \to g^k_\infty(1)$ as $c \to \infty$. Since $g_\infty$ has a Siegel disk with quasi-circle boundary which passing through 1, it follows that the cross ratio $C(g^k_\infty(1), g^l_\infty(1), g^m_\infty(1), g^n_\infty(1))$ is defined and is not equal to 0. All of these imply that
$$\lim_{c \to \infty} \lambda_{k,l,m,n}(c) = C(g^k_\infty(1), g^l_\infty(1), g^m_\infty(1), g^n_\infty(1))$$
is a finite non-zero complex value. The lemma follows. □

Let us summarize the above lemmas as the following,

Proposition 4.1. For any integers $0 \leq k < l < m < n$, $\lambda_{k,l,m,n}(c)$ is a non-zero and holomorphic function in $\Omega_\infty$. Moreover, it can be continuously extended to $\partial \Omega_\infty$.

Remark 4.5. Note that the distortion of a cross ratio by a $K-$quasiconformal homeomorphism of the sphere is bounded by some constant dependent only on $K$. This is an important fact which will be used in the proof of Theorem C.

4.8. Proof of Theorem C. Recall that $\Omega_\infty$ is the unbounded component of $\hat{\mathbb{C}} - \xi$. For each $c \in \Omega_\infty$, let $\gamma_c$ be the closure of $\{g^k_c(1), k = 0, 1, \cdots \}$.

Lemma 4.25. For each $c \in \Omega_\infty$, $\gamma_c$ is a Jordan curve.

Proof. When $c = \infty$, $g_c$ is a quadratic polynomial with a bounded type Siegel disk, and the lemma follows from the well-known theorem of Douady, Ghys, Herman, and Shishikura [19]. Let us assume that $c \neq \infty$.

First let us show that $\gamma_c$ is contained in some compact set of $\mathbb{C}$. In fact, if this were not true, there would be a subsequence, say $k_1, k_2, \cdots$, such that $g^{k_i}_c(1) \to \infty$ and $e^{2\pi ik_i \theta} \to t$ for some $t \in \partial \Delta$ as $i \to \infty$. Since $g_c$ fixes the infinity, it follows that $g^{k_i+1}_c(1) \to \infty$ also. Take integers $m, n \geq 0$ such that $e^{2\pi im \theta}, e^{2\pi in \theta}, e^{2\pi i t \theta}$ are all distinct with each other. On the one hand, we have
$$\frac{(g^m_c(1) - g^n_c(1))(g^{k_i}_c(1) - g^{k_i+1}_c(1))}{(g^m_c(1) - g^{k_i}_c(1))(g^n_c(1) - g^{k_i+1}_c(1))} \to 0$$
as $i \to \infty$. On the other hand, By Proposition 4.11, the cross ratio function on the left hand of (77) is holomorphic in $c$ and has no zeros in $\Omega_\infty$, and moreover, it can be continuously extended to $\partial \Omega_\infty = \xi$. It follows that its
minimum of its modulus is obtained on \( \partial \Omega_\infty = \xi \). Since \( e^{2\pi ik_0} \to t \) and \( e^{2\pi im_0}, e^{2\pi in_0}, t, e^{2\pi i\theta}t \) are all distinct, the cross ratio
\[
\frac{(e^{2\pi im_0} - e^{2\pi in_0})(e^{2\pi ik_0} - e^{2\pi i(k_1+1)\theta})}{(e^{2\pi im_0} - e^{2\pi ik_0})(e^{2\pi in_0} - e^{2\pi i(k_1+1)\theta})}
\]
is uniformly bounded away from zero for all \( i \) large enough. By Lemma 4.3 and Remark 4.5, it follows that the modulus of the cross ratio function on the left hand of (77) has a positive lower bound when restricted on \( \xi \). This is a contradiction.

Now for \( c \in \Omega_\infty \), define a map \( T_c : \{e^{2\pi ik\theta}, k = 0, 1, \cdots \} \to \hat{\mathbb{C}} \) by
\[
T_c(e^{2\pi ik\theta}) = g^\alpha_k(1).
\]
First let us show that \( T_c \) is uniformly continuous. To see this, note that \( \theta \) is irrational, and therefore there is an \( M > 0 \) dependent only on \( \theta \) such that for any \( 0 < \delta < 1/100 \), and any \( k, l \) with
\[
|e^{2\pi ik\theta} - e^{2\pi il\theta}| < \delta,
\]
there exist integers \( 0 \leq m, n \leq M \) such that
\[
|e^{2\pi im\theta} - e^{2\pi in\theta}| < 1/4, |e^{2\pi im\theta} - e^{2\pi ik\theta}| > 1/4, \text{ and } |e^{2\pi in\theta} - e^{2\pi il\theta}| > 1/4.
\]
The existence of such \( M \) is obvious since for \( M \) large, the orbit segment \( \{e^{2\pi im\theta}, 0 \leq t \leq M \} \) will be dense enough in \( \partial \Delta \) so that one can find two elements \( e^{2\pi im\theta} \) and \( e^{2\pi in\theta} \) in this orbit segment which satisfy the above three inequalities.

For such \( m \) and \( n \), we have
\[
(e^{2\pi im\theta} - e^{2\pi in\theta})(e^{2\pi ik\theta} - e^{2\pi il\theta}) < 4\delta.
\]
By Remark 4.3 and Lemma 4.3, it follows that there is a positive function \( k(\delta) \) satisfying \( k(\delta) \to 0 \) as \( \delta \to 0 \), such that for all \( t \in \xi \),
\[
|g^m_t(1) - g^n_t(1)| < k(\delta)
\]
From (79), Proposition 4.1, and the maximal modulus principle, we have
\[
|g^m_t(1) - g^n_t(1)| < k(\delta)
\]
for all \( c \in \Omega_\infty \).

Since \( 0 \leq m < n \) are bounded by \( M \) which depends only on \( \theta \), for any given \( c \),
\[
|g^m_c(1) - g^n_c(1)|
\]
has a positive lower bound. Since we have proved that the absolute value of the denominator of the above fraction has an upper bound in the beginning of the proof, it follows that
\[
|g^k_c(1) - g^l_c(1)| < Ck(\delta)
\]
for some uniform \( C > 0 \). This implies the uniform continuity of \( T_c \). Now we can continuously extend \( T_c \) to the unit circle.
We now need only to prove that $T_c$ is injective. We prove this by contradiction. Assume that there exist $x, y \in \partial \Delta$ such that $T_c(x) = T_c(y)$ and $x \neq y$. Take subsequences $k_i, l_i$ such that $e^{2\pi i k_i \theta} \to x$ and $e^{2\pi i l_i \theta} \to y$ as $i \to \infty$. Take integers $m, n$ such that $T_c(e^{2\pi i m \theta}), T_c(e^{2\pi i n \theta})$ and $T_c(x)$ are all distinct. It follows that $e^{2\pi i m \theta}, e^{2\pi i n \theta}, x$ and $y$ are all distinct. Then there is a uniform $\delta > 0$ such that
\[
\left| \frac{(e^{2\pi i m \theta} - e^{2\pi i l \theta})(e^{2\pi i k \theta} - e^{2\pi i l \theta})}{(e^{2\pi i m \theta} - e^{2\pi i k \theta})(e^{2\pi i l \theta} - e^{2\pi i n \theta})} \right| \geq \delta
\]
for all $i$ large enough. By Lemma 4.3 and Remark 4.5, it follows that there is a constant $C(\delta) > 0$ which depends only on $\delta$ such that
\[
\left| \frac{g^{m_1}_i(1) - g^{n_1}_i(1))(g^{k_1}_i(1) - g^{l_1}_i(1))}{(g^{m_1}_i(1) - g^{k_1}_i(1))(g^{n_1}_i(1) - g^{l_1}_i(1))} \right| > C(\delta)
\]
for all $t \in \xi$. This, together with Proposition 4.1 and the minimal modulus principle, implies
\[
\left| \frac{g^{m_1}_i(1) - g^{n_1}_i(1))(g^{k_1}_i(1) - g^{l_1}_i(1))}{(g^{m_1}_i(1) - g^{k_1}_i(1))(g^{n_1}_i(1) - g^{l_1}_i(1))} \right| > C(\delta).
\]
Since $T_c(e^{2\pi i m \theta}), T_c(e^{2\pi i n \theta})$ and $T_c(x)$ are distinct with each other and $T(x) = T(y)$, the absolute value of the denominator of the above fraction has a positive lower bound. But the numerator goes to zero as $i \to \infty$. This is a contradiction. The lemma follows.

Using the above argument in the proof of the uniform continuity of $T_c$, the reader shall easily supply a proof of the following lemma,

**Lemma 4.26.** Define $T : \Omega_\infty \times \partial \Delta \to \mathbb{C}$ by $T(c, x) = T_c(x)$. Then $T$ is continuous.

The following lemma characterizes a quasi-circle in the complex plane by the lower bound of the cross ratios of every four ordered points on it (Lemma 9.8 [23]).

**Lemma 4.27.** For each $\delta > 0$, there is a $K(\delta) > 1$ such that for any simple closed curve $\gamma \subset \mathbb{C}$, if for every four ordered points $z_1, z_2, z_3, z_4 \in \gamma$, the following inequality hold,
\[
\left| \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)} \right| \geq \delta,
\]
then $\gamma$ is a $K(\delta)$–quasi-circle. Similarly, for each $K > 1$, there is a $\delta(K) > 0$ such that for any $K$–quasi-circle $\gamma \subset \mathbb{C}$, the following inequality hold for every four ordered points $z_1, z_2, z_3, z_4$ on $\gamma$,
\[
\left| \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)} \right| \geq \delta(K).
\]
Now let us prove Theorem C. By Lemma 4.25, \( \gamma_c \) is a simple closed curve. By Lemma 4.3 there exists a uniform \( 1 < K < \infty \) such that the boundary of the Siegel disk of \( g_c \) for every \( c \in \xi \) is a \( K \)-quasi-circle. By Lemma 4.27 the inequality (82) holds for every \( c \in \xi \) and any four ordered points \( z_1, z_2, z_3, z_4 \) on \( \gamma_c \). By Proposition 4.1 and minimal modulus principle, it also holds for every \( c \in \Omega_\infty \). By Lemma 4.27 again, it follows that there is a uniform \( 1 < K' < \infty \) such that \( \gamma_c \) is a \( K' \)-quasi-circle for every \( c \in \Omega_\infty \).

Now we need only to show that \( \gamma_c \) is the boundary of the Siegel disk of \( g_c \) which is centered at the origin. By Lemma 4.26, \( \gamma_c \) moves continuously as \( c \) varies in \( \Omega_\infty \). Let \( D_c \) be the bounded component of \( \mathbb{C} - \gamma_c \). We will show that \( D_c \) is the Siegel disk of \( g_c \). First let us show that \( g_c \) is holomorphic on \( D_c \). When \( c = \infty \), this is obviously true. As \( c \) varies from the infinity to any value in \( \Omega_\infty \), the finite pole of \( g_c \), say \( p_c \), varies continuously. But \( g_c(\gamma_c) = \gamma_c \) and \( \infty \notin \gamma_c \), it follows that \( \gamma_c \) does not meet the pole \( p_c \). It follows that \( p_c \notin D_c \) for otherwise there is some \( c \) such that \( \gamma_c \) meets \( p_c \), which is a contradiction. This implies that \( g_c \) is holomorphic on \( D_c \). Since \( g_c(\partial D_c) = g_c(\gamma_c) = \partial D_c \), it follows that
\[
g_c(D_c) = D_c.
\]
This implies that \( D_c \) is a periodic Fatou component of \( g_c \). Since \( 0 \notin \gamma_c \) for every \( c \in \Omega_\infty \), and \( 0 \in D_c \), by the same argument as above, it follows that \( 0 \in D_c \) for every \( c \in \Omega_\infty \). Because \( g_c(0) = e^{2\pi i \theta} \) and \( g_c(0) = 0 \), it follows that \( D_c \) is the Siegel disk of \( g_c \) which is centered at the origin, and in particular, \( \partial D_c \) passes through the critical point 1 of \( g_c \). This completes the proof of Theorem C.

Let \( \xi \) be the simple closed curve in Lemma 4.10. Recall that \( \Omega_0 \) is the bounded component of \( \hat{\mathbb{C}} - \xi \) and \( \Omega_\infty \) the unbounded one. For \( c \in \hat{\mathbb{C}} - \{0, 1, -1\} \), let \( D_c \) be the Siegel disk of \( g_c \) which is centered at the origin. Based on the proof of Theorem C, the reader shall easily draw the following conclusion,

**Proposition 4.2.** Let \( c \in \hat{\mathbb{C}} - \{0, 1, -1\} \). We have (1) if \( c \in \xi \), \( \partial D_c \) passes through both of the critical points 1 and \( c \), (2) if \( c \in \Omega_\infty \), \( \partial D_c \) passes through 1 only, (3) if \( c \in \Omega_0 \), \( \partial D_c \) passes through \( c \) only.

**Corollary 4.1.** Let \( f \) be a degree-3 rational map with a bounded type Herman ring. Then each boundary component of the Herman ring is a quasi-circle which passes through at least one but at most two of the critical points of \( f \).

**Proof.** This is because for any boundary component \( \gamma \) of the Herman ring, by using a quasi-conformal surgery, one can get a quadratic rational map with a Siegel disk which has the same rotation number as the Herman ring and which has \( \gamma \) as its boundary. We leave the details to the reader. \( \square \)
5. Appendix

5.1. Thurston’s characterization theory on postcritically finite rational maps. Since the Thurston’s characterization theorem used in this paper is slightly different from the one presented in [9], we will give a brief introduction of this theory, which has been adapted to our situation: we use a larger invariant set $X \supseteq P_f$, instead of the postcritical set $P_f$. The proof is completely the same as the one presented in [9].

Let $f : S^2 \to S^2$ be a postcritically finite branched covering map. Let $X \subset S^2$ be a finite set such that $f(X) \subseteq X$ and $P_f \subseteq X$. A simple closed curve in $S^2 - X$ is said to be non-peripheral if $\gamma$ is not homotopic to a point in $S^2 - X$. A multi-curve of $f$ in $S^2 - X$ is a family of disjoint, non-homotopic and non-peripheral curves. We say a multi-curve $\Gamma$ is $f$–stable if for any $\gamma \in \Gamma$, any non-peripheral component of $f^{-1}(\gamma)$ is homotopic in $S^2 - X$ to one of the elements in $\Gamma$.

Two branched covering maps $f$ and $g$ are said to be combinatorially equivalent with respect to the set $X$ if there are two homeomorphisms of the sphere $\varphi, \psi$ which are isotopic to each other rel $X$ such that $f = \varphi^{-1} \circ g \circ \psi$.

Let $\Gamma = \{\gamma_1, \cdots, \gamma_n\}$ be a $f$–stable multi-curve. For each $\gamma_j$, let $\gamma_{i,j,\alpha}, \alpha \in \Lambda$ be the non-peripheral components of $f^{-1}(\gamma_j)$ which is homotopic to $\gamma_i$. Define

$$a_{i,j} = \frac{1}{d_{i,j,\alpha}}.$$

The matrix $A = (a_{i,j})_{n \times n}$ is called the Thurston linear transformation matrix of $f$. $\Gamma$ is called a Thurston obstruction if the maximal eigenvalue of $A$ is greater than 1.

Associated to each postcritically finite rational map $f$, one can construct an orbifold $O_f = (S^2, \nu_f)$ by defining $\nu_f : S^2 \to \mathbb{Z}^+ \cup \{\infty\}$ to be the minimal function satisfying the following two conditions,

1. $\nu_f(x) = 1$ for $x \notin P_f$,
2. $\nu_f(x)$ is a multiple of $\nu_f(y) \deg_y f$ for each $y \in f^{-1}(x)$.

An orbifold is called hyperbolic if

$$\chi_f = 2 - \sum_{\nu_f(x) \geq 2} (1 - \frac{1}{\nu_f(x)}) < 0.$$

Thurston’s Characterization Theorem. Let $f$ be a postcritically finite branched covering map of the sphere. Let $X$ be a finite set such that $f(X) \subseteq X$, and $P_f \subseteq X$. Assume that the orbifold $O_f$ is hyperbolic. Then $f$ is combinatorially equivalent to a rational map with respect to $X$ if and only if $f$ has no Thurston obstructions in $S^2 - X$.

5.2. Short simple closed geodesics. In this appendix, we present a few results on the simple closed geodesics in a hyperbolic Riemann surface, and for detailed proofs of these results, we refer the reader to §6 and §7 of [DH].
Theorem A.1 (Corollary 6.6, [9]). Let $X$ be a hyperbolic Riemann surface and $\gamma_1, \gamma_2$ be two simple closed geodesics with length $< \log(\sqrt{2} + 1)$. Then either $\gamma_1 = \gamma_2$ or $\gamma_1 \cap \gamma_2 = \emptyset$.

Theorem A.2 (Corollary 6.7, [9]). Let $X$ be a hyperbolic Riemann surface. Let $\gamma$ be a geodesic in $X$ which intersects itself transversally at least once. Then $l_X(\gamma) > 2 \log(\sqrt{2} + 1)$.

Theorem A.3 (Theorem 7.1, [9]). Let $X$ be a hyperbolic Riemann surface, $P \subset X$ a finite set, with $|P| = p > 0$. Choose $L < \log(\sqrt{2} + 1)$. Let $X' = X - P$. Let $\gamma$ be a simple closed geodesic in $X$ and $\{\gamma_1', \cdots, \gamma_s'\}$ be the simple closed geodesic in $X'$ which is homotopic to $\gamma$ in $X$ with length $< L$. Then

$$\frac{1}{l} - \frac{2}{\pi} - \frac{p + 1}{L} < \sum_{1 \leq i \leq s} \frac{1}{l_i} < \frac{1}{l} + \frac{2(p + 1)}{\pi}.$$

Theorem A.4 (Proposition 7.2, [9]). Let $P \subset S^2$ be a finite set, and $\gamma$ be a non-peripheral curve in $S^2 - P$. Let $\phi, \psi : S^2 \to \mathbb{P}^1$ be quasiconformal homeomorphisms. If $\text{dist}_{T(S^2, P)}(\phi, \psi) < K$, then

$$e^{-2K} \|\gamma\|_{\phi, P} \leq \|\gamma\|_{\psi, P} \leq e^{2K} \|\gamma\|_{\phi, P}.$$ 

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