Improved bounds on the Radio degree of a cycle

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Abstract. A labeling $f : V(G) \rightarrow \mathbb{Z}^+$ such that $|f(u) - f(v)| \geq \text{diam}(G) + 1 - d(u, v)$ holds for every pair of vertices, $u, v \in V(G)$, is called a radio labeling of a graph, $G$. The radio degree of a labeling, $f : V(G) \rightarrow \{1, 2, \ldots |V(G)|\}$ in a graph, was defined by the same authors as the number of pairs of vertices $u, v \in V(G)$ satisfying the condition $|f(u) - f(v)| \geq \text{diam}(G) + 1 - d(u, v)$ and was denoted by $rdeg(f)$. The maximum value of $rdeg(f)$ taken over all such labelings was defined as the radio degree of the graph, denoted by $rdeg(G)$. The radio degree of some standard graphs like paths, complete graphs, complete bipartite graphs, wheel graph and fan graph was completely determined and a lower bound on the radio degree of cycles was obtained. In this paper, the authors have obtained better bounds on the radio degree of a cycle.

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1. Introduction

Simple, finite, connected and undirected graphs are considered in this paper. Standard graph terms are used and the terms not defined here may be found in \cite{1, 2}. Extensive applications of wireless networks brought back channel assignment problem or frequency assignment problem into limelight among the researchers worldwide. Assigning a non-negative integer(channel) to each of the given set of stations such that, there is no interference between any two of them, keeping the highest frequency used to the minimum is the well known channel assignment problem.

Channel assignment problem can be modelled as a graph labeling problem. In 1980, Hale \cite{13} has modelled Channel assignment problem as a generalised graph coloring problem. Each station is denoted by a vertex and each edge denotes a possible interference of transmission between the corresponding vertices (stations). Since the interference is bound to be directly proportional to the geographical proximity of the stations, the separation of the channels assigned to them must be large enough. This assignment should also satisfy the requirement that the highest integer used, called the span of the labeling must be minimised. Chartrand et al. introduced the concept of
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Radio labeling [3] in 2001. In [3], they defined radio labeling of a graph $G$ as a function $f : V(G) \rightarrow \mathbb{Z}^+$ such that for all $u, v \in V(G)$, $|f(u) - f(v)| \geq \text{diam}(G) + 1 - d(u, v)$.

Some practical applications of radio labeling have been discussed in [17, 18]. A radio labeling of a simple graph of diameter 2 is shown in Figure 1 below:

![Figure 1. A radio labeling of a graph $G$](image)

As shown in the Figure 1 the span of the labeling is 7. The least value of the span over all such radio labelings is called the radio number of the graph [3]. The radio number for paths and cycles was studied by Chartrand et al. [14, 4] but the exact value remained open until it was solved by Liu and Zhu in [6]. They have determined the exact value for the radio number of path $P_n$, $n \geq 4$, as $2p^2 - 2p + 2$ if $n = 2p$ and $2p^2 + 3$ if $n = 2p + 1$. They have also determined the exact value of radio number of cycle, $C_n$ for $n \geq 3$. If radio number of the graph is equal to $n$, the number of vertices of the graph, then the graph is called radio graceful [3]. The graph shown below in Figure 2 is an example of a radio graceful graph.

The problem of obtaining an optimal assignment of channels for a specified set of radio stations, according to some prescribed restrictions on the distances between the stations, led to the introduction of radio $k$-colorings of graphs by Chartrand et al. [14]. A radio $k$-coloring $f$ of graph $G$ assigns positive integers to the vertices of $G$ in such a way that $d(u, v) + |f(u) - f(v)| \geq 1 + k$ holds for every pair of distinct vertices $u, v$ of $G$, where $1 \leq k \leq \text{diam}(G)$. The maximum color assigned in this process is called the span of a radio $k$-coloring and the minimum value taken over all the radio $k$-colorings of a graph, $G$ is called the radio $k$-chromatic number of $G$. The radio 1-chromatic number is simple the chromatic number $\chi(G)$. When $k = \text{diam}(G)$, it is same as the radio coloring of $G$, see [3]. In this case, the radio $k$-chromatic number is called the radio number of $G$. The radio number for paths and cycles was investigated by [3, 4] and was completely solved by Liu and Zhu in [6]. Liu [7] proved a lower bound for the radio number of trees and characterized the trees achieving this bound.

The case of $k = \text{diam}(G) - 1$ in radio $k$-coloring, is known as the **radio antipodal**
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Figure 2. A radio graceful graph

coloring of $G$. Two vertices of a graph $G$ are said to be antipodal of they are at a distance equal to diameter of the graph. The span in such a case is called the radio antipodal number $ac(G)$ of $G$. Radio antipodal chromatic numbers of paths and cycles are discussed in [15, 16]. Upper and lower bounds are presented for the same. Finding radio $k$-chromatic number of $G$ is highly non-trivial.

The square of a graph is obtained by adding edges between vertices of distance two apart. The radio number for square cycles were studied in [5], where, the radio numbers for the square of even cycles were completely determined whereas a lower bound was obtained in the case of odd cycles. Square paths (paths obtained by adding edges between vertices of distance two apart) were studied in [8]. Sooryanarayana and Raghunath [9] have determined the radio number of the cube of a cycle (cycles obtained by adding edges between vertices of distance two and distance three apart) for all $n \leq 20$ and for $n \equiv 0$ or $2$ or $4 \pmod{6}$.

If the labelling is defined as $f : V(G) \to \{1, 2, \ldots |V(G)|\}$, the minimum number of such functions which together satisfy the radio labeling condition was defined as radiatic dimension and has been studied in [10]. The cardinality of the largest subset of $V(G)$ satisfying the radio labeling condition was defined as radio secure number of a graph and has been studied in [11].

In [12], the authors gave a different perspective to radio gracefulness by defining a new concept called radio degree of a graph. Radio degree is a measure of how close a given graph is to radio gracefulness. The radio degree of a labeling $f : V(G) \to \{1, 2, \ldots |V(G)|\}$ was defined as the number of pairs of vertices $u, v \in V(G)$ satisfying the condition $|f(u) - f(v)| \geq diam(G) + 1 - d(u,v)$ and was denoted by $rdeg(f)$. The maximum value of $rdeg(f)$ taken over all such labelings was defined as the radio degree of the graph, denoted by $rdeg(G)$. We give an example below to illustrate this.

In Figure 3, a wheel graph is given, whose diameter is 2. Let $v_1$ be the full vertex
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and \( v_i, 2 \leq i \leq 5 \) be the vertices on the rim of the wheel in the clock wise direction. If the label 1 is assigned to \( v_1 \), then, as the labeling is one-one the label 2 should be assigned to some vertex on the rim, say \( v_2 \). Now \( v_2 \) is adjacent to \( v_3 \) and \( v_5 \) and the label 3 cannot be assigned to both of them. Hence 3 must be assigned to \( v_4 \). Now assigning 4 to any of the remaining vertices leads to a violation between that vertex and \( v_4 \). Hence there are two pairs of vertices which do not satisfy the radio labeling condition, namely \((v_1, v_2)\) and \((v_3, v_4)\) or \((v_3, v_5)\).

A second possible scenario is, if 1 is given to a vertex on the rim say \( v_2 \), then 2 can be assigned only to \( v_4 \). Assigning 3 to any of the remaining vertices violates the condition between that vertex and \( v_4 \). Let 3 be assigned to say \( v_3 \). Also any assignment of labels 4 and 5 to the remaining vertices leads to another violation. So, the maximum number of pairs of vertices that satisfy the Radio labeling condition will be 8. Hence, the radio degree of the graph \( w_{(1,4)} \) is 8.

![Diagram](image-url)

**Figure 3.** A labeling to show \( rdeg(W_{1,4}) = 8 \)

In [12], the radio degree of some standard graphs like paths, complete graphs and complete bipartite graphs were completely determined in terms of \( n \), the number of vertices in the graph. Also a lower bound on the radio degree of cycles was obtained. In this paper, this bound has been considerably improved and a proof of the same is given.

2. Main Results

All complete graphs are radio graceful as shown in [12], so is \( C_3 \). The labeling in the case of \( n = 5 \) is shown in the Figure 4 below clearly indicating \( C_5 \) is radio graceful:

We now consider cycles \( C_n, n \geq 4, n \neq 5 \) with vertex set \( \{v_1, v_2 \ldots v_n\} \) taken in the clockwise direction.

**Theorem 2.1.** For \( n \geq 4, n \neq 5 \),
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$rdeg(C_n) \geq \begin{cases} 
\frac{13n^2-12n}{36}, & \text{if } n \equiv 0 \ (mod \ 12) \\
\frac{13n^2-8n-5}{36}, & \text{if } n \equiv 1 \ (mod \ 12) \\
\frac{13n^2-4n-8}{36}, & \text{if } n \equiv 2, 8 \ (mod \ 12) \\
\frac{13n^2-9}{36}, & \text{if } n \equiv 3, 9 \ (mod \ 12) \\
\frac{13n^2-8n+4}{36}, & \text{if } n \equiv 4 \ (mod \ 12) \\
\frac{13n^2-4n+19}{36}, & \text{if } n \equiv 5 \ (mod \ 12) \\
\frac{13n^2}{36}, & \text{if } n \equiv 6 \ (mod \ 12) \\
\frac{13n^2+4n-17}{36}, & \text{if } n \equiv 7 \ (mod \ 12) \\
\frac{13n^2+4n-8}{36}, & \text{if } n \equiv 10 \ (mod \ 12) \\
\frac{13n^2+8n-5}{36}, & \text{if } n \equiv 11 \ (mod \ 12) 
\end{cases}$

Proof. Case 1: If $n \equiv 1 \ (mod \ 2)$

Define:
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\[
f(v_i) = \begin{cases} 
1, & \text{if } i = 1 \\
2, & \text{if } i = \lceil \frac{n}{2} \rceil \\
3, & \text{if } i = \frac{n+3}{2} \\
n, & \text{if } i = n \\
2i, & \text{if } i \equiv 0 \pmod{2} \text{ and } 2 \leq i \leq \lfloor \frac{n}{2} \rfloor \\
2i - 1, & \text{if } i \equiv 1 \pmod{2} \text{ and } 2 \leq i \leq \lfloor \frac{n}{2} \rfloor \\
f(v_{i-1}) + 3, & \text{if } (i - \frac{n+3}{2}) \equiv 1 \pmod{2} \text{ and } \frac{n+5}{2} \leq i \leq n - 1 \\
f(v_{i-1}) + 1, & \text{if } (i - \frac{n+3}{2}) \equiv 0 \pmod{2} \text{ and } \frac{n+5}{2} \leq i \leq n - 1 
\end{cases}
\]

The labeling \( f \) in the case of \( n = 11 \) is illustrated in the Figure 5 below. We give the proof in the following 5 subcases:

![Figure 5](image_url)

**Figure 5.** Labeling \( f \) for \( C_{11} \)

**Subcase 1:** If \( n \equiv 1 \pmod{12} \)

By this labeling we can see the following:
The pairs of vertices which are at a distance \( k \), \( 1 \leq k \leq \frac{n-1}{6} \), satisfying the radio labeling condition are \( 2, 4, 6, \ldots, \frac{n-1}{6} \) respectively. Also, every pair of vertices at a distance \( k \), where \( \frac{n+5}{6} \leq k \leq \frac{n-1}{2} \), satisfies the radio labeling condition. Hence, we can see that there are \( \frac{n+5}{n(n-1)} \) such pairs. Summing them all, we get \( \frac{13n^2 - 8n - 5}{36} \) pairs on the whole.

**Subcase 2:** If \( n \equiv 3, 9 \pmod{12} \)

By this labeling we can see the following:
The pairs of vertices which are at a distance \( k \), \( 1 \leq k \leq \frac{n-3}{6} \), satisfying the radio labeling condition are \( 2, 4, 6 \ldots, \frac{n-3}{3} \) respectively. Also, every pair of vertices at a distance...
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$k$, where $\frac{n+3}{9} \leq k \leq \frac{n-1}{2}$, satisfies the radio labeling condition. Hence, we can see that there are $\frac{n}{3}$ such pairs. Summing them all, we get $\frac{13n^2-9}{36}$ pairs on the whole.

Subcase 3: If $n \equiv 5 \pmod{12}$

By this labeling we can see the following:
The pairs of vertices which are at a distance $k$, $1 \leq k \leq \frac{n-5}{6}$, satisfying the radio labeling condition are $2, 4, 6, \ldots \frac{n-5}{2}$ respectively. At a distance of $\frac{n+1}{6}$, we have $\frac{2(n+1)}{3}$ pairs of vertices satisfying the required condition. Also, every pair of vertices at a distance $k$, where $\frac{n+7}{6} \leq k \leq \frac{n-1}{2}$, satisfies the radio labeling condition. Hence, we can see that there are $\frac{n}{n(n-2)}$ such pairs. Summing them all, we get $\frac{13n^2-4n+19}{36}$ pairs on the whole.

Subcase 4: If $n \equiv 7 \pmod{12}$

By this labeling we can see the following:
The pairs of vertices which are at a distance $k$, $1 \leq k \leq \frac{n-7}{6}$, satisfying the radio labeling condition are $2, 4, 6, \ldots \frac{n-7}{2}$ respectively. At a distance of $\frac{n+1}{6}$, we have $\frac{2(n-1)}{3}$ pairs of vertices satisfying the required condition. Also, every pair of vertices at a distance, $k$ where $\frac{n+5}{6} \leq k \leq \frac{n-1}{2}$, satisfies the radio labeling condition. Hence, we can see that there are $\frac{n}{n(n-1)}$ such pairs. Summing them all, we get $\frac{13n^2+4n-17}{36}$ pairs on the whole.

Subcase 5: If $n \equiv 11 \pmod{12}$

By this labeling we can see the following:
The pairs of vertices which are at a distance $k$, $1 \leq k \leq \frac{n-5}{6}$, satisfying the radio labeling condition are $2, 4, 6, \ldots \frac{n-5}{2}$ respectively. Also, every pair of vertices at a distance, $k$ where $\frac{n+1}{6} \leq k \leq \frac{n-1}{2}$, satisfies the radio labeling condition. Hence, we can see that there are $\frac{n}{n(n+1)}$ such pairs. Summing them all, we get $\frac{13n^2+5n-5}{36}$ pairs on the whole.
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Case 2 : \( n \equiv 0 (\text{mod} \ 2) \)

Define:

\[
f(v_i) = \begin{cases} 
1, & \text{if } i = 1 \\
2, & \text{if } i = \frac{n}{2} + 1 \\
3, & \text{if } i = \frac{n+4}{2} \\
n, & \text{if } i = n, \text{ when } n \equiv 2 (\text{mod} \ 4) \\
n-1, & \text{if } i = n, \text{ when } n \equiv 0 (\text{mod} \ 4) \\
2i, & \text{if } i \equiv 0 (\text{mod} \ 2) \text{ and } 2 \leq i \leq \frac{n}{2} \\
2i - 1, & \text{if } i \equiv 1 (\text{mod} \ 2) \text{ and } 2 \leq i \leq \frac{n}{2} \\
f(v_{i-1}) + 3, & \text{if } (i - \frac{n+4}{2}) \equiv 1 (\text{mod} \ 2) \text{ and } \frac{n+6}{2} \leq i \leq n - 1 \\
f(v_{i-1}) + 1, & \text{if } (i - \frac{n+4}{2}) \equiv 0 (\text{mod} \ 2) \text{ and } \frac{n+6}{2} \leq i \leq n - 1 
\end{cases}
\]

The labeling \( f \) in the case of \( n = 10 \) -is illustrated in the Figure 6. We now give the proof in 5 subcases as follows:

![Figure 6. Labeling f for C_{10}](image-url)
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Subcase 1: $n \equiv 0(\text{mod } 12)$

By this labeling we can see the following:
The pairs of vertices which are at a distance $k$, $1 \leq k \leq \frac{n}{6}$, satisfying the radio labeling condition are $2, 4, 6, \frac{n-6}{3}$ respectively. Also, every pair of vertices at a distance, $k$ where $\frac{n-6}{3} \leq k \leq \frac{n-2}{2}$, satisfies the radio labeling condition, and we can see that there are $\frac{n(n-3)}{3}$ such pairs. Taking note that there are only $\frac{n}{2}$ pairs of vertices at a distance $\frac{n}{2}$, and all of them satisfy the condition. Summing them all, we get $\frac{13n^2-12n-2}{36}$ pairs on the whole.

Subcase 2: $n \equiv 2, 8(\text{mod } 12)$

By this labeling we can see the following:
The pairs of vertices which are at a distance $k$, $1 \leq k \leq \frac{n-2}{6}$, satisfying the radio labeling condition are $2, 4, 6, \frac{n-2}{3}$ respectively. Also, every pair of vertices at a distance, $k$ where $\frac{n-2}{3} \leq k \leq \frac{n-2}{2}$, satisfies the radio labeling condition, and we can see that there are $\frac{n(n-2)}{3}$ such pairs. Taking note that there are only $\frac{n}{2}$ pairs of vertices at a distance $\frac{n}{2}$, and all of them satisfy the condition. Summing them all, we get $\frac{13n^2-4n-8}{36}$ pairs on the whole.

Subcase 3: $n \equiv 4(\text{mod } 12)$

By this labeling we can see the following:
The pairs of vertices which are at a distance $k$, $1 \leq k \leq \frac{n-4}{6}$, satisfying the radio labeling condition are $2, 4, 6, \frac{n-4}{3}$ respectively. At a distance of $\frac{n-2}{3}$, we have $2\frac{n+1}{3}$ pairs of vertices satisfying the required condition. Also, every pair of vertices at a distance, $k$ where $\frac{n+4}{6} \leq k \leq \frac{n-2}{2}$, satisfies the radio labeling condition, and we can see that there are $\frac{n(n-4)}{3}$ such pairs. Taking note that there are only $\frac{n}{2}$ pairs of vertices at a distance $\frac{n}{2}$, and all of them satisfy the condition. Summing them all, we get $\frac{13n^2-8n+4}{36}$ pairs on the whole.

Subcase 4: $n \equiv 6(\text{mod } 12)$

By this labeling we can see the following:
The pairs of vertices which are at a distance $k$, $1 \leq k \leq \frac{n-6}{6}$, satisfying the radio labeling condition are $2, 4, 6, \frac{n-6}{3}$ respectively. At a distance of $\frac{n}{6}$, we have $2\frac{n}{3}$ pairs of vertices satisfying the required condition. Also, every pair of vertices at a distance, $k$ where $\frac{n+6}{6} \leq k \leq \frac{n-2}{2}$, satisfies the radio labeling condition, and we can see that there are $\frac{n(n-3)}{3}$ such pairs. Taking note that there are only $\frac{n}{2}$ pairs of vertices at a distance $\frac{n}{2}$, and all of them satisfy the condition. Summing them all, we get $\frac{13n^2}{36}$ pairs on the whole.

Subcase 5: $n \equiv 10(\text{mod } 12)$

By this labeling we can see the following:
The pairs of vertices which are at a distance $k$, $1 \leq k \leq \frac{n-4}{6}$, there are $2, 4, 6, \frac{n-4}{3}$ respectively. Also, every pair of vertices at a distance, $k$ where $\frac{n+2}{6} \leq k \leq \frac{n-2}{2}$, satisfies the radio labeling condition, and we can see that there are $\frac{n(n-1)}{3}$ such pairs. Taking note that there are only $\frac{n}{2}$ pairs of vertices at a distance $\frac{n}{2}$, and all of them satisfy the condition. Summing them all, we get $\frac{13n^2+4n-8}{36}$ pairs on the whole.
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3. Open Problem

By this labeling, the bounds on the Radio degree of a cycle has been observed to be considerably improved, compared to our results in [12] especially for larger $n$. Finding the upper bound for the radio degree of cycles is highly non-trivial and the authors strongly feel that the lower bound is also the upper bound and are working towards developing proof for the same.

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