Lynch-Morawska Systems on Strings

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Abstract

We investigate properties of convergent and forward-closed string rewriting systems in the context of the syntactic criteria introduced in [8] by Christopher Lynch and Barbara Morawska (we call these \(LM\)-Systems).

Since a string rewriting system can be viewed as a term-rewriting system over a signature of purely monadic function symbols, we adapt their definition to the string rewriting case. We prove that the subterm-collapse problem for convergent and forward-closed string rewriting systems is effectively solvable. Therefore, there exists a decision procedure that verifies if such a system is an \(LM\)-System. We use the same construction to prove that the \textit{cap problem} from the field of cryptographic protocol analysis, which is undecidable for general \(LM\)-systems, is decidable when restricted to the string rewriting case.

1 Introduction

In this paper we investigate the properties of convergent and forward-closed string rewriting systems. Our motivation comes from the syntactic criteria defined by Christopher Lynch and Barbara Morawska in [8]. They showed that for any term-rewriting system \(R\) that satisfies their criteria (which we call \(LM\)-Systems), the unification problem modulo \(R\) is solvable in polynomial time. In [6] it was shown that these conditions are tight, i.e., relaxing any of them leads to NP-hard unification problems. It was also shown in [6] that the subterm-collapse problem for term-rewriting systems that satisfy all of the other conditions of \(LM\)-Systems is undecidable.

In this current work, we show that the subterm-collapse problem is decidable when restricted to convergent and forward-closed string rewriting systems. These string rewriting systems can be viewed as term rewriting systems over a signature of purely monadic function symbols. We give an analogous definition of \(LM\)-Systems for string rewriting systems. Thus, given a forward-closed and convergent string rewriting system \(T\) there is an algorithm that decides if \(T\) is an \(LM\)-System.

The construction used to show the decidability of the subterm-collapse problem for forward-closed and convergent string rewriting systems is also used to show that the \textit{cap problem}, an important problem from the field of cryptographic protocol analysis [1], is also decidable for such string rewriting systems. This is in contrast with some of our recent work that shows that the cap problem, which is undecidable in general, remains undecidable when restricted to general \(LM\)-Systems.

2 Definitions

We present here some notation and definitions. Only a few essential definitions are given here; for more details, the reader is referred to [3] for term rewriting systems, and to [4] for string rewriting systems.
Let $\Sigma$ be a finite alphabet. As is usual, $\Sigma^*$ stands for the set of all strings over $\Sigma$. The empty string is denoted by $\lambda$. For a string $x$, $|x|$ denotes its length and $x^R$ denotes its reversal. A string $u$ overlaps with a string $v$ iff there is a non-empty proper suffix of $u$ which is a prefix of $v$. For instance, $aba$ overlaps with $acc$, but $aba$ does not overlap with $cca$. However, $aba$ overlaps with itself since $a$ is both a prefix and a suffix of $aba$. (See Fig[1])

![Figure 1: overlap](image)

A string rewriting (rewrite) system (SRS) $R$ over this alphabet is a set of rewrite rules of the form $l \rightarrow r$ where $l, r \in \Sigma^*$; $l$ and $r$ are respectively called the left- and right-hand-side (lhs and rhs) of the rule. The rewrite relation on strings defined by the rewrite system $R$, denoted $\rightarrow_R$, is

$$\{(xly, xry) \mid x, y \in \Sigma^* \text{ and } (l, r) \in R\}$$

The reflexive and transitive closure of this relation is $\Rightarrow_R$. An SRS $R$ is terminating iff there is no infinite chain of strings $s_i, i \in \mathbb{N}$, such that $s_i R$-rewrites to $s_{i+1}$, that is to say $s_i \rightarrow_R s_{i+1}$. An SRS $R$ is confluent iff for all strings $t, s_1, s_2$ such that $s_1 \leftarrow_R t \rightarrow_R s_2$ there exists a string $t'$ such that $s_1 \rightarrow_R t' \leftarrow_R s_2$. An SRS $R$ is convergent iff it is both terminating and confluent.

A string is irreducible with respect to $R$ iff no rule of $R$ can be applied to it. The set of strings that are irreducible modulo $R$ is denoted by $IRR(R)$. Note that this set is a regular language, since $IRR(R) = \Sigma^* \setminus \{\Sigma^* l_1 \Sigma^* \cup \ldots \cup \Sigma^* l_m \Sigma^*\}$, where $l_1, \ldots, l_m$ are the lhs of the rules in $R$. A string $w'$ is an $R$-normal form (or a normal form if the rewrite system is obvious from the context) of a string $w$ for an SRS $R$ if and only if $w \rightarrow_R w'$ and $w'$ is irreducible. We write this as $w \rightarrow_R^* w'$. An SRS $R$ is right-reduced if every right-hand side is in normal form. An SRS $T$ is said to be canonical if and only if it is convergent and inter-reduced, i.e., it is right-reduced and, besides, no lhs is a substring of another lhs.

Given a rewrite system $R$ and a set of strings $L$, $R^*(L)$ is the set of all descendants of strings from $L$, i.e., $\{x \mid \exists y \in L : y \rightarrow^* x\}$, and $R^i(L)$ the set of normal forms of strings in $L$ for the rewrite system $R$. Thus $R^i(L) = R^*(L) \cap IRR(R)$.

String rewriting systems can be viewed as a restricted class of term rewriting systems where all functions are unary. As in [2] a string $u$ over a given alphabet $\Sigma$ is viewed as a term over one variable derived from the reversed string of $u$; i.e., if $g, h \in \Sigma$, the string $gh$ corresponds to the term $h(g(x))$. (In other words, the unary operators defined by the symbols of a string are applied successively in the order in which these symbols appear in that string.) A string of the form $w\ell$ where $w \in \Sigma^*$ and $\ell$ is a left-hand side is called a redex. A redex is innermost if no proper prefix of it is a redex. The longest suffix of an innermost redex that is a left-hand side in $R$ is called its $l$-part and the remaining prefix is referred to as its $s$-part.
We will also need a special kind of normal form for strings, modulo any given SRS \( T \). With that purpose, we define, following Sénizergues \([9]\), a leftmost-largest reduction as follows: let \( \succ \) be a given total ordering on the alphabet \( \Sigma \) and \( \succ_L \) be its length + lexicographic extension\([1]\). A rewrite step \( xly \rightarrow xry \) is leftmost-largest if and only if (a) \( x \) is an innermost redex, (b) any other left-hand side that is a suffix of \( x \) is a suffix of \( l \) as well, (i.e., \( l \) is the \( l \)-part of this redex) and (c) if \( l \rightarrow l' \) is another rule in the rewrite system, then \( l' \succ_L r \). A string \( w' \) is said to be a leftmost-largest (ll-) normal form of a string \( w \) iff \( w \overset{1}{\rightarrow} w' \) using only leftmost-largest rewrite steps. Given a terminating system \( T \), it holds that any string \( w \) has a unique normal form produced by leftmost-largest rewrite steps alone, since every rewrite step is unique; this unique normal form will be denoted as \( \rho_T(w) \).

Next, we define what it means for a string \( x \in \Sigma^+ \) to cause a subterm collapse.

**Definition 2.1.** Let \( R \) be a convergent string rewriting system. A string \( x \) is said to cause a subterm-collapse if and only if there is a non-empty string \( y \) such that \( xy \rightarrow^*_R x \).

Throughout the rest of the paper, \( a, b, c, \ldots, h \) will denote elements of the alphabet \( \Sigma \), and strings over \( \Sigma \) will be denoted as \( l, r, u, v, w, x, y, z \), along with subscripts and superscripts.

A string rewrite system \( T \) is said to be forward-closed iff every innermost redex can be reduced to its normal form in one step.

We now give some preliminary results on convergent and forward-closed string rewriting systems. This first lemma shows that reducing all right-hand sides of rules in \( R \) will preserve the equivalence generated by \( R \) as well as the properties that we are interested in.

**Lemma 2.2.** Let \( R \) be a convergent and forward-closed string rewriting system, and let \( l \rightarrow r \) be a rule in \( R \). Then \((R \setminus \{l \rightarrow r\}) \cup \{l \rightarrow \rho_R(r)\}\) is convergent, forward-closed and equivalent to \( R \).

**Proof.** Let \( R' = (R \setminus \{l \rightarrow r\}) \cup \{l \rightarrow \rho_R(r)\} \) where \( R \) is convergent and forward-closed. We make a few observations first. First of all, since \( R' \) contains the same left-hand sides as \( R \), \( \text{IRR}(R') = \text{IRR}(R) \). The set of redexes are the same too. Besides, it is not hard to see that \( l \rightarrow_R r \rightarrow_R^* \rho_R(r) \) for all rules \( l \rightarrow r \) in \( R \).

We first show that \( R' \) and \( R \) are equivalent. This is straightforward since for every rule \( l \rightarrow r \in R \), \( l \) and \( r \) are joinable modulo \( R' \) and vice versa. Thus \( \leftrightarrow_R^* = \leftrightarrow_{R'}^* \).

We next show that \( R' \) is terminating given that \( R \) is convergent. For the sake of deriving a contradiction, assume that \( R' \) is not terminating. Then \( \exists t \in \Sigma^* : (t_i)_{i=0}^m \) and \( t_i \rightarrow_R t_{i+1} \) where \( t_0 = t \). Consider any \( t_i \rightarrow_R t_{i+1} \) step in the above sequence. Then, by definition of reduction, there must be a rule \( l \rightarrow r \in R' \) such that:

\[
t_i = xly \rightarrow xry = t_{i+1}
\]

Since no left-hand sides of rules in \( R \) were altered in the construction of \( R' \), we can apply a corresponding rule in \( R \). If the rule \( l \rightarrow \rho_R(r) \) was used, then we could replace the above step with at most two reduction steps. Thus, we could construct an infinite descending chain modulo \( R \), which is a contradiction.

Next, we show that \( R' \) is confluent. Suppose it is not. Then since \( R' \) is terminating, there must be a string \( t \) with two distinct normal forms \( t_1 \) and \( t_2 \). But since \( R \) is confluent and equivalent to \( R' \), one of \( t_1 \) and \( t_2 \) must be reducible modulo \( R \). This is clearly a contradiction since \( \text{IRR}(R) = \text{IRR}(R') \).

Thus, \( R' \) is convergent given that \( R \) is convergent.

\( ^1\)Sénizergues refers to this as the short-lex ordering
It remains to show that $R'$ is forward-closed. For this it is enough to show that every innermost redex can be reduced to its normal form in a single reduction step. Let $x = x' l$ be an innermost redex modulo $R'$ where $x, x' \in \Sigma^*$. Then $x$ is also an innermost redex modulo $R$. Since $R$ is forward-closed $x' r \in \text{IRR}(R)$ for $l \rightarrow r \in R$. Thus, $x' r \in \text{IRR}(R')$ as well, again since $\text{IRR}(R) = \text{IRR}(R')$.

We next show that no left-hand sides of rules of a forward closed and convergent string-rewriting system can be the same.

**Corollary 2.3.** Let $R$ be a convergent, forward-closed and right-reduced string rewriting system. Then no two distinct rules have the same left-hand side.

**Proof.** Suppose not. Let $l_i \rightarrow r_i \in R$ for $i \in \{1, 2\}$ such that $l_1 = l_2$ but $r_1 \neq r_2$, but then $l \rightarrow r_1$ and $l \rightarrow r_2$ as trivial reductions would not be joinable, as $r_1$ and $r_2$ are in normal form.

The next preliminary result shows that we can use leftmost-largest reduction steps to reduce an innermost redex to its normal form in a single step.

**Lemma 2.4.** Let $R$ be a convergent, forward-closed and right-reduced string rewriting system, and let $w$ be an innermost redex. Then $w \rightarrow \rho_{R}(w)$, i.e., $w$ reduces to its normal form in one leftmost-largest reduction step.

**Proof.** Let $w \in \Sigma^*$ be an innermost redex. Then $w = w'l$ for $w' \in \Sigma^*$ and by forward closure there must be some rule $l \rightarrow r \in R$ such that $l \rightarrow r$ reduces $w$ to its normal form in a single step. If this were not a leftmost-largest reduction, then there must be some other rule $l' \rightarrow r' \in R$ such that $w = w''l'$ is also an innermost-redex. By Corollary 2.3, $l$ must be a proper suffix of $l'$ and $l'$ must be unique, then $w \rightarrow w''r' \in \text{IRR}(R)$ and $w \rightarrow w'r \in \text{IRR}(R)$, which contradicts the convergence of $R$.

## 3 LM-Conditions for String Rewriting Systems

We now give an equivalent definition of quasi-determinism for string rewriting systems $R$. This definition is adapted from that of [8]. We also define a right-hand side critical pair for string-rewriting systems. Thus, we are able to formulate the conditions of [8] in the context of string rewriting systems.

A string rewriting system $R$ is quasi-deterministic if and only if

1. No rule has $\lambda$ as its right-hand side

2. No rule in $R$ is end-stable—i.e., no rule has the same rightmost symbol on its left- and right-hand sides, and

3. $R$ has no end pair repetitions—i.e., no two rules in $R$ have the same unordered pair of rightmost symbols on their sides.

We define a right-hand-side critical pair as follows: if $l_1 \rightarrow r_1$ and $l_2 \rightarrow r_2$ are two distinct rewrite rules and $r_2 = xr_1$ for some $x$ (i.e., $r_1$ is a suffix of $r_2$) then $\{xl_1, l_2\}$ is a right-hand-side critical pair. The set of all right-hand-side critical pairs is referred to as $\text{RHS}(R)$.

It can be shown [6] that
**Lemma 3.1.** Suppose $R$ is a convergent quasi-deterministic string rewriting system. Then $\text{RHS}(R)$ is not quasi-deterministic if and only if $\text{RHS}(R)$ has an end pair repetition.

A string-rewriting system is *deterministic* if and only if it is non-subterm-collapsing and $\text{RHS}(R)$ is quasi-deterministic.

A *Lynch-Morawska string rewriting system* or LM-system is a convergent right-reduced string rewriting system $R$ which satisfies the following conditions:

(i) $R$ is non-subterm-collapsing,
(ii) $R$ is forward-closed, and
(iii) $\text{RHS}(R)$ is quasi-deterministic.

In light of the results of [7], a convergent string rewriting system $R$ is an LM-system if and only if $\text{RHS}(R)$ is quasi-deterministic and

(a) $R$ is almost-left reduced (see [7]).
(b) There are no overlaps among the left-hand sides of $R$.
(c) No lhs overlaps with a rhs.

We now work towards proving the main results of this paper. Namely, we will show in the sequel below that the subterm-collapse problem for convergent and forward-closed string rewriting systems is decidable.

The first of our results towards the above goal is below:

**Lemma 3.2.** Let $R$ be a convergent, forward-closed and quasi-deterministic string rewriting system and $x,y,z \in \text{IRR}(R)$ such that $xy \rightarrow^1 z$. Then there exist irreducible strings $x = x_1, x_2, \ldots, x_n, x_{n+1}$, $y_1, y_2, \ldots, y_n, y_{n+1}$ such that

1. $y = y_1 \cdots y_{n+1}$,
2. $x_i y_i$ is an innermost redex for all $1 \leq i \leq n$,
3. $x_i y_i \rightarrow x_{i+1}$ for all $1 \leq i \leq n$, and
4. $x_{n+1} y_{n+1} = z$.

*Proof.* The proof proceeds by induction on the number of rewrite steps along the path from $xy$ to the normal form $z$.

**Basis.** Suppose $x, y, z \in \text{IRR}(R)$ such that $xy \rightarrow^1 z$ in $k = 1$ steps. That is, $xy \rightarrow z$. Since $x, y \in \text{IRR}(R)$ there cannot be a redex that is a substring of either $x$ or $y$ alone. Hence there must be strings $x', y' \in \Sigma^*$ and $l_1, l_2 \in \Sigma^+$ such that

$$x = x'l_1, \quad y = l_2y'$$

and $l_1l_2 = l$ for some $l \rightarrow r \in R$. Note that, since $R$ is convergent we may assume that $x'l_1l_2$ is the shortest such redex.

**:** We can construct the following sequence: $x_1 = x'l_1$, $x_2 = x'r$, $y_1 = l_2$, $y_2 = y'$ such that,
this normal form must be a unique leftmost-largest normal form modulo a terminating string-rewriting system, and since \( y \in \Sigma \)

We construct such a string as follows: let \( (x',y') \) be any prefix of \( xy \) from the left and crossing the boundary between \( x \) and \( y \), it must be an innermost redex. Thus, we have established (2).

Now, since \( R \) is forward-closed, \( x'r \) can be assumed to be in normal form. Since \( y' \) is a proper suffix of \( x \) \( \in IRR(R) \), we get that \( y' \in IRR(R) \). Note that the above reductions are leftmost-largest. Since every string has a unique leftmost-largest normal form modulo a terminating string-rewriting system, and since \( R \) is convergent, this normal form must be \( z \).

**Inductive Hypothesis.** Assume that the result holds for all \( x,y,z \in IRR(R) \) such that \( xy \rightarrow^1 z \) in \( k > 1 \) steps.

We show that it holds for strings \( x,y,z \in IRR(R) \) such that \( xy \rightarrow^1 z \) in \( k + 1 \) steps.

Since \( k > 1 \), \( \exists w \in \Sigma^+ \) such that \( xy \rightarrow w \rightarrow^1 z \). Note that \( w \rightarrow^1 z \) must take exactly \( k \) rewrite steps. As in the base case, since \( x,y \in IRR(R) \) and \( xy \) is reducible, we have that \( xy = x'l_1 l_2 y' \) where \( x = x'l_1 \), \( y = l_2 y' \), and \( l_1 l_2 = l \) for some \( l \rightarrow r \in R \) and \( x',y' \in \Sigma^\ast \). Since \( R \) is convergent, we assume that \( x'l \) is the leftmost prefix of \( xy \) that is a redex.

We thus form the sequence: \( x_1 = x'l_1 \), \( y_1 = l_2 \), \( x_2 = x'r \), \( y_2 = y' \). Since \( x_1 y_1 \) is the leftmost redex of \( xy \) it must be the case that \( x_1 y_1 \) is an innermost redex. Therefore, \( x'r \) can be assumed to be in normal form. Then \( w = x'r y' \), and since \( x'r \in IRR(R) \) and \( y \in IRR(R) \) we get that \( w = uv \) for some \( u,v \in IRR(R) \). We can then apply the induction hypothesis to \( u,v \), and \( z \) to fill in the rest of the sequence with the desired properties.

Therefore we can conclude that the result holds for all \( x,y,z \in IRR(R) \) such that \( xy \rightarrow^1 z \).

An immediate consequence of the definition of subterm-collapse given below.

**Lemma 3.3.** Let \( R \) be a convergent forward-closed string rewriting system and \( x,y \in IRR(R) \) such that \( xy \rightarrow^1 x \) and \( y \neq \lambda \). (Thus \( x \) causes a subterm-collapse.) Let \( y_1 \) be a prefix of \( y \). Then \( xy_1 \) causes a subterm-collapse.

**Proof.** Let \( x,y \in IRR(R) \) such that \( xy \rightarrow^1 x \). Let \( y' \) be any prefix of \( y \). Thus \( y = y'y' \) for some string \( y' \).

In order to generate a subterm-collapse with respect to \( xy' \), we must have a string \( w \in \Sigma^+ \) such that \( xyw \rightarrow^* xy \). We construct such a string as follows: let \( w = y'y' \).

Therefore, \( xyw = xy'y'y = xy \). Since \( xy \rightarrow^1 x \) we get \( xyw = xy \rightarrow^* xy \).
We now prove that $R$ is subterm-collapsing if and only if there is a right-hand side of a rule in $R$ that causes a subterm collapse in the sense of the above definition. This lemma will be key in showing the decidability of the subterm-collapse problem as it allows us only to consider right-hand sides of rules for possible sources of subterm-collapse.

**Lemma 3.4.** Suppose $R$ is a convergent forward-closed string rewriting system. Then $R$ is subterm-collapsing if and only if there is a right-hand side of a rule in $R$ that causes a subterm-collapse.

**Proof.** If there is a right-hand side that causes a subterm-collapse, then $R$ is subterm-collapsing. Towards proving the “only if” direction, assume for the sake of deriving a contradiction that the result doesn’t hold, i.e., $R$ is subterm-collapsing but no right-hand side causes a subterm-collapse. Then, let $w$ be one of the shortest strings that causes a subterm-collapse.

Since $w \neq \lambda$, it must be the case that $(\exists a \in \Sigma)(\exists w' \in \Sigma^+): w = aw'$. Also, since $w$ is assumed to cause a subterm-collapse, $(\exists z \in \Sigma^+): wz = aw'z \rightarrow^* w = aw'$. There are thus two cases to consider: either $a$ is involved in the reduction, i.e., $a$ is in the $l$-part of a redex, or it is not.

Suppose $a$ is involved in the reduction. By Lemma 2.2 without loss of generality we can assume that $R$ is right-reduced. Since $a$ is the first letter of $w$ and $a$ is involved in some reduction step, there must be a prefix $z'$ and a corresponding suffix $z''$ of $z$ such that $w'z' = aw'z' \rightarrow^* aw'' \rightarrow r$ and $rz'' \rightarrow^* w$ for some $w''$. That is, $aw''$ is a redex as well as its $l$-part, i.e., $aw'' = l$ for some $l \rightarrow r \in R$. But by the previous lemma (Lemma 3.3), $w'z'$ and hence $r$ causes a subterm-collapse. This contradicts our assumption.

Now, suppose $a$ is not involved in the reduction sequence. Then it must be that $w'z \rightarrow^* w'$. Thus, $w'$ causes a subterm-collapse and $|w'| < |w|$, which contradicts the minimality of $w$.

The main lemma of this section appears below. It gives us that a certain language, parameterized by two strings $u, v \in \Sigma^*$, is a deterministic context-free language. We prove this by constructing a deterministic pushdown automaton to recognize this language.

**Lemma 3.5.** Let $R$ be a convergent, right-reduced, and forward-closed string rewriting system, $u, v \in IRR(R)$, and $\# \not\in \Sigma$. Then the language 
\[
L_{u, v} = \{ w\# | uw\rightarrow^* v, w \neq \lambda \}
\]
is a deterministic context-free language over $(\Sigma \cup \{\#\})^*$.

**Proof.** We design a deterministic pushdown automaton (DPDA) $M$ that recognizes $L_{u, v}$. In the sequel, let $x$ denote the contents of $M$’s stack from bottom to top.

Initially, $M$ pushes a special symbol, $\$$, onto the stack (which serves as a bottom marker) and then pushes $u$. Thus, the contents of the stack after the initialization steps are $\$$u.

Then, we design a transition system based on two cases. Either pushing the symbol $a \in \Sigma$ completes a redex or it does not. That is,

1. $(x, a) \mapsto (xa, \lambda)$ if $xa$ is not a redex, or
2. $(x, a) \mapsto (x'r_0, \lambda)$ if $xa = x'l_0$ where $x'$ is the $s$-part and $l_0$ the $l$-part of $xa$ (i.e., $l_0$ the longest left-hand side in $R$ that is a suffix of $xa$).
\( \mathcal{M} \) will carry out the above transitions by pushing symbols of \( w \) (which is initially on the tape) and reducing each redex that appears. When \( \mathcal{M} \) reaches the \# symbol, by Lemma 3.2 if \( uw \rightarrow^1 v \) then the contents of the stack must be \$\!v\$. This can be checked by \( \mathcal{M} \).

Finally, \( \mathcal{M} \) can be created by building an Aho-Corasick automaton, \( \mathcal{K} \), for the set \{\( l_1, l_2, \ldots, l_n \)\} as given, for instance, in [5]. Then \( \mathcal{M} \) can simulate \( \mathcal{K} \) on its stack by essentially restarting \( \mathcal{K} \) whenever \( \mathcal{K} \) accepts. \( \square \)

As a consequence of the above Lemma 3.5 the subterm collapse problem is decidable for convergent, forward-closed, string-rewriting systems.

**Corollary 3.6.** The following decision problem:

**Given:** A convergent, forward-closed, right-reduced SRS \( R \).

**Question:** Is \( R \) subterm-collapsing?

is effectively solvable.

**Proof.** A decision procedure can be constructed by creating, for each \( l \rightarrow r \in R \), a DPDA \( \mathcal{M}_r \) such that \( L(\mathcal{M}_r) = L_r \) by lemma 3.5. \( \mathcal{M}_r \) can then be converted into an equivalent context-free grammar \( G_r \). Then \( L(G_r) = \emptyset \) if and only if \( r \) does not cause a subterm-collapse. By Lemma 3.4 this is enough to conclude that \( R \) is not subterm-collapsing in general. Finally, deciding if a CFG generates the empty language is decidable, therefore, the overall problem is decidable as well. \( \square \)

Note also, that the construction outlined above can be carried out in *polynomial time*. Thus, not only is the above subterm-collapse problem for convergent, forward-closed string rewriting systems decidable, it is efficiently decidable. This is in contrast to the results of [6] where it was shown that checking if a given term-rewriting system is subterm-collapsing, even when the system satisfies all of the other Lynch-Morawska conditions, is undecidable.

We can therefore conclude that the problem of verifying if a convergent and forward-closed string rewriting system (or a term rewriting system over a signature of monadic function symbols) is an LM-system is decidable.

As another corollary of the above result, we get that the cap problem for convergent, forward-closed, string-rewriting systems is also decidable. This problem, also known as the deduction problem, is often studied in the field of symbolic cryptographic protocol analysis.

**Corollary 3.7.** The Cap Problem:

**Given:** A convergent, forward-closed string-rewriting system \( R \), a string \( u \in \Sigma^+ \) (representing the intruder knowledge) and a secret \( v \in \Sigma^+ \).

**Question:** Does there exists a string \( w \in \Sigma^+ \) (called a cap term) such that \( uw \rightarrow^1_R v \)?

is decidable.

**Proof.** The construction is essentially the same as that in the proof of Corollary 3.6. This time a DPDA is constructed, using Lemma 3.5 for the language \( L_{u,v} \). \( \square \)
The result of Corollary 3.7 is contrasted with the fact that, for general term-rewriting systems, the cap problem is known to be undecidable. The cap problem remains undecidable even when restricted to $LM$-Systems. The above results shows, in the monadic case, if $R$ is convergent and forward-closed, then the problem becomes decidable.
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