Continuum mechanics with torsion

ILYA PESHKOV,¹,² EVGENIY ROMENSKI,²,³,⁴ MICHAEL DUMBSER,⁴

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Abstract

This paper is an attempt to introduce methods and concepts of the Riemann-Cartan geometry largely used in such physical theories as general relativity, gauge theories, solid dynamics, etc. to fluid dynamics in general and to studying and modeling turbulence in particular. Thus, in order to account for the rotational degrees of freedom of the irregular dynamics of small scale vortexes, we further generalize our unified first-order hyperbolic formulation of continuum fluid and solid mechanics which treats the flowing medium as a Riemann-Cartan manifold with zero curvature but non-vanishing torsion. We associate the rotational degrees of freedom of the main field of our theory, the distortion field, to the dynamics of microscopic (unresolved) vortexes. The distortion field characterizes the deformation and rotation of the material elements and can be viewed as anholonomic basis triad with non-vanishing torsion. The torsion tensor is then used to characterize distortion’s spin and is treated as an independent field with its own time evolution equation. This new governing equation has essentially the structure of the non-linear electrodynamics in a moving medium and can be viewed as a Yang-Mills-type gauge theory. The system is closed by providing an example of the total energy potential. The extended system describes not only irreversible dynamics (which raises the entropy) due to the viscosity or plasticity effect but it also has dispersive features which are due to the reversible energy exchange (which conserves the entropy) between micro and macro scales. Both the irreversible and dispersive processes are represented by relaxation-type algebraic source terms so that the overall system remains first-order hyperbolic. The turbulent state is then treated as an excitation of the equilibrium laminar state due to the non-linear interplay between dissipation and dispersion.

1 Introduction

A unified formulation for continuum fluid and solid mechanics was proposed recently in [83, 31, 32, 30]. In a one system of governing partial differential equations (PDEs), it is able to describe the irreversible dynamics of viscous fluids, either Newtonian [31] or non-Newtonian [55], and elastoplastic [46, 85, 30, 44, 6] solids as well as the reversible dynamics of inviscid fluids and elastic solids.
Such a formulation relies on the theory of Symmetric Hyperbolic Thermodynamically Compatible equations (SHTC) \cite{84, 46, 91, 43, 41}, which implies that the governing PDEs are first-order hyperbolic equations. Also, the model is grounded on geometrical principles and Hamiltonian’s principle of stationary action \cite{84} which are known to be valid in diverse physical theories. For instance, this opens attractive new possibilities for designing of models in a first-principle-based fashion in contrast to the classical/modern phenomenology prevailing Navier-Stokes-based models in fluid dynamics in general and turbulence modeling in particular.

The first goal of this paper is to demonstrate that the original unified formulation of continuum mechanics can be generalized in a thermodynamically consistent way in order to take into account the rotational degrees of freedom (DoF) of the microstructure (either emerging like in turbulence or plasticity or preset like in microstructured solids) encoded in the key field of the theory, the distortion field. Our second goal is to introduce powerful and universally valid methods of differential geometry to fluid dynamics which is hardly possible in the framework of classical continuum mechanics of viscous fluids. We then discuss a possible application of the theory to modeling of turbulent flows. Herewith, we propose a viewpoint according to which the turbulence is treated as a non-equilibrium state emerging from the laminar (equilibrium) state as a result of non-linear interplay between dissipation and dispersion.

Despite the basic model \cite{83, 31, 32, 30} is applicable to both fluid and solid dynamics problems, its use in the solid dynamics context has been known for half a century already since the works of Godunov and Romenski in 1970s \cite{45, 42, 90}, see also historical remarks in \cite{85}. Therefore, in this paper, we mainly focus on the fluid dynamics applications and, in particular, on turbulence modeling. However, to the best of our knowledge, the proposed system of governing equations is the first genuinely non-linear dynamical model which can be used to describe dynamics of continuously distributed defects in solids (dislocations, disclinations). So far, either linear dynamical or non-linear static theories have been developed starting from the works of Nye \cite{78}, Kondo, Bilby \cite{9} and Kröner \cite{62}, e.g. see a historical review in \cite{102} (see also \cite{61, 58, 48}).

Apart of the conventional mass density $\rho$, momentum density $m = \rho \mathbf{v}$ ($\mathbf{v}$ is the material velocity), entropy density $s$ and total energy density $E$, another important field of our theory is the so-called distortion field $\mathbf{A} = [A^a_i]$, $a, i = 1, 2, 3$. The distortion field can be viewed as the local basis triad attached to each material element. Thus, the evolution of $\mathbf{A}$ provides complete information about the deformation and rotation of material elements. For further discussion, it is important to emphasize that the material elements are assumed to have finite length scale $\ell$ and thus, are deformable, that is they possess a certain structure in contrast to the classical point of view according to which the material elements are treated as scale-less points. For example, for Newtonian fluids, the length scale $\ell$ can be estimated as $\ell \sim \tau c_{sh}$ \cite{30}, where $\tau$ is the characteristic dissipation time scale (characteristic time for the material element rearrangements, e.g. see \cite{30, 31}), and $c_{sh}$ is the characteristic velocity of propagation of shear perturbations.

Treatment of the material elements as the finite volumes assumes that the material elements may have rotational degrees of freedom in general. This means that the material elements may rotate
Figure 1: (a): Laminar boundary layer for a Newtonian fluid (inflow is from the left, the wall is at the bottom), see details in [31]. The horizontal velocity profile (top) identical to the Navier-Stokes and is homogeneous. However, the distribution of the internal rotations (spin) is heterogeneous (bottom, $A_{11}$ component). (b): double shear layer problem at two time instants (columns), from [31]. Vorticity (top raw). Apparently, the distortion field (bottom, $A_{11}$ component) provides more information about the flow structure than available in the velocity field (vorticity). (c): 3D Taylor-Green vortex, $A_{11}$ component, see details in [31].

independently enough relative to each other. It is important to understand that such a spin is not related to the velocity field via vorticity because the distortion field $\mathbf{A}$ and the fluid velocity $\mathbf{v}$ are two independent state variables. Thus, the internal spin of the fluid elements was clearly observed in the extensive numerical studies [31, 32] of our unified theory in the context of Newtonian flows. For example, Fig.1, a) shows the numerical solution of the laminar boundary layer problem for a viscous gas. The top figure depicts the horizontal velocity distribution which is homogeneous and perfectly matches the Navier-Stokes solution [31]. On the other hand, the bottom figure in Fig.1 a) shows $A_{11}$ component of the distortion field which is clearly heterogeneous and is due to the rotation of the triad $\mathbf{A}$. Such a heterogeneous distribution cannot be derived by any means from the homogeneous distribution of the velocity field of the Navier-Stokes solution. Fig.1 b) compares the vorticity distribution (top) with one of the components of the distortion field (bottom). Much more flow structure is visible in the distortion field. Fig.1 c) shows a 3D structure in the three-dimensional Taylor-Green vortex problem ($A_{11}$ component is shown), see details in [31]. We note that so far the spin of the triad field was computed as a byproduct in our simulations and it does not affect the solution due to the gauge freedom in the relation between the distortion field and the stress tensor, see Section 2. It is thus a principal goal of this paper to account for the material element spin in a thermodynamically consistent way.

Note that in the presence of the material spin, the basis triad field $\mathbf{A}$ becomes anholonomic triad (see definitions in Section 4), which means that it has the non-vanishing torsion tensor $\partial_i A^a_k - \partial_k A^a_i$, where $\partial_i$ are partial space derivatives. It also means that the internal geometry of the flow is non-Euclidean, see Section 4. Recall that the use of anholonomic basis triads (or actually tetrads) is well known in such geometrical theories as Einstein-Cartan theory of relativity [52] or teleparallel equivalent of general relativity (TEGR) [60, 1, 18, 47] in which the gravity interaction is due to not only the curvature of the spacetime but also due to torsion which is coupled to the intrinsic angular momentum (spin) of the matter. This paper, therefore, can be considered as an attempt to look at
the turbulence problem from the standpoint of classical geometrical field theories such as general relativity or Yang-Mills gauge theory [3].

Finally, note that, in principle, one may want to try to account for the rotational DoF of the distortion $A$ via the polar decomposition $A = R\sqrt{G}$, where $R$ is the rotation matrix, and $G = A^tA$ is the deformation tensor. Such an approach was developed by the Cosserat brothers [23, 36]. In the non-linear settings as in this paper, the time evolution for $R$ can be obtained [21, 93] and $R$ can be considered as a natural choice of the state variable to represent the internal rotations. However, in the context of multiphysics modeling and development of efficient computational methods this choice is not that obvious. For example, the PDE for $R$ is a nonlinear equation with no apparent structure. It is therefore not clear if it can be derived from a variational principle or if it admits a Hamiltonian formulation via Poisson brackets [84]. Without such a formulation, it is not clear how the PDE for $R$ can be consistently (mechanically and thermodynamically) coupled with other physical fields, for example, in the presence of electromagnetic forces, non-equilibrium mass and heat transfer, etc. On the other hand, a variational or Hamiltonian formulation for a model provides a quite universal way for model extensions towards nonlinear coupling with other physical fields. Therefore, instead of following the original Cosserat approach to describe internal rotational DoF via the polar decomposition of $A$, we follow a less straightforward (at first sight) but, in fact, more geometrical and elegant way proposed by Cartan [19, 51, 27] who, in fact, was inspired by the work of Cosserat brothers and developed a continuum model based on Riemann-Cartan manifold (see historical remarks in [94]), endowed with not only the metric (measuring the shape change) but also a non-symmetric affine connection with non-vanishing torsion (a measure of anholonomy). It has appeared that the use of Cartan’s geometrical approach allows us to build a non-linear thermodynamically and mechanically consistent model with internal spin which shares some common features with the Yang-Mills gauge theory, posses a variational formulation, and has a rather elegant structure of governing equations with good mathematical properties (symmetric hyperbolicity, well-posedness of the initial value problem). Also, most likely these governing equations also have a Hamiltonian structure (i.e they can be generated by the corresponding Poisson brackets) but this requires some further considerations and will be done elsewhere. Also, we note that the presence of a clear structure of governing equations is beneficial for the development of the structure-preserving numerical methods (i.e. when the PDE structure is respected at the discrete level), e.g. symplectic integrators [72].

2 Basic SHTC equations for fluids and solids

The basic SHTC system describing flow of a continuous medium governs the evolution of the mass density $\rho$, momentum density $m = \rho v = [m_i]$ (where $v = [v^i]$ is the medium velocity), entropy density $s$, the distortion field $A$, and the total energy density $E$ and can be written in a Cartesian
coordinate system as

\[
\frac{\partial \rho}{\partial t} + \partial_k (\rho v^k) = 0,
\]

(1a)

\[
\frac{\partial m_i}{\partial t} + \partial_k m_i v^k + \rho \delta^k_i + A^a_i E A^{a_k} = 0,
\]

(1b)

\[
\frac{\partial A^a_i}{\partial t} + \partial_k (A^a_i v^k) + \rho \delta^k_i (\partial_j A^a_j - \partial_k A^a_j) = -\theta^{-1} E A^{a_k},
\]

(1c)

\[
\frac{\partial s}{\partial t} + \partial_k (sv^k) = \frac{1}{E_A} E A^a_i E A^{a_k} \geq 0,
\]

(1d)

\[
\frac{\partial E}{\partial t} + \partial_k (Ev^k + v^i (p \delta^k_i + A^a_i E A^{a_k})) = 0,
\]

(1e)

where the notations \(E_\rho\), \(E_s\), \(E_{A^a_i}\), etc. stand for \(\frac{\partial E}{\partial \rho}\), \(\frac{\partial E}{\partial s}\), \(\frac{\partial E}{\partial A^a_i}\) and have the meaning of thermodynamic forces, \(p = \rho E_\rho + s E_s + m_i E_{m_i} - E = \rho^2 \varepsilon_\rho\) is the thermodynamic pressure, \(E = E(\rho, s, v, A)\) is the total energy density and should be provided in order to close the system, \(E_s = T\) is the temperature, \(\varepsilon(\rho, s)\) is the internal specific energy, \(\theta \sim \rho \tau c_{sh}^2\) where \(c_{sh}\) is the mentioned above shear sound speed, and \(\tau\) is the relaxation time scale which, in our approach, replaces the classical viscosity coefficient and characterizes the ability of the medium to flow. For simple flows the energy can be taken as

\[
E = \rho \varepsilon(\rho, s) + \rho \frac{c_{sh}^2}{4} ||G'||^2 + \frac{1}{2\rho} ||m||^2,
\]

(2)

where \(G = A^T A\), and \(G' = G - \frac{\text{tr}(G)}{3} I\) is the deviatoric part of \(G\).

The key parameter of the system (1) controlling the transition from the solid to fluid regime is the relaxation time \(\tau\). Thus, the medium responses elastically if the imposed perturbations have characteristic time scale \(T_{\text{macro}}\) much shorter than the relaxation time, \(\tau \gg T_{\text{macro}}\) [31, 16, 85].

The medium flows like a Newtonian fluid if \(\tau \ll T_{\text{macro}}\), and the medium response is of a viscoelastic type if \(\tau\) and \(T_{\text{macro}}\) are of comparable orders. One may think that the same is true for the Navier-Stokes-based fluid dynamics where in some simulations the solid state is represented by an infinite (very large) viscosity. This however leads to an infinitely rigid undeformable solid. In contrast, the solid state in our approach is deformable and can be as soft as needed. The softness is controlled by the shear sound speed \(c_{sh}\) (or shear modulus \(\rho c_{sh}^2\)).

Therefore, by providing a suitable model for \(\tau(\rho, s, A)\), one can obtain the whole range of fluid and solid regimes in one simulation. This is, in particular, an attractive feature of the model for problems where the solid and fluid state coexist. For example, granular flows [57, 2], flows of viscoplastic fluids [4], melting and solidification problems in metallurgy and additive manufacturing (3D printing) [70, 73, 101], polymer processing, etc.

An extensive validation of the model in the Newtonian regime and comparison against the Navier-Stokes equations has been done in [31]. It was demonstrated analytically and numerically that Navier-Stokes solutions are indeed realized in system (1). In particular, it was shown that for the
choice \( \theta = \frac{1}{3|A|^2} \rho \tau c_{sh}^2 \), \( |A| = \det(A) \) of relaxation source term (the source of anholonomy) in the distortion PDE (1c) and energy \( E \) of the form (2), the effective shear viscosity is

\[
\mu = \frac{1}{6} \rho \tau c_{sh}^2.
\]

Hence, the Reynolds number

\[
\text{Re} = \frac{\rho v L}{\mu} \sim \frac{L}{\ell}
\]

scales as the ratio of the macroscopic length scale \( L \) to the dissipation length scale \( \ell \sim \tau c_{sh} \).

### 3 Macroscopic observer

Despite the physical values for \( \tau \) and \( c_{sh} \) can be recovered for a given fluid from sound wave propagation data [30, 17, 12, 13, 15, 14] they might be irrelevant for the practical use of the model (1) for Newtonian flows of weakly viscous fluids such as air or water\(^1\). For example, typical values of \( \tau \) may range from \( 10^{-7} \) s to \( 10^{-10} \) s for fluids with shear viscosity of the order \( 10^{-3} - 10^{-5} \) Pa·s which, if applied to problems having \( T^{\text{macro}} \) even of the order of seconds, makes the system (1) extremely stiff. The good news is that the solutions corresponding to different relaxation times \( \tau' = 6\mu/(\rho c_{sh}^2) \) and \( \tau'' = 6\mu/(\rho c_{sh}^2) \) (say \( \tau' = 10^{-10} \) s and \( \tau'' = 10^{-3} \) s) but to the same value of the effective shear viscosity \( \mu = \rho \tau' c_{sh}^2/6 = \rho \tau'' c_{sh}^2/6 \) are practically indistinguishable as long as both relaxation times satisfy \( \tau' \ll T^{\text{macro}} \) and \( \tau'' \ll T^{\text{macro}} \). In this case, the macroscopic observer sees the same stress state, e.g. see the stress relaxation profiles in Fig. 2. Obviously, the corresponding dissipation length scales \( \ell' = \tau' c_{sh} \) and \( \ell'' = \tau'' c_{sh} \) are also different (\( c_{sh} \) and \( c_{sh} \) should be computed from (3)). Therefore, one may rightfully question what is then the physical meaning of the length scale \( \ell \) if it may differ by several orders of magnitude?

In order to address this issue, we reject using the physical values for \( \tau \) in practical simulations of Newtonian flows and follow the methodology similar to those adopted in Large Eddy Simulations (LES) of turbulent flows and in particular in implicit LES (ILES) [68]. Thus, following Margolin’s terminology [68], we introduce the notion of the macroscopic observer which is “a length scale \( L \) that separates what is known or can be measured about a flow from what is unknown or unresolved”. Importantly, that the observer length is not determined by the flow, but by the questions that are being asked about the flow and so varies from problem to problem. In practice, this is the computational cell size \( L = \Delta x \). The events that happen at scales smaller than \( \Delta x \) (or equivalently, faster than the corresponding time scale) are not available for a direct observation by the macroscopic observer but emerge as indirect evidences in macroscopic quantities (e.g. momentum density) and have to be modeled.

\(^1\)Note that the difference between gases and liquids is encoded in the hydrodynamic part \( \varepsilon(\rho, s) \) of the total energy left unspecified in (2). For instance, one may use the ideal gas equation of state for gases, while the stiffened gas or Mie-Grüneisen equation of state give a good approximation for liquids.
We then define the relaxation time $\tau$ such that, for laminar flows, the corresponding dissipation length scale $\ell < L = \Delta x$ is the largest length scale for which Newton’s law of viscosity still provides a good approximation of the stress state inside the computational cells of characteristic length $\Delta x$. In the terminology of [68], $\ell$ is the physical length scale at which dissipation is significant. Note that such an adaptive strategy for selecting the relaxation time scale is similar to the one adopted by Nishikawa in his “hyperbolic Navier-Stokes solver” [76, 77].

Once the hierarchy of basic length scales has been defined, we may focus on the main goal of the paper, that is describing how the spin of the material elements can be taken into account in a thermodynamically compatible way. But first, we provide some motivation from the differential geometry in the following section.

4 Kinematics of the continuum from the differential geometry standpoint

This section intends to recall basic definitions from the Riemannian and non-Riemannian differential geometry and provides a motivation for how the distortion spin can be taken into account. Also, from what follows, it will be clear that the distortion field can be treated as the field of Cartan’s moving frame and thus, provides a basis for introduction of the internal non-Riemannian (Riemann-Cartan) flow geometry.

We treat the flowing continuous medium (either fluid or solid) as a 3D material manifold $M^t$ evolving in time $t$ in the three-dimensional Euclidean space $R^3$. For simplicity, the ambient Euclidean space is
is equipped with the Cartesian coordinate system \( x^i, i = 1, 2, 3 \), that is the metric \( g_{ij} \) of \( R^3 \) is constant and identified with the Kronecker delta \( \delta_{ij} \) (in principle, arbitrary curvilinear coordinate systems \( x^i \) can be considered with a location dependent metric \( g_{ij}(x^k) \)). In what follows, we shall consistently use letters \( i, j, k = 1, 2, 3 \) to index objects living in the manifold \( M^t \) and its stressed tangent space \( T_x M^t \), capital letters \( A, B, C = 1, 2, 3 \) to index objects living in the globally relaxed (globally stress-free) manifold \( M^0 \) and its tangent space \( T_{\xi(t)} M^0 \) at a point \( \xi^A \), and \( a, b, c = 1, 2, 3 \) to index objects in the locally relaxed tangent space (defined below) \( T_{\chi}^{rel} M^t \) at a point \( x^i \) of \( M^t \).

### 4.1 Holonomic case

There are two intrinsic notions in the Lagrangian viewpoint on the continuum mechanics. Firstly, it is implied that it is possible to introduce a field of labels \( \xi^A, A = 1, 2, 3 \) which label individual material elements and do not change in time. The material labels \( \xi^A \) form the coordinate system in the globally relaxed matter manifold denoted as \( M^0 \) and are called the Lagrangian coordinates. Usually, they can be identified with the location of the medium with respect to the coordinate system \( x^i \) at the initial time instant \( t = 0 \). Secondly, the laws of motion, or simply the motion, of the continuum is identified with a map

\[
x^i = \chi^i(t, \xi^A),
\]

which, for every time instant \( t \), establishes a one-to-one correspondence between the manifolds \( M^0 \) and \( M^t \). The motion \( \chi^i(t, \xi^A) \) thus induces naturally a coordinate system on the manifold \( M^t \). Such a coordinate system is deforming with the medium and is curvilinear in general. Furthermore, it induces the coordinate, or holonomic, basis \( \partial_i \) and the dual basis (coordinate co-basis) \( d\chi^i \) which constitute the bases in the stressed tangent space \( T_x M^t \). Similarly, at every point \( \xi^A \) of \( M^0 \), there is a tangent space, called globally relaxed tangent space \( T_{\xi(t)} M^0 \) with the coordinate basis \( \partial_A \) and co-basis \( d\xi^A \) induced by the Lagrangian coordinate system \( \xi^A \).

Because the motion \( \chi^i \) is a single-valued map (i.e. its inverse \( \xi^A(t, \chi^i) \) exists), its gradients \( F^i_A := \frac{\partial \chi^i}{\partial \xi^A} \) and \( F^A_i := \frac{\partial \xi^A}{\partial \chi^i} \) are well-defined at every time instant. Thus, the bases of the tangent spaces \( T_{\chi}^{rel} M^t \) and \( T_{\xi(t)} M^0 \) are related by

\[
F^i_A d\xi^A = d\chi^i, \quad d\xi^A = F^A_i d\chi^i, \tag{6a}
\]

\[
\partial_A = F^i_A \partial_i, \quad F^A_i \partial_A = \partial_i. \tag{6b}
\]

Furthermore, in order to measure distances on \( M^t \) and \( M^0 \), one needs a metric. For isotropic media, it is naturally to assume that the globally relaxed manifold \( M^0 \) is initiated with the metric \( g_{AB} \) which coincides with the metric \( g_{ij} \) of the ambient space (in general, the specification of the material metric \( g_{AB} \) should be done based on the physical properties of the medium, e.g. anisotropy, presence of residual stresses, etc. [95]) and hence, the material metric \( G_{ij}(t, x^k) \) on \( M^t \) is defined as

\[
G_{ij} := \frac{\partial \xi^A}{\partial \chi^i} \frac{\partial \xi^B}{\partial \chi^j} g_{AB} = F^A_i F^B_j g_{AB}, \tag{7}
\]
which in general (if the motion is not a pure rotation or translation) does not equal to the space metric $g_{ij}$. The material metric $G_{ij}$ thus defines the distances on $M^t$ and can be used to construct other strain measures, e.g. \( E_{ij} := \frac{1}{2}(g_{ij} - G_{ij}) \).

For the further discussion, it is important to remark that, because \( F^A_i \) is the gradient of \( \xi^A(t,\chi^i) \), the partial derivatives \( \partial_i = \frac{\partial}{\partial \chi^i} = \frac{\partial}{\partial x^i} \) commute\(^2\)

\[
\partial_i F^A_j - \partial_j F^A_i = \partial_i \partial_j \xi^A(x^k) - \partial_j \partial_i \xi^A(x^k) \equiv 0,
\]

that is the quantity, called torsion tensor below (see (19)),

\[
T^i_{jk} := F^i_A(\partial_j F^A_k - \partial_k F^A_j) \equiv 0
\]

identically vanishes. In this case, the gradients \( F^i_A \) and \( F^A_i \) are called holonomic triads, while the coordinate bases \( \partial_i, \partial_A \) and co-bases \( d\chi^i, d\xi^A \) are called holonomic frames and co-frames, respectively.

The above description intrinsically relies on the notion of the motion \( x^i = \chi^i(t,\xi^A) \) as a single-valued map between the two manifolds \( M^0 \) and \( M^t \). Such a description, however, is limited and can be only applied to describe reversible deformations of the continuum. In order to be able to deal with irreversible deformations (flows), that is to deal with the material element rearrangements, the one-to-one correspondence between \( M^0 \) and \( M^t \) should be “destroyed”. The material element rearrangements implies that the single-valued map \( \chi^i(t,\xi^A) \) should be replaced with a multi-valued and/or non-smooth map \([60, 87]\). In fact, it is rather natural to assume that the current state of a flowing medium should not depend on the entire flow history encoded in the laws of motion (5). This implies that the map \( x^i = \chi^i(t,\xi^A) \) and the globally relaxed material manifold \( M^0 \) should be eliminated from the mathematical formulation of the governing equations for media deforming irreversibly. As we shall demonstrate in what follows, such a theory requires the replacement of the holonomic triads (6) by anholonomic triads with non-zero torsion which constitute the basis of the Riemann-Cartan geometry.

### 4.2 Non-holonomic case

Geometrically speaking, the motion of the continuum described in the previous section can be viewed as the tangent bundle with two tangent spaces at each point of the base space, which is \( M^t \). The fiber \( T_x M^t \) is a natural tangent space spanned by the holonomic triad \( \partial_i \). The second tangent space is \( T_{\xi(x)} M^0 \) spanned by the triad \( \partial_A \). It can be viewed not as the tangent space to \( M^0 \) at a point \( \xi^A \) but directly as a tangent space to the base space \( M^t \) soldered at the point \( x^i = \chi^i(t,\xi^A) \).

Such a construct of a manifold with two tangent spaces is conditioned by the following reasoning. In principle, it is possible to introduce an evolving in time locally relaxed manifold, say \( M^{t,\text{rel}} \), instead

\(^2\)The partial derivative notation \( \partial_i \) should not be confused with the coordinate bases vector notation \( \partial_i \) (written in bold) of the curvilinear coordinates \( \chi^i \). Remark that the coordinate bases \( \partial_{x^i} \) and \( \partial_{\chi^i} = \partial_i \) of the coordinates \( x^i \) and \( \chi^i \), correspondingly, are different in general.
of the globally relaxed manifold $M^0$, even in the case of a medium deformed irreversibly. However, due to the material element rearrangements, it is no longer possible to establish a global one-to-one correspondence between the manifolds $M^t$ and $M^t_{\text{rel}}$. The absence of such a map implies that the geometry of $M^t_{\text{rel}}$ is incompatible with the Euclidean geometry of the ambient space and $M^t_{\text{rel}}$ cannot be realized in $\mathbb{R}^3$ (otherwise the geometry of $M^t_{\text{rel}}$ would be equivalent to the Euclidean geometry). This also can be illustrated by the following thought experiment. If one partitions the material manifold $M^t$ into small pieces (material elements) along the coordinate lines $x^i$ and allows each piece individually and instantaneously to relax, it would be impossible to realize the relaxed material as a whole from the relaxed pieces (the pieces would not fit to each other globally without gaps). However, one may imagine that the small relaxed material elements lie tightly in a non-Riemannian manifold with nonzero curvature and/or torsion (and even non-metricity) [102].

Despite the global geometry of the locally relaxed manifold $M^t_{\text{rel}}$ is not accessible (e.g. we do not need and actually, we do not know how to introduce a coordinate system on it), its local geometrical structure can be still recovered via the availability of the tangent spaces to $M^t_{\text{rel}}$ which, however, are now soldered to the manifold $M^t$ directly, and a proper definition of affine connection. It has appeared that this is pretty enough to build a consistent theory of irreversible deformation. We emphasize that the detailed knowledge of the geometry of $M^t_{\text{rel}}$ is not of interest on its own, but it is necessary to define the stress field in the flowing medium.

Motivated by this reasoning, we shall consider the flowing medium as the manifold $M^t$ with two tangent spaces at each point $x^i$ with the only difference that the holonomic frames $\partial_i$ and $\partial_a$ will be replaced by the anholonomic frames $e_i$ and $e_a$. The tangent space spanned by the anholonomic basis $e_i$ is called the stressed tangent space and denoted as $T_x M^t$. The tangent space spanned by the anholonomic basis $e_a$ (recall that the objects living in this space are indexed with small Latin letters $a, b, c = 1, 2, 3$) is called the locally relaxed tangent space and denoted as $T^r_x M^t$.

We then introduce the field of non-coordinate (or anholonomic) basis triad $A^a_i$ and its inverse $E^i_a$ as the matrices of a linear map between the anholonomic co-frame fields $e^i$ and $e^a$

$$ E^i_a e^a = e^i, \quad e^a = A^a_i e^i. \quad (10) $$

Here, the co-frame field $e^i$ is dual to $e_i$, and $e^a$ is dual to $e_a$. In the case of isotropic media, there is a metric field $g_{ab}(x^i)$ defined on $M^t$ which is used to measure lengths in the relaxed tangent space $T^r_x M^t$. The metric $g_{ab}$ is not a dynamical parameter of the theory and is assumed to be globally constant, $g_{ab}(x^i) = \delta_{ab}$ so that $e_a \cdot e_b = g_{ab}$ and $e^a \cdot e^b = g^{ab}$ with $g^{ab}$ being the inverse of $g_{ab}$. The distortion field defined as the solution to PDE (1c) is identified with the anholonomic triads $A^a_i$ in (10). In the following, we shall directly work with the matrices $A^a_i$ and $E^i_a$ instead of the co-frames $e^i$ and $e^a$.

At each point of $M^t$ there is also another metric field $G_{ij}$ which is used to measure lengths in the stressed tangent space $T_x M^t$ and which is defined analogously to (7)

$$ G_{ij} = A^a_i A^b_j g_{ab}. \quad (11) $$

It is the metric $G_{ij}$ and not $g_{ab}$ which should be used to measure distances on $M^t$. Note that such
a metric is defined up to an arbitrary local rotation $R^a_i(x^i)$ of the triad $A^a_i = R^a_i(x^i)A^a_i$ in the tangent space $T^x_M$, i.e. $G_{ij} = g_{ab}A^a_iA^b_j = g_{ab}Q^a_aA^b_iQ^b_jA^a_j$, where $Q^a_a$ is the inverse of $R^a_i$. Thus, if one is interested in only the deformation DoF of the material elements then such rotational DoF of the triad field should be considered as redundant (gauge invariance). However, it is the main objective of this paper to try to relate such a gauge invariance in the dynamics of the triad field (distortion field) to the rotational DoF of the microstructure of the flowing medium. Thus, in what follows, we shall discuss how the presence of rotational DoF of the triad field defines the non-Riemannian geometry of the relaxed material manifold $M_{t,rel}$ which, in turn, defines the stress field in the medium.

We now proceed in defining the geometry of $M_{t,rel}$. However, because $M_{t,rel}$ is not explicitly available, we shall work directly on the manifold $M_t$ and use the fact that the tangent spaces to $M_{t,rel}$ are now soldered to $M_t$. This means that the objects defined in the tangent spaces $T^x_M$ and $T_xM$ with respect to the frames $e_a$ and $e_i$, respectively, can be locally transformed from one space into another in the manner of (11) using the triads $A^a_i$ and $E^i_a$, e.g. for the vector fields $v^a(x^i)$ and $v_a(x^i)$ defined in $T^x_M$ we have their representations $v^i(x^i)$ and $v^i(x^i)$ in $T_xM$

\[ v^i = E^i_av^a, \quad v^i = A^a_i v_a. \quad (12) \]

So far, we have defined only metric $g_{ab}(x^i)$ which only partially characterizes the geometry of $M_{t,rel}$. Note that, despite $g_{ab}(x^i) = \text{const}$ globally, it is, in general, not enough to conclude that $M_{t,rel}$ is flat\(^3\) in the settings of non-Riemannian geometry. In addition, one needs to define an affine connection in order to define the parallel transport and thus, the covariant differentiation of the vector fields on $M_{t,rel}$.

Because the orthonormal bases $e^a$ are the only admissible bases in the tangent space $T^x_M$, the local rotations $R^a_i(x^i)$ form the symmetry group, or the gauge group, on $T^x_M$ of admissible transformations of the bases. Hence, $\Omega^a_{bc}$ defined as

\[ \Omega^a_{bc} := R^a_i\partial_b Q^a_c \quad (13) \]

is a natural connection in $M_{t,rel}$ which is also called the spin connection and which defines the covariant differentiation of a contravariant $v^a(x^i)$ and covariant vector field $v_a(x^i)$ as

\[ \nabla_a v^b = \partial_a v^b + \Omega^b_{ac}v^c, \quad \nabla_a v_b = \partial_a v_b - \Omega^b_{ac}v^c, \quad (14) \]

where $\partial_a$ is nothing else but

\[ \partial_a := E^i_a \partial_i. \quad (15) \]

In this paper, however, we shall consider only a particular realization of such a geometry. Namely, we shall assume that the frames $e^a_i(x^i)$ are all fixed to be globally aligned which implies that the only admissible transformations of the frame field are global (point-independent) rotations

\(^3\)If one, however, works in the settings of the Riemannian geometry, $g_{ab}(x^i) = \delta_{ab}$ is, of course, equivalent to the flatness of the manifold.
\( R^i_a(x^i) = \text{const}. \) The rotation of the frame field \( e^i \) is thus measured with respect to the chosen global orientation of the frame field \( e^a \). Fixing a preferable global frame results in that the spin connection \( \Omega_{ab} \) vanishes in this frame and the covariant derivatives \( \nabla_a \) coincide with \( \partial_a, \nabla_a = \partial_a \). The general case with no preferable orientations of the frames and thus with non-vanishing spin connection will be considered in a later publication. In this case, the local rotation field \( R^a(x) \) should play the role of an independent state variable with its own time evolution equation. In principle, this introduces an extra degree of freedom and add extra inertia.

Since we are intending to develop an Eulerian system, i.e. written in the coordinates \( x^i \), it is necessary to rewrite the covariant derivatives \( \partial_a v^b \) and \( \partial_a v_b \) (written in the co-basis \( e^a \)) in the co-basis \( e_i \). Thus, one obtains

\[
\partial_a v^b = E^i_a A^b_j \nabla_i v^j, \quad \partial_a v_b = E^i_a E^j_b \nabla_i v^j, \tag{16}
\]

where the covariant derivatives \( \nabla_i \) are defined as

\[
\nabla_i v^j = \partial_i v^j + W^j_{ik} v^k, \quad \nabla_i v_b = \partial_i v^j - W^j_{ik} v^k, \tag{17}
\]

with \( W^i_{jk} \) being the so-called Weitzenböck connection

\[
W^i_{jk} := E^i_a \partial_j A^a_k. \tag{18}
\]

An important property of the Weitzenböck connection is that if the triad field \( A^i \) is anholonomic then \( W^i_{jk} \) is not symmetric in the lower indices and hence, it has the non-vanishing torsion tensor (the field strength of the triad) which is defined as

\[
T^i_{jk} := W^i_{jk} - W^i_{kj} = E^i_a (\partial_j A^a_k - \partial_k A^a_j) \tag{19}
\]

and can be viewed as the measure of anholonomy. If rewritten in the co-basis \( e^a \), the torsion \( T^i_{jk} \) is

\[
T^i_{ab} = \partial_a E^i_b - \partial_b E^i_a = -E^j_a E^k_b T^i_{jk} = -E^j_a E^k_b (\partial_j A^c_k - \partial_k A^c_j). \tag{20}
\]

Therefore, even though the metric \( g_{ab}(x^i) \) is globally constant, \( M^{t,\text{rel}} \) has non-zero torsion \( T^i_{ab} \) in general. This means that the geometry of the manifold \( M^{t,\text{rel}} \) is non-Euclidean. The manifold with non-zero torsion (and curvature) constitute the subject of the Riemann-Cartan geometry. In such a theory, the manifold is said to be flat, or Euclidean, if only both the torsion and curvature are zero. Therefore, the manifold \( T^i_{x,\text{rel}} M^t \) having the constant metric \( g_{ab}(x^i) = \text{const} \) but non-zero torsion \( T^i_{ab} \) is not flat.

Another important feature of the Weitzenböck connection is that the curvature tensor (the field strength of the connection)

\[
R^i_{jkl} = \partial_k W^i_{lj} - \partial_l W^i_{kj} + W^i_{km} W^m_{lj} - W^i_{lm} W^m_{kj} = E^i_a (\partial_j \partial_k A^a_l - \partial_k \partial_l A^a_j) \equiv 0 \tag{21}
\]

identically vanishes, e.g. see [60, 35, 102]. This, however, happens due to fixing of a global preferable orientation of the frames \( e^a(x^i) \) in \( T^i_{x,\text{rel}} M^t \). In the general case, when the spin connection \( \Omega_{ab} \) does not vanish, the curvature tensor does not vanish too. This case will be considered elsewhere.
Finally, note that the triads $A^a_i$ and $E^i_a$ are covariantly constant, i.e. they are the fields for which the covariant derivative in an arbitrary direction vanishes

$$\nabla_j A^a_i = \partial_j A^a_i - W^k_{ji} A^a_k = 0, \quad \nabla_j E^i_a = \partial_j E^i_a + W^i_{jk} E^k_a = 0. \quad (22)$$

Not that, if written in the frame $e^a_i$, the triad $A^a_i$ and $E^i_a$ become $\delta^a_b$ and hence, the vanishing of the covariant derivatives of triads can also be seen from the transformation rule (16) and the fact that $\partial_a \delta^b_c = 0$. This means that the manifold $M^{t,rel}$, despite being non-flat, possess absolute parallelism, or teleparallelism (i.e. parallelism “at a distance”), allowing the path independent parallel transport of vectors [35]. Also, note that due to (11) and the assumption that $g_{ab}(x^i) = \delta_{ab}$, the Weitzenbök connection has vanishing non-metricity $\nabla G_{jk} = 0$, that is it is the metric compatible connection.

Remarkably, the discussed properties of the Weitzenbök connection is in the basis of the so-called teleparallel reformulations of Einstein’s general relativity (the latter is a torsion free theory), e.g. see [60, 1, 18, 47, 87]. Thus, the theory considered in this paper, i.e. with fixing the orientation for the frame field $e^a_i$, is equivalent to the “pure tetrad” teleparallel gravity (in the terminology of [47, 63]) while the general case with non-vanishing spin connection (13) corresponds to the fully covariant version of teleparallel gravity. The geometrical settings of our theory may thus help one to see some analogies between the gravity interaction and interactions between distant points in a turbulent flow due to rotational DoF of unresolved scales.

We thus have discussed that the non-Riemannian geometry of the relaxed manifold $M^{t,rel}$ can be determined by defining a field of anholonomic frames on it (Cartan’s moving frames). The evolution of non-Euclidean properties of such a geometry is then defined by the evolution of the spin intensity (torsion) of the basis triads. The motivation for this paper is thus to adopt the apparatus of the Riemann-Cartan geometry for describing (modeling) the internal structure of turbulent flows whose nature is essentially in the rotational dynamics of small scales eddies.

In the rest of the paper, instead of $T^i_{jk}$, we shall use its mixed-indices counterpart $T^a_{jk} := \partial_j A^a_k - \partial_k A^a_j = A^i_j T^i_{jk}$, also called the field strength of the triad [18]. Moreover, we shall restrict ourselves to the Cartesian coordinates $x^i$ so that the placement of the indices (up or down) will not play an essential role. Thus, in 3D, there are only 9 independent components of the torsion $T^a_{jk}$

$$
\begin{pmatrix}
0 & T^a_{12} & T^a_{13} \\
-T^a_{12} & 0 & T^a_{23} \\
-T^a_{13} & -T^a_{23} & 0
\end{pmatrix}, \quad (23)
$$

which can be cast to the 3-by-3 matrix

$$B^{ai} := \varepsilon^{ijk} \partial_j A^a_k, \quad (24)$$

where $\varepsilon^{mjk}$ is the Levi-Civita symbol.
5 Extended SHTC equations for media with internal spin

We now return to the modeling part and derive an extended system of PDEs which is the main result of this paper. Thus, in order to account for the rotational degrees of freedom of the distortion field \( A \), we shall use the torsion tensor, which, in 3D, reduces to a 3-by-3 matrix

\[
B = \alpha \nabla \times A \quad \text{or} \quad B^{ai} = \alpha \varepsilon^{ijk} \partial_j A^a_k,
\]

where \( \alpha \sim L^{-1} \) is just a scaling constant, 'L' is a length unit so that \( B \sim L^{-2} \). In the fluid dynamics context, the torsion field can be understood as the number density of unresolved vortex lines piercing through unit areas oriented normal to the coordinate directions, while in the solid dynamics context it is treated as the number density of continuously distributed defects such as dislocations and disclinations\(^4\) \cite{46, 59, 75, 60, 102}. Before to discuss the evolution equation for \( B \) let us first recall our motivation for the choice of \( B \) to represent internal rotations instead of the conventional micropolar rotation \( R = AG^{-1/2} \) and its gradients \( \nabla R \) (micropolar curvature), e.g. see \cite{96, 33}. Indeed, the matrices \( R \) and \( \nabla R \) contain only the information about the rotations of the triad field while \( B \) also incorporates some excessive information (contained in \( G \)) on the stretch of the triads. At first sight, this may say in favor of utilizing \( R \) as the proper micropolar state variable. However, our long-term goal is to build a consistent nonlinear multi-physics continuous theory for modeling of internal rotations in the presence of electromagnetic fields or non-equilibrium mass and heat transfer, and it has appeared that the torsion field \( B \) is more beneficial in the multi-physics modeling context. This is due to the fact that governing PDEs for \( R \) and \( \nabla R \) are very nonlinear and do not have an apparent structure of Euler-Lagrange equations and therefore, their variational nature is unknown. Without a variational formulation, a consistent extension of the conventional micropolar continuum towards nonlinear coupling with new fields is unclear. On the other hand, the governing equation for \( B \), as it is discussed below, is simpler and admits a variational formulation. Therefore, our preferences are given to those state variables which have a better structure of the governing equations while their intuitive interpretation might be less evident. Another important argument in favor of \( B \) is that it comes with the underlying geometrical structure of the Riemann-Cartan geometry as it has been discussed in the previous section.

The time evolution for \( B \) can be obtained \cite{84, 46} easily by means of applying the curl operator to (1c). As the result, one gets

\[
\frac{\partial B^{ai}}{\partial t} + \frac{\partial}{\partial x^k} \left( B^{ai} v^k - v^i B^{ak} + \varepsilon^{ijk} \alpha \theta^{-1} E_A a \right) + v^i \frac{\partial B^{ak}}{\partial x^k} = 0.
\]

\(^4\)Note that in the dynamics of solids, linear theories of defects are usually used, e.g. see historical review in \cite{102}. In the linear regime, it is possible to separate rotational degrees of freedom of the distortion field from the pure deformation part (displacement) and thus, it is possible to use two tensors one of which represents the density of dislocations (curl of the displacement field) while the second one represents the density of disclinations (curl of the rotation), e.g. see \cite{59}. In a non-linear formulation like the one used in this paper, such a separation is hardly possible in general because both degrees of freedom, the rotation and deformation, are coupled multiplicatively which can be seen in the polar decomposition of the distortion, see Section 1.
It, in particular, follows immediately from this PDE that the torsion is an intrinsic property of the viscous dissipation in fluids or irreversible deformation in solids. Indeed, if the dissipative source term in (1c) is absent, \( \theta^{-1} E_A \equiv 0 \), then, from (26), we automatically have \( B \equiv 0 \) for all time instants if it was zero initially.

**Remark 1.** Note that even though \( \nabla \cdot B = 0 \) (as follows automatically from the definition (25)), the term \( \nu^i \partial_k B^{ak} \) should be retained in (26) in order to preserve the proper characteristic structure and Galilean/Lorentz invariance property [32, 84].

So far, PDE (26) does not add any new physics to the basic model because it does not contribute neither to the momentum flux nor energy. To relate \( B \) to a certain physics we have to treat it as a real *independent* state variable, i.e. we should include \( B \) into the set of arguments of the total energy. This, however, is not a straightforward task and should be done without violating the thermodynamic consistency of the basic model (1).

In this paper, we follow a route similar to those taken in the framework of SHTC equations [84]. Thus, we extend the set of state variables, so that the new (extended) energy potential is

\[
E = E(\rho, s, M_i, A_i^a, B^{ai}, D^a_i),
\]

where \( M_i \) is the generalized momentum density, \( B^{ai} \) is the torsion field, and \( D^a_i \) is a field *complimentary* in a certain sense to \( B^{ai} \). In fact, as we shall see in Section 11, the fields \( B^{ai} \) and \( D^a_i \) can be viewed as parts of the only one rank-3 four-torsion tensor. In Section 11, one may, in particular, see some similarities between these fields and electric and magnetic fields in electrodynamics where the three-dimensional electromagnetic vector fields are also parts of the single four-dimensional rank-2 electromagnetic tensor.

Therefore, if one wishes to recover the SHTC structure of the extended system, a complimentary to \( B \) field has to be introduced in such a way that the constitutive flux in (26), \( \alpha \theta^{-1} E_A \), should be identified with the Legendre conjugate \( \mathcal{E}_D \) of the complimentary field \( D \) so that (26) takes the form

\[
\frac{\partial B^{ai}}{\partial t} + \partial_k \left( B^{ai} \nu^k - \nu^i B^{ak} + \varepsilon^{ikj} E_{D^a_j} \right) + \nu^i \partial_k B^{ak} = 0.
\]

After introducing fields \( B \) and \( D \), distortion PDE (1c) reads

\[
\frac{\partial A^a_k}{\partial t} + \partial_k (A^a_i \nu^i) + \nu^j \left( \partial_j A^a_k - \partial_k A^a_j \right) = -\alpha^{-1} \mathcal{E}_{D^a_k}.
\]

**Remark 2.** It is important to remark that, in fact, we will not require the equality

\[
\alpha^{-1} \mathcal{E}_D = \theta^{-1} E_A
\]

to be fulfilled at every time instants but we will construct the extended system in a more general way so that this equality is satisfied and thus, the basic model (1) is recovered, only in the relaxation limit of the extended model (i.e when a relaxation parameter goes to zero), see Section 9.

Furthermore, the time evolution for \( D \) has to be provided. Actually, the PDE for \( D \) is known. Thus, from the studying the structure of SHTC equations, the torsion PDE (28) is consistent with
the energy conservation if only it is accompanied by an equation for $D$ with a very precise structure, see [84]. Alternatively, this PDE can be derived from the variational principle, see Section 11.

The system of governing equations thus reads

$$
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v^k)}{\partial x^k} = 0, \tag{31a}
$$

$$
\frac{\partial M_i}{\partial t} + \frac{\partial}{\partial x^k} \left( M_i v^k + P \delta_i^k + A^a_i \mathcal{E}_A^a_k - B^a_k \mathcal{E}_B^a_i - D^a_k \mathcal{E}_D^a_i \right) = 0, \tag{31b}
$$

$$
\frac{\partial A^a_k}{\partial t} + \frac{\partial (A^a_i v^i)}{\partial x^k} + \nu^j \left( \frac{\partial A^a_k}{\partial x^j} - \frac{\partial A^a_j}{\partial x^k} \right) = -\frac{1}{\alpha} \mathcal{E}_D^a_k, \tag{31c}
$$

$$
\frac{\partial B^{ai}}{\partial t} + \frac{\partial}{\partial x^k} \left( B^{ai} v^k - v^i B^{ak} + \varepsilon^{ijk} \mathcal{E}_D^a_j \right) + v^i \frac{\partial B^a_k}{\partial x^k} = 0, \tag{31d}
$$

$$
\frac{\partial D^a_i}{\partial t} + \frac{\partial}{\partial x^k} \left( D^a_i v^k - v^i D^a_k - \varepsilon^{ijk} \mathcal{E}_B^a_j \right) + v^i \frac{\partial D^a_k}{\partial x^k} = \frac{1}{\alpha} \mathcal{E}_A^a_i - \frac{1}{\eta} \mathcal{E}_D^a_i, \tag{31e}
$$

$$
\frac{\partial s}{\partial t} + \frac{\partial (sv^k)}{\partial x^k} = \frac{1}{\xi} \mathcal{E}_D^a_i E_{D^a_i} \geq 0. \tag{31f}
$$

Here, the generalized momentum density $M$ does not equal to $m = \rho v$ but may include the contribution from the torsion fields $B$ and $D$. For example in the simplest case, which depends on the energy specification (see equation (37) below), $M = \rho v + D^a_a \times B^a$ where $D^a_a$ and $B^a_a$ are $a$-th rows of $D$ and $B$ (summation over $a$ is implied). In general, the Cauchy stress tensor

$$
\Sigma_i^k = -P \delta_i^k - A^a_i \mathcal{E}_A^a_k + B^a_k \mathcal{E}_B^a_i + D^a_k \mathcal{E}_D^a_i \tag{32}
$$

is non-symmetric ($\Sigma_i^k \neq \Sigma_i^k$) in media with internal rotations [96]. Nevertheless, in order to guaranty the conservation of the total angular momentum (matter + fields, see the discussion in Section 10) the full flux of the total momentum

$$
T^k_i = v^k M_i - \Sigma_i^k \tag{33}
$$

has to be symmetric because the equations (31b) represents not solely the balance of the macroscopic momentum $\rho v$ but the conservation of the total momentum $M$, see Section 10. The symmetry of the momentum flux $T^k_i$ has to be taken into account in the construction of the energy potential $\mathcal{E}$, which is discussed in Section 7. Also, $P = \rho \mathcal{E}_\rho + s \mathcal{E}_s + M_i \mathcal{E}_M_i + B^{ai} \mathcal{E}_B^a_i + D^a_i \mathcal{E}_D^a_i - \mathcal{E}$ is the thermodynamic pressure which includes the contribution due to the torsion fields $B$ and $D$.

Remark 3. As well as in the basic model (1), where we have $v = \mathcal{E}_m$, one can show that in the extended model (31), we have $v = \mathcal{E}_M$, that is the velocity is always the Legendre conjugate to the total momentum in the SHTC framework [84].

Note that the extra terms in the pressure and stress tensor in (31b) are not arbitrary. Their appearance is conditioned by the structure of the evolution equations (31d) and (31e) and the requirement that the system should respect the energy conservation law. Thus, only this choice
of the momentum flux is compatible with the energy conservation. Indeed, if one sums up the equations multiplied by the corresponding factors

$$E_\rho \cdot (31a) + E_{M_i} \cdot (31b) + E_{A^a_k} \cdot (31c) + E_{B^{ai}} \cdot (31d) + E_{D_{a,i}} \cdot (31e) + E_s \cdot (31f)$$

the following energy conservation law can be obtained

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x^k} \left( v^k E + v^i \Sigma_{ij}^k + \varepsilon^{ijk} E_{D_{a,i}} E_{B_{a,j}} \right) = 0,$$

where the last term in the energy flux, $\varepsilon^{ijk} E_{D_{a,i}} E_{B_{a,j}}$, is the contribution due to torsion.

In the following section, we discuss the structure of the algebraic source terms on the right hand-side of (31).

Lastly, we note that the left hand-side of the governing equations for $B$ and $D$ share a common structure with the equations of electrodynamics of moving media [32, 84, 91, 92]. In particular, system (31) can be symmetrized in the way discussed in [84] and thus, its hyperbolicity depends on the convexity of the energy potential (27). Nevertheless, we underline that the Hamiltonian structure (i.e. Poisson brackets generating the extended system) of system (31) is not completely understood since it is not fully equivalent to the Hamiltonian structure of the electrodynamics equations due to the source terms $-\alpha^{-1}E_D$ and $\alpha^{-1}E_A$ which are reversible (i.e. they do not change sign with respect to the time reversal transformation [82, 81]) in contrast to the irreversible term $-\eta^{-1}E_D$ in (31e)) and therefore, they are also a part of the reversible time evolution (i.e. they are a part of the corresponding Poisson brackets). This, however, requires further investigations and we plan to study the Hamiltonian structure of the extended model in detail in a subsequent publication.

6 Irreversibility, dispersion and non-locality

Recall that in the SHTC framework as well as in the more general GENERIC formulation of non-equilibrium thermodynamics [84, 81, 79], the reversible and irreversible\(^5\) parts of the time evolution are treated separately. The reversible part conserves both the energy and entropy and can be derived from Hamilton’s principle as well as can be generated from corresponding Poisson brackets [84]. The irreversible part of the time evolution raises the entropy but still conserves the total energy.

In particular, the reversible part of the time evolution of system (31) is represented by the left hand-side and by the reversible source terms $-\alpha^{-1}E_D$ and $\alpha^{-1}E_A$. The irreversible part is represented by the source term $\eta^{-1}E_D$ in (31c) and $(\xi_s \eta)^{-1}E_D : E_D$ in (31f). Either reversible or irreversible source terms are constructed in such a way that after the summation (34), their contribution to

\(^5\)Here, we follow the definition of reversibility and irreversibility as given with respect to the time reversal transformation (TRT), e.g. see [82, 81]. Thus, to distinguish between reversible and irreversible terms, we apply TRT to the evolution equations, and if a term does not change its sign it is regarded to the reversible part. The irreversible terms change their signs under TRT.
the energy conservation (35) vanishes. Thus, the irreversible nature of the source term \( \eta^{-1} \mathcal{E}_D \) in (31e) is now more obvious because it is coupled with the source term in the entropy PDE (31f) (entropy production). They annihilate after the summation (34). On the other hand, the terms \(-\alpha^{-1} \mathcal{E}_D\) and \(\alpha^{-1} \mathcal{E}_A\) in (31c) and (31e) are indeed of the reversible nature and they annihilate with each other in the summation (34) and thus, they do not contribute to the entropy production. The presence of the reversible source terms can be also justified from the physical ground. Thus, the extended system intends to account for impact of the small scale dynamics on the macroscopic dynamics and it belongs to the class of non-local models because the torsion is the first spatial gradient of the distortion field while the field \(D\) characterizes non-locality in time (micro-inertia), see Section 11. In multi-scale systems with a strong inter-scale interaction, i.e. when the characteristic wave length of the perturbations is comparable with the characteristic length of the small scale flow structures, the dispersion phenomena are observed. Dispersive process can be viewed as process with reversible energy exchange between the small and large scales. In the extended system (31), the reversible source terms, therefore, responsible for the emergence of dispersion. For example, in the context of turbulence modeling, the reversible source terms can be related to the reversible energy exchange between large and small turbulent scales, the so-called energy cascade. Remark that the antisymmetric-like structure of the reversible source terms is essentially the same as in the recently proposed first-order hyperbolic reformulation of the dispersive nonlinear Schrödinger equation by Dhaouadi et al [28].

Note that in classical continuum mechanics, dispersive phenomena are usually modeled with high order PDE systems. However, recent results obtained with first-order hyperbolic equations [89, 69, 28] demonstrate that, in fact, dispersive phenomena can be also successfully modeled with first-order hyperbolic PDEs with relaxation type source terms.

### 7 Closure: an example of equation of state

In order to close a system of PDEs in the SHTC framework, one has to provide the energy potential [84]. In particular, as it is clear from the structure of system (31), we have to define the total energy \( \mathcal{E} \) as the function of all the state variables in order to define constitutive fluxes and source terms. An additional constraint is that the provided energy potential should guaranty the symmetry of the total momentum flux \( T^k_i \), see (33), in order to guaranty the conservation of the total angular momentum, see Section 10.

In this paper, we introduce only a particular but quite general example for \( \mathcal{E} \). Thus, we define \( \mathcal{E} \)

\[
\mathcal{E} = \rho \varepsilon (\rho, s) + \rho \frac{c_{\text{th}}^2}{4} ||G'||^2 + \frac{1}{2\rho} M^2 + \mathcal{E}^{\xi}(M, B, D)
\]

(36)

as a sum of the basic energy (2) and an additional energy \( \mathcal{E}^{\xi} \) which accounts for the contribution due to torsion, and can be identified with the energy contained in the small scale unresolved eddies of a turbulent flow or the energy carried by small scale defects in solids. As a starting point, one
can start with
\[ \mathcal{E}^t = \frac{1}{2} \left( \frac{1}{\epsilon} \| D \|^2 + \frac{1}{\mu} \| B \|^2 \right) - \frac{1}{\rho} \sum_{a=1}^{3} M_a D_a^1 B_{a1}^1 - \frac{c_{sp}}{2} \sum_{a=1}^{3} A_a^1 D_a^2 B_{a2}^2 - \frac{c_{sp}}{2} \sum_{a=1}^{3} A_a^2 D_a^3 B_{a3}^3 \]  

(37)

where the coefficients \( \epsilon = \epsilon(\rho, s) \) and \( \mu = \mu(\rho, s) \) depend on \( \rho \) and \( s \), in general, but are assumed to be constant in this paper. They are some transport parameters such that they scale as square of velocity, \( (\epsilon\mu)^{-1} \sim v^2 \), see Section 8. The parameter \( c_{sp} \) has the dimension of velocity and characterizes the propagation of rotational degrees of freedom. In general, one may try other energies \( \mathcal{E}^t \), which, of course, will affect all terms in (31) via the thermodynamic forces \( \mathcal{E}_A, \mathcal{E}_B, \mathcal{E}_D \).

Note that if \( \mathcal{E}^t \) is given by (37), the momentum density is \( (D_a \text{ and } B_a \text{ are } a\text{-th rows of } D \text{ and } B, \text{ the summation over } a \text{ is implied}) \)

\[ M = \rho v + D_a \times B^a \]  

(38)

and hence, the convective part \( v^k M_i \) of the momentum flux \( T^k_i \), as well as the Cauchy stress tensor \( \Sigma^k_i \), is not symmetric. Nevertheless, the entire total momentum flux \( T^k_i = v^k M_i - \Sigma^k_i \) corresponding to the potential (37) is symmetric as can be verified by direct calculations.

### 8 Dimensional analysis

In this section, we discuss the physical units of the parameters and state variables of the extended model. From this, we shall see that the new fields \( B \) and \( D \) have the meanings close to the number density and angular momentum (spin), respectively.

Let us denote the time unit as 'T', the length unit as 'L', and the mass unit as 'M'. It is convenient to chose the scaling parameter \( \alpha \) in (25) to scale as \( L^{-1} \). In such a way, the torsion \( B \) scales as \( L^{-2} \) which can be interpreted as the number of dislocation lines (in solids) or vortex lines (in fluids) crossing a unit area perpendicular to each coordinate direction.

Thus, according to our choice \( [\alpha] = L^{-1} \) and because \( [\theta] = [\rho c_{sp}^2] = \frac{M}{L^T} \), we have

\[ [B] = L^{-2}, \quad [\mathcal{E}_D] = L^{-1} \cdot T^{-1}. \]  

(39)

From that the quantity \( \mathcal{E}_D \otimes D \) should have the dimension of stress, i.e. \( M \cdot L^{-1} \cdot T^{-2} \), we get that

\[ [D] = M \cdot T^{-1}. \]  

(40)

Thus, \( B \) has the units of the number density while \( D \) scales as the angular momentum, \( M \cdot L^2 \cdot T^{-1} \), divided by \( L^2 \). Also, it follows from (40) that \( B_i \times D_i \sim \rho V \), i.e. it indeed scales as the momentum density, where \( V = L/T \) is some velocity.

Furthermore, transport coefficients \( \epsilon \) and \( \mu \) in (37) and relaxation parameter \( \eta \) have the following units

\[ [\epsilon] = M \cdot L, \quad [\mu] = T^2 \cdot L^{-3} \cdot M^{-1}, \quad [\eta^{-1}] = M \cdot L \cdot T^{-1}. \]  

(41)
Note that similar to the electrodynamics \[32\], the inverse product \((\varepsilon \mu)^{-1}\) scales as the velocity square
\[
\frac{1}{\varepsilon \mu} \sim V^2.
\] (42)

9 Turbulence as a non-equilibrium state

In this section, we discuss a non-equilibrium thermodynamics point of view on the emergence of turbulence as an excitation of the laminar flow, which is treated as a near equilibrium state of the fluid in a certain sense specified in what follows. In the discussion, we shall distinguish three length scales: the scale of interest (the problem length scale) \(\Lambda\), the macroscopic observer length scale \(L\) which we identify with the resolution scale \(\Delta x\) (see Section 3), and the viscosity dominated microscale which can be identified with the so-called Taylor microscale. The latter has been denoted by \(\ell\) in Section 2. In particular, the extended model (31) is essentially formulated in the settings when \(L = \Delta x > \ell\), that is we are not in the direct numerical simulation (DNS) conditions (recall that \(\text{Re} \sim L/\ell\), see (4)). Therefore it is implied that
\[
\ell < L < \Lambda.
\] (43)

We say that, for a given macroscopic observer \(L = \Delta x\), the flow is turbulent or non-equilibrium if at this scale, the following local equilibrium condition is violated. Otherwise, the flow is treated as near equilibrium or laminar. Thus, the extended model (31) has equilibrium states of two types:

1. **Global equilibrium state**, which is the state when all thermodynamic forces vanish (it corresponds to the Euler equations of ideal fluids):
   \[
   \mathcal{E}_A = 0, \quad \mathcal{E}_B = 0, \quad \mathcal{E}_D = 0.
   \] (44)

2. **Local equilibrium state**, which is a state when the thermodynamic forces do not identically vanish but the non-equilibrium source term in (31e) vanishes (see also Remark 2):
   \[
   \eta^{-1}\mathcal{E}_D - \alpha^{-1}\mathcal{E}_A = 0.
   \] (45)

In other words, in the vicinity of local equilibrium state, i.e. when the left hand-side of (45) is small, we say that the flow is laminar at the scale \(L = \Delta x\) even though it may look highly irregular at the scale of interest \(\Lambda\) (the largest scale). In this case, the flow can be well approximated by the basic model (1) or equivalently, by the Navier-Stokes equations. Moreover, we shall explicitly demonstrate that, in this terminology, the DNS conditions, \(L = \ell\) or \(\text{Re} = 1\), correspond to near-equilibrium conditions (small left hand-side in (45)). But first, an important remark is in order.

**Remark 4.** The condition (45) is scale dependent because the relaxation parameter \(\alpha\) scales as inverse length \(\alpha \sim L^{-1}\). It is therefore not surprisingly that the condition (45) is adjustable. For example, if we identify \(\alpha\) with the macroscopic observer length scale \(\Delta x^{-1}\) then, for an observer \(\Delta x'\), exactly the same flow may look “more equilibrium” (more “laminar”) than for another observer \(\Delta x'' > \Delta x'\). In other words, the extent to what the flow can be regarded as equilibrium (laminar-like) or non-equilibrium (turbulent-like) is the matter of at what scale we observe the flow (e.g. it is the matter of resolution of the computational mesh).
We now note that $\eta \sim \theta^{-1} \alpha^2$ (their physical units are the same, see Section 8), and it is convenient to assume that
\[ \eta = \theta^{-1} \alpha^2. \] (46)
Furthermore, we define the new relaxation parameter $\lambda = \alpha/\theta$ which, after identifying the scaling parameter $\alpha$ with the inverse length scale of the macroscopic observer $L = \Delta x$, scales as Reynolds number (4):
\[ \lambda = \frac{\alpha}{\theta} \sim \frac{L^{-1}}{\rho \tau \bar{c}_m} \sim \frac{T}{M} \cdot \frac{L}{\ell} \approx \frac{L}{\ell} = \text{Re}, \] (47)
where $T$ is the time scale of the macroscopic observer.

Hence, the source term in (31e) can be rewritten as
\[ -\frac{\theta}{\alpha} (\alpha^{-1} \mathcal{E}_D - \theta^{-1} \mathcal{E}_A) \sim -\frac{1}{\text{Re}} (\alpha^{-1} \mathcal{E}_D - \theta^{-1} \mathcal{E}_A). \] (48)

Therefore, in the relaxation limit $\lambda \to 0$ ($\text{Re} \to 0$), i.e. when approaching local equilibrium (45), the basic system (1) is recovered (see Remark 2) which does contain the Navier-Stokes solutions as a particular case [31]. In particular, the DNS conditions, $\text{Re} \sim L/\ell \leq 1$, correspond to the near-equilibrium settings and hence, both the basic model (1) and the Navier-Stokes model can be still used to describe turbulent flows under the DNS conditions. On the other hand, if $\text{Re} \to \infty$, or equivalently $L \gg \ell$, the local equilibrium condition (45) is not respected in general, and the basic model, as well as the Navier-Stokes one, is not applicable.

One may note that in this framework, the turbulence is a non-linear interplay of the irreversible and dispersive processes governed by the thermodynamic forces $\mathcal{E}_A$ and $\mathcal{E}_D$ and by the new relaxation parameter $\lambda$, which is essentially the Reynolds number of the macroscopic observer $L = \Delta x$.

10 Angular momentum conservation

There are two aspects regarding the angular momentum conservation. The first one is due to the choice of the local (point-dependent) observer and related to such an observer *spurious* Coriolis forces. This is, however, not about the material modeling but about the proper choice of the spacetime geometry and can be addressed in a very similar fashion as it is done in the special relativity via the adoption of the spin connection (or inertial connection) similar to the one in (13), e.g. see [1]. Such a covariant theory will take into account the spurious inertial forces related to the local observer by construction (including the conservation of angular momentum). In this paper, however, we are not considering such a fully covariant extension and planning to do it in the future.

The second aspect is due to the presence of the microstructure with a certain characteristic length scale associated with $\alpha^{-1}$ in Section 9. Under many circumstances, the presence of microstructure results in non-negligible inertial effects (microinertia) which have to be taken into account in the macroscopic balance equations (balance of linear and angular momentum).
Usually, in continuous theories of media with a microstructure that possesses rotational degrees of freedom (micropolar continua), the balance of linear momentum is written only for the macroscopic momentum (the average of the center-of-mass momenta of the microstructure elements) $m = \rho v$:

$$\frac{\partial m_i}{\partial t} + \frac{\partial (v^k m_i - \sigma^k i)}{\partial x^k} = 0. \tag{49}$$

This implies that the momentum flux $t^k_i = v^k m_i - \sigma^k i$ is not symmetric (equivalently, $\sigma^k i$ is not symmetric due to the symmetry of $v^k m_i = \rho v^k v_i$) in general because (49) does not account for the inertial forces experienced by the elements of the microstructure due to their micropolar rotations.

In such theories, in order to account for the inertial effects due to the micropolar rotations, the concept of the couple stress is introduced which emerges as the constitutive flux in the balance of micropolar angular momentum, e.g. see [34, 96, 33].

On the other hand, the balance of linear momentum (31b) of our theory

$$\frac{\partial M_i}{\partial t} + \frac{\partial (v^k M_i - \Sigma^k i)}{\partial x^k} = 0 \tag{50}$$

is the conservation law of the total momentum $M = m + \ldots$ of the system which also includes the momentum of the torsion fields associated with the microstructure, see (38). Therefore, because the total angular momentum, that is the sum of the angular momenta of all the constituents of the system including any fields, is a fundamentally conserved quantity, the total momentum flux $T^k_i = v^k M_i - \Sigma^k i$ has to be symmetric. This is similar to the electrodynamics of moving medium where the sum of the angular momentum of the medium and electromagnetic field is conserved [56]. Yet, we recall that the parts of the momentum flux, that is the convective part $v^k M_i$ and the constitutive part $\Sigma^k i$, are not symmetric in general, see Section 7.

11 Variational nature of the extended system and relation to micromorphic continua

In this section, we demonstrate that the reversible part (which does not raise the entropy), i.e. the left hand-side of (31) and the reversible source terms can be obtained from the variational principle. Also, the variational formulation allows to clarify the physical meaning of the new state variables. Thus, from what follows, it becomes clear that $B$ represents the non-local interaction in space while $D$ represents non-locality in time and can be associated to microinertia.

11.1 The case of holonomic triad

We first need to recall the standard variational formulation of the continuum mechanics in order to see how it should be generalized for the case of anholonomic triads. In the following, we shall demonstrate how to derive the left hand-side of the basic model (1) from the variational principle.

\[\text{For example, see [99, 25, 34, 96]}\]
Similar to the GENERIC formulation of non-equilibrium thermodynamics [81, 79], in the SHTC framework, the reversible and irreversible parts of the time evolution are treated separately. Thus, the reversible part of (1) is Hamiltonian and can be obtained either from Hamilton’s principle of stationary action or can be generated by corresponding Poisson brackets, e.g. see [84]. The irreversible part is represented by local (algebraic) terms of relaxation type, \(-\frac{1}{\theta} E_A\) and \(\frac{1}{\theta} E_s \theta E_A\): 

\[ E_A, \]

and is derived from the second law of thermodynamics [84]. Hence, in the absence of dissipation, the deformations are reversible and the Lagrangian formalism can be utilized, that is the variational principle in the SHTC framework is formulated on the globally relaxed matter manifold \(M^0\) parameterized with the Lagrangian coordinates \(\xi^a, a = 1, 2, 3\). Also, despite we are working in the non-relativistic settings and the time is absolute, it is convenient to distinguish notations for the time coordinate. Thus, we use the chart \((\tau, \xi^a)\) for the four Lagrangian coordinates, while the chart \((t, x^i)\) stands for the Eulerian (laboratory) coordinate system even though \(\tau = t\). For example, in this notations, the so-called material time derivative is 

\[ \frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i}, \]

with \(v^i\) being the material element velocity.

The existence of motion \(x^i = \chi^i(\tau, \xi^a)\) connecting in an one-to-one manner the manifolds \(M^0\) and \(M^t\) plays the central role because the Lagrangian density is defined as

\[ \tilde{\Lambda}(\tau, \xi^a, \chi^i, \partial_\tau \chi^i, \partial_\xi \chi^i) = \Lambda(\partial_\tau \chi^i, \partial_\xi \chi^i), \quad (51) \]

where \(\partial_\tau := \frac{\partial}{\partial \tau} = \partial_t + v^i \partial_i, \partial_\xi := \frac{\partial}{\partial \xi^a}\) are the Lagrangian time and spatial derivatives. Variation of the action \(\int \Lambda d\tau d\xi\) with respect to the motion \(\chi^i\) gives the Euler-Lagrange equations (momentum conservation)

\[ \partial_\tau \Lambda_{v^i} + \partial_\xi \Lambda_{F^i_A} = 0, \quad (52) \]

where the notations

\[ v^i := \partial_\tau \chi^i, \quad F^i_A := \partial_\xi \chi^i, \quad (53) \]

were introduced and have the meaning of the velocity and the deformation gradient respectively. The Euler-Lagrange equations (52) are under-determined equations (we have 12 unknowns \(v^i\) and \(F^i_A\) and only three equations) and thus, have to be supplemented with nine more evolution equations. These equations are given by the time evolution of the deformation gradient

\[ \partial_\tau F^i_A - \partial_\xi v^i = 0, \quad (54) \]

which are the trivial consequences of the definitions (53) and thus, they play the role of the integrability conditions for the Euler-Lagrange equations (52). Equations (52) and (53) together form a closed system of twelve evolution PDEs for twelve unknowns \((v^i, F^i_A)\) if only the Lagrangian density \(\Lambda(v^i, F^i_A)\) is specified (closure). These are however the equations written in the Lagrangian coordinates \((\tau, \xi^a)\) and have to be transformed into the Eulerian coordinates in order to obtain left hand-side of the equations (1b) and (1c), see details in [84].

\[ ^7 \text{Note that, in principle, the equation of motion (the energy-momentum conservation) can be also obtained in the Eulerian coordinates. However, the time evolution of the triad field (deformation gradient in this case) can be only obtained in an ad hoc manner, e.g. see [39]. While, if formulated in the Lagrangian coordinates, the triad evolution is derived rigorously [84].} \]
11.2 The case of anholonomic triad

The variational scheme described in the previous section is applicable only to the case when the triad $F^i_\gamma$ is holonomic, i.e. it is the gradient of the single-valued map $\chi(\tau, \xi^A)$. Because our main intention is to deal with irreversible deformations of fluids (or solids), the original globally relaxed configuration of the continuum $M^0$ plays no role in the evolution of flowing matter. Instead, a new locally relaxed configuration $M^{t, rel}$ was introduced in Section 4. As was discussed there, such a locally relaxed manifold $M^{t, rel}$ can not be connected with $M^t$ in a one-to-one manner and hence, a map, like $\chi^i(\tau, \xi^A)$ between $M^0$ and $M^t$, does not exist in general. In turn, this poses a serious problem in the classical way of the velocity field definition, that is via the time derivative of the motion (53).

In fact, our Riemann-Cartan formulation of fluid dynamics admits a variational formulation which does not require the motion $\chi^i(\tau, \xi^A)$. This can be naturally explained in the four-dimensional formalism (we still stay in the non-relativistic settings) because the velocity field is, in fact, a part of the anholonomic tetrad (4-distortion). Indeed, in the four-dimensional formalism, the classical definition of the velocity field (53) can be viewed as a part of the 4-deformation gradient. Thus, let us consider the extended Eulerian coordinates $x^\mu = (t, x^i), \mu = 0, 1, 2, 3$ and the Lagrangian coordinates $\xi^A = (\tau, \xi^1, \xi^2, \xi^3), A = 0, 1, 2, 3$ which now include also the time coordinate. Yet recall that, in the non-relativistic settings, the time is assumed to be absolute, i.e. $t = \tau$. Hence, the velocity field $v^\mu = \frac{\partial \chi^\mu}{\partial \xi^0} = (1, v^1, v^2, v^3)$ is just a part of the 4-deformation gradient $F^\mu_\gamma = \frac{\partial \chi^\mu}{\partial \xi^A}$

$$F^\mu_\gamma := \begin{pmatrix} 1 & 0 & 0 & 0 \\ v^1 & F^1_1 & F^1_2 & F^1_3 \\ v^2 & F^2_1 & F^2_2 & F^2_3 \\ v^3 & F^3_1 & F^3_2 & F^3_3 \end{pmatrix}. \quad (55)$$

Motivated by this fact, we thus consider the velocity $v^\mu$ and the anholonomic triad field $E^i_a$, $a = 1, 2, 3$ (inverse of $A^a_i$) as being parts of the anholonomic tetrad $E^\mu_\gamma$, $a = 0, 1, 2, 3$. In other words, the velocity field will be not defined as the time derivative of a motion but it is postulated axiomatically and is defined as the solution to a certain evolution equation, namely the momentum conservation law.

The Lagrangian density is then considered as a function of the tetrad $E^\mu_\gamma$ and its first-order derivatives $\partial_a E^\mu_\gamma$

$$\tilde{\Lambda}(E^\mu_\gamma, \partial_a E^\mu_\gamma) = \Lambda(E^\mu_\gamma, T^{*\mu\nu}), \quad (56)$$

where $T^{*\mu\nu} := \varepsilon^{abcd} \partial_c E^\mu_d = \frac{1}{2} \varepsilon^{abcd} T^{ab}_{\mu b}$ is the Hodge dual of the torsion $T^{\mu}_{ab} := \partial_a E^\mu_b - \partial_b E^\mu_a$, and $\varepsilon^{abcd}$ is the 4D Levi-Civita symbol. Recall that we do not need to introduce a coordinate system in $M^{t, rel}$ in order to define the derivatives $\partial_a$. Instead, we use that fact that the tangent spaces to $M^{t, rel}$ are soldered to $M^t$ and the material derivatives $\partial_a$ are treated as the space derivatives $\partial_\mu$ written in the local Cartan frame $e^a$, that is $\partial_a := E^\mu_a \partial_\mu$, see (15). Recall that, due to fixing gauge degrees of freedom (local rotations $R^a_{\gamma}(x^i)$ in the tangent spaces $T^x M^i$), the derivatives $\partial_a$ are, in fact, the covariant derivatives in $M^{t, rel}$, see the discussion right after equation (15).
The first variation of the action gives us the Euler-Lagrange equations
\[
\partial_d (\varepsilon^{abcd} \Lambda_{\mathbf{T}^*\mu_{bc}}) = \Lambda_{E^\mu_a},
\]
which have to be supplemented by the integrability conditions
\[
\partial_b T^{*\mu ab} = 0.
\]
So far, we obtain only equations for the torsion field \( T^\mu_{ab} \) which give us equations (31d) and (31e), while we still do not have evolution PDEs for the tetrad field \( E^\mu_a \). In fact, the tetrad evolution PDE
\[
\partial_a \Lambda_{E^\mu_a} = 0
\]
is exactly the momentum conservation law (the energy-momentum conservation to be more precise). This PDE, however, is not obtained by means of variation of the action like in the classical mechanics but it is an identity which is obtained after applying the divergence operator to (57) (the left hand-side then identically vanishes). Remark that such a variation scheme, though formulated in the matter manifold, is fully equivalent to the variation scheme in the teleparallel gravity formulated in the spacetime manifold, e.g. see [18].

Because the targeting reader, we believe, is used to work in the three-dimensional formalism instead of the four-dimensional one described above, we shall rewrite now this variational formulation in the 3D notations.

Thus, we now treat the Lagrangian density (56) as the function of the potentials \( v^i \) and \( E^i_a \) and their first derivatives (in what follows \( a, b, c = 1, 2, 3 \) again)
\[
L(v^i, E^i_a, e^i_a, h^{ai}),
\]
where
\[
e^i_a := \partial_\tau E^i_a - \partial_a v^i, \quad h^{ai} := -\varepsilon^{abc} \partial_b E^i_c,
\]
where, in turn, \( \partial_\tau \) is the material time derivative and \( \partial_a = E^i_a \partial_i, \ a = 1, 2, 3 \) are the material space derivatives. Note that the PDE (61)\(_1\) looks as the PDE (54) for the total deformation gradient (which is holonomic triad) in the Lagrangian coordinates apart that (61)\(_1\) has the source of non-holonomy \( e^i_a \).

Then, the first variation of the action with respect to \( v^i \) and \( E^i_a \) gives the Euler-Lagrange equations:
\[
\partial_\tau L_{e^i_a} + \varepsilon^{abc} \partial_b L_{h^{ci}} = L_{E^i_a}, \quad -\partial_a L_{e^i_a} = L_{v^i}.
\]
Exactly as for (59), the momentum conservation (compare with (52))
\[
\partial_\tau L_{v^i} + \partial_a L_{E^i_a} = 0
\]
is obtained not by means of variations but it is the identity which is obtained by applying \( \partial_\tau \) to (62)\(_2\) and \( \partial_a \) to (62)\(_1\) and summing up the results.
Equations of motion (62) and (63) have to be supplemented with the integrability conditions (they are trivial consequences of the definitions (61)):

\[ \partial_t h^{ai} + \varepsilon^{abc} \partial_b h^{ci} = 0, \quad \partial_b h^{bi} = 0, \quad \partial_t E^i_a - \partial_a v^i = e^i_a. \]  

(64)

After the introduction of the new variables

\[ m_i = L v^i, \quad D^a_i := L e^i_a, \quad B^{ai} := h^{ai}, \]  

(65)

and new potential

\[ U := v^i L v^i + e^i_a L e^i_a - L \]  

(66)
as a partial Legendre transformation of \( L \), the equations (62) and (64) can be rewritten as (note that \( U^B = -L^B, U^E = -L^E \))

\[ \partial_t m_i - \partial_a U^E_i a = 0, \quad \partial_t E^i_a - \partial_a U_{m_i} = U_{D^i_a}, \]  

(67a)

\[ \partial_t D^a_i - \varepsilon^{abc} \partial_b U^B_{ci} = -U^E_{a_i}, \quad \partial_t B^{ai} + \varepsilon^{abc} \partial_b U_{D^a_i} = 0. \]  

(67b)

These are the equations written in the relaxed matter manifold \( M_{t,rel} \). Their structure may remind the structure of the electrodynamics equations studied in [32, 84]. In particular, one can use the series of cumbersome coordinate, state variable and thermodynamic potential transformations from [84] in order to transform these equations into their Eulerian counterparts (31). Nevertheless, we remark that the Hamiltonian structure of (67b) (i.e. the underlined Poisson brackets) is not fully equivalent to the Hamiltonian structure of the electrodynamics equations due to the presence of the reversible source terms, see Section 5 which are the part of the reversible part of the time evolution and therefore have to be generated by the Poisson brackets. Remark that the energy potential \( \mathcal{E} \) in (31) and the potential \( U \) in (67b) are related as \( \mathcal{E} = wU \) if \( w = \det(A) \) [84], and hence \( \mathcal{E} \) is directly related to the Lagrangian \( L \) via the Legendre transformation (66).

Finally, we note that the variational viewpoint on the extended model (31) allows to see more clearly the physical meaning of the new fields \( D \) and \( B \). Thus, from the definition (61) it becomes clear that the complimentary filed \( D \) represents the non-locality in time and can be associated to the micro-inertia of the microstructure (either emerging as in turbulence or preset from the beginning as in the micro-structured solids). In particular, such an observation can be useful for modeling of microstructurally heterogeneous materials which are called acoustic metamaterials and/or phononic crystals, which, in the last years, raised a lot of interest in mathematical modeling in solid mechanics and show exotic response with respect to perturbations of certain wave lengths [24]. To model such materials, higher gradients (in space and time) of a total and microscopic deformation field are adopted. A quite general class of such models, called micromorphic or generalized continua models, has been developed by many authors starting from the work by Cosserat brothers [23], Mindlin [71] and Eringen [34], in the context of linear deformations, e.g. see [36, 38, 67, 5, 26] and the references therein, and finite deformations [8, 7, 11, 10]. The proposed non-linear model (31) can be also attributed to the class of micromorphic models since the higher spatial and time gradients of the
local distortion field are used as independent state variables. In particular, we expect that the nonlinear dispersive models like the one proposed in this paper can be applied to modeling of genuinely nonlinear waves such as solitons and shock waves in microstructured solids.

12 Galilean and Lorentz invariance

In Newtonian mechanics, we used to deal with material equations which have to respect the Galilean invariance principle. The basic equations (1) as well as other equations belonging to the SHTC class such as hyperbolic mass and heat conduction [84] are Galilean invariant by construction [43]. This is due to the Hamiltonian nature of the SHTC equations.

The extended system (31) is however not a true material system in the sense that the new fields $B$ and $D$ are like the classical fields such as electric and magnetic fields, for example, that do not have a material carrier. Moreover, as we could see, there is a straightforward analogy between the structure of the Maxwell equations in moving medium [32] and the extended model (31). It also immediately follows from this analogy that the equations for $B$ and $D$ fields are not Galilean invariant but Lorentz invariant. For example, the characteristic analysis in [32] shows that the maximum sound speed in such a medium is $c = 1/\sqrt{\epsilon\mu}$ instead of one would expect $v \pm c$ which is inherent to Galilean invariant systems, where $v$ denotes the medium velocity in the given coordinate direction.

13 Conclusion and perspectives

We have presented an extension of the unified hyperbolic formulation of continuum fluid and solid mechanics recently proposed in [83, 31, 30] mentioned above as the basic system (1). The basic system, in turn, is based on the Godunov-Romenski model for elastoplastic deformation in metals developed in 1970s [45, 42, 90]. The extended system aims in accounting for the rotational degrees of freedom of the main field of the basic model, the distortion field, denoted as $A$.

In Section 4, the distortion field is considered as the Cartan moving frame and thus it represents anholonomic basis triad field in general. As the characteristic of anholonomy, we use the torsion tensor. The distortion and torsion are therefore define completely the internal non-Euclidean (Riemann-Cartan) flow geometry with Weitzenböck connection as a linear affine connection. The paper is therefore can be treated as an attempt to introduce methods of differential geometry to fluid dynamics in general and turbulence modeling in particular. Moreover, the proposed theory is essentially an analog of the teleparallel gravity (torsional gravity) [60, 1, 18, 47], which can be viewed as a Yang-Mills-type translational gauge theory [50, 22, 49, 65]. It is therefore would be also interesting to consider the turbulence phenomena from the standpoint of a gauge field theory. In this paper, we however have considered a version of the model with a fixed gauge. Namely, we assumed that the anholonomic frames in the relaxed tangent space are all co-aligned. The general
case, without a gauge fixing and hence, with non-zero spin connection will be considered elsewhere.

In the fluids dynamics context, the anholonomy of the distortion field is mainly due to its rotational degrees of freedom (spin) which can be used to characterize the kinetic energy of the unresolved eddies in a turbulent flow. The torsion field $B$ is therefore was used to account for the distortion spin in the model and thus, $B$ plays the role of an independent state variable in the extended model along side with the complimentary field $D$. The former has the meaning of the number density of unresolved eddies while the latter has the meaning similar to the angular momentum and characterizes micro inertia as discussed in Section 11. We then observed that in the extended model, the contribution due to torsion $B$ and $D$ field affects the definitions of the total momentum density, stress tensor, total energy and the flux of the total energy.

Furthermore, in order to fulfill the thermodynamic consistency, the dispersive terms (which conserve the entropy) have had to be introduced. These terms are of algebraic relaxation type and do not affect hyperbolicity of the model.

In Section 9, we identify the laminar-to-turbulent transition as the excitation of the laminar state, which, in turn, is considered as a local equilibrium state. This transition is highly non-linear and is a result of the interplay between irreversible and dispersive mechanisms in the new model. The new relaxation parameter (47) governing such a transition is essentially the Reynolds number of the macroscopic observer.

We then observed that the governing equations for the new fields $B$ and $D$ share some elements of the common structure with the equations for electrodynamics of moving media recently studied in [32, 84]. Also, similar to the electrodynamics, the fields $B$ and $D$ like the magnetic and electric fields are true classical fields because their carriers are mass-less entities like vortexes in fluids or defects in solids. In the solid dynamics context, the straightforward analogy between the equations governing the dislocation dynamics and those governing the electromagnetic fields was observed by several authors, e.g. [61, 58, 48, 64]. Also, similar to the electrodynamics equations, the momentum density $M$ is treated as the total momentum (matter momentum + field’s momentum). Due to this fact, the conservation of the total angular momentum is guaranteed by the symmetry of the full momentum flux tensor and thus, no additional equation for the balance of angular momentum is required in our theory, see Section 10.

In Section 11, we discussed the variational nature of the governing equations (31). In the absence of dissipation, we formulated the variational principle without referencing to the initial globally relaxed configuration $M^0$. Instead, only locally relaxed $M^{t,\text{rel}}$ and current $M^t$ configurations were used. This is an important result showing that the notion of motion encoding the full history of the medium evolution is not required for formulation of the variational principle.

Importantly, the new system is a nonlocal PDE system due to the fact that the torsion tensor is defined as higher gradients of the triad $A$. The nonlocality at small scales (both in space and time), i.e. when behavior at a point can be strongly influenced by the flow remote from that point, is within the main features of turbulent flows and is within the most difficult features to be
modeled [86, 100]. Also recall that the torsion is specifically used to describe nonlocal interaction (action at a distance) in tetrad formulations of relativistic theories such as teleparallel theories of gravity. One could therefore expect that the same geometrical concepts can be used to describe nonlocal effects in fluid dynamics. Besides, note that mathematical analogies between the non-linear electromagnetism and tetrad (or torsion) theories of gravity similar to those observed in this paper was recently discussed in [53].

So far, we did not discuss possible numerical implementation issues of the proposed model. We expect that solving the extended model numerically is not a trivial task due to the presence of stiff relaxation terms of both natures, irreversible and dispersive, and due to the strong non-linearity of the model. However, despite the strong non-linearity of the extended model, one can guaranty the local well-posedness (hyperbolicity) of the Cauchy problem if the energy potential is convex [84]. Nevertheless, one may expect that the standard numerical methods for hyperbolic PDEs can be used only exceptionally but methods with the asymptotic preserving and well-balanced properties should be employed [29, 20, 80]. Thus, as in the case of the basic model (1), the family of ADER (Arbitrary high-order DERivatives) Finite Volume (ADER-FV) or Discontinuous Galerkin (ADER-DG) schemes, see [31, 32] and references therein, can successfully handle such issues in the asymptotic preserving and well-balanced manner. We therefore plan to study the extended model numerically and compare its solution against some experimental data as well as the results of Direct Numerical Simulation (DNS) of turbulent flows in subsequent papers.

One of the natural possibilities for the further extension of the present results might be obtaining of general/special relativistic versions of the basic and extended models. Recall that the classical Navier-Stokes fluid dynamics can not be applied to modeling relativistic flows due to the intrinsic causality and stability issues of the Navier-Stokes equations [88]. To fix these issues of the classical fluid dynamics, several alternatives have been developed [74, 66, 54, 97, 40, 37, 98] to name just a few. The geometrical nature of our theory with the underlying Riemann-Cartan geometry also suggests that it would be not difficult to obtain the relativistic version of the theory which should fit well to the geometrical settings of the general relativity or its teleparallel equivalent [1]. This work is currently in progress.

The diversity of physical theories where the geometrical methods and concepts are used (e.g. gravity, differential forms in electromagnetism, Lagrangian and Hamiltonian mechanics, Yang-Mills gauge theory) may witness about their universal validity and independence on the physical context. We thus hope that this paper will help to reformulate modern Navier-Stokes-based fluid dynamics in a more universal language of differential geometry. This, in turn, may help to look at the turbulence problem from a fundamentally different angle.

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