COHOMOLOGY COMPUTATIONS OF SOLVMANIFOLDS WITH LOCAL COEFFICIENTS

HISASHI KASUYA

Abstract. Let $G$ be a simply connected solvable Lie group with a lattice $\Gamma$ and $N$ the nilradical of $G$. For a complex valued representation $\rho : G \to GL(V_\rho)$ such that the restriction $\rho|_N$ is unipotent, as an advanced variation of cohomology computation of solvmanifolds by using Lie algebra cohomology, we construct an explicit finite dimensional cochain complex whose cohomology is isomorphic to the cohomology of $G/\Gamma$ with twisted by $\rho$.

1. Introduction

Let $G$ be a simply connected solvable Lie group with a lattice $\Gamma$. $G$ has the maximal normal nilpotent subgroup $N$ called the nilradical of $G$. Let $\rho : G \to V_\rho$ be a finite dimensional linear representation. Suppose $G$ has a lattice $\Gamma$. We consider the flat bundle $E_{\rho|\Gamma} = (G \times V_\rho)/\Gamma$ given by the equivalent relation $(\gamma g, \rho(\gamma)v) \sim (g, v)$ for $g \in G$, $v \in V_\rho$, $\gamma \in \Gamma$. In this paper we will study the cohomology $H^*(G/\Gamma, E_{\rho|\Gamma})$ with values in $E_{\rho|\Gamma}$. Since $G$ is diffeomorphic to the euclidian space $\mathbb{R}^{\dim G}$, the cohomology $H^*(G/\Gamma, E_{\rho|\Gamma})$ is isomorphic to the group cohomology $H^*(\Gamma, \rho|\Gamma)$.

Main result 1. Suppose the restriction $\rho|_N$ is unipotent. Then we can construct an explicit finite dimensional cochain complex $A^*_\Gamma$ (constructed in Section 6) to compute the cohomology $H^*(G/\Gamma, E_{\rho|\Gamma})$ by using the techniques of Lie groups and Lie algebras.

Let $\mathfrak{g}$ be the Lie algebra of $G$. It is known that under some conditions of $G, \Gamma, \rho$ the cohomology $H^*(G/\Gamma, E_{\rho|\Gamma})$ is isomorphic to the Lie algebra cohomology $H^*(\mathfrak{g}, V_\rho)$ (see [4], [7] and [9]). However in general the isomorphism $H^*(G/\Gamma, E_{\rho|\Gamma}) \cong H^*(\mathfrak{g}, V_\rho)$ does not holds. The techniques of computations of the cohomology $H^*(G/\Gamma, E_{\rho|\Gamma})$ in this paper are effective even if such isomorphism does not holds.

Consider the adjoint representation $\text{Ad} : G \to \text{Aut}(\mathfrak{g}_C)$. The cohomology $H^*(\Gamma, \text{Ad})$ is important for studying the deformation of lattice $\Gamma$ in $G$. Since we have $N = \{g \in G|\text{Ad}_g \text{ is unipotent}\}$ (see [9]), we can compute the cohomology $H^*(\Gamma, \text{Ad})$ in general case.

In earlier work [6], we consider the case $\rho$ is trivial. The main result of this paper is a twisted version of [6, Corollary 7.6].

2. Jordan decompositions of representations

Let $A \in GL_n(\mathbb{C})$. We denote by $A_s$ (resp. $A_u$) the semi-simple (resp. unipotent) part of $A$ for the Jordan decomposition (see [3] for the definition). We will use the following facts.

Lemma 2.1. Let $N$ be a simply connected nilpotent Lie group and $\varphi : N \to GL(V_\phi)$ a representation. Then the map $\varphi' : N \ni g \to (\varphi(g))_s$ is also a representation (see [1]). Since $\varphi'(N)$ is connected nilpotent group and consists of semi-simple elements, the Zariski-closure of $\varphi'(N)$ is an algebraic torus (see [5, Section 19]) and hence $\varphi'$ is diagonalizable.

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3. Lie algebra cohomology

Let $G$ be a simply connected Lie group, $g$ the Lie algebra of $G$ and $\rho : G \to GL(V_\rho)$ a representation. We consider the $G$-action on the cochain complex $A^*(G) \otimes V_\rho$ induced by the left multiplication. Let $(A^*(G) \otimes V_\rho)^G$ be the subcomplex consisting of the $G$-invariant elements. Then $(A^*(G) \otimes V_\rho)^G$ is isomorphic to the cochain complex $\bigwedge g_C^* \otimes V_\rho$ of Lie algebra (see [8] for the definition). Suppose $G$ has a lattice $\Gamma$. Let $(A^*(G) \otimes V_\rho)^\Gamma$ be the subcomplex consisting of the $\Gamma$-invariant elements. Then $(A^*(G) \otimes V_\rho)^\Gamma$ is isomorphic to $A^*(G/\Gamma, E_{\rho\Gamma})$. Hence we have the canonical inclusion

$$\bigwedge g_C^* \otimes V_\rho \to A^*(G/\Gamma, E_{\rho\Gamma}).$$

**Theorem 3.1.** ([8]) Let $N$ be a simply connected nilpotent Lie group with a lattice and $\rho : N \to GL(V_\rho)$ a unipotent representation. Then the inclusion

$$\bigwedge n_C^* \otimes V_\rho \to A(N/\Gamma, E_{\rho\Gamma})$$

induces a cohomology isomorphism.

We will use the triviality of the adjoint action on the cohomology. We consider the adjoint action $Ad^* \otimes \rho : G \to Aut(\bigwedge g_C^* \otimes V_\rho)$ induced by the conjugation. The adjoint action on $\bigwedge g_C^* \otimes V_\rho$ induces the trivial action on the cohomology $H^*(g, V_\rho)$.

4. Cohomology of tori

Let $A$ be a simply connected abelian group with a lattice $\Gamma$ and $\mathfrak{a}$ the Lie algebra.

**Lemma 4.1.** Let $\varphi : \Gamma \to GL(V_\varphi)$ be a representation. Suppose $\varphi = \beta \otimes \phi$ such that $\beta$ is a character of $\Gamma$ and $\phi$ is a unipotent representation. If $\beta$ is non-trivial, then we have

$$H^*(A/\Gamma, E_{\varphi}) = 0.$$

**Proof.** Suppose dim $A = 1$. Then we have

$$H^0(A/\Gamma, E_{\varphi}) \cong H^0(\Gamma, V_\varphi) = \{m \in V_{\beta \otimes \phi}\beta(g)\phi(g)m = m, \text{ for all } g \in \Gamma\} = 0.$$

By the Poincaré duality we have

$$H^1(\Gamma, V_\varphi) \cong H^0(\Gamma, V_\varphi^*)^* = 0,$$

and obviously $H^p(\Gamma, V_\varphi) = 0$ for $p \geq 2$.

In general, we take a decomposition $\Gamma = \Gamma' \oplus \Gamma''$ such that $\Gamma'$ is a rank 1 subgroup and the restriction $\beta|_{\Gamma'}$ is also non-trivial. Then we have the Hochshild-Serre spectral sequence $E_r$ such that

$$E_2^{p,q} = H^p(\Gamma/\Gamma', H^q(\Gamma', V_{\varphi})).$$

and this converges to $H^{p+q}(\Gamma, V_{\varphi})$. Since $H^q(\Gamma, V_{\varphi}) = 0$ for any $q$, we have $E_2 = 0$ and hence the lemma follows. \hfill $\square$

Similarly we have:

**Lemma 4.2.** Let $\varphi : A \to GL(V_\varphi)$ be a representation. Suppose $\varphi = \alpha \otimes \phi$ such that $\alpha$ is a character of $A$ and $\phi$ is a unipotent representation. If $\alpha$ is non-trivial, then we have

$$H^*(\mathfrak{a}, V_{\varphi}) = 0.$$

Let $\rho : A \to GL(V_\rho)$ be a representation. Let $\{V_\alpha\}_{\alpha \in Hom(A, C^*)}$ be the set of all 1-dimensional representations of $A$ and $\{V_\beta\}_{\beta \in Hom(\Gamma, C^*)}$ the set of all 1-dimensional representations of $\Gamma$. We consider the flat bundles $\{E_\beta\}_{\beta \in Hom(\Gamma, C^*)}$. We consider the direct sum

$$\bigoplus_{\alpha \in Hom(A, C^*)} \bigwedge a_C^* \otimes V_\alpha \otimes V_\rho$$

of dual complexes of Lie algebra. We also consider the direct sum

$$\bigoplus_{\beta \in Hom(\Gamma, C^*)} A^*(A/\Gamma, E_\beta \otimes E_\rho).$$
We have an inclusion
\[ \bigoplus_{\alpha \in \text{Hom}(A, C^*)} \bigwedge \mathfrak{a}_C^* \otimes V_\alpha \otimes V_\rho \to \bigoplus_{\beta \in \text{Hom}(\Gamma, C^*)} A^* (A/\Gamma, E_\beta \otimes E_\rho). \]

**Lemma 4.3.** We have a decomposition
\[ \rho = \bigoplus_{i=1}^k \alpha_i \otimes \phi_i \]
such that \( \alpha_i \) are characters and \( \phi_i \) are unipotent representations.

**Proof.** For a character \( \alpha \), we denote by \( W_\alpha \) the subspace of \( V_\rho \) consisting of the elements \( w \in V_\rho \) such that for some positive integer \( n \) we have \( (\rho(a) - \alpha(a)I)^n w = 0 \) for any \( a \in A \). Since \( A \) is abelian, we have a decomposition
\[ V_\rho = W_{\alpha_1} \oplus \cdots \oplus W_{\alpha_k} \]
by generalized eigenspace decomposition of \( \rho(a) \) for all \( a \in A \). Let \( \rho_1(a) = (\rho(a)|_{W_{\alpha_1}} \). Then we have \( \rho = \rho_1 \oplus \cdots \oplus \rho_k \). We have \( (\rho_1(a))_s = \alpha_1 I \). Let \( \phi_1(a) = (\rho_1(a))_u \). By Lemma 2.1, \( \phi_1 \) is a unipotent representation and we have \( \rho_1(a) = (\rho_1(a))_s (\rho_1(a))_u = (\alpha_1 \otimes \phi_1)(a) \). Hence the Lemma follows.

**Proposition 4.4.** The inclusion
\[ \bigoplus_{\alpha \in \text{Hom}(A, C^*)} \bigwedge \mathfrak{a}_C^* \otimes V_\alpha \otimes V_\rho \to \bigoplus_{\beta \in \text{Hom}(\Gamma, C^*)} A^* (A/\Gamma, E_\beta \otimes E_\rho) \]
induces a cohomology isomorphism.

**Proof.** Consider the decomposition
\[ \rho = \bigoplus_{i=1}^k \alpha_i \otimes \phi_i \]
as the above lemma. Then we have
\[ \bigoplus_{\alpha \in \text{Hom}(A, C^*)} \bigwedge \mathfrak{a}_C^* \otimes V_\alpha \otimes V_\rho = \bigoplus_{\alpha \in \text{Hom}(A, C^*)} \bigwedge \mathfrak{a}_C^* \otimes \bigoplus_{i=1}^k V_{\alpha \alpha_i} \otimes V_{\phi_i} \]
and
\[ \bigoplus_{\beta \in \text{Hom}(\Gamma, C^*)} A^* (A/\Gamma, E_\beta \otimes E_\rho) = \bigoplus_{\beta \in \text{Hom}(\Gamma, C^*)} A^* (A/\Gamma, \bigoplus_{i=1}^k E_{\beta \alpha_i}) \otimes E_{\phi_i}. \]
By Theorem 3.1 and Lemma 4.1, we have
\[ H^* \left( \bigoplus_{\beta \in \text{Hom}(\Gamma, C^*)} A^* (A/\Gamma, E_\beta \otimes E_\rho) \right) \cong H^* (A/\Gamma, \bigoplus_{i=1}^k E_{\phi_i}). \]
By Lemma 4.2, we have
\[ H^* \left( \bigoplus_{\alpha \in \text{Hom}(A, C^*)} \bigwedge \mathfrak{a}_C^* \otimes V_\alpha \otimes V_\rho \right) \cong H^* (a, \bigoplus_{i=1}^k V_{\phi_i}). \]
Hence the proposition follows.

**Proposition 4.5.** The inclusion
\[ \bigoplus_{\alpha \in \text{Hom}(A, C^*)} \bigwedge \mathfrak{a}_C^* \otimes V_\alpha \otimes V_\rho \to \bigoplus_{\beta \in \text{Hom}(\Gamma, C^*)} A^* (A/\Gamma, E_\beta \otimes E_\rho) \]
induces a cohomology isomorphism.
5. COHOMOLOGY OF SOLVMANIFOLDS

Let $G$ be a simply connected solvable Lie group with a lattice $\Gamma$ and $\mathfrak{g}$ be the Lie algebra of $G$. Let $N$ be the nilradical of $G$. Let $\mathcal{A}(G,N) = \{ \alpha \in \text{Hom}(G, \mathbb{C}^*) | \alpha|_N = 1 \}$ and $\mathcal{A}(\Gamma, G/N) = \{ \alpha \in \text{Hom}(\Gamma, \mathbb{C}^*) | \alpha|_{\Gamma \cap N} = 1 \}$. For $\alpha \in \mathcal{A}(G,N)$ (resp. $\mathcal{A}(\Gamma, G/N)$), we denote by $V_\alpha$ the 1-dimensional representation via $\alpha$. Let $\rho : G \to GL(V_\rho)$ be a representation. We consider the direct sum

$$ \bigoplus_{\alpha \in \mathcal{A}(G,N)} \bigwedge g_\mathcal{C} \otimes V_\alpha \otimes V_\rho $$

of dual complexes of Lie algebra. We also consider the direct sum

$$ \bigoplus_{\beta \in \mathcal{A}(\Gamma, G/N)} A^*(G/\Gamma, E_\beta \otimes E_\rho). $$

Then we have an inclusion

$$ \bigoplus_{\alpha \in \mathcal{A}(G,N)} \bigwedge g_\mathcal{C} \otimes V_\alpha \otimes V_\rho \to \bigoplus_{\beta \in \mathcal{A}(\Gamma, G/N)} A^*(G/\Gamma, E_\beta \otimes E_\rho). $$

**Theorem 5.1.** Suppose the restriction $\rho|_N$ is unipotent. Then the inclusion

$$ \bigoplus_{\alpha \in \mathcal{A}(G,N)} \bigwedge g_\mathcal{C} \otimes V_\alpha \otimes V_\rho \to \bigoplus_{\beta \in \mathcal{A}(\Gamma, G/N)} A^*(G/\Gamma, E_\beta \otimes E_\rho). $$

induces a cohomology isomorphism.

**Proof.** Consider the quotient $q : G \to G/N$. $\Gamma \cap N$ is a lattice of $N$, $q(\Gamma)$ is a lattice of the abelian group $G/N$ and we have the fiber bundle

$$ p : G/\Gamma \to (G/N)/q(\Gamma) $$

with nilmanifold $N/\Gamma \cap N$ as the fiber. We use the spectral sequence of de Rham complex induced by the fiber bundle $p : G/\Gamma \to (G/N)/q(\Gamma)$ (see [4]). For a representation $\rho : G \to GL(V_\rho)$ we have a filtration of $\mathcal{A}^*(G/\Gamma, E_\rho)$ which gives the spectral sequence $E_1^{*,*}$ such that

$$ E_1^{*,*} \cong A^*(G/N/q(\Gamma), H^*(N/\Gamma \cap N, E_{\rho|_\Gamma})). $$

where $H^*(N/\Gamma \cap N, E_{\rho|_\Gamma})$ is the flat vector bundle

$$ \bigoplus_{x \in (G/N)/q(\Gamma)} H^*(p^{-1}(x), E_{\rho|_\Gamma}). $$

over $(G/N)/q(\Gamma)$. This filtration gives the filtration of the subcomplex $\bigwedge g_\mathcal{C} \otimes V_\rho$ which gives the spectral sequence $E_1^{*,*}(\mathfrak{g})$ such that

$$ E_1^{*,*}(\mathfrak{g}) = \bigwedge (g/n)_\mathcal{C} \otimes H^*(n, V_\rho) $$

and we have the commutative diagram

$$ \begin{array}{ccc}
E_1^{*,*}(\mathfrak{g}) & \xrightarrow{\cong} & E_1^{*,*} \\
\bigwedge (g/n)_\mathcal{C} \otimes H^*(n, V_\rho) & \xrightarrow{\cong} & A^*(G/N/q(\Gamma), H^*(N/\Gamma \cap N, E_{\rho|_\Gamma})).
\end{array} $$

We consider the spectral sequence of cochain complex

$$ \bigoplus_{\beta \in \mathcal{A}(\Gamma, G/N)} A^*(G/\Gamma, E_\beta \otimes E_\rho). $$

We have

$$ E_1^{*,*} = \bigoplus_{\beta \in \mathcal{A}(\Gamma, G/N)} A^*(G/N/q(\Gamma), H^*(N/\Gamma \cap N, E_{\beta \otimes E_{\rho|_\Gamma}})). $$

Since $\beta|_{\Gamma \cap N} = 1$, we have

$$ H^*(N/\Gamma \cap N, E_{\beta \otimes E_{\rho|_\Gamma}}) = E_{\beta} \otimes H^*(N/\Gamma \cap N, E_{\rho|_\Gamma}). $$
Since the restriction $\rho_{|N}$ is unipotent, we have $H^*(N/\Gamma \cap N, E_{g_{N}}) = H^*(n, V_{\rho_{|N}})$. Hence we have $H^*(N/\Gamma \cap N, E_{|N}) = E_{\Psi}$ where we denote by $\Psi : G/N \to \text{Aut}(H^*(n, V_{\rho_{|N}}))$ the $G/N$-action on the cohomology $H^*(n, V_{\rho_{|N}})$ induced by the extension $\mathfrak{g} \to \mathfrak{g}/n$.

Consider the spectral sequence $E_{1}^{*,*}(g)$ of
$$
\bigoplus_{\alpha \in \mathcal{A}(G,N)} \bigoplus_{\rho \in \mathcal{A}(G,N)} \bigwedge g_{C}^* \otimes V_{\alpha} \otimes V_{\rho}.
$$

We have
$$
E_{1}^{*,*}(g) = \bigoplus_{\alpha \in \mathcal{A}(G,N)} \bigoplus_{\rho \in \mathcal{A}(G,N)} \bigwedge (g/n)^* \otimes H^*(n, V_{\alpha} \otimes V_{\rho})
= \bigoplus_{\alpha \in \mathcal{A}(G,N)} \bigoplus_{\rho \in \mathcal{A}(G,N)} \bigwedge (g/n)^* \otimes V_{\alpha} \otimes H^*(n, V_{\rho})
= \bigoplus_{\alpha \in \mathcal{A}(G,N)} \bigoplus_{\rho \in \mathcal{A}(G,N)} \bigwedge (g/n)^* \otimes V_{\alpha} \otimes V_{\rho}.
$$

Since we can identify $\mathcal{A}(G,N)$ (resp. $\mathcal{A}(\Gamma, \Gamma\cap N)$) with $\text{Hom}(G/N, \mathbb{C}^*)$ (resp. $\text{Hom}(\Gamma/\Gamma \cap N, \mathbb{C}^*)$), we have the commutative diagram
$$
\begin{array}{ccc}
E_{1}^{*,*}(g) & \cong & E_{1}^{*,*} \\
\bigoplus_{\alpha \in \text{Hom}(G/N, \mathbb{C}^*)} \bigwedge (g/n)^* \otimes V_{\alpha} \otimes V_{\rho} & \longrightarrow & \bigoplus_{\beta \in \text{Hom}(\Gamma/\Gamma \cap N, \mathbb{C}^*)} A^*((G/N)/\gamma(\Gamma), E_{\beta} \otimes E_{\rho_{|\Gamma}}).
\end{array}
$$

By Proposition 4.5, the homomorphism $E_{1}^{*,*}(g) \to E_{1}^{*,*}$ induces a cohomology isomorphism and hence we have an isomorphism $E_{2}^{*,*}(g) \cong E_{2}^{*,*}$. Hence the theorem follows. \qed

6. Construction of finite cochain complex

Let $G$ be a simply connected solvable Lie group and $\mathfrak{g}$ be the Lie algebra of $G$. Let $N$ be the nilradical of $G$. Let $\rho : G \to GL(V_{n})$ be a representation. We consider the direct sum
$$
\bigoplus_{\alpha \in \mathcal{A}(G,N)} \bigwedge g_{C}^* \otimes V_{\alpha} \otimes V_{\rho}
$$
of dual complexes of Lie algebra. Then we have the $G$-action on this cochain complex via $\bigoplus \text{Ad} \otimes \alpha \otimes \rho$. Consider the semi-simple part
$$
((\bigoplus_{\alpha \in \mathcal{A}(G,N)} \text{Ad} \otimes \alpha \otimes \rho)(g))_{s} = \bigoplus_{\alpha \in \mathcal{A}(G,N)} \text{Ad}_{g} \otimes \alpha (g) \otimes (\rho(g))_{s}.
$$

By [1] Proposition 3.3, we have a simply connected nilpotent subgroup $C \subset G$ such that $G = C \cdot N$. Since $C$ is nilpotent, the map
$$
\Phi : C \ni c \mapsto \bigoplus_{\alpha \in \mathcal{A}(G,N)} \text{Ad}_{g} \otimes \alpha (g) \otimes (\rho(g))_{s} \in \text{Aut}(\bigoplus_{\alpha \in \mathcal{A}(G,N)} \bigwedge g_{C}^* \otimes V_{\alpha} \otimes V_{\rho})
$$
is a homomorphism. We denote by
$$
((\bigoplus_{\alpha \in \mathcal{A}(G,N)} \bigwedge g_{C}^* \otimes V_{\alpha} \otimes V_{\rho})_{\Phi(C)}
$$
the subcomplex consisting of the $\Phi(C)$-invariant elements.

**Lemma 6.1.** The inclusion
$$
((\bigoplus_{\alpha \in \mathcal{A}(G,N)} \bigwedge g_{C}^* \otimes V_{\alpha} \otimes V_{\rho})_{\Phi(C)} \subseteq \bigoplus_{\alpha \in \mathcal{A}(G,N)} \bigwedge g_{C}^* \otimes V_{\alpha} \otimes V_{\rho}
$$
induces a cohomology isomorphism.
Proof. Since the induced $G$-action on the cohomology $H^*(\bigoplus_{\alpha \in A(G,N)} \Lambda g_C^* \otimes V_\alpha \otimes V_\rho)$ is trivial and $\Phi(C)$-action is semi-simple part of $G$-action, the induced $\Phi(C)$-action on the cohomology $H^*(\bigoplus_{\alpha \in A(G,N)} \Lambda g_C^* \otimes V_\alpha \otimes V_\rho)$ is also trivial and hence

$$H^*(\bigoplus_{\alpha \in A(G,N)} \Lambda g_C^* \otimes V_\alpha \otimes V_\rho)^{\Phi(C)} = H^*(\bigoplus_{\alpha \in A(G,N)} \Lambda g_C^* \otimes V_\alpha \otimes V_\rho).$$

Since $\Phi$ is diagonalizable, we have

$$H^*((\bigoplus_{\alpha \in A(G,N)} \Lambda g_C^* \otimes V_\alpha \otimes V_\rho)^{\Phi(C)}) = H^*(\bigoplus_{\alpha \in A(G,N)} \Lambda g_C^* \otimes V_\alpha \otimes V_\rho)^{\Phi(C)}.$$

Hence the lemma follows.

We suppose $G$ has a lattice $\Gamma$. We consider the cochain complex

$$\bigoplus_{\beta \in A(\Gamma, G \cap N)} \Lambda^*(G/\Gamma, E_\beta \otimes E_\rho).$$

Corollary 6.2. Suppose the restriction $\rho|_N$ is unipotent. The inclusion

$$\iota : (\bigoplus_{\alpha \in A(G,N)} \Lambda g_C^* \otimes V_\alpha \otimes V_\rho)^{\Phi(C)} \rightarrow \bigoplus_{\beta \in A(\Gamma, \Gamma \cap N)} \Lambda^*(G/\Gamma, E_\beta \otimes E_\rho)$$

induces a cohomology isomorphism.

We have a basis $X_1, \ldots, X_n$ of $g_C$ such that $(Ad_s)_s = \text{diag}(\alpha_1(c), \ldots, \alpha_n(c))$ for $c \in C$. Let $x_1, \ldots, x_n$ be the basis of $g_C$ which is dual to $X_1, \ldots, X_n$. We have a basis $v_1, \ldots, v_m$ of $V_\rho$ such that $(\rho(c))_s = \text{diag}(\alpha'_1(c), \ldots, \alpha'_m(c))$ for any $c \in C$. Let $v_\alpha$ be a basis of $V_\alpha$ for each character $\alpha \in A(G,N)$. By $G = C \cdot N$, we have $G/N = C/C \cap N$ and hence we have $A(G,N) = A_{C,C \cap N} = \{\alpha \in \text{Hom}(C, C) | \alpha|_{C \cap N} = 1\}$.

For a multi-index $I = \{i_1, \ldots, i_p\}$ we write $x_I = x_{i_1} \wedge \cdots \wedge x_{i_p}$, and $\alpha_I = \alpha_{i_1} \cdots \alpha_{i_p}$. We consider the basis

$$\{ x_I \otimes v_\alpha \otimes v_k \}_{I \subseteq \{1, \ldots, n\}, \alpha \in A_{C,C \cap N}, k \in \{1, \ldots, m\}}$$

of $\bigoplus_{\alpha \in A_{C,C \cap N}} \Lambda g_C^* \otimes V_\alpha \otimes V_\rho$. Since the action

$$\Phi : C \rightarrow \text{Aut}(\bigoplus_{\alpha} \Lambda g_C^* \otimes V_\alpha \otimes V_\rho)$$

is the semi-simple part of $(\bigoplus \text{Ad} \otimes \alpha \otimes \rho)|_C$, we have

$$\Phi(\alpha)(x_I \otimes v_\alpha \otimes v_k) = \alpha_I^{-1} \alpha_{i_k} x_I \otimes v_\alpha \otimes v_k.$$

Hence we have

$$((\bigoplus_{\alpha} \Lambda g_C^* \otimes V_\alpha \otimes V_\rho)^{\Phi(C)})$$

$$= \langle x_I \otimes v_{\alpha_1} \otimes v_k \otimes v_{\alpha_{k-1}} \rangle_{I \subseteq \{1, \ldots, n\}, k \in \{1, \ldots, m\}}$$

$$= \Lambda \langle x_1 \otimes v_{\alpha_1}, \ldots, x_n \otimes v_{\alpha_n} \rangle \otimes \langle v_1 \otimes v_{\alpha_1^{-1}}, \ldots, v_m \otimes v_{\alpha_m^{-1}} \rangle.$$

Remark 1. Let $c$ be the Lie algebra of $C$. Take a subvector $V \subseteq c$ (not necessarily Lie algebra) such that $g = V \oplus n$. Then we define the map

$$\text{ad}_s : g = V \oplus n \triangleright A + X \mapsto (\text{ad}_A)_s \in D(g)$$

where $(\text{ad}_A)_s$ is the semi-simple part of $\text{ad}_A$ and $D(g)$ is the Lie algebras of derivations of $g$. This map is a Lie algebra homomorphism and a diagonalizable representation (see [2] and [3]). Let $\text{Ad}_s : G \rightarrow \text{Aut}(g)$ be the extension of $\text{ad}_s$. Then this map is identified with the map

$$G = C \cdot N \triangleright c \cdot n \mapsto (\text{Ad}_s) \in \text{Aut}(g).$$

Consider the above basis $\{x_1, \ldots, x_n\}$ of $g_C^*$. Then in [3] the author showed that we have an isomorphism

$$\Lambda \langle x_1 \otimes v_{\alpha_1}, \ldots, x_n \otimes v_{\alpha_n} \rangle \cong \Lambda u_G^*.$$
where $u_G$ is the nilpotent Lie algebra defined as

$$u_G = \{X - \text{ad}_s X | X \in \mathfrak{g}\}.$$  
(This fact gives the new developments of de Rham homotopy theory on solvmanifolds. See [3].)

Hence we can regard $\bigwedge \langle x_1 \otimes v_{a_1}, \ldots, x_n \otimes v_{a_n} \rangle \otimes \langle v_1 \otimes v_{a_1^{-1}}, \ldots, v_m \otimes v_{a_m^{-1}} \rangle$ as the cochain complex of nilpotent Lie algebra of $u_G$ with values in some nilpotent representation.

Consider the inclusion

$$\iota : \left( \bigoplus_{\alpha \in A_{G,N}} \mathfrak{g}_\mathbb{C}^\ast \otimes V_{\alpha} \otimes V_{\rho} \right)^{\Phi(C)} \to \bigoplus_{\beta \in A_{G,N}} A^\ast(G/\Gamma, E_\beta \otimes E_{\rho}).$$

$\iota(x_1 \otimes v_{a_1} \otimes v_k \otimes v_{a_k^{-1}}) \in A^\ast(G/\Gamma, E_{\rho})$ if and only if $(\alpha_1 \alpha_1^{-1} \rho)_{|k} = \rho_{|k}$. Let $A^\ast_r$ be the subcomplex of $(\bigoplus_{\alpha} \mathfrak{g}_\mathbb{C}^\ast \otimes V_{\alpha} \otimes V_{\rho})^{\Phi(C)}$ defined as

$$A^\ast_r = (x_1 \otimes v_{a_1} \otimes v_k \otimes v_{a_k^{-1}} | (\alpha_1 \alpha_1^{-1})_{|k} = 1).$$

Then we have $\iota^{-1}(A^\ast(G/\Gamma, E_{\rho})) = A^\ast_r$. Hence we have a finite dimensional cochain complex which can compute the cohomology $H^\ast(G/\Gamma, E_{\rho})$.

**Corollary 6.3.** We have an isomorphism

$$H^\ast(A^\ast_r) \cong H^\ast(G/\Gamma, E_{\rho}).$$

By Remark[4] $A^\ast_r$ is subcomplex of the cochain complex of the nilpotent Lie algebra $u_G$ with values in some nilpotent representation.

## 7. Demonstration

Let $G = \mathbb{C} \ltimes \phi \mathbb{C}^2$ such that $\phi(x + \sqrt{-1}y) = \begin{pmatrix} e^x + \sqrt{-1}y & 0 \\ 0 & e^{-x} - \sqrt{-1}y \end{pmatrix}$. Then for a coordinate $(w, z_1, z_2) \in \mathbb{C} \ltimes \phi \mathbb{C}^2$ we have the basis $\{v_1, \ldots, v_6\}$ of $\mathfrak{g}_\mathbb{C}$ such that

$$v_1 = e^w \frac{\partial}{\partial z_1}, v_2 = e^w \frac{\partial}{\partial z_2}, v_3 = e^{-w} \frac{\partial}{\partial z_2}, v_4 = e^{-w} \frac{\partial}{\partial z_2}, v_5 = \frac{\partial}{\partial w}, v_6 = \frac{\partial}{\partial w}.$$  

Consider the dual basis

$$e^{-w}dz_1, e^{-w}dz_2, e^w dz_2, e^w dz_2, dw, dw.$$  

As we consider $\mathfrak{g}_\mathbb{C}$ as a representation of $\mathfrak{g}$ via $\text{Ad}$, we have the cochain complex $\bigwedge \mathfrak{g}^\ast \otimes \mathfrak{g}_\mathbb{C}$ whose differential is given by

$$dv_1 = dw \otimes v_1, dv_2 = dw \otimes v_2, dv_3 = -dw \otimes v_3, dv_4 = -dw \otimes v_4$$

$$dv_5 = -e^w dz_1 \otimes v_1 + e^w dz_2 \otimes v_3, dv_6 = -e^w dz_1 \otimes v_2 + e^w dz_2 \otimes v_4.$$  

For $(w, 0, 0) \in \mathbb{C}$, we have $(\text{Ad}_{e^w, 0, 0})_s = \text{diag}(e^w, e^w, e^{-w}, e^{-w}, 1, 1)$ for the basis $\{v_1, \ldots, v_6\}$. Consider the cochain complex

$$\left( \bigoplus_{\alpha} \mathfrak{g}^\ast \otimes V_{\alpha} \otimes V_{\rho} \right)^{\Phi(C)}$$  

as Section[5] where $C = \mathbb{C}$. Then we have

$$\left( \bigoplus_{\alpha} \mathfrak{g}^\ast \otimes V_{\alpha} \otimes V_{\rho} \right)^{\Phi(C)}$$

$$= \bigwedge \left( e^{-w}dz_1 \otimes v_{e^w}, e^{-w}dz_2 \otimes v_{e^{-w}}, e^w dz_2 \otimes v_{e^{-w}}, dw, dw \right) \otimes \left( v_1 \otimes v_{e^w}, v_2 \otimes v_{e^{-w}}, v_3 \otimes v_{e^w}, v_4 \otimes v_{e^{-w}}, v_5, v_6 \right).$$

We have $a + \sqrt{-1}b, c + \sqrt{-1}d \in \mathbb{C}$ such that $\mathbb{Z}(a + \sqrt{-1}b) + \mathbb{Z}(c + \sqrt{-1}d)$ is a lattice in $\mathbb{C}$ and $\phi(a + \sqrt{-1}b)$ and $\phi(c + \sqrt{-1}d)$ are conjugate to elements of $SL(4, \mathbb{Z})$ where we regard $SL(2, \mathbb{C}) \subset SL(4, \mathbb{R})$ (see [3]). Hence we have a lattice $\Gamma = (\mathbb{Z}(a + \sqrt{-1}b) + \mathbb{Z}(c + \sqrt{-1}d)) \ltimes \phi \Gamma^\prime$.
such that $\Gamma''$ is a lattice of $\mathbb{C}^2$. For any lattice $\Gamma$ we have $b_1(G/\Gamma) = b_1(\mathfrak{g}) = 2$. But we will see that $\dim H^1(G/\Gamma, V_{Ad})$ varies for a choice of $\Gamma$. If $b, d \in \pi \mathbb{Z}$, then we have
\begin{align*}
A^0_1 &= \langle v_5, v_6 \rangle,
A^1_1 &= \langle e^{-w}dz_1 \otimes v_1, e^{-w}dz_1 \otimes ve_\omega \otimes v_2 \otimes v_{e^{-\omega}},
&\quad e^{-\overline{w}}d\overline{z}_1 \otimes v_{e^\omega} \otimes v_2, \quad e^{-w}dz_2 \otimes v_3, \quad e^w dz_2 \otimes v_{e^{-w}} \otimes v_4 \otimes v_{e^w},
&\quad e^{\overline{w}}d\overline{z}_2 \otimes v_{e^{-\omega}} \otimes v_3 \otimes v_{e^\omega}, \quad e^{w}d\overline{z}_2 \otimes v_4,
&\quad dw \otimes v_5, \quad dw \otimes v_6, \quad d\overline{w} \otimes v_5, \quad d\overline{w} \otimes v_6 \rangle.
\end{align*}
Hence we have $\dim H^1(G/\Gamma, V_{Ad}) = \dim H^1(A^*_1) = 6$.

On the other hand, if $b \not\in \pi \mathbb{Z}$ or $d \not\in \pi \mathbb{Z}$, then we have
\begin{align*}
A^0_1 &= \langle v_5, v_6 \rangle,
A^1_1 &= \langle e^{-w}dz_1 \otimes v_1, e^{-\overline{w}}d\overline{z}_1 \otimes v_2, \quad e^{-w}dz_2 \otimes v_3, \quad e^w dz_2 \otimes v_{e^{-w}} \otimes v_4 \otimes v_{e^w},
&\quad e^{\overline{w}}d\overline{z}_2 \otimes v_{e^{-\omega}} \otimes v_3 \otimes v_{e^\omega}, \quad e^{w}d\overline{z}_2 \otimes v_4,
&\quad dw \otimes v_5, \quad dw \otimes v_6, \quad d\overline{w} \otimes v_5, \quad d\overline{w} \otimes v_6 \rangle.
\end{align*}
Hence we have $\dim H^1(G/\Gamma, V_{Ad}) = \dim H^1(A^*_1) = 2$.

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(H.kasuya) Graduate School of Mathematical Science University of Tokyo Japan
E-mail address: khsc@ms.u-tokyo.ac.jp