Strong anomalous diffusion in two-state process with Lévy walk and Brownian motion

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Strong anomalous diffusion phenomena are often observed in complex physical and biological systems, which are characterized by the nonlinear spectrum of exponents \( qv(q) \) by measuring the absolute \( q \)th moment \( \langle |x|^q \rangle \). This paper investigates the strong anomalous diffusion behavior of a two-state process with the Lévy walk and Brownian motion, which usually serves as an intermittent search process. The sojourn times in the Lévy walk and Brownian phases are taken as power-law distributions with exponents \( \alpha_+ \) and \( \alpha_- \), respectively. Detailed scaling analyses are performed for the coexistence of three kinds of scalings in this system. Different from the pure Lévy walk, the phenomenon of strong anomalous diffusion can be observed for this two-state process even when the distribution exponent of the Lévy walk phase satisfies \( \alpha_+ < 1 \), provided that \( \alpha_- < \alpha_+ \). When \( \alpha_- < 2 \), the probability density function (PDF) in the central part becomes a combination of stretched Lévy distribution and Gaussian distribution due to the long sojourn time in the Brownian phase, whereas the PDF in the tail part (in the ballistic scaling) is still dominated by the infinite density of the Lévy walk.

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I. INTRODUCTION

In the recent decades, it has been widely recognized that anomalous diffusion is a very general phenomenon in the natural world, which is characterized by the nonlinear evolution of mean-squared displacement (MSD) with respect to time, i.e., \( \langle x^2(t) \rangle \propto t^\beta \) with \( \beta \neq 1 \) [1–3]. The common examples are \( \beta < 1 \) for a subdiffusive continuous-time random walk (CTRW) with a divergent first moment of waiting time [4,5] and \( \beta > 1 \) for a Lévy flight with a divergent second moment of jump length [6,7]. The common feature of the two typical anomalous diffusive processes is their single mode of the motions. However, a particle moving in a complex or even seemingly simple structures might present simultaneous modes [8,9], such as the tracing particle under the effect of a flow acting in the phase space of chaotic Hamiltonian systems [10,11]. Such a system is not easy to be analyzed since it exhibits, at least, two modes of the motion. The common tool to analyze it is the spectrum of exponents \( qv(q) \) [9] by measuring the absolute \( q \)th moment \( \langle |x|^q \rangle \) of the displacement of the particles,

\[
\langle |x(t)|^q \rangle \propto t^{\nu(q)}.
\]

(1)

For the motions with single modes, \( \nu(q) \) is a constant being independent of \( q \), such as \( \nu(q) \equiv 1/2 \) for the Brownian motion. Otherwise, one can find a nonlinear function \( \nu(q) \) of \( q \) for the motions with multiple modes; this phenomenon is named strong anomalous diffusion [9].

There have been vast systems exhibiting strong anomalous diffusion, such as the nonlinear dynamical systems [9,12–15], the annealed or quenched Lévy walk [16–20], sand pile models [21–23], the active transport of polymeric particles in living cells [24], and the spreading of cold atoms in optical lattices [25–27]. The mechanisms of the strong anomalous diffusion for the Lévy walk are studied in detail in Refs. [28–30] where the probability density function (PDF) consists of two kinds of distributions—Lévy distribution in the central part and infinite density in the tail part. The infinite density is non-normalizable, the concept of which was thoroughly investigated as mathematical issues [31,32] and has been successfully applied to physics; for the Lévy walk, it aims at characterizing the ballistic scaling \( (x \sim t) \), which is complementary to the Lévy scaling in the central part of the Lévy walk. In contrast, the propagators of subdiffusive CTRW and Lévy flight only have a single mode, being the stretched Gaussian asymptotics and Lévy distribution [3], respectively. Compared with Lévy flight with divergent MSD, the infinite density characterizes the strong correlation between long jump and long rest in the Lévy walk, resulting in a finite MSD. In addition, the infinite density can be used to study the rare fluctuations of occupation time statistics in ergodic CTRW [33] and renewal theory [34]. It is also discussed together with infinite-ergodic theory, for example, the Brownian motion in a logarithmic potential [35] and the Langevin system with multiplicative noise [36,37].

In this paper, we are considering a two-state process alternating between the Lévy walk and the Brownian motion, which serves as an intermittent search process [38–40]. The searcher displays a slow active motion in the Brownian phase, during which the hidden target can be detected. Although in the Lévy walk phase, the searcher aims to relocate into some unvisited region to reduce oversampling. This kind of intermittent search process has wide applications in physical
or the anomalous problems [41–43]. The theoretical analyses of the anomalous and nonergodic behavior of this intermittent search process have been investigated in Ref. [44]. It shows that the ensemble-averaged and time-averaged MSDs are both the combination of two parts representing the Lévy walk phase and the Brownian phase, respectively. The weights of two parts are time dependent and determined by the occupation fraction of two states.

Here, we turn our attention to the strong anomalous diffusion behavior of such a two-state process. Since the case of a pure Lévy walk has been fully studied in Refs. [28–30], the main objective of this paper is to try to discover new phenomena after introducing the Brownian phase in the Lévy walk and to uncover the intrinsic mechanism by clear theoretical analysis. Intuitively, one can expect that, at least, three modes coexist in this two-state process. It is true, and furthermore, the newly appearing mode in the Brownian phase could bring in many interesting phenomena. The pure Lévy walk shows the strong anomalous diffusion phenomenon only in the case of power-law exponent \( \alpha > 1 \). Now, the two-state process could exhibit the strong anomalous diffusion even for \( \alpha < 1 \) if the sojourn time in the Brownian phase is longer than the one of the Lévy walk phase. Although the particle in the Brownian phase could move an arbitrarily long distance, the infinite density which characterizes the ballistic scale of the Lévy walk phase with a finite velocity still plays a leading role compared with the Gaussian distribution resulting from the Brownian phase. In particular, another observation different from the pure Lévy walk is an accumulation effect found at the end of infinite density (\( x = \pm v_0 t \)).

This paper is organized as follows. In Sec. II, we first introduce the two-state process with different power-law exponents (\( \alpha_+ \) and \( \alpha_- \)) of sojourn time in the Lévy walk and Brownian phases, respectively. Then, we derive the corresponding propagator \( p_{\pm}(x, t) \) in two phases in Sec. III. The detailed scaling analyses for the cases of \( 0 < \alpha_- < \alpha_+ < 1 \) and \( 0 < \alpha_- < 1 < \alpha_+ \) are presented in Secs. IV and V, respectively. Then, in Sec. VI, we show how these different scaling regimes are complementary and their consistency in the intermediate region. The ensemble-averaged absolute fractional-order moments of the displacement are given in Sec. VII. A summary of the key results is provided in Sec. VIII. Some mathematical details are collected.

II. MODEL

We consider the process with its motion alternating between two different states—the standard Lévy walk and the Brownian motion. For the standard Lévy walk, the particle moves with constant velocity \( v_0 \) and then changes its direction at a random time. The running times of each unidirectional flight are independent and drawn from the same distribution. Whereas for the Brownian motion, the particle undergoes normal diffusion with diffusivity \( D \). Now, we assume that the sojourn time distributions of the two-state process switching between the Lévy walk and the Brownian phase are \( \psi_+(t) \) and \( \psi_-(t) \), respectively. The subscripts + and − are introduced to represent the Lévy walk and the Brownian phase, respectively.

This process can be explicitly described by means of the velocity process \( v(t) \) which also consists of two states: \( v_+(t) \) for the Lévy walk and \( v_-(t) \) for the Brownian motion. The PDF of \( v_+(t) \) is \( \delta(|v| - v_0)/2 \), whereas \( v_-(t) = \sqrt{2D} \xi(t) \) with \( \xi(t) \) is a Gaussian white noise satisfying \( \langle \xi(t) \rangle = 0 \) and \( \langle \xi(t_1) \xi(t_2) \rangle = \delta(t_1 - t_2) \).

Let the sojourn time distributions in the two states be a power-law form with exponents \( \alpha_{\pm} \), i.e.,

\[
\psi_{\pm}(t) \sim |\Gamma(-\alpha_{\pm})|^{1+\alpha_{\pm}}
\tag{2}
\]

for large \( t \), where \( \alpha_{\pm} \) are scale factors and \( \Gamma(\cdot) \) is the Gamma function. The exponents \( \alpha_{\pm} \in (0, 2) \) in the two states can be the same or different. As usual, we define the Laplace transform

\[
\psi_{\pm}(s) := \int_{0}^{\infty} dt \ e^{-st} \psi_{\pm}(t)
\]

and obtain the asymptotic behavior of \( \psi_{\pm}(s) \) for small \( s \) as [45]

\[
\psi_{\pm}(s) \simeq 1 - \alpha_{\pm} s^{\alpha_{\pm}}, \quad \alpha_{\pm} \in (0, 1),
\psi_{\pm}(s) \simeq 1 - \mu_{\pm} s + \alpha_{\pm} s^{\alpha_{\pm}}, \quad \alpha_{\pm} \in (1, 2).
\tag{3}
\]

For the case of \( \alpha_{\pm} \in (1, 2) \), the mean sojourn time, denoted as \( \mu_{\pm} \), for the two states, is finite. In particular, the term \( s^{\alpha_{\pm}} \) of \( \alpha_{\pm} \in (1, 2) \) is saved to characterize the rare fluctuations of the Lévy walk, i.e., the information in its tail part. The survival probability of finding the sojourn time in state ± exceeding \( t \) is defined as

\[
\Psi_{\pm}(t) = \int_{t}^{\infty} dt' \psi_{\pm}(t')
\]

with its Laplace transform,

\[
\Psi_{\pm}(s) = \frac{1 - \psi_{\pm}(s)}{s}.
\tag{4}
\]

It is well known that the dynamical behaviors of a standard Lévy walk vary significantly for different regimes of power-law exponents, which naturally motivates us to study the properties of this two-state process with different values of \( \alpha_{\pm} \in (0, 2) \).

III. PROPAGATOR OF THE TWO-STATE PROCESS

The propagator \( p(x, t) \) represents the PDF of finding the particle at position \( x \) at time \( t \), supposing that the particles are initialized at the origin. For this two-state process, we use \( p_{\pm}(x, t) \) to denote the joint PDF for finding the particle at position \( x \) and state ± at time \( t \). They are associated with the propagator as \( p(x, t) = p_+(x, t) + p_-(x, t) \). Besides, the notation \( G_{\pm}(x, t) \) denotes the conditional PDF of making a displacement \( x \) for a complete step in state ± within sojourn time \( t \), defined as [44]

\[
G_+(x, t) = \delta(|x| - v_0 t)/2,
\]

\[
G_-(x, t) = \frac{1}{\sqrt{4\pi Dt}} \exp \left\{ - \frac{x^2}{4Dt} \right\},
\tag{5}
\]

respectively. Based on these notations and the method of master equations for CTRWs, the integral equations for \( p_{\pm}(x, t) \) can be built as [44]

\[
\gamma_+(x, t) = \int_{0}^{t} dt' \int_{-\infty}^{\infty} dx' \psi_+(t')G_+(x', t') \gamma_+(x - x', t - t') + p_0^0 \psi_+(t)G_+(x, t),
\tag{6}
\]

\[
\gamma_-(x, t) = \int_{0}^{t} dt' \int_{-\infty}^{\infty} dx' \psi_-(t')G_-(x', t') \gamma_-(x - x', t - t') + p_0^0 \psi_-(t)G_-(x, t),
\tag{7}
\]

\[
\gamma_+(x, t) = \int_{0}^{t} dt' \int_{-\infty}^{\infty} dx' \psi_+(t')G_+(x', t') \gamma_+(x - x', t - t') + p_0^0 \psi_+(t)G_+(x, t),
\tag{8}
\]

\[
\gamma_-(x, t) = \int_{0}^{t} dt' \int_{-\infty}^{\infty} dx' \psi_-(t')G_-(x', t') \gamma_-(x - x', t - t') + p_0^0 \psi_-(t)G_-(x, t),
\tag{9}
\]
and

\[ p_{\pm}(x, t) = \int_0^t dt' \int_{-\infty}^{\infty} dx' \Psi_{\pm}(x', t') G_{\pm}(x', t - t') + p_{\pm}^0 \Psi_{\pm}(t) G_{\pm}(x, t), \]

(7)

where the flux of particles \( \gamma_{\pm}(x, t) \) defines how many particles leave the position \( x \) and change from state \( \mp \) to state \( \pm \) per unit time and we have taken the initial condition as \( p_{\pm}(x, t = 0) = p_{\pm}^0 \delta(x) \) with the constant \( p_{\pm}^0 \) being the initial fraction of two states. By using the techniques of Laplace and Fourier transforms,

\[ p_{\pm}(k, s) = \int_0^t dt \int_{-\infty}^{\infty} dx e^{-st} e^{ikx} p_{\pm}(x, t), \]

(8)

and performing the transforms on Eqs. (6) and (7), there are

\[ \gamma_{\pm}(k, s) = p_{\pm}^0 \Phi_{\mp}(k, s) + \Phi_{\pm}(k, s) \gamma_{\mp}(k, s), \]

(9)

\[ p_{\pm}(k, s) = p_{\pm}^0 \Phi_{\mp}(k, s) + \Phi_{\pm}(k, s) \gamma_{\mp}(k, s), \]

where

\[
\begin{align*}
\Phi_+(k, s) &= [\psi_+(s + ikv_0) + \psi_+(s - ikv_0)]/2, \\
\Phi_-(k, s) &= \psi_-(s + Dk^2), \\
\Phi_+(k, s) &= [\psi_+(s + ikv_0) + \psi_+(s - ikv_0)]/2, \\
\Phi_-(k, s) &= \psi_-(s + Dk^2).
\end{align*}
\]

(10)

Solving Eq. (9) yields

\[ p_{\pm}(k, s) \simeq \frac{\Phi_{\pm}(k, s)}{1 - \Phi_+(k, s) \Phi_-(k, s)}. \]

(11)

Based on Eq. (11), the explicit expression of the propagator \( p(k, s) = p_+(k, s) + p_-(k, s) \) can be obtained. It can be found that the main ingredients of \( p(k, s) \) in Eq. (11) are the sojourn time-distributions \( \psi_{\pm}(t) \). So the further analyses on \( p(k, s) \) will be developed for the specific \( \psi_{\pm}(t) \) with fixed \( \alpha_{\pm} \).

Since \( \alpha_{\pm} \) are both in the range of (0, 2), it can be divided into almost six situations for different values of \( \alpha_{\pm} \): \( 0 < \alpha_- < \alpha_+ < 1, 0 < \alpha_+ < \alpha_- < 1, 0 < \alpha_- = \alpha_+ < 1, 0 < \alpha_+ < 1 < \alpha_- < 2, 0 < \alpha_- < 1 < \alpha_+ < 2, \) and \( 1 < \alpha_+ < \alpha_- < 2 \). The anomalous and nonergodic behaviors for these six situations have been demonstrated in Ref. [44]. The main result therein is that the state with a smaller power exponent will dominate the whole process in a power-law rate as \( t \to \infty \). In contrast to that, the aim of this paper is to investigate the complementary PDFs and the strong anomalous diffusion behavior of this two-state process. Comparing with the thoroughly investigated strong anomalous diffusion behavior of the standard Lévy walk, a larger exponent \( \alpha_- \) in our two-state process makes no difference on the diffusion behavior. Therefore, we only focus on the cases of \( 0 < \alpha_- < \min(\alpha_+, 1) \) in this paper, whereas another four cases will present the same results as the pure Lévy walk.

IV. SCALING ANALYSES FOR \( 0 < \alpha_- < \alpha_+ < 1 \)

In this case, it holds that \( \psi_{\pm}(s) \simeq 1 - \alpha_{\pm} s^{\alpha_{\pm}} \). Substituting it into Eq. (11), we obtain the asymptotic form as

\[ p(k, s) \simeq a_-(s + Dk^2)^{\alpha_- - 1} + \frac{a_+}{2} \left[ (s + ikv_0)^{\alpha_+ - 1} + (s - ikv_0)^{\alpha_+ - 1} \right], \]

(12)

The normalization of the propagator \( p(x, t) \) can be verified by taking \( k = 0 \) in Eq. (12), which yields \( p(0, s) \simeq 1/s \). The direct inverse Fourier-Laplace transform of Eq. (12) is infeasible, which implies extra efforts are needed to deal with Eq. (12). Actually, the information contained in the asymptotic form of \( p(k, s) \) could be extracted through some appropriate scaling analyses. By carefully looking at the denominator in Eq. (12), three kinds of scaling (\( s \sim k, s \sim |k|^2, \) and \( s \sim |k|^\alpha/|k|^\alpha \)) can be observed. We will first consider the scaling \( s \sim k \) since it characterizes the ballistic scaling of the Lévy walk due to its unidirectional flight at each sojourn in this phase.

The outmost distance that particles can arrive at is \( \pm v_0 t \) in the Lévy walk phase, linear with time, which truncates the PDF \( p(x, t) \) at \( \pm v_0 t \). Although the particle in the Brownian phase might go farther than the distance \( v_0 t \), the corresponding distribution decays exponentially when \( x \gg t \) as the propagator \( G_-(x, t) \) shows in Eq. (5). So we omit the contributions of the Brownian phase in the scaling \( s \sim k \). The explicit tail information is described by the ballistic scaling \( s \sim k \) in Eq. (12), corresponding to \( x \sim t \) in the space-time domain. In contrast to \( s \sim k \), the other two kinds of scalings aim at characterizing the central part of the graph of the PDF \( p(x, t) \).

A. Infinite density of rare fluctuations

To consider the scaling \( s \sim k \), we let \( s, k \to 0 \) and \( s/k \) be fixed. Then Eq. (12) can be rewritten as

\[ p(k, s) \simeq \frac{1}{s} + \frac{a_+}{2s} a_{\alpha_- - \alpha_+} \left[ \left( 1 + \frac{ikv_0}{s} \right)^{\alpha_- - 1} + \left( 1 - \frac{ikv_0}{s} \right)^{\alpha_- - 1} \right], \]

(13)

after neglecting the higher-order term \( k^2 \). The two terms in Eq. (13) containing \( s^{\alpha_- - \alpha_+} \) tend to zero since \( \alpha_+ > \alpha_- \). Thus, we further have the asymptotic form as

\[ p(k, s) \simeq \frac{1}{s} + \frac{a_+}{2s} a_{\alpha_- - \alpha_+} \left[ \left( 1 + \frac{ikv_0}{s} \right)^{\alpha_- - 1} + \left( 1 - \frac{ikv_0}{s} \right)^{\alpha_- - 1} \right] \times \left[ \left( 1 + \frac{ikv_0}{s} \right)^{\alpha_+ - 1} + \left( 1 - \frac{ikv_0}{s} \right)^{\alpha_+ - 1} \right] \]

\[ - \left( 1 + \frac{ikv_0}{s} \right)^{\alpha_+} - \left( 1 - \frac{ikv_0}{s} \right)^{\alpha_+} \]

\[ = \frac{1}{s} + \frac{a_+}{2s} \left[ R_+(k, s) + R_-(k, s) \right], \]

(14)
where, for convenience, we use the notation,
\[ R_d(k, s) := s^{α−a−1} \left[ \left( 1 + \frac{i k v_0}{s} \right)^{a_{−1}} - \left( 1 + \frac{i k v_0}{s} \right)^{a_+} \right] \]
\[ = -\frac{i k v_0}{s^{α−1}} (s + i k v_0)^{a_{−1}}. \tag{15} \]

The leading term in Eq. (14) is $1/s$, the inverse Fourier-Laplace transform of which is $δ(x)$. It contributes to a normalized PDF in this scaling whereas the latter terms provide the information on the tail of the PDF $p(x, t)$. We consider the ballistic scaling $s \sim k$ here, which compresses all the information in the central part into the origin and, thus, yields the normalized term $δ(x)$. Since we are focusing on the information in the tail $|x| > 0$, we omit the term $δ(x)$ and pay attention to the inverse of $R_d(±k, s)$ in Eq. (14). With some technical calculations in Appendix A, the inversion of $R_d(±k, s)$ is obtained, and there is
\[ p(x, t) \simeq \frac{a_+}{2a_−−a_+−1} \frac{e^{−a−1}−1}{Γ(α_−+1)Γ(1−α_+)} \int_0^1 \frac{\text{d}z}{v_0} I(z), \tag{16} \]
where
\[ I(z) = 1(0<z<1)^{−a−1}−(1−z)^{−a−1}[α_+ + (α_−−α_+)z]. \tag{17} \]

There is a truncation at $z = 1$ in the expression of $I(z)$, which implies $|x| \leq v_0 t$, consistent with the previous analysis that the particle will not go beyond the distance $±v_0 t$. On the other hand, regarding $z = x/v_0 t$ as a new variable, the integral of the auxiliary function $I(z)$ diverges due to its singularity at the origin $z = 0$, which gives it a name—infinitely dense. Therefore, the infinite density $I(z)$ is not a real physical PDF. Despite this, it reveals the long-time asymptotic behavior of the propagator $p(x, t)$ through the relationship in Eq. (16). When calculating moments, we multiply $|x|^q$ on both sides of Eq. (16) and integrate with respect to $x$. Then, we obtain
\[ \int_{−∞}^{∞} |x|^q p(x, t) dx \propto t^{α−a−q} \int_0^1 z^q I(z) dz. \tag{18} \]

For $q < α_−$, the integral on the right-hand side of Eq. (18) diverges. However, the infinite density $I(z)$ is valid for high-order moments with $q > α_+$, which cures the singularity at $z = 0$. Therefore, the main functions of the infinite density $I(z)$ is to characterize the tail information of PDF $p(x, t)$ and to calculate the high-order moments.

We also observe another interesting phenomenon—an accumulation at $z = 1$ due to $α_− < 1$ in Eq. (17). This accumulation even exists for $D = 0$ (i.e., the Lévy walk interrupted by rest [11,42,46]). Therefore, this accumulation is not contributed by the particles in the Brownian phase, which vanishes when $α_− = 1$. Although, for $α_− < 1$ and big $t$, it can be balanced by the prefactor $t^{α−a−1}$ in Eq. (16). Actually, this phenomenon implies that the PDF of the pure Lévy walk is dropped down by the long sojourn time in the Brownian phase except for the end point at $z = 1$. The end point of the infinite density is not affected since it results from the particles running in its first step for the whole time. Once the particle renews and turns into the second step in the Brownian phase with longer sojourn time, it is less likely for the particle to go back to the Lévy walk phase again.

### B. Dual scaling regimes in the central part

After obtaining the tail information of $p(x, t)$ by introducing an infinite density $I(z)$, we turn our attention to the central part of $p(x, t)$ where the scaling relation $s \ll k$ is valid. This scaling helps to simplify Eq. (12) into
\[ p(k, s) \simeq \frac{a_−(s + D k^2)^{α−1}−1 + \frac{π}{α_−} [i k v_0]^{α−1}−1} {a_−(s + D k^2)^{α−1}−1 + [i k v_0]^{α−1}−1} \]. \tag{19} \]

It can be found that two different scalings coexist in Eq. (19), i.e., $s \sim |k|^{v_0/α_−}$ and $s \sim |k|^2$. This phenomenon is different from the standard Lévy walk where only Lévy scaling is observed at the central part [29]. Now, the Gaussian shape with scaling $s \sim |k|^2$ cannot be omitted due to the longer sojourn time in the Brownian phase.

Therefore, it is necessary to consider the magnitude relation between $α_+$ and $2α_−$ for further analyses. If $α_+ < 2α_−$, the Lévy scaling $s \sim |k|^{−v_0/α_−}$ dominates the PDF $p(k, s)$. In this case, after omitting $D k^2$ and the second term in the numerator of Eq. (19) due to $|k|^{α−1} \ll s^{α−1}$, we obtain
\[ p(k, s) \simeq \frac{a_−s^{α−1}−1 + a_+ \cos(π α_+/2) v_0^{α+} |k|^{α_+}} {s^{α−1}−1 + K_α |k|^{α_+}} \] \tag{20} \]

where the generalized diffusion coefficient is $K_α = a_+ \cos(π α_+/2) v_0^{α+}/α_−$.

When $α_− = 1$, the inverse of $p(k, s)$ is a normalized symmetric Lévy stable PDF, which recovers the central part of the PDF of a standard Lévy walk. For $α_− < 1$, the PDF is like a Lévy flight coupled with an inverse subordinator. The displacement in the Brownian phase can be neglected, and thus, it acts like a trap event with power-law exponent $α_− < 1$. The running time in the Lévy walk phase is far less than the one in the Brownian phase and, thus, can be neglected so that the displacement in this phase acts like a jump obeying power-law distribution with exponent $α_+$. The corresponding Langevin system can be found in Ref. [47].

The PDF $p(x, t)$ in Eq. (20) is a stretched Lévy distribution, the closed form of which can be expressed by the Fox $H$ function,
\[ p(x, t) \simeq \frac{1}{\sqrt{π |x|}} H_{2,1}^{1,2} \left( \frac{|x|^{α_+}} {2K_α}, \frac{z}{α_−}, \frac{1}{α_−}; 1, 1, 1, 1, \frac{α_−}{α_+} \right) \]

Based on the asymptotic form of the Fox $H$ function [48], there is
\[ p(x, t) \simeq \tilde{K}_α \frac{e^{−x/|x|^{1+α_−}}} {\Gamma(1 + α_−)π}. \tag{21} \]

for large $|x|$, where the coefficient,
\[ \tilde{K}_α = \frac{Γ(1 + α_+) \sin(π α_+/2) K_α} {Γ(1 + α_−)π}. \]

On the contrary, if $α_+ > 2α_−$, the dominant part of $p(k, s)$ is in the scaling $s \sim |k|^2$. Similarly, the second terms in the numerator and denominator of Eq. (19) are both higher order than the corresponding first terms. We neglect them and obtain
\[ p(k, s) \simeq \frac{1}{s + D k^2}. \tag{22} \]
displaying the classical behavior of the Brownian motion. The inverse Fourier-Laplace transform of Eq. (22) yields the Gaussian shape,

\[ p(x, t) \sim \frac{1}{\sqrt{4\pi D t}} \exp \left( -\frac{x^2}{4Dt} \right), \]

in the central part of \( p(x, t) \).

Compared with the infinite density \( \mathcal{I}(z) \), the different asymptotic forms of \( p(x, t) \) on the central part within different scaling regimes are both normalized since taking \( k = 0 \) both yield \( p(0, s) \approx 1/s \) in Eqs. (20) and (22). However, the high-order (bigger than \( \alpha_+ \)) moments will diverge if we use the asymptotic forms of \( p(x, t) \) in the central part since the large-\( x \) behavior in Eq. (21) is heavy tailed with exponent \( 1 + \alpha_- \). On the contrary, the high-order moments with Gaussian PDF in Eq. (23) exponentially decay and can be neglected compared with the infinite density \( \mathcal{I}(z) \) on the tail part.

V. SCALING ANALYSES FOR \( 0 < \alpha_- < 1 < \alpha_+ \)

In this case, it holds that \( \psi_+(s) \approx 1 - \mu_+ s + a_+ s^{\alpha_+} \) and \( \psi_-(s) \approx 1 - a_- s^{\alpha_-} \). Substituting them into Eq. (11), we obtain the asymptotic form as \( s, k \to 0 \),

\[ p(k, s) \approx \frac{\mu_+ + a_- (s + Dk^2)^{\alpha_- - 1} - \frac{a_-}{\pi} \left[ (s + ikv_0)^{\alpha_-} + (s - ikv_0)^{\alpha_-} \right]}{\mu_+ + a_+ (s + Dk^2)^{\alpha_+} - \frac{a_+}{\pi} \left[ (s + ikv_0)^{\alpha_+} + (s - ikv_0)^{\alpha_+} \right]}, \]

which is also normalized. It can be found that the slight difference between Eqs. (24) and (12) are the terms containing \( \mu_+ \). Considering \( \alpha_- < 1 \), there is \( \mu_+ \ll (s + Dk^2)^{\alpha_- - 1} \). Therefore, \( \mu_+ \) can be omitted in the denominator and numerator of Eq. (24). Then, the asymptotic form in Eq. (24) is almost the same as the one in Eq. (12) except for the minuses in front of the last terms in the denominator and numerator.

Although the asymptotic forms are similar in Eqs. (12) and (24), the details of scaling analyses are slightly different due to \( \alpha_+ > 1 \). More precisely, let us first focus on the scaling \( s \sim k \). A result similar to Eq. (14) can be obtained as

\[ p(k, s) = \frac{1}{s} \frac{2a_-}{\alpha_-} [R_+(k, s) + R_-(k, s)]. \]

Since \( \alpha_+ > 1 \), we need to split \( R_+(k, s) \) into two parts to perform inverse Fourier-Laplace transforms, that is,

\[ R_+(k, s) = \left[ \frac{-iv_0}{s^{\alpha_+}} + \frac{v_0^2 k^2}{s^{\alpha_+ - 1}} \right] (s + ikv_0)^{\alpha_- - 2}. \]

With similar procedures as in Appendix A, we finally get

\[ p(x, t) \sim t^{\alpha_- - \alpha_+ - 1} g_{cen}(\alpha_- \Gamma(\alpha_- + 1)/\Gamma(1 - \alpha_+)) \mathcal{I}(|x|/v_0 t), \]

where \( \mathcal{I}(z) \) is defined in Eq. (17). To replace \( \Gamma(1 - \alpha_-) \) by \( |\Gamma(1 - \alpha_+)| \) for positivity preserving, one can get Eq. (27) from Eq. (16), which is the only difference. Similarly, for the scaling analyses when \( s \ll k \), Eqs. (20) and (22) are also valid if we replace \( \cos(\pi \alpha_+/2) \) by \( |\cos(\pi \alpha_+/2)| \).

VI. COMPLEMENTARITY AMONG DIFFERENT SCALING REGIMES

For both cases of \( 0 < \alpha_- < \alpha_+ < 1 \) and \( 0 < \alpha_- < 1 < \alpha_+ \), the PDFs \( p(x, t) \) are studied in different scaling regimes. The tail part \( x \sim t^\beta \) can be well approximated by the infinity density \( \mathcal{I}(z) \) as

\[ p(x, t) \sim t^{\alpha_- - \alpha_+ - 1} g_{tail}(x/t), \]

where the scaling function,

\[ g_{tail}(z) = \frac{a_+}{2a_- v_0 \Gamma(\alpha_- + 1) |\Gamma(1 - \alpha_+)|} \mathcal{I}(|z|/v_0 t), \]

On the other hand, the central part of \( p(x, t) \) is well approximated by two densities as

\[ p(x, t) \sim \begin{cases} t^{\alpha_- \alpha_+/a_-} g_{cen}(\alpha_- \Gamma(\alpha_- + 1)/\Gamma(1 - \alpha_+)), & \alpha_- < 2\alpha_-, \\ t^{-1/2} g_{cen}(x/t^{1/2}), & \alpha_- > 2\alpha_-, \end{cases} \]

where the scaling functions,

\[ g_{cen1}(z) = \frac{1}{\sqrt{\pi |z|}} H_{\frac{1}{2},1}^{1,1} \left[ \frac{|z|^{\alpha_-}}{2\pi \Gamma(1, 1, (1, \alpha_-) \frac{1, 1, (1, \alpha_+) \Gamma(1, 1, \frac{\Gamma(1, \alpha_+)}{\Gamma(1 - \alpha_+)}))}\]

and

\[ g_{cen2}(z) = \frac{1}{\sqrt{4\pi D}} \exp \left( -\frac{z^2}{4Dt} \right). \]

The central part of \( p(x, t) \) is with the scaling \( x \sim t^\beta \), where

\[ \beta = \max(\alpha_-/\alpha_+, 1/2) < 1. \]

Therefore, the intermediate region between central part \( t^\beta \) and tail part \( t \) is very large as \( t \to \infty \). For convenience, we simplify Eq. (30) as

\[ p(x, t) \sim t^{-\beta} g_{cen}(x/t^\beta). \]

where \( g_{cen} = g_{cen1} \) when \( \alpha_- < 2\alpha_- \) and \( g_{cen} = g_{cen2} \) when \( \alpha_- > 2\alpha_- \). To verify the results of PDF \( p(x, t) \) in Eqs. (28) and (30), we simulate the PDF with different scalings for several pairs of \( \alpha_\pm \). The simulation results are presented in Fig. 1, showing the agreement with theoretical results very well.

A natural expectation on the analyses in different scales is that the different distributions should be consistent in the intermediate region. The intermediate region is described by \( x \to 0 \) for the tail part and \( x \to \infty \) for the central part. By taking the corresponding limits in Eqs. (28) and (30), respectively, we obtain the same asymptotic form as

\[ p(x, t) \sim t^{\alpha_- - \alpha_+ - 1} g_{tail}(x/t^\beta) \]

\[ \approx t^{-\alpha_-/\alpha_+} g_{cen1}(x/t^{\alpha_-/\alpha_+}) \]

\[ \approx C_0 t^{\alpha_-/\alpha_+} \]

\[ t^{\alpha_-/\alpha_+} \]

\[ C_0 t^{\alpha_-/\alpha_+} \]

\[ \]
for $t^\beta \ll x \ll t$, where the coefficient,

$$c_0 = \frac{a_+ + v_0^{\alpha_+}}{2a_+ - \Gamma(1 + \alpha_-) \Gamma(1 - \alpha_+)}.$$ 

The different scaling regimes are complementary here, and they together depict the whole graph of PDF $p(x, t)$. Note that we only use the first density in the central part which is the power-law decay in Eq. (30) since another one decays exponentially and can be omitted. Apart from the consistence of two distributions in the intermediate region, the previous discussions of different dominant roles in Eq. (30) make sense when calculating low-order moments.

**VII. ENSEMBLE AVERAGES**

Now, we pay attention to the absolute moments of all orders for the displacement. Since the different scaling regimes approximate the different parts of $p(x, t)$, they together yield the entire information on the long-time asymptotics and, thus, determine the moments of displacement. We introduce an auxiliary function $c(t)$ which satisfies

$$t^\beta \ll c(t) \ll t \quad (33)$$

to divide the central part and the tail part. Then, we can split the integral into two parts where different scaling regimes well approximate $p(x, t)$, that is,

$$\langle |x(t)|^q \rangle = \int_{|x| \leq c(t)} |x|^q p(x, t) dx + \int_{|x| > c(t)} |x|^q p(x, t) dx$$

$$= \int_{|x| \leq c(t)} |x|^q t^{-q} g_{cen}(\frac{x}{t^\beta}) dx + \int_{|x| > c(t)} |x|^q t^{\alpha_- - \alpha_+ - 1} g_{tail}(\frac{x}{t^{\alpha_+ + q}}) dx$$

$$= t^\beta g_{cen}(\frac{c(t)}{t^\beta}) + t^{\alpha_- - \alpha_+ + q} \int_{|z| > c(t)/t^\beta} |z|^q g_{tail}(z) dz. \quad (34)$$

Therefore, the central and tail parts have different contributions to the absolute $q$th moments, which are $t^\beta g_{cen}$ and $t^{\alpha_- - \alpha_+ + q}$, respectively, the critical value of which is

$$q_c = \frac{\alpha_+ - \alpha_-}{1 - \beta}, \quad (35)$$

implying the piecewise linear behavior of the spectrum of exponents $q \nu(q)$ in Eq. (1). When $q < q_c$, the former one plays a leading role, otherwise, the latter one dominates. The two integrals in Eq. (34) are both finite by choosing appropriate $c(t)$ for different order $q$. For example, for low-

**FIG. 1.** Scaled PDF of the two-state process. The color symbols represent the simulation results whereas the black solid lines are the theoretical results with parameters $v_0 = 1$, $D = 0.1$, and $\nu = 0.1$. All the simulations agree with the theoretical results very well. In (a) and (d), $\alpha_+ = 1.2$, $\alpha_- = 0.8$ and $\alpha_+ = 1.2$, $\alpha_- = 0.5$ are chosen for the cases of $\alpha_+ < 2\alpha_-$ and $\alpha_+ < 2\alpha_-$ to verify the infinity density $g_{tail}(z)$ in Eq. (29), respectively. In (b) and (e), $\alpha_+ = 1.2$, $\alpha_- = 0.8$ and $\alpha_+ = 0.8$, $\alpha_- = 0.6$ are taken for the case of $\alpha_+ < 2\alpha_-$ to verify $g_{cen}(z)$ in Eq. (30). The solid line describes the asymptotic result of the Fox $H$ function for large $z$, i.e., $g_{cen}(z) \simeq K_\nu|z|^{-\nu}$ in Eq. (31). In (c) and (f), $\alpha_+ = 1.8$, $\alpha_- = 0.7$ and $\alpha_+ = 1.8$, $\alpha_- = 0.5$ for the case of $\alpha_+ > 2\alpha_-$ to verify $g_{cen}(z)$ in Eq. (30).
order moments with \( q < q_\star \), choosing \( c(t) = c_1 t \) with \( c_1 \ll 1 \), the two integrals become
\[
\int_{-\infty}^{\infty} |z|^{q} g_{cen}(z) dz, \quad \int_{|z|>c_1} |z|^{q} g_{tail}(z) dz < \infty, \tag{36}
\]
as \( t \to \infty \). The singular point \( z = 0 \) of the latter integral is excluded by a short-distance \( c_1 \). Whereas for high-order moments with \( q > q_\star \), we choose \( c(t) = c_2 t^2 \) with \( 1 \ll c_2 \). Then, the two integrals are
\[
\int_{|z|<c_1} |z|^{q} g_{cen}(z) dz, \quad \int_{|z|>c_1} |z|^{q} g_{tail}(z) dz < \infty. \tag{37}
\]
The high-order moments for the infinity density \( g_{tail}(z) \) will not diverge.

Considering \( \beta = \max(\alpha_-/\alpha_+, 1/2) \), the absolute \( q \)th moments are given in two different cases. If \( \alpha_- < 2\alpha_+ \),
\[
\langle |x(t)|^q \rangle \simeq \begin{cases} 
M^\star_1 |q^{\alpha_-/\alpha_+}|, & q < \alpha_+; \\
M^\star_{-1} q^{\alpha_+ - \alpha_-}, & q > \alpha_+.
\end{cases} \tag{38}
\]
If \( \alpha_+ \geq 2\alpha_- \),
\[
\langle |x(t)|^q \rangle \simeq \begin{cases} 
M^\star_2 |q^{\beta/2}|, & q < 2(\alpha_- - \alpha_+); \\
M^\star_{-1} q^{\alpha_- - \alpha_+}, & q > 2(\alpha_- - \alpha_+).
\end{cases} \tag{39}
\]
The results in Eqs. (38) and (39) imply that this system exhibits strong anomalous diffusion behavior with a bilinear spectrum of exponents, which has been verified by simulations in Fig. 2. The diffusion coefficients \( M^\star_1 \), \( M^\star_2 \), and \( M^\star_\alpha \) can be obtained from the derivations in Eq. (34) as
\[
M^\star_1 = \int_{-\infty}^{\infty} |z|^{q} g_{cen}(z) dz = (\kappa_0)^{\beta/\alpha_+} \Gamma(1 - q/\alpha_+) \Gamma(1 + q/\alpha_- - \alpha_+), \\
M^\star_2 = \int_{-\infty}^{\infty} |z|^{q} g_{cen2}(z) dz = \frac{(4D)^{q/2} \Gamma(q+1)}{\sqrt{\pi}},
\]
\[
M^\star_\alpha = \int_{-\infty}^{\infty} |z|^{q} g_{tail}(z) dz = \frac{a_+ q \Gamma(q - \alpha_+)}{2 a_- \Gamma(q - \alpha_+ + \alpha_- + 1) \Gamma(1 - \alpha_+ - \alpha_- + 1)}. \tag{40}
\]
The coefficients \( M^\star_1 \) and \( M^\star_2 \) can be directly obtained by using the expressions of \( g_{cen2}(z) \) and \( g_{cen1}(z) \), respectively. Whereas it is not easy to get \( M^\star_\alpha \) from the expression of \( g_{cen1}(z) \), a Fox \( H \) function. So, we resort to the method of subordination and present the details in Appendix B.

VIII. SUMMARY

The intermittent search strategy has been widely applied in the real world. The most powerful and representative one is an alternating process with two states: Lévy walk and Brownian motion. In this paper, we mainly investigate the anomalous diffusion with multiple modes for the two-state process. It is well known that the pure Lévy walk exhibits the strong anomalous diffusion when the power-law exponent of the running time is bigger than one. The intrinsic mechanism is that two kinds of distributions are complementary in the PDF of the Lévy walk, i.e., the Lévy distribution in the central part and the infinite density in the tail part. Here, the two-state process becomes more complicated since three kinds of scales coexist in this system. The usual method to deal with the system with multiple modes is scaling analysis on the PDF. If \( \alpha_- > \alpha_+ \) or \( \alpha_+ > 1 \), the Lévy walk phase will dominate for long times in this system, and thus, the strong anomalous diffusion phenomenon will be the same as the pure Lévy walk. Therefore, we only consider the case of \( \alpha_- < \min(\alpha_+, 1) \) in this paper.

Based on the technique of the master equation, we build the integral equations for this two-state process and, thus, obtain the explicit expression of the PDF in the Fourier-Laplace space \( p(k, s) \) by solving the integral equations. Consistent with the intuitive understanding of this system, three kinds of scaling regimes can be found in the expression of \( p(k, s) \), which are \( s \sim k \) for ballistic scaling in the Lévy walk phase, \( s \sim |k|^{\alpha_-/\alpha_+} \) for Lévy scaling in the Lévy walk phase, and \( s \sim |k|^2 \) for Gaussian scaling in the Brownian phase. By applying the detailed scaling analyses within these regimes, respectively, we obtain the infinite density in the tail part and a combination of stretched Lévy and Gaussian distributions in the central part.

The relationships among the three distributions are abundant. (i) In the central part, the leading role (with respect to moments) of the stretched Lévy distribution, and the Gaussian distribution is determined by the magnitude size of \( \alpha_+ \) and \( 2\alpha_- \). The former distribution dominates when \( \alpha_+ < 2\alpha_- \), otherwise, the latter one dominates. (ii) Whatever the magnitude sizes of \( \alpha_+ \) and \( 2\alpha_- \) are, it is the stretched Lévy distribution rather than the Gaussian distribution, which is consistent with the infinite density in the intermediate region since the Gaussian distribution decays exponentially and can be omitted. (iii) There is a seeming accumulation effect at the end of the infinite density (\( z = 1 \)). The end of the infinite density is contributed by the particles running in its first step for the whole time. Once it renews and turns into the second step in the Brownian phase, it is less likely for the particle...
Finally, considering the relationship between the PDF of the Lévy walk and the PDF of the Lévy flight, we can derive the expressions for the moments of the displacement. The PDF of the Lévy flight is given by

\[ p(x, t) = \int_{0}^{\infty} p_{0}(x, \tau) h(\tau, t) d\tau, \]  

where \( p_{0}(x, \tau) \) is the PDF of displacement of the Lévy flight with Fourier transform \( x \rightarrow k \) being

\[ p_{0}(k, \tau) = e^{-\tau K_{0}(k)^{\alpha_{-}}}, \]  

and \( h(\tau, t) \) is the PDF of the inverse \( \alpha_{-} \)-stable subordinator with Laplace transform \( t \rightarrow s \) being

\[ h(s, \tau) = e^{-\tau s^{\alpha_{-}}} \]  

Equation (20) can be obtained by substituting Eqs. (B2) and (B3) into Eq. (B1). Multiplying \( |x|^{q} \) on both sides of Eq. (B1), we obtain

\[ \langle |x(t)|^{q} \rangle = \int_{0}^{\infty} \langle |x(t)|^{q} \rangle h(\tau, t) d\tau, \]  

where

\[ \langle |x(\tau)|^{q} \rangle = K_{1} \tau^{q/\alpha_{+}}, \]  

is the absolute \( q \)th moment of the Lévy flight. Here [29,52],

\[ K_{1} = (K_{0})^{q/\alpha_{+}} \int_{-\infty}^{\infty} |x|^{q} l_{\alpha_{+},0}(x) dx \]

with \( l_{\alpha_{+},0}(x) \) being the symmetric \( \alpha_{+} \)-stable Lévy noise [53]. Substituting Eq. (B5) into Eq. (B4) and performing Laplace transform \( t \rightarrow s \), we obtain

\[ \mathcal{L} \langle |x(t)|^{q} \rangle = K_{1} \Gamma(1 + q/\alpha_{+}) s^{-1-q\alpha_{-}/\alpha_{+}}, \]  

the inverse Laplace transform of which is

\[ \langle |x(t)|^{q} \rangle = M_{1}^{-\alpha_{-}/\alpha_{+}}, \]  

with

\[ M_{1}^{-\alpha_{-}/\alpha_{+}} = \frac{K_{1} \Gamma(1 + q/\alpha_{+})}{\Gamma(1 + q\alpha_{-}/\alpha_{+})} \]  

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APPENDIX A: INVERSE FOURIER-LAPLACE TRANSFORM OF \( R_{a}(k, s) \) IN EQ. (15)

First, the inverse Laplace transform \( (s \rightarrow t) \) of \( R_{a}(k, s) \) is

\[ R_{a}(k, t) = -i k v_{0} \int_{0}^{t} e^{-ikv_{0}t' \Gamma(1-\alpha_{-})} (t-t')^{\alpha_{+}} dt'. \]  

(A1)

By performing the substitutions \( t' = tx \) and \( v_{0}t k = \xi \), one arrives at

\[ \tilde{R}_{a}(\xi, t) = -i \xi \Gamma(1-\alpha_{+}) \int_{0}^{1} e^{-i\xi x \Gamma(1-\alpha_{+})} (1-x)^{\alpha_{+}} dx. \]  

(A2)

Then, taking inverse Fourier transform \( (\xi \rightarrow z) \) leads to

\[ \tilde{R}_{a}(z, t) = - \Gamma(1-\alpha_{+}) \frac{1}{\Gamma(1-\alpha_{+})} \frac{\partial}{\partial z} \int_{0}^{1} \delta(z-x) x^{\alpha_{+}} (1-x)^{\alpha_{+}} dx \]

\[ = - \Gamma(1-\alpha_{+}) \frac{\partial}{\partial z} \left[ \Gamma(\alpha_{+}) (1-z)^{\alpha_{+}} - 1 \right] \]

\[ = \Gamma(1-\alpha_{+}) \frac{\partial}{\partial z} \left[ I_{0}(\alpha_{+}) (1-z)^{\alpha_{+}} \right], \]  

(A3)

where

\[ I_{0}(\alpha_{+}) = \Gamma(\alpha_{+}) (1-z)^{\alpha_{+}} \]  

(A4)

Finally, considering the relationship \( v_{0} t k = \xi \), the inverse Fourier transform \( (k \rightarrow x) \) of \( R_{a}(k, t) \) is

\[ R_{a}(x, t) = \frac{1}{v_{0} t} \tilde{R}_{a} \left( \frac{x}{v_{0} t}, t \right) \]

\[ = \frac{1}{v_{0} t} e^{-x^{\alpha_{+}} / \Gamma(1-\alpha_{+})} \frac{1}{v_{0} t} \frac{\partial}{\partial x} \left[ I_{0}(\alpha_{+}) x^{\alpha_{+}} \right]. \]  

(A5)

Similarly, the inverse Fourier-Laplace transform \( (k \rightarrow x, s \rightarrow t) \) of \( R_{a}(-k, s) \) is

\[ R_{a}(x, t) = \frac{1}{v_{0} t} e^{-x^{\alpha_{+}} / \Gamma(1-\alpha_{+})} \frac{1}{v_{0} t} \frac{\partial}{\partial x} \left[ I_{0}(\alpha_{+}) x^{\alpha_{+}} \right]. \]  

(A6)

APPENDIX B: DERIVATION OF THE COEFFICIENT \( M_{1}^{-\alpha_{-}/\alpha_{+}} \)

It is not easy to directly obtain \( M_{1}^{-\alpha_{-}/\alpha_{+}} \) in Eq. (40) from the expression of \( g_{\text{cenl}}(z) \) since \( g_{\text{cenl}} \) is a Fox H function. However, we find that the PDF \( p(x, t) \) in Eq. (20) corresponds to the model—Lévy flight coupled with an inverse subordinator, the Langevin picture of which is discussed in Ref. [47]. Based on the method of subordination [49–51], \( p(x, t) \) can be written into an integral form as
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