A NOTE ON THE CONCORDANCE INVARIANTS EPSILON AND UPSILON

JENNIFER HOM

Abstract. Ozsváth-Stipsicz-Szabó [OSS14] recently defined a one-parameter family $\Upsilon_K(t)$ of concordance invariants associated to the knot Floer complex. We compare their invariant to the $\{-1,0,1\}$-valued concordance invariant $\varepsilon(K)$, which is also associated to the knot Floer complex. In particular, we give an example of a knot $K$ with $\Upsilon_K(t) \equiv 0$ but $\varepsilon(K) \neq 0$.

1. Introduction

Beginning with the $\mathbb{Z}$-valued concordance homomorphism $\tau(K)$ [OS03], the knot Floer homology package [OS04, Ras03] has yielded an abundance of concordance invariants. One of the benefits of these invariants, as opposed to classical concordance invariants such as signature, is that they can be non-vanishing on topologically slice knots. For example, we have the following theorem.

Theorem 1 ([Hom13, Theorem 1]). The subgroup of the smooth concordance group given by topologically slice knots contains a direct summand isomorphic to $\mathbb{Z}^\infty$.

The proof of the above theorem relies on the $\{-1,0,1\}$-valued concordance invariant $\varepsilon(K)$ associated to the knot Floer complex [Hom11a, Definition 3.1]. The quotient of the concordance group by the subgroup $\{K \mid \varepsilon(K) = 0\}$ is totally ordered, and properties of the order structure can be used to construct linearly independent concordance homomorphisms.

Ozsváth-Stipsicz-Szabó [OSS14, Theorem 1.20] recently gave a new proof of Theorem 1, using a one-parameter family $\Upsilon_K(t)$ of $\mathbb{R}$-valued concordance homomorphisms also associated to the knot Floer complex. Both $\varepsilon$ and $\Upsilon$ are strictly stronger than $\tau$ in that

$$\varepsilon(K) = 0 \text{ implies } \tau(K) = 0 \quad \text{and} \quad \Upsilon_K(t) \equiv 0 \text{ implies } \tau(K) = 0,$$

but there exist knots $K$ with $\tau(K) = 0$ while $\varepsilon(K) \neq 0$ and $\Upsilon_K(t) \neq 0$. One such example is the knot $T_{3,4} \# -T_{2,7}$, where $T_{p,q}$ denotes the $(p,q)$-torus knot and $-K$ denotes the reverse of the mirror image of $K$.

The knot Floer complex $CFK^\infty(K)$ is a bifiltered chain complex associated to the knot $K$. We call the two filtrations the vertical and horizontal filtrations. The invariants $\varepsilon$ and $\Upsilon$ are both defined using the bifiltration, while the definition of $\tau$ uses only one of the two filtrations. Roughly, $\varepsilon(K)$ is a measure of how the vertical filtration interacts with the horizontal filtration: the so-called vertical homology has rank one, and $\varepsilon$ measures whether this homology class is a boundary, cycle, or neither in the horizontal homology. On the other hand, the idea behind $\Upsilon_K(t)$ is to apply a linear transformation to the bifiltration on the knot Floer complex and then look at the grading of a certain distinguished generator in the homology of the resulting complex.

More generally, both $\varepsilon$ and $\Upsilon$ are invariants of not just knots, but of (suitable) bifiltered chain complexes. In [OSS14, Proposition 9.4], Ozsváth-Stipsicz-Szabó give an example of a complex $C$ with $\varepsilon(C) = 0$ but $\Upsilon_C(t) \neq 0$, although it is currently unknown if the complex $C$ is realized as $CFK^\infty$ of a knot. Conversely, we prove the following.

Theorem 2. There exist knots $K$ with $\Upsilon_K(t) \equiv 0$ but $\varepsilon(K) \neq 0$. 

The author was partially supported by NSF grant DMS-1307879.
The knots used in the above theorem are connected sums of certain (iterated) torus knots.

An interesting question to consider is what obstructions to sliceness can be extracted from $CFK^\infty(K)$ when $\Upsilon_K(t) \equiv 0$ and $\varepsilon(K) = 0$.

Recall that the concordance genus of $K$, $g_c(K)$, is the minimal Seifert genus of any knot $K'$ which is concordant to $K$. The function $\Upsilon_K(t)$ is a piecewise-linear function of $t$ whose slope has finitely many discontinuities [OSS14, Proposition 1.4]. Let $s$ denote the maximum of the finitely many slopes appearing in the graph of $\Upsilon_K(t)$. Ozsváth-Stipsicz-Szabó [OSS14, Theorem 1.13] prove that $s \leq g_c(K)$.

There is also a concordance genus bound $\gamma(K)$, defined using $\varepsilon$ [Hom12].

Corollary 3. There exist knots $K$ for which the concordance genus bound given by $\Upsilon_K(t)$ is zero, but $\gamma(K) \neq 0$.

Acknowledgements. I would like to thank Peter Ozsváth for useful correspondence, and Tye Lidman for helpful comments on an earlier draft.

2. The example

We will let $T_{p,q; s, t}$ denote the $(s, t)$-cable of $T_{p,q}$, where $s$ denotes the longitudinal winding. We assume the reader is familiar with the knot Floer complex; see, for example, [Hom11a, Section 2] and [OSS14, Section 2].

Lemma 2.1. Let $K = T_{1,5}# - T_{2,3;2,5}$. Then $CFK^\infty(K)$ contains a direct summand generated over $\mathbb{F}[U, U^{-1}]$ by $x, y,$ and $z$ with

\[
\begin{align*}
M(x) &= 0 & A(x) &= 2 \\
M(y) &= -3 & A(y) &= 0 \\
M(z) &= -4 & A(z) &= -2
\end{align*}
\]

and differential

\[
\begin{align*}
\partial x &= 0 & \partial y &= U^2 x + z & \partial z &= 0.
\end{align*}
\]

Here, $M$ and $A$ denote the Maslov grading and Alexander filtration, respectively.

Proof. The knot $T_{2,3;2,5}$ is an L-space knot [Hed09, Theorem 1.10]; see also [Hom11b]. The Alexander polynomial of $T_{2,3;2,5}$ is

\[
\Delta_{T_{2,3;2,5}}(t) = \Delta_{T_{2,3}}(t^2) \cdot \Delta_{T_{2,5}}.
\]

Then by [OS05] (as restated in [OSS14, Theorem 2.10]), the complex $CFK^\infty(T_{2,3;2,5})$ is generated over $\mathbb{F}[U, U^{-1}]$ by $a, b, c, d,$ and $e$ with

\[
\begin{align*}
M(a) &= 0 & A(a) &= 4 \\
M(b) &= -1 & A(b) &= 3 \\
M(c) &= -2 & A(c) &= 0 \\
M(b) &= -7 & A(b) &= -3 \\
M(c) &= -8 & A(c) &= -4
\end{align*}
\]

and differential

\[
\begin{align*}
\partial a &= \partial c &= \partial e = 0 & \partial b &= Ua + c & \partial d &= U^3 c + e.
\end{align*}
\]
In the language of [HHN13, Section 2.4], we have that $C\text{FK}^\infty(T_{2,3,2,5})$ can be denoted $[1,3]$, and the summand $C$ specified in the statement of Lemma 2.1 can be denoted $[2]$. This notation refers to the lengths of the horizontal and vertical arrows in a graphical depiction of $C\text{FK}^\infty$, beginning from the generator of vertical homology and continuing to the point of symmetry. See Figures 1(a) and 1(b). It then follows from [HHN13, Lemma 3.1] that we have that $C\text{FK}^\infty(T_{2,3,2,5}) \otimes C$ is of the form $[1,3,2]$. See Figure 1(c).

The Alexander polynomial of $T_{4,5}$ is
$$\Delta_{T_{4,5}}(t) = t^6 - t^5 + t^2 - 1 + t^{-2} - t^{-5} + t^6.$$ Since $T_{4,5}$ admits a lens space surgery, it an L-space knot. Thus, we may apply [OSS14, Theorem 2.10] to obtain a description of $C\text{FK}^\infty(T_{4,5})$, and we see that, in the notation of [HHN13, Section 2.4], this complex is of the form $[1,3,2]$. See Figure 1(c).

**Figure 1.** Left, $C\text{FK}^\infty(T_{2,3,2,5})$. Center, the relevant summand of $C\text{FK}^\infty(T_{4,5}\#-T_{2,3,2,5})$ from the statement of Lemma 2.1. Right, $C\text{FK}^\infty(T_{4,5})$. More precisely, $C\text{FK}^\infty$ is generated over $\mathbb{F}[U,U^{-1}]$ by the generators depicted.

It follows from [HHN13, Section 2.4] that since $C\text{FK}^\infty(T_{2,3,2,5}) \otimes C$ has the same form as $C\text{FK}^\infty(T_{4,5})$, the complex $C$ is a direct summand of $C\text{FK}^\infty(T_{4,5}) \otimes C\text{FK}^\infty(T_{2,3,2,5})^*$, or, equivalently, $C\text{FK}^\infty(T_{4,5}\#-T_{2,3,2,5})$. □

**Lemma 2.2.** Let $K = T_{4,5}\#-T_{2,3,2,5}$. Then
$$\Upsilon_K(t) = \begin{cases} -2t & \text{if } 0 \leq t \leq 1 \\ 2t - 4 & \text{if } 1 < t \leq 2 \end{cases}.$$ 

**Proof.** The summand of $C\text{FK}^\infty(K)$ described in Lemma 2.1 generates the homology of the total complex $C\text{FK}^\infty(K)$. In particular, this summand determines $\Upsilon_K(t)$. Although this summand is not itself $C\text{FK}^\infty$ of an L-space knot [HW14, Corollary 9], the calculation in [OSS14, Proof of Theorem 6.2] still applies, yielding the desired result. □

**Lemma 2.3.** For the $(2,5)$-torus knot, we have
$$\Upsilon_{T_{2,5}}(t) = \begin{cases} -2t & \text{if } 0 \leq t \leq 1 \\ 2t - 4 & \text{if } 1 < t \leq 2 \end{cases}.$$
Proof. The result follows immediately from [OSS14, Theorem 1.15]. \qed

With these lemmas in place, we are now ready to prove Theorem 2.

Proof of Theorem 2. By [OSS14, Propositions 1.8 and 1.9],
$$\Upsilon_{K_1 \# K_2}(t) = \Upsilon_{K_1}(t) + \Upsilon_{K_2}(t) \quad \text{and} \quad \Upsilon_{-K}(t) = -\Upsilon_K(t).$$

Combined with Lemmas 2.2 and 2.3, it follows that
$$\Upsilon_{T_{2,5}\# T_{4,5}\# T_{2,3,2,5}}(t) \equiv 0.$$

We consider the invariant $a_1(K)$ defined in [Hom11a, Section 6]. For complexes such as the ones in Figure 1, the invariant $a_1(K)$ is equal to the length of the horizontal arrow coming in to the generator of vertical homology. From the partial description of $\text{CFK}^\infty(T_{4,5}\# - T_{2,3,2,5})$ in Lemma 2.1, it follows that
$$a_1(T_{4,5}\# - T_{2,3,2,5}) = 2.$$

By [Hom11a, Lemma 6.5] we have that
$$a_1(T_{2,5}) = 1.$$

Lastly, by [Hom11a, Lemma 6.3] we have that if $a_1(J) > a_1(K)$, then $\epsilon(K\# - J) = 1$. Thus
$$\epsilon(T_{2,5}\# - T_{4,5}\# T_{2,3,2,5}) = 1,$$
as desired.

Recall from [Hom14, Proposition 3.6] that for $n > 0$, we have
$$\epsilon(nK) = \epsilon(K) \quad \text{and} \quad \epsilon(-K) = -\epsilon(K),$$

It follows that any non-zero multiple $nK$ of the knot $K = T_{2,5}\# - T_{4,5}\# T_{2,3,2,5}$ will also have the property that $\Upsilon_{nK}(t) \equiv 0$ and $\epsilon(nK) \neq 0$. \qed

Proof of Corollary 3. The invariant $\gamma(K)$ vanishes if and only if $\epsilon(K) = 0$. Hence $K = T_{2,5}\# - T_{4,5}\# T_{2,3,2,5}$ (or any non-zero multiple thereof) has the desired property. \qed

Remark 2.4. Let $K = T_{2,5}\# - T_{4,5}\# T_{2,3,2,5}$. By computing $\text{CFK}^\infty(K)$ using the Künneth formula [OS04, Theorem 7.1], one can determine that $\gamma(K) = 4$. More generally, we expect that $\gamma(nK) = 4n$, giving knots for which the concordance genus bound obtained from $\Upsilon_K(t)$ is zero, but the bound obtained from $\gamma$ is arbitrarily large.

References

[Hed09] Matthew Hedden, On knot Floer homology and cabling II, Int. Math. Res. Not. IMRN (2009), no. 12, 2248–2274.

[HHN13] Stephen Hancock, Jennifer Hom, and Michael Newman, On the knot Floer filtration of the concordance group, J. Knot Theory Ramifications 22 (2013), no. 14.

[Hom11a] Jennifer Hom, The knot Floer complex and the smooth concordance group, preprint (2011), to appear in Comment. Math. Helv., available at arXiv:1111.6635v1.

[Hom11b] , A note on cabling and L-space surgeries, Algebr. Geom. Topol. 11 (2011), no. 1, 219–223.

[Hom12] , On the concordance genus of topologically slice knots, preprint (2012), to appear in Int. Math. Res. Not. IMRN, available at arXiv:1203.4594v1.

[Hom13] , An infinite rank summand of topologically slice knots, preprint (2013), to appear in Geom. Topol., available at arXiv:1310.4476v1.

[Hom14] , Bordered Heegaard Floer homology and the tau-invariant of cable knots, J. Topol. 7 (2014), no. 2, 287–326.

[HW14] Matthew Hedden and Liam Watson, On the geography and botany of knot Floer homology, preprint (2014), arXiv:1404.6915v2.
A NOTE ON THE CONCORDANCE INVARIANTS EPSILON AND UPSILON

[OS03] Peter Ozsváth and Zoltán Szabó, Knot Floer homology and the four-ball genus, Geom. Topol. 7 (2003), 615–639.

[OS04] ———, Holomorphic disks and knot invariants, Adv. Math. 186 (2004), no. 1, 58–116.

[OS05] ———, On knot Floer homology and lens space surgeries, Topology 44 (2005), no. 6, 1281–1300.

[OSS14] Peter Ozsváth, András Stipsicz, and Zoltán Szabó, Concordance homomorphisms from knot Floer homology, preprint (2014), arXiv:1407.1795.

[Ras03] Jacob Rasmussen, Floer homology and knot complements, Ph.D. thesis, Harvard University, 2003.

Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027

E-mail address: hom@math.columbia.edu