The Path Integral Quantization corresponding to the Deformed Heisenberg Algebra

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Abstract

In this paper, we analyze a deformation of the Heisenberg algebra consistent with both the generalized uncertainty principle and doubly special relativity. We observe that this algebra can give rise to fractional derivatives terms in the corresponding quantum mechanical Hamiltonian. However, a formal meaning can be given to such fractional derivative terms, using the theory of harmonic extensions of functions. Thus we obtain the expression of the propagator of path integral corresponding to this deformed Heisenberg algebra. In fact, we explicitly evaluate this expression for a free particle in one dimension and check its consistency.

1 Introduction

According to the usual Heisenberg uncertainty principle, a particle position can be measured to an arbitrary accuracy if the momentum is kept unrestricted. Thus, the concept of position measurement within a minimum length scale, beyond which one can not measure the position accurately, is not present within the usual Heisenberg uncertainty principle. In other words, the concept of minimum measurable length is inconsistent with Heisenberg uncertainty principle. However, almost every approaches of quantum gravity has an universal predication of the existence of a minimum measurable length scale. This is because, in most of these approaches to quantum gravity the continuum picture of spacetime breaks down at Planck scale, and this in turn implies the restriction that the Planck length acts like a minimum measurable length scale for almost every approaches to quantum gravity. Indeed, there are strong indications from the physics of black holes, that a minimum length of the order of the Planck length arises as an universal feature of all models of quantum gravity [1,2]. In fact, the energy needed to probe spacetime below Planck length scale, will give rise to a mini black hole in that region of spacetime. Hence, it is not possible to probe spacetime beyond Planck length scale. This concept is consistent with string theory because even in perturbative string theory it is not possible to probe spacetime below string length scale. Thus, the idea of minimum length scale is also present in string theory [3–7]. Moreover, a minimum length scale scenario also occurs...
in loop quantum gravity, with important phenomenological consequence. In fact, in loop quantum gravity, the existence of the minimum length scale turns big bang into a big bounce [8].

The existence of a minimum measurable length scale modifies the usual Heisenberg uncertainty principle into a Generalized Uncertainty Principle (GUP) [3–7, 9–13]. However, this GUP deforms the usual Heisenberg algebra in such a way that the quantum commutator of position and momentum operators become momentum dependent. The deformed Heisenberg algebra modifies the coordinate representation of the momentum operator, and this in turn produces correction terms in all quantum mechanical systems. A different kind of deformation of the Heisenberg algebra is motivated by the study of a theory where both the velocity of light and Planck energy are universal constants. This theory is called Doubly Special Relativity (DSR) [14–16]. The deformation of the Heisenberg algebra studied in DSR theory has been predicted from many consequences, such as discrete spacetime [17], spontaneous symmetry breaking of Lorentz invariance in string field theory [18], spacetime foam models [19], spin-network in loop quantum gravity [20], non-commutative geometry [21], and Horava-Lifshitz gravity [22]. Latter on the extension of DSR to double general relativity (DGR), has also been accomplished and the resultant theory is called gravity’s rainbow [23,24]. Interestingly, it is possible to combine both these deformations into a single deformation of the Heisenberg algebra [25,26]. However, the resultant algebra gives rise to non-local fractional derivative terms in all the quantum mechanical Hamiltonian except in one dimensional case. In one dimension, this deformation of the Heisenberg algebra does not give rise to such terms, and so, various interesting one dimensional quantum mechanical systems have been studied using this deformed Heisenberg algebra. For example, the effect of this deformation on the transition rate of ultra cold neutrons in gravitational field has been studied [27]. Besides, this deformation has been investigated in cosmology to yield a big bounce picture [28]. Furthermore, the Lamb shift and Landau levels have also been analyzed using this deformation of the Heisenberg algebra [29]. One of the most interesting result derived from this algebra is that the space actually has to be a discrete lattice, if both the DSR and GUP are undertaken [25]. Similar results were obtained by repeating this analysis using the relativistic wave equation [26]. Thermodynamical properties of simple quantum mechanical systems have also been analysed using this deformed Hamiltonian [30]. It was observed that the correction to these thermodynamic properties caused by this deformation of the Hamiltonian has the same form as the correction caused by the polymer quantization. This suggests that this deformation of the Hamiltonian might be related to the polymer quantization. In a series of papers, various applications of the new model of GUP were investigated [31–39]. For a recent detailed review along the mentioned lines of minimal length theories and quantum gravity phenomenology can be found in [40–42].

All most all the consequences of both the deformed Heisenberg algebra, one corresponding to GUP and another corresponding to combination of GUP with DSR, have been studied using canonical quantization. However, the link to classical mechanics in path integral formalism is very intuitive. This is because in path integral formalism, the quantum theory is constructed by adding quantum corrections to the classical results. The path integral formalism can also be used for analysing non-perturbative phenomena like instantons in quantum mechanics. The generalization of such non-perturbative phenomena to field theory has led to the discovery of some very important results in quantum gravity [43,44]. Furthermore, sometimes in certain systems, like superconductors [47] and superfields [48], the degrees of freedom of these systems lock to form a single collective variables. Path integral is ideally suited to deal with such situations [49,50]. So, it is desirable to analyse the deformed Heisenberg algebra using the path integral formalism. In fact, recently deformed Heisenberg algebra corresponding to GUP has been analyzed using the path integral formalism [51]. As there are theoretical reasons suggesting the existence of both GUP and DSR, it is important to extend this work to the deformed Heisenberg algebra which is consistent with both GUP and
DSR. In order to do that, we first give a formal meaning to the fractional derivative terms that occur in this Hamiltonian. Then we explicitly derive the expression of one dimensional free particle propagator of path integral. After that the consistency property of this one dimensional free particle propagator will be analyzed.

2 Deformed Heisenberg algebra

In this section, we analyze the non-local Heisenberg algebra using harmonic extension of functions. We start with the deformed Heisenberg algebra corresponding to the GUP that can be written as [1-7]

$$[\tilde{x}^i, \tilde{p}_j] = i\hbar[\delta^i_j(1 + \beta\tilde{p}^2) + 2\beta\tilde{p}_i\tilde{p}_j],$$

(1)

where $\beta$ is a parameter of deformation. On the other hand, the DSR produces another deformed Heisenberg algebra which is linear in $p$,

$$[\tilde{x}^i, \tilde{p}_j] = i\hbar[(1 - \alpha\sqrt{p^i\tilde{p}_k})\delta^i_j + \alpha^2\tilde{p}^i\tilde{p}_j].$$

(2)

It is possible to unify both the above two deformed Heisenberg algebra into a single deformed algebra [26],

$$[\tilde{x}^i, \tilde{p}_j] = i\hbar\left[\delta^i_j - \alpha\sqrt{p^i\tilde{p}_k}\delta^i_j + \alpha(\sqrt{p^k\tilde{p}_k})^{-1}\tilde{p}^i\tilde{p}_j + \alpha^2(p^k\tilde{p}_k)\delta^i_j + 3\alpha^2\tilde{p}^i\tilde{p}_j\right],$$

(3)

where $\alpha$ is the deformation parameter defined by $\alpha = \alpha_0/M_{Pl}c = \alpha_0\ell_{Pl}/\hbar$, $M_{Pl}$ is the Planck mass, $\ell_{Pl} \approx 10^{-35}$ m is the Planck length, $M_{Pl}c^2 \approx 10^{19}$ GeV is the Planck energy. The upper bounds on the parameter $\alpha_0$ has been calculated in [29] and it was proposed that GUP may introduce an intermediate length scale between Planck scale and electroweak scale. Recent proposals suggested that these bounds can be measured using quantum optics techniques in [52] and using gravitational wave techniques [53] which may be considered as a milestone in quantum gravity phenomenology.

Note that the deformed uncertainty relation corresponding to the above combined algebra can be written in one dimension as $\Delta \tilde{x}\Delta \tilde{p} \geq \hbar/2[1 - 2\alpha(\tilde{p}) + 4\alpha^2(\tilde{p}^2)]$, which in turn implies the existence of a minimum length $\Delta \tilde{x}_{\min} \approx \alpha_0\ell_{Pl}$ and a maximum momentum $\Delta \tilde{p}_{\max} \approx \alpha_0^{-1}M_{Pl}c$. The deformed commutation relation (4) modifies the coordinates representation of the momentum operator to $\tilde{p}_i = p_i(1 - \alpha\sqrt{\tilde{p}^i\tilde{p}_k} + 2\alpha^2\tilde{p}^i\tilde{p}_k)$, where $\tilde{x}_i = x_i$, and $[x^i, p_i] = i\hbar\delta^i_j$. Here $p_i$ can be interpreted as the momentum at low energies, and it is possible to use the standard coordinate representation of this momentum as $p_i = -i\hbar\partial_i$. Thus the coordinate representation for $\tilde{p}_i$ can be written as

$$\tilde{p}_i = -i\hbar\left(1 - \alpha\hbar\sqrt{-\partial^i\partial_j} - 2\alpha^2\hbar^2\partial^i\partial_j\right)\partial_i.$$  

(4)

This deformation of the momentum operator deforms the original Hamiltonian

$$H = \frac{1}{2m}\tilde{p}_i\tilde{p}_i + V(\tilde{x}),$$

(5)

to $H = H_o + H_d$, where the correction Hamiltonian $H_d$ is given by

$$H_d = -\frac{\alpha}{m}p_i\partial_i\sqrt{p^j\tilde{p}_j} + \frac{5\alpha^2}{2m}(p^i\tilde{p}_i)(p^j\tilde{p}_j).$$

(6)

So, we can write the deformed Schrödinger equation as

$$-\frac{\hbar^2}{2m}\partial^i\partial_i\psi + \frac{\alpha\hbar^3}{m}\partial^i\partial_i\sqrt{-\partial^j\partial_j}\psi + \frac{5\alpha^2\hbar^4}{2m}\partial^i\partial_i\partial^j\partial_j\psi + V(x)\psi = i\hbar\partial_t\psi.$$  

(7)
It may be noted that the fractional derivative term which occur in this deformed Schrödinger equation can be given a formal meaning in the framework of harmonic extensions of a function $[54–58]$. We start by defining a harmonic function $u : R^n \times (0, \infty) \rightarrow R$, such that its restriction to $R^n$ coincides with the wave function of the deformed Schrödinger equation, $\psi : R^n \rightarrow R$. It is possible to find $u$ by solving the Dirichlet problem defined by $u(x,0) = \psi(x)$ and $\partial^2_{n+1}u(x,y) = 0$, for a given wave function $\psi$. Here $\partial^2_{n+1}$ is defined to be the Laplacian in $R^{n+1}$, such that $x \in R^n$ and $y \in R$. So, there is a unique harmonic extension $u \in C^{\infty}(R^n \times (0, \infty))$ for a smooth function $C_0^\infty(R^n)$. We can now give a formal meaning to $\sqrt{-\partial^j \partial_j}$ by analyzing its action of the wave function, $\psi : R^n \rightarrow R$, such that the harmonic extension of the wave function $u : R^n \times (0, \infty) \rightarrow R$ satisfies,

$$
\sqrt{-\partial^j \partial_j} \psi(x) = -\frac{\partial u(x,y)}{\partial y} \bigg|_{y=0} .
$$

(8)

It may be noted that $u_y(x,y)$ is the harmonic extension of $\sqrt{-\partial^j \partial_j} \psi(x)$ to $R^n \times (0, \infty)$, this is because $u : R^n \times (0, \infty) \rightarrow R$ is a harmonic extension of the wave function $\psi : R^n \rightarrow R$. The successive application of $\sqrt{-\partial^j \partial_j}$ on the wave function is given by

$$
\left[ \sqrt{-\partial^j \partial_j} \right]^2 \psi(x) = \frac{\partial^2 u(x,y)}{\partial y^2} \bigg|_{y=0} = -\partial^2 u(x,y) \bigg|_{y=0} .
$$

(9)

This equation can be used to give a formal meaning to $\sqrt{-(\partial^k \partial_k)}$. Now as $u(x,y)$ is the harmonic extension of the wave function $\psi(x)$, and $\sqrt{-\partial^j \partial_j} \psi = -u_y(x,y)$, therefore the harmonic extension of $\partial_1 \psi(x)$ can be written as $\partial_1 u(x,y)$. Now if $u \in C^2(R \times (0, \infty))$, we also have

$$
\sqrt{-\partial^j \partial_j} \partial_1 \psi(x) = -\partial_1 u_y(x,y) \bigg|_{y=0} .
$$

(10)

Thus, this non-local operator $\sqrt{-\partial^j \partial_j}$ commutes with a derivative,

$$
\sqrt{-\partial^j \partial_j} \partial_1 \psi(x) = \partial_1 \sqrt{-\partial^j \partial_j} \psi(x).
$$

(11)

As an example, we can analyze the action of $\sqrt{-\partial^j \partial_j}$ on $\exp(ikx)$, using the framework of harmonic extensions of a function. First, let we consider $\psi(x) = \cos kx$. Its bounded harmonic extension can be written as $u(x,y) = \exp -|k|y \cos kx$, where $y \in (0, \infty)$. Therefore, we can write

$$
\frac{\partial^2 u(x,y)}{\partial^2 x} + \frac{\partial^2 u(x,y)}{\partial^2 y} = 0
$$

(12)

Thus, the action of $\sqrt{-\partial^j \partial_j}$ on the wave function $\psi$ can be evaluated as

$$
\sqrt{-\partial^j \partial_j} \cos kx = -u_y(x,y) \bigg|_{y=0} .
$$

(13)

So, we obtain the following result $\sqrt{-\partial^j \partial_j} \cos kx = |k| \cos kx$. A similar result can be obtained by considering, $\psi = \sin(kx)$. This is because its bounded harmonic on $R^n \times (0, \infty)$ can be written as
u(x, y) = \exp -|k|y \sin kx, and so, we obtain \( \sqrt{-\partial^i \partial_j} \sin kx = |k| \sin (kx) \). Finally, the action of \( \sqrt{-\partial^i \partial_j} \) on \( \exp i kx \) can be written as

\[
\sqrt{-\partial^i \partial_j} \exp i kx = \sqrt{-\partial^i \partial_j}(\cos kx + i \sin kx).
\] (14)

Thus, we obtain the desired result, \( \sqrt{-\partial^i \partial_j} \exp i kx = |k| \exp i kx \).

Now we define \( u_1(x, y) \) as the harmonic extensions of \( \psi_1(x) \) to \( C = R^n \times (0, \infty) \), and \( u_2(x, y) \) as the harmonic extensions of \( \psi_2(x) \) to \( C = R^n \times (0, \infty) \). We also assume that both these harmonic extensions vanish for \( |x| \to \infty \) and \( |y| \to \infty \). It is now possible to write [55],

\[
\int_C u_1(x, y) \partial_{n+1} u_2(x, y) dx dy - \int_C u_2(x, y) \partial_{n+1} u_1(x, y) dx dy = 0.
\] (15)

So, we obtain the following expression

\[
\int_{R^n} (u_1(x, y) \partial_y u_2(x, y) - u_2(x, y) \partial_y u_1(x, y)) |_{y=0} dx = 0.
\] (16)

This expression can be written in terms of the functions \( \psi_1(x) \) and \( \psi_2(x) \) as,

\[
\int_{R^n} (\psi_1(x) \partial_y \psi_2(x) - \psi_2(x) \partial_y \psi_1(x)) dx = 0.
\] (17)

Thus, the non-local differential operator \( \sqrt{-\partial^i \partial_j} \) can be moved from \( \psi_2(x) \) to \( \psi_1(x) \),

\[
\int_{R^n} \psi_1(x) \sqrt{-\partial^i \partial_j} \psi_2(x) dx = \int_{R^n} \psi_2(x) \sqrt{-\partial^i \partial_j} \psi_1(x) dx.
\] (18)

Now we have analyzed a general deformation of the Heisenberg algebra which was consistent with both GUP and DSR. We have shown that even though this algebra could give rise to fractional derivative terms in the Hamiltonian, it was possible to formally deal with these derivatives using harmonic extension of functions. We will now derive an expression for the quantum mechanical propagator corresponding to this deformed Heisenberg algebra. First note that in the deformed Schrödinger equation the modified Hamiltonian does not have explicit time dependence,

\[
H \psi = i\hbar \frac{\partial}{\partial t} \psi,
\] (19)

where

\[
H = \frac{1}{2m} \hat{p}^2 \hat{p}_i - \frac{\alpha}{m} \hat{p}^2 \hat{p}_i \sqrt{\hat{p}^2 \hat{p}_j} + \frac{5\alpha^2}{2m} \hat{p}^2 \hat{p}_i \hat{p}^2 \hat{p}_j + V(x).
\] (20)

So, if we know the wave function \( \psi(x, t') \), then we can explicitly write the wave function \( \psi(x, t'') \), using the propagation relation

\[
\psi(x, t'') = \exp [-iH(t'' - t') / \hbar] \psi(x, t').
\] (21)

Now for small time interval \( \Delta t = t'' - t' \), we can write

\[
\langle x | \exp (-iH \Delta t / \hbar) | x \rangle = \int dp |x e^{-\frac{i}{\hbar} \left( \frac{1}{2m} \hat{p}^2 \hat{p}_i - \frac{\alpha}{m} \hat{p}^2 \hat{p}_i \sqrt{\hat{p}^2 \hat{p}_j} + \frac{5\alpha^2}{2m} \hat{p}^2 \hat{p}_i \hat{p}^2 \hat{p}_j + V(x) \right) \Delta t} | p \rangle | p \exp (-\frac{i}{\hbar} V(x) \Delta t | x \rangle
\]

\[
= \int \frac{dp}{2\pi \hbar} \left[ e^{-\frac{i}{\hbar} \left( \frac{1}{2m} \hat{p}^2 \hat{p}_i - \frac{\alpha}{m} \hat{p}^2 \hat{p}_i \sqrt{\hat{p}^2 \hat{p}_j} + \frac{5\alpha^2}{2m} \hat{p}^2 \hat{p}_i \hat{p}^2 \hat{p}_j + V(x) \right) \Delta t} \right].
\] (22)
So, we can now write the corresponding quantum mechanical propagator for small time interval $\Delta t = t'' - t'$, for this non-local Hamiltonian as

$$K(x'', t''; x', t') = \int e^{i\int_{t'}^{t''} \mathcal{L} dt} \frac{dp}{2\pi \hbar},$$

where

$$\mathcal{L} = p^i \dot{x}_i - \frac{\dot{p}_i}{2m} + \frac{\alpha}{m} p^i p_j \sqrt{p^j p_j} - \frac{5\alpha^4}{2m} p^i p_j p^j - V(x).$$

### 3 Explicit form of propagator

In the previous section, we analysed the propagator for the deformation of the Heisenberg algebra which was consistent with both GUP and DSR. However, even though we have given a formal meaning to this quantum mechanical propagator, it is computationally very difficult to calculate the full three dimensional form of this propagator. So, in this section, we explicitly calculate the one dimensional free particle propagator corresponding to this deformed Hamiltonian. The above form of propagator can be expanded up to the second order of $\alpha$ as,

$$K(x'', t''; x', t') = \int \frac{dp}{2\pi \hbar} e^{-i\Delta t \left(p \cdot \left(\frac{x'' - x'}{\Delta t}\right)^2 + \frac{i m (x'' - x')^3}{\Delta t}\right)} \times \left(1 + \frac{i \alpha \Delta t}{\hbar} p^3 - \frac{5 i \alpha^2 \Delta t}{2 \hbar} p^4 - \frac{\alpha^2 \Delta t^2}{m^2 \hbar^2} p^6\right).$$

Integrating we finally get the propagator for small time interval $\Delta t$ as,

$$K(x'', t''; x', t') = \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \left[1 + \alpha \left(\frac{3m(x'' - x')}{\Delta t} + \frac{i m^2 (x'' - x')^3}{\Delta t^2}\right) + \alpha^2 \left(\frac{15m^2 (x'' - x')^2}{2 \Delta t^2} + \frac{5 i m^3 (x'' - x')^4}{\Delta t^3} - \frac{m^4 (x'' - x')^6}{2 \hbar^2 \Delta t^4}\right)\right] \times e^{-\frac{i m (x'' - x')^2}{2 \hbar \Delta t}}.\tag{26}$$

If we consider the terms up to third order of $O(\alpha)$, then the one dimensional deformed Hamiltonian can be written as

$$H = \frac{1}{2m} p^2 - \frac{\alpha}{m} p^3 + \frac{5 \alpha^2}{2m} p^4 + \frac{2 \alpha^3}{m} p^5 + V(x).\tag{27}$$

Using this deformed Hamiltonian, the propagator to the third order of $O(\alpha)$, can be written as

$$K(x'', t''; x', t') = \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \left[1 + \alpha \left(\frac{3m(x'' - x')}{\Delta t} + \frac{i m^2 (x'' - x')^3}{\Delta t^2}\right) + \alpha^2 \left(\frac{15m^2 (x'' - x')^2}{2 \Delta t^2} + \frac{5 i m^3 (x'' - x')^4}{\Delta t^3} - \frac{m^4 (x'' - x')^6}{2 \hbar^2 \Delta t^4}\right) + \alpha^3 \left(\frac{-135ihm^2 (x'' - x')^3}{\Delta t^2} - \frac{145m^3 (x'' - x')^3}{2 \Delta t^3} + \frac{25m^4 (x'' - x')^5}{2 \hbar^2 \Delta t^4} + \frac{7 m^5 (x'' - x')^7}{2 \hbar^2 \Delta t^5} - \frac{i m^6 (x'' - x')^9}{6 \hbar^3 \Delta t^6}\right)\right] \times e^{-\frac{i m (x'' - x')^2}{2 \hbar \Delta t}}.\tag{28}$$
Proceeding in this way one can construct more general form of the GUP-DSR corrected free particle propagator. The path integral corresponding to polynomial term \( p^4 \), i.e. the terms comes from GUP, has already been analyzed [51]. Therefore in this paper, we are mainly interested in analyzing the effect of the linear corrections to the Heisenberg algebra, that comes from DSR theory. Looking at the Eq. (20), one can check that the terms containing linear corrections of \( \alpha \) comes from DSR theory. So, we neglect terms proportional to \( \alpha^2 \) for explicit calculations. Thus, here we concentrate on the part of the propagator, containing first order of \( \alpha \), i.e.

\[
K(x'', t'' ; x', t') = \sqrt{\frac{m}{2\pi i\hbar \Delta t}} \left[ 1 + \frac{3\alpha m(x'' - x')}{\Delta t} + \frac{i\alpha m^2(x'' - x')^3}{\hbar \Delta t^2} \right] e^{i\pi 2 \alpha m (x'' - x')^2}. \tag{29}
\]

In order to obtain free particle propagator for a finite time interval \( t'' - t' \), we divide the interval into \( N \) subintervals of equal length \( \Delta t \) such that \( t'' - t' = N\Delta t \) and use the above result given by Eq. (29) for each subinterval. Therefore the propagator for finite time interval can be written as:

\[
P(x'', t'' ; x', t') = \left( \sqrt{\frac{m}{2\pi i\hbar \Delta t}} \right)^N \int dx_1 dx_2 ... dx_{N-1} e^{i\pi 2 \alpha m [(x_1 - x_0)^2 + (x_2 - x_1)^2 + ... + (x_{N-1} - x_N)^2]} \times \left\{ 1 + \frac{3\alpha m(x_1 - x_0)}{\Delta t} + \frac{i\alpha m^2(x_1 - x_0)^3}{\hbar \Delta t^2} \right\} \times \left\{ 1 + \frac{3\alpha m(x_2 - x_1)}{\Delta t} + \frac{i\alpha m^2(x_2 - x_1)^3}{\hbar \Delta t^2} \right\} \times \cdots \times \left\{ 1 + \frac{3\alpha m(x_N - x_{N-1})}{\Delta t} + \frac{i\alpha m^2(x_N - x_{N-1})^3}{\hbar \Delta t^2} \right\}. \tag{30}
\]

Neglecting the term containing higher order of \( \alpha \), we obtain the following result,

\[
P(x'', t'' ; x', t') = \left( \sqrt{\frac{m}{2\pi i\hbar \Delta t}} \right)^N \int dx_1 ... dx_{N-1} \times e^{i\pi 2 \alpha m [(x_1 - x_0)^2 + (x_2 - x_1)^2 + ... + (x_{N-1} - x_N)^2]} \times \left[ 1 + \frac{3\alpha m}{\Delta t} (x_N - x_0) + \frac{i\alpha m^2}{\hbar \Delta t^2} \{(x_1 - x_0)^3 + ... + (x_N - x_{N-1})^3\} \right]. \tag{31}
\]

Now using the expressions,

\[
\int e^{i\lambda [(x_1 - x_0)^2 + ... + (x_{N-1} - x_N)^2]} \, dx_1 \, dx_2 \, ... \, dx_{N-1} = \frac{1}{\sqrt{\pi N}} \left( \frac{2N\pi}{\lambda} \right)^{N-1/2} e^{i\lambda (x_N - x_0)^2}, \tag{32}
\]

and

\[
\int e^{i\lambda [(x_1 - x_0)^2 + ... + (x_{N-1} - x_N)^2]} \left[ (x_1 - x_0)^3 + ... + (x_N - x_{N-1})^3 \right] \, dx_1 \, dx_2 \, ... \, dx_{N-1} = \frac{1}{N^2 \sqrt{N}} \frac{1}{i\sqrt{i\pi \lambda}} \left( \frac{i\pi}{\lambda} \right)^{N/2} (x_N - x_0) \left[ -\frac{3}{2} N(N - 1) + i\lambda (x_N - x_0)^2 \right] e^{i\lambda (x_N - x_0)^2}, \tag{33}
\]
the propagator can be expressed as

\[ K(x''', t''; x', t') = \sqrt{\frac{m}{2\pi i\hbar N\Delta t}}\left(1 + 3\alpha m \frac{x_N - x_0}{\Delta t} + \frac{3\alpha m (1 - N)(x_N - x_0)}{N\Delta t}\right) + \frac{i\alpha m^2}{\hbar N^2\Delta t^2} (x_N - x_0)^3\right) \]

Finally, by replacing \( x_N, x_0 \) by \( x'', x' \) and using \( N\Delta t = t'' - t' \), we obtain the following expression,

\[ K(x'', t''; x', t') = \sqrt{\frac{m}{2\pi i\hbar (t'' - t')}}\left[1 + 3\alpha m \frac{x'' - x'}{t'' - t'} + \frac{i\alpha m^2 (x'' - x')^3}{\hbar (t'' - t')^2}\right] e^{i\alpha m^2 (x'' - x')^2/(2\hbar (t'' - t'))}. \]

It may be noted that the final form of propagator given by Eq. (36) has exactly the same form as the propagator for the infinitesimal interval given by Eq. (29).

Now we can calculate the corresponding probability which takes the form

\[ P(x'') = K^*(x'', t''; x', t')K(x'', t''; x', t') = \left[1 + 6\alpha m \frac{x'' - x'}{t'' - t'}\right] \frac{m}{2\pi \hbar (t'' - t')}. \]

It is well known that this probability for the ordinary case is given by

\[ P(x'') = \frac{m}{2\pi \hbar (t'' - t')} . \]

Thus, this probability increases with the incorporation of GUP and DSR in the path integral. This probability obtains a maximum value for the maximum momentum,

\[ m \frac{x'' - x'}{t'' - t'} = (\Delta p)_{\text{max}} = \frac{1}{\alpha}. \]

Furthermore, as the measurement is made at scales greater than the Planck scale, the corresponding momentum also decreases away from this maximum value, and this in turn makes the probability to decreased and tend back to its original value. Now we can using the expression for probabilities, we can calculate the expression for \(|K(x'', t''; x', t')|^2 dx \). If \( K(x'', t''; x', t')_{GUP} \) is the Kernal obtained from considering only the effects of GUP, \( K(x'', t''; x', t')_{DSR} \) is the the Kernal obtained from considering only the effects of DSR, and \( K(x'', t''; x', t')_{GUP+DSR} \) is the the Kernal obtained from considering the effects of both GUP and DSR, then we can write,

\[ |K(x'', t''; x', t')_{GUP}|^2 dx = \frac{m}{2\pi \hbar (t'' - t')} \left(1 - 12\alpha^2 p^2\right) dx, \]

\[ |K(x'', t''; x', t')_{DSR}|^2 dx = \frac{m}{2\pi \hbar (t'' - t')} \left(1 + 6\alpha p\right) dx, \]

\[ |K(x'', t''; x', t')_{GUP+DSR}|^2 dx = \frac{m}{2\pi \hbar (t'' - t')} \left(1 + 6\alpha p + 24\alpha^2 p^2\right) dx, \]

\[ |K(x'', t''; x', t')_{GUP+DSR}|^2 dx = \frac{m}{2\pi \hbar (t'' - t')} \left(1 + 6\alpha p + 24\alpha^2 p^2 - 100\alpha^3 p^3\right) dx, \]
where the momentum for a particle has been expressed as \( p = m(x'' - x')/(t'' - t') \). Here we have taken contributions up to \( \alpha^3 \) for the Kernal obtained from considering the effects of both GUP and DSR. It will be very interesting to plot the above expressions graphically. Since we have taken \( \alpha \) as a small parameter then we can choose \( \alpha_0 \) within \( 0 < \alpha_0 \leq 1 \). This inequality gives the bound for \( \alpha_0 \) as \( 0 < \alpha_0 < 6.52(= M_{pl}c) \). But since the minimum length is of the order of \( \Delta x_{min} \approx \alpha_0 l_{pl} \), we can not take \( \alpha_0 < 1 \). So the allowed region for \( \alpha_0 \) is \( 1 \leq \alpha_0 \leq 6.52 \). This inequality implies the allowed region for \( \alpha_0 \) as \( 0 < \alpha_0 \leq 1.53(= 1/M_{pl}c) \leq \alpha \leq 1 \).

Since \( |K(x'', t''; x', t')|^2 \geq 0 \), then from the above expressions (43) it will be justified if we take the particle momentum \( p \) from \( 0 \leq p \leq 0.28/\alpha \). The corresponding plot is as follow:

![Figure 1](image)

**Figure 1**: Here the pink plot, the green plot, the blue plot and the red plot are corresponding to normal GUP case, normal DSR case, combined GUP and DSR case, combined GUP and DSR case up to third order of \( \alpha \), respectively.

One can see from the above plot that for very high momentum \((\approx 0.28/\alpha)\), the difference between probability amplitude of normal GUP and normal DSR cases increases as \( \alpha \) takes more and more higher value. But, whenever both of them are incorporated the combined probability amplitude founds near to normal DSR case. A probable reason behind this is that, the DSR and the GUP are introduced in combined uncertainty relation to the first and the second order of \( \alpha \) respectively. Basically, the main observation from the above plot is that, even in free particle case, the GUP and DSR shows very much different result.

Now we show that the propagator corresponding to the deformed Heisenberg algebra satisfies the basic properties of a propagator. Differentiating the above propagator given by Eq. (36) two, three times with respect to \( x'' \) and one time with respect to \( t'' \), one can see that the above propagator given by Eq. (36), satisfy the modified Schrödinger equation at the final state \((x'', t'')\),

\[
-\frac{\hbar^2}{2m} \frac{\partial^2 K(x'', t''; x', t')}{\partial x''^2} + \frac{i\alpha \hbar^3}{m} \frac{\partial^3 K(x'', t''; x', t')}{\partial x''^3} = i\hbar \frac{\partial K(x'', t''; x', t')}{\partial t''}.
\]  

(44)

Up to the first order of \( \alpha \), the part of the deformed Schrödinger equation (7) can be written as

\[
-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2} + \frac{i\alpha \hbar^3}{m} \frac{\partial^3 \psi(x, t)}{\partial x^3} = i\hbar \frac{\partial \psi(x, t)}{\partial t}.
\]  

(45)

Now the solution of the above Schrödinger equation given by Eq. (45) can be written as

\[
\psi(x, t) = \left( Ae^{ik(1+\alpha \hbar) x - \frac{\hbar t}{2m}} + Be^{-ik(1-\alpha \hbar) x - \frac{\hbar t}{2m}} + Ce^{ikx - \frac{\hbar t}{2m}} \right).
\]  

(46)

Using this solution given by Eq. (46), it is possible to show that the propagator given by Eq. (24) propagates the wave function \( \psi(x, t) \) from an initial state \((x', t')\) to the slightly latter state \((x'', t'')\), for very
short time period $\Delta t = t'' - t'$. This means the following relation is satisfied to the first order of $O(\alpha)$:

$$\psi(x'', t'') = \int K(x''', t'''; x', t') \psi(x', t') dx'. \quad (47)$$

In order to prove the propagation for any finite time interval, we divide the time interval $(t'' - t')$ into $N$ subintervals of equal length $\Delta t$ and then apply Eq. (47) for each subinterval as,

$$\int K(x'', t''; x', t') \psi(x', t') \, dx' \quad = \quad \int \int \ldots \int K(x'', t''; x_{N-1}, t_{N-1}) K(x_{N-1}, t_{N-1}; x_{N-2}, t_{N-2}) \ldots K(x_1, t_1; x', t')$$

$$\times \psi(x', t') \, dx' \, dx_1 \ldots \, dx_{N-1} \quad = \quad \int \int \ldots \int K(x'', t''; x_{N-1}, t_{N-1}) K(x_{N-1}, t_{N-1}; x_{N-2}, t_{N-2}) \ldots K(x_2, t_2; x_1, t_1)$$

$$\times \psi(x_1, t_1) \, dx_1 \ldots \, dx_{N-1} \quad = \quad \ldots = \quad \int K(x'', t''; x_{N-1}, t_{N-1}) \psi(x_{N-1}, t_{N-1}) \, dx_{N-1}$$

$$\quad = \quad \psi(x'', t''). \quad (48)$$

Furthermore, one can see that our propagator given by Eq. (36) satisfy the normalization criteria for the propagator. This can be shown as follows: taking the complex conjugate of the Eq. (36), we obtain

$$K^*(x'', t''; x', t') = \sqrt{-\frac{m}{2\pi \hbar (t'' - t')}} \left[ 1 + \frac{3\alpha m (x'' - x')}{(t'' - t')} + \frac{i\alpha m^2 (x''^2 - x'^2)}{\hbar (t'' - t')^2} \right] \times e^{-\frac{i\alpha m (x'' - x')^2}{2(t'' - t')}}. \quad (49)$$

Now, using Eq. (49), we can demonstrate,

$$\int K^*(x'', t''; x', t') K(x'', t''; x', t') \, dx''$$

$$\quad = \quad \frac{m}{2\pi \hbar (t'' - t')} \int \frac{e^{-\frac{i\alpha m}{2\hbar(t'' - t')} (x'' - x')^2}}{2\hbar(t'' - t')} \left[ x'' - x' \right] \left[ \left( x'' - x' \right)^2 - \left( x'' - x' \right) \right]$$

$$\quad \times \left[ 1 + \frac{3\alpha m}{(t'' - t')} \left( (x'' - x')^2 - \left( x'' - x' \right) \right) + \frac{i\alpha m^2}{\hbar (t'' - t')^2} \left[ (x'' - x')^3 - \left( (x'' - x') \right)^3 \right] \right]$$

$$\quad = \quad \frac{m}{2\pi \hbar (t'' - t')} \int e^{-\frac{i\alpha m}{2\hbar(t'' - t')} (x'' - x')^2} \left[ x'' - x' \right] \left[ 1 - \frac{3\alpha m}{(t'' - t')} (x'' + x') \right]$$

$$\quad \times \left[ \frac{i\alpha m^2}{\hbar (t'' - t')^2} (x''^2 - x'^2) + \left( \frac{6\alpha m}{(t'' - t')} \right) + \frac{3i\alpha m^2}{\hbar (t'' - t')^2} (x''^2 - x'^2) \right]$$

$$\quad - \frac{3i\alpha m^2}{\hbar (t'' - t')^2} (x'' - x') x'' \right] dx'' \quad (50)$$

We substitute $u = mx''/\hbar (t'' - t')$, and use the identity,

$$\int e^{-u(x' - x_1)u^n} \, du = 2\pi (i)^n \frac{d^n(x' - x_1)^n}{d(x' - x_1)^n}. \quad (52)$$
to obtain the following expression,

\[
\int K^*(x'', t'' ; x'_1, t') K(x'', t'' ; x', t') \, dx'' = e^{\frac{im}{2\hbar(t'' - t')}} (1 - \frac{3\alpha m}{(t'' - t')} (x' + x'_1) - \frac{i\alpha m^2}{\hbar(t'' - t')} (x'^3 - x''_1^3)) \delta(x' - x'_1) + \left(6i\alpha\hbar - \frac{3\alpha m}{(t'' - t')} (x'^2 - x''_1^2) \right) \delta'(x' - x'_1) + 3i\alpha\hbar x' \psi^*(x', t') \delta''(x' - x'_1)
\]

\[ (53) \]

Now first multiplying both sides of Eq. (53) by \( \psi^*(x'_1, t') \), and then we then integrating \( x'_1 \) out by using the identity,

\[
\int \delta^{(n)}(y - x) f(y) dy = - \int \frac{\partial f}{\partial x} \delta^{(n-1)}(y - x) dy,
\]

we obtain the following result,

\[
\int \int K^*(x'', t'' ; x'_1, t') K(x'', t'' ; x', t') \psi^*(x'_1, t') \, dx'_1 \, dx'' = \left[ \psi^*(x', t') - \frac{6\alpha m x'}{t} \psi^*(x', t') \right] + \left[ \frac{12\alpha m x'}{t} \psi^*(x', t') + 6i\alpha\hbar \psi^*(x', t') \right] - \left[ \frac{6\alpha m x'}{t} \psi^*(x', t') + 6i\alpha\hbar \psi^*(x', t') \right] = \psi^*(x', t').
\]

\[ (55) \]

\[ (56) \]

Therefore the free particle propagator given by Eq. (56) is consistent with all the basic properties of a propagator.

4 Conclusion

In this paper, we have analyzed the deformation of the Heisenberg algebra consistent with both the GUP and DSR theory. It has been shown that this deformed Heisenberg algebra modifies the coordinate representation of the momentum operator, which further modifies the Hamiltonian for all quantum mechanical systems. We also noted that this deformed Hamiltonian could contain fractional derivative terms. However, it was possible to give a formal meaning to these fractional derivative terms by using the theory of harmonic extension of functions. In fact, we also have constructed a formal expression for the quantum mechanical propagator corresponding to this Hamiltonian by using path integration. We have obtained the explicit form of quantum mechanical free particle propagator for one dimensional system. With this propagator in hand one can now study other quantities of path integral, like the free particle partition function.

It has been observed that the propagator corresponding to the deformed Hamiltonian satisfies all the basic properties of a quantum mechanical propagator. It may be noted that recently this deformation of Heisenberg algebra has been used for analyzing various systems, such as, transition rate of ultra cold neutrons in gravitational field [27] and the Lamb shift and Landau levels [29]. In fact, it has been argued using this algebra that the space is a discrete lattice [76]. It would be interesting to analyze these systems by using path integral formalism of quantum mechanics. It may also be observed, that in absence of
matter fields, the Wheeler-DeWitt equation looks like the Schrödinger wave equation for a one dimensional particle with time dependent frequency \[ 59, 60 \]. The modification of the Wheeler-DeWitt equation by deforming the Heisenberg algebra has already been done \[ 61 \]. The big bang singularity gets avoided in deformed Wheeler-DeWitt equation. It would be interesting to calculate the wave function of the universe corresponding to this deformed Wheeler-DeWitt equation using path integral formalism. An advantage of using path integral formalism for analysing the deformed Heisenberg algebra is that this formalism can be generalized to field theory, and then used for constructing a deformed partition function Euclidean quantum gravity. This in turn could be used for studying the deformation of spacetime foam. It has been known that it can be argued the cosmological constant vanishes due to the effects coming from spacetime foam \[ 62, 63 \]. However, we know from data obtained from type I supernovae that our universe is an accelerating in its expansion, and this mean that our universe has a small but finite cosmological constant \[ 60, 71 \]. It would be interesting to repeat the analysis with a deformed partition function for Euclidean quantum gravity. It might be possible to relate this small but finite cosmological constant to the deformation parameter of the Heisenberg algebra. An alternative deformation of the Heisenberg algebra generated by noncommutative spacetime has also been studied. In fact, this deformation has been generalized to non-anticommutativity, and this has in turn been applied to constructing models of non-local perturbative quantum gravity \[ 72, 74 \]. It would be interesting to study a combination non-anticommutativity with the deformation of the Heisenberg algebra consistent with both GUP and DSR. It would also been interesting to study the effects of this combined deformation on a free particle using path integral formalism. It may also be noted that deformation of field theories, consistent with GUP \[ 75, 76 \], and a combination of GUP and DSR \[ 77, 78 \], have been studied. So, it would be interesting to analyse the combination of these deformations of field theories with noncommutativity and non-anticommutativity.

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