A REMARK ON GENERALIZED COMPLETE INTERSECTIONS

ALICE GARBAGNATI, BERT VAN GEEMEN

ABSTRACT. We observe that an interesting method to produce non-complete intersection subvarieties, the generalized complete intersections from L. Anderson and coworkers, can be understood and made explicit by using standard Cech cohomology machinery. We include a worked example of a generalized complete intersection Calabi-Yau threefold.

INTRODUCTION

Calabi-Yau varieties, in particular those of dimension three, are of great interest in string theory. Since there are not many general results yet on their classification, but see [W], the explicit construction of CY threefolds is a quite important enterprise. For example, Kreuzer and Starke classified the toric fourfolds which have CY threefolds as (anticanonical) hypersurfaces [KS], [AGHJN]. Besides generalizations to complete intersection CYs in certain ambient toric varieties, like products of projective spaces, there are various other examples of CY threefolds constructed with more sophisticated algebro-geometrical methods. Recent examples include [MOU], [CGKK], [KK].

In the recent paper [AAGGL], L. Anderson, F. Apruzzi, X. Gao, J. Gray and S-J. Lee found a very nice method to construct many more CY threefolds. The basic idea is to take a hypersurface $Y$ in an ambient variety $P$ and to consider hypersurfaces $X$ in $Y$. These hypersurfaces need not be complete intersections in $P$, that is, there need not exist two sections of two line bundles on $P$ whose common zero locus is $X$. There are various generalizations of this method, but we will stick to this basis case. As in [AAGGL], we refer to these varieties as generalized complete intersections (gCIs).

A particularly interesting and accessible case that was found and studied by Anderson and coworkers is when the ambient variety is a product of two varieties, one of which is $P^1$, so $P = P_2 \times P^1$. The variety $P_2$ they consider is a product of projective spaces, but this is not essential, one could consider any toric variety or even more general cases. The factor $P^1$ is important since there are line bundles on $P^1$ with non-trivial first cohomology group and this is essential to find generalized complete intersections. We review this construction in Section 1.1.

We provide a proposition, proven with standard Cech cohomology methods, that allows one, under a certain hypothesis, to find three equations (more precisely, three sections of three line bundles on $P$) that define $X$. In Section 2 we work out a detailed example, with explicit equations, of a CY threefold which was already considered in [AAGGL]. The explicit example $X$ has an automorphism of order two and the quotient of $X$ by the involution provides, after desingularization, another CY threefold. More generally, we think that among the gCIs found in [AAGGL] one could find more examples of CY threefolds with non-trivial automorphisms. It might be hard though to implement a systematic search as was done in [CGL] for complete intersection CY threefolds in products of projective spaces. We did not find new CY threefolds.
with small Hodge numbers (see [CCM] for an update on these), but the gCICY seem to be a promising class of CYs to search for these. The recent paper [BH] by Berglund and Hübsch provides further techniques to deal with gCICYs whereas [AAGGL2] explores string theoretical aspects of gCICYs.

1. The construction of generalized complete intersections

1.1. The general setting. Let $P_2$ be a projective variety of dimension $n$ and let $P := P_2 \times \mathbb{P}^1$. We denote the projections to the factors of $P$ by $\pi_1, \pi_2$ respectively. For a coherent sheaf $\mathcal{F}$ on $P$ and an integer $d$ we define a coherent sheaf on $P$ by:

$$\mathcal{F}[d] := \pi_1^* \mathcal{F} \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(d).$$

The Künneth formula gives

$$H^r(P, \mathcal{F}[d]) = \bigoplus_{p+q=r} H^p(P_2, \mathcal{F}) \otimes H^q(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)).$$

Recall that the only non-zero cohomology of $\mathcal{O}_{\mathbb{P}^1}(d)$ is: $h^0(\mathcal{O}_{\mathbb{P}^1}(d)) = h^1(\mathcal{O}_{\mathbb{P}^1}(-2-d)) = d+1$ for $d \geq 0$ and a basis for $H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ is given by the monomials $z_0^i z_1^{d-i}$, $i = 0, \ldots, d$, where $(z_0 : z_1)$ are the homogeneous coordinates on $\mathbb{P}^1$.

Let $L$ be a line bundle on $P_2$ and assume that $L[d]$, for some $d \geq 1$, has a non-trivial global section $F$. Using the Künneth formula, we can write $F = \sum_i f_i z_0^i z_1^{d-i}$ for certain sections $f_i \in H^0(P_2, L)$. Let $Y = (F)$ be the zero locus of $F$ in $P$. We assume that $Y$ is a (reduced, irreducible) variety, although this will not be essential in this section.

To define a codimension two subvariety of $P$, we consider another line bundle $M$ on $P_2$. The Künneth formula shows that $M[-e]$ has no global sections if $e \geq 1$. But upon restricting to $Y$, the vector space $H^0(Y, M[-e]|_Y)$ could still be non-trivial. In fact, from the exact sequence

$$0 \longrightarrow (L^{-1} \otimes M)[-d-e] \xrightarrow{F} M[-e] \longrightarrow M[-e]|_Y \longrightarrow 0$$

we deduce the exact sequence

$$0 \longrightarrow H^0(Y, M[-e]|_Y) \xrightarrow{d^0} H^1(P, (L^{-1} \otimes M)[-d-e]) \xrightarrow{F_1} H^1(P, M[-e])$$

thus $H^0(Y, M[-e]|_Y) \cong \ker(F_1)$, where we denote by $F_1$ the map induced by multiplication by $F$ on the first cohomology groups. Since now

$$H^1(P, (L^{-1} \otimes M)[-d-e]) \cong H^0(P_2, L^{-1} \otimes M) \otimes H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d-e)) \quad (d+e \geq 2)$$

the domain of $F_1$ is non trivial if and only if $h^0(P_2, L^{-1} \otimes M) \neq 0$. So for suitable choices of line bundles on $P_2$ we might find interesting, non-complete intersection, codimension two subvarieties of $P$ in this way. In the proof of Proposition 1.4 we explain how to compute $F_1$.

1.2. Example. Let $P_2 = \mathbb{P}^n$, $L = \mathcal{O}_{\mathbb{P}^n}(k)$, $M = \mathcal{O}_{\mathbb{P}^n}(k+l)$ with $l \geq 0$, let $d \geq 1$ and $e = 1$. Then $h^0(\mathbb{P}^n, L^{-1} \otimes M) = h^0(\mathcal{O}_{\mathbb{P}^n}(l)) \neq 0$ so $h^1(P, (L^{-1} \otimes M)[-d-e]) \neq 0$, but $h^1(P, M[-e]) = 0$ since $\mathcal{O}_{\mathbb{P}^1}(-1)$ has no cohomology. Thus $H^0(Y, M[-e]|_Y) \cong H^1(P, (L^{-1} \otimes M)[-d-e])$ is indeed non-trivial.
1.3. Generalized complete intersections. Given a variety $Y \subset P$ that is the zero locus of $F \in H^0(L[d])$ as in Section 1.2.1 and given a global section $\tau \in H^0(M[-e]|_Y)$, its zero locus $X := (\tau) \subset Y$ is called a generalized complete intersection.

The scheme $X$ may not be defined by two global sections $\sigma_1, \sigma_2$ of line bundles $L_1, L_2$ on $P$. However in certain cases we can find three sections of line bundles on $P$.

1.4. Proposition. Let $F \in H^0(P, L[d])$, let $Y = (F)$, let $\tau \in H^0(Y, M[-e]|_Y)$ with $d, e \geq 1$ be as above and assume that $H^1(P_2, L^{-1} \otimes M) = 0$.

Then there are two global sections $G, H \in H^0(P, M[d-1])$ such that the generalized complete intersection subscheme $X$ of $P$ defined by $\tau$ in $Y$ can also be defined as

$$X = \{x \in P : F(x) = G(x) = H(x) = 0\}$$

(the equality is of schemes). Moreover, there is a global section $A \in H^0(P, (L^{-1} \otimes M)[d+e-2])$ such that $AF = z_1^{d+e-1}G + z_0^{d+e-1}H$, so that on the open subset of $P_2 \times P^1$ where $z_0 \neq 0$ the subscheme $X$ of $P$ is defined by the two equations $F = G = 0$.

Proof. We use Cech cohomology to make the isomorphism $H^0(Y, M[-e]|_Y) \cong \ker(F_1)$, see exact sequence (2), explicit. Let $U_i \subset P^1$ be the open subset where $z_i \neq 0$. For a coherent sheaf $G$ on $P^1$ we have the exact sequence

$$0 \rightarrow H^0(P^1, G) \rightarrow G(U_0) \oplus G(U_1) \xrightarrow{\delta} G(U_0 \cap U_1) \rightarrow H^1(P^1, G) \rightarrow 0,$$

where $\delta(t_0, t_1) = t_0 - t_1$. The Cech groups we consider are computed with the Künneth formula. Note that after tensoring this exact sequence by a vector space $W \otimes H^0(P^1, G) = \ker(1_W \otimes \delta)$ and $W \otimes H^1(P^1, G) = \text{coker}(1_W \otimes \delta)$.

For an affine open subset $V \subset P^1$, the cohomology of the exact sequence (1) on $P_2 \times V$ gives the exact sequence, where we extend $M[-e]|_Y$ by zero to $P_2 \times V$.

$$H^0(P_2 \times V, M[-e]) \rightarrow H^0(P_2 \times V, M[-e]|_Y) \rightarrow H^1(P_2 \times V, (L^{-1} \otimes M)[-d-e]).$$

The Künneth formula, combined with the assumption $H^1(P_2, L^{-1} \otimes M) = 0$ and the fact that $H^2(V, \mathcal{F}) = 0$ for any coherent sheaf $\mathcal{F}$ since $V$ is affine, implies that the last group is zero.

Taking $V = U_0, U_1$, the exact sequence (1) on $P_2 \times V$ thus gives two exact sequences whose sum (term by term) is

$$0 \rightarrow \oplus_{i=0}^1 H^0(L^{-1} \otimes M) \otimes (\mathcal{O}_{P^1}(-d-e)(U_i)) \xrightarrow{F} \oplus_{i=0}^1 H^0(M) \otimes (\mathcal{O}_{P^1}(-e)(U_i)) \rightarrow \oplus_{i=0}^1 (M[-e]|_Y)(P_2 \times U_i) \rightarrow 0.$$

Similarly taking $V = U_0 \cap U_1$ one has the exact sequence:

$$0 \rightarrow H^0(L^{-1} \otimes M) \otimes (\mathcal{O}_{P^1}(-d-e)(U_0 \cap U_1)) \xrightarrow{F} H^0(M) \otimes (\mathcal{O}_{P^1}(-e)(U_0 \cap U_1)) \rightarrow (M[-e]|_Y)(P_2 \times (U_0 \cap U_1)) \rightarrow 0.$$

Next we use the Cech boundary map $\delta$ to map sequence (3) to sequence (4) and we obtain a commutative diagram with three complexes as columns. The first two columns are Cech complexes for the covering $\{U_i\}_{i=0,1}$ of $P^1$, their cohomology groups are respectively

$$H^0(L^{-1} \otimes M) \otimes H^q(\mathcal{O}_{P^1}(-d-e)) \cong H^q(P, (L^{-1} \otimes M)[-d-e]),$$

$$H^0(M) \otimes H^q(\mathcal{O}_{P^1}(-e)) \cong H^q(P, M[-e]), \quad (q = 0, 1).$$

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The zero-th cohomology group of the last column is $H^0(Y, M[-e]|_Y)$. So we conclude that the maps $q$ and $F_1$ can be computed with the long exact cohomology sequence associated to this diagram.

We observe, but will not use, that the Künneth formula implies that $H^2(P, (L^{-1} \otimes M)[-d-e]) = 0$ and thus the cohomology sequence of (11) gives a six term exact sequence with the zero-th and first cohomology groups. The first 5 terms are the same as those of the long exact sequence associated to the diagram, so we conclude that the first cohomology group of the last column is $H^1(Y, M[-e]|_Y)$.

Given $\tau \in H^0(Y, M[-e]|_Y)$, let $q := d^0(\tau) \in \ker(F_1)$. Since the first row (3) of the complex is exact, the section $\tau$ is locally given by restricting sections $\tau_i \in M[-e](P_2 \times U_i)$ to $Y$. By the snake lemma, they satisfy $\tau_0 - \tau_1 = Fq$ on $P_2 \times (U_0 \cap U_1)$, in particular $\tau_0 = \tau_1$ on $Y \cap (P_2 \times (U_0 \cap U_1))$ since $F = 0$ on $Y$.

The images of the $z_{-j}^{-d+e+1-j} \in O_{P_1}(-d-e)(U_0 \cap U_1)$, $j = 1, \ldots, d + e - 1,$ form a basis of $H^1(P, O(-d-e))$. A cohomology class $q \in H^1(P, (L^{-1} \otimes M)[-d-e]) \cong H^0(P, L^{-1} \otimes M) \otimes H^1(P, O(-d-e))$ can thus be represented by $q = \sum_j q_j z_{-j}^{-d+e+1-j}$ with $q_j \in H^0(P, L^{-1} \otimes M)$. Let $F = \sum f_i (d-1)$, where $f_i \in H^0(P_2, L)$, then $Fq$ is homogeneous of degree $d - (d + e) = -e$ and it is a sum of terms $r_k z_{0}^{-e-k}$ with $r_k \in H^0(P_2, M)$. Writing

$$Fq = \sum_{k=-d+e+1}^{d-1} r_k z_{0}^{-1-e-k} = \left( \sum_{k=-d+e+1}^{-e} r_k z_{0}^{-1-e-k} \right) + \left( \sum_{k=e+1}^{-1} r_k z_{0}^{-1-e-k} \right) + \left( \sum_{k=0}^{d-1} r_k z_{0}^{-1-e-k} \right),$$

the first summand lies in $M[-e](P_2 \times U_0)$ (where $z_0 \neq 0$) and the last summand lies in $M[-e](P_2 \times U_1)$, we denote these summand by $\tau_0$ and $-\tau_1$ respectively. The middle summand has monomials $z_{0}^a z_{1}^b$ with both $a, b < 0$. Thus $Fq$ represents a class in $q' \in H^1(P, M[-e])$, which is the same as the class represented by the middle summand. By definition, one has $q' = F_1(q)$ and thus $q \in \ker(F_1)$ when all coefficients $r_k$, $k = -e+1, \ldots, -1$, are zero.

Since $q \in \ker(F_1)$ this middle summand is zero, so that $Fq = \tau_0 - \tau_1$ as desired. Now we define $G := z_{0}^{d+e-1} \tau_0$ and $H := -z_{1}^{d+e-1} \tau_1$ so that all their monomials $z_{0}^a z_{1}^b$ have $a, b \geq 0$ and $a + b = d - 1$, thus both $G, H \in H^0(P, M[d-1])$. Then $(z_{0} z_{1})^{d+e-1} Fq = z_{1}^{d+e-1} G + z_{0}^{d+e-1} H$ and with $A := (z_{0} z_{1})^{d+e-1} q \in H^0(P, (L^{-1} \otimes M)[d-e-2])$ we find the desired relation. \hfill $\Box$

1.5. Example. With the choices of $P_2, L, M$ as in Example (12) and if $X$ is a smooth variety (of dimension $n - 1$), then $H^1(P_2, L^{-1} \otimes M) = H^1(P^n, O_{P^n}(l)) = 0$, for any $l$, if $n > 1$. The adjunction formula implies that $X$ has trivial canonical bundle if we choose $l = n + 1 - 2k$ and $d = 3$. In that case $P = P^n \times P^1$ and $F$ is homogeneous of bidegree $(k, 3)$ whereas $G, H$ have bidegree $(n + 1 - k, 2)$.

1.6. A fibration on $X$. Given $X$ as in the proposition, the projection $\pi_2 : P_2 \times P^1 \to P^1$ restricts to $X$ to give a fibration denoted by $\pi_2 : X \to P^1$. For a point $p = (z_0 : z_1) \in P^1$, we denote by $F_p \in H^0(P_2, L)$, $H_p \in H^0(P_2, M)$ the restrictions of $F$ and $H$ to the fiber $X_p$. The equation $AF = z_{1}^{d+e-1} G + z_{0}^{d+e-1} H$ shows that if $z_1 \neq 0$ then $F_p$ and $H_p$ define the fiber $X_p$, which is thus a complete intersection in $P_2$.

1.7. Example. This example illustrates that $X$, as in Proposition (14), might be reducible, even if $h^0(Y, M[-e]|_Y)$ is rather large. The example is taken from [AAGGL, Table 4], third
item (with \(i = 2\)) where it is in fact observed that no smooth varieties arise in that case. We take
\[ P_2 := \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1, \quad L := \mathcal{O}(0, 1, 1), \quad M := \mathcal{O}(3, 1, 1), \quad d = 4, \ e = -2. \]
Notice that \( H^1(P_2, L^{-1} \otimes M) = H^1(\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(3, 0, 0)) = 0 \) by the Künneth formula, so we can, but will not, apply Proposition 1.4. Since \( h^1(L^{-1} \otimes M)[-d-e] = h^1(\mathcal{O}(3, 0, 0)[-6]) = 10 \cdot 1 \cdot 5 = 50 \) and \( h^1(M[-e]) = 10 \cdot 2 \cdot 2 \cdot 1 = 40 \), we find \( h^0(M[-e]|_Y) \geq 10 \). We will show that, for general \( Y \), \( h^0(M[-e]|_Y) = 10 \) but that all sections of \( M[-e]|_Y \) define reducible subvarieties of \( Y \).

Due to the first zero in \( L = \mathcal{O}(0, 1, 1) \), the variety \( Y = \mathbb{P}^2 \times S \subset P \), with \( S \subset (\mathbb{P}^1)^3 \) the surface defined by a section of \( \mathcal{O}(1, 1, 4) \). Then we have \( h^0(M[-e]|_Y) = h^0(\mathbb{P}^2 \times S, \pi_1^* \mathcal{O}_{\mathbb{P}^2}(3) \otimes \pi_2^* \mathcal{O}_S(1, 1, -2)) \) and using the Künneth formula we find \( h^0(M[-e]|_Y) = h^0(\mathcal{O}_{\mathbb{P}^2}(3))h^0(\mathcal{O}_S(1, 1, -2)) = 10h^0(\mathcal{O}_S(1, 1, -2)) \). The exact sequence
\[ 0 \rightarrow \mathcal{O}_{(\mathbb{P}^1)^3}(0, 0, -6) \rightarrow \mathcal{O}_{(\mathbb{P}^1)^3}(1, 1, -2) \rightarrow \mathcal{O}_S(1, 1, -2) \rightarrow 0, \]
where \( f \) is the equation of \( S \), shows that (with \( f_1 \) the map induced by \( f \) on \( H^1)\):
\[ h^0(\mathcal{O}_S(1, 1, -2)) = \dim \ker \left( f_1 : H^1(\mathcal{O}_{(\mathbb{P}^1)^3}(0, 0, -6)) \rightarrow H^1(\mathcal{O}_{(\mathbb{P}^1)^3}(1, 1, -2)) \right). \]
Since these spaces have dimensions \( 1 \cdot 1 \cdot 5 = 5 \) and \( 2 \cdot 2 \cdot 1 = 4 \) respectively, one expects \( h^0(\mathcal{O}_S(1, 1, -2)) = 1 \). In that case any section \( \tau \in H^0(M[-e]|_Y) \) would be the product \( \tau = gs \) with \( g \in H^0(\mathcal{O}_{\mathbb{P}^2}(3)) \) and \( s \in H^0(\mathcal{O}_S(1, 1, -2)) \) the unique (up to scalar multiple) section, hence \( X \) would be reducible.

To see that indeed \( h^0(\mathcal{O}_S(1, 1, -2)) = 1 \) for a general equation \( f \), take a smooth (genus one) curve \( C \) of bidegree \((2, 2)\) in \( \mathbb{P}^1 \times \mathbb{P}^1 \) and choose eight distinct points on \( C \) which are not cut out by another curve of bidegree \((2, 2)\). As curves of bidegree \((1, 4)\) depend on \(2 \cdot 5 = 10\) parameters, we can find two polynomials \( g_0, g_1 \) of bidegree \((1, 4)\) such that \( g_0 = g_1 = 0 \) consists of these eight points on \( C \). Take \( f = x_0g_0 + x_1g_1 \) with \((x_0 : x_1) \in \mathbb{P}^1 \), the first copy of \( \mathbb{P}^1 \) in \((\mathbb{P}^1)^3 \), and the \( g_i \) on the last two copies of \( \mathbb{P}^1 \). The surface \( S \subset (\mathbb{P}^1)^3 \) defined by \( f \) is thus the blow up of \( \mathbb{P}^1 \times \mathbb{P}^1 \) in the eight points where \( g_0 = g_1 = 0 \). The adjunction formula shows that the line bundle \( \mathcal{O}_S(1, 1, -2) \) is the anticanonical bundle of \( S \). The effective anticanonical divisors are the strict transforms of bidegree \((2, 2)\)-curves on passing through these eight points. Hence the strict transform of \( C \) in \( S \) will be the unique effective anticanonical divisor on \( S \) and therefore \( h^0(\mathcal{O}_S(1, 1, -2)) = 1 \).

2. An example: a generalized complete intersection Calabi-Yau threefold

2.1. We illustrate the use of Proposition 1.4 (and its proof) for the generalized complete intersection Calabi-Yau discussed in [AAGGL, Section 2.2.2]. We also consider an explicit example which has a non-trivial involution and we compute the Hodge numbers of a desingularization of the quotient threefold which is again a CY.

2.2. The varieties \( P_2 \) and \( Y \). We consider the case that \( P_2 = \mathbb{P}^4 \), we choose the line bundle \( L := \mathcal{O}_{\mathbb{P}^4}(2) \) and we let \( d = 3 \). Then the line bundle \( L[d] = \mathcal{O}_{\mathbb{P}^2}(2, 3) \) is very ample on \( P = \mathbb{P}^4 \times \mathbb{P}^1 \) and thus a general section \( F \) will define a smooth fourfold \( Y \) of \( P \). To obtain a CY threefold in \( Y \), we consider global sections of the anticanonical bundle of \( Y \). By adjunction, \( \omega_Y = (\mathcal{O}_{\mathbb{P}^4}(-5, -2) \otimes \mathcal{O}_{\mathbb{P}^2}(2, 3))|_Y = \mathcal{O}_Y(-3, 1) \). Thus we take \( M = \mathcal{O}_{\mathbb{P}^4}(3) \) and \( e = 1 \), so that
\[ M[-\epsilon]|_Y = \mathcal{O}_Y(3, -1) = \omega_Y^{-1}. \] As the \( H^1 \) of any line bundle on \( \mathbb{P}^4 \) is trivial, we can use (the proof of) Proposition 3.4 to find polynomials \( G, H \in H^0(P, \mathcal{O}_P(3, 2)) \) which together with \( F \) define a generalized complete intersection \( X \).

As in Example 1.2, we get
\[
H^0(\mathcal{O}_Y(3, -1)) \xrightarrow{z} H^1(\mathcal{O}_P(1, -4)) .
\]
To find explicit elements of \( H^0(\mathcal{O}_Y(3, -1)) \), we write the defining equation of \( Y \) as
\[
F = P_0z_0^3 + P_1z_0^2z_1 + P_2z_0z_1^2 + P_3z_1^3 \quad (\in H^0(P, \mathcal{O}_P(2, 3))) ,
\]
with \( P_i \in H^0(\mathbb{P}^4, \mathcal{O}(2)) \) homogeneous polynomials of degree two in \( y = (y_0 : \ldots : y_4) \). As \( H^1(\mathcal{O}_P(1, -4)) \cong H^0(\mathcal{O}_P(1)) \otimes H^1(\mathcal{O}_P(-4)) \), a basis of this \( 5 \cdot 3 = 15 \) dimensional vector space are the products of one of \( y_0, \ldots, y_4 \) with one of \( z_0^{-3}z_1^{-1}, z_0^{-2}z_1^{-2}, z_0^{-1}z_1^{-3} \). Thus any class \( q \in H^1(\mathcal{O}_P(1, -4)) \) has a representative
\[
q = Q_0z_0^{-3}z_1^{-1} + Q_1z_0^{-2}z_1^{-2} + Q_2z_0^{-1}z_1^{-3} \quad (\in H^1(\mathcal{O}_P(1, -4))) ,
\]
with linear forms \( Q_i \in H^0(\mathbb{P}^4, \mathcal{O}(1)) \). As in the proof of Proposition 3.4 we must write:
\[
Fq = \tau_0 - \tau_1, \quad G := z_0^3\tau_0, \quad H := -z_1^3\tau_1 ,
\]
with \( \tau_i \in \mathcal{O}_P(3, -1)(\mathbb{P}^4 \times U_i) \). So we find
\[
G = z_0^2(P_1Q_0 + P_2Q_1 + P_3Q_2) + z_0z_1(P_2Q_0 + P_3Q_1) + z_1^2P_3Q_0 , \quad H = -\left(z_0^2P_0Q_2 + z_0z_1(P_0Q_1 + P_1Q_2) + z_1^2(P_0Q_0 + P_1Q_1 + P_2Q_2)\right) .
\]

2.3. The base locus of \( |-K_Y| \). In Section 2.2 we showed how to find the global sections of \( \omega_Y^{-1} = \mathcal{O}_Y(3, -1) \) explicitly, locally such a section is given by the polynomials \( G \) and \( H \). From the formula for \( F \) we see that if \( x \in \mathbb{P}^4 \) and \( P_0(x) = \ldots = P_3(x) = 0 \), then the curve \( \{x\} \times \mathbb{P}^4 \) lies in \( Y \). This curve also lies in the zero loci of \( G \) and \( H \), for any choice of \( Q_0, Q_1, Q_2 \in H^0(\mathcal{O}_P(1)) \), hence it lies in the base locus of anticanonical system \( |-K_Y| \). Since the four quadrics \( P_i = 0 \) in \( \mathbb{P}^4 \) intersect in at least \( 2^4 \) points, counted with multiplicity, we see that this base locus is non-empty. Thus we cannot use Bertini’s theorem to guarantee that there are smooth CY threefolds \( X \subset Y \), but we resort to an explicit example, see below.

2.4. The CY threefold \( X \). To obtain an explicit example, we choose
\[
P_0 := y_0^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2 , \quad P_1 := y_0^2 + y_1^2 , \quad P_2 := y_1^2 + y_3^2 , \quad P_3 := y_0^2 + y_1^2 - y_2^2 - y_3^2 - y_4^2 ,
\]
and
\[
Q_0 := y_0, \quad Q_1 := y_1, \quad Q_2 := y_2 .
\]
Using a computer algebra system (we used Magma [M]), one can verify that \( Y := (F = 0) \) and \( X := (F = G = H = 0) \) are smooth varieties in \( P \). The variety \( X \) is a Calabi-Yau threefold since it is an anticanonical divisor on \( Y \). In [AAGGL] (2.27), (2.28) one finds that the Hodge numbers of \( X \) are \((h^{1,1}(X), h^{2,1}(X)) = (2, 46)\), in particular, \( h^2(X) = 2, h^3(X) = 94.\)
2.5. Parameters. The CY threefold $X$ is defined by a section $F \in H^0(P, \mathcal{O}_P(2, 3))$ and a section $\tau \in H^0(Y, \mathcal{O}_Y(3, -1))$. The first is a vector space of dimension
\[ h^0(P, \mathcal{O}_P(2, 3)) = h^0(P^4, \mathcal{O}_{P^4}(2)) \cdot h^0(P^1, \mathcal{O}_{P^1}(3)) = 15 \cdot 4 = 60 , \]
whereas the second has dimension 15. The group $GL(5, \mathbb{C}) \times GL(2, \mathbb{C})$ acts on $H^0(\mathcal{O}_P(2, 3))$ and has dimension $5^2 + 2^2 = 29$. The subgroup of elements $(\lambda I_5, \mu I_2)$ with $\lambda^2 \mu^3 = 1$ acts trivially, so we get $60 - 28 = 32$ parameters for $P$ and next $15 - 1 = 14$ parameters for $\tau$, so we do get $32 + 14 = 46 = h^{2,1}(X)$ parameters for $X$. So the general deformation of $X$ seems to be again a gCICY of the same type as $X$. (In [AAGGL], just below (2.28), the dependence of the CY threefold on $\mathbb{P}$, which gives 32 parameters, seems to have been overlooked.)

2.6. A CY quotient. A well-known method to obtain Calabi-Yau threefolds is to consider desingularizations of quotients of such threefolds by finite groups, see for example [CGL]. In the example above, we see that $X$ is a CY quotient.

\[ X \]

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2.6. A CY quotient. A well-known method to obtain Calabi-Yau threefolds is to consider desingularizations of quotients of such threefolds by finite groups, see for example [CGL]. In the example above, we see that $X \subset \mathbb{P}^4 \times \mathbb{P}^1$ has a subgroup $(\mathbb{Z}/2\mathbb{Z})^2 \subset Aut(X)$ given by the sign changes of $y_3$ and $y_4$. We consider the involution
\[ \iota : X \longrightarrow X, \quad \left( (y_0 : \ldots : y_4), (z_0 : z_1) \right) \longmapsto \left( (y_0 : y_1 : y_2 : -y_3 : -y_4), (z_0 : z_1) \right) . \]
Its fixed point locus has two components, one defined by $y_3 = y_4 = 0$ and the other by $y_0 = y_1 = y_2 = 0$ in $X$. The first is a curve in $\mathbb{P}^2 \times \mathbb{P}^1 \subset \mathbb{P}$, which is smooth, irreducible and reduced of genus 8 according to Magma. Similarly, the other component is a genus 2 curve in $\mathbb{P}^1_{(y_0:y_4)} \times \mathbb{P}^1_{(z_0:z_1)} \subset \mathbb{P}$. In fact, only $F = 0$ provides a non-trivial equation for this curve since $y_0 = y_1 = y_2 = 0$ implies $Q_0 = Q_1 = Q_2 = 0$ and hence $G = H = 0$ on this $\mathbb{P}^1 \times \mathbb{P}^1$. As $F = 0$ defines a smooth curve of bidegree $(2, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, this curve has genus $(2 - 1)(3 - 1) = 2$.

In particular, the singular locus of the quotient $X/\iota$ consists of two curves of $A_1$-singularities. Since the fixed point locus $X^\iota$ consists of two curves, we conclude that locally on $X$ the involution is given by $(t_1, t_2, t_3) \mapsto (-t_1, -t_2, t_3)$ in suitable coordinates. Hence $\iota$ acts trivially on the nowhere vanishing holomorphic 3-form on the CY threefold $X$. Thus the blow up $Z$ of $X/\iota$ in the singular locus will again be a CY threefold.

We determine the Hodge numbers of $Z$. To do so, it is more convenient to consider the blow up $\tilde{X}$ of $X$ in the fixed point locus $X^\iota$. The involution extends to an involution $\tilde{\iota}$ on $\tilde{X}$, the fixed point set of $\tilde{\iota}$ consists of the two exceptional divisors and the quotient $\tilde{X}/\tilde{\iota}$ is the same $Z$. Moreover, $H^i(Z, \mathbb{Q}) \cong H^i(\tilde{X}, \mathbb{Q})^\iota$, the $\iota$-invariant subspace.

Standard results on the blow up of smooth varieties in smooth subvarieties (cf. [V] Thm 7.31) show that $h^2(\tilde{X}) = h^2(X) + 2 = 4$ (due to the two exceptional divisors over the two fixed curves) and $h^3(\tilde{X}) = h^3(X) + 2 \cdot 8 + 2 \cdot 2 = 114$ (the contribution of the $H^4$ of the fixed curves to $H^3$ of the blow up). The Lefschetz fixed point formula for $\iota$ gives
\[ \chi(\tilde{X}^\iota) = \sum_{i=0}^{6} (-1)^i tr(\iota^* H^i(\tilde{X}, \mathbb{Q})) . \]

Notice that $\iota^*$ is the identity on $H^0, H^2, H^4, H^6$, in particular $h^2(Z) = \dim H^2(\tilde{X}, \mathbb{Q})^\iota = 4$. The fixed points of $\iota$ are the two exceptional divisors, these are $\mathbb{P}^1$-bundles over the exceptional curves hence
\[ 2(2 - 2 \cdot 2) + 2(2 - 2 \cdot 8) = 1 - 0 + 4 - t_3 + 4 - 0 + 1 \implies t_3 = 42 . \]
If the $+, -$ eigenspaces of $\tilde{\iota}$ on $H^3(\tilde{X}, Q)$ have dimensions $a, b$ respectively, then $a + b = 114$ and $a - b = 42$, thus $a = 78$ and $a = \dim H^3(\tilde{X}, Q) = h^3(Z)$. As $Z$ is a CY threefold it has $h^{3,0}(Z) = 1$ and thus $h^{2,1}(Z) = (78 - 2)/2 = 38$. Other examples of CY threefolds with $(h^{1,1}, h^{2,1}) = (4, 38)$ are already known.

2.7. A (singular) projective model of $Z$. The fibers of $\pi_2 : X \to P^1$ are K3 surfaces, complete intersections of a quadric and a cubic hypersurface in $P^4$. The involution $\iota$ on $X$ restricts to a Nikulin involution on each smooth fiber. The quotient of such a fiber by the involution will in general be isomorphic to a K3 surface in $P^2 \times P^1$, defined by an equation of bidegree $(3, 2)$ (see [GS, Section 3.3]). Using the same method as in that reference, we found that the rational map

$$P^4 \times P^1 \dasharrow P^2 \times P^1 \times P^1, \quad \left((y_0 : \ldots : y_4), (z_0 : z_1)\right) \mapsto \left((y_0 : y_1 : y_2), (y_3 : y_4), (z_0 : z_1)\right)$$

factors over $X/\iota$ and the image, defined by an equation of multidegree $(3, 2, 2)$, is birational with $Z$. Using the explicit equation for the image and Magma, we found that the image has 38 singular points.

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Dipartimento di Matematica, Università di Milano, via Saldini 50, 20133 Milano, Italia
E-mail address: alice.garbagnati@unimi.it

Dipartimento di Matematica, Università di Milano, via Saldini 50, 20133 Milano, Italia
E-mail address: lambertus.vangeemen@unimi.it