NON-HOMOGENEOUS DENSITY FUNCTIONS TO STRENGTHEN RECTANGULAR PARTIALLY HINGED PLATES

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Abstract. We consider a non-homogeneous partially hinged rectangular plate having structural engineering applications. In order to study possible remedies for torsional instability phenomena we consider the gap function as a measure of the torsional performances of the plate. We treat different configurations of load and we study which density function is optimal for our aims. The analysis is in accordance with some results obtained studying the corresponding eigenvalue problem in terms of maximization of the ratio of specific eigenvalues. Some numerical experiments complete the analysis.

1. Introduction

We study a long narrow rectangular thin plate $\Omega \subset \mathbb{R}^2$, hinged at short edges and free on the remaining two, see [11]. This plate may model the deck of a bridge; since this kind of structures manifest problems of flutter instability, e.g. see [12, 13, 17], we would optimize its design in order to reduce the phenomenon. To this aim one may vary the shape of the plate, see [4], or modify the materials composing it, see [5, 6, 7].

Here we fix the geometry of the plate, assuming that it has length $\pi$ and width $2\ell$ with $2\ell \ll \pi$ so that

$$\Omega = (0, \pi) \times (-\ell, \ell) \subset \mathbb{R}^2;$$

hence, we study the structural performance of the plate with respect to a density function $p = p(x, y)$, representing the non-homogeneity of the plate. For which purpose is the density function optimal? In a rectangular plate it is possible to distinguish vertical and torsional oscillations; the most problematic are the second one, that may cause the collapse of the structure, see [12]. Then we consider a functional, named gap function, able to measure the torsional performances of the plate, see also [3]. In particular this functional measures the gap between the displacements of the two free edges of the deck; higher is the gap higher is the torsional motion of the plate. In this paper we compute the gap function for some external forces, that seem to be the most prone to generate torsional instability in the structure. For every forcing term we consider different density functions $p(x, y)$ in order to understand how the gap function varies. We want to find the density function that reduces as much as possible the maximum of the absolute value of the gap function; this will be our optimal density function with respect to the choice of the external force. In [7] the authors present a study of the correspondent weighted eigenvalue problem and they compare different density functions in order to find the optimal one with respect to the ratio of critical eigenvalues; they tested some density functions proposing theoretical and numerical justifications. We point out that the study of a ratio of eigenvalues has some limits; first of all it requires to consider two specific eigenvalues, moreover the direct optimization of the ratio is very involved. As a consequence the question is often dealt with in terms of minimization or maximization of a single eigenvalue, see [7] for details. Here we compare the same density functions proposed in [7] and we see that $p(x, y)$ optimal for [7] is optimal also with respect to the reduction of the gap function. This result confirms that the gap function is a reliable measure for the torsional performances of rectangular plates; furthermore, it is a useful tool to get information on optimal reinforces in order to reduce torsional instability phenomena.

The paper is organized as follows. In Section 2 we introduce some preliminaries and notations and we define longitudinal and torsional modes of vibration. In Section 3 we define the gap function, we write the minmaxmax problem we are interested in and we state the existence results, proved in Section 6.
In this framework we minimize the energy functional obtaining this constant in the density function; then we write the corresponding stationary weighted problem:

\[ E[u] = \int_{\Omega} \frac{1}{2} p u_t^2 \, dxdy + \frac{Eh^3}{12(1-\sigma^2)} \int_{\Omega} \left( \frac{(\Delta u)^2}{2} + (1-\sigma)(u_{xyy}^2 - u_{xxx} u_{yyy}) \right) dxdy - \int_{\Omega} pf \, dxdy, \]

where \( h \) is its constant thickness, see also [11].

To proceed with the classical minimization of the functional, we need some information on the regularity of the functions representing the materials composing the plate, i.e. \( p(x,y) \), \( E(x,y) \), \( \sigma(x,y) \). We consider the possibility that the plate is composed by different materials, hence we cannot assume the continuity of the previous functions. In general discontinuous Young modulus and Poisson ratio generates some mathematical troubles in finding the minimization problem in strong form. For the civil engineering applications, which we are interested, we point out that the Poisson ratio do not vary so much with respect to the possible choice of the materials; therefore, as a first approach, we suppose \( E \) and \( \sigma \) constant in space, while the density of the plate is in general variable and possibly discontinuous.

Hence we obtain

\[ E(u) = \int_{\Omega} \frac{1}{2} p u_t^2 \, dxdy + \frac{Eh^3}{12(1-\sigma^2)} \int_{\Omega} \left( \frac{(\Delta u)^2}{2} + (1-\sigma)(u_{xyy}^2 - u_{xxx} u_{yyy}) \right) dxdy - \int_{\Omega} pf \, dxdy, \]

In this framework we minimize the energy functional obtaining

\[
\begin{align*}
\begin{cases}
p(x,y)u_t + \frac{Eh^3}{12(1-\sigma^2)} \Delta^2 u = p(x,y)f(x,y,t) & \text{in } \Omega \times (0, +\infty) \\
u(0,y,t) = u_{xx}(0,y,t) = u(\pi,y,t) = u_{xx}(\pi,y,t) = 0 & \text{for } (y,t) \in (-\ell, 0) \times (0, +\infty) \\
u_{y}(x,\pm\ell,t) + \sigma u_{xx}(x,\pm\ell,t) = 0 & \text{for } (x,t) \in (0, \pi) \times (0, +\infty) \\
u_{yy}(x,\pm\ell,t) + (2-\sigma)u_{xyy}(x,\pm\ell,t) = 0 & \text{for } (x,t) \in (0, \pi) \times (0, +\infty) \\
u(x,y,0) = u_{0}(x,y) & \text{for } (x,y) \in \Omega \\
u_{y}(x,y,0) = u_{1}(x,y) & \text{for } (x,y) \in \Omega
\end{cases}
\end{align*}
\]

The boundary conditions on short edges are of Navier type, see [16], and model the situation in which the plate is hinged on \( \{0, \pi\} \times (-\ell, \ell) \). Instead, the boundary conditions on large edges are of Neumann type, modeling the fact that the deck is free to move vertically; for the Poisson ratio we shall assume

\[ \sigma \in \left(0, \frac{1}{2}\right), \]

since most of the materials have values in this range.

We divide the differential equation in (2.1) for the flexural rigidity of the plate \( \frac{Eh^3}{12(1-\sigma^2)} \) and we include this constant in the density function; then we write the corresponding stationary weighted problem:

\[
\begin{align*}
\begin{cases}
\Delta^2 u = p(x,y)f(x,y) & \text{in } \Omega \\
u(0,y) = u_{xx}(0,y) = u(\pi,y) = u_{xx}(\pi,y) = 0 & \text{for } y \in (-\ell, \ell) \\
u_{yy}(x,\pm\ell) + \sigma u_{xx}(x,\pm\ell) = u_{yy}(x,\pm\ell) + (2-\sigma)u_{xyy}(x,\pm\ell) = 0 & \text{for } x \in (0, \pi).
\end{cases}
\end{align*}
\]

In this paper we study (2.3): in particular we test the behaviour of the plate with respect to different weight functions \( p \) and external forcing terms \( f \).
2.2. Families of forcing terms and weight functions. We denote by
\[\mathcal{F}_V := \{ f \in V(\Omega) : \| f \|_V = 1 \}\]
the set of admissible forcing terms, fixed a certain functional space \(V(\Omega)\). We introduce a family of weights to which \(p\) belongs
\[(2.4) \quad \mathcal{P}^{\alpha,\beta}_{L^\infty} := \left\{ p \in L^\infty(\Omega) : \alpha \leq p \leq \beta , \ p(x,y) = p(x, -y) \text{ a.e. in } \Omega, \int_\Omega p \, dx \, dy = |\Omega| \right\}\]
where \(\alpha, \beta \in \mathbb{R}^+\) with \(\alpha < \beta\) fixed. When \(f\) belongs to certain functional spaces, we need further regularity on the weight functions; therefore we introduce a second family
\[\mathcal{P}^{\alpha,\beta}_{H^2} := \left\{ p \in H^2(\Omega) : \alpha \leq p \leq \beta , \ p(x,y) = p(x, -y) \text{ in } \Omega, \int_\Omega p \, dx \, dy = |\Omega|, \ \| p \|_{H^2} \leq (\beta + \alpha)\sqrt{|\Omega|} \right\},\]
with \(\alpha, \beta \in \mathbb{R}^+\) and \(\alpha < \beta\) fixed. The integral condition in (2.4) represents the preservation of the total mass of the plate; this is our fixed parameter, useful to compare the results between different weights. We will always assume
\[0 < \alpha < 1 < \beta,\]
studying the effect of a non-constant weight on the solution of (2.3). The assumption \(\alpha < 1 < \beta\) is not restrictive; if we assume \(\beta = 1\), it must be \(p = 1\) a.e. in \(\Omega\), since otherwise we would have \(\int_\Omega p \, dx \, dy < |\Omega|\); similarly, if we consider \(\alpha = 1\).

Moreover, we are interested in designs which are symmetric with respect to the mid-line of the roadway, being \(\ell\) very small with respect to \(\pi\). From a mathematical point of view, this assures two classes of eigenfunctions for the correspondent eigenvalue problem, respectively, even or odd in the \(y\)-variable; we shall clarify this question in Section 2.4.

2.3. Existence and uniqueness result. We denote by \(\| \cdot \|_q\) the norm related to the Lebesgue spaces \(L^q(\Omega)\) with \(1 \leq q \leq \infty\) and we refer to \(q'\) as the conjugate of \(q\), i.e. \(1/q + 1/q' = 1\). We shall often omit the set \(\Omega\), being fixed, in the notation of the functional spaces, e.g. \(V := V(\Omega)\).

We introduce the space
\[H^2_\alpha(\Omega) = \{ u \in H^2(\Omega) : u = 0 \text{ on } \{0, \pi\} \times (-\ell, \ell) \},\]
where to study the weak solution of (2.3). Let us observe that the condition \(u = 0\) has to be meant in a classical sense because \(\Omega \subset \mathbb{R}^2\) and the energy space \(H^2_\alpha\) embeds into continuous functions. Furthermore, \(H^2_\alpha\) is a Hilbert space when endowed with the scalar product
\[(u, v)_{H^2_\alpha} := \int_\Omega [\Delta u \Delta v + (1 - \sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})] \, dx \, dy\]
and associated norm
\[\| u \|_{H^2_\alpha}^2 := (u, u)_{H^2_\alpha},\]
which is equivalent to the usual norm in \(H^2\), see [11, Lemma 4.1]. We denote by \(H^{-2}_\alpha\) the dual space of \(H^2_\alpha\) and \(\langle \cdot, \cdot \rangle\) its dual product. We write the problem (2.3) in weak sense
\[(2.5) \quad (u, v)_{H^2_\alpha} = \langle pf, v \rangle \quad \forall v \in H^2_\alpha.\]
Let us clarify what we mean for the dual product in (2.5) with respect to the choice of \(f\) and \(p\).

If \(f \in \mathcal{F}_V\) with \(q \in (1, \infty]\) and \(p \in \mathcal{P}^{\alpha,\beta}_{L^\infty}\), we write \(\int_\Omega pf \, dx \, dy\) instead of \(\langle pf, v \rangle\).

If \(f \in H^{-2}_\alpha\) we need further regularity on \(p\), e.g. \(p \in \mathcal{P}^{\alpha,\beta}_{H^2}\). We introduce the linear functional
\[T_f : H^2_\alpha(\Omega) \to \mathbb{R} \text{ such that } T_f(v) = \langle f, v \rangle \text{ for all } v \in H^2_\alpha \text{ and we define}\]
\[(2.6) \quad pT_f(v) = \langle pf, v \rangle =: \langle f, pv \rangle = T_f(pv) \quad \forall v \in H^2_\alpha(\Omega).\]
Indeed \(H^2_\alpha(\Omega)\) is a Banach algebra, being the \(H^2\)-norm equivalent to the \(H^2\)-norm, see [11, Theorem 5.23] applied to the Sobolev space \(W^{m,p}(\Omega)\) with \(m = p = 2\) and \(\Omega \subset \mathbb{R}^2\) convex with Lipschitz boundary. Therefore, if \(p \in \mathcal{P}^{\alpha,\beta}_{H^2}\) we get \(K > 0\) such that
\[pv \in H^2_\alpha(\Omega) \quad \| pv \|_{H^2_\alpha} \leq K \| p \|_{H^2} \| v \|_{H^2_\alpha} \quad \forall v \in H^2_\alpha(\Omega).\]
We state

**Proposition 2.1.** Let $f \in \mathcal{F}_V$ and $0 < \alpha < \beta$ if

1. $V = L^q(\Omega)$ with $q \in (1, \infty)$ and $p \in \mathcal{P}^{\alpha,\beta}_{L^\infty}$,
2. $V = H^{-2}_g(\Omega)$ and $p \in \mathcal{P}^{\alpha,\beta}_{H^2}$,

then the problem (2.5) admits a unique weak solution $u \in H^2(\Omega) \subset C^0(\Omega)$.

**Proof.** By [11] we have that the bilinear form $(u,v)_{H^2}$ is continuous and coercive, hence to apply Lax Milgram Theorem we consider the functional $(pf,v)$.

1. If $p \in \mathcal{P}^{\alpha,\beta}_{L^\infty}$ and $f \in \mathcal{F}_V$ with $q \in (1, \infty)$ then $pf \in L^q$; moreover we have $\Omega \subset \mathbb{R}^2$ so that $H^2(\Omega)$ is embedded in $C^0(\Omega)$. Therefore, applying Hölder inequality, we obtain

$$|\langle pf,v \rangle| = \left| \int_\Omega pf \, dx \, dy \right| \leq \|pf\|_q \|v\|_{q'} \leq C \|v\|_{H^2} \quad \forall v \in H^2,$$

so that $(pf,v)$ is a linear and continuous functional.

2. By (2.6) we observe that $pTf(v) = Tf(pv)$ is linear and continuous, indeed we have

$$|\langle Tf(pv) \rangle| = |\langle f,pv \rangle| \leq \|f\|_{H^{-2}} \|pv\|_{H^2} \leq C \|v\|_{H^2} \quad \forall v \in H^2,$$

being $H^2$ a Banach algebra.

The solution $u$ is continuous since the space $H^2(\Omega)$ embeds into $C^0(\Omega)$.

2.4. **Definition of longitudinal and torsional modes.** To tackle (2.3) we need some preliminary information on the associated eigenvalue problem:

$$
\begin{cases}
\Delta^2 u = \lambda p(x,y) u & \text{in } \Omega \\
\alpha u(0,y) = u_{xx}(0,y) = u(\pi,y) = u_{xx}(\pi,y) = 0 & \text{for } y \in (-\ell,\ell) \\
u_{yy}(x,\pm\ell) + \sigma u_{xx}(x,\pm\ell) = u_{yyy}(x,\pm\ell) + (2 - \sigma) u_{xy}(x,\pm\ell) = 0 & \text{for } x \in (0,\pi).
\end{cases}
$$

As in [5, 9], we introduce the subspaces of $H^2$:

$$
H^2 := \{ u \in H^2 : u(x,-y) = u(x,y) \quad \forall (x,y) \in \Omega \},
H_0^2 := \{ u \in H^2 : u(x,-y) = -u(x,y) \quad \forall (x,y) \in \Omega \},
$$

where

$$H^2 \perp H_0^2, \quad H^*_2 = H^2 \oplus H_0^2.
$$

We say that the eigenfunctions in $H^2_2$ are *longitudinal* modes and those in $H^2_0$ are *torsional* modes.

When $p \equiv 1$ the whole spectrum of (2.7) was determined explicitly in [11] and gives two class of eigenfunctions belonging respectively to $H^2_2$ or $H^2_0$. Thanks to the symmetry assumption on $p$ we obtain the same distinction for all the linearly independent eigenfunctions of the weighted eigenvalue problem (2.7). We denote by $\mu_m(p)$ and $\nu_m(p)$ respectively the ordered longitudinal and torsional eigenvalues of (2.7), repeated with their multiplicity; moreover, we denote by $z_m^p(x,y) \in \mathcal{H}^2_2$ and $\theta_m^p(x,y) \in \mathcal{H}^2_0$, respectively, the corresponding (ordered) longitudinal and torsional linear independently eigenfunctions of (2.7). We consider the eigenfunctions normalized in $L^2_p$ ($L^2$-weighted), i.e.

$$
\int_\Omega p(z_m^p)^2 \, dx \, dy = 1 \quad \int_\Omega p(\theta_m^p)^2 \, dx \, dy = 1.
$$

3. **Gap function**

In real structures the most problematic motions are related to the torsional oscillations, i.e. those in which prevail torsional modes. How can we measure the torsional behaviour? By Proposition 2.1, the solution of (2.3) is continuous; hence we define the *gap function*, see also [3],

$$G_{f,p}(x) := u(x,\ell) - u(x,-\ell) \quad \forall x \in [0,\pi],$$
depending on the weight $p$ and on the external load $f$. This function gives for every $x \in [0, \pi]$ the difference between the vertical displacements of the free edges, providing a measure of the torsional response. The maximal gap is given by

\[ G_{f,p}^\infty := \max_{x \in [0,\pi]} |G_{f,p}(x)|. \]

In this way we introduce the map $G_{f,p}^\infty : F_{V} \times P_{W}^{\alpha,\beta} \to [0, +\infty)$ with $(f,p) \mapsto G_{f,p}^\infty$, that we study respectively in the cases

\[ (V, W) = (L^q, L^\infty) \text{ with } q \in (1, \infty], \]

\[ (V, W) = (H^{-2}_s, H^2) \]

for which Proposition 2.1 assures the uniqueness of a solution to (2.3).

Our aim is to find the worst $f \in F_V$, i.e. the forcing term that maximizes $G_{f,p}^\infty$, and, in correspondence, the best weight $p \in P_{W}^{\alpha,\beta}$ that minimizes $G_{f,p}^\infty$. More precisely we want to solve the minmaxmax problem

\[ G^\infty := \min_{p \in P_{W}^{\alpha,\beta}} \max_{f \in F_V} \max_{x \in [0,\pi]} |G_{f,p}(x)|, \]

in the cases (3.2).

In Section 6 we prove the following existence results

**Theorem 3.1.** Given $p \in P_{W}^{\alpha,\beta}$ with $0 < \alpha < \beta$ if

i) $W = L^\infty(\Omega)$ and $f \in F_V$ with $V = L^q(\Omega)$ $q \in (1, \infty]$,

ii) $W = H^2(\Omega)$ and $f \in F_V$ with $V = H^{-2}_s(\Omega)$,

then the problem

\[ G_p^\infty := \max_{f \in F_V} G_{f,p}^\infty \]

admits solution.

**Theorem 3.2.** Given $f \in F_V$ if

i) $V = L^q(\Omega)$ with $q \in (1, \infty]$ and $p \in P_{W}^{\alpha,\beta}$ $(0 < \alpha < \beta)$ with $W = L^\infty(\Omega)$,

ii) $V = H^{-2}_s(\Omega)$ and $p \in P_{W}^{\alpha,\beta}$ $(0 < \alpha < \beta)$ with $W = H^2(\Omega)$,

then the problem

\[ \min_{p \in P_{W}^{\alpha,\beta}} G_p^\infty, \]

admits solution.

The choice of the external force $f$ is not trivial at all. Which kind of $f$ favourites torsional oscillations in the structure? For instance, we may think that the torsional performance of the plate gets worse if we consider $f$ odd with respect to the $y$ variable, e.g. $f \in F_{L^q}$ such that $f(x, -y) = -f(x, y)$ with $q \in (1, \infty]$. This idea is confirmed by [5, Theorem 4.1-4.2] when the weights $p$ are symmetric with respect to $y$ as in our case. This result and others in [2] seem to suggest that odd loads are favourite in attaining the maximum (3.3).

4. THE CHOICE OF THE WEIGHT FUNCTION $p \in P_{L^\infty}^{\alpha,\beta}$

About the choice of the weight function $p \in P_{W}^{\alpha,\beta}$ we are mainly interested in density functions not necessarily continuous, hence we consider $W = L^\infty$; therefore, in the rest of the paper we focus on (3.3)-(3.4) in the case $(V, W) = (L^q, L^\infty)$ with $q \in (1, \infty]$. We refer to some results obtained on the correspondent eigenvalue problem (2.7) presented in [7]. Here the authors find the best rearrangement of materials in $\Omega$ which maximizes the ratio between two
selected eigenvalues of (2.7), considering the optimization problem:

\[(4.1) \quad \mathcal{R} = \sup_{p \in \mathcal{P}_{L^\infty}^{\alpha,\beta}} \frac{\nu(p)}{\mu(p)},\]

where \(\nu(p)\) and \(\mu(p)\) are respectively a torsional and a longitudinal eigenvalue. The direct study of (4.1) is very involved, then there are some theoretical results on the problem of maximization of the first torsional eigenvalue or minimization of the first longitudinal eigenvalue with respect to \(p\); these results give suggestions on (4.1) and support some conjectures also thanks to numerical experiments. More precisely, in [7] the authors proved theoretically that optimal weights \(p(x, y)\) in increasing or reducing the first torsional or longitudinal eigenvalue must be of bang-bang type, i.e.

\[p(x, y) = \alpha \chi_{S}(x, y) + \beta \chi_{\partial S}(x, y) \quad \text{for a.e. } (x, y) \in \Omega,\]

for a suitable set \(S \subset \Omega\), \(\alpha, \beta \in \mathbb{R}^+\) and \(\chi_{S}\) is the characteristic function of \(S \subset \mathbb{R}^2\). In other words, the plate must be composed by two different materials properly located in \(\Omega\); this is useful in engineering terms, since the manufacturing of two materials with constant density is simpler than the assemblage of a material having variable density. On the other hand this produces some mathematical troubles, for instance when we consider as external forcing term \(f\) of a material having variable density. This is a particular case when \(\alpha = \beta = 1\) that corresponds to the homogeneous plate; we do not apply reinforcements, but we consider this case to compare it with the non-homogeneous ones.

ii) \(\hat{p}(x, y)\)

This choice comes out from the study of the problem

\[(4.2) \quad \nu_1^{\alpha,\beta} := \sup_{p \in \mathcal{P}_{L^\infty}^{\alpha,\beta}} \nu_1(p).\]

We call optimal pair for (4.2) a couple \((\hat{p}, \hat{\theta}_1)\) such that \(\hat{p}\) achieves the supremum in (4.2) and \(\hat{\theta}_1\) is an eigenfunction of \(\nu_1(\hat{p})\). In [7] is proved the

**Proposition 4.1.** [7] Problem (4.2) admits an optimal pair \((\hat{p}, \hat{\theta}_1)\) \(\in \mathcal{P}_{L^\infty}^{\alpha,\beta} \times H^2_{\Omega}\). Furthermore, \(\hat{\theta}_1\) and \(\hat{p}\) are related as follows

\[\hat{p}(x, y) = \beta \chi_{\hat{S}}(x, y) + \alpha \chi_{\partial \hat{S}}(x, y) \quad \text{for a.e. } (x, y) \in \Omega,\]

where \(\hat{S} = \{(x, y) \in \Omega : (\hat{\theta}_1)^2(x, y) \leq \hat{t}\}\) for some \(\hat{t} > 0\) such that \(|\hat{S}| = \frac{1}{\beta - \alpha} |\Omega|\).

Since we do not know explicitly \(\hat{\theta}_1\), the function \(\hat{\theta}_1\) is replaced by the torsional eigenfunction \(\theta_1(x, y)\) of (2.7) with \(p \equiv 1\), i.e. an eigenfunction corresponding to \(\nu_1(1)\). This is explicitly known, see [11]; for details on this choice see [7]. Therefore we consider

\[p^*(x, y) := \beta \chi_{S^*}(x, y) + \alpha \chi_{\partial S^*}(x, y) \quad \text{for a.e. } (x, y) \in \Omega,\]

where \(S^* = \{(x, y) \in \Omega : (\theta_1)^2(x, y) \leq t^*\}\) for \(t^* > 0\) such that \(|S^*| = \frac{1}{\beta - \alpha} |\Omega|\).

iii) \(\bar{p}(y)\)

In order to find a reinforce more suitable for practical reproduction, inspired by \(p^*(x, y)\), we consider a weight depending only on \(y\) and concentrated around the mid-line \(y = 0\), i.e.

\[\bar{p}(x, y) = \bar{p}(y) := \beta \chi_{\bar{I}}(y) + \alpha \chi_{(-\bar{t}, \bar{t})}(y) \quad \text{for a.e. } (x, y) \in \Omega,\]

where \(\bar{I} := (-\frac{(\beta - 1)}{\beta - \alpha}, \frac{(\beta - 1)}{\beta - \alpha})\).
iv) \( \overline{p}_i(x), i \in \mathbb{N}^+ \)

The reasons of this choice are quite involved. We give here only the main idea and for details we refer to [7].

For \( i \in \mathbb{N}^+ \), we set the minimum problem

\[
\mu_{i}^{\alpha, \beta} := \inf_{p \in \mathcal{P}_{L_{2}^\infty}} \mu_i(p),
\]

where \( \mu_i(p) \) is the \( i \)-th longitudinal eigenvalue of (2.7). We call optimal pair for (4.3) a couple \((\overline{p}_i, z^{\overline{p}_i})\) such that \( \overline{p}_i \) achieves the infimum in (4.3) and \( z^{\overline{p}_i} \) is an eigenfunction of \( \mu_i(\overline{p}_i) \). In [6] Theorem 3.2 it is proved the

**Proposition 4.2.** [6] Set \( i = 1 \), then problem (4.3) admits an optimal pair \((\overline{p}_1, z^{\overline{p}_1})\) \( \in \mathcal{P}_{L_{2}^\infty}^{\alpha, \beta} \times H^2_{\Omega} \). Furthermore, \( z^{\overline{p}_1} \) and \( \overline{p}_1 \) are related as follows

\[
\overline{p}_1(x, y) = \alpha \chi_{S_1(x, y)} + \beta \chi_{\Omega \setminus S_1(x, y)}
\]

for a.e. \((x, y) \in \Omega\), where \( S_1 = \{(x, y) \in \Omega : (z^{\overline{p}_1})^2(x, y) \leq t_1\} \) for some \( t_1 > 0 \) such that \( |S_1| = \frac{\beta-1}{\beta-\alpha} |\Omega| \).

Things become more involved for higher longitudinal eigenvalues and we do not find an analytical expression as for \( i = 1 \). Focusing on upper bounds for \( \mu_i(p) \), see [7], we propose the following approximated optimal weight for \( \mu_i^{\alpha, \beta} \):

\[
\overline{p}_i(x, y) = \overline{p}_i(x) := \beta \chi_{I_i(x)}(x, y) + \alpha \chi_{(0, \pi) \setminus I_i(x)}(x, y)
\]

for a.e. \((x, y) \in \Omega\), where

\[
I_i := \left( \frac{\pi}{2t_1}(2h-1) - \frac{(1-\alpha)}{i} \frac{\pi}{2t_1}(2h-1) + \frac{(1-\alpha)}{i} \frac{\pi}{2(\beta-\alpha)} \right).
\]

v) \( \overline{p}(x) \)

We consider a weight concentrated near the short edges of the plate:

\[
\overline{p}(x, y) = \alpha \chi_{I}(x) + \beta \chi_{(0, \pi) \setminus I}(x, y)
\]

for a.e. \((x, y) \in \Omega\), where

\[
I := \left( \frac{\pi}{2} - \frac{\pi}{2(\beta-\alpha)} \right).
\]

This weight seems to be simple for practical reproduction and reasonable in order to increase \( R \).

We denote by

\[
\hat{P}_{\alpha, \beta} := \{p \in \mathcal{P}_{L_{2}^\infty}^{\alpha, \beta} : p(x, y) \text{ coincides with 1 or } p^*(x, y) \text{ or } \hat{p}(y) \text{ or } \overline{p}_{10}(x) \text{ or } \overline{p}(x) \quad \forall (x, y) \in \Omega \};
\]

we shall explain in the next section why we are interested for the fourth case to \( \overline{p}_{10}(x) \).

5. \( L^2(\Omega) \) external forcing terms

When \( f \in \mathcal{F}_{L^2} \) it is possible to obtain more information on the solution of (2.5) and, in turn, on the gap function. In this case we expand \( u \) in Fourier series, adopting an orthonormal basis of \( L^p_{\Omega} \) composed by the eigenfunctions of (2.7). In Section 6 we prove the

**Proposition 5.1.** For \( m \in \mathbb{N}^+ \), we denote by \( \nu_m(p) \) and \( \mu_m(p) \) the eigenvalues of (2.7) and, respectively, \( \theta_{m}^p(x, y) \) and \( \zeta_{m}^p(x, y) \) the corresponding normalized eigenfunctions, see (2.8).

If \( f \in \mathcal{F}_{L^2} \) and \( p \in \mathcal{P}_{L_{2}^\infty}^{\alpha, \beta} \) then the unique solution of (2.5) reads

\[
u_m(p) \theta_{m}^p(x, y) + \frac{b_m}{\mu_m(p)} \zeta_{m}^p(x, y)
\]

and

\[
G_{f,p}(x) = 2 \sum_{m=1}^{\infty} \frac{a_m}{\nu_m(p)} \theta_{m}(x, \ell) \quad \forall x \in [0, \pi],
\]
where
\[ a_m := \int_{\Omega} f \theta^p_m \, dx \, dy \quad \text{and} \quad b_m := \int_{\Omega} f \, z^p_m \, dx \, dy. \]

If \( f \in \mathcal{F}_{L^2} \) and \( f(x, -y) = -f(x, y) \) a.e. in \( \Omega \) then \( u(x, y) = \sum_{m=1}^{\infty} \frac{a_m}{\nu_m(p)} \theta^p_m(x, y) \).

We shall consider \( y \)-odd forcing terms; in [2] the authors conjectured as worst forcing term
\[ f_0(x, y) = \begin{cases} 1 & y \in [0, \ell] \\ -1 & y \in [-\ell, 0) \end{cases}. \]

Since \( \|f_0\|_2 = \sqrt{|\Omega|} \) and we are interested in \( f \in \mathcal{F}_{L^2} \), we normalize \( f_0 \), i.e.
\[ f_0(x, y) = \begin{cases} 1/\sqrt{|\Omega|} & y \in [0, \ell] \\ -1/\sqrt{|\Omega|} & y \in [-\ell, 0) \end{cases}. \]

We refer to Table 1 for numerical results about \( f_0 \).

A physical interesting case is when \( f \) is in resonance with the structure, i.e. when \( f \) is a multiple of an eigenfunction of (2.7). The case in which \( f \) is proportional to a longitudinal mode is not interesting from our point of view since the gap function vanishes. Hence we consider \( f \) proportional to the \( j \)-th torsional mode, i.e.
\[ f_j(x, y) = \theta^p_j(x, y); \]

since \( \|f_j\|_2 \neq 1 \), we consider \( \mathcal{F}_j(x, y) = \theta^p_j(x, y)/\|\theta^p_j\|_2 \) so that \( \mathcal{F}_j \in \mathcal{F}_{L^2} \) for all \( j \in \mathbb{N}^+ \). Through Proposition 5.1, we readily obtain
\[ a_m = \begin{cases} 1/\|\theta^p_j\|_2 & m = j \\ 0 & m \neq j \end{cases} \quad u(x, y) = \frac{\theta^p_j(x, y)}{\nu_j(p)\|\theta^p_j\|_2} \quad G_{f_j, p}(x) = 2\frac{\theta^p_j(x, \ell)}{\nu_j(p)\|\theta^p_j\|_2}. \]

We provide now some numerical results obtained considering a narrow plate, as it may be the deck of a suspension bridge, composed by typical materials adopted for these structures, i.e.

\[ (5.3) \quad \ell = \frac{\pi}{150} \quad \sigma = 0.2, \]

for details see [8, 10]. We point out that with these parameters the eigenvalues of the homogeneous plate \( (p \equiv 1) \) are ordered in the following sequence
\[ \mu_1(1) < ... < \mu_{10}(1) < \nu_1(1) < \mu_{11}(1) < ... \]

Hence the longitudinal eigenvalue closest to the first torsional from below is \( \mu_{10}(1) \); for this reason we consider \( p \in \hat{P}_{\alpha, \beta} \) fixing \( i = 10 \) for the fourth reinforce \( p_{10} \) proposed in Section 4.

| \( p \equiv 1 \) | \( p^*(x, y) \) | \( \hat{p}(y) \) | \( \mathcal{P}_{10}(x) \) | \( \mathcal{P}(x) \) |
|---|---|---|---|---|
| \( \nu_1(p) \cdot 10^{-4} \) | 1.09 | 1.98 | 1.75 | 1.09 | 1.56 |
| \( G_{f_0, p}^{\infty} \cdot 10^4 \) | 9.32 | 6.09 | 6.99 | 9.32 | 7.00 |
| \( G_{f_1, p}^{\infty} \cdot 10^4 \) | 1.23-10 | 6.74 | 7.71 | 1.23-10 | 8.21 |
| \( G_{f_2, p}^{\infty} \cdot 10^4 \) | 3.08 | 1.93 | 1.93 | 3.11 | 3.38 |

Table 1. The first torsional weighted eigenvalue \( \nu_1(p) \) and \( G_{f_j, p}^{\infty} \) defined in (3.1), assuming (5.3)-(5.4) and \( N = 30 \).
the values $\alpha < 1 < \beta$ related to the family $\mathcal{P}^{\alpha,\beta}_{L^\infty}$, for the applicative purpose we may strengthen the plate with steel and we may consider the other material composed by a mixture of steel and concrete; therefore, the denser material has approximately triple density with respect to the weaken. Thus we assume

\begin{equation}
\alpha = 0.5 \quad \beta = 1.5.
\end{equation}

The numerical computation of the gap function in $\{5.2\}$ is obtained truncing the Fourier series at a certain $N \geq 1$, integer; we compute the weighted eigenvalues and eigenfunctions of $\{2.7\}$, exploiting the explicit information we have in the case $p \equiv 1$, see $[11]$, and adopting the same numerical procedure described in $[7]$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Plots of the gap functions $\mathcal{G}_{J_0,p}(x)$ and $\mathcal{G}_{J_1,p}(x)$ for $x \in [0,\pi]$, varying $p$, assuming $\{5.3\}$-$\{5.4\}$ and $N = 30$.}
\end{figure}

In Table $1$ we present the maximum values assumed by the gap function with respect to the choice of $f \in \mathcal{F}_{L^2}$ and $p \in \mathcal{P}_{\alpha,\beta}$; as one can expect, for $f = J_0$ the absolute maximum is always attained in $x = \pi/2$, while for $f = J_j$ is assumed where $\sin(jx)$ has stationary points; indeed, $\theta_p^j(x,\pm\ell)$ is qualitatively similar to $\pm A \sin(jx)$ ($A \in \mathbb{R}^+$, $j \in \mathbb{N}^+$), see Figure $1$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Plots of $j \mapsto \mathcal{G}_{J_j,1}^\infty$ and $j \mapsto \mathcal{G}_{J_j,p^*}^\infty$, assuming $\{5.3\}$-$\{5.4\}$ and $N = 30$.}
\end{figure}
In Figure 2 we plot \(j \mapsto G_{f,p}^\infty\) when the plate is homogeneous and \(p = p^*\); through this result we conjecture that the gap function reduces in amplitude when \(\overline{f}_j\) is in resonance with higher torsional modes.

The worst situation among the tested external forces appears when \(f = \overline{f}_1\) followed by \(f = \overline{f}_0\); this suggests that the forces \(f \in \mathcal{F}_{L^2}\) which maintain the same (and opposite) sign along the two free edges of the plate seem to be the candidate solutions of (3.3). In any case the possible optimal reinforce of (3.3) is \(p^*(x,y)\) followed by \(\hat{p}(y)\), see Figure 1; this agrees with the results obtained in [7], in which the problem is dealt with a different point of view, based on the maximization of the eigenvalues ratio \(R\) in (4.1). We point out that reinforces reducing the longitudinal eigenvalues as \(\overline{p}_{10}(x)\), are not so useful in lowering the gap function for the external forcing considered; indeed, in Figure 1 the corresponding gap function is very close to the gap function of the homogeneous plate.

6. Proofs

6.1. Proof of Theorem 3.1-i). Fixed \(p \in \mathcal{P}^{\alpha,\beta}_{L^\infty}\), we proof the continuity of the map \(f \mapsto G_{f,p}^\infty\) in

**Lemma 6.1.** The map \(G_{f,p}^\infty: L^q \to [0, +\infty)\) with \(q \in (1, \infty]\) is continuous when \(L^q\) is endowed with the weak* topology.

**Proof.** We recall that the weak* topology related to \(L^q\) with \(q \in (1, \infty]\) coincides with the weak topology except when \(q = \infty\).

Let \(\{f_n\}_n \subset L^q\) be such that \(f_n \xrightarrow{\ast} f\) in \(L^q\) for \(n \to +\infty\). Denoting by \(u_n\) the solution of (2.5) corresponding to \(f_n\), we have

\[
(u_n, v)_{H^2_q} = \int_{\Omega} f_n pv \, dx \, dy \quad \forall v \in H^2_q;
\]

since \(f_n \xrightarrow{\ast} f\) in \(L^q\), its \(L^q\) norm is bounded then the above equality with \(v = u_n \in H^2_q(\Omega) \subset C^0(\overline{\Omega})\) gives

\[
\|u_n\|_{H^2_q}^2 = \left| \int_{\Omega} f_n pv u_n \, dx \, dy \right| \leq \beta \int_{\Omega} |f_n u_n| \, dx \, dy \leq \beta \|f_n\|_q \|u_n\|_{H^2_q} \leq C \|f_n\|_q \|u_n\|_{H^2_q} \leq \overline{C} \|u_n\|_{H^2_q},
\]

so that \(\|u_n\|_{H^2_q} \leq \overline{C}\) for some \(\overline{C} > 0\); thus we obtain, up to a subsequence, \(u_n \rightharpoonup \overline{u}\) in \(H^2_q\). Being \(pv \in L^q\), we pass to the limit (6.1)

\[
(\overline{u}, v)_{H^2_q} = \int_{\Omega} f \overline{p} v \, dx \, dy \quad \forall v \in H^2_q,
\]

obtaining by the uniqueness \(\overline{u} = u\).

The embedding \(H^2_q(\Omega) \subset C^0(\overline{\Omega})\) is compact, therefore \(u_n \rightharpoonup u\) in \(C^0(\overline{\Omega})\), implying that the gap function \(G_{f_n,p}(x)\) converges uniformly to \(G_{f,p}\) as \(n \to +\infty\) for all \(x \in [0, \pi]\). Therefore \(G_{f_n,p} \to G_{f,p}^\infty\) as \(n \to +\infty\). \(\square\)

**Proof of Theorem 3.1-i) completed.** Let \(p \in \mathcal{P}^{\alpha,\beta}_{L^\infty}\) fixed and \(\{f_n\}_n \subset \mathcal{F}_{L^q}\) a maximizing sequence for (3.3): since \(\|f_n\|_q = 1\), we have, up to a subsequence, \(f_n \rightharpoonup \overline{f}\) in \(L^q\) if \(q \in (1, \infty]\) and \(f_n \xrightarrow{\ast} \overline{f}\) if \(q = \infty\). By the lower semi continuity of the norms we have \(\|\overline{f}\|_q \leq \|f_n\|_q = 1\). Through Lemma 6.1 we obtain

\[
\max_{f \in \mathcal{F}_{L^q}} G_{f,p}^\infty = G_{\overline{f},p}^\infty;
\]

we prove that \(\|\overline{f}\|_q = 1\). For contradiction we suppose \(\|\overline{f}\|_q < 1\); hence we set \(\hat{f} = \overline{f}/\|\overline{f}\|_q\) and by linearity we obtain

\[
G_{\hat{f},p}^\infty = G_{\overline{f},p}^\infty /\|\overline{f}\|_q > G_{\overline{f},p}^\infty. \quad \square
\]
6.2. Proof of Theorem 3.1-(ii). We need the following

**Lemma 6.2.** The map \( G_{f,p}^\infty : H_s^{-2} \to [0, +\infty) \) is continuous when \( H_s^{-2} \) is endowed with the weak* topology.

**Proof.** We follow the lines of the proof of Lemma 6.1 we consider \( \{f_n\}_n \subset H_s^{-2} \) be such that \( f_n \xrightarrow{\star} f \) in \( H_s^{-2} \) for \( n \to +\infty \).

We denote by \( u_n \) the solution of (2.5) corresponding to \( f_n \), hence

\[
(6.2) \quad (u_n, v)_{H_s^2} = \langle pf_n, v \rangle \quad \forall v \in H_s^2;
\]

the above equality with \( v = u_n \) gives

\[
\|u_n\|_{H_s^2}^2 = \|pf_n, u_n\| = \|f_n, pu_n\| \leq \|f_n\|_{H_s^{-2}}\|pu_n\|_{H_s^2} \leq C\|u_n\|_{H_s^2},
\]

since \( H_s^2 \) is a Banach algebra and \( f_n \xrightarrow{\star} f \) in \( H_s^{-2} \). Therefore \( \|u_n\|_{H_s^2} \leq C \) for some \( C > 0 \); thus we obtain, up to a subsequence, \( u_n \rightharpoonup \bar{u} \) in \( H_s^2 \). Being \( pv \in H_s^2 \), we pass to the limit (6.2)

\[
(u, v)_{H_s^2} = \langle f, pv \rangle \quad \forall v \in H_s^2,
\]

obtaining by the uniqueness \( \bar{u} \equiv u \).

We use the compact embedding \( H_s^2(\Omega) \subset C^0(\overline{\Omega}) \) to obtain the thesis as in proof of Lemma 6.1 \( \square \)

**Proof of Theorem 3.1-(ii) completed.** Let \( p \in \mathcal{P}_{H_s^2}^{\alpha,\beta} \) fixed and \( \{f_n\}_n \subset \mathcal{F}_{H_s^{-2}} \) a maximizing sequence for (3.3); being \( \|f_n\|_{H_s^{-2}} = 1 \), we get, up to a subsequence, \( f_n \xrightarrow{\star} \bar{f} \) in \( H_s^{-2} \). Thanks to Lemma 6.2 we obtain

\[
\max_{f \in \mathcal{F}_{H_s^{-2}}} G_{f,p}^\infty = G_{\bar{f},p}^\infty;
\]

we prove that \( \|\bar{f}\|_{H_s^{-2}} = 1 \) by contradiction as in proof of Theorem 3.1-(i). \( \square \)

6.3. Proof of Theorem 3.2-(i). Fixed \( f \in \mathcal{F}_{L_s^r} \), we proof the continuity of the map \( p \mapsto G_p^\infty \) in

**Lemma 6.3.** The map \( G_p^\infty : \mathcal{P}_{L_s^\infty}^{\alpha,\beta} \to [0, +\infty) \) is continuous when \( L_s^\infty \) is endowed with the weak* topology.

**Proof.** Let \( \{p_n\}_n \subset \mathcal{P}_{L_s^\infty}^{\alpha,\beta} \) be such that \( p_n \rightharpoonup p \) in \( L_s^\infty \) for \( n \to +\infty \); in \([7]\) Lemma 5.2 is proved that \( \mathcal{P}_{L_s^\infty}^{\alpha,\beta} \) is compact for the \( L_s^\infty \) weak* topology, then \( p \in \mathcal{P}_{L_s^\infty}^{\alpha,\beta} \).

We denote by \( u_n \) the solution of (2.5) corresponding to \( p_n \) and we get

\[
(6.3) \quad (u_n, v)_{H_s^2} = \int_\Omega p f v \, dx \, dy \quad \forall v \in H_s^2;
\]

since \( f \in L^q \) with \( q \in (1, \infty] \), then \( fv \in L^1 \); moreover \( \|p_n\|_\infty \leq \beta \) so that the above equality with \( v = u_n \in H_s^2(\Omega) \subset C^0(\overline{\Omega}) \) gives

\[
\|u_n\|_{H_s^2}^2 = \int_\Omega p f u_n \, dx \, dy \leq \|p_n\|_\infty \|f u_n\|_1 \leq \beta \|f\|_q \|u_n\|_q' \leq C \|u_n\|_{H_s^2},
\]

implying \( \|u_n\|_{H_s^2} \leq C \) for some \( C > 0 \); thus we obtain, up to a subsequence, \( u_n \rightharpoonup \bar{u} \) in \( H_s^2 \) and we pass to the limit (6.3)

\[
(\bar{u}, v)_{H_s^2} = \int_\Omega p f v \, dx \, dy \quad \forall v \in H_s^2,
\]

obtaining by the uniqueness \( \bar{u} \equiv u \).

As in Lemma 6.1 we use the compact embedding \( H_s^2(\Omega) \subset C^0(\overline{\Omega}) \), implying that the gap function \( G_{p_n} \) converges uniformly to \( G_p \) as \( n \to +\infty \) for all \( x \in [0, \pi] \). \( \square \)

**Proof of Theorem 3.2-(i) completed.** By Lemma 6.3 we have that \( p \mapsto G_p^\infty \) is continuous on \( \mathcal{P}_{L_s^\infty}^{\alpha,\beta} \) with respect to the \( L_s^\infty \) weak* topology. Moreover by \([7]\) Lemma 5.2 the set \( \mathcal{P}_{L_s^\infty}^{\alpha,\beta} \) is compact for the \( L_s^\infty \) weak* topology; this readily implies the existence of the minimum (3.4). \( \square \)
6.4. Proof of Theorem 3.2-ii). We state the preliminary

**Lemma 6.4.** The set $\mathcal{P}^{\alpha,\beta}_{H^2}$ with $0 < \alpha < \beta$ is compact for the $H^2$ weak topology.

**Proof.** Let $\{p_n\}_n \subset \mathcal{P}^{\alpha,\beta}_{H^2}$, then by definition $\|p_n\|_{H^2} \leq (\beta + \alpha)\sqrt{|\Omega|}$, hence, up to a subsequence, we have $p_n \rightarrow p$ in $H^2$ (as $n \rightarrow +\infty$) for some $p \in H^2$ and
\[
\|p\|_{H^2} \leq \liminf_{n \rightarrow +\infty} \|p_n\|_{H^2} \leq (\beta + \alpha)\sqrt{|\Omega|};
\]
due to the compact embedding $H^2(\Omega) \subset C^0(\Omega)$, we obtain $p_n \rightarrow p$ uniformly as $n \rightarrow \infty$. This implies $\alpha \leq p \leq \beta$ and $p(x, -y) = p(x, y)$ for all $(x, y) \in \Omega$; moreover, passing the limit under the integral, we obtain $|\Omega| = \int_{\Omega} p_n \, dx \, dy \rightarrow \int_{\Omega} p \, dx \, dy$, implying $\int_{\Omega} p \, dx \, dy = |\Omega|$.

Therefore the limit point $\bar{p} \in \mathcal{P}^{\alpha,\beta}_{H^2}$ and $\mathcal{P}^{\alpha,\beta}_{H^2}$ is compact for the $H^2$ weak topology. □

Fixed $f \in \mathcal{F}_{H^{-2}}$, we prove the continuity of the map $p \mapsto \mathcal{G}^\infty_p$ in

**Lemma 6.5.** The map $\mathcal{G}^\infty_p : \mathcal{P}^{\alpha,\beta}_{H^2} \rightarrow [0, +\infty)$ is continuous when $H^2$ is endowed with the weak topology.

**Proof.** We follow the lines of the proof of Lemma 6.3. Let $\{p_n\}_n \subset \mathcal{P}^{\alpha,\beta}_{H^2}$ be such that $p_n \rightarrow p$ in $H^2$ for $n \rightarrow +\infty$; through Lemma 6.4 we have $p \in \mathcal{P}^{\alpha,\beta}_{H^2}$.

We denote by $u_n$ the solution of (2.5) corresponding to $p_n$ and we get
\[
(6.4) \quad (u_n, v)_{H^2} = \langle p_n f, v \rangle \quad \forall v \in H^2_*;
\]
the above equality with $v = u_n$ gives
\[
\|u_n\|_{H^2}^2 = |\langle f, p_n u_n \rangle| \leq \|f\|_{H^{-2}} \|p_n u_n\|_{H^2} \leq C \|u_n\|_{H^2},
\]
in which we use that $H^2$ is a Banach algebra, (2.6) and $p_n \rightarrow p$ in $H^2$. Hence we have $\|u_n\|_{H^2} \leq C$ for some $C > 0$, we pass to the limit (6.4) and we conclude as in final part of the proof of lemma 6.3. □

**Proof of Theorem 3.2-ii) completed.** By Lemma 6.5 we have that $p \mapsto \mathcal{G}^\infty_p$ is continuous on $\mathcal{P}^{\alpha,\beta}_{H^2}$ with respect to the $H^2$ weak topology. By Lemma 6.4 the set $\mathcal{P}^{\alpha,\beta}_{H^2}$ is compact for the $H^2$ weak topology, implying the existence of the minimum (3.4). □

6.5. Proof of Proposition 5.1. We choose $\{z^p_m, \theta^p_m\}_{m=1}^\infty$ as orthonormal base of $L^2_p$ (and orthogonal base of $H^2_*$). Since $f \in L^2 \subset L^2_p$ we expand it in Fourier series
\[
f(x, y) = \sum_{m=1}^\infty \left[ a_m \theta^p_m(x, y) + b_m z^p_m(x, y) \right],
\]
with $a_m, b_m \in \mathbb{R}$ defined as
\[
a_m := \int_\Omega f \, \theta^p_m \, dx \, dy \quad b_m := \int_\Omega f \, z^p_m \, dx \, dy.
\]
We write
\[
u(x, y) = \sum_{m=1}^\infty \left[ \alpha_m \theta^p_m(x, y) + \beta_m z^p_m(x, y) \right],
\]
where $\alpha_m, \beta_m \in \mathbb{R}$ are defined as
\[
\alpha_m := \int_\Omega \nu \, \theta^p_m \, dx \, dy \quad \beta_m := \int_\Omega \nu \, z^p_m \, dx \, dy.
\]
For all $m \in \mathbb{N}^+$, $z^p_m$ and $\theta^p_m$ solve:
\[
(6.5) \quad (z^p_m, v)_{H^2} = \mu_m(p) \langle z^p_m, v \rangle_{L^2} \quad \forall v \in H^2_*
\]
\[
(\theta^p_m, v)_{H^2} = \nu_m(p) \langle \theta^p_m, v \rangle_{L^2} \quad \forall v \in H^2_*.
\]
Then considering (2.5) with \( v = \theta^p_m, v = z^p_m \) and putting \( v = u \) in (6.5) we have

\[
\alpha_m = \frac{a_m}{\nu_m(p)} \quad \beta_m = \frac{b_m}{\mu_m(p)}
\]

and (5.1).

Now we verify that \( u(x, y) \) written in Fourier series as (5.1) belongs to \( H^2_{2} \). Through (6.5) we obtain that \( \left\{ \frac{\theta^p_m}{\sqrt{\nu_m(p)}}, \frac{z^p_m}{\sqrt{\mu_m(p)}} \right\}_{m=1}^{\infty} \) is an orthonormal base in \( H^2_{2} \); therefore, if \( \left\{ \frac{a_m}{\sqrt{\nu_m(p)}}, \frac{b_m}{\sqrt{\mu_m(p)}} \right\}_{m} \subset \ell^2(\mathbb{N}^+) \) we infer \( u \in H^2_{2} \). We recall the variational representation of the eigenvalues of (2.7): for every \( m \in \mathbb{N}^+ \) it holds

\[
\lambda_m(p) = \inf_{W_m \subset H^2_{2}} \sup_{\dim W_m = m} \frac{\|u\|^2_{H^2_{2}}}{\|\sqrt{\nu_m(p)} u\|^2_{2}},
\]

implying the stability inequality

\[
\frac{\lambda_m(1)}{\beta} \leq \lambda_m(p) \leq \frac{\lambda_m(1)}{\alpha},
\]

for every \( m \in \mathbb{N}^+ \). In [11, Theorem 7.6] the authors find explicit bounds for the eigenvalues when the plate is homogeneous (\( p \equiv 1 \)); in general it holds \( \lambda_m(1) > (1 - \sigma)^2 m^4 \), where \( \sigma \) is the Poisson ratio, see (2.2). Then we obtain

\[
\lambda_m(p) \geq \frac{\lambda_m(1)}{\beta} > \frac{(1 - \sigma)^2 m^4}{\beta}
\]

so that

\[
\frac{|a_m|}{\sqrt{\nu_m(p)}} \leq \frac{\beta \|f\|_2 \|\sqrt{\nu_m(p)}\|_2}{(1 - \sigma)m^2} = \frac{\beta}{(1 - \sigma)m^2} \quad \frac{|b_m|}{\sqrt{\mu_m(p)}} \leq \frac{\beta}{(1 - \sigma)m^2}
\]

and

\[
\sum_{m=1}^{\infty} \frac{|a_m|^2}{\nu_m(p)} + \frac{|b_m|^2}{\mu_m(p)} \leq \frac{2\beta^2}{(1 - \sigma)^2} \sum_{m=1}^{\infty} \frac{1}{m^4} < \infty.
\]

Through (5.1) we get

\[
\mathcal{G}_{f,p}(x) = 2 \sum_{m=1}^{\infty} \frac{a_m}{\nu_m(p)} \theta^p_m(x, \ell) \quad \forall x \in [0, \pi],
\]

since \( z^p_m(x, y) \) is \( y \)-even.

If \( f \) is \( y \)-odd then \( b_m = 0 \).

\[ \square \]

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