COMPOSITION
OF ORDINARY GENERATING
FUNCTIONS

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Abstract
A solution is proposed for the problem of composition of ordinary generating functions. A new class of functions that provides a composition of ordinary generating functions is introduced; main theorems are presented; compositae are written for polynomials, trigonometric and hyperbolic functions, exponential and log functions. It is shown that the composition holds true for many integer sequences.

Abstract A solution is proposed for the problem of composition of ordinary generating functions. A new class of functions that provides a composition of ordinary generating functions is introduced; main theorems are presented; compositae are written for polynomials, trigonometric and hyperbolic functions, exponential and log functions. It is shown that the composition holds true for many integer sequences.

1 Introduction
Generating functions are an efficient tool of solving mathematical problems. Given the ordinary generating functions $F(x) = \sum_{n \geq 1} f(n)x^n$ and $R(x) = \sum_{n \geq 0} r(n)x^n$, the operation of composition of generating functions $A(x) = R(F(x))$ is defined correctly. [3, 1, 6, 2]. However, coefficients of the composition of generating functions are difficult to find. Stanley [3] came close to the solution of the problem and proposed a formula for the composition of exponential generating functions based on ordered partitions of a finite set. Let us show that the basis for the composition of ordinary generating functions is ordered partitions of a positive integer $n$ and put forward basic formulae for the coefficients of the composition of ordinary generating functions. For this purpose, we introduce several definitions.
Definition 1. An ordinary generating function $F(x)$ is a series that belongs to the ring of formal power series in one variable $K[[x]]$:

$$F(x) = \sum_{n \geq 0} f(n)x^n,$$

where $f(n) : P \to K$, $P$ is a set of nonnegative numbers; $K$ is a commutative field.

Further we consider only ordinary generating functions. The known generating functions are denoted as $F(x)$, $R(x)$, $G(x)$, and the desired generating function as $A(x)$.

Definition 2. An ordered partition (composition) of a positive integer $n$ is an ordered sequence of positive integers $\lambda_i$ such that

$$\sum_{i=1}^{k} \lambda_i = n,$$

where $\lambda_i \geq 1$ and $k = \overline{1, n}$ are parts of the ordered partition.

$C_n$ is a set of all ordered partitions of $n$.

$\pi_k \in C_n$ is an ordered partition of $C_n$ with $k$ parts.

The ordered partitions of $n$ have been much studied [4, 5].

2 Compositae and their properties

Let there be functions $f(n)$ and $r(n)$ and their generating functions $F(x) = \sum_{n \geq 1} f(n)x^n$, $R(x) = \sum_{n \geq 0} r(n)x^n$. Then, calculating the composition of the generating functions $A(x) = R(F(x))$ requires [2]

$$[F(x)]^k = \sum_{n \geq k} \sum_{\lambda_1 + \lambda_2 + \ldots + \lambda_k = n} f(\lambda_1)f(\lambda_2)\ldots f(\lambda_k)x^n. \quad (1)$$

Hence it follows that for the function $a(n)$ of the composition of generating functions with $n > 0$, the formula

$$a(0) = r(0),\quad a(n) = \sum_{k=1}^{n} \left[ \sum_{\lambda_1 + \lambda_2 + \ldots + \lambda_k = n} f(\lambda_1)f(\lambda_2)\ldots f(\lambda_k) \right] r(k) \quad (2)$$

holds true. Further the composition of generating functions is written implying that $a(0) = r(0)$.

Remark 3. It should be noted that the summation in formulae (1),(2) is over all ordered partitions of $n$ that have exactly $k$ parts, because $\{\lambda_1 + \lambda_2 + \ldots + \lambda_k = n\}$, $\lambda_i > 0$, $i = \overline{1, k}$ (further we use the reduction $\pi_k \in C_n$).
Thus, the ordered partitions of \( n \) are the basis for calculation of the composition of generating functions.

Let us consider the following example. Assume that \( f(0) = 0, f(n) = 1 \) for all \( n > 0 \). This function is defined by the generating function \( F(x) = \frac{x}{1-x} \). Then, the expression

\[
\sum_{\pi_k \in C_n} f(\lambda_1) f(\lambda_2) \ldots f(\lambda_k)
\]

gives the number of ordered partitions of \( n \) with exactly \( k \) parts; this number is equal to \( \binom{n-1}{k-1} \) [4]. Thus,

\[
\sum_{\pi_k \in C_n} f(\lambda_1) f(\lambda_2) \ldots f(\lambda_k) = \binom{n-1}{k-1}.
\]

Hence it follows that the formula valid for any generating function \( R(x) = \sum_{n \geq 0} r(n)x^n \) and \( A(x) = R \left( \frac{x}{1-x} \right) \) is

\[
a(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} r(k).
\]

**Example 4.** For \( R(x) = \frac{x}{1-x} \), we have the composition \( A(x) = \frac{x}{1-2x} \) and

\[
a(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1}.
\]

Thus, we calculate the total number of ordered partitions of \( n \).

**Example 5.** We have \( R(x) = e^x \), then for the composition \( A(x) = e^{\frac{x}{1-x}} \) we can write

\[
a(n) = \sum_{k=1}^{n} \left( \frac{n-1}{k-1} \right) \frac{1}{k!}
\]

(see A000262 formula Herbert S. Wilf).

**Example 6.** We have \( R(x) = \frac{x}{1-x-x^2} \), then for the composition \( A(x) = R \left( \frac{x}{1-x} \right) \) we can write

\[
a(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} F(k),
\]

where \( F(k) \) is the Fibonacci numbers (see A001519, formula Benoit Cloitre).

**Definition 7.** A composita of the generating function \( F(x) = \sum_{n>0} f(n)x^n \) is the function

\[
F^\Delta(n, k) = \sum_{\pi_k \in C_n} f(\lambda_1) f(\lambda_2) \ldots f(\lambda_k).
\]
Calculation of $F^\Delta(n, k)$ is of prime importance to obtain a composition of generating functions, because from formula (2) it follows that the formula valid for the composition $A(x) = R(F(x))$ is

$$a(n) = \sum_{k=1}^{n} F^\Delta(n, k)r(k).$$

The basis for the derivation of a composita is calculation of the ordered partition $\pi_k$ of $C_n$. From formula (1) it follows that the generating function of the composita is equal to

$$[F(x)]^k = \sum_{n \geq k} F^\Delta(n, k)x^n.$$

For $F(x)$, the condition $f(0) = 0$ holds true, and hence numbering for the composita begins with $k = 1, n = 1$. For $k = 1, F^\Delta(n, k) = f(n)$. For $k > n, F^\Delta(n, k)$ is equal to zero. This statement stems from the fact that there is no ordered partition of $n$ in which the number of parts is larger than $n$.

The above example demonstrates that the Pascal triangle is a composita for the generating function $x_1 - x$ and deriving the composition $A(x) = R\left(x_1 - x\right)$ requires the use of

$$F^\Delta(n, k) = \binom{n-1}{k-1}.$$

Let us derive a recurrence formula for the composita of a generating function.

**Theorem 8.** For the composita $F^\Delta(n, k)$ of the generating function $F(x) = \sum_{n>0} f(n)x^n$, the following relation holds true:

$$F^\Delta(n, k) = \begin{cases} 
    f(n), & k = 1, \\
    [f(1)]^n, & k = n, \\
    \sum_{i=0}^{n-k} f(i + 1)F^\Delta(n - i - 1, k - 1) & k < n.
\end{cases}$$

**Proof.** Let us derive a recurrence formula for the $c_{n,k}$ number of ordered partitions of $n$ that have exactly $k$ parts. Let us introduce the operation $\text{pos}[\lambda^*, \pi_k]$ of adjunction of the new part $\lambda^*$ on the left to a certain ordered partition $\pi_k \in C_n$ providing that $\lambda^* > 0$. From the ordered partition $\pi_k \in C_n$ this operation obtains an ordered partition $\pi_{k+1} \in C_{\lambda^*+n}$. Let us extend this operation to sets. Assume that $C_{n,k} = \{\pi_k | \pi_k \in C_n\}$, then the set $\hat{C} = \text{pos}[\lambda^*, C_{n,k}]$ is a subset $C_{\lambda^*+n,k+1}$. Thus, we can write

$$C_{n,k} = \text{pos}[1, C_{n-1,k-1}] \cup \text{pos}[2, C_{n-1,k-1}] \cup \ldots \cup \text{pos}[n - k - 1, C_{k-1,k-1}].$$

In this case, the condition

$$\text{pos}[i, C_{n-i,k-1}] \cap \text{pos}[j, C_{n-j,k-1}] = \emptyset$$

is fulfilled for all $i \neq j$, because the first parts of the ordered partitions $\pi_k$ do not coincide. Hence,

$$c_{n,k} = \sum_{i=0}^{n-k} c_{n-i-1,k-1},$$

Theorem 8. For the composita $F^\Delta(n, k)$ of the generating function $F(x) = \sum_{n>0} f(n)x^n$, the following relation holds true:

$$F^\Delta(n, k) = \begin{cases} 
    f(n), & k = 1, \\
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$$C_{n,k} = \text{pos}[1, C_{n-1,k-1}] \cup \text{pos}[2, C_{n-1,k-1}] \cup \ldots \cup \text{pos}[n - k - 1, C_{k-1,k-1}].$$

In this case, the condition

$$\text{pos}[i, C_{n-i,k-1}] \cap \text{pos}[j, C_{n-j,k-1}] = \emptyset$$

is fulfilled for all $i \neq j$, because the first parts of the ordered partitions $\pi_k$ do not coincide. Hence,

$$c_{n,k} = \sum_{i=0}^{n-k} c_{n-i-1,k-1},$$

4
and $c_{k,k} = 1$ because we have the only ordered partition $\pi_k = \{1 + 1 + \ldots + 1 = n\}$, and $c_{n,1} = 1$ because $\pi_1 = \{n = n\}$.

Let us now consider expression (3). Using expression (6), we can write

$$\begin{align*}
F^\Delta(n, k) &= f(1)F^\Delta(n-1, k-1) + f(2)F^\Delta(n-2, k-1) + \ldots + \\
&+ f(n-k+1)F^\Delta(k-1, k-1).
\end{align*}$$

The set $C_{n,n}$ consists of the only ordered partition $\{1 + 1 + \ldots + 1\}$, and then $F^\Delta_{n,n} = [f(1)]^n$; the set $C_{n,1}$ consists of $\{n\}$, and then $F^\Delta_{n,1} = f(n)$. Thus, the theorem is proved.

Consideration of formula (4) allows the conclusion that the composita does not depend on $R(x)$ and characterizes the generating function $F(x)$. In tabular form, the composita is represented as

$$\begin{array}{cccccccc}
& & F^\Delta_{1,1} & & F^\Delta_{2,1} & & F^\Delta_{2,2} & \\
& F^\Delta_{3,1} & & F^\Delta_{3,2} & & F^\Delta_{3,3} & \\
F^\Delta_{4,1} & & F^\Delta_{4,2} & & F^\Delta_{4,3} & & F^\Delta_{4,4} & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
F^\Delta_{n,1} & F^\Delta_{n,2} & \ldots & \ldots & F^\Delta_{n,n-1} & F^\Delta_{n,n} & \\
\end{array}$$

or, knowing that $F^\Delta_{1,n} = f(n), F^\Delta_{n,n} = [f(1)]^n$, as

$$\begin{array}{cccccccc}
f(1) & f(2) & f^2(1) & f^3(1) & f^4(1) & \ldots & \ldots & f^n(1) \\
f(3) & F^\Delta_{3,2} & F^\Delta_{3,3} & \ldots & \ldots & \ldots & \ldots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
f(n) & F^\Delta_{n,2} & \ldots & \ldots & F^\Delta_{n,n-1} & f^n(1) & \\
\end{array}$$

Below are the terms of the composite of the generating function $F(x) = \frac{x}{1-x}$:

$$\begin{array}{cccccccc}
1 & & & & & & & \\
1 & 1 & & & & & & \\
1 & 2 & 1 & & & & & \\
1 & 3 & 3 & 1 & & & & \\
1 & 4 & 6 & 4 & 1 & & & \\
1 & 5 & 10 & 10 & 5 & 1 & & \\
\end{array}$$

**Theorem 9.** For a given ordinary generating function $F(x) = \sum_{n \geq 1} f(n)x^n$, the composita $F^\Delta(n, k)$ always exists and is unique.

*Proof.* Without proof. \(\square\)
3 Calculation of compositae

Calculation of compositae is based on derivation of the generating function of a composita

\[ [A(x)]^k = \sum_{n \geq k} A^\Delta(n, k)x^n \]

and operation on them.

**Theorem 10.** Let there be a generating function \( F(x) = \sum_{n \geq 0} f(n)x^n \), its composita \( F^\Delta(n, k) \), and a constant \( \alpha \). Then, the generating function \( A(x) = \alpha F(x) \) has the composita

\[ A^\Delta(n, k) = \alpha^k F^\Delta(n, k). \]

**Proof.**

\[ [A(x)]^k = [\alpha F(x)]^k = \alpha^k [F(x)]^k. \]

\[ \square \]

**Theorem 11.** Let there be a generating function \( F(x) = \sum_{n \geq 0} f(n)x^n \), its composita \( F^\Delta(n, k) \), and a constant \( \alpha \). Then, the generating function \( A(x) = F(\alpha x) \) has the composita

\[ A^\Delta(n, k) = \alpha^n F^\Delta(n, k). \]

**Proof.** By definition, we have

\[ A^\Delta(n, k) = \sum_{\pi_k \in C_n} \alpha^{\lambda_1}f(\lambda_1)\alpha^{\lambda_2}f(\lambda_2)\ldots\alpha^{\lambda_k}f(\lambda_k) = \]

\[ = \alpha^n \sum_{\pi_k \in C_n} f(\lambda_1)f(\lambda_2)\ldots f(\lambda_k) = \alpha^n F^\Delta(n, k). \]

\[ \square \]

**Theorem 12.** Let there be a generating function \( F(x) = \sum_{n \geq 0} f(n)x^n \), its composita \( F^\Delta(n, k) \), a generating function \( B(x) = \sum_{n \geq 0} b(n)x^n \) and \( [B(x)]^k = \sum_{n \geq 0} B(n, k)x^n \). Then, the generating function \( A(x) = F(x)B(x) \) has the composita

\[ A^\Delta(n, k) = \sum_{i=k}^n F^\Delta(i, k)B(n - i, k). \]

**Proof.** Because \( a(0) = f(0)b(0) = 0 \), \( A(x) \) has the composita \( A^\Delta(n, k) \). On the other hand,

\[ [A(x)]^k = [F(x)]^k[B(x)]^k. \]

This, reasoning from the rule of product of generating functions, gives

\[ A^\Delta(n, k) = \sum_{i=k}^n F^\Delta(i, k)B(n - i, k). \]

\[ \square \]
For \( B(x) b(0) = 0 \), the formula has the form:

\[ A^\Delta(n, k) = \sum_{i=k}^{n-k} F^\Delta(i, k) B^\Delta(n - i, k). \]

**Theorem 13.** Let there be generating functions \( F(x) = \sum_{n>0} f(n)x^n \), \( G(x) = \sum_{n>0} g(n)x^n \) and their compositae \( F^\Delta(n, k) \), \( G^\Delta(n, k) \). Then, the generating function \( A(x) = F(x) + G(x) \) has the composita

\[ A^\Delta(n, k) = F^\Delta(n, k) + \sum_{j=1}^{k-1} \binom{k}{j} \sum_{i=j}^{n-k+j} F^\Delta(i, j)G^\Delta(n - i, k - j) + G^\Delta(n, k). \]

**Proof.** According to the binomial theorem, we have

\[ [A(x)]^k = \sum_{j=0}^{k} \binom{k}{j} [F(x)]^j [G(x)]^{k-j}, \]

\[ [F(x)]^j = \sum_{n \geq j} F^\Delta(n, j), \]

and

\[ [G(x)]^{k-j} = \sum_{n \geq k-j} G^\Delta(n, k-j). \]

According to the rule of multiplication of series, we obtain

\[ A^\Delta(n, k) = F^\Delta(n, k) + \sum_{j=1}^{k-1} \binom{k}{j} \sum_{i=j}^{n-k+j} F^\Delta(i, j)G^\Delta(n - i, k - j) + G^\Delta(n, k). \]

**Definition 14.** Let there be a composition of generating functions \( A(x) = R(F(x)) \). Then, the product of two compositae will be termed a composite of the composition \( A(x) \) and denoted as \( A^\Delta(n, k) = F^\Delta(n, k) \circ R^\Delta(n, k) \).

**Theorem 15.** Let there be two generating functions \( F(x) = \sum_{n>0} f(n)x^n \) and \( R(x) = \sum_{n>0} r(n)x^n \), and their compositae \( F^\Delta(n, k) \) and \( R^\Delta(n, k) \). Then, the expression valid for the product of the compositae \( A^\Delta = F^\Delta \circ R^\Delta \) is

\[ A^\Delta(n, m) = \sum_{k=m}^{n} F^\Delta(n, k) R^\Delta(k, m). \]  

(7)

**Proof.**

\[ [A(x)]^m = [G(F(x))]^m = G^m(F(x)). \]

Hence, according to the composition rule and taking into account that the nonzero terms \( G^\Delta(n, m) \) begin with \( n \geq m \), we have

\[ A^\Delta(n, m) = \sum_{k=m}^{n} F^\Delta(n, k) G^\Delta(k, m). \]  

\[ \square \]
Corollary. Because the composition of generating functions is an associative operation and
\[ F(x) \circ (R(x) \circ G(x)) = (F(x) \circ R(x)) \circ G(x), \]
the product of compositae is also an associative operation and
\[ \sum_{k=m}^{n} \sum_{i=k}^{n} F^\Delta(n, i) R^\Delta(i, k) G^\Delta(k, m) = \sum_{k=m}^{n} \sum_{i=k}^{n} R^\Delta(n, i) G^\Delta(i, k) F^\Delta(k, m). \]

4 Compositae of generating functions

4.1 Identical composita

Definition 16. An identical composita \( Id^\Delta(n, k) \) is a composita of the generating function \( F(x) = x \).

By definition, \( [F(x)]^k = x^k \). Then
\[ F^\Delta(n, k) = \begin{cases} 1, & n = k, \\ 0, & n \neq k. \end{cases} \] (8)

Thus, \( F^\Delta(n, k) = \delta_{n,k} \), where \( \delta_{n,k} \) is the Kronecker delta. It is easily seen that for any generating function \( A(x) \), we have the identity
\[ a(n) = \sum_{k=1}^{n} Id^\Delta(n, k)a(k). \]

The composita of the function \( F(x) = x^m \) is
\[ F^\Delta(n, k, m) = \delta_{m,k}, \mod (n, m) = 0 \text{ or } n = km. \] (9)

4.2 Compositae of polynomials

4.2.1 Composita for \( P_2(x) = (ax + bx^2) \)

Let us consider \( P_2(x) = (ax + bx^2) \). Then, \( p_2(0) = 0 \), \( p_2(1) = a \) and \( p_2(2) = b \), and the rest are \( p_2(n) = 0 \), \( n > 2 \). The composita of the function \( F(x) = ax \) is equal to \( a^k \delta_{n,k} \), and the composita of the function \( G(x) = bx^2 \) is equal to \( b^k \delta_{n,k} \). Using sum theorem (13), we obtain
\[ P_2^\Delta(n, k) = \sum_{j=0}^{k} \binom{k}{j} \sum_{i=j}^{n-k+j} a^i \delta_{i,j} b^{k-j} \delta_{n-i,k-j}, \]
\[ \delta_{n-i,k-j} = 1 \text{ for } \frac{n-i}{2} = k-j, \text{ whence } i = n - 2k + 2j. \] So we have
\[ P_2^\Delta(n, k) = \sum_{j=0}^{k} \binom{k}{j} a^i \delta_{n-2k+2j} b^{k-j}. \]
Now $\delta_{n-2k+2j,j} = 1$ for $n - 2k + 2j = j$, whence $j = 2k - n$. So we obtain

$$P_2^\Delta(n, k, a, b) = \left(\begin{array}{c} k \\ n - k \end{array}\right) a^{2k-n} b^{n-k}$$

(10)

for $\left\lceil \frac{n}{2} \right\rceil \leq k \leq n$.

Thus, the composition $A(x) = R(ax + bx^2)$ can be found using the expression:

$$a(n) = \sum_{k=\left\lceil \frac{n}{2} \right\rceil}^{n} \left(\begin{array}{c} k \\ n - k \end{array}\right) a^{2k-n} b^{n-k} r(k).$$

For example, let us derive an expression for the coefficients of the generating function $A(x) = e^{x+x^2}$ (see A000085). Taking into account that this function is an exponential generating function, we obtain

$$a(n) = n! \sum_{k=\left\lceil \frac{n}{2} \right\rceil}^{n} \left(\begin{array}{c} k \\ n - k \end{array}\right) \frac{1}{2^{n-k}} \frac{1}{k!}.$$ 

Another example is $A(x) = R(F(x))$, where $R(x) = \frac{x}{1-x}$ and $F(x) = x + x^2$, $A(x) = \frac{x+x^2}{1-x-x^2}$. Hence

$$a(n) = \sum_{k=\left\lceil \frac{n}{2} \right\rceil}^{n} \left(\begin{array}{c} k \\ n - k \end{array}\right)$$

(see A000045).

**4.2.2 Composita for $P_3(x)$**

The polynomial $P_3(x) = ax + bx^2 + cx^3$ can be expressed as

$$P_3(x) = ax + xP_2(x, b, c).$$

The composita $ax$ is equal to $\delta(n,k)a^k$, and the composita $xP_2(x)$ to $A_2\Delta(n-k,k)$. Then, on the strength of the theorem on the composita of the sum of generating functions, we have

$$A_3^\Delta(n, k, a, b, c) = \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array}\right) \sum_{i=j}^{n-k+j} A_2\Delta(i-j, j, b, c) \delta(n-i, k-j) a^{k-j}.$$ 

Simplification gives $\delta(n-i, k-j) = 1$ for $n-i = k-j$, whence we have $i = n - k + j$ and

$$A_3^\Delta(n, k, a, b, c) = \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array}\right) A_2(n-k, j, b, c) a^{k-j},$$

where $A_2^\Delta(n-k, j, b, c) = \left(\begin{array}{c} j \\ n-k-j \end{array}\right) b^{2j+k-n} b^{n-k-j}$. Hence,

$$A_3^\Delta(n, k, a, b, c) = \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array}\right) \left(\begin{array}{c} j \\ n-k-j \end{array}\right) a^{k-j} b^{2j+k-n} b^{n-k-j}. $$
Then, for the generating function \( A(x) = \frac{1}{1-ax-bx^2-cx^3} \), the following expression holds true:

\[
a(n) = \sum_{k=1}^{n} \sum_{j=0}^{k} \binom{k}{j} \binom{j}{n-k-j} a^{k-j} b^{2j+k-n} c^{n-k-j}.
\]

### 4.2.3 Composita for \( P(x) = ax + cx^3 \)

An important condition in the foregoing examples is that \( a, b, c \neq 0 \). Therefore, if \( b = 0 \) the formula for the composita should be sought for over again. For example,

\[
P(x) = ax + cx^3.
\]

In this case, the expression for the composita is

\[
P^\Delta(n, k) = \binom{k}{3k-n} a^{\frac{3k-n}{2}} c^{\frac{n-k}{2}},
\]

where \((n-k)\) is exactly divisible by 2. For example, for the generating function \( A(x) = \frac{1}{1-x-x^3} \) the following expression holds true:

\[
a(n) = \sum_{k=1}^{n} \binom{k}{3k-n}
\]

(see A000930).

### 4.2.4 Composita for \( P_4(x) = ax + bx^2 + cx^3 + dx^4 \)

At \( n = 4 \), the polynomial \( P_4(x) = ax + bx^2 + cx^3 + dx^4 \) can be expressed as

\[
P_4(x) = P_2(x) + x^2 P_2(x).
\]

The generating function of the composita for \( x^2 P_2(x) \) is equal to

\[
x^{2k} \binom{k}{n-k} c^{2k-n} d^{n-k} x^n = \binom{k}{n-k} c^{2k-n} d^{n-k} x^{n+2k},
\]

and hence the expression for the composita is

\[
\binom{k}{n-3k} c^{4k-n} d^{3k}.
\]

Then the composita \( P_4(x) \) has the following expression:

\[
\sum_{j=0}^{k} \binom{k}{j} \sum_{i=j}^{n-k+j} \binom{i}{j} a^{2j-i} b^{i-j} \binom{k-j}{n-i-3(k-j)} c^{A(k-j)-(n-i)} d^{n-i-k+j}.
\]
For example, for the generating function  
\[ A(x) = \frac{1}{1-ax-bx^2-cx^3-dx^4} \]  
the following expression holds true:  
\[
a(n) = \sum_{k=1}^{n} \sum_{j=0}^{k} \binom{k}{j} \sum_{i=j}^{n-k+j} \binom{j}{i-j} a^{2j-i} b^{i-j} \left( \frac{k-j}{n-i-3(k-j)} \right) c^{4(k-j)-(n-i)} d^{n-i-k+j}.
\]

At  
\[ a = b = c = d = 1, \]
we obtain the generating function  
\[ A(x) = \frac{1}{1-x^2-x^3-x^4} \]. Hence  
\[
a(n) = \sum_{k=1}^{n} \sum_{j=0}^{k} \binom{k}{j} \sum_{i=j}^{n-k+j} \binom{j}{i-j} \left( \frac{k-j}{n-i-3(k-j)} \right).
\]

### 4.2.5 Composita for  \( P_5(x) = ax + bx^2 + cx^3 + dx^4 + ex^5 \)  

For finding the composita of the \( m \)th power polynomial, we can propose the recurrent algorithm  
\[
A^\Delta_m(n, k) = \sum_{j=0}^{k} A^\Delta_{m-1}(n-k, j) a^{k-j},
\]

providing that  
\[ A_{m-1}(n, 0) = 1. \]

Using this recurrent algorithm, we obtain the composita of the 5th power polynomial:  
\[
\sum_{r=0}^{k} a^{k-r} \binom{k}{r} \sum_{m=0}^{r} b^{r-m} \left( \sum_{j=0}^{m} c^{m-j} d^{r-n+m+k+2j} e^v \binom{j}{v} \binom{m}{j} \right) \binom{r}{m},
\]

where  
\[ v = -r + n - m - k - j. \]

### 4.3 Composita for  \( A(x) = \left( \frac{ax}{1-bx} \right) \)  

For the generating function  
\[ F(x) = \frac{x}{(1-x)}, \]
\[ F^\Delta(n, k) = \binom{n-1}{k-1}, \]
and  
\[ A(x) = (ab^{-1} \frac{bx}{1-bx}). \]

Using theorems (10,11), we obtain  
\[
A^\Delta(n, k) = \binom{n-1}{k-1} a^{k} b^{n-k}.
\]

### 4.4 Compositae of the exponent  

Let us find the expression for the coefficients of the generating function  
\[ [B(x)]^k = e^{kx} \]  
\[
B(x)^k = e^{xk} = \sum_{n \geq 0} \frac{k^n}{n!}.
\]
whence it follows that

$$B(n, k) = \frac{k^n}{n!}.$$ 

Now, for $A(x) = xe^x$ the composita is equal to

$$A^\Delta (n, k) = B(n - k, k) = \frac{k^{n-k}}{(n-k)!}.$$ (11)

Let us write the composita for the generating function $A(x) = e^x - 1$:

$$A(x)^k = \sum_{m=0}^{k} \binom{k}{m} e^{mx} (-1)^{k-m},$$

whence it follows that the composita is

$$A^\Delta (n, k) = \sum_{m=0}^{k} \binom{k}{m} \frac{m^n}{n!} (-1)^{k-m} = \frac{k!}{n!} S_2(n, k),$$ (12)

where $S_2(n, k)$ is the Stirling numbers of the second kind. For the generating functions of the Bell numbers $A(x) = e^{e^x-1}$, we have

$$a(n) = n! \sum_{k=1}^{n} S_2(n, k) \frac{k!}{n!} \frac{1}{k!} = \sum_{k=1}^{n} S_2(n, k)$$ (see A000110).

### 4.5 Composita for $\ln(1 + x)$

Let $F(x) = \ln(x + 1)$. Then, knowing the relation [6]

$$\sum_{n=k}^{\infty} S_1(n, k) \frac{x^n}{n!} = \frac{[\ln(1 + x)]^k}{k!},$$

where $S_1(n, k)$ is the Stirling numbers of the first kind, and using formula (1), we obtain the expression for the composita of the generating function $\ln(1 + x)$:

$$F^\Delta (n, k) = \frac{k!}{n!} S_1(n, k).$$ (13)

### 4.6 Composita for the generating function of the Bernoulli numbers

The generating function of the Bernoulli numbers is

$$A(x) = \frac{x}{e^x - 1}.$$
This function can be represented as the composition \( B(F(x)) \), where \( B(x) = \frac{\ln x}{x} \), \( F(x) = e^x - 1 \). Let us find the expression for the coefficients of the generating function \([B(x)]^k\):

\[
[B(x)]^k = \sum_{n \geq 0} S_1(n, k) \frac{k!}{n!} x^{n-k},
\]

whence

\[
B(n, k) = S_1(n + k, k) \frac{k!}{(n + k)!}.
\]

Knowing the composita of the function \( F(x) \) (see 12),

\[
F^\Delta(n, k) = \frac{k!}{n!} S_2(n, k).
\]

Let us write the composition of the generating functions \( A(x) = [B(e^x - 1)]^k \):

\[
A(n, m) = \begin{cases} 1, & n = 0, \\ \sum_{k=1}^{n} S_2(n, k) \frac{k!}{n!} S_1(k + m, m) \frac{m!}{(k + m)!}, & n > 0. \end{cases}
\]

Then the composita of \( xA(x) \) is

\[
A^\Delta(n, m) = \begin{cases} 1, & n = m, \\ \frac{m!}{(n-m)!} \sum_{k=1}^{n-m} \frac{k!}{(k+m)!} S_1(k + m, m) S_2(n - m, k), & n > m. \end{cases}
\]

### 4.7 Composita for the generating function of the Fibonacci numbers

Let us find the composita for the generating function of the Fibonacci numbers:

\[
A(x) = \frac{x}{1 - x - x^2}.
\]

The function can be represented as the composition of the generating functions \( A(x) = R(F(x)) \), where \( R(x) = \frac{x}{1-x} \), \( F(x) = \frac{x}{1-x} \). Let us find the composita for \( F(x) \):

\[
F^\Delta(n, k) = \begin{cases} \binom{n+k-1}{k-1}, & \text{at } n + k \text{ - even}, \\ 0, & \text{at } n + k \text{ - odd}. \end{cases}
\]

Now, using the operation of product of compositae, we find the composita of the generating function \( A(x) \):

\[
A^\Delta(n, m) = \sum_{k=m}^{n} \binom{n+k-1}{k-1} \binom{k-1}{m-1}, \quad \text{at } n + k \text{ - even}.
\]

Below are the first terms of the composita for the generating function of the Fibonacci numbers:

\[
\begin{array}{ccccccccccc}
1 & 1 & & & & & & & & \\
2 & 2 & 1 & & & & & & & \\
3 & 5 & 3 & 1 & & & & & & \\
5 & 10 & 9 & 4 & 1 & & & & & \\
8 & 20 & 22 & 14 & 5 & 1 & & & & \\
13 & 38 & 51 & 40 & 20 & 6 & 1 & & & \\
\end{array}
\]
4.8 Composita for the generalized Fibonacci numbers

Let us find the composita of the generating function:

\[ F(x) = x + x^2 + \ldots + x^m = \frac{x - x^{m+1}}{1 - x}. \]

Let us write \( F(x) \) as the product of the functions \( G(x) = x - x^{m+1} \) and \( R(x) = \frac{1}{1-x} \).

Let us find the composita for \( G(x) \). For this purpose, we consider the compositae of the functions \( y(x) = x \) and \( z(x) = -x^m \). For \( y(x) \), the composita is equal to \( Id(n,k) = \delta_{n,k} \).

For \( z(x) = -x^m \), the composita is

\[ Z^\Delta(n, k) = (-1)^k \delta_{\frac{n}{m}, k}. \]

Then, on the strength of the theorem on the composite of sum of generating functions \( y(x) + z(x) \), we have

\[ G^\Delta(n, k) = \sum_{j=0}^{\infty} \binom{k}{j} \sum_{i=j}^{n-k+j} Id(i, j) Z^\Delta(n - i, k - j) = \]

\[ = \sum_{j=0}^{\infty} \binom{k}{j} \sum_{i=j}^{n-k+j} \delta_{i,j} \delta_{\frac{n-i}{m}, k-j} (-1)^{k-j}. \]

The function \( \delta_{i,j} = 1 \) is only for \( i = j \), and hence

\[ G^\Delta(n, k) = \sum_{j=0}^{\infty} \binom{k}{j} \delta_{\frac{n-j}{m}, k-j} (-1)^{k-j}. \]

The function \( \delta_{\frac{n-j}{m}, k-j} = 1 \) is only for \( \frac{n-j}{m} = k - j \), and hence

\[ G^\Delta(n, k) = \left( \frac{k}{(m+1)k-n} \right) (-1)^{n-k/m}. \]

It is known that \( R(n, k) = \binom{n+k-1}{k-1} \). Then, with regard to the rule of finding the composita of the product of generating functions (case 2), we obtain

\[ F^\Delta(n, k) = \sum_{i=k}^{n} \binom{k}{(m+1)i+k-1} (-1)^{i-k/m} \binom{n-i+k-1}{k-1}. \]

Let us consider the composita of the generating functions:

\[ A(x) = \frac{F(x)}{1 - F(x)} = \frac{x - x^{m+1}}{1 - 2x - x^{m+1}}. \]
Hence, using the theorem on the product of compositae, we obtain the composita of the generating function $A(x)$:

$$A^\Delta(n, l) = \sum_{k=1}^{\infty} F^\Delta(n, k) \binom{k-1}{m-1} =$$

$$= \sum_{k=m}^{n} \sum_{i=k}^{n} \left( \frac{k}{(m+1)k-i} \right) (-1)^{\frac{i-k}{m}} \binom{n-i+k-1}{k-1} \binom{k-1}{l-1}.$$ 

For $l = 1$, we derive the formula for the generalized Fibonacci numbers:

$$F^{(m)}_n = \sum_{k=1}^{n} \sum_{i=k}^{n} \left( \frac{k}{(m+1)k-i} \right) (-1)^{\frac{i-k}{m}} \binom{n-i+k-1}{k-1}. \quad (14)$$

### 4.9 Composita of the generating function for the Catalan numbers

Let $F(x) = x \frac{1 - \sqrt{1-4x}}{2x}$, then the composita has the form

$$F^\Delta(n, k) = \sum_{i=0}^{n-k} C(i) F^\Delta_{n-i-1, k-1},$$

where $C(i)$ is the Catalan numbers. The composita $F^\Delta(n, k)$ has the following triangular form:

1
1 1
2 1 1
5 5 3 1
14 14 9 4 1

Let us consider the sequence A009766 called the Catalan triangle. This triangle is given by the formula

$$a(n, m) = \binom{n+m}{n} \frac{n-k+1}{n+1}.$$ 

Below are the initial values of the triangle, and $n$ and $m$ begin with zero.

1
1 1
2 1 2
1 3 5 5
1 4 9 14 14

Comparison of two triangles suggests that $a(n, k) = F^\Delta(n+1, n-k+1)$. Hence, the composita for the Catalan generating function is equal to

$$F^\Delta(n, k) = \binom{2n-k-1}{n-1} \frac{k}{n}.$$
Thus, the expression valid for the coefficients of the composition \( A(x) = R(\frac{1 - \sqrt{1 - 4x}}{2}) \) is

\[
a(n) = \left(\frac{2n - k - 1}{n - 1}\right) \frac{k}{n} r(k).
\]

### 4.10 Composita of the generating function \( \frac{x}{\sqrt{1-x}} \)

This generating function can be represented as the composition of the functions:

\[
\frac{x}{\sqrt{1-x}} = x \frac{1}{1 - \left(2 \sqrt{1 - \frac{4x}{2}} - 1\right)} = x \frac{1}{1 - 2C(\frac{1}{4}x)},
\]

where \( C(x) = \frac{1 - \sqrt{1 - 4x}}{2} \).

Using the formula of composition, we finally obtain

\[
A^\Delta(n, m) = \begin{cases} 
1, & n = m, \\
\sum_{k=1}^{n-m} \left(2n-2m-k-1\right) \frac{k}{n-m} 2^{k-2n+2m} \binom{k+m-1}{m-1}, & n > m.
\end{cases}
\]

### 4.11 Compositae of trigonometric functions

#### 4.11.1 Composita of the sine

Using the expression

\[
\sin(x) = e^{ix} - e^{-ix} \]

we obtain \( \sin(x)^k \):

\[
\sin(x)^k = \frac{1}{2^k i^k} \sum_{m=0}^{k} \binom{k}{m} e^{imx} e^{-i(k-m)x} (-1)^{k-m} = \frac{1}{2^k i^k} \sum_{m=0}^{k} \binom{k}{m} e^{i(2m-k)x} (-1)^{k-m}.
\]

Hence the composita is

\[
\frac{1}{2^k i^{n-k}} \sum_{m=0}^{k} \binom{k}{m} \frac{(2m-k)^n}{n!} (-1)^{k-m}.
\]

Taking into account that \( n - k \) is an even number and the function is symmetric about \( k \), we obtain the composita of the generating function \( \sin(x) \):

\[
A^\Delta(n, k) = \begin{cases} 
1, & n - k \text{ is even} \\
0, & n - k \text{ is odd}
\end{cases}
\]

**Example 17.** For the Euler numbers we know the exponential generating function \( \frac{1}{1 - \sin(x)} \).

Hence,

\[
E_{n+1} = \sum_{k=1}^{n} \sum_{m=0}^{\left[\frac{k}{2}\right]} \binom{k}{m} (2m-k)^n (-1)^{\frac{n-k-m}{2}}
\]

(see A000111).
Example 18. For the generating function \( A(x) = e^{\sin(x)} \), the valid expression is

\[
an = \sum_{n+k=1}^{n} \frac{1}{2^{k-1}k!} \sum_{m=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k}{m} (2m-k)^n (-1)^{\frac{n+k}{2}} m
\]

(see A002017).

4.11.2 Compositae of the cosine

Knowing that

\[
\cos(x) = \frac{e^{ix} + e^{-ix}}{2},
\]

We have

\[
[\cos x]^k = \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} e^{(2j-k)ix} = \frac{1}{2^k} \sum_{n \geq 0} \sum_{j=0}^{k} \binom{k}{j} (2j-k)^n i^n x^n n!.
\]

Hence

\[
B(n, k) = \frac{1}{2^k n!} (-1)^{\frac{n-k}{2}} \sum_{j=0}^{k} \binom{k}{j} (2j-k)^n.
\]

Then, the composita of the generating function \( x \cos(x) \) is

\[
A^\Delta(n, k) = \begin{cases} \frac{1}{2^{n-k} (n-k)!} (-1)^{\frac{n-k}{2}} \sum_{j=0}^{k} \binom{k}{j} (2j-k)^{n-k}, & n-k \text{ even} \\ 0, & n-k \text{ odd} \end{cases}
\]

The composita of the function \( \cos(x) - 1 \) is equal to

\[
A^\Delta(n, k) = \sum_{i=0}^{k} B(n, i) (-1)^{k-i}.
\]

Let us consider the following example. Let there be a generating function \( A(x) = \sec(x) = \frac{1}{\cos(x)} = \frac{1}{1+\sin(x)} \). Hence, on the strength of the formula of composition and composita \( (\cos(x) - 1) \), we obtain

\[
a(n) = \sum_{k=1}^{2n} \sum_{m=0}^{k} \binom{k}{m} 2^{1-m} \left( \sum_{j=0}^{m} (2j-m)^2 n \binom{m}{j} \right) (-1)^{n+m}
\]

(see A000364).
4.11.3 Composita for $\tan(x)$

For the tangent, we know the identity

$$\tan(x) = \frac{e^{ix} - e^{-ix}}{i(e^{ix} - e^{-ix})}.$$  

Division of the numerator and denominator by $e^{ix}$ gives

$$\tan(x) = \frac{1 - e^{-2ix}}{i(1 - e^{-2ix})}.$$  

Multiplication of the numerator and denominator by $i$, and addition and then subtraction of unity gives

$$\tan(x) = i \frac{e^{-2ix} - 1}{2 - (e^{-2ix} - 1)}.$$  

Whence it follows that

$$\tan(x) = \frac{i}{2} \frac{e^{-2ix} - 1}{1 - \frac{1}{2}(e^{-2ix} - 1)}.$$  

Thus, the function $\tan(x)$ is expressed as the composition of the functions

$$F(x) = \frac{i}{2} \frac{x}{1 + \frac{1}{2}x}$$  

and functions $R(x) = e^{-2ix} - 1$. The composita for $F(x)$ is equal to

$$F^\Delta(n, k) = \frac{1}{2n}(-1)^{n-k} \binom{n-1}{k-1} i^k.$$  

The composita for $R(x)$ is equal to

$$R^\Delta(n, k) = (-2i)^n \frac{k!}{n!} S_2(n, k),$$  

where $S_2(n, k)$ is the Stirling numbers of the second kind. Then, on the strength of the theorem on the product of compositae, we obtain the composita of the function $\tan(x)$:

$$A^\Delta(n, m) = \sum_{k=m}^{n} (-2i)^n S_2(n, k) \frac{k!}{n!} \frac{1}{2^k} (-1)^{k-m} \binom{k-1}{m-1} i^m.$$  

After transformation, we obtain

$$A^\Delta(n, m) = (-1)^{n+m} \sum_{k=m}^{n} (2)^{n-k} S_2(n, k) \frac{k!}{n!} (-1)^{n+k-m} \binom{k-1}{m-1}.$$  

Then at $k = 1$, the expression for the tangential numbers is

$$a(n) = (-1)^{n+1} \sum_{j=1}^{2n+1} (-1)^j j! 2^{2n-j+1} S_2(2n + 1, j)$$
Let us consider the example $A(x) = \tan(x)$:

$$a(n) = \sum_{k=1}^{n} \frac{(-1)^{n+k}}{k!} \sum_{j=k}^{n} \frac{j!}{(i-j)^{n-k+j}} S_{n-k+j}(n,j)$$

(see A006229). For more examples, see A000828, A000831, A003707

### 4.11.4 Composita for $x^2 \cot(x)$

It is known that

$$x^2 \cot(x) = ixe^{-ix} + e^{ix} = ix^2 + \frac{2ix^2}{e^{2ix} - 1}.$$  

The composita $ix^2$ is equal to $\delta\left(\frac{n}{2}, k\right)i^k$, and the composita for $\frac{2ix^2}{e^{2ix} - 1}$ is equal to

$$(2i)^{n-k}B^{\Delta}(n, k),$$

where $B^{\Delta}(n, k)$ is the composita for the generating function of the Bernoulli numbers. Using the theorem on the composita of the sum of generating functions, we obtain the composita of the function $x^2 \cot(x)$:

$$A^{\Delta}(n,k) = \delta\left(\frac{n}{2}, k\right)i^k + \sum_{j=1}^{k} B^{\Delta}(n - 2k + 2j) (2i)^{n-2k+j} i^{k-j} =$$

$$= \delta\left(\frac{n}{2}, k\right)i^k + i^{n-k} \sum_{j=1}^{k} B^{\Delta}(n - 2k + 2j) 2^{n-2k+j}$$

### 4.11.5 Composita of the arc tangent $F(x) = \arctan(x)$

Let us consider the generating function of the arc tangent:

$$\arctan(x) = \sum_{n \geq 0} \frac{(-1)^n}{(2n + 1)} x^{2n+1}.$$

Let us find an expression for the composita of the arc tangent from the operation of product of compositae. For this purpose, the expression

$$\arctan(x) = \frac{i}{2} (\ln(1 - ix) - \ln(1 + ix))$$

is written as follows:

$$\arctan(x) = \frac{i}{2} \ln(1 - \frac{2ix}{1+ix}).$$

The composita of the function $f(x) = \frac{2ix}{1+ix}$ is equal to

$$F^{\Delta}(n,k) = 2^k \binom{n-1}{k-1} i^n,$$
whence it follows that

\[ A^\Delta_z(n, m) = \frac{i^m}{2^m} \sum_{k=m}^n 2^k \binom{n-1}{k-1} i^n \frac{m!}{k!} S_1(k, m). \]  

(15)

\[ A^\Delta_z(n, m) = \frac{(-1)^{m+n}}{2^m} \sum_{k=m}^n 2^k \binom{n-1}{k-1} \frac{m!}{k!} S_1(k, m). \]  

(16)

Below are the first terms of the composita of the arc tangent \( A^\Delta_z(n, k) \) in the triangular form:

|   | 1   | 0   | 1   |
|---|-----|-----|-----|
| 0 | -\frac{1}{3} | 0   | 1   |
| 0 | 0   | -\frac{2}{3} | 0   |
| \frac{1}{5} | 0   | 0   | -1  | 0   |
| 0   | \frac{23}{45} | 0   | -\frac{4}{3} | 0   |
| -\frac{1}{7} | 0   | \frac{14}{15} | 0   | -\frac{5}{3} | 0   |

**Example 19.** Let there be \( R(x) = \frac{1}{1-x} \), then the coefficients of the generating function

\[ A(x) = \frac{1}{1 - \arctan(x)} \]

are expressed by the formula:

\[ a(n) = \sum_{m=1}^n \frac{(-1)^{m+n}}{2^m} \sum_{k=m}^n 2^k \binom{n-1}{k-1} \frac{m!}{k!} S_1(k, m). \]

Hence, summation of rows of the composita of the arc tangent gives the following series:

\[ A(x) = 1 + x + x^2 + \frac{2}{3} x^3 + \frac{1}{3} x^4 + \frac{1}{5} x^5 + \frac{8}{45} x^6 + \ldots. \]

**Example 20.** Let \( A(x) = e^{\arctan(x)} \), then the valid expression is

\[ a(n) = n! \sum_{m=1}^n \frac{(-1)^{m+n}}{2^m} \sum_{i=m}^n 2^i \binom{n}{i} S_1(i, m) \frac{(n-1)!}{i!} \]

(see [A002019](https://oeis.org/A002019)).

### 4.12 Compositae of hyperbolic functions

For the hyperbolic sine, we have the known expression:

\[ \sinh(x) = \frac{e^x - e^{-x}}{2}. \]
Let us find the composita of this generating function. For this purpose, we write
\[
\left( \frac{e^x - e^{-x}}{2} \right)^k = \frac{1}{2^k} (e^x + e^{-x})^k = \frac{1}{2^k} \sum_{i=0}^{k} \binom{k}{i} e^{(k-i)x} (-1)^i e^{-ix} = \frac{1}{2^k} \sum_{i=0}^{k} (-1)^i \binom{k}{i} e^{(k-2i)x}.
\]
Let us write \(e^x\) as a series, then we obtain
\[
\frac{1}{2^k} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \sum_{n \geq 0} \frac{(k-2i)^n}{n!} x^n.
\]
Hence, the composita is
\[
F^\Delta(n, k) = \frac{1}{2^k} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{(k-2i)^n}{n!}.
\]
For example, for \(A(x) = e^{\sinh x}\) the valid expression is
\[
a(n) = \sum_{k=1}^{n} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{(k-2i)^n}{2^k k!}.
\]
(see \(A002724\)). For the hyperbolic cosine, we have
\[
cosh(x) = \frac{e^x + e^{-x}}{2}.
\]
Then,
\[
cosh^k(x) = \left( \frac{e^x + e^{-x}}{2} \right)^k = \frac{1}{2^k} \sum_{i=0}^{k} \binom{k}{i} e^{(k-2i)x} = \frac{1}{2^k} \sum_{i=0}^{k} \binom{k}{i} \sum_{n \geq 0} \frac{(k-2i)^n}{n!} x^n,
\]
and hence the composita of the generating function \(x \cosh(x)\) is
\[
F^\Delta(n, k) = \frac{1}{2^k} \sum_{i=0}^{k} \binom{k}{i} \frac{(k-2i)^n}{(n-k)!}.
\]
For example, for \(A(x) = e^{\cosh x}\) the valid expression is
\[
\sum_{k=1}^{n} \left( \sum_{i=0}^{k} (k-2i)^{n-k} \binom{k}{i} \right) \frac{(n)}{2^k} \binom{n}{k}
\]
see \(A003727\).
5 Conclusion

The operation of the composition \( A(x) = R(F(x)) \) of ordinary generating functions requires:
1. Finding the composita \( F^\Delta(n,k) \) of the generating function \( F(x) \) with the use of theorems (10,11,12,13,15)
2. Writing the composition in the form
\[
a(n) = \sum_{k=1}^{n} F^\Delta(n,k)r(n).
\]

References

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