Highest Cusped Waves for the Burgers–Hilbert Equation

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Abstract

In this paper we prove the existence of a periodic highest, cusped, traveling wave solution for the Burgers–Hilbert equation \( f_t + ff_x = H[f] \), and give its asymptotic behaviour at 0. The proof combines careful asymptotic analysis and a computer-assisted approach.

1. Introduction

The Burgers–Hilbert equation [25] is a nonlinear wave model, in the periodic setting given by

\[
    f_t + ff_x = H[f], \quad \text{for } (x, t) \in \mathbb{T} \times \mathbb{R}.
\]

Here \( H \) is the Hilbert transform which, for \( f : \mathbb{T} \to \mathbb{R} \), is defined by

\[
    H[f](x) = \frac{1}{2\pi} \text{p.v.} \int_{-\pi}^{\pi} \cot \left( \frac{x - y}{2} \right) f(y) \, dy, \quad \hat{H}[f](k) = -i \text{sgn}(k) \hat{f}(k).
\]

The equation was first used by Marsden and Weinstein in 1983 as a second order approximation for the evolution of the boundary of a simply connected vortex patch in two dimensions [44]. More recently Biello and Hunter used it to serve as an approximation for small slope vorticity fronts [4]. The validity of this approximation was recently proved [28].

For small initial data in \( H^2(\mathbb{R}) \), estimates for the lifespan were proved by Hunter and Ifrim [26], see also [27].

The global existence of weak solutions for initial data in \( L^2(\mathbb{R}) \) was established by Bressan and Nguyen, in which case the solution lies in \( L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) for \( t > 0 \) [5]. Bressan and Zhang constructed locally in time piecewise smooth solutions with a single, logarithmic, shock [6]. Stability and uniqueness of these solutions in a larger class of solutions were shown by Krupa and Vasseur [42].
Numerical simulations have shown formation of shocks in finite time [4,25]. Castro, Córdoba and Gancedo proved finite time blow up of the $C^{1,\delta}$-norm with $0 < \delta < 1$ for initial data $f_0 \in L^\infty(\mathbb{R}) \cap C^{1,\delta}(\mathbb{R})$ satisfying that there exists a point $x_0$ with $H[f_0](x_0) > 0$ and $u_0(x_0) \geq (32\pi \|u_0\|_{L^2(\mathbb{R})}^2)^{1/\delta}$ [8]. Saut and Wang proved shock formation in finite time [51]. Solutions that develop an asymptotic self-similar shock at a single point with an explicitly computable blowup profile were constructed by Yang [56].

The Burgers–Hilbert equation occurs as a special case in the family of fractional KdV equations given by

$$f_t + ff_x = |D|^{\alpha} f_x. \quad (2)$$

Here

$$\widehat{|D|^{\alpha} f}(\xi) = |\xi|^{\alpha} \widehat{f}(\xi),$$

and the parameter $\alpha$ may in general take any real value. For $\alpha = 2$ and $\alpha = 1$ we get the classical KdV and Benjamin–Ono equations. For $\alpha = -1$ it reduces to the Burgers–Hilbert equation.

For $\alpha \in (-1, 0)$ the fractional KdV equation exhibits finite time blow [8,29]. That this blowup happens in terms of wave breaking was proved for $\alpha \in (-1, -1/3)$ by Hur and Tao [30,31] and for $\alpha \in (-1, 0)$ by Oh and Pasqualotto [48]; see [40] for a numerical study and also [11]. For other results see, e.g., [19] for existence time, [50] for well-posedness and [39,41] for some results for related equations.

In this work we are concerned with traveling wave solutions of the Burgers–Hilbert equation (1). The study of traveling waves is an important topic in fluid dynamics, see, e.g., [23] for a recent overview of traveling water waves. The traveling wave assumption $f(x, t) = \varphi(x - ct)$, where $c > 0$ denotes the wave speed, gives us

$$-c\varphi' + \varphi\varphi' = H[\varphi]. \quad (3)$$

The Burgers–Hilbert equation has an analytic branch of even, zero-mean, $2\pi$-periodic, smooth traveling wave solutions bifurcating from constant solutions [25]. If we let $\epsilon$ be the bifurcation parameter and $\varphi_\epsilon$, $c_\epsilon$ be the solution with its corresponding wave speed, then as $\epsilon \to 0$ we have

$$\varphi_\epsilon(x) = \epsilon \cos(x) + O(\epsilon^2),$$

$$c_\epsilon = 1 + O(\epsilon^2).$$

Castro et al. [10] proved that this branch exists in the range $(0, \epsilon^*)$ with $\epsilon^* \sim 0.23$ and fails to exist for $\epsilon > \frac{2}{\epsilon}$. Moreover they proved an enhanced lifespan estimate for perturbations of $\varphi_\epsilon$ compared to the results in [26]. In [25], Hunter remarks that this branch presumably ends in a highest wave which is not smooth at its crest, as is common for equations of this type. In this paper we prove the existence of a highest cusped wave and give its behaviour at the crest. More precisely, we prove the following theorem:
Theorem 1.1. There is a $2\pi$-periodic traveling wave $\varphi$ of (3), which behaves asymptotically at $x = 0$ as
\[ \varphi(x) = c + \frac{1}{\pi} |x| \log |x| + O(|x| \sqrt{\log |x|}). \]

Remark 1.2. We don’t expect the remainder term $O(|x| \sqrt{\log |x|})$ in Theorem 1.1 to be sharp. The estimate follows from the choice of our space.

The notion of highest traveling waves exist for a large number of equations. For the free boundary Euler equation Stokes argued that if there exists a singular solution with a steady profile it must have an interior angle of $120^\circ$ at the crest [52]. This is known as the Stokes conjecture and was proved in 1982 [1]. For the Whitham equation [55] the existence of a highest cusped traveling wave was conjectured by Whitham in [54]. Its existence, together with its $C^{1/2}$ regularity, was recently proved by Ehrnström and Wahlén [18]. They conjectured that the wave is convex between its crests and also its precise asymptotic behaviour. This conjecture was proved by Enciso, Gómez-Serrano and Vergara [20]. See also the recent paper [17] where they construct the full family of solitary waves. For the fractional KdV equations the traveling waves assumption allows us to write the equation as
\[ -c\varphi' + \varphi\varphi' = |D|^\alpha \varphi'. \] (4)

There has recently been much progress related to highest waves to this family of equations. For $\alpha < -1$ Bruell and Dhara proved the existence of highest traveling waves which are Lipschitz at their cusp [7]. Very recently Ørke proved their existence for $-1 < \alpha < 0$ for the (inhomogeneous) fractional KdV equations as well as the fractional Degasperis-Procesi equations, together with their optimal $-\alpha$-Hölder regularity [49]. Hildrum and Xue prove a similar result for another class of equations, including the (homogeneous) fractional KdV equations for $-1 < \alpha < 0$ [24].

The results in [7,18,24,49] are all based on global bifurcation arguments, bifurcating from the constant solution and proving that the branch must end in a highest wave which is not smooth at its crest. The proof of the convexity of the highest cusped wave for the Whitham equation in [20] uses a completely different approach. The problem is first rephrased as a fixed point problem, the existence and properties of the fixed point is then related to inequalities for certain constants that appear in the reduction. These inequalities are then checked using a a computer assisted proof. In this paper we use a similar approach for proving Theorem 1.1.

The ansatz $\varphi(x) = c - u(x)$ allows us to rewrite (3) as an equation that does not explicitly depend on the wave speed $c$. Proving the existence of a solution $u$ can be rewritten as a fixed point problem by considering the ansatz
\[ u(x) = u_0(x) + w(x)v(x) \]
where $u_0(x)$ is an explicit, carefully chosen, approximate solution and $w(x)$ is an explicit weight factor. Proving the existence of a fixed point can be reduced to checking an inequality involving three constants, $D_0$, $\delta_0$ and $n_0$, that only depend on...
Fig. 1. An enclosure of the function $u(x)$ on the interval $[-\pi, \pi]$. The thin line is the approximation $u_0(x)$ and the width of the thick line is computed using the bound of $\|v\|_{L^\infty(T)}$ the choice of $u_0$ and $w$, see Proposition 2.2. This inequality is checked by bounding $D_0, \delta_0$ and $n_0$ using a computer assisted proof, see Lemmas 7.1, 7.2 and 7.3. These bounds are highly non-trivial, in particular $\delta_0$ is given by the supremum of a function on the interval $[0, \pi]$ which attains its maximum around $10^{-5000}$. A plot of the function $u$ is given in Fig. 1.

One of the key difficulties is the construction of the approximate solution $u_0$. Due to the singularity at $x = 0$ it is not possible to use a trigonometric polynomial alone, it would converge very slowly and have the wrong asymptotic behaviour. Pure products of powers and logarithms, $|x|^a \log^b |x|$, have the issue that they are not periodic and do not interact well with the operator $H$. Instead we take inspiration from the construction in [20] and consider a combination of trigonometric polynomials and Clausen functions of different orders, defined as

$$C_s(x) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^s}, \quad S_s(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^s},$$

for $s > 1$ and by analytic continuation otherwise. We also make use of their derivatives with respect to the order, for which we use the notation

$$C_s^{(\beta)}(x) := \frac{d^\beta}{ds^\beta} C_s(x), \quad S_s^{(\beta)}(x) := \frac{d^\beta}{ds^\beta} S_s(x).$$

These functions are $2\pi$-periodic, non-analytic at $x = 0$ and behave well with respect to $H$. In particular $C_2^{(1)}(x) - C_2^{(1)}(0) \sim |x| \log |x|$, which corresponds to the behaviour we expect in Theorem 1.1. However, directly applying the ideas from [20] for the construction does not work since it is not easy to determine the next term in the expansion. Instead we go through the fractional KdV equations for $-1 < \alpha < 0$, for this family of equations the same ideas do work and we then study the limit $\alpha \to -1$.

One benefit with our approach of working with an explicit approximation of the function is that it is possible to compute approximations of many properties of it.
Combining this with the bounds for \( v \), given in Section 8, it is possible to compute quantitative bounds for properties of the solution.

**Remark 1.3.** It is possible to enclose the first Fourier coefficient of the solution \( u \), and hence also \( \phi \). For \( u_0 \) the first Fourier coefficient is given by \(-0.54771600\ldots\), taking into account the bounds for \( v \) we can compute the enclosure \([0.534, 0.561]\) for the first Fourier coefficient of \( \phi \). This agrees with the results from Castro, Córdoba and Zheng [10] that the branch of solutions break down for \( \epsilon \) somewhere between \( \epsilon^* \sim 0.23 \) and \( \frac{2}{3} \). It also agrees with their (unpublished) numerical results indicating that the branch ends around \( \epsilon = 0.548 \) [9].

**Remark 1.4.** It is possible to enclose the mean of \( u \), which gives the wavespeed \( c \) for a corresponding zero-mean \( \phi \). From the expression of \( u_0 \) (see (11)) it is straightforward to compute the mean \( 1.11041\ldots \) of \( u_0 \). Taking into account the bounds for \( v \) we get the enclosure \([1.1, 1.121]\) for the mean of \( u \).

An important part of our work is the interplay between traditional mathematical tools and rigorous computer calculations. Traditional numerical methods typically only compute approximate results, to be able to use the results in a proof we need the results to be rigorously verified. The basis for rigorous calculations is interval arithmetic, pioneered by Moore in the 1970’s [46]. Due to improvements in both computational power as well as great improvements in software it has become possible to use computer assisted tools in many more problems. The main idea with interval arithmetic is to do arithmetic not directly on real numbers but on intervals with computer representable endpoints. Given a function \( f : \mathbb{R} \to \mathbb{R} \), an interval extension of \( f \) is an extension to intervals satisfying that for an interval \( x \), \( f(x) \) is an interval satisfying \( f(x) \in f(x) \) for all \( x \in x \). In particular this allows us to prove inequalities for the function \( f \), for example the right endpoint of \( f(x) \) gives an upper bound of \( f \) on the interval \( x \). For an introduction to interval arithmetic and rigorous numerics we refer the reader to the books [46,53] and to the survey [21] for a specific treatment of computer assisted proofs in PDE. For all the calculations in this paper we make use of the Arb library [36] for ball (intervals represented as a midpoint and radius) arithmetic. It has good support for many of the special functions we use [32,37,38], Taylor arithmetic (see e.g. [34]) as well as rigorous integration [33].

The rest of this paper is organized as follows. In Section 2 we reduce the proof of Theorem 1.1 to a fixed point problem: in Section 3 we give a brief overview of properties of the Clausen functions that are relevant for the construction of \( u_0 \), in Section 4 we give the construction of \( u_0 \). Section 5 is devoted to the approach for bounding \( n_0 \) and \( \delta_0 \), Section 6 to studying the linear operator that appears in the construction of the fixed point problem. The computer assisted proofs giving bounds for \( n_0, \delta_0 \) and \( D_0 \) are given in Section 7. Finally we give the proof of Theorem 1.1 in Section 8.

Three appendices are given at the end of the paper. “Appendix A” gives some technical details for how to compute enclosures of functions around removable singularities. “Appendix B” is concerned with computing enclosures of the Clausen functions and “Appendix C” with the rigorous numerical integration needed for bounding \( D_0 \).
2. Reduction to a Fixed Point Problem

In this section we reduce the problem of proving Theorem 1.1 to proving the existence of a fixed point for a certain operator. We start with the following lemma which motivates the notion of highest wave (see also [24, Theorem 3.4])

**Lemma 2.1.** Let \( \varphi \in C^1 \) be a nonconstant, even solution of (3) which is nondecreasing on \((-\pi, 0)\), then

\[
\varphi' > 0 \quad \text{and} \quad \varphi < c
\]
on \((-\pi, 0)\).

**Proof.** We start by proving that under these assumptions \( H[\varphi] < 0 \). Since \( \varphi \) is even we can write the Hilbert transform of \( \varphi \) as

\[
H[\varphi]\,(x) = \frac{1}{2\pi} \int_{-\pi}^{0} \left( \cot\left(\frac{x-y}{2}\right) + \cot\left(\frac{x+y}{2}\right) \right) \varphi(y) \, dy.
\]

Integration by parts gives us

\[
H[\varphi]\,(x) = \frac{1}{2\pi} \int_{-\pi}^{0} \left( \log \left| \sin\left(\frac{x-y}{2}\right) \right| - \log \left| \sin\left(\frac{x+y}{2}\right) \right| \right) \varphi'(y) \, dy.
\]

For \( x \in (-\pi, 0) \) and \( y \in (-\pi, 0) \) we have

\[
\left| \sin\left(\frac{x-y}{2}\right) \right| < \left| \sin\left(\frac{x+y}{2}\right) \right|.
\]

For \( y > x \) this follows from that \( |\sin(x)| \) is increasing as a distance to a multiple of \( \pi \) and that \( \frac{x-y}{2} \) is closer to zero than \( \frac{x+y}{2} \) is to zero or \(-\pi\), the case \( y < x \) can be reduced to the previous case by switching \( x \) and \( y \) and using that \( \sin \) is odd. It follows that

\[
\log \left| \sin\left(\frac{x-y}{2}\right) \right| - \log \left| \sin\left(\frac{x+y}{2}\right) \right| < 0.
\]

Since \( \varphi' \geq 0 \) we get \( H[\varphi] \leq 0 \). Furthermore \( \varphi \) is nonconstant and continuous so we have \( \varphi' > 0 \) in some open set, giving us \( H[\varphi] < 0 \).

Now, writing (3) as

\[
\varphi'(\varphi - c) = H[\varphi]
\]

and using \( H[\varphi] < 0 \) we get

\[
\varphi'(\varphi - c) < 0.
\]

Which implies that \( \varphi' > 0 \) and \( \varphi < c \). \(\square\)
As a consequence, any continuous, nonconstant, even function which is non-decreasing on \((-\pi, 0)\) that satisfy (3) almost everywhere must satisfy \(\varphi \leq c\). The maximal possible height is thus given by \(c\) and due to the function being even and nondecreasing on \((-\pi, 0)\) the maximal height has to be attained at \(x = 0\).

Now, the ansatz \(\varphi(x) = c - u(x)\) inserted in (3) gives an equation which does not explicitly depend on the wave speed \(c\). Indeed, inserting this gives us

\[
 uu' = -\mathcal{H}[u].
\] (5)

Note that a solution of this equation gives a solution of (3) for any wave speed \(c\). This is to be expected due to the Galilean change of variables

\[
 \varphi \mapsto \varphi + \gamma, \quad c \mapsto c + \gamma
\]

which leaves (3) invariant. In particular, taking \(c\) equal to the mean of \(u\) gives a zero mean solution. For a highest wave we expect to have \(\varphi(0) = c\), giving us \(u(0) = 0\). Integrating (5) gives us

\[
 \frac{1}{2} u^2 = -\mathcal{H}[u].
\] (6)

Here \(\mathcal{H}\) is the operator

\[
 \mathcal{H}[f](x) = \frac{1}{\pi} \text{p.v.} \int_{-\pi}^{\pi} \left( \log \left| \sin \left( \frac{x - y}{2} \right) \right| - \log \left| \sin \left( \frac{y}{2} \right) \right| \right) f(y) \, dy.
\] (7)

It is the integral of the Hilbert transform with the constant of integration taken such that \(\mathcal{H}[f](0) = 0\), this ensures that any solution of (6) satisfies \(u(0) = 0\). Note that any solution of (6) is a solution of (5) and hence gives a solution to (3).

To reduce the problem to a fixed point problem we write \(u\) as one, explicit, approximate solution of (6) and one unknown term. More precisely we make the ansatz

\[
 u(x) = u_0(x) + w(x)v(x)
\] (8)

where \(u_0(x)\) is an explicit, carefully chosen, approximate solution of (6) and \(w(x) = x \sqrt{\log \left( 1 + \frac{1}{|x|} \right)}\). By taking \(u_0(x) \sim \frac{1}{\pi} |x| \log |x|\), proving Theorem 1.1 reduces to proving existence of \(v \in L^\infty(\mathbb{T})\) such that the given ansatz is a solution of (6).

Inserting the ansatz (8) into (6) gives us

\[
 \frac{1}{2} (u_0 + wv)^2 = -\mathcal{H}[u_0 + wv]
\]

\[
 \iff \frac{1}{2} u_0^2 + u_0wv + \frac{1}{2} w^2v^2 = -\mathcal{H}[u_0] - \mathcal{H}[wv].
\]

By collecting all the linear terms in \(v\) we can write this as

\[
 u_0wv + \mathcal{H}[wv] = -\mathcal{H}[u_0] - \frac{1}{2} u_0^2 - \frac{1}{2} w^2v^2
\]
\[ v + \frac{1}{wu_0} \mathcal{H}[wv] = -\frac{1}{wu_0} \left( \mathcal{H}[u_0] + \frac{1}{2} u_0^2 \right) - \frac{w}{2u_0} v^2. \]

Now let \( T \) denote the operator
\[ T[v] = -\frac{1}{wu_0} \mathcal{H}[wv]. \tag{9} \]

Denote the weighted defect of the approximate solution \( u_0(x) \) by
\[ F(x) = \frac{1}{w(x)u_0(x)} \left( \mathcal{H}[u_0](x) + \frac{1}{2} u_0(x)^2 \right), \]
and let
\[ N(x) = \frac{w(x)}{2u_0(x)}. \]

Then we can write the above as
\[ (I - T)v = -F - Nv^2. \]

Assuming that \( I - T \) is invertible we can rewrite this as
\[ v = (I - T)^{-1} (-F - Nv^2) =: G[v]. \tag{10} \]

Hence proving the existence of \( v \) such that that \( u_0 + wv \) is a solution to (6) reduces to proving the existence of a fixed point of the operator \( G \).

Next we reduce the problem of proving that \( G \) has a fixed point to checking an inequality for three numbers that depend only on the choice of \( u_0 \) and \( w \). We let \( \|T\| \) denote the \( L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T}) \) norm of a linear operator \( T \).

**Proposition 2.2.** Let \( D_0 = \|T\|, \delta_0 = \|F\|_{L^\infty(\mathbb{T})} \) and \( n_0 = \|N\|_{L^\infty(\mathbb{T})} \). If \( D_0 < 1 \) and they satisfy the inequality
\[ \delta_0 < \frac{(1 - D_0)^2}{4n_0} \]
then for
\[ \epsilon = \frac{1 - D_0 - \sqrt{(1 - D_0)^2 - 4\delta_0n_0}}{2n_0} \]

and
\[ X_\epsilon = \{ v \in L^\infty(\mathbb{T}) : v(x) = v(-x), \|v\|_{L^\infty(\mathbb{T})} \leq \epsilon \} \]
we have
1. \( G(X_\epsilon) \subseteq X_\epsilon \);
2. \( \|G[v] - G[w]\|_{L^\infty(\mathbb{T})} \leq k_0 \|v - w\|_{L^\infty(\mathbb{T})} \) with \( k_0 < 1 \) for all \( v, w \in X_\epsilon \).
Proof. Using that $N$ and $F$ are even it can be checked that

$$G(X\epsilon) \subseteq (I - T)^{-1}X_{\delta_0 + n_0 \epsilon^2}.$$ 

Since $\|T\| < 1$ the operator $I - T$ is invertible and an upper bound of the norm of the inverse is given by $\frac{1}{1 - D_0}$, moreover $T$ takes even functions to even functions and hence so will $(I - T)^{-1}$. This gives us

$$G(X\epsilon) \subseteq (I - T)^{-1}X_{\delta_0 + n_0 \epsilon^2} \subseteq X_{\frac{\delta_0 + n_0 \epsilon^2}{1 - D_0}}.$$

The choice of $\epsilon$ then gives

$$\frac{\delta_0 + n_0 \epsilon^2}{1 - D_0} = \epsilon.$$ 

Next we have $G[v] - G[w] = (I - T)^{-1}(-N(v^2 - w^2))$ and hence

$$\|G[v] - G[w]\|_{L^\infty(T)} \leq \frac{n_0}{1 - D_0} \|v^2 - w^2\|_{L^\infty(T)} \leq \frac{2n_0 \epsilon}{1 - D_0} \|v - w\|_{L^\infty(T)}.$$ 

Where $k_0 = \frac{2n_0 \epsilon}{1 - D_0} < 1$ since $\epsilon < \frac{1}{2n_0 - D_0}$.

\[
3. \text{ Clausen Functions}
\]

We here give definitions and properties of the Clausen functions that are used in Sects. 4, 5 and 6. For more details about the Clausen functions see “Appendix B”.

The Clausen functions are related to the polylogarithm through

$$Cs(x) = \frac{1}{2} \left( Li_s(e^{ix}) + Li_s(e^{-ix}) \right) = \text{Re} \left( Li_s(e^{ix}) \right),$$

$$S_s(x) = \frac{1}{2} \left( Li_s(e^{ix}) - Li_s(e^{-ix}) \right) = \text{Im} \left( Li_s(e^{ix}) \right).$$

They behave nicely with respect to the Hilbert transform, for which we have

$$H[Cs](x) = S_s(x), \quad H[S_s](x) = -Cs(x).$$

In many cases we want to work with functions which are normalised to be zero at $x = 0$, for which we use the notation

$$\tilde{C}_s(x) = C_s(x) - C_s(0), \quad \tilde{C}_s^{(\beta)}(x) = C_s^{(\beta)}(x) - C_s^{(\beta)}(0).$$

With this notation we get for the operator $\mathcal{H}$,

$$\mathcal{H}[\tilde{C}_s](x) = -\tilde{C}_{s+1}(x), \quad \mathcal{H}[S_s](x) = -S_{s+1}(x).$$

From [20] we have the following expansion for $C_s(x)$ and $S_s(x)$, valid for $|x| < 2\pi$:

$$C_s(x) = \Gamma(1 - s) \sin \left(\frac{\pi}{2} s\right) |x|^{s-1} + \sum_{m=0}^{\infty} (-1)^m \zeta(s - 2m) \frac{x^{2m}}{(2m)!};$$
Lemma 4.1. Let $u$ be a bounded, even solution of (6) with the asymptotic behaviour
\[ v(x) = \frac{\pi}{2} \text{sgn}(x) |x|^{s-1} + \sum_{m=0}^{\infty} (-1)^m \zeta(s - 2m - 1) \frac{x^{2m+1}}{(2m+1)!}. \]

For the functions $C^{(\beta)}_x$ and $S^{(\beta)}_x$ we will mainly make use of $C^{(1)}_2(x)$ and $C^{(1)}_3(x)$, for which we have the expansions [2, Eq. 16], valid for $|x| < 2\pi$,
\[
C^{(1)}_2(x) = \zeta^{(1)}(2) - \frac{\pi}{2} |x| \log |x| - (\gamma - 1) \frac{\pi}{2} |x| + \sum_{m=1}^{\infty} (-1)^m \zeta^{(1)}(2 - 2m) \frac{x^{2m}}{(2m)!};
\]
\[
C^{(1)}_3(x) = \zeta^{(1)}(3) + \frac{1}{4} x^2 \log^2 |x| - \frac{3}{4 \pi} x^2 \log |x| - \frac{36 \gamma - 12 \gamma^2 - 24 \gamma_1 - 42 + \pi^2}{48} x^2
\]
\[
+ \sum_{m=2}^{\infty} (-1)^m \zeta^{(1)}(3 - 2m) \frac{x^{2m}}{(2m)!},
\]
where $\gamma_n$ is the Stieltjes constant and $\gamma = \gamma_0$. Bounds for the tails are given in Lemmas B.3 and B.4.

4. Construction of the Approximate Solution

In this section we give the construction of the approximate solution $u_0$. As a first step we determine the coefficient for the leading term in the asymptotic expansion. See also [16] for similar-flavoured results for a big class of equations.

Lemma 4.1. Let $u$ be a bounded, even solution of (6) with the asymptotic behaviour
\[ u(x) = -\frac{2v}{\pi} \tilde{C}^{(1)}_2(x) + o(|x||\log |x||) = v|x| \log |x| + o(|x||\log |x||) \]
close to zero, with $v \neq 0$. Then the coefficient is given by $v = -\frac{1}{\pi}$.

Proof. We directly get
\[
\frac{1}{2} u(x)^2 = \frac{v^2}{2} \log^2 |x| + o(|x|^2 \log^2 |x|).
\]

From the asymptotic expansion of $\tilde{C}^{(1)}_3(x)$ we obtain
\[
\mathcal{H}[u](x) = \frac{2v}{\pi} \tilde{C}^{(1)}_3(x) + \mathcal{H}[o(|x| \log |x||)](x) = \frac{v}{2\pi} x^2 \log^2 |x| + o(x^2 \log^2 |x|)
\]
\[
+ \mathcal{H}[o(|x| \log |x||)](x)
\]
\[
= \frac{v}{2\pi} x^2 \log^2 |x| + o(x^2 \log^2 |x|).
\]

We now focus on proving the last equality, which implies the lemma. Let $v(x) = o(|x| \log |x||).$ To prove that $\mathcal{H}[v](x) = o(x^2 \log^2 |x|)$ we need to prove that for any $\varepsilon > 0$, there exists $x_0 > 0$ such that
\[
|\mathcal{H}[v](x)| \leq \varepsilon x^2 \log^2 x \quad \text{for all} \quad 0 < x < x_0.
\]
For any $\epsilon'$, let $x_0'(\epsilon')$ be such that $|v(x)| < \epsilon'x \log x$ for all $0 < x < x_0'$ and let $E = \|v\|_{L^n}$. Using the integral representation of $H$ from (14) we then have
\[
|H[v](x)| \leq \frac{1}{\pi} \int_0^\pi |I(x, y)v(y)|\,dy \leq \frac{\epsilon'}{\pi} \int_0^{x_0'} |I(x, y)y\log y|\,dy + \frac{E}{\pi} \int_{x_0'}^\pi |I(x, y)|\,dy
\]
which implies
\[
E \hat{V}_1(x) + \frac{E}{\pi} V_2(x).
\]
with $I(x, y)$ as in (17).

The change of coordinates $t = y/x$ gives us
\[
V_1(x) = x^2 \int_0^{x_0'/x} |\hat{I}(x, t)\log(xt)|\,dt \quad \text{and} \quad V_2(x) = x \int_{x_0'/x}^\pi |\hat{I}(x, t)|\,dt,
\]
with $\hat{I}(x, t)$ as in (18). For $x < \frac{x_0'}{2}$ we can split $V_1$ at $t = 2$ and using that $|\log(xt)| \leq |\log x| + |\log t|$ this allows us to bound $V_1$ as
\[
V_1(x) \leq x^2 |\log x|(V_{1,1,1}(x) + V_{1,1,2}(x)) + x^2(V_{1,2,1}(x) + V_{1,2,2}(x))
\]
with
\[
V_{1,1,1}(x) = \int_0^2 |\hat{I}(x, t)|\,dt, \quad V_{1,1,2}(x) = \int_2^{x_0'/x} |\hat{I}(x, t)|\,dt,
\]
\[
V_{1,2,1}(x) = \int_0^2 |\hat{I}(x, t)\log t|\,dt, \quad V_{1,2,2}(x) = \int_2^{x_0'/x} |\hat{I}(x, t)\log t|\,dt.
\]
From Corollary 6.6 we have for $|x| \leq 1$
\[
|\hat{I}(x, t)| \leq \left|\log \left|1 - \frac{1}{t^2}\right|\right| + R,
\]
which implies $V_{1,1,1}(x) = O(1), V_{1,2,1}(x) = O(1)$.

For $t \geq 2$ we get from Corollary 6.6 that $|\hat{I}(x, t)| \leq \frac{2}{t^2} + Rx^2$. Using this together with $|x| \leq 1$ gives us the bounds
\[
V_{1,2,2}(x) \leq \int_2^{x_0'/x} \left(\frac{2}{t^2} + Rx^2\right)\log t\,dt = 2 \log \left(\frac{x_0'}{2x}\right) + R \left(\frac{x_0^2}{2} - 2x^2\right) \leq 2|\log x| + R.
\]
and
\[
V_{1,2,2}(x) \leq \int_2^{x_0'/x} \left(\frac{2}{t^2} + Rx^2\right)\log t\,dt
\]
\[
= \left(\log^2 \left(\frac{x_0'}{x}\right) - \log^2 2\right) + \frac{R}{2} \left(\frac{x_0^2}{2} \log \left(\frac{x_0'}{x}\right) - \frac{1}{2}\right) - x^2(4\log 2 - 2)
\]
\[
\leq \log^2 x + R|\log x|.
\]
Combining the above bounds we get that $V_1(x) \leq C_1x^2 \log^2 x$ for $x$ sufficiently small and $C_1$ a universal constant. For $V_2$ we can use the same approach as for $V_{1,1,2}$ and $V_{1,2,2}$ to get that $V_2(x) \leq C_2x^2$. This gives us
\[
|H[v](x)| \leq \frac{\epsilon'C_1}{\pi}x^2 \log^2 x + \frac{EC_2}{\pi}x^2
\]
and the result follows from taking $\epsilon'$ and $x$ sufficiently small. \qed
In addition to having the correct asymptotic behaviour we want \( u_0 \) to be a good approximate solution of (6), in the sense that we want the defect,

\[
F(x) = \frac{1}{w(x)u_0(x)} \left( \mathcal{H}[u_0](x) + \frac{1}{2} u_0(x)^2 \right),
\]
to be small for \( x \in \mathbb{T} \). The hardest part is to make \( F(x) \) small locally near the singularity at \( x = 0 \), this is done by studying the asymptotic behaviour of \( \mathcal{H}[u_0](x) + \frac{1}{2} u_0(x)^2 \). Ones the defect is sufficiently small near \( x = 0 \) it can be made small globally by adding a suitable trigonometric polynomial.

The construction is similar to that in [20], the main difference is that the asymptotic behaviour is more complicated in our case. We take \( u_0 \) to be a combination of three parts:

1. The first part is the term \(-\frac{2}{\pi^2} \tilde{C}_2^{(1)}\), where the coefficient is chosen to give the right asymptotic behaviour according to Lemma 4.1.
2. The second part is chosen to make the defect small near \( x = 0 \) and, similarly to in [20], it is given by a sum of Clausen functions.
3. The third part is chosen to make the defect small globally and is given by a trigonometric polynomial.

More precisely the approximation is given by

\[
u_0(x) = \frac{2}{\pi^2} \tilde{C}_2^{(1)}(x) + \sum_{j=1}^{N_0} a_j \tilde{C}_{s_j}(x) + \sum_{n=1}^{N_1} b_n \cos(nx) - 1,
\]

(11)

To ensure that the leading asymptotics are determined by \( \tilde{C}_2^{(1)}(x) \) we require that \( s_j \geq 2 \).

We want to chose the values of \( a_j \) and \( s_j \) to make the defect small near \( x = 0 \). Taking \( u_0 = \frac{2}{\pi^2} \tilde{C}_2^{(1)} \) gives us that the leading term in the expansion of \( \mathcal{H}[u_0] + \frac{1}{2} u_0^2 \) is of order \( |x|^2 \log|x| \). A natural choice would then be to take the next term as a multiple of \( \tilde{C}_2 \), for which \( \mathcal{H}[\tilde{C}_2] \) behaves like \( |x|^2 \log|x| \). However, its contribution to the \( |x|^2 \log|x| \) term of \( \mathcal{H}[u_0] \) and \( \frac{1}{2} u_0^2 \) turns out to exactly cancel out and we are left with no improvement to the asymptotic behaviour.

Instead we take inspiration from the fractional KdV equations (4), which for \( \alpha = -1 \) reduces to the Burgers–Hilbert equation. To chose \( a_j \) and \( s_j \) we study the limit \( \alpha \rightarrow -1^+ \). For \(-1 < \alpha < 0\) the fractional KdV equations, like the Burgers–Hilbert equation, admits a highest cusped traveling wave solution, as recently proved in [49]. In a forthcoming [12] work we show that the traveling waves asymptotically at \( x = 0 \) behave like \( c - \nu_\alpha |x|^{-\alpha} \), with

\[
\nu_\alpha = \frac{2 \Gamma(2 \alpha) \cos(\pi \alpha)}{\Gamma(\alpha) \cos(\frac{\pi}{2} \alpha)}.
\]

Following the same approach as in Section 2 proving the existence of a highest cusped wave for the fractional KdV equations can be reduced to studying the equation

\[
\frac{1}{2} u^2 = -\mathcal{H}^\alpha[u],
\]
where $\mathcal{H}^\alpha[u](x) = |D|^\alpha[u](x) - |D|^\alpha[u](0)$. By studying the asymptotic behaviour of $\mathcal{H}^\alpha[u] + \frac{1}{2}u^2$ and following the same reasoning as in [20] a suitable approximation for this equation is given by

$$
a_{\alpha,0} \tilde{C}_{1-\alpha}(x) + \sum_{j=1}^{N_0} a_{\alpha,j} \tilde{C}_{1-\alpha+jp_\alpha}(x)
$$

with

$$
a_{\alpha,0} = \frac{2\Gamma(2\alpha) \cos(\pi\alpha)}{\Gamma(\alpha)^2 \cos(\frac{\pi}{2}\alpha)^2}
$$

and $p_\alpha$ a solution of

$$
\Gamma(\alpha) \cos\left(\frac{\pi}{2}\alpha\right) a_{\alpha,0} \Gamma(\alpha - p_\alpha) \cos\left(\frac{\pi}{2}(\alpha - p_\alpha)\right) - \Gamma(2\alpha - p_\alpha) \cos\left(\frac{\pi}{2}(2\alpha - p_\alpha)\right) = 0.
$$

One can numerically solve for $a_{\alpha,j}$ by considering the asymptotic expansion of the defect. As $\alpha \to -1$ we have $a_{\alpha,0} \to -\infty$ and $p_\alpha \to 0$. Numerically one also sees that $a_{\alpha,1} \to \infty$, in such a way that $a_{\alpha,0} + a_{\alpha,1}$ remains bounded. The other coefficients, $a_{\alpha,j}$ for $j \geq 2$, all remain bounded. Furthermore we note that as $\alpha$ approaches $-1$, the function $a_{\alpha,0}(\tilde{C}_{1-\alpha}(x) - \tilde{C}_{1-\alpha+p_\alpha})$ approaches $\frac{2}{\pi^2} \tilde{C}_2^{(1)}(x)$. The remaining part,

$$
(a_{\alpha,0} + a_{\alpha,1}) \tilde{C}_{1-\alpha+p_\alpha}(x) + \sum_{j=2}^{N_0} a_{\alpha,j} \tilde{C}_{1-\alpha+jp_\alpha}(x),
$$

is numerically seen to converge to a function as $\alpha \to -1$, as long as $N_0$ is increased as we approach $\alpha = -1$. This gives a hint on how to choose $a_j$ and $s_j$ for $u_0$, we take it according to

$$
(a_{\alpha,0} + a_{\alpha,1}) \tilde{C}_{1-\alpha+p_\alpha}(x) + \sum_{j=2}^{N_0} a_{\alpha,j} \tilde{C}_{1-\alpha+jp_\alpha}(x).
$$

For this we fix some $\alpha$ close to $-1$ and compute $p_\alpha$ as well as the coefficients $a_{\alpha,j}$. We then take $a_1 = a_{\alpha,0} + a_{\alpha,1}$, $a_j = a_{\alpha,j}$ for $j \geq 2$ and

$$
s_j = 1 - \alpha + jp_\alpha.
$$

As long as $\alpha$ is sufficiently close to $-1$ and $N_0$ is sufficiently high this gives a small enough defect close to $x = 0$. While not obvious from the expression of $p_\alpha$ we do have $1 - \alpha + p_\alpha > 2$, as required.

The coefficients $b_n$ are then taken to make the defect small globally. This is done by taking $N_1$ equally spaced points $\{x_n\}_{1 \leq n \leq N_1}$ on the interval $(0, \pi)$ and numerically solving the non-linear system

$$\mathcal{H}[u_0](x_n) + \frac{1}{2}u_0(x_n)^2 = 0 \quad \text{for} \quad 1 \leq n \leq N_1.$$
Remark 4.2. The approximation $u_0$ only needs to be an approximate solution to the equation, as opposed to the coefficient for $\tilde{C}_2^{(1)}$, which has to match exactly to obtain the right asymptotic behaviour.

With this approximation we get

$$
\mathcal{H}[u_0](x) = -\frac{2}{\pi^2} \tilde{C}_3^{(1)}(x) - \sum_{j=1}^{N_0} a_j \tilde{C}_{1+s_j}(x) - \sum_{n=1}^{N_1} b_n \frac{(\cos(nx) - 1)}{n}.
$$

(12)

Furthermore we have the following asymptotic expansions for the approximation, which we give without proof since they follow directly from the expansions of the Clausen and trigonometric functions.

Lemma 4.3. The approximation $u_0$ given by Eq. (11) has the asymptotic expansions

$$
u_0(x) = -\frac{1}{\pi} |x| \log |x| - \frac{\gamma - 1}{\pi} |x| + \sum_{j=1}^{N_0} a_j^0 |x|^{-\alpha + j p_\alpha}$$

$$+ \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m)!} \left( \frac{2}{\pi^2} \xi(1)(2 - 2m) + \sum_{j=1}^{N_0} a_j \xi(1 - \alpha + j p_\alpha - 2m) + \sum_{n=1}^{N_1} b_n n^{2m} \right) x^{2m}$$

and

$$
\mathcal{H}[u_0(x)] = -\frac{1}{2\pi^2} x^2 \log^2 |x| + \frac{3 - 2\gamma}{2\pi^2} x^2 \log |x| - \sum_{j=1}^{N_0} A_j^0 |x|^{1-\alpha + j p_\alpha}$$

$$+ \frac{1}{2} \left( \frac{36\gamma - 12\gamma^2 - 24\gamma_1 - 42 + \pi^2}{12\pi^2} + \sum_{j=1}^{N_0} a_j \xi(-\alpha + j p_\alpha) + \sum_{n=1}^{N_1} b_n n \right) x^2$$

$$- \sum_{m=2}^{\infty} \frac{(-1)^m}{(2m)!} \left( \frac{2}{\pi^2} \xi(1)(3 - 2m) + \sum_{j=1}^{N_0} a_j \xi(2 - \alpha + j p_\alpha - 2m) + \sum_{n=1}^{N_1} b_n n^{2m-1} \right) x^{2m}$$

valid for $|x| < 2\pi$, where

$$a_j^0 = \Gamma(\alpha - j p_\alpha) \cos \left( (\alpha - j p_\alpha) \frac{\pi}{2} \right) a_j;$$

$$A_j^0 = \Gamma(\alpha - 1 - j p_\alpha) \cos \left( (\alpha - 1 - j p_\alpha) \frac{\pi}{2} \right) a_j.$$
away from $x = 0$ and is therefore easier to approximate there. The second reason is that although the Clausen functions in the approximation are chosen to make the defect near $x = 0$ small, they also turn out to give a fairly good approximation away from zero.

5. Bounding $n_0$ and $\delta_0$

Recall that

$$n_0 := \|N\|_{L^\infty(T)} = \sup_{x \in [0, \pi]} |N(x)|,$$

$$\delta_0 := \|F\|_{L^\infty(T)} = \sup_{x \in [0, \pi]} |F(x)|,$$

where

$$N(x) = \frac{x \sqrt{\log(1 + 1/x)}}{2u_0(x)} \quad \text{and} \quad F(x) = \frac{\mathcal{H}[u_0](x) + \frac{1}{\pi} u_0(x)^2}{x \sqrt{\log(1 + 1/x) u_0(x)}}.$$

For fixed $x > 0$ which is not too small we can compute accurate enclosures of $N(x)$ as well as $F(x)$ using interval arithmetic. This allows us to compute

$$\sup_{x \in [\epsilon, \pi]} |N(x)| \quad \text{and} \quad \sup_{x \in [\epsilon, \pi]} |F(x)|$$

for some fixed $\epsilon > 0$. As $x \to 0$ the numerators and denominators tend to zero in both $N$ and $F$ and we need to handle the removable singularities. We start with the following lemma:

**Lemma 5.1.** The function $\frac{-|x| \log |x|}{u_0(x)}$ is positive and bounded at $x = 0$ and for $|x| < 1$ it has the expansion

$$-\frac{|x| \log |x|}{u_0(x)} = -\left(-\frac{1}{\pi} - \frac{\gamma - 1}{\pi} \frac{1}{\log |x|} + \sum_{j=1}^{N_0} a_j |x|^{-1 - \alpha + j \rho_\alpha} \frac{1}{\log |x|} \right)$$

$$+ \sum_{m=1}^\infty \frac{(-1)^m}{(2m)!} \left( \frac{2}{\pi^2} \zeta(1) (2 - 2m) + \sum_{j=1}^{N_0} a_j \zeta(1 - \alpha + j \rho_\alpha - 2m) \right)$$

$$+ \sum_{n=1}^{N_1} b_n n^{2m} \left( \frac{|x|^{2m-1}}{\log |x|} \right)^{-1}.$$

**Proof.** The expansion follows directly from the expansion of $u_0$ given in Lemma 4.3 and canceling the $|x| \log |x|$ factor.

The function $\frac{1}{\log |x|}$ goes to zero at $x = 0$ and is bounded nearby and by construction $\alpha$ and $\rho$ are taken such that $-1 - \alpha + j \rho > 0$ for $j \geq 1$. This means that all terms except $-\left(-\frac{1}{\pi}\right)^{-1} = \pi$ tend to zero as $x \to 0$. The value at $x = 0$ will hence be given by $-\left(-\frac{1}{\pi}\right)^{-1} = \pi$. □
In light of this lemma we do the split

\[ N(x) = \frac{\sqrt{\log (1 + 1/x)}}{-2 \log x} \cdot \frac{-x \log x}{u_0(x)}. \]

The second factor can then be bounded using the lemma. For the first factor it is enough to notice that it tends to zero at \( x = 0 \) and is increasing in \( x \) on \((0, 1)\). For \( F(x) \) we do the split

\[ F(x) = -\frac{1}{\sqrt{\log(1 + 1/x)}} \cdot \frac{-x \log x}{u_0(x)} \cdot \frac{\mathcal{H}[u_0](x) + \frac{1}{2}u_0(x)^2}{x^2 \log x} \quad (13) \]

The first factor can be handled by noticing that it tends to zero at \( x = 0 \) and is increasing in \( x \), the second factor is handled using the above lemma. What remains is to handle the third factor, which is done in the following lemma:

**Lemma 5.2.** The function

\[ \frac{\mathcal{H}[u_0](x) + \frac{1}{2}u_0(x)^2}{x^2 \log |x|} \]

is non-zero and bounded at \( x = 0 \) and for \( |x| < 1 \) it has the expansion

\[
\frac{\mathcal{H}[u_0](x) + \frac{1}{2}u_0(x)^2}{x^2 \log x} = \frac{1}{2\pi^2} + \frac{1}{2} \left( \frac{(\gamma - 1)^2}{\pi^2} + \frac{36\gamma - 12\gamma^2 - 24\gamma_1 - 42 + \pi^2}{12\pi^2} + \sum_{j=1}^{N_0} a_j \xi(-\alpha + jp_\alpha) + \sum_{n=1}^{N_1} b_{n\alpha} \right) \frac{1}{\log |x|} \\
- \frac{1}{\pi} \left( 1 + \frac{\gamma - 1}{\log |x|} \right) \sum_{j=1}^{N_0} a_j^0 |x|^{-1-\alpha+jp_\alpha} + \frac{1}{2 \log |x|} \left( \sum_{j=1}^{N_0} a_j^0 |x|^{-1-\alpha+jp_\alpha} \right)^2 \\
- \frac{1}{\log |x|} \sum_{j=1}^{N_0} A_j^0 |x|^{-1-\alpha+jp_\alpha} \\
- \frac{1}{\pi} \left( 1 + \frac{\gamma - 1}{\log |x|} \right) \frac{S_1}{|x|} + \frac{1}{\log |x|} \left( \sum_{j=1}^{N_0} a_j^0 |x|^{-1-\alpha+jp_\alpha} \right) \frac{S_1}{|x|} + \frac{1}{2} \frac{S_1^2}{x^2 \log |x|} - \frac{S_2}{x^2 \log |x|},
\]

where

\[
S_1 = \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m)!} \left( \frac{2}{\pi^2} \zeta^{(1)}(2 - 2m) + \sum_{j=1}^{N_0} a_j \xi(1 - \alpha + jp_\alpha - 2m) + \sum_{n=1}^{N_1} b_{n\alpha} n^{2m} \right) |x|^{2m} \\
S_2 = \sum_{m=2}^{\infty} \frac{(-1)^m}{(2m)!} \left( \frac{2}{\pi^2} \zeta^{(1)}(3 - 2m) + \sum_{j=1}^{N_0} a_j \xi(2 - \alpha + jp_\alpha - 2m) + \sum_{n=1}^{N_1} b_{n\alpha} n^{2m-1} \right) x^{2m}
\]
Proof. From the expansions of \( u_0 \) in Lemma 4.3 we get
\[
\frac{1}{2} u_0(x)^2 = \frac{1}{2\pi^2} x^2 \log^2 |x| + \frac{\gamma - 1}{\pi^2} x^2 \log |x| + \frac{(\gamma - 1)^2}{2 \pi^2} x^2 \\
- \frac{1}{\pi} (\log |x| + \gamma - 1) \sum_{j=1}^{N_0} a_j^0 |x|^{1-\alpha+jp_0} + \frac{1}{2} \left( \sum_{j=1}^{N_0} a_j^0 |x|^{-\alpha+jp_0} \right)^2 \\
- \frac{1}{\pi} (\log |x| + \gamma - 1) |x| S_1 + \left( \sum_{j=1}^{N_0} a_j^0 |x|^{-\alpha+jp_0} \right) S_1 + \frac{1}{2} S_1^2.
\]
This together with the expansion for \( \mathcal{H}[u_0] \) gives us
\[
\mathcal{H}[u_0](x) + \frac{1}{2} u_0(x)^2 = \frac{1}{2 \pi} x^2 \log |x| + \frac{(\gamma - 1)^2}{12 \pi^2} x^2 \\
+ \frac{1}{\pi} \left( \log |x| + \gamma - 1 \right) \sum_{j=1}^{N_0} a_j^0 |x|^{1-\alpha+jp_0} + \frac{1}{2} \left( \sum_{j=1}^{N_0} a_j^0 |x|^{-\alpha+jp_0} \right)^2 \\
- \frac{1}{\pi} \left( \log |x| + \gamma - 1 \right) |x| S_1 + \left( \sum_{j=1}^{N_0} a_j^0 |x|^{-\alpha+jp_0} \right) S_1 + \frac{1}{2} S_1^2 - S_2.
\]
Division by \( x^2 \log |x| \) gives the required expansion. Note that the two leading terms in \( \frac{1}{2} u_0 \) and \( \mathcal{H}[u_0] \) exactly cancel out, this is needed for the result to be bounded near \( x = 0 \).

6. Analysis of \( T \) and Bounding \( D_0 \)

In this section we give more details about the operator \( T \) defined by
\[
T[v] = -\frac{1}{wu_0} \mathcal{H}[wv]
\]
and show how to bound \( D_0 := \|T\| \).

For an even function \( v(x) \) and \( 0 < x < \pi \) we can write (7) as
\[
\mathcal{H}[v](x) = \frac{1}{\pi} \int_0^\pi \log \left( \frac{\sin((x - y)/2) \sin((x + y)/2)}{\sin(y/2)^2} \right) v(y) \, dy.
\] (14)
Using that \( C_1(x) = -\log(2 \sin(|x|/2)) \) this can alternatively be written as
\[
\mathcal{H}[v](x) = \frac{1}{\pi} \int_0^\pi - (C_1(x - y) + C_1(x + y) - 2C_1(y)) v(y) \, dy.
\] (15)
From (14) we have
\[
T[v](x) = \frac{1}{\pi w(x)u_0(x)} \int_0^\pi \log \left( \frac{\sin(|x - y)/2) \sin((x + y)/2)}{\sin(y/2)^2} \right) w(y)v(y) \, dy.
\]
The norm of $T$ is then given by
\[
D_0 = \|T\| = \sup_{x \in [0, \pi]} \left| \frac{1}{\pi w(x)u_0(x)} \int_0^\pi \log \left( \frac{\sin(|x-y|/2) \sin((x+y)/2)}{\sin(y/2)^2} \right) w(y) \ dy \right|.
\]

(16)

Let
\[
I(x, y) = \log \left( \frac{\sin(|x-y|/2) \sin((x+y)/2)}{\sin(y/2)^2} \right),
\]

and
\[
U(x) = \int_0^\pi |I(x, y)w(y)| \ dy = \int_0^\pi |I(x, y)|y\sqrt{\log(1+1/y)} \ dy.
\]

We are then interested in computing
\[
D_0 = \|T\| = \sup_{x \in [0, \pi]} \left| \frac{U(x)}{\pi w(x)u_0(x)} \right|.
\]

We use the notation
\[
T(x) = \frac{U(x)}{\pi w(x)u_0(x)}.
\]

The integrand of $U(x)$ has a singularity at $y = x$. It is therefore natural to split the integral into two parts
\[
U_1(x) = \int_0^x |I(x, y)|y\sqrt{\log(1+1/y)} \ dy = x^2 \int_0^1 |\hat{I}(x, t)|t\sqrt{\log(1+1/(xt))} \ dt,
\]
\[
U_2(x) = \int_x^\pi |I(x, y)|y\sqrt{\log(1+1/y)} \ dy,
\]

where
\[
\hat{I}(x, t) = \log \left( \frac{\sin(|x(1-t)|/2) \sin(x(1+t)/2)}{\sin(xt/2)^2} \right).
\]

(18)

The next two lemmas give information about the sign of $\hat{I}(x, t)$ and $I(x, y)$, allowing us to remove the absolute value.

**Lemma 6.1.** For all $x \in (0, \pi)$ the function $\hat{I}(x, t)$ is decreasing and continuous in $t$ for $t \in (0, 1)$ and has the limits
\[
\lim_{t \to 0^+} \hat{I}(x, t) = \infty,
\]
\[
\lim_{t \to 1^-} \hat{I}(x, t) = -\infty.
\]

Moreover, the unique root, $r_x$, is decreasing in $x$ and satisfies the inequality
\[
\frac{1}{2} < r_x < \frac{1}{\sqrt{2}}.
\]
**Proof.** The left and right limits are easily checked and it is clear that the function is continuous in $t$ on the interval. To show that it is decreasing in $t$ we split the log into three terms and differentiate, giving us

$$\frac{d}{dt} \hat{I}(x, t) = \frac{1}{2} x (-\cot(x(1-t)/2) + \cot(x(1+t)/2) - 2 \cot(xt/2)).$$

We want to prove that this is negative. Note that $0 < xt/2 < \pi/2$ and that cot is positive on the interval $(0, \pi/2)$, hence it is enough to check

$$-\cot(x(1-t)/2) + \cot(x(1+t)/2) < 0,$$

which follows immediately from the monotonicity of cot on the interval $[0, \pi]$. This proves the existence of a unique root $r_x$ on the interval $(0, 1)$.

To prove that $r_x$ is decreasing in $x$ it is enough to prove that $\hat{I}(x, t)$ is decreasing in $x$. Differentiating with respect to $x$ gives us

$$\frac{d}{dx} \hat{I}(x, t) = \frac{1}{2} (((1-t) \cot(x(1-t)/2) + (1+t) \cot(x(1+t)/2) - 2t \cot(xt/2)),$$

which we want to prove is negative. Letting $g(t) = t \cot(xt/2)$ and using that $g(1) \leq g(t)$ for $t \in (0, 1)$ it is enough to show that

$$g(1-t) + g(1+t) \leq 2g(1).$$

This holds since $g$ is concave, as

$$g''(t) = \frac{x}{2 \sin^2(xt/2)} (xt \cot(xt/2) - 2) < 0$$

following from $\frac{x}{2 \sin^2(xt/2)} > 0$ and $xt \cot(xt/2) \leq 2$.

Finally, to see that $\frac{1}{2} < r_x < \frac{1}{\sqrt{2}}$ it is enough to check that for $x = \pi$ the root is given by $t = \frac{1}{2}$ and that

$$\lim_{x \to 0^+} \hat{I}(x, t) = \lim_{x \to 0^+} \log \left( \frac{x(1-t)/2 \cdot x(1+t)/2}{(xt/2)^2} \right) = -\log \left( \frac{1}{t^2} - 1 \right)$$

together with the monotonicity in $x$. 

**Lemma 6.2.** For $x \in (0, \pi)$ we have $I(x, y) < 0$ for all $y \in (x, \pi)$.

**Proof.** The function $f(y) = -\log(\sin(y/2))$ is strictly convex on the domain $(0, 2\pi)$. This follows from the fact that $-\log(y)$ is strictly convex and decreasing and $\sin(y/2)$ is strictly concave on the interval. It immediately follows that

$$I(x, y) = \log(\sin((y-x)/2))) + \log((\sin((y+x)/2))) - 2\log((\sin(y/2))) < 0.$$
With these two lemmas we can slightly simplify $U_1$ and $U_2$, we get

\[ U_1(x) = x^2 \int_0^x \log \left( \frac{\sin(x(1-t)/2) \sin(x(1+t)/2)}{\sin(xt/2)^2} \right) t \sqrt{\log(1 + 1/(xt))} \, dt \]

\[ -x^2 \int_{r_x}^1 \log \left( \frac{\sin(x(1-t)/2) \sin(x(1+t)/2)}{\sin(xt/2)^2} \right) t \sqrt{\log(1 + 1/(xt))} \, dt \]

\[ = x^2 (U_{1,1}(x) + U_{1,2}(x)) \]

and

\[ U_2(x) = - \int_{r_x}^\pi \log \left( \frac{\sin((y-x)/2) \sin((x+y)/2)}{\sin(y/2)^2} \right) y \sqrt{\log(1 + 1/y)} \, dy. \]

Similarly to in the previous section we divide the interval $[0, \pi]$, on which we take the supremum, into two parts, $[0, \epsilon]$ and $[\epsilon, \pi]$.

For the interval $[\epsilon, \pi]$ we split $T(x)$ as

\[ T(x) = \frac{U(x)}{\pi x \sqrt{\log(1 + 1/x)u_0(x)}} = \frac{1}{u_0(x)} \left( \frac{x(U_{1,1}(x) + U_{1,2}(x))}{\pi \sqrt{\log(1 + 1/x)}} + \frac{U_2(x)}{\pi x \sqrt{\log(1 + 1/x)}} \right). \]

See “Appendix C” for details on how $U_{1,1}$, $U_{1,2}$ and $U_2$ are computed.

For the interval $[0, \epsilon]$ write $T(x)$ as

\[ T(x) = \frac{1}{\pi} \cdot \frac{-x \log x}{u_0(x)} \cdot \left( \frac{U_1(x)}{-x^2 \log x \sqrt{\log(1 + 1/x)}} + \frac{U_2(x)}{-x^2 \log x \sqrt{\log(1 + 1/x)}} \right). \]

The factor $\frac{-x \log x}{u_0(x)}$ is handled by Lemma 5.1 as before. For the two remaining terms we have the following two lemmas:

**Lemma 6.3.** For $x \in [0, \epsilon]$ with $\epsilon < 1$ we have

\[ \frac{U_1(x)}{-x^2 \log x \sqrt{\log(1 + 1/x)}} \leq \frac{1}{\sqrt{\log(1 + 1/x)}} \left( \frac{\log 2}{\sqrt{\log(1/x)}} + \frac{c_1 + \log 2 \sqrt{\log(1 + x)}}{\log(1/x)} \right) + \frac{3R_1}{8} x^2 \left( \frac{2}{\sqrt{\log(1/x)}} + \frac{\sqrt{\pi/2} + 2\sqrt{\log(1 + x)}}{\log(1/x)} \right) \]

where

\[ c_1 = \int_0^1 \log(1/t^2 - 1) |t| \sqrt{\log(1/t)} \, dt, \]

\[ R_1 = \sup_{y \in [0, \epsilon]} \frac{1}{2} \left| \frac{d^2}{dy^2} \log \left( \frac{\sin(y)}{y} \right) \right|. \]

**Proof.** As a first step we split $I(x, t)$ into one main term and one remainder term. We can write $I(x, t)$ as

\[ I(x, t) = \log \left( \sin \left( \frac{x(1-t)}{2} \right) \right) + \log \left( \sin \left( \frac{x(1+t)}{2} \right) \right) - 2 \log \left( \sin \left( \frac{xt}{2} \right) \right). \]
Using that
\[
\log \left( \sin \left( \frac{x(1-t)}{2} \right) \right) = \log \left( \text{sinc} \left( \frac{x(1-t)}{2} \right) \right) + \log \left( \frac{x(1-t)}{2} \right),
\]
where \( \text{sinc} x = \frac{\sin x}{x} \), and similarly for the other log-sin terms, we can split \( \hat{I}(x, t) \) as
\[
\hat{I}(x, t) = \log \left( \frac{x(1-t)}{2} \right) + \log \left( \frac{x(1+t)}{2} \right) - 2 \log \left( \frac{xt}{2} \right)
+ \log \left( \text{sinc} \left( \frac{x(1-t)}{2} \right) \right) + \log \left( \text{sinc} \left( \frac{x(1+t)}{2} \right) \right) - 2 \log \left( \text{sinc} \left( \frac{xt}{2} \right) \right).
\]
where
\[
\log \left( \frac{x(1-t)}{2} \right) + \log \left( \frac{x(1+t)}{2} \right) - 2 \log \left( \frac{xt}{2} \right)
= \log(1-t) + \log(1+t) - 2 \log(t)
= \log(1/t^2 - 1).
\]
Note that for small \( x \), \( \text{sinc}(x) \) is close to one, and the corresponding log-terms will therefore be small. We split \( U_1(x) \) as
\[
U_1(x) \leq x^2 \int_0^1 \left| \log(1/t^2 - 1) \right| t \sqrt{\log(1 + 1/(xt))} \, dt
+ x^2 \int_0^1 \left| \log \left( \text{sinc} \left( \frac{x(1-t)}{2} \right) \right) + \log \left( \text{sinc} \left( \frac{x(1+t)}{2} \right) \right) \right|
- 2 \log \left( \text{sinc} \left( \frac{xt}{2} \right) \right) \, dt
= U_{1,m}(x) + U_{1,r}(x).
\]
Focusing in \( U_{1,m} \) we note that for \( t \in (0, 1) \) we have
\[
\sqrt{\log(1 + 1/(xt))} = \sqrt{\log(1/x) + \log(1/t) + \log(1 + xt)}
\leq \sqrt{\log(1/x) + \sqrt{\log(1/t)} + \sqrt{\log(1 + xt)}}
\leq \sqrt{\log(1/x) + \sqrt{\log(1/t)} + \sqrt{\log(1 + x)}}.
\]
Hence
\[
U_{1,m}(x) \leq x^2 \left( \sqrt{\log(1/x)} \int_0^1 \left| \log(1/t^2 - 1) \right| t \, dt + \int_0^1 \left| \log(1/t^2 - 1) \right| t \sqrt{\log(1/t)} \, dt \right.
+ \sqrt{\log(1 + x)} \int_0^1 \left| \log(1/t^2 - 1) \right| t \, dt \right).
\]
We have \( \int_0^1 \left| \log(1/t^2 - 1) \right| t \, dt = \log 2 \) and if we let
\[
c_1 = \int_0^1 \left| \log(1/t^2 - 1) \right| t \sqrt{\log(1/t)} \, dt
\]
this gives us

\[ U_{1,m}(x) \leq x^2 \left( \sqrt{\log(1/x)} \log 2 + c_1 + \log 2 \sqrt{\log(1 + x)} \right). \]

For \( U_{1,r} \) we will give a uniform bound of

\[ \log \left( \sin \left( \frac{x(1-t)}{2} \right) \right) + \log \left( \sin \left( \frac{x(1+t)}{2} \right) \right) - 2 \log \left( \sin \left( \frac{xt}{2} \right) \right) \]

in \( t \) on the interval \([0, 1]\) and use this to simplify the integrand. Note that \( \frac{x(1-t)}{2}, \frac{x(1+t)}{2} \) and \( \frac{xt}{2} \) all lie on the interval \([0, x]\). The function \( \log(\sin(y)) \) is analytic around \( y = 0 \) and the first two terms in the Taylor expansion are zero, by Taylor’s Theorem we hence have

\[ |\log(\sin(y))| \leq R_1 y^2 \]

where

\[ R_1 = \sup_{y \in [0,x]} \frac{1}{2} \left| \frac{d}{dy^2} \log(\sin(y)) \right|. \]

This gives us

\[ \left| \log \left( \sin \left( \frac{x(1-t)}{2} \right) \right) + \log \left( \sin \left( \frac{x(1+t)}{2} \right) \right) - 2 \log \left( \sin \left( \frac{xt}{2} \right) \right) \right| \]

\[ \leq R_1 x^2 \left( \frac{(1-t)^2}{4} + \frac{(1+t)^2}{4} + \frac{t^2}{2} \right) \leq \frac{3R_1}{2} x^2. \]

With this we get

\[ U_{1,r}(x) \leq \frac{3R_1}{2} x^4 \int_0^1 t \sqrt{\log(1 + 1/(xt))} \, dt. \]

Using the inequality (19) we have

\[ \int_0^1 t \sqrt{\log(1 + 1/(xt))} \, dt \leq \sqrt{\log(1/x)} \int_0^1 t \, dt \]

\[ + \int_0^1 t \sqrt{\log(1/t)} \, dt + \sqrt{\log(1 + x)} \int_0^1 t \, dt. \]

With

\[ \int_0^1 t \, dt = \frac{1}{2}, \quad \int_0^1 t \sqrt{\log(1/t)} \, dt = \frac{\sqrt{\pi/2}}{4} \]

this gives

\[ U_{1,r}(x) \leq \frac{3R_1}{8} x^4 \left( 2\sqrt{\log(1/x)} + \sqrt{\pi}/2 + 2\sqrt{\log(1 + x)} \right) \]

Combining the bound for \( U_{1,m} \) with that for \( U_{1,r} \) gives us the result. \( \square \)
Lemma 6.4. For $x \in [0, \epsilon]$ with $\epsilon < \frac{1}{2}$ we have

\[
\frac{U_2(x)}{-x^2 \log x \sqrt{\log(1 + 1/x)}} \leq \frac{\log \left( \frac{16}{3\sqrt{3}} \right)}{\log(1/x)} + \left( \frac{2}{\log(1/(2x))} \right)^{\frac{3}{2}} + \frac{R_2 \sqrt{\log(1/(2x))}}{8 \log(1/x) \sqrt{\log(1 + 1/x)}} + \frac{\sqrt{\log 2}}{\sqrt{\log(1 + 1/x)}} \]

\[
- \frac{\log(2)^{\frac{3}{2}}}{\log(1/x) \sqrt{\log(1 + 1/x)}} + \frac{R_2 \sqrt{\log 2(1 - 4x^2)}}{8 \log(1/x) \sqrt{\log(1 + 1/x)}} \]

\[
+ \frac{\sqrt{\log 2}}{\log(1/x) \sqrt{\log(1 + 1/x)}} \left( \frac{1}{2} \log \left( \frac{\pi^2 - x^2}{1 - x^2} \right) + \frac{\log(1 - x^2)}{2x^2} - \frac{\pi^2 \log \left( \frac{1 - x^2}{\pi^2} \right)}{2x^2} \right) \]

\[
+ \frac{D_1 c_2}{\log(1/x) \sqrt{\log(1 + 1/x)}}
\]

where

\[
c_2 = \int_0^\pi y \sqrt{\log(1 + 1/y)} \, dy,
\]

\[
R_2 = \sup_{y \in [0, 1/4]} \left\{ \frac{1}{2} \left| \frac{d^2}{dy^2} \log(1 - y) \right| \right\},
\]

\[
D_1 = \sup_{x \in [0, \epsilon]} \left( \frac{1}{x^2} \right) \left( \frac{\log(\text{sinc}(\frac{\pi - x}{2})) + \log(\text{sinc}(\frac{\pi + x}{2})) - 2 \log(\text{sinc}(\frac{\pi}{2}))}{\log(1/(1/y^2))} \right).
\]

Proof. Similarly as in the previous lemma we split $I(x, y)$ into one main term and one remainder term. In this case we get

\[
I(x, y) = \log((y - x)/2) + \log((y + x)/2) - 2 \log(y/2) + \log(\text{sinc}((y - x)/2)) + \log(\text{sinc}((y + x)/2)) - 2 \log(\text{sinc}(y/2)).
\]

where

\[
\log((y - x)/2) + \log((y + x)/2) - 2 \log(y/2) = \log(1 - (x/y)^2).
\]

Note that the log-sinc terms are not individually small since $y$ is not in general small, but for small values of $x$ they mostly cancel out. We split $U_2$ as

\[
U_2(x) = \int_x^\pi - \log(1 - (x/y)^2)y \sqrt{\log(1 + 1/y)} \, dy
\]

\[
+ \int_x^\pi -(\log(\text{sinc}((y - x)/2)) + \log(\text{sinc}((y + x)/2)) - 2 \log(\text{sinc}(y/2)))y \sqrt{\log(1 + 1/y)} \, dy
\]

\[
= U_{2,m}(x) + U_{2,r}(x)
\]

Due to the occurrence of $x/y$ in $U_{2,m}$ it is natural to switch coordinates to $t = y/x$, as was done for $U_1$. This gives us

\[
U_{2,m}(x) = x^2 \int_{1}^{\pi/x} - \log(1 - 1/t^2)t \sqrt{\log(1 + 1/(xt))} \, dt.
\]
Next we split the interval $[1, \pi/x]$ into three parts, $[1, 2], [2, 1/x]$ and $[1/x, \pi/x]$, and treat each of them separately. Let

$$U_{2,m,1}(x) = x^2 \int_1^2 -\log(1 - 1/t^2)t\sqrt{\log(1 + 1/(xt))} \, dt,$$

$$U_{2,m,2}(x) = x^2 \int_2^{1/x} -\log(1 - 1/t^2)t\sqrt{\log(1 + 1/(xt))} \, dt,$$

$$U_{2,m,3}(x) = x^2 \int_{1/x}^{\pi/x} -\log(1 - 1/t^2)t\sqrt{\log(1 + 1/(xt))} \, dt.$$

For $U_{2,m,1}$ we note that $\sqrt{\log(1 + 1/(xt))}$ is decreasing in $t$ and hence upper bounded by the value at $t = 1$, which is $\sqrt{\log(1 + 1/x)}$. We therefore have

$$U_{2,m,1}(x) \leq x^2 \sqrt{\log(1 + 1/x)} \int_1^2 -\log(1 - 1/t^2)t \, dt \leq x^2 \sqrt{\log(1 + 1/x)} \log \left( \frac{16}{3\sqrt{3}} \right).$$

For $U_{2,m,2}$ we use a Taylor expansion of $\log(1 - 1/t^2)$ at $t = \infty$ and explicitly bound the remainder term. For $y \in [0, 1/4]$ we have

$$-\log(1 - y) \leq y + R_2 y^2$$

with

$$R_2 = \sup_{y \in [0, 1/4]} \left| \frac{1}{2} \frac{d^2}{dy^2} \log(1 - y) \right|.$$ 

This gives us that for $t > 2$ we have

$$-\log(1 - 1/t^2) \leq \frac{1}{t^2} + \frac{R_2}{t^4}.$$ 

We also use the inequality

$$\sqrt{\log(1 + 1/(xt))} = \sqrt{\log(1 + xt) + \log(1/(xt))} \leq \sqrt{\log(1 + xt)} + \sqrt{\log(1/(xt))}$$

together with the bound $\sqrt{\log(1 + xt)} < \sqrt{\log 2}$ for $2 \leq t \leq 1/x$ to split $U_{2,m,2}$ as

$$U_{2,m,2}(x) \leq \frac{x^2}{2} \int_2^{1/x} \frac{\sqrt{\log(1/(xt))}}{t} \, dt + R_2 \int_2^{1/x} \frac{\sqrt{\log(1/(xt))}}{t^3} \, dt + \sqrt{\log 2} \int_2^{1/x} \frac{1}{t} \, dt + R_2 \sqrt{\log 2} \int_2^{1/x} \frac{1}{t^3} \, dt.$$ 

For the integrals we have

$$\int_2^{1/x} \frac{\sqrt{\log(1/(xt))}}{t} \, dt = \frac{2}{3} \log(1/(2x))^{3/2}.$$
\[ \int_{2}^{\infty} \frac{\sqrt{\log(1/(x^2))}}{t^3} \, dt = \frac{\sqrt{\log(1/(2x^2))}}{8} - \frac{\sqrt{2\pi x^2} \operatorname{erfi}(\sqrt{\log(1/(4x^2))})}{8}, \]

\[ \int_{2}^{\infty} \frac{1}{t} \, dt = \log(1/x) - \log 2, \]

\[ \int_{2}^{1/x} \frac{1}{t^3} \, dt = \frac{1 - 4x^2}{8}. \]

From the integral representation

\[ \operatorname{erfi}(\sqrt{\log(1/(4x^2))}) = \frac{2}{\sqrt{\pi}} \int_{0}^{\sqrt{\log(1/(4x^2))}} e^{t^2} \, dt \]

we see that \( \operatorname{erfi}(\sqrt{\log(1/(4x^2))}) \) is positive and we can get an upper bound by removing the term with it. This gives us

\[ U_{2,m,2}(x) \leq x^2 \left( \frac{2}{3} \log(1/(2x)) + R_2 \frac{\sqrt{\log(1/(2x))}}{8} + \sqrt{2 \log(1/x) - \log(2)^2} + R_2 \sqrt{\log 2 \frac{1 - 4x^2}{8}} \right). \]

For \( U_{2,m,3} \) we use that \( \sqrt{\log(1 + 1/(xt))} < \sqrt{\log 2} \) for \( 1/x < t < \pi/x \), hence

\[ U_{2,m,3}(x) \leq x^2 \sqrt{\log 2} \int_{1/x}^{\pi/x} - \log(1 - 1/t^2) t \, dt. \]

The integral can be explicitly computed to be

\[ \int_{1/x}^{\pi/x} - \log(1 - 1/t^2) t \, dt = \left( \frac{1}{2} \left( 1 - \frac{\pi^2}{x^2} \right) \log \left( 1 - \frac{x^2}{\pi^2} \right) + \log \left( \frac{\pi}{x} \right) \right) - \left( \frac{1}{2} \left( 1 - \frac{1}{x^2} \right) \log (1 - x^2) + \log \left( \frac{1}{x} \right) \right). \]

Reordering and simplifying this gives us

\[ \int_{1/x}^{\pi/x} - \log(1 - 1/t^2) t \, dt = \frac{1}{2} \log \left( \frac{\pi^2 - x^2}{1 - x^2} \right) + \log(1 - x^2) - \frac{\pi^2 \log \left( 1 - \frac{x^2}{\pi^2} \right)}{2 \pi^2} \]

and hence

\[ U_{2,m,3}(x) \leq x^2 \sqrt{\log 2} \left( \frac{1}{2} \log \left( \frac{\pi^2 - x^2}{1 - x^2} \right) + \log(1 - x^2) - \frac{\pi^2 \log \left( 1 - \frac{x^2}{\pi^2} \right)}{2 \pi^2} \right). \]

Putting \( U_{2,m,1}, U_{2,m,2} \) and \( U_{2,m,3} \) together we arrive at

\[ U_{2,m}(x) \leq x^2 \left( \sqrt{\log(1 + 1/x)} \log \left( \frac{16}{3\sqrt{3}} \right) \right) \]

\[ + \left( \frac{2}{3} \log(1/(2x)) \right)^3 + R_2 \frac{\sqrt{\log(1/(2x))}}{8} + \sqrt{\log 2 \log(1/x)} \]
\[ -\log(2)^{\frac{3}{2}} + R_2\sqrt{\log 2 \frac{1 - 4x^2}{8}} \]
\[ + \sqrt{\log 2} \left( \log \left( \frac{\pi^2 - x^2}{2} \right) + \log \left( \frac{1 - x^2}{2x^2} \right) - \frac{\pi^2 \log \left( \frac{1 - x^2}{\pi^2} \right)}{2x^2} \right) \].

For \( U_{2,r} \) we use the fact that
\[ -\log \left( \frac{\text{sinc} \left( \frac{y - x}{2} \right)}{x^2} \right) + \log \left( \frac{\text{sinc} \left( \frac{y + x}{2} \right)}{x^2} \right) - 2 \log \left( \frac{\text{sinc} \left( \frac{y}{2} \right)}{x^2} \right) \]
is bounded for \( y \in [x, \pi] \) uniformly in \( x \). If we let
\[ D_1 = \sup_{x \in [0, \epsilon]} \sup_{y \in [x, \pi]} -\log \left( \frac{\text{sinc} \left( \frac{y - x}{2} \right)}{x^2} \right) + \log \left( \frac{\text{sinc} \left( \frac{y + x}{2} \right)}{x^2} \right) - 2 \log \left( \frac{\text{sinc} \left( \frac{y}{2} \right)}{x^2} \right) \]
(20)
we get
\[ U_{2,r}(x) \leq D_1 x^2 \int_x^\pi \sqrt{y \log(1 + 1/y)} \, dy. \]

If we also let
\[ c_2 = \int_0^\pi \sqrt{y \log(1 + 1/y)} \, dy \]
we have \( U_{2,r}(x) \leq D_1 c_2 x^2 \).

To easier bound \( D_1 \) we note that the function in (20) is increasing in \( y \) for \( y \in [x, \pi] \). To see this focus on the part
\[ \log \left( \frac{\text{sinc} \left( \frac{y - x}{2} \right)}{x^2} \right) + \log \left( \frac{\text{sinc} \left( \frac{y + x}{2} \right)}{x^2} \right) - 2 \log \left( \frac{\text{sinc} \left( \frac{y}{2} \right)}{x^2} \right) \]
which we want to show is decreasing in \( y \). If we let \( f(y) = \log \left( \frac{\text{sinc} \left( \frac{y}{2} \right)}{x^2} \right) \) we can write the above as
\[ f(y - x) + f(y + x) - 2f(y). \]
Differentiating we have
\[ f'(y - x) + f'(y + x) - 2f'(y), \]
since \( f' \) is concave on \((0, 2\pi)\) this is non-positive. To see that \( f' \) indeed is concave on this interval it is enough to check that
\[ f''''(y) = \frac{\cot \left( \frac{y}{2} \right)}{4 \sin^2 \left( \frac{y}{2} \right)} - \frac{2}{y^3}. \]
is negative on \((0, 2\pi)\), which follows from that the coefficients in the Taylor expansion at \(y = 0\) all are negative. This means that the supremum for \(D_1\) is attained at \(y = \pi\) and it reduces to

\[
D_1 = \sup_{x \in [0, \epsilon]} - \frac{\log \left( \sin \left( \frac{x - \pi}{2} \right) \right) + \log \left( \sin \left( \frac{x + \pi}{2} \right) \right)}{x^2} - 2 \log \left( \sin \left( \frac{\pi}{2} \right) \right).
\]

To use these lemmas in the computer assisted proof the first step is to compute enclosures of \(c_1, c_2, R_1, R_2\) and \(D_1\). For \(R_2\) this can be done directly using Taylor arithmetic. For \(R_1\) and \(D_1\) we have removable singularities that need to be dealt with, this can be done using the approach described in “Appendix A”. For \(c_1\) and \(c_2\) we refer to “Appendix C”.

For Lemma 6.3 it is straightforward to compute enclosures of all the terms in the upper bound by using monotonicity properties in \(x\). For Lemma 6.4 the expression for the upper bound is more complicated. In particular several of the terms contain removable singularities at \(x = 0\), these are handled using the approach described in “Appendix A”. As an example we show how to bound

\[
\frac{2}{3} \log \left( \frac{1}{2x} \right)^{3/2} \frac{\log(1/x)}{\sqrt{\log(1 + 1/x)}},
\]

the other terms can be done in a similar way.

**Lemma 6.5.** The function

\[
f(x) = \frac{2}{3} \log \left( \frac{1}{2x} \right)^{3/2} \frac{\log(1/x)}{\sqrt{\log(1 + 1/x)}},
\]

is bounded from above by \(2/3\) and is decreasing in \(x\) on the interval \((0, 1/2)\).

**Proof.** Taking the limit \(x \to 0^+\) gives the value \(\frac{2}{3}\), it is hence enough to prove that it is decreasing in \(x\). Differentiating with respect to \(x\) gives us

\[
3x(x + 1) \log(1 + 1/x)^{3/2} \log(1/x)^2 \left( \log(1/(2x)) \log(1/x) - (x + 1) \log(1 + 1/x) \log(4/x) \right).
\]

The sign is given by that of

\[
\log(1/(2x)) \log(1/x) - (x + 1) \log(1 + 1/x) \log(4/x).
\]

For \(x \in (0, 1/2)\) we get the upper bound

\[
\log(1/(2x)) \log(1/x) - (x + 1) \log(1 + 1/x) \log(4/x)
\leq \log(1/(2x)) \log(1/x) - \log(1 + 1/x) \log(4/x)
\leq \log(1/(2x)) \log(1/x) - \log(1/x) \log(4/x)
= \log(1/x)(\log(1/(2x)) - \log(4/x))
= \log(1/x) \log(1/8),
\]

which is negative. \(\square\)
From the proofs of Lemma 6.3 we can also get the following bound for \( \hat{I} \), which is used in the proof of Lemma 4.1.

**Corollary 6.6.** There exists \( R > 0 \) such that for all \( 0 < x < \pi \) and \( 0 < t < \pi / x \) we have

\[
|\hat{I}(x, t)| \leq \left| \log \left| 1 - \frac{1}{t^2} \right| \right| + Rx^2.
\]

Furthermore, if we also have \( t \geq 2 \) then \( |\hat{I}(x, t)| \leq \frac{2}{t^2} + Rx^2 \).

**Proof.** As in the proof of Lemma 6.3 we have

\[
\hat{I}(x, t) = \log \left| 1 - \frac{1}{t^2} \right| + \log \left( \text{sinc} \left( \frac{x(1-t)}{2} \right) \right) + \log \left( \text{sinc} \left( \frac{x(1+t)}{2} \right) \right)
\]

\[
-2 \log \left( \text{sinc} \left( \frac{xt}{2} \right) \right).
\]

The function \( \log(\text{sinc}(y)) \) is analytic for \( |y| < 2\pi \) and the first two terms in the Taylor expansions are zero, hence there exists \( R > 0 \) such that for all \( |y| \leq \pi \) we have \( |\log(\text{sinc}(y))| \leq Ry^2 \). Giving us

\[
|\hat{I}(x, t)| \leq \left| \log \left| 1 - \frac{1}{t^2} \right| \right| + Rx^2 \left( \frac{(1-t)^2}{4} + \frac{(1+t)^2}{4} - \frac{t^2}{2} \right) = \left| \log \left| 1 - \frac{1}{t^2} \right| \right| + \frac{R}{2} x^2.
\]

This proves the first statement. The second statement follows from the inequality \( |\log(1 - y)| < 2y \) for \( 0 < y < \frac{1}{4} \).

\[ \square \]

### 7. Bounds for \( D_0, \delta_0 \) and \( n_0 \)

We are now ready to give bounds for \( n_0, \delta_0 \) and \( D_0 \). Recall that they are given by

\[
n_0 = \sup_{x \in [0, \pi]} |N(x)|, \quad \delta_0 = \sup_{x \in [0, \pi]} |F(x)|, \quad D_0 = \sup_{x \in [0, \pi]} |\mathcal{T}(x)|.
\]

In each case we split the interval \([0, \pi]\) into two parts, \([0, \epsilon]\) and \([\epsilon, \pi]\), with \( \epsilon \) varying for the different cases, and threat them separately. For the interval \([0, \epsilon]\) we use the asymptotic bounds for the different functions that were introduced in the previous two sections. For the interval \([\epsilon, \pi]\) we evaluate the functions directly using interval arithmetic. For the direct evaluation the only complicated part is the computation of \( U(x) \) in \( \mathcal{T}(x) \), where we proceed as discussed in Section 6 and “Appendix C”.

Consider the problem of enclosing the maximum of \( f \) on some interval \( I \) to some predetermined tolerance. The main idea is to iteratively bisect the interval \( I \) into smaller and smaller subintervals. At every iteration we compute an enclosure of \( f \) on each subinterval. From these enclosures a lower bound of the maximum can be computed. We then discard all subintervals for which the enclosure is less than the lower bound of the maximum, the maximum cannot be attained there.
For the remaining subintervals we check if their enclosure satisfies the required tolerance, in that case we don’t bisect them further. If there are any subintervals left we bisect them and continue with the next iteration. In the end, either when there are no subintervals left to bisect or we have reached some maximum number of iterations (to guarantee that the procedure terminates), we return the maximum of all subintervals that were not discarded. This is guaranteed to give an enclosure of the maximum of $f$ on the interval.

If we are able to compute Taylor series of the function $f$ we can improve the performance of this procedure significantly (see e.g. [13,14] where a similar approach is used). Consider a subinterval $I_i$, instead of computing an enclosure of $f(I_i)$ we compute a Taylor polynomial $P$ at the midpoint and an enclosure $R$ of the remainder term such that $f(x) \in P(x) + R$ for $x \in I_i$. We then have

$$\sup_{x \in I_i} f(x) \in \sup_{x \in I_i} P(x) + R. \tag{21}$$

To compute $\sup_{x \in I_i} P(x)$ we isolate the roots of $P'$ on $I_i$ and evaluate $P$ on the roots as well as the endpoints of the interval. In practice the computation of $R$ involves computing an enclosure of the Taylor series of $f$ on the full interval $I_i$. Since this includes the derivative we can as an extra optimization check if the derivative is non-zero, in which case $f$ is monotone and it is enough to evaluate $f$ on either the left or the right endpoint of $I_i$, depending on the sign of the derivative.

The above procedures can easily be adapted to instead compute the minimum of $f$ on the interval, joining them together we can thus compute the extrema on the interval. In some cases we don’t care about computing an enclosure of the maximum, but only to prove that it is bounded by some value. Instead of using a tolerance we then discard any subintervals for which the enclosure of the maximum is less than the bound.

In most cases we bisect the subintervals at the midpoint, meaning that the interval $[\bar{x}, \bar{x}]$ would be bisected into the two intervals $[\bar{x}, (\bar{x} + \bar{x})/2]$ and $[(\bar{x} + \bar{x})/2, \bar{x}]$. However, when the magnitude of $\bar{x}$ and $\bar{x}$ are very different it can be beneficial to bisect at the geometric midpoint (see e.g. [22]), in that case we split the interval into $[\bar{x}, \sqrt{\alpha \bar{x}}]$ and $[\sqrt{\alpha \bar{x}}, \bar{x}]$, where we assume that $\bar{x} > 0$.

We split the computation of the bounds for $n_0$, $\delta_0$ and $D_0$ into three lemmas. The proof of the lemmas are computer assisted and we give some details about the process. The lemmas are stated as upper bounds, but in the process of proving them we do compute actual enclosures of the values.

The code\(^1\) for the computer assisted part is implemented in Julia [3]. The main tool for the rigorous numerics is Arb [35] which we use through the Julia wrapper Arblib.jl.\(^2\) Many of the basic interval arithmetic algorithms, such as isolating roots or enclosing maximum values, are implemented in a separate package, ArbExtras.jl.

---

\(^1\) Available at https://github.com/Joel-Dahne/BurgersHilbertWave.jl, the results in the paper are from commit fccf726c3771132d9f4f00df0ce6ef96b277d4666.

\(^2\) https://github.com/kalmarek/Arblib.jl.
For finding the coefficients \( \{a_j\} \) and \( \{b_n\} \) of \( u_0 \) we make use of non-linear solvers from NLsolve.jl \([45]\). The computations were done on an AMD Ryzen 9 5900X processor with 32 GB of RAM using 12 threads and when timings are given it refers to this configuration. In most cases the computations are multithreaded and make use of all available threads. The computations were done using 100 bits of precision.

**Lemma 7.1.** The constant \( n_0 \) satisfies the inequality \( n_0 \leq \tilde{n}_0 = 0.53682 \).

**Proof.** A plot of \( N(x) \) on the interval \([0, \pi]\) is given in Fig. 2a and hints at the maximum being attained at \( x = \pi \). A good guess for the maximum value is thus given by \( N(\pi) \).

We take \( \epsilon = 0.5 \). For the interval \([0, \epsilon]\) we don’t compute an enclosure of the maximum but only prove that it is bounded by \( N(\pi) \).

For the interval \([\epsilon, \pi]\) we compute an enclosure of the maximum. This gives us

\[
n_0 \in [0.536815354831687609478 \pm 8.94 \cdot 10^{-23}],
\]

which is upper bounded by \( \tilde{n}_0 \).

We are able to compute Taylor expansions of \( N(x) \) in both the asymptotic and non-asymptotic case as long as the subinterval doesn’t contain zero. This allows us to use the better version of the algorithm, based on the Taylor polynomial, for enclosing the maximum and fall back to the naive version, where no information about the derivatives is used, for the subintervals containing zero. We use a Taylor expansion of degree 0, which is enough to pick up the monotonicity after only a few bisections in most cases. The runtime is about 5 s, most of it for handling the interval \([\epsilon, \pi]\). \(\square\)

**Lemma 7.2.** The constant \( \delta_0 \) satisfies the inequality \( \delta_0 \leq \tilde{\delta}_0 = 8.4976 \cdot 10^{-4} \).

**Proof.** A plot of \( F(x) \) on the interval \([0, \pi]\) is given in Figure 3a, however this plot doesn’t reveal the full picture of what happens close to \( x = 0 \). Figures 3b and 3c show a log-plot of \( F(x) \) on the intervals \([10^{-100}, 10^{-1}]\) and \([10^{-20000}, 10^{-100}]\) respectively. In the latter of these figures we can see that \( F \) has an extremum between \(10^{-10000}\) and \(10^{-5000}\), this turns out to be where the maximum of \(|F|\) is attained. Note that these numbers are extremely small, too small to be represented even in standard quadruple precision (binary128), though Arb has no problem handling them.

We take \( \epsilon = 0.1 \). Handling the interval \([\epsilon, \pi]\) is straightforward and we get the enclosure \([0.00022669 \pm 2.82 \cdot 10^{-9}]\). The interval \([0, \epsilon]\) is more delicate due to the extrema being attained for such an extremely small \( x \).

In general evaluation with the asymptotic version of \( F \) is much faster than the non-asymptotic version. However, for the interval \([0, \epsilon]\) we have to do a huge number of subdivisions and it is therefore beneficial to optimize it slightly more.

---

\(^3\) https://github.com/Joel-Dahne/ArbExtras.jl.
The terms in the expansions used for computing $F$ are given in Lemmas 5.1 and 5.2, they contain factors of the form $c|x|^d$ and $c\frac{|x|^d}{\log |x|}$ with varying coefficients $c$ and exponents $d$. When $x$ gets smaller more and more of these terms become negligible. To reduce the number of terms in the expansions we can collapse all negligible terms into one remainder term. If we fix some $d_0 > 0$ then on the interval $[0, \epsilon]$ with $\epsilon < 1$ we have for $d \geq d_0$ and $c > 0$

$$cx^d \in [0, c] \cdot x^{d_0}, \quad c\frac{x^d}{\log x} \in \left[\frac{c}{\log \epsilon}, 0\right] \cdot x^{d_0},$$

with a similar expression for $c < 0$. In this way we can take all terms with an exponent greater than $d_0$ and put them into one single term with the exponent $d_0$. If $x$ is small enough so that $x^{d_0}$ is negligible this will still give a good enclosure.

For this we split the interval $[0, \epsilon]$ into four parts, $[0, \epsilon_1]$, $[\epsilon_1, \epsilon_2]$, $[\epsilon_2, \epsilon_3]$ and $[\epsilon_3, \epsilon]$, with $\epsilon_1 = 10^{-10000000}$, $\epsilon_2 = 10^{-100000}$ and $\epsilon_3 = 10^{-100}$. For the first two intervals we take $d_0 = 10^{-4}$, on the third we take it to be 1/4 and on the fourth we keep all terms. For the expansion from Lemma 5.1 this leaves us with 4, 837 and 1936 terms respectively. For the expansion from Lemma 5.2 we get 6, 2505 and 17381 terms.

The interval $[0, \epsilon_1]$ is taken such that a single evaluation of $F$ gives a good enough enclosure, we have $F([0, \epsilon_1]) \subseteq [\pm 4.27 \cdot 10^{-4}]$. The bulk of the work is for the second interval, it needs to be split into more than $2^{24}$ subintervals and the final enclosure for the maximum is $[0.000345 \pm 3.11 \cdot 10^{-7}]$. The third interval, $[\epsilon_2, \epsilon_3]$, is split into more than $2^{18}$ subintervals, with the enclosure $[0.0008497 \pm 5.37 \cdot 10^{-8}]$. For $[\epsilon_3, \epsilon]$ it suffices to split it in around $2^{10}$ subintervals and we get the enclosure $[0.00038601 \pm 4.5 \cdot 10^{-9}]$.

We are able to compute Taylor expansions of $F(x)$ in both the asymptotic and non-asymptotic case as long as the subinterval doesn’t contain zero, this allows us to use the better version of the algorithm for enclosing the maximum and fall back to the naive version for the subintervals containing zero. We use a Taylor expansion of degree 4 in all cases except for the interval $[\epsilon_1, \epsilon_2]$ where we use degree 2. On the intervals $[\epsilon_1, \epsilon_2]$, $[\epsilon_2, \epsilon_3]$ and $[\epsilon_3, \epsilon]$ we bisect at the geometric midpoint instead of bisecting at the arithmetic midpoint. The runtime is about 10 min. \hfill $\Box$

**Lemma 7.3.** The constant $D_0$ satisfies the inequality $D_0 \leq \bar{D}_0 = 0.94589$.

**Proof.** A plot of $T(x)$ on the interval $[0, \pi]$ is given in Fig. 2b. It hints at the maximum being attained around $x \approx 2.3$.

We take $\epsilon = 0.01$ and start by computing the maximum on $[\epsilon, \pi]$, this gives us the enclosure $[0.945126, 0.94589]$.

For the interval $[0, \epsilon]$ we make use of the asymptotic expansions from Lemmas 6.3 and 6.4. Since these only give upper bounds we are not able to compute an enclosure of $T$ on this interval. However, since the maximum is attained on the interval $[\epsilon, \pi]$ we only need prove that $T$ is bounded by this value on $[0, \epsilon]$. Hence the maximum for the full interval is given by that on $[\epsilon, \pi]$ and we get

$$D_0 \in [0.945126, 0.94589],$$
Fig. 2. Plot of the functions $N$ and $T$ on the interval $[0, \pi]$. The dashed green lines show the upper bounds $\bar{n}_0$ and $\bar{D}_0$ as given in Lemmas 7.1 and 7.3.

Fig. 3. Plot of the function $F$ on the intervals $[0, \pi]$, $[10^{-100}, 10^{-1}]$ and $[10^{-20000}, 10^{-100}]$. The dashed green line shows the upper bound $\bar{\delta}_0$ as given in Lemma 7.2. The dotted red line shows $\left(1 - \bar{D}_0\right)^2 / 4\bar{n}_0$, which is the value we want $\delta_0$ to be smaller than which is upper bounded by $\tilde{D}_0$.

In this case we do not have access to Taylor series of $T(x)$ in either the asymptotic or non-asymptotic case. This means we have to rely on the naive version for bounding the maximum. On the interval $[\epsilon, \pi]$ there is one optimization that we can do. $T(x)$ involves a division by $u_0(x)$ and for this function we have access to Taylor series, we therefore compute a tighter enclosure of this using a $C^1$ bound.

The total runtime for the computation is around 100 s, the majority for handling the interval $[\epsilon, \pi]$. 

\[ \square \]

8. Proof of Theorem 1.1

We are now ready to give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Consider the operator $G$ from (10) given by

\[ G[v] = (I - T)^{-1}(-F - Nv^2). \]

By Lemma 7.3 we have $\|T\| \leq \tilde{D}_0 < 1$ so the inverse of the operator $I - T$ is well defined. Combining Lemmas 7.1, 7.2 and 7.3 gives us the inequality

\[ \delta_0 \leq \bar{\delta}_0 < \frac{(1 - \tilde{D}_0)^2}{4\bar{n}_0} \leq \frac{(1 - D_0)^2}{4n_0}. \]
This, together with Proposition 2.2 and Banach fixed-point theorem proves that for
\[ \epsilon = \frac{1 - D_0 - \sqrt{(1 - D_0)^2 - 4\delta_0n_0}}{2n_0} \]
the operator $G$ has a unique fixed-point $v_0$ in $X_\epsilon \subseteq L^\infty(\mathbb{T})$.

By the construction of the operator $G$ this means that the function
\[ u(x) = u_0(x) + w(x)v_0(x) \]
solves (6), given by
\[ \frac{1}{2}u^2 = -\mathcal{H}[u]. \]
For any wavespeed $c \in \mathbb{R}$ we then have that the function
\[ \varphi(x) = c - u(x) \]
is a traveling wave solution to (1). This proves the existence of a $2\pi$-periodic highest cusped traveling wave.

To get the asymptotic behaviour we note that
\[ u_0(x) = -\frac{1}{\pi} |x| \log |x| + O(|x|) \quad \text{and} \quad w(x)v_0(x) = O(|x|\sqrt{\log |x|}). \]
Hence
\[ u(x) = -\frac{1}{\pi} |x| \log |x| + O(|x|\sqrt{\log |x|}) \]
and
\[ \varphi(x) = c + \frac{1}{\pi} |x| \log |x| + O(|x|\sqrt{\log |x|}), \]
as we wanted to show. \qed

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Data Availability  The datasets generated during and/or analysed during the current study are available in github, https://github.com/Joel-Dahne/BurgersHilbertWave.jl and included in this published article and its supplementary information files.

Declarations  
Conflict of interest  The authors declare that they have no conflict of interest.

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A. Removable Singularities

In several cases we have to compute enclosures of functions with removable singularities. For example the function 
\[ \Gamma(1 - s) \cos(\pi(1 - s)/2) \]
comes up when computing \(C_s\) through equation (26) and has a removable singularity whenever \(s\) is a positive even integer. In this appendix we explain how to compute rigorous enclosures at and around these points. For this we need a way to handle the removable singularity. Let
\[ f_n(x) = \frac{f^{(n)}(x)}{n!}. \]
We have the following lemma for handling functions with removable singularities:

**Lemma A.1.** Let \(m \in \mathbb{Z}_{\geq 0}\) and let \(I\) be an interval containing zero. Consider a function \(f(x)\) with a zero of order \(n\) at \(x = 0\) and such that \(f^{(m+n)}(x)\) is absolutely continuous on \(I\). Then for all \(x \in I\) we have
\[
\frac{f(x)}{x^n} = \sum_{k=0}^{m} f_{k+n}(0)x^k + f_{m+n+1}(\xi)x^{m+1}
\]
for some \(\xi\) between 0 and \(x\). Furthermore, if \(f^{m+n+p}(x)\) is absolutely continuous for \(p \in \mathbb{Z}_{\geq 0}\) we have
\[
\frac{d^p}{dx^p} \frac{f(x)}{x^n} = \sum_{k=0}^{m} \frac{(k + p)!}{k!} f_{k+n+p}(0)x^k + \frac{(m + p + 1)!}{(m + 1)!} f_{m+n+p+1}(\xi)x^{m+1}
\]
for some \(\xi\) between 0 and \(x\).
Proof. The first statement follows directly from expanding \( f \) in a Taylor series with a remainder term on Lagrange form and dividing by \( x^n \), using that \( f_k(0) = 0 \) for \( k < n \).

For the second statement we start by noting that

\[
\frac{d^p}{dx^p} \frac{f(x)}{x^n} = \sum_{l=0}^{p} \binom{p}{l} f^{(p-l)}(x)(-1)^l \frac{(n+l-1)!}{(n-1)!} \frac{1}{x^{n+l}}
\]

\[
= \frac{1}{x^{n+p}} \sum_{l=0}^{p} (-1)^l \binom{p}{l} \frac{(n+l-1)!}{(n-1)!} f^{(p-l)}(x)x^{p-l}.
\]  \((22)\)

The Taylor expansion of \( f^{(p-l)}(x) \) with the remainder term in integral form is

\[
f^{(p-l)}(x) = \sum_{k=0}^{m+n+l} (f^{(p-l)})_k(0)x^k
\]

\[
+ x^{m+n+l+1}(m+n+l+1) \int_0^1 (f^{(p-l)})_{m+n+l+1}(tx)(1-t)^{m+n+l} \, dt.
\]  \((23)\)

Using that

\[
(f^{(p-l)})_k(x) = \frac{f^{(p-l+k)}(x)}{k!} = \frac{(p-l+k)!}{k!} \frac{f^{(p-l+k)}(x)}{(p-l+k)!} = \frac{(p-l+k)!}{k!} f_{p-l+k}(x)
\]

we can write \((23)\) as

\[
f^{(p-l)}(x) = \sum_{k=0}^{m+n+l} \frac{(p-l+k)!}{k!} f_{p-l+k}(0)x^k
\]

\[
+ x^{m+n+l+1}(m+n+l+1) \int_0^1 (p-l+(m+n+l+1))! f_{p-l+1+m+n+l+1}(tx)(1-t)^{m+n+l} \, dt
\]

\[
= \sum_{k=0}^{m+n+l} \frac{(p-l+k)!}{k!} f_{p-l+k}(0)x^k + x^{m+n+l+1} \frac{(p+m+n+1)!}{(m+n+l)!} \int_0^1 f_{p-m+n+1+1}(tx)(1-t)^{m+n+l} \, dt.
\]

Multiplying by \( x^{p-l} \) we get

\[
f^{(p-l)}(x)x^{p-l} = \sum_{k=p-l}^{m+n+p} \frac{k!}{(k-p+l)!} f_k(0)x^k
\]

\[
+ x^{m+n+p+1} \frac{(p+m+n+1)!}{(m+n+l)!} \int_0^1 f_{p-m+n+1+1}(tx)(1-t)^{m+n+l} \, dt.
\]  \((24)\)
Inserting the main term of (24) into (22) gives us
\[
\frac{1}{x^{n+p}} \sum_{l=0}^{p} (-1)^l \left( \begin{array}{c} p \\ l \end{array} \right) \frac{(n+l-1)!}{(n-1)!} \sum_{k=p-l}^{m+n+p} \frac{k!}{(k-p+l)!} f_k(0)x^k
\]

Splitting into \( k < p \) and \( k \geq p \) we have
\[
\sum_{k=0}^{p-1} \sum_{l=p-k}^{p} (-1)^l \left( \begin{array}{c} p \\ l \end{array} \right) \frac{(n+l-1)!}{(n-1)!} \frac{k!}{(k-p+l)!} f_k(0)x^{k-n-p}
\]
\[
+ \sum_{k=p}^{m+n+p} \sum_{l=0}^{p} (-1)^l \left( \begin{array}{c} p \\ l \end{array} \right) \frac{(n+l-1)!}{(n-1)!} \frac{k!}{(k-p+l)!} f_k(0)x^{k-n-p} \quad (25)
\]

Since \( f_k(0) = 0 \) for \( k < n \) the first sum reduces to
\[
\sum_{k=n}^{p-1} \sum_{l=p-k}^{p} (-1)^l \left( \begin{array}{c} p \\ l \end{array} \right) \frac{(n+l-1)!}{(n-1)!} \frac{k!}{(k-p+l)!} f_k(0)x^{k-n-p}
\]

Using [47, Sec. 15.2.4] we have
\[
\sum_{l=p-k}^{p} (-1)^l \left( \begin{array}{c} p \\ l \end{array} \right) \frac{(n+l-1)!}{(n-1)!} \frac{k!}{(k-p+l)!}
\]
\[
= (-1)^{p-k} \left( \begin{array}{c} p \\ p-k \end{array} \right) \frac{k!(n+p-k-1)!}{(n-1)!} _2F_1(-k, n + p - k, p - k + 1, 1).
\]

From [47, Sec. 15.4.24] we get that this is zero for \( k \geq n \). The second sum in (25) we can rewrite as
\[
\sum_{k=-n}^{m} \sum_{l=0}^{p} (-1)^l \left( \begin{array}{c} p \\ l \end{array} \right) \frac{(n+l-1)!}{(n-1)!} \frac{(k+n+p)!}{(k+n+l)!} f_{k+n+p}(0)x^k
\]

For \( k < -p \) the terms are zero since \( f_{k+n+p}(0) = 0 \). For \( k \geq -p \) we use [47, Sec. 15.2.4] to write the inner sum as
\[
\sum_{l=0}^{p} (-1)^l \left( \begin{array}{c} p \\ l \end{array} \right) \frac{(n+l-1)!}{(n-1)!} \frac{(k+n+p)!}{(k+n+l)!} = \frac{(k+n+p)!}{(k+n)!} _2F_1(-p, n, k + n + 1, 1).
\]

From [47, Sec. 15.4.24] we get that this is zero for \( -p \leq k < 0 \) and for \( 0 \leq k \leq m \) it is given by \( \frac{(k+p)!}{k!} \). This gives us that (25) can be written as
\[
\sum_{k=0}^{m} \frac{(k+p)!}{k!} f_{k+n+p}(0)x^k,
\]
which is what we wanted to show.
We are now interested in the remainder term of (24) when inserted into (22), we get

\[
\frac{1}{x^{n+p}} \sum_{l=0}^{p} (-1)^l \binom{p}{l} \frac{(n+l-1)!}{(n-1)!} \frac{(p+m+n+1)!}{(m+n+l)!} x^{m+n+p+1} (p+m+n+1)! \\
\int_{0}^{1} f_{p+m+n+1}(tx)(1-t)^{m+n+l} \, dt \\
= x^{m+1} \int_{0}^{1} f_{p+m+n+1}(tx) \sum_{l=0}^{p} (-1)^l \binom{p}{l} \frac{(n+l-1)!}{(n-1)!} \frac{(p+m+n+1)!}{(m+n+l)!} (1-t)^l \\
\frac{(p+m+n+1)!}{(m+n+l)!} (1-t)^{m+n+l} \, dt.
\]

We first show that the sum is non-negative. Using [47, Sec. 15.2.4] we have

\[
\sum_{l=0}^{p} (-1)^l \binom{p}{l} \frac{(n+l-1)!}{(n-1)!} \frac{(p+m+n+1)!}{(m+n+l)!} (1-t)^{m+n+l} \\
= (1-t)^{m+n} \frac{(p+m+n+1)!}{(m+n)!} \sum_{l=0}^{p} (-1)^l \binom{p}{l} \frac{(n+l-1)!}{(n-1)!} \frac{(m+n)!}{(m+n+l)!} (1-t)^l \\
= (1-t)^{m+n} \frac{(p+m+n+1)!}{(m+n)!} \frac{\text{2F1}(-p, n, m+n+1, 1-t)}{1-t^{m+n+l}}.
\]

By [43, Theorem 3.2] (see also [15, Theorem 2]), \( \text{2F1}(-p, n, m+n+1, 1-t) \) have no roots on the interval \([0, 1]\) and for \( t = 1 \) it is positive, it is hence positive on \([0, 1]\). Since the sum is positive the integral can be written as

\[
f_{p+m+n+1}(\xi) x^{m+1} \int_{0}^{1} \sum_{l=0}^{p} (-1)^l \binom{p}{l} \frac{(n+l-1)!}{(n-1)!} \frac{(p+m+n+1)!}{(m+n+l)!} (1-t)^{m+n+l} \, dt.
\]

for some \( \xi \) between 0 and x. Simplifying further we get

\[
f_{m+n+p+1}(\xi) x^{m+1} \sum_{l=0}^{p} (-1)^l \binom{p}{l} \frac{(n+l-1)!}{(n-1)!} \frac{(p+m+n+1)!}{(m+n+l)!} \int_{0}^{1} (1-t)^{m+n+l} \, dt \\
= f_{m+n+p+1}(\xi) x^{m+1} \sum_{l=0}^{p} (-1)^l \binom{p}{l} \frac{(n+l-1)!}{(n-1)!} \frac{(p+m+n+1)!}{(m+n+l)!} \frac{1}{m+n+l+1} \\
= f_{m+n+p+1}(\xi) x^{m+1} \sum_{l=0}^{p} (-1)^l \binom{p}{l} \frac{(n+l-1)!}{(n-1)!} \frac{(p+m+n+1)!}{(m+n+l+1)!} \\
= \frac{(m+p+1)!}{(m+1)!} f_{m+n+p+1}(\xi) x^{m+1},
\]

which is exactly the expression given for the remainder term. \( \square \)

Using this lemma we can compute enclosures at and around removable singularities as long as we have good control over \( f \) and its derivatives.
**Example A.2.** Consider the function $\Gamma(1-s)\cos(\pi(1-s)/2)$ with a removable singularity at $s = 2$. If we let $t = s - 2$ we can write the function as
\[
\frac{\cos(\pi(-1-t)/2)}{t} \cdot (t\Gamma(-1-t))
\]
If we let $f(t) = \cos(\pi(-1-t)/2)$ and take $m \geq 0$, then for the second factor the lemma then tells us that
\[
\frac{\cos(\pi(-1-t)/2)}{t} = \sum_{k=0}^{m} \left( (k + 1) f_{k+1}(0) t^k + (m + 2) f_{m+2}(\xi) t^{m+1} \right)
\]
for some $\xi$ between 0 and $x$. Using interval arithmetic we can easily compute an enclosure of the coefficients for the polynomial as well as $f_{m+2}(\xi)$. For the second factor we can’t directly apply the lemma, instead we rewrite it in terms of the reciprocal gamma function, $1/\Gamma(s)$. If we let $g(t) = 1/\Gamma(-1-t)$ we have
\[
t\Gamma(-1-t) = \left( \frac{g(t)}{t} \right)^{-1}.
\]
The function $g(t)$ has a root at $t = 0$ and we can thus apply the lemma on $\frac{g(t)}{t}$. An enclosure of the coefficients for the polynomial as well as the remainder term can be computed using the implementation of the reciprocal gamma function in Arb.

**B. Computing Enclosures of Clausen Functions**

To be able to compute bounds of $D_{\alpha}, \delta_{\alpha}$ and $n_{\alpha}$ it is critical that we can compute accurate enclosures of $C_s(x)$ and $S_s(x)$, including expansions in the argument and derivatives in the parameter. We here go through how these enclosures are computed. We make use of several different special functions, most of them with implementations in Arb (see e.g. [32,38]). In several cases we encounter removable singularities, they are all dealt with as explained in “Appendix A”, see in particular example A.2.

We start by going through how to compute $C_s(x)$ and $S_s(x)$ for $s, x \in \mathbb{R}$. Since both $C_s(x)$ and $S_s(x)$ are $2\pi$-periodic we can reduce it to $x = 0$ or $0 < x < 2\pi$.

For $x = 0$ and $s > 1$ we get directly from the defining sum that $C_s(0) = \zeta(s)$ and $S_s(0) = 0$. For $s \leq 1$ both functions typically diverge at $x = 0$.

For $0 < x < 2\pi$ we can compute the Clausen functions by going through the polylog function,
\[
C_s(x) = \text{Re} \left( \text{Li}_s(e^{ix}) \right), \quad S_s(x) = \text{Im} \left( \text{Li}_s(e^{ix}) \right).
\]
However it is computationally beneficial (about 40% faster in general) to instead go through the periodic zeta function [47, Sec. 25.13],
\[
F(x, s) := \text{Li}_s(e^{2\pi ix}) = \sum_{n=1}^{\infty} \frac{e^{2\pi i nx}}{n^s},
\]
for which we have
\[
C_s(x) = \text{Re } F\left(\frac{x}{2\pi}, s\right), \quad S_s(x) = \text{Im } F\left(\frac{x}{2\pi}, s\right).
\]
For \(0 < x < 1\) the periodic zeta function can be written as [47, Eq. 25.13.2]
\[
F(x, s) = \frac{\Gamma(1 - s)}{(2\pi)^{1-s}} \left( e^{\pi i (1-s)/2} \zeta(1 - s, x) + e^{-\pi i (1-s)/2} \zeta(1 - s, 1 - x) \right).
\]
Taking the real and imaginary part we get, for \(0 < x < 2\pi\),
\[
C_s(x) = \frac{\Gamma(1 - s)}{(2\pi)^{1-s}} \cos(\pi(1 - s)/2) \left( \zeta(1 - s, \frac{x}{2\pi}) + \zeta(1 - s, 1 - \frac{x}{2\pi}) \right),
\]
\[
S_s(x) = \frac{\Gamma(1 - s)}{(2\pi)^{1-s}} \sin(\pi(1 - s)/2) \left( \zeta(1 - s, \frac{x}{2\pi}) - \zeta(1 - s, 1 - \frac{x}{2\pi}) \right).
\]
This formulation works well as long as \(s\) is not a non-negative integer. For non-negative integers we have to handle some removable singularities.

For \(s = 0\) the functions \(\zeta(1 - s, \frac{x}{2\pi})\) and \(\zeta(1 - s, 1 - \frac{x}{2\pi})\) diverge and needs to be handled differently. The Laurent series of the zeta function gives us
\[
\zeta(s, x) = \frac{1}{s - 1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n(x)(s - 1)^n,
\]
where \(\gamma_n\) is the Stieltjes constant. The sum is referred to as the deflated zeta function, it has an implementation in Arb and we denote it by \(\zeta(s, x)\). Writing the Clausen functions in terms of the deflated zeta function gives us
\[
C_s(x) = \frac{\Gamma(1 - s)}{(2\pi)^{1-s}} \cos(\pi(1 - s)/2) \left( \zeta(1 - s, \frac{x}{2\pi}) + \zeta(1 - s, 1 - \frac{x}{2\pi}) - \frac{2}{s} \right),
\]
\[
S_s(x) = \frac{\Gamma(1 - s)}{(2\pi)^{1-s}} \sin(\pi(1 - s)/2) \left( \zeta(1 - s, \frac{x}{2\pi}) - \zeta(1 - s, 1 - \frac{x}{2\pi}) \right).
\]
The expression for \(S_s(x)\) is now well defined, all terms are finite, for \(s\) around zero. For \(C_s(x)\) we also have to handle the removable singularity of \(\frac{2 \cos(\pi(1-s)/2)}{s}\).

For \(s\) equal to a positive integer, \(\Gamma(1 - s)\) has a pole. For \(C_s(x)\) with even \(s\) we get a removable singularity in \(\Gamma(1 - s) \cos(\pi(1 - s)/2)\) and for \(S_s(x)\) with odd \(s\) we get that \(\Gamma(1 - s) \sin(\pi(1 - s)/2)\) has a removable singularity. For the other parity of \(s\) we instead have to look at the zeta function. For integer \(s \geq 1\) we have
\[
\zeta(1 - s, x) = \frac{-B_s(x)}{s}
\]
where \(B_s(n)\) is the Bernoulli polynomial [47, Eq. 25.6.3]. Since \(B_s(1 - x) = (-1)^s B_s\) [47, Eq. 24.4.3] we have
\[
\zeta\left(1 - s, \frac{x}{2\pi}\right) + \zeta\left(1 - s, 1 - \frac{x}{2\pi}\right) = 0
\]
for odd \( s \) and

\[
\zeta\left(1 - s, \frac{x}{2\pi}\right) - \zeta(1 - s, 1 - \frac{x}{2\pi}) = 0
\]

for even \( s \). This means that

\[
\Gamma(1 - s) \left( \zeta\left(1 - s, \frac{x}{2\pi}\right) + \zeta(1 - s, 1 - \frac{x}{2\pi}) \right)
\]

has a removable singularity for odd \( s \) and

\[
\Gamma(1 - s) \left( \zeta\left(1 - s, \frac{x}{2\pi}\right) - \zeta(1 - s, 1 - \frac{x}{2\pi}) \right)
\]

has a removable singularity for even \( s \). This allows us to handle all removable singularities in (26) and (27) when \( s \) is a positive integer.

### B.1. Interval Arguments

We are now ready to compute enclosures for interval arguments. Let \( x = [x, \bar{x}] \) and \( s = [s, \bar{s}] \) be two finite intervals, we are interested in computing an enclosure of \( C_s(x) \) and \( S_s(x) \). Due to the periodicity we can reduce it to three different cases for \( x \)

1. \( x \) doesn’t contain a multiple of \( 2\pi \), by adding or subtracting a suitable multiple of \( 2\pi \) we can assume that \( 0 < x < 2\pi \);
2. \( x \) has a diameter of at least \( 2\pi \), it then covers a full period and can without loss of generality be taken as \( x = [0, 2\pi] \);
3. \( x \) contains a multiple of \( 2\pi \) but has a diameter less than \( 2\pi \), by adding or subtracting a suitable multiple of \( 2\pi \) we can take \( x \) such that \( -2\pi < x \leq 0 \leq \bar{x} < 2\pi \).

We begin by considering the case when \( 0 < x < 2\pi \). It is possible to evaluate (26) and (27) directly, treating \( x \) as an interval, however this gives huge overestimations as soon as \( x \) is not a very tight interval. For \( C_s \) we make use of the following lemma, see also [20, Lemma B.1].

**Lemma B.1.** For all \( s \in \mathbb{R} \) the Clausen function \( C_s(x) \) is monotone in \( x \) on the interval \((0, \pi)\). For \( s > 0 \) it is non-increasing.

**Proof.** We have \( C'_s(x) = -S_{s-1}(x) \). For \( s > 1 \) we have [47, Eq. 25.12.11]

\[
S_{s-1}(x) = \text{Im} \int_0^\infty t^{s-2} \frac{1}{e^t - e^{-i\pi}} dt
\]

\[
= \text{Im} \int_0^\infty t^{s-2} \frac{e^t - e^{-i\pi}}{(e^t - \cos(x))^2 + \sin^2(x)} dt
\]

\[
= \frac{\sin(x)}{\Gamma(s-1)} \int_0^\infty t^{s-2} \frac{e^t}{(e^t - \cos(x))^2 + \sin^2(x)} dt.
\]
Which for $s - 1 > 0$ and $x \in (0, \pi)$ is positive, $C_s(x)$ is hence decreasing.

For $s < 1$ we use equation (27) together with [47, Eq. 25.11.25] to get

$$S_{s-1}(x) = \frac{\sin \left( \frac{\pi}{2} (2 - s) \right)}{(2\pi)^{2-s}} \int_0^\infty t^{1-s} e^{-\frac{st}{2\pi}} \frac{1 - e^{(x/\pi - 1)t}}{1 - e^{-t}} \, dt,$$

which sign depends only on the value of $s$. In particular, for $0 < s \leq 1$ we have $\sin \left( \frac{\pi}{2} (2 - s) \right) > 0$ so $C_s(x)$ is decreasing for $s > 0$. \hfill $\square$

Since $C_s(x)$ is even around $x = \pi$ it follows that if $0 < x < 2\pi$ then the extrema of $C_s(x)$ are attained at either $x = x$, $x = \pi$ or, if $\pi \in x$, $x = \pi$. This allows us to compute very tight enclosures of $C_s(x)$ in the argument $x$. The function $S_s(x)$ is in general not monotone. We handle it by computing and enclosure of the derivative $S_s'(x) = C_{s-1}(x)$, if the enclosure of the derivative doesn’t contain zero then the function is monotone and we evaluate it at the endpoints, if the derivative contains zero we instead use the midpoint approximation $S_s(x) = S_s(x_0) + (x - x_0)C_{s-1}(x)$ where $x_0$ is the midpoint of $x$.

For $x = [0, 2\pi]$ we assume that $s > 1$ since the value is typically unbounded otherwise (in which case we return an indeterminate result). We get an enclosure of $C_s(x)$ by evaluating at the critical points $x = 0$ and $x = \pi$. For $S_s(x)$ we use the trivial bound

$$|S_s(x)| = \left| \sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n^s} \right| \leq \sum_{n=1}^{\infty} \left| \frac{\sin(n\pi x)}{n^s} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s)$$

to get the enclosure $S_s(x) \subseteq [-\zeta(s), \zeta(s)]$.

For the final case, when $-2\pi < x \leq 0 \leq \pi < 2\pi$, we also assume that $s > 1$. We handle $C_s(x)$ by using the eveness and monotonicity in $x$. If $\max(-\pi, \pi) < \pi$ then the extrema are attained at $x = 0$ and $x = \max(-\pi, \pi) < \pi$, otherwise the extrema are attained at $x = 0$ and $x = \pi$. For $S_s(x)$ we check if it is increasing on $x$ by checking if $\max(-\pi, \pi) < \pi$ and $S_s'(\max(-\pi, \pi)) = C_{s-1}(\max(-\pi, \pi)) \geq 0$, where we have used that $C_s(x)$ is decreasing for $s > 0$. If it is increasing we evaluate at the endpoints, otherwise we have the trivial bound $[-\zeta(s), \zeta(s)]$.

In all the above cases we could reduce the problem of enclosing the Clausen functions on $x$ to evaluating them on, a subset of, $x = 0$, $x = \frac{\pi}{2}$ (or $-\pi$ if $\pi < 0$), $x = \pi$, $x = x_0$ (the midpoint) and $x = \frac{\pi}{2}$. In all cases except $x = 0$ we have $0 < x < 2\pi$ so we can use (26) and (27).

For $s$ we get two different cases depending on if $s$ contains a non-negative integer or not.

When $s$ doesn’t contain a non-negative integer the formulations (26) and (27) are well defined. However, similarly as for $x$, direct evaluation gives large over-estimations when $s$ is not very tight. In this case there is no monotonicity to use, instead we compute a tighter enclosure using a Taylor expansion as in (21). One exception to this is the modified Clausen function $\tilde{C}_s(x)$, for which we have the following result

**Lemma B.2.** For $s > 1$ and $x \in \mathbb{R}$ the function $\tilde{C}_s(x)$ is non-decreasing in $s$. 
Proof. We have
\[ \tilde{C}_s(x) = C_s(x) - \zeta(s) = \sum_{n=1}^{\infty} \frac{\cos(nx) - 1}{n^s}. \]
Since \( \cos(nx) - 1 \leq 0 \) for \( x \in \mathbb{R} \) all terms in the sum are negative. That it is non-decreasing in \( s \) then follows easily from the monotonicity of \( n^s \). \( \square \)

This means that for \( s > 1 \) we only have to evaluate on \( s \) and \( \bar{s} \) to get an enclosure. In practice \( \tilde{C}_s(x) \) is only every used with \( s > 1 \) so this is enough to get good bounds.

When \( s \) contains a non-negative integer we have to handle the removable singularities in (26) and (27) as described in the beginning of the section. The required derivatives can generally be computed directly with Arb, this is the case for the reciprocal gamma function, \( \sin, \cos \) as well as \( \zeta(s, x) \). There is an implementation of the deflated zeta function \( \zeta(s, a) \) in Arb, however it only supports \( s \) exactly equal to 1, and not intervals containing 1, and can therefore not be used directly. In our application this case does however not occur, so it is not an issue.

B.2. Expansion in \( x \)

We now go through how to compute expansions of the Clausen functions in the argument \( x \).

Around any point \( 0 < x < 2\pi \) the functions are analytic and it is straightforward to compute the Taylor expansions by differentiating directly, using that \( \frac{d}{dx} C_s(x) = -S_{s-1}(x) \) and \( \frac{d}{dx} S_s(x) = C_s - 1(x) \).

At \( x = 0 \) we then have the following asymptotic expansions [20]:

\[ C_s(x) = \Gamma(1 - s) \sin \left( \frac{\pi}{2} s \right) |x|^{s-1} + \sum_{m=0}^{\infty} (-1)^m \zeta(s - 2m) \frac{x^{2m}}{(2m)!}; \] \hspace{1cm} (29)

\[ S_s(x) = \Gamma(1 - s) \cos \left( \frac{\pi}{2} s \right) \text{sgn}(x)|x|^{s-1} + \sum_{m=0}^{\infty} (-1)^m \zeta(s - 2m - 1) \frac{x^{2m+1}}{(2m + 1)!}. \] \hspace{1cm} (30)

These expressions work well as long as \( s \) is not a positive integer. For positive integers we have to handle the poles of \( \Gamma(s) \) at non-positive integers and the pole of \( \zeta(s) \) at \( s = 1 \).

For \( C_s(x) \) with positive even integers \( s \) the only problematic term is
\[ \Gamma(1 - s) \sin \left( \frac{\pi}{2} s \right) |x|^{s-1} \]
which has a removable singularity. Similarly for \( S_s(x) \) with positive odd integers \( s \). For \( C_s(x) \) with positive odd integers \( s \) and \( S_s \) with positive even integers \( s \) the singularities are not removable, however in this paper we don’t encounter this case.

With the above we are able to compute expansions at \( x = 0 \) to arbitrarily high degree. What remains is to bound the tails of the sums in (29) and (30). For this we have the following lemma, see also [20, Lemma 2.1]. We omit the proof since it is very similar to that in [20].
Lemma B.3. Let \( s \geq 0, 2M \geq s + 1 \) and \( |x| < 2\pi \), we then have the following bounds for the tails in Eqs. (29) and (30)

\[
\begin{align*}
\sum_{m=M}^{\infty} (-1)^m \zeta(s - 2m) & \frac{x^{2m}}{(2m)!} \\
& \leq 2(2\pi)^1 + s - 2M \left| \sin \left( \frac{\pi}{2} s \right) \right| \zeta(2M + 1 - s) \frac{x^{2M}}{4\pi^2 - x^2}, \\
\sum_{m=M}^{\infty} (-1)^m \zeta(s - 2m + 1) & \frac{x^{2m+1}}{(2m + 1)!} \\
& \leq 2(2\pi)^s - 2M \left| \cos \left( \frac{\pi}{2} s \right) \right| \zeta(2M + 2 - s) \frac{x^{2M+1}}{4\pi^2 - x^2}.
\end{align*}
\]

B.3. Derivatives in \( s \)

For \( C_s^{(\beta)}(x) \) and \( S_s^{(\beta)}(x) \) we use (26) and (27) and differentiate directly in \( s \). When \( s \) is not an integer this is handled directly using Taylor arithmetic, for integers we use the approach in “Appendix A” to handle the removable singularities.

To get asymptotic expansions in \( x \) we take the expansions (29) and (30) and differentiate them with respect to \( s \). Giving us

\[
\begin{align*}
C_s^{(\beta)}(x) &= \frac{d}{ds^\beta} \left[ \Gamma(1 - s) \sin \left( \frac{\pi}{2} s \right) \right] |x|^{s-1} + \sum_{m=0}^{\infty} (-1)^m \zeta^{(\beta)}(s - 2m) \frac{x^{2m}}{(2m)!}; \\
S_s^{(\beta)}(x) &= \frac{d}{ds^\beta} \left[ \Gamma(1 - s) \cos \left( \frac{\pi}{2} s \right) \right] \text{sgn}(x)|x|^{s-1} \\
&+ \sum_{m=0}^{\infty} (-1)^m \zeta^{(\beta)}(s - 2m + 1) \frac{x^{2m+1}}{(2m + 1)!}.
\end{align*}
\]

These formulas work well when \( s \) is not a positive odd integer for \( C_s^{(\beta)}(x) \) or a positive even integer for \( S_s^{(\beta)}(x) \) and the derivatives can be computed using Taylor series. We mostly make use of the functions \( C_2^{(1)}(x) \) and \( C_3^{(1)}(x) \), in which case the expansions can be computed explicitly using [2, Eq. 16], for \( |x| < 2\pi \) we have

\[
\begin{align*}
C_2^{(1)}(x) &= \zeta^{(1)}(2) - \frac{\pi}{2} |x| \log |x| - (\gamma - 1) \frac{\pi}{2} |x| + \sum_{m=1}^{\infty} (-1)^m \zeta^{(1)}(2 - 2m) \frac{x^{2m}}{(2m)!} \\
C_3^{(1)}(x) &= \zeta^{(1)}(3) - \frac{1}{4} x^2 \log^2 |x| + \frac{3 - 2\gamma}{4} x^2 \log |x| \\
&- \frac{36\gamma - 12\gamma^2 - 24\gamma_1 - 42 + \pi^2}{48} x^2 \\
&+ \sum_{m=2}^{\infty} (-1)^m \zeta^{(1)}(3 - 2m) \frac{x^{2m}}{(2m)!}.
\end{align*}
\]
Where \( \gamma_n \) is the Stieltjes constant and \( \gamma = \gamma_0 \). To bound the tails we have the following lemma:

**Lemma B.4.** Let \( \beta \geq 1, s \geq 0, 2M \geq s + 1 \) and \( |x| < 2\pi \), we then have the following bounds:

\[
\sum_{m=M}^{\infty} (-1)^m \zeta(\beta)(s - 2m) \frac{x^{2m}}{(2m)!} \leq \sum_{j_1 + j_2 + j_3 = \beta} \left( \begin{array}{c} \beta \\ j_1, j_2, j_3 \end{array} \right) 2 \left( \log(2\pi) + \frac{\pi}{2} \right)^{j_1} (2\pi)^{s - 1} |\zeta(j_3)(1 + 2M - s) | \sum_{m=M}^{\infty} \left| p_{j_2} (1 + 2m - s) \left( \frac{x}{2\pi} \right)^{2m} \right|,
\]

\[
\sum_{m=M}^{\infty} (-1)^m \zeta(\beta)(s - 2m - 1) \frac{x^{2m+1}}{(2m + 1)!} \leq \sum_{j_1 + j_2 + j_3 = \beta} \left( \begin{array}{c} \beta \\ j_1, j_2, j_3 \end{array} \right) 2 \left( \log(2\pi) + \frac{\pi}{2} \right)^{j_1} (2\pi)^{s - 2} |\zeta(j_3)(1 + 2M - s) | \sum_{m=M}^{\infty} \left| p_{j_2} (2 + 2m - s) \left( \frac{x}{2\pi} \right)^{2m+1} \right|.
\]

Here \( p_{j_2} \) is given recursively by

\[
p_{k+1}(s) = \psi^{(0)}(s) p_k(s) + p_k'(s), \quad p_0 = 1,
\]

where \( \psi^{(0)} \) is the polygamma function. It is given by a linear combination of terms of the form

\[
(\psi^{(0)}(s))^{q_0} (\psi^{(1)}(s))^{q_1} \cdots (\psi^{(j_2-1)}(s))^{q_{j_2-1}}.
\]

If \( x^2 < e^{-\frac{q_0}{1+2M}} \) we have the bounds

\[
\sum_{m=M}^{\infty} \left| (\psi^{(0)}(1 + 2m - s))^{q_0} \cdots (\psi^{(j_2-1)}(1 + 2m - s))^{q_{j_2-1}} \left( \frac{x}{2\pi} \right)^{2m} \right| \leq \left| (\psi^{(1)}(1 + 2M - s))^{q_1} \cdots (\psi^{(j_2-1)}(1 + 2M - s))^{q_{j_2-1}} (1 + 2M - s) \right| \frac{1}{2^{q_0/2}} (2\pi)^{-2M} \Phi \left( \frac{x^2}{4\pi^2}, -\frac{q_0}{2}, M + \frac{1}{2} \right) x^{2M}
\]

and

\[
\sum_{m=M}^{\infty} \left| (\psi^{(0)}(2 + 2m - s))^{q_0} \cdots (\psi^{(j_2-1)}(2 + 2m - s))^{q_{j_2-1}} \left( \frac{x}{2\pi} \right)^{2m+1} \right| \leq \left| (\psi^{(1)}(2 + 2M - s))^{q_1} \cdots (\psi^{(j_2-1)}(2 + 2M - s))^{q_{j_2-1}} (2 + 2M - s) \right| \frac{1}{2^{q_0/2}} (2\pi)^{-2M} \Phi \left( \frac{x^2}{4\pi^2}, -\frac{q_0}{2}, M + \frac{1}{2} \right) x^{2M}.
\]
where

\[
\Phi(z, s, a) = \sum_{m=0}^{\infty} \frac{z^m}{(a + m)^s},
\]

is the Lerch transcendent.

**Proof.** We give the proof for the first case, the other one is similar. We have the functional identity

\[
\zeta(s) = 2(2\pi)^{s-1} \sin\left(\frac{\pi}{2}s\right) \Gamma(1 - s) \zeta(1 - s).
\]

If we let

\[
f(s) = 2(2\pi)^{s-1} \sin\left(\frac{\pi}{2}s\right),
\]

\[
g(s) = \Gamma(1 - s),
\]

\[
h(s) = \zeta(1 - s),
\]

we have

\[
\zeta^{(\beta)}(s) = \sum_{j_1 + j_2 + j_3 = \beta} \left( \begin{array}{c} \beta \\ j_1, j_2, j_3 \end{array} \right) f^{(j_1)}(s) g^{(j_2)}(s) h^{(j_3)}(s).
\]

This gives us

\[
\sum_{m=M}^{\infty} (-1)^m \zeta^{(\beta)}(s - 2m) \frac{x^{2m}}{(2m)!} = \sum_{m=M}^{\infty} (-1)^m \frac{x^{2m}}{(2m)!} \sum_{j_1 + j_2 + j_3 = \beta} \left( \begin{array}{c} \beta \\ j_1, j_2, j_3 \end{array} \right) f^{(j_1)}(s - 2m) g^{(j_2)}(s - 2m) h^{(j_3)}(s - 2m)
\]

\[
= \sum_{j_1 + j_2 + j_3 = \beta} \left( \begin{array}{c} \beta \\ j_1, j_2, j_3 \end{array} \right) \sum_{m=M}^{\infty} (-1)^m f^{(j_1)}(s - 2m) g^{(j_2)}(s - 2m) h^{(j_3)}(s - 2m) \frac{x^{2m}}{(2m)!}.
\]

We are thus interested in bounding

\[
\left| \sum_{m=M}^{\infty} (-1)^m f^{(j_1)}(s - 2m) g^{(j_2)}(s - 2m) h^{(j_3)}(s - 2m) \frac{x^{2m}}{(2m)!} \right| \leq \sum_{m=M}^{\infty} \left| f^{(j_1)}(s - 2m) g^{(j_2)}(s - 2m) h^{(j_3)}(s - 2m) \frac{x^{2m}}{(2m)!} \right|.
\]

for \(j_1, j_2, j_3 \geq 0\).

For \(f^{(j_1)}(s - 2m)\) we start by noting that

\[
f^{(j_1)}(s) = 2 \sum_{k=0}^{j_1} \binom{j_1}{k} d^k (2\pi)^{s-1} \frac{d^{j_1-k}}{ds^{j_1-k}} \sin\left(\frac{\pi}{2}s\right).
\]
\[
= 2(2\pi)^{s-1} \sum_{k=0}^{j_1} \left( \frac{j_1}{k} \log(2\pi)^k \frac{d^{j_1-k} \sin \left( \frac{\pi}{2} s \right)}{d s^{j_1-k}} \right).
\]

Hence
\[
|f^{(j_1)}(s)| \leq 2(2\pi)^{s-1} \sum_{k=0}^{j_1} \left( \frac{j_1}{k} \log(2\pi)^k \left( \frac{\pi}{2} \right)^{k-j_1} = 2 \left( \log(2\pi) + \frac{\pi}{2} \right)^{j_1} (2\pi)^{s-1}
\]

and, in particular,
\[
|f^{(j_1)}(s - 2m)| \leq 2 \left( \log(2\pi) + \frac{\pi}{2} \right)^{j_1} (2\pi)^{s-1} (2\pi)^{-2m}.
\]

For \(g^{(j_2)}(s - 2m)\) we start from \(\Gamma^{(1)}(s) = \Gamma(s) \psi^{(0)}(s)\). Differentiating this we get
\[
\Gamma^{(j_2)}(s) = \Gamma(s) p_{j_2}(s),
\]
where \(p_{j_2}\) is given recursively by
\[
p_{k+1} = \psi^{(0)} p_k + p_k', \quad p_0 = 1.
\]

This gives us
\[
|g^{(j_2)}(s - 2m)| = \Gamma(1 + 2m - s)|p_{j_2}(1 + 2m - s)|.
\]

For \(h^{(j_3)}(s - 2m)\) we have
\[
h^{(j_3)}(s - 2m) = (-1)^{j_3} \zeta^{(j_3)}(1 + 2m - s).
\]

Since \(2m \geq 2 M \geq s + 1\) we have \(1 + 2m - s > 1\) and hence
\[
\zeta^{(j_3)}(1 + 2m - s) = \sum_{k=1}^{\infty} \frac{\log^{j_3}(k)}{k^{1+2m-s}},
\]
which is decreasing in \(m\). Giving us
\[
|h^{(j_3)}(s - 2m)| \leq |\zeta^{(j_3)}(1 + 2M - s)|.
\]

Combining the bounds for \(|f^{(j_1)}|\), \(|g^{(j_2)}|\), and \(|h^{j_3}|\), we have
\[
\sum_{m=M}^{\infty} \left| f^{(j_1)}(s - 2m) g^{(j_2)}(s - 2m) h^{(j_3)}(s - 2m) \frac{x^{2m}}{(2m)!} \right| \leq 2 \left( \log(2\pi) + \frac{\pi}{2} \right)^{j_1} (2\pi)^{s-1} |\zeta^{(j_3)}(1
\]
\[
+ 2M - s) \right) \sum_{m=M}^{\infty} \left| (2\pi)^{-2m} \Gamma(1 + 2m - s) p_{j_2}(1 + 2m - s) \frac{x^{2m}}{(2m)!} \right| = 2 \left( \log(2\pi) + \frac{\pi}{2} \right)^{j_1} (2\pi)^{s-1} |\zeta^{(j_3)}(1
\]
Using that \((2m)! \geq \Gamma(2m + 1 - s)\) this simplifies to
\[
\sum_{m=M}^{\infty} \left| f^{(j_1)}(s-2m)g^{(j_2)}(s-2m)h^{(j_3)}(s-2m) \frac{x^{2m}}{(2m)!} \right| \leq 2 \left( \log(2\pi) + \frac{\pi}{2} \right)^{j_1} (2\pi)^{s-1} |\psi^{(j_3)}(1+2m - s)| \sum_{m=M}^{\infty} |p_{j_2}(1+2m - s)| \left( \frac{x}{2\pi} \right)^{2m}.
\]

This proves the first part of the statement.

Using that \(\frac{d}{dx} \psi^{(k)}(s) = \psi^{(k+1)}(s)\) we can compute \(p_k\) for any fixed \(k\). It is clear \(p_{j_2}(s)\) will be a linear combination of terms of the form
\[
(\psi^{(0)}(s))^{q_0} \psi^{(1)}(s))^{q_1} \ldots (\psi^{(j_2-1)}(s))^{q_{j_2-1}}.
\]

From the integral representation [47, Eq. 5.9.15] it follows that for \(s > 0\) we have \(\psi^{(0)}(s) < \log s\), using that \(\log s < \sqrt{s}\) we get \(\psi^{(0)}(s) < \sqrt{s}\). For \(s > 0\) and \(k \geq 1\) it follows from the integral representation
\[
\psi^{(k)}(s) = (-1)^{k+1} \int_{0}^{\infty} \frac{t^k e^{-st}}{1 - e^{-t}} \, dt
\]
that \(|\psi^{(k)}(s)|\) is decreasing in \(s\). For \(m \geq M\) we then have the bound
\[
|((\psi^{(0)}(1+2m - s))^{q_0} (\psi^{(1)}(1+2m - s))^{q_1} \ldots (\psi^{(j_2-1)}(1+2m - s))^{q_{j_2-1}}) \leq (1+2m)^{q_0/2} (\psi^{(1)}(1+2M - s))^{q_1} \ldots (\psi^{(j_2-1)}(1+2M - s))^{q_{j_2-1}}.\]

This gives us
\[
\sum_{m=M}^{\infty} \left| (\psi^{(0)}(1+2m - s))^{q_0} \ldots (\psi^{(j_2-1)}(1+2m - s))^{q_{j_2-1}} \left( \frac{x}{2\pi} \right)^{2m} \right| \leq |(\psi^{(1)}(1+2M - s))^{q_1} \ldots (\psi^{(j_2-1)}(1+2M - s))^{q_{j_2-1}}| \sum_{m=M}^{\infty} (1+2m)^{q_0/2} \left( \frac{x}{2\pi} \right)^{2m}.
\]

Focusing on the sum we rewrite it as
\[
\frac{1}{2^{q_0/2}} \left( \frac{x}{2\pi} \right)^{2M} \sum_{m=0}^{\infty} \left( m + M + \frac{1}{2} \right)^{q_0/2} \left( \frac{x}{2\pi} \right)^{2m}.
\]

Since \(|x| < 2\pi\) the sum can be written using the Lerch transcendent as \(\Phi \left( \frac{x^2}{4\pi^2}, -\frac{q_0}{2}, M + \frac{1}{2} \right)\). Hence
\[
\sum_{m=M}^{\infty} (1+2m)^{q_0/2} \left( \frac{x}{2\pi} \right)^{2m} \leq \frac{1}{2^{q_0/2}} (2\pi)^{-2M} x^{2M} \Phi \left( \frac{x^2}{4\pi^2}, -\frac{q_0}{2}, M + \frac{1}{2} \right).
Putting all of this together we arrive at the bound

\[
\sum_{m=M}^{\infty} \left| (\psi^{(0)}(1 + 2m - s))^{q_0} \cdots (\psi^{(j_2-1)}(1 + 2m - s))^{q_{j_2-1}} \left( \frac{x}{2\pi} \right)^{2m} \right|
\]

\[
\leq |(\psi^{(1)}(1 + 2M - s))^{q_1} \cdots (\psi^{(j_2-1)}(1 + 2M - s))^{q_{j_2-1}}| \left( \frac{1}{2^{q_0/2}} \right)^{2M} \Phi \left( \frac{x^2}{4\pi^2}, -\frac{q_0}{2}, M + \frac{1}{2} \right) x^{2M}.
\]

\[
\square
\]

C. Rigorous Integration with Singularities

We here explain how to compute enclosures of the integrals \(U_{1,1}(x), U_{1,2}(x)\) and \(U_2(x)\) in the non-asymptotic case, as well as how to enclose \(c_1\) and \(c_2\) occurring in Lemmas 6.3 and 6.4. Recall that

\[
U_{1,1}(x) = \int_0^{r_x} \log \left( \frac{\sin(x(1 - t)/2) \sin(x(1 + t)/2)}{\sin(xt/2)^2} \right) t \sqrt{\log(1 + 1/(xt))} \, dt;
\]

\[
U_{1,2}(x) = -\int_{r_x}^1 \log \left( \frac{\sin(x(1 - t)/2) \sin(x(1 + t)/2)}{\sin(xt/2)^2} \right) t \sqrt{\log(1 + 1/(xt))} \, dt;
\]

\[
U_2(x) = -\int_{x}^{\pi} \log \left( \frac{\sin((y - x)/2) \sin((x + y)/2)}{\sin(y/2)^2} \right) y \sqrt{\log(1 + 1/y)} \, dy.
\]

The integrand for \(U_{1,2}\) has a (integrable) singularity at \(t = 1\) and the integrand for \(U_2\) has one at \(y = x\). As a first step we split these off to handle them separately. Let

\[
U_{1,2}(x) = -\int_{r_x}^{1-\delta_1} \log \left( \frac{\sin(x(1 - t)/2) \sin(x(1 + t)/2)}{\sin(xt/2)^2} \right) t \sqrt{\log(1 + 1/(xt))} \, dt
\]

\[-\int_{1-\delta_1}^1 \log \left( \frac{\sin(x(1 - t)/2) \sin(x(1 + t)/2)}{\sin(xt/2)^2} \right) t \sqrt{\log(1 + 1/(xt))} \, dt
\]

\[= U_{1,2,1}(x) + U_{1,2,2}(x),
\]

\[
U_2(x) = -\int_{x}^{x+\delta_2} \log \left( \frac{\sin((y - x)/2) \sin((x + y)/2)}{\sin(y/2)^2} \right) y \sqrt{\log(1 + 1/y)} \, dy
\]

\[-\int_{x+\delta_2}^{\pi} \log \left( \frac{\sin((y - x)/2) \sin((x + y)/2)}{\sin(y/2)^2} \right) y \sqrt{\log(1 + 1/y)} \, dy
\]

\[= U_{2,1}(x) + U_{2,2}(x).
\]

For \(U_{1,2,2}\) and \(U_{2,1}\) we have

**Lemma C.1.** We have

\[
U_{1,2,2}(x) = \frac{\xi_1}{x} (-S_2(0) + S_2(2x) - 2S_2(x) - (\xi_2 x \delta_1) + S_2(x(2 - \delta_1)) - 2S_2(x(1 - \delta_1)))
\]
for some \( \xi_1 \) in the image of \( t \sqrt{\log(1 + 1/(xt))} \) for \( t \in [1 - \delta_1, 1] \). Furthermore

\[
U_{2,1}(x) = \xi_2 \left( (S_2(\delta_2) + S_2(2x + \delta_2) - 2S_2(x + \delta_2)) - (S_2(0) + S_2(2x) - 2S_2(x)) \right)
\]

for some \( \xi_2 \) in the image of \( y \sqrt{\log(1 + 1/y)} \) for \( y \in [x, x + \delta_2] \).

**Proof.** We give the proof for \( U_{1,2,2}(x) \), the other one is similar.

The factor \( t \sqrt{\log(1 + 1/(xt))} \) in the integrand of \( U_{1,2,2} \) is bounded in \( t \) on the interval \([1 - \delta_1, 1]\) and the integrand has a constant sign, hence there exists \( \xi_1 \) in the range of \( t \sqrt{\log(1 + 1/(xt))} \) on this interval such that

\[
U_{1,2,2}(x) = \xi_1 \int_{1 - \delta_1}^{1} -\log \left( \frac{\sin((1 - t)/2) \sin((1 + t)/2)}{\sin(x t/2)^2} \right) \, dt. \tag{33}
\]

It is therefore enough to compute an enclosure of this integral. For this we use that

\[
C_1(x) = -\log(2 \sin(|x|/2))
\]

to write it as

\[
\int_{1 - \delta_1}^{1} C_1(x(1 - t)) + C_1(x(1 + t)) - 2C_1(xt) \, dt.
\]

Using that \( \int C_1(t) \, dt = S_2(t) \) we get

\[
\int C_1(x(1 - t)) + C_1(x(1 + t)) - 2C_1(xt) \, dt = \frac{1}{x} \left( -S_2(x(1 - t)) + S_2(x(1 + t)) - 2S_2(xt) \right).
\]

Integrating from \( 1 - \delta_1 \) to 1 gives us

\[
U_{1,2,2}(x) = \xi_1 \frac{1}{x} \left( -S_2(0) + S_2(2x) - 2S_2(x) \right.
\]

\[
- \left( -S_2(x(\delta_1)) + S_2(x(2 - \delta_1)) - 2S_2(x(1 - \delta_1)) \right),
\]

which proves the result.

What remains is computing \( U_{1,1}, U_{1,2,1} \) and \( U_{2,2} \). In this case the integrands are bounded everywhere on the intervals of integration and we enclose the integrals using the rigorous numerical integrator implemented by Arb [33]. For functions that are analytic on the interval the integrator uses Gaussian quadratures with error bounds computed through complex magnitudes, we therefore need to evaluate the integrands on complex intervals. When the function is not analytic it falls back to naive enclosures using interval arithmetic. The only non-trivial part is bounding the integrand for \( U_{1,1} \) near \( t = 0 \), where it is bounded but not analytic. For this we start by splitting it as

\[
\log \left( \frac{\sin(x(1 - t)/2) \sin(x(1 + t)/2)}{\sin(x t/2)^2} \right) t \sqrt{\log(1 + 1/(xt))}
\]

\[
= \log(\sin(x(1 - t)/2)) t \sqrt{\log(1 + 1/(xt))} + \log(\sin(x(1 + t)/2)) t \sqrt{\log(1 + 1/(xt))} - 2 \log(\sin(x t/2)) t \sqrt{\log(1 + 1/(xt))}.
\]
For the first two terms the only problematic part is the factor \( t \sqrt{\log(1 + 1/(xt))} \), we note that it is zero at \( t = 0 \) and differentiating it gives us

\[
\frac{d}{dt} t \sqrt{\log(1 + 1/(xt))} = \frac{2x t \log(1 + 1/(xt)) + 2 \log(1 + 1/(xt)) - 1}{(2xt + 1) \sqrt{\log(1 + 1/(xt))}}.
\]

If

\[
2 \log(1 + 1/(xt)) - 1 > 0 \iff t < \frac{1}{x(e^{1/2} - 1)}.
\]

then the derivative is positive and we can thus get an enclosure by checking that this inequality holds and using monotonicity. This leaves us with the third term,

\[
\log(\sin(xt/2)) t \sqrt{\log(1 + 1/(xt))},
\]

which is also zero at \( t = 0 \) and for the monotonicity we have

**Lemma C.2.** For \( 0 < x < \pi \) and \( 0 < t < \frac{1}{10x} \) the function

\[
\log(\sin(xt/2)) t \sqrt{\log(1 + 1/(xt))}
\]

is decreasing in \( t \).

**Proof.** Differentiating we have

\[
\frac{d}{dt} \log(\sin(xt/2)) t \sqrt{\log(1 + 1/(xt))} = \frac{(2xt + 1) \log(1 + 1/(xt)) - 1) \log(\sin(xt/2)) + xt(x + 1) \log(1 + 1/(xt)) \cot(xt/2)}{2(xt + 1) \sqrt{\log(1 + 1/(xt))}}.
\]

Since the denominator is positive it is enough to show that

\[
(2xt + 1) \log(1 + 1/(xt)) - 1) \log(\sin(xt/2)) + xt(x + 1) \log(1 + 1/(xt)) \cot(xt/2) \leq 0.
\]

Since \( t < \frac{1}{10x} < \frac{1}{x(e^{1/2} - 1)} \) the factor \( (2xt + 1) \log(1 + 1/(xt)) - 1) \) is positive, multiplication by \( \log(\sin(xt/2)) \) makes it negative. Using that \( \log(\sin(xt/2)) < \log(xt/2) < \log(xt) \) an upper bound is hence given by

\[
(2 \log(1 + 1/(xt)) - 1) \log(xt) + xt(x + 1) \log(1 + 1/(xt)) \cot(xt/2).
\]

If we let \( r = xt \in [0, \frac{1}{4}] \) this becomes

\[
(2 \log(1 + 1/r) - 1) \log(r) + r(r + 1) \log(1 + 1/r) \cot(r/2).
\]

Using that \( r \cot(r/2) \leq 2 \) for \( |r| < \pi \) and \( \log(1 + 1/r) > -\log(r) \) we get the upper bound

\[
(-2 \log(r) - 1) \log(r) + 2(r + 1) \log(1 + 1/r).
\]

Splitting \( \log(1 + 1/r) = \log(1 + r) - \log(r) \) gives

\[
(-2 \log(r) - 1) \log(r) + 2(r + 1) \log(1 + r) - 2(r + 1) \log(r).
\]
Using that $2(r + 1) \leq \frac{11}{5}$ and $\log(1 + r) \leq \log(11/10)$ it reduces to
\[
(-2 \log(r) - 1) \log(r) + \frac{11}{5} \log(11/10) - \frac{11}{5} \log(r)
\]
which is a quadratic expression in $\log(r)$, which can easily be checked to be negative on $[0, \frac{1}{10}]$. \hfill \square

For enclosing $c_1$ and $c_2$ recall that
\[
c_1 = \int_0^1 |\log(1/t^2 - 1)|t\sqrt{\log(1/t)} \, dt;
\]
\[
c_2 = \int_0^\pi y\sqrt{\log(1 + 1/y)} \, dy.
\]
Both integrands are bounded, by splitting $c_1$ as
\[
c_1 = \int_0^{\sqrt{3}/2} \log(1/t^2 - 1)t\sqrt{\log(1/t)} \, dt - \int_{\sqrt{3}/2}^1 \log(1/t^2 - 1)t\sqrt{\log(1/t)} \, dt,
\]
they are also analytic except at $t = 0$ and $t = 1$ for $c_1$ and $y = 0$ for $c_2$. For $c_2$ the integrand is easily seen to be increasing for $y > 0$ and it can be bounded near $y = 0$ using that. The integrand for $c_1$ is not monotone, to bound it we split it into three terms
\[
\log(1/t^2 - 1)t\sqrt{\log(1/t)} = tf_1(t) + tf_2(t) - 2f_3(t),
\]
with
\[
f_1(t) = \log(1 - t)\sqrt{\log(1/t)}, \quad f_2(t) = \log(1 + t)\sqrt{\log(1/t)},
\]
\[
f_3(t) = t \log(t)\sqrt{\log(1/t)}.
\]
It is then enough to bound $f_1$, $f_2$ and $f_3$. Differentiating we have
\[
f_1'(t) = \frac{1}{2\sqrt{\log(1/t)}} \left( \frac{2\log t}{1 - t} - \frac{\log(1 - t)}{t} \right),
\]
\[
f_2'(t) = -\frac{1}{2\sqrt{\log(1/t)}} \left( \frac{2\log t}{1 + t} + \frac{\log(1 + t)}{t} \right),
\]
\[
f_3'(t) = \frac{1}{2\sqrt{\log(1/t)}}(3 + 2\log(t)).
\]
For $f_3'$ we get the unique root $e^{-3/2}$ on the interval $(0, 1)$, we can thus use monotonicity of $f_3$ as long as we avoid this point. For $f_1'$ and $f_2'$ we are looking for the zeros of
\[
g_1(t) = \frac{2\log t}{1 - t} - \frac{\log(1 - t)}{t} \quad \text{and} \quad g_2(t) = \frac{2\log t}{1 + t} + \frac{\log(1 + t)}{t}
\]
respectively. For $g_1$ both $\frac{2\log t}{1 - t}$ and $-\frac{\log(1 - t)}{t}$ are increasing, hence $g_1$ is increasing. From the limits $\lim_{t \to 0^+} g_1(t) = -\infty$ and $\lim_{t \to 1^-} g_1(t) = \infty$ it follows that
$f_1'$ has exactly one root on $(0, 1)$. This root can easily be isolated using interval arithmetic and we can then use monotonicity of $f_1$ as long as we avoid this root. The function $g_2$ is also monotone, to see this we differentiate, giving us
\[ g_2'(t) = -\frac{2t^2 \log(t) + (1 + t)(t(\log(1 + t) - 3) + \log(1 + t))}{t^2(1 + t)^2}. \]

The sign is determined by
\[-(2t^2 \log(t) + (1 + t)(t(\log(1 + t) - 3) + \log(1 + t))),(\]

A lower bound is given by
\[(1 + t)(3t - t \log(1 + t) - \log(1 + t)),\]

that this is positive follows from the inequality $t > \log(1 + t)$. This proves that $f_2'$ is monotone, we can then use the same approach as for $f_1$ to isolate the unique critical point.

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