A CONTROL PROBLEM RELATED TO THE PARABOLIC DOMINATIVE $p$-LAPLACE EQUATION

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Abstract. We show that value functions of a certain time-dependent control problem in $\Omega \times (0, T)$, with a continuous payoff $F$ on the parabolic boundary, converge uniformly to the viscosity solution of the parabolic dominative $p$-Laplace equation

$$2(n+p)u_t = \Delta u + (p-2)\lambda_n(D^2u),$$

with the boundary data $F$. Here $2 < p < \infty$, and $\lambda_n(D^2u)$ is the largest eigenvalue of the Hessian $D^2u$.

1. Introduction

In this paper we give a control problem interpretation for the parabolic dominative $p$-Laplace equation

$$2(n+p)u_t = D_p u \quad \text{in} \quad \Omega_T. \hspace{1cm} (1.1)$$

Here $\Omega_T := \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain satisfying a uniform exterior sphere condition, and

$$D_p u := (\lambda_1 + ... + \lambda_{n-1}) + (p-1)\lambda_n = \Delta u + (p-2)\lambda_n,$$

where $2 < p < \infty$, and $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ are the eigenvalues of the Hessian $D^2u$. The operator $D_p$ is called the dominative $p$-Laplacian, introduced by Brustad [Bru17, Bru18] and later studied by Brustad, Lindqvist and Manfredi [BLM18] and Høeg [Hoe19] in the elliptic case. The dominative $p$-Laplacian explains the superposition principle of the $p$-Laplace equation, see [CZ03, LM08] for more about this property. The operator $D_p$ is sublinear, so it is convex, and equation $(1.1)$ is uniformly parabolic. By Theorem 3.2 in [Wan92], viscosity solutions of $(1.1)$ are in $C^{2+\alpha, 2+\alpha/2}(\Omega_T)$ for some $\alpha > 0$.

Let $u$ be a viscosity solution of $(1.1)$ with a given continuous boundary data $F$ on $\partial_p \Omega_T := (\Omega \times \{0\}) \cup (\partial \Omega \times [0, T])$. By [CIL92], the solution is unique. In Section 3 we see that for $\varepsilon > 0$ and the boundary data $F$, there is a unique Borel-measurable function $u_\varepsilon$ satisfying a dynamic programming...
principle (hereafter DPP)
\[ u_\varepsilon(x,t) = \frac{n+2}{p+n} \int_{B_\varepsilon(x)} u_\varepsilon(y, t^2) \, dy 
+ \frac{p-2}{p+n} \sup_{|\sigma|=1} \left( \frac{u_\varepsilon(x+\varepsilon\sigma, t^2) + u_\varepsilon(x-\varepsilon\sigma, t^2)}{2} \right) \] in \( \Omega_T \). \quad (1.2)

Here \( B_\varepsilon(x) \subset \mathbb{R}^n \) is a ball centered at \( x \) with the radius \( \varepsilon \), in the first term we have an average integral, and in the second term the supremum is taken over all unit vectors in \( \mathbb{R}^n \). In Theorem 4.3 we show that \( u_\varepsilon \to u \) uniformly when \( \varepsilon \to 0 \). The idea of the proof is to first show that the family \( \{u_\varepsilon\}_{\varepsilon>0} \) is uniformly bounded and asymptotically equicontinuous, and use a variant of the Arzelà-Ascoli theorem to see that solutions of the DPP converge uniformly to some continuous function. To show that the uniform limit is the viscosity solution of (1.1), we make use of an asymptotic mean value formula
\[ \frac{n+2}{p+n} \int_{B_\varepsilon(x)} v(y, t^2) \, dy 
+ \frac{p-2}{p+n} \sup_{|\sigma|=1} \left( \frac{v(x+\varepsilon\sigma, t^2) + v(x-\varepsilon\sigma, t^2)}{2} \right) 
= v(x,t) + \frac{\varepsilon^2}{2(n+p)} (D_p v(x,t) - 2(n+p)v_t(x,t)) + o(\varepsilon^2), \quad (1.3) \]
which is valid for all functions \( v \in C^{2,1}(\Omega_T) \), see Theorem 2.1.

It turns out that the solution \( u_\varepsilon \) of DPP (1.2) is the value of the following time-dependent control problem. Let us denote \( \alpha = \frac{p-2}{p+n}, \beta = \frac{n+2}{p+n} \), and place a token at \( (x_0,t_0) \in \Omega_T \). The controller tosses a biased coin with probabilities \( \alpha \) and \( \beta \). If she gets tails (with probability \( \beta \)), the game state moves according to the uniform probability density to a point \( x_1 \in B_\varepsilon(x_0) \). If the coin toss is heads (with probability \( \alpha \)), the controller chooses a unitary vector \( \sigma \in \mathbb{R}^n \). The position of the token is then moved to \( x_1 = x_0 + \varepsilon\sigma \) or \( x_1 = x_0 - \varepsilon\sigma \) with equal probabilities. After this step, the position of the token is now at \( (x_1,t_1) \), where \( t_1 = t_0 - \varepsilon^2 \). The game continues from \( (x_1,t_1) \) according to the same rules yielding a sequence of game states
\[ (x_0,t_0), (x_1,t_1), (x_2,t_2), ... \]

The game is stopped when the token is moved outside of \( \Omega_T \) for the first time and we denote this point by \( (x_\tau,t_\tau) \). The controller is then paid the amount \( F(x_\tau,t_\tau) \). Naturally, the controller aims to maximize her payoff, and heuristically, the rules of the game can be read from the DPP (1.2).

We remark that the scaling of the time derivative in equation (1.1) is just a matter of convenience. For the equation \( u_t = D_p u \) we would define a game with the same rules as before, except that we would have \( t_{j+1} = t_j - \frac{\varepsilon^2}{2(n+p)} \) for every step in the game, see also Remark 2.4.
This control problem has some similarities with two-player zero-sum tug-of-war games, which were introduced by Peres, Schramm, Sheffield and Wilson [PSSW09, PS08] and later studied from different perspectives, see e.g. [AS12, MPR12, Lew18]. Time-dependent tug-of-war games, having connections to parabolic equations with the normalized $p$-Laplacian, were studied in [MPR10, PR16, Han18], whereas two-player games for equations $u_t = \lambda_j(D^2 u)$, $j \in \{1, \ldots, n\}$, were recently formulated in [BER19]. For a deterministic game-theoretic approach to parabolic equations, we refer to [KS10].

This paper is organized as follows. In Section 2 we prove the asymptotic mean value formula (1.3). In Section 3 we show that the value of the control problem satisfies the DPP (1.2). Finally, in Section 4 we show that value functions converge uniformly to the viscosity solution of (1.1) when $\varepsilon \to 0$.

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2. Asymptotic mean value formula

**Theorem 2.1.** Let $v : \Omega_T \to \mathbb{R}$ be in $C^{2,1}(\Omega_T)$. Then it satisfies the asymptotic mean value formula (1.3).

**Proof.** Averaging the Taylor expansion

$$v(y, t - \varepsilon^2) = v(x, t) + \langle Dv(x, t), (y - x) \rangle + \frac{1}{2} \langle D^2 v(x, t)(y - x), (y - x) \rangle$$

$$- \varepsilon^2 v_t(x, t) + o(|y - x|^2 + \varepsilon^2)$$

over the ball $B_\varepsilon(x)$ and calculating

$$\int_{B_\varepsilon(x)} \langle Dv(x, t), (y - x) \rangle \, dy = 0$$

and

$$\int_{B_\varepsilon(x)} \langle D^2 v(x, t)(y - x), (y - x) \rangle \, dy = \frac{\varepsilon^2}{n + 2} \Delta v(x, t),$$

we obtain

$$\int_{B_\varepsilon(x)} v(y, t - \varepsilon^2) \, dy$$

$$= v(x, t) + \frac{\varepsilon^2}{2(n + 2)} \Delta v(x, t) - \varepsilon^2 v_t(x, t) + o(\varepsilon^2). \quad (2.4)$$

Next we take an arbitrary unit vector $\sigma$ and write the Taylor expansions for $v(x + h, t - \varepsilon^2)$ with $h = \varepsilon \sigma$ and $h = -\varepsilon \sigma$ to obtain

$$v(x + \varepsilon \sigma, t - \varepsilon^2) = v(x, t) + \langle Dv(x, t), \varepsilon \sigma \rangle + \frac{1}{2} \langle D^2 v(x, t) \varepsilon \sigma, \varepsilon \sigma \rangle$$

$$- \varepsilon^2 v_t(x, t) + o(\varepsilon^2),$$
\[ v(x - \varepsilon \sigma, t - \varepsilon^2) = v(x,t) - (Dv(x,t), \varepsilon \sigma) + \frac{1}{2} (D^2 v(x,t)(-\varepsilon \sigma), (-\varepsilon \sigma)) - \varepsilon^2 v_t(x,t) + o(\varepsilon^2), \]

which yield
\[ \frac{v(x + \varepsilon \sigma, t - \varepsilon^2) + v(x - \varepsilon \sigma, t - \varepsilon^2)}{2} \]
\[ = v(x,t) + \frac{\varepsilon^2}{2} (D^2 v(x,t) \sigma, \sigma) - \varepsilon^2 v_t(x,t) + o(\varepsilon^2). \]

Taking the supremum over all \(|\sigma| = 1\) gives
\[ \sup_{|\sigma| = 1} \left[ \frac{v(x + \varepsilon \sigma, t - \varepsilon^2) + v(x - \varepsilon \sigma, t - \varepsilon^2)}{2} \right] \]
\[ = v(x,t) + \frac{\varepsilon^2}{2} \lambda_n - \varepsilon^2 v_t(x,t) + o(\varepsilon^2). \] (2.5)

By multiplying equations (2.4) and (2.5) by \(\frac{n+2}{p+n}\) and \(\frac{p-2}{p+n}\) respectively, we get
\[ \frac{n+2}{p+n} \int_{B_\varepsilon(x)} v(y,t - \varepsilon^2) \, dy \]
\[ + \frac{p-2}{p+n} \sup_{|x| = 1} \left[ \frac{v(x + \varepsilon \sigma, t - \varepsilon^2) + v(x - \varepsilon \sigma, t - \varepsilon^2)}{2} \right] \]
\[ = v(x,t) + \frac{\varepsilon^2}{2(n+p)} (D_p v(x,t) - 2(n+p) v_t(x,t)) + o(\varepsilon^2). \] \(\square\)

Next we define viscosity solutions for equation (1.1).

**Definition 2.2.** An upper semicontinuous function \(u : \Omega_T \rightarrow \mathbb{R}\) is a viscosity subsolution to equation 2 if for all \((x_0, t_0) \in \Omega_T\) and \(\phi \in C^2(\Omega_T)\) such that
\begin{enumerate}
  \item \(u(x_0, t_0) = \phi(x_0, t_0)\),
  \item \(\phi(x, t) > u(x, t)\) for \((x, t) \in \Omega_T, \ (x, t) \neq (x_0, t_0)\),
\end{enumerate}

it holds \(2(n+p) \phi_t(x_0, t_0) \leq D_p \phi(x_0, t_0)\).

A lower semicontinuous function \(u : \Omega_T \rightarrow \mathbb{R}\) is a viscosity supersolution to equation 2 if for all \((x_0, t_0) \in \Omega_T\) and \(\phi \in C^2(\Omega_T)\) such that
\begin{enumerate}
  \item \(u(x_0, t_0) = \phi(x_0, t_0)\),
  \item \(\phi(x, t) < u(x, t)\) for \((x, t) \in \Omega_T, \ (x, t) \neq (x_0, t_0)\),
\end{enumerate}

it holds \(2(n+p) \phi_t(x_0, t_0) \geq D_p \phi(x_0, t_0)\).

A continuous function \(u : \Omega_T \rightarrow \mathbb{R}\) is a viscosity solution to equation 2 if it is both a subsolution and a supersolution.

Because viscosity solutions of (1.1) are in \(C^{2+\alpha, \frac{2+\alpha}{2}}(\Omega_T)\) for some \(\alpha > 0\) (see Section 1), we get the following corollary.
Corollary 2.3. Let $u$ be a viscosity solutions of (1.1). Then it satisfies an asymptotic mean value formula

$$u(x, t) = \frac{n + 2}{p + n} \int_{B_\varepsilon(x)} u(y, t - \varepsilon^2) \, dy + \frac{p - 2}{p + n} \sup_{|\sigma| = 1} \left[ \frac{u(x + \varepsilon \sigma, t - \varepsilon^2) + u(x - \varepsilon \sigma, t - \varepsilon^2)}{2} \right] + o(\varepsilon^2).$$

(2.6)

Remark 2.4. Our scaling of the time variable is for convenience. The same idea would give for viscosity solutions of $u_t = D^p u$ an asymptotic mean value formula

$$u(x, t) = \frac{n + 2}{p + n} \int_{B_\varepsilon(x)} u(y, t - \frac{\varepsilon^2}{2(n + p)}) \, dy + \frac{p - 2}{p + n} \sup_{|\sigma| = 1} \left[ \frac{u(x + \varepsilon \sigma, t - \frac{\varepsilon^2}{2(n + p)}) + u(x - \varepsilon \sigma, t - \frac{\varepsilon^2}{2(n + p)})}{2} \right] + o(\varepsilon^2).$$

3. Control problem formulation

In this section we show that the value of the control problem described in Section 1 satisfies the DPP (1.2). Since the game token may be placed outside of $\Omega_T$, we denote the compact parabolic boundary strip of width $\varepsilon > 0$ by

$$\Gamma_\varepsilon = (S_\varepsilon \times [-\varepsilon^2, 0]) \cup (\Omega \times [-\varepsilon^2, 0]),$$

where

$$S_\varepsilon = \{ x \in \mathbb{R}^n \setminus \Omega : \text{dist}(x, \partial \Omega) \leq \varepsilon \}.$$

Throughout this section, we are given a continuous function $F : \Gamma_\varepsilon \to \mathbb{R}$.

Our control problem with the payoff $F$ was formulated in Section 1. The process is stopped when the token hits the boundary strip $\Gamma_\varepsilon$ for the first time at, say $(x_\tau, t_\tau) \in \Gamma_\varepsilon$, and then the controller earns the amount $F(x_\tau, t_\tau)$.

Next we define the stochastic vocabulary for the control problem. A strategy is a rule which gives, at each step of the game, a direction $\sigma$,

$$S(t_0, x_0, x_1, ..., x_k) = \sigma \in \mathbb{R}^n, \quad |\sigma| = 1.$$

Here, $S$ is a Borel measurable function. Let $A \subset \Omega_T \cup \Gamma_\varepsilon$ be a measurable set. Given a sequence of token positions $(x_0, t_0), (x_1, t_1), ..., (x_k, t_k)$ and a
strategy $S$, the next position of the token is distributed according to the transition probability

$$\pi_S((x_0, t_0), (x_1, t_1), \ldots, (x_k, t_k), A) = \beta \left| \frac{A \cap \left( B_\varepsilon(x_k) \times \{t_k - \varepsilon^2\} \right)}{|B_\varepsilon(x_k) \times \{t_k - \varepsilon^2\}|} \right| + \frac{\alpha}{2} \delta_{(x_k + \varepsilon \sigma, t_k - \varepsilon^2)}(A) + \frac{\alpha}{2} \delta_{(x_k - \varepsilon \sigma, t_k - \varepsilon^2)}(A)$$

where in the first term we use the $n$-dimensional Lebesgue measure, and in the last terms $\delta_{(y,s)}(B) = 1$ if $(y,s) \in B$ and 0 otherwise.

For a starting point $(x_0, t_0)$, a strategy $S$ and the corresponding transition probabilities, we can use Kolmogorov’s extension theorem to determine a unique probability measure $\mathbb{P}^{(x_0,t_0)}_S$ in the space of all game sequences denoted $H^\infty$. The expected payoff is then

$$\mathbb{E}^{(x_0,t_0)}_S[F(x_\tau, t_\tau)] = \int_{H^\infty} F(x_\tau, t_\tau) \, d\mathbb{P}^{(x_0,t_0)}_S,$$

and the value of the game for the controller is

$$u^\varepsilon(x_0, t_0) = \sup_S \mathbb{E}^{(x_0,t_0)}_S[F(x_\tau, t_\tau)].$$

Since $F$ is bounded and

$$\tau \leq \frac{T}{\varepsilon^2} + 1,$$

the value of the game is well defined. From the definition we immediately get the following comparison principle.

**Proposition 3.1.** Fix $\varepsilon > 0$. Let $u^\varepsilon$ be the value of the game with the payoff $F_1$, and $v^\varepsilon$ the value of the game with the payoff $F_2$. Assume that $F_1 \geq F_2$ on $\Gamma_\varepsilon$. Then $u^\varepsilon \geq v^\varepsilon$ in $\Omega_T$.

Our aim is to show that the value function $u^\varepsilon$ satisfies the DPP with the boundary data $F$.

**Definition 3.2.** A Borel measurable function $u^\varepsilon$ satisfies the dynamic programming principle, abbreviated DPP, in $\Omega_T$, with the boundary data $F$, if

$$u^\varepsilon(x,t) = \frac{n+2}{p+n} \int_{B_\varepsilon(x)} u^\varepsilon(y, t-\varepsilon^2) \, dy + \frac{p-2}{p+n} \sup_{|\sigma|=1} \left[ \frac{u^\varepsilon(x+\varepsilon \sigma, t-\varepsilon^2) + u^\varepsilon(x-\varepsilon \sigma, t-\varepsilon^2)}{2} \right] \text{ in } \Omega_T$$

$$u^\varepsilon(x,t) = F(x,t) \text{ on } \Gamma_\varepsilon.$$

**Lemma 3.3.** There is a unique Borel measurable function $u^\varepsilon$ satisfying the DPP. Moreover, $u^\varepsilon$ is lower semi-continuous.
Proof. The existence and uniqueness of such a function \( u_\varepsilon \) can be seen from the following argument. Given \( F \) on \( \Gamma_\varepsilon \), we can determine \( u_\varepsilon(x, t) \) for all \( x \in \Omega \) and \( 0 < t < \varepsilon^2 \). We want to continue this process, but we need to make sure that the function is lower semi-continuous or at least Borel measurable. The following argument is from personal communication with Brustad, Lindqvist, and Manfredi. In general, when \( u \) is any bounded and lower semi-continuous function, then by using Fatou's lemma,

\[
\frac{n + 2}{p + n} \int_{B_\varepsilon(x)} u(y, t - \varepsilon^2) \, dy + \frac{p - 2}{p + n} \sup_{|\sigma| = 1} \left[ \frac{u(x + \varepsilon\sigma, t - \varepsilon^2) + u(x - \varepsilon\sigma, t - \varepsilon^2)}{2} \right]
\]

is again bounded and lower semi-continuous. This gives a lower semi-continuous function \( u_\varepsilon \) defined for all \( x \in \Omega \) and \( 0 < t < \varepsilon^2 \). Continuing this process until \( t = T \) gives the desired function.

\[ \square \]

**Lemma 3.4.** Let \( u_\varepsilon \) be the unique function satisfying the DPP of definition \( \mathcal{A} \) with the boundary data \( F \) on \( \Gamma_\varepsilon \), and let \( u^\varepsilon \) be the value of the game with the payoff \( F \). Then

\[ u_\varepsilon = u^\varepsilon. \]

Proof. Let \( (x_0, t_0) \in \Omega_T \). We aim to show that \( u_\varepsilon(x_0, t_0) = u^\varepsilon(x_0, t_0) \).

Assume that the game starts at \( (x_0, t_0) \in \Omega_T \).

First we assume that the controller uses an arbitrary strategy \( S \). Then we have for the function \( u_\varepsilon \) satisfying the DPP,

\[
\mathbb{E}_S^{(x_0, t_0)}[u_\varepsilon(x_{k+1}, t_{k+1})|(t_0, x_0, x_1, \ldots, x_k)] = \beta \int_{B_\varepsilon(x_k)} u_\varepsilon(y, t_k - \varepsilon^2) \, dy + \alpha \frac{u_\varepsilon(x_k + \varepsilon\sigma, t_k - \varepsilon^2) + u_\varepsilon(x_k - \varepsilon\sigma, t_k - \varepsilon^2)}{2}
\]

\[
\leq \beta \int_{B_\varepsilon(x_k)} u_\varepsilon(y, t_k - \varepsilon^2) \, dy + \alpha \sup_{|\sigma| = 1} \left[ \frac{u_\varepsilon(x_k + \varepsilon\sigma, t_k - \varepsilon^2) + u_\varepsilon(x_k - \varepsilon\sigma, t_k - \varepsilon^2)}{2} \right]
\]

\[
= u_\varepsilon(x_k, t_k).
\]

This shows that \( M_k := u_\varepsilon(x_k, t_k) \) is a supermartingale, so

\[
\mathbb{E}_S^{(x_0, t_0)}[F(x_\tau, t_\tau)|(t_0, x_0, x_1, \ldots, x_{\tau - 1})] \leq u_\varepsilon(x_0, t_0)
\]

by the optimal stopping theorem. Hence

\[ u^\varepsilon(x_0, t_0) = \sup_S \mathbb{E}_S^{(x_0, t_0)}[F(x_\tau, t_\tau)] \leq u_\varepsilon(x_0, t_0). \]
To prove the reverse inequality, we choose a strategy $S_0$ giving a corresponding $\sigma(x,t)$ for the controller that almost maximizes $u_\varepsilon(x,t)$. To be more precise, for arbitrary $\eta > 0$, the controller chooses

$$u_\varepsilon(x_k + \varepsilon\sigma(x_k,t_k), t_k - \varepsilon^2) + u_\varepsilon(x_k - \varepsilon\sigma(x_k,t_k), t_k - \varepsilon^2)$$

$$\geq \sup_{|\sigma|=1} \left[ \frac{u_\varepsilon(x_k + \varepsilon\sigma, t_k - \varepsilon^2) + u_\varepsilon(x_k - \varepsilon\sigma, t_k - \varepsilon^2)}{2} \right] - \eta 2^{-(k+1)}.$$  

The function $S_0$ can be taken to be a Borel function, see Lemma 3.4 in [LM17].

We obtain

$$\mathbb{E}^{(x_0,t_0)}_{S_0}[u_\varepsilon(x_{k+1}, t_{k+1}) - \eta 2^{-(k+1)}](t_0, x_0, x_1, ..., x_k)$$

$$\geq \beta \int_{B_\varepsilon(x_k)} u_\varepsilon(y, t_k - \varepsilon^2) \, dy$$

$$+ \alpha \sup_{|\sigma|=1} \left[ \frac{u_\varepsilon(x_k + \varepsilon\sigma, t_k - \varepsilon^2) + u_\varepsilon(x_k - \varepsilon\sigma, t_k - \varepsilon^2)}{2} \right]$$

$$- \alpha \eta 2^{-(k+1)} - \eta 2^{-(k+1)}$$

$$\geq u_\varepsilon(x_k, t_k) - \eta 2^{-k}.$$  

Hence

$$M_k = u_\varepsilon(x_k, t_k) - \eta 2^{-k}$$

is a submartingale. Using the optimal stopping theorem for this submartingale we find

$$u^\varepsilon(x_0, t_0) = \sup_S \mathbb{E}^{(x_0,t_0)}_S[F(x_\tau, t_\tau)] \geq \mathbb{E}^{(x_0,t_0)}_{S_0}[F(x_\tau, t_\tau)]$$

$$\geq \mathbb{E}^{(x_0,t_0)}_{S_0}[u_\varepsilon(x_\tau, t_\tau) - \eta 2^{-k}]$$

$$\geq \mathbb{E}^{(x_0,t_0)}_{S_0}[u_\varepsilon(x_0, t_0) - \eta 2^{-0}] = u_\varepsilon(x_0, t_0) - \eta.$$  

Since $\eta > 0$ was arbitrary, this proves the lemma.

4. Convergence to the viscosity solution

In this section, we are given a continuous payoff function $F : \Gamma_1 \to \mathbb{R}$. Our goal is to show that with this payoff, value functions of our game converge uniformly to the unique viscosity solution of

$$\begin{cases}
2(n+p)u_t = \mathcal{D}_p u & \text{in } \Omega_T, \\
u = F & \text{on } \partial_p \Omega_T.
\end{cases}$$  

(4.7)

We will make use of the following Arzelá-Ascoli-type lemma, which has been previously used e.g. in [MPR10, PR16, BER19]. We omit the proof, which is a modification of [MPRT12, Lemma 4.2].
Lemma 4.1. Let \( \{ f_\varepsilon : \overline{\Omega}_T \to \mathbb{R} \} \) be a uniformly bounded family of functions such that for a given \( \eta > 0 \), there are constants \( r_0 \) and \( \varepsilon_0 \) such that for every \( \varepsilon < \varepsilon_0 \) and any \( (x, t), (y, s) \in \overline{\Omega}_T \) with 
\[
| (x, t) - (y, s) | < r_0,
\]
it holds 
\[
| f_\varepsilon(x, t) - f_\varepsilon(y, s) | < \eta.
\]
Then there exists a uniformly continuous function \( f : \overline{\Omega}_T \to \mathbb{R} \) and a subsequence, still denoted by \( (f_\varepsilon) \), such that \( f_\varepsilon \to f \) uniformly in \( \overline{\Omega}_T \) as \( \varepsilon \to 0 \).

For the next lemma, we assume that the domain \( \Omega \) satisfies a uniform exterior sphere condition. That is, we assume that there is \( \delta > 0 \) such that for any \( y \in \partial \Omega \), there is an open ball \( B_\delta \subset \mathbb{R}^n \setminus \Omega \) with the radius \( \delta \) so that \( \overline{B}_\delta \cap \overline{\Omega} = \{ y \} \).

Lemma 4.2. The family \( \{ u_\varepsilon \} _{\varepsilon \in (0, 1)} \) of value functions of the game satisfies the assumptions of Lemma 4.1.

Proof. Since \( | u_\varepsilon(x, t) | \leq \max_{\Gamma_1} |F| \) for all \( (x, t) \in \overline{\Omega}_T \) and \( \varepsilon \in (0, 1) \), the family \( \{ u_\varepsilon \} _{\varepsilon \in (0, 1)} \) is uniformly bounded.

Fix \( \eta > 0 \). Since the payoff function \( F \) is uniformly continuous on \( \Gamma_1 \), there is \( \gamma > 0 \) so that when \( (x, t), (y, s) \in \Gamma_1 \) with \( |(x, t) - (y, s)| < \gamma \), it holds 
\[
| F(x, t) - F(y, s) | < \frac{\eta}{4}.
\]
We prove the asymptotic equicontinuity of the family \( \{ u_\varepsilon \} _{\varepsilon \in (0, 1)} \) in four steps. In all steps we have \( \varepsilon < \varepsilon_0 \) and \( |(x, t) - (y, s)| < r_0 \). The precise choices of \( \varepsilon_0 \) and \( r_0 \) clarify during the proof. We will denote by \( C_1, C_2, \ldots \) constants larger than 1 which may depend only on \( n, \delta \), and the diameter of \( \Omega \).

Step 1. If \( (x, t), (y, s) \in \partial_\varepsilon \Omega_T \), then 
\[
| u_\varepsilon(x, t) - u_\varepsilon(y, s) | = | F(x, t) - F(y, s) | < \eta
\]
when \( r_0 < \gamma \).

Step 2. Suppose that \( (x, t) \in \Omega_T \) and \( (y, 0) \in \Gamma_\varepsilon \). Let us start the game from \( (x_0, t_0) = (x, t) \) with an arbitrary strategy \( S \). We obtain 
\[
\mathbb{E}_S^{(x_0, t_0)}[|x_k - x_0|^2 | (t_0, x_0, \ldots, x_{k-1})] 
\]
\[
= \frac{\alpha}{2} (| (x_{k-1} + \sigma \varepsilon) - x_0 |^2 + | (x_{k-1} - \sigma \varepsilon) - x_0 |^2) + \beta \int_{B_\varepsilon(x_{k-1})} | y - x_0 |^2 dy 
\]
\[
\leq \alpha (| x_{k-1} - x_0 |^2 + \varepsilon^2) + \beta (| x_{k-1} - x_0 |^2 + C_1 \varepsilon^2) 
\]
\[
\leq | x_{k-1} - x_0 |^2 + C_1 \varepsilon^2.
\]
Hence, 
\[
M_k := | x_k - x_0 |^2 - C_1 k \varepsilon^2
\]
is a supermartingale, and the optimal stopping theorem gives 
\[
\mathbb{E}_S^{(x_0, t_0)}[ | x_\tau - x_0 |^2 ] \leq | x_0 - x_0 |^2 + C_1 \varepsilon^2 \mathbb{E}_S^{(x_0, t_0)}[ \tau ] \leq C_1 (r_0 + \varepsilon_0^2).
\]
Here, we used the fact that the stopping time $\tau \leq \frac{t}{\varepsilon^2} + 1$ for a game starting at $t_0$ and in this case $t_0 \leq r_0$. Since this is true for all strategies, it holds

$$\sup S |E^{(x_0, t_0)} S_g [|x_\tau - x_0|^2] \leq C_1 (r_0 + \varepsilon_0^2),$$

which yields

$$|u_\varepsilon (x_0, t_0) - u_\varepsilon (x_0, 0)| = |\sup S E^{(x_0, t_0)} S_g [F(x_\tau, t_\tau)] - F(x_0, 0)| < \frac{\eta}{2},$$

when $r_0, \varepsilon_0$ are chosen so that $C_1 (r_0 + \varepsilon_0^2) < \gamma^2$.

The triangle inequality finishes the argument. Recalling that $(x_0, t_0) = (x, t)$, we have

$$|u_\varepsilon (x, t) - u_\varepsilon (y, 0)| \leq |u_\varepsilon (x, t) - F(x, 0)| + |F(x, 0) - F(y, 0)| < \eta.$$

**Step 3.** Suppose that $(x, t) \in \Omega_T$ and $(y, s) \in \partial_p \Omega_T$ with $y \in \partial \Omega$. Since the domain $\Omega$ satisfies the uniform exterior sphere condition with $\delta$, there is a ball $B_\delta (z) \subset \mathbb{R}^n \setminus \Omega$ with $\partial B_\delta (z) \cap \overline{\Omega} = \{y\}$. We use a barrier argument. In an annulus of $\mathbb{R}^n$, define a function $w$ as

$$\begin{cases}
    w(x) = -a|x - z|^2 - b|x - z|^{-\xi} + c & \text{in } B_R (z) \setminus \overline{B_\delta (z)}, \\
    w = 0 & \text{on } \partial B_\delta (z), \\
    \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial B_R (z),
\end{cases}$$

where $\frac{\partial w}{\partial \nu}$ is the normal derivative, and $R$ is chosen so that $\Omega \subset B_R (z)$. The exponent $\xi = n + p - 4 > 0$, since $p > 2$ and we may assume that $n \geq 2$ (1-dimensional case is essentially a random walk in an open interval). The positive constants $a, b, c$ are specified below. The function $w$ satisfies

$$\Delta w(x) = -2an + b\xi n|x - z|^{-\xi - 2} - b\xi (\xi + 2)|x - z|^{-\xi - 2},$$

$$\lambda_n (D^2 w(x)) = -2a + b\xi |x - z|^{-\xi - 2},$$

hence

$$D_p w = -2a(n + p - 2) \quad \text{in } B_R (z) \setminus \overline{B_\delta (z)},$$

(4.8) and it can be extended as a solution to the same equations in $B_{R+\varepsilon} (z) \setminus \overline{B_{\delta - \varepsilon}} (z)$ so that equation (1.8) holds also near the boundaries. It satisfies an estimate

$$w(x) \leq C_2 (R/\delta) \text{dist} (\partial B_\delta (z), x) + o(1)$$

for any $x \in B_R (z) \setminus B_\delta (z)$. Here $o(1) \to 0$ when $\varepsilon \to 0$.

Let us consider for a moment an elliptic game starting at $x_0 = x$ and played by the rules of our game without a time-dependence in the annulus $B_R (z) \setminus \overline{B_\delta (z)}$, with a special rule that if we are at, say $x_k$, a possible random move is chosen from $B_\varepsilon (x_k) \cap B_R (z)$ according to the uniform probability density, and also the controller cannot exit $B_R (z)$. The game ends when the token enters the ball $\overline{B_\delta (z)}$. Because of the random moves, the game ends almost surely in a finite time. Define a stopping time for this game as $\tau^*$,

$$\tau^* = \inf\{k : x_k \in \overline{B_\delta (z)}\}.$$
Let $S$ be an arbitrary strategy for the controller. The Taylor expansion for $w$ gives

\[
\frac{1}{2}(w(x_{k-1} + \varepsilon \sigma) + w(x_{k-1} - \varepsilon \sigma))
\]

\[
= w(x_{k-1}) + \frac{1}{2}\varepsilon^2 \langle D^2 w(x_{k-1}) \sigma, \sigma \rangle + o(\varepsilon^2)
\]

\[
\leq w(x_{k-1}) + \frac{1}{2}\varepsilon^2 \lambda_n(D^2 w(x_{k-1})) + o(\varepsilon^2),
\]

since the first order terms vanish,

\[
\langle Dw(x_{k-1}), \varepsilon \sigma \rangle + \langle Dw(x_{k-1}), -\varepsilon \sigma \rangle = 0.
\]

Moreover, since $w$ is radially increasing, it holds

\[
\int_{B(x_{k-1}) \cap B_R(z)} w(y) \, dy \leq w(x_{k-1}) + \frac{\varepsilon^2}{2(n+2)} \Delta w(x_{k-1}) + o(\varepsilon^2).
\]

By choosing the constant $a$ properly,

\[
M_k := w(x_k) + k \varepsilon^2
\]

is a supermartingale. Indeed, we have

\[
\mathbb{E}_S^{x_0}[M_k | x_0, \ldots, x_{k-1}] = \frac{\alpha}{2}(w(x_{k-1} + \varepsilon \sigma) + w(x_{k-1} - \varepsilon \sigma))
\]

\[
+ \beta \int_{B(x_{k-1}) \cap B_R(z)} w(y) \, dy + k \varepsilon^2
\]

\[
\leq w(x_{k-1}) + \frac{\varepsilon^2}{2(p+n)} D_p w(x_{k-1}) + k \varepsilon^2 + o(\varepsilon^2)
\]

\[
= w(x_{k-1}) - \frac{n+p-2}{n+p} \varepsilon \sigma^2 + k \varepsilon^2 + o(\varepsilon^2)
\]

\[
\leq w(x_{k-1}) + (k-1) \varepsilon^2,
\]

by choosing for example $a = 2\frac{n+p}{n+p-2}$ and assuming that $o(\varepsilon^2) < \varepsilon^2$. The choice of $a$ determines the other constants $b$ and $c$: The Neumann and Dirichlet boundary conditions of the barrier function $w$ are satisfied by choosing $b = (2a/\xi) R^{k+2}$ and $c = a \delta^2 + b \delta^{-k}$.

By the optimal stopping theorem, we have

\[
\mathbb{E}_S^{x_0}[w(x_{\tau^*}) + \tau^* \varepsilon^2] \leq w(x_0),
\]

that is,

\[
\mathbb{E}_S^{x_0}[\tau^*] \leq \frac{w(x_0)}{\varepsilon^2} \leq \frac{C_2(R/\delta)}{\varepsilon^2} \text{dist}(\partial B_\delta(z), x_0) + o(1),
\]

where we used $|\mathbb{E}_S^{x_0}[w(x_{\tau^*})]| \leq o(1)$.

Now we come back to our game, starting at $(x_0, t_0) = (x, t)$, again with an arbitrary strategy $S$. Since it holds $|x_0 - y| \geq \text{dist}(\partial B_\delta(z), x_0)$, for the
stopping time of our game we now have an estimate
\[
E_S^{(x_0,t_0)}[\tau] \leq E_S^{(x_0,t_0)}[\tau^*] \\
\leq \frac{C_2(R/\delta) \text{dist}(\partial B_\delta(z), x_0) + o(1)}{\varepsilon^2} \\
\leq \frac{C_2(R/\delta)|x_0 - y| + o(1)}{\varepsilon^2}.
\]
By using the same martingale argument as in Step 2 but replacing \(x_0\) by \(y\), we have
\[
E_S^{(x_0,t_0)}[|x_\tau - y|^2] \leq |x_0 - y|^2 + C_1 \varepsilon^2 E_S^{(x_0,t_0)}[\tau] \\
\leq |x_0 - y|^2 + C_1 \varepsilon^2 \frac{C_2(R/\delta)|x_0 - y| + o(1)}{\varepsilon^2} \\
\leq |x_0 - y|^2 + C_3(|x_0 - y| + o(1)) \\
\leq r_0^2 + C_3(r_0 + o(1)) < \left(\frac{\gamma}{4}\right)^2,
\]
when \(\varepsilon_0, r_0\) are chosen so that \(C_3(r_0 + o(1)) < \left(\frac{\gamma}{4}\right)^2\) and \(r_0^2 < \left(\frac{\gamma}{4}\right)^2\). This also gives
\[
|E_S^{(x_0,t_0)}[t_\tau] - t_0| < \left(\frac{\gamma}{4}\right)^2.
\]
Hence, we have
\[
|u_\varepsilon(x_0,t_0) - u_\varepsilon(y,t_0)| = |\sup_S E_S^{(x_0,t_0)}[F(x_\tau, t_\tau)] - F(y,t_0)| < \eta/2,
\]
and recalling that \((x_0,t_0) = (x,t)\) the triangle inequality gives
\[
|u_\varepsilon(x,t) - u_\varepsilon(y,s)| \leq |u_\varepsilon(x,t) - F(y,t)| + |F(y,t) - F(y,s)| < \eta.
\]
**Step 4.** Finally, suppose that \((x,t), (y,s) \in \Omega_T\). This is an argument based on translation invariance and comparison principle. Let \(r_0, \varepsilon_0\) satisfy the conditions of the previous steps. Define an inner \(\varepsilon\)-strip \(I_\varepsilon\) by
\[
I_\varepsilon := \{(z,r) \in \overline{\Omega_T} : \text{dist}((z,r), \partial_\rho \Omega_T) \leq r_0\}.
\]
If \((x,t) \in I_\varepsilon\), there is a point \((x',t') \in \partial_\rho \Omega_T\) such that \(|(x,t) - (x',t')| \leq r_0\). Then from the conclusions of the previous steps we obtain
\[
|u_\varepsilon(x,t) - u_\varepsilon(y,s)| \leq |u_\varepsilon(x,t) - F(x',t')| + |F(x',t') - u_\varepsilon(y,s)| < \eta.
\]
The argument is identical if \((y,s) \in I_\varepsilon\), so it remains to study the case \((x,t), (y,s) \in \Omega_T \setminus I_\varepsilon\). We may assume that \(t \leq s\). Define functions \(F_1, F_2\) on the strip \(I_\varepsilon\) as follows,
\[
F_1(z,r) = u_\varepsilon(z - x + y, r - t + s) - \eta, \quad F_2(z,r) = u_\varepsilon(z - x + y, r - t + s) + \eta.
\]
Then
\[
F_1(z,r) \leq u_\varepsilon(z,r) \leq F_2(z,r)
\]
for all \((z,r) \in I_\varepsilon\). Let \(u_1^\varepsilon\) be the value function of the game in \(\Omega_T \setminus I_\varepsilon\) with the payoff \(F_1\) on \(I_\varepsilon\), and \(u_2^\varepsilon\) the value function of the game in \(\Omega_T \setminus I_\varepsilon\) with
the payoff $F_2$ on $I_\varepsilon$. By the uniqueness of the value function, we have for all $(z, r) \in \Omega_T \setminus I_\varepsilon$

$$
\begin{align*}
    u^1_\varepsilon(z, r) &= u_\varepsilon(z - x + y, r - t + s) - \eta, \\
    u^2_\varepsilon(z, r) &= u_\varepsilon(z - x + y, r - t + s) + \eta.
\end{align*}
$$

By the comparison principle, see Proposition 3.1, we have

$$
\begin{align*}
    u_\varepsilon(x, t) &\geq u^1_\varepsilon(x, t) = u_\varepsilon(y, s) - \eta, \\
    u_\varepsilon(x, t) &\leq u^2_\varepsilon(x, t) = u_\varepsilon(y, s) + \eta.
\end{align*}
$$

From the previous lemmas it follows that if $(u_{\varepsilon_j})$ is a sequence of value functions with $\varepsilon_j \to 0$ and $(u_{\varepsilon_{jk}})$ is an arbitrary subsequence, then this subsequence has a subsequence converging uniformly to $v$. Hence, the sequence $(u_{\varepsilon_j})$ converges to $v$ uniformly, and we write $u_\varepsilon \to v$ to simplify the notation.

**Theorem 4.3.** The uniform limit $v = \lim_{\varepsilon \to 0} u_\varepsilon$ is the unique viscosity solution of (4.7).

**Proof.** By uniqueness of viscosity solutions (see [CIL92]), it is sufficient to show that $v$ is a viscosity solution of (4.7). To this end, let $\phi \in C^2$ touch $v$ from above at $(x_0, t_0) \in \Omega_T$,

$$
0 = (v - \phi)(x_0, t_0) > (v - \phi)(x, t)
$$

for all $(x, t)$ close to $(x_0, t_0)$. From the definition of supremum, given $\delta_\varepsilon > 0$, there are points $(x_\varepsilon, t_\varepsilon)$ close to $(x_0, t_0)$ such that

$$
    u_\varepsilon(x_\varepsilon, t_\varepsilon) - \phi(x_\varepsilon, t_\varepsilon) \geq u_\varepsilon(y, s) - \phi(y, s) - \delta_\varepsilon
$$

for all $(y, s)$ in a neighborhood of $(x_\varepsilon, t_\varepsilon)$. Using the fact that $u_\varepsilon \to v$ uniformly and $v - \phi$ is a continuous function with a maximum point at $(x_0, t_0)$, we see that $(x_\varepsilon, t_\varepsilon) \to (x_0, t_0)$ as $\varepsilon \to 0$.

Since $\phi \in C^2(\Omega_T)$, Theorem 2.1 gives

$$
\begin{align*}
    &\beta \int_{B_{\varepsilon}(x_\varepsilon)} \phi(y, t_\varepsilon - \varepsilon^2) \, dy \\
    &\quad + \alpha \sup_{|\sigma| = 1} \left[ \frac{\phi(x_\varepsilon + \varepsilon\sigma, t_\varepsilon - \varepsilon^2) + \phi(x_\varepsilon - \varepsilon\sigma, t_\varepsilon - \varepsilon^2)}{2} \right] \\
    &= \phi(x_\varepsilon, t_\varepsilon) + \frac{\varepsilon^2}{2(n + p)} (D_p\phi(x_\varepsilon, t_\varepsilon) - 2(n + p)\phi_t(x_\varepsilon, t_\varepsilon)) + o(\varepsilon^2).
\end{align*}
$$
We can now estimate

\[
\beta \int_{B_\varepsilon(x_\varepsilon)} u_\varepsilon(y, t_\varepsilon - \varepsilon^2) \, dy \\
+ \alpha \sup_{|\sigma|=1} \left[ \frac{u_\varepsilon(x_\varepsilon + \varepsilon \sigma, t_\varepsilon - \varepsilon^2)}{2} + u_\varepsilon(x_\varepsilon - \varepsilon \sigma, t_\varepsilon - \varepsilon^2) \right] \\
\leq u_\varepsilon(x_\varepsilon, t_\varepsilon) - \phi(x_\varepsilon, t_\varepsilon) + \delta_\varepsilon + \beta \int_{B_\varepsilon(x)} \phi(y, t_\varepsilon - \varepsilon^2) \, dy \\
+ \alpha \sup_{|\sigma|=1} \left[ \frac{\phi(x_\varepsilon + \varepsilon \sigma, t_\varepsilon - \varepsilon^2) + \phi(x_\varepsilon - \varepsilon \sigma, t_\varepsilon - \varepsilon^2)}{2} \right] \\
= u_\varepsilon(x_\varepsilon, t_\varepsilon) + \delta_\varepsilon + \frac{\varepsilon^2}{2(n+p)} (D_p \phi(x_\varepsilon, t_\varepsilon) - 2(n+p) \phi_t(x_\varepsilon, t_\varepsilon)) + o(\varepsilon^2).
\]

As the function $u_\varepsilon$ satisfies the DPP, we are left with

\[
0 < \delta_\varepsilon + \frac{\varepsilon^2}{2(n+p)} (D_p \phi(x_\varepsilon, t_\varepsilon) - 2(n+p) \phi_t(x_\varepsilon, t_\varepsilon)) + o(\varepsilon^2).
\]

Choose now $\delta_\varepsilon = o(\varepsilon^2)$. Dividing by $\varepsilon^2$ and letting $\varepsilon \to 0$ gives

\[
2(n+p) \phi_t(x_0, t_0) \leq D_p \phi(x_0, t_0),
\]

which shows that $v$ is a viscosity subsolution. To show that $v$ is a viscosity supersolution is analogous. \qed

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