New non-diagonal solutions to the $a_n^{(1)}$ boundary Yang-Baxter equation

G. M. Gandenberger

Department of Mathematical Sciences
Durham University
Durham DH1 3LE, U.K.

ABSTRACT

Extending previous work on $a_2^{(1)}$, we present a set of reflection matrices, which are explicit solutions to the $a_n^{(1)}$ boundary Yang-Baxter equation. Unlike solutions found previously these are multiplet-changing $K$-matrices, and could therefore be used as soliton reflection matrices for affine Toda field theories on the half-line.
1 Introduction

In the study of two-dimensional integrable models on the whole line an important role is being played by the Yang-Baxter-equation (YBE), which arises from the factorisability of the $S$-matrix. The YBE has been studied for a long time and a large number of solutions, the so-called $R$-matrices, are known.

The boundary Yang-Baxter equation (BYBE) is the analogue of the YBE in two-dimensional models on a half-line, i.e. models with one reflecting boundary. Factorisability and integrability on a half-line imply a highly non-trivial relation between $R$-matrices and reflection matrices, which are the so-called $K$-matrices. Much less work has been done on the BYBE and only a small number of solutions for some specific cases have been found so far. (For more details on the BYBE and its connection to integrable models on a half-line see [1], [2] and references therein.)

Here we are interested in BYBEs related to trigonometric $R$-matrices. These are intertwining maps of the form

\[ \tilde{R}_{a,b}(x) : V_a \otimes V_b \rightarrow V_b \otimes V_a, \]

in which $x$ is a spectral parameter and the $V$’s are the representation spaces of the fundamental representations of some quantized universal enveloping algebra $U_q(\hat{g})$ of an affine Lie algebra $\hat{g}$. These $R$-matrices satisfy the YBE in the form

\[ \tilde{R}_{b,c}(x) \otimes I_a \cdot I_b \otimes \tilde{R}_{a,c}(xy) \cdot \tilde{R}_{a,b}(y) \otimes I_c = I_c \otimes \tilde{R}_{a,b}(y) \cdot \tilde{R}_{a,c}(xy) \otimes I_b \cdot I_a \otimes \tilde{R}_{b,c}(x), \]

in which $I_a$ denotes the identity on $V_a$, such that both sides of the equation map $V_a \otimes V_b \otimes V_c$ into $V_c \otimes V_b \otimes V_a$. (Note that the indices $a,b,c$ denote the type of representation.)

Here we are only considering the case of $\hat{g} = a_n^{(1)}$, for which two different types of the BYBE exist. These two types are distinguished by whether the $K$-matrices map a representation space $V_a$ into itself or into the conjugate space $V_{n+1-a}$. These two cases could be related to integrable models in which the particles either remain in the same multiplet or transform into a particle in the charge conjugate multiplet after reflection from the boundary. The non-multiplet changing BYBE for the lowest $R$-matrices can be written in the following form:

\[ I_1 \otimes \tilde{K}(y) \cdot \tilde{R}_{1,1}(xy) \cdot I_1 \otimes \tilde{K}(x) \cdot \tilde{R}_{1,1}(x) = \tilde{R}_{1,1}(x) \cdot I_1 \otimes \tilde{K}(x) \cdot \tilde{R}_{1,1}(xy) \cdot I_1 \otimes \tilde{K}(y), \]

in which the reflection matrices are maps

\[ \tilde{K}(x) : V_1 \rightarrow V_1. \]

Diagonal solutions to equation (1.1) were found some years ago in [3] and they have been used in connection with spin chains (see for instance [4]).

However, more recently it has been discovered that in order to describe the reflection of solitons in imaginary coupled $a_n^{(1)}$ affine Toda field theory a different type of $K$-matrices is needed. In [2] it was realised that affine Toda solitons reflect into antisolitons, which are the states in the charge conjugate multiplet. This property has also been confirmed in a semiclassical study of affine Toda theories on a half-line in [5]. Therefore, we require $K$-matrices of the general form

\[ K(x) : V_1 \rightarrow V_n, \]
which now have to be solutions to the following multiplet changing BYBE:

\[ I_n \otimes K(y) \cdot \tilde{R}_{1,n}(xy) \cdot I_1 \otimes K(x) \cdot \tilde{R}_{1,1}(\frac{x}{y}) = \tilde{R}_{n,n}(\frac{x}{y}) \cdot I_n \otimes K(x) \cdot \tilde{R}_{1,n}(xy) \cdot I_1 \otimes K(y). \tag{1.3} \]

Solutions to this equation for the case of \( g = a^{1}_{2} \) were first found in \[3\] and the solutions for the general case of \( g = a^{1}_{n} \) are the subject of this letter.

2 The \( a^{1}_{n} \) R-matrix

In order to solve the BYBE (1.3) we first need an explicit expression of the R-matrix corresponding to the first fundamental representation of \( \mathbb{U}_{q}(a^{1}_{n}) \). These R-matrices were originally given by Jimbo in \[6\]. We use a slightly modified notation here:

\[ \tilde{R}_{1,1}(x) = (x^{-h} q^{4} - x^{h} q^{-4}) \sum_{i=1}^{n} E_{ii} \otimes E_{ii} + (x^{h} - x^{-h}) \sum_{i \neq j} E_{ij} \otimes E_{ji} + (q^{4} - q^{-4}) \left( \sum_{i < j} x^{\nu(i,j)} E_{ii} \otimes E_{jj} + \sum_{i > j} x^{\nu(i,j)} E_{ij} \otimes E_{ji} \right), \tag{2.1} \]

in which the \( E_{i,j} \) are \( n \times n \) matrices with the only entry being a 1 in the \( i \)th row and \( j \)th column. Alternatively, we can write this R-matrix as a \( n^{2} \times n^{2} \)-matrix \[\tilde{R}_{1,1}(x)_{i,j}^{k,l}\]. For the sake of convenience we omit the multiplet labelling from now on and write \[\tilde{R}_{1,1}(x)_{i,j}^{k,l} \equiv \tilde{R}_{i,j}^{k,l}(x)\]. All the non-zero elements of this matrix are then given by

\[
\begin{align*}
\tilde{R}_{i,i}^{i,i}(x) &= x^{-h} q^{4} - x^{h} q^{-4}, \\
\tilde{R}_{i,j}^{i,j}(x) &= x^{h} - x^{-h}, \\
\tilde{R}_{i,j}^{i,j}(x) &= (q^{4} - q^{-4})x^{\nu(i,j)}, \quad \text{(for } i, j = 1, ..., n \text{ and } i \neq j) \tag{2.2}
\end{align*}
\]

in which \( h = n + 1 \) is the Coxeter number of \( a^{1}_{n} \), and \( \nu(i,j) \) is determined by the gradation of the R-matrices. (For more details about R-matrices and their gradations see for instance \[7\], \[8\].) For the two gradations we are interested in, \( \nu(i,j) \) is given as follows:

- **homogeneous gradation:**
  
  \[ \nu(i, j) = \begin{cases} -h & (i < j) \\ h & (i > j) \end{cases}, \tag{2.3} \]

- **principal gradation:**
  
  \[ \nu(i, j) = \begin{cases} 2(i - j) + h & (i < j) \\ 2(i - j) - h & (i > j) \end{cases}. \tag{2.4} \]

\[\text{Compared to those in [3], we have changed } x \rightarrow x^{2h} \text{ and } k \rightarrow -q^{4}. \text{ Note also that this is the intertwining R-matrix which is related to Jimbo's R-matrix by } R = PR, \text{ in which } P \text{ is the permutation matrix.} \]
An important aspect of these $R$-matrices is the fact that they are crossing symmetric, and one can therefore choose the bases of the representation spaces such that the crossed $R$-matrices are given as

\[
\hat{R}_{ij}^{kl}(x) = \hat{R}_{ij}^{kl}(q^2 x^{-1}),
\]
\[
\hat{R}_{ik}^{jl}(x) = \hat{R}_{ij}^{kl}(x),
\]
\[
\hat{R}_{ik}^{jl}(x) = \hat{R}_{ij}^{kl}(q^2 x^{-1}).
\]

Here we have used a short hand notation in which barred indices denote the labels in $V_n$ and unbarred indices those in $V_1$, which means for instance

\[
\hat{R}_{ij}^{kl}(x) \equiv [\hat{R}_{i,n}(x)]_{ij}^{kl} : V_1 \otimes V_n \to V_n \otimes V_1.
\]

3 The $K$-matrices

Since the dimension of the spaces $V_1$ and $V_n$ are both equal to $n$, we can write the $K$-matrices as $n \times n$ matrices $K_i^j(x)$. Using this explicit matrix form we can rewrite the BYBE (1.3) into the following form:

\[
K_j^k(y) \hat{R}_{ij}^{lm}(xy) K_m^n(x) \hat{R}_{i,m}^{p,r}(xy) = \hat{R}_{ij}^{kl}(xy) K_l^m(x) \hat{R}_{k,m}^{p,r}(xy) K_n^p(y),
\]

in which summation over repeated indices is implied.

In analogy to the $a_2^{(1)}$ case there is one very simple solution, which is a diagonal matrix

\[
K_1^{(d)}(x) = \begin{pmatrix}
1 & 0 & \ldots \\
0 & 1 & \\
& & \ddots \\
& & & 1
\end{pmatrix}.
\]

It is straightforward to show that this is indeed a solution to equation (1.1). In fact, it is not even necessary to know the exact form of the $R$-matrix elements, because the diagonal matrix $K_1^{(d)}$ solves the BYBE corresponding to any $R$-matrix which satisfies the crossing relations (2.5). If this $K$-matrix was used as a reflection matrix it would mean that the states in the first multiplet are reflected into their charge conjugate partners in the $n$th multiplet. In a recent paper [9] this solution was used as a starting point in the construction of the reflection factors for $a_2^{(1)}$ affine Toda field theory.

However, the main result of this letter is the fact that we have also found two explicit non–diagonal solutions to equation (1.1), for both the homogeneous and the principal gradation. These two solutions are only distinguished by some signs and can be written in the following form:
\[ K^i_j(x) = \begin{cases} \frac{x h q^{-h+2} - x h q^{h-2}}{q^2 - q^{-2}}, & \text{if } i = \overline{j}, \\ \left(\frac{x}{q}\right)^{\nu(i, \overline{j})}, & \text{if } i \neq \overline{j}, \end{cases} \]

and

\[ K^\overline{i}_j(x) = \begin{cases} \frac{x h q^{-h+2} + x h q^{h-2}}{q^2 - q^{-2}}, & \text{if } i = \overline{j}, \\ \text{sign}(\overline{j} - i) \left(\frac{x}{q}\right)^{\nu(i, \overline{j})}, & \text{if } i \neq \overline{j}. \end{cases} \]

For the case of \( n = 2 \) these solutions are the same as those found in [2]. Unfortunately, we do not know any elegant way to prove that these expression do satisfy the BYBE, but they have been checked in detail using algebraic software. Some details of this check are provided in the appendix.

In both solutions (3.3) and (3.4) there still remains a \( n \)-fold freedom, namely these K-matrices remain solutions to the same BYBE, if we multiply them from the left and from the right by a diagonal matrix

\[ \mathcal{E} = \begin{pmatrix} E_1 & 0 & \ldots \\ 0 & E_2 & \vdots \\ \vdots & \ddots & \ddots \\ 0 & \ldots & \ldots & E_n \end{pmatrix} \]

in which \( E_i = E_i(q) \) can be arbitrary functions of the deformation parameter \( q \) (but not of \( x \)). In other words, the BYBE appears to be invariant under transformations of the form

\[ K^i_j(x) \to E_i(q) E^\overline{j}\overline{i}(q) K^i_j(x). \]

If these K-matrices are to be used as reflection matrices for an integrable model, then the parameters \( E_i \) could be related to different integrable boundary conditions. However, as it was shown in [2], the additional conditions imposed on the reflection matrices by boundary unitarity and boundary crossing, can restrict significantly the possible number of free boundary parameters.

In addition, the K-matrices are only determined up to an overall scalar factor \( \mathcal{A}(x) \). In order to use these solutions as reflection matrices for the reflection of solitons in \( a_n^{(1)} \) affine Toda field theories, it would be necessary to find explicit expressions for the overall scalar factor, which is determined by the boundary-unitarity and boundary bootstrap conditions.

Finally, note that we have only provided solutions related to the first fundamental representation of \( U_q(a_n^{(1)}) \). In general, there are of course \( n \) possible sets of K-matrices \( K^{(a)}: V_a \to V_{n+1-a} \) (for \( a = 1, \ldots, n \)). These higher K-matrices can in principle be constructed from the solutions (3.3) and (3.4) by using the bootstrap equations. This, however, goes beyond the scope of this letter. However,
we do hope that the explicit solutions found here will shed some light on the general structure of $K$-matrices and may help in the construction of $K$-matrices for other algebras.

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A A sketch of proof

It would be desirable to have a proof in terms of quantum algebra properties, but all we can do at this stage is to check the above solutions using some algebraic software such as MapleV. Fortunately, however, due to the particular form of the $a_n^{(1)}$ $R$-matrices this check can be performed for general $n$, rather than just for some particular example. The details of this proof itself are not very illuminating and we therefore just mention the main idea.

In general, in order to prove equation (3.1), we would have to check $n^4$ different equations. However, because of the fact that most of the entries in the $R$-matrix (2.2) are equal to zero, this number can be decreased substantially and we can perform the check for general $i, j, k, l = 1, \ldots, n$. We will demonstrate this for one example in detail.

Consider equation (3.1) for the case $i = j$ and $p \neq r$, and also $i \neq p$ and $i \neq r$. Let us first examine the last term on the left hand side of (3.1). Because of the fact that $p \neq r$ there are only two possible non-zero terms, namely $\tilde{R}^{p,r}_{i,i}(x y)\tilde{R}^{r,p}_{i,j}(x)$ and $\tilde{R}^{p,r}_{i,r}(x y)\tilde{R}^{r,p}_{i,p}(x)$. Thus, the left hand side of the equation reduces to

$$\sum_{k,m} K^\overline{i}_k(x) \tilde{R}^{p,m}_{i,k}(xy) K^\overline{j}_m(x) \tilde{R}^{r,p}_{k,p}(x y) + K^\overline{i}_m(x) \tilde{R}^{p,m}_{i,k}(xy) K^\overline{j}_m(x) \tilde{R}^{r,p}_{k,p}(x y) =$$

$$= \left[ K^\overline{i}_m(x) \tilde{R}^{p,m}_{i,k}(xy) K^\overline{j}_m(x) \tilde{R}^{r,p}_{k,p}(x y) + K^\overline{i}_m(x) \tilde{R}^{p,m}_{i,k}(xy) K^\overline{j}_m(x) \tilde{R}^{r,p}_{k,p}(x y) \right] \tilde{R}^{p,r}_{k,p}(x y)$$

$$= \left( \frac{x}{q} \right)^{2i-2p} \left( \frac{y}{q} \right)^{2j-2r} \left[ q^4 \left( \frac{x}{y} \right)^{-h} - q^{-4} \left( \frac{x}{y} \right)^h \right] \left[ \left( \frac{q}{x} \right)^{2h} - \left( \frac{q}{y} \right)^{-2h} + (q^8 - 1)(x^h q^{-h-2} - x^{-h} q^{h+2}) \right]$$

$$= \left( \frac{x}{q} \right)^{2p-2i} \left( \frac{y}{q} \right)^{2j-2r} \left[ \left( q^{-2h-4} y^h - (1 + q^{-4}) y^{-h} \right) x^{-h} q^{2h} y^h \right]. \quad (A.1)$$
Analogously, considering the right hand side of (3.1), we see that the first term is the $R$-matrix element $R^{k,l}_{i,i}(x)$, which is zero for all but $k = l = i$. Therefore, the right hand side is reduced to
\[
\sum_{m,n} \tilde{R}^{i,i}_{m,n}(xy) K^m_i(x) \tilde{R}^{m,n}_{i,i}(xy) K^m_i(y) = 
\]
\[
= \tilde{R}^{i,i}_{1,1}(xy) \left[ K_1^m(x) \tilde{R}^{m,n}_{i,i}(xy) K^n_i(y) + K^n_i(x) \tilde{R}^{m,n}_{i,i}(xy) K^m_i(y) \right] 
\]
\[
= \left( \left( \frac{x}{y} \right)^h - \left( \frac{x}{q} \right)^{-h} \right) \left( \frac{q}{x} \right)^{2i-2p} \left( \frac{y}{q} \right)^{2i-2r} \left[ \left( \frac{q}{x} \right)^{2h} - \left( \frac{q}{y} \right)^{-2h} + x^{-h}(q^4 + 1)\left( \frac{y}{q} \right)^{2h} - q^{-4} \right] 
\]
\[
+ (q^4 - q^{-4}) \left( \frac{x}{q} \right)^{2i-2p} \left( \frac{y}{q} \right)^{2i-2r} \left[ \left( \frac{q^2}{xy} \right)^h - \left( \frac{q^2}{y} \right)^{-h} + \left( \frac{y}{x} \right)^h (q^4 + 1)(1 - \left( \frac{q}{y} \right)^{2h} q^{-4}) \right] 
\]
\[
= \left( \frac{x}{q} \right)^{2i-2p} \left( \frac{y}{q} \right)^{2i-2r} \left[ x^h \left( q^{-2h+4y^h} - (1 + q^{-4})y^{-h} \right) + x^{-h} \left( q^{2h-8y^h} - q^{-2h+4y^h} + (q^8 + q^4)y^h \right) - x^{-3h} q^{2h} y^h \right],
\]
(A.2)

which proves the BYBE for this particular case. We saw that we did not need to know the explicit values of $i$, $p$, $r$ or $n$. It was sufficient to know which indices differ from each other. It is therefore fairly straightforward to implement this in form of an algorithm and check every case for general $n$. Here we have used MapleV to check that (3.3) and (3.4) both solve (3.1) for the homogeneous as well as the principal gradation. This procedure does not lend itself to finding new solutions easily, but it is straightforward to check whether a given ansatz is a solution.

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