Constructive expressive power of population protocols

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Abstract
Population protocols are a model of distributed computation intended for the study of networks of independent computing agents with dynamic communication structure. Each agent has a finite number of states, and communication opportunities occur nondeterministically, allowing the agents involved to change their states based on each other’s states. Population protocols are often studied in terms of reaching a consensus on whether the input configuration satisfied some predicate.

In the present paper we propose an alternative point of view. Instead of studying the properties of inputs that a protocol can recognise, we study the properties of outputs that a protocol eventually ensures. We define constructive expressive power. We show that for general population protocols and immediate observation population protocols the constructive expressive power coincides with the normal expressive power.

Immediate observation protocols also preserve their relatively low verification complexity in the constructive expressive power setting.

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1 Introduction
Population protocols have been introduced in [1, 2] as a restricted yet useful subclass of general distributed protocols. Each agent in a population protocol has a fixed amount of local storage, and an execution consists of selecting pairs of agents and letting them update their states based on an interaction. The choice of pairs is assumed to be performed by an adversary subject to a fairness condition. The fairness condition ensures that the adversary must allow the protocol to progress.

Typically, population protocols are studied from the point of view of recognising some properties of an input configuration. In this context population protocols and their subclasses have been studied from the point of view of expressive power [4], verification complexity [6, 10, 9], time to convergence [3, 7], necessary state count [5], etc.

The original target application of population protocols and related models is modelling networks of restricted sensors, starting from the original paper [1] on population protocols. Of course, in the modern applications the cheapest microcontrollers typically have tens of thousands of bits of volatile memory permitting the use of simpler and faster algorithms for recognising properties of an input configuration. So on the one hand, the original motivation for the restrictions in the population protocol model seems to have less relevance. On the other hand, verifying distributed systems benefits from access to a variety of restricted models.
with a wide range of trade-offs between the expressive power and verification complexity, as most problems are undecidable in the unrestricted case. Complex, unrestricted, and impossible to verify distributed deployments lead to undesirable and hard to predict and sometimes even diagnose situations such as so called gray failures [12] and similar. On the other hand, desirable behaviours of distributed systems go beyond consensus about a single property of initial configuration. We find it natural to study what classes of properties a distributed system can eventually reach and then maintain indefinitely.

In the present paper we introduce a notion of constructive expressive power of a protocol model, formalising this question. We show that for population protocols, as well as for a useful subclass of population protocols, immediate observation (or one-way) population protocols, the constructive expressive power coincides with the classical expressive power.

We also show that immediate observation population protocols preserve their relatively low verification complexity in the constructive expressive power setting.

The rest of the present paper is organised as follows. First we provide the basic definitions and the previously known results about expressive power from the point of view of computing predicates. We continue by defining constructive expressive power. In the next section we establish the constructive expressive power of population protocols. Then in the following section we establish the constructive expressive power of immediate observation protocols and show that verification remains in \textbf{PSPACE} in this setting just like in the setting of computing predicates. The paper ends with conclusion and future directions.

2 Basic definitions

First we define the population protocols, as well as their subclass, immediate observation population protocols.

\textbf{Definition 1.} A population protocol is defined by a finite set of states $Q$ and a step relation $\text{Step} \subset Q^2 \times Q^2$. When there is no ambiguity about the protocol, we abbreviate $((q_1, q_2), (q_1', q_2')) \in \text{Step}$ as $(q_1, q_2) \mapsto (q_1', q_2')$ and call the quadruple $(q_1, q_2) \mapsto (q_1', q_2')$ a transition.

A population protocol is an immediate observation population protocol if $(q_1, q_2) \mapsto (q_1', q_2')$ implies $q_2 = q_2'$. We say that an agent in the state $q_1$ changes its state to $q_1'$ by observing $q_2$ and denote it $q_1 \xrightarrow{q_2} q_1'$.

A configuration of a population protocol is a multiset of states $C : Q \rightarrow \mathbb{N}$. We use the notation $\{q_1, \ldots, q_k\}$ for a multiset $C$ such that $C(q)$ is the number of times $q$ occurs among $q_1, \ldots, q_k$. In some cases it is also convenient to interpret a multiset as a tuple or a vector with nonnegative integer coordinates and denote it e.g. $C \in \mathbb{N}^Q$. Arithmetic operations and predicates apply to multisets pointwise (coordinatewise). The size of a configuration $C$ is the sum of images of all states, $|C| = \sum q C(q)$. An execution of a population protocol is a finite or infinite sequence $(C_0, C_1, \ldots)$ of configurations such that for each $j$ between 1 and the execution length we have $C_j \geq \{q_1, q_2\}$ and $C_{j+1} = C_j - \{q_1, q_2\} + \{q_1', q_2'\}$ where $((q_1, q_2), (q_1', q_2')) \in \text{Step}$. In other words, we let two agents with states $q_1$ and $q_2$ interact in some way permitted by the Step relation.

\textbf{Example 2.} Consider the set of states $\{q_0, q_1, q_2, q_3\}$. The step relation described by $(q_1, q_1) \mapsto (q_0, q_2), (q_2, q_1) \mapsto (q_0, q_3), (q_2, q_2) \mapsto (q_1, q_3), (q_0, q_1) \mapsto (q_3, q_3), (q_1, q_1) \mapsto (q_3, q_1), (q_2, q_1) \mapsto (q_3, q_3)$ is a population protocol but not an immediate observation population protocol. An example execution is: $\{q_1, q_1, q_1\} = (0.3, 0, 0), (1, 1, 1, 0), (2, 0, 0, 1), (1, 0, 0, 2), (0, 0, 0, 3)$. Here we use the first two steps, then twice the fourth step.
On the other hand, \((q_1, q_1) \mapsto (q_2, q_1), (q_2, q_2) \mapsto (q_3, q_2), (q_0, q_1) \mapsto (q_3, q_3), (q_1, q_3) \mapsto (q_3, q_3), (q_2, q_3) \mapsto (q_3, q_3)\) is an immediate observation population protocol that can also be described as \(q_1 \xrightarrow{q_2} q_2 \xrightarrow{q_3} q_3, q_0 \xrightarrow{q_4} q_3, q_1 \xrightarrow{q_5} q_3, q_2 \xrightarrow{q_6} q_3\). An example execution is: \((0, 3, 0, 0), (0, 2, 1, 0), (0, 1, 2, 0), (0, 1, 1, 1), (0, 0, 1, 2), (0, 0, 0, 3)\). Here we use the first two possible steps, then the last two possible steps.

\[\text{Remark 3. Note that all the configurations in an execution have the same size.}\]

We often consider executions with the steps chosen by an adversary. However, we need to restrict adversary to ensure that some useful computation remains possible. To prevent the adversary from e.g. only letting one pair of agents to interact, we require the executions to be fair. The fairness condition can also be described by comparison with random choice of steps to perform: fairness is a combinatorial way to exclude a zero-probability set of bad executions.

\[\text{Definition 4. Consider a population protocol } (Q, \text{Step}).\]

A configuration \(C'\) is reachable from configuration \(C\) iff there is a finite execution with the initial configuration \(C\) and the final configuration \(C'\).

A finite execution is fair if it is not a beginning of any longer execution.

An infinite execution \(C_0, C_1, \ldots\) is fair if for every configuration \(C'\) either \(C\) is not reachable from some \(C_j\) (and all the following configurations), or \(C\) occurs among \(C_j\) infinitely many times.

\[\text{Example 5. The finite executions in the example 2 are fair.}\]

The most popular notion of expressive power for population protocols is computing predicates, defined in the following way.

\[\text{Definition 6. Consider a population protocol } (Q, \text{Step}) \text{ with additionally defined nonempty set of input states } I \subset Q, \text{ output alphabet } O \text{ and output function } o : Q \to O.\]

Support of a configuration \(C\) is the set of all states with nonzero images, \(\text{The states belonging to the support of a configuration are also called inhabited in the configuration.}\)

\(\text{supp} C = Q \setminus C^{-1}(0).\)

A configuration \(C\) is an input configuration if its support is a subset of the set of the input states, \(\text{supp} C \subset I.\)

A configuration \(C\) is a b-consensus for some \(b \in O\) if the output function yields \(b\) for all the inhabited states, A configuration is a stable b-consensus if it is a b-consensus together with all the configurations reachable from it. A configuration is called just a consensus or a stable consensus if it is a b-consensus (respectively stable b-consensus) for some \(b.\)

A protocol computes a function \(\varphi : \mathbb{N}^I \to O\) iff for each input configuration \(C\) every fair execution with initial configuration \(C\) contains a stable \(\varphi(C)\)-consensus. We usually use the protocols computing predicates, which corresponds to \(O = \{\text{true}, \text{false}\}\).

\[\text{Example 7. If we define the set of input states } I = \{q_1\}, \text{ output set } O = \{\text{true}, \text{false}\} \text{ and the output function } o(q) = (q = q_3), \text{ both protocols in the example 2 compute the predicate } \varphi(C) = (C(q_1) \geq 3).\]

The expressive power of population protocols and immediate observation population protocols has been studied in \([4]\).

\[\text{Definition 8. A cube is the set of configurations defined by a lower and an upper bound for the number of agents in each state. The lower bound can be zero, the upper bound can be infinite. A counting set is a finite union of cubes.}\]
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An integer cone is the set of multisets defined by a base multiset (or just base) $B$ and a finite number of period multisets (periods) $v_j$. A multiset belongs to the cone if it can be represented as a sum of the base multiset and a non-negative integer combination of periods. A semilinear set is a finite union of integer cones.

Theorem 9. Population protocols can compute membership in semilinear sets and no other predicates.

Immediate observation population protocols can compute membership in counting sets and no other predicates.

We now define constructive expressive power of protocols.

Definition 10. A configuration $C$ satisfies an output condition $\psi : N^O \rightarrow \{true, false\}$ iff $\psi(x \mapsto \sum_{q \in C(q)} x)$ is true. In other words, we consider the multiset of the outputs corresponding to the states of individual agents then apply $\psi$ to this multiset. A configuration $C$ ensures an output condition $\psi$ (given the protocol $P$) if every configuration $C'$ reachable from $C$ (including $C$) satisfies $\psi$. A protocol ensures $\psi$ from configuration $C$, if every fair execution of $P$ starting from $C$ reaches a configuration $C'$ that ensures $\psi$. A protocol ensures $\psi$, if it ensures $\psi$ from every input configuration.

An output condition $\psi$ and size $n$ are compatible if there exists a multiset $D \in N^O$ of size $|D| = n$ satisfying $\psi$, i.e. $\psi(D)$ holds. An output condition $\psi$ is size-flexible if it is compatible with every (non-negative integer) size.

We interpret each multiset $D \subset N^O$ as an output condition $\psi : D \mapsto D \in D$.

Example 11. Both protocols from the example 2 ensure the condition $D \mapsto D(true) = 0 \lor D(false) = 0$. This condition is ensured by any protocol computing a predicate.

Remark 12. Only size-flexible expressive conditions can be ensured.

Note that the same a protocol that ensures a predicate $\psi$ also ensures every predicate $\psi'$ such that $\psi \Rightarrow \psi'$. Therefore defining constructive expressive power of a class of protocols requires an extra step.

Definition 13. A class $\mathcal{P}$ of population protocols ensures a class $\Psi$ of output conditions if for each size-flexible $\psi \in \Psi$ there is a protocol $P \in \mathcal{P}$ that ensures $\psi$.

A class $\mathcal{P}$ ensures at most a class $\Psi$ of output conditions if for each $\psi' \in \Psi$, for some protocol $P \in \mathcal{P}$ there is a size-flexible condition $\psi \in \Psi$ implying $\psi'$, i.e. $\psi \Rightarrow \psi'$.

Example 14. The class of protocols consisting of a single protocol, namely the immediate observation protocol from the example 2 with output function satisfying $o(q_3) = large$ and $o(q_0) = o(q_1) = o(q2) = small$, ensures the class of output conditions with a single condition $D \mapsto ((D(large) = 0 \land D(small) \leq 2) \lor (D(large) \geq 3 \land D(small) = 0))$, and at most that class.

Remark 15. Note that for a given class $\mathcal{P}$ of protocols there is more than one class $\Psi$ of output conditions such that $\mathcal{P}$ ensures $\Psi$ and at most $\Psi$. For example, adding a condition $\psi'$ that follows from some $\psi \in \Psi$ to the class $\Psi$ yields a class $\Psi'$ such that $\mathcal{P}$ ensures $\Psi'$ and at most $\Psi'$. We tolerate this and do not require minimality of $\Psi$ because convenient classes of conditions, such as semilinear sets and counting sets, are not minimal.

Constructive power of population protocols

In this section we establish the constructive expressive power of general population protocols.
3.1 Constructing semilinear predicates

We start by providing a positive result, showing that every predicate that can be computed, can also be ensured.

**Theorem 16.** The class of all population protocols ensures the class of semilinear output conditions.

The theorem follows from the following lemmas:

**Lemma 17.** Each integer cone $C$ can be represented as a finite union of cones such that all the periods of each cone have the same size.

**Proof.** Let the cone $C$ have base $B$ and periods $v_j$ for $1 \leq j \leq n$. Let $L$ be the least common multiple of $|v_j|$. Consider all the bases $B + \sum r_j v_j$ where $0 \leq r_j \leq \frac{L}{|v_j|}$. We consider all the cones $C_{r_1,\ldots,r_n}$ with such bases and periods $\frac{L}{|v_j|} v_j$. The size of all periods is $L$; it is easy to see that $C = \bigcup C_{r_1,\ldots,r_n}$. \hfill $\Box$

**Lemma 18.** For each integer cone $C \subset \mathbb{N}^O$ with all periods having the same size there is a protocol $P$ such that the protocol $P$ ensures membership in $C$ from each input configuration $C$ of size compatible with $C$.

**Proof.** Let the cone $C$ have base $B$ and periods $v_j$. Let $B^{(1)}, \ldots, B^{(|B|)}$ and $v_j^{(1)}, \ldots, v_j^{(|v_j|)}$ for each $1 \leq j \leq n$ be sequences of output values enumerating the corresponding multisets with correct multiplicities.

Note that the sizes compatible with $C$ are sums of $|B|$ and a non-negative multiple of $|v_1|$ as all the periods have the same length. In the special case of no periods, only size $|B|$ is compatible with $C$. The idea of the construction is to select $|B|$ agents to output the base multitset, then split the remaining agents in groups of $|v_1|$ agents with the outputs forming the multitset $v_1$. The split is performed by allowing some agents to recruit unassigned agents to positions in groups of size $|v_1|$. If two such agents interact, one of them disbands the current group and becomes unassigned. Eventually we will have just one active agent having recruited every other agent in some group, and the last group is complete iff the total number of agents available is divisible by $|v_1|$.

The protocol has the following states: $q_{B,j}$ for $1 \leq j \leq |B|$, and $q_{v,i,1}$, $q_{v,i,4}$, $q_{v,i,\diamond}$, for $1 \leq i \leq |v_1|$. The state $q_{v,1,1}$ plays a special role in the protocol and will also be denoted as $q_\bot$. If there are no periods, we formally define $q_{B,1}$ to be also called $q_{v,1,1}$ and $q_{v,1,\diamond} = q_\bot$.

The only input state is $q_{B,1}$ if $|B| > 0$ and $q_{v,1,\diamond}$ otherwise. The transitions are as follows:

- $(q_{B,j}, q_{B,j}) \rightarrow (q_{B,j}, q_{B,j+1})$ for all $1 \leq j < |B|$
- $(q_{B,B}, q_{B,B}) \rightarrow (q_{B,B}, q_{v,1,1})$
- $(q_{v,i}, q_{v,j}) \rightarrow (q_{v,i}, q_{v,j})$ for $1 \leq i \leq |v_1|$ and $1 \leq j \leq |v_1|$ (for $j = 1$ we produce $q_{v,1,1} = q_\bot$);
- $(q_{v,i}, q_\bot \rightarrow q_{v,(i+1)\uparrow}, q_{v,i})$ for $1 \leq i < |v_1|$;
- $(q_{v,i,1\uparrow}, q_\bot \rightarrow q_{v,i,1\uparrow}, q_{v,i,|v_1|\uparrow})$
- $(q_{v,i,\diamond}, q_{v,\diamond} \rightarrow q_{v,(i-1)\downarrow}, q_\bot)$ for $2 \leq i \leq |v_1|$ (for $i = 2$ the first agent also switches to the state $q_{v,1\uparrow}$ at some times during the execution).

The output function yields $B^{(j)}$ for $q_{B,j}$ and $v_1^{(j)}$ for $q_{v,i,1\uparrow}$, $q_{v,i,4}$, and $q_{v,i,\diamond}$.

It is easy to see that if there are at least $|B|$ agents, all $|B|$ states $q_{B,j}$ will eventually have exactly one agent, with the remaining agents switching to the state $q_{v,1\uparrow}$ at some times during the execution.
Claim. At every moment all the agents not in the states $q_{v,j}$ can be split into groups with one agent in each state $q_{v,j}$ for $j$ from 1 to some $k$ unless $k = |v_1|$. Some of such groups contain only one agent in the state $q_{v,1}$ or $q_{v,1} = q_\perp$.

Indeed, this is true initially as there are either no agents except in the state $q_{B,1}$ or each agent forms a group being in the state $q_{v,1}$, and each transition preserves the desired property.

In the following we will use the fairness condition to claim that if some property of configuration can always be destroyed by some transitions, it will eventually stop holding in any fair execution.

Note that once there are only $|B|$ agents in the states $q_{B,j}$, the number of agents in the states $q_{v,j}$ cannot increase anymore, but will sometimes decrease until there is exactly one such agent. After that point, the sum of the indices of all the agents in the states $q_{v,j}$ with $j > 1$ will only decrease until it becomes zero. Afterwards the number of agents in the state $q_\perp$ will only decrease until it reaches zero.

At that moment, if the difference between the number of agents and $|B|$ is divisible by $|v_1|$, all agents will be divided into one group of $|B|$ agents with the multiset of outputs $B$ and some groups of $|v_1|$ agents each having multisets of outputs equal to $v_1$. As all other sizes are not compatible with $C$, this concludes the proof.

Lemma 19. The set of sizes compatible with an integer cone is a one-dimensional integer cone.

Proof. We identify the multisets with one-element domain with natural numbers. The base of the cone is the size $|B|$ of the base configuration, and the periods are the sizes $|v_j|$ of the periods. We observe that using the same coefficients for non-negative integer combinations proves that the constructed one-dimensional cone contains exactly the sizes compatible with the original cone.

Proof of the theorem. Consider a size-flexible semilinear set $S \subset \mathbb{N}^O$. We construct our protocol as a synchronous product of multiple sub-protocols.

The set $S$ can be represented as a union of cones $\bigcup_{j=1}^n C_j \subset S$ each having periods of the same size. We run synchronous product of $2n$ protocols, $P_{con}$ ensuring membership in $C_j$ for all compatible sizes (using the lemma 18), and $P_{rec}$ computing compatibility of configuration size with $C_j$ (this predicate is semilinear by the lemma 19) thus it can be computed by the theorem 9. The global output function is the output corresponding to construction of the first cone that is expected to be compatible with configuration size, $o((q_1, q_2, \ldots, q_n, q_1^{rec}, q_2^{rec}, \ldots, q_n^{rec})) = o_{con}(q_{con})$ where $j = \min k : o_{rec}(q_k^{rec}) = true$. If there is no such cone, we return the first element of the output set.

Eventually, all the protocols $P_{rec}$ will converge to a stable consensus representing the true value of size compatibility. Therefore from some time on we will just use the output of $P_{con}$ corresponding to the first size-compatible cone, which will be in $S$ from some point on by the lemma 18.

3.2 Upper bound on constructive expressive power of population protocols

In this section we provide a matching upper bound for constructive expressive power.

Theorem 20. The class of all population protocols ensures at most the class of semilinear output conditions.
The proof uses the fact from [13], describing the structure of reachability sets of VAS, a more general model than population protocols.

Definition 21. An asymptotic integer cone is a set of multisets defined by a base multiset (or just base) $B$ and a possibly infinite set of period multisets (periods) $v_j$. We require that the domain of multisets $B$ and $v_j$ is finite; let its size be $n$. We require that the convex hull of the origin and all the periods interpreted as vectors in $\mathbb{Q}^n$ is definable in $(\mathbb{Q},+,\succ)$. A multiset belongs to the cone if it can be represented as a sum of the base multiset and a non-negative integer combination of periods. An almost semilinear set is a finite union of asymptotic integer cones.

Definition 22. The pre-image of a set of configurations $X$ is the set $\text{pre}^*(X)$ such that $C \in \text{pre}^*(X)$ iff there is some $C' \in X$ reachable from $C$. The post-image of $X$ is the set $\text{post}^*(X)$ such that $C \in \text{post}^*(X)$ iff it is reachable from some $C' \in X$.

Theorem 23 ([13], restriction of Corollary 6.3). For any semilinear sets of configurations $X$ and $Y$, the sets $\text{post}^*(X) \cap Y$ and $X \cap \text{pre}^*(Y)$ are almost semilinear.

We also use the results on structure of mutual reachability from [8].

Definition 24. A configuration $C$ is a bottom configuration if for each configuration $C'$ reachable from $C$, the configuration $C$ is reachable from $C'$.

Theorem 25 ([8], lemma 3 and proposition 14). Each fair execution of a population protocol reaches a bottom configuration. The set of bottom configurations is semilinear.

Proof of the theorem 25. Each fair execution reaches a reachable bottom configuration and then reaches it infinitely many times. Thus any output condition ensured by the protocol is satisfied by the output corresponding to any bottom configuration that is reachable from some input configuration. It suffices to find a size-flexible semilinear set of reachable bottom configurations $\mathcal{B}$, as its image under the output function will also be size-flexible and semilinear.

The proof idea is to consider under-approximations of the set of reachable bottom configuration using finite subsets of of periods. Observe that compatibility with any specific size can be demonstrated using just a finite number of periods; a simple divisibility argument shows that covering a finite set of sizes is sufficient.

We know that the set of bottom configuration is semilinear, and therefore the set of bottom configuration reachable from input configurations is almost semilinear. Let $B_1, \ldots, B_s$ be the bases of corresponding asymptotic integer cones. Fix some enumeration $(v_{i,j})$ of the periods of these cones, where $v_{i,j}$ is the $j$-th period of the $i$-th cone. Let $M = \max_j |B_j|$ be the maximum size of a base. Let $L$ be the least common multiple of $|v_{i,1}|$ for all $i$ from 1 to $s$ corresponding to the cones with at least one period.

Note that compatibility with any given size can be demonstrated using a finite number of periods. Let $K$ be the maximal number of a period used to demonstrate compatibility with any size up to $M + L$. We show that the semilinear set $\mathcal{B}$ consisting of the integer cones with bases $B_i$ and periods $v_{i,j}$ for $j \leq K$ is size-flexible. Indeed, consider any size $S > M + L$. Let $r$ be the remainder of $S - M - 1$ modulo $L$. Consider the size $M + 1 + r > M$. As this size is strictly larger than all the base sizes, its compatibility with $\mathcal{B}$ has to be shown using an integer cone with base $B_i$ and a nonempty set of periods $\{v_{i,j}\}$. Moreover, it can be demonstrated using only the periods $v_{i,1}, \ldots, v_{i,K}$, as $r \leq L - 1$ and thus $M + 1 + r \leq M + L$. We have $C = B_i + \sum_{j=1}^{K} v_{i,j} \in \mathcal{B}$, $|C| = M + 1 + r$. As $L$ is divisible by $|v_{i,1}|$, we can add the period $v_{i,1}$ to the configuration $C$ exactly $\frac{S - |C|}{|v_{i,1}|}$ times: $C' = C + \sum_{j=1}^{\frac{S - |C|}{|v_{i,1}|}} v_{i,1} \in \mathcal{B}$ and $|C'| = S$. This concludes the proof. ◀
Constructive expressive power of immediate observation protocols

In this section we switch to the study of constructive expressive power for a subclass of population protocols, namely immediate observation population protocols.

From the point of view of computing predicates, they have lower but still significant expressive power, but benefit from a much lower verification computational complexity than the general protocols, namely PSPACE-complete. We show that these properties also hold in the context of ensuring protocols.

4.1 Constructing counting sets

Just like in the case of general population protocols, we start by providing the feasibility result.

Theorem 26. The class of immediate observation population protocols ensures the class of counting output conditions.

The proof is similar to the proof of the theorem 16.

Lemma 27. For every multiset \( D \in \mathbb{N}^O \) there is an immediate observation protocol with a single input state that ensures equality to \( D \) from the input configuration of size at least \( |D| \).

Proof. Let an enumeration \( D^{(j)} \) for \( j \) from 1 to \( |D| \) contain each element \( x \in O \) exactly \( D(x) \) times. The protocol has the input state \( q_1 \) and other states \( q_2, \ldots, q_{|D|} \). The transitions are \( q_j \xrightarrow{\omega_j} q_{j+1} \), and the output function is \( o(q_j) = D^{(j)} \). It is easy to see that in a fair execution all \( |D| \) agents will have different states and therefore produce output \( D \).

Lemma 28. For a multiset \( D \in \mathbb{N}^O \) and output value \( x \in O \) consider the cube with the lower bounds specified by \( D \) and the upper bounds specified by \( D \) except for infinite upper bound for the value \( x \). Then there is an immediate observation protocol with a single input state that ensures membership in that cube from each input configuration of size at least \( |D| \).

Proof. Again, let an enumeration \( D^{(j)} \) for \( j \) from 1 to \( |D| \) contain each element \( x \in O \) exactly \( D(x) \) times. The protocol has the input state \( q_1 \) and other states \( q_2, \ldots, q_{|D|}, q_{|D|+1} \). The transitions are \( q_j \xrightarrow{\omega_j} q_{j+1} \), and the output function is \( o(q_j) = D^{(j)} \), \( o(q_{|D|+1}) = x \). It is easy to see that in a fair execution with at least \( |D| \) agents, states \( q_1, \ldots, q_{|D|} \) will contain one agent each with the rest of the agents in the state \( q_{|D|+1} \). Such a configuration will produce the output differing from \( D \) only by increasing the multiplicity of \( x \), as required.

Proof of the theorem 26. Consider a size-flexible counting constraint \( \psi \). It has to contain a cube with at least one infinite upper bound. Consider the smallest multiset \( D \) in that cube, and the output value \( x \) having an infinite upper bound. Let \( P^{\infty}_\psi \) be the protocol corresponding to \( D \) and \( x \) by the lemma 28. Let \( P^{\infty}_j \) for \( 0 \leq j < |D| \) be the protocol constructed by the lemma 27 for some multiset of size \( j \) satisfying the constraint \( \psi \).

By the theorem 26 there are immediate observation population protocols \( P^{\infty}_j \), \( j \) recognising equality of input size to \( j \) respectively.

We consider the synchronous of all these protocols and define the output function to be the output of \( P^{\infty}_j \), for the minimal \( j \) such that \( P^{\infty}_j \) outputs \( true \), or the output of \( P^{\infty}_\infty \) if none does. Eventually, all the protocols \( P^{\infty}_j \) provide correct configuration size information to each agent, and thus the outputs of the same \( P^{\infty}_j \) or \( P^{\infty}_\infty \) protocol are used by all the agents. By construction, the multiset of these outputs satisfies \( \psi \) from some moment on.
4.2 Structure of bottom configurations of immediate observation protocols

In this section we prove a structural lemma about the structure of bottom configurations of immediate observation protocols, which also implies an upper bound on constructive expressive power.

Theorem 29. The set of bottom configurations of an immediate observation population protocol is a counting set.

We use the pruning techniques from [11]. The pruning approach is based on deanonymisation of the agents, giving agents identities and arbitrarily picking which specific agent performs the observation at each step in the execution. Note that there are usually many ways to deanonymise a single execution.

Lemma 30 (Pruning Lemma). Consider an immediate observation population protocol with the set of states $Q$, and a configuration $C'$ reachable from another configuration $C$. Consider an execution $E$ from $C$ to $C'$ and its deanonymisation such that more than $|Q|$ agents go from some state $q$ to a state $q'$, where $q$ and $q'$ might be the same state. Then there is an execution from $C - \{q\}$ to $C' - \{q'\}$ and its deanonymisation $E'$ where one less agent goes from $q$ to $q'$ and for all other pairs of states the same number of agents go between them.

Proof of the theorem 29. Consider an immediate observation population protocol with the set of states $Q$.

We prove the claim in the following equivalent form. For each bottom configuration $B$, each configuration $C \geq B$ such that for each $q$ in the support of $C - B$ we have $B(q) \geq |Q|^4$ is also a bottom configuration.

The reformulated claim is proven by induction over the size $|C|$ if the configuration $C$. If $|C| = |B|$ we have $C = B$ and the claim is obviously true. Otherwise let $q_0$ be such a state that $C(q_0) > B(q_0) \geq |Q|^4$. Consider any configuration $C'$ reachable from $C$ and a deanonymised execution $E$ from $C'$ to $C$. By the pigeonhole principle, there is a state $q_1$ such that more than $|Q|^3$ agents go from $q_0$ to $q_1$.

We can prune the execution to obtain an execution $E_1$ from $C - \{q_0\}$ to $C' - \{q_1\}$. Note that $C - \{q_0\} \geq B$. By the induction hypothesis, $C - \{q_0\}$ is a bottom configuration. Thus there is an execution from $C' - \{q_1\}$ to $C - \{q_0\}$. Let’s deanonymise it: if going from $C - \{q_0\}$ to $C' - \{q_1\}$ and back permutes the agents, repeat this procedure the order of the permutation times. This yields an execution $E_{k+1}$ from $C' - \{q_1\}$ to $C - \{q_0\}$ such that it moves all the agents back to the same states, undoing the execution $E_k$ from $C - \{q_0\}$ to $C' - \{q_1\}$.

Adding a non-interacting agent in the state $q_1$, provides an execution from $C'$ to $C - \{q_0\} + \{q_1\}$ with most of the agents going in the opposite direction compared to $E$. Combining this with the execution $E$, we obtain an execution from $C$ to $C - \{q_0\} + \{q_1\}$ with only one agent going from $q_0$ to $q_1$ and the rest eventually going from their states back to the same states. Now we have at least $C(q_0) - 1 \geq |Q|^4$ agents going from $q_0$ to $q_0$. We apply the lemma 30 for obtain an execution from $C - \{q_0\}$ to $C - 2 \times \{q_0\} + \{q_1\}$. We use again that $C - \{q_0\}$ is a bottom configuration to obtain an execution from $C - 2 \times \{q_0\} + \{q_1\}$ to $C - \{q_0\}$. Adding a non-interacting agent in the state $q_0$ provides an execution from $C - \{q_0\} + \{q_1\}$ to $C$, proving that we can reach $C$ from $C'$ via $C - \{q_0\} + \{q_1\}$. As $C'$ was an arbitrary configuration reachable from $C$, this concludes the proof that $C$ is a bottom configuration.

This structural result implies the desired constructive expressive power upper bound using one more lemma from [11].
Lemma 31 (11). The set of configurations reachable from a given counting set of configurations is also a counting set.

Theorem 32. The class of immediate observation population protocols ensures at most the class of counting output conditions.

Proof. We observe that the set of reachable bottom configurations is a counting set as an intersection of the counting set of bottom configurations and the counting set of configurations reachable from input configuration. Then its image under the output function is a size-flexible counting set implying the ensured output condition.

4.3 Verification complexity for constructive immediate observation protocols

In this section we show that the relatively low verification complexity for immediate observation protocols is also applicable in the case of constructive expressive power.

Theorem 33. The problem of verifying whether a given immediate observation protocol $P$ ensures a given counting output condition $\psi$ given as a list of cubes with bounds written in unary is in PSPACE.

Here we use a convenient complexity claim from [9].

Lemma 34 ([9], claim in the proof of Theorem 4.50). Given two functions that produce counting sets with membership and emptiness in PSPACE and at most exponential constants from a counting set and a protocol, their boolean combinations as well as pre-image and post-image also have the same properties.

Let $S_1$ and $S_2$ be two functions that take as arguments an IO protocol $P$ and a counting constraint $X$, and return counting sets $S_1(P, X)$ and $S_2(P, X)$ respectively. Assume that $S_1(P, X)$ and $S_2(P, X)$ use bounds at most exponential in the size of the $(P, X)$, and have PSPACE-decidable membership (given input $(C, P, X)$, decide whether $C \in S_1(P, X)$) and emptiness.

Then the same is true about the counting sets $S_1(P, X) \cap S_2(P, X)$, $S_1(P, X) \cup S_2(P, X)$, $\overline{S_1(P, X)}$, $\text{pre}^*(S_1(P, X))$, $\text{post}^*(S_1(P, X))$.

Proof of the theorem 33. Given the output condition $\psi$ and a protocol, we have a counting set $\hat{\psi}$ of configurations satisfying $\psi$. The protocol ensures $\hat{\psi}$ if each reachable configuration can still reach a configuration in $\hat{\psi}$. In other words, no input configuration can reach a configuration outside the pre-image of $\hat{\psi}$. This can be expressed as emptiness of $\mathcal{I} \cap \text{pre}^*(\overline{\text{post}^*(\hat{\psi})})$ where $\mathcal{I}$ is the set of input configurations. As $\hat{\psi}$ and $\mathcal{I}$ are decidable in PSPACE, repeated application of the lemma yields PSPACE-decidability of the desired emptiness. This concludes the proof.

Remark 35. The proof of PSPACE-hardness of verification of immediate observation protocols given in [11] uses the constantly false protocol, thus it can be interpreted as hardness of verifying whether a protocol ensures $D \Rightarrow D(\text{true}) = 0$.

5 Conclusion

We have introduced a notion of constructive expressive power for population protocols and have shown that both for general population protocols and for immediate observation
population protocols it coincides with the expressive power in the classical setting of computing predicates. We have also shown that the relatively low verification complexity for immediate observation protocols is preserved.

The aim of being able to verify deployment strategies suggests further work in the direction of modelling failures, as well as self-stabilisation (i.e. making all states input states). On the other hand, deployment strategies often operate on heterogeneous fleets, requiring input-output conditions instead of pure output conditions to verify (e.g. we want to assign a file-server role to some server with sufficient storage attached).

Input-output conditions also seem to be a promising direction for achieving generic composition of population protocol.

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