Field Theories Found Geometrically from Embeddings in Flat Frame Bundles

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Abstract

We present two families of exterior differential systems (EDS) for non-isometric embeddings of orthonormal frame bundles over Riemannian spaces of dimension $q = 2, 3, 4, 5, ...$ into orthonormal frame bundles over flat spaces of sufficiently higher dimension. We have calculated Cartan characters showing that these EDS satisfy Cartan's test and are well-posed dynamical field theories. The first family includes a constant-coefficient (cc) EDS for classical Einstein vacuum relativity ($q = 4$). The second family is generated only by cc 2-forms, so these are integrable (but nonlinear) systems of partial differential equations. These latter field theories apparently are new, although the simplest case $q = 2$ turns out to embed a ruled surface of signature (1,1) in flat space of signature (2,1). Cartan forms are found to give explicit variational principles for all these dynamical theories.

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I. INTRODUCTION

We discuss two families of geometric field theories. By “geometric” we mean that these theories are given as exterior differential systems (EDS) for embedding of q-dimensional submanifolds $R^q$ in flat homogeneous isotropic metric spaces $E^N$ of higher dimension, say $N$. To formulate these EDS we in fact embed the orthonormal frame bundles over the submanifolds into the orthonormal frame bundles over the flat spaces, that is, into the groups $ISO(N)$, which have dimension $N(N + 1)/2$. E.g., the $q = 4$ dimensional EDS are set using the 55 basis 1-forms of $ISO(10)$. The fibers of the embedded bundles are subgroups of the $O(N)$ fibers of $ISO(N)$, thus inducing embedding maps of their $q$-dimensional bases $R^q$ into the $E^N$ bases of $ISO(N)$.

By “field theories” we mean that each of these various EDS is shown, by an explicit numerical calculation of its Cartan characteristic integers [1] to have the property of being well-posed or, with the correct signature, of being “causal”. This calculation uses a suite of Mathematica programs for EDS written by H. D. Wahlquist, and evaluates the Cartan characteristic integers with a Monte Carlo program to compute the ranks of the large matrices that arise [2]. The successive integers determine the dimension and well-posedness of the general solutions, and the Wahlquist programs also confirm the “involutory” property of certain fields in a solution, viz. those that can be adopted as independent variables.

A properly set EDS in a space with $N$ variables is equivalent to a set of first order partial differential equations in $N-q$ dependent variables, functions of $q$ independent variables, and Cartan’s technique of EDS is a deep approach to the Cauchy-Kowalewska analysis of such field theories.

Cartan’s theory considers construction of a sequence of regular integral manifolds (of successively higher dimensions) of an EDS. His characteristic integers $s_i, i = 0, 1, \ldots, q - 1$, are calculated from the ranks of matrices that arise, and are diagnostic; they must pass Cartan’s test [1] if the EDS (or an equivalent set of partial differential equations) is well posed. Then the final construction of the solution is determined solely from gauge and boundary data, and, at least in the analytic category, Cauchy existence and uniqueness are proved. We believe that, with proper attention to signature, the sets of partial differential equations following from such a well-posed EDS are those of a canonical field theory. Non-trivial embedding EDS that are well-posed, or causal, are not common. A key to their existence may be that for all the EDS we consider here we are also able to find Cartan $q$-forms from which the EDS may be derived by arbitrary variation.

There is a large literature, beginning with Lepage and Dedecker, on the use of Cartan $q$-forms and their closure $q + 1$-forms (“multisymplectic” forms). These are respectively the multidimensional field-theoretic extensions of classical Hamiltonian theory (the Cartan 1-form $Ldt$) and symplectic geometry. A short but essential bibliography can be found in Gotay [3]; cf. also Gotay et al [4], Bryant et al [5], Hermann [6] and Estabrook [7] [8]. The differential geometric setting for that work was for the most part (the structure equations of basis forms on) the first or second jet bundle over a base of $q$ independent variables. Our use instead of basis forms and structure equations for embedding geometry, orthonormal frame bundles over flat metric geometries, is more in the spirit of string theory. It allows application of the variational techniques of field theory to the movable frames of general relativity, and can lead to interesting extensions.

In both these families of EDS we adapt the method usually used in the mathematical literature for isometric embedding, cf. e.g. [9], [10], in that we do not begin with a prior
framing of the solution and prolong to higher bundles, but rather only use the bases of the embedding bundle. Such EDS have also been used in the theory of calibrated subspaces \[1\]. Perhaps this generalization of the customary isometric embedding can be called “dynamic embedding”. The EDS that naturally arise are considerably more elegant, interpretable as field theories.

The Lie group ISO(\(N\)) (or one of its signature siblings ISO(\(N - 1, 1\)) etc.) is the isometry group of \(N\)-dimensional flat space \(E^N\) (or a signature sibling). The group space is spanned by \(P = N(N + 1)/2\) canonical vector fields, and by a dual basis of left-invariant 1-forms that we first denote by \(\theta^\mu, \mu = 1...N\), corresponding to translations, and \(\omega^\mu_\nu\), that will correspond to rotations. Now the structure equations for general movable frames over an \(N\)-dimensional manifold are usually written covariantly (on the second frame bundle) as

\[
\begin{align*}
  d\theta^\mu + \omega^\mu_\nu \wedge \theta^\nu &= 0 \quad (1) \\
  d\omega^\mu_\nu + \omega^\mu_\sigma \wedge \omega^\sigma_\nu + R^\mu_\nu &= 0. \quad (2)
\end{align*}
\]

These become the Cartan-Maurer equations of ISO(\(N\)) or one of its siblings when the curvature 2-forms \(R^\mu_\nu\) are put equal to zero, and upper indices are systematically lowered using (for signature) a non-singular matrix of constants \(\eta_{\mu\nu}\), after which imposing antisymmetry (orthonormality) \(\omega_{\mu\nu} = -\omega_{\nu\mu}\). These structure equations then describe \(N(N - 1)/2\)-dimensional rotation groups as fibers over \(N\)-dimensional homogeneous spaces \(E^N\). (The \(\eta_{\mu\nu}\), and other possible signatures in \(E^N\), are often conveniently ignored in the following, and can be inserted later.)

We will write the two families of EDS using partitions \((n, m), n + m = N\), of the basis forms of ISO(\(N\)) into classes labeled respectively by the first \(n\) indices \(i, j\), etc. = 1, 2, ...\(n\) and the remaining indices \(A, B\), etc. = \(n + 1, n + 2, ...N\). So the basis forms are \(\theta^i, \theta^A\), and, after lowering an index, \(\omega_{ij} = -\omega_{ji}\), \(\omega_{AB} = -\omega_{BA}\), \(\omega_{iA} = -\omega_{Ai}\). Summation conventions on repeated indices will be used separately on each partition. The structure equations (1) (2) before lowering become

\[
\begin{align*}
  d\theta^i + \omega^i_j \wedge \theta^j &= -\omega^i_A \wedge \theta^A \quad (3) \\
  d\theta^A + \omega^A_B \wedge \theta^B &= -\omega^A_i \wedge \theta^i \quad (4) \\
  d\omega^i_j + \omega^i_k \wedge \omega^k_j &= -\omega^i_A \wedge \omega^A_j \quad (5) \\
  d\omega^A_B + \omega^A_C \wedge \omega^C_B &= -\omega^A_i \wedge \omega^i_B \quad (6) \\
  d\omega^i_A + \omega^i_j \wedge \omega^j_A + \omega^i_B \wedge \omega^B_A &= 0. \quad (7)
\end{align*}
\]

The terms we have put on the right are interpreted as torsions and curvatures induced by an embedding; we will use them to set the EDS.

In Sections 2 and 3 we calculate the Cartan characteristic integers of the embedding EDS for the two families. We will report the results in a short tabular form: \(P\{s_0, s_1, ...s_{q-1}\}q + CC\). This gives first the dimension \(P\) of the space in which we set the EDS, i.e. the total number of basis forms with whose structure equations we begin, then the series of Cartan integers found, \(\{s_0, s_1, ...s_{q-1}\}\). \(q\) is the dimensionality of the base space of a general solution. Finally CC is the number of Cartan characteristic vectors (the number of auxiliary fields allowing us to write a cc system, fibers corresponding to variables that could in principle be eliminated from the EDS). Cartan denotes \(q + CC\), by \(g\), the genus. The ultimately simple Cartan test showing the EDS to be well-posed and causal, calculated from these, is derived in \[1\] and the literature cited there. Here the test is simply that these integers
satisfy \( P - \sum_{i=0}^{q-1} s_i - q - CC := s_q \geq 0 \). We will always have \( s_q = 0 \), which a physicist interprets as absence of a gauge group, and so according to Cartan solutions will depend on \( s_{q-1} \) functions of \( q-1 \) variables. We denote these theories as causal but that requires also adjusting the signatures, so that the final integration of solutions from this boundary data is hyperbolic.

It is a classic result \([10]\) that smooth local embedding of Riemannian geometries of dimension \( q = 3, 4, 5 \ldots \) is always possible into flat spaces of dimension respectively \( N = q(q + 1)/2 = 6, 10, 15 \ldots \), which motivates the partitions of our first family, viz. \((n, m) = (3, 3), (4, 6), (5, 10), \ldots \). The causal EDS we give determine submanifolds of \( ISO(N) \) which are themselves \( O(n) \otimes O(m) \) bundles fibered over \( q = n \)-dimensional base spaces, say \( R^q \) and induce maps of these into \( E^N \). The \( n \theta_i \) remain independent (“in involution”) when pulled back to a solution bundle, satisfying the structure equations of an orthonormal basis in any cross section, and Equations (5) and (7) express embedding relations that go back to Gauss and Codazzi. The solution bundle metric is the pullback of \( \theta_i \theta_i \). We will present in Section 2 the family of Einstein-Hilbert Cartan forms from which the EDS of this first family are derived by variation. The EDS will require zero torsion for the \( \theta_i \) but not insist on aligning the solutions with these orthonormal frames (the \( \theta_A \) are not included in the EDS so it is not necessarily “isometric”), and from the induced map of bundle bases there is also a less interesting “ghost” metric which is the pullback of \( \theta_i \theta_i + \theta_A \theta_A \). The induced curvature 2-forms are required by the EDS to satisfy “horizontality” 3-form conditions and also to have vanishing Ricci \( n-1 \)-forms.

The field theories of our second family, of dimension \( q = 2, 3, 4, 5 \ldots \) also arise from embeddings into flat spaces \( E^N \) of dimension \( N = 3, 6, 10, 15, \ldots \) but the EDS use different partitions, viz. \((n, m) = (1, 2), (2, 4), (3, 7), (4, 11), \ldots \). Solutions are \((O(n) \otimes O(m) \) bundles over) geometries of dimension \( q = n + 1 \) and can be called n-branes. They have rulings that are flat \( n \)-spaces. The EDS are generated only by cc sets of 2-forms (for vanishing torsion of both partitions) and are so-called “integrable systems”. Again the embedding is dynamic, the partitioned frames are not required to be an orthonormal framing of the solution manifolds. In Section 3 we give the EDS and report the calculated Cartan characters showing them to be causal. The \( n \theta_i \) when pulled back into a solution both determine a Riemannian submersion and geodesic slicing. This is either a theory of relativistic rigidity or perhaps of a Kaluza-Klein gravitational field, depending on \( N \) and the signature adopted. Cartan forms for those EDS are easily found.

As a sole illustration of introduction of explicit coordinates into such a frame bundle EDS, the simplest of these non-isometric geometric field theories, that based on partition \((1, 2)\), is integrated in Section 4. Its solutions turn out to be classically known, in the guise of geodesically ruled surfaces in \( E^3 \). We have only changed signature to show it as a stringy field causally evolving in time.

**II. EINSTEIN-HILBERT ACTION**

The EDS of our first family arise from Cartan \( n \)-forms on \( ISO(N) \) expressing the Ricci scalars of \( q = n \) dimensional submanifolds of \( E^N \),

\[
\Lambda = R_{ij} \wedge \theta_k \wedge \ldots \theta_p \epsilon_{ijk \ldots p},
\]  

(8)

4
where from the Gauss structure equation, Eq. (5), $2R_{ij} := -\omega_{iA} \wedge \omega_{jA}$ is the induced Riemann 2-form. The exterior derivative of the $n$-form field $\Lambda$ on $\text{ISO}(N)$, using Eq. (3) and (7), is quickly calculated to be the $n + 1$-form (closed, multisymplectic)

$$d\Lambda = \theta_A \wedge \omega_{Ai} \wedge R_{jk} \wedge \theta_l \wedge ... \theta_p \epsilon_{ijkl...p}. \quad (9)$$

This $n + 1$-form is a sum of products of the $m$ 1-forms $\theta_A$ and the $m$ $n$-forms $\omega_{Ai} \wedge R_{jk} \wedge \theta_l \wedge ... \theta_p \epsilon_{ijkl...p}$. A variational isometric embedding EDS is generated by the $m$ $\theta_A$, their exterior derivatives for closure, and the $m$ $n$-forms, since any vector field contracted on $d\Lambda$ yields a form in the EDS. That is, up to boundary terms, the arbitrary variation of $\Lambda$ vanishes on solutions. We previously calculated Cartan’s characteristic integers for these isometric embedding EDS showing them to be well set and causal [11] [12]. We denoted them as being “constraint free” geometries. Isometric embedding formulations of the Ricci-flat field equations then are obtained by adding in the closed $n - 1$-forms for Ricci-flatness as constraints. The augmented EDS are again calculated to be causal. To be explicit, for partition $(4, 6)$ the constraint-free Cartan character table was $55 \{6, 6, 6, 12\} q = 4 + 21$ which, with the augmentation with four 3-forms became $55 \{6, 6, 10, 8\} q = 4 + 21$. We now see that formulation as nevertheless somewhat unsatisfactory as field theory, since the Einstein-Hilbert action appears to have lead to equations which in fact mostly follow from the imposed constraints.

We have however noticed that there is another variational EDS belonging to a different quadratic factoring of the multisymplectic forms $d\Lambda$, Eq. (9). The $\theta_i$ will frame a Riemannian metric on an embedded space of dimension $n$ so long as the induced torsion 2-forms of Eq.(2), $\omega_{iA} \wedge \theta_A$, vanish, and these factor Eq.(9) term-by-term, as products with the $n - 1$-forms for Ricci-flatness. The exterior derivatives of the torsion terms must be included; these are sometimes called conditions for horizontality. In sum, we have considered the following closed EDS (which now do not include the mathematically customary isometric embedding 1-forms $\theta_A$)

$$(\omega_{iA} \wedge \theta_A, R_{ij} \wedge \theta_j, R_{ij} \wedge \theta_k \wedge ... \theta_l \epsilon_{ijkl...p}) \quad (10)$$

When $n=4$ this EDS is an exact parallel to the EDS for a moving frame formulation of vacuum relativity that used the 44 traditional intrinsic coordinates of tetrad frames and connections over 4-space, and had 10 gauge freedoms [13]. It had the same Cartan character table but no CC. The present formulation is set with more variables, viz. 55, but its solutions have 21 CC fibers (since $\omega_{ij}$ and $\omega_{AB}$ do not enter explicitly) and no gauge freedom; moreover it has the elegance of a cc EDS (no coordinate functions appear in the generating forms) [14] [15] [16].

The calculation shows the EDS Eq.(10) to be well set and causal systems for embedding of $O(n) \otimes O(m)$ bundles over $n$ space, for the partitions $(3, 3), (4, 6), (5, 10)$ etc. as stated in the introduction. The embedding dimension, the computed Cartan characters, dimensionality and $O(n) + O(m)$ fiber dimension (CC) of the solutions for these cases are respectively $21\{0, 6, 3\} 3 + 9, 55\{0, 4, 12, 14\} 4 + 21, 120\{0, 5, 10, 20, 25\} 5 + 55$, etc. The base spaces of the fibered solution manifolds are spanned by the 1-forms $\theta_i$; evidently a solution is a bundle of orthonormal frames belonging to the Ricci-flat Riemannian connection $\omega_{ij}$. The metric is $\theta_i \theta_i$. There is also present in the base space $R^m$ another metric pulled back from the induced embedding of it in the base space $E^N$ about which we know little: $\theta_i \theta_i + \theta_A \theta_A$. It is a ghost tensor field, perhaps with only indirect influence. The ideals we are writing are set on $\text{ISO}(N)$, and their solutions are frame bundles embedded in $\text{ISO}(N)$, and the induced embeddings of the base spaces seem to be of less interest.
The ideal Eq.(10) is contained in the augmented embedding ideal we have previously used, so solutions of the latter will be solutions of the former. This would seem to imply that our new dynamic embedding ideal will have additional solutions; indeed it implies fewer partial differential equations than does the isometric embedding ideal augmented with constraints for Ricci-flat geometry. Perhaps so-called singular solutions of the isometric embedding ideal—solutions which are not regular, that is, obtained by Cartan’s sequential integrations—now appear as regular solutions, which could make this new formulation important for local numerical computation from boundaries.

III. TORSION-FREE N-BRANE EMBEDDING

We have searched whether the torsion 2-forms induced in both the local partitions can together be taken as an EDS: \((\omega_A \wedge \theta_A, \omega_i A \wedge \theta_i)\). It can easily be checked that it is closed, and calculation of the characteristic integers indeed showed that for just the values of \((n, m)\) of the second family described in the introduction these EDS are causal, with \(q = n + 1\) and fibers \(O(n) \otimes O(m), \dim (n-1)/2 + m(m-1)/2\). The results for the first five EDS are: \((n, m) = (1, 2), (2, 4), 21\{0, 6, 5\}3 + 7; (3, 7), 55\{0, 10, 9, 8\}4 + 24; (4, 11), 120\{0, 15, 14, 13, 12\}5 + 61; (5, 16), 231\{0, 21, 20, 19, 18, 17\}6 + 130;\) and the pattern seems evident.

Now well set EDS for geodesic flat dimension \(n\) submanifolds of flat \(N\) spaces are generated, using the partition \((n, m)\) by the closed ideal of 1-forms \((\theta_A, \omega_Ai)\). For example, if \(N = 3\) and \(n = 1\) and \(m = 2\), geodesic lines in flat 3-space, the Cartan characteristic integers are 6\{4\}1 + 1. If \(N = 4\), for partition \((1, 3)\) we find 10\{6\}1 + 3 (in all cases \(\omega_{ij}\) and \(\omega_{AB}\) give the Cauchy characteristic fibers). Similarly, the EDS for flat 2-dimensional submanifolds of flat \(N\) spaces are generated by the 1-forms with partitions \((2, N - 2)\). For example if \(N = 5\), \((n, m) = (2, 3)\), and the character table is 15\{9, 0\}2 + 4. When \(N = 6\), \((n, m) = (2, 4)\) and 21\{12, 0\}2 + 7. The zeros can be ascribed to a gauge freedom. These constructions clearly continue. Our new torsion-free EDS \((\omega_A \wedge \theta_A, \omega_Ai \wedge \theta_i)\) are contained in \((\theta_A, \omega_Ai)\), so we see that the \(q\)-dimensional solutions of the torsion-free embedding theory must contain flat geodesic fibers of dimension \(n = q - 1\). Thus the solutions are ruled spaces.

In a solution the \(\theta_i\) remain independent (are “in involution”) but fall short by one of being a complete basis. In addition to the slicing, they define there a vector field, say \(V\), of arbitrary normalization (a congruence), by the relations \(V \cdot \theta_i = V \cdot \omega_{ij} = V \cdot \omega_{AB} = 0\). Contracting \(V\) on the second torsion 2-form, since the \(\theta_i\) remain linearly independent, gives also \(V \cdot \omega_Ai = 0\). It follows that the Lie derivatives with respect to \(V\) of \(\theta_i, \omega_{ij}\) and \(R_{ij}\) vanish on solutions. They live in (and are lifted from) an \(n\) dimensional quotient space of the solution, with metric \(\theta_i \theta_j\) and Riemann tensor \(\omega_Ai \wedge \omega_Aj\). Cross sections of this quotient map are the rulings, geodesic \(n\)-dimensional subspaces calibrated by the volume form \(\theta_i \wedge \theta_j \ldots \wedge \theta_k\).

In an earlier time we have discussed the problem of defining a rigid body in special and general relativity \[17\]. The kinematic quotient-space definition of rigidity due originally to Max Born (Riemannian submersion) was shown by Herglotz and Noether to have only three degrees of freedom: the only Born-rigid congruences which were rotating (had vorticity) in Minkowski space were isometries of the space-time without time evolution. We showed this to be the case also for kinematic or “test” rigid bodies moving in vacuum Einstein spaces. It seemed to be impossible then to sensibly discuss the so-called “dynamic” rigid bodies envisioned by Pirani, which were to carry their own 3-dimensional geometry while distorting
space-time. We are charmed by having now arrived at space-times, using dynamic embedding in the (3, 7) partition, having the greater dynamical freedom allowed by separation of the rôles of the induced 3-metrics in the cross sections and quotient space of a solution.

In the (4, 11) partition, the solutions are five dimensional, with a dynamically rigid congruence that projects to a metric quotient 4-space. This may be a well-posed causal variant of Kaluza-Klein theory, and merits further investigation.

Closed EDS generated only by cc 2-forms have a special structure, inasmuch as they can be equivalent to dual infinite Lie algebras of Kać-Moody type and lead to hierarchies of so-called integrable systems. Lie groups have a duality between 2-form Cartan-Maurer structure equations for basis 1-forms and Lie commutator products of dual basis vector fields. This duality persists when the additional cc 2-forms of an EDS are imposed, but then the vector commutator table is incomplete. New vectors can be introduced in terms of the unknown commutators, and then more commutators calculated using the Jacobi identities. These allow adding 2-form structure equations for new dual 1-forms in higher dimensional spaces. If this expansion terminates, an embedding in a group has been found, the new 1-forms being potentials that integrate the original EDS. If the expansion continues, it leads to a Kać-Moody algebra of finite growth. Such EDS belong to so-called integrable systems of partial differential equations. The prototype of this is the well-known Korteweg-de Vries equation, which both leads to [18], and belongs to, the hierarchy of the infinite Lie algebra $A_1^{(1)}$ derived from SL$(2, \mathbb{R})$. The Kać-Moody algebras dual to our embedding EDS remain to be worked out.

Finally, although we did not derive these EDS variationally, Cartan forms are easily found, at least for even dimensions. In particular, in the (3, 7) theory either the 2-forms $\tau_A = \omega_{Ai} \land \theta_i$ or $\sigma_i = \omega_{iA} \land \theta_A$ can be used to write a quadratic Cartan form as in some Yang-Mills theories:

$$\Lambda = \tau_A \land \tau_A, \text{ so } d\Lambda = 2\tau_A \land \omega_{Ai} \land \sigma_i$$  \hfill (11)

Every term of $d\Lambda$ contains both a $\tau_A$ and a $\sigma_i$ so arbitrary variation yields the EDS. We also note that $\tau_A \land \tau_A + \sigma_i \land \sigma_i$ is exact.

IV. THE PARTITION (1, 2)

We will set this EDS on the frame bundle ISO(1, 2) over a flat 3-space with signature (-, +, -), so the structure equations of the bases are

$$d\theta_1 + \omega_{12} \land \theta_2 + \omega_{31} \land \theta_3 = 0$$ \hfill (12)
$$d\theta_2 + \omega_{12} \land \theta_1 - \omega_{23} \land \theta_3 = 0$$ \hfill (13)
$$d\theta_3 - \omega_{31} \land \theta_1 - \omega_{23} \land \theta_2 = 0$$ \hfill (14)
$$d\omega_{12} - \omega_{31} \land \omega_{23} = 0$$ \hfill (15)
$$d\omega_{23} - \omega_{12} \land \omega_{31} = 0$$ \hfill (16)
$$d\omega_{31} + \omega_{23} \land \omega_{12} = 0,$$ \hfill (17)
and the EDS to be integrated is generated by the three 2-forms $\omega_i A \wedge \theta^A, \omega_A i \wedge \theta^i, i = 1, A = 2, 3$:

$$\omega_{12} \wedge \theta_2 + \omega_{31} \wedge \theta_3 \tag{18}$$
$$\omega_{12} \wedge \theta_1 \tag{19}$$
$$\omega_{31} \wedge \theta_3 \tag{20}$$

The characteristic integers are $6\{0, 3\} q = 2$ and CC = 1; O(2) fiber (since $\omega_{23}$ is not present). To introduce coordinates - scalar fields - we will successively prolong the EDS with potentials or pseudopotentials, checking at each step that it remains well-set and causal.

First, it is obvious that there is a conservation law, a closed 2-form that is zero mod the EDS, viz. $d\theta_1$. So we adjoin the 1-form

$$\theta_1 + dv \tag{21}$$

introducing the scalar potential $v$. The characters are now $7\{1, 3\} 2 + 1$. Next we specialize to a particular, convenient, fiber cross-section making a choice of frame: we introduce two new fields $\zeta$ and $\eta$ while prolonging with three 1-forms taken so that the original 2-forms in the EDS vanish (they have been “factored”)

$$\omega_{12} - \zeta \theta_1 \tag{22}$$
$$\omega_{13} - \eta \theta_1 \zeta \tag{23}$$
$$\zeta \theta_2 - \eta \theta_3 + (\eta + \zeta) \theta_1 \tag{24}$$

To maintain closure, however, three new 2-forms, exterior derivatives of these or algebraically equivalent, must also be adjoined:

$$(d\zeta - \eta \omega_{23}) \wedge dv \tag{25}$$
$$(d\eta - \zeta \omega_{23}) \wedge dv \tag{26}$$
$$(\eta d\zeta - \zeta d\eta) \wedge (\theta_2 + \theta_3) - (\eta + \zeta) \omega_{23} \wedge (\eta \theta_2 - \zeta \theta_3) \tag{27}$$

Now we have $9\{4, 3\} 2$ with no CC. $\omega_{23}$ now appears in the EDS, but is conserved, $d\omega_{23} = 0$ mod EDS. Thus, we can introduce a pseudopotential variable $x$, and then further find another conserved 1-form and a final pseudopotential $u$. Which is to say we can adjoin

$$\omega_{23} - dx \tag{28}$$
$$\theta_2 + \theta_3 - e^x du \tag{29}$$

without adding any 2-forms to the EDS. We have a total of 11 basis 1-forms: six in $\theta_i, \theta_A, \omega_{AB}, \omega_{iA}$, plus $d\zeta, d\eta, dx, du, dv$, and an EDS with $11\{6, 3\} 2$. The pulled-back original six bases are now all solvable in terms of coordinate fields on the solutions, and can be eliminated: $5\{0, 3\} 2$. We have eliminated the CC. This is equivalent to a set of first order partial differential equations in 3 dependent variables and 2 independent variables. From the character table, we expect solutions to involve 3 arbitrary functions of 1 variable.

Taking $x$ and $v$ as independent in the solution, we can solve the first two 2-forms in Eq. (25) and (26) for $\eta$ and $\zeta$:

$$\eta = ae^x + be^{-x} \tag{30}$$
$$\zeta = ae^x - be^{-x} \tag{31}$$
where $a$ and $b$ are arbitrary functions of $v$. The third 2-form then amounts to

$$e^x = 1/2(b/a)' \partial_x u,$$

which integrates to

$$e^x = 1/2B'(u - A(v)).$$

We have put $b/a = B(v)$ and prime is derivation with respect to $v$.

The three arbitrary functions of $v$, $a$, $b$ and $A$, give the general solution. On it the pulled-back bases of $E^3$ (no longer orthonormal or independent) are

$$\theta_1 = -dv$$

$$\theta_2 = dv + \frac{ae^x + be^{-x}}{2a}du$$

$$\theta_3 = -dv + \frac{ae^x - be^{-x}}{2a}du,$$

and the induced 2-metric from $E^3$ is

$$g = -\theta_1 \theta_1 + \theta_2 \theta_2 - \theta_3 \theta_3$$

$$= Bdu^2 + B'(u - A)dvdu - dv^2.$$

This is, up to signature, the metric found classically from the construction of geodesically ruled surfaces in $E^3$, cf, e.g., Eisenhardt [10]. The surfaces are intrinsically characterized by a “line of striction”, the locus $u - A(v) = 0$, and a “parameter of distribution” $2B/B'$. The geodesic rulings, on which $\theta_2$, $\theta_3$, $\omega_{12}$, and $\omega_{13}$ pull back to vanish, are the set of lines $u = \text{const}$. The rigid congruence is the set of lines on which $V$ contracted with $\theta_1$, $\omega_{12}$ and $\omega_{13}$ vanishes, hence $v = \text{const}$.

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