Nonholonomic Ricci Flows: 
Exact Solutions and Gravity

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Abstract

In a number of physically important cases, the nonholonomically (nonintegrable) constrained Ricci flows can be modelled by exact solutions of Einstein equations with nonhomogeneous (anisotropic) cosmological constants. We develop two geometric methods for constructing such solutions: The first approach applies the formalism of nonholonomic frame deformations when the gravitational evolution and field equations transform into systems of nonlinear partial differential equations which can be integrated in general form. The second approach develops a general scheme when one (two) parameter families of exact solutions are defined by any source-free solutions of Einstein’s equations with one (two) Killing vector field(s). A successive iteration procedure results in a class of solutions characterized by an infinite number of parameters for a non–Abelian group involving arbitrary functions on one variable. We also consider nonlinear superpositions of some mentioned classes of solutions in order to construct more general integral varieties of the Ricci flow and Einstein equations depending on infinite number of parameters and three/ four coordinates on four/ five dimensional (semi) Riemannian spaces.

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1 Introduction

In recent years, much work has been done on Ricci flow theory and fundamental problems in mathematics [1, 2, 3] (see [4, 5, 6] for reviews...
and references therein). In this context, a number of possible applications in modern gravity and mathematical physics were proposed, for instance, for low dimensional systems and gravity \[7, 8, 9, 10\] and black holes and cosmology \[11, 12\]. Such special cases were investigated following certain low dimensional or approximative solutions of the evolution equations.

There were also examined possible connections between Ricci flows, solitonic configurations and Einstein spaces \[13, 14, 15\]. In our works, we tackled the problem of constructing exact solutions in Ricci flow and gravity theories in a new way. Working with general (pseudo) Riemannian spaces and moving frames, we applied certain methods from the geometry of Finsler–Lagrange spaces and nonholonomic manifolds provided with nonlinear connection structure (\(N\)-connection) \[16, 17, 18, 19\].

Prescribing on a manifold some preferred systems of reference and symmetries, it is equivalent to define some nonintegrable (nonholonomic, equivalently, anholonomic) distributions with associated \(N\)-connections. From this point of view, of the geometry of so–called nonholonomic manifolds, it is possible to elaborate a unified formalism for locally fibred manifolds and vector/tangent bundles when the geometric constructions are adapted to the \(N\)-connection structure. We can consider different classes of metric and \(N\)-connection ansatz and model, for instance, a Finsler, or Lagrange, geometry in a (semi) Riemannian (in particular, Einstein) space. Inversely, we can define some effective Lagrangian, or Finsler like, fundamental functions for lifts of geometric objects for a theory of gravity to tangent bundles in order to elaborate a geometric mechanics model for such gravitational and/or gauge field interactions, see examples and details in Refs. \[19, 20, 21, 22, 23\]. It was also proved that constraining some classes of Ricci flows of (semi) Riemannian metrics we can model Finsler like geometries and, inversely, we can transform Finsler–Lagrange metrics and connections into Riemannian, or Riemann–Cartan ones \[13, 14, 24\].

The most important idea in constructing exact solutions by geometric methods is that we can consider such nonholonomic deformations of the frame and connection structures when the Cartan structure equations, Ricci flow and/or Einstein equations transform into systems of partial differential equations which can be integrated in general form, or one can be derived certain bi–Hamilton and solitonic equations with corresponding hierarchies and conservation laws, see \[19, 15\] and references therein.

The first examples of physically valuable exact solutions of nonholonomic Ricci flow evolution equations and gravitational field equations were constructed following the so–called anholonomic frame method \[25, 26, 27\]. We analyzed two general classes of solutions of evolution equations on time like
and/or extra dimension coordinate (having certain nontrivial limits to exact solutions in gravity theories): The first class was elaborated for solitonic and pp–wave nonholonomic configurations. The second class was connected to a study of nonholonomic Ricci flow evolutions of three and four dimensional (in brief, 3D and 4D) Taub–NUT metrics. Following those constructions and further geometric developments in Refs. [13, 14], we concluded that a number of important for physical considerations solutions of Ricci flow equations can be defined by nonholonomically generalized Einstein spaces with effective cosmological constant running on evolution parameter, or (for more general and/or normalized evolution flows) by ‘nonhomogeneous’ (locally anisotropic) cosmological constants.

This is the forth paper in a series of works on nonholonomic Ricci flows modelled by nonintegrable constraints on the frame structure and evolution of metrics [13, 14, 15]. It is devoted to geometric methods of constructing generic off–diagonal exact solutions in gravity and Ricci flow theory.

The goal is to elaborate a general scheme when starting with certain classes of metrics, frames and connections new types of exact solutions are constructed following some methods from nonholonomic spaces geometry [20, 21, 22, 23, 15] and certain group ideas [31, 32]. The approach to generating vacuum Einstein metrics by parametric nonholonomic transforms was recently formulated in Ref. [33] (this article proposes a “Ricci flow development” of sections 2 and 3 in that paper). Such results seem to have applications in modern gravity and nonlinear physics: In the fifth partner paper [28], we show how nonholonomic Ricci flow evolution scenario of physically valuable metrics can be modelled by parametric deformations of solitonic pp–waves and Schwarzschild solutions.

One should be noted that even there were found a large number of exact solutions in different models of gravity theory [29, 30, 19, 20, 21, 22, 23], and in certain cases in the Ricci flow theory [7, 8, 9, 10, 25, 26, 27, 15], one has been elaborated only a few general methods for generating new physical solutions from a given metric describing a real physical situation. For quantum fields, there were formulated some approximated approaches when (for instance, by using Feynman diagrams, the formalism of Green’s functions, or quantum integrals) the solutions are constructed to represent a linear or nonlinear prescribed physical situation. Perhaps it is unlikely that similar computation techniques can be elaborated in general form in gravity theories and related evolution equations. Nevertheless, certain new

\[^1\text{We shall follow the conventions from the first two partner works in the series; the reader is recommended to study them in advance.}\]
possibilities seem to be opened after formulation of the anholonomic frame method with parametric deformations for the Ricci flow theory. Although many of the solutions resulting from such methods have no obvious physical interpretation, one can be formulated some criteria selecting explicit classes of solutions with prescribed symmetries and physical properties.

The paper has the following structure: In section 2, we outline some results on nonholonomic manifolds and Ricci flows. Section 3 is devoted to the anholonomic frame method for constructing exact solutions of Einstein and Ricci flow equations. There are analyzed the conditions when such solutions define four and five dimensional foliations related to Einstein spaces and Ricci flows for the canonical distinguished connection and the Levi Civita connection. In section 4, we consider how various classes of metrics can be subjected to nonholonomic deformations and multi–parametric transforms (with Killing symmetries) resulting in new classes of solutions of the Einstein/ Ricci flow equations. We consider different ansatz for metrics and two examples with multi–parametric families of Einstein spaces and related Ricci flow evolution models. The results are discussed in section 5. The reader is suggested to see Appendices before starting the main part of the paper: Appendix A outlines the geometry of nonlinear connections and the anholonomic frame method of constructing exact solutions. Appendix B summarizes some results on the parametric (Geroch) transforms of vacuum Einstein equations.

**Notation remarks:** It is convenient to use in parallel two types of denotations for the geometric objects subjected to Ricci flows by introducing "left–up" labels like $\gamma(\ldots, \chi)$. Different left–up labels will be also considered for some classes of metrics defining Einstein spaces, vacuum solutions and so on. We shall also write "boldface" symbols for geometric objects and spaces adapted to a nonholonomic (N–connection) structure, for instance, $V, E, \ldots$ A nonholonomic distribution with associated N–connection structure splits the manifolds into conventional horizontal (h) and vertical (v) subspaces. The geometric objects, for instance, a vector $X$ can be written in abstract form $X = (hX, vX)$, or in coefficient forms, $X^\alpha = (X^i, X^a) = (X^\underline{x}, X^\underline{a})$, equivalently decomposed with respect to a general nonholonomic frame $e_\alpha = (e_i, e_a)$ or coordinate frame $\partial_\underline{x} = (\partial_x, \partial_y)$ for local h- and v–coordinates $u = (x, y)$, or $u^\alpha = (x^i, y^a)$ when $\partial_\underline{x} = \partial/\partial u^\underline{x}$ and $\partial_\underline{a} = \partial/\partial u^\underline{a}$, $\partial_\underline{x} = \partial/\partial x^\underline{a}$. The h–indices $i, j, k, \ldots = 1, 2, \ldots n$ will be used for nonholonomic vector components and the v–indices $a, b, c, \ldots = n + 1, n + 2, \ldots n + m$ will be used for holonomic vector components. Greek indices of type $\alpha, \beta, \ldots$ will be used as cumulative ones. We shall omit labels, indices and parametric/
coordinate dependencies if it does not result in ambiguities.

2 Preliminaries

In this section we present some results on nonholonomic manifolds and Ricci flows \[13, 14\] selected with the aim to outline a new geometric method of constructing exact solutions. The anholonomic frame method and the geometry of nonlinear connections (N–connections) are considered, in brief, in Appendix A. The ideas on generating new solutions from one/ two Killing vacuum Einstein spacetimes \[31, 32\] are summarized in Appendix B.

2.1 Nonholonomic (pseudo) Riemannian spaces

We consider a spacetime as a (necessary smooth class) manifold \(V\) of dimension \(n + m\), when \(n \geq 2\) and \(m \geq 1\) (a splitting of dimensions being defined by a N–connection structure, see (A.3)). Such manifolds (equivalently, spaces) are provided with a metric, \(g = g_{\alpha\beta}e^\alpha \otimes e^\alpha\), of any (pseudo) Euclidean signature and a linear connection \(D = \{\Gamma^\alpha_{\beta\gamma}e^\beta\}\) satisfying the metric compatibility condition \(Dg = 0\). The components of geometrical objects, for instance, \(g_{\alpha\beta}\) and \(\Gamma^\alpha_{\beta\gamma}\), are defined with respect to a local base (frame) \(e_\alpha\) and its dual base (co–base, or co–frame) \(e_\beta\) for which \(e_\alpha \lrcorner e_\tau = \delta^\tau_\alpha\); the indices run correspondingly values of type: \(i,j,... = 1,2,...,n\) and \(a,b,... = n+1,n+2,...,n+m\) for any conventional splitting \(\alpha = (i,a),\beta = (j,b),...\)

Any local (vector) basis \(e_\alpha\) and dual basis \(e^\beta\) can be decomposed with respect to other local bases \(e_\alpha'\) and \(e^\beta'\) by considering frame transforms,

\[
e_\alpha = A^\alpha_{\alpha'}(u)e_\alpha' \quad \text{and} \quad e^\beta = A^\beta_{\beta'}(u)e^\beta',
\]

where the matrix \(A^\beta_{\beta'}\) is the inverse to \(A^\alpha_{\alpha'}\). It should be noted that an arbitrary basis \(e_\alpha\) is nonholonomic (equivalently, anholonomic) because, in

\[\text{in this work, the Einstein’s summation rule on repeating "upper–lower" indices will be applied if the contrary will not be stated}\]
general, it satisfies certain anholonomy conditions
\[ e_\alpha e_\beta - e_\beta e_\alpha = W^\gamma_{\alpha\beta} e_\gamma \] (2)

with nontrivial anholonomy coefficients \( W^\gamma_{\alpha\beta}(u) \). For \( W^\gamma_{\alpha\beta} = 0 \), we get holonomic frames: for instance, if we fix a local coordinate basis, \( e_\alpha = \partial_\alpha \).

Denoting the covariant derivative along a vector field \( X = X^\alpha e_\alpha \) as \( D_X = X^\alpha \nabla_\alpha \), we can define the torsion
\[ T(X,Y) \doteq D_X Y - D_Y X - [X,Y], \] (3)

and the curvature
\[ R(X,Y)Z \doteq D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z, \] (4)
tensors of connection \( D \), where we use "by definition" symbol "\( \doteq \)" and \( [X,Y] \doteq XY - YX \). The components \( T = \{ T^\alpha_{\beta\gamma} \} \) and \( R = \{ R^\alpha_{\beta\gamma\tau} \} \) are computed by introducing \( X \rightarrow e_\alpha, Y \rightarrow e_\beta, Z \rightarrow e_\gamma \) into respective formulas (3) and (4).

The Ricci tensor is constructed \( \text{Ric}(D) = \{ R_{\beta\gamma} \doteq R^\alpha_{\beta\gamma\alpha} \} \). The scalar curvature \( R \) is by definition the contraction with \( g^{\alpha\beta} \) (being the inverse to the matrix \( g_{\alpha\beta} \)), \( R \doteq g^{\alpha\beta} R_{\alpha\beta} \), and the Einstein tensor is \( \mathcal{E} = \{ E_{\alpha\beta} \doteq R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \} \). The vacuum (source–free) Einstein equations are
\[ \mathcal{E} = \{ E_{\alpha\beta} = R_{\alpha\beta} \} = 0. \] (5)

In general relativity theory, one chooses a connection \( D = \nabla \) which is uniquely defined by the coefficients of a metric, \( g_{\alpha\beta} \), following the conditions of metric compatibility, \( \nabla g = 0 \), and of zero torsion, \( \mathcal{T} = 0 \). This is the so–called Levi Civita connection \( \nabla = \nabla \). We shall respectively label its curvature tensor, Ricci tensor, scalar curvature and Einstein tensor in the form \( \mathcal{R} = \{ R^\alpha_{\beta\gamma\tau} \}, \ \text{Ric}(\nabla) = \{ R_{\alpha\beta} \doteq R^\alpha_{\beta\gamma\alpha} \}, \ \mathcal{R} \doteq g^{\alpha\beta} R_{\alpha\beta} \) and \( \mathcal{E} = \{ E_{\alpha\beta} \} \).

Modern gravity theories consider extra dimensions and connections with nontrivial torsion. For instance, in string gravity [35, 36], the torsion coefficients are induced by the so–called anti–symmetric \( H \)–fields and contain additional information about additional interactions in low–energy string limit. A more special class of gravity interactions are those with effective torsion when such fields are induced as a nonholonomic frame effect in a unique form [39] by prescribing a nonholonomic distribution [40, 41], defining a nonlinear connection structure, \( N \)–connection, on a (pseudo) Riemannian
manifold $V$ enabled with a metric structure (A.9). Such spaces with local fibred structure are called nonholonomic (in more special cases, when the nonholonomy is defined by a $N$–connection structures, the manifolds are called $N$–anholonomic) [17, 19]. The $N$–anholonomic spaces can be described in equivalent form by two linear connections $\nabla$ (A.16) and $\hat{\nabla}$ (A.17), both metric compatible and completely stated by a metric (A.8). As a matter of principle, the general relativity theory can be formulated in terms of both connections, $\nabla$ and $\hat{\nabla}$; the last variant being with nonholonomic constraints on geometrical objects. One must be emphasized that the standard approach follows the formulation of gravitational field equations just for the Einstein tensor $\mathcal{E}$ for $\nabla$, which, in general, is different from the Einstein tensor $\hat{\mathcal{E}}$ for $\hat{\nabla}$.

A surprising thing found in our works is that for certain classes of generic off–diagonal metric ansatz (A.9) it is possible to construct exact solutions in general form by using the connection $\hat{\nabla}$ but not the connection $\nabla$. Here we note that having defined certain integral varieties for a first class of linear connections we can impose some additional constraints and generate solutions for a class of Levi Civita connections, for instance, in the Einstein and string, or Finsler like, generalizations of gravity. Following a geometric $N$–adapted formalism (the so–called anholonomic frame method), such solutions were constructed and studied in effective noncommutative gravity [18], various locally anisotropic (Finsler like and more general ones) extensions of the Einstein and Kaluza–Klein theory, in string an brane gravity [20, 21, 23] and for Lagrange–Fedosov manifolds [34], see a summary in [19].

The anholonomic frame method also allows us to construct exact solutions in general relativity: One defines a more general class of solutions for $\hat{\nabla}$ and then imposes certain subclasses of nonholonomic constraints when such solutions solve the four dimensional Einstein equations for $\nabla$. Here we note that by nonholonomic deformations we were able to study nonholonomic Ricci flows of certain classes of physically valuable exact solutions like solitonic pp–waves [25] and Taub NUT spaces [26, 27]. In this work, we develop the approach by applying new group methods.

Certain nontrivial limits to the vacuum Einstein gravity can be selected if we impose on the nonholonomic structure such constraints when

$$E = \mathcal{E}$$ (6)

even, in general, $D \neq \nabla$. We shall consider such conditions when $D$ and $\nabla$

---

3In this work, we shall use only $\nabla$ and the canonical $d$–connection $\hat{D}$ and, for simplicity, we shall omit "hat" writing $D$ if that will not result in ambiguities.
have the same components with respect to certain preferred bases and the equality (6) can be satisfied for some very general classes of metric ansatz.

We shall use left–up labels "\(\circ\)" or "\(\lambda\)" for a metric,

\[
\circ g = \circ g_{\alpha\beta} e^\alpha \otimes e^\beta \quad \text{or} \quad \lambda g = \lambda g_{\alpha\beta} e^\alpha \otimes e^\beta
\]

being (correspondingly) a solution of the vacuum Einstein (or with cosmological constant) equations \(\mathcal{E} = 0\) or of the Einstein equations with a cosmological constant \(\lambda\), \(R_{\alpha\beta} = \lambda g_{\alpha\beta}\), for a linear connection \(D\) with possible torsion \(T \neq 0\). In order to emphasize that a metric is a solution of the vacuum Einstein equations, in any dimension \(n + m \geq 3\), for the Levi Civita connection \(\nabla\), we shall write

\[
\circ g = \circ g_{\alpha\beta} e^\alpha \otimes e^\beta \quad \text{or} \quad \lambda g = \lambda g_{\alpha\beta} e^\alpha \otimes e^\beta,
\]

where the left–low label "\(\bar{}\)" will distinguish the geometric objects for the Ricci flat space defined by a Levi Civita connection \(\nabla\).

Finally, in this section, we note that we shall use "boldface" symbols, for instance, if \(\lambda g = \lambda g_{\alpha\beta} e^\alpha \otimes e^\beta\) defines a nonholonomic Einstein space as a solution of

\[
R_{\alpha\beta} = \lambda g_{\alpha\beta}
\]

for the canonical \(\bar{D}\)-connection \(\bar{D}\).

2.2 Evolution equations for nonholonomic Ricci flows

The normalized (holonomic) Ricci flows \([3, 4, 5, 6]\) for a family of metrics \(g_{\alpha\beta}(\chi) = g_{\alpha\beta}(u^\alpha, \chi)\), parametrized by a real parameter \(\chi\), with respect to the coordinate base \(\partial_\alpha = \partial/\partial u^\alpha\), are described by the equations

\[
\frac{\partial}{\partial \chi} g_{\alpha\beta} = -2 R_{\alpha\beta} + 2 r g_{\alpha\beta},
\]

where the normalizing factor \(r = \int RdV/dV\) is introduced in order to preserve the volume \(V\).

\[\text{footnote text}\]

\[\text{footnote text}\]
For N–anholonomic Ricci flows, the coefficients $g_{\alpha \beta}$ are parametrized in the form (A.9), see proofs and discussion in Refs. [13, 25, 26, 27]. With respect to N–adapted frames (A.4) and (A.5), the Ricci flow equations (8), redefined for $\nabla \rightarrow \hat{D}$ and, respectively, $\bar{\Gamma}_{\alpha \beta} \rightarrow \hat{\Gamma}_{\alpha \beta}$ are

$$\frac{\partial}{\partial \chi} g_{ij} = 2 \left[ N_i^a N_j^b \left( \hat{R}_{ab} - \lambda g_{ab} \right) - \hat{R}_{ij} + \lambda g_{ij} \right] - g_{cd} \frac{\partial}{\partial \chi} (N_i^c N_j^d), \quad (9)$$

$$\frac{\partial}{\partial \chi} g_{ab} = -2 \left( \hat{R}_{ab} - \lambda g_{ab} \right), \quad (10)$$

$$\hat{R}_{ia} = 0 \text{ and } \hat{R}_{ai} = 0, \quad (11)$$

where $\lambda = r/5$ the Ricci coefficients $\hat{R}_{ij}$ and $\hat{R}_{ab}$ are computed with respect to coordinate coframes. The equations (11) constrain the nonholonomic Ricci flows to result in symmetric metrics.

Nonholonomic deformations of geometric objects (and related systems of equations) on a N–anholonomic manifold $\mathbf{V}$ are defined for the same metric structure $g$ by a set of transforms of arbitrary frames into N–adapted ones and of the Levi Civita connection $\nabla$ into the canonical d–connection $\hat{D}$, locally parametrized in the form

$$\partial_\alpha = (\partial_i, \partial_a) \rightarrow e_\alpha = (e_i, e_a); \quad g_{\alpha \beta} \rightarrow [g_{ij}, g_{ab}, N_1^a]; \quad \bar{\Gamma}_{\alpha \beta} \rightarrow \hat{\Gamma}_{\alpha \beta}.$$

A rigorous proof for nonholonomic evolution equations is possible following a N–adapted variational calculus for the Perelman’s functionals presented in Refs. [14]. For a five dimensional space with diagonal d–metric ansatz (A.10), when $g_{ij} = \text{diag}[\pm 1, g_2, g_3]$ and $g_{ab} = \text{diag}[g_4, g_5]$, we considered [25] the nonholonomic evolution equations

$$\frac{\partial}{\partial \chi} g_{ii} = -2 \left[ \hat{R}_{ii} - \lambda g_{ii} \right] - g_{cc} \frac{\partial}{\partial \chi} (N_i^c)^2, \quad (12)$$

$$\frac{\partial}{\partial \chi} g_{aa} = -2 \left( \hat{R}_{aa} - \lambda g_{aa} \right), \quad (13)$$

$$\hat{R}_{\alpha \beta} = 0 \text{ for } \alpha \neq \beta, \quad (14)$$

for a set of Riemannian metrics $g_{\alpha \beta}(\chi)$ and corresponding Ricci tensors $\hat{R}_{\alpha \beta}(\chi)$ parametrized by a real $\chi$ (we shall underline symbols or indices in order to emphasize that certain geometric objects/ equations are given with the components defined with respect to a coordinate basis). For our further purposes, on generalized Riemann–Finsler spaces, it is convenient to use a different system of denotations than those considered by R. Hamilton or Grisha Perelman on holonomic Riemannian spaces.

In Refs. [13, 24], we discuss this problem related to the fact that the tensor $\hat{R}_{\alpha \beta}$ is not symmetric which results, in general, in Ricci flows of nonsymmetric metrics.
with the coefficients defined with respect to N–adapted frames \((A.4)\) and \((A.5)\). This system can be transformed into a similar one, like \((9)\)–\((11)\), by nonholonomic deformations.

3 Off–Diagonal Exact Solutions

We consider a five dimensional (5D) manifold \(V\) of necessary smooth class and conventional splitting of dimensions \(\dim V = n + m\) for \(n = 3\) and \(m = 2\). The local coordinates are labelled in the form \(u^\alpha = (x^i, y^a) = (x^1, x^i, y^4 = v, y^5)\), for \(i = 1, 2, 3\) and \(\tilde{i} = 2, 3\) and \(a, b, \ldots = 4, 5\). Any coordinates from a set \(u^\alpha\) can be a three dimensional (3D) space, time, or extra dimension (5th) one. Ricci flows of geometric objects will be parametrized by a real \(\chi\).

3.1 Off–diagonal ansatz for Einstein spaces and Ricci flows

The ansatz of type \((A.10)\) is parametrized in the form

\[
\begin{aligned}
g &= g_1 dx^1 \otimes dx^1 + g_2(x^2, x^3)dx^2 \otimes dx^2 + g_3(x^2, x^3)dx^3 \otimes dx^3 \\
&\quad + h_4(x^k, v) \delta v \otimes \delta v + h_5(x^k, v) \delta y \otimes \delta y, \\
\delta v &= dv + w_i(x^k, v) dx^i, \\
\delta y &= dy + n_i(x^k, v) dx^i
\end{aligned}
\]  

(15)

with the coefficients defined by some necessary smooth class functions

\[
\begin{aligned}
g_1 &= \pm 1, g_{2,3} = g_{2,3}(x^2, x^3), h_{4,5} = h_{4,5}(x^i, v), \\
w_i &= w_i(x^i, v), n_i = n_i(x^i, v).
\end{aligned}
\]

The off–diagonal terms of this metric, written with respect to the coordinate dual frame \(du^\alpha = (dx^i, dy^a)\), can be redefined to state a N–connection structure \(N = [N^4_i = w_i(x^k, v), N^5_i = n_i(x^k, v)]\) with a N–elongated co–frame \((A.5)\) parametrized as

\[
\begin{aligned}
e^1 &= dx^1, e^2 = dx^2, e^3 = dx^3, \\
e^4 &= \delta v = dv + w_i dx^i, e^5 = \delta y = dy + n_i dx^i.
\end{aligned}
\]

(16)

This coframe is dual to the local basis

\[
\begin{aligned}
e_i &= \frac{\partial}{\partial x^i} - w_i(x^k, v) \frac{\partial}{\partial v} - n_i(x^k, v) \frac{\partial}{\partial y^5}, \\
e_4 &= \frac{\partial}{\partial v}, e_5 = \frac{\partial}{\partial y^5}.
\end{aligned}
\]

(17)
We emphasize that the metric (15) does not depend on variable \( y^5 \), i.e., it possesses a Killing vector \( e_5 = \partial/\partial y^5 \), and distinguishes the dependence on the so-called “anisotropic” variable \( y^4 = v \).

The above considered ansatz and formulas can be generalized in order to model Ricci flows,

\[
\chi g = g_1 dx^1 \otimes dx^1 + g_2(x^2, x^3, \chi) dx^2 \otimes dx^2 + g_3(x^2, x^3, \chi) dx^3 \otimes dx^3
\]

\[+ h_4 \left( x^k, v, \chi \right) \chi \delta v \otimes \chi \delta v + h_5 \left( x^k, v, \chi \right) \chi \delta y \otimes \chi \delta y,\]

\[
\chi \delta v = dv + w_i \left( x^k, v, \chi \right) dx^i, \quad \chi \delta y = dy + n_i \left( x^k, v, \chi \right) dx^i \tag{18}
\]

with corresponding flows for \( N \)-adapted bases,

\[
e_{\alpha} = (e_i, e_\alpha) \rightarrow x e_{\alpha} = (x e_i, e_\alpha) = e_{\alpha}(\chi) = (e_i(\chi), e_\alpha),
\]

\[
e^\alpha = (e^i, e^\alpha) \rightarrow x e^\alpha = (e^i, x e^\alpha) = e^\alpha(\chi) = (e^i, e^\alpha(\chi))
\]

defined by \( w_i \left( x^k, v \right) \rightarrow w_i \left( x^k, v, \lambda \right), n_i \left( x^k, v \right) \rightarrow n_i \left( x^k, v, \lambda \right) \) in (17), (16).

Computing the components of the Ricci and Einstein tensors for the metric (18) (see main formulas in Appendix and details on tensors components’ calculus in Refs. [18] [19]), one proves that the corresponding family of Ricc tensors for the canonical d–connection with respect to \( N \)-adapted frames are compatible with the sources (they can be any matter fields, string corrections, Ricci flow parameter derivatives of metric, ...)

\[
\Upsilon^\alpha_\beta = \left[ \Upsilon^1_1 = \Upsilon_2 + \Upsilon_4, \Upsilon^2_2 = \Upsilon_2(x^2, x^3, v, \chi), \Upsilon^3_3 = \Upsilon_2(x^2, x^3, v, \chi), \Upsilon^4_4 = \Upsilon_4(x^2, x^3, \chi), \Upsilon^5_5 = \Upsilon_4(x^2, x^3, \chi) \right] \tag{19}
\]

transform into this system of partial differential equations:

\[
R^2_2 = R^3_3(\chi) \tag{20}
\]

\[
= \frac{1}{2g_2g_3} g^*_2 g^*_3 + \frac{\left( g_3^* \right)^2}{2g_2} - g^*_3 \frac{g^*_2}{2g_3} + \left( \frac{g^*_2}{2g_2} \right)^2 - g^*_2 \frac{g^*_2}{2g_2} - g^*_2 = -\Upsilon_4(x^2, x^3, \chi),
\]

\[
S^4_i = S^5_5(\chi) = \frac{1}{2h_5 h_4} \left[ h^*_5 \left( \ln \sqrt{|h_4 h_5|} \right)^* - h^*_5 \right] = -\Upsilon_2(x^2, x^3, v, \chi), \tag{21}
\]

\[
R^4_i = -w_i(\chi) \frac{\beta(\chi)}{2h_5(\chi)} - \frac{\alpha_i(\chi)}{2h_5(\chi)} = 0, \tag{22}
\]

\[
R^5_i = -\frac{h_5(\chi)}{2h_4(\chi)} \left[ n^*_i(\chi) + \gamma(\chi) n^*_i(\chi) \right] = 0, \tag{23}
\]

where, for \( h^*_{4,5} \neq 0, \)

\[
\alpha_i(\chi) = h^*_5(\chi) \partial_i \phi(\chi), \quad \beta(\chi) = h^*_5(\chi) \phi(\chi), \tag{24}
\]

\[
\gamma(\chi) = \sqrt{\frac{3h^*_5(\chi)}{2h_5(\chi)} - \frac{h^*_4(\chi)}{h_4(\chi)}}, \quad \phi(\chi) = \ln \left| \frac{h^*_5(\chi)}{\sqrt{|h_4(\chi)h_5(\chi)|}} \right|. \tag{25}
\]
when the necessary partial derivatives are written in the form $a^* = \partial a / \partial x^2$, $a' = \partial a / \partial x^3$, $a^* = \partial a / \partial v$. In the vacuum case, we must consider $\Upsilon_{2,4} = 0$.

We note that we use a source of type (19) in order to show that the anholonomic frame method can be applied also for non-vacuum configurations, for instance, when $\Upsilon_2 = \lambda_2 = \text{const}$ and $\Upsilon_4 = \lambda_4 = \text{const}$, defining local anisotropies generated by an anisotropic cosmological constant, which in its turn, can be induced by certain ansatz for the so-called $H$–field (absolutely antisymmetric third rank tensor) in string theory [18, 19]. We note that the off–diagonal gravitational interactions and Ricci flows can model locally anisotropic configurations even if $\lambda_2 = \lambda_4$, or both values vanish.

Summarizing the results for an ansatz (15) with arbitrary signatures $\epsilon_\alpha = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5)$ (where $\epsilon_\alpha = \pm 1$) and $h_4^* \neq 0$ and $h_5^* \neq 0$, for a fixed value of $\chi$, one proves, see details in [18, 19], that any off–diagonal metric

$$\delta v = dv + w_k (x^i, v) dx^k, \delta y^5 = dy^5 + n_k (x^i, v) dx^k, \ (26)$$

with the coefficients being of necessary smooth class and the indices with "hat" running the values $\hat{i}, \hat{j}, \ldots = 2, 3$, where $g_k (x^i)$ is a solution of the 2D equation (20) for a given source $\Upsilon_4 (x^i)$,

$$\varsigma (x^i, v) = \varsigma_0 (x^i) - \frac{\epsilon_4}{8} h_0^2 (x^i) \int \Upsilon_2 (x^k, v) f^* (x^i, v) [f (x^i, v) - f_0 (x^i)] \, dv,$$

and the N–connection coefficients $N^4_i = w_i (x^k, v), N^5_i = n_i (x^k, v)$ are computed following the formulas

$$w_i = - \frac{\partial \varsigma (x^k, v)}{\varsigma^* (x^k, v)} \ (27)$$

$$n_k = n_k[1] (x^i) + n_k[2] (x^i) \int \frac{[f^* (x^i, v)]^2}{[f (x^i, v) - f_0 (x^i)]^3} \varsigma (x^i, v) \, dv, \ (28)$$

define an exact solution of the system of Einstein equations (7). It should be emphasized that such solutions depend on arbitrary functions $f (x^i, v)$, with $f^* \neq 0, f_0 (x^i), h_0^2 (x^i), \varsigma_0 (x^i), n_k[1] (x^i), n_k[2] (x^i)$ and $\Upsilon_2 (x^k, v), \Upsilon_4 (x^i)$.
Such values for the corresponding signatures $\epsilon_\alpha = \pm 1$ have to be stated by
certain boundary conditions following some physical considerations.

The ansatz of type (15) with $h^*_4 = 0$ but $h^*_5 \neq 0$ (or, inversely, $h^*_4 \neq 0$ but
$h^*_5 = 0$) consist more special cases and request a bit different methods for
constructing exact solutions. Nevertheless, such solutions are also generic
off–diagonal and they may be of substantial interest (the length of paper
does not allow us to include an analysis of such particular cases).

3.2 Generalization of solutions for Ricci flows

For families of solutions parametrized by $\chi$, we consider flows of the
generating functions, $g_2(x^i, \chi)$, or $g_3(x^i, \chi)$, and $f(x^i, v, \chi)$, and various
types of integration functions and sources, for instance, $n_{k[1]}(x^i, \chi)$ and
$n_{k[2]}(x^i, \chi)$ and $\Upsilon_2(x^k, v, \chi)$, respectively, in formulas (27) and (28). Let us
analyze an example of exact solutions of equations (12)–(14):

We search a class of solutions of with

$g_2 = \epsilon_2 \varpi(x^2, x^3, \chi),
g_3 = \epsilon_3 \varpi(x^2, x^3, \chi),
h_4 = h_4(x^2, x^3, v),
h_5 = h_5(x^2, x^3, v),$

for a family of ansatz (18) with any prescribed signatures $\epsilon_\alpha = \pm 1$
and non–negative functions $\varpi$ and $h$. Following a tensor calculus, adapted to
the N–connection, for the canonical d–connection, we express the integral
variety for a class of nonholonomic Ricci flows as

$$\epsilon_2 (\ln |\varpi|)^{**} + \epsilon_3 (\ln |\varpi|)^{''} = 2\lambda - h_5 \partial_\chi (n_2)^2,$$
$$h_4 = h_{\varsigma_4}$$

for

$$\varsigma_4(x^2, x^3, v) = \varsigma_{4[0]}(x^2, x^3) - \frac{\lambda}{4} \int \frac{h_5}{h^*_5} dv$$
$$\sqrt{|h|} = h_{[0]}(x^i) \left( \sqrt{|h_5(x^2, x^3, v)|} \right)^s,$$

\footnote{Our classes of solutions depending on integration functions are more general than
those for diagonal ansatz depending, for instance, on one radial like variable like in the
case of the Schwarzschild solution (when the Einstein equations are reduced to an effective
nonlinear ordinary differential equation, ODE). In the case of ODE, the integral varieties
depend on integration constants to be defined from certain boundary/ asymptotic and
symmetry conditions, for instance, from the constraint that far away from the horizon the
Schwarzschild metric contains corrections from the Newton potential. Because our ansatz
transforms (7) in a system of nonlinear partial differential equations transforms, the
solutions depend not on integration constants but on integration functions.

\footnote{Similar computations are given in [13] and Chapter 10 of [19]}
and, for \( \varphi = -\ln \left| \frac{\sqrt{h_4h_5}}{|h_5^*|} \right| \),

\[
\begin{align*}
   w_1 &= 0, w_2 = (\varphi^*)^{-1}\varphi^*, w_3 = (\varphi^*)^{-1}\varphi', \\
n_1 &= 0, n_2 = n_3 = n_{1[1]}(x^2, x^3, \chi) + n_{2[2]}(x^2, x^3, \chi) \int dv \frac{h_4}{\left( \sqrt{|h_5|} \right)^3},
\end{align*}
\]

where the partial derivatives are denoted in the form \( \varphi^* = \partial \varphi/\partial x^2 \), \( \varphi' = \partial \varphi/\partial v \), \( \varphi = \partial \varphi/\partial x^2 \), and arbitrary \( h_5 \) when \( h_5^* \neq 0 \).

For \( \lambda = 0 \), we shall consider \( \varsigma_4[0] = 1 \) and \( h_4[0](x^i) = \text{const} \) in order to solve the vacuum Einstein equations. There is a class of solutions when

\[
h_5 \int dv \frac{h_4}{\left( \sqrt{|h_5|} \right)^3} = C(x^2, x^3),
\]

for a function \( C(x^2, x^3) \). This is compatible with the condition (30), and we can chose such configurations, for instance, with \( n_{1[1]} = 0 \) and any \( n_{2[2]}(x^2, x^3, \chi) \) and \( \varpi(x^2, x^3, \chi) \) solving the equation (29).

Putting together (29)–(31), we get a class of solutions of the system (12)–(14) (the equations being expressed equivalently in the form (20)–(23)) for nonholonomomorphic Ricci flows of metrics of type (18),

\[
\begin{align*}
   \gamma_g & = \epsilon_1 dx^1 \otimes dx^1 + \varpi(x^2, x^3, \chi) \left[ \epsilon_2 dx^2 \otimes dx^2 + \epsilon_3 dx^3 \otimes dx^3 \right] + h_4 \left( x^2, x^3, v \right) \delta v \otimes \delta v + h_5 \left( x^2, x^3, v \right) \lambda \delta y \otimes \lambda \delta y, \\
   \delta v & = dv + w_2 \left( x^2, x^3, v \right) dx^2 + w_3 \left( x^2, x^3, v \right) dx^3, \\
   \lambda \delta y & = dy + n_2 \left( x^2, x^3, v, \chi \right) [dx^2 + dx^3].
\end{align*}
\]

Such solutions describe in general form the Ricci flows of nonholonomic Einstein spaces constrained to relate in a mutually compatible form the evolution of horizontal part of metric, \( \varpi(x^2, x^3, \chi) \), with the evolution of N–connection coefficients \( n_2 = n_3 = n_2 \left( x^2, x^3, v, \chi \right) \). We have to impose certain boundary/ initial conditions for \( \chi = 0 \), beginning with an explicit solution of the Einstein equations, in order to define the integration functions and state an evolution scenario for such classes of metrics and connections.

### 3.3 4D and 5D Einstein foliations and Ricci flows

The method of constructing 5D solutions can be restricted to generate 4D nonholonomic configurations and generic off–diagonal solutions in general relativity. In order to consider reductions \( 5D \to 4D \) for the ansatz (15), we can eliminate from the formulas the variable \( x^1 \) and consider a 4D space \( V^4 \) (parametrized by local coordinates \( (x^2, x^3, v, y^5) \)) trivially embedded
into a 5D spacetime $\mathbf{V}$ (parametrized by local coordinates $(x^1, x^2, x^3, v, y^5)$ with $g_{11} = \pm 1, g_{1\alpha} = 0, \alpha = 3, 4, 5$). In this case, there are possible 4D conformal and anholonomic transforms depending only on variables $(x^2, x^3, v)$ of a 4D metric $g_{\alpha\beta}(x^2, x^3, v)$ of arbitrary signature. To emphasize that some coordinates are stated just for a 4D space we might use "hats" on the Greek indices, $\hat{\alpha}, \hat{\beta}, ...$ and on the Latin indices from the middle of the alphabet, $\hat{i}, \hat{j}, ... = 2, 3$; local coordinates on $\mathbf{V}^4$ are parametrized $u^\alpha = (x^i, y^a) = (x^2, x^3, y^4 = v, y^5)$, for $a, b, ... = 4, 5$. The ansatz

$$g = g_2 \, dx^2 \otimes dx^2 + g_3 \, dx^3 \otimes dx^3 + h_4 \, \delta v \otimes \delta v + h_5 \, \delta y^5 \otimes \delta y^5,$$

(33)

is written with respect to the anholonomic co–frame $(dx^i, \delta v, \delta y^5)$, where

$$\delta v = dv + w_i \, dx^i$$

and

$$\delta y^5 = dy^5 + n_i \, dx^i$$

(34)

is the dual of $(\delta_i, \partial_4, \partial_5)$, for

$$\delta_i = \partial_i + w_i \, \partial_4 + n_i \, \partial_5,$$

(35)

and the coefficients are necessary smoothly class functions of type:

$$g_j^i = g_j^i(x^\kappa), h_{4,5} = h_{4,5}(x^\kappa, v), w_i^\alpha = w_i^\alpha(x^\kappa, v), n_i^\alpha = n_i^\alpha(x^\kappa, v); \hat{i}, \hat{k} = 2, 3.$$

In the 4D case, a source of type (19) should be considered without the component $\Upsilon_{11}$ in the form

$$\Upsilon_{\hat{\alpha}\beta} = \text{diag}[\Upsilon_2^2 = \Upsilon_3^3 = \Upsilon_2(x^\kappa, v), \Upsilon_4^i = \Upsilon_5^5 = \Upsilon_4(x^\kappa)].$$

(36)

The Einstein equations with sources of type (36) for the canonical d–connection (A.16) defined by the ansatz (33) transform into a system of nonlinear partial differential equations very similar to (20)–(23). The difference for the 4D equations is that the coordinate $x^1$ is not contained into the equations and that the indices of type $i, j, .. = 1, 2, 3$ must be changed into the corresponding indices $\hat{i}, \hat{j}, .. = 2, 3$. The generated classes of 4D solutions are defined almost by the same formulas (26), (27) and (28).

Now we describe how the coefficients of an ansatz (33) defining an exact vacuum solution for a canonical d–connection can be constrained to generate
a vacuum solution in Einstein gravity: We start with the conditions (A.20) written (for our ansatz) in the form

\[ \frac{\partial h_4}{\partial x^k} - w^*_k h_4 - 2 w^*_k h_4 = 0, \]  
\[ \frac{\partial h_5}{\partial x^k} - w^*_k h_5^* = 0, \]  
\[ n^*_k h_5 = 0. \]  

These equations for nontrivial values of \( w^*_k \) and \( n^*_k \) constructed for some defined values of \( h_4 \) and \( h_5 \) must be compatible with the equations (21)–(23) for \( \Upsilon_2 = 0 \). One can be taken nonzero values for \( w^*_k \) in (22) if and only if \( \alpha \) = 0 because the the equation (21) imposes the condition \( \beta = 0 \). This is possible, for the sourceless case and \( h_5^* \neq 0 \), if and only if

\[ \phi = \ln \left| \frac{h_5^*}{\sqrt{|h_4 h_5|}} \right| = \text{const}, \]  

see formula (25). A very general class of solutions of equations (37), (38) and (40) can be represented in the form

\[ h_4 = \epsilon_4 h_0^2 (b^*)^2, h_5 = \epsilon_5 (b + b_0)^2, \]  
\[ w^*_k = (b^*)^{-1} \frac{\partial (b + b_0)}{\partial x^k}, \]  

where \( h_0 = \text{const} \) and \( b = b(x^k, v) \) is any function for which \( b^* \neq 0 \) and \( b_0 = b_0(x^k) \) is an arbitrary integration function.

The next step is to satisfy the integrability conditions (A.18) defining a foliated spacetimes provided with metric and N–connection and d–connection structures [18, 19, 34] (we note that (pseudo) Riemannian foliations are considered in a different manner in Ref. [17]) for the so–called Schouten – Van Kampen and Vranceanu connections not subjected to the condition to generate Einstein spaces). It is very easy to show that there are nontrivial solutions of the constraints (A.18) which for the ansatz (33) are written in the form

\[ w'_2 - w'_3 + w_3 w'_2 - w_2 w'_3 = 0, \]  
\[ n'_2 - n'_3 + w_3 n'_2 - w_2 n'_3 = 0. \]  

We solve these equations for \( n^*_2 = n^*_3 = 0 \) if we take any two functions \( n_{2,3}(x^k) \) satisfying

\[ n'_2 - n'_3 = 0 \]  

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restricted to solve the equations defining vacuum exact solutions in general relativity if the coefficients are for signatures $\epsilon$ local coordinates when a 2D metric can be written in a conformally flat form, i.e., we can choose such the conditions (6) of equality of the Einstein tensors. Here we note that any coefficients both for the Levi Civita and canonical $d$-connection being satisfied satisfying (43), the generic off-diagonal metric (33) possess the same coefficients. We conclude that for any sets of coefficients

$$h_4(x^\hat{k}, v), h_5(x^\hat{k}, v), w_\hat{k}(x^\hat{k}, v), n_{2,3}(x^\hat{k})$$

respectively generated by functions $b(x^\hat{k}, v)$ and $n_{\hat{k}[1]}(x^\hat{k})$, see (41), and satisfying (43), the generic off-diagonal metric (33) possess the same coefficients both for the Levi Civita and canonical $d$-connection being satisfied the conditions (6) of equality of the Einstein tensors. Here we note that any 2D metric can be written in a conformally flat form, i.e., we can chose such local coordinates when

$$g_2(dx^2)^2 + g_3(dx^3)^2 = e^{\psi(x)} \left[ \epsilon_2(dx^2)^2 + \epsilon_3(dx^3)^2 \right],$$

for signatures $\epsilon_\hat{k} = \pm 1$, in (33).

Summarizing the results of this section, we can write down the generic off-diagonal metric (it is a 4D dimensional reduction of (26))

$$\overset{\odot}{g} = e^{\psi(x^2,x^3)} \left[ \epsilon_2 \, dx^2 \otimes dx^2 + \epsilon_3 \, dx^3 \otimes dx^3 \right] + \epsilon_4 h_0^2 \left[ b^* (x^2, v) \right]^2 \, \delta v \otimes \delta v + \epsilon_5 b (x^2, x^3, v) - b_0(x^2, x^3) \right]^2 \, \delta y^5 \otimes \delta y^5,$$

$$\delta v = dv + w_2 (x^2, x^3, v) \, dx^2 + w_3 (x^2, x^3, v) \, dx^3,$$

$$\delta y^5 = dy^5 + n_2 (x^2, x^3) \, dx^2 + n_3 (x^2, x^3) \, dx^3,$$

defining vacuum exact solutions in general relativity if the coefficients are restricted to solve the equations

$$\epsilon_2 \psi^\bullet + \epsilon_3 \psi'' = 0,$$

$$w_2' - w_3 + w_3 w_2' - w_2 w_3' = 0,$$

$$n_2' - n_3 = 0,$$

for $w_2 = (b^*)^{-1} (b + b_0)^*$ and $w_3 = (b^*)^{-1} (b + b_0)'$, where, for instance, $n_3 = \partial_2 n_3$ and $n_2' = \partial_3 n_2$. 

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We can generalize (44) similarly to (26) in order to generate solutions for nontrivial sources (36). In general, they will contain nontrivial anholonomically induced torsions. Such configurations may be restricted to the case of Levi Civita connection by solving the constraints (37)–(39) in order to be compatible with the equations (21) and (22) for the coefficients $\alpha_i$ and $\beta$ computed for $h_5^* \neq 0$ and $\ln |h_5^*/\sqrt{h_4 h_5}| = \phi(x^2, x^3, v) \neq \text{const}$, see formula (23), resulting in more general conditions than (40) and (41).

Roughly speaking, all such coefficients are generated by any $h_4$ (or $h_5$) defined from (22) for prescribed values $h_5$ (or $h_5^*$) and $\Upsilon_2(x^k, v)$. The existence of a nontrivial matter source of type (36) does not change the condition $n^*_k = 0$, see (39), necessary for extracting torsionless configurations. This means that we have to consider only trivial solutions of (23) when two functions $n_k = n_k^*(x^2, x^3)$ are subjected to the condition (42). We conclude that this class of exact solutions of the Einstein equations with nontrivial sources (36), in general relativity, is defined by the ansatz

$$g = e^{\psi(x^2, x^3)} \left[ \epsilon_2 \, dx^2 \otimes dx^2 + \epsilon_3 \, dx^3 \otimes dx^3 \right] + h_4(x^2, x^3, v) \, \delta v \otimes \delta v + h_5(x^2, x^3, v) \, \delta y^5 \otimes \delta y^5,$$

$$\delta v = dv + w_2(x^2, x^3, v) \, dx^2 + w_3(x^2, x^3, v) \, dx^3,$$

$$\delta y^5 = dy^5 + n_2(x^2, x^3) \, dx^2 + n_3(x^2, x^3) \, dx^3,$$

where the coefficients satisfy the conditions

$$\epsilon_2 \psi^{\bullet \bullet} + \epsilon_3 \psi^{\prime \prime} = \Upsilon_2,$$

$$h_5^* \phi/h_4 h_5 = \Upsilon_2,$$

$$w_2^* - w_3 w_2^* - w_2 w_3^* = 0,$$

$$n_2^* - n_3^* = 0,$$

for $w_2 = \partial_2 \phi^*/\phi^{*}$, see (25), being compatible with (37) and (38), for given sources $\Upsilon_4(x^k)$ and $\Upsilon_2(x^k, v)$. We emphasize that the second equation in (47) relates two functions $h_4$ and $h_5$. In references [20, 21, 22, 23, 18], we investigated a number of configurations with nontrivial two and three-dimensional solitons, reductions to the Riccati or Abbel equation, defining off-diagonal deformations of the black hole, wormhole or Taub NUT spacetimes. Those solutions where constructed to be with trivial or nontrivial torsions but if the coefficients of the ansatz (46) are restricted to satisfy the conditions (47) in a compatible form with (37) and (38), for sure, such metrics will solve the Einstein equations for the Levi Civita connection. We
emphasize that the ansatz (46) defines Einstein spaces with a cosmological constant \( \lambda \) if we put \( \Upsilon_2 = \Upsilon_4 = \lambda \) in (47).

Let us formulate the conditions when families of metrics (46) subjected to the conditions (47) will define exact solutions of the Ricci flows of usual Einstein spaces (for the Levi Civita connection). We consider the ansatz

\[
\lambda g(\chi) = e^{\psi(x^2, x^3, \chi)} \left[ \epsilon_2 \, dx^2 \otimes dx^2 + \epsilon_3 \, dx^3 \otimes dx^3 \right] + h_4 \left( x^2, x^3, v, \chi \right) \, \delta v \otimes \delta v + h_5 \left( x^2, x^3, v, \chi \right) \, x^1 \delta y^5 \otimes x^1 \delta y^5,
\]

\[
\delta v = dv + w_2 \left( x^2, x^3, v \right) \, dx^2 + w_3 \left( x^2, x^3, v \right) \, dx^3,
\]

\[
x^1 \delta y^5 = dy^5 + n_2 \left( x^2, x^3, \chi \right) \left[ dx^2 + dx^3 \right],
\]

which is a subfamily of (32), when \( \omega = e^{\psi(x^2, x^3, \chi)} \) and \( n_2 = n_3 \) does not depend on variable \( v \) and the coefficients satisfy the conditions (29) and (30), when \( n_2 = 0 \) but \( n_1 \) can be nontrivial in (31), and (additionally)

\[
\epsilon_2 \psi^{\bullet\bullet}(\chi) + \epsilon_3 \psi''(\chi) = \lambda,
\]

\[
h_4' \phi/h_4 - h_5 = \lambda,
\]

\[
w_2' - w_3' + w_3 w_2' - w_2 w_3' = 0,
\]

\[
n_2'(\chi) - n_2^*(\chi) = 0,
\]

for \( w_7 = \partial \phi/\phi^* \), see (25), being compatible with (37) and (38), for given sources \( \Upsilon_4 = \lambda \) and \( \Upsilon_2 = \lambda \). The family of metrics (48) define a self-consistent evolution as a class of general solutions of the Ricci flow equations (12)–(14) transformed equivalently in the form (20)–(23). The additional constraints (49) define an integral subvariety (foliation) of (32) when the evolution is selected for the Levi Civita connection.

4 Nonholonomic and Parametric Transforms

Anholonomic deformations can be defined for any primary metric and frame structures on a spacetime \( V \) (as a matter of principle, the primary metric can be not a solution of the gravitational field equations). Such deformations always result in a target spacetime possessing one Killing vector symmetry if the last one is constrained to satisfy the vacuum Einstein equations for the canonical \( d \)-connection, or for the Levi Civita connection. For such target spacetimes, we can always apply a parametric transform and generate a set of generic off-diagonal solutions labelled by a parameter \( \theta \) (15.2). There are possible constructions when the anholonomic frame transforms are applied to a family of metrics generated by the parametric
method as new exact solutions of the vacuum Einstein equations, but such primary metrics have to be parametrized by certain type ansatz admitting anholonomic transforms to other classes of exact solutions. Additional constraints and parametrizations are necessary for generating exact solutions of holonomic or nonholonomic Ricci flow equations.

4.1 Deformations and frame parametrizations

Let us consider a \((n + m)\)-dimensional manifold (spacetime) \(V\), \(n \geq 2, m \geq 1\), enabled with a metric structure \(\tilde{g} = \tilde{g} \oplus \tilde{h}\) distinguished in the form

\[
\tilde{g} = \tilde{g}_i(u)(dx^i)^2 + \tilde{h}_a(u)(\tilde{c}^a)^2, \tag{50}
\]

\[
\tilde{c}^a = dy^a + \tilde{N}_i^a(u)dx^i.
\]

The local coordinates are parametrized \(u = (x, y) = \{u^\alpha = (x^i, y^a)\}\), for the indices of type \(i, j, k, \ldots = 1, 2, \ldots, n\) (in brief, horizontal, or \(h\)-indices/components) and \(a, b, c, \ldots = n + 1, n + 2, \ldots n + m\) (vertical, or \(v\)-indices/components). We suppose that, in general, the metric (50) is not a solution of the Einstein equations but can be nonholonomically deformed in order to generate exact solutions. The coefficients \(\tilde{N}_i^a(u)\) from (50) state a conventional \((n + m)\)-splitting \(\oplus \tilde{N}\) in any point \(u \in V\) and define a class of ‘\(N\)-adapted’ local bases

\[
\tilde{e}_\alpha = \left( \tilde{e}_i = \frac{\partial}{\partial x^i} - \tilde{N}_i^a(u) \frac{\partial}{\partial y^a}, e_a = \frac{\partial}{\partial y^a} \right), \tag{51}
\]

and local dual bases (co–frames) \(\tilde{c} = (c, \tilde{c})\), when

\[
\tilde{c}^a = \left( c^i = dx^i, \tilde{c}^b = dy^b + \tilde{N}_i^b(u)dx^i \right), \tag{52}
\]

for \(\tilde{c} \mid \tilde{e} = I\), i.e. \(\tilde{e}_\alpha \mid \tilde{c}^\beta = \delta_\alpha^\beta\), where the inner product is denoted by ‘\(\mid\)’ and the Kronecker symbol is written \(\delta_\alpha^\beta\). The frames (51) satisfy the nonholonomy (equivalently, anholonomy) relations

\[
\tilde{e}_\alpha \tilde{e}_\beta - \tilde{e}_\beta \tilde{e}_\alpha = \tilde{W}^\gamma_{\alpha\beta} \tilde{e}_\gamma
\]

with nontrivial anholonomy coefficients

\[
\tilde{W}^a_{\ ji} = -\tilde{W}^a_{\ ij} = \tilde{\Omega}^a_{\ ij} \equiv \tilde{e}_j \left( \tilde{N}^a_i \right) - \tilde{e}_i \left( \tilde{N}^a_j \right), \tag{53}
\]

\[
\tilde{W}^b_{\ ia} = -\tilde{W}^b_{\ ai} = e_a \left( \tilde{N}^b_i \right).
\]
A metric $\mathbf{g} = g \oplus_N h$ parametrized in the form
\begin{align*}
g &= g_i(u) (e^i)^2 + g_a(u) (c^a), \\
c^a &= dy^a + N^a_i(u) dx^i, \\
(54)
\end{align*}
is a nonholonomic transform (deformation), preserving the $(n+m)$–splitting, of the metric, $\mathbf{g} = \mathbf{\hat{g}} \oplus_N \mathbf{\hat{h}}$ if the coefficients of (50) and (54) are related by formulas
\begin{align*}
g_i = \eta_i(u) \mathbf{\hat{g}}_i, \quad h_a = \eta_a(u) \mathbf{\hat{h}}_a \quad \text{and} \quad N^a_i = \eta^a_i(u) \mathbf{\hat{N}}^a_i, \\
(55)
\end{align*}
where the summation rule is not considered for the indices of gravitational ‘polarizations’ $\eta^\alpha = (\eta_i, \eta_a)$ and $\eta^\beta = (\eta_i^2, \eta_i^3)$ in (55). For nontrivial values of $\eta_i^\alpha(u)$, the nonholonomic frames (51) and (52) transform correspondingly into
\begin{align*}
\mathbf{e}_\alpha &= \left( e_i = \frac{\partial}{\partial x^i} - N^a_i(u)e_a, \quad e_a = \frac{\partial}{\partial y^a} \right) \\
(56)
\end{align*}
and
\begin{align*}
\mathbf{c}^\alpha &= (c^i = dx^i, c^a = dy^a + N^a_i(u) dx^i) \\
(57)
\end{align*}
with the anholonomy coefficients $W_{\alpha\beta}^\gamma$ defined by $N^a_i$ (A.7).

We emphasize that in order to generate exact solutions, the gravitational ‘polarizations’ $\eta^\alpha = (\eta_i, \eta_a)$ and $\eta^\beta = (\eta_i^2, \eta_i^3)$ in (55) are not arbitrary functions but restricted in a such form that the values
\begin{align*}
\pm 1 &= \eta_1(u^\alpha) \mathbf{\hat{g}}_1(u^\alpha), \\
g_2(x^2, x^3) &= \eta_2(u^\alpha) \mathbf{\hat{g}}_2(u^\alpha), \quad g_3(x^2, x^3) = \eta_3(u^\alpha) \mathbf{\hat{g}}_3(u^\alpha), \\
h_4(x^i, v) &= \eta_4(u^\alpha) \mathbf{\hat{h}}_4(u^\alpha), \quad h_5(x^i, v) = \eta_5(u^\alpha) \mathbf{\hat{h}}_5(u^\alpha), \\
w_i(x^i, v) &= \eta_i^4(u^\alpha) \mathbf{\hat{N}}^4_i(u^\alpha), \quad n_i(x^i, v) = \eta_i^5(u^\alpha) \mathbf{\hat{N}}^5_i(u^\alpha), \\
(58)
\end{align*}
define an ansatz of type (26), or (44) (for vacuum configurations) and (46) for nontrivial matter sources $\Upsilon_2(x^2, x^3, v)$ and $\Upsilon_4(x^2, x^3)$.

Any nonholonomic deformation
\begin{align*}
\mathbf{\hat{g}} = \mathbf{\hat{g}} \oplus_N \mathbf{\hat{h}} \longrightarrow \mathbf{g} = g \oplus_N h \quad (59)
\end{align*}
can be described by two frame matrices of type (A.11),
\begin{align*}
\mathbf{\tilde{A}}_{\alpha \beta}(u) &= \begin{bmatrix}
\delta^i_1 & -\mathbf{\hat{N}}^b_i \delta^a_\beta \\
0 & \delta^a_\beta
\end{bmatrix}, \\
(60)
\end{align*}
generating the d–metric $\mathbf{\tilde{g}}_{\alpha \beta} = \mathbf{\tilde{A}}_{\alpha \beta} \mathbf{\tilde{A}}_{\beta \gamma} \mathbf{\tilde{g}}_{\gamma \gamma}$, see formula (A.11), and
\begin{align*}
\mathbf{A}_{\alpha \beta}(u) &= \begin{bmatrix}
\sqrt{|\eta_1|} \delta^i_1 & -\eta_i^a \mathbf{\hat{N}}^b_i \delta^a_\beta \\
0 & \sqrt{|\eta_a|} \delta^a_\beta
\end{bmatrix}, \\
(61)
\end{align*}
generating the d–metric $g_{\alpha\beta} = A^\alpha_\alpha A^\beta_\beta \tilde{g}_{\alpha\beta}$ \[58\].

If the metric and N–connection coefficients \[55\] are stated to be those from an ansatz \[26\] (or \[44\]), we should write $g = g \oplus g_N$ (or $\tilde{g} = g \oplus g_N$) and say that the metric $\tilde{g} = \tilde{g} \oplus \tilde{g}_N$ \[50\] was nonholonomically deformed in order to generate an exact solution of the Einstein equation for the canonical d–connection (or, in a restricted case, for the Levi Civita connection). In general, such metrics have very different geometrical and physical properties. Nevertheless, at least for some classes of ’small’ nonsingular nonholonomic deformations, it is possible to preserve a similar physical interpretation by introducing small polarizations of metric coefficients and deformations of existing horizons, not changing the singular structure of curvature tensors. Explicit examples are constructed in Ref. [28].

4.2 The Geroch transforms as parametric nonholonomic deformations

We note that any metric $g_{\alpha\beta}$ defining an exact solution of the vacuum Einstein equations can be represented in the form \[50\]. Then, any metric $\tilde{g}_{\alpha\beta}(\theta)$ \[15.2\] from a family of new solutions generated by the first type parametric transform can be written as \[54\] and related via certain polarization functions of type \[55\], in the parametric case depending on parameter $\theta$, i.e. $\eta_\alpha(\theta) = (\eta_i(\theta), \eta_a(\theta))$ and $\eta^\alpha_i(\theta)$. Roughly speaking, any parametric transform can be represented as a generalized class of anholonomic frame transforms additionally parametrized by $\theta$ and adapted to preserve the \((n + m)\)-splitting structure. The corresponding frame transforms \[15.5\] and \[15.6\] are parametrized, respectively, by matrices of type \[60\] and \[61\], also ”labelled” by $\theta$. Such nonholonomic parametric deformations

$$^\circ g = ^\circ g \oplus ^\circ h \rightarrow ^\circ g(\theta) = ^\circ g(\theta) \oplus ^\circ N(\theta) \tilde{h}(\theta) \quad \text{\[62\]}$$

are defined by the frame matrices,

$$^\circ A^\alpha_\alpha(u) = \begin{bmatrix} \delta^\lambda_\alpha - ^\circ N^b_i(u) \delta^\lambda_b \\ \delta^\alpha_b \end{bmatrix}, \quad \text{\[63\]}$$

generating the d–metric $^\circ g_{\alpha\beta} = A^\alpha_\alpha A^\beta_\beta \tilde{g}_{\alpha\beta}$ and

$$\tilde{A}^\alpha_\alpha(u, \theta) = \begin{bmatrix} \sqrt{\eta_i(u, \theta)} \delta^\lambda_i - \eta^\alpha_i(u, \theta) \eta^b_i(u, \theta) \delta^\lambda_b \\ \sqrt{\eta_i(u, \theta)} \delta^\alpha_b \end{bmatrix}, \quad \text{\[64\]}$$

It should be emphasized that such constructions are not trivial, for usual coordinate transforms, if at least one of the primary or target metrics is generic off–diagonal.
generating the d–metric \( \hat{g}_{\alpha\beta}(\theta) = \hat{A}_\alpha^\varphi \hat{A}_\beta^\beta \circ \hat{g}_{\alpha\beta} \). Using the matrices (63) and (64), we can compute the matrix of parametric transforms

\[
\hat{B}_\alpha^\alpha = \hat{A}_\alpha^\varphi \circ \hat{A}_\alpha'^\varphi,
\]

like in (B.7), but for "boldfaced" objects, where \( \hat{A}_\alpha^\alpha' \) is inverse to \( \hat{A}_\alpha'^\varphi \),

10 and define the target set of metrics in the form

\[
\hat{g}_{\alpha\beta} = \hat{B}_\alpha^\alpha(u, \theta) \hat{B}_\beta^\beta(u, \theta) \circ \hat{g}_{\alpha'\beta'}.
\]

There are two substantial differences from the case of usual anholonomic frame transforms (59) and the case of parametric deformations (62). The first one is that the metric \( \hat{g} \) was not constrained to be an exact solution of the Einstein equations like it was required for \( \hat{g} \). The second one is that even \( g \) can be restricted to be an exact vacuum solution, generated by a special type of deformations (58), in order to get an ansatz of type (44), an arbitrary metric from a family of solutions \( \hat{g}_{\alpha\beta}(\theta) \) will not be parametrized in a form that the coefficients will satisfy the conditions (45).

Nevertheless, even in such cases, we can consider additional nonholonomic frame transforms when \( \hat{g} \) is transformed into an exact solution and any particular metric from the set \( \{ \hat{g}_{\alpha\beta}(\theta) \} \) will be deformed into an exact solution defined by an ansatz (44) with additional dependence on \( \theta \).

By superpositions of nonholonomic deformations, we can parametrize a solution formally constructed following by the parametric method (from a primary solution depending on variables \( x^2, x^3 \)) in the form

\[
\hat{g}(\theta) = e^{\psi(x^2, x^3, \theta)} \left[ \epsilon_2 \, dx^2 \otimes dx^2 + \epsilon_3 \, dx^3 \otimes dx^3 \right]
+ \epsilon_4 b_0 \left[ b^* \left( x^1, v, \theta \right) \right]^2 \delta v \otimes \delta v
+ \epsilon_5 \left[ b(x^2, x^3, v, \theta) - b_0(x^2, x^3, \theta) \right]^2 \delta y^5 \otimes \delta y^5,
\]

\[
\delta v = dv + u_2(x^2, x^3, v, \theta) \, dx^2 + w_3(x^2, x^3, v, \theta) \, dx^3,
\]

\[
\delta y^5 = dy^5 + n_2(x^2, x^3, \theta) \, dx^2 + n_3(x^2, x^3, \theta) \, dx^3,
\]

with the coefficients restricted to solve the equations (45) but depending additionally on parameter \( \theta \).

\[
\epsilon_2 \psi''''(\theta) + \epsilon_3 \psi''(\theta) = 0,
\]

\[
w'_2(\theta) - w_3^* (\theta) + w_3 w_2^* (\theta) - w_2(\theta) w_3^* (\theta) = 0,
\]

\[
n'_2(\theta) - n_3^* (\theta) = 0,
\]

10 We use a "boldface" symbol because in this case the constructions are adapted to a \((n + m)\)-splitting.
for $w_2(\theta) = (b^*(\theta))^{-1}(b(\theta) + b_0(\theta))^\bullet$ and $w_3 = (b^*(\theta))^{-1}(b(\theta) + b_0(\theta))'$, where, for instance, $n_3^\bullet(\theta) = \partial_2 n_3(\theta)$ and $n_2' = \partial_3 n_2(\theta)$.

One should be noted that even, in general, any primary solution $\tilde{g}$ can not be parametrized as an ansatz (44), it is possible to define nonholonomic deformations to such type generic off–diagonal ansatz $\tilde{g}$ or any $\tilde{g}$, defined by an ansatz (50), which in its turn can be transformed into a metric of type (66) without dependence on $\theta$.

Finally, we emphasize that in spite of the fact that both the parametric and anholonomic frame transforms can be parametrized in very similar forms by using frame transforms there is a criteria distinguishing one from another: For a "pure" parametric transform, the matrix $\tilde{B}_\alpha^\alpha'(u, \theta)$ and related $\tilde{A}_\alpha^\beta$ and $\tilde{A}_\alpha^\alpha'$ are generated by a solution of the Geroch equations (B.4). If the "pure" nonholonomic deformations, or their superposition with a parametric transform, are introduced into consideration, the matrix $\tilde{A}_\alpha^\beta(u)$ (61), or its generalization to a matrix $\tilde{A}_\alpha^\alpha'(64)$, can be not derived only from solutions of (B.4). Such transforms define certain, in general, nonintegrable distributions related to new classes of Einstein equations.

### 4.3 Two parameter transforms of nonholonomic solutions

As a matter of principle, any first type parameter transform can be represented as a generalized anholonomic frame transform labelled by an additional parameter. It should be also noted that there are two possibilities to define superpositions of the parameter transforms and anholonomic frame deformations both resulting in new classes of exact solutions of the vacuum Einstein equations. In the first case, we start with a parameter transform and, in the second case, the anholonomic deformations are considered from the very beginning. The aim of this section is to examine such possibilities.

Firstly, let us consider an exact vacuum solution $\tilde{g}$ in Einstein gravity generated following the anholonomic frame method. Even it is generic off–diagonal and depends on various types of integration functions and constants, it is obvious that it possess at least a Killing vector symmetry because the metric does not depend on variable $y^5$. We can apply the first type parameter transform to such metric generated by anholonomic deformations (59).

If we work in a coordinate base with the coefficients of $\tilde{g}$ defined in the form $\tilde{g}_{\alpha\beta} = \tilde{g}_{\alpha\beta}$, we generate a set of exact solutions

\[
\tilde{g}_\alpha^\beta(\theta') = \tilde{B}_\alpha^\alpha'(\theta') \tilde{B}_\alpha^\beta(\theta') \tilde{g}_\alpha^\beta',
\]

\[\text{in our formulas we shall not point dependencies on coordinate variables if that will not result in ambiguities} \]

---

25
see (B.2), were the transforms (B.7), labelled by a parameter $\theta'$, are not adapted to a nonholonomic $(n+m)$–splitting. We can elaborate $N$–adapted constructions starting with an exact solution parametrized in the form (54), for instance, like \( \tilde{g}_{\alpha'\beta'} = A_{\alpha'}^{\alpha} A_{\beta'}^{\beta} \tilde{g}_{\alpha \beta} \), with $A_{\alpha'}^{\alpha}$ being of type (61) with coefficients satisfying the conditions (58). The target ‘boldface’ solutions are generated as transforms

\[
\tilde{g}_{\alpha\beta}(\theta') = \tilde{B}_{\alpha}^{\alpha'}(\theta') \tilde{B}_{\beta}^{\beta'}(\theta') \tilde{g}_{\alpha'\beta'},
\]

where

\[
\tilde{B}_{\alpha}^{\alpha'} = \tilde{A}_{\alpha}^{\alpha'} \circ A_{\alpha'}^{\alpha},
\]

like in (B.7), but for “boldfaced’ objects, the matrix $\tilde{A}_{\alpha}^{\alpha'}$ is inverse to

\[
\tilde{A}_{\alpha'}^{\alpha}(u) = \begin{bmatrix}
\sqrt{|\eta_i|} \delta_{i\hat{i}} & -\eta_{\hat{i}}^{\beta'} N_k^{\beta'} \delta_{\hat{i}a} \\
0 & \sqrt{|\eta_a|} \delta_{a\hat{a}}
\end{bmatrix},
\]

and there is considered the matrix

\[
\tilde{A}_{\alpha'}^{\alpha}(u, \theta') = \begin{bmatrix}
\sqrt{|\eta_i|} \tilde{\eta}_i(\theta') \delta_{i\hat{i}} & -\eta_{\hat{i}}^{\beta'} \tilde{\eta}_k(\theta') N_k^{\beta'} \delta_{\hat{i}a} \\
0 & \sqrt{|\eta_a|} \tilde{\eta}_a(\theta') \delta_{a\hat{a}}
\end{bmatrix},
\]

where $\tilde{\eta}_i(u, \theta'), \tilde{\eta}_a(u, \theta')$ and $\eta_{\hat{i}}^{\beta'}(u, \theta')$ are gravitational polarizations of type (55). Here it should be emphasized that even \( \tilde{g}_{\alpha\beta}(\theta') \) are exact solutions of the vacuum Einstein equations they can not be represented by ansatz of type (66), with $\theta \to \theta'$, because the mentioned polarizations were not constrained to be of type (58) and satisfy any conditions of type (67).

Now we prove that by using superpositions of nonholonomic and parameter transforms we can generate two parameter families of solutions. This is possible, for instance, if the metric \( \tilde{g}_{\alpha'\beta'} \) form (68), in its turn, was generated as an ansatz of type (66), from another exact solution \( \tilde{g}_{\alpha''\beta''} \). We write

\[
\tilde{g}_{\alpha'\beta'}(\theta) = \tilde{B}_{\alpha'}^{\alpha''}(u, \theta) \tilde{B}_{\beta'}^{\beta''}(u, \theta) \tilde{g}_{\alpha''\beta''},
\]

and define the superposition of transforms

\[
\tilde{g}_{\alpha\beta}(\theta', \theta) = \tilde{B}_{\alpha}^{\alpha'}(\theta') \tilde{B}_{\beta}^{\beta'}(\theta') \tilde{B}_{\alpha'}^{\alpha''}(\theta) \tilde{B}_{\beta'}^{\beta''}(\theta) \tilde{g}_{\alpha''\beta''}. \tag{69}
\]

It can be considered an iteration procedure of nonholonomic parameter transforms of type (69) when an exact vacuum solution of the Einstein
equations is related via a multi $\theta$–parameters frame map with another prescribed vacuum solution. Using anholonomic deformations, one introduces (into chains of such transforms) certain classes of metrics which are not exact solutions but nonholonomically deformed from, or to, some exact solutions.

4.4 Multi–parametric Einstein spaces and Ricci flows

Let us denote by $\theta = (k\theta = \theta', 2\theta, ..., \theta = \theta_0, ..., 1\theta_0)$ a chain of nonholonomic parametric transforms (it can be more general as (69), beginning with an arbitrary metric $g_{\alpha\beta'}$ resulting in a metric $g_{\alpha\beta}$). Any step of nonholonomic parametric and/or frame transforms are parametrize matrices of type (64), (65) or (68). Here, for simplicity, we consider two important examples when $g_{\alpha\beta}$ will generate solutions of the nonholonomic Einstein equations or Ricci flow equations.

4.4.1 Example 1:

We get a multi–parametric ansatz of type (15) with $h^*_4 \neq 0$ and $h^*_5 \neq 0$ if $g_{\alpha\beta}$ is of type

$$g_{\alpha\beta}\bigl(\theta\bigr) = \epsilon_1 \delta x^1 \otimes \delta x^1 + \epsilon_2 \delta x^2 \otimes \delta x^2 + \epsilon_3 \delta x^3 \otimes \delta x^3 + \epsilon_4 h^0_4 \left[f^*\left(\theta, x^i, v\right) - f_0\left(\theta, x^i\right)\right]^2 \delta v \otimes \delta v + \epsilon_5 \left[f\left(\theta, x^i, v\right) - f_0\left(\theta, x^i\right)\right]^2 \delta y^5 \otimes \delta y^5,$$

$$\delta v = dv + k\delta \left(\theta, x, v\right) dx^k,$$

$$\delta y^5 = dy^5 + k\delta \left(\theta, x, v\right) dx^k,$$

the indices with "hat" running the values $\hat{i}, \hat{j}, ..., = 2, 3$, where $g_{k\beta}\left(\theta, x^i\right)$ are multi–parametric families of solutions of the 2D equation (20) for given sources $\Upsilon_4\left(\theta, x^i\right)$,

$$\varsigma\left(\theta, x^i, v\right) = \varsigma_0\left(\theta, x^i\right) - \frac{\epsilon_4 h^2_0\left(\theta, x^i\right)}{8} \times \int \Upsilon_2\left(\theta, x^i, v\right) f^*\left(\theta, x^i, v\right) \left[f\left(\theta, x^i, v\right) - f_0\left(\theta, x^i\right)\right] dv,$$
and the N–connection \( N^4_i = w_i(\overrightarrow{\theta}, x^k, v) \), \( N^5_i = n_i(\overrightarrow{\theta}, x^k, v) \) computed
\[
\begin{align*}
w_i(\overrightarrow{\theta}, x^k, v) &= -\frac{\partial_i \varsigma(\overrightarrow{\theta}, x^k, v)}{\varsigma^*(\overrightarrow{\theta}, x^k, v)}, \quad (71) \\
n_k(\overrightarrow{\theta}, x^k, v) &= n_k[1](\overrightarrow{\theta}, x^i) + n_k[2](\overrightarrow{\theta}, x^i) \times \int \frac{[f^*(\overrightarrow{\theta}, x^i, v)]^2}{[f(\overrightarrow{\theta}, x^i, v) - f_0(x^i)]^3} \varsigma(\overrightarrow{\theta}, x^i, v) \, dv, \quad (72)
\end{align*}
\]
define an exact solution of the Einstein equations (7). We emphasize that such solutions depend on an arbitrary nontrivial function \( f(\overrightarrow{\theta}, x^i, v) \), with \( f^* \neq 0 \), integration functions \( f_0(\overrightarrow{\theta}, x^i), h_0^2(\overrightarrow{\theta}, x^i), \varsigma_0^0(\overrightarrow{\theta}, x^i), \)
\( n_k[1](\overrightarrow{\theta}, x^i), n_k[2](\overrightarrow{\theta}, x^i) \) and sources \( Y_2(\overrightarrow{\theta}, x^k, v), Y_4(\overrightarrow{\theta}, x^i) \). Such values for the corresponding signatures \( \epsilon_\alpha = \pm 1 \) have to be defined by certain boundary conditions and physical considerations. We note that formulas (71) and (72) state symbolically that at any intermediary step from the chain \( \overrightarrow{\theta} \) one construct the solution following the respective formulas (27) and (28). The final aim, is to get a set of metrics (70), parametrized by \( \overrightarrow{\theta} \), when for fixed values of \( \theta \)-parameters, we get solutions of type (26), for the vacuum Einstein equations for the canonical d–connection.

4.4.2 Example 2:

We consider a family of ansatz, labelled by a set of parameters \( \overrightarrow{\theta} \) and \( \chi \) (as a matter of principle, we can identify the Ricci flow parameter \( \chi \) with any \( \theta \) from the set \( \overrightarrow{\theta} \) considering that the evolution parameter is also related to the invariance of Killing equations, see Appendix [13],
\[
\begin{align*}
\chi g(\overrightarrow{\theta}, \chi) &= e^{\psi(\overrightarrow{\theta}, x^2, x^3, \chi)} \left[ \epsilon_2 \, dx^2 \otimes dx^2 + \epsilon_3 \, dx^3 \otimes dx^3 \right] + \epsilon_4 \, \delta v \otimes \delta v + \epsilon_5 \, \chi \delta y^5 \otimes \chi \delta y^5, \quad (73) \\
\delta v &= dv + w_2(\overrightarrow{\theta}, x^2, x^3, v) \, dx^2 + w_3(\overrightarrow{\theta}, x^2, x^3, v) \, dx^3, \\
\chi \delta y^5 &= dy^5 + n_2(\overrightarrow{\theta}, x^2, x^3, \chi) \, [dx^2 + dx^3],
\end{align*}
\]
which for any fixed set \( \vec{\theta} \) is of type (48) with the coefficients are subjected to the conditions (49), in our case generalized in the form

\[
\epsilon_2 \psi^{\bullet\bullet}(\vec{\theta}, \chi) + \epsilon_3 \psi''(\vec{\theta}, \chi) = \lambda, \\
h_5^2(\vec{\theta}) \phi(\vec{\theta})/h_4(\vec{\theta})h_5(\vec{\theta}) = \lambda, \\
w'_2(\vec{\theta}) - w_3^*(\vec{\theta}) + w_3(\vec{\theta})w_2^*(\vec{\theta}) - w_2(\vec{\theta})w_3^*(\vec{\theta}) = 0, \\
n'_2(\vec{\theta}, \chi) - n_2^*(\vec{\theta}, \chi) = 0,
\]

for \( w_i = \partial_i \phi/\phi^* \), see (25), being compatible with (37) and (38) and considered that finally on solve the Einstein equations for given sources \( \Upsilon_4 = \lambda \) and \( \Upsilon_2 = \lambda \). The metrics (73) define self–consistent evolutions of a multi–parametric class of general solutions of the Ricci flow equations (12)–(14) transformed equivalently in the form (20)–(23). The additional constraints (74) define multi–parametric integral subvarieties (foliations) when the evolutions are selected for the Levi Civita connections.

5 Summary and Discussion

In this work, we have developed an unified geometric approach to constructing exact solutions in gravity and Ricci flow theories following superpositions of anholonomic frame deformations and multi–parametric transforms with Killing symmetries.

The anholonomic frame method, proposed for generalized Finsler and Lagrange theories and restricted to the Einstein and string gravity, applies the formalism of nonholonomic frame deformations (20, 21, 23, 18) (see outline of results in [19] and references therein) when the gravitational field equations transform into systems of nonlinear partial differential equations which can be integrated in general form. The new classes of solutions are defined by generic off–diagonal metrics depending on integration functions on one, two and three/ four variables (if we consider four or five dimensional, in brief, 5D or 4D, spacetimes). The important property of such solutions is that they can be generalized for effective cosmological constants induced by certain locally anisotropic matter field interactions, quantum fluctuations and/or string corrections and from Ricci flow theory.

In general relativity, there is also a method elaborated in Refs. [31, 32] as a general scheme when one (two) parameter families of exact solutions are defined by any source–free solutions of Einstein’s equations with one (two) Killing vector field(s) (for nonholonomic manifolds, we call such transforms to be one-, two- or multi–parameter nonholonomic deformations/
transforms). A successive iteration procedure results in a class of solutions characterized by an infinite number of parameters for a non–Abelian group involving arbitrary functions on one variable.

Both the parametric deformation techniques combined with nonholonomic transforms state a number of possibilities to construct "target" families of exact solutions and evolution scenarios starting with primary metrics not subjected to the conditions to solve the Einstein equations. The new classes of solutions depend on group like and flow parameters and on sets of integration functions and constants resulting from the procedure of integrating systems of partial differential equations to which the field equations are reduced for certain off–diagonal metric ansatz and generalized connections. Constraining the integral varieties, for a corresponding subset of integration functions, the target solutions are determined to define Einstein spacetimes and their Ricci flow evolutions. In general, such configurations are nonholonomic but can constrained to define geometric evolutions for the Levi Civita connections.

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A The Anholonomic Frame Method

We outline the geometry of nonholonomic frame deformations and nonlinear connection (N–connection) structures \[18, 19\]. Let us consider a \((n + m)\)–dimensional manifold \(V\) enabled with a prescribed frame structure \([1]\) when frame transforms are linear on \(N^b_i(u)\),

\[
\begin{align*}
A_\alpha^\beta(u) &= \begin{bmatrix}
e_i^\beta(u) & -N_i^b(u)e_\beta^a(u) \\
0 & e_\alpha^a(u)
\end{bmatrix}, \\
A_\beta^\alpha(u) &= \begin{bmatrix}
e_i^\alpha(u) & N_i^b(u)e_\alpha^k(u) \\
0 & e_\beta^a(u)
\end{bmatrix},
\end{align*}
\]

(A.1)

(A.2)

where \(i, j, \ldots = 1, 2, \ldots, n\) and \(a, b, \ldots = n + 1, n + 2, \ldots n + m\) and \(u = \{u^\alpha = (x^i, y^a)\}\) are local coordinates. The geometric constructions will be adapted to a conventional \(n + m\) splitting stated by a set of coefficients \(N = \{N_i^b(u)\}\) defining a nonlinear connection (N–connection) structure as a nonintergrable distribution

\[TV = hV \oplus vV\]

(A.3)

with a conventional horizontal (h) subspace, \(hV\), (with geometric objects labelled by "horizontal" indices \(i, j, \ldots\)) and vertical (v) subspace \(vV\) (with
geometric objects labelled by indices $a, b, ...$). The "boldfaced" symbols will be used to emphasize that certain spaces (geometric objects) are provided (adapted) with (to) a $N$–connection structure $\mathbf{N}$.

The transforms (A.1) and (A.2) define a $N$–adapted frame structure

$$
e_\nu = (e_i = \partial_i - N_i^a(u)\partial_a, e_a = \partial_a), \tag{A.4}$$

and the dual frame (coframe) structure

$$
e^\mu = (e^i = dx^i, e^a = dy^a + N_i^a(u)dx^i). \tag{A.5}$$

The frames (A.5) satisfy the certain nonholonomy (equivalently, anholonomy) relations of type (2),

$$[e_\alpha, e_\beta] = e_\alpha e_\beta - e_\beta e_\alpha = W^\gamma_{\alpha\beta} e_\gamma, \tag{A.6}$$

with anholonomy coefficients

$$W^b_{ia} = \partial_a N_i^b \text{ and } W^a_{ji} = \Omega_{ij} = e_j(N_i^a) = e_j(N_i^a). \tag{A.7}$$

A distribution (A.3) is integrable, i.e. $\mathbf{V}$ is a foliation, if and only if the coefficients defined by $\mathbf{N} = \{N_i^a(u)\}$ satisfy the condition $\Omega_{ij} = 0$. In general, a spacetime with prescribed nonholonomic splitting into $h$- and $v$–subspaces can be considered as a nonholonomic manifold [18, 17, 34].

Let us consider a metric structure on $\mathbf{V}$,

$$\hat{g} = g_{\alpha\beta}(u) du^\alpha \otimes du^\beta \tag{A.8}$$

defined by coefficients

$$g_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b h_{ab} & N_i^e h_{ae} \\ N_i^e h_{be} & h_{ab} \end{bmatrix}. \tag{A.9}$$

This metric is generic off–diagonal, i.e. it can not be diagonalized by any coordinate transforms if $N_i^a(u)$ are any general functions. We can adapt the metric (A.8) to a $N$–connection structure $\mathbf{N} = \{N_i^a(u)\}$ induced by the off–diagonal coefficients in (A.9) if we impose that the conditions

$$\hat{g}(e_i, e_a) = 0, \text{ equivalently, } g_{ia} - N_i^b h_{ab} = 0,$$

where $g_{ia} \equiv g(\partial/\partial x^i, \partial/\partial y^a)$, are satisfied for the corresponding local basis (A.4). In this case $N_i^b = h^{bc} g_{ca}$, where $h^{ab}$ is inverse to $h_{ab}$, and we can write
the metric \( \hat{g} \) \[^{A.9}\] in equivalent form, as a distinguished metric (d–metric) adapted to a N– connection structure, 
\[
g = g_{\alpha\beta}(u)e^\alpha \otimes e^\beta = g_{ij}(u)e^i \otimes e^j + h_{ab}(u)e^a \otimes e^b, \tag{A.10}
\]
where \( g_{ij} \equiv g(e_i,e_j) \) and \( h_{ab} \equiv g(e_a,e_b) \). The coefficients \( g_{\alpha\beta} \) and \( g_{\alpha\beta} = g_{\alpha\beta} \) are related by formulas 
\[
g_{\alpha\beta} = A_{\alpha}^{\gamma} A_{\beta}^{\delta} g_{\gamma\delta}, \tag{A.11}
\]
or 
\[
g_{ij} = e_i^j e_j^i g_{ij} \text{ and } h_{ab} = e_a^a e_b^b g_{ab},
\]
where the frame transform is given by matrices \[^{A.1}\] with \( e_i^j = \delta_i^j \) and \( e_a^b = \delta_a^b \). We shall call some geometric objects, for instance, tensors, connections,..., to be distinguished by a N–connection structure, in brief, d–tensors, d–connections,... if they are stated by components computed with respect to N–adapted frames \[^{A.4}\] and \[^{A.5}\]. In this case, the geometric constructions are elaborated in N–adapted form, i.e. they are adapted to the nonholonomic distribution \[^{A.3}\].

Any vector field \( X = (hX, vX) \) on \( TV \) can be written in N–adapted form as a d–vector 
\[
X = X^\alpha e_\alpha = (hX = X^i e_i, vX = X^a e_a).
\]
In a similar form, we can ’N–adapt’ any tensor object and get a d–tensor.

By definition, a d–connection is adapted to the distribution \[^{A.3}\] and splits into h– and v–covariant derivatives, \( D = hD + vD \), where \( hD = \{D_k = (L^i_{jk}, L^a_{bk}) \} \) and \( vD = \{D_c = (C^i_{jk}, C^a_{bc}) \} \) are correspondingly introduced as h– and v–parametrizations of the coefficients 
\[
L^i_{jk} = (D_k e_j)|e^i, \quad L^a_{bk} = (D_k e_b)|e^a, \quad C^i_{jk} = (D_c e_j)|e^i, \quad C^a_{bc} = (D_c e_b)|e^a.
\]
The components \( \Gamma_{\alpha\beta}^\gamma(u) = L^i_{jk}, L^a_{bk}, C^i_{jk}, C^a_{bc} \), with the coefficients defined with respect to \[^{A.5}\] and \[^{A.4}\], completely define a d–connection \( D \) on a N–anholonomic manifold \( V \).

The simplest way to perform a local covariant calculus by applying d–connections is to use N–adapted differential forms and to introduce the d–connection 1–form \( \Gamma_{\alpha\beta}^\gamma = \Gamma_{\alpha\beta}^\gamma e^\gamma \), when the N–adapted components of d–connection \( D_\alpha = (e_\alpha|D) \) are computed following formulas 
\[
\Gamma_{\alpha\beta}^\gamma (u) = (D_\alpha e_\beta)|e^\gamma, \tag{A.12}
\]
where "\( \cdot \)" denotes the interior product. We define in N–adapted form the torsion \( T = \{ T^\alpha \} \),

\[
T^\alpha \equiv D e^\alpha = de^\alpha + \Gamma^\alpha_\beta \wedge e_\alpha,
\]

and curvature \( R = \{ R^\alpha_\beta \} \),

\[
R^\alpha_\beta \equiv D \Gamma^\alpha_\beta = d\Gamma^\alpha_\beta - \Gamma^\alpha_\gamma \wedge \Gamma^\gamma_\beta,
\]

The coefficients of torsion \( T \) of a d–connection \( D \) (in brief, d–torsion) are computed with respect to N–adapted frames \((A.5)\) and \((A.4)\),

\[
T^i_{jk} = L^i_{jk} - L^k_{ij}, \quad T^i_{ja} = -T^i_{aj} = C^i_{ja}, \quad T^a_{ji} = \Omega^a_{ji},
\]

\[
T^a_{bi} = T^a_{ib} = \frac{\partial N^a_i}{\partial y^b} - L^a_{bi}, \quad T^a_{bc} = C^a_{bc} - C^a_{cb},
\]

where, for instance, \( T^i_{jk} \) and \( T^a_{bc} \) are respectively the coefficients of the \( h(hh)\)–torsion \( hT(hX, hY) \) and \( v(vv)\)–torsion \( vT(vX, vY) \). In a similar form, we can compute the coefficients of a curvature \( R \), d–curvatures.

There is a preferred, canonical d–connection structure, \( \hat{D} \), on a N–anholonomic manifold \( V \) constructed only from the metric and N–connection coefficients \([g_{ij}, h_{ab}, N^a_i]\) and satisfying the conditions \( \hat{D} g = 0 \) and \( \hat{T}^i_{jk} = 0 \) and \( \hat{T}^a_{bc} = 0 \). It should be noted that, in general, the components \( \hat{T}^i_{ja}, \hat{T}^a_{ji} \) and \( \hat{T}^a_{bi} \) are not zero. This is an anholonomic frame (equivalently, off–diagonal metric) effect. Hereafter, we consider only geometric constructions with the canonical d–connection which allow, for simplicity, to omit "hats" on d–objects. We can verify by straightforward calculations that the linear connection \( \Gamma^\gamma_{\alpha \beta} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc}) \) with the coefficients defined

\[
D_{e_i}(e_j) = L^i_{jk} e_i, \quad D_{e_b}(e_a) = L^a_{bk} e_a, \quad D_{e_b}(e_j) = C^i_{ja} e_i, \quad D_{e_c}(e_b) = C^a_{bc} e_a,
\]

where

\[
L^i_{jk} = \frac{1}{2} g^{ij} (e_k g_{jr} + e_j g_{kr} - e_r g_{jk}),
\]

\[
L^a_{bk} = e_b (N_k^b) + \frac{1}{2} h^{ac} \left( e_k h_{bc} - h_{dc} e_b N_k^d - h_{db} e_c N_k^d \right),
\]

\[
C^i_{jc} = \frac{1}{2} g^{ik} e_c g_{jk}, \quad C^a_{bc} = \frac{1}{2} h^{ad} (e_c h_{bd} + e_c h_{cd} - e_d h_{bc}),
\]

uniquely solve the conditions stated for the canonical d–connection.
The Levi Civita linear connection $\nabla = \{, \Gamma^\alpha_{\beta\gamma} \}$, uniquely defined by the conditions $\nabla T = 0$ and $\nabla \hat{g} = 0$, is not adapted to the distribution $\mathcal{A}$, for $\mathcal{A}$.

Denoting $\Gamma^\alpha_{\beta\gamma} = (, L_{jk}^i, L_{jk}^a, L_{bk}^i, L_{bk}^a, C_{jb}^i, C_{jb}^a, C_{bc}^i, C_{bc}^a)$, for

$$
\nabla e_k(e_j) = L_{jk}^i e_i + L_{jk}^a e_a, \quad \nabla e_k(e_b) = L_{bk}^i e_i + L_{bk}^a e_a,
$$

$$
\nabla e_a(e_j) = C_{jb}^i e_i + C_{jb}^a e_a, \quad \nabla e_a(e_b) = C_{bc}^i e_i + C_{bc}^a e_a,
$$
after a straightforward calculus we get

$$
L_{jk}^i = L_{jk}^i, \quad L_{jk}^a = -C^a_{ijk} h^{ab} - \frac{1}{2} \Omega^a_{jk}, \quad (A.17)
$$

$$
L_{bk}^i = \frac{1}{2} \Omega^c_{jk} h_{cb} g^{ji} - \frac{1}{2} (\delta_j^i \delta_k^h - g_{jk} g^{ih}) C^j_{hi},
$$

$$
L_{jk}^a = \frac{1}{2} \Omega^c_{jk} h_{cb} g^{ja} + \frac{1}{2} (\delta_j^i \delta_k^h - g_{jk} g^{ih}) C^j_{hi},
$$

$$
C_{jb}^i = C_{kb}^i + \frac{1}{2} \Omega^c_{jk} h_{cb} g^{ji} + \frac{1}{2} (\delta_j^i \delta_k^h - g_{jk} g^{ih}) C^j_{hi},
$$

$$
C_{jb}^a = -\frac{1}{2} (\delta_c^a \delta_d^j - h_{cb} h^{ad}) [L^e_{dj} - e_b(N^e_j)],
$$

$$
C_{ab}^i = -\frac{g^{ij}}{2} \{ [L^c_{aj} - e_a(N^c_j)] h_{cb} + [L^c_{bj} - e_b(N^c_j)] h_{ca} \},
$$

where $\Omega^a_{jk}$ are computed as in the second formula in $\mathcal{A}$.

For our purposes, it is important to state the conditions when both the Levi Civita connection and the canonical d–connection may be defined by the same set of coefficients with respect to a fixed frame of reference. Following formulas $\mathcal{A}$, we obtain equality $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$ if

$$
\Omega^c_{jk} = 0 \quad (A.18)
$$

(there are satisfied the integrability conditions and our manifold admits a foliation structure),

$$
C^i_{kb} = C^i_{bk} = 0 \quad (A.19)
$$

and $L^c_{aj} - e_a(N^c_j) = 0$, which, following the second formula in $\mathcal{A}$, is equivalent to

$$
e_k h_{bc} - h_{dc} e_b N^d_k - h_{db} e_c N^d_k = 0. \quad (A.20)
$$

We conclude this section with the remark that if the conditions $\mathcal{A}$, $\mathcal{A}$ and $\mathcal{A}$ hold true for the metric $\mathcal{A}$, equivalently $\mathcal{A}$, the torsion coefficients $\mathcal{A}$ vanish. This results in respective equalities of the coefficients of the Riemann, Ricci and Einstein tensors (the conditions $\mathcal{A}$ being satisfied) for two different linear connections.

34
B The Killing Vectors Formalism

The first parametric method (on holonomic (pseudo) Riemannian spaces, it is also called the Geroch method [31]) proposes a scheme of constructing a one–parameter family of vacuum exact solutions (labelled by tilde "~" and depending on a real parameter $\theta$

\[ \tilde{g}_{\alpha\beta}(\theta) = g_{\alpha\beta} e^\alpha \otimes e^\beta \] (B.1)

beginning with any source–free solution $g_{\alpha\beta}$ with Killing vector $\xi = \{\xi_\alpha\}$ symmetry satisfying the conditions $E = 0$ (Einstein equations) and $\nabla \xi(\tilde{g}) = 0$ (Killing equations). We denote this ‘primary’ spacetime $(V, \tilde{g}, \xi_\alpha)$ and follow the conventions: The class of metrics $\tilde{g}$ is generated by the transforms

\[ \tilde{g}_{\alpha\beta} = \tilde{B}_\alpha^{\alpha'}(u, \theta) \tilde{B}_\beta^{\beta'}(u, \theta) g_{\alpha'\beta'} \] (B.2)

where the matrix $\tilde{B}_\alpha^{\alpha'}$ is parametrized in the form when

\[ \tilde{g}_{\alpha\beta} = \lambda^{-1} g_{\alpha\beta} - \lambda^{-1} \xi_\alpha \xi_\beta + \mu_\alpha \mu_\beta \] (B.3)

for

\[ \lambda = \lambda((\cos \theta - \omega \sin \theta)^2 + \lambda^2 \sin^2 \theta)\]
\[ \mu_\tau = \lambda^{-1} \xi_\tau + \alpha_\tau \sin 2\theta - \beta_\tau \sin^2 \theta. \]

A rigorous proof [31] states that the metrics (B.1) define also exact vacuum solutions with $\tilde{E} = 0$ if and only if the values $\xi_\alpha, \alpha_\tau, \mu_\tau$ from (B.3), subjected to the conditions $\lambda = \xi_\alpha \xi_\beta g^{\alpha\beta}, \omega = \xi^\gamma \alpha_\gamma, \xi^\gamma \mu_\gamma = \lambda^2 + \omega^2 - 1$, solve the equations

\[ \nabla_\alpha \omega = \epsilon_{\alpha\beta\gamma\tau} \xi^\beta \nabla^\gamma \xi^\tau, \quad \nabla_{[\alpha} \xi_{\beta]} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\tau} \nabla^\gamma \xi^\tau, \] (B.4)

\[ \nabla_{[\alpha} \mu_{\beta]} = 2\lambda \nabla_\alpha \xi_\beta + \omega \epsilon_{\alpha\beta\gamma\tau} \nabla^\gamma \xi^\tau, \]

where the Levi Civita connection $\nabla$ is defined by $\tilde{g}$ and $\epsilon_{\alpha\beta\gamma\tau}$ is the absolutely antisymmetric tensor. The existence of solutions for (B.4) (Geroch’s equations) is guaranteed by the Einstein’s and Killing equations.

The first type of parametric transforms (B.2) can be parametrized by a matrix $\tilde{B}_\alpha^{\alpha'}$ with the coefficients depending functionally on solutions for (B.4). Fixing a signature $g_{\alpha\beta} = diag[\pm 1, \pm 1, \ldots, \pm 1]$ and a local coordinate
system on $(V, \mathring{\circ}g, \xi_\alpha)$, we define a local frame of reference $e_\alpha' = A_\alpha^\alpha(u) \partial_\alpha$, like in (1), for which

$$\mathring{\circ}g_{\alpha'\beta'} = A_\alpha^\alpha A_\beta^\beta g_{\alpha\beta}.$$  \hspace{1cm} (B.5)

We note that $A_\alpha^\alpha$ have to be constructed as a solution of a system of quadratic algebraic equations (B.5) for given values $g_{\alpha\beta}$ and $\mathring{\circ}g_{\alpha'\beta'}$. In a similar form, we can write $\mathring{\mathring{e}}_\alpha = A_\alpha^\alpha(\theta, u) \partial_\alpha$ and

$$\mathring{\mathring{\circ}}g_{\alpha\beta} = A_\alpha^\alpha A_\beta^\beta g_{\alpha\beta}.$$  \hspace{1cm} (B.6)

The method guarantees that the family of spacetimes $(V, \mathring{\circ}\mathring{\circ}g)$ is also vacuum Einstein but for the corresponding families of Levi Civita connections $\nabla$. In explicit form, the matrix $\mathring{\tilde{B}}_\alpha^\alpha(u, \theta)$ of parametric transforms can be computed by introducing the relations (B.5), (B.6) into (B.2),

$$\mathring{\tilde{B}}_\alpha^\alpha = A_\alpha^\beta A_\beta^\alpha$$  \hspace{1cm} (B.7)

where $A_\alpha^\alpha$ is inverse to $A_\alpha^\alpha$.

The second parametric method [32] was similarly developed which yields a family of new exact solutions involving two arbitrary functions on one variables, beginning with any two commuting Killing fields for which a certain pair of constants vanish (for instance, the exterior field of a rotating star). By successive iterating such parametric transforms, one generates a class of exact solutions characterized by an infinite number of parameters and involving arbitrary functions. For simplicity, in this work we shall consider only a nonholonomic version of the first parametric method.

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