COMBINATORICS OF THE SUBSHIFT ASSOCIATED TO
GRIGORCHUK’S GROUP

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Abstract. We study combinatorial properties of the subshift induced by the substitution
that describes Lysenok’s presentation of Grigorchuk’s group of intermediate growth by gen-
erators and relators. This subshift has recently appeared in two different contexts: on one
hand, it allowed to embed Grigorchuk’s group in a topological full group, and on the other
hand, it was useful in the spectral theory of Laplacians on the associated Schreier graphs.

In memory of Dmitry Victorovich Anosov

Substitutional dynamical systems constitute an important class studied in symbolic dy-
amics. Such systems are defined by a substitution over the underlying alphabet. They
appear naturally in various branches of mathematics and applications. In particular, certain
substitutional systems provide important models for the theory of ‘aperiodic order’, and the
spectral theory of the associated Schrödinger operators becomes a major tool in understanding
the quantum mechanics of quasi-crystals [1, 6]. Recently it was discovered that substitutional
subshifts are also useful in the study of groups of intermediate growth [8, 4, 5, 9].

This note is devoted to the study of a particular substitution associated to the first group
of intermediate growth constructed by the first author in 1980 in [2] and generally known
as Grigorchuk’s group. The remarkable properties of this group described in [3] were first
reported at Anosov’s seminar in the Moscow State University in 1982-83. The group, defined
by its action by automorphisms on the rooted binary tree, is 3-generated but not finitely
presented, that is, does not admit a presentation with finitely many relations. However,
Lysenok found in [7] the following recursive presentation of this group by generators and
relators
\[ J = \langle a, b, c, d \mid 1 = a^2 = b^2 = c^2 = d^2 = bed = \kappa^k((ad)^4) = \kappa^k((adacac)^4), k = 0, 1, 2, \ldots \rangle \]

with
\[ \kappa : \{a, b, c, d\} \rightarrow \{a, b, c, d\}^* \]
given by \( \kappa(a) = aca, \kappa(b) = d, \kappa(c) = b, \kappa(d) = c \).

In [4] the authors showed that the substitution appearing in Lysenok’s presentation can
be used to determine the spectral type of the discrete Laplacian on the Schreier graphs
naturally associated with \( J \) via the action of \( J \) on the boundary of the rooted binary tree.
Another interesting fact observed in [5] (that also follows from [4]), is that \( J \) embeds into the
topological full group of a related substitutional subshift. Various properties of this subshift
were described in [4, 5]. In particular, it was shown that the subshift is linearly repetitive.

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Schroedinger operators.

1This is how we will refer to it in spite of the first author’s reluctance.
Linear repetitivity has various consequences including linear bounds on the growth of the word complexity function. The purpose of this note is to give a more refined treatment of this complexity function.

The word complexity function $C$ counts, for each $n$, the number of words of length $n$ that appear as subwords in sequences from the subshift (see below for formal definition). It is an important combinatorial characteristic of the dynamical system. For example, the topological entropy of the system can be computed as

$$h := \lim_{n \to \infty} \frac{\ln C(n)}{n}$$

(see e.g. [10]). Hence, linearly repetitive subshifts have zero entropy. The main results of the note are the following.

Consider the alphabet $\mathcal{A} = \{a, x, y, z\}$ and let $\tau$ be the substitution mapping $a \mapsto axa$, $x \mapsto y$, $y \mapsto z$, $z \mapsto x$. Let $\text{Sub}_\tau$ be the associated set of finite words given by $\text{Sub}_\tau = \bigcup_{s \in \mathcal{A}, n \in \mathbb{N} \cup \{0\}} \text{Sub}(\tau^n(s))$, where $\text{Sub}(w)$ denotes the set of finite subwords of the string $w$.

**Theorem 1 (Complexity Theorem).** The complexity function $C$ of the subshift associated with the substitution $\tau$ satisfies $C(1) = 4, C(2) = 6, C(3) = 8$ and, for any $n \geq 2$ and $L = 2^n + k$ with $0 \leq k < 2^n$,

$$C(L) = \begin{cases} 
2^{n+1} + 2^{n-1} + 3k : 0 \leq k < 2^{n-1} \\
2^{n+1} + 2^n + 2k : 2^{n-1} \leq k < 2^n \end{cases}$$

To prove the theorem, we determine the difference $C(L) - C(L - 1)$ for each natural number $L$. In fact, we obtain even more detailed information and determine all right-special words of each length. Here, a word in $\text{Sub}_\tau$ is called right-special if it can be extended in more than one way to the right by a letter of the alphabet to form another word in $\text{Sub}_\tau$.

**Theorem 2 (Right-special words).** Consider $n \geq 2$ and $L = 2^n + k$ with $0 \leq k < 2^n$.

- If $0 \leq k < 2^{n-1}$, then there exist among words of length $L$ in $\text{Sub}_\tau$, exactly two right-special words: the suffix of length $L$ of the word $\tau^n(a)$, which can be extended by $x, y, z$; and the suffix of length $L$ of the word $\tau^{n-2}(a)\tau^{n-2}(x)\tau^{n-1}(a)$ which can be extended by $\tau^{n-2}(x)$ and by $\tau^{n-1}(x)$.
- If $2^{n-1} \leq k < 2^n$, then there exists a unique right-special word of length $L$, namely, the suffix of $\tau^n(a)$ of length $L$ which can be extended by $x, y, z$.

Recall that we consider the alphabet $\mathcal{A} = \{a, x, y, z\}$ and denote by $\tau$ be the substitution mapping $a \mapsto axa$, $x \mapsto y$, $y \mapsto z$, $z \mapsto x$. As above, $\text{Sub}_\tau$ denotes the associated set of finite words given by $\text{Sub}_\tau = \bigcup_{s \in \mathcal{A}, n \in \mathbb{N} \cup \{0\}} \text{Sub}(\tau^n(s))$.

The following three properties obviously hold:

- The letter $a$ is a prefix of $\tau^n(a)$ for all $n \in \mathbb{N} \cup \{0\}$.
- The lengths $|\tau^n(a)|$ converge to $\infty$ for $n \to \infty$.
- Every letter of $\mathcal{A}$ occurs in $\tau^n(a)$ for some $n$. 

By the first two properties, $\tau^n(a)$ is a prefix of $\tau^{n+1}(a)$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, there exists a unique one-sided infinite word $\eta$ such that $\tau^n(a)$ is a prefix of $\eta$ for all $n \in \mathbb{N} \cup \{0\}$. This $\eta$ is a fixed point of $\tau$, i.e., $\tau(\eta) = \eta$. We will refer to it as the fixed point of the substitution $\tau$.

By the third property, we have

$$\text{Sub}_r = \text{Sub}(\eta).$$

We can now associate to $\tau$ the set

$$\Omega_\tau := \{\omega \in A^\mathbb{Z} : \text{Sub}(\omega) \subset \text{Sub}_r\}.$$ 

The study of $\Omega_\tau$ can be based on the investigation of the $\tau^n(a)$.

A direct calculation gives the following recursion formula

$$RF \qquad \tau^{n+1}(a) = \tau^n(a)\tau^n(x) \tau^n(a) \text{ with } \tau^n(x) = \begin{cases} x : & n = 3k, k \in \mathbb{N} \cup \{0\} \\ y : & n = 3k + 1, k \in \mathbb{N} \cup \{0\} \\ z : & n = 3k + 2, k \in \mathbb{N} \cup \{0\} \end{cases}$$

It is not hard to see that any other letter of $\eta$ is the letter $a$, and $\eta$ can be written as

$$\eta = ar_1ar_2a...$$

with a unique sequence $(r_n)$ in $\{x, y, z\}$. As $\eta$ is a fixed point of $\tau$, this gives

$$\eta = \tau^n(\eta) = \tau^n(a)\tau^n(r_1)\tau^n(a)\tau^n(r_2)\tau^n(a)...$$

This way of writing $\eta$ is the basis for our further analysis. While we do not need it here, we mention in passing that the sequence $\eta$ can be generated by an automaton. The automaton is given in Figure 1 (see [4] for details).

![Figure 1. The automaton generating $\eta$](image)

We define the word complexity of $(\Omega_\tau, T)$ as

$$C : \mathbb{N} \cup \{0\} \rightarrow \mathbb{N} \quad C(L) = \text{number of elements of Sub}_r \text{ of length } L.$$ 

The word $ws$ with $s \in \{a, x, y, z\}$ and $ws \in \text{Sub}_r$ will be called an extension of $w \in \text{Sub}_r$ and we will say in this case that $w$ can be extended by $s$. A word $w \in \text{Sub}_r$ is called right-special if the set of its extensions has more than one element. Left-special words are defined in a similar way.
Proposition 3. Any word \( w \in \text{Sub}_r \) with \( |w| \leq |\tau^n(a)| = 2^{n+1} - 1 \) (for some \( n \in \mathbb{N} \cup \{0\} \)) appears in \( \tau^{n+3}(a) \).

Proof. Recall that \( \text{Sub}_r = \text{Sub}(\eta) \). As discussed above, there exist \( r_j^{(n)} := \tau^n(r_j) \in \{x, y, z\} \), with
\[
\eta = \tau^n(a)r_1^{(n)}\tau^n(a)r_2^{(n)}\tau^n(a)\ldots
\]
Thus, any word of length \( L \leq |\tau^n(a)| \) is a subword of one of the three words
\[
\tau^n(a)x\tau^n(a), \tau^n(a)y\tau^n(a), \tau^n(a)z\tau^n(a).
\]
These three words can easily be seen to appear in \( \tau^{n+3}(a) \). \( \square \)

We can use the previous proposition to obtain the values of \( C(L) \) for small values of \( L \) by inspection of \( \tau^k(a) \) for suitable small \( k \). This gives
\[
C(1) = 4, C(2) = 6, C(3) = 8, C(4) = 10.
\]
From the previous proposition we also directly obtain an upper bound for the word complexity.

Lemma 4 (Upper bound). Let \( L = |\tau^n(a)| = 2^{n+1} - 1 \) for some \( n \in \mathbb{N} \). Then,
\[
C(L) \leq 2L + \frac{L + 1}{2} = 2^{n+2} + 2^n - 2.
\]

Proof. By the previous proposition, it suffices to give an upper bound for the number of subwords of length \( L \) in \( \tau^{n+3}(a) \). In order to be specific, we assume that \( n \) is divisible by three. (The other cases can be treated analogously.) Thus, we obtain
\[
\tau^{n+3}(a) = \tau^n(a)x\tau^n(a)y\tau^n(a)x\tau^n(a)y\tau^n(a)x\tau^n(a).
\]
Here, the \( z \) in the 'middle' is at position \( |\tau^{n+2}(a)| + 1 \). We will count the words of length \( L \) appearing in \( \tau^{n+3}(a) \) starting from the left and dismissing words we obviously have already encountered: this will give us the desired upper bound. We note that all subwords of length \( L \) or less starting after the \( z \) at the position \( |\tau^{n+2}(a)| + 1 \) must have already appeared to the left of this position. Thus, we can focus on subwords appearing in the first
\[
|\tau^{n+2}(a)| + 1 = 4L + 4
\]
positions. We then see the following:

- The subword of length \( L \) appearing at position \( \tau^n(a) + 2 \) is \( \tau^n(a) \) and has already appeared at the first position.
- The subwords of length \( L \) at the positions \( P \in [|\tau^{n+1}(a)| + 2, |\tau^{n+1}(a)| + |\tau^n(a)| + 3] \) are subwords of \( \tau^n(a)x\tau^n(a) \), which have already occurred in the prefix \( \tau^n(a)x\tau^n(a) \) of \( \tau^{n+3}(a) \).
- There is the word \( v = \tau^n(a)z\tau^n(a) \) appearing at position \( |\tau^{n+2}(a)| + 1 \) of \( \tau^{n+3}(a) \). We can decompose this as
\[
v = \tau^{n-1}(a)z\tau^{n-1}(a)z\tau^{n-1}(a)z\tau^{n-1}(a)z\tau^{n-1}(a).
\]
Then, \( v \) starts with three copies of \( \tau^{n-1}(a)z \). Clearly, the subwords of length \( L \) starting in the second copy of \( \tau^{n-1}(a)z \) have already appeared starting in the first copy of \( \tau^{n-1}(a)z \).
Taking these double occurrences into account we obtain
\[ C(L) \leq 4L + 4 - (1 + (|\tau^n(a)| + 2) + (|\tau^{n-1}(a)| + 1)) = 2L + \frac{L + 1}{2}. \]
This finishes the proof. \(\square\)

We now complement this by a lower bound on the complexity difference.

**Lemma 5** (Lower bound). (a) The inequality \( C(L + 1) - C(L) \geq 2 \) holds for all \( L \in \mathbb{N} \).

(b) For \( L \in \mathbb{N} \) with \( 2^n \leq L \leq 2^n + 2^{n-1} - 1 \) (for some \( n \geq 2 \)) \( C(L + 1) - C(L) \geq 3 \) holds.

**Proof.** (a) It suffices to show that there exists a word \( w \) of length \( L \) such that \( wx, wy, wz \) all appear in \( \text{Sub}_\tau \). Consider an arbitrary \( k \in \mathbb{N} \) with \( L \leq |\tau^k(a)| \). It is not hard to see that \( \tau^k(a)x, \tau^k(a)y, \tau^k(a)z \) all appear in \( \text{Sub}_\tau \). Thus, \( w \) can be chosen as the suffix of \( \tau^k(a) \) of length \( L \).

(b) By assumption, we have
\[ 2^n = |\tau^{n-1}(a)| + 1 \leq |\tau^{n-1}(a)| + 1 + |\tau^{n-2}(a)| = 2^n + 2^{n-1} - 1. \]
In order to be specific, let us assume that \( n \) is such that \( \tau^n(a) = \tau^{n-1}(a)x\tau^{n-1}(a) \). As already noted in the proof of (a), any suffix of \( \tau^n(a) \) has three different extensions. This is in particular true for the suffix of \( \tau^n(a) \) of length \( L \). Note that this suffix has \( x\tau^{n-1}(a) \) as a suffix (due to \( L \geq |\tau^{n-1}(a)| + 1 \)).

We are going to find another word with two different extensions. Considering \( \tau^{n+3}(a) \), we find that \( w = \tau^n(a)z\tau^n(a)y \) belongs to \( \text{Sub}_\tau \). Now, a short calculation gives
\[ w = \tau^{n-1}(a)x\tau^{n-2}(a)z\tau^{n-2}(a)z\tau^{n-2}(a)x\tau^{n-1}(a)y. \]
Thus,
\[ \tau^{n-2}(a)z\tau^{n-2}(a)z\tau^{n-2}(a)z = \tau^{n-2}(a)z\tau^{n-1}(a)z \]
belongs to \( \text{Sub}_\tau \). This shows that every suffix of \( u \) of \( \tau^{n-2}(a)z\tau^{n-1}(a) \) can be extended by \( z \). On the other hand, the above formula also gives that \( \tau^{n-1}(a)z\tau^{n-1}(a)x \) occurs in \( w \). Thus, \( u \) can also be extended by \( x \) and, hence, has two extensions. Taking into account that the suffixes of length \( L \) of \( \tau^{n-2}(a)z\tau^{n-1}(a) \) have \( z\tau^{n-1}(a) \) as a suffix, (which is different from \( x\tau^{n-1}(a) \),) we see that these extensions are different from the previously encountered extensions. \(\square\)

Clearly, the previous lemma implies a lower bound on the complexity function via
\[ C(L) \geq C(L_0) + \sum_{k=L_0}^{L-1} (C(k + 1) - C(k)) \]
for arbitrary \( L_0 \leq L \) in \( \mathbb{N} \). We are going to use this for a particular \( L \) and then for \( L_0 \).

**Proposition 6.** For \( L \) with \( L = |\tau^n(a)| = 2^{n+1} - 1 \) with \( n \geq 2 \), we have \( C(L) = 2^{n+2} + 2^n - 2 \).

**Proof.** We decompose the interval \([4, 2^{n+1} - 1]\) according to powers of 2 as
\[
\bigcup_{k=1}^{n-2} \left( [2^{k+1}, 2^{k+1} + 2^k - 1] \cup [2^{k+1} + 2^k - 1, 2^{k+2}] \right) \cup ([2^n, 2^n + 2^{n-1} - 1] \cup [2^n + 2^{n-1}, 2^{n+1} - 1]).
\]
We then apply (a) and (b) of the previous lemma to obtain

\[ C(L) \geq C(4) - 2 + \sum_{k=1}^{n-1} (3 \cdot 2^k + 2 \cdot 2^k) = 2^{n+2} + 2^n - 2. \]

Here, $-2$ arises in the first step, as our sum deals with the full interval $[4, 2^{n+1}]$, whereas we actually only need the interval $[4, 2^n + 1]$. In the last step, we use the already checked $C(4) = 10$. Comparing this lower bound with the upper bound of Lemma 4, we see that both bounds agree, and this gives the desired statement.

After these preparations we can now state the main technical ingredient of our investigation.

**Lemma 7** (Growth of complexity). For any $n \geq 2$, and for $L = 2^n + k$ with $0 \leq k < 2^n$, we have

\[ C(L + 1) - C(L) = \begin{cases} 3 & 0 \leq k < 2^{n-1} \\ 2 & 2^{n-1} \leq k < 2^n \end{cases} \]

**Proof.** It suffices to show that for any $n \in \mathbb{N}$, and for $L = 2^{n+1} - 1$, the lower bound on the complexity function coming from Lemma 5 gives the correct value of the complexity function. This is, however, just the content of the previous proposition. This finishes the proof.

Based on this lemma we can now determine, in Theorem 1 and in Theorem 2, the complexity and the set of right-special words.

**Proof of Theorem 1.** This follows easily from Lemma 7 and Proposition 6.

**Proof of Theorem 2.** From Lemma 7 we know that the lower bounds in Lemma 5 are sharp. Inspecting the proof of that lemma, we find exactly the words which have more than one extension.

**Remark.**

(a) We note that Sub$_\tau$ is closed under reflections, hence, the statements in Theorem 2 about right-special words easily translate on corresponding statements about left-special words. As a consequence one obtains that the words $\tau^n(a)$, $n \in \mathbb{N} \cup \{0\}$, are both right-special and left-special (and are the only words with this property).

(b) Similar subshifts of low complexity can be associated to other groups of intermediate growth. This more general framework and other related topics will be discussed in a forthcoming paper.

(c) To any subshift one can associate a family of graphs, indexed by the natural numbers, known as Rauzy graphs (or de Bruijn graphs in the case of the full shift) in the following way: For each natural number $n$ the vertices are the words of length $n$ and the edges are the words of length $n + 1$. Specifically, there is an edge from the $u$ to $v$ if there exists a word $w$ of length $n + 1$ such that $u$ is a prefix of $w$ and $v$ is a suffix of $w$.

Then, these graphs basically store the information on the right-special words. Thus, given that we have determined these words, it is possible to determine the Rauzy graphs based on the material presented in this section, compare Figure 3. Further details will be provided elsewhere.
Figure 2. The Rauzy graph containing $\tau^n(a)$. The numbers give the number of vertices for each loop (without counting the middle vertex).

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