A NOTE ON THE AREA AND COAREA FORMULAS

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ABSTRACT. We present the area and coarea formulas for Lipschitz maps between euclidean spaces and $C^1$ maps between differentiable manifolds which are valid for general volume densities. As applications we give a short, "euclidean" proof of the anisotropic Sobolev inequality and describe an anisotropic tube formula for hypersurfaces in $\mathbb{R}^n$.

1. INTRODUCTION

The area and coarea formulas are well-established tools in analysis and geometry. The purpose of this note is to take another look at these formulas but in the context of Finsler geometry, although the results obtained below are applicable outside this framework as well. The main result we prove here is a general coarea formula valid for general volume densities. Whether these volume densities arise in some functorial way from an underlying Minkowski norm as in [2] is of secondary importance.

The main application we give is a "euclidean" proof of the anisotropic Sobolev inequality which was first proven by Gromov using techniques of optimal transport (see the Appendix in [8]). The proof we present follows the lines of the one given by Federer and Fleming in [5]. For that to work we needed an anisotropic isoperimetric inequality which is an easy consequence of the Brunn-Minkowski inequality and we also needed a result that puts the equal sign between the anisotropic outer Minkowski content and the anisotropic area of the boundary. While such a result, under very weak regularity conditions appears in the recent article of Chambolle, Lisini and Lussardi (see [3]) we found worthwhile to give here a geometric proof at the expense of allowing more regularity. In fact, our efforts were rewarded since this proof lead us immediately to an anisotropic half-tube formula for hypersurfaces which involves the expected ingredients, namely the symmetric polynomials in the eigenvalues of the (positively oriented) anisotropic second fundamental form.

A coarea formula for real valued maps in Finsler geometry was proven by Shen in [10]. For a fixed volume form, the area density that Shen uses is the contraction of the volume form with the Finslerian normal vector field. We obtain Shen’s theorem as a corollary of the main result (see Corollary 4.5 and Remark 4.6). In their study of rectifiable subsets of metric spaces and Lipschitz maps between them, Ambrosio and Kirchheim in [1] present area and coarea formulas in which the measures used are the Hausdorff measures. In the reversible Finsler context, the Hausdorff measure is induced by the Busemann volume density. In this note, the jacobians and cojacobians are given by relatively simple expressions that involve only the differential of the map and the underlying norms/densities, very much in the spirit of the familiar euclidean jacobian. The cojacobians introduced by Ambrosio and Kirchheim do not have such simple expressions as they are meant to be employed in the general context of metric spaces.

One paper that motivated and influenced us is the beautifully written article of Álvarez-Paiva and Thompson [2]. One finds there a quite extensive description of the important volume densities in Finsler geometry.

The article is structured as follows. We start by presenting the linear picture. The main novelty here is the notion of a codensity that is used in the definition of the cojacobian. This is a density on the dual vector space obtained, in a certain sense, by taking the quotient of a density of complementary
dimension on the original vector space and a top-degree density. We next describe the change of variables and the Fubini formula in the smooth \((C^1)\) framework. In Section 4, we present the area and coarea formulas for Lipschitz maps between the euclidean spaces. The proofs use the well-known Riemannian analogues and some canonical relations between the corrections factors derived in Section 2. The last section is dedicated to the proof of the anisotropic Sobolev inequality and the derivation of the tube formula.

2. The Linear Side

Let \(V\) be a real vector space of dimension \(n\). The cone of simple \(k\)-multivectors is the subset \(\Lambda^k_s V\) of elements of \(\Lambda^k V\) which can be written as a wedge product of \(k\)-vectors from \(V\).

**Definition 2.1.** A \(k\)-volume density on \(V\) is a map \(F : \Lambda^k_s V \rightarrow \mathbb{R}_{\geq 0}\) which takes the value 0 only in 0 and is homogeneous of degree 1, i.e.

\[
F(a\xi) = |a| F(\xi), \quad \forall a \in \mathbb{R}, \xi \in \Lambda^k_s V
\]

Denote by \(D_n(V)\) the space of \(n\)-volume densities.

It follows that an \(n\)-volume density is just a norm on the line \(\Lambda^n V\). For this reason we will occasionally use the norm symbol \(\|\cdot\|\) to denote a \(k\)-density, even if this might not arise from a norm on \(\Lambda^k V\).

**Example 2.2.** If \(\Omega : \Lambda^n \rightarrow \mathbb{R}\) is linear non-zero, then

\[
\|\Omega\| := |\cdot| \circ \Omega
\]

is an \(n\)-density and each top degree volume density can be obtained from a top degree form in exactly two ways.

There is one important observation to be made regarding the previous definition. Notice that the volume density is required to be symmetric, i.e. \(F(\xi) = F(-\xi)\) and this is because we want our formulas to be insensitive to orientation. However, on a Finsler manifold, each tangent space at a point is endowed with a Minkowski norm, which might not be reversible, meaning that it might lack symmetry. This implies, in particular, that one has a forward length and a possibly different backward length for the same curve. That is why when asymmetry is present, orientations are a necessity especially when one is contemplating change of variables formulas. We will have more to say about this later.

**Definition 2.3.** Let \((V,F)\) and \((W,G)\) be two vector spaces, with \(F \in D_n(V)\) and \(G \in D_n(W)\). Suppose \(\dim V = n\) and let \(A : V \rightarrow W\) be a linear map. The jacobian of \(A\) with respect to the densities \(F\) and \(G\) is

\[
J(A; F,G) = G(\Lambda^n A(\xi)),
\]

where \(\xi \in \Lambda^n V\) satisfies \(F(\xi) = 1\). We will occasionally write it \(J(A)\) when the densities are clear.

Obviously this definition is not trivial only when \(A\) is injective. If \(G\) is the restriction of a norm on \(\Lambda^n W\) then the jacobian is the norm of the operator \(\Lambda^n A\).

**Example 2.4.** The Holmes-Thompson volume density of a normed space \(V\) is defined by (see [2])

\[
\mu(v_1 \wedge \ldots \wedge v_n) = \epsilon_n^{-1} \int_{B(V^*)} dv_1^* \ldots dv_n^*,
\]

where \(B(V^*)\) is the dual unit ball, \(v_1^*, \ldots, v_n^*\) is the dual basis to \(v_1,\ldots, v_n\) and \(\epsilon_n\) is the volume of the euclidean unit ball. The notation \(dv_1^* \ldots dv_n^*\) is a substitute for the Lebesgue measure induced by the basis \(\{v_1^* \ldots v_n^*\}\). One can rewrite (2.1) as:

\[
\mu(v_1 \wedge \ldots \wedge v_n) = \epsilon_n^{-1} \int_{B(V^*)} dv_1 \wedge \ldots \wedge dv_n,
\]
where now, \( dv_1 \wedge \ldots \wedge dv_n \) is an element of \( V^{**} \) corresponding to \( v_1 \wedge \ldots \wedge v_n \) under the canonical isomorphism \( V \simeq V^{**} \) and the orientation on \( V^* \) is the one given by \( v_1^* \wedge \ldots \wedge v_n^* \).

The jacobian of a bijective linear map \( A : V \to W \) between two normed vector spaces endowed with the Holmes-Thompson volumes is:

\[
J_{HT}(A) = \left| \frac{\int_{B(W^*)} \Lambda^n A(\xi)}{\int_{B(V^*)} \xi} \right|,
\]

where \( \xi \) is any non-zero element of \( \Lambda^n V \), considered as an \( n \)-form over \( V^* \). If \( W = V \), possibly with a different norm, then this simplifies to:

\[
J_{HT}(A) = | \det A | \frac{\int_{B(W^*)} \xi}{\int_{B(V^*)} \xi} = | \det A | \frac{\Vol (B(W^*))}{\Vol (B(V^*))}.
\]

In order to define the cojacobian we need the next lemma.

**Lemma 2.5.** Let \( V \) be a vector space of dimension \( n + m \) and let \( F \in \mathcal{D}_n(V) \) and \( \mu \in \mathcal{D}_{n+m}(V) \). Let \( v_1^*, \ldots, v_m^*, w_1, \ldots, w_n \) be basis of \( V^* \) dual to the basis \( v_1, \ldots, v_m, w_1, \ldots, w_n \) of \( V \). Then

\[
F^*(v_1^* \wedge \ldots \wedge v_m^*) := \frac{F(w_1 \wedge \ldots \wedge w_n)}{\mu(v_1 \wedge \ldots \wedge v_m \wedge w_1 \wedge \ldots \wedge w_n)}
\]

gives a well defined \( m \)-volume density on \( V^* \).

**Proof.** It suffices to show that the right hand side does not depend on the choice of \( w_1^*, \ldots, w_n^* \). Let \( U^* := \langle v_1^*, \ldots, v_m^* \rangle \subset V^* \) with dual space \( U \). We obviously have an exact sequence:

\[
0 \to W \to V \to U \to 0,
\]

where \( W \) is the kernel of the projection \( V \to U \). The \( n \)-dimensional subspace \( W \) is endowed with an \( n \)-density by restricting \( F \). Together with the \( n + m \) density on \( V \) one gets an \( m \)-density \( H \) on \( U \). Indeed, if \( u_1, \ldots, u_m \) is a basis of \( U \), then we choose some lifts of the \( u \)'s in \( V \), call them \( e_1, \ldots, e_m \) and a basis \( \{ f_1, \ldots, f_n \} \) of \( W \) and we define

\[
H(u_1 \wedge \ldots \wedge u_m) := \frac{\mu(e_1 \wedge \ldots \wedge e_m \wedge f_1 \wedge \ldots \wedge f_n)}{F(f_1 \wedge \ldots \wedge f_n)}.
\]

One checks immediately that the definition does not depend on the basis \( f_1, \ldots, f_n \) of \( W \) or on the lifts \( e_1, \ldots, e_m \). Indeed if one makes different choices \( \{ e_1', \ldots, e_m' \} \) and \( \{ f_1', \ldots, f_n' \} \) then the change of basis matrix going from \( \{ e_1, \ldots, e_m, f_1, \ldots, f_n \} \) to \( \{ e_1', \ldots, e_m', f_1', \ldots, f_n' \} \) has the block decomposition:

\[
X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

with \( D \) the matrix obtained by going from \( \{ f_1, \ldots, f_n \} \) to \( \{ f_1', \ldots, f_n' \} \). Therefore \( \det X = \det D \).

From \( H \) we get an \( n \)-density on \( U^* \):

\[
H^*(u_1^* \wedge \ldots \wedge u_m^*) := \frac{1}{H(u_1 \wedge \ldots \wedge u_m)}.
\]

Notice now that \( W = \langle w_1, \ldots, w_n \rangle \) and that \( v_1, \ldots, v_m \in V \) are lifts of the vectors in \( U \) which represent the dual basis to \( v_1^*, \ldots, v_m^* \).

**Definition 2.6.** The \( m \)-densities on \( V^* \) resulting by taking the “quotient” of an \( n \) density \( F \) on \( V \) and an \( n + m \)-density \( \mu \) as in Lemma 2.5 will be called \( m \)-codensities on \( V \).
If \( n = 0 \) then the codensity associated to \( \mu \in \mathcal{D}_n(V) \) is
\[
\mu^* (v_1^* \wedge \ldots \wedge v_n^*) = \frac{1}{\mu(v_1 \wedge \ldots \wedge v_n)}.
\]
The space of \( m \)-codensities on \( V \) corresponding to the same top degree volume density \( \mu \) will be denoted by \( \mathcal{D}_m^m(V) \). The notation \( F^*_\mu \in \mathcal{D}_m^m(V) \) will mean that a top degree density \( \mu \) on \( V \) has been fixed and \( F^*_\mu \) is a codensity whose "denominator" is \( \mu \).

**Remark 2.7.** A codensity should not be confused with a dual density. Given \( F \in \mathcal{D}_n(V) \) one defines \( F^\sharp \in \mathcal{D}_n(V^*) \) by putting
\[
F(\omega)^\sharp := \sup_{\substack{\xi \text{ simple} \\zeta \geq 1 \\text{ such that} \zeta(\xi) = 1}} |\zeta(\omega)|
\]
However, if \( n = \dim V \) and \( \mu \in \mathcal{D}_n(V) \), then \( \mu^\sharp = \mu^* \).

It is useful to have another description of codensities. Fix a density \( \mu \) of top degree \( n + m \). Let \( \Omega \) be a top degree form on \( V \) such that
\[
\mu = |\Omega|.
\]
Notice that \( \Omega \) induces a natural isomorphism of vector spaces.
\[
\iota_\Omega : \Lambda^n V \to \Lambda^m V^*, \quad \iota_\Omega(\theta) = \{ \eta \to \Omega(\theta \wedge \eta) \}.
\]
This isomorphism coincides with the classical Hodge duality, once one picks an inner product on \( V \) whose volume form coincides with \( \Omega \) and identifies \( \Lambda^m V^* \) with \( \Lambda^m V \) in the usual fashion.

For reasons which have to do with orientation conventions and will appear in Section 5 we will use
\[
\iota^*_\Omega : \Lambda^n V \to \Lambda^m V^*, \quad \iota^*_\Omega(\theta) = \{ \eta \to \Omega(\eta \wedge \theta) \}.
\]
Notice that \( \iota_\Omega \) and \( \iota^*_\Omega \) are also isomorphisms between the cones of simple vectors. Indeed, if \( w_1 \wedge \ldots \wedge w_n \in \Lambda^n V \) then choose \( v_1, \ldots, v_m \in V \) such that
\[
\Omega(v_1 \ldots \wedge v_m \wedge w_1 \wedge \ldots \wedge w_n) = 1
\]
It is straightforward to check that \( \iota^*_\Omega(w_1 \wedge \ldots \wedge w_n) = v_1^* \wedge \ldots \wedge v_m^* \), where \( v_1^*, \ldots, v_m^* \) are those vectors in the dual basis of \( \{ w_1, \ldots, w_n, v_1, \ldots, v_m \} \) which vanish on \( \langle w_1, \ldots, w_n \rangle \). On the other hand
\[
\iota_\Omega(w_1 \wedge \ldots \wedge w_n) = (-1)^{mn} v_1^* \wedge \ldots \wedge v_m^*.
\]
We can say more.

**Proposition 2.8.** Let \( F \) be an \( n \)-density on \( V \) and let \( F^*_\Omega \) be the induced \( m \)-codensity. Then
\[
F(w_1 \wedge \ldots \wedge w_n) = F^*_\Omega(\iota^*_\Omega(w_1 \wedge \ldots \wedge w_n))
\]

**Proof.** Using the notation preceding the proposition we have:
\[
F^*_\Omega(\iota^*_\Omega(w_1 \wedge \ldots \wedge w_n)) = F^*_\Omega(v_1^* \wedge \ldots \wedge v_m^*) = \frac{F(w_1 \wedge \ldots \wedge w_n)}{|\Omega|(v_1 \ldots \wedge v_m \wedge w_1 \ldots \wedge w_n)}
\]

The next corollary says among other things that the \( m \)-codensity can be extended to a norm if and only if the \( n \)-density it comes from can. For the relevant definitions we refer the reader to [2].

**Corollary 2.9.** An \( n \)-density on \( V \) is weakly/extendibly/totally convex if and only if the \( m \)-codensity it determines has the same property.
Remark 2.10. One can take Proposition 2.8 as the definition of a codensity once the vector space $V$ comes with a distinguished non-vanishing, top degree form $\Omega$. However, when working over manifolds, finding a non-vanishing differential form $\Omega$ is possible if and only if the manifold is orientable.

Definition 2.11. Let $V$ and $W$ be two vector spaces of dimension $n + m$ and $m$ respectively and suppose that $V$ is endowed with an $m$-codensity $F^*_\mu$, while $W$ is given an $m$-density $\nu$. Let $A : V \to W$ be a linear map. The cojacobian of $A$ is defined by the expression:

$$C(A; \nu^*, F^*_\mu) := F^*_\mu(\Lambda^n A^*(\omega)),$$

where $\omega \in \Lambda^m W^*$ is one of the two vectors of norm 1, i.e $\nu^*(\omega) = 1$. We will write it simply $C(A)$ when the densities are clear.

Notice that the co-jacobian is trivial when $A$ is not surjective.

Example 2.12. In practice, one encounters situations when the vector space $V$ of dimension $n + 1$ comes endowed with a top density $\vert \Omega \vert$ and either a dimension 1-convex density $\bar{F}$ or a codimension 1-convex density, meaning a function $F : \Lambda^n V \to \mathbb{R}$ such that $F$ is positively homogeneous and convex, also called an anisotropic functional. One goes from one situation to the other via the functional dual $F^* : V^* \to \mathbb{R}$ of $F$ and Proposition 2.8 (see also Section 5). Clearly, the codensity $F^*_\vert [\Omega]$ equals $F^*$ and the cojacobian of a linear map $f : V \to \mathbb{R}$ is

$$C(f, dt, F^*_\vert [\Omega]) = \bar{F}^*(f).$$

If $\bar{F}$ is strictly convex, one can define the Legendre isomorphism $V^* \setminus \{0\} \to V \setminus \{0\}$ by associating to $f$ the unique vector $v$ with $F^*(f) = \bar{F}(v)$ such that

$$f(v_f / \bar{F}(v_f)) = \sup_{\bar{F}(w) = 1} f(w).$$

When $f$ is the differential of some $C^1$-function $g : V \to \mathbb{R}$ at a point $p$, then $v_f$ is usually called the Finslerian gradient of $g$ at $p$ and is denoted $\nabla g(p)$. We therefore have

$$C(dg, dt, F^*_\vert [\Omega]) = \bar{F}^*(dg) = \bar{F}(\nabla g).$$

Remark 2.13. In order for the jacobian $J(A; F, G)$ to be defined one needs

$$\deg F = \deg G = \dim \text{Dom}(A).$$

It is convenient to set $J(A; F, G) = 0$ if this dimension condition is not satisfied. Similarly for the cojacobian $C(A; \nu^*, F^*_\mu)$ one needs

$$\deg \nu = \deg \mu - \deg F = \dim \text{Codom} A.$$

We set $C(A; \nu^*, F^*_\mu) = 0$ if this is not fulfilled.

We list now some straightforward properties of the jacobian and cojacobian which we will need for the proof of the area and coarea formulas.

Proposition 2.14. Let $V, W, Z$ be three vector spaces and $A : V \to W$ and $T : W \to Z$ be linear maps.

(a) If $\dim V = \dim W = n$, $\mu \in \mathcal{D}_n(V)$ and $\nu \in \mathcal{D}_n(W)$ then

$$J(A; \mu, \nu) = C(A; \nu^*, \mu^*)$$

(b) If $\dim V = n$, $\mu \in \mathcal{D}_n(V)$, $G \in \mathcal{D}_n(W)$, $H \in \mathcal{D}_n(Z)$ then

$$J(T \circ A ; \mu, H) = J(A ; \mu, G) \cdot J(T \vert_{\text{Im} A} ; G, H).$$
(c) If \( \dim Z = n, \lambda \in \mathcal{D}_n(Z), F_{\lambda}^* \in \mathcal{D}_n^*(V), G_{\mu}^* \in \mathcal{D}_n^*(W) \) then
\[
C(T \circ A; \lambda^*, F_{\mu}^*) = C(T; \lambda^*, G_{\nu}^*) \cdot J(A^*)|_{\text{Im} T^*}; G_{\nu}^*, F_{\mu}^*).
\]

d) If \( \dim W = \dim Z = n, \nu \in \mathcal{D}_n(W), \lambda \in \mathcal{D}_n(Z), F_{\mu}^* \in \mathcal{D}_n^*(V) \) then:
\[
C(T \circ A; \lambda^*, F_{\mu}^*) = C(T; \lambda^*, \nu^*) \cdot C(A; \nu^*, F_{\mu}^*) = J(T; \nu, \lambda) \cdot C(A; \nu^*, F_{\mu}^*)
\]

(e) If \( \dim V = \dim W = n, \dim Z = k, \mu \in \mathcal{D}_n(V), \nu \in \mathcal{D}_n(W), F \in \mathcal{D}_{n-k}(V), G \in \mathcal{D}_{n-k}(W), \lambda \in \mathcal{D}_k(Z), A \) is an isomorphism and \( T \) is surjective then
\[
J(A|_{\text{Ker} T \circ A}; F, G) \cdot C(T \circ A; \lambda^*, F_{\mu}^*) = C(T; \lambda^*, G_{\nu}^*) \cdot J(A; \mu, \nu),
\]
where by convention the jacobian of the unique map on the null vector space is equal to 1.

**Proof.** (a) It is easy to check that \( J(A)J(A^{-1}) = 1 \). Now if \( \{w_1, \ldots, w_n\} \) is a basis of \( W \), then \( \{A^*w_1^*, \ldots, A^*w_n^*\} \) and \( \{A^{-1}w_1, \ldots, A^{-1}w_n\} \) are dual basis of \( V^* \) and \( V \) respectively. Clearly \( \nu(w_1 \wedge \ldots \wedge w_n) = 1 \iff \nu^*(A^*w_1^* \wedge \ldots \wedge w_n^*) = 1 \). Therefore
\[
C(A) = \mu^*(A^*w_1^* \wedge \ldots \wedge A^*w_n^*) = \frac{1}{\mu(A^{-1}w_1 \wedge \ldots \wedge A^{-1}w_n)} = \frac{1}{J(A^{-1})} = J(A).
\]

(b) If \( A \) is not injective both sides are zero. For \( A \) injective it is a straightforward check once one chooses a basis.

c) If \( T \) is not surjective both sides are zero. Either way, one applies (b) to \( T^* \) and \( A^* \).

d) If \( T \) is not an isomorphism then \( T \) is not surjective and both sides are zero. The result follows directly from (c) and (a).

e) This follows from (c), (a) and the next lemma applied to \( U = \text{Ker} T \). Notice that \( \text{Im} T^* = \mathcal{U}^\perp \). □

**Lemma 2.15.** Let \( A : V \to W \) be an isomorphism of vector spaces and \( U \subset W \) a vector subspace. Let \( U^\perp := \{ \alpha \in W^* \mid \alpha\vert_U \equiv 0 \} \). Suppose both \( V, W \) are endowed with top degree densities and also with densities of degree equal to \( \dim U \). Then
\[
J(A) = J\left( A\big|_{A^{-1}(U)} \right) J\left( A^*\big|_{U^\perp} \right),
\]
where \( J(A^*\big|_{U^\perp}) \) is computed using the induced \((\dim W - \dim U)\)-codensities on \( V \) and \( W \).

**Proof.** Let \( u_1^*, \ldots, u_n^* \) be a basis of \( U^\perp \) which we complete to a basis \( u_1^*, \ldots, u_n^* \) of \( W^* \). It follows quickly that \( \mathcal{U} \) is the span of \( u_{k+1}, \ldots, u_n \). We will use the norm symbol \( \| \cdot \| \) for all densities involved and \( \| \cdot \|^* \) for codensities.

Clearly \( \|u_1^* \wedge \ldots \wedge u_k^*\|^* = 1 \) is equivalent with \( \|u_{k+1} \wedge \ldots \wedge u_n\| = \|u_1 \wedge \ldots \wedge u_n\| \).

The dual basis to \( \{A^*u_1^*, \ldots, A^*u_n^*\} \) is \( \{A^{-1}u_1, \ldots, A^{-1}u_n\} \). Now
\[
J(A^*\big|_{U^\perp}) = \|A^*u_1^* \wedge \ldots \wedge A^*u_k^*\|^* = \frac{\|A^{-1}u_{k+1} \wedge \ldots \wedge A^{-1}u_n\|}{\|A^{-1}u_1 \wedge \ldots \wedge A^{-1}u_n\|} = \frac{J(A^{-1}\big|_{\mathcal{U}})}{J(A^{-1})} \cdot \frac{J(A)\big|_{A^{-1}(U)}}{J(A)\big|_{A^{-1}(U)}}.
\]

For the conclusion of this section let us check that the jacobian defined above coincides with the one introduced by Kirchheim ([1],[7]) when the relevant densities arise from the Busemann-Hausdorff definition of volume.
Definition 2.16. Let $V$ be a normed vector space of dimension $n$. Then the Busemann $n$-volume density on $V$ is

$$
\mu_B : \Lambda^n V \to \mathbb{R}, \quad \mu_B(v_1 \wedge \ldots \wedge v_n) := \frac{\epsilon_n}{\int_B dv_1 \ldots dv_n},
$$

where $\epsilon_n$ is the volume of the Euclidean unit ball and the integral in the denominator is the volume of the unit ball $B$ in $V$ measured with respect to the Lebesgue measure resulting by considering the basis $\{v_1, \ldots, v_n\}$ to be orthonormal.

It is clear that one can define a $k$-Busemann volume density on $V$ by considering $k$-dimensional subspaces of $V$ with the induced norm.

Definition 2.17. Let $V$ be a normed vector space of dimension $n$. The K-jacobian of a linear map $A : V \to W$ between normed spaces is

$$
J^K(A) := \frac{\epsilon_n}{\mathcal{H}_n(\{x \mid \|Ax\| \leq 1\})},
$$

where $\mathcal{H}_n$ is the $n$-Hausdorff measure on $V$ seen as a metric space with respect to its norm.

Notice that if $A$ is not injective then the $K$-jacobian is 0 as the denominator represents the volume of an infinite cylinder.

Proposition 2.18. The $K$-jacobian coincides with the jacobian of Definition 2.3 when the $n$-volume densities on $V$ and $W$ are the Busemann densities induced by the norms.

Proof. The Busemann $n$-density induces a translation invariant measure on $V$ (a multiple of any non-canonical Lebesgue measure on $V$) and a result of Busemann says that the volume of a measurable set $K$ with respect to this measure equals its Hausdorff measure $\mathcal{H}_n(K)$. Let $K := \{x \mid \|Ax\| \leq 1\}$.

Let $v_1, \ldots, v_n$ be vectors such that $\mu_B(v_1 \wedge \ldots \wedge v_n) = 1$. Then

$$
\mathcal{H}_n(K) = \int_K dv_1 \ldots dv_n,
$$

where $dv_1 \ldots dv_n$ represents the Lebesgue measure $\lambda$ determined by $v_1, \ldots, v_n$. Indeed if $\mu$ and $\nu$ are two translation invariant measures on $V$ then

$$
\frac{\text{Vol}(K, \mu)}{\text{Vol}(P, \mu)} = \frac{\text{Vol}(K, \nu)}{\text{Vol}(P, \nu)}
$$

for any non-zero measure set $P$. If we let $P$ be the parallelogram spanned by $v_1, \ldots, v_n$, $\mu := \mu_B$ and $\nu = \lambda$ we get (2.4).

If $A : V \to W$ is injective, then $A$ is an isometry between the euclidean space $V$ with orthonormal basis $v_1, \ldots, v_n$ and $A(V)$ with orthonormal basis $Av_1, \ldots, Av_n$. We therefore get

$$
\mathcal{H}_n(K) = \int_K dv_1 \ldots dv_n = \int_{A(K)} dA(v_1) \wedge \ldots \wedge dA(v_n) = \int_{A(V) \cap B} dA(v_1) \wedge \ldots \wedge dA(v_n) = \frac{\epsilon_n}{\mu_B(A(v_1) \wedge \ldots \wedge A(v_n))} = \frac{\epsilon_n}{J(A)}.
$$
3. THE DIFFERENTIABLE SIDE

As is well-known, the area formula in its simplest case is a change of variables while the coarea is Fubini’s theorem. We state the analogues of these results for volume densities in the abstract manifold case.

**Definition 3.1.** Let $M$ be $C^1$-manifold of dimension $n$. A $k$-volume density on $M$ is a continuous map $F : \Lambda^k TM \to \mathbb{R}$, which is a $k$-volume density on each $\Lambda^k T_m M$, where $m$ is a point on $M$. Another name for $F$ is $k$-dimensional positive parametric integrand.

A $k$-volume density restricted to a $C^1$-submanifold $M$ of dimension $k$, determines a measure on this set via a standard process which we review. First let us assume that $k = n$ and that $M$ is an open subset of $\mathbb{R}^n$. Then for every Borel subset $B \subset M$ one defines

$$\int_B dF = \int_B F_x(e_1 \wedge \ldots \wedge e_n) d\lambda(x),$$

where $e_1 \wedge \ldots \wedge e_n$ is the vector of Euclidean norm 1 (positively oriented) and $\lambda$ is the Lebesgue measure. Already in this definition we notice the appearance of the jacobian since $F_x(e_1 \wedge \ldots \wedge e_n) = J(id_{T_x \mathbb{R}^n}; \lambda, F_x)$.

One extends this definition to Borel subsets of an abstract manifold $M$ of dimension $k$, by putting

$$\int_B dF = \int_{\alpha^{-1}(B)} d\alpha^* F$$

where $\alpha : \Omega \to M$ is a chart, $\Omega \subset \mathbb{R}^n$ open, bounded and $B \subset \alpha(\Omega)$ is a Borel set. This is well defined because of the standard change of variables formula. In other words, if $\phi : M \to \mathbb{R}^n$ is a diffeomorphism with $M \subset \mathbb{R}^n$ open then for every Borel set $B$ we have

$$\int_B d\phi^* F = \int_B \phi^* F_x(e_1 \wedge \ldots \wedge e_n) d\lambda(x) = \int_B F_{\phi(x)}(d\phi_x(e_1) \wedge \ldots \wedge d\phi_x(e_n)) d\lambda(x) =$$

$$= \int_B F_{\phi(x)}(\det (d\phi_x)(e_1 \wedge \ldots \wedge e_n)) = \int_B |\det (d\phi_x)| F_{\phi(x)}(e_1 \wedge \ldots \wedge e_n) d\lambda(x) =$$

$$= \int_{\phi(B)} F_x(e_1 \wedge \ldots \wedge e_n) d\lambda(x) = \int_{\phi(B)} dF.$$

Using the jacobian, we can rewrite this as

$$\int_B J(d\phi_x; \lambda, F_x) d\lambda(x) = \int_{\phi(B)} dF.$$

So without further ado

**Proposition 3.2.** Let $(B, F)$ and $(P, G)$ be two $C^1$ manifolds endowed with $n$-volume densities and let $\phi : B \hookrightarrow P$ be an embedding. Suppose furthermore that $\dim B = n$. Let $f : P \to \mathbb{R}$ be a continuous function. Then

$$\int_{\phi(B)} f dG = \int_B J(d\phi_x; F, G) f \circ \phi dF.$$

**Proof.** One considers $\alpha : \Omega \to B$ to be a chart on $B$ with $\Omega \subset \mathbb{R}^n$ open bounded and rewrites the two quantities as integrals over $\Omega$. The result is a simple application of Proposition 2.14 part (b). $\square$

Let us give a simple application of Proposition 3.2. We recall the following from [2].
**Definition 3.3.** A definition of volume in dimension \( n \) associates in a natural (functorial) way to every normed vector space \( V \) of dimension \( n \) a norm on \( \Lambda^n V \) and satisfies a few axioms one of which is the next one.

\(^(*)\) If \( T : (V, \| \cdot \|_V) \rightarrow (W, \| \cdot \|_W) \) is a linear map between vector spaces of dimension \( n \) such that \( \|T\| \leq 1 \) then \( \|\Lambda^n T\| \leq 1 \).

By a Finsler manifold we will understand a manifold such that the tangent space at each point is endowed with a norm, which varies continuously with the point. Strictly speaking the definition should allow for non-reversible (asymmetric) norms and should include some differentiability assumption.

**Proposition 3.4.** Let \( \pi : P \rightarrow B \) be a fiber bundle of Finsler manifolds with \( B \) compact. Suppose a definition of volume has been fixed, e.g. the Hausdorff-Busemann. Suppose furthermore that \( \|d\pi\| \leq 1 \). Then

\[ \text{Vol}(s(B)) \geq \text{Vol}(B), \]

for all sections \( s : B \rightarrow P \).

**Proof.** Let \( \dim B = n \) and let \( F \) and \( G \) be the \( n \)-volume densities on \( B \) and \( P \) that the definition of volume fixes. Then

\[ \text{Vol}(s(B)) = \int_{s(B)} dG = \int_B J(ds; F, G) dF \geq \int_B dF = \text{Vol}(B), \]

because \( J(ds; F, G) = 1/J(d\pi|_{\text{Im} d\phi}; G, F) \geq 1 \) since the definition of volume satisfies \(^(*)\). \( \square \)

The proposition generalizes thus the next well known result.

**Corollary 3.5.** Let \( \pi : P \rightarrow B \) be a Riemannian submersion with \( B \) compact. Then

\[ \text{Vol}(s(B)) \geq \text{Vol}(B), \]

for all sections \( s : B \rightarrow P \).

**Proof.** The differential \( d\pi \) is an orthogonal projection. \( \square \)

Let us now recall how fiber-integration of densities works (see Section 7.12 in [6]). Suppose \( \pi : P \rightarrow B \) is a submersion of \( C^1 \)-manifolds with \( \dim P = n + m \) and \( \dim B = m \) and let \( \mu \) be a top degree density over \( P \). The push-forward \( \pi_* \mu \) of the measure determined by \( \mu \) exists always and in this situation is a top degree density on \( B \) which is constructed by first defining the “retrenchment” \( R_\mu \) of \( \mu \). This is an \( n \)-density on each fiber \( P_b := \pi^{-1}(b) \) with values in the space of \( m \)-densities on \( T_b B \) which at a point \( p \in P_b \) has the expression

\[ R_\mu(p)(w_1 \wedge \ldots \wedge w_n)(v_1 \wedge \ldots \wedge v_m) = \mu_p(w_1 \wedge \ldots \wedge w_n \wedge \tilde{v}_1 \wedge \ldots \wedge \tilde{v}_m), \]

for all \( w_1 \wedge \ldots \wedge w_n \in \Lambda^n V_p P \) where \( V_P := \text{Ker } d\pi_p \) and \( v_1 \wedge \ldots \wedge v_m \in \Lambda^m T_{b} B \). Here \( \tilde{v}_1, \ldots, \tilde{v}_m \) are lifts of the \( v_i \)'s to \( T_p P \). The retrenchment does not depend on the lifts \( \tilde{v}_1, \ldots, \tilde{v}_m \). Then the push-forward density of \( \mu \) is defined

\[ \pi_* \mu(b) := \int_{P_b} dR_\mu. \]

Integration on the right hand-side works as follows. The space of volume densities on \( T_b B \) is in bijection (non-canonically) with the half line \((0, \infty)\), by evaluating the density on a fixed vector \( v_1 \wedge \ldots \wedge v_m \in \Lambda^m T_b B \). So we use one such bijection to integrate the density \( R_\mu \) as if it was a real valued density and interpret the result via the inverse of the same bijection as a density on \( T_b B \).

**Remark 3.6.** Nothing changes if we consider negatively valued densities or even complex ones.
The following relation is an application of the standard Fubini theorem, after one chooses trivializing charts on $B$ and $P$:

$$\int_P \mu = \int_B \pi_* \mu. \tag{3.2}$$

**Proposition 3.7.** Let $(P, F, \mu)$ be an $n+m$ dimensional $C^1$ manifold endowed with an $n+m$-volume density $\lambda$ and an $n$-density $F$. Let $\pi: P \to B$ be a $C^1$ submersion over an $m$-dimensional $C^1$ manifold $B$ also endowed with a top degree volume density $\nu$. Then

$$\int_P C(d\pi, \Lambda^*, F\nu_\pi) \, d\mu = \int_B \text{Vol}_F(\pi^{-1}(b)) \, d\nu(b),$$

where $\text{Vol}_F(\pi^{-1}(b))$ is the volume of the fiber with respect to the density $F$.

**Proof.** We use (3.2) for the density $\nu := C(d\pi, \Lambda^*, F\nu_\pi) \, d\mu$. Then the retrenchment of $\nu$ at $p \in P$ is

$$R_b(p)(\omega)(\xi) = F_p(\omega)\lambda_b(\xi), \quad \forall \omega \in \Lambda^m V_p P, \forall \xi \in \Lambda^n T_b B,$$

which implies that

$$\int_{P_b} dR_{b} = \text{Vol}_F(P_b)\lambda_b.$$

To see that (3.3) is true, notice that for every $v_1 \wedge \ldots \wedge v_m \in \Lambda^n T_b B$

$$C(d\pi, \Lambda^*, F\nu_\pi)_p = \frac{F\mu_\pi(p)(d_p \pi^*(v_1^*) \wedge \ldots \wedge d_p \pi^*(v_n^*))}{\lambda_b(v_1^* \wedge \ldots \wedge v_m^*)} = \frac{F_p(w_1 \wedge \ldots \wedge w_n)\lambda_b(v_1 \wedge \ldots \wedge v_m)}{\mu_p(\tilde{v}_1 \wedge \ldots \wedge \tilde{v}_m \wedge w_1 \wedge \ldots \wedge w_n)},$$

where $w_1, \ldots, w_n$ is a basis of $V_p P$ and $\tilde{v}_i$’s are lifts of the $v_i$’s. The last equality holds because

$$d\pi^*(v_1^*)(\tilde{v}_j) = \delta^i_j \quad \text{and} \quad d\pi^*(v_i^*)(w_j) = 0 \quad \forall i, j.$$

$\square$

**Remark 3.8.** In order for the result of Proposition 3.7 to hold, the $n$-density $F$ need only be defined on $\Lambda^n V P \to \Lambda^n T P$ hence one can see it as a fiber volume density.

4. **THE LIPSCHITZ SIDE**

Let us recall the classical Theorem 3.2.3 from [4].

**Theorem 4.1 (Area Formula).** Suppose $f: \mathbb{R}^m \to \mathbb{R}^n$ is a Lipschitz function with $m \leq n$.

(a) If $A$ is an Lebesgue measurable set, then

$$\int_A J(d_x f) \, d\mathcal{L}(x) = \int_{\mathbb{R}^m} N(f \big|_A, y) \, d\mathcal{H}^m(y),$$

where $J(d_x f) = \sqrt{\det (d_x f)^* d_x f}$, $N(f \big|_A, y)$ is the cardinality of the set $f^{-1}(y) \cap A$ and $\mathcal{L}$, $\mathcal{H}^m$ represent the Lebesgue respectively the Hausdorff measures.

(b) If $u$ is an $\mathcal{L}$ integrable function, then

$$\int_{\mathbb{R}^m} u(x) J(d_x f) \, d\mathcal{L}_m = \int_{\mathbb{R}^n} \left( \sum_{x \in f^{-1}(y)} u(x) \right) \, d\mathcal{H}^m(y).$$

We would like to replace $\mathcal{L}$ and $\mathcal{H}_m$ by two general $m$-volume densities, one on $\mathbb{R}^m$ and one on $\mathbb{R}^n$. In order to do that it is convenient to write the integral on the right as an integral over $f(A)$ which is the support of the function $y \to N(f \big|_A, y)$. It is worth noting that $f(A)$ is an $\mathcal{H}_m$ measurable set, which is an implicit statement in the area formula. For the sake of completeness we state
Proposition 4.2. Let \( X \) and \( Y \) be metric spaces such that \( X \) is separable, complete. Let \( f : X \to Y \) be a Lipschitz map. Then \( f \) takes \( \mathcal{H}^m \) measurable sets into \( \mathcal{H}^m \) measurable sets.

Proof. Since \( \mathcal{H}^m \) is Borel regular every measurable set \( A \) can be written as \( A = B \cup N \) with \( B \) Borel and \( \mathcal{H}^m(N) = 0 \). Now [2.2.10-13] in [4] explains why the set \( f(B) \) is \( \mathcal{H}^m \)-measurable for every continuous function \( f \). On the other hand, Corollary 2.10.11 in [4], saying that

\[
(4.1) \quad (\text{Lip } f)^m \mathcal{H}^m(C) \geq \int N(f|_C, y) \, d\mathcal{H}^m(y), \quad \forall C
\]

implies that \( f(N) \) is a set of measure zero. \( \square \)

Now \( f(A) \) is a measurable and an \( m \)-rectifiable set. This means that we can define for every \( m \)-density \( G \) on \( \mathbb{R}^n \)

\[
\int_{f(A)} dG(y) := \int_{f(A)} G(y, \xi_y) \, d\mathcal{H}^m(y),
\]

where \( \xi_y \in \Lambda^m T_y f(A) \) is one of the two vectors of euclidean norm 1. Recall that the approximate tangent space \( T_y f(A) \) exists \( \mathcal{H}^m \)-a.e. This is of course the definition of the integral of an \( m \)-parametric integrand over an \( m \)-dimensional integer-multiplicity current (see page 515 in [4]) which in the case that \( f(A) \) is a \( C^1 \) manifold coincides with the one given in Section 3.

We can extend this definition to obtain a measure on \( f(A) \) denoted \( G \), which is absolutely continuous with respect to \( \mathcal{H}^m \) and such that the Radon-Nykodim derivative of \( G \) by \( \mathcal{H}^m \) is a jacobian:

\[
\frac{dG}{d\mathcal{H}^m}(y) = J(\text{id}_{T_y f(A)}; \mathcal{H}^m, G) = G(y, \xi_y)
\]

Above, the notation \( \mathcal{H}^m \) plays also the role of the euclidean \( m \)-density on \( \mathbb{R}^n \). A measurable function \( v \) is called \( G \)-integrable if \( \int_{f(A)} |v| \, dG < \infty \).

Theorem 4.3. Suppose \( f : \mathbb{R}^m \to \mathbb{R}^n \) is a Lipschitz function with \( m \leq n \) and let \( F \) be an \( m \)-density on \( \mathbb{R}^m \) and \( G \) be an \( m \)-density on \( \mathbb{R}^n \).

(a) If \( A \) is a Lebesgue measurable set, then

\[
\int_A J(d_x f; F, G) \, dF(x) = \int_{f(A)} N(f|_A, y) \, dG(y),
\]

(b) If \( u \) is a Lebesgue measurable function such that \( u \cdot J(df; F, G) \) is an \( F \)-integrable function over \( A \), then

\[
\int_A u(x)J(d_x f; F, G) \, dF = \int_{f(A)} \left( \sum_{x \in f^{-1}(y)} u(x) \right) \, dG(y).
\]

Proof. Notice first that by (4.1) both sides vanish on the set of points where \( df \) does not have maximal rank. It is therefore enough to consider

\[
A \subset \{ x \mid df_x \text{ exists and has maximal rank} \}.
\]

In this situation for \( y = f(x) \) we have \( T_y f(A) = d_x f(\mathbb{R}^n) \), (at the points where the tangent space exists). The next relation together with part (b) of Theorem 4.1 finishes the proof.

\[
J(\text{id}_{T_y \mathbb{R}^m}; \mathcal{H}^m, F) \cdot J(d_x f; F, G) = J(d_x f; \mathcal{H}^m, \mathcal{H}^m) \cdot J(\text{id}_{\text{Im} d_x f}; \mathcal{H}^m, G)
\]

Due to Proposition 2.14, part (b), both sides equal \( J(d_x f; \mathcal{H}^m, G) \). \( \square \)

The situation is not much different for the coarea formula. However, this time we have to deal with 3 densities. We state it first and then explain.
Theorem 4.4. Suppose $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is a Lipschitz function and let $F$, $\mu$ be volume densities on $\mathbb{R}^{n+m}$ of degree $m$ and $m+n$ respectively. Let $\lambda$ be an $n$ volume density on $\mathbb{R}^n$ and let $A \subset \mathbb{R}^{n+m}$ be a Lebesgue measurable set. Then

(a) $$\int_A C(df; \lambda^*, F^*_\mu) \, d\mu(x) = \int_{\mathbb{R}^n} \text{Vol}_F(A \cap f^{-1}(y)) \, d\lambda(y).$$

(b) If $g : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is a Lebesgue measurable function such that $g \cdot C(df; \lambda^*, F^*_\mu)$ is $\mu$ integrable then

$$\int_A g(x) \cdot C(df; \lambda^*, F^*_\mu) \, d\mu(x) = \int_{\mathbb{R}^n} \left( \int_{A \cap f^{-1}(y)} g(x) \, dF \right) \, d\lambda(y).$$

Proof. It is essential for the good definition of the right hand side that $A \cap f^{-1}(y)$ is $\mathcal{H}^m$-countably rectifiable for $\mathcal{L}^n$ almost all $y \in \mathbb{R}^n$. This is Theorem 3.2.15 in [4].

Again we are allowed to make the simplifying assumption that the differential of $f$ has maximal rank at all points of $A$. This is because both sides vanish on the rest of the points since the integrals are zero with respect to the Hausdorff measure as in the proof of Theorem 3.2.11 from [4].

We claim now that

$$C(df; \lambda^*, F^*_\mu) \cdot J(id_{T_x \mathbb{R}^{n+m}}; \mathcal{L}, \mu) \overset{(\ast)}{=} C(df; \lambda^*, (\mathcal{H}^m)_g) \cdot J(id_{\text{Ker} df}; \mathcal{H}^m, F) \cdot J(id_{\text{Im} df}; \mathcal{L}, \lambda).$$

This happens on one hand because part (d) of Proposition 2.14 implies that

$$C(df; \lambda^*, (\mathcal{H}^m)_g) \cdot J(id_{\text{Im} df}; \mathcal{L}, \lambda) = C(df; \lambda^*, (\mathcal{H}^m)_g),$$

while part (e) of the same proposition implies that

$$C(df; \lambda^*, (\mathcal{H}^m)_g) \cdot J(id_{\text{Ker} df}; \mathcal{H}^m, F) = C(df; \lambda^*, F^*_\mu) \cdot J(id_{T_x \mathbb{R}^{n+m}}; \mathcal{L}, \mu).$$

One can use the standard Coarea formula (Theorem 3.2.11 in [4]) to finish the proof since

$$\int_A C(df; \lambda^*, F^*_\mu) \, d\mu(x) = \int_A C(df; \lambda^*, F^*_\mu) \cdot J(id_{T_x \mathbb{R}^{n+m}}; \mathcal{L}, \mu) \, d\mathcal{L} =$$

$$= \int_A C(df; \lambda^*, (\mathcal{H}^m)_g) \cdot J(id_{\text{Ker} df}; \mathcal{H}^m, F) \cdot J(id_{\text{Im} df}; \mathcal{L}, \lambda) \, d\mathcal{L} =$$

$$= \int_{\mathbb{R}^n} \left( \int_{A \cap f^{-1}(y)} J(id_{\text{Ker} df}; \mathcal{H}^m, F) \, d\mathcal{H}^m \right) \cdot J(id_{\text{Im} df}; \mathcal{L}, \lambda) \, d\lambda(y) =$$

$$= \int_{\mathbb{R}^n} \left( \int_{A \cap f^{-1}(y)} dF \right) \, d\lambda.$$

\[\Box\]

We can improve now Proposition 3.7 by removing the submersion condition. Notice that on a general smooth manifold $P$ it makes sense about measurable sets. By this we understand that the set is measurable in every chart of fixed atlas with respect to the Lebesgue measure induced by the chart. Clearly this does not depend on the particular atlas used. Moreover this is equivalent to saying that the set is measurable with respect to any top degree volume density on the manifold.

Corollary 4.5. Let $\pi : P \to B$ be a $C^1$ map between smooth manifolds with $\dim P = m + n$ and $\dim B = n$. Let $F$ and $\mu$ be volume densities of degrees $m$ and $n + m$ on $P$ and $\lambda$ be an $n$-density on $B$. Let $A$ be a measurable set. Then
with an inner product \( \langle \cdot \rangle \). In other words, \( \bar{\lambda} \) where \( W \) function of \( F \) in general is not assumed to be origin symmetric but is assumed to be extendibly convex, i.e. it is the \( (5.1) \)

\[ \int_{\mathbb{R}^n} C(d_y \pi; \lambda^*, F_{\mu}^*) \, d\mu(y) = \int_{\mathbb{R}^n} \text{Vol}_F(A \cap \pi^{-1}(y)) \, d\lambda(y). \]

(b) If \( g : P \to B \) is a measurable function such that \( g \cdot C(d\pi; \lambda^*, F_{\mu}^*) \) is \( \mu \) integrable then

\[ \int_{\mathbb{R}^n} g(x) \cdot C(d_y \pi; \lambda^*, F_{\mu}^*) \, d\mu(x) = \int_{\mathbb{R}^n} \left( \int_{A \cap \pi^{-1}(y)} g(x) \, dF \right) \, d\lambda(y). \]

Proof. By choosing a partition of unity \((D_i, \tau_i)\) on \( B \) and using the result for \( \pi \big|_P \) where \( P = \pi^{-1}(D_i) \) and \( g = \tau_i \) one is reduced to prove the claim for \( B = \mathbb{R}^n \).

Choose a covering of \( P \) with open sets \( U_i \) diffeomorphic with \( \mathbb{R}^{n+m} \) and a partition of unity \( \rho_i \) subordinate to \( U_i \). Since \( \pi \) is \( C^1 \), by shrinking the \( U_i \)'s if necessary, we can assume that \( \pi \big|_{U_i} \) is Lipschitz (as a map \( \mathbb{R}^{n+m} \to \mathbb{R}^n \)). One then uses Theorem 4.4 for \( \pi \big|_{U_i} \) and \( h = \rho_i \) and sums up.

Remark 4.6. An important particular case of Corollary 4.5 occurs when \( \pi = f : P \to \mathbb{R}, \lambda = dt \), \( \mu = |\Omega| \) a volume form and \( F \) arises by “contracting” the volume form \( \Omega \) with a convex 1-density \( \bar{F} \), meaning that \( F = \bar{F}^* \circ t_1^* \) where \( \bar{F}^* \) is the functional dual of \( \bar{F} \). In this case, the cojacobian is \( \bar{F}^*(df) = \bar{F}(\nabla f) \) (see Example 2.12) and one recovers Shen’s Theorem 3.3.1 in [10].

5. The anisotropic outer Minkowski content and the Sobolev inequality

The codensities introduced in Section 2 are, at least in degree 1, important objects in anisotropic geometry. The starting point is an oriented \((n+1)\)-dimensional vector space \( V \), endowed with a volume form \( \Omega \in \Lambda^{n+1}V^* \) consistent with the orientation of \( V \) and an \( n \)-volume density \( F \), which in general is not assumed to be origin symmetric but is assumed to be extendibly convex, i.e. it is the restriction of a convex, positively homogeneous function \( F : \Lambda^n V \to \mathbb{R} \). To make the distinction we call such an object a parametric integrand of degree \( n \). Then \( F \) and \( \Omega \) induce a Minkowski functional \( \bar{F}^* \) on \( V^* \) by putting

\[ \bar{F}^* := F \circ (t_1^*)^{-1}, \quad \text{see (2.2)}. \]

One relates this to what was done in Section 2 as follows. Start with a parametric integrand \( F \) of degree \( n \) and a volume form \( \Omega \). We can easily modify the Definition-Lemma 2.5, asking for the basis \( \{v_1 \ldots v_m, w_1, \ldots w_n\} \) to be positively oriented in order to obtain a well-defined parametric integrand \( F_{\Omega}^* \) on \( \Lambda^m V^* \) which coincides with \( \bar{F} \) when \( m = 1 \) since the proof of Proposition 2.8 is the same.

We will stay with the case \( m = 1 \). The set \( W_f := \{v \in V \mid \bar{F} \leq 1\} \) is called the Wulff shape, where

\[ \bar{F}(v) = \sup_{F^*(f) \leq 1} f(v). \]

In other words, \( \bar{F}^* \) is the support function of \( W_f \) (see [9]). To be more precise, suppose \( V \) is endowed with an inner product \( \langle \cdot, \cdot \rangle \) such that \( \Omega \) is the volume form induced by \( \langle \cdot, \cdot \rangle \). Let \( h_{W_f} \) be the support function of \( W_f \) on \( V \). Then

\[ h_{W_f}(u) = \bar{F}^*(\langle u, \cdot \rangle) \quad \text{(5.1)} \]

Let us mention next a multiplication property of the densities involved. Suppose that \( v_1 \ldots v_n \) are \( n \) linearly independent vectors and \( w \) is a vector such that the hyperplane spanned by \( \{v_1, \ldots, v_n\} \) is supporting for \( \lambda W \) where \( \lambda > 0 \) is such that \( w \in \partial(\lambda W) \). Then

\[ |\Omega|(w \wedge v_1 \wedge \ldots v_n) = \bar{F}(w) \cdot F(v_1 \wedge \ldots v_n) \quad \text{(5.2)} \]
Indeed we can reinterpret this equality as
\begin{equation}
\bar{F}(w)\bar{F}^*(w^*) = 1,
\end{equation}
where $w^*$ is the dual of $w$ with respect to the basis $w, v_1, \ldots, v_n$. Let $\tilde{w} := \frac{1}{\lambda}w \in W_F$. We will prove
\begin{equation}
\bar{F}^*(w^*) = w^*(\tilde{w}),
\end{equation}
from which (5.3) follows.

Now the fact that $H := \langle v_1, \ldots, v_n \rangle$ is supporting for $W_F$ at $\tilde{w}$ can be rephrased as saying that
\begin{equation}
\bar{F}(\tilde{w} + v) \geq \bar{F}(\tilde{w}), \quad \forall v \in H.
\end{equation}
Every $z \in W_F$ can be written as $z = a\tilde{w} + v$, with $v \in H$. Since we are looking for those $z$ that realize $\sup_{z \in W_F} w^*(z)$, we can assume that $a > 0$. From
\begin{equation}
1 \geq \bar{F}(z) = a\bar{F}(\tilde{w} + \frac{1}{a}v) \geq a\bar{F}(\tilde{w}) = a,
\end{equation}
we conclude that $w^*(z) \leq w^*(\tilde{w})$ for all $z \in W_F$, hence (5.4).

For the next result we use an oriented jacobian which is defined for linear maps between oriented spaces simply by taking in Definition 2.3 the vector $\xi$ to be positively oriented.

**Lemma 5.1.** Let $U \subset V$ be an oriented subspace of dimension $n$. Then
\begin{equation}
J(\text{id}_U; \mathcal{H}^n|_U, F|_U) = h_{W_F}(\nu_U),
\end{equation}
where $\nu_U$ is the unit normal to $U$ with respect to $\langle \cdot, \cdot \rangle$, such that $\nu, u_1, \ldots, u_n$ is a positively oriented basis of $V$ for every oriented basis $u_1, \ldots, u_n$ of $U$. In particular if $\Sigma$ is an oriented $C^1$ hypersurface in $V$, then
\begin{equation}
\int_{\Sigma} dF = \int_{\Sigma} h_{W_F}(\nu_{\Sigma}) \, d\mathcal{H}^n.
\end{equation}

**Proof.** For $\{v_1, \ldots, v_n\}$ an oriented basis of $U$ one has
\begin{equation}
J(\text{id}_U; \mathcal{H}^n|_U, F|_U) = \frac{F(v_1 \wedge \ldots \wedge v_n)}{\mathcal{H}^n(v_1 \wedge \ldots \wedge v_n)} = \frac{F(v_1 \wedge \ldots \wedge v_n)}{\Omega(\nu_U \wedge v_1 \wedge \ldots \wedge v_n)} = \bar{F}(\nu_U^*) = \bar{F}(\langle \nu_U, \cdot \rangle) = h_{W_F}(\nu_U).
\end{equation}
We used the fact that $\nu_U^*$, which is the element of the dual basis to $\{\nu_U, v_1, \ldots, v_n\}$ that returns 1 when evaluated on $\nu_U$ is in fact equal to $\langle \nu_U, \cdot \rangle$ independently of the choice of $v_1, \ldots, v_n$. □

**Remark 5.2.** If $\Sigma$ is the smooth boundary of an open set in $V$ then we use the ”outer normal first” convention to orient $\Sigma$ and in this case $\nu$ of the previous lemma is the unit outer normal.

Let $B$ be a subset of $V$. All volumes of sets below are with respect to the volume form $\Omega$. The anisotropic outer Minkowski content is the quantity
\begin{equation}
SM^F_B(B) = \lim_{t \searrow 0} \frac{\text{Vol}(B + tW_F) - \text{Vol}(B)}{t},
\end{equation}
when the limit exists.

**Theorem 5.3.** Let $B$ be a bounded domain with $C^2$ boundary. Then
\begin{equation}
SM^F_B(B) = \int_{\partial B} dF.
\end{equation}
Proof. We will assume first that \( W_F \) has support function of class \( C^\infty \) (although \( \partial W_F \) might still have singularities). This implies in particular that \( W_F \) is strictly convex (see Corollary 1.7.3 in [9]).

Consider the \( \bar{F} \)-distance from \( \Sigma := \partial B \) to a point \( p \) outside \( B \). This is by definition the infimum over the set of all piecewise smooth curves \( \gamma : [0, 1] \rightarrow V \) that connect a point on \( \Sigma \) with \( p \) of the following integral

\[
\int_{\gamma} d\bar{F} = \int_{0}^{1} F(\gamma'(t)) \, dt.
\]

Due to the fact that \( \bar{F} \) is convex (and constant) the geodesics of this action functional are straight lines. Consider the anisotropic Gauss map \( n : \Sigma \rightarrow \partial W_F \) which associates to a point \( b \) the unique vector \( n(b) \in \partial W_F \) such that \( T_b \Sigma \) is a supporting hyperplane for \( W_F \) at \( n(b) \) and \( n(b) \wedge \text{orient} \, T_b \Sigma = \text{orient} \, V \). With a choice of an inner product as above, this Gauss map can be seen as the composition of the standard Gauss map with the gradient of the support function \( h_{W_F} \). Clearly \( n \) is of class \( C^1 \) and due to the compactness of \( \Sigma \) the map

\[
\phi : [0, t] \times \Sigma \rightarrow V, \quad (b, s) \mapsto b + sn(b)
\]

is injective for \( t \) sufficiently small. This would be true for every non-vanishing vector field \( n \) along \( \Sigma \). The points in the image of \( \phi \) are those such that the \( \bar{F} \)-distance from \( \Sigma \) to them is at most \( t \). Moreover, for such a \( t \) we have

\[
\bar{B}_t := B + tW_F \setminus \bar{B} = \text{Im} \, \phi.
\]

Indeed the \( \supset \) inclusion is trivial while \( \subset \) uses the fact that a point \( x + sy \in B + tW_F \setminus B \) with \( x \in B \), \( y \in W_F \) and \( s \leq t \) is at an \( F^2 \) distance at most \( t \) from \( \Sigma \).

Therefore \( \phi \) is an oriented \( C^1 \) diffeomorphism onto \( \bar{B}_t \). Using the density \( \lambda \times F \) on \([0, t] \times \Sigma \) where \( \lambda \) is the Lebesgue measure on the line and \( \Omega \) on \( \bar{B}_t \), one gets:

\[
\text{Vol}(\bar{B}_t) = \int_{[0, t] \times \Sigma} J(d\phi; \lambda \times F, \Omega) \, d\lambda \otimes dF,
\]

If we can prove that for \( t \) sufficiently small \( |J(d\phi; \lambda \times F, \Omega)_{\phi(b,s)} - 1| \leq Mt \) for all \( s \leq t \) for some \( M > 0 \) then we would have

\[
\frac{1}{t} \int_{[0, t] \times \Sigma} 1 - J(d\phi; \lambda \times F, \Omega) \, d\lambda \otimes dF \leq \int_{[0, t] \times \Sigma} M \, d\lambda \otimes dF \rightarrow 0.
\]

Now

\[
J(d\phi; \lambda \times F, \Omega)_{\phi(b,s)} = \frac{|\Omega|(n(b) \wedge d\phi_s(v_1) \wedge \ldots \wedge d\phi_s(v_n))}{F(v_1 \wedge \ldots \wedge v_n)},
\]

for every oriented basis \( \{v_1, \ldots, v_n\} \) of \( T_b \Sigma \). Observe that by the multiplication property (5.2) for \( s = 0 \), \( J(d\phi; \lambda \times F, \Omega) = 1 \) and in fact the numerator is a polynomial in the variable \( s \) and therefore \( |J(d\phi; \lambda \times F, \Omega)_{\phi(b,s)} - 1| \leq Ms \) for \( s \) small.

In order to prove (5.5) for a general convex body \( W_F \) which contains the origin in the interior we use the fact that \( W_F \) can be approximated with respect to the Hausdorff metric \( \delta \) from the inside and from the outside with convex bodies of the type used in the first part of the proof. This follows from Theorem 1.8.13, Lemma 1.8.4 and Theorem 3.3.1 of [9]. This means in particular that there exist sequences of support functions \( h_{K_n} \) that uniformly (over every compact subset of \( V \)) approximate \( h_{W_F} \) from below and from above. Let \( K_2 \subset W_F \subset K_1 \) be two convex sets as above such that

\[
\sup_{\Sigma} |h_{K_1} - h_{K_2}| < \frac{\epsilon}{6H^n(\Sigma)},
\]
This gives
\[ \left| \int_{\Sigma} h_{K_1} - h_{K_2} \right| < \frac{\epsilon}{6} \]
and a similar inequality with \( h_{W_F} \) instead of \( h_{K_1} \). Now choose \( t \) with
\[ \left| \frac{\text{Vol}(B + tK_1 \setminus B)}{t} - \int_{\Sigma} h_{K_i} \, d\mathcal{H}^n \right| < \frac{\epsilon}{12}, \quad i = 1, 2. \]
Combining (5.7) and (5.8) we get that
\[ \left| \frac{V(B + tK_1 \setminus B) - V(B + tK_2 \setminus B)}{t} \right| < \frac{\epsilon}{3}. \]
We put everything together
\[
\left| \frac{V(B + tW_F \setminus B)}{t} - \int_{\Sigma} h_{W_F} \, d\mathcal{H}^n \right| \leq \left| \frac{V(B + tK_1 \setminus B) - V(B + tK_2 \setminus B)}{t} \right| + \\
+ \left| \frac{\text{Vol}(B + tK_2 \setminus B)}{t} - \int_{\Sigma} h_{K_2} \, d\mathcal{H}^n \right| + \int_{\Sigma} \left| h_{K_2} - h_{W_F} \right| \, d\mathcal{H}^n < \frac{\epsilon}{3} + \frac{\epsilon}{12} + \frac{\epsilon}{6} < \epsilon
\]
\[ \blacksquare \]
Recall now the Brunn-Minkowski inequality (Theorem 3.2.41 in [4]). Let \( A, B \subset V \) be two non-empty sets. Then
\[ \text{Vol}(A + B)^{\frac{1}{n+1}} \geq \text{Vol}(A)^{\frac{1}{n+1}} + \text{Vol}(B)^{\frac{1}{n+1}} \]
Taking \( A = tW_F \) we get:
\[ \frac{\text{Vol}(B + tW_F)^{\frac{1}{n+1}} - \text{Vol}(B)^{\frac{1}{n+1}}}{t} \geq \text{Vol}(W_F)^{\frac{1}{n+1}} \]
and letting \( t \to 0 \) we arrive at the anisotropic isoperimetric inequality:
\[ \frac{1}{n+1} \text{Vol}(B)^{-\frac{n}{n+1}} \int_{\partial B} dF \geq \text{Vol}(W_F)^{\frac{1}{n+1}}. \]
This can be rewritten in the more familiar form
\[ (5.9) \quad \text{Area}_F(\partial B) \geq (n+1) \text{Vol}(W_F)^{\frac{1}{n+1}} \text{Vol}(B)^{1-\frac{1}{n+1}}. \]
We should mention that a proof of this inequality for the case of convex bodies using mixed volumes appears in [2]. The anisotropic Sobolev inequality follows immediately using the same trick as in [5]. To wit, let \( f \) be a smooth function with compact support and let
\[ A_t := |f|^{-1}(t, \infty), \quad B_t := |f|^{-1}\{t\} \]
By Sard’s theorem \( B_t \) is a smooth hypersurface that bounds the domain \( A_t \) for almost all \( t > 0 \). At a regular point \( x \in \mathbb{R}^{n+1} \), let \( \tilde{\partial}_t \) be a lift of the canonical vector \( 1 \in \mathbb{R} \), i.e. \( d_x f|_{\tilde{\partial}_t} = 1 \). Let \( v_1, \ldots, v_n \) be an oriented basis of \( T_x B_t \), orienting \( B_t \) as a boundary of \( A_t \). The cojacobian (see (4.4)) of \( |f|: \mathbb{R}^{n+1} \to \mathbb{R} \) is then:
\[
C(d_x f; \lambda^*, \tilde{F}^*) = \frac{F(v_1 \wedge \ldots \wedge v_n)}{\Omega(\tilde{\partial}_t \wedge v_1 \wedge \ldots \wedge v_n)} = \frac{F(v_1 \wedge \ldots \wedge v_n)}{\Omega(-\tilde{\partial}_t \wedge v_1 \wedge \ldots \wedge v_n)} = \tilde{F}^*( -d_x f ) \quad (5.1)
\]
where \( F \) is the map defined in (5.1).
The reason for the appearance of the minus sign is that \( \bar{\partial_t} \wedge v_1 \wedge \ldots \wedge v_n \) is negatively oriented as \( \bar{\partial_t} \) points towards the interior of \( A_t \) and this happens because as \( t \) increases, \( A_t \) decreases. We therefore have:

\[
\int_V h_{W_F}(-\text{grad}_x f) \, d\Omega = \int_0^\infty \text{Area}_F(|f|^{-1}\{t\}) \, dt.
\]

It follows from (5.9) that

\[
\int_V h_{W_F}(-\text{grad}_x f) \, d\Omega \geq (n + 1) \text{Vol}(W_F) \frac{1}{n+1} \int_0^\infty \text{Vol}(A_t)^{1-\frac{1}{n+1}} \, dt
\]

and using

\[
\int_0^\infty \text{Vol}(A_t)^{\frac{1}{n+1}} \, dt \geq \left( \int_V |f|^{\frac{n+1}{n}} \, d\Omega \right)^{\frac{n}{n+1}}
\]

proved in [5], pag 488 one gets the anisotropic Sobolev inequality:

(5.10)

\[
\int_V h_{W_F}(-\text{grad}_x f) \, d\Omega \geq (n + 1) \text{Vol}(W_F)^{\frac{1}{n+1}} \|f\|_{L^{(n+1)'}(\mathbb{R}^n)}.
\]

where \( (n + 1)' = \frac{n+1}{n} \). This inequality implies of course to the anisotropic Sobolev embedding

\[
W^{1,1}_{F'}(V) \hookrightarrow L^{(n+1)'}(V, \Omega),
\]

where \( W^{1,1}_{F'}(V) \) is the completion of the normed vector space \( C^\infty_c(V) \) with the norm:

\[
\|f\|_{W^{1,1}_{F'}} := \int_V F^*(-df) \, d\Omega + \int_V |f| \, d\Omega.
\]

Applying (5.10) to \( f^p \) with \( v = \frac{pm}{n+1-p}, \ p > 1 \) and using Holder’s inequality just like in [5], pay 488 one gets the anisotropic Gagliardo-Nirenberg inequality:

\[
\left( \int |f|^p \right)^{\frac{1}{p'}} \leq C \left( \int |h_{W_F}(-\text{grad}_x f)|^p \right)^{\frac{1}{p}},
\]

where \( \frac{1}{p'} = \frac{1}{p} - \frac{1}{n+1} \) and \( C = \frac{pm}{(n+1-p)(n+1)} \text{Vol}(W_F)^{-\frac{1}{n+1}} \). This induces the embedding \( W^{1,p}_{F'} \hookrightarrow L^{p'} \).

6. A TUBE FORMULA

The proof of Theorem 5.3 gives us for free the following anisotropic "half-tube" formula for hypersurfaces. We will restrict attention to those Wulff shapes \( W_F \) for which the support function \( h_{W_F} \) is smooth which implies the differentiability of the Gauss map \( \Sigma \rightarrow \partial W_F \).

**Definition 6.1.** The (positively oriented) anisotropic second fundamental form of an oriented hypersurface \( \Sigma \) is the bundle endomorphism \( \Pi^F : T\Sigma \rightarrow T\Sigma \), which at a point \( b \) is defined by

\[
\Pi^F_b(v) = -P_b(d_b n(v)),
\]

where \( n : \Sigma \rightarrow \partial W_F \) is the anisotropic Gauss map corresponding to the orientation and \( P_b \) is the projection to \( T_b \Sigma \) along the vector \( n(b) \). The fundamental symmetric polynomials in the eigenvalues of \( \Pi^F \) are denoted by \( c_k(\Pi^F) \).

With this definition we see that (5.6) equals \( |\det(1 - s \Pi^F)| \) and for \( s \) small one can forget about \( |\cdot| \). We therefore get
Proposition 6.2. Let \( \Sigma \subset V \) be a smooth hypersurface oriented using the normal that points to the unbounded component of \( V \setminus \Sigma \). Let \( \text{Vol}^+(\varepsilon) \) be the volume of the set of points that lie in the unbounded component at an \( \bar{F} \)-distance at most \( \varepsilon \) from \( \Sigma \). Then for \( \varepsilon \) sufficiently small the following holds:

\[
\text{Vol}^+(\varepsilon) = \sum_{k=1}^{n+1} (-1)^{k-1} \frac{\varepsilon^k}{k} \int_{\Sigma} c_k(\bar{\Pi}^F) \, dF.
\]

Of course this is only a half-tube formula. To get the other half we need to consider the negatively oriented anisotropic Gauss map \( n^- : \Sigma \to \partial W_F \) which associates to every \( b \in \Sigma \) the unique vector \( n^-(b) \in \partial W_F \) such that \( T_{n^-(b)} \partial W_F = T_b \Sigma \) and \( n^-(b) \wedge \text{orient} \Sigma = - \text{orient} V \). The (negatively oriented) second fundamental form \( \bar{\Pi}^F \) is defined analogously to 6.1. Notice that when \( \bar{F} \) is reversible, i.e. \( \bar{F}(v) = \bar{F}(-v) \), one has the relation

\[
\bar{\Pi}^F = -\bar{\Pi}^F.
\]

Corollary 6.3. Let \( \Sigma \subset V \) be a smooth hypersurface as above and let \( T_{\varepsilon} = \Sigma + \varepsilon W_F \). Then for \( \varepsilon \) sufficiently small the following holds:

\[
\text{Vol}(T_{\varepsilon}) = \sum_{k=1}^{n+1} (-1)^{k-1} \frac{\varepsilon^k}{k} \int_{\Sigma} c_k(\bar{\Pi}^F) + c_k(\bar{\Pi}^F) \, dF.
\]

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