EXAMPLES OF NON-FORMAL CLOSED \((k - 1)\)-CONNECTED MANIFOLDS OF DIMENSIONS \(4k - 1\) AND MORE

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Abstract. We construct closed \((k - 1)\)-connected manifolds of dimensions \(\geq 4k - 1\) that possess non-trivial rational Massey triple products. We also construct examples of manifolds \(M\) such that all the cup-products of elements of \(H^k(M)\) vanish, while the group \(H^{3k-1}(M; \mathbb{Q})\) is generated by Massey products: such examples are useful for theory of systols.

For every \(k\) we construct closed \((k - 1)\)-connected manifolds of dimensions \(\geq 4k - 1\) that possess non-trivial rational Massey triple products and therefore are non-formal. For \(k = 1\) such manifolds can be obtained as the products of Heisenberg manifold with circles. For \(k = 2\) such examples are also known, see e.g. [4, 2], but even in this case our construction seems more direct and simple.

Miller [3] proved that every closed \((k - 1)\)-connected manifold \(M\) of dimension \(\leq 4k - 2\) is formal. In particular, all rational Massey products in \(M\) vanish. So, neither Miller’s nor our results can be improved.

Given a diagram

\[ B \supset A \xrightarrow{f} Y \]

we denote by \(Z_f\) its double mapping cylinder.

Recall that a subset \(S\) of a space \(\mathbb{R}^m\) is called radial if, for all points \(s \in S\), the linear segment \([0, s]\) contains precisely one point of \(S\) (namely, \(s\)).

1. Proposition. Let \(B\) be a finite polyhedron in \(\mathbb{R}^m, m > 1\), let \(A\) be a subpolyhedron of \(B\) such that \(A \setminus \{0\}\) is radial in \(\mathbb{R}^m\), and let \(Y\) be a finite polyhedron in \(\mathbb{R}^n\). Then the double cylinder \(Z_f\) of any simplicial map \(f : A \to Y\) admits a PL embedding in \(\mathbb{R}^{m+n}\).

Proof. We denote by \(0_m\) and \(0_n\), the origins of spaces \(\mathbb{R}^m\) and \(\mathbb{R}^n\), respectively. We first consider the case when \(0_m \notin A\). We assume that \(Y\) is far away from \(0_n\). Let \(\Gamma \in \mathbb{R}^m \times \mathbb{R}^n\) be the graph of the map \(f\). We join every point \((x, f(x)) \in \Gamma, x \in A\) with the point \((0_m, f(x)) \in \mathbb{R}^m \times Y \subset \mathbb{R}^{m+n}\) by the linear segment. Then, since \(A\) is radial, we get an embedding of the mapping cylinder \(M_f\) of \(f\) to \(\mathbb{R}^{m+n}\).

Moreover, if we join the points \((x, 0_n)\) with \((x, f(x))\) by the linear segment, we still have an embedding \(M_f \hookrightarrow \mathbb{R}^{m+n}\). Here (the image of) \(M_f\) is formed by segments \([x, 0_n], (x, f(x)]\) and \([x, f(x)), (0_m, f(x))\]. Finally, we get an embedding of the double mapping cylinder \(Z_f\) to \(\mathbb{R}^{m+n}\) by adding the space \(B\) to the embedded mapping cylinder \(M_f\).

The case \(0_m \in A\) can be considered similarly. We can assume that there is a point \(y_0 \in Y\) which is the closest to \(0_n \in \mathbb{R}^n\), i.e. \(||y_0|| < ||y||\) if \(y \neq y_0\) and \(y \in Y\). We can also assume that \(f(0_m) = y_0\). Consider the map \(f' = f|(A \setminus \{0\})\) and the
therefore the indeterminacy of the Massey product is zero. Furthermore, the map \( H \) has zero indeterminacy and is non-zero.

**Proof.** Choose a base point on the boundary of each disc \( D_i^n, i = 1, \ldots, k \) and consider the wedge \( \vee_{i=1}^n D_i^n \). We can regard this wedge as a polyhedron in \( \mathbb{R}^m \) such that the base point is the origin and \( \vee S_i^{m-1} \setminus \{0\} \) is a radial set. Now the claim follows from Proposition 1.

Consider the wedge \( K = S^{k_1} \vee S^{k_2} \vee S^{k_3} \) of spheres with \( k_i \geq 2 \) and let \( \iota_r \in \pi_{k_r}(K) \) be represented by the inclusion map \( S^{k_r} \subset K \). Set \( m = k_1 + k_2 + k_3 - 1 \), let \( f : S^{m-1} \to K \) represent the element \( [\iota_1, [\iota_2, \iota_3]] \), and let \( X \) be the cone of the map \( f \). Let \( \alpha_i \in H^{k_i}(X) \) be the cohomology class which takes the value 1 on the cell \( S_i^k \) of \( X \) and 0 on other cells. We recall the following classical result

**Theorem.** The Massey product \( \langle \alpha_1, \alpha_2, \alpha_3 \rangle \in H^{k_1+k_2+k_3-1}(X) \) has the zero indeterminacy and takes the value \((-1)^{k_1} \) on the \((m-1)\)-dimensional cell of \( X \).

**Proof.** See [5, Lemma 7].

Now let \( k_1 = k_2 = k_3 = k \) and consider the corresponding space \( X \). According to Proposition 1, \( X \) admits a PL embedding in \( \mathbb{R}^N \) with \( N \geq 4k \). Fix such an embedding and let \( W \) be a closed regular neighborhood of \( X \) in \( \mathbb{R}^N \). So, \( W \) is a manifold with the boundary \( V = \partial W \). Furthermore, \( W \) has the homotopy type of \( X \). (Notice that \( W \) is a PL manifold by the construction, but without loss of generality we can assume that \( W \) is smooth.)

**Proposition.** The manifold \( V \) is \((k-1)\)-connected.

**Proof.** Consider a sphere \( S_i, i < k \) in \( V \). Since \( W \) is \((k-1)\)-connected, there exists a disk \( D_{i+1} \) in \( W \) with \( \partial D_{i+1} = S_i \). Since \( i + 1 + \dim X \leq 4k - 1 < N \), we can assume that \( D_{i+1} \cap X = \emptyset \). But \( V \) is a retract of \( W \setminus X \), and thus \( S_i \) bounds a disk in \( V \).

**Proposition.** \( H^i(W, V) = H_{N-i}(X) \).

**Proof.** We have

\[
H^i(W, V) = H_{N-i}(W) = H_{N-i}(X)
\]

where the first equality holds by the Poincaré duality, see e.g. Dold [1].

Consider the map

\[
g : V \xrightarrow{i} W \xrightarrow{r} X
\]

where \( i \) is the inclusion and \( r \) is a deformation retraction.

**Theorem.** If \( N \neq 5k - 1, 6k - 2 \), then the Massey product \( \langle g^*\alpha_1, g_*\alpha_2, g^*\alpha_3 \rangle \) has zero indeterminacy and is non-zero.

**Proof.** Notice that \( H_i(X) = 0 \) for \( i \neq 0, k, 3k - 1 \). We have \( H^{2k-1}(W) = H^{2k-1}(X) = 0 \) and \( H^{2k}(W, V) = H_{n-2k}(X) = 0 \). Now, in view of the exactness of the sequence \( H^{2k-1}(W) \to H^{2k-1}(V) \to H^{2k}(W, V) \) we have \( H^{2k-1}(V) = 0 \), and therefore the indeterminacy of the Massey product is zero. Furthermore, the map \( i^* : H^{3k-1}(W) \to H^{3k-1}(V) \) is injective since \( H^{3k-1}(W, V) = H_{n-3k+1}(X) = 0 \).
Thus, the map $g^* : H^{3k-1}(X) \to H^{3k-1}(V)$ is injective. But $g^*(\langle \alpha_1, g_*\alpha_2, g_*\alpha_3 \rangle) = \langle g^*\alpha_1, g_*\alpha_2, g^*\alpha_3 \rangle$ because both parts of the equality have zero indeterminacies. □

Thus, we have examples of $(k-1)$-connected manifolds with non-trivial triple Massey product of dimensions $d \geq 4k-1$ but $d \neq 5k-2, 6k-3$. In order to construct an example in exceptional dimensions, just take the double of the manifold $W$ (or multiple by the sphere of the correspondent dimension if $k \neq 2$).

When we put the first version of the paper into the e-archive, Mikhail Katz asked us if we can construct a closed manifold $M$ such that all the cup-products of elements of $H^k(M)$ vanish, while the group $H^{3k-1}(M; \mathbb{Q})$ is generated by Massey products. Now we present such an example.

**7. Lemma.** Consider a wedge $X \vee Y$ and three elements $u, v, w \in H^*(X)$ such that $uv = 0$, $u|Y = 0 = v|Y$ and $w|X = 0$. Then all the Massey products $\langle u, v, w \rangle$ and $\langle u, w, v \rangle$ are trivial, i.e. they contain the zero element.

**Proof.** This follows from the following fact: If $A \in C^*(X \vee Y)$ and $B \in C^*(X \vee Y)$ are cochains with the supports in $X$ and $Y$, respectively, then their product is equal to zero. We leave the details to the reader. □

Consider the wedge $S^1_k \vee S^2_k \vee S^3_k \vee S^4_k$ of $k$-dimensional spheres, $k > 1$. Let $\iota_m \in \pi_k(S^m_k)$ be the generator. Set

$$Z = (\bigvee_{i=1}^4 S^k_i) \cup f_1 e^{3k-1}$$

where $f_1 : S^{3k-2} \to \bigvee_{i=1}^4 S^k_i$ represents the homotopy class $[\iota_1, [\iota_2, \iota_3]]$. Let $\alpha_i \in H^k(Z)$ be the cohomology class which takes the value $1$ on the cell $S^k_i$ of $Z$ and $0$ on other cells.

**8. Corollary.** If at least one of the indices $i, j, k$ is equal to $4$, then $\langle \alpha_i, \alpha_j, \alpha_k \rangle = 0$ in $X$.

**Proof.** This follows directly from Lemma □ since

$$Z = (\bigvee_{i=1}^4 S^k_i) \cup f_1 e^{3k-1} \lor S^k_i.$$

For convenience of notation, we set $\iota_5 = \iota_1$ and $\iota_6 = \iota_2$. Let $f_m : S^{3k-2} \to \bigvee_{i=1}^4 S^k_i$ be the map which represents $[\iota_m, [\iota_{m+1}, \iota_{m+2}]], m = 1, 2, 3, 4$. Consider the map

$$f : (\bigvee_{i=1}^4 S^{3k-2}) \to \bigvee_{i=1}^4 S^k_i$$

such that $f|S^{3k-2} = f_i$ and set $X = C_f$. We define $\alpha_m \in H^k(X)$ the cohomology class which takes the value $1$ on the cell $S^k_i$ of $X$ and $0$ on other cells. For convenience of notation, we set $\alpha_5 = \alpha_1$ and $\alpha_6 = \alpha_2$.

**9. Lemma.** The homology classes $\langle \alpha_m, \alpha_{m+1}, \alpha_{m+2} \rangle$ are linearly independent in $H^{3k-1}(X)$.

**Proof.** First, notice all these Massey products are defined and have zero indeterminacies. Now, suppose that $\sum_{m=1}^4 c_m \langle \alpha_m, \alpha_{m+1}, \alpha_{m+2} \rangle = 0$ for some $c_m \in \mathbb{R}$. Consider the space $Z$ as in □ and the obvious inclusion $j : Z \to X$. Then $j^* \langle \alpha_m, \alpha_{m+1}, \alpha_{m+2} \rangle = 0$ for $m = 2, 3, 4$ by Corollary □ while $j^* \langle \alpha_1, \alpha_2, \alpha_3 \rangle \neq 0$ by Theorem □ Therefore $c_1 = 0$. Similarly, we can prove that $c_m = 0$ for all $m$. □

Now, because of Proposition □ $X$ can be regarded as a polyhedron in $\mathbb{R}^N$ with $N \geq 4k$. Let $W$ be a regular neighborhood of $X$ in $\mathbb{R}^N$ and set $M = \partial W$. 


10. Theorem. If \( N \neq 4k, 5k-1, 6k-2, 6k-1 \) then \( H^{3k-1}(M; \mathbb{Q}) \) is generated by triple Massey products, while all the cup-products of elements of \( H^k(M) \) vanish.

Proof. Consider the map

\[
g: V \xrightarrow{i} W \xrightarrow{r} X
\]

where \( i \) is the inclusion and \( r \) is a deformation retraction. Using the isomorphisms

\[
H^i(W, M) \cong H_{N-i}(X) \quad \text{and} \quad H^i(W) \cong H^i(X),
\]

and the exactness of the sequence

\[
H^i(W, M) \xrightarrow{i} H^i(W) \xrightarrow{i^*} H^i(M) \xrightarrow{r^*} H^{i+1}(W, M).
\]

we conclude that \( H^{2k-1}(M) = 0 \) and

\[
g^*: H^{3k-1}(X) \to H^{3k-1}(M)
\]

is an isomorphism. Now, the equality \( H^{2k-1}(M) = 0 \) implies that all the Massey products \( \langle \alpha_i, \alpha_j, \alpha_k \rangle \) have zero indeterminacies. Furthermore, since \( g^* \) is an isomorphism, Lemma 9 implies that the \( g^* \)-images of the classes \( \langle \alpha_m, \alpha_{m+1}, \alpha_{m+2} \rangle \), \( m = 1, 2, 3, 4 \) in \( M \) form a basis of \( H^{3k-1}(M; \mathbb{Q}) \). Finally, the map \( i^*: H^k(W) \to H^k(M) \) is surjective for \( N \neq 4k \), and so the cup-products of elements of \( H^k(M) \) vanish. \( \square \)

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