Research Article

Dynamics on Effect of Prey Refuge Proportional to Predator in Discrete-Time Prey-Predator Model

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Prey-predator models with refuge effect have great importance in the context of ecology. Constant refuge and refuge proportional to prey are the most popular concepts of refuge in the existing literature. Now, there are new different types of refuge concepts attracting researchers. This study considers a refuge concept proportional to the predator due to the fear induced by predators. When predators increase, fears also increase and that is why prey refuges also increase. Here, we examine the influence of prey refuge proportional to predator effect in a discrete prey-predator interaction with the Holling type II functional response model. Is this refuge stabilizing or destabilizing the system? That is the central question of this study. The existence and stability of fixed points, Period-Doubling Bifurcation, Neimark–Sacker Bifurcation, the influence of prey refuge, and chaos are analyzed. This work provides the bifurcation diagrams and Lyapunov exponents to analyze the refuge parameter of the model. The proposed discrete model indicates rich dynamics as the effect of prey refuge through numerical simulations.

1. Introduction

In the current biomathematical literature, prey-predator interaction has become an exciting subject matter due to its influence on the environment. Many researchers [1–4] have devoted considerable time to explore several perspectives of the dynamical behaviour of this subject matter in ecology and the associated growth of population models [5–7]. Nature offers some amount of protection to some prey populations in the form of refuge dimensions. In some instances, refugia give a prolonged prey-predator interaction [8, 9] to reduce the probability of extinction due to predation activity. This phenomenon has been investigated extensively by many researchers in the discrete domain of refuse concepts [10–12].

In the context of populations characterized by overlapping generations, the nature of the birth process is a continuous matter; therefore, the predator-prey relationship is frequently developed through different models utilizing a deterministic approach such as ordinary differential equations. Several species, including monocarpic plants and semelparous animals, possess a discrete nonoverlapping generation character and have birth and regular breeding seasons. Their interactions are presented in the form of difference equations or other forms such as discrete-time mappings [13–17]. A discrete-time prey-predator model is usually characterized by more complicated dynamics behaviour than the associated continuous-time models [18–21]. Several researchers have focused on this concept to present a comprehensive rich dynamics of this phenomenon, including stability analysis of equilibria [22–27], Period-Doubling Bifurcation [28], Neimark–Sacker bifurcation [29], and chaos control [30]. Liu and Cai [31] explored the presence of bifurcation and chaos in a discrete-time prey-predator system. Elettreby et al. [32] used mixed functional response to investigate the dynamics of a discrete
prey-predator model. AlSharawi et al. [33] studied a discrete prey-predator model with a nonmonotonic functional response and strong Allee effect in prey. Wang and Fečkan [34] studied the dynamics of a discrete nonlinear prey-predator model. Reich [35] investigated two discrete-time counterparts of a continuous-time prey-predator model. Khan and Khalique [36] discussed Neimark–Sacker bifurcation and hybrid control in a discrete-time Lotka–Volterra model. For some more dynamical investigations related to different versions of prey-predator models, we refer to Elsadany et al. [37]; Baydemir et al. [38]; Rozikov and Shoyimardonov [39]; Singh and Deolia [40]; and references therein.

In this paper, we consider a refuge term proportional to predator density due to fear induced by predators. The earlier literature has pointed out that the use of refuge by a fixed number or a small percentage of prey exerts a stabilizing effect on the dynamics of the interacting populations. This article is concerned with the above statement assuming that the quantity of prey in refugia is proportional to that of predators due to the fear induced by the predator. We analyze the dynamic properties of such a prey-predator model with prey self-limitation. The rest of the paper is partitioned as follows. A discrete-time prey-predator model with refuge is formulated in the second section. Section 3 is devoted to the local stability analysis of the model. Bifurcation analysis of the proposed model is presented in Section 4. Section 5 deals with the influence of prey refuge on the proposed model. Section 6 gives a chaos control procedure. Section 7 presents numerical simulations with support of the presented dynamical analysis of the proposed model. Section 8 presents the conclusion that is the last aspect of the findings from this study.

2. Development of a Discrete Prey-Predator System

The densities of prey and predator populations vary with time, have a uniform distribution over space, and have no stage structure for neither prey nor predators. The proposed model takes into consideration a generalised prey-predator model incorporating the logistic growth of prey with functional response \( \varphi(x) \) of the predator population as \( (dx/dt) = rx(1 - (x/k)) - c\varphi(x)y \), and the predator as a population is epitomised by the expression \( (dy/dt) = d\varphi(x)y - fy \), where \( x \) and \( y \) are the density of prey and predator populations correspondingly at any time \( t \). The parameters \( r, k, c, d, \) and \( f \) are all taken to be positive constants and have its appropriate biological interpretations, respectively, \( r \) denotes the intrinsic per capita growth rate of prey population, \( k \) represents the environmental carrying capacity of prey population, \( c \) denotes the maximal per capita consumption rate of predators, \( d \) considers the efficiency with which predators transform consumed prey into new predators, and \( f \) represents the per capita death rate of predators. \( \varphi(x) \) deals with the functional response of the predator population and fulfils the assumption \( \varphi(0) = 0 \) and \( \varphi'(x) > 0 \) for \( x > 0 \). There exists a quantity \( x_1 \) of prey population that incorporates refuges, and therefore we have modified the functional response to incorporate prey refuges, and we consider \( \varphi(x) = (x/(e + x)) \), and \( x_1 = by \). The mathematical form of the proposed predator-prey model is as follows:

\[
\begin{align*}
\frac{dx}{dt} &= rx(1 - \frac{x}{k}) - \frac{c(x - by)y}{e + x - by}, \\
\frac{dy}{dt} &= \frac{d(x - by)y}{e + x - by} - fy. 
\end{align*}
\]

To arrive at a discrete-time model from (1), let \( (dx/dt) = ((x_{t+1} - x_t)/h) \) and \( (dy/dt) = ((y_{t+1} - y_t)/h) \), where \( x_t \) and \( y_t \) are the densities of the prey and predator populations, respectively, in discrete time \( t \). Let \( h \rightarrow 1 \) and \( f = 1 \), we obtain the equations for the \( (n + 1)th \) generation of the prey and predator populations substituting \( t \) by \( n \) as follows:

\[
\begin{align*}
x_{n+1} &= (r + 1)x_n\left(1 - \frac{r}{k(r + 1)x_n}\right) - \frac{c(x_n - by_n)y_n}{e + x_n - by_n}, \\
y_{n+1} &= \frac{d(x_n - by_n)y_n}{e + x_n - by_n}. 
\end{align*}
\]

Let \( (r/k(r + 1)) = 1 \) and \( (r + 1) = a \), we obtain the discrete-time predator-prey system from (2) as follows:

\[
\begin{align*}
x_{n+1} &= ax_n(1 - x_n) - \frac{c(x_n - by_n)y_n}{e + x_n - by_n}, \\
y_{n+1} &= \frac{d(x_n - by_n)y_n}{e + x_n - by_n}. 
\end{align*}
\]

where \( a, b, c, d, \) and \( e \) are all positive constants. By the biological meaning of the model variables, we only consider the system in the region \( \Omega = \{(x, y) : x \geq 0, y \geq 0\} \) in the \((x, y)\) – plane.

3. Fixed Points and Stability Analysis

Fixed points of the discrete system (3) are determined by solving the following nonlinear system of equations:

\[
\begin{align*}
x &= ax(1 - x) - \frac{c(x - by)y}{e + x - by}, \\
y &= \frac{d(x - by)y}{e + x - by}. 
\end{align*}
\]

We get three nonnegative fixed points by solving above equations: (i) \( P_0 = (0, 0) \), (ii) \( P_1 = ((a - 1)/a, 0) \), \( a > 1 \), and (iii) \( P_2 = (x_2, y_2) \); here, \( x_2 \) is positive root of the equation \( x^2 + Ax + B = 0 \), where \( A = ((1 - a)/a) + (c/abcd) \), \( B = -((ce/bd)(d - 1)) \), and \( y_2 = (x_2/b) - (e/b(d - 1)) \).

3.1. Dynamic Behaviour of the Discrete System. In this part, we examine the local behaviour of the discrete system (3) for every steady point of the model. The stability of system (3) is accomplished by deriving a Jacobian matrix connected to
each equilibrium point. The Jacobian Matrix $J$ for system (3) is

$$J = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$  \hspace{1cm} (5)

where $a_{11} = a(1 - 2x) - (c + e)(e + x - by)^2$, $a_{12} = - (c + e)(e + x - by)^2(e + x - 2by)/(e + x - by)^2$, and $a_{22} = (d(e - 2by)/(e + x - by)^2)/(e + x - by)^2$.

The characteristic equation of the matrix $J$ is $\lambda^2 - \text{Tr}(J)\lambda + \text{Det}(J) = 0$, where $\text{Tr}(J)$ = trace of matrix $= a(1 - 2x) + ((d(e - 2by)/(e + x - by)^2))/\text{Det}(J) = \text{determinant}$ of matrix $= a(1 - 2x)(d(e - 2by)/(e + x - by)^2)/(e + x - by)^2$.

Therefore, the discrete-time system (3) possesses the following:

(i) A dissipative dynamical system if $|a(1 - 2x)(d(e - 2by)/(e + x - by)^2)/(e + x - by)^2| < 1$

(ii) A conservative dynamical system if and only if $|a(1 - 2x)(d(e - 2by)/(e + x - by)^2)/(e + x - by)^2| = 1$

(iii) An undissipated dynamical system otherwise

3.1.1 Stability and Dynamic Behaviour of the Model at $P_0$.

The dynamical behaviour of the system is studied utilizing the Jacobian matrix at the fixed point $P_0(0,0)$. The Jacobian matrix at $P_0(0,0)$ is $J = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$. The equilibrium point $P_0(0,0)$ is called (i) sink if $|a| < 1$; (ii) saddle if $|a| > 1$; and (iii) nonhyperbolic if $|a| = 1$.

3.1.2 Stability and Dynamic Behaviour at $P_1$. The Jacobian matrix at the fixed point $P_1((a - 1)/a, 0)$ is $J = \begin{bmatrix} 2 - a - (c(a - 1)/(ae + a - 1)) \\ 0 \end{bmatrix}$. The equilibrium point $P_1$ is said to be

(i) Sink if $|2 - a| < 1$ and $|d(a - 1)/(ae + a - 1)| < 1$

(ii) Source if $|2 - a| > 1$ and $|d(a - 1)/(ae + a - 1)| > 1$

(iii) Saddle if $|2 - a| > 1$ and $|d(a - 1)/(ae + a - 1)| < 1$

(iv) Nonhyperbolic if $|2 - a| = 1$ or $|d(a - 1)/(ae + a - 1)| = 1$

3.1.3 Dynamic Behaviour around the Interior Fixed Point.

Following the Jacobian matrix at the interior fixed point $P_2(x_2, y_2)$, we get

\begin{align*}
1 - \text{Tr}(L) + \text{Det}(L) &= 1 - a(1 - 2x_2) - \frac{d(e(x_2 - 2by_2)/(x_2 - 2by_2)^2) - cey_2}{(e + x_2 - by_2)^2} + a(1 - 2x_2)\frac{d(e(x_2 - 2by_2)/(x_2 - 2by_2)^2)}{(e + x_2 - by_2)^2}, \\
1 + \text{Tr}(L) + \text{Det}(L) &= 1 + a(1 - 2x_2) + \frac{d(e(x_2 - 2by_2)/(x_2 - 2by_2)^2) - cey_2}{(e + x_2 - by_2)^2} + a(1 - 2x_2)\frac{d(e(x_2 - 2by_2)/(x_2 - 2by_2)^2)}{(e + x_2 - by_2)^2}.
\end{align*}

If $1 - \text{Tr}(L) + \text{Det}(L) > 0$, then interior equilibrium point $P_2(x_2, y_2)$ is as follows:

(i) Sink if $1 + \text{Tr}(L) + \text{Det}(L) > 0$ and $\text{Det}(L) < 1$

(ii) Source if $1 + \text{Tr}(L) + \text{Det}(L) > 0$ and $\text{Det}(L) > 1$

(iii) Saddle if $1 + \text{Tr}(L) + \text{Det}(L) < 0$

(iv) Nonhyperbolic if $1 + \text{Tr}(L) + \text{Det}(L) = 0$ and $\text{Tr}(L) \neq 0, 2$ or $|\text{Tr}(L)|^2 - 4\text{Det}(L) < 0$ and $\text{Det}(L) = 1$

If $1 - \text{Tr}(L) + \text{Det}(L) > 0$, $1 + \text{Tr}(L) + \text{Det}(L) = 0$, and $\text{Tr}(L) \neq 0, 2$, then $P_2(x_2, y_2)$ can undergo Period-Doubling Bifurcation (PDB).

If $1 - \text{Tr}(L) + \text{Det}(L) > 0$, $(\text{Tr}(L))^2 - 4\text{Det}(L) < 0$, and $\text{Det}(L) = 1$, then $P_2(x_2, y_2)$ can undergo Neimark–Sacker Bifurcation (NSB).

4. Bifurcation Analysis

In this section, we investigate the existence of NSB and PDB at the positive fixed point $P_2(x_2, y_2)$ of the discrete-time of

(3). Countless numbers of dynamical properties of the system can be analyzed with regard to the emergence of NSB and PDB. Bifurcation frequently occurs when there is a change in the stability of a fixed point. Thus, the qualitative properties of a dynamical system vary. The NSB and PDB are explored in the positive fixed point direction $P_2(x_2, y_2)$ of system (3) considering refuge parameter $b$ as a bifurcation parameter. Utilizing the emergence of NSB, closed invariant circles are obtained. The bifurcation can be classified as supercritical or subcritical, leading to a stable or unstable closed invariant curve, in that order. A PDB in a discrete dynamical system is a bifurcation where a small perturbation in a parameter value brings about the system moving to a new behaviour with the double period of the original system.

4.1. Neimark–Sacker Bifurcation. Here, we examine the NSB of the discrete prey-predator model (3) at $P_2(x_2, y_2)$ for the parameters lying in the following set $A_{\text{NSB}} = [(a, b = b_{\text{NSB}}, c, d, e): 1 - \text{Tr}(L) + \text{Det}(L) > 0, (\text{Tr}(L))^2 - 4\text{Det}(L) < 0, \text{Det}(L) = 1]$. 
In exploring the NSB, \( b \) is utilised as the bifurcation parameter. Further, \( b' \) is the perturbation of \( b \) where \( |b'| \ll 1 \), and perturbation of the model is considered as follows:

\[
x_{n+1} = ax_n(1 - x_n) - \frac{c(x_n - (b + b')y_n)y_n}{e + x_n - (b + b')y_n} = f(x_n, y_n, b'),
\]

\[
y_{n+1} = \frac{d(x_n - (b + b')y_n)}{e + x_n - (b + b')y_n} = g(x_n, y_n, b').
\]

(7)

Let \( u_n = x_n - x_2 \) and \( v_n = y_n - y_2 \), then equilibrium \( P_2(x_2, y_2) \) is converted into the origin, and further making expansion \( f \) and \( g \) as a Taylor series at \((u_n, v_n) = (0, 0)\) to the third order, model (7) turns to

\[
u_{n+1} = \alpha_1 u_n + \alpha_2 v_n + \alpha_3 u_n^2 + \alpha_4 u_n v_n + \alpha_5 v_n^2 + \alpha_6 u_n^3 + \alpha_7 u_n^2 v_n + \alpha_8 u_n v_n^2 + \alpha_9 v_n^3 + O\left(\left(|u_n| + |v_n|\right)^4\right),
\]

\[
v_{n+1} = \beta_1 u_n + \beta_2 v_n + \beta_3 u_n^2 + \beta_4 u_n v_n + \beta_5 v_n^2 + \beta_6 u_n^3 + \beta_7 u_n^2 v_n + \beta_8 u_n v_n^2 + \beta_9 v_n^3 + O\left(\left(|u_n| + |v_n|\right)^4\right).
\]

(8)

where \( \alpha_1 = f_x(x_2, y_2, 0) \), \( \alpha_2 = f_y(x_2, y_2, 0) \), \( \alpha_3 = f_{xx}(x_2, y_2, 0) \), \( \alpha_4 = f_{xy}(x_2, y_2, 0) \), \( \alpha_5 = f_{yy}(x_2, y_2, 0) \), \( \alpha_6 = f_{xxx}(x_2, y_2, 0) \), \( \alpha_7 = f_{xxy}(x_2, y_2, 0) \), \( \alpha_8 = f_{xyy}(x_2, y_2, 0) \), \( \alpha_9 = f_{yyy}(x_2, y_2, 0) \), \( \beta_1 = g_x(x_2, y_2, 0) \), \( \beta_2 = g_y(x_2, y_2, 0) \), \( \beta_3 = g_{xx}(x_2, y_2, 0) \), \( \beta_4 = g_{xy}(x_2, y_2, 0) \), \( \beta_5 = g_{yy}(x_2, y_2, 0) \), \( \beta_6 = g_{xxx}(x_2, y_2, 0) \), \( \beta_7 = g_{xxy}(x_2, y_2, 0) \), \( \beta_8 = g_{xyy}(x_2, y_2, 0) \), \( \beta_9 = g_{yyy}(x_2, y_2, 0) \).

Note that the characteristic equation associated with the linearization of model (8) at \((u_n, v_n) = (0, 0)\) is given by

\[
\lambda^2 - \text{Tr}(L_1(b'))\lambda + \text{Det}(L_1(b')) = 0.
\]

The roots of the characteristic equation are \( \lambda_{1,2}(b') = \left(\frac{\text{Tr}(L_1(b')) ± \sqrt{\text{Det}(L_1(b'))}}{2}\right) \).

From \( |\lambda_{1,2}(b')| = 1 \), and \( b' = 0 \), we have \( |\lambda_{1,2}(b')| = |\text{Det}(L_1(b'))|^{1/2} \) and \( l = |\text{det}(\lambda_{1,2}(b'))|\).

In addition, it is required that when \( b^* = 0 \), \( \lambda_{1,2} \neq 1 \), \( i = 1, 2, 3, 4 \), which is equivalent to \( \text{Tr}(L_1(0)) \neq -2, -1, 1, 2 \).

To study the normal form, let \( y = \text{Im}(\lambda_{1,2}) \) and \( \delta = \text{Re}(\lambda_{1,2}) \). We define \( T = \begin{bmatrix} 0 & 1 \\ -1 & \delta \end{bmatrix} \), and using the transformation

\[
\begin{bmatrix} u_n \\ v_n \end{bmatrix} = T\begin{bmatrix} x_n \\ y_n \end{bmatrix},
\]

model (8) becomes

\[
x_{n+1} = \delta x_n - y y_n + f_1(x_n, y_n),
\]

\[
y_{n+1} = y x_n + \delta y_n + g_1(x_n, y_n),
\]

(9)

where the functions \( f_1 \) and \( g_1 \) denote the terms in model (9) in variables \((x_n, y_n)\) with the order at least two.

To undergo NSB, we require that the following discriminatory quantity \( \Omega \) be nonzero:

\[
\Omega = -\text{Re}\left(\frac{(1 - 2\lambda^2)\xi_{20}}{1 - \lambda} \xi_{11} \xi_{20} - \frac{1}{2} |\xi_{11}|^2 - |\xi_{02}|^2 + \text{Re}(\bar{\lambda} \xi_{21})\right),
\]

(10)

where
Finally, the above analysis leads to the following associated results.

**Theorem 1.** If $\Omega \neq 0$, then model (3) undergoes NSB at $P_2(x_2, y_2)$ when the parameter $b$ changes in a small neighbourhood of $b_{NSB}$. If $\Omega < 0$ ($\Omega > 0$), then an attracting (repelling) invariant closed curve bifurcates from $P_2(x_2, y_2)$.

### 4.2. Period-Doubling Bifurcation.

Here, we consider one of the eigenvalues of the positive fixed point that $P_2(x_2, y_2)$ is $\lambda_1 = -1$, and the other ($\lambda_2$) is neither 1 nor $-1$, whenever parameters of the model are located in the following set $A_{PDDB} = \{(a, b = b_{PDDB}, c, d, e): 1 - Tr(J) + Det(J) > 0, 1 + Tr(J) + Det(J) = 0, Tr(J) \neq 0, 2\}$. Here, we discuss PDB of model (3) at $P_2(x_2, y_2)$ when parameters change in a small neighbourhood of $A_{PDDB}$. In studying the flip bifurcation, $b$ is made use as the bifurcation parameter. Furthermore, $b^*$ is the perturbation of $b$ where $|b^*| \ll 1$, and perturbation of the model is considered as follows:

$$
\begin{align*}
x_{n+1} &= ax_n(1-x_n) - \frac{c(x_n - (b + b^*)y_n)y_n}{e + x_n - (b + b^*)y_n} = f(x_n, y_n, b^*), \\
y_{n+1} &= \frac{d(x_n - (b + b^*)y_n)y_n}{e + x_n - (b + b^*)y_n} = g(x_n, y_n, b^*).
\end{align*}
$$

(12)

Let $u_n = x_n - x_2$ and $v_n = y_n - y_2$; then, the equilibrium $P_2(x_2, y_2)$ is transformed into the origin, and further expanding $f$ and $g$ as a Taylor series at $(u_n, v_n, b^*) = (0, 0, 0)$ up to the third order, system (12) becomes

$$
\begin{align*}
u_{n+1} &= \alpha_1 u_n + \alpha_2 v_n + \alpha_1 u_n^2 + \alpha_2 u_n v_n + \alpha_3 u_n b^* + \alpha_2 v_n b^* + \alpha_1 u_n^3 + \alpha_2 u_n^2 v_n + \alpha_1 u_n v_n b^* + \alpha_1 u_n v_n b^* + O\left(\left(|u_n| + |v_n| + |b^*|\right)^4\right), \\
v_{n+1} &= \beta_1 u_n + \beta_2 v_n + \beta_1 u_n^2 + \beta_2 u_n v_n + \beta_2 v_n b^* + \beta_1 u_n^3 + \beta_1 u_n^2 v_n + \beta_1 u_n v_n b^* + \beta_1 u_n v_n b^* + O\left(\left(|u_n| + |v_n| + |b^*|\right)^4\right).
\end{align*}
$$

(13)

where $\alpha_1 = f_x(x_2, y_2, 0), \alpha_2 = f_y(x_2, y_2, 0), \alpha_11 = f_{xx}(x_2, y_2, 0), \alpha_12 = f_{xy}(x_2, y_2, 0), \alpha_21 = f_{yx}(x_2, y_2, 0), \alpha_22 = f_{xx}(x_2, y_2, 0), \beta_1 = g_x(x_2, y_2, 0), \beta_2 = g_y(x_2, y_2, 0), \beta_{11} = g_{xx}(x_2, y_2, 0), \beta_{12} = g_{xy}(x_2, y_2, 0), \beta_{21} = g_{yx}(x_2, y_2, 0), \beta_{22} = g_{yy}(x_2, y_2, 0)$.

We define $T = \begin{bmatrix} \alpha_2 & \alpha_1 \\ -1 & \lambda_2 - \alpha_1 \end{bmatrix}$, and it is obvious that $T$ is invertible. Using the transformation

$$
\begin{bmatrix} u_n \\ v_n \end{bmatrix} = T \begin{bmatrix} \bar{x}_n \\ \bar{y}_n \end{bmatrix},
$$

system (13) becomes

$$
\begin{align*}
\bar{x}_{n+1} &= \alpha_2 \left[ (1 + \alpha_1)\alpha_{11} + \alpha_2 \beta_{11} \right] + \frac{\beta_{22} (1 + \alpha_1)^2 - (1 + \alpha_1) \alpha_{11} (1 + \alpha_1) + \alpha_2 \beta_{12}}{1 - \lambda_2^2} \left[ \bar{x}_n + \frac{\alpha_2 \alpha_{11} (1 + \alpha_1) + \alpha_2 \beta_{11}}{\alpha_2 (1 + \lambda_2)^2} \right], \\
\bar{y}_{n+1} &= \frac{\alpha_1 \alpha_{11} (1 + \alpha_1) + \alpha_2 \beta_{21}}{\alpha_2 (1 + \lambda_2)^2} - \left[ \bar{x}_n + \frac{\alpha_2 \alpha_{11} (1 + \alpha_1) + \alpha_2 \beta_{11}}{\alpha_2 (1 + \lambda_2)^2} \right].
\end{align*}
$$

(16)

where $\lambda_2$ is the eigenvalue of the fixed point $P_2(x_2, y_2)$. The centre manifold theorem guarantees that there exists a centre manifold $W^c(0, 0, 0)$ of system (14) at $0, 0, 0$ in a small neighbourhood of $b^* = 0$, which can be approximated as follows:

$$
W^c(0, 0, 0) = \left\{ (\bar{x}_n, \bar{y}_n, b^*) : \bar{y}_{n+1} = \bar{x}_n^2 + \bar{x}_n b^* + O\left(\left(|\bar{x}_n| + |b^*|\right)^3\right) \right\},
$$

(15)

where the functions $f_1$ and $g_1$ denote the terms in model (14) in variables $(u_n, v_n, b^*)$ with order at least two.
System (14) which is restricted to the centre manifold $W^c(0,0,0)$ has the following form:

$$
\mathbf{x}_{n+1} = -\mathbf{x}_n + h_1 \mathbf{x}_n b^* + h_2 \mathbf{x}_n b^* + h_3 \mathbf{x}_n^2 + h_4 \mathbf{x}_n^3 + O\left((|\mathbf{x}_n| + |b|^4)^3\right) = F(\mathbf{x}_n, b^*),
$$

where

$$
\begin{align*}
Wc & = P \left( \begin{array}{c}
PDB \at \text{at}
\end{array} \right)
\end{align*}
$$

For PDB, the two discriminatory quantities $\xi_1$ and $\xi_2$ be nonzero,

$$
\begin{align*}
\xi_1 &= \left( \frac{\partial^2 F}{\partial x \partial y} + \frac{1}{2} \frac{\partial F}{\partial x} \frac{\partial^2 F}{\partial x^2} \right)_{(0,0)}, \\
\xi_2 &= \left( \frac{1}{6} \frac{\partial^2 F}{\partial x^3} + \left( \frac{1}{2} \frac{\partial F}{\partial x} \right)^2 \right)_{(0,0)}.
\end{align*}
$$

Finally, from the above analysis, we have the following result.

**Theorem 2.** If $\xi_1 \neq 0$ and $\xi_2 \neq 0$, then model (3) undergoes PDB at $P_2(x_2, y_2)$ when the parameter $b$ varies in a small neighbourhood of $b_{PDB}$. If $\xi_2 > 0(\xi_2 < 0)$, then the period-two orbits that bifurcate from $P_2(x_2, y_2)$ are stable (unstable).

### 5. Influence of Refuge

This section presents the impact of prey refuge on the model when the coexistence steady-state point $P_2(x_2, y_2)$ exists and is stable. It is simple to observe that $x$ and $y$ are continuous differential functions of parameter $b$.

Here, $x_2$ is the positive root of the equation $x^2 + Ax + B = 0$ where $A = (1-a)/a + (c/abD)$, $B = -((ce/abD)(d-1))$, and $y_2 = (x_2/b) - (e/b(d-1))$.

If $dx_2/db > y_2$, then $(dy_2/db) > 0$; hence, $y_2$ is strictly increasing function of the parameter $b$.

If $dx_2/db < y_2$, then $(dy_2/db) < 0$; hence, $y_2$ is strictly decreasing function of the parameter $b$. 


Clearly, if \(0 < (dx/dy) < y_2\), then \(x_2\) is strictly increasing function of the parameter \(b\) and \(y_2\) is strictly decreasing function of the parameter \(b\).

If \((dx/dy) < 0\), then \(x_2\) and \(y_2\) are strictly decreasing function of the parameter \(b\).

If \((dx/dy) > 0\), then \((dy/dx) > 0\); hence, \(x_2\) and \(y_2\) are strictly increasing function of the parameter \(b\).

### 6. Chaos Control

In discrete-time models, several approaches can be used in chaos control. These include the following: hybrid control method, pole-placement technique, and the state feedback method [9, 14, 18, 30]. In this regard, a feedback control method is employed to stabilize the chaotic orbits at an unstable positive fixed point of model (3) as follows:

\[
\begin{align*}
    x_{n+1} &= ax_n(1 - x_n) - c(x_n - by_n)y_n + S, \\
    y_{n+1} &= d(x_n - by_n)y_n / (e + x_n - by_n)
\end{align*}
\]  

(20)

With the feedback control law as the control force, \(S = -q_1(x_n - x_2) - q_2(y_n - y_2)\), where \(q_1\) and \(q_2\) are the feedback gain and \((x_2, y_2)\) is a positive fixed point of the model.

The Jacobian matrix \(J\) for system (20) at \((x_2, y_2)\) is

\[
J = \begin{bmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{bmatrix}, \quad \text{where} \quad a_{11} = a(1 - 2x) - (cey)/
(e + (x - by)^2), \quad a_{12} = -(c(x - 2by) + (x - by)^2)(e + (x - by)^2), \quad a_{21} = (d ey)/((e + (x - by)^2)), \quad \text{and} \quad a_{22} = (d(e(x - 2by) + (x - by)^2))/((e + (x - by)^2)).
\]

The characteristic equation of the matrix \(J\) is given by

\[
\lambda^2 - (a_{11} + a_{22} - q_1)\lambda + a_{22}(a_{11} - q_1) - a_{21}(a_{12} - q_2) = 0.
\]

Let \(\lambda_1\) and \(\lambda_2\) be the eigenvalues, then

\[
\begin{align*}
    \lambda_1 + \lambda_2 &= a_{11} + a_{22} - q_1, \\
    \lambda_1\lambda_2 &= a_{22}(a_{11} - q_1) - a_{21}(a_{12} - q_2).
\end{align*}
\]

(21)

The lines of marginal stability are determined by solving the equation \(\lambda_1 = \pm 1\) and \(\lambda_1\lambda_2 = 1\). These conditions guarantee that the eigenvalues \(\lambda_1\) and \(\lambda_2\) have modulus less than \(1\).

Suppose \(\lambda_1\lambda_2 = 1\), we have a line \(l_1\) from (21) as \(a_{22}q_1 - a_{21}q_2 = a_{22}a_{11} - a_{21}a_{12} - 1\).

Suppose \(\lambda_1 = \pm 1\), we have the line \(l_1\) and \(l_2\) from (21) as \((1 - a_{22})q_1 + a_{21}q_2 = a_{11} + a_{22} - 1 - a_{22}a_{11} + a_{21}a_{12}\) and \((1 + a_{22})q_1 - a_{21}q_2 = a_{11} + a_{22} + 1 + a_{21}a_{11} - a_{22}a_{12}\).

The stable eigenvalues lie within a triangular region by the lines \(l_1, l_2,\) and \(l_3\).

### 7. Numerical Simulation

To support the analytical results, this section presents the numerical simulation based on system (3) through the following examples.

**Example 1.** The equilibrium points and their nature for \(a = 4, b = 0.4, c = 3, d = 3,\) and \(e = 0.7\) of system (3) are derived.

The discrete prey-predator system (3) has three equilibrium points. One is the trivial equilibrium point \((0, 0)\), another one is axial equilibrium point \((0.75, 0)\), and the last one is unique positive equilibrium point \((0.53, 0.46)\). The characteristic equation of the Jacobian matrix of system (3) evaluated at the trivial equilibrium point \((0, 0)\) is given by

\[
F(\lambda) = \lambda^2 - 4\lambda = 0.
\]

(22)
Now, $F(1) = -3 < 0$, $F(-1) = 5 > 0$, and $F(0) = 0 < 1$. The roots of (22) are given by $\lambda_1 = 0$ and $\lambda_2 = 4$. Therefore, the trivial equilibrium point $(0, 0)$ is saddle.

The characteristic equation of the Jacobian matrix of system (3) evaluated at the axial equilibrium point $(0.75, 0)$ is given by

$$F(\lambda) = \lambda^2 + 0.4483\lambda - 3.1034 = 0.$$

Now, $F(1) = -1.6551 < 0$, $F(-1) = -2.5517 < 0$, and $F(0) = -3.1034 < 1$. The roots of (23) are given by $\lambda_1 = -2$ and $\lambda_2 = 1.5517$. Therefore, the axial equilibrium point $(0.75, 0)$ is source. The characteristic equation of the Jacobian matrix of system (3) evaluated at the positive equilibrium point $(0.53, 0.46)$ is given by

$$F(\lambda) = \lambda^2 + 0.4837\lambda - 0.1534 = 0.$$

Example 2. We consider the system parameters $a = 4$, $b \in [0, 1]$, $c = 3$, $d = 3$, $e = 0.7$, and initial population $(0.6, 0.3)$. For $a = 4$, $b = 0.0355$, $c = 3$, $d = 3$, and $e = 0.7$,
system (3) has three equilibrium points. One is the trivial equilibrium point \((0,0)\), another one is axial equilibrium point \((0.75,0)\), and the last one is unique positive equilibrium point \((0.37,0.56)\). Now, we will study the dynamics of the interior equilibrium point \((0.37,0.56)\). Furthermore, the characteristic equation of the Jacobian matrix of system (3) for \(a = 4\), \(b = 0.0355\), \(c = 3\), \(d = 3\), and \(e = 0.7\) evaluated at positive steady-state \((0.37,0.56)\) is given by

\[
\lambda^2 - 0.9359\lambda + 1.0009 = 0. \tag{25}
\]

The roots of (25) are given by \(\lambda_{1,2} = 0.4680 \pm 0.8842i\) with \(|\lambda_{1,2}| = 1\). Thus, \(A_{NSB} = (a = 4, b = 0.0355, c = 3, d = 3, e = 0.7)\).

Example 3. We consider parameters \(a = 4\), \(b \in [0,1]\), \(c = 3\), \(d = 3\), \(e = 0.7\), and initial population \((0.6,0.3)\). Then, system (3) undergoes the Period-Doubling Bifurcation as \(b\) varies in a small neighbourhood of \(b = 0.52\). The corresponding bifurcation diagram and Lyapunov exponents are shown in Figures 1 and 2, respectively. For \(a = 4\), \(b = 0.52\), \(c = 3\), \(d = 3\), and \(e = 0.7\), system (3) has three equilibrium points. One is the trivial equilibrium point \((0,0)\), another one is axial equilibrium point \((0.75,0)\), and the last one is unique positive equilibrium point \((0.56,0.42)\). Now, we will study the dynamics of the interior equilibrium point \((0.56,0.42)\). Furthermore, the characteristic equation of the Jacobian matrix of system (3) for \(a = 4\), \(b = 0.52\), \(c = 3\), \(d = 3\), and \(e = 0.7\) evaluated at positive steady-state \((0.56,0.42)\) is given by

\[
\lambda^2 + 0.7318\lambda - 0.2693 = 0. \tag{26}
\]

The roots of (26) are given by \(\lambda_1 = -1\) and \(\lambda_2 = 0.269\) with \(|\lambda_2| \neq 1\). Thus, \(A_{PDB} = (a = 4, b = 0.52, c = 3, d = 3, e = 0.7)\).
With the comparison of bifurcation analysis through the pictorial representation in Figure 3 with Figure 1 for the same parameter value with only the difference in functional response parameter $e$, we observed that Neimark–Sacker bifurcation vanishes from Figure 3, but Period-Doubling Bifurcation occurs. Figure 4 is the corresponding Lyapunov exponent diagram of Figure 3. Furthermore, the bifurcation diagram by changing the parameter value $a$ is shown in Figure 5, and compared with Figures 1 and 3, here both bifurcations are vanished.

**Example 4.** For the parameter set $a = 4$, $b = 0.95$, $c = 3$, $d = 3$, $e = 0.75$, and the initial population $(0.6, 0.3)$, we observe the chaotic behaviour of prey predator shown in Figure 6(a). In this case, the fixed point $(0.64, 0.28)$ is unstable. For feedback gain $q_1 = -0.5$ and $q_2 = 0.5$, we observe that fixed point $(0.64, 0.28)$ is stable as shown in Figure 6(b). Model (3) with the feedback control law as the control force $S = 0.5(x_n - 0.64) - 0.5(y_n - 0.28)$ is as follows:
$x_{n+1} = ax_n(1 - x_n) - \frac{c(x_n - by_n)y_n}{e + x_n - by_n} + S,$
\[y_{n+1} = \frac{d(x_n - by_n)y_n}{e + x_n - by_n}.
\] (27)

The Jacobian matrix $J$ for system (27) at $(0.64, 0.28)$ is

\[J = \begin{bmatrix}
-1.1187 & -1.0245 \\
0.4987 & 0.5245
\end{bmatrix}.
\] The characteristic equation of the matrix $J$ is given by

$$\lambda^2 + 0.5942\lambda - 0.0759 = 0.$$ (28)

The roots of (28) are given by $\lambda_1 = -0.7023$ and $\lambda_2 = 0.1081$. Now, line $l_1$ is $0.5245q_1 - 0.4987q_3 = -1.5874$, line $l_2$ is $0.4755q_1 + 0.4987q_3 = -1.5068$, and line $l_3$ is $1.5245q_1 - 0.4987q_3 = -0.6816$.

The stable eigenvalues lie within a triangular region by the lines $l_1, l_2$, and $l_3$ shown in Figure 7. The system is stable for the triangular region $ABC$ bounded by the marginal lines $l_1$, $l_2$, and $l_3$.

8. Conclusion

In this paper, the impacts of refuges used by prey on a prey-predator interaction are studied through the analytical approach and graphical representation. This study considered the prey refuge proportional to predator density due to fear induced by a predator on prey. The detailed effects of the local stability of the interior steady-state point and the long-term dynamics of the interacting populations on the model have been investigated. The results show that the impact of refuges can stabilize and destabilize the interior equilibrium point of the proposed prey-predator model for different parameter sets.

Different observations have been made on the types of refuge and their impacts on the proposed system. Generally, in the existing literature, we always found the stabilizing effect of refuge. However, here, we observed an unstable to stable to unstable behaviour in Figure 1. For a different parameter set in Figure 3, we have also found a stable to unstable behaviour of the system. In Figure 5, another interesting observation is that refuge is profitable for prey and adverse effect on the predator. We have studied the bifurcations of the proposed discrete prey-predator model with refuge. It is shown that the model exhibits various bifurcations of codimension one, including PDB and NSB, as the values of the parameters vary.

Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

G. S. Mahapatra conceptualized the study, provided software, and supervised and visualized the study. P. K. Santra conceptualized the study, developed methodology, provided software, and investigated the study. Ebenezer Bonyah visualized the study and reviewed and edited the manuscript.

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