Mild solution of fractional order differential equations with not instantaneous impulses

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Abstract: In this paper, we investigate the boundary value problems of fractional order differential equations with not instantaneous impulse. By some fixed-point theorems, the existence results of mild solution are established. At last, one example is also given to illustrate the results.

Keywords: Mild solution, Fractional order, Not instantaneous impulse

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1 Introduction

The impulsive differential equations arise from the real world problems to describe the dynamics of processes in which sudden, discontinuous jumps occur. Such processes are naturally seen in biology, physics, engineering, etc. Due to their significance, many authors have established the solvability of impulsive differential equations. For the general theory and applications of such equations we refer the interested readers to see the papers [1-4] and references therein. However, in almost all the papers concerning impulsive differential equations, the impulses are all instantaneous impulses, and the classical models with instantaneous impulses can not characterize many practical problems, for example, the dynamics of evolution processes in pharmacotherapy. Let us consider the hemodynamic equilibrium of a person. The introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous process. In fact, this situation should be characterized by a new case of impulsive action, which starts at an arbitrary fixed point $t_i$ and stays active on a finite time interval $[t_i, s_i]$. To this end, Hernandez and O’Regan [5] initially offered to study a new class of abstract semilinear impulsive differential equations with not instantaneous impulses in a PC-normed Banach space. In [5], the authors discussed the following problems.

\[
\left\{
\begin{array}{l}
u'(t) = Au(t) + f(t, u(t)), t \in (s_i, t_{i+1}], i = 0, \cdots, N \\
u(t) = g_i(t, u(t)), t \in (t_i, s_i], i = 1, \cdots, N \\
u(0) = x_0
\end{array}
\right.
\]

where $A : D(A) \subset X \to X$ is the generator of a $C_0$ -semigroup of bounded linear operators $(T(t))_{t \geq 0}$ defined on a Banach space $(X, \| . \|)$, $x_0 \in X, 0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 \leq \cdots \leq s_N \leq t_N \leq t_{N+1} = a$ are pre-fixed numbers, $g_i \in C([t_i, s_i) \times X, X)$ for all $i = 1, \cdots, N$ and $f : [0, a] \times X \to X$ is a suitable function. Meanwhile, Pierri et al. [6] continue the work and development in [5] in a $PC_{\alpha}$ -normed Banach space.

On the one hand, the absorption of drugs has a memory effect, thus, the new class of impulsive conditions introduced by [5] may not explain this phenomenon very well. On the other hand, fractional calculus provide a powerful tool for the description of hereditary properties of various materials and memory processes [7-8].
differential equations have recently proved to be strong tools in the modeling of medical, physics, economics and technical sciences. For more details on fractional calculus theory, one can see the monographs of Diethelm [9], Kilbas et al. [10], Lakshmikantham et al. [11], Miller and Ross [12], Podlubny [13] and Tarasov [14]. Fractional differential equations involving the Riemann-Liouville fractional derivative or the Caputo fractional derivative have been paid more and more attentions (see for examples [7,8,15-19]). In [20], the authors considered the following problem:

\[
\begin{align*}
\{ cD_{0+}^{q}u(t) & : = cD_{0+}^{q}u(t) = f(t, u(t)), t \in J := J \setminus \{t_1, t_2, \cdots, t_m\}, J := [0, T], \\
\Delta u(t_k) & := u(t_k^+) - u(t_k^-) = I_k(u(t_k^-)), k = 1, 2, \cdots, m \\
\alpha u(0) + bu(T) & = c
\end{align*}
\]  

(2)

where \( cD_t^q \) is the Caputo fractional derivative of order \( q \in (0, 1) \) with the lower limit zero, \( f : J \times R \rightarrow R \) is jointly continuous and \( t_k \) satisfy \( 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T \), \( u(t_k^+) = \lim_{h \rightarrow 0^+} u(t_k + h) \) and \( u(t_k^-) = \lim_{h \rightarrow 0^-} u(t_k + h) \) represent the right and left limits of \( u(t) \) at \( t = t_k \). \( I_k \in C(R, R) \), and \( a, b, c \) are real constants with \( a + b \neq 0 \). Obviously, the impulses in (2) are instantaneous. Motivated by the work in [5-6,20], in this article, we consider the following impulsive fractional differential equations which impulses are not instantaneous.

\[
\begin{align*}
\{ cD_{0+}^{q}u(t) & : = cD_{0+}^{q}u(t) = f(t, u(t)), t \in (s_i, t_{i+1}], i = 0, \cdots, N \\
\alpha u(t) & = g_i(t, u(t)), t \in (t_i, s_i], i = 1, \cdots, N \\
\alpha u(0) + bu(T) & = c
\end{align*}
\]  

(3)

where \( cD_t^q \) is the Caputo fractional derivative of order \( q \in (0, 1) \) with the lower limit zero, \( 0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 \leq \cdots \leq t_N \leq s_N \leq t_{N+1} = T \) are pre-fixed numbers, \( J = [0, T], g_i \in C((t_i, s_i] \times R, R) \), for all \( i = 1, \cdots, N, f : [0, T] \times R \rightarrow R \) is a continuous, and \( a, b, c \) are real constants with \( a + b \neq 0 \).

The rest of this paper is organized as follows. In Section 2, some lemmas which are essential to prove our main results are stated. In Section 3, we give the main results. In Section 4, one examples is offered to demonstrate the application of our main results.

## 2 Preliminaries

At first, we present the necessary definitions for the fractional calculus theory.

**Definition 2.1** ([10]). The Riemann-Liouville fractional integral of order \( \alpha > 0 \) of a function \( y : (0, \infty) \rightarrow R \) is given by

\[
I_0^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds,
\]

where the right side is pointwise defined on \((0, +\infty)\).

**Definition 2.2** ([10]). The Caputo fractional derivative of order \( \alpha > 0 \) of a function \( y : (0, \infty) \rightarrow R \) is given by

\[
\begin{align*}
cD_0^\alpha y(t) & = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y^{(n)}(s)ds,
\end{align*}
\]

where \( n = [\alpha] + 1 \), \([\alpha]\) denotes the integer part of number \( \alpha \), the right side is pointwise defined on \((0, +\infty)\).

**Lemma 2.3** ([10]). Let \( \alpha > 0 \), then the fractional differential equation \( cD_0^\alpha u(t) = 0 \) has solutions

\[
u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},
\]

where \( c_i \in R, i = 0, 1, \cdots, n-1, n = [\alpha] + 1 \).
Lemma 2.4 ([10]). Let $\alpha > 0$. Then we have
\[ I_{0+}^\alpha c D^\alpha u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \]
where $c_i \in \mathbb{R}, i = 0, 1, \cdots, n - 1, n = [\alpha] + 1$.

Lemma 2.5 (Krasnoselskii’s fixed point theorem [21]). Let $M$ be a closed convex and nonempty subset of a Banach space $X$. Let $A$ and $B$ be two operators such that:
1. $Ax + By \in M$ whenever $x, y \in M$;
2. $A$ is compact and continuous;
3. $B$ is a contraction mapping.
Then there exists $z \in M$ such that $z = Az + Bz$.

We define
\[ PC(J, \mathbb{R}) = \{ x : J \to \mathbb{R}; x \in C((t_k, t_{k+1}], \mathbb{R}), k = 0, 1, \cdots, N, \} \]
and $x(t_k^+), x(t_k^-)$, exist with $x(t_k^-) = x(t_k^+), k = 1, \cdots, N$.

Obviously, $PC(J, \mathbb{R})$ is a Banach space with the norm $\|x\|_{PC} = \sup_{t \in J} |x(t)|$.

If $u \in PC(J, \mathbb{R})$ satisfies impulsive problem (3), if $t \in [0, t_1]$, then integrating the first equation in (3) from 0 to $t$ by virtue of the Definition 2.1, one can obtain
\[ u(t) = u(0) + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s, u(s))ds. \]
If $t \in (s_i, t_{i+1}], i = 1, \cdots, N$, one obtains
\[ u(t) = g_i(s_i, u(s_i)) + \frac{1}{\Gamma(q)} \int_{s_i}^t (t - s)^{q-1} f(s, u(s))ds. \]
Then we have
\[ u(T) = g_N(s_N, u(s_N)) + \frac{1}{\Gamma(q)} \int_{s_N}^T (T - s)^{q-1} f(s, u(s))ds. \]
By the boundary condition in (3), we can get
\[ u(0) = \frac{1}{a} \{ c - b[g_N(s_N, u(s_N)) + \frac{1}{\Gamma(q)} \int_{s_N}^T (T - s)^{q-1} f(s, u(s))ds] \}. \]
Then similarly to Definition 2.1 in [6], we can define the mild solution for problems (3).

Definition 2.6. A function $u \in PC(J, \mathbb{R})$ is a mild solution of problems (3) if
\[ \alpha u(0) + bu(T) = c, \]
\[ u(t) = g_i(t, u(t)), t \in (t_i, s_i], i = 1, \cdots, N, \]
and
\[ u(t) = \frac{1}{a} \{ c - b[g_N(s_N, u(s_N)) + \frac{1}{\Gamma(q)} \int_{s_N}^T (T - s)^{q-1} f(s, u(s))ds] \} + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s, u(s))ds \]
for all $t \in [0, t_1]$ and
\[ u(t) = g_i(s_i, u(s_i)) + \frac{1}{\Gamma(q)} \int_{s_i}^t (t - s)^{q-1} f(s, u(s))ds \]
for all $t \in (s_i, t_{i+1}], i = 1, \cdots, N$. 
3 Main results

This section deals with the existence of mild solutions for problem (3). Before stating and proving the main results, we make the following hypotheses.

\( (H_1) \). \( f : J \times R \to R \) is jointly continuous. Let \( W \in PC(J, R) \) be a open subset and \( 0 \in W \) (0 is the zero function), there exists a function \( k \in C(J, R) \) such that

\[
\| f(t, x(t)) - f(t, y(t)) \| \leq k(t)\| x - y \|_{PC}, \forall x, y \in W, \text{ for all } t \in J.
\]

\( (H_2) \). \( g_i \in C((t_i, s_i] \times R, R) \) and there exist \( l_i(t) \in C[J, R] \) such that

\[
\| g_i(t, x(t)) - g_i(t, y(t)) \| \leq l_i(t)\| x - y \|_{PC}, \forall x, y \in PC(J, R), \text{ for all } t \in J.
\]

Let

\[
L = \max_{1 \leq i \leq N} \sup_{t \in J} |l_i(t)|, M = \sup_{t \in J} |k(t)|.
\]

**Theorem 3.1.** Assume \( (H_1) - (H_2) \) hold and \( n < 1 \), then problem (3) has a unique mild solution, where

\[
n = \max\{L + \frac{T^q M}{\Gamma(q + 1)} \cdot \frac{bL}{a} + \frac{T^q M}{\Gamma(q + 1)} \cdot \frac{b}{a} + 1\}.
\]

**Proof.** Let \( A : PC(J, R) \to PC(J, R) \) be the map defined by

\[
Au(t) = g_i(t, u(t)), \text{ for } t \in (t_i, s_i], i = 1, \cdots, N
\]

and

\[
Au(t) = \frac{1}{a} \{c - b[g_N(s_N, u(s_N))] + \frac{1}{\Gamma(q)} \int_{s_N}^{T} (T-s)^{q-1} f(s, u(s))ds\} + \frac{1}{\Gamma(q)} \int_{0}^{t} (t-s)^{q-1} f(s, u(s))ds,
\]

for all \( t \in [0, t_1] \) and

\[
Au(t) = g_i(s_i, u(s_i)) + \frac{1}{\Gamma(q)} \int_{s_i}^{t} (t-s)^{q-1} f(s, u(s))ds
\]

for all \( t \in (s_i, t_{i+1}], i = 1, \cdots, N \). Clearly, \( A \) is well defined.

Next we show that \( A \) is contraction on \( B_r \in PC(J, R) \), where

\[
B_r = \{x \in PC(J, R) : \| x \|_{PC} \leq r \} \subset W
\]

\( \forall x, y \in B_r \), for \( t \in (s_i, t_{i+1}], i = 1, \cdots, N \), by the assumption \( (H_1), (H_2) \), we have

\[
\| Ax(t) - Ay(t) \| \leq \| g_i(s_i, x(s_i)) - g_i(s_i, y(s_i)) \| + \frac{1}{\Gamma(q)} \int_{s_i}^{t} (t-s)^{q-1} \| f(s, x(s))ds - f(s, y(s)) \| ds
\]

\[
\leq \{L + \frac{T^q M}{\Gamma(q + 1)}\} \| x - y \|_{PC}
\]

\( \forall x, y \in B_r, \) if \( t \in [0, t_1] \), also by \( (H_1), (H_2) \), one can obtain

\[
\| Ax(t) - Ay(t) \| \leq \frac{b}{a} \{g_N(s_N, x(s_N)) - g_N(s_N, y(s_N))\}
\]

\[
+ \frac{1}{\Gamma(q)} \int_{s_N}^{T} (T-s)^{q-1} \| f(s, x(s))ds - f(s, y(s)) \| ds
\]
\[ + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \| f(s, x(s)) ds - f(s, y(s)) \| ds \]

\[ \leq \left( \frac{bL}{a} + \frac{T^q M}{\Gamma(q+1)} (\frac{b}{a} + 1) \right) \| x - y \|_{PC} \]

\[ \forall x, y \in B_r, \text{if } t \in (t_i, s_i], i = 1, \ldots, N, \text{then again using the assumption } (H_2), \text{we get} \]

\[ \| Ax(t) - Ay(t) \| \leq L \| x - y \|_{PC}. \]

Hence we can prove that

\[ \| Ax(t) - Ay(t) \| \leq n \| x - y \|_{PC} \]

which implies that \( A \) is a contraction mapping and there exists a unique mild solution of (3).

In order to get the second main result, we give assumption \((H_3)\).

\((H_3)\). For \( x \in PC(J, R) \), the function \( f : J \times R \to R \) is jointly continuous and strongly measurable on \( J \). There are \( m_f \in L^1(J; R^+) \) and a nondecreasing function \( h_f \in C([0, \infty); R^+) \), such that

\[ \| f(t, x) \| \leq m_f(t) h_f(\| x \|), \text{for all } (t, x) \in J \times PC(J, R). \]

Let \( d = \max \{ 1, \frac{b}{a} \}, B_r = \{ x \in PC(J, R) : \| x \|_{PC} \leq r \} \).

**Theorem 3.2.** Assume that \((H_2), (H_3)\) hold. Let the functions \( g_i(t, 0) \) be bounded, \( dL \leq 1 \) and there exists a constant \( r > 0 \) such that

\[ r(1-dL) \geq \frac{c}{a} + d[e + \frac{2T^q}{\Gamma(q+1)} \| m_f \|_{L^1(0,T)} h_f(r)] \] (4)

where \( d = \max \{ 1, \frac{b}{a} \}, e = \max_{i = 1, \ldots, N} \| g_i(t, 0) \|, \text{then the problem } (3) \text{ has a mild solution.} \)

**Proof.** Let \( Au(t) \) be the map introduced in the proof of Theorem 3.1. We introduce the decomposition \( Au(t) = A^1 u(t) + A^2 u(t) \), where

\[ A^1 u(t) = \sum_{i=1}^N A^1_i u(t), A^2 u(t) = \sum_{i=1}^N A^2_i u(t) \]

\[ A^1_i u(t) = \begin{cases} \frac{1}{\Gamma(q)} \int_{s_i}^t (t-s)^{q-1} f(s, u(s)) ds, & \text{if } t \in (s_i, s_{i+1}], i \geq 1, \\ -\frac{b}{\Gamma(q)} \int_{s_i}^t (T-s)^{q-1} f(s, u(s)) ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds, & \text{if } t \in [0, t_i], \end{cases} \]

\[ A^2_i u(t) = \begin{cases} \frac{1}{2} c - b[g_N(s_N, u(s_N))], & \text{if } t \in [0, t_i]. \\ 0, & \text{if } t \notin [t_i, t_{i+1}], i \geq 1. \end{cases} \]

In order to use the Krasnoselski fixed-point theorem (Lemma 2.5), we divide our proof into three steps.

**Step 1.** First we show that \( A^1 x + A^2 y \in B_r \) whenever \( x, y \in B_r \).

Let \( \forall x \in B_r, \) we have

\[ \| A^1_i x(t) \| \leq \frac{T^q}{\Gamma(q+1)} (1 + \frac{b^L}{a}) \| m_f \|_{L^1(s_i, s_{i+1})} h_f(r) \leq \frac{2dT^q}{\Gamma(q+1)} \| m_f \|_{L^1(s_i, s_{i+1})} h_f(r). \]

\[ \forall y \in B_r, \text{if } t \in (t_i, s_i], i \geq 1, \text{one can get} \]

\[ \| A^2_i y(t) \| \leq \| g_i(t, y(t) - g_i(t, 0)) \| + \| g_i(t, 0) \| \leq Lr + e. \]
Proceeding as above, we obtain that \( \left\| A^2_t y(t) \right\| \leq Lr + e, \forall y \in B_R, \) if \( t \in (s_i, t_{i+1}], i \geq 1. \) \( \forall y \in B_R, \) if \( t \in [0, t_1], \) we have
\[
\left\| A^2_t y(t) \right\| \leq \frac{c}{a} + \frac{b}{a} \left\| g_N(s_N, u(s_N)) \right\| \leq \frac{c}{a} + \frac{b}{a}(Lr + e).
\]
Then, \( \forall x, y \in B_R, \) we find that
\[
\left\| Ax + By \right\|_{PC} \leq \frac{c}{a} + d(Lr + e) + \frac{2d T^q}{\Gamma(q + 1)} \left\| m_f \right\|_{L^1(0, T)} h_f(r) \leq r.
\]

**Step 2.** We show \( A^2 = \sum_{i=1}^N A_i^2 \) is a contraction mapping.

From the definition of \( A^2 u(t), A_i^2 u(t) \) and the assumption \((H_2), \) we can easily get
\[
\left\| A^2_t x(t) - A^2_t y(t) \right\| \leq dL \left\| x - y \right\|_{PC}, \forall x, y \in B_T, \forall t \in J,
\]
which implies that \( A^2 \) is a contraction mapping.

**Step 3.** Next we will prove that \( A^1 \) is compact and continuous.

We also divide the proof into 3 steps.

I. \( A^1 \) is continuous.

Let \( \{x_n\} \) be a sequence such that \( x_n \to x \) in \( PC(J, R). \) Then \( \forall t \in J, \) by the definition of \( A^1 u(t), A^1 u(t) \), we have
\[
\left\| A^1 x_n(t) - A^1 x(t) \right\| \leq \frac{2T^q d}{\Gamma(q + 1)} \left\| f(t, x_n(t)) - f(t, x(t)) \right\|.
\]
Due to \((H_3), \) \( f : J \times R \to R \) is jointly continuous, then we know that
\[
\left\| A^1 x_n - A^1 x \right\|_{PC} \to 0, \text{as} \ x_n \to x \ (n \to \infty),
\]
which shows the operator \( A^1 \) is continuous.

II. \( A^1 \) maps bounded set into bounded sets in \( PC(J, R). \)

Indeed, it is enough to show that for any \( R > 0, \) there exists a \( R' > 0 \) such that for each \( x \in B_R \) \( \{ u \in PC(J, R) : \left\| u \right\|_{PC} \leq R \}, \) we have \( \left\| A^1 u \right\|_{PC} \leq R'. \)

From the definition of \( A^1 u(t), A_i^2 u(t) \) and the assumption \((H_3), \) \( \forall t \in J, \) one can obtain
\[
\left\| A^1 x \right\| \leq \frac{2T^q d}{\Gamma(q + 1)} \left\| m_f \right\|_{L^1(0, T)} h_f(R) := R'.
\]
Then we conclude that \( A^1 \) maps bounded set into bounded sets in \( PC(J, R). \)

III. \( A^1 \) maps bounded set into equicontinuous sets in \( PC(J, R). \)

For interval \( t \in (s_i, t_{i+1}], s_i \leq l_1 < l_2 \leq l_{i+1}, i = 1, \ldots, N, \forall x(t) \in B_R, \) using the definition of \( A^1 u(t) \) and the assumption \((H_3), \) we have
\[
\left\| \left( A^1 x \right) (l_2) - \left( A^1 x \right) (l_1) \right\| \\
= \frac{1}{\Gamma(q)} \int_{s_i}^{l_2} (l_2 - s)^{q-1} f(s, u(s)) ds - \frac{1}{\Gamma(q)} \int_{s_i}^{l_1} (l_1 - s)^{q-1} f(s, u(s)) ds \\
\leq \frac{1}{\Gamma(q)} \int_{s_i}^{l_2} (l_2 - s)^{q-1} f(s, u(s)) ds + \frac{1}{\Gamma(q)} \int_{s_i}^{l_1} f(s, u(s)) \left[ (l_1 - s)^{q-1} - (l_2 - s)^{q-1} \right] ds \\
\leq \frac{m_f \| \mathcal{L}^{1, (s_i, t_{i+1})}_{(q)} \| h_f(r)}{\Gamma(q)} \left\{ \int_{s_i}^{l_2} (l_2 - s)^{q-1} ds + \int_{s_i}^{l_1} (l_1 - s)^{q-1} ds \right\} \left[ (l_2 - l_1)^q + \left| (l_1 - s_1)^q - (l_2 - s_1)^q \right| \right].
\]

As \( l_1 \to l_2, \) the right-hand side of the above inequality tends to zero, therefore \( A^1 \) is equicontinuous on interval \( (s_i, t_{i+1}], i \geq 1. \)

Proceeding as above, we can similarly prove that \( A^1 \) is equicontinuous for the time interval \([0, t_1].\) And, it is also easily to see that \( A^1 \) is equicontinuous for the time interval \( (t_i, s_i], i \geq 1. \)

As a consequence of step I-III together with the PC-type Arzela-Ascoli Theorem, we can conclude that \( A^1 \) is continuous and compact.

Then we complete the proof of Steps 1-3. As a consequence of Lemma 2.5, we deduce that the operator \( A \) has a fixed point \( z \in B_R \) which is a mild solution of the problem (3). \( \square \)
4 Example

In this section we give an example to illustrate the usefulness of our main result. Consider the following impulsive system of fractional differential equations.

Example 4.1.

\[ \begin{align*}
  &c D^\frac{1}{2} u(t) = \frac{u(t) \sin t^2}{16(1+e^t)(e^{t^2}+|u(t)|)}, t \in (s_i, t_{i+1}], i = 1, \ldots, N \\
  &u(t) = \frac{u(t)}{18e^{t^2}(1+|u(t)|)}, t \in (t_i, s_i], i = 1, \ldots, N \\
  &u(0) + u(1) = \frac{1}{2}
\end{align*} \]  

(5)

where \( 0 = t_0 = s_0 < s_1 \leq t_2 \leq \cdots \leq s_N \leq t_{N+1} = 1 \) are pre-fixed numbers, \( J = [0, 1], a = b = 1, c = \frac{1}{2}, q = \frac{1}{2}. \)

First we prove that Example 4.1 satisfy all the assumptions of Theorem 3.1.

In Example 4.1, it is easy to see that \( f(t, u) = \frac{u(t) \sin t^2}{16(1+e^t)(e^{t^2}+|u(t)|)} \in C([0, 1] \times R, R). \) For \( t \in [0, 1], i = 1, \cdots, N, u \in PC^1(J, R), \) we have

\[ |f(t, u) - f(t, v)| \leq \frac{1}{32} \| u - v \|, \]

So \( (H_1) \) is satisfied, where \( k(t) = \frac{1}{32} \).

For \( t \in [0, 1], u \in PC^1(J, R), g_t(u) = \frac{u(t)}{18e^{t^2}(1+|u(t)|)}, \) then we know

\[ |g_t(u) - g_t(v)| \leq \frac{1}{18} \| u - v \|, \]

with \( l_t(t) = \frac{1}{18}, \) so \( (H_2) \) is also satisfied.

From Example 4.1, we also have \( L = \frac{1}{18}, T = 1, M = \frac{1}{32}, a = b = 1, q = \frac{1}{2}, \Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}, \) then

\[ n = \max\left\{ \frac{1}{18} + \frac{1}{16\sqrt{\pi}}, \frac{1}{18} + \frac{1}{8\sqrt{\pi}} \right\} < 1. \]

So all the conditions of Theorem 3.1 are satisfied, as a consequence of Theorem 3.1, Example 4.1 has a unique mild solution.

Second, we verify that all the assumptions of Theorem 3.2 are satisfied.

Obviously, from (5), we know that \( g_t(0) = \frac{1}{18} \) is bounded and

\[ \| f(t, u) \| \leq \frac{u(t) \sin t^2}{16(1+e^t)(e^{t^2}+|u(t)|)} \leq \frac{\| u \|}{32}, \]

with \( m_f = \frac{1}{32} \in L^1(J; R^+), h_f(\| u \|) = \| u \| \in C([0, \infty); R^+) \) is nondecreasing. Then the assumption \( (H_3) \) is satisfied.

Also by (5), we can find \( d = 1, dL = \frac{1}{18} < 1, e = \frac{1}{18}. \) Then the inequality (4) becomes

\[ (\frac{17}{18} - \frac{\sqrt{\pi}}{8})r \geq \frac{5}{9}. \]

Hence, it is easy to choose \( r > 0 \) which satisfies the inequality (4).

Thus, all the assumptions in Theorem 3.2 hold, our results can be applied to the Example 4.1, i.e., Example 4.1 has at least one mild solution.

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References

[1] Mophou G. M., Existence and uniqueness of mild solutions to impulsive fractional differential equations, Nonlinear Anal., 2010, 72 (3-4), 1604–1615
[2] Tai Z., Wang X., Controllability of fractional-order impulsive neutral functional infinite delay integrodifferential systems in Banach spaces, Appl. Math. Lett., 2009, 22 (11), 1760–1765
[3] Shu X., Lai Y., Chen Y., The existence of mild solutions for impulsive fractional partial differential equations, Nonlinear Anal., 2011, 74, 2003–2011
[4] Zhang X., Huang X., Liu Z., The existence and uniqueness of mild solutions for impulsive fractional equations with nonlocal conditions and infinite delay, Nonlinear Anal. Hybrid Syst., 2010, 4, 775–781
[5] Hernandez E., O’Regan D., On a new class of abstract impulsive differential equations, Proc. Amer. Math. Soc., 2013, 141, 1641–1649
[6] Pierri M., O’Regan D., Rolnik V., Existence of solutions for semi-linear abstract differential equations with not instantaneous impulses, Appl. Math. Comput., 2013, 219, 6743–6749
[7] Zhou Y., Jiao F., Nonlocal Cauchy problem for fractional evolution equations, Nonlinear Anal: RWA, 2010, 11, 4465–4475
[8] Zhou Y., Jiao F., Li J., Existence and uniqueness for fractional neutral differential equations with infinite delay, Nonlinear Anal: TMA, 2009, 7, 3249–3256
[9] Diethelm K., The analysis of fractional differential equations, Lect. Notes Math., 2010
[10] Kilbas A.A., Srivastava M.H., Trujillo J.J., Theory and Applications of Fractional Differential Equations, in: North-Holland Mathematics studies, vol.204, Elsevier Science B.V., Amsterdam, 2006
[11] Lakshmikantham V., Leela S., Vasundhara Devi J., Theory of fractional dynamic systems, Cambridge Scientific Publishers, Cambridge, 2009
[12] Miller K.S., Ross B., An introduction to the fractional calculus and differential equations, John Wiley, New York, 1993
[13] Podlubny I., Fractional differential equations, Academic Press, New York, 1999
[14] Tarasov VE., Fractional dynamics: application of fractional calculus to dynamics of particles, fields and media, Springer, HEP, 2011
[15] Agarwal R. P., Benchohra M., Hamani S., A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, Acta. Appl. Math., 2010, 109, 973–1033
[16] Benchohra M., Henderson J., Ntouyas S.K., Ouahab A., Existence results for fractional order functional differential equations with infinite delay. J. Math. Anal. Appl., 2008, 338, 1340–1350
[17] Wang J., Zhou Y., A class of fractional evolution equations and optimal controls, Nonlinear Anal: RWA., 2011, 12, 262–272
[18] Wang J., Zhou Y., Wei W., A class of fractional delay nonlinear integrodifferential controlled systems in Banach spaces, Commun Nonlinear Sci Numer Simulat, 2011, 16, 4049–4059
[19] Zhang S., Existence of positive solution for some class of nonlinear fractional differential equations, J. Math. Anal. Appl., 2003, 278, 136–148
[20] Guo T., Jiang W., Impulsive problems for fractional differential equations with boundary value conditions, Comput. Math. Appl., 2012, 64, 3281–3291
[21] Krasnoselskii Ma., Topological methods in the theory of nonlinear integral equation., Pergamon Press, New York, 1964