QUARKONIUM AND POWER LAW POTENTIALS

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ABSTRACT

The spectra and decay rates of $c\bar{c}$ and $b\bar{b}$ levels are well described, for the most part, by a power-law potential of the form $V(r) = \lambda r^{\alpha - 1} / \alpha + \text{const.}$, where $\alpha \simeq 0$. The results of an up-to-date fit to the data on spin-averaged levels are presented. Results on electric dipole transitions in systems bound by power law potentials are also presented, with applications to the bottomonium system.

Charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$) systems provide a rich source of information on the interquark force at distances ranging from less than 0.1 fm to greater than 1 fm. At short distances our theoretical prejudices favor a potential which should act like a Coulomb potential $V(r) = \alpha_s(r)/r$, with $\alpha_s$ becoming smaller at shorter distances owing to the asymptotic freedom of QCD. At long distances, there are both experimental and theoretical reasons to believe that the interquark force in QCD becomes approximately distance-independent, corresponding to a linear potential $V(r) \sim r$. The $c\bar{c}$ and $b\bar{b}$ systems appear to lie in an intermediate range, where a power-law potential $V(r) \sim r^{\alpha}$ provides a convenient interpolating form between the short-distance Coulomb-like and long-distance linear behavior.

An early fit to quarkonia spectra found a power $\alpha \simeq 0.1$. Since then, data on $P$-wave levels have appeared and information on leptonic widths has improved. It is appropriate to update the earlier fit for a number of reasons. The power-law method can be of use in estimating properties of systems containing $b$ and $c$ quarks, and in interpolating between these cases to get estimates of $b\bar{c}$ properties. The power law method also gives an estimate of the mass difference between the $b$ and $c$ quarks, which can be of use in attempts to extract the Cabibbo-Kobayashi-Maskawa matrix element $V_{cb}$ from data on semi-leptonic $b$ decays. One would also like to see if there is a consistent pattern of data signaling a departure from a single effective power at short distances, as one would expect from the short-distance behavior of QCD. Finally, power law potentials can be of use in explaining patterns of electric dipole transition rates in the $b\bar{b}$ system.

Since the power-law description does not give an adequate description of spin-dependent effects, we fit spin-averaged levels. For $S$ waves, we use

$$ M(S) = [M(^1S_0) + 3M(^3S_1)]/4 \quad , $$

(1)

\[\text{Presented by Aaron K. Grant at DPF 92 Meeting, Fermilab, November, 1992.}\]
a combination which eliminates hyperfine splittings. For $P$ waves, we use

$$M(P) = \overline{M}^P = \frac{M(3P_0) + 3M(3P_1) + 5M(3P_2)}{9},$$

which eliminates spin-orbit and tensor force splittings. In the latter case we assume hyperfine effects are small. In the case of $b\bar{b}$ levels, the masses of the $1^S_0$ levels are not experimentally known. However, the hyperfine splitting can still be estimated using information from the leptonic widths.

We obtain theoretical values for the masses and leptonic widths of the levels using non-relativistic quantum mechanics. We find the energies and wavefunctions of the radial Schrödinger equation by solving the dimensionless equation numerically and then rescaling the dimensionless quantities by appropriate powers of the mass and coupling constant. We take $V(r) = \lambda(r^\alpha - 1)/\alpha + C$. Particle masses are then given by $M = E + 2m_Q$, where $E$ is the binding energy and $m_Q$ is the quark mass.

We present results from three fits. In each case we minimize $\chi^2$, with the standard deviations for the energies set equal to 10 MeV, and the standard deviations for the leptonic widths set equal to the experimental errors. In the first fit, we consider the levels only. In this case, we find that the best fit is given by a potential $V(r) \sim r^{-0.045}$, and that the quark mass difference is $m_b - m_c = 3.19$ GeV. However, the fit gives little preference for quark masses, and in fact the best fit is given by $m_b, m_c \to \infty$. In the second fit, summarized in Table I, we remedy this by including the leptonic widths, which are given by the formula

$$\Gamma(Q\bar{Q} \to e^+e^-) = \frac{16\pi e_Q^2 \alpha^2}{M^2} |\Psi(0)|^2 \left[ 1 - \frac{16\alpha_s(m_Q)}{3\pi} \right],$$

and where we use $\alpha_s(m_b) = 0.189 \pm 0.008$, $\alpha_s(m_c) = 0.29 \pm 0.02$. In this case, we find

$$m_b = 5.24 \text{ GeV} \quad m_c = 1.86 \text{ GeV} \quad m_b - m_c = 3.38 \text{ GeV}$$

for the quark masses, while in the potential $V(r) = \lambda(r^\alpha - 1)/\alpha + C$, we have

$$\alpha = -0.14 \quad \lambda = 0.808 \quad C = -1.305 \text{ GeV}.$$ (5)

We find that the fitted masses are correct to within 10 MeV or so, but the leptonic widths are off, typically, by 20 to 30 percent. In an attempt to remedy this, in the third fit, we include an ad hoc relativistic correction to the leptonic widths. We correct Eq. (3) by introducing a factor of the form $(1 + Kv^2/c^2)$, and treat $K$ as a free parameter. In this case, we find $\alpha = -0.12$, $\lambda = 0.801$, and $C = -0.772$ GeV, while for the quark masses we have $m_b = 4.96$ GeV, $m_c = 1.56$ GeV, and $m_b - m_c = 3.40$ GeV. The constant $K$ is found to be 1.25. The fitted quantities are given in Table I. The relativistic correction gives only a marginal improvement in the fit. In the fits to masses and leptonic widths, we find that $m_b$ and $m_c$ are not particularly well determined. However, the mass difference $m_b - m_c$ appears to be more stable, with a value around 3.39 GeV.

We can use the fitted potentials to estimate the centers of gravity of a few low-lying levels of $b\bar{c}$, $b\bar{s}$ and $c\bar{s}$ levels. For the $b\bar{c}$ system, we find

$$M_{b\bar{c}}(1S) = 6.304 \text{ GeV}, \quad M_{b\bar{c}}(2S) = 6.898 \text{ GeV}, \quad M_{b\bar{c}}(1P) = 6.764 \text{ GeV},$$

(6)
Table 1: \(J/\psi\) and \(\Upsilon\) masses and leptonic widths

| Particle | Mass (GeV) | Width (keV) |
|----------|------------|-------------|
|          | (Expt.)    | (NR\(^a\)) | (RC\(^b\)) | (Expt.) | (NR\(^a\)) | (RC\(^b\)) |
| \(J/\psi\)(1S) | 3.068 | 3.077 | 3.079 | 5.36±0.29 | 6.41±0.43 | 6.29±0.43 |
| \(J/\psi\)(2S) | 3.663 | 3.654 | 3.654 | 2.14±0.21 | 2.03±0.13 | 2.04±0.13 |
| \(J/\psi\)(1P) | 3.525 | 3.524 | 3.522 | – | – | – |
| \(\Upsilon\)(1S) | 9.449 | 9.420 | 9.423 | 1.34±0.04 | 1.21±0.02 | 1.18±0.02 |
| \(\Upsilon\)(2S) | 10.018 | 10.044 | 10.042 | 0.563±0.14 | 0.477±0.009 | 0.475±0.009 |
| \(\Upsilon\)(3S) | 10.351 | 10.358 | 10.358 | 0.24±0.05 | 0.197±0.004 | 0.200±0.004 |
| \(\Upsilon\)(4S) | 10.578 | 10.564 | 10.567 | – | – | – |
| \(\Upsilon\)(1P) | 9.900 | 9.903 | 9.900 | – | – | – |
| \(\Upsilon\)(2P) | 10.260 | 10.269 | 10.267 | – | – | – |
| \(\Upsilon\)(1D) | – | 10.181 | 10.177 | – | – | – |
| \(\Upsilon\)(2D) | – | 10.436 | 10.435 | – | – | – |

\(^a\)No relativistic corrections \(^b\)With relativistic corrections

while for the \(b\bar{s}\) and \(c\bar{s}\) levels, we have

\[
M_{c\bar{s}}(1S) = 2.085 \text{ GeV}, \quad M_{c\bar{s}}(1P) = 2.509 \text{ GeV}, \quad M_{b\bar{s}}(1S) = 5.401 \text{ GeV}.
\] (7)

The experimental spin-averaged masses\(^{10,11}\) of the \(c\bar{s}\) states are 2.075 GeV for the 1S level, and 2.536 GeV for the 1P level. We have also estimated the 1S - 2S splitting in toponium. Taking \(m_t = 130\) GeV, we find the splitting to be roughly 0.8 GeV. We expect that this is a conservative lower bound, since the short-distance behavior of QCD (which will make this splitting larger) should be important in this case.

Power law potentials also offer some insight into the strengths of various E1 transitions in the \(b\bar{b}\) system\(^{12}\). Experimentally\(^{10}\), we find that the 3S-1P transition is suppressed relative to the 3S-2P, and that the 2P-1S transition is suppressed relative to the 2P-2S. We can gain some insight into this by considering the radial dipole matrix elements \(\langle u_{n\ell}|r|u_{n\ell}\pm 1\rangle\), where \(n\) denotes the number of nodes in the radial wavefunction \(u_{n\ell}\). We can approximately evaluate these matrix elements by considering the large \(\ell\) limit. In this case, we expand the potential in the radial Schrödinger equation about the point \(\bar{r} = \ell^2/(2+\alpha)\). The potential then has the form of a harmonic oscillator potential, plus anharmonic terms which we include perturbatively. For the dipole matrix elements, we find

\[
\langle u_{n\ell}|r|u_{n\ell-1}\rangle = \frac{\ell^2/(\alpha+2)}{\sqrt{n} \Upsilon_\pm(\alpha)\ell^{(2-\alpha)/(4+2\alpha)}},
\]

\[
\langle u_{n\ell-1}|r|u_{n\ell-1}\rangle = \sqrt{n} \Upsilon_\pm(\alpha)\ell^{(2-\alpha)/(4+2\alpha)},
\]

\[
\langle u_{n-1\ell-1}|r|u_{n\ell}\rangle = \Phi(\alpha)\ell^{-\alpha/(2+\alpha)},
\]

\[
\langle u_{n\ell}|r|u_{2\ell-1}\rangle = \Phi(\alpha)\ell^{-\alpha/(2+\alpha)},
\]

where \(\Upsilon_\pm\) and \(\Phi\) are functions which depend only on \(\alpha\). We see from Eqs. \(^{8}\) that transitions with \(\Delta n = 0\) are dominant, and that the others are suppressed by a factor \(\ell^{-(\Delta n/2)}\) in the limit of large \(\ell\). Furthermore, in the region of interest for quarkonium, \(\alpha \simeq 0\), the coefficient \(\Phi\) is quite small: \(\Phi(0) \simeq -0.04\). This further suppresses the E1 rate for \(\Delta n = 2\) transitions. Direct numerical calculations show that this is also the
case for small $\ell$: we find that the $3S-1P$ rate is dramatically suppressed relative to the $3S-2P$ rate. Consequently, it appears that the suppression of E1 transitions with nonzero $\Delta n$ is a general property of power law potentials.

We can use the ratios of radial dipole matrix elements for certain transitions in the $b\bar{b}$ system to place limits on the power $\alpha$. From the experimental data\textsuperscript{6,10} we deduce the ratios

\[
\frac{\langle 3S| r |1P \rangle}{\langle 3S| r |2P \rangle} = 0.016 \pm 0.004, \quad \frac{\langle 2P| r |1S \rangle}{\langle 2P| r |2S \rangle} = 0.117 \pm 0.014. \tag{9}
\]

Computing these ratios numerically and comparing with experiment, we find that $\alpha$ is constrained to lie in the region $-0.2 < \alpha < 0$, consistent with what was found from the fit to levels and leptonic widths. Finally, we can estimate $m_b$ using the rates of the $3S-1P$, $3S-2P$, and $2S-1P$ electric dipole transitions. Extracting the radial dipole matrix elements from the data, we find that the potential model favors a value $m_b \simeq 4.0 \pm 0.9$ GeV. This is smaller than expected, and in particular we find that the $2S-1P$ transition yields the surprisingly small value $m_b = 3.45 \pm 0.44$ GeV. This is an indication that either the potential model is inadequate for describing E1 rates, or that the $\Upsilon(2S)$ total width has been overestimated.

We thank Eric Rynes for collaboration on the results of Ref. 3. This work was supported in part by the United States Department of Energy through Grant No. DE FG02 90ER 40560.

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