The generating pairs of the 2-transitive groups

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Abstract: Given a finite group $G$. The generating pair $(H, a)$ of $G$, that is, $H < G$ and $a \in G$ such that $\langle a, H \rangle = G$. In this paper, we introduce the definition of FF-subgroup to characterize the generating pairs of the symmetric groups, alternating groups and projective groups $PSL(2, q)$. This gives a partial answer to an open problem of J. André and J. Araújo and P. J. Cameron.

Keywords: Generating pair; FF-subgroup; Symmetric groups; Alternating groups; $PSL(2, q)$.

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1 Introduction

It is well-known that the generating sets for groups are far more complicated than generating sets for vector spaces. So it has always been an interesting subject to study the generating sets of groups. Recently, J. André and J. Araújo and P. J. Cameron proposed the following open problem about the generating sets for groups in [1].

**Question 1.1** ([1, Problem 1]) Let $G \leq S_n$ be a 2-transitive group. Classify the generating pairs $(a, H)$, where $a \in S_n$ and $H \leq S_n$, such that $\langle a, H \rangle = G$.

In fact, the maximal subgroups are closely related to generating sets, and the research of this direction has attracted the attention of some scholars, such as the references [3,10]. Thereby, we also try to solve the Question 1.1 by applying the maximal subgroups. Note that all elements of Frattini subgroup are not generating elements, and thus $H$ is far from the Frattini subgroup if $(a, H)$ is a generating pair. On the other hand, it is evident that for each maximal subgroup $M$, the $(a, M)$ is a generating pair where $a \notin M$. Motivated by these two features, we introduce the following notions and notations.

Given a proper subgroup $H$ of a group $G$. We denote by $\Delta_H(G)$ the union of all maximal subgroups of $G$ containing $H$, and further we say $\Delta_H(G)$ is the maximal cover of $H$ in $G$.

**Definition 1.2** Let $H$ be a proper subgroup of a finite group $G$. If $\Delta_H(G) \neq G$, then we say $H$ is a FF-subgroup of $G$.

Obviously, every maximal subgroup of a finite group is a FF-subgroup, however, the Frattini
subgroup and its subgroups of a finite non-cyclic group are not FF-subgroups. Most critically, the FF-subgroup has the following property.

**Lemma 1.3** Let $G$ be a finite group with a proper subgroup $H$. Then $\langle H, a \rangle = G$ if and only if $a \in G \setminus \Delta_H(G)$ and $H$ is a FF-subgroup of $G$.

**Proof** It is simple to show the sufficiency holds. Assume $\langle H, a \rangle = G$ and $H$ is not a FF-subgroup of $G$. Then $\Delta_H(G) = G$, and thus there exists a maximal subgroup $M$ in $G$ such that $H \leq M$ and $a \in M$, a contradiction. Similarly, we obtain $a \in G \setminus \Delta_H(G)$. The proof of this lemma is completed.

According to Definition 1.2 and Lemma 1.3 we see FF-subgroups are not only closely related to maximal subgroups but also to generating pairs, and further the Question 1.1 is equivalent to classify the FF-subgroups of the 2-transitive groups in some sense.

Let’s go back to see if the Definition 1.2 is trivial. In other words, are there some groups which have some subgroups are neither the subgroups of their Frattini subgroups nor FF-subgroups? Actually, there are many such groups, and then we give an example.

**Example 1.4** (6) Consider the 3-generator group $G = \langle x, y, z \rangle | y^{-1}xy = y^{b-2}x^{-1}y^{b+2}, z^{-1}yz = z^{c-2}y^{-1}z^{c+2}, x^{-1}zx = x^{a-2}z^{-1}x^{a+2} \rangle$ where $a, b, c$ are even integers distinct from zero. Then the subgroups $\langle x \rangle, \langle y \rangle, \langle z \rangle$ are neither contained in the Frattini subgroup nor FF-subgroups.

**Proof** Since $G$ is a 3-generator group, it follows that there do not exist any element $g \in G$ such that $\langle g, \langle x \rangle \rangle = G$ or $\langle g, \langle y \rangle \rangle = G$ or $\langle g, \langle z \rangle \rangle = G$. Then by Lemma 1.3 we see the subgroups $\langle x \rangle, \langle y \rangle, \langle z \rangle$ are not FF-subgroups. On the other hand, it is easy to see the maximal subgroup containing $y$ and $z$ does not contain $x$, and thus $\langle x \rangle$ is not the subgroup of Frattini subgroup of $G$. Similarly, $\langle y \rangle$ and $\langle z \rangle$ are also not the subgroups of Frattini subgroup of $G$. The proof of this example is completed.

In this paper we will show that all nontrivial subgroups of the finite symmetric and alternating groups and projective groups $PSL(2, q)$ are FF-subgroups.

## 2 Symmetric and Alternating Groups

In this section, a permutation has cycle-type $(d_1, d_2, ..., d_t)$ where the $d_i$ are distinct, that is, the permutation is the product of $t$ disjoint cycles whose lengths are $d_i$ for $i = 1, 2, ..., t$. Moreover, $S_n$ is the symmetric group acts on $[n]$, where $[n] = \{1, 2, ..., n\}$.

In [9], M. W. Liebecka, C. E. Praeger and J. Saxl show that if $X$ is $A_n$ or $S_n$, acting on a set $\Omega$ of size $n$, and $G$ is any maximal subgroup of $X$ with $G \neq A_n$, then $G$ satisfies one of the following:

(a) $G = (S_m \times S_k) \cap X$, with $n = m + k$ and $m \neq k$ (intransitive case);
(b) $G = (S_k \wr S_m) \cap X$, with $n = km$, $m > 1$ and $k > 1$ (imprimitive case);
(c) $G = AGL_k(p) \cap X$, with $n = p^k$ and $p$ is a prime (affine case);
(d) $G = (T^k \cdot (OutT \times S_k)) \cap X$, with $T$ a nonabelian simple group, $k \geq 2$ and $n = |T|^{k-1}$ (diagonal case);
(e) $G = (S_k \wr S_m) \cap X$, with $n = k^m$, $k \geq 5$ and $m \geq 2$, excluding the case where $X = A_n$ and $G$ is imprimitive on $\Omega$;

(f) $T \triangleleft G \leq Aut(T)$, with $T$ a nonabelian simple group, $T \neq A_n$ and $G$ acting primitively on $\Omega$ (almost simple case).

Given a subgroup $H$ of $S_n$. Then we use $Max(S_n, H)$ to denote the set of all maximal subgroups of $S_n$ containing $H$, see [11]. Obviously, $H$ is the subgroup of the intersection of all the maximal subgroups in $Max(S_n, H)$, and thus if the intersection of some maximal subgroups in $Max(S_n, H)$ is the identity then $H$ is the identity. Let’s first deal with the symmetric groups along this line.

**Lemma 2.1** Let $n$ be an odd number. Then all nontrivial subgroups of $S_n$ are FF-subgroups.

**Proof** It is clear that there is no nontrivial subgroup in $S_1$, and all nontrivial subgroups of $S_3$ are maximal subgroups. Hence, it suffices to consider the case $n > 3$. Furthermore, we know that the maximal subgroup $S_m \times S_k$ contains the element has cycle-type $(n - 2, 2)$ if and only if $m = 2$ or $k = 2$. In other words, the element has cycle-type $(n - 2, 2)$ is contained in a maximal subgroup $S_{n-2} \times S_2$. We claim that every element has cycle-type $(n - 2, 2)$ is only contained in one maximal subgroup $S_{n-2} \times S_2$.

Since $n$ is an odd number, we see $n - 2$ and $2$ are coprime, therefore, if a maximal subgroup contains an element has cycle-type $(n - 2, 2)$, then the maximal subgroup contains a 2-cycle. Applying [7, Corollary 1.3], it follows that the primitive maximal subgroup contains an element has cycle-type $(n - 2, 2)$ is $A_n$, however, the element has cycle-type $(n - 2, 2)$ is an odd permutation. Hence, all primitive maximal subgroups do not contain the element has cycle-type $(n - 2, 2)$. Considering the imprimitive maximal subgroup $S_k \wr S_m$ with $n = mk$, $m > 1$ and $k > 1$. Then there exists a partition of $[n]$ into $m$ sets of size $k$ such that $S_k \wr S_m$ is the stabiliser of this partition. Noticing that $S_k \wr S_m$ contains the element has cycle-type $(n - 2, 2)$ if and only if $k = 2$ or $m = 2$, and this is contradict to the $n$ is an odd number. Therefore, our claim holds.

Assume that the nontrivial subgroup $H$ of $S_n$ is not a FF-subgroup. Then by Definition [1,2] we see $\Delta_H(S_n) = S_n$. On the other hand, we note that for each element has cycle-type $(n - 2, 2)$, there exists unique maximal subgroup $S_{n-2} \times S_2$ contains it. According to our claim, it follows that all the maximal subgroups $S_{n-2} \times S_2$ are contained in $Max(S_n, H)$. Obviously, the intersection of all the maximal subgroups $S_{n-2} \times S_2$ is the identity, and thus $H$ is the identity, a contradiction. We have thus proved this lemma. □

**Lemma 2.2** Let $n$ be an even number. Then all nontrivial subgroups of $S_n$ are FF-subgroups.

**Proof** Obviously, it suffices to prove the case $n \geq 4$. Since $n$ is an even number, we see $n - 3$ and $2$ are coprime. An argument similar to the one used in proving Lemma [2,1] shows that the element has cycle-type $(n - 3, 2, 1)$ is only contained in the maximal subgroup $S_{n-1}$ or $S_3 \times S_{n-3}$ or $S_2 \times S_{n-2}$. Note that if $n = 4$, then the element has cycle-type $(n - 3, 2, 1)$ is only contained in the maximal subgroup $S_3$, and the proof of Lemma [2,1] indicates all nontrivial subgroups of $S_4$ are FF-subgroups.

Consider the case $n > 4$. Assume that the nontrivial subgroup $H$ of $S_n$ is not a FF-subgroup. However, we observe that $H$ is the identity, in other words, all $x \in [n]$ are fixed points of $H$. Now
we start to confirm our observation.

Note that if \( x \in [n] \) is a fixed point of a maximal subgroup in \( Max(S_n, H) \), then \( x \) is the fixed point of \( H \). So we suppose \( x \) is not the fixed point of all maximal subgroups in \( Max(S_n, H) \). Then we see the element with cycle-type \((n - 3, 2, 1)\) and the fixed point \( x \) is contained in the maximal subgroup \( S_3 \times S_{n-3} \) or \( S_2 \times S_{n-2} \). If all elements with cycle-type \((n - 3, 2, 1)\) and the fixed point \( x \) are contained in the maximal subgroups \( S_2 \times S_{n-2} \), then all these maximal subgroups are contained in \( Max(S_n, H) \) and further the intersection of these maximal subgroups is the identity, and so \( x \) is the fixed point of \( H \). Similarly, \( x \) is the fixed point of \( H \) if all elements with cycle-type \((n - 3, 2, 1)\) and the fixed point \( x \) are contained in the maximal subgroups \( S_3 \times S_{n-3} \). Now we may assume \( \alpha \) with cycle-type \((n - 3, 2, 1)\) and the fixed point \( x \) is contained in the maximal subgroup \( S_3 \times S_{n-3} \), and \( \Delta \) with cycle-type \((n - 2, 2)\) and the fixed point \( x \) is the identity. This leads to our observation. The proof of this lemma is complete. \( \square \)

Using Lemma 2.1 and Lemma 2.2 we obtain the following theorem immediately.

**Theorem 2.3** All nontrivial subgroups of \( S_n \) are FF-subgroups.

We are now turning to study the alternating groups. The proof of alternating group is similar to that given earlier for symmetric group and so we will not go into details here.

**Theorem 2.4** All nontrivial subgroups of \( A_n \) are FF-subgroups.

**Proof** Assume that the nontrivial subgroup \( H \) of \( A_n \) is not a FF-subgroup. Case 1: \( n \) is an odd number. Proceeding as in the proof of Lemma 2.2, we see all elements with cycle-type \((n - 3, 2, 1)\) are contained in the maximal subgroups \( A_{n-1} \) or \( A_3 \times A_{n-3} \) or \( A_2 \times A_{n-2} \). Using the same argument as in the proof of Lemma 2.2 to show that \( H \) is the identity. Case 2: \( n \) is an even number. According to the proof of Lemma 2.1, it follows that all elements with cycle-type \((n - 2, 2)\) are contained in the maximal subgroups \( A_2 \times A_{n-2} \). Similarly, we can obtain that \( H \) is the identity. The proof of this theorem is complete. \( \square \)

## 3 Projective Groups

Recall that the projective group \( PSL(d, q) \) has a faithful 2-transitive action of degree \( \frac{q^d - 1}{q - 1} \) on the set of 1-dimensional subspaces of \( F_q \). However, the maximal subgroups of \( PSL(d, q) \) are rather complicated except \( d = 2 \), see the references [2, 8]. So we mainly investigate \( PSL(2, q) \) in this section. See the reference [4], the maximal subgroups of \( PSL(2, q) \) are as follows.

**Lemma 3.1** ([4]) (A). Let \( q = 2^f \geq 4 \). Then the maximal subgroup of \( PSL(2, q) \) is one of the following:
1) \( Z_2^f \rtimes Z_{q-1} \), that is, the stabilizer subgroup of the projective line \( PG(1, q) \);
2) \( D_{2(q-1)} \);
3) \( D_{2(q+1)} \).
4) $PGL(2, q_0)$, $q = q_0^r$ and $q_0 \neq 2$ where $r$ is a prime number.

(B). Let $q = p^f \geq 5$, where $p$ is an odd prime number. Then the maximal subgroup of $PSL(2, q)$ is one of the following:

1) $Z_p^f \cong Z_{(q-1)/2}$, that is, the stabilizer subgroup of the projective line $PG(1, q)$;
2) $D_q-1$ with $q \geq 13$;
3) $D_{q+1}$, $q \neq 7, 9$;
4) $PGL(2, q_0)$, $q = q_0^2$;
5) $PSL(2, q_0)$, $q = q_0^r$ where $r$ is an odd prime number;
6) $A_5$, $q = p$, $q \equiv \pm 1 (mod 10)$ or $q = p^2$, $p \equiv \pm 3 (mod 10)$;
7) $A_4$, $q = p \equiv \pm 3 (mod 8)$ and $q \equiv \pm 1 (mod 10)$;
8) $S_4$, $q = p \equiv 1 (mod 8)$.

Now we start to study the FF-subgroups of $PSL(2, q)$ and further give the following lemmas.

**Lemma 3.2** Let $q = 2^f \geq 4$. Then all nontrivial subgroups of $PSL(2, q)$ are FF-subgroups.

**Proof** We claim that all elements of order $q + 1$ are contained in the maximal subgroups $D_{2(q+1)}$. It follows from Lemma 3.1 (A) that there are four classes of maximal subgroups in $PSL(2, q)$, and then we verify our claim one by one.

Obviously, there exist some elements of order $q + 1$ in the maximal subgroup $D_{2(q+1)}$. Consider other classes of maximal subgroups. Since $q = 2^f \geq 4$, it follows that $q + 1$ and $q - 1$ are coprime, and thus there is no element of order $q + 1$ in the maximal subgroups $Z_2^f \times Z_q - 1$ and $D_{2(q-1)}$. It is well-known that $|PGL(2, q_0)| = (q_0^2 - 1)(q_0^2 - q_0)/q_0 - 1 = q_0(q_0^2 - 1)$. However, $q = q_0$ and $q_0 \neq 2$ indicate $q + 1 \nmid |PGL(2, q_0)|$, and so there is no element of order $q + 1$ in $PGL(2, q_0)$. So far, we have obtained our claim.

Suppose the nontrivial subgroup $H$ of $PSL(2, q)$ is not a FF-subgroup. Thus $\Delta_H(PSL(2, q)) = PSL(2, q)$. Then by our claim and [5] Theorem 8.3, 8.5 we see that all maximal subgroups of the type $D_{2(q+1)}$ are in Max($PSL(2, q), H$) and further $H$ is the identity, a contradiction. This leads to the lemma.

**Lemma 3.3** Let $q = p^f \geq 5$, where $p$ is an odd prime number. Then all nontrivial subgroups of $PSL(2, q)$ are FF-subgroups.

**Proof** Consider the case that $q < 13$. In this case, we have $q = 5, 7, 9, 11$. According to [5] Chapter 2, Theorem 6.14 and Theorem 2.4, it follows that $PSL(2, 9) \cong A_6$ and $PSL(2, 5) \cong A_5$, and further the lemma holds for $q = 9$ and $q = 5$. If $q = p = 7$, then the maximal subgroup is 1) or 8) of Lemma 3.1 (B), and further all elements of order 4 are contained in the maximal subgroup 8); If $q = p = 11$, then the maximal subgroup is 1) or 3) or 6) or 7) of Lemma 3.1 (B), and further all elements of order 6 are contained in the maximal subgroup 3). Then by using the similar argument as in proof of Lemma 3.2, we see the lemma is also true for $q = p = 7$ and $q = p = 11$.

In the case of $q \geq 13$, we claim that all elements of order $\frac{q^2 - 1}{2}$ are contained in the maximal subgroup $D_{(q+1)}$. One easily checks that there is no element of order $\frac{q^2 + 1}{2}$ in the maximal subgroup 1), 2), 6), 7), 8) of Lemma 3.1 (B). Moreover, it is well-known that $|PGL(2, q_0)| = (q_0^2 - 1)(q_0^2 - q_0)/q_0 - 1 = \frac{(q_0^2 - 1)(q_0^2 - q_0)}{q_0 - 1}$.
and so \( q \mid |PGL(2, q_0)| \) and \( q + 1 \mid |PSL(2, q_0)| \). Hence, our claim holds. An argument similar to the one used in the proof of Lemma \[3.2\] shows this lemma is true for \( q \geq 13 \). The proof of this lemma is complete. □

Applying the Lemma \[3.2\] and Lemma \[3.3\] we obtain the following theorem.

**Theorem 3.4** All nontrivial subgroups of \( PSL(2, q) \) are FF-subgroups.

### 4 Final remarks

Note that for the 2-generator group, the probability of its nontrivial subgroups are FF-subgroups is very high. On the other hand, the Frattini subgroup of finite simple group is trivial. So we propose the following conjecture and open problem to finish this paper.

**Conjecture 4.1** All nontrivial subgroups of a finite simple group are FF-subgroups.

**Question 4.2** Is there a 2-generator group has a nontrivial subgroup is neither the subgroup of its Frattini subgroup nor a FF-subgroup.

### References

1. J. André, J. Araújo and P. J. Cameron, The classification of partition homogeneous groups with applications to semigroup theory, *J. Algebra.*, 452 (2016), 288-310.

2. M. Aschbacher, On the maximal subgroups of the finite classical groups, *Inv. Math.*, 76 (1984), 469-514.

3. P. J. Cameron, A. Lucchini and C. M. Roney-Dougal, Generating sets of finite groups, *Trans. Amer. Math. Soc.*, 370:9 (2018), 6751-6770.

4. L. E. Dickson, *Linear Groups: With an Exposition of the Galois Field Theory*. New York: Dover Publications Inc., 1958.

5. B. Huppert, *Endliche Gruppen I*. Springer, Berlin-Heidelberg, 1967.

6. D. L. Johnson, A New Class of 3-Generator Finite Groups of Deficiency Zero, *J. London Math. Soc.*, 19:2 (1979), 59-61.

7. G. A. Jones, Primitive permutation groups containing a cycle, *Bull. Aust. Math. Soc.*, 89 (2014), 159-165.

8. P. B. Kleidman and M. W. Liebecka, *The Subgroup Structure of the Finite Classical Groups*. Graduate Texts in Mathematics 129. London: Springer-Verlag, 1990.

9. M. W. Liebecka, C. E. Praeger and J. Saxl, Classification of the Maximal Subgroups of the Finite Alternating and Symmetric Groups, *J. Algebra.*, 111 (1987), 365-383.

10. A. Lucchini and P. Spiga, Maximal subgroups of finite groups avoiding the elements of a generating set, *Monatsh. Math.*, 185 (2018), 455-472.
[11] Hangyang Meng and Xiuyun Guo, Overgroups of weak second maximal subgroups, *Bull. Aust. Math. Soc.*, 99 (2019), 83-88.