K₁(S₁) and the group of automorphisms of the algebra S₂ of one-sided inverses of a polynomial algebra in two variables

V. V. Bavula

Abstract

Explicit generators are found for the group G₂ of automorphisms of the algebra S₂ of one-sided inverses of a polynomial algebra in two variables over a field. Moreover, it is proved that

\[ G₂ \simeq S₂ \times \mathbb{T}^2 \times \mathbb{Z} \rtimes ((K^* \ltimes E_\infty(S₁)) \mathbb{S}_{\mathbb{GL}_{\infty}(K)}(K^* \ltimes E_\infty(S₁))) \]

where S₂ is the symmetric group, \( \mathbb{T}^2 \) is the 2-dimensional algebraic torus, E_\( \infty \)(S₁) is the subgroup of GL_\( \infty \)(S₁) generated by the elementary matrices. In the proof, we use and prove several results on the index of operators, and the final argument in the proof is the fact that K₁(S₁) \( \simeq \) K* proved in the paper. The algebras S₁ and S₂ are noncommutative, non-Noetherian, and not domains. The group of units of the algebra S₂ is found (it is huge).

Key Words: the group of automorphisms, the inner automorphisms, the Fredholm operators, the index of an operator, K₁(S₁), the semi-direct product of groups, the minimal primes.

Mathematics subject classification 2000: 14E07, 14H37, 14R10, 14R15.

1 Introduction

Throughout, ring means an associative ring with 1; module means a left module; \( \mathbb{N} := \{0, 1, \ldots \} \) is the set of natural numbers; K is a field and K* is its group of units; \( P_n := K[x_1, \ldots, x_n] \) is a polynomial algebra over K; \( \partial_1 := \frac{\partial}{\partial x_1}, \ldots, \partial_n := \frac{\partial}{\partial x_n} \) are the partial derivatives (K-linear derivations) of \( P_n \); End₀K(\( P_n \)) is the algebra of all K-linear maps from \( P_n \) to \( P_n \) and Aut₀K(\( P_n \)) is its group of units (i.e. the group of all the invertible linear maps from \( P_n \) to \( P_n \)); the subalgebra \( A_n := K(x_1, \ldots, x_n, \partial_1, \ldots, \partial_n) \) of End₀K(\( P_n \)) is called the n’th Weyl algebra.

Definition, [4]. The algebra \( S_n \) of one-sided inverses of \( P_n \) is an algebra generated over a field K by 2n elements \( x_1, \ldots, x_n, y_n, \ldots, y_n \) that satisfy the defining relations:

\[
y_1x_1 = \cdots = y_nx_n = 1, \quad [x_i, y_j] = [x_i, x_j] = [y_i, y_j] = 0 \quad \text{for all } i \neq j,
\]

where \([a, b] := ab - ba\), the commutator of elements a and b.

By the very definition, the algebra \( S_n \) is obtained from the polynomial algebra \( P_n \) by adding commuting, left (but not two-sided) inverses of its canonical generators. The algebra \( S_1 \) is a well-known primitive algebra [8], p. 35, Example 2. Over the field \( \mathbb{C} \) of complex numbers, the completion of the algebra \( S_1 \) is the Toeplitz algebra which is the \( \mathbb{C}^* \)-algebra generated by a unilateral shift on the Hilbert space \( l^2(\mathbb{N}) \) (note that \( y_1 = x_1^* \)). The Toeplitz algebra is the universal \( \mathbb{C}^* \)-algebra generated by a proper isometry.

Example, [4]. Consider a vector space \( V = \bigoplus_{\alpha \in \mathbb{N}^n} Ke_\alpha \) and two shift operators on V, X : \( e_i \mapsto e_{i+1} \) and Y : \( e_i \mapsto e_{i-1} \) for all \( i \geq 0 \) where \( e_{-1} := 0 \). The subalgebra of End₀K(\( V \)) generated by the operators X and Y is isomorphic to the algebra \( S_1 \) (\( X \mapsto x, \ Y \mapsto y \)). By taking the n’th tensor power \( V^\otimes n = \bigoplus_{\alpha \in \mathbb{N}^n} Ke_\alpha \) of V we see that the algebra \( S_n \simeq S_1^\otimes n \) is isomorphic to the subalgebra of End₀K(\( V^\otimes n \)) generated by the 2n shifts \( X_1, Y_1, \ldots, X_n, Y_n \) that act in different directions.
Let $G_n := \text{Aut}_K^\text{-alg}(S_n)$ be the group of automorphisms of the algebra $S_n$, and $S_n^*$ be the group of units of the algebra $S_n$.

Theorem 1.1

1. $G_n = S_n \times T^n \rtimes \text{Inn}(S_n)$.

2. $G_1 = T^1 \ltimes \text{GL}_\infty(K)$.

where $S_n = \{ s(x_i) = x_{s(i)}, s(y_i) = y_{s(i)} \}$ is the symmetric group, $T^n := \{ t_\lambda | t_\lambda(x_i) = \lambda x_i, t_\lambda(y_i) = \lambda^{-1} y_i, \lambda = (\lambda_i) \in K^{*n} \}$ is the $n$-dimensional algebraic torus, $\text{Inn}(S_n)$ is the group of inner automorphisms of the algebra $S_n$ (which is huge), and $\text{GL}_\infty(K)$ is the group of all invertible infinite dimensional matrices of the type $1 + M_\infty$ where the algebra (without 1) of infinite dimensional matrices $M_\infty := \lim_{d \to \infty} M_d(K) = \bigcup_{d \geq 1} M_d(K)$ is the injective limit of matrix algebras. A semi-direct product $H_1 \ltimes H_2 \ltimes \cdots \ltimes H_m$ of several groups means that $H_1 \ltimes (H_2 \ltimes (\cdots \ltimes (H_{m-1} \ltimes H_m) \cdots )).

The results of the papers [1, 3, 4, 5, 6] and the present paper show that (ignoring non-Noetherian property) the algebra $S_n$ belongs to the family of algebras like the $n$’th Weyl algebra $A_n$, the polynomial algebra $P_{2n}$ and the Jacobian algebra $A_n$ (see [1–6]). The structure of the group $G_1 = T^1 \ltimes \text{GL}_\infty(K)$ is another confirmation of ‘similarity’ of the algebras $P_2$, $A_1$, and $S_1$. The groups of automorphisms of the polynomial algebra $P_2$ and the Weyl algebra $A_1$ (when $\text{char}(K) = 0$) were found by Jung [10], Van der Kulk [11], and Dixmier [7] respectively. These two groups have almost identical structure, they are ‘infinite GL-groups’ in the sense that they are generated by the algebraic torus $T^1$ and by the obvious automorphisms: $x \mapsto x + \lambda y$, $y \mapsto y$; $x \mapsto x$, $y \mapsto y + \lambda x$, where $i \in N$ and $\lambda \in K$; which are sort of ‘elementary infinite dimensional matrices’ (i.e. ‘infinite dimensional transvections’). The same picture as for the group $G_1$. In prime characteristic, the group of automorphism of the Weyl algebra $A_1$ was found by Makar-Limanov [9] (see also Bavula [2] for a different approach and for further developments).

Theorem 1.2

1. $S_n^* = K^* \times (1 + a_n)^*$ where the ideal $a_n$ of the algebra $S_n$ is the sum of all the height 1 prime ideals of the algebra $S_n$.

2. The centre of the group $S_n^*$ is $K^*$, and the centre of the group $(1 + a_n)^*$ is $\{1\}$.

3. The map $(1 + a_n)^* \to \text{Inn}(S_n)$, $u \mapsto \omega_u$, is a group isomorphism ($\omega_u(a) := uau^{-1}$ for $a \in S_n$).

The proof of this theorem is given at the end of Section 2 (another proof via the Jacobian algebras is given in [6]).

To save on notation, we identify the groups $(1 + a_n)^*$ and $\text{Inn}(S_n)$ via $u \mapsto \omega_u$. Clearly, $S_2 = S_1(1) \otimes S_1(2)$ where $S_1(1) := K(x_1, y_1) \simeq S_1$. The algebra $S_2$ has only two height one prime ideals $p_1$ and $p_2$. Let $F_2 := p_1 \cap p_2$. The aim of the paper is to find generators for the group $G_2$ (see the end of the paper) and to prove the following theorem.

- (Theorem 2.13)

1. $G_2 = S_2 \times T^2 \ltimes \Theta \ltimes ((U_1(K) \ltimes E_\infty(S_1(2))) \boxtimes (U_2(K) \ltimes E_\infty(S_1(1))))$.

2. $G_2 \simeq S_2 \times T^2 \rtimes \mathbb{Z} \ltimes ((K^* \ltimes E_\infty(S_1(1))) \boxtimes \text{GL}_\infty(K) (K^* \ltimes E_\infty(S_1(1))))$.

For a ring $R$, let $E_\infty(R)$ be the subgroup of $\text{GL}_\infty(R)$ generated by all the elementary matrices $1 + rE_{ij}$ (where $i, j \in N$ with $i \neq j$, and $r \in R$), and let $U(R) := \{ uE_{00} + 1 - E_{00} | u \in R^* \} \approx R^*$ where $R^*$ is the group of units of the ring $R$. The group $E_\infty(R)$ is a normal subgroup of $\text{GL}_\infty(R)$. The group $\Theta \simeq \mathbb{Z}$ is generated by a single element $\theta$ (see Section 2).

If a group $G$ is equal to the product $AB := \{ab | a \in A, b \in B \}$ of its normal subgroups $A$ and $B$ then we write $G = A \boxtimes B = A \boxtimes A \cap B$. So, each element $g \in G$ is a product $ab$ for some elements $a \in A$ and $b \in B$; and $ab = a'b'$ (where $a' \in A$ and $b' \in B$) iff $a' = ac$ and $b' = c^{-1}b$ for some element $c \in A \cap B$. Clearly, $A \boxtimes B = B \boxtimes A$.

At the last stage of the proof of Theorem 2.13 we use the fact that

- (Theorem 2.11) $K_1(S_1) \simeq K^*$.
The group of units $S_2^*$ of the algebra $S_2$ is found.

- (Corollary 2.14) $S_2^* = K^* \times \Theta \ltimes ((U_1(K) \ltimes E_\infty(S_1(2))) \boxtimes_{(1+F_2)^*} (U_2(K) \ltimes E_\infty(S_1(1)))$.

**The structure of the proof of Theorem 2.13** The $S_2$-module $P_2$ is faithful, and so $S_2 \subseteq \text{End}_K(P_2)$. By Theorem 1.1 and Theorem 1.2, the question of finding the group $G_2 = S_2 \ltimes T^2 \ltimes (1+a_2)^*$ is equivalent to the question of finding the group $(1+a_2)^*$ or $S_2^* = K^* \times (1+a_2)^*$. Difficulty in finding the group $S_2^*$ stems from two facts: (i) $S_2^* \subseteq S_2 \cap \text{Aut}_K(P_2)$, i.e. there are non-units of the algebra $S_2$ that are invertible linear maps in $P_2$; and (ii) some units of the algebra $S_2$ are product of non-units with non-zero indices (each unit has zero index). To eliminate (ii) the group $\Theta$ is introduced, and it is proved that $(1+a_2)^* = \Theta \times K$ and for the normal subgroup $K$ of $(1+a_2)^*$ the situation (ii) does not occur. The group $K$ is the common kernel of group epimorphisms $\text{ind}_i : (1+a_2)^* \to \mathbb{Z}$ (see 2.1) where $i = 1, 2$. In order to construct the maps $\text{ind}_i$ and to prove that they are well defined group homomorphisms we need several results on the index of operators which are collected at the beginning of Section 2. Some of these are new (Theorem 2.3 and Corollary 2.5). Briefly, using indices of operators is the main tool in finding the group $S_2^*$ and to prove that $K_1(S_1) \simeq K^*$. Using indices and the fact that $(1+F_2)^* = (1+F_2) \cap \text{Aut}_K(P_2)$ we show that $K = (1+p_1)^* \boxtimes_{(1+p_2)^*} (1+ p_2)^*$ (Proposition 2.9). Then using indices, we prove that $(1+p_1)^* = U_1(K) \times E_\infty(S_1(i+1))$ (Proposition 2.10). This fact is equivalent to the fact that $K_1(S_1) \simeq K^*$ (Theorem 2.11).

## 2 The groups $G_2$ and $K_1(S_1)$

In this section, the groups $G_2$, $S_2^*$, and $K_1(S_1)$ are found (Theorems 2.13 and 2.11). The proofs are constructive.

We mentioned already in the Introduction that the key idea in finding the group $G_2$ is to use indices of operators. That is why we start this section with collecting known results on indices and prove new ones. These results are used in all the proofs that follow.

**The index ind of linear maps and its properties.** Let $\mathcal{C}$ be the family of all $K$-linear maps with finite dimensional kernel and cokernel, i.e. $\mathcal{C}$ is the family of *Fredholm* linear maps/operators. For vector spaces $V$ and $U$, let $\mathcal{C}(V,U)$ be the set of all the linear maps from $V$ to $U$ with finite dimensional kernel and cokernel. So, $\mathcal{C} = \bigcup_{V,U} \mathcal{C}(V,U)$ is the disjoint union.

*Definition.* For a linear map $\varphi \in \mathcal{C}$, the integer $\text{ind}(\varphi) := \dim \ker(\varphi) - \dim \text{coker}(\varphi)$ is called the index of the map $\varphi$.

For vector spaces $V$ and $U$, let $\mathcal{C}(V,U)_i := \{ \varphi \in \mathcal{C}(V,U) | \text{ind}(\varphi) = i \}$. Then $\mathcal{C}(V,U) = \bigcup_{i \in \mathbb{Z}} \mathcal{C}(V,U)_i$ is the disjoint union, and the family $\mathcal{C}$ is the disjoint union $\bigcup_{i \in \mathbb{Z}} \mathcal{C}_i$ where $\mathcal{C}_i := \{ \varphi \in \mathcal{C} | \text{ind}(\varphi) = i \}$. When $V = U$, we write $\mathcal{C}(V) := \mathcal{C}(V,V)$ and $\mathcal{C}(V)_i := \mathcal{C}(V,V)_i$.

*Example.* Note that $S_1 \subset \text{End}_K(P_1)$ $(x \cdot x^i = x^{i+1}, \ y \cdot x^i = x^i, \ i \in \mathbb{N}, \ \text{and} \ y \cdot 1 = 0)$. The map $x^i$ acting on the polynomial algebra $P_1$ is an injection with $P_1 = (\bigoplus_{j=0}^{i-1} Kx^j) \bigoplus \text{im}(x^j)$; and the map $y^i$ acting on $P_1$ is a surjection with $\ker(y^i) = \bigoplus_{j=0}^{i-1} Kx^j$, and so

\[
\text{ind}(x^i) = -i \ \text{and} \ \text{ind}(y^i) = i, \ i \geq 1.
\]

(1)

Lemma 2.1 shows that $\mathcal{C}$ is a multiplicative semigroup with zero element (if the composition of two elements of $\mathcal{C}$ is undefined we set their product to be zero).

*Lemma 2.1.* Let $\psi : M \to N$ and $\varphi : N \to L$ be $K$-linear maps. If two of the following three maps: $\psi$, $\varphi$, and $\varphi \psi$, belong to the set $\mathcal{C}$ then so does the third; and in this case,

\[
\text{ind}(\varphi \psi) = \text{ind}(\varphi) + \text{ind}(\psi).
\]

By Lemma 2.1 $\mathcal{C}(N,L)_i \mathcal{C}(M,N)_j \subseteq \mathcal{C}(M,L)_{i+j}$ for all $i, j \in \mathbb{Z}$. 3
Lemma 2.2 Let

\[
\begin{array}{cccc}
0 & \rightarrow & V_1 & \rightarrow V_2 & \rightarrow V_3 & \rightarrow 0 \\
\downarrow{\varphi_1} & & \downarrow{\varphi_2} & & \downarrow{\varphi_3} & \\
0 & \rightarrow & U_1 & \rightarrow U_2 & \rightarrow U_3 & \rightarrow 0
\end{array}
\]

be a commutative diagram of \(K\)-linear maps with exact rows. Suppose that \(\varphi_1, \varphi_2, \varphi_3 \in C\). Then

\[\text{ind}(\varphi_2) = \text{ind}(\varphi_1) + \text{ind}(\varphi_3).\]

Let \(V\) and \(U\) be vector spaces. Define \(I(V,U) := \{\varphi \in \text{Hom}_K(V,U) \mid \dim \ker(\varphi) < \infty\}\), and when \(V = U\) we write \(I(V) := I(V,V)\).

Theorem 2.3 Let \(V\) and \(U\) be vector spaces. Then \(C(V,U)_i + I(V,U) = C(V,U)_i\) for all \(i \in \mathbb{Z}\).

Proof. The theorem is obvious if the vector spaces \(V\) and \(U\) are finite dimensional since in this case the index of each linear map from \(V\) to \(U\) is equal to \(\dim(V) - \dim(U)\). We deduce the general case from this one. Let \(u \in C(V,U)_i\) and \(f \in I(V,U)\). Using the fact that the kernel \(\ker(f)\) has finite codimension in \(V\), i.e. \(\dim(V/\ker(f)) < \infty\) (since \(V/\ker(f) \cong \text{im}(f)\)), we can easily find subspaces \(V_1, V_2 \subseteq V\) and \(W, U_1, U_2 \subseteq U\) such that \(\dim(V_1) < \infty\), \(\dim(U_1) < \infty\), \(\dim(W) < \infty\),

\[V = \ker(u) \bigoplus V_1 \bigoplus V_2, \quad u = W \bigoplus U_1 \bigoplus U_2, \quad u|_{V_1} : V_1 \cong U_1, \quad u|_{V_2} : V_2 \cong U_2, \quad f(V_2) = 0\]

and \(f(\ker(u) \bigoplus V_1) \subseteq W \bigoplus U_1\). Note that \(\text{im}(u) = U_1 \bigoplus U_2\) and \(U/\text{im}(u) \cong W\). Consider the restrictions, say \(u'\) and \(f'\), of the maps \(u\) and \(f\) to the finite dimensional subspace \(\ker(u) \bigoplus V_1\) of \(V\), i.e.

\[u', f' : \ker(u) \bigoplus V_1 \rightarrow W \bigoplus U_1.\]

Then it is obvious that \(\text{ind}(u') = \text{ind}(u)\) and \(\text{ind}(u' + f') = \text{ind}(u + f)\) (since \(u + f|_{V_2} = u|_{V_2} : V_2 \cong U_2\)). On the other hand, \(\text{ind}(u' + f') = \text{ind}(u')\) since the vector spaces in \(\bigoplus\) are finite dimensional. Therefore, \(\text{ind}(u + f) = \text{ind}(u)\). □

Lemma 2.4 Let \(V\) and \(V'\) be vector spaces, and \(\varphi : V \rightarrow V'\) be a linear map such that the vector spaces \(\ker(\varphi)\) and \(\text{coker}(\varphi)\) are isomorphic. Fix subspaces \(U \subseteq V\) and \(V' \subseteq V'\) such that \(V = \ker(\varphi) \bigoplus U\) and \(V' = W \bigoplus \text{im}(\varphi)\) and fix an isomorphism \(f : \ker(\varphi) \rightarrow W\) (this is possible since \(\ker(\varphi) \cong \text{coker}(\varphi) \cong W\) and extend it to a linear map \(f : V \rightarrow V'\) by setting \(f(U) = 0\). Then the map \(\varphi + f : V \rightarrow V'\) is an isomorphism.

Proof. The map \(\varphi + f\) is a surjection since \((\varphi + f)(V) = (\varphi + f)(\ker(\varphi) + U) = W + \text{im}(\varphi) = V'\). The map \(\varphi + f\) is an injection: if \(v \in \ker(\varphi + f)\) then \(\varphi(v) = f(-v) \in W \cap \text{im}(\varphi) = 0\), and so \(v \in \ker(\varphi) \cap \ker(f) = \ker(\varphi) \cap U = 0\). Therefore, the map \(\varphi + f\) is an isomorphism. □

Lemma 2.5 Let \(V\) and \(V'\) be vector spaces, \(\varphi \in C(V,V')_i\) for some \(i \in \mathbb{Z}\), \(V = \ker(\varphi) \bigoplus U\) and \(V' = W \bigoplus \text{im}(\varphi)\) for some subspaces \(U \subseteq V\) and \(W \subseteq V'\).

1. If \(\dim \ker(\varphi) \leq \dim \text{coker}(\varphi)\) then fix an injective linear map \(f : \ker(\varphi) \rightarrow W\) and extend it to a linear map \(f : V \rightarrow V'\) by setting \(f(U) = 0\). Then the map \(\varphi + f\) is an injection that belongs to \(C(V,V')_i\).

2. If \(\dim \ker(\varphi) \geq \dim \text{coker}(\varphi)\) then fix a surjective linear map \(f : \ker(\varphi) \rightarrow W\) and extend it to a linear map \(f : V \rightarrow V'\) by setting \(f(U) = 0\). Then the map \(\varphi + f\) is a surjection that belongs to \(C(V,V')_i\).

Proof. 1. An arbitrary element \(v \in V = \ker(\varphi) \bigoplus U\) is a unique sum \(k + u\) where \(k \in \ker(\varphi)\) and \(u \in U\). If \(v \in \ker(\varphi + f)\) then \(0 = (\varphi + f)(k + u) = f(k) + \varphi(u)\) and so \(f(k) = 0\) and \(\varphi(u) = 0\) (since \(V' = W \bigoplus \text{im}(\varphi), f(k) \in W\) and \(\varphi(u) \in \text{im}(\varphi)\)) hence \(k = 0\) (\(f\) is an injection) and \(u = 0\)
The algebras $S_1$ and $S_2$. We collect some results without proofs on the algebras $S_1$ and $S_2$ from [4] that will be used in this paper, their proofs can be found in [4]. Clearly, $S_2 = S_1(1) \otimes S_1(2) \simeq S_1^2$, where $S_1(i) := K\langle x_i, y_i \mid y_i x_i = 1 \rangle \simeq S_1$ and $S_2 = \bigoplus_{\alpha, \beta \in \mathbb{N}^2} Kx^\alpha y^\beta$ where $x^\alpha := x_1^{i_1} x_2^{i_2}$, $\alpha = (\alpha_1, \alpha_2)$, $y^\beta := y_1^{j_1} y_2^{j_2}$, $\beta = (\beta_1, \beta_2)$. In particular, the algebra $S_2$ contains two polynomial subalgebras $P_2$ and $Q_2 := K[y_1, y_2]$ and is equal, as a vector space, to their tensor product $P_2 \otimes Q_2$.

When $n = 1$, we usually drop the subscript ‘1’ if this does not lead to confusion. So, $S_1 = K\langle x, y \mid y x = 1 \rangle = \bigoplus_{i,j \geq 0} Kx^i y^j$. For each natural number $d \geq 1$, let $M_d(K) := \bigoplus_{i,j=0}^{d-1} KE_{ij}$ be the algebra of $d$-dimensional matrices where $\{E_{ij}\}$ are the matrix units, $M_\infty(K) := \lim_{d \to \infty} M_d(K) = \bigoplus_{i,j \in \mathbb{N}} KE_{ij}$ be the algebra (without 1) of infinite dimensional matrices, and $GL_\infty(K)$ be the group of units of the monoid $1 + M_\infty(K)$. The algebra $S_1$ contains the ideal $F := \bigoplus_{i,j \in \mathbb{N}} KE_{ij}$, where
\[
E_{ij} := x^i y^j - x^{i+1} y^{j+1}, \quad i, j \geq 0.
\]
(3)

For all natural numbers $i, j, k,$ and $l$, $E_{ij} E_{kl} = \delta_{jk} E_{il}$ where $\delta_{jk}$ is the Kronecker delta function. The ideal $F$ is an algebra (without 1) isomorphic to the algebra $M_\infty(K)$ via $E_{ij} \mapsto E_{ij}$.

The polynomial algebra $P_n$ is the only (up to isomorphism) faithful, simple $S_n$-module.

In more detail, $S_n P_n \simeq S_n / \sum_{i=1}^n \overline{S_n y_i} = \bigoplus_{\alpha \in \mathbb{N}^n} Kx^\alpha \mathbb{T}, \quad \mathbb{T} := 1 + \sum_{i=1}^n \overline{S_n y_i}$; and the action of the canonical generators of the algebra $S_n$ on the polynomial algebra $P_n$ is given by the rule:

\[
x_i * x^\alpha = x^{\alpha + e_i}, \quad y_i * x^\alpha = \begin{cases} x^{\alpha + e_i} & \text{if } \alpha_i > 0, \\ 0 & \text{if } \alpha_i = 0, \end{cases} \quad E_\beta * x^\alpha = \delta_{\gamma \alpha} x^\beta,
\]
where the set of elements $e_1 := (1,0,\ldots,0),\ldots,e_n := (0,\ldots,0,1)$ is the canonical basis for the free $\mathbb{Z}$-module $\mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z}e_i$. We identify the algebra $\mathbb{S}_n$ with its image in the algebra $\text{End}_K(P_n)$ of all the $K$-linear maps from the vector space $P_n$ to itself, i.e. $\mathbb{S}_n \subseteq \text{End}_K(P_n)$.

**Corollary 2.7**  
1. $1 + F_2 \subseteq C(P_2)_0$.  
2. $\mathbb{S}_2^* + F_2 \subseteq C(P_2)_0$.

**Proof.** Both statements follows from Theorem 2.3 $\mathbb{S}_2^* \subseteq C(P_2)_0$ and $F_2 \in \mathcal{I}(P_2)$, but we give short independent proofs (that do not use Theorem 2.3).

1. Since $1 + F_2 \simeq 1 + M_{\infty}(K)$, statement 1 is obvious.

2. Let $u \in \mathbb{S}_2^*$ and $f \in F_2$. Then $u^{-1}f \in F_2$. By statement 1, the element $1 + u^{-1}f \in C(P_2)_0$. Since $u \in C(P_2)_0$, we have $u + f = u(1 + u^{-1}f) \in C(P_2)_0$, by Lemma 2.1. □

The subgroup $\Theta$ of $(1 + a_2)^*$. The element $\theta := (1 + (y_1 - 1)E_{00}(2))(1 + E_{00}(1)(x_2 - 1)) \in (1 + a_2)^*$ is a unit and

$$\theta^{-1} = (1 + E_{00}(1)(y_2 - 1))(1 + (x_1 - 1)E_{00}(2)) \in (1 + a_2)^*. \quad (6)$$

This is obvious since

$$\theta \star x^\alpha = \begin{cases} 
    x^\alpha & \text{if } \alpha_1 > 0, \alpha_2 > 0, \\
    x_1^{\alpha_1 - 1} & \text{if } \alpha_1 > 0, \alpha_2 = 0, \\
    x_2^{\alpha_2 + 1} & \text{if } \alpha_1 = 0, \alpha_2 > 0,
\end{cases} \quad \text{and} \quad \theta^{-1} \star x^\alpha = \begin{cases} 
    x^\alpha & \text{if } \alpha_1 > 0, \alpha_2 > 0, \\
    x_1^{\alpha_1 + 1} & \text{if } \alpha_1 \geq 0, \alpha_2 = 0, \\
    x_2^{\alpha_2 - 1} & \text{if } \alpha_1 = 0, \alpha_2 > 0.
\end{cases}$$

Let $\Theta$ be the subgroup of $(1 + a_2)^*$ generated by the element $\theta$. Then $\Theta \simeq \mathbb{Z}$ since $\theta^* \star 1 = x_2^i$ for all $i \geq 1$. It follows from

$$(1 + (y_1 - 1)E_{00}(2)) \star x^\alpha = \begin{cases} 
    x^\alpha & \text{if } \alpha_2 > 0, \\
    x_1^{\alpha_1 - 1} & \text{if } \alpha_1 > 0, \alpha_2 = 0, \\
    0 & \text{if } \alpha_1 = 0, \alpha_2 = 0,
\end{cases}$$

that the map $1 + (y_1 - 1)E_{00}(2) \in \text{End}_K(P_2)$ is a surjection with kernel equal to $K$, and so

$$\text{ind}(1 + (y_1 - 1)E_{00}(2)) = 1. \quad (7)$$

Similarly, it follows from

$$(1 + E_{00}(1)(x_2 - 1)) \star x^\alpha = \begin{cases} 
    x^\alpha & \text{if } \alpha_1 > 0, \\
    x_2^{\alpha_2 + 1} & \text{if } \alpha_1 = 0,
\end{cases}$$

that the map $1 + E_{00}(1)(x_2 - 1) \in \text{End}_K(P_2)$ is an injection with $P_2 = K \bigoplus \text{im}(1 + E_{00}(1)(x_2 - 1))$, and so

$$\text{ind}(1 + E_{00}(1)(x_2 - 1)) = -1. \quad (8)$$

We see that the unit $\theta$ of the algebra $\mathbb{S}_2$ is the product of two non-units having nonzero indices the sum of which is equal to zero since $\text{ind}(\theta) = 0$. Lemma 2.8 shows that this is a general phenomenon, and so the group $(1 + a_2)^*$ is a sophisticated group in the sense that in construction of units non-units are involved.

**Lemma 2.8** Let $u = 1 + a_1 + a_2 \in (1 + a_2)^*$ where $a_i \in p_i$. Then

1. $1 + a_1, 1 + a_2 \in C(P_2)$ and $\text{ind}(1 + a_1) + \text{ind}(1 + a_2) = 0$.
2. If $u = 1 + a_1' + a_2'$ where $a_i' \in p_i$ then $\text{ind}(1 + a_1) = \text{ind}(1 + a_1')$ and $\text{ind}(1 + a_2) = \text{ind}(1 + a_2')$. 

6
Proof. 1. Since $a_i \in p_i$, we have $a_1a_2, a_2a_1 \in F_2$. By Corollary \ref{corollary}(2), $u + a_1a_2, u + a_2a_1 \in C(P_2)_0$. Then, it follows from the equalities $u + a_1a_2 = (1 + a_1)(1 + a_2)$ and $u + a_2a_1 = (1 + a_2)(1 + a_1)$, that

\[
\text{im}(1 + a_1) \supseteq \text{im}(u + a_1a_2), \quad \ker(1 + a_1) \subseteq \ker(u + a_2a_1),
\]

\[
\text{im}(1 + a_2) \supseteq \text{im}(u + a_2a_1), \quad \ker(1 + a_2) \subseteq \ker(u + a_1a_2).
\]

This means that $1 + a_1, 1 + a_2 \in C(P_2)$. By Corollary \ref{corollary}(2) and Lemma \ref{lemma},

\[
0 = \text{ind}(u + a_1a_2) = \text{ind}(1 + a_1)(1 + a_2) = \text{ind}(1 + a_1) + \text{ind}(1 + a_2).
\]

2. It is obvious that $a' = a_1 + f$ and $a'' = a_2 - f$ for an element $f \in p_1 \cap p_2 = F_2$. Since $F_2 \subseteq I(P_2)$, we see that $\text{ind}(1 + a') = \text{ind}(1 + a_1 + f) = \text{ind}(1 + a_1)$ and $\text{ind}(1 + a'') = \text{ind}(1 + a_2 - f) = \text{ind}(1 + a_2)$, by Theorem \ref{theorem} \hfill \Box

By Lemma \ref{lemma} for each number $i = 1, 2$, there is a well-defined map,

\[
\text{ind}_i : (1 + a_2)^* \rightarrow \mathbb{Z}, \quad u = 1 + a_1 + a_2 \mapsto \text{ind}(1 + a_i), \tag{9}
\]

which is a group homomorphism:

\[
\text{ind}_i(uu') = \text{ind}_i((1 + a_1 + a_2)(1 + a'_1 + a''_2)) = \text{ind}(1 + a_i + (a'_1 + a''_2)) = \text{ind}_i((1 + a_i)(1 + a'_i)) = \text{ind}(1 + a_i) + \text{ind}(1 + a'_i) \]

\[
= \text{ind}_i(u) + \text{ind}_i(u').
\]

The restriction of the homomorphism $\text{ind}_i$ to the subgroup $\Theta$ is an isomorphism: $\text{ind}_1 : \Theta \rightarrow \mathbb{Z}$, $\theta \mapsto -1$; $\text{ind}_2 : \Theta \rightarrow \mathbb{Z}$, $\theta \mapsto 1$. Therefore, the homomorphisms $\text{ind}_i$ are epimorphisms which have the same kernel (Lemma \ref{lemma}(1)) which we denote by $K$. Then,

\[
(1 + a_2)^* = \Theta \ltimes K. \tag{10}
\]

It is obvious that $(1 + p_1)^* \otimes_{(1 + F_2)^*} (1 + p_2)^* \subseteq K$ since $(1 + p_1)^* \cap (1 + p_2)^* = (1 + F_2)^*$.

Proposition \ref{proposition}. $K = (1 + p_1)^* \otimes_{(1 + F_2)^*} (1 + p_2)^*$.

Proof. It suffices to show that each element $u = 1 + a_1 + a_2$ of the group $K$ is a product $u_1u_2$ for some elements $u_i \in (1 + p_i)^*$. Note that $1 + a_1 \in C(P_2)_0$. Fix a subspace, say $W$, of $P_2$ such that $P_2 = \ker(1 + a_1) \oplus W$ and $W = \bigoplus_{a \in J} Kx^a$ where $J$ is a subset of $\mathbb{N}$. By Lemma \ref{lemma}, we can find an element $f_1 \in F_2$ (since $\dim \ker(1 + a_1) < \infty$, $W$ has a monomial basis, and $f_1(W) = 0$) such that $u_1 := 1 + a_1 + f_1 \in \text{Aut}_K(P_2)$. We claim that $u_1 \in (1 + p_1)^*$. It is a subtle point since not all elements of the algebra $S_2$ that are invertible linear maps in $P_2$ are invertible in $S_2$, i.e. $S_2^* \not\subseteq S_2 \cap \text{Aut}_K(P_2)$ but $(1 + F_2)^* = (1 + F_2) \cap \text{Aut}_K(P_2)$, \emph{\textbf{[5]}}. The main idea in the proof of the claim is to use this equality. Similarly, we can find an element $f_2 \in F_2$ such that $v := 1 + a_2 + f_2 \in \text{Aut}_K(P_2)$. Then $u = u_1v + g_1$ and $u = vu_1 + g_2$ for some elements $g_i \in F_2$. Hence,

\[
u_{1} vu^{-1} = 1 - g_1 u^{-1} \quad \text{and} \quad u^{-1} vu_1 = 1 - u^{-1} g_2,
\]

and so $1 - g_1 u^{-1}, 1 - u^{-1} g_2 \in (1 + F_2) \cap \text{Aut}_K(P_2) = (1 + F_2)^*$. It follows that $u_1^{-1} = v u^{-1} (1 - g_1 u^{-1})^{-1} \in (1 + p_1)^*$ since

\[
1 \equiv 1 - g_1 u^{-1} \equiv u_1 vu^{-1} \equiv v u^{-1} \mod p_1.
\]

This proves the claim. Clearly, $u_2 := v + u_1^{-1} g_1 \in 1 + p_2$. Then, it follows from the equality $u = u_1v + g_1 = u_1 (v + u_1^{-1} g_1) = u_1 u_2$ that $u_2 = u_1^{-1} u \in (1 + p_2)^*$. This finishes the proof of the proposition. \hfill \Box
In order to save on notation, it is convenient to treat the set of indices \{1, 2\} as the group \(\mathbb{Z}/2\mathbb{Z} = \{1, 2\}\) where \(1 + 1 = 2\) and \(1 + 2 = 1\). For each number \(i = 1, 2\), the group of units of the monoid \(1 + p_i = 1 + F(i) \otimes S_1(i + 1) = 1 + M_\infty(S_1(i + 1))\) is equal to \((1 + p_i)^* = \text{GL}_\infty(S_1(i + 1))\). It contains the semi-direct product \(U_i(K) \ltimes E_\infty(S_1(i + 1))\) of its two subgroups, where

\[ U_i(K) := \{ \lambda E_{00}(i) + 1 - E_{00}(i) \mid \lambda \in K^* \} \cong K^* \]

and the group \(E_\infty(S_1(i + 1))\) is generated by all the elementary matrices \(1 + aE_{kl}(i)\) where \(k \neq l\) and \(a \in S_1(i + 1)\). Note that the group \(E_\infty(S_1(i + 1))\) is a normal subgroup of \(\text{GL}_\infty(S_1(i + 1))\).

The set \(F_2\) is an ideal of the algebra \(K + p_i = K(1 + p_i)\) which is a subalgebra of the algebra \(S_2\), and \((K + p_i)/F_2 = K(1 + p_i/F_2) \cong K(1 + M_\infty(L_{i+1}))\) where \(L_{i+1} := K[x_{i+1}, x_{i+1}^{-1}] \cong S_1(i + 1)/F(i + 1)\) is the Laurent polynomial algebra. The algebra \(L_{i+1}\) is a Euclidian domain, hence \(\text{GL}_\infty(L_{i+1}) = U(L_{i+1}) \ltimes E_\infty(L_{i+1})\) where

\[ U(L_{i+1}) := \{ aE_{00}(i) + 1 - E_{00}(i) \mid a \in L_{i+1}^* \} \cong L_{i+1}^* \cong \{ x_{i+1}^m \mid m \in \mathbb{Z} \} \]

and \(E_\infty(L_{i+1})\) is the subgroup of \(\text{GL}_\infty(L_{i+1})\) generated by all the elementary matrices.

The group of units of the algebra \((K + p_i)/F_2\) is equal to \(K^* \times \text{GL}_\infty(L_{i+1}) = K^* \times (U(L_{i+1}) \ltimes E_\infty(L_{i+1}))\). The algebra epimorphism \(\psi_i : K + p_i \to (K + p_i)/F_2\), \(a \mapsto a + F_2\), induces the exact sequence of groups,

\[ 1 \to (1 + F_2)^* \to (1 + p_i)^* \xrightarrow{\psi_i} \text{GL}_\infty(L_{i+1}) = U(L_{i+1}) \ltimes E_\infty(L_{i+1}), \]

which yields the short exact sequence of groups,

\[ 1 \to (1 + F_2)^* \to U_i(K) \times E_\infty(S_1(i + 1)) \to U(K) \times E_\infty(L_{i+1}) \to 1 \]

since \((1 + F_2)^* \subseteq E_\infty(S_1(i + 1))\), by Proposition 2.10. Note that \(U_i(K) \times E_\infty(S_1(i + 1)) \subseteq (1 + p_i)^*\). In fact, the equality holds.

**Proposition 2.10**: \((1 + p_i)^* = U_i(K) \times E_\infty(S_1(i + 1))\).

**Proof.** In view of the exact sequences (11) and (12), it suffices to show that the image of the map \(\psi_i\) in (11) is equal to \(U(K) \times E_\infty(L_{i+1})\). Since

\[ U(L_{i+1}) = U(K) \times \{ E_{00}(i)x_{i+1}^m + 1 - E_{00}(i) \mid m \in \mathbb{Z} \} \]

this is equivalent to show that that if \(\psi_i(u) := E_{00}(i)x_{i+1}^m + 1 - E_{00}(i)\) for some element \(u \in (1 + p_i)^*\) and an integer \(m \in \mathbb{Z}\) then \(m = 0\). Let \(u(m) := E_{00}(i)v_{i+1}(m) + 1 - E_{00}(i)\) where \(v_{i+1}(m) := \begin{cases} x_{i+1}^m & \text{if } m \geq 0, \\ y_{i+1}^{\lfloor m \rfloor} & \text{if } m < 0, \end{cases}\). Then \(u(m) \in 1 + p_i\) and \(\psi_i(u(m)) = \psi_i(u)\). Hence, \(u(m) = u + f_m\) for some element \(f_m \in F_2\). Note that

\[ u(m) = \begin{cases} u(1)^m & \text{if } m \geq 0, \\ u(-1)^{|m|} & \text{if } m < 0, \end{cases}\]

and, by (7) and (8), \(\text{ind}(u(m)) = -m\). By Corollary 2.11 (2).

\[ 0 = \text{ind}(u) = \text{ind}(u + f_m) = \text{ind}(u(m)) = -m, \]

and so \(m = 0\), as required. \(\Box\)

Proposition 2.10 is equivalent to the next theorem.

**Theorem 2.11**: \(K_1(S_1) \cong U(K) \cong K^*\) and \(\text{GL}_\infty(S_1) = U(K) \ltimes E_\infty(S_1)\).
Proof. Recall that $K_1(S_1) := \text{GL}_\infty(S_1)/E_\infty(S_1)$ where $E_\infty(S_1)$ is the subgroup of $\text{GL}_\infty(S_1)$ generated by the elementary matrices. The group $E_\infty(S_1)$ is a normal subgroup of $\text{GL}_\infty(S_1)$. It follows from $(1 + p_1)^* \cong \text{GL}_\infty(S_1)$ and Proposition 2.10 that

$$K_1(S_1) \cong U(K) \times E_\infty(S_1)/E_\infty(S_1) \cong U(K) \cong K^*.$$  

\[ \square \]

The determinant $\text{det}$. The algebra epimorphism $S_1 \to S_1/F \simeq L_1$, $a \to a + F$, yields the group homomorphism $\psi : \text{GL}_\infty(S_1) \to \text{GL}_\infty(L_1)$. By Proposition 2.10 the image of the group homomorphism $\text{det} \circ \psi : \text{GL}_\infty(S_1) \to \text{GL}_\infty(L_1) \to \text{det} \circ L_1^* \cong K^*$. Therefore, there is a well determined group epimorphism:

$$\text{det} := \text{det} \circ \psi : \text{GL}_\infty(S_1) \to K^*.$$  

By the very definition, $\text{det}(E_\infty(S_1)) = 1$ and $\text{det}(\mu(\lambda)) = \lambda$ for all elements $\mu(\lambda) = \lambda E_{00} + 1 - E_{00} \in U(K)$ where $\lambda \in K^*$. Therefore, there is the exact sequence of groups:

$$1 \to E_\infty(S_1) \to \text{GL}_\infty(S_1) \to K^* \cong K_1(S_1) \to 1.$$  

Corollary 2.12 Each element $a$ of the group $\text{GL}_\infty(S_1) = U(K) \times E_\infty(S_1)$ is a unique product $a = \mu(\lambda)e$ where $\mu(\lambda) \in U(K)$ and $e \in E_\infty(S_1)$. Moreover, $\lambda = \text{det}(a)$ and $e = \mu(\mathbf{det}(a))a$.

Recall that $G_2 = S_2 \times T^2 \rtimes \text{Inn}(S_2)$ (Theorem 1.1(1)) and $(1 + a_2)^* \simeq \text{Inn}(S_2)$, $u \leftrightarrow \omega_u$ (Theorem 1.2(3)). We identify the groups $(1 + a_2)^*$ and $\text{Inn}(S_2)$ via $u \leftrightarrow \omega_u$.

Theorem 2.13

1. $G_2 = S_2 \times T^2 \rtimes \Theta \times ((U_1(K) \times E_\infty(S_1(2))) \boxtimes (1 + F_2)^*) \rtimes (U_2(K) \times E_\infty(S_1(1)))$.

2. $G_2 \simeq S_2 \times T^2 \rtimes \mathbb{Z} \rtimes ((K^* \times E_\infty(S_1)) \boxtimes GL_\infty(K)) (K^* \times E_\infty(S_1(1))).$

Proof. By [10], Proposition 2.9 and Proposition 2.10

$$(1 + a_2)^* = \Theta \times ((U_1(K) \times E_\infty(S_1(2))) \boxtimes (1 + F_2)^*) \rtimes (U_2(K) \times E_\infty(S_1(1))),$$

and the statements follow since $(1 + F_2)^* \simeq GL_\infty(K)$. \[ \square \]

Corollary 2.14 $S_2^* = K^* \rtimes \Theta \times ((U_1(K) \times E_\infty(S_1(2))) \boxtimes (1 + F_2)^*) \rtimes (U_2(K) \times E_\infty(S_1(1))).$

Generators for the group $G_2$. Using Theorem 2.13 and [11], we can easily write down a set of generators for the group $G_2$:

$$s : \quad x_i \mapsto x_{i+1}, \quad y_i \mapsto y_{i+1}, \quad \lambda \in K^*;$$

$$t_{(\lambda, 1)} : \quad x_1 \mapsto \lambda x_1, \quad y_1 \mapsto \lambda y_1, \quad x_2 \mapsto x_2, \quad y_2 \mapsto y_2, \quad \lambda \in K^*;$$

$$\omega_\theta, \omega_\lambda E_{00}(1 + 1 - E_{00}(1), \omega_\lambda E_{ij}(1) E_{ij}(2) + 1 - E_{ij}(1) E_{ij}(2), \omega_1 + \mu x_{m} E_{ij}(1), \omega_1 + \mu x_{m} E_{ij}(1), \omega_1 + \mu x_{m} E_{ij}(1)$$

and where $\lambda \in K^*$, $\mu \in K$, $i, j, k, l \in \mathbb{N}$, $m \geq 1$, $i \neq j$ (note that $s \omega_1 + \lambda x_{m} E_{ij}(1)$).

Each element $\sigma \in G_n = S_n \rtimes T^m \rtimes \text{Inn}(S_n)$ is a unique product $st_\lambda \omega_u$ where $s \in S_n$, $t_\lambda \in T^m$ and $\omega_u \in \text{Inn}(S_n)$. In [1], for each element $\sigma \in G_n$, using the elements $\sigma(x_i), \sigma(y_i) \in S_n$, $i = 1, \ldots, n$ explicit algebraic formulae are found for the components $s, t_\lambda, \omega_u$ of $\sigma$. So, the automorphism $\sigma$ can be effectively (in finitely many steps) decomposed into the product $st_\lambda \omega_u$.

Proposition 2.15 $(1 + F_2)^* \subseteq E_\infty(S_1(i))$ for $i = 1, 2$.

Proof. Due to symmetry it suffices to show that $(1 + F_2)^* \subseteq E_\infty(S_1(2))$. Recall that the group $(1 + F_2)^* = (1 + \sum_{\alpha, \beta} K E_{\alpha \beta})$ is non-canonically isomorphic to the group $G_2$. To see this we have to choose a bijection $b : \mathbb{N}^2 \to \mathbb{N}$. Then the matrix units $E_{\alpha \beta}$ can be seen as the usual matrix units $E_{b(\alpha)b(\beta)}$ and so $(1 + F_2)^* \simeq GL_\infty(K)$. This isomorphism depends on
the choice of the bijection. Since $(1 + F_2)^* \simeq \text{GL}_\infty(K)$, the group $(1 + F_2)^*$ is generated by the elements $a = 1 + \lambda E_{ij}(1)E_{kl}(2)$ where $\lambda \in K$ and $(i, k) \neq (j, l)$, and $b = 1 + \lambda E_{00}(1)E_{00}(2)$ where $\lambda \in K \setminus \{-1\}$. It suffices to show that these generators belong to the group $E_\infty(S_1(2))$.

First, let us show that $a \in E_\infty(S_1(2))$. If $i \neq j$ then obviously the inclusion holds. If $i = j$, i.e. $a = 1 + \lambda E_{ii}(1)E_{kk}(2)$, then necessarily $k \neq l$ since $(i, k) \neq (i, l)$. For an element $g$ and $h$ of a group, $[g, h] := ghg^{-1}h^{-1}$ is their group commutator. For any natural number $l$ such that $l \neq i$, the elements $1 + E_{ii}(1)E_{kk}(2)$ and $1 + \lambda E_{ii}(1)E_{kk}(2)$ belong to the group $E_\infty(S_1(2))$. Then so does their commutator

\[
[1 + E_{ii}(1)E_{kk}(2), 1 + \lambda E_{ii}(1)E_{kk}(2)] = 1 + \lambda E_{ii}(1)E_{kk}(2).
\]

Therefore, all the generators $a$ belong to the group $E_\infty(S_1(2))$.

It remains to prove that $b \in E_\infty(S_1(2))$. In the $2 \times 2$ matrix ring $M_2(S_1(2))$ with entries in the algebra $S_1(2)$ we have the equality, for all scalars $\lambda \in K \setminus \{-1\}$:

\[
\begin{pmatrix}
\frac{1}{1 + \lambda} & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\lambda x_2^2 & 1 \\
y_2 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & \frac{1}{1 + \lambda}
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
\lambda x_2^2 & 1 \\
1 & 0
\end{pmatrix}
= \begin{pmatrix}
1 + \lambda & 0 \\
0 & 1
\end{pmatrix}.
\]

This can be checked by direct multiplication using the equalities $y_2 x_2 = 1$, $x_2 y_2 = 1 - E_{00}(2)$, $y_2 E_{00}(2) = 0$ and $E_{00}(2) x_2 = 0$ in the algebra $S_1(2)$.

Since the first six matrices in the equality belong to the group $E_\infty(S_1(2))$, the last matrix $c = \begin{pmatrix} 1 - \frac{\lambda E_{00}(2)}{1 + \lambda} & 0 \\ 0 & 1 \end{pmatrix}$ belongs to the group $E_\infty(S_1(2))$ as well and can be written as

\[
c = E_{00}(1)(1 - \frac{\lambda}{1 + \lambda} E_{00}(2)) + 1 - E_{00}(1) = 1 - \frac{\lambda}{1 + \lambda} E_{00}(1)E_{00}(2) \in (1 + F_2)^*.
\]

Since the map $\varphi : K \setminus \{-1\} \to K \setminus \{-1\}$, $\lambda \mapsto -\frac{\lambda}{1 + \lambda}$, is a bijection ($\varphi^{-1} = \varphi$), all the elements $b$ belong to the group $E_\infty(S_1(2))$. The proof of the proposition is complete. □

**Proof of Theorem 1.2** The third statement follows at once from the second one. To prove the remaining two statements we use induction on $n$. The case when $n = 1$ is Theorem 4.6, [5].

So, let $n > 1$ and we assume that the first two statements hold for all natural numbers $n' < n$. Clearly, $S_n = \bigotimes_{i=1}^n S_1(i)$ where $S_1(i) := K(x_i, y_i) \simeq S_1$. Consider the following ideals of the algebra $S_n$:

\[
p_1 := F \otimes S_{n-1}, \quad p_2 := S_1 \otimes F \otimes S_{n-2}, \ldots, \quad p_n := S_{n-1} \otimes F, \quad a_n := p_1 + \cdots + p_n.
\]

Let $K(x_n)$ be the field of fractions of the polynomial algebra $K[x_n]$. It follows from the chain of algebra homomorphisms

\[
S_n \to S_n/p_n \simeq S_{n-1} \otimes S_1/F \simeq S_{n-1} \otimes K[x_n, x_n^{-1}] \to S_{n-1} \otimes K(x_n) \simeq S_{n-1}(K(x_n))
\]

and from the induction on $n$ that $S_n^{*} \subseteq S_1(n) + a_n = \sum_{i \geq 1} K y_i^i + K + \sum_{i \geq 1} K x_i^n + a_n$ (since $S_{n-1}(K(x_n))^* = K(x_n)^*$, by induction). By symmetry of the indices $1, \ldots, n$, we have the inclusion $S_n^{*} \subseteq \bigcap_{i=1}^n (\sum_{i \geq 1} K y_i^i + K + \sum_{i \geq 1} K x_i^n + a_n) = K + a_n$ and so

\[
K^*(1 + a_n)^* \subseteq S_n^* \subseteq (K + a_n)^* = K^* \cdot (S_n^* \cap (1 + a_n)) = K^*(1 + a_n)^* = K^* \times (1 + a_n)^*
\]

since $a_n$ is an ideal of the algebra $S_n$ (and so $S_n^* \cap (1 + a_n) = (1 + a_n)^*$). This proves statement 1.

It follows from statement 1, [13] and induction that $Z(S_n) \subseteq K^*(1 + p_n)^*$, hence, by symmetry,

\[
Z(S_n^*) \subseteq \bigcap_{i=1}^n K^*(1 + p_i)^* = K^*(1 + \bigcap_{i=1}^n p_i)^* = K^*(1 + F_n)^*
\]

where $F_n := \bigcap_{i=1}^n p_i$. Since $(1 + F_n)^* \simeq \text{GL}_\infty(K)$ (see Section 2, [4]) and the centre of the group $\text{GL}_\infty(K)$ is $\{1\}$, statement 2 follows. □
3 Normal subgroups of $GL_\infty(S_1)$ and the centres of the groups $GL_\infty(S_1)/SL$ and $E_\infty(S_1)/SL$

In this section, several normal subgroups of the group $GL_\infty(S_1)$ are introduced, see [11] and Propositions 8.2 (4). The most important (and non-obvious) is the normal subgroup $SL$ (Proposition 8.2 (4)). The group $(1 + F_2)^*$ is non-canonically isomorphic to the group $GL_\infty(K)$, hence it inherits the determinant homomorphism $\det$, see (19), and $SL := \{ u \in (1 + F_2)^* \mid \det(u) = 1 \}$.

We will show that the determinant $\det$ and the group $SL$ does not depend on the isomorphism $(1 + F_2)^* \simeq GL_\infty(K)$. Moreover, the determinant is invariant under the conjugation of its argument by the elements of the group $GL_\infty(S_1)$, Proposition 3.2 (3). This is the central point of this section. It implies that the group $SL$ is a normal subgroup of $GL_\infty(S_1)$ and is a key fact in finding the centres of the groups $GL_\infty(S_1)/SL$ and $E_\infty(S_1)/SL$ (Theorem 3.3).

In order to prove Theorem 3.3 we use results and notations of Section 2. In particular, we use the following group isomorphism:

$$(1 + p_1)^* = (1 + M_\infty(S_1))^* \simeq GL_\infty(S_1), \quad 1 + \sum a_{ij} E_{ij}(1) \mapsto 1 + \sum a_{ij} E_{ij},$$

where $a_{ij} \in S_1 = S_1(2)$ and $E_{ij}$ are the matrix units. It is convenient to identify the groups $(1 + p_1)^*$ and $GL_\infty(S_1)$ via this isomorphism, i.e. $E_{ij}(1) = E_{ij}$. Then, by Proposition 2.10

$$GL_\infty(S_1) = U \ltimes E_\infty(S_1)$$

where $U := \{ \mu(\lambda) := \lambda E_{00} + 1 - E_{00} \mid \lambda \in K^* \} \simeq K^*$, $\mu(\lambda) \leftrightarrow \lambda$, and the groups $E_\infty(S_1)$ and $(1 + F_2)^*$ are normal subgroups of $GL_\infty(S_1)$.

As we have seen in the proof of Proposition 2.15 the group $(1 + F_2)^*$ is isomorphic to the group $GL_\infty(K)$. This isomorphism depends on the choice of the bijection $b$. For the group $GL_\infty(K)$, we have the determinant (group epimorphism) $\det : GL_\infty(K) \to K^*$, the short exact sequence of groups $1 \to SL_\infty(K) \to GL_\infty(K) \xrightarrow{\det} K^* \to 1$, and the decomposition $GL_\infty(K) = U(K) \ltimes SL_\infty(K)$ where $U(K) = \{ \mu(\lambda) \mid \lambda \in K^* \}$. Therefore, for the group $(1 + F_2)^*$ we have the determinant (group epimorphism) $\det : (1 + F_2)^* \to K^*$, the short exact sequence of groups

$$1 \to SL \to (1 + F_2)^* \xrightarrow{\det} K^* \to 1 \quad (19)$$

and the decomposition $(1 + F_2)^* = U' \ltimes SL$ where

$$U' := \{ \mu'(\lambda) := \lambda E_{00}(1)E_{00}(2) + 1 - E_{00}(1)E_{00}(2) \mid \lambda \in K^* \} \simeq K^*, \quad \mu'(\lambda) \leftrightarrow \lambda,$$

$$SL := \{ u \in (1 + F_2)^* \mid \det(u) = 1 \}.$$ 

The group $SL$ is generated by the elements $1 + \lambda E_{\alpha\beta}$ where $\lambda \in K$ and $\alpha, \beta \in \mathbb{N}^2$ such that $\alpha \neq \beta$. Theorem 8.1, [5], says that the map $\det : (1 + F_2)^* \to K^*$ does not depend on the choice of the bijection $b$.

**Theorem 3.1** (Theorem 8.1, [5]) Let $V = \{ V_i \}_{i \in \mathbb{N}}$ be a finite dimensional vector space filtration on $P_2$ (i.e. $V_0 \subseteq V_1 \subseteq \cdots$ and $P_2 = \bigcup_{i \in \mathbb{N}} V_i$) and $a \in (1 + F_2)^*$. Then $a(V_i) \subseteq V_i$ and $\det(a|_{V_i}) = \det(a|_{V_j})$ for all $i, j \gg 0$. Moreover, this common value of the determinants does not depend on the filtration $V$ and, therefore, coincides with the determinant in (19).

By Theorem 3.1, the group $SL$ does not depend on the choice of the bijection $b$. The algebra $S_2$ admits the involution:

$$\eta : S_2 \to S_2, \quad x_i \mapsto y_i, \quad y_i \mapsto x_i, \quad i = 1, 2,$$

i.e. it is a $K$-algebra anti-isomorphism ($\eta(ab) = \eta(b)\eta(a)$ for all elements $a, b \in S_2$) such that $\eta^2 = \text{id}_{S_2}$, the identity map on $S_2$. It follows that

$$\eta(E_{ij}(k)) = E_{ji}(k)$$ and $\eta(E_{\alpha\beta}) = E_{\beta\alpha}$

(20)
Theorem 3.3

Using the fact that \((1 + \alpha_\mathbb{N})^4. By statement 3, the group \(SL\) is a normal subgroup of \(GL_\infty\).

The polynomial algebra \(P_{\ge 1}\) is equipped with the cubic filtration \(C := \{C_m := \sum_{\alpha \in C_m} Kx^\alpha\}_{m \in \mathbb{N}}\) where \(C_m := \{\alpha \in \mathbb{N}^2 | \alpha_i \le m\}\). The filtration \(C\) is an ascending, finite dimensional filtration such that \(P_2 = \bigcup_{m \ge 0} C_m\) and \(C_mC_l \subseteq C_{m+l}\) for all \(m, l \ge 0\).

Proposition 3.2

1. For all elements \(a \in (1 + F_2)^*\), \(\det(\eta(a)) = \det(a)\).

2. Let \(a \in GL_\infty(S_1) = (1 + p_1)^*\) and let \(V = \{V_i\}_{i \in \mathbb{N}}\) be an ascending finite dimensional filtration on \(P_2\) such that \(a(V_i) \subseteq V_i\) for all \(i \gg 0\). Then \(\det(aba^{-1}) = \det(a)\) for all elements \(b \in (1 + F_2)^*\).

3. For all elements \(a \in GL_\infty(S_1)\) and \(b \in (1 + F_2)^*\), \(\det(aba^{-1}) = \det(a)\).

4. The group \(SL\) is a normal subgroup of \(GL_\infty(S_1)\). Moreover, each subgroup \(N\) of the group \((1 + F_2)^*\) that contains the group \(SL\) is a normal subgroup of \(GL_\infty(S_1)\).

Proof. 1. Recall that \((1 + F_2)^* = U' * SL\). It is obvious that \(\eta(u) = u\) for all elements \(u \in U'\) and \(\eta(SL) \subseteq SL\) (by [20], the set of standard generators of the group \(SL\) is invariant under the action of the involution \(\eta\)). An element \(a \in (1 + F_2)^*\) is a unique product \(a = us\) for some elements \(u \in U'\) and \(s \in SL\). Now, \(\det(\eta(a)) = \det(\eta(us)) = \det(\eta(s)u) = \det(u) = \det(a)\).

2. Applying Theorem 3.1 to the element \(aba^{-1} \in (1 + F_2)^*\) we have the result, for all \(i \gg 0\):

\[\det(aba^{-1}) = \det(aba^{-1}|_{V_i}) = \det(a|_{V_i} \cdot b|_{V_i} \cdot a^{-1}|_{V_i}) = \det(b|_{V_i}) = \det(b).\]

3. We deduce statement 3 from the previous two. Recall that we have identified the groups \((1 + p_1)^*\) and \(GL_\infty(S_1)\). By Theorem 2.11 \(GL(S_1) = U \times E_\infty(S_1)\). Since the subgroup \((1 + F_2)^*\) of \(GL_\infty(S_1)\) is normal, it suffices to show that statement 3 holds for generators of the groups \(U\) and \(E_\infty(S_1)\). Since \(U = \{\mu(\lambda) | \lambda \in K^*\}\) and \(\mu(\lambda)(C_m) \subseteq C_m\) for all \(m \ge 0\) where \(C = \{C_m\}_{m \in \mathbb{N}}\) is the cubic filtration on the polynomial algebra \(P_2\), we see that \(\det(\mu(\lambda)b\mu(\lambda)^{-1}) = \det(b)\), by statement 2.

2. The group \(E_\infty(S_1)\) is generated by the elements \(\{\lambda \in K, i \neq j\}: a_{ij} = 1 + E_{ij}\lambda x_{ij}^2\) where \(n \ge 1\); \(b_{ij} = 1 + E_{ij}\lambda x_{ij}^m\) where \(m \ge 0\); and \(1 + E_{ij}f\) where \(f \in F_2\). Statement 3 holds for the elements \(1 + E_{ij}f\) since they belong to the group \((1 + F_2)^*\).

4. By Proposition 2.2 (4), there is a chain of normal subgroups of the group \(GL_\infty(S_1)\):

\[SL \subset (1 + F_2)^* \subset E_\infty(S_1) \subset GL_\infty(S_1).\]

Using the fact that \((1 + F_2)^* = U' * SL\), we have the chain of normal subgroups of the factor group \(GL_\infty(S_1)/SL\):

\[U' \subset E_\infty(S_1)/SL \subset GL_\infty(S_1)/SL.\]

Theorem 3.3

The group \(U' \simeq K^*\) is the centre of both groups \(GL_\infty(S_1)/SL\) and \(E_\infty(S_1)/SL\).
Proof. By Proposition 4.2 (3), the group $U'$ belongs to the centres of both groups. The algebra homomorphism $S_1 \to S_1/F \cong K[x_2, x_2^{-1}]$ yields the group homomorphisms: $GL_{\infty}(S_1) \xrightarrow{\psi} GL_{\infty}(K[x_2, x_2^{-1}])$ and $E_{\infty}(S_1) \xrightarrow{\phi} E_{\infty}(K[x_2, x_2^{-1}])$. By Proposition 2.10, $\text{im}(\phi) = U(K) \ltimes E_{\infty}(K[x_2, x_2^{-1}])$ and $\text{im}(\psi) = E_{\infty}(K[x_2, x_2^{-1}])$. Since the groups $\text{im}(\phi)$ and $\text{im}(\psi)$ have trivial centre and $\text{ker}(\phi) = \text{ker}(\psi) = (1 + F_2^*/\text{SL})$, the result follows. \qed

References

[1] V. V. Bavula, The Jacobian algebras, J. Pure Appl. Algebra 213 (2009), 664-685; Arxiv:math.RA/0704.3850.

[2] V. V. Bavula, The group of order preserving automorphisms of the ring of differential operators on a Laurent polynomial algebra in prime characteristic, Proc. Amer. Math. Soc. 137 (2009), 1891-1898, (Arxiv:math.RA/0806.1038).

[3] V. V. Bavula, The group of automorphisms of the first Weyl algebra in prime characteristic and the restriction map, Glasgow Math. J. 5 (2009), 263-274; (Arxiv:math.RA/0708.1620).

[4] V. V. Bavula, The algebra of one-sided inverses of a polynomial algebra, J. Pure Appl. Algebra 214 (2010), 1874–1897; (Arxiv:math.RA/0903.0641).

[5] V. V. Bavula, The group of automorphisms of the algebra of one-sided inverses of a polynomial algebra, ArXiv:math.AG/0903.3049.

[6] V. V. Bavula, The group of automorphisms of the Jacobian algebra $A_n$, Arxiv:math.AG/0910.0999.

[7] J. Dixmier, Sur les algèbres de Weyl, Bull. Soc. Math. France 96 (1968), 209–242.

[8] N. Jacobson, “Structure of rings,” Am. Math. Soc. Colloq., Vol. XXXVI (rev. ed.), Am. Math. Soc., Providence, 1968.

[9] L. Makar-Limanov, On automorphisms of Weyl algebra, Bull. Soc. Math. France 112 (1984), 359–363.

[10] H. W. E. Jung, Über ganze birationale Transformationen der Ebene, J. Reine Angew. Math. 184 (1942), 161-174.

[11] W. Van der Kulk, On polynomial rings in two variables, Nieuw Arch. Wisk. (3) 1 (1953), 33-41.

Department of Pure Mathematics
University of Sheffield
Hicks Building
Sheffield S3 7RH
UK
email: v.bavula@sheffield.ac.uk