We analyze propagation equations for the polar modes of gravitational waves in cosmological space-times. We prove that polar gravitational waves must perturb the density and non-azimuthal components of the velocity of material medium of the Friedman–Lemaitre–Robertson–Walker spacetimes. Axial gravitational waves can influence only the azimuthal velocity, leading to local cosmological rotation. The whole gravitational dynamics reduces to the single ‘master equation’ that has the same form for polar and axial modes. That allows us to conclude that the status of the Huygens principle is the same for axial and polar gravitational waves. In particular, this principle is valid exactly in radiation spacetimes with the vanishing cosmological constant, and it is broken otherwise.

Keywords: gravitational waves, cosmology, Huygens principle, polar mode’s equations, cosmological rotation

1. Introduction

The canonical way to establish the topological type of the Universe is through the analysis of the Hubble relation [1]. There would seem to exist an alternative—it is known that gravitational waves can backscatter on the curvature of the spacetime. The effects of the backscatter (or equivalently, the breakdown of the Huygens principle) would give, in principle at least, a new local way to see the global topology of a cosmological spacetime in a way analogous to effects known in the Schwarzschild spacetime. Quasinormal modes and tails [2, 3] that are present in the radiation that originated beneath the light sphere of a black hole would allow its identification. These effects can be vigorous, as seen in the recent detection of gravitational
waves \cite{4}. They have been discussed in theoretical studies of \cite{5} and \cite{6}; the latter have shown a positive correlation between the strength of backscattering and the intensity of ringing. The results of \cite{7} have shown that while axial modes of gravitational waves indeed do backscatter (with the exception of purely radiative spacetimes), but this effect becomes significant only for length-waves comparable to the Hubble radius, and thus cannot be seen in local observations.

The original motivation of this paper was to complete the analysis of \cite{7} by discussing polar modes. That constitutes a technically demanding task, but the conclusion is the same as in \cite{7}: backscattering effects cannot be seen locally. There is, however, a feature, that (we think) is more interesting: it appears that polar gravitational waves must perturb the material content—both its density and poloidal components of the velocity—of the cosmological universe. This should be contrasted with the case of axial gravitational waves, where one may choose initial data in such a way, that gravitational waves completely decouple from matter. The new result is that the azimuthal component of the velocity—and only azimuthal one—of the background fluid can be affected; moreover, it is expected to be perturbed by generic initial data. Thus axial modes can lead to a local (but cosmologically significant) rotation.

Regge and Wheeler extracted two gauge-independent linearized modes (axial and polar) of gravitational waves in the Schwarzschild spacetime \cite{8} and derived an evolution equation for axial modes. They made minor mistakes in equations describing propagation of polar modes, corrected by Zerilli \cite{9}. In the cosmological context, Malec and Wyleżek \cite{7} adapted the method of Regge and Wheeler and obtained a propagation equation for axial modes in Friedman–Lemaître–Robertson–Walker (FLRW thereafter) spacetimes, with vanishing cosmological constant. They found that the Huygens principle is strictly obeyed in radiation spacetimes and approximately—for relatively short length-waves—in spacetimes filled by dust.

The content of this paper is following. Next section specifies the FRLW geometries with a non-zero cosmological constant, and their (Regge–Wheeler) perturbations. The material is composed of perfect fluids, and we write down both the relevant energy-momentum tensor, and its linearization. Section 3 supplements formerly known results obtained in \cite{7}. It appears that matter density cannot be perturbed by travelling axial gravitational waves, and the description of waves reduces to a simple propagation equation (a ‘master equation’). Clarkson, Clifton and February discussed the perturbations in the context of dust dominated Lemaître–Tolman cosmologies \cite{10}; their equations, restricted to the FLRW metrics deformed by axial modes, coincide with those of \cite{7}. The new element of the present analysis, absent in \cite{7} and \cite{10}, is that axial gravitational waves can interact with matter and change its azimuthal velocity—azimuthal velocity perturbations can be non-zero.

The case with polar modes is discussed in section 4. The calculation was done from scratch, with the use of Mathematica. The full evolution system consists of six equations; one can extract a part of the metric that is described by the master equation, that turned out to have exactly the same form as for the axial model and that agrees with the corresponding result of \cite{10}. There was only one minor difference in one of the remaining equations, comparing to \cite{10}; see a comment later on.

We should mention here an earlier analysis of Gundlach and Martín-García \cite{11} that uses the approach of Gerlach and Sengupta \cite{12}, and that describes propagation of gravitational modes within time-dependent stellar enviroments. We understand that one can get from these the cosmological (FLRW) propagation equations—see a discussion in \cite{10}—but our analysis is independent.

Section 5 deals with the universe filled by perturbed perfect fluid. We demonstrate that equations are consistent and we prove that mass density and velocity (radial and poloidal
components) of the material (perfect fluid) must be perturbed by a travelling polar gravitational wave pulse.

The vacuum case, de Sitter spacetime, is considered in section 6. The whole system of 6 equations is consistent, reduces to 3 independent equations, the master equation and two other conditions that completely determine the linearized metric.

The next section shows that the Huygens principle is strictly valid only in radiation spaces-times (with $\Lambda = 0$), both for axial and polar modes.

The final section gives a concise description of main results and comments on their significance.

We assume standard gravitational units $c = 8\pi G = 1$.

2. FLRW geometry and its perturbations

We shall use standard coordinates, but with the conformal time coordinate; the FLRW metric tensor reads

$$\begin{pmatrix}
-a^2(\eta) & 0 & 0 & 0 \\
0 & a^2(\eta) & 0 & 0 \\
0 & 0 & a^2(\eta)f^2 & 0 \\
0 & 0 & 0 & a^2(\eta)f^2 \sin^2 \theta
\end{pmatrix},$$

where $f = f(r) = r, \sin r, \sinh r$ for the flat, closed, and open universes, respectively. $d\Omega^2 = d\theta^2 + \sin^2 \theta \ d\phi^2$ is the line element on a unit sphere.

We shall use below the ‘conformal’ Hubble constant,

$$H(\eta) := \frac{\partial_\eta a}{a};$$

it is related to the ‘ordinary’ Hubble constant $H$ by $H = \frac{H}{a}$.

2.1. Metric Regge–Wheeler perturbations

We shall employ results of Regge and Wheeler [8]. Below we put for notational simplicity $Y = Y(\theta) := Y_0(\theta)$; thus

$$Y'' = -l(l + 1)Y - \cot \theta \ Y',$$

where $Y' = \partial_\theta Y$.

For completeness we shall consider both polar and axial modes; in the latter case we again arrive at conclusions already known from [7].

2.1.1. Axial modes. The axially perturbed components of the metric read

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + e \cdot h_{\mu\nu}^{(o)} + o(e^2),$$

where

$$(h_{\mu\nu}^{(o)}) = \begin{pmatrix}
0 & 0 & 0 & h_0 \sin \theta \ Y' \\
0 & 0 & 0 & h_1 \sin \theta \ Y' \\
0 & 0 & 0 & 0 \\
h_0 \sin \theta \ Y' & h_1 \sin \theta \ Y & 0 & 0
\end{pmatrix}.$$
Here $h_0 = h_0(\eta, r), h_1 = h_1(\eta, r)$. The small parameter $e$ is introduced for convenience; it measures the strength of perturbations.

2.1.2. Polar modes. The perturbed components of the metric corresponding to polar modes are

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + e \cdot h_{\mu\nu}^{(p)} + o(e^2),$$

where we used the convention of Clarkson et al [10]:

$$\left(h_{\mu\nu}^{(p)}\right) = \begin{pmatrix} (\chi + \varphi) Y & \sigma Y & 0 & 0 \\ \sigma Y & (\chi + \varphi) Y & 0 & 0 \\ 0 & 0 & f^2 \varphi Y & 0 \\ 0 & 0 & 0 & f^2 \sin^2 \theta \varphi Y \end{pmatrix};$$

as before, we do not write arguments of perturbing functions explicitly. Thus $\chi = \chi(\eta, r), \varphi = \varphi(\eta, r), \sigma = \sigma(\eta, r)$.

We need to say in this place, that the relevant perturbation matrix of [10] contains an additional function $\psi$ in the metric element $h_{00}$, but we checked that it must vanish for the FLRW background.

2.2. The energy-momentum tensor

In the former calculation concerning axial gravitational waves [7] there was no need to perturb the cosmological background. This is in contrast, as we shall see, with the polar modes. Below we shall perturb the background material quantities in both cases, but finally it appears that this may be avoided in the situation discussed in [7].

The tensor $T_{\mu\nu}$ has the form:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} - \Lambda g_{\mu\nu}. $$

The mass density is given by

$$\rho = \rho_0 (1 + e \cdot \Delta(\eta, r)Y) + o(e^2),$$

where $\rho_0$ is the background mass density and $e \cdot \Delta(\eta, r)Y$ is the mass density contrast.

The pressure splits onto $p_0$, the background pressure, and $e \cdot \Pi(\eta, r)Y$—the pressure contrast:

$$p = p_0 (1 + e \cdot \Pi(\eta, r)Y) + o(e^2).$$

The pressure and mass density perturbations $\Pi$ and $\Delta$ are related, since $\rho$ and $p$ are related by an equation of state.

Finally, one should allow the possibility that matter is not necessarily comoving with the unperturbed cosmological expansion. Thus the 4-velocity of matter reads

$$u_0 = 2g_{00}^{(0)} + e \cdot h_{00} + o(e^2),$$

$$u_1 = e \cdot a(\eta) w(\eta, r) Y + o(e^2),$$

$$u_2 = e \cdot v(\eta, r) Y' + o(e^2),$$

$$u_3 = e \cdot \sin \theta \cdot u(\eta, r) Y' + o(e^2)$$

This ensures that $u_\mu u^\mu = -1 + o(e^2)$. 

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2.3. Background Friedman–Lemaitre solution

The (background) isotropic and homogeneous solution of Friedman equations satisfies the following relations

\[ \rho_0 = \frac{3}{a^2} H^2 + \frac{3}{a^2} \frac{1 - f'^2}{f^2} - \Lambda, \]  
\[ p_0 = \Lambda - \frac{1}{a^2} \frac{1 - f'^2}{f^2} - \frac{1}{a^2} H^2 - \frac{2}{a^2} \dot{H}. \]

From these two equations one arrives at

\[ \frac{a^2}{2} \left( \frac{1}{3} \rho_0 - p_0 + \frac{4}{3} \Lambda \right) = H^2 + \dot{H} + \frac{1 - f'^2}{f^2}; \]

this relation is used later.

3. Main calculations: axial modes

Linearized Einstein equations corresponding to the metric (5), read

\[ \Delta \cdot \rho_0 = 0 \quad (15) \]
\[ \Pi \cdot p_0 = 0 \quad (16) \]
\[ w \cdot (\rho_0 + p_0) = 0 \quad (17) \]
\[ v \cdot (\rho_0 + p_0) = 0 \quad (18) \]
\[ h'_1 = h_0 \quad (19) \]
\[ \dot{h}'_1 - h''_1 - 2Hh'_1 + 2\frac{f'}{f} h'_1 - 4H^2 f' h_1 + \frac{l(l+1) + 4f'^2}{f^2} h_0 = -2a^3 (\rho_0 + p_0) u \quad (20) \]
\[ \ddot{h}_1 - h'_1 - 2Hh'_1 + 2\frac{f'}{f} h_0 - 2\dot{H}h_1 + \frac{l(l+1) - 2}{f^2} h_1 = 0 \quad (21) \]

The first four equations immediately imply \( \Delta = \Pi = w = v = 0 \). In what follows we form the following linear combination of equations (19)–(21)

\[ \partial_r (f^2 \cdot (21)) - \partial_h (f^2 \cdot (20)) - (l(l+1) - 2) \cdot (19) \]
\[ = f^2 \partial_h (2a^3 (\rho_0 + p_0) u). \]

The expression \( \partial_r (f^2 \cdot (21)) \) means that the left hand side of equation no (21) is multiplied by \( f^2 \) and then differentiated with respect \( r \). The same convention is applied in the remaining part of this paper.
It appears that the left hand side of (22) vanishes identially, which in turn implies that the angular perturbing term \( u \) acquires following form:

\[
\dot{u} = \frac{C(r)}{a^4(\rho_0 + p_0)}. \tag{23}
\]

One may choose \( C(r) = 0 \)—that was implicitly assumed in [7] and [10]. Then the material content of the background cosmological spacetime does not feel the propagation of axial modes. In the generic case initial data for axial modes yield nonzero \( C(r) \)—see equation (20). Therefore a pulse of an axial radiation enforces infinitesimal rotation of a cosmological fluid around \( z \)-axis. One may expect, judging from the form of the right hand side of (23), that this perturbation increases, \( u \propto a \), during radiation era and remains constant during dust-dominated era. This issue requires a more careful analysis that will be done elsewhere [13].

Furthermore, inserting (19) into (21), one arrives at

\[
\ddot{h}_1 - h_1'' - 2\dot{H}h_1 + 2\frac{f'}{f}h_1' - 2\dot{H}h_1 + \frac{l(l+1) - 2}{f^2}h_1 = 0. \tag{24}
\]

Defining now a new quantity \( \tilde{Q}(\eta, r) \) by

\[
h_1(\eta, r) = f(r)a(\eta)Q(\eta, r), \tag{25}
\]

and using (14), one arrives at a single wave equation

\[
\ddot{Q} - Q'' + \frac{l(l+1)}{f^2}Q - \frac{1}{2a^2} \left( \frac{1}{3} \rho_0 - p_0 + \frac{4}{3} \Lambda \right) Q = 0. \tag{26}
\]

This agrees with the corresponding result of [7] and [10].

The master equation (26) can be solved independently of the remaining equations—it constitutes one of the two independent gravitational modes.

From equation (19) one gets \( h_0 \):

\[
h_0(\eta, r) = A(r) + \int_{\eta_0}^{\eta} h_1'(\tau, r) d\tau. \tag{27}
\]

The function \( A(r) \) is arbitrary, but if \( h_0(\eta, r) \) vanishes at the initial hypersurface, then \( A(r) = 0 \).

4. Polar modes: equations

The linearization of Einstein equation yields, after straightforward calculations, three groups of equations.

Two equations describe the evolution of the mass density contrast

\[
\Delta(\eta, r) = \frac{1}{2a^4a_0} \left[ \left( \frac{l(l+1) + 6f'^2 - 4}{f^2} + 2H^2 \right) \chi 
+ 2 \left( \frac{l(l+1) + 3f'^2 - 3}{f^2} - 3H^2 \right) \varphi 
- 8Hf' \sigma + 2f' (\chi' - 2\varphi') + 2H(\chi + 3\dot{\varphi} - 2\sigma') - 2\varphi'' \right]. \tag{28}
\]

and the pressure contrast,
\[ \Pi(\eta, r) = \frac{1}{2a^4 p_0} \left[ \left( \frac{l(l + 1) - 2f'^2}{f^2} + 2H^2 - 4\dot{H} \right) \chi \right. \\
+ \left. 2 \left( \frac{1 - f'^2}{f^2} + H^2 \right) \varphi + 2 \frac{f'}{f} \chi' + 2H(\dot{\varphi} - \dot{\chi}) - 2\ddot{\varphi} \right]. \] (29)

The second group describes deformation of the 2 components of the 4-velocity:

\[ w(\eta, r) = \frac{1}{2a^4 (\rho_0 + p_0)} \left[ 4Hf' \chi - \frac{l(l + 1) + 4f'^2 - 4}{f^2} \sigma \right. \\
- \left. 2 \frac{f'}{f} \ddot{\chi} + 2H(\chi' - \varphi') + 2\varphi' \right]; \] (30)

\[ v(\eta, r) = -\frac{2H \varphi - \sigma' + 2\dot{\varphi} + \dot{\chi}}{2a^3 (\rho_0 + p_0)} \] (31)

The last pair of equations does not depend explicitly on matter:

\[ \chi' = \dot{\sigma} \] (32)

\[ \ddot{\chi} - \chi'' - 2H \dot{\chi} + 2 \frac{f'}{f} \chi' - 2H \chi + \frac{l(l + 1)}{f^2} - \frac{2}{\chi} = 0 \] (33)

Notice that the equation (33) onto the function \( \chi \) separates from the rest. With the substitution

\[ \chi(\eta, r) = f(r)a(\eta)Q(\eta, r), \] (34)

one arrives at a result that coincides with the equation (26).

Again it appears that the master equation can be solved independently of the remaining equations. This implies that the metric functions \( \chi \) constitute one gravitational degree of freedom. Moreover, the evolution equations of axial and polar modes have the same form. The remaining metric functions, \( \sigma \) and \( \varphi \) are not arbitrary. For instance one has

\[ \sigma(\eta, r) = B(r) + \int_{\eta_0}^\eta \chi'(\tau, r)d\tau; \] (35)

the function \( B(r) \) that appears here is arbitrary, but if one assumes that initially \( \sigma(\eta, r) \) vanishes, then \( B(r) = 0 \).

One of our equations that describes behaviour of the radial velocity perturbation, equation (30), disagrees with the Friedmannian limit of equation (44) in [10]; in order to achieve agreement, there should be \( 8\pi\rho \) instead of \( 4\pi\rho \) in equation (44). Below we show the (wrong) equation implied by equation (44):

\[ w(\eta, r) = \frac{1}{2a^4 (\rho_0 + p_0)} \left[ 4Hf' \chi - \frac{l(l + 1) + 4f'^2 - 4}{f^2} \sigma \right. \\
+ \left. 3\sigma \left( \frac{1 - f'^2}{f^2} + H^2 \right) - 2 \frac{f'}{f} \ddot{\chi} + 2H(\chi' - \varphi') + 2\varphi' \right]; \] (36)
the spurious term is \(3\sigma \left( \frac{1-f^2}{f} + H^2 \right)\).

We checked that equations (32), (33) and (37) (see the next section) agree with corresponding equations of [11].

5. Polar perturbations and material inhomogeneities

5.1. Metric and material perturbations

The part of the perturbing metric that contains the function \(\varphi\) represents a conformally flat perturbation. The evolution of \(\varphi\) is influenced by other metric polar perturbation functions—\(\sigma\) and \(\chi\). We show that the full set of equations is consistent and fully describes the propagation of polar modes and the creation of inhomogeneities in the universe filled by perfect fluid.

Assume \(p = K \rho^3\) as the equation of state. Then the speed of sound \(c_s\) is given by \(c_s^2 = \frac{\partial_p}{\rho}\) and \(\Delta \rho_0 = \Pi p_0/c_s^2\). Multiply both sides of equation (29) by \(2a^2 p_0/c_s^2\) and equation (28) by \(2a^2 p_0\) and subtract the obtained equations. After an easy calculation one arrives at

\[
\frac{2}{c_s^2} \frac{d^2 \varphi}{dr^2} - 2\varphi'' - 4\varphi' \frac{f'}{f} + 2 \left( \frac{l(l+1)}{c_s^2} \frac{f^2 c_s^2}{c_s^2} - 4 \right) \varphi
\]

\[
+ 2 \left( \frac{l(l+1)}{f^2} \chi ' \right) + 2 \left( \frac{f'}{f} \right) \left( \frac{c_s^2}{c_s^2} - 1 \right) \chi
\]

\[
+ 2H \left( \frac{1 + c_s^2}{c_s^2} + \varphi \frac{3c_s^2}{c_s^2} - 1 \right) = 0
\]

(37)

This is a hyperbolic differential equation, whose initial data—\(\varphi\) and \(\dot{\varphi}\)—are dictated by initial values of the mass density contrast \(\Delta\) and the velocity (\(v\)) perturbation. Indeed, one finds from equation (31)

\[
2\ddot{\varphi} = 2v(\eta, r) a^3 \left( \rho_0 + p_0 \right) + 2H \varphi + \sigma' - \chi;
\]

(38)

inserting that into equation (28) yields an ordinary differential equation of second order onto the unknown \(\varphi\). All other data are known from the background geometry (\(H\) and \(a\)) or from solving of the master equation (\(\chi, \chi', \sigma, \sigma'\)). The appropriate boundary conditions are \(\varphi'(r = 0) = 0\) and \(\varphi(r = \infty) = 0\). If in addition the initial mass density contrast \(\Delta\) is prescribed, then the initial values of \(\varphi\) can be determined uniquely. The initial ‘velocity’ \(\dot{\varphi}\) is then obtained from the initial velocity perturbation dictated by (38). These initial data for the equation (37) give rise to an evolving solution \(\varphi\): this in turn, determines in later times the mass density contrast \(\Delta\) and the pressure contrast \(\Pi\).

Deformations of the velocity components \(w(\eta, r)\) and \(v(\eta, r)\) are dictated by equations (30) and (31).

We would like to point out that the above reasoning proves consistency of equations. It suggests also that even if initially density and velocity perturbations do vanish, then they are expected to appear in later times. Next subsection proves a stronger statement.
5.2. Polar waves must influence cosmological matter

In the remainder of this section we show, that if a cosmological spacetime is nonvacuous, \( p_0 > 0 \) and \( \varrho_0 > 0 \), then propagating polar waves must disturb the material medium. We shall use the method of contradiction—assume that matter perturbations are absent, i.e. \( \Pi = \Delta = w = v \equiv 0 \) and derive from that \( \sigma = \chi = \varphi \equiv 0 \)—i.e., that polar modes are absent. This means that nonzero polar modes induce matter disturbances.

The propagation equations (28)–(33) read, assuming \( \Pi = \Delta = w = v \equiv 0 \):

\[
\left( \frac{l(l+1)}{f^2} + 6f'^2 - 4 + 2H^2 \right) \chi + 2 \left( \frac{l(l+1)}{f^2} + 3f'^2 - 3H^2 \right) \varphi - 8Hf' \sigma + 2\frac{f'}{f} (\chi' - 2\varphi') + 2H(\dot{\chi} + 3\dot{\varphi} - 2\sigma') - 2\varphi'' = 0 \tag{39}
\]

\[
\left( \frac{l(l+1)}{f^2} - 2f'^2 + 2H^2 - 4\dot{H} \right) \chi + 2 \left( \frac{1-f'^2}{f^2} + H^2 \right) \varphi + 2\frac{f'}{f} \chi' + 2H(\dot{\varphi} - \dot{\chi}) - 2\dot{\varphi} = 0 \tag{40}
\]

\[
\frac{l(l+1) + 4f'^2 - 4}{f^2} \sigma - 4Hf' \chi + 2\frac{f'}{f} \chi' + 2H(\chi' - \varphi') - 2\varphi' = 0 \tag{41}
\]

\[
2H\sigma + \sigma' - 2\dot{\varphi} - \ddot{\chi} = 0 \tag{42}
\]

\[
\chi' - \dot{\sigma} = 0 \tag{43}
\]

\[
\ddot{\chi} - \chi'' - 2H\ddot{\chi} + 2\frac{f'}{f} \chi' - 2H' \chi + \frac{l(l+1) - 2}{f^2} \chi = 0 \tag{44}
\]

It is easy to show

**Lemma 1.** If \( p_0 + \varrho_0 > 0 \) then \( \varphi(\eta, r) = -\chi(\eta, r) \).

Indeed, performing on equations (40), (42)–(44) the following operations (40) – (44) – \( \partial_r(43) \) – \( \partial_\eta(42) \), one arrives at

\[
a^2(p_0 + \varrho_0)(\varphi + \chi) = 0. \tag{45}
\]

The set of the foregoing equations reduces now to the following system

\[
\left( \frac{2 - l(l+1)}{f^2} + 8H^2 \right) \chi - 8Hf' \sigma + 6f' \chi' - 4H(\ddot{\chi} + \sigma') + 2\chi'' = 0 \tag{46}
\]

\[
\left( \frac{l(l+1) - 2}{f^2} - 4\dot{H} \right) \chi + 2\frac{f'}{f} \chi' - 4H\ddot{\chi} + 2\ddot{\chi} = 0 \tag{47}
\]
\[
\frac{l(l+1) + 4f'^2 - 4}{f^2} \sigma - 4Hf' \chi + 2f' \dot{\chi} - 4H\chi' + 2\chi'' = 0
\]

(48)

\[
\sigma' + \dot{\chi} - 2H\chi = 0
\]

(49)

\[
\chi' - \dot{\sigma} = 0
\]

(50)

The combination of operations (47) \(-2(\partial_r(49) + \partial_r(50))\) yields

\[
0 = \frac{l(l+1) - 2}{f^2} \chi + 2f' \frac{\chi'}{f} - 2\chi''.
\]

(51)

This equation can be written as

\[
0 = \frac{l(l+1) - 2}{2f^3} \chi - \frac{d}{dr} \frac{\chi'}{f}.
\]

(52)

We show that

**Lemma 2.** Equation (52) does not possess nonzero solutions of compact support.

**Proof.** Indeed, let there exist two closest points \(r_1\) and \(r_2\) \((r_1 < r_2)\) such that \(\chi(r_1) = \chi(r_2) = 0\), and integrate (52) over the interval \((r_1, r_2)\). We arrive at

\[
0 = \int_{r_1}^{r_2} dr \frac{l(l+1) - 2}{2f^3} \chi - \frac{\chi'(r_2)}{f(r_2)} + \frac{\chi'(r_1)}{f(r_1)}.
\]

(53)

Let \(\chi\) be strictly positive in the open interval \((r_1, r_2)\). Then necessarily \(\frac{\chi'(r_2)}{f(r_2)} \gg \frac{\chi'(r_1)}{f(r_1)}\); but that means that \(\chi\) is monotonically increasing, which contradicts the assumption that \(\chi(r_2)\) vanishes. In the same manner one considers the case with \(\chi\) being strictly negative in the open interval \((r_1, r_2)\). That ends the proof of the lemma 2. 

This allows us to conclude, that the vanishing of material cosmological perturbations \(\Pi = \Delta = w = \nu \equiv 0\) enforces the absence of polar gravitational waves, \(\sigma = \chi = \varphi \equiv 0\).

This in turn means that travelling gravitational polar pulses must leave inhomogeneous and nonisotropic tracks in the cosmological background.

### 6. Polar modes in the de Sitter universe

In the de Sitter universe matter is absent, \(\rho_0 = p_0 = 0\). Viaggiu has shown, using Laplace transforms, that polar perturbations can also be expressed in terms of four independent integrable differential equations [14]. We have six equations describing the propagation of gravitational waves, while there are only two free initial data, corresponding to the one degree of freedom carried by polar modes. We show in this section that these equations are consistent and that the whole dynamics is described by the master equation.

The evolution equation (14) describing the background geometry reads now

\[
\frac{1}{3} a^2 \Lambda = H^2 + \frac{1 - f'^2}{f^2} = \dot{H}.
\]

(54)

It is useful to notice that \(\frac{d}{dr} \frac{1 - f'^2}{f^2} = 0\), that is
\[ f'' + \frac{1 - (f')^2}{f} = 0. \]  

(55)

This identity can be derived from (54)—notice that \( H \) depends only on \( \eta \)—or directly from the definition of \( f \).

Equations (28)–(33) take following form:

\[
\left( \frac{l(l+1) + 6r'^2 - 4}{f^2} + 2H^2 \right) \chi 
+ 2 \left( \frac{l(l+1) + 3f'^2 - 3}{f^2} - 3H^2 \right) \varphi - 8H \frac{f'}{f} \sigma 
+ 2 \frac{f'}{f} (\chi' - 2\varphi') + 2H(\chi + 3\varphi - 2\sigma') - 2\varphi'' = 0 
\]

(56)

\[
\left( \frac{l(l+1) + 2f'^2 - 4}{f^2} - 2H^2 \right) \chi 
+ 2 \left( \frac{1 - f'^2}{f^2} + H^2 \right) \varphi 
+ 2 \frac{f'}{f} \chi' + 2H(\dot{\varphi} - \ddot{\chi}) - 2\ddot{\varphi} = 0 
\]

(57)

\[
4H \frac{f'}{f} \chi - \frac{l(l+1) + 4f'^2 - 4}{f^2} \sigma - 2 \frac{f'}{f} \dot{\chi} + 2H(\chi' - \varphi') + 2\dot{\varphi}' = 0 
\]

(58)

\[
2(\ddot{\varphi} - H \varphi) - \sigma' + \ddot{\chi} = 0 
\]

(59)

\[
\chi' - \dot{\sigma} = 0 
\]

(60)

\[
\ddot{\chi} - \chi'' - 2H \ddot{\chi} + 2 \frac{f'}{f} \chi' + \left( \frac{l(l+1) + 2f'^2 - 4}{f^2} - 2H^2 \right) \chi = 0 
\]

(61)

In the first step we shall express equations (56)–(58) as linear combinations of the remaining three equations (59)–(61). We assume that at some initial time \( \eta_0 \) the left hand side of equations (58) does vanish.

One can check by straightforward calculation that

\[
(57) = (61) + \partial_{t}(60) - \partial_{\eta}(59); 
\]

in the course of calculations we replaced \( H \), using equation (54).

Direct calculation shows that

\[
\partial_{\eta}(58) = - \frac{l(l+1) + 4f'^2 - 4}{f^2} (60) + \partial_{t}(57) + \frac{2f'}{f} (61) 
\]

(63)

Here we have to use the identity (55).

It is clear that if (59)–(61) are satisfied, then equation (57) also holds. Equation (58) is also satisfied modulo a time-independent function, say \( f_3(\eta) \); but now we use the assumption that initially the velocity perturbation vanishes; thus \( f_3(\eta) = 0 \) and (58) is also valid.
The last equation (56), satisfies following relation:

$$-a \partial_\eta \left( \frac{1}{a} \right) = H(61) - H_\eta(60) - 3H \partial_\eta(59)$$

$$-8H f'(60) + 2f'(58) - \frac{l(l+1)}{f^2}(59) + \partial_\eta(58)$$

(64)

We know, from the preceding analysis, that the right hand side of (64) vanishes. Therefore (56) holds true modulo a time-independent function $f_1(r)$; but $f_1(r) = 0$, since we assumed that the mass density and its perturbation are absent. That means that (56) is valid. In summary, this reasoning shows that the only relevant equations are (59)–(61).

In the second step one shows that metric functions $\varphi$ and $\sigma$ can be derived from equations (59) and (60). They become functionally dependent on solutions $\chi$ of the master equation—see explicit formulae in preceding sections—modulo a time-independent function; but the latter can be set to zero, with appropriate initial conditions. Thus the whole dynamics reduces to the single master equation, as in the analysis of Zerilli in the Schwarzschild spacetime [9].

7. The Huygens principle

We adopt the following version of the Huygens principle. Purely outgoing compact pulses of radiation, that at $\eta_0$ occupy a shell $r_{in} \leq r \leq r_{out}$, have to move within space-time regions bounded by a pair of outgoing null cones $\eta = r + \text{const}$. They should not leave any tails trailing behind the main pulse. Analogous property is obeyed by purely ingoing waves. It is well known that solutions of the massless scalar field equation $= 0$ satisfy the above Huygens principle in $1 + 3$ dimensional Minkowski spacetime, while massive scalar fields do not obey it. Radiation shocks—with initial data located strictly on a sphere—should move strictly along null cones. Various versions are discussed in the monograph of Hadamard [16].

During the radiation epoch and assuming that $\Lambda = 0$, the Huygens principle holds. Indeed, the form of the master equation describing the propagation of axial and polar waves

$$\ddot{Q} - \frac{\partial^2}{\partial r^2}Q = 0,$$

(65)

where $Q = Q(\eta, r)$ is exactly the same as that describing the propagation of electromagnetic fields in FLRW spacetimes. Therefore its general solution has the form (see, for instance, [17])

$$\phi(r, \eta) = f^l \delta_\eta \frac{1}{f} \delta_r \frac{1}{f} \cdots \delta_r \left( \frac{g + h}{f} \right)$$

(66)

where functions $h$ and $g$ depend on the combinations $r - \eta$ or $r + \eta$, respectively. The multi-

pole index $l$ exceeds 1 for the wave-like solution. This solution obviously satisfies the Huygens principle. Another argument for the validity of the Huygens principle employs the conformal flatness of the FLRW models and conformal invariance of the Maxwell equations in vacuum.

We should point that there exist other forms of explicit solutions of equation (65) [18].

In all remaining cases—when the dark energy ($\Lambda \neq 0$), and/or nonradiational matter are present—the Huygens principle is broken. That means that tail terms—delayed signals tracking the main wave pulse—should be present. This effect depends, however, on the ratio of the length-wave to the Hubble radius and it is weak when this ratio is small [7, 17]. Therefore it
should not be observed on subhorizon scales. This in turn implies that it cannot be observed in present or presently planned gravitational waves detectors.

8. Summary

It was not our intention to focus on the derivation of equations describing propagation of gravitational waves in cosmological spacetimes. It appeared, however, that there is no full consensus as to the form of these equations. We have got almost the same results as in [10] (after specialization to dust-like FLRW cosmologies)—in particular ‘master equations’ are the same and there is only one minor difference in one of the remaining equations that describes velocity perturbations. Take a note, however, that Clarkson et al. assume dust while we deal with more general polytropic fluids. Our derivation is completely independent of the analysis of Gundlach and Martin-Garcia [11], but our main equations agree with those 3 equations describing polar modes (including the master equation) that they give explicitly.

One of the results of this paper is the proof that polar gravitational modes lead inevitably—even in the linearized approximation—to matter inhomogeneities and anisotropies. These latter must emerge in later evolution, even if matter is distributed isotropically and homogeneously at the initial hypersurface. Polar gravitational waves perturb both density and non-azimuthal components of the velocity. We would like to point out that this conclusion is consistent with earlier studies [19–21] and also with the investigation of Clarkson et al [10]. We show also that axial gravitational waves can perturb—in the linear approximation—only azimuthal velocity of cosmological fluid, leaving intact its density. It is not clear to us whether these properties of axial and polar modes would have observational consequences, but in principle they would also, in subhorizon scales.

The Huygens principle is strictly obeyed by polar and axial modes during the radiation epoch and broken in other cosmological epochs. This breakdown effect is small when the ratio of the length-wave to the Hubble radius is small [7, 15, 17]. Therefore it is of no significance in local astronomical observations, but it may manifest on superhorizon scales. Primordial gravitational waves can be produced [22] during formation of putative primordial black holes [23]; whether they may leave detectable imprints in the cosmic microwave background radiation is an open question [24].

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