Modification of Moment-Based Tail Index Estimator: Sums versus Maxima

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Abstract In this paper, we continue the investigation of the SRCEN estimator of the extreme value index $\gamma$ (or the tail index $\alpha = 1/\gamma$) proposed in [12] for $\gamma > 1/2$. We propose a new estimator based on the local maximum. This, in fact, is a modification of the SRCEN estimator to the case $\gamma > 0$. We establish the consistency and asymptotic normality of the newly proposed estimator for i.i.d. data. Additionally, a short discussion on the comparison of the estimators is included.

Key words: asymptotic normality, extreme value index, mean squared error, tail index

1 Introduction and main results

Let $X_k$, $k \geq 1$ be non-negative independent, identically distributed (i.i.d.) random variables (r.v.s) with the distribution function (d.f.) $F$. Suppose that $F$ belongs to the domain of attraction of the Fréchet distribution

$$\Phi_\gamma(x) = \begin{cases} 0, & x \leq 0, \\ \exp\{-x^{-1/\gamma}\}, & x > 0, \end{cases} \Phi := \Phi_1,$$

which means that there exists normalizing constants $a_m > 0$ such that

$$\lim_{m \to \infty} \mathbb{P}\left(\frac{L_m}{a_m} \leq x\right) = \lim_{m \to \infty} F^m(a_m x) = \Phi_\gamma(x),$$

(1)
for all \( x > 0 \), where \( L_u,v = \max \{ X_u, \ldots, X_v \} \) for \( 1 \leq u \leq v \) and \( L_v = L_{1,v} \). The parameter \( \gamma > 0 \) is referred to as positive extreme-value index in the statistical literature.

Meerschaert and Scheffler [13] introduced the estimator for \( \gamma \geq 1/2 \), which is based on the growth rate of the logged sample variance of \( N \) observations \( X_1, \ldots, X_N \):

\[
\hat{\gamma}_N = \frac{1}{2 \ln(N)} \ln \left( N s_N^2 \right),
\]

where \( s_N^2 = \frac{1}{N-1} \sum_{i=1}^{N} (X_i - \bar{X}_N)^2, \bar{X}_N = (X_1 + \ldots + X_N)/N \) and \( \ln_{+}(x) = 0 \lor \ln x \).

McElroy and Politis [12] divided the observations \( X_1, \ldots, X_N \) into non-intersecting blocks \( \{ X_{(k-1)m^2+1}, \ldots, X_{km^2} \} \), \( 1 \leq k \leq \lceil N/m^2 \rceil \) of the width \( m^2 \), while each such block was divided into non-intersecting sub-blocks of the width \( m \). To estimate \( \gamma > 1/2 \) the so-called SRCEN estimator was proposed as the sample mean over all blocks:

\[
\hat{\gamma}_N^{(1)}(m) = \frac{1}{\lceil N/m^2 \rceil} \sum_{i=1}^{\lceil N/m^2 \rceil} \xi_i(m),
\]

where

\[
\xi_i(m) = \frac{\ln \left( \sum_{j=(i-1)m^2+1}^{im^2} X_j^2 \right)}{2 \ln(m)} - \frac{1}{m} \sum_{k=1}^{m} \ln \left( \frac{\sum_{j=(k-1)m^2+(k-1)m+1}^{km^2} X_j^2}{\sum_{j=(k-1)m^2+(k-1)m+1}^{km^2} X_j^2} \right),
\]

and \( \lfloor \cdot \rfloor \) denotes the integer part. In applications a simple heuristic rule for the choice of sub-block width \( m = \lfloor N^{1/3} \rfloor \), provided in [12], works quite well, see the Monte-Carlo simulation studies in [12], [17] and [18].

Using the inequality of arithmetic and geometric means we obtain that for sample \( X_1, \ldots, X_N \), \( \hat{\gamma}_N^{(1)}(m) \geq 1/2 \) holds with equality if and only if \( X_2^{(i-1)m^2+1} = \ldots = X_{im^2}^{(i-1)m^2+1}, 1 \leq i \leq \lceil N/m^2 \rceil \).

In this paper we provide an estimator similar to the SRCEN estimator but one that can be used for \( \gamma > 0 \), not only for \( \gamma > 1/2 \). Namely, we replace the sums in (2) by corresponding maxima and introduce the new estimator

\[
\hat{\gamma}_N^{(2)}(m) = \frac{1}{\lceil N/m^2 \rceil} \sum_{i=1}^{\lceil N/m^2 \rceil} \tilde{\xi}_i(m)
\]

where

\[
\tilde{\xi}_i(m) = \frac{\ln \left( L_{(i-1)m^2+1,im^2} \right)}{\ln(m)} - \frac{1}{m} \sum_{j=1}^{m} \ln \left( \frac{L_{(i-1)m^2+(j-1)m+1,im^2+jm}}{L_{(i-1)m^2+(j-1)m+1,im^2+jm}} \right).
\]

In fact, the estimator \( \hat{\gamma}_N^{(2)}(m) \) is based on the convergence \( E \ln (L_u)/\ln(m) \to \gamma \) as \( m \to \infty \), which implies
we obtain
\[
2E \left( \frac{\ln(L_{m^2})}{\ln(m^2)} \right) - E \left( \frac{\ln(L_m)}{\ln(m)} \right) \to \gamma, \quad m \to \infty.
\] (3)

Thus, the estimator \( \hat{\gamma}_N^{(2)}(m) \) is nothing else, but a moment-type estimator for the left hand side in (3).

Note that \( \hat{\gamma}_N^{(2)}(m) \) as well as \( \hat{\gamma}_N^{(1)}(m) \) are scale-free, i.e., they do not change when \( X_i \) is replaced by \( cX_i \) with \( c > 0 \).

Typically, the estimators, whose constructions are based on the grouping of the observations into the blocks, are well suited for recursive on-line calculations. In particular, if \( \hat{\gamma}_N^{(1)}(m) = \hat{\gamma}_N^{(1)}(m;X_1,\ldots,X_N) \) denotes the estimate of \( \gamma \) obtained from observations \( X_1,\ldots,X_N \) and we get the next group of updates \( X_{N+1},\ldots,X_{N+2m^2} \), then we obtain
\[
\hat{\gamma}_N^{(1)}(m;X_1,\ldots,X_{N+2m^2}) = \frac{1}{N+1} \sum_{i=1}^{N+1} \xi_i(m) = \frac{1}{N+1} \left( \hat{N} \hat{\gamma}_N^{(1)}(m) + \xi_{N+1}(m) \right),
\]
denoting \( \hat{N} = \lfloor N/m^2 \rfloor \). After getting \( L \) additional groups \( \{X_{N+(k-1)m^2+1},\ldots,X_{N+km^2}\} \), \( k = 1,\ldots,L \), we have
\[
\hat{\gamma}_N^{(1)}(m;X_1,\ldots,X_{N+Lm^2}) = \frac{1}{N+L} \sum_{i=1}^{N+L} \xi_i(m) = \frac{1}{N+L} \left( \hat{N} \hat{\gamma}_N^{(1)}(m) + \xi_{N+1}(m) + \cdots + \xi_{N+L}(m) \right).
\]
It is important that \( \hat{\gamma}_N^{(1)}(m;X_1,\ldots,X_{N+Lm^2}) \) is obtained using \( \hat{\gamma}_N^{(1)}(m) \) after \( O(1) \) calculations. The same is valid for \( \hat{\gamma}_N^{(2)}(m) \) substituting \( \xi_i(m) \) by \( \hat{\xi}_i(m) \). The discussion on on-line estimation of the parameter \( \gamma > 0 \) can be found in Section 1.2.3 of [11].

There are situations when data can be divided naturally into blocks but only the largest observations within blocks (the block-maxima) are available. Several such examples are mentioned in [15], see also [1], where battle deaths in major power wars between 1495 and 1975 were analyzed. Then the estimator \( \hat{\gamma}_N^{(2)}(m) \) can be applied while the estimators \( \hat{\gamma}_N \) and \( \hat{\gamma}_N^{(1)}(m) \) are not applicable.

We will formulate our assumptions in terms of a so-called quantile function \( V \) of the d.f. \( F \), which is defined as the left continuous generalized inverse:
\[
V(t) := \inf \left\{ x \geq 0 : -\frac{1}{\ln F(x)} \geq t \right\}.
\]
The domain of attraction condition (1) can be stated in the following way in terms of \( V \): regarding the d.f. \( F \), (1) holds if and only if for all \( x > 0 \),
\[
\lim_{t \to \infty} \frac{V(tx)}{V(t)} = x^\gamma,
\] (4)
i.e. the function $V$ varies regularly at infinity with the index $\gamma > 0$ (written $V \in RV_\gamma$), see, e.g., [3, p.34].

First our result states that $\hat{\gamma}^2_N(m)$ is a weakly consistent estimator for $\gamma > 0$. For the sake of completeness we include a corresponding result (as a direct consequence of Prop. 1 in [12]) for the SRCEN estimator $\hat{\gamma}^1_N(m)$.

**Theorem 1.** Let observations $X_1, \ldots, X_N$ be i.i.d. r.v.s with d.f. $F$.

(i) Suppose $F$ satisfies the first-order condition (4) with $\gamma > 1/2$. Suppose, in addition, that the probability density function $p(x)$ of $F$ exists and is bounded, and also that $p(x)/x$ is bounded in a neighborhood of zero. Then for the sequence $m = m(N)$ satisfying

$$m(N) \to \infty, \quad \frac{N \ln^2 m}{m^2} \to \infty, \quad N \to \infty, \quad (5)$$

it holds

$$\hat{\gamma}^1_N(m) \xrightarrow{P} \gamma, \quad (6)$$

where $\xrightarrow{P}$ denotes convergence in probability.

(ii) Suppose $F$ satisfies (4) with $\gamma > 0$. Suppose, in addition,

$$F(\delta) = 0 \quad (7)$$

for some $\delta > 0$. Then for the sequence $m = m(N)$ satisfying (5) it holds

$$\hat{\gamma}^2_N(m) \xrightarrow{P} \gamma. \quad (8)$$

As usual, in order to get asymptotic normality for estimators the so-called second-order regular variation condition in some form is assumed. We recall that the function $V$ is said to satisfy the second-order condition if for some measurable function $A(t)$ with the constant sign near infinity, which is not identically zero, and $A(t) \to 0$ as $t \to \infty$,

$$\lim_{t \to \infty} \frac{V(tx) - x^\gamma}{A(t)} = x^\gamma x^\rho - 1$$

holds for all $x > 0$ with $\rho < 0$, which is a second order parameter. The function $A(t)$ measures the rate of convergence of $V(tx)/V(t)$ towards $x^\gamma$ in (4), and $|A(t)| \in RV_\rho$, see [8].

In this paper, we assume a second order condition stronger than (9). Namely, we assume that we are in Hall’s class of models (see [9]), where

$$V(t) = Ct^\gamma \left(1 + \frac{\rho^{-1} A(t) (1 + o(1))}{t} \right), \quad t \to \infty \quad (10)$$

with $A(t) = \gamma \beta \rho$, where $C > 0$, $\beta \in \mathbb{R} \setminus \{0\}$ and $\rho < 0$. The relation (10) is equivalent to

$$F(x) = \exp \left\{ - \left( \frac{X}{C} \right)^{-1/\gamma} \left( 1 + \frac{\beta (x/C)^{\rho/\gamma}}{\rho} + o \left( (x^{\rho/\gamma}) \right) \right) \right\}, \quad x \to \infty. \quad (11)$$
Theorem 2. Let the observations $X_1, \ldots, X_N$ be i.i.d. r.v.s with d.f. $F$.

(i) Suppose $F$ satisfies the second-order condition (11) with $\gamma > 1/2$ and, in addition, that the probability density function $p(x)$ of $F$ exists and it is bounded, and also that $p(x)/x$ is bounded in a neighborhood of zero. Then for the sequence $m = m(N)$ satisfying $m \to \infty$ and

$$N^{1/2}m^{-2(1+\rho)/(1-\rho)} \ln(m) \to 0, \quad \text{if } -1 \lor \rho \neq 1 - 2\gamma,$$
$$N^{1/2}m^{-2\gamma} \ln^2(m) \to 0, \quad \text{if } -1 \lor \rho = 1 - 2\gamma,$$

holds, where $d \to$ stands for the convergence in distribution.

(ii) Suppose $F$ satisfies (7) and (11) with $\gamma > 0$. Then, for the sequence $m = m(N)$ satisfying (5) and

$$N^{1/2}A(m) \to \nu \in (-\infty, +\infty),$$

it follows

$$N^{1/2} \left( \frac{\hat{\gamma}^{(2)}(m) - \gamma}{6} \right) \to \mathcal{N} \left( 0, \frac{\nu \Gamma(1-\rho)}{\rho}, \frac{\gamma^2 \pi^2}{6} \right), \quad N \to \infty.$$ 

The rest of the paper is organized as follows. In the next section we investigate the asymptotic mean squared error (AMSE) of the introduced estimator, and compare this estimator with several classical estimators, using the same methodology as in [4]. The last section contains the proofs of the results.

2 Comparison

The AMSE of the estimator $\hat{\gamma}^{(2)}(m)$ is given by

$$\text{AMSE} \left( \hat{\gamma}^{(2)}_N(m) \right) := \frac{1}{\ln^2(m)} \left( \frac{\Gamma^2(1-\rho)A^2(m)}{\rho^2} + \frac{\gamma^2 \pi^2 m^2}{6N} \right). \quad (15)$$

Regular variation theory, provided in [5] (see also [4]), allows us to perform the minimization of the sum in the curly brackets of (15). Namely, under the choice

$$\bar{m}(N) = \left( \frac{6\Gamma^2(1-\rho)\beta^2}{-\rho \pi^2} \right)^{1/(2(1-\rho))} N^{1/(2(1-\rho))}(1 + o(1)), \quad N \to \infty,$$

we have
AMSE \left( \hat{\gamma}_N^{(2)}(\bar{m}) \right) \sim \Gamma^2(-\rho)\beta^2 \left( \frac{6\beta^22\pi^2(1-\rho)}{\pi^2(-\rho)} \right)^{1/(1-\rho)} \frac{N^{\rho/(1-\rho)}}{\ln^2(N)}, \quad N \to \infty.

Probably, the Hill’s estimator

\hat{\gamma}_N^{(H)}(k) = \frac{1}{k} \sum_{j=0}^{k-1} \ln \left( \frac{X_{N-j,N}}{X_{N-k,N}} \right),

is the most popular, [10]. Here, \(1 \leq k \leq N\) is a tail sample fraction, while \(X_{1,N} \leq X_{2,N} \leq \cdots \leq X_{N,N}\) are order statistics from a sample \(X_1, \ldots, X_N\). Let us denote \(r = -1 \vee \rho\) and

\(v = \begin{cases} \beta, & -1 < \rho < 0, \\ \beta + (1/2), & \rho = -1, \\ 1/2, & \rho < -1. \end{cases} \)

From [4] it follows that the minimal AMSE of the Hill’s estimator under assumption (11) satisfies the relation

\[
\text{AMSE} \left( \hat{\gamma}_N^{(H)}(\bar{k}) \right) \sim \frac{1-2r}{-2r} \left( -\frac{2rv^2\rho^2-4r}{(1-r)^2} \right)^{1/(1-2r)} N^{2r/(1-2r)}, \quad N \to \infty,
\]

where

\[
\bar{k}(N) = \left( \frac{(1-r)^2}{-2rv^2} \right)^{1/(1-2r)} N^{-2r/(1-2r)} \left( 1 + o(1) \right), \quad N \to \infty.
\]

Now we can compare the estimators \(\hat{\gamma}_N^{(2)}(\bar{m})\) and \(\hat{\gamma}_N^{(H)}(\bar{k})\). Denote the relative minimal AMSE in the same way as in [4]:

\[
\text{RMAMSE}(\gamma, \beta, \rho) = \lim_{N \to \infty} \frac{\text{AMSE} \left( \hat{\gamma}_N^{(H)}(\bar{k}) \right)}{\text{AMSE} \left( \hat{\gamma}_N^{(2)}(\bar{m}) \right)}.
\]

Following [4] we may conclude that \(\hat{\gamma}_N^{(H)}(\bar{k})\) dominates \(\hat{\gamma}_N^{(2)}(\bar{m})\) at the point \((\gamma, \beta, \rho)\) if \(\text{RMAMSE}(\gamma, \beta, \rho) < 1\) holds. Note that \(\text{RMAMSE}(\gamma, \beta, \rho) = 0\) holds for \(-2 < \rho < 0\), i.e. \(\hat{\gamma}_N^{(H)}(\bar{k})\) dominates \(\hat{\gamma}_N^{(2)}(\bar{m})\), while for \(\rho \leq -2\) we have \(\text{RMAMSE}(\gamma, \beta, \rho) = \infty\) and thus, \(\hat{\gamma}_N^{(2)}(\bar{m})\) outperforms \(\hat{\gamma}_N^{(H)}(\bar{k})\) in this region of the parameter \(\rho\). It is worth to note that the same conclusion holds if we replace Hill’s estimator by another estimator investigated in [4].

Unfortunately, it is impossible to compare the performance of \(\hat{\gamma}_N^{(1)}(m)\) and other estimators taking the AMSE as a measure. By taking \(v = 0\) in (14) one can compare the estimators \(\hat{\gamma}_N^{(1)}(m)\) and \(\hat{\gamma}_N^{(2)}(m)\) under the same block width \(m^2\). By comparing variances in the limit laws (12) and (14) we conclude that \(\hat{\gamma}_N^{(1)}(m)\) outperforms \(\hat{\gamma}_N^{(2)}(m)\) for \(\gamma > 1/2\).
3 Proofs

Let us firstly provide preliminary results that are useful in our proofs.

**Lemma 1.** Let $X_1, \ldots, X_N$ be i.i.d. r.v.s with d.f. $F$. Suppose $F$ satisfies (4) with $\gamma > 0$ and (7). Then

\[
\lim_{m \to \infty} \mathbb{E} \ln \left( \frac{L_m}{V(m)} \right) = \chi \gamma, \quad (16)
\]

\[
\lim_{m \to \infty} \mathbb{E} \ln^2 \left( \frac{L_m}{V(m)} \right) = \gamma^2 \left( \chi^2 + \frac{\pi^2}{6} \right), \quad (17)
\]

\[
\lim_{m \to \infty} \mathbb{E} \ln^4 \left( \frac{L_m}{V(m)} \right) = \gamma^4 \left( \chi^4 + \chi^2 \pi^2 + \frac{3\pi^4}{20} + 8\chi \zeta(3) \right), \quad (18)
\]

\[
\lim_{m \to \infty} \mathbb{E} \left( \ln \left( \frac{L_m}{V(m)} \right) \ln \left( \frac{L_m}{V(m)} \right) \right) = \chi^2 \gamma^2, \quad (19)
\]

holds, where $\chi \approx 0.5772$ is the Euler-Mascheroni constant defined by $\chi = -\int_0^\infty \ln(t) \exp(-t) \, dt$, while $\zeta(t)$ denotes the Riemann zeta function, $\zeta(3) \approx 1.202$.

**Proof of Lemma 1.** We shall prove (16). Let $Y$ be a r.v. with d.f. $\Phi$. It is easy to check that it holds

\[
\ln \left( \frac{L_m}{V(m)} \right) \overset{d}{=} \ln \left( \frac{V(mY)}{V(m)} \right).
\]

By Theorem B.1.9 in [3], the assumption $V \in RV_{\gamma}, \gamma > 0$ implies that for arbitrary $\varepsilon_1 > 0, \varepsilon_2 > 0$ there exists $m_0 = m_0(\varepsilon_1, \varepsilon_2)$ such that for $m \geq m_0, my \geq m_0$,

\[
(1 - \varepsilon_1) \gamma \min \{y^{\varepsilon_2}, y^{-\varepsilon_2}\} < \frac{V(my)}{V(m)} < (1 + \varepsilon_1) \gamma \max \{y^{\varepsilon_2}, y^{-\varepsilon_2}\}
\]

holds. Whence we get that under restriction $0 < \varepsilon_1 < 1$ it follows

\[
\ln(1 - \varepsilon_1) + (\gamma - u(y)) \ln(y) < \ln \left( \frac{V(my)}{V(m)} \right) < \ln(1 + \varepsilon_1) + (\gamma + u(y)) \ln(y), \quad (20)
\]

where $u(y) = -\varepsilon_2 I\{y < 1\} + \varepsilon_2 I\{y \geq 1\}$ and $I\{\cdot\}$ denotes the indicator function.

We write for $m > m_0$,

\[
\mathbb{E} \left( \ln \left( \frac{V(mY)}{V(m)} \right) \right) = J_{1,m} + J_{2,m},
\]

where

\[
J_{1,m} = \int_{0}^{m/m_0} \ln \left( \frac{V(my)}{V(m)} \right) \, d\Phi(y), \quad J_{2,m} = \int_{m/m_0}^{\infty} \ln \left( \frac{V(my)}{V(m)} \right) \, d\Phi(y).
\]

The statement (16) follows from
\[ \lim_{m \to \infty} J_{1,m} = 0, \quad \lim_{m \to \infty} J_{2,m} = \chi \gamma. \] (21) (22)

Substituting \( my = t \) we get

\[ |J_{1,m}| \leq \int_0^{m_0} \left| \int_0^t \frac{V(t)}{V(m)} \right| \, d\Phi(t/m) = \int_0^{m_0} |\ln V(t)| \, d\Phi(t/m) + \Phi(m_0/m) |\ln V(m)|. \]

By using \( d\Phi(t/m) = m\Phi(t/(m-1)) \, d\Phi(t) \) we obtain

\[ |J_{1,m}| \leq m\Phi(m_0/(m-1)) \int_0^{m_0} |\ln V(t)| \, d\Phi(t) + \Phi(m_0/m) |\ln V(m)|. \]

Assumption (7) ensures \( V(0) \geq \delta \), which implies \( \int_0^{m_0} |\ln V(t)| \, d\Phi(t) < \infty \). Since the sequence \( V(n) \) is of a polynomial growth and \( \Phi(m_0/m) = \exp\{-m/m_0\} \) tends to zero exponentially fast, then relation (21) follows.

To prove (22) we use inequality (20). Then we obtain

\[ |J_{2,m} - \chi \gamma| \leq \max \{-\ln(1-\varepsilon_1), \ln(1+\varepsilon_1)\} + \varepsilon_2 E[|\ln(Y)|] + \gamma \int_0^{m_0/m} |\ln(y)| \, d\Phi(y). \]

One can check that \( E[|\ln(Y)|] = \chi - 2E(1) \), where \( E(x), x \in \mathbb{R} \setminus \{0\} \) denotes the exponential integral function. \( E(-1) \approx -0.219384 \).

Since \( \varepsilon_1 > 0 \) and \( \varepsilon_2 > 0 \) may be taken arbitrary small, the proof of relation (22) will be finished if we show that \( \int_0^{m_0/m} |\ln(y)| \, d\Phi(y) \to 0, m \to \infty \). Substituting \( t = my \) we get

\[ \int_0^{m_0/m} |\ln(y)| \, d\Phi(y) = \int_0^{m_0} |\ln(t/m)| \, d\Phi(t/m) = m \int_0^{m_0} |\ln(t/m)| \, d\Phi(t/(m-1)) \, d\Phi(t) \leq m\Phi(m_0/(m-1)) (\ln(m) + E[|\ln(Y)|]) \to 0, \]

as \( m \to \infty \). This completes the proof of (22), and also of relation (16).

Proofs of relations (17) and (18) are similar and thus are skipped. It remains to prove (19). We note that \( L_m \) and \( L_{m+1,m^2} \) are independent r.v.s and \( L_{m^2} = L_m \lor L_{m+1,m^2} \). Let \( Y_1 \) and \( Y_2 \) be independent r.v.s with d.f. \( \Phi \). Then it holds

\[ \ln \left( \frac{L_{m^2}}{V(m^2)} \right) \ln \left( \frac{L_m}{V(m)} \right) \overset{d}{=} \ln \left( \frac{V(mY_1) \lor V(m(m-1)Y_2)}{V(m^2)} \right) \ln \left( \frac{V(mY_1)}{V(m)} \right), \]

and consequently,
Let us recall that \( V(t), t \geq 0 \) is a non-decreasing function, see, e.g., Prop. 2.3 in [6]. By using this property we obtain

\[
E \left( \ln \left( \frac{L_m}{V(m^2)} \right) \ln \left( \frac{L_m}{V(m)} \right) \right) = E \left( \ln \left( \frac{V(mY_1) \lor V(m(m-1)Y_2)}{V(m^2)} \right) \ln \left( \frac{V(mY_1)}{V(m)} \right) \right).
\]

We apply the Hölder’s inequality to get

\[
J_3,m = E \left( \ln \left( \frac{V(mY_1)}{V(m)} \right) \ln \left( \frac{V(mY_1)}{V(m)} \right) \right),
\]

\[
J_4,m = E \left( \ln \left( \frac{V(m(m-1)Y_2)}{V(m^2)} \right) \right),
\]

\[
J_5,m = E \left( \ln \left( \frac{V(m(m-1)Y_2)}{V(m^2)} \right) \right).
\]

Let us rewrite quantity \( J_4,m \) as follows:

\[
J_4,m = \left\{ \ln \left( \frac{V(m(m-1))}{V(m^2)} \right) + E \ln \left( \frac{L_m}{V(m(m-1))} \right) \right\} E \ln \left( \frac{L_m}{V(m)} \right).
\]

For any \( \varepsilon > 0 \) there exists natural \( \tilde{m}_0 \) such that \( 1/m < \varepsilon \) for \( m \geq \tilde{m}_0 \). Then \( V(m^2(1-\varepsilon))/V(m^2) \leq V(m^2(1-1/m))/V(m^2) \leq 1 \). By (4) we get \( V(m^2(1-\varepsilon))/V(m^2) \to (1-\varepsilon)^2, m \to \infty \). Since \( \varepsilon > 0 \) can be taken arbitrary small, the relation \( V(m(m-1))/V(m^2) \to 1, m \to \infty \) holds. By using the last relation and (16) we deduce that \( J_4,m \to \chi^2 \gamma^2 \) holds as \( m \to \infty \).

Next, we have

\[
J_{3,m} = E \left( \ln^2 \left( \frac{V(mY_1)}{V(m)} \right) I\{Y_1 > (m-1)Y_2\} \right)
\]

\[
+ E \ln \left( \frac{V(m)}{V(m^2)} \right) E \left( \ln \left( \frac{V(mY_1)}{V(m)} \right) I\{Y_1 > (m-1)Y_2\} \right).
\]

We apply the Hölder’s inequality to get

\[
|J_{3,m}| \leq \left\{ E \ln^4 \left( \frac{L_m}{V(m)} \right) \right\}^{1/2} \{P(Y_1 > (m-1)Y_2)\}^{1/2}
\]

\[
+ \left| E \ln \left( \frac{V(m)}{V(m^2)} \right) \right| \left\{ E \ln^2 \left( \frac{L_m}{V(m)} \right) \right\}^{1/2} \{P(Y_1 > (m-1)Y_2)\}^{1/2}.
\]

We find that \( P(Y_1 > (m-1)Y_2) = 1/m \) holds. Let us recall the well-known property of regularly varying functions: if \( V \in RV_\gamma \), then
where, e.g., Prop. B.1.9 in [3]. By using (23) we obtain \( \ln V(m^2) / \ln(m) \sim \gamma \ln(m) \), \( m \to \infty \). Thus, keeping in mind (17) and (18) we obtain \( |J_{3,m}| = O(m^{-1/2} \ln(m)) \), \( m \to \infty \). By a similar argument we obtain \( |J_{5,m}| = O(m^{-1/2}) \), \( m \to \infty \). This finishes the proof of (19) and Lemma 1.

**Proof of Theorem 1.** First we prove (8). Let us rewrite

\[
\hat{\gamma}_N^{(2)}(m) = \gamma + \left\{ \frac{\ln V(m^2) - \ln V(m)}{\ln(m)} \right\} + S_N(m),
\]

where

\[
\frac{\ln V(m^2) - \ln V(m)}{\ln(m)} = \frac{1}{\ln(m)} \left( \text{E} \ln \left( \frac{L_{m^2}}{V(m)} \right) - \text{E} \ln \left( \frac{L_m}{V(m)} \right) \right)
\]

and

\[
S_N(m) = \frac{1}{\ln(m)} \sum_{i=1}^{[N/m]} \left\{ \ln \left( \frac{L_{(i-1)m^2 + 1,i,m^2}}{V(m^2)} \right) - \text{E} \ln \left( \frac{L_{m^2}}{V(m^2)} \right) \right\}
\]

By combining (16) and (23) we deduce that \( \hat{\gamma}_N^{(2)}(m) - \gamma \to 0 \), \( m \to \infty \). Thus, it is enough to prove that \( S_N(m) \to 0 \) as \( N \to \infty \). By Chebyshev’s inequality, for any \( \varepsilon > 0 \) it holds \( P(|S_N(m)| > \varepsilon) \leq \varepsilon^{-2} \text{E}(S_N(m))^2 \). We have

\[
\text{E} \left( S_N(m) \right)^2 = \frac{1}{[N/m]^2 \ln(m)} \left\{ \text{Var} \left( \ln \left( \frac{L_{m^2}}{V(m^2)} \right) \right) - 2 \text{Cov} \left( \ln \left( \frac{L_{m^2}}{V(m^2)} \right), \ln \left( \frac{L_m}{V(m)} \right) \right) \right\}.
\]

Use (16)-(17) and (19) to deduce that the sum in the curly brackets has a finite limit as \( m \to \infty \). Thus, assumption (5) ensures \( \text{E} \left( S_N(m) \right)^2 \to 0 \), \( m \to \infty \). This finishes the proof of (8).

Consider now (6), where the restriction \( \gamma > 1/2 \) holds. Assumption (4) is equivalent to \( 1 - F \in RV_{-1/\gamma} \). By the Representation Theorem (see, Thm. B.1.6. in [3]), there exists a function \( \ell \in RV_0 \), such that

\[
1 - F(x^{1/2}) = x^{-1/(2\gamma)} \ell \left( x^{1/2} \right), \quad x \to \infty.
\]
Following the Mijnheer Theorem (see, Thm. 1.8.1 in [16]), we determine the norming function \( a(m) \in RV_{2\gamma} \) from

\[
\lim_{m \to \infty} \frac{m \ell \left( a^{1/2}(m) \right)}{(a(m))^{1/(2\gamma)}} = d(\gamma), \quad d(\gamma) = \Gamma(1 - 1/(2\gamma)) \cos(\pi/(4\gamma)). \tag{28}
\]

Put \( Q(m) = (X_1^2 + \ldots + X_m^2)/a_m \). Then \( Q(m) \overset{d}{\to} Z \), as \( m \to \infty \), where \( Z \) is totally skewed to the right \( 1/(2\gamma) \)-stable r.v. with characteristic function

\[
E \exp(i\theta Z) = \exp \left\{ -|\theta|^{1/(2\gamma)} \left( 1 - i \, \text{sgn}(\theta) \tan \left( \frac{\pi}{4\gamma} \right) \right) \right\}. \tag{29}
\]

Similarly to (24) we use the decomposition

\[
\tilde{\gamma}^{(1)}_{\hat{N}}(m) = \gamma + \left\{ E \gamma^{(1)}_{\hat{N}}(m) - \gamma \right\} + \tilde{S}_N(m),
\]

where

\[
\tilde{S}_N(m) = \frac{1}{2[N/m^2] \ln(m)} \sum_{i=1}^{[N/m^2]} \left\{ \ln \left( \frac{\sum_{j=(i-1)m^2+1}^{im^2} X_j^2}{a(m^2)} \right) - E \ln Q(m^2) \right\} - \frac{1}{m} \sum_{j=1}^{m} \left\{ \ln \left( \frac{\sum_{j=(i-1)m^2+(i-1)m^2+1}^{(i+1)m^2-1} X_j^2}{a(m)} \right) - E \ln Q(m) \right\}.
\]

The bias of the estimator \( \tilde{\gamma}^{(1)}_{\hat{N}}(m) \) is given by \( E \gamma^{(1)}_{\hat{N}}(m) - \gamma = \Delta(m^2) - (1/2)\Delta(m) \), where

\[
\Delta(m) = \frac{\ln a(m)}{\ln m} - 2\gamma + E \ln Q(m) - E \ln Z.
\]

In Prop. 1-2 of [12] it is proved

\[
E \ln Q(m) \to E \ln Z, \quad E \ln^2 Q(m) \to E \ln^2 Z, \quad \text{Cov} \left( \ln Q(m^2), \ln Q(m) \right) \to 0, \quad m \to \infty. \tag{30}
\]

It is worth to note that the moments \( E \ln Z \) and \( E \ln^2 Z \) can be found explicitly. Indeed, there is a direct connection between moments of order \( r < 1/(2\gamma) \) and log-moments of order \( k \in \mathbb{N} \):

\[
E \ln^k Z = \left. \frac{d^k}{dr^k} E Z^r \right|_{r=0}, \tag{32}
\]

see [19]. Regarding the moments \( EZ^r \), the following relation is proved in Section 8.3 of [14]:

\[
EZ^r = \frac{\Gamma(1 - 2\gamma r)}{\Gamma(1 - r)} \left( 1 + \tan^2 \left( \frac{\pi}{4\gamma} \right) \right)^{\gamma r}, \quad -1 < r < 1/(2\gamma). \tag{33}
\]
By using (32) and (33) we obtain

\[
E \ln Z = -\chi + 2\chi \gamma + \gamma \ln \left( \tan^2 \left( \frac{\pi}{4\gamma} \right) + 1 \right),
\]

(34)

\[
E \ln^2 Z = \chi^2 - \frac{\pi^2}{6} + 4\chi^2 \gamma^2 - 4\chi^2 \gamma + \frac{2\pi^2 \gamma^2}{3} + \gamma^2 \log^2 \left( \tan^2 \left( \frac{\pi}{4\gamma} \right) + 1 \right)
+ 4\chi \gamma^2 \log \left( \tan^2 \left( \frac{\pi}{4\gamma} \right) + 1 \right) - 2\chi \gamma \log \left( \tan^2 \left( \frac{\pi}{4\gamma} \right) + 1 \right).
\]

(35)

We combine (23) and the first relation in (30) to deduce that

\[\Delta (m) \to 0, \quad m \to \infty,\]

which implies \(E \hat{\gamma} \left( \mathcal{N}(m) \right) \to 0, \quad m \to \infty.\) Thus, relation (6) will be proved if we show that under assumptions (5), \(E \left( \mathcal{S}_N(m) \right)^2 \to 0.\) The last relation can be verified by using (30) and (31), and

\[
E \left( \mathcal{S}_N(m) \right)^2 = \frac{\text{Var} \left( \ln Q(m) \right)}{4N/m^2 \ln^2(m)}.
\]

(36)

This completes the proof of Theorem 1.

**Proof of Theorem 2.** In view of decomposition (24), the assertion (14) follows from

\[
E \left( \mathcal{S}_N(m) \right)^2 \sim \frac{\pi^2 \gamma^2 m^2}{6N \ln^2(m)},
\]

(37)

\[
\left\{ \frac{E \left( \mathcal{S}_N(m) \right)^2} {m / N^{1/2} \ln(m)} \right\}^{-1/2} \mathcal{S}_N(m) \xrightarrow{d} \mathcal{N}(0,1),
\]

(38)

\[
\frac{N^{1/2} \ln(m)}{m} \left( E \left( \mathcal{S}_N(m) \right)^2 \right)^{1/2} \mathcal{S}_N(m) - \gamma \xrightarrow{d} \nu \Gamma(1-\rho) \rho, \quad N \to \infty,
\]

(39)

where \(\nu\) is the same as in (13).

Relation (37) follows from (26) by applying (16)-(17) and (19). To prove (38), by using (16)-(19) we check the 4-th order Lyapunov condition for i.i.d. random variables forming a triangular array. We skip standard details.

By using (10) we obtain

\[
\frac{\ln V(m)}{\ln(m)} - \gamma = \frac{1}{\ln(m)} \left\{ \ln(C) + \frac{A(m)}{\rho} (1 + o(1)) \right\}, \quad m \to \infty.
\]

Following the proof of Lemma 2 in [18] one can obtain

\[
E \ln \left( \frac{L_m}{V(m)} \right) - \chi \gamma = \frac{\Gamma(1-\rho) - 1}{\rho} A(m) (1 + o(1)), \quad m \to \infty.
\]

We combine the last two relations, assumption (13) and decomposition (25) to verify (39).
Let us discuss the proof of (12) now. Relations (30), (31), (34)-(36) imply $E(\hat{S}_N(m))^2 \sim m^2 N^{-1} \ln^{-2}(m) (\gamma^2 - (1/4)) \pi^2 / 6, N \to \infty$. In view of the last relation it is enough to prove that

$$\left\{ \text{Var}(\hat{S}_N(m)) \right\}^{-1/2} \hat{S}_N(m) \xrightarrow{d} N(0,1),$$

(40)

$$E^{\hat{\gamma}(1)}_N(m) - \gamma = \begin{cases} O(m^{-1/\rho + (1 - 2\gamma)}), & -1 \lor \rho \neq 1 - 2\gamma, \\ O(m^{1 - 2\gamma} \ln(m)), & -1 \lor \rho = 1 - 2\gamma. \end{cases}$$

(41)

We skip a standard proof of (40) and focus on the investigation of the bias $E^{\hat{\gamma}(1)}_N(m) - \gamma$. Firstly, we prove that

$$\frac{\ln a(m^2) - \ln a(m)}{2 \ln(m)} - \gamma = O\left(\frac{m^{-1/\rho}}{\ln(m)}\right), \quad m \to \infty.$$  

(42)

The relation (11) can be written in the form $1 - F(x) = x^{-1/\gamma} \ell(x), x \to \infty$, where function $\ell \in RV_0$ has the form

$$\ell(x) = C^{1/\gamma} \left(1 + C(\beta, \rho)(x/C)^{(-1/\rho)/\gamma} + o(x^{(-1/\rho)/\gamma})\right), \quad x \to \infty,$$

(43)

where

$$C(\beta, \rho) = \begin{cases} \beta / \rho, & -1 < \rho < 0, \\ -(2\beta - 1)/\rho, & \rho = -1, \beta \neq 1/2, \\ -1/2, & \rho < -1. \end{cases}$$

Now, by using (28), one can find that under assumption (11) the norming function satisfies the asymptotic relation

$$a(m) = \left(C^{1/\gamma} / d(\gamma)\right)^{2\gamma} m^{2\gamma} \left(1 + 2\gamma C(\beta, \rho)d(-1/\rho)\gamma m^{-1/\rho} + o(m^{-1/\rho})\right)$$

as $m \to \infty$, while the last relation implies (42).

We claim that

$$\frac{E \ln Q(m) - E \ln Z}{\ln m} = \begin{cases} O(m^{-1/\rho + (1 - 2\gamma)}), & -1 \lor \rho \neq 1 - 2\gamma, \\ O(m^{1 - 2\gamma} \ln(m)), & -1 \lor \rho = 1 - 2\gamma \end{cases}$$

(44)

as $m \to \infty$.

Then terms $\ln^{-1}(m^2) \{E \ln Q(m^2) - E \ln Z\}$ and $(2 \ln(m))^{-1} \{\ln a(m^2) - \ln a(m)\}$ are negligible with respect to $\ln^{-1}(m) \{E \ln Q(m) - E \ln Z\}$ and thus, the relation (41) follows.

To verify (44) we use the similar decomposition $E \ln Q(m) - E \ln Z = R_{1,m} - R_{2,m} - R_{3,m}$ as in the proof of Prop. 3 in [12], where

$$R_{1,m} = \int_0^\infty \{P(\ln Q(m) > x) - P(\ln Z > x)\} \, dx,$$
\[ \begin{aligned}
R_{2,m} &= \int_{-\ln m}^{0} \{ P(\ln Q(m) < x) - P(\ln Z < x) \} \, dx, \\
R_{3,m} &= \int_{-\infty}^{-\ln m} \{ P(\ln Q(m) < x) - P(\ln Z < x) \} \, dx.
\end{aligned} \]

By using substitution \( t = \exp\{x\} \) we obtain

\[ R_{1,m} = \int_{1}^{\infty} t^{-1} \{ P(\{Q(m) > t\}) - P(\{Z > t\}) \} \, dt. \]

Similarly we get \( R_{2,m} = \int_{1/m}^{1} t^{-1} \{ P(\{Q(m) < t\}) - P(\{Z < t\}) \} \, dt \). From Corollary 2 in [2] it follows

\[ \sup_{t \geq 0} f_{t}(t) \{ P(\{Q(m) > t\}) - P(\{Z > t\}) \} = O(\lambda \{m^{2\gamma} + m^{-2\gamma}\}), \quad m \to \infty, \]

where \( f_{t}(t) = 1 + t^{2\gamma}\ln^{-2}(e + t) \) and \( \lambda(R) = \lambda_{1}(R) + R^{-1+1/(2\gamma)}\lambda_{2}(R), \quad R > 0, \)

where \( \lambda_{1}(R) = \sup_{u \geq R} u^{1/(2\gamma)} |P(X_{1}^{2} > u) - P(Z > u)| \),

\[ \lambda_{2}(R) = \int_{R}^{\infty} |P(X_{1}^{2} > u) - P(Z > u)| \, du. \]

It is well-known that \( P(Z > x) = C_{1}x^{-1/(2\gamma)} \left( 1 + C_{2}x^{-1/(2\gamma)} + o \left( x^{-1/(2\gamma)} \right) \right), \quad x \to \infty \)
holds, where \( C_{k} = C_{k}(\gamma) \) are some constants. The asymptotic of \( P(X_{1}^{2} > u) \) is given in (27), where a function \( \ell \) slowly varying at infinity is given in (43). Recall that \( \hat{\gamma}^{(1)}_{N}(m) \) is a scale-free estimator. Thus, without loss of generality, we may assume that the scale parameter \( C \) in (43) satisfies \( C^{1/(2\gamma)} = C_{1} \). Then we have

\[ P(X_{1}^{2} > x) - P(Z > x) = Dx^{-1/(2\gamma) + o \left( x^{-1/(2\gamma)} \right)}, \quad x \to \infty, \quad (45) \]

where \( D \neq 0 \) is some constant. By applying (45) we obtain immediately \( \lambda_{1} \{m^{2\gamma}\} = O \{m^{-1/p}\}, \quad m \to \infty. \) If \( -2 \vee (\rho - 1) > -2\gamma \), by ex. 1.2 in [7], a relation \( f(x) \sim x^{r} \), \( x \to \infty \) implies

\[ \int_{0}^{x} f(t) \, dt \sim \begin{cases} x^{r+1}/(r+1), & r > -1, \\
\ln(x), & r = -1, \\
x \to \infty \end{cases} \quad (46) \]

and thus we obtain \( m^{1-2\gamma}\lambda_{2}(m^{2\gamma}) = O \{m^{-1/p}\}, \quad m \to \infty. \) In the case \( -2 \vee (\rho - 1) = -2\gamma \), by applying (46) one more time we get \( m^{1-2\gamma}\lambda_{2}(m^{2\gamma}) = O \{m^{1-2\gamma}\ln(m)\}, \quad m \to \infty. \) As for the case \( -2 \vee (\rho - 1) < -2\gamma \), we have \( m^{1-2\gamma}\lambda_{2}(m^{2\gamma}) = O \{m^{1-2\gamma}\}, \quad m \to \infty. \) By putting the obtained results together we get

\[ \sup_{t \geq 0} f_{t}(t) \{ P(\{Q(m) > t\}) - P(\{Z > t\}) \} = \begin{cases} O \{m^{-1/p \vee (1-2\gamma)}\}, & -1 \vee \rho \neq 1 - 2\gamma, \\
O \{m^{1-2\gamma}\ln(m)\}, & -1 \vee \rho = 1 - 2\gamma \end{cases} \]
as \( m \to \infty \).

Applying the last asymptotic relation we obtain immediately

\[
|R_{2,m}| = \begin{cases} 
O \left( m^{-1/\rho \vee (1-2\gamma)} \ln(m) \right), & -1 \vee \rho \neq 1 - 2\gamma, \\
O \left( m^{1-2\gamma} \ln^2(m) \right), & -1 \vee \rho = 1 - 2\gamma
\end{cases}
\]

and \(|R_{1,m}| = o(|R_{2,m}|)\) as \( m \to \infty \). Since the relation \(|R_{3,m}| = O(m^{-1}) = o(|R_{2,m}|)\), \( m \to \infty \) holds (see proof of Prop. 3 in [12]), the statement of Theorem 2 follows.

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