Segal Operations in the Algebraic $K$-theory of Topological Spaces

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Abstract

We extend earlier work of Waldhausen which defines operations on the algebraic $K$-theory of the one-point space. For a connected simplicial abelian group $X$ and symmetric groups $\Sigma_n$, we define operations $\theta^n: A(X) \to A(X \times B\Sigma_n)$ in the algebraic $K$-theory of spaces. We show that our operations can be given the structure of $E_\infty$-maps. Let $\phi_n: A(X \times B\Sigma_n) \to A(X \times E\Sigma_n) \simeq A(X)$ be the $\Sigma_n$-transfer. We also develop an inductive procedure to compute the compositions $\phi_n \circ \theta^n$, and outline some applications.
1 Introduction

Let \( X \) be a connected simplicial abelian group, let \( \Sigma_n \) be the symmetric group on \( n \) letters, and let \( B\Sigma_n \) be the classifying space. Our goal is to define a family of Segal operations

\[
\theta^n : A(X) \longrightarrow A(X \times B\Sigma_n)
\]

satisfying the properties listed in theorems 1.1 and 1.2 below. We follow Waldhausen \cite{12} in our naming convention, which can be explained as follows. Around 1972 Graeme Segal \cite{10} defined a set of operations in stable homotopy theory \( \theta^n : \pi^*_s(S^0) \longrightarrow \pi^*_s((B\Sigma_n)_+) \), verified certain properties and used the information to give a proof of the Kahn-Priddy theorem. The key to the Kahn-Priddy proof is a certain relation satisfied by the composition of an operation followed by a transfer homomorphism.

Waldhausen \cite{12} adapted the construction in \cite{10} to define operations \( \theta^n : A(\ast) \longrightarrow A(B\Sigma_n) \) and proved these new operations have properties precisely analogous to fundamental properties of Segal’s original operations. Consequently, Waldhausen used the same notation and called the operations “Segal operations.”

We obtain the following result.

**Theorem 1.1.** For a connected simplicial abelian group \( X \), there are maps \( \theta^n : A(X) \rightarrow A(X \times B\Sigma_n) \) which have the following properties.

1. The map \( \theta^1 \) is the identity.

2. The combined map

\[
\theta = \prod_{n \geq 1} \theta^n : A(X) \longrightarrow \{1\} \times \prod_{n \geq 1} A(X \times B\Sigma_n)
\]

has the structure of an \( E_\infty \)-map if the target is equipped with the \( E_\infty \)-structure arising from pairings \( A(X \times B\Sigma_m) \times A(X \times B\Sigma_n) \rightarrow A(X \times B\Sigma_{n+m}) \) defined in \cite[4.6]{4}.

The first property is a normalization condition, as satisfied by the constructions of Segal and Waldhausen. The second property implies that for every \( j > 0 \) the operations induce homomorphisms \( \pi_j A(X) \rightarrow \{1\} \times \prod_{n \geq 1} \pi_j A(X \times B\Sigma_n) \), when the target is given a particular algebraic structure. A third, algebraic, property of Waldhausen’s operations is recalled in proposition 8.1. This third property is crucial in the applications made by Segal and Waldhausen. Our extended operations exhibit a more technical algebraic property stated in theorem 1.2 and theorem 8.5.

A large part of our work follows \cite{5} in which many results are developed for quite general situations, satisfying certain technical conditions. Part of this paper verifies these conditions. In order to explain the necessity of this technical work, we repeat several definitions from \cite{5} and quote many results.

In section 2 the main results are proposition 2.9 and theorem 2.1. For the purposes of algebraic \( K \)-theory we verify exactness properties of certain constructions; to prepare for the \( E_\infty \)-structure we verify coherence properties.

In section 3 we recall the \( G_\ast \)-construction for algebraic \( K \)-theory \cite{4}, \cite{2} and prepare the constructions underlying the definition of the operations in definition 3.19.
In section 4 we set up to apply general machinery, taking the first step toward a main result: For $X$ a connected simplicial abelian group, there is an operation

$$\omega = \prod_{n \geq 1} \omega^n: A(X) \to \{1\} \times \prod_{n \geq 1} A_{\Sigma_n,\{\text{all}\}}(X)$$

which is a map of $E_\infty$-spaces with respect to specific algebraic structures described in section 4. The target of $\omega$ is the algebraic $K$-theory of $\Sigma_n$-spaces retracting to $X$ (with the trivial $\Sigma_n$-action) and relatively finite with respect to $X$. See definition 3.4. In the first step the $E_\infty$-structure is only visible if we restrict to spherical objects. The next section addresses this problem.

In section 5 we study how the functors from definition 3.19 interact with suspension operators. At the end of the section we complete the construction of the operation displayed in equation (1.1).

In theorem 6.1, we split $A_{\Sigma_n,\{\text{all}\}}(X)$ as a product of the algebraic $K$-theory of other spaces, one of which is $A(X \times B\Sigma_n)$. This corresponds to the subcategory of $\Sigma_n$-spaces retracting to $X$ (with the trivial $\Sigma_n$ action), relatively finite with respect to $X$, and with $\Sigma_n$ acting freely outside of $X$. We also obtain an expression for the composite functors “projecting to the free part”

$$\theta^n: A(X) \xrightarrow{\omega^n} A_{\Sigma_n,\{\text{all}\}}(X) \to A(X \times B\Sigma_n).$$

This expression is used in section 8.

In section 7 we establish equivalences among various models for equivariant $K$-theory and discuss the functors that induce transfer operations.

In section 8 our main computational result evaluates the composition

$$A(X) \xrightarrow{\theta^n} A(X \times B\Sigma_n) \xrightarrow{\phi_n} A(X),$$

where $\phi_n$ is the transfer map.

**Theorem 1.2** (Theorem 8.5). Let $X$ be a connected simplicial abelian group, thinking of the group operation as a multiplication, and let $\tau^n: X \to X$ be the homomorphism that raises elements to the $n$th power. Then

$$\phi_n \theta^n = (-1)^{n-1} \cdot (n-1)! \cdot \tau^n_*: \pi_j A(X) \to \pi_j A(X)$$

for $j > 0$.

We conclude this introduction with some comments on applications. First, we recall one formulation of the Kahn-Priddy theorem in stable homotopy theory. Let $Q(X) = \Omega^\infty S^\infty(X_+)$ denote unreduced stable homotopy theory and define reduced stable homotopy theory $\tilde{Q}(X) = \text{fiber}(Q(X) \to Q(\ast))$, the homotopy fibre. For each $n$ there is a transfer map $Q(B\Sigma_n) \to Q(E\Sigma_n) \simeq Q(\ast)$, and, by composition, there results a map $\tilde{Q}(B\Sigma_n) \to Q(\ast)$. The formulation of Kahn-Priddy theorem that we prefer is that the map

$$\pi_j(\tilde{Q}B\Sigma_p(\ast)) \to \pi_j(Q(\ast))_{(p)}$$
of homotopy groups localized at a prime $p$ is surjective for $j > 0$.

Walhausen’s analogue of this result applies to the algebraic $K$-theory of the one-point space. For the formulation we let $A(X)$ denote the algebraic $K$-theory of the space $X$ and let $\tilde{A}(X) = \text{fiber}(A(X) \to A(\ast))$ be the algebraic $K$-theory of $X$ reduced relative to a point. Manipulations formally similar to those above yield a map $\tilde{A}(B\Sigma_n) \to A(\ast)$ and the analogue of the Kahn-Priddy theorem is that the induced map

$$
\pi_j(\tilde{A}(B\Sigma_p)_{(p)} \longrightarrow \pi_j(A(\ast))_{(p)}
$$

of homotopy groups localized at $p$ is surjective for $j > 0$. In [14] these operations are further developed and used to prove that the third factor $\mu(X)$ in the splitting

$$
A(X) \simeq Q(X_+) \times Wh^{Diff}(X) \times \mu(X)
$$

is contractible, yielding the final result $A(X) \simeq Q(X_+) \times Wh^{Diff}(X)$. The significance of this fact is developed in [15].

In our situation we fix as base space a connected simplicial abelian group $X$ and define reduced algebraic $K$-theory relative to $X$ as $\tilde{A}(X \times B\Sigma_n \text{ rel } X) = \text{fiber}(A(X \times B\Sigma_n) \to A(X))$. The inclusion of a point into $B\Sigma_n$ combined with the definition of the algebraic $K$-theory of $X \times B\Sigma_n$ reduced relative to $X$ yields a splitting

$$
\pi_j A(X \times B\Sigma_n) \cong \pi_j \tilde{A}(X \times B\Sigma_n \text{ rel } X) \oplus \pi_j A(X)
$$

(1.2)

for any $j \geq 0$. We have transfer maps $\phi_n : A(X \times B\Sigma_n) \to A(X \times E\Sigma_n) \simeq A(X)$ and a basic calculation in lemma [7.7] that the composition

$$
A(X) \to A(X \times B\Sigma_n) \overset{\phi_n}{\longrightarrow} A(X)
$$

is multiplication by $n! = |\Sigma_n|$, where the first map is induced by inclusion of a point into $B\Sigma_n$.

When we specialize $n$ to a prime number $p$, we have the following observations. Make the following diagram of homotopy groups reduced mod $p$, where the splitting [12] appears as the middle column. The diagonal arrow from the bottom row is multiplication by $p! = |\Sigma_p|$, which is 0 modulo $p$. Thus, in terms of the splitting of $\pi_j A(X \times B\Sigma_p) / p\mathbb{Z}$ given above, on the second component of the image of $\theta^p_\ast$, the map $\phi_{p!}$ is zero.

$$
\text{Diagram}
$$

\[\begin{array}{cccc}
\pi_j \tilde{A}(X \times B\Sigma_p \text{ rel } X) / p\mathbb{Z} \\
\pi_j A(X) / p\mathbb{Z} & \phi^p \downarrow & \pi_j A(X \times B\Sigma_p) / p\mathbb{Z} & \phi_{p!} \\
& \phi_{p!} \downarrow & \pi_j A(X) / p\mathbb{Z} \\
\pi_j A(X) / p\mathbb{Z} & i_\ast \\
\end{array}\]

Applying theorem [8.6] $\phi_{p!}$ applied to the first component $\pi_j \tilde{A}(X \times B\Sigma_p \text{ rel } X) / p$ of the splitting contains the image of $\phi_{p!} \theta^p_\ast = (-1)^{p-1} \cdot (p-1)! \cdot \tau^p_\ast$, where $\tau^p : X \to X$ raises elements to the $p$th power. The numerical factors are invertible mod $p$ so that

$$
\phi_{p!} (\pi_j \tilde{A}(X \times B\Sigma_p \text{ rel } X) / p\mathbb{Z}) \supset \text{Image } \tau^p_\ast,
$$
viewing $\tau_p^p$ as an endomorphism of $\pi_j A(X)/p\mathbb{Z}$.

From these calculations one extracts various additional observations. It may happen that
the $p$th power homomorphism $\tau^p$ is an isomorphism, as in the case when $X$ is a connected
simplicial abelian group, finite in each simplicial dimension and $p$ is relatively prime to the
order of $X_n$ for each $n$. Then for $j > 0$,

$$\phi_{pr}: \pi_j \tilde{A}(X \times B\Sigma_p \operatorname{rel} X)/p\mathbb{Z} \longrightarrow \pi_j A(X)/p\mathbb{Z}$$

is surjective. The next input is the following theorem.

**Theorem 1.3** (Theorem, [1]). If $\pi_1(X)$ is a finite group, and $\pi_i(X)$ is finitely generated for
all $i \geq 2$, then $\pi_j(A(X))$ is finitely generated for all $j$.

Then Nakayama’s lemma applies as in [12] to lift the result on mod $p$ homotopy to a
result on $p$-localized homotopy. We obtain the following theorem of Kahn-Priddy type.

**Theorem 1.4.** Let $X$ be a connected simplicial abelian group, finite in each dimension, such
that the order of $X_n$ is prime to $p$. For $j > 0$ and $p$ an odd prime, the transfer induces surjections

$$\pi_j \tilde{A}(X \times B\Sigma_p \operatorname{rel} X)/(p) \longrightarrow \pi_j A(X)/(p).$$

on homotopy groups localized at $p$. □

In particular, take $X = BC_2 = \mathbb{R}P^\infty$ and $p$ an odd prime. There are similar statements
for all the lens spaces $BC_q$, $q$ prime to $p$.

A very interesting case is $X = BC_\infty$, the classifying space of the infinite cyclic group $C_\infty$. Of course $X \simeq S^1$, and there are splittings-up-to-homotopy of infinite loop spaces

$$A^{fd}(S^1) \simeq A^{fd}(\ast) \times BA^{fd}(\ast) \times N_-A^{fd}(\ast) \times N_+A^{fd}(\ast)$$

and

$$A^{fd}(S^1 \times B\Sigma_n) \simeq A^{fd}(B\Sigma_n) \times BA^{fd}(B\Sigma_n) \times N_-A^{fd}(B\Sigma_n) \times N_+A^{fd}(B\Sigma_n).$$

These are studied in [6] and the first is examined in great detail in [3]. In future work we
would like to understand the operations we have constructed in terms of these splittings. As
a first step in this direction we have shown in section 4 that the operations we construct are
morphisms of infinite loop spaces. Should the $\theta$ operations be compatible with the splitting,
one must then investigate whether or not the $\theta$ operations commute with the Frobenius and
Verschiebung operations on the Nil-terms defined in [3].

Our work also admits a generalization where $X$ may be any connected space. This result
is a total operation

$$\tilde{\omega}: A(X) \longrightarrow \{1\} \times \prod_{n \geq 1} A_{\Sigma_n,(all)}(X^n),$$

about which we know little at this point. Our experiments have also lead to the observation
that if $G$ is a simplicial group, not necessarily abelian, whose realization is homotopy
equivalent to a finite CW-complex then there is a product structure on $A(BG)$. This will be
the subject of a later paper. Finally, reversing the progression from Segal’s original idea to
Waldhausen’s generalization, we can develop operations $\theta^n: \pi_+^n(X_+) \rightarrow \pi_+^n(X \times B\Sigma_n)$, where $X$ is again a connected simplicial abelian group.
2 The symmetric bimonoidal category of retractive spaces over a connected simplicial abelian group

The category $\mathcal{R}(X)$ is the category of retractive simplicial sets $(Y, r, s)$ over the simplicial set $X$, where $r: Y \to X$ is a retraction, $s: X \to Y$ is a section for $r$ and morphisms $(Y_1, r_1, s_1) \to (Y_2, r_2, s_2)$ respect all the data. A cofibration $(Y_1, r_1, s_1) \mapsto (Y_2, r_2, s_2)$ in $\mathcal{R}_f(X)$ is a map such that $Y_1 \to Y_2$ is injective. A weak equivalence is a map $(Y_1, r_1, s_1) \to (Y_2, r_2, s_2)$ whose realization $|Y_1| \to |Y_2|$ is a homotopy equivalence of spaces. For algebraic $K$-theory we use the full subcategory $\mathcal{R}_f(X)$ of relatively finite retractive simplicial sets with cofibrations and weak equivalences. “Relatively finite” means that there are only finitely many non-degenerate simplices in $Y-X$. For background on the terminology, see [13, Section 1.1].

We aim to construct a total operation

$$\theta: A(X) \to \{1\} \times \prod_{n \geq 1} A(X \times B\Sigma_n)$$

for $X$ a connected simplicial abelian group with multiplication $\mu: X \times X \to X$ and to prove the operation has an $E_\infty$-structure. In order to achieve this, the elements from which the construction is developed must be of high quality. The necessary qualities are recorded in the first part of theorem 2.1 the second part of the theorem records algebraic properties of the product operation $\wedge_\mu$. We discuss first the definition of the product operation and then prove the second part of the proposition.

Concerning the first part of the theorem, our constructions require a coherence result for diagrams involving sum and product operations, as provided by LaPlaza [7, Proposition 10]. His coherence theorem takes as input the commutativity of 24 diagrams, reducible to a smaller, but still relatively large, subset [7, pp. 40–41]. We will see that the coherence properties we need rest on the well-understood coherence properties of the one-point union and smash product of pointed sets. On the other hand, the second part of the theorem involves properties of the operations not reducible to dimension-wise considerations.

**Theorem 2.1.** Let $X$ be a connected simplicial abelian group.

1. The triple $(\mathcal{R}(X), \vee_X, \wedge_\mu)$, where $\vee_X$ denotes the operation of union along the common subspace $X$ and $\wedge_\mu$ denotes the pairing (2.2), is a symmetric bimonoidal category.

2. The pairing $\wedge_\mu$ restricts to $\mathcal{R}_f(X)$, where it is biexact, meaning exact in each variable separately. Explicitly, the functors defined by $- \wedge_\mu Y$ and $Y \wedge_\mu -$ preserve cofibrations, pushouts along cofibrations, and weak equivalences.

Our product operation $\wedge_\mu$ derives from an exterior smash product $\wedge_e$ of retractive simplicial sets, following the exterior smash product of retractive spaces as described in [8]. Since we are working with simplicial sets, our version of the exterior smash product has a description in terms of operations on discrete sets, applied dimensionwise. See the discussion at the start of the proof of part one of theorem 2.1.
**Definition 2.2.** Let \((Y_i, r_i, s_i)\) be objects of \(\mathcal{R}(X_i)\), for \(i = 1, 2\). The exterior smash product of \((Y_1, r_1, s_1)\) with \((Y_2, r_2, s_2)\) is in \(\mathcal{R}(X_1 \times X_2)\), and the underlying space \(Y_1 \wedge_e Y_2\) completes the following square to a pushout.

\[
\begin{array}{ccc}
Y_1 \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_2 & \longrightarrow & Y_1 \times Y_2 \\
\downarrow & & \downarrow \\
X_1 \times X_2 & \longrightarrow & Y_1 \wedge_e Y_2 \\
\end{array}
\]

Example 2.3. For any \(Y_2 \in R(X_2)\), note that \(X_1 \wedge_e Y_2 \cong X_1 \times X_2\), the “zero” object in \(\mathcal{R}(X_1 \times X_2)\). Colloquially, the exterior smash product of a terminal object with any object yields a terminal object. Explicitly, a natural isomorphism \(\lambda_2^*: (X_1 \wedge_e Y_2) \to X_1 \times X_2\) arises from the following diagram by mapping the pushout of the top row to the pushout of the bottom row.

Example 2.4. The bifunctor \(\mathcal{R}(\cdot) \times \mathcal{R}(X) \to \mathcal{R}(X)\) given on objects by \((Y_1, Y_2) \mapsto Y_1 \wedge_e Y_2\) defines an action of \(\mathcal{R}(\cdot)\) on \(\mathcal{R}(X)\) after identifying \(\{\ast\} \times X\) with \(X\) in the canonical way. This bifunctor also restricts to an action \(\mathcal{R}_f(\cdot) \times \mathcal{R}_f(X) \to \mathcal{R}_f(X)\).

**Definition 2.5.** Let \(X\) be a connected simplicial abelian group. We operate on the category \(\mathcal{R}(X)\), using the pairing

\[
\wedge_\mu = \mu_* \circ \wedge_e: \mathcal{R}(X) \times \mathcal{R}(X) \xrightarrow{\wedge_e} \mathcal{R}(X \times X) \xrightarrow{\mu_*} \mathcal{R}(X)
\]

where \(\wedge_e\) is the external smash product pairing defined in \((2.1)\), and \(\mu_*\) is the functor induced by the multiplication \(\mu: X^2 \to X\). Explicitly, \((Y_1, r_1, s_1) \wedge_\mu (Y_2, r_2, s_2)\) completes the following
diagram to a pushout.

\[
\begin{array}{c}
Y_1 \times X \cup_{X \times X} X \times Y_2 \rightarrow \rightarrow Y_1 \times Y_2 \\
\mu(r_1, \text{id}) \cup \mu(\text{id}, r_2) \downarrow \downarrow \\
X \rightarrow \rightarrow Y_1 \wedge_\mu Y_2
\end{array}
\] (2.3)

We use these notations to bring this section close to conformity with [5]. Perfect conformity is not possible, for we must use both the one-point union of pointed spaces ∨ and the union of two spaces along a common subspace \( X \), denoted \( \vee_X \). We also point out that the usual notation \( \wedge \) has been used in [8] for a product defined by restricting the external smash product of two spaces over \( X \) to the diagonal of \( X \times X \).

The following lemma is used to develop properties of the smash products; the proof will be given later.

**Lemma 2.6.** Let \( \mathcal{C} \) be a category with cofibrations and let

\[
\begin{array}{c}
A_2 \leftarrow B_2 \rightarrow C_2 \\
A_1 \leftarrow B_1 \rightarrow C_1 \\
A_0 \leftarrow B_0 \rightarrow C_0
\end{array}
\] (2.4)

be a commutative diagram in which the canonical map from \( B_2 \cup_{B_1} C_1 \rightarrow C_2 \) is a cofibration.

Passing to pushouts by columns results in a diagram in which the right-pointing arrow is a cofibration:

\[
\begin{array}{c}
A_0 \cup_{A_1} A_2 \leftarrow B_0 \cup_{B_1} B_2 \rightarrow C_0 \cup_{C_1} C_2.
\end{array}
\] (2.5)

The diagram

\[
\begin{array}{c}
A_0 \cup_{B_0} C_0 \leftarrow A_1 \cup_{B_1} C_1 \rightarrow A_2 \cup_{B_2} C_2
\end{array}
\] (2.6)

obtained by passing to pushouts by rows has a similar property.

As an easy application we have the following proposition.

**Proposition 2.7.** The exterior smash product \( \wedge_\epsilon \) is functorial for pairs of maps. That is, given \( f_1: X_1 \rightarrow X'_1 \) and \( f_2: X_2 \rightarrow X'_2 \), the diagram

\[
\begin{array}{c}
\mathcal{R}_f(X_1) \times \mathcal{R}_f(X_2) \xrightarrow{\wedge_\epsilon} \mathcal{R}_f(X_1 \times X_2) \\
(f_1 \times f_2) \downarrow \downarrow (f_1 \times f_2)_*$
\end{array}
\] (2.7)

commutes up to natural isomorphism.
Proof. For the naturality property of the external smash product, consider the diagram

\[
\begin{array}{ccc}
X_1 \times X_2 & \longleftrightarrow & Y_1 \times X_2 \cup X_1 \times X_2 \rightarrow X_1 \times Y_2 \\
\uparrow & & \uparrow \\
X_1 \times X_2 & \longleftrightarrow & X_1 \times X_2 \\
{f_1 \times f_2} & \downarrow & {f_1 \times f_2} \\
X'_1 \times X'_2 & \longleftrightarrow & X'_1 \times X'_2 \\
\end{array}
\]  

(2.8)

which fulfills the hypotheses of lemma 2.6. Computing the colimits of the columns in this diagram yields the diagram

\[
\begin{array}{ccc}
X'_1 \times X'_2 & \xrightarrow{r_1 \times r_2} (f_1 \ast Y_1) \times X'_2 \cup X'_1 \times f_2, Y_2) & \longrightarrow f_1 \ast Y_1 \times f_2, Y_2; \\
\end{array}
\]

whose pushout is by definition \( f_1 \ast Y_1 \land_e f_2, Y_2 \).

On the other hand computing the colimits of the rows in the diagram yields the diagram

\[
\begin{array}{ccc}
X'_1 \times X'_2 & \xrightarrow{f_1 \times f_2} X_1 \times X_2 & \longrightarrow Y_1 \land_e Y_2, \\
\end{array}
\]

whose pushout is \((f_1 \times f_2)_*(Y_1 \land_e Y_2)\). Since both iterative procedures compute the colimit of diagram (2.8), they are canonically isomorphic:

\[ f_1 \ast Y_1 \land_e f_2, Y_2 \cong (f_1 \times f_2)_*(Y_1 \land_e Y_2). \]

\[ \square \]

As a consequence, we have the following result.

**Proposition 2.8.** Let \( X \) be a connected simplicial abelian group. The action of \( \mathcal{R}(\ast) \) on \( \mathcal{R}(X) \) set up in example 2.4 may be made internal to \( \mathcal{R}(X) \). Diagramatically, the diagram

\[
\begin{array}{ccc}
\mathcal{R}(\ast) \times \mathcal{R}(X) & \xrightarrow{i_{\ast \ast} \times \text{id}} & \mathcal{R}(X) \times \mathcal{R}(X) \\
\downarrow \mathcal{R}(X) & \mathcal{R}(X) & \mathcal{R}(X) \\
\mathcal{R}(\ast) \times \mathcal{R}(X) & \xrightarrow{\land_e} & \mathcal{R}(\{\ast\} \times X) \\
\end{array}
\]

commutes up to natural isomorphism.

**Proof.** Let \( i_e : \{\ast\} \to X \) be the inclusion of the one point space to the identity element of the abelian group \( X \). The functor \( i_{\ast \ast} : \mathcal{R}(\ast) \to \mathcal{R}(X) \) sends a pointed retractive space \( Y \) to \( X \lor Y \), where the base point of \( Y \) is identified to the unit element of \( X \). The new retraction collapses \( Y \subset X \lor Y \) to the identity \( \{\ast\} \) in \( X \). We have the diagram

\[
\begin{array}{ccc}
\mathcal{R}(\ast) \times \mathcal{R}(X) & \xrightarrow{\land_e} & \mathcal{R}(\{\ast\} \times X) \\
\downarrow & & \downarrow \\
\mathcal{R}(X) \times \mathcal{R}(X) & \xrightarrow{\land_e} & \mathcal{R}(X \times X) \\
\end{array}
\]

The lefthand square commutes by proposition 2.7 and the righthand triangle commutes because \( e \) is the group identity. The bottom row defines \( \land_\mu \) and the trip across the top defines the action of \( \mathcal{R}(\ast) \) on \( \mathcal{R}(X) \). \[ \square \]
Thus, the pairing contains only finitely many non-degenerate simplices, then the same is true of their product.

Proposition 2.9. The external smash product functor

$$\land_e : \mathcal{R}_f(X_1) \times \mathcal{R}_f(X_2) \rightarrow \mathcal{R}_f(X_1 \times X_2)$$

is biexact. That is, the functors $Z_1 \land_e -$ and $- \land_e Z_2$ carry cofibrations to cofibrations and preserve pushouts of cofibrations. For example,

$$Z_1 \land_e (Y_2 \cup_{Y_1} Y_0) \cong (Z_1 \land_e Y_2) \cup_{(Z_1 \land e Y_1)} (Z_1 \land e Y_0).$$

Moreover, $\land_e$ carries a pair of weak equivalences to a weak equivalence.

Remark 2.10. In the approach of [13] the external smash product is shown to preserve all colimits by exhibiting a left adjoint functor. Their approach uses properties of convenient categories of topological spaces.

For our applications in algebraic $K$-theory it seems more reasonable to give arguments modeled on those of [13, Lemma 1.1.1], which serve to illuminate other constructions we make.

Proof of proposition 2.9. For simplicial sets, cofibrations are precisely the injections. Given a pair of cofibrations

$$(W_1, r_1, s_1) \xrightarrow{\sim} (W'_1, r'_1, s'_1) \quad \text{and} \quad (W_2, r_2, s_2) \xrightarrow{\sim} (W'_2, r'_2, s'_2)$$

in $\mathcal{R}_f(X_1)$ and $\mathcal{R}_f(X_2)$, respectively, the maps of differences of simplicial sets $W_1 - X_1 \rightarrow W'_1 - X_1$ and $W_2 - X_2 \rightarrow W'_2 - X_2$ are injective maps of sets in each simplicial dimension. The product of these maps is also injective. Since $(W_1 \land_e W_2) - X_1 \times X_2 = (W_1 - X_1) \times (W_2 - X_2)$, it follows that $W_1 \land_e W_2 \xrightarrow{\sim} W'_1 \land_e W'_2$ is also a cofibration. Finally, if $W_1 - X_1$ and $W_2 - X_2$ contain only finitely many non-degenerate simplices, then the same is true of their product. Thus, the pairing $\land_e$ restricts to a pairing of $\mathcal{R}_f(X_1) \times \mathcal{R}_f(X_2)$ to $\mathcal{R}_f(X_1 \times X_2)$.

To prove that the functor $Z \land_e (-) : \mathcal{R}_f(X_2) \rightarrow \mathcal{R}_f(X_1 \times X_2)$ preserves pushouts of cofibrations, start by considering the diagram

$$\xymatrix{ X_1 \times X_2 \ar@{^{(}->}[r] \ar@{=}[d] & Z \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_2 \ar@{^{(}->}[r] \ar@{=}[d] & Z \times Y_2 \ar@{=}[d] \\
X_1 \times X_2 \ar@{=}[d] & Z \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_1 \ar@{^{(}->}[r] \ar@{=}[d] & Z \times Y_1 \ar@{=}[d] \\
X_1 \times X_2 & Z \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_0 \ar@{^{(}->}[r] & Z \times Y_0 } \tag{2.9}$$

where the right-pointing arrows are induced from the retractions and the left-pointing arrows are induced by inclusions. We verify the cofibration hypothesis of lemma 2.6 using the
Pass to pushouts in the columns, apply the universal mapping properties of the pushouts, and use isomorphism (2.13) to simplify the pushout of the middle column to obtain the following commuting diagram.

\[
\begin{array}{c}
Z \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_1 \\
\downarrow \\
Z \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_1 \\
\end{array} \cong \begin{array}{c}
( Z \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_1 ) \cup_{X_1 \times Y_1} X_1 \times Y_2 \\
\downarrow \\
Z \times Y_1 \\
\end{array} \cong \begin{array}{c}
Z \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_1 \\
\downarrow \\
Z \times Y_1 \\
\end{array} \cong \begin{array}{c}
Z \times Y_1 \\
\downarrow \\
Z \times Y_1 \\
\end{array} 
\]

The space \( Z \times Y_1 \cup_{X_1 \times Y_1} X_1 \times Y_2 \) is a subspace of \( Z \times Y_2 \), so the downward arrow on the right is a cofibration. Since isomorphisms are cofibrations, it follows that the lower arrow is also a cofibration. Thus, we have verified the cofibration condition of lemma 2.6 for diagram (2.9).

We may now calculate the colimit of diagram (2.9) by two different iterative procedures. Computing the pushouts of the rows first and applying lemma 2.6 gives a diagram

\[
\begin{array}{c}
Z \times Y_1 \cup_{X_1 \times Y_1} X_1 \times Y_2 \\
\downarrow \cong \downarrow \\
Z \times Y_1 \cup_{X_1 \times Y_1} X_1 \times Y_2 \\
\end{array} 
\]

and calculating the pushouts of the columns first and applying lemma 2.6 again gives a another diagram

\[
\begin{array}{c}
X_1 \times X_2 \\
\downarrow \\
X_1 \times X_2 \\
\end{array} \leftarrow \begin{array}{c}
Z \times X_2 \cup_{X_1 \times X_2} X_1 \times ( Y_0 \cup_{Y_1} Y_2 ) \\
\downarrow \\
Z \times ( Y_0 \cup_{Y_1} Y_2 ) \\
\end{array} 
\]

To see this formula for the middle object in diagram (2.11) make the following considerations. We have the diagram

\[
\begin{array}{c}
Z_2 \times X_2 \\
\downarrow = \downarrow \\
Z_2 \times X_2 \\
\end{array} \leftarrow \begin{array}{c}
X_1 \times X_2 \\
\downarrow = \downarrow \\
X_1 \times X_2 \\
\end{array} \rightarrow \begin{array}{c}
X_1 \times Y_2 \\
\downarrow \\
X_1 \times Y_2 \\
\end{array} 
\]

meeting the conditions of lemma 2.6 whose colimit we also compute iteratively. Computing the pushouts of the rows first gives precisely the middle column in diagram (2.9), whose pushout we are now evaluating. On the other hand, computing the pushouts along the columns first gives a diagram

\[
\begin{array}{c}
Z_2 \times X_2 \\
\downarrow \\
Z_2 \times X_2 \\
\end{array} \leftarrow \begin{array}{c}
X_1 \times X_2 \\
\downarrow \\
X_1 \times X_2 \\
\end{array} \rightarrow \begin{array}{c}
X_1 \times ( Y_2 \cup_{Y_1} Y_0 ) \\
\downarrow \\
X_1 \times ( Y_2 \cup_{Y_1} Y_0 ) \\
\end{array} 
\]
whose pushout is the middle term displayed in diagram (2.11). As the iterated pushouts of diagram (2.12) are isomorphic to the colimit of the entire diagram, the iterated pushouts are isomorphic. This justifies diagram (2.11).

Completing the analysis of diagram (2.9), the pushouts of these diagrams (2.10) and (2.11) are isomorphic, because they both represent the colimit of the original diagram (2.9). Interpreting this statement, we have the result that $Z \wedge e$ preserves pushouts of cofibrations.

Suppose $f_1 : Y_1 \to Y_1'$ and $f_2 : Y_2 \to Y_2'$ are weak equivalences in $R_f(X_1)$ and $R_f(X_2)$, respectively. That is, the geometric realizations $|f_1|$ and $|f_2|$ are homotopy equivalences. Then $|f_1| \times \text{id}$ and $\text{id} \times |f_2|$ are homotopy equivalences. By the ordinary gluing lemma for homotopy equivalences applied to the diagram

$$
\begin{tikzcd}
|Y_1| \times |X| & |X| \times |X| \times |X| \times |Y_2| \\
|Y_1'| \times |X| & |X| \times |X| \times |X| \times |Y_2'|
\end{tikzcd}
$$

the central arrow in the diagram below is also a homotopy equivalence.

$$
\begin{tikzcd}
|X| \times |X| & |Y_1| \times |X| \times |X| \times |X| \times |X| \times |Y_2| \\
|X| \times |X| & |Y_1'| \times |X| \times |X| \times |X| \times |X| \times |Y_2'|
\end{tikzcd}
$$

Since the pushout of the last diagram is homeomorphic to $|Y_1 \wedge Y_2| \to |Y_1 \wedge Y_2'|$ ("colimits commute"), $Y_1 \wedge e Y_2 \to Y_1' \wedge e Y_2'$ is a weak equivalence. By a final application of the gluing lemma, the map on pushouts induced by $|\mu| : |X| \times |X| \to |X|$ is a homotopy equivalence. That is, $Y_1 \wedge e Y_2 \to Y_1' \wedge e Y_2'$ is a weak homotopy equivalence.

**Remark 2.11.** The external smash product will also preserve many colimits. However, our applications principally involve the special colimits that are pushouts of cofibration squares.

**Proof of the second part of theorem 2.7.** Since the functor $\mu_* : R_f(X \times X) \to R_f(X)$ is exact [13, Lemma 2.1.6], and we have seen that $\wedge e$ is biexact in proposition 2.9, the composite $\wedge \mu = \mu_* \circ \wedge e$ is biexact.

Before we consider the coherence of the product $\wedge \mu$, we prove lemma 2.6. The reader interested in the coherence properties may skip ahead.

**Proof of lemma 2.6.** We make frequent use of the isomorphism

$$
(A \cup_B C) \cup_C D \cong A \cup_B D.
$$

The canonical arrow $B_2 \cup_{B_1} B_0 \to C_2 \cup_{C_1} C_0$ factors into the composition of canonical arrows induced by passing to pushouts of the columns in the map of diagrams

\[
\begin{array}{c}
\begin{tikzcd}
B_2 \ar[r] \ar[d] & B_2 \ar[r] \ar[d] & C_2 \\
B_1 \ar[r] & B_1 \ar[r] & C_1 \\
B_0 \ar[r] & C_0 \ar[r] & C_0
\end{tikzcd}
\end{array}
\]
and we show each arrow in the factorization is a cofibration, as follows. The first arrow in
the factorization appears as the lower row in the completed pushout diagram

\[
\begin{array}{ccc}
B_0 & \rightarrow & C_0 \\
\downarrow & & \downarrow \\
B_2 \cup B_1 \rightarrow (B_2 \cup B_1 \cup B_0) \rightarrow B_2 \cup B_1 C_0
\end{array}
\]

augmented by an isomorphism, so the first arrow is a cofibration, as claimed. From the
hypothesis on the canonical map from \(B_2 \cup B_1 C_1 \rightarrow C_2\), the upper arrow in the next diagram
is a cofibration, so the lower arrow in the completed pushout diagram is as well.

\[
\begin{array}{ccc}
B_2 \cup B_1 C_1 & \rightarrow & C_2 \\
\downarrow & & \downarrow \\
B_2 \cup B_1 C_0 & \cong & (B_2 \cup B_1 C_1 \cup C_1 C_0) \rightarrow C_2 \cup C_1 C_0
\end{array}
\]

Augmenting the completed pushout diagram by the two isomorphisms, the second arrow
\(B_2 \cup B_1 C_0 \rightarrow C_2 \cup C_1 C_0\) in the factorization is also a cofibration. Then the composition
\(B_2 \cup B_1 B_0 \rightarrow B_2 \cup B_0 C_0 \rightarrow C_2 \cup C_1 C_0\) is a cofibration and this arrow is isomorphic to
the arrow in diagram (2.5).

To obtain the result for the row-wise pushouts from the result for column-wise pushouts,
observe that the properties of the arrows in the diagram are symmetric with respect to
reflection in the diagonal \(A_0 B_1 C_2\). Therefore, it suffices to reflect the diagram in this diagonal
and apply the column-wise result.

\[
\begin{proof}
\[\text{Proof of the first part of theorem 2.1.}\]
It is well-known that disjoint union of sets and the one point union \(\vee\) of pointed sets are categorical sum operations, so that all coherence
conditions for these operations are automatically met. For the category of sets containing a
fixed set \(S\) the union \(\vee_S\) of two sets along the common subset is also the categorical sum,
so \(\vee_S\) fulfills all coherence conditions. Concerning products, the cartesian product of sets
and the smash product of pointed sets are operations also meeting coherence conditions.
When these operations of sum and product are considered together, they are related by
distributivity isomorphisms, and the combined systems exhibit the coherence properties
discussed in [7]. It is possible to develop the coherence properties we need for operations
on retractive spaces from these basic elements by developing the operations \(\vee_X\) and \(\wedge_e\)
dimensionwise and pointwise over \(X\), resp., \(X_1 \times X_2\) in the case of products, from one point
union and smash product of pointed sets. Compare the remark following definition 2.2. We
take a different approach here.

For the sum \(\vee_X\), we need a slight extension of the union of sets along a common subset to
cover the case of the disjoint union of two simplicial sets along a common simplicial subset.
Let \(\mathcal{T}\) be the category of triples \((T, r: T \rightarrow S, s: S \rightarrow T)\), where \(S\) and \(T\) are sets and the
functions satisfy \(r \circ s = \text{id}_S\). Occasionally, it is convenient to view \(S\) as a subset of \(T\). A
morphism

\[
(f, f'): (T_1, r_1: T_1 \rightarrow S_1, s_1: S_1 \rightarrow T_1) \rightarrow (T_0, r_0: T_0 \rightarrow S_0, s_0: S_0 \rightarrow T_0)
\]
is a pair of maps \( f: T_1 \to T_0 \) and \( f': S_1 \to S_0 \) such that \( s_0f' = fs_1 \) and \( r_0f = f'r_1 \). A object \((Y, r, s)\) of \( \mathcal{R}(X) \) can be viewed as a functor \( \Delta^\text{op} \to \mathcal{T} \), and conversely. There is a functor \( u: \mathcal{T} \to \text{Set} \) that selects the subset \( S \) and morphisms \( f': S_1 \to S_0 \). On the pullback category

\[
\begin{array}{ccc}
\mathcal{T} \times \text{Set} \mathcal{T} & \longrightarrow & \mathcal{T} \times \mathcal{T} \\
\downarrow & & \downarrow \mathbf{u} \times \mathbf{u} \\
\text{Set} & \longrightarrow & \Delta \to \text{Set} \times \text{Set}
\end{array}
\]

define the operation \((T_1, r_1: T_1 \to S, s_1: S \to T_1) \vee_S (T_2, r_2: T_2 \to S, s_2: S \to T_2)\), abbreviated \((T_1, r_1, s_1) \vee_S (T_2, r_2, s_2)\), or even \( T_1 \vee_S T_2 \). Set

\[ T_1 \vee_S T_2 = T_1 \amalg T_2/ \sim, \]

where \( \sim \) is the equivalence relation generated by setting \( s_1(x) \sim s_2(x) \) for \( x \in S \). Set \( i_j: T_j \to T_1 \vee_S T_2 \) to be the inclusion \( T_j \to T_1 \amalg T_2 \) followed by the quotient map to \( T_1 \vee_S T_2 \). For the rest of the structure, set

\[ r: T_1 \vee_S T_2 \to S, \]

to be the unique function satisfying \( ri_j = r_j \), for \( j = 1, 2 \) and let

\[ s: S \to T_1 \vee_S T_2 \]

satisfy \( s(x) = i_1s_1(x) = i_2s_2(x) \) for \( x \in S \). Define \((i_1', i_1^\prime = \text{id}): (T_1, s_1, r_1) \to (T_1 \vee_S T_2, r, s)\) to obtain a morphism in \( \mathcal{T} \). The identities \( ri_1 = i_1'r_1 \) and \( si_1' = i_1s_1 \) are satisfied by definition and by the condition \( r_1s_1 = \text{id} \). Define \((i_2', i_2^\prime): (T_2, s_2, r_2) \to (T_1 \vee_S T_2, r, s)\) similarly. If \((T', r', s')\) is another object of \( \mathcal{T} \), and suppose \((f_i, f_i'): (T_i, r_i, s_i) \to (T', r', s')\) is a morphism in \( \mathcal{T} \times \text{Set} \mathcal{T} \) for \( i = 1, 2 \). This just means that \( f_i' = f'_i: S \to S' \). Then the categorical sum properties of the disjoint union on the category \text{Set} \ and the quotient construction deliver a unique morphism

\[ (h, h'): (T_1 \vee_S T_2, r', s') \to (T', r', s') \]

such that \( (h, h') \circ (i_1, i_1') = (f_1, f_1') \) and \( (h, h') \circ (i_2, i_2') = (f_2, f_2') \). When the base set is fixed, we obtain a categorical sum; in general, when the base set varies, we obtain a (partially defined) categorical sum on \( \mathcal{T} \).

We have observed that an object of the category \( \mathcal{R}(X) \) is a simplicial object in the category \( \mathcal{T} \), that is, a functor \( \Delta^\text{op} \to \mathcal{T} \). A pair of objects \((Y_1, r_1, s_1)\) and \((Y_2, r_2, s_2)\) in \( \mathcal{R}(X) \) defines a functor \( \Delta^\text{op} \to \mathcal{T} \times \text{Set} \mathcal{T} \). We obtain the operation \((Y_1, r_1, s_1) \vee_X (Y_2, r_2, s_2)\) based on the dimension-wise operation \((Y_1)_p \vee_X (Y_2)_p\). This makes \( \vee_X \) a categorical sum in \( \mathcal{R}(X) \), with unit (zero element, thinking additively) the space \( X \). The commutativity isomorphisms \( \gamma' \), associativity isomorphisms \( \alpha' \), left and right unit isomorphisms \( \lambda' \) and \( \rho' \) are straightforward consequences of the analogous properties of the disjoint union operation on sets. Essentially, all the basic properties required for coherence of the sum operation \( \vee_X \) are automatically fulfilled. That \( \vee_X \) is the categorical sum simplifies almost all coherence considerations involving diagrams involving both \( \vee_X \) and \( \wedge_\mu \).
To complete the input for LaPlaza’s coherence result we need to identify in $\mathcal{R}(X)$ an additive identity, a multiplicative zero element, a multiplicative identity, commutativity and associativity isomorphisms for $\land_\mu$, and, finally, distributivity isomorphisms.

Clearly $(X, \text{id}, \text{id})$ is the identity for $\lor_X$. Example~2.3 implies that $(X, \text{id}, \text{id})$ is a zero object from the left and the right for $\land_\mu$, in the sense that there are natural isomorphisms $\lambda_Y : X \land_\mu Y \to X$ and $\rho_Y : Y \land_\mu X \to X$. Example~2.4 combined with proposition 2.8 delivers the fact that $i_{e*}(S^0) = X \lor S^0$, where the base point of $S^0$ is identified with the multiplicative identity of $X$ and the retraction collapses $S^0$ to the identity of $X$, is a multiplicative identity in the sense that there are natural isomorphisms $\lambda_Y : (X \lor S^0) \land_\mu Y \to Y$ and $\rho_Y : Y \land_\mu (X \lor S^0) \to Y$.

For commutativity of the product $\land_\mu = \mu_* \circ \land_e$, we have the following considerations. Use commutativity for cartesian products and apply the definitions from (2.3) of the internal smash product to obtain the following diagram.

\[
\begin{array}{c}
  X \xleftarrow{\mu} X \times X \xleftarrow{\gamma} Y_1 \times X \cup X \times Y_2 \xrightarrow{\gamma} Y_1 \times Y_2 \\
  X \xleftarrow{\mu} X \times X \xleftarrow{\gamma} Y_2 \times X \cup X \times Y_1 \xrightarrow{\gamma} Y_2 \times Y_1
\end{array}
\]

In the diagram the arrows labeled $\gamma$ are the isomorphisms switching the factors in the cartesian products. Note that $r_1 \land_\mu r_2 = \mu_*(r_1 \land_e r_2) = r_1 \cdot r_2 = r_2 \cdot r_1 = \mu_*(r_2 \land_e r_1) = r_2 \land_\mu r_1$, since $X$ is abelian. Passage to pushouts yields an isomorphism

$\gamma_{Y_1,Y_2} : (Y_1 \land_\mu Y_2, r_1 \land_\mu r_2, s_1 \land_\mu s_2) \xrightarrow{\cong} (Y_2 \land_\mu Y_1, r_2 \land_\mu r_1, s_2 \land_\mu s_1)$

It is easily seen that $\gamma_{Y_2,Y_1} \gamma_{Y_1,Y_2} = \text{id}$ holds, often written “$\gamma^2 = \text{id}$” and called the inverse law, and that the left and right unit laws are compatible. These facts are recorded in the following commuting diagrams.

\[
\begin{array}{ccc}
  Y_1 \land_\mu Y_2 & \xrightarrow{\gamma_{Y_1,Y_2}} & Y_2 \land_\mu Y_1 \\
  \gamma_{Y_2,Y_1} & \xrightarrow{\gamma_{Y_2,Y_1}} & \gamma_{Y_1,Y_2}
\end{array}
\]

\[
\begin{array}{ccc}
  Y_1 \land_\mu Y_2 & \xrightarrow{\lambda_\mu} & Y_1 \land_\mu Y_2 \\
  \rho_Y & \xrightarrow{\rho_Y} & \lambda_Y
\end{array}
\]

Consider now associativity. The point is that the associativity for $\land_\mu$ rests on associativity for $\times$, $\cup$, and associativity of the multiplication $\mu$ on $X$. By passage to colimits we obtain associativity for $\land_\mu$. For the usual smash product, associativity for cartesian products passes to associativity for smash products; our argument is similarly structured.

The first step is to obtain an expression for $(Y_1 \land_\mu Y_2) \land_\mu Y_3$ that involves only cartesian
products and colimits. Consider the following diagram

\[
\begin{array}{c}
\mu(r_1, r_2, id) \cup \mu(id, id, r_3) \cup \mu(id, r_2, r_3) \\
\mu(id, id, r_3) \cup \mu(id, r_2, r_3) \\
\mu(id, id, r_3) \cup \mu(id, r_2, r_3)
\end{array}
\]

\[
\begin{array}{cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc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and so on, induce an isomorphism of diagram (2.17) with diagram (2.16) and an associativity isomorphism

$$\alpha_{Y_1,Y_2,Y_3} : Y_1 \land \mu (Y_2 \land \mu Y_3) \to (Y_1 \land \mu Y_2) \land \mu Y_3.$$ (2.18)

In Laplaza’s framework [7] left distributivity of the product over the sum operation is encoded by a monomorphism $$\delta_{Y_0,Y_1,Y_2} : Y_0 \land \mu (Y_1 \vee X Y_2) \to (Y_0 \land \mu Y_1) \vee X (Y_0 \land \mu Y_2)$$. That $$\vee X$$ is a categorical sum enables us to construct an isomorphism $$\delta_{Y_0,Y_1,Y_2}^{-1} : (Y_0 \land \mu Y_1) \vee X (Y_0 \land \mu Y_2) \to Y_0 \land \mu (Y_1 \vee X Y_2)$$ quite easily, as follows. Applying the functor $$\land \mu$$ to the sum diagram $$Y_1 \to Y_1 \vee X Y_2 \leftarrow Y_2$$ provides a diagram $$Y_0 \land \mu Y_1 \to Y_0 \land \mu Y_1 \vee X Y_2 \leftarrow Y_0 \land \mu Y_2$$. Since $$\vee X$$ is a categorical sum, there results a map $$(Y_0 \land \mu Y_1) \vee X (Y_0 \land \mu Y_2) \to Y_0 \land \mu (Y_1 \vee X Y_2)$$. To check that this map is an isomorphism observe that in a simplicial dimension $$p$$ the $$p$$-simplices outside of $$X$$ in the domain are $$(Y_0-X)_p \times (Y_1-X)_p \coprod (Y_0-X)_p \times (Y_2-X)_p$$, the $$p$$-simplices outside of $$X$$ in the target are $$(Y_0-X)_p \times ((Y_1-X)_p \coprod (Y_2-X)_p)$$, and the induced map is a one-to-one correspondence. Thus, we obtain the isomorphism $$\delta_{Y_0,Y_1,Y_2} : (Y_0 \land \mu Y_1) \vee X (Y_0 \land \mu Y_2) \to Y_0 \land \mu (Y_1 \vee X Y_2)$$, whose inverse

$$\delta_{Y_0,Y_1,Y_2}^{-1} : Y_0 \land \mu (Y_1 \vee X Y_2) \cong (Y_0 \land \mu Y_1) \vee X (Y_0 \land \mu Y_2)$$ (2.19)

can be shown to meet LaPlaza’s conditions. Similarly, we obtain an isomorphism

$$\delta_{Y_0,Y_1,Y_2}^{\#} : (Y_0 \vee X Y_1) \land \mu Y_2 \cong (Y_0 \land \mu Y_2) \vee X (Y_1 \land \mu Y_2)$$ (2.20)

This concludes the catalog of basic inputs for LaPlaza’s theorem.

Given the basic inputs, the next step is to establish the commutativity of certain diagrams, twenty-four in number. Because $$\vee X$$ is a categorical sum and $$\land \mu$$ is biexact, preserving sums, checking the commutativity of seventeen of the diagrams is routine. The other seven diagrams involve the multiplicative or additive neutral objects or the multiplicative zero object and are straightforward to verify. LaPlaza’s main theorem applies and “all diagrams that should commute do, in fact, commute.” These remarks complete the proof of part one of theorem 2.1. $\square$

### 3 Defining the operations

The ingredients for the operations take values in categories of retractive spaces on which groups are acting. We first establish language and notation following [5] Definitions 5.1, 5.2, 5.3, and 5.4] for the following definitions.

**Definition 3.1.** A set $$F$$ of subgroups of $$\Sigma_n$$ is called a family of subgroups if it contains at most one member from each conjugacy class of subgroups.

**Definition 3.2.** For a finite group $$G$$, a $$G$$-simplicial set $$Y$$ has orbit types in a family $$F$$ relative to another $$G$$-simplicial set $$W$$ if $$Y$$ may be obtained from $$W$$ by direct limit and by formation of pushouts of diagrams of the form

$$Y \longleftarrow \partial \Delta^n \times (G/H) \longrightarrow \Delta^n \times (G/H),$$ (3.1)

where $$\Delta^n$$ is the standard simplicial $$n$$-simplex, $$\partial \Delta^n$$ is the simplicial boundary, and $$H \in F$$. 

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Definition 3.3. For a $\Sigma_n$-simplicial set $W$ let $\mathcal{R}(W, \Sigma_n, F)$ denote the category whose objects are the triples $(Y, r, s)$ where $Y$ is a $\Sigma_n$-simplicial set with orbit types in $F$ relative to a $\Sigma_n$-section $s: W \to Y$. The map $r: Y \to W$ is $\Sigma_n$-retraction of $Y$ to $W$, that is, $r \circ s = \text{id}_W$. Morphisms are $\Sigma_n$-equivariant maps commuting with the retractions and sections.

Definition 3.4. Let $\mathcal{R}_f(W, \Sigma_n, F)$ denote the full subcategory of $\mathcal{R}(W, \Sigma_n, F)$ whose objects are the triples $(Y, r, s)$ such that $Y$ is built from $W$ by formation of finitely many pushouts of the form of diagram (3.1). The category $\mathcal{R}_f(W, \Sigma_n, F)$ is also equipped with cofibrations and weak equivalences. A cofibration $(W_1, r_1, s_1) \to (W_2, r_2, s_2)$ is an injective $\Sigma_n$-map and a weak equivalence $(Y_1, r_1, s_1) \to (Y_2, r_2, s_2)$ is a morphism for which the geometric realization of the underlying map $Y_1 \to Y_2$ is a $\Sigma_n$-equivariant homotopy equivalence.

For $X$ a connected simplicial abelian group on which $\Sigma_n$ acts trivially, we will need the categories $\mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$ of retractive left $\Sigma_n$-spaces $Y$ over $X$ which are finite relative to $X$. In principle, we may also allow $X$ to be a connected commutative simplicial monoid with unit element. We will write $\Omega|wS\mathcal{R}_f(X, \Sigma_n, \{\text{all}\})| = A_{\Sigma_n, \{\text{all}\}}(X)$. The category of retractive left $\Sigma_n$-spaces on which $\Sigma_n$ acts with trivial isotropy outside of $X$ is then $\mathcal{R}_f(X, \Sigma_n, \{e\})$. In proposition (7.4) we will justify the notation $\Omega|wS\mathcal{R}_f(X, \Sigma_n, \{e\})| = A(X \times BS\Sigma_n)$.

There are two constructions underlying our approach to the Segal operations. First is a family of bi-exact functors

$$\boxtimes_{k, \ell}: \mathcal{R}_f(X, \Sigma_k, \{\text{all}\}) \times \mathcal{R}_f(X, \Sigma_\ell, \{\text{all}\}) \to \mathcal{R}_f(X, \Sigma_{k+\ell}, \{\text{all}\})$$

defined for $k, \ell \geq 0$, called box-tensor operations (Definition 3.5). Second is a family of functors

$$\diamondsuit_{n,k} \cdot \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})^{[k]} \to \mathcal{R}_f(X, \Sigma_{kn}, \{\text{all}\}),$$

called diamond operations (Definition 3.6). Here $\mathcal{R}_f(X, \Sigma_n, \{\text{all}\})^{[k]}$ is the category of filtered objects

$$Y_1 \to Y_2 \to \cdots \to Y_k$$

with $Y_i$ in $\mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$ and natural transformations of such sequences.

First we set up the box-tensor operation. For a connected simplicial abelian group $X$, let $n = k+\ell$ and define an induction functor

$$\text{Ind}_{\Sigma_k \times \Sigma_\ell}^{\Sigma_n}: \mathcal{R}_f(X, \Sigma_k \times \Sigma_\ell, \{\text{all}\}) \to \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}). \quad (3.2)$$

Let $n$ be a finite set of cardinality $n$ (for example, the standard example), let $k \cup l$ be the disjoint union of finite sets of cardinality $k$ and $l$, respectively, and let $\text{Iso}(n, k \cup l)$ be the set of isomorphisms from $n$ to the disjoint union. Let $\text{Iso}(n, k \cup l) = \text{Iso}(n, k \cup l) \cup \{\ast\}$ be viewed as an object of $\mathcal{R}_f(\ast)$, with the obvious section and with the retraction the constant map to $\{\ast\}$. The group $\Sigma_n$ acts from the left on $\text{Iso}(n, k \cup l)$ by fixing the basepoint and by the rule $\sigma \cdot f = f \circ \sigma^{-1}$ for $\sigma \in \Sigma_n$ and $f: n \to k \cup l$. Normally $\Sigma_k \times \Sigma_\ell$ also acts from the left by post-composition, but we find it convenient to use the right action defined by
\( f \cdot (\sigma_1, \sigma_2) = (\sigma_1^{-1}, \sigma_2^{-1}) \circ f \). For \((Y, r, s) \in \mathcal{R}_f(X, \Sigma_k \times \Sigma_\ell, \{\text{all}\})\) we unwind the defining pushout square

\[
\begin{array}{c}
\text{(Iso(n, k\mathcal{U})_+ \times X) \cup_{\ast \times X} (\ast \times Y) } \\
\downarrow \\
\ast \times X \\
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\end{array}
\text{Iso(n, k\mathcal{U})_+ \times Y}
\]

(3.3)

\( X \rightarrow \text{Iso(n, k\mathcal{U})_+} \wedge _e Y \xrightarrow{r} X \) (3.4)

This completes the definition of the induction functor \( \text{Ind}_{\Sigma_n}^{\Sigma_k} \times \Sigma_\ell} : \mathcal{R}_f(X, \Sigma_k \times \Sigma_\ell, \{\text{all}\}) \rightarrow \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})\).

Next we need an elementary pairing functor.

\[
\mathcal{R}_f(X, \Sigma_k, \{\text{all}\}) \times \mathcal{R}_f(X, \Sigma_\ell, \{\text{all}\}) \rightarrow \mathcal{R}_f(X, \Sigma_k \times \Sigma_\ell, \{\text{all}\})
\]

(3.5)

The pairing sends \((f_1, f_2)\) to \((f_1, f_2) \wedge (f_1, f_2)\).

**Definition 3.5.** Define the box-tensor operations following the pairing functor (3.5) with the induction functor (3.4).

\[
\otimes_{k, \ell} : \mathcal{R}_f(X, \Sigma_k, \{\text{all}\}) \times \mathcal{R}_f(X, \Sigma_\ell, \{\text{all}\})
\]

\[
\xrightarrow{\wedge} \mathcal{R}_f(X, \Sigma_k \times \Sigma_\ell, \{\text{all}\}) \xrightarrow{\text{Ind}_{\Sigma_k}^{\Sigma_\ell \times \Sigma_\ell}} \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})
\]

(3.6)

The diamond operation \( \diamondsuit_{k,1} = \diamondsuit_k \) requires some preliminary definitions. First recall the category of filtered objects \( F_k \mathcal{R}_f(X) \) from [13] section 1.1; this is a category with cofibrations and weak equivalences. Let

\[
\underline{P} = (P_1 \rightarrow P_2 \rightarrow \cdots \rightarrow P_k)
\]

be an object of \( F_k \mathcal{R}_f(X) \). For functions \( f, g : k \rightarrow k \) we say \( f \leq g \) if \( f(i) \leq g(i) \) for all \( i \in k \). Let \( I(k) = \{ f : k \rightarrow k \mid \text{There is } \sigma \in \Sigma_k \text{ such that } f \leq \sigma. \} \).

The set \( I(k) \) is partially ordered by \( \leq \), and the sequence \( \underline{P} \) defines a functor \( \overbrace{P : I(k)} \rightarrow \mathcal{R}_f(X) \) by the rule \( P(f) = P_{f(1)} \wedge \mu P_{f(2)} \wedge \mu \cdots \wedge \mu P_{f(k)} \) on objects. We apply the convention
that parentheses in iterated products are collected to the left. In particular, \( P_f(1) \land_\mu P_f(2) \land_\mu P_f(3) \), and, in general,
\[
P_f(1) \land_\mu P_f(2) \land_\mu \cdots \land_\mu P_f(k) = \left( \cdots (P_f(1) \land_\mu P_f(2)) \land_\mu \cdots \land_\mu P_f(k) \right).
\]
For arrows we observe that \( f \leq g \) implies there are cofibrations \( P_f(i) \rightarrow P_g(i) \) which induce a cofibration \( P(f) \rightarrow P(g) \). This depends on the exactness of \( \land_\mu \), proved in theorem 2.1.

**Definition 3.6.** Define the functor \( \diamondsuit_k : \mathcal{F}_k \mathcal{R}_f(X) \rightarrow \mathcal{R}(X, \Sigma_k, \{\text{all}\}) \) on objects by making a choice of \( \operatorname{colim}_{I(k)} P \) and setting
\[
\diamondsuit_k(P) = \operatorname{colim}_{I(k)} P.
\]
The definition extends to arrows by the universal property of the colimit. The \( \Sigma_k \) action is induced by the permutation of factors.

**Example 3.7.** Applied to a constant cofibration sequence \( Y = (Y \xrightarrow{=} Y \xrightarrow{=} \cdots \xrightarrow{=} Y) \) of length \( k \), we obtain simply
\[
\diamondsuit_k(Y) = Y \land_\mu Y \land_\mu \cdots \land_\mu Y
\]
with the group \( \Sigma_k \) permuting the factors. Thus, the purpose of \( \diamondsuit_k \) is to extend \( \land_\mu \)-powers to filtered objects.

**Definition 3.8.** The generalized diamond operation
\[
\diamondsuit_{n,k} : \mathcal{F}_k \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \rightarrow \mathcal{R}_f(X, \Sigma_{nk}, \{\text{all}\})
\]
is a composition
\[
\diamondsuit_{n,k} : \mathcal{F}_k \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \xrightarrow{\diamondsuit_k} \mathcal{R}_f(X, B_{n,k}, \{\text{all}\}) \xrightarrow{\operatorname{Ind}_{B_{n,k}}^{\Sigma_{nk}}} \mathcal{R}_f(X, \Sigma_{nk}, \{\text{all}\}),
\]
with a basic diamond operation \( \diamondsuit_k \) followed by an induction construction \( \operatorname{Ind}_{B_{n,k}}^{\Sigma_{nk}} \). The intermediate group \( B_{n,k} \) is the group of block permutations of \( nk \) objects blocked into \( k \) groups of \( n \) objects. Thus, the group \( B_{n,k} \) is a wreath product: \( B_{n,k} = \Sigma_k \wr \Sigma_n \). Explicitly, there is a short exact sequence of groups \( 1 \rightarrow (\Sigma_n)^k \rightarrow B_{n,k} \rightarrow \Sigma_k \rightarrow 1 \).

To ensure that these operations behave properly we make the following definition and observe that cofibrations in \( \mathcal{R}_f(X) \) do have this property.

**Definition 3.9.** (Compare Definition 4.3, [1, p.274].) A category \( \mathcal{C} \) with cofibrations has the extension property if for all commutative diagrams of cofibration sequences
\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow i \\
A' & \rightarrow & B'
\end{array}
\begin{array}{ccc}
& & C \\
& \downarrow & \\
& C'
\end{array}
\]
in \( \mathcal{C} \), with vertical cofibrations as indicated, it follows that the middle arrow \( i \) is also a cofibration.
Lemma 3.10. The categories $\mathcal{R}_f(X, \Sigma_n, \mathcal{F})$ with cofibrations have the extension property.

Proof. Because we are working inside $\mathcal{R}(X)$ with simplicial sets, cofibrations are the injective maps. Therefore, the extension property holds.

Recall $G_\bullet$ briefly here, following [4]. For a simplicial set $Z$ the corresponding simplicial path set $PZ$ is defined by $PZ_n = Z_{n+1}$. The face operator $d_i: PZ_n \to PZ_{n-1}$ coincides with $d_{i+1}: Z_{n+1} \to Z_n$; the degeneracy operator $s_i: PZ_n \to PZ_{n+1}$ coincides with $s_{i+1}: Z_{n+1} \to Z_{n+2}$. The face operator $d_0: Z_{n+1} \to Z_n$ induces a simplicial map $d_0: PZ \to Z$. The simplicial set $PZ$ is simplicially homotopy equivalent to the constant simplicial set $Z_0$ [13, Lemma 1.5.1, p.328]. Viewing $Z_1 = PZ_0$ as another constant simplicial set provides a simplicial map $Z_1 \to PZ$.

Definition 3.11. [4, p.257] For a category $\mathcal{C}$ with cofibrations and weak equivalences the simplicial category $G_\bullet \mathcal{C}$ is defined by the cartesian square

$$
\begin{array}{ccc}
wG\mathcal{C} & \longrightarrow & PwS\mathcal{C} \\
\downarrow & & \downarrow \text{d_0} \\
PwS\mathcal{C} & \longrightarrow & wS\mathcal{C}
\end{array}
$$

(3.7)

Recalling a few more details from [4], $G_\bullet \mathcal{C}$ has cofibrations and weak equivalences. Since $G_n\mathcal{C} = (P\mathcal{C} \times \mathcal{C})_n = \mathcal{C}_{n+1} \times \mathcal{C}$, the weak equivalences and cofibrations in $wG_\bullet \mathcal{C}$ are given by pullback. There is also a stabilization map $j: \mathcal{C} \to G_\bullet \mathcal{C}$, where $\mathcal{C}$ is viewed as a constant simplicial category with cofibrations and weak equivalences, defined as follows. We have the map $\mathcal{C} = (Pw\mathcal{C})_0 \to Pw\mathcal{C}$ and the constant map $\mathcal{C} \to Pw\mathcal{C}$ carrying $\mathcal{C}$ to the terminal object. These two combine to give an inclusion $j: \mathcal{C} \to G_\bullet \mathcal{C}$. After passing to diagonals, the construction may be iterated so there results a stabilization sequence

$$
\mathcal{C} \to G_\bullet \mathcal{C} \to G_\bullet (G_\bullet \mathcal{C}) \to \cdots \to G_\bullet \mathcal{C} := G(G_\bullet \mathcal{C}) \to \cdots \to \operatorname{colim}_n G_\bullet \mathcal{C} := G_\infty \mathcal{C}
$$

of simplicial categories with cofibrations and weak equivalences. Returning to the square (3.7), after passage to nerves in the $w$-direction, diagonalization, and geometric realization, there results a natural map

$$
|wG_\bullet \mathcal{C}| \to \Omega|wS_\bullet \mathcal{C}|.
$$

This may not always be a homotopy equivalence, but it is a homotopy equivalence when $\mathcal{C}$ has a property called pseudo-additivity [4, Definition 2.3 and Theorem 2.6]. In our case, with $\mathcal{C} = \mathcal{R}_f(X)$ we follow [4] to achieve the pseudo-additivity property by using the cylinder functor defined in [13, Section 1.6]. The cylinder functor induces a cone functor $c: \mathcal{R}_f(X) \to \mathcal{R}_f(X)$ and a suspension functor $\Sigma: \mathcal{R}_f(X) \to \mathcal{R}_f(X)$ so that we may define a category of prespectra

$$
\Sigma^\infty \mathcal{R}_f(X) = \operatorname{colim}(\mathcal{R}_f(X) \xrightarrow{\Sigma} \mathcal{R}_f(X) \xrightarrow{\Sigma} \mathcal{R}_f(X) \xrightarrow{\Sigma} \cdots).
$$

Then $\Sigma^\infty \mathcal{R}_f(X)$ has the pseudo-additivity property [4, Remark 2.4 and Lemma 2.5, p.258–259] so

$$
|wG_\bullet \Sigma^\infty \mathcal{R}_f(X)| \to \Omega|wS_\bullet \Sigma^\infty \mathcal{R}_f(X)|
$$

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is a weak homotopy equivalence. Additionally, by [13, Proposition 1.6.2], \(|wS_\bullet R_f(X)| \rightarrow |wS_\bullet \Sigma^\infty R_f(X)|\) is a weak homotopy equivalence.

Additionally, we need the fact that there are weak homotopy equivalences

\[ |wG^\infty C| \rightarrow \Omega|wG^\infty S_\bullet C| \leftarrow \Omega|wS_\bullet C| \]  

for any category \(C\) with cofibrations and weak equivalences [14, Theorem 4.1, p.268].

The \(G_\bullet C\) construction has an explicit description as a category of exact functors. For full details refer to [5] and [2]. First, extend the partially ordered set \(A \in \Delta\) to the set \(\gamma(A) = \{L, R\} \coprod A\) with the ordering in which \(L\) and \(R\) are not comparable; \(L < a\) and \(R < a\) for every \(a \in A\); and, for \(a, a' \in A\), \(a < a'\) in \(\gamma(A)\) if and only if \(a < a'\) in \(A\). Pictorially, for \(A = [n]\), \(\gamma(A)\) looks like

\[
\begin{array}{c}
L \\
\nearrow
\end{array}
0 \rightarrow 1 \rightarrow \cdots \rightarrow n
\begin{array}{c}
R \\
\searrow
\end{array}
\]

The category \(\Gamma(A)\) is the category of arrows in \(\gamma(A)\), omitting the identity arrows on \(L\) and \(R\). Diagrammatically, \(\Gamma(A)\) looks like

\[
\begin{array}{c}
0/L \rightarrow 1/L \rightarrow \cdots \rightarrow n/L \\
0/R \rightarrow 1/R \rightarrow \cdots \rightarrow n/R \\
0/0 \rightarrow 1/0 \rightarrow \cdots \rightarrow n/0 \\
1/1 \rightarrow \cdots \rightarrow n/1 \\
\cdots \\
n/n.
\end{array}
\]

Here \(a/b\) stands for \(b \rightarrow a\) (or \(b < a\)), and an arrow \(a/b \rightarrow c/d\) stands for a square \(a \rightarrow c\) \(b \rightarrow d\) in \(\gamma(A)\). The exact sequences in \(\Gamma(A)\) are sequences \(j/k \rightarrow i/k \rightarrow i/j\) where \(k \rightarrow j \rightarrow i\) in \(\gamma(A)\). Then, for \(A \in \Delta\),

\[ G_A C = \text{Exact}(\Gamma(A), C). \]

Since \(\Gamma(A)\) is functorial in \(A\), preserving exact sequences \(j/k \rightarrow i/k \rightarrow i/j\), we have another description of \(G_\bullet C : \Delta^{op} \rightarrow \text{Cat}\).

Several more constructions are necessary before we can define the operations. For full details consult [2] and [5, Section 2].

**Definition 3.12** (Compare 2.1, p.268, [5] and [2]). Let \(Z\) be a simplicial object in a category \(D\). Define a concatenation operation \(\text{con} : \Delta^k \rightarrow \Delta\). For a sequence \((A_1, \ldots, A_k)\) of finite non-empty ordered sets, order their disjoint union \(A_1 \coprod \cdots \coprod A_k\) so that the subset \(A_i\) inherits

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the original order and so that, if \(i \leq j\) and \(a_i \in A_i\) and \(a_j \in A_j\), then \(a_i < a_j\). Then define the \(k\)-fold edgewise subdivision of a simplicial object \(Z\) to be the composite functor

\[
\text{sub}_k Z : \Delta^k \rightarrow \Delta \rightarrow Z \rightarrow D
\]

For a simplicial set \(Z\) there is a natural homeomorphism \(|\text{sub}_k Z| \rightarrow |Z|\), [2, section 4].

**Definition 3.13** (Section 4, [2]). For \(A \in \Delta\), define inductively a sequence of categories \(\Gamma^k(A)\). Put \(\Gamma^1(A) = \Gamma(A)\). For \(k > 1\) define \(\text{Ob}(\Gamma^k(A)) \subset \text{Ob}(\Gamma(A)) \times (\{\odot, \boxdot\} \times \Gamma(A))^{k-1}\) to be the subset consisting of elements \(\alpha = (i_1/\ell_1, *_{2}/\ell_2, *_{3}, \ldots, *_{k}, i_{k}/\ell_{k})\) where

\[
(\text{A1}): \ i_r \in \gamma(A), \ \ell_r \in \gamma(A), \ \text{and} \ \ast_r \in \{\odot, \boxdot\}.
\]

for all \(r \in \{1, \ldots, k\}\).

\[
(\text{A2}): \ \ell_r \leq i_r \ \text{and} \ i_r \in A.
\]

\[
(\text{A3}): \ \text{if} \ r = \odot \ \text{and} \ \text{r} > 1, \ \text{then} \ \ell_{r-1} = \ell_r \ \text{and} \ \ell_{r-1} \leq i_r.
\]

The arrows in \(\Gamma^k(A)\) are given by the requirements that for \(\alpha, \alpha' \in \text{Ob}(\Gamma^k(A))\) there is one and only one arrow \(\alpha \rightarrow \alpha'\), if

\[
(\text{B1}): \ i_r \leq i'_r,
\]

for all \(r \in \{1, \ldots, k\}\).

\[
(\text{B2}): \ \ell_r \leq \ell'_r,
\]

\[
(\text{B3}): \ \text{if} \ \ast_r = \odot \ \text{and} \ \ast_r' = \boxdot, \ \text{then} \ i_{r-1} \leq \ell'_r.
\]

There is no map if \(\ast_r = \boxdot\) and \(\ast_r' = \odot\). This definition implies that \(\Gamma^k(A)\) is also a partially ordered set.

A sequence \(\alpha' \rightarrow \alpha \rightarrow \alpha''\) in \(\Gamma^k(A)\) is exact, if there exist integers \(r \leq s\) such that

\[
(\text{C1}): \ \text{for any} \ p < r \ \text{or} \ s < p \ \text{we have} \ i'_p = i_p = i''_p, \ \ell'_p = \ell_p = \ell''_p \ \text{and} \ \ast'_p = \ast_p = \ast''_p,
\]

\[
(\text{C2}): \ \text{for any} \ p \ \text{satisfying} \ r < p \leq s, \ \text{we have} \ i'_p = \ast_p = i''_p = \odot \ \text{and} \ i'_p = i_p = i''_p,
\]

\[
(\text{C3}): \ \ell_r = \ell'_r \leq i'_r = \ell''_r, \ \ast_r = \ast_r', \ \text{and} \ \ast''_r = \boxdot.
\]

The operations we seek are extracted from \(\Gamma^k(A)\) using the following functor.

**Definition 3.14** (Section 5, [2]). For \(A_1, \ldots, A_k \in \Delta\), let \(A_1 \ldots A_k\) be the concatenation, and define

\[
\Xi = \Xi_k : \Gamma(A_1) \times \cdots \times \Gamma(A_k) \rightarrow \Gamma^k(A_1 \ldots A_k)
\]

on objects by \(\Xi_k(i_1/j_1, \ldots, i_k/j_k) = (i_1/\ell_1, *_{2}/\ell_2, *_{3}, \ldots, *_{k}, i_{k}/\ell_{k})\), where \(\ell_1 = j_1\) and, inductively for \(r > 1\), we put

\[
(\text{D1}): \ \text{if} \ j_r = L, \ \text{then} \ \ast_r = \odot \ \text{and} \ \ell_r = \ell_{r-1},
\]

\[
(\text{D2}): \ \text{if} \ j_r \neq L, \ \text{then} \ \ast_r = \boxdot \ \text{and} \ \ell_r = j_r.
\]

**Theorem 3.15** (Section 6 of [2]). The category \(\Gamma^k(A)\) is a category with exact sequences, natural in the variable \(A\); the functor \(\Xi_k\) is multi-exact, i.e., exact in each variable separately, and is natural in each of the variables \(A_1, \ldots, A_k \in \Delta\). \(\square\)
The operations $\{\boxtimes_{k,\ell}\}$ for $k, \ell \geq 0$ and $\{\nabla_{n,k}\}$ for $n, k \geq 0$ satisfy a number of compatibility conditions codified by Grayson [2]. We abbreviate

$$\nabla_{n,k}(Y_1 \rightarrow \cdots \rightarrow Y_k) = Y_1 \cdots \nabla Y_k$$

• (E1) Given $V \rightarrow \cdots \rightarrow W \rightarrow X \rightarrow \cdots \rightarrow Y$ there is a natural map

$$(V \circ \cdots \circ W) \boxtimes (X \circ \cdots \circ Y) \rightarrow V \circ \cdots \circ W \circ X \circ \cdots \circ Y$$

The family of such maps is associative in the obvious sense.

• (E2) Given $V \rightarrow \cdots \rightarrow W \rightarrow X \rightarrow \cdots \rightarrow Y$ there is a natural map

$$V \circ \cdots \circ W \circ X \circ \cdots \circ Y \rightarrow (V \circ \cdots \circ W) \boxtimes (X/W \circ \cdots \circ Y/W)$$

This family of maps is also associative in the obvious sense. Moreover, the map exists for any choice of quotient objects.

• (E3) Given $U \rightarrow \cdots \rightarrow V \rightarrow W \rightarrow \cdots \rightarrow X \rightarrow Y \rightarrow \cdots \rightarrow Z$ the following diagram commutes.

$$\begin{array}{ccc}
(U \circ \cdots \circ V \circ W \circ \cdots \circ X) \boxtimes (Y \circ \cdots \circ Z) & \rightarrow & (U \circ \cdots \circ V \circ W \circ \cdots \circ X \circ Y \circ \cdots \circ Z) \\
\downarrow & & \downarrow \\
(U \circ \cdots \circ V) \boxtimes ((W/V \circ \cdots \circ X/V) \boxtimes (Y/V \circ \cdots \circ Z/V)) & \rightarrow & U \circ \cdots \circ V \boxtimes (W/V \circ \cdots \circ X/V \circ Y/V \circ \cdots \circ Z/V)
\end{array}$$

• (E4) Given $U \rightarrow \cdots \rightarrow V \rightarrow W \rightarrow \cdots \rightarrow X \rightarrow Y \rightarrow \cdots \rightarrow Z$ the following diagram commutes.

$$\begin{array}{ccc}
(U \circ \cdots \circ V) \boxtimes (W \circ \cdots \circ X \circ Y \circ \cdots \circ Z) & \rightarrow & (U \circ \cdots \circ V \circ W \circ \cdots \circ X \circ Y \circ \cdots \circ Z) \\
\downarrow & & \downarrow \\
(U \circ \cdots \circ V \circ W \circ \cdots \circ X) \boxtimes (Y/X \circ \cdots \circ Z/X) & \rightarrow & U \circ \cdots \circ V \circ W \circ \cdots \circ X \circ Y \circ \cdots \circ Z
\end{array}$$

• (E5) Given $U \rightarrow \cdots \rightarrow V \rightarrow W' \rightarrow W \rightarrow X \rightarrow Y$ the sequence

$$U \circ \cdots \circ V \circ W' \circ X \circ \cdots \circ Y \rightarrow U \circ \cdots \circ V \circ W \circ X \circ \cdots \circ Y$$

is an exact sequence.
Proposition 3.16 (Proposition 4.5 of [5]). The box tensor operations are associative up to natural isomorphism. The box tensor operations and the diamond operations fulfill properties (E1) through (E5).

Proof. The associativity of the box tensor operations is a consequence of the symmetric monoidal structure on $\mathcal{R}_f(X)$ associated with $\wedge_\mu$, along with properties of the cartesian product of groups and disjoint union of sets. In detail, the assertion is that the diagram

$$
\begin{array}{ccc}
\mathcal{R}_f(X, \Sigma_{k_1}, \{\text{all}\}) \times \mathcal{R}_f(X, \Sigma_{k_2}, \{\text{all}\}) \times \mathcal{R}_f(X, \Sigma_{k_3}, \{\text{all}\}) & \xrightarrow{\varepsilon_{k_1,k_2} \times \text{id}} & \mathcal{R}_f(X, \Sigma_{k_1+k_2}, \{\text{all}\}) \\
\mathcal{R}_f(X, \Sigma_{k_1+k_2}, \{\text{all}\}) \times \mathcal{R}_f(X, \Sigma_{k_3}, \{\text{all}\}) & \xrightarrow{\text{id} \times \varepsilon_{k_2,k_3}} & \mathcal{R}_f(X, \Sigma_{k_1+k_2+k_3}, \{\text{all}\}) \\
\mathcal{R}_f(X, \Sigma_{k_1+k_2+k_3}, \{\text{all}\}) & \xrightarrow{\varepsilon_{k_1+k_2+k_3}} & \mathcal{R}_f(X, \Sigma_{k_1+k_2+k_3}, \{\text{all}\})
\end{array}
$$

commutes up to canonical isomorphism. Given a triple $(Y_1, Y_2, Y_3)$ in the category at top of the diagram, the value of the lefthand sequence of arrows is

$$
\text{Iso}(k_1+k_2+k_3, k_1+k_2 \cup k_3) + \wedge_{\Sigma_{k_1+k_2} \times \Sigma_{k_3}} \left( \left( \text{Iso}(k_1+k_2, k_1 \cup k_2) + \wedge_{\Sigma_{k_1} \times \Sigma_{k_2}} Y_1 \wedge_\mu Y_2 \right) \wedge_\mu Y_3 \right),
$$

and we claim this space is isomorphic to

$$
\text{Iso}(k_1+k_2+k_3, (k_1 \cup k_2) \cup k_3) + \wedge_{(\Sigma_1 \times \Sigma_2) \times \Sigma_3} (Y_1 \wedge_\mu Y_2) \wedge_\mu Y_3. \tag{3.9}
$$

To clarify the notations, $k_1+k_2+k_3$ denotes the standard finite set of cardinality $k_1+k_2+k_3$, $(k_1+k_2 \cup k_3)$ denotes the disjoint union of finite sets of cardinality $k_1+k_2$ and $k_3$, and so on. Similarly, the value of the righthand sequence of arrows is

$$
\text{Iso}(k_1+k_2+k_3, k_1 \cup (k_2 \cup k_3)) + \wedge_{\Sigma_{k_1} \times \Sigma_{k_2+k_3}} \left( Y_1 \wedge_\mu \left( \text{Iso}(k_2+k_3, k_2 \cup k_3) + \wedge_{\Sigma_{k_2} \times \Sigma_{k_3}} Y_2 \wedge_\mu Y_3 \right) \right),
$$

which we claim is isomorphic to

$$
\text{Iso}(k_1+k_2+k_3, k_1 \cup (k_2 \cup k_3)) + \wedge_{\Sigma_1 \times (\Sigma_2 \times \Sigma_3)} Y_1 \wedge_\mu (Y_2 \wedge_\mu Y_3). \tag{3.10}
$$

The spaces in (3.9) and (3.10) are isomorphic by combining the associativity isomorphisms for disjoint union, cartesian products of groups, and the smash product $\wedge_\mu$. Thus, we have proved that the box tensor operations are naturally associative, granting the two isomorphisms.

To see one of these isomorphisms requires several steps. We concentrate on the first case, since the second is completely parallel. First, since $\text{Iso}(k_3, k_3) = \Sigma_{k_3}$, there is an isomorphism

$$
\text{Iso}(k_3, k_3) + \wedge_{\Sigma_{k_3}} Y_3 \xrightarrow{\sigma} Y_3 \tag{3.11}
$$

in $\mathcal{R}_f(X, \Sigma_3)$ induced by the formula $[f_3, y] \mapsto f_3^{-1} y$. With the right action of $\Sigma_{k_3}$ on $\text{Iso}(k_3, k_3)$ given by $f \cdot \sigma = \sigma^{-1} \circ f$, we have $[f_3 \cdot \sigma, y] \mapsto (\sigma^{-1} f_3)^{-1} y = f_3^{-1} \sigma y$; starting from
\([f_3, \sigma y]\), we also arrive at \(f_3^{-1}\sigma y\). Thus, a map \(\text{Iso}(k_3, k_3) \circ \Sigma_{k_3} Y_3 \to Y_3\) exists. Surjectivity is clear. For injectivity, if \([f_3, y]\) and \([f_3', y']\) map to the same element of \(Y\), we have \(f_3^{-1}y = (f_3')^{-1}y'\). Putting \(\sigma = f_3f_3^{-1}\), we have \(y' = \sigma y\) and \(f_3' = f_3(f_3')^{-1}f_3' = f_3\), so \([f_3, y] = [f_3', \sigma y] \sim [f_3', y']\). For equivariance, recall that the left action of \(\Sigma_{k_3}\) on \(\text{Iso}(k_3, k_3)\) is given by \(\sigma \cdot f_3 = f \circ \sigma^{-1}\). Thus,

\[
[\sigma \circ f, y] = [f \circ \sigma^{-1}, y] \mapsto (f \circ \sigma^{-1})^{-1}y = \sigma(f^{-1}y)
\]

shows equivariance.

Consequently,

\[
\text{Iso}(k_1 + k_2 + k_3, k_1 + k_2 \cup k_3) \circ \wedge_{\Sigma_{k_1 + k_2} \times \Sigma_{k_3}} \left( (\text{Iso}(k_1 + k_2, k_1 \cup k_2) \circ \wedge_{\Sigma_{k_1} \times \Sigma_{k_2}} Y_1 \wedge_{\mu} Y_2) \wedge_{\mu} \text{Iso}(k_3, k_3) \circ \wedge_{\Sigma_{k_3}} Y_3 \right)
\]

is isomorphic to

\[
\text{Iso}(k_1 + k_2 + k_3, k_1 + k_2 \cup k_3) \circ \wedge_{\Sigma_{k_1 + k_2} \times \Sigma_{k_3}} \left( (\text{Iso}(k_1 + k_2, k_1 \cup k_2) \circ \wedge_{\Sigma_{k_1} \times \Sigma_{k_2}} Y_1 \wedge_{\mu} Y_2) \wedge_{\mu} \text{Iso}(k_3, k_3) \circ \wedge_{\Sigma_{k_3}} Y_3 \right)
\]

Applying a commutativity isomorphism of the product \(\wedge_{\Sigma}\), this object is isomorphic to

\[
\left( (\text{Iso}(k_1 + k_2, k_1 \cup k_2) \circ \wedge_{\Sigma_{k_1} \times \Sigma_{k_2}} Y_1 \wedge_{\mu} Y_2) \wedge_{\mu} \text{Iso}(k_3, k_3) \circ \wedge_{\Sigma_{k_3}} Y_3 \right)
\]

\[
\circ (\text{Iso}(k_1 + k_2 + k_3, k_1 + k_2 \cup k_3) \circ \wedge_{\Sigma_{k_1 + k_2} \times \Sigma_{k_3}} \text{Iso}(k_3, k_3) \circ \wedge_{\Sigma_{k_3}} Y_3)
\]

Now we claim there is an isomorphism

\[
\left( (\text{Iso}(k_1 + k_2 + k_3, k_1 + k_2 \cup k_3) \circ \wedge_{\Sigma_{k_1 + k_2} \times \Sigma_{k_3}} \text{Iso}(k_3, k_3) \circ \wedge_{\Sigma_{k_3}} Y_3) \right)
\]

\[
\cong \text{Iso}(k_1 + k_2 + k_3, k_1 + k_2 \cup k_3) \circ \wedge_{\Sigma_{k_1 + k_2} \times \Sigma_{k_3}} \text{Iso}(k_3, k_3) \circ \wedge_{\Sigma_{k_3}} Y_3
\]

induced by the formula \([f_{123}, [f_{12}, f_3]] \mapsto (f_{12}, f_3) \circ f_{123}\). We check that balanced expressions in

\[
(\text{Iso}(k_1 + k_2 + k_3, k_1 + k_2 \cup k_3) \circ \wedge_{\Sigma_{k_1 + k_2} \times \Sigma_{k_3}} \text{Iso}(k_3, k_3) \circ \wedge_{\Sigma_{k_3}} Y_3)
\]

map to the same element of the target.

\[
[f_{123} \cdot (g_{12}, g_3), [f_{12}, f_3]] = [(g_{12}^{-1}, g_3^{-1}) \circ f_{123}, [f_{12}, f_3]] \mapsto (f_{12}, f_3) \circ ((g_{12}^{-1}, g_3^{-1}) \circ f_{123})
\]

On the other hand,

\[
[f_{123}, (g_{12}, g_3) \cdot [f_{12}, f_3]] = [f_{123}, [f_{12}, g_{12}^{-1}, f_3 \circ g_3^{-1}]] \mapsto (f_{12} \circ g_{12}^{-1}, f_3 \circ g_3^{-1}) \circ f_{123}
\]

and these expressions are the same, by associativity of composition of functions. Now suppose that \([f_{123}, [f_{12}, f_3]]\) and \([f'_{123}, [f'_{12}, f'_3]]\) map to the same isomorphism. The equation \((f_{12}, f_3) \circ \)
The collection of operations

Definition 3.17. The collection of operations \( \circ \) and \( \otimes \) define functors

\[
\Lambda^k_{\circ,\otimes} : \text{Exact}(\Gamma(A), \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})) \rightarrow \text{Exact}(\Gamma^k(A), \mathcal{R}_f(X, \Sigma_{n,k}, \{\text{all}\}))
\]

Given \( M \in \text{Exact}(\Gamma(A), \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})) \) and given \( \alpha = (i_1/\ell_1, *_2, i_2/\ell_2, *_3, \ldots, *_k, i_k/\ell_k) \in \text{Ob}(\Gamma^k(A)) \), define \( \Lambda^k M = \Lambda^k_{\circ,\otimes} M \) on objects by the formula

\[
\Lambda^k M((i_1/\ell_1, *_2, i_2/\ell_2, *_3, \ldots, *_k, i_k/\ell_k)) = M(i_1/\ell_1) *_2 M(i_2/\ell_2) *_3 \cdots *_k M(i_k/\ell_k)
\]

where \( \circ \) has precedence over \( \otimes \) and left association is used for \( \otimes \). The extension to arrows is natural.

Remark 3.18. By property (A3) of definition 3.13 the formula for the value of \( \Lambda^k_{\circ,\otimes} \) on an object makes sense. Properties (E1) through (E4) in theorem 3.16 ensure that the formulas on arrows yield a well-defined functor. Property (E5) of the same theorem ensures that the functors \( \Lambda^k_{\circ,\otimes} \) carry an exact functor \( M \) to another exact functor. Compare [p.256–257, [2]].

Definition 3.19. The ingredients \( \omega^k \) for the total Segal operation are defined as follows.

\[
\begin{array}{c}
\text{Exact}(\Gamma(A_1 \ldots A_k), \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})) \\
\omega^k \\
\text{Exact}(\Gamma(A_1) \times \cdots \times \Gamma(A_k), \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}))
\end{array}
\]

The result is a family of functors

\[
\omega^k : w \text{sub}_k G_* \mathcal{R}_f(X) \rightarrow w G_*^k \mathcal{R}_f(X, \Sigma_k, \{\text{all}\})
\]
Consequently, for each $k$ which sends $(G_0$ can be identified with $k$ indicates the constant filtration of the zero object. Since sub where the top row indicates constant filtered object in $\mathcal{R}_f(X) \rightarrow \text{sub}_k G_\bullet \mathcal{R}_f(X)$

which sends $(Y, r, s)$ to the diagram

\[
\begin{array}{c}
Y = Y = \cdots = Y \\
X = X = \cdots = X
\end{array}
\]

where the top row indicates constant filtered object in $(PS_\bullet \mathcal{R}_f(X))_k$ and the bottom row indicates the constant filtration of the zero object. Since $\text{sub}_k G_\bullet \mathcal{C}$ in simplicial dimension 0 can be identified with $G_k \mathcal{C}$ this object also admits the description $(s_0)^k(j(Y))$, where $s_0^k : G_0 \mathcal{R}_f(X) \rightarrow G_k \mathcal{R}_f(X)$ is the iterated degeneracy.

**Example 3.20.** The formula for the composite

\[
\tilde{\alpha}^k_1 : \mathcal{R}_f(X) \rightarrow \text{sub}_k G_\bullet \mathcal{R}_f(X) \xrightarrow{\omega^k} G_\bullet \mathcal{R}_f(X, \Sigma_k, \{\text{all}\})
\]

is the functor $\Gamma([0])^k \rightarrow \mathcal{R}_f(X, \Sigma_k, \{\text{all}\})$ given by

\[
\begin{cases}
Y \wedge_\mu Y \wedge_\mu \cdots \wedge_\mu Y, & \text{in positions } 0/L, 0'/L', \ldots, 0^{(k)}/L^{(k)}, \\
X, & \text{in all other positions}.
\end{cases}
\]

## 4 \(E_\infty\)-structure and restriction to spherical objects

We have already seen that, in order to obtain the algebraic $K$-theory of spaces using the $G_\bullet$-model, one uses a category of prespectra $\Sigma^\infty \mathcal{R}_f(X)$ obtained from $\mathcal{R}_f(X)$ by passage to a limit using a suspension operation. We are now going to deal with natural transformations of semi-group valued functors $[-, \mathcal{R}_f(X)] \rightarrow [-, \{1\} \times \prod_{n \geq 1} A_{\Sigma_n, \{\text{all}\}}(X)]$, where the target is an abelian-group-valued functor. First we restrict to categories of $n$-spherical objects $\mathcal{R}_f^n(X)$, whose definition is recalled below. Segal’s group completion theorem [9 Proposition 4.1] provides a unique natural transformation of abelian-group-valued functors $[-, \Omega hN_\Gamma \mathcal{R}_f^n(X)] \rightarrow [-, \{1\} \times \prod_{n \geq 1} A_{\Sigma_n, \{\text{all}\}}(X)]$. In the domain, $hN_\Gamma \mathcal{R}_f^n(X)$ is the simplicial category arising from the categorical sum operation $\vee$, as described in [13 Section 1.8], and maps are weak homotopy equivalences. The following diagram displays this result as the diagonal arrow.

\[
\begin{array}{ccc}
[-, \Omega hN_\Gamma \mathcal{R}_f^n(X)] & \leftarrow & [-, h\mathcal{R}_f^n(X)] \\
\downarrow & & \downarrow \\
[-, A(X)] & \rightarrow & [-, \{1\} \times \prod_{n \geq 1} A_{\Sigma_n, \{\text{all}\}}(X)]
\end{array}
\]

\[
\begin{array}{c}
\xrightarrow{\omega}
\end{array}
\]

\[
\begin{array}{c}
\xrightarrow{(4.1)}
\end{array}
\]
In this section we show that the diagonal arrow is induced by an $E_\infty$-map $\Omega|hN_1R^n_f(X)| \to \{1\} \times \prod_{n \geq 1} A_{\Sigma_n,\{\text{all}\}}(X)$. But we want a natural transformation of abelian-group-valued functors $[-, A(X)] \to [-, 1 \times \prod_{n \geq 1} A_{\Sigma_n,\{\text{all}\}}(X)]$ as displayed by the lower horizontal arrow in the diagram, and we want it to be induced by an $E_\infty$-map $A(X) \to \{1\} \times \prod_{n \geq 1} A_{\Sigma_n,\{\text{all}\}}(X)$. There is a natural chain of equivalences
\[
\lim_{n \to \infty} hN_\bullet R^n(X) \simeq hS_\bullet R_f(X) \simeq hS_\bullet \Sigma\infty R_f(X),
\]
where the colimit is taken over suspension relative to $X$ [13, Theorems 1.7.1 and 1.8.1]. This implies we have to examine the behavior of our constructions as they relate to suspension, which we analyse in section 5.

We recall from [13, Section 1.7, p.360] a definition of spherical objects in the category $\mathcal{R}_f(X)$, where $X$ is a connected space. On this category we have the homology theory $H_n(Y, r, s) = H_n(Y, s(X); r^*(\mathbb{Z}[\pi_1 X]))$ (homology with local coefficients), and we say $(Y, r, s)$ is $n$-spherical if $H_n(Y, r, s) = 0$ for $q \neq n$ and $H_n(Y, r, s)$ is a stably-free $\mathbb{Z}[\pi_1 X]$-module. For $n \geq 0$ denote by $\mathcal{R}^n_f(Y)$ the full subcategory of $\mathcal{R}_f(X)$ whose objects are $n$-spherical. For example, in case $X$ is a connected simplicial abelian group, $\mathcal{R}^n_f(X)$ contains spaces homotopy equivalent to retractive spaces $(Y, r, s)$ obtained by completing to pushouts diagrams of the form
\[
X \xleftarrow{\phi_1} \bigvee_{i=1}^N \partial \Delta^n \xrightarrow{\phi_i} \Delta^n
\]
where the attaching maps $\phi_i$ are constant maps to the identity element of $X$.

Let $\mathbb{N}$ be the natural numbers and $F$ the category of finite subsets of $\mathbb{N}$ and injections. Let $F_+ \subset F$ be the full subcategory of non-empty finite subsets. Let $\prod$ denote the associative sum on $F_+$ given by
\[
\{x_i|1 \leq i \leq m\} \prod \{y_j|1 \leq j \leq n\} = \{x_i|1 \leq i \leq m\} \cup \{y_j + x_m - y_1 + 1|1 \leq j \leq n\},
\]
where we assume $x_1 < \cdots < x_m$ and $y_1 < \cdots < y_n$.

**Lemma 4.1.** The category $F_+$ is contractible. 

We work under the assumption that the category $\mathcal{C}$ and the categories $\mathcal{C}_{\Sigma_n}$ satisfy the extension property for cofibrations. This has been verified for $\mathcal{R}_f(X)$ and $\mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$ in lemma 3.10. Under this assumption we may identify the iterated $G_\bullet$-construction $G^n_\bullet \mathcal{C}$ with $\text{Exact}(\Gamma(-)^n, \mathcal{C})$. We extend $n \mapsto \text{Exact}(\Gamma(-)^n, \mathcal{C}) = G^n_\bullet \mathcal{C}$ to $G_\bullet^x : F \to \cat{Alt}^{\Delta^\text{op}}$ following the recipe in [5]. Thus, on an object $x \in \text{Ob}(F_+)$ put $G^x_A := \text{Exact}(\Gamma(A)^x, \mathcal{C})$. We use the adjoinness relation, or diagonals, so that
\[
G^3_A(\mathcal{C}) := \text{Exact}(\Gamma(A)^x, \mathcal{C}) = \text{Exact}(\Gamma(A) \times \Gamma(A), \text{Exact}(\Gamma(A), \mathcal{C})) = \cdots = G_A(G_A(G_A(\mathcal{C}))),
\]
for example. To obtain the extension to $F$, identify $\Gamma(A)^\emptyset$ with the one point category, so that $G^0_\bullet \mathcal{C} := \text{Exact}(\Gamma(A)^\emptyset, \mathcal{C}) = \mathcal{C}$.

For the behavior on morphisms we distinguish cases. An isomorphism $x \to x'$ in $F$ induces a natural morphism $G^0_\bullet \mathcal{C} \to G^0_\bullet \mathcal{C}$ by permuting coordinates. An injection $i : x \to y$
induces $G^*_aC \to G^*_bC$ using stabilization

$$
\begin{array}{ccc}
\Gamma(A)^{i(x)} & \xrightarrow{\cong} & \Gamma(A)^{i(x)} \times \{L/0\}^{y\setminus i(x)} \\
\equiv & & \Gamma(A)^{i(x)} \times \Gamma(0)^{y\setminus i(x)} \\
\Gamma(A)^x & \xrightarrow{i_*} & \Gamma(A)^y
\end{array}
$$

where we recall $\Gamma(0) = \{L/0, R/0\}$ is the two point discrete category, and we define $X'$ to be zero outside $\Gamma(A)^{i(x)} \times \Gamma(0)^{y\setminus i(x)}$. This is the $j$-stabilization given by inclusion of $C$ on the $L$-line in $G_0C$.

Let $F_+G_A\mathcal{C}$ denote the category consisting of objects $(x, X; \Gamma(A)^x \to \mathcal{C})$ and morphisms $(x, X) \to (y, Y)$ given by $i: x \to y$ in $F_+$ and a natural transformation $i_*X \to Y$ in $G^*_b\mathcal{C}$. \[\text{[11 Def 1.2.2]}\]

**Theorem 4.2** (Compare [5], 10.3 Theorem.). The construction $F\int wG_*\mathcal{C}$ gives a model for $K$-theory.

**Lemma 4.3.** If $T: F \to \text{Top}_*$ is a functor with values in pointed topological spaces such that $T(i)$ is a homotopy equivalence for every morphism $i$ in $F$, then for every pair of morphisms $i, j: x \to y$ in $F$, $T(i)$ and $T(j)$ are homotopic. \[\text{[\[11\] Def 1.2.2]}\]

**Remark 4.4.** In particular, each $\sigma \in \Sigma_n$ induces a self map of $|G^*_aG^\infty\mathcal{R}_f(X, \Sigma_n, \{\text{all}\})|$ that is homotopic to the identity. Moreover, all the stabilization maps $|G^*_a\mathcal{R}_f(X, \Sigma_n, \{\text{all}\})| \to |G^\infty\mathcal{R}_f(X, \Sigma_n, \{\text{all}\})|$ are homotopic.

As in [5] the $E_\infty$-structure on the total Segal operation will be described in terms of the following diagram.

$$
\begin{array}{ccc}
\mathcal{R}^n_f(X) & \xrightarrow{\alpha_1} & \prod_{n \geq 1} \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \\
\beta_2 \downarrow & \alpha_2 & \beta_3 \\
\prod_{n \geq 1} F\int G_*\mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) & \xrightarrow{\beta_2} & \prod_{n \geq 1} F\int G^*_a\mathcal{R}_f(X, \Sigma_n, \{\text{all}\})
\end{array}
$$

The components of the map $\tilde{\alpha}_1$ are defined in example [3.20]. The other maps in diagram (4.2) are defined as follows.

**Definition 4.5.** For $\alpha_1$ the $n$th component $\alpha_1(Y)_n = Y \wedge_\mu \cdots \wedge_\mu Y$, where $\Sigma_n$ acts by permuting factors using the coherence data.

The maps $\beta_1$ and $\beta_2$ come from stabilizations $j^n_\mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \to G^*_a\mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$.

The maps $\alpha_2$ and $\beta_3$ are given by the identification

$$
\mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \cong \{\emptyset\} \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \cong G^\emptyset_\mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) = \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}).
$$
The category $R_f^j(X)$ has the pairing derived from the categorical sum $\vee_X$. This feature allows us to dispense with the subdivision construction. Each of the four categories in the lower part of the diagram also has a natural pairing. Underlying these pairings are the facts established in proposition 3.11 that $R_f(X)$ is a category with cofibrations and weak equivalences, with a categorical sum $\vee$ and a symmetric monoidal biexact product $\wedge_\mu$.

Definition 4.6. 1. A product $\boxtimes$ on $\{1\} \times \prod_{n \geq 1} R_f(X, \Sigma_n, \{\text{all}\})$:

For $(y_n)$ and $(z_n)$ in $1 \times \prod_{n \geq 1} R_f(X, \Sigma_n, \{\text{all}\})$, put

$$(y_n)\boxtimes (z_n) = \left(\vee_{s+t=n} y_s \boxtimes z_t\right)_{0 \leq n}$$

where $\boxtimes$ is as in definition 3.6. In this connection $\{1\}$ has the representation $S^0$, $(y_n) = (Y_0 = S^0, Y_1, \ldots)$ and $(z_n) = (Z_0 = S^0, Z_1, \ldots)$. Since $S^0 \vee S^0 = S^0$, this is consistent with the symbolism.

2. A product also denoted $\boxtimes$ on $\{1\} \times \prod_{n \geq 1} G^n A^\bullet R_f(X, \Sigma_n, \{\text{all}\})$:

For $(Y_n)$ and $(Z_n)$ in $\{1\} \times \prod_{n \geq 1} G^n A^\bullet R_f(X, \Sigma_n, \{\text{all}\})$, that is, for sequences of functors $Y_n: \Gamma(A)^n \rightarrow R_f(X, \Sigma_n, \{\text{all}\})$ and $Z_n: \Gamma(A)^n \rightarrow R_f(X, \Sigma_n, \{\text{all}\})$, put

$$(Y_n)\boxtimes (Z_n) = \left(\vee_{s+t=n} (\Gamma^{s+t}(A) \rightarrow X, \Sigma_n, \{\text{all}\}) \times R_f(X, \Sigma_n, \{\text{all}\}) \boxtimes R_f(X, \Sigma_n, \{\text{all}\})\right)_{n \geq 0}$$

3. A product $\boxtimes$ on $\{1\} \times \prod_{n \geq 1} F \{G^\bullet R_f(X, \Sigma_n, \{\text{all}\})$:

We have sequences of pairs $(y_n, Y_n)$ and $(z_n, Z_n)$, where $y_n, z_n$ are in $F$ and, for each simplicial dimension $A$, functors $Y_n: \Gamma(A)^y_n \rightarrow R_f(X, \Sigma_n, \{\text{all}\})$ and $Z_n: \Gamma(A)^z_n \rightarrow R_f(X, \Sigma_n, \{\text{all}\})$. For indices $s, t$ satisfying $s+t=n$ we have the construction

$$Y_s \boxtimes Z_t: \Gamma(A)^y_n \Pi^2 z_t \rightarrow R_f(X, \Sigma_n, \{\text{all}\})$$

and an insertion $\Pi_{u+v=n} y_u \Pi^2 z_v: Y_s \Pi z_t \rightarrow \Pi_{u+v=n} y_u \Pi z_v$. Combine these ingredients in the following formula.

$$(y_n, Y_n)\boxtimes (z_n, Z_n) = \left(\prod_{u+v=n} y_u \Pi z_v, (\Pi_{u+v=n} y_u \Pi^2 z_v, Y_s \boxtimes Z_t)\right)_{n \geq 0},$$

where $\Pi_{u+v=n} y_u \Pi^2 z_v: G^\bullet \Pi^2 \rightarrow R_f(X, \Sigma_n, \{\text{all}\}) \rightarrow G^\bullet \Pi_{u+v=n} y_u \Pi^2 z_v R_f(X, \Sigma_n, \{\text{all}\})$.

4. A product $\boxtimes$ on $\{1\} \times \prod_{n \geq 1} F \{G^\bullet G^n A^\bullet R_f(X, \Sigma_n, \{\text{all}\})$

This will be nearly the same as in the previous item. We have sequences of pairs $(y_n, Y_n)$ and $(z_n, Z_n)$, where $y_n, z_n$ are in $F$ and, for each simplicial dimension $A$, functors $Y_n: \Gamma(A)^y_n \rightarrow G^A R_f(X, \Sigma_n, \{\text{all}\})$ and $Z_n: \Gamma(A)^z_n \rightarrow G^A R_f(X, \Sigma_n, \{\text{all}\})$. For indices $s, t$ satisfying $s+t=n$ we have the construction

$$Y_s \boxtimes Z_t: \Gamma(A)^y_n \Pi^2 z_t \rightarrow G^A R_f(X, \Sigma_n, \{\text{all}\})$$

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applying naturality or functoriality of $G^n$ to the previous construction. We have an insertion $\ell_{y_u z_t}: y_u \coprod z_t \to \bigcup_{u+v=n} y_u \coprod v$. Define

$$(y_n, Y_n) \hat{\otimes} (z_n, Z_n) = \left( \bigcup_{u+v=n} y_u \otimes z_v, (\ell_{y_u z_t})_* (Y_s \otimes Z_t) \right)$$

where $(\ell_{y_u z_t})_*: G^n y_s \otimes z_t \to G^n \bigcup_{u+v=n} y_u \otimes z_v$.

**Theorem 4.7** (Compare [5], 10.7 Theorem, p.292.).

1. The categories $\{1\} \times \prod_{n \geq 1} R_f(X, \Sigma_n, \{\text{all}\})$ and $\{1\} \times \prod_{n \geq 1} F G^n R_f(X, \Sigma_n, \{\text{all}\})$ with composition laws as in definition 4.6 inherit symmetric monoidal structures from the coherence data on $R_f(X)$.

2. The categories $\{1\} \times \prod_{n \geq 1} G^n R_f(X, \Sigma_n, \{\text{all}\})$ and $\{1\} \times \prod_{n \geq 1} F G^n G^n R_f(X, \Sigma_n, \{\text{all}\})$ with composition laws as in definition 4.6 inherit monoidal structures from the coherence data on $R_f(X)$.

3. The maps $\alpha_1$ and $\alpha_2$ in diagram 4.2 are maps of symmetric monoidal categories.

4. The maps $\beta_1$, $\beta_2$, and $\beta_3$ are maps of monoidal categories.

5. The map $\beta_2$ is a homotopy equivalence and in the pseudo-additive case $\beta_3$ is also a homotopy equivalence.

6. The diagram 4.2 is commutative in the category of monoidal categories.

**Theorem 4.8.** Let $X$ be a connected simplicial abelian group. The functor

$$Z \mapsto [Z, \{1\} \times \prod_{n \geq 1} A_{\Sigma_n, \{\text{all}\}}(X)]$$

takes values in the category of abelian groups.

**Proof.** By theorem 4.2 we take

$$\{1\} \times \prod_{n \geq 1} A_{\Sigma_n, \{\text{all}\}}(X) = \{1\} \times \prod_{n \geq 1} [F G^n R_f(X, \Sigma_n, \{\text{all}\})].$$

Since the category $\{1\} \times \prod_{n \geq 1} F G^n R_f(X, \Sigma_n, \{\text{all}\})$ has a symmetric monoidal structure by part 1 of theorem 4.7, the functor $[-, \{1\} \times \prod_{n \geq 1} A_{\Sigma_n, \{\text{all}\}}(X)]$ takes values in the category of abelian monoids. Repeating the argument of [12] Lemma 2.3, p.404] shows that values taken are actually in the category of abelian groups. \qed

**Remarks on the proof of theorem 4.7.** The entire proof of the analogous result in [5] is essentially a formal appeal to LaPlaza’s coherence theorem [7], so it carries over completely.
The very interesting parts of the proof are the claims about the maps $\alpha_1$ and $\alpha_2$, which we repeat here for clarity. The biexactness and coherence of $\wedge_\mu$ give canonical natural isomorphisms $\gamma^k_n$ called Cartan multinomial formulas:

$$\gamma^k_n: (\wedge_\mu)_n \left( \bigvee_{i=1}^k c_i \right) \xrightarrow{\cong} \bigvee_{s_1 + \cdots + s_k = n} \text{Ind}_{\Sigma_{s_1} \times \cdots \times \Sigma_{s_k}}^\mathbb{Z} (\wedge_\mu)^k_{s_1} \left( (\wedge_\mu)_{s_i} c_i \right).$$

These induce natural isomorphisms

$$\gamma^k: \alpha_1 \circ \vee^k_X \xrightarrow{\cong} (\mathbb{E})^k \circ \alpha_1^k.$$ 

Then the coherence theorem implies that $\alpha_1$ has a (lax) symmetric monoidal structure. The functor $\alpha_2$ is the inclusion of a symmetric monoid subcategory, so the assertion is immediate.

In contrast to the algebraic roles played by $\alpha_1$ and $\alpha_2$, the roles of $\beta_1$, $\beta_2$, and $\beta_3$ are to assure us that we are ending in the correct target. Since the proof that $\beta_3$ is a homotopy equivalence requires the pseudo-additivity condition, which is fulfilled by suspension, this part of the argument actually depends on the next section.

\[\square\]

5 Suspension

The main theorem of this section is

**Theorem 5.1.** Let $X$ be a simplicial abelian group. The total Segal operation

$$\omega: A(X) \longrightarrow \{1\} \times \prod_{n \geq 1} A_{\Sigma_n, \text{all}}(X)$$

carries an infinite loop map structure.

Section 4 has delivered an infinite loop map $\Omega| hN_f \mathcal{R}_f^n(X) | \rightarrow \{1\} \times \prod_{n \geq 1} A_{\Sigma_n, \text{all}}(X)$ with domain the $K$-theory of a category of $n$-spherical objects. To obtain theorem 5.1 we have to examine the passage to the limit over suspension in view of Waldhausen’s result

$$\lim_{n \rightarrow \infty} hN_f \mathcal{R}_f^n(X) \simeq hS_f \mathcal{R}_f(X).$$

The technically challenging part is the compatibility of the operations with suspension. Fortunately, the machinery set up in [5, section 10] is sufficiently general that we need only extend some definitions and quote a sequence of results to prove our generalization.

First we need a description of the suspension operation that is amenable to coherence considerations. To this end, we go step-by-step through Waldhausen’s cone and suspension constructions and identify the result with a construction involving the operation $\wedge_e$. The cone construction for $(Y, r, s)$ in $\mathcal{R}_f(X)$ takes the ordinary mapping cylinder of the retraction $M(r)$ and collapses out the cylinder $\Delta^1 \times X$ so that end result is in $\mathcal{R}_f(X)$. To amplify the
definition, consider the following diagram, which fulfills the hypotheses of lemma 2.6.

\[ \begin{array}{ccc}
Y \coprod X & \xleftarrow{id \coprod r} & \partial \Delta^1 \times Y \\
\downarrow & & \downarrow \\
X \coprod X & \xleftarrow{\partial \Delta^1 \times X} & \Delta^1 \times X
\end{array} \]

(5.1)

Taking the pushouts of the rows produces a diagram

\[ X \xleftarrow{\Delta^1 \times X} \xrightarrow{M(r)} Y \]

where \(M(r)\) is the usual mapping cylinder of \(r\) and the pushout of the top row. As described above, taking the pushout of this diagram produces \(cY\), the underlying space of the cone construction. The retraction to \(X\) arises from a map of diagram (5.1) to a trivial diagram of identity maps on \(X\); the section \(X \to cY\) and a cofibration \(i: Y \to cY\) arise from canonical maps of ingredients of the diagram to the colimit. Then the suspension \(\Sigma Y\) is defined as the pushout of the diagram \(X \xleftarrow{r} \coprod Y \xrightarrow{\eta} cY\).

**Lemma 5.2.** For \(Y \in \mathcal{R}_f(X)\) there is a commuting diagram

\[ \begin{array}{ccc}
\{0\} \times Y & \xrightarrow{i_0} & \Delta^1 \wedge_e Y \\
\downarrow & \approx & \downarrow \\
Y & \xrightarrow{i} & cY
\end{array} \]

(5.2)

where \(\Delta^1 \in \mathcal{R}(\ast)\) is the standard simplicial one-simplex given the base point 1, and \(i_0\) is induced from the inclusion \(\{0\} \to \Delta^1\).

Moreover,

\[ \Sigma Y := cY / Y \cong S^1 \wedge_e Y, \]

where \(S^1 = \Delta^1 / \partial \Delta^1\) is the standard simplicial circle.

**Proof.** Pass to pushouts in the commutative diagram

\[ \begin{array}{ccc}
X & \xleftarrow{p_2 \cup r p_2} & \Delta^1 \times X \cup \{1\} \times X \xrightarrow{(1)} \Delta^1 \times Y \\
\downarrow & \xrightarrow{id \cup r} & \downarrow \\
X & \xleftarrow{p_2} & \Delta^1 \times X \xrightarrow{\eta_1} M(r)
\end{array} \]

(5.3)

to obtain a unique natural map \(\eta_1: \Delta^1 \wedge_e Y \to cY\) making the following diagram commute.

\[ \begin{array}{ccc}
\Delta^1 \times Y & \xleftarrow{\eta_1} & \Delta^1 \wedge_e Y \\
\downarrow & \approx & \downarrow \\
\Delta^1 \times cY & \xrightarrow{\eta_1} & cY
\end{array} \]

(5.4)
Restricting $Δ^1_1 × Y \to Δ^1_1 \wedge_e Y$ to $∂Δ^1_1 × Y$ yields a diagram

$$
\begin{array}{ccc}
∂Δ^1_1 × Y & \xrightarrow{r'} & Δ^1_1 × Y \\
\downarrow{r'} & & \downarrow{} \\
Y \coprod X & \xrightarrow{i'} & Δ^1_1 \wedge_e Y,
\end{array}
$$

where $r'(0, y) = y$, $r'(1, y) = r(y)$ and $i'(y) = i_0(y)$, $i'(x) = s'(x)$. There results a canonical arrow $M(r) \to Δ^1_1 \wedge_e Y$ such that the following square commutes.

$$
\begin{array}{ccc}
Δ^1_1 \times X & \xrightarrow{p_2} & Δ^1_1 \wedge_e Y \\
\downarrow{s'} & & \downarrow{} \\
X & \xrightarrow{i'} & Δ^1_1 \wedge_e Y.
\end{array}
$$

In turn, there is a unique map $\bar{η}_1 : cY \to Δ^1_1 \wedge_e Y$ such that the following diagram commutes.

$$
\begin{array}{ccc}
Δ^1_1 \times Y & \xrightarrow{cY} & X \\
\downarrow{\bar{η}_1} & & \downarrow{} \\
Δ^1_1 \wedge_e Y & \xrightarrow{i'} & Δ^1_1 \wedge_e Y
\end{array}
$$

(5.5)

Combining diagrams (5.4) and (5.5) shows that $η_1$ and $\bar{η}_1$ are mutually inverse isomorphisms, relative to the common subspace $X$ and compatible with the retractions.

Restricting the left half of diagram (5.4) to $\{0\} \times Y \subset Δ^1_1 × Y$ gives diagram (5.2):

$$
\begin{array}{ccc}
\{0\} \times Y & \equiv Y & \xrightarrow{m} cY \\
\downarrow{i} & & \downarrow{} \\
& & cY
\end{array}
$$

(5.6)

Replace $S^0 = \{*, *'\}$ with basepoint $*$ in example 2.4 by $∂Δ^1_1$ with basepoint 1, and obtain the diagram

$$
\begin{array}{ccc}
X & \xleftarrow{r} \partialΔ^1_1 \wedge_e Y & \xrightarrow{r} Δ^1_1 \wedge_e Y \\
\downarrow{\cong} & \downarrow{\cong} & \downarrow{} \\
X & \xleftarrow{i_0} \partialΔ^1_1 \wedge_e Y & \xrightarrow{i} cY.
\end{array}
$$

(5.7)

Passage to pushouts shows that the quotient $(Δ^1 \wedge_e Y)/(∂Δ^1 \wedge_e Y)$ is isomorphic to $ΣY$ in $R(X)$. According to proposition 2.9 the functor $- \wedge_e Y : R_f(*) \to R_f(X \times \{\cdot\}) \cong R_f(X)$ preserves colimits so we deduce

$$
(Δ^1 \wedge_e Y)/(∂Δ^1 \wedge_e Y) \cong (Δ^1/∂Δ^1) \wedge_e Y \equiv S^1 \wedge_e Y,
$$

where we define $S^1 := Δ^1/∂Δ^1$ in $R(*)$. □
According to proposition 2.8, the action of $\mathcal{R}_f(*)$ on $\mathcal{R}(X)$ may be made internal. Explicitly, there is a natural isomorphism $i_\ast \epsilon^1 \wedge \mu \equiv S^1 \wedge \epsilon Y$. In the following we abuse notation slightly and write simply $S^1 \wedge \mu Y$ leaving $i_\ast \epsilon$ understood, where $i_\ast: \{\ast\} \to X$ is the inclusion of the one point space as the identity element of $X$. We do this to emphasize the dependence of the rest of this section on the coherence of the operation $\wedge \mu$.

**Proposition 5.3** (Compare [5], 6.1 Proposition, p. 283). *The following diagram commutes up to natural isomorphism.*

\[
\begin{array}{ccc}
\omega^k \circ (S^1 \wedge \mu (-)) & \mu & \omega^k \circ (S^1 \wedge \mu (-)) \\
\downarrow{\phi_k S^1 \wedge \mu} & & \downarrow{\phi_k S^1 \wedge \mu} \\
\omega^k \circ (S^1 \wedge \mu (-)) & \mu & \omega^k \circ (S^1 \wedge \mu (-)) \\
\end{array}
\]

*Proof.* Write $F_1$ for the composite functor $\omega^k \circ (S^1 \wedge \mu (-))$ and $F_2$ for the composite $\phi_k S^1 \wedge \mu (-)$. Although $\omega^k(S^1) = \phi_k S^1 = S^1 \wedge \mu$ $\cdot \cdot \cdot \wedge \mu S^1$ we use the $\phi_k$-notation for orientation purposes. Following [5, p. 297], given a functor $M: \Gamma(A_1 \cdots A_k) \to \mathcal{R}_f(X)$ representing an object of $\text{sub}_k \mathcal{G} \mathcal{R}_f(X)$, the value of $\omega^k(M)$ on a typical element

\[
(i_1/j_1, 2, \cdots, i_{n_1}/j_1, i_{n_1+1}/j_1+1, \cdots, i_{n_k+1}/j_n+1, \cdots, i_{n_1+\cdots+n_k}/j_{n_1+\cdots+n_k+1})
\]

of $\Gamma^k(A_1 \cdots A_k)$ has the form

\[
(\phi_{n_1} M(-)) \times (\phi_{n_2} M(-)) \times \cdots \times (\phi_{n_k} M(-)) = Z_{n_1} \times \cdots \times Z_{n_k},
\]

where $Z_{n_i} := \phi_{n_i} M(-)$ is an object of $\mathcal{R}_f(X, \Sigma n_i, \{\text{all}\})$. Extending the formulas in the argument of theorem 3.16 for the associativity of $\times$, we write

\[
Z_{n_1} \times \cdots \times Z_{n_k} = \text{Ind}_{\Sigma n_1 \times \cdots \times \Sigma n_k}^{\Sigma n_1+\cdots+n_k} (Z_{n_1} \wedge \mu \cdots \wedge \mu Z_{n_k})
\]

Write $n = n_1 + \cdots + n_k$.

Then a typical value of $F_1(M)$ has the form

\[
\text{Ind}_{\Sigma n_1 \times \cdots \times \Sigma n_k}^{\Sigma n_1+\cdots+n_k} ((S^1 \wedge \mu \cdots \wedge \mu S^1 \wedge \mu Z_{n_1}) \wedge \mu \cdots \wedge \mu (S^1 \wedge \mu \cdots \wedge \mu S^1 \wedge \mu Z_{n_k}))
\]

\[
\approx \text{Ind}_{\Sigma n_1 \times \cdots \times \Sigma n_k}^{\Sigma n_1+\cdots+n_k} ((S^1 \wedge \mu \cdots \wedge \mu S^1) \wedge \mu \cdots \wedge \mu (S^1 \wedge \mu \cdots \wedge \mu S^1)) \wedge \mu (Z_{n_1} \wedge \mu \cdots \wedge \mu Z_{n_k}))
\]

applying commutativity and associativity isomorphisms. Now proposition 5.3 applies to deliver an isomorphism of $\Sigma_{n_1+\cdots+n_k}$-spaces.

\[
\text{Ind}_{\Sigma n_1 \times \cdots \times \Sigma n_k}^{\Sigma_n} ((S^1 \wedge \mu \cdots \wedge \mu S^1) \wedge \mu (S^1 \wedge \mu \cdots \wedge \mu S^1)) \wedge \mu (Z_{n_1} \wedge \mu \cdots \wedge \mu Z_{n_k}))
\]

\[
\approx \text{Ind}_{\Sigma n_1 \times \cdots \times \Sigma n_k}^{\Sigma n} (Z_{n_1} \wedge \mu \cdots \wedge \mu Z_{n_k}))
\]

This final expression is the value of $F_2$ on the same typical element $M$, so we have a natural isomorphism of functors $\epsilon: F_1 \Rightarrow F_2$. $\square$

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Now we prove the general lemma 5.4 and its specialization proposition 5.5.

**Lemma 5.4.** Let $H$ be a subgroup of $G$, let $Y \in \mathcal{R}(X, G)$, and let $Z \in \mathcal{R}(X, H)$. By restricting the $G$-action on $Y$ to $H$, we obtain $Y \wedge_H Z \in \mathcal{R}(X, H)$, where the action is diagonal. Then there is a natural isomorphism of left $G$-spaces

\[ G_+ \wedge_e^H (Y \wedge_H Z) \cong Y \wedge\mu (G_+ \wedge_e^H Z) \]

where the $G$-action on the righthand space is diagonal.

**Proof.** First define a $G$-map $f : G_+ \wedge_e (Y \wedge_H Z) \to Y \wedge\mu (G_+ \wedge_e^H Z)$ by the formula

\[ f(g, (y, z)) = (gy, [g, z]). \]

Applying the equivalence relation defining $Y \wedge\mu (G_+ \wedge_e^H Z)$,

\[ f(g, (hy, hz)) = (ghy, [gh, z]) = f(g, (y, z)). \]

Therefore, there is an induced $G$-map

\[ f' : G_+ \wedge_e^H (Y \wedge_H Z) \to Y \wedge\mu (G_+ \wedge_e^H Z). \]

To reverse this map, define $F : Y \wedge\mu (G_+ \wedge_e Z) \to G_+ \wedge_e^H (Y \wedge\mu Z)$ by the formula

\[ F(y, [g, z]) = (g, (g^{-1}y, z)]. \]

Now

\[ F(y, [gh, z]) = [gh, (h^{-1}g^{-1}y, z)] = [g, (h^{-1}g^{-1}y, hz)] = [g, (g^{-1}y, hz)] = F(y, [g, hz]), \]

so there is an induced $G$-map

\[ F' : Y \wedge\mu (G_+ \wedge_e^H Z) \to G_+ \wedge_e^H (Y \wedge\mu Z) \]

Clearly the composites $f' F'$ and $F' f'$ are the respective identities. \hfill \square

**Proposition 5.5.** Let $n = n_1 + \cdots n_k$. Let $Z \in \mathcal{R}(X, \Sigma_{n_1} \times \cdots \times \Sigma_{n_k}, \{\text{all}\})$. There is a natural isomorphism of $\Sigma_n$-spaces

\[ \text{Iso}(n, n_1 \cup \cdots \cup n_k)_+ \wedge_{e_{n_1} \times \cdots \times e_{n_k}} ((S \circ n_1 \text{ terms} \circ S) \wedge\mu \cdots \wedge\mu (S \circ n_k \text{ terms} \circ S) \wedge\mu Z) \cong (S \circ n \text{ terms} \circ S) \wedge\mu (\text{Iso}(n, n_1 \cup \cdots \cup n_k)_+ \wedge_{e_{n_1} \times \cdots \times e_{n_k}} Z) \]

**Proof.** Apply lemma 5.4 and observe that the operation $\circ$ is defined in terms of $\wedge\mu$, which is coherently associative. Collect all parentheses in expressions $(S \circ n_1 \text{ terms} \circ S) \wedge\mu \cdots \wedge\mu (S \circ n_k \text{ terms} \circ S)$ to the left. Note that we need only the map $f' : G_+ \wedge_e^H (Y \wedge\mu Z) \to Y \wedge\mu (G_+ \wedge_e^H Z)$ from the lemma. We do require the choice of an identification of $\text{Iso}(n, n_1 \cup \cdots \cup n_k)$ with $\Sigma_n$, to make sense of $f'$. This amounts to identifying $n_1 \cup \cdots \cup n_k$ with $\{1, \ldots, n_1, n_1+1, \ldots, n_1+n_2, \ldots, n_1 + \ldots + n_k\}$. \hfill \square

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We use Grothendieck constructions on functors defined on the category $F$ to pass to the limit with suspension constructions. To treat suspension by $S^1$ on $\text{sub}_k wG \cdot \mathcal{R}_f(X)$ define an op-lax functor $\Phi_1: F \to \text{Cat}^{\Delta^{op}}$ as follows.

$$\Phi_1(x) = \text{sub}_k wG \cdot \mathcal{R}_f(X), \quad \text{for an object } x \in F,$$
$$\Phi_1(\sigma) = \text{id}, \quad \text{for an isomorphism } \sigma: x \to x,$$
$$\Phi_1(i: y \to x) \quad \text{is induced by suspension by } x \setminus i(y) \text{ factors } S^1.$$

Interpreting the smash product with an empty number of factors as $S^0$, the definitions coincide on isomorphisms. For $x \xleftarrow{i} y \xleftarrow{j} z$ we need to produce the natural transformation $\Phi_1(ij) \Rightarrow \Phi_1(i) \circ \Phi_1(j)$. On $(Y, r, s)$ the value of $\Phi_2(j)$ is $\bigl((S^1)^{\eta \circ (y)} \land_e Y, r', s'\bigr)$ and the value of $\Phi_1(i)$ applied to this is $\bigl((S^1)^{\eta \circ (y)} \land_e Y, r'', s''\bigr)$. Since $i$ is injective, the set $y \setminus j(z)$ is identified with $i(y \setminus j(z))$. Since $x \setminus i j(z) = x \setminus i(y) \cup i(y \setminus j(z))$ we use associativity isomorphisms of the $\land_e$-action to write $\Phi_1(i \circ j) \cong \Phi_1(i) \circ \Phi_1(j)$. The coherence properties of the action imply commutativity of the necessary diagrams [11, Definition 3.1.1, p.99].

In a similar way we treat $\circ_k S^1 \land_\mu -$ on $wG \cdot \mathcal{R}_f(X, \Sigma_n\{\text{all}\})$, defining an op-lax functor $\Phi_2: F \to \text{Cat}^{\Delta^{op}}$.

$$\Phi_2(x) = wG \cdot \mathcal{R}_f(X, \Sigma_n\{\text{all}\}), \quad \text{for an object } x \in F,$$
$$\Phi_2(\sigma) = \text{id}, \quad \text{for an isomorphism } \sigma: x \to x,$$
$$\Phi_2(i: y \to x) \quad \text{is induced by suspension by } x \setminus i(y) \text{ factors } \circ_k S^1.$$

The natural transformation $\Phi_2(i \circ j) \cong \Phi_2(i) \circ \Phi_2(j)$ is treated in the same manner.

The results are two categories

$$\text{hocolim}_{\text{sub}_k wG \cdot \mathcal{R}_f(X)} := F\int \text{hocolim}_{\text{sub}_k wG \cdot \mathcal{R}_f(X)}$$

**Remark 5.6.** There are a number of constructions in [11] that may justifiably termed homotopy colimits. The Grothendieck construction is the essential one for us, but the standard notation for a Grothendieck construction gives little clue about its function in context.

Now we explain how proposition 5.3 promotes $\omega^k: \text{sub}_k wG \cdot \mathcal{R}_f(X) \to wG \cdot \mathcal{R}_f(X)$ to a left-op natural transformation (lont) $\epsilon: \Phi_1 \Rightarrow \Phi_2$. First, we need for an object $x$ of $F$, a functor $\epsilon(x): \Phi_1(x) \to \Phi_2(x)$. This is just $\omega^k$. Then we need for each arrow $i: y \to x$ in $F$ a natural transformation $\epsilon(i): \epsilon(x) \circ \Phi_1(i) \Rightarrow \Phi_2(i) \circ \epsilon(y)$. For any morphism $i$ such that $x \setminus i(y)$ has cardinality 1, we obtain $\epsilon(i)$, by inverting the isomorphism of functors provided by proposition 5.3. For the general case, one just goes back to the proof and replaces the symbol 1 by $x \setminus i(y)$ everywhere it occurs. The coherence results of section 2 guarantee that the necessary diagrams commute, so $\epsilon$ is a lont. By [11] Definition 3.1.4, p.101] $\epsilon$ induces a functor

$$F\int \epsilon: F\int \Phi_1 \to F\int \Phi_2.$$

We have now proved the following result.

**Theorem 5.7.** The operations $\omega^k$ pass through the homotopy colimit (Grothendieck) construction to deliver operations

$$F\int \epsilon := \omega^k: \text{hocolim}_{\text{sub}_k wG \cdot \mathcal{R}_f(X)} \to \text{hocolim}_{\text{sub}_k wG \cdot \mathcal{R}_f(X)}$$

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Proof of theorem 5.1. The main result of section 4 is that

\[ \Omega|hN \Gamma R^n_f(X)| \to \{1\} \times \prod_{n \geq 1} A\Sigma_{n,\text{all}}(X) \]

is an infinite loop map, and this section shows these maps are compatible with suspension. Likewise for the equivalence \( \Omega|hN \Gamma R^n_f(X)| \to \Omega|wS \Gamma R^n_f(X)| \). The maps obtained by passing to the limit over suspension remain infinite loop maps and we know \( \Omega \text{colim}|wS \Gamma R^n_f(X)| \simeq \Omega|wS \Gamma f(X)| = A(X) \).

6 Projecting to the free part

As stated in theorem 5.1 the constructions of \([5]\) as modified in section 5 deliver a total operation

\[ \omega = \prod \omega^n: A(X) \to \prod_{n \geq 1} A\Sigma_{n,(\text{all})}(X), \]

where \( A\Sigma_{n,(\text{all})}(X) = \Omega|hS \Gamma R_{hf}(X, \Sigma_n, \{\text{all}\})| \). We examine the target of this map, and introduce the Weyl group notation \( W_{\Sigma_n} H = N_{\Sigma_n} H / H \), where \( H \) is a subgroup of the permutation group \( \Sigma_n \) and \( N_{\Sigma_n} H \) is the normalizer in \( \Sigma_n \) of \( H \).

Theorem 6.1. For each \( n \) there is a homotopy equivalence

\[ h_n: A\Sigma_{n,(\text{all})}(X) \longrightarrow \prod_{H \in \{\text{all}\}} A(X \times BW_{\Sigma_n} H) \]

of infinite loop spaces. Here \( A(X \times B(W_{\Sigma_n} H)) = \Omega|hS \Gamma R_f(X, W_{\Sigma_n} H, \{e\})| \) is the \( K \)-theory of the category of retractive \( W_{\Sigma_n} H \)-spaces relative to \( X \) with the action being free outside of \( X \).

Proof. The argument is largely formal, based on some well-known facts. Let \( \mathcal{F} \) be the set of conjugacy classes \( (H_i) \) of subgroups of \( \Sigma_n \). The set is a finite set partially ordered in the usual way: \( (H_i) \preceq (H_j) \) if some conjugate of \( H_i \) is contained in \( H_j \). The partial ordering may be extended to a linear ordering, or enumeration \( \{(H_0), (H_1), \ldots, (H_N)\} \), so that \( (H_i) \preceq (H_j) \) implies \( i < j \). Observe that \( (H_0) = \{e\} \), we may take \( (H_1) \) as the class of transpositions, and \( (H_N) = \Sigma_n \).

For any \( \Sigma_n \)-space \( Z \) we may define

\[ \mathcal{F}_{\succeq (H)} Z = \text{colim}_{(K) \preceq (H)} Z^{(K)}, \]

essentially the union of the fixed point sets of the conjugates of all the subgroups properly containing a conjugate of \( H \). \( \mathcal{F}_{\succeq (H)} Z \) is by definition a \( \Sigma_n \)-invariant subspace of \( Z \). If \( (H_i) \preceq (H_{i+1}) \) in the enumeration then we may compute \( \mathcal{F}_{\succeq (H_{i+1})}(\mathcal{F}_{\succeq (H_i)} Z) \), essentially the fixed points of conjugates of \( H_{i+1} \) inside the fixed points of \( H_i \). On the complement \( \mathcal{F}_{\succeq (H_i)} Z \setminus (\mathcal{F}_{\succeq (H_{i+1})}(\mathcal{F}_{\succeq (H_i)} Z)) \) the group \( \Sigma_n \) acts and the Weyl group \( W_{\Sigma_n} H_i = N_{\Sigma_n} H_i / H_i \) acts freely.
Inductively define exact functors

\[ S_i, Q_j : \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \longrightarrow \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}), \quad -1 \leq i \leq N, \ 0 \leq j \leq N \]

by \( S_{-1} \) is the identity functor, and for \( i \geq 0 \) put \( S_i(Y) = \mathcal{F}_{\sim(H_i)}(S_{i-1}(Y)) \). Then the functors \( Q_j \) are defined by the natural cofibration sequences

\[ S_j(Y) \xrightarrow{\sim} S_{j-1}(Y) \longrightarrow Q_j(Y), \quad 0 \leq j \leq N. \]

For us, the important case will be \( S_0 \): Since \( H_0 = \{e\} \), \( S_0(Y) \) is going to be the union of the fixed point sets of all the non-identity subgroups of \( G \). Then the quotient \( Q_0(Y) \) can be thought of as extracting the part of \( G \) on which \( G \) acts freely.

Let \( i_k : \mathcal{R}_f(X, \Sigma_n, \{H_k\}) \rightarrow \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \) be the inclusion. Since \( Q_k(Y) \) actually lies in \( \mathcal{R}_f(X, \Sigma_n, \{H_k\}) \), we may formally write \( Q_k = i_k \circ \overline{Q}_k \) where \( \overline{Q}_k : \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \rightarrow \mathcal{R}_f(X, \Sigma_n, \{H_k\}) \) is a retraction. We want to make an inductive application of the additivity theorem for the \( \mathcal{G}_* \) construction, but this requires that the input be pseudo-additive. Passing to prespectra \( \Sigma^\infty \mathcal{R}_f(X) \), \([\Pi]\) there results a splitting

\[ \text{hocolim } wG_* \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \rightarrow \prod_{H \in \{\text{all}\}} \text{hocolim } wG_* \mathcal{R}_f(X, \Sigma_n, \{H\}) \]

induced by the functors \( \overline{Q}_k \) for \( 0 \leq k \leq N \). Recalling that \( W_{\Sigma_n}H = N_{\Sigma_n}H/H \) is the Weyl group of \( H \), consider the exact functor

\[ \mathcal{R}_f(X, \Sigma_n, \{H\}) \longrightarrow \mathcal{R}_f(X, W_{\Sigma_n}H, \{e\}) \quad \text{taking } Y \mapsto Y^H. \]

The induction construction \( Z \mapsto Z \times^{W_{\Sigma_n}} \Sigma_n \) provides an exact functor going the other way and the composites in either order are equivalent to the identities. Putting these equivalences together and specializing the notation establishes a chain of homotopy equivalences

\[ \text{hocolim } wG_* \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \rightarrow \prod_{H \in \{\text{all}\}} \text{hocolim } wG_* \mathcal{R}_f(X, \Sigma_n, \{H\}) \]

\[ \rightarrow \prod_{H \in \{\text{all}\}} \text{hocolim } wG_* \mathcal{R}_f(X, W_{\Sigma_n}H, \{e\}) \]

\[ \boxdot \]

This completes the proof of theorem 1.1 to explain theorem 1.2 is the object of the next two sections. We are focusing on the composition

\[ \theta^n : A(X) \xrightarrow{\omega^n} A_{\Sigma_n,\{\text{all}\}}(X) \xrightarrow{h_n} \prod_{H \in \{\text{all}\}} \Omega[hS_* \mathcal{R}_f(X, N_{\Sigma_n}H/H, \{e\})] \]

\[ \xrightarrow{p_e} \Omega[hS_* \mathcal{R}_h f(X, \Sigma_n, \{e\})]. \]

In section 7 we justify the interpretation \( \Omega[hS_* \mathcal{R}_h f(X, \Sigma_n, \{e\})] = A(X \times B\Sigma_n) \). Then we want to understand what happens when we follow this composition by the transfer
\( \phi_n : A(X \times B\Sigma_n) \to A(X \times E\Sigma_n) \simeq A(X) \). We start by introducing notation for the composition

\[
\mathcal{R}_f(X) \longrightarrow \text{sub}_n G_* \mathcal{R}_f(X) \xrightarrow{\omega^n} G_*^n \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \xrightarrow{Q_0=S_{-1}/S_0} G_*^n \mathcal{R}_f(X, \Sigma_n, \{e\}).
\]

On \((Y, r, s) \in \mathcal{R}_f(X)\), the composition of the first two maps in the chain is \(\tilde{\alpha}_n(Y)\) in the notation of example 3.21 so we want to evaluate the functor \(Q_0 = S_{-1}/S_0 \circ \tilde{\alpha}_n\) on the object \((Y, r, s)\). By the terminology used in the proof of theorem 6.1, \(S_{-1}\) is the identity and \(S_0\) is the union of subobjects that are fixed by some non-identity subgroup of \(\Sigma_n\). The interpretation and transfer issues are taken up in the next section 7 to prepare for the analysis of \(\phi_n \circ \theta_n\) in section 8 we introduce some notation.

The definitions of the Segal operations in [12] use certain subfunctors \(P^m_j\) of the smash power functor \(P^m\) on pointed sets. We extend the considerations to define certain subfunctors of \(\wedge_\epsilon\) and \(\wedge_\mu\) powers. For \((Y, r, s) \in \mathcal{R}(X)\), the set \(Y^{\wedge_\epsilon n}\) is a quotient of the cartesian product \(Y^n\). In a fixed simplicial dimension, we view this as the set of functions \(y: n \to Y\). The pushout construction identifies any such function \(y\) with at least one value \(y_i\) in \(X\) with the composite function \(r \circ y\). Thus, to represent points of \(Y^{\wedge_\epsilon n}\) in a given dimension, we just need to look at functions all of whose values are in \(Y - X\) and functions all of whose values are in \(X\). For \(0 \leq j \leq n\) we define \(\tilde{P}^m_j Y\) to be the subset of functions \(y: i \mapsto y_i\) such that the cardinality of \(y^{-1}(Y - X)\) is less than or equal to \(j\), if the image of \(y\) is contained in \((Y - X)\).

Said another way, \(\tilde{P}^m_j Y\) is the set of \(n\)-tuples where at most \(j\) distinct elements of \(Y - X\) are involved. For example, \(\tilde{P}^m_0 Y = X^n\) and \(\tilde{P}^m_1 Y\) is the union of \(X^n\) with the diagonal of \((Y - X)^n\). Most important for us, the subset \(\tilde{P}^m_{n-1} Y\) consists of all \(n\)-tuples involving no more than \(n-1\) distinct elements of \(Y\), so that if no member of \((y_1, \ldots, y_n)\) is in \(X\), then there are at least two distinct indices \(i, j\) with \(y_i = y_j\).

When \(X\) is a connected abelian group, then we can push out along the iterated multiplication \(X^n \to X\), obtaining functors \(\tilde{P}^m_j Y\) relative to \(X\). In particular, \(\tilde{P}^m_{n-1} Y\) is the subset of \(P^m Y\) consisting of points fixed by some non-trivial subgroup of \(\Sigma_n\), so not all members of an \(n\)-tuple can be distinct. Thus \(\tilde{P}^m_{n-1} Y = S_0 \tilde{\alpha}_n(Y)\). In terms of functions \(y: n \to Y\), \(\tilde{P}^m_{n-1} Y\) is the set of functions where the cardinality of \(y^{-1}(Y - X)\) is at most \(n-1\).

**Definition 6.2.** Define

\[
\begin{array}{ccc}
\tilde{P}^m_{n-1} Y & \longrightarrow & \tilde{P}^m Y \\
\wedge_\epsilon \, \circ \, r & \downarrow & \\
X^n & \longrightarrow & \tilde{\theta}^m Y
\end{array}
\]

\[
\begin{array}{ccc}
P^m_{n-1} Y & \longrightarrow & P^m Y \\
n \circ \, r & \downarrow & \\
X & \longrightarrow & \theta^m Y
\end{array}
\]

Letting \(j^n : \mathcal{R}_f(X, \Sigma_n, \{e\}) \to G_*^n \mathcal{R}_f(X, \Sigma_n, \{e\})\) be the iterated stabilization, we combine the preceding observations with the definitions to immediately obtain the following proposition.

**Proposition 6.3.** As functors from \(\mathcal{R}_f(X)\) to \(G_*^n \mathcal{R}_f(X, \Sigma_n, \{e\})\), \(Q_0 \circ \tilde{\alpha}_n = j^n \circ \theta^n. \)

**7 Transfer constructions**

Our immediate goals are to interpret \(\Omega|hS_* \mathcal{R}_f(X^n, \Sigma_n)|\) and \(\Omega|hS_* \mathcal{R}_f(X, \Sigma_n)|\) in terms of the algebraic \(K\)-theory of topological spaces. There are two steps to these goals and each
We let $G$ be a finite group and let $Z$ be a $G$-space. Let $EG$ be the canonical contractible free left $G$-space. We prefer the model $EG_n = G^{n+1}$ with the $G$-action given by multiplication on the left in each factor, face maps defined by projecting away from a coordinate, and degeneracies defined by repeating a coordinate. An isomorphism of the quotient space $* \times^G EG \cong BG$ is induced by $(g_0, \ldots, g_{i-1}, g_i, \ldots, g_n) \mapsto (g_0^{-1}g_1, \ldots, g_i^{-1}g_i, \ldots, g_{n-1}g_n)$.

Lemma 7.1. The projection $EG \times Z \to Z$ induces a homotopy equivalence

$$hS_* R_{hf}(EG \times Z, G) \to hS_* R_{hf}(Z, G).$$

Proof. The argument here is similar to that given to prove Proposition 2.1.4 [13, p.382]. In detail, let $(Y', r', s') \in R_{hf}(EG \times Z, G)$. Completing the diagram

$$Y' \leftarrowEG \times Z \begin{array}{c} \downarrow p_2 \end{array} \begin{array}{c} \downarrow p_2 \end{array} Z$$

provides an exact functor $R_{hf}(EG \times Z, G) \to R_{hf}(Z, G)$. Certainly, homotopy finite objects are carried to homotopy finite objects, and, incidentally, finite objects are carried to finite objects. Also, weak equivalences are mapped to weak equivalences.

Taking the product with $EG$ provides an exact functor $R_{hf}(Z, G) \to R_{hf}(EG \times Z, G)$. In this case, when $G$ is nontrivial, finite objects are carried to homotopy finite objects, since $EG$ is contractible.

For $(Y, r, s)$ in $R_{hf}(Z, G)$, taking the induced map of pushouts in the following diagram provides a natural transformation from the composite endofunctor on $R_{hf}(Z, G)$ to the identity functor. This natural transformation is a weak equivalence.

$$EG \times Y \leftarrow EG \times Z \begin{array}{c} \downarrow p_2 \end{array} \begin{array}{c} \downarrow p_2 \end{array} Z \begin{array}{c} \downarrow \id \end{array}$$

For $(Y', r', s')$ in $R_{hf}(EG \times Z, G)$, taking the induced map of pushouts in the next diagram provides a natural transformation from the identity functor on $R_{hf}(EG \times Z, G)$ to the other composite endofunctor. Again, this natural transformation is a weak equivalence.

$$Y' \leftarrow EG \times Z \begin{array}{c} \downarrow \id \end{array} \begin{array}{c} \downarrow \id \end{array} EG \times Z \begin{array}{c} \downarrow \id \end{array}$$

By Proposition 1.3.1 [13, p.330] $hS_* R_{hf}(Z \times EG, G) \to hS_* R_{hf}(Z, G)$ is a homotopy equivalence.

Next, we simply observe that $EG \times Z$ is the total space of a principal $G$-bundle with base $EG \times^G Z$. To make this transparent, and for later use, we replace the notation $EG \times^G Z$ by $* \times^G (EG \times Z)$. To explain the connection, $* \times^G (EG \times Z)$ is the orbit space of $EG \times Z$ under the diagonal left $G$-action, thought of as the balanced product of $EG \times Z$ with the
trivial right $G$-space $\ast$. We can turn the left action of $G$ on $EG$ into a right action by setting $e \cdot r g = g^{-1} \cdot e$. Then left $G$-orbits in $EG \times Z$ are seen to correspond to equivalence classes in $EG \times Z$ under the equivalence relation generated by $(e \cdot r g) \sim (e, g z)$. The associated quotient space is usually denoted $EG \times^G Z$.

Then Lemma 2.1.3 [13, p.381] applies to yield the following result.

**Lemma 7.2.** There is an equivalence of categories $\mathcal{R}(EG \times^G Z) \sim \mathcal{R}(EG \times Z, G)$.

For reference, pullback along the projection

$$EG \times Z \to EG \times^G Z$$

defines a functor $\mathcal{R}(EG \times^G Z) \to \mathcal{R}(EG \times Z, G)$; the orbit map defines a functor in the opposite direction. The composites in either order are isomorphic to the respective identity functors. Moreover, these functors preserve weak equivalences, finite, and homotopy finite objects.

Substitute for $G$ the symmetric group $\Sigma_n$, and let $Z$ be $X$ with the trivial $\Sigma_n$ action. We obtain the following results.

**Proposition 7.3.** Let $B \Sigma_n \times X$ be the quotient of $E \Sigma_n \times X$ by the action of $\Sigma_n$ on the first factor. There is a homotopy equivalence $hS_* \mathcal{R}_f(B \Sigma_n \times X) \simeq hS_* \mathcal{R}_f(E \Sigma_n \times X, \Sigma_n)$.

Putting lemma 7.1 and proposition 7.3 together with the definition $A(B \Sigma_n \times X) = \Omega|hS_* \mathcal{R}_f(B \Sigma_n \times X)|$ yields

**Proposition 7.4.** The space $\Omega|wS_* \mathcal{R}_f(X, \Sigma_n)|$ is homotopy equivalent to $A(B \Sigma_n \times X)$.

Substitute for $G$ the symmetric group $\Sigma_n$ and take $Z$ to be the space $X^n$ with the permutation action, we may draw the next conclusion.

**Proposition 7.5.** Let $D_n X = E \Sigma_n \times X^n$ be the quotient of $E \Sigma_n \times X^n$ by the diagonal action of $\Sigma_n$. There is a homotopy equivalence $hS_* \mathcal{R}_f(D_n X) \simeq hS_* \mathcal{R}_f(E \Sigma_n \times X^n, \Sigma_n)$.

Similarly, putting lemma 7.1 and proposition 7.5 together with the definition $A(D_n X) = \Omega|hS_* \mathcal{R}_f(D_n X)|$ yields

**Proposition 7.6.** The space $\Omega|hS_* \mathcal{R}_f(X^n, \Sigma_n)|$ is homotopy equivalent to $A(D_n X)$.

We recall here basic facts about the transfer in the algebraic $K$-theory of spaces adapted to our context. We are actually interested in two cases of transfer operations. For the first case the transfer operations are associated with finite subgroups of the symmetric groups $\Sigma_n$. In the second case the operations are associated with (injective) homomorphisms of simplicial abelian groups $\widetilde{X} \to X$, where the fiber is homotopy-finite.

In terms of the description $A(X) = \Omega|hS_* \mathcal{R}_f(X)|$, we have the following direct transfer construction. A fiber bundle projection $p: E \to B$ with finite fiber, resp., homotopy-finite fiber, induces by pullback a functor $\mathcal{R}_f(B) \to \mathcal{R}_f(E)$, resp., $\mathcal{R}_f(B) \to \mathcal{R}_f(E)$. We then obtain a transfer morphism $p^*: A(B) \to A(E)$. In terms of equivariant models for algebraic $K$-theory, there are other descriptions of the transfer, as given below. We need to relate the various descriptions.
Eventually we need the transfer operations \( A(B\Sigma_n \times X) \to A(BH \times X) \), where \( H \) is a subgroup of \( \Sigma_n \). Our working definition is \( A(B\Sigma_n \times X) = \Omega [hS_n R(X, \Sigma_n)] \) but, in view of the equivalences \( hS_n R_h f(X, \Sigma_n) \simeq hS_n R_h f(E\Sigma_n \times X, \Sigma_n) \) and \( hS_n R_h f(E\Sigma_n \times X, \Sigma_n) \simeq hS_n R_h f(B\Sigma_n \times X) \), we have to compare three definitions in each context.

To this end, let \( G \) be a discrete group, \( H \) be a subgroup of finite index, and let \( Z \) be a trivial \( G \)-space. We consider the following diagram where the vertical arrows represent transfer constructions.

\[
\begin{array}{ccc}
\mathcal{R}(Z, H) & \xrightarrow{p_1^*} & \mathcal{R}(EG \times Z, H) \\
\downarrow p_1 & \quad & \downarrow p_1^* \\
\mathcal{R}(Z, G) & \xrightarrow{p_2^*} & \mathcal{R}(EG \times Z, G) \\
\downarrow p_3 & \quad & \downarrow p_3^* \\
\mathcal{R}(EG \times H Z) & \xleftarrow{p_3^*} & \mathcal{R}(EG \times G Z)
\end{array}
\] (7.1)

The forgetful functor \( p_1^*: \mathcal{R}(Z, G) \to \mathcal{R}(Z, H) \) just restricts the action to the subgroup \( H \). This provides the simplest path to \( p_1^*: A(BG \times Z) \to A(BH \times Z) \), using the basic model \( A(BG \times Z) = \Omega [hS_n R_h f(Z, G)] \). In the middle, the functor \( p_2^*: \mathcal{R}(EG \times Z, G) \to \mathcal{R}(EG \times Z, H) \) is also a forgetful functor. At the right, the functor \( p_3^*: \mathcal{R}(EG \times G Z) \to \mathcal{R}(EG \times H Z) \) is given by a pullback functor construction, explained in detail below.

To reach the categories in the middle column from those in the left column we compute products with \( EG \). Along the top, the fact that \( EG \) is a non-standard contractible \( H \)-space is an insignificant detail. Comparing with \( p_1^* \) on the left, the transfer \( p_2^* \) in the middle column is also obtained by restricting the action of \( G \) to \( H \). Thus, the lefthand square in diagram (7.1) obviously commutes.

Before we compare \( p_3^* \) with \( p_2^* \), we discuss \( p_3 \), the rightmost column in diagram (7.1), in detail. In order to manipulate pullback squares efficiently we replace the notation \( EG \times G Z \) by \( * \times ^G (EG \times Z) \) as discussed before lemma (7.2). Suppose \( H \) is a subgroup of the group \( G \), and let \( EG \) be the standard model for a contractible \( G \)-space on which \( G \) acts freely from the right. The space \( EG \) plays a similar role relative to the subgroup \( H \). In order to compare situations, we take the standard model \( X = * \times ^H (EG \times Z) \) and a modified model \( \tilde{X} = * \times ^H (EG \times Z) \). In this situation we have the basic pullback square

\[
\begin{array}{ccc}
(\ast \times ^H G) \times (EG \times Z) & \xrightarrow{p_2} & (\ast \times ^H EG \times Z) = \tilde{X} \\
\downarrow & \quad & \downarrow \\
EG \times Z & \xrightarrow{} & \ast \times ^G (EG \times Z) = X
\end{array}
\] (7.2)

This displays the comparison map \( \tilde{X} \to X \) of the chosen models as a fiber bundle, with fiber \( \ast \times ^H G \). One may identify \( \ast \times ^H (EG \times Z) \cong (\ast \times ^H G) \times ^G (EG \times Z) \) and then the righthand vertical arrow is isomorphic to the map \( (\ast \times ^H G) \times ^G (EG \times Z) \to \ast \times ^G (EG \times Z) \) induced by projecting the coset space \( \ast \times ^H G \) to a point. This replacement also displays the upper horizontal map as the quotient projection \( (\ast \times ^H G) \times (EG \times Z) \to (\ast \times ^H G) \times ^G (Z \times EG) \).

The direct construction \( p^*: \mathcal{R}(X) \to \mathcal{R}(\tilde{X}) \) maps \((Y, r, s)\) to \((\tilde{Y}, \tilde{r}, \tilde{s})\) derived from the
following pullback square.

\[ \begin{array}{ccc}
\tilde{Y} & \xrightarrow{\tilde{r}} & \tilde{X} = * \times^H (EG \times Z) \\
\downarrow & & \downarrow p \\
Y & \xrightarrow{r} & X = * \times^G (EG \times Z)
\end{array} \] (7.3)

Augmenting the righthand column of (7.3) to the square of (7.2) shows that $\tilde{Y} \to Y$ is a fiber bundle with fiber $* \times^H G$.

Now we address commutativity of the righthand square in diagram (7.1). To reach the categories in the middle column from those in the right column we also compute pullbacks. Recalling lemma 7.2, the equivalence of categories $\mathcal{R}(EG \times^G Z) \simeq \mathcal{R}(EG \times Z, G)$ [13, Lemma 2.1.3] describes the functor moving left to the middle column. This functor assigns to a retractive space $(Y, r, s)$ over $EG \times^G Z$ the retractive $G$-space $(Y', r', s')$ over $EG \times Z$ defined as the pullback in the following diagram.

\[ \begin{array}{ccc}
Y' & \xrightarrow{r'} & EG \times Z \\
\downarrow & & \downarrow \\
Y & \xrightarrow{r} & X = * \times^G (EG \times Z)
\end{array} \]

Then moving up to $\mathcal{R}(EG \times Z, H)$ amounts to restricting the $G$-action in this pullback to $H$.

On the other hand, to move from the lower right to the upper middle by going up and then to the left, compute first the pullback (7.3) and then compute

\[ \begin{array}{ccc}
\tilde{Y}' & \xrightarrow{\tilde{r}'} & EG \times Z \\
\downarrow & & \downarrow \\
\tilde{Y} & \xrightarrow{\tilde{r}} & \tilde{X} = * \times^H (EG \times Z).
\end{array} \]

The composition of the two functors may be displayed in the stacked diagram

\[ \begin{array}{ccc}
\tilde{Y}' & \xrightarrow{\tilde{r}'} & EG \times Z \\
\downarrow & & \downarrow \\
\tilde{Y} & \xrightarrow{\tilde{r}} & \tilde{X} = * \times^H (EG \times Z) \\
\downarrow & & \downarrow p \\
Y & \xrightarrow{r} & X = * \times^G (EG \times Z)
\end{array} \]

The end result is that $(\tilde{Y}', \tilde{r}', \tilde{s}')$ is simply the $G$-space $(Y', r', s')$ with the action restricted to $H$. Therefore, the righthand square commutes.

**Lemma 7.7** (Compare [12], Lemma 1.3, p.399). Let $G$ be a finite group, $EG$ a universal $\Sigma$-bundle, $BG = * \times^G EG$ a classifying space, and let $Z$ be a space with a trivial $G$-action. Then the composition

\[ A(Z) \xrightarrow{\text{inclusion}} A(BG \times Z) \xrightarrow{\text{transfer}} A(EG \times Z) \simeq A(Z) \]

is given by multiplication by the order of $G$, in the sense of the additive $H$-space structure.
Proof. This can be proven in two ways. Model \( A(Z) \) by \( hS\cdot R_f(Z, \{e\}) \), and use lemmas \[7.1\] and \[7.2\] to model \( A(BG \times Z) \) by \( hS\cdot R_f(Z, G) \). The inclusion-induced map sends a retractive space \( (Y, r, s) \) over \( Z \) to the retractive relatively free \( G \)-space \( (Y', r', s') \) over \( Z \), which is the quotient of \( G \times Y \) by the equivalence relation generated by \( (g, y) \sim (g', y) \). Interpreting the transfer as forgetting the action, this new space is just \(|G|\) copies of \( Y \) glued together along \( Z \). This is the first proof.

The second proof uses the pullback diagram of spaces

\[
\begin{array}{ccc}
G \times EG \times Z & \longrightarrow & EG \times Z \\
\downarrow & & \downarrow \\
EG \times Z & \longrightarrow & \ast \times^G (EG \times Z)
\end{array}
\]

which gives rise to a diagram

\[
\begin{array}{ccc}
hS\cdot R_hf(G \times EG \times Z) & \longrightarrow & hS\cdot R_hf(EG \times Z) & \text{modeling} & A(G \times EG) & \longrightarrow & A(Z) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
hS\cdot R_hf(EG \times Z) & \longrightarrow & hS\cdot R_hf(\ast \times^G (EG \times Z)) & & & & A(Z) & \longrightarrow & A(BG \times Z)
\end{array}
\]

where the horizontal maps are induced by taking pushouts and the vertical maps are induced by taking pullbacks.

It is a general fact that, given a pullback diagram

\[
\begin{array}{ccc}
E_1 & \overset{g}{\longrightarrow} & E_2 \\
p_1 \downarrow & & \downarrow p_2 \\
B_1 & \overset{f}{\longrightarrow} & B_2
\end{array}
\]

where \( p_2 \) is a fiber bundle with finite fiber, the functor \( p_2^*f_*: R_f(B_1) \rightarrow R_f(E_2) \), push forward followed by transfer, coincides with \( p_1^*g_* \), transfer followed by push forward. In the present situation, moving up from the lower left and then to the right sends a retractive space \( (Y, r, s) \) over \( Z \times EG \) first to \((Y \times G, r \times \text{id}, s \times \text{id})\) over \( Z \times EG \times G \). Then the pushout identifies the \(|G|\) copies of \( Z \times EG \) inside \( Y \times G \) to one. This computes the map of the lemma, seen in the diagram as moving right and then up.

\[\square\]

8 A fundamental cofibration sequence

Waldhausen’s main result is this proposition.

**Proposition 8.1** (Compare \[12\], Proposition 2.7, p.407). The composition of the operation \( \theta^n: A(*) \rightarrow A(B\Sigma_n \times *) \) with the transfer map \( \phi_n: A(B\Sigma_n \times *) \rightarrow A(*) \) is the same, up to weak homotopy, as the polynomial map on \( A(*) \) given by the polynomial

\[ p(x) = x(x - 1) \cdots (x - n + 1). \]  

\[\square\]
The analogous result for the present situation with the one point space replaced by a simplicial abelian group $X$ is more complicated to formulate and to work with. To prepare for the analogue of Waldhausen’s result, we develop the following constructions, taking up where we left off with definition [6.2] and proposition [6.3]. We make use of the maps

\[ \delta_{n-1}^{n,k} : X^{n-1} \rightarrow X^n \quad \text{given by} \quad \delta_{n-1}^{n,k}(x_1, \ldots, x_{n-1}, x_k) \]

and the respective induced functors $\tilde{\delta}_{n-1}^{n,k} : R_f(X^{n-1}) \rightarrow R_f(X^n)$. The pushout construction

\[
\begin{array}{ccc}
X^{n-1} & \xrightarrow{s} & Z \\
\downarrow \delta_{n-1}^{n,k} & & \downarrow \delta_{n-1}^{n,k} \\
X^n & \rightarrow & \tilde{\delta}_{n-1}^{n,k}Z
\end{array}
\]

defines an exact functor $\tilde{\delta}_{n-1}^{n,k} : R_f(X^{n-1}) \rightarrow R_f(X^n)$. For a retractive space $(Z, r, s)$ over $X^{n-1}$ with retraction $r : Z \rightarrow X^{n-1}$ written in terms of components as $r = (r_1, \ldots, r_{n-1})$, the composition of the canonical map $i_{n-1}^{n,k}$ followed by the retraction $\delta_{n-1}^{n,k}r$ is given by the formula $\left( \delta_{n-1}^{n,k}r \right) \circ i_{n-1}^{n,k}(z) = \delta_{n-1}^{n,k} \circ r(z) = (r_1(z), \ldots, r_k(z), \ldots, r_{n-1}(z), r_k(z))$. Note that in the special case $Z = \tilde{P}^{n-1}Y = (\wedge_e)^{n-1}Y$, we have, for each $k, 1 \leq k \leq n-1$,

\[ (\delta_{n-1}^{n,k}(\tilde{P}^{n-1}r)) \circ i_{n-1}^{n,k}(y_1, \ldots y_{n-1}) = (r(y_1), \ldots, r(y_k), \ldots, r(y_{n-1}), r(y_k)). \] (8.1)

Next we assemble these functors by gluing along the common space $X^n$, obtaining

\[ \tilde{\Delta}^n_{n-1} : R_f(X^{n-1}) \rightarrow R_f(X^n) \]

given on objects by $\tilde{\Delta}^n_{n-1}(Z) = \delta_{n-1}^{n-1}Z \cup X^n \ldots \cup X^n \delta_{n-1}^{n-1}Z$, which can be viewed as an iterated pushout or as the colimit of a diagram modeled on the cone on $n-1$ points. We also need to push this construction forward to $R_f(X)$ by $\mu_*$, the iterated multiplication, obtaining

\[ \Delta^n_{n-1} = \mu_* \circ \tilde{\Delta}^n_{n-1} : R_f(X^{n-1}) \rightarrow R_f(X) \]

given on objects by $\Delta^n_{n-1}(Z) = \mu_*(\delta_{n-1}^{n-1}Z) \cup X \ldots \cup X \mu_*(\delta_{n-1}^{n-1}Z)$. If we start with $Z = Y \wedge_e \ldots \wedge_e \mu_*(\delta_{n-1}^{n-1}Z) \cup X \ldots \cup X \mu_*(\delta_{n-1}^{n-1}Z)$, then the formula for the retraction on the $k$th summand is

\[ (\mu_*\delta_{n-1}^{n,k}(\tilde{P}^{n-1}r)) \circ i_{n-1}^{n,k}(y_1, \ldots y_{n-1}) = \mu(r(y_1), \ldots, r(y_k), \ldots, r(y_{n-1}), r(y_k)), \] (8.2)

where $\mu$ is the iterated multiplication.

We can now succinctly state our general results. Let

\[ \tilde{\phi}_k : R_f(X^k, \Sigma_k, \{\text{all}\}) \rightarrow R_f(X^k) \quad \text{and} \quad \phi_k : R_f(X, \Sigma_k, \{\text{all}\}) \rightarrow R_f(X) \]

be the functors that forget the group action.
Proposition 8.2 (Compare [12], Proposition 2.7, page 407). There is a cofibration sequence of functors \( \mathcal{R}_f(X) \to \mathcal{R}_f(X^n) \)

\[ \tilde{\Delta}_{n-1} \phi_{n-1} \theta^{n-1} Y \twoheadrightarrow \tilde{\phi}_{n-1} \phi^{n-1} Y \wedge \varepsilon Y \twoheadrightarrow \phi_n \theta^n Y \]  

(8.3)

In the case \( X \) is a connected simplicial abelian group, we have the cofibration sequence

\[ \Delta_{n-1} \phi_{n-1} \theta^{n-1} Y \twoheadrightarrow \phi_{n-1} \theta^{n-1} Y \wedge \mu \theta^1 Y \twoheadrightarrow \phi_n \theta^n Y \]  

(8.4)

of functors \( \mathcal{R}_f(X) \to \mathcal{R}_f(X) \).

Remark 8.3. The second cofibration sequence is obtained by applying the exact functor induced by the iterated multiplication \( \mu : X^n \to X \) to the first sequence. The result in the middle term of the second sequence is open to interpretation. The formulation chosen amounts to interpretation of the factorization \( \mu = \mu \circ (\mu \times \text{id}) \) along with the facts that \( \mu \circ \wedge_e = \wedge_\mu \) and \( \theta^1 Y = \theta^1 Y = Y \).

Proof. Following section 7, we interpret the transfer maps

\[ \phi_n : A(D_n X) \to A(X^n) \quad \text{and} \quad \phi_n : A(X \times B \Sigma_n) \to A(X) \]

as induced by the forgetful functors

\[ \mathcal{R}_f(X^n, \Sigma_n, \{\text{all}\}) \to \mathcal{R}_f(X^n, \{e\}) \quad \text{and} \quad \mathcal{R}_f(X, \Sigma_n \{\text{all}\}) \to \mathcal{R}_f(X, \{e\}), \]

respectively. This means we have to make non-equivariant analyses of the functors \( \tilde{\theta}^n \) and \( \theta^n \), respectively.

To obtain the surjections, we consider the following diagram

\[ \begin{array}{cccccc}
X & \xleftarrow{} & X^{n-1} \times X & \xrightarrow{((\lambda_e)_{n-1}) \times r} & \tilde{P}^{n-2} Y \wedge_e Y & \xrightarrow{} & \tilde{P}^{n-1} Y \wedge_e Y \\
X & \xleftarrow{} & X^n & \xrightarrow{r^n} & \tilde{P}^n Y & \xrightarrow{} & \tilde{P}^n Y \\
\end{array} \]

(8.5)

Clearly, \( \tilde{P}^{n-1}_n \wedge_e Y \) maps into \( \tilde{P}^{n-1}_n \), because, if there are two indices \( i, j \) with \( 1 \leq i, j \leq n-1 \) and \( i \neq j \) and with \( y_i = y_j \), then this still holds for \( (y_1, \ldots, y_{n-1}, y) \) rebracketed as \( (y_1, \ldots, y_{n-1}, y) \). Taking the pushouts along the rows using the columns two, three, and four produces a surjection

\[ \phi_{n-1} \tilde{\theta}^{n-1} Y \wedge_e \theta^1 Y \twoheadrightarrow \phi_n \theta^n Y \]

in \( \mathcal{R}_f(X^n) \) and pushing out along the rows using columns one, three and four yields

\[ \phi_{n-1} \theta^{n-1} Y \wedge_\mu \theta^1 Y \twoheadrightarrow \mu_*(\phi_{n-1} \theta^{n-1} Y \wedge_\mu \theta^1 Y) \twoheadrightarrow \phi_n \theta^n Y, \]

the surjection in \( \mathcal{R}_f(X) \). Now we have to identify the “kernels.”

Reviewing the remarks at the end of section 7, \( \tilde{P}^{n-1} Y \wedge_e Y = \tilde{P}^n Y \) is the space whose simplices outside of \( X^n \) are \( n \)-tuples of simplices from \( Y-X \); \( \tilde{P}^{n-1}_{n-2} Y \wedge_e Y \) is the space whose
simplices outside of \(X^n\) are \(n\)-tuples \(((y_1, \ldots, y_{n-1}), y)\) with the condition that there are at least two distinct indices \(1 \leq i, j \leq n-1\) with \(y_i = y_j\); and \(\tilde{P}^{n-1}_n Y\) is the space whose simplices outside of \(X^n\) are \(n\)-tuples \(((y_1, \ldots, y_{n-1}, y_n)\) with the condition that there are at least two distinct indices \(1 \leq i, j \leq n\) with \(y_i = y_j\). Then the simplices of \(\tilde{P}^{n-1}_n Y\) not in the image of \(\tilde{P}^{n-1}_{n-2} Y \cup_e Y\) are those \(n\)-tuples where the first \(n-1\) are distinct but \(y_n = y_k\) for some \(1 \leq k \leq n-1\).

Using this observation we extend the diagram \(8.5\) by means of the following constructions. For \(1 \leq k \leq n-1\), consider the diagrams

\[
\begin{array}{ccccc}
X^n & \xleftarrow{\delta_{n-1}^{n,k}} & X^{n-1} & \xrightarrow{\delta_{n-1}^{n,k}} & \tilde{P}^{n-1} Y \\
\downarrow & & \downarrow & & \downarrow \\
X^n & \xrightarrow{\delta_{n-1}^{n,k}} & \tilde{P}^{n-1} Y \times X \cup X^n X^{n-1} \times Y & \xrightarrow{\delta_{n-1}^{n,k}} & \tilde{P}^{n-1} Y \times Y \\
\end{array}
\]

where \(\delta_{n-1}^{n,k} : X^{n-1} \to X^n\) is given by \(\delta_{n-1}^{n,k}(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{n-1}, x_k)\) and the other maps labeled \(\delta_{n-1}^{n,k}\) are given by similar formulas. For each \(k\), taking the pushout of the first row extends \(\tilde{P}^{n-1} Y\) over \(X^{n-1}\) to the space \(\delta_{n-1}^{n,k} \tilde{P}^{n-1} Y\) over \(X^n\); taking the pushout of the second row yields \(\tilde{P}^{n-1} Y \cup_e Y\). Since the diagram commutes, we obtain a family of maps over \(X^n\)

\[
\delta_{n-1}^{n,k} : \delta_{n-1}^{n,k} \phi_{n-1} \tilde{P}^{n-1} Y \to \phi_{n-1} \tilde{P}^{n-1} Y \cup_e Y
\]

with \(\delta_{n-1}^{n,k}(y_1, \ldots, y_{n-1}) = (y_1, \ldots, y_k, \ldots, y_{n-1}, y_k)\).

Now we are ready to augment diagram \(8.5\) after which we can compute the desired cofibration sequence. Having established the notation

\[
\tilde{\Delta}_n^{n} \phi_{n-1} \tilde{P}^{n-1} Y = \delta_{n-1}^{n,k} \phi_{n-1} \tilde{P}^{n-1} Y \cup X^n \ldots \cup X^n \delta_{n-1}^{n,k} \phi_{n-1} \tilde{P}^{n-1} Y
\]

write \(\Delta_{n-1}^{n} : \Delta_{n-1}^{n} \phi_{n-1} \tilde{P}^{n-1} Y \to \phi_{n-1} \tilde{P}^{n-1} Y \cup_e Y\) for the union of the maps \(\delta_{n-1}^{n,k}\) just defined. Add this map above the upper right corner of the diagram \(8.5\) and fill out the following diagram:

\[
\begin{array}{ccc}
X & \xleftarrow{\mu} & X^n \\
\downarrow & & \downarrow \\
X & \xleftarrow{\mu} & X^{n-1} \times X \\
\downarrow & & \downarrow \cong \\
X & \xleftarrow{\mu} & X^n \\
\end{array}
\]

\[
\begin{array}{ccccc}
\Delta_{n-1}^{n} \phi_{n-1} \tilde{P}^{n-1} Y & \xrightarrow{\phi_{n-1} \tilde{P}^{n-1} Y \cup_e Y} & \Delta_{n-1}^{n} \phi_{n-1} \tilde{P}^{n-1} Y \\
\downarrow & & \downarrow \\
\tilde{\phi}_{n-1} \tilde{P}^{n-1} Y \cup_e Y & \xrightarrow{i} & \phi_{n-1} \tilde{P}^{n-1} Y \cup_e Y \\
\downarrow & & \downarrow \cong \\
\tilde{P}^{n-1} Y & \xrightarrow{i''} & \tilde{P}^{n-1} Y
\end{array}
\]  

(8.6)

To explain the entry at the top of the third column, we identify the conditions on

\[(z_1, \ldots, z_n) \in (\phi_{n-1} \tilde{P}^{n-1} Y \cup_e Y)\quad \text{and} \quad (y_1, \ldots, y_{n-1}) \in \Delta_{n-1}^{n} \phi_{n-1} \tilde{P}^{n-1} Y\]

such that \(i(z_1, \ldots, z_n) = \Delta_{n-1}^{n}(y_1, \ldots, y_{n-1})\). We see that \(z_j = y_j\) for \(1 \leq j \leq n-1\) and that there is \(k\) between \(1\) and \(n-1\) such that \(z_n = y_k\). Moreover, since no more than \(n-2\) of the
first \( n-1 \) simplices \( z_j \) are distinct, no more than \( n-2 \) of the simplices \( y_j \) are distinct. Hence, we obtain the description of the term at the top of the third column. Additionally we obtain the fact that the induced map

\[
(\phi_{n-1} - P_{n-2}^n) \cup (\Delta_{n-1} \phi_{n-1} - P_{n-2}^n) (\Delta_{n-1} \phi_{n-1} - P_{n-2}^n) \to \phi_{n-1} - P_{n-2}^n \land e Y
\]

is a cofibration, so lemma 2.6 applies to diagram (8.6).

One takes the row-wise pushout of the three columns on the right and obtains the following cofibration sequence in \( R_f(X^n) \).

\[
\Delta_{n-1} \phi_{n-1} - \theta_{n-1}^n \to \phi_{n-1} - \theta_{n-1}^n \land \theta^1 Y \to \phi_{n} - \theta^n Y,
\]

which is (8.3) from the statement.

One also compose the arrows pointing to the left in each row and take the row-wise pushout of the resulting diagram, which consists of columns one, three, and four of the diagram (8.6), obtaining

\[
\Delta_{n-1} \phi_{n-1} - \theta_{n-1}^n \to \phi_{n-1} - \theta_{n-1}^n \land \theta^1 Y \to \phi_{n} - \theta^n Y,
\]

which is the second cofibration sequence (8.4) in the statement.

We want to apply the cofibration sequence (8.4) to evaluate the composite \( \phi_{n} - \theta^n \) on a homotopy class in \( \pi_j A(X) \), where the basepoint is taken in the zero component. Two features of algebraic \( K \)-theory make this possible. The first feature is essentially a consequence of the additivity theorem and says that cofibration sequences imply additive relations.

**Lemma 8.4.** Let \( Z \) be a space. The two composite maps

\[
|hS_2R_f(Z)| \xrightarrow{t} |hR_f(Z)| \to \Omega|hS\cdot R(Z)|
\]

are homotopic, where the right hand arrow is the canonical map

\[
|hR_f(Z)| \to \Omega|hS\cdot R_f(Z)|.
\]

The second feature is the triviality of products in higher homotopy groups, explained as follows. Since \( X \) is a simplicial abelian group, the homotopy functor \( Y \mapsto [Y, A(X)] \) has a ring structure induced from the bi-exact pairing

\[
R(X) \times R(X) \to R(X \times X) \to R(X).
\]

Now suppose \( Y = \Sigma Y' \) is a suspension. Under this ring structure the product of two elements \([f_1]\) and \([f_2]\) in \([Y, A(X)]\) is zero, because \([f_1]\) may be represented by a map taking the upper cone \( C_+ Y' \) in \( \Sigma Y' \) to the point in \( A(X) \) represented by the zero element in \( R_f(X) \), while \([f_2]\) is represented by a map taking the lower cone \( C_- Y' \) in \( \Sigma Y' \) to the zero element. In a similar manner, there are pairings

\[
R(X^{n-1}) \times R(X) \to R(X^{n-1} \times X) = R(X^n)
\]
and these are also zero on higher homotopy groups. Combining these observations means we have a chance to compute by induction the action of $\phi_n \theta^n$ on higher homotopy groups, because at each stage of the induction the middle term of the relevant cofibration contributes nothing to the final answer.

To start the induction, we compute $(\phi_2 \theta^2)_*[f]$ for $f : S^j \to A(X)$. Applying the additivity theorem to the cofibration sequence (8.4), we can write

$$(\phi_2 \theta^2)_*[f] = (\theta_1^1[f] \wedge_\mu \theta_1^1[f]) - (\Delta_1^2 \theta_1^1)_*[f]$$

For the first term on the right side of the equation, we have observed that this product is zero. So we first obtain

$$(\phi_2 \theta^2)_*[f] = - (\Delta_1^2 \theta_1^1)_*[f]$$  \hspace{1cm} (8.7)$$

We analyse this expression as follows. First, $\phi_1$ and $\theta_1^1$ are identity functors. For $n = 2$, there is one diagonal map $\delta_{1}^{2,1} : Z \to \delta_{1}^{2,1}Z$ so $\Delta_1^2 \phi_1 \theta_1^1 Y = \delta_{1}^{2,1} \phi_1 \theta_1^1 Y = \delta_{1}^{2,1} \theta_1^1 Y$. Then $\Delta_1^2 \phi_1 \theta_1^1 = \mu_* \circ \Delta_1^2 \phi_1 \theta_1^1 = \mu_* \circ \delta_{1}^{2,1}$, and the point is to see what is happening with the retraction $r : Y \to X$. Applying formula (8.2), the composition

$$\mu \circ (\Delta_1^2 r) \circ \delta_{1}^{2,1} = \mu(r(y), r(y)) = (r(y))^2 = (\tau_2 \circ r)(y),$$

where $\tau_2 : X \to X$ is the squaring homomorphism. That is, the action of $\Delta_1^2 = \mu_* \tilde{\Delta}_1^2$ on homotopy is the same as the action on homotopy induced by the squaring homomorphism $\tau_2$. Consequently,

$$(\phi_2 \theta^2)_*[f] = - \tau_2^2 [f].$$

The general result is

**Theorem 8.5.** Let $\tau^n : X \to X$ be the homomorphism that raises elements to the $n$th power, thinking of the operation in $X$ as multiplication. Then

$$\phi_n \theta^n_* = (-1)^{n-1} \cdot (n-1)! \cdot \tau^n_* : \pi_j A(X) \to \pi_j A(X)$$

for $j > 0$.

**Proof.** First we observe that on higher homotopy groups

$$(\phi_n \theta^n)_* = (-1)^{n-1} \cdot (\Delta_{n-1}^n \tilde{\Delta}_{n-2}^{n-1} \cdots \tilde{\Delta}_1^2)_*$$

An application of the cofibration sequence (8.4) and the vanishing product principle gives $(\phi_n \theta^n)_* = (-1) \cdot (\Delta_{n-1}^n \phi_{n-1} \theta^{n-1})_*$. Then one continues with applications of the cofibration sequence (8.4) and the vanishing pairing principle.

$$(\phi_n \theta^n)_* = (-1)^2 \cdot (\Delta_{n-1}^n \tilde{\Delta}_{n-2}^{n-1} \cdots \tilde{\Delta}_2^0)_* = \cdots = (-1)^{n-1} \cdot (\Delta_{n-1}^n \tilde{\Delta}_{n-2}^{n-1} \cdots \tilde{\Delta}_1^2)_*,$$

recalling that $\tilde{\phi}_1$ and $\tilde{\theta}_1$ are identity functors.

Since the functors $\tilde{\Delta}_{n-1}^p$ are built by unions from functors $\delta_{p-1}^{n,k}$ we have to analyse composites

$$\delta_{n-1}^{n,k,n-1} \circ \delta_{n-2}^{n-1,k,n-2} \circ \cdots \circ \delta_{1}^{2,1} : R_f(X) \to R_f(X^n)$$
for all choices of indices $1 \leq k_{n-1} \leq n-1$, $1 \leq k_{n-2} \leq n-2$, \ldots, $1 \leq k_2 \leq 2$. On $(Y, r, s)$ the value of the chain is $(Y \cup_X X^n, r^n, s)$, where the retraction $r^n: Y \to X^n$ is evaluated by repeated application of formula (8.1). When we apply $\mu_\ast$ to this object, the value on $(Y, r, s)$ is seen to be $(Y, \tau^n \circ r, s)$. Finally, we identify the numerical coefficient $(n-1)!$ by counting the number of terms in the composites $\tilde{\Delta}_n \cdots \tilde{\Delta}_2$ according to the description above.

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