ADDITIVE UNITS OF PRODUCT SYSTEM

B. V. RAJARAMA BHAT, MARTIN LINDSAY, AND MITHUN MUKHERJEE

Abstract

We introduce the notion of additive units and roots of a unit in a spatial product system. The set of all roots of any unit forms a Hilbert space and its dimension is the same as the index of the product system. We show that a unit and all of its roots generate the type I part of the product system. Using properties of roots, we also provide an alternative proof of the Powers’ problem that the cocycle conjugacy class of Powers sum is independent of the choice of intertwining isometries. In the last section, we introduce the notion of cluster of a product subsystem and establish its connection with random sets in the sense of Tsirelson ([27]) and Liebscher ([11]).

1. Introduction

A fundamental goal of quantum dynamics is the classification of semigroups of unital \( \ast \)-endomorphisms of the algebra of all bounded operators on a separable Hilbert space up to cocycle conjugacy. Associated with every such ‘\( E_0 \)-semigroup’, is a (tensor) product system of Hilbert spaces ([1]). This translates the problem of classification of \( E_0 \)-semigroups up to cocycle conjugacy into the problem of classification of the product systems up to isomorphism. A product system is a measurable family of separable Hilbert spaces \( (\mathcal{E}_s)_{s>0} \) with associative identification \( \mathcal{E}_{s+t} \cong \mathcal{E}_s \otimes \mathcal{E}_t \) through unitaries. A unit is a measurable section of non-zero vectors \( (u_s)_{s>0}, u_s \in \mathcal{E}_s \) which factorises: \( u_{s+t} = u_s \otimes u_t, s,t > 0 \). Depending on the existence of units, product systems are classified into three categories. A product system is said to be of type I if units exist and they ‘generate’ the product system. A product system is said to be of type II if it has a unit but they fail to ‘generate’ the product system. Product systems having units are also known as spatial product systems. A product system is said to be of type III or non-spatial if it does not have any unit. Spatial product systems have an index. The index is a complete invariant for type I product systems and each is cocycle conjugate to a CCR flow ([2]). There is an operation of tensoring on the category of product systems. The index is additive under the tensor product of spatial product systems. Product systems of type II and type III exist in abundance but their classification theory is far from complete. It was shown that there are uncountably many cocycle conjugacy classes of type II and type III product systems ([17],[18],[29],[28]) but we still lack good invariants to distinguish them.

Tsirelson ([27],[26]) established interesting new examples of type II product systems coming from measure types of random sets or generalized random (Gaussian) processes. Liebscher, ([11]) then made a systematic study of measure types of random sets. Given a pair of product systems, one contained in the other, one associates a measure type of random (closed) sets of the interval \([0,1]\). These measure types are stationary and factorizing over disjoint intervals. The corresponding measure type is an invariant of the product system. See [11] for more details.

Contractive semigroups of completely positive maps are known as quantum dynamical semigroups. The dilation theory of quantum dynamical semigroups ([4]) reveals a new approach to understand \( E_0 \)-semigroups. Every unital quantum dynamical semigroup dilates to an \( E_0 \)-semigroup and the minimal dilation is unique up to conjugacy.

Similarly, \( E_0 \) semigroups on general \( C^* \)-algebras or von Neumann algebras correspond to product systems of Hilbert modules, ([14],[20],[21]). Much of the theory of product system of Hilbert spaces...
and the theory of $E_0$-semigroups acting on $\mathcal{B}(H)$ can be carried through also for the product systems of Hilbert modules and $E_0$ semigroups acting on $\mathcal{B}^\infty(E)$, the algebra of all adjointable operators on a Hilbert module. However there is no natural tensor product operation on the category of product systems of Hilbert modules. Skeide ([23]) overcame this by introducing the spatial product of spatial product systems of Hilbert modules in which the reference units (normalized) are identified and under which the index of the spatial product system of Hilbert module is additive. Restricting to the case of spatial product systems of Hilbert spaces, we have another operations on the category of spatial product systems. Suppose $\mathcal{E}$ and $\mathcal{F}$ are two spatial product systems with normalized units $u$ and $v$ respectively. The spatial product can be identified with the product subsystem of the tensor product, generated by the two subsystems $\mathcal{E} \otimes v$ and $u \otimes \mathcal{F}$. This raises the question whether the spatial product is the tensor product or not. Powers ([19]) answered this in the negative sense by solving the seemingly different but equivalent following problem:

Suppose $\phi = \{ \phi_t : t \geq 0 \}$ and $\psi = \{ \psi_t : t \geq 0 \}$ are two $E_0$ semigroups on $\mathcal{B}(H)$ and $\mathcal{B}(K)$ respectively and $U = \{ U_t : t \geq 0 \}$ and $V = \{ V_t : t \geq 0 \}$ are two strongly continuous semigroups of isometries which intertwine $\phi_t (\phi_t(A)U_t = U_t A, \forall A \in \mathcal{B}(H), t \geq 0)$ and $\psi_t$ respectively. Note that the intertwining isometries of $E_0$-semigroups correspond bijectively to the normalized units of the associated product systems. Consider the CP semigroup (Powers sum) $\tau_t$ on $\mathcal{B}(H \oplus K)$ defined by

$$\tau_t \left( \begin{array}{cc} X & Y \\ Z & W \end{array} \right) = \left( \begin{array}{cc} \phi_t(X) & U_t Y V_t^* \\ V_t Z U_t^* & \psi_t(W) \end{array} \right).$$

How is the product system of the minimal dilation (in the sense of [9],[4]) of $\tau$ related to the product systems of $\phi$ and $\psi$? Skeide ([22]) identified the product system as a spatial product through normalized units. The definition of Powers’ sum easily extends to CP semigroups and the product system of Powers’ sum in that case also is the spatial product of the product systems of its summands ([7],[24]). Motivated by this problem and its straightforward generalization to more general ‘corner’, amalgamated product (see Section 2) through general contractive morphism of two product systems (not necessarily spatial) was introduced in [8] which generalizes the spatial product. The spatial product may be viewed as an amalgamated product through the contractive morphism defined through normalized units. This answers Powers’ problem for the Powers’ sum obtained from not necessarily isometric intertwining semigroups.

The structure of the spatial product, a priori depends on the choice of the reference units in their respective factors. In fact, Tsirelson ([30]) showed that the group of all automorphisms of a product system may not act transitively on the set of all units. It raises another question whether the isomorphism class of the spatial product depends on the choice of the reference units. Equivalently, whether the cocycle conjugacy class of the minimal dilation of Powers sum depends on the choice of the intertwining isometries. This was answered in the negative sense in [5]. See also [6].

In this paper, we start with a brief overview of the theory of inclusion systems and amalgamated products to make the readers familiar with these notions which we use repeatedly. Readers are referred to [8], [16] for more details. In Section 3, we introduce the notion of additive units and roots of a unit in a spatial product system. Additive units are measurable sections of product system which are ‘additive with respect to a given unit’. Roots are the special additive units such that for each $t > 0$, the sections are orthogonal to the unit. The set of all additive units forms a Hilbert space and the set of all roots is a subspace of co-dimension one. We compute all the roots of the vacuum unit in CCR flows $(\Gamma_{sym}(L^2[0,t], K))$. They are given by the set of all $c \chi_c, c \in K$ almost surely. From this, we establish that a unit and all of its roots ‘generate’ the type I part of the product system and the dimension of the Hilbert space of the set of all roots of a unit is the same for every unit and coincides with the index of the product system. We also generalize the notion of additive units and roots of a unit on the level of inclusion systems (see Section 2). We show that the set of all additive units of a unit in an inclusion system are in a bijective correspondence with the set of all additive units of the ‘lifted’ unit in the generated algebraic product system. The behaviour of the roots under amalgamated product is also studied. Using the properties of roots, we have an alternating proof of the fact that the Powers sum is independent of the choice of the intertwining isometries or equivalently that the isomorphism class of the amalgamated product through normalized units is independent of the choice of the units (see Section 4). In fact, we have an improvement of this result which says that the isomorphism class of the amalgamated product
through strictly contractive units is also independent of the choice of the units. This fact will be explained elsewhere ([15]).

In Section 5, given any product subsystem $\mathcal{F}$ of a product system $\mathcal{E}$, we construct an intermediate subsystem called the cluster subsystem of $\mathcal{F}$. A product subsystem corresponds to an ‘adapted’ family of commutative projections satisfying some relation. The commutative von Neumann algebra generated by them is uniquely determined by a measure type of random closed sets of the interval $[0, 1]$. The distribution of the random mapping which sends a closed set to its limit points is the measure type of the cluster system of the original product subsystem. In a special case, the measure type corresponding to a single unit and the measure type corresponding to the type I part, both share the same relation. See Proposition 3.33, Chapter 3, [11]. Liebscher’s proofs of those facts use heavy machinery from measure theory of random sets and the direct integral construction. Here we explicitly construct the cluster subsystem without involving any heavy machinery. We show that the measure type corresponding to the subsystem and the measure type of its cluster are related by the above random mapping. Without using any random sets theory, we also compute that the cluster of the subsystem generated by a single unit in a spatial product system is the type I part of the product system.

2. Inclusion system and amalgamation

An inclusion system is a parametrized family of Hilbert spaces exactly like product system but the connecting maps are now only isometries. These objects seem to be ubiquitous in the field of product system. They are the recurrent theme of studying quantum dynamics, in particular CP semigroups. (See [10],[14],[12],[20],[8]). Even while associating product systems to CP semigroups what one gets first are inclusion systems, and then an inductive limit procedure gives product systems ([10],[8]). The notion of inclusion systems is introduced in [8]. It was also introduced by Shalit and Shoel (20) under the name subproduct system. The following definition is taken from [8].

**Definition 1.** An inclusion System $(E, \beta)$ is a family of Hilbert spaces $E = \{E_t, t \in (0, \infty)\}$ together with isometries $\beta_{s,t}: E_{s+t} \to E_s \otimes E_t$, for $s, t \in (0, \infty)$, such that $\forall r, s, t \in (0, \infty), (\beta_{r,s} \otimes 1_{E_t})\beta_{r+s,t} = (1_E \otimes \beta_{s,t})\beta_{r,s+t}$. It is said to be an algebraic product system if further every $\beta_{s,t}$ is a unitary.

**Definition 2.** Suppose $(E, \beta)$ is an inclusion system. Then a family $F = (F_t)_{t>0}$ of closed subspaces, $F_t \subset E_t$ is said to be an inclusion subsystem of $(E, \beta)$ if $\beta_{s,t}|_{F_{s+t}}(F_{s+t}) \subset F_s \otimes F_t$ for every $s, t > 0$.

For each $t \in \mathbb{R}_+$, we set

$$J_t = \{ (t_1, t_2, \ldots, t_n) : t_i > 0, \sum_{i=1}^n t_i = t, n \geq 1 \}.$$ 

For $s = (s_1, s_2, \ldots, s_m) \in J_s$, and $t = (t_1, t_2, \ldots, t_n) \in J_t$, we define $s \sim t := (s_1, s_2, \ldots, s_m, t_1, t_2, \ldots, t_n) \in J_{s+t}$. Now fix $t \in \mathbb{R}_+$. On $J_t$, define a partial order $t \succeq s = (s_1, s_2, \ldots, s_m)$ if for each $i, (1 \leq i \leq m)$ there exists (unique) $s_i \in J_{s_i}$, such that $t = s_1 \sim s_2 \sim \cdots \sim s_m$. The order relation $\succeq$ makes $J_t$ a directed set.

Suppose $(E, \beta)$ is an inclusion system. For $s = (s_1, \ldots, s_n) \in J_s$, we set $E_s = E_{s_1} \otimes \cdots \otimes E_{s_n}$. For $s = (s_1, \ldots, s_n) \leq t = s_1 \sim \cdots \sim s_n \in J_t$, define $\beta_{s,t} : E_s \to E_t$ by $\beta_{s,t} = \beta_{s_1,s_1} \otimes \cdots \otimes \beta_{s_n,s_n}$, where $\beta_{s,s} = I_{E_s}$ and for $s = (s_1, \ldots, s_n) \in J_s$, inductively define

$$\beta_{s,s} = (I \otimes \beta_{s_{n-1},s_n}) \cdots (I \otimes \beta_{s_2,s_3}) \cdots \beta_{s_1,s_1}.$$

Proof of the following theorem can be found in Theorem 5, [8].

**Theorem 3.** Suppose $(E, \beta)$ is an inclusion system. Let $\mathcal{E}_t = \text{indlim}_{s \succeq t} E_s$ be the inductive limit of $E_s$ over $J_t$ for $t > 0$. Then $\mathcal{E} = \{ \mathcal{E}_t : t > 0 \}$ has the structure of an algebraic product system.

Let $(\mathcal{E}, B)$ be the generated algebraic product system of the inclusion system $(E, \beta)$. Note that the unitary map $B_{s,t}$ goes from $\mathcal{E}_{s+t}$ to $\mathcal{E}_s \otimes \mathcal{E}_t$ for every $s, t > 0$. In other words, algebraic product systems are inclusion systems with all the linking maps are unitaries. Observe that any product system is an algebraic product system, but not the converse may not be true. The multiplication operation of a product
system $E$ gives rise to the unitary maps which goes from $E_s \otimes E_t$ to $E_{s+t}$ for every $s, t > 0$. Adjoints of these unitary maps obviously associative and makes it into an algebraic product system. Therefore we can assume that a product system is a special algebraic product system. Though the linking maps implement ‘co-product’ rather than ‘product’ but abusing of terminology, we call it an algebraic product system. Nevertheless, we can talk about an inclusion subsystem of a product system. The following important fact that an inclusion subsystem in a product system generates a product subsystem is used throughout without reference. For the proof, see Lemma 33, Appendix A. The following definition is taken from [8].

**Definition 4.** Let $(E, \beta)$ be an inclusion system. Let $u = \{u_t : t > 0\}$ be a family of vectors such that (1) for all $t > 0$, $u_t \in E_t$ (2) there is a $k \in \mathbb{R}$, such that $\|u_t\| \leq \exp(tk)$, for all $t > 0$. and (3) $u_t \neq 0$ for some $t > 0$. Then $u$ is said to be a unit if

$$u_{s+t} = \beta_{s,t}^+ (u_s \otimes u_t) \forall s, t > 0.$$

Let $i_t : E_t \to E_t$ be the canonical embedding.

**Theorem 5.** Let $(E, \beta)$ be an inclusion system and let $(E, B)$ be the algebraic product system generated by it. Then the map $i^*$ provides a bijection between the set of all units of $(E, B)$ and the set of all units of $(E, \beta)$ by letting it acts point-wise on units.

For the proof, readers are referred to Theorem 10, [8].

Fix a unit $u$ of $(E, \beta)$. Then by the above theorem there is a unique unit $u$ in $(E, B)$ such that for every $t > 0$, $i^*_t (u_t) = u_t$. We say $u$ as the ‘lift’ of $u$. Note that if $u$ is normalized, then $u$ is also normalized.

**Amalgamation**

The amalgamated product of two product systems over a contractive morphism is introduced in [8]. The index of the amalgamated product over general contractive morphism is computed in [16]. The following theorem characterizes the amalgamated product. See Theorem 2.7, [16].

**Theorem 6.** Suppose $(E, W_E)$ and $(F, W_F)$ are two product systems and let $C : (F, W_F) \to (E, W_E)$ be a contractive morphism. Then there exist an algebraic product system $(\mathcal{G}, W_\mathcal{G})$ and isometric product system morphisms $I : \mathcal{E} \to \mathcal{G}$ and $J : F \to \mathcal{G}$ such that the following holds:

1. $(I_s(x), I_s(y)) = (x, C_s y)$ for all $x \in E_s$ and $y \in F_s$.
2. $I = I(\mathcal{E}) \cup J(F)$.

$\mathcal{G}$ is said to be the amalgamated product of $E$ and $F$ over the contractive morphism $C$ and denoted by $\mathcal{G} = E \otimes_C F$. For the details of construction, we refer to Section 3, [8].

### 3. Additive units

Suppose $E$ is a product system. The multiplication operation in $E$ is as follows: For $s, t > 0, a \in E_s, b \in E_t$, we have $a \cdot b \in E_{s+t}$ and $E_{s+t} = \text{span} E_s \cdot E_t$. Also for $a, a' \in E_s, b, b' \in E_t$, we have

$$\langle a \cdot a', b \cdot b' \rangle_{E_{s+t}} = \langle a, b \rangle_{E_s} \langle a', b' \rangle_{E_t}.$$

In this section, we abbreviate the multiplication $a \cdot b$ as $ab$.

**Definition 7.** Let $E$ be a spatial product system and let $u = (u_t)_{t>0}$ be a unit of $E$. A measurable section $(a_t)_{t>0}$ of $E$ is said to be an additive unit of $u$ if for all $s, t > 0$,

$$a_{s+t} = a_s u_t + u_s a_t.$$

**Definition 8.** An additive unit $a = (a_t)_{t>0}$ of a unit $u = (u_t)_{t>0}$ is said to be a root if $\langle a_t, u_t \rangle = 0$ for all $t > 0$.

**Remark 9.** It is clear that the set of all additive units of a given unit forms a vector space under point wise addition and point wise scalar multiplication. The set of all roots forms a vector subspace of it. Indeed if $a = (a_s)_{s>0}$ and $b = (b_s)_{s>0}$ are two additive units(roots) of a unit $u$, then clearly $\lambda a := (\lambda a_s)_{s>0}$ and $(a + b) := (a_s + b_s)_{s>0}$ are additive units(roots) of $u = (u_s)_{s>0}$. Also note that, if $a$ is an additive
unit (roots) of $u$, then $(a')_{s>0}$ which is defined by $a'_s = \exp(\lambda s)a_s$, is an additive unit (roots) of $(u'_s)_{s>0}$, where $u'_s = \exp(\lambda s)u_s$. In other words, the additive units of a unit are completely determined by the additive units of the normalized unit.

**Example 10.** Let $u = (u_s)_{s>0}$ be a unit in a product system $\mathcal{E}$. Then the measurable section $b = (b_s)_{s>0}$ given by $b_s = \lambda s u_s$, for some $\lambda \in \mathbb{C}$, $s > 0$, is an additive unit of the unit $u$. We call them as the trivial additive units of the unit $u$.

Let $a$ be an additive unit of a unit $u$. For $s > 0$, consider the measurable function

$$f : \mathbb{R}_+ \rightarrow \mathbb{C}$$

given by

$$f(s) = \langle u_s, a_s \rangle \|u_s\|^{-2}.$$  

Then a simple computation shows that $f(s + t) = f(s) + f(t)$, $s, t > 0$. This implies $f(s) = sf(1)$.

Decomposing $a_s = b_s + b'_s$, where

$$b_s = \langle u_s, a_s \rangle \|u_s\|^{-2}u_s$$

and

$$b'_s = a_s - \langle u_s, a_s \rangle \|u_s\|^{-2}u_s,$$

we find that $b_s = (\lambda s u_s)_{s>0}$ for some $\lambda \in \mathbb{C}$ and $b'$ is a root of $u$. In other words, every additive unit decomposes uniquely as a trivial additive unit and a root. From the remark, we may assume without loss of generality that our unit $u$ is normalized, i.e. $\|u_s\| = 1$, for every $s > 0$. Let $a$ and $b$ be two roots of the normalized unit $u$. Then a similar computation shows that

$$\langle a_s, b_s \rangle = s \langle a_1, b_1 \rangle, \quad s > 0.$$  

Now consider $a, b$ two additive units of $u$. Then we can decompose

$$a_s = c_s + c'_s, \quad b_s = d_s + d'_s, \quad s > 0,$$

where

$$c_s = s\langle u_1, a_1 \rangle u_s, \quad d_s = s\langle u_1, b_1 \rangle u_s, \quad s > 0,$$

and $c', d'$ are roots of $u$ with

$$\langle c'_s, d'_s \rangle = s\langle c'_1, d'_1 \rangle.$$  

Now $\langle c'_s, d'_s \rangle = \langle (a_1 - (u_1, a_1)u_1), (b_1 - (u_1, b_1)u_1) \rangle = \langle a_1, b_1 \rangle - \langle a_1, u_1 \rangle \langle u_1, b_1 \rangle$. From this, a simple computation shows $\langle a_s, b_s \rangle = s^2\langle a_1, u_1 \rangle \langle u_1, b_1 \rangle + s\langle a_1, b_1 \rangle - s\langle a_1, u_1 \rangle \langle u_1, b_1 \rangle$. In other words,

$$\langle a_s, b_s \rangle = \langle \theta_s a_1, \theta_s b_1 \rangle,$$

where $\theta_s : \mathcal{E}_1 \rightarrow \mathcal{E}_1$ is given by $\theta_s = [sI + (s^2 - s)|u_1]^{\frac{1}{2}}$.

**Proposition 11.** Let $u$ be a normalized unit of a product system $\mathcal{E}$. Then the set of all additive units of $u$ forms a Hilbert space under the inner product $(a, b) = : (a_1, b_1)_{\mathcal{E}_1}$ and the set of all roots of $u$ is a closed subspace of co-dimension one.

**Proof:** Let us denote by $A^u_\mathcal{E}$ and $R^u_\mathcal{E}$ be the vector spaces of all additive units and roots of $u$ respectively. For $a, b \in A^u_\mathcal{E}$, define an inner product on $A^u_\mathcal{E}$ by $\langle a, b \rangle = \langle a_1, b_1 \rangle$. Let $(a^n)_{n \geq 1}$ be a Cauchy sequence. i.e. $\|a^n - a^m\| \rightarrow 0$ as $m, n \rightarrow \infty$. Now from Equation 3.1, we get for each $s > 0 \|a^n_s - a^m_s\| = \|\theta_s(a^n_s - a^m_s)\| \leq \|\theta_s\|\|a^n - a^m\| \rightarrow 0$. Let for each $s > 0$, $a_s = \lim_{n \rightarrow \infty} a^n_s$. The section $(a_s)_{s>0}$ is clearly measurable as being point-wise limit of measurable sections. Now we will show that $(a_s)_{s>0}$ is an additive unit of $u$. Let $\epsilon > 0$ be given. For $s, t > 0$ choose $N$ such that for $n > N$, $\|a^n_s - a_s\| \leq \frac{1}{N} \epsilon$, $\|a^n_t - a_t\| \leq \frac{1}{N} \epsilon$ and $\|a^n_{s+t} - a_{s+t} - a_s a_t\| \leq \frac{1}{N} \epsilon$. Then

$$\|a_{s+t} - a_s u_t - u_s a_t\| \leq \|a^n_{s+t} - a^n_s u_t - a^n_t u_s + a^n_s u_t - a_s u_t + u_s a^n_t - u_s a_t\| \leq \epsilon.$$  

So $a \in A^u_\mathcal{E}$ and $\|a^n - a\| \rightarrow 0$. This proves that $A^u_\mathcal{E}$ is complete with respect to the inner product. Other part is trivial. \(\square\)
Proposition 12. The set of all roots of the vacuum unit in CCR flow of index $k$ is given by the set 
\{c\chi_t : c \in K\} almost everywhere.

Proof: It is easy to see that $c\chi_t, c \in K$ are the roots of the vacuum unit. To prove the converse, if $a$ is a root of the vacuum unit, then in Guichardet’s picture described in Appendix B, the following identity is valid almost everywhere,

\[ a_s(\sigma) = \begin{cases} 
  a_s(\sigma \cap [t, s + t] - t) & \text{if } \sigma \cap [0, t] = \emptyset \\
  a_s(\sigma \cap [0, t]) & \text{if } \sigma \cap [t, s + t] = \emptyset \\
  0 & \text{otherwise.}
\end{cases} \]

Fix $s \in \mathbb{R}_+$. Let $k$ be any natural number. Denote by $[a, b]'$ the complement of $[a, b]$ in $[0, s]$. Then we have the identity, almost everywhere,

\[ a_s(\sigma) = \begin{cases} 
  a_s(\sigma \cap [\frac{(k-1)s}{k}, s] - \frac{(k-1)s}{k}) & \text{if } \sigma \cap [\frac{(k-1)s}{k}, s]' = \emptyset \\
  a_s(\sigma \cap [\frac{(k-2)s}{k}, \frac{(k-1)s}{k}] - \frac{(k-2)s}{k}) & \text{if } \sigma \cap [\frac{(k-2)s}{k}, \frac{(k-1)s}{k}]' = \emptyset \\
  \vdots & \text{if } \sigma \cap [0, \frac{t}{k}]' = \emptyset \\
  0 & \text{otherwise.}
\end{cases} \]

Suppose that $\# \sigma = n$, then the subset of $\Delta_n(s)$, where $a_s$ is non zero except on a set of measure zero, is contained in

\[ \bigcup_{i=0}^{k-1} \Delta_n(s/k) + is/k, \text{ for all } k = 1, 2, \ldots. \]

The Lebesgue measure of the set $\bigcup_{i=0}^{k-1} \Delta_n(s/k) + is/k$ is $s^n/nk^{n-1}$. So the Lebesgue measure of the set $\bigcap_{k \geq 1} \bigcup_{i=0}^{k-1} \Delta_n(s/k) + is/k$ is zero for $n \geq 2$. It follows that $a_s$ vanishes on $\Delta_n(s)$, for $n \geq 2$. As it is a root, it is orthogonal to the vacuum unit, we conclude that, $a_s$ is a measurable function in $L^2([0, s], K)$ with the property, a.e.

\[ a_s = a_r + S_r a_{s-r}, \forall r, 0 < r < s. \]

where $S_t$ on $L^2(\mathbb{R}_+, K)$ defined by

\[ (S_t f)(s) = \begin{cases} 
  f(s-t) & \text{if } s \geq t \\
  0 & \text{otherwise.}
\end{cases} \]

For every $x \in K$, define the measurable function $A_x : \mathbb{R}_+ \to \mathbb{C}$, by $A_x(s) = \langle a_s, x\chi_s \rangle$. An easy calculation shows that $A_x(s + t) = A_x(s) + A_x(t)$. Its measurable solution is given by $A_x(s) = sA_x(1)$. Let us define the linear functional $f : K \to \mathbb{C}$ by $f(x) = \langle a_1, x\chi_1 \rangle$, for $x \in K$. It is bounded as $\|f\| \leq \|a_1\|_L^2$. So by Riesz representation theorem there is a unique $y \in K$ such that $f(x) = \langle y, x \rangle$. Now for $r \leq s, z \in K,$

\[ \langle a_s - y\chi_s, z\chi_r \rangle = \langle a_r, z\chi_r \rangle - r(y, z) = rA_r(1) - r(y, z) = 0. \]

As the set $\{z\chi_r : z \in K, 0 \leq r \leq s\}$ is total in $L^2([0, s], K)$, we have $a_s = y\chi_s$. \hfill \Box

Let us denote by $R^\infty_u$, the Hilbert space of roots of the unit $u$ in $\mathcal{E}$.

Theorem 13. Suppose $\mathcal{E}, W$ is a product system and $u$ is a normalized unit of $\mathcal{E}$. Then $\dim R^\infty_u = index \mathcal{E}$.

Proof. First we claim that roots of $u$ are in $\mathcal{E}',$ the type I part of $\mathcal{E}$. Given a root $a$ of $u$, $\|a\| = 1$, set $E_u = \sigma \{a_u, a_s\}$. Then it is easy to see that $(E, W|_E)$ is an inclusion system. Let $\Gamma_{sym}(L^2[0, t])$ be the symmetric Fock product system. Define $\phi_s : E_u \to \Gamma_{sym}(L^2[0, t])$ by $\phi_s(a_u + \beta a_s) = a\Omega_s + \beta \chi_s$. Then $\phi = (\phi_s)_{s > 0}$ is an isometric morphism of inclusion system. So the product system generated by $u$ and $a$ is isomorphic to a type I product system. This proves the claim. Any isomorphism of $\mathcal{E}'$ to $\Gamma_{sym}(L^2[0, t], K)$ $(\dim K = index \mathcal{E})$ sending $u$ to vacuum unit, sends roots to roots. This implies every root of $u$ under this will be mapped to a $(c\chi_s)_{s > 0}$ and vice versa. The result now follows. \hfill \Box
Corollary 14. Let \( a \) be a root of a unit \( u \) in a spatial product system \((E, B)\). Then \( a \in \mathcal{E}^I \).

Corollary 15. Suppose \((E, B)\) is a spatial product system and \( u \) is a unit. Then the product system generated by the unit \( u \) and all roots of it, is the type \( I \) part of \((E, B)\).

We shall now define all these notions on the level of inclusion system. We quote the following definition from [8].

Definition 16. Let \((E, \beta)\) be an inclusion system and let \( u \) be a normalized unit of \((E, \beta)\). A section \((a_t)_{t \geq 0} \) of \((E, \beta)\) is said to be an additive unit of the unit \( u \) if

\[
a_{s+t} = \beta_{s,t}^*(a_s \otimes u_t + a_t \otimes u_s) \text{ and } \|a_s\|^2 \leq k(s + s^2), s \geq 0, \text{ for some } k \geq 0.
\]

Definition 17. An additive unit \( a = (a_t)_{t \geq 0} \) of a unit \( u = (u_t)_{t \geq 0} \) is said to be a root if \( \langle a_t, u_t \rangle = 0 \) for all \( t > 0 \).

Proposition 18. Let \((E, \beta)\) be an inclusion system and let \((E, B)\) be the algebraic product system generated by it. Then \( i^* \) provides a bijection between the set of all additive units of \( u \) in \((E, B)\) and the set of all additive units of \( i^*(u) \) in \((E, \beta)\) by letting it acts point-wise on units. More over if \( i^*(a) \) is a root of \( i^*(u) \), then \( a \) is a root of \( u \).

Proof: Suppose \( u \) is a unit of the algebraic product system \((E, B)\). Then by Theorem 5, \( i^*(u) \) is a unit of the of the inclusion system and \( i^*(u) = u \). Let \( a \) be an additive unit of \( u \). Consider \( i^*(a) \). Now

\[
\beta_{s,t}^*[i^*_s(a_s) \otimes i^*_t(u_t) + i^*_t(u_s) \otimes i^*_s(a_t)] = [i_s \otimes i_t]_s^*[a_s \otimes u_t + u_s \otimes a_t]
\]

\[
= [B_{s,t}i_{s+t}]^*[a_s \otimes u_t + u_s \otimes a_t]
\]

\[
= i^*_{s+t}a_{s+t}.
\]

Hence \( i^*(a) \) is an additive unit of the unit \( i^*(u) \).

Now we prove the injectivity of \( i^* \). Consider two additive units \( a \) and \( b \) of the unit \( u \) in \((E, B)\) such that \( i^*_t a_t = i^*_t b_t \) for all \( t > 0 \). Fix \( t > 0 \). For \( s = (s_1, s_2, ..., s_n) \in J_t \), Define \( a_s = \sum_{j=1}^{n} u_{s_1} \otimes u_{s_2} \otimes \cdots \otimes u_{s_{j-1}} \otimes a_{s_j} \otimes u_{s_{j+1}} \otimes \cdots \otimes u_{s_n} \) and \( b_s = \sum_{j=1}^{n} u_{s_1} \otimes u_{s_2} \otimes \cdots \otimes u_{s_{j-1}} \otimes b_{s_j} \otimes u_{s_{j+1}} \otimes \cdots \otimes u_{s_n} \). Now for \( s \in J_t \),

\[
i^*_s a_s = i^*_s B^*_s a_s
\]

\[
= (B_t i_s a_s)^s
\]

\[
= (i^*_s \otimes \cdots \otimes i^*_s) \sum_{j=1}^{n} u_{s_1} \otimes u_{s_2} \otimes \cdots \otimes u_{s_{j-1}} \otimes a_{s_j} \otimes u_{s_{j+1}} \otimes \cdots \otimes u_{s_n}
\]

\[
= (i^*_s \otimes \cdots \otimes i^*_s) \sum_{j=1}^{n} u_{s_1} \otimes u_{s_2} \otimes \cdots \otimes u_{s_{j-1}} \otimes b_{s_j} \otimes u_{s_{j+1}} \otimes \cdots \otimes u_{s_n}
\]

\[
= (B_t i_s a_s) b_s
\]

\[
i^*_s b_s
\]

\[
i^*_s b_s.
\]

This implies \( i_s i^*_s a_s = i_s i^*_s b_s \). The net of projection \( \{i_s i^*_s : s \in J_t\} \) converges strongly to the identity. So we get \( a_s = b_s \).

Conversely, let \( u \) be a unit and \( a \) be an additive unit of \( u \) in \((E, \beta)\). Fix \( t > 0 \). For \( s = (s_1, s_2, ..., s_n) \in J_t \), Define \( a_s = \sum_{j=1}^{n} u_{s_1} \otimes u_{s_2} \otimes \cdots \otimes u_{s_{j-1}} \otimes a_{s_j} \otimes u_{s_{j+1}} \otimes \cdots \otimes u_{s_n} \). Now the family \( \{i_s a_s : s \in J_t\} \) is bounded as

\[
\|i_s a_s\|^2 \leq \sum_{i=1}^{n} k(s_i + s_i^2)
\]

\[
\leq k(s + s^2).
\]
It follows from the hypothesis that, for \( s \leq t \in J_t \),
\[
a_s = \beta_{s,t}^* a_t.
\]
Now for \( s \leq t \in J_t \),
\[
i_s i_s^* i_t a_t = i_s \beta_{s,t}^* a_t = i_s a_s.
\]
Given \( \epsilon > 0 \), \( x \in E_t \), choose \( s \in J_t \) such that \( \| (I - i_s^* i_s) x \| < \epsilon \). Then for \( t \geq s \), we have
\[
\langle i_t a_t - i_s a_s, x \rangle = \langle (I - i_s^* i_s) i_t a_t, x \rangle = \langle i_t a_t, (I - i_s^* i_s) x \rangle \leq \| i_t a_t \| \| (I - i_s^* i_s) x \| \leq [k(s + s^2)]^{1/2} \epsilon.
\]
So for each \( x \in E_t \), \{ \langle i_s a_s, x \rangle : s \in J_t \} \) is a weakly Cauchy net. Set \( \phi(x) = \lim_{s \in J_t} \langle i_s a_s, x \rangle \). Then \( \phi : E_t \to \mathbb{C} \) is a bounded linear functional with \( \| \phi \| \leq k(s + s^2) \). So there is a unique vector \( \hat{a}_t \in E_t \) such that \( \phi(x) = \langle \hat{a}_t, x \rangle \). This implies for every \( x \in E_t \), \( \langle i_s a_s, x \rangle = \langle \hat{a}_t, x \rangle \). Now for \( s \in J_t \),
\[
i_s i_s^* \hat{a}_t = \lim_{t \in J_t} i_s i_s^* i_t a_t = \lim_{t \in J_t} i_s \beta_{s,t}^* a_t = i_s a_s.
\]
This shows that \{ \( i_s a_s : s \in J_t \) \} converges to \( \hat{a}_t \) in the Hilbert space norm. Let \( \hat{u} \) be the lift of \( u \) in the algebraic product system. Our claim is that \( \hat{a} = (\hat{a}_t)_{t \geq 0} \) is an additive unit of the unit \( \hat{u} = (\hat{u}_t)_{t \geq 0} \) in the algebraic product system. For \( x \in E_s \), \( y \in E_t \),
\[
\langle \hat{a}_s \otimes \hat{u}_t + \hat{a}_s \otimes \hat{u}_t, x \otimes y \rangle = \lim_{s \in J_s, t \in J_t} \langle (i_s \otimes i_t)[a_s \otimes u_t + u_s \otimes a_t], x \otimes y \rangle = \lim_{s \in J_s, t \in J_t} \langle (i_s \otimes i_t)a_{s-t}, (x \otimes y) \rangle = \lim_{s \in J_s, t \in J_t} \langle B_{s,t} i_s i_t a_{s-t}, (x \otimes y) \rangle = \langle B_{s,t} i_s i_t a_{s-t}, (x \otimes y) \rangle = \langle B_{s,t} i_s i_t a_{s-t}, (x \otimes y) \rangle.
\]
This proves the claim.
For \( x \in E_t \), we have
\[
\langle i_t^* \hat{a}_t, x \rangle = \langle \hat{a}_t, i_t x \rangle = \lim_{r \in J_t} \langle i_r a_r, i_t x \rangle = \lim_{r \in J_t} \langle i_t^* i_r a_r, x \rangle = \langle \beta_{r,t}^* \beta^* a_r, x \rangle = \langle \hat{a}_t, x \rangle.
\]
This implies \( i_t^* \hat{a}_t = a_t \).
Finally, if \( b \) is a root of a unit \( v \) in the inclusion system \((E, \beta)\), then
\[
\langle \hat{b}_t, \hat{v}_t \rangle = \lim_{r \in J_t} \langle i_r b_r, i_r v_r \rangle = \lim_{r \in J_t} \langle b_r, v_r \rangle = \lim_{r \in J_t} \sum_{j=1}^n \langle v_{r_1} \otimes v_{r_2} \otimes \cdots \otimes v_{r_{j-1}} \otimes b_{r_j} \otimes v_{r_{j+1}} \otimes \cdots \otimes v_{r_n}, v_{r_1} \otimes v_{r_2} \otimes \cdots \otimes v_{r_n} \rangle.
\]
Therefore it is enough to show that note that for every normalized unit of $\forall v \in (v_1)_{t>0}$ be a normalized unit of $\forall C_t v_t = v_t$ for all $t > 0$. Then $Cv := (C_t v_t)_{t>0}$ is a normalized unit of $\forall E$. We have for every $t > 0$, $I_t C_t v_t = J_t v_t$, where $I_t: \forall E_t \to (\forall E \otimes \forall F_t)$ and $J_t: \forall F_t \to (\forall E \otimes \forall F_t)$ are injection morphisms. Let us denote the common unit by $u$ in $\forall E \otimes \forall F$. i.e. $u_t = I_t (C_t v_t) = J_t (v_t)$ for all $t > 0$. Denoting $\forall R^E_{C_u} := \{ a_1 \in \forall E_1 : a \in \forall R^E_{C_u} \}$ and $\forall R^F_{u} := \{ b_1 \in \forall F : b \in \forall R^F_u \}$. Given two closed subspaces $H$ and $H'$ of a Hilbert space $G$, denote by $H \sqcup H'$ the smallest closed subspace of $G$ containing $H$ and $H'$.

**Theorem 19.** Suppose $(\forall E, W^E)$ and $(\forall F, W^F)$ are two spatial product systems and suppose $C = (C_t)_{t>0} : \forall F_t \to \forall E_t$ is a morphism of partial isometry. Also suppose $v = (v_t)_{t>0}$ is a normalized unit of $\forall F$ such that $C_t^* C_t v_t = v_t$ for all $t > 0$. Then $Cv := (C_t v_t)_{t>0}$ is a normalized unit of $\forall E$. We have for every $t > 0$, $I_t C_t v_t = J_t v_t$, where $I_t : \forall E_t \to (\forall E \otimes \forall F_t)$ and $J_t : \forall F_t \to (\forall E \otimes \forall F_t)$ are injection morphisms. Let us denote the common unit by $u$ in $\forall E \otimes \forall F$. i.e. $u_t = I_t (C_t v_t) = J_t (v_t)$ for all $t > 0$. Denoting $\forall R^E_{C_u} := \{ a_1 \in \forall E_1 : a \in \forall R^E_{C_u} \}$ and $\forall R^F_{u} := \{ b_1 \in \forall F : b \in \forall R^F_u \}$. Given two closed subspaces $H$ and $H'$ of a Hilbert space $G$, denote by $H \sqcup H'$ the smallest closed subspace of $G$ containing $H$ and $H'$.

Proof: We may assume from Theorem 27, [16], that $\forall E$ and $\forall F$ are subsystems of the amalgamated product $\forall E \otimes \forall F$. As $C$ is a morphism of partial isometry, we get from [16], Proposition 2.10, that for each $t > 0$, $P_{\forall E_t} \sqcup P_{\forall F_t}$ commute as elements in $B(\forall E \otimes \forall F_t)$. So $P_{(\forall E \sqcup \forall F_t)} = P_{\forall E_t} P_{\forall F_t}$, which implies $\forall E \sqcup \forall F := (\forall E \sqcup \forall F_t)$ is a product subsystem. In this identification, we have $u = v = Cv$. Hence $u$ is a normalized unit of $\forall E \sqcup \forall F$ and $\forall R^E_{u} \sqcup \forall R^F_{u}$ coincides with $\forall R^E_{u} \sqcup \forall R^F_{u}$ inside $\forall R^E \sqcup \forall R^F$. So to prove the theorem, it is enough to show that

$$\forall R^E_{u} \sqcup \forall R^F_{u} = \forall R^E \sqcup \forall F.$$ 

Clearly $\forall R^E_{u} \sqcup \forall R^F_{u} \subset \forall R^E \sqcup \forall F$. Now for $a \in \forall R^E_{u} \sqcup \forall F$, consider $b = (b_t)_{t>0}$ where $b_t = P_{\forall E_t} a_t$, $b' = (b'_t)_{t>0}$ where $b'_t = P_{\forall F_t} a_t$. We claim that

$$b = P_{\forall E} b', \quad b' = P_{\forall E} b''.$$ 

Note that for every $s > 0$,

$$u_s = P_{\forall E_s} = P_{\forall F_s} u_s = P_{\forall F_s} u_s.$$ 

As $P_{\forall E} = (P_{\forall E_s})_{s>0}$ is a projection morphism from $(\forall E \otimes \forall F, W^{\forall E \otimes \forall F})$ to $(\forall E, W^E)$, we have

$$(P_{\forall E} \sqcup P_{\forall E_s}) W^{\forall E \otimes \forall F}_{s,t} = W^{\forall E}_{s,t} P_{\forall E,t+} s,t > 0.$$ 

This implies

$$W^{\forall E}_{s,t} b_{s+t} = W^{\forall E}_{s,t} b_{s+t} = (P_{\forall E} \sqcup P_{\forall E_s}) W^{\forall E \otimes \forall F}_{s,t} a_{s+t} = (P_{\forall E} \sqcup P_{\forall E_s})(a \otimes u_t + u_s \otimes a_t) = (b \otimes u_t + u_s \otimes b_t).$$ 

This shows $b \in \forall R^E_u$. Similarly we have, $b' \in \forall R^F_u$ and $b'' \in \forall R^E \sqcup \forall F$. Also note that $b, b', b'' \in \forall R^E \sqcup \forall F$. 

Therefore

$$P_{\forall F} (a_t - c_t) = b_t - b_t = 0, \quad P_{\forall F} (a_t - c_t) = 0.$$ 

Note that $(\forall E \sqcup \forall F_t)_{t>0}$ is an inclusion system which generates the product system $\forall E \otimes \forall F$. Also note that $(P_{\forall E \sqcup \forall F_t} (a_t - c_t))_{t>0}$ is a root of $u$ in the inclusion system $(\forall E \sqcup \forall F_t)_{t>0}$ while $(a_t - c_t)_{t>0}$ is a root of $u$ in the product system $(\forall E \otimes \forall F)$. As $(\forall E \sqcup \forall F_t)_{t>0}$ generates the product system $(\forall E \otimes \forall F)$, we have from the injectivity of the map $i^*$ described in Theorem 18, for all $t > 0$, $a_t = c_t$. So $a_1 = b_1 - b''_1 + b'_1$, where $b_1 - b''_1 \in \forall R^E_u$ and $b'_1 \in \forall R^F_u$. Hence $\forall R^E_u \sqcup \forall R^F_u = \forall R^E \sqcup \forall F$. □
Suppose \((E, W^E)\) and \((F, W^F)\) are two product systems. Let \(u^0\) and \(v^0\) be two normalized units of \(E\) and \(F\) respectively. Consider \(E \otimes_C F\), where \(C_t = |u^0_t⟩⟨v^0_t|\). In the amalgamated product system \(E \otimes_C F\), \(u^0\) and \(v^0\) are identified. We denote the common unit by \(σ\).

**Corollary 20.** Let \(E, F, u^0, v^0, σ\) be as above. Then \(R^E_σ \otimes_C R^F_σ = R^E_{u^0} \oplus R^F_{v^0}\).

Proof: For \(x \in R^E_{u^0}, y \in R^F_{v^0}\),

\[
⟨x, y⟩_{C_1} = ⟨x, C_1 y⟩ = ⟨x, |u^0⟩⟨v^0| y⟩ = ⟨x, u^0⟩⟨v^0, y⟩ = 0.
\]

\(\square\)

**Remark 21.** It is noted that the condition on \(C\) that it is a partial isometry in Theorem 19 is a necessary condition. It may not be true for general contractive morphism. Let \(E_t = C u_t\) and \(F_t = C v_t\) be two type \(I_0\) product systems with \(∥u_t∥∥v_t∥ < 1\) for some \(t > 0\). Let \(C_t = |u_t⟩⟨v_t|\). Then \(R^E_{u^0} = 0\) and \(R^F_{v^0} = 0\). On the other side, we have \(E \otimes_C F\) is a type \(I_1\) product system. Though a priori, it is not clear whether in this case, \(E \otimes_C F\) is a product system. But this is indeed true \(([15])\). Therefore \(R^E_{σ} \otimes_C R^F_σ \neq \{0\}\) for every unit \(σ\) in \(E \otimes_C F\). Hence \(R^E_{u^0} \otimes_C R^F_{v^0} \neq R^E_{σ} \otimes_C R^F_σ\).

4. **Amalgamation through normalized units is independent of the choice of units**

In this section, we will show that the amalgamation through normalized units does not depend on the choice of the units. Proof of this fact is almost visible when we use the theory of random sets \(([11])\). In \([5]\), a short and self-contained proof has been presented. Also see \([3]\). Here we will prove this fact using roots.

First, we show that the amalgamation of two spatial product systems through normalized units can be identified with the product subsystem of the tensor product of the two systems. Let \(E\) and \(F\) be two spatial product systems and \(u\) and \(v\) be two normalized units of \(E\) and \(F\) respectively. Define a contractive morphism \(C = (C_t)_{t > 0} : E_t \to E\) by \(C_t = |u_t⟩⟨v_t|\). Denote \(E \otimes_{u,v} F := E \otimes_C F\). For two product subsystems \(G\) and \(G'\) of the product system \(H\), we denote by \(G \lor G'\) the smallest product subsystem of \(H\) containing \(G\) and \(G'\).

**Proposition 22.** Suppose \(E\) and \(F\) are two spatial product systems and \(u\) and \(v\) are two normalized units of \(E\) and \(F\) respectively. Then \(E \otimes_{u,v} F\) is isomorphic to the product system generated by \(E \otimes v\) and \(u \otimes F\) inside \(E \otimes F\). i.e. \(E \otimes_{u,v} F \simeq (E \otimes v) \lor (u \otimes F)\).

Proof: As \(u\) and \(v\) are normalized, we see that \(I : E \to E \otimes v\) and \(J : F \to u \otimes F\) are isometric morphisms of product system. Also note that for \(x \in E_s\) and \(y \in F_t\), \(⟨I(x), J(y)⟩ = ⟨x, |u_t⟩⟨v_t| y⟩\). Now from the property of amalgamation \((\text{Theorem 2.7, [16]}\)\) we conclude that \(E \otimes_{u,v} F \simeq (E \otimes v) \lor (u \otimes F)\) as algebraic product systems. Now transferring the measurable structure of \((E \otimes v) \lor (u \otimes F)\) onto \(E \otimes_{u,v} F\) via the isomorphism, we can make \(E \otimes_{u,v} F\) into a product system and the isomorphism becomes the isomorphism of product systems.

Suppose \(E\) is a product system and \(u = (u_t)_{t > 0}\) is a normalized unit of \(E\). Then for every interval \([s, t]\), \(0 < s < t < 1\), we may identify, \(E_t \simeq E_s \otimes E_{t-s} \otimes E_{1-t}\). Let \(P_{s,t} = P_{E_t} \otimes_{u_{t-s}} \otimes 1_{E_{1-t}}\). From Proposition 3.18, \([11]\), we know that \((s, t) \to P_{s,t}\) is jointly continuous. So in the compact simplex \(\{0 \leq s \leq t \leq 1\}\), it is uniformly continuous. i.e. \(P_{s,t}\) goes to identity strongly as \((t - s) \to 0\). In this section, we denote the multiplication operation of the product system by \(\circ\) i.e. \(a \in E_s, b \in E_t\), we have \(a \circ b \in E_{s+t}\). We write \(P_{s,t}\) as \(1_{E_s} \circ P_{E_{t-s}} \circ 1_{E_{1-t}}\). This is to differentiate the multiplication operation of the product system with the tensor product operation on the category of product systems. Though note that this is not the usual operator multiplications as they are not acting on the same space. We hope these notations do not lead any confusion.

For \(n \geq 1\), we have \(P_{\frac{1}{n}, \frac{1}{n}} = 1_{E_{\frac{1}{n}}} \circ \cdots \circ 1_{E_{\frac{1}{n}}} \circ P_{E_{\frac{1}{n}}} \circ \cdots \circ P_{E_{\frac{1}{n}}},\) where \(P_{E_{\frac{1}{n}}}\) on the \(i\)-th place.
Theorem 23. Suppose $\mathcal{E}$ and $\mathcal{F}$ are two spatial product systems with normalized units $u$ and $v$ respectively. Then $\mathcal{E} \otimes_{u,v} \mathcal{F}$ is isomorphic to the product system generated by $\mathcal{E} \otimes \mathcal{F}^I$ and $\mathcal{F}^I \otimes \mathcal{E}$ inside $\mathcal{E} \otimes \mathcal{F}$, i.e. $\mathcal{E} \otimes_{u,v} \mathcal{F} \cong (\mathcal{E} \otimes \mathcal{F}^I) \bigvee (\mathcal{E}^I \otimes \mathcal{F})$.

Proof: We know from Proposition 22, that $\mathcal{E} \otimes_{u,v} \mathcal{F} \cong (\mathcal{E} \otimes v) \bigvee (u \otimes \mathcal{F}) \subset \mathcal{E} \otimes \mathcal{F}$. So to prove the theorem, it is enough to show that $\mathcal{E} \otimes \mathcal{F}^I \subset (\mathcal{E} \otimes v) \bigvee (u \otimes \mathcal{F})$, as the proof of $\mathcal{F}^I \otimes \mathcal{E} \subset (\mathcal{E} \otimes v) \bigvee (u \otimes \mathcal{F})$ is identical. We fix the time point $t = 1$. Now from Theorem 15, it is enough to show that for $z \in \mathcal{E}_1$ and for any root $a$ of $v$ with $\|a\|_1 = 1$, $z \otimes a \in ((\mathcal{E} \otimes v) \bigvee (u \otimes \mathcal{F}))_1$. For other time point, proof goes identically. Let $\epsilon > 0$ be given. From uniform continuity of $P_{s,t}$, choose $N$ such that $n \geq N$, $\|z - P_{s,t} \frac{1}{n} z\| \leq \epsilon$, for every $i = 1, 2, \ldots, n$. Choose and fix $n \geq N$. Decompose $a_1 = \sum_{i=1}^{n} x_i$, where $x_i = v_1^{i} \cdots v_n^{i} a_1^{i} \cdots v_n^{i}$, with $a_1$ at $i$-th place. Clearly $\|x_i\|_1 = 1/\sqrt{n}$.

$$ \|z \otimes a_1 - \sum_{i=1}^{n} (P_{s,t} \frac{1}{n} z \otimes x_i)\|_2^2 = \|z \otimes x_i - \sum_{i=1}^{n} (P_{s,t} \frac{1}{n} z \otimes x_i)\|_2^2 = \sum_{i=1}^{n} \|z \otimes x_i\|_2^2 \|x_i\|_2^2 \leq 1/n \sum_{i=1}^{n} \|z - P_{s,t} \frac{1}{n} z\|_2^2 \leq \epsilon^2. $$

Now the vector

$$ P_{s,t} \frac{1}{n} z \otimes x_i = \sum_{j} (c_j^{i} \circ c_j^{i} \circ \cdots \circ c_j^{i-1} \circ u_1^{i} \circ \cdots \circ u_1^{i} \circ c_j^{i} \circ \cdots \circ c_j^{n}) \otimes (v_1^{i} \cdots v_n^{i} \circ a_1^{i} \circ v_1^{i} \cdots v_n^{i} \circ v_1^{i} \cdots v_n^{i}). $$

Note that, for every $1 \leq j \leq n$, $c_j^{i} \circ v_1^{i} \in (\mathcal{E} \otimes v)_1$ and $(u_1^{i} \otimes a_1^{i}) \in (u \otimes \mathcal{F})_1$.

This implies that $z \otimes a_1 \in ((\mathcal{E} \otimes v) \bigvee (u \otimes \mathcal{F}))_1$. \hfill $\square$

Corollary 24. Suppose $\mathcal{E}$ and $\mathcal{F}$ are two spatial product systems with normalized units $u$ and $v$ respectively. Then $(\mathcal{E}^I \otimes v) \bigvee (u \otimes \mathcal{F}^I) = (\mathcal{E}^I \otimes \mathcal{F}^I) = (\mathcal{E} \otimes \mathcal{F})^I$.

Theorem 25. Suppose $\mathcal{E}$ and $\mathcal{F}$ are two spatial product systems with normalized units $u$ and $v$ respectively. Then $R_n^{\mathcal{E} \otimes \mathcal{F}} = (R_v^{\mathcal{E}} \otimes \mathcal{F}) \oplus (u \otimes R_v^{\mathcal{F}})$.

Proof: For $a \in R_u^{\mathcal{E}}$ and $b \in R_v^{\mathcal{F}}$, define for each $s > 0$, $d_s = a_s \otimes v_s + u_s \otimes b_s$. Then for $s, t > 0$, we have

$$ d_s \circ (u_t \otimes v_t) + (u_s \otimes v_s) \circ d_t = (a_s \otimes v_s + u_s \otimes b_s) \circ (u_t \otimes v_t) + (u_s \otimes v_s) \circ (a_t \otimes v_t + u_t \otimes b_t) = [(a_s \otimes u_t) \circ (v_s \otimes v_t) + (u_s \otimes u_t) \circ (b_s \otimes v_t)] + [(u_s \otimes a_t) \circ (v_s \otimes v_t) + (u_s \otimes u_t) \circ (v_s \otimes b_t)] = [(a_s \otimes u_t + u_s \otimes a_t) \circ (v_{s+t} \otimes v_t + u_{s+t} \otimes (b_s \otimes v_t + v_s \otimes b_t)] $$
This implies $d \in R^{\mathcal{E} \otimes F}_{u \otimes v}$ and we obtain $R^{\mathcal{E} \otimes F}_{u \otimes v} \supset (R^{\mathcal{E} \otimes F}_{u} \otimes (u \otimes R^{\mathcal{F}}_{v}))$. Observe that $\{ (a_s \otimes v_s) + (u_s \otimes b_s) : a \in R^{\mathcal{E}}_{u}, b \in R^{\mathcal{F}}_{v} \}$ is a closed subspace of $\{ c_s : c \in R^{\mathcal{E} \otimes F}_{u \otimes v} \}$. We claim that for every $s > 0$, $\{ c_s : c \in R^{\mathcal{E} \otimes F}_{u \otimes v} : (a_s \otimes v_s) + (u_s \otimes b_s) : a \in R^{\mathcal{E}}_{u}, b \in R^{\mathcal{F}}_{v} \}$ = 0. Suppose $c \in R^{\mathcal{E} \otimes F}_{u \otimes v}$. Also suppose that for all $a \in R^{\mathcal{E}}_{u}$ and $b \in R^{\mathcal{F}}_{v}$, and for every $s > 0$, $\langle c_s, a_s \otimes v_s + u_s \otimes b_s \rangle = 0$. As $c \in R^{\mathcal{E} \otimes F}_{u \otimes v}$, we have for every $s > 0$, $\langle c_s, a_s \otimes v_s \rangle = 0$. Now from Corollary 15, we have for every $s > 0$, $c_s$ belong to the ortho-complement of $((F^{1} \otimes v) \setminus (u \otimes F^{1}))_{s}$. Now from Corollary 24, we get, for every $s > 0$, $c_s$ is in the ortho-complement of $((F^{1} \otimes F^{1})_{s}$. But as $c \in R^{\mathcal{E} \otimes F}_{u \otimes v}$, we have from Corollary 14, for every $s > 0$, $c_s \in (\mathcal{E} \otimes \mathcal{F})_{s}$. This shows that for every $s > 0$, $c_s = 0$. This proves the claim. Hence we have the equality $R^{\mathcal{E} \otimes F}_{u \otimes v} = (R^{\mathcal{E} \otimes F}_{u} \otimes (u \otimes R^{\mathcal{F}}_{v})$. □

5. Cluster construction

Here we introduce a new construction called the cluster construction. Given any product subsystem $\mathcal{F}$ in a product system $\mathcal{E}$, we attach a product subsystem $\mathcal{F}' \supset \mathcal{F}$. We call the product subsystem $\mathcal{F}'$ as the cluster of $\mathcal{F}$ in $\mathcal{E}$. The name ‘cluster’ comes from the following connection of random sets discussed in [11]. Every product subsystem corresponds to a unique probability measure on the closed subsets of $[0,1]$. The set of all closed sets of $[0,1]$ can be topologized by hit and miss topology (see Page 2, [11], Section 1-4, [13] for details). The mapping ‘cluster’ which sends a closed set to its limit points is a measurable map on this space. We show here that the probability measure corresponding to the cluster subsystem is the distribution of the cluster map. We compute the ‘cluster’ of the product subsystem of a spatial product system given by a single unit and show that it is the type I part of the product system.

Suppose $(\mathcal{E}, B)$ is a product system and $(F, B|_{F})$ is an inclusion subsystem. Define $\tilde{F}$ by

$$\tilde{F}_{s,t} = \operatorname{span}\{x \otimes y : x \in \mathcal{E}_{r} \otimes F_{r}, y \in \mathcal{E}_{t-r} \otimes F_{t-r}, \text{ for some } r,0 < r < t\}.$$

Set $F'_{s,t} = E_{t} \otimes \tilde{F}_{s,t}$.

**Lemma 26.** With the notation as above, $(F'_{t}, B_{s,t}|_{F'_{t}})$ is an inclusion system.

**Proof:** Let $x \in F'_{s+t}$. First note that

$$F'_{s} \otimes F'_{t} = (E_{s} \otimes F'_{t}) \cap (F'_{s} \otimes E_{t}).$$

Now

$$F_{s+r} \subset F_{s} \otimes F_{r} \subset E_{s} \otimes F_{r},$$

implies

$$E_{s} \otimes (E_{r} \otimes F_{r}) \subset E_{s+r} \otimes F_{s+r}.$$ 

Now for $y \in E_{s}$, $z_{1} \in E_{r} \otimes F_{r}$, $z_{2} \in E_{t-r} \otimes F_{t-r}$, for some $0 < r < t$, we get $y \otimes z_{1} \in E_{s+r} \otimes F_{s+r}$. So $\langle x, y_{1} \otimes z_{1} \otimes z_{2} \rangle = 0$. This shows

$$x \in E_{s+t} \otimes (E_{s} \otimes \tilde{F}_{t}).$$

i.e. $x \in E_{s} \otimes F'_{t}$. Similarly we get for $0 < r' < s$,

$$E_{s-r'} \otimes F_{s-r'} \subset E_{s} \otimes F'_{t} \subset E_{s+t-r'} \otimes F_{s+t-r'}.$$ 

So for $z'_{1} \in E_{s-r'} \otimes F_{s-r'}$, $z'_{2} \in E_{s-r'} \otimes F_{s-r'}$, $y' \in E_{t}$, we have $z'_{2} \otimes y' \in E_{s+t-r'} \otimes F_{s+t-r'}$. This shows

$$x \in E_{s+t} \otimes (E_{t} \otimes \tilde{F}_{s}).$$

i.e. $x \in F'_{s+t} \otimes E_{t}$. Associativity property follows from the associativity of the product system. □

Given a product subsystem $\mathcal{F}$ of a product system $\mathcal{E}$, denote by $\tilde{\mathcal{F}}$ the product system generated by the inclusion system $(\mathcal{F}', W|_{\mathcal{F}'})$. We call this product subsystem as the cluster of $\mathcal{F}$ in $\mathcal{E}$. Now our present task is to relate the cluster construction with the theory of random sets described in [11]. Recall that the random closed sets are characterized by the random variables $X_{s,t} = \chi_{\{z \in \mathcal{Z} : \mathcal{F}_{t,r} = \emptyset\}}(Z)$, fulfilling $X_{s,t}X_{s,t} = X_{r,s}$, $0 \leq r \leq s \leq t \leq 1$. Theorem 3.16, [Lie] shows that the embedding of the product subsystem into the whole product system, i.e. the structure encoded in the algebraic properties of the
Moreover the correspondence
\[ \mu^F_\eta \{ Z : Z \cap [s_i, t_i] = \emptyset, i = 1, 2, \ldots, k \} = \eta(P^F_{s_1, t_1} \cdots P^F_{s_k, t_k}) \quad ((s_i, t_i) \in I). \]

Moreover the correspondence
\[ \chi_{\{ Z : Z \cap [s_i, t_i] = \emptyset \}} \to P^F_{s_i, t_i}, \quad ((s, t) \in I) \]
extends to an injective normal representation \( J^F_\eta \) of \( L^\infty(\mu^F_\eta) \) on \( \mathcal{E}_1 \). Its image is \( \{ P^F_{s,t}(s, t) \in I \}'' \). For any \( Z \in \mathcal{F}_K \), denote \( \hat{Z} \) the set of its cluster points:
\[ \hat{Z} = \{ t \in Z : t \in Z \setminus \{ t \} \}. \]

Suppose \( l : \mathcal{F}_K \to \mathcal{F}_K \) is the measurable map defined by \( l(Z) = \hat{Z} \). With these preparations in hand, we can derive an interesting relation between measure types \( \mathcal{M}^F \) and \( \mathcal{M}^{F'} \).

**Theorem 27.** Suppose \( \mathcal{E}, F, \mu^F_\eta \) are as above. Then
\[ J^F_\eta(\chi_{\{ Z : Z \cap [s_i, t_i] = \emptyset \}}) = P^F_{s,t}, \quad ((s, t) \in I). \]

Therefore
\[ \mu^F_\eta \{ Z : \hat{Z} \cap [s_i, t_i] = \emptyset, i = 1, 2, \ldots, k \} = \eta(P^F_{s_1, t_1} \cdots P^F_{s_k, t_k}) \quad ((s_i, t_i) \in I). \]

Consequently, \( \mathcal{M}^{F'} = \mathcal{M}^F \circ l^{-1} \).

Proof: First note that, Equation (5.6) implies that \( P^F \in \{ P^F_{s,t} : (s, t) \in I \}'' \). Now it is enough to prove Equation (5.6). Indeed
\[
J^F_\eta(\chi_{\{ Z : \hat{Z} \cap [s_i, t_i] = \emptyset, i = 1, 2, \ldots, k \}}) = J^F_\eta(\Pi_{i=1}^k(\chi_{\{ Z : Z \cap [s_i, t_i] = \emptyset \}})) = \Pi_{i=1}^k J^F_\eta(\chi_{\{ Z : Z \cap [s_i, t_i] = \emptyset \}}) = \Pi_{i=1}^k P^F_{s_i, t_i}.
\]

Now applying the states \( f(\cdot) \mu^F_\eta \) and \( \eta \) on \( L^\infty(\mu^F_\eta) \) and \( \{ P^F_{s,t} : (s, t) \in I \}'' \) respectively, we obtain Equation (5.7). Now for a closed random set \( Z \), and the interval \([s, t],\) if we have \( \sharp [Z \cap [s, t]] \geq 2 \), i.e. if \( Z \) intersects \([s, t]\) at more than one point, then there exists a rational \( q \in \mathbb{Q} \), such that \( Z \cap [s, q] \neq \emptyset \) and \( Z \cap [q, t] \neq \emptyset \). So we have the identity,
\[
\{ Z : \sharp [Z \cap [s, t]] \leq 1 \} = \{ [u \in \mathbb{Q} : (s, t)] \{ Z : Z \cap [s, q] \neq \emptyset, Z \cap [q, t] \neq \emptyset \} \}.
\]

Applying \( J^F_\eta \) on the indicator function of the above two sets, we get
\[
J^F_\eta(\chi_{\{ Z : Z \cap [s, q] \leq 1 \}}) = J^F_\eta(\chi_{\{ [u \in \mathbb{Q} : (s, t)] \{ Z : Z \cap [s, q] \neq \emptyset, Z \cap [q, t] \neq \emptyset \} \}}).
\]

Now let \( A_q = \{ Z : Z \cap [s, q] \neq \emptyset \} \), \( B_q = \{ Z : Z \cap [q, t] \neq \emptyset \} \). Then \( A^c_q = \cap [Z : Z \cap [s, q - 1/n] = \emptyset \}. \) Further from Equation 5.5 and continuity of \( P^F_{s,t} \), Proposition 3.18, [11], we get \( J^F_\eta(\chi_{A^c_q}) = \lim_{n} P^F_{s, q - 1/n} \). Hence
\[
J^F_\eta(\chi_{A_q}) = 1_{\mathcal{E}_s} \otimes P_{s, q - 1/n} \otimes 1_{\mathcal{E}_t}.
\]

Similarly,
\[
J^F_\eta(\chi_{B_q}) = 1_{\mathcal{E}_q} \otimes P_{q - 1, t} \otimes 1_{\mathcal{E}_t}.
\]

Now
\[
J^F_\eta(\chi_{A_q \cap B_q}) = J^F_\eta(\chi_{A_q} \chi_{B_q}) = (1_{\mathcal{E}_s} \otimes P_{s, q - 1/n} \otimes 1_{\mathcal{E}_t}) (1_{\mathcal{E}_q} \otimes P_{q - 1, t} \otimes 1_{\mathcal{E}_t}) = 1_{\mathcal{E}_s} \otimes P_{s, q - 1, t} \otimes 1_{\mathcal{E}_t}.
\]
For measurable sets $D_1, D_2, \cdots$, we claim that

$$J^F_\eta (\chi_{\cup_i D_i}) = \bigvee_i J^F_\eta (\chi_{D_i}).$$

Indeed $J^F_\eta (\chi_{\cup_i D_i}) \geq J^F_\eta (\chi_{D_i})$ for all $i$, implying $J^F_\eta (\chi_{\cup_i D_i}) \geq \bigvee_i J^F_\eta (\chi_{D_i})$. For the reverse inequality, let $D'_1, D'_2, \cdots$ be the sets constructed from $D_1, D_2, \cdots$ by $D'_1 = D_1, D'_i = D_i \setminus \bigcup_{k \leq i-1} D_k, i \geq 2$. Then $\cup_i D'_i = \cup_i D_i, D'_i \subset D_i$ and $D'_1, D'_2, \cdots$ are mutually disjoint. Then

$$J^F_\eta (\chi_{\cup_i D_i}) = \bigvee_i J^F_\eta (\chi_{D'_i})$$

$$= \bigvee_i J^F_\eta (\chi_{D_i}).$$

Now it only remains to prove the claim. Clearly

$$J^F_\eta (\chi_{\cup_i D_i}) \leq \bigvee_i J^F_\eta (\chi_{D_i}).$$

So we have,

$$J^F_\eta (\chi_{\{Z: Z \cap [s,t] \leq 1\}}) = J^F_\eta (\chi_{\bigcup_{q \in \mathbb{Q}^\vee (s,t)} (A_q \cap B_q)})$$

$$= I_{\xi_1} - \bigvee_{q \in \mathbb{Q}^\vee (s,t)} J^F_\eta (\chi_{A_q \cap B_q})$$

$$= I_{\xi_1} - \bigvee_{q \in \mathbb{Q}^\vee (s,t)} 1_{\xi_s} \otimes (F_{F-q}^\perp \otimes F_{F-q}) \otimes 1_{\xi_{1-t}}$$

$$= I_{\xi_1} - 1_{\xi_s} \otimes P_{\bigcup_{q \in \mathbb{Q}^\vee (s,t)} (F_{F-q}^\perp \otimes F_{F-q})} \otimes 1_{\xi_{1-t}}$$

$$= 1_{\xi_s} \otimes P_{\bigcup_{q \in \mathbb{Q}^\vee (s,t)} (F_{F-q}^\perp \otimes F_{F-q})} \otimes 1_{\xi_{1-t}}.$$

We claim that

$$\bigvee_{q \in \mathbb{Q}^\vee (s,t)} [F_{q-s}^\perp \otimes F_{q-t}]^\perp = \bigvee_{s < r < t} [F_{r-s}^\perp \otimes F_{r-t}]^\perp.$$

For the moment let us assume the claim. Then using the definition of $F'$, we get that

$$J^F_\eta (\chi_{\{Z: Z \cap [s,t] \leq 1\}}) = P_{s,t} F'.$$

For any partition $P = \{s = r_1 < r_2 < \cdots < r_k = t\}$ of $[s,t]$, define the set $A_P = \{Z: Z \cap [r_i, r_{i+1}] \leq 1, 1 \leq i \leq k\}$. Then the following identity holds:

$$\{Z: \sharp Z \cap [s,t] < \infty\} = \bigcup_{P} A_P.$$

As $F$ is the product system generated by the inclusion system $F'$, we have

$$P_{s,t} F' = \bigvee_{P} \prod_{i=1}^{k} P_{r_i, r_{i+1}} F'$$

$$= \bigvee_{P} J^F_\eta (\chi_{A_P})$$

$$= J^F_\eta (\chi_{\cup_P A_P})$$

$$= J^F_\eta (\chi_{\{Z: Z \cap [s,t] \leq 1\}}).$$

Now from the identity

$$\{Z: Z \cap (s, t) = \emptyset\} = \{Z: \sharp Z \cap [s,t] \leq \infty\},$$

we get

$$J^F_\eta (\chi_{\{Z: Z \cap (s,t) = \emptyset\}}) = J^F_\eta (\chi_{\{Z: Z \cap [s,t] = \emptyset\}}) = P_{s,t} F' , \ (s, t) \in I).$$

Therefore

$$\mu^F_\eta (\{Z: Z \cap [s,t] = \emptyset\} = \eta (P_{s,t} F') , \ ([s,t] \subset I).$$

Now it only remains to prove the claim. Clearly

$$\bigvee_{q \in \mathbb{Q}^\vee (s,t)} [F_{q-s}^\perp \otimes F_{q-t}]^\perp \supset \bigvee_{s < r < t} [F_{r-s}^\perp \otimes F_{r-t}]^\perp.$$
Without loss of generality, we may assume $0 < s < t < 1$. We will use the continuity properties as in Proposition 3.18, [11]. Let

$$P^0 = 1_{E} \otimes P_{\bigwedge_{0 \leq r < t} F^+_{r,t} \otimes F^-_{r,t}} \otimes 1_{E_{t-1}}.$$ 

Clearly

$$P^0 \leq Q^0.$$ 

To prove the claim, it is enough to show that $P^0 \geq Q^0$. We have,

$$P_0 = \bigwedge_{s < r < t} \left[ P_{s,r}(1 - P_{r,t}) + (1 - P_{s,r})P_{r,t} + P_{s,t} \right]$$

$$= \bigwedge_{s < r < t} \left[ P_{s,r} + P_{r,t} - P_{s,t} \right].$$

Fix $x \in \text{range } Q^0$. Fix $r \in (s, t)$. Given $\epsilon > 0$, there is a $q \in \mathbb{Q}$ such that

$$\|P_{s,q}x - P_{s,q}x\| < \epsilon/2$$

and

$$\|P_{r,q}x - P_{r,q}x\| < \epsilon/2.$$ 

Now

$$\|x - [P_{s,r} + P_{r,t} - P_{s,t}]x]\| = \|[P_{s,q} + P_{r,t} - P_{s,t}]x - [P_{s,q} + P_{r,t} - P_{s,t}]x\|$$

$$< \epsilon/2 + \epsilon/2$$

$$< \epsilon.$$ 

So

$$x \in \text{range } [P_{s,r} + P_{r,t} - P_{s,t}]$$

for all $s < r < t$.

This implies

$$x \in \text{range } P^0.$$ 

This shows $P^0 \geq Q^0$ and completes the proof.

Suppose $(E, W)$ is a product system and $u$ is a unit of $(E, W)$. For the product subsystem $F_t = Cu_t$, we wish to show that $F$ is the type 1 part of $E$. To prove the result, we need the following lemmas.

**Lemma 28.** Suppose $(E, W)$ is a product system and $u$ is a normalized unit of $(E, W)$. Then $u_s \otimes F'_t \subset F'_{s+t}$ and $F'_s \otimes u_t \subset F'_{s+t}$.

**Proof:** Suppose $x \in F'_t$, consider the set

$$A := \{(z_1 \otimes z_2) : (z_1, u_r) = 0 = (z_2, u_{s+t-r}) \},$$

for some $r$, $0 < r < s + t$.

Then we claim that $\text{span } A = \text{span } (A_1 \cup A_2 \cup A_3)$, where

$$A_1 = \{(y_1 \otimes y_2 \otimes y_3) : (y_1, u_r) = 0 = (y_2, u_{s-r}) (y_3, u_t) = 0, \text{ for some } 0 < r < s \},$$

$$A_2 = \{(y_1 \otimes y_2 \otimes y_3) : (y_1, u_s) (y_2, u_{s-r}) = 0 = (y_3, u_{s+t-r}) = 0, \text{ for some } s < r < s + t \}$$

and

$$A_3 = \{(z_1 \otimes z_2) : (z_1, u_s) = 0, (z_2, u_t) = 0 \}.$$ 

Suppose $y_1 \otimes y_2 \otimes y_3 \in A_1$. That means for some $0 < r < s$, $(y_1, u_r) = 0 = (y_2, u_{s-r}) (y_3, u_t) = 0$. This implies $y_1 \in E_r \otimes Cu_r$ and $y_2 \otimes y_3 \in E_{s+t-r} \otimes Cu_{s+t-r}$. This shows $y_1 \otimes y_2 \otimes y_3 \in A$. We obtain $A_1 \subset A$. Similarly, $A_2, A_3 \subset A$. We obtain, $\text{span } A \supset \text{span } (A_1 \cup A_2 \cup A_3)$. For the converse, let $z_1 \otimes z_2 \in A$, with $(z_1, u_r) = 0, (z_2, u_{s+t-r}) = 0, 0 < r < s$. This implies $z_2 \in \text{span } \{x_1 \otimes x_2 : x_1 \in E_{s-r}, x_2 \in E_t, (x_1 \otimes x_2, u_{s+t-r}) = 0 \}$. Clearly $z_1 \otimes x_1 \otimes x_2 \in A$. We get $z_1 \otimes z_2 \in \text{span } A_1$. Similarly, for $z_1 \otimes z_2 \in A$ with $(z_1, u_r) = 0, z_2, u_{s+t-r}) = 0, s < r < s + t$, we have $z_1 \otimes z_2 \subset \text{span } A_2$. Therefore $\text{span } A \subset \text{span } (A_1 \cup A_2 \cup A_3)$. This proves the claim. Now suppose $y_1 \otimes y_2 \otimes y_3 \in A_1$, be an arbitrary vector. Then there is some $r_0, 0 < r_0 < s$, such that $y_1, u_{r_0}) = 0 = (y_2, u_{s-r_0}) (y_3, u_t) = 0$, and

$$\langle u_s \otimes x, y_1 \otimes y_2 \otimes y_3 \rangle = \langle u_{r_0} \otimes u_{s-r_0} \otimes x, y_1 \otimes y_2 \otimes y_3 \rangle.$$
\[= \langle x, y \rangle = 0.\]

This shows that \(u_s \otimes x \in A_1^\perp\). Now let \(y_1 \otimes y_2 \otimes y_3 \in A_2\) be arbitrary. Then there is some \(r_s\), \(s < r_s < s + t\), such that \(\langle y_1, u_s \rangle \langle y_2, u_{r_s} \rangle = 0\), \(\langle y_3, u_{s+t-r_s} \rangle = 0\). Now if \(\langle u_s, y_1 \rangle \neq 0\), then the inner product \(\langle u_s \otimes x, y_1 \otimes y_2 \otimes y_3 \rangle = 0\) and if \(\langle u_s, y_1 \rangle = 0\), then \(\langle y_2, u_{r_s} \rangle = 0\) and \(\langle y_3, u_{s+t-r_s} \rangle = 0\). This is equivalent to \(y_2 \otimes y_3 \in F_t\). As \(x \in F_t\), the inner product \(\langle u_s \otimes x, y_1 \otimes y_2 \otimes y_3 \rangle = 0\). For \(z_1 \otimes z_2 \in A_3\), it is easily seen that \(\langle u_s \otimes x, z_1 \otimes z_2 \rangle = 0\). Thus for arbitrary vector \(z \in \overline{\text{span}}A_t\), we have \(\langle u_s \otimes x, z \rangle = 0\). Hence \(u_s \otimes x \in F_{s+t}\). Similarly \(F_s \otimes u_t \subseteq F_{s+t}\).

Define \(X_t = F_t' \ominus C u_t\). From the previous lemma, it follows easily that \(u_s \otimes X_t \subseteq X_{s+t}\) and \(X_s \otimes u_t \subseteq X_{s+t}\). We identify the space \(X_s\) as a subspace of \(X_{s+t}\) by \(x \mapsto x \otimes u_t\). This is an isometric embedding. Set \(X = \text{ind limits}_{s \geq 0} X_s\). Denote the image of \(x \in X_s\) in \(X\) via \(\pi\). For \(t > 0\), define \(S_t : X \rightarrow X\) via \(S_t(\pi(x)) = \pi(x \otimes u_t)\), and set \(S_0 = \text{id}\).

**Lemma 29.** Suppose \(S_t\) is defined as above. Then \((S_t)_{t \geq 0}\) forms a strongly continuous semigroup of isometries. Also \((S_t)_{t \geq 0}\) is a pure semigroup i.e. \((S_t)_{t \geq 0}\) does not have any unitary part.

**Proof:** Clearly \((S_t)_{t \geq 0}\) is a semigroup of isometries. Now to prove strong continuity of \(t \mapsto S_t\), it is enough to show that for \(x \in X_p, y \in X_q, 0 \leq t \leq 1, t \mapsto \langle \pi(x), S_t y \rangle\) is continuous. Set \(T > 0\) such that \(p < T\), \(q + t < T\). Now
\[
\langle \pi, S_t y \rangle = \langle \pi, u_t \otimes y \rangle = \langle x \otimes u_{T-p}, u_t \otimes y \otimes u_{T-q-t} \rangle.
\]
Set \(z = x \otimes u_{T-p}\) and \(w = y \otimes u_{T-q}\). Then \(z, w \in \mathcal{E}_T\). Let \(U_{t}^T\) in \(\mathcal{B}(\mathcal{E}_T)\) be the unitary group \((U_t)_{t \in (0,T)} \subseteq \mathcal{B}(\mathcal{E}_T)\) acting with regard to the representations \(\mathcal{E}_T - t \otimes \mathcal{E}_t \cong \mathcal{E}_T \cong \mathcal{E}_t \otimes \mathcal{E}_{-t}\) as flip:
\[
\langle x \otimes y, U_t^T(x \otimes y) \rangle = \langle x, y \otimes u_{T-q-t} \rangle = \langle x, y \otimes u_{T-q-t} \rangle,
\]
Thus \(U_t^T(x \otimes y) = u_t \otimes y \otimes u_{T-q-t}\). So we have,
\[
\langle \pi, S_t y \rangle = \langle z, U_t^T w \rangle.
\]
Now an identical argument to the Proposition 3.11, [11], shows the map \(t \mapsto U_t^T\) is strongly continuous. Hence our result follows.

Now for the last part, it is equivalent to show that \(\cap_{t \geq 0} S_t(X) = \{0\}\). We claim that \(S_t(X_s)\) is orthogonal to \(X_t\) for every \(s, t > 0\). Indeed, the claim follows from the fact that \(u_t \otimes X_s\) and \(X_t \otimes u_s\) are orthogonal for every \(s, t > 0\). This implies that \(S_t(X_s)\) is orthogonal to \(X_t\). Now if \(z \in \cap_{t \geq 0} S_t(X_s)\), we have \(z \in X_s^\perp\) for every \(t > 0\) \(\Rightarrow z \in X_t^\perp \Rightarrow z = 0\). □

**Theorem 30.** Suppose \((\mathcal{E}, B)\) is a product system and \(u\) is a normalized unit of \((\mathcal{E}, B)\). Let \(F = (F_t)_{t > 0}\) be the product subsystem given by \(F_t := C u_t\). Then \((\tilde{F}, B)\) is the type I part of \((\mathcal{E}, B)\).

**Proof:** From the definition of \(F_t\), it is clear that if \(a\) is a root of the unit \(u\), then \(a \in \tilde{F}_t\). Now it follows from corollary 15, that \((\tilde{F}, B)\) contains the type I part of \((\mathcal{E}, B)\). On the other hand, we claim that
\[X_{s+t} = u_s \otimes X_t \oplus X_s \otimes u_t.\]
Indeed, from Lemma 28, we get \(X_{s+t} \supset u_s \otimes X_t \oplus X_s \otimes u_t\). For the reverse containment, we observe that \(F_{s+t} \subseteq F_s' \otimes F'_t\) implies \(X_{s+t} \subseteq (X_s \otimes u_t) \oplus (u_s \otimes X_t) \oplus (X_s \otimes X_t)\). So it is enough to show that \(X_{s+t} \subseteq \mathcal{E}_{s+t} \oplus (X_s \otimes X_t)\). This follows from the fact that \(X_s \otimes X_t \subseteq F_{s+t}\). So under the identification of \(X_s\) inside \(X_{s+t}\), we have for every \(s, t > 0\),
\[X_{s+t} = X_s \oplus S_s(X_t).\]
Taking limit as \(t \uparrow \infty\), we get for every \(s > 0\),
\[X = X_s \oplus S_s(X).\]

Theorem 9.3, Chapter III, [25] states that every pure strongly continuous semigroup of isometries is unitarily equivalent to the unilateral shift semigroup on \(L^2((0, t], K)\), for some Hilbert space \(K\). From
Lemma 29, there is a unitary $U : X \to L^2([0, \infty), K)$ for some separable Hilbert space $K$, such that $US_tU^* = S'_t$, where $S'_t$ is the unilateral shift semigroup on $L^2([0, \infty], K)$.

\[
(S'_t f)(x) = \begin{cases} f(x-t) & \text{if } x \geq t \\ 0 & \text{otherwise.} \end{cases}
\]

Now $L^2(0, \infty), K$) decomposes for every $s > 0$, as

\[
L^2([0, \infty], K) = U(X)
= U(X_s) \oplus US_s(X)
= U(X_s) \oplus US_sU^*U(X)
= U(X_s) \oplus S'_sL^2([0, \infty], K)
= U(X_s) \oplus L^2([s, \infty), K).
\]

Therefore $U(X_s) = L^2([0, s], K)$. For each $t > 0$, consider the set $E_t = \mathbb{C} \Omega_t \oplus L^2([0, t], K) \subset \Gamma_{sym}(L^2([0, t], K)$, where $\Omega_t = \Omega$ for all $t > 0$, is the vacuum vector. Suppose $f \in L^2([0, s+t], K)$.

Let $g \in L^2([0, s], K)$ be defined as $g = f|_{[0,s]}$ and $h \in L^2([0, t], K)$ be defined as

\[
h(r) = f(r+s), 0 \leq r \leq t.
\]

Note that $S'_s h = f|_{[s,s+t]}$. From the equation $f = g + S'_s h$, we have $W^\Gamma_{s,t} : \Gamma_{sym}(L^2([0, s+t], K) \to \Gamma_{sym}(L^2([0, s], K) \oplus \Gamma_{sym}(L^2([0, t], K)$ is given by

\[
W^\Gamma_{s,t} f = g \otimes \Omega_t + \Omega_s \otimes h.
\]

For $\alpha \in \mathbb{C}$ and $f \in L^2([0, s+t], K)$,

\[
W^\Gamma_{s,t}|_{E_{s+t}}(\alpha \Omega_{s+t} \oplus f) = \alpha(\Omega_s \otimes \Omega_t) \oplus (g \otimes \Omega_t + \Omega_s \otimes h) \in E_s \oplus E_t.
\]

So $(E_t, W^\Gamma_{s,t}|_{E_{s+t}})$ is an inclusion system. Define $\Phi_t : F'_t \to E_t$ by

\[
\Phi_t(\lambda u_s \otimes x_t) = \lambda \Omega_s \otimes U|x_t(x)
\]

for $x \in X_t$. We claim that $(\Phi_t)_{t>0}$ is an isometric morphism of inclusion system. For $x \in X_{s+t}$, there are $y \in X_t$ and $z \in X_s$ such that $W^\Gamma_{s,t} x = u_s \otimes y + z \otimes u_t$. Under the identification on $X$, we have $x = S_t y + z$. So $U x = US_tU^*U y + U z = S'_tU y + U z$. Under the map $W^\Gamma_{s,t}$ on $\Gamma_{sym}(L^2([0, s+t], K)$, we get $S'U y + U z = W^\Gamma_{s,t}(\Omega_s \otimes U|X_t(x))$. This implies

\[
W^\Gamma_{s,t}(\Phi_s \otimes \Phi_t)W^\xi_{s,t}(\lambda u_{s+t} + x) = W^\Gamma_{s,t}(\Phi_s \otimes \Phi_t)(\lambda u_s \otimes u_t + u_s \otimes y + z \otimes u_t)
= W^\Gamma_{s,t}(\lambda(\Omega_s \otimes \Omega_t) \oplus (\Omega_s \otimes U|x_t,y + U|x_t,z \otimes \Omega_t))
= \lambda \Omega_{s+t} \otimes S'_t U|x_t,y + U|x_t,z
= \lambda \Omega_{s+t} \otimes U|x_t,y + U|x_t,z
= \lambda \Omega_{s+t} \otimes U|x_{s+t},x
= \Phi_{s+t}(\lambda u_{s+t} + x).
\]

So $F'$ and $E$ are isomorphic as inclusion systems. So their generated product systems are isomorphic. As $E$ generates a type I product system, $\Gamma_{sym}(L^2([0, t], K)$, we have $F'$ generated by $F^\prime$ is a type I product system of index $\text{dim}(K)$.

Here we do a similar construction which generalize the cluster construction. Suppose $E$ is a product system and $F^1$ and $F^2$ are two inclusion subsystems of the product system $E$. Consider for each $t > 0$, the space

\[
G_t = \text{span}\{x \otimes y : x \in E_r \otimes F^1_r, y \in E_{t-r} \otimes F^2_{t-r}, \text{ for some } 0 < r < t\}.
\]

Define $G'_t = E_t \otimes G_t$.

**Proposition 31.** Let $G'_t$ be defined as above. Then $G'$ is an inclusion system containing $F^1$ and $F^2$. 

Proof: First we will show that $F^1$ and $F^2$ is contained in $G'$. Now fix $x \otimes y \in G_t$. So $\langle x, F^1_t \rangle = 0 \langle y, F^2_{t-r} \rangle = 0$, for some $0 < r < t$. This implies that $\langle F^1_t \otimes F^1_{t-r}, x \otimes y \rangle = 0$, which with the fact $F^1_t \subset F^1_t \otimes F^1_{t-r}$, for every $0 < r < t$ proves that $F^1$ is contained in $G'$. Similarly $F^2 \subset G'$. Suppose $x \in G'_{s+t}$. Now observe that

$$G'_s \otimes G'_t = (E_s \otimes G'_t) \cap (G'_s \otimes E_t).$$

Now from the containment $F^1_{s+t} \subset F^1_s \otimes F^1_{t-r} \subset E_s \otimes F^1_t$, we get

$$E_{s+r} \otimes F^1_{s+t} \supset E_s \otimes (E_r \otimes F^1_t).$$

For $y \in E_{s+t} \otimes F^1_t$, we get $y \otimes z_1 \in E_{s+r} \otimes F^1_{s+t}$. Consequently $(x, y \otimes z_1 \otimes z_2) = 0$. We get $x \in E_s \otimes G'_t$. Similarly, from the containment $F^2_{s+t-r} \subset F^2_s \otimes F^2_t \subset F^2_{s-r} \otimes E_t$, we get

$$E_{s+r} \otimes F^2_{s+t-r} \supset (E_s \otimes F^2_{s-r}) \otimes E_t.$$
ADDITIVE UNITS OF PRODUCT SYSTEM

\[ = V_{s,t}j_s + t s_t. \]

From Equation (5.11), we get \( (j_s \otimes j_t)B_{s,t}i_{s-t} = V_{s,t}j_s + t i_{s-t}. \) As by property (iv) above,

\[
\text{span} \{ i_{s-t}(a) : s \sim t \in J_s \sim J_t, a \in F_{s-t} \} = F_{s+t},
\]

we get \( (j_s \otimes j_t)B_{s,t} = V_{s,t}j_s + t. \)

This proves the claim. We may thus identify \( F \) as an algebraic product subsystem of \( E. \) Now suppose \( U_T = (U_t)_{t \in \mathbb{R}} \) is the unitary group on \( E_T \) as in (5.10). Note that \( U_t [F, \mathcal{F}] = U_t [F, \mathcal{F}] \) from Proposition 3.11, [Lie] we get that \( t \rightarrow U_t \) is strongly continuous for every \( T > 0. \) Therefore being a restriction map, \( t \rightarrow U_t \) is also strongly continuous for every \( T > 0. \) Now an application of Theorem 7.7, [11] shows that \( (F, B) \) is a product subsystem of \( (E, V). \)

**Appendix B: Guichardet’s picture of symmetric Fock space**

Suppose \( K \) is a separable Hilbert space. Set \( H = L^2(\mathbb{R}, K), H_t = L^2([0, t], K). \) We denote by \( \Gamma_{sym}(H), \Gamma_{sym}(H_t) \) the symmetric Fock spaces over \( H \) and \( H_t \) respectively. For \( f \in \Gamma_{sym}(H_s) \) and \( g \in \Gamma_{sym}(H_t), \) we define \( W^f_t(e(f) \otimes e(g)) = e(S_t f + g), \) where \( S_t \) is defined as in Equation (3.3). Then \( \Gamma_{sym}(K) := (\Gamma_{sym}(H), W^T_t) \) is a product system. Set \( \Delta_n(t) \) is the set of all subsets of the interval \([0, t]\) of cardinality \( n \) and \( \Delta(t) \) is the set of all finite subsets of the interval \([0, t].\)

From the Lebesgue measure on the real line, we induce the Poisson measure \( P \) on \( \Delta(t) \) by

\[
P(E) = \delta_{\emptyset}(E) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^t \cdots \int_0^t dt_1 \cdots dt_n \delta_{\{t_1, t_2, \ldots, t_n\}}(E).
\]

Set \( F_t = L^2(\Delta(t), K, P) = \{ g : \Delta(t) \rightarrow \Gamma(K) : g(\sigma) \in K^{\otimes |\sigma|}, \| g \|^2 dP < \infty \}, \) where \( \Gamma(K) \) is the full Fock space over \( K \) and \(|\sigma|\) is the cardinality of \( \sigma. \) For \( f \in L^2([0, t], K), \) denote \( f \in F_t \) by

\[
f_\sigma = \left\{ \begin{array}{ll} 1 & \text{if } \sigma = \emptyset \\ f(t_1) \otimes f(t_2) \otimes \cdots \otimes f(t_n) & \text{if } \sigma = \{t_1 \leq t_2 \leq \cdots \leq t_n\}. \end{array} \right.
\]

We claim that \( \{ f : f \in L^2([0, t], K) \} \) is dense in \( F_t. \) Indeed, suppose \( g \in F_t \) and \( \int \overline{f} g dP = 0, \) for all \( f \in L^2([0, t], K). \) Then \( \int \overline{f} g dP = 0, \) for all intervals \([a, b] \subset [0, t].\) It implies \( \int g dP = 0 \) for \( E = \{ \sigma \in \Delta(t) : \sigma \subset [a, b] \}. \) As these sets are the cylinder sets for the sigma filed, we get \( g = 0. \) Now under the map \( e(f) \rightarrow \hat{f} \) we have the Hilbert space isomorphism \( \Gamma_{sym}(H_t) \simeq F_t. \) For \( f \in F_t, \) and \( g \in F_t, \) define \( W^T(f \otimes g) \in F_{s+t} \) by

\[
W^T(f \otimes g)(\sigma) = f(\sigma \cap [t, s + t] - t) \otimes g(\sigma \cap [0, t]), \, \sigma \in \Delta(s + t).
\]

Then it is easily verified that the product systems \( \Gamma_{sym}(K) := (\Gamma_{sym}(H), W^T) \) and \( F = (F_t, W^T) \) are isomorphic. Under this isomorphism, the vacuum vector \( \Omega_t = e(0) \) is identified as

\[
\hat{\Omega}(\sigma) = \left\{ \begin{array}{ll} 1 & \text{if } \sigma = \emptyset \\ 0 & \text{otherwise.} \end{array} \right.
\]

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(Bhat, B. V. R.) Indian Statistical Institute, R. V. College Post, Bangalore-560059, India
E-mail address: bhat@isibang.ac.in

(Lindsay, J. M.) Lancaster University, Lancaster- LA1 4YW, UK
E-mail address: jmartinlindsay@googlemail.com

(Mukherjee, M.) IISER Kolkata, Mohanpur-741 252, India
E-mail address: mithun.mukherjee@iiserkol.ac.in