Some Two–Step and Three–Step Nilpotent Lie Groups with Small Automorphism Groups

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Vienna, Preprint ESI 1150 (2002) April 3, 2002

Supported by the Austrian Federal Ministry of Education, Science and Culture
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Some two-step and three-step nilpotent Lie groups with small automorphism groups

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April 5, 2002

Let $G$ be a connected Lie group and $\text{Aut}(G)$ be the group of all (continuous) automorphisms of $G$. When does the action of $\text{Aut}(G)$ on $G$ have a dense orbit? It turns out that if this holds then $G$ is a nilpotent Lie group; (see [4]). However it does not hold for all nilpotent Lie groups, and a characterization of the class of groups for which it holds seems to be a remote possibility. When $G$ is a vector space then $\text{Aut}(G)$ is its general linear group, and the action has an open dense orbit, namely the complement of the zero. On the other hand there exist 3-step simply connected nilpotent Lie groups such that every orbit of the action is a proper closed subset; this holds for the simply connected Lie group corresponding to the Lie algebra described in [8]; see also § 4 below. Among the simply connected groups this brings us to considering the question for 2-step nilpotent Lie groups. In this note we construct an example of a 2-step simply connected nilpotent Lie group $G$ for which the action of the automorphism group has no dense orbits; see Theorem 2.1.

We discuss the question also for Lie groups which are not simply connected, and show that while for all connected abelian groups other than the circle the automorphism group action has dense orbits, among the quotients of the group $G$ as above by discrete central subgroups there are nilpotent Lie groups $G'$ such that every orbit of $\text{Aut}(G')$ on $G'$ consists of either one or two cosets of the commutator subgroup $[G',G']$, the latter being a closed subgroup; see Corollary 5.2.

The method involved also enables us to give an example of a 3-step simply connected Lie group whose automorphism group is nilpotent (its action on the Lie algebra is by unipotent transformations); see § 4; the condition implies in particular that all orbits of its action on $G$ are closed. An example of a 6-step simply connected nilpotent Lie group with this property was given earlier in [9].

Another motivation for studying the automorphism groups of nilpotent Lie groups comes from the question of understanding which compact nilmanifolds support Anosov diffeomorphisms and which do not. It is known that if $G$ is a free $k$-step nilpotent Lie group over a $n$-dimensional vector space $V$ with $k < n$, then
for any lattice \( \Gamma \) in \( G \) the nilmanifold \( G/\Gamma \) admits Anosov automorphisms; see [3], pp. 558; see [6] for another approach to the question. There are also other classes of compact nilmanifolds for which this holds; see [1], [3], [6], [7]. However in general compact nilmanifolds may not admit Anosov automorphisms. Clearly, if \( G \) is a 2-step simply connected nilpotent Lie group such that \( [G, G] \) is one-dimensional then \( G/\Gamma \) can not admit an Anosov automorphism, for any lattice \( \Gamma \). Only a few other examples of 2-step simply connected Lie groups are known with this property; see [1], [6], [11]. We shall show that for the 2-step nilpotent Lie group \( G \) in our example, if \( \Gamma \) is a lattice in \( G \) then the nilmanifold \( G/\Gamma \) has no Anosov automorphism, and also no ergodic automorphism; see Corollary 6.1.

1 Lie algebras and Lie groups

We recall that a Lie algebra \( \mathfrak{g} \) is said to be nilpotent if the central series \( \{ \mathfrak{g}_i \} \) defined by \( \mathfrak{g}_0 = \mathfrak{g} \) and \( \mathfrak{g}_{i+1} = [\mathfrak{g}, \mathfrak{g}_i] \) for all \( i \geq 0 \), terminates, namely there exists \( k \geq 1 \) such that \( \mathfrak{g}_k = 0 \); if \( k \) is the smallest integer for which this holds then \( \mathfrak{g} \) is said to be \( k \)-step nilpotent. We say that a connected Lie group \( G \) is a \( k \)-step nilpotent Lie group if the Lie algebra of \( G \) is a \( k \)-step nilpotent Lie algebra.

For a Lie group \( G \) we shall denote by \( \text{Aut}(G) \) the group of all continuous automorphisms of \( G \), and similarly for a Lie algebra \( \mathfrak{g} \) we denote by \( \text{Aut}(\mathfrak{g}) \) the group of all Lie automorphisms of \( \mathfrak{g} \).

Let \( G \) be a connected nilpotent Lie group. Let \( G^{(1)} = [G, G] \) and \( Z(G) \) denote the center of \( G \). For any (continuous) homomorphism \( \psi : G/G^{(1)} \to Z(G) \) the map \( \tau : G \to G \) defined by \( \tau(g) = g\psi(gG^{(1)}) \) for all \( g \in G \) is an automorphism of \( G \); we shall call an automorphism arising in this way a shear automorphism. The class of shear automorphisms forms a normal subgroup of \( \text{Aut}(G) \). Every automorphism \( \tau \) of \( G \) factors to an automorphism of \( G/G^{(1)} \); we denote the factor of \( \tau \) on \( G/G^{(1)} \) by \( \overline{\tau} \). Clearly \( \tau \mapsto \overline{\tau} \) is a homomorphism of \( \text{Aut}(G) \) into \( \text{Aut}(G/G^{(1)}) \). We denote by \( \mathcal{A}(G) \) the image of the homomorphism, namely,

\[
\mathcal{A}(G) = \{ \overline{\tau} \in \text{Aut}(G/G^{(1)}) \mid \tau \in \text{Aut}(G) \}.
\]

We note that for all shear automorphisms as well as all inner automorphisms of \( G \) the factor on \( G/G^{(1)} \) is trivial.

Suppose now that \( G \) is a simply connected nilpotent Lie group and let \( \mathfrak{g} \) be the Lie algebra of \( G \). Then \( G^{(1)} = [G, G] \), namely \( [G, G] \) is closed, and \( G/[G, G] \) and \( [G, G] \) are (topologically isomorphic to) vector spaces; they may be identified canonically with \( \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \) and \( [\mathfrak{g}, \mathfrak{g}] \) respectively. Also in this case \( \text{Aut}(G) \) can be realised as \( \text{Aut}(\mathfrak{g}) \) identifying each automorphism with its derivative on \( \mathfrak{g} \). We note also the following:
Lemma 1.1. Let $G$ be a simply connected nilpotent Lie group and $\mathcal{G}$ be the Lie algebra of $G$. Let $\tau \in \text{Aut}(\mathcal{G})$ and suppose that $\tau$ is unipotent (as an element of $GL(\mathcal{G}/[\mathcal{G},\mathcal{G}])$). Then $\tau$ is unipotent (as an element of $GL(\mathcal{G})$).

Proof: It can be seen that the largest $\tau$-invariant subspace, say $\mathcal{G}_0$, of $\mathcal{G}$ on which the restriction of $\tau$ is unipotent (namely the generalized eigenspace for 1 as the eigenvalue) is a Lie subalgebra of $\mathcal{G}$. Since $\tau$ is unipotent $\mathcal{G}_0 + [\mathcal{G},\mathcal{G}] = \mathcal{G}$. Substituting from the equation for $\mathcal{G}$ on the left hand side successively and using that $\mathcal{G}$ is nilpotent we deduce that $\mathcal{G}_0 = \mathcal{G}$. Therefore $\tau$ is unipotent.

Now let $V$ be a (finite dimensional) vector space and let $W$ be a subspace of $\wedge^2 V$, the second exterior power of $V$. We can associate to these canonically a 2-step nilpotent Lie group as follows. Let $V' = (\wedge^2 V)/W$ and $\mathcal{G} = V \oplus V'$. We set $[v_1, v_2] = v_1 \wedge v_2 \mod W$ for all $v_1, v_2 \in V$, and $[x, y] = 0$ for all $x \in \mathcal{G}$ and $y \in V'$. These relations extend uniquely to a Lie bracket operation on $\mathcal{G}$. Furthermore $\mathcal{G}$ is a 2-step nilpotent Lie algebra, with $\mathcal{G}/[\mathcal{G},\mathcal{G}] = V$. Conversely every 2-step nilpotent Lie algebra $\mathcal{G}$ can be realised as a Lie algebra associated to a datum as above, with $V = \mathcal{G}/[\mathcal{G},\mathcal{G}]$, and $W$ a suitable subspace of $\wedge^2 V$.

Let $G$ be the simply connected Lie group corresponding to the Lie algebra $\mathcal{G}$ associated to a pair $V, W$ as above. Then $G/[G, G]$ may be realised canonically as $V$; this identifies the subgroup $\mathcal{A}(G)$ with a subgroup of $GL(V)$. It is easy to see that $x \in GL(V)$ is of the form $\tau$ for some $\tau \in \text{Aut}(G)$ if and only if the subspace $W$ of $\wedge^2 V$ is invariant under the action induced by $x$ on $\wedge^2 V$. It follows in particular that $\mathcal{A}(G)$ is an algebraic subgroup of $GL(V)$.

We note also the following:

Proposition 1.2. Let $G$ be a simply connected nilpotent Lie group. Then every orbit of $\text{Aut}(G)$ on $G$ is open in its closure. In particular every dense orbit is open. Consequently, if the $\mathcal{A}(G)$-action on $V$ has no open orbit then the $\text{Aut}(G)$-action on $G$ has no dense orbit.

Proof: Let $\mathcal{G}$ be the Lie algebra of $G$. Then $\text{Aut}(\mathcal{G})$ is an algebraic subgroup of $GL(\mathcal{G})$ and this implies that the orbits of its action on $\mathcal{G}$ are open in their closures (see [2]). Since $G$ is simply connected and nilpotent the exponential map is a diffeomorphism of $\mathcal{G}$ to $G$ and it is equivariant under the actions of the respective automorphism groups. Therefore for the action of $\text{Aut}(G)$ on $G$ also every orbit is open in its closure. The second assertion is immediate from the first. The last assertion follows from the fact that the $\mathcal{A}(G)$-action on $V$ is a factor of the $\text{Aut}(G)$-action on $G$. 

3
2 A representation

In this section we describe a construction of a representation and prove some properties. The results will be used in the later sections to give examples of nilpotent Lie algebras.

Let $S$ be the vector space of $3 \times 3$ symmetric matrices with real entries. For each $k, l = 1, 2, 3$ we denote by $E_{kl}$ the $3 \times 3$ matrix in which the $(k, l)$-entry (in the $k$th row and $l$th column) is 1 and all other entries are 0. Let $V$ be the subspace of $S$ defined by

$$V = \{ \Sigma \sigma_{kl} E_{kl} \in S \mid \sigma_{22} = \sigma_{13} + \sigma_{31} = 2 \sigma_{13} \}.$$ 

Let $\sigma_1, \ldots, \sigma_5$ be the elements defined as follows, forming a basis of $V$:

$$\sigma_1 = 2E_{11}, \quad \sigma_2 = E_{12} + E_{21}, \quad \sigma_3 = E_{13} + 2E_{22} + E_{31}, \quad \sigma_4 = E_{23} + E_{32}, \quad \text{and} \quad \sigma_5 = 2E_{33}.$$ 

Let $W$ be the subspace of $\wedge^2 V$ spanned by the three elements $\sigma_1 \wedge \sigma_4 - \sigma_2 \wedge \sigma_3$, $\sigma_1 \wedge \sigma_5 - \sigma_2 \wedge \sigma_4$ and $\sigma_2 \wedge \sigma_5 - \sigma_3 \wedge \sigma_4$.

Let $\delta, \nu^+$ and $\nu^-$ be the matrices defined by

$$\delta = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \nu^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \nu^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$ 

The space spanned by the three matrices is a Lie subalgebra of the Lie algebra of $3 \times 3$ matrices of trace 0, namely the Lie algebra of $SL(3, \mathbb{R})$. We denote the Lie subalgebra by $\mathcal{H}$. We note that it is isomorphic to the Lie algebra of $SL(2, \mathbb{R})$, with $\delta, \nu^+$ and $\nu^-$ corresponding to the standard basis of the latter; in particular the subspace spanned by $\delta$ is a Cartan subalgebra.

Let $H$ be the connected Lie subgroup of $SL(3, \mathbb{R})$ corresponding to $\mathcal{H}$. Consider the action of $H$ on $\mathbb{R}^3$ given by restriction of the natural action of $SL(3, \mathbb{R})$ on $\mathbb{R}^3$. We see that there is no proper nonzero subspace on $\mathbb{R}^3$ invariant under both $\exp \nu^+$ and $\exp \nu^-$. It follows therefore that the action of $H$ on $\mathbb{R}^3$ is irreducible. Since $H$ contains an element, viz $\exp \delta$, with distinct eigenvalues this implies that every element of the center of $H$ acts by scalar multiplication by real numbers, and considering the determinant we see that it must be trivial. Thus $H$ has trivial center. This shows that $H$ is Lie isomorphic to $PSL(2, \mathbb{R})$, the adjoint group of $SL(2, \mathbb{R})$.

Consider the action of $SL(3, \mathbb{R})$ on $S$ given by $(x, \sigma) \mapsto x \sigma x^t$ for all $x \in SL(3, \mathbb{R})$ and $\sigma \in S$, where $x^t$ denotes the transpose of $x$. It is straightforward to verify that the subspace $V$ is invariant under the action of $H$ (obtained by restriction); it suffices to verify that $V$ is invariant under the corresponding action of the Lie algebra, given by $(\xi, \sigma) \mapsto \xi \sigma + \sigma \xi^t$, for all $\xi \in \mathcal{H}$ and $\sigma \in \Sigma$, and furthermore it is enough to consider $\xi = \delta, \nu^+$ and $\nu^-$. The elements $\sigma_1, \ldots, \sigma_5$ are
weight vectors with respect to $\delta$ with weights $4, 2, 0, -2$ and $-4$ respectively, and the latter being distinct shows that the representation of $H$ over $V$ is irreducible.

We shall realize $H$ as a subgroup of $GL(V)$ by identifying each $h$ in $H$ with its action on $V$ (this is indeed an injective correspondence). We denote by $D$ the one-parameter subgroup of $GL(V)$ corresponding to the one-parameter subgroup $\{\exp t\delta \mid t \in \mathbb{R}\}$. We note that for any nontrivial element of $D$, $\sigma_1, \ldots, \sigma_5$ are eigenvectors with distinct eigenvalues, and hence the centralise of $D$ in $GL(V)$ consists of diagonal matrices with respect to the basis $\sigma_1, \ldots, \sigma_5$.

Let $A$ be the subgroup of consisting of all $x$ in $GL(V)$ such that $W$ is invariant under the action induced by $x$ on $\wedge^2 V$. A straightforward computation shows that $H$ (viewed as a subgroup of $GL(V)$) is contained in $A$. We note that $A$ is an algebraic subgroup of $GL(V)$. Let $A_u$ be the unipotent radical of $A$, namely the largest normal subgroup consisting of unipotent elements. Then the set of common fixed points of the action of $A_u$ on $V$ is a nonzero subspace invariant under $H$, and since the $H$-action on $V$ is irreducible it follows that $A_u$ fixes all points of $V$, which means that $A_u$ is trivial. Therefore $A$ is reductive.

**Theorem 2.1.** $A = ZH$, where $Z$ is the subgroup of $GL(V)$ consisting of scalar multiplications by real numbers. The $A$-action on $V$ has no open orbit.

**Proof:** Let $A^0$ be the connected component of the identity in $A$ and let $S = [A^0, A^0]$, the commutator subgroup. As $A^0$ is reductive it follows that $S$ is a semisimple subgroup of $GL(V)$; (see [10], [14]). Since $H$ is a simple Lie subgroup of $A^0$ it follows that it is contained in $S$. We shall show that $S = H$. Firstly let $C$ be the maximal compact connected normal subgroup of $S$. Since $S$ is semisimple, there exists a closed connected normal subgroup $S'$ such that $S = S'C$ and $S' \cap C$ is a finite subgroup contained in the center of $S$; both $S'$ and $C$ being connected subgroups this implies in particular that every element of $S'$ commutes with every element of $C$. Since $H$ is a noncompact connected simple Lie group it has no nontrivial homomorphism into a compact Lie group. Applying this to the quotient homomorphism of $S$ onto $S/S' = C/(C \cap S')$ we conclude that $H$ is contained in $S'$. Therefore $C$ is contained in the centralizer of $D$. Since the centralizer is diagonalizable and $C$ is a compact connected subgroup it follows that $C$ is trivial. Thus $S$ has no compact normal subgroups of positive dimension.

Now recall that the subspace $W$ of $\wedge^2 V$ is invariant under the action of $A$ and in particular that of $S$, and let $\varphi : S \to GL(W)$ be the representation of $S$ induced by the action. Let $K$ be the connected component of the identity in the kernel of $\varphi$. Then $K$ is a connected normal subgroup of $S$. Let $A$ be a maximal subgroup of $S$ such that $D$ is contained in it and its adjoint action on the Lie algebra of $S$ is diagonalisable over $\mathbb{R}$. Then $A$ intersects (by Aswan decomposition) any noncompact connected normal subgroup nontrivially, and since $S$ has no compact normal subgroups it follows that $A \cap K$ is nontrivial, unless $K$ is trivial. Let $\alpha \in A \cap K$. Then $\alpha$ commutes with all elements of $D$, and hence $\sigma_1, \ldots, \sigma_5$ are
eigenvectors of \( \alpha \); let \( \lambda_1, \ldots, \lambda_5 \) be the corresponding eigenvalues. Since \( \alpha \in K \) the action induced by \( \alpha \) on \( W \) is trivial. This means that the vectors \( \sigma_1 \wedge \sigma_4 - \sigma_2 \wedge \sigma_3 \), \( \sigma_1 \wedge \sigma_5 - \sigma_2 \wedge \sigma_4 \) and \( \sigma_2 \wedge \sigma_5 - \sigma_3 \wedge \sigma_4 \) are all fixed under the action of \( \alpha \). Each \( \sigma_i \wedge \sigma_j \) is an eigenvector of the action of \( \alpha \) on \( \wedge^2 V \) and hence the three vectors as above being fixed implies that the vectors \( \sigma_1 \wedge \sigma_4, \sigma_1 \wedge \sigma_5, \sigma_2 \wedge \sigma_3, \sigma_2 \wedge \sigma_4, \sigma_3 \wedge \sigma_5 \) and \( \sigma_3 \wedge \sigma_4 \) are fixed individually. Therefore for the eigenvalues we get the relations \( \lambda_1 \lambda_4 = \lambda_1 \lambda_5 = \lambda_2 \lambda_3 = \lambda_2 \lambda_4 = \lambda_3 \lambda_4 = 1 \), which is possible only if \( \lambda_i = 1 \) for all \( i = 1, \ldots, 5 \). Since \( \alpha \) is diagonalizable this shows that \( \alpha \) is the identity. Thus \( A \cap K \) is trivial and hence, as noted above, \( K \) is trivial. Therefore the kernel of \( \varphi \) is discrete.

Clearly \( \varphi(S) \) is a connected semisimple Lie subgroup of \( GL(W) \) containing \( \varphi(H) \). Since \( W \) is 3-dimensional this implies that \( \varphi(S) \) is either \( \varphi(H) \) or \( SL(W) \); (inspection of the weights for the action of \( \varphi(D) \) on the Lie algebra of \( SL(W) \), via the adjoint representation of \( SL(W) \), shows that the \( \varphi(H) \)-action has only two irreducible components, one of them being the Lie subalgebra of \( \varphi(H) \) itself; this means that there is not even a subspace invariant under the \( \varphi(H) \)-action lying strictly between the Lie algebras of \( \varphi(H) \) and \( SL(W) \), which in particular implies the assertion here; the assertion can also be proved in other ways). Since \( \varphi \) has discrete kernel this shows that the Lie algebra of \( S \) is isomorphic to that of either \( H \) or \( SL(3, \mathbb{R}) \). We note that the Lie algebra of \( SL(3, \mathbb{R}) \) has no irreducible 5-dimensional representation; this can be deduced from the corresponding statement over the field of complex numbers, the latter being easy to see from the classification theory of representations of semisimple Lie algebras; see [13]. Recall that the \( H \)-action on \( V \) is irreducible. Since \( S \) contains \( H \), its action on \( V \) is irreducible, and since \( V \) is 5-dimensional the preceding observation implies that the Lie algebra of \( S \) can not be isomorphic to that of \( SL(3, \mathbb{R}) \). Therefore from the alternatives above we now get that \( S \) is locally isomorphic to \( H \). Since \( S \) is connected and \( H \) is a subgroup of \( S \) this implies that \( S = H \).

Since \( H = S = [\mathcal{A}^0, \mathcal{A}^0] \), in particular \( H \) is normal in \( \mathcal{A} \). Now let \( \alpha \in \mathcal{A} \) be arbitrary and consider the automorphism of \( H \) induced by the conjugation action of \( \alpha \). Since \( H \) is Lie isomorphic to \( PSL(2, \mathbb{R}) \) every automorphism of \( H \) is inner. Hence there exists \( h_0 \in H \) such that \( \alpha h_0 \alpha^{-1} = h_0 \alpha h_0^{-1} \) for all \( h \in H \). Then \( h_0^{-1} \alpha \) commutes with every element of \( H \) and particular with all elements of \( D \). Therefore \( h_0^{-1} \alpha \) is diagonalisable and in particular all its eigenvalues are real. Since the action of \( H \) on \( V \) is irreducible it now follows that \( h_0^{-1} \alpha \) acts by scalar multiplication by a (nonzero) real number. Thus \( h_0^{-1} \alpha \in Z \) in the notation as in the hypothesis. Then \( \alpha \in HZ = ZH \), and since \( \alpha \in \mathcal{A} \) was arbitrary we get that \( \mathcal{A} \) is contained in \( ZH \). On the other hand \( H \) and \( Z \) are contained in \( \mathcal{A} \) and therefore we have \( \mathcal{A} = ZH \). This proves the first assertion in the theorem. The second assertion follows from the fact that \( ZH \) is 4-dimensional while \( V \) is 5-dimensional.

We note also the following simple fact about representations of \( PSL(2, \mathbb{R}) \):
Lemma 2.2. Let $\rho : \text{PSL}(2, \mathbb{R}) \to \text{GL}(Q)$ be a representation of $\text{PSL}(2, \mathbb{R})$ over a $\mathbb{R}$-vector space $Q$ of dimension $n \geq 5$, such that no nonzero point of $Q$ is fixed by the image of $\rho$. Let $E$ be the set of points $p$ in $Q$ such that $p$ is an eigenvector of $\rho(g)$ for some nontrivial $g \in \text{PSL}(2, \mathbb{R})$. Then $E$ is a union of countably many smooth submanifolds of dimension at most $(n+4)/2$. In particular if $L$ is a subspace of $Q$ of dimension $m > (n+4)/2$ and $\lambda$ is the Lebesgue measure on $L$ then $\lambda(E \cap L) = 0$.

Proof: We realize $\text{PSL}(2, \mathbb{R})$ as the group $H$ as above (this is just for notational convenience). For each $t \in \mathbb{R}$ let $d_t = \exp t\delta$ and $u_t = \exp t\nu^+$, where $\delta$ and $\nu^+$ are the matrices as defined earlier. Let $\{k_t\}$ be a periodic one-parameter subgroup of $H$, say with period $2\pi$. Then every element of $H$ is conjugate to one of $d_t$, $u_t$ or $k_t$ for some $t \in \mathbb{R}$. Let $R$ be the countable subset of $H$ consisting of $d_1, u_1, k_1$ and $k_r$ for all $r$ of the form $2\pi/m$ with $m$ a positive integer.

For any $h \in H$ let $E(h)$ denote the set of points in $L$ which are eigenvectors of $\rho(h)$. Then $E(h)$ is a union of finitely many vector subspaces of $L$; we shall denote by $d(h)$ the maximum of the dimension of the subspaces contained in $E(h)$. Now consider any nontrivial element $h$ in $H$. We can find a conjugate $h'$ of $h$ such that the subgroup generated by $h$ and $h'$ is dense in $H = \text{PSL}(2, \mathbb{R})$. Then every point of $E(h) \cap E(h')$ is an eigenvector of all elements of $\rho(H)$, and since the latter is a simple Lie group the points must be fixed by $\rho(H)$. The condition in the hypothesis therefore implies that $E(h) \cap E(h') = 0$. Since $h$ and $h'$ are conjugates $d(h) = d(h')$ and hence the preceding conclusion implies that $d(h) \leq n/2$.

Now let $p \in E$ and $h \in H$ be such that $p$ is an eigenvector of $\rho(h)$. Suppose first that $h$ is of infinite order. There exists $t \in \mathbb{R}$ and $g \in H$ such that $ghg^{-1}$ is one of $d_t$, $u_t$ or $k_t$. Then $\rho(g)(p)$ is an eigenvector of $d_t$, $u_t$ or $k_t$. This implies that it is an eigenvector of one of the elements $d_1$, $u_1$ or $k_1$ respectively; in respect of the last alternative note that since $k_t$ is of infinite order the closure of the cyclic subgroup generated by it contains $k_1$. Thus there exists $x \in R$ such that $\rho(g)(p) \in E(x)$, and so $p \in \rho(H)E(x)$. Now suppose that $h$ as above is of finite order. Then $h$ is conjugate to $k_r$ for some $r$ of the form $2\pi/m$ for a positive integer $m$. Therefore in this case also we get that $\rho(g)(p) \in E(x)$ for some $x \in R$. Thus $E = \cup_{x \in R} \rho(H)(E(x))$. Each $E(x)$ is a finite union of vector subspaces which are invariant under a one-parameter subgroup of $H$ and have dimension at most $n/2$. It follows that each $\rho(H)(E(x))$ is a finite union smooth submanifolds of dimension at most $(n/2) + 2 = (n + 4)/2$. This proves the Lemma.

In respect of the conclusion as in the lemma the situation in low dimensions is as follows. For dimensions 2 and 4 there are no representations of $\text{PSL}(2, \mathbb{R})$ with no nonzero common fixed points. In dimension 3 the realization of $H$ (as above) as a subgroup of $\text{SL}(3, \mathbb{R})$ is (up to equivalence) the only representation with no common fixed points. For this representation the set of points which are eigenvectors of nontrivial elements form a cone in $\mathbb{R}^3$ with nonempty interior.
3 A 2-step nilpotent Lie group

We now use Theorem 2.1 to give an example of a simply connected 2-step nilpotent Lie group $G$ such that the action of Aut$(G)$ on $G$ has no dense orbit.

Let $V$ and $W \subset \wedge^2 V$ be as in the preceding section and let $\mathcal{G}$ be the 2-step nilpotent Lie algebra associated with the pair $V, W$ as in §1; namely $\mathcal{G} = V \oplus V'$, where $V' = (\wedge^2 V)/W$ and the Lie bracket operation is determined by the conditions $[v_1, v_2] = (v_1 \wedge v_2) \pmod{W}$ for all $v_1, v_2 \in V$, and $[x, y] = 0$ for $x \in \mathcal{G}$ and $y \in V'$. Let $G$ be the simply connected nilpotent Lie group corresponding to $\mathcal{G}$. Then we have

Corollary 3.1. The action of Aut$(G)$ on $G$ has no dense orbit.

Proof: For the Lie algebra $\mathcal{G}$ as above, corresponding to $G$, the subgroup $\mathcal{A}(\mathcal{G})$ as defined in §1 is the same as the subgroup $\mathcal{A}$ as in Theorem 2.1. Since by Theorem 2.1 the action of $\mathcal{A}$ on $V$ has no open orbit, by Proposition 1.2 the Aut$(\mathcal{G})$-action on $G$ has no dense orbit.

Using Theorem 2.1 one can also describe Aut$(G)$ as follows. To each $h \in H$ corresponds a Lie automorphism of $\mathcal{G}$ such that the restrictions to $V$ and $V' = (\wedge^2 V)/W$ are respectively given by the $h$-action on $V$ and the factor of the $h$-action on $\wedge^2 V$. We denote by $\tilde{H}$ the group of automorphisms of $G$ for which the associated Lie automorphism corresponds to some $h \in H$ as above. Then $\tilde{H}$ is a Lie subgroup of Aut$(G)$, canonically Lie isomorphic $H$. Similarly each $z \in Z$ corresponds canonically to an automorphism of $G$, and we shall denote by $\tilde{Z}$ the subgroup of Aut$(G)$ corresponding to $Z$. Also let $\Psi$ be the group of shear automorphisms of $G$ (see §1).

Corollary 3.2. Aut$(G) = (\tilde{Z}\tilde{H})\Psi$, semidirect product (with $\Psi$ as the normal subgroup).

Proof: We note that $(\tilde{Z}\tilde{H})\Psi$ is a subgroup of Aut$(G)$. Now consider the homomorphism $\tau \mapsto \tau_\Psi$ of Aut$(G)$ into $GL(V)$. It is easy to see that its kernel is $\Psi$, and its image is $\mathcal{A}$ which by Theorem 2.1 is $ZH$. As the subgroup $(\tilde{Z}\tilde{H})\Psi$ of Aut$(G)$ contains $\Psi$, and its image under the homomorphism as above is $ZH$, this shows that it must be the whole. Since the map $\tau \mapsto \tau_\Psi$ is injective on $\tilde{Z}\tilde{H}$ it follows that $(\tilde{Z}\tilde{H}) \cap \Psi$ is trivial. This proves the corollary.

Remark 3.3. The action of $H$ on $\wedge^2 V$ (notation as before) decomposes into two irreducible components, one being on the subspace $W$ as above and another on the subspace, say $W'$, spanned by the $H$-orbit of $\sigma_1 \wedge \sigma_2$; this can be deduced by inspection of the set of weights. Let $\mathcal{G}'$ be the 2-step nilpotent Lie algebra associated to the pair $V$ and $W'$, as described in §1. Then with some modifications in the above argument it can be shown that Aut$(G')$ has no open orbit on $G'$; in this
respect we note mainly that $W$ is also invariant, along with $W'$, under the action of $\mathcal{A}(G')$ on $\wedge^2 V$. This yields an 8-dimensional example (as against 12 of $G$) for which the automorphism group action has no dense orbit. The example as in Theorem 2.1 on the other hand admits a simpler presentation.

4 Lie groups with nilpotent automorphism groups

An example of a 6-step simply connected nilpotent Lie group such that $\text{Aut}(G)$ is nilpotent was given in [9]. Earlier in [8] an example of a 3-step nilpotent Lie algebra $G$ was constructed for which all derivations are nilpotent; the condition implies that $\text{Aut}(G)^0$, the connected component of the identity in $\text{Aut}(G)$, consists of unipotent elements (viewed as elements in $GL(G)$), and hence is a nilpotent Lie group; $\text{Aut}(G)$ itself is however not nilpotent for that example, as has been remarked in [9]. In this section we shall describe a class of examples of simply connected 3-step nilpotent Lie groups, related to the results in the preceding sections, for which the all automorphisms are unipotent, and the automorphism groups are nilpotent. The following implication of this to orbits of the automorphism group action may be borne in mind.

**Remark 4.1.** Let $G$ be a nilpotent Lie algebra such that $\text{Aut}(G)^0$ consists of unipotent elements. Then $\text{Aut}(G)^0$ is a unipotent algebraic subgroup of $GL(G)$ and hence all its orbits on $G$ are closed (see [10]). Since $\text{Aut}(G)$ is an algebraic group it has only finitely many connected components and hence we get furthermore that all orbits of $\text{Aut}(G)$ on $G$ are closed. As the $\text{Aut}(G)$-action on $G$ and the $\text{Aut}(G)$-action on $G$ are topologically equivalent (via the exponential map) it follows that all orbits of $\text{Aut}(G)$ on $G$ are closed. Thus in this case we have a situation stronger than the orbits not being dense, that was obtained in §3.

We note also the following, which shows that the 3-step condition in the above assertions is optimal.

**Remark 4.2.** Let $G$ be a 2-step simply connected nilpotent Lie group. Then for the action of $\text{Aut}(G)$ on $G$ the closure of every orbit contains the identity element; in particular orbits other than that of the identity element are not closed; (in this case the automorphism group, or even its connected component of the identity, does not consist entirely of unipotent elements (viewed as automorphisms of the Lie algebra). The assertion is immediate from the fact that all scalar transformations of $V = G/[G,G]$ belong to $\mathcal{A}(G)$; this shows that the closure of every orbit of $\text{Aut}(G)$ on $G$, where $G$ is the Lie algebra of $G$, contains the zero element of $G$; this is equivalent to the statement as above.

We now proceed to describe the 3-step nilpotent Lie algebras. We shall follow the notation as in §2. For $1 \leq i < j \leq 5$ let $p_{ij} = \sigma_i \wedge \sigma_j \mod(W)$. We note that
Proof. Let \( \tau \in \text{Aut}(\mathcal{N}) \). Then \( \tau \) factors to a linear transformation \( x = \tau \in \text{GL}(V) \). Since \([\sigma_i, \sigma_j] = \sigma_i \wedge \sigma_j \) (mod \( W \)) for all \( i, j = 1, \ldots, 5 \), it follows that \( x \) on \( \wedge^2 V \) leaves invariant the subspace \( W \). Hence by Theorem 2.1 \( x \in ZH \). Let \( z \in Z \) and \( h \in H \) be such that \( x = zh \). Since the subspace spanned by \( p \) equals \([\mathcal{N}, [\mathcal{N}, \mathcal{N}]]\) it is invariant under all automorphisms of \( \mathcal{N} \). Thus \( p \) is an eigenvector of \( x \) (the action on \( V' \)). Since the elements of \( Z \) act as scalars on \( V' \), it follows that \( p \) is an eigenvector of \( h \). Since by choice \( p \) is not an eigenvector of any nontrivial element of \( H \), we get that \( h \) is trivial. Thus \( \tau = x = z \), a scalar transformation, say multiplication by \( \lambda \). Then the action of \( \tau \) on \( V' \) is given by multiplication by \( \lambda^2 \). Since \([\sigma_1, p_{12}] = p \), we have \( \lambda^2 p = \tau(p) = \tau([\sigma_1, p_{12}]) = [\tau(\sigma_1), \tau(p_{12})] = \lambda^3 [\sigma_1, p_{12}] = \lambda^3 p \). Since \( p \) is a nonzero element this implies that \( \lambda = 1 \). This means that \( z \) is trivial, and therefore \( \tau \) is trivial. Thus \( \tau \) is trivial for all \( \tau \in \text{Aut}(\mathcal{N}) \). This shows that \( \mathcal{A}(\mathcal{N}) \) is trivial. The second assertion follows from this, together with Lemma 1.1. Thus proves the corollary.

We deduce also the following, showing that under a slight further condition the automorphism group is ‘minimum possible’.

Corollary 4.4. Let \( N \) and \( \mathcal{N} \) be as above. Suppose the point \( p \) in the definition of the Lie algebra structure on \( \mathcal{N} \) is not contained in the subspace spanned by \( \{p_{14}, p_{15}, p_{25}, p_{35}, p_{45}\} \). Let \( N^* \) denote subgroup of \( \text{Aut}(N) \) consisting of all inner automorphisms and \( \Psi \) be the subgroup consisting of all shear automorphisms of \( N \). Then \( \text{Aut}(N) = N^* \Psi \).

Proof: We shall identify \( \text{Aut}(N) \) with \( \text{Aut}(\mathcal{N}) \) as before. As an algebraic subgroup consisting of unipotent elements \( \text{Aut}(N) \) is a connected Lie group. The Lie algebra
\(\mathcal{D}\) consisting of all derivations of \(\mathcal{N}\) is the Lie algebra of \(\text{Aut}(\mathcal{N})\). The subgroups \(\mathcal{N}^*\) and \(\Psi\) are normal Lie subgroups of \(\text{Aut}(\mathcal{N})\). Therefore to prove the corollary it suffices to show that the Lie subalgebras of \(\mathcal{D}\) corresponding to \(\mathcal{N}^*\) and \(\Psi\) span \(\mathcal{D}\). The Lie subalgebra corresponding to \(\Psi\) consists of all derivations \(\delta\) such that \(\delta(\mathcal{N})\) is contained in \(L\). We shall show that this Lie subalgebra and \(\text{ad}\sigma_1\) and \(\text{ad}\sigma_2\) together span \(\mathcal{D}\) as a vector space. This would complete the proof.

Let \(\delta\) be any derivation of \(\mathcal{N}\). In view of Corollary 4.3 its factor on \(\mathcal{N}/[\mathcal{N},\mathcal{N}]\) is trivial, and hence \(\delta(\mathcal{N})\) is contained in \(V'\). Hence there exist \(a_1, a_2 \in \mathbb{R}\) such that \(\delta(\sigma_i) \in a_ip_{12} + L, \ i = 1, 2\). Then \(\delta' = \delta - a_2(\text{ad}\sigma_1) + a_1(\text{ad}\sigma_2)\) is a derivation such that \(\delta'(\sigma_1), \delta'(\sigma_2) \in L\). It suffices to show that \(\delta'\) belongs to the Lie subalgebra of \(\Psi\), namely that \(\delta'(\mathcal{N})\) is contained in \(L\). Changing notation we shall assume that \(\delta(\sigma_1), \delta(\sigma_2) \in L\) and deduce that \(\delta(\mathcal{N}) \subset L\).

As \(\delta(\mathcal{N}) \subset V'\), for \(i, j \geq 2\) we have \(\delta(p_{ij}) = \delta([\sigma_i, \sigma_j]) = [\delta\sigma_i, \sigma_j] + [\sigma_i, \delta\sigma_j] = 0\). Since \(p_{14} = p_{23}\) and \(p_{15} = p_{24}\), it follows that \(\delta(p_{14}) = \delta(p_{15}) = 0\). We note that \(\delta(p_{12}) = \delta([\sigma_1, \sigma_2]) = [\delta\sigma_1, \sigma_2] + [\sigma_1, \delta\sigma_2] \in [\mathcal{N}, [\mathcal{N}, \mathcal{N}]] \subset L\). Therefore \(\delta(p) = \delta([\sigma_1, p_{12}]) = [\delta\sigma_1, p_{12}] + [\sigma_1, \delta p_{12}] = 0\). Let \(L'\) be the subspace spanned by \(\{p_{14}, p_{15}, p_{25}, p_{35}, p_{45}\}\). Then from what we have seen \(\delta(x) = 0\) for all \(x \in L'\). By the assumption on \(p\) there exists \(\lambda \neq 0\) such that \(p = \lambda p_{13} + x\) for some \(x \in L'\). As \(\delta(p) = 0\) and \(\delta(x) = 0\), this implies that \(\delta(p_{13}) = 0\). Thus we have \([\sigma_1, \delta(\sigma_2)] = \delta([\sigma_1, \sigma_j]) = \delta(p_{1j}) = 0\) for \(j = 3, 4, 5\). This shows that \(\delta(\sigma_j) \in L\) for \(j = 3, 4, 5\). Since by assumption \(\delta(\sigma_1), \delta(\sigma_2) \in L\), we now have \(\delta(\sigma_i) \in L\) for all \(i = 1, \ldots, 5\). Therefore \(\delta(\mathcal{N})\) is contained in \(L\). This completes the proof.

5 Non-simply connected Lie groups

In this section we shall discuss the analogous questions for Lie groups which are not necessarily simply connected. We begin with the following observation.

Proposition 5.1. Let \(G\) be a connected abelian Lie group. Then the action of \(\text{Aut}(G)\) on \(G\) has a dense orbit if and only if \(G\) is not (topologically isomorphic to) the circle group.

Proof. It is well known that for \(m \geq 2\) the \(m\)-dimensional torus admits automorphisms with dense orbits (see [15] for instance). Now let \(G = V \times C\), where \(V\) is a vector space of dimension \(n \geq 1\), \(C\) is the torus of dimension \(m \geq 0\). Let \(\mathcal{H}\) be the set of all continuous homomorphisms of \(V\) into \(C\). For each \(\tau \in GL(V)\) and \(\psi \in \mathcal{H}\) we get a continuous automorphism of \(G\) defined by \((v, c) \mapsto (\tau(v), c\psi(v))\) for all \(v \in V\) and \(c \in C\). The automorphisms arising in this way form a subgroup of \(\text{Aut}(G)\) (it is in fact the identity component of the latter). We recall that there exist homomorphisms \(\psi : V \to C\) such that \(\psi(V)\) is dense in \(C\). Using this it is straightforward to verify that under the action of the group of automorphisms as above the orbit of any element of the form \((v, c)\) with \(v \neq 0\) is dense in \(G\). In
particular the Aut($G$)-action on $G$ has dense orbits. Finally, the circle group has only two automorphisms and hence in this case the Aut($G$)-action has no dense orbit. This proves the proposition.

For Lie groups $G$ of the form $\mathbb{R}^m \times \mathbb{T}^m$ with $m \neq 1$ there exist abelian subgroups of Aut($G$), with finite rank, whose action on $G$ has dense orbits; see [5] for more precise results in this respect.

In the context of the results in the last section and Proposition 5.1 one may ask whether there exist connected 2-step nilpotent Lie groups $G$ such that all orbits of the Aut($G$)-action on $G$ are closed. By Remark 4.2 such a group is necessarily non-simply connected. The following corollary shows that there are groups with this property among quotients of the group $G$ as in §3, by discrete central subgroups.

**Corollary 5.2.** Let $G$ be the simply connected 2-step nilpotent Lie group as in Corollary 3.1. Then there exists a subset $E$ of Lebesgue measure 0 in $[G,G]$ such that if $\theta \in [G,G]$, $\theta \notin E$ and $\Theta$ is the cyclic subgroup of $G$ generated by $\theta$, then for the Lie group $G' = G/\Theta$ the following holds: for any automorphism $\tau$ of $G'$ the factor of $\tau$ on $G'/[G',G']$ is $\pm I$, where $I$ is the identity transformation; furthermore, orbits of the action of Aut($G'$) on $G'$ consist of either one or two cosets of $[G',G']$ in $G'$.

**Proof:** Let $\hat{H}$ be the group of automorphisms as in §3 and consider the $\hat{H}$-action on $[G,G]$. Recall that the latter may be viewed as a vector space and that the action is irreducible; in particular there is no nonzero point fixed by the whole of $\hat{H}$. Since $\hat{H}$ is Lie isomorphic to $PSL(2,\mathbb{R})$ and $[G,G]$ is 7-dimensional, by Lemma 2.2 there exists a subset $E$ of Lebesgue measure 0 such that points outside $E$ are not eigenvectors of any nontrivial element of $\hat{H}$. Now let $\theta \in [G,G]$, $\theta \notin E$ and $\Theta$ be the cyclic subgroup generated by $\theta$; since $[G,G]$ is central in $G$, $\Theta$ is a normal subgroup. Let $G' = G/\Theta$. Since $\Theta$ is discrete, the automorphisms of $G'$ are precisely factors on $G'$ of automorphisms $\tau$ of $G$ such that $\tau(\Theta) = \Theta$. We note that the shear automorphisms, namely those from $\Psi$ in the notation as in §3, satisfy the condition. Therefore by Corollary 3.2 Aut($G'$) is the semidirect product of $\Psi$ with the subgroup $\{\tau \in \hat{Z}\hat{H} \mid \tau(\Theta) = \Theta\}$. Now let $\tau = zh \in \hat{Z}\hat{H}$, where $z \in \hat{Z}$ and $h \in \hat{H}$, be such that $\tau(\Theta) = \Theta$. Since $z$ acts by scalar multiplication on $[G,G]$, this implies that $\theta$ is an eigenvector of $h$. Since $\theta \notin E$ it follows that $h$ is the identity element. Hence $\tau = z \in \hat{Z}$, and since $\tau(\Theta) = \Theta$ it follows that the corresponding scalar transformation of $V = G/[G,G]$ is $\pm I$, where $I$ denotes the identity transformation. This implies the first assertion in the Corollary. The second one follows from this together with the fact that automorphisms from $\Psi$ factor to $G'$.
6 Anosov automorphisms

As mentioned in the introduction study of the automorphism group can be applied also to the general question of understanding the class of compact nilmanifolds supporting Anosov automorphisms; see [1], [3], [6]. In this respect we note the following consequence of Theorem 2.1.

**Corollary 6.1.** Let $G$ be the 2-step simply connected nilpotent Lie group as in Theorem 2.1 and $V = G/[G, G]$ (as before, realised as a vector space). If $\tau \in \text{Aut}(G)$ and the factor $\overline{\tau}$ on $V$ has determinant 1 then $\overline{\tau}$ has a nonzero fixed point on $V$. Consequently, if $\Gamma$ is a lattice (a discrete cocompact subgroup) in $G$ and $\tau \in \text{Aut}(G)$ is such that $\tau(\Gamma) = \Gamma$ then the factor automorphism $\pi(\tau) : G/\Gamma \to G/\Gamma$ is not an Anosov automorphism, and furthermore it is not ergodic.

**Proof:** We follow the notation as before. By Theorem 2.1 $\overline{\tau} \in ZH$, and furthermore if its determinant is 1 then $\overline{\tau} \in H$. The first assertion therefore follows from the fact that for the $H$-action on $V$ every element of $h$ has a nonzero fixed point in $V$; for hyperbolic elements this follows from consideration of weights, for parabolic elements by unipotence, and for elliptic elements by odd-dimensionality of $V$ (analogous assertion holds for any irreducible representation of $H$ on an odd-dimensional vector space).

If $\Gamma$ is a lattice in $G$ then $[G, G] \Gamma$ is closed and $[G, G] \Gamma/[G, G]$ is a lattice in $V = G/[G, G]$; see [12]. Therefore for any $\tau \in \text{Aut}(G)$ such that $\tau(\Gamma) = \Gamma$, $\overline{\tau}$ has determinant $\pm 1$. Hence by the first part $\overline{\tau}^2$ has nonzero fixed points in $V$. This implies that the factor of $\tau$ on $G/\Gamma$ is not an Anosov automorphism, and also that it is not ergodic. This proves the Corollary.

Compact nilmanifolds covered by the 3-step simply connected nilpotent Lie group $G$ as in §4 also do not support Anosov automorphisms; this is immediate from the fact that there are no hyperbolic automorphisms, as $[G, [G, G]]$ is one-dimensional.

**Acknowledgement:** The author would like to thank Karel Dekimpe for useful comments on an earlier version of this manuscript.

**References**

[1] L. Auslander and J. Scheuneman, On certain automorphisms of nilpotent Lie groups, 1970 Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif. 1968) pp. 9-15, Amer. Math. Soc., Providence, 1970.

[2] A. Bialynicki-Birula and M. Rosenlicht, Injective morphisms of real algebraic varieties, Proc. Amer. Math. Soc. 13 (1962), 200-203.
[3] S.G. Dani, Nilmanifolds with Anosov automorphisms, J. London Math. Soc. (2) 18 (1978), 553-559.

[4] S.G. Dani, On automorphism groups acting ergodically on connected locally compact groups, Ergodic Theory and Harmonic Analysis (Mumbai, 1999), Sankhya, Ser. A, 62 (2000), 360-366.

[5] S.G. Dani, On ergodic $\mathbb{Z}^d$-actions on Lie groups by automorphisms, Israel J. Math. 126 (2001), 327-344.

[6] K. Dekimpe, Hyperbolic automorphisms and Anosov diffeomorphisms on nilmanifolds, Trans. Amer. Math. Soc. 353 (2001), 2859-2877.

[7] K. Dekimpe and W. Malfait, A special class of nilmanifolds admitting Anosov diffeomorphism, Proc. Amer. Math. Soc. 128 (2000), 2171-2179.

[8] J. Dixmier and W.G. Lister, Derivations of nilpotent Lie algebras, Proc. Amer. Math. Soc. 8 (1957), 155-158.

[9] J. L. Dyer, A nilpotent Lie algebra with nilpotent automorphism group, Bull. Amer. Math. Soc. 76 (1970), 52-56.

[10] G. P. Hochschild, The Basic Theory of Algebraic Groups and Lie Algebras, Graduate Texts in Mathematics 75, Springer Verlag, 1981.

[11] W. Malfait, Anosov diffeomorphisms on nilmanifolds of dimension at most six, Geometriae Dedicata (3) 79 (2000), 291-298.

[12] M.S. Raghunathan, Discrete Subgroups of Lie Groups, Springer Verlag, 1972.

[13] J.-P. Serre, Complex Semisimple Lie Algebras, Springer Monographs in Mathematics, Springer Verlag, 2001 (reprinted edition).

[14] V.S. Varadarajan, Lie Groups, Lie Algebras and their Representations, Graduate Texts in Mathematics 102, Springer Verlag, 1984 (reprinted edition).

[15] P. Walters, An Introduction to Ergodic Theory, Graduate Texts in Mathematics 79, Springer Verlag, 1982.

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