Weil-Petersson Completion of Teichmüller Spaces and Mapping Class Group Actions

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Abstract

Given a surface of higher genus, we will look at the Weil-Petersson completion of the Teichmüller space of the surface, and will study the isometric action of the mapping class group on it. The main observation is that the geometric characteristics of the setting bear strong similarities to the ones in semi-simple Lie group actions on noncompact symmetric spaces.

1 Introduction

It is well known [23] that the Weil-Petersson metric is not complete on the Teichmüller space over a closed surface of higher genus. When a Weil-Petersson geodesic cannot be further extended, a non-trivial closed geodesic shrinks in length (with respect to the hyperbolic metric) to zero, thus developing a node. Take the Weil-Petersson completion $\mathcal{T}$ of the Teichmüller space $\mathcal{T}$. It was shown by Masur [15] that the Weil-Petersson metric extends to $\mathcal{T}$. In this paper, we show that the space $(\mathcal{T},d)$ is an NPC (or CAT(0)) space in the sense of Toponogov [11], even though the distance function $d$ induced by the Weil-Petersson metric is no longer smooth (with respect to geometric quantities such as the hyperbolic length of closed geodesics.) By construction, the mapping class group (Teichmüller modular group) acts isometrically on the Teichmüller space $\mathcal{T}$. One can extend the isometric action of the mapping class group to the completion $\mathcal{T}$. It will be noted that the geometry of $\mathcal{T}$ is closely related to the isometric actions of various subgroups of the mapping class group. Although $\mathcal{T}$ is no longer a manifold, it still has many geometric characteristics shared with the so called Cartan-Hadamard manifolds; complete simply-connected manifolds with nonpositive sectional curvature. The similarities with the action of semi-simple Lie group $G$ on the symmetric space $G/K$ will be noted. To be more specific, the aim of this paper is to rewrite the paper [3] of Lipman Bers’ where he characterizes, after Thurston [17], the elements of mapping class group in terms of their translation distances with respect to the Teichmüller metric, only to replace the Teichmüller metric by the Weil-Petersson metric.

This paper is motivated to provide a geometric approach to the subject of super/strong rigidity where lattices of Lie groups are represented in the mapping class group of a surface. As in the papers of Corlette [5], Gromov-Schoen [10], the rigidity questions can be transcribed into the study of equivariant harmonic maps into the NPC space on which the isometry group acts. In this approach, the negative curvature condition is crucial to controlling analytic properties of the harmonic maps. In the case of strong rigidity, the representation arises as the monodromy of some fibration where the fiber is the Riemann surfaces of varying conformal structures. The monodromy is created by

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existence of singular surfaces/fibers, or equivalently vanishing cycles. It should be noted that the super rigidity of lattices of rank two and higher in mapping class groups have been studied recently by Farb and Masur via a group theoretic approach.

Also it should be pointed out that there has been much work done on so-called argumented Teichmüller space, and its mapping class group action on it (see for example). One should note that the Weil-Petersson completion of a Teichmüller space can be identified with the augmented Teichmüller space set-theoretically.

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2 Background

Let $\Sigma^2$ be a closed (compact and without boundary) surface of genus $g$ with $g > 1$. Denote the set of all smooth Riemannian metrics on $\Sigma$ by $\mathcal{M}$. Denote the set of all hyperbolic metrics on $\Sigma$ by $\mathcal{M}_{-1}$. Note that by the uniformization theorem, $\mathcal{M}_{-1}$ can be identified with the set of all conformal structures on $\Sigma^2$. Let $\mathcal{D}$ be the group of smooth orientation-preserving diffeomorphisms of $\Sigma$, and $\mathcal{D}_0$ the subgroup of diffeomorphisms homotopic to the identity map from a fixed Riemann surface $\tilde{\Sigma}$ (this gives markings to all the points in $\mathcal{M}_{-1}$.)

Define the Teichmüller space $T_g$ of $\Sigma$ to be

$$T_g = \mathcal{M}_{-1}/\mathcal{D}_0.$$ 

Define the moduli space $M_g$ of $\Sigma$ to be

$$M_g = \mathcal{M}_{-1}/\mathcal{D}.$$ 

The discrete group $\mathcal{D}/\mathcal{D}_0$ is called the mapping class group, or the Teichmüller modular group, which we will denote by $\text{Map}(\Sigma)$.

The space $\mathcal{M}$ of all Riemannian metrics has a natural $L^2$-metric defined by

$$< h, k >_{L^2(\mathcal{G})} = \int_N < h(x), k(x) >_{\mathcal{G}(x)} \, d\mu_G(x)$$

where $h$ and $k$ are symmetric $(0,2)$-tensors, which belong to $T_G\mathcal{M}$. Knowing that $\mathcal{M}_{-1}$ is smoothly imbedded in $\mathcal{M}$ with the induced $L^2$-metric, and also that $\mathcal{M}_{-1} \to \mathcal{M}_{-1}/\mathcal{D}_0$ is a Riemannian submersion (see ), it makes sense to restrict the $L^2$-metric defined on $\mathcal{M}$ to $\mathcal{M}_{-1}/\mathcal{D}_0$. Thus the Teichmüller space has a $L^2$-inner product structure, and it is called Weil-Petersson metric. It should be noted that the Weil-Petersson conormetric was introduced (Ahlfors ) as an $L^2$ pairing of two cotangent vectors, or equivalently two holomorphic quadratic differentials on the surface. It was then identified with the $L^2$ metric defined as above by Fischer and Tromba . Recall the standard geometric fact that any Weil-Petersson geodesic in $\mathcal{T}$ can be lifted horizontally once the initial point of the lift is specified, and the lift is then itself a geodesic in $\mathcal{M}_{-1}$ with respect to the $L^2$ metric. In what follows, we will not distinguish a Weil-Petersson geodesic in $\mathcal{T}$ and its horizontal lift in $\mathcal{M}_{-1}$ unless it is necessary.
With respect to this metric, the Teichmüller space $T$ has non-positive sectional curvature (see Tromba [18] or Wolpert [22]) and though the metric is incomplete (Wolpert [23]) —not every Weil-Petersson geodesic can be extended indefinitely— $T$ is still geodesically convex, that is, every pair of points can be joined by a unique length minimizing geodesic (Wolpert [21]). It is also known that the space is simply connected, diffeomorphic to the $6g - 6$ dimensional Euclidean ball, where $g$ is the genus of the surface $\Sigma$ (see [18] for references.)

We will first show that the incompleteness is always caused by pinching of (at least) one neck of the Riemann surface. Since the proof (as presented in [18]) is short and elementary, we will include it here.

**Proposition 1** Suppose that $\sigma : [0, T) \to T$, where $T < +\infty$ is a Weil-Petersson geodesic, which cannot be extended beyond $T$. Then for any sequence $\{t_n\}$ with $\lim n t_n = T$, the hyperbolic length of the shortest closed geodesic(s) on $(\Sigma, \sigma(t_n))$ converges to zero.

**Proof** Suppose not. Then there is some lower bound $\varepsilon$ for the length of all closed geodesics in $\Sigma$ on $\sigma([0, T))$. Then the compactness theorem of Mumford and Mahler says that there exists a subsequence of $\{t_n\}$ again, and a sequence of diffeomorphisms $\{\phi_n\}$ of $\Sigma$ such that $\phi_n^* \sigma(t_n)$ converges to a hyperbolic metric $G$. Note $\phi_n^* \sigma$ is a horizontal lift of a Weil-Petersson geodesic defined on $(0, T]$ for each $n$. (Here we are using the fact that $M_{g-1} \to T$ is a Riemannian submersion.)

In the meantime, the existence theorem of ordinary differential equation says that given $G$ in the space $M_{g-1}$ of hyperbolic metrics, there exist an open neighborhood $U$ of $G$ and $\delta > 0$ such that any geodesic with an initial point $G'$ in $U$ is defined on $(-\delta, \delta)$. Choose $n$ sufficiently large so that $\phi_n^* \sigma(t_n)$ is in $U$, and $T - t_n < \delta/2$. Then the geodesic $\phi_n^* \sigma(t)$ can be extended to the interval $(t_n - \delta, t_n + \delta)$, which is a contradiction since $T < t_n + \delta$. Q.E.D.

**Definition 1** Let $T$ be the Weil-Petersson completion of the Teichmüller space of a Riemann surface of genus greater than one. Denote by $\partial T$ the frontier set $T \setminus T$.

The preceding proposition states that every point in $\partial T$ represent a nodal surface, that is, a surface with a node or equivalently a pinched neck. H. Masur has shown in [3] that $\partial T$ consists of a union of Teichmüller spaces of topologically reduced Riemann surfaces, created by neck pinching as the conformal structure degenerates toward the frontier points. Masur also showed that the Weil-Petersson metric tensor of $T$ restricted to the directions tangent to the frontier set $\partial T$, spanned by the holomorphic quadratic differentials with poles of order one or less over the pinching neck, converges to the Weil-Petersson metric tensor of the Teichmüller space of the topologically reduced Riemann surface. In this sense the Weil-Petersson metric extends to $T$. The Weil-Petersson metric tensor evaluated in the directions spanned by holomorphic quadratic differentials with order two poles over the pinching neck, blows up at various rates (also in [3]), which we will carefully analyze in the following section.

Lastly in this section we prove the following theorem, which was first proved by the author by a different argument. It was pointed out later by M. Wolf and H. Masur that the statement can be obtained by a direct application of a result (Corollary II 3.11) in the book [4] by Birdson and Heafliger.

**Theorem 1** The Weil-Petersson completed Teichmüller space $T$ is an NPC space (or equivalently a CAT(0) space.)
Remark NPC stands for “non-positively curved” as defined in [11]. It is a length space \((X, d)\), in which any pair of points \(p\) and \(q\) can be connected by a rectifiable curve whose length realizes the distance \(d(p, q)\), and in which any triangle satisfies the length comparison in the sense of Toponogov with a comparison triangle in \(\mathbb{R}^2\).

Proof The result (Corrolary 3.11) cited in [3] says that the metric completion of an NPC space is an NPC space. The Teichmüller space equipped with the Weil-Petersson metric is an NPC space, since it is simply connected, non-positively curved, geodesically convex, open manifold as described above. Hence it follows that its Weil-Petersson metric completion \(\overline{T}\) is an NPC space. Q.E.D.

3 Singular Behavior of Weil-Petersson Metric

We will first consider the case where \(P\) in \(\partial T\) represents a Riemann surface \(\Sigma_0\) with one node. It belongs to a copy of a Teichmüller space \(T_{c_1}\) of a topological surface with a node (or equivalently a surface with two punctures.) Suppose that this \(\Sigma_0\) is obtained by pinching a closed geodesic \(c_1\) of a non-singular surface \(\Sigma\) (i.e. without nodes) to a point. Now introduce a complex coordinate system, as demonstrated in [15], \(t = (t_1, ..., t_{3g-3})\) where \(g\) is the genus of the non-singular surface \(\Sigma\) such that the origin 0 is \(\Sigma_0\), where \(t_2, ..., t_{3g-3}\) parametrize the Teichmüller space \(T_{c_1}\), and \(t_1\) is induced by the local coordinates near the node \(N\) as follows.

At the node \(N\), \(\Sigma_0\) has a neighborhood isomorphic to \(\{|z| < 1, |w| < 1, zw_1 = 0\}\) in \(\mathbb{C}^2\). Remove two discs \(\{z : 0 < |z| \leq |t_1|\}\) and \(\{w : 0 < |w| \leq |t_1|\}\) from \(\Sigma_0\), and then identify \(z\) with \(t_1/w\). We denote by \(\Sigma_t\) the Riemann surface thus obtained. Given the complex structure of \(\Sigma_t\), we will assume that \(\Sigma_t\) is uniformized, that is, equipped with the hyperbolic metric. As \(|t_1| \to 0\), the surface \(\Sigma\) develops a node \(N\).

Observe that by pinching a closed geodesic to a point, one can have two topologically distinct pictures depending on whether \([c_1]\) is homologically nontrivial or not. One is when the resulting surface \(\Sigma_0\) has one path-connected component, with genus \(g - 1\) and with two punctures. The other is that the surface \(\Sigma_0\) consists of two disconnected surfaces, of genus \(g_1\) and \(g_2\) with \(g_1 + g_2 = g\) and each surface has one puncture.

In the first case, the frontier component \(T_{c_1}\) is the Teichmüller space of surfaces of genus \(g - 1\) with two punctures. The complex dimension of \(T_{c_1}\) then is \(3[(g - 1) - 1] + 2 = 3g - 3 - 1\), where the extra two real dimensions is due to the freedom to choose the positioning of the two punctures.

In the second case, \(T_{c_1}\) is a product space of two Teichmüller spaces \(T^1_{c_1}\) and \(T^2_{c_1}\), where \(T^i_{c_1}\) represents the set of Riemann surfaces of genus \(g_i\) with one puncture. Then the dimension of the product space is

\[
[3(g_1 - 1) + 1] + [3(g_2 - 1) + 1] = 3(g_1 + g_2 - 1) - 3 + 2 = 3g - 3 - 1.
\]

Hence in either case the dimension of the frontier Teichmüller space \(T_1\) is of complex codimension one. Similarly when \(\Sigma_0\) has \(n\) nodes, the frontier component that parametrizes the nodal surfaces is of complex codimension \(n\).

H. Masur [15] showed that the Weil-Petersson metric tensor blows up as \(|t_1| \to 0\). In particular, he showed that

\[
0 < \lim inf_{t = (t_1, ..., t_{3g-3}) \to 0} |t_1|^2(- \log |t_1|)^3G_{1T} < \lim sup_{t = (t_1, ..., t_{3g-3}) \to 0} |t_1|^2(- \log |t_1|)^3G_{1T} < C
\]
where \( t = 0 \in \mathbb{C}^{3,-3} \) represent the surface with the node \( P \).

We will refine Masur’s result and show the following.

**Proposition 2** As \(|t_1|\) goes down to zero, that is, as a node develops, one has the following description of the blowing up of the Weil-Petersson metric component.

\[
|G_{1\mathbb{T}}(t)| = \frac{1}{(C + O((-\log |t_1|)^{-2}))|t_1|^2(-\log |t_1|)^3}.
\]

for some constant \( C > 0 \).

**Proof** We will introduce a one-parameter family of hyperbolic surfaces which models the development of the node as \(|t_1|\) goes to zero.

Denote by \( A_{|t_1|} \) the annulus \( \{z : |t_1| < |z| < 1\} \) in \( \mathbb{C} \). One can uniformize the annulus by assigning the following conformal factor to the conformal structure of the annulus.

\[
ds^2_{|t_1|} = \left( \frac{\pi}{\log |t_1|} \csc \frac{\pi \log |z|}{\log |t_1|} \left| \frac{dz}{z} \right| \right)^2.
\]

As the neck pinches \((|t_1| \to 0)\), for each fixed \( z \), the above converges to the hyperbolic metric on two copies of the punctured disc \( \{0 < |z| < 1\} \);

\[
ds^2_0 = \left( \frac{|dz|}{|z| \log |z|} \right)^2,
\]

which models the standard hyperbolic cusp.

Let \( ds^2_{|t_1|} \) be the hyperbolic metric of \( \Sigma_t \) restricted to the annulus region \( A_{|t_1|} \) with respect to the coordinates defined near the closed geodesic \( \{|z| = \sqrt{|t_1|}\} \). Now we quote the following result of Wolf-Wolpert [20].

\[
\|ds^2_{|t_1|} - ds^2_0\|_{C^2} = O((-\log |t_1|)^{-2})
\]

on the smaller annulus \( A_{|t_1|}^\delta = \{|t| \leq |z| \leq 1 - \delta\} \) for a given \( \delta > 0 \).

Indeed \( ds^2_{|t_1|} \) has the following expansion

\[
ds^2_{|t_1|} = \left( \frac{|dz|}{|z| \log |z|} \right)^2 (\Theta \csc \Theta)^2
\]

\[
= ds^2_0 \left( 1 + \frac{1}{3} \Theta^2 + \frac{1}{15} \Theta^4 + \ldots \right).
\]

where

\[
\Theta = \pi \log |z|/\log |t_1|,
\]

and for each \( z \) with \(|z| < 1 - \delta \), we have \( \Theta(z) \to 0 \) as \(|t_1| \to 0\).

Now it follows from the Wolf-Wolpert estimate that

\[
ds^2_{|t_1|} = (1 + O((-\log |t_1|)^{-2})) \frac{\Theta^2}{\sin^2 \Theta} ds^2_0
\]

as \(|t_1|\) goes down to zero. This in turn implies that \( ds^2_{|t_1|} \) converges pointwise to \( ds^2_0 \) as \(|t_1|\) goes down to zero.
Now we will follow very closely the method of Masur’s to compute the asymptotics of the Weil-Petersson metric. Let \( \phi \) be the cotangent vector dual to the tangent vector \( \partial_t \) against the Weil-Petersson pairing. Then we have the following expression for \( \phi \)

\[
\phi_t(z) = -\frac{t}{\pi} a(z) \left( \frac{dz}{z} \right)^2
\]

near the pinching neck, where \( a(z) = 1 + O(|z|) \) for \( |t_1| > 0 \) (see [13]).

We also need the following estimates [15]. For any \( \delta > 0 \), we have the following expression for \( |\phi|^2 \)

\[
\frac{1}{C} r^2 (-\log r)^2 \leq 1/\rho^2(z) \leq C r^2 (-\log r)^2
\]

for \( \sqrt{|t|} \leq r \leq 1 - \delta \), where \( \rho^2(z) = \rho^2(z,t) \) is the conformal factor as above of the complex coordinate \( z \) for the hyperbolic metric \( ds^2 = \rho^2 dz \otimes d\bar{z} \).

Then take the Weil-Petersson pairing of the cotangent vector \( \phi \) with itself, over the modified annulus \( A^\delta_{|t_1|} = \{|t| \leq |z| \leq 1 - \delta \} \).

\[
\int_{A^\delta_{|t_1|}} |\phi|^2 |z|d\theta.
\]

We first note that this quantity is convergent as \( |t_1| \) goes down to zero. To see this, first note that as Masur shows, the quantity is bounded above and below by \( C_1 |t_1|^2 (-\log |t_1|)^3 \) and \( C_2 |t_1|^2 (-\log |t_1|)^3 \), with \( C_1 > C_2 > 0 \).

We claim that the Weil-Petersson pairing above is described by \( \{ C + O((-\log |t_1|)^{-2}) \} |t_1|^2 (-\log |t_1|)^3 \) as \( |t_1| \) goes down to zero for some positive constant \( C \).

Substituting the expansion of the conformal factor \( \rho^2(z,t_1) \), we obtain

\[
\int_{A^\delta_{|t_1|}} \frac{|\phi|^2(z)}{\rho^2(z)} |z|d\theta = \{ 1 + O((-\log |t_1|)^{-2}) \} |t_1|^2 \int_{A^\delta_{|t_1|}} \frac{(1+O(r)) r^2 (-\log r)^2 \sin^2 \Theta^2}{\Theta^2} d\theta
\]

Define \( s = \sin \left( \frac{\pi(-\log r)}{\pi \log |t_1|} \right) \) and change the variable. Then the above is equal to

\[
\{ 1 + O((-\log |t_1|)^{-2}) \} |t_1|^2 \int_0^{2\pi} \int_{|t_1|}^1 \sin \left( \frac{\pi(-\log r)}{\pi \log |t_1|} \right) \frac{s^2}{\sqrt{1 - s^2}} ds d\theta
\]

for some positive number \( C \). Note here that the number \( C \) above does not depend on \( t_1 \)'s for it is determined by the value of the integral \( \int_0^1 \frac{s^2}{\sqrt{1 - s^2}} ds \).

As described in Masur’s paper, it is known that on any compact set \( K \) we have

\[
\int_K \frac{|\phi|^2(z)}{\rho^2(z)} dxdy = O(|t_1|^2).
\]
which vanishes faster than the integral of the same integrand over the annulus. Therefore we have

\[ G^{ij}(t_1) = \int_{\Sigma(t_1)} \frac{\phi_i(z)\phi_j(z)}{\rho^2(z)} \, dx \, dy = \{C + O((- \log |t_1|)^{-2})\}|t_1|^2(- \log |t_1|)^3 \]

The statement of the proposition follows by inverting the matrix \( G^{ij} \) with \( j \neq 1 \) as in the argument given by Masur [13]. Other diagonal terms are bounded away from zero. One point which needs to be addressed in inverting the matrix is the fact that all the off-diagonal terms \( G^{ij} \) vanish at faster rates than the diagonal terms, and hence the matrix behaves as if it were diagonal. Q.E.D.

Let \( z \) be the local coordinate near the node employed in the previous argument. Recall that this particular coordinate was chosen so that the pinching of the neck is closely approximated by the hyperbolic cylinder with the hyperbolic metric \( ds^2_{\Sigma(t_1)} \) defined above. In particular, it is shown by Wolpert [24] that the hyperbolic length \( \lambda_1 \) of the closed geodesic around the pinching neck is given by

\[ \lambda_1(t_1) = \frac{2\pi^2}{- \log |t_1|} + O\left(\frac{1}{(- \log |t_1|)^4}\right) \]

for \( t_1 \) sufficiently small. Now let \( l_1 \) be a new coordinate defined by

\[ l_1 = \frac{2\pi^2}{- \log |t_1|} \]

We would like to know the rate at which \( G_{ij}(t) \) converges to \( G_{ij}(0, t_2, ..., t_{3g-3}) \). We will first quote a result of Wolf [14], which says that the hyperbolic metric \( g_{ij} = \rho^2_{ij}(z)dz \otimes d\bar{z} \) \((t_1 \neq 0)\) on the non-degenerate surface \( \Sigma_{t_1} \) is real analytic in the lengths \( \lambda = (\lambda_1, ..., \lambda_n) \) of the closed geodesics around the pinching necks, and as \( l_1 \) goes to zero, it converges to the cuspidal hyperbolic metric \( g_0 = \rho^2_0(z)dz \otimes d\bar{z} \) on \( \Sigma_0 \). Moreover when there is only one neck pinching Wolf [19] has shown that

\[ |\rho^2_{ij}(z) - \rho^2_0(z)| = O((\lambda_1)^2) = O(\{|l_1|\}^{-2}) \]

over \( \Sigma \). We now claim that the Weil-Petersson metric restricted to the directions \( \partial_{\bar{z}} \) with \( i, j > 1 \) behaves as follows.

\[ |G_{ij}(t_1, t_2, ..., t_{3g-3}) - G_{ij}(0, t_2, ..., t_{3g-3})| = O(\{|l_1|\}^{-2}). \]

To see this, take two cotangent vectors \( dt_{i}, dt_{j} \) with \( i, j > 1 \), each identified with a meromorphic quadratic differentials \( \phi_i \) and \( \phi_j \) respectively, with at most simple poles at \( z = 0 \) (see [15]). \( \phi_i \) and \( \phi_j \) have no other poles away from \( z = 0 \).

Then the Weil-Petersson cometric tensor \( G^{ij} \) is given by

\[ \int_{\Sigma_{t_1}} \frac{\phi_i \phi_j}{\rho^2_{ij}} \, dx \, dy. \]

As before, we consider the region containing the pinching neck and the rest separately. Let \( A_{|t_1|} \) be the annulus \( \{z : |t_1| < |z| < 1\} \) in \( \Sigma_{t_1} \). Then recall the Wolf-Wolpert estimate [20] quoted above, which implies with respect to the complex coordinate \( z \)

\[ \|\rho^2_{ij} - \rho^2_0\|_{C^2} = O(\{|l_1|\}^{-2}) \]
on the smaller annulus $A^\delta_{|t_1|} = \{|t| \leq |z| \leq 1 - \delta\}$ for a given $\delta > 0$. Hence we have

$$\int_{A^\delta_{|t_1|}} \frac{\overline{\phi_i \phi_j}}{\rho_0^2} dxdy - \int_{A^\delta_{|t_1|}} \frac{\overline{\phi_i \phi_j}}{\rho_0^2} dxdy = \int_{A^\delta_{|t_1|}} \phi_i \overline{\phi_j} \left[ \frac{1}{1 + O\left( \frac{-\log |t_1|}{|\rho_0(z)|} \right)} \rho_0^2(z) - \rho_0^2(z) \right] dxdy$$

$$= \int_{A^\delta_{|t_1|}} O\left( \frac{-\log |t_1|}{|\rho_0(z)|} \right)^{-2} dxdy$$

$$= O\left( \frac{-\log |t_1|}{|\rho_0(z)|} \right)^{-2}$$

where the term $O\left( \frac{-\log |t_1|}{|\rho_0(z)|} \right)^{-2}$ is bounded in terms of $\{ -\log |t_1| \}^{-2}$, uniformly in $z$ in $A^\delta_{|t_1|}$ due to the fact that the Wolf-Wolpert estimate is a $C^2$ (in particular $C^0$) estimate on $A^\delta_{|t_1|}$. The last equality follows from the fact that the part of the integrand $\phi_i \overline{\phi_j}$ is a term which as $z \to 0$ can blow up no faster than the rate of $1/|z|^2$, which in turn implies that the integral $\int_{A^\delta_{|t_1|}} \frac{\overline{\phi_i \phi_j}}{\rho_0^2} dxdy$ is a term $O(1)$ as $t_1$ goes to zero.

On a compact set $K$ away from the pinching neck, we have

$$\int_K \frac{\overline{\phi_i \phi_j}}{\rho_1^2} dxdy - \int_K \frac{\overline{\phi_i \phi_j}}{\rho_0^2} dxdy = \int_K \frac{\phi_i \overline{\phi_j} \rho_1^2(z) - \rho_0^2(z)}{\rho_0^2(z) \rho_1^2(z)} dxdy$$

$$= O\left( \frac{-\log |t_1|}{|\rho_0(z)|} \right)^{-2} \int_K \frac{\overline{\phi_i \phi_j}}{\rho_0^2(z)} dxdy$$

$$= O\left( \frac{-\log |t_1|}{|\rho_0(z)|} \right)^{-2}$$

where the second equality follows from the fact shown by Wolf [19] as already described above, that $\rho_1^2$ is real analytic in $\lambda_1 = 2\pi^2 \{ -\log |t_1| \}^{-1} + O\left( \frac{-\log |t_1|}{|\rho_0(z)|} \right)^{-1}$ and $\rho_1(z) - \rho_0(z) = O(\lambda_1^2)$ pointwise on $\Sigma$. The last equality follows from the fact that the integrand of the previous line is continuous in $z$ over $K$.

Combining those estimates, we see that the difference between $G^\Sigma(t_1, t_2, ..., t_{3g-3})$ and $G^\Sigma(0, t_2, ..., t_{3g-3})$ is a term of $O\left( \frac{-\log |t_1|}{|\rho_0(z)|} \right)^{-2}$.

Introduce a new variable $u_1 = \sqrt{t_1}$ here. Then the description of the Weil-Petersson metric near the frontier point is written down as

$$ds^2 = \{ C + O((u_1)^4) du_1^2 + \frac{1}{4}(C + O((u_1)^4)) (u_1)^6 d\theta_1^2 + 4C + O((u_1)^4)) (u_1)^3 \times \left[ \text{cross terms of } du_1 \text{ and } dt_i \text{'s (or } dt_i \text{'s)} \right] \}$$

$$+ \{ C + O((u_1)^4)) (u_1)^6 \times \left[ \text{terms of } d\theta \text{ and } dt_i \text{'s (or } dt_i \text{'s)} \right] \}$$

$$+ \sum_{1 < j \leq 3g-3} \left( 1 + O((u_1)^4) \right) |dt_j|^2$$

with $i > 1$. Here we have used the following relations due to the change of variables.

$$t_1 = |t_1| e^{i \delta_1}, \quad l_1 = \frac{2\pi^2}{-\log |t_1|} \text{ and } u_1 = \sqrt{t_1}$$

$$dt_1 = e^{i \delta_1} dt_1 + it_1 d\theta_1, \quad d\theta_1 = e^{-i \delta_1} dt_1 - it_1 d\theta_1$$

$$\Re \left[ \frac{1}{|t_1|^2 (-\log |t_1|)^3} dt_1 \otimes d\theta_1 \right] = \frac{1}{|t_1|^2 (-\log |t_1|)^3} \left[ \frac{(dt_1)^2}{|t_1|^2} + (|t_1|^2 d\theta_1)^2 \right]$$

$$= \frac{1}{|t_1|^2 (-\log |t_1|)^3} \left( \frac{(dt_1)^2}{|t_1|^2} + (|t_1|^2 d\theta_1)^2 \right)$$

$$= \frac{(du_1)^2}{|t_1|^2} + (u_1)^3 (d\theta_1)^2$$

$$= 4 (du_1)^2 + (u_1)^6 (d\theta_1)^2$$

where $\Re$ denotes the real part of the complex-valued tensor.
Proposition 3  The Weil-Petersson metric tensor near the frontier $T_a$ is continuously differentiable in $u_1, \theta_1$ and $t_i$'s.

Proof  We will show that the Weil-Petersson cometric tensor $G^{ij}$ is continuously differentiable by showing that the differentiations with respect to the parameters $(-\log |t_1|)^{-1}$, $\theta_1$, and $t_i$'s commute with the integration over the hyperbolic surface. Since $u_1$ is defined to be $\sqrt{2\pi^2(-\log |t_1|)^{-1}}$, the statement of the proposition then follows.

In [20] Wolf and Wolpert showed that the hyperbolic metric is sector-real-analytic in $(-\log |t_1|)^{-1}$, $\theta_1$, and $t_i$'s, that is, for any ray from the origin $t = 0$ there is a sector of the neighborhood at the origin containing that ray in which the tensor has a convergent expansion in the variables.

As for $G^{ij}$ the expression for Weil-Petersson pairing of a deformation tensor $\phi$ with itself over the annulus region $A_{[t_1]}$:

\[
\int_{A_{[t_1]}} \frac{|\phi|^2(z)}{\rho^2(z)} |d|d\theta = |t_1|^2 \int_{A_{[t_1]}} \{1 + O((-\log |t_1|)^{-2})\} \frac{1+O(1)}{|z|^2} (1+O(|z|)) \sin^2 \left( \frac{\pi(-\log |t_1|)}{1-\log |t_1|} \right) dxdy
\]

where the term $\{1 + O((-\log |t_1|)^{-2})\}$ is sector-real-analytic in $(-\log |t_1|)^{-1}$, $\theta_1$ and $t_i$'s. Denote the integrand of the last line above by $f(z, (-\log |t_1|)^{-1}, \theta_1, t_i)$. Formally we can differentiate the integral

\[
\int_{0}^{2\pi} \int_{|t_1|}^{1-\delta} f(z, (-\log |t_1|)^{-1}, \theta_1, t_i) |d|d\theta_1
\]

with respect to $(-\log |t_1|)^{-1}$ and obtain

\[
\frac{\partial}{\partial (-\log |t_1|)^{-1}} \int f |d|d\theta_1 = \int_{0}^{2\pi} \int_{|t_1|}^{1-\delta} \frac{\partial f}{\partial (-\log |t_1|)^{-1}} |d|d\theta_1 - \int_{0}^{2\pi} f \bigg|_{z=|t_1|} d\theta_1
\]

This expression is justified as follows. The first term exists since the difference quotient

\[
f(z, (-\log |t_1|)^{-1} + \varepsilon) - f(z, (-\log |t_1|)^{-1})
\]

is uniformly bounded as $\varepsilon$ goes down to zero, so that one can apply the Lebesgue dominated convergence theorem (note here we need to extend the domain of $f$ in $z$ suitably to take the difference.) The second term is zero since when $|z| = |t_1|$, we have $f(z) = 0$.

Similarly we can differentiate the same term with respect to $\theta_1$ and $t_i$'s. Note that the only dependence of $f$ on $\theta_1$ and $t_i$'s comes from the term $\{1 + O((-\log |t_1|)^{-2})\}$ and from the result of [20] we know that this term is differentiable in those variables.

On the complement of the pinching neck, the sector-real-analytic dependence of the hyperbolic metric on all the variables once again induces the differentiability of the Weil-Petersson pairing.

The differentiability of $G^{ij}$ can be checked analogously. Q.E.D.
Now we turn our attention to the case where the frontier point \( P \) in \( \mathcal{T} \) represents a Riemann surface with more than one node. Let \( p > 1 \) be the number of nodes. Recall that \( p \) is bounded by \( 3g - 3 \); the maximal number of mutually disjoint closed geodesics on \( \Sigma \).

Having the convergence of the hyperbolic metrics as a node develops as studied in the proof of the previous proposition, we can now improve the estimates of Masur’s and get the following blow-up rates of the Weil-Petersson metric tensor.

**Proposition 4** In the neighborhood of \( t = (0, 0) \) in \( \mathbb{C}^{3g-3} = \mathbb{C}^p \times \mathbb{C}^{3g-3-p} \), where \( t_i \) (\( 1 \leq i \leq p \)) parametrizes the sizes of the pinching necks, the Weil-Petersson metric is parametrized as follows; as \( t = (t_1, t_2, ..., t_{3g-3}) \rightarrow 0 \),

1) \( |G_i(t)| = \left( C + O((- \log |t_1|)^{-2}) \right) \left[ |t_1|^3(- \log |t_1|)^3 \right]^{-1} \) for \( 1 \leq i \leq p \)

2) \( |G_j(t)| = \left( C + O((- \log |t_i|)^{-2}) \right) \left[ |t_i|^3(- \log |t_i|)^3(- \log |t_j|)^3 \right]^{-1} \) for \( 1 \leq i, j \leq p \) and \( i \neq j \).

3) \( |G_i(t_1, ..., t_{3g-3} - G_j(0, ..., 0, t_{p+1}, ..., t_{3g-3})| = O \left( \sum_{k=1}^{p} (- \log |t_k|)^{-2} \right) \) for \( i, j > p \)

4) \( |G_i(t)| = \left( C + O((- \log |t_i|)^{-2}) \right) \left[ |t_i|^3(- \log |t_i|)^3 \right]^{-1} \) for \( i \leq p \) and \( j > p \).

In proving the proposition, Masur’s proof is modified at two technical points: the first is whenever there is an integration over a pinching neck the finer convergence of the hyperbolic metric parametrized by the \( t_i \)’s is used, and the second technical improvement is to use the convergence of the hyperbolic metric away from the nodes using the estimates by Wolf as quoted above.

As before we perform the change of variables, this time set \( u_i = \sqrt{t_i} \) (\( 1 \leq i \leq p \)). Then the Weil-Petersson metric near the origin \((0, ..., 0)\) in \( \mathbb{C}^{3g-3} \) representing the nodal surface \( P \) with \( p \) nodes has the following expression:

\[
\begin{align*}
    ds^2 &= \sum_{i=1}^{p} \left( C_i + O((u_i)^4) \right) du_i^2 \\
    &\quad + \sum_{p < j \leq 3g-3} \left( 1 + O((u_j)^4) \right) |dt_j|^2 \\
    &\quad + \sum_{1 \leq i, j \leq p} \left( C_{ij} + O((u_i)^4) + O((u_j)^4) \right)(u_i)^3(u_j)^3 \times \left[ \text{cross terms of } du_i \text{ and } du_j \right] \\
    &\quad + \sum_{k \leq p, \ i > p} \left( C_{ik} + O((u_k)^4) \right)(u_k)^3 \times \left[ \text{cross terms of } du_k \text{ and } dt_i \text{ (or } dt_j) \right] \\
    &\quad + \sum_{1 \leq i \leq p, \ p < j \leq 3g-3} \left( C_i + O((u_i)^4) \right)(u_i)^6 \times \left[ \text{cross terms of } d\theta_i \text{ and } dt_j \text{ (or } d\theta_j) \right] \\
    &\quad + \sum_{i=1}^{p} \frac{1}{4} \left( C_i + O((u_i)^4) \right)(u_i)^6 d\theta_i^2 \\
    &\quad + \sum_{1 \leq i, j \leq p} O((u_i)^6(u_j)^6) d\theta_i \otimes d\theta_j.
\end{align*}
\]

### 4 Geometry of the Frontier Set \( \partial \mathcal{T} \)

We start this section with a theorem which describes how each boundary component is embedded in \( \partial \mathcal{T} \).

**Theorem 2** Each component of the boundary Teichmüller spaces is totally geodesic; that is, given any pair of points \( p \) and \( q \) in a Teichmüller space \( \mathcal{T}_C \) representing a collection of nodal surfaces \( \Sigma_C \) obtained by pinching a collection \( C \) of mutually disjoint simple closed geodesics \( c_i \) of the angular surface \( \Sigma \), a length minimizing geodesic connecting \( p \) and \( q \) are totally contained in \( \mathcal{T}_C \) and it is unique.

**Proof** Suppose \( C = \bigcup_{i=1}^{\lvert C \rvert} c_i \). Let \( l_{c_i}(x) \) be the hyperbolic length of the simple closed geodesic \( c_i \).
with respect to the hyperbolic metric $x$ on $\Sigma$. The domain of the functional $l_{c_i}$ can be continuously extended to $\cup_{A \subset C} \mathcal{T}_A$ from $\mathcal{T}$ by defining $l_{c_i}|_{\mathcal{T}_A} \equiv 0$ if $c_i \in A$.

Define a new functional $L_C : \cup_{A \subset C} \mathcal{T}_A \to \mathbb{R}$ by

$$L_C(x) = \sum_{i=1}^{\left| C \right|} l_{c_i}(x).$$

Note that $L_C|_{\mathcal{T}_C} \equiv 0$, hence that $L_C(p) = L_C(q) = 0$.

We now construct a length minimizing geodesic connecting $p$ and $q$. Let $\{p_i\}$ and $\{q_i\}$ be Cauchy sequences in $\mathcal{T}$ converging to $p$ and $q$ respectively. Let $\sigma_i(t)$ be the unique length minimizing Weil-Petersson geodesic connecting $p_i = \sigma_i(0)$ and $q_i = \sigma_i(1)$. Note that $\sigma_i$ lies entirely in $\mathcal{T}$ due to the geodesic convexity of $\mathcal{T}$ [23]. Then by the strictly negative sectional curvature of the Weil-Petersson metric on $\mathcal{T}$, we know that

$$d(\sigma_i(t), \sigma_j(t)) \leq \max \left( d(p_i, p_j), d(q_i, q_j) \right).$$

The right hand side of the inequality converges to zero, and hence it follows that $\sigma_i(t)$ converges to a point in $\overline{\mathcal{T}}$, which we call $\sigma(t)$.

Consider the composite function $f(t) = L_C(\sigma(t))$. We now use the fact that the length functional $l_{c_i}$ is convex with respect to the Weil-Petersson metric on $\mathcal{T}$ and $\mathcal{T}_{c_i}$ (a result of S. Wolpert [23], see [23] for generalizations.) Then it follows that $L_C$ is convex on $\mathcal{T}$ and $\mathcal{T}_C$, and hence that $f(t)$ is convex in $t$. Suppose that $M = \max f(t) > 0$. Then it follows that $f(t) \equiv M > 0$ which contradicts with $f(0) = f(1) = 0$.

We have so far shown that $f(t) \equiv 0$, which then implies that $\sigma$ lies in $\overline{\mathcal{T}_C}$. To see $\sigma$ lies in $\mathcal{T}_C$, note that there exists a Weil-Petersson length realizing geodesic $\sigma'$ connecting $p$ and $q$ lying entirely in the Teichmüller space $\mathcal{T}_C$ due to the Weil-Petersson geodesic convexity of the $\mathcal{T}_C$. Since given two points in an NPC/CAT(0) space $X$ a length-realizing geodesic is unique (here we take $X$ to be $\mathcal{T}_C$), we know that $\sigma'$ is the geodesic $\sigma$ connecting $p$ and $q$.

Q.E.D.

We are in a position to present a series of results which illustrate the geometry of the frontier set $\partial \mathcal{T}$.

**Proposition 5** Let $q$ be a nodal surface, and $p$ a non-degenerate surface. Then the Weil-Petersson open geodesic segment connecting $p$ and $q$ lies entirely in the interior Teichmüller space $\mathcal{T}$.

**Proof** We suppose the contrary; that we have a length minimizing geodesic $\sigma$ connecting $p$ and $q$ with a part of $\sigma$ lying in the frontier, and then moves into the interior Teichmüller space to reach $p$.

We will consider the case that $q$ represent a degenerate surface with one node with a simple closed geodesic $c$ pinched first. Choose the coordinates so that the origin corresponds to the point $\sigma(0) = r$ at which $\sigma$ leaves the frontier. Then the geodesic segment $\overline{pq}$ lies in the frontier and the rest is in $\mathcal{T}$ by the geodesic convexity of the spaces $\mathcal{T}$ and $\mathcal{T}_c$. Let the $3g - 4$ complex dimensional linear space $(t_2, \ldots, t_{3g-3})$ be the geodesic normal coordinate centered at the origin (hence the geodesic segment $\overline{rt}$ is a ray starting from the origin.) Let the vertical axis be parametrized by $u = C(-\log |t_1|)^{-1/2}$, where $C$ is chosen so that the Weil-Petersson metric tensor at the origin restricted to the plane spanned by $\partial x$ and $\partial u$. is approximated by a model metric $du_1^2 + 1/4(u_1)^6d\theta_1^2 + \sum_{i=2}^{3g-3} dt_i \otimes dt_i$ defined on the upper-half space $\{u_1 \geq 0\}$.
Now consider the geodesic connecting \( p \) and \( q \) as an one-dimensional harmonic map \( v : [-1, 1] \to \mathcal{T} \) with the Dirichlet boundary condition \( v(-1) = p \) and \( v(1) = q \). The next step is the following lemma.

**Lemma 1** The pull-back \( v^* u_1 = u_1(v) \) of the coordinate function \( u_1 = C/\sqrt{-\log|t_1|} \) by the harmonic map \( v \) satisfies the differential inequality

\[
\frac{d^2}{dt^2}(v^* u_1)(t) \leq C_1 (v^* u_1)(t)
\]

on \((-1, 1)\) distributionally for sufficiently small values of \( v^* u_1 \) for some constant \( C_1 \).

Once we have this estimate, we can proceed to prove the theorem as follows. The next lemma is the one dimensional version of Harnack inequality for \( W^{1,2} \) functions (functions whose derivatives are in \( L^2 \)).

**Lemma 2** Suppose that non-negative \( W^{1,2} \) function \( u \), defined on \( B_{2 R_0} := \{ -2R_0 < x < 2R_0 \} \), satisfies the differential inequality \( u'' \leq C_1 u \) weakly. Then \( u \) satisfies

\[
\sup_{B_{R_0}} u < C_2 \inf_{B_{R_0}} u
\]

for some constant \( C_2 > 0 \) independent of \( u \) and \( R_0 \).

Note that \( \inf_{B_{R_0}(0)} (v^* u_1) = 0 \) since \( v(0) = r \) denote a nodal surface, and the inequality then implies \( v^* u_1 \equiv 0 \) on \( B_{R_0}(0) \) which is a contradiction to the initial supposition that \( (v^* u_1)(t) > 0 \) for \( t > 0 \).

**Proof** [of the Harnack-type inequality] Modify \( u \) so that

\[
w_{\varepsilon}(x) = u(x) + \varepsilon.
\]

We have \( \varepsilon > 0 \) fixed, and later let it go to zero. In the meantime, we suppress the dependence on \( \varepsilon; w = w_{\varepsilon} \). Note that \( w \) satisfies the same differential inequality \( w'' \leq C_1 w \) as \( f \). Choose \( \zeta \) to be a compactly supported cut-off function on \( B_{2 R_0} \) such that \( \zeta \equiv 1 \) on \( B_{R_0} \), and \( \|
\zeta(x)\| < 2/R_0 \) on \( B_{2 R_0} \setminus B_{R_0} \).

\[
\log \frac{\sup_{B_{R_0} \setminus B_{R_0}} w}{\inf_{B_{R_0}} w} = \log(\sup_{B_{R_0}} w) - \log(\inf_{B_{R_0}} w) \\
\quad = \sup_{B_{R_0}} \log w - \inf_{B_{R_0}} \log w \\
\quad \leq \int_{B_{R_0}} |(\log w)'(x)| dx \\
\quad \leq \left( \int_{B_{R_0}} |(\log w)'(x)|^2 dx \right)^{1/2} \sqrt{2R_0} \\
\quad \leq \left( \int_{B_{2 R_0}} \zeta^2(x) |(\log w)'(x)|^2 dx \right)^{1/2} \sqrt{2R_0} \\
\]

We now claim that

\[
\int_{B_{2 R_0}} \zeta^2(x) |(\log w)'(x)|^2 dx < C_3/R_0
\]

for any \( 0 < R_0 < M \) and \( C_3 = C_3(M) \). Once we have this estimate, we combine the two inequalities above we get for \( 0 < R_0 < M \),

\[
\log \frac{\sup_{B_{R_0}} w}{\inf_{B_{R_0}} w} < C_2.
\]
Finally we let \( \varepsilon \) goes to zero and the statement of the lemma follows.

To see the claim hold true, note that
\[
\int_{B_{2R_0}} C_1 w(\zeta^2 w^{-1})dx \geq \int_{B_{2R_0}} C_1 w''(\zeta^2 w^{-1})dx
\]
\[
= -\int_{B_{2R_0}} [-w''(w')^2 \zeta^2 + 2w' w^{-1} \zeta \zeta' ]dx
\]
\[
= \int_{B_{2R_0}} (\log w)'^2 \zeta^2 - 2(\log w)' \zeta \zeta' dx.
\]

The second integral should be regarded as \( w'' \) being integrated against a test function \( (\zeta^2 w^{-1}) \), and the first inequality is due to the hypothesis \( u'' \leq C_1 u \). By reorganizing the terms, we get
\[
\int_{B_{2R_0}} (\log w)'^2 \zeta^2 dx \leq \int_{B_{2R_0}} 2(\log w)' \zeta \zeta' + C_1 \zeta^2 dx
\]
\[
\leq \int_{B_{2R_0}} \frac{1}{2} (\log w)'^2 \zeta^2 + 2(\zeta')^2 + C_1 \zeta^2 dx
\]

The second inequality is obtained by the inequality \( 2ab \leq a^2 + b^2 \).

Hence it follows that
\[
\int_{B_{2R_0}} (\log w)'^2 \zeta^2 dx \leq 2 \int_{B_{2R_0}} 2(\zeta')^2 + C_1 \zeta^2 dx < \frac{C_3}{R_0}
\]
since \( |\zeta'| < 2/R_0 \) and \( 0 \leq \zeta \leq 1 \) on \( B_{2R_0} \).

Q.E.D.

Proof [of Lemma 1] The harmonic map equation (or the geodesic equation) for \( v : [-1, 1] \to \mathcal{T} \) for the first coordinate function \( v^1 \) is
\[
(v^1)'' + \Gamma^1_{\alpha \beta} (v(t)) (v^\alpha)'(v^\beta)' = 0
\]
where \( \Gamma^1_{\alpha \beta} \) is the Christoffel symbol for the Weil-Petersson metric. Note that \( v^1(t) \) is nothing but the pulled-back function \( (v^*u_1)(t) \). (Also \( v^2(t) = (v^*\bar{t}_1)(t) \).) Hence to show the inequality, it is equivalent to showing that the terms
\[
\sum_{\alpha, \beta} \Gamma^1_{\alpha \beta} (v(t)) (v^\alpha)'(v^\beta)' < C(v^*u_1)(t)
\]
for some \( C > 0 \). In fact, we will show
\[
\sum_{\alpha, \beta} \Gamma^1_{\alpha \beta} (v(t)) (v^\alpha)'(v^\beta)' = O(v^*u_1(t))
\]
which allow us to choose \( C > 0 \) for sufficiently small \( v^*u_1 \).

Recall from the previous section that with respect to the coordinate system \( t = (u_1, \theta_1, t_2, ..., t_{3g-3}) \) near \( t = 0 \), the Weil-Petersson metric tensor has the following form:

\[
G_{11}(t) = 1 + O((u_1)^4)
\]
\[
G_{12}(t) = 0
\]
\[
G_{1j}(t) = O((u_1)^3) \quad (j > 2)
\]
\[
G_{22}(t) = \frac{1+O((u_1)^4)}{4}(u_1)^6
\]
\[
G_{2j}(t) = O((u_1)^6) \quad (j > 2)
\]
\[
G_{ij}(t) = (1 + O((u_1)^4))G_{ij}(0, 0, t_2, ..., t_{3g-3}) \quad \text{for } i, j > 2
\]
as well as
\[ G^{11} = O(1), \ G^{12} = O(1) \text{ and } G^{1j} = O((u_1)^3) \text{ for } j > 2. \]

The Christoffel symbols are obtained from the metric tensor by the following formula (see [7] for example)
\[ \Gamma^l_{ij} = \frac{1}{2} \sum_l G^{1l}(G_{il,j} + G_{lj,i} - G_{ij,l}). \]

Recall here that in Proposition 3 we showed that the metric tensor is continuously differentiable with respect to the variables \( u_1, \theta_1, t_s' \) and \( t_i' \). Hence in differentiating the \( O((u_1)^k) \) terms appearing the description of \( G_{ij}' \)s, the first derivatives behave as follows;
\[ \frac{\partial O((u_1)^k)}{\partial u_1} = O((u_1)^{k-1}) \]
\[ \frac{\partial O((u_1)^k)}{\partial x} = O((u_1)^k). \]

where \( x \) is any one of the variables other than \( u_1 \). This holds because \( O((u_1)^k) = f(x)(u_1)^k + o((u_1)^k) \)
where \( f(x) \) is a continuously differentiable function independent of \( u_1 \) (though it depends on other variables \( x \).)

Consequently we have
\[ \Gamma^1_{11} = O((u_1)^3), \ \Gamma^1_{12} = O((u_1)^4), \ \Gamma^1_{22} = O((u_1)^5) \]
\[ \Gamma^1_{ij} = O((u_1)^2), \ \Gamma^1_{2j} = O((u_1)^3) \text{ and } \Gamma^1_{ij} = O((u_1)^3) \text{ for } i, j > 2. \]

Note that the energy density of the harmonic map \( v \) is uniformly bounded, say by \( M^2 > 0 \) (in fact constant, since the geodesic is parametrized by the arc-length) we know that
\[ |(v^\alpha)'| < M < \infty \]
for \( \alpha \neq 2 \) and for \( v^2 = \theta_1(v) \) we have
\[ (u_1)^6(\theta')^2 < M < \infty. \]

It is easy to see that \( |(\theta_1)'| = O(1) \) as \( u_1 \) goes to zero, for if \( |(\theta_1)'| = O((u_1)^-\kappa) \) for some \( \kappa > 0 \), a comparison map \( \bar{v} \) where \( \bar{v}^2 = (\bar{\theta}_1)(\bar{v}) \) is constant and otherwise \( \bar{v} \) is defined identical to \( v \) has less energy, which contradicts \( v \) being energy minimizing.

Now with the estimates above note that the term out of the geodesic equation
\[ \sum_{\alpha,\beta} \Gamma^1_{\alpha\beta}(v(t))(v^\alpha)'(v^\beta)' \]
is of the order \( O((u_1)^2) \). Hence for \( u_1 < \varepsilon \) for a sufficiently small \( \varepsilon > 0 \), we have the differential inequality
\[ \frac{d^2}{dt^2}(v^*u_1)(t) \leq C(v^*u_1)(t) \]
for \( (v^*u_1)(t) > 0 \) where \( v \) is a smooth map.

We will show that the inequality is valid over the extended region \( u_1 \geq 0 \) distributionally, that is for any non-negative smooth compactly supported test function \( \phi \) on \([-1, 1]\)
\[ \int_{-1}^{1} (v^*u_1)\phi'' dt \leq \int_{-1}^{1} C(v^*u_1)\phi dt. \]
Recall that \((v^* u_1)(t) = 0\) for \(-1 \leq t \leq 0\) and that \((v^* u_1)(t) > 0\) for \(0 < t \leq 1\). Hence
\[
\int_{-1}^{1} (v^* u_1)(t)\phi''(t) dt = \int_{-1}^{0} (v^* u_1)(t)\phi''(t) dt + \int_{0}^{1} (v^* u_1)(t)\phi''(t) dt
\]
\[
= -\int_{0}^{1} (v^* u_1)'(t)\phi'(t) dt + (v^* u_1)\phi\bigg|_{0}^{1}
\]
\[
= \int_{0}^{1} (v^* u_1)'\phi dt + (v^* u_1)'\phi\bigg|_{0}^{1}
\]
\[
\leq \int_{0}^{1} C(v^* u_1)\phi dt
\]
\[
= \int_{-1}^{1} C(v^* u_1)\phi dt.
\]
where we have used the facts that \((v^* u_1)(0) = 0\), \(\phi'(1) = \phi(1) = 0\), \((v^* u_1)'(0) = 0\) and that \(v\) is Lipschitz continuous. The last equality \((v^* u_1)'(0) = 0\) is due to the following argument taken out of \([10]\). By taking a sequence of scalings of both the domain metric and the target distance function at 0 and \(v(0)\) respectively, one obtains the homogeneous map \(v_\star(s)\) from \(\mathbb{R}\) to the tangent cone of \(\mathcal{T}\) at \(v(0)\), which is by itself harmonic since dilations preserves the harmonicity of the map. The tangent cone is isomorphic to \(\mathbb{R}^{3q-8} \times \mathbb{R}^+\). Now since \(t \in \mathbb{R}\), \(v_\star(t) \in \mathbb{R}^{3q-8} \times \{0\}\), it follows that \((v^* u_1)'(0) = 0\). This is because if \((v^* u_1)'(0) > 0\), then after taking the limit of the scaling, the resulting map \(v_\star\) would not be harmonic, for the image of \(\mathbb{R}\) will have a corner at the origin of the tangent cone, (namely a jump discontinuity in its first derivative) a contradiction to the fact that on \(\mathbb{R}\) linear functions are the only harmonic functions.

The fact that \(v\) is Lipschitz continuous was also shown in \([11]\) by showing that the Bochner’s formula holds distributionally, which then implies the DiGiorgi-Nash-Moser type estimate applied to the energy density of the geodesic map \(v\), showing that the modulus of continuity of \(v\) is bounded.

This proves the lemma.

Q.E.D.

To complete the proof of the proposition, we need to consider the case when the point \(q\) represents a nodal surface \(\Sigma_C\) where \(C = \cup_{i=1}^{N} c_i\) is a collection of more than one simple closed mutually disjoint geodesics \(c_i\)’s. Recall that \(p\) is a point in the interior Teichmüller space \(\mathcal{T}\). Suppose now that the geodesic \(\sigma\) starting at \(p = \sigma(-1)\) travels within \(\mathcal{T}_C\) till leaves it at \(r = \sigma(0)\) in \(\mathcal{T}_C\). As before, denote by \(v\) the map \(v : [-1, 1] \to \mathcal{T}\) with the Dirichlet condition \(\sigma(-1) = q\) and \(\sigma(1) = p\). Now consider the pull-back function \(v^*(\sum_{i=1}^{N} u_i) = \sum_{i=1}^{N} u_i(v(t))\). Since each function \(v^* u_i\) satisfies the weak differential inequality
\[
\frac{d}{dt^2} (v^* u_i)(t) \leq C_i (v^* u_i)(t)
\]
over any open interval \((a, b) \subset [-1, 1]\) for some \(C_i > 0\). By taking the sum over \(i\) of the inequalities,
\[
\frac{d^2}{dt^2} \bigg[ \sum_{i=1}^{N} (v^* u_i) \bigg](t) \leq C \bigg[ \sum_{i=1}^{N} (v^* u_i) \bigg](t)
\]
for \(C = \max C_i\).

Therefore as before we have the following Harnack type inequality
\[
\sup_{(-1/2, 1/2)} \bigg[ \sum_{i=1}^{N} (v^* u_i) \bigg] \leq C \inf_{(-1/2, 1/2)} \bigg[ \sum_{i=1}^{N} (v^* u_i) \bigg]
\]

Given that \(\sigma(0)\) represent a surface with multiple nodes, or equivalently \([v^*(\sum_{i=1}^{N} u_i)](0) = 0\), the inequality implies \([v^*(\sum_{i=1}^{N} u_i)] = 0\) over \((-1/2, 1/2)\), a contradiction to the supposition that \([v^*(\sum_{i=1}^{N} u_i)](t) > 0\) for \(t > 0\). Hence we have \([v^*(\sum_{i=1}^{N} u_i)](t) > 0\) over \((-1, 1)\).
The next theorem says there is no kink/corner in any length minimizing geodesic in $\mathcal{T}$.

**Theorem 3** Every open Weil-Petersson geodesic segment in $\mathcal{T}$ is entirely contained in a single copy of Teichmüller space.

**Remark** Given an open geodesic segment, the particular copy of Teichmüller space it lies in may be $\mathcal{T}$ itself, or one component of the frontier $\partial \mathcal{T}$. The theorem states that the image of a harmonic/energy-minimizing map from an open interval to $\mathcal{T}$ respects the stratified structure of the Weil-Petersson completed Teichmüller space $\tilde{\mathcal{T}}$, in the sense that the interior of the geodesic segment meeting but a stratum of $\tilde{\mathcal{T}}$.

**Proof** We will first prove that a path goes through two distinct divisors has a kink, and therefore it cannot be a Weil-Petersson geodesic. We will show this by comparing the lengths of two paths: one through the frontier $\partial \mathcal{T}$, the other through the interior $\mathcal{T}$.

Now let $P_1$ and $P_2$ be two nodes which are obtained by pinching two distinct non-intersecting closed curves $c_1$ and $c_2$ on $\Sigma$. Let $z_i, w_i$ with $i = 1, 2$ be the coordinate systems such that the set $\{|z_i| < 1, |w_i| < 1, z_1w_1 = 0\}$ describes a neighborhood of the node $P_1$ of the surface $\Sigma_{c_1 \cup c_2}$. Recall from the previous argument that $\{|z_i| < 1, |w_i| < 1, z_iw_i = t_i\}$ describes a fattened node, and thus $t_i$ gives us a local coordinate of the completed Teichmüller space near the point $x_0$ representing the nodal surface $\Sigma_{c_1 \cup c_2}$. We assume that the coordinate systems $z_i, w_i$ and $t_i$ are the same as the one used in the discussion of the degenerating family of hyperbolic metrics. Then the hyperbolic lengths $\lambda_i$ of the closed geodesic $c_i$ is given by

$$\lambda_i = \frac{2\pi^2}{-\log |t_i|} + O\left(\frac{1}{(-\log |t_i|)^2}\right),$$

as $(-\log |t_i|)^{-1}$ goes down to zero. From now on, we will denote $\frac{2\pi^2}{-\log |t_1|}$ by $l_1$ and $\frac{2\pi^2}{-\log |t_2|}$ by $l_2$. Define a functional $L$ defined locally in the neighborhood of $x_0$ in $\mathcal{T}$ by

$$L = l_1 + l_2.$$

Note that as the value $\varepsilon$ of $L$ goes to zero, the value of $L$ approximates the values of $\lambda_1 + \lambda_2$.

Now we will proceed to calculate Weil-Petersson lengths of two distinct paths near the point $x_0$ representing the nodal surface with two nodes $P_1$ and $P_2$. The first path $\sigma_1$ describes a deformation of Riemann surfaces along which the approximate length functional $L$ remains constant $\varepsilon > 0$, i.e. it starts at a Riemann surface $x_1$ with a node $P_2$ and with a closed geodesic $c_1$ whose length is approximately $\varepsilon$, moves through a family of surfaces where the sum of the hyperbolic lengths of the two closed geodesics $c_1$ and $c_2$ are approximately $\varepsilon$, and ends at the surface $x_2$ with the node $P_1$ and with the closed geodesic $c_2$ of length approximately $\varepsilon$.

The second path $\sigma_2$ is chosen to be the path connecting $x_1$ and $x_2$, which goes through a point $x_0$ in $\mathcal{T}_{c_1 \cup c_2}$. In other words, first pinch off the closed geodesic $c_1$ to a point while keeping the node $P_2$, and then secondly fatten the node $P_2$ till it becomes a closed geodesic of hyperbolic length approximately $\varepsilon$ while keeping the node $P_1$.

We need to justify the choice of the second path $\sigma_2$ among all other paths connecting the two nodal surfaces, which traverse within the frontier $\partial \mathcal{T}$. As $\sigma_2$ is required to go through the frontier Teichmüller space $\mathcal{T}_{c_1 \cup c_2}$, it is necessary to show that there is no open geodesic segment of $\sigma_2$ lying
Lemma 3. Near the nodal surface $r$, the Weil-Petersson distance function $d$ is approximated by the distance $d_0$ induced by the model metric $dx^2 + du_1^2 + 1/4(u_1)^6 d\theta_1^2$ as follows:

$$|d(p, q) - d_0(p, q)| = O\left([u_1(p)]^3\right)$$

where $p$ has coordinates $(u_1(p), \theta_1(p), t_2(p), ..., t_{3g-3}(p))$.

**Remark** In the coordinates above, $\theta_1$ varies over the entire $\mathbb{R}$. Any two points in the upper half space $\{u_1 \geq 0\}$ whose $\theta_1$ coordinate differs by an integral multiple of $2\pi$ represent the same point in the moduli space $M_g$. Furthermore a point $(0, \theta_1, t_2, ..., t_{3g-3})$ represents a single nodal surface in $\mathcal{T}$ regardless of the value of $\theta_1 \in \mathbb{R}$.

**Proof** [of the lemma] We first choose an arbitrary smooth path $\sigma(s) = (u_1(s), \theta_1(s), t_1(s), ..., t_{3g-3}(s))$ parametrized by arc-length with respect to the model distance function $d_0$. We claim then that the difference between the Weil-Petersson length $L_0(\sigma)$ of the path $\sigma$ and the $d_0$ length $L_0(\sigma)$ are a term
of size $O(U_1)^3$ where $U_1$ is the maximum $u_1$ coordinate $\sigma$ reaches.

$$L(\sigma) - L_0(\sigma) = \int_0^{L_0} [\|\sigma\| - \|\sigma\|_0] ds$$

$$= \int_0^{L_0} \left[ \left(1 + O((u_1)^4)\right)(u_1')^2 + \{1 + O((u_1)^4)\} \sum_i |t'_i|^2 \right.$$

$$+ \frac{1}{4} O((u_1)^4)(u_1)^6(\theta_1')^2 + \sum_i O((u_1)^3)|u_i'|^2 + \sum_i \int O((u_1)^6)|\theta_i'||t'_i| \right]^{1/2}$$

$$\left. - \left[|u_1'|^2 + \sum_i |t'_i|^2 + \frac{1}{4}(u_1)^6(\theta_1')^2 \right]^{1/2} ds \right)$$

$$= \int_0^{L_0} \left[ (u_1')^2 + \sum_i |t'_i|^2 + \frac{1}{4}(u_1)^6(\theta_1')^2 + O((u_1)^4) \left\{ (u_1')^2 + \sum_i |t'_i|^2 + \frac{1}{4}(u_1)^6(\theta_1')^2 \right\} \right]^{1/2}$$

$$+ \sum_i O((u_1)^3)|u_i'|^2 + \sum_i \int O((u_1)^6)|\theta_i'||t'_i| ds$$

$$= \int_0^{L_0} \left[ O((u_1)^4) + \sum_i O((u_1)^3)|u_i'|^2 + \sum_i \int O((u_1)^6)|\theta_i'||t'_i| \right] ds$$

$$= O((u_1)^3)$$

where we have used the equality $\sqrt{1 + x} = 1 + x/2 + o(x)$ as well as the fact that $(u_1')^2 + \sum_i |t'_i|^2 + \frac{1}{4}(u_1)^6(\theta_1')^2 \equiv 1$ for all $s$, since $\sigma(s)$ is parametrized by arc-length with respect to $d_0$.

Now $d(p, q) = L(\sigma)$ where $\sigma$ is the Weil-Petersson geodesic connecting $p$ and $q$, while $d_0(p, q) = L_0(\sigma_0)$ where $\sigma_0$ is the $d_0$ length minimizing geodesic. The estimate above shows that

$$d_0(p, q) = L_0(\sigma_0) = L(\sigma_0) + O(u_1)^3 \geq d(p, q) + O(u_1)^3$$

as well as

$$d(p, q) = L(\sigma) = L_0(\sigma) + O(u_1)^3 \geq d_0(p, q) + O(u_1)^3.$$
with $0 \leq s \leq 1$.

And let $\sigma_2$ be a path consisting of two parts
\[
\left( (1 - t) \varepsilon, \theta^1, 0, *, \sqrt{\varepsilon}(1 - t) T_1 \right)
\]
with $0 \leq t \leq 1$ and
\[
\left( 0, *, u \varepsilon, \theta^2, \sqrt{\varepsilon} u T_2 \right)
\]
with $0 \leq u \leq 1$.

Note that each of the two parts of $\sigma_2$ is composed of is a $d_0$ length-minimizing geodesic, which implies that the $d_0$ length of each path is the same as the $d_0$ distance between the two end-points. Also note that on $\sigma_1$ the quantity $L = l_1 + l_2$ is held constant $\varepsilon$, while on $\sigma_2$, we have $L \leq \varepsilon$.

First calculate the model metric norm of the tangent vector $v(s)$ to the path $\sigma_1(s)$.
\[
v(s) = \left( -\varepsilon, 0, 0, \sqrt{\varepsilon} (T_2 - T_1) \right) = -\varepsilon \partial_{t_1} + \varepsilon \partial_{t_2} + \sqrt{\varepsilon} (t_2^2 - t_1^2) \partial_t,
\]
and thus
\[
\|v(s)\|_{0}^2 = <v(s), v(s)> = g_{11}(-\varepsilon)^2 + 2g_{12} \varepsilon(-\varepsilon) + |T_2 - T_1|^2 \varepsilon = \frac{C_1}{T_1} \varepsilon^2 + \frac{C_2}{T_2} \varepsilon^2 + |T_2 - T_1|^2 \varepsilon
\]
where $0 \leq s \leq 1$.

Now we integrate the norm of the tangent vector while $s$ changes from 0 to $1/2$. This corresponds to the length of the first half of the path $\sigma_1$.
\[
\int_0^{1/2} \|v\|_{0} ds = \int_0^{1/2} \left[ \frac{C_1}{T_1} \varepsilon + \frac{C_2}{T_2} \varepsilon + |T_2 - T_1|^2 \varepsilon \right]^{1/2} ds
\]
\[
= \sqrt{\varepsilon} \int_0^{1/2} \left[ \frac{C_1}{T_1} + \frac{C_2}{T_2} + |T_2 - T_1|^2 \right]^{1/2} ds
\]
\[
\leq \sqrt{\varepsilon} \int_0^{1/2} \left[ \frac{C_1}{T_1} + \frac{C_2}{T_2} + |T_1|^2 + |T_1|^2 \right]^{1/2} ds
\]
\[
< \sqrt{\varepsilon} \int_0^{1/2} \left[ \frac{C_1}{T_1} + |T_1|^2 \right]^{1/2} ds + \sqrt{\varepsilon} \int_0^{1/2} \left[ \frac{C_2}{T_2} + |T_2|^2 \right]^{1/2} ds
\]
\[
= \sqrt{\varepsilon} \int_0^{1/2} \left[ \frac{C_1}{T_1} + |T_1|^2 \right]^{1/2} ds + \sqrt{\varepsilon} \int_0^{1/2} \left[ \frac{C_2}{T_2} + |T_2|^2 \right]^{1/2} ds
\]
where the strict inequality comes from the fact that $\sqrt{A + B} < \sqrt{A} + \sqrt{B}$ for $A, B > 0$, and the last equality is due to a change of variable. By symmetry of the setting with respect to $C$ and $\tilde{C}$, we have the following,

the $d_0$ length of the second part $< \sqrt{\varepsilon} \int_0^{1/2} \left[ \frac{C_2}{T_2} + |T_1|^2 \right]^{1/2} ds + \sqrt{\varepsilon} \int_0^{1/2} \left[ \frac{C_1}{T_1} + |T_2|^2 \right]^{1/2} ds.$

Hence we have the following estimate for the $d_0$ length $L_0(\sigma_1)$ of $\sigma_1$.
\[
L_0(\sigma_1) < \sqrt{\varepsilon} \int_0^{1/2} \left[ \frac{C_1}{T_1} + |T_1|^2 \right]^{1/2} ds + \sqrt{\varepsilon} \int_0^{1/2} \left[ \frac{C_2}{T_2} + |T_2|^2 \right]^{1/2} ds.
\]
On the other hand the $d_0$ length of the path $\sigma_2$ is calculated to be

$$L_0(\sigma_2) = \int_0^1 \|v\|dt + \int_0^1 \|v\|du$$

$$= \int_0^1 \left[ C_s + |T_1|^2 \right]^{1/2} dt + \int_0^1 \left[ C_s + |T_2|^2 \right]^{1/2} du$$

$$= \sqrt{\varepsilon} \int_0^1 \left[ C_s + |T_1|^2 \right]^{1/2} ds + \sqrt{\varepsilon} \int_0^1 \left[ C_s + |T_2|^2 \right]^{1/2} ds$$

Note here that by construction, both $L(\sigma_1)$ and $L(\sigma_2)$ are quantities homogeneous in $\varepsilon$ of degree 1/2. Therefore we have $L(\sigma_1) < L(\sigma_2)$ for any value of $\varepsilon$.

We claim that any path connecting $p$ and $q$ going through the frontier Teichmüller space $T_{c_1\cup c_2}$ cannot be length minimizing. To see this, notice the infimum of the Weil-Petersson distance of such paths is given by

$$L_0(\sigma_2) + O(\varepsilon^{3/2}) = \sqrt{\varepsilon} \left[ \int_0^1 \left( \frac{C}{s} + |T_1|^2 \right)^{1/2} ds + \int_0^1 \left( \frac{C}{s} + |T_2|^2 \right)^{1/2} ds \right] + O(\varepsilon^{3/2})$$

which follows from the length comparison argument above. While the Weil-Petersson length of the path $\sigma_1$ is bounded strictly from above by the quantity

$$\sqrt{\varepsilon} \left[ \int_0^1 \left( \frac{C}{s} + |T_1|^2 \right)^{1/2} ds + \int_0^1 \left( \frac{C}{s} + |T_2|^2 \right)^{1/2} ds \right] + O(\varepsilon^{3/2}).$$

This shows that for sufficiently small $\varepsilon$ (or equivalently for $p$ and $q$ chosen sufficiently close to the frontier Teichmüller space $T_{c_1\cup c_2}$) the Weil-Petersson length of $\sigma_1$ is strictly less than the Weil-Petersson length of any path going entirely through the frontier set $T_{c_1} \cup T_{c_2} \cup T_{c_1\cup c_2}$.

We have so far shown that given two mutually disjoint simple closed curves $c_1$ and $c_2$, the frontier sets $\overline{T_{c_1}}$ and $\overline{T_{c_2}}$ intersect transversely in the sense that there is no length minimizing geodesic originating in $\overline{T_{c_1}}$ ending in $\overline{T_{c_2}}$ going through $\overline{T_{c_1\cup c_2}}$.

To prove the statement of the theorem, this observation needs to be generalized in the cases where we have two sets $C_1$ and $C_2$ of simple closed curves, where $C_1 \cup C_2$ represents a collection of mutually disjoint simple closed curves on $\Sigma$.

There are two distinct cases, the first being when $C_1$ is totally contained in $C_2$. Let $p$ be in $T_{C_1}$, $q$ in $T_{C_2}$. Since $T_{C_2}$ belongs to the frontier set of $T_{C_1}$, by Lemma 1, we know that the open geodesic segment connecting $p$ and $q$ lies entirely in $T_{C_1}$.

The second case is where $C_1 \setminus C_2 \neq \emptyset$ and $C_2 \setminus C_1 \neq \emptyset$. Then we claim that given $p$ in $T_{C_1}$ and $q$ in $T_{C_2}$, the open geodesic segment connecting $p$ and $q$ lies in entirely $T_{C_1\cup C_2}$. If not, it has to go through the frontier Teichmüller space $T_{C_1\cup C_2}$ at a single point $r$. The Weil-Petersson metric can be approximated by the model metric

$$\sum_{i=1}^{|C_1|} C_i \left( du_i^2 + \frac{1}{4} (u_i)^6 d\theta_i^2 \right) + \sum_{j=1}^{|C_2|} C_j \left( du_j^2 + \frac{1}{4} (u_j)^6 d\theta_j^2 \right) + \sum_{k>|C_1|+|C_2|}^{3g-3} dt_k \otimes d\bar{t}_k,$$

where $u_i = \sqrt{-2\varepsilon} e^{-\arg(t_i)}$ and $\theta_i = \arg t_i$. By following the argument for the case when $C_1 = c_1$ and $C_2 = c_2$, it can be checked that the path through $r \in T_{C_1\cup C_2}$ cannot be length minimizing.

Finally we have to consider the case when we have $C_1$ and $C_2$ be sets of mutually disjoint simple closed curves, but $C_1 \cup C_2$ is not a set of mutually disjoint closed curves, for $p \in T_{C_1}$ and $q \in T_{C_2}$, note a length minimizing geodesic lies in $T_{C_1\cup C_2}$. If not, the open geodesic segment connecting $p$
and \( q \) has to go through at least another frontier Teichmüller space \( \mathcal{T}_{C_3} \) with \( C_3 \) strictly containing \( C_1 \cap C_2 \). Then by the argument of the previous paragraph, there has to be a corner the length minimizing geodesic segment has to go around, a contradiction.

Q.E.D.

The next theorem had been known, in particular it is a consequence of a statement (Theorem 6) which appears in [3], due to the fact the the Teichmüller distance dominates the Weil-Petersson distance. The proof is based on the fact that the Dehn twist can be arbitrarily localized in the presence of a pinching neck. The proof is presented here for the sake of completeness and also to make this idea of localizing the Dehn twist explicit in the Weil-Petersson geometric terms.

It is of particular interest when one studies a local monodromy around a singular fiber (a nodal surface \( \Sigma_0 \).) (See for example papers of Matsumoto-Montesinos-Amilibia [14], Earle-Sipe [6]).

First define the following functional on \( \mathcal{T} \)
\[
\delta_\gamma(x) = d(x, \gamma x).
\]
Since the Weil-Petersson distance functional \( d : \mathcal{T} \times \mathcal{T} \) is strictly convex in both entries, it follows that \( \delta_\gamma \) is a convex functional as well.

**Theorem 4** Suppose that \( \gamma \) is a Dehn twist around a simple closed geodesic \( c \) in \( \Sigma \). Let \( \mathcal{T}_c \) be the Teichmüller space of the surface \( \Sigma_0 \) obtained by pinching \( c \) of non-singular surface \( \Sigma \) to a node. Then the set of points in \( \mathcal{T} \) fixed by \( \gamma \) is \( \mathcal{T}_c \).

The statement says that the local monodromy is caused by having a map from the universal cover of a punctured disc to the Weil-Petersson completed Teichmüller space, which is \( \rho \) equivariant where \( \rho : \mathbb{Z} \to \langle \gamma \rangle \subset \text{Map}(\Sigma) \).

**Proof** Suppose \( \gamma \) is a Dehn twist around a closed geodesic \( c \) on \( \Sigma \). Suppose \( \Sigma_0 \) is a Riemann surface with a node \( N \) which is obtained by pinching the closed geodesic \( c \). Then at the node \( N \), \( \Sigma_0 \) has a neighborhood isomorphic to \( |z| < 1, |w| < 1, zw = 0 \) in \( \mathbb{C}^2 \).

Remove the discs \( \{z : 0 < |z| \leq |t|\} \) and \( \{w : 0 < |w| \leq |t|\} \) from \( \Sigma_0 \), and then attach \( z \) to \( t/w \). Let \( A_t = \{z : |t| < |z| < 1\} \) and \( \alpha \) be the curve \( |z| = |w| = |t|^{1/2} \). Uniformize this new non-singular Riemann surface \( \Sigma_t \) such that for a fixed value of \( \delta \) over the annular region \( A_t^{\delta} = \{z : |t|^{1/2} < |z| < 1 - \delta\} \) \( \subset A_t \) the conformal factor \( \rho \) of the hyperbolic metric \( \rho(z)|dz| \) on \( \Sigma_t \) satisfy the following uniform estimate;
\[
\frac{1}{C} |z|^{-2} (\log |z|)^{-2} \leq \rho^2(z) \leq C|z|^{-2} (\log |z|)^{-2}.
\]
for \( z \) in \( A_t^{\delta} \), that is \( |t|^{1/2} < |z| < 1 - \delta \). This estimate has been improved in the previous section, but here it suffices to have the original estimate from [15]. Now construct a one-parameter family of maps \( w_\theta \) of \( A_t \subset \Sigma_t \) as follows.
\[
w_\theta = \begin{cases} 
  z & \text{if } |z| < |t|^{3/4} \\
  z \exp \left( i \theta \int_{|t|^{3/4}}^{s} \phi(s)ds \right) & \text{if } |t|^{3/4} \leq |z| < |t|^{1/4} \\
  z \exp(i\theta) & \text{if } |z| > |t|^{1/4}
\end{cases}
\]
where the function \( \phi(s) \) is a smooth non-negative function supported on \( |t|^{1/2} < s < |t|^{1/4} \) with \( \int_{|t|^{1/2}}^{s} \phi ds = 1 \). Here the number \( \theta \) represents the amount of angle the neck has been twisted by. By
where we have used the fact that \( \nu \) is differential. Let \( B \) of conformal structure given by this infinitesimal Beltrami differential \( \nu \) of this vector field, to obtain the infinitesimal Beltrami differential \( B \phi \) where

\[
d_{\sigma} \phi = \frac{\nu}{2} z^{1/2} \phi \left( \frac{\nu}{\overline{\nu}} \right)^{1/2}.
\]

We want to estimate the Weil-Petersson norm of the tangent vector \( \nu \) induced by the deformation of conformal structure given by this infinitesimal Beltrami differential \( \frac{\partial}{\partial \theta} = \mu_0 \).

We recall the geometry of the space of deformations of a given hyperbolic metric on a surface. We will use the upper half plane model of the hyperbolic two space \( \mathbb{H}^2 \). Then the hyperbolic metric has the conformal factor

\[
p^2(z) = \frac{-4}{(z - \overline{z})^2}.
\]

Let \( \Gamma \) be the Fuchsian group representing the Riemann surface \( \Sigma \) whose tangent space (to the space of all hyperbolic structures on the topological surface) the infinitesimal Beltrami differential \( \mu_0 \) belong to. Let \( B(\Gamma) \) be the complex Banach space of Beltrami differentials of finite \( L^\infty \) norm which are \( \Gamma \) invariant. A Beltrami differential \( \mu \) is called harmonic if \( \mu = (z - \overline{z})^2 \phi \) for a holomorphic quadratic differential. Let \( B(\Gamma) \) be the subspace of \( B(\Gamma) \) consisting of harmonic Beltrami differentials. Given \( \mu \) in \( B(\Gamma) \), there is a projection map (see [22] for example)

\[
P[\mu] = -\frac{3(z - \overline{z})^2}{\pi} \int_{\mathbb{H}^2} \frac{\mu(\zeta)}{(|\zeta - z|^2)^4} d\sigma(\zeta)
\]

where \( d\sigma \) is the Euclidean area element. Now we claim that the Weil-Petersson norm of \( P[\mu_0] \) is bounded by the \( L^2 \) norm of \( \mu_0 \). Recall that the Weil-Petersson norm \( \| \mu_0 \|_{WP} \) here is the pairing

\[
\langle \mu_0, \phi \rangle = \int_{\mathbb{H}^2} \mu_0 \phi d\sigma(z)
\]

where \( \phi \) is the holomorphic quadratic differential \( P[\mu_0]/(z - \overline{z})^2 = P[\mu_0] \rho^2(z) \), and the \( L^2 \) norm \( \| \mu_0 \|_{L^2}^2 \) is given by

\[
\langle \mu_0, \mu_0 \rangle = \int_{\mathbb{H}^2} |\mu_0|^2 \rho^2(z) d\sigma(z).
\]

We denote the term \( \mu_0 - P[\mu_0] \) by \( \nu_0 \). It was shown by Ahlfors (see [22] for example) that \( \nu \) belongs to the kernel \( N(\Gamma) \) of the pairing \( B(\gamma) \times QD(\Gamma) \rightarrow \mathbb{C} \) given by \( (\cdot, \cdot) \) defined as above. To show the claim above, observe

\[
\int_{\mathbb{H}^2} |\mu_0|^2(z) \rho^2(z) d\sigma(z) = \int_{\mathbb{H}^2} \left[ (\mu_0 - \nu_0) + \nu_0 \right] \left[ \frac{\mu_0 - \nu_0}{\overline{\nu_0}} \right] \rho^2 d\sigma(z) = \int_{\mathbb{H}^2} \left[ P[\mu_0]^2 + |\nu_0|^2 + \nu_0 P[\mu_0] + P[\mu_0] \overline{\nu_0} \right] \rho^2(z) d\sigma(z) = \| P[\mu_0] \|^2_{WP} + \| \nu_0 \|^2_{L^2} = \| \nu_0 \|^2_{WP} + \| \nu_0 \|^2_{L^2}
\]

where we have used the fact that \( \nu_0 \) is perpendicular with \( P[\mu_0] \) with respect to the \( L^2 \) pairing, or equivalently that \( \nu_0 \) is in the kernel of the pairing \( (\cdot, \phi) \).

22
Now the Weil-Petersson norm of the tangent vector $v_0$ is estimated as follows;

\[
\|v_0\|_{WP}^2 \leq \|\mu_0\|_{WP}^2 = \int_{A_t} |\mu_0|^2(z)\rho^2(z)d\sigma(z) = \int_{A_t} \frac{1}{4}|z|^2\phi^2(z)\rho^2(z)d\sigma(z) \leq \int_{A_t} \frac{1}{4}|z|^2\phi^2(z)C_\frac{|z|^2}{(\log |z|^2)^2}d\sigma(z) \leq C \int_0^{2\pi} \int_{|t|/2}^{1/4} \phi^2(z)\rho^2(z)|z|drd\theta \\
\leq C \int_0^{2\pi} \int_{|t|/2}^{1/4} \phi^2(z)d\sigma(z) \leq C \int_0^{2\pi} \int_{|t|/2}^{1/4} \phi^2(z)drd\theta \\
\leq C \frac{|t|^{1/4}}{(\log (|t|^{1/4}))^2} \left( \int_{|t|/2}^{1/4} \phi^2(z)dr \right)^2 \\
= C \frac{|t|^{1/4}}{(\log (|t|^{1/4}))^2} \to 0 \quad \text{as} \quad |t| \to 0.
\]

One can also check that the Weil-Petersson norm of the tangent vector $v_\theta$ for any $\theta \in [0, 2\pi]$ also goes down to zero as $|t|$ goes to zero.

Therefore, the path of a Dehn twist $\gamma$ around a “fattened” node of size $|t|$ has a Weil-Petersson length $o\left(\frac{|t|^{1/4}}{(\log |t|^{1/4})^2}\right) \to 0$, which in turn implies that $\delta(\gamma) \to 0$ and it is realized on all nodal surfaces with the closed geodesic $c$ pinched.

We have shown now that $\mathcal{T}_c \subset \{x \in \mathcal{T} : \gamma x = x\}$. We will now show the other inclusion.

Suppose $x \notin \mathcal{T}_c$ is fixed by $\gamma$. Choose a point $y \in \mathcal{T}_c$, and let $\sigma$ be the geodesic connecting $x$ and $y$. Then note that $\delta_\sigma(x) = d(x, \gamma x) = 0 = d(y, \gamma y) = \delta_\sigma(y)$. Since the number $d(z, \gamma z)$ is a convex functional on $\mathcal{T}_c$, for any point $w$ on $\sigma$ we have $d(w, \gamma w) = 0$. This immediately indicates that $\sigma \subset \partial \mathcal{T}$, for the Dehn twist has no fixed point in $\mathcal{T}$. Since $\sigma$ is a geodesic, it has to lie entirely in one component $\mathcal{T}_c$ of the frontier. However since $x$ is not in $\mathcal{T}_c$, this is not possible, a contradiction. Hence every points fixed by $\gamma$ belongs to $\mathcal{T}_c$.

Therefore the statement

$$\mathcal{T}_c = \{x \in \mathcal{T} : \gamma x = x\}$$

follows.

\[\text{Q.E.D.}\]

**Corollary 1** Suppose that $\gamma$ in $\text{Map}(\Sigma)$ represents a product of Dehn twists around a set of mutually disjoint nontrivial closed geodesics $c_1, \ldots, c_n$. Then the set of fixed points by $\gamma$ is the frontier $\mathcal{T}_{1 \ldots n}$ which represents the collection of Riemann surfaces obtained by pinching the closed geodesics $c_i$’s.

The next theorem had been essentially known (see [3]) since the geometry with respect to the Teichmüller metric coincides with that of Weil-Petersson metric in this particular situation. It should be stated for the sake of completeness of the picture.

**Theorem 5** Given an element $\gamma$ of $\text{Map}(\Sigma)$, there exists a unique point $x$ in $\mathcal{T}$ with $\delta_\gamma(x) = 0$ if and only if $\gamma$ is an element of finite order.

The proof follows from the proof of the same statement for Teichmüller distance, once one notes that $d(x, \gamma x) = 0$ implies the Teichmüller distance between $x$ and $\gamma x$ is also zero.
Weil-Petersson Isometric Action of Mapping Class Groups

Let us recall Thurston’s classification \[17\] of diffeomorphisms of a Riemann surface. We will assume the surface is uniformized to have the hyperbolic metric. An element of Map(Σ) = D/D₀ is classified as one of the following three types:

1) it can be represented by a diffeomorphism of finite order, also called periodic or elliptic;
2) it can be represented by a reducible diffeomorphism, that is, the diffeomorphism leaves a tubular neighborhood of a collection \(C\) of closed geodesics \(c_1, \ldots, c_n\) invariant;
3) it can be represented by a pseudo-Anosov diffeomorphism (also called irreducible), that is, there is \(r > 1\) and transverse measured foliations \(F_+, F_-\) such that \(\gamma(F_+) = rF_+\) and \(\gamma(F_-) = r^{-1}F_-\). In this case the fixed point set of \(\gamma\) action in \(\mathcal{P}\mathcal{M}\mathcal{F}(\Sigma)\) (the Thurston boundary of \(\mathcal{T}\)) is precisely \(F_+, F_-\).

As for classification of subgroups, McCarthy and Papadopoulos \[16\] have shown that the subgroups of Map(Σ) is classified into four classes:

1) subgroup containing a pair of independent pseudo-Anosov elements (called sufficiently large subgroups);
2) subgroups fixing the pair \(\{F_+(\gamma), F_-(\gamma)\}\) of fixed points in \(\mathcal{P}\mathcal{M}\mathcal{F}(\Sigma)\) for a certain pseudo-Anosov element \(\gamma \in \text{Map}(\Sigma)\) (such groups are virtually cyclic);
3) finite subgroups;
4) infinite subgroups leaving invariant a finite, nonempty, system of disjoint, non-peripheral, simple closed curves on \(\Sigma\) (such subgroups are called reducible.)

We now will relate those classification results with the stratification structure of the space \(\mathcal{T}\). What one should bear in mind is the correspondence between various subgroups of a semi-simple Lie group \(G\) and totally geodesic submanifolds of the symmetric space \(G/K\) (where \(K\) is the maximal compact subgroup of \(G\)) the subgroups stabilize.

**Theorem 6** Given a reducible element \(\gamma\) of the mapping class group Map(Σ), leaving a collection \(C\) of mutually disjoint closed geodesics \(c_i, i = 1, \ldots, n\) invariant, where \(n\) is chosen to be maximal. Then there is a positive integer \(m\) such that \(\gamma^m\) stabilizes the divisor \(D\) which represents the collection of nodal surfaces with all the \(c_i\)’s pinched.

**Remark** Note that the action of \(\gamma^m\) on each \(\Sigma_i\) is either finite or irreducible, for if not, one can introduce an additional node which is kept invariant by \(\gamma^m\).

**Theorem 7** Given a subgroup \(\Gamma\) of the mapping class group Map(Σ), every element of which fixes a set \(C\) of mutually disjoint closed geodesics \(\{c_i\}\), there is a subgroup \(\Gamma' \subset \Gamma\) of a finite index which stabilizes the divisor which represents the collection of nodal surfaces with all the \(c_i\)’s pinched.

Note that the first theorem is a special case of the second theorem, when the subgroup is the cyclic group generated by the reducible element \(\gamma\). We will prove hence the second theorem now.

**Proof** Suppose the action of \(\Gamma\) is reducible and is completely reduced by a set of mutually disjoint closed geodesics \(C_1, \ldots, C_r\). Then \(\Gamma\) acts on

\[\Sigma_1 \times \ldots \times \Sigma_n\]

where each \(\Sigma_i\) is a component of \(\Sigma \backslash \{C_1 \cup \ldots \cup C_r\}\). Now each element of \(\Gamma\) gives a permutation of the connected components \(\{\Sigma_1, \ldots, \Sigma_n\}\). Hence the representation \(\rho : \Gamma \to \text{Map}(\Sigma)\) induces a homomorphism

\[\phi : \Gamma \to \mathcal{S}_n\]

24
where $S_n$ is the symmetric group of $n$ elements. Let $\Gamma'$ be the kernel of $\phi$. Then $\Gamma'$ is a subgroup of $\Gamma$ of finite index, each element of which leaves each punctured surface $\Sigma_i$ invariant. This means that regarding the collection of $\Sigma_i$'s as a single surface $\Sigma_0$ connected by nodes, $\Gamma'$ acts on $\Sigma_0$ leaving the partition by the nodes invariant, and therefore the action of $\Gamma'$ on $\Pi$ leaves the divisor $D$ representing the nodal surfaces $\Sigma_0$ of various conformal structures invariant. Note here that we have used the fact that the action of $\text{Map}(\Sigma)$ on $\mathcal{T}$ leaves $\partial \mathcal{T}$ and $\mathcal{T}$ invariant, consequently that a nodal surface is sent to another nodal surface. Q.E.D.

Finally we prove the following statement, which characterizes the action of pseudo-Anosov element analogous to the isometric action of a hyperbolic element in $\text{SL}(2, \mathbb{R})$.

**Theorem 8** Suppose $\gamma$ is a pseudo-Anosov element in $\text{Map}(\Sigma)$ where $\Sigma$ is a surface possibly with punctures. Then there exists a $\gamma$-invariant Weil-Petersson geodesic in $\mathcal{T}$ of $\Sigma$.

**Proof** First we demonstrate, after the argument used in [3], that it suffices to show that there exists a point $q$ in $\mathcal{T}$ such that
\[
  d(q, \gamma q) = \inf_{x \in \mathcal{T}} d(x, \gamma x).
\]
Suppose we have such a point $q$. Then since $\gamma$ is of infinite order, $q$, $\gamma q$ and $\gamma^2 q$ are all distinct points. Let $m_1$ be the mid-point of the geodesic segment $q \gamma q$, $m_2$ that of $(\gamma q)(\gamma^2 q)$. Then
\[
  d(q, m_1) = d(m_1, \gamma q) = \frac{1}{2} d(q, \gamma q) = \frac{1}{2} \inf_{x \in \mathcal{T}} d(x, \gamma x).
\]
Similarly
\[
  d(\gamma q, m_2) = d(m_2, \gamma^2 q) = \frac{1}{2} d(\gamma q, \gamma^2 q) = \frac{1}{2} \inf_{x \in \mathcal{T}} d(x, \gamma x).
\]
and
\[
  \gamma m_1 = m_2.
\]
Then by the triangle inequality,
\[
  d(m_1, m_2) \leq d(m_1, \gamma q) + d(\gamma q, m_2) = \inf_{x \in \mathcal{T}} d(x, \gamma x).
\]
while we have
\[
  d(m_1, m_2) = d(m_1, \gamma m_1) \geq \inf_{x \in \mathcal{T}} d(x, \gamma x)
\]
due to the facts that $m_1$, $m_2$ are in $\mathcal{T}$ and that $m_1 m_2$ is contained in $\mathcal{T}$. (Recall $\mathcal{T}$ is geodesically convex [2].) Therefore
\[
  d(m_1, m_2) = \inf_{x \in \mathcal{T}} d(x, \gamma x)
\]
which in turn implies that $\gamma q$ is the mid-point of the geodesic segment $m_1 m_2$, and that $q$, $\gamma q$ and $\gamma^2 q$ lie on a line $l_1$, where line here means a harmonic image of $\mathbb{R}^1$ into $\mathcal{T}$. The line $l_1$ thus obtained is invariant under the isometric action of $\gamma$.

To show that the infimum is achieved at some point $q$ in $\mathcal{T}$, first let $\{p_i\}$ be a sequence in $\mathcal{T} = M_{-1}/D_0$ such that
\[
  \lim_{i \to \infty} d(p_i, \gamma p_i) = \inf_{x \in \mathcal{T}} d(x, \gamma x).
\]
Now let $[p_i]$ be the sequence in the moduli space $M_g = M_{-1}/D$. In other words, let $[p_i]$ be the image of the projection map $P: \mathcal{T}_g \to M_g$. Now within the Deligne-Mumford compactification $\overline{M}$ of $M$, find a convergent subsequence $[p_k]$ of $[p_i]$ which converges to $[p_\infty]$ in $\overline{M}$.
Let \( l_i \) be the geodesic loop in \( \overline{M} \) based at \( p_i \) which lies in the free homotopy class represented by the pseudo-Anosov element \( \gamma \).

We state the following technical lemma.

**Lemma 4** There is some number \( \delta > 0 \) such that for every \( i \), there is a portion of \( l_i \) which lies more than \( \delta \) away from the divisors of the compactified moduli space \( \overline{M} \).

**Proof** [of the Lemma] Suppose not. Then the sequence of the loops \( l_i \) converges to the divisors as \( i \) increases. In particular, \( \lim [p_i] = [p] \) lies in the set \( \mathcal{D} \) of the divisors in \( \overline{M} \). Let \( q_i \) be a lift of \([p_i]\) in \( \mathcal{T} \), and suppose \( q_i \) converges to \( q \), a lift of \( p \). Furthermore, as parametrized sets, \( l_i(t) \) converges to a loop \( l(t) \) in the boundary set \( \mathcal{D} \subset \overline{M} \). Then there exists a collection \( C \) of simple closed geodesics such that \( q \) lies in \( \mathcal{T}_C \). Define \( g_i \) so that \( p_i = g_i q_i \). Then we have

\[
d(p_i, \gamma p_i) = d(g_i q_i, \gamma g_i p_i) = d(q_i, g_i^{-1} q_i g_i).
\]

Since \( \gamma \) is pseudo-Anosov, so is \( g_i^{-1} \gamma g_i \) for each \( i \). We shall denote \( g_i^{-1} \gamma g_i \) by \( \gamma_i \).

Under the supposition with which we started the proof, for any \( \delta > 0 \) there exists \( N_\delta \) so that for any \( i > N_\delta \), the loop \( l_i \) lies entirely in the \( \delta \)-neighborhood of the divisors. The geodesic loop \( l_i \) whose initial point as well as end point are \([p_i]\) can be lifted to \( \mathcal{T} \), so that it becomes a Weil-Petersson geodesic segment \( \sigma_i \) connecting \( q_i \) and \( \gamma_i q_i \). Note that \( q_i \) converges to \( q \) in \( \mathcal{T}_C \), while \( \gamma_i q_i \) converges to \( \gamma q \) in \( \mathcal{T}_{\gamma C} \). The fact that \( \gamma_i \) is pseudo-Anosov implies that for the collection \( C \) of mutually disjoint simple closed geodesics \( c_i \)

\[
\left( \bigcup_{i=1}^{[C]} \mathcal{T}_{c_i} \right) \cap \gamma_i \left( \bigcup_{i=1}^{[C]} \mathcal{T}_{c_i} \right) = \emptyset.
\]

This in turn implies that for each \( i > N_\delta \), there exists a set \( \hat{C}_i \) of mutually disjoint simple closed curves and some nonempty subset \( \hat{C}_i \) of \( C \) which satisfy the following conditions:

1) \( \hat{C}_i \setminus \hat{C}_i = \emptyset \)
2) \( \hat{C}_i \setminus \hat{C}_i = \emptyset \)
3) the set of curves \( \hat{C}_i \cup \hat{C}_i \) can be shrunk to nodes concurrently
4) the set of curves \( C \cup \hat{C}_i \) cannot be shrunk to nodes concurrently,

such that the Weil-Petersson geodesic segment \( \sigma_i \) is contained in the \( \delta \) neighborhood of \( \mathcal{T}_{\hat{C}_i} \cup \mathcal{T}_{\hat{C}_i} \cup \mathcal{T}_{\hat{C}_i} \), where \( \mathcal{T}_{\hat{C}_i} \cup \mathcal{T}_{\hat{C}_i} \) is a “corner” the geodesic passes through the \( \delta \)-neighborhood of.

The condition \( \hat{C}_i \setminus \hat{C}_i = \emptyset \) above says that a point in \( \mathcal{T}_{\hat{C}_i} \) represents a hyperbolic surface where at least one of the curves represented in \( \hat{C}_i \) has a strictly positive hyperbolic length (a node has been “fattened.”) And the condition that the set of curves \( \hat{C}_i \cup \hat{C}_i \) can be shrunk to nodes concurrently implies that the frontier sets \( \mathcal{T}_{\hat{C}_i} \) and \( \mathcal{T}_{\hat{C}_i} \) meet at \( \mathcal{T}_{\hat{C}_i \cup \hat{C}_i} \). The condition 4) is equivalent to saying that \( \mathcal{T}_{\hat{C}_i} \) and \( \mathcal{T}_{\hat{C}_i} \) are disjoint. It then follows that for sufficiently small \( \delta \), and sufficiently large \( i \), \( q_i \) lies in the \( \delta \)-neighborhood of \( \mathcal{T}_{\hat{C}_i} \), though not in the \( \delta \)-neighborhood of \( \mathcal{T}_{\hat{C}_i} \). Similarly note that \( \gamma q_i \) lies in the \( \delta \)-neighborhood of \( \mathcal{T}_{\hat{C}_i} \), though not in the \( \delta \)-neighborhood of \( \mathcal{T}_{\hat{C}_i} \).

Let \( u_k \)'s with \( 1 \leq k \leq |\hat{C}_i \cup \hat{C}_i| \) be the coordinate functions as used in the previous sections defined near the frontier Teichmüller space \( \mathcal{T}_{\hat{C}_i \cup \hat{C}_i} \).

Let \( v_i \) be the one-dimensional harmonic maps \( v_i : [0, 1] \rightarrow \mathcal{T} \) whose image is the geodesic segment \( \sigma_i \). Define the following pulled-back functional \( F_i \)

\[
F_i(t) = \sum_{k=1}^{|\hat{C}_i \cup \hat{C}_i|} v_i^* u_k.
\]
As shown before, for $u_k$’s sufficiently small (less than some $\varepsilon > 0$) a constant only depends on the genus $g$) this function $F_i(t)$ defined on $[0, 1]$ satisfies the differential inequality

$$F''_i(t) \leq K_i F_i(t)$$

(1)

for $t$ satisfying $F_i(t) < \varepsilon$. Now we claim that the constant $K_i = K(\hat{C}_i \cup \hat{C}_i)$ in the inequality does not depend on $i$. This is because that $\hat{C}_i \cup \hat{C}_i$ is determined by the Weil-Petersson geometry of the compactified moduli space $\overline{M}_g$ near the intersection/corner $[C]$ of divisors that the loop $l_i$ passes by for large $i$’s, where $[C]$ is the equivalence class of all the $\hat{C}_i \cup \hat{C}_i$’s since all the $\hat{C}_i \cup \hat{C}_i$’s are conjugates of others by elements of the mapping class group. Denote the constant by $K$

Choose $t_0$ so that $\lim_{t\to\infty} F_i(t_0) = 0$. Note that such $t_0$ exists by the construction of $F_i$’s.

Choose $a_i$ and $b_i$ be the numbers so that $[a_i, b_i]$ is the largest connected interval in $[0, 1]$ containing the number $t_0$ with $F_i(a_i) = F_i(b_i) = \varepsilon$. Reparametrize the domain of $v_i$ via translations and dilations of $\mathbb{R}^1$ so that $F_i(0) = F_i(1) = \varepsilon$. Note that the inequality (1) is the geodesic equation in disguise, and that the geodesic equation is invariant under the affine change of coordinate of the domain $\mathbb{R}$. Define the set $S$ to be

$$S = \{t \in [0, 1] : \lim_{i \to \infty} F_i(t) = 0\},$$

which is nonempty. Note that for each $t \in S$ the Harnack-type inequality from Lemma 2

$$\inf_{(a_i, b_i)} F_i \geq L \sup_{(a_i, b_i)} F_i$$

holds for some open interval $(a_i, b_i) \subset [0, 1]$ containing $t$, which in turn implies that $S$ is open. Recall $L$ in the estimate is independent of $i$. On the other hand one can check that $S$ is a closed set using the facts that $F_i$ are continuous in $t$ and that $F_i$ converges uniformly to some $F_0(t)$ since the loop $l_i$ converges uniformly to some loop in $l_0$ in the boundary $\overline{M}_g \setminus M_g$ of the compactified moduli space $\overline{M}_g$.

Now we have shown that $S \subset [0, 1]$ is open and closed, which implies $S = [0, 1]$, a contradiction for $F_i(0) = F_i(1) = \varepsilon$ for all $i$.

Q.E.D.

Here we modify the $p_i$’s (and $p$ subsequently) so that for a sufficiently large $N$ the points $[p_i]$’s on the loops $l_i$’s with $i > N$ lie outside the $\delta$-neighborhood of the divisors for the $\delta > 0$ from the previous lemma, and that $[p_i]$ converges to $[p_{\infty}]$ which again lies outside the $\delta$-neighborhood of the divisors. Let $q_i$ be a lift of $[p_i]$ in $\mathcal{T}$, chosen so that $\{q_i\}$ converges to a lift $q$ of $[p]$. Then let $g_i$ be defined by $p_i = g_i q_i$. Then we have

$$d(p_i, \gamma p_i) = d(g_i q_i, \gamma g_i q_i) = d(q_i, g_i^{-1} \gamma g_i q_i) \to \inf_{x \in \mathcal{T}} d(x, \gamma x)$$

as $i \to \infty$. Denote $g_i^{-1} \gamma g_i$ by $\gamma_i$. Now note that for large $i$’s, $q$ is translated by $\gamma_i$ by a bounded distance;

$$d(q, \gamma q) \leq d(q, q_i) + d(q_i, \gamma_i q_i) + d(\gamma_i q_i, \gamma q) \leq \varepsilon + (\inf_{x \in \mathcal{T}} d(x, \gamma x) + \varepsilon) + \varepsilon = \inf_{x \in \mathcal{T}} d(x, \gamma x) + 3 \varepsilon$$

Hence the set of points $\{\gamma_i q\}$ lies in a ball of radius $\left(\inf_{x \in \mathcal{T}} d(x, \gamma x) + 3 \varepsilon\right)$ centered at $q$. Also recall from Lemma 3 that the set of points $\{\gamma_i q\}$ lies outside the $\delta$ neighborhood $N_\delta$ of the frontier sets.
Now denote the unit tangent vector at \( q \) of the geodesic \( \sigma_i \), connecting \( q \) and \( \gamma_i q \) by \( w_i \). The sequence \( \{w_i\} \) has a convergent subsequence on the unit tangent sphere at \( q \), which we denote by \( \{w_i\} \) again. Let \( w_0 \) be the direction \( \{w_i\} \) converges to. We now claim that the geodesic \( \sigma_0 \) obtained by exponentiating \( w_0 \) at \( q \) lying in the ball \( B_{(\inf d(x,\gamma x)+3\varepsilon)}(q) \) does not hit the frontier sets \( \partial T \).

Suppose the contrary. Then \( \sigma_0 \) hits a frontier Teichmüller \( T_C \) for some \( C \neq \emptyset \). Define \( F_i \) to be the pulled-back function

\[
F_i(x) = \sum_{k} v_i^* u_k
\]

where \( v_i : [0,1] \to T \) is the one dimensional harmonic map whose image is \( \sigma_i \), and \( u_k \) with \( 1 \leq k \leq |C| \) is the coordinate function used in the previous sections defined near the frontier set \( T_C \). Now choose \( t_0 \) so that \( \lim_{t \to \infty} F_i(t_0) = 0 \). For a sufficiently small \( \varepsilon > 0 \) as well as sufficiently large \( i \)'s, let \( a_i \) and \( b_i \) be the numbers so that \( [a_i, b_i] \) is the largest connected interval in \( R \) containing \( t_0 \) with \( F_i(a_i) = F_i(b_i) = \varepsilon \). Such \( a_i \) and \( b_i \) exist because the end points of each geodesic segment \( \sigma_i \) lies at least \( \delta > 0 \) distance away from the boundary set \( \partial T \). Note that by the geodesic convexity of \( T \), each \( \sigma_i \) lies in \( T \), and thus \( F_i > 0 \) for each \( i \). Reparametrize \( v_i \) via dilations and translations of the domain \( R \) so that \( F_i(0) = F_i(1) = \varepsilon \).

Let \( S \) be the set

\[
S = \{ t : \lim_{i \to \infty} F_i(t) = 0. \}
\]

a nonempty subset of \([0,1]\). Since \( F_i \) satisfies the Harnack-type inequality as noted in the proof of Lemma 4, and since \( F_i \)'s are equicontinuous in \( t \), the set \( S \) is open and closed in \([0,1]\), and hence \( S = [0,1] \), a contradiction to the fact that \( F_i(0) = F_i(1) = \varepsilon \) for all \( i \).

Now we know that the sequence of the directions \( \{\sigma'_i(0) = w_i\} \) converges to a direction \( w_0 \) and that the geodesic obtained by exponentiating \( w_0 \) does not hit the boundary set \( \partial T \) within the ball \( B_{(\inf d(x,\gamma x)+3\varepsilon)}(q) \). Then there exists a sufficiently small neighborhood \( N \) of \( w_0 \) on the unit tangent sphere at \( q \), such that the exponential map \( \exp : N \times [0,(\inf d(x,\gamma x)+3\varepsilon)] \to T \) is a diffeomorphism. Thus it follows that the sequence \( \{\gamma_i q\} \) has a convergent subsequence within the image of the exponential map above, which is a subset of \( T \). In particular the convergent subsequence, which we again denote by \( \{\gamma_i q\} \), is a Cauchy sequence and for a given \( \varepsilon_0 > 0 \), there exists an integer \( N \) such that

\[
d(\gamma_i q, \gamma_j q) = d(q, \gamma_i^{-1}\gamma_j q) \leq \varepsilon_0
\]

for \( i, j > N \).

Recall here that the mapping class group \( \text{Map}(\Sigma) \) acts properly discontinuously away from the frontier set \( \partial T \). That is, for \( q \) there exists \( r_q > 0 \) such that

\[
B_{r_q}(q) \cap B_{r_q}(qq) = \emptyset
\]

for all but finitely many \( g \)'s in \( \text{Map}(\Sigma) \). We choose \( \varepsilon_0 \) above to be smaller than the \( r_q \), and fix \( i \) to be \( I > N \). Choose a subsequence \( \{\gamma_j\} \) (which we denote also by \( \{\gamma_j\} \)) such that \( \gamma_i^{-1}\gamma_j = \gamma_0 \) where \( \gamma_0 \) is one of the finite set of elements in \( \text{Map}(\Sigma) \) which moves \( B_{r_q}(q) \) no more than \( 2r_q \).

Then for all \( j > N \), we have \( \gamma_j \equiv \gamma_i \gamma_0 \). Or equivalently

\[
\gamma_i^{-1}\gamma_j = \gamma_i \gamma_0.
\]

Suppose \( j, k > N \). Then note that

\[
\gamma_i^{-1}\gamma_j = \gamma_i^{-1}\gamma_k
\]

and that

\[
(\gamma_i \gamma_j^{-1}) \gamma = \gamma_0 (\gamma_i \gamma_j^{-1})
\]

28
so that \((g_k g_j^{-1})\) commutes with \(\gamma\). Since \(\gamma\) is pseudo-Anosov, it then follows that \((g_k g_j^{-1}) = \gamma^n\) for some \(n\). Therefore \(g_j = \gamma^n \mathcal{T}\) for some \(\mathcal{T}\). Recall how \(p_j\)'s were chosen;

\[
d(p_j, \gamma p_j) \to \inf_{x \in \mathcal{T}} d(x, \gamma x)
\]
as \(j \to \infty\). Therefore

\[
d(p_j, \gamma p_j) = d(g_j q_j, \gamma g_j q_j) = d(\gamma^n \mathcal{T} q_j, \gamma^n \gamma \mathcal{T} q_j) = d(\mathcal{T} q_j, \gamma \mathcal{T} q_j) \to \inf_{x \in \mathcal{T}} d(x, \gamma x).
\]

Since \(\{q_j\}\) converges to \(q\), it follows that \(\{\mathcal{T} q_j\}\) converges to \(\mathcal{T} q\) in \(\mathcal{T}\). Therefore we have

\[
d(q, \gamma q) = \inf_{x \in \mathcal{T}} d(x, \gamma x).
\]

Q.E.D.

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