Positivity bounds on parton distributions at large $N_c$

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Abstract

Positivity bounds for twist-two parton distributions in multicolored QCD are stronger than their analogs at finite $N_c$. They include the enhanced large $N_c$ version of Soffer inequality. These bounds are compatible with the DGLAP evolution to higher normalization points.

1 Introduction

The spectrum of the applications of the $1/N_c$ expansion to the analysis of the dynamics of strong interactions is rather rich. However, the parton distributions of hadrons remained a sort of exception for a long time. Historically the large $N_c$ behavior of parton distributions was first analyzed in the context of the chiral quark soliton model but the general structure of the $N_c$ counting in this model is the same as in QCD.

Among recent applications of the large $N_c$ expansion to parton distributions one should mention the phenomenological analysis of the antiquark polarized distribution, of the polarized gluon distribution and the large $N_c$ relations between the quark distributions in the nucleon and in the $\Delta$ resonance. Here we shall consider another interesting result: combining the large $N_c$ expansion with the probabilistic interpretation of parton distributions one can derive certain inequalities for parton distributions which are stronger than the corresponding inequalities known in the case of finite $N_c$.

Our analysis will be based on the standard picture of the large $N_c$ baryons as states that can be described by mean field (Hartree) equations of some effective theory. Although the exact action of this effective theory equivalent to the large $N_c$ QCD is not known certain conclusions can be made using only the assumption about the symmetry of the solutions of the corresponding Hartree equations.

The large $N_c$ inequalities described here are not mere products of the large $N_c$ pure art per se. For example, they include the enhanced large $N_c$ version of
Soffer inequality \[12\] for the transversity distribution that can be of interest for the phenomenological applications. Actually some of the results considered here were noticed earlier \[3\] in the context of the quark-soliton model \[2\]. Since the derivation was based only on the spin-flavor symmetry of the large \(N_c\) baryons the generalization to the case of the large \(N_c\) QCD is rather straightforward.

2 Positivity bounds at finite \(N_c\)

Let us first recall the well-known positivity properties of parton distributions in the real world at finite number of colors. The unpolarized parton distribution \(q_f(x)\) is defined as the probability to find a quark of type \(f\) carrying fraction \(x\) of the nucleon momentum in the infinite momentum frame. The positivity of this probability leads us to the simplest inequality \(q_f(x) \geq 0\). If one considers the longitudinally polarized nucleon then \(q_f(x)\) can be decomposed into two pieces coming from quarks with the longitudinal spin collinear with nucleon momentum (\(q_f^\uparrow(x)\)) and with anticollinear spin (\(q_f^\downarrow(x)\))

\[
q_f(x) = q_f^\uparrow(x) + q_f^\downarrow(x). \tag{1}
\]

Since both functions \(q_f^\uparrow(x)\) and \(q_f^\downarrow(x)\) have a probabilistic interpretation they are positive

\[
q_f^\uparrow(x) \geq 0, \quad q_f^\downarrow(x) \geq 0. \tag{2}
\]

The (longitudinally) polarized parton distribution is defined as

\[
\Delta L q_f(x) = q_f^\uparrow(x) - q_f^\downarrow(x). \tag{3}
\]

Now it follows from (1), (2), (3)

\[
|\Delta L q_f(x)| \leq q_f. \tag{4}
\]

Next, one can consider the nucleon and quarks with the transverse polarization and define by analogy with \(\Delta L q_f\) the transversity distribution \(\Delta T q_f\) which will obey inequality

\[
|\Delta T q_f(x)| \leq q_f. \tag{5}
\]

One more inequality can be obtained by choosing arbitrary directions for the polarizations of quarks and nucleons. The positivity of the corresponding probabilities leads to Soffer inequality \[12\]

\[
q_f + \Delta L q_f \geq 2|\Delta T q_f|. \tag{6}
\]

3 Positivity bounds at large \(N_c\)

Thus in the real world with \(N_c = 3\) (and at any other finite \(N_c\)) we have the following set of the positivity bounds for the unpolarized (\(q_f\), longitudinally
polarized \((\Delta_{Lq_f})\) and transversity \((\Delta_{Tq_f})\) quark distributions

\[
\begin{align*}
|\Delta_{Lq_f}| &\leq q_f, \\
|\Delta_{Tq_f}| &\leq q_f, \\
2|\Delta_{Tq_f}| &\leq q_f + \Delta_{Lq_f}.
\end{align*}
\] (finite \(N_c\)) \(\tag{7}\)

Below we shall derive the following analogs of these inequalities in the large \(N_c\) limit:

\[
\begin{align*}
|\Delta_{Lq_f}| &\leq \frac{1}{3}q_f \left[1 + O(N_c^{-1})\right], \\
|\Delta_{Tq_f}| &\leq \frac{1}{2}q_f \left[1 + O(N_c^{-1})\right], \quad (N_c \to \infty) \tag{8}
\end{align*}
\]

2\(|\Delta_{Tq_f}| \leq \left(\frac{1}{3}q_f + \Delta_{Lq_f}\right) \left[1 + O(N_c^{-1})\right].
\]

Note that these inequalities for the large \(N_c\) case are stronger than the corresponding inequalities at finite \(N_c\) \((7)\) because of the factors 1/3 and 1/2 appearing in the right-hand sides. The origin of these factors can be traced to the spin-flavor symmetry of the large \(N_c\) baryons; in particular, the factor of 1/3 has nothing to do with the fact that 1/\(N_c\) = 1/3 in the real world.

This enhancement of the positivity bounds in the large \(N_c\) limit is rather a striking result. Naively one could expect that in the leading order of the large \(N_c\) limit some part of information about parton distributions would be lost but instead in this limit we have new stronger bounds. The reason of this phenomenon can be explained in simple terms without appealing to the technical derivation of \((8)\). In the leading order of the large \(N_c\) approximation the nucleon is degenerate with the \(\Delta\) resonance and other baryonic resonances with \(T = J = 1/2, 3/2, 5/2, \ldots\) For all of these particles one can analyze the twist-two parton distributions and derive the corresponding positivity bounds. Moreover, since all these particles are degenerate at the level of the leading order of the 1/\(N_c\) expansion one can consider the formal superposition of states with different values of \(T = J\) and derive the positivity bounds for the parton distribution function of these formal superpositions. Although the superpositions of states with different \(T = J\) make no physical sense this formal trick allows us to derive the new enhanced positivity bounds \((8)\) valid in the large \(N_c\) limit.

### 4 Large \(N_c\) counting for parton distributions

In order to derive the large \(N_c\) inequalities \((8)\) we should start from the analysis of the large \(N_c\) behavior of the parton distributions. This analysis can be performed using standard large \(N_c\) counting rules of QCD (see e.g. \([2, 5, 6]\)). On the other hand, it is well known that the same large \(N_c\) behavior of various quantities can be also extracted from the nonrelativistic quark model, Skyrme
model, quark soliton model etc. Below we briefly describe the large $N_c$ behavior of parton distributions appealing mainly to the naive quark model and using sum rules for parton distributions.

Note the twist-two parton distributions $q_f(x)$, $\Delta_L q_f(x)$, $\Delta_T q_f(x)$ depend on Bjorken variable $x$ so that the analysis of the large $N_c$ behavior should involve a convention about the behavior of $x$ in this limit. From the physical point of view the nucleon consists of $N_c$ quarks (with an admixture of quark-antiquark pairs) so that it is natural to expect that at large $N_c$ limit the nucleon momentum in the infinite momentum frame is more or less uniformly distributed between $O(N_c)$ quarks and antiquarks. Then in the large $N_c$ limit it makes sense to concentrate on $x = O(1/N_c)$. This physical argument shows that the large $N_c$ behavior of various parton distributions should have the form

$$ q_f(x) = N_c^2 \left[ \phi_f(N_c x) + O(N_c^{-1}) \right], \quad (9) $$

$$ \Delta_L q_f(x) = N_c^2 \left[ \Delta_L \phi_f(N_c x) + O(N_c^{-1}) \right]. \quad (10) $$

Functions $\phi_f$, $\Delta_L \phi_f$, $\Delta_T \phi_f$ are stable in the large $N_c$ limit so that the $x$ dependence of the form $\phi_f(N_c x)$ implies that $x \sim N_c^{-1}$ at large $x$. The factor of $N_c^2$ ensures the consistency of the large $N_c$ behavior for various sum rules for parton distributions. For example for the baryon charge sum rule

$$ \sum_f \int_0^1 dx \left[ q_f(x) - \bar{q}_f(x) \right] = \sum_f \int_{-1}^1 dx q_f(x) = N_c \quad (11) $$

we find in the large $N_c$ limit

$$ \int_{-1}^1 dx q_f(x) \xrightarrow{N_c \to \infty} N_c^2 \int_{-1}^1 dx \phi_f(N_c x), $$

$$ \int_{-1}^N dy \phi_f(y) \xrightarrow{N_c \to \infty} N_c \int_{-\infty}^{\infty} dy \phi_f(y) = O(N_c) \quad (12) $$

which really agrees with the sum rule (11).

In the leading order of the large $N_c$ limit we have for the $u$ and $d$ distribution functions of the proton

$$ q_u = q_d \left[ 1 + O(N_c^{-1}) \right], \quad \Delta_L q_u = -\Delta_L q_d \left[ 1 + O(N_c^{-1}) \right], $$

$$ \Delta_T q_u = -\Delta_T q_d \left[ 1 + O(N_c^{-1}) \right]. \quad (13) $$

These large $N_c$ relations have a simple interpretation in the nonrelativistic quark model: at large odd $N_c$ the proton consists of $(N_c+1)/2$ $u$-quarks and $(N_c-1)/2$ $d$-quarks. In the leading order of large $N_c$ we have the same amount of $u$ and $d$ quarks which means that $q_u = q_d [1 + O(1/N_c)]$. Next, since we want to have the spin of the proton $1/2$, at large $N_c$ there must be a cancelation between the $O(N_c)$ parts of the spin of $u$ and $d$ quarks, which is expressed by the last two equations (13).
5 Derivation of large $N_c$ inequalities

Below we sketch the main ideas of the derivation of inequalities (8). The complete description can be found in ref. [7].

Although we cannot compute parton distributions in the large $N_c$ QCD, still it is possible to write certain general relations using only the spin-flavor symmetry of the large $N_c$ baryons. The twist-two quark distributions $q_u, \Delta_L q_u, \Delta_T q_u$ in proton can be represented in the form

$$q_u(x) = \frac{1}{2} S p \rho(x),$$
$$\Delta_L q_u(x) = -\frac{1}{6} S p \left[ \gamma^5 \tau^3 \rho(x) \right],$$
$$\Delta_T q_u(x) = \frac{1}{12} S p \left[ \gamma^5 (\gamma^1 \tau^1 + \gamma^2 \tau^2) \rho(x) \right].$$

Here $\rho_{f', fs}(x)$ is an $8 \times 8$ matrix with SU(2) flavor indices $f', f$ and with Dirac spin indices $s', s$. Matrix $\rho(x)$ depending on $x$ is determined by the dynamics of the large $N_c$ QCD and is not known. Nevertheless we are aware of the following general properties:

1) $\rho$ is a hermitean positive matrix (i.e. all eigenvalues are positive or zero)

$$\rho^+ = \rho,$$
$$\rho \geq 0,$$

2) $\rho$ lives in the subspace of the projector $(1 + \gamma^0 \gamma^3)/2$

$$\rho(\gamma^0 \gamma^3) = (\gamma^0 \gamma^3) \rho = \rho,$$

3) $\rho$ commutes with $i \gamma^1 \gamma^2 + \tau^3 \equiv i \gamma^1 \gamma^2 \otimes 1_{\text{flavor}} + 1_{\text{spin}} \otimes \tau_3$

$$[\rho, (i \gamma^1 \gamma^2 + \tau^3)] = 0.$$

These properties allow us to write the following general representation for $\rho(x)$

$$\rho(x) = \frac{1 + \gamma^0 \gamma^3}{2} \left[ c_1(x) \cdot 1 + c_2(x) \gamma^5 \tau^3 + c_3(x) \gamma^5 (\gamma^1 \tau^1 + \gamma^2 \tau^2) \right]$$

(19)

where $c_i(x)$ are some scalar functions of $x$.

The positivity property (14) leads to the following constraints on coefficients $c_i$

$$c_1 - c_2 \geq 2 |c_3|,$$
$$c_1 + c_2 \geq 0.$$

(20)
(21)

Inserting the explicit representation for $\rho$ (19) into the general equations for parton distributions (14) we immediately express the parton distributions $f_i(x)$ in terms the coefficients $c_i(x)$. Then the constraints (20), (21) on these coefficients immediately lead to bounds (8) on parton distributions.
6 Phenomenological status

Concerning the practical applications of the large $N_c$ inequalities we have to keep in mind two competing factors:

1) large $N_c$ inequalities (8) are stronger than the standard inequalities (7) valid at finite $N_c$.

2) at finite $N_c$ inequalities (8) can be violated by large $1/N_c$ corrections.

In the case of $\Delta_L q_f$ the $1/N_c$ corrections are known to be rather large.

Indeed, the Bjorken sum rule relates $\Delta_L q_u - \Delta_L q_d$ with the axial constant $g_A$ and it is well known that $g_A$ is usually underestimated in various model calculations based on the leading order of the $1/N_c$ expansion, which is usually attributed to sizable $1/N_c$ corrections. Hence the $1/N_c$ corrections to $\Delta_L q_f$ should be also large. Therefore one should not wonder that our large $N_c$ inequality for $\Delta_L q_u - \Delta_L q_d$

$$\frac{|\Delta_L q_u - \Delta_L q_d|}{q_u + q_d} \leq \frac{1}{3} [1 + O(N_c^{-1})]$$

(22)

following from (8), (13) is in conflict with experimental data. For example, in the GRSV parametrization (13)

$$\max_x |\Delta_L q_u - \Delta_L q_d| \sim 0.6$$

(23)

On the other hand, the large $N_c$ bounds (7) can be of use in the case of the transversity distribution and polarized antiquark distribution where the experimental knowledge is rather poor.

7 Concluding remarks

In the large $N_c$ limit one has the positivity bounds (3) for twist-two parton distributions that are stronger than the standard positivity bounds at finite $N_c$. However, the practical applications of the large $N_c$ inequalities are sensitive to the size of the $1/N_c$ corrections.

In the case of finite $N_c$ it is well known that positivity bounds (7) are compatible with the DGLAP evolution equations in the following sense. If inequalities (7) hold at some normalization point then the one-loop evolution to higher normalization points will not violate these inequalities. In ref. (7) it was shown that the large $N_c$ inequalities (8) have the same property: if one uses the evolution kernels only in the leading order of the large $N_c$ expansion then inequalities (8) are preserved by the evolution upwards.

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