On the distribution of the Rudin-Shapiro function for finite fields

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In memory of Christian Mauduit

Abstract

Let $q = p^r$ be the power of a prime $p$ and $(\beta_1, \ldots, \beta_r)$ be an ordered basis of $\mathbb{F}_q$ over $\mathbb{F}_p$. For

$$\xi = \sum_{j=1}^{r} x_j \beta_j \in \mathbb{F}_q$$

with digits $x_j \in \mathbb{F}_p$,

we define the Rudin-Shapiro function $R$ on $\mathbb{F}_q$ by

$$R(\xi) = \sum_{i=1}^{r-1} x_i x_{i+1}, \quad \xi \in \mathbb{F}_q.$$ 

For a non-constant polynomial $f(X) \in \mathbb{F}_q[X]$ and $c \in \mathbb{F}_q$, we study the number of solutions $\xi \in \mathbb{F}_q$ of $R(f(\xi)) = c$. If the degree $d$ of $f(X)$ is fixed, $r \geq 6$ and $p \to \infty$, the number of solutions is asymptotically $p^{r-1}$ for any $c$. The proof is based on the Hooley-Katz Theorem.

MSC 2020. 11A63, 11T23, 11T30

Keywords. finite fields, digit sums, Hooley-Katz Theorem, polynomial equations, Rudin-Shapiro function
1 Introduction

In recent years, many spectacular results have been obtained on important problems combining some arithmetic properties of the integers and some conditions on their digits in a given basis, see for example [12,13,15,17,19,23]. In particular, Drmota, Mauduit and Rivat [8] and Müllner [17] showed that Thue-Morse sequence and Rudin-Shapiro sequence along squares are both normal.

A natural question is to study analog problems in finite fields, see for example [4,5,7,9,12,18,20–22]. Many of these problems can be solved for finite fields although their analogs for integers are actually out of reach.

In particular, it is conjectured but not proved yet that the subsequences of the Thue-Morse sequence and Rudin-Shapiro sequence along any polynomial of degree \( d \geq 3 \) are normal, see [8, Conjecture 1]. Even the weaker problem of determining the frequency of 0 and 1 in the subsequence of the Thue-Morse sequence and Rudin-Shapiro sequence along any polynomial of degree \( d \geq 3 \) seems to be out of reach, see [8, above Conjecture 1]. However, the analog of the latter weaker problem for the Thue-Morse sequence in the finite field setting was settled by the first author and Sárközy [4].

This paper deals with the following analog of the frequency problem for the Rudin-Shapiro sequence along polynomials.

Let \( q = p^r \) be the power of a prime \( p \) and \( \mathcal{B} = (\beta_1, \ldots, \beta_r) \) be an ordered basis of the finite field \( \mathbb{F}_q \) over \( \mathbb{F}_p \). Then any \( \xi \in \mathbb{F}_q \) has a unique representation

\[
\xi = \sum_{j=1}^{r} x_j \beta_j \quad \text{with} \quad x_j \in \mathbb{F}_p, \quad j = 1, \ldots, r.
\]

The coefficients \( x_1, \ldots, x_r \) are called the digits with respect to the basis \( \mathcal{B} \).

In order to consider the finite field analogue of the Rudin-Shapiro sequence along polynomial values, we define the Rudin-Shapiro function \( R(\xi) \) for the finite field \( \mathbb{F}_q \) with respect to the basis \( \mathcal{B} \) by

\[
R(\xi) = \sum_{i=1}^{r-1} x_i x_{i+1}, \quad \xi = x_1 \beta_1 + \cdots + x_r \beta_r \in \mathbb{F}_q, \quad r \geq 2.
\]

For \( f(X) \in \mathbb{F}_q[X] \) and \( c \in \mathbb{F}_p \) we put

\[
\mathcal{R}(c, f) = \{ \xi \in \mathbb{F}_q : R(f(\xi)) = c \}.
\]

Our goal is to prove that the size of \( \mathcal{R}(c, f) \) is asymptotically the same for all \( c \).

Our main result is the following theorem.

**Theorem 1.** Let \( f(X) \in \mathbb{F}_q[X] \) be of degree \( d \geq 1 \). For \( c \in \mathbb{F}_p \), we have

\[
| | \mathcal{R}(c, f) | - p_{r-1}^r | \leq C_{d, r} p^{(3r+1)/4-h_{r,c}},
\]
where $h_{r,c}$ is defined by

$$
\begin{align*}
    h_{r,c} &= \begin{cases} 
      3/4, & r \text{ even and } c \neq 0, \\
      1/2, & r \text{ odd and } c \neq 0, \\
      1/4, & r \text{ even and } c = 0, \\
      0, & r \text{ odd and } c = 0,
    \end{cases}
\end{align*}
$$

and $C_{d,r}$ is a constant depending only on $d$ and $r$.

In particular, we have for fixed $d$,

$$
\lim_{p \to \infty} \frac{|R(c, f)|}{p^{r-1}} = 1 \quad \text{for } c \neq 0 \text{ and } r \geq 4 \text{ or } c = 0 \text{ and } r \geq 6.
$$

For $d = 1$, or more generally, for any permutation polynomial $f(X)$ of $\mathbb{F}_q$, it is easy to see that

$$
|R(c, f)| = \begin{cases} 
      p^{r-1} - p^{\lfloor (r-1)/2 \rfloor}, & c \neq 0, \\
      p^{r-1} + p^{\lfloor (r+1)/2 \rfloor} - p^{\lfloor (r-1)/2 \rfloor}, & c = 0, \\
      r \geq 2.
    \end{cases}
$$

For the convenience of the reader we will provide a very short proof in Section 2. Hence, it remains to prove Theorem 1 for $d \geq 2$.

A commonly used idea, for example in [4], to estimate the number of solutions of certain equations over finite fields is to apply the Weil bound. In some special situations the Deligne bound [6, Théorème 8.4] provides stronger results. The Weil bound has the only condition $d \geq 1$ but is too weak for our purpose. The Deligne bound needs some more intricate technically conditions which are not satisfied in our situation, see Section 6. Our main tool is a generalization of Deligne’s Theorem for projective surfaces [6], the Hooley-Katz Theorem [10], see Lemma 1 in Section 3 below. The crucial steps in the proof are:

1. Identify $R(f(X))$ with a multivariate polynomial of the form

   $$
   Q(Y_0, \ldots, Y_{r-1}) = \sum_{j,k=0}^{r-1} a_{j,k} f_j(Y_j) f_k(Y_k)
   $$

   in Section 4. Note that this polynomial has coefficients in $\mathbb{F}_q$.

2. Estimate the dimensions of the singular loci, defined in Section 3 below, of $Q - c$ and its homogeneous part of largest degree, see Lemma 2 below.

3. We complete the proof in Section 4. After some linear variable substitution, $Q$ is transformed to a polynomial $F$ of the same degree as $Q$ but with coefficients in $\mathbb{F}_p$. In particular, the dimensions of the singular loci are invariant under this linear transformation. Then we apply the Hooley-Katz Theorem to $F - c$. 

3
2 The case of permutation polynomials

For a permutation polynomial $f(X)$ of $\mathbb{F}_q$, $|R(c,f)|$ is the number $N_r(c)$ of solutions $(x_1, \ldots, x_r) \in \mathbb{F}_p^r$ of the equation

$$x_1x_2 + \ldots + x_{r-1}x_r = c.$$

We have

$$N_r(c) = \begin{cases} p^{r-1} - p^{(r-1)/2}, & c \neq 0, \\ p^{r-1} + p^{(r+1)/2} - p^{(r-1)/2}, & c = 0, \end{cases} \quad r \geq 2,$$

which can be easily verified using the recursion

$$N_r(c) = pN_{r-2}(c) + (p-1)p^{r-2}, \quad r \geq 4.$$

This recursion is obtained by distinguishing the cases $x_{r-1} = 0$ and $x_{r-1} \neq 0$.

3 The Hooley-Katz Theorem

We denote by $\overline{\mathbb{F}_p}$ the algebraic closure of $\mathbb{F}_p$.

The (affine) singular locus $\mathcal{L}(F)$ of a polynomial $F$ over $\mathbb{F}_p$ in $r$ variables is the set of common zeros in $\overline{\mathbb{F}_p}$ of the polynomials

$$F, \frac{\partial F}{\partial X_1}, \ldots, \frac{\partial F}{\partial X_r}.$$

Our main tool is the following result, see [16, Theorem 7.1.14], which is the affine version of the Hooley-Katz Theorem [10].

Lemma 1 (Hooley-Katz). Let $Q$ be a polynomial over $\mathbb{F}_p$ in $r$ variables of degree $D \geq 1$ such that the dimensions of the singular loci of $Q$ and its homogeneous part $Q_D$ of degree $D$ satisfy

$$\max \{ \dim(\mathcal{L}(Q)), \dim(\mathcal{L}(Q_D)) - 1 \} \leq s.$$

Then the number $N$ of zeros of $Q$ in $\mathbb{F}_p^r$ satisfies

$$|N - p^{r-1}| \leq C_{D,r} p^{(r+s)/2},$$

where $C_{D,r}$ is a constant depending only on $D$ and $r$.

4 Proof of Theorem [1]

First, we express the Rudin-Shapiro function $R(\xi)$ of $\mathbb{F}_q$ in terms of trace and dual basis.

Let $\varphi$ be the Frobenius automorphism defined by

$$\varphi(\xi) = \xi^p \quad \text{for} \ \xi \in \mathbb{F}_q.$$
We extend \( \varphi \) to the polynomial ring \( \mathbb{F}_q[X_1, \ldots, X_r] \) by
\[
\varphi(X_i) = X_i, \quad i = 1, \ldots, r.
\]

Let
\[
\text{Tr}(\xi) = \xi + \varphi(\xi) + \cdots + \varphi^{r-1}(\xi) \in \mathbb{F}_p
\]
denote the (absolute) trace of \( \xi \in \mathbb{F}_q \). Let \( (\delta_1, \ldots, \delta_r) \) denote the (existent and unique) dual basis of the basis \( B = (\beta_1, \ldots, \beta_r) \) of \( \mathbb{F}_q \), see for example [11], that is,
\[
\text{Tr}(\delta_i \beta_j) = \begin{cases} 
1 & \text{if } i = j, \\
0 & \text{if } i \neq j,
\end{cases} \quad 1 \leq i, j \leq r. \tag{1}
\]

Then we have
\[
\text{Tr}(\delta_i \xi) = x_i \quad \text{for any} \quad \xi = \sum_{j=1}^r x_j \beta_j \in \mathbb{F}_q \quad \text{with } x_j \in \mathbb{F}_p.
\]

For \( f(X) \in \mathbb{F}_q[X] \) we obtain that
\[
R(f(\xi)) = \sum_{i=1}^{r-1} \text{Tr}(\delta_i f(\xi)) \text{Tr}(\delta_{i+1} f(\xi))
= \sum_{i=1}^{r-1} \sum_{j,k=0}^{r-1} \delta_i^{p^j} \delta_{i+1}^{p^k} \varphi^j(f(\xi)) \varphi^k(f(\xi)).
\]

Write
\[
F(X_1, \ldots, X_r)
= \sum_{j,k=0}^{r-1} a_{j,k} f_j(\beta_1^{p^j} X_1 + \cdots + \beta_r^{p^j} X_r) f_k(\beta_1^{p^k} X_1 + \cdots + \beta_r^{p^k} X_r), \tag{2}
\]
where
\[
a_{j,k} = \sum_{i=1}^{r-1} \delta_i^{p^j} \delta_{i+1}^{p^k}, \quad j, k = 0, \ldots, r - 1, \tag{3}
\]
and \( f_j = \varphi^j(f) \in \mathbb{F}_q[X] \). Then \( F \in \mathbb{F}_p[X_1, \ldots, X_r] \) and
\[
R(f(\xi)) = F(x_1, \ldots, x_r) \quad \text{for} \quad \xi = \sum_{i=1}^r x_i \beta_i, \quad x_i \in \mathbb{F}_p.
\]

Theorem [1] follows from Lemma [1] and the following lemma which we prove in the next section.
Lemma 2. Let \( f(X) \in \mathbb{F}_q[X] \) be of degree \( d \) with \( 2 \leq d < p \) and \( F \in \mathbb{F}_p[X_1, \ldots, X_r] \) be defined by (2). Then \( F \) has degree \( 2d \). Moreover, for any \( c \in \mathbb{F}_p \) we have

\[
\dim(\mathcal{L}(F - c)) \leq \begin{cases} 
\frac{r}{2} - 1, & r \text{ even and } c \neq 0, \\
\frac{(r - 1)/2}{2}, & r \text{ odd and } c \neq 0, \\
\frac{r}{2}, & r \text{ even and } c = 0, \\
\frac{(r + 1)/2}{2}, & r \text{ odd and } c = 0.
\end{cases}
\]

Furthermore, if \( F_{2d} \in \mathbb{F}_p[X_1, \ldots, X_r] \) is the homogeneous part of \( F \) of degree \( 2d \), then

\[
\dim(\mathcal{L}(F_{2d})) \leq \begin{cases} 
\frac{r}{2}, & r \text{ even,} \\
\frac{(r + 1)/2}{2}, & r \text{ odd.}
\end{cases}
\]

5 Proof of Lemma 2

Consider the linear transformation on \( \mathbb{F}_p^r \)

\[
y_i = \sum_{j=1}^{r} \beta_j^p x_j, \quad i = 0, \ldots, r - 1. \tag{4}
\]

It is regular with inverse

\[
x_k = \sum_{i=0}^{r-1} \delta_k^p y_i, \quad k = 1, \ldots, r, \tag{5}
\]

by (1).

Then we denote by \( Q \) the polynomial obtained from \( F \), defined by (2), with the corresponding variable transformation,

\[
F(X_1, \ldots, X_r) = \sum_{j,k=0}^{r-1} a_{j,k}f_j(Y_j)f_k(Y_k) = Q(Y_0, \ldots, Y_{r-1}), \tag{6}
\]

where

\[
Y_i = \sum_{j=1}^{r} \beta_j^p X_j, \quad i = 0, \ldots, r - 1. \tag{7}
\]

As the degree and the dimension of singular loci are invariant under the regular transformation (7), it is enough to show the results for the polynomial \( Q \).

We may assume that \( f(X) \) is monic since otherwise we multiply the basis \( \mathcal{B} \) element-wise with the leading coefficient of \( f(X) \). The degree \( 2d \) homogeneous part of \( Q \) is

\[
Q_{2d}(Y_0, \ldots, Y_{r-1}) = \sum_{j,k=0}^{r-1} a_{j,k}Y_j^d Y_k^d.
\]
By the definition (3) of $a_{j,k}$ we have
\[ \sum_{j=0}^{r-1} a_{j,0} \beta_i^{p_j} = \sum_{i=1}^{r-1} \delta_{i+1} \text{Tr}(\beta_1 \delta_i) = \delta_2 \neq 0. \]

Hence, $a_{j,0} \neq 0$ for some $j$. Since $Y_j^d Y_k^d$, $0 \leq j, k < r$, are linearly independent over $\mathbb{F}_q$, we get that $Q_{2d}$ is not the zero polynomial. In particular we have
\[ \deg(F) = \deg(Q) = \deg(Q_{2d}) = 2d. \]

We estimate the dimension of the singular locus $L(Q - c)$. The bound for the dimension of $L(Q_{2d})$ corresponds to the special case $f(X) = X^d$ and $c = 0$.

To estimate $\dim(L(Q - c))$, consider the partial derivatives
\[ \frac{\partial(Q - c)}{\partial Y_\ell}(Y_0, \ldots, Y_{r-1}) = f'_\ell(Y_\ell) \sum_{k=0}^{r-1} (a_{k,\ell} + a_{\ell,k}) f_k(Y_k), \quad \ell = 0, \ldots, r - 1. \]

The condition $2 \leq d < p$ implies that $f'(X) = f'_0(X)$ is not constant and so $f'_0(X)$ is not constant for $\ell = 0, \ldots, r - 1$.

Note that
\[ L(Q - c) = \bigcup_{L \subseteq \{0, \ldots, r - 1\}} (V_L \cap C_L), \]

where $V_L$ is the (affine) variety in $\mathbb{F}_p^r$ of solutions of the equation system
\[ Q(Y_0, \ldots, Y_{r-1}) = c, \]
\[ \sum_{k=0}^{r-1} (a_{k,\ell} + a_{\ell,k}) f_k(Y_k) = 0, \quad \ell \in L, \]

and $C_L$ the variety of solutions of
\[ Q(Y_0, \ldots, Y_{r-1}) = c, \]
\[ f'_\ell(Y_\ell) = 0, \quad \ell \in \{0, 1, \ldots, r - 1\} \setminus L. \]

Hence,
\[ \dim(L(Q - c)) \leq \max\{\min\{\dim(V_L), \dim(C_L)\} : L \subseteq \{0, \ldots, r - 1\}\}, \quad (8) \]

since
\[ \dim(U \cup V) = \max\{\dim(U), \dim(V)\} \]

and
\[ \dim(U \cap V) \leq \min\{\dim(U), \dim(V)\}, \]

see for example [3, Chapter 9].

It remains to estimate the dimensions of $V_L$ and $C_L$. 

7
Lemma 3. For \( L \subseteq \{0, 1, \ldots, r - 1\} \) the (affine) variety \( V_L \) is of dimension at most
\[
\begin{cases}
  r - |L| - 1, & c \neq 0, \\
  r - |L|, & r \text{ even and } c = 0, \\
  r - |L| + 1, & r \text{ odd and } c = 0.
\end{cases}
\]

Proof. First we prove the result for \( L = \{0, 1, \ldots, r - 1\} \). In this case for any \((\eta_0, \ldots, \eta_{r - 1}) \in V_L\) we have
\[
\sum_{k=0}^{r-1} (a_{\ell, k} + a_{k, \ell}) f_k(\eta_k) = 0, \quad \ell = 0, \ldots, r - 1,
\]
and thus
\[
\sum_{k=0}^{r-1} \sum_{\ell=0}^{r-1} \beta_m^p (a_{\ell, k} + a_{k, \ell}) f_k(\eta_k) = 0, \quad m = 1, \ldots, r.
\]
Since
\[
\sum_{\ell=0}^{r-1} \beta_m^p (a_{\ell, k} + a_{k, \ell}) = \sum_{i=1}^{r-1} \left( \delta_i^p \text{Tr}(\beta_m \delta_i) + \delta_i^p \text{Tr}(\beta_m \delta_{i+1}) \right) = \begin{cases} 
\delta_m^p, & m = 1, \\
\delta_m^p \delta_{m+1}, & m = 2, \ldots, r - 1, \\
\delta_{r-1}^p, & m = r,
\end{cases}
\]
for \( k = 0, \ldots, r - 1 \), we get
\[
\sum_{k=0}^{r-1} \delta_2^p f_k(\eta_k) = 0,
\]
\[
\sum_{k=0}^{r-1} \left( \delta_{m-1}^p + \delta_{m+1}^p \right) f_k(\eta_k) = 0, \quad m = 2, \ldots, r - 1, \quad (9)
\]
\[
\sum_{k=0}^{r-1} \delta_{r-1}^p f_k(\eta_k) = 0.
\]
For even \( r \) this implies
\[
\sum_{k=0}^{r-1} \delta_m^p f_k(\eta_k) = 0, \quad m = 1, \ldots, r,
\]
and since the transformation \([5]\) is regular we get \( f_k(\eta_k) = 0 \) for all \( k \). Note that \( Q(\eta_0, \ldots, \eta_{r-1}) = 0 \) and thus \( V_L \) is empty, that is of dimension \(-1\), for \( c \neq 0 \). Since a polynomial of degree \( d \) has exactly \( d \) zeros in \( \mathbb{F}_p \) (counted with multiplicities), for \( c = 0 \) we have
\[
1 \leq |V_{\{0, \ldots, r-1\}}| \leq d^r,
\]
8
that is, the dimension of $V_{\{0,\ldots,r-1\}}$ is 0, see again [3, Chapter 9].

For odd $r$, (9) implies

$$
\sum_{k=0}^{r-1} \delta_m^k f_k(\eta_k) = \begin{cases} 
0, & m \text{ even, } m = 1, \ldots, r, \\
(-1)^{(m-1)/2} \lambda, & m \text{ odd, }
\end{cases}
$$

(10)

for any $\lambda \in \mathbb{F}_p$. We get by the definition (6) of $Q$ and its coefficients (3) that

$$
Q(\eta_0, \ldots, \eta_{r-1}) = \sum_{i=1}^{r-1} \sum_{j=0}^{r-1} \delta_i^j f_j(\eta_j) \sum_{k=0}^{r-1} \delta_{i+1}^k f_k(\eta_k) = 0
$$

since by (10) the sum over $j$ vanishes for even $i$ and the sum over $k$ for odd $i$.

Hence, for $c \neq 0$, $V_{\{0,\ldots,r-1\}}$ is empty and of dimension $-1$.

We continue with the case $c = 0$. Applying (4) and (5) to (10) we get

$$
f_k(\eta_k) = \lambda \sum_{m=1}^{r} \beta_m^k (-1)^{(m-1)/2}, \quad k = 0, \ldots, r-1.
$$

(11)

Note that

$$
\sum_{m=1}^{r} \beta_m^k (-1)^{(m-1)/2} \neq 0
$$

for some $m$ by (11) and (12). For fixed

$$
\lambda = \lambda_0
$$

(12)

the system (11) and (12) has at most $d'$ solutions $(\eta_0, \ldots, \eta_{r-1})$, that is, the solution set is finite and of dimension 0. Deleting the condition (12) the dimension increases by 1.

Now for any proper subset $L$ of $\{0, \ldots, r-1\}$ the variety is defined by deleting $j = r - |L|$ equations from the definition of $V_{\{0,\ldots,r-1\}}$. That is, its dimension is increased by at most $r - |L|$, see e.g. [3], and the result follows.

It is easy to see that

$$
\dim(C_L) \leq |L|.
$$

(13)

Combining (8), Lemma 3 and (13) we get

$$
\dim(L(Q - c)) \leq \begin{cases} 
\frac{r}{2} - 1, & r \text{ even and } c \neq 0, \\
\frac{r-1}{2}, & r \text{ odd and } c \neq 0, \\
\frac{r}{2}, & r \text{ even and } c = 0, \\
\frac{r+1}{2}, & r \text{ odd and } c = 0.
\end{cases}
$$
6 Final remarks

Some cases with singular locus $\mathcal{L}(Q_{2d})$ of positive dimension

It is clear from the proof of Lemma \ref{lemma} that for odd $r$ the singular locus of $Q_{2d}$ is of dimension at least 1. For some special choices of the dual basis, $\mathcal{L}(Q_{2d})$ has also positive dimension for any $r \geq 4$.

Namely, if

$$\sum_{i=1}^{r-1} \delta_i \delta_{i+1} = 0,$$

the coefficients $a_{j,j}, j = 0, \ldots, r-1$, defined by \ref{definition} vanish. Then each $(\eta_0, \ldots, \eta_{r-1}) \in \mathbb{F}_p$ with only one non-zero coordinate is a singular point of $Q_{2d}$.

Now we construct such a dual basis. Let $\alpha$ be a defining element of $\mathbb{F}_q$ over $\mathbb{F}_p$, that is, $\mathbb{F}_q = \mathbb{F}_p(\alpha)$. Then, for sufficiently large $p$ with respect to $r$, $(\delta_1, \ldots, \delta_r)$ defined by

$$\begin{align*}
\delta_{2i+1} &= \alpha^{r-1-i}, & i &= 0, 1, \ldots, \lfloor r/2 \rfloor - 1, \\
\delta_{2i+2} &= \alpha^i, & i &= 0, 1, \ldots, \lfloor (r-1)/2 \rfloor - 1, \\
\delta_r &= - \begin{cases} (r/2 - 1)(\alpha^{r/2-1} + \alpha^{r/2-2}), & r \text{ even}, \\
& r \geq 4,
\end{cases}
\end{align*}$$

is a basis of $\mathbb{F}_q$ over $\mathbb{F}_p$, since $\alpha^{\lfloor (r-1)/2 \rfloor}$ appears only in $\delta_r$, satisfying \ref{equation}.

The Thue-Morse function of $\mathbb{F}_q$ for monomials

The Thue-Morse function $T$ for $\mathbb{F}_q$ with respect to the basis $B$ is

$$T(\xi) = \sum_{i=1}^{r} x_i, \quad \xi = x_1 \beta_1 + \ldots + x_r \beta_r \in \mathbb{F}_q,$$

where $x_1, \ldots, x_r \in \mathbb{F}_p$. For $f(X) \in \mathbb{F}_q[X]$ of degree $d \geq 1$ and $c \in \mathbb{F}_p$ we put

$$\mathcal{T}(c, f) = \{ \xi \in \mathbb{F}_q : T(f(\xi)) = c \}.$$

The first author and Sárközy \cite{Sarkozy} Theorem 1.2 proved

$$||\mathcal{T}(c, f)| - p^{r-1}| \leq (d-1)p^{r/2}, \quad \gcd(d, p) = 1.$$

For monomials $f(X) = X^d$, $c \neq 0$, fixed $d \geq 2$ and fixed $r$, the Hooley-Katz Theorem provides the improvement

$$||\mathcal{T}(c, X^d)| - p^{r-1}| \leq C_{d,r}p^{(r-1)/2}, \quad c \neq 0.$$

In particular, we get

$$\lim_{p \to \infty} \frac{|\mathcal{T}(c, X^d)|}{p^{r-1}} = 1, \quad c \neq 0,$$
also for $r = 2$.

The crucial step is to show that the singular locus of

$$Q(Y_0, \ldots, Y_{r-1}) = \sum_{\ell=0}^{r-1} \delta^p \varphi^d Y_{\ell}^d - c$$

is of dimension $-1$, where we used the same notation as before and

$$\delta = \sum_{i=1}^{r} \delta_i \neq 0,$$

since $\delta_1, \ldots, \delta_r$ are linearly independent. Now the partial derivatives are

$$\frac{\partial Q}{\partial Y_{\ell}} = \delta^p d Y_{\ell}^{d-1}, \quad \ell = 0, \ldots, r - 1.$$

We may assume $d < p$. Then the only common zero of all partial derivatives is $(0, \ldots, 0)$. However, $(0, \ldots, 0)$ is not a zero of $Q$ for $c \neq 0$.

The Hooley-Katz Theorem can also be applied for general $f(X) \in \mathbb{F}_q[X]$ of degree $d \geq 2$ but would give an improvement of \cite{3} Theorem 1.2 only for $c \in \mathbb{F}_p \setminus \mathcal{C}$ where $\mathcal{C}$ is a subset of $\mathbb{F}_p$ with at most $(d - 1)^r$ elements, where $d$ and $r$ are fixed and $p$ is sufficiently large. The polynomial $Q$ for a general $f$ becomes

$$Q(Y_0, \ldots, Y_{r-1}) = \sum_{\ell=0}^{r-1} \delta^p f_{\ell}(Y_{\ell}) - c,$$

with $f_{\ell} = \varphi^d(f)$ as in Section 4.

A singular point $(\eta_0, \ldots, \eta_{r-1}) \in \overline{\mathbb{F}_p}$ satisfies

$$f'_{\ell}(\eta_{\ell}) = 0 \quad \text{for } \ell = 0, \ldots, r - 1. \quad (15)$$

This singular point has to be also a zero of $Q$, that is,

$$c = \sum_{i=0}^{r-1} \delta^p f_{i}(\eta_{i}).$$

For all other $c \in \mathbb{F}_p$ there are no singular points. Since (15) has at most $(d - 1)^r$ solutions in $\overline{\mathbb{F}_p}$ we have $|\mathcal{C}| \leq (d - 1)^r$.

Acknowledgments

The second and third author are partially supported by the Austrian Science Fund FWF, Projects P 31762 and P 30405, respectively.

The authors wish to thank Igor Shparlinski for pointing to Deligne’s bound for projective surfaces and related theorems.
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