REFINEMENTS OF EXACT COVERING SYSTEMS

OFIR SCHNABEL

Abstract. An exact covering system $A$ which admits a maximal modulus of the form $p_1^{k_1}p_2^{k_2}$ primely refines an exact covering system $B$.

1. Introduction

An exact covering system (ECS) is a partition of $Z$ into finitely many arithmetic progressions

\[ A = \{a_s(n_s)\}_{s=1}^k, \]

where $a(t)$ is the arithmetic progression $a + Zt$. $t$ is called the modulus of the arithmetic progression $a(t)$. An ECS \([1]\) admits multiplicity if there exist $1 \leq i < j \leq k$ such that $n_i = n_j$. The ECS $A = \{0(1)\}$ is called the trivial ECS.

The concept of ECS was firstly introduced by P. Erdős in the early 1930’s. A main concern in the research of ECS is restrains on the number of times a modulus occurs in an ECS. Erdős conjectured the following: Every non-trivial ECS admits multiplicity. Erdős’ conjecture was proved in the beginning of the 1950’s independently by H. Davenport, L. Mirsky, D. Newman and R. Rado (see \([2]\)). In fact, the proof shows that such multiplicity occurs in the greatest modulus. This result was generalized by Š. Znám \([3]\) and later by Y.G. Chen and Š. Porubský \([1]\). All proofs use generating functions of an ECS and a deep relation between the number of times the greatest difference $m$ occurs in an ECS and minimal vanishing sums of $m$-th roots of unity (see \([5]\)). For a more comprehensive study of ECS the reader is referred to a monograph by Š. Porubský \([4]\) and to a review by Š. Porubský and J. Schönhheim \([8]\).

Our main concern in this note is refinements of ECS. Notice that for any natural number $n$, there is a natural ECS

\[ \{i(n)\}_{i=0}^{n-1}. \]

This is a refinement of the trivial ECS. In a similar way we can refines any ECS by splitting an arithmetic progression $a(t)$ into $n$ arithmetic progressions

\[ \{a + it(n)\}_{i=0}^{n-1}. \]

Definition 1.1. An ECS $A$ primely refines an ECS $B$, (denote $A \models B$), if there exists a prime number $p$ such that

\[ B = \{a_i(n_i)\}_{i=1}^k, \quad A = \{a_i(n_i)\}_{i=1}^{k-1} \cup \{a_k + jn_k(pn_k)\}_{j=0}^{p-1}. \]

In other words: $A$ is obtained from $B$ by splitting one of the arithmetic progressions into $p$ arithmetic progressions.
Throughout this note, maximality will be with respect to the division partial order. In particular, when given an ECS, a modulus is maximal if it is not dividing any other modulus is this ECS. Our main theorem is the following

**Theorem A.** Suppose that an ECS \( A = \{a_s(n_s)\}_{s=1}^k \) has a maximal modulus of the form \( p_1^{k_1} p_2^{k_2} \), where \( p_1, p_2 \) are primes and \( k_1, k_2 \geq 0 \). Then \( A \) primely refines an ECS \( B \).

In a sense, Theorem A discuss reducibility of ECS. Denote the least common multiple of \( B = \{n_1, n_2, \ldots, n_k\} \subseteq \mathbb{N} \) by \( N(B) \). For an ECS \( \{A_k\} \), denote the least common multiple of all the moduli by \( N(A) \). Throughout this not \( p_1 \) stands for prime number. In two papers \([3,4]\) I. Korec investigate ECS \( \{A_k\} \) with \( N(A) = p_1^{k_1} p_2^{k_2} \) and \( N(A) = p_1^{k_1} p_2^{k_2} p_3^{k_3} \). In particular, he discus the reducibility (see \( [3] \)) of such ECS. In \([3]\) I. Polách generalize some of Korec’s results. The approach adopted by Korec an Polách is essentially different than the classical approach of generating functions. Theorem A is in the same spirit as Korec and Polách results. However, our methods are similar to the classical methods and relay heavily on the above mentioned relation between ECS and vanishing sums roots of unity. As a corollary of Theorem A we get classification of all the ECS \( \{A_k\} \) with \( N(A) = p_1^{k_1} p_2^{k_2} \).

**Corollary 1.2. (Korec, see \([3]\))** Let \( A = \{a_s(n_s)\}_{s=1}^k \) be an ECS with \( N(A) = p_1^{k_1} p_2^{k_2} \) where \( k_1, k_2 \geq 0 \). Then there exists a sequence of ECS

\[
A = A_1 \models A_2 \models \ldots \models A_{n-1} \models \{0(1)\}.
\]

Another corollary of Theorem A is a restraint on ECS \( A \) with \( N(A) = p_1^{k_1} p_2^{k_2} p_3^{k_3} \).

**Corollary 1.3.** Let \( A = \{a_s(n_s)\}_{s=1}^k \) be an ECS such that \( N(A) = p_1^{k_1} p_2^{k_2} p_3^{k_3} \). Assume also there exist modulus \( n_1, n_2, n_3 \) such that

\[
p_1 \nmid n_1 n_2, \quad p_2 \nmid n_1 n_3, \quad p_3 \nmid n_2 n_3,
\]

\[
p_1 \nmid n_3, \quad p_2 \nmid n_2, \quad p_3 \nmid n_1
\]

Then \( p_1 p_2 p_3 \) divides some modulus \( n_j \).

2. Preliminaries

Given an ECS \( \{A_k\} \) we may always assume that

\[
0 \leq a_s < n_s \quad \text{for all} \quad 1 \leq s \leq k.
\]

The classical approach for investigating multiplicity is to consider the following generating function. For \( |z| < 1 \) we have:

\[
\sum_{s=1}^k \frac{z^{a_s}}{1 - z^{n_s}} = \sum_{s=1}^k \sum_{q=0}^\infty z^{a_s + q n_s} = \sum_{n=0}^\infty z^n = \frac{1}{1 - z}.
\]

Let \( B = \{n_1, n_2, \ldots, n_k\} \) be a set of natural numbers. Assume that \( n_r \) is a maximal element in \( B \). Denote the least common multiple of \( n_1, n_2, \ldots, n_s \) by \( N \). Let \( C_{n_r} \) be the cyclic group of order \( n_r \) generated by \( \tau \) and let \( \zeta \) be a primitive \( n_r \)-th root of unity. Consider the following ring homomorphisms:

\[
\gamma : \mathbb{Z}[z] \to \mathbb{Z}C_{n_r}
\]

\[
z \mapsto \tau
\]

\[
\varphi : \mathbb{Z}C_{n_r} \to \mathbb{C}
\]

\[
\tau \mapsto \zeta.
\]
Here, $\mathbb{Z}[z]$ is the ring of polynomials with integral coefficients in the variable $z$ and $\mathbb{Z}C_{n_r}$ is the integral group algebra. The following two lemmas will be used in the proof of Theorem A.

**Lemma 2.1.** With the above notation, let $t$ be a divisor of $N(B)$. Then $\varphi \circ \gamma \left( \frac{1 - z^{N(B)}}{1 - z} \right) \neq 0$ if and only if $n_r$ divides $t$. In particular,

1. \[\varphi \circ \gamma \left( \frac{1 - z^{N(B)}}{1 - z} \right) = 0.\]
2. By the maximality of $n_r$, for $1 \leq i \leq k$
   \[\varphi \circ \gamma \left( \frac{1 - z^{N(B)}}{1 - z^{n_i}} \right) = 0\]
   if and only if $n_i \neq n_r$.

**Proof.** First, since $n_r$ divides $N(B)$,

\[\varphi \circ \gamma (1 - z^{N(B)}) = 1 - \zeta^{N(B)} = 0.\]

Now, $t$ admits the following decomposition $t = qn_r + r$ such that $q, r$ are natural numbers and $0 \leq r < n_r$. Then, $\varphi \circ \gamma (1 - z^t) = 1 - \zeta^t = 1 - \zeta^r$. Hence, if $n_r$ is not a divisor of $t$ we get that

\[\varphi \circ \gamma \left( \frac{1 - z^{N(B)}}{1 - z^t} \right) \neq 0.\]

Assume now that $n_r$ is a divisor of $t$ and recall that if $r_1 | r_2$ then

\[\frac{1 - z^{r_2}}{1 - z^{r_1}} = \sum_{i=0}^{r_2 - 1} z^{i \cdot r_1}.\]

Then, if we denote $N(B) = cn_r$ and $t = qn_r$ we get that

\[\frac{1 - z^{N(B)}}{1 - z^t} = \sum_{i=0}^{c-1} z^{i \cdot n_r} \sum_{i=0}^{q-1} z^{i \cdot n_r}.\]

Hence,

\[\varphi \circ \gamma \left( \frac{1 - z^{N(B)}}{1 - z^t} \right) = \underbrace{1 + 1 + \ldots + 1}_{e \text{ times}} \neq 0.\]

Let $P_i = \{ z^{\frac{i \cdot n_r}{p_i}} \}_{j=0}^{p_i - 1}$ be the unique subgroup of order $p_i$ in $C_{n_r}$. Denote

\[\sigma_{n_r}(P_i) := \sum_{g \in P_i} q \in \mathbb{Z}C_{n_r} = \sum_{j=0}^{p_i-1} z^{\frac{j \cdot n_r}{p_i}}.\]
Lemma 2.2. (see [5, Theorem 3.3]) Let \( n_r = p_1^{k_1} p_2^{k_2} \) and let \( 0 \neq x \in NC_n \cap \ker(\varphi) \). Then \( x \) admits one of the following decompositions

\[
x = z^d \cdot \sigma_{n_r}(P_1) + \sum_{i=0}^{n_r-1} b_i z^i, \quad b_i \geq 0, \quad 0 \leq d < \frac{n_r}{p_1}.
\]

Or

\[
x = z^d \cdot \sigma_{n_r}(P_2) + \sum_{i=0}^{n_r-1} b_i z^i, \quad b_i \geq 0, \quad 0 \leq d < \frac{n_r}{p_2}.
\]

3. Main part

Proof of Theorem A.

Let \( n_r = p_1^{t_1} p_2^{t_2} \) be a maximal modulus in \( A \). Equation (6) can be written in the following way:

\[
\sum_{s=1}^{k} \frac{z^{a_s}}{1 - z^{n_s}} = \sum_{\{s: n_s = n_r\}} \frac{z^{a_s}}{1 - z^{n_r}} + \sum_{\{s: n_s \neq n_r\}} \frac{z^{a_s}}{1 - z^{n_s}} = \sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}.
\]

Both sides of (10) are elements in \( \mathbb{Q}(z) \), the field of rational functions with rational coefficients in the variable \( z \). As before, denote the least common multiple of \( n_1, n_2, \ldots, n_k \) by \( N \). By multiplying both sides of (10) by \( 1 - z^N \) we get

\[
\sum_{s=1}^{k} \frac{z^{a_s} \cdot (1 - z^N)}{1 - z^{n_s}} = \frac{1 - z^N}{1 - z}.
\]

Hence, both sides of (11) are in \( \mathbb{Z}[z] \).

Consequently, by Lemma 2.1 the right hand side of (11) is in the kernel of \( \varphi \circ \gamma \). Therefore,

\[
\sum_{\{s: n_s = n_r\}} \frac{z^{a_s} \cdot (1 - z^N)}{1 - z^{n_s}} = \frac{1 - z^N}{1 - z} \sum_{\{s: n_s = n_r\}} z^{a_s} \in \ker(\varphi \circ \gamma).
\]

Thus, by Lemma 2.1 we get:

\[
\sum_{\{s: n_s = n_r\}} z^{a_s} \in \ker(\varphi \circ \gamma).
\]

Hence,

\[
\gamma \left( \sum_{\{s: n_s = n_r\}} z^{a_s} \right) = \sum_{\{s: n_s = n_r\}} z^{a_s} \in \ker(\varphi).
\]

Now, by Lemma 2.2 we may assume without loss of generality that

\[
\gamma \left( \sum_{\{s: n_s = n_r\}} z^{a_s} \right) = \sum_1 + \sum_2,
\]

where

\[
\sum_1 = \sigma^d \cdot \sigma_{n_r}(P_1) = z^d \cdot \sum_{j=0}^{p_1-1} z^j \cdot \sigma_{n_r}(P_2),
\]

where \( d < \frac{n_r}{p_1} \). And

\[
\sum_2 = \sum_{i=0}^{n_r-1} b_i z^i, \quad b_i \geq 0.
\]
Notice that \( \ker \gamma = ( z^{n_r} - 1) \mathbb{Z}[z] \). Consequently, the restriction of \( \gamma \) to polynomials in \( \mathbb{Z}[z] \) with degree smaller than \( n_r \) is 1-1. Let

\[
g(z) = z^d \cdot \sum_{j=0}^{p_1 - 1} z^{\frac{j n_r}{p_1}}.
\]

Then \( \gamma(g(z)) = \sum_1 \). Hence

\[
\gamma \left( \sum_{\{s: n_s = n_r\}} z^{a_s} - g(z) \right) = \gamma \left( \sum_{i=0}^{n_r-1} b_i z^i \right) = \sum_2.
\]

By (18), the degree of

\[
\sum_{\{s: n_s = n_r\}} z^{a_s} - g(z)
\]

is smaller than \( n_r \). Therefore by the 1-1 property on such polynomials,

\[
\sum_{\{s: n_s = n_r\}} z^{a_s} - g(z) = \sum_{i=0}^{n_r-1} b_i z^i.
\]

Consequently,

\[
\sum_{\{s: n_s = n_r\}} z^{a_s} = g(z) + \sum_{i=0}^{n_r-1} b_i z^i = z^d \cdot \sum_{j=0}^{p_1 - 1} z^{\frac{j n_r}{p_1}} + \sum_{i=0}^{n_r-1} b_i z^i.
\]

By (10) and (22),

\[
\sum_{\{s: n_s = n_r\}} z^{a_s} = g(z) + \sum_{i=0}^{n_r-1} b_i z^i = \frac{z^d \cdot \sum_{j=0}^{p_1 - 1} z^{\frac{j n_r}{p_1}}}{1 - z^{n_r}} + \sum_{\{s: n_s \neq n_r\}} \frac{z^{a_s}}{1 - z^{n_s}} = \frac{1}{1 - z}.
\]

Notice that

\[
\frac{z^d \cdot \sum_{j=0}^{p_1 - 1} z^{\frac{j n_r}{p_1}}}{1 - z^{n_r}} = \frac{z^d}{1 - z^{\frac{n_r}{p_1}}}.
\]

So, by (23) and by (24)

\[
\sum_{\{s: n_s \neq n_r\}} \frac{z^{a_s}}{1 - z^{n_s}} + \frac{z^d}{1 - z^{\frac{n_r}{p_1}}} + \sum_{i=0}^{n_r-1} b_i z^i = \frac{1}{1 - z}.
\]

Since \( b_i \geq 0 \), every summand \( z^{a_s} \) of the left hand side of equation (22) is either a summand of \( g(z) \) if \( a_s \equiv d(\text{mod} \ \frac{n_r}{p_1}) \) or a summand in \( \sum_{i=0}^{n_r-1} b_i z^i \). So, (25) is a generating function of a new ECS \( B \), where \( B \) is obtained by consolidation of the \( p_1 \) arithmetic progressions:

\[
\left\{ d(n_r), d + \frac{n_r}{p_1}(n_r), \ldots, d + \frac{(p_1 - 1)n_r}{p_1}(n_r) \right\} \subset A,
\]

into one arithmetic progression \( d(\frac{n_r}{p_1}) \) in \( B \). Hence \( A \) is primely finer than the ECS \( B \), and this complete the proof. \( \square \)

It is important to notice that ECS which are not prime refinement of any ECS exist. By theorem A the following example which is a particular case of [5, Example 2.5] is minimal.
Example 3.1. The following ECS is not a prime refinement of any ECS.

\[ A = \{2(6), 4(6), 1(10), 3(10), 7(10), 9(10), 0(15), 5(30), 6(30), 12(30), 18(30), 24(30), 25(30)\} \]  

Notice that the maximal modulus is 30. The reason that \( A \) is not a primely refinement of any ECS follows from the fact that there is no way to split the following vanishing sum

\[ \xi^5 + \xi^6 + \xi^{12} + \xi^{18} + \xi^{24} + \xi^{25} = 0, \]

where \( \xi \) is a 30-th primitive root of unity, into two vanishing sums. See [5].

Proof of Corollary 1.2

By applying Theorem A to each maximal modulus we proceed by induction noticing that all the maximal moduli are of the form \( p_1^{l_1}p_2^{l_2} \) until we get the trivial ECS. □

Remark 3.2. Note that Corollary 1.2 gives a way to construct all ECS with \( N = p_1^{s_1}p_2^{s_2} \) for two given primes \( p_1, p_2 \).

For the next proof note that by the Chinese remainder theorem, an ECS cannot contain coprime moduli.

Proof of Corollary 1.3

Assume that there is no modulus \( n_j = p_1^{d_1}p_2^{d_2}p_3^{d_3}, d_1, d_2, d_3 > 0 \).

Let \( A = A_1 \). By the hypothesis of the corollary and by the above assumption there is a maximal modulus \( p_1^{l_1}p_2^{l_2}(l_1, l_2 \geq 1) \). Hence by Theorem A there is an ECS \( A_1 \models A_2 \). We proceed by induction. As long as there is a modulus \( p_1^{l_1}p_2^{l_2}(l_1, l_2 \geq 1) \) there is a maximal modulus of the same form. The sequence \( A_1 \models A_2 \ldots \models A_l \), must terminate. The terminal ECS \( A_l \) has no modulus of the form \( p_1^{l_1}p_2^{l_2}(l_1, l_2 \geq 1) \). Hence there is either a modulus \( p_1^{l_1}(l_1 > 0) \) or a modulus \( p_2^{l_2}(l_2 > 0) \). Since we assumed that \( A \) contains moduli \( p_1^{m_1}p_3^{m_3} \) and \( p_2^{m_2}p_3^{m_3} \), then in both cases \( A_l \) contain coprime moduli. This cannot happen by the Chinese remainder theorem. □

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Department of Mathematics, University of Haifa, Haifa 31905, Israel

E-mail address: os2519@yahoo.com