Comment on “$w$ and $w'$ of scalar field models of dark energy”

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We comment on the calculation mistake in the paper “$w$ and $w'$ of scalar field models of dark energy” by Takeshi Chiba, where $w$ is the dark energy equation of state and $w'$ is the time derivative of $w$ in units of the Hubble time. The author made a mistake while rewriting the phantom equation of motion, which led to an incorrect generic bound for the phantom model and an incorrect bound for the tracker phantom model on the $w$–$w'$ plane.

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In the paper “$w$ and $w'$ of scalar field models of dark energy” [1], Takeshi Chiba derived bounds on $w'$ as a function of $w$ for scalar field models of dark energy including quintessence, phantom and k-essence. This provides very useful means to discriminate between dark energy models and to confront dark energy models with the observational results (see [2] for example). In [1], we find that there is a mistake in calculation for the phantom case, which leads to the incorrect bounds on $w'$. Following the framework in [1], we derive the corrected bounds for the phantom model. In addition, we also correct an argument leading to the bound for the tracker quintessence.

I. LIMITS OF PHANTOM

A. Generic bound

The phantom model is scalar field dark energy having negative kinetic term and its equation of state $w < -1$. The energy density and the pressure are given by $\rho = -\dot{\phi}^2/2 + V$ and $p = -\dot{\phi}^2/2 - V$, respectively. The equation of motion is given by

$$\ddot{\phi} + 3H\dot{\phi} - V_{,\phi} = 0 .$$

(1)

Therefore, the phantom field tends to roll up the potential $V$. The phantom equation of motion can be rewritten as [3]

$$\pm \frac{V_{,\phi}}{V} = \sqrt{-\frac{3\kappa^2(1 + w)}{\Omega_{\phi}} \left(1 + \frac{x'}{6}\right)} ,$$

(2)

where the plus sign corresponds to $\dot{\phi} > 0$ and the minus sign to the opposite, $\kappa^2 = 8\pi G$ and $\Omega_{\phi}$ is the fractional energy density of the phantom field. The variable $x$ is defined as

$$x = \ln \left(\frac{1 + w}{1 - w}\right) ,$$

(3)

and $x'$ is the derivative of $x$ with respect to $\ln a$, which is related with $w'$ as

$$x' = \frac{2w'}{(1 - w)(1 + w)} .$$

(4)

Since the left-hand side of Eq. (2) is positive for the uprolling phantom field, we have $1 + x'/6 > 0$. Therefore, using Eq. (4), the upper bound on $w'$ is obtained as

$$w' < -3(1 + w).$$

(5)

Note that the upper bound on $w'$ for phantom is smoothly connected to the lower bound on $w'$ for quintessence, the latter of which is shown as Eq. (2.6) in [1].

In [1], there is a calculation mistake that the rewritten equation of motion is obtained as

$$\pm \frac{V_{,\phi}}{V} = \sqrt{-\frac{3\kappa^2(1 + w)}{\Omega_{\phi}} \left(-1 + \frac{x'}{6}\right)} .$$

(6)

As a consequence, an incorrect bound on $w'$ is obtained as

$$w' > 3(1 - w)(1 + w).$$

(7)

B. Tracker phantom

Following the approach in [1], the bound can be tightened for the tracker field. Taking the derivative of Eq. (2) with respect to $\phi$, we obtain the tracker equation for the phantom field

$$\Gamma - 1 = \frac{3(w_B - w)(1 - \Omega_{\phi})}{(1 + w)(6 + x')} - \frac{(1 - w)x'}{2(1 + w)(6 + x')} - \frac{2x' - 2}{(1 + w)(6 + x')^2} ,$$

(8)
The minimum value of $w$ is nearly constant and $\Omega_\phi$ is initially negligible, $w$ is given by

$$w = \frac{w_B - 2(\Gamma - 1)}{2(\Gamma - 1) + 1}. \quad (9)$$

Thus $\Gamma < 1/2$ is required for tracker phantom, which has $w < -1$.

Following [1], we consider a solution in which initially $w$ follows the tracker solution in Eq. (9) and then evolves toward $-1$. Therefore, the tracker $w$ in Eq. (9) is a lower bound of $w$. In such solution, $x'$ eventually stops decreasing and then increases back to a value near zero. The minimum value of $x'$, $x'_m$, gives an upper bound on $w'$ via Eq. (4). To find $x'_m$, we put $x'' = 0$ and $w_B = 0$ in Eq. (5) and find that

$$x'_m = -6\frac{w(1 - \Omega_\phi) + 2(1 + w)(\Gamma - 1)}{(1 - w) + 2(1 + w)(\Gamma - 1)}. \quad (10)$$

Since $x'_m$ is an increasing function of $w$, a lower bound on $x'_m$ is given by that of $w$, for which we take the tracker $w$ in Eq. (9) and obtain

$$x'_m > \frac{6w\Omega_\phi}{1 - 2w} > \frac{6w}{1 - 2w}. \quad (11)$$

With Eq. (11), we then obtain the upper bound on $w'$

$$w' < \frac{3w(1 - w)(1 + w)}{1 - 2w}. \quad (12)$$

In [1], there is a mistake that the tracker equation for the phantom field is given as

$$\Gamma - 1 = \frac{3(w_B - w)(1 - \Omega_\phi)}{(1 + w)(6 - x')} - \frac{(1 - w)x'}{2(1 + w)(6 - x')} + \frac{2x''}{(1 + w)(6 - x')^2}. \quad (13)$$

As a consequence, the resulting $x'_m$ is

$$x'_m = -6\frac{w(1 - \Omega_\phi) + 2(1 + w)(\Gamma - 1)}{(1 - w) - 2(1 + w)(\Gamma - 1)} \quad (14)$$

$$> \frac{6w\Omega_\phi}{1 - 2w}. \quad (15)$$

Therefore, an incorrect bound on $w'$ is given as

$$w' < 3w(1 - w)(1 + w). \quad (16)$$

The bounds for the phantom model from [1] are shown in Fig. 1 and the corrected ones we obtain are shown in Fig. 2.
II. LIMITS OF TRACKER QUINTESSENCE

For tracker quintessence the Eqs. (8) – (11) remain valid. However, we have \( w > -1 \) and \( \Gamma > 1 \) in this case. In addition, instead of an upper bound on \( w' \), the minimum value of \( x', x'_m \), gives a lower bound on \( w' \) via Eq. (11). Since \( x'_m \) is a decreasing function of \( w \), a lower bound on \( x'_m \) is given by an upper bound of \( w \), for which we take the tracker \( w \) in Eq. (9) and obtain Eq. (11). As a result, the lower bound on \( w' \) is given as

\[
  w' > \frac{3w(1+w)(1+w)}{1-2w}.
\]  

(17)

Note that our argument which leads to the lower bound on \( x'_m \) is different from that in [1], between Eq. (2.9) and Eq. (2.10): “since \( x_m \) is an increasing function of \( w(-1) \), a lower bound is given by \( w \) of the tracker solution Eq. (2.8)”. However, the resulting bound is the same.

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