On the boundedness of the type of an almost Gorenstein monomial curve in $\mathbb{A}^5$

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ABSTRACT

We prove that the Cohen-Macaulay type of an almost Gorenstein monomial curve $C \subseteq \mathbb{A}^5$ is bounded.

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1. Introduction

Almost Gorenstein rings are a class of Cohen-Macaulay rings, introduced by Barucci and Fröberg in 1997 (cf. [3]), which are close to Gorenstein rings. This class of rings was originally defined in the context of analytically unramified rings of dimension one, however this definition has been later generalized in the context of one-dimensional rings (cf. [8]), and then to the higher dimension case (cf. [11]).

It is well-known that a local ring is Gorenstein if and only if it is Cohen-Macaulay and its Cohen-Macaulay type is equal to one; in a sense, almost Gorenstein rings are meant to extend Gorenstein rings to arbitrary type. The properties of this class of rings have been studied by several authors in the last two decades (cf. [2, 5, 6, 10, 12, 15, 19] for example); in particular, curious patterns started to appear in works dealing with the Cohen-Macaulay type of almost Gorenstein rings. In the original context of one-dimensional analytically unramified rings, including local rings associated to monomial curves, it is well-known that if the embedding dimension is at most 3 then the Cohen-Macaulay type is either 1 (and thus the ring is Gorenstein) or 2 (cf. [7]), while if the embedding dimension is at least 4, there is no upper bound for the Cohen-Macaulay type (cf. [7, Example p. 75]). On the other hand, if we restrict ourselves to almost Gorenstein rings of embedding dimension 4, Numata (cf. [17]) asked if the Cohen-Macaulay type is at most 3, which turned out to be true (cf. [13]). Families of almost Gorenstein rings with arbitrarily large Cohen-Macaulay type $t$ are present in the literature (cf. [9, 10]); however, for these families of rings, the embedding dimension $e$ is at least $\frac{t}{2}$ (actually, in this context there is no example in the literature of an almost Gorenstein ring satisfying $t > 2e$). Therefore, contrary to the general context, for almost Gorenstein rings and curves the following question (cf. [13, 17]) naturally arises.

Question 1. Let $C$ be an almost Gorenstein monomial curve. Is the Cohen-Macaulay type $t(C)$ bounded by a function of the embedding dimension $e$?
Since the cases $e \leq 4$ have already been solved (cf. [7, 13]), we investigate the case $e = 5$. Interestingly, we obtain the following result.

**Theorem 2.** The Cohen-Macaulay type of an almost Gorenstein monomial curve $C \subseteq \mathbb{A}^5$ is bounded.

We provide an explicit bound, that is, $t(C) \leq 473$. However, this bound does not seem optimal; computational evidence, included in the last part of this work, suggests that the bound can be largely improved. Finally, we give an example of almost Gorenstein ring with embedding dimension 6 and type 14, satisfying $t > 2e$; this suggests that if $e = 6$ it might be harder to bound the Cohen-Macaulay type $t$.

Our approach relies on the correspondence between numerical semigroups and monomial curves ([1]); we study the class of numerical semigroups associated to almost Gorenstein monomial curves, which are called almost symmetric.

2. **Numerical semigroups**

A numerical semigroup is a submonoid $S$ of $(\mathbb{N}, +)$ such that the set $\mathbb{N} \setminus S$ is finite. Every numerical semigroup is finitely generated and admits a unique minimal system of generators $\{g_1, \ldots, g_r\}$, such that $\gcd(g_1, \ldots, g_r) = 1$. The integer $e = e(S)$ is called the embedding dimension of $S$. The largest natural number not belonging to $S$ is called the Frobenius number of $S$, denoted by $F(S) = \max \mathbb{N} \setminus S$. Elements of $S$ which are smaller than the Frobenius number are called small elements, and the set of small elements of a numerical semigroup is denoted by $N(S)$, while its cardinality is usually denoted by $n(S)$. On the other hand, natural numbers not belonging to $S$ are called gaps of the numerical semigroup $S$, while the set of gaps is denoted by $G(S) = \mathbb{N} \setminus S$.

We say that an integer $f$ is a pseudo-Frobenius number of a numerical semigroup $S = \langle g_1, \ldots, g_r \rangle$ if $f \not\in S$ and $f + g_i \in S$ for every $i = 1, \ldots, e$. We denote the set of pseudo-Frobenius numbers by $PF(S)$, and define the type of $S$ as the cardinality of $PF(S)$, denoted by $t(S)$. Clearly $F(S) = \max PF(S)$. Given a numerical semigroup $S$, we can define a partial order relation $\leq_S$ on $\mathbb{Z}$ in this way: $x \leq_S y$ if $y - x \in S$. By definition, $PF(S)$ is the set of maximal elements of $\mathbb{Z} \setminus S$ with respect to $\leq_S$.

**Remark 3.**

1. In [7] it is proved that the type of a numerical semigroup with embedding dimension 3 is at most 2.
2. In [7, Example p. 75] it is shown that there exist families of numerical semigroups with embedding dimension 4 and arbitrarily large type. Namely, if $n \geq 2$ and $r \geq 3n + 2$ are given, $s = r(3n + 2) + 3$ and $S = \langle s, s + 3, s + 3n + 1, s + 3n + 2 \rangle$, then $t(S) = 3n + 2$.

A numerical semigroup $S$ is said to be symmetric if for every $x \not\in S$, $F(S) - x \in S$; it is well-known that symmetric numerical semigroups are exactly those with type equal to 1. Further, we say that $S$ is almost symmetric if for every $x \not\in S$ we have either $F(S) - x \in S$ or $\{x, F(S) - x\} \subseteq PF(S)$. Almost symmetric numerical semigroups can have arbitrarily large type (cf. [9]); however, contrary to the general case (see Remark 3), it is known that almost symmetric numerical semigroups with embedding dimension 4 have type at most 3 (cf. [13]). It is not clear if, once the embedding dimension $e$ of an almost symmetric numerical semigroup is fixed, there is some sort of restriction which bounds $t$ in function of $e$ (see Question 1).

The monograph [18] is a good reference on numerical semigroups. As one might guess from the terminology, there is a strict relation between numerical semigroups and commutative rings. In fact, the value semigroups associated to analytically unramified one dimensional local domains are actually numerical semigroups, and there is a correspondence between the invariants of these mathematical objects (cf. [1]). For our purposes, it is worth remembering that the Cohen-Macaulay type $t(C)$ of an almost Gorenstein monomial curve coincides with the type $t(S)$ of the associated numerical semigroup, which is almost symmetric.
3. Type of almost symmetric numerical semigroups

Let \( S = \langle g_1, \ldots, g_e \rangle \) be a numerical semigroup.

Let \( f \in \mathbb{Z} \). We say that a square matrix \( A = (a_{ij}) \) of order \( e \) is a RF-matrix (short for row-factorization matrix) for \( f \) if, for every \( i = 1, \ldots, e, f - a_{ii}g_i \in S, f - (a_{ii} + 1)g_i \not\in S \), \( a_{ij} \geq 0 \) for every \( j \neq i \) and \( f = \sum_{j=1}^{e} a_{ij}g_j \).

**Remark 4.** It is straightforward to check that \( f \in S \) if and only if there is a RF-matrix for \( f \) such that \( a_{ii} \geq 0 \) for every \( i = 1, \ldots, e \), while \( f \in PF(S) \) if and only if there is a RF-matrix for \( f \) such that \( a_{ii} = -1 \) for every \( i = 1, \ldots, e \).

**Example 5.** Consider the numerical semigroup \( S = \langle 5, 12, 13 \rangle \), and let \( f = 19 \). Since we can write \( 19 = -5 + 2 \cdot 12 = 5 - 12 + 2 \cdot 13 = 4 \cdot 5 + 12 - 13 \), we have that

\[
A = \begin{pmatrix}
-1 & 2 & 0 \\
1 & -1 & 2 \\
4 & 1 & -1
\end{pmatrix}
\]

is a RF-matrix for \( f \), and furthermore we can see that \( f \in PF(S) \).

**Proposition 6.** [13, Proposition 4] Let \( f, F(S) - f \in PF(S) \), \( A = (a_{ii}) \) be a RF-matrix for \( f \) and \( B = (b_{ij}) \) a RF-matrix for \( F(S) - f \). Then for every \( i \neq j \) we have \( a_{ij}b_{ji} = 0 \). Therefore, there are at least \( e(e-1) \) entries of \( A \cup B \) equal to zero.

For every \( i, j \in \{1, \ldots, e\}, i \neq j \), define \( \lambda_{ij} = \max\{k \in \mathbb{N} \mid kg_i - gi \not\in S\} \geq 2 \) and \( \Lambda_{ij} = \lambda_{ij}g_i - g_i \), and consider the multiset \( \Lambda := \{\Lambda_{ij} \mid i \neq j\} \). Notice that \( |\Lambda| = e(e-1) \).

**Remark 7.** For every \( i, j \in \{1, \ldots, e\}, i \neq j \), if \( kg_j - g_i \) is contained in \( PF(S) \), then \( k \) must be \( \lambda_{ij} \), which implies that such an element is unique and belongs to \( \Lambda \).

**Remark 8.** As a direct consequence of Proposition 6 we can see that, if \( S \) is an almost-symmetric numerical semigroup with embedding dimension 4 and \( A, B \) are RF-matrices for two pseudo-Frobenius numbers \( f, F(S) - f \), there are at least \( 4 \cdot 3 = 12 \) entries of \( A \cup B \) equal to zero, therefore there is at least one row of \( A \) or \( B \) with exactly one positive entry. This implies that either \( f \) or \( F(S) - f \) (or both) has to be equal to an element of \( \Lambda \), meaning that there are a bounded number of couples \( \{f, F(S) - f\} \) of elements of \( PF(S) \). This simple argument shows that the type of an almost symmetric numerical semigroup with embedding dimension 4 is bounded (cf. [13]).

Let \( S = \langle g_1, \ldots, g_5 \rangle \) be an almost-symmetric numerical semigroup with embedding dimension five.

Since \( S \) is almost symmetric, all its pseudo-Frobenius numbers \( f \) besides \( F(S) \) are such that \( F(S) - f \in PF(S) \). We say that a pseudo-Frobenius number \( f \in PF(S) \setminus \{F(S)\} \) is good if either \( f \) or \( F(S) - f \) is of the form \( kg_i - g_i \) for some \( i \neq j \) and some positive integer \( k \), otherwise we say that \( f \) is bad. We denote by \( PF_g(S) \) the set of good pseudo-Frobenius numbers, and by \( PF_b(S) \) the set of bad pseudo-Frobenius numbers.

**Remark 9.** By definition \( f \in PF_g(S) \) if and only if \( F(S) - f \in PF_g(S) \), and \( t(S) = |PF_g(S)| + |PF_b(S)| + 1 \).

**Proposition 10.** Let \( S \) be an almost-symmetric numerical semigroup with embedding dimension five. Then \( |PF_g(S)| \leq 40 \).
Proof. In light of Remark 9 we can partition the set $PF_{g}(S)$ in couples $\{f, F(S) − f\}$ (eventually pairing the element $F(S)\over 2$ with itself). On the other hand, by Remark 7 every couple $\{f, F(S) − f\}$ is associated to (at least) one element of $\Lambda$, thus $|PF_{g}(S)| \leq 2|\Lambda| = 40$. □

Let $f \in PF_{b}(S)$, $A = (a_{ij})$ be a RF-matrix for $f$, and $B = (b_{ij})$ a RF-matrix for $F(S) − f$. Notice that if at least one row of either $A$ or $B$ has at least 3 entries equal to zero, then by definition either $f$ or $F(S) − f$ would be of the form $Kg_{j} − g_{i}$ for some $i, j$, contradicting $f \in PF_{b}(S)$. But since by Proposition 6 there are at least 20 zeroes among the entries of $A$ and $B$, we deduce that $A$ and $B$ must satisfy the following properties:

1. There are exactly 20 entries of $A \cup B$ equal to zero, and for every $i \neq j$ exactly one of $a_{ij}$ and $b_{ij}$ is zero.
2. Each row and column of $A$ and $B$ has exactly two positive entries and exactly two zeroes.

Example 11. Let $S = \{64, 67, 91, 138, 150\}$, $PF(S) = \{209, 327, 445, 654\}$. It is simple to check that 209 is a good pseudo-Frobenius number, since $209 = 3 \cdot 91 - 64$. On the other hand, $f = 327$ is a bad pseudo-Frobenius number, and its RF-matrix is

$$A = \begin{pmatrix}
-1 & 0 & 1 & 0 & 2 \\
4 & -1 & 0 & 1 & 0 \\
0 & 4 & -1 & 0 & 2 \\
3 & 0 & 3 & -1 & 0 \\
0 & 3 & 0 & 2 & -1
\end{pmatrix}.$$  

Since $F(S) = 654$, we have $f = F(S) − f$ and $B = A$ in the previous argument; it is easy to see that the couple of RF-matrices $A$ and $B$ is not satisfied by the properties (a) and (b).

Definition 12. Let $M = (m_{ij}), M' = (m'_{ij})$ be two square matrices of order $n$. We say that $M$ and $M'$ have the same $0$-configuration if $m_{ij} = 0$ if and only if $m'_{ij} = 0$.

In our context, if $A = (a_{ij})$ and $A' = (a'_{ij})$ are two RF-matrices for the same bad pseudo-Frobenius number $f \in PF_{b}(S)$ and $B = (b_{ij})$ is a RF-matrix for $F(S) − f$, then by (a) in both couples $(a_{ij}, b_{ij})$ and $(a'_{ij}, b_{ij})$ there is exactly one zero: therefore we have that $a_{ij} = 0$ if and only if $a'_{ij} = 0$. Then, all RF-matrices of a bad pseudo-Frobenius $f$ have the same 0-configuration.

Lemma 13. Let $f \in PF_{b}(S)$ and $A = (a_{ij})$ be a RF-matrix for $f$. Then there are two row vectors $R_{1}$ and $R_{2}$ of $A$ such that, for exactly one index $j$, the $j$-th entries of $R_{1}$ and $R_{2}$ are both positive.

Proof. The RF-matrix $A$ satisfies the two properties (a) and (b), hence each column of $A$ contains exactly two positive entries and two zeroes. Then, for every $j = 1, \ldots, 5$, there exists a couple of rows $\{R_{1j}, R_{2j}\}$ which $j$-th entries are both positive. However, if the other positive entry of $R_{1j}, R_{2j}$ corresponds to the same index $k$, then the two indices $j$ and $k$ would be associated to the same couple $\{R_{1j}, R_{2j}\}$, and they are the only two indices associated to these two rows (each row has exactly two positive entries). However, since the number of indices is odd, this cannot happen for all values; therefore there must be one index $j$ such that the two rows $R_{1j}, R_{2j}$ have exactly one positive common component, proving our thesis. □

Lemma 14. Let $f, f' \in PF_{b}(S)$ be such that

$$f = a_{ij}g_{j} + a_{ik}g_{k} - g_{i} = a_{pj}g_{j} + a_{pq}g_{q} - g_{p} \quad (1)$$

$$f' = b_{ij}g_{j} + b_{ik}g_{k} - g_{i} = b_{pj}g_{j} + b_{pq}g_{q} - g_{p} \quad (2)$$
where the indices \( i, j, k, p, q \in \{1, \ldots, 5\} \) are such that \( i \neq j, i \neq k, j \neq k, p \neq j, p \neq q, j \neq q \) (i.e. all indices on the same side of an equation are distinct), \( q \neq k \), and all coefficients \( a_{mn} \) and \( b_{mn} \) are positive integers, with \( a_{ij} \geq a_{pj}, b_{ij} \geq b_{pj} \).

Then \( f = f' \).

**Proof.** Assume without loss of generality that \( a_{pq} \geq b_{pq} \). From equation (2) we obtain

\[
b_{pq}g_q = (b_{ij} - b_{pj})g_j + b_{ik}g_k + g_p - g_i.
\]

By substitution in \( f \) we get

\[
f = a_{pj}g_j + a_{pq}g_q - g_p = a_{pj}g_j + (a_{pq} - b_{pq})g_q - g_p + (b_{ij} - b_{pj})g_j + b_{ik}g_k + g_p - g_i = (a_{pj} + b_{ij} - b_{pj})g_j + b_{ik}g_k + (a_{pq} - b_{pq})g_q - g_i. \tag{3}
\]

Since \( j \neq k, j \neq q \) and \( q \neq k \), and \( a_{pj} + b_{ij} - b_{pj} \geq a_{pq} > 0, b_{ik} > 0 \) and \( a_{pq} - b_{pq} \geq 0 \), equation (3) is associated to a row of a RF-matrix for \( f \). Since \( f \in PF_b(S) \), by property (b) this row must contain exactly 2 positive entries and two zeroes, thus we must necessarily have \( a_{pq} = b_{pq} \) (notice that if \( q = i \) we still have \( a_{pq} = b_{pq} \) because \( f \notin S \)).

Therefore \( f - f' = (a_{pj} - b_{pj})g_j \), and thus we have either \( f \preceq_S f' \) or \( f' \preceq_S f \). In both cases, since \( f, f' \in PF(S) \) are maximals, we conclude that \( f = f' \). \(\square\)

**Proposition 15.** Let \( S \) be an almost-symmetric numerical semigroup with embedding dimension five, and let \( N \) be the number of possible 0-configurations satisfying the two properties (a) and (b). Then \( |PF_b(S)| \leq 2N \).

**Proof.** We argue by contradiction: assume that \( |PF_b(S)| \geq 2N + 1 \). By definition of \( N \), there must be three distinct pseudo-Frobenius numbers \( f_1, f_2, f_3 \in PF_b(S) \) which RF-matrices have the same 0-configuration. Let \( A \) be a RF-matrix for \( f_1 \); by Lemma 13 there exist two rows \( R_1, R_2 \) of \( A \) such that for exactly one index \( j \) the \( j \)-th entries of \( R_1 \) and \( R_2 \) are both positive. Since the RF-matrices of \( f_2 \) and \( f_3 \) have the same 0-configuration, we obtain

\[
f_1 = a_{ij}g_j + a_{ik}g_k - g_i = a_{pj}g_j + a_{pq}g_q - g_p
\]

\[
f_2 = b_{ij}g_j + b_{ik}g_k - g_i = b_{pj}g_j + b_{pq}g_q - g_p
\]

\[
f_3 = c_{ij}g_j + c_{ik}g_k - g_i = c_{pj}g_j + c_{pq}g_q - g_p
\]

and since the expressions correspond to positive entries of \( R_1 \) and \( R_2 \) we deduce that all indices appearing on the same side of these equations are distinct, while the hypothesis that \( R_1 \) and \( R_2 \) have exactly one positive entry in the same index \( j \) means that \( q \neq k \).

Consider now the three couples of elements \( \{a_{ij}, a_{pj}\}, \{b_{ij}, b_{pj}\}, \{c_{ij}, c_{pj}\} \). Obviously, at least two of these couples must be ordered in the same way, therefore (up to a change of indices) we obtain that \( a_{ij} \geq a_{pj} \) and \( b_{ij} \geq b_{pj} \). Hence \( f_1, f_2 \) fall under the hypotheses of Lemma 14, implying \( f_1 = f_2 \), yielding a contradiction. \(\square\)

Combining Propositions 10 and 15 we obtain \( t(S) \leq 2N + 41 \), and since \( N \) is bounded (for instance it is easy to check that \( N < 6^5 \)), we prove the following Theorem, which is equivalent to Theorem 2.

**Theorem 16.** The type of an almost symmetric numerical semigroup \( S \) with embedding dimension five is bounded.

**Remark 17.** We computed with GAP all 0-configurations of square matrices \( A = (a_{ij}) \) of order 5 satisfying property (b)—i.e. each row and column of \( A \) contains exactly two zeroes and two positive entries, and \( a_{ii} = -1 \). We obtained that there are exactly 216 such configurations, thus \( N \leq 216 \). In particular, this yields \( t(S) \leq 2 \cdot 216 + 41 = 473 \).
The next remark suggests that the bound obtained in Remark 17 can be largely improved, though it would require new techniques.

**Remark 18.** An extensive computation performed by P.A. García-Sánchez, using GAP and the package Numericalsgps (see [4]), found that, if $S = \langle g_1, \ldots, g_5 \rangle$ is an almost symmetric numerical semigroup and $g_1 < \cdots < g_5 \leq 200$ (this family consists of more than $7 \cdot 10^6$ examples!), then $t(S) \leq 5$. Moreover, there is always at most one bad pseudo-Frobenius number, which is exactly $\frac{F(S)}{2}$. If this finding turns out to be true for all almost symmetric numerical semigroups $S$ with embedding dimension 5, the bound could be vastly improved.

The first key point of our argument is that RF-matrices associated to bad pseudo-Frobenius numbers must have some very special properties. In the higher dimension case, these matrices are hard to control. To see this, let $S$ be an almost symmetric numerical semigroup with embedding dimension 6, $f, F(S) - f$ be bad pseudo-Frobenius numbers and $A, B$ be RF-matrices for $f, F(S) - f$ respectively. Then Proposition 6 infers that there are at least 30 entries equal to zero on $A \cup B$; however, contrary to the embedding dimension 5 case, it could be possible to have more than 30 without having too many zeroes on each row (a pseudo-Frobenius number is good if at least one row contains $e - 2$ zeroes). Moreover, the key idea behind Proposition 15 is that, under our hypotheses, there cannot be more than two bad pseudo-Frobenius numbers sharing the same 0-configuration. This might not be true in the higher dimension case.

**Example 19.** Consider the numerical semigroup $S = \langle 455, 497, 574, 589, 631, 708 \rangle$. We have

$$PF(S) = \{3079, 3289, 3521, 3657, 3789, 3923, 4057, 4172, 4191, 4325, 4557, 4767, 7846\},$$

thus $S$ has embedding dimension $e = 6$ and type $t(S) = 14 > 2e$. Further, $PF(S)$ contains the arithmetic progression $\{3521, 3657, 3789, 3923, 4057, 4191, 4325\}$ of ratio 134. Notice that, if $A_f$ is a RF-matrix associated to $f \in PF(S)$, we have

$$A_{3521} = \begin{pmatrix}
-1 & 8 & 0 & 0 & 0 & 0 \\
0 & -1 & 7 & 0 & 0 & 0 \\
9 & 0 & -1 & 0 & 0 & 0 \\
0 & 7 & 0 & -1 & 1 & 0 \\
0 & 0 & 6 & 0 & -1 & 1 \\
8 & 0 & 0 & 1 & 0 & -1
\end{pmatrix},$$

$$A_{3521+\lambda} = \begin{pmatrix}
-1 & 8 - \lambda & 0 & 0 & \lambda & 0 \\
0 & -1 & 7 - \lambda & 0 & 0 & \lambda \\
9 - \lambda & 0 & -1 & \lambda & 0 & 0 \\
0 & 7 - \lambda & 0 & -1 & \lambda + 1 & 0 \\
0 & 0 & 6 - \lambda & 0 & -1 & \lambda + 1 \\
8 - \lambda & 0 & 0 & \lambda + 1 & 0 & -1
\end{pmatrix}$$

for $\lambda = 0, 1, \ldots, 6$.

The 5 pseudo-Frobenius numbers $\{3521, 3657, 3789, 3923, 4057, 4191\}$ (obtained for $\lambda = 1, \ldots, 5$) all have RF-matrices sharing the same 0-configuration.

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