On the Boundedness of a Generalized Fractional Integral on Generalized Morrey Spaces

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Abstract

In this paper we extend Nakai’s result on the boundedness of a generalized fractional integral operator from a generalized Morrey space to another generalized Morrey or Campanato space.

keywords: Generalized fractional integrals, generalized Morrey spaces, generalized Campanato spaces

1. Introduction and Main results

For a given function $\rho : (0, \infty) \longrightarrow (0, \infty)$, let $T_\rho$ be the generalized fractional integral operator, given by

$$T_\rho f(x) = \int_{\mathbb{R}^n} \frac{f(y)\rho(|x-y|)}{|x-y|^n} dy,$$

and put

$$\tilde{T}_\rho f(x) = \int_{\mathbb{R}^n} f(y) \left( \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|y|)(1 - \chi_{B_0}(y))}{|y|^n} \right) dy,$$

where $\chi_{B_0}(y)$ is the characteristic function of the unit ball $B_0$.
the modified version of $T_\rho$, where $B_0$ is the unit ball about the origin, and $\chi_{B_0}$ is the characteristic function of $B_0$.

In [4], Nakai proved the boundedness of the operators $\tilde{T}_\rho$ and $T_\rho$ from a generalized Morrey space $M_{1,\phi}$ to another generalized Morrey space $M_{1,\psi}$ or generalized Campanato space $L_{1,\psi}$. More precisely, he proved that

$$\|T_\rho f\|_{M_{1,\psi}} \leq C \|f\|_{M_{1,\phi}} \quad \text{and} \quad \|\tilde{T}_\rho f\|_{L_{1,\psi}} \leq C \|f\|_{M_{1,\phi}},$$

where $C > 0$, with some appropriate conditions on $\rho, \phi$ and $\psi$. Using the techniques developed by Kurata et al. [1], we investigate the boundedness of these operators from generalized Morrey spaces $M_{p,\phi}$ to generalized Morrey spaces $M_{p,\psi}$ or generalized Campanato spaces $L_{p,\psi}$ for $1 < p < \infty$.

The generalized Morrey and Campanato spaces are defined as follows. For a given function $\phi: (0, \infty) \to (0, \infty)$, and $1 < p < \infty$, let

$$\|f\|_{M_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left( \frac{1}{|B|} \int_B |f(y)|^p dy \right)^{\frac{1}{p}},$$

and

$$\|f\|_{L_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left( \frac{1}{|B|} \int_B |f(y) - f_B|^p dy \right)^{\frac{1}{p}},$$

where the supremum is taken over all open balls $B = B(a, r)$ in $\mathbb{R}^n$, $|B|$ is the Lebesgue measure of $B$ in $\mathbb{R}^n$, $f_B = \frac{1}{|B|} \int_B f(y) dy$. We define the Morrey space $M_{p,\phi}$ by

$$M_{p,\phi} = \{ f \in L_{p,loc}^p(\mathbb{R}^n) : \|f\|_{M_{p,\phi}} < \infty \},$$

and the Campanato space $L_{p,\phi}$ by

$$L_{p,\phi} = \{ f \in L_{p,loc}^p(\mathbb{R}^n) : \|f\|_{L_{p,\phi}} < \infty \}.$$

Our results are the following:

**Theorem 1.1** If $\rho, \phi, \psi: (0, \infty) \to (0, \infty)$ satisfying the conditions below:

\[
\begin{align*}
\frac{1}{2} \leq \frac{t}{r} \leq 2 & \Rightarrow \frac{1}{A_1} \leq \frac{\phi(t)}{\phi(r)} \leq A_1, \quad \text{and} \quad \frac{1}{A_2} \leq \frac{\rho(t)}{\rho(r)} \leq A_2, \\
\int_0^1 \rho(t) \frac{dt}{t} < \infty, \text{and for all } r > 0, \text{we have} \int_r^\infty \frac{\phi(t)}{t} dt \leq A_3 \phi(r)^p, \\
\phi(r) \int_0^r \frac{\rho(t)}{t} dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t^2} dt & \leq A_4 \psi(r), \text{for all } r > 0,
\end{align*}
\]

where $A_i > 0$ are independent of $t, r > 0$, then for each $1 < p < \infty$ there exists $C_p > 0$ such that

$$\|T_\rho f\|_{M_{p,\phi}} \leq C_p \|f\|_{M_{p,\phi}}.$$
Theorem 1.2 If \( \rho, \phi, \psi : (0, \infty) \rightarrow (0, \infty) \) satisfying the conditions below

\[
\frac{1}{2} \leq \frac{t}{r} \leq 2 \Rightarrow \frac{1}{A_1} \leq \frac{\phi(t)}{\phi(r)} \leq A_1, \quad \text{and} \quad \frac{1}{A_2} \leq \frac{\rho(t)}{\rho(r)} \leq A_2 \quad (4)
\]

\[
\int_0^1 \frac{\rho(t)}{t} dt < \infty, \text{and for all} r > 0, \text{we have} \int_r^\infty \frac{\phi(t)^p}{t} dt \leq A_3 \phi(r)^p, \quad (5)
\]

\[
\frac{\rho(r)}{r^n} - \frac{\rho(t)}{t^n} | \leq A_4 |r - t| \frac{\rho(r)}{r^{n+1}}, \quad \text{for} \quad \frac{1}{2} \leq \frac{t}{r} \leq 2,
\]

\[
\phi(r) \int_0^r \frac{\rho(t)}{t} dt + r \int_r^\infty \frac{\rho(t) \phi(t)}{t^2} dt \leq A_5 \psi(r), \text{for all} r > 0, \quad (7)
\]

where \( A_i > 0 \) are independent of \( t, r > 0 \), then for each \( 1 < p < \infty \) there exists \( C_p > 0 \) such that

\[
\| \mathcal{T}_p f \|_{\mathcal{L}_{p,0}} \leq C_p \| f \|_{\mathcal{M}_{p,0}}.
\]

2. Proof of the Theorems

To prove the theorems, we shall use the following result of Nakai [2] (in a slightly modified version) about the boundedness of the standard maximal function \( Mf \) on a generalized Morrey space \( \mathcal{M}_{p,\phi} \). The standard maximal function \( Mf \) is defined by

\[
Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy, \quad x \in \mathbb{R}^n,
\]

where the supremum is taken over all open balls \( B \) containing \( x \).

Theorem 2.1 (Nakai). If \( \phi : (0, \infty) \rightarrow (0, \infty) \) satisfying the conditions below :

(a). \( \frac{1}{2} \leq \frac{t}{r} \leq 2 \Rightarrow \frac{1}{A_1} \leq \frac{\phi(t)}{\phi(r)} \leq A_1, \)

(b). \( \int_r^\infty \frac{\phi(t)^p}{t} dt \leq A_2 \phi(r)^p, \text{for all} r > 0, \)

where \( A_i > 0 \) are independent of \( t, r > 0 \), then for each \( 1 < p < \infty \) there exists \( C_p > 0 \) such that

\[
\| Mf \|_{\mathcal{M}_{p,\phi}} \leq C_p \| f \|_{\mathcal{M}_{p,\phi}}.
\]

From now on, \( C \) and \( C_p \) will denote positive constants, which may vary from line to line. In general, these constants depend on \( n \).
Proof of Theorem 1.1
For $x \in \mathbb{R}^n$, and $r > 0$, write

$$T_{\rho}f(x) = \int_{|x-y|<r} \frac{f(y)\rho(|x-y|)}{|x-y|^n} dy + \int_{|x-y|\geq r} \frac{f(y)\rho(|x-y|)}{|x-y|^n} dy = I_1(x) + I_2(x).$$

Note that, for $t \in [2^k r, 2^k+1 r]$, there exist constants $C_i > 0$ such that

$$\rho(2^k r) \leq C_1 \int_{2^k r}^{2^{k+1} r} \frac{\rho(t)}{t} dt$$

and

$$\rho(2^k r) \phi(2^k r) \leq C_2 \int_{2^k r}^{2^{k+1} r} \frac{\rho(t)\phi(t)}{t} dt.$$

So, we have

$$|I_1(x)| \leq \int_{|x-y|<r} \frac{|f(y)|\rho(|x-y|)}{|x-y|^n} dy$$

$$\leq \sum_{k=-\infty}^{-1} \int_{2^k r \leq |x-y|<2^{k+1} r} \frac{|f(y)|\rho(|x-y|)}{|x-y|^n} dy$$

$$\leq C \sum_{k=-\infty}^{-1} \frac{\rho(2^k r)}{(2^k r)^n} \int_{|x-y|<2^{k+1} r} |f(y)| dy$$

$$\leq C \sum_{k=-\infty}^{-1} \rho(2^k r) M f(x)$$

$$\leq C M f(x) \sum_{k=-\infty}^{-1} \int_{2^k r}^{2^{k+1} r} \frac{\rho(t)}{t} dt$$

$$\leq C M f(x) \int_0^r \frac{\rho(t)}{t} dt$$

$$\leq C \frac{\psi(r)}{\phi(r)} M f(x).$$
Meanwhile,

\[
|I_2(x)| \leq \int_{|x-y| \geq r} \frac{|f(y)| \rho(|x-y|)}{|x-y|^n} \, dy
\]

\[
\leq \sum_{k=0}^{\infty} \int_{2^kr \leq |x-y| < 2^{k+1}r} \frac{|f(y)| \rho(|x-y|)}{|x-y|^n} \, dy
\]

\[
\leq C \sum_{k=0}^{\infty} \frac{\rho(2^{k+1}r)}{(2^k r)^n} \int_{|x-y| < 2^{k+1}r} |f(y)| \, dy
\]

\[
\leq C \sum_{k=0}^{\infty} \rho(2^{k+1}r) \phi(2^{k+1}r) \|f\|_{M_{p,\phi}}
\]

\[
\leq C\|f\|_{M_{p,\phi}} \sum_{k=0}^{\infty} \int_{2^k r}^{2^{k+1} r} \frac{\phi(t) \rho(t)}{t} \, dt
\]

\[
\leq C\|f\|_{M_{p,\phi}} \int_{r}^{\infty} \frac{\phi(t) \rho(t)}{t} \, dt
\]

\[
\leq C\psi(r)\|f\|_{M_{p,\phi}}.
\]

Now, for \(1 \leq p < \infty\), we have

\[
|T_\rho f(x)|^p \leq 2^{p-1}(|I_1(x)|^p + |I_2(x)|^p),
\]

and by Nakai’s Theorem, we have for all balls \(B = B(a, r)\)

\[
\frac{1}{\psi(r)^p |B|} \int_B |I_1(x)|^p \, dx \leq C \frac{\|f\|_{M_{p,\phi}}^p}{\phi(r)^p |B|} \int_B M f(x)^p \, dx \leq C \|f\|_{M_{p,\phi}}^p \leq C_p \|f\|_{M_{p,\phi}}^p,
\]

and

\[
\frac{1}{\psi(r)^p |B|} \int_B |I_2(x)|^p \, dx \leq C\|f\|_{M_{p,\phi}}^p.
\]

Combining the two estimates, we obtain

\[
\frac{1}{\psi(r)^p |B|} \int_B |T_\rho f(x)|^p \, dx \leq C_p \|f\|_{M_{p,\phi}}^p,
\]

and the result follows. \(\square\)

**Proof of Theorem 1.2**

Let \(B = B(a, 2r)\). For \(x \in B = B(a, r)\), we have

\[
\tilde{T}_\rho f(x) - C_B = E_1^B(x) + E_2^B(x),
\]

where

\[
C_B = \int_{\mathbb{R}^n} f(y) \left( \frac{\rho(|a - y|)(1 - \chi_B(y))}{|a - y|^n} - \frac{\rho(|y|)(1 - \chi_B(y))}{|y|^n} \right) \, dy,
\]
\[ E_1^B(x) = \int_B f(y) \frac{\rho(|x-y|)}{|x-y|^n} \, dy, \]

and

\[ E_2^B(x) = \int_B f(y) \left( \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)}{|a-y|^n} \right) \, dy. \]

From (6), we have

\[ |C_B| \leq C \left( \int_{|a-y|<k} |f(y)| \, dy + |a| \int_{|a-y|\geq k} \frac{|f(y)|}{|a-y|^{n+1}} \, dy \right), \]

where \( k = \max(2|a|, 2r) \), and so we know that \( C_B \) is finite for every ball \( B = B(a, r) \).

With the same technique as in the proof of the previous theorem, we have

\[ |E_1^B(x)| \leq \int_{|a-y|<2r} \frac{|f(y)|}{|x-y|^n} \, dy \]
\[ \leq \int_{|x-y|<3r} \frac{|f(y)|}{|x-y|^n} \, dy \]
\[ \leq CMf(x) \int_0^{3r} \frac{\rho(t)}{t} \, dt \]
\[ \leq CMf(x) \int_0^r \frac{\rho(t)}{t} \, dt, \]

and by (6)

\[ |E_2^B(x)| \leq \int_{|a-y|\geq 2r} \frac{|f(y)|}{|x-y|^n} \, dy \]
\[ \leq C|x-a| \int_{|a-y|\geq 2r} \frac{|f(y)|}{|a-y|^{n+1}} \, dy \]
\[ \leq C \|f\|_{L^{p,\psi}} \int_0^\infty \frac{\rho(t)\phi(t)}{t^2} \, dt, \]

and the result follows as before. \( \Box \)

### 3. Remark

We also suspect that \( \tilde{T}_\rho \), the modified version of \( T_\rho \), is bounded from \( L^{p,\phi} \) to \( L^{p,\psi} \) under the same hypothesis on \( \rho, \phi \) and \( \psi \) as in Theorem 1.2. However, we have not obtained the proof and the research in this direction is still ongoing.
References

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