RG Analysis for Quantum Gravity with A Single Dimensionless Coupling

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Abstract

We study the quantum conformal gravity whose dynamics is governed by a single dimensionless gravitational coupling with negative beta function. Since the Euler term is not dynamical classically, the constant in front of it is not an independent coupling. Quantum mechanically, however, it induces the Riegert conformal-factor dynamics with BRST conformal symmetry representing background free nature. In this paper, we propose how to handle the Euler term systematically incorporating such a dynamics on the basis of renormalization group analysis using dimensional regularization. As a non-trivial test of renormalization, we explicitly calculate the three-loop anomalous dimension of the cosmological constant operator and show that it agrees with the exact expression derived using the BRST conformal symmetry. The physical significance to inflation and CMB is also discussed.

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1 Introduction

Recent observations of CMB anisotropies by various groups [1, 2, 3] suggest that the Universe began to expand at very high energies beyond the Planck scale. There spacetime would be totally fluctuating according to the laws of quantum mechanics. Thus, now is a good opportunity to develop the study of quantum gravity.

On the other hand, at first glance, it seems to be contradictory that we trace the origin of primordial fluctuations to quantum gravity, because the observations show that scalar fluctuations are more significant than tensor ones in the early Universe. This implies that if we wish to explain such fluctuations using quantum gravity only without adding a phenomenological scalar field, we have to construct a model whose physical states become scalar-like in the UV limit.

In order to resolve this problem, we propose the model of quantum gravity with a certain gauge symmetry imposing that all spacetimes related to each other under conformal transformations are equivalent, called the BRST conformal invariance here, as a representation of the background-metric independence. It has been known that the Riegert action plays a significant role to realize it [4, 5, 6, 7, 8]. We then have shown that the Riegert theory including the kinetic term of the Weyl action has such a gauge symmetry as a part of diffeomorphism invariance, which is so strong that physical fields are restricted to “real composite scalars” (called primary scalars) only [9, 10, 11].

The model we consider here is the renormalizable quantum theory of gravity expanded just from this background-free system by a single dimensionless coupling constant that brings the dynamics of traceless tensor fields [12, 13].

\[\text{Due to the presence of this symmetry, the ghost modes in 4th-order gravitational fields, which are necessary for the conformal algebra to close, are not gauge invariant. So, we will be able to begin a argument toward the resolution of the non-perturbative issue of unitarity.}\]
2 Renormalization Group Analysis

From the analysis of the renormalization group (RG) equations using dimensional regularization [14, 15, 16, 17, 18], it has been recently shown that the gravitational counterterms for massless QED in curved space can be determined at all orders of the perturbation [18], which are given by only two forms: the square of the Weyl tensor $C_{\mu\nu\lambda\sigma}$ in $D$ dimensions and the modified Euler density defined by

$$G_D = G_4 + (D - 4)\chi(D)H^2,$$

where $G_4$ is the usual Euler combination and $H = R/(D - 1)$ is the rescaled scalar curvature. The coefficient $\chi$ is a finite function of $D$ only, which can be determined order by order in a series of $D - 4$ as $\chi = \sum_{n=1}^{\infty} \chi_n(D - 4)^{n-1}$ by solving the RG equations. The first three terms of $\chi$ are explicitly calculated as [18]

$$\chi_1 = \frac{1}{2}, \quad \chi_2 = \frac{3}{4}, \quad \chi_3 = \frac{1}{3}. \quad (1)$$

We expect that $\chi$ is a universal function for dimensionless conformal couplings. Really, these values are not changed even when any number of fermions is added below, and also partially confirmed for $\lambda \phi^4$-scalar theory [16] and Yang-Mills theory [17]. In this paper, we will see that these values are also approved when gravity is quantized.

Based on this result, we study the quantum gravity model described by the action

$$S_g = \int d^D x \sqrt{g} \left\{ \frac{1}{t_0^2} C^2_{\mu\nu\lambda\sigma} + b_0 G_D \right\}$$

beyond the Planck scale, where $t_0$ is a dynamical coupling constant, while $b_0$ is not so, as discussed below. The lower-derivative actions such as the Einstein action are suppressed here.

We consider the perturbation theory in $t_0$ expanding about a conformally flat space defined by $C_{\mu\nu\lambda\sigma} = 0$, which is characterized by the expansion of
the metric field:

\[ g_{\mu\nu} = e^{2\phi}(\hat{g}e^{t_0h_0})_{\mu\nu} = e^{2\phi}(\hat{g}_{\mu\nu} + t_0 h_{0\mu\nu} + \cdots), \]

where \( \hat{g}_{\mu\nu} h_{0\mu\nu} = 0 \) and \( \hat{g}_{\mu\nu} \) is the background metric. Thus, the quantum gravity model can be described as a quantum field theory on the curved background. At this time, it is significant that the conformal factor \( e^{2\phi} \) is treated exactly without introducing its own coupling constant, because the conformally flat condition does not give any restrictions on it.

As in the previous study [12, 18], we consider the model coupled to massless QED: \( S = S_g + S_{\text{QED}}, \) where \( S_{\text{QED}} = \int d^Dx \sqrt{g}\{(1/4)F_{0\mu\nu}F_{0}^{\mu\nu} + \sum_{j=1}^{n_F} i\bar{\psi}_j D\psi_j\}. \) The renormalization factors are defined by \( A_0 = Z_0^{1/2} A_\mu, \) \( \psi_j = Z_0^{1/2} \psi_j \) and \( h_{0\mu\nu} = Z_h^{1/2} h_{\mu\nu} \) for photon, \( n_F \) fermions and traceless tensor fields, respectively, and \( e_0 = \mu^{2-D}/2 Z_3^{-1/2} e \) and \( t_0 = \mu^{2-D/2} Z_t t \) for coupling constants, where the Ward-Takahashi identity holds even when QED couples with quantized gravity. On the other hand, \( \phi \) is not renormalized such that \( Z_\phi = 1 \) because there is no coupling constant for this field.

The non-renormalization theorem of \( \phi \) is related to the geometrical property of \( G_D \) [19, 20]. Since its volume integral becomes topological at four dimensions, it is not dynamical at the classical level. Therefore, \( b_0 \) does not be the independent coupling that governs the dynamics of gravity. So, we expand the bare parameter \( b_0 \) in a pure-pole series as [12]

\[ b_0 = \frac{\mu^{D-4}}{(4\pi)^{D/2}} L_b, \quad L_b = \sum_{n=1}^{\infty} \frac{b_n}{(D-4)^n}. \]

Since the field-dependence of the volume integral of \( G_D \) appears at \( o(D-4) \), this expansion indicates that the dynamics is induced at the quantum level by canceling out the pole with the \( D \)-dependence of the action. The residue \( b_n (n \geq 2) \) depends on the coupling constants, while the simple-pole residue is divided as

\[ b_1 = b + b'_1, \]

where \( b'_1 \) is coupling-dependent and \( b \) is a constant part.
In order to carry out the RG analysis systematically incorporating the dynamics induced quantum mechanically, we propose the following procedure. For the moment, $b$ is regarded as a new coupling constant, instead of the finite term usually introduced in $b_0$. The effective action is then finite up to the topological term as follows:

$$\Gamma = \frac{\mu^{D-4}}{(4\pi)^{D/2}} \frac{b - b_c}{D - 4} \int d^D x \sqrt{\hat{g}} \hat{G}_4 + \Gamma_R(e, t, b),$$

where $\Gamma_R$ is the renormalized quantity that depends on the coupling constants. The divergent term exists in a curved background only. The constant $b_c = 11n_F/360 + 40/9$ comes from the sum of direct one-loop calculations of QED [19] and gravitational fields [21, 6, 8]. After the renormalization procedure is carried out, we take $b = b_c$. In this way, we can obtain the finite effective action $\Gamma_R(e, t, b_c)$ whose dynamics is governed by a single gravitational coupling $t$.

It has been checked that the renormalization procedure above goes well at higher loops up to $o(e^6)$, $o(t^2)$ and $o(1/b)$ through direct calculations using the values of $\chi_{1,2}$ in (1) [12], in which the RG equation is also discussed, but it is somewhat defective. One of the aims of this paper is to justify the procedure through the RG analysis more sufficiently. In the following, we proceed the argument without specifying the value of $\chi_3$.  

The perturbative calculation can be performed as follows. Expanding the volume integral of $G_D$ in flat background, we obtain

$$\int d^D x \sqrt{g} G_D = \sum_{n=0}^{\infty} \frac{(D - 4)^n}{n!} \int d^D x \{ 4\chi(D) \phi^n \partial^4 \phi + (D - 4)\zeta(D) \phi^n \partial^2 \phi \partial_\lambda \phi \partial_\lambda \phi + (D - 4)\eta(D) \phi^n (\partial_\lambda \phi \partial_\lambda \phi)^2 \},$$

where $\zeta = 8\chi - 2(D-2)(D-3)$ and $\eta = (D-2)^2\chi - (D-2)(D-3)^2$. The couplings with traceless tensor fields are not presented here. As a convention in

\footnote{It is because the values of $\chi_{1,2}$ agree with those conjectured in the model of quantum gravity [12], but $\chi_3$ unfortunately disagrees. It seems that the condition imposed to determine the $G_D$ action there may be somewhat strong.}
this paper, the same lower indices denote contraction by the flat background metric. Substituting the explicit values of $\chi_{1,2}$ and neglecting $o((D-4)^3)$, the coefficients reduce to $\zeta = (8\chi_3 - 2)(D-4)$ and $\eta = (8\chi_3 - 1)(D-4)^2/2$. Therefore, the action is expanded as

$$b_0 \int d^D x \sqrt{g} G_D = \frac{\mu^{D-4}}{(4\pi)^{D/2}} \int d^D x \left\{ 2b \phi \partial^4 \phi + (D-4)b_0 \phi^2 \partial^4 \phi 
+ (D-4)^2 \left[ \frac{1}{3} \phi^3 \partial^4 \phi + \frac{1}{2} (8\chi_3 - 1) (\partial \phi \partial \phi)^2 \right] + \cdots \right\}. \tag{2}$$

The first term of r.h.s is just the induced Riegert action [4] in flat background, which gives the propagator of $\phi$. Thus, quantum corrections from this field are expanded in $1/b$, which corresponds to the large-$n_F$ expansion. The other terms are novel vertices. The vertices from $\zeta$ are disregarded here because they are not the leading terms of each order of $D-4$ that yield the most singular diagrams. The terms denoted by dots include the vertices and counterterms coming from the residues $b'_1$ and $b_n (n \geq 2)$. Similarly, the Weyl and QED actions are expanded in powers of $\phi$.

The beta functions for the coupling constants $\alpha = \frac{e^2}{4\pi}$ and $\alpha_t = \frac{t^2}{4\pi}$ are defined by $\beta = (\mu/\alpha)d\alpha/d\mu = D-4 + \bar{\beta}$ and $\beta_t = (\mu/\alpha_t)d\alpha_t/d\mu = D-4 + \bar{\beta}_t$, respectively. Using the renormalization factors, they are expressed as $\bar{\beta} = \mu d(\log Z_3)/d\mu$ and $\bar{\beta}_t = \mu d(\log Z_{\phi^2})/d\mu$.

The pure-pole term $L_b$ is divided as $L_b = b/(D-4) + L'_b$. From the RG equation $\mu db_0/d\mu = 0$, we obtain the expression

$$\frac{db}{d\mu} = (D-4)\bar{\beta}_b,$$

where $\bar{\beta}_b = -b - (D-4)L'_b - \mu dL'_b/d\mu$. In order that we can set $b$ to the constant $b_c$ at the end, this equation should satisfy the condition $\mu db/d\mu \rightarrow 0$ in the $D \rightarrow 4$ limit, and thus $\bar{\beta}_b$ should be finite. Imposing this condition, we obtain the RG equation

$$\left( \alpha \frac{\partial}{\partial \alpha} + \alpha_t \frac{\partial}{\alpha_t} + \bar{\beta}_b \frac{\partial}{\phi} + 1 \right) b_{n+1} + \left( \bar{\beta}_\alpha \frac{\partial}{\partial \alpha} + \bar{\beta}_t \alpha_t \frac{\partial}{\partial \alpha_t} \right) b_n = 0$$
for \( n \geq 1 \), where note that \( \partial b / \partial \alpha = \partial b / \partial \alpha_t = 0 \) for \( n = 1 \). This equation reduces to that in curved space \([15, 18]\) when the dependence on \( \alpha_t \) and \( b \) turns off. Thus, the present renormalization procedure is consistent with the results in curved space. The finite expression is then given by

\[
\bar{\beta}_b = - \left( \frac{\partial b_1}{\partial b} \right)^{-1} \left( b_1 + \alpha \frac{\partial b_1}{\partial \alpha} + \alpha_t \frac{\partial b_1}{\partial \alpha_t} \right) .
\]

The \( \alpha \)-dependent terms of \( b_1 \) and \( b_2 \) have been calculated up to \( o(\alpha^3) \) and \( o(\alpha^4) \), respectively \([15, 18]\). We only present \( b_1 = b - (n_F^2/6)(\alpha/4\pi)^2 \) and \( b_2 = (2n_F^3/9)(\alpha/4\pi)^3 \) here, and then obtain

\[
\bar{\beta}_b = -b + \frac{n_F^2}{2} \left( \frac{\alpha}{4\pi} \right)^2 .
\]

The residues of \( \log Z_3 = \sum_{n=1}^{\infty} f_n/(D-4)^n \) satisfy the RG equation

\[
\alpha \frac{\partial f_{n+1}}{\partial \alpha} + \alpha_t \frac{\partial f_{n+1}}{\partial \alpha_t} + \bar{\beta}_b \frac{\partial f_{n+1}}{\partial b} + \bar{\beta}_t \frac{\partial f_n}{\partial \alpha_t} + \bar{\beta}_t \frac{\partial f_n}{\partial \alpha_t} = 0 .
\]

From the direct loop calculations, the simple and double poles are given by \([12]\)

\[
f_1 = \frac{8n_F}{3} \frac{\alpha}{4\pi} + \left( \frac{4n_F - 16n_F^2}{27b} \right) \left( \frac{\alpha}{4\pi} \right)^2 ,
\]

\[
f_2 = -\frac{32n_F^2}{9} \left( \frac{\alpha}{4\pi} \right)^2 - \left( \frac{128n_F^2}{9} - \frac{160n_F^3}{81b} \right) \left( \frac{\alpha}{4\pi} \right)^3 .
\]

Here, the corrections including \( 1/b \) come from the diagrams with an internal line of \( \phi \). It has been shown that the \( o(\alpha_t) \) correction to \( f_1 \) vanishes. Noting that \( \bar{\beta}_b = -b + o(\alpha^2) \), we find that \( f_{1,2} \) are consistent with the RG equation. The beta function is now expressed as

\[
\bar{\beta} = \alpha \partial f_1 / \partial \alpha + \alpha_t \partial f_1 / \partial \alpha_t + \bar{\beta}_b \partial f_1 / \partial b ,
\]

and thus

\[
\bar{\beta} = \frac{8n_F}{3} \frac{\alpha}{4\pi} + \left( \frac{8n_F - 16n_F^2}{9b} \right) \left( \frac{\alpha}{4\pi} \right)^2 .
\]

Replacing \( b \) with \( b_c \), we obtain the final expression of the beta function. Note that the \( o(\alpha^2) \) term becomes negative for \( n_F \geq 24 \) as the standard model is
so, and in this case we can expect the existence of the UV fixed point about the Planck scale.

In the same way, expanding as $\log Z^t = \sum_{n=1}^{\infty} g_n/(D-4)^n$, we obtain the similar RG equations for $g_n$. The beta function has been calculated as [21, 6, 8, 12]

$$\bar{\beta}_t = -\left(\frac{n_F}{20} + \frac{20}{3}\right) \frac{\alpha_t}{4\pi} - \frac{7n_F}{36} \frac{\alpha\alpha_t}{(4\pi)^2}.$$

Thus, the coupling constant $\alpha_t$ indicates the asymptotic freedom, which justifies the perturbation theory about conformally flat spacetime.

3 Background-Metric Independence

Reflecting the shift-invariance of the measure of $\phi$, the background-metric independence is now expressed as $\int D\phi \delta(Oe^{-S})/\delta \phi = 0$. Taking $\sqrt{g}\theta = \delta S/\delta \phi$ as an operator $O$, we obtain

$$\langle \sqrt{g}\theta(x)\sqrt{g}\theta(y) \rangle - \langle \frac{\delta \sqrt{g}\theta(x)}{\delta \phi(y)} \rangle = 0. \tag{3}$$

Here, $\theta = \theta_A + \theta_\psi + \theta_g$ is merely the trace of the energy-momentum tensor. Each part is given by $\theta_A = (D-4)F_{0\mu\nu}F_0^{\mu\nu}/4$, $\theta_\psi = (D-1)\sum_{j=1}^{n_F} i\vec{\psi}_0\frac{\partial}{\partial \psi_0} \bar{\psi}_0 j$ and $\theta_g = (D-4)(C_{\mu\nu\lambda\sigma}^2/t_0^3 + b_0 E_D)$, where $E_D = G_D - 4\chi(D)\nabla^2 H$. The vanishing of r.h.s is significant here unlike in curved space.

Let us consider Eq.(3) focusing the role of the $\phi$ field. The traceless tensor fields are neglected for simplicity, which have the similar structure to the photon field. We expand $\sqrt{g}\theta$ in a series of $D - 4$ in flat background. The photon part is then expressed as

$$\sqrt{g}\theta_A = (D-4)Z_3 \frac{1}{4} e^{(D-4)\phi} F_{\mu\nu}F_{\mu\nu}$$

$$= \frac{D - 4}{4} \left[ 1 + \frac{f_1}{D - 4} + (D - 4 + f_1) \phi + \cdots \right] F_{\mu\nu}F_{\mu\nu}.$$
Figure 1: Diagrams representing the background-metric independence
The gravity part is expanded as
\[
\sqrt{g} \theta = \frac{\mu^{D-4}}{(4\pi)^{D/2}} \left\{ 4b \partial^4 \phi + (D - 4)b \left[ 2\phi \partial^4 \phi + \partial^4 (\phi^2) \right] 
+ (D - 4)^2 b \left[ \phi^2 \partial^4 \phi + \frac{1}{3} \partial^4 (\phi^3) - 2 (8\chi_3 - 1) \partial_\alpha (\partial_\sigma \phi \partial_\sigma \phi) \partial_\alpha \phi \right] + \cdots \right\}, \tag{4}
\]
where we write down the \( b \)-dependent leading terms only. The two-point function of \( \sqrt{g} \theta \) at \( o(b) \) can be easily calculated as \( 4b \mu^{D-4} p^4 / (4\pi)^{D/2} \) in momentum space from the first term of (4) using the propagator \( \langle \phi(p)\phi(-p) \rangle = \mu^{D-4} (4\pi)^{D/2} / 4bp^4 \). It cancels out the second term of Eq.(3).

Furthermore, using (4) and the vertices (2), we can show that Eq.(3) is satisfied at \( o(1) \) and \( o(\alpha) \) as in Fig.1, where the wavy, dashed-arrow and solid lines denote the propagators of photon, fermions and \( \phi \), respectively, and the double line denotes \( \sqrt{g} \theta \). The last three tadpole-type diagrams of \( o(1) \) cannot be neglected in 4th-order theories unlike massless 2nd-order theories. The last one of these comes from the second term of Eq.(3). At \( o(\alpha) \), there is no contribution from the second term of Eq.(3), which is consistent with the RG analysis deducing that there is no \( o(\alpha) \)-term in the residue \( b_1 \) [15].

If the energy-momentum tensor is defined by the variation of the action with respect to the background metric as \( \sqrt{\hat{g}} \hat{\theta} = \delta S / \delta \sigma \), where \( \delta / \delta \sigma = 2\hat{g}_{\mu\nu} \delta / \delta \hat{g}_{\mu\nu} \), it satisfies the relationship \( \sqrt{\hat{g}} \hat{\theta} = \sqrt{g} \theta \), up to the gauge-fixing origin term which vanishes in physical correlation functions. Thus, Eq.(3) represents the background-metric independence for \( \sigma \) as expected.

4 Cosmological Constant Operator

As a non-trivial test of the method proposed here, we calculate the anomalous dimension of the cosmological constant operator at \( \alpha = \alpha_t = 0 \), for which the exact solution derived using the BRST conformal invariance has been known.

Since \( \phi \) is not renormalized, the cosmological constant term \( \Lambda_0 \int d^D x \sqrt{g} \) can be simply renormalized by \( \Lambda_0 = \mu^{D-4} Z_\Lambda \Lambda \), where \( \sqrt{g} = e^{D\phi} = \sum_{n=0}^{\infty} D^n \phi^n / n! \)
in flat background. The anomalous dimension is defined by \( \gamma_\Lambda = - (\mu/\Lambda) d\Lambda/d\mu \).

Expanding the renormalization factor as \( \log Z_\Lambda = \sum_{n=1}^{\infty} u_n/(D-4)^n \), the anomalous dimension is given by \( \gamma_\Lambda = D-4 + \bar{\gamma}_\Lambda \) and \( \bar{\gamma}_\Lambda = \mu d(\log Z_\Lambda)/d\mu = -b \partial u_1/\partial b \).

The \( o(1/b) \) and \( o(1/b^2) \) corrections to \( u_1 \) are calculated from 1- and 2-loop diagrams in Fig.2, respectively. In order to handle IR divergences, we introduce an infinitesimal fictitious mass in the 4th-order propagator as \( 1/p^4 \rightarrow 1/p^4 + \varepsilon \equiv 1/(p^2 + z^2)^2 \).\(^4\) Extracting UV divergences, we obtain \( u_1 = 4/b + 4/b^2 \) [12].

The \( o(1/b^3) \) correction is obtained from two 3-loop diagrams in Fig.2 as

\[
\left(-J + \frac{K}{8}\right)(4\pi)^{D/2} \mu^{3(D-4)} \left(\frac{D-4}{b^4}\right)^2 \mu^{D-4} \Lambda^D n! \int d^D x \phi^n, \tag{5}
\]

where \( J \) and \( K \), defined below, are the momentum integrals from the diagrams including a 4-point vertex and two 3-point vertices, respectively. The factor \( (D-4)^2 \) comes from these vertices in (2).

First, we calculate \( J \), which is divided into two parts as \( J = (1/3)J_A + (4\chi_3 - 1/2)J_B \) according to the two types of 4-point vertex in (2).

\(^4\)Since this mass is not gauge invariant, the \( z \)-dependences will be canceled out after summing up all contributions [13]. Incidentally, the lower-derivative action such as the Einstein action cannot be regarded as a usual mass term because there is an exponential conformal-factor in this case.
integrals are then defined by

\[ \begin{align*}
J_A &= \frac{1}{4} \int dV_4 \frac{p^4 + q^4 + r^4 + s^4}{p_z^4 q_z^4 r_z^4 s_z^4}, \\
J_B &= \frac{1}{3} \int dV_4 \frac{(p \cdot q)(r \cdot s) + (p \cdot r)(q \cdot s) + (p \cdot s)(q \cdot r)}{p_z^4 q_z^4 r_z^4 s_z^4},
\end{align*} \]

where \( dV_4 = [dpdqdrds] \delta(p + q + r + s) \), \([dp] = d^Dp/(2\pi)^D\) and \( \delta(p) = (2\pi)^D \delta^D(p) \). The most singular term of \( J_A \) is simply given by the cube of the tadpole integral \( I = \int [dp]/p_z^4 = \Gamma(2 - D/2)z^{D-4}/(4\pi)^{D/2} \), and thus \( J_A = I^3 = 1/(4\pi)^6 \epsilon^3 \), where \( D = 4 - 2\epsilon \). The less singular terms and the IR divergence arising through \( z^{3(D-4)} = 1 - 6\epsilon \log z \) are neglected. Taking into account the overall factor \((D - 4)^2\) in (5), the contribution from \( J_A \) gives a simple pole, which can be renormalized by taking \( u_1 = -8/3b^3 \). On the other hand, the integral \( J_B \) is given by

\[ J_B = \int [dk] \int [dp] \frac{p \cdot (p - k)}{p_z^4 (p - k)_z^4} \int [dq] \frac{q \cdot (q + k)}{q_z^4 (q + k)_z^4} \]

\[ = \frac{z^{3(D-4)}}{(4\pi)^{3D/2}} \frac{1}{4} \Gamma \left( 6 - \frac{3D}{2} \right) \int_0^1 dx \int_0^1 dy \int_0^1 dt \int \left[ x(1 - x)y(1 - y) \right]^{1 - D} Y^{\frac{3D}{2} - 6} \]

\[ \times [t(1 - t)]^{2 - \frac{D}{2}} \left\{ D^2 - 2D^2(1 - t) + D(2 - D)(1 - t) \right\}, \]

where \( Y = ty(1 - y) + (1 - t)x(1 - x) \). This integral becomes finite apart from the factor \( \Gamma(6 - 3D/2) \), and so multiplying \( J_B \) by \((D - 4)^2\), the simple pole vanishes. Thus, there is no contribution to \( u_1 \) from \( J_B \), namely \( \chi_3 \).

The momentum integral \( K \) is defined by

\[ K = \int dV_4 \frac{(p^4 + r^4 + (p + r)^4)(q^4 + s^4 + (p + r)^4)}{p_z^4 q_z^4 r_z^4 s_z^4 (p + r)_z^4}. \]

As before, the integral is evaluated as \( K = 8I^3 + \tilde{K} \), where \( I \) is the tadpole integral and

\[ \tilde{K} = \int [dk] k^4 \int [dp] \frac{[dp]}{p_z^4 (p - k)_z^4} \int \frac{[dq]}{q_z^4 (q + k)_z^4} \]

\(^5\)Since we focus on UV divergences, we can expand \( p^4/p_z^4 = 1 - 2z^2/p^2 + \cdots \) and disregard the \( z \)-dependent terms here.
\[
\frac{z^{3(D-4)} D(D+2)}{(4\pi)^{2/2}} \frac{D}{4} \Gamma \left( 6 - \frac{3D}{2} \right) \int_0^1 dx \int_0^1 dy \int_0^1 dt \\
\times \left[ x(1-x)y(1-y) \right]^{3-D[t(1-t)]^{\frac{3}{2}}} \left( 6 - \frac{3D}{2} \right) - \frac{D}{2} \frac{4}{3}.
\]

Here, the multiple parameter integrals are numerically evaluated using the Monte Carlo method by Maple software and we find that \( \tilde{K} \) has the triple-pole term as \((4\pi)^6 c^3 \tilde{K} = 1.33 \) at the \( \epsilon \to 0 \) limit.\(^6\) The numerical number is consistent with 4/3. Therefore, substituting this value, we obtain \((4\pi)^6 c^3 \tilde{K} = 8 + 4/3 = 28/3\). Thus, the last diagram in Fig.2 yields the simple pole \( u_1 = 28/3b^3 \).

Combining the results of \( J \) and \( K \), we obtain \( u_1 = 20/3b^3 \) as the 3-loop correction. Thus, the anomalous dimension is given by

\[
\tilde{\gamma}_\Lambda = 4 \frac{b}{b} + 8 \frac{b}{b^2} + \frac{20}{b^3}
\]

with taking \( b = b_c \) at last. This result agrees with the first three terms in the large-\( b_c \) expansion of the exact expression \( \tilde{\gamma}_\Lambda = \gamma - 4 = \gamma^2/4b_c \), where \( \gamma = 2b_c(1 - \sqrt{1 - 4/b_c}) \) is the charge for the cosmological constant operator defined by \( \int d^4x :e^{2\phi} : \) \([5, 9, 10, 11]\).

In this way, we have seen that the renormalization manner proposed here can reproduce the exact anomalous dimension. Consequently, the result is independent of the value of \( \chi_3 \) and it suggests that vertices of the type \( b(D-4)^n \phi^{n+1} \partial^4 \phi \) only contribute to \( u_1 \). We expect that \( \chi_n \)'s are independent of the theory as mentioned before, but those of \( n \geq 3 \) may depend on it.

5 Conclusion and Cosmological Implications

The goal of quantum gravity is to break the wall of Planck scale and

\(^6\)Since the integrals of \( x \) and \( y \) are numerically dangerous, we change these variables as \( dx/x(1-x) = dX \) such that \( x(1-x) = 1/4 \cosh^2(X/2) \) and evaluate it within a finite range (\( |X| \leq 8/\epsilon \) here), and similarly for \( y \). Since the precision is not so good when \( \epsilon \) is small, we evaluate \((4\pi)^3 D/2 c^3 \tilde{K} \) (apart from overall \( z \)-dependence) at many values of \( \epsilon \) more than 100 between 0.02 and 0.1 and read off it as a polynomial function of \( \epsilon \) using the LeastSquares method. In this manner, we extract the value at \( \epsilon \to 0 \).
to reveal the laws of physics there. We imagine that once we go beyond the Planck scale, there spreads a harmonious space without scale and singularity. The quantum gravity model with a single dimensionless gravitational coupling has a lot of desirable properties to describe such a spacetime. In this paper, we have studied the model by developing how to treat the conformal factor on the basis of the RG method using dimensional regularization, and then have seen that it passes several theoretical tests of renormalization.

The asymptotically free behavior of traceless tensor fields indicates that quantum fluctuations of the conformal factor become more significant than tensor in the early epoch of the Universe. There a novel spacetime phase with background free nature called the BRST conformal symmetry will emerge. Such a phase can be imaged by four-dimensional simplicial manifold with varying connectivity of simplices, called dynamical triangulation [22]. It provides a scale-invariant scalar spectrum primordially.

The induced Riegert action has a stable inflationary solution that can explain the results of CMB experiments well [23, 24, 25]. When the energy is going down to about the Planck mass scale $m_{\text{pl}}$, the Einstein action becomes effective and the Universe evolves to the inflationary phase. The inflation will terminate eventually at the dynamical energy scale $\Lambda_{\text{QG}}$ indicated from the asymptotic freedom because a conformally flat spacetime is no longer significant there, and the Universe turns to the classical Einstein phase. The number of e-foldings is then given by the ratio of two mass scales as $N_e \simeq m_{\text{pl}}/\Lambda_{\text{QG}}$ and the amplitude of scalar fluctuation is roughly estimated to be $\delta R/R \simeq (\Lambda_{\text{QG}}/m_{\text{pl}})^2$. Thus, $\Lambda_{\text{QG}}$ is predicted to be the order of $10^{17}\text{GeV}$. This scale can also explain the sharp falloff of CMB angular power spectra at very large angles.

In order to achieve the overall fit to the CMB data, more detail consideration on the evolution process may be necessary. For a substance to make up the Universe, if there is a stable gravitational soliton, it will be a candidate for dark matter. They are left in future study.
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