Solitonic solutions of Faddeev model

Chang-Guang Shi1 and Minoru Hirayama2
1) College of Mathematics and Physics, Shanghai University of Electric Power, Pinglian Road 2103, Shanghai 200090, China
2) Department of Physics, University of Toyama, Gofuku 3190, Toyama, Japan

An application of the equation proposed by the present authors, which is equivalent to the static field equation of the Faddeev model, is discussed. Under some assumptions on the space and on the form of the solution, the field equation is reduced to a non-linear ODE of second order. By solving this equation numerically, some solitonic solutions are obtained. It is discussed that the product of two integers specifying solutions may be identified with the Hopf topological invariant.

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I. INTRODUCTION

Hopf solitons are classified by the Hopf topological invariant characterizing the knot structure of system. These topological solutions play an important role in many areas of physics1–5. The minimal model possessing solutions with stable knot structures seems to be the Faddeev model6 which concerns the real scalar fields

\[ n(x) = (n^1(x), n^2(x), n^3(x)) \]  
(1)

and is expected to describe the low energy behavior of the SU(2) gauge field6. Although its numerical solutions exhibit quite interesting knot-soliton properties6,9, the analytic analysis of the model does not seem to have shown much progress because of the high nonlinearity of the model.

The Lagrangian density of the Faddeev model is given by

\[ \mathcal{L}_F(x) = c_2 l_2(x) + c_4 l_4(x), \]  
(3)

\[ l_2(x) = \partial_\mu n(x) \cdot \partial^\nu n(x), \]  
(4)

\[ l_4(x) = -H_{\mu \nu}(x) H^{\mu \nu}(x), \]  
(5)

\[ H_{\mu \nu}(x) = n(x) \cdot [\partial_\mu n(x) \times \partial_\nu n(x)] ] = \epsilon_{abc} n^a(x) \partial_\mu n^b(x) \partial_\nu n^c(x), \]  
(6)

where \( c_2 \) and \( c_4 \) are constants. The field \( n \) can be expressed by a complex function \( u \) as

\[ n = \left( \frac{u + u^*}{|u|^2 + 1}, -i(u - u^*), \frac{|u|^2 - 1}{|u|^2 + 1}, \frac{|u|^2 + 1}{|u|^2 + 1} \right). \]  
(7)

We define \( R, \Phi, X \) and \( q \) by

\[ R = |u|, \quad u = Re^{i\Phi}, \]  
(8)

\[ X = 2 \sqrt{\frac{c_4}{c_2}} \frac{1}{\sqrt{1 + R^2}} = \frac{1}{1 + R^2}, \]  
(9)

\[ q = X \nabla u, \]  
(10)

where \( 2\sqrt{c_4/c_2} \) of the dimension of length has been set equal to 1 and \( q \) is dimensionless. If we define a complex 3-vector \( \alpha \) and a real 3-vector \( \beta \) by

\[ \alpha = q^* - q^* \times (q \times q^*), \]  
(11)

\[ \beta = \frac{1}{2}(u^* q - u q^*) = B \nabla \Phi, \quad B = \frac{2R^2}{1 + R^2}, \]  
(12)

the static field equation can be written as

\[ \nabla \cdot \alpha + i \beta \cdot \alpha = 0 \]  
(13)

and its complex conjugate. It was found in10 that \( \alpha \) defined by

\[ \alpha = \nabla \Phi \times \nabla \mu + e^{-iB\Phi} (\nabla R \times \nabla \nu) \]  
(14)

satisfies for arbitrary complex functions \( \mu \) and \( \nu \). Although it is not clear whether the above form of \( \alpha \) is general enough or not, we see that the static field equation of the Faddeev model possesses the above kind of linearity.

II. REDUCTION OF FIELD EQUATION

We regard \( \mu \) and \( \nu \) in (14) as functions of \( R, \Phi \) and \( \zeta \) satisfying \( \partial(R, \Phi, \zeta) \neq 0 \), where \( \zeta \) is a real function.

We represent the gradient of \( \zeta \) as

\[ \nabla \zeta = \Gamma \nabla R \times R \nabla \Phi + \Xi \nabla R + \Sigma \nabla \Phi \]  
(15)
with $\Gamma, \Xi$ and $\Sigma$ being real functions. Defining real functions $a, b, c$ and $Y$ by

$$\mu_\zeta = \frac{e^{i(B-1)\Phi}}{\Gamma} [b + Y + ia],$$  \hspace{1cm} (16)

$$\nu_\zeta = \frac{e^{i(B-1)\Phi}}{\Gamma} [a + i(c + Y)],$$  \hspace{1cm} (17)

$$Y = 2X^3$$  \hspace{1cm} (18)

with $\nu_\zeta = \frac{\partial}{\partial \zeta}$, etc., we find that the two expressions (11) and (12) for $\alpha$ coincide irrespectively of the directions of the 3-vectors $\nabla R$ and $\nabla \Phi$ if the following relations are satisfied:

$$(\nabla R)^2 = \frac{-bX}{a^2 - bc},$$  \hspace{1cm} (19)

$$\nabla R \cdot R \nabla \Phi = \frac{aX}{a^2 - bc},$$  \hspace{1cm} (20)

$$(R \nabla \Phi)^2 = \frac{-cX}{a^2 - bc},$$  \hspace{1cm} (21)

$$\nu_\Phi + R \Sigma \nu_\zeta = e^{iB}\Phi (\mu_R + \Xi \mu_\zeta).$$  \hspace{1cm} (22)

We also obtain the inequalities $b, c \geq 0$, $a^2 - bc \leq 0$. On the basis of (22), it was shown that $a, b$ and $c$ should be determined by the following PDEs:

$$\begin{align*}
\left( \frac{d_1}{d_2} \right) (\ln \Gamma) &= \frac{1}{G} \begin{pmatrix} -(c + Y) & a \\ -Ra & R(b + Y) \end{pmatrix} \begin{pmatrix} E \\ F \end{pmatrix},
\end{align*}$$

(23)

where

$$E = (B - 1)(c + Y) + D_1(b + Y) - D_2(a),$$  \hspace{1cm} (24)

$$F = (1 - B)a + D_1(a) - D_2(c + Y),$$  \hspace{1cm} (25)

$$G = R[a^2 - (b + Y)(c + Y)],$$  \hspace{1cm} (26)

$$d_1 = \frac{\partial}{\partial R} + \Xi \frac{\partial}{\partial \zeta}; \quad d_2 = \frac{\partial}{\partial \Phi} + R \Sigma \frac{\partial}{\partial \zeta},$$  \hspace{1cm} (27)

$$D_1 = Rd_1 + 1 + R\Xi \zeta; \quad D_2 = d_2 + R\Sigma \zeta.$$  \hspace{1cm} (28)

After we find $a, b$ and $c$ satisfying the relation (23), we are left with the first order partial differential equations of the following form:

$$\begin{align*}
(\nabla R)^2 &= S(R, \Phi, \zeta),
\nabla R \cdot R \nabla \Phi &= T(R, \Phi, \zeta),
(\nabla \Phi)^2 &= U(R, \Phi, \zeta),
\end{align*}$$

(29)

where $S, T$ and $U$ are the functions fixed by (13), (20), (24).

In this paper, we consider the case with a given function $\zeta(x)$. The functions $\Gamma, \Xi$ and $\Sigma$ are now given by

$$\Gamma = \frac{(\nabla R \times \nabla \Phi) \cdot \nabla \zeta}{(R \nabla R \times \nabla \Phi)^2},$$  \hspace{1cm} (30)

$$\Xi = \frac{[\nabla \Phi \times (\nabla R \times \nabla \Phi)] \cdot \nabla \zeta}{(\nabla R \times \nabla \Phi)^2},$$  \hspace{1cm} (31)

$$\Sigma = \frac{[\nabla R \times (\nabla \Phi \times \nabla R)] \cdot \nabla \zeta}{R(\nabla R \times \nabla \Phi)^2}.$$  \hspace{1cm} (32)

In the following, we consider the scheme (28) supplemented with (30) (31) (32) to present the example illustrating how the scheme works.

### III. SOLITONIC SOLUTIONS

It can be seen that all the known exact Hopf solitons of the Nicole model\(^{11}\) defined by the Lagrangian $\mathcal{L}_N(x) = \frac{1}{2} x^2$ and the Aratyn-Ferreira Zimerman model\(^{11}\) defined by $\mathcal{L}_{AFZ}(x) = -l_0(x)^\frac{2}{3}$ are solutions of the first order PDEs of the type of (13) and (14).

We here consider the Paddeen model in the space

$$M = \{ x = (\rho, \varphi, z) | 2 \pi \geq z \geq 0, 2 \pi \geq \varphi \geq 0, \rho_0 \geq \rho \geq 0 \},$$

(33)

where $\rho, \varphi$ and $z$ are cylindrical coordinates. We impose the boundary condition

$$\begin{align*}
u_\rho(\rho, \varphi, 0) &= u(\rho, \varphi, 0),
\nu_\rho(\rho_0, 0) &= u(\rho_0, 2 \pi, z),
\nu_0(0, \varphi, z) &= 0, \quad |u(\rho_0, \varphi, z)| = \infty.
\end{align*}$$

(34)

We further assume

$$\begin{align*}
R &= g(\rho),
\Phi &= m \varphi + \ell(z), \quad \ell(0) = 0, \quad \ell(2\pi) = 2 \pi n,
\end{align*}$$

(35)

where $m$ and $n$ are non-vanishing integers. As for $\zeta$, we adopt a simple choice

$$\zeta = z.$$  \hspace{1cm} (36)

Anticipating that the nonlinearity of the differential equation for $g(\rho)$ may cause a singularity of $g(\rho)$ at a certain point $\rho_0$, we have introduced the restriction $\rho_0 \geq 0$. We have also restricted the range of $z$ so that it is an appropriate variable to describe the angular part $\Phi$ of $u(\rho, \varphi, z)$. In this case, we have

$$\Gamma = \frac{m \rho}{\sqrt{m^2 + (\ell')^2 \rho^2}}, \quad \Xi = 0, \quad \Sigma = \frac{\ell' \rho^2}{\sqrt{m^2 + (\ell')^2 \rho^2}} g,$$

(37)

and

$$a = 0, \quad b = \frac{\rho^2}{\sqrt{m^2 + (\ell')^2 \rho^2}} g (g^2 + 1), \quad c = \frac{1}{m^2 + (\ell')^2 \rho^2} (g f)^2,$$

(38)

with $g' = \frac{d_1}{d_0}$ and $\ell' = \frac{d_2}{d_0}$. Then, the relation among $\left( \frac{\partial}{\partial \rho}, \frac{\partial}{\partial \varphi}, \frac{\partial}{\partial z} \right)$ and $\left( \frac{\partial}{\partial R}, \frac{\partial}{\partial \Phi}, \frac{\partial}{\partial \zeta} \right)$ is given by

$$\begin{align*}
\left( \frac{\partial}{\partial R} \right) &= \begin{pmatrix} g' & 0 & 0 \\ 0 & m & 0 \\ 0 & \ell' & 1 \end{pmatrix} \left( \frac{\partial}{\partial \rho} \right),
\end{align*}$$

(39)

or

$$\begin{align*}
\left( \frac{\partial}{\partial R} \right) &= \begin{pmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & \frac{1}{m} & 1 \end{pmatrix} \left( \frac{\partial}{\partial \rho} \right),
\end{align*}$$

(40)
From (23), we have
\[
\frac{\partial}{\partial R} \ln \left( \frac{R(b+Y)}{\Gamma} \right) = \frac{1}{1 + R^2} \frac{c + Y}{R(b+Y)},
\]
\[
\left( \frac{\partial}{\partial \Phi} + R\Sigma \frac{\partial}{\partial \zeta} \right) \ln \left( \frac{c + Y}{\Gamma} \right) = -R\Sigma \zeta.
\]
Noting that \( \Gamma \) and \( \Sigma \) are independent of \( \varphi \), we obtain
\[
\frac{\partial}{\partial z} \left( \frac{\Sigma(c + Y)}{\Gamma} \right) = 0
\]
from (1) and (40). This equality is realized only in the case
\[
\ell(z) = nz,
\]
from which we obtain \( \Phi = m\varphi + nz \).

In this case, (11) becomes the equation to determine \( g(\rho) \):
\[
\frac{d}{d\rho} \left\{ \frac{2[m^2 + n^2\rho^2]g^2 + \rho^2(g^2 + 1)^2|g'|}{\rho(g^2 + 1)^3} \right\} = \frac{(m^2 + n^2\rho^2)g(1 - g^2)(g^2 + 1)^2 + 2g^2}{\rho(g^2 + 1)^4}.
\]
Through the leading order analysis of (15), we see that the allowed behaviors of \( g(\rho) \) near \( \rho = 0 \) and \( \rho = \rho_0 \) are given by
\[
g(\rho) \sim \text{const}, \rho^{|m|} \quad \text{or} \quad g(\rho) \sim \text{const}, \rho^{-|m|} : \rho \sim 0,
\]
\[
g(\rho) \sim \text{const}, \rho_0^{-|m|} \quad \text{or} \quad g(\rho) \sim \text{const}, (\rho_0 - \rho) : \rho \lesssim \rho_0.
\]
If there exists a solution satisfying \( g(\rho) \sim \text{const}, \rho^{|m|} \) near \( \rho = 0 \) and \( \rho \sim 0 \) and \( g(\rho) \sim \text{const}, \rho_0^{-|m|} : \rho \gtrsim \rho_0 \), it is the desired one satisfying the boundary condition (34). The parameter \( \rho_0 \) denotes the moving singularity of the solution of the nonlinear ODE (15). Its value depends on the input, e.g. \( g'(0) \).

In accordance with the behavior \( g(\rho) \sim \text{const}, \rho^{|m|} \) near \( \rho = 0 \), we assume \( g(0) = 0 \), \( g'(0) = 1 \) for the case \( m = 1 \) and \( g(0) = g'(0) = 0 \) for \( m = 2, 3, \ldots \). We show in Fig.1 three examples of numerical estimation of \( g(\rho) \) for the cases \( (m, n) = (1, 1), (2, 1), (2, 2) \). The values of \( \rho_0 \) in the three cases are given by 2.34, 0.47 and 0.49, respectively. Conversely, if we fix \( \rho_0 \), the value of \( g'(0) \) is determined by \( m \) and \( n \).

We note that there exist cases in which we cannot obtain finite and positive \( \rho_0 \). In Fig.2, we show the behavior of \( g(\rho) \) for the cases \( (m, n) = (1, 2), (1, 3) \) with \( g(0) = 0 \), \( g'(0) = 1 \). We see that \( g(\rho) \) in these cases is finite for any positive \( \rho \).

**IV. SIMILARITY TO HOPF SOLITON**

We now mention on the degree of mapping for \( n : M \to S^2 \). We hereafter consider only the case that the conditions \( g(0) = 0 \) and \( g(\rho_0) = \infty \) are satisfied. We first define \( \Phi_\alpha(x) (\alpha = 1, 2, 3, 4) \) and \( Z(x) \) by
\[
Z(x) = \begin{pmatrix} Z_1(x) \\ Z_2(x) \end{pmatrix},
\]
\[
Z_1(x) = \Phi_1(x) + i\Phi_2(x) = \frac{g(\rho)}{\sqrt{1 + g(\rho)^2}} e^{-inz},
\]
\[
Z_2(x) = \Phi_3(x) + i\Phi_4(x) = \frac{1}{\sqrt{1 + g(\rho)^2}} e^{inz}.
\]
classifying the mapping $S^3 \rightarrow S^2$, by
\[
Q = \frac{1}{16\pi^2} \int_M dV \mathbf{A}(x) \cdot \mathbf{B}(x)
\]
\[
= \frac{1}{12\pi^2} \int_M dV \epsilon_{\alpha\beta\gamma\delta} \Phi_\alpha \frac{\partial(\Phi_\beta, \Phi_\gamma, \Phi_\delta)}{\partial(x, y, z)},
\]  
(53)
\[B_i(x) = \frac{1}{2} \epsilon_{ijk} \partial_j A_k(x) - \partial_k A_j(x),
\]  
(54)
\[A_i(x) = \frac{1}{i} \{Z^\dagger(x)[\partial_i Z(x)] - [\partial_i Z^\dagger(x)]Z(x)\},
\]  
(55)
where $\epsilon_{\alpha\beta\gamma\delta}$ is the four-dimensional Levi-Civita symbol satisfying $\epsilon_{1234} = 1$. From (55), it is straightforward to obtain
\[
Q = -mn\left(\frac{1}{|g(\rho_0)|^2 + 1} - \frac{1}{|g(0)|^2 + 1}\right) = mn.
\]  
(56)
In the conventional discussion of Hopf solitons, $n$ satisfying $n(x) = \text{const.}$ for $|x| = \infty$ is regarded as a field defined in the three-dimensional Euclidean space with the points at $|x| = \infty$ identified, which is isomorphic to $S^3$. In our case, recalling the conditions $u(\rho, \varphi, 0) = u(\rho, \varphi, 2\pi)$, $u(\rho, 0, z) = u(\rho, 2\pi, z)$ and $g(\rho_0) = \infty$, our $n$ may be regarded as a field on a solid torus with the points on the surface $\rho = \rho_0$ identified, which we denote by $M$. Thus, we have seen that the mapping $n : M \rightarrow S^2$ can be classified similarly to the mapping $n : S^3 \rightarrow S^2$.

V. SUMMARY

We considered an application of the equation for the 3-component vector $\alpha = q^\ast - q^\ast \times (q \times q^\ast)$, which is equivalent to the static field equation of the Faddeev model. Under some special assumptions, the equation was reduced to a nonlinear ODE of second order. Solutions of this equation was investigated numerically. The similarity to the case of Hopf soliton was discussed.

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