DESTA: A Framework for Safe Reinforcement Learning with Markov Games of Intervention

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Abstract

Reinforcement learning (RL) involves performing exploratory actions in an unknown system. This can place a learning agent in dangerous and potentially catastrophic system states. Current approaches for tackling safe learning in RL simultaneously trade-off safe exploration and task fulfillment. In this paper, we introduce a new generation of RL solvers that learn to minimise safety violations while maximising the task reward to the extent that can be tolerated by the safe policy. Our approach introduces a novel two-player framework for safe RL called Distributive Exploration Safety Training Algorithm (DESTA). The core of DESTA is a game between two adaptive agents: Safety Agent that is delegated the task of minimising safety violations and Task Agent whose goal is to maximise the environment reward. Specifically, Safety Agent can selectively take control of the system at any given point to prevent safety violations while Task Agent is free to execute its policy at any other states. This framework enables Safety Agent to learn to take actions at certain states that minimise future safety violations, both during training and testing time, while Task Agent performs actions that maximise the task performance everywhere else. Theoretically, we prove that DESTA converges to stable points enabling safety violations of pretrained policies to be minimised. Empirically, we show DESTA’s ability to augment the safety of existing policies and secondly, construct safe RL policies when the Task Agent and Safety Agent are trained concurrently. We demonstrate DESTA’s superior performance against leading RL methods in Lunar Lander and Frozen Lake from OpenAI gym.

1 Introduction

Reinforcement learning (RL) is a framework that enables autonomous agents to learn complex behaviours from interactions with the environment in domains such as robotics and video games. During its training phase, an RL agent explores using a trial and error approach to determine the best actions. This process can lead to the agent selecting actions that in some states may result in critical damage. For example, an aerial robot attempting to fly at high velocities can result in the helicopter crashing and subsequent permanent system failure. Consequently, a major challenge in RL is to produce methods that solve the task and ensure safety both during and after training.

A common approach to tackle the problem of safe exploration in RL is to use a constrained Markov decision process (c-MDP) formulation. In this framework, the agent seeks to maximise a single objective subject to various safety constraints. Although c-MDP can be solved if the model is known, extending this formalism to settings in which the model is unknown remains a challenge. One of the main tools for tackling the c-MDP problem setting in RL is the Lagrangian approach for solving a constrained problem, that is, solving \( \max_\theta \min_\lambda f(\theta) - \lambda g(\theta) \) by gradient descent in \( \lambda \) and ascent in \( \theta \). In contrast, to ensure their safety, animals exhibit a vast number of safety reflexes: reactive intervention systems designed to override and assume control in dangerous situations to prevent injury. In this way, the procedures to maintain safety are managed by isolated systems. One such example is the diving reflex, a sequence of physiological responses to the threat of oxygen deprivation (asphyxiation) that overrides the body’s basic behavioural and regulatory (homeostatic) systems.
Inspired by naturally occurring systems, in this paper, we tackle the challenge of learning safely in RL with a new two-agent framework for safe exploration and learning, DESTA. The framework entails an interdependent interaction between an agent, Task Agent whose policy maximises the set of environment rewards and an RL agent, Safety Agent whose goal is to ensure the safety of the system. The Safety Agent has the authority to override the Task Agent and assume control of the system and, using a deterministic policy to avoid unsafe states while Task Agent can perform its actions everywhere else to maximise rewards. Transforming these components into a workable framework requires a formalism known as Markov games (MGs). To bridge the gap between the interventionist approach to safety solutions and RL machinery, we develop a new type of MG namely a nonzero-sum MG of interventions. In this MG, two agents exchange control of the system to minimise safety violations while maximising the task objective. Our framework confers several key advantages:

1. Decoupled Objectives & Safety Planning. The tasks of maximising the environment reward and minimising safety violations are fully decoupled. This means Safety Agent pursues its safety objectives without trading-off safety for environment rewards. DESTA has a nested sequential structure in which the Safety Agent first observes the Task Agent’s proposed action. Since the Safety Agent performs safe planning, it prevents the Task Agent from performing unsafe actions and anticipates future actions to minimise future safety violations states after and during training (see Experiment 1).

2. Safety Enhancement Tool. An important use case for DESTA is an enhancement tool for pretrained policies. This enables DESTA to transform unsafe pretrained policies that maybe especially equipped to solve specific tasks into safe policies. Since the Safety Agent makes selective interventions, DESTA can preserve the ability of the base algorithm to solve the task (see Sec. 4).

3. Selective (deterministic) interventions: Safety Agent acts only at states in which its action ensures the safety criteria. At states in which the safety criterion is irrelevant only actions that are relevant to the task are played. Moreover Safety Agent uses a deterministic policy which eliminates inadvertent actions that may lead to safety violations (see Lunar Lander experiment).

4. Plug & play. Unlike various approaches in which safety contingencies are manually embedded into the policy e.g. [38], DESTA accommodates any RL policy and various notions of safety.

We establish the following theoretical results that ensure DESTA’s convergence to stable solutions:

i) A Bellman equation for our game which is solved using a new Q learning variant (Theorem 2).

ii) The conditions for the Safety Agent to perform interventions are characterised by a simple ‘obstacle condition’ (Prop. 1) involving the Safety Agent’s value function and the agents’ policies.

iii) Our Q learning variant converges to a solution using (linear) function approximators (Theorem 3).

Related Work. Recent works in safe RL include assumptions from knowing the set of safe states and access to a safe policy [6, 30], knowing the environment model [18, 23], having access to the cost function [14], having a continuous safety cost function [16] and using reversibility as a criterion for safety [22]. Constrained Policy Optimization (CPO) [1] extends trust region policy optimisation (TRPO) [38] with the aim of ensuring that a feasible policy stays within the constraints in expectation. Similarly in the context of multi-agent RL [46], [26] develops MACPO by extending multi-agent TRPO [31] with safe constraints. However, the convergence of these methods is still challenging: the learning dynamics tend to oscillate [41], and the methodology does not readily accommodate general RL algorithms [12]. In [43], a reward-shaping approach is used to guide the learned policy to satisfy the constraints, however their approach provides no guarantees during the learning process. In [17], a safety layer is introduced that acts on top of an RL agent’s possibly unsafe actions to prevent safety violations though their framework does not deal with negative long-term consequences of an action. In [8], a similar framework to CPO is used with a sparse binary safety signal where the Q function is overestimated to provide tuneable safety. Recently, [41] investigated the oscillation issue from a dynamical system point of view and introduced a treatment by applying a PID controller on the dual variable. [4] proposed a Bayesian world model approach to safety, [52] extended “RL as inference” framework to safety. Finally, [11, 49] solved RL with constraints imposed almost surely.

To combat the limitations of the c-MDP formulation, several methods transform the original constraint to a more conservative one to ease the problem. For example [21] replace the constraint cost with a conservative step-wise surrogate constraint. A significant drawback of these approaches is their conservativeness undermines task performance (the extent of the sub-optimality has yet to be characterised). Other approaches manually embed engineered safety-responses that are executed near safety-violating regions [22, 45]. For example, in [45], a safe teacher-student framework in which the teacher’s objective is the value of the student’s final policy, the agent is endowed with a pre-specified library of reset controls that it activates close to danger. These approaches can require time-consuming human input contrary to the goal of autonomous learning. They also do not allow the safety response to anticipate future the behaviour of the RL agent and do not perform safety planning.
Safety Agent switched off allowing the Safety Agent stochastic policies. Therefore, whenever the respective policies are concerned. Importantly, the set \( \pi \) policies of safety violations since \( v \) function:

\[
P(d_i) \in \text{system's dynamics,} \quad \text{maximise the environment reward everywhere else. Different to the c-MDP formulation, the goal of minimising frequency of failure in each training episode as small as possible.}
\]

\[
\text{violations (or catastrophic failures) during training in an unknown system} \quad \text{here, the aim is to keep the}
\]

\[
\text{indicator function i.e. takes values} \quad \text{is to find a policy} \quad \text{the accumulative safety costs can be represented using hard constraints, this captures for example avoiding subregions} \quad \text{Safe exploration in RL seeks to address the challenge of learning an optimal policy for a task while minimising the occurrence of safety violations (or catastrophic failures) during training in an unknown system.}
\]

Safety in RL. A key concern for RL in control and robotics settings is the idea of safety. This is handled in two main ways [25]: using prior knowledge of safe states to constrain the policy during learning or modifying the objective to incorporate appropriate penalties or safety constraints. The constrained MDP (c-MDP) framework [3] is a central formalism for tackling safety within RL. This involves maximising reward while maintaining costs within certain bounds which restricts the set of allowable policies for the MDP. Formally, a c-MDP consists of an MDP \( \langle S, A, P, R, \gamma \rangle \) and \( C = \{(L_i : S \times A \to \mathbb{R}, d_i \in \mathbb{R})| i = 1, 2, \ldots n\} \), which is a set of safety constraint functions \( L := (L_1, \ldots, L_n) \) that the agent must satisfy and \( \{d_i\} \) which describe the extent to which the constraints are allowed to be not satisfied. Given a set of allowed policies \( \Pi_C := \{\pi \in \Pi : v_{L_i}^\pi \leq d_i, \forall i = 1, \ldots, n\} \) where \( v_{L_i}^\pi(s) = \mathbb{E}[\sum_{t=0}^{\infty} \gamma^t L_i(s_t, a_t)|s_0 = s] = \mathbb{P}(\text{violation}) \), Safe exploration in RL seeks to address the challenge of learning an optimal policy for a task while minimising the occurrence of safety violations (or catastrophic failures) during training in an unknown system [27]. Here, the aim is to keep the frequency of failure in each training episode as small as possible.

Markov games. Our framework involves a system of two agents each with their individual objectives. Settings of this kind are formalised by MGs which model self-interested agents that act over time [20]. In the standard MG setup, the actions of both agents influence both each agent’s rewards and the system dynamics. Therefore, each agent \( i \in \{1, 2\} \) has its own reward function \( R_i : S \times (\times_{i=1}^2 A_i) \to \mathbb{R} \) and action set \( A_i \) and its goal is to maximise its own expected returns. The system dynamics, influenced by both agents, are described by a probability kernel \( P : S \times (\times_{i=1}^2 A_i) \times S \to [0, 1] \).

3 Our Framework

We now describe our framework which consists of two core components: firstly a game between two agents, the Task Agent and a second agent, the Safety Agent and, an impulse control component which is used by the Safety Agent. As we later explain, the impulse control component allows the Safety Agent to be selective about the set of states that it assumes control (and in doing so influence the transition dynamics and reward). With this, actions geared towards safety concerns are performed only at relevant states. This leaves the Task Agent to maximise the environment reward everywhere else. Different to the c-MDP formulation, the goal of minimising safety violations and maximising the task reward are now delegated to two individual agents that now have distinct objectives. Unlike classical MGs, using a form of control known as impulse control [33, 35, 47], one of the agents, Safety Agent does not intervene at each state but assumes control at states that it decides.

Our framework is modelled by an MG \( G = (N, S, A, A^2_{\text{safe}}, P, R_1, R_2, \gamma) \) where \( N = \{\text{Safety Agent, Task Agent}\} \), \( A^2_{\text{safe}} \subseteq A \) is the action set for the Safety Agent and \( R_i : S \times A \times A^2_{\text{safe}} \to \mathbb{R} \) are the one-step reward functions for agent \( i \in \{1, 2\} \) which we will define shortly. The transition probability \( P : S \times A \times A^2_{\text{safe}} \times S \to [0, 1] \) takes the state and action of both agents as inputs. The Task Agent and the Safety Agent use the Markov policies \( \pi : S \times A \to [0, 1] \) and \( \pi^2_{\text{safe}} : S \times A^2_{\text{safe}} \to [0, 1] \) which are contained in the sets \( \Pi \) and \( \Pi_{2_{\text{safe}}} \subseteq \Pi \) respectively. Importantly, the set \( \Pi_{2_{\text{safe}}} \subseteq \Pi \) consists of deterministic policies whereas the set \( \Pi \) consists of stochastic policies. Therefore, whenever the Safety Agent assumes control, random exploratory actions are switched off allowing the Safety Agent to exercise precise actions to avoid unsafe states. Lastly, the Safety Agent also has a policy \( g_2 : S \to \{0, 1\} \) which it uses to determine whether or not it should intervene.

In this setup, whenever the Safety Agent decides to act, the transition dynamics are affected by only the
Safety Agent while the Task Agent is allowed to affect the dynamics at all other times. Therefore the system transitions according to the probability kernel $P : S \times \mathcal{A} \times \mathcal{A}^2_{safe} \times S \rightarrow [0, 1]$ given by:

$$P(s', a, a^2_{safe}, s) = P(s', a, s) \left(1 - 1_{\mathcal{A}^2_{safe}}(a^2_{safe})\right) + P(s', a^2_{safe}, s) 1_{\mathcal{A}^2_{safe}}(a^2_{safe}),$$

where $1_Y(y)$ is the indicator function which is 1 whenever $y \in Y$ and 0 otherwise.

The Task Agent Objective is to maximise its expected cumulative reward set by the environment (note that this does not include safety which is delegated to the Safety Agent). To construct the objective for the Task Agent, we begin by defining the function $R_1(s_t, a_t, a^2_{safe}) = R(s_t, a_t) (1 - 1_{\mathcal{A}^2_{safe}}(a^2_{safe})) + R(s, a^2_{safe}) 1_{\mathcal{A}^2_{safe}}(a^2_{safe})$. The Task Agent seeks to maximise the following:

$$v_1(\pi(\pi^2_{safe}, g_2))(s) = \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t R_1(s_t, a_t, a^2_{safe}) \mid s_0 \equiv s \right],$$

where $a_t \sim \pi(\cdot | s_t)$ is Task Agent’s action and $a^2_{safe} \sim \pi^2_{safe}(\cdot | s_t)$ is an action chosen by the Safety Agent. Therefore, the reward received by Task Agent is $R(s_t, a^2_{safe})$ whenever the Task Agent decides to take an action and $R(s_t, a_t)$ otherwise. The Task Agent can be either a pretrained policy (which can be trained using RL or some other method) or a policy which is trained concurrently with the Safety Agent within the DESTA framework.

The Safety Agent Objective is to minimise safety violations during and after training. Unlike the Task Agent, each time the Safety Agent performs an action, it incurs a cost. This ensures any interventions by the Safety Agent are warranted by an increase in safety and the Safety Agent is selective about when it acts. The Safety Agent’s objective which it aims to maximise is:

$$v_2(\pi(\pi^2_{safe}, g_2))(s) = \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t (-\bar{L}(s_t, a_t, a^2_{safe}) - c1_{\mathcal{A}^2_{safe}}(a^2_{safe})) \mid \right].$$

where $c \in \mathbb{R}$ is a fixed positive constant and the function $\bar{L}$, which is provided by the environment, is defined by $\bar{L}(s_t, a_t, a^2_{safe}) = L(s_t, a_t)(1 - 1_{\mathcal{A}^2_{safe}}(a^2_{safe})) + L(s, a^2_{safe}) 1_{\mathcal{A}^2_{safe}}(a^2_{safe})$ and $\mathbb{L} : = (L_1, \ldots, L_n)$ is a set of constraint functions that indicate how much a given constraint has been violated. Each function $L_i$ can represent a (possibly binary) signal that indicates a visitation to an unsafe state. Therefore to maximise its objective, the Safety Agent must determine the sequence of points at which the benefit of performing a precise action overcomes the cost of doing so.

We now mention some important features. The task of ensuring safety is delegated to the Safety agent whose sole objective is minimise safety violations. The intervention cost it pays for its actions induces selectivity about where it intervenes that is, it does so only at states at which intervening leads to an appreciable reduction in expected total safety violations. At all other states (where the Safety agent sees no potential for safety violations), the task agent is free to play actions that deliver task rewards. A key aspect of DESTA is the safety agent gets to observe the state the task agent would take before it is allowed to do so. Since the Safety agent acts in response to the Task Agent, the Safety agent learns to anticipate the future joint actions and do planning for maintaining safety.

The Safety Agent Impulse Control Mechanism

The problem for the Safety Agent is to determine at which states it should assume control and which actions to take. We now describe how at a given state, using an impulse control policy [33, 36], how Safety Agent decides whether to override the Task Agent and its choice of intervention. At each state, the Safety Agent first makes a binary decision to decide to assume control. An important feature of DESTA is the sequential decision process. The policies $\pi$ and $\pi^2_{safe}$ first propose actions $a$ and $a^2_{safe}$ (resp.), which are observed by the policy $g_2$. The role of $g_2$ is to switch to the action suggested by $\pi^2_{safe}$ whenever the Task Agent’s action may incur a safety violation (at present or in future). Unlike [22, 43], our approach enables learning a safe intervention policy during training without the need to preprogram manually engineered safety responses and minimises safety violations during training unlike [17, 43]. Unlike [14, 18, 23, 24], which require access to information which is not available without a priori knowledge of the environment, our framework does not require a priori knowledge of the model of the environment or the unsafe states.

Denote by $\{\tau_k\}_{k \geq 0}$ the intervention times when the Safety Agent decides to take an action e.g. if the Safety Agent chooses first to intervene at state $s_0$ and again at $s_8$, then $\tau_1 = 6$ and $\tau_2 = 8$. Since $\tau_k = \inf \{t > \tau_{k-1} | s_t \in T, g_2(s_t) = 1\} \{\tau_k\}$ are rules that depend on the state where $T$ is the trajectory induced by the joint actions and kernel $P$. By learning an optimal $g_2$, the Safety Agent learns the optimal states to perform an
intervention. As we later explain, these intervention times are determined by an easy to evaluate condition on the state and $\pi, \pi^{2,\text{safe}}$ proposals (see Prop. [1]).

Learning to solve our system involves finding a solution in which the Safety Agent learns to assume control at a subset of states and minimise safety violations given the Task Agent’s policy while at all other states the Task Agent executes actions intended to maximise its task objective. In Sec. [5] we prove the existence of a stable point of our MG and the convergence of our learning method to the solution. As we later show, the learning process for converges under our variant Q-learning method.

4 Use Cases for DESTA

DESTA as a Safety Enhancement Tool. A core usage for DESTA is to act as a framework for enhancing the safety of existing (pretrained) (Markov) policies. Used in this way, DESTA is agnostic to the training procedure that generated the underlying policy. With this DESTA can be used to transform unsafe pretrained or expert policies that may be specially equipped to solve specific tasks into safe policies by using the Safety Agent to make selective interventions to maintain safety at present and future states while preserving the ability of the base algorithm to apply actions elsewhere. This confers a benefit of DESTA which is unlike various approaches in which safety contingencies are manually embedded into the policy e.g. [38]. DESTA permits easily plugging in any combination of RL policies e.g. model-free or model-based methods and, various notions of safety and generic expert policies. We demonstrate this in Sec. [6] by incorporating (a) a Dijkstra-based expert policy [29] as the Task Agent to exploit domain knowledge in a graph environment, while using the DESTA framework to handle the safety aspect and (b) a pretrained unsafe RL policy as the Task Agent which we make safe while preserving its ability to solve the task.

DESTA as a Safe RL Framework. DESTA can be used as a framework for performing Safe RL. In this case, DESTA is used to train both a Task Agent and the Safety Agent concurrently. As we demonstrate in Sec. [6] this leads to a policy which minimises safety violations both during and after training (through the interventions of the Safety Agent) while performing the underlying RL task.

Implementing both the safety policy and the task policy can be done using any off-the-shelf policy gradient methods without modifications. To summarise, we will learn three policies: the first learning the actions of the task agent, the second learning the action of the method without modifications. To summarise, we will learn three policies: the first one learning the actions of the task agent, the second learning the action of the methods without modifications. Since we learn off-policy every agent can collect the same triplets $s, a, a_{t+1}$, where $a$ is the current action (i.e., either the task action $a_t$ or the safety action $a_t^{2,\text{safe}}$). The rewards that the policies receive differ. The task policy receives $R_1 = R(s_t, a_t)$, the safety policy gets $R_2 = -L(s_t, a_t) - c_1 A_{2,\text{safe}}(a_t^{2,\text{safe}})$, and the intervention policy rewards are $R^{\text{int}}_t = -L(s_t, a_t) - c_1 A_{2,\text{safe}}(a_t^{2,\text{safe}})$. Note we chose to use the same triplets for all policies since we use off-policy algorithms, leading to efficient usage of the acquired data.

5 Theoretical Analysis of DESTA

A key aspect of our framework is the presence of multiple RL processes that make decisions in a sequential order. In order to determine when to intervene, using the policy $g$ the Safety Agent must learn the states to allow the policy $\pi^{2,\text{safe}}$ to perform a safe action which the policy $\pi^{2,\text{safe}}$ must learn to select safe actions whenever it is allowed to execute an action. At a stable point of the learning processes the Safety Agent minimises safety violations while Task Agent maximises the environment reward. Additionally, Safety Agent learns the set of states in which to perform an action to maintain safety at the current or future states. In this section, we prove that DESTA converges to an optimal solution of the system. Central to DESTA is a Q-learning type method which is adapted to handle RL settings in which the the Safety Agent must also learn the optimal intervention criteria for the times $\{r_k\}$. We then extend the result to allow for (linear) function approximators. We provide a result that shows the optimal intervention times are characterised by an ‘obstacle condition’ which can be evaluated online therefore allowing the $g_2$ to be learned online.

We begin by defining a key object in preparation for our main results. Given a function $Q^\pi, (\pi^{2,\text{safe}}, g_2) : S \times A \to \mathbb{R}$, $\forall \pi \in \Pi, \forall g_2, \forall s_{r_k} \in S$, the intervention operator $M^{(\pi^{2,\text{safe}}, g_2)}$ is given by $M^{(\pi^{2,\text{safe}}, g_2)}Q^\pi, (\pi^{2,\text{safe}}, g_2)(s_{r_k}, a) := R(s_{r_k}, a_{r_k}) - c + y \int_S ds' P(s'; a_{r_k}, s) \pi^{(\pi^{2,\text{safe}}, g_2)}(s') a_{r_k} \sim \pi$, where $r_k$ is an intervention time. The quantity $MQ^{(\pi^{2,\text{safe}}, g_2)}$ measures the expected future stream of safety costs after an immediate intervention minus its
cost for intervening. We define the operator $T$, for all $s \in \mathcal{S}$, $\forall \pi \in \Pi$, $\forall \pi_{2,\text{safe}} \in \Pi_{2,\text{safe}}$ by:

$$T_{\pi_{2,\text{safe}}(g_2)}(s) := \max \left\{ \mathcal{M}(\pi_{2,\text{safe}},g_2)Q_{2,\text{safe}}(s,a), R(s,a) + \gamma \mathbb{E}_{s'} P(s'|a,s)Q_{2,\text{safe}}(s')|a_t \sim \pi(\cdot|s) \right\}$$

The Bellman operator $T$ captures the nested sequential structure of DESTA - the structure consists of an inner component with two terms: the first is the expected future stream of safety costs for the Safety Agent given it makes an intervention at the current state using its policy $\pi_{2,\text{safe}}$. The second term is the expected future stream of safety costs for the Safety Agent given no intervention and that the Task Agent executes the action $a_t \sim \pi(\cdot|s_t)$.

Theorem 1. Given any $v_2(\pi_{2,\text{safe}},g_2) : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$, the optimal value function $v_2$ is given by $\lim_{k \to \infty} T^{k} v_2(\pi_{2,\text{safe}},g_2) = \max_{\pi_{2,\text{safe}}(g_2)} v_2(\pi_{2,\text{safe}},g_2)$, $\forall \pi \in \Pi$.

The result of Theorem 1 enables the solution to the Safety Agent to be determined using a value iteration procedure. Moreover, Theorem 1 enables a Q-learning approach for finding the solution:

Theorem 2. Consider the following Q learning variant:

$$Q_{2,t}^\pi(\pi_{2,\text{safe}},g_2)(s_t,a_t) = Q_{2,t}^\pi(\pi_{2,\text{safe}},g_2)(s_t,a_t) - \alpha_t(s_t,a_t)Q_{2,t}^\pi(\pi_{2,\text{safe}},g_2)(s_t,a_t) + \alpha_t(s_t,a_t) \max_{a_t'} \left\{ \mathcal{M}(\pi_{2,\text{safe}},g_2)Q_{2,t}^\pi(\pi_{2,\text{safe}},g_2)(s_t,a') + \gamma Q_{2,t}^\pi(\pi_{2,\text{safe}},g_2)(s',a_t) \right\}$$

where $a_t \sim \pi(\cdot|s_t)$, then $Q_{2,t}(s)$ converges to $Q_2^\pi$ with probability 1 where $s_t, s_{t+1} \in \mathcal{S}$ and $a_t \in \mathcal{A}$.

We now extend the result to (linear) function approximators:

Theorem 3. Given a set of linearly independent basis functions $\Phi = \{\phi_1, \ldots, \phi_p\}$ with $\phi_k \in L_2$, $\forall k$. Algorithm 1 converges to a limit point $r^* \in \mathbb{R}^p$ which is the unique solution to $\Pi \Phi r^* = \Phi r^*$ where $\Pi \Lambda := \hat{R} + \gamma P \max \{M \Lambda, \Lambda\}$. Moreover, $r^*$ satisfies: $\|\Phi r^* - Q^\pi\| \leq (1 - \gamma^2)^{-1/2} \|\Pi Q - Q^\pi\|$. 

The theorem establishes the convergence of Algorithm 1 to a stable point with the use of linear function approximators. The second statement bounds the proximity of the convergence point by the smallest approximation error that can be achieved given the choice of basis functions. The following result characterises the Safety Agent policy $g_2$ and when the Safety Agent must intervene:

Algorithm 1: Distirbutive Exploration Safety Training Algorithm (DESTA)

```python
1: Inputs: Replay buffers $D_{2,\text{safe}} = \{\emptyset\}, D_{\text{int}} = \{\emptyset\}$, AGENT - base agent for policy learning
2: for $N_{\text{episodes}}$ do
3:   State $s_0$
4:   for $t = 0, 1, \ldots$ do
5:     Sample a task action $a_t \sim \pi(\cdot|s_t)$, a safe action $a_{t,\text{safe}}^2 \sim \pi(\cdot|s_t)$, and an intervention action $a_{t,\text{int}} \sim g_2(\cdot|s_t) \in \{0, 1\}$
6:     if $a_{t,\text{int}} = 0$ then
7:       Apply task action $a_t$ so $s_{t+1} \sim P(a_t, s_t)$. Set $a_t = a_{t,\text{safe}}$
8:     else if $a_{t,\text{int}} = 1$ then
9:       Apply safe action $a_{t,\text{safe}}$ so $s_{t+1} \sim P(a_{t,\text{safe}}, s_t)$. Set $a_t = a_{t,\text{safe}}$
10:    end if
11:   Receive reward $R(s_t,a)$ and cost $L(s_t,a)$
12:   Set $R_1 = R(s_t,a)$, $R_2 = -L(s_t,a) - c(a)$, $R_{\text{int}} = -L(s_t,a) - c(a)a_{t,\text{int}}$
13:   Add the sample $(s_t,a,s_{t+1},R_2)$ to $D_{2,\text{safe}}$, the sample $(s_t,a_{t,\text{int}},s_{t+1},R_{\text{int}})$ to $D_{\text{int}}$
14: end for
15: // Learn the individual policies
16: Update the policy $\pi_{2,\text{safe}}$ using $D_{2,\text{safe}}$ and the base agent AGENT, the policy $g_2$ using $D_{\text{int}}$ and the base agent AGENT.
17: end for
```
Proposition 1. The policy \( g_2 \) is given by: 
\[
g_2(s_t) = H(M(\pi^{s, \text{safe}}, g_2)Q_2 - Q_2)(s_t, a_t), \quad \text{where} \quad Q_2 = \pi^s(\gamma^{s, \text{safe}}, g_2)
\]

Prop. 1 characterises the (categorical) distribution \( g_2 \). The times \( \{\tau_k\} \) are determined by evaluating when \( MQ_2 = Q_2 \). The result yields a key aspect of DESTA for executing Safety agent’s activations.

While implementing the intervention policy appears to be straightforward by comparing value functions, it requires the optimal value functions in question. Furthermore, learning the intervention policy and these value functions simultaneously resulted in an unstable procedure. As a solution we propose to learn \( g_2 \) using an off-the-shelf policy gradient algorithm (such as TRPO, PPO or SAC). This policy is categorical with values \( \{0, 1\} \) and has a reward \( R^{\text{int}} \) equal to \(-L(s_t, a) + c(a)\) if the safety agent intervenes with an action \( a = a_t^{2, \text{safe}}\), and \(-L(s_t, a)\) if the safety agent does not intervene, i.e., the task agent applies an action \( a = a_t\).

6 Experiments

We performed a series of challenges to see if DESTA 1) learns to perform safe planning 2) learns to select the appropriate state to perform safe override interventions which avoids a trade-off between safety and task performance 3) learns to use deterministic controls to ensure precise actions. Here we We compared the performance of DESTA with leading RL methods for safe learning: SAC, PPO [39], Lagrangian PPO, TRPO [38], Lagrangian TRPO and CPO [1]. We then compared DESTA against these baselines on performance benchmarks in challenging high dimensional problems in Frozen lake and Lunar Lander from Open AI gym [9]. Further experiment details are in the Appendix.

Case I: Augmenting the Safety of a pretrained Task Agent policy.

We tested a method DESTA-SAC-EXPERT which uses a pretrained policy as the Task Agent that makes use of domain knowledge. In this case, we use the Dijkstra algorithm to find the next node on the shortest path from the current node to the goal, leaving the safety considerations to RL. Though this convincingly outperformed the baselines, we note that DESTA-SAC without domain knowledge yielded almost the same level of performance. Finally, we pretrained SAC on just the reward (i.e. an unsafe policy) and then plugged it in as the task agent to a training round of DESTA-SAC, resulting in a policy that solved the environment safely and quickly. These results validate the claim that DESTA is a flexible framework which augments the safety of both generic and specialised learning and control methods.

Case II: Learning the Task Agent policy and Safety Agent policy

Experiment 1. Safe Planning: Having a dedicated agent for safety enables our method to learn to plan ahead to minimise safety violations at future states. To demonstrate this, we designed a graph-structured environment with many routes to a shared goal (Shortest Safe Route). The agent must go from the start node to the goal node whilst avoiding the unsafe zone, where there will be a cost on each node with some (constant) probability. There is a guaranteed reward at the goal. The agent cannot go backwards so the agent’s decision commits it to that path (invalid actions leave the agent unmoved). Moreover, there is a constant negative reward at each step. These features require the agent plan its route and choose the shortest route that does not incur a cost (the shortest path traverses the unsafe zone so the agent is required to make a trade-off). We compared DESTA-SAC to PPO, Lagrangian PPO, TRPO, Lagrangian TRPO, SAC and CPO, where the (scaled) cost was deducted from the reward for PPO, TRPO and SAC. As shown in Fig. 2, DESTA-SAC successfully learned the shortest safe route to the goal acquiring the maximum safe reward (70) faster than all baselines. Other methods instead traversed unsafe zone due to the higher reward. The poor performance of SAC (the base learner of DESTA-SAC) proves DESTA is able to successfully augment safety.

Figure 1: Environments and tasks: Shortest Safe Route: traverse a one-way system. Lunar lander: Land on the pad between two flags. Frozen Lake: Reach the goal while avoiding dangers.
Figure 2: **Top row:** 5 seeds evaluation of DESTA-SAC and baselines on Shortest Safe Route environment. (a) Average return of evaluation episodes. 70 is the maximum safe return per episode, achieved by following the green route. Goal reward is 100 and per-step reward is -5. Cost value is 100 and cost prob in unsafe zone is 0.5. (b) Average cost of evaluation episodes. **Bottom row:** Comparison of a pretrained unsafe SAC policy instance (SAC-UNSAFE) with the resulting safe DESTA policy (SAC-PRETRAINED) (c) Evaluation return (d) Evaluation cost

Experiment 2. Safe precision control using the Lunar Lander [9]: Since the Safety Agent uses deterministic controls to perform its actions, we claim DESTA is able to perform precise actions to ensure the safety of the system. To verify this claim, we tested DESTA’s performance in the Lunar Lander environment in OpenAI gym [9]. As there is no strict definition of safety violation this environment [9], we defined a safety violation to be whenever the spacecraft transitions outside a fixed horizontal threshold radius from the origin. By introducing the safety definition, our goal was to test if DESTA can override actions that aim to exploit rewards or explore in instances when such behaviour can incur safety violations while ensuring the spacecraft is always on a near-optimal path.

In Fig. 3(a) we observe DESTA outperforms all the baselines in terms of evaluation costs and the overall score. We observe DESTA enables more stable training. In Fig. 3(b), we again observe that DESTA yields the lowest cumulative cost among all the evaluated methods while maintaining the stability of the costs across different random seeds. This indicates the Safety Agent has learned to better avoid the safety-violation states in comparison to the baseline methods. Fig. 3(c) shows DESTA maintains the lowest cost rate throughout the training process. In this experiment, 32 evaluation episodes were run every 1000 steps and 5 seeds were used per algorithm. We note the consistent low variance across various independent training runs indicates that DESTA enables low sensitivity with respect to the randomness given different random seeds which is an issue for some RL algorithms [15].
Experiment 3. Frozen lake: In this grid world environment (depicted in Fig. 1(c)) the agent’s aim is to arrive at a goal as quickly as possible while avoiding unsafe and danger states. The agent receives a penalty of $-1$ for each visit to an unsafe state and a safety cost of 10 for each visit to a dangerous state and a reward of 100 for reaching the goal state at which point the episode terminates. The agent reward signal is observed with noise so the agent has only a 70% probability of getting the reward on that grid. In this setup, there are multiple paths from the initial state to the goal — behaving deterministically enables that agent to ensure its safety by avoiding safety violations associated with random exploration. For methods that do not decouple rewards and safety the probabilistic noise of the reward can hinder this process. Conversely, for DESTA, control of the system is transferred to safe policy which is is concerned only by safety and therefore is not affected by such noise. This allows DESTA to avoid safety violations in instances where precision is required. In this setting, at the initial state, although the both go down or right can reach the goal with the same reward, the agent will fall into a safety disaster when going down. Visualising the interventions (shown in Fig. 4), we observe that DESTA intervenes to encourage the agent to take immediately move right direction at the first step. Additionally, as depicted in the heatmap, DESTA becomes more active at upper left region where it intervenes to protect the agent from falling into dangerous states which may occur due to random exploration. As shown in Fig. 5, DESTA solves the problem faster and than other algorithms, does so with greater stability and produces the lowest safety cost.

7 Conclusion

We presented a novel two-player Markov game (MG) framework for solving the problem of learning safely. Our MG framework decouples the tasks of ensuring safety and maximising the task reward and assigns these tasks to a Task Agent and a Safety agent. This enables the Task Agent to learn complex behaviours to assume control at specific states to minimise safety violations and behave adaptively to the Task Agent whose policy is aimed at maximising the environment reward. By presenting a theoretically-grounded and high performing approach to the safe RL problem, our method opens up the applicability of RL to a range of real-world control problems with complex safety constraints.
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Part I
Appendix

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## 8 Hyperparameter Settings

Table 1: Hyperparameters for DESTA.

|                  | Shortest Safe Route | Lunar lander | Frozen Lake | DynamicEnv |
|------------------|---------------------|--------------|-------------|-------------|
| **Runner**       |                     |              |             |             |
| # gradient steps | 10                  | 10           | 8           | 8           |
| # environment steps | 70k               | 100k         | 600k        | 600k        |
| Agent frequency update | 0                | 0            | 4           | 4           |
| Agent batch size | 1024                | 1024         | 64          | 64          |
| **Agents**       |                     |              |             |             |
| Policy network dimensions | {0}             | {256, 256}  | {256, 256} | {256, 256} |
| Policy networks activations | ReLU          | ReLU         | ReLU        | ReLU        |
| Value network layer dims | {256, 256}   | {256, 256}  | {256, 256} | {256, 256} |
| Value networks activations | ReLU          | ReLU         | ReLU        | ReLU        |
| Discount factor  | 0.99                | 0.99         | 0.99        | 0.99        |
| Polyak update scale | 0                | 0            | 0.005       | 0.005       |
| Intervention cost | 5                 | 0.5          | 0.1         | 0.5         |
| **Optimiser**    |                     |              |             |             |
| Opt Algorithm    | Adam               | Adam         | Adam        | Adam        |
| Policy learning rate | $10^{-3}$    | $10^{-3}$    | $10^{-4}$   | $10^{-4}$   |
| Value function learning rate | $10^{-3}$    | $10^{-3}$    | $10^{-4}$   | $10^{-4}$   |
| Temperature learning rate | $10^{-3}$    | $10^{-3}$    | $10^{-4}$   | $10^{-4}$   |
9 Notation & Assumptions

We assume that $S$ is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and any $s \in S$ is measurable with respect to the Borel $\sigma$-algebra associated with $\mathbb{R}^p$. We denote the $\sigma$-algebra of events generated by $\{s_t\}_{t \geq 0}$ by $\mathcal{F}_t \subset \mathcal{F}$. In what follows, we denote by $(\mathcal{V}, ||||)$ any finite normed vector space and by $\mathcal{H}$ the set of all measurable functions.

The results of the paper are built under the following assumptions which are standard within RL and stochastic approximation methods:

**Assumption 1.** The stochastic process governing the system dynamics is ergodic, that is the process is stationary and every invariant random variable of $\{s_t\}_{t \geq 0}$ is equal to a constant with probability 1.

**Assumption 2.** The function $R$ is in $L_2$.

**Assumption 3.** For any positive scalar $c$, there exists a scalar $\mu_c$ such that for all $s \in S$ and for any $t \in \mathbb{N}$ we have: $E[1 + ||s_t||^c | s_0 = s] \leq \mu_c (1 + ||s||^c)$.

**Assumption 4.** There exists scalars $C_1$ and $c_1$ such that for any function $J$ satisfying $|v(s)| \leq C_2 (1 + ||s||^{c_2})$ for some scalars $c_2$ and $C_2$ we have that: $\sum_{t=0}^{\infty} |E[v(s_t)|s_0 = s] - E[v(s_0)]| \leq C_1 C_2 (1 + ||s||^{c_1 + c_2})$.

**Assumption 5.** There exists scalars $c$ and $C$ such that for any $s \in S$ we have that: $|R(s, \cdot)| \leq C (1 + ||s||^c)$.

10 Proof of Technical Results

We begin the analysis with some preliminary lemmata and definitions which are useful for proving the main results.

**Definition 1.** A.1 An operator $T : \mathcal{V} \to \mathcal{V}$ is a **contraction** w.r.t a norm $|| \cdot ||$ if $\exists \lambda \in [0, 1]$ such that for any $J_1, J_2 \in \mathcal{V}$ the following inequality holds: $||TJ_1 - TJ_2|| \leq \lambda ||J_1 - J_2||$.

**Definition 2.** A.2 An operator $T : \mathcal{V} \to \mathcal{V}$ is said to be **non-expansive** if for any $J_1, J_2 \in \mathcal{V}$ the following bound holds: $||TJ_1 - TJ_2|| \leq ||V_1 - V_2||$.

**Lemma 1.** For any maps $f : \mathcal{V} \to \mathbb{R}$, $g : \mathcal{V} \to \mathbb{R}$, the following expression holds:

$$\left|\max_{a \in \mathcal{V}} f(a) - \max_{a \in \mathcal{V}} g(a)\right| \leq \max_{a \in \mathcal{V}} \| f(a) - g(a) \|.$$

**Proof.** We restate the proof given in [34]:

$$f(a) \leq \| f(a) - g(a) \| + g(a) \quad \text{(5)}$$

$$\implies \max_{a \in \mathcal{V}} f(a) \leq \max_{a \in \mathcal{V}} \| f(a) - g(a) \| + g(a) \leq \max_{a \in \mathcal{V}} \| f(a) - g(a) \| + \max_{a \in \mathcal{V}} g(a). \quad \text{(6)}$$

Deducting $\max_{a \in \mathcal{V}} g(a)$ from both sides of (6) yields:

$$\max_{a \in \mathcal{V}} f(a) - \max_{a \in \mathcal{V}} g(a) \leq \max_{a \in \mathcal{V}} \| f(a) - g(a) \|. \quad \text{(7)}$$

After reversing the roles of $f$ and $g$ and redoing steps (5) - (6), we deduce the desired result since the RHS of (7) is unchanged. □

**Lemma 2.** A.4 The probability transition kernel $P$ is non-expansive, that is:

$$||PV_1 - PV_2|| \leq ||V_1 - V_2||.$$

**Proof.** The result is well-known e.g. [44]. We give a proof using the Tonelli-Fubini theorem and the iterated law of expectations, we have that:

$$\|PJ\|^2 = E[(PJ)^2|s_0] = \sum_{s_0} \left( \sum_{s_1} \left( \sum_{s_2} \left( \sum_{s_3} \left( \sum_{s_4} \ldots \right) \right) \right) \right) \leq E \left[ \sum_{s} \left( \sum_{s_1} \left( \sum_{s_2} \left( \sum_{s_3} \ldots \right) \right) \right) \right] = \sum_{s} \left( \sum_{s_1} \left( \sum_{s_2} \left( \sum_{s_3} \ldots \right) \right) \right) = ||J||^2,$$

where we have used Jensen’s inequality to generate the inequality. This completes the proof. □

Proof of Theorem\[1\]

Lemma 3. The Bellman operator $T$ is a contraction, that is the following bound holds:

$$\|Tv - Tv'\| \leq \gamma \|v - v'\|,$$

where $v, v'$ are elements of a finite normed vector space. Lemma[3] establishes the contraction property of the Bellman operator for the problem.

Proof. Recall we define the Bellman operator $T$ acting on a function $\Lambda : S \times \mathbb{N} \rightarrow \mathbb{R}$ by

$$T\Lambda(s_{\tau_k}) := \max \left\{ \mathcal{M}\Lambda(s_{\tau_k}), \left[ \mathcal{R}(s_{\tau_k}, a) + \gamma \max_{a \in A} \sum_{s' \in S} P(s'; a, s_{\tau_k})\Lambda(s', I(\tau_k)) \right] \right\} \tag{9}$$

For ease of exposition we use the following shorthands in the forthcoming proofs:

$$\mathcal{P}_{ss'}^a := \sum_{s' \in S} P(s'; a, s), \quad \mathcal{P}_{ss'}^\pi := \sum_{a \in A} \pi(a|s)\mathcal{P}_{ss'}^a, \quad \mathcal{R}_\pi(s_t) := \sum_{a_t \in A} \pi(a_t|s)R(s_t, a_t)$$

To prove that $T$ is a contraction, we consider the three cases produced by (9), that is to say we prove the following statements:

i)  $\left| \mathcal{R}(s_t, a_t) + \gamma \max_{a \in A} \mathcal{P}_{ss't}^a \psi(s', \cdot) - \left( \mathcal{R}(s_t, a_t) + \gamma \max_{a \in A} \mathcal{P}_{ss't}^a \psi(s', \cdot) \right) \right| \leq \gamma \|\psi - \psi'\|$

ii) $\|\mathcal{M}\pi \psi - \mathcal{M}\pi \psi'\| \leq \gamma \|\psi - \psi'\|$, (and hence $\mathcal{M}$ is a contraction).

iii) $\|\mathcal{M}\pi \psi - \left( \mathcal{R}(\cdot, a) + \gamma \max_{a \in A} \mathcal{P}_{ss'}^a \psi' \right)\| \leq \gamma \|\psi - \psi'\|$ ..

We begin by proving i).

Indeed, for any $a \in A$ and $\forall s_t \in S, \forall \theta_t, \theta_{t-1} \in \Theta, \forall s' \in S$ we have that

$$\left| \mathcal{R}(s_t, a_t) + \gamma \max_{a \in A} \mathcal{P}_{ss't}^a \psi(s', \cdot) - \left( \mathcal{R}(s_t, a_t) + \gamma \max_{a \in A} \mathcal{P}_{ss't}^a \psi(s', \cdot) \right) \right|$$

$$\leq \max_{a \in A} \left| \gamma \mathcal{P}_{ss't}^a \psi(s', \cdot) - \gamma \mathcal{P}_{ss't}^a \psi(s', \cdot) \right|$$

$$\leq \gamma \max |P\psi - P\psi'|$$

$$\leq \gamma \|\psi - \psi'\|,$$

again using the fact that $P$ is non-expansive and Lemma[4]

We now prove ii).

For any $\tau \in \mathcal{F}$, define by $\tau' = \inf \{ t > \tau | s_t \in A, \tau \in \mathcal{F}_t \}$. Now using the definition of $\mathcal{M}$ we have that for any $s_\tau \in S$

$$\|(\mathcal{M}\pi \psi - \mathcal{M}\pi \psi')(s_\tau)\|$$

$$\leq \max_{a_t \in A} \left| \mathcal{R}(s_\tau, a_\tau) + c(s_\tau, a_\tau) + \gamma \mathcal{P}_{ss't}^a \mathcal{P}_{ss't}^a \mathcal{P}_{ss't}^a \psi(s_\tau) - \left( \mathcal{R}(s_\tau, a_\tau) + c(s_\tau, a_\tau) + \gamma \mathcal{P}_{ss't}^a \mathcal{P}_{ss't}^a \psi(s_\tau) \right) \right|$$

$$= \gamma \left| \mathcal{P}_{ss't}^a \mathcal{P}_{ss't}^a \mathcal{P}_{ss't}^a \psi(s_\tau) - \mathcal{P}_{ss't}^a \mathcal{P}_{ss't}^a \psi(s_\tau) \right|$$

$$\leq \gamma \|P\psi - P\psi'\|$$

$$\leq \gamma \|\psi - \psi'\|,$$

using the fact that $P$ is non-expansive. The result can then be deduced easily by applying max on both sides.

We now prove iii). We split the proof of the statement into two cases:

Case 1:

$$\mathcal{M}\pi \psi(s_\tau) - \left( \mathcal{R}(s_\tau, a_\tau) + \gamma \max_{a \in A} \mathcal{P}_{ss't}^a \psi(s') \right) < 0. \tag{10}$$
We now observe the following:

\[
\mathcal{M}^\tau \psi(s_r) - \mathcal{R}(s_r, a_r) + \gamma \max_{a \in A} \mathcal{P}_{s_r}^a \psi'(s') \\
\leq \max \{ \mathcal{R}(s_r, a_r) + \gamma \mathcal{P}_{s_r}^a \psi(s'), \mathcal{M}^\tau \psi(s_r) \} - \mathcal{R}(s_r, a_r) + \gamma \max_{a \in A} \mathcal{P}_{s_r}^a \psi'(s')
\]

Case 2:

\[
\mathcal{M}^\tau \psi(s_r) - \left( \mathcal{R}(s_r, a_r) + \gamma \max_{a \in A} \mathcal{P}_{s_r}^a \psi'(s') \right) \geq 0.
\]

For this case, first recall that for any \( \tau \in \mathcal{F}, -c(s_r, a_r) > \lambda \) for some \( \lambda > 0 \).

\[
\mathcal{M}^\tau \psi(s_r) - \left( \mathcal{R}(s_r, a_r) + \gamma \max_{a \in A} \mathcal{P}_{s_r}^a \psi'(s') \right) \\
\leq \mathcal{M}^\tau \psi(s_r) - \left( \mathcal{R}(s_r, a_r) + \gamma \max_{a \in A} \mathcal{P}_{s_r}^a \psi'(s') \right) - c(s_r, a_r) \\
\leq \mathcal{R}(s_r, a_r) + c(s_r, a_r) + \gamma \mathcal{P}_{s_r}^a \psi'(s') \\
- \left( \mathcal{R}(s_r, a_r) + \gamma \max_{a \in A} \mathcal{P}_{s_r}^a \psi'(s') \right) \\
\leq \gamma \max_{a \in A} \left\{ \mathcal{P}_{s_r}^a \psi(s') - \psi'(s') \right\} \\
\leq \gamma \left\| \psi(s') - \psi'(s') \right\| \\
\leq \| \psi - \psi' \|,
\]

again using the fact that \( P \) is non-expansive. Hence we have succeeded in showing that for any \( \Lambda \in L_2 \) we have that

\[
\left\| \mathcal{M}^\tau \Lambda - \max_{a \in A} \left[ \psi(a) + \gamma \mathcal{P}^a \Lambda' \right] \right\| \leq \| \Lambda' \|.
\]

Gathering the results of the three cases gives the desired result.

To prove part ii), we make use of the following result:

**Theorem 4** (Theorem 1, pg 4 in [28]). Let \( \Xi_l(s) \) be a random process that takes values in \( \mathbb{R}^n \) and given by the following:

\[
\Xi_{l+1}(s) = (1 - \alpha_l(s)) \Xi_l(s) + \alpha_l(s) L_l(s),
\]

then \( \Xi_l(s) \) converges to \( 0 \) with probability 1 under the following conditions:
i) $0 \leq \alpha_t \leq 1, \sum_t \alpha_t = \infty$ and $\sum_t \alpha_t < \infty$

ii) $\|\mathbb{E}[L_t|\mathcal{F}_t]\| \leq \gamma \|\hat{\Xi}_t\|$, with $\gamma < 1$;

iii) $\text{Var}[L_t|\mathcal{F}_t] \leq c(1 + \|\Xi_t\|^2)$ for some $c > 0$.

To prove the result, we show (i) - (iii) hold. Condition (i) holds by choice of learning rate. It therefore remains to prove (ii) - (iii). We first prove (ii). For this, we consider our variant of the Q-learning update rule:

$$Q_{t+1}(s_t, a_t) = Q_t(s_t, a_t) + \alpha_t(s_t, a_t) \left[ \max \left\{ M^\tau Q_t(s_t, a_t), R(s_t, a_t) + \gamma \max_{a' \in A} Q_t(s_{t+1}, a') \right\} - Q_t(s_t, a_t) \right].$$

After subtracting $Q^*(s_t, a_t)$ from both sides and some manipulation we obtain that:

$$\Xi_{t+1}(s_t, a_t) = (1 - \alpha_t(s_t, a_t)) \Xi_t(s_t, a_t) + \alpha_t(s_t, a_t) \left[ \max \left\{ M^\tau Q_t(s_t, a_t), R(s_t, a_t) + \gamma \max_{a' \in A} Q_t(s_{t+1}, a') \right\} - Q^*(s_t, a_t) \right],$$

where $\Xi_t(s_t, a_t) := Q_t(s_t, a_t) - Q^*(s_t, a_t)$.

Let us now define by

$$L_t(s_{r_k}, a) := \max \left\{ M^\tau Q(s_{r_k}, a), R(s_{r_k}, a) + \gamma \max_{a' \in A} Q(s', a') \right\} - Q^*(s_{r_k}, a).$$

Then

$$\Xi_{t+1}(s_t, a_t) = (1 - \alpha_t(s_t, a_t)) \Xi_t(s_t, a_t) + \alpha_t(s_t, a_t) [L_t(s_{r_k}, a)].$$

We now observe that

$$\mathbb{E}[L_t(s_{r_k}, a)|\mathcal{F}_t] = \sum_{s' \in S} P(s'; s_{r_k}) \max \left\{ M^\tau Q(s_{r_k}, a), R(s_{r_k}, a) + \gamma \max_{a' \in A} Q(s', a') \right\} - Q^*(s_{r_k}, a)$$

$$= T_{\phi}Q_t(s, a) - Q^*(s, a).$$

Now, using the fixed point property that implies $Q^* = T_{\phi}Q^*$, we find that

$$\mathbb{E}[L_t(s_{r_k}, a)|\mathcal{F}_t] = T_{\phi}Q_t(s, a) - T_{\phi}Q^*(s, a)$$

$$\leq \|T_{\phi}Q_t - T_{\phi}Q^*\|$$

$$\leq \gamma \|Q_t - Q^*\| = \gamma \|\Xi_t\|_{\infty}. \quad (15)$$

using the contraction property of $T$ established in Lemma 3. This proves (ii).

We now prove iii), that is

$$\text{Var}[L_t|\mathcal{F}_t] \leq c(1 + \|\Xi_t\|^2). \quad (16)$$

Now by [4] we have that

$$\text{Var}[L_t|\mathcal{F}_t] = \text{Var} \left[ \max \left\{ M^\tau Q_t(s_t, a_t), R(s_t, a_t) + \gamma \max_{a' \in A} Q_t(s_{t+1}, a') \right\} - Q^*(s_t, a) \right]$$

$$= \mathbb{E} \left[ \max \left\{ M^\tau Q(s_{r_k}, a), R(s_{r_k}, a) + \gamma \max_{a' \in A} Q(s', a') \right\} \right] - Q^*(s_t, a) - (TQ_t(s, a) - Q^*(s, a)) \right]^2$$

$$= \mathbb{E} \left[ \max \left\{ M^\tau Q(s_{r_k}, a), R(s_{r_k}, a) + \gamma \max_{a' \in A} Q(s', a') \right\} - TQ_t(s, a) \right]^2$$

$$= \text{Var} \left[ \max \left\{ M^\tau Q_t(s_t, a_t), R(s_t, a_t) + \gamma \max_{a' \in A} Q_t(s_{t+1}, a') \right\} - TQ_t(s, a) \right]^2$$

$$\leq c(1 + \|\Xi_t\|^2).$$
for some $c > 0$ where the last line follows due to the boundedness of $Q$ (which follows from Assumptions 2 and 4). This concludes the proof of the Theorem.

**Proof of Convergence with Linear Function Approximation**

First let us recall the statement of the theorem:

**Theorem 3.** Algorithm 1 converges to a limit point $r^*$ which is the unique solution to the equation:

$$\Pi \Phi^{r^*} = \Phi^{r^*}, \quad a.e.$$  \hspace{1cm} \text{(17)}

where we recall that for any test function $\Lambda \in \mathcal{V}$, the operator $\Phi$ is defined by $\Phi \Lambda := \Theta + \gamma P \max \{ M\Lambda, \Lambda \}$.

Moreover, $r^*$ satisfies the following:

$$\| \Phi^{r^*} - Q^* \| \leq \gamma \| Q - Q^* \|.$$  \hspace{1cm} \text{(18)}

The theorem is proven using a set of results that we now establish. To this end, we first wish to prove the following bound:

**Lemma 4.** For any $Q \in \mathcal{V}$ we have that

$$\| \Phi^{r^*} - Q^* \| \leq \gamma \| Q - Q^* \|,$$  \hspace{1cm} \text{(19)}

so that the operator $\Phi$ is a contraction.

**Proof.** Recall, for any test function $\psi$, a projection operator $\Pi$ acting $\Lambda$ is defined by the following

$$\Pi \Lambda := \arg \min_{\Lambda \in \{ \Phi \in \mathbb{R} \}} \| \Lambda - \hat{\Lambda} \|.$$  \hspace{1cm} \text{(20)}

Now, we first note that in the proof of Lemma 3, we deduced that for any $\Lambda \in L^2$ we have that

$$\| M\Lambda - \left[ \psi(\cdot, a) + \gamma \max_{a \in A} P^a \Lambda \right] \| \leq \gamma \| \Lambda - \Lambda' \|,$$  \hspace{1cm} \text{(c.f. Lemma 3)}

Setting $\Lambda = Q$ and $\Lambda = \Theta$, it can be straightforwardly deduced that for any $Q, \hat{Q} \in L^2$: $\| MQ - \hat{Q} \| \leq \gamma \| Q - \hat{Q} \|$. Hence, using the contraction property of $M$, we readily deduce the following bound:

$$\max \left\{ \| MQ - \hat{Q} \|, \| MQ - \hat{\hat{Q}} \| \right\} \leq \gamma \| Q - \hat{Q} \|.$$  \hspace{1cm} \text{(20)}

We now observe that $\Phi$ is a contraction. Indeed, since for any $Q, Q' \in L^2$ we have that:

$$\| \Phi Q - \Phi Q' \| = \| \Theta + \gamma P \max \{ MQ, Q \} - (\Theta + \gamma P \max \{ MQ', Q' \}) \|$$

$$\leq \| \gamma \max \{ MQ, Q \} - \gamma \max \{ MQ', Q' \} \|$$

$$\leq \gamma \| \max \{ MQ - MQ', Q - MQ', MQ - Q', Q - Q' \} \|$$

$$\leq \gamma \max \{ \| MQ - MQ' \|, \| Q - MQ' \|, \| MQ - Q' \| \}$$

$$\leq \gamma \| Q - Q' \|,$$

using (20) and again using the non-expansiveness of $P$. \hfill \Box

We next show that the following two bounds hold:

**Lemma 5.** For any $Q \in \mathcal{V}$ we have that

\begin{itemize}
  \item[i)] $\| \Pi \Phi^{r^*} - \Pi \Phi^{\hat{Q}} \| \leq \gamma \| Q - \hat{Q} \|.$
  \item[ii)] $\| \Phi^{r^*} - Q^* \| \leq \frac{1}{\sqrt{1 - \gamma^2}} \| \Pi \Phi^{r^*} - Q^* \|.$
\end{itemize}
Proof. The first result is straightforward since as \( \Pi \) is a projection it is non-expansive and hence:

\[
\| \Pi \bar{\Phi} Q - \Pi \bar{\Phi} \bar{Q} \| \leq \| \bar{\Phi} Q - \bar{\Phi} \bar{Q} \| \leq \gamma \| Q - \bar{Q} \|,
\]

using the contraction property of \( \bar{\Phi} \). This proves i). For ii), we note that by the orthogonality property of projections we have that \( (\Phi r^* - \Pi Q^*, \Phi r^* - \Pi Q^*) \), hence we observe that:

\[
\| \Phi r^* - Q^* \| = \| \Phi r^* - \Pi Q^* \|^2 + 2 \gamma \| \Phi r^* - \Pi Q^* \| \| Q - \bar{Q} \|
\]

\[
\leq \| \bar{\Phi} \Phi r^* - Q^* \|^2 + 2 \gamma \| \Phi r^* - \Pi Q^* \| \| Q - \bar{Q} \|
\]

\[
\leq \gamma \| \Phi r^* - Q^* \|^2 + \| \Phi r^* - \Pi Q^* \|^2,
\]

after which we readily deduce the desired result.

Lemma 6. Define the operator \( H \) by the following: \( H Q(z) = \begin{cases} MQ(z), & \text{if } MQ(z) > \Phi r^*, \\ Q(z), & \text{otherwise}, \end{cases} \)

and \( \bar{\Phi} \) by: \( \bar{\Phi} Q := \Theta + \gamma P HQ \).

For any \( Q, \bar{Q} \in L_2 \) we have that

\[
\| \bar{\Phi} Q - \bar{\Phi} \bar{Q} \| \leq \gamma \| Q - \bar{Q} \|
\]

(21)

and hence \( \bar{\Phi} \) is a contraction mapping.

Proof. Using (20), we now observe that

\[
\| \bar{\Phi} Q - \bar{\Phi} \bar{Q} \| = \| \Theta + \gamma PHQ - (\Theta + \gamma PHQ) \|
\]

\[
\leq \gamma \| H Q - H \bar{Q} \|
\]

\[
\leq \gamma \max \{ MQ - M \bar{Q}, Q - \bar{Q}, MQ - \bar{Q}, Q, MQ - Q \}
\]

\[
\leq \gamma \max \{ MQ - M \bar{Q}, MQ - \bar{Q}, MQ - Q, M \bar{Q} - Q \}
\]

\[
\leq \gamma \max \{ \| MQ - M \bar{Q} \|, \| Q - \bar{Q} \|, \| MQ - \bar{Q} \|, \| M \bar{Q} - Q \| \}
\]

\[
\leq \gamma \| Q - \bar{Q} \|, \| Q - \bar{Q} \|, \| MQ - \bar{Q} \|, \| M \bar{Q} - Q \|
\]

\[
= \gamma \| Q - \bar{Q} \|
\]

again using the non-expansive property of \( P \).

\[\square\]

Lemma 7. Define by \( \tilde{Q} := \Theta + \gamma P v^\tilde{x} \) where

\[
v^\tilde{x}(z) := R(s_{r_x}, a) + \gamma \max_{a \in A} \sum_{s' \in S} P(s'; a, s_{r_x}) \Phi r^*(s'), \]

(22)

then \( \tilde{Q} \) is a fixed point of \( \bar{\Phi} Q \), that is \( \bar{\Phi} \tilde{Q} = \tilde{Q} \).

Proof. We begin by observing that

\[
H \tilde{Q}(z) = H (\Theta(z) + \gamma P v^\tilde{x})
\]

\[
= \begin{cases} MQ(z), & \text{if } MQ(z) > \Phi r^*, \\ Q(z), & \text{otherwise}, \end{cases}
\]

\[
= \begin{cases} MQ(z), & \text{if } MQ(z) > \Phi r^*, \\ \Theta(z) + \gamma P v^\tilde{x}, & \text{otherwise}, \end{cases}
\]

= \( v^\tilde{x}(z) \).

Hence,

\[
\bar{\Phi} \tilde{Q} = \Theta + \gamma P H \tilde{Q} = \Theta + \gamma P v^\tilde{x} = \tilde{Q}.
\]

(23)

which proves the result.

\[\square\]
Lemma 8. The following bound holds:
\[
    E \left[ v^\pi(s_0) \right] - E \left[ v^{\pi}(s_0) \right] \leq 2 \left[ 1 - \gamma \right] \sqrt{(1 - \gamma^2)}^{-1} \| Q^* - Q^* \|.
\] (24)

Proof. By definitions of \( v^\pi \) and \( v^{\pi} \) (c.f. (22)) and using Jensen’s inequality and the stationarity property we have that,
\[
    E \left[ v^\pi(s_0) \right] - E \left[ v^{\pi}(s_0) \right] = E \left[ P v^\pi(s_0) \right] - E \left[ P v^{\pi}(s_0) \right] \\
    \leq \left| E \left[ P v^\pi(s_0) \right] - E \left[ P v^{\pi}(s_0) \right] \right| \\
    \leq \| P v^\pi - P v^{\pi} \|.
\] (25)

Now recall that \( \bar{Q} := \Theta + \gamma P v^{\pi} \) and \( Q^* := \Theta + \gamma P v^{\pi^*} \), using these expressions in (25) we find that
\[
    E \left[ v^\pi(s_0) \right] - E \left[ v^{\pi}(s_0) \right] \leq \frac{1}{\gamma} \| \bar{Q} - Q^* \|.
\]

Moreover, by the triangle inequality and using the fact that \( \bar{Q} \) and \( Q^* \) hold for all \( \pi \), we have that
\[
    \| \bar{Q} - Q^* \| \leq \| \bar{Q} - \Phi r^* \| + \| Q^* - \Phi r^* \| \\
    \leq \gamma \| \bar{Q} - \Phi r^* \| + \gamma \| Q^* - \Phi r^* \| \\
    \leq 2\gamma \| \bar{Q} - \Phi r^* \| + \gamma \| Q^* - \bar{Q} \|,
\]

which gives the following bound:
\[
    \| \bar{Q} - Q^* \| \leq 2 \left( 1 - \gamma \right)^{-1} \| \bar{Q} - \Phi r^* \|,
\]

from which, using Lemma 8, we deduce that \( \| \bar{Q} - Q^* \| \leq 2 \left( 1 - \gamma \right) \sqrt{(1 - \gamma^2)}^{-1} \| \bar{Q} - \Phi r^* \| \), after which by (26), we finally obtain
\[
    E \left[ v^\pi(s_0) \right] - E \left[ v^{\pi}(s_0) \right] \leq 2 \left( 1 - \gamma \right) \sqrt{(1 - \gamma^2)}^{-1} \| \bar{Q} - \Phi r^* \|,
\]

as required. \( \square \)

Let us rewrite the update in the following way:
\[
    r_{t+1} = r_t + \gamma_t \Xi(w_t, r_t),
\]

where the function \( \Xi : \mathbb{R}^{2d} \times \mathbb{R}^p \rightarrow \mathbb{R}^p \) is given by:
\[
    \Xi(w, r) := \phi(z) (\Theta(z) + \gamma \max \{ (\Phi r)(z'), \mathcal{M}(\Phi r)(z) \} - (\Phi r)(z)),
\]

for any \( w = (z, z') \in (\mathbb{N} \times \mathcal{S})^2 \) where \( z = (t, s) \in \mathbb{N} \times \mathcal{S} \) and \( z' = (t, s') \in \mathbb{N} \times \mathcal{S} \) and for any \( r \in \mathbb{R}^p \). Let us also define the function \( \Xi : \mathbb{R}^p \rightarrow \mathbb{R}^p \) by the following:
\[
    \Xi(r) := E_{w_0 \sim (p, p)} [\Xi(w_0, r)]; w_0 := (z_0, z_1).
\]

Lemma 9. The following statements hold for all \( z \in \{ 0, 1 \} \times \mathcal{S} \):
\begin{itemize}
    \item[i)] \( (r - r^*) \Xi_L(r) < 0, \quad \forall r \neq r^*, \)
    \item[ii)] \( \Xi_L(r^*) = 0. \)
\end{itemize}
Theorem 5 (Th. 17, p. 239 in [5]). Consider a stochastic process \( r_t : \mathbb{R} \times \{\infty\} \times \Omega \rightarrow \mathbb{R}^k \) which takes an initial value \( r_0 \) and evolves according to the following:

\[
r_{t+1} = r_t + \alpha \Xi(s_t, r_t),
\]

for some function \( s : \mathbb{R}^{2d} \times \mathbb{R}^k \rightarrow \mathbb{R}^k \) and where the following statements hold:

1. \( \{s_t| t = 0, 1, \ldots\} \) is a stationary, ergodic Markov process taking values in \( \mathbb{R}^{2d} \)
2. For any positive scalar \( q \), there exists a scalar \( \mu_q \) such that \( \mathbb{E}[1 + \|s_t\|^q|s = s_0] \leq \mu_q \) \((1 + \|s\|^q)\)
3. The step size sequence satisfies the Robbins-Monro conditions, that is \( \sum_{t=0}^{\infty} \alpha_t = \infty \) and \( \sum_{t=0}^{\infty} \alpha_t^2 < \infty \)
4. There exists scalars \( c \) and \( q \) such that \( \|\Xi(w, r)\| \leq c (1 + \|w\|^q)(1 + \|r\|) \)
5. There exists scalars \( c \) and \( q \) such that \( \sum_{t=0}^{\infty} \mathbb{E}\|\Xi(w_t, r)\|z_0 \equiv z - \mathbb{E}[\Xi(w_0, r)]\| \leq c (1 + \|w\|^q)(1 + \|r\|) \)
6. There exists a scalar \( c > 0 \) such that \( \|\Xi(w_0, r) - \mathbb{E}[\Xi(w_0, r)]\| \leq c\|c - r\| \)
7. There exists scalars \( c > 0 \) and \( q > 0 \) such that \( \sum_{t=0}^{\infty} \mathbb{E}[\Xi(w_t, r)|w_0 \equiv w] - \mathbb{E}[\Xi(w_0, r)]\| \leq c\|c - r\| (1 + \|w\|^q) \)
8. There exists some \( r^* \in \mathbb{R}^k \) such that \( r^*(r - r^*) < 0 \) for all \( r \neq r^* \) and \( \bar{s}(r^*) = 0 \).
Then \( r_t \) converges to \( r^* \) almost surely.

In order to apply the Theorem we show that conditions 1 - 7 are satisfied.

**Proof.** Conditions 1-2 are true by assumption while condition 3 can be made true by choice of the learning rates. Therefore it remains to verify conditions 4-7 are met.

To prove 4, we observe that

\[
\|\Xi(w, r)\| = \|\phi(z) (\Theta(z) + \gamma \max \{\Phi r(z'), \mathcal{M}\Phi(z')\} - (\Phi r)(z))\|
\]

\[
\leq \|\phi(z)\| \|\Theta(z) + \gamma (\|\phi(z')\| \|r\| + \mathcal{M}\Phi(z'))\| + \|\phi(z)\| \|r\|
\]

\[
\leq \|\phi(z)\| (\|\Theta(z)\| + \gamma \|\mathcal{M}\Phi(z')\|) + \|\phi(z)\| (\|\gamma \phi(z')\| + \|\phi(z)\|) \|r\|.
\]

Now using the definition of \( \mathcal{M} \), we readily observe that \( \|\mathcal{M}\Phi(z')\| \leq \|\Theta\| + \gamma \|\mathcal{P}_{s_x, s_t}\phi\| \leq \|\Theta\| + \gamma \|\phi\| \) using the non-expansiveness of \( P \).

Hence, we lastly deduce that

\[
\|\Xi(w, r)\| \leq \|\phi(z)\| (\|\Theta(z)\| + \gamma \|\mathcal{M}\Phi(z')\|) + \|\phi(z)\| (\gamma \|\phi(z')\| + \|\phi(z)\|) \|r\|
\]

\[
\leq \|\phi(z)\| (\|\Theta(z)\| + \gamma \|\Theta\| + \gamma \|\psi\|) + \|\phi(z)\| (\gamma \|\phi(z')\| + \|\phi(z)\|) \|r\|
\]

we then easily deduce the result using the boundedness of \( \phi \), \( \Theta \) and \( \psi \).

Now we observe the following Lipschitz condition on \( \Xi \):

\[
\|\Xi(w, r) - \Xi(w, \bar{r})\| = \|\phi(z) (\gamma \max \{\Phi r(z'), \mathcal{M}\Phi(z')\} - \gamma \max \{\Phi \bar{r}(z'), \mathcal{M}\Phi(z')\}) - ((\Phi r)(z) - (\Phi \bar{r})(z))\|
\]

\[
\leq \gamma \|\phi(z)\| \|\max \{\phi'(z')r, \mathcal{M}\Phi(z')\} - \max \{\phi'(z')\bar{r}, \mathcal{M}\Phi(z')\}\| + \|\phi(z)\| \|\phi'(z')r - \phi'(z')\bar{r}\|
\]

\[
\leq \gamma \|\phi(z)\| \|\phi'(z')r - \phi'(z')\bar{r}\| + \|\phi(z)\| \|\phi'(z')r - \phi'(z')\bar{r}\|
\]

\[
\leq \gamma \|\phi(z)\| \|\phi'(z')r - \phi'(z')\bar{r}\| + \|\phi(z)\| \|\phi'(z')r - \phi'(z')\bar{r}\|
\]

\[
\leq c \|r - \bar{r}\|,
\]

using Cauchy-Schwarz inequality and that for any scalars \( a, b, c \) we have that \( |\max\{a, b\} - \max\{b, c\}| \leq |a - c| \).

Using Assumptions 3 and 4, we therefore deduce that

\[
\sum_{t=0}^{\infty} \|E[\Xi(w, r) - \Xi(w, \bar{r})|w_0 = w] - E[\Xi(w_0, r) - \Xi(w_0, \bar{r})]\| \leq c \|r - \bar{r}\| (1 + \|w\|).
\]

Part 2 is assured by Lemma while Part 4 is assured by Lemma and lastly Part 8 is assured by Lemma.

This result completes the proof of Theorem.

**Proof of Proposition**

**Proof.** First let us recall that the *intervention time* \( \tau_k \) is defined recursively \( \tau_k = \inf \{ t > \tau_{k-1} | s_t \in A, \tau_k \in \mathcal{F}_t \} \) where \( A = \{ s \in S, g(s_t) = 1 \} \). The proof is given by establishing a contradiction. Therefore suppose that \( \mathcal{M}\psi(s_{\tau_k}) \leq \psi(s_{\tau_k}) \) and suppose that the intervention time \( \tau'_1 > \tau_1 \) is an optimal intervention time. Construct the \( \pi' \in \Pi \) and \( \tilde{\pi} \in \Pi \) policy switching times by \( (\tau'_0, \tau'_1, \ldots) \) and \( (\tau'_0, \tau_1, \ldots) \) respectively. Define by \( l = \inf \{ t > 0; \mathcal{M}\psi(s_t) = \psi(s_t) \} \) and \( m = \sup \{ t < \tau'_1 \} \). By construction we have that

\[
v^{\pi'}(s) = E \left[ \mathcal{R}(s_0, a_0) + E \left[ \ldots + \gamma^{l-1} E \left[ \mathcal{R}(s_{\tau_1-1}, a_{\tau_1-1}) + \ldots + \gamma^{m-l-1} E \left[ \mathcal{R}(s_{\tau'_1-1}, a_{\tau'_1-1}) + \gamma \mathcal{M}^{\tilde{\pi}} v^{\pi'}(s', I(\tau'_1)) \right] \right] \right] \right]
\]

\[
< E \left[ \mathcal{R}(s_0, a_0) + E \left[ \ldots + \gamma^{l-1} E \left[ \mathcal{R}(s_{\tau_1-1}, a_{\tau_1-1}) + \gamma \mathcal{M}\psi(s_{\tau_1}) \right] \right] \right]
\]

We now use the following observation

\[
E \left[ \mathcal{R}(s_{\tau_1-1}, a_{\tau_1-1}) + \gamma \mathcal{M}\psi(s_{\tau_1}) \right] \leq \max \left\{ \mathcal{M}\psi(s_{\tau_1}), \max_{a_{\tau_1} \in A} \mathcal{R}(s_{\tau_1}, a_{\tau_1}) + \gamma \sum_{s' \in S} P(s'; a_{\tau_1}, s_{\tau_1}) v^\pi(s') \right\}.
\]
Using this we deduce that

\[ v^{\pi'}(s) \leq \mathbb{E}\left[ R(s_0, a_0) + \mathbb{E}\left[ \ldots \right. \right. \right. \]

\[ + \gamma^{l-1}\mathbb{E}\left[ R(s_{\tau_{l-1}}, a_{\tau_{l-1}}) + \gamma \max \left\{ M \tilde{v}^{\pi'}(s_{\tau_l}), \max_{a_{\tau_l} \in A} \mathbb{E}\left[ R(s_{\tau_k}, a_{\tau_k}) + \gamma \sum_{s' \in S} P(s'; a_{\tau_1}, s_{\tau_1}) v^{\pi}(s') \right] \right\} \right] \]

\[ = \mathbb{E}\left[ R(s_0, a_0) + \mathbb{E}\left[ \ldots + \gamma^{l-1}\mathbb{E}\left[ R(s_{\tau_{l-1}}, a_{\tau_{l-1}}) + \gamma \left[ T \tilde{v}^{\pi'}(s_{\tau_l}) \right] \right] \right] = v^{\tilde{\pi}}(s) \],

where the first inequality is true by assumption on \( \mathcal{M} \). This is a contradiction since \( \pi' \) is an optimal policy for Player 2. Using analogous reasoning, we deduce the same result for \( \tau'_{k} < \tau_k \) after which deduce the result. Moreover, by invoking the same reasoning, we can conclude that it must be the case that \( (\tau_0, \tau_1, \ldots, \tau_{k-1}, \tau_k, \tau_{k+1}, \ldots) \) are the optimal switching times. \( \square \)