Abstract

The trade-off between the cost of acquiring and processing data, and uncertainty due to a lack of data is fundamental in machine learning. A basic instance of this trade-off is the problem of deciding when to make noisy and costly observations of a discrete-time Gaussian random walk, so as to minimise the posterior variance plus observation costs. We present the first proof that a simple policy, which observes when the posterior variance exceeds a threshold, is optimal for this problem. The proof generalises to a wide range of cost functions other than the posterior variance.

This result implies that optimal policies for linear-quadratic-Gaussian control with costly observations have a threshold structure. It also implies that the restless bandit problem of observing multiple such time series, has a well-defined Whittle index. We discuss computation of that index, give closed-form formulae for it, and compare the performance of the associated index policy with heuristic policies.

The proof is based on a new verification theorem that demonstrates threshold structure for Markov decision processes, and on the relation between binary sequences known as mechanical words and the dynamics of discontinuous nonlinear maps, which frequently arise in physics, control and biology.

Keywords: restless bandits, Whittle index, mechanical words, Kalman filter, linear-quadratic-Gaussian control

1. Introduction

This paper answers three closely-related questions about discrete-time filtering of scalar time series with costly observations, where the nature of the observations is controlled through a query action. The first two questions concern the structure of optimal policies for observing a single time series so as to minimise either a function of the posterior variance (Theorem 1) or a quadratic function of the system state and control input (Corollary 2). The third question concerns the observation of several such time series with a constraint on the number of time series that can be observed simultaneously. This is an instance of a restless bandit problem and it is interesting to know that the problem has a well-defined Whittle index (Theorem 4).

This introduction begins with the time-series model (Section 1.1) that the three questions have in common. It then motivates, formulates and states the key results for each question in turn (Sections 1.2 to 1.4). It concludes with an intuitive guide to the main concepts involved in the proof (Section 1.5) and a description of the structure of the rest of the paper (Section 1.6).
1.1 Time-Series Model

We consider the classic discrete-time scalar normally-distributed state-space model. In this model, the state is partially observed through measurements as fully described by the conditional dependencies

\[
\begin{align*}
X_0 &\sim \mathcal{N}(x_0, v_0) \\
X_{t+1}|X_t, u_t &\sim \mathcal{N}(AX_t + Bu_t, \Sigma_X) \\
Y_{t+1}|X_{t+1}, a_t &\sim \mathcal{N}(X_{t+1}, \Sigma_Y(a_t))
\end{align*}
\]

for \( t \in \mathbb{Z}_+ \). (1)

(In this paper \( \mathbb{Z}_+, \mathbb{R}_+ \) include zero, unlike \( \mathbb{Z}_{++}, \mathbb{R}_{++} \).) The state \( X_t \) is a real-valued random variable with initial mean \( x_0 \) and variance \( v_0 \). The sequence of states depends on the control or exogenous input \( u_t \in \mathbb{R} \). The measurement \( Y_{t+1} \) is a real-valued random variable which depends on a query action \( a_t \in \{0, 1\} \). The variances \( \Sigma_X, \Sigma_Y(0), \Sigma_Y(1) > 0 \) and real-valued parameters \( A, B \) are known. Query action \( a_t = 1 \) is assumed to correspond to a higher-quality observation than query action \( a_t = 0 \), so that \( \Sigma_Y(1) < \Sigma_Y(0) \) and it is possible that \( \Sigma_Y(0) = \infty \) which represents a totally uninformative observation or no observation at all.

The observed history \( H_t \) at time \( t \), is \( x_0, v_0, a_0, a_1, \ldots, a_{t-1}, u_0, u_1, \ldots, u_{t-1}, Y_1, Y_2, \ldots, Y_t \). Under the Bayesian filter, the information state is given by the posterior mean \( x_t := \mathbb{E}[X_t|H_t] \) and variance \( v_t := \mathbb{E}[(X_t - x_t)^2|H_t] \). In this case, the Bayesian filter is the Kalman filter (Thiele, 1880; Kalman, 1960) and it follows that the information state undergoes the following Markovian transitions:

\[
\begin{align*}
x_{t+1}|x_t, v_t, a_t, u_t &\sim \mathcal{N}(Ax_t + Bu_t, A^2v_t + \Sigma_X - \phi_{a_t}(v_t)) \\
v_{t+1}|x_t, v_t, a_t, u_t &\sim \phi_{a_t}(v_t)
\end{align*}
\]

for \( t \in \mathbb{Z}_+ \) (2)

where \( \phi_a : \mathbb{R}_+ \to \mathbb{R}_+ \) for \( a \in \{0, 1\} \) is the Möbius transformation

\[
\phi_a(v) := \frac{(A^2v + \Sigma_X) \times \Sigma_Y(a)}{(A^2v + \Sigma_X) + \Sigma_Y(a)}.
\]

(3)

1.2 Optimal Policies for Observing a Single Time Series

The simplest problem addressed here involves a measurement cost \( c(a_t) \in \mathbb{R} \) and uncertainty cost \( C(v_t) \). Cost \( c(a_t) \) might reflect costs of energy, labour, communication, computational processing, hardware or risks associated with each measurement. Recall that a policy is non-anticipative if it selects actions at time \( t \) based only on information available up-to and including time \( t \). The objective is to find a non-anticipative policy \( \pi \) that selects query actions \( a_t \) so as to minimise the \( \beta \)-discounted performance functional, for \( \beta \in [0, 1) \),

\[
\mathbb{E} \left( \sum_{t=0}^{\infty} \beta^t(c(a_t) + C(v_t)) \right| \pi, x_0, v_0)
\]

where the expectation is over the Markovian transitions (2). As the transitions of the posterior variance are given by the Möbius transformation (3), this problem reduces to the following deterministic dynamic program for value function \( V : \mathbb{R}_+ \to \mathbb{R}_+ \),

\[
V(v_t) = \min_{a_t \in \{0, 1\}} \left\{ c(a_t) + C(v_t) + \beta V(\phi_{a_t}(v_t)) \right\}.
\]

(4)
The first question addressed in this paper is: for what cost functions is a threshold policy optimal for this problem? For instance, one may intuitively guess that optimal policies for variance minimisation with \( C(v) = v \), for entropy minimisation with \( C(v) = \log(v) \), or for precision maximisation with \( C(v) = -1/v \), might involve making expensive observations at time \( t \) when the variance \( v_t \) exceeds a threshold. The following condition on \( C(\cdot) \) covers these examples.

\textbf{Condition C.} The state space \( I \) is either \([0, \infty)\) or \((0, \infty)\). The first cost function \( c : \{0, 1\} \rightarrow \mathbb{R} \), representing data acquisition costs, has \( c(0) < c(1) \). Also, the second cost function \( C : I \rightarrow \mathbb{R} \), representing the cost of uncertainty, is of the form \( C(x) = \sum_{i=1}^{n_C} C_i(x) \) for some \( n_C \in \mathbb{Z}_{++} \), where each of the functions \( C_i : I \rightarrow \mathbb{R} \) satisfies one of the following conditions:

\begin{enumerate}
  \item \textit{C1.} For \( x \in I \), the derivatives \( C'_i(x) := \frac{d}{dx} C_i(x) \) and \( C''_i(x) := \frac{d^2}{dx^2} C_i(x) \) exist and
    \begin{itemize}
      \item the function \( C_i(x) \) is concave,
      \item the function \( \frac{1}{x} C''_i \left( \frac{1}{x} \right) \) is non-decreasing,
      \item and the function \( \frac{1}{x^2} C'_i \left( \frac{1}{x} \right) \) is non-increasing and convex.
    \end{itemize}
  \item \textit{C2.} For \( x \in I \), the function \( C_i(x) \) is non-decreasing, convex and differentiable.
\end{enumerate}

Note that we may work with the interval \( I = (0, \infty) \) in cases where the cost function \( C(v) \) is not a real number for \( v = 0 \), in order to include cases like \( \log(v) \) and \( -1/v \), but the results of the paper continue to hold in cases where \( C(0) \) is defined. Also, note that \( C(\cdot) \) need not be convex or concave, for instance in the case \( C(v) = (v^2 - 1)/v \), and it is possible that \( C(\cdot) \) is bounded, for instance in the case \( C(v) = v/(v + 1) \).

The above condition requires that functions \( C_i \) satisfying C2 have a derivative \( C'_i \). This is simply for convenience in the proofs of Propositions 26 and 31. As such functions are real-valued convex functions, one can instead set \( C'_i \) equal to any subderivative at points where the derivative is not defined.

\textbf{Theorem 1} Suppose the state space \( I \) and the cost functions \( c : \{0, 1\} \rightarrow \mathbb{R} \) and \( C : I \rightarrow \mathbb{R} \) satisfy Condition C. Then a threshold policy is optimal for the dynamic program (4).

\textbf{Proof} This result is an immediate consequence of Theorem 6 whose hypotheses hold according to Propositions 19, 22, 26, 27, 31 and 33. \qed

From one perspective, this answer is a rare example of an explicit solution to a real-state partially-observed Markov decision process (POMDP). From another perspective, this answer is a rare example of an explicit solution to the problem of observation selection in sensor management (Hero and Cochran, 2011). Indeed, given a collection of variables which can (in principle) be observed and a single variable to predict, which are jointly Gaussian with known covariance, even the problem of deciding whether there exists a subset of \( k \) observations that reduces the prediction variance below a given threshold is NP-hard (Davis et al., 1997). Work has therefore focused on finding covariance structures for which the problem is tractable, for instance Das and Kempe (2008) show that selection of Gaussian
observations with an exponential covariance can be solved by a simple discrete dynamic program, and on finding appropriate choices of cost functions for which there are guaranteed approximation algorithms (Krause et al., 2008, 2011; Badanidiyuru et al., 2014; Chen et al., 2014).

1.3 The Linear Quadratic Gaussian Problem with Costly Observations

The second question addressed by this paper is: when are threshold policies optimal for making observations in a generalisation of the linear-quadratic-Gaussian control problem in which observations are costly but controlled through a query action? This is an old but unsolved problem (Meier et al., 1967; Wu and Arapostathis, 2005; Molin and Hirche, 2009). Specifically, suppose the states and observations are as in (1) but the objective is to find a non-anticipative policy \( \pi \) that selects a feedback-control action \( u_t \in \mathbb{R} \) and a sensor-query action \( a_t \in \{0, 1\} \) so as to minimise the \( \beta \)-discounted performance functional

\[
E \left( \sum_{t=0}^{\infty} \beta^t (DX_t^2 + Fu_t^2 + c(a_t)) \middle| \pi, x_0, v_0 \right),
\]

where \( D, F \in \mathbb{R}_+ \) and the expectation is over the Markovian transitions (2).

An immediate corollary of Theorem 1 is the following answer to the above question.

**Corollary 2** Suppose that \( A \in [-1, 1], D \in \mathbb{R}_+, F \in \mathbb{R}_+, \beta \in (0, 1), \Sigma_Y(q) \in [0, \infty] \) for \( q \in \{0, 1\} \) with \( \Sigma_Y(0) \geq \Sigma_Y(1) \), \( c(q) \in \mathbb{R} \) for \( q \in \{0, 1\} \) with \( c(0) \leq c(1) \), where \( A \) and \( \Sigma_Y(\cdot) \) are as in equation (1). Then an optimal policy for linear-quadratic-Gaussian control with costly observations is to set

\[
a_t = \begin{cases} 
1 & \text{if } v_t \geq z \\
0 & \text{if } v_t < z 
\end{cases}\quad \text{and} \quad u_t = -Lx_t
\]

for some \( L \in \mathbb{R} \) and \( z \in [0, \infty] \).

A proof of Corollary 2 is presented in Appendix D.

1.4 Multi-Target Tracking and Restless Bandits

This paper also addresses the problem of monitoring multiple time series so as to maintain a precise belief while minimising the cost of sensing. This problem is often called the *multi-target tracking* problem.

To formulate the problem, suppose there are \( n \in \mathbb{Z}_{++} \) independent time series of the form (1), indexed by \( i \in \{1, 2, \ldots, n\} \), and time series \( i \) has state \( X_{i,t} \) at time \( t \in \mathbb{Z}_+ \). Each time series may have its own parameters \( x_{i,0}, v_{i,0}, A_{i,0}, B_{i,0}, \Sigma_{X_i} \), its own input \( u_{i,t} \) and its own uncertainty cost \( C_i : I \to \mathbb{R} \), where the interval \( I \) is as in Condition C. Corresponding to these time series there are \( n \) query actions \( a_{i,t} \in \{0, 1\} \) which specify the nature of the observation \( Y_{i,t} \) of time series \( i \). These observations have their own parameters \( \Sigma_{Y_i} : \{0, 1\} \to [0, \infty] \) and costs \( c_i : \{0, 1\} \to \mathbb{R} \). However, these actions are subject to the constraint that only \( m \in \mathbb{Z}_{++} \) with \( m < n \) expensive observations can be made at each time.
The problem is to minimise the total $\beta$-discounted observation cost and uncertainty cost
\[
\sum_{i=1}^{n} \sum_{t=0}^{\infty} \beta^t (c_i(a_{i,t}) + C_i(v_{i,t}))
\]
subject to the constraint on the number of observations
\[
\sum_{i=1}^{n} a_{i,t} = m \quad \text{for } t \in \mathbb{Z}_+.
\]
Continuous-time versions of this problem were previously addressed by Le Ny et al. (2011) and versions of the discrete-time problem given above have attracted considerable attention (Gupta et al., 2006; Mourikis and Roumeliotis, 2006; La Scala and Moran, 2006; Washburn, 2008; Niño-Mora and Villar, 2009; Villar, 2012; Zhao et al., 2014; Dance and Silander, 2015; Niño-Mora, 2016). One example of a real-world application of the discrete-time problem, which was our original motivation for studying the problems in this paper, is the measurement of on-street parking occupancy (Dey, 2014), in a setting where cheap-but-low-quality observations are available through payment data (at parking meters or through mobile phones), expensive-but-high-quality observations are available through portable cameras, which are moved daily or weekly (and thus in discrete time), and there are a limited number of portable cameras with which to observe many streets.

Restless Bandits. The multi-target tracking problem is an instance of a restless bandit problem (Whittle, 1988). Typically, such problems are defined in terms of a set of $n \in \mathbb{Z}_{++}$ two-action Markov decision processes (MDPs), although generalisations to a time-varying number of MDPs (Verloop, 2016) and to more than two actions per MDP (Glazebrook et al., 2011) have been explored. The two actions are usually referred to as active or play versus inactive or passive and each of the MDPs is referred to as an arm or project.

In a restless bandit problem, these $n$ MDPs are coupled into a single MDP as follows. The state space is the Cartesian product of the state spaces of the arms, and the state of each arm transitions independently of the other arms given the actions taken on that arm. Thus the transitions of an arm depend only on the actions taken on that arm and on that arm’s current state. The objective is to find a non-anticipative policy that minimises the sum over the arms of each arm’s cost-to-go.

However, the action space is only a subset of the Cartesian product of the action spaces of the arms, as there is a constraint on the number $m$ of arms that are simultaneously active at each time, where $m \in \mathbb{Z}_{++}$ with $m < n$. Typically, the constraint is that exactly $m$ arms are active at each time, but this is readily relaxed to a constraint that at most $m$ arms are active by including “dummy arms”, whose cost is always zero, in the population of $n$ arms. More general constraints have been explored (Niño-Mora, 2015), in which each arm consumes resources as a function of both its state and the action taken, and the total cost of the resources consumed at each time is constrained. In the absence of any such action constraint, the problem would be solved by applying an optimal policy for each arm independently. Moreover, it turns out that if the constraint were only on the (discounted) time-average number of arms that are simultaneously active, rather than a constraint at each time, the problem could again be separated into $n$ smaller problems after introducing a Lagrange multiplier.
Let us relate the above definition to the typical usage of the term bandit in the machine-learning literature. In that context, multi-armed bandits are reinforcement-learning problems involving a set of arms whose reward distributions are unknown. At each time, the learner must select which arm to play. Such bandits involve a trade-off between exploring arms to acquire information about their expected payoffs and exploiting arms with the highest expected payoffs. In the simplest versions of such problems, where the prior on the reward distributions is independent over arms, each arm can be viewed as an MDP whose state corresponds to the belief about that arm’s payoff distribution. Each time the arm is played, its reward is observed and this belief is updated. Such updates correspond to state transitions. Each time the arm is inactive, its state does not change.

If we allow arms to make general Markovian state transitions, not just transitions corresponding to belief updates, while preserving the requirement that an arm only changes state when it is played, then we arrive at a more general class of problems known as ordinary or classical bandits (Gittins et al., 2011). In turn, restless bandits generalise ordinary bandits in two ways. Firstly, restless bandits allow more than one arm to be simultaneously active (if \( m > 1 \)). Secondly, restless bandits allow the state of an arm to change even when the arm is not active, which is why they are called restless.

While this additional generality is important in modelling real-world problems, it comes at a price. On the one hand, the Gittins index policy is optimal for ordinary bandit problems and can be computed in polynomial time for problems with finite state spaces (Niño-Mora, 2007). On the other hand, it is in general PSPACE-hard (Papadimitriou and Tsitsiklis, 1999; Guha et al., 2010) to find policies that approximate optimal policies for restless bandit problems with finite state spaces to any non-trivial factor. At first glance, this might suggest that the multi-target tracking problem addressed here, with uncountable state-space \( \mathbb{R}_+ \) or \( \mathbb{R}_{++} \), is impossibly difficult. At second glance, this poses an interesting question: for which restless bandit problems can we find approximately-optimal policies efficiently?

**Whittle Index Policy.** Whittle (1988) proposed a policy which generalises the Gittins index policy to restless bandit problems. This policy associates a real (or in some definitions an extended-real) number \( \lambda^*_i(x_i) \) called the Whittle index with the state \( x_i \) of each arm \( i \) and plays the \( m \) arms with the largest Whittle index at each time. Ties are usually broken uniformly at random or according to a predefined priority ordering.

The literature contains many definitions of the Whittle index \( \lambda^*_i(x_i) \) of arm \( i \), of which we describe only three. These definitions are not equivalent in general, although they turn out to be equivalent for the problem addressed in this paper. All the definitions involve a modified version of arm \( i \)'s MDP, which we call the \( \lambda \)-MDP, in which the cost \( C_i(x_i, a_i) \) for taking action \( a_i \) in state \( x_i \) is replaced by \( C_i(x_i, a_i) + \lambda a_i \) where \( \lambda \in \mathbb{R} \) represents a price for taking the active action \( a_i = 1 \). Verloop (2016) then defines \( \lambda^*_i(x_i) \) as the least price \( \lambda \) for which action \( a_i = 0 \) is optimal for the \( \lambda \)-MDP in state \( x_i \). Meanwhile, Guha et al. (2010) define \( \lambda^*_i(x_i) \) as the largest price \( \lambda \) for which the actions \( a_i = 0 \) and \( a_i = 1 \) are both optimal for the \( \lambda \)-MDP in state \( x_i \). In this paper, we use the following definition from Niño-Mora (2015).

**Definition 3** The Whittle index of arm \( i \) in state \( x_i \) is a price \( \lambda^*_i(x_i) \) for which

1. Action \( a_i = 1 \) is optimal in state \( x_i \) of the \( \lambda \)-MDP if and only if \( \lambda \leq \lambda^*_i(x_i) \),
2. Action \( a_i = 0 \) is optimal in state \( x_i \) of the \( \lambda \)-MDP if and only if \( \lambda \geq \lambda^*_i(x_i) \).

**Arm \( i \) is indexable** if the Whittle index \( \lambda^*_i(x_i) \) exists for all states \( x_i \) in arm \( i \)'s state space.

For all of the above definitions, it is immediate that the Whittle index is unique if it exists. Verloop’s definition has the advantage that the Whittle index, and hence the Whittle index policy, exist for a wider range of arms. On the other hand, if we know arm \( i \) is indexable, the definition used in this paper has the advantage that we know we have found the Whittle index when we find a price \( \lambda \in \mathbb{R} \) for which actions \( a_i = 0 \) and \( a_i = 1 \) are both optimal in state \( x_i \) of the \( \lambda \)-MDP.

Whittle’s index policy has been the subject of great interest for computational, empirical and theoretical reasons. The policy is potentially attractive in terms of computational cost as it reduces the original restless bandit problem, whose state space is the Cartesian product of the state spaces of the arms, to the computation of \( n \) Whittle indexes for individual arms. The policy is also attractive from a systems-architecture point-of-view, as it allows one to mix-and-match different types of arms, and it naturally accommodates the arrival or departure of arms in the sense that the Whittle index does not depend on the number of arms \( n \). Additionally, extensive numerical tests of Whittle’s policy in different applications repeatedly demonstrate that it performs remarkably well when the arms are all indexable. Indeed, 12 references to such empirical work are cited in Section 8 of Verloop (2016).

Although Whittle’s policy is not an optimal policy for general restless bandits, under certain sufficient conditions and for a certain limit, it is an *asymptotically* optimal policy. Specifically, in the limit as the number of arms \( n \) tends to infinity, while the number of arms that can be simultaneously active \( m \) varies in such a way that \( m/n \) is as constant as possible, the ratio of the cost-rate of Whittle’s policy to the cost-rate of an optimal policy for the given \( n,m \) tends to one. Assuming an average-cost setting, for collections of identical arms whose size \( n \) does not vary with time, where each arm has a finite state space, Whittle (1988) originally conjectured that it was sufficient that the identical arm was indexable for such an asymptotic optimality result to hold. However, Weber and Weiss (1990) found counterexamples to this conjecture. Nevertheless, Weber and Weiss also found sufficient conditions for asymptotic optimality to hold, and those sufficient conditions imply that the arms are indexable (Lemma 2 of that paper). Under similar sufficient conditions, this asymptotic optimality result has recently been generalised by Verloop (2016) to restless bandits with dynamic populations of non-identical arms. Both the results of Weber and Weiss and the results of Verloop assume an average-cost setting and arms with finite state spaces. So new theoretical work may be required to understand asymptotic optimality for arms with uncountable state spaces, as studied here.

**Whittle Index for the Multi-Target Tracking Problem.** The above discussion prompts the third question addressed in this paper: *is the multi-target tracking problem indexable, and if so, what is a computationally-convenient expression for its Whittle index?* The answer is given by the following Theorem.

**Theorem 4** Suppose the state space \( I \) and the cost functions \( c : \{0,1\} \to \mathbb{R} \) and \( C : I \to \mathbb{R} \) satisfy Condition C. Let the price \( \nu \in \mathbb{R} \), the discount factor \( \beta \in [0,1) \) and let the transitions \( \phi_a : I \to I \) for \( a \in \{0,1\} \) be given by (3). Then the family of dynamic programs

\[
V(x; \nu) = \min_{a \in \{0,1\}} \{ \nu c(a) + C(x) + \beta V(\phi_a(x); \nu) \}
\]
is indexable. Furthermore, for each \( x \in \mathcal{I} \) the Whittle index is

\[
\lambda(x) := \frac{\sum_{t=0}^{\infty} \beta^t (C(X_t(x,0;x)) - C(X_t(x,1;x)))}{\sum_{t=0}^{\infty} \beta^t (c(A_t(x,1;x)) - c(A_t(x,0;x)))}
\]

where for any \( s \in [-\infty, \infty] \) we define \( X_t(x,a;s) \) to be the state at time \( t = 0, 1, \ldots \) if the system starts in state \( x \in \mathcal{I} \) at time \( t = 0 \), then action \( a \in \{0,1\} \) is taken and a policy which takes the actions \( A_t(x,a;s) := 1_{X_t(x,a;s) \geq s} \) is followed thereafter (this is the \( x \)-threshold policy).

**Proof** This result is an immediate consequence of Theorem 6 whose hypotheses hold according to Propositions 19, 22, 26, 27, 31 and 33.

This paper thus generalises the work of Dance and Silander (2015) by demonstrating that threshold policies are in fact optimal for the single arm problem, which was Assumption A1 of (Dance and Silander, 2015). It also generalises by considering the case of multipliers \( A < 1 \) rather than only considering \( A = 1 \), where \( A \) is as in equation (1), and by considering cost functions \( C(v_t) \neq v_t \) other than the posterior variance.

### 1.5 Intuitive Guide to the Paper

As with other work on Markov decision processes, we work with the cost-to-go \( Q(x,a) \) when starting in initial state \( x \) and taking initial action \( a \), but then following an optimal policy. A common way to prove that threshold policies are optimal when the state \( x \) is real-valued, is to show that the difference \( Q(x,1) - Q(x,0) \) is a non-increasing function of \( x \). Such approaches have been studied by Serfozo (1976), Altman and Stidham Jr. (1995) and Altman et al. (2000). Unfortunately, as shown in Figure 1, such an approach fails for the process considered in this paper, even when the cost equals the variance.

Instead, this paper proves the optimality of threshold policies using a new verification theorem by Niño-Mora (2015). This theorem applies to Markov decision processes that satisfy the so-called partial conservation law indexability (PCLI) conditions (Section 2). The central concept underlying the verification theorem is the marginal productivity index which turns out to be equal to the ratio \( \lambda \) given in Theorem 4.

One of the PCLI conditions requires that the marginal productivity index \( \lambda(x) \) is a non-decreasing function of the state \( x \). This is the most challenging of the conditions to verify. As a quick check, we plot \( \lambda(x) \) in Figure 2. Although \( \lambda(x) \) is increasing, the numerator and denominator have a fractal structure, so it is surprising that the index is continuous. Furthermore, if we subtract a cubic fit to \( \lambda(x) \), the residual has a complicated sequence of cusps. Therefore the paper then focuses on characterising the sequence of actions \( A_t(x,a;x) \) that give rise to this fractal pattern. We prove that these sequences are special binary strings called mechanical words (Section 3).

Another key to proving that the index is non-decreasing is the fact that the mappings \( \phi_0(x) \) and \( \phi_1(x) \) are Möbius transformations of the form

\[
\mu_A(x) := \frac{A_{11}x + A_{12}}{A_{21}x + A_{22}}
\]
Figure 1: Counterexample to monotonicity of the difference in $Q$-functions. The functions $Q(x, 0)$ and $Q(x, 1)$ cross only a single time at $x = 1.1$ (left plot). However, the difference $Q(x, 1) - Q(x, 0)$ is increasing for some $x$ (right plot, for $x$ in the left plot’s grey box). The model has $\beta = 0.95$, $C(x) = x$, $\phi_0(x) = x + 1$, $\phi_1(x) = 1/(a_1 + 1/(x + 1))$ with $a_1 = 0.1$ and $\nu = 0.7647$.

Figure 2: The numerator (left) and denominator (mid-left) of the index (mid-right), and the error in a cubic fit to the index (right). The model has cost $C(x) = x$, discount factor $\beta = 0.99$, map-with-a-gap $\phi_0(x) = rx + 1$ and $\phi_1(x) = 1/(a_1 + 1/(rx + 1))$ with $r = 0.9$ and $a_1 = 0.01$. 
for some $A \in \mathbb{R}^{2 \times 2}$. Now the composition of Möbius transformations is homeomorphic to matrix multiplication, so that

$$\mu_A(\mu_B(x)) = \mu_{AB}(x)$$

for any $A, B \in \mathbb{R}^{2 \times 2}$. Further, if $\det(A) = 1$ then the gradient of the corresponding Möbius transformation is

$$\frac{d}{dx}\mu_A(x) = \frac{1}{(A_{21}x + A_{22})^2}$$

which is a convex function for $x \in \mathbb{R}^+$ and $A_{21}, A_{22} \in \mathbb{R}^+$. So the gradient of the numerator of the index, in the case $C(x) = x$, is the difference of the sums of a convex function of a sequence of linear functions of $x$. Such sums can be addressed by the theory of majorisation (Marshall et al., 2010), provided the sequence of linear functions of $x$ satisfy certain majorisation conditions. It turns out that those conditions are satisfied because of a special palindromic property of mechanical words.

### 1.6 Structure of the Paper

First we give Niño-Mora’s theorem about the optimality of threshold policies, which is based on four partial conservation law indexability (PCLI) conditions (Section 2). Then we relate the sequence of actions under threshold policies to mechanical words (Section 3). We use the properties of those words to demonstrate that each PCLI condition holds. These conditions concern bounded variation (Section 4), the positivity of so-called marginal work (Section 5), the non-decreasing nature of the marginal productivity index (Section 6), the continuity of that index (Section 7) and a condition that characterises the index as a Radon-Nikodym derivative (Section 8).

Having completed the proof, we then turn to closed-form expressions for the index and numerical methods for evaluating it when such closed forms are not available (Section 9). We demonstrate the accuracy of such numerical methods and show how the index varies as its parameters change. Also, we compare the performance of Whittle’s index policy with other well-known heuristics (Section 9).

Finally, we discuss interesting avenues for further work (Section 10). The appendices contain detailed proofs about the relation of itineraries to mechanical words (Appendix A), of a key majorisation inequality (Appendix B), about the linear systems orbits to which this majorisation result is applied (Appendix C) and of the optimality of threshold policies for the LQG problem with costly observations (Appendix D).

### 2. Verification Theorem for the Optimality of Threshold Policies

We present a theorem which guarantees the optimality of threshold policies for two-action Markov decision problems under certain hypotheses. This is a special case of a theorem due to Niño-Mora (2015) which extends previous work on countable state spaces (Niño-Mora, 2001, 2006) to problems where the state space is an interval of the real line. Niño-Mora calls the theorem’s hypotheses the partial conservation law indexability (PCLI) conditions. This terminology was chosen to contrast with the strong conservation law conditions of Shan-thikumar and Yao (1992) and the generalised conservation law conditions of Bertsimas and
Niño-Mora (1996), which have their roots in conservation laws for queueing systems under which waiting times are invariant to the queueing discipline (Kleinrock, 1965).

The theorem relates to a family of dynamic programming equations with a single parameter \( \nu \in \mathbb{R} \), which might be interpreted as a wage, a tax, the cost of activating a sensor or a Lagrange multiplier. For each \( \nu \in \mathbb{R} \), we consider the following simple dynamic program for value function \( V(\cdot; \nu) : \mathcal{I} \to \mathbb{R} \), where the state space \( \mathcal{I} \) is an interval of \( \mathbb{R} \):

\[
V(x; \nu) = \min_{a \in \{0, 1\}} \{ \nu c(x, a) + C(x, a) + \beta V(\phi_a(x); \nu) \}
\]

(5)

where \( c : \mathcal{I} \times \{0, 1\} \to \mathbb{R} \) is called the work, \( C : \mathcal{I} \to \mathbb{R} \) is called the cost, the discount factor is \( \beta \in [0, 1) \) and the state transitions are given by \( \phi_a : \mathcal{I} \to \mathcal{I} \) for each action \( a \in \{0, 1\} \). We discuss generalisations of this dynamic program after the proof of the verification theorem.

To state the PCLI conditions, we first recall the well-known definitions of càdlàg, càglàd and bounded-variation functions. Let \( \mathcal{J} \subseteq \mathbb{R} \) and let \( (\mathcal{M}, d) \) be a metric space. A function \( f : \mathcal{J} \to \mathcal{M} \) is a càdlàg function if both the left limit \( f(x^-) := \lim_{u \downarrow x} f(u) \) and the right limit \( f(x^+) := \lim_{u \uparrow x} f(u) \) exist and \( f(x^-) = f(x) \), for all \( x \in \mathcal{J} \). A function \( f : \mathcal{J} \to \mathcal{M} \) is càglàd if both of the limits \( f(x^-), f(x^+) \) exist and \( f(x^+) = f(x) \), for all \( x \in \mathcal{J} \). (Càdlàg is an abbreviation of the French description “continue à droite, limite à gauche”, which means “right continuous, left limit”.)

Let \( \mathcal{I} \) be an interval of \( \mathbb{R} \). A partial subdivision of \( \mathcal{I} \) is a collection \( \{\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n\} \) of closed intervals of \( \mathcal{I} \), where \( n \in \mathbb{Z}_{++} \), such that the set \( \mathcal{I}_j \cap \mathcal{I}_k \) is either empty or consists of a single point that is an endpoint of both \( \mathcal{I}_j \) and \( \mathcal{I}_k \), for all \( 1 \leq j, k \leq n \). Let \( \mathcal{S} \) be the set of partial subdivisions of \( \mathcal{I} \). A function \( f : \mathcal{I} \to \mathbb{R} \) has bounded variation if

\[
\sup_{\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\} \in \mathcal{S}} \sum_{i=1}^{n} |f(b_i) - f(a_i)| < \infty.
\]

Now we need some definitions concerning \( s \)-threshold policies. For state \( x \in \mathcal{I} \), initial action \( a \in \{0, 1\} \) and threshold \( s \in \mathbb{R} = [-\infty, \infty] \), let \( X_t(x, a; s) \) and \( A_t(x, a; s) \) be the state and action at time \( t \in \mathbb{Z}_+ \) when the initial state is \( X_0(x, a; s) = x \), the initial action is \( A_0(x, a; s) = a \) and the \( s \)-threshold policy is followed for \( t \in \mathbb{Z}_+ \), which is the policy that takes action \( A_t(x, a; s) = 1 \) if and only \( X_t(x, a; s) \geq s \). Also, for \( t \in \mathbb{Z}_+ \), let \( X_t(x; s) := X_t(x, 1_{x \geq s}; s) \) and \( A_t(x; s) := A_t(x, 1_{x \geq s}; s) \) denote the state and action when all actions are taken according to the \( s \)-threshold policy. The cost-to-go \( f(x, a; s) \) and the work-to-go \( g(x, a; s) \) are

\[
f(x, a; s) := \sum_{t=0}^{\infty} \beta^t C(X_t(x, a; s), A_t(x, a; s)), \quad g(x, a; s) := \sum_{t=0}^{\infty} \beta^t c(X_t(x, a; s), A_t(x, a; s)).
\]

We also define \( f(x; s) := f(x, 1_{x \geq s}; s) \) and \( g(x; s) := g(x, 1_{x \geq s}; s) \) as the cost-to-go and work-to-go when the first action is taken according to the \( s \)-threshold policy.

We are now ready to state the partial conservation law indexability (PCLI) conditions.

**Definition 5** Consider the family of dynamic programs \( (5) \). For state \( x \in \mathcal{I} \), action \( a \in \{0, 1\} \) and threshold \( s \in \mathbb{R} \), the marginal cost \( c_x(s) \) and the marginal work \( w_x(s) \) are

\[
c_x(s) := f(x, 0; s) - f(x, 1; s), \quad w_x(s) := g(x, 1; s) - g(x, 0; s),
\]
and the marginal productivity index $\lambda(x)$ is

\[ \lambda(x) := c_x(x)/w_x(x). \]

Family (5) is partial conservation law indexable if for all $x \in \mathcal{I}$ and all $s \in \mathbb{R}$:

- PCLI0. The marginal work $w_x(s)$ is a càdlàg function of $s$ with bounded variation.
- PCLI1. The marginal work $w_x(s)$ is positive.
- PCLI2. The marginal productivity index $\lambda(x)$ is non-decreasing and continuous.
- PCLI3. The marginal cost satisfies $c_x(b) - c_x(a) = \int_{[a,b]} \lambda dw_x$ for all $[a,b] \subseteq \mathcal{I}$.

Niño-Mora (2015) uses three rather than four conditions, as he derives an equivalent to our condition PCLI0 from additional assumptions and with a different definition of the $s$-threshold policy resulting in the marginal work $w_x(s)$ being a càdlàg rather than càglàd function of $s$. Condition PCLI3 requires that $\lambda$ is a Radon-Nikodym derivative of the signed Lebesgue-Stieltjes measure (Carter and van Brunt, 2000) corresponding to the marginal cost $c_x(\cdot)$ with respect to the marginal work $w_x(\cdot)$. In analyses of discrete-state Markov decision processes, condition PCLI3 is not required as it is implied by PCLI1 and PCLI2 (Niño-Mora, 2015).

**Theorem 6** Suppose PCLI0-PCLI3 hold for the family (5). Let $x \in \mathcal{I}$ and $\nu \in \mathbb{R}$.

1. If $\lambda(s) = \nu$ for some $s \in \mathcal{I}$, then the $s$-threshold policy is an optimal policy and actions 0 and 1 are both optimal in state $x$ if and only if $\lambda(x) = \nu$.
2. If $\lambda(s) > \nu$ for all $s \in \mathcal{I}$, then the always-active policy is the unique optimal policy.
3. If $\lambda(s) < \nu$ for all $s \in \mathcal{I}$, then the always-passive policy is the unique optimal policy.
4. The family is indexable and its Whittle index is the marginal productivity index $\lambda$.

**Proof** Let $Q(x, a; s, \nu)$ be the total-cost-to-go under the $s$-threshold policy from initial state $x$ when the initial action is $a$, so that

\[ Q(x, a; s, \nu) = f(x, a; s) + \nu g(x, a; s). \]

**Claim 1.** Suppose that $s \in \mathcal{I}$ with $\lambda(s) = \nu$. We shall show that

\[ Q(x, 0; s, \nu) \leq Q(x, 1; s, \nu) \quad \text{if} \quad x \leq s \]
\[ Q(x, 1; s, \nu) \leq Q(x, 0; s, \nu) \quad \text{if} \quad x \geq s \]

and these inequalities are strict if and only if $\lambda(x) \neq \lambda(s)$. Thus the value function of the $s$-threshold policy, given by

\[ V(x; s, \nu) := \begin{cases} Q(x, 0; s, \nu) & \text{if} \quad x \leq s \\ Q(x, 1; s, \nu) & \text{if} \quad x \geq s \end{cases} \]
satisfies the dynamic program (5). Therefore the $s$-threshold policy is optimal and actions 0 and 1 are both optimal if and only if $\lambda(x) = \nu$.

Say $x \leq s$. Noting that $w_x$ has bounded variation (by PCLI0) and $\lambda$ is a continuous function (by PCLI2), Lebesgue-Stieltjes integration-by-parts (Carter and van Brunt, 2000) gives

$$\int_{[x,s]} \lambda dw_x + \int_{[x,s]} w_x d\lambda = \lambda(s^-)w_x(s^-) - \lambda(x^-)w_x(x^-).$$

Now the second integral is non-negative, as $\lambda$ is non-decreasing (by PCLI2) and the integrand $w_x$ is non-negative (by PCLI1). Also, $\lambda(s^-)w_x(s^-) = \lambda(s)w_x(s)$ and $\lambda(x^-)w_x(x^-) = \lambda(x)w_x(x)$, as $\lambda$ is continuous (by PCLI2) and $w_x$ is a càdlàg function (by PCLI0). Therefore

$$\int_{[x,s]} \lambda dw_x \leq \lambda(s)w_x(s) - \lambda(x)w_x(x).$$

Using PCLI3, it follows that the marginal cost is bounded by

$$c_x(s) = c_x(x) + \int_{[x,s]} \lambda dw_x \leq c_x(x) + \lambda(s)w_x(s) - \lambda(x)w_x(x) = \lambda(s)w_x(s)$$

where we cancelled two terms as the definition of $\lambda$ gives $c_x(x) = \lambda(x)w_x(x)$. Finally, using the definitions of $c_x$ and $w_x$ in conjunction with this bound gives

$$Q(x, 1; s, \nu) - Q(x, 0; s, \nu) = f(x, 1; s) + \nu g(x, 1; s) - f(x, 0; s) - \nu g(x, 0; s)$$

$$= \nu w_x(s) - c_x(s)$$

$$\geq \nu w_x(s) - \lambda(s)w_x(s)$$

$$= 0$$

where the last line follows from the hypothesis that $\lambda(s) = \nu$.

Otherwise, $x \geq s$, and the claim follows easily from a symmetric argument. However, for additional insight, let us reorder the argument for $x \geq s$ as a single chain of equalities:

$$Q(x, 0; s, \nu) - Q(x, 1; s, \nu) = -\nu w_x(s) + c_x(s)$$

$$= -\nu w_x(s) + c_x(x) - \int_{[s,x]} \lambda dw_x$$

$$= -\nu w_x(s) + c_x(x) - \lambda(x)w_x(x) + \lambda(s)w_x(s) + \int_{[s,x]} w_x d\lambda$$

$$= (\lambda(s) - \nu)w_x(s) + \int_{[s,x]} w_x d\lambda$$

$$= \int_{[s,x]} w_x d\lambda.$$
Therefore the always-active policy is the unique optimal policy.

We consider two cases: either the interval \( I \) is left-open, being of the form \((l, h)\) or \([l, h]\), or it is left-closed, being of the form \([l, h)\) or \((l, h]\). Say the interval is left-open and consider any \( s \in \mathbb{R} \) with \( s \leq l \). As \( \phi_a : I \to I \) for \( a \in \{0, 1\} \), we have \( X_t(x, a; s') > l \) for all \( t \in \mathbb{Z}_+ \) and all \( s' > l \). Thus \( A_t(x, a; l^+) = 1 = A_t(x, a; s) \) for all \( t \in \mathbb{Z}_+ \). Therefore

\[
c_x(l^+) = c_x(s) \quad \text{and} \quad w_x(l^+) = w_x(s). \tag{6}
\]

Recalling Theorem 6.1.3 (i) of Carter and van Brunt (2000), which says that if \( w_x \) is continuous at \( u \) then \( \int_{[u, x)} \lambda dw_x = \int_{(u, x)} \lambda dw_x \), we have

\[
c_x(x) - c_x(s) = \lim_{u \downarrow l} (c_x(x) - c_x(u)) \quad \text{by (6)}
= \lim_{u \downarrow l} \int_{[u, x)} \lambda dw_x \quad \text{by PCLI3}
= \lim_{u \downarrow l} \int_{(u, x)} \lambda dw_x \quad \text{by (6)}
= \int_{(l, x)} \lambda dw_x.
\]

Now \( \lambda(l^+), \lambda(x) \in \mathbb{R} \), and \( \lambda \) is non-decreasing, so \( \lambda \) has bounded variation on the interval \((l, x)\). Thus, the integration-by-parts argument used for Claim 1 gives

\[
Q(x, 0; s, \nu) - Q(x, 1; s, \nu) = c_x(s) - \nu w_x(s)
= c_x(x) - \int_{(l, x)} \lambda dw_x - \nu w_x(s)
= c_x(x) - \lambda(x^-) w_x(x^-) + \lambda(l^+) w_x(l^+) + \int_{(l, x)} \lambda dw_x - \nu w_x(s)
= (\lambda(l^+) - \nu) w_x(l) + \int_{(l, x)} \lambda dw_x
> 0
\]

as claimed.

Now say the interval is left-closed, with lower limit \( l \) and consider any \( s \in \mathbb{R} \) with \( s < l \). As for left-open intervals, we argue that \( c_x(l) = c_x(s) \) and \( w_x(l) = w_x(s) \). The argument for Claim 1 (without needing to consider any \( \lim_{u \downarrow l} \), since PCLI3 immediately covers intervals of the form \([l, x))\) then gives \( Q(x, 0; s, \nu) - Q(x, 1; s, \nu) > 0 \) as claimed.

**Claim 3.** Suppose \( \lambda(s') < \nu \) for all \( s' \in I \). Then an argument similar to for Claim 2 gives

\[
Q(x, 0; \infty, \nu) < Q(x, 1; \infty, \nu)
\]

for all \( x \in I \).

Therefore the always-passive policy is the unique optimal policy.

**Claim 4.** If \( \lambda(x) < \nu \) then either \( \lambda(s) = \nu \) for some \( s \in I \) in which case Claim 1 shows that action 0 is strictly optimal, or \( \lambda(s) < \nu \) for all \( s \in I \) in which case Claim 3 shows that
action 0 is strictly optimal. If \( \lambda(x) > \nu \) then a similar argument using Claims 1 and 2 shows that action 1 is strictly optimal. If \( \lambda(x) = \nu \) then Claim 1 shows that actions 0 and 1 are both optimal. Thus

\[
\begin{align*}
\text{action 0 is optimal in state } x & \text{ if and only if } \lambda(x) \leq \nu \\
\text{action 1 is optimal in state } x & \text{ if and only if } \lambda(x) \geq \nu 
\end{align*}
\]

for all \( x \in \mathcal{I} \).

Therefore, the family is indexable with Whittle index \( \lambda \).

This completes the proof.

**Remark 7** The verification theorem presented by Niño-Mora (2015) is considerably more general than that given here as it covers the case of stochastic rather than deterministic transition kernels, but the proof is considerably longer. Further generalisation of that theorem may be interesting, for instance to state spaces that are subsets of \( \mathbb{R}^n \) and to semi-Markov problems.

3. Itineraries and Mechanical Words

The transitions from state-to-state under an \( s \)-threshold policy are given by a discontinuous mapping known as a map-with-a-gap, and the corresponding action sequences are known as the itinerary of that map. The purpose of this section is to introduce a central result of the paper (Theorem 16) which show that these itineraries are given by special binary strings known as mechanical words. Before giving that result, we must first describe some important properties of maps-with-gaps and mechanical words.

3.1 Maps-with-Gaps

Many phenomena involve the iterated application of discontinuous maps. Such phenomena are important in control problems (Haddad et al., 2014), in physics, electronics and mechanics (Bernardo et al., 2008; Makarenkov and Lamb, 2012), economics (Tramontana et al., 2010), biology and medicine (Aihara and Suzuki, 2010). Such maps either arise directly from a discrete-time model or they may arise as the Poincaré maps of continuous-time systems.

For the purposes of this paper, we shall call a function \( \psi : \mathcal{I} \rightarrow \mathcal{I} \), where \( \mathcal{I} \) is an interval of \( \mathbb{R} \), a map-with-a-gap if

\[
\psi(x) = \begin{cases} 
\phi_0(x) & \text{if } x < z \\
\phi_1(x) & \text{otherwise}
\end{cases}
\]

for some functions \( \phi_0 : \mathcal{I} \rightarrow \mathcal{I} \) and \( \phi_1 : \mathcal{I} \rightarrow \mathcal{I} \) and some threshold \( z \in [-\infty, \infty] \). This allows for thresholds \( z \notin \mathcal{I} \), so the map \( \psi \) may not really have a discontinuity, but this is helpful to allow for a general analysis of threshold policies. The key concepts associated with such maps are given in the following definition, as illustrated in Figure 3.
Figure 3: Map-with-a-gap. The orbit traces the path $ABCD\ldots$ corresponding to the itinerary $10010\ldots$. The map has $\phi_0(x) = x + 1$, $\phi_1(x) = 1/(a_1 + 1/(x + 1))$ with $a_1 = 0.1$ and threshold $z = 5$.

**Definition 8** Suppose $\psi : \mathcal{I} \to \mathcal{I}$ is a map-with-a-gap with threshold $z \in [-\infty, \infty]$. The orbit of $\psi$ from initial state $x \in \mathcal{I}$ is the sequence $(x_t : t \in \mathbb{Z}_+)$ with $x_1 = x$ and $x_{t+1} = \psi(x_t)$ for $t \in \mathbb{Z}_+$.

The itinerary of $\psi$ from state $x \in \mathcal{I}$ is the infinite binary string $\sigma(x|z)$ with $t^{th}$ letter $\sigma(x|z)_t := 1_{x_t \geq z}$ for $t \in \mathbb{Z}_+$.

It is helpful to view such itineraries as words as we now explain.

### 3.2 Mechanical Words and $M$-Words

In this paper, a word $w$ is a string on the alphabet $\{0, 1\}$ and the empty word is denoted by $\epsilon$. The length of a word $w$ is the number of letters in the string, which is finite or countably infinite, and is denoted by $|w|$. The $k^{th}$ letter of word $w$ is $w_k$ for $k \in \mathbb{Z}_+$ with $k \leq |w|$. Letters $i$-through-$j$ of word $w$ are denoted by $w_{i:j} := w_i w_{i+1} \ldots w_j$ for $i, j \in \mathbb{Z}_+$ with $i \leq j \leq |w|$. For $j < i$, we treat $w_{i:j}$ as the empty word. The reverse word of a finite word $w$ is denoted by $w^R := w_{|w|} \ldots w_2 w_1$. A finite word satisfying $w^R = w$ is called a palindrome.

The concatenation of a finite word $u$ and a word $v$ is denoted by $uv$. For $n \in \mathbb{Z}_+$, the $n$-fold concatenation of a finite word $w$ is denoted by $w^n$, with the convention that $w^0 = \epsilon$. 
and the word resulting from infinitely concatenating the word \(w\) is denoted by \(w^\omega\). For an infinite word \(w\) and \(n \in \mathbb{Z}_{++}\) we define \(w^n = w^\omega = w\).

A finite word \(f\) is a factor of a word \(w\) if \(w = ufv\) for some finite word \(u\) and some word \(v\). The number of times that word \(f\) appears in \(w\), overlapping appearances included, is denoted by \(|w|_f\). A finite word \(p\) is a prefix of word \(w\) if \(w = ps\) for some word \(s\) and a word \(s\) is a suffix of word \(w\) if \(w = ps\) for some finite word \(p\).

We say that a word \(u\) is lexicographically less than a word \(v\), written \(u < v\), if either \(u\) is a finite word and \(v = ua\) for some non-empty word \(a\), or if \(u = a0b\) and \(v = a1c\) for some finite word \(a\) and some words \(b\) and \(c\). We use \(>,\leq\) and \(\geq\) for the other lexicographic ordering relations.

We say an infinite word \(w\) is the limit of a sequence of words \((x^{(n)} : n \in \mathbb{Z}_{++})\) and write

\[
\lim_{n \to \infty} x^{(n)} = w
\]

if for each \(i \in \mathbb{Z}_{++}\) there is an \(n \in \mathbb{Z}_{++}\) such that \(w_i = x_i^{(m)}\) for all \(m \in \mathbb{Z}_{++}\) with \(m \geq n\).

**Example.** For \(w = 010111\) we have \(|w| = 6, w_3 = 0, w_{2:4} = 101, |w|_00 = |w|_11 = 2\) and \(w^2 = wuw = 010111010111\). Also for \(a = 01, b = 11\) we have \(w = aab\) and \(a < w < b\).

One can view the itineraries of maps-with-gaps as words.

**Definition 9** Sequence \((x_k : k \in \mathbb{Z}_{++})\) is the \(x\)-threshold orbit for \(\phi_0, \phi_1\) and \(x \in I\) if

\[
x_1 = \phi_1(x), \quad x_{k+1} = \begin{cases} 
\phi_1(x_k) & \text{if } x_k \geq x \\
\phi_0(x_k) & \text{if } x_k < x 
\end{cases} \quad \text{for } k \in \mathbb{Z}_{++}.
\]

The \(x\)-threshold word for \(\phi_0\) and \(\phi_1\), denoted by \(\pi(x, \phi_0, \phi_1)\), is the shortest word \(w\) with

\[
x_{k+1} = \phi_{(w^\omega)_k}(x_k) \quad \text{for } k \in \mathbb{Z}_{++}.
\]

We shall just say “\(x\)-threshold orbit”, “\(x\)-threshold word” and write \(\pi(x)\) in place of \(\pi(x, \phi_0, \phi_1)\) when \(\phi_0\) and \(\phi_1\) are obvious from the context. Clearly, the itinerary is related to the \(x\)-threshold word by \(\sigma(x|x) = 1\pi(x)^\omega\).

The rate of any any finite non-empty word \(w\) is the ratio

\[
\text{rate}(w) := |w|_1/|w|
\]

whereas for an infinite word \(w\), when the limit exists, we define

\[
\text{rate}(w) := \lim_{n \to \infty} |w_{1:n}|_1/n.
\]

While some authors refer to such ratios as the “slope” of a word, we use the term “rate” as the “slope” of a word \(w\) is sometimes defined as the ratio \(|w|_1/|w|_0\) and this seems justified from a geometrical point of view in terms of digital straight lines (Berstel et al., 2008).

We characterise itineraries of maps-with-gaps in terms of following type of words.

**Definition 10** The \(M\)-word of rate \(\alpha \in [0, 1]\) is the shortest word \(w\) such that

\[
(w^\omega)_n = \lfloor \alpha n \rfloor - \lfloor \alpha(n - 1) \rfloor \quad \text{for } n \in \mathbb{Z}_{++}.
\]

If \(\alpha\) is rational then \(w\) is called a Christoffel word. If \(\alpha\) is irrational then \(w\) is called a Sturmian \(M\)-word.
Example. The shortest $M$-words are the words 0, 1 and 01 with rates 0, 1 and $\frac{1}{2}$.

We call such words $M$-words as our definition is closely related to the set of mechanical words. For a given slope $\alpha \in [0, 1]$ and intercept $\rho \in \mathbb{R}$, Morse and Hedlund (1940) defined the upper and lower mechanical words to be the infinite sequences, for $n \in \mathbb{Z}_+$,

$$u_n = \lceil \alpha(n + 1) + \rho \rceil - \lceil \alpha n + \rho \rceil$$

$$l_n = \lfloor \alpha(n + 1) + \rho \rfloor - \lfloor \alpha n + \rho \rfloor$$

Lothaire (2002) and Berstel et al. (2008) offer rich introductions to the mathematics of mechanical words, while Bousch and Mairesse (2002) and Altman et al. (2000) explore other optimisation problems that give rise to such words. Our $M$-words are prefixes of lower-mechanical-words-of-zero-intercept up to a change of indexing from $n \in \mathbb{Z}_+$ to $n \in \mathbb{Z}_{++}$.

It is not hard to see that the Christoffel word of rate $a/b$, where $a, b$ are relatively-prime integers, has length $b$. In contrast, Sturmian $M$-words are infinite and aperiodic.

In general the $M$-word $w$ of rate $\alpha$ does have rate($w$) = $\alpha$. Indeed if $w$ is the $M$-word of rate $a/b$ for some $a, b \in \mathbb{Z}_{++}$, then

$$\text{rate}(w) = |w|_1/|w| = |(a/b)|w|/|w| = a/b,$$

whereas, if $w$ is an $M$-word of irrational rate $\alpha$, then

$$\text{rate}(w) = \lim_{n \to \infty} |w_{1:n}|_1/n = \lim_{n \to \infty} \lfloor \alpha n \rfloor/n = \alpha.$$

Furthermore, as remarked by Christoffel (1875), all Christoffel words other than the words 0 and 1 are of the form $0p1$ where the word $p$ is a palindrome. Indeed for relatively-prime positive integers $m < n$, the letters of the Christoffel word $w$ of rate $m/n$ satisfy

$$w_{n-k} = \left\lfloor \frac{m}{n} (n-k) \right\rfloor - \left\lfloor \frac{m}{n} (n-k-1) \right\rfloor = \left\lfloor -\frac{m}{n} k \right\rfloor - \left\lfloor -\frac{m}{n} (k+1) \right\rfloor = w_{k+1}$$

for $k = 1, 2, \ldots, n-2$.

The Christoffel words can be defined in other ways. In this paper the most important alternative-but-equivalent definition is in terms of the Christoffel tree (Figure 4), which is an infinite complete binary tree (Berstel et al., 2008) in which each node is labelled with a pair $(u, v)$ of words, called a Christoffel pair. The root of the tree is labelled with the pair $(0, 1)$ and the left and right children of node $(u, v)$ are the nodes $(u, uv)$ and $(uv, v)$. 

![Figure 4: Part of the Christoffel tree.](image-url)
respectively. In fact the Christoffel words are the words 0, 1 and the set of concatenations $uv$ for all $(u, v)$ in the Christoffel tree.

Another definition of Christoffel words is in terms of modular arithmetic, as in the following Lemma, where we use a bar to denote the remainder modulo the length $n = |w|$ of a Christoffel word $w$, so that $\overline{x} := x \mod n$ for $x \in \mathbb{Z}$.

**Lemma 11** Suppose $w$ is a Christoffel word of length $n$. Let $m := |w|_1$ and $p := |w|_0$. Then

$$w_{i+1} = 1_{\overline{mi} \geq p} \quad (i \in \mathbb{Z}_n).$$

**Proof** As $n|\overline{mi}/n = \overline{mi} - \overline{mi}$, the definition of Christoffel words gives

$$w_{i+1} = -|\overline{mi}/n| + \lfloor m(i+1)/n \rfloor$$

$$= (-\overline{mi} + \overline{mi} + m(i+1) - \overline{mi}/n)$$

$$= (-\overline{mi} + \overline{mi} + m(i+1) - \overline{mi}/n - m \overline{1_{mi \geq n}})/n,$$

which simplifies to $1_{\overline{mi} \geq p}$, as claimed. $\blacksquare$

Finally, we give two results about $\mathcal{M}$-words that play a key role elsewhere in the paper. The first result is about conjugacy and lexicographic order. In particular, we say two finite words $a$ and $b$ are conjugate if $a = uv$ and $b = vu$ for some words $u$ and $v$. For instance, the words $a = 00011$ and $b = 01100$ are conjugate.

**Lemma 12** Suppose $w$ is a Christoffel word of length $n$ and that $l$ satisfies $\overline{1m} = 1$ where $m = |w|_1$. Then the conjugates $u(i) := w_{(i+1)n}w_{1:z}$ satisfy

$$w = u(0) \prec u(l) \prec u(2l) \prec \cdots \prec u((n-1)l) = w^R.$$ 

**Proof** Let $x_i := \overline{mi - 1}$, $y_i := \overline{mi}$ and $p := n - m$. Then $x_0 = n - 1$ and $x_{n-1} = p - 1$. As $\gcd(m, n) = 1$, the sequence $x_0, \ldots, x_{n-1}$ is a permutation of $\mathbb{Z}_n$. So, $x_i \notin \{p - 1, n - 1\}$ for $i \in \{1, \ldots, n - 2\}$. As $y_i = x_i + 1$ these results give

$$1_{x_i \geq p} > 1_{y_i \geq p} \quad \text{for } i = 0$$

$$1_{x_i \geq p} = 1_{y_i \geq p} \quad \text{for } i = 1, \ldots, n - 2$$

$$1_{x_i \geq p} < 1_{y_i \geq p} \quad \text{for } i = n - 1.$$

But Lemma 11 gives $u(0)_{j+1} = 1_{y_j \geq p}$ and $u((n-1)l)_{j+1} = 1_{x_j \geq p}$ for $j \in \mathbb{Z}_n$. Thus $u(0) = 0a1$ and $u((n-1)l) = 1a0$ for some word $a$. But $u(0) = w$ and $w$ is a Christoffel word, so $a$ is a palindrome. Therefore $u((n-1)l) = w^R$.

Now for $i = 0, \ldots, n - 2$, the conjugates $u(il)$ and $u((i+1)l)$ are related to $u((n-1)l)$ and $u(0)$ respectively by the same non-zero cyclic rotation. Thus $u(il) = c01d$ and $u((i+1)l) = c10d$ for some words $c$ and $d$ with $dc = a$. Therefore $u(il) \prec u((i+1)l)$ $\blacksquare$

The second result shows how the prefixes of $\mathcal{M}$-words vary as a function of their rates. It requires one more definition.
Definition 13 For each positive integer \( n \), the Farey sequence \( F_n \) is the sequence of rational numbers on \([0,1]\) whose denominator is at most \( n \).

For example, the Farey sequence \( F_5 \) is \( 0, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1 \).

Lemma 14 Suppose \( n \in \mathbb{Z}_{++} \) and \( q \in [0,1] \). Let \( q_1 < q_2 < \cdots < q_m \) be the Farey sequence \( F_n \). Let \( p(s) \) be the first \( n \) letters of the word \( w^\omega \) where \( w \) is the \( \mathcal{M} \)-word of rate \( s \in [0,1] \). Then \( p(q) = p(q_i) \) if and only if either \( q = q_i = 1 \) or \( q \in [q_i, q_{i+1}) \) for some \( 1 \leq i < m \).

Proof Let \( b(q) := ([q], [2q], \ldots, [nq]) \) and consider the intervals \( Q_i := [q_i, q_{i+1}) \) for \( i < m \) and \( Q_m := \{1\} \). As the line \( y = qx \) hits an integer point \((x,y) \in \mathbb{Z}^2 \) with \( 1 \leq x \leq n \) and \( 0 \leq y \leq x \) if and only if \( q \) is an element of \( F_n \), it follows that \( b(q) = b(q_i) \) if and only if \( q \in Q_i \). Let \( g(x_1, x_2, \ldots, x_n) := (x_1, x_2 - x_1, \ldots, x_n - x_{n-1}). \) By definition of \( \mathcal{M} \)-words, \( p(q) = g(b(q)) \). As \( g \) is invertible it follows that \( p(q) = p(q_i) \) if and only if \( q \in Q_i \). 

3.3 Characterising Itineraries as \( \mathcal{M} \)-Words

Our aim here is to characterise the itineraries of maps-with-gaps. We first set up some notation and then demonstrate a simple result about the lexicographical ordering of itineraries. Then we state an assumption under which itineraries are guaranteed to be specific \( \mathcal{M} \)-words whose proof is given in Appendix A.

Let \( \mathcal{I} \) be an interval of \( \mathbb{R} \) and consider two mappings \( \phi_0 : \mathcal{I} \rightarrow \mathcal{I} \) and \( \phi_1 : \mathcal{I} \rightarrow \mathcal{I} \). For any finite word \( w \), the composition \( \phi_w : \mathcal{I} \rightarrow \mathcal{I} \) is the mapping

\[
\phi_w(x) := \phi_{w|_w} \circ \cdots \circ \phi_{w_2} \circ \phi_{w_1}(x) \quad \text{and} \quad \phi_e(x) := x.
\]

A simple application of compositions gives the following result about lexicographic ordering of the itineraries \( \sigma(\cdot|z) \) of a map-with-a-gap given by mappings \( \phi_0 : \mathcal{I} \rightarrow \mathcal{I} \) and \( \phi_1 : \mathcal{I} \rightarrow \mathcal{I} \) and threshold \( z \).

Lemma 15 Suppose \( \phi_0, \phi_1 \) are increasing mappings and that \( x, y \in \mathcal{I} \) with \( \sigma(x|z) < \sigma(y|z) \). Then \( x < y \).

Proof If \( \sigma(x|z) < \sigma(y|z) \) then \( \sigma(x|z) = a0b \) and \( \sigma(y|z) = a1c \) for some finite word \( a \) and some infinite words \( b, c \), by the definition of lexicographic order. So, the definition of \( \sigma(\cdot|z) \) gives \( \phi_a(x) < z \leq \phi_a(y) \). But \( \phi_a(\cdot) \) increasing as it is a finite composition of increasing functions. It follows that \( x < y \). 

However, without additional assumptions about the mappings \( \phi_0, \phi_1 \), it is not possible to precisely characterise the itineraries of the associated maps-with-gaps.

Assumption A1 Functions \( \phi_0 : \mathcal{I} \rightarrow \mathcal{I} \) and \( \phi_1 : \mathcal{I} \rightarrow \mathcal{I} \), where \( \mathcal{I} \) is an interval of \( \mathbb{R} \), are increasing, contractive and have unique fixed points \( y_0 \) and \( y_1 \) on \( \mathcal{I} \) which satisfy \( y_1 < y_0 \).
Equivalently, for all \( x, y \in I \) such that \( x < y \) and for \( a \in \{0, 1\} \) we have
\[
\phi_a(x) < \phi_a(y) \quad \text{increasing} \quad \text{and} \quad \phi_a(y) - \phi_a(x) < y - x \quad \text{contractive}.
\]

A fixed point for a finite word \( w \), is a solution to the equation \( x = \phi_w(x) \). If Assumption A1 holds, then it turns out that there is a unique such fixed point on \( I \) for any finite non-empty word \( w \) and we shall denote it by \( y_w \).

In general, it is not clear what a “fixed point” corresponding to an infinite word \( w \) might mean. One approach might be to consider a sequence \( (w^{(n)} : n \in \mathbb{Z}^{++}) \) of words with \( w = \lim_{n \to \infty} w^{(n)} \) and to define “\( y_w \)” as \( \lim_{n \to \infty} y_{w^{(n)}} \) if that limit exists. However, for any word \( a \), the sequence with elements \( w^{(n)}a \) also converges to \( w \) and it is not hard to find examples where
\[
\lim_{n \to \infty} y_{w^{(n)}} \neq \lim_{n \to \infty} y_{w^{(n)}}a.
\]

Therefore we shall only define fixed points for a particular class of infinite words, as follows. Let \( 0s \) be the Sturmian \( M \)-word of rate \( \alpha \). Consider the sequence of Christoffel words \( 0w^{(n)}1 \) that lie on the following path through the Christoffel tree. We start from the root, so that \( w^{(1)} = \epsilon \). Then for \( n \in \mathbb{Z}^{++} \), we set \( 0w^{(n+1)}1 \) equal to the left child of \( 0w^{(n)}1 \) if the slope of \( 0w^{(n)}1 \) exceeds \( \alpha \) and equal to the right child otherwise. We call
\[
y_s := \lim_{n \to \infty} y_{01w^{(n)}} = \lim_{n \to \infty} y_{10w^{(n)}}
\]
the fixed point of the Sturmian \( M \)-word \( 0s \). The fact that these limits exist and are equal is proved in Appendix A.

We are now ready to fully characterise the itineraries of maps-with-gaps when the initial point equals the threshold.

**Theorem 16** Suppose A1 holds, \( 0p1 \) is a Christoffel word and \( 0s \) is a Sturmian \( M \)-word. Then the fixed points \( y_{01p}, y_{10p}, y_s \) exist in \( I \). Also, the itinerary \( \sigma(z|z) \) is a lexicographically non-increasing function of \( z \in I \) and is of the form \( \sigma(z|z) = 1\pi(z)^\omega \) for some mapping \( \pi : I \to \{0, 1\}^* \) whose image is the set of \( M \)-words. Specifically,
\[
\sigma(z|z) = \begin{cases} 
1^\omega & \text{if and only if } z \leq y_1 \\
(10p)^\omega & \text{if and only if } z \in [y_{01p}, y_{10p}] \\
10s & \text{if and only if } z = y_s \\
10^\omega & \text{if and only if } z \geq y_0.
\end{cases}
\]

This result is previously known for linear maps-with-gaps (Rajpathak et al., 2012), although those authors do not draw any relation to mechanical words. Dance and Silander (2015) previously extended those authors’ proof to the nonlinear case under Assumption A1. The proof presented in Appendix A of this paper can be seen as a simplification of that extension. On the other hand, it is known that itineraries of a broader class of nonlinear maps-with-gaps that do not necessarily satisfy Assumption A1 also correspond to
mechanical words (Kozyakin, 2003). However such generality comes at a cost, as it is not clear in that work which range of thresholds gives rise to which words.

Finally, not all the maps-with-gaps considered in this paper satisfy Assumption A1. However, this does not always prevent the application of Theorem 16. Notably for $\mathcal{I} := [0, \infty)$ and $a \in (0, \infty)$, the pair

$$\phi_0(x) = x + 1,$$
$$\phi_1(x) = 1/(a + 1/(x + 1))$$

involves the non-contractive map $\phi_0$. Nevertheless, after the change of coordinates

$$g : x \mapsto x/(x + 1),$$

the transformed functions

$$\tilde{\phi}_0(x) := g(\phi_0(g^{-1}(x))) = 1/(2 - x),$$
$$\tilde{\phi}_1(x) := g(\phi_1(g^{-1}(x))) = 1/(2 + a - x)$$

and the interval $\tilde{\mathcal{I}} := [0, 1]$ do satisfy Assumption A1. Indeed

$$\frac{d\tilde{\phi}_1(x)}{dx} = 1/(2 + a - x)^2 \in (0, 1]$$

for $x \in \tilde{\mathcal{I}}$ and $a \in [0, \infty)$, and this derivative only equals 1 for the endpoint $x = 1$. Thus $\tilde{\phi}_1$ is increasing and contractive on $\tilde{\mathcal{I}}$. Noting that $\tilde{\phi}_0(x) = \lim_{a \to 0} \tilde{\phi}_1(x)$, the same holds for $\tilde{\phi}_0$. Also $\tilde{\phi}_1$ has a fixed point at $y(a) = (2 + a - \sqrt{a^2 + 4a})/2$ which lies in $\tilde{\mathcal{I}}$ for $a \in [0, \infty)$, and $\tilde{\phi}_0$ has a fixed point at $y(0) = 1 > y(a)$. As $g$ is an increasing function, all conclusions of Theorem 16 still hold for the original functions $\phi_0, \phi_1$.

We conclude our discussion of mechanical words by showing that the itinerary, viewed as a function of the threshold, has at most a polynomial number of discontinuities. This result is important for changing the order of certain summations when showing that conditions PCLI0 and PCLI3 hold.

**Theorem 17** Suppose $\phi_0, \phi_1$ satisfy A1, that $n \in \mathbb{Z}_{++}$ and $x \in \mathcal{I}$. Then $\sigma(x|s)$ is a lexicographically non-increasing function of $s \in \mathcal{I}$. Also, for any fixed $x, s \in \mathcal{I}$, we have

$$\sigma(x|s)_{1:n} = l^m w$$

for some $l \in \{0, 1\}$, some $m \in \{0, 1, \ldots, n\}$, and some factor $w$ of a lower mechanical word. Furthermore, for any $x \in \mathcal{I}$, the mapping $s \mapsto \sigma(x|s)_{1:n}$ for $s \in \mathcal{I}$ has at most a polynomial number $p(n)$ of discontinuities.

The proof of this result is given at the end of Appendix A.
4. Bounded Variation Condition (PCLI0)

The next few sections of this paper prove that conditions PCLI0-PCLI3 hold for the systems considered in the introduction, that is, systems satisfying the following condition.

**Condition D.** For the tuple \((I, C, \phi_0, \phi_1, \beta)\),

- The interval \(I\) and cost function \(C\) satisfy Condition C.
- The transitions \(\phi_0 : I \to I\) and \(\phi_1 : I \to I\) are of the form
  \[
  \phi_0(x) := \frac{r^2 x + 1}{a_0 r^2 x + a_0 + 1}, \quad \phi_1(x) := \frac{r^2 x + 1}{a_1 r^2 x + a_1 + 1},
  \]
  for some \(0 \leq a_0 < a_1 < \infty\) and some \(r \in (0, 1]\).
- The discount \(\beta\) is in \([0, 1)\).

Associated with any such tuple, we consider the family of dynamic programs with a single parameter \(\nu \in \mathbb{R}\), of the form (5), in which the work is of the form \(c(x, a) := a\) and where the cost is independent of the action:

\[
V(x; a) = \min_{a \in \{0, 1\}} \{\nu a + C(x) + \beta V(\phi_0(x); \nu)\}.
\]

(In fact, the results of the paper cover a slightly more general set of systems than those satisfying Condition D. It is possible to work with a smaller interval \(I\) as long as the initial state is contained in \(I\) and the open interval containing the fixed points \((y_1, y_0)\) is in \(I\).)

After a simple Lemma, we give Proposition 19 which shows that condition PCLI0 holds.

**Lemma 18** Suppose \(\phi_0, \phi_1 : I \to I\) are continuous functions, \(t \in \mathbb{Z}_+, x \in I\) and \(a \in \{0, 1\}\). Then \(X_t(x, a; s)\) and \(A_t(x, a; s)\) are piecewise-constant càglàd functions of \(s \in \mathbb{R}\).

**Proof** Let \(\mathcal{P}\) denote the set of piecewise-constant càglàd functions from \(\mathbb{R}\) to \(\mathbb{R}\) and recall the following easily-demonstrated facts about limits.

(i) If \(a \in \mathcal{P}\) and \(f : \mathbb{R} \to \mathbb{R}\) is continuous then \(f \circ a\) is in \(\mathcal{P}\).

(ii) If \(a \in \mathcal{P}\) then both \(s \mapsto 1_{a(s) < s}\) and \(s \mapsto 1_{a(s) \geq s}\) are in \(\mathcal{P}\).

(iii) If \(a, b \in \mathcal{P}\) then \(s \mapsto a(s) b(s)\) is in \(\mathcal{P}\).

(iv) If \(a, b \in \mathcal{P}\) then \(s \mapsto a(s) + b(s)\) is in \(\mathcal{P}\).

Consider the claim about \(X_t(x, a; s)\). We use induction on \(t \in \mathbb{Z}_+\). In the base case, \(X_0(x, a; s) = x\) by definition, and the mapping \(s \mapsto x\) is in \(\mathcal{P}\). For the inductive step, suppose \(s \mapsto X_t(x, a; s)\) is in \(\mathcal{P}\). Then, by definition

\[
X_{t+1}(x, a; s) = \phi_0(X_t(x, a; s))1_{X_t(x, a; s) < s} + \phi_1(X_t(x, a; s))1_{X_t(x, a; s) \geq s}.
\]
As $\phi_0$ is continuous, the induction hypothesis and (i) show that $s \mapsto \phi_0(X_t(x; a; s))$ is in $\mathcal{P}$. Also, the induction hypothesis and (ii) show that $s \mapsto 1_{X_t(x, a; s) \leq s}$ is in $\mathcal{P}$. Thus (iii) shows that $T_0$ is in $\mathcal{P}$. A similar argument shows that $T_1$ is in $\mathcal{P}$. Therefore (iv) shows that $s \mapsto T_0(s) + T_1(s) = X_{t+1}(x, a; s)$ is in $\mathcal{P}$.

The claim about $A_t(x, a; s) = 1_{X_t(x, a; s) \leq s}$ follows from (ii) and the fact that $s \mapsto X_t(x, a; s)$ is in $\mathcal{P}$. This completes the proof. 

Recalling the definitions of work-to-go and marginal work (Definition 5), we are now ready to show that condition PCLI0 holds.

**Proposition 19** Suppose $\langle I, C, \phi_0, \phi_1, \beta \rangle$ satisfy Condition D, $t \in \mathbb{Z}_+$, $x \in I$ and $a \in \{0, 1\}$. Then the work-to-go $g(x, a; s)$ and the marginal work $w_x(s)$ are càdlàg functions of $s \in \mathbb{R}$ with bounded variation.

**Proof** First we show that $s \mapsto g(x, a; s)$ is càdlàg. For any $b \in \mathbb{R}$, we have

$$g(x, a; b-) = \lim_{s \uparrow b} \sum_{t=0}^{\infty} \beta^t A_t(x, a; s)$$

by definition

$$= \lim_{n \to \infty} \sum_{t=0}^{\infty} \beta^t A_t(x, a; s_n)$$

for any sequence with $s_n \uparrow b$.

Now $|\beta^t A_t(x, a; s)| \leq \beta^t$ for $t \in \mathbb{Z}_+$ and $\sum_{t=0}^{\infty} \beta^t < \infty$. Thus the dominated convergence theorem (treating sums as integrals with respect to the counting measure) gives

$$\lim_{n \to \infty} \sum_{t=0}^{\infty} \beta^t A_t(x, a; s_n) = \sum_{t=0}^{\infty} \lim_{n \to \infty} \beta^t A_t(x, a; s_n)$$

$$= \sum_{t=0}^{\infty} \beta^t A_t(x, a; b)$$

by Lemma 18

$$= g(x, a; b).$$

Therefore $s \mapsto g(x, a; s)$ is a càdlàg function.

Next we show that $s \mapsto g(x, a; s)$ has bounded variation. Let $D_t$ be the set of discontinuities of $A_t(x, a; s)$ as a function of $s \in \mathbb{R}$. As $s \mapsto A_t(x, a; s)$ is a piecewise-constant and càdlàg with values in $\{0, 1\}$, for any $b, c \in \mathbb{R}$ with $b \leq c$, we can write

$$A_t(x, a; c) - A_t(x, a; b) = \sum_{d \in D_t} a_t(d) 1_{d \in [b, c)}$$

for some $a_t(d) \in \{-1, 1\}$. So for any partial subdivision $S_n = \{[b_1, c_1], \ldots, [b_n, c_n]\}$ of $I$,

$$\sum_{i=1}^{n} |g(x, a; c_i) - g(x, a; b_i)|$$

$$= \sum_{i=1}^{n} \left| \sum_{t=0}^{\infty} \beta^t A_t(x, a; c_i) - \sum_{t=0}^{\infty} \beta^t A_t(x, a; b_i) \right|$$

...
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\[
= \sum_{i=1}^{n} \left| \sum_{t=0}^{\infty} \beta^t \left( A_t(x, a; c_i) - A_t(x, a; b_i) \right) \right| \\
= \sum_{i=1}^{n} \sum_{t=0}^{\infty} \sum_{d \in D_t} \beta^t a_t(d) 1_{d \in [b_i, c_i]} \\
\leq \sum_{i=1}^{n} \sum_{t=0}^{\infty} \sum_{d \in D_t} \beta^t 1_{d \in [b_i, c_i]} \\
= \sum_{t=0}^{\infty} \sum_{d \in D_t} \sum_{i=1}^{n} \beta^t 1_{d \in [b_i, c_i]} \\
\leq \sum_{t=0}^{\infty} \sum_{d \in D_t} \beta^t \\
\leq \sum_{t=0}^{\infty} \beta^t p(t)
\]

using Tonelli’s theorem, the fact that $S_n$ is a partial subdivision, and the polynomial function $p(t)$ of Theorem 17. But the bound $\sum_{t=0}^{\infty} \beta^t p(t)$ is finite and independent of the choice of $S_n$. Therefore $s \mapsto g(x, a; s)$ has bounded variation.

Now $w_x(s)$ is defined as $g(x, 1; s) - g(x, 0; s)$. But the difference of càglàd functions is càglàd and the difference of functions of bounded variation has bounded variation. Therefore $s \mapsto w_x(s)$ is a càglàd function of bounded variation. This completes the proof. ■

5. Positivity of Marginal Work (PCLI1)

We prove that condition PCLI1 holds. The argument is based on the following notion of swapping, which is partly inspired by results about the Burrows-Wheeler transform of Christoffel words (Berstel et al., 2008, Chapter 6).

**Definition 20** A finite word $a$ swaps to a finite word $b$ if either $a = b$ or there exist words $p_1, q_1, p_2, q_2, \ldots, p_n, q_n$ for some $n \in \mathbb{Z}_{++}$ with

\[
\begin{align*}
    a &= p_1 10 q_1, & p_1 01 q_1 &= p_2 10 q_2, & \ldots, & p_n 01 q_n &= b.
\end{align*}
\]

We call a transformation $p_k 10 q_k \to p_k 01 q_k$ an exchange.

**Example.** The word $1100$ swaps to the word $0101$ via the exchanges

\[
1100 \to 1010 \to 0110 \to 0101
\]

for which $p_1 = 1, q_1 = 0, p_2 = \varepsilon, q_2 = 10, p_3 = 01$ and $q_3 = \varepsilon$.

The idea of our proof is as follows. First we find conditions on the number of 1’s in prefixes of two words that make it possible to swap one word for another (Lemma 21). Proposition 22 shows that those conditions are satisfied by the dynamical system (Claim 1), and they imply positivity of marginal work (Claim 2).
**Lemma 21** Suppose $a, b$ are words of common length $|a| = |b| = n \in \mathbb{Z}_+$ with

$$|a|_1 = |b|_1 \quad \text{and} \quad |a_{1:k}|_1 \geq |b_{1:k}|_1 \quad \text{for} \ k < n.$$  

Then $a$ swaps to $b$.

**Proof** Given any words $u, v$ of length $n$, consider the distance

$$d(u, v) := \sum_{i=1}^n ||u_{1:i}|_1 - |v_{1:i}|_1|.$$  

If $a = b$ then $a$ swaps to $b$ after $d(a, b) = 0$ exchanges. Otherwise $a \neq b$ and we shall show that there exists a word $a'$ such that

- $a$ and $a'$ differ by a single exchange,  
  $$d(a', b) = d(a, b) - 1,$$  
- and $a'$ and $b$ satisfy the hypotheses of this Lemma.

Repeating this argument shows that $a$ swaps to $b$ after $d(a, b)$ exchanges.

We now define an appropriate word $a'$. As $a \neq b$ and $|a_{1:i}|_1 \geq |b_{1:i}|_1$ for $i = 1, 2, \ldots, n-1$, there must exist a first index $i$ such that $|a_{1:i}|_1 > |b_{1:i}|_1$. Also, as $|a|_1 = |b|_1$, there must exist a first index $j > i$ such that $a_j = 0$. As $i, j$ are the first such indices, it follows that $a_k = 1$ for $i \leq k < j$. Thus

$$a = a_{1:(j-2)}01a_{(j+1):n}$$  

with the convention that $a_{1:0} = a_{(n+1):n} = \epsilon$. Now consider the word

$$a' := a_{1:(j-2)}01a_{(j+1):n}.$$  

It is immediate that (7) holds. Furthermore, as $a_k = 1$ for $i \leq k < j$, we have

$$|a_{1:(j-1)}|_1 - |b_{1:(j-1)}|_1 \geq |a_{1:i}|_1 - |b_{1:i}|_1 > 0$$  

where the second inequality follows from the definition of $i$. Thus the definition of $a'$ gives

$$|a'_{1:l}|_1 = |a_{1:l}|_1 - 1_{l=j-1} \geq |b_{1:l}|_1$$  

for $l = 1, 2, \ldots, n,$  

so that

$$d(a', b) = \sum_{i=1}^n (|a'_{1:i}|_1 - |b_{1:i}|_1) = d(a, b) - 1.$$  

Therefore (8) holds.

Finally, combining (10) with the fact that $|a'|_1 = |a|_1 = |b|_1$, we conclude that (9) holds. This completes the proof.

The hypotheses of the following proposition are clearly satisfied if $\langle I, C, \phi_0, \phi_1, \beta \rangle$ satisfy Condition D, therefore PCLI1 holds.
Proposition 22 Suppose $I$ is an interval of $\mathbb{R}$ and that $\phi_0 : I \rightarrow I, \phi_1 : I \rightarrow I$ satisfy

(i) $\phi_0(\cdot), \phi_1(\cdot)$ are increasing functions

(ii) $\phi_{01}(z) < \phi_{10}(z)$ for all $z \in I$.

Also suppose that $x \in I$, $s \in \mathbb{R}$ and consider the itineraries

\[ a := 1\sigma(\phi_1(x)|s)_{1:(n-1)} \quad \text{and} \quad b := 0\sigma(\phi_0(x)|s)_{1:(n-1)}. \]

Then

1. For any $n \in \mathbb{Z}_{++}$, we have $|a_{1:n}|_1 \geq |b_{1:n}|_1$.

2. For any $\beta \in (0, 1)$, the marginal work $w_x(s)$ is positive.

Proof We prove Claim 1 by induction. In the base case $|a_1|_1 = 1 \geq 0 = |b_1|_1$. For the inductive step, suppose $|a_{1:k}|_1 \geq |b_{1:k}|_1$ for all $k \leq m$ for some $m \in \mathbb{Z}_{++}$. This induction hypothesis shows that either $|a_{1:m}|_1 > |b_{1:m}|_1$ or $|a_{1:m}|_1 = |b_{1:m}|_1$. In the first case, $|a_{1:(m+1)}|_1 \geq |b_{1:(m+1)}|_1$ as we are only adding one letter to $a_{1:m}$ and $b_{1:m}$. In the second case, the induction hypothesis shows that the words $a_{1:m}$ and $b_{1:m}$ satisfy the assumptions of Lemma 21, so there is a sequence of swaps that transforms $a_{1:m}$ into $b_{1:m}$. Consider any swap $p10q$ to $p01q$ in this sequence. Then hypothesis (ii) gives $\phi_{01}(\phi_p(x)) > \phi_{10}(\phi_p(x))$ and hypothesis (i) implies that $\phi_q(\cdot)$ is increasing, so

\[ \phi_{p01q}(x) > \phi_{p10q}(x). \]

Repeating this argument over the sequence of swaps gives

\[ \phi_{a_{1:m}}(x) > \phi_{b_{1:m}}(x). \]

Thus, it follows from the definition of itineraries that the last letters of $a_{1:(m+1)}, b_{1:(m+1)}$ have $a_{m+1}b_{m+1} \in \{00, 10, 11\}$. Hence $|a_{1:(m+1)}|_1 \geq |b_{1:(m+1)}|_1$. This proves Claim 1.

To prove Claim 2, note that the definition of $w_x(s)$ gives

\[ w_x(s) = \sum_{k=1}^{\infty} \beta^{k-1} (a_k - b_k) \]

\[ = \sum_{k=1}^{\infty} \beta^{k-1} (|a_{1:k}|_1 - |a_{1:(k-1)}|_1 - |b_{1:k}|_1 + |b_{1:(k-1)}|_1) \]

\[ = (1 - \beta) \sum_{k=1}^{\infty} \beta^{k-1} (|a_{1:k}|_1 - |b_{1:k}|_1) \]

\[ \geq 1 - \beta \]

where the last line follows from Claim 1 and the fact that $a_1 = 1, b_1 = 0$. As $\beta < 1$, this completes the proof. \[ \blacksquare \]
6. Non-Decreasing Marginal Cost (PCLI2, First Part)

Condition PCLI2 requires that the marginal productivity \( \lambda(x) = c_x(x)/w_x(x) \) is non-decreasing for \( x \in I \). In view of Theorem 16, the interval \( I \) can be divided up into intervals corresponding to Christoffel words on which \( w_x(x) \) is constant and points corresponding to Sturmian \( \mathcal{M} \)-words. The main result of this section is Proposition 26, which shows that the marginal cost \( c_x(x) \) is non-decreasing for \( x \in I \). In view of Theorem 16, the interval \( I \) can be divided up into intervals corresponding to Christoffel words on which \( w_x(x) \) is constant and points corresponding to Sturmian \( \mathcal{M} \)-words. The main result of this section is Proposition 26, which shows that the marginal cost \( c_x(x) \) is non-decreasing for \( x \) in the interval corresponding to any given Christoffel word of the form \( 0p1 \), for systems satisfying Condition D. This result is complemented by Proposition 27, which shows that \( c_x(x) \) is also increasing for \( x \leq y_1 \) and \( x \geq y_0 \) where \( y_0, y_1 \) are the fixed points of \( \phi_0, \phi_1 \). As the marginal work \( w_x(x) \) is positive by PCLI1, this implies that \( \lambda(x) \) is non-decreasing on such intervals. In the Section 7, we show that \( \lambda(x) \) is continuous for \( x \in I \), so that PCLI2 is satisfied.

A related proof was given by Dance and Silander (2015). However, that proof only covers systems for which the multiplier \( r \) in Condition D is \( r = 1 \), rather than multipliers \( r \in (0,1] \) as addressed here. For \( r = 0 \), the sum in Proposition 26 is a constant, so the analysis presented here is unnecessary. Also, the proof of Dance and Silander (2015) only addressed the cost function \( C(x) = x \), whereas here we generalise to any cost function satisfying Condition C, which includes any cost function of the form \( x^q/q \) for \( q \in [-1, \infty) \). A counterexample presented in Section 9, shows that marginal cost is not necessarily increasing for \( C(x) = x^q/q \) with \( q < -1 \).

We use the following well-known result about majorisation (Marshall et al., 2010). A proof is given in Appendix B.

Lemma 23 Suppose that:

1. The sequences \( a_1:n \) and \( b_1:n \) are non-decreasing sequences on \( \mathbb{R}^{++} \)
2. The inequality \( \sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i \) holds for \( k = 1, 2, \ldots, n \)
3. For \( i = 1, 2, \ldots, n \), the function \( f_i : \mathbb{R}^{++} \rightarrow \mathbb{R} \) is non-increasing and convex
4. For \( i = 2, 3, \ldots, n \), the difference \( f_{i-1}(x) - f_i(x) \) is non-increasing for \( x \in \mathbb{R}^{++} \).

Then

\[
\sum_{i=1}^{n} f_i(a_i) \geq \sum_{i=1}^{n} f_i(b_i).
\]

We apply this majorisation result to the sequences appearing in Lemma 25 below. To state that Lemma, we first define some matrices that are motivated by the form of the Kalman Filter variance updates.

Definition 24 Let \( I \) be the 2-by-2 identity matrix. For \( r \in (0,1] \) and \( 0 \leq a \leq b \), let

\[
F := \begin{pmatrix} r & 1/r \\ ar & (a+1)/r \end{pmatrix}, \quad G := \begin{pmatrix} r & 1/r \\ br & (b+1)/r \end{pmatrix}.
\]

Let \( M(\epsilon) = I, M(0) = F, M(1) = G \) and for any finite non-empty word \( w \) let

\[
M(w) = M(w_{|w|}) \cdots M(w_2)M(w_1).
\]
Thus, if \( a = a_0, b = a_1 \) and \( \phi_0, \phi_1 \) are as in Condition D, we have
\[
\phi_0(x) = \frac{F_{11}x + F_{12}}{F_{21}x + F_{22}}, \quad \phi_1(x) = \frac{G_{11}x + G_{12}}{G_{21}x + G_{22}}.
\]

As remarked in Section 3, the central portion of Christoffel words are palindromes. The following result holds for any palindromes, not just palindromes generated by Christoffel words.

**Lemma 25** Suppose \( p \) is a palindrome, \( r \in (0, 1] \), \( n \in \mathbb{Z}_+ \) and \( x \) satisfies
\[
\phi_p(0) \leq x \leq \phi_p \left( \frac{1}{1 - r^2} \right).
\]

Let \( m := |01p| \) and for \( k = 1, 2, \ldots, m \) let
\[
\begin{align*}
(a_k(x)) := M \left( (01p)^n(01p)_{1:k} \right) \left( \frac{x}{1} \right), & \quad (b_k(x)) := M \left( (10p)^n(10p)_{1:k} \right) \left( \frac{x}{1} \right).
\end{align*}
\]

Then
1. The sequences \( a_{1:m}(x), b_{1:m}(x), c_{1:m}(x) \) and \( d_{1:m}(x) \) are non-decreasing and positive.
2. The inequality \( a_k(x) \leq b_k(x) \) holds for \( k = 1, 2, \ldots, m \).
3. The inequality \( \sum_{i=1}^k c_i(x) \leq \sum_{i=1}^k d_i(x) \) holds for \( k = 1, 2, \ldots, m \).
4. The inequalities \( c_1(x) \leq d_1(x) \) and \( c_k(x) \geq d_k(x) \) hold for \( k = 2, 3, \ldots, m \).
5. The fixed points \( y_{01p} \) and \( y_{10p} \) satisfy
\[
\phi_p(0) \leq y_{01p} < y_{10p} \leq \phi_p \left( \frac{1}{1 - r^2} \right)
\]
so Claims 1-4 hold if \( x \in [y_{01p}, y_{10p}] \).

The proof of the above Lemma is given in Appendix C.

Now we are ready to demonstrate that marginal cost is non-decreasing for a wide range of cost functions on the intervals \([y_{01p}, y_{10p}]\) corresponding to a Christoffel word \( 0p1 \) appearing in Theorem 16. While we state the result for functions satisfying Conditions C1 and C2 separately, as sums of non-decreasing functions are non-decreasing, the result holds for any function satisfying Condition C.

**Proposition 26** Suppose the interval \( \mathcal{I} \) is either \( \mathbb{R}_+ \) or \( \mathbb{R}_{++} \) and that \( C : \mathcal{I} \to \mathbb{R} \) has one of the following two properties:
1. For all \( x \in \mathcal{I} \),
   - the derivatives \( C'(x) := \frac{d}{dx} C(x) \) and \( C''(x) := \frac{d^2}{dx^2} C(x) \) exist,
   - the function \( C(x) \) is concave,
Dance and Silander

• the function \( \frac{1}{x} C'(\frac{1}{x}) \) is non-increasing and convex
• and the function \( \frac{1}{x^2} C''(\frac{1}{x}) \) is non-decreasing.

2. For all \( x \in I \),

• the derivative \( C'(x) := \frac{d}{dx} C(x) \) exists,
• the function \( C(x) \) is non-decreasing and convex.

Further suppose that \( 0p1 \) is a Christoffel word, \( \beta \in [0, 1] \), \( r \in (0, 1] \) and \( N \in \mathbb{Z}_+ \). Then

\[
\sum_{k=1}^n \beta^k (C(\phi(01p)^N(01p)_{1:k}(x)) - C(\phi(10p)^N(10p)_{1:k}(x)))
\]

is a non-decreasing function of \( x \) for \( y_{01p} \leq x \leq y_{10p} \), where \( n = |0p1| \).

Proof [Proof when \( C(\cdot) \) satisfies Property 1.] Let \( a_k(x), b_k(x), c_k(x), d_k(x) \) be as defined in Lemma 25. Then the proposition is proved by the following inequalities (as justified immediately below):

\[
\frac{d}{dx} \sum_{k=1}^n \beta^k (C(\phi(01p)^N(01p)_{1:k}(x)) - C(\phi(10p)^N(10p)_{1:k}(x)))
\]

\[
= \sum_{k=1}^n \left( \frac{\beta^k}{c_k(x)^2} C'(\frac{a_k(x)}{c_k(x)}) - \frac{\beta^k}{d_k(x)^2} C'(\frac{b_k(x)}{d_k(x)}) \right)
\]  

\[
\geq \sum_{k=1}^n \left( \frac{\beta^k}{c_k(x)^2} C'(\frac{b_k(x)}{c_k(x)}) - \frac{\beta^k}{d_k(x)^2} C'(\frac{b_k(x)}{d_k(x)}) \right)
\]

\[
= \sum_{k=1}^n (f_k(c_k(x)) - f_k(d_k(x))) \quad \text{where } f_k(u) := \beta^k C'(b_k(x)/u)/u^2
\]

\[
\geq 0.
\]

Step (11) follows from the chain rule as the homeomorphism of matrix multiplication and composition of Möbius transformations gives

\[
\phi(01p)^n(01p)_{1:k}(x) = \left[ \begin{array}{c} M((01p)^N(01p)_{1:k}) \left( \frac{x}{1} \right)_1 \\ M((01p)^N(01p)_{1:k}) \left( \frac{x}{1} \right)_2 \end{array} \right] = \frac{a_k(x)}{c_k(x)}
\]

while the fact that matrices \( F \) and \( G \) have unit determinant implies that the matrix \( M((01p)^N(01p)_{1:k}) \) also has unit determinant so that

\[
\frac{d}{dx} \phi(01p)^n(01p)_{1:k}(x) = \frac{1}{c_k(x)^2}.
\]

Step (12) follows as \( \beta^k, c_k(x) > 0 \), as \( C(\cdot) \) is concave and as \( a_k(x) \leq b_k(x) \) by Lemma 25.
Step (13) follows from Lemma 23. In particular, Lemma 25 shows that the sequences \( c_{1:n}(x) \) and \( d_{1:n}(x) \) satisfy the hypotheses of Lemma 23. Also, the fact that \( C(\cdot) \) satisfies Property 1 shows that the functions \( f_i(\cdot) \) for \( i = 1, \ldots, n \) satisfy hypotheses 1 and 2 of Lemma 23. Indeed, \( f_i(\cdot) \) is non-increasing and convex as \( \frac{1}{u} C'( \frac{1}{u} ) \) is non-increasing and convex and \( \beta^i, b_i(x) > 0 \). Also, as \( \frac{1}{u} C''( \frac{1}{u} ) \) is non-decreasing in \( u \) for \( b > 0 \) and \( 0 < b_i(x) \leq b_i(x) \) for \( i = 2, \ldots, n \), by Claim 1 of Lemma 25, the following integral is also non-decreasing in \( u \):

\[
\int_{b_{i-1}(x)}^{b_i(x)} \frac{1}{u^3} C'' \left( \frac{b}{u} \right) \, db = \frac{1}{u^2} C'( \frac{b_i(x)}{u} ) - \frac{1}{u^2} C'( \frac{b_{i-1}(x)}{u} ) = \frac{1}{\beta^i} (f_i(u) - f_{i-1}(u))
\]

So \( f_{i-1}(u) - f_i(u) \) is the sum of the non-increasing functions \( \beta f_{i-1}(u) - f_i(u) \) and \( 1 - \beta f_{i-1}(u) \).

This completes the proof. \( \blacksquare \)

**Proof** [Proof when \( C(\cdot) \) satisfies Property 2.] For \( k \in \mathbb{Z}_n \) let

\[
a_k' := \beta^{k+1} \frac{d}{dx} \phi_{(01)p}^N(01p)_{1:1}(x), \quad b_k' := \beta^{k+1} \frac{d}{dx} \phi_{(10)p}^N(10p)_{1:1}(x),
\]

\[
a_k := C'(y_{(1p0)_{1:1},(1p0)})(x), \quad b_k := C'(y_{(0p1)_{1:1},(0p1)})(x).
\]

Let \( r_0 \geq \cdots \geq r_{n-1} \) denote real numbers \( r_0, \ldots, r_{n-1} \) in non-increasing numerical order. Let \( \pi \) denote \( x \) modulo \( n \) and let \( l \) satisfy \( \frac{l}{0p1} = 1 \). Then the proposition is proved by the following inequalities (as justified immediately below):

\[
\frac{d}{dx} \sum_{k=1}^{n} \beta^k \left( C \left( \phi_{(01)p}^N(01p)_{1,k}(x) \right) - C \left( \phi_{(10)p}^N(10p)_{1,k}(x) \right) \right)
\]

\[
= \sum_{k=1}^{n} \left( C' \left( \phi_{(01)p}^N(01p)_{1,k}(x) \right) a_{k-1}' - C' \left( \phi_{(10)p}^N(10p)_{1,k}(x) \right) b_{k-1}' \right) \]

\[
\geq \sum_{k=1}^{n} \left( C' \left( y_{(1p0)_{k,n}(1p0)}(x) \right) a_{k-1}' - C' \left( y_{(0p1)_{k,n}(0p1)}(x) \right) b_{k-1}' \right) \quad (14)
\]

\[
= \sum_{k=1}^{n} \left( a_k a_k' - b_k b_k' \right) \]

\[
= \sum_{k=0}^{n-1} a_k \left( a_k' \frac{l}{l(n-k)} - b_k' \frac{l}{l(n-k)} \right) \quad (15)
\]

\[
\geq \sum_{k=0}^{n-1} a_k \left( a_k' \frac{l}{l(n-k)} - b_k' \frac{l}{l(n-k)} \right) \quad (16)
\]

\[
\geq \sum_{k=0}^{n-1} a_k \left( a_k' \frac{l}{l(n-k)} - b_k' \frac{l}{l(n-k)} \right) \quad (17)
\]

\[
\geq \sum_{k=0}^{n-1} a_k \left( a_k' \frac{l}{l(n-k)} - b_k' \frac{l}{l(n-k)} \right) \quad (18)
\]

\[
\geq \sum_{k=0}^{n-1} a_k \left( a_k' \frac{l}{l(n-k)} - b_k' \frac{l}{l(n-k)} \right) \quad (19)
\]

\[
\geq \sum_{k=0}^{n-1} a_k \left( a_k' \frac{l}{l(n-k)} - b_k' \frac{l}{l(n-k)} \right) \quad (20)
\]
Step (14) follows from the chain rule and definition of \(a'_k, b'_k\).

Step (15) follows as \(C(\cdot)\) is convex, as \(a'_k, b'_k \geq 0\), and as, for \(k = 1, 2, \ldots, n\), we have

\[
\phi(01p)^N(01p)_{1:k}(x) = \phi(01p)^N(\phi(01p)^N(x)) \\
\geq \phi(01p)^N(y01p) \\
= y(01p)_{(k+1):n}(01p)_{1:k} \\
= y(10p)_{k:n}(10p)_{1:(k-1)}
\]

where the inequality holds as \(x \geq y01p\) so that \(\phi(01p)^N(x) \geq y01p\), and as \(\phi(01p)_{1:k}(\cdot)\) is increasing. The same argument using \(x \leq y10p\) gives an upper bound on \(C'(\phi(10p)^N(10p)_{1:k}(x))\).

Step (16) follows by shifting the summation indices and from the definition of \(a_k, b_k\).

Step (17) follows from Lemmas 12 and 15 and the convexity of \(C(\cdot)\). Let \(w_{[0]} \geq \cdots \geq w_{[n-1]}\) denote words \(w(0), \ldots, w(n-1)\) in non-increasing lexicographic order and let \(c_k := (1p0)_{(k+1):n}(10p)_{1:k}, d_k := (0p1)_{(k+1):n}(01p)_{1:k}\) for \(k \in \mathbb{Z}_n\). Then Lemma 12 shows that \(c_{[i]} = d_{[i]} = c_{(n-i)} = d_{(n-i)-1}\). Thus Lemma 15 gives \(y_{c_{[i]}} = y_{d_{[i]}} = y_{c_{(n-i)}} = y_{d_{(n-i)-1}}\). Therefore the convexity of \(C(\cdot)\) gives

\[
a_{[i]} = b_{[i]} = a_{(n-i)} = b_{(n-i)-1}.
\]

Step (18) follows as \(C(\cdot)\) is non-decreasing, so that \(a_{[i]} \geq 0\) and as \(\phi_0(\cdot), \phi_1(\cdot)\) are non-decreasing and non-expansive, so that \(b'_k\) is a product of derivatives where each derivative is in \([0, 1]\). Thus \(a_{[0]}, \ldots, a_{[n-1]}\) and \(b'_{0}, \ldots, b'_{n-1}\) are non-negative non-increasing sequences. Therefore the rearrangement inequality \(a_{[i]}b'_j + a_{[i+1]}b'_0 \leq a_{[i]}b'_0 + a_{[i+1]}b'_j\) holds for all \(i \in \mathbb{Z}_{n-1}\) and all \(j \in \mathbb{Z}_n\). But \(b'_{(n-(n-i)-1)} = b'_0\) so repeated application of the rearrangement inequality gives

\[
\sum_{k=0}^{n-1} a_{[k]}b'_{(n-k)-1} \leq \sum_{k=0}^{n-1} a_{[k]}b'_{(n-k)}.
\]

Step (19) follows from Claim 4 of Lemma 25, as for the \(c_i(x), d_i(x)\) defined in that Lemma, we have

\[
a'_{i-1} - b'_{i-1} = \frac{\beta^i}{c_i(x)^2} - \frac{\beta^i}{d_i(x)^2} \begin{cases} \leq 0 & \text{for } i = 1 \\ \geq 0 & \text{for } i = 2, 3, \ldots, n. \end{cases}
\]

Step (20) follows from this Proposition using the function \(\tilde{C}(x) = x\) which satisfies Property 1.

This completes the proof. \(\blacksquare\)

It is much simpler to show that the marginal work is non-decreasing when the itinerary is \(0^\omega\) (that is, \(x \geq y_0\)) or when the itinerary is \(1^\omega\) (that is, \(x \leq y_1\)).

**Proposition 27** Suppose Condition D holds. Then

\[
\sum_{k=1}^{\infty} \beta^k (C(\phi_{0k}(x)) - C(\phi_{1k-1}(x))) \quad \text{and} \quad \sum_{k=1}^{\infty} \beta^k (C(\phi_{0k-1}(x)) - C(\phi_{1k}(x)))
\]

are non-decreasing functions of \(x \in \mathcal{I}\).
Proof Consider the first sum. As in Lemma 25, for $k \in \mathbb{Z}_{++}$, we define
\[
\begin{pmatrix}
    a_k(x) & b_k(x) \\
    c_k(x) & d_k(x)
\end{pmatrix} := \begin{pmatrix}
    (0^k)(x) \\
    (10^{k-1})(x)
\end{pmatrix}.
\]
As $G \geq F$, the entries of $F$ are non-negative and $x \geq 0$, it follows that
\[
\begin{pmatrix}
    b_k(x) - a_k(x) \\
    d_k(x) - c_k(x)
\end{pmatrix} = F^{k-1}(G - F) \begin{pmatrix}
    x \\
    1
\end{pmatrix} \geq 0.
\]
(21)

If $C$ satisfies Condition C1, then for any $k \in \mathbb{Z}_{++}$,
\[
\begin{aligned}
    \frac{d}{dx} (C(\phi_k(x)) - C(\phi_{10^{k-1}}(x))) &= \frac{1}{c_k(x)^2} C' \left( \frac{a_k(x)}{c_k(x)} \right) - \frac{1}{d_k(x)^2} C' \left( \frac{b_k(x)}{d_k(x)} \right) \\
    &\geq \frac{1}{c_k(x)^2} C' \left( \frac{b_k(x)}{c_k(x)} \right) - \frac{1}{d_k(x)^2} C' \left( \frac{b_k(x)}{d_k(x)} \right) \geq 0.
\end{aligned}
\]

The first inequality holds as $C'$ is concave and $a_k(x) \leq b_k(x)$ by (21). The second inequality holds as $C'(1/x)/x^2$ is non-increasing and $c_k(x) \leq d_k(x)$ by (21).

If $C$ satisfies Condition C2, then for any $k \in \mathbb{Z}_{++}$,
\[
\begin{aligned}
    \frac{d}{dx} (C(\phi_k(x)) - C(\phi_{10^{k-1}}(x))) &= \frac{1}{c_k(x)^2} C' (\phi_k(x)) - \frac{1}{d_k(x)^2} C' (\phi_{10^{k-1}}(x)) \geq 0.
\end{aligned}
\]

The inequality is justified as follows. As $a \leq b$ in the definition of $F, G$, we have $\phi_0(x) \geq \phi_1(x)$. As $\phi_{10^{k-1}}(\cdot)$ is an increasing function, it follows that $\phi_{10^{k-1}}(x) \geq \phi_{10^{k-1}}(x)$. As $C$ is convex, it follows that $C'(\phi_k(x)) \geq C'(\phi_{10^{k-1}}(x))$. Furthermore, $c_k(x) \leq d_k(x)$, by (21).

Thus, if $C$ satisfies Condition C, the sum
\[
\sum_{k=1}^{\infty} \beta^k (C(\phi_k(x)) - C(\phi_{10^{k-1}}(x)))
\]
is the sum of non-decreasing functions. Therefore this sum is non-decreasing.

The proof for the second sum is similar. This completes the proof.

7. Continuity (PCLI2, Second Part)

We demonstrate Proposition 31 which shows that the marginal productivity index $\lambda$ for systems satisfying Condition D is a continuous function.

Definition 28 For $k \in \mathbb{Z}_+$ and $x \in \mathcal{I}$, let $f_k(x) := (\pi(x)^\omega)_{1:k}$, where $\pi(x)$ is the $x$-threshold word for $\phi_0, \phi_1$.

Lemma 29 Suppose $k \in \mathbb{Z}_{++}$ and $d \in \mathbb{R}_+$ with $f_k(d) \neq f_k(d^+)$. Then $\lambda(d) = \lambda(d^+)$. 

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Proof For $x \in \mathbb{R}_+$, let $\pi(x)$ be the $x$-threshold word and let $s(x)$ be the rate of $\pi(x)$. The rate $s(x)$ is a non-increasing function and our characterisation of the $x$-threshold word shows that we can find a range of $x$ corresponding to the word of rate $q$ for any $q \in [0,1]$. So, Lemma 14 implies that $f_k(d) \neq f_k(d^+)$ if and only if $d$ is the upper fixed point of a Christoffel word of length at most $k$. That is, if $d = y_1$ or $d = y_{10b}$ for some Christoffel word $0b1$ with $|0b1| \leq k$.

Let $S(w, x) := \sum_{n=1}^{|w|} \beta^{n-1} C(\phi_{w_1,n}(x))$ for any word $w$.

Say $d = y_{10b}$. Let $(0a1, 0c1)$ be the Christoffel pair for $0b1$. Then $\pi(d) = 0b1$ and by going left in the Christoffel tree then repeatedly turning right we get $\pi(d^+) = 0a1(0b1)^\omega$. But $(0a1, 0b1) = (0a1, 0a10c1)$ is also a Christoffel pair and as $0a1, 0b1, 0a10b1$ are Christoffel words, $a, b, a10b$ are palindromes. So $a10b = b01a$. Repeated application of this result gives $a(10b)^\omega = b01a(10b)^\omega = (b01)^\omega a$. Thus putting $m := |0b1|$ and noting that $\phi_{10b}(d) = d$ gives

$$(1 - \beta)\lambda(d^+) = S(01a1(0b1)^\omega, d) - S(10a1(0b1)^\omega, d)$$

$$= S((01b)^\omega, d) - S((10b, 01b)^\omega, d)$$

$$= S((01b)^\omega, d) - S(10b, d) - \beta^m S((01b)^\omega, \phi_{10b}(d))$$

$$= (1 - \beta^m) S((01b)^\omega, d) - (1 - \beta^m) S((10b)^\omega, d)$$

$$= (1 - \beta)\lambda(d).$$

Now say $d = y_1$ then $\pi(d) = 1$ and $\pi(d^+) = 01^\omega$. Then

$$(1 - \beta)\lambda(d^+) = S(01^\omega, d) - S(101^\omega, d)$$

$$= S(01^\omega, d) - S(1, d) - \beta S(01^\omega, \phi_1(d))$$

$$= (1 - \beta) S(01^\omega, d) - (1 - \beta) S(1^\omega, d)$$

$$= (1 - \beta)\lambda(d).$$

This completes the proof. \(\blacksquare\)

Lemma 30 Suppose Condition D holds, that $k \in \mathbb{Z}_{++}$ and $0 \leq x \leq y$ with $f_k(x) = f_k(y)$. Let $K := \sup_{z \in \{\phi_1(x), \phi_0(y)\}} C'(z)$ where $C'(\cdot)$ is the derivative of $C(\cdot)$. Then

$$|\lambda(x) - \lambda(y)| \leq \frac{K(3\beta^k(y + 1) + 2(y - x))}{(1 - \beta)^2}.$$
follow from the facts that: (i) the lowest and highest points on the $z$-threshold orbit are $\phi_1(z)$ and $\phi_0(z)$ (by Lemma 41); (ii) $y \geq x$; (iii) function $C(\cdot)$ is non-decreasing and either convex or concave; (iv) function $\phi_w(\cdot)$ is non-expansive for any word $w$; (v) for any $z \in \mathbb{R}_+$, $\phi_0(z) \leq z + 1$ and $\phi_1(z) \geq 0$.

Also, as $|0p| = k$ it follows that $|e - f| \leq \beta^k/(1 - \beta)$. Therefore

\[
|\beta| \lambda(x) - \lambda(y)|
\]

\[
= \left( e - f \right) \sum_{n=1}^{\infty} \beta^n a_n + f \sum_{n=1}^{k} \beta^n (a_n - b_n) + f \sum_{n=k+1}^{\infty} \beta^n (a_n - b_n)
\]

\[
\leq |e - f| \sum_{n=1}^{\infty} \beta^n \sup_{m \geq 1} a_m | + f \sum_{n=1}^{k} \beta^n \sup_{m \leq k} |a_m - b_m| + f \sum_{n=k+1}^{\infty} \beta^n \sup_{m > k} |a_m - b_m|
\]

\[
\leq \frac{\beta^k}{1 - \beta} \frac{\beta}{1 - \beta} K(y + 1) + \frac{1}{1 - \beta} \frac{\beta}{1 - \beta} 2K(y - x) + \frac{1}{1 - \beta} \frac{\beta^k+1}{1 - \beta} 2K(y + 1)
\]

which rearranges to the inequality claimed.

\[\blacksquare\]

**Proposition 31** Suppose Condition D holds. Then the marginal productivity index $\lambda(s)$ is a continuous function of $s \in I$.

**Proof** We show that for any $\epsilon > 0$ there is a $\delta > 0$ such that $|\lambda(x) - \lambda(y)| < \epsilon$ for any $x, y$ in the domain of $\lambda(\cdot)$ with $|x - y| < \delta$. Without loss of generality we assume that $y \geq x$.

For $k \in \mathbb{Z}_+$, let $l_k$ be the distance between the closest pair of discontinuities of $f_k(\cdot)$. By Lemma 14, these discontinuities are at the upper fixed points of Christoffel words of length at most $k$. But the upper fixed points of distinct Christoffel words are distinct. Therefore $l_k > 0$. Also, the words $01^{k-1}$ and $01^k$ have rates that are adjacent in the Farey sequence $F_k$. But the sequence $y_{101^{k-1}}$ converges to $y_1$. Thus $l_k \leq y_{101^{k-1}} - y_{101^k}$ is a non-increasing function of $k$ that converges to 0. Therefore, for any $\epsilon > 0$ and $x$ in the domain of $\lambda(\cdot)$ it is always possible to select a $k < \infty$ such that

\[
\frac{3\beta^k(x + l_k + 1) + 2l_k}{(1 - \beta)^2} \sup_{z \in \{\phi_0(x+l_k), \phi_1(x)\}} C'(z) < \frac{\epsilon}{2}
\]

where $C'(\cdot)$ is the gradient of $C(\cdot)$ (which exists as Condition C is satisfied).

Let $\delta = l_k$ for such a $k$ and note that the above argument shows that $\delta > 0$. Then for $0 \leq x \leq y \leq x + \delta$, the definition of $l_k$ shows that $f_k(\cdot)$ has at most one discontinuity in $(x, y]$. If there is no discontinuity, let $d$ be an arbitrary point in $(x, y]$, otherwise let $d$ be the point of discontinuity. Thus Lemmas 29 and 30 give

\[
|\lambda(x) - \lambda(y)| \leq |\lambda(x) - \lambda(d)| + |\lambda(d) - \lambda(d^+)| + |\lambda(d^+) - \lambda(y)| < \frac{\epsilon}{2} + 0 + \frac{\epsilon}{2} = \epsilon.
\]

This completes the proof. 

\[\blacksquare\]
8. Radon-Nikodym Condition (PCLI3)

We demonstrate that PCLI3 holds provided that s-threshold policies result in itineraries that are mechanical words. The argument is based on the fact that the work-to-go and cost-to-go are discrete measures, as defined just below, because the number of factors of mechanical words is polynomially bounded, by Theorem 17.

**Definition 32** Let $\mu$ be a signed measure defined on the Lebesgue measurable sets of $\mathbb{R}$ and taking values in $[-\infty, \infty]$. Then measure $\mu$ is **discrete** if there is a countable set $\mathcal{S} = \{(a_1, s_1), (a_2, s_2), \ldots \}$ of pairs of real numbers such that

$$\mu(\mathcal{X}) = \sum_{(a,s)\in \mathcal{S}} a 1_{s\in \mathcal{X}}$$

for any Lebesgue measurable set $\mathcal{X}$ of $\mathbb{R}$.

**Proposition 33** Suppose $(I, C, \phi_0, \phi_1, \beta)$ satisfy Condition D. Then PCLI3 holds.

**Proof** Let $D_t(x)$ be the set of discontinuities of $A_t(x; s)$ as a function of $s \in I$. As $A_t(x; s)$ is in $\{0,1\}$ for any $s \in I$, it follows that

$$A_t(x; s^+) = A_t(x; s^-) + \sum_{d \in D_t(x)} (A_t(x; d^+) - A_t(x; d^-)) 1_{d \in [s_0, s]}$$

for any $s_0 \leq s$ with $s_0 \in I$. Thus the definition of the work-to-go gives

$$g(x; s^+) = g(x; s^-) + \sum_{t=0}^{\infty} \sum_{d \in D_t(x)} \beta^t (c(A_t(x; d^+)) - c(A_t(x; d^-))) 1_{d \in [s_0, s]}.$$ As $\text{card}(D_t(x)) \leq p(t)$ for some polynomial function $p(t)$ by Theorem 17 and $A_t(x; s) \in \{0,1\}$, the series on the right-hand side of this expression is absolutely summable. Indeed,

$$\sum_{t=0}^{\infty} \sum_{d \in D} |\beta^t (c(A_t(x; d^+)) - c(A_t(x; d^-))) 1_{d \in [s_0, s]}| \leq |c(1) - c(0)| \sum_{t=0}^{\infty} \beta^t p(t) < \infty$$

where $D := \bigcup_{t=0}^{\infty} D_t(x)$. Therefore Fubini’s theorem gives

$$g(x; s^+) = g(x; s^-) + \sum_{d \in D} a_g(d) 1_{d \in [s_0, s]}$$

$$a_g(d) := \sum_{t=0}^{\infty} \beta^t (c(A_t(x; d^+)) - c(A_t(x; d^-)))$$

which corresponds to a discrete measure.

Now for any $d \in [s_0, s]$, the sequence of states $X_t(x; d)$ only depends on $d$ via the sequence of actions $A_{t-1}(x; d)$. Also, an argument similar to that of Lemma 41 shows that both $X_t(x; d^+)$ and $X_t(x; d^-)$ lie in the interval

$$[\min\{\phi_1(d), x\}, \max\{\phi_0(d), x\}]$$
for all \( t \in \mathbb{Z}_+ \). As the cost function \( C(x) \) is bounded and continuous on that interval (as Condition D requires that Condition C is satisfied), a similar argument to that given above for the work-to-go \( g \) gives

\[
f(x; s^+) = f(x; s_0^-) + \sum_{d \in D} a_f(d) \mathbf{1}_{d \in [s_0, s]}
\]

\[
a_f(d) := \sum_{t=0}^{\infty} \beta^t (C(X_t(x; d^+)) - C(X_t(x; d^-)))
\]

which is also a discrete measure.

Noting that \( X_0(s; s^+) = s \), let \( \tau \) be the next time that \( X_{\tau}(s; s^+) = s \) or let \( \tau = \infty \) if there is no such time. Thus \( X_{\tau}(s; s^+) \) is a periodic sequence if \( \tau < \infty \). From the definition of the policies it follows that \( X_t(s; s) = X_t(s, 1; s) \) and \( A_t(s; s) = A_t(s, 1; s) \) for \( t = 0, 1, \ldots \), so that

\[
g(s; s) = \sum_{t=0}^{\infty} \beta^t c(A_t(s; s)) = g(s, 1; s). \tag{22}
\]

Using the definition of \( g(\cdot; \cdot) \), it follows that

\[
g(s; s^+) = \sum_{t=0}^{\infty} \beta^t c(A_t(s; s^+))
\]

\[
= \beta^0 g(s, 0; s) + \beta^\tau \sum_{t=0}^{\infty} \beta^t (c(A_t(s; s^+)) - c(A_t(s, 1; s)))
\]

\[
= g(s, 0; s) + \beta^\tau (g(s; s^+) - g(s, 1; s))
\]

so that

\[
g(s; s^+) = \frac{g(s, 0; s) - \beta^\tau g(s, 1; s)}{1 - \beta^\tau}. \tag{23}
\]

Let \( \tau_1 \) be the first time that \( X_{\tau_1}(x; s^+) = s \) or \( \tau_1 = \infty \) if there is no such time. For \( t = 0, \ldots, \tau_1 - 1 \), the definition of the policies then gives \( X_t(x; s^+) = X_t(x; s) \) and \( A_t(x; s^+) = A_t(x; s) \). Thus

\[
g(x; s) - g(x; s^+) = \beta^{\tau_1} (g(s; s) - g(s; s^+))
\]

\[
= \beta^{\tau_1} \left( g(s, 1; s) - \frac{g(s, 0; s) - \beta^\tau g(s, 1; s)}{1 - \beta^\tau} \right)
\]

\[
= \frac{\beta^{\tau_1}}{1 - \beta^\tau} (g(s, 1; s) - g(s, 0; s))
\]

\[
= \frac{\beta^{\tau_1}}{1 - \beta^\tau} w_s(s) \tag{24}
\]

where the second equality follows from (23) and (22) and the last from the definition of the marginal work \( w(\cdot; \cdot) \).
A similar argument for the cost-to-go gives
\[
f(x; s^+) - f(x; s) = \frac{\beta T_1}{1 - \beta x} c_x(s).
\]
Combining this equation with (24) and recalling that \(\lambda(s) = c_x(s)/w_x(s)\) gives
\[
f(x; s^+) - f(x; s) = -\lambda(s)(g(x; s^+) - g(x; s)). \tag{25}
\]
Now by definition the marginal cost and marginal work are
\[
c_x(s) = \beta(f(\phi_0(x); s) - f(\phi_1(x); s)) \tag{26}
\]
\[
w_x(s) = 1 + \beta(g(\phi_1(x); s) - g(\phi_0(x); s)).
\]
Combined with (25) these give
\[
c_x(s^+) - c_x(s) = \beta(f(\phi_0(x); s^+) - f(\phi_1(x); s^+) - f(\phi_0(x); s) + f(\phi_1(x); s))
\]
\[
= -\beta \lambda(s)(g(\phi_0(x); s^+) - g(\phi_1(x); s^+) - g(\phi_0(x); s) + g(\phi_1(x); s))
\]
\[
= \lambda(s)(w_x(s^+) - w_x(s)).
\]
As \(f, g\) are discrete measures, it follows that \(c_x, w_x\) are also discrete measures, so the Lebesgue-Stieltjes integral of this expression over any interval \([a, b] \subseteq \mathcal{I}\) is
\[
c_x(b) - c_x(a) = \int_{[a, b]} \lambda \, dw_x.
\]
Noting that \(f(x; s)\) is a càglàd function of \(s\) and \(\phi_0, \phi_1\) are continuous, it follows from (26) that \(c_x(s)\) is also a càglàd function of \(s\), so that the left-hand side of this expression is \(c_x(b) - c_x(a)\). Thus Condition PCLI3 holds. \(\blacksquare\)

9. Numerical Experiments

We discuss algorithms for computing the Whittle index given in Theorem 4, we present closed-form expressions for that index and compare the performance of the Whittle index policy with two previously-proposed heuristics.

9.1 Approximating the Index for Discount Factor \(\beta \leq 0.999\)

Truncating the sums defining the marginal productivity index \(\lambda(x)\) after a suitably large number of terms \(T\) suggests the approximation
\[
\hat{\lambda}(x) = \frac{\sum_{t=0}^{T} \beta^t (C(X_t(x, 0; x)) - C(X_t(x, 1; x)))}{\sum_{t=0}^{T} \beta^t (A_t(x, 1; x) - A_t(x, 0; x))}. \tag{27}
\]
Assuming accurate calculation of the terms in the numerator and denominator, as well as continuity of the cost function \(C\), this approximation requires \(O(T)\) basic arithmetical and comparison operations, and setting \(T\) to \(\Omega(\log(\epsilon)/\log(\beta))\) guarantees absolute errors in the
Figure 5: Errors in approximating the index. The cost $C(x)$ corresponds to variance ($C(x) = x$, red), to entropy ($C(x) = \log(x)$, green) or to negative precision ($C(x) = -1/x$, blue). The discount factor $\beta$ is either 0.9 (open circles) or 0.999 (filled circles). The map-with-a-gap has $\phi_0(x) = x + 1$ and $\phi_1(x) = 1/(a_1 + 1/(x + 1))$ with $a_1$ equal to 0.001 (solid lines) or 1 (dotted lines).

numerator and denominator of $O(\epsilon)$. Of course, the constants hidden by the $O(\cdot)$ or $\Omega(\cdot)$ depend on the detailed properties of $C$ and choice of $x$.

However, this approximation faces a potential complication. Indeed, some of the iterates $X_t(x,a;x)$ may be so close to the threshold $x$ that an arbitrarily small tolerance is required to correctly decide whether $X_t(x,a;x) \geq x$. This might be problematic as errors in such decisions can result in large changes to the numerator and denominator of this approximation.

Dance and Silander (2015) overcame this problem by constraining the sequence of decisions to correspond to mechanical words. This resulted in a polynomial-time algorithm for approximating the index $\lambda(x)$ using variable-precision arithmetic. Specifically, suppose that basic arithmetic operations to tolerance $2^{-m}$ on positive numbers less than $2^n$ require at most $M(n + m)$ operations, and that $C(x) = x$. Then, the approximate index $\hat{\lambda}(x)$ output by their algorithm satisfies $|\hat{\lambda}(x) - \lambda(x)| < \epsilon$ when $x < 2^n$ and $\epsilon > 2^{-m}$ and is computed in $O((n + m)^3 M(n + m))$ operations. Nevertheless, that algorithm requires the tabulation of the fixed points of all Christoffel words of at most a given length.

Here we suggest that such tabulation is an unnecessary expense, and conjecture that standard floating-point approximations of the decision sequences $A_t(x,a;x)$ correspond to mechanical words (at least for the vast majority of floating point values of $x$). Perhaps such a conjecture might be proven by extending results by Kozyakin (2003). His results concern mappings $\phi_0, \phi_1$ which are strictly increasing but potentially discontinuous. In
the floating-point case one would only require those mappings to be non-decreasing and piecewise constant, but might perhaps impose additional conditions.

Rather than attempting to prove such a conjecture here, we simply evaluate the accuracy of approximation (27). Figure 5 shows the accuracy based on comparing double-precision and quadruple-precision implementations, with $T = \lceil \log 10^{-17}/\log \beta \rceil$ and $T = \lceil \log 10^{-34}/\log \beta \rceil$ respectively. As the difference between these approximations is a highly variable function of $x$, we only show points that are local maxima of the error. Specifically, we used a logarithmically-spaced grid with $10^{-2} = x_1, x_2, \ldots, x_{1000} = 10^4$ and plot the error $e(x_i)$ only for points with $e(x_i) = \max_{-20 \leq j \leq i+20} e(x_j)$. The plot shows no line for $x$ less than the first such point or greater than the last such point.

The worst absolute and relative errors are below $10^{-6}$ and $10^{-11}$ respectively. In any practical application, such errors would be swamped by imprecision in the time-series models. The absolute error remains small as $x$ increases to the fixed point $y_1$ of the mapping $\phi_1$, and then it increases due to roundoff in computing iterates of the map-with-a-gap for large $x$. Overall, the worst results are for large discount factors and for variance as the cost function.

Finally, it is possible to substantially accelerate the convergence of the numerator, for instance with Aitken acceleration (Brezinski, 2000), particularly if one has high accuracy requirements. For instance, if the $x$-threshold word has period $n$, one may accumulate $n$ terms of the sum at a time and apply acceleration methods to such partial sums. Having experimented with such approaches, we find that further work is required in selecting appropriate termination conditions if one is interested in accuracy guarantees for a wide range of problem instances. The difficulty we encountered is that two types of linear convergence are going on simultaneously, namely the convergence due to $\phi_0$ (when $a_0 > 0$ or $r < 1$) and $\phi_1$, and the convergence due to $\beta$. In such situations, what looks like a healthy stopping time to existing termination criteria can actually be a misleading and unhealthy prematurity.

9.2 Approximating the Index for Discount Factor $\beta \to 1$

For discount factors $\beta > 0.999$ the number of terms $T$ required for accuracy in approximation (27) becomes prohibitively large. In such cases, it makes sense to Taylor expand the numerator of $\lambda$ as a function of $\beta$. For brevity, we only do so here for the case $\beta \to 1$.

Suppose that the itineraries $A_t(x, a; x)$ appearing in the definition of the index $\lambda(x)$ correspond to a Christoffel word with period $n$. Then, in the limit $\beta \to 1$, the results presented elsewhere in this paper show that the denominator of the index tends to $1/n$. Also, letting $X_t^{a,\infty} := \lim_{k \to \infty} X_{kn+t}(x, a; x)$, the numerator of index has the limit

$$\sum_{t=0}^{\infty} \beta^t \left( C(X_t(x, 0; x)) - C(X_t(x, 1; x)) \right)$$

$$\to \sum_{t=0}^{\infty} \left( C(X_t(x, 0; x)) - C(X_t^{0,\infty}) - C(X_t(x, 1; x)) + C(X_t^{1,\infty}) \right) + \frac{1}{n} \sum_{t=0}^{n-1} t \left( C(X_t^{1,\infty}) - C(X_t^{0,\infty}) \right).$$
Now, the sequences $C(X_t(x,a;x)) - C(X_t^{a,\infty})$ in this expression converge to zero, for $C$ satisfying Condition C. This suggests approximating $X_{kn+1}^{a,\infty}$ by $X_{Tn-n+1}(x,a;x)$ for a suitably large positive integer $T$ and for $k = 0,1,\ldots$. This also suggests approximating the first sum by truncating it after $Tn$ terms.

While the itinerary for $x$ might have a very large and possibly infinite period $n$, we did not encounter such situations when tabulating the index. If this were an issue, it is possible to find a good rational approximation to the slope of the $x$-threshold word, for instance as in Dance and Silander (2015).

9.3 Closed Form Expressions and Graphs

We analyse the behaviour of the index as the cost function $C(\cdot)$ and parameters $\beta, r, a_0, a_1$ vary.

Given noise free observations for action $a = 1$ and totally uninformative observations for action $a = 0$, it is easy to find a closed form for the index.

**Proposition 34** Suppose the cost function is $C(x) = x$ and the precision $a_0 = 0$. Then

$$
\lim_{a_1 \to \infty} \lambda(x) = \frac{1 - \beta^{n+1}}{1 - \beta} \left( rx + 1 - \frac{\beta}{1 - \beta^{n+1}} \left( 1 - (\beta r)^n - \beta^n + \frac{1 - r^n}{1 - r} \right) \right),
$$

for all $\beta \in [0,1), r \in [0,1)$ and $x \in [0,1/(1-r))$, where $n := \left\lceil \frac{\log(1-(1-r)x)}{\log r} \right\rceil$. Thus

$$
\lim_{\beta \to 1} \lim_{r \to 1} \lim_{a_1 \to \infty} \lambda(x) = [x + 1] \left( x + 1 - \frac{[x]}{2} \right) = \int_0^{x+1} \lfloor u \rfloor \, du, \quad \text{for all } x \in \mathbb{R}_+.
$$

**Proof** As $\lim_{a_1 \to \infty} \phi_1(x) = 0$, the orbits involve the sequence $0, 1, r+1, r^2+r+1, \ldots, r^{n-1} + \cdots + r + 1$, where $n$ is the least integer for which $r^{n-1} + \cdots + r + 1 \geq x$. The result then follows from the definition of the index, using well-known summation formulae.

Other closed forms exist, for instance in the limit $\beta \to 0$, whenever the cost is a polynomial function of $x$ or whenever the process tends to the continuous-time process analysed in Le Ny et al. (2011). However, even for $C(x) = x, \beta \to 1, r \to 1$ and $a_0 = 0$, the integral in the above proposition only gives an approximation to $\lambda(x)$ with a relative error of under 25% for $a_1 > 4$.

Therefore, Figure 6 graphs the index using the algorithms of the previous subsection. Looking at these graphs, one notices that the index is increasing in all-but-one of the cases shown: indeed $C(x) = -x^{-3/2}$ is not covered by Condition C. Also, the index has cusps at the fixed point $x = \phi_0(x)$ which are clearly visible as $a_0$ and $r$ vary. Finally, the index becomes increasingly serrated as $\beta \to 1$ and $a_1 \to 0$. One would anticipate such serrations on the basis of the above proposition as

$$
\int_0^{x+1} \lfloor u \rfloor \, du - \frac{1}{2}(x+1)(x+2) = \frac{1}{2}([x] - x)(x - [x])
$$

and they are visible in the plot of the residual after subtracting a cubic fit, for $a_1 = 1$ and $\beta \to 1$. However, in general, the serrations have a complex non-periodic pattern and give a slightly ragged appearance to the plot with $a_1 = 0.01$.  

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Figure 6: Behaviour of the index. The index as a function of the discount factor $\beta$ (top-left) and the residual after fitting a cubic to this curve in the case $\beta = 1$ (top-right). The index, normalised by $\lambda(50)$, as the cost function $C(x)$, the multiplier $r$ and the observation precisions $a_0$ and $a_1$ are varied (other plots). In all plots, all parameters (or function) other than that varied are set to $\beta = 0.99$, $C(x) = x$, $\phi_0(x) = 1/(a_0+1/(rx+1))$ and $\phi_1(x) = 1/(a_1+1/(rx+1))$, with $a_0 = 0$, $a_1 = 1$ and $r = 1$. 
9.4 Performance Relative to Heuristic Policies

Two heuristic policies have been commonly used for the problem of multi-sensor scheduling. The myopic policy observes the \( m \) time series with the highest current cost \( C_i(x_{i,t}) \) and has been used in radar systems Moran et al. (2008). Meanwhile, the round robin policy chooses an ordering of the \( n \) arms and observes the time series \( m \) at a time, while respecting this order.

Figure 7 compares the costs incurred by these heuristics in a simple scenario in which estimation errors for one of the arms (time series) are more expensive than for the other arms. In detail, there are \( n = 10 \) arms, \( m = 1 \) observations per round, the cost at time \( t \) is \( 10x_{1,t} + \sum_{i=1}^{n} x_{i,t} \), observations have zero cost, and the initial posterior variance is \( x_{i,0} = 4 \) for all arms. The arms have the same map-with-a-gap given by \( \phi_0(x) = x + 1 \) and \( \phi_1(x) = 1/(0.1 + 1/(x + 1)) \).

Clearly, the myopic policy is over-eager to observe arm \( i = 1 \) and does so at the expense of arms \( i = 2, \ldots, 9 \). In contrast, the round robin policy makes no special effort to observe arm \( i = 1 \) and incurs substantial expense for that time arm. Meanwhile, the Whittle policy takes a just medium between these extremes, and is by far the least costly policy.

10. Further Work

This paper presented conditions under which threshold policies are optimal for observing a single time series with costly observations. It also explored the implications of this result by showing that it leads to optimal policies for the linear-quadratic Gaussian problem with costly observations and that it demonstrates the indexability of related restless bandit problems, which were both long-standing open questions.

It would be natural to extend this work to situations where more than two observation actions are available, perhaps using known generalisations of mechanical words (Glen and Justin, 2009). There are also truly-stochastic versions of the one-dimensional problem considered here, for instance situations where the costs depend on the posterior mean rather than just the posterior variance, situations where the quality of the observation is a random variable and situations involving non-Gaussian time series.
Dance and Silander

It is also important to understand the structure of optimal policies for making costly observations with discrete-time Kalman filters in multiple dimensions. One attack on this problem would begin by extending the verification theorem of Niño-Mora (2015) to multi-dimensional state spaces.

Finally, we cannot claim the asymptotic optimality of Whittle’s index policy for the problem studied here as the results of Verloop (2016) only apply to countable state spaces. Furthermore, little is known about the performance of policies for restless bandits in non-asymptotic situations involving finite numbers of arms and finite time horizons.

Acknowledgments

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Appendix A. Itineraries as Mechanical Words

We begin with some elementary properties of compositions of functions satisfying Assumption A1 and their fixed points (Subsection A.1). This enables us to present a proof of Theorem 16 which is based on the Christoffel tree. First we consider the case where itineraries correspond to Christoffel words (Subsection A.2), then the case where itineraries correspond to Sturmian $\mathcal{M}$-words (Subsection A.3) and finally we couple these results together to prove the theorem (Subsection A.4). We also present a proof of Theorem 17 about itineraries from initial points not equal to the threshold (Subsection A.5).

A.1 Compositions and Fixed Points

The following fundamental result about compositions is well known.

Lemma 35 Suppose A1 holds and that $w$ is a finite non-empty word. Then $\phi_w$ is increasing, contractive, continuous and has a unique fixed point $y_w$ on $I$.

Proof First we show that $\phi_w(x)$ is increasing, by induction on the length of word $w$. In the base case, as $w$ is non-empty, we suppose that $|w| = 1$. The claim then follows immediately from A1. For the inductive step, assume $\phi_u(x)$ is increasing, where $w = au$ for some letter $a \in \{0, 1\}$ and some finite non-empty word $u$. Then for any $x, y \in I$ such that $x < y$,

$\phi_w(y) = \phi_u(\phi_a(y))$

$> \phi_u(\phi_a(x))$ as $\phi_a(y) > \phi_a(x)$ and $\phi_u$ is increasing

$= \phi_w(x)$.

Therefore $\phi_w$ is increasing.

Now we show that $\phi_w(x)$ is contractive, again by induction on $|w|$. If $|w| = 1$ then this follows immediately from A1. Else, say $\phi_u(x)$ is contractive where $w = ua$ and $a \in \{0, 1\}$. Then for any $x, y \in I$ such that $x < y$,

$\phi_w(y) - \phi_w(x) = \phi_a(\phi_u(y)) - \phi_a(\phi_u(x))$
Suppose A1 holds, that Lemma 36 applied to the function $g$ gives necessary and sufficient conditions for $\phi$ as $\phi_a(y) > \phi_a(x)$ and $\phi_a$ is contractive.

Therefore $\phi_w$ is contractive.

As $\phi_w$ is contractive, for any $\epsilon > 0$ and $c \in \mathcal{I}$, it follows that

$$|\phi_w(x) - \phi_w(c)| < |x - c| < \epsilon$$

for any $x \in \mathcal{I}$ with $|x - c| < \epsilon$.

Therefore $\phi_w$ is continuous.

Now we show that the fixed point $y_w$ exists, using the intermediate value theorem applied to the function $g(x) := x - \phi_w(x)$. First we show that $g(y_0) \geq 0$. Indeed, as $\phi_1$ is contractive, the definition of $y_1$ gives

$$y_0 - y_1 > \phi_1(y_0) - \phi_1(y_1) = \phi_1(y_0) - y_1, \text{ so that } \phi_1(y_0) < y_0.$$ 

So, it follows from the definition of $y_0$ that the upper bound $\psi(x) := \max\{\phi_0(x), \phi_1(x)\}$ satisfies $\psi(y_0) = \phi_0(y_0) = y_0$. As $\phi_u(x)$ is increasing for any finite word $u$, it follows that

$$\phi_u(y_0) = \phi_{w_2|w|} \circ \phi_{w_1}(y_0) \leq \phi_{w_2|w|} \circ \psi(y_0) = \phi_{w_2|w|}(y_0) \leq \cdots \leq y_0,$$

so that

$$g(y_0) = y_0 - \phi_w(y_0) \geq y_0 - y_0 = 0.$$

A similar argument, using the lower bound $\min\{\phi_0(x), \phi_1(x)\}$, gives $g(y_1) \leq 0$. In summary, $g(y_1) \leq 0 \leq g(y_0)$, where $y_1 < y_0$ by A1, and $g(x)$ is continuous as $\phi_w(x)$ is continuous. So the intermediate value theorem shows that $g(y) = 0$ for some $y \in [y_1, y_0]$.

Therefore a fixed point $y_w$ exists on $\mathcal{I}$.

Now we show that the fixed point $y_w$ is unique. Suppose both $y$ and $z$ are fixed points of $\phi_w$ with $y > z$. This leads to the following contradiction: as $\phi_w$ is contractive we have

$$\frac{\phi_w(y) - \phi_w(z)}{y - z} < 1,$$

yet as $\phi_w(y) = y$ and $\phi_w(z) = z$ we have

$$\frac{\phi_w(y) - \phi_w(z)}{y - z} = 1.$$

Therefore the fixed point is unique. This completes the proof. ■

We make widespread use of the following simple Lemma. Given a word $w$, this Lemma gives necessary and sufficient conditions for $\phi_w(x)$ to be greater than or less than $x$.

**Lemma 36** Suppose A1 holds, that $x \in \mathcal{I}$ and $w$ is a finite non-empty word. Then

$$x < \phi_w(x) \iff \phi_w(x) < y_w \iff x < y_w \text{ and } x > \phi_w(x) \iff \phi_w(x) > y_w \iff x > y_w,$$

This completes the proof.
Lemma 36 shows that \( y \) increasing gives \( \phi \) where the last step follows by repeating the same argument. But if \( \phi \) is increasing and therefore injective. So if \( \phi(x) < y \) then \( x < y \). Therefore

\[
x < y \iff \phi(x) < y.
\]

As \( \phi \) is contractive, if \( x < y \) then \( \phi(y) - \phi(x) < y - x \). As \( \phi(y) = y \), this rearranges to give \( x < \phi(x) \). Similarly, if \( x > y \) then \( x > \phi(x) \). Thus if \( x \leq \phi(x) \) then \( x \leq y \), by the contrapositive. But if \( x \neq \phi(x) \) then \( x \) is not a fixed point, so \( x \neq y \). So if \( x < \phi(x) \) then \( x < y \). Therefore

\[
x < y \iff x < \phi(x).
\]

A similar argument shows that \( x > y \iff \phi(x) > y \) and \( x > y \iff x > \phi(x) \).

Lemma 37 Suppose A1 holds and \( w \) is a finite word with \( |w|_0 |w|_1 > 0 \). Then

\[
y_1 < y_w < y_0.
\]

Proof As \( |w|_0 > 0 \) we have \( w = s01^q \) for some finite word \( s \) and some \( q \in \mathbb{Z}_+ \). As \( y_0 > y_1 \) by A1, Lemma 36 gives \( \phi_0(y_1) > y_1 \). Thus the definition of \( y_1 \) and the fact that \( \phi_s \) is increasing gives

\[
\phi_w(y_1) = \phi_{s01^q}(y_1) = \phi_{s0}(y_1) = \phi_s(\phi_0(y_1)) > \phi_s(y_1) \geq y_1
\]

where the last step follows by repeating the same argument. But if \( \phi_w(y_1) > y_1 \) then Lemma 36 shows that \( y_w > y_1 \).

A similar argument leads to the conclusion that \( y_w < y_0 \).

Lemma 38 Suppose A1 holds, \( x \in \mathcal{I} \) and \( w \) is a finite non-empty word. Then

\[
\lim_{n \to \infty} \phi_w^n(x) = y_w.
\]

Proof The sequence with elements \( x_0 := \phi_w(x) \) for \( n \in \mathbb{Z}_+ \) is monotone and bounded, by Lemma 36. So the monotone convergence theorem for sequences of real numbers shows that \( l := \lim_{n \to \infty} x_n \) exists. But as \( \phi_w \) is continuous, by Lemma 35, the limit \( l \) satisfies

\[
\phi_w(l) = \phi_w\left( \lim_{n \to \infty} x_n \right) = \lim_{n \to \infty} \phi_w(x_n) = \lim_{n \to \infty} x_{n+1} = l.
\]

So \( l \) is a fixed point of \( \phi_w \). By Lemma 35, \( y_w \) is the unique such fixed point.

As a contractive function is not necessarily a contraction mapping, some additional work is required to prove the following result which is essential in the Sturmian case addressed in Subsection A.3.
Lemma 39 Suppose A1 holds, $w$ is an infinite word and $y_0 \geq a > b \geq y_1$. Then

$$\lim_{n \to \infty} (\phi_{w_1,n}(a) - \phi_{w_1,n}(b)) = 0.$$ 

Proof Let $a_n := \phi_{w_1,n}(a)$ and $b_n := \phi_{w_1,n}(b)$ for $n \in \mathbb{Z}^{++}$. By Assumption A1, $\phi_{w_n}$ is a contractive function, so $(a_n - b_n : n \in \mathbb{Z}^{++})$ is a decreasing sequence and as $a, b \in [y_1, y_0]$, Lemma 36 shows that this is also a bounded sequence. Therefore the monotone convergence theorem for real-valued sequences shows that

$$\lim_{n \to \infty} (a_n - b_n)$$

exists.

Now we argue that the limit is zero, by contradiction. Assume that $a_n - b_n \geq \epsilon$ for all $n \in \mathbb{Z}^{++}$, where $\epsilon$ is a positive real number. Let the domain $D \subset \mathbb{R}^2$ be

$$D := \{(h, l) : h, l \in [y_1, y_0], \ h \geq l + \epsilon\},$$

let the functions $f_c : D \to \mathbb{R}$ for letters $c \in \{0, 1\}$ be

$$f_c(h, l) := \frac{\phi_c(h) - \phi_c(l)}{h - l}$$

where $(h, l) \in D$ and define the number $q \in \mathbb{R}$ by

$$q := \sup_{(h, l) \in D} \left\{ \max_{c \in \{0, 1\}} f_c(h, l) \right\}.$$ 

Now the functions $f_0, f_1$ are continuous on their domain $D$ by Lemma 35. Also, the domain $D$ is a non-empty, bounded and closed set. So the extreme value theorem for functions of several variables shows that the maximum equals the supremum. Thus

$$q = \max_{(h, l) \in D} \left\{ \max_{a \in \{0, 1\}} f_a(h, l) \right\}.$$ 

As $\phi_c$ is contractive for $c \in \{0, 1\}$ it follows that

$$q < 1$$

and as $\phi_c$ is increasing we have $q > 0$. So the definition of $q$ and hypothesis that $a > b$ give

$$a_n - b_n \leq q^n(a - b) < \epsilon \quad \text{for} \ n > \log((a - b)/\epsilon)/\log(1/q).$$

Thus there is an $n \in \mathbb{Z}^{++}$ with $a_n - b_n < \epsilon$. This contradicts the assumption that $a_n - b_n \geq \epsilon$ for all $n \in \mathbb{Z}^{++}$. Since $\epsilon > 0$ was arbitrary, we conclude that

$$\lim_{n \to \infty} (a_n - b_n) = 0.$$ 

This completes the proof. \[\qed\]
A.2 $x$-Threshold Words as Christoffel Words

We begin with some definitions that go beyond the main text.

The set $\{0,1\}^*$ consists of all finite words on the alphabet $\{0,1\}$, including the empty string $\epsilon$. The morphism $\mathcal{M}: \{0,1\}^* \to \{0,1\}^*$ generated by a mapping $s: \{0,1\} \to \{0,1\}^*$, substitutes $s(w_k)$ for each letter $w_k$ of a word $w$, so that $\mathcal{M}(\epsilon) = \epsilon$ and $\mathcal{M}(w) = s(w_1)s(w_2)\cdots s(w_{|w|})$.

We work with the morphisms $\mathcal{L}: \{0,1\}^* \to \{0,1\}^*$ and $\mathcal{R}: \{0,1\}^* \to \{0,1\}^*$ given by

\[
\mathcal{L} : \begin{cases} 
0 \mapsto 0 \\
1 \mapsto 01 
\end{cases} \quad \mathcal{R} : \begin{cases} 
0 \mapsto 01 \\
1 \mapsto 1 
\end{cases}.
\]

Let $\circ$ denote composition of morphisms, for example $\mathcal{R} \circ \mathcal{L}(1) = \mathcal{R}(01) = 011$.

Remark 40 These morphisms generate the Christoffel tree through pre-composition. Say $(u,v)$ is a Christoffel pair and consider the morphism

$$\mathcal{M} : \begin{cases} 
0 \mapsto u \\
1 \mapsto v 
\end{cases}.$$ 

Then pre-composition maps the node $(u,v)$ of the Christoffel tree to its children:

$$(\mathcal{M} \circ \mathcal{L}(0), \mathcal{M} \circ \mathcal{L}(1)) = (\mathcal{M}(0), \mathcal{M}(01)) = (u, uv)$$

$$(\mathcal{M} \circ \mathcal{R}(0), \mathcal{M} \circ \mathcal{R}(1)) = (\mathcal{M}(01), \mathcal{M}(1)) = (uv, v).$$

(28)

Now let us give a simple upper and lower bound on the $x$-threshold orbit.

Lemma 41 Suppose A1 holds and $(x_k : k \in \mathbb{Z}_{++})$ is the $x$-threshold orbit. Then

$$x \in [y_1, y_0] \Rightarrow \phi_1(x) \leq x_k < \phi_0(x) \quad \text{for} \quad k \in \mathbb{Z}_{++}.$$ 

Proof Say $z \in [y_1, y_0]$. Then Lemma 36 gives

$$y_1 \leq \phi_1(z) \leq z < \phi_0(z) < y_0.$$ 

An induction using this fact, immediately shows that for $k \in \mathbb{Z}_{++}$,

$$y_1 \leq x_k < y_0.$$ 

Now we prove the claim by induction with hypothesis

$$H_k : \phi_1(x) \leq x_k < \phi_0(x)$$

for $k \in \mathbb{Z}_{++}$. The base case $H_1$ is true as $x_1 = \phi_1(x)$ by definition of the $x$-threshold orbit. For the inductive step, say $H_k$ is true for some $k \in \mathbb{Z}_{++}$. Then there are two cases to consider: $x_k \notin [\phi_1(x), x)$ and $x_k \in [x, \phi_0(x))$. If $x_k \in [\phi_1(x), x)$ then $x_{k+1} = \phi_0(x_k)$ and $\phi_1(x) \leq x_k$.
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< φ₀(xₖ) by Lemma 36 as xₖ < y₀
< φ₀(x) as xₖ < x and φ₀(·) is increasing,

so Hₖ₊₁ is true. If xₖ ∈ [x, φ₀(x)) then xₖ₊₁ = φ₁(xₖ) and

φ₁(x) ≤ φ₁(xₖ) as x ≤ xₖ and φ₁(·) is increasing
≤ xₖ by Lemma 36 as y₁ ≤ xₖ
< φ₀(x),

so Hₖ₊₁ is true. This completes the proof.

Now we show that x-threshold words are Christoffel words in three important special cases and then we find the general conditions on x for which x-threshold words are Christoffel words (Lemma 46).

Lemma 42 Suppose π is the x-threshold word for φ₀, φ₁ satisfying A1. Then

π = 1 ⇔ x ≤ y₁
π = 01 ⇔ x ∈ [y₀₁, y₁₀]
π = 0 ⇔ x ≥ y₀.

Proof If π = 1 then the definition of the x-threshold word shows that φ₁(x) ≥ x. Therefore Lemma 36 shows that x ≤ y₁.

If x ≤ y₁ then Lemma 36 shows that x ≤ φ₁ⁿ(x) ≤ y₁ for all n ∈ ℤ⁺. Therefore π = 1.

If π = 01 then the next letter of πω after (01)ⁿ is 0 for any n ∈ ℤ⁺. So x > φ₁ⁿ(φ₁(x)). Therefore Lemma 38 gives

x ≥ limₙ→∞ φ₁ⁿ(φ₁(x)) = y₀₁.

If π = 01 then π₂ = 1. So x ≤ φ₁₀(x). Therefore Lemma 36 gives

x ≤ y₁₀.

If x ≥ y₀₁ then x > y₁ by Lemma 37. So φ₁(x) < x by Lemma 36. As φ₀₁ is increasing by Lemma 35, it follows that for any n ∈ ℤ⁺,

φ₀₁ⁿ(φ₁(x)) < φ₀₁ⁿ(x) ≤ x

where the second inequality follows from Lemma 36 as x ≥ y₀₁. Therefore, if πω begins with (01)ⁿ then the next letter is 0.

If x ≤ y₁₀ then Lemma 36 shows that φ₁₀ⁿ(x) ≥ x for all n ∈ ℤ⁺. Therefore, if πω begins with (01)ⁿ₀ then the next letter is 1.

The proof for the case π = 0 is symmetric to that for π = 1. This completes the proof.
Lemma 43 Suppose \( \phi_0, \phi_1 \) satisfy A1, \( x \in \mathcal{I} \) and \( \pi \) is the \( x \)-threshold word. Then
\[
\begin{cases}
|\pi|_{11} > 0 & \Rightarrow \ x < y_01 \\
|\pi|_{00} > 0 & \Rightarrow \ x > y_{10}.
\end{cases}
\]

Proof If \( |\pi|_{11} > 0 \) then \( x < y_0 \) by Lemma 42. So, either \( x \leq y_1 \), in which case Lemma 37 shows that \( x < y_{01} \), or \( x \in (y_1, y_0) \). In the latter case, let \( (x_k : k \in \mathbb{Z}_{++}) \) be the \( x \)-threshold orbit. As \( |\pi|_{11} > 0 \) there exists a \( k \in \mathbb{Z}_{++} \) with \( \phi_1(x_k) \geq x \) by definition of the \( x \)-threshold word. Now \( x_k < \phi_0(x) \) by Lemma 41, so that \( \phi_1(\phi_0(x)) > \phi_1(x_k) \geq x \) as \( \phi_1 \) is increasing. But \( \phi_01(x) > x \) implies that \( x < y_{01} \) by Lemma 36.

The proof for \( |\pi|_{00} > 0 \) is symmetric. This completes the proof. \( \blacksquare \)

Lemma 44 Suppose \( \phi_0, \phi_1 \) satisfy A1. Then for \( x \in [y_{10}, y_0] \), there is a unique \( z \in \mathcal{I} \) with \( \phi_0(z) = x \).

Furthermore
\[
\pi(x, \phi_0, \phi_1) = \begin{cases} 
\mathcal{L}(\pi(\phi_0^{(-1)}(x), \phi_0, \phi_{01})) & \text{if } x \in [y_{10}, y_0] \\
\mathcal{R}(\pi(x, \phi_{01}, \phi_1)) & \text{if } x \in [y_1, y_{01}].
\end{cases}
\]

Proof In the first claim, existence of \( z \) follows from the intermediate value theorem, as \( \phi_0 \) is continuous by Lemma 35, as \( y_{01} \in [y_0, y_1] \subseteq \mathcal{I} \) by Lemma 37, as \( \phi_0(y_{01}) = y_{10} \leq x \) by definition of \( y_{01} \), and as \( \phi_0(y_0) = y_0 \geq x \) by definition of \( y_0 \). Uniqueness follows as \( \phi_0 \) is increasing.

Now say \( x \in [y_{10}, y_0] \) consider the claim involving the morphism \( \mathcal{L} \). Let
\[
V := \pi(\phi_0^{(-1)}(x), \phi_0, \phi_{01})^\omega \quad \text{and} \quad W := \pi(x, \phi_0, \phi_1)^\omega.
\]

We show by induction that the hypothesis
\[
H_i : \mathcal{L}(V_{i;i}) \text{ is a prefix of } W
\]
is true for all \( i \in \mathbb{Z}_+ \), noting that this proves the claim.

In the base case, \( \mathcal{L}(e) = e \) is a prefix of \( W \), so \( H_0 \) is true. For the inductive step, say \( H_{i-1} \) is true for some \( i \in \mathbb{Z}_+ \). Let \( (x_n : n \in \mathbb{Z}_{++}) \) be the \( x \)-threshold orbit for \( \phi_0, \phi_1 \), let \( (\tilde{x}_n : n \in \mathbb{Z}_{++}) \) be the \( \phi_0^{(-1)}(x) \)-threshold orbit for \( \psi_0 := \phi_0, \psi_1 := \phi_{01} \) and let \( k := |\mathcal{L}(V_{1:(i-1)})| + 1 \). Then
\[
\tilde{x}_1 = \psi_1(\phi_0^{(-1)}(x)) = \phi_1(x) = x_1
\]
and, letting \( \psi_w \) denote the composition of \( \psi_0, \psi_1 \) corresponding to a word \( w \),
\[
\tilde{x}_i = \psi_{V_{1:(i-1)}}(\tilde{x}_1) \quad \text{by definition of } V
\]
\[
= \phi_{\mathcal{L}(V_{1:(i-1)})}(\tilde{x}_1) \quad \text{by definition of } \psi_0, \psi_1
\]
Hence Lemma 44 gives

\[
\phi_{W_1(k-1)}(\tilde{x}_1) = \phi_{W_1(k-1)}(x_1) = x_k
\]

as \(H_{i-1}\) is true

\[L_0 \quad \text{and} \quad L_0 \] by definition of \(W_0\).

As \(x \in [y_{10}, y_0]\), we have \(\tilde{x} := \phi_{0}^{-1}(x) \in [y_{01}, y_0]\). Letting \(\tilde{y}_0, \tilde{y}_1\) be the fixed points of \(\psi_0, \psi_1\), this reads \(\tilde{x} \in [\tilde{y}_1, \tilde{y}_0]\). But \(\psi_0, \psi_1\) satisfy A1, as Lemma 35 shows that these functions are increasing and contractive, and Lemma 37 shows that \(\tilde{y}_1 < \tilde{y}_0\). Thus Lemma 41 shows that \(\tilde{x}_i \in \psi_0(\tilde{x}) = \phi_{0}^{-1}(x) = x\).

But we already showed that \(x_k = \tilde{x}_i\), so this gives \(x_k < x\). Therefore \(W_k = 0\), by definition of the \(x\)-threshold word. If \(V_i = 0\) then we can conclude that \(H_i\) is true. Otherwise \(V_i = 1\) so that \(\tilde{x}_i \geq \phi_{0}^{-1}(x)\). But we already showed that \(W_k = 0\) and \(x_k = \tilde{x}_i\), so \(x_{k+1} = \phi_{0}(x_k) = \phi_{0}(\tilde{x}_i) \geq x\). Therefore \(W_{k+1} = 1\) and we conclude that \(H_i\) is true.

The proof for the claim involving \(R\) is similar. This completes the proof. \(\blacksquare\)

**Lemma 45** Suppose \(\phi_0, \phi_1\) satisfy A1, \(x \in I\) and \(0v1\) is a finite word. Then

\[
\begin{align*}
\pi(\phi_{0}(x), \phi_{0}, \phi_{1}) &= \mathcal{L}(0v1) \iff \pi(x, \phi_{0}, \phi_{01}) = 0v1 \\
\pi(x, \phi_{0}, \phi_{1}) &= \mathcal{R}(0v1) \iff \pi(x, \phi_{01}, \phi_{1}) = 0v1.
\end{align*}
\]

**Proof** Consider the claim involving the morphism \(\mathcal{L}\).

If \(\pi(\phi_{0}(x), \phi_{0}, \phi_{1}) = \mathcal{L}(0v1)\) then \(|\pi(\phi_{0}(x), \phi_{0}, \phi_{1})|_{00} > 0\), as \(|0v1|_{01} > 0\) for any finite word \(v\) and \(\mathcal{L}(01) = 001\). So Lemma 43 shows that \(\phi_{0}(x) > y_{10}\). Thus \(x > y_{01}\). Also, \(|\pi(\phi_{0}(x), \phi_{0}, \phi_{1})|_{1} = |\mathcal{L}(0v1)|_{1} > 0\), so Lemma 42 shows that \(\phi_{0}(x) < y_{0}\). Thus \(x < y_{0}\). Hence Lemma 44 gives

\[
\mathcal{L}(0v1) = \mathcal{L}(\pi(x, \phi_{0}, \phi_{01})).
\]

As \(\mathcal{L}\) is injective, it follows that

\[
0v1 = \pi(x, \phi_{0}, \phi_{01}).
\]

If \(\pi(x, \phi_{0}, \phi_{01}) = 0v1\) then \(|\pi(x, \phi_{0}, \phi_{01})|_{0} > 0\) and \(|\pi(x, \phi_{0}, \phi_{01})|_{1} > 0\). So Lemma 42 shows that \(x \in (y_{01}, y_{0})\). Hence Lemma 44 gives

\[
\pi(\phi_{0}(x), \phi_{0}, \phi_{1}) = \mathcal{L}(0v1).
\]

The argument for the claim involving \(\mathcal{R}\) is symmetric. This completes the proof. \(\blacksquare\)

**Lemma 46** Suppose \(\phi_0, \phi_1\) satisfy A1 and \(0w1\) is a Christoffel word. Then

\[
x \in [y_{01w}, y_{10w}] \iff \pi(x, \phi_{0}, \phi_{1}) = 0w1.
\]

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**Proof** We use induction on the depth of $0w1$ in the Christoffel tree, with hypothesis

$$H_n : \begin{cases} 
\text{If } 0w1 \text{ is at depth } n \text{ of the tree and } \phi_0, \phi_1 \text{ satisfy } A1, \text{ then } \\
x \in [y_{0w}, y_{10w}] \iff \pi(x, \phi_0, \phi_1) = 0w1.
\end{cases}$$

Lemma 42 shows that the base case ($H_1$) with $0w1 = 01$ is true. For the inductive step, let $0w1$ be a word at depth $n+1$ of the tree and assume $H_n$ is true. Then either $0w1 = L(0v1)$ or $0w1 = R(0v1)$ for some word $0v1$ which is at depth $n$ of the tree. If $0w1 = L(0v1)$ then Lemma 45 gives

$$\pi(x, \phi_0, \phi_1) = 0w1 \iff \pi(\phi_0^{-1}(x), \phi_0, \phi_01) = 0v1.$$ 

Now $\phi_01$ is increasing and contractive by Lemma 35 and $y_{01} < y_0$ by Lemma 37. Thus $\phi_0, \phi_01$ satisfy A1. So the assumption that $H_n$ is true shows that

$$\pi(\phi_0^{-1}(x), \phi_0, \phi_01) = 0v1 \iff \phi_0^{-1}(x) \in [y_{L(01v)}, y_{L(10v)}].$$

But as $0w1 = L(0v1) = 0L(v)01$, we have

$$\phi_0(y_{L(01v)}) = \phi_0(y_{001L(v)}) = y_{01L(v)}0 = y_{01w}$$

$$\phi_0(y_{L(10v)}) = \phi_0(y_{010L(v)}) = y_{10L(v)}0 = y_{10w}.$$ 

As $\phi_0$ is increasing and continuous, it follows that

$$\phi_0^{-1}(x) \in [y_{L(01v)}, y_{L(10v)}] \iff x \in [y_{01w}, y_{10w}].$$

Therefore $H_{n+1}$ is true.

If $0w1 = R(0v1)$ then the proof is similar. This completes the proof. 

**A.3 $x$-Threshold Words as Sturmian $\mathcal{M}$-Words**

It turns out that Lemma 46 characterises $x$-threshold words for nearly all $x \in (y_1, y_0)$. However, its proof cannot be extended to all values of $x$ as it is based on induction on the depth $n \in \mathbb{Z}_{++}$ in the Christoffel tree, and we need to take the limit as $n \to \infty$ to address the remaining values of $x$. Those remaining values correspond to Sturmian $\mathcal{M}$-words, as we show in this subsection.

First we show how Sturmian $\mathcal{M}$-words can be written as the limit of a sequence of words.

**Lemma 47** A word $w$ is a Sturmian $\mathcal{M}$-word if and only if it is of the form

$$w = \lim_{n \to \infty} \mathcal{M}_1 \circ \mathcal{M}_2 \circ \cdots \circ \mathcal{M}_n(01)$$

for some sequence of morphisms $(\mathcal{M}_n : n \in \mathbb{Z}_{++})$ with $\mathcal{M}_n \in \{L, R\}$ which is not eventually constant (i.e. there is no $m \in \mathbb{Z}_{++}$ such that $\mathcal{M}_n = \mathcal{M}_m$ for all $n \in \mathbb{Z}_{++}$ with $n > m$).
The above result is well known, for instance, noting the isomorphism of the Christoffel tree and the Stern-Brocot tree, see Chapter 4 of Graham et al. (1994).

**Remark 48** The requirement that the sequence of morphisms generating Sturmian $M$-words is not eventually constant rules out words such as
\[
\lim_{n \to \infty} L^n(01) = 0^\omega.
\]
Indeed, this corresponds to the Christoffel word $0$. It also rules out words such as
\[
\lim_{n \to \infty} R^n(01) = 01^\omega.
\]
Indeed, this is not an $M$-word because there is no $\alpha \in [0,1]$ with
\[
0 = \lfloor \alpha n \rfloor - \lfloor \alpha (n-1) \rfloor \text{ for } n = 1 \text{ and } 1 = \lfloor \alpha n \rfloor - \lfloor \alpha (n-1) \rfloor \text{ for } n = 2, 3, \ldots.
\]

Now we show that every $x$-threshold word corresponds to an $M$-word.

**Lemma 49** Suppose $\phi_0, \phi_1$ satisfy A1 and $x \in I$. Then $\pi(x, \phi_0, \phi_1)$ is an $M$-word.

**Proof** Lemma 42 shows that $\pi(x, \phi_0, \phi_1)$ is an $M$-word for $x \notin (y_1, y_0)$. So, for the rest of this proof we assume that $x \in (y_1, y_0)$. We shall now define procedure which generates a sequence of morphisms $M_k$, thresholds $x_k \in I$, mappings $\phi_{k,a} : I \to I$ for $a \in \{0, 1\}$ and we denote the compositions of those mappings by
\[
\phi_{k,w} = \phi_{k,w_1} \circ \cdots \circ \phi_{k,w_2} \circ \phi_{k,w_1}
\]
for any finite word $w$. The procedure is as follows:

1. let $k \leftarrow 1$
2. let $(x_k, \phi_{k,0}, \phi_{k,1}) \leftarrow (x, \phi_0, \phi_1)$
3. let $y_{k,01}, y_{k,10}$ be the fixed points of $\phi_{k,01}, \phi_{k,10}$
4. while $x_k \notin [y_{k,01}, y_{k,10}]$
5. if $x_k < y_{k,01}$
6. set $(x_{k+1}, \phi_{k+1,0}, \phi_{k+1,1}, M_k) \leftarrow (x_k, \phi_{k,01}, \phi_{k,1, M})$
7. else
8. set $(x_{k+1}, \phi_{k+1,0}, \phi_{k+1,1}, M_k) \leftarrow (\phi_{k,0}^{-1}(x), \phi_{k,0}, \phi_{k,01}, L)$
9. end
10. let $k \leftarrow k + 1$
11. let $y_{k,01}, y_{k,10}$ be the fixed points of $\phi_{k,01}, \phi_{k,10}$
12. end

The sequence of morphisms $M_1, M_2, \ldots$ generated by this procedure is either empty, of finite length $n \in \mathbb{Z}_{++}$ or of infinite length. We consider each of these cases in turn.

If the sequence is empty, then $x \in [y_{1,01}, y_{1,10}] = [y_{01}, y_{10}]$. So Lemma 42 shows that $\pi(x, \phi_0, \phi_1) = 01$, which is an $M$-word.
If the sequence has finite length, let \( n \) be that length. As the morphisms \( L \) and \( R \) generate the Christoffel tree by pre-composition, as remarked at (28), the word
\[
w := \mathcal{M}_1 \circ \cdots \circ \mathcal{M}_n(01)
\]
is a Christoffel word. Also, when the procedure sets \( \mathcal{M}_k = R \) for some \( k \in \mathbb{Z}^+ \), we have \( x_k < y_{k,01} < y_{k,10} \) so Lemma 44 shows that
\[
\pi(x_k, \phi_{k,0}, \phi_{k,1}) = R(\pi(x_{k+1}, \phi_{k+1,0}, \phi_{k+1,1})).
\]
Similarly, when the procedure sets \( \mathcal{M}_k = L \) we have
\[
\pi(x_k, \phi_{k,0}, \phi_{k,1}) = L(\pi(x_{k+1}, \phi_{k+1,0}, \phi_{k+1,1})).
\]
Therefore the \( x \)-threshold word is \( \pi(x, \phi_0, \phi_1) = w \) which is an \( M \)-word.

Finally, if the sequence does not terminate, then Lemma 47 shows that the word
\[
w := \lim_{n \to \infty} \mathcal{M}_1 \circ \cdots \circ \mathcal{M}_n(01)
\]
is a Sturmian \( M \)-word provided there is no \( m \in \mathbb{Z}^+ \) such that \( \mathcal{M}_n = \mathcal{M}_m \) for all \( n \in \mathbb{Z}^+ \) with \( n \geq m \). Also, the argument given for finite sequences above shows that \( w = \pi(x, \phi_0, \phi_1) \). If there were such an \( m \) and \( \mathcal{M}_m = R \), then
\[
w = \mathcal{M}_1 \circ \cdots \circ \mathcal{M}_{m-1} \circ \lim_{k \to \infty} R^k(01)
= \mathcal{M}_1 \circ \cdots \circ \mathcal{M}_{m-1}(01^\omega).
\]
Thus the word at stage \( m \) is \( \pi(x_m, \phi_{m,0}, \phi_{m,1}) = 01^\omega \). But this is impossible, as the fact that the first letter is 0 requires \( \phi_{m,1}(x_m) < x_m \), so that \( x_m > y_{m,1} \), whereas the fact that remaining letters are 1 requires \( \phi_{m,101^\omega}(x_m) \geq x_m \) for all \( n \in \mathbb{Z}^+ \), so that \( y_{m,1} \geq x_m \), which is a contradiction. If there were such an \( m \) and \( \mathcal{M}_m = L \), then
\[
w = \mathcal{M}_1 \circ \cdots \circ \mathcal{M}_{m-1} \circ \lim_{k \to \infty} L^k(01)
= \mathcal{M}_1 \circ \cdots \circ \mathcal{M}_{m-1}(0^\omega)
\]
Therefore at stage \( m \) the word \( \pi(x_m, \phi_{m,0}, \phi_{m,1}) = 0 \), but this is impossible as \( x_m < y_{m,0} \).

In conclusion, \( \pi(x, \phi_0, \phi_1) = w \) is an \( M \)-word.

This completes the proof.

\[\square\]

**Lemma 50** Suppose \( 0 \leq \alpha < \beta \leq 1 \) and let \( a, b \) be the \( M \)-words of rate \( \alpha, \beta \) respectively. Then \( a^\omega \nless b^\omega \).

**Proof** Consider the first \( n \in \mathbb{Z}^+ \) with \( (a^\omega)_n \neq (b^\omega)_n \). Then
\[
|\alpha(n-1)| = |(a^\omega)_{1:(n-1)}|_1 = |(b^\omega)_{1:(n-1)}|_1 = |\beta(n-1)|,
\]

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by definition of $\mathcal{M}$-words, so that
\[(a^\omega)_n = \lceil \alpha n \rceil - \lceil \alpha(n-1) \rceil = \lceil \beta n \rceil - \lceil \beta(n-1) \rceil = (b^\omega)_n\]
where the inequality holds as $(a^\omega)_n \neq (b^\omega)_n$, as $\alpha < \beta$ and as the floor function is non-decreasing. Therefore $a^\omega < b^\omega$.

Consider a tree and a node $x$ of the tree. Recall that the subtree rooted at $x$ is the tree of all descendents of $x$ that has node $x$ as a root. If the tree is a binary tree, the left subtree of $x$ is the subtree rooted at the left child of $x$, and the right subtree of $x$ is the subtree rooted at the right child of $x$. Thus the subtree rooted at $x$ contains node $x$, but the left and right subtrees of $x$ do not contain node $x$.

**Lemma 51** Suppose $(u,v)$ is a Christoffel pair. Considering $uv$ as a node of the Christoffel tree, let $l$ and $r$ be Christoffel words in the left and right subtrees of $uv$. Then
\[\text{rate}(u) < \text{rate}(l) < \text{rate}(uv) < \text{rate}(r) < \text{rate}(v).\]

**Proof** First we use induction to show that for any Christoffel pair $(u,v)$, we have
\[\text{rate}(u) < \text{rate}(v). \tag{29}\]
In the base case $(u,v) = (0,1)$ and (29) is true. For the inductive step, say $(u,v)$ is the left child of a Christoffel pair $(a,b)$ with $\text{rate}(a) < \text{rate}(b)$. Then $(u,v) = (a,ab)$ so that
\[\text{rate}(u) = \frac{|a|_1}{|a|} < \frac{|a|_1 + |b|_1}{|a| + |b|} = \text{rate}(v)\]
by the mediant inequality. The proof for right children is similar. Therefore (29) is true.

As $l$ is in the left subtree of $uv$, it follows from the construction of the Christoffel tree that $l$ consists of $m$ copies of $u$ and $n$ copies of $uv$ concatenated in some order, for some $m,n \in \mathbb{Z}_{++}$. Thus the mediant inequality and (29) give
\[\text{rate}(l) = \text{rate}(u^m(vu)^n) = \frac{m|u|_1 + n|uv|_1}{m|u| + n|uv|} \in \left(\frac{|u|_1}{|u|}, \frac{|uv|_1}{|uv|}\right)\]
Therefore $\text{rate}(u) < \text{rate}(l) < \text{rate}(uv)$, as claimed.

The proof for a node $r$ in the right subtree of $uv$ is similar. This completes the proof. ■

**Lemma 52** Suppose $(0a1,0b1)$ is a Christoffel pair. Then $a10b = b01a$.

**Proof** As $a,b,a10b$ are palindromes, we have $a10b = (a10b)^R = bR01aR = b01a$. ■

**Lemma 53** Suppose the word $0c1$ is in the subtree of the Christoffel tree rooted at $0p1$. Then $p$ is both a prefix and suffix of $c$. 55
Proof There are four cases to consider:

1. The word $0c1$ has no parent, in which case $c = p = \epsilon$ and the claim holds.

2. The word $c$ is of one of the forms $0^m$ or $0^m10^m$ for some $m \in \mathbb{Z}_{++}$. In that case either $p = c$ or $p = 0^l$ for some $l \in \mathbb{Z}_+$ with $l < m$ and the claim holds.

3. The word $c$ is of one of the forms $1^m$ or $1^m01^m$ for some $m \in \mathbb{Z}_{++}$. This is similar to the previous case.

4. The Christoffel pair of the parent of $0c1$ is of the form $(0^a1, 0^b1)$.

In the last case, we use induction on the length of the path $n \in \mathbb{Z}_+$ through the Christoffel tree from $0p1$ to $0c1$. In the base case, $n = 0$, we have $c = p$ and the claim is true. For the inductive step, say $0c1$ is a child of the node with Christoffel pair $(0^a1, 0^b1)$, that $0a10b1$ is $n$ steps along the path from $0p1$, and that $p$ is both a prefix and suffix of $a10b$, so that $a10b = pq = rp$ for some words $q, r$. If $0c1$ is a left-child, then $c = a10a10b = a10p01a = pq01a$ so $p$ is a prefix and suffix of $c$. Similarly, if $0c1$ is a right child, then $c = a10b10b$ and $pq10b = a10b10b = b01a10b = b01rp$.

This completes the proof.

Lemma 54 Suppose $\phi_0, \phi_1$ satisfy A1. Then

$$\pi(x, \phi_0, \phi_1)^\omega$$

is a lexicographically non-increasing function of $x \in \mathcal{I}$.

Proof If $a, b \in \mathcal{I}$ and $\pi(a, \phi_0, \phi_1)^\omega > \pi(b, \phi_0, \phi_1)^\omega$, then there is a finite word $u$ such that $\pi(a, \phi_0, \phi_1)^\omega = u1v$, $\pi(b, \phi_0, \phi_1)^\omega = u0w$, for some words $v, w$. So the definition of threshold orbits and Lemma 36 give

$$y_{1u} \geq \phi_{1u}(a) \geq a, \quad y_{1u} < \phi_{1u}(b) < b.$$

Therefore $a < b$. This completes the proof.

In the main text, we defined the fixed point $y_s$ of a Sturmian $\mathcal{M}$-word as the limit of a sequence of fixed points $y_{01w^{(n)}}$ or $y_{10w^{(n)}}$ where the words $(0w^{(n)}1 : n \in \mathbb{Z}_{++})$ correspond to a particular path in the Christoffel tree. We now define a subsequence associated with each of these sequences of words and fixed points. Consider a Sturmian $\mathcal{M}$-word

$$0s = \lim_{n \to \infty} L^{a_1} \circ R^{b_1} \circ \ldots \circ L^{a_n} \circ R^{b_n}(01)$$
where \( a_1 \in \mathbb{Z}_+ \) and \( a_{n+1}, b_n \in \mathbb{Z}_{++} \) for \( n \in \mathbb{Z}_{++} \). We define the sequences \((u^{(n)} : n \in \mathbb{Z}_{++})\) and \((l^{(n)} : n \in \mathbb{Z}_{++})\) of the central portions of the Christoffel words as

\[
0u^{(n)} := \mathcal{L}^{a_1} \circ \mathcal{R}^{b_1} \circ \cdots \circ \mathcal{L}^{a_n}(01), \quad 0l^{(n)} := \mathcal{L}^{a_1} \circ \mathcal{R}^{b_1} \circ \cdots \circ \mathcal{L}^{a_n} \circ \mathcal{R}^{b_n}(01). \tag{30}
\]

These words form a subsequence of \((0w^{(n)} : n \in \mathbb{Z}_{++})\), so if \( \lim_{n \to \infty} y_{01w^{(n)}} \) exists, then so does \( \lim_{n \to \infty} y_{01l^{(n)}} \) and these limits are equal.

**Lemma 55** Suppose A1 holds, \( n \in \mathbb{Z}_{++} \) and that \( 0u^{(n)} \) and \( 0l^{(n)} \) are as in (30). Then

\[
y_{01l^{(n)}} < y_{01l^{(n+1)}} < y_{10u^{(n+1)}} < y_{10u^{(n)}}.
\]

**Proof** By definition, \( 0l^{(n+1)} \) is in the left subtree of \( 0l^{(n)} \), so Lemma 51 gives

\[
\text{rate}(0l^{(n+1)}) < \text{rate}(0l^{(n)}).
\]

Hence Lemma 46 and Lemma 50 give

\[
\pi(y_{01l^{(n+1)}})^\omega = (0l^{(n+1)})^\omega < (0l^{(n)})^\omega = \pi(y_{10l^{(n)}})^\omega.
\]

Therefore Lemma 54 shows that

\[
y_{01l^{(n)}} < y_{01l^{(n+1)}}.
\]

But \( 0l^{(n+1)} \) is in the right subtree of \( 0u^{(n+1)} \). So the same argument gives

\[
y_{01l^{(n+1)}} < y_{10u^{(n+1)}}.
\]

Similarly, \( 0u^{(n+1)} \) is in the right subtree of \( 0u^{(n)} \), so

\[
y_{10u^{(n+1)}} < y_{10u^{(n)}}.
\]

This completes the proof. \( \blacksquare \)

Note that the argument in the above proof also shows that any word \( 0u^{(m)} \) lying strictly between \( 0u^{(n)} \) and \( 0l^{(n)} \) on the path through the Christoffel tree from \( 0u^{(n)} \) to \( 0l^{(n)} \) has

\[
y_{10l^{(n)}} < y_{01l^{(m)}} < y_{10u^{(m)}} < y_{01u^{(n)}}.
\]

Similarly, any word \( 0u^{(m)} \) lying strictly between \( 0l^{(n)} \) and \( 0u^{(n+1)} \) on the path from \( 0l^{(n)} \) to \( 0u^{(n+1)} \) has

\[
y_{10l^{(n)}} < y_{01u^{(m)}} < y_{10u^{(m)}} < y_{01u^{(n+1)}}.
\]

Thus if the subsequences \((y_{01l^{(n)}} : n \in \mathbb{Z}_{++})\) and \((y_{10u^{(n)}} : n \in \mathbb{Z}_{++})\) have limits, then so do the full sequences \((y_{01u^{(m)}} : m \in \mathbb{Z}_{++})\) and \((y_{10u^{(m)}} : m \in \mathbb{Z}_{++})\).
Lemma 56 Suppose \((0w^{(n)}1 : n \in \mathbb{Z}_+)\) is the sequence of words traversed along an infinite path down the Christoffel tree. Let \((n_i : i \in \mathbb{Z}_+)\) be an increasing sequence on \(\mathbb{Z}_+\). Then there exists an infinite word \(s\) and an increasing sequence \((k_i : i \in \mathbb{Z}_+)\) on \(\mathbb{Z}_+\) such that \(s_{1:k_i}\) is a suffix of both \(w^{(n_i)}\) and \(w^{(n_{i+1})}\), for all \(i \in \mathbb{Z}_+\).

Proof We show that \(s_{1:k_i} := w^{(n_i)}\) for \(i \in \mathbb{Z}_+\), is well-defined and satisfies this claim. As \(w^{(n_i)}\) is a prefix of all of its descendents by Lemma 53, it follows that \(s_{1:k_i}\) is a prefix of \(s_{1:k_{i+1}}\). Also, as \(w^{(n_i)}\) is a suffix of all of its descendents, it follows that \(s_{1:k_i}\) is a suffix of both \(w^{(n_i)}\) and \(w^{(n_{i+1})}\).

Lemma 57 Suppose \(\phi_0, \phi_1\) satisfy A1 and \(0s\) is a Sturmian \(M\)-word. Consider the sequence of Christoffel words \((0u^{(n)}1 : n \in \mathbb{Z}_+)\) traversed on the infinite path through the Christoffel tree towards \(0s\) (as defined in the main text just before Theorem 16). Then the fixed points

\[ y_{01s} := \lim_{n \to \infty} y_{01u^{(n)}} \quad \text{and} \quad y_{10s} := \lim_{n \to \infty} y_{10u^{(n)}} \]

exist and are equal.

Proof Let \(a_n, b_n, 0u^{(n)}1, 0l^{(n)}1\) for \(n \in \mathbb{Z}_+\) be as in (30).

Existence of the fixed points follows from the monotone convergence theorem for real-valued sequences. Indeed, Lemma 55 shows that \((y_{01l^{(n)}} : n \in \mathbb{Z}_+)\) is an increasing sequence, that \((y_{10u^{(n)}} : n \in \mathbb{Z}_+)\) is a decreasing sequence, and that these sequences are bounded.

By Lemma 56 we have \(u^{(n)} = c^{(n)}w^{1:k_n}l^{(n)} = d^{(n)}w^{1:k_n}\) for all \(n \in \mathbb{Z}_+\) for some sequences of finite words \((c^{(n)} : n \in \mathbb{Z}_+)\) and \((d^{(n)} : n \in \mathbb{Z}_+)\), for some infinite word \(w\) and for some increasing sequence \((k_n : n \in \mathbb{Z}_+)\) on \(\mathbb{Z}_+\).

As \(\phi_a(x)\) is an increasing function of \(x \in I\) for any finite word \(a\), by Lemma 35,

\[ y_{10u^{(n)}} = \phi_{10u^{(n)}}(y_{10u^{(n)}}) \]

\[ < \phi_{10u^{(n)}}(y_0) \]

\[ = \phi_{10c^{(n)}w^{1:k_n}}(y_0) \]

\[ = \phi_{w^{1:k_n}}(\phi_{10c^{(n)}}(y_0)) \]

\[ < \phi_{w^{1:k_n}}(y_0). \]

Similarly, we have

\[ y_{01l^{(n)}} > \phi_{w^{1:k_n}}(y_1). \]

Thus

\[ \lim_{n \to \infty} (y_{10u^{(n)}} - y_{01l^{(n)}}) \leq \lim_{n \to \infty} (\phi_{w^{1:k_n}}(y_0) - \phi_{w^{1:k_n}}(y_1)) = 0 \]

where the last step is Lemma 39. But \(y_{10u^{(n)}} > y_{01l^{(n)}}\) for \(n \in \mathbb{Z}_+\) by Lemma 55. Therefore

\[ y_{10s} = \lim_{n \to \infty} y_{10u^{(n)}} = \lim_{n \to \infty} y_{01l^{(n)}} = y_{01s}. \]
This completes the proof. □

In view of the above Lemma, from now on we shall write \( y_s = y_{10s} = y_{01s} \).

**Lemma 58** Suppose \( \phi_0, \phi_1 \) satisfy A1 and 0s is a Sturmian \( \mathcal{M} \)-word. Then

\[
\pi(x, \phi_0, \phi_1) = 0s \iff x = y_s.
\]

**Proof** Let \((u^{(n)} : n \in \mathbb{Z}_{++})\) and \((l^{(n)} : n \in \mathbb{Z}_{++})\) be the sequences (30) appearing in the definition of the fixed point of 0s. For fixed \( \phi_0, \phi_1 \) let us write \( \pi(x) \) in place of \( \pi(x, \phi_0, \phi_1) \).

Say \( x = y_s \). Recall that \( x = y_{10s} = y_{01s} \) by Lemma 57. As \( y_{10u^{(n)}} > y_{10s} \) for all \( n \in \mathbb{Z}_{++} \), by Lemma 55, and \( \pi(z)^\omega \) is a lexicographically non-increasing function of \( z \), by Lemma 54, it follows that

\[
\pi(x)^\omega > \pi(y_{10u^{(n)}})^\omega = (0u^{(n)}1)^\omega
\]

where the equality follows from Lemma 46. A similar argument gives \( \pi(x)^\omega < (0l^{(n)}1)^\omega \) for \( n \in \mathbb{Z}_{++} \). Therefore

\[
0s = \lim_{n \to \infty} (0u^{(n)}1)^\omega \preceq \pi(x)^\omega \preceq \lim_{n \to \infty} (0l^{(n)}1)^\omega = 0s.
\]

Now say \( \pi(x) = 0s \) for some \( x \in \mathcal{I} \). Then \( y_{01l^{(n)}} < x < y_{10u^{(n)}} \) as \( \pi(z)^\omega \) is a lexicographically non-increasing function of \( z \) and \( (0u^{(n)}1)^\omega < 0s < (0l^{(n)}1)^\omega \) for all \( n \in \mathbb{Z}_{++} \).

Therefore

\[
y_s = \lim_{n \to \infty} y_{01l^{(n)}} \leq x \leq \lim_{n \to \infty} y_{10u^{(n)}} = y_s.
\]

This completes the proof. □

**A.4 Proof of Theorem 16**

**Proof** The existence of fixed points \( y_{01p}, y_{10p} \) follows from Lemma 35 and the existence of \( y_s \) follows from Lemma 57.

The fact that \( \sigma(z|z) = 1\pi(z, \phi_0, \phi_1)^\omega \) is a lexicographically non-decreasing function of \( z \in \mathcal{I} \) follows from Lemma 54.

Lemma 42 addresses the value of \( \sigma(z|z) \) for \( z \leq y_1 \) and \( z \geq y_0 \), Lemma 46 addresses the case \( z \in [y_{01p}, y_{10p}] \) and Lemma 58 address the case \( z = y_s \). Thus the image of \( \pi(z, \phi_0, \phi_1) \) as \( z \) ranges over \( \mathcal{I} \) contains all \( \mathcal{M} \)-words. Using Lemma 49, it follows that this image is exactly the set of \( \mathcal{M} \)-words.

This completes the proof. □
A.5 Proof of Theorem 17

Based on the work of Kozyakin (2003), we begin by showing that maps-with-gaps formed from functions satisfying Assumption A1 are locally-growing relaxation functions (Lemmas 60 and 61) and that the itineraries of such functions are 1-balanced words (Lemmas 63 and 64). As 1-balanced words correspond to factors of lower mechanical words (Lemma 65), this enables us to describe the itineraries of maps-with-gaps, for arbitrary initial states and thresholds (Lemma 66). We couple this description with an easy result about lexicographic ordering (Lemma 67) and with a result about the number of factors of mechanical words (Lemma 68) to bound the number of discontinuities of the itinerary of a map-with-a-gap as a function of its threshold (Lemma 69). Finally, we prove Theorem 17.

**Definition 59** Let \( I \) be an interval of \( \mathbb{R} \). A function \( f: I \to I \) is a locally-growing relaxation function with threshold \( z \in I \) if

1. \( f(z) < z < f(z^-) < \infty \)
2. \( f(f(z^-)) \leq f(f(z)) \)
3. \( f(x) \) is increasing for \( x \in [f(z), z) \)
4. \( f(x) \) is increasing for \( x \in [z, f(z^-)) \).

In Kozyakin (2003), this terminology was used for a smaller class of functions \( f \). In particular, the domain and range were restricted to the interval \([0, 1)\) and there was a requirement that \( f \) is continuous on each of the intervals \([f(z), z)\) and \([f(z^-), f(z))\).

The following two Lemmas show that maps-with-gaps whose parts satisfy Assumption A1 lead to locally-growing relaxation functions.

**Lemma 60** Suppose \( \phi_0, \phi_1 \) satisfy A1 and \( x \in [y_1, y_0] \). Then \( \phi_{01}(x) < \phi_{10}(x) \).

**Proof** Suppose that \( x \in [y_1, y_0] \). Then using Lemma 36 gives

\[
\phi_1(\phi_0(x)) - \phi_1(x) < \phi_0(x) - x \quad \text{as } \phi_1 \text{ is contractive and } x < y_0
\]

\[
\phi_0(x) - \phi_0(\phi_1(x)) \leq x - \phi_1(x) \quad \text{as } \phi_0 \text{ is contractive and } x \geq y_1
\]

\[
\phi_{01}(x) < \phi_{10}(x) \quad \text{by adding these inequalities.}
\]

A symmetric argument holds if \( x \in (y_1, y_0] \). This completes the proof. \( \blacksquare \)

**Lemma 61** Suppose \( \phi_0, \phi_1 \) satisfy A1, that \( z \in (y_1, y_0) \) and let \( f(x) := \phi_{1_{x \geq z}}(x) \) for \( x \in I \). Then \( f \) is a locally-growing relaxation function with threshold \( z \).

**Proof** By definition \( f(z^-) = \phi_0(z^-) = \phi_0(z) \), as \( \phi_0 \) is continuous by Lemma 35. Also \( f(z) = \phi_1(z) \), \( f(f(z^-)) = \phi_{01}(z) \) and \( f(f(z)) = \phi_{10}(z) \). So the conditions defining a locally-growing relaxation function read as follows:

1. \( \phi_1(z) < z < \phi_0(z) < \infty \), which is true for \( z \in (y_1, y_0) \) by Lemma 36
2. $\phi_{01}(z) < \phi_{10}(z)$, which is the result of Lemma 60

3. $\phi_0(x)$ is increasing for $x \in [\phi_1(z), z)$, which holds by A1

4. $\phi_1(x)$ is increasing for $x \in [z, \phi_0(z))$, which holds by A1.

This completes the proof.

The following definition is due to Morse and Hedlund (1940).

**Definition 62** An word $w$ is **1-balanced** if

$$| |u|_1 - |v|_1| \leq 1$$

for all factors $u, v$ of $w$ with $|u| = |v|$.

The next two Lemmas use an argument from Kozyakin (2003), to show that the itineraries of a locally-growing relaxation function are 1-balanced. We denote the fractional part of a real number $x$ by \(\{x\} := x - \lfloor x \rfloor\).

**Lemma 63** Suppose $f$ is a locally-growing relaxation function with threshold $z$. Let

$$F(x) := \frac{f(t\{x\} - t1_{\{x\} \in [0,1] + z} - z}{t} + |x| + 1 \quad \text{for } x \in \mathbb{R}$$

where $t := f(z^-) - f(z)$ and $\alpha := (f(z^-) - z)/t$. Then

**F1.** $F(0) \in [0,1)$

**F2.** $F(x + 1) = F(x) + 1$ for all $x \in \mathbb{R}$

**F3.** $F$ is increasing

**F4.** The itineraries of $f$ and $F$ agree in the sense that

$$f^{(n-1)}(x) \geq z \iff \left\{ F^{(n-1)} \left( \frac{x - z}{t} \right) \right\} \in [0, F(0))$$

for all $n \in \mathbb{Z}^{++}$ and all $x \in [f(z), f(z^-))$.

**Proof** First, note that the function $F$ is well-defined as $f(z) \in \mathbb{R}$ and $f(z) < z < f(z^-) < \infty$ so that $t \in \mathbb{R}^{++}$ and

$$\alpha = \frac{f(z^-) - z}{f(z^-) - f(z)} \in (0, 1). \quad (31)$$

It is easy to see that F1 and F2 hold. Indeed

$$F(0) = \frac{f(z) - z}{t} + 1 = \frac{f(z) - z + f(z^-) - f(z)}{f(z^-) - f(z)} = \alpha \in (0, 1)$$

and as the ratio in the definition of $F(x)$ only depends on $\{x\}$, we have

$$F(x + 1) - F(x) = |x + 1| - |x| = 1 \quad \text{for } x \in \mathbb{R}. \quad (32)$$

Now we show that F3 holds.
• If $x \in [0, \alpha)$ then $tx + z \in [z, f(z))$. But $f(\cdot)$ is increasing on $[z, f(z))$ and $t \in \mathbb{R}_{++}$. It follows that $F(x) = (f(tx + z) - z)/t$ is increasing for $x \in [0, \alpha)$.

• As $f(f(z^-)) \leq f(f(z))$ and $f$ is increasing on $[z, f(z^-))$, it follows that

$$F(\alpha^-) = \lim_{x \uparrow f(z^-)} f(u) - z = \frac{f(f(z^-)) - z}{t} + 1 \leq \frac{f(f(z)) - z}{t} + 1 = F(\alpha).$$

• If $x \in [\alpha, 1)$ then $tx - t + z \in [f(z^-), z)$. But $f(\cdot)$ is increasing on $[f(z^-), z)$ and $t \in \mathbb{R}_{++}$. It follows that $F(x) = (f(tx - t + z) - z)/t$ is increasing for $x \in [\alpha, 1)$.

• The definition of $F$ gives

$$F(1^-) = \frac{f(z^-) - z}{t} + 1 = \alpha + 1 = F(1).$$

In summary, $F(x)$ is increasing for $x \in [0, 1]$. But as $F(x + 1) = F(x) + 1$ for $x \in \mathbb{R}$ we conclude that $F$ is increasing for $x \in \mathbb{R}$. Therefore F3 holds.

Now, we note that for any $m \in \mathbb{Z}_+$ and any $x \in [\alpha - 1, \alpha)$ we have

$$\frac{f(m)(tx + z) - z}{t} \in [\alpha - 1, \alpha). \quad (33)$$

Indeed, for $m = 0$ we have $(f(m)(tx + z) - z)/t = x \in [\alpha - 1, \alpha)$. Also if $y \in [f(z), f(z^-))$ then the assumptions about $f$ show that $f(y) \in [f(z), f(z^-))$, while if $x \in [\alpha - 1, \alpha)$ then the definitions of $t$ and $\alpha$ show that $tx + z \in [f(z), f(z^-))$. Thus $(f(m)(tx + z) - z)/t \in [\alpha - 1, \alpha)$.

Now, we show by induction that for any for any $x \in [\alpha - 1, \alpha)$ and $n \in \mathbb{Z}_+$ we have

$$F^{(n)}(\{x\}) = \frac{f^{(n)}(tx + z) - z}{t} \in \mathbb{Z}. \quad (34)$$

In the base case $n = 0$ we have $F^{(0)}(\{x\}) - (f^{(0)}(tx + z) - z)/t = \{x\} - ((tx + z) - z)/t \in \mathbb{Z}$. For the inductive step, let $p := F^{(n)}(\{x\})$ and $q := (f^{(n)}(tx + z) - z)/t$ and suppose that $p - q \in \mathbb{Z}$. Then we have

$$F^{(n+1)}(\{x\}) = F(p) = F(q) + p - q$$

as $F$ satisfies F2 and by the assumption that $p - q \in \mathbb{Z}$. Furthermore, $q \in [\alpha - 1, \alpha)$, by (33) so that for some $k \in \mathbb{Z},$

$$F^{(n+1)}(\{x\}) = p - q + \begin{cases} \frac{f(tq + z) - z}{t} + 1 & \text{if } q \in [0, \alpha) \\ \frac{f(t(q + 1) - t + z) - z}{t} & \text{if } q \in [\alpha - 1, 0) \end{cases}$$

$$= \frac{f(tx + z) - z}{t} + k$$

$$= \frac{f(tf^{(n)}(tx + z) - z)}{t} + k$$

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\[
= \frac{f^{(n+1)}(tx + z) - z}{t} + k.
\]

Therefore (34) is true.

From (33) and (34) it follows that for any \( x \in [\alpha - 1, \alpha) \) and \( n \in \mathbb{Z}_+ \) we have

\[
\{ F^{(n-1)}(\{x\}) \} \in [0, \alpha) \Leftrightarrow \left\{ \frac{f^{(n-1)}(tx + z) - z}{t} \right\} \in [0, \alpha)
\]

\[
\Leftrightarrow \frac{f^{(n-1)}(tx + z) - z}{t} \in [0, \alpha)
\]

\[
\Leftrightarrow f^{(n-1)}(tx + z) \geq z.
\]

(35)

Therefore F4 holds. This completes the proof. □

**Lemma 64** Suppose \( F : \mathbb{R} \to \mathbb{R} \) satisfies Claims F1-F3 of Lemma 63. Let \( \alpha := F(0) \) and for \( x \in \mathbb{R} \) let \( s(x) \) denote the infinite word with letters

\[
s_n(x) := \begin{cases} 
1 & \text{if } \{ F^{(n-1)}(x) \} \in [0, \alpha) \\
0 & \text{if } \{ F^{(n-1)}(x) \} \in [\alpha, 1) 
\end{cases}
\]

for \( n \in \mathbb{Z}_+ \). Then \( s(x) \) is 1-balanced.

**Proof** For any \( n \in \mathbb{Z}_+ \) and any \( x \in [0, 1) \) we have

\[
\begin{align*}
[F^{(n)}(x) - \alpha] - [F^{(n-1)}(x) - \alpha] &= [F(r + z) - \alpha] - [r + z - \alpha] \\
&= [F(r) + z - \alpha] - [r + z - \alpha] \\
&= [F(r) - \alpha] - [r - \alpha] \\
&= -[r - \alpha] \\
&= \lceil \alpha - r \rceil \\
&= \begin{cases} 
1 & \text{if } r \in [0, \alpha) \\
0 & \text{if } r \in [\alpha, 1) 
\end{cases} \\
&= s_n(x).
\end{align*}
\]

Therefore, for any \( m \in \mathbb{Z}_+ \),

\[
\sum_{k=1}^{m} s_k(x) = \sum_{k=1}^{m} \left( [F^{(k)}(x) - \alpha] - [F^{(k-1)}(x) - \alpha] \right) = [F^{(m)}(x) - \alpha] - [x - \alpha].
\]

(36)

Now we show by induction that for any \( m \in \mathbb{Z}_+ \),

\[
F^{(m)}(z) - F^{(m)}(y) \in [0, 1) \text{ for any } y, z \in \mathbb{R} \text{ with } z - y \in [0, 1).
\]

(37)
The base case with \( m = 0 \) reads \( z - y \in [0, 1) \) which is true. For the inductive step, let \( z' := F^{(m-1)}(z) \) and \( y' := F^{(m-1)}(y) \) for \( m - 1 \in \mathbb{Z}_+ \) and assume that \( z' - y' \in [0, 1) \). Then \( F(y') \leq F(z') \) as \( y' \leq z' \) and \( F \) is increasing. Also \( F(z') < F(y' + 1) = F(y') + 1 \) as \( z' - y' < 1 \), as \( F \) is increasing and as \( F(u + 1) = F(u) + 1 \) for any \( u \in \mathbb{R} \). Therefore \( F^{(m)}(z) - F^{(m)}(y) = F(z') - F(y') \in [0, 1) \).

For any choice of \( m \in \mathbb{Z}_+ \) and \( x, y \in [0, 1) \), we wish to prove that \( |\Delta| \leq 1 \) for

\[
\Delta := |s_1(x)s_2(x)\ldots s_m(x)|_1 - |s_1(y)s_2(y)\ldots s_m(y)|_1
\]

using equation (36). But if \( x \geq y \) then \( x - y \in [0, 1) \) so (37) shows that \( A \) and \( B \) are both of the form \( |a| - |b| \) for some \( a - b \in [0, 1) \) and it follows that both \( A \) and \( B \) are in \( \{0, 1\} \). Whereas if \( x \leq y \) then \( y - x \in [0, 1) \) so both \( A \) and \( B \) are in \( \{-1, 0\} \). We conclude that \( |\Delta| = |A - B| \leq |1 - 0| = 1 \). 

The following is Theorem 3.1 of Dulucq and Gouyou-Beauchamps (1990) and provides the missing link between 1-balanced words and mechanical words.

**Lemma 65** Suppose \( w \) is a word of length \( n \in \mathbb{Z}_+ \) with \( |w|_0|w|_1 > 0 \). Then \( w \) is 1-balanced if and only if \( w_k = \left\lfloor \frac{pk+r}{q} \right\rfloor - \left\lfloor \frac{p(k-1)+r}{q} \right\rfloor \) for \( k = 1, 2, \ldots, n \), where \( p, q, r \in \mathbb{Z} \) satisfy \( 0 < p < q \leq n \), \( 0 \leq r < q \leq n \) and \( \gcd(p, q) = 1 \).

Now we are ready to describe the itineraries of maps-with-gaps, for arbitrary initial states and thresholds.

**Lemma 66** Suppose \( \phi_0, \phi_1 \) satisfy A1 and that \( n \in \mathbb{Z}_+ \), \( x \in \mathbb{I} \), \( s \in \mathbb{R} \) and \( a \in \{0, 1\} \). Then

\[
\sigma(x|s)_{1:n} = l^m w
\]

for some \( l \in \{0, 1\} \), some \( m \in \{0, 1, \ldots, n\} \), and some factor \( w \) of a lower mechanical word.

**Proof** First, note that if \( a, b \in \mathbb{I} \), then Lemma 38 shows that

\[
b > y_1 \Rightarrow \phi_1^m(a) < b \text{ for some } m \in \mathbb{Z}_+ \tag{38}
\]

\[
b \leq y_0 \Rightarrow \phi_0^m(a) \geq b \text{ for some } m \in \mathbb{Z}_+ \tag{39}
\]

We consider seven cases.

1. Say \( s \leq y_1 \) and \( x \geq s \). Then \( \sigma(x|s) = 1\sigma(\phi_1(x)|s) \). But if \( x > y_1 \) then Lemma 36 gives \( \phi_1(x) > y_1 \geq s \), whereas if \( x \leq y_1 \) then \( \phi_1(x) \geq x \geq s \). In both cases, \( \sigma(\phi_1(x)|s) = 1\sigma(\phi_1(x)|s) \). Repeating this argument gives \( \sigma(x|s) = 1^\omega \), so the claim is true.
2. Say $s \leq y_1$ and $x < s$. Then $\sigma(x|s)$ begins with 0, and as $s < y_0$ it follows from (39) that there is a least $m \in \mathbb{Z}_{++}$ with $\phi_{0m}(x) \geq s$. Thus $\sigma(x|s) = 0^m \sigma(\phi_{0m}(x)|s) = 0^m \omega$ by Case 1, so the claim is true.

3. Say $s \in (y_0, y_1)$ and $x \in [\phi_1(s), \phi_0(x))$. As $\phi_0, \phi_1$ satisfy A1, Lemmas 61, 63 and 64 together show that $\sigma(x|s)$ is a 1-balanced word. Thus Lemma 65 shows that $\sigma(x|s)_{1:n} = w$ for some factor $w$ of a lower mechanical word, so the claim is true.

4. Say $s \in (y_0, y_1)$ and $x < \phi_1(s)$. Then Lemma 36 shows that $\phi_1(s) < s$. Thus $x < s$, so $\sigma(x|s)$ begins with 0, and $\phi_1(s) < y_0$, so (39) shows that there is a least $m \in \mathbb{Z}_{++}$ with $\phi_{0m}(x) \geq \phi_1(s)$. Thus $\sigma(x|s) = 0^m \sigma(\phi_{0m}(x)|s)$ where $\sigma(\phi_{0m}(x)|s)_{1:n} = w$ for some factor $w$ of a lower mechanical word, by Case 3, so the claim is true.

5. Say $s \in (y_0, y_1)$ and $x \geq \phi_0(s)$. Then arguing as in Case 4, but using (38), shows that $\sigma(x|s)_{1:n} = 1^m w$ for some $m \in \mathbb{Z}_{++}$ and some factor $w$ of a lower mechanical word.

6. Say $s \geq y_0$ and $x < s$. Then arguing as in Case 1 shows that $\sigma(x|s) = 0^\omega$.

7. Say $s \geq y_0$ and $x \geq s$. Then arguing as in Case 2 shows that $\sigma(x|s) = 1^m 0^\omega$ for some $m \in \mathbb{Z}_{++}$.

This completes the proof.

The following is an analogue of Lemma 54 in which only the threshold varies.

**Lemma 67** Suppose $\phi_0, \phi_1$ satisfy A1 and $x \in \mathcal{I}$. Then $\sigma(x|s)$ is a lexicographically non-increasing function of $s \in \mathbb{R}$.

**Proof** If $s, t \in \mathbb{R}$ and $\sigma(x|s) \succeq \sigma(x|t)$, then for some finite word $u$ and words $v, w$,

$$\sigma(x|s) = uv, \quad \quad \sigma(x|t) = uw.$$  

So the definition of the itinerary gives $t > \phi_u(x) \geq s$. This completes the proof.

The following is Theorem 17 and part of Corollary 18 of Mignosi (1991).

**Lemma 68** Let $\mathcal{A}_n$ be the set of factors of length $n \in \mathbb{Z}_{++}$ of lower mechanical words. Then

$$\text{card}(\mathcal{A}_n) = 1 + \sum_{i=1}^{n} (n - i + 1) \text{EulerTotient}(i) = \frac{2n^3}{\pi^2} + O(n^2 \log n).$$

Now we are ready to bound the number of discontinuities of the itinerary of a map-with-a-gap as a function of its threshold.

**Lemma 69** Suppose $\phi_0, \phi_1$ satisfy A1, that $t \in \mathbb{Z}_{++}$, $x \in \mathcal{I}$ and $a \in \{0, 1\}$. Then the mapping $s \mapsto A_{1:t}(x, a; s)$ for $s \in \mathbb{R}$ has at most a polynomial number $p(t)$ of discontinuities.
Proof Let $A_n$ be the set of all factors of length $n$ of lower mechanical words. Let $F_t$ be the set of all words of the form $l^m w$ for some $l \in \{0, 1\}$, some $m \in \{1, \ldots, n\}$ and some factor $w$ of a lower mechanical word. By Lemma 66, the word $A_{1,t}(x, a; s)$ is in $F_t$. Also, Lemma 67, shows that the mapping $s \mapsto A_{1,t}(x, a; s)$ is lexicographically non-increasing. Thus, the number of discontinuities of this mapping is at most

$$\text{card}(F_t) = \text{card}\left(\{l^m w : l \in \{0, 1\}, m \in \mathbb{Z}_+, m \leq t, w \in A_{t-m}\}\right) = 2 \sum_{m=0}^{t} \text{card}(A_{t-m}).$$

Finally, Lemma 68 shows the right-hand side is $O(t^4)$. This completes the proof.

Proof [Proof of Theorem 17] The theorem simply couples together Lemmas 66, 67 and 69.

Appendix B. Proof of Lemma 23

Demonstrating Lemma 23 by combining results in Marshall et al. (2010) requires as much text as a direct proof.

Proof Let $X$ be the set of sequences $X$ with components $X_k = \sum_{i=1}^{k} x_i$ for $k = 1, 2, \ldots, n$ where $x_1, x_2, \ldots, x_n$ is a non-decreasing sequence on $\mathbb{R}^+$. Let $g : X \to \mathbb{R}$ be the function

$$g(X) := f_1(X_1) + \sum_{i=2}^{n} f_i(X_i - X_{i-1}).$$

Let $f'_i(\cdot)$ denote the (sub)gradient of $f_i(\cdot)$. For $i = 1, 2, \ldots, n - 1$, the (sub)gradients of $g(\cdot)$ are

$$\frac{\partial g(X)}{\partial X_i} = f'_i(x_i) - f'_{i+1}(x_{i+1}) \leq f'_{i+1}(x_i) - f'_{i+1}(x_{i+1}) \leq 0$$

where the first inequality holds as $f'_i(x) \leq f'_{i+1}(x)$ for $x \in \mathbb{R}^+$ (by Hypothesis 4) and the second holds as $x_i \leq x_{i+1}$ and $f'_{i+1}(\cdot)$ is convex (by Hypothesis 3). Also,

$$\frac{\partial g(X)}{\partial X_n} = f'_n(x_n) \leq 0$$

as $f_n(\cdot)$ is non-increasing (by Hypothesis 3). Therefore $g(\cdot)$ is non-increasing in all of its arguments. As the sequences $A, B$ with components $A_k := \sum_{i=1}^{k} a_i, B_k := \sum_{i=1}^{k} b_k$ are in $X$ (by Hypothesis 1) and $A_k \leq B_k$ for $k = 1, 2, \ldots, n$ (by Hypothesis 2), it follows that $g(A) \geq g(B)$. So the definition of $g(\cdot)$ gives

$$\sum_{i=1}^{n} f(a_i) = g(A) \geq g(B) = \sum_{i=1}^{n} f(b_i)$$

as claimed.
Appendix C. Proof of Lemma 25

We start by recalling the definition of the matrix \( M(w) \) corresponding to a given finite word \( w \), which corresponds to the composition of Kalman-Filter variance updates, and introducing some related matrices \( K, S(w) \) and \( X \). We then prove Claims 1 to 5 of Lemma 25 in turn.

**Definition 70** Let \( I \) be the 2-by-2 identity matrix. For \( r \in (0, 1] \) and \( 0 \leq a \leq b \), let
\[
F := \begin{pmatrix} r & 1/r \\ ar & (a+1)/r \end{pmatrix}, \quad G := \begin{pmatrix} r & 1/r \\ br & (b+1)/r \end{pmatrix}, \quad K := \begin{pmatrix} r & 1/r \\ r-r^3 & -r \end{pmatrix}.
\]
Let \( M(\epsilon) = I, M(0) = F, M(1) = G \) and for any finite word \( w \) let
\[
M(w) = M(w_1) \cdots M(w_2)M(w_1), \quad S(w) = \sum_{j=0}^{|w|} M(w_{1:j}).
\]
Let
\[
X := \begin{pmatrix} -r/(1-r^2) & 0 \\ 0 & 1/r \end{pmatrix}.
\]

**Remark 71** We use the following facts repeatedly without mention. Clearly \( \det(F) = \det(G) = 1 \), so that \( \det(M(w)) = 1 \) for any word \( w \). Also, \( KF = F^{-1}K, KG = G^{-1}K \) and \( K^2 = I \). Thus for \( A \in \{KF, KG, K\} \) we have \( A^2 = I \), so \( A \) is an involutory matrix.

Thus \( KM(w)^{-1}K = M(w^R) \), where \( w^R \) denotes the reverse \( w_n \cdots w_2w_1 \) of a word \( w = w_1w_2\cdots w_n \).

**Notation.** For a vector \( v \in \mathbb{R}^m \) where \( m \in \mathbb{Z}_{++} \), we write \( v > 0 \) if \( v_i > 0 \) for \( i = 1, 2, \ldots, m \) and \( v \geq 0 \) if \( v_i \geq 0 \) for \( i = 1, 2, \ldots, m \). Similarly, for a matrix \( P \in \mathbb{R}^{m \times n} \) where \( m, n \in \mathbb{Z}_{++} \), we write \( P > 0 \) (\( P \geq 0 \)) if \( P_{ij} > 0 \) (\( P_{ij} \geq 0 \)) for \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n \).

C.1 Claim 1

We require one simple Lemma.

**Lemma 72** Suppose \( a \in \mathbb{R}_+ \) and \( r \in (0, 1] \). Then the fixed point \( y_0 \) satisfies
\[
y_0 \leq \frac{1}{1-r^2}.
\]

**Proof** As the positive root of
\[
y_0 = \frac{r^2y_0 + 1}{ar^2y_0 + 1+a}
\]
is a decreasing function of \( a \) for \( a \in \mathbb{R}_+ \), setting \( a = 0 \) gives an upper bound. This upper bound \( u \) satisfies \( u = r^2u + 1 \), so that \( u = 1/(1-r^2) \). \( \blacksquare \)
Proof [Proof of Claim 1 of Lemma 25] We prove the claim for \( x \) satisfying
\[ \phi_p(0) \leq x \leq \frac{1}{r - r^2}, \]
noting that
\[ \phi_p \left( \frac{1}{1 - r^2} \right) \leq \frac{1}{1 - r^2} \leq \frac{1}{r - r^2}, \]
where the first inequality follows from Lemma 36 (in Appendix A) as Lemma 72 gives \( 1/(1 - r^2) \geq y_0 \geq y_p \), and the second inequality holds as \( r \in (0, 1) \).

For any word \( w \) and for \( k = 1, 2, \ldots, |w| \), let
\[ \begin{pmatrix} u_k \\ v_k \end{pmatrix} := M(w_{1:k}) \begin{pmatrix} x \\ 1 \end{pmatrix}. \]
Clearly \( u_k, v_k \) are positive as \( x \geq \phi_p(0) \geq 0 \) and
\[ M(w_{1:k}) \begin{pmatrix} x \\ 1 \end{pmatrix} \geq \begin{pmatrix} r \\ 0 \end{pmatrix}^k \begin{pmatrix} 0 \\ 1 \end{pmatrix} > 0. \]

Now, for any \( 0 \leq z \leq 1/(r - r^2) \) and \( H = M(w_k) \), noting that \( a, b, r \geq 0 \) and \( r \leq 1 \) gives
\[ \frac{H_{11}z + H_{12}}{H_{21}z + H_{22}} \leq r^2 z + 1 \leq \frac{r^2}{r - r^2} + 1 \leq \frac{1}{r - r^2}. \]

For \( k = 1, 2, \ldots, |w| \), induction using this inequality proves that
\[ \frac{u_k}{v_k} \leq \frac{1}{r - r^2}. \]
(For \( k = 1 \), put \( z = x \leq 1/(r - r^2) \). For \( k > 1 \), assume that \( z = u_{k-1}/v_{k-1} \leq 1/(r - r^2) \).)

Thus
\[ \begin{pmatrix} u_{k+1} - u_k \\ v_{k+1} - v_k \end{pmatrix} = (M(w_{k+1}) - I) \begin{pmatrix} u_k \\ v_k \end{pmatrix} \geq \begin{pmatrix} r - 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ r - 1 \end{pmatrix} \begin{pmatrix} 1 \\ r - r^2 \end{pmatrix} v_k \geq 0. \]

But both \( a_k(x), b_k(x) \) are of the form \( u_k \) and \( c_k(x), d_k(x) \) are of the form \( v_k \) for appropriate \( w \). Thus \( a_{1:m}(x), b_{1:m}(x), c_{1:m}(x) \) and \( d_{1:m}(x) \) are non-decreasing and positive.

C.2 Claim 2
To prove Claim 2 we need two simple Lemmas.

Lemma 73 If \( w \) is a word, then \( M(w) - M(w^R) = tr(KM(w))K \).

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Proof For any $C \in \mathbb{R}^{2 \times 2}$, direct calculation gives
$C - K \text{adj}(C)K = \text{tr}(KC)K$. But
$\det(M(w)) = 1$, so
$M(w) - M(w^R) = M(w) - KM(w)^{-1}K = M(w) - \text{adj}(M(w))K = \text{tr}(KM(w))K$. ■

Lemma 74 Suppose $p$ is a palindrome, $r \in (0, 1]$ and $n \in \mathbb{Z}_+$. Then for some $x \geq 0$,
$M((10p)^n10) - M((01p)^n01) = xK$.

Proof For any matrices $P, Q \in \mathbb{R}^{2 \times 2}$ with $\det(P) = 1$ and $Q \geq 0$, direct calculation gives
$[QGFP]_{22} [QFGP]_{21} - [QFGP]_{22} [QGFP]_{21}
= (b - a) \det(P)((1 - r^2 + a + b + ab)Q_{22}^2 + (2 + a + b)Q_{22}Q_{21} + Q_{21}^2) \geq 0$
as $1 \geq r^2, b \geq a \geq 0$. Therefore, for any words $w, w'$,
$\frac{M(w01w')_{22}}{M(w01w')_{21}} \geq \frac{M(w10w')_{22}}{M(w10w')_{21}}. (40)$

Let $A = M((10p)^n10), B = M((01p)^n01)$. Repeated application of (40) gives
$\frac{B_{22}}{B_{21}} = \frac{M(01p101\cdots01)_{22}}{M(01p101\cdots01)_{21}} \geq \frac{M(10p101\cdots10)_{22}}{M(10p101\cdots10)_{21}} \geq \frac{M(10p01p\cdots01)_{22}}{M(10p01p\cdots01)_{21}} = A_{22} / A_{21}.$

As $A, B \geq 0$ and Lemma 73 gives $A = B + xK$ for some $x \in \mathbb{R}$, it follows that
$(B + xK)_{21}B_{22} \geq (B + xK)_{22}B_{21} \Rightarrow K_{21}B_{22}x \geq K_{22}B_{21}x.$

Finally, the fact that $K_{22} \leq 0 < K_{21}$ and $B \geq 0$ gives $x \geq 0$. ■

Proof [Proof of Claim 2 of Lemma 25 ] For some $t \geq 0$, Lemma 74 gives
$(d/dx)(b_1(x) - a_1(x)) = [M((10p)^n11) - M((01p)^n00)]_{11}
= [(F^{-1} - G^{-1})M((01p)^n01) + tF^{-1}K]_{11}
= [(0, 0)M((01p)^n01)]_{11} + t[KF]_{11}
= t(rF_{11} + (1/r)F_{21})
\geq 0$

and
$b_1(\phi_p(0)) - a_1(\phi_p(0)) = [M(p(10p)^n10) - M(p(01p)^n00)]_{12}
= [(F^{-1} - G^{-1})M(p(01p)^n01) + tF^{-1}KM(p)]_{12}
= t[KFM(p)]_{12}
= t(rM(p)_{12} + (1/r)M(p)_{22})
\geq 0.$
Also, if \( k > 1 \) and \( w = p_{1:(k-2)} \), then
\[
\frac{d}{dx}(b_k(x) - a_k(x)) = [M(((10p)^n10w) - M((01p)^n01w))]_{11}
= t[M(w)K]_{11}
= t(M(w)_{11}r + M(w)_{12}r(1 - r^2))
\geq 0
\]
and as \( p \) is a palindrome, \( p = sw^R \) for some word \( s \), so
\[
b_k(\phi_p(0)) - a_k(\phi_p(0)) = [M(p(10p)^n10w) - M(p(01p)^n01w)]_{12}
= t[M(w)KM(p)]_{12}
= t[LM(w)^{-1}M(w^R)M(s)]_{12}
= t[LM(s)]_{12}
= t(rM(s)_{12} + (1/r)M(s)_{22})
\geq 0.
\]

This completes the proof. \( \blacksquare \)

C.3 Claim 3

Claim 3 of Lemma 25 is more challenging than Claims 1 and 2. We begin with six Lemmas.

**Lemma 75** Suppose \( p \) is any palindrome and \( r \in (0, 1] \). Then for any \( k \in \mathbb{Z}_+ \),
\[
\Delta_k := [M(((10p)^\omega)_{1:k}) - M(((01p)^\omega)_{1:k})]_{21} \geq 0.
\]

**Proof** If \( k = 1 \) then
\[
\Delta_k = [M(1) - M(0)]_{21}
= [G - F]_{21}
= (b - a)r
\geq 0.
\]

If \( k = (n + 1)|01p| + 1 \) for some \( n \in \mathbb{Z}_+ \) then Lemma 74 shows there is an \( x \geq 0 \) such that
\[
\Delta_k = [M(((10p)^n10p1) - M((01p)^n01p0)]_{21}
= [M(p1)(xK + M((01p)^n01)) - M(p0)M((01p)^n01)]_{21}
= [xM(p1)K + (G - F)M((01p)^{n+1})_{21}
= x(M(p1)_{21}r + M(p1)_{22}(r - r^3))
+ (b - a)(rM((01p)^{n+1})_{12} + (1/r)M((01p)^{n+1})_{22})
\geq 0.
\]
Otherwise, there is a prefix \( w \) of \( p \) and an \( n \in \mathbb{Z}_+ \) such that
\[
\Delta_k = [M((10p)^n10w) - M((01p)^n01w)]_{21}
\]
\[
= [M(w)(M((10p)^n10) - M((01p)^n01))]_{21}
\]
\[
= [xM(w)K]_{21} \quad \text{for some } x \geq 0 \text{ by Lemma 74}
\]
\[
= x(M(w)_{21}r + M(w)_{22}(r - r^3))
\]
\[
\geq 0.
\]
This completes the proof.  

\[\blacksquare\]

**Lemma 76** Suppose \( p = ws \) is a palindrome, \( n \in \mathbb{Z}_+ \) and \( r \in (0, 1] \). Then
\[
[M(p(10p)^n10w) - M(p(01p)^n01w)]_{22} \leq 0.
\]

**Proof** First note that for any finite word \( u \),
\[
M(u)_{22} \geq M(u)_{21}. \quad (41)
\]
Indeed, if \( u = \epsilon \) then \( M(\epsilon) = I \) so the inequality holds. Otherwise, for some \( c \in \{a, b\} \)
\[
M(u)_{22} - M(u)_{21} = \left[M(u_{2:|u|})M(u_1) \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\right]_2 = \left[M(u_{2:|u|})\frac{1}{r} \begin{pmatrix} 1 - r^2 & 1 - r^2 \\ 1 + (1 - r^2)c \end{pmatrix}\right]_2 \geq 0
\]
as the definition of \( M(\cdot) \) assumes that \( 0 \leq a < b \) so that \( c \geq 0 \), and as \( r \in (0, 1] \) and
\( M(u_{2:|u|}) \geq 0 \).

Also, by Lemma 74, for some \( x \geq 0 \)
\[
[M(p(10p)^n10w) - M(p(01p)^n01w)]_{22} = [M(s)^{-1}(M((10p)^{n+1}) - M((01p)^{n+1}))M(p)]_{22}
\]
\[
= x[M(s)^{-1}KM(p)]_{22}
\]
\[
= x[KM(ps^R)]_{22}
\]
\[
= xr ((1 - r^2)M(ps^R)_{21} - M(ps^R)_{22})
\]
\[
\leq xr ((1 - r^2)M(ps^R)_{22} - M(ps^R)_{22})
\]
\[
\leq 0
\]
where the penultimate line is (41).  

\[\blacksquare\]

**Lemma 77** Suppose \( p \) is a palindrome, \( n \in \mathbb{Z}_+ \) and \( r \in (0, 1] \). Then
\[
\left[\left(M((10p)^n10)X - M((01p)^n01)X + M((10p)^n1) - M((01p)^n0)\right)M(p)\right]_{22} = 0.
\]
Proof Let $P = M(p)$. Solving $KP = P^{-1}K$ shows that there exist $f, h \in \mathbb{R}$ such that

$$P = \begin{pmatrix}
1 - f^2 r^2 + f h r^2 + f^2 r^4 & f \\
fr^2 + h & -1 - fh + h^2 + f h r^2 - fr^2 & h
\end{pmatrix}.
$$

Directly substituting this expression into $Q_n := \left( (FG(PFG)^n X - GF(PGF)^n X + G(PFG)^n - F(PGF)^n) P \right)_{22}$ shows that $Q_0 = Q_1 = 0$. (Showing that $Q_1 = 0$ directly is algebraically tedious and we had to check this with computer algebra. The authors would be interested in a short demonstration that $Q_1 = 0$ as this may give insight into related problems.)

Lemma 73, then the fact that $\text{tr}(K) = 0$, then the cyclic property of the trace give

$$\text{tr}(PFG) = \text{tr}(GFP + \text{tr}(KPFG)K) = \text{tr}(GFP) = \text{tr}(PGF).$$

So $PFG$ and $PGF$ are 2-by-2 matrices with unit determinant whose traces are equal. For some matrices of eigenvectors $U, V$ and some eigenvalue $\lambda \geq 1$, such matrices may be written in the form

$$PFG = U\Lambda U^{-1}, \quad PGF = V\Lambda V^{-1}, \quad \Lambda := \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}.$$

Thus, for some $\alpha, \beta \in \mathbb{R}$,

$$Q_n = \alpha \lambda^n + \beta \lambda^{-n}$$

But $Q_0 = 0$ implies that $\beta = -\alpha$. Thus $Q_1 = \alpha(\lambda - 1/\lambda) = 0$ implies that either $\alpha = 0$ or $\lambda = 1$. In either case, $Q_n = \alpha(\lambda^n - \lambda^{-n}) = 0$ for all $n \geq 0$.

Lemma 78 Suppose $p$ is a palindrome. Then $S(p)KM(p) = KS(p)$.

Proof We proceed by induction on the length of $p$. In the base case, $S(\epsilon)KM(\epsilon) = IKI = KS(\epsilon)$ and if $a$ is a letter then $S(a)KM(a) = (I + M(a))KM(a) = K(I + M(a)^{-1})M(a) = KS(a)$. Otherwise, say $p = aqa$ where $a$ is a letter and $S(q)KM(q) = KS(q)$. Then

$$S(p)KM(p) = (I + M(a) + S(q)M(a) + M(aqa))KM(aqa)
= (I + M(a) + M(aqa))KM(aqa) + S(q)KM(a)^{-1}M(a)M(q)M(a)
= K(I + M(a)^{-1} + M(aqa)^{-1})M(aqa) + S(q)KM(q)M(a)
= K(M(aqa) + M(aq) + I) + KS(q)M(a)
= KS(p).$$

\[72\]
Lemma 79 Suppose $w$ is a finite word and $r \in (0, 1]$. Then $[K(S(w) - XM(w))]_{22} \geq 0$.

Proof We proceed by induction on the length of $w$. In the base case,
$$[K(S(\epsilon) - XM(\epsilon))]_{22} = [K(I - X)]_{22} = 1 - r \geq 0.$$ 
Otherwise, say $w = ul$ where $|l| = 1$, $|u| < \infty$ and $[K(S(u) - XM(u))]_{22} \geq 0$. Then
$$[K(S(w) - XM(w))]_{22} = [K(S(u) + M(w) - X(M(w) - M(u)) - XM(u))]_{22}$$ 
$$= [K(S(u) - XM(u))]_{22} + [K(I - X(I - M(l)^{-1}))M(w)]_{22}$$ 
$$\geq [K(I - X(I - M(l)^{-1}))M(w)]_{22}$$ 
$$= (1 - r)(r^2M(w)_{12} + M(w)_{22})$$ 
$$\geq 0$$
where in the penultimate line we substituted the definitions of $K$ and $X$, noting that $M(l) \in \{F, G\}$, and where the final inequality follows as $r \leq 1$ and $M(w) \geq 0$.  

Lemma 80 Suppose $p$ is a palindrome and $r \in (0, 1]$. Then for any $n \in \mathbb{Z}_+$
$$[(S(10p) - I)M(p(10p)^n) - (S(01p) - I)M(p(01p)^n)]_{22} \geq 0.$$ 

Proof Let $P := M(p)$. Then
$$[(S(10p) - I)M(p(10p)^n) - (S(01p) - I)M(p(01p)^n)]_{22}$$ 
$$= [S(p)(FG(PFG)^n - GF(PGF)^n)P + (G(PFG)^n - F(PGF)^n)P]_{22}$$ 
$$= [S(p)(FG(PFG)^n - GF(PGF)^n)P - (FG(PFG)^n - GF(PGF)^n)XP]_{22}$$ 
$$= [S(p)xKP - xKPXP]_{22}$$ 
$$= [xK(S(p) - X)P]_{22}$$ 
$$\geq 0$$
where the second equality uses Lemma 77, the third holds for some $x \geq 0$ by Lemma 74, the fourth follows from Lemma 78 and the final inequality is Lemma 79.  

Proof [Proof of Claim 3 of Lemma 25.] Say $1 \leq k \leq m$. Using Lemmas 75, 76 and 80 successively gives
$$\sum_{i=1}^{k}(d_i(x) - c_i(x)) = \sum_{i=1}^{k}[M((10p)^n(10p)_{1,i}) - M((01p)^n(01p)_{1,i})]_{21}x$$ 
$$+ \sum_{i=1}^{k}[M((10p)^n(10p)_{1,i}) - M((01p)^n(01p)_{1,i})]_{22}$$ 
$$\geq \sum_{i=1}^{k}[M((10p)^n(10p)_{1,i}) - M((01p)^n(01p)_{1,i})]_{21} \frac{M(p)_{12}}{M(p)_{22}}$$
\[ + \sum_{i=1}^{k} [M((10p)^n(10p)_{1:i}) - M((01p)^n(01p)_{1:i})]_{22} \]
\[ = \frac{1}{M(p)_{22}} \sum_{i=1}^{k} [M(p(10p)^n(10p)_{1:i}) - M(p(01p)^n(01p)_{1:i})]_{22} \]
\[ \geq \frac{1}{M(p)_{22}} \sum_{i=1}^{m} [M(p(10p)^n(10p)_{1:i}) - M(p(01p)^n(01p)_{1:i})]_{22} \]
\[ = \frac{1}{M(p)_{22}} [(S(10p) - I)M(p(10p)^n) - (S(01p) - I)M(p(01p)^n)]_{22} \]
\[ \geq 0. \]

This completes the proof.

**C.4 Claim 4**

The proof of Claim 4 requires only one preparatory Lemma.

**Lemma 81** Suppose \( w \) is any finite word. Then

\[ [M(w)^{-1}]_{21} \leq 0 \leq [M(w)^{-1}]_{22}. \]

**Proof** We use induction on the length of \( w \). In the base case, \( M(\epsilon) = I \), for which

\[ [I^{-1}]_{21} = 0 \leq 1 = [I^{-1}]_{22}. \]

Otherwise, suppose \( w = 0u \) (the case \( w = 1u \) is similar), let \( U := M(u)^{-1} \) and assume that

\[ U_{21} \leq 0 \leq U_{22}. \]

Then the induction assumption and and fact that \( a, r \geq 0 \) give

\[ [M(w)^{-1}]_{21} = [UF^{-1}]_{21} = \frac{1}{r}((1 + a)U_{21} - ar^2U_{22}) \leq 0 \]
\[ [M(w)^{-1}]_{22} = [UF^{-1}]_{22} = \frac{1}{r}(U_{22}r^2 - U_{21}) \geq 0. \]

This completes the proof.

**Proof** [Proof of Claim 4 of Lemma 25.] Claim 3 of Lemma 25 applied for \( k = 1 \) shows that

\[ c_1(x) \leq d_1(x). \]

For \( k = 2, 3, \ldots, m \), as Lemma 75 shows that \( c_k(x) - d_k(x) \) is a decreasing function of \( x \), it suffices to prove that

\[ \left[ M((01p)^n(01p)_{1:k}) \left( \phi_p \left( \frac{1}{1-r} \right) \right) \right]_2 \geq \left[ M((10p)^n(10p)_{1:k}) \left( \phi_p \left( \frac{1}{1-r^2} \right) \right) \right]_2. \]
The left-hand side minus the right-hand side, up to a positive factor, is

\[
(M((01p)^n(01p)_{1:k}) - M((10p)^n(10p)_{1:k})) M(p) \left( \frac{1}{1 - r^2} \right)
\]

\[
= -z \left[ M(p_{1:k-2}) K M(p) \left( \frac{1}{1 - r^2} \right) \right]_2
\]

\[
= -z \left[ M(p_{1:|p|-k+2})^{-1} K \left( \frac{1}{1 - r^2} \right) \right]_2
\]

\[
= -z \left[ M(p_{1:|p|-k+2})^{-1} \left( \frac{1}{r} \right) \right]_2
\]

\[
\geq 0
\]

where the last line is Lemma 81. Therefore

\[c_k(x) \geq d_k(x) \text{ for } k = 2, 3, \ldots, m \text{ and } x \leq \phi_p \left( \frac{1}{1 - r^2} \right).
\]

This completes the proof.

**C.5 Claim 5**

**Proof** [Proof of Claim 5 of Lemma 25] We show that

\[\phi_w(0) \leq y_{01w} < y_{10w} \leq \phi_w \left( \frac{1}{1 - r^2} \right)\]

for any finite word \(w\) (not just for palindromes \(p\)).

The first inequality follows as

\[\phi_w(0) \leq \phi_w(\phi_0(0)) = \phi_{01w}(0) \leq y_{01w}\]

as \(0 \leq \phi_0(0)\), as \(\phi_w\) is increasing, as \(0 \leq y_{01w}\) and by Lemma 36 (in Appendix A).

The second inequality holds as \(\phi_{01}(x) < \phi_{10}(x)\) for \(x \in \mathbb{R}_+\). Thus

\[y_{01w} = \phi_w(\phi_{01}(y_{01w})) < \phi_w(\phi_{10}(y_{01w})) = \phi_{10w}(y_{01w})\]

so applying Lemma 36 gives

\[y_{01w} < y_{10w}\]

The third inequality holds as

\[y_{10w} = \phi_w(y_{w10}) < \phi_w(y_0) \leq \phi_w \left( \frac{1}{1 - r^2} \right)\]

by definition of \(y_{10w}\), as \(y_{w10} < y_0\), as \(\phi_w\) is increasing and by Lemma 72.

This completes the proof.
Appendix D. LQG Control with Costly Observations

Problem. The analysis of this paper gives optimal policies for a version of the classic linear-quadratic-Gaussian (LQG) control problem in which observations are costly and the nature of each observation is controlled through a query action. In this problem, the state is partially observed through measurements as described by the system equations

\[ X_0 \sim \mathcal{N}(x_0, v_0), \quad X_{t+1} | X_t, u_t \sim \mathcal{N}(AX_t + Bu_t, \Sigma_X), \quad Y_{t+1} | X_{t+1}, a_t \sim \mathcal{N}(X_{t+1}, \Sigma_Y(a_t)) \]

for \( t \in \mathbb{Z}_+ \), where \( X_t \in \mathbb{R} \) is the state with initial mean \( x_0 \) and variance \( v_0 \), \( u_t \in \mathbb{R} \) is the control, \( Y_{t+1} \in \mathbb{R} \) is a measurement which depends on a query action \( a_t \in \{0, 1\} \) and where \( \Sigma_X, \Sigma_Y(a_t) > 0 \) are variances. For measurement cost \( c(a_t) \in \mathbb{R} \), the objective is to find a non-anticipative policy \( \pi \) that selects actions \( u_t, a_t \) so as to minimise the \( \beta \)-discounted performance functional

\[ \mathbb{E}\left( \sum_{t=0}^{\infty} \beta^t (DX_t^2 + Fu_t^2 + c(a_t)) \mid \pi, x_0, v_0 \right). \]

Thus the policy can select \( u_t, a_t \) based only on the observed history \( H_t \) at time \( t \), which consists of \( x_0, v_0, a_0, a_1, \ldots, a_{t-1}, u_0, u_1, \ldots, u_{t-1}, Y_1, Y_2, \ldots, Y_t \). Under the Bayesian filter, the information state is given by the posterior mean \( x_t := \mathbb{E}[X_t | H_t] \) and variance \( v_t := \mathbb{E}[(X_t - x_t)^2 | H_t] \). As \( \mathbb{E}[X_t^2 | H_t] = x_t^2 + v_t \), it is not hard to see that the problem reduces to the dynamic program

\[ V(x_t, v_t) = \min_{u_t \in \mathbb{R}, a_t \in \{0, 1\}} \left\{ D x_t^2 + D v_t + F u_t^2 + c(a_t) + \beta \mathbb{E}[V(x_{t+1}, v_{t+1})] | x_t, v_t, a_t, u_t \right\} \] (42)

where the expectation is over the following Markovian transitions of the information state:

\[ x_{t+1} | x_t, v_t, a_t, u_t \sim \mathcal{N}(Ax_t + Bu_t, A^2 v_t + \Sigma_X - \phi_a(v_t)) \]

\[ v_{t+1} | x_t, v_t, a_t, u_t = \phi_a(v_t). \]

Corollary 82 Suppose \( A \in [-1, 1], B \in \mathbb{R} \) with \( B \neq 0 \), \( D \in \mathbb{R}_{++}, F \in \mathbb{R}_{++}, \beta \in (0, 1), \Sigma_Y(a) \in [0, \infty) \) for \( a \in \{0, 1\} \) with \( \Sigma_Y(0) \geq \Sigma_Y(1) \), and that \( c(a) \in \mathbb{R} \) for \( a \in \{0, 1\} \) with \( c(0) \leq c(1) \). Then an optimal policy for the problem of linear-quadratic-Gaussian control with costly observations is to set

\[ a_t = \begin{cases} 1 & \text{if } v_t \geq z \\ 0 & \text{if } v_t < z \end{cases} \quad \text{and} \quad u_t = -L x_t \]

for some \( L \in \mathbb{R} \) and \( z \in \mathbb{R} \). In particular

\[ L = \frac{A}{B + \frac{F}{\beta BR}} \]

where \( R \) is the unique positive root of the quadratic equation

\[-\beta B^2 R^2 + (\beta B^2 D + \beta A^2 F - F)R + DF = 0.\]
Proof For trial solutions of the form \( V(x, v) = Rx^2 + Rv + g(v), \) where \( R \in \mathbb{R} \) and \( g : \mathbb{R}_+ \to \mathbb{R} \) are to be determined, the expectation in (42) is

\[
\mathbb{E}[V(x_{t+1}, v_{t+1})|x_t, v_t, a_t, u_t] = R((Ax_{t+1} + Bu_t)^2 + A^2v_t + \Sigma_X - \phi_{a_t}(v_t) + \phi_{a_t}(v_t)) + g(\phi_{a_t}(v_t)).
\]

Thus (42) is solved if

\[
Rx_t^2 + Rv_t + g(v_t) = \min_{a_t \in \mathbb{R}} \left\{ Dx_t^2 + Fu_t^2 + \beta R(Ax_{t+1} + Bu_t)^2 \right\} + \min_{a_t \in \{0, 1\}} \left\{ c(a_t) + \beta R\Sigma_X + (D + \beta RA^2)v_t + \beta g(\phi_{a_t}(v_t)) \right\}.
\] (43)

Now the minimum with respect to \( u_t \) is achieved if the coefficient \((F + \beta B^2 R)\) of \( u_t^2 \) is positive, in which case the minimiser is

\[
u_t = -\frac{\beta ABR}{F + \beta B^2 R} x_t.
\]

So (43) is solved if \( R \) satisfies

\[
R = D + F \left( \frac{\beta ABR}{F + \beta B^2 R} \right)^2 + \beta R \left( A - B \frac{\beta ABR}{F + \beta B^2 R} \right)^2
\]

and if \( g(\cdot) \) satisfies the dynamic program

\[
g(v) = \min_{a_t \in \{0, 1\}} \left\{ c(a_t) + \beta R\Sigma_X + \alpha v + \beta g(\phi_{a_t}(v)) \right\}
\] (44)

where \( \alpha := D - (1 - \beta A^2)R. \)

After simple algebra, the condition on \( R \) is equivalent to the quadratic equation

\[-\beta B^2 R^2 + (\beta B^2 D + \beta A^2 F - F) R + DF = 0.
\]

Using Descartes’ rule of signs and considering the cases \( F = 0 \) and \( F > 0 \) separately, we see that this equation has a unique positive root for \( \beta B^2 > 0 \) and \( D > 0. \)

To apply Theorem 1 to the dynamic program for \( g(\cdot) \) we must ensure that \( \alpha \geq 0. \) Noting that \( m := 1 - \beta A^2 > 0 \) by the hypotheses about \( A, \beta, \) we see that \( \alpha \geq 0 \) if \( R \leq D/m. \) But, substituting \( R = y + (D/m) \) in the equation for \( R \) gives

\[
0 = [-\beta B^2 R^2 + (\beta B^2 D + \beta A^2 F - F) R + DF]_{R=y+(D/m)} = -\beta B^2 y^2 - ((\beta^2 A^2 + \beta)B^2(D/m) + mF)y - \beta^2 A^2 B^2(D/m)^2
\]

in which the coefficients of \( y^0, y^1, y^2 \) are all negative by the hypotheses about \( A, B, D, F \) and \( \beta. \) So this quadratic equation for \( y \) has no positive roots and it follows that \( \alpha \geq 0. \) Therefore Theorem 1 shows that there is an optimal policy for (44) that sets \( a_t = 1 \) if and only if \( v_t \geq z \) for some \( z \in \mathbb{R}. \) This completes the proof.
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