An Indeterminate Universe
Dark Energy and Norton’s Dome
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Abstract
We describe a case of indeterminacy in general relativity for homogeneous and isotropic cosmologies for a class of dark energy fluids. The cosmologies are parametrized by an equation of state variable, with one instance giving the same solution as Norton’s mechanical dome. Our example goes beyond previously studied cases in that indeterminacy lies in the evolution of spacetime itself.

1 Introduction and Background
The publication in 2003 of what became known as Norton’s Dome [1, 2] generated an interesting debate on determinism in classical dynamics. Related examples had been discussed by other authors: Bhat and Bernstein [3] consider a mathematically similar system, and Earman [4] surveys several classes of examples described as “the fault modes of determinism” in (non-quantum) physics. What distinguishes Norton’s approach is the attractiveness of an easily visualized and unexpectedly simple Newtonian system that involves only elementary mathematics and may appear to have a natural physical interpretation.

The example involves a point-mass placed at absolute rest at the apex of a dome-shaped surface with radial symmetry. The profile of the surface of revolution is chosen so that the equation of motion under gravity of the mass sliding (without friction) along the dome has the form \( \ddot{r} \sim \sqrt{r} \) with initial conditions \( r(0) = 0 \) and \( \dot{r}(0) = 0 \), where \( r(t) \) denotes the distance at time \( t \geq 0 \) from the summit, measured along the surface of the dome. The expected solution of the equation of motion would be for the mass to remain at rest \( (r(t) \equiv 0) \); however, the equation does not fulfil the Lipschitz condition this solution is not unique. Other solutions suggest that the mass can spontaneously start sliding down the dome at an arbitrary time in any direction.

Norton [2] contends that the dome demonstrates a failure of determinism in Newtonian physics, or at least raises questions regarding the widespread views on determinism in classical mechanics and broadly accepted mathematical idealizations in that context. Malament [5] concludes his extensive analysis on a more cautious note: “we do not have a sufficiently clear idea in the first place what should count as a ‘Newtonian system’ (or count as falling within the ‘domain of application’ of Newtonian theory). My inclination is to avoid labels here and direct attention, instead, to a rich set of issues that the example raises.”
The equation of motion itself, non-uniqueness of its solutions, as well as a slightly more general class of examples, had been considered as early as 1806 by Poisson, and discussions of questions this raises about determinism unfolded at various times during the nineteenth century involving Poisson, Duhamel and others. Van Strien [6] provides historical background, while Fletcher [7] also includes a detailed technical review of arguments surrounding Norton’s Dome.

The debate in a sense reflects ambiguities in the understanding of determinism and its intended meaning in classical physics. This is illustrated by no less of an authority than Arnol’d, who asserts in one standard text, *Mathematical Methods of Classical Mechanics* [8], that “the initial positions and velocities of all the particles of a mechanical system uniquely determine all of its motion”; only to state in another standard text, *Ordinary Differential Equations* [9],

the form of the differential equation of the process, and also the very fact of determinacy […] can be established only by experiment, and consequently only with limited accuracy. In what follows we shall not emphasize that circumstance every time, and we shall talk about real processes as if they coincided exactly with our idealized mathematical models.

In response to critiques to the effect that the dome is “unphysical”, or that it cannot be supported (or refuted) by experiment, Norton [2] argues that his example is not a statement about the world but about our theories of the world:

The dome is not intended to represent a real physical system. The dome is purely an idealization within Newtonian theory. On our best understanding of the world, there can be no such system. For an essential part of the setup is to locate the mass *exactly* at the apex of the dome and *exactly* at rest. Quantum mechanics assures us that cannot be done. What the dome illustrates is indeterminism within Newtonian theory in an idealized system that we do not expect to be realized in the world.

Quantum mechanics aside, it is instructive to consider a more complex example of interest in this context, not as frequently discussed. In 1895 Painlevé exhibited a dynamical system in classical contact mechanics, involving a box sliding with friction on an inclined plane, which for some values of the coefficient of friction is indeterministic in the sense of having multiple solutions for the same initial conditions. Painlevé constructed another example in 1905 and left it at that, but the associated Painlevé “paradox” remains an active area of research. In their 2016 survey of results in this field, Champneys and Várkonyi [10] note the practical relevance of these seemingly theoretical pursuits:

One of the purposes of this paper is to show that the Painlevé paradox is *not* just a theoretical curiosity manifest only in “toy” mathematical models of pencils on unrealistically rough surfaces. The consequences of the Painlevé paradox are in fact ubiquitous, even in everyday phenomena. For example, robotic manipulators, pieces of chalk or even your finger are known to *judder* when they are being pushed across a rough surface […].
Several paragraphs later, however, they express a view similar to Norton’s:

Philosophically speaking, the Painlevé paradox is not a puzzle about the real world, but a failure of a theory based on rigid body mechanics and Coulomb friction to provide complete unequivocal descriptions of dynamics. [...] In effect, the Painlevé paradox provides insight into points of extreme sensitivity in rigid body mechanics where additional physics, possibly at the microscale, is required in order to accurately capture the dynamics."

It is perhaps in this spirit that systems analogous to Norton’s Dome have been constructed for electric charge motion, and for special relativistic and geodesic motion in curved spacetime (Fletcher [11]). In each of these cases the resulting equations of motion are equivalent to those for a particle mass on the dome, although they lack the elegant simplicity of Norton’s example.

2 Example of Indeterminism in Cosmology

One of the purposes of our note is to provide a similar example in cosmology that is a solution of Einstein’s field equations. Considerations analogous to Norton’s dome arise naturally through Friedmann’s cosmological equations. Our example also serves to highlight Malament’s [5] doubts regarding distinctions among smoothness of constraints: “Do we allow constraint systems in which the defining constraints involve singularities? If so, how bad can the latter be?” A curvature singularity such as the one at the apex of Norton’s Dome does not seem to have resulted in the dismissal of cosmological models as being “unphysical”.

With that said, we make no claims regarding the “physicality” of our cosmological dome, although the fluid equation of state we use corresponds to the so-called dark energy models [12]; we view it as a potentially interesting and attractively simple case of indeterminism. While our assumptions are purely theoretical and intended to simplify, the methodology follows standard discussions. Whether or not a more realistic example can be constructed is beyond the scope of this note.

Einstein’s equation 

\[ G_{ab} = 8\pi GT_{ab} \]

for a flat universe with metric 

\[ ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2) \]

and fluid \( p = w\rho \) leads to the Friedmann equations

\[ \frac{\dot{a}^2}{a^2} = 2C\rho \quad (1) \]

\[ \frac{\ddot{a}}{a} = C(\beta + 2)\rho, \quad (2) \]

with \( C = 4\pi G/3 \) and \( \beta = -3(1 + w) \). These equations are not defined at \( a = 0 \) and so do not permit specification of initial conditions such as \( a(0) = 0 \) and \( \dot{a}(0) = 0 \).

There is however an alternative “densitized” form of the Einstein equations, 

\[ \sqrt{-g} G_{ab} = 8\pi G\sqrt{-g} T_{ab}, \]

that do permit these conditions. This form may be viewed as more natural when derived from an action \( S[g, \rho] \), since the variational equation \( \delta S/\delta g_{ab} = 0 \) yields a tensor density. The commonly used form of the Einstein
equations is recovered by dividing the result by $\sqrt{-g}$ assuming this quantity is non-vanishing. The densitized form of the equations yields the same solution space, but permits the imposition of initial conditions at the cosmological singularity. This version has been discussed in the literature with application to cosmological solutions [13], and implicitly for quantum fields on an extended FLRW spacetime [14]. In the following we consider such an extended spacetime for $t \geq 0$.

For the FLRW universe with $k = 0$, $\sqrt{-g} = a^3$, and with the equation of state $\hat{p} = w \hat{\rho}$, the densitized Friedmann equations are

\begin{align*}
a \ddot{a}^2 &= 2C\hat{\rho}, \\
a^2 \ddot{a} &= C(\beta + 2)\hat{\rho},
\end{align*}

(3) (4)

where $\hat{\rho} = a^3 \rho$. We take $w$ in the range $-1 < w < -\frac{2}{3}$, that is, $\beta \in (-1, 0)$. This range of $w$ corresponds to quintessence or another kind of dark energy [12], but excludes “phantom energy” defined by $w < -1$.

Since $(\beta + 2) \neq 0$ equations (3) and (4) give

\[ a^2 \ddot{a} - \left(1 + \frac{\beta}{2}\right) a\dot{a}^2 = 0. \]

(5)

For $n = \lfloor -\frac{2}{\beta} \rfloor \geq 2$ this has $C^n$ solutions

\[ a(t) = \alpha(t - \kappa)^{-2/\beta}, \quad t \geq 0 \]

(6)

for all $\alpha, \kappa \in \mathbb{R}$. Given any $\alpha \neq 0$, $\hat{\rho}$ is determined from this solution using (3) and it follows that (4) is identically satisfied. (This results in $\hat{\rho} \sim a^{\beta+3}$, which for $a \neq 0$ agrees with the familiar $\rho \sim a^{\beta+3(1+w)}$.)

Multiple solutions of the densitized Friedman equations (3) and (4), parametrized by $\kappa \geq 0$, may be exhibited for initial conditions $a_\kappa(0) = 0$ and $\dot{a}_\kappa(0) = 0$:

\[ a_\kappa(t) = \begin{cases} 0 & 0 \leq t \leq \kappa \\ \alpha(t - \kappa)^{-2/\beta} & t > \kappa \end{cases}. \]

(7)

Such a universe, apparently, can spontaneously enter an acceleration phase at any time $\kappa \geq 0$.

With $\beta \in (-1, 0)$, or $w \in (-1, -\frac{2}{3})$, solutions (7) correspond to those of the class of equations discussed by Poisson (see [6, 7]). In our notation Poisson’s equations would be of the form $\ddot{a} = a^{\beta+1}$, which do not fulfill the Lipschitz condition at $a = 0$ and have multiple solutions with the initial conditions $a(0) = \dot{a}(0) = 0$. In particular, in the case $\beta = -\frac{1}{2}$ ($w = -\frac{5}{6}$) the solutions (7) correspond to those of the equation of motion for Norton’s dome.

It is perhaps interesting to note that the non-uniqueness of solutions persists with initial conditions

\[ \frac{d^m a}{dt^m}(0) = 0 \]

for each $m \leq n = \lfloor -\frac{2}{\beta} \rfloor$, and that $n$ can be made arbitrarily large by taking $w$ arbitrarily close to $-1$. 
It may also be worth noting that extended FLRW spacetimes [13, 14] involve solutions $a(t)$ for $t \in \mathbb{R}$. Solutions (7) can be extended to that setting by taking $a(t) = 0$ for $t < 0$, and additional solutions over $\mathbb{R}$ can be exhibited, e.g., for $\kappa \geq 0$ and $\mu \leq 0$:

$$
\bar{a}_{\kappa, \mu}(t) = \begin{cases} 
\alpha(t - \mu)^{-2/\beta} & t < \mu \\
0 & \mu \leq t \leq \kappa \\
\alpha(t - \kappa)^{-2/\beta} & t > \kappa 
\end{cases}.
$$

The previously studied examples of indeterminism in Newtonian mechanics, special relativity, and geodesic motion on curved spacetime all arise in the context of a pre-determined spacetime stage, whether it is Newton’s absolute space and time, or an externally provided pseudo-Riemannian metric. The cosmological model considered here is of an ontologically different nature in that it provides an example where the evolution of spacetime itself is indeterminate.

Additionally the model avoids many of the objections that might be raised for a Newtonian system, such as violation of the First Law, loss of contact with the dome, or an incomplete formulation of Newtonian theory. Perhaps the remaining objection might be ”physicality.” However, the purpose of our model is not to give an accurate description of the observable universe, but rather to highlight that a kind of matter used to model a phase of it leads to indeterminacy if the equation of state parameter is a constant in the range $w \in (-1, -2/3)$. It should also be noted that the physical universe is expected to be quantum in nature in its early stages and that classical GR is not expected to be applicable near the singularity at $a = 0$.

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