Optimal entanglement swapping in quantum repeaters

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We formulate the problem of finding the optimal entanglement swapping scheme in a quantum repeater chain as a Markov decision process and present its solution for different repeater’s sizes. Based on this, we are able to demonstrate that the commonly used “doubling” scheme for performing probabilistic entanglement swapping of probabilistically distributed entangled qubit pairs in quantum repeaters does not always produce the best possible raw rate. Focussing on this figure of merit, without considering additional probabilistic elements for error suppression such as entanglement distillation on higher “nesting levels”, our approach reveals that a power-of-two number of segments has no privileged position in quantum repeater theory; the best scheme can be constructed for any number of segments. Moreover, classical communication can be included into our scheme, and we show how this influences the raw waiting time for different number of segments, confirming again the optimality of “non-doubling” in some relevant parameter regimes. Thus, our approach provides the minimal possible waiting time of quantum repeaters in a fairly general physical setting.

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Introduction. A long-standing problem in the theory of quantum repeaters is to determine the optimal entanglement distribution time as a function of a repeater’s characteristics like the distribution success probability \( p \) for a single segment and the entanglement swapping probability \( a \) between two segments [1]. In particular, it is commonly assumed that the number of segments \( n \) is a power of two, \( n = 2^d \), and the only scheme considered is “doubling”, where the segments are divided into two equal halves, which are then treated as independent smaller repeaters. When both halves have finally distributed an entangled state the last swapping is attempted. Such a doubling scheme can be useful in a “nested quantum repeater” allowing for a systematic inclusion of entanglement distillation [2] or, exploiting the repeater’s self-similarity, for a recursive and approximate calculation of repeater rates in certain regimes [3]. However, it is unknown whether “doubling” gives the optimal rates, i.e. the shortest repeater waiting times – in general, for general \((n = 2^d, p, a)\) without entanglement distillation, or at least in a certain regime such as that of \( p \) and \( a \) both being small. Furthermore, one may ask: is there an optimal scheme if \( n \neq 2^d \)?

Rate calculations for quantum networks so far have focussed either on the ultimate, information-theoretical limits independent of experimental constraints such as non-deterministic gate operations [4] or on more realistic systems under simplifying assumptions, such as specific parameter regimes allowing for an approximate treatment [3] or to determine bounds [5] and certain shapes [6] and sizes [7,8] of the network reducing its complexity. In this work, we bridge these two approaches for the case of a sufficiently small quantum repeater chain up to about ten segments and present its exact, optimal solutions, generalizing and optimizing our previous, exact results on the statistics of repeater waiting times in various settings [9] [10].

It turns out that, depending on \( p \) and \( a \), the “doubling” scheme does not always deliver the highest raw rate, and for some values of parameters other schemes perform better. The corresponding rate enhancement seems to increase for larger repeaters with growing \( n = 2^d \). Moreover, the assumption that \( n \) is a power of two is superfluous, the optimal scheme is defined for all combinations of \( n, p \), and \( a \) [11]. In this work, we show how this scheme can be found. We will also include the physically relevant case where the memory qubits have to wait for classical signals to obtain information regarding the distributions in other segments [12]. We assume that all segments have identical properties [13] and put no restriction on how long a state can be kept in memory [14].

Markov chains. We use the formalism of Markov decision processes, which provides a method to choose an optimal action in a discrete-time stochastic system. Before we formulate our general method, we solve a simpler problem. Consider a finite Markov chain with a single absorbing state. The set of states we denote as \( S \) and the transition probability matrix as \( P = (p_{ss'}) \), where \( p_{ss'} \) is the transition probability from \( s \) to \( s' \). With every state \( s \in S \) we associate a cost \( r_s \geq 0 \) of making a transition from this state. Assuming that the cost of the absorbing state is zero, we define the total cost of absorption \( T_s \) from any state \( s \in S \) as a sum of all costs \( r_s \) from \( s \) to the absorbing state. Clearly, \( T_s \) is a random variable whose distribution depends on the transition probabilities of the chain. What is the average value of this variable? Denoting \( v_s = \mathbb{E}[T_s] \), it can be shown that these quantities satisfy the system of linear equations

\[
v_s = \sum_{s' \in S} p_{ss'} v_{s'} + r_s. \tag{1}
\]

Following the convention that the absorbing state is the last one let \( Q \) be the stripped \( P \), i.e. the matrix obtained from \( P \) by removing its last row and last column. The
system \( \mathbf{v} = Q\mathbf{v} + \mathbf{r} \), and its solution is given by \( \mathbf{v} = (I - Q)^{-1}\mathbf{r} \), where \( I \) is the identity matrix of order \( n - 1 \), \( n = |\mathcal{S}| \), and \( \mathbf{r} \) is the \( (n - 1) \)-vector of transition costs (except the last component, which we assumed to be zero). It is known that \( I - Q \) is invertible and thus the system \( \mathbf{v} = Q\mathbf{v} + \mathbf{r} \) has a unique solution given by this expression, see [1]. This formula gives an exact analytical expression for the average absorption cost. For large \( n \) such an expression is impractical to deal with, so \( \mathbf{r} \) must be computed numerically. In this case solving the system of linear equations \( (I - Q)\mathbf{v} = \mathbf{r} \) is numerically more robust than inverting the matrix \( I - Q \) and multiplying the inverse by \( \mathbf{r} \).

**Markov decision problems.** Now consider a more complicated case. What if the transition probabilities and the associated transition cost in each state depend on a parameter, so-called action? These actions can be freely chosen at will and any such a choice is referred to as a policy in the reinforcement learning parlance. A (finite) Markov decision process (MDP) consists of a finite set \( \mathcal{S} \) of states and a finite set of actions \( \mathcal{A}_s \) for each state \( s \in \mathcal{S} \). For every policy we have an instance of the problem considered above, with the transition probabilities \( p_{ss'}^\alpha \) and transition costs \( r_{ss'}^\alpha \) now depending on the actions \( \alpha \in \mathcal{A}_s \) chosen for all \( s \in \mathcal{S} \). One can say that an MDP embeds many Markov chain absorption problems into one framework. What is the optimal (minimal or maximal) average absorption cost of these problems and how can we determine the corresponding optimization problem? We are interested in minimizing the cost, but all the results below can be easily modified for the opposite direction.

A straightforward approach is to compute the average cost of all embedded problems \([1]\) and take the best value. Totally, there are \( N = \prod_{s \in \mathcal{S}} |\mathcal{A}_s| \) embedded problems, and this number becomes ridiculously large even for problems of moderate size, so this method is feasible only for very small systems (see SM). We show that there is a more practical approach based on solving a proper linear optimization problem. This statement is based on the following

**Theorem 1.** Any solution of the linear optimization problem which maximizes the sum \( \sum_{s \in \mathcal{S}} v_s \) under the constraints

\[
v_s = \sum_{s' \in \mathcal{S}} p_{ss'}^\alpha v_{s'}, \quad \alpha \in \mathcal{A}_s
\]

is a solution of the following system of nonlinear equations:

\[
v_s = \min_{\alpha \in \mathcal{A}_s} \left[ \sum_{s' \in \mathcal{S}} p_{ss'}^\alpha v_{s'} + r_{ss'}^\alpha \right].
\]

As the objective function one can use any linear combination \( \sum_{s \in \mathcal{S}} c_s v_s \) with positive coefficients \( c_s \). The problem \([2]\) has at least one solution.

The proof of this theorem is given in the Supplemental Material. If for any concrete choice of \( \alpha \in \mathcal{A}_s \) for all \( s \in \mathcal{S} \), i.e., for any policy \( \pi \), we leave just one equation in Eq. \([3]\), we get a system of linear equations of the form given by Eq. \([1]\). The solution \( \mathbf{v}^\pi \) of this system is the vector of average costs of the absorption problem corresponding to the policy \( \pi \). It is in this sense that an MDP embeds many absorption problems — every choice of actions produces a problem and all these problems are contained in one framework described by Eq. \([3]\). Note that any solution of Eq. \([3]\) (which has at least one solution according to the previous theorem) corresponds to a policy — for any \( s \in \mathcal{S} \) take an action \( \alpha \in \mathcal{A}_s \) that minimizes the right-hand side of Eq. \([3]\). For some \( s \) there can be more than one minimizing action, so the policy corresponding to a solution may not be unique. We now show that any solution of Eq. \([3]\) is at least as optimal as the solution for any policy.

**Theorem 2.** Let \( \mathbf{v}^\ast \) be a solution of Eq. \([3]\). Then for any policy \( \pi \) we have \( \mathbf{v}^\ast \leq \mathbf{v}^\pi \), where this inequality is meant componentwise.

The proof is also given in the Supplemental Material. As it was noted before, any optimal solution corresponds to some policy, so from this theorem we derive the following property of optimal solutions: \( v^\ast_s = \min_{\alpha \in \mathcal{A}_s} v^\pi_s \), where the minimum is taken over all possible policies. We conclude that the system of nonlinear equations Eq. \([3]\) has a unique solution, which can be obtained by solving the linear optimization problem given by Eq. \([2]\). Having found the optimal solution \( \mathbf{v}^\ast \) we can obtain an optimal policy corresponding to this solution by taking an action \( \alpha \in \mathcal{A}_s \) that minimizes the right-hand side in Eq. \([3]\) for all \( s \in \mathcal{S} \). Such a scheme may not be unique.

**Application to quantum repeaters.** We now apply the presented theory to the problem of finding the minimal waiting time in quantum repeaters. In a state where there are segments not ready yet (which are trying to distribute an entangled state) and those that are ready (which have already distributed entanglement), there is always a choice — either wait for non-ready segments or try to swap a pair of neighboring ready segments. Clearly, different actions have different probabilistic evolutions, so the entanglement distribution process in a quantum repeater fits into an MDP model.

First, we need to list all possible states of a quantum repeater. We use a simple model where an attempt to distribute entanglement takes one unit of time and an attempt to swap segments takes no time at all. Under these assumptions a state of a repeater can be characterized by a string of nonnegative numbers, where 0 marks a segment trying to distribute entanglement, and \( i > 0 \) marks a group of \( i \) successfully distributed and swapped segments. For the simplest case of a 2-segment repeater the states are 00 (the initial state), 01, 10, 11 and 2 (the final, absorbing state). We are interested in an optimal
strategy, and such a strategy must have identical actions on the states which are mirror images of each other, like the states 01 and 10 above. It means that we can apply the so-called lumpability trick — we lump the mirror images into one new state and recompute the transition probabilities. This allows us to compress the size of the problem by reducing the number of states and actions, which will be very helpful for larger repeaters. In the case above from the two states 01 and 10 we form a new state \{01, 10\}. So, in this simple case we have four states \(s_1 = 00, s_2 = \{01, 10\}, s_3 = 11\) and \(s_4 = 2\). In each of these states only one action is possible, so our MDP reduces to the Markov chain problem of the form [1]:

\[
\begin{align*}
v_1 &= q^2 v_1 + 2p q v_2 + p^2 v_3 + 1 \\
v_2 &= q v_2 + p v_3 + 1 \\
v_3 &= (1-a) v_1 + a v_4,
\end{align*}
\]

where \(v_4 = 0\) (and we set \(q = 1 - p\)). Note that the transition probability \(p_{12} = P(s_1 \rightarrow s_2) = 2pq\) has a factor 2, since \(s_2 = \{01, 10\}\) and \(s_1 = 00\) can go to \(s_2\) in two ways — when either of the segments distributes entanglement. The probability of each path is \(pq\), so the total transition probability is \(2pq\). The constant terms on the right-hand side of the system (4) express our assumption that a distribution attempt costs one unit of time and a swapping attempt costs zero. Solving this system of linear equations, we obtain \(v_1 = (3 - 2p)/((ap(2 - p))\), which is a well-known expression for the waiting time of a 2-segment repeater. Note that for the total repeater waiting time we are generally interested in component \(v_1\) from the optimal solution vector.

**Example.** Now consider a more interesting case, a 3-segment repeater. In this case there are nine states: \(s_1 = 000, s_2 = 001, s_3 = 010, s_4 = 011, s_5 = 101, s_6 = 111, s_7 = 02, s_8 = 12\) and \(s_9 = 3\), where any non-symmetric sequence like 001 denotes the corresponding class \(001, 100\) not to overload the notation. In the state \(s_4 = 011\) (which denotes \(011, 110\)) two actions are possible — waiting while the last segment distributes entanglement, which costs one time unit per attempt, or trying to swap the other two segments, which costs nothing (in the state 111 the two possible swappings represent one action in the compressed system). The MDP equations in this case read as

\[
\begin{align*}
v_1 &= q^2 (q v_1 + 2p q v_2 + p v_3) + p^2 (2q v_4 + q v_5 + p v_6) + 1 \\
v_2 &= q^2 v_2 + pq v_3 + pq v_4 + p^2 v_5 + 1 \\
v_3 &= q^2 v_3 + 2 pq v_4 + pq v_5 + p^2 v_6 + 1 \\
v_4 &\leq q v_4 + p v_5 + 1 \\
v_5 &\leq (1-a) v_1 + a v_7 \\
v_6 &\leq q v_5 + p v_6 + 1 \\
v_7 &\leq (1-a) v_2 + a v_8 \\
v_8 &\leq q v_7 + p v_8 + 1 \\
v_9 &\leq (1-a) v_1,
\end{align*}
\]

where we took into account that \(v_9 = 0\). If we remove the first inequality for \(v_4\), we get the scheme where we always swap in the states 011 and 110, removing the second inequality we get the scheme where we always try to distribute entanglement in these states [15]. Optimizing the sum \(\sum_{i=1}^{8} v_i\) under the constraints given above, for each \(p\) and \(a\) we obtain the best waiting time \(v_1^*\) and the optimal scheme (which may depend on \(p\) and \(a\)). It turns out that for all \(p\) and \(a\) the former scheme (always swapping when ready) is better and the analytical expression for the waiting time is the same as we have already given in [9], where it is denoted as \(K^{3(dyn)}\).

**Non-doubling optimal schemes.** The next case of a 4-segment repeater is even more interesting. The corresponding MDP has 20 variables (excluding the variable for the absorbing state, whose value is zero) and 29 constraints, so we do not present it explicitly. For every state there is at most one “wait for distribution” action and zero or more “swapping” actions. One of the possible schemes is “doubling”, where the repeater is divided into two halves which are treated as independent 2-segment repeaters. Only when both halves are ready can we try to perform the last swapping. This scheme has been most commonly considered in the literature. Somewhat surprisingly, this scheme is not always the best one. We have solved the MDP for all \(0.01 \leq p, a \leq 1.0\) and for each pair of probabilities \(p\) and \(a\) we determined the best scheme for these parameters. We found that in this case of \(n = 4\) there are three schemes that are optimal in different regions of the probability square, see Fig. [1]

In the lower-left corner of the square, which corresponds to small \(p\) and \(a\), the optimal scheme is “doubling”, denoted as \(\pi_0\) (thus confirming that for such quantum repeaters “doubling” is indeed optimal). A practically more relevant range of parameters may be at small \(p\) and large \(a\), which corresponds to the upper-left corner of the square, and the optimal scheme there differs from “doubling”, denoted as \(\pi_2\). In between these two regions there is a third optimal scheme, \(\pi_1\). These schemes are de-
FIG. 2: Optimal raw waiting time as a function of $n$. For some values of $p$ and $a$ it is approximately linear.

FIG. 3: Ratio of the “doubling” to the optimal waiting time for $n = 4$ (left) and $n = 8$ (right).

scribed in the Supplemental Material. For some relevant fixed $p$ and $a$ values, Fig. 2 illustrates that in an intermediate regime of $a$ (neither too small nor too close to one), the optimized raw waiting time is a linear function of the repeater size $n$.

Classical communication. We can extend our model to include classical communication (CC), assuming that it takes one unit of time to restart a segment (and the swapping process itself takes no time). For example, for $n = 4$ in the state 0110 we can try to swap the inner pair of segments. If the swapping fails, then in the previous model the system transitions to the initial state 0000, but in this model it goes to a new state (0)(1)(1)0, where the number in brackets denotes the number of time units after which this segment returns to the initial state 0. If we make a swapping in the state 012 and fail, this state goes to (0)(1)(1)2. With probability $q$ the next state is 000(1), and with probability $p$ it is 100(1). In the former case the next state will be ***0, where *** is any combination of three zeros and ones, and in the latter case it will be 1 # 0. So, the general rule is: $(i) \rightarrow (i - 1)$ if $i > 1$ and $(1) \rightarrow 0$. Let us illustrate possible transitions from the initial state 13: $13 \rightarrow (1)(1)(2)(3) \rightarrow 00(1)(2) \rightarrow 11(1)(1)0(1) \rightarrow 0000$. We first try to swap and fail, restarting all the segments. Then, two segments are in the initial state and the other two are in progress still waiting for classical signals. Next, the ready segments both distribute entanglement, simultaneously succeeding here, another segment goes to the initial state and the last segment is still in progress. Then, we try to perform swapping and fail, restarting the first two segments (the other two are in the same state since swapping itself takes no time). Finally, all segments are in the initial state, since the third segment failed to distribute entanglement. Note that this is only one of the possible transition sequences between repeater states in our model. These transitions illustrate that in multisegment repeaters several “waves” of restarting are possible — an earlier restart signal still in progress when a newer one starts to propagate. There are many more states and transitions by comparison with the previous model, but it still fits into the MDP approach. The influence of the classical communication on the raw waiting time is illustrated in the Supplementary Material for various repeater sizes. It is interesting to compare the optimal waiting time with the “doubling” waiting time. The ratio of the two quantities is shown in Fig. 3 for the model including CC. This figure shows that there is a small but noticeable advantage of the optimal scheme. This advantage becomes more visible for larger repeaters, as Fig. 3(right) demonstrates for an 8-segment repeater, in the practically highly relevant regime of small $p$ and large $a$ ($\sim 1.5\%$ for $n = 4$ and $\sim 5\%$ for $n = 8$).

Conclusion. In conclusion, we presented a method to determine the most efficient entanglement swapping scheme in a quantum repeater and demonstrated that the “doubling” scheme is not always the best. Moreover, our approach shows that when additional elements such as entanglement distillation on higher levels are excluded the power-of-two number of segments is not a distinguished case, since the best scheme can be constructed for any number of segments. We showed that for small repeaters the best scheme has a tiny, but noticeable advantage over the “doubling” scheme, but this advantage seems to increase with the repeater’s size. Our most general model leading to this conclusion includes all necessary classical communication times, while we were able to treat repeater sizes up to the order of ten segments. It is currently intractable to treat 16 repeater segments or more for a direct comparison.

Our algorithm has exponential complexity and thus is applicable to fairly “small” repeaters only, but even these repeaters are still beyond current technological capabilities and so our approach here is fully applicable to current experiments to have meaningful physical benchmarks. Moreover, a ten-segment repeater can cover a distance of around 1000km, which is already of practical interest. On the other hand, great progress has been made in algorithms for solving linear optimization problems. A study of different versions of an optimiza-
tion software, CPLEX, performed in Ref. [18], shows a speedup of a factor of 29,000 only due to algorithmic advantages. Combined with hardware advances that happened during this time (two decades), we get an even more impressive performance boost factor. Of course, no technological advance can turn an exponential algorithm to a subexponential one, but what seems intractable now could become feasible in the near future.

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Proof of Theorem 1

We start with the proof of Theorem 1. Let $v^*_s$ be a solution of the optimization problem [2]. We have to prove that for any $s \in S$ at least one of the inequalities [2] for $\alpha \in \mathcal{A}_s$ is equality. Let us assume that for some state $s_0$ all these inequalities are strict for the solution $v^*_s$. These inequalities can be written as

$$\left(1 - p^\alpha_{s_0,s_0}\right)v^*_s < \sum_{s' \neq s_0} p^\alpha_{s_0,s'}v^*_s + r^\alpha_{s_0},$$

for all $\alpha \in \mathcal{A}_{s_0}$. Due to the relation $1 - p^\alpha_{s_0,s_0} \geq 0$ we can slightly increase $v^*_s$ without violating these inequalities (and the inequalities for $s \neq s_0$ too, since $v^*_s$ appears there on the right-hand side with a nonnegative coefficient), so we can get a larger value of the objective function $\sum_{s \in S} c_s v_s$, which contradicts the assumption that $v^*_s$ is the optimal solution (it is at point that we use the condition $c_{s_0} > 0$). This proves that $v^*_s$ is a solution of Eq. (3).

Now we prove that the optimization problem [2] has a solution. We need to prove that the constraints are feasible and that the objective function is bounded under the given constraints. The former statement is trivial, since the point $v = \mathbf{0}$ satisfies all the constraints, so we need only to prove the boundness. Because all sets of actions $\mathcal{A}_s$ are non-empty, we can arbitrarily take an action for each state, which is equivalent to choosing some scheme $\pi$. Taking only the conditions corresponding to the chosen actions in the system [2], we see that any feasible point $v$ satisfies the inequalities $v \leq Q_\pi v + r_\pi$, where $Q_\pi$ is obtained from $Q$ by leaving the rows corresponding to the actions of $\pi$ and $r_\pi$ is the cost vector of these actions. The matrix $Q_\pi$ is the stripped transition probability matrix of some absorbing Markov chain. It is well-known that $Q_\pi^k \to 0$ elementwise when $k \to +\infty$ and thus $I - Q_\pi$ is invertible. Since all elements of $Q_\pi$ are nonnegative, we can multiply both sides of this inequality by $Q_\pi$ and the inequality still holds row-wise, so for all $k \geq 1$ we have $v \leq Q_\pi v + Q_\pi^{k-1}r_\pi + \ldots + Q_\pi r_\pi + r_\pi$. As before, in the limit $k \to +\infty$ we obtain the inequality $v \leq (I - Q_\pi)^{-1}r_\pi$, from which we conclude that any feasible point is bounded and thus the objective function is bounded. So, we have verified that the problem [2] is feasible and bounded, so it has a solution, which, as we have proved before, is also a solution of the nonlinear equations [3].

Proof of Theorem 2

Let $v^*$ be any solution of the system [3]. For any scheme $\pi$ the solution $v^\pi$ satisfies the system of linear equations $v^\pi = Q_\pi v^\pi + r_\pi$, where $Q_\pi$ and $r_\pi$ are defined above. The optimal solution $v^*$ satisfies the inequality $v^* \leq Q_\pi v^\pi + r_\pi$. Subtracting from this inequality the equality for $v^\pi$, we obtain the series of relations

$$v^* - v^\pi \leq Q_\pi (v^* - v^\pi) \leq \ldots \leq Q_\pi^k (v^* - v^\pi),$$

for all $k \geq 1$. As we know, $Q_\pi^k \to 0$ elementwise when $k \to +\infty$. We conclude that $v^* \leq v^\pi$ componentwise, which proves the desired relation.

MDP of a quantum repeater

At any moment in time a state of a repeater can be characterized by a string of numbers, where 0 always denotes a segment that is still trying to distribute entanglement, 1 denotes a segment with entanglement already distributed, and $i > 1$ denotes a group of $i$ successfully swapped segments. The set of all such strings we express by $S'_n$. It has been shown in [9] that the number of such states is given by the odd Fibonacci number, $|S'_n| = F_{2n+1}$. As mentioned in the main text, we can combine two states which are mirror images of each other into one state. This trick reduces the number of states and constraints, thus reducing the size of the problem and time to solve it. The reduced set of the strings corresponding to the new states is denoted as $S_n$. The number of states in $S_n$ is given by the following theorem.

Theorem 3. The size of the set $S_n$ is given by

$$|S_n| = \frac{F_{2n+1} + F_{n+2}}{2}. \tag{7}$$

For example, all $|S_4| = (F_9 + F_6)/2 = 21$ relevant states of a 4-segment quantum repeater are

$$
\begin{array}{cccc}
0000 & 0001 & 0010 & 0011 \\
0101 & 0110 & 0111 & 012 \\
021 & 03 & 1001 & 1011 \\
1111 & 112 & 121 & 13 & 22 \\
\end{array}
$$

and the terminating state 4, where all four segments have successfully distributed and swapped entanglement.

Proof. We first prove that the number of symmetric states in $S'_n$ is the $(n + 2)$-th Fibonacci number, $F_{n+2}$. The set of symmetric states we denote as $S_n \subset S_n$. It is
easy to check that the statement is valid for \( n = 0, 1, 2, 3 \). In fact, we have \( S_0 = \{ \emptyset \} \), \( S_1 = \{ 0, 1 \} \), \( S_2 = \{ 00, 11, 02 \} \) and \( S_3 = \{ 000, 010, 101, 111, 012 \} \), so \( |S_n| = F_{n+2} \) for \( n = 0, 1, 2, 3 \). We show that the numbers \( N_n = |S_n| \) satisfy the relation
\[
N_n = 3N_{n-2} - N_{n-4},
\]
for \( n > 3 \). In fact, we can partition the set \( S_n \) as
\[
S_n = S'_n \cup S''_n,
\]
where \( S'_n \) is the set of symmetric states with the first (and thus the last) element equal to 0 or 1, and \( S''_n \) is the set of states which start and end with an element larger than 1. Since \( S'_n \cap S''_n = \emptyset \), we have
\[
N_n = |S_n| = |S'_n| + |S''_n|.
\]
Clearly, \( |S'_n| = 2N_{n-2} \), since any state from \( S'_n \) can be obtained from a state of \( S_{n-2} \) in two ways — by prefixing and suffixing with 0 or 1. To compute \( |S''_n| \) note that the states of \( S''_n \) are in one-to-one correspondence with the states of \( S_{n-2} \) that start and end with 1. It is easier to compute the number of states of \( S_{n-2} \) that end with 0 since this number is just \( N_{n-4} = |S_{n-4}| \). We thus have that \( |S''_n| = N_{n-2} - N_{n-4} \). Adding the two numbers, we get the relation (8).

Fibonacci numbers \( F_{n+2} \) also satisfy this relation, as one can easily check from their defining relation \( F_{n+1} = F_n + F_{n-1} \). Since \( N_n = F_{n+2} \) for \( n = 0, 1, 2, 3 \) and for \( n > 3 \) these numbers satisfy the same recurrence relation, we conclude that the statement is valid for all \( n \).

We can now compute \( |S_n| \). We have
\[
|S_n| = |S_n| + \frac{|S'_n| - |S_n|}{2} = F_{2n+1} + F_{n+2}.
\]

**Example of 4-segment repeater**

In our first model (that without including classical communication times on the level of the entanglement swapping) we count only the time needed to distribute entanglement. For any state with no neighboring ready segments we have no other choice except waiting when some segments distribute entanglement. If a state has several consecutive ready segments (or several consecutive groups of segments), then the situation is more interesting since now we have several possibilities. For example, for \( n = 4 \), consider the state 0111. We have three possibilities — either we wait for the first segment to successfully distribute entanglement, or we try to swap segments 2 and 3, or we try to swap segments 3 and 4. The first action costs one time unit, while the other two cost nothing. Different choices have different statistical properties, and in this work we present a method to determine the fastest way to distribute entanglement.

For any state \( s \in S \), we introduce the set of actions \( A_s \), that are possible in this state. Any action \( \alpha \in A_s \) is either waiting for some segments to distribute entanglement (if not all segments have done it), which we denote by “distribute”, or trying to swap consecutive groups of ready segments, which we denote by “swap consecutive groups of ready segments”, we conclude that the statement is valid for \( n = 0 \) and \( n = 1 \). Since \( F_n = F_{n+1} + F_{n-1} \), we get the relation (8). Fibonacci numbers \( F_{n+2} \) also satisfy this relation, as one can easily check from their defining relation \( F_{n+1} = F_n + F_{n-1} \). Since \( N_n = F_{n+2} \) for \( n = 0, 1, 2, 3 \) and for \( n > 3 \) these numbers satisfy the same recurrence relation, we conclude that the statement is valid for all \( n \).

We can now compute \( |S_n| \). We have
\[
|S_n| = |S_n| + \frac{|S'_n| - |S_n|}{2} = F_{2n+1} + F_{n+2}.
\]

\( \square \)

**Classical communication**

The classical communication times on the level of the entanglement swapping can be also included into this scheme, assuming that the time needed to restart a group of segments is equal to the number of segments in this group. This significantly increases the size of the problem. The number of states and constraints of the problem with and without CC is given in the table below for several values of \( n \).

Note that the numbers in the column “N. of states” without CC are given by
\[
\frac{F_{2n+1} + F_{n+2}}{2} - 1,
\]

since we do not include the absorbing state. Explicit expressions for the numbers in the other columns are not known.

**Table:**

| \( n \) | \( |S_n| \) | \( F_{2n+1} + F_{n+2} \) | \( |S_n| \) without CC |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 1 | 1 | 2 | 1 |
| 2 | 2 | 3 | 1 |
| 3 | 3 | 5 | 1 |
| 4 | 5 | 8 | 1 |
| 5 | 8 | 13 | 1 |
| 6 | 13 | 21 | 1 |
| 7 | 21 | 34 | 1 |
| 8 | 34 | 55 | 1 |
| 9 | 55 | 89 | 1 |
| 10 | 89 | 144 | 1 |

**Example of 4-segment repeater**

In our first model (that without including classical communication times on the level of the entanglement swapping) we count only the time needed to distribute entanglement. For any state with no neighboring ready segments we have no other choice except waiting when some segments distribute entanglement. If a state has several consecutive ready segments (or several consecutive groups of segments), then the situation is more interesting since now we have several possibilities. For example, for \( n = 4 \), consider the state 0111. We have three possibilities — either we wait for the first segment to successfully distribute entanglement, or we try to swap segments 2 and 3, or we try to swap segments 3 and 4. The first action costs one time unit, while the other two cost nothing. Different choices have different statistical properties, and in this work we present a method to determine the fastest way to distribute entanglement.

For any state \( s \in S \), we introduce the set of actions \( A_s \), that are possible in this state. Any action \( \alpha \in A_s \) is either waiting for some segments to distribute entanglement (if not all segments have done it), which we denote by “distribute”, or trying to swap consecutive groups of ready segments, which we denote by “swap consecutive groups of ready segments”, we conclude that the statement is valid for \( n = 0 \) and \( n = 1 \). Since \( F_n = F_{n+1} + F_{n-1} \), we get the relation (8). Fibonacci numbers \( F_{n+2} \) also satisfy this relation, as one can easily check from their defining relation \( F_{n+1} = F_n + F_{n-1} \). Since \( N_n = F_{n+2} \) for \( n = 0, 1, 2, 3 \) and for \( n > 3 \) these numbers satisfy the same recurrence relation, we conclude that the statement is valid for all \( n \).

We can now compute \( |S_n| \). We have
\[
|S_n| = |S_n| + \frac{|S'_n| - |S_n|}{2} = F_{2n+1} + F_{n+2}.
\]

\( \square \)
TABLE I: Size of the MDP with and without CC depending on the number of repeater segments $n$. 

| $n$ | N. of states Without CC | N. of constr. Without CC | N. of states With CC | N. of constr. With CC |
|-----|-------------------------|---------------------------|----------------------|-----------------------|
| 4   | 20                      | 29                        | 45                   | 56                    |
| 5   | 50                      | 86                        | 150                  | 206                   |
| 6   | 126                     | 261                       | 525                  | 791                   |
| 7   | 321                     | 763                       | 1,795                | 2,922                 |
| 8   | 825                     | 2,234                     | 6,265                | 10,922                |
| 9   | 2,134                   | 6,424                     | 21,877               | 40,502                |
| 10  | 5,544                   | 18,398                    | 76,814               | 150,328               |

The three schemes for $n = 4$

For a uniform description of the optimal actions in each scheme, only those states are presented that allow for more than a single action. The “doubling” scheme, $\pi_0$, is characterized by the following choice of actions:

- $0011 \rightarrow$ swap
- $0110 \rightarrow$ distribute
- $0111 \rightarrow$ swap 3 and 4
- $1011 \rightarrow$ swap 3 and 4

The scheme $\pi_1$ is characterized by the following actions:

- $0011 \rightarrow$ swap 3 and 4
- $0110 \rightarrow$ swap 2 and 3
- $0111 \rightarrow$ swap 3 and 4
- $1011 \rightarrow$ swap 3 and 4

The scheme $\pi_2$ is characterized by the following actions:

- $0011 \rightarrow$ swap 3 and 4
- $0110 \rightarrow$ swap 2 and 3
- $0111 \rightarrow$ swap 3 and 4
- $1011 \rightarrow$ swap 3 and 4

Policy $\pi_2$ can be simply interpreted as “swap as soon as possible” (as soon as there are ready pairs), whereas for $\pi_0$ and $\pi_1$ this is not always the case (e.g. see the state 012). However, note that in general, for larger $n$, there will be many different schemes, each optimal in a corresponding small region of the unit square $(p,a)$ and having a tiny advantage over each other in neighboring regions; so, in the general case, a simple interpretation as “swap as soon as possible” for the optimal schemes is not applicable. Further note that there is always strictly only one action in any given state and the optimization finds the best action in each state. If in some state two or more actions have the same optimal behavior one may choose any of them randomly.

For $n = 4$, policy $\pi_1$ swaps as soon as possible on the lowest level (on the level of connecting the initial segments), unlike $\pi_0$, see the state 0110. A possible explanation of these different optimal policies is the following. The higher the chance for a failed swapping attempt (smaller $a$), the better it is to directly swap only small groups of segments, or even only groups of segments that belong to “independent” smaller sub-repeaters on the lower levels. Trying to connect these sub-repeaters as soon as possible is the best strategy for larger $a$.

As a final remark we notice that for a given scheme some states may not be reachable from the initial state, e.g. see 021 for $\pi_0$ (“doubling”), which nonetheless is listed above. As the optimal scheme is not known beforehand, inequalities have to be included for all states. In practice, once the optimal solution has been found, we would care only about those states reachable from some initial state (which is 0000 for $n = 4$ in our model) and we can ignore some parts of the information provided by the optimal solution. If the system would start in some different initial state, the optimal policy would also become available. Would we know beforehand what states are unreachable, the optimization problem would become smaller, but this information is only accessible afterwards together with the solution. Nonetheless, reachability of states is a meaningful element of the finally obtained optimal policies.

Supplementary figures

The two additional figures below show a comparison between optimal and “doubling” waiting times for the simplest model without all CC times and between the optimal waiting times with and without CC times for various repeater sizes. From the latter comparison, we can infer that the impact of CC grows with an increasing number of repeater segments. In Fig. 2, this is not always easy to see, but, for example, for $n = 9$ the maximal impact (ratio) is 2.41, for $n = 10$ it is 2.46.

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[1] In a typical quantum repeater system, the parameter $p$ is primarily given by the probability that a photonic qubit is successfully transmitted via a fiber channel of length $L_0$ connecting two stations, $e^{-L_0/22km}$. It also includes local state preparation/detection, fiber coupling, frequency conversion, and memory write-in efficiencies. The parameter $a$ can be related to the memory read-out or an optical Bell measurement efficiency.
FIG. 4: Ratio of the “doubling” to the optimal waiting time for $n = 4$ (left) and $n = 8$ (right), model without CC.

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[10] See also [19] and [20]. In our previous work [9], we have shown how the exact repeater rates can be obtained for arbitrary $(n, p, a)$ and arbitrary fixed (e.g. “doubling”) or dynamical (e.g. “swap as soon as possible”) protocols. However, even including methods compressing the Markov state space (“lumpability”), the resulting set of linear equations to be solved grows exponentially with the size of the repeater $n$, and so explicit expressions can be given in a compact form only for smaller repeaters such as $n = 4$; for larger repeaters, the linear system must be solved directly. Note that in terms of their possible Markov states, a quantum repeater chain is remarkably complex. The rates of other classes of quantum networks, for example, a star-shaped “entanglement switch” with a single center node [10] may be computed more easily and efficiently (e.g. by counting the number of successfully distributed segments where actions like Bell measurements do not depend on the availability of specific segments [9]).

[11] If memory and gate errors are included and state fidelities or bit error rates in quantum key distribution are considered, it depends on the corresponding hardware parameter regime whether “doubling” with entanglement distillation or swapping as soon as possible performs best, which can be found through numerical simulations (see [21] and also references therein). The present work instead shows via rigorous and exact analytical optimizations of statistical quantities in a physical setting that even with sufficiently good hardware, but assuming non-deterministic entanglement swapping (which actually may come along with an imperfect lossy memory system such as a collective bosonic spin mode of an atomic ensemble [22]), a different performance ordering of the various protocols can occur depending on the parameter triple $(n, p, a)$. For a nice review on the different approaches to treat quantum repeaters and networks quantitatively, either analytically under ideal or more realistic conditions or, even more realistically but by omitting the possible general insights that primarily an analytical treatment may provide, via numerical simulations, see [23].

[12] The classical communication times are an important limiting factor in a realistic quantum network and the corresponding schemes are hard to optimize. Unlike previous treatments of nested “doubling” schemes based on approximations for small $p$ and $a$ [24] we exactly derive the optimal schemes with classical communication independent of the $n = 2^a$ assumption for any $(n, p, a)$ up to about $n = 10$.

[13] Similarly, all intermediate stations can perform entanglement swapping with the same probability. Considering equal segment lengths $L_0$ is then a reasonable assumption for a linear quantum repeater chain. The ultimate repeater-assisted capacity [1] is determined by the capacity of the longest segment and hence is also maximized for a repeater with equal segment lengths.

[14] The schemes we consider for our rate optimizations are error-free. In particular, memory qubits can be held as long as needed. However, we do not assume an unlimited availability of quantum memories. If a segment is filled with a single entangled pair there are no further distribution attempts in that segment which is a common assumption [3, 7–9]. While it can be useful to continue distributing and load additional memory qubits or just replace an already held one by a fresh, newly distributed pair (in case of a memory state fidelity decaying with time), provided the extra resources are available, the corresponding (secret key) rate analysis gets again more involved requiring simplifying assumptions on the network topology to allow for more general “memory buffers” [9] or a numerical treatment [21]. Our schemes have unit buffer size and it would not affect the raw rates if we continued distributing pairs in a loaded segment to replace the existing pair. An infinite buffer size makes most efficient use of the quantum communication channel. In this kind of unphysical case, assuming unlimited resources, the optimal rates can be calculated even for a repeater chain and it turns out that “doubling” is always optimal when $n = 2^a$. Besides memory buffers, a “memory cutoff” can be introduced where a memory state is discarded after waiting for too long [20], reducing the raw communication rates but preventing state fidelities from dropping below a predetermined value. In the exact Markov chain approach, a finite memory cutoff can be included at
FIG. 5: Ratio between the optimal raw waiting times with CC and without for different numbers of segments $n$. The relative impact of CC is small for small $p$ and becomes maximal for $p$ approaching one and $a$ at around one half.

the expense of computational efficiency due to a further increase of the linear equation system [13]. In principle, it could also be included into the optimizations of the present work. Finally, simple multiplexing with several repeater chains [26] could also be incorporated into our schemes by replacing the corresponding $p$ value in each segment.

[15] For the present $n = 3$ case, there is only one state, $s_4$, that allows for more than a single action, namely two, and so we have two inequalities for $v_4$. For the optimal solution, one becomes an equation (the one corresponding to swapping), the other (the one corresponding to distributing) can be removed, and the remaining system has 8 linear equations for 8 unknowns. This system of linear equations corresponds to a Markov chain, like in Eq. (1), and from any state we can go to some other states with some probabilities. However, prior to optimization, we have several actions, and the probabilities depend on the action chosen. In this sense generally the MDP embeds many Markov chains and only when we choose one action for each state, we get back a standard Markov chain. The relevant question here is what actions to choose to get the optimal behavior.

[16] The value $p = 0.01$ corresponds to a segment length of $L_0 = 100km$, only considering fiber channel loss. Thus, through segment numbers $n = 2...12$ we can cover total distances from 200km to 1200km. A swapping probability value of, for example, $a = 2/3$ is meaningful, because it corresponds to the efficiency of a linear-optics Bell measurement enhanced by means of 4 auxiliary photons [27].

[17] The typical repeater regime is at small $p$ where the rates are generally low and so the relative impact of CC is small. However, note that the waiting times here are the inverse rates per total channel use, independent of the actual duration of one time unit, i.e. the combined CC and local processing time per segment (due to the blocking of successfully distributed segments, the total channel use is overestimated in our model). As small $p$ values correspond to larger segment lengths and so larger CC times, the rates per second drop significantly when including CC. Growing CC times also have a negative effect on the quantum memories which must store qubits for longer.

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