On the support of the Grover walk on higher-dimensional lattices

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Abstract. This paper presents the minimum supports of states for stationary measures of the Grover walk on the \(d\)-dimensional lattice by solving the corresponding eigenvalue problem. The numbers of the minimum supports for moving and flip-flop shifts are \(2^d\) \((d \geq 1)\) and \(4\) \((d \geq 2)\), respectively.

1 Introduction

The quantum walk was introduced by Aharonov et al. \cite{1} as a generalization of the random walk on graphs. On the one-dimensional lattice \(Z\), where \(Z\) is the set of integers, the properties of quantum walks are well studied, see Konno \cite{6}, for example. There are some results on the Grover walk on \(Z^2\), such as weak limit theorem by Watabe et al. \cite{8} (moving shift case) and Higuchi et al. \cite{2} (flip-flop shift case), and localization shown by Imui et al. \cite{3} (moving shift case) and Higuchi et al. \cite{2} (flip-flop shift case).

In this paper, we present the minimum support of states for the stationary measures of the Grover walk on \(Z^d\) by solving the corresponding eigenvalue problem. As for the number of the support of the Grover walk on \(Z^d\) with moving shift, \(2^2\) \((Z^2\) case) and \(3^d\) \((Z^d\) case with \(d \geq 2)\) were given in Stefanak et al. \cite{7} and Komatsu and Konno \cite{4} by the Fourier analysis, respectively. Compared with the above-mentioned previous results, the number of our minimum support for \(Z^d\) case with \(d \geq 1\) is \(2^d\) (Theorem 1). Moreover, concerning the number of the support of the Grover walk on \(Z^d(d \geq 2)\) with flip-flop shift, 4 was obtained in Higuchi et al. \cite{2} by the spectral mapping theorem, which coincides with our result (Theorem 2). Remark that any finite support does not exist for \(Z^1\) case.

The rest of the paper is as follows. Section 2 is devoted to the definition of the discrete-time quantum walks on \(Z^d\). Section 3 deals with the stationary measure of the Grover walk on \(Z^d\). We give main results on minimum support for the Grover walk on \(Z^d\) with moving shift (Theorem 1) in Section 4 and flip-flop shift (Theorem 2) in Section 5, respectively. Section 6 summarizes our paper.

2 Discrete-time quantum walks on \(Z^d\)

In this section, we give the definition of \(2d\)-state discrete-time quantum walks on \(Z^d\). The quantum walk is defined by using a shift operator and a unitary matrix. Let \(\mathbb{C}\) be the set of complex numbers. For \(i \in \{1, 2, \ldots, d\}\), the shift operator \(\tau_i\) is given by

\[
(\tau_i f)(x) = f(x - e_i), \quad (f : Z^d \rightarrow \mathbb{C}^{2d}, \ x \in Z^d),
\]

where \(\{e_1, e_2, \ldots, e_d\}\) denotes the standard basis of \(Z^d\). Let \(A = (a_{ij})_{i,j=1,2,\ldots,2d}\) be a \(2d \times 2d\) unitary matrix. We call this unitary matrix the coin matrix. To describe the time evolution of the quantum walk, decompose the unitary matrix \(A\) as

\[
A = \sum_{i=1}^{2d} P_i A,
\]

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Then for \( \Psi \in \text{stationary amplitude} \), \( \Psi \) is called the \emph{stationary measure}. We introduce the set of solutions of Eq. (3.2) for \( \lambda \) given by the following 2-dimensional unitary operator \( U \):

\[
U_A = \sum_{i=1}^d \left( P_{2i-1} A \tau_i^{-1} + P_{2i} A \tau_i \right).
\]

(2.3)

Let \( \mathbb{Z}_> = \{0,1,2,\ldots\} \). The state at time \( n \in \mathbb{Z}_> \) and location \( x \in \mathbb{Z}^d \) can be expressed by a 2d-dimensional vector:

\[
\Psi_n(x) = T \left[ \Psi_n^2(x), \Psi_n^1(x), \Psi_n^3(x), \cdots, \Psi_n^{2d}(x) \right] \in \mathbb{C}^{2d},
\]

(2.4)

where \( T \) denotes a transposed operator. For \( \Psi_n : \mathbb{Z}^d \rightarrow \mathbb{C}^{2d} \), \( (n \in \mathbb{Z}_>) \), it follows from Eq. (2.3) that

\[
\Psi_{n+1}(x) \equiv (U_A \Psi_n)(x) = \sum_{i=1}^d \left( P_{2i-1} A \Psi_n(x + e_i) + P_{2i} A \Psi_n(x - e_i) \right).
\]

(2.5)

where \( \| \cdot \|_{\mathbb{C}^{2d}} \) denotes the standard norm on \( \mathbb{C}^{2d} \). Let \( \mathbb{R}_\geq = [0, \infty) \). Here we introduce a map \( \phi : (\mathbb{C}^{2d})^{\mathbb{Z}_+} \rightarrow (\mathbb{R}_\geq)^{\mathbb{Z}_+} \), such that if \( \Psi_n : \mathbb{Z}^d \rightarrow \mathbb{C}^{2d} \) and \( x \in \mathbb{Z}^d \), thus we get

\[
\phi(\Psi_n)(x) = \sum_{j=1}^{2d} |\Psi_n^j(x)|^2 = \mu_n(x),
\]

(2.7)

namely this map \( \phi \) has a role to transform from amplitudes to measures.

### 3 Stationary measure of the Grover walk on \( \mathbb{Z}^d \)

In this section, we give the definition of the stationary measure for the quantum walk. We define a set of measures, \( \mathcal{M}_s(U_A) \), by

\[
\mathcal{M}_s(U_A) = \left\{ \mu \in \mathbb{R}_{\geq}^{\mathbb{Z}_+} \setminus \{0\} : \text{there exists } \Psi_0 \in (\mathbb{C}^{2d})^{\mathbb{Z}_+} \text{ such that } \phi(U_A^n \Psi_0) = \mu \quad (n \in \mathbb{Z}_+) \right\},
\]

(3.1)

where \( 0 \) is the zero vector. Here \( U_A \) is the time evolution operator of quantum walk associated with a unitary matrix \( A \). We call this measure \( \mu \in \mathcal{M}_s(U_A) \) the stationary measure for the quantum walk defined by the unitary operator \( U_A \). If \( \mu \in \mathcal{M}_s(U_A) \), then \( \mu_n = \mu \) for \( n \in \mathbb{Z}_+ \), where \( \mu_n \) is the measure of quantum walk given by \( U_A \) at time \( n \).

Next we consider the following eigenvalue problem of the quantum walk determined by \( U_A \):

\[
U_A \Psi = \lambda \Psi \quad (\lambda \in \mathbb{C}, \ |\lambda| = 1).
\]

(3.2)

We introduce the set of solutions of Eq. (3.2) for \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \) as follows.

\[
W(\lambda) = \{ \Psi \neq 0 : U_A \Psi = \lambda \Psi \}.
\]

(3.3)

Then for \( \Psi \in W(\lambda) \), we see that \( \phi(\Psi) \in \mathcal{M}_s(U_A) \). If the function \( \Psi \) satisfied with \( \lambda = 1 \) in Eq. (3.2), then \( \Psi \) is called the \emph{stationary amplitude}. From now on, we focus on the Grover Walk on \( \mathbb{Z}^d \) which is defined by the following \( 2d \times 2d \) coin matrix \( U_G = (g_{ij})_{i,j=1,2,\ldots,2d} \) with

\[
g_{ij} = \frac{1}{d} - \delta_{ij}.
\]

(3.4)

Remark that Komatsu and Tate [5] showed that the eigenvalue of Eq. (3.2) is only \( \lambda = \pm 1 \) for the \( d \)-dimensional Grover walk. Our purpose of this paper is to investigate the support of the \( 2d \)-state Grover walk on \( \mathbb{Z}^d \).
4 Grover walk on $\mathbb{Z}^d$ with moving shift

In this section, we present our main results on the support of the Grover walk on $\mathbb{Z}^d$ with moving shift. To do so, we begin with the eigenvalue problem $U_G \Psi = \lambda \Psi$ ($\lambda \in \mathbb{C}$ with $|\lambda| = 1$), which is equivalent to

$$
\begin{align*}
\lambda \Psi^1(x) &= \frac{1-d}{d} \Psi^1(x + e_1) + \frac{1}{d} \Psi^2(x + e_1) + \cdots + \frac{1}{d} \Psi^{2d-1}(x + e_1), \\
\lambda \Psi^2(x) &= \frac{1}{d} \Psi^1(x - e_1) + \frac{1-d}{d} \Psi^2(x - e_1) + \cdots + \frac{1}{d} \Psi^{2d-1}(x - e_1), \\
\vdots & \\
\lambda \Psi^{2d-1}(x) &= \frac{1}{d} \Psi^1(x + e_d) + \frac{1}{d} \Psi^2(x + e_d) + \cdots + \frac{1-d}{d} \Psi^{2d-1}(x + e_d), \\
\lambda \Psi^{2d}(x) &= \frac{1}{d} \Psi^1(x - e_d) + \frac{1}{d} \Psi^2(x - e_d) + \cdots + \frac{1-d}{d} \Psi^{2d-1}(x - e_d),
\end{align*}
$$

(4.1)

where $\Psi(x) = T[\Psi^1(x), \Psi^2(x), \ldots, \Psi^{2d}(x)]$ ($x \in \mathbb{Z}^d$). Put $\Gamma(x) = \sum_{j=1}^{2d} \Psi_j(x)$ for $x \in \mathbb{Z}^d$. By using $\Gamma(x)$, Eq. (4.1) can be written as

$$
\begin{align*}
\lambda \Psi^{2k-1}(x - e_k) + \Psi^{2k-1}(x) &= \frac{1}{d} \Gamma(x), \\
\lambda \Psi^{2k}(x + e_k) + \Psi^{2k}(x) &= \frac{1}{d} \Gamma(x),
\end{align*}
$$

(4.2) (4.3)

for any $k = 1, 2, \ldots, d$ and $x \in \mathbb{Z}^d$. From Eqs. (4.2) and (4.3), we get immediately

$$
\lambda \Psi^{2k-1}(x - e_k) + \Psi^{2k-1}(x) = \lambda \Psi^{2k}(x + e_k) + \Psi^{2k}(x),
$$

(4.4)

for any $k = 1, 2, \ldots, d$ and $x \in \mathbb{Z}^d$. In order to state the following lemma, we introduce the support of $\Psi : \mathbb{Z}^d \to \mathbb{C}^{2d}$ as follows.

$$
S(\Psi) = \{ x \in \mathbb{Z}^d : \Psi(x) \neq 0 \}.
$$

(4.5)

**Lemma 1** Let $\Psi \in W(\lambda)$ with $\lambda = \pm 1$. Suppose $\#(S(\Psi)) < \infty$, where $\#(A)$ is the cardinality of a set $A$. If there exist $k \in \{1, 2, \cdots, d\}$ and $x \in \mathbb{Z}^d$ such that

$$
\begin{bmatrix}
\Psi^{2k-1}(x - e_k) \\
\Psi^{2k}(x - e_k)
\end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},
$$

(4.6)

then we have

$$
\begin{bmatrix}
\Psi^{2k-1}(x - e_k) \\
\Psi^{2k}(x - e_k)
\end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix}
\Psi^{2k-1}(x + e_k) \\
\Psi^{2k}(x + e_k)
\end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
$$

(4.7)

**Proof** First we assume that

$$
\begin{bmatrix}
\Psi^{2k-1}(x) \\
\Psi^{2k}(x)
\end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},
$$

(4.8)

for some $k \in \{1, 2, \cdots, d\}$ and $x \in \mathbb{Z}^d$. Moreover we suppose

$$
\begin{bmatrix}
\Psi^{2k-1}(x - e_k) \\
\Psi^{2k}(x - e_k)
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix}
\Psi^{2k-1}(x + e_k) \\
\Psi^{2k}(x + e_k)
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
$$

(4.9)

that is,

$$
\Psi^{2k-1}(x - e_k) = 0,
$$

(4.10)
\begin{align}
\Psi^{2k}(x - e_k) &= 0, \quad (4.11) \\
\Psi^{2k-1}(x + e_k) &= 0, \quad (4.12) \\
\Psi^{2k}(x + e_k) &= 0. \quad (4.13)
\end{align}

Combining Eq. (4.4) with Eqs. (4.10), (4.11) and (4.15), we have
\begin{align}
\Psi^{2k-1}(x) &= \Psi^{2k}(x). \quad (4.14)
\end{align}

From the assumption Eqs. (4.8) and (4.14), we put
\begin{align}
\Psi^{2k-1}(x) = \Psi^{2k}(x) = \eta, \quad (4.15)
\end{align}
where \( \eta \in \mathbb{C} \) with \( \eta \neq 0 \). Furthermore, by Eq. (4.4) for \( x - e_k \), we obtain
\begin{align}
\lambda \Psi^{2k-1}(x - 2e_k) + \Psi^{2k-1}(x - e_k) = \lambda \Psi^{2k}(x) + \Psi^{2k}(x - e_k). \quad (4.16)
\end{align}
Combining Eq. (4.16) with Eqs. (4.11) and (4.15) implies
\begin{align}
\Psi^{2k-1}(x - 2e_k) = \eta, \quad (4.17)
\end{align}
since \( \lambda \neq 0 \). In a similar way, Eq. (4.4) for \( x - 2e_k \) becomes
\begin{align}
\lambda \Psi^{2k-1}(x - 3e_k) + \Psi^{2k-1}(x - 2e_k) &= \lambda \Psi^{2k}(x - e_k) + \Psi^{2k}(x - 2e_k). \quad (4.18)
\end{align}
From Eq. (4.18) with Eqs. (4.11) and (4.17), we have
\begin{align}
\Psi^{2k-1}(x - 3e_k) = \lambda \{ \Psi^{2k}(x - 2e_k) - \eta \}, \quad (4.19)
\end{align}
since \( \lambda = \pm 1 \). Similarly, Eq. (4.4) for \( x - 3e_k \) becomes
\begin{align}
\lambda \Psi^{2k-1}(x - 4e_k) + \Psi^{2k-1}(x - 3e_k) &= \lambda \Psi^{2k}(x - 2e_k) + \Psi^{2k}(x - 3e_k). \quad (4.20)
\end{align}
From Eq. (4.20) with Eq. (4.19), we get
\begin{align}
\Psi^{2k-1}(x - 4e_k) = \lambda \Psi^{2k}(x - 3e_k) + \eta. \quad (4.21)
\end{align}
Continuing this argument repeatedly, we finally obtain
\begin{align}
\Psi^{2k-1}(x - (j + 1)e_k) &= \lambda \Psi^{2k}(x - je_k) + (-\lambda)^{j+1} \eta, \quad (4.22)
\end{align}
for any \( j = 0, 1, 2, \cdots \). Assumption \#(S(\Psi)) < \infty implies that there exists \( J \) such that
\begin{align}
\Psi^{2k-1}(x - j'e_k) = \Psi^{2k}(x - j'e_k) = 0, \quad (4.23)
\end{align}
for any \( j' \geq J \). Combining Eq. (4.22) with Eq. (4.23) gives \( \eta = 0 \) since \( \lambda \neq 0 \). Therefore contradiction occurs, so the proof is complete.

**Lemma 2** Let \( \Psi \in W(\lambda) \) with \( \lambda = \pm 1 \). Suppose \#(S(\Psi)) < \infty. If there exist \( k \in \{1, 2, \cdots, d\} \) and \( x \in \mathbb{Z}^d \) such that
\begin{align}
\begin{bmatrix}
\Psi^{2k-1}(x) \\
\Psi^{2k}(x)
\end{bmatrix} \neq \begin{bmatrix}
0 \\
0
\end{bmatrix},
\end{align}
then there exist \( m^-(\leq 0) \) and \( m^+(\geq 0) \) with \( m^- < m^+ \) and \( \alpha, \beta \in \mathbb{C} \) with \( \alpha \beta \neq 0 \) such that
\begin{align}
\begin{bmatrix}
\Psi^{2k-1}(x + me_k) \\
\Psi^{2k}(x + me_k)
\end{bmatrix} = \begin{cases}
T \begin{bmatrix} 0, 0 \end{bmatrix} \ (m < m^-) \\
T \begin{bmatrix} \alpha, 0 \end{bmatrix} \ (m = m^-) \\
T \begin{bmatrix} 0, \beta \end{bmatrix} \ (m = m^+) \\
T \begin{bmatrix} 0, 0 \end{bmatrix} \ (m > m^+)
\end{cases}.
\end{align}
Moreover, we have
\[
\begin{bmatrix}
\Psi^{2l-1}(x + m^{(-)}e_k) \\
\Psi^{2l}(x + m^{(-)}e_k)
\end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]
and
\[
\begin{bmatrix}
\Psi^{2l-1}(x + m^{(+)}e_k) \\
\Psi^{2l}(x + m^{(+)}e_k)
\end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]
for any \(l \in \{1, 2, \cdots, d\} \setminus \{k\} \).

**Proof**  From Lemma 1, we get \#\(S(\Psi) \geq 2\). Therefore we see that there exist \(m^{(-)}(\leq 0)\) and \(m^{(+)}(\geq 0)\) with \(m^{(-)} < m^{(+)}\) and \(\alpha, \beta, \gamma, \delta \in \mathbb{C}\) with \(|\alpha| + |\gamma| > 0\) and \(|\beta| + |\delta| > 0\) such that
\[
\begin{bmatrix}
\Psi^{2k-1}(x + me_k) \\
\Psi^{2k}(x + me_k)
\end{bmatrix} = \begin{cases} 
T \begin{bmatrix} 0, 0 \end{bmatrix} & (m < m^{(-)}) \\
T \begin{bmatrix} \alpha, \gamma \end{bmatrix} & (m = m^{(-)}) \\
T \begin{bmatrix} \delta, \beta \end{bmatrix} & (m = m^{(+)}) \\
T \begin{bmatrix} 0, 0 \end{bmatrix} & (m > m^{(+)})
\end{cases}.
\]

By Eq. (4.8) for \(x + (m^{(-)} - 1)e_k\), we have
\[
\lambda \Psi^{2k-1}(x + (m^{(-)} - 2)e_k) + \Psi^{2k-1}(x + (m^{(-)} - 1)e_k)
= \lambda \Psi^{2k}(x + m^{(-)}e_k) + \Psi^{2k}(x + (m^{(-)} - 1)e_k).
\]
Combining Eq. (4.28) with Eq. (4.29) gives
\[
\Psi^{2k}(x + m^{(-)}e_k) = \gamma = 0,
\]
since \(\lambda \neq 0\). In a similar fashion, from Eq. (4.14) for \(x + (m^{(+)} + 1)e_k\), we have
\[
\Psi^{2k-1}(x + m^{(+)}e_k) = \delta = 0.
\]
Thus combining Eqs. (4.28), (4.30) and (4.31) implies Eq. (4.25).

By Eq. (4.2) for \(x + m^{(-)}e_k\), we have
\[
\lambda \Psi^{2k-1}(x + (m^{(-)} - 1)e_k) + \Psi^{2k-1}(x + m^{(-)}e_k) = \frac{1}{d} \Gamma(x + m^{(-)}e_k).
\]
Then combining Eq. (4.32) with Eq. (4.33) gives
\[
\frac{1}{d} \Gamma(x + m^{(-)}e_k) = \alpha.
\]
Similarly, by Eq. (4.3) for \(x + m^{(+)}e_k\) and Eq. (4.29), we get
\[
\frac{1}{d} \Gamma(x + m^{(+)}e_k) = \beta.
\]
From now on, we assume that there exists \(l \in \{1, 2, \cdots, d\} \setminus \{k\}\) such that
\[
\begin{bmatrix}
\Psi^{2l-1}(x + m^{(-)}e_k) \\
\Psi^{2l}(x + m^{(-)}e_k)
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]
or
\[
\begin{bmatrix}
\Psi^{2l-1}(x + m^{(+)}e_k) \\
\Psi^{2l}(x + m^{(+)}e_k)
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]
First we consider Eq. (4.35) case. We now use Eq. (4.2) with \( k \to l \) and \( x \to x + m(-)e_k \) to get
\[
\lambda \Psi^{2l-1}(x + m(-)e_k) - e_l + \Psi^{2l-1}(x + m(-)e_k) = \frac{1}{d} \Gamma(x + m(-)e_k).
\] (4.37)

Using the equation just derived and Eq. (4.33), we have
\[
\lambda \Psi^{2l-1}(x + m(-)e_k) - e_l + \Psi^{2l-1}(x + m(-)e_k) = \alpha.
\] (4.38)

By assumption \( \Psi^{2l-1}(x + m(-)e_k) = 0 \) in Eq. (4.35), we see that Eq. (4.35) becomes
\[
\Psi^{2l-1}(x + m(-)e_k) - e_l = \lambda \alpha,
\] (4.39)

since \( \lambda = \pm 1 \). Next we see Eq. (4.4) with \( k \to l \) and \( x \to x + m(-)e_k - e_l \) to get
\[
\lambda \Psi^{2l-1}(x + m(-)e_k) - 2e_l + \Psi^{2l-1}(x + m(-)e_k) - e_l = \lambda \Psi^2(x + m(-)e_k) + \Psi^2(x + m(-)e_k) - e_l.
\] (4.40)

Combining this equation with Eq. (4.39) and assumption \( \Psi^2(x + m(-)e_k) = 0 \) in Eq. (4.35) gives
\[
\Psi^{2l-1}(x + m(-)e_k) - 2e_l = \lambda \Psi^2(x + m(-)e_k) - e_l - \lambda^2 \alpha,
\] (4.41)

since \( \lambda = \pm 1 \). By the similar argument repeatedly, we obtain,
\[
\Psi^{2l-1}(x + m(-)e_k) - (j+1)e_l = \lambda \Psi^2(x + m(-)e_k) - je_l - (-\lambda)^j+1 \alpha,
\] (4.42)

for any \( j = 1, 2, \ldots \). Assumption \( \#(S(\Psi)) < \infty \) implies that there exists \( J \) such that
\[
\Psi^{2l-1}(x + m(-)e_k) - j'e_k = \Psi^2(x + m(-)e_k) - j'e_l = 0,
\] (4.43)

for any \( j' \geq J \). Combining Eq. (4.42) with Eq. (4.43) gives \( \alpha = 0 \) since \( \lambda \neq 0 \). Thus we have a contradiction.

Next we consider Eq. (4.36) case. In a simiar fashion, we get \( \beta = 0 \) and have a contradiction. Therefore the proof of Lemma 2 is complete.

**Theorem 1** For the Grover walk on \( \mathbb{Z}^d \) with moving shift, we have
\[
\#(S(\Psi)) \geq 2^d,
\] (4.44)

for any \( \Psi \in W(\lambda) \) with \( \lambda = \pm 1 \). In particular, there exists \( \Psi^*(\lambda) \in W(\lambda) \) such that
\[
\#(S(\Psi^*(\lambda))) = 2^d,
\] (4.45)

for \( \lambda = \pm 1 \). In fact, we obtain
\[
\Psi^*(\lambda)(x) = \lambda^{x_1 + x_2 + \cdots + x_d} \times T [x_1, x_2, \ldots, x_d] \quad (x \in S(\Psi^*(\lambda))),
\] (4.46)

where
\[
S(\Psi^*(\lambda)) = \{x = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d : x_k \in \{0, 1\} \ (k = 1, 2, \ldots, d)\}.
\] (4.47)

Here \( |0\rangle = T[1, 0] \) and \( |1\rangle = T[0, 1] \).

**Proof.** For \( \Psi \in W(\lambda) \) with \( \lambda = \pm 1 \), there exist \( k \in \{1, 2, \ldots, d\} \) and \( x \in \mathbb{Z}^d \) such that
\[
\begin{bmatrix}
\Psi^{2k-1}(x) \\
\Psi^{2k}(x)
\end{bmatrix} \neq \begin{bmatrix}
0 \\
0
\end{bmatrix}.
\] (4.48)

Thus, we have \( x \in S(\Psi) \).

First we consider \( d = 1 \) case. From Lemma 1, we see that \( x - e_1 \in S(\Psi) \) or \( x + e_1 \in S(\Psi) \), so \( \#(S(\Psi)) \geq 2 \). If fact, we can construct a \( \Psi^*(\lambda) \in W(\lambda) \) with \( \lambda = \pm 1 \) satisfying \( \#(S(\Psi^*(\lambda))) = 2 \) as follows.
By Lemma 2 with Eq. (4.50), we can also assume
\[
\begin{bmatrix}
\Psi_1(x) \\
\Psi_2(x)
\end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\] and
\[
\begin{bmatrix}
\Psi_1(x + e_1) \\
\Psi_2(x + e_1)
\end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\tag{4.50}
\]

By Lemma 2 with Eq. (4.50), we can also assume \(m(-) = 0\) and \(m(+) = 1\) to minimize the \(#(S(Ψ))\), then we have
\[
\begin{bmatrix}
\Psi_3(x) \\
\Psi_4(x)
\end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\tag{4.51}
\]
and
\[
\begin{bmatrix}
\Psi_3(x + e_1) \\
\Psi_4(x + e_1)
\end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\tag{4.52}
\]

From Lemma 1 with Eqs. (4.51) and (4.52), we obtain “\(x - e_2 \in S(Ψ)\) or \(x + e_2 \in S(Ψ)\)” and “\(x + e_1 - e_2 \in S(Ψ)\) or \(x + e_1 + e_2 \in S(Ψ)\)” respectively, so \(#(S(Ψ)) \geq 4\). In fact, we can construct a \(Ψ^{(λ)} \in W(λ)\) with \(λ = \pm 1\) satisfying \(#(S(Ψ^{(λ)})) = 4\) as follows.

\[
Ψ(x + m_1 e_1 + m_2 e_2) = \begin{cases}
λ_{m_1 + m_2} x^T [1, 0, 1, 0] & (m_1, m_2) = (0, 0) \\
λ_{m_1 + m_2} x^T [0, 1, 1, 0] & (m_1, m_2) = (1, 0) \\
λ_{m_1 + m_2} x^T [1, 0, 0, 1] & (m_1, m_2) = (0, 1) \\
λ_{m_1 + m_2} x^T [0, 1, 0, 1] & (m_1, m_2) = (1, 1) \\
T [0, 0, 0, 0] & (otherwise)
\end{cases}
\tag{4.53}
\]

for \(m_1, m_2 \in \mathbb{Z}\). Remark that Eq. (4.53) has been introduced in Stefanak et al. [7]. Continuing a similar argument for \(d = 3, 4, \cdots\), we have the desired conclusion.

From Eq. (4.53), we obtain the following equation as a stationary measure of Grover walk on \(\mathbb{Z}^2\) when \(λ = 1\).

\[
Ψ(x, y) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} g(x, y) + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} g(x - 1, y) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} g(x, y - 1) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} g(x - 1, y - 1),
\tag{4.54}
\]

for \((x, y) \in \mathbb{Z}^2\). Here \(g : \mathbb{Z}^2 \rightarrow \mathbb{C}\). Let \(g(x, y)\) as follows.

\[
g(x, y) = \begin{cases}
δ_{(x,y)} & (x, y) \in \{0, -1\}
\\
0 & (otherwise).
\end{cases}
\tag{4.55}
\]

Then combining Eq. (4.51) with Eq. (4.55), we easily get \(#(S(Ψ)) = 9\) such that

\[
Ψ = \begin{bmatrix}
\delta_{(0,0)} & \delta_{(0,1)} & \delta_{(1,0)} & \delta_{(1,-1)} \\
\delta_{(1,1)} & \delta_{(1,-1)} & \delta_{(0,-1)} & \delta_{(-1,0)} \\
\delta_{(2,0)} & \delta_{(2,1)} & \delta_{(1,2)} & \delta_{(2,-1)} \\
\delta_{(1,1)} & \delta_{(1,-1)} & \delta_{(0,1)} & \delta_{(-1,1)}
\end{bmatrix}
\tag{4.56}
\]

Remark that Eq. (4.50) has been introduced in Komatsu and Konno [3].
5 Grover walk on $\mathbb{Z}^d$ with flip-flop shift

In this section, we consider the case of the $d$-dimensional Grover walk with flip-flop shift. The eigenvalue problem $U_G\Psi = \lambda\Psi$ ($\lambda \in \mathbb{C}$ with $|\lambda| = 1$) is equivalent to

$$
\begin{align*}
\lambda\Psi^1(x) &= \frac{1}{d}\Psi^1(x + e_1) + \frac{1}{d}\Psi^2(x + e_1) + \cdots + \frac{1}{d}\Psi^{2d-1}(x + e_1) + \frac{1}{d}\Psi^{2d}(x + e_1), \\
\lambda\Psi^2(x) &= \frac{1}{d}\Psi^1(x - e_1) + \frac{1}{d}\Psi^2(x - e_1) + \cdots + \frac{1}{d}\Psi^{2d-1}(x - e_1) + \frac{1}{d}\Psi^{2d}(x - e_1), \\
& \vdots \\
\lambda\Psi^{2d-1}(x) &= \frac{1}{d}\Psi^1(x + e_d) + \frac{1}{d}\Psi^2(x + e_d) + \cdots + \frac{1}{d}\Psi^{2d-1}(x + e_d) + \frac{1}{d}\Psi^{2d}(x + e_d), \\
\lambda\Psi^{2d}(x) &= \frac{1}{d}\Psi^1(x - e_d) + \frac{1}{d}\Psi^2(x - e_d) + \cdots + \frac{1}{d}\Psi^{2d-1}(x - e_d) + \frac{1}{d}\Psi^{2d}(x - e_d),
\end{align*}
$$

(5.1)

where $\Psi(x) = T[\Psi^1(x), \Psi^2(x), \ldots, \Psi^{2d}(x)]$ ($x \in \mathbb{Z}^d$). Put $\Gamma(x) = \sum_{j=1}^{2d}\Psi^j(x)$ for $x \in \mathbb{Z}^d$. By using $\Gamma(x)$, Eq. (5.1) can be written as

$$
\lambda\Psi^{2k-1}(x - e_k) + \Psi^{2k}(x) = \frac{1}{d}\Gamma(x),
$$

(5.2)

$$
\lambda\Psi^{2k}(x + e_k) + \Psi^{2k-1}(x) = \frac{1}{d}\Gamma(x),
$$

(5.3)

for any $k = 1, 2, \ldots, d$ and $x \in \mathbb{Z}^d$. From Eqs. (5.2) and (5.3), we get immediately

$$
\lambda\Psi^{2k-1}(x - e_k) + \Psi^{2k}(x) = \lambda\Psi^{2k}(x + e_k) + \Psi^{2k-1}(x),
$$

(5.4)

for any $k = 1, 2, \ldots, d$ and $x \in \mathbb{Z}^d$.

**Lemma 3** Let $\Psi \in W(\lambda)$ with $\lambda = \pm 1$. Suppose $(S(\Psi)) < \infty$. If there exist $k \in \{1, 2, \ldots, d\}$ and $x \in \mathbb{Z}^d$ such that

$$
\begin{bmatrix}
\Psi^{2k-1}(x) \\
\Psi^{2k}(x)
\end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},
$$

(5.5)

then we have

$$
\begin{bmatrix}
\Psi^{2k-1}(x - e_k) \\
\Psi^{2k}(x - e_k)
\end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix}
\Psi^{2k-1}(x + e_k) \\
\Psi^{2k}(x + e_k)
\end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
$$

(5.6)

**Proof** First we assume that

$$
\begin{bmatrix}
\Psi^{2k-1}(x) \\
\Psi^{2k}(x)
\end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},
$$

(5.7)

for some $k \in \{1, 2, \ldots, d\}$ and $x \in \mathbb{Z}^d$. Furthermore we suppose

$$
\begin{bmatrix}
\Psi^{2k-1}(x - e_k) \\
\Psi^{2k}(x - e_k)
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix}
\Psi^{2k-1}(x + e_k) \\
\Psi^{2k}(x + e_k)
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
$$

(5.8)

By a similar calculation as in Lemma 1, we get the following equation corresponding to Eq. (4.22).

$$
\Psi^{2k-1}(x - (j + 1)e_k) = -\lambda\Psi^{2k}(x - je_k) + \lambda^{j+1}\eta,
$$

(5.9)

where $\eta = \Psi^{2k-1}(x) = \Psi^{2k}(x)$ for any $j = 0, 1, 2, \ldots$. Assumption $(S(\Psi)) < \infty$ implies that there exists $J$ such that

$$
\Psi^{2k-1}(x - j'e_k) = \Psi^{2k}(x - j'e_k) = 0,
$$

(5.10)

for any $j' > J$. Combining Eq. (5.9) with Eq. (5.10) gives $\eta = 0$ since $\lambda \neq 0$. Therefore contradiction occurs, so the proof is complete.
Lemma 4  Let $\Psi \in W(\lambda)$ with $\lambda = \pm 1$. Suppose $\#(S(\Psi)) < \infty$. If there exist $k \in \{1, 2, \cdots, d\}$ and $x \in \mathbb{Z}^d$ such that
\[
\left[ \begin{array}{c}
\Psi^{2k-1}(x) \\
\Psi^{2k}(x)
\end{array} \right] \neq \left[ \begin{array}{c}
0 \\
0
\end{array} \right],
\]  (5.11)
then there exist $m(\cdot)(\leq 0)$ and $m(\cdot)(\geq 0)$ with $m(\cdot) < m(\cdot)$ and $\alpha, \beta \in \mathbb{C}$ with $\alpha \beta \neq 0$ such that
\[
\left[ \begin{array}{c}
\Psi^{2k-1}(x + me_k) \\
\Psi^{2k}(x + me_k)
\end{array} \right] = \left[ \begin{array}{c}
T[0, 0] \\
T[0, 0]
\end{array} \right] (m < m(\cdot))
\]  (5.12)
\[
\left[ \begin{array}{c}
T[0, 0] \\
T[0, 0]
\end{array} \right] (m = m(\cdot))
\]  (5.13)
\[
\left[ \begin{array}{c}
T[0, 0] \\
T[0, 0]
\end{array} \right] (m > m(\cdot))
\]  (5.14)
Moreover, we have
\[
\Gamma(x + m(\cdot)e_k) = 0,
\]  (5.15)
and
\[
\Gamma(x + m(\cdot)e_k) = 0.
\]  (5.16)

Proof  From Lemma 3, we get $\#S(\Psi) \geq 2$. Therefore we see that there exist $m(\cdot)(\leq 0)$ and $m(\cdot)(\geq 0)$ with $m(\cdot) < m(\cdot)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $|\alpha| + |\gamma| > 0$ and $|\beta| + |\delta| > 0$ such that
\[
\left[ \begin{array}{c}
\Psi^{2k-1}(x + me_k) \\
\Psi^{2k}(x + me_k)
\end{array} \right] = \left[ \begin{array}{c}
T[0, 0] \\
T[0, 0]
\end{array} \right] (m < m(\cdot))
\]  (5.17)
\[
\left[ \begin{array}{c}
T[0, 0] \\
T[0, 0]
\end{array} \right] (m = m(\cdot))
\]  (5.18)
\[
\left[ \begin{array}{c}
T[0, 0] \\
T[0, 0]
\end{array} \right] (m > m(\cdot))
\]  (5.19)
By Eq. (5.3) for $x + (m(\cdot) - 1)e_k$, we have
\[
\alpha \Psi^{2k-1}(x + (m(\cdot) - 2)e_k) + \psi^{2k}(x + (m(\cdot) - 1)e_k)
\]  (5.20)
\[
= \alpha \Psi^{2k}(x + m(\cdot)e_k) + \Psi^{2k-1}(x + (m(\cdot) - 1)e_k).
\]  (5.21)
Combining Eq. (5.15) with Eq. (5.16) gives
\[
\Psi^{2k}(x + m(\cdot)e_k) = \gamma = 0,
\]  (5.22)
since $\lambda \neq 0$. In a similar fashion, from Eq. (5.4) for $x + (m(\cdot) + 1)e_k$, we have
\[
\Psi^{2k-1}(x + m(\cdot)e_k) = \delta = 0.
\]  (5.23)
Thus combining Eqs. (5.15), (5.17) and (5.18) implies Eq. (5.12).

By Eq. (5.2) for $x + m(\cdot)e_k$, we have
\[
\alpha \Psi^{2k-1}(x + (m(\cdot) - 1)e_k) + \psi^{2k}(x + m(\cdot)e_k) = \frac{1}{d} \Gamma(x + m(\cdot)e_k).
\]  (5.24)
Then combining Eq. (5.19) with Eq. (5.12) gives
\[
\frac{1}{d} \Gamma(x + m(\cdot)e_k) = 0.
\]  (5.25)
Similarly, by Eq. (5.3) for $x + m(\cdot)e_k$ and Eq. (5.12), we get
\[
\frac{1}{d} \Gamma(x + m(\cdot)e_k) = 0.
\]  (5.26)
Therefore the proof of Lemma 4 is complete.

**Theorem 2** For the Grover walk on $\mathbb{Z}^d$ with flip-flop shift, we have

$$
\begin{cases}
\#(S(\Psi)) = 0 & (d = 1) \\
\#(S(\Psi)) \geq 4 & (d \geq 2)
\end{cases}
$$

(5.22)

for any $\Psi \in W(\lambda)$ with $\lambda = \pm 1$. In particular, there exists $\Psi^{(\lambda)}_\ast \in W(\lambda)$ such that

$$\#(S(\Psi)) = 4 \quad (d \geq 2)$$

(5.23)

for $\lambda = \pm 1$. In fact, we obtain

$$\Psi^{(\lambda)}_\ast(x) = \lambda^{x_1+x_2} \times T \left[ (-1)^{x_1+x_2}x_1, (-1)^{x_1+x_2+1}x_2, 0, \ldots, 0 \right] \quad (x \in S(\Psi^{(\lambda)}_\ast)),$n

where

$$S(\Psi^{(\lambda)}_\ast) = \{ x = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d : x_1, x_2 \in \{0, 1\}, x_3 = x_4 = \cdots = x_d = 0 \}.$$n

Here $|0\rangle = T[1, 0], |1\rangle = T[0, 1]$ and $0 = T[0, 0]$.

**Proof** First, we consider $d = 1$ case. For $\Psi \in W(\lambda)$ with $\lambda = \pm 1$, there exists $x \in \mathbb{Z}$ such that

$$\begin{bmatrix} \Psi^1(x) \\ \Psi^2(x) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (5.26)$$

From Lemma 4, we have $m_1^{(-)} \leq 0$ and $m_1^{(+)} > 0$ with $m_1^{(-)} < m_1^{(+)}$ and $\alpha, \beta \in \mathbb{C}$ with $\alpha \beta \neq 0$ such that

$$\begin{bmatrix} \Psi^1(x + m_1) \\ \Psi^2(x + m_1) \end{bmatrix} = \begin{cases} T \left[ \begin{bmatrix} 0, 0 \end{bmatrix} \right] \quad (m_1 < m_1^{(-)}) \\
T \left[ \begin{bmatrix} \alpha, 0 \end{bmatrix} \right] \quad (m_1 = m_1^{(-)}) \\
T \left[ \begin{bmatrix} 0, \beta \end{bmatrix} \right] \quad (m_1 = m_1^{(+)}) \\
T \left[ \begin{bmatrix} 0, 0 \end{bmatrix} \right] \quad (m_1 > m_1^{(+)}).
\end{cases} \quad (5.27)$$

and

$$\begin{cases}
\Gamma(x + m_1^{(-)}) = 0 \\
\Gamma(x + m_1^{(+)}) = 0.
\end{cases} \quad (5.28)$$

By definition of $\Gamma$ and Eq. (5.27), we have

$$\begin{cases}
\Gamma(x + m_1^{(-)}) = \alpha \\
\Gamma(x + m_1^{(+)}) = \beta.
\end{cases} \quad (5.29)$$

Combining Eq. (5.28) with Eq. (5.29), we get $\alpha = \beta = 0$. So we see that the finite support for $d=1$ does not exist.

Next we deal with $d = 2$ case. For $\Psi \in W(\lambda)$ with $\lambda = \pm 1$, we assume that there exists $x \in \mathbb{Z}^2$ such that

$$\begin{bmatrix} \Psi^1(x) \\ \Psi^2(x) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (5.30)$$

and we put

$$\begin{cases}
m^{(-)} = 0 \\
m^{(+)} = 1.
\end{cases} \quad (5.31)$$

for Eq. (5.12) on Lemma 4 to minimize $\#(S(\Psi))$. By using (5.12) with Eq. (5.31), we have

$$\begin{bmatrix} \Psi^1(x) \\ \Psi^2(x) \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}. \quad (5.32)$$
By definition of $\Gamma$ with Eqs. (5.13), (5.31) and (5.32), we get

\[
\begin{bmatrix}
\Psi^3(x) \\
\Psi^4(x)
\end{bmatrix} \neq \begin{bmatrix}
0 \\
0
\end{bmatrix},
\]

since $\alpha \neq 0$.

Similarly, from Lemma 3 with Eq. (5.30), we can assume

\[
\begin{bmatrix}
\Psi^1(x + e_1) \\
\Psi^2(x + e_1)
\end{bmatrix} \neq \begin{bmatrix}
0 \\
0
\end{bmatrix},
\]

and we obtain

\[
\begin{bmatrix}
\Psi^3(x + e_1) \\
\Psi^4(x + e_1)
\end{bmatrix} \neq \begin{bmatrix}
0 \\
0
\end{bmatrix},
\]

since $\beta \neq 0$. From Lemma 3 with Eqs. (5.33) and (5.35), we obtain “$x - e_2 \in S(\Psi)$ or $x + e_2 \in S(\Psi)$” and “$x + e_1 - e_2 \in S(\Psi)$ or $x + e_1 + e_2 \in S(\Psi)$” respectively, so $\#(S(\Psi)) \geq 4$. In fact, we can construct a $\Psi^{(\lambda)} \in W(\lambda)$ with $\lambda = \pm 1$ satisfying $\#(S(\Psi^{(\lambda)})) = 4$ as follows.

\[
\Psi(x + m_1 e_1 + m_2 e_2) = \begin{cases}
T [1, 0, -1, 0] & (m_1, m_2) = (0, 0) \\
T [0, -\lambda, \lambda, 0] & (m_1, m_2) = (1, 0) \\
T [-\lambda, 0, 0, \lambda] & (m_1, m_2) = (0, 1) \\
T [0, 1, 0, -1] & (m_1, m_2) = (1, 1) \\
T [0, 0, 0, 0] & (\text{otherwise})
\end{cases}
\]

for $m_1, m_2 \in \mathbb{Z}$.

Finally, we consider $d \geq 3$ case by continuing the argument on $d = 2$ case. To expand Eq. (5.36) to $d \geq 3$, we focus on the fact that $\Gamma(x + m_1 e_1 + m_2 e_2) = 0$ for any $x \in \mathbb{Z}^2$ and $m_1, m_2 \in \mathbb{Z}$ in Eq. (5.36). By assuming $\Psi^{2k-1}(x + m_1 e_1 + m_2 e_2) = \Psi^{2k}(x + m_1 e_1 + m_2 e_2) = 0$ for any $k \in \{3, 4, \cdots, d\}$, we can construct a $\Psi^{(\lambda)} \in W(\lambda)$ with $\lambda = \pm 1$ satisfying $\#(S(\Psi^{(\lambda)})) = 4$ as follows.

\[
\Psi(x + m_1 e_1 + m_2 e_2) = \begin{cases}
T [1, 0, -1, 0, \cdots, 0] & (m_1, m_2) = (0, 0) \\
T [0, -\lambda, \lambda, 0, \cdots, 0] & (m_1, m_2) = (1, 0) \\
T [-\lambda, 0, 0, \lambda, \cdots, 0] & (m_1, m_2) = (0, 1) \\
T [0, 1, 0, -1, \cdots, 0] & (m_1, m_2) = (1, 1) \\
T [0, 0, 0, 0, \cdots, 0] & (\text{otherwise})
\end{cases}
\]

Theorem 2 can be derived from another approach based on the spectral mapping theorem, see Corollary 2 in Higuchi et al. [2].

6 Summary

We presented the minimum supports of states for the Grover walk on $\mathbb{Z}^d$ with moving and flip-flop shifts, respectively, by solving the eigenvalue problem $U_G \Psi = \lambda \Psi$. Results on the moving shift model was obtained by Theorem 1 which coincides with result in Stefának et al. [7] ($\mathbb{Z}^2$ case) and improves result in Komatsu and Konno [3] ($\mathbb{Z}^d$ case). Moreover, results on the flip-flop shift model shown by Higuchi et al. [2] was given by Theorem 2. One of the interesting future problems might be to clarify a relationship between the stationary measure and the time-averaged limit measure of the Grover walk on $\mathbb{Z}^d$. 

11
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