ON TRIANGULAR SIMILARITY OF NILPOTENT TRIANGULAR MATRICES

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Abstract. Let $B_n$ (resp. $U_n$, $N_n$) be the set of $n \times n$ nonsingular (resp. unit, nilpotent) upper triangular matrices. We use a novel approach to explore the $B_n$-similarity orbits in $N_n$. The Belitskii’s canonical form of $A \in N_n$ under $B_n$-similarity is in $QU_n$ where $Q$ is the subpermutation such that $A \in B_nQB_n$. Using graph representations and $U_n$-similarity actions stabilizing $QU_n$, we obtain new properties of the Belitskii’s canonical forms and present an efficient algorithm to find the Belitskii’s canonical forms in $N_n$. As a consequence, we construct new Belitskii’s canonical forms in all $N_n$’s, list all Belitskii’s canonical forms for $n = 7, 8$, and show examples of 3-nilpotent Belitskii’s canonical forms in $N_n$ with arbitrary numbers of parameters up to $O(n^2)$.

1. Introduction

Let $\mathbb{F}$ be a fixed field. Let $M_{m,n}$ (resp. $M_n$, $GL_n$) be the set of $m \times n$ (resp. $n \times n$, $n \times n$ nonsingular) matrices over $\mathbb{F}$. Let $B_n$ (resp. $U_n$, $N_n$) be the set of $n \times n$ nonsingular (resp. unit, nilpotent) upper triangular matrices, and $D_n$ the set of $n \times n$ nonsingular diagonal matrices, over $\mathbb{F}$.

The main goal of this paper is to describe the $B_n$-similarity orbits in $N_n$ through the Belitskii’s canonical forms. We link a $B_n$-similarity orbit to the corresponding $(B_n, B_n)$ double coset. Given $A \in N_n$, let $Q$ be the unique subpermutation such that $A \in B_nQB_n$. The Belitskii’s canonical form of $A$ under $B_n$-similarity is in $QU_n$. We improve the Belitskii’s algorithm to efficiently search for the Belitskii’s canonical forms using graph representations and graph operations on matrices in $QU_n$. As a consequence, all indecomposable Belitskii’s canonical forms for $n = 7$ and $n = 8$ are given, which extends the works of D. Kobal [10] and Y. Chen et al [5]. Moreover, we discover a way to obtain new indecomposable Belitskii’s canonical forms of any order $n$; we also present examples of 3-nilpotent Belitskii’s canonical forms in $N_n$ with arbitrary number of parameters up to $O(n^2)$, which improves the $O(n)$ result in [5].

The $B_n$-similarity orbits in $N_n$ is a special case of the $\Lambda$-similarity matrix problem explored by V. Sergeichuk in [15]. Sergeichuk showed how the $\Lambda$-similarity
can be used to formulate the representations of quivers and matrix problems [15, Examples 1.1, 1.2], and presented the Belitski˘ı’s algorithm to obtain so called the Belitski˘ı’s canonical form under Λ-similarity. The strengthen Tame-Wild theorem for matrix problem ([15, Theorem 3.1]) and the existing classification on the Belitski˘ı’s canonical forms with two parameters [5] indicate that the $B_n$-similarity problem on $N_n$ is of wild type.

In 1978, M. Roitman discovered that if $F$ is infinite, the number of $B_n$-similarity orbits in $N_n$ is infinite for $n \geq 12$ [14]. D. Djokovi´c and J. Malzan improved the result to $n \geq 6$ in 1980 [7]. D. Kobal in 2005 listed all Belitski˘ı’s canonical forms of the $B_n$-similarity orbits in $N_n$ for $n \leq 5$ [10]. P. Thijsse showed in 1997 that every upper triangular matrix is $B_n$-similar to a generalized direct sum of irreducible blocks, and gave a classification of indecomposable (non-Belitski˘ı’s) canonical forms for $n \leq 6$ [16]. Besides, Thijsse showed that if an upper triangular matrix $A$ is non-derogatory or $A$ has Jordan block sizes no more than 2, then $A$ is $B_n$-similar to a generalized Jordan canonical form. In 2016, Y. Chen et al classified the indecomposable Belitski˘ı’s canonical forms for $n = 6$ and for $n = 7$ which admits a parameter, and showed that there exists an indecomposable Belitski˘ı’s canonical form which admits at least $\left\lfloor \frac{n}{2} \right\rfloor - 2$ parameters [5].

When $F = \mathbb{C}$ or $\mathbb{R}$, the conjugacy orbits on nilpotent matrices or Lie algebra elements were also intensively investigated by Lie theorists and representation theorists. In the book [6] of D. Collingwood and W. McGovern, nilpotent $G$-orbits in semisimple Lie algebras $\mathfrak{g}$ are bijectively corresponding to the $G$-orbits of the standard $\mathfrak{sl}_2$-triples, and are parameterized by weighted Dynkin diagrams. L. Fresse gave sufficient and necessary conditions for the intersection of a nilpotent $GL_n$-orbit with $N_n$ to be a union of finitely many $B_n$-orbits [8]. A. Melnikov described the $B_n$-orbits and their geometry on upper triangular 2-nilpotent matrices by link patterns in [11, 12, 13]. M. Boos and M. Reineke described the $B_n$-orbits and their closure relations of all 2-nilpotent matrices [4]. N. Barnea and A. Melnikov described the Borel orbits of 2-nilpotent elements in nilradicals for the symplectic algebra in 2017 [1]. M. Boos et al described the parabolic orbits of 2-nilpotent elements for classical groups [2, 3].

The structure of this paper is as follows.

In Section 2, we review the classification and invariants of $(B_n, B_n)$ double cosets and the Belitski˘ı’s algorithm for the $B_n$-similarity. We show that the Belitski˘ı’s canonical form of $A \in N_n$ is necessarily in $QU_n$ in which $Q$ is the subpermutation such that $A \in B_nQB_n$ (Theorem 2.5). As a by product, we can construct new Belitski˘ı’s canonical forms $\begin{bmatrix} A_1 & Q_{12} \\ A_2 \end{bmatrix}$ when $A_1 \in Q_1U_p$ and $A_2 \in Q_2U_q$ are Belitski˘ı’s canonical forms and $\begin{bmatrix} Q_1 & Q_{12} \\ Q_2 \end{bmatrix}$ is a subpermutation in $N_{p+q}$ (Theorem 2.7). The criteria for $D_n$-similarity is given in Theorem 2.9. Finally, every matrix in $B_nQB_n$ for a subpermutation $Q \in N_n$ can be transformed via $B_n$-similarity to a matrix in
and this matrix can be transformed via elementary $U_n$-similarity operations (ESOs) stabilizing $QU_n$ to a matrix which is $D_n$-similar to the Belitskiĭ’s canonical form (Theorem 2.16).

Section 3 introduces the graph representations of matrices, and the graph operations corresponding to ESOs stabilizing $QU_n$. The graph operations visualize the $U_n$-similarity reduction process on $QU_n$ and help obtain the Belitskiĭ’s canonical forms efficiently.

Section 4 is devoted to explore the properties of the Belitskiĭ’s canonical form through its graph. The graph of a Belitskiĭ’s canonical form in $N_n$ with $m$ connected components and $N$ arcs has exactly $m$ indecomposable components and $N - n + m$ parameters (Theorem 4.1). Theorem 4.4 determines the places of parameters in a Belitskiĭ’s canonical form. Theorems 4.6 and 4.9 prove that some entries in a Belitskiĭ’s canonical form must be zero, and Theorem 4.8 describes the possible places of nonzero entries. Finally, Theorem 4.11 constructs indecomposable 3-nilpotent Belitskiĭ’s canonical forms with $r$ parameters for all $r \leq \left\lfloor \frac{n-2}{3} \right\rfloor$ if $n \equiv 0 \mod 3$, and $r \leq \left\lfloor \frac{n-2}{3} \right\rfloor - 1$ if $n \equiv 1 \mod 3$.

In Section 5, we give an efficient graphical algorithm to search for the Belitskiĭ’s canonical forms based on Theorems 4.8 and 4.9. The algorithm significantly improves the Belitskiĭ’s algorithm. The indecomposable Belitskiĭ’s canonical forms for $n = 7$ is given in Theorem 5.4, and those for $n = 8$ is given in Theorem 5.5 and the Appendix. Examples of the algorithm, graph illustrations of Theorem 2.7, and connections to the $B_n$-similarity orbits of upper triangular matrices are also included in this section.

2. Preliminary

2.1. $B_n \times B_n$ action on $N_n$. Given a subgroup $G$ of $GL_n$, two matrices $A, C \in M_n$ are called $G$-similar, denoted by $A \sim_G C$, if there exists $B \in G$ such that $C = BAB^{-1}$. The $A$ and $C$ are in the same $(B_n, B_n)$ double coset if there exist $B, B' \in B_n$ such that $C = BAB'$. The $B_n$-similarity orbit of $A \in M_n$ is contained in the $(B_n, B_n)$ double coset of $A$:

$$\{ BAB^{-1} \mid B \in B_n \} \subseteq B_n AB_n := \{ BAB' \mid B, B' \in B_n \}.$$

The $(B_n, B_n)$ double cosets on $M_n$ are well classified as an extension of both the Bruhat decomposition in semisimple Lie groups and Gelfand-Naimark decomposition in matrix theory. We review the results here.

Definition 2.1. A matrix $Q \in M_{m,n}$ is called a subpermutation if each of the rows and columns of $Q$ has at most one nonzero entry, which equals 1.

Let $[n] := \{1, 2, \ldots, n\}$. Given $i, j \in [n]$, let $E_{i,j}^{(n)} \in M_n$ (or $E_{ij}^{(n)}$ for simplicity) be the matrix that has 1 on the $(i, j)$ entry and 0’s elsewhere, and let $e_i^{(n)} \in \mathbb{F}^n$ be the vector that has 1 on the $i$th entry and 0’s elsewhere. They are abbreviated
as $E_{i,j}$ (or $E_{ij}$ for simplicity) and $e_i$, respectively, if the size $n$ is clear. Every subpermutation $Q \in M_n$ can be determined by a bijective map $\sigma : I \to \sigma(I)$ between two subsets $I$ and $\sigma(I)$ of $[n]$ of the same cardinality:

$$Q = \sum_{i \in I} E_{i,\sigma(i)}; \quad Q := 0 \text{ if } I = \emptyset.$$  

Given $A \in M_n$ and $I, J \subseteq [n]$, let $A[I,J]$ denote the submatrix of $A$ with rows indexed by $I$ and columns indexed by $J$. Moreover, given $i,j \in [n]$, let

$$(2.2) \quad r^{i,j}(A) := \text{rank } A[[n] \setminus [n-i],[j]] = \text{rank } A[[n-i+1,\ldots,n],[1,\ldots,j]]$$

be the rank of the lower left $i \times j$ submatrix of $A$; define $r^{0,j}(A) = r^{i,0}(A) := 0$.

The following characterization of $(B_n, B_n)$ double cosets on $M_n$ is classical. Analogic double coset results on $GL_n$ can be found in [9, Theorem 3.5.14].

**Lemma 2.2.** The $(B_n, B_n)$ double coset of $A \in M_n$ is completely determined by the set of invariants:

$$Q = \{r^{i,j}(A) : i, j \in [n]\}.$$  

There is a unique subpermutation $Q \in M_n$ such that $A \in B_nQB_n$. The entries of $Q = [q_{ij}]$ are determined by:

$$q_{n-i+1,j} = r^{i,j}(A) - r^{i-1,j}(A) - r^{i,j-1}(A) + r^{i-1,j-1}(A), \quad i, j \in [n].$$

**Proof.** Given arbitrary $B, B' \in B_n$ and $i, j \in [n]$, we look at $BAB'$ from the following partitions:

$$BAB' = \begin{bmatrix}
B_{11} & B_{12} \\
0 & B_{22}
\end{bmatrix} \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} \begin{bmatrix}
B'_{11} & B'_{12} \\
0 & B'_{22}
\end{bmatrix}$$

$$= \begin{bmatrix}
B_{22}A_{21}B'_{11} & * \\
* & *
\end{bmatrix}.$$

Both $B_{22} \in M_i$ and $B'_{11} \in M_j$ are nonsingular. Therefore, $r^{i,j}(BAB') = r^{i,j}(A)$.

Next, we illustrate how to transform $A = [a_{ij}] \in M_n$ to a subpermutation $Q$ through elementary row and column operations associated with multiplications of matrices in $B_n$.

1. Start from the last row of $A$. If it is a zero row, we are done for the row.
2. Otherwise, let $\sigma(n) \in [n]$ such that $a_{n,\sigma(n)}$ is the first nonzero entry of the row. For each $j \in [n] \setminus [\sigma(n)]$, add a multiple $(-a_{nj}/a_{n,\sigma(n)})$ of the $\sigma(n)$th column of $A$ to the $j$th column of $A$. These elementary column operations result in multiplying $A$ from the right by a matrix $B'_{(1)} \in B_n$. Denote $A'_1 = AB'_{(1)}$. Then for each $i \in [n-1]$, add a multiple $(-a_{i,\sigma(n)}/a_{n,\sigma(n)})$ of the $n$th row of $A'_1$ to the $i$th row of $A'_1$. These elementary row operations
result in multiplying \( A'_1 \) from the left by a matrix \( B_{(1)} \in B_n \). Denote a new matrix \( A_1 = \left[ a_{ij}^{(1)} \right] = B_{(1)}A'_1 = B_{(1)}AB'_1 \). Then \( e_{n,\sigma(n)}^{(1)} = a_{n,\sigma(n)} \) is the only nonzero entry of its row and column in \( A_1 \).

(2) Repeat the same strategy on the other rows of the new matrix in the reversing row order until all rows are done.

The above process produces a matrix \( Q' = B_sAB''_s \) in which each of the rows and columns has at most one nonzero entry. By multiplying an appropriate nonsingular diagonal matrix \( D' \) from the right, we get a subpermutation \( Q = B_sAB''_sD' = B_sAB'_s \) for some \( B_s, B'_s \in B_n \).

Clearly, \( r^{i,j}(Q) = r^{i,j}(A) \) for \( i, j \in [n] \cup \{0\} \). Moreover, given \( i, j \in [n] \), \( Q[[n] \setminus [n-i], [j]] \) has exactly one of the following forms \((k \in [j-1], l \in [i-1]):\)

\[
\begin{bmatrix}
0 & 1 \\
Q[[n \setminus [n-i+1], [j-1]] & 0 \\
Q[[n \setminus [n-i+1], [j-1]] & e_{l}^{(j-1)}
\end{bmatrix}, \quad \begin{bmatrix}
(e_{k}^{(j-1)})^T & 0 \\
Q[[n \setminus [n-i+1], [j-1]] & 0 \\
0 & 0
\end{bmatrix}.
\]

In all cases, the entries of subpermutation \( Q = [q_{ij}] \) can be obtained by:

\[
q_{n-i+1,j} = r^{i,j}(Q) - r^{i-1,j}(Q) - r^{i,j-1}(Q) + r^{i-1,j-1}(Q), \quad i,j \in [n].
\]

Therefore, the set of invariants \( \{r^{i,j}(A) : i,j \in [n]\} \) completely determines the unique subpermutation \( Q \) and the corresponding \( (B_n, B_n) \) double coset of \( A \). \( \square \)

If two matrices are similar and in the same \( (B_n, B_n) \) double coset, are they necessarily \( B_n \)-similar? The answer is no.

**Example 2.3.** Let \( A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \). Both \( A \) and \( B \) have the only eigenvalue 0, and \( \text{rank}(A^m) = \text{rank}(B^m) \) for all \( m \in \mathbb{Z}^+ \). So \( A \) and \( B \) are similar. They are also in the same \( (B_n, B_n) \) double coset represented by the subpermutation \( B \). However, \( A \) and \( B \) are not \( B_n \)-similar \([10, \text{Theorem 2}]\).

The \( (B_n, B_n) \) double coset provides a good direction to explore the \( B_n \)-similarity orbits it includes. Suppose \( A = BQB' \in N_n \) where \( B, B' \in B_n \) and \( Q \in N_n \) is a subpermutation. Then \( A \overset{B_2}{\sim} QB'B \). Write \( B'B = DU \) for \( D \in D_n \) and \( U \in U_n \). Since \( Q \in N_n \), there exists \( D' \in D_n \) such that \( D'QD(D')^{-1} = Q \). Then

\[
A \overset{B_2}{\sim} QB'B = QDU \overset{B_2}{\sim} D'QDU(D')^{-1} = QD'U(D')^{-1} \in QU_n.
\]

The coset \( QU_n \) takes the following form.

**Lemma 2.4.** Suppose \( Q = \sum_{i \in I} F_{i,\sigma(i)} \in M_n \) is a subpermutation. Then \( A = [a_{ij}] \in QU_n \) if and only if \( A \) meets the following conditions:

1. \( A \) and \( Q \) have the same places of nonzero rows indexed by \( I \);
(2) \( A \) and \( Q \) have the same places and values of the first nonzero entry in each nonzero row; precisely, for each \( i \in I \), \( a_{i,\sigma(i)} = 1 \) is the first nonzero entry of the \( i \)th row of \( A \).

The proof can be done by direct computation.

### 2.2. The Belitskiĭ’s algorithm for the \( B_n \)-similarity in \( \mathbb{N}_n \)

V. Sergeichuk presented the Belitskiĭ’s algorithm to find a canonical form, the Belitskiĭ’s canonical form, for the \( \Lambda \)-similarity matrix problem [15]. On the \( B_n \)-similarity of \( A \in \mathbb{N}_n \), the algorithm can be described as follows.

1. List the matrix entry positions above the diagonal in a reversal row lexicographical order “\( \prec \)” called the Belitskiĭ’s order:

\[
(n - 1, n) \prec (n - 2, n - 1) \prec (n - 2, n) \prec (n - 3, n - 2) \prec \cdots \prec (1, n).
\]

The strictly upper triangular entries will be normalized through \( B_n \)-similarity one-by-one in this order.

2. (Normalizing the first entry) Let \((A^{(0)}, B^{(0)}) := (A, B_n)\). Find \( A^{(1)} \) in the \( B^{(0)} \)-similarity orbit of \( A^{(0)} = [a_{ij}] \) such that the \((n - 1, n)\) entry of \( A^{(1)} \) is either 0 or 1. For example,

\[
A^{(1)} := \begin{cases} 
A & \text{if } a_{n-1,n} = 0, \\
(I_{n-1} \oplus [a_{n-1,n}])A(I_{n-1} \oplus [a_{n-1,n}])^{-1} & \text{if } a_{n-1,n} \neq 0.
\end{cases}
\]

Denote the group

\[
B^{(1)} := \{ g \in B^{(0)} | gA^{(1)}g^{-1} \text{ fixes the value of the } (n - 1, n) \text{ entry of } A^{(1)} \}.
\]

3. (Normalizing the consequent entries) Suppose \((A^{(k)}, B^{(k)}) \) has been determined, and the group \( B^{(k)} \) fixes the first \( k \) entries of \( A^{(k)} = [a'_{ij}] \) in the Belitskiĭ’s order. Let \((p, q)\) be the \((k + 1)\)th entry position. There are three situations for the \((p, q)\) entry of matrices \( C = [c_{ij}] \) in the \( B^{(k)} \)-similarity orbit of \( A^{(k)} \):

   \(a\) \( c_{p,q} \) is always 0, or \( c_{p,q} \) could take any value of \( F \): we find \( A^{(k+1)} = [a''_{ij}] \sim B^{(k)} A^{(k)} \) such that \( a''_{p,q} = 0 \);

   \(b\) \( c_{p,q} \) could take any value of \( F \setminus \{0\} \): we find \( A^{(k+1)} = [a''_{ij}] \sim B^{(k)} A^{(k)} \) such that \( a''_{p,q} = 1 \);

   \(c\) otherwise, \( c_{p,q} \equiv \lambda \) for a fixed \( \lambda \in F \setminus \{0\} \): we choose \( A^{(k+1)} = A^{(k)} \) with \( a''_{p,q} = \lambda \).

Let \( B^{(k+1)} \) denote the subgroup of \( B^{(k)} \) that fixes the \((k + 1)\)th entry value as well as the first \( k \) entry values of \( A^{(k+1)} \).

4. Repeat the preceding step until the last position in the Belitskiĭ’s order is reached. Denote the last pair \((A^\infty, B^\infty)\). The matrix \( A^\infty \) is called the Belitskiĭ’s canonical form of \( A \) under the \( B_n \)-similarity.
The above algorithm shows that each upper triangular entry of the Belitskiǐ’s canonical form $A^\infty$ is 0 or 1 or a parameter $\lambda$ in which different $\lambda$ values correspond to different $B_n$-similarity orbits. This property is similar to that of a Jordan canonical form. Moreover, the Belitskiǐ’s canonical form $A^\infty$ has the following connection to the subpermutation $Q$ in the $(B_n, B_n)$ double coset of $A$ and $A^\infty$.

**Theorem 2.5.** Given a Belitskiǐ’s canonical form $A \in N_n$, if $A \in B_nQB_n$ in which $Q$ is a subpermutation, then $A \in QU_n$.

**Proof.** The proof is done by induction on $n$. $n = 1$ is obviously true. Suppose the statement holds for all $n < m$. Given $A \in B_mB_m$ where $Q \in N_m$ is a subpermutation, write $A = \begin{bmatrix} 0 & a^T \\ A_1 \end{bmatrix}$ for $A_1 \in N_{m-1}$ and $a \in F^{m-1}$. By the Belitskiǐ’s algorithm, $A_1$ is a Belitskiǐ’s canonical form in $N_{m-1}$. Write $Q = \begin{bmatrix} 0 & b^T \\ Q_1 \end{bmatrix}$ in which $Q_1$ is a subpermutation in $N_{m-1}$ and $b \in F^{m-1}$. Then $A_1 \in B_{m-1}Q_1B_{m-1}$. So by induction hypothesis $A_1 = Q_1\hat{U}$ for $\hat{U} \in U_{m-1}$.

1. If $a = 0$, then Lemma 2.2 implies that $Q = \begin{bmatrix} 0 \\ Q_1 \end{bmatrix}$. Hence $A = \begin{bmatrix} 0 & 0 \\ Q_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \hat{U} \end{bmatrix} \in QU_m$.

2. If $a \neq 0$, let $A = [a_{ij}]$ and let $a_{1q}$ ($q \in \{2, \ldots, m\}$) be the leading nonzero entry in the first row of $A$. Then

$$A \sim B_m \begin{bmatrix} a_{1q} & 0 \\ I_{m-1} \end{bmatrix}^{-1} A \begin{bmatrix} a_{1q} & 0 \\ I_{m-1} \end{bmatrix} = \begin{bmatrix} 0 & a_{1q}^{-1}a^T \\ A_1 \end{bmatrix}$$

in which the last matrix has the leading entry 1 on the $(1, q)$ position. By the Belitskiǐ’s algorithm $a_{1q} = 1$. We claim that there is no $p \in \{2, \ldots, m\}$ such that $a_{pq}$ is the leading nonzero entry of the $p$th row of $A$ (i.e. the $(p-1)$th row of $A_1$). Otherwise,

$$A \sim B_m \begin{bmatrix} 1 & \frac{1}{a_{pq}}(e_{p-1}^{(m-1)})^T \\ I_{m-1} \end{bmatrix}^{-1} A \begin{bmatrix} 0 & a^T \\ A_1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{a_{pq}}(e_{p-1}^{(m-1)})^T \\ I_{m-1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & a^T - \frac{1}{a_{pq}}(e_{p-1}^{(m-1)})^TA_1 \\ A_1 \end{bmatrix}$$

where the first row of the last matrix has at least $q$ leading zeros; contradicting the Belitskiǐ’s algorithm. By Lemma 2.4, the $(q-1)$th column of $Q_1$ is zero. Using (2.4), we have $Q = \begin{bmatrix} 0 \\ (e_{q-1}^{(m-1)})^T \\ Q_1 \end{bmatrix}$. Let $\hat{U}(a^T)$ denote
the matrix obtained by replacing the \((q - 1)\)th row of \(\hat{U}\) by \(a^T\). Then

\[ A = \begin{bmatrix} 0 & (e_{q-1})^T \vspace{10pt} \\
Q_1 & 1 \\
\hat{U}(a^T) & 0 \end{bmatrix} \in QU_m. \]

Overall, the statement holds for \(n = m\) and the induction process is completed. □

**Remark 2.6.** A Belitskii’s canonical form needs not be in \(U_nQ\) or \(B_nQ\). See the examples in Theorems 5.2, 5.4, and 5.5.

The direct sums of Belitskii’s canonical forms are obviously Belitskii’s canonical forms. Moreover, Theorem 2.5 implies a way to combine Belitskii’s canonical forms together through certain subpermutations to form a new Belitskii’s canonical form, as shown below.

**Theorem 2.7.** Suppose \(A_1 \in N_p\) and \(A_2 \in N_q\) are Belitskii’s canonical forms, in which \(A_1 \in Q_1U_p\) and \(A_2 \in Q_2U_q\) for subpermutations \(Q_1 \in N_p\) and \(Q_2 \in N_q\). If \(Q_{12} \in M_{p,q}\) such that \(\begin{bmatrix} Q_1 & Q_{12} \\
0 & Q_2 \end{bmatrix}\) is a subpermutation, then \(\begin{bmatrix} A_1 & Q_{12} \\
0 & A_2 \end{bmatrix}\) is a Belitskii’s canonical form in \(N_{p+q}\).

**Proof.** Let \(A = \begin{bmatrix} A_{11} & A_{12} \\
0 & A_{22} \end{bmatrix}\) (\(A_{11} \in N_p\)) be the Belitskii’s canonical form of \(A' := \begin{bmatrix} A_1 & Q_{12} \\
0 & A_2 \end{bmatrix}\). Then \(A_{22} = A_2\) by the Belitskii’s algorithm.

Let \(Q := \begin{bmatrix} Q_1 & Q_{12} \\
0 & Q_2 \end{bmatrix}\). Write \(A_1 = Q_1U'\) and \(A_2 = Q_2U''\) for \(U' \in U_p\) and \(U'' \in U_q\). Then the nonzero rows of \(A' = \begin{bmatrix} Q_1U' & Q_{12} \\
0 & Q_2U'' \end{bmatrix}\) have the same places and values (i.e., 1) of leading nonzero entries as the nonzero rows of \(Q\) do. Therefore, \(A' \in QU_{p+q}\) by Lemma 2.4, and \(A \in QU_{p+q}\) by Theorem 2.5.

Now consider \(A_{11}\) and \(A_{12}\). One one hand, each nonzero entry of the subpermutation \(Q_{12}\) equals the corresponding row leading nonzero entry of \(A_{12}\). On the other hand, \(A'_{11} \sim_{p+q} A\) implies that \(A_1 \sim_{p} A_{11}\); \(A_{11}\) cannot be further reduced from the Belitskii’s canonical form \(A_1\) in the Belitskii’s algorithm. Therefore, \(A_{11} = A_1\) and \(A_{12} = Q_{12}\) by the Belitskii’s algorithm, so that \(A = \begin{bmatrix} A_1 & Q_{12} \\
0 & A_2 \end{bmatrix}\) is a Belitskii’s canonical form. □

**Remark 2.8.** In Theorem 2.7, the form of the Belitskii’s canonical form \(\begin{bmatrix} A_1 & Q_{12} \\
0 & A_2 \end{bmatrix}\) could have more parameters in nonzero entries of \(A_1\) and \(A_2\) than in the original Belitskii’s canonical forms \(A_1\) and \(A_2\). For an example, see the case \(A_1 = A_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{bmatrix}\) and \(Q_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \end{bmatrix}\) in Example 5.6.
2.3. $B_n$, $D_n$, and $U_n$ Similarities. On the group level, $B_n = D_n \ltimes U_n$. Two matrices $A \cong C$ if and only if $C = BAB^{-1}$ for $B \in B_n$ and $B = UD$ such that $D \in D_n$ and $U \in U_n$, so that $A \cong DAD^{-1}U \cong C$. The $D_n$-similarity on $M_n$ is easy to classify.

In this paper, $A \in M_n$ is called indecomposable if no permutation matrix $P \in M_n$ satisfies that $PAP^T$ can be written as a direct sum of two proper principal submatrices. The notation is different from that in [5], but they are identical when referring to an indecomposable Belitskii’s canonical form.

Given $A \in M_n$ and $i, j \in [n]$, let us define

$$f_{ij}(A) := \begin{cases} a_{ij} & \text{if } a_{ij} \neq 0, \\ \frac{1}{a_{ji}} & \text{if } a_{ij} = 0 \text{ but } a_{ji} \neq 0, \\ 0 & \text{if } a_{ij} = a_{ji} = 0. \end{cases}$$

(2.7)

**Theorem 2.9.** Two matrices $A = [a_{ij}], C = [c_{ij}] \in M_n$ have $A \cong C$ if and only if the following two conditions hold:

1. $A$ and $C$ have the same places of nonzero entries, namely, $a_{ij} \neq 0$ if and only if $c_{ij} \neq 0$; and
2. for every sequence $(i_1, \ldots, i_p)$ of distinct elements in $[n]$ such that at least one of $a_{i_1 i_2}$ and $a_{i_{k+1} i_k}$ is nonzero for each $k \in [p]$ (let $i_{p+1} := i_1$), we have the identity

$$f_{i_1 i_2}(A) \cdots f_{i_{p-1} i_p}(A) f_{i_p i_1}(A) = f_{i_1 i_2}(C) \cdots f_{i_{p-1} i_p}(C) f_{i_p i_1}(C).$$

(2.8)

**Proof.** Suppose $C = DAD^{-1}$ where $D = \text{diag}(d_1, \ldots, d_n)$ is nonsingular. Then $c_{ij} = \frac{d_i}{d_j} a_{ij}$ for $i, j \in [n]$. Conditions (1) and (2) in the theorem obviously hold.

Conversely, we use induction on $n$ to prove that (1) and (2) imply $A \cong C$. $n = 1$ is true. Suppose the claim holds for all cases of $n < m$. Now for $n = m$, let $A, C \in M_n$ satisfy (1) and (2). If $A$ is not indecomposable, then there is a permutation matrix $P$ such that $PAP^T$ and $PCP^T$ are direct sums of respective proper principal submatrices. So by induction hypothesis $PAP^T \cong PCP^T$ and $A \cong C$. Otherwise, $A$ is indecomposable. We find $d_1, \ldots, d_n \in F \setminus \{0\}$ as follows such that $c_{ij} = \frac{d_i}{d_j} a_{ij}$ for $i, j \in [n]$. Let $S_0 := \{1\}$ and $d_1 := 1$.

1. Since $A$ is indecomposable, there are $j \in [n] \setminus S_0$ such that $a_{1j} \neq 0$ or $a_{j1} \neq 0$, in which we define

$$d_j := \begin{cases} d_1 a_{1j} & \text{if } a_{1j} \neq 0, \\ d_1 a_{j1} & \text{if } a_{1j} = 0 \text{ and } a_{j1} \neq 0. \end{cases}$$

(2.9)

In the case $a_{1j} \neq 0$ and $a_{j1} \neq 0$, (2.8) gives $a_{1j} a_{j1} = c_{1j} c_{j1}$ so that the $d_j$ defined by (2.9) satisfies both $c_{1j} = \frac{d_j}{d_1} a_{1j}$ and $c_{j1} = \frac{d_i}{d_1} a_{j1}$. Let

$$S_1 := S_0 \cup \{j \in [n] \setminus S_0 : a_{1j} \neq 0 \text{ or } a_{j1} \neq 0\}.$$
Then $S_1 \supseteq S_0$ and $c_{ij} = \frac{d_i}{d_j} a_{ij}$ for $i, j \in S_1$.

(2) If $S_1 \neq [n]$, then $A$ being indecomposable implies that $a_{ij} \neq 0$ or $a_{ji} \neq 0$ for some $(i, j) \in S_1 \times ([n] \setminus S_1)$, in which we define

$$d_j := \begin{cases} \frac{d_i}{a_{ij}} & \text{if } a_{ij} \neq 0, \\ \frac{d_i}{a_{ji}} & \text{if } a_{ij} = 0, a_{ji} \neq 0. \end{cases}$$

Let

$$S_2 := S_1 \cup \{j \in [n] \setminus S_1 : a_{ij} \neq 0 \text{ or } a_{ji} \neq 0 \text{ for some } i \in S_1\}.$$ 

Then $S_2 \supseteq S_1$ and $c_{ij} = \frac{d_i}{d_j} a_{ij}$ for $i, j \in S_2$ by (2.8).

(3) Repeat the process until we reach $S_m = [n]$, where all $d_j$ for $j \in [n]$ are well-defined. Let $D := \text{diag} (d_1, \ldots, d_n)$ then $C = DAD^{-1}$ as desired. \hfill \Box

Theorem 2.9 shows that: if $A \in M_n$ is transformed via a $B_n$-similarity action to $C \in M_n$, the zero places of $C$ are determined by the associated $U_n$-similarity transformation. The identities (2.8) in Theorem 2.9 will also be used to determine the places of parameters in a Belitskii’s canonical form.

The matrix group $U_n$ is generated by

$$\{I_n + \lambda E_{pq} : \lambda \in \mathbb{F}, \ p, q \in [n], \ p < q \}.$$ 

**Definition 2.10.** Given $\lambda \in \mathbb{F}$, $(p, q) \in [n] \times [n]$ and $p < q$, we define an elementary $U_n$-similarity operation (ESO) to be the function $O_{p,q}^\lambda : N_n \to N_n$ such that for $A = [a_{ij}] \in N_n$:

$$O_{p,q}^\lambda (A) := (I_n + \lambda E_{pq}) A (I_n + \lambda E_{pq})^{-1} = (I_n + \lambda E_{pq}) \left( \sum_{i,j=1}^n a_{ij} E_{ij} \right) (I_n - \lambda E_{pq}) = A + \sum_{j \in [n]} \lambda a_{jq} E_{pj} - \sum_{i \in [n]} \lambda a_{ip} E_{iq}.$$ 

(2.12)

Each $O_{p,q}^\lambda$ is also called an $O_{p,q}$-operation.

The ESOs will be described by graph operations in Section 3.

**Lemma 2.11.** Given $U \in U_n$, write $U = I_n + \sum_{k=1}^m u_{ik,jk} E_{ik,jk}$ where $(i_1, j_1) \prec (i_2, j_2) \prec \cdots \prec (i_m, j_m)$ in the Belitskii’s order (2.6). Then

$$U = (I_n + u_{i_1,j_1} E_{i_1,j_1}) \cdots (I_n + u_{i_m,j_m} E_{i_m,j_m}).$$ 

(2.13)

**Proof.** Left multiply $(I_n + u_{i_1,j_1} E_{i_1,j_1})^{-1}$ onto $U$. The matrix $(I_n + u_{i_1,j_1} E_{i_1,j_1})^{-1} U = (I_n - u_{i_1,j_1} E_{i_1,j_1}) U$ is the one that eliminates the $(i_1, j_1)$ entry of $U$. Keep left multiplying $(I_n + u_{i_2,j_2} E_{i_2,j_2})^{-1}, \ldots, (I_n + u_{i_m,j_m} E_{i_m,j_m})^{-1}$ in order. We will have $(I_n + u_{i_1,j_1} E_{i_1,j_1})^{-1} \cdots (I_n + u_{i_m,j_m} E_{i_m,j_m})^{-1} U = I_n$. So (2.13) holds. \hfill \Box
Remark 2.12. Given $U \in U_n$, if we write $U^{-1} = I_n - \sum_{k=1}^m u'_{ik,jk} E_{ik,jk}$ where $(i_1,j_1) \prec (i_2,j_2) \prec \cdots \prec (i_m,j_m)$, then (2.13) implies that

$$U^{-1} = (I_n - u'_{i_1,j_1} E_{i_1,j_1}) \cdots (I_n - u'_{i_m,j_m} E_{i_m,j_m})$$

so that

$$U = (I_n + u'_{i_1,j_1} E_{i_1,j_1}) \cdots (I_n + u'_{i_m,j_m} E_{i_m,j_m}).$$

Lemma 2.13. Let $S \subseteq \{(i,j) \in [n] \times [n] : i < j \}$ such that

$$U_S := \{I_n + \sum_{(i,j) \in S} a_{ij} E_{ij} : a_{ij} \in \mathbb{F} \}$$

is a subgroup of $U_n$. Then $U_S$ is generated by $\{I_n + \lambda E_{ij} : (i,j) \in S, \lambda \in \mathbb{F} \}$, and each element of $U_S$ can be written as a product of no more than $|S|$ elements in $\{I_n + \lambda E_{ij} : (i,j) \in S, \lambda \in \mathbb{F} \}$.

Proof. It is a direct consequence of Lemma 2.11. \hfill \square

Given a subpermutation $Q$, the coset $QU_n$ is not closed under the $U_n$-similarity. However, the following result indicates that $U_n$-similar matrices in $QU_n$ can be transformed to each other via finitely many ESOs stabilizing $QU_n$.

Theorem 2.14. Let $Q \in N_n$ be a subpermutation. Let $A, C \in QU_n$ such that $A \overset{U_n}{\sim} C$. Then there exist a sequence of ESOs $\{O_{i_1,j_1}^{\lambda_1}, \ldots, O_{i_m,j_m}^{\lambda_m} \}$, $\lambda_k \in \mathbb{F}$ and $1 \leq i_k < j_k \leq n$ for $k \in [m]$, such that the followings conditions hold:

1. $(i_1,j_1) \prec (i_2,j_2) \prec \cdots \prec (i_m,j_m)$ in the Belitskii’s order (2.6).
2. Let $A_0 := A$ and for $k \in [m]$:

$$A_k := O_{i_k,j_k}^{\lambda_k}(A_{k-1}) = (I_n + \lambda_k E_{i_k,j_k})A_{k-1}(I_n + \lambda_k E_{i_k,j_k})^{-1}.$$

Then $A_0, A_1, \ldots, A_m \in QU_n$ and $A_m = C$.

Proof. Let $Q = \sum_{i \in I} E_{i,\sigma(i)}$ as in (2.1). Let $A = QU'$ and $C = UAU^{-1} = UQU'U^{-1}$ for $U, U' \in U_n$. Write $U = I_n + [u_{ij}]$ where $[u_{ij}] \in N_n$. By direct computation, $C \in QU_n$ if and only if $UQ \in QU_n$, and if only if the nonzero $u_{ij}$ entries have the pairs $(i,j)$ in the set

$$S_Q := \{(i,j) \in I \times I : i < j, \sigma(i) < \sigma(j)\} \cup \{(i,j) \in [n] \times ([n] \setminus I) : i < j\}.$$ 

Therefore, the group $\{T \in U_n : TAT^{-1} \in QU_n\} = U_{S_Q}$ which is generated by $\{I_n + \lambda E_{ij} : (i,j) \in S_Q, \lambda \in \mathbb{F} \}$ according to Lemma 2.13. Moreover, $U^{-1} \in U_{S_Q}$. If we write

$$U^{-1} = I_n - \sum_{k=1}^m \lambda_k E_{i_k,j_k},$$

in which $\lambda_k \in \mathbb{F} \setminus \{0\}$, $(i_k,j_k) \in S_Q$ and $(i_1,j_1) \prec \cdots \prec (i_m,j_m)$ in the Belitskii’s order, then by Lemma 2.11, $U^{-1} = (I_n - \lambda_1 E_{i_1,j_1}) \cdots (I_n - \lambda_m E_{i_m,j_m})$ and

$$C = (I_n + \lambda_m E_{i_m,j_m}) \cdots (I_n + \lambda_1 E_{i_1,j_1})A(I_n + \lambda_1 E_{i_1,j_1})^{-1} \cdots (I_n + \lambda_m E_{i_m,j_m})^{-1}.$$
So Theorem 2.14 (1) and (2) are proved. □

**Remark 2.15.** Theorem 2.14 also holds if we replace condition (1) by the condition: \((i_1, j_1) \succ (i_2, j_2) \succ \cdots \succ (i_m, j_m)\) in the Belitskii’s order (2.6).

**Theorem 2.16.** If \(A \in QU_n\) and \(Q \in N_n\) is a subpermutation, then \(A\) can be transformed via a finite number of ESOs stabilizing \(QU_n\) to a matrix \(\widetilde{A}^\infty \in QU_n\) which is \(D_n\)-similar to the Belitskii’s canonical form \(A^\infty \in QU_n\).

Proof. Since \(B_n = D_n \times U_n\), there exist \(D \in D_n\) and \(U \in U_n\) such that \(A^\infty = (DU)A(DU)^{-1} = D(UAU^{-1})D^{-1}\). Let \(\widetilde{A}^\infty := UAU^{-1}\). We first prove that \(\widetilde{A}^\infty \in QU_n\). Notice that \(A^\infty \in QU_n\) by Theorem 2.5, and \(\widetilde{A}^\infty \) and \(A^\infty = DA\overline{A}D^{-1}\) have the same places of nonzero entries by Theorem 2.9. Using Lemma 2.4, it suffices to show that the leading nonzero entry of each nonzero row of \(A^\infty\) equals 1. Let \(R_i(C)\) denote the \(i\)th row of a matrix \(C\). By \(A \in QU_n\), we have \(AU^{-1} \in QU_n\) so that all nonzero rows \(R_i(AU^{-1})\) have distinct places of leading nonzero entries 1.

Suppose \(R_i(\tilde{A}^\infty)\) is a nonzero row for a given \(i\). Then \(R_i(A^\infty)\), \(R_i(A^\infty)\), and \(R_i(AU^{-1})\) have the same places of leading nonzero entries as \(Q\) does. Moreover, every \(u_{i,j} \neq 0\) for \(i < j \leq n\) implies that either \(R_j(AU^{-1})\) is zero or the place of leading nonzero entry of \(R_j(AU^{-1})\) is after that of \(R_i(AU^{-1})\). Therefore, the leading nonzero entry of \(R_i(\tilde{A}^\infty)\) equals that of \(R_i(AU^{-1})\), namely 1. We get \(\widetilde{A}^\infty \in QU_n\).

Finally, Theorem 2.14 shows that \(A\) can be transformed via a finite number of ESOs stabilizing \(QU_n\) to \(\widetilde{A}^\infty\), and \(\widetilde{A}^\infty\) is \(D_n\)-similar to \(A^\infty\). □

In summary, here is a simplification process to get the Belitskii’s canonical form \(A^\infty\) of a given \(A \in N_n\) under the \(B_n\)-similarity:

1. Use elementary row and column operations (cf. the proof of Lemma 2.2) to factorize \(A = BQB'\) for \(B, B' \in B_n\) and \(Q \in N_n\) is the subpermutation determined by \(\{r^{i,j}(A) : i, j \in [n]\}\). Then \(A \overset{B_n}{\sim} QB'B\).
2. Write \(B'B = DU\) for \(D \in D_n\) and \(U \in U_n\). Find \(D' \in D_n\) such that \(D'QU(D')^{-1} = Q\). Then
   \[QB'B = QDU \overset{D_n}{\sim} D'QDU(D')^{-1} = QD'U(D')^{-1} \in QU_n.\]
3. Use a sequence of ESOs stabilizing \(QU_n\) to simplify \(QD'U(D')^{-1}\) to a matrix \(\widetilde{A}^\infty\) which is \(D_n\)-similar to \(A^\infty\) (cf. Theorem 2.14 and Theorem 2.16). Then determine \(A^\infty\) (cf. Theorem 2.9).

We will explore the details of step (3) above in the coming sections.
3. Graph representations and graph operations

In this section, given a subpermutation \( Q \in N_n \), we use graph representations to visualize matrices in \( QU_n \), then use graph operations to visualize ESOs on matrices in \( QU_n \).

3.1. Graph representation of matrices in \( QU_n \). Every \( A = [a_{ij}] \in M_n \) is the adjacency matrix of a directed graph \( G_A = (V_A, E_A) \) with a weight function \( w_A : [n] \times [n] \to \mathbb{F} \setminus \{0\} \) whose support is \( E_A \). Precisely,

\[
(3.1) \quad V_A = [n]; \quad E_A = \{(i, j) \in [n] \times [n] : a_{ij} \neq 0\}; \quad w_A(i, j) = a_{ij}.
\]

Each element of \( V_A \) (resp. \( E_A \)) is called a vertex (resp. an arc) of the graph \( G_A \). Each arc \((i, j) \in E_A\) is visualized as \( i \to j \), in which \( i \) (resp. \( j \)) is called the tail (resp. the head) of the arc \((i, j)\), and \( w_A(i, j) \) is called the weight of the arc \((i, j)\).

Call \( G_A = (V_A, E_A) \) the graph of \( A \), and \( \tilde{G}_A = (V_A, E_A, w_A) \) the weighted graph of \( A \), respectively.

When \( A \in N_n \), the graph of \( A \) is simple and it consists of some arcs \((i, j) \in [n] \times [n] \) with \( i < j \).

A partition of \( [n] \) has the form \([n] = S_1 \cup \cdots \cup S_m \) where each partition subset \( S_i \neq \emptyset \). For the uniqueness of expression, we assume that the minimal elements of \( S_1, \ldots, S_m \) are in ascending order, and write the partition as \( \tilde{S}_1 | \cdots | \tilde{S}_m \) where \( \tilde{S}_i \) is the list of elements of \( S_i \) in ascending order. For example, the partition \( \{5, 6\} \cup \{7, 3\} \cup \{2, 4, 1\} \) of \( [7] \) will be expressed as \( 1243756 \) (for \( n > 9 \), we will add spaces between neighboring numbers).

**Lemma 3.1.** Given a subpermutation \( Q \in N_n \), the graph \( G_Q \) of \( Q \) consists of finite connected components, each of which is a directed path of the form:

\[
(3.2) \quad i_1 \to i_2 \to \cdots \to i_p, \quad i_1 < \cdots < i_p, \quad p \in \mathbb{Z}^+.
\]

There is a bijective correspondence between the set of all subpermutations in \( N_n \) and the set of all partitions of \([n]\), in which \( Q \) corresponds to the partition \( \mathcal{P}_Q \) of the union of the sets \( \{i_1, \ldots, i_p\} \), namely, the \((i, j)\) entry of \( Q \) is nonzero if and only if \( i < j \) are sequential elements in a partition subset of \( \mathcal{P}_Q \).

**Proof.** Since \( Q \in N_n \), the graph \( G_Q \) only contains arcs \((i, j)\) with \( i < j \). Since \( Q \) is a subpermutation, each row and column of \( Q \) has at most one nonzero entry, so that each vertex \( i \) of \( G_Q \) is the head (resp. the tail) of at most one arc. Therefore, each connected component of \( G_Q \) must have the form \((3.2)\). The rest is obvious. \( \square \)

We call each connected component subgraph \((3.2)\) of \( G_Q \) a chain of \( G_Q \). So the graph \( G_Q \) is a union of finite disconnected chains. \( G_Q \) is connected if and only if \( Q \) is indecomposable. When \( Q \) is fixed, in the chain \((3.2)\):

- \( i_1 \) (resp. \( i_p \)) is called the chain tail (resp. the chain head) of the chain \((3.2)\);
• for each $k \in [p - 1]$, $i_{k+1}$ is called the \textit{chain successor} of $i_k$, denoted by $i_k^+ = i_{k+1}$; and $i_k$ is called the \textit{chain predecessor} of $i_{k+1}$, denoted by $i_{k+1}^- = i_k$.

We call the partition $\mathcal{P}_Q$ in Lemma 3.1 the \textit{partition of $Q$}. $\mathcal{P}_Q$ also determines the permutation matrices $P$ in which each $PQP^T$ is a direct sum of indecomposable submatrices.

Lemma 2.4 for a subpermutation $Q \in N_n$ can be rephrased in graphs as follows.

**Lemma 3.2.** $A \in M_n$ is in $QU_n$ for a subpermutation $Q \in N_n$ if and only if the weighted graph of $A$ satisfies the following conditions:

1. $E_A \supseteq E_Q$ and the weights $w_A(i, j) = w_Q(i, j) = 1$ for all $(i, j) \in E_Q$;
2. each $(i, j) \in E_A \setminus E_Q$ satisfies that $i$ is not the chain head of any chain of $G_Q$, and $i < i^+ < j$ where $(i, i^+) \in E_Q$.

**Proof.** Suppose $A \in QU_n$ in which $Q \in N_n$ is a subpermutation. Write $Q = \sum_{i \in I} E_{i, \sigma(i)}$ for $I \subseteq [n]$. Then $i^+ = \sigma(i)$ for all $i \in I$. Moreover, $i$ is a chain head of $G_Q$ if and only if $i \in [n] \setminus I$. Lemma 2.4 (2) shows that $G_A$ contains $G_Q$ as a weighted subgraph. Given $(i, j) \in E_A \setminus E_Q$, Lemma 2.4 (1) shows that $i$ is not a chain head of $G_Q$, and Lemma 2.4 (2) and the assumption $Q \in N_n$ show that $i < \sigma(i) = i^+ < j$.

The converse statement also holds by Lemma 2.4. \hfill \Box

In Lemma 3.2, $G_A$ contains $G_Q$ as a subgraph. When $A \in QU_n$ for a subpermutation $Q$, we call each element of $E_A \setminus E_Q$ an \textit{extra arc} of $G_A$. We denote the \textit{graph type} of $A$ as $\mathcal{P}_Q : i_1j_1|\cdots|i_tj_t$ where $\mathcal{P}_Q$ is the partition corresponding to $Q$ and $(i_1, j_1), \ldots, (i_t, j_t)$ are the extra arcs of $G_A$ listed in ascending Belitskiǐ’s order (2.6). If $A$ has no extra arc (i.e., $A = Q$), its graph type is denoted as $\mathcal{P}_Q : \emptyset$. The graph type of $A$ is a concise expression of the graph $G_A$.

**Example 3.3.** Let $n = 7$. Let

\[
Q = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \in N_7, \quad A = Q \begin{bmatrix}
1 & * & * & * & * & * & * \\
1 & 0 & 3 & -2 & 0 & 1 & * \\
1 & * & * & * & 1 & -1 & 0 \\
1 & * & * & * & 1 & 1 & 0 \\
0 & 1 & 0 & 3 & -2 & 0 & 1 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 
\end{bmatrix}.
\]

The subpermutation $Q$ corresponds to the partition $\mathcal{P}_Q = 124|37|56$ of [7]. The graph of $Q$ is $G_Q = ([7], E_Q)$ in which $E_Q = \{(1, 2), (2, 4), (3, 7), (5, 6)\}$. The graph of $A$ is $G_A = ([7], E_A)$ in which $E_A = E_Q \cup \{(1, 4), (1, 5), (1, 7), (2, 5)\}$. So the graph type of $A$ is $124|37|56 : 25|14|15|17$.

In the graph on the right, $\overline{G}_Q$ consists of three chains formed by black arcs with weights 1, and $\overline{G}_A$ has the extra arcs with weights marked in red. By Lemma 3.2, an arc like $(4, 7)$ or $(3, 5)$ cannot be an extra arc of $G_A$. 

\[
\begin{array}{c}
\text{2} \quad 3 \quad 4 \\
\text{1} \\
\text{5} \quad 6 \\
\end{array}
\]
3.2. The elementary $U_n$-similarity graph operations on $QU_n$. The ESO in Definition 2.10 can be rephrased using graph operations. Let $\lambda \in F$ and $(p,q) \in [n] \times [n]$ with $p < q$. For $A = [a_{ij}] \in N_n$, (2.12) shows that

$$O_{p,q}^\lambda(A) = A + \sum_{j \in [n]} \lambda a_{qj} E_{pj} - \sum_{i \in [n]} \lambda a_{ip} E_{iq}.$$  

Let $A' = [a'_{ij}] := O_{p,q}^\lambda(A)$. The changes made by $O_{p,q}^\lambda$ from $G_A$ to $G_{A'}$ are below:

1. whenever $(i,p) \in E_A$ (i.e. $a_{ip} \neq 0$),
   $$w_{A'}(i,q) = a'_{iq} = a_{iq} - \lambda a_{ip} = w_A(i,q) - \lambda w_A(i,p);$$
2. whenever $(q,j) \in E_A$ (i.e. $a_{qj} \neq 0$),
   $$w_{A'}(p,j) = a'_{pj} = a_{pj} + \lambda a_{qj} = w_A(p,j) + \lambda w_A(q,j).$$

These changes are visualized as follows, in which a red arc indicates a change of the weight, and a dashed arc indicates that the weight may be zero:

$$\begin{align*}
\text{By abuse of language, we also call transformation (3.3) the elementary $U_n$-similarity operation (ESO) } O_{p,q}^\lambda \text{ on the weighted graph } G_A, \text{ denoted by } O_{p,q}^\lambda(G_A) = G_{A'}.
\end{align*}$$

Given a matrix $A = [a_{ij}] \in QU_n$ where $Q \in N_n$ is a subpermutation, Theorem 2.16 shows that $A$ could be transformed via a sequence of ESOs stabilizing $QU_n$ to a matrix $A^\infty \in QU_n$ which is $D_n$-similar to the Belitskii’s canonical form $A^\infty$. By Theorem 2.9, $A^\infty$ and $A^\infty$ have the same places of nonzero entries, that is, $G_{A^\infty} = G_{A^\infty}$, and the relation of weight functions (2.8) holds on each undirected cycle of $G_{A^\infty}$. Therefore, we will use ESOs (3.3) to eliminate “redundant arcs” on $G_A$ following the Belitskii’s order until we reach $G_{A^\infty}$; then we adjust the weights on undirected cycles of $G_{A^\infty}$ following the Belitskii’s order and get $G_{A^\infty}$.

**Lemma 3.4.** Let $A = [a_{ij}] \in N_n$. An arc $(i,j)$ of $G_A = ([n], E_A, w_A)$ can be eliminated by an ESO only if one of the following two cases happens:

1. there is $p$ such that $i < p < j$ and $(i,p) \in E_A$, in which $O_{p,j}^\lambda(G_A)$ for $\lambda = \frac{a_{ip}}{a_{ip}}$ has no arc $(i,j);$  
2. there is $q$ such that $i < q < j$ and $(q,j) \in E_A$, in which $O_{q,j}^\lambda(G_A)$ for $\lambda = \frac{a_{iq}}{a_{iq}}$ has no arc $(i,j);$

**Proof.** The statement is a direct consequence of (3.3). \[\square\]
By Theorem 2.16, when \( A \in QU_n \) for a subpermutation \( Q \in N_n \), we should try to eliminate the extra arcs of \( \tilde{G}_A \) by ESOs stabilizing \( QU_n \). So in practice, not every ESO satisfying conditions in Lemma 3.4 will be considered.

**Example 3.5.** Let subpermutation \( Q \in N_7 \) and matrix \( A \in QU_7 \) be given in Example 3.3. The extra arcs in \( \tilde{G}_A \) sorted by the Belitskiï’s order are: \((2, 5) \prec (1, 4) \prec (1, 5) \prec (1, 7)\). We test and eliminate them by ESOs stabilizing \( QU_7 \) in this order. The first arc \((2, 5)\) cannot be eliminated, since the only type of ESOs that can modify the weight of \((2, 5)\) is \( O_{4,5} \) which creates the arc \((4, 6)\) and does not stabilize \( QU_7 \). Then:

\[
\begin{align*}
\begin{array}{c}
1 \rightarrow 2 & 4 \\
2 \rightarrow 3 & 5 \rightarrow 6 \\
1 \rightarrow 2 \rightarrow 4 & 1 \rightarrow 2 & 4 \rightarrow 5 \rightarrow 6
\end{array}
\end{align*}
\]

Hence \( O_{2,7}^2 O_{4,6}^2 O_{2,5}^2 O_{2,4}(A) = \tilde{A}^\infty \) in which \( \tilde{G}_{\tilde{A}^\infty} \) is the last weight graph above. There is no undirected cycle on \( \tilde{G}_{\tilde{A}^\infty} \). So \( \tilde{A}^\infty \) is \( D_7 \)-similar to the Belitskiï’s canonical form \( A^\infty \) whose weighted graph has weight 1 on the arc \((2, 5)\). In other words,

\[
A \Upsilon_{2,7} A^\infty = Q - E_{2,5} \Upsilon_{2,7} A^\infty = Q + E_{2,5}.
\]

The elimination process in Example 3.5 may be roughly abbreviated as changes on graph types as below, where an appropriate operation on the first row causes the changes of arcs listed on the second row.

\[
(3.4) \quad O_{2,4} \quad O_{2,5} \quad O_{4,6} \quad O_{2,7}
\]

So the Belitskiï’s canonical form of \( A \) has the type 124|37|56 : 25|14|15|17 \(-14 \quad -15 + 26 \quad -26 \quad -17 \quad = 25\)

Another observation about Example 3.5 is that: unlike those ESOs in Theorem 2.14, in \( O_{2,7}^2 O_{4,6}^2 O_{2,5}^2 O_{2,4}(A) = \tilde{A}^\infty \), the pairs \((2, 4), (2, 5), (4, 6), (2, 7)\) do not completely follow the Belitskiï’s order \( \prec \). However, we check and (if possible) eliminate
the extra arcs of $\tilde{G}_A$ following the Belitskiǐ’s order; the success of this process is guaranteed by the combination of the Belitskiǐ’s algorithm and Theorem 2.14.

4. Properties of the Belitskiǐ’s canonical forms under $B_n$-similarity

Given a matrix $A \in B_n Q B_n$ or $Q U_n$ where $Q \in N_n$ is a subpermutation, Theorem 2.5 shows that the Belitskiǐ’s canonical form $A^\infty \in Q U_n$. Here we investigate the nonzero entries in $A^\infty$, or equivalently, what extra arcs and weights could be in $\tilde{G}_{A^\infty}$. For simplicity, we assume that $A$ is already a Belitskiǐ’s canonical form.

4.1. Characterization of the Belitskiǐ’s canonical form. The (2.8) in Theorem 2.9 indicates that if the graph of a Belitskiǐ’s canonical form has an undirected cycle, then at least one arc of this undirected cycle has a parameter weight. It derives the following results.

**Theorem 4.1.** Let $A \in N_n$ be a Belitskiǐ’s canonical form. If $G_A$ has $m$ connected components and $|E_A| = N$, then $A$ has $m$ indecomposable components and $N - n + m$ parameters.

**Proof.** If $G_A$ has $m$ connected components, then the vertex sets of these $m$ connected subgraphs form a partition of $[n]$. For each permutation matrix $P \in M_n$, the graphs $G_{P A P^T}$ and $G_A$ are isomorphic. There is a permutation matrix $P$ such that the vertex set of each connected component of $G_{P A P^T}$ contains sequential integer(s). Then $P A P^T$ is a direct sum of $m$ principal submatrices, each of which is indecomposable. In other words, $A$ has $m$ indecomposable components.

If a connected component of $G_A$ has $n_1$ vertices and $r_1$ arcs, then $r_1 \geq n_1 - 1$. When $r_1 = n_1 - 1$, the connected component contains no undirected cycle so that all weights of its arcs are 1 by Theorem 2.9 and the Belitskiǐ’s algorithm. When $r_1 > n_1 - 1$, the connected component can be obtained by adding $r_1 - n_1 + 1$ arcs to a connected subgraph with $n_1 - 1$ arcs, and adding each arc creates an undirected cycle on the union of this arc and the subgraph. Therefore, by Theorem 2.9, there are $r_1 - n_1 + 1$ parameter weights on the arcs of this connected component.

Summing over all $m$ connected components of $G_A$, we see that $A$ has $N - n + m$ parameters. \(\square\)

**Remark 4.2.** In matrix way, Theorem 4.1 says that: if a Belitskiǐ’s canonical form $A \in N_n$ is permutation similar to a direct sum of $m$ indecomposable squared submatrices, and $A$ has $N$ nonzero entries, then $A$ has $N - n + m$ parameters.

The following two results describe an indecomposable Belitskiǐ’s canonical form and its graph. They show that if the graph type or the places of nonzero entries of a Belitskiǐ’s canonical form are known, then we can determine the amount and the places of parameters among these nonzero entries.
Corollary 4.3. Let \( A \in N_n \) be a Belitskiǐ's canonical form. Then \( A \) is indecomposable if and only if the graph \( G_A = ([n], E_A) \) is connected. Moreover, if \( A \) is indecomposable with \( N \) nonzero entries, then \( A \) has \( N - n + 1 \) parameters.

Theorem 4.4. Let \( A \in QU_n \) be an indecomposable Belitskiǐ's canonical form in which \( Q \in N_n \) is a subpermutation. List the extra arcs of \( G_A \) (i.e. the elements of \( E_A \setminus E_Q \)) in the Belitskiǐ's order:

\[(i_1, j_1) \prec (i_2, j_2) \prec \cdots \prec (i_t, j_t).\]

Then the places of parameters of \( A \) (if any) correspond to the marked extra arcs determined by the following steps, starting at the graph \( G := G_Q \) in which all arcs in \( E_Q \) are unmarked:

1. add the extra arcs of \( G_A \) one at a time to \( G \) according to the Belitskiǐ's order (4.1).
2. when adding an extra arc \((i, j)\) to \( G \) creates an undirected cycle in which none of the arcs is marked, mark the extra arc \((i, j)\) and continue;
3. repeat the steps (1) and (2) until all extra arcs of \( G_A \) are gone through.

Proof. Since \( A \in QU_n \), the parameters of \( A \) appear only in the entries corresponding to extra arcs.

In step (2), when adding an extra arc \((i, j)\) results in an undirected cycle in which none of the arcs is marked, we may assume that the undirected cycle has distinct vertices by removing redundant subcycles. By Theorem 2.9 (2), the undirected cycle contains at least one arc with a parameter weight to represent the scalar in (2.8). Moreover, \((i, j)\) is the last arc in the Belitskiǐ’s order in this undirected cycle. So by the Belitskiǐ’s algorithm, the parameter weight in the undirected cycle should be on \((i, j)\).

After step (3), if we remove all marked arcs from \( G_A \) then the remaining subgraph does not have any undirected cycle. By Theorem 2.9, \( A \) is \( D_n \)-similar (and thus \( B_n \)-similar) to a matrix whose unmarked arcs have weights 1 and marked arcs have parameter weights.

The normalization steps (1), (2), (3) allow us to place the parameters of \( A \) in accordance with the Belitskiǐ’s algorithm. So these steps determine the places of parameters.

Example 4.5. An analysis similar to Example 3.5 shows that: every matrix \( A \in QU_8 \) of the graph type 123678|45 : 46|24|14 has no extra arc in \( G_A \) that can be eliminated by ESOs stabilizing \( QU_8 \). So \( A \) is a Belitskiǐ’s canonical form, which is indecomposable since \( G_A \) is connected. Corollary 4.3 shows that \( A \) has 2 parameters, and Theorem 4.4 shows that the parameters appear in the \((2, 4)\) and \((1, 4)\) entries. So \( \tilde{G}_A \) and \( A \) have the forms \((\lambda, \mu \in \mathbb{F} \setminus \{0\})\):
\[ \tilde{G}_A : 1 \rightarrow 2 \rightarrow 3 \quad 4 \rightarrow i \rightarrow 6 \rightarrow 7 \rightarrow 8 \]

All such Belitskii’s canonical forms may be represented by the graph type with additional underlines indicating parameters, namely [123678: 46: 24: 14].

4.2. Extra arcs in the graph of the Belitskii’s canonical form. Fix a subpermutation \( Q = \sum_{i \in I} E_{i, \sigma(i)} \in N_n \). Using the notations in Section 3.1, the graph \( G_Q \) consists of \( n - |I| \) chains in which the set \( S_h \) of chain heads and the set \( S_t \) of chain tails are

\begin{equation}
S_h = [n] \setminus I, \quad S_t = [n] \setminus \sigma(I).
\end{equation}

We also denote the maps

\begin{equation}
I \rightarrow \sigma(I), \quad i \mapsto i^+ := \sigma(i), \quad \text{and} \quad \sigma(I) \rightarrow I, \quad j \mapsto j^- := \sigma^{-1}(j).
\end{equation}

Given a Belitskii’s canonical form \( A \in QU_n \), Lemma 2.4 and its graph version Lemma 3.2 give a description of the entries of \( A \). In this subsection, we further explore what entries of \( A \) should be zero, namely, what extra arcs should not be in \( G_A \).

**Theorem 4.6.** Let \( A = [a_{ij}] \in QU_n \) be a Belitskii’s canonical form in which \( Q = \sum_{i \in I} E_{i, \sigma(i)} \in N_n \) is a subpermutation. Then for \( i \in I \), \( (i, j) \notin E_A \) (i.e. \( a_{ij} = 0 \)) when one of the following situations happen:

1. \( i^+ < j \in S_h \);
2. \( j \notin S_t \) and \( i < j^- \).

In particular, if \( i \) and \( j \) are on the same chain of \( G_Q \) but \( j \neq i^+ \), then \( (i, j) \notin E_A \).

**Proof.** We prove by contradictions that \( A \) cannot be a Belitskii’s canonical form if an arc \( (i, j) \in E_A \) satisfies (1) or (2) of Theorem 4.6. The idea is to find a matrix \( A' \in QU_n \) such that \( A' \sim A \) and

\begin{equation}
E_A \subseteq (E_A \setminus \{(i, j)\}) \cup \{(i', j') \in [n] \times [n] : i' < j', \ (i, j) \prec (i', j')\},
\end{equation}

which contradicts the Belitskii’s algorithm to get the Belitskii’s canonical form \( A \).

1. Suppose \( (i, j) \in E_A \) such that \( i^+ < j \in S_h \). By Lemma 3.4, there is \( \lambda_1 \in F \) such that the graph of \( A_1 := O^{\lambda_1}_{i^+, j}(A) \) contains no arc \( (i, j) \). By Lemma 3.2 (2), \( j \in S_h \) is not the tail of any arc of \( \tilde{G}_A \). Thus by (3.3), \( E_{A_1} \setminus E_A \) only contains some \( (i_1, j) \) in which \( (i_1, i^+) \in E_A \) and \( i_1 \neq i \) so that \( i_1 \in I \)
and $i_1^+ < i_1^+$ by Lemma 3.2 (2).

The changes from $G_A$ to $G_{A_1}$ are illustrated on the right, in which the dashed blue arc is removed and some solid blue arcs are added.

Similarly, for each $(i_1, j) \in E_{A_1} \setminus E_A$, an appropriate $O_{i_1^+, j}$-operation will remove the arc $(i_1, j)$ from the graph and add the arcs $(i_2, j)$ in which $(i_2, i_1^+) \in E_A$ and $i_2 \neq i_1$ so that $i_2 < i_1$. Repeating the process results in $i_1^+ > i_2^+ > i_3^+ > \cdots$. However, the process cannot go on forever. Hence by a finite steps of ESOs we can remove $(i, j)$ from $G_A$ as well as all arcs created by these ESOs. In other words, we get $A' = A - a_{ij}E_{ij}$ such that $A' \sim U A$. This contradicts the assumption that $A$ is a Belitskii’s canonical form. Therefore, $(i, j) \notin E_A$.

(2) Suppose $(i, j) \in E_A$ such that $j \notin S_h$ and $i < j^-$. By Lemma 3.4, there is $\lambda \in F$ such that the graph of $A' := O_{i,j^-}^\lambda (A)$ contains no arc $(i, j)$. By (3.3), $E_{A'} \setminus E_A$ only contains the following possible arcs:

(a) $(h, j^-) \in E_{A'} \setminus E_A$ in which $(h, i) \in E_A$. In such a case, $h < i$ so that $(h, j^-) > (i, j)$ in the Belitskii’s order.

(b) $(i, k) \in E_A \setminus E_A$ in which $(j^-, k) \in E_A$ and $k \neq j$. In such a case, $j^- < j < k$ by Lemma 3.2 (2) so that $(i, k) > (i, j)$ in the Belitskii’s order.

Overall, we get $A' \sim U A$ in which $E_{A'}$ satisfies (4.4). It contradicts the assumption that $A$ is a Belitskii’s canonical form. Therefore, $(i, j) \notin E_A$. □

The less intuitive matrix version of Theorem 4.6 is as follows.

**Theorem 4.7.** Let $A = [a_{ij}] \in QU_n$ be a Belitskii’s canonical form where $Q = \sum_{i \in I} E_{i, \sigma(i)} \in N_n$ is a subpermutation. Then $a_{ij} = 0$ whenever:

1. $j \in [n] \setminus (I \cup \{\sigma(i)\})$, or
2. $j \in \sigma(I)$ and $i < \sigma^{-1}(j)$.

In particular, $a_{ij} = 0$ if $j = \sigma^k(i)$ for some integer $k > 1$.

Theorem 4.6 is equivalent to the following result which gives a characterization of the possible arcs in a Belitskii’s canonical form.

**Theorem 4.8.** Let $A = [a_{ij}] \in QU_n$ be a Belitskii’s canonical form in which $Q = \sum_{i \in I} E_{i, \sigma(i)} \in N_n$ is a subpermutation. Then $(i, j) \in E_A$ (i.e. $a_{ij} \neq 0$) implies $i \in I = [n] \setminus S_h$ and one of the following:
\( \text{(1) } j = i^+ \text{ (where } a_{ij} = 1). \quad \begin{array}{c}
 i \quad \xrightarrow{i} \quad i^+ = j \\
 \end{array} \)

\( \text{(2) } j \in S_i \setminus S_h \text{ and } i^+ < j. \quad \begin{array}{c}
 i \quad \xrightarrow{i} \quad i^+ < j \quad \in S_i \setminus S_h \quad \xrightarrow{j} \quad j^+ \\
\end{array} \)

\( \text{(3) } j \notin S_i \cup S_h \text{ and } j^- < i < i^+ < j. \quad \begin{array}{c}
 i \quad \xrightarrow{i} \quad i^+ < j \quad \notin S_i \cup S_h \quad \xrightarrow{j} \quad j \notin S_i \cup S_h \\
\end{array} \)

In particular, given \( i \in I \), there is at most one vertex \( j \) in each chain of \( G_Q \) such that \( (i, j) \in E_A \).

\textbf{Proof.} The case (1) is \( (i, j) \in E_Q \). The cases (2) and (3) cover those extra arcs \( (i, j) \) not included in Theorem 4.6 (1) and (2).

It remains to prove the last claim. Suppose \( (i, j) \in E_A \) and the vertex \( j \) is in a chain \( G' \) of \( G_Q \). If the chain \( G' \) contains the vertex \( i \), then \( j = i^+ \). Otherwise, \( j \) is the lowest vertex number in the chain \( G' \) such that \( j > i^+ \).

\textbf{Theorem 4.9.} Let \( A = [a_{ij}] \in QU_n \) be a Belitskiǐ’s canonical form in which \( Q = \sum_{i \in I} E_{i, \sigma(i)} \in N_n \) is a subpermutation. Given \( (i, j) \in [n] \times [n] \) with \( i < j \), suppose there exist \( m \in \mathbb{N} \) and sequences \( i_0 = i, i_1, \ldots, i_m \in [n] \) and \( j_0 = j, j_1, \ldots, j_m \in [n] \) such that all of the following conditions hold for \( p \in [m] \):

\( i = i_0 \quad \xrightarrow{i} \quad i_1 \quad \xrightarrow{i} \quad i_2 \quad \cdots \quad i_{m-1} \quad \xrightarrow{i} \quad i_m \)

\( j = j_0 \quad \xrightarrow{j} \quad j_1 \quad \xrightarrow{j} \quad j_2 \quad \cdots \quad j_{m-1} \quad \xrightarrow{j} \quad j_m \)

\( 1 \) \( (i_{p-1}, i_p) \) is the only arc in \( G_A \) whose head is \( i_p \).

\( 2 \) \( (j_{p-1}, j_p) = (j_{p-1}', j_{p-1}') \) is the only arc in \( G_A \) whose tail is \( j_{p-1} \).

\( 3 \) \( i_p < j_{p-1} \).

\( 4 \) \( i_m \notin S_h \) but \( j_m \in S_h \).

Then \( (i, j) \notin E_A \).

\textbf{Proof.} Suppose on the contrary \( (i, j) = (i_0, j_0) \in E_A \). There exists \( \lambda_1 \in \mathbb{F} \) such that the graph of \( A_1 := O_{i_0, j_0}^\lambda(A) \) does not contain the arc \( (i, j) \). Then either \( E_{A_1} = E_A \setminus \{(i, j)\} \) or \( E_{A_1} = (E_A \setminus \{(i, j)\}) \cup \{(i_1, j_1)\} \). However, \( E_{A_1} = E_A \setminus \{(i, j)\} \) is impossible since \( A \) is a Belitskiǐ’s canonical form. So \( E_{A_1} = (E_A \setminus \{(i, j)\}) \cup \{(i_1, j_1)\} \). Similarly, applying a sequence of appropriate \( O_{i_2, j_1}, \ldots, O_{i_{m-1}, j_{m-1}} \) operations to \( A_1 \), we will get \( A_m \overset{\cup_n}{\sim} A_1 \overset{\cup_n}{\sim} A \) such that

\( E_{A_m} = (E_A \setminus \{(i, j), (i_1, j_1), \ldots, (i_{m-1}, j_{m-1})\}) \cup \{(i_m, j_m)\} \).

Since \( i_m \notin S_h \) and \( j_m \in S_h \), the proof of Theorem 4.6 indicates that there is \( A_{m+1} \overset{\cup_n}{\sim} A_m \overset{\cup_n}{\sim} A \) such that

\( E_{A_{m+1}} = E_{A_m} \setminus \{(i_m, j_m)\} = E_A \setminus \{(i, j), (i_1, j_1), \ldots, (i_m, j_m)\} \).

It contradicts the assumption that \( A \) is a Belitskiǐ’s canonical form. \( \square \)
Example 4.10. Let $Q \in N_8$ be the subpermutation with $P_Q = 12368|457$. Consider the possible Belitskii’s canonical forms $A \in QU_8$. Theorem 4.8 implies that the possible extra arcs in $E_A \setminus E_Q$ are $(1, 4), (2, 4)$, and $(4, 6)$. By Theorem 4.9, neither $(4, 6)$ nor $(1, 4)$ can be in a Belitskii’s canonical form. Therefore, the only indecomposable Belitskii’s canonical form in $QU_8$ is of the graph form $12368|457: 24$.

4.3. Possible numbers of parameters in a Belitskii’s canonical form. In [5, Theorem 2.4], Chen et al showed that for $n \geq 6$, there exists an indecomposable Belitskii’s canonical form in $N_n$ which admits at least $\left\lfloor \frac{n^2}{3} \right\rfloor - 2$ parameters. Note that a matrix in $N_n$ has up to $n(n-1)$ nonzero entries. We show below the existence of indecomposable 3-nilpotent Belitskii’s canonical forms with arbitrary number of parameters up to $O(n^2)$.

Theorem 4.11. Let $n, r \in \mathbb{N}$ such that $n \geq 6$ and

$$r \leq \begin{cases} \frac{1}{2} \left\lfloor \frac{n-2}{3} \right\rfloor \left(\left\lfloor \frac{n-2}{3} \right\rfloor - 1 \right) & \text{if } n \equiv 0, 2 \mod 3, \\ \frac{1}{2} \left\lfloor \frac{n-2}{3} \right\rfloor \left(\left\lfloor \frac{n-2}{3} \right\rfloor - 1 \right) - 1 & \text{if } n \equiv 1 \mod 3. \end{cases}$$

Then there exists an indecomposable Belitskii’s canonical form $A \in N_n$ with $r$ parameters, and $A$ has the minimal polynomial $x^3$.

Proof. We construct the desired Belitskii’s canonical forms for $n \geq 6$ according to $n \mod 3$:

1. When $n = 3m$, choose the subpermutation $Q \in N_{3m}$ with

$$P_Q = 1 \ 2m \ 2m + 1 \ | \ 2 \ 2m - 1 \ 2m + 2 \ | \ \cdots \ | \ m \ m + 1 \ 3m.$$  

Let $G = ([3m], E)$ be the graph containing $G_Q$ as a subgraph and

$$E \setminus E_Q = \{(i, j) \in [3m] \times [3m] : 2 \leq i \leq m, \ 2m + 2 - i \leq j \leq 2m\}.$$  

The graph $G$ is illustrated as below ($1 \leq i_1 < i_2 < i_3 < \ldots < i_n$):

\[\begin{array}{c}
  i_1 \\
  i_2 \\
  i_3 \\
  i_4 \\
  i_5 \\
  i_6 \\
  i_7 \\
  i_8 \\
  i_9 \\
  i_{10} \\
  i_{11} \\
  i_{12}
\end{array}\]

Let $A \in QU_n$ such that $G_A = G$ and the places of parameters in $A$ are given by Theorem 4.4. We claim that $A$ is a Belitskii’s canonical form.

By (2.17), the ESOs stabilizing $QU_n$ are those $Q^\lambda_{p,q}$ in which either

(a) $(p, q) \in [m] \times ([2m] \setminus [m])$, or
(b) \( p < q \) and \( q \in [3m] \setminus [2m] \).

In both cases, these ESOs only changes the weights of \((i, j)\) such that \(i < j\) and \( j \in [3m] \setminus [2m] \). None of the weights of extra arcs in \( E \setminus E_Q \) can be modified by the ESOs stabilizing \( QU_n \). So \( A \) is a Belitskii’s canonical form.

The number of extra arcs of \( A \) is \( 1 + 2 + \cdots + (m - 1) = \frac{1}{2}(m - 1)m \). By Theorem 4.1, the number of parameters in \( A \) is

\[
\frac{1}{2} (m - 1)m + 2m - 3m + 1 = \frac{1}{2} (m - 1)(m - 2).
\]

Given \( r \in \{0, 1, \ldots, \frac{1}{2}(m - 1)(m - 2) - 1\} \), we can remove \( \frac{1}{2}(m - 1)(m - 2) - r \) extra arcs from the graph \( G = G_A \) and keep the remaining graph connected. The resulting graph is the graph of an indecomposable Belitskii’s canonical form with \( r \) parameters. So (4.5) is true for \( n \equiv 0 \pmod 3 \).

(2) When \( n = 3m - 1 \), let \( G \) be the subgraph of the graph in \( n = 3m \) case, obtained by removing the vertex \( 3m \) and the arc \((m + 1, 3m)\). See the illustrated graph below. Similar argument shows that there is a Belitskii’s canonical form \( A \) with \( G_A = G \), and (4.5) is true for \( n \equiv 2 \pmod 3 \).

(3) When \( n = 3m - 2 \), let \( G \) be the subgraph of the graph in \( n = 3m - 1 \) case, obtained by removing the vertex \( 3m - 1 \) and the arcs \((m + 2, 3m - 1)\) and \((m, m + 2)\). See the illustrated graph below. Similarly, there is a Belitskii’s canonical form \( A \) with \( G_A = G \), and (4.5) is true for \( n \equiv 1 \pmod 3 \).

The graphs of the above Belitskii’s canonical forms show that the minimal polynomials of these Belitskii’s canonical forms are \( x^3 \). \( \square \)

5. Searches of the Belitskii’s canonical forms

5.1. Algorithms to search for the Belitskii’s canonical forms. We apply the results in Sections 2-4 to get the following efficient algorithm to obtain the Belitskii’s canonical forms under the \( B_n \)-similarity for a given \( n \).

Algorithm:

1. List all subpermutations \( Q \) in \( N_n \) (by the set partitions \( \mathcal{P}_Q \) of \([n]\)).
(2) For each subpermutation \( Q \), apply Theorems 4.8 and 4.9 to filter out a set \( S \) of possible extra arcs of the Belitskii’s canonical forms in \( QU_n \) and list them in the Belitskii’s order, say, \( S := \{(i_1, j_1) \prec \cdots \prec (i_m, j_m)\} \).

(3) Explore all possible combinations of the above extra arcs that produce Belitskii’s canonical forms. To do this, let \( S_0 := \emptyset \) and start at \( p = 1 \):  
(a) Determine whether there exists an \( U_n \)-similarity operation \( g \) composed by ESOs stabilizing \( QU_n \) such that \( g \) changes the graph \( ([n], E_Q \cup S_{p−1} \cup \{(i_p, j_p)\}) \) to a graph \( ([n], E') \) where 
\[
E' \subseteq E_Q \cup S_{p−1} \cup \{(i, j) \in S : (i, j) \succ (i_p, j_p)\}.
\]
If such a \( g \) exists, then the arc \( (i_p, j_p) \) can be removed by the \( g \) operation from the graph of any \( A \in QU_n \) whose set of extra arcs no greater than \( (i_p, j_p) \) is \( S_{p−1} \cup \{(i_p, j_p)\} \); we let \( S_p := S_{p−1} \). Otherwise, divide the upcoming process into two cases \( S_p := S_{p−1} \cup \{(i_p, j_p)\} \) and \( S_p := S_{p−1} \).
(b) Increase \( p \) by 1. If \( p \leq n \), repeat the preceding process.
(c) The outcoming sets \( S_m \) are the sets of extra arcs of all Belitskii’s canonical forms in \( QU_n \). Apply Theorem 4.4 to determine the places of parameters for each graph type of the Belitskii’s canonical forms.

The above algorithm can be restricted to search for only the indecomposable Belitskii’s canonical forms. Moreover, the algorithm can be slightly modified to obtain the Belitskii’s canonical form of a given matrix \( A \in N_n \) after finding \( A' \in QU_n \) such that \( A \sim_A A' \) by steps (1) and (2) of the simplification process at the end of Section 2. It is much more efficient than the Belitskii’s algorithm.

**Example 5.1.** Let us search the Belitskii’s canonical forms in \( QU_8 \) in which \( \mathcal{P}_Q = 123|478|56 \). By Theorems 4.8 and 4.9, the possible extra arcs of a Belitskii’s canonical form in \( QU_8 \) are listed in the Belitskii’s order as follows:

\[
(5, 7) \prec (2, 4) \prec (2, 5) \prec (1, 4) \prec (1, 5).
\]

Let \( S \) be the set of these arcs. The graph \( G_Q \) (in solid arcs) and the set \( S \) (in dashed arcs) are shown on the right.

Let \( S_0 := \emptyset \). The arc \((5, 7)\) cannot be removed by a composition of ESOs stabilizing \( QU_8 \), since the only type of ESOs that changes the weight of the arc \((5, 7)\) is \( O_{6,7} \) which does not stabilize \( QU_8 \). There are two cases \( S_1 := \{(5, 7)\} \) and \( S_1 := \emptyset \). Consider the case \( S_1 := \{(5, 7)\} \). Similarly, we can reach one of the outcoming cases \( S_2 := \{(5, 7), (2, 4)\} \) and \( S_3 := \{(5, 7), (2, 4), (2, 5)\} \). The next arc in consideration is \((1, 4)\), which can be removed as illustrated below (cf. (3.4)):

\[
\mathcal{P}_Q : 57|24|25|14| \cdots \ -14 + 13 + 15 -13 \ = 57|24|25|15.
\]
Note that (1, 5) created in the above process satisfies (1, 5) ≻ (1, 4). Now $S_4 := \{(5, 7), (2, 4), (2, 5)\}$. The next arc (1, 5) can be removed as below:

$$O_{25} \quad O_{36} \quad O_{47} \quad O_{78} \quad O_{68}$$

$$P_Q : 57|24|25|15 \quad -15 + 26 + 27 \quad -26 - 27 + 48 \quad -48 + 58 \quad -58 = 57|24|25.$$ 

So $S_5 := \{(5, 7), (2, 4), (2, 5)\}$. We get a Belitskiǐ’s canonical form of the type $P_Q : 57|24|25$ in which the underline indicates the place of a parameter. Explicitly, the $8 \times 8$ Belitskiǐ’s canonical form is:

$$E_{12} + E_{23} + E_{47} + E_{78} + E_{56} + E_{57} + E_{24} + \lambda E_{25}.$$ 

The table on the right lists the forms of Belitskiǐ’s canonical forms in $QU_8$ for $P_Q = 123|478|56$. We use “Y” (resp. “N”) to mark the presence (resp. absence) of an extra arc, and “−” to indicate that the arc can be removed given the combination of preceding extra arcs. Totally there are 10 forms of Belitskiǐ’s canonical forms in the double coset $B_8QB_8$, and 5 of them are indecomposable.

| $57$ | $24$ | $25$ | $14$ | $15$ | type | indecomp. |
|------|------|------|------|------|-------|-----------|
| Y    | Y    | Y    | −    | −    | 57|24|25 | Yes    |
| Y    | Y    | N    | −    | −    | 57|24 | Yes    |
| Y    | N    | Y    | −    | −    | 57|25 | Yes    |
| Y    | N    | N    | Y    | 57|15 | Yes    |
| Y    | N    | N    | N    | 57|57 | No      |
| N    | Y    | Y    | −    | 24|25 | Yes    |
| N    | Y    | N    | −    | 24|24 | No      |
| N    | N    | Y    | −    | 25|25 | No      |
| N    | N    | N    | Y    | 14|14 | No      |
| N    | N    | N    | N    | 0 |0  | No      |

5.2. The indecomposable Belitskiǐ’s canonical forms for $n \leq 8$. In this subsection, we describe the indecomposable Belitskiǐ’s canonical forms in $N_n$ under the $B_n$-similarity for $n \leq 8$ using their graph types together with underlines indicating nonzero parameters (see Example 5.1). The classifications for $n \leq 6$ have been done by Kobal [10] and Chen et al [5] (see Theorem 5.2). We apply MAPLE programs to filter out possible extra arcs using the algorithm in the preceding subsection and obtain all classifications for $n \leq 8$.

**Theorem 5.2** (Kobal [10], Chen et al [5]). The indecomposable Belitskiǐ’s canonical forms in $N_n$ under the $B_n$-similarity for $n \leq 6$ are listed by their graph types together with underlines indicating nonzero parameters as follows (29 forms, separated by commas):

1: $\emptyset$, $\emptyset$, $\emptyset$, $\emptyset$, $\emptyset$, $\emptyset$, $\emptyset$

12: $\emptyset$, $123|45|24$, $1234|56|35$, $1235|46|24$

123: $\emptyset$, $125|34|13$, $1234|5|34|13$, $1235|4|35|13$, $1236|5|24$, $1237|4|36|13$

124: $\emptyset$, $145|23|24$, $1234|56|35|13|35|13$, $1235|46|24$, $1236|57|35|13$, $1237|47|36|13$

125: $\emptyset$, $125|34|13$, $1234|5|34|13$, $1235|4|35|13$, $1236|5|24$, $1237|4|36|13$

126: $\emptyset$, $126|34|13$, $1234|5|34|13$, $1235|4|35|13$, $1236|5|24$, $1237|4|36|13$

127: $\emptyset$, $127|34|13$, $1234|5|34|13$, $1235|4|35|13$, $1236|5|24$, $1237|4|36|13$
Remark 5.3. For $n = 6$, Theorem 5.2 lists 19 forms instead of 18 forms shown in \cite{5}, Theorem 2.2, since we use nonzero parameters in our classifications.

The results for $n = 7$ are as follows, including the 8 forms with a parameter discovered in \cite{5}, Theorem 2.3).

**Theorem 5.4.** The indecomposable Belitskii’s canonical forms in $N_n$ under the $B_n$-similarity for $n = 7$ are of the graph types (85 forms in 58 subpermutations, separated by commas):

\begin{align*}
134|256 : 35, & 124|356 : 13, & 12|34|56 : 35|13, \\
145|236 : 24, & 125|346 : 13, & 12|36|45 : 13|14, \\
156|234 : 35, & 126|345 : 13, & 14|23|56 : 25|15, \\
123|456 : 24, 14, & 12|3456 : 13, &
\end{align*}

| 134|2567 : 0, & 123|4567 : 35, 25, & 147|23|56 : 24|25|15, |
| 1456|2346 : 24, & 123|4567 : 24, 14, & 24|25, 24|15, 25|15, |
| 1256|345 : 35, 13, 35, 13, & 124|3567 : 13, & 156|23|47 : 24|25, |
| 1236|457 : 46|24, 46, 24, & 125|3467 : 13, & 167|23|45 : 46|24, |
| 1234|567 : 35, & 126|3457 : 13, & 167|25|34 : 36|26, |
| 1234|567 : 35, & 127|3456 : 13, & 12|34567 : 13, |
| 1567|234 : 35, 25, & 134|2567 : 35, & 15|23|46 : 36|16, |
| 1467|235 : 24, & 145|2367 : 46|24, 46, 24, & 14|23|75 : 25|15, |
| 1457|236 : 24, & 156|2347 : 35, & 13|24|75 : 25|15, |
| 1456|237 : 24, & 167|2345 : 46, & 13|26|74 : 46|14, |
| 1367|245 : 46, & 123|45|67 : 46|24, 46|14, & 12|34|56 : 46|13, |
| 1347|256 : 35, & 123|47|56 : 24|25, & 12|34|567 : 35|13, 35|15, |
| 1345|267 : 46, & 124|35|67 : 36|13, & 12|36|74 : 46|13|14, |
| 1267|345 | 46|13, 46, 13, & 125|34|67 : 36|26|13, 46|13, 46|14, 13|14, |
| 1257|346 : 13, & 36|26, 36|13, 26|13, & 12|37|456 : 13|14, |
| 1256|347 : 35|13, 35, 13, & 126|34|57 : 35|13, & 12|36|457 : 13|14, |
| 1247|356 : 13, & 127|34|56 : 35|13, & 12|35|467 : 13|14, |
| 1245|367 : 46, & 127|36|45 : 13|14, & 12|34|567 : 35|13, 13|15, |
| 1237|456 : 24, 14, & 134|25|67 : 36|26, & 14|23|567 : 25|15, |
| 1236|457 : 24, & 145|23|67 : 46|24, 24|16, |
| 1235|467 : 24, & 146|23|57 : 24|15, |

Theorem 5.5. For $n = 8$, there are 481 forms (in 245 subpermutations) of indecomposable Belitskii’s canonical forms in $N_n$ under the $B_n$-similarity. The graph types of these forms will be listed in the Appendix section.

The number of subpermutations in $N_n$ equals the number of partitions of $[n]$, which is called a Bell or exponential number. The first few Bell numbers starting
at $n = 1$ are:

$$1, 2, 5, 15, 52, 203, 877, 4140, 21147, \ldots$$

Many properties of the Bell numbers have been studied (cf. [http://oeis.org](http://oeis.org)). Both Theorem 2.7 and the Bell numbers imply that the numbers of indecomposable Belitskii’s canonical forms in $N_n$ grow in a rate greater than any exponential function of $n$.

5.3. **Create new Belitskii’s canonical forms.** Theorem 2.7 can be used to obtain new Belitskii’s canonical forms. Let $A_1 \in Q_1 U_p$ and $A_2 \in Q_2 U_q$ be Belitskii’s canonical forms. Theorem 2.7 claims that if

$$\begin{bmatrix}
Q_1 & Q_{12} \\
0 & Q_2
\end{bmatrix} \in N_{p+q}
$$

is a subpermutation, then

$$\begin{bmatrix}
A_1 & Q_{12} \\
0 & A_2
\end{bmatrix}
$$

is a Belitskii’s canonical form in $N_{p+q}$. Here we rephrase Theorem 2.7 in the language of graphs. Let $G_{A_1} + p$ denote the graph with the vertex set $\{1 + p, \ldots, q + p\}$ and the edge set $\{(i + p, j + p) : (i, j) \in E_{A_2}\}$. If we add some arcs from chain heads of $G_{A_1}$ to chain tails of $G_{A_2} + p$ such that each involving vertex is on at most one such arc, then the resulting graph represents a Belitskii’s canonical form in $N_{p+q}$. The following are two examples.

**Example 5.6.** Let $A_1 = A_2$ be the Belitskii’s canonical forms of the graph type $12|34 : 13$. So $P_{Q_1} = P_{Q_2} = 12|34$. By Theorem 2.7, we can obtain Belitskii’s canonical forms by adding to the graph $G_{A_1} \cup (G_{A_2} + 4)$ some arcs from the chain heads 2 and 4 of $G_{A_1}$ to the chain tails 5 and 7 of $G_{A_2} + 4$ such that each vertex is on at most one arc. The illustrated graph is as below, in which dashed arcs are possible arcs added to the graph $G_{A_1} \cup (G_{A_2} + 4)$:

There are 6 indecomposable Belitskii’s canonical forms obtained from this way:

$1256|3478 : 57|13, 1278|3456 : 57|13, 12|3456|78 : 57|13, 1256|3478 : 57|13, 1278|3456 : 57|13$. All of them can be found in Theorem 5.5. Note that a parameter is added in each of the last two cases for the general forms. The direct sum $A_1 \oplus A_2$ is the only non-indecomposable Belitskii’s canonical form given by Theorem 2.7 here.

**Example 5.7.** Let $A_1 \in Q_1 U_6$ and $A_2 \in Q_2 U_3$ be the Belitskii’s canonical forms of the graph types $12|36|45 : 13|14$ and $13|2 : \emptyset$, respectively. The Belitskii’s canonical forms $\begin{bmatrix} A_1 & Q_{12} \\ A_2 \end{bmatrix}$ constructed in Theorem 2.7 are constructed by adding the following possible dashed arcs to the graph $G_{A_1} \cup (G_{A_2} + 6)$ such that each vertex is on at most one such arc:
There are 6 indecomposable Belitskii’s canonical forms constructed in this way:

\[
1279 | 368 | 45 : 13 \quad 1279 | 36 | 458 : 13 \quad 128 | 36 | 4579 : 13 \quad 12 | 368 | 4579 : 13 \quad 12 | 3679 | 45 : 13 \quad 12 | 368 | 458 : 13 \quad 12 | 36 | 4579 : 13.
\]

Similarly, the Belitskii’s canonical forms \( \begin{bmatrix} A_2 & Q'_{12} \\ A_1' \end{bmatrix} \) constructed in Theorem 2.7 are constructed by adding the following possible dashed arcs to the graph \( G_{A_2} \cup (G_{A_1} + 3) \) such that each vertex is on at most one such arc:

There are also 6 indecomposable Belitskii’s canonical forms constructed in this way:

\[
1345 | 269 | 78 : 46 \quad 1345 | 278 | 69 : 46 \quad 1369 | 245 | 78 : 46 \quad 1369 | 278 | 45 : 46 \quad 1378 | 245 | 69 : 46 \quad 1378 | 269 | 45 : 46.
\]

5.4. The \( B_n \)-similarity of upper triangular matrices. Sylveste’s theorem (cf. [9, Theorem 2.4.4.1]) says that if \( M \in M_p \) and \( N \in M_q \) have no eigenvalue in common, then the equation \( MX - XN = R \) has a unique solution \( X \in M_{p,q} \) for each \( R \in M_{p,q} \), that is,

\[
\begin{bmatrix} M & R \\ 0 & N \end{bmatrix} = \begin{bmatrix} I_p & X \\ 0 & I_q \end{bmatrix}^{-1} \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} I_p & X \\ 0 & I_q \end{bmatrix} U_{\mathbb{Z}^+} \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}.
\]

Similarly, every \( n \times n \) upper triangular matrix \( A = [a_{ij}] \) is \( B_n \)-similar to the matrix \( C = [c_{ij}] \) such that \( c_{ij} = a_{ij} \) if \( a_{ii} = a_{jj} \), and \( c_{ij} = 0 \) otherwise; \( C \) is permutation similar to a direct sum of matrices of the form \( \lambda I_k + C' \) for \( k \in [n] \) and \( C' \in N_k \). See [14] or [16, Section 1] for more details. The \( B_n \)-similarity problem of upper triangular \( A \) is transformed to the upper triangular similarity problems of nilpotent upper triangular matrices.

In [16, Theorem 1.5], Thijsse showed that if an \( n \times n \) upper triangular matrix \( A \) satisfies one of the following two conditions:

1. \( A \) is nonderogatory;
2. \( \dim \ker(A - \lambda I)^2 = \dim \ker(A - \lambda I)^3 \) for each \( \lambda \in \mathbb{C} \).

Then \( A \) is \( B_n \)-similar to a matrix which is permutation similar to a direct sum of Jordan blocks. We provide a new proof here. By the argument in the preceding paragraph, it suffices to consider the case \( A \in N_n \):
• Condition (1) means that $A \in JU_n$ where $J$ is the nilpotent Jordan block of size $n$. The only Belitskii's canonical form in $JU_n$ is $J$.

• Condition (2) means that each Jordan block of $A$ has size no more than 2. So $A \sim A' \in QU_n$ in which $Q \in N_n$ is a subpermutation, $A^2 = (A')^2 = 0$, and each extra arc $(i, j) \in E_{A'} \setminus E_Q$ of $A'$ has $j$ a chain head of a chain of $G_Q$. So $(i, j)$ is not in the Belitskii's canonical form of $A$. Therefore, the Belitskii's canonical form of $A$ is exactly $Q$, which is permutation similar to a direct sum of nilpotent Jordan blocks of sizes one or two.

Another observation is [9, 2.5.P49] which states that: an $n \times n$ upper triangular matrix $A$ is similar to a diagonal matrix if and only if it is $B_n$-similar to a diagonal matrix. In particular, if $A$ has distinct diagonal entries, then $A$ is $B_n$-similar to its diagonal. However, the result is not quite useful for nilpotent upper triangular matrices since the only diagonalizable nilpotent upper triangular matrix is the zero matrix.

APENDIX

As a complement to Theorem 5.5, the list of indecomposable Belitskii's canonical forms in $N_n$ under the $B_n$-similarity for $n = 8$ is as follows (481 forms in 245 subpermutations):

\begin{align*}
12345678 : \emptyset, & \quad 15678|234 : 35, 25, & \quad 12367|458 : 46|24, 46, 24, \\
12|345678 : 13, & \quad 14678|235 : 24, & \quad 12358|467 : 24, \\
145678|23 : 24, & \quad 14578|236 : 24, & \quad 12356|478 : 57, \\
125678|34 : 35|13, 35, 13, & \quad 14568|237 : 24, & \quad 12348|567 : 35, 25, \\
12367|8 \sim : 46|24|14, & \quad 14567|238 : 24, & \quad 12347|568 : 35, \\
46|24, 46, 24, & \quad 13678|245 : 46, & \quad 12346|578 : 35, \\
123478|56 : 57|35, 57, 35, & \quad 13478|256 : 57|25, 57, 35, & \quad 12345|678 : 46, 36, \\
123458|67 : 46, & \quad 13458|267 : 46, & \quad 12345|678 : 35, 25, 15, \\
123456|78 : 57, & \quad 13456|278 : 57, & \quad 1235|4678 : 24, 14, \\
123|45678 : 24, 14, & \quad 12678|345 : 46|13, 46, & \quad 1236|4578 : 24, 14, \\
124|35678 : 13, & \quad 36|13, 36, 13, & \quad 1237|4568 : 24, 14, \\
125|34678 : 13, & \quad 12578|346 : 35|13, 35, 13, & \quad 1238|4567 : 24, 14, \\
126|34578 : 13, & \quad 12568|347 : 35|13, 35, 13, & \quad 1245|3678 : 46, 13, \\
127|34568 : 13, & \quad 12567|348 : 35|13, 35, 13, & \quad 1246|3578 : 13, \\
128|34567 : 13, & \quad 12478|356 : 57|13, 57, 13, & \quad 1247|3568 : 13, \\
134|25678 : 35, & \quad 12458|367 : 46, & \quad 1248|3567 : 13, \\
145|23678 : 46|24, 46, 24, & \quad 12456|378 : 57, & \quad 1256|3478 : 57|35|13, \\
156|23478 : 57|35, 57, 35, & \quad 12378|456 : 57|24, 57|14, 57, 35, 57, 35, 35|13, 57, 35, 35|13, \\
167|23458 : 46, & \quad 57, 24, 14, & \quad 35, 13, \\
178|23456 : 57, & \quad 12368|457 : 24, & \quad 1257|3468 : 13,
\end{align*}
1258|3467 : 13,
1267|3458 : 46|13, 46|13,
1268|3457 : 13,
1278|3456 : 57|13, 57|13,
1345|2678 : 46, 36, 57|13, 57|13, 13|14,
1346|2578 : 35,
1347|2568 : 35,
1348|2567 : 35,
1356|2478 : 57,
1367|2458 : 46,
1378|2456 : 57,
1456|2378 : 57|24,57|24,
1457|2368 : 24,
1458|2367 : 46|24, 46|24,
1467|2358 : 24,
1478|2356 : 57,
1567|2348 : 35, 25,
1568|2347 : 35,
1578|2346 : 35,
1678|2345 : 46, 36,
12|34|5678 : 35|13, 13|15,
12|35|4678 : 13|14,
12|36|4578 : 13|14,
12|37|4568 : 13|14,
12|38|4567 : 13|14,
12|3678|45 : 46|13|14,
12|34|5678 : 35|13, 13|15,
12|34|5678 : 35|13, 13|15,
12|34|5678 : 35|13, 13|15,
12|34|5678 : 35|13, 13|15,
12|34|5678 : 35|13, 13|15,
12|34|5678 : 35|13, 13|15,
12|34|5678 : 35|13, 13|15,
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36|26, 36|13, 26|13, 13|16, 47|37, 47|14, 37|14, 125|34|867 : 36|26, 13,
126|34|578 : 35|13, 13|15, 137|26|845 : 46|14, 128|34|578 : 46|13,
127|34|568 : 35|13, 13|15, 138|26|745 : 46|14, 134|26|845 : 35|26,
128|34|567 : 35|13, 13|15, 167|26|345 : 46|24, 145|26|867 : 46|24, 46|16,
156|278|34 : 57|35, 168|26|745 : 46|24, 35|26, 35|26, 36|26,
158|26|745 : 35|36, 178|26|745 : 47|37|24, 145|26|867 : 46|24, 46|16,
167|258|34 : 35|36|26, 47|37, 47|24, 37|24, 158|25|867 : 35|36|16,
35|36, 35|26, 36|26, 123|46|78 : 47|24, 35|36, 35|16, 36|16,
168|25|745 : 35|36, 123|46|78 : 24|25, 123|45|78 : 26|14,
178|25|678 : 35|35, 35|27, 125|278|46 : 47|13|14, 124|35|678 : 26|13,
127|35|46 : 47|13|14, 178|23|546 : 47|24, 127|35|46 : 46|13,
147|35|678 : 35|13, 123|46|78 : 24|25, 145|23|78 : 46|24,
126|35|478 : 47|13|14, 156|23|846 : 24|25, 157|23|468 : 35|16,
35|12|467 : 13|14, 123|48|567 : 24|25, 24|15, 123|45|678 : 57|24, 57|14,
27|14,
127|36|45 : 47|37|37|14, 123|47|856 : 57|24, 27|13,
47|37|13, 47|37|14, 47|37, 57|24, 57|15, 27|13,
47|13|14 47|13, 37|13|14, 124|37|856 : 57|25, 57|15,
37|14, 13|14, 127|34|568 : 35|13, 35|15,
124|34|567 : 35|13, 134|27|856 : 57|35, 57|15,
125|278|46 : 47|13|14, 134|27|856 : 24|25, 145|23|78 : 47|24,
145|278|36 : 47|37, 147|23|856 : 24|25|15, 47|37, 47|24, 47|37,
178|24|567 : 35|37, 24|25, 24|15, 24|15, 134|23|678 : 57|35, 57|25,
128|34|567 : 35|13|14, 148|23|785 : 25|15, 156|23|468 : 57|35, 57|25,
123|45|678 : 46|24, 46|14, 178|23|468 : 57|35, 57|25,
35|17,
123|45|68 : 46|14, 126|34|867 : 35|13, 123|45|68 : 57|35, 123|45|68 : 35|36|13,
123|45|78 : 47|37|24, 146|23|856 : 24|25, 35|36|16,
47|37, 47|24, 37|24, 123|46|758 : 25|14, 123|46|758 : 46|13|14,
24|26, 124|36|758 : 25|13, 123|45|678 : 47|37|13|14,
126|37|85 : 47|13|14, 134|26|758 : 35|36, 47|37|13, 47|37|14,
47|13, 47|14, 13|14, 167|23|458 : 35|36, 47|13|14, 37|13|14,
127|36|85 : 46|13|14, 123|45|678 : 46|24, 123|45|678 : 46|13|14,
46|13, 46|14, 13|14, 46|26|14, 46|26|14, 46|14, 46|14,
128|36|75 : 46|13|14, 26|14, 123|45|678 : 13|14, 13|
46|13, 46|14, 13|14, 124|35|867 : 36|26|13, 13|24|58|67 : 25|26|16,
136|27|85 : 47|37|14, 36|26, 36|13, 26|13, 13|6|24|5678 : 47|27|14,
References

[1] Nurit Barnea and Anna Melnikov. B-orbits of square zero in nilradical of the symplectic algebra. *Transform. Groups*, 22(4):885–910, 2017.

[2] Magdalena Boos. Finite parabolic conjugation on varieties of nilpotent matrices. *Algebr. Represent. Theory*, 17(6):1657–1682, 2014.

[3] Magdalena Boos, Giovanni Cerulli Irelli, and Francesco Esposito. Parabolic orbits of 2-nilpotent elements for classical groups. *J. Lie Theory*, 29(4):969–996, 2019.

[4] Magdalena Boos and Markus Reineke. B-orbits of 2-nilpotent matrices and generalizations. In *Highlights in Lie algebraic methods*, volume 295 of *Progr. Math.* Birkhäuser/Springer, New York, 2012.

[5] Yuan Chen, Yunge Xu, Huanhuan Li, and Wenhao Fu. Belitskiǐ’s canonical forms of upper triangular nilpotent matrices under upper triangular similarity. *Linear Algebra Appl.*, 506:139–153, 2016.

[6] David H. Collingwood and William M. McGovern. *Nilpotent orbits in semisimple Lie algebras*. Van Nostrand Reinhold Mathematics Series. Van Nostrand Reinhold Co., New York, 1993.

[7] Dragimir Ž. Djoković and J. Malzan. Orbits of nilpotent matrices. *Linear Algebra Appl.*, 32:157–158, 1980.

[8] Lucas Fresse. Upper triangular parts of conjugacy classes of nilpotent matrices with finite number of B-orbits. *J. Math. Soc. Japan*, 65(3):967–992, 2013.

[9] Roger A. Horn and Charles R. Johnson. *Matrix analysis*. Cambridge University Press, Cambridge, second edition, 2013.

[10] Damjan Kobal. Belitskii’s canonical form for 5 × 5 upper triangular matrices under upper triangular similarity. *Linear Algebra Appl.*, 403:178–182, 2005.

[11] Anna Melnikov. B-orbits in solutions to the equation $X^2 = 0$ in triangular matrices. *J. Algebra*, 223(1):101–108, 2000.

[12] Anna Melnikov. Description of B-orbit closures of order 2 in upper-triangular matrices. *Transform. Groups*, 11(2):217–247, 2006.

[13] Anna Melnikov. B-orbits of nilpotency order 2 and link patterns. *Indag. Math. (N.S.)*, 24(2):443–473, 2013.

[14] Moshe Roitman. A problem on conjugacy of matrices. *Linear Algebra and Appl.*, 19(1):87–89, 1978.

[15] Vladimir V. Sergeichuk. Canonical matrices for linear matrix problems. *Linear Algebra Appl.*, 317(1-3):53–102, 2000.

[16] Philip Thijsse. Upper triangular similarity of upper triangular matrices. *Linear Algebra Appl.*, 260:119–149, 1997.
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