Integral balance methods applied to a non-classical Stefan problem

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Abstract

In this paper we consider two different Stefan problems for a semi-infinite material for the non classical heat equation with a source which depends on the heat flux at the fixed face \(x = 0\). One of them (with constant temperature on \(x = 0\)) was studied in [4] where it was found a unique exact solution of similarity type and the other (with a convective boundary condition at the fixed face) is presented in this work. Due to the complexity of the exact solution it is of interest to obtain some kind of approximate solution. For the above reason, the exact solution of each problem is compared with approximate solutions obtained by applying the heat balance integral method and the refined heat balance integral method, assuming a quadratic temperature profile in space. In all cases, a dimensionless analysis is carried out by using the parameters: Stefan number (Ste) and the generalized Biot number (Bi). In addition it is studied the case when Bi goes to infinity, recovering the approximate solutions when a Dirichlet condition is imposed at the fixed face. Some numerical simulations are provided in order to verify the accuracy of the approximate methods.

Keywords: Stefan problem, convective condition, heat balance integral method, refined heat balance integral method, similarity solution.

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1 Introduction

Stefan problems model heat transfer processes that involve a change of phase. They constitute a broad field of study since they arise in a great number of mathematical and industrial significance problems [10, 11, 13, 16, 17]. A large bibliography on the subject was given in [28] and a review on analytical solutions is given in [30]. Nonclassical heat conduction problem for a semi-infinite material was studied in [2, 6, 7, 15, 16, 32, 33]. A problem of this type involves equations of the form:

\[
\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = -F(W(t), t), \quad x > 0, \quad t > 0,
\]

where \(F\) is a given function of two variables. A particular and interesting case is the following

\[
F(W(t), t) = \frac{\lambda_0}{\sqrt{t}} W(t) \quad (\lambda_0 > 0)
\]

where \(W = W(t)\) represents the heat flux on the boundary \(x = 0\), that is \(W(t) = \frac{\partial U}{\partial x}(0, t)\).

This kind of problems can be thought of by modelling of a system of temperature regulation in isotropic mediums [32, 33], with a nonuniform source term which provides a cooling or heating
effect depending upon the properties of $F$ related to the course of the heat flux at the boundary $x = 0$. Other references on the subject are in [7, 8, 13].

In this paper, firstly we consider a free boundary problem which consists in determining the temperature $u = u(x, t)$ and the free boundary $x = s(t)$, that we call:

**Problem $(P)$:** Find the temperature $U = U(x, t)$ at the liquid region $0 < x < S(t)$ and the location of the free boundary $x = S(t)$ such that:

$$
\rho c \frac{\partial U}{\partial t} - k \frac{\partial^2 U}{\partial x^2} = -\gamma F \left( \frac{\partial U}{\partial x}(0, t), t < 0, 0 < x < S(t), t > 0, \right)
$$

$$
U(0, t) = u_\infty > 0, \quad t > 0,
$$

$$
U(S(t), t) = 0, \quad t > 0,
$$

$$
k \frac{\partial U}{\partial x}(S(t), t) = -\rho l \dot{S}(t), \quad t > 0,
$$

$$
S(0) = 0.
$$

where the thermal conductivity $k$, the mass density $\rho$, the specific heat $c$ and the latent heat per unit mass $l$, are given constants, $h$ characterizes the heat transfer coefficients [29, 35] and the control function $F$ depend on the the evolution of the heat flux at the boundary $x = 0$ as follows:

$$
F \left( \frac{\partial U}{\partial x}(0, t), t \right) = \lambda_0 \frac{\partial U}{\sqrt{1} \partial x}(0, t)
$$

where $\lambda_0$ is a given constant. This problem was studied in [4].

The problem is also considered with another condition at the fixed face $x = 0$: the convective condition [31]. This condition states that heat flux at the fixed face is proportional to the difference between the material temperature and the neighbourhood temperature, that is:

$$
k \frac{\partial U}{\partial x}(0, t) = H(t) (U(0, t) - u_\infty), \quad H(t) \text{ characterizes the heat transfer at the fixed face and } 0 < U(0, t) < u_\infty.
$$

In this paper we consider a free boundary problem with a convective condition of the form $H(t) = \frac{1}{\sqrt{1}} h > 0$.

More precisely, we consider a free boundary problem which consists in determining the temperature $u_{h} = u_{h}(x, t)$ and the free boundary $x = s_{h}(t)$, that we call:

**Problem $(P_h)$:** Find the temperature $U = U(x, t)$ at the solid region $0 < x < S(t)$ and the location of the free boundary $x = S(t)$ such that are satisfied equation [3], conditions [34] instead condition [4] of problem $(P)$ and the condition

$$
k \frac{\partial U}{\partial x}(0, t) = \frac{h}{\sqrt{1}} (U(0, t) - u_\infty), \quad t > 0,
$$

instead condition [4] of problem $(P)$.

Due to the non-linear nature of this type of problems exact solutions are limited to a few cases when the exact solution can be find, it is necessary to solve them either numerically or approximately. Despite having the exact solution to the problem that we will study, it is very complicated to find the exact solution. The heat balance integral method introduced by Goodman in [9] is a well-known approximate mathematical technique for solving the location of the free front in heat-conduction problems involving a phase of change. This method consists in transforming the heat equation into an ordinary differential equation over time by assuming a quadratic temperature profile in space. In [3, 11, 12, 13, 19, 20, 21] and in [25] this method is applied using different accurate temperature profiles such as: exponential, potential, etc.

Recently, various papers has been published applying integral methods to a variety of thermal and moving boundary problems, especially to non-linear heat conduction and fractional diffusion: [10, 22, 23, 36, 37, 38, 39, 40, 41].

Different alternative pathways to develop the heat balance integral method were established in [34]. In this paper, we obtain approximate solutions through integral heat balance methods and variants obtained thereof proposed in [34] for the problems $(P)$ and $(P_h)$. As one of the mechanisms for the heat conduction is the diffusion, the excitation at the fixed face $x = 0$ (for
example, a temperature, a flux or a convective condition) does not spread instantaneously to the material \( x > 0 \). However, the effect of the fixed boundary condition can be perceived in a bounded interval \([0, \delta(t)]\) (for every time \( t > 0 \)) outside of which the temperature remains equal to the initial temperature. The heat balance integral method presented in [9] established the existence of a function \( \delta = \delta(t) \) that measures the depth of the thermal layer. In problems with a phase of change, this layer is assumed to be the free boundary, i.e \( \delta(t) = s(t) \).

From equation (5) we obtain the new condition:

\[
\left( \frac{\partial U}{\partial x} \right)^2 (S(t), t) = -\frac{l}{kc} (\frac{\partial^2 U}{\partial x^2} (S(t), t) - \gamma \lambda_0 \frac{\partial U}{\partial x} (S(t), t)) .
\]

(10)

From equation (3) and conditions (5)-(6) we obtain the integral condition:

\[
\frac{d}{dt} \int_0^{S(t)} U(x, t) dx = -\frac{\partial U}{\partial x} (0, t) \rho c \left[ \gamma \lambda_0 \frac{S(t)}{\sqrt{t}} + k \right] - \frac{l}{c} \frac{\partial U}{\partial x} (0, t) \delta(t).
\]

(11)

The classical heat balance integral method introduced in [9] to solve problem \((P)\) or \((P_h)\) proposes the resolution of a problem that arises by replacing the equation (3) by the condition (11), and the condition (6) by the condition (10), keeping all others conditions of the problem \((P)\) or \((P_h)\) equals.

In [34], a variant of the classical heat balance integral method was proposed by replacing only equation (3) by condition (11), keeping all others conditions of the problem \((P)\) or \((P_h)\) equals.

From equation (3) and conditions (11) and (5) we can also obtain:

\[
\int_0^{S(t)} \int_0^x \frac{\partial U}{\partial \xi} (\xi, t) d\xi dx = \frac{1}{\rho c} \left[ -\gamma \lambda_0 \frac{S^2(t)}{2\sqrt{t}} \frac{\partial U}{\partial x} (0, t) - ku_{\infty} - k \frac{\partial U}{\partial x} (0, t) \delta(t) \right] .
\]

(12)

The refined heat balance integral method introduced in [26] to solve the problem \((P)\) proposes the resolution of the approximate problem that arises by replacing equation (3) by condition (12), keeping all others conditions of the problem \((P)\) or \((P_h)\) equals.

For solving the approximate problems previously defined we propose a quadratic temperature profile in space as follows:

\[
U(x, t) = \hat{A} \left( 1 - \frac{x}{S(t)} \right) + \hat{B} \left( 1 - \frac{x}{S(t)} \right)^2 , \quad 0 < x < S(t) \quad t > 0 ,
\]

(13)

where \( \hat{A} \) and \( \hat{B} \) are unknown constants to be determined.

The goal of this paper is to study different approximations for one-dimensional one phase Stefan problems with a source function that depends on the flux. It is considered two different problems, which differ from each other in the boundary condition imposed at the fixed face \( x = 0 \): temperature (Dirichlet) condition or convective (Robin) condition. In Section 2 we present the exact solution of the problem \((P)\) which was given in [4]. Taking advantage of the exact solution of \((P)\), we obtain approximate solutions using the heat balance integral method, an alternative method of it and the refined integral method, comparing each approach with the exact one. A similar study is done in Section 3 for the problem with a convective condition at the fixed face, \((P_h)\). In order to make this analysis, we obtain previously the exact solution of \((P_h)\). We also study the limit cases of the obtained approximate solutions when \( h \to \infty \), recovering the approximate solutions when a temperature condition at the fixed face is imposed.

2 Explicit and approximate solutions to the one-phase Stefan problem for a non-classical heat equation with a source and a temperature condition at the fixed face

In this section we present the exact solution of the problem \((P)\) and we obtain approximate solutions by using heat balance integral methods, comparing each approach with the exact one.
2.1 Exact solution

In [4], it has been proved that for each dimensionless parameter:

$$\lambda = \frac{\gamma \lambda_0}{\sqrt{k \rho c}} > 0 \quad (14)$$

the free boundary problem \((P)\), where \(F\) defined by [5], has a unique similarity solution of the type

$$u(x, t) = u_\infty \left(1 - \frac{E(\eta, \lambda)}{E(\xi, \lambda)}\right), \quad 0 < \eta = \frac{x}{2a \sqrt{t}} < \xi \quad (15)$$

where

$$E(x, \lambda) = \text{erf}(x) + 4 \frac{\lambda}{\sqrt{\pi}} \int_0^x f(r)dr , \quad f(x) = \exp(-x^2) \int_0^x \exp(r^2)dr \quad (17)$$

and \(\xi > 0\) is the unique solution of

$$W_1(x) = 2\lambda W_2(x), \quad x > 0 \quad (18)$$

where the real functions \(W_1\) and \(W_2\) are defined by

$$W_1(x) = \text{Ste} \exp(-x^2) - \sqrt{\pi} x \text{erf}(x), \quad W_2(x) = 2x \int_0^x f(r)dr - \text{Ste} f(x), \quad (19)$$

and the dimensionless parameter defined by:

$$\text{Ste} = \frac{cu_\infty}{l} \quad (20)$$

represent the Stefan number. We remark that function \(f\) defined in [17], is called the Dawson’s integral.

From now on, we will consider the case \(\text{Ste} \in (0, 1)\), due to the fact that for most phase-change materials candidates over a realistic temperature, the Stefan number will not exceed \(1(27)\).

2.2 Approximate solution using the classical heat balance integral method

The classical heat balance integral method in order to solve the problem \((P)\) proposes the resolution of the approximate problem \((P_1)\) defined as follows:

Find the temperature \(u_1 = u_1(x, t)\) at the solid region \(0 < x < s_1(t)\) and the location of the free boundary \(x = s_1(t)\) such that satisfy the following conditions: \((4)-(5)-(7)-(10)-(11)\).

A solution to this problem will be an approximate one of the problem \((P)\). Proposing the following quadratic temperature profile in space:

$$u_1(x, t) = A_1 u_\infty \left(1 - \frac{x}{s_1(t)}\right) + B_1 u_\infty \left(1 - \frac{x}{s_1(t)}\right)^2, \quad 0 < x < s_1(t), \quad t > 0, \quad (21)$$

the free boundary takes the form:

$$s_1(t) = 2a \xi_1 \sqrt{t}, \quad t > 0, \quad (22)$$
where the constants $A_1$, $B_1$ and $\xi_1$ will be determined from the conditions (4), (10) and (11). Because of (21) and (22), the conditions (5) and (7) are immediately satisfied. From conditions (5) and (11) we obtain:

$$A_1 = \frac{-2(3 + \text{Ste})\xi_1^2 + 12\lambda \text{Ste} \xi_1 + 6\text{Ste}}{\text{Ste} (\xi_1^2 + 6\lambda \xi_1 + 3)},$$  
(23)

$$B_1 = \frac{3(2 + \text{Ste})\xi_1^2 - 6\lambda \text{Ste} \xi_1 - 3\text{Ste}}{\text{Ste} (\xi_1^2 + 6\lambda \xi_1 + 3)}.$$  
(24)

Since $A_1$ and $B_1$ are defined from the parameters $\xi_1$, $\lambda$ and Ste, condition (10) will be used to find the value of $\xi_1$. In this way, it turns out that $\xi_1$ must be a positive solution of the fifth degree polynomial equation:

$$-4\lambda (3 + 2\text{Ste}) z^5 + 2 (12 + 9\text{Ste} + 2\text{Ste}^2 - 12\lambda^2 (3 + 2\text{Ste})) z^4 - 12\lambda (-9 + 16\text{Ste} + 4\text{Ste}^4) z^3 +$$

$$+12(1 + 2\text{Ste}) (-3 + (6\lambda^2 - 1)\text{Ste}) z^2 + 72\lambda \text{Ste} (1 + 2\text{Ste}) z + 18\text{Ste} + 3\text{Ste}^2 = 0, \quad z > 0. \quad (25)$$

It is easy to see that (25) has at least one solution. Descartes’ rule of signs states that if the terms of a single-variable polynomial with real coefficients are ordered by descending variable exponent, then the number of positive roots of the polynomial is either equal to the number of sign differences between consecutive nonzero coefficients, or is less than it by an even number. Therefore, in our case, to have a unique root of (25) is enough to take $\lambda$ such that $12 + 9\text{Ste} + 2\text{Ste}^2 - 12\lambda^2 (3 + 2\text{Ste}) < 0$, that is

$$\lambda > \sqrt{\frac{2\text{Ste}^2 + 9\text{Ste} + 12}{36 + 24\text{Ste}}} \equiv f(\text{Ste}) $$  
(26)

As $f$ is an increasing function then for $0 < \text{Ste} < 1$ it is sufficient to take $\lambda > f(1) \cong 0.6191391874$. Then we have proved the following result:

**Theorem 2.1.** The solution of problem (P1), for a quadratic profile in space, is given by (21) and (22), where the constants $A_1$ and $B_1$ are defined by (23) and (24), respectively and $\xi_1$ is the unique positive solution of the polynomial equation (25) if $0 < \text{Ste} < 1$ and $\lambda > 0.62$.

As the approximate methods we are working with are designed as a technique for tracking the location of the free boundary, the comparisons between the approximate solutions and the exact one will be done on the free boundary thought the coefficients that characterizes them. That is to say, we will compare the known exact solution of the Stefan problem (P) and the approximate solution of the problem (P1) by computing the coefficients $\xi$ and $\xi_1$ that characterizes the free boundaries of each problem, which are obtained by solving (18) and (24), respectively. In Figure 1, we plot the dimensionless coefficients $\xi$ and $\xi_1$ against Stefan number, fixing $\lambda = 0.7$.

![Figure 1: Plot of the dimensionless coefficients $\xi$ and $\xi_1$ against Stefan number, for $\lambda = 0.7$.](image-url)

Then we have proved the following result:
2.3 Approximate solution using a modified method of the classical heat balance method

An alternative method of the classical heat balance integral method in order to solve the problem \((P)\) proposes the resolution of the approximate problem \((P_2)\) defined as follows:

Find the temperature \(u_2 = u_2(x,t)\) at the solid region \(0 < x < s_2(t)\) and the location of the free boundary \(x = s_2(t)\) such that satisfy the following conditions: (4)-(5)-(6)-(7)-(11).

A solution to this problem will be an approximate one of the problem \((P)\). Proposing the following quadratic temperature profile in space:

\[
u_2(x,t) = A_2u_\infty \left(1 - \frac{x}{s_2(t)}\right) + B_2u_\infty \left(1 - \frac{x}{s_2(t)}\right)^2, \quad 0 < x < s_2(t), \quad t > 0, \quad (27)
\]

the free boundary takes the form:

\[
s_2(t) = 2a\xi_2\sqrt{t}, \quad t > 0, \quad (28)
\]

where the constants \(A_2\), \(B_2\) and \(\xi_2\) will be determined from the conditions (4), (6) and (11). Because of (27) and (28), the conditions (5) and (7) are immediately satisfied. From conditions (5) and (11) we obtain:

\[
A_2 = \frac{2}{\text{St}\xi_2^2}, \quad (29)
\]

\[
B_2 = 1 - \frac{2}{\text{St}\xi_2^2}. \quad (30)
\]

Since \(A_2\) and \(B_2\) are defined from the parameters \(\xi_2\) and \(\text{St}\), condition (11) will be used to find the value of \(\xi_2\). In this way, it turns out that \(\xi_2\) must be a positive solution of the fourth degree polynomial equation:

\[
z^4 + 6\lambda z^3 + (6 + \text{St}) z^2 - 6\text{St}\lambda z - 3\text{St} = 0, \quad z > 0. \quad (31)
\]

It is easy to see, using the Descartes’ rule, that (31) has a unique positive solution.

Therefore, the following theorem holds:

**Theorem 2.2.** The solution of problem \((P_2)\), for a quadratic profile in space, is given by (27) and (28), where the constants \(A_2\) and \(B_2\) are defined by (29) and (30), respectively and \(\xi_2\) is the unique positive solution of the polynomial equation (31).

To compare the free boundaries obtained in problem \((P)\) and the approximate problem \((P_2)\), we compute the coefficient that characterizes the free boundaries. The exact value of \(\xi\) and the approach \(\xi_2\) are the unique roots of equations (18) and (31), respectively.

Figure 2 shows, for Stefan values up to 1, how the dimensionless coefficient \(\xi_2\), which characterizes the location of the free boundary \(s_2\), approaches the coefficient \(\xi\), corresponding to the exact free boundary \(s\), when the dimensionless parameter is \(\lambda = 0.7\).

\[
\begin{array}{c}
\text{Stefan number (Ste)} \\
0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1
\end{array}
\]

\[
\begin{array}{c}
\text{Dimensionless front location} \\
0 \quad 0.1 \quad 0.2 \quad 0.3 \quad 0.4 \quad 0.5 \quad 0.6 \quad 0.7 \quad 0.8
\end{array}
\]

\[
\begin{array}{c}
\xi \quad \text{Exact solution (P)} \\
\xi_2 \quad \text{Approximate solution (P_2)}
\end{array}
\]

\[
\text{Figure 2: Plot of the dimensionless coefficients } \xi \text{ and } \xi_2 \text{ against Ste number, for } \lambda = 0.7.
\]
2.4 Approximate solution using the refined integral method

The refined heat balance integral method in order to solve the problem (P), proposes the resolution of an approximate problem (P₃) formulated as follows:

Find the temperature \( u₃(x, t) \) at the solid region \( 0 < x < s₃(t) \) and the location of the free boundary \( x = s₃(t) \) such that satisfy the following conditions: (4)-(5)-(6)- (7)-(12).

A solution to this problem will be an approximate one of the problem (P). Proposing the following quadratic temperature profile in space:

\[
\begin{align*}
  u₃(x, t) &= A₃ u_∞ \left( 1 - \frac{x}{s₃(t)} \right) + B₃ u_∞ \left( 1 - \frac{x}{s₃(t)} \right)^2, \quad 0 < x < s₃(t), \quad t > 0, \\
  s₃(t) &= 2aξ₃√t, \quad t > 0,
\end{align*}
\]

the free boundary takes the form:

\[
  s₃(t) = 2aξ₃√t, \quad t > 0,
\]

where the constants \( A₃, B₃ \) and \( ξ₃ \) will be determined from the conditions (4), (6) and (12). Because of (32) and (33), the conditions (5) and (7) are immediately satisfied. From conditions (5) and (12) we obtain:

\[
\begin{align*}
  A₃ &= \frac{2}{Ste} ξ₃^2, \\
  B₃ &= 1 - \frac{2}{Ste} ξ₃^2.
\end{align*}
\]

Since \( A₃ \) and \( B₃ \) are defined from the parameters \( ξ₃ \) and \( Ste \), condition (12) will be used to find the value of \( ξ₃ \). In this way, it turns out that \( ξ₃ \) must be a positive solution of the third degree polynomial equation:

\[
-6λz^3 - (6 + Ste) z^2 + 6λSte z + 3Ste = 0, \quad z > 0.
\]

It is easy to see, using the Descartes’ rule, that (36) has a unique positive solution.

Therefore, the following theorem holds:

**Theorem 2.3.** The solution of problem (P₃), for a quadratic profile in space, is given by (32) and (33), where the constants \( A₃ \) and \( B₃ \) are defined by (34) and (35), respectively and \( ξ₃ \) is the unique positive solution of the polynomial equation (36).

To compare the free boundaries obtained in problem (P) and the approximate problem (P₃), we compute the coefficient that characterizes the free boundaries. The exact value of \( ξ \) and the approach \( ξ₃ \) is obtained by solving the equations obtained in (13) and (36), respectively.

For every \( Ste < 1 \), we plot the numerical value of the dimensionless coefficient \( ξ₃ \) obtained by applying the refined integral method, against the exact coefficient \( ξ \) (Figure 3).
2.5 Comparisons between solutions

In this subsection, for different Ste numbers, we make comparisons between the numerical value of the coefficient $\xi$ given by (18) and the approximations $\xi_1, \xi_2$ and $\xi_3$ given by (25), (31), (36), respectively. In order to compare the approximate solution with the exact one, and to obtain which technique gives the best agreement, we display in Table 1, varying $\text{Ste}$ between 0 and 1, the exact dimensionless free front, and its different approaches, showing also the percentage relative error committed in each case being $E_{\text{rel}}(\xi_i) = 100 \left| \frac{\xi_i - \xi}{\xi} \right|$.

Table 1: Dimensionless free front coefficients and its relative errors for $\lambda = 0.7$.

| Ste   | $\xi$ (P) | $\xi_1$ (P$_1$) | $E_{\text{rel}}(\xi_1)$ | $\xi_2$ (P$_2$) | $E_{\text{rel}}(\xi_2)$ | $\xi_3$ (P$_3$) | $E_{\text{rel}}(\xi_3)$ |
|-------|-----------|-----------------|--------------------------|-----------------|--------------------------|-----------------|--------------------------|
| 0.1   | 0.2351    | 0.2401          | 2.139 %                  | 0.2363          | 1.145 %                  | 0.2373          | 0.963 %                  |
| 0.2   | 0.3342    | 0.3486          | 4.289 %                  | 0.3381          | 2.351 %                  | 0.3408          | 1.956 %                  |
| 0.3   | 0.4091    | 0.4348          | 6.284 %                  | 0.4162          | 4.577 %                  | 0.4211          | 2.932 %                  |
| 0.4   | 0.4708    | 0.5089          | 8.109 %                  | 0.4818          | 6.753 %                  | 0.4890          | 5.666 %                  |
| 0.5   | 0.5238    | 0.5750          | 9.776 %                  | 0.5392          | 8.549 %                  | 0.5489          | 4.788 %                  |
| 0.6   | 0.5706    | 0.6350          | 11.30 %                  | 0.5905          | 10.23 %                  | 0.6029          | 5.666 %                  |
| 0.7   | 0.6125    | 0.6903          | 12.69 %                  | 0.6373          | 11.98 %                  | 0.6524          | 6.150 %                  |
| 0.8   | 0.6507    | 0.7417          | 13.98 %                  | 0.6805          | 13.18 %                  | 0.6983          | 7.322 %                  |
| 0.9   | 0.6857    | 0.7897          | 15.16 %                  | 0.7206          | 14.48 %                  | 0.7413          | 8.102 %                  |
| 1.0   | 0.7181    | 0.8348          | 16.25 %                  | 0.7582          | 15.57 %                  | 0.7817          | 8.854 %                  |

It may be noticed that the relative error committed in each approximate technique increases when the Stefan number becomes greater, reaching the percentages 16.25 %, 5.579 % and 8.854 % for the problems (P$_1$), (P$_2$) and (P$_3$) respectively.

3 Explicit and approximate solutions to the one-phase Stefan problem for a non-classical heat equation with a source and a convective condition at the fixed face

In this section we present the exact solution of the problem (P$_h$) and we obtain different approaches by using heat balance integral methods, comparing them with the exact one.

3.1 Exact solution

In this subsection we will obtain the exact solution of the problem (P$_h$) given by (3),(5)-(7) and (9) instead of condition (4) of problem (P).

In similar way as [4], if we define the similarity variable $\eta = \frac{x}{2\sqrt{\text{at}}}$ and $\Phi(\eta) = u_h(x,t)$, then (P$_h$) turns equivalent to the following ordinary differential problem:

\[
\begin{align*}
\Phi''(\eta) + 2\eta \Phi'(\eta) &= 2\lambda \Phi'(0), \\
\Phi'(0) &= 2\text{Bi} (\Phi(0) - u_\infty), \\
\Phi(\xi_h) &= 0, \\
\Phi' (\xi_h) &= \frac{-u_\infty}{\text{Ste}} \xi_h 
\end{align*}
\]

where the dimensionless parameter defined by

\[\text{Bi} = \frac{ha}{k}\]

represent the Biot number and $\xi_h$ is the coefficient that characterizes the free boundary $s_h$.

It is a simple matter to find the solution to (37)-(40) and thus the solution to (P$_h$) which is given by

\[
\begin{align*}
u_h(x,t) &= \Phi(\eta) = \frac{\text{Bi}u_\infty \sqrt{\pi}}{1 + \text{Bi} \sqrt{\pi} E(\xi_h, \lambda)} \left[ E(\xi_h, \lambda) - E(\eta, \lambda) \right], \\
s_h(t) &= 2a \xi_h \sqrt{t}
\end{align*}
\]
where the function $E$ is given by (17) and $\xi_h$ must be a solution of

$$W_{1h}(x) = 2\lambda W_2(x), \quad x > 0$$

(44)

with $W_{1h}(x) = W_1(x) - \frac{kx}{h}$, and $W_1$, $W_2$ given by (19).

It can be proved that $W_{1h}$ has the same properties as $W_1$, therefore we can apply the results obtained in [4] to conclude that there exists a unique solution $\xi_h$ of (44).

Notice that in problem (P$_h$) a convective boundary condition (9) characterized by the coefficient $h$ at the fixed face $x = 0$ is imposed. This condition constitutes a generalization of the Dirichlet one in the sense that if we take de limit when $h \to \infty$ we must obtain $U(0,t) = u_\infty$. From definition (11), studying the limit behaviour of the solution to our problem (P$_{h}$) when $h \to \infty$ is equivalent to study the case when $Bi \to \infty$.

If for every $h$, we define $\xi_h$ as the unique solution to (44) then, it can be observed that $\{\xi_h\}$ is increasing and bounded, and so convergent. In addition, it can be easily seen that $\xi_h \to \xi$ where $\xi$ is the unique solution to (18). Then, we can state the following result:

**Theorem 3.1.** The solution to problem (P$_{h}$) converges to the solution to problem (P) when $Bi \to \infty$ (i.e. $h \to \infty$), that is:

$$\lim_{h \to \infty} s_h(t) = s(t), \quad t > 0$$

$$\lim_{h \to \infty} u_h(x,t) = u(x,t), \quad 0 < x < s(t), \; t > 0$$

### 3.2 Approximate solution using the classical heat balance integral method

The classical heat balance integral method in order to solve the problem (P$_{h}$) proposes the resolution of the approximate problem (P$_{h_1}$) defined as follows:

Find the temperature $u_{h_1} = u_{h_1}(x,t)$ at the solid region $0 < x < s_{h_1}(t)$ and the location of the free boundary $x = s_{h_1}(t)$ such that satisfy the following conditions: (3), (4), (6), (7), (10), (11).

A solution to this problem will be an approximate one of the problem (P$_{h}$). Proposing the following quadratic temperature profile in space:

$$u_{h_1}(x,t) = A_{h_1} u_\infty \left(1 - \frac{x}{s_{h_1}(t)}\right) + B_{h_1} u_\infty \left(1 - \frac{x}{s_{h_1}(t)}\right)^2, \quad 0 < x < s_{h_1}(t), \; t > 0,$$

(45)

the free boundary takes the form:

$$s_{h_1}(t) = 2a\xi_{h_1} \sqrt{t}, \quad t > 0,$$

(46)

where the constants $A_{h_1}$, $B_{h_1}$ and $\xi_{h_1}$ will be determined from the conditions (9), (10) and (11). Because of (15) and (16), the conditions (5) and (7) are immediately satisfied. From conditions (5) and (11) we obtain:

$$A_{h_1} = \frac{-2(3 + Ste)\xi_{h_1}^2 + (12\lambda Ste - \frac{6}{Bi}) \xi_{h_1} + 6Ste}{Ste \left(\frac{\xi_{h_1}^2}{\xi_{h_1} + (6\lambda + \frac{2}{Bi}) \xi_{h_1} + 3}\right)},$$

(47)

$$B_{h_1} = \frac{3(2 + Ste)\xi_{h_1}^2 + (\frac{3}{Bi} - 6Ste) \xi_{h_1} - 3Ste}{Ste \left(\frac{\xi_{h_1}^2}{\xi_{h_1} + (6\lambda + \frac{2}{Bi}) \xi_{h_1} + 3}\right)}.$$  

(48)

Since $A_{h_1}$ and $B_{h_1}$ are defined from the parameters $\xi_{h_1}$, $\lambda$, Ste and Bi, condition (10) will be used to find the value of $\xi_{h_1}$. In this way, it turns out that $\xi_{h_1}$ must be a positive solution of the
fifth degree polynomial equation:

\[-4\lambda (3 + 2\text{Ste}) z^5 + 2 \left(12 + 9\text{Ste} + 2\text{Ste}^2 - 12\lambda^2 (3 + 2\text{Ste}) - \frac{4\lambda}{\text{Bi}} (3 + 2\text{Ste})\right) z^4 +
\]
\[+ 12 \left(1 + 2\text{Ste})(-3 + (6\lambda^2 - 1)\text{Ste}) + \frac{2}{\text{Bi}^2} - \frac{3\lambda}{\text{Bi}} (1 + \text{Ste})\right) z^3 +
\]
\[+ 12 \left(72\lambda\text{Ste}(1 + 2\text{Ste}) - \frac{6}{\text{Bi}} (3 + 10\text{Ste})\right) z + 18\text{Ste} + 3\text{Ste}^2 = 0, \quad z > 0. \quad (49)\]

It is easy to see that equation (49) has at least one solution. In order to prove uniqueness, we are going to use Descartes rule. Therefore, if we rewrite (49) as

\[\sum_{i=0}^{5} a_i z^i = 0,\]

we have to analyse the sign of each coefficient \(a_i\). Clearly, \(a_5 < 0\) and \(a_0 > 0\). For \(0 < \text{Ste} < 1\) and \(\lambda > 0.62\), as in problem (P\(_1\)), \(a_4 < 0\) for all \(\text{Bi}\). Under these hypothesis: \(a_3 < 0\) if and only if \(\text{Bi} > \frac{7 + 2\text{Ste}}{2(9\lambda + 16\lambda\text{Ste} + 4\lambda^2\text{Ste}^2)}\), and \(a_1 > 0\) if and only if \(\text{Bi} > \frac{3 + 10\text{Ste}}{12\lambda\text{Ste}(1 + 2\text{Ste})}\). Summarizing what has been discussed, the following theorem holds:

**Theorem 3.2.** The solution of problem (P\(_{h1}\)), for a quadratic profile in space, is given by (45) and (46), where the constants \(A_{h1}\) and \(B_{h1}\) are defined by (47) and (48), respectively and \(\xi_{h1}\) is the unique positive solution of the polynomial equation (49) if

\[0 < \text{Ste} < 1, \quad \lambda > 0.62 \quad \text{and} \quad \text{Bi} > \frac{3 + 10\text{Ste}}{12\lambda\text{Ste}(1 + 2\text{Ste})}.\]

In addition, the solution to problem (P\(_{h1}\)) converges to the solution to problem (P\(_1\)) when \(\text{Bi} \to \infty\).

To compare the solutions obtained in (P\(_h\)) and (P\(_{h1}\)), we compute the coefficient that characterizes the free boundary in each problem. The exact value of \(\xi_h\) and the approach \(\xi_{h1}\) are obtained by solving the equations obtained in (14) and (49), respectively. In Figure 4 we plot the coefficients \(\xi_h\) and \(\xi_{h1}\) against \(\text{Bi}\) in order to visualize the behaviour of the approximate solution, fixing \(\text{Ste} = 0.5\) and \(\lambda = 0.7\). In order that the convergence mentioned above of \(\xi_h \to \xi\) and \(\xi_{h1} \to \xi_1\) when \(\text{Bi} \to \infty\), could be appreciated, we also plot \(\xi\) and \(\xi_1\) given by the solution of (18) and (25), respectively.

![Figure 4: Plot of the dimensionless coefficients \(\xi_h\) and \(\xi_{h1}\) against Bi number, for \(\text{Ste} = 0.5\) and \(\lambda = 0.7\).](image)
3.3 Approximate solution using a modified method of the classical heat balance method

An alternative method of the classical heat balance integral method in order to solve the problem ($P_h$) proposes the resolution of the approximate problem ($P_{h2}$) defined as follows:

Find the temperature $u_{h2} = u_{h2}(x, t)$ at the solid region $0 < x < s_h(t)$ and the location of the free boundary $x = s_{h2}(t)$ such that satisfy the following conditions: (3)-(9)-(11).

A solution to this problem will be an approximate one of the problem ($P_h$). Proposing the following quadratic temperature profile in space:

$$u_{h2}(x, t) = A_{h2}u_\infty \left(1 - \frac{x}{s_{h2}(t)}\right) + B_{h2}u_\infty \left(1 - \frac{x}{s_{h2}(t)}\right)^2, \quad 0 < x < s_{h2}(t), \quad t > 0,$$

the free boundary takes the form:

$$s_{h2}(t) = 2a\xi_{h2}\sqrt{t}, \quad t > 0,$$

where the constants $A_{h2}$, $B_{h2}$ and $\xi_{h2}$ will be determined from the conditions (6), (9) and (11). Because of (50) and (51), the conditions (5) and (7) are immediately satisfied. From conditions (5) and (11) we obtain:

$$A_{h2} = \frac{2}{\text{Ste}}\xi_{h2}^2,$$

$$B_{h2} = -\frac{2\xi_{h2}^3}{\text{Ste}} - \frac{1}{\text{Ste}}\frac{\xi_{h2}^2}{\text{Bi}} + \frac{\xi_{h2}}{\text{Bi}}.$$

Since $A_{h2}$ and $B_{h2}$ are defined from the parameters $\xi_{h2}$, Ste and Bi, condition (9) will be used to find the value of $\xi_{h2}$. In this way, it turns out that $\xi_{h2}$ must be a positive solution of the fourth degree polynomial equation:

$$z^4 + \left(6\lambda + \frac{2}{\text{Bi}}\right)z^3 + (6 + \text{Ste})z^2 - \left(6\lambda\text{Ste} + \frac{3}{\text{Bi}}\right)z - 3\text{Ste} = 0, \quad z > 0.$$  

Existence and uniqueness of solution for equation (54) can be easily seen by Descartes’ rule. Therefore, the following theorem can be stated:

**Theorem 3.3.** The solution of problem ($P_{h2}$), for a quadratic profile in space, is given by (50) and (51), where the constants $A_{h2}$ and $B_{h2}$ are defined by (52) and (53), respectively and $\xi_{h2}$ is the unique positive solution of the polynomial equation (54).

In addition, the solution to problem ($P_{h2}$) converges to the solution to problem ($P_2$) when $\text{Bi} \to \infty$.

Comparisons between the exact solution $\xi_h$ with the approximate one $\xi_{h2}$ are shown in Figure 5. We plot them against $\text{Bi}$ for $\text{Ste} = 0.5$ and $\lambda = 0.7$. In order that the convergence of $\xi_h \to \xi$ and $\xi_{h2} \to \xi_2$ when $\text{Bi} \to \infty$, could be appreciated, we also plot $\xi$ and $\xi_2$.

![Figure 5: Plot of the dimensionless coefficients $\xi_h$ and $\xi_{h2}$ against $\text{Bi}$ number, for $\text{Ste} = 0.5$ and $\lambda = 0.7$.](image-url)
3.4 Approximate solution using the refined integral method

The refined heat balance integral method in order to solve the problem \((P_h)\), proposes the resolution of an approximate problem \((P_{h3})\) formulated as follows:

Find the temperature \(u_{h3} = u_{h3}(x, t)\) at the solid region \(0 < x < s_{h3}(t)\) and the location of the free boundary \(x = s_{h3}(t)\) such that satisfy the following conditions: \((5)-(6)-(7)-(12)\).

A solution to this problem will be an approximate one of the problem \((P_h)\). Proposing the following quadratic temperature profile in space:

\[
u_{h3}(x, t) = A_{h3} u_{\infty} \left(1 - \frac{x}{s_{h3}(t)}\right) + B_{h3} u_{\infty} \left(1 - \frac{x}{s_{h3}(t)}\right)^2, \quad 0 < x < s_{h3}(t), \quad t > 0 , \tag{55}\]

the free boundary takes the form:

\[
s_{h3}(t) = 2a\xi_{h3}\sqrt{t}, \quad t > 0 , \tag{56}\]

where the constants \(A_{h3}, B_{h3}\) and \(\xi_{h3}\) will be determined from the conditions \((6), (9)\) and \((12)\). Because of \((55)\) and \((56)\), the conditions \((5)\) and \((7)\) are immediately satisfied. From conditions \((5)\) and \((12)\) we obtain:

\[
A_{h3} = \frac{2}{\text{Ste}} \xi_{h3}^2, \tag{57}
\]

\[
B_{h3} = \left(\frac{1}{\text{Bi}} - \frac{1}{\text{Ste}}\right) \xi_{h3}^2 + \frac{\xi_{h3}}{\text{Bi}}. \tag{58}\]

Since \(A_{h3}\) and \(B_{h3}\) are defined from the parameters \(\xi_{h3}, \text{Ste}\) and \(\text{Bi}\), condition \((5)\) will be used to find the value of \(\xi_{h3}\). In this way, it turns out that \(\xi_{h3}\) must be a positive solution of the third degree polynomial equation:

\[- \left(\frac{\lambda + 1}{\text{Bi}}\right) z^3 - \left(6 + \text{Ste}\right) z^2 + \left(6\lambda\text{Ste} - \frac{3}{\text{Bi}}\right) z + 3\text{Ste} = 0 , \quad z > 0 . \tag{59}\]

Clearly, by Descartes' rule of signs, we can assure that \((59)\) has a unique positive solution. So, the following result holds:

**Theorem 3.4.** The solution of problem \((P_{h3})\), for a quadratic profile in space, is given by \((55)\) and \((56)\), where the constants \(A_{h3}\) and \(B_{h3}\) are defined by \((57)\) and \((58)\), respectively and \(\xi_{h3}\) is the unique positive solution of the polynomial equation \((59)\).

In addition, the solution to problem \((P_{h3})\) converges to the solution to problem \((P_3)\) when \(\text{Bi} \to \infty\).

In Figure 6, the coefficient that characterizes the free boundary of the exact solution \(\xi_h\) of problem \((P_h)\) is compared with the coefficient \(\xi_{h3}\) that characterizes the free boundary of the approximate problem \((P_{h3})\), when we fix \(\text{Ste} = 0.5\) and \(\lambda = 0.7\). We also show the value of \(\xi\) and \(\xi_{h3}\) in order to visualize the mentioned convergence when \(\text{Bi} \to \infty\).

![Figure 6: Plot of the dimensionless coefficients \(\xi\) and \(\xi_{h3}\) against Bi number, for Ste = 0.5 and \(\lambda = 0.7\).](image)
3.5 Comparisons between solutions

Let us compare, for different Bi numbers, the numerical value of the coefficient $\xi_h$ given by (44) and the approximations $\xi_{h1}$, $\xi_{h2}$, and $\xi_{h3}$ given by (49), (54), (59), respectively. In order to obtain which technique gives the best agreement, we display in Table 2, varying Bi between 1 and 100, the exact dimensionless free front, and its different approaches, showing also the percentage relative error committed in each case $E_{rel}(\xi_h) = 100 \left| \frac{\xi_h - \xi}{\xi_h} \right|$

Table 2: Dimensionless free front coefficients and its relative errors.

| Bi  | $\xi_h$ (P$_h$) | $\xi_{h1}$ (P$_{h1}$) | $E_{rel}(\xi_{h1})$ | $\xi_{h2}$ (P$_{h2}$) | $E_{rel}(\xi_{h2})$ | $\xi_{h3}$ (P$_{h3}$) | $E_{rel}(\xi_{h3})$ |
|-----|----------------|----------------------|---------------------|----------------------|---------------------|----------------------|---------------------|
| 1   | 0.3598         | 0.3729               | 3.633 %             | 0.3633               | 0.991 %             | 0.3717               | 3.325 %             |
| 10  | 0.5051         | 0.5504               | 8.965 %             | 0.5185               | 2.657 %             | 0.5286               | 4.655 %             |
| 20  | 0.5144         | 0.5626               | 9.365 %             | 0.5287               | 2.793 %             | 0.5387               | 4.723 %             |
| 30  | 0.5175         | 0.5667               | 9.501 %             | 0.5322               | 2.840 %             | 0.5421               | 4.745 %             |
| 40  | 0.5191         | 0.5687               | 9.569 %             | 0.5339               | 2.863 %             | 0.5438               | 4.756 %             |
| 50  | 0.5206         | 0.5700               | 9.610 %             | 0.5350               | 2.878 %             | 0.5446               | 4.763 %             |
| 60  | 0.5206         | 0.5708               | 9.638 %             | 0.5357               | 2.887 %             | 0.5455               | 4.767 %             |
| 70  | 0.5211         | 0.5714               | 9.657 %             | 0.5362               | 2.894 %             | 0.5459               | 4.770 %             |
| 80  | 0.5214         | 0.5719               | 9.672 %             | 0.5365               | 2.899 %             | 0.5463               | 4.772 %             |
| 90  | 0.5217         | 0.5722               | 9.684 %             | 0.5368               | 2.903 %             | 0.5466               | 4.774 %             |
| 100 | 0.5219         | 0.5725               | 9.693 %             | 0.5371               | 2.906 %             | 0.5468               | 4.776 %             |

From Table 2, for the fixed values Ste = 0.5 and $\lambda$ = 0.7, we can appreciate that the error committed in each approximation increases when Bi becomes greater. We can notice that for the problems (P$_{h1}$), (P$_{h2}$) and (P$_{h3}$) the percentage errors do not exceed 9.693 %, 2.906 % and 4.776 % respectively.

4 Conclusion

In this paper we have considered two different Stefan problems for a semi-infinite material for the non classical heat equation with a source which depends on the heat flux at the fixed face $x = 0$. We have defined the problem (P) with a prescribed constant temperature on $x = 0$, which has been studied in [4] and where it was found a unique exact solution of similarity type. Also we have considered the problem (P$_h$) with a convective boundary condition at the fixed face which was studied in this article, proving existence and uniqueness of an exact solution. The aim of this paper was to apply the classical heat balance integral method, an alternative of it and the refined integral method to those two problems in order to obtain approximate solutions. Comparisons with known exact solutions have been made in all cases and all solutions have been presented in graphical form, providing an overview of popular approaches considered in recent literature. We emphasise that the fact of having the exact solution of problems (P) and (P$_h$) allows us to measure the percentage relative committed error by the approximate techniques applied throughout this paper. In all cases, a dimensionless analysis was carried out by using the parameters: Stefan number (Ste) and the generalized Biot number (Bi), considering Stefan number up to 1 due to the fact that it covers most of the phase change materials.

We have obtained, for $\lambda$ = 0.7 that the best approximate solution to problem (P) was given by (P$_2$) obtaining a relative percentage error that does not exceed 5%. Furthermore the best approximation to problem (P$_h$) was obtained by (P$_{h2}$) obtaining a relative error of 2.9%. Therefore it can be said that in general the optimal approximate technique for solving (P) and (P$_h$) was given by the alternative form of the heat balance integral method, in which the Stefan condition is not removed and remains equal to the exact problem.

In addition it was studied the case when Bi goes to infinity in the solution to the exact problem (P$_h$) an the approximate problems (P$_{h1}$), (P$_{h2}$), (P$_{h3}$), recovering the solutions to the exact problem (P) and the approximate problems (P$_1$), (P$_2$), (P$_3$). Some numerical simulations were also provided in order to visualize this asymptotic behaviour.
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References

1. Alexiades V., Solomon A.D. Mathematical Modelling of Melting and Freezing Processes. Hemisphere-Taylor: Francis, Washington, 1993.

2. Berrone L. R., Tarzia D. A., Villa L. T., Asymptotic behaviour of non-classical heat conduction problem for a semi-infinite material, Mathematical in the Applied Sciences, 23 No.13 (2000), 1161-1177.

3. Bollati J., Sentiel, J.A., Tarzia D. A., Heat balance integral methods applied to the one-phase Stefan problem with a convective boundary condition at the fixed face, Applied Mathematics and Computation, 331 (2018), 1-19.

4. Briozzo, A., Tarzia D. A., Exact solutions for Nonclassical Stefan problems, International Journal of Differential Equations, ID 868059 (2010), doi:10.1155/2010/868059. Article ID 868059 (2018) 1-19.

5. Cannon, J.R., The one-dimensional heat equation, Addison-Wesley: Menlo Park, California, 1984.

6. Cannon, J.R., Yin H. M., A class of nonlinear nonclassical parabolic equations, Journal of differential equations, 79 No. 2 (1989), 266-288.

7. Glashoff K., Sprekels J., An application of Glicksberg’s theorem to set-valued integral equations arising in the theory of thermostats, SIAM Journal on Mathematical Analysis, 12 No. 3 (1981), 477-486.

8. Glashoff K., Sprekels J., The regulation of temperature by thermostats and set-valued integral equations, Journal of integral equations, 4 No. 2 (1982), 95-112.

9. Goodman, T.R., The heat balance integral methods and its application to problems involving a change of phase, Transactions of the ASME, 80 (1958) 335-342.

10. Gupta S.C. The classical Stefan problem. Basic concepts, modelling and analysis. Elsevier: Amsterdam, 2003.

11. Hristov, J., The heat-balance integral method by a parabolic profile with unspecified exponent: analysis and benchmark exercises, Thermal Science, 13 (2009) 27-48.

12. Hristov, J., Research note on a parabolic heat-balance integral method with unspecified exponent: An Entropy Generation Approach in Optimal Profile Determination, Thermal Science, 13 (2009) 49-59.

13. Hristov, J., Multiple integral-balance method: Basic Idea and Example with Mullin’s Model of Thermal Grooving, Thermal Science 21 (2017) 1555-1560.

14. Kenmochi N., Heat conduction with a class of automatic heat-source controls, Free Boundary Problems: Theory and Applications, 186 of Pitman Research Notes in Mathematics Series (1990) 471-474.

15. Kenmochi N., Primicerio M., One-dimensional heat conduction with a class of automatic heat-source controls,IMA Journal of Applied Mathematics, 40 No. 3 (1998) 205-216.

16. Koleva M. N., Numerical solution of heat-conduction problems on a semi-infinite strip with nonlinear localized flow source, in Proceedings of the International Conference ”Pioneers of Bulgarian Mathematics, Sofia, Bulgaria, July 2006.

17. Lunardini V.J., Heat transfer with freezing and thawing. Elsevier: London, 1991.
18. Mitchell, S., Applying the combined integral method to one-dimensional ablation, Applied Mathematical Modelling, 36 (2012) 127-138.
19. Mitchell, S., Myers, T., Improving the accuracy of heat balance integral methods applied to thermal problems with time dependent boundary conditions, Int. J. Heat Mass Transfer, 53 (2010) 3540-3551.
20. Mitchell, S., Myers, T., Application of Standard and Refined Heat Balance Integral Methods to One-Dimensional Stefan Problems, SIAM Review, 52 (2010) 57-86.
21. Mitchell, S., Myers, T., Application of Heat Balance Integral Methods to One-Dimensional Phase Change Problems, Int. J. Diff. Eq., 2012 (2012) 1-22.
22. MacDevette, M., Myers, T., Nanofluids: An innovative phase change material for cold storage systems?, International Journal of Heat and Mass Transfer 92 (2016) 550-557.
23. Mitchell, S. L., O’Brien, B. G., Asymptotic and numerical solutions of a free boundary problem for the sorption of a finite amount of solvent into a glassy polymer, SIAM Journal on Applied Mathematics 74 (2014) 697-723.
24. Mitchell, S. L., Applying the combined integral method to two-phase Stefan problems with delayed onset of phase change, Journal of Computational and Applied Mathematics 28 (2015) 58-73.
25. Mosally, F., Wood, A., Al-Fhaid, A., An exponential heat balance integral method, Applied Mathematics and Computation, 130 (2002) 87-100.
26. Sadoun, N., Si-Ahmed, E.K., Colinet, P., On the refined integral method for the one-phase Stefan problem with time-dependent boundary conditions, Applied Mathematical Modelling, 30 (2006) 531-544.
27. A. D. Solomon, An easily computable solution to a two-phase Stefan problem, Solar energy 33 (1979) 525-528.
28. Tarzia, D.A., A bibliography on moving-free boundary problems for the heat-diffusion equation, The Stefan and related problems, MAT-Serie A: Conferencias, Seminarios Y Trabajos de Matemática, 2 (2000) 1-297.
29. Tarzia, D.A., An explicit solution for a two-phase unidimensional Stefan problem with a convective boundary condition at the fixed face, MAT-Serie A: Conferencias, Seminarios Y Trabajos de Matemática, 8 (2004) 21-27.
30. Tarzia, D.A., Explicit and approximated solutions for heat and mass transfer problem with a moving interface, Chapter 20, in Advanced Topics in Mass Transfer, M. El-Amin (Ed.), InTech Open Access Publisher, Rijeka, (2011) 439-484.
31. Tarzia, D.A., Relationship between Neumann solutions for two phase Lamé-Clapeyron-Stefan problems with convective and temperature boundary conditions, Thermal Science, 21 (2017) 1-11.
32. Tarzia, D.A., Villa L. T., Some non-linear heat conduction problems for a semi-infinite strip with a non-uniform heat source, Revista de la Unión Mtemática Argentina, 41 No. 1 (1998) 99-114.
33. Villa L. T., Problemas de control para una ecuación unidimensional del calor, Revista de la Unión Mtemática Argentina, 32 No. 1 (1986) 163-169.
34. Wood, A.S., A new look at the heat balance integral method, Applied Mathematical Modelling, 25 (2001) 815-824.
35. Zubair S. M., Chaudhry M. A., Exact solutions of solid-liquid phase-change heat transfer when subjected to convective boundary conditions, Heat and Mass Transfer, 30 No. 2 (1994) 77-81.
36. Hristov, J. An approximate analytical (integral-balance) solution to a non-linear heat diffusion equation, Thermal Science 19 (2015) 723-733.
37. Fabre, A., Hristov, J., On the integral-balance approach to the transient heat conduction with linearly temperature-dependent thermal diffusivity, Heat Mass Transfer 53 (2017) 177-204.

38. Hristov, J., Integral solutions to transient nonlinear heat (mass) diffusion with a power-law diffusivity: a semi-infinite medium with fixed boundary conditions, Heat Mass Transfer 52 (2016) 635-655.

39. Hristov, J., Double integral-balance method to the fractional subdiffusion equation: Approximate solutions, optimization problems to be resolved and numerical simulations, Journal of Vibration and Control 23 (2015) 1-24.

40. Hristov, J., Fourth-order fractional diffusion model of thermal grooving: Integral approach to approximate closed form solution of the Mullins model, Mathematical Modelling of Natural Phenomena. doi:10.1051/m2np/2017080.

41. Hristov, J., Integral-balance solution to a nonlinear subdiffusion equation, Ed: Sachin Bhalekar. Bentham Publishing, 2017, Ch. 3, In Frontiers in Fractional Calculus, 71-106.