Characterization of triple $\chi^3$ sequence spaces via Orlicz functions

N. Subramanian and A. Esi

Abstract. In this paper we study of the characterization and general properties of triple gai sequence via Orlicz space of $\chi^3_M$ of $\chi^3$ establishing some inclusion relations.

1. Introduction

Throughout $w$, $\chi$ and $\Lambda$ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write $w^3$ for the set of all complex sequences $(x_{m,n,k})$, where $m, n, k \in \mathbb{N}$, the set of positive integers. Then, $w^3$ is a linear space under the coordinate wise addition and scalar multiplication.

We can represent triple sequences by matrix. In case of double sequences we write in the form of a square. In the case of a triple sequence it will be in the form of a box in three dimensional case.

Some initial work on double series is found in Apostol [1] and double sequence spaces is found in Hardy [5], Subramanian et al. [10-12], and many others. Later on investigated by some initial work on triple sequence spaces is found in Sahiner et al. [9], Esi et al. [2-4], Subramanian et al. [13-19] and many others.

Let $(x_{m,n,k})$ be a triple sequence of real or complex numbers. Then the series $\sum_{m,n,k=1}^\infty x_{m,n,k}$ is called a triple series. The triple series $\sum_{m,n,k=1}^\infty x_{m,n,k}$ is said to be convergent if and only if the triple sequence $(S_{m,n,k})$ is convergent, where

$$S_{m,n,k} = \sum_{i,j,q=1}^{m,n,k} x_{ijq} (m, n, k = 1, 2, 3, \ldots).$$

A sequence $x = (x_{m,n,k})$ is said to be triple analytic if

$$\sup_{m,n,k} |x_{m,n,k}|^{\frac{1}{m+n+k}} < \infty.$$
The vector space of all triple analytic sequences are usually denoted by $\Lambda^3$. A sequence $x = (x_{m,n,k})$ is called triple entire sequence if
$$|x_{m,n,k}|^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.$$The vector space of all triple entire sequences are usually denoted by $\Gamma^3$. The space $\Lambda^3$ and $\Gamma^2$ is a metric space with the metric
$$d(x, y) = \sup_{m,n,k} \{|x_{m,n,k} - y_{m,n,k}|^{\frac{1}{m+n+k}} : m, n, k : 1, 2, 3, \ldots \}.$$for all $x = \{x_{m,n,k}\}$ and $y = \{y_{m,n,k}\}$ in $\Gamma^3$.

Let $\phi = \{\text{finite sequences}\}$. Consider a double sequence $x = (x_{m,n,k})$. The $(m,n,k)$th section $x^{[m,n,k]}$ of the sequence is defined by
$$x^{[m,n,k]} = \sum_{i,j,q} x_{ijq} \delta_{ijq}$$for all $m, n, k \in \mathbb{N}$, where $\delta_{ijq}$ denotes the triple sequence whose only non zero term is $1$ in the $(i,j,k)$th position for each $i, j, k \in \mathbb{N}$.

Consider a triple sequence $x = (x_{m,n,k})$. The $(m,n,k)^{th}$ section $x^{[m,n,k]}$ of the sequence is defined by
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An FK-space(or a metric space) $X$ is said to have AK property if $(\delta_{mnk})$ is a Schauder basis for $X$, or equivalently $x^{[m,n,k]} \to x$.

An FDK-space is a triple sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings are continuous.

A sequence $x = (x_{m,n,k})$ is said to be triple gai sequence if
$$((m + n + k)! |x_{m,n,k}|)^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.$$The triple gai sequences will be denoted by $\chi^3$.

$$\delta_{mnk} = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 & \ldots \\ 0 & 0 & \ldots & 0 & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ 0 & 0 & \ldots & 1 & 0 & \ldots \\ 0 & 0 & \ldots & 0 & 0 & \ldots \end{bmatrix}$$with 1 in the $(m,n,k)^{th}$ position and zero other wise.

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A sequence $x = (x_{m,n,k})$ is said to be triple gai sequence if
$$((m + n + k)! |x_{m,n,k}|)^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.$$The triple gai sequences will be denoted by $\chi^3$. 

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A sequence $x = (x_{m,n,k})$ is said to be triple gai sequence if
$$((m + n + k)! |x_{m,n,k}|)^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.$$The triple gai sequences will be denoted by $\chi^3$. 

$$\delta_{mnk} = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 & \ldots \\ 0 & 0 & \ldots & 0 & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ 0 & 0 & \ldots & 1 & 0 & \ldots \\ 0 & 0 & \ldots & 0 & 0 & \ldots \end{bmatrix}$$with 1 in the $(m,n,k)^{th}$ position and zero other wise.
Consider a triple sequence \( x = (x_{mnk}) \). The \((m, n, k)^{th}\) section \( x^{m,n,k} \) of the sequence is defined by
\[
x^{m,n,k} = \sum_{i,j,q=0}^{m,n} x_{ijq} \mathcal{S}_{ijq}
\]
for all \( m, n, k \in \mathbb{N} \); where \( \mathcal{S}_{ijq} \) denotes the triple sequence whose only non-zero term is \( a_{ijq} \) in the \((i, j, k)^{th}\) place for each \( i, j, k \in \mathbb{N} \).

An FK-space (or a metric space) \( X \) is said to have AK property if \( (\mathcal{S}_{m,n,k}) \) is a Schauder basis for \( X \), or equivalently \( x^{m,n,k} \to x \).

An FDK-space is a triple sequence space endowed with a complete, metrizable, locally convex topology under which the coordinate mappings are continuous.

If \( X \) is a sequence space, we give the following definitions:

(i) \( X' \) is continuous dual of \( X \);
(ii) \( X^\alpha = \{ a = (a_{m,n,k}) : \sum_{m,n,k=1}^{\infty} |a_{m,n,k} x_{m,n,k}| < \infty, \text{ for each } x \in X \} \);
(iii) \( X^\beta = \{ a = (a_{m,n,k}) : \sum_{m,n,k=1}^{\infty} a_{m,n,k} x_{m,n,k} \text{ is convergent, for each } x \in X \} \);
(iv) \( X^\gamma = \{ a = (a_{m,n,k}) : \sup_{m,n \geq 1} \left| \sum_{m,n,k=1}^{M,N,K} a_{m,n,k} x_{m,n,k} \right| < \infty, \text{ for each } x \in X \} \);
(v) Let \( X \) be an FK-space \( \supset \phi \); then \( X^f = \{ f(\mathcal{S}_{m,n,k}) : f \in X' \} \);
(vi) \( X^\delta = \{ a = (a_{m,n,k}) : \sup_{m,n,k} |a_{m,n,k} x_{m,n,k}|^{1/(m+n+k)} < \infty, \text{ for each } x \in X \} \).

\( X^\alpha, X^\beta, X^\gamma \) are called \( \alpha \)- (or Köthe-Toeplitz) dual of \( X \), \( \beta \)- (or generalized-Köthe-Toeplitz) dual of \( X \), \( \gamma \)-dual of \( X \), \( \delta \)-dual of \( X \) respectively. \( X^\alpha \) is defined by Gupta and Kamptan [10]. It is clear that \( X^\alpha \subset X^\beta \) and \( X^\alpha \subset X^\gamma \), but \( X^\alpha \subset X^\gamma \) does not hold.

### 2. Definitions and Preliminaries

A sequence \( x = (x_{m,n,k}) \) is said to be triple analytic if \( \sup_{m,n,k} |x_{m,n,k}|^{\frac{1}{m+n+k}} < \infty \). The vector space of all triple analytic sequences is usually denoted by \( \Lambda^3 \).

A sequence \( x = (x_{m,n,k}) \) is called triple entire sequence if \( |x_{m,n,k}|^{\frac{1}{m+n+k}} \to 0 \) as \( m, n, k \to \infty \). The vector space of triple entire sequences is usually denoted by \( \Gamma^3 \). A sequence \( x = (x_{m,n,k}) \) is called triple gai sequence if \((m+n+k)! |x_{m,n,k}|^{\frac{1}{m+n+k}} \to 0 \) as \( m, n, k \to \infty \). The vector space of triple gai sequences is usually denoted by \( \chi^3 \). The space \( \chi^3 \) is a metric space with the metric
\[
d(x, y) = \sup_{m,n,k} \left\{ \left( (m+n+k)! |x_{m,n,k} - y_{m,n,k}| \right)^{\frac{1}{m+n+k}} \right\},
\]
for all \( x = \{x_{m,n,k} \} \) and \( y = \{y_{m,n,k} \} \) in \( \chi^3 \).

Let \( w^3 \) denote the set of all complex double sequences \( x = (x_{m,n,k})_{m,n,k=1}^\infty \) and \( M : [0, \infty) \to [0, \infty) \) be an Orlicz function. Given a triple sequence,
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$x \in w^3$. Define the sets

$$\chi_M^3 = \{ x \in w^3 : \left( M \left( \frac{(m+n+k)! |x_{mnk}|}{\rho^{\frac{1}{m+n+k}}} \right) \right) \to 0, \quad \text{as } m, n, k \to \infty \text{ for some } \rho > 0 \}$$

and

$$\Lambda_M^3 = \{ x \in w^3 : \sup_{m,n,k \geq 1} \left( M \left( \frac{|x_{mnk}|}{\rho^{\frac{1}{m+n+k}}} \right) \right) < \infty \text{ for some } \rho > 0 \}.$$ 

The space $\Lambda_M^3$ is a metric space with the metric

$$d(x,y) = \inf \left\{ \rho > 0 : \sup_{m,n,k \geq 1} \left( M \left( \frac{|x_{mnk} - y_{mnk}|}{\rho^{\frac{1}{m+n+k}}} \right) \right) \leq 1 \right\}$$

The space $\chi_M^3$ is a metric space with the metric

$$\tilde{d}(x,y) = \inf \left\{ \rho > 0 : \sup_{m,n,k \geq 1} \left( M \left( \frac{(m+n+k)!(x_{mnk} - y_{mnk})}{\rho^{\frac{1}{m+n+k}}} \right) \right) \leq 1 \right\}.$$ 

This paper is a study of the characterization and general properties of gai sequences via triple Orlicz space of $\chi_M^3$ of $\chi^3$ establishing some inclusion relations.

3. Main Results

**Proposition 3.1.** If $M$ is a Orlicz function, then $\chi_M^3$ is a linear set over the set of complex numbers $\mathbb{C}$.

**Proof.** It is trivial. Therefore, the proof is omitted. \qed

**Proposition 3.2.** $(\chi_M^3)^\delta \subsetneq \Lambda_M^3$

**Proof.** Let $y \in \delta$– dual of $\chi_M^3$. Then $\left( M \left( \frac{|x_{mnk}y_{mnk}|}{\rho^{\frac{1}{m+n+k}}} \right) \right) \leq M^{m+n+k}$ for some constant $M > 0$ and for each $x \in \chi_M^3$. Therefore, $\left( M \left( \frac{|y_{mnk}|}{\rho^{\frac{1}{m+n+k}}} \right) \right) \leq M^{m+n+k}$ for each $m,n,k$ by taking $x = (\exists_{mnk})$. This implies that $y \in \Lambda_M^3$. Thus,

$$(3) \quad (\chi_M^3)^\delta \subset \Lambda_M^3.$$ 

We now choose $M = \text{id}$ and define the triple sequences $(y_{mnk})$ and $(x_{mnk})$ by $(y_{mnk}) = 1$ for all $m,n$ and $k$, and by

$$(m+2)!x_{m1} = 2^{(m+2)^2} \quad \text{and} \quad (m+n+k)!x_{mnk} = 0(n, k \geq 2) \text{ for all } m = 1, 2, \ldots$$

Obviously, $y \in \Lambda_M^3$ and since $(m+n+k)!x_{mnk} = 0$ for all $m,n,k \geq 0$, $(m+n+k)! (x_{mnk})$ converges to zero. Hence, $x \in \chi_M^3$. But

$$((m+2)!|a_{m1}x_{m11}|)\frac{1}{\rho^{\frac{1}{m+n+k}}} = 2^{m+2} \to \infty \text{ as } m \to \infty,$$
hence
\begin{equation}
(4) \quad x \notin (\chi^3_M)^\delta.
\end{equation}

From (3) and (4), we are granted \((\chi^3_M)^\delta \subseteq \Lambda^3_M\).
This completes the proof. \hfill \square

**Proposition 3.3.** The dual space of \(\chi^3_M\) is \(\Lambda^3_M\). In other words \((\chi^3_M)^* = \Lambda^3_M\).

**Proof.** We recall that
\[
\mathfrak{I}_{mnk} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ 0 & 0 & \cdots & \frac{1}{(m+n+k)!} & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots 
\end{pmatrix}
\]
with \(\frac{1}{(m+n+k)!}\) in the \((m,n,k)th\) position and zero’s else where, with
\[
x = \mathfrak{I}_{mnk},
\]
\[
\left\{ M \left( \frac{(m+n+k)! |x_{mnk}|}{\rho} \right) \right\} = 
\begin{pmatrix}
M(0^{1/3}/\rho) & M(0^{1/1+n+k}/\rho) \\
\vdots & \ddots \\
M(0^{1/m+4}/\rho) & M(\left(\frac{1}{(m+n+k)!}\right)^{1/m+n+k}/\rho) & M(0^{1/m+n+k+2}/\rho) \\
M(0^{1/m+4}/\rho) & \cdots & M(0^{1/m+n+k+4}/\rho)
\end{pmatrix}
\]
\[
\begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & M(\left(\frac{1}{(m+n+k)!}\right)^{1/m+n+k}/\rho) & 0 \\
0 & \cdots & 0
\end{pmatrix}
\]
which is a triple gai sequence. Hence, \(\mathfrak{I}_{mnk} \in \chi^3_M\), \(f(x) = \sum_{m,n,k=1}^{\infty} x_{mnk}y_{mnk}\)
with \(x \in \chi^3_M\) and \(f \in (\chi^3_M)^*\), where \((\chi^3_M)^*\) is the dual space of \(\chi^3_M\).

Take \(x = (x_{mnk}) = \mathfrak{I}_{mnk} \in \chi^3_M\). Then,
\begin{equation}
(5) \quad |y_{mnk}| \leq ||f|| d(\mathfrak{I}_{mnk},0) < \infty \quad \forall m,n,k.
\end{equation}

Thus, \((y_{mnk})\) is a bounded sequence and hence an triple analytic sequence. In other words, \(y \in \Lambda^3_M\). Therefore \((\chi^3_M)^* = \Lambda^3_M\). This completes the proof. \hfill \square
Proposition 3.4. \((\Lambda_M^3)\beta \subsetneq \chi_M^3\)

Proof. Step 1: Let \((x_{mnk}) \in (\Lambda_M^3)\beta\),

\[
\sum_{m,n,k=1}^{\infty} |x_{mnk}y_{mnk}| < \infty \forall (y_{mnk}) \in \Lambda_M^3.
\]

Let us assume that \((x_{mnk}) \notin \chi_M^3\). Then, there exists a strictly increasing sequence of positive integers \((m_p + n_p + k_p)\) such that

\[
\left( M \left( \frac{(m_p + n_p + k_p)! |x_{(m_p+n_p+k_p)}|}{\rho} \right) \right) > \frac{1}{2(m_p+n_p+k_p)}, \quad (p = 1, 2, 3, \ldots).
\]

Let

\[
\begin{align*}
(m_p + n_p + k_p)! y_{(m_p+n_p+k_p)} &= 2^{(m_p+n_p+k_p)} & \text{for } (p = 1, 2, 3, \ldots), \\
y_{mnk} &= 0 & \text{otherwise}.
\end{align*}
\]

Then \((y_{mnk}) \in \Lambda_M^3\). However,

\[
\sum_{m,n,k=1}^{\infty} \left( M \left( \frac{|x_{mnk}y_{mnk}|}{\rho} \right) \right) = \\
= \sum_{p=1}^{\infty} \left( M \left( \frac{(m_p + n_p + k_p)! |x_{(m_p+n_p+k_p)}y_{(m_p+n_p+k_p)}|}{\rho} \right) \right) > \\
> 1 + 1 \ldots
\]

We know that the infinite series \(1 + 1 + 1 + \ldots\) diverges. Now we choose \(M = id\), where id is the identity and hence \(\sum_{m,n,k=1}^{\infty} (M (|x_{mnk}y_{mnk}|/\rho))\) diverges. This contradicts (6). Hence \((x_{mnk}) \in \chi_M^3\). Therefore,

\[
(\Lambda_M^3)\beta \subset \chi_M^3.
\]

If we now choose \(M = id\), where id is the identity and \(y_{1nk} = x_{1nk} = 1\) and \(y_{mnk} = x_{mnk} = 0\) \((m > 1)\) for all \(n, k\), then obviously \(x \in \chi_M^3\) and \(y \in \Lambda_M^3\), but \(\sum_{m,n,k=1}^{\infty} x_{mnk}y_{mnk} = \infty\). Hence,

\[
y \notin (\Lambda_M^3)\beta.
\]

From (8) and (9), we are granted \((\Lambda_M^3)\beta \subsetneq \chi_M^3\).

This completes the proof. \(\square\)
**Definition 3.5.** Let \( p = (p_{mnk}) \) be a triple sequence of positive real numbers. Then

\[
\chi_M^3(p) = \left\{ x = (x_{mnk}) : \left( M \left( \frac{((m+n+k)!|x_{mnk}|)^{1/m+n+k}}{\rho} \right) \right)^{p_{mnk}} \to 0, \quad (m, n, k \to \infty) \right\}
\]

(10)

for some \( \rho > 0 \). Suppose that \( p_{mnk} \) is a constant for all \( m, n, k \) then \( \chi_M^3(p) = \chi_M^3 \).

**Proposition 3.6.** Let \( 0 \leq p_{mnk} \leq q_{mnk} \) for all \( m, n, k \in \mathbb{N} \) and let \( \left\{ \frac{q_{mnk}}{p_{mnk}} \right\} \) be bounded. Then \( \chi_M^3(q) \subset \chi_M^3(p) \).

**Proof.** Let

\[
x \in \chi_M^3(q),
\]

then

\[
\left( M \left( \frac{((m+n+k)!|x_{mnk}|)^{1/m+n+k}}{\rho} \right) \right)^{q_{mnk}} \to 0, \text{ as } m, n, k \to \infty.
\]

(11)

Let \( t_{mnk} = \left( M \left( \frac{((m+n+k)!|x_{mnk}|)^{1/m+n+k}}{\rho} \right)^{q_{mnk}} \right), \) and let \( \gamma_{mnk} = \frac{p_{mnk}}{q_{mnk}} \). Since \( p_{mnk} \leq q_{mnk} \), we have \( 0 \leq \gamma_{mnk} \leq 1 \). Let \( 0 < \gamma < \gamma_{mnk} \), then

\[
u_{mnk} = \begin{cases} \gamma_{mnk} & \text{if } (t_{mnk} \geq 1) \\ 0 & \text{if } (t_{mnk} < 1) \end{cases}
\]

(12)

\[
u_{mnk} = \begin{cases} t_{mnk} & \text{if } (t_{mnk} \geq 1) \\ 0 & \text{if } (t_{mnk} < 1) \end{cases}
\]

(13)

\[
u_{mnk} = \begin{cases} u_{mnk} & \text{if } (t_{mnk} \geq 1) \\ v_{mnk} & \text{if } (t_{mnk} < 1) \end{cases}
\]

Now, it follows that

\[
u_{mnk}^{\gamma_{mnk}} \leq u_{mnk} \leq t_{mnk}, \quad v_{mnk}^{\gamma_{mnk}} \leq u_{mnk}^{\gamma_{mnk}}.
\]

(14)

Since \( t_{mnk}^{\gamma_{mnk}} = u_{mnk}^{\gamma_{mnk}} + v_{mnk}^{\gamma_{mnk}} \), we have \( t_{mnk}^{\gamma_{mnk}} \leq t_{mnk} + v_{mnk}^{\gamma_{mnk}} \). Thus,

\[
\left( M \left( \frac{((m+n+k)!|x_{mnk}|)^{1/m+n+k}}{\rho} \right)^{q_{mnk}} \right) \gamma_{mnk}^{\gamma_{mnk}} \leq \left( M \left( \frac{((m+n+k)!|x_{mnk}|)^{1/m+n+k}}{\rho} \right)^{q_{mnk}} \right) \gamma_{mnk},
\]

(15)

\[
\left( M \left( \frac{((m+n+k)!|x_{mnk}|)^{1/m+n+k}}{\rho} \right)^{q_{mnk}} \right)^{p_{mnk}/q_{mnk}} \leq \left( M \left( \frac{((m+n+k)!|x_{mnk}|)^{1/m+n+k}}{\rho} \right)^{q_{mnk}} \right)^{q_{mnk}},
\]

which yields

\[
\left( M \left( \frac{((m+n+k)!|x_{mnk}|)^{1/m+n+k}}{\rho} \right)^{p_{mnk}} \right)^{q_{mnk}}.
\]
\[ \leq \left( M \left( ((m + n + k)! |x_{mnk}|)^{1/m+n+k} / \rho \right) \right)^{q_{mnk}}. \]

However,
\[ \left( M \left( ((m + n + k)! |x_{mnk}|)^{1/m+n+k} / \rho \right) \right)^{q_{mnk}} \to 0 \quad \text{(by (12))}. \]

Thus,
\[ \left( M \left( ((m + n + k)! |x_{mnk}|)^{1/m+n+k} / \rho \right) \right)^{p_{mnk}} \to 0 \text{ as } m, n, k \to \infty. \]

Hence,
\[ (16) \quad x \in \chi^3_M(p). \]

Hence (11) and (16), we are granted
\[ (17) \quad \chi^3_M(q) \subset \chi^3_M(p). \]

This completes the proof. \( \square \)

**Proposition 3.7.** (a) Let \( 0 < \inf p_{mnk} \leq p_{mnk} \leq 1 \), then \( \chi^3_M(p) \subset \chi^3_M \).

(b) If \( 1 \leq p_{mnk} \leq \sup p_{mnk} < \infty \), then \( \chi^3_M \subset \chi^3(p) \).

**Proof.** The above statements are special cases of Proposition 3.6. Therefore, it can be proved by similar arguments. \( \square \)

**Proposition 3.8.** If \( 0 < p_{mnk} \leq q_{mnk} < \infty \) for each \( m, n, k \) then \( \chi^3_M(p) \subset \chi^3(q) \).

**Proof.** Let \( x \in \chi^3_M(p) \), then
\[ (18) \quad \left( M \left( ((m + n + k)! |x_{mnk}|)^{1/m+n+k} \right) \right)^{p_{mnk}} \to 0, \text{ as } m, n, k \to \infty. \]

This implies that \( \left( M \left( ((m + n + k)! |x_{mnk}|)^{1/m+n+k} / \rho \right) \right) \leq 1 \) for sufficiently large \( m, n, k \). Since \( M \) is non-decreasing, we get
\[ (19) \quad \left( M \left( ((m + n + k)! |x_{mnk}|)^{1/m+n+k} / \rho \right) \right)^{q_{mnk}} \leq \left( M \left( ((m + n + k)! |x_{mnk}|)^{1/m+n+k} / \rho \right) \right)^{p_{mnk}}, \]

then \( \left( M \left( ((m + n + k)! |x_{mnk}|)^{1/m+n+k} / \rho \right) \right)^{q_{mnk}} \to 0 \text{ as } m, n, k \to \infty \) (by using (18)).

Let \( x \in \chi^3_M(q) \). Hence, \( \chi^3_M(p) \subset \chi^3(q) \). This completes the proof. \( \square \)

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N. Subramanian  
Department of Mathematics  
SASTRA University  
Thanjavur-613 401  
India  
*E-mail address*: nsmaths@yahoo.com

A. Esi  
Adiyaman University  
Adiyaman  
Turkey  
*E-mail address*: aesi23@hotmail.com