Stopping Set Distributions of Some Linear Codes *

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Abstract

Stopping sets and stopping set distribution of an low-density parity-check code are used to determine the performance of this code under iterative decoding over a binary erasure channel (BEC). Let $C$ be a binary $[n,k]$ linear code with parity-check matrix $H$, where the rows of $H$ may be dependent. A stopping set $S$ of $C$ with parity-check matrix $H$ is a subset of column indices of $H$ such that the restriction of $H$ to $S$ does not contain a row of weight one. The stopping set distribution $\{T_i(H)\}_{i=0}^{n}$ enumerates the number of stopping sets with size $i$ of $C$ with parity-check matrix $H$. Note that stopping sets and stopping set distribution are related to the parity-check matrix $H$ of $C$. Let $H^*$ be the parity-check matrix of $C$ which is formed by all the non-zero codewords of its dual code $C^\perp$. A parity-check matrix $H$ is called BEC-optimal if $T_i(H) = T_i(H^*)$, $i = 0, 1, \ldots, n$ and $H$ has the smallest number of rows. On the BEC, iterative decoder of $C$ with BEC-optimal parity-check matrix is an optimal decoder with much lower decoding complexity than the exhaustive decoder. In this paper, we study stopping sets, stopping set distributions and BEC-optimal parity-check matrices of binary linear codes. Using finite geometry in combinatorics, we obtain BEC-optimal parity-check matrices and then determine the stopping set distributions for the Simplex codes, the Hamming codes, the first order Reed-Muller codes and the extended Hamming codes.

*This research is supported in part by the National Natural Science Foundation of China under the Grants 60972011, 60872025 and 10990011. The material in this work was presented in part at the IEEE Information Theory Workshop, Chengdu, China, Oct. 2006.

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Keywords: Low-density parity-check (LDPC) codes, binary erasure channel, iterative decoding, stopping sets, stopping set distribution, finite geometry.

I Introduction

It is well known that the performance of an low-density parity-check (LDPC) code under iterative decoding over a binary erasure channel (BEC) is completely determined by certain combinatorial structures, called stopping sets, of the parity-check matrix of the LDPC code \cite{2,23}. The weight distribution of a linear code plays an important role in determining the performance of this linear code under maximum likelihood decoding over a binary symmetric channel. The so-called stopping set distribution characterizes the performance of an LDPC code under iterative decoding over BEC. Stopping sets and stopping set distributions of linear codes have been studied recently by a number of researchers, for examples, see \cite{1,14,16-17,19-23} and \cite{20-32}.

Let $C$ be a binary $[n,k,d]$ linear code with length $n$, dimension $k$ and minimum distance $d$. Let $H$ be an $m \times n$ parity-check matrix of $C$, where the rows of $H$ may be dependent. Let $I = \{1, 2, \ldots , n\}$ and $J = \{1, 2, \ldots , m\}$ denote the sets of column indices and row indices of $H$, respectively. The Tanner graph $G_H$ \cite{25} corresponding to $H$ is a bipartite graph comprising of $n$ variable nodes labelled by the elements of $I$, $m$ check nodes labelled by the elements of $J$, and the edge set $E \subseteq \{(i, j) : i \in I, j \in J\}$, where there is an edge $(i, j) \in E$ if and only if $h_{ji} = 1$. The girth $g$ of $G_H$, or briefly the girth of $H$, is defined as the minimum length of circles in $G_H$. A stopping set $S$ of $H$ is a subset of column indices $\{1, 2, \ldots , n\}$ such that the restriction of $H$ to $S$, say $H(S)$, does not contain a row of weight one. The smallest size of a
nonempty stopping set, denoted by \( s(H) \), is called the stopping distance of \( C \). The codewords with minimum weight \( d \) are called the minimum codewords of \( C \). Let \( W(x) = \sum_{i=0}^{n} A_i x^i \) denote the weight enumerator of \( C \), where \( A_i \) is the number of codewords with weight \( i \). \( \{A_i\}_{i=0}^{n} \) is called the weight distribution of \( C \). The stopping sets with size \( s(H) \) are called the smallest stopping sets of \( H \). Let \( T^{(H)}(x) = \sum_{i=0}^{n} T_i(H) x^i \) denote the stopping set enumerator of \( C \) with parity-check matrix \( H \), where \( T_i(H) \) is the number of stopping sets of \( H \) with size \( i \). Note that \( \emptyset \) is defined as a stopping set and \( T_0(H) = 1 \). \( \{T_i(H)\}_{i=0}^{n} \) is called the stopping set distribution (SSD) of \( C \) with parity-check matrix \( H \). Note that the stopping sets and stopping set distribution dependent on the choice of the parity-check matrix \( H \) of \( C \).

Schwartz and Vardy \cite{23} defined the stopping redundancy of the binary linear code \( C \) as the minimum number of rows of \( H \) such that \( s(H) = d \). Etzion \cite{4} studied the stopping redundancy of Reed-Muller codes. In particular, the stopping redundancies are determined respectively for the Hamming codes \cite{23}, the Simplex codes and the extended Hamming codes \cite{4}, and an upper bound on the stopping redundancy of the first order Reed-Muller codes was obtained in \cite{4}. In this paper, we study a similar concept of the binary linear code \( C \), BEC-optimal parity-check matrix, in which both the number of stopping sets and the number of rows are minimal among all parity-check matrices of \( C \).

Suppose a codeword \( c = (c_1, c_2, \ldots, c_n) \in C \) is transmitted over the BEC. Let \( r = (r_1, r_2, \ldots, r_n) \) be the received word. The erasure set is defined by \( E_r = \{ j : r_j \neq 0, 1 \} \). An incorrigible set of \( C \) is an erasure set which contains the support of a non-zero codeword of \( C \). As noted by Weber and Abdel-Ghaffar in \cite{28}, the received word \( r \) can be decoded unambiguously if and only if it matches exactly one codeword of \( C \) on all its nonerased
positions. This is equivalent to the condition that the erasure set $E_r$ is not an incorrigible set since $C$ is a linear code. A decoder is said to be optimal for the BEC if it can achieve unambiguous decoding whenever the erasure set is not incorrigible. Note that an exhaustive decoder searching the complete set of codewords is optimal. Let $H^*$ be formed by rows which are all the non-zero codewords of the dual code $C^\perp$, and denote its stopping set enumerator by $T^*(x) = \sum_{i=0}^n T^*_i x^i$. The iterative decoder with parity-check matrix $H^*$ achieves the best possible performance, but has the highest decoding complexity. It is also known from [28] and [12] that the iterative decoder with parity-check matrix $H^*$ is an optimal decoder for the BEC. For fixed parity-check matrix $H$, since $H$ is a sub-matrix formed by some rows of $H^*$, any stopping set of $H^*$ is a stopping set of $H$, but the converse proposition may not be true in general. Hence, we have $T_i(H) \geq T^*_i$ for every $0 \leq i \leq n$.

A parity-check matrix $H$ is called BEC-optimal if $T(H)(x) = T^*(x)$ and $H$ has the smallest number of rows. Since a BEC-optimal parity-check matrix has the same SSD with $H^*$, the iterative decoder with BEC-optimal parity-check matrix must be an optimal decoder and it has lower decoding complexity than $H^*$. Moreover, it achieves the best possible performance as the iterative decoder with parity-check matrix $H^*$.

For the binary $[2^m - 1, 2^m - m - 1, 3]$ Hamming code, say $H(m)$, it is known from [23] that for any parity-check matrix, the stopping distance is equal to the minimum distance. In the 2004 Shannon lecture, McEliece [19] gave an exact expression for the number of smallest stopping sets of $H(m)$ with the full rank parity-check matrix $F$, i.e.,

$$T_3(F) = \frac{1}{6}(5^m - 3^{m+1} + 2^{m+1}). \quad (1)$$

Recently, Abdel-Ghaffar and Weber [1] further determined the whole SSD.
of \( H(m) \) with the parity-check matrix \( F \). From [18] we know that

\[ A_3 = \frac{1}{3}(2^m - 1)(2^{m-1} - 1) \quad (2) \]

and \( A_3 < T_3(F) \), i.e., \( F \) is not BEC-optimal. Weber and Abdel-Ghaffar [27] showed that for the parity-check matrix \( H^* \), \( T_3(H^*) = A_3 \) and \( T_4(H^*) = A_4 \), but they did not determine the whole SSD of \( H^* \). In this paper, we obtain BEC-optimal parity-check matrices and then determine their SSDs for the Simplex codes, the Hamming codes, the first order Reed-Muller codes and the extended Hamming codes by using finite geometry theory. Moreover, the above BEC-optimal parity-check matrices are unique up to the equivalence. The rest of this paper is arranged as follows. In Section II, we give some notations and results in combinatorics that are needed in this paper. In Section III, we obtain the BEC-optimal matrices for the Simplex codes, the Hamming codes, the first order Reed-Muller codes and the extended Hamming codes. In Section IV, in order to determine the SSDs for these BEC-optimal parity-check matrices, the stopping generators of finite geometries are introduced. In Section V, we determine the SSDs for the corresponding BEC-optimal parity-check matrices of these codes. Finally, some conclusions are given in Section VI.

II Preliminaries

In this section, we introduce some notations and results of finite geometry and Gaussian binomial coefficients that will be used in this paper.

II.1 Finite Geometries

Let \( \mathbb{F}_q \) be a finite field of \( q \) elements and \( \mathbb{F}_q^m \) be the \( m \)-dimensional vector space over \( \mathbb{F}_q \), where \( m \geq 2 \).
Let $EG(m, q)$ be the $m$-dimensional Euclidean geometry over $\mathbb{F}_q$ [18, pp. 692-702]. $EG(m, q)$ has $q^m$ points, which are vectors of $\mathbb{F}_q$. The $\mu$-flat in $EG(m, q)$ is a $\mu$-dimensional subspace of $\mathbb{F}_q^m$ or its coset. A point is a 0-flat, a line is a 1-flat, a plane is a 2-flat, and an $(m-1)$-flat is called a hyperplane.

Let $PG(m, q)$ be the $m$-dimensional projective geometry over $\mathbb{F}_q$ [18, pp. 692-702]. $PG(m, q)$ is defined in $\mathbb{F}_q^{m+1} \setminus \{0\}$. Two nonzero vectors $p, p' \in \mathbb{F}_q^{m+1}$ are said to be equivalent if there is $\lambda \in \mathbb{F}_q$ such that $p = \lambda p'$. It is well known that all equivalence classes of $\mathbb{F}_q^{m+1} \setminus \{0\}$ form points of $PG(m, q)$. $PG(m, q)$ has $(q^{m+1} - 1)/(q - 1)$ points. The $\mu$-flat in $PG(m, q)$ is simply the set of equivalence classes contained in a $(\mu + 1)$-dimensional subspace of $\mathbb{F}_q^{m+1}$. 0-flat, 1-flat, and $(m-1)$-flat are also called point, line and hyperplane respectively.

In this paper, in order to present a unified approach, we use $FG(m, q)$ to denote either $EG(m, q)$ or $PG(m, q)$. Let $n$ denote the number of points of $FG(m, q)$. All points of $FG(m, q)$ are indexed from 1 to $n$. We will use $i$ to denote the $i$-th point of $FG(m, q)$ for convenience if there is no confusion. For any two different points $i, i' \in FG(m, q)$, there is one and only one line, say $L(i, i')$, passing through them; for any three distinct points $i, i', i'' \in FG(m, q)$ which are not collinear, there is one and only one plane, say $M(i, i', i'')$, passing through them. For a set of points $\Pi \subseteq FG(m, q)$, let $\chi(\Pi) = (x_1, x_2, \ldots, x_n)$ denote the incidence vector of $\Pi$, i.e., $x_i = 1$ if $i \in \Pi$ and $x_i = 0$ otherwise. For $u > 0$, a $u$-set means a set of $u$ points of $FG(m, q)$. For a non-empty subset $S$ of $FG(m, q)$, define $\langle S \rangle$ as the flat generated by the points in $S$, i.e., $\langle S \rangle$ is the flat containing $S$ with the minimum dimension. Clearly, $\langle S \rangle$ solely exists and for any flat $F \supseteq S$, $\langle S \rangle \subseteq F$. The next lemma is obvious.

**Lemma 1** Let $\Pi$ be a non-empty subset of $FG(m, q)$. Then $\Pi$ is a flat if
and only if \( \langle S \rangle \subseteq \Pi \) for any non-empty \( S \subseteq \Pi \). Moreover,

(i). Let \( \Pi \subseteq PG(m-1,2) \) and \( |\Pi| \geq 2 \). Then \( \Pi \) is a flat if and only if \( L(i, i') \subseteq \Pi \) for any two different points \( i, i' \in \Pi \);

(ii). Let \( \Pi \subseteq EG(m,2) \) and \( |\Pi| \geq 3 \). Then \( \Pi \) is a flat if and only if \( M(i, i', i'') \subseteq \Pi \) for any three distinct points \( i, i', i'' \in \Pi \).

For \( 0 \leq \mu_1 < \mu_2 \leq m \), there are \( N(\mu_2, \mu_1) \) \( \mu_1 \)-flats contained in a given \( \mu_2 \)-flat and \( A(\mu_2, \mu_1) \) \( \mu_2 \)-flats containing a given \( \mu_1 \)-flat, where for \( EG(m, q) \) and \( PG(m, q) \) respectively (see [24])

\[
N_{EG}(\mu_2, \mu_1) = q^{\mu_2-\mu_1} \prod_{i=1}^{\mu_1} \frac{q^{\mu_2-i+1} - 1}{q^{\mu_1-i+1} - 1}, \tag{3}
\]
\[
N_{PG}(\mu_2, \mu_1) = \prod_{i=0}^{\mu_1} \frac{q^{\mu_2-i+1} - 1}{q^{\mu_1-i+1} - 1}, \tag{4}
\]
\[
A_{EG}(\mu_2, \mu_1) = A_{PG}(\mu_2, \mu_1) = \prod_{i=\mu_1+1}^{\mu_2} \frac{q^{m-i+1} - 1}{q^{\mu_2-i+1} - 1}. \tag{5}
\]

For \( 1 \leq \mu \leq m \), let \( n = N(m, 0) \) and \( J = N(m, \mu) \) be the numbers of points and \( \mu \)-flats in \( FG(m, q) \) respectively. The points and \( \mu \)-flats are indexed from 1 to \( n \) and 1 to \( J \) respectively. Let

\[
H = H_{FG}(m, \mu) = (h_{ji})_{J \times n} \tag{6}
\]

be the point-\( \mu \)-flat incidence matrix, where \( h_{ji} = 1 \) for \( 1 \leq j \leq J \) and \( 1 \leq i \leq n \) if and only if the \( j \)th \( \mu \)-flat contains the \( i \)th point. The rows of \( H \) correspond to all the \( \mu \)-flats in \( FG(m, q) \) and have the same weight \( N(\mu, 0) \). The columns of \( H \) correspond to all the points and have the same weight \( A(\mu, 0) \). The binary linear code with the parity-check matrix \( H \) is a class of LDPC codes based on finite geometries [24][15][30], denoted by \( C_{FG}(m, \mu) \). Clearly, the girth of \( H \) is 6 if \( \mu = 1 \) and 4 otherwise [24]. Xia and Fu [30]
proved that
\[ d \geq s(H) \geq A(\mu, \mu - 1) + 1 = \frac{q^{m-\mu+1} - 1}{q - 1} + 1. \] (7)

Clearly, for \( q = 2 \) and \( 2 \leq \mu \leq m \), \( C_{EG}(m, \mu) \) is the \((\mu - 1)\)-th order Reed-Muller code \( RM(m, \mu - 1) \) [18][24]. Since the minimum distance of \( RM(m, \mu - 1) \) is \( 2^{m-\mu+1} \), by (7), the stopping distance is equal to the minimum distance.

II.2 Gaussian binomial coefficients

For non-negative integers \( m \leq n \), let
\[ \left[ \begin{array}{c} n \\ m \end{array} \right]_q = \prod_{i=0}^{m-1} \frac{q^{n-i} - 1}{q^m - 1} \] (8)
denote the \( q \)-binomial coefficient or Gaussian binomial coefficient [18, pp.443-444]. In this paper, we will omit the subscript \( q \) when \( q = 2 \). It is easy to check that
\[ \left[ \begin{array}{c} n \\ 0 \end{array} \right]_q = 1, \quad \left[ \begin{array}{c} n \\ m \end{array} \right]_q = \left[ \begin{array}{c} n \\ n - m \end{array} \right]_q, \] (9)
\[ \left[ \begin{array}{c} n \\ m \end{array} \right]_q \left[ \begin{array}{c} m \\ r \end{array} \right]_q = \left[ \begin{array}{c} n \\ r \end{array} \right]_q \left[ \begin{array}{c} n - r \\ m - r \end{array} \right]_q. \] (10)

The well-known Cauchy Binomial Theorem states that
\[ \prod_{i=1}^{m} (1 + q^i x) = \sum_{i=0}^{m} \left[ \begin{array}{c} m \\ i \end{array} \right]_q q^{i(i+1)/2} x^i. \] (11)

From now on, we will always assume that \( q = 2 \). As usual, we define
\( \binom{0}{0} = 1, \binom{i_2}{i_1} = 0, \binom{0}{i} = 1, \binom{i_2}{i_1} = 0, \sum_{i=i_1}^{i_2} a_i = 0 \) and \( \prod_{i=i_1}^{i_2} a_i = 1 \) if \( i_1 > i_2 \).

Letting \( x = -1/2 \) in (11), we have that
\[ \sum_{i=0}^{m} \left[ \begin{array}{c} m \\ i \end{array} \right] q^{i(i+1)/2} (-1)^i = \delta_{m,0}. \] (12)
where $\delta_{m,n} = 1$ if $m = n$ and $\delta_{m,n} = 0$ otherwise. It is easy to check by (3)-(5) and (8)-(10) that

$$N_{PG}(\mu_2, \mu_1) = \begin{bmatrix} \mu_2 + 1 \\ \mu_1 + 1 \end{bmatrix},$$

$$N_{EG}(\mu_2, \mu_1) = 2^{\mu_2-\mu_1} \begin{bmatrix} \mu_2 \\ \mu_1 \end{bmatrix},$$

$$A(\mu_2, \mu_1) = \begin{bmatrix} m - \mu_1 \\ \mu_2 - \mu_1 \end{bmatrix},$$

$$N(l, l-j)N(l-j, k) = \begin{bmatrix} l-k \\ j \end{bmatrix} N(l, k).$$

### III  BEC Optimal Parity-Check Matrices

In this section, using finite geometry theory, we obtain the BEC-optimal matrices for the Simplex codes, the Hamming codes, the first order Reed-Muller codes and the extended Hamming codes.

The points of $PG(m-1,2)$ are simply the nonzero vectors of $\mathbb{F}_2^m$. A $\mu$-flat of $PG(m-1,2)$ is simply the nonzero linear combination of $\mu + 1$ linearly independent points. By (4) and (5), $PG(m-1,2)$ has $2^m - 1$ points, $(2^m - 1)(2^{m-1} - 1)/3$ lines and $2^m - 1$ hyperplanes. Moreover, every line contains three points.

The points of $EG(m, 2)$ are simply the vectors of $\mathbb{F}_2^m$. A $\mu$-flat of $EG(m, 2)$ is simply a $\mu$-dimensional subspace of its coset. By (3) and (5), $EG(m, 2)$ has $2^m$ points, $2^{m-1}(2^m - 1)$ lines, $2^{m-2}(2^m - 1)(2^{m-1} - 1)/3$ planes and $2^{m+1} - 2$ hyperplanes. Moreover, every line contains two points, every plane contains 4 points.

Let $RM(m, r)$ be the $r$-th order binary Reed-Muller code [18] Ch. 13. By puncturing a fixed coordinate from all codewords of $RM(m, r)$, we obtain the punctured Reed-Muller code $RM(m, r)^*$.
Lemma 2 [18] p. 381, Th. 10] The incidence vectors of all the \((m - r - 1)\)flats of \(PG(m - 1, 2)\) generate \(RM(m, r)^*\).

Lemma 3 [18] p. 385, Th. 12] The incidence vectors of all the \((m - r)\)-flats of \(EG(m, 2)\) generate \(RM(m, r)\).

It is well known that \(RM(m, m - 2)\) is the binary \([2^m, 2^m - m - 1, 4]\) extended Hamming code, which is also denoted by \(\hat{H}(m)\); \(RM(m, 1)\) is the dual code of \(\hat{H}(m)\) and a binary \([2^m, m + 1, 2^{m-1}]\) linear code; \(RM(m, m - 2)^*\) is the binary \([2^m - 1, 2^m - m - 1, 3]\) Hamming code, which is denoted by \(H(m)\); the shortened \(RM(m, 1)\), or the Simplex code \(S(m)\), is the dual code of \(H(m)\) and a binary \([2^m - 1, m, 2^{m-1}]\) linear code.

In \(PG(m - 1, 2)\), by (6), let

\[
H^{(1)} = H_{PG}(m - 1, 1)
\]

be the \((2^m - 1)(2^{m-1} - 1)/3 \times (2^m - 1)\) point-line incidence matrix. Clearly, \(H^{(1)}\) has uniform row weight 3 and uniform column weight \(2^{m-1} - 1\) and girth 6. By (6), let

\[
H^{(2)} = H_{PG}(m - 1, m - 2) + J,
\]

where \(H_{PG}(m - 1, m - 2)\) is the \((2^m - 1) \times (2^m - 1)\) point-hyperplane incidence matrix and \(J\) is a \((2^m - 1) \times (2^m - 1)\) all-1 matrix. It is obvious that for any hyperplane \(P\), the incidence vector of \(\bar{P} = PG(m - 1, 2) \setminus P\) is a row of \(H^{(2)}\) and vice versa. Clearly, \(H^{(2)}\) has uniform row weight \(2^{m-1}\), uniform column weight \(2^{m-1}\) and girth 4.

Lemma 4 \(H^{(1)}\) is a parity-check matrix of \(S(m)\) and the rows of \(H^{(1)}\) form all minimum codewords of \(H(m)\). \(H^{(2)}\) is a parity-check matrix of \(H(m)\) and the rows of \(H^{(2)}\) form all nonzero codewords of \(S(m)\).
Proof: By Lemma\cite{2} the lines of $PG(m - 1, 2)$ generate $RM(m, m - 2)^*$ or $\mathcal{H}(m)$, which implies that $H^{(1)}$ is a parity-check matrix of $S(m)$. Since the number of weight 3 codewords of $\mathcal{H}(m)$ is exactly $(2^m - 1)(2^{m-1} - 1)/3$\cite{18} p. 64, Cor. 16], the rows of $H^{(1)}$ form all minimum codewords of $\mathcal{H}(m)$.

For the second part, since there are $2^m - 1$ rows in $H^{(2)}$ and $S(m)$ has $2^m - 1$ non-zero codewords, it is enough to show that every row of $H^{(2)}$ is orthogonal to all rows of $H^{(1)}$. Let $\chi(\bar{P})$ be a row of $H^{(2)}$, where $P$ is a hyperplane of $PG(m - 1, 2)$. By \cite{18} p. 697, problem (8)], any line $L$ either intersects $P$ on a unique point or lies in $P$. Since $L$ has three points, $L$ can intersect $P$ on either one or three points, i.e., $L$ can only intersect $\bar{P}$ on zero or two points, which implies that $\chi(L)$ is orthogonal to $\chi(\bar{P})$. This finishes the proof. \hfill \Box

In $EG(m, 2)$, by \cite{6}, let

$$H^{(3)} = H_{EG}(m, 2)$$  \hspace{1cm} (19)

be the $2^{m-2}(2^m - 1)(2^{m-1} - 1)/3 \times 2^m$ point-plane incidence matrix. By Lemma\cite{3} $H^{(3)}$ generates $\hat{\mathcal{H}}(m)$, which implies that $H^{(3)}$ is a parity-check matrix of $RM(m, 1)$. Clearly, $H^{(3)}$ has uniform row weight 4 and uniform column weight $(2^m - 1)(2^{m-1} - 1)/3$ and girth 4. By \cite{6}, let

$$H^{(4)} = H_{EG}(m, m - 1),$$ \hspace{1cm} (20)

be the $(2^{m+1} - 2) \times 2^m$ point-hyperplane incidence matrix. By Lemma\cite{3} $H^{(4)}$ generates $RM(m, 1)$, which implies that $H^{(4)}$ is a parity-check matrix of $\hat{\mathcal{H}}(m)$. Clearly, $H^{(4)}$ has uniform row weight $2^{m-1}$, uniform column weight $2^m - 1$ and girth 4.

Hence, $H^{(1)}, H^{(2)}, H^{(3)}, H^{(4)}$ are respectively the parity-check matrices of $S(m), \mathcal{H}(m), RM(m, 1), \hat{\mathcal{H}}(m)$, and their rows are formed by all minimum...
codewords of the dual codes. For convenience, we list the results in the next table, where $\chi(\cdot)$ denotes an incidence vector, $L$ a line, $M$ a plane, $P$ a hyperplane, and $\overline{P} = PG(m-1,2) \setminus P$.

| $S(m)$ | $PG(m-1,2)$ | $H^{(1)}$ has rows formed by all $\chi(L)$ | $H^{(1)*}$ |
|-------|-------------|---------------------------------|------------|
| $\mathcal{H}(m)$ | $PG(m-1,2)$ | $H^{(2)}$ has rows formed by all $\chi(\overline{P})$ | $H^{(2)*}$ |
| $RM(m,1)$ | $EG(m,2)$ | $H^{(3)}$ has rows formed by all $\chi(M)$ | $H^{(3)*}$ |
| $\hat{\mathcal{H}}(m)$ | $EG(m,2)$ | $H^{(4)}$ has rows formed by all $\chi(P)$ | $H^{(4)*}$ |

Moreover, $H^{(1)*}$, $H^{(2)*}$, $H^{(3)*}$, $H^{(4)*}$ have rows formed by all non-zero codewords of $\mathcal{H}(m)$, $S(m)$, $\hat{\mathcal{H}}(m)$, $RM(m,1)$, respectively. Clearly, $H^{(2)} = H^{(2)*}$ and $H^{(4)}$ is formed by all rows except the all-1 row of $H^{(4)*}$.

**Proposition 1** Let $C$ be a binary linear code with parity-check matrix $H$. Let $C^\perp$ be the dual code of $C$. The minimum distance $d^\perp$ of $C^\perp$ is at least 3. Then a necessary condition of $T^{(H)}(x) = T^*(x)$ is that all minimum codewords of $C^\perp$ are contained in rows of $H$.

**Proof:** Assuming the contrary that there is a minimum codeword of $C^\perp$, say $y_0$, is not in the rows of $H$, it is enough to show that there is a stopping set $S$ of $H$ such that $S$ is not a stopping set of $H^*$. Fixing a coordinate $i_0 \in supp(y_0)$, let $S = supp(y_0) \cup \{i_0\}$, where $supp(y_0) = \{1,2,\ldots,n\} \setminus supp(y_0)$. Since $S \cap supp(y_0) = \{i_0\}$, $S$ is not a stopping set of $H^*$. On the other hand, for any non-zero row $y$ of $H$, we will show that $|S \cap supp(y)| \geq 2$ which implies that $S$ is a stopping set of $H$. Clearly, $y$ is a non-zero codeword of $C^\perp$ other than $y_0$. We claim that $|supp(y_0) \cap supp(y)| \geq 2$. Assume the contrary that $|supp(y_0) \cap supp(y)| \leq 1$. Then

$$|supp(y_0) \cap supp(y)| = |supp(y)| - |supp(y_0) \cap supp(y)| \geq |supp(y)| - 1.$$

Clearly, $d_H(y,y_0) \geq d^\perp$ and $w_H(y) = |supp(y)| \geq d^\perp = w_H(y_0)$. Hence,

$$d_H(y,y_0) = w_H(y) + w_H(y_0) - 2|supp(y) \cap supp(y_0)|$$
\[
\leq w_H(y) + w_H(y_0) - 2(w_H(y) - 1) \\
= w_H(y_0) - w_H(y) + 2 \leq 2,
\]

which leads a contradiction. Hence, \(|\text{supp}(y_0) \cap \text{supp}(y)| \geq 2\), which implies that \(|S \cap \text{supp}(y)| \geq |\text{supp}(y_0) \cap \text{supp}(y)| \geq 2\) and thus \(S\) is a stopping set of \(H\). This completes the proof. \(\square\)

**Proposition 2** Let \(C\) be a binary linear code with parity-check matrix \(H\). Then a sufficient condition of \(T^H(x) = T^*(x)\) is that for any non-zero stopping set \(S\) of \(H\),

\[
S = \bigcup_{x \in C, \text{supp}(x) \subseteq S} \text{supp}(x). \tag{21}
\]

**Proof:** Let \(S\) be a stopping set of \(H\) and \(S = \bigcup_{x \in C, \text{supp}(x) \subseteq S} \text{supp}(x)\). We only need to show that \(S\) is also a stopping set of \(H^*\), i.e., for any fixed row of \(H^*\), say \(y\), \(|S \cap \text{supp}(y)| \neq 1\), or

\[
\left| \bigcup_{x \in C, \text{supp}(x) \subseteq S} \left[ \text{supp}(x) \cap \text{supp}(y) \right] \right| \neq 1. \tag{22}
\]

Since \(y\) represents a parity-check equation of \(C\), \(\text{supp}(x) \cap \text{supp}(y)\) must have even number elements for any \(x \in C\). Thus (22) holds, which finishes the proof. \(\square\)

**Remark 1** Suppose a parity-check matrix \(H\) of \(C\) is formed by all minimum codewords of \(C^\perp\) with \(d^\perp \geq 3\). It is easy to see by Propositions 1 and 2 that \(H\) is BEC-optimal provided that \(H\) satisfies the condition of Proposition 2.

**Lemma 5** Let \(S(m)\) be the \([2^m - 1, m, 2^{m-1}]\) Simplex code with parity-check matrix \(H^{(1)}\). Then \(S \subseteq PG(m - 1, 2)\) is a stopping set if and only if \(S = PG(m - 1, 2)\) or \(\bar{S} = PG(m - 1, 2) \setminus S\) is a flat of \(PG(m - 1, 2)\).
Proof: By the definition of stopping set, $S \subseteq PG(m-1, 2)$ is a stopping set if and only if $H^{(1)}(S)$ has no rows with weight one, i.e., $|L \cap S| \neq 1$ for any line $L$. Since $L$ has only three points, $|L \cap S| \neq 1$ is equivalent to $|L \cap \tilde{S}| \neq 2$. Hence, $S$ is a stopping set if and only if any line $L$ intersects $\tilde{S}$ on $0$, or $1$, or $3$ points. Clearly, if $|\tilde{S}| \leq 1$, this is equivalent to $\tilde{S}$ is a flat of $PG(m-1, 2)$ or $S = PG(m-1, 2)$ or $\tilde{S}$ is a flat of $PG(m-1, 2)$. Otherwise, if $|\tilde{S}| \geq 2$, this is equivalent to $L(i, j) \in \tilde{S}$ for any different $i, j \in S$. Hence, the lemma follows by (i) of Lemma 1.

By using (ii) of Lemma 1 and the similar arguments used in the proof of Lemma 5, it is easy to obtain the next lemma.

Lemma 6 Let $RM(m, 1)$ be the first order Reed-Muller code with parity-check matrix $H^{(3)}$. Then $S \subseteq EG(m, 2)$ is a stopping set if and only if $S = EG(m, 2)$ or $\tilde{S} = EG(m, 2) \setminus S$ is a flat of $EG(m, 2)$.

Theorem 1 $H^{(1)}, H^{(2)}, H^{(3)}, H^{(4)}$ are the BEC-optimal parity-check matrices for $S(m), H(m), RM(m, 1), \hat{H}(m)$, respectively. Moreover, for each of the above four cases, there is no other BEC-optimal parity-check matrix up to the permutation of rows.

Proof: Note that the rows of $H^{(1)}, H^{(2)}, H^{(3)}, H^{(4)}$ are formed by all minimum codewords of the dual codes of $S(m), H(m), RM(m, 1), \hat{H}(m)$, respectively. By Proposition 1 we only need to show that $T^{(H)}(x) = T^*(x)$ for $H = H^{(1)}, H^{(2)}, H^{(3)}, H^{(4)}$.

(i) $H = H^{(1)}$: We show that $H^{(1)}$ satisfies the sufficient condition given in Proposition 2. Let $S$ be a non-empty stopping set of $H^{(1)}$. We need to show that $S$ satisfies (21). If $|S| = n$, it is true since there is no codewords of weight 1 in the dual code of $S(m)$. If $1 \leq |S| \leq n - 1$, by Lemma 5 $\tilde{S}$ is a $\mu$-flat of $PG(m - 1, 2)$, where $0 \leq \mu \leq m - 2$. Let $P_1, P_2, \ldots, P_{A(m-2, \mu)}$ be all the hyperplanes which contain $\tilde{S}$, then $\tilde{S} = \bigcap_{j=1}^{A(m-2, \mu)} P_j$, or $S =$
\[ \bigcup_{j=1}^{A(m-2,\mu)} P_j. \]

Since every \( P_j \) is the support of a codeword of \( S(m) \), (21) holds for \( S \).

(ii) \( H = H^{(2)} \): It follows from the fact that \( H^{(2)} = H^{(2)*} \).

(iii) \( H = H^{(3)} \): It is totally similar to the case (i).

(iv) \( H = H^{(4)} \): It follows from the fact that \( H^{(4)} \) is formed by all rows except the all-1 row of \( H^{(4)*} \).

\[ \square \]

IV Generators in Finite Geometries

In this section, we introduce the concept of stopping generators of finite geometries and give some enumeration results that will be used to determine the SSDs for the BEC-optimal parity-check matrices \( H^{(1)}, H^{(2)}, H^{(3)}, H^{(4)} \) of \( S(m), H(m), R\!M(m, 1), \hat{H}(m) \).

Let \( S \) be a non-empty subset of \( FG(m, 2) \). For any \( j \in S \), denote \( S_j = S \setminus \{j\} \). A point \( i \) is said to be independent to \( S \) if \( i \notin \langle S \rangle \). \( S \) is said to be independent if for any \( j \in S \), \( j \) is independent to \( \langle S_j \rangle \). The empty set \( \emptyset \) is defined as an independent set. It is known from [18] that the dimension of a flat \( F \) of \( FG(m, 2) \) is equal to \( |J| - 1 \), where \( J \) is an independent subset of \( F \) with maximum size. Clearly, for a non-empty set \( S \), \( S \) is independent if and only if \( \langle S \rangle \) is an \((|S| - 1)\)-flat.

For an integer \( 0 \leq l \leq m \), let \( F^{(l)} \) denote an \( l \)-flat of \( FG(m, 2) \). \( F^{(l)} \) has \( N(l, 0) \) points. Let \( u \geq 1 \), if a \( u \)-set generates \( F^{(l)} \), we call it a \( u \)-generator of \( F^{(l)} \). If a \( u \)-generator \( S \) of \( F^{(l)} \) satisfies \( \langle S_j \rangle = \langle S \rangle = F^{(l)} \) for any \( j \in S \), we call \( S \) a stopping \( u \)-generator of \( F^{(l)} \). Define \( B(u, l) \) as the number of \( u \)-generators of \( F^{(l)} \) and \( G(u, l) \) as the number of stopping \( u \)-generators of \( F^{(l)} \), i.e., for \( u \geq 1 \) and \( l \geq 0 \),

\[ B(u, l) = |\{ S \subseteq F^{(l)} : |S| = u, \langle S \rangle = F^{(l)} \}|, \quad (23) \]
\[ G(u, l) = |\{ S \subseteq F(l) : |S| = u, \forall j \in S, \langle S_j \rangle = F(l) \}|. \quad (24) \]

Define \( B(u, l) = 0 \) if \( u \leq 0 \) or \( l < 0 \). Clearly,
\[ G(u, l) \leq B(u, l), \quad (25) \]
\[ B(u, l) = 0 \text{ if } u \leq l. \quad (26) \]

For a \( u \)-set \( S \), where \( u \geq 1 \), \( S \) is a \( u \)-generator of a \((u-1)\)-flat if and only if \( S \) is independent. A non-empty independent set \( S \) could not be a stopping generator, this is because for any \( j \in S \), \( \langle S_j \rangle \subset \langle S \rangle \). Hence, \( G(u, u - 1) = 0 \) for any \( u \geq 1 \). Combining this fact with (25)-(26), we have
\[ G(u, l) = 0 \text{ if } u \leq l + 1. \quad (27) \]

**Lemma 7** For any \( u \geq 1 \) and \( l \geq 0 \), \( B(u, l) \) satisfies the following recursive equation
\[ B(1, 0) = 1, \quad B(u, 0) = 0 \text{ if } u \geq 2, \quad (28) \]
\[ \binom{N(l, 0)}{u} = \sum_{i=0}^{l} N(l, i) B(u, i), \quad l \geq 0. \quad (29) \]

*Proof:* (28) is obvious by (23) and (26). In \( F(l) \), there are \( \binom{N(l, 0)}{u} \) \( u \)-subsets, and each of which generates an \( i \)-flat, where \( 0 \leq i \leq l \). There are \( N(l, i) \) \( i \)-flats in \( F(l) \), and each of which contains \( B(u, i) \) \( u \)-generators of this \( i \)-flat \( F^{(i)} \). Clearly, these \( u \)-sets are distinct, which implies the lemma. \( \square \)

**Lemma 8**
\[ B(u, l) = \sum_{j=0}^{l} (-1)^j 2^{j(j-1)/2} N(l, l-j) \binom{N(l-j, 0)}{u}, \quad (30) \]
\[ B_{PCG}(u, l) = \sum_{j=0}^{l} (-1)^j 2^{j(j-1)/2} \left[ \binom{l+1}{j} \right] \binom{2^{l-j+1}-1}{u}, \quad (31) \]
\[ B_{EG}(u, l) = \sum_{j=0}^{l} (-1)^j 2^{j(j+1)/2} \binom{l}{j} \binom{2^{l-j}}{u}, \quad (32) \]

Proof: By Lemma 7,

\[ \binom{N(l-j, 0)}{u} = \sum_{k=0}^{l-j} N(l-j, k) B(u, k). \]

Hence, by (16) and (12),

\[
\begin{align*}
&\sum_{j=0}^{l} (-1)^j 2^{j(j-1)/2} N(l, l-j) \binom{N(l-j, 0)}{u} \\
&= \sum_{j=0}^{l-j} \sum_{k=0}^{j} (-1)^j 2^{j(j-1)/2} N(l,j) N(l-j,k) B(u,k) \\
&= \sum_{k=0}^{l} \sum_{j=0}^{l-k} (-1)^j 2^{j(j-1)/2} \binom{l-k}{j} N(l,k) B(u,k) \\
&= \sum_{k=0}^{l} N(l,k) B(u,k) \cdot \sum_{j=0}^{l-k} (-1)^j 2^{j(j-1)/2} \binom{l-k}{j} \\
&= \sum_{k=0}^{l} N(l,k) B(u,k) \cdot \delta_{l-k,0} \\
&= N(l,l) B(u,l) = B(u,l).
\end{align*}
\]

Moreover, (31) and (32) follow from (30) and (13)-(14).

**Lemma 9** Let \( l \geq 0 \), \( u \geq l+1 \), and \( S \) be a \( u \)-generator of \( F^{(l)} \), where \( F^{(l)} \) is an \( l \)-flat. Let \( J = \{ j \in S : j \notin \langle S_j \rangle \} \). Then

(i) \( J \) is an independent set;

(ii) \( J = \emptyset \) if and only if \( S \) is a stopping \( u \)-generator of \( F^{(l)} \);

(iii) \( J = S \) if and only if \( S \) is an independent set;

(iv) otherwise, suppose \( J \) is a non-empty proper subset of \( S \) and \( |J| = k \), then \( \langle S \setminus J \rangle \) is an \((l-k)\)-flat, \( 1 \leq k \leq l-1 \), and \( S \setminus J \) is a stopping \((u-k)\)-generator of \( \langle S \setminus J \rangle \).
Proof: Note that $\emptyset$ is an independent set according to the definition. If $J \neq \emptyset$, for any $j \in J$, $J \subseteq S$ implies $J_j \subseteq S_j$ and $\langle J_j \rangle \subseteq \langle S_j \rangle$. Hence, $j \in J$ implies $j \notin \langle S_j \rangle$ and $j \notin \langle J_j \rangle$. This completes the proof of (i). By the definition of stopping generator, (ii) is obvious. (iii) follows from (i) and the definition of independent set. Next, we suppose $J = \{j_1, j_2, \ldots, j_k\}$ ($k \geq 1$) is a non-empty proper subset of $S$ and give the proof of (iv).

Note that $\langle S \rangle = F^{(l)}$ is an $l$-flat. Since $j_1 \in \langle S \rangle$ and $j_1 \notin \langle S \setminus \{j_1\} \rangle$, $\langle S \setminus \{j_1\} \rangle$ is an $(l - 1)$-flat in $\langle S \rangle$. Since $j_2 \in S \setminus \{j_1\}$, $j_2 \in \langle S \setminus \{j_1\} \rangle$. Moreover, $j_2 \notin \langle S \setminus \{j_1, j_2\} \rangle$ since $j_2 \notin \langle S \setminus \{j_2\} \rangle$. Hence, $\langle S \setminus \{j_1, j_2\} \rangle$ is an $(l - 2)$-flat in $\langle S \setminus \{j_1\} \rangle$. Repeating the above procedure, we have that $\langle S \setminus \{j_1, j_2, j_3\} \rangle$ is an $(l - 3)$-flat in $\langle S \setminus \{j_1, j_2\} \rangle$, $\ldots$, $\langle S \setminus J \rangle$ is an $(l - k)$-flat in $\langle S \setminus \{j_1, \ldots, j_k\} \rangle$. Since $S \setminus J$ is non-empty, $l - k \geq 0$. If $k = l$, $\langle S \setminus J \rangle$ is a single point set, say $\{i\}$. Then $S \setminus J = \{i\}$ or $S_i = J$. Hence, by (i), $\langle S_i \rangle$ is an $(l - 1)$-flat, which implies $\langle S_i \rangle \subseteq \langle S \rangle$ and $i \notin \langle S_i \rangle$. This means $i \in J$ and leads a contradiction. Therefore $1 \leq k \leq l - 1$.

Now, we show that $S \setminus J$ is a stopping generator, i.e., for any $j \in S \setminus J$, $\langle S \setminus J \rangle = \langle S_j \setminus J \rangle$. Assume by contrary that there exists $j^* \in S \setminus J$ such that $S_{j^*} \setminus J$ generates an $(l - k - 1)$-flat in $\langle S \setminus J \rangle$. By using the inverse procedure given in the last paragraph, it is not difficult to see that $\langle S_{j^*} \setminus \{j_1, \ldots, j_k\} \rangle$ is an $(l - k)$-flat, $\langle S_{j^*} \setminus \{j_1, \ldots, j_{k-1}\} \rangle$ is an $(l - k + 1)$-flat, $\ldots$, $\langle S_{j^*} \setminus \{j_1\} \rangle$ is an $(l - 2)$-flat, $\langle S_{j^*} \rangle$ is an $(l - 1)$-flat, which implies that $j^* \notin \langle S_{j^*} \rangle$ or $j^* \in J$. This gives a contradiction.

Combining these results, the lemma follows. \hfill $\Box$

**Lemma 10** For any $l \geq 1$ and $0 \leq k \leq l$, let $F^{(l)}$ be an $l$-flat. Then there are exactly $\alpha(l, k)$ pairs $(F^{(l-k)}, J^{(k)})$ such that $F^{(l-k)} \subseteq F^{(l)}$ is an $(l - k)$-flat, $J^{(k)} \subseteq F^{(l)}$ is an independent $k$-set, and $\langle J^{(k)} \cup F^{(l-k)} \rangle = F^{(l)}$.  

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where

\[ \alpha(l, k) = \frac{N(l, l - k)}{k!} \prod_{i=1}^{k} [N(l, 0) - N(l - k + i, 0)], \quad (33) \]

\[ \alpha_{PG}(l, k) = \frac{1}{k!} \prod_{i=1}^{k} 2^{l-i+1}(2^{l-i+2} - 1), \quad (34) \]

\[ \alpha_{EG}(l, k) = \frac{1}{k!} \prod_{i=1}^{k} 2^{l-i+1}(2^{l-i+1} - 1). \quad (35) \]

**Proof:** Clearly, \( \alpha(l, 0) = 1 \) which implies that (33) holds for \( k = 0 \). It is easy to verify (34) and (35) from (33) and (3)-(4). Hence, it is enough to show (33) for \( 1 \leq k \leq l \). Suppose \( F(l-k) \subseteq F(l) \) is a fixed \((l-k)\)-flat. We enumerate all suitable independent \( k \)-set \( J(k) \) as follows. Choosing the first point from \( F(l) \setminus F(l-k) \), there are \( N(l, 0) - N(l - k, 0) \) choices. \( F(l-k) \) and the first point generate an \((l-k+1)\)-flat, say \( F(l-k+1) \). Choosing the second point from \( F(l) \setminus F(l-k+1) \), there are \( N(l, 0) - N(l - k+1, 0) \) choices. \( F(l-k+1) \) and the second point generate an \((l-k+2)\)-flat, say \( F(l-k+2) \). Repeating the above procedure, we have \( N(l, 0) - N(l - 1, 0) \) choices when choosing the \( k \)-th point. It is easy to see that there are exactly \( k! \) repetitions for the above choosing procedure. Hence, there are totally

\[ \frac{1}{k!} \prod_{i=0}^{k-1} [N(l, 0) - N(l - k + i, 0)] \]

independent sets \( J(k) = \{ j_1, j_2, \ldots, j_k \} \) to form a suitable pair \((F(l-k), J(k))\) for fixed \((l-k)\)-flat \( F(l-k) \). Hence, (33) follows from the fact that there are \( N(l, l - k) \) \((l-k)\)-flats in \( F(l) \).

\[ \square \]

**Lemma 11** For any \( u \geq 1 \) and \( l \geq 0 \), \( G(u, l) \) satisfies the following recursive equation

\[ G(u, 0) = 0 \quad \text{for any } u; \quad G(u, l) = 0 \quad \text{for any } u \leq l + 1; \quad (36) \]
\[ B(u, l) = \sum_{k=0}^{l-1} \alpha(l, k)G(u - k, l - k), \quad u \geq l + 2. \quad (37) \]

**Proof:** It is easy to check that (36) holds by the definition (24) of \( G(u, l) \) and (27). Below we suppose \( l \geq 1 \) and \( u \geq l + 2 \). Since \( u \geq l + 2 \), by Lemma 9 each \( u \)-generator of \( F^{(l)} \) is 1-1 corresponding to a \((u - k)\)-stopping generator of an \((l - k)\)-flat of \( F^{(l)} \), where \( 0 \leq k \leq l - 1 \). For fixed \( 0 \leq k \leq l - 1 \), by Lemma 10 there are \( \alpha(l, k)G(u - k, l - k) \) such \( u \)-generators of \( F^{(l)} \). Hence, (37) follows by counting these \( u \)-generators where \( k \) is from 0 to \( l - 1 \). \( \square \)

**Lemma 12** Let \( u \geq l + 2 \). Then

\[ G(u, l) = \sum_{k=0}^{l-1} (-1)^k \alpha(l, k)B(u - k, l - k), \quad (38) \]
\[ G_{PG}(u, l) = \sum_{k=0}^{l-1} B_{PG}(u - k, l - k) \left( \frac{(-1)^k}{k!} \prod_{i=1}^{k} 2^{l-i+1}(2^{l-i+2} - 1) \right), \quad (39) \]
\[ G_{EG}(u, l) = \sum_{k=0}^{l-1} B_{EG}(u - k, l - k) \left( \frac{(-1)^k}{k!} \prod_{i=1}^{k} 2^{l-i+1}(2^{l-i+1} - 1) \right). \quad (40) \]

**Proof:** It is easy to check by (34)-(35) that

\[ \alpha(l, 0) = 1, \]
\[ \alpha(l, k)\alpha(l - k, j - k) = \binom{j}{k} \alpha(l, j). \]

Clearly, \( \sum_{k=0}^{j} (-1)^k \binom{j}{k} = \delta_{j,0} \). Moreover, by Lemma 11

\[ B(u - k, l - k) = \sum_{j=0}^{l-k-1} \alpha(l - k, j)G(u - k - j, l - k - j) \]
\[ = \sum_{j=k}^{l-1} \alpha(l - k, j - k)G(u - j, l - j). \]

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Hence, using these equations, we have

\[
\sum_{k=0}^{l-1} (-1)^k \alpha(l, k) B(u - k, l - k)
\]

\[
= \sum_{k=0}^{l-1} \sum_{j=k}^{l-1} (-1)^k \alpha(l, k) \alpha(l - k, j - k) G(u - j, l - j)
\]

\[
= \sum_{j=0}^{l-1} \sum_{k=0}^{j} (-1)^{j} \binom{j}{k} \alpha(l, j) G(u - j, l - j)
\]

\[
= \sum_{j=0}^{l-1} \delta_{j, 0} \alpha(l, j) G(u - j, l - j)
\]

\[
= \alpha(l, 0) G(u, l) = G(u, l).
\]

Moreover, (39) and (40) follow from (38) and (34)-(35). \qed

\section{Stopping Set Distributions}

In this section, we determine the SSDs for the Simplex codes $S(m)$, the Hamming codes $H(m)$, the first order Reed-Muller codes $RM(m, 1)$ and the extended Hamming codes $\hat{H}(m)$ with the BEC-optimal parity-check matrices $H^{(1)}, H^{(2)}, H^{(3)}, H^{(4)}$, respectively.

\subsection{Simplex Codes $S(m)$}

Throughout this subsection, $n = 2^m - 1$ and $PG(m - 1, 2) = \{1, 2, \ldots, 2^m - 1\}$. By (4), there are $N_{PG}(m - 1, \mu) \mu$-flats in $PG(m - 1, 2)$ and a $\mu$-flat has exactly $2^{\mu+1} - 1$ points. The next theorem follows from Lemma 5 immediately.

Theorem 2 Let $S(m)$ be the $[2^m - 1, m, 2^{m-1}]$ Simplex code with parity-
check matrix $H^{(1)}$. Let $\{T_i(H^{(1)})\}_{i=0}^{n}$ be the SSD of $S(m)$. Then

$$T_i(H^{(1)}) = \begin{cases} 1 & \text{if } i = 0 \text{ or } 2^m - 1, \\ N_{PG}(m - 1, \mu) & \text{if } i = 2^m - 2^{\mu+1}, \\ 0 & \mu = 0, \ldots, m - 2, \\ \text{otherwise}, \end{cases}$$

where

$$N_{PG}(m - 1, \mu) = \prod_{i=0}^{\mu} \frac{2^{m-i} - 1}{2^{\mu-i+1} - 1}.$$  

Remark 2 Let $\mu = m - 2$, by Theorem 2, it is easy to check that the number of smallest stopping sets $T_{2m-1}(H^{(1)}) = 2^m - 1$, which coincides with the number of minimum codewords of $S(m)$.

Example 1 By Theorem 2 we can easily calculate the SSDs of $S(m)$ with parity-check matrix $H^{(1)}$ by Mathematica software. Here are some examples for $m = 3, 4, 5$.

For $S(3)$,

$$T(x) = 1 + 7x^4 + 7x^6 + x^7.$$  

For $S(4)$,

$$T(x) = 1 + 15x^8 + 35x^{12} + 15x^{14} + x^{15}.$$  

For $S(5)$,

$$T(x) = 1 + 31x^{16} + 155x^{24} + 155x^{28} + 31x^{30} + x^{31}.$$  

It is worthy to note that all examples in this section besides the above one are calculated through two ways, one of which uses the derived formula, and the other of which uses the exhaust computer search for verification.
V.2 Hamming Codes $\mathcal{H}(m)$

Throughout this subsection, $n = 2^m - 1$ and $PG(m-1, 2) = \{1, 2, \ldots, 2^m - 1\}$. Note that $H^{(2)} = H^{(2)*}$ and $P$ is a hyperplane if and only if $\chi(\bar{P})$ is a row of $H^{(2)}$.

Lemma 13 Let $\mathcal{H}(m)$ be the $[2^m - 1, 2^m - m - 1, 3]$ Hamming code with parity-check matrix $H^{(2)}$. Then $S \subseteq PG(m-1, 2)$ is a non-empty stopping set if and only if $\langle S \rangle = \langle S_j \rangle$ for any $j \in S$, where $S_j = S \setminus \{j\}$.

Proof: By the definition of stopping sets, a non-empty subset $S \subseteq PG(m-1, 2)$ is a stopping set if and only if $H^{(2)}(S)$ has no rows with weight one, i.e., $|\bar{P} \cap S| \neq 1$ for any hyperplane $P$ of $PG(m-1, 2)$. Clearly, $|\bar{P} \cap S| \neq 1$ is equivalent to $|P \cap S| \neq |S| - 1$. Hence, we only need to show that $|P \cap S| \neq |S| - 1$ for any hyperplane $P$ of $PG(m-1, 2)$ if and only if $\langle S \rangle = \langle S_j \rangle$ for any $j \in S$.

Firstly, we will prove the necessary condition. Suppose that $S$ satisfies $|P \cap S| \neq |S| - 1$ for any hyperplane $P$. Clearly, $\langle S_j \rangle \subseteq \langle S \rangle$. Assume by contrary that there exists $j \in S$ such that $\langle S_j \rangle \subset \langle S \rangle$, i.e., $d_j = d - 1$, where $d_j$ and $d$ are the dimensions of $\langle S_j \rangle$ and $\langle S \rangle$ respectively. If $d = m - 1$, then $\langle S_j \rangle$ is a hyperplane not including $S$, i.e., $|\langle S_j \rangle \cap S| = |S_j| = |S| - 1$, which leads a contradiction. Otherwise, if $d < m - 1$, by (5), there are $A(m-2, d_j)$ hyperplanes containing $S$, and there are $A(m-2, d)$ hyperplanes containing $S_j$. It is easy to check that for $PG(m-1, 2)$

$$\frac{A(m-2, d_j)}{A(m-2, d)} = \prod_{i=d_j+1}^{d} \frac{2^m-i-1}{2^{m-i-1}-1} = \frac{2^{m-d}-1}{2^{m-d-1}-1} > 1,$$

which implies that there exists a hyperplane, say $P^*$, such that $S_j \subseteq P^*$ and $S \not\subseteq P^*$. Hence, $|P^* \cap S| = |S| - 1$, which leads a contradiction.
On the other hand, suppose that $S$ satisfies $\langle S \rangle = \langle S_j \rangle$ for any $j \in S$. Assume by contrary that there exists a hyperplane $P^*$ such that $|P^* \cap S| = |S| - 1$, i.e., there exists a point $j^* \in S$ such that $P^* \cap S = S_{j^*}$. Then $S_{j^*} \subseteq P^*$ and $S \not\subseteq P^*$, i.e., $\langle S_{j^*} \rangle \subseteq P^*$ and $\langle S \rangle \not\subseteq P^*$, which leads a contradiction.

Combining these claims, the lemma follows.

\[ \square \]

**Remark 3** It is easy to see from Lemma 13 that when $u \geq 2^{m-1} + 1$, any $u$-set is a stopping set since any set with at least $2^{m-1}$ points generates $PG(m-1, 2)$.

**Theorem 3** Let $\mathcal{H}(m)$ be the $[2^m-1, 2^m-m-1, 3]$ Hamming code with parity-check matrix $H^{(2)}$. Let $\{T_i(H^{(2)})\}_{i=0}^n$ be the SSD of $\mathcal{H}(m)$. Then

\[
T_u(H^{(2)}) = \begin{cases} 
1, & u = 0, \\
0, & u = 1, 2, \\
\sum_{l=\lceil \log u \rceil}^{\min\{u-2, m-1\}} N_{PG}(m-1, l) G_{PG}(u, l), & u = 3, \ldots, 2^{m-1}, \\
\binom{2^m-1}{u}, & u = 2^{m-1} + 1, \ldots, 2^m - 1,
\end{cases}
\]

\[
(41)
\]

where $N_{PG}(m-1, l)$ and $G_{PG}(u, l)$ are defined in (4) and (39) respectively.

**Proof:** Clearly, $T_0 = 1$. By Lemma 13 and the definition $\text{(24)}$ of $G_{PG}(u, l)$, it is easy to see that

\[
T_u = \sum_{l=0}^{m-1} N_{PG}(m-1, l) G_{PG}(u, l).
\]

\[
(42)
\]

Since any $u$-set in $PG(m-1, 2)$ generates a flat with dimension at least $\lceil \log u \rceil$,

\[
B_{PG}(u, l) = G_{PG}(u, l) = 0 \quad \text{if } l < \lceil \log u \rceil.
\]

\[
(43)
\]
Combining (42), (43) and (27), we have that
\[
T_u = \min_{m-2, m-1} \left\{ u - 2, m - 1 \right\} \sum_{l=[\log u]}^{N_{PG}(m-1, l)} G_{PG}(u, l), \quad 1 \leq u \leq 2^m - 1.
\] (44)

Let \( u = 1, 2 \), we have \( T_1 = T_2 = 0 \). Combining these facts and Remark 3, (41) follows.

**Remark 4** By Theorem 3, we have that
\[
T_3(H^{(2)}) = (2^m - 1)(2^{m-1} - 1)/3,
\]
\[
T_4(H^{(2)}) = (2^m - 1)(2^{m-1} - 1)(2^{m-2} - 1)/3.
\]

It is easy to see from [18] that \( A_3 = T_3(H^{(2)}) \) and \( A_4 = T_4(H^{(2)}) \) for \( H(m) \), which were also obtained by Weber and Abdel-Ghaffar [27].

**Example 2** By Theorem 3, we can easily calculate the SSDs for \( H(m) \) by Mathematica software. Here are some examples for \( m = 3, 4, 5 \).

For \( H(3) \),
\[
T(x) = 1 + 7x^3 + 7x^4 + 21x^5 + 7x^6 + x^7.
\]

For \( H(4) \),
\[
T(x) = 1 + 35x^3 + 105x^4 + 483x^5 + 2485x^6 + 5595x^7 + 6315x^8
+5005x^9 + 3003x^{10} + 1365x^{11} + 455x^{12} + 105x^{13} + 15x^{14} + x^{15}.
\]

For \( H(5) \),
\[
T(x) = 1 + 155x^3 + 1085x^4 + 8463x^5 + 88573x^6 + 798095x^7 + 4909005x^8
+16998075x^9 + 41869685x^{10} + 83182827x^{11}
+140443485x^{12} + 206027395x^{13} + 265130445x^{14} + 300532755x^{15}
\]
V.3 The First Order Reed-Muller Codes $RM(m,1)$

Throughout this subsection, $n = 2^m$ and $EG(m,2) = \{1,2,\ldots,2^m\}$. By (3), there are $N_{EG}(m,\mu)$ $\mu$-flats in $EG(m,2)$ and a $\mu$-flat has exactly $2^\mu$ points. The next theorem follows from Lemma 6 immediately.

**Theorem 4** Let $RM(m,1)$ be the first order Reed-Muller code with parity-check matrix $H^{(3)}$. Let $\{T_i(H^{(3)})\}_{i=0}^n$ be the SSD of $RM(m,1)$. Then

$$T_i(H^{(3)}) = \begin{cases} 1 & \text{if } i = 0 \text{ or } 2^m, \\ N_{EG}(m,\mu) & \text{if } i = 2^m - 2^\mu, \\ 0 & \mu = 0,1,\ldots,m-1, \\ \text{otherwise} & \end{cases}$$

(45)

where

$$N_{EG}(m,\mu) = 2^{m-\mu} \prod_{i=1}^\mu \frac{2^{m-i+1} - 1}{2^{\mu-i+1} - 1}.$$  

**Remark 5** Let $\mu = m-1$, by Theorem 4 it is easy to check that the number of smallest stopping sets $T_{2m-1}(H^{(3)}) = 2^{m+1} - 2$, which coincides with the number of minimum codewords of $RM(m,1)$.

**Example 3** By Theorem 4 we can easily calculate the SSDs of $RM(m,1)$ with parity-check matrix $H^{(3)}$ by Mathematica software. Here are some examples for $m = 3,4$.

For $RM(3,1)$,

$$T(x) = 1 + 14x^4 + 28x^6 + 8x^7 + x^8.$$
For $RM(4,1)$,

$$T(x) = 1 + 30x^8 + 140x^{12} + 120x^{14} + 16x^{15} + x^{16}.$$  

V.4 The Extended Hamming Codes $\hat{H}(m)$

Throughout this subsection, $n = 2^m$ and $EG(m, 2) = \{1, 2, \ldots, 2^m\}$. Note that $P$ is a hyperplane if and only if $\chi(P)$ is a row of $H^{(4)}$, and if and only if $\bar{P} = EG(m, 2) \setminus P$ is a hyperplane.

**Lemma 14** Let $\hat{H}(m)$ be the $[2^m, 2^m - m - 1, 4]$ extended Hamming code with parity-check matrix $H^{(4)}$. Then $S \subseteq EG(m, 2)$ is a non-empty stopping set if and only if $\langle S \rangle = \langle S_j \rangle$ for any $j \in S$.

**Proof:** By the definition of stopping sets, a non-empty subset $S \subseteq EG(m, 2)$ is a stopping set if and only if $H^{(4)}(S)$ has no rows with weight one, i.e., $|P \cap S| \neq 1$ or $|\bar{P} \cap S| \neq |S| - 1$ for any hyperplane $P$ of $EG(m, 2)$. Since $P$ is a hyperplane in $EG(m, 2)$ if and only if $\bar{P}$ is also a hyperplane, we only need to show that $|P \cap S| \neq |S| - 1$ for any hyperplane $P$ of $EG(m, 2)$ if and only if $\langle S \rangle = \langle S_j \rangle$ for any $j \in S$. With the same arguments used in the proof of Lemma 13, the lemma follows.  

**Remark 6** It is easy to see from Lemma 14 that when $u \geq 2^{m-1} + 2$, any $u$-set of $EG(m, 2)$ is a stopping set since any set with at least $2^{m-1} + 1$ points generates $EG(m, 2)$.

Since $H^{(4)}$ is formed by all rows except the all-1 row of $H^{(4)*}$, they have the same SSDs.
Theorem 5  Let $\hat{H}(m)$ be the $[2^m, 2^m - m - 1, 4]$ extended Hamming code with parity-check matrix $H^{(4)}$. Let $\{T_i(H^{(4)})\}_{i=0}^n$ be the SSD of $\hat{H}(m)$. Then

$$T_u(H^{(4)}) = \begin{cases} 1, & u = 0, \\ 0, & u = 1, 2, 3, \\ \sum_{l=\lfloor \log u \rfloor}^{\min\{u-2, m\}} N_{EG}(m, l) G_{EG}(u, l), & u = 4, \ldots, 2^{m-1} + 1, \\ (2^m_u), & u = 2^{m-1} + 2, \ldots, 2^m, \end{cases}$$

where $N_{EG}(m, l)$ and $G_{EG}(u, l)$ are defined in (39) and (40) respectively.

Proof:  Clearly, $T_0 = 1$. By Lemma 14 and the definition (24) of $G_{EG}(u, l)$, it is easy to see that

$$T_u = \sum_{l=0}^{m} N_{EG}(m, l) G_{EG}(u, l).$$

Since any $u$-set in $EG(m, 2)$ generates a flat with dimension at least $\lfloor \log u \rfloor$,

$$B_{EG}(u, l) = G_{EG}(u, l) = 0 \quad \text{if } l < \lfloor \log u \rfloor.$$ 

Combining (47)-(48) and (27), we have that

$$T_u = \sum_{l=\lfloor \log u \rfloor}^{\min\{u-2, m\}} N_{EG}(m, l) G_{EG}(u, l), \quad 1 \leq u \leq 2^m.$$ 

Let $u = 1, 2, 3$, we have $T_1 = T_2 = T_3 = 0$. Combining these results and Remark 6 (46) follows. \qed

Remark 7  By Theorem 5, we have that

$$T_4(H^{(4)}) = 2^{m-2}(2^m - 1)(2^{m-1} - 1)/3, \quad T_5(H^{(4)}) = 0.$$ 

It is easy to see from [18] that $A_4 = T_4(H^{(4)})$ and $A_5 = 0$ for $\hat{H}(m)$, which were also obtained by Weber and Abdel-Ghaffar [27].
VI  Conclusions

Let $C$ be a binary $[n, k]$ linear code. Let $H^*$ be the parity-check matrix of $C$ which is formed by all the non-zero codewords of its dual code $C^\perp$. On the BEC, the iterative decoder with parity-check matrix $H^*$ achieves the best possible performance, but has the highest decoding complexity. The stopping set distribution of $C$ with the parity-check matrix $H^*$ is used to determine the performance of $C$ under iterative decoding with the parity-check matrix $H^*$ over a BEC. In general, it is difficult to determine the stopping set distribution $\{T_i(H^*)\}_{i=0}^n$ of $C$ with the parity-check matrix $H^*$. Let $H$ be a parity-check matrix of $C$. Let $\{T_i(H)\}_{i=0}^n$ be the stopping set distribution of $C$ with the parity-check matrix $H$. Since $H$ is a sub-matrix formed by some rows of $H^*$, any stopping set of $H^*$ is a stopping set of $H$. This implies that $T_i(H) \geq T_i(H^*)$ for every $0 \leq i \leq n$. A parity-check matrix $H$ is called BEC-optimal if $T_i(H) = T_i(H^*)$ for every $0 \leq i \leq n$ and $H$ has the smallest number of rows. On the BEC, the iterative decoder with BEC-optimal parity-check matrix $H$ achieves the best possible performance as the iterative decoder with parity-check matrix $H^*$ and it has lower decoding complexity than $H^*$. In general, it is difficult to obtain BEC-optimal parity-check matrix for a general linear code. It is interesting to construct BEC-optimal parity-check matrices and then determine the corresponding stopping set distributions for LDPC codes and well known linear codes. In this paper, we obtain BEC-optimal parity-check matrices and then determine the corresponding stopping set distributions for the Simplex codes, the Hamming codes, the first order Reed-Muller codes and the extended Hamming codes.
References

[1] K. A. S. Abdel-Ghaffar and J. H. Weber, “Complete enumeration of stopping sets of full-rank parity-check matrices of Hamming codes,” IEEE Trans. Inform. Theory, vol. 53, no. 9, pp. 3196-3201, 2007.

[2] C. Di, D. Proietti, I. E. Telatar, T. J. Richardson and R.L. Urbanke, “Finite-length analysis of low-density parity-check codes on the binary erasure channel,” IEEE Trans. Inform. Theory, vol. 48, no. 6, pp. 1570-1579, 2002.

[3] M. Esmaeili and M. J. Amoshahy, “On the stopping distance of array code parity-check matrices,” IEEE Trans. Inform. Theory, vol. 55, no. 8, pp. 3488-3493, Aug. 2009.

[4] T. Etzion, “On the stopping redundancy of Reed-Muller codes,” IEEE Trans. Inform. Theory, vol. 52, no. 11, pp. 4867-4879, Sep. 2006.

[5] J. Feldman, Decoding Error-Correcting Codes via Linear Programming, Ph.D. Thesis, Massachusetts Institute of Technology, Sep. 2003.

[6] J. Feldman, M. J. Wainwright, and D. R. Karger, “Using linear programming to decode binary linear codes,” IEEE Trans. Inform. Theory, vol. 51, no. 3, pp. 954-972, 2005.

[7] J. Han and P. H. Siegel, “Improved upper bounds on stopping redundancy,” IEEE Trans. Inform. Theory, vol. 53, no. 1, pp. 901-104, Jan. 2007.

[8] J. Han, P. H. Siegel, and A. Vardy, “Improved probabilistic bounds on stopping redundancy,” IEEE Trans. Inform. Theory, vol. 54, no. 4, pp. 1749-1753, Apr. 2008.
[9] J. Han, P. H. Siegel, and R. M. Roth, “Single-exclusion number and the stopping redundancy of MDS codes,” *IEEE Trans. Inform. Theory*, vol. 55, no. 9, pp. 4155-4166, Sep. 2009.

[10] T. Hehn, O. Milenkovic, S. Laendner, and J. B. Huber, “Permutation decoding and the stopping redundancy hierarchy of cyclic and extended cyclic codes,” *IEEE Trans. Inform. Theory*, vol. 54, no. 12, pp. 5308-5331, Dec. 2008.

[11] H. Hollmann and L. Tolhuizen, “Erasure correcting sets: bounds and constructions,” *Journal of Combinatorial Theory, Series A*, vol. 113, pp. 1746-1759, 2006.

[12] H. Hollmann and L. Tolhuizen, “On parity-check collections for iterative erasure decoding that correct all correctable erasure patterns of a given size,” *IEEE Trans. Inform. Theory*, vol. 53, no. 2, pp. 823-828, Feb. 2007.

[13] N. Kashyap and A. Vardy, “Stopping sets in codes from designs,” in *Proc. IEEE Int. Symp. Inform. Theory*, Yokohama, Japan, Jun./Jul. 2003, p. 122.

[14] R. Koetter and P. O. Vontobel, “Graph covers and iterative decoding of finite-length codes,” *Proc. 3rd Int. Conf. Turbo Codes and Related Topics*, Brest, France, Sep. 2003, pp. 75-82.

[15] Y. Kou, S. Lin, and M. P. C. Fossorier, “Low-density parity-check codes based on finite geometries: A rediscovery and new results,” *IEEE Trans. Inform. Theory*, vol. 47, no. 7, pp. 2711-2736, 2001.

[16] K. M. Krishnan and P. Shankar, “Computing the stopping distance of a Tanner graph is NP-hard,” *IEEE Trans. Inform. Theory*, vol. 53, no. 6, pp. 2278-2280, Jun. 2007.
[17] S. Laendner and O. Milenkovic, “LDPC codes based on Latin squares: cycle structure, stopping set, and trapping set analysis,” IEEE Trans. Communications, Vol. 55, No. 2, pp. 303-312, Feb. 2007.

[18] F. J. MacWilliams and N. J. A. Sloane, The Theory of Error-Correcting Codes. Amsterdam, The Netherlands: North-Holland, 1981 (3rd printing).

[19] R. J. McEliece, “Are there turbo-codes on Mars?” Shannon Lecture, Proc. IEEE Int. Symp. Inform. Theory, Chicago, IL, USA, Jun./Jul. 2004. The slides are available at the web site [http://www.systems.caltech.edu/EE/Faculty/rjm/](http://www.systems.caltech.edu/EE/Faculty/rjm/)

[20] O. Milenkovic, E. Soljanin, and P. Whiting, “Asymptotic spectra of trapping sets in regular and irregular LDPC code ensembles,” IEEE Trans. Inform. Theory, vol. 53, no. 1, pp. 39-55, 2007.

[21] A. Orlitsky, K. Viswanathan, and J. Zhang, “Stopping set distribution of LDPC code ensembles,” IEEE Trans. Inform. Theory, vol. 51, no. 3, pp. 929-953, Mar. 2005.

[22] V. Rathi, “On the asymptotic weight and stopping set distribution of regular LDPC ensembles,” IEEE Trans. Inform. Theory, vol. 52, no. 9, pp. 4212-4218, Sep. 2006.

[23] M. Schwartz and A. Vardy, “On the stopping distance and the stopping redundancy of codes,” IEEE Trans. Inform. Theory, vol. 52, no. 3, pp. 922-932, 2006.

[24] H. Tang, J. Xu, S. Lin, and K. A. S. Abdel-Ghaffar, “Codes on finite geometries,” IEEE Trans. Inform. Theory, vol. 51, no. 2, pp. 572-596, 2005.

[25] R. M. Tanner, “A recursive approach to low complexity codes,” IEEE Trans. Inform. Theory, vol. 27, no. 5, pp. 533-547, Sep. 1981.
[26] T. Wadayama, “Average stopping set weight distributions of redundant random ensembles,” IEEE Trans. Inform. Theory, vol. 54, no. 11, pp. 4991-5004, Nov. 2008.

[27] J. H. Weber and K. A. S. Abdel-Ghaffar, “Stopping set analysis for Hamming codes,” Proc. 2005 IEEE Information Theory Workshop,Rotorua, New Zealand, Aug./Sep. 2005, pp. 244-247.

[28] J. H. Weber and K. A. S. Abdel-Ghaffar, “Results on parity-check matrices with optimal stopping and/or dead-end set enumerators,” IEEE Trans. Inform. Theory, vol. 54, no. 3, pp. 1368-1374, 2008.

[29] S.-T. Xia and F.-W. Fu, “On the minimum pseudo-codewords of LDPC codes,” IEEE Communications Letters, vol. 10, no. 5, pp. 363-365, May 2006.

[30] S.-T. Xia and F.-W. Fu, “On the stopping distance of finite geometry LDPC Codes,” IEEE Communications Letters, vol. 10, no. 5, pp. 381-383, May 2006.

[31] S.-T. Xia and F.-W. Fu, “Stopping set distributions of some linear codes,” Proc. IEEE Inform. Theory Workshop, Chengdu, China, Oct. 2006, pp. 47-51.

[32] S.-T. Xia and F.-W. Fu, “Minimum pseudoweight and minimum pseudocodewords of LDPC codes,” IEEE Trans. Inform. Theory, vol. 54, no. 1, pp. 480-485, Jan. 2008.