EXPONENTIALLY CONVERGENT TRAPEZOIDAL RULES TO APPROXIMATE FRACTIONAL POWERS OF OPERATORS

LIDIA ACETO AND PAOLO NOVATI

Abstract. In this paper we are interested in the approximation of fractional powers of self-adjoint positive operators. Starting from the integral representation of the operators, we apply the trapezoidal rule combined with a single-exponential and a double-exponential transform of the integrand function. For the first approach our aim is only to review some theoretical aspects in order to refine the choice of the parameters that allow a faster convergence. As for the double exponential transform, in this work we show how to improve the existing error estimates for the scalar case and also extend the analysis to operators. We report some numerical experiments to show the reliability of the estimates obtained.

1. Introduction

In this work we are interested in the numerical approximation of $\lambda^{-\alpha}$, $\alpha \in (0, 1)$. Here $\mathcal{L}$ is a self-adjoint positive operator acting in an Hilbert space $\mathcal{H}$ in which the eigenfunctions of $\mathcal{L}$ form an orthonormal basis of $\mathcal{H}$, so that $\mathcal{L}^{-\alpha}$ can be written through the spectral decomposition of $\mathcal{L}$. In other words, for a given $g \in \mathcal{H}$, we have

$$\mathcal{L}^{-\alpha} g = \sum_{j=1}^{+\infty} \mu_j^{-\alpha} \langle g, \varphi_j \rangle \varphi_j$$

where $\mu_j$ and $\varphi_j$ are the eigenvalues and the eigenfunctions of $\mathcal{L}$, respectively, and $\langle \cdot, \cdot \rangle$ denotes the $\mathcal{H}$-inner product. Throughout the paper we also assume $\sigma(\mathcal{L}) \subseteq [1, +\infty)$, where $\sigma(\mathcal{L})$ denotes the spectrum of $\mathcal{L}$.

Applications of (1) include the numerical solution of fractional equations involving the anomalous diffusion, in which $\mathcal{L}$ is related to the Laplacian operator, and this is the main reason for which in recent years a lot of attention has been placed on the efficient approximation of fractional powers. Among the approaches recently introduced we quote here the methods based on the best uniform rational approximations of functions closely related to $\lambda^{-\alpha}$ that have been studied in [6, 7, 8, 9]. Another class of methods relies on quadrature rules arising from the Dunford-Taylor integral representation of $\lambda^{-\alpha}$ [1, 2, 3, 4, 5, 17, 18]. Very recently, time stepping methods for a parabolic reformulation of fractional diffusion equations, proposed in [19], have been interpreted by Hofreither in [10] as rational approximations of $\lambda^{-\alpha}$.

Key words and phrases. Matrix functions, Single-exponential transform, Double-exponential transform, Trapezoidal rule, Fractional Laplacian.

This work was partially supported by GNCS-INdAM and FRA-University of Trieste. The authors are members of the INdAM research group GNCS.
In this work, starting from the integral representation

\[ \mathcal{L}^{-\alpha} = \frac{2 \sin(\alpha \pi)}{\pi} \int_0^{+\infty} t^{2\alpha-1} (I + t^2 \mathcal{L})^{-1} dt, \quad \alpha \in (0, 1), \]

where \( I \) is the identity operator in \( \mathcal{H} \), we consider the trapezoidal rule applied to the single and the double-exponential transform of the integrand function. The former approach has been extensively studied in \([5]\), where the authors also provide reliable error estimates. The rate of convergence has been shown to be of type

\[ \exp(-c\sqrt{n}), \]

where \( n \) is closely related to the number of nodes. Our aim here is just to review some theoretical aspects in order to refine the choice of the parameters that allow faster convergence, even if, still of type \([8]\). As for the double-exponential transform, widely investigated in \([12, 13, 14, 15, 16]\) for general scalar functions, in this work we show how to improve the existing error estimates for the function \( \lambda^{-\alpha} \). We also extend the analysis to operators, showing that it is possible to reach a convergence rate of type

\[ \exp\left(-c \sqrt{\frac{n}{\ln n}}\right). \]

While theoretically disadvantageous with respect to the single-exponential approach, we show that the double-exponential approach is actually faster at least for \( \alpha \in [1/2, 1) \).

The paper is organized as follows. In Section 2 we make a short background concerning the trapezoidal rule with particular attention to functions that decay exponentially at infinity. In Section 3 we review the existing convergence analysis for the trapezoidal rule combined with a single-exponential transform and we refine the choice of the parameters that allow a faster convergence. Section 4 is devoted to the trapezoidal rule combined with a double-exponential transform. Here the convergence analysis is derived for the approximation of the scalar function \( \lambda^{-\alpha} \) and is then extended to the case of the operator \( \mathcal{L}^{-\alpha} \). Some concluding remarks are finally reported in Section 5.

2. A GENERAL CONVERGENCE RESULT FOR THE TRAPEZOIDAL RULE

Given a generic continuous function \( f : \mathbb{R} \to \mathbb{R} \), in this section we make a short background concerning the trapezoidal approximation

\[ I(f) = \int_{-\infty}^{+\infty} f(x) dx \approx h \sum_{\ell=-\infty}^{+\infty} f(\ell h), \]

where \( h \) is a suitable positive value. Given \( M \) and \( N \) positive integers, we denote the truncated trapezoidal rule by

\[ T_{M,N,h}(f) = h \sum_{\ell=-M}^{N} f(\ell h). \]

For the error we have

\[ E_{M,N,h}(f) := |I(f) - T_{M,N,h}(f)| \leq E_D + E_{TL} + E_{TR}, \]
where

\[ \mathcal{E}_D = \left| \int_{-\infty}^{+\infty} f(x)dx - h \sum_{\ell=-\infty}^{+\infty} f(\ell h) \right|, \]

\[ \mathcal{E}_{TL} = h \sum_{\ell=-\infty}^{-M-1} |f(\ell h)|, \quad \mathcal{E}_{TR} = h \sum_{\ell=N+1}^{+\infty} |f(\ell h)|. \]

The quantities \( \mathcal{E}_D \) and \( \mathcal{E}_T := \mathcal{E}_{TL} + \mathcal{E}_{TR} \) are referred to as the discretization error and the truncation error, respectively.

**Definition 1.** \( [11, \text{Definition 2.12}] \) For \( d > 0 \), let \( \mathcal{D}_d \) be the infinite strip domain of width \( 2d \) given by

\[ \mathcal{D}_d = \{ \zeta \in \mathbb{C} : |\text{Im}(\zeta)| < d \}. \]

Let \( \mathcal{B}(\mathcal{D}_d) \) be the set of functions analytic in \( \mathcal{D}_d \) that satisfy

\[ \int_{-d}^{d} |f(x+iy)|dy = O(|x|^a), \quad x \to \pm\infty, \quad 0 \leq a < 1, \]

and

\[ N(f, d) = \lim_{\eta \to d^-} \left\{ \int_{-\infty}^{+\infty} |f(x+iy)|dx + \int_{-\infty}^{+\infty} |f(x-iy)|dx \right\} < +\infty. \]

The next theorem gives an estimate for the discretization error of the trapezoidal rule when applied to functions in \( \mathcal{B}(\mathcal{D}_d) \).

**Theorem 1.** \( [11, \text{Theorem 2.20}] \) Assume \( f \in \mathcal{B}(\mathcal{D}_d) \). Then

\[ (4) \quad \mathcal{E}_D \leq \frac{N(f, d)}{2 \sinh(\pi d/h)} e^{-\pi d/h}. \]

**Theorem 2.** Assume \( f \in \mathcal{B}(\mathcal{D}_d) \) and that there are positive constants \( \beta, \gamma \) and \( C \) such that

\[ |f(x)| \leq C \left\{ \begin{array}{ll} \exp(-\beta |x|), & x < 0, \\ \exp(-\gamma |x|), & x \geq 0. \end{array} \right. \]

Then,

\[ (6) \quad \mathcal{E}_{M,N,h}(f) \leq \frac{N(f, d)}{2 \sinh(\pi d/h)} e^{-\pi d/h} + \frac{C}{\beta} e^{-\beta M h} + \frac{C}{\gamma} e^{-\gamma N h}. \]

**Proof.** By (4) we immediately have

\[ \mathcal{E}_{TL} \leq \frac{C}{\beta} e^{-\beta M h}, \quad \mathcal{E}_{TR} \leq \frac{C}{\gamma} e^{-\gamma N h}. \]

Using Theorem 1, we obtain (6).

The above result states that for functions that decay exponentially for \( x \to \pm\infty \) it may be possible to have exponential convergence after a proper selection of \( h \). When working with the more general situation

\[ (7) \quad I(g) := \int_a^b g(t)dt, \]

one can consider a suitable conformal map

\[ \psi : (-\infty, +\infty) \to (a, b), \]
and, through the change of variable \( t = \psi(x) \), transform (7) to
\[
I(g) := \int_{-\infty}^{+\infty} g_{\psi}(x) dx, \quad g_{\psi}(x) = g(\psi(x))\psi'(x).
\]
A suitable choice of the mapping \( \psi \) may allow to work with a function \( g_{\psi} \) that fulfills the hypothesis of Theorem 2 so that \( I(g) \) can be evaluated with an error that decays exponentially.

Since the aim of the paper is the computation of \( L^{-\alpha} \) with \( \sigma(L) \subseteq [1, +\infty) \), for \( \lambda \geq 1 \) we consider now the integral representation (2)
\[
\lambda^{-\alpha} = \frac{2\sin(\alpha\pi)}{\pi} \int_{0}^{+\infty} t^{2\alpha-1}(1+t^2\lambda)^{-1} dt, \quad \alpha \in (0, 1).
\]
Defining
\[
g_{\lambda}(t) := t^{2\alpha-1}(1+t^2\lambda)^{-1},
\]
and a change of variable \( t = \psi(x) \), \( \psi: (-\infty, +\infty) \to (0, +\infty) \), let
\[
g_{\lambda,\psi}(x) = g_{\lambda}(\psi(x))\psi'(x).
\]
Let moreover
\[
Q_{M,N,h}^{\alpha}(g_{\lambda,\psi}) = \frac{2\sin(\alpha\pi)}{\pi} h \sum_{\ell=-M}^{N} g_{\lambda,\psi}(\ell h)
\]
be the truncated trapezoidal rule for the computation of \( \lambda^{-\alpha} \), that is, for the computation of
\[
\frac{2\sin(\alpha\pi)}{\pi} \int_{0}^{+\infty} g_{\lambda}(t) dt = \frac{2\sin(\alpha\pi)}{\pi} \int_{-\infty}^{+\infty} g_{\lambda,\psi}(x) dx.
\]
We denote the error by
\[
E_{M,N,h}(\lambda) = \left| \lambda^{-\alpha} - Q_{M,N,h}^{\alpha}(g_{\lambda,\psi}) \right|
\]
(11)
and for operator argument
\[
E_{M,N,h}(L) = \| L^{-\alpha} - Q_{M,N,h}^{\alpha}(gL,\psi) \|_{H \to H}.
\]
The remainder of the paper is devoted to the analysis of two special choices for \( \psi \), the single-exponential (SE) and the double-exponential (DE) transforms.

3. THE SINGLE-EXPONENTIAL TRANSFORM

The SE transform is defined by
\[
\psi_{SE}(x) = \exp(x),
\]
so that from (9) and (10) we get
\[
g_{\lambda,\psi_{SE}}(x) = e^{2\alpha x}(1 + e^{2x}\lambda)^{-1}.
\]
Since the poles of this function are given by
\[
x_{k} = -\frac{1}{2}\log \lambda - i(2k+1)\frac{\pi}{2}, \quad k \in \mathbb{Z},
\]
we have that \( g_{\lambda,\psi_{SE}} \) is analytic in \( \mathcal{D}_{\psi_{SE}} = \{ \zeta \in \mathbb{C} : |\Im(\zeta)| < \frac{\pi}{2} \} \),
that is, the strip domain with 
\( d = \frac{\pi}{2} \), independently of \( \alpha \) and \( \lambda \). Now, in order to prove that \( g_{\lambda,\psi SE} \) belongs to \( B(D_{\psi SE}) \) (see Definition \([1]\)), following the analysis given in \([5]\) we first note that for \( \eta \in \mathbb{R} \), \( |\eta| < \frac{\pi}{2} \) and \( \lambda \geq 1 \),

\[
\left| (1 + e^{2(x+i\eta)\lambda})^{-1} \right| \leq \begin{cases} 
1, & x < 0, \\
e^{-2x}, & x \geq 0.
\end{cases}
\]

Therefore,

\[
|g_{\lambda,\psi SE}(x + i\eta)| \leq \begin{cases} 
e^{2\alpha x}, & \text{for } x < 0, \\
e^{-2(1-\alpha)x}, & \text{for } x \geq 0.
\end{cases}
\]

This implies that

\[
\mathcal{N}(g_{\lambda,\psi SE}, \pi/2) = \lim_{\eta \to (\pi/2)^-} \left\{ \int_{-\infty}^{+\infty} |g_{\lambda,\psi SE}(x + i\eta)| \, dx + \int_{-\infty}^{+\infty} |g_{\lambda,\psi SE}(x - i\eta)| \, dx \right\}
\]

\[\leq \frac{1}{\alpha(1-\alpha)},\]

and also that

\[
\int_{-\pi/2}^{\pi/2} |g_{\lambda,\psi SE}(x + i\eta)| \, d\eta = \mathcal{O}(1), \quad x \to +\infty.
\]

Therefore, by \([1]\) we can conclude that \( g_{\lambda,\psi SE} \) belongs to \( B(D_{\psi SE}) \). By \([4]\), for the discretization error we have

\[
\left| \int_{-\infty}^{+\infty} g_{\lambda,\psi SE}(x) \, dx - h \sum_{\ell=-\infty}^{+\infty} g_{\lambda,\psi SE}((\ell h) \right| \leq \frac{1}{\alpha(1-\alpha)} \frac{e^{-\pi d/h}}{2 \sinh(\pi d/h)}, \quad d = \pi/2.
\]

Since

\[
e^{-t} \frac{2}{\sinh(t)} = e^{-2t} (1 + \mathcal{O}(e^{-2t})) \quad \text{as } t \to +\infty,
\]

by \([2]\), for \( h \leq 2\pi d \) we obtain

\[
\mathcal{E}_{M,N,h}(g_{\lambda,\psi SE}) = \left| \int_{-\infty}^{+\infty} g_{\lambda,\psi SE}(x) \, dx - h \sum_{\ell=-\infty}^{N} g_{\lambda,\psi SE}((\ell h) \right| \\
\leq \frac{1}{\alpha(1-\alpha)} e^{-2\pi d/h} + \frac{1}{2\alpha} e^{-2\alpha Mh} + \frac{1}{2(1-\alpha)} e^{-2(1-\alpha)Nh}.
\]

After choosing \( h \), the contribute of the three exponentials can be equalized by taking \( M \) and \( N \) such that

\[
\pi d/h \approx \alpha Mh \approx (1-\alpha)Nh,
\]

that is

\[
M = \left[ \frac{\pi d}{\alpha h^2} \right], \quad N = \left[ \frac{\pi d}{(1-\alpha)h^2} \right],
\]

where \([\cdot]\) is the ceiling function. Denoting by \( n = M + N + 1 \) the total number of inversions we have that

\[
n \approx \frac{\pi d}{h^2} \frac{1}{\alpha(1-\alpha)},
\]

and therefore by \([15]\) we obtain

\[
\mathcal{E}_{M,N,h}(g_{\lambda,\psi SE}) \leq \frac{3}{2} \frac{1}{\alpha(1-\alpha)} \exp \left( -2\sqrt{\pi \alpha(1-\alpha) \sqrt{n}} \right).
\]
By [12], since \( L \) is assumed to be self-adjoint and \( \sigma(L) \subseteq [1, +\infty) \), we have that
\[
E_{M,N,h}(L) \leq \frac{2\sin(\alpha \pi)}{\pi} \max_{\lambda \geq 1} \mathcal{E}_{M,N,h}(g_{\lambda, \psi_{SE}}),
\]
and then, finally, taking \( d = \pi/2 \) we obtain
\[
(16) \quad E_{M,N,h}(L) \leq \frac{\sin(\alpha \pi)}{\pi} \frac{3}{\alpha(1 - \alpha)} \exp \left( -\pi \sqrt{2\alpha(1 - \alpha)} \sqrt{n} \right), \quad n = M + N + 1.
\]
The analysis just presented is almost identical to the one given in [5]. Nevertheless in that paper the authors define \( d = \pi/4 \), while here we have shown that one can take \( d = \pi/2 \). This choice produces a remarkable speedup, as shown in Figure 1, where for different values of \( \alpha \) we have considered the error in the computation of \( L^{-\alpha} \) for the artificial operator
\[
(17) \quad L = \text{diag}(1, 2, \ldots, 100)^8, \quad \sigma(L) \subseteq [1, 10^{16}].
\]

![Figure 1. Error for the trapezoidal rule applied with the single-exponential transform with \( d = \pi/4 \) and \( d = \pi/2 \), and error estimate given by (16).](image)

4. Double-Exponential Transformation

The DE transform we use here is given by
\[
(18) \quad \psi_{DE}(x) = \exp \left( \frac{\pi}{2} \sinh(x) \right).
\]
We consider in (8) the change of variable
\[ \tau^2 = (\psi_{DE}(x))^2 = \exp(\pi \sinh(x)), \quad \tau > 0. \]

The function involved in this case is
\[ g_{\lambda,\psi_{DE}}(x) = \frac{\pi}{2} \tau^{1-\alpha} \frac{\exp(\alpha \pi \sinh(x))}{\tau + \lambda \exp(\pi \sinh(x))} \cosh(x) \]
\[ = \frac{\pi}{2} \lambda^{-\alpha} \left( \frac{\lambda/\tau \exp(\pi \sinh(x))}{1 + \lambda/\tau \exp(\pi \sinh(x))} \right)^\alpha \cosh(x), \]
and we employ the trapezoidal rule to compute
\[ \lambda^{-\alpha} = \frac{2 \sin(\alpha \pi)}{\pi} \int_{-\infty}^{+\infty} g_{\lambda,\psi_{DE}}(x) dx. \]

The parameter \( \tau \) needs to be selected in some way and the analysis is provided in Section 4.5. Its introduction is motivated by the fact that, when moving from \( \lambda \) to \( \mathcal{L} \), the method (the choice of \( M, N \) and \( h \)) and the error estimates have to be derived by working uniformly in the interval \([1, +\infty)\) containing \( \sigma(\mathcal{L}) \). As in the SE case, the function \( g_{\lambda,\psi_{DE}}(x) \) exhibits a fast decay for \( x \to \pm\infty \), but the definition of the strip of analyticity is now much more difficult to handle since everything now depends on \( \lambda \) and \( \tau \).

4.1. Asymptotic behavior of the integrand function. In order to apply Theorem 2 we need to study \( |g_{\lambda,\psi_{DE}}(x + i\eta)| \). From (19) we have
\[ |g_{\lambda,\psi_{DE}}(x + i\eta)| = \frac{\pi}{2} \lambda^{-\alpha} \left( \frac{\lambda/\tau \exp(\pi \sinh(x + i\eta))}{1 + \lambda/\tau \exp(\pi \sinh(x + i\eta))} \right)^\alpha |\cosh(x + i\eta)|. \]

After simple manipulations based on standard relations we find
\[ |\cosh(x + i\eta)| = \sqrt{\cosh^2 x - \sin^2 \eta}, \]
and therefore
\[ |\cosh(x + i\eta)| \leq \cosh x. \]

Moreover
\[ |(\lambda/\tau \exp(\pi \sinh(x + i\eta)))^\alpha| = \left( \frac{\lambda}{\tau} \right)^\alpha |\exp(\alpha \pi \sinh x \cos \eta)|. \]

In addition, we can bound the denominator using the results given in [13, p. 388], that is,
\[ \frac{1}{1 + \lambda/\tau \exp(\pi \sinh(x + i\eta))} \leq \frac{1}{1 + \lambda/\tau \exp(\pi \sinh x \cos \eta) \cos(\pi/2 \sin \eta)}. \]

From the above relations we find
\[ |g_{\lambda,\psi_{DE}}(x + i\eta)| \leq \frac{\pi}{2} \lambda^{-\alpha} \frac{\cosh x}{\cos(\pi/2 \sin \eta)} G_\alpha(x, \eta), \]
where
\[ G_\alpha(x, \eta) = \frac{(\lambda/\tau \exp(\pi \sinh x \cos \eta))^\alpha}{1 + \lambda/\tau \exp(\pi \sinh x \cos \eta)}. \]

Let \( x^* \) be such that
\[ \pi \sinh x^* \cos \eta = \ln(\tau/\lambda); \]
we have
\[ G_\alpha(x, \eta) \leq \begin{cases} 
(\lambda/\tau)^\alpha \exp(\alpha \pi \cos \eta \sinh x), & x \leq x^*, \\
(\lambda/\tau)^{\alpha-1} \exp(-(1-\alpha)\pi \cos \eta \sinh x), & x > x^*. 
\end{cases} \]

4.2. Error estimate for the scalar case. The bound (20) implies that
\[ N(g_{\lambda, \psi_{DE}}, d) = \lim_{\eta \to d^-} \left\{ \int_{-\infty}^{+\infty} |g_{\lambda, \psi_{DE}}(x + i\eta)| \, dx + \int_{-\infty}^{+\infty} |g_{\lambda, \psi_{DE}}(x - i\eta)| \, dx \right\} \]
\[ \leq \lim_{\eta \to d^-} \frac{\pi \lambda^{-\alpha}}{\cos(\pi/2 \sin \eta)} \int_{-\infty}^{+\infty} G_\alpha(x, \eta) \cosh x \, dx \]
\[ \leq \lim_{\eta \to d^-} \frac{\pi \lambda^{-\alpha}}{\cos(\pi/2 \sin \eta)} \left\{ (\lambda/\tau)^\alpha \int_{-\infty}^{\lambda/\tau} \exp(\alpha \pi \cos \eta \sinh x) \cosh x \, dx + \right. \]
\[ + (\lambda/\tau)^{\alpha-1} \int_{\lambda/\tau}^{+\infty} \exp(-(1-\alpha)\pi \cos \eta \sinh x) \cosh x \, dx \right\} \]
\[ \leq \frac{1}{\alpha(1-\alpha)} \frac{2}{\cos d \cos(\pi/2 \sin d)} \lambda^{-\alpha}. \]

In addition, assuming \( d = d(\lambda, \tau) < \pi/2 \), it can be observed that
\[ \int_{-d(\lambda, \tau)}^{d(\lambda, \tau)} |g_{\lambda, \psi_{DE}}(x + i\eta)| \, d\eta \leq \frac{\pi}{2} \lambda^{-\alpha} \int_{-d(\lambda, \tau)}^{d(\lambda, \tau)} \frac{G_\alpha(x, \eta) \cosh x}{\cos(\pi/2 \sin \eta)} \, d\eta \]
\[ = O(1) \quad \text{for} \quad x \to \pm \infty. \]

Using (11), for the discretization error we have
\[ \left| \int_{-\infty}^{+\infty} g_{\lambda, \psi_{DE}}(x) \, dx - h \sum_{\ell=-\infty}^{\infty} g_{\lambda, \psi_{DE}}(\ell h) \right| \leq \xi(d) \frac{1}{\alpha(1-\alpha)} \lambda^{-\alpha} \frac{e^{-\pi d/h}}{2 \sinh(\pi d/h)}, \]
where
\[ \xi(d) = \frac{2}{\cos d \cos(\pi/2 \sin d)}. \]

The remaining task is to estimate the truncation error. Using (20) we obtain
\[ h \sum_{\ell=-\infty}^{-M-1} |g_{\lambda, \psi_{DE}}(\ell h)| \leq \frac{\pi}{2} \frac{\tau^{-\alpha}}{\tau} h \sum_{\ell=-\infty}^{-M-1} \exp(\alpha \pi \sinh(\ell h)) \cosh(\ell h) \]
\[ \leq \frac{\pi}{2} \frac{\tau^{-\alpha}}{\tau} \int_{-\infty}^{-M h} \exp(\alpha \pi \sinh x) \cosh(x) \, dx \]
\[ \leq \frac{\tau^{-\alpha}}{2\alpha} \exp(-\alpha \pi \sinh(M h)) \]
\[ \leq \frac{\tau^{-\alpha}}{2\alpha} \exp \left( \frac{\alpha \pi}{2} \right) \exp \left( -\frac{(1-\alpha)\pi}{2} \exp(N h) \right). \]

Similarly,
\[ h \sum_{\ell=N}^{+\infty} |g_{\lambda, \psi_{DE}}(\ell h)| \leq \frac{\pi}{2} \lambda^{-1} \frac{\tau^{-1-\alpha}}{\tau} h \sum_{\ell=N+1}^{+\infty} \exp(-(1-\alpha)\pi \sinh(\ell h)) \cosh(\ell h) \]
\[ \leq \frac{\pi}{2} \lambda^{-1} \frac{\tau^{-1-\alpha}}{\tau} h \int_{Nh}^{+\infty} \exp(-(1-\alpha)\pi \sinh x) \cosh(x) \, dx \]
\[ \leq \frac{\lambda^{-1} \tau^{-1-\alpha}}{2(1-\alpha)} \exp \left( \frac{(1-\alpha)\pi}{2} \right) \exp \left( -\frac{(1-\alpha)\pi}{2} \exp(N h) \right). \]
The above results are summarized as follows.

**Proposition 1.** Using the double-exponential transform, for the quadrature error it holds

\[
\begin{align*}
E_{M,N,h}(g_{\lambda,\psi_{DE}}) & \leq \frac{1}{\alpha(1-\alpha)}\xi(d)\lambda^{-\alpha}e^{-\pi d/h} + \\
& \frac{\tau^{-\alpha}}{2\alpha}\exp\left(\frac{\alpha\pi}{2}\right)\exp\left(-\frac{\alpha\pi}{2}\exp(Mh)\right) + \\
& \frac{\lambda^{-1}\tau^{1-\alpha}}{2(1-\alpha)}\exp\left(\frac{(1-\alpha)\pi}{2}\right)\exp\left(-\frac{(1-\alpha)\pi}{2}\exp(Nh)\right),
\end{align*}
\]

where \(\xi(d)\) is defined by (22).

Defining

\[
h = \ln\left(\frac{4dn}{\mu}\right) \frac{1}{n}, \quad \text{for} \quad n \geq \frac{\mu e}{4d}, \quad \mu = \min(\alpha, 1-\alpha)
\]
as in [13, Theorem 2.14], we first observe that (see (23))

\[
\exp\left(-\frac{\pi d}{h}\right) \leq \frac{1}{1-e^{-\frac{\pi d}{h}}} \exp\left(-2\pi dn\ln\left(\frac{4dn}{\mu}\right)\right).
\]

Setting \(M = N = n\), the choice of \(h\) as in (26) leads to a truncation error that decays faster than the discretization one, because for an arbitrary constant \(c\) (see (24)-(25))

\[
\exp(-c\exp(nh)) = \exp\left(-\frac{4cdn}{\mu}\right).
\]

As consequence the idea is to assume the discretization error as estimator for the global quadrature error, that is, using (23) and (27),

\[
E_{n,h}(g_{\lambda,\psi_{DE}}) = E_{M,N,h}(g_{\lambda,\psi_{DE}}) \approx K_\alpha\xi(d)\lambda^{-\alpha}\exp\left(-\frac{2\pi dn}{\ln\left(\frac{4dn}{\mu}\right)}\right),
\]

where

\[
K_\alpha = \frac{1}{\alpha(1-\alpha)}\frac{1}{1-e^{-\frac{\pi d}{h}}\mu}.
\]

Formula (28) is very similar to the one given in [13, Theorem 2.14], that reads

\[
\hat{E}_{M,N,h}(g_{\lambda,\psi_{DE}}) \approx \frac{\tau^{-\alpha}}{\mu}\alpha(1-\alpha)\left(K_\alpha\xi(d) + e^{\hat{\tau}^\nu}\right)\exp\left(-\frac{2\pi dn}{\ln\left(\frac{4dn}{\mu}\right)}\right)
\]

where \(\nu = \max(\alpha, 1-\alpha)\) and \(M = n, N = n - \chi\) (or viceversa depending on \(\alpha\)), where \(\chi > 0\) is defined in order to equalize the contribute of the truncation errors [13, Theorem 2.11]. The important difference is given by the factor \(\lambda^{-\alpha}\) that replaces \(\tau^{-\alpha}\), and this is crucial to correctly handle the case of \(\lambda \to +\infty\). In this situation the error of the trapezoidal rule goes 0 because \(g_{\lambda,\psi_{DE}}(x) \to 0\) as \(\lambda \to +\infty\) (see (19)). Anyway, as we shall see, \(d \to 0\) as \(\lambda \to +\infty\), so that the exponential term itself is not able to reproduce this situation. An example is given in Figure 2 in which we consider \(\lambda = 10^{12}\) and \(\tau = 100\).
4.3. The poles of the integrand function. All the analysis presented so far is based on the assumption that the integrand function

\[ g_{\lambda,\psi,DE}(x) = \frac{\pi}{2^\alpha} - \frac{1}{\lambda} \exp(\alpha \pi \sinh x) \frac{\exp(\pi \sinh x)}{\tau + \lambda \exp(\pi \sinh x)} \cosh x \]

is analytic in the strip \( D_d \), for a certain \( d = d(\lambda, \tau) \). Therefore we have to study the poles of this function, that is, we have to study the equation

\[ \tau + \lambda \exp(\pi \sinh x) = 0. \]

We have

\[ \exp(\pi \sinh x) = \frac{\tau}{\lambda} e^{i\pi}, \]

\[ \sinh x = \frac{1}{\pi} \ln \frac{\tau}{\lambda} + i(2k + 1), \quad k \in \mathbb{Z}. \]

By solving the above equation for each \( k \), we obtain the complete set of poles. Assuming to work with the principal value of the logarithm and taking \( k = 0 \), we
obtain the poles closest to the real axis $x_0$ and its conjugate $\overline{x_0}$, where

$$x_0 = \sinh^{-1} \left( \frac{1}{\pi} \ln \frac{\tau}{\lambda} + i \right)$$

$$= \ln \left( \frac{1}{\pi} \ln \frac{\tau}{\lambda} + i + \sqrt{\left( \frac{1}{\pi} \ln \frac{\tau}{\lambda} \right)^2 + 2i \frac{1}{\pi} \ln \frac{\tau}{\lambda}} \right).$$

(31)

In order to apply the bound on the strip we have to define

$$d = d(\lambda, \tau) = r \Im x_0, \quad 0 < r < 1.$$  

(32)

The introduction of the factor $r$ is necessary to avoid $\xi(d) \to +\infty$ as $\Im x_0 \to \pi/2$, which verifies for $\lambda \to \tau$ (see (22)).

4.3.1. Asymptotic behaviors. Setting

$$s = \frac{1}{\pi} \ln \frac{\lambda}{\tau},$$

we have

$$\frac{1}{\pi} \ln \frac{\tau}{\lambda} = -s,$$

and therefore we can write (31) as

$$x_0 = \ln \left( s \left( -1 + \frac{i}{s} + \sqrt{1 - \frac{2i}{s}} \right) \right).$$

Assuming $\lambda \gg \tau$, that is, $s \gg 1$, and using

$$\sqrt{1 - x} \approx 1 - \frac{1}{2} x - \frac{1}{8} x^2 - \frac{1}{16} x^3, \quad x \approx 0,$$

we obtain

$$\sqrt{1 - \frac{2i}{s}} \approx 1 - \frac{i}{s} + \frac{1}{2s^2} + \frac{i}{2s^3}.$$

Using also $\ln(1 + x) \approx x$,

$$x_0 \approx \ln \left( s \left( -1 + \frac{i}{s} + 1 - \frac{i}{s} + \frac{1}{2s^2} + \frac{i}{2s^3} \right) \right)$$

$$= \ln \left( s \left( \frac{1}{2s^2} + \frac{i}{2s^3} \right) \right)$$

$$= \ln \left( \frac{1}{2s} \right) + \ln \left( 1 + \frac{i}{s} \right)$$

$$\approx \ln \left( \frac{1}{2s} \right) + \frac{i}{s}.$$

Therefore, for $\lambda \gg \tau$,

$$\Im x_0 \approx \frac{1}{s} = \frac{\pi}{\ln \frac{\tau}{\lambda}}.$$  

(34)

Assume now $\lambda = 1$ and $\tau \gg 1$. By (31) we have

$$x_0 = \ln \left( \frac{1}{\pi} \ln \tau + i + \sqrt{\left( \frac{1}{\pi} \ln \tau \right)^2 + 2i \frac{1}{\pi} \ln \tau} \right).$$
Setting $$s = \frac{1}{\pi} \ln \tau,$$
we have
\[
x_0 = \ln \left( s \left( 1 + \frac{i}{s} + \sqrt{1 + \frac{2i}{s}} \right) \right)
\approx \ln \left( s \left( 1 + \frac{i}{s} + 1 + \frac{i}{s} \right) \right) = \ln \left( 2s \left( 1 + \frac{i}{s} \right) \right)
\approx \ln (2s) + \frac{i}{s},
\]
that finally leads to
\[
(35) \quad \text{Im} x_0 \approx \frac{1}{s} = \frac{\pi}{\ln \tau}.
\]

4.4. The minimax problem. Let us define the function
\[
\varphi(\lambda, \tau) = \xi(d) \lambda^{-\alpha} \exp \left( \frac{-2\pi dn}{\ln \left( \frac{4dn}{\mu} \right)} \right), \quad d = d(\lambda, \tau),
\]
representing the $(\lambda, \tau)$-dependent factor of the error estimate given by (28), that is,
\[
E_{n,h}(g_{\lambda,\psi,DE}) \approx K_\alpha \varphi(\lambda, \tau),
\]
where $K_\alpha$ is defined by (29). Since our aim is to work with a self-adjoint operator with spectrum contained in $[1, +\infty)$ the problem consists in defining properly the parameter $\tau$. This can be done by solving
\[
(36) \quad \min_{\tau \geq 1} \max_{\lambda \geq 1} \varphi(\lambda, \tau).
\]
As for the true error, experimentally one observes that $\tau$ must be taken much greater than 1, independently of $\alpha$. Therefore, from now on the analysis will be based on the assumption $\tau \gg 1$. Regarding the function $\varphi(\lambda, \tau)$, by taking $d = d(\lambda, \tau)$ as in (32) and $n$ sufficiently large, again, one experimentally observes that with respect to $\lambda$ the function initially decreases, reaches a local minimum (for $\lambda = \tau$ in which $d = r\pi/2$), then a local maximum (much greater than $\tau$), and finally goes to 0 for $\lambda \to +\infty$ (see Figure 3). In this view, denoting by $\lambda$ the local maximum, for $n$ sufficiently large the problem (36) reduces to the solution of
\[
(37) \quad \varphi(1, \tau) = \varphi(\lambda, \tau).
\]
4.4.1. Evaluating the local maximum. Since $0 < d \leq r\pi/2$, $0 < r < 1$, we have
\[
0 < C \leq \cos d \cos \left( \frac{\pi}{2} \sin d \right) < 1,
\]
where $C$ is a constant depending on $r$. Therefore by (22),
\[
2 < \xi(d) \leq \frac{2}{C},
\]
so that we neglect the contribute of this function in what follows.
Since the maximum is seen to be much larger than $\tau$, we consider the approximation (34). Therefore we have to solve

$$\frac{d}{d\lambda} \lambda^{-\alpha} \exp \left( -2\pi r \frac{\alpha \ln(n)}{n} \frac{\lambda}{\ln(n)} \right) = 0,$$

that, after some manipulation leads to

$$\frac{d}{d\lambda} \lambda^{-\alpha} \exp \left( -\frac{c_1 n}{\ln \frac{\alpha}{q(\lambda)}} \right) = 0,$$

where

1. $c_1 = 2\pi^2 r$, $q(\lambda) = \ln (c_2 n) - \ln \left( \frac{\lambda}{\tau} \right)$, $c_2 = \frac{4}{\mu \pi r}$.

We find the equation

$$-\alpha \lambda^{-1} - \frac{d}{d\lambda} \left( \frac{c_1 n}{\ln \frac{\alpha}{q(\lambda)}} \right) = 0,$$

and since

$$\frac{d}{d\lambda} \left( \frac{c_1 n}{\ln \frac{\alpha}{q(\lambda)}} \right) = \frac{c_1 n}{\lambda} \frac{1 - q(\lambda)}{(\ln \frac{\alpha}{q(\lambda)})^2 q(\lambda)^2},$$

we finally have to solve

$$\alpha + c_1 n \frac{1 - q(\lambda)}{(\ln \frac{\alpha}{q(\lambda)})^2 q(\lambda)^2} = 0.$$

For large $n$ we have

$$q(\lambda) \approx \ln (c_2 n),$$

$$\frac{q(\lambda) - 1}{q(\lambda)^2} \approx \frac{1}{q(\lambda)} \approx \frac{1}{\ln (c_2 n)},$$

so that the solution of (39) can be approximated by

$$\lambda^* = \tau \exp \left( \sqrt{\frac{c_1 n}{\alpha \ln (c_2 n)}} \right).$$

For any given $\tau \geq 1$, it can be observed experimentally that $\lambda^*$ is a very good approximation of the local maximum (see Figure 3).

We also remark that the assumption on $n$ stated in (26), that leads to the error estimate (28), is automatically fulfilled for $\lambda = \lambda^*$, at least for $\alpha$ not too small. Indeed, using (32) and (34) we first observe that

$$d(\lambda^*, \tau) \approx \frac{r \pi}{\ln \frac{\lambda^*}{\tau}} = r \pi \sqrt{\frac{\alpha \ln (c_2 n)}{c_1 n}}.$$. 
Then by (38), using \( \mu \leq \frac{1}{2} \) and assuming for instance \( 0.9 < r < 1 \), after some simple computation we find

\[
\frac{\mu e}{4d(\lambda^*, \tau)} \approx \frac{\mu e}{4\pi r} \sqrt{\frac{c_1 n}{\alpha \ln (c_2 n)}} \leq \frac{1}{3} \sqrt{\frac{n}{\alpha}}.
\]

Therefore the condition (26) holds true for \( n \geq \frac{1}{9\alpha} \).

### 4.4.2. The error at the local maximum.

By (41) clearly \( d(\lambda^*, \tau) \to 0 \) for \( n \to +\infty \), and therefore from (22) we deduce that \( \xi(d(\lambda^*, \tau)) \to 2 \) for \( n \to +\infty \). As consequence

\[
\varphi(\lambda^*, \tau) \approx 2(\lambda^*)^{-\alpha} \exp \left( \frac{-2\pi d(\lambda^*, \tau)n}{\ln \left( \frac{4d(\lambda^*, \tau)n}{\mu} \right)} \right).
\]

By defining (42)

\[
s_n = \sqrt{\frac{c_1 n}{\ln (c_2 n)}},
\]

from (40) and (41) we have

\[
\lambda^* = \tau \exp \left( \frac{s_n}{\sqrt{\alpha}} \right),
\]

\[
d(\lambda^*, \tau) \approx \frac{\sqrt{\alpha \pi}}{s_n},
\]

and hence, after some computation

\[
(\lambda^*)^{-\alpha} \exp \left( \frac{-2\pi d(\lambda^*, \tau)n}{\ln \left( \frac{4d(\lambda^*, \tau)n}{\mu} \right)} \right) \approx \tau^{-\alpha} \exp \left( -\sqrt{\alpha} s_n \right) \exp \left( \frac{-2\pi \sqrt{\alpha} \tau n}{\ln \left( \frac{4\sqrt{\alpha} \tau n}{\mu} \right)} \right)
\]

\[
= \tau^{-\alpha} \exp \left( -\sqrt{\alpha} \left( s_n + \frac{c_1 n}{s_n \ln \left( \frac{c_2 n \sqrt{\alpha}}{s_n} \right)} \right) \right).
\]

By (22) we have

\[
s_n + \frac{c_1 n}{s_n \ln \left( \frac{c_2 n \sqrt{\alpha}}{s_n} \right)} = \sqrt{\frac{c_1 n}{\ln (c_2 n)}} + \frac{c_1 n}{\ln (c_2 n)} \ln \left( \frac{c_2 n \sqrt{\alpha}}{s_n} \right)
\]

\[
= \sqrt{\frac{c_1 n}{\ln (c_2 n)}} \left( 1 + \frac{\ln (c_2 n)}{\ln \left( \frac{c_2 n \sqrt{\alpha}}{s_n} \right)} \right)
\]

\[
\approx 3 \sqrt{\frac{c_1 n}{\ln (c_2 n)}},
\]

because

\[
\frac{\ln (c_2 n)}{\ln \left( \frac{c_2 n \sqrt{\alpha}}{s_n} \right)} \to 2 \quad \text{for} \quad n \to +\infty.
\]
Joining the above approximations we finally obtain
\[
\varphi(\lambda^*, \tau) \approx 2\tau^{-\alpha} \exp \left( -3\sqrt{\alpha} \sqrt{\frac{c_1 n}{\ln (c_2 n)}} \right) 
\]
(43)
\[
= 2\tau^{-\alpha} \exp \left( -3\sqrt{\alpha}s_n \right). 
\]

4.4.3. Error at \( \lambda = 1 \). By (35), that is,
\[
d(1, \tau) \approx \frac{\pi}{\ln \tau}, \quad \tau \gg 1,
\]
we have again \( \xi(d(1, \tau)) \approx 2 \) and therefore
\[
\varphi(1, \tau) \approx 2 \exp \left( -\frac{2\pi d(1, \tau)n}{\ln \left( \frac{4d(1, \tau)n}{\mu} \right)} \right) \]
(44)
Using (38) we find
\[
\varphi(1, \tau) \approx 2 \exp \left( -\frac{2\pi r}{\ln \tau} \right) \approx 2 \exp \left( -\frac{s_n^2}{2\alpha \ln \tau} \right). 
\]

4.5. Approximating the optimal value for \( \tau \). We need to solve (37). Using the approximations (44) and (43) we impose
\[
\exp \left( -\frac{s_n^2}{\ln \tau} \right) = \tau^{-\alpha} \exp \left( -3\sqrt{\alpha}s_n \right) 
\]
\[
= \exp \left( -3\sqrt{\alpha}s_n - \alpha \ln \tau \right), 
\]
that is,
\[
-\frac{s_n^2}{\ln \tau} = -3\sqrt{\alpha}s_n - \alpha \ln \tau. 
\]
Solving the above equation we find
\[
\ln \tau = \frac{(-3 + \sqrt{13}) \sqrt{\alpha}s_n}{2\alpha} 
\]
\[
\approx 0.3 \frac{s_n}{\sqrt{\alpha}}, 
\]
so that
\[
\tau^* = \exp \left( 0.3 \frac{s_n}{\sqrt{\alpha}} \right) 
\]
(45)
represents an approximate solution of (37).

In Figure 3 we plot the function \( \varphi(\lambda, \tau^*) \) for \( \lambda \in [1, 10^2] \), in an example in which \( n = 40, \alpha = 1/2 \), and \( \tau^* \approx 84.4 \) defined by (45). Moreover we show the results of the approximations (40), (43) and (44), for \( \tau = \tau^* \). Clearly the ideal situation would be to have \( \tau^* \) such that \( \varphi(1, \tau^*) = \varphi(\lambda, \tau^*) \), but notwithstanding
Figure 3. Plot of the function \( \varphi(\lambda, \tau^*) \) for \( n = 40 \) and \( \alpha = 1/2 \). The asterisk represents the approximation of the local maximum given by (40), that is, the point \( (\lambda^*, \varphi(\lambda^*, \tau^*)) \). The diamond represents the approximation of \( \varphi(\lambda^*, \tau^*) \) stated in (43). Finally, the circle is the approximation of \( \varphi(1, \tau^*) \) given in (44).

all the approximations used, the results are fairly good and allow to have a simple expression for \( \tau^* \).

By using (45) in (43) we obtain

\[
(46) \quad \varphi(\lambda^*, \tau^*) \approx 2 \exp \left( -3.3 \sqrt{\alpha} \sqrt{\frac{c_1 n}{\ln(c_2 n)}} \right).
\]

Remembering that

\[
E_{n,h}(L) \leq 2 \frac{\sin(\alpha \pi)}{\pi} \max_{\lambda \geq 1} \mathcal{E}_{n,h}(g_{\lambda,\psi_E}),
\]

using (46) we finally obtain the error estimate

\[
(47) \quad E_{n,h}(L) \approx \overline{K}_\alpha \exp \left( -3.3 \sqrt{\alpha} \sqrt{\frac{c_1 n}{\ln(c_2 n)}} \right),
\]

where

\[
\overline{K}_\alpha = 4 \frac{\sin(\alpha \pi)}{\pi} K_\alpha
= 4 \frac{\sin(\alpha \pi)}{\pi} \frac{1}{\alpha(1-\alpha)} \frac{1}{1 - e^{-\beta}}.
\]
In Figure 4 we show the behavior of the method for the computation of $\mathcal{L}^{-\alpha}$, with $\mathcal{L}$ defined in (17), together with the estimate (47). For comparison, in the same pictures we also plot the error of the SE approach. As mentioned in the introduction, the DE approach appears to be faster for $1/2 \leq \alpha < 1$.

Figure 4. Error for the trapezoidal rule applied with the double-exponential transform (error DE), with the single-exponential transform (error SE) and error estimate given by (47).

5. Conclusions

In this work we have analyzed the behavior of the trapezoidal rule for the computation of $\mathcal{L}^{-\alpha}$, in connection with the single and the double-exponential transformations. All the analysis has been based on the assumption of $\mathcal{L}$ unbounded, so that the results can be applied even to discrete operators, with spectrum arbitrarily large, without the need to know its amplitude, that is, the largest eigenvalue. In particular we have revised the analysis for the single-exponential transform and we have introduced new error estimates for the scalar and the operator case for the double-exponential transform. The sharp estimate obtained for the scalar case has been fundamental for the proper selection of the parameter $\tau$ that is necessary to obtain good results also for the operator case.
References

[1] L. Aceto, D. Bertaccini, F. Durastante, P. Novati. Rational Krylov methods for functions of matrices with applications to fractional partial differential equations. J. Comput. Phys. 396, 470-482 (2019)

[2] L. Aceto, P. Novati. Padé-type approximations to the resolvent of fractional powers of operators. J. Sci. Comput. 83 (2020): 13, DOI:10.1007/s10915-020-01198-w

[3] L. Aceto, P. Novati. Rational approximations to fractional powers of self-adjoint positive operators. Numer. Math. 143, 1-16 (2019)

[4] L. Aceto, P. Novati. Fast and accurate approximations to fractional powers of operators. IMA J. Numer. Anal. (2021), drab002, https://doi.org/10.1093/imanum/drab002

[5] A. Bonito, J.E. Pasciak. Numerical approximation of fractional powers of elliptic operators. Math. Comp. 84, 2083-2110 (2015)

[6] S. Harizanov, R. Lazarov, S. Margenov, P. Marinov, Y. Vutov. Optimal solvers for linear systems with fractional powers of sparse SPD matrices. Numer. Linear Algebra Appl. 25(5), e2167 (2018)

[7] S. Harizanov, R. Lazarov, P. Marinov, S. Margenov, J.E. Pasciak. Analysis of numerical methods for spectral fractional elliptic equations based on the best uniform rational approximation. arXiv:1905.08155v2 (2019)

[8] S. Harizanov, R. Lazarov, P. Marinov, S. Margenov, J.E. Pasciak. Comparison analysis of two numerical methods for fractional diffusion problems based on the best rational approximations of $t^\gamma$ on $[0, 1]$. In: Apel T., Langer U., Meyer A., Steinbach O. (eds) Advanced Finite Element Methods with Applications. FEM 2017. Lecture Notes in Computational Science and Engineering, vol 128. Springer, Cham, 2019

[9] S. Harizanov, S. Margenov. Positive approximations of the inverse of fractional powers of SPD M-Matrices. In: Feichtinger G., Kovacevic R., Trager G. (eds) Control Systems and Mathematical Methods in Economics. Lecture Notes in Economics and Mathematical Systems, vol 687. Springer, Cham, 2018

[10] C. Hofreither. A unified view of some numerical methods for fractional diffusion. Comput. Math. Appl. 80(2), 351-366 (2020)

[11] J. Lund, K.L. Bowers. Sinc Methods for Quadrature and Differential Equations. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992. MR1171217 (93i:65004)

[12] T. Okayama, K. Tanaka, T. Matsuo, M. Sugihara. DE-Sinc methods have almost the same convergence property as SE-Sinc methods even for a family of functions fitting the SE-Sinc methods Part I: definite integration and function approximation Numer. Math. 125, 511-543 (2013)

[13] T. Okayama, T. Matsuo, M. Sugihara. Error estimates with explicit constants for Sinc approximation, Sinc quadrature and Sinc indefinite integration. Numer. Math. 124, 361-394 (2013)

[14] M. Mori. Discovery of the double exponential transformation and its developments. Publ. RIMS, Kyoto Univ. 41, 897-935 (2005)

[15] K. Tanaka, M. Sugihara, K. Murota, M. Mori. Function classes for double exponential integration formulas. Numer. Math. 111, 631-655 (2009)

[16] L.N. Trefethen, J.A.C. Weideman. The exponentially convergent trapezoidal rule. SIAM Review 56(3), 385-458 (2014)

[17] P.N. Vabishchevich. Approximation of a fractional power of an elliptic operator. CoRR abs/1905.10838 (2019)

[18] P.N. Vabishchevich. Numerical solution of time-dependent problems with fractional power elliptic operator. Comput. Meth. in Appl. Math. 18(1), 111-128 (2018)

[19] P.N. Vabishchevich. Numerically solving an equation for fractional powers of elliptic operators. J. Comput. Phys. 282, 289-302 (2015)

Lidia Aceto, Università del Piemonte Orientale, Dipartimento di Scienze e Innovazione Tecnologica, viale Teresa Michel 11, 15121 Alessandria, Italy
Email address: lidia.aceto@uniupo.it

Paolo Novati, Università di Trieste, Dipartimento di Matematica e Geoscienze, via Valerio 12/1, 34127 Trieste, Italy
Email address: novati@units.it