Abstract. Regular integer lattices are characterized by $k$ unit vectors that build up their generator matrices. These have rank $k$ for $D_k$-lattices, and are rank-deficient for $A_k$-lattices, for $E_6$ and $E_7$. We count lattice points inside hypercubes centered at the origin for all three types, as if classified by maximum infinity norm in the host lattice. The results assume polynomial format as a function of the hypercube edge length.

1. Scope

We consider infinite translationally invariant point lattices set up by generator matrices $G$

$$p_i = \sum_{j=1}^{k} G_{ij} \alpha_j$$

which select point coordinates $p$ given a vector of integers $\alpha$. In a purely geometric enumerative manner we count all points that reside inside a hypercube defined by $|p_i| \leq n$, $\forall i$. These numbers shall be called $A_b^k(n)$, $D_b^k(n)$ and $E_b^k(n)$ for the three lattice types dealt with. In the incremental version of boxing the hypercubes, the points that are on the surface of the hypercube are given the upper index $s$,

$$A_s^k(n) = A_b^k(n) - A_b^k(n-1), ~ D_s^k(n) = D_b^k(n) - D_b^k(n-1), ~ E_s^k(n) = E_b^k(n) - E_b^k(n-1),$$

the first differences of the “bulk” numbers with respect to the edge size $n$.

There is vague resemblance to volume computation of the polytope defined in $\alpha$-space by other straight cuts in $p$-space [11, 10].

In all cases discussed, the generating functions $D_b^k(x)$, $A_b^k(x)$ or $E_b^k(x)$ are rational functions with a factor $(1 - x)^k$ in the denominator. They count sequences starting with a value of 1 at $n = 0$. The generating functions of the first differences, $D_s^k(x)$ etc., are therefore obtained by decrementing the exponent of $1 - x$ in these denominators by one [14, 19], and have not been written down individually for that reason.

The manuscript considers first the $D$-lattices $D_6-D_4$ in tutorial detail in sections 2–4, then the case of general $k$ in Section 5. The points in $A_2$–$A_4$ are counted in sections 6–8 by examining sums over the $\alpha$-coefficients, and the general value of $k$ is addressed by summation over $p$-coordinates in Section 9. The cases $E_6$–$E_8$ are reduced to the earlier lattice counts in sections 10–12.

Date: April 22, 2010.
2010 Mathematics Subject Classification. Primary 52B05, 06B05; Secondary 05B35, 52B20.
Key words and phrases. root lattices, polytopes, infinity norm, hypercube, centered multilinear coefficient.
2. Lattice $D_2$

In the $D_2$ lattice, the expansion coefficients $\alpha_i$ and Cartesian coordinates $p_i$ are connected by

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}. \tag{3}$$

If we read the two lines of this system of equations separately, points inside the square $|p_i| \leq n$ ($i = 1, 2$) are constrained to $\alpha$-coordinates inside a tilted square, as shown in Figure 1.

![Figure 1](image)

**Figure 1.** The conditions $|\alpha_1 \pm \alpha_2| \leq n$ select two orthogonal diagonal stripes in the $(\alpha_1, \alpha_2)$-plane. Their intersection is a tilted square centered at the origin.

The point count inside the square is

$$D^b_2 = \sum_{|\alpha_1 - \alpha_2| \leq n} \sum_{|\alpha_1 + \alpha_2| \leq n} 1. \tag{4}$$

Resummation considering the two non-overlapping triangles below and above the horizontal axis yields

$$D^b_2 = \sum_{\alpha_2 = -n}^{\alpha_2 = n} \sum_{\alpha_1 = -n - \alpha_2}^{\alpha_1 = n - \alpha_2} 1 + \sum_{\alpha_2 = 1}^{\alpha_2 = n} \sum_{\alpha_1 = \alpha_2}^{\alpha_1 = n - \alpha_2} 1$$

$$= \sum_{\alpha_2 = -n}^{\alpha_2 = n} (2\alpha_2 + 2n + 1) + \sum_{\alpha_2 = 1}^{\alpha_2 = n} (2n - 2\alpha_2 + 1). \tag{5}$$

We will frequently sum over low order multinomials of this type with a basic formula in terms of Bernoulli Polynomials $B_j, [9$ (0.121)$][20$ (1.2.11)$][7$

$$\sum_{m=1}^{j} m^k = \frac{B_{1+k}(j+1) - B_{1+k}(0)}{1+k}. \tag{6}$$

Application to (5) and its first differences yields essentially sequences A001844 and A008586 of the Online Encyclopedia of Integer Sequences (OEIS) [17]:

**Theorem 1.** (Lattice points in the bulk and on the surface of $D_2$)

$$D^b_2(n) = 2n^2 + 2n + 1 = 1, 5, 13, 25, \ldots; \quad D^s_2(n) = \begin{cases} 1, & n = 0; \\ 4n, & n > 0. \end{cases} \tag{7}$$
3. Lattice $D_3$

The relation between expansion coefficients $\alpha_i$ and Cartesian coordinates $p_i$ for the $D_3$ lattice is

$$
\begin{pmatrix}
1 & 1 & 0 \\
1 & -1 & 1 \\
0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{pmatrix}
=
\begin{pmatrix}
p_1 \\
p_2 \\
p_3
\end{pmatrix}.
$$

The determinant of the Generator Matrix is non-zero; by multiplication with the inverse matrix, a form more suitable to the counting problem results:

$$
\begin{pmatrix}
1/2 & 1/2 & 1/2 \\
1/2 & -1/2 & -1/2 \\
0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
p_1 \\
p_2 \\
p_3
\end{pmatrix}
=
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{pmatrix}.
$$

$D_3^b(n)$ is the number of integer solutions restricted to the cube $-n \leq p_i \leq n$. This is the full triple sum $(2n+1)^3$—where $2n+1$ sizes the edge length of the cube—minus the number of solutions of (9) that result in non-integer $\alpha_i$. The structure of the three equations in (9) suggests to separate the cases according to the parities of $p_3$ and $p_1 + p_2$:

$$
D_3^b(n) = \sum_{\substack{|p_1| \leq n, |p_2| \leq n, |p_3| \leq n \text{ even} \atop p_1 + p_2 + p_3 \text{ even}}} 1 = \sum_{|p_1| \leq n, |p_2| \leq n, |p_3| \leq n} \sum_{|p_3| \leq n} \sum_{|p_1| \leq n, |p_2| \leq n} \sum_{|p_3| \leq n} 1.
$$

The auxiliary sums are examined separately for even and odd $n$ [17, A109613, A052928]:

$$
\sum_{|p_3| \leq n} 1 = n + \frac{1 + (-1)^n}{2} = 1, 1, 3, 3, 5, 5, 7, 7, 9, 9, \ldots;
$$

$$
\sum_{|p_3| \leq n, p_3 \text{ odd}} 1 = n + \frac{1 - (-1)^n}{2} = 0, 2, 2, 4, 4, 6, 6, 8, 8, \ldots.
$$

The parity-filtered double sum of (11) over the square in $(p_1, p_2)$-space selects points on lines parallel to the diagonal.

**Definition 1.** (Order of even ($g$) and odd ($u$) point sets in $k$-dimensional hypercube planes)

$$
V_k^g(n) \equiv \sum_{|p_i| \leq n, p_1 + p_2 + \cdots + p_k \text{ even}} 1; 
V_k^u(n) \equiv \sum_{|p_i| \leq n, p_1 + p_2 + \cdots + p_k \text{ odd}} 1.
$$

This decomposition applies to higher dimensions recursively:

$$
V_k^g(n) = V_{k-1}^u(n)V_k^u(n) + V_{k-1}^g(n)V_k^g(n);
$$

$$
V_k^u(n) = V_{k-1}^u(n)V_k^g(n) + V_{k-1}^g(n)V_k^u(n).
$$

Starting from $V_2^g(n)$ and $V_2^u(n)$ given in (11)–(12), the recurrences provide Table 1. The two disjoint sets of lattice points complement the hypercube:

$$
V_k^g(n) + V_k^u(n) = (2n+1)^k.
$$
Theorem 3. and A175109 in the OEIS [17]. The proof is simple by induction with the aid of (14) and (16), using Proof.

\[
V_k(n) = \begin{cases} 
\frac{(2n+1)^k}{2} + \frac{1}{2}, & k \text{ even; } \\
\frac{(2n+1)^k}{2} + \frac{(-1)^n}{2}, & k \text{ odd. }
\end{cases}
\]

\(D_k^b(n)\) in (10) equals \(V_k^n(n)\) by definition. \(D_k^b(n)\) and \(D_k^s(n)\) are sequences A110907 and A175109 in the OEIS [17].

Theorem 2. (fcc lattice counts for edge measure 2\(n+1\))

\[
D_k^b(n) = 4n^3 + 6n^2 + 3n + \frac{1+(-1)^n}{2} = 1, 13, 63, 171, 365, 665 \ldots
\]

\[
D_k^s(n) = \begin{cases} 
1, & n = 0; \\
12n^2 + 1 + (-1)^n & n > 0.
\end{cases}
\]

The corresponding recurrences and generating function are

\[
D_k^b(n) = 3D_k^b(n-1) - 2D_k^b(n-2) - 2D_k^b(n-3) + 3D_k^b(n-4) - D_k^b(n-5);
\]

\[
D_k^b(x) = \frac{(1 + 6x + x^2)(1 + 4x + x^2)}{(1 + x)(1 - x)^4};
\]

\[
D_k^s(n) = 2D_k^s(n-1) - 2D_k^s(n-3) + D_k^s(n-4); \quad (n > 3).
\]
4. LATTICE $D_4$

The transformation between expansion coefficients and Cartesian coordinates in the $D_4$ case reads

$$
\begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & -1 & 1 & 0 \\
0 & 0 & -1 & 1 \\
0 & 0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4
\end{pmatrix}
= 
\begin{pmatrix}
p_1 \\
p_2 \\
p_3 \\
p_4
\end{pmatrix}.
$$

The technique of counting points inside cubes is the same as in the previous section. Inversion of the $4 \times 4$ matrix yields

$$
\begin{pmatrix}
1/2 & 1/2 & 1/2 & 1/2 \\
1/2 & -1/2 & -1/2 & -1/2 \\
0 & 0 & -1 & 1 \\
0 & 0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
p_1 \\
p_2 \\
p_3 \\
p_4
\end{pmatrix}
= 
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4
\end{pmatrix}.
$$

We wish to count all lattice points subject to the constraint $|p_i| \leq n$ ($i = 1, \ldots, 4$), and the first two lines of the previous equation require in addition that the sum over all four $p_i$ is even to keep all four $\alpha_i$ in the integer domain:

$$
D^b_4(n) = \sum_{|p_1| \leq n, |p_2| \leq n, |p_3| \leq n, |p_4| \leq n} \sum_{|p_1| + |p_2| + |p_3| + |p_4| \text{ even}} 1.
$$

This expression is $V^\beta_b(n)$ already computed above. $D^s_4(n)$ is OEIS sequence A117216; $D^b_4(n)$ is A175110 \[17].

**Theorem 4. (Lattice points in the bulk and on the surface of $D_4$)**

$$
\begin{align*}
D^b_4(n) &= 1 + 4n + 12n^2 + 16n^3 + 8n^4 \\
&= 1, 41, 313, 1201, 3281, 7321, 14281, 25313, 41761, 65161, 97241, \ldots;
\end{align*}
$$

$$
\begin{align*}
D^s_4(n) &= \begin{cases} 
1 & n = 0; \\
8n(1 + 4n^2) & n > 0;
\end{cases} \\
&= 1, 40, 272, 888, 2080, 4040, 6960, 11032, 16448, 23400, 32080, \ldots
\end{align*}
$$

The associated generating function and recurrences are

$$
D^b_4(x) = \frac{1 + 36x + 118x^2 + 36x^3 + x^4}{(1 - x)^5};
$$

$$
D^b_4(n) = 5D^b_4(n - 1) - 10D^b_4(n - 2) + 10D^b_4(n - 3) - 5D^b_4(n - 4) + D^b_4(n - 5); \\
D^s_4(n) = 4D^s_4(n - 1) - 6D^s_4(n - 2) + 4D^s_4(n - 3) - D^s_4(n - 4); \quad (n > 4).
$$
5. LATTICES $D_k$, GENERAL $k$

No new aspect arises in comparison to the previous two sections \[16\]. The $D^b_k(n)$ equal the $V^b_k(n)$ and their first differences constitute the $D^s_k(n)$:

$$D^b_k(n) = 16n^5 + 40n^4 + 40n^3 + 20n^2 + 5n + \frac{1 + (-)^n}{2};$$

$$D^s_k(n) = \begin{cases} 
1, & n = 0; \\
1 + 40n^2 + 80n^4 + (-)^n, & n > 0;
\end{cases}$$

$$D^b_k(n) = 32n^6 + 96n^5 + 120n^4 + 80n^3 + 30n^2 + 6n + 1;$$

$$D^s_k(n) = \begin{cases} 
1, & n = 0; \\
4n(1 + 12n^2)(3 + 4n^2), & n > 0;
\end{cases}$$

$$D^b_k(n) = 64n^7 + 225n^6 + 336n^5 + 280n^4 + 130n^3 + 43n^2 + 7n + \frac{1 + (-)^n}{2};$$

$$D^s_k(n) = \begin{cases} 
1, & n = 0; \\
1 + 84n^2 + 560n^4 + 448n^6 + (-)^n, & n > 0;
\end{cases}$$

$D_5$ and $D_6$ are materialized as sequences A175111 to A175114 \[17\]. All cases are summarized in a Corollary to Theorem 2:

**Corollary 1.** ($D_k$ Lattice points inside the hypercube)

$$D^b_k(n) = \begin{cases} 
\frac{(2n+1)^k}{2} + \frac{1}{2}, & k \text{ even}; \\
\frac{(2n+1)^k}{2} + \frac{(-)^n}{2}, & k \text{ odd};
\end{cases}$$

$$D^s_k(n) = \begin{cases} 
\frac{(2n+1)^k}{2} - \frac{(2n-1)^k}{2}, & k \text{ even}, n > 0; \\
\frac{(2n+1)^k}{2} - \frac{(2n-1)^k}{2} + (-)^n, & k \text{ odd}, n > 0.
\end{cases}$$

The generating functions are

$$D^b_k(x) = \begin{cases} 
\sum_{i=0}^{k} \beta^b_i x^i (1-x)^{k+1}, & k \text{ even}; \\
1 + \sum_{i=1}^{k} \beta^b_i x^i (1+x)(1-x)^{k+1}, & k \text{ odd};
\end{cases}$$

where

$$2\beta^b_i = i \sum_{t=0}^{i} ((2i-2t+1)^k + 1) \binom{k+1}{t} (-1)^t,$$

$$2\beta^s_i = \sum_{t=0}^{i} ((2i-2t+1)^k + (-)^{i-t}) \binom{k+1}{t} (-1)^t + \sum_{t=0}^{i-1} ((2i-2t-1)^k + (-)^{i-t}) \binom{k+1}{t} (-1)^t.$$
Remark 1. The $D_k^*$ lattices are characterized by

$$
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 1/2 \\
0 & 1 & 0 & \cdots & 0 & 1/2 \\
0 & 0 & 1 & \ddots & 0 & 1/2 \\
\vdots & \vdots & 0 & 1 & \ddots & 1/2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & 1/2 \\
0 & 0 & 0 & \cdots & 0 & 1/2
\end{pmatrix}.
$$

Matrix inversion gives

$$
\begin{pmatrix}
1 & 0 & \cdots & 0 & -1 \\
0 & 1 & 0 & \vdots & -1 \\
\vdots & 0 & 1 & \vdots & -1 \\
0 & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 2
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{pmatrix}
= 
\begin{pmatrix}
p_1 \\
p_2 \\
p_3
\end{pmatrix}.
$$

which shows that there is no constraint on generating any $p_i$ inside the regions $|p_i| \leq n$: The number of lattice points up to infinity norm $n$ is simply $D_k^*(n) = (2n+1)^k$.

6. Lattice $A_2$

$A_2^b(n)$ is the number of integer solutions to

$$
\begin{pmatrix}
1 & 0 \\
-1 & 1 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2
\end{pmatrix}
= 
\begin{pmatrix}
p_1 \\
p_2 \\
p_3
\end{pmatrix}
$$

in the range $|p_i| \leq n$. The three requirements from the three lines of this equation become

$$
A_2^b(n) = \sum_{|\alpha_1| \leq n} \sum_{|\alpha_2| \leq n} 1.
$$

As outlined in Figure 2, decomposition of the conditions allows resummation over the quadrangles above and below the $\alpha_1$ axis:

$$
A_2^b(n) = \sum_{\alpha_2=-n}^{n} \sum_{\alpha_1=-n}^{n} 1 + \sum_{\alpha_2=1}^{n} \sum_{\alpha_1=-n}^{n} 1 = \sum_{\alpha_2=-n}^{n} (2n+1 + \alpha_2) + \sum_{\alpha_2=1}^{n} (2n+1 - \alpha_2).
$$

further evaluated with [13].

Theorem 5. (Lattice points in the bulk and on the surface of $A_2$, [17, A003215])

$$
A_2^s(n) = 1 + 3n(n+1) = 1, 7, 19, 37, 61, 91, 127, 169, 217, 271, 331, 397, 469, \ldots
$$

The first differences are [17, A008458]

$$
A_2^s(n) = \begin{cases} 1, & n = 0; \\ 6n, & n > 0. \end{cases}
$$
Figure 2. The conditions $|\alpha_1| \leq n$ and $|\alpha_2| \leq n$ select a square in the $(\alpha_1, \alpha_2)$-plane. The requirement $| - \alpha_1 + \alpha_2| \leq n$ admits only values inside a diagonal stripe. The intersection is the dotted hexagon.

7. LATTICE $A_3$

The generator matrix sets

$$
\begin{pmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
0 & -1 & 1 \\
0 & 0 & -1 \\
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\end{pmatrix}
=
\begin{pmatrix}
p_1 \\
p_2 \\
p_3 \\
p_4 \\
\end{pmatrix}
$$

This translates the four bindings $|p_1| \leq n$ into four constraints on the three $\alpha$:

$$
A_3^b(n) = \sum_{|\alpha_1| \leq n} \sum_{-|\alpha_1+\alpha_2| \leq n} \sum_{-|\alpha_3| \leq n} 1.
$$

Figure 3 illustrates resummation of the format

$$
\sum_{|\alpha_1| \leq n} \sum_{-|\alpha_1+\alpha_2| \leq n} 1 = \sum_{\alpha_2=-2n}^{\alpha_2+n} \sum_{\alpha_1=-n}^{\alpha_1+n} 1 + \sum_{\alpha_2=1}^{2n} \alpha_1 = \alpha_2 - n.
$$

This is applied twice (note this factorization generates quad-sums which are a
convenient notation to keep track of the limits. The sums actually remain triple sums):

\[
A_{3}^{b}(n) = \left( \sum_{\alpha_{2}=-2n}^{0} \sum_{\alpha_{1}=-n}^{\alpha_{2}+n} 1 + \sum_{\alpha_{2}=1}^{2n} \sum_{\alpha_{1}=-n}^{\alpha_{2}-n} 1 \right) \left( \sum_{\alpha_{2}=-2n}^{0} \sum_{\alpha_{3}=-n}^{\alpha_{2}+n} 1 + \sum_{\alpha_{2}=1}^{2n} \sum_{\alpha_{3}=-n}^{\alpha_{2}-n} 1 \right)
\]

\[
= \sum_{\alpha_{2}=-2n}^{0} \sum_{\alpha_{1}=-n}^{\alpha_{2}+n} \sum_{\alpha_{3}=-n}^{\alpha_{2}+n} 1 + 2n \sum_{\alpha_{2}=-2n}^{0} \sum_{\alpha_{1}=-n}^{\alpha_{2}-n} \sum_{\alpha_{3}=-n}^{\alpha_{2}+n} 1
\]

\[
= \sum_{\alpha_{2}=-2n}^{0} \left( (2n + 1 + \alpha_{2})^{2} + \sum_{\alpha_{2}=1}^{2n} (2n + 1 - \alpha_{2})^{2} \right).
\]

After binomial expansion, both remaining sums are reduced with (6):

**Theorem 6.** *(Lattice points in the bulk and on the surface of $A_{3}$)*

\[
A_{3}^{b}(n) = 1 + \frac{2}{3}n(7 + 12n + 8n^{2}) = 1, 19, 85, 231, 489, 891, 1469, 2255, 3281, \ldots
\]

\[
A_{3}^{s}(n) = \begin{cases} 
1, & n = 0 \\
2 + 16n^{2}, & n > 0
\end{cases} = 1, 18, 66, 146, 258, 402, 578, \ldots
\]

These are sequences A063496 and A010006 in the OEIS [17].

8. Lattice $A_{4}$

$A_{4}$ is characterized by a quad-sum over $\alpha_{i}$ with five constraints on the $p_{i}$ set up by the hypercube:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & -1 & 1 \\
0 & 0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
\alpha_{1} \\
\alpha_{2} \\
\alpha_{3} \\
\alpha_{4}
\end{pmatrix}
= \begin{pmatrix}
p_{1} \\
p_{2} \\
p_{3} \\
p_{4} \\
p_{5}
\end{pmatrix}.
\]

\[
A_{4}^{b}(n) = \sum_{|\alpha_{1}| \leq n} \sum_{|\alpha_{1} + \alpha_{2}| \leq n} \sum_{|\alpha_{2} + \alpha_{3}| \leq n} \sum_{|\alpha_{3} + \alpha_{4}| \leq n} \sum_{|\alpha_{4}| \leq n} 1.
\]
The resummation (51) is separately applied to \((\alpha_1, \alpha_2)\) and \((\alpha_3, \alpha_4)\); the entanglement between \(\alpha_2\) and \(\alpha_3\) is noted in the second factor:

\[
A_3^n(n) = \left( \sum_{\alpha_2=-2n}^{0} \sum_{\alpha_1=-n}^{\alpha_2+n} 1 + \sum_{\alpha_3=1}^{2n} \sum_{\alpha_4=\alpha_2-n}^{n} 1 \right) \times \left( \sum_{\alpha_3=-2n}^{0} \sum_{\alpha_4=-n}^{\alpha_3+n} 1 + \sum_{\alpha_3=1}^{2n} \sum_{\alpha_4=\alpha_3-n}^{n} 1 \right)
= \left( \sum_{\alpha_2=-2n}^{0} (2n + 1 + \alpha_2) + \sum_{\alpha_2=1}^{2n} (2n + 1 - \alpha_2) \right) \times \left( \sum_{\alpha_3=-2n}^{0} (2n + 1 + \alpha_3) + \sum_{\alpha_3=1}^{2n} (2n + 1 - \alpha_3) \right).
\]

Product expansion generates 4 terms. The coupling between \(\alpha_2\) and \(\alpha_3\) is rewritten individually in their 4 different quadrants facilitated with Figure 4.

**Figure 4.** The conditions \(|\alpha_2| \leq 2n\) and \(|\alpha_3| \leq 2n\) define the large square, and \(|-\alpha_2 + \alpha_3| \leq n\) narrows the region down to the dotted hexagon.
This is derived by adding a k-tation of the multinomial coefficient A

Balancing the accumulated powers as required for So

Theorem 7. (Lattice points in the bulk and on the surface of A4) A

So A

Selecting values for the coefficient α matrix. (This simple format suffices; laminations are not involved.) An associated zero—with the exception of the last component—as a final column to the generator A

Over the d- lattices. Theorem 7 translates into six elementary double sums over products of the form (2n + 1 ± α2)(2n + 1 ± α3), eventually handled with (6).

9. Lattices A

Direct summation over the polytopes in αi-space becomes increasingly laborious in higher dimensions; we switch to summation in p-space based on the alternative

This is derived by adding a k + 1-st unit vector with all components equal to zero—with the exception of the last component—as a final column to the generator matrix. (This simple format suffices; laminations are not involved.) An associated coefficient αk+1 embeds the lattice into full space, while the condition αk+1 = 0 is maintained for the counting process. Inversion of the matrix generator equation demonstrates that this zero-condition translates into the requirement on the sum over the pi shown above. This point of view is occasionally used to define the A-lattices.

Counting the points subjected to some fixed \sum p_i = m is equivalent to computation of the multinomial coefficient

Balancing the accumulated powers as required for A

Selecting values for the \pi is equivalent to a Motzkin-path, picking one term of each of the k instances of the 1 + x + x^-1 of the trinomial, for example [5]. First, the
formula is a route to quick numerical evaluation (Table 2). Second, it proves that \( A_b^k(n) \) is a polynomial of order \( \leq k \) in \( n \), because each of the binomial factors in the \( j \)-sum is a polynomial of order \( k-1 \). This is easily made more explicit by invocation of the Stirling numbers of the first kind \([13][1, (24.1.3)]\).

**Remark 2.** This scheme of polynomial extension has been used for coordination sequences before \([6]\), and is found in growth series as well \([2]\).

**Table 2.** \( A_b^k(n) \) displaying columns of central 3-nomial, 5-nomial, 7-nomial etc. numbers \([17, A002426,A005191,A025012,A025014,A163269]\)

| \( k \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8  |
|-------|---|---|---|---|---|---|---|---|----|
| 1     | 1 | 3 | 5 | 7 | 9 | 11| 13| 15| 17 |
| 2     | 1 | 7 | 19| 37| 61| 91|127|169|217 |
| 3     | 1 | 19| 85|231|489|891|1469|2255|3281|
| 4     | 1 | 51|381|1451|3951|8801|17151|30381|50101|
| 5     | 1 |141|1751|9331|32661|88913|204763|418503|782153|
| 6     | 1 |393|8135|60691|273127|908755|2473325|5832765|12354469|
| 7     | 1 |1107|38165|398567|2306025|9377467|30162301|82073295|197018321|
| 8     | 1 |3139|180325|2636263|19610233|97464799|370487485|116320547|3164588407|

By computing the initial terms of any \( A_k \) numerically, the others follow by the recurrence obeyed by \( k \)-th order polynomials \([8]\):

\[
A_b^k(n) = \sum_{j=1}^{k+1} \binom{k+1}{j} (-)^{j+1} A_b^k(n-j).
\]

**Theorem 8.** (Lattice points in the bulk and on the surface of \( A_5 \))

\[
A_b^5(n) = \frac{1}{5} (2n+1)(5+27n+71n^2+88n^3+44n^4);
\]

\[
A_s^5(n) = \begin{cases} 1, & n = 0, \\ 2 + 50n^2 + 88n^4, & n > 0, \end{cases} = 1, 140, 1610, 7580, 23330, \ldots
\]

\( A_b^5 \) is a bisection of sequence A071816 of the OEIS \([17]\). \( A_b^6 \) is a bisection of sequence A133458 \([17]\).

**Theorem 9.** (\( A_6 \) and \( A_7 \) point counts)

\[
A_b^6(n) = 1 + \frac{7}{180} n(n+1)(222 + 727n + 1568n^2 + 1682n^3 + 841n^4).
\]

\[
A_b^7(n) = \begin{cases} 1, & n = 0, \\ \frac{7}{30} n(74 + 765n^2 + 841n^4), & n > 0, \end{cases} = 1, 392, 7742, 52556, 212436 \ldots
\]

\[
A_b^8(n) = \frac{2n+1}{315} (315 + 2568n + 10936n^2 + 26400n^3 + 37360n^4 + 28992n^5 + 9664n^6).
\]
Remark 3. The $A_k^n(n)$ can be phrased as k-th order polynomials of $L \equiv 2n + 1$ with the same parity as $k$:

\begin{equation}
A_3^1(L) = \frac{3}{4} L^2;
\end{equation}
\begin{equation}
A_3^2(L) = \frac{1}{4} + \frac{3}{4} L^2;
\end{equation}
\begin{equation}
A_3^3(L) = \frac{1}{3} L + \frac{2}{3} L^3;
\end{equation}
\begin{equation}
A_3^4(L) = \frac{9}{64} + \frac{25}{96} L^2 + \frac{115}{192} L^4;
\end{equation}
\begin{equation}
A_3^5(L) = \frac{1}{5} L + \frac{1}{4} L^3 + \frac{11}{20} L^5;
\end{equation}
\begin{equation}
A_3^6(L) = \frac{25}{256} + \frac{539}{2304} L^4 + \frac{5887}{11520} L^6;
\end{equation}
\begin{equation}
A_3^7(L) = \frac{1}{7} L + \frac{7}{45} L^3 + \frac{2}{9} L^5 + \frac{151}{315} L^7;
\end{equation}
\begin{equation}
A_3^8(L) = \frac{1225}{16384} + \frac{3229}{28672} L^2 + \frac{6063}{40960} L^4 + \frac{867}{4096} L^6 + \frac{259723}{573440} L^8.
\end{equation}

If we rewrite (66) \[15\] the multiplication formula of the $\Gamma$-function converts this to terminating Saalschützian Hypergeometric Series:

\begin{equation}
A_{k-1}^b(n) = \sum_{j=0}^{\lfloor (k+2)/n \rfloor} (-1)^j \frac{k}{\Gamma(k-j+1) \Gamma(kn-j(2n+1)+1)} \Gamma(kn-j(2n+1)+1),
\end{equation}
and

\begin{equation}
A_{k-1}^b(n) = \frac{\Gamma((n+1)k)}{\Gamma(k) \Gamma(nk+1)} \left( \frac{\prod_{j=1}^{n}(j)^{k-j}}{\prod_{j=0}^{n-1}(j)^{n-j}} \right) F_{2n+1}^{2n+2} \left( \begin{array}{c}
-k, -\frac{n}{2n+1}, -\frac{nk-1}{2n+1}, -\frac{nk-2}{2n+1}, \ldots, -\frac{nk-2n}{2n+1} \\
2n+1, 2n+1, 2n+1, \ldots, 2n+1
\end{array} \right) 
\end{equation}

The functional equation $\Gamma(m+1) = m \Gamma(m)$ presumably induces a non-linear recurrence along each column of Table \[2\] as shown by Sulanke for column $n = 1$ \[18\]. Numerical experimentation rather than proofs \[12\] suggest:

**Conjecture 1.** (Recurrents of centered 3-nomial, 5-nomial, 7-nomial coefficients)

\begin{equation}
(k+1)A_{k-1}^b(1) - (2k+1)A_{k-1}^b(1) - 3kA_{k-2}^b(1) = 0;
\end{equation}
\begin{equation}
2(k+1)(2k+1)A_{k}^b(2) + (k^2 - 49k - 2)A_{k-1}^b(2) + 5(-21k^2 + 37k - 18)A_{k-2}^b(2) - 25(k - 1)(k - 4)A_{k-3}^b(2) + 125(k - 1)(k - 2)A_{k-4}^b(2) = 0.
\end{equation}
\begin{equation}
3(3k+2)(3k+1)(k+1)A_{k}^b(3) + (41k^3 - 600k^2 - 191k - 6)A_{k-1}^b(3) + 7(-383k^3 + 1458k^2 - 1927k + 840)A_{k-2}^b(3) + 49(-83k^3 + 1068k^2 - 4321k + 5040)A_{k-3}^b(3) + 343(199k^3 - 1890k^2 + 6017k - 6390)A_{k-4}^b(3) + 2401(k - 3)(43k^2 - 351k + 722)A_{k-5}^b(3) - 16807(k - 3)(k - 4)(5k - 19)A_{k-6}^b(3) - 117649(k - 5)(k - 4)(k - 3)A_{k-7}^b(3) = 0.
\end{equation}
Table 3. Binomial coefficients $\eta_{k,j}$ of (88).

| $k \setminus j$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|
| 1               | 1   |     |     |     |     |     |     |     |
| 2               |     | 3   | 3   |     |     |     |     |     |
| 3               |     | 9   | 24  | 16  |     |     |     |     |
| 4               |     | 25  | 140 | 230 | 115 |     |     |     |
| 5               |     | 70  | 735 | 2250| 2640| 1056|     |     |
| 6               |     | 196 | 3675| 18732| 38801| 35322| 11774|     |
| 7               |     | 553 | 17976| 143696| 468160| 728448| 541184| 154624|
| 8               |     | 1569| 87024| 1052352| 5067288| 11994354| 14906484| 9350028| 2337507|

Remark 4. Inverse binomial transformations of the $A_k^b(n)$ define coefficients $\eta_{k,j}$ via

$$A_k^b(n) \equiv 1 + 2 \sum_{j=1}^{n} \binom{n}{j} \eta_{k,j},$$

$$\eta_{k,j} = \frac{1}{2} \sum_{l=0}^{j} (-1)^{j+l} \binom{j}{l} \left(\frac{k+1}{l(k+1)}\right)^{2l},$$

as demonstrated in Table 3. They are related to the partial fractions of the rational generating functions :

$$A_k^b(x) = \frac{1}{1-x} + 2 \sum_{j=1}^{k} \eta_{k,j} \frac{x^j}{(1-x)^{j+1}} \equiv \sum_{l=0}^{k} \gamma_{k,l} x^l \frac{1}{(1-x)^{k+1}}.$$

The first column and the diagonal of Table 3 appear to be sequences A097861 and A011818 of the OEIS, respectively [17].

Remark 5. From (66) we deduce the numerator coefficients defined in (89):

$$\gamma_{k,l} = \sum_{n=0}^{l} \binom{k+1}{l-n} (-1)^{l-n} \binom{k+1}{n(k+1)} 2n.$$

Some of these are shown in Table 4. Caused by the mirror symmetry of the coefficients, $-1$ is a root of the polynomial $\sum l \gamma_{k,l} x^l$ if $k$ is odd; a factor $1 + x$ may then be split off.

Formula (2) converts Table 2 into Table 5. And similar to Conjecture 1 we formulate recurrences along columns of this derived table:

Conjecture 2. (Recurrences of $A_k^b$)

$$(k+1)(k-1)A_k^b(1) - (3k^2 - k - 1)A_{k-1}^b(1) - k(k-2)A_{k-2}^b(1) + 3k(k-1)A_{k-3}^b(1) = 0,$$
Table 4. Synopsis of the numerators $\gamma_{k,l}$ of the generating functions (89).

| $k \setminus l$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  |
|-----------------|----|----|----|----|----|----|----|----|
| 1               | 1  | 1  |    |    |    |    |    |    |
| 2               | 1  | 4  | 1  |    |    |    |    |    |
| 3               | 1  | 15 | 15 | 1  |    |    |    |    |
| 4               | 1  | 46 | 136| 46 | 1  |    |    |    |
| 5               | 1  | 135| 920| 920| 135| 1  |    |    |
| 6               | 1  | 386| 5405|11964|5405|386|1    |    |
| 7               | 1  | 1099|29337|124187|124187|29337|1099|1    |
| 8               | 1  | 3130|152110|1126258|2112016|1126258|152110|3130|
| 9               | 1  | 8943|767460|9371472|29836764|9371472|767460|    |
| 10              | 1  | 25642|3809367|73628622|372715542|372715542|73628622|    |

Table 5. $A^*_k(n)$ derived from Table 2, building differences between adjacent columns [17, A175197].

| $k \setminus n$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|-----------------|----|----|----|----|----|----|----|----|----|
| 1               | 1  | 2  | 2  | 2  | 2  | 2  | 2  | 2  |    |
| 2               | 1  | 6  | 12 | 18 | 24 | 30 | 36 | 42 | 48 |
| 3               | 1  | 18 | 66 | 146| 258| 402| 578| 786|1026|
| 4               | 1  | 50 | 330|1070|2500|4850|8350|13230|19720|
| 5               | 1  | 140|1610|7580|23330|56252|115850|213740|363650|
| 6               | 1  | 392|7742|52556|212436|635628|1564570|3359440|6521704|
| 7               | 1  | 1106|37058|360402|1907458|7071442|20784834|51910994|114945026|
| 8               | 1  | 3138|177186|2455938|16973970|77854566|273022686|792717990|2001382932|

(92) $2(k - 1)(2k + 1)(k + 1)(65576k - 74475)A^*_k(2) + (262304k^4 - 10121201k^3 + 21353744k^2 - 8959001k - 149490)A^*_{k-1}(2) + 2(-6440305k^4 + 44418225k^3 - 87651471k^2 + 52631106k - 4105233)A^*_{k-2}(2) + 20(811225k^4 - 3988621k^3 + 5814523k^2 + 2441684k - 8566578)A^*_{k-3}(2) + 2(24847058k^4 - 190384802k^3 + 480247197k^2 - 462996527k + 158679414)A^*_{k-4}(2) - (k - 3)(20387704k^3 - 72824267k^2 - 29485137k + 331041750)A^*_{k-5}(2) - 10(k - 3)(k - 4)(3707581k^2 - 5729012k + 3352341)A^*_{k-6}(2) + 150(k - 3)(k - 4)(k - 5)(26006k + 104375)A^*_{k-7}(2) = 0.$
10. Lattice $E_6$

The task is to sum over the 6-dimensional representation with limits set by the 8-dimensional cube:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1/2 \\
-1 & 0 & 0 & 0 & 0 & 1/2 \\
1 & -1 & 0 & 0 & 0 & 1/2 \\
0 & 1 & -1 & 0 & 0 & 1/2 \\
0 & 0 & 1 & -1 & 0 & -1/2 \\
0 & 0 & 0 & 1 & -1 & -1/2 \\
0 & 0 & 0 & 0 & 1 & -1/2 \\
0 & 0 & 0 & 0 & 0 & -1/2
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\alpha_6 \\
\alpha_7 \\
\alpha_8
\end{pmatrix}
= 
\begin{pmatrix}
p_1 \\
p_2 \\
p_3 \\
p_4 \\
p_5 \\
p_6 \\
p_7 \\
p_8
\end{pmatrix}.
\]

(93)

This is extended to an 8-dimensional representation

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 1/2 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 1/2 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & -1/2 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & -1/2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1/2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\alpha_6 \\
\alpha_7 \\
\alpha_8
\end{pmatrix}
= 
\begin{pmatrix}
p_1 \\
p_2 \\
p_3 \\
p_4 \\
p_5 \\
p_6 \\
p_7 \\
p_8
\end{pmatrix}.
\]

(94)

maintaining the count of $E_6^8$ by adding the condition $\alpha_7 = \alpha_8 = 0$ to the lattice sum. Inversion of this matrix equation yields

\[
\begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
3 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\
2 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\
1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
p_1 \\
p_2 \\
p_3 \\
p_4 \\
p_5 \\
p_6 \\
p_7 \\
p_8
\end{pmatrix}
= 
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\alpha_6 \\
\alpha_7 \\
\alpha_8
\end{pmatrix}.
\]

(95)

The first but last equation of this linear system argues that 6 components of $p_i$ are confined to $\sum_{i=2,\ldots,6} p_i = 0$ while summing over $|p_i| \leq n$ to ensure $\alpha_7 = 0$; the same sum regulated the 6-dimensional cube $A_6^8$. The last equation represents the confinement $p_1 + p_8 = 0$ to ensure $\alpha_8 = 0$. Since this is not entangled with the requirement on the other 6 components, the associated double sum emits a factor $2n + 1$. (Imagine counting points in a square of edge size $2n + 1$ along two coordinates $p_1$ and $p_8$, where $p_1 + p_8 = 0$ admits only points on the diagonal.)
The inverse of this equation is
\[ \alpha \]

The inverse of this equation is
\[ \alpha \]

The inverse of this equation is
\[ \alpha \]

The inverse of this equation is
\[ \alpha \]

The inverse of this equation is
\[ \alpha \]

11. Lattice \( E_7 \)

The \( E_7 \) lattice is spanned by
\[
\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 1/2 \\
1 & -1 & 0 & 0 & 0 & 0 & 1/2 \\
0 & 1 & -1 & 0 & 0 & 0 & 1/2 \\
0 & 0 & 1 & -1 & 0 & 0 & 1/2 \\
0 & 0 & 0 & 1 & -1 & 0 & -1/2 \\
0 & 0 & 0 & 0 & 1 & -1 & -1/2 \\
0 & 0 & 0 & 0 & 0 & 0 & -1/2
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\alpha_6 \\
\alpha_7
\end{pmatrix}
= 
\begin{pmatrix}
p_1 \\
p_2 \\
p_3 \\
p_4 \\
p_5 \\
p_6 \\
p_7 \\
p_8
\end{pmatrix}
\]

Again we consider only the sublattice with even \( \alpha_7 \), that is, integer \( p_i \).

Theorem 11. (Point counts of \( E_7 \))

\[ E_7^b(n) = A_7^b(n). \]

Proof. We reach out into a direction of the \( p_8 \) axis adding a unit vector with axis section \( \alpha_8 \): \( E_7^b(n) \) counts only points with \( \alpha_8 = 0 \).

\[
\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 1/2 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 1/2 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1/2 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & -1/2 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & -1/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1/2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 1
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\alpha_6 \\
\alpha_7 \\
\alpha_8
\end{pmatrix}
= 
\begin{pmatrix}
p_1 \\
p_2 \\
p_3 \\
p_4 \\
p_5 \\
p_6 \\
p_7 \\
p_8
\end{pmatrix}
\]

The inverse of this equation is
\[
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 2 & 2 & 2 & 2 & 2 & 0 \\
2 & 2 & 2 & 3 & 3 & 3 & 3 & 0 \\
3 & 3 & 3 & 3 & 4 & 4 & 4 & 0 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
p_1 \\
p_2 \\
p_3 \\
p_4 \\
p_5 \\
p_6 \\
p_7 \\
p_8
\end{pmatrix}
= 
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\alpha_6 \\
\alpha_7 \\
\alpha_8
\end{pmatrix}
\]
and—reading the last line—the restriction on the \( \alpha_8 \) coordinate implied by the embedding translates into \( \sum_i p_i = 0 \). In comparison, we can also embed the \( A_7 \) lattice into its 8-dimensional host,

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\alpha_6 \\
\alpha_7 \\
\alpha_8 \\
\end{pmatrix}
= \begin{pmatrix}
p_1 \\
p_2 \\
p_3 \\
p_4 \\
p_5 \\
p_6 \\
p_7 \\
p_8 \\
\end{pmatrix},
\]

(103)

and invert this representation, too:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
p_1 \\
p_2 \\
p_3 \\
p_4 \\
p_5 \\
p_6 \\
p_7 \\
p_8 \\
\end{pmatrix}
= \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\alpha_6 \\
\alpha_7 \\
\alpha_8 \\
\end{pmatrix},
\]

(104)

The implied slice \( \alpha_8 = 0 \) and the last line of this equation leads to the same condition \( \sum_i p_i = 0 \) as derived from (102). Since both cases select from the \((2n + 1)^8\) points in the hypercube subject to the same condition, both counts are the same. \( \square \)

12. Lattice \( E_8 \)

The \( E_8 \) coordinates are mediated by

\[
\begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1/2 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1/2 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1/2 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1/2 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & -1/2 \\
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\alpha_6 \\
\alpha_7 \\
\alpha_8 \\
\end{pmatrix}
= \begin{pmatrix}
p_1 \\
p_2 \\
p_3 \\
p_4 \\
p_5 \\
p_6 \\
p_7 \\
p_8 \\
\end{pmatrix},
\]

(105)

Explicit numbers are found with the formula in Theorem 2.

**Theorem 12.** (Lattice points in the bulk and on the surface of \( E_8 \))

\[
E_8^B(n) = V_8^B(n) = 1, 3281, 195313, 2882401, 21523361 \ldots
\]

(106)

\[
E_8^s(n) = \begin{cases} 
1, & n = 0; \\
16n(4n^2 + 1)(16n^4 + 24n^2 + 1), & n > 0; 
\end{cases}
\]

(107)

\[
= 1, 3280, 192032, 2687088, 18640960, 85656080, \ldots
\]

**Proof.** The inverse of the generator matrix in (105) has exactly one row filled with the value 1/2, all other entries are integer. As already argued for the \( D \)-lattices
in sections [3][4] this leads to the constraint that the sum over the $p_i$ must remain even, which matches Definition 1.

\[ \square \]

13. Summary

For $D_k$ lattices, the number of lattice points inside a hypercube is essentially a $k$-th order polynomial of the edge length, summarized in Eq. (37). For $A_k$ lattices, explicit polynomials have been computed for $k \leq 5$ in Eqs. (47), (53), (62) and (68). For higher dimensions, the numbers are centered multinomial coefficients (66) which can be quickly converted to $k$-th order polynomials in $n$. The counts for $E_6$, $E_7$ and $E_8$ are closely associated with the counts for $A_5$, $A_7$ and $D_8$, respectively.

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