The consistency of arithmetic from a point of view of constructive tableau method with strong negation, Part I: the system without complete induction

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Abstract

In this Part I, we shall prove the consistency of arithmetic without complete induction from a point of view of strong negation, using its embedding to the tableau system \( SN \) of constructive arithmetic with strong negation without complete induction, for which two types of cut elimination theorems hold. One is \( SN \)-cut elimination theorem for the full system \( SN \). The other is \( PCN \)-cut elimination theorem for a proposed subsystem \( PCN \) of \( SN \). The disjunction property and the E-theorem (existence property) for \( SN \) are also proved. As a novelty, we shall give a simple proof of a restricted version of \( SN \)-cut elimination theorem as an application of the disjunction property, using \( PCN \)-cut elimination theorem.

Keywords: the consistency of arithmetic, complete induction, strong negation, constructive logic, embedding, cut elimination theorem, tableau system, disjunction property, E-theorem, existence property.

1 Introduction

We shall be concerned in this paper with the consistency of arithmetic without complete induction, namely, Peano’s fifth axiom. The consistency of arithmetic with or without complete induction has been studied rather intensively by a number of authors such as Ackermann [1924, 1940] 2, 3 Neumann [1927] 17, Herbrand [1931] 20, Gentzen [1934 and 1936] 22, 23, 24, Ono [1938] 51, and Novikov [1959] 49 and in post-war times Hlodovskii (or Kholdovskii) [1959] 37 and Schütte [1960] 42. This is the second version of the paper, August 13, 2021. One typographical mistake was corrected in the rule 3.6g on page 8. My current affiliations were modified.
and 1977) [54, 57]. (Also refer to Hilbert and Bernays [1970, Vol.2, 2nd. ed.] [27].) (For the details of the recent development, refer to Aczel [1992] [1], Arai [2011] [6], Buchholz et al [1989] [11], Buss [1998] [12], Buss and Ignjatović [1995] [13], Feferman [1981] [15], Feferman and Sieg [1981a and 1981b] [16, 17], Kahle and Rathjen [2015] [35], Pohlers [2009] [53], Schütte [1977] [57], Takeuti [1975] [64] and Toledo [1975] [65]. (For Novikov’s proof theory, refer to Bellotti [7]. For the history, see Murawski [15].)

All throughout these works, the logical or mathematical system, the consistency of which is at issue, is reduced to (or embedded in) another system which is susceptible to the tableau method. As well-known, the system to be developed by the tableau method is cut-free, and this property thereof is made use of for proving the consistency.

In what follows, the tableau method to be employed is that for arithmetic based on first-order constructive predicate logic with strong negation. Since the proposed predicate logic is constructive, the proposed reduction constitutes a Kolmogorov-Gentzen-Gödel-type interpretation of classical arithmetic by its constructive counterpart (Kolmogorov [1925] [41], Gödel [1932-3] [25] and Gentzen [1936] [24]). (For tableau methods, refer to Fitting [20], Smullyan [60], etc.)

Now the purpose of this paper is to give a consistency proof of arithmetic without complete induction on the basis of the principle as described. As already mentioned the constructive system to which classical arithmetic is reduced is not traditional intuitionistic or Heyting arithmetic, but one to be developed on the basis of constructive predicate logic with strong negation which takes place of intuitionistic or Heyting negation. As well known, strong negation was first introduced by Nelson [1949] [46] in connection with recursive realizability. The negation was incorporated into constructive logic involving not only strong negation, but also intuitionistic one as well by Markov [1950] [43], Vorob’ev [1952 and 1964] [68, 69], Rasiowa [1958] [52] and others. The proposed arithmetic is based on these studies. The paper to follow consists of ten sections inclusive of this introduction. §2 and §3 concern the preparatories for classical arithmetic and its constructive counterpart $SN$ without complete induction to be developed on the basis of constructive predicate logic with strong negation. (For a recent literature for strong negation, refer to Odintsov [50].)

More specifically, classical arithmetic is transformed into a proper subsystem of the cut-free arithmetic, namely, its tableau version, and the consistency of the whole system of constructive arithmetic is straightforward as is the case with any cut-free logic. And from this follows the consistency of classical arithmetic. As a result, the actual infinity involved, in the intuitive interpretation of classical arithmetic by way of the law of excluded middle is completely done away with by this embedding, since constructive logic does not need any notion of actual infinity.

In §4, we shall introduce two subsystems $PCN$ and $FN$ of $SN$. Especially, $PCN$ will play essential roles in this paper.

In §5, a number of theorems will be proved for facilitating the subsequent development. The disjunction property and the E-theorem (existence property) for $SN$ are also proved. §6 will concern two cut elimination theorems, one called $PCN$-
cut elimination theorem is indispensable for reducing classical arithmetic to its constructive version with the help of the embedding theorem of the former to the latter. (Refer to Inoué [1984 and 1984] [28, 29].) In §7, we deal with the reduction, i.e., embedding theorem, which constitutes the core of this paper giving rise to the consistency result. In §8, we prove the consistency of arithmetic without complete induction.

In §9, we shall present the proof of SN-cut elimination theorem for the full system SN. We shall more concentrate on the restricted version of SN-cut elimination theorem, We shall give a simple proof of it, using the disjunction property of SN and PCN-cut elimination theorem.

The last §10 suggests future studies in the direction of this paper.

2 Classical arithmetic without complete induction

Here we wish to present classical arithmetic without complete induction called CN in its Hilbert-type version.

CN could be defined in a number of ways, but here it is defined as an axiomatic system with the axiom schemata of the following kinds.

2.1. The axiom schemata for classical predicate logic to be developed in terms of six logical symbols, namely, ∧ (conjunction), ∨ (disjunction), ⊃ (implication), ∼ (classical negation), ∀ (universal quantifier), and ∃ (existential quantifier):

2.11a \( A \supset .B \supset .A \),
2.11b \( A \supset .B \supset .(A \supset .B \supset C) \supset (A \supset C) \),
2.12 \( A \supset .B \supset (A \land B) \),
2.13a \( A \land B. \supset A \),
2.13b \( A \land B. \supset B \),
2.14a \( A \supset .A \lor B \),
2.14b \( B \supset .A \lor B \),
2.15 \( A \supset C. \supset (B \supset C) \supset (A \lor B. \supset C) \),
2.16 \( A \supset B. \supset (A \supset \sim B) \supset \sim A \),
2.17 \( \sim \sim A \supset A \),
2.18 \( \forall x.A(x). \supset A(t) \),
2.19 \( A(t) \supset \exists x.A(x) \),

where \( t \) is a term such that no free occurrence of \( x \) in \( A(x) \) is in the scope of a quantifier \( \forall y \) or \( \exists y \) with being a variable of \( t \).

2.2. The axiom schemata for arithmetic in terms of ‘ (successor function), ; (addition), ∙ (multiplication), = (equality):

2.21 \( a' = b' \supset a = b \),
2.22 \( \sim a' = 0 \),
2.23 \( a = b \supset .b = c \supset .a = c \),
2.24 \( a = b \supset a' = b' \),
2.25 \( a + 0 = a \),
2.26 \( a + b' = (a + b)' \),
2.27 \( a \cdot 0 = 0 \),
2.28 \( a \cdot b' = a \cdot b + a \).

Hereby, following Schütte [1977] [57], \( \text{CN} \vdash^n A \ (n \geq 0) \) means that \( A \) is provable in \( \text{CN} \) by a proof of the length \( n \). (\( A \) is an axiom if \( n = 0 \).)

2.3 The rules of inference are as follows:

2.31
\[
\text{CN} \vdash^{n_1} A, \quad \text{CN} \vdash^{n_2} A \supset B \\
\hline
\text{CN} \vdash^n B
\]
where \( n = \max(n_1, n_2) + 1 \).

2.32
\[
\text{CN} \vdash^n C \supset A(x) \quad \text{CN} \vdash^{n+1} C \supset \forall A(x)
\]
where \( C \) is a formula which does not contain \( x \) free.

2.33
\[
\text{CN} \vdash^n A(x) \supset C \\
\hline
\text{CN} \vdash^{n+1} \exists A(x) \supset C
\]
where \( C \) is a formula which does not contain \( x \) free. This is an arithmetic taken over from Kleene [1952] [39] if we suppress the axiom of complete induction. The well-formed expressions including terms and formulas are defined in the well-known way in terms of the primitive symbols. And we do not use different syntactic symbols respectively for free and bound variables. Now, two theses of \( \text{CN} \) will be presented without proof.

2.41 \( \text{CN} \vdash^1 x = x \),
2.42 \( \text{CN} \vdash^1 x = y \supset A(x) \supset A(y) \).

Theorem 2.1 Every general recursive predicate is completely representable in \( \text{CN} \).

For the proof of Theorem 2.1, refer to Kleene [1952] [39].

3 Constructive arithmetic without complete induction

Constructive arithmetic without complete induction denoted by \( \text{SN} \) is now developed by way of the tableau method, or the cut-free Gentzen-type formulation. It is again emphasized that the tableau method is a cut-free system, and it is one of our main
results that classical arithmetic in its Hilbert-type version is embedded in this cut-
free system, of which the consistency is forthcoming outright. Now, we introduce
the formal language of SN. As primitive symbols we use:

3.11 A denumerably infinite number variables. (Different syntactic symbols are
not used respectively for free and bound variables.)

3.12 The symbols $0$ (zero), $'$ (successor function), $\sim$ (strong negation), $\land$ (con-
junction), $\lor$ (disjunction), $\supset$ (implication), $\forall$ (universal quantifier), and $\exists$ (existen-
tial quantifier): We note that negation in SN is always regarded as strong negation.
We shall use the same negation symbol $\sim$ for strong negation as classical one.

3.13 Symbols for $n$-place general recursive function and $n$-place general recursive
predicates ($n \geq 1$).

3.14 Round brackets and comma.

The terms of SN are defined as usual in terms of 0, number variables and
functionsymbols. 0 and the expressions to be obtained by successively applying
successor function $'$ to 0 are called numerals. $0 (n)$ stands for

\[ 0 \underbrace{\prime \cdots \prime}_{\text{n times}} , \]

which is the result to be obtained by applying successor function $'$ to 0 times. A
term is numerical if it contains no free number variables. The prime formulas of
SN are $P(t_1, \ldots, t_n)$ where $P$ is a symbol for an $n$-place general recursive predicate
($n \geq 1$) and $t_1, \ldots, T_n$ are terms. A prime formula is said to be constant if it
contains no free number variable. We define the value of a numerical term as
follows.

3.21 The term 0 has value 0.

3.22 If $t$ is a numerical term of value $m$, then $t'$ has value $m'$.

3.23 If $f$ is a symbol for an $n$-place general recursive function ($n \geq 1$) and
t$_1, \ldots, t_n$ are terms with values $m_1, \ldots, m_n$, then $f(t_1, \ldots, t_n)$ has the value given
as $f(m_1, \ldots, m_n)$. Clearly every numerical term has a uniquely determined value
which is a numeral. We now proceed to the definition of the truth-value of a constant
prime formula in order to introduce the axiom of SN.

3.3 If $P$ is a symbol for an $n$-place general recursive predicate ($n \geq 1$) and
t$_1, \ldots, t_n$ are numerical terms with values $m_1, \ldots, m_n$, then $P(t_1, \ldots, t_n)$ is true
or false according as $P(m_1, \ldots, m_n)$ is decided to be true or false. The formula is
decidably determined whether it is true or not.

It is necessary that every constant prime formula is decidably determined
whether it is true or not. The formula $A(s_1, \ldots, s_n)$ is said to be equivalent to
$A(r_1, \ldots, r_n)$ if $s_1, \ldots, s_n, r_1, \ldots, r_n$ are numerical terms with $s_1, \ldots, s_n$, respec-
tively, having the same values as $r_1, \ldots, r_n$. It isImmediate that equivalence of
formulas is an equivalence relation. (Outermost round brackets will be suppressed
wherever no ambiguity arises therefrom.) This language is essentially due to Schütte [1977] [27, p.169 for the system $\Delta^1_1$-analysis, DA]. Before presenting the axioms
and reduction rules of SN, the notion of the positive and negative parts of a formula
(of SN) is in order. The notion is not indispensable, but will have the effect for simplifying the subsequent development of SN. The notion here to be developed following Schütte [1960] [54] is somewhat more complicated than its classical counterpart, since we need to distinguish between antecedent and succedent formulas in view of the notion of sequents which is assumed here from the outset. The sequents of SN have the form $\Gamma \rightarrow \Delta$ with $\Gamma$ and $\Delta$ being a formula or the empty expression. Unlike the traditional Gentzen-type formulation, $\Gamma$ and $\Delta$ represent at most one formula, thus. Since we are implying sequents in place of formulas, the positive and negative parts of a formula are defined separately for the antecedent and succedent formulas of a sequent. The antecedent (succedent ) positive and negative parts of a sequent $A \rightarrow \Delta$ ($\Gamma \rightarrow B$) is defined recursively as follows:

3.41 $A$ ($B$) is an antecedent (succedent) positive part of $A \rightarrow \Delta$ ($\Gamma \rightarrow B$).
3.42 If $A_1 \land A_2$ ($B_1 \lor B_2$) is an antecedent (succedent) positive parts of $\Gamma \rightarrow \Delta$, then $A_1$ and $A_2$ ($B_1$ and $B_2$) are antecedent (succedent) positive parts of the sequent.
3.43 $A_1 \lor A_2$ ($B_1 \land B_2$) is an antecedent (succedent) negative parts of $\Gamma \rightarrow \Delta$, then $A_1$ and $A_2$ ($B_1$ and $B_2$) are antecedent (succedent) negative parts of the sequent.
3.44 $A_1 \supset A_2$ ($B_1 \land B_2$) is an antecedent negative parts of $\Gamma \rightarrow \Delta$, then $A_1$ and $A_2$ ($B_1$ and $B_2$) are respectively an antecedent positive and an antecedent negative part of the sequent.
3.45 $\neg A_1$ ($\neg B_1$) is an antecedent (succedent) positive part of $\Gamma \rightarrow \Delta$, then $A_1$ ($B_1$) are antecedent (succedent) negative part of the sequent.
3.46 $\neg A_1$ ($\neg B_1$) is an antecedent (succedent) negative part of $\Gamma \rightarrow \Delta$, then $A_1$ ($B_1$) are antecedent (succedent) positive part of the sequent.

Again, following Schütte, $F[A^+]$ ($F[A^-]$) means that $A$ occurs there as an antecedent positive (negative) part of a sequent, while $G[A_+]$ ($G[A_-]$) means that $A$ signifies that $A$ is an succedent positive (negative) part of a sequent. Such expressions as $F[A^+, B^-] \rightarrow G[C_-, D_-]$ are understood similarly if the specified ocurrences of the formulas do not overlap with one another.

For the purpose of illustration, we present some example of antecedent (succedent) positive or negative parts of a sequent.

$$
\begin{align*}
F[A^+] & \rightarrow G[B_+] = A \rightarrow B, \\
F[A^+] & \rightarrow G[B_-] = \neg\neg\neg A \rightarrow \neg\neg\neg\neg B, \\
F[A^-, B^+] & \rightarrow G[B_-, C_-] = \neg A \land \neg B \rightarrow \neg C \land \neg (B \land A), \\
F[A^+, A^-, B^-] & \rightarrow G[A_-, A_+] = \neg (B \lor (A \supset A)) \rightarrow A \lor \neg A,
\end{align*}
$$

where $A$, $B$, and $C$ are formulas different from each other.

We, further, need the notion of deleting a formulas form another again after Schütte [1960] [54]. The notion is dispensed with as shown by Schütte [1977] [57]. The deletion of $A$ from an antecedent formula $F[A^+]$ (succedent formula $G[A_+]$), of which the result is expression as $F[-] \rightarrow G[-]$ is defined recursively as follows:

3.51 If $F[A^+] = A$ ($G[A_+] = A$), then $F[-]$ ($G[-]$) is the empty expression.
3.52 If $F[A^+] = F_1[A \land B^+]$ or $= F_1[B \land A^+]$ ($G[A_+] = G_1[A \land B_+]$ or $= G_1[B \land A_+]$), then $F[-] = F_1[-] \land B$ ($G[-] = G_1[-] \lor B$).
3.53 If $F[A^{-}] = F_{1}[A \lor B^{-}]$ or $= F_{1}[B \lor A^{-}] \ (G[A^{-}] = G_{1}[A \land B_{-}])$ or $= G_{1}[B \land A_{-}])$, then $F[\_^{-}] = F_{1}[-\_^{-}] \land B \ (G[\_^{-}] = G_{1}[\_^{-}] \lor \sim B)$.

3.54 If $F[A^{+}] = F_{1}[A \supset B^{-}]$, then $F[\_^{+}] = F_{1}[\_^{-}] \land B$.

3.55 If $F[A^{-}] = F_{1}[B \supset A^{-}]$, then $F[\_^{+}] = F_{1}[\_^{-}] \land B$.

3.56 If $F[A^{+}] = F_{1}[A \sim A^{-}]$, then $F[\_^{+}] = F_{1}[\_^{-}]$.

3.57 If $F[A^{-}] = F_{1}[A \lor B^{+}]$, then $F[\_^{-}] = F_{1}[\_^{+}]$.

3.58 If $G[A_{+}] = G_{1}[A \sim A_{-}]$, then $G[\_^{+}] = G_{1}[\_^{-}]$.

3.59 If $G[A_{-}] = G_{1}[A \sim A_{+}]$, then $G[\_^{-}] = G_{1}[\_^{+}]$.

An example will be given of the above definition.

If $F[A^{+}] \rightarrow G[B_{+}] = F_{1}[A \land C^{+}] \rightarrow G_{1}[B \lor D_{+}] = A \land C \rightarrow B \lor D$, then $F[\_^{+}] \rightarrow G[\_^{+}] = F_{1}[\_^{+}] \land C \rightarrow G_{1}[\_^{+}] \lor D = C \rightarrow D$.

We, next, present the axioms of $SN$. The formulation below is an adaptation from Schütte [1977] [57].

Axiom 1: $F[A^{-}] \rightarrow \Delta$ if $A$ is a true constant prime formula.

Axiom 2: $F[A^{+}] \rightarrow \Delta$ if $A$ is a false constant prime formula.

Axiom 3: $\Gamma \rightarrow G[A_{+}]$ if $A$ is a true constant prime formula.

Axiom 4: $\Gamma \rightarrow G[A_{-}]$ if $A$ is a false constant prime formula.

Axiom 5: $F[A^{+}, B^{-}] \rightarrow \Delta$ if $A$ and $B$ are equivalent formula of length 0.

Axiom 6: $F[A^{+}] \rightarrow G[B_{+}]$ if $A$ and $B$ are equivalent formula of length 0.

Axiom 7: $F[A^{-}] \rightarrow G[B_{-}]$ if $A$ and $B$ are equivalent formula of length 0.

Axiom 8: $A(x_{1}, \ldots, x_{n}) \rightarrow B(y_{1}, \ldots, y_{m})$ where $n \geq 1$ and $m \geq 0$, or $n \geq 0$ and $m \geq 1$, if for every $(n+m)$ numerals $k_{1}, \ldots, k_{n}, l_{1}, \ldots, l_{m}, A(k_{1}, \ldots, k_{n}) \rightarrow B(l_{1}, \ldots, l_{m})$ is one of the axiom 1.7.

Hereby, the length of a formula is defined to be the number of logical symbols.

The well-formed formulas of $SN$, on the other hand, are defined exactly as in the classical case. The (minimal) parts as indicated in the above axiom 1-7 by $A$ or $B$ are called the principal parts of these axioms again following Schütte [1977] [57].

The following reduction rules for $SN$ are proposed by Ishimoto [1970 and 1977] [32, 31] and Matsuda and A. Ishimoto [1984] [44] (refer to Inoué [1984] [28] and Shimizu [1990] [58]). The propositional logic part is equivalent to Markov [1950] [43] and Vorob’ev [1952] [65]. Probably the system is equivalent to the one proposed by Almukdad and Nelson [1984] [5].

The reduction rules of $SN$ are the following. (Below $\Gamma$ and $\Delta$ are only one formula or empty, respectively.)

3.6a

$$\frac{F[A \land B^{-}] \rightarrow \Delta}{F[A^{-}] \rightarrow \Delta \quad | \quad F[B^{-}] \rightarrow \Delta}, \quad (\land \rightarrow)$$

3.6b

$$\frac{\Gamma \rightarrow G[A \land B_{+}]}{\Gamma \rightarrow G[A_{+}] \quad | \quad \Gamma \rightarrow G[B_{+}]}}, \quad (\rightarrow \land)$$

3.6c
\[
\frac{F[A \lor B^\nu] \to \Delta}{F[A^\nu] \to \Delta \ | \ F[B^\nu] \to \Delta}, \quad (\lor^+ \to)
\]

3.6d
\[
\frac{\Gamma \to G[A \lor B^-]}{\Gamma \to G[A^-] \ | \ \Gamma \to G[B^-]}, \quad (\to \lor)
\]

3.6e
\[
\frac{F[A \supset B^\nu] \to \Delta}{F[A \supset B^\nu] \to \Delta \lor A \ | \ F[A \supset B^\nu] \land B \to \Delta}, \quad (\supset^+ \to)
\]

3.6f
\[
\frac{\Gamma \to G[A \supset B^\nu]}{\Gamma \land A \to B}, \quad (\to \supset)
\]

3.6g
\[
\frac{\Gamma \to G[A \supset B^-]}{\Gamma \to G[B^-] \lor A \ | \ \Gamma \to G[B^-]}, \quad (\supset^+ \to)
\]

3.6h
\[
\frac{F[\forall x A(x)^\nu] \to \Delta}{F[\forall x A(x)^\nu] \land A(t) \to \Delta}, \quad (\forall^+ \to)
\]

3.6i
\[
\frac{F[\forall x A(x)^\nu] \to \Delta}{F[A(b)^\nu] \to \Delta}, \quad (\forall^- \to)
\]

3.6j
\[
\frac{\Gamma \to G[\forall x A(x)^+]}{\Gamma \to A(b)}, \quad (\to \forall)
\]

3.6k
\[
\frac{\Gamma \to G[\forall x A(x)^-]}{\Gamma \to G[\forall x A(x)^-] \lor \sim A(t) \lor \sim A(t)}, \quad (\forall^- \to)
\]

3.6l
\[
\frac{F[\exists x A(x)^+] \to \Delta}{F[A(b)^+] \to \Delta}, \quad (\exists^+ \to)
\]

3.6m
\[
\frac{F[\exists x A(x)^-] \to \Delta}{F[\exists x A(x)^-] \land \sim A(t) \to \Delta}, \quad (\exists^- \to)
\]

8
where the b’s in (∃+ →), (∃∀+), (∃+ →) and (∃− →) are variables not occurring free in sequent to be reduced, i.e., proper (eigen) variables, and t is a term. (In the sequent, we will often use the words a principal formula and a quasi-principal formula of a reduction as usual.) A sequent Γ → Δ of SN is provable in SN if each branch of the tableau obtained from Γ → Δ by applying reduction rules 3.6a–3.6o ends with one of the axioms of SN. SN ⊢ Γ → Δ means that it is provable in SN. A formula of A of SN is a theorem of SN if SN ⊢ → A. We also write SN ⊢ A if A is a theorem of SN in the case where no ambiguity arises.

We have to mention the notion of normal tableaus for SN. In order to prove the completeness of SN, we need normal tableaus with the notion of Hintikka formula. However, this paper is not intended to develop semantical study. So in this paper, we shall not go further on this direction.

For a semantic study for the simpler predicate logic part of SN, refer to Shimizu [1990] [58]. The predicate logic part of SN is due to Prof. Arata Ishimoto. It is also used in Matsuda and Ishimoto [1984] [44] and Inoué [1984, 202?] [29, 30] for logic programming and Prolog.

4 Subsystems PCN and FN of SN

Now, we propose two subsystems FN and PCN of SN. The first subsystem FN is obtained from SN by supressing the reduction rule (⊃+ →). The second one PCN may be called a pseudo-classical subsystem of SN. The axioms and reduction rules of PCN are presented as follows:

(Axioms of PCN)
3.71 \( F[A^-] \rightarrow \) if A is a true constant prime formula.
3.72 \( F[A^+] \rightarrow \) if A is a false constant prime formula.
3.73 \( F[A^+, B^-] \rightarrow \) if A and B are equivalent formulas of length 0.
3.74 \( A(x_1, \ldots, x_n) \rightarrow \) where \( n \geq 1 \) and for every numerals \( k_1, \ldots, k_n \) \( A(k_1, \ldots, k_n) \rightarrow \) is one of the axioms 3.71, 3.72 ad 3.73.

(Reduction rules of PCN)
3.81
\[
\frac{F[A \land B^-] \rightarrow}{F[A^-] \rightarrow \mid F[B^-] \rightarrow}, \quad (\land^- \rightarrow_c)
\]
3.82
\[
\frac{F[A \lor B^+] \rightarrow}{F[A^+] \rightarrow \mid F[B^+] \rightarrow}, \quad (\lor^+ \rightarrow^c)
\]

3.83
\[
\frac{F[\forall x A(x)^+] \rightarrow}{F[\forall x A(x)^+] \land A(t) \rightarrow}, \quad (\forall^+ \rightarrow^c)
\]

3.84
\[
\frac{F[\forall x A(x)^-] \rightarrow}{F[A(b)^-] \rightarrow}, \quad (\forall^- \rightarrow^c)
\]

3.85
\[
\frac{F[\exists x A(x)^+] \rightarrow}{F[A(b)^+] \rightarrow}, \quad (\exists^+ \rightarrow^c)
\]

3.86
\[
\frac{F[\exists x A(x)^-] \rightarrow}{F[\exists x A(x)^-] \land \sim A(t) \rightarrow}, \quad (\exists^- \rightarrow^c)
\]

where \(b\)'s are proper (eigen) variables and \(t\) is a term.

We shall continue to use the name of axioms and reduction rules of \(\text{SN}\) in place of the above ones from now on because axioms 3.71–3.74 and 3.81–3.86 are their special cases of their counterparts in \(\text{SN}\) with the lack of succedent formula.

For example, we will use the Axiom 8 in place of the Axiom 3.74, the reduction rule \((\land \rightarrow\rightarrow\rightarrow^c)\) in place of \((\land \rightarrow\rightarrow\rightarrow^c)\). It is remarked that \(\text{PCN}\) is classical in its appearance as long as we confine ourselves to antecedent formulas. A formula of \(\text{PCN}\) may contain implication sign though reduction rules for implication are not available in \(\text{PCN}\). The point will be crucial in the sequel to show that the cut elimination theorem is provable with in \(\text{PCN}\). The fact will lead us to a proof of the embedding theorem.

The subsystems \(\text{FN}\) and \(\text{PCN}\) here introduced are very special systems. It is, however, believed that these logics are worthy of more careful studies.

**Proposition 4.1**

1. \(\text{FN} \vdash \Gamma \rightarrow \Delta \Rightarrow \text{SN} \vdash \Gamma \rightarrow \Delta.\)
2. \(\text{PCN} \vdash \Gamma \rightarrow \Rightarrow \text{SN} \vdash \Gamma \rightarrow.\)

*Proof.* Trivial from the definitions. \(\square\)
5 Theorems and metatheorems of SN

We wish to obtain a number of theorems and metatheorems of SN which will be in order for proving main theorems to be stated §5 and §6.

**Theorem 5.1** (Simultaneous substitution theorem)

\[ \text{SN} \vdash \Gamma \to \Delta \Rightarrow \text{SN} \vdash (\Gamma \to \Delta)[t_1/x_1, \ldots, t_n/x_n], \]

where \( \Gamma \to \Delta \)[\( t_1/x_1, \ldots, t_n/x_n \)] stands for the result of simultaneously replacing \( x_1, \ldots, X_n \) in \( \Gamma \to \Delta \) by terms \( t_1, \ldots, t_n \), respectively, if variables occurring free in \( \Gamma \to \Delta \) are among \( x_1, \ldots, X_n \) and no terms \( t_1, \ldots, t_n \) are variables occurring bound in \( \Gamma \to \Delta \) and contain any variables occurring bound in \( \Gamma \to \Delta \).

*Proof.** This is easily proved by induction on the length of the tableau. \( \Box \)

**Theorem 5.2** (Inversion theorems)

\[
\begin{align*}
(1) & \quad \text{SN} \vdash F[A \land B^{-}] \to \Delta \iff (\text{SN} \vdash F[A^{-}] \to \Delta \text{ and } \text{SN} \vdash F[B^{-}] \to \Delta). \\
(2) & \quad \text{SN} \vdash \Gamma \to G[A \land B_{+}] \iff (\text{SN} \vdash \Gamma \to G[A_{+}] \text{ and } \text{SN} \vdash \Gamma \to G[B_{+}]). \\
(3) & \quad \text{SN} \vdash F[A \lor B^{+}] \to \Delta \iff (\text{SN} \vdash F[A^{+}] \to \Delta \text{ and } \text{SN} \vdash F[B^{+}] \to \Delta). \\
(4) & \quad \text{SN} \vdash \Gamma \to G[A \lor B_{-}] \iff (\text{SN} \vdash \Gamma \to G[A_{-}] \text{ and } \text{SN} \vdash \Gamma \to G[B_{-}]). \\
(5) & \quad \text{SN} \vdash \Gamma \to G[A \supset B_{-}] \iff (\text{SN} \vdash \Gamma \to G[A_{-}] \lor A \text{ and } \text{SN} \vdash \Gamma \to G[B_{-}]). \\
(6) & \quad \text{SN} \vdash F[\forall x A(x)^{-}] \to \Delta \iff \text{SN} \vdash F[A(b)^{-}] \to \Delta. \\
(7) & \quad \text{SN} \vdash \Gamma \to G[\forall x A(x)^{+}] \iff \text{SN} \vdash \Gamma \to A(b). \\
(8) & \quad \text{SN} \vdash F[\exists x A(x)^{+}] \to \Delta \iff \text{SN} \vdash F[A(b)^{+}] \to \Delta. \\
(9) & \quad \text{SN} \vdash \Gamma \to G[\exists x A(x)^{-}] \iff \text{SN} \vdash \Gamma \to A(b).
\end{align*}
\]

*Proof.** We only prove \( \Rightarrow \) of (1) on the basis of the given sequent, contain the principal formula \( A \land B \) of the reduction rule in the tableau for the given sequent \( F[A \land B^{-}] \to \Delta \).

*Case 1: Assume that \( F[A \land B^{-}] \to \Delta \) is by no means a principal part of the axiom.*

*Case 2: Suppose that \( F[A \land B^{-}] \to \Delta \) is reduced by applying \( (\land^{-} \to) \) in the following way:

\[
\frac{F[A \land B^{-}] \to \Delta}{F[A^{-}] \to \Delta \quad | \quad F[B^{-}] \to \Delta} \quad (\land^{-} \to)
\]

It, then, follows that \( F[A^{-}] \to \Delta \) and \( F[B^{-}] \to \Delta \) are both provable in \( \text{SN} \).

*Induction steps* Case 1: Assume that \( F[A \land B^{-}] \to \Delta \) is reduced by applying reduction rules of which the principal formula is not \( A \land B \). For example, let us assume that the reduction rule applied to the given sequent is \( (\lor^{+} \to) \):

\[
\frac{F_1[A \land B^{-}, C \lor D^{+}] \to \Delta}{F_1[A \land B^{-}, C^{+}] \to \Delta \quad | \quad F_1[A \land B^{-}, D^{+}] \to \Delta} \quad (\lor^{+} \to)
\]
where $F_1[A \land B^-, C \lor D^+] \rightarrow \Delta = F[A \land B^-] \rightarrow \Delta$. These two sequents obtained as a result of the reduction are, then, subject to the hypothesis of induction, since the corresponding to the given sequent. These sequents are provable in $SN$ by the hypothesis of induction. We, thus, obtain $SN \vdash F_1[A^-, C \lor D^+] \rightarrow \Delta$ and $SN \vdash F_1[B^-, C \lor D^+] \rightarrow \Delta$.

Case 2: Assume that the given sequent is reduced by applying $(\land^- \rightarrow)$ and that the principal formula of the rule is $A \land B$. The case is similarly proved as in the case 2 of the basis case.

In the cases that the rules to be inverted be succedent rules, there are more basis cases than in the antecedent cases, but this does not present any difficulties. In addition, it is remarked that the tableau obtained as a result of inversion is not longer than the original one.

The converse, namely, $\Leftarrow$ holds obviously, since $F[A \land B^-] \rightarrow \Delta$ is reduced to $F[A^-] \rightarrow \Delta$ and $F[B^-] \rightarrow \Delta$ by applying $(\land^- \rightarrow)$. □

**Theorem 5.3** (Contraction theorems)

(1) $SN \vdash F[A^\pm, A^\pm] \rightarrow \Delta \Rightarrow SN \vdash F[A^\pm, \pm] \rightarrow \Delta$.

(2) $SN \vdash \Gamma \rightarrow G[A^\pm, A^\pm] \Rightarrow SN \vdash \Gamma \rightarrow G[A^\pm, \pm]$.

This is proved by induction on the length of the tableau. In the proof, the inversion theorems are indispensable for the treatment of the induction steps. Use is made of the case that the tableau of any sequent obtained by applying the inversion theorem is not longer than the original one. In addition, it is noticed that the presence of the quasi-principal formula for five rules $(\lor^+ \rightarrow)$, $(\forall^+ \rightarrow)$, $(\rightarrow \forall^+)$, $(\exists^- \rightarrow)$ and $(\rightarrow \exists^+)$ is required in view of the failure of the inversion theorems for them.

We are, now, proceeding to metatheorems of $SN$ corresponding to structural rules. In what follows, those theorems of which we do not give proofs are demonstrated with ease by induction on the length of the tableau.

**Theorem 5.4** (Thinning theorems)

(1) $SN \vdash F[\pm] \rightarrow \Delta \Rightarrow SN \vdash F[A^\pm] \rightarrow \Delta$.

(2) $SN \vdash \Gamma \rightarrow G[A^\pm, \pm] \Rightarrow SN \vdash \Gamma \rightarrow G[A^\pm]$.

**Theorem 5.5** (Translation theorems)

(1) $SN \vdash F[A^\pm, \pm] \rightarrow \Delta \Rightarrow SN \vdash F[\pm, A^\pm] \rightarrow \Delta$.

(2) $SN \vdash \Gamma \rightarrow G[A^\pm, \pm] \Rightarrow SN \vdash \Gamma \rightarrow G[\pm, A^\pm]$.

*Proof.* We prove (1) only. The proof of (2) is similar to that of (1). Assume $SN \vdash F[A^\pm, \pm] \rightarrow \Delta$. By the thinning theorem, $SN \vdash F[A^\pm, A^\pm] \rightarrow \Delta$. We obtain $SN \vdash F[\pm, A^\pm] \rightarrow \Delta$ by the contraction theorem. □

**Theorem 5.6** (Interchange theorems)

(1) $SN \vdash F[A^\pm, B^\pm] \rightarrow \Delta \Rightarrow SN \vdash F[B^\pm, A^\pm] \rightarrow \Delta$.

(2) $SN \vdash \Gamma \rightarrow G[A^\pm, B^\pm] \Rightarrow SN \vdash \Gamma \rightarrow G[B^\pm, A^\pm]$. 

12
Proof.
\[
\text{SN} \vdash F[A^+, B^+] \rightarrow \Delta
\]
\[
\Rightarrow \text{SN} \vdash F[+, B^+] \land A \rightarrow \Delta \quad \text{(Translation theorem)}
\]
\[
\Rightarrow \text{SN} \vdash F[B^+, A^+] \rightarrow \Delta. \quad \text{(Translation theorem)}
\]
The other cases are analogously dealt with. □

We shall present the disjunction and existence properties of \text{SN} as rather more general forms.

Theorem 5.7 (Disjunction property of \text{SN})

\[
\text{SN} \vdash \rightarrow G[A_+] \Leftrightarrow (\text{SN} \vdash \rightarrow G[+] \text{ or } \text{SN} \vdash \rightarrow A).
\]

Proof of \(\Rightarrow\) for Theorem 5.7. If \(\rightarrow G[A_+]\) is empty, then \(\text{SN} \vdash \rightarrow A\) is trivial by 3.51. So assume that \(\rightarrow G[A_+]\) is not empty. We prove the theorem by induction on the number of the sequents which are successively obtained by reducing the given \(\rightarrow G[A_+]\) down to that where \(A\) occurs for the first time.

Basis Case 1: Let \(\rightarrow G[A_+]\) be an axiom. If \(A\) does not contain the principal part of the axiom, then \(\rightarrow G[+]\) is also an axiom. Otherwise \(\rightarrow A\) constitutes an axiom.

Case 2: Let \(G[A_+]\) be \(G_1[B \supset C_+]\) with \(B \supset C\) being the principal formula. The sequent, then, is reduced as follows:

\[
\rightarrow G_1[B \supset C_+] \quad B \rightarrow C.
\]

We, then, obtain \(\text{SN} \vdash \rightarrow G_1[+, B \supset C_+], \text{ i.e., } \rightarrow G[+]\) which is forthcoming by:

\[
\rightarrow G_1[+, B \supset C_+] \quad B \rightarrow C.
\]

Case 3: Let \(G[A_+]\) be \(G_1[B \supset C_+]\), where \(B \supset C\) constitutes a positive part of \(A\). The given tableau, then, is of the form:

\[
\rightarrow G_1[+, B \supset C_+] \quad B \rightarrow C.
\]

It is clear that \(\rightarrow A\) is provable in \(\text{SN}\), since \(\rightarrow A\) is reduced to \(B \rightarrow C\) by applying \((\rightarrow \supset)\).

Case 4: Let \(G[A_+]\) be \(G_1[A_+, \forall x B(x)_+]\), and consider the following reduction:

\[
\rightarrow G_1[A_+, \forall x B(x)_+] \quad \rightarrow B(b).
\]

We obtain \(\text{SN} \vdash \rightarrow G_1[+, \forall x B(x)_+]\), since the sequent is reduced to \(\rightarrow B(b)\) by applying the same rule.

Case 5: Let \(G[A_+]\) be \(G_1[\forall x B(x)_+]\). Assume, further, that \(A\) contains \(\forall x B(x)\) as its positive part. If \(\rightarrow G_1[\forall x B(x)_+]\) has the following reduction:
\[
\frac{\to G_1[\forall x B(x)\_x]}{\to B(b)\_x}. \quad (\to \forall_+) 
\]

Then, \(\to A\) is provable in SN, since the sequent is reduced to \(\to B(b)\) by applying \((\to \forall_+)\).

Case 6: Let \(G[A_+]\) be \(G_1[\exists x B(x)\_x]\), and consider that \(\to G[A_+]\) is reduced to \(\to B(b)\) by applying \((\to \exists_+)\). Although the case splits up into two subcases like the case 4 and 5 of the baseis case, they are similarly dealt with as in the cases 4 and 5.

**Induction steps**

Case 1: Let \(G[A_+]\) be \(G_1[A_+\_B \land C_+]\), which results by way of \((\to \land_+)\) in the following way:

\[
\frac{\to G[A_+\_B \land C_+]}{\to G[A_+\_B \land C_+] \quad \|
\to G[A_+\_C_+]}. \quad (\to \land_+) 
\]

By H.I. (we shall abbreviate the hypothesis of induction as H.I.), we obtain \((\text{SN} \vdash \to A \text{ or SN} \vdash \to G_1[\_B \land C_+])\) and \((\text{SN} \vdash \to A \text{ or SN} \vdash \to G_1[\_B \land C_+])\). If \(\to A\) is not provable in SN, then \(\text{SN} \vdash \to G_1[\_B \land C_+]\) and \(\text{SN} \vdash \to G_1[\_B \land C_+]\). From this we obtain \(\text{SN} \vdash \to G_1[\_B \land C_+]\) by \((\to \land_+)\).

Case 2: Let \(G[A_+]\) be of the form \(G[A_+\_B \land C_+]\) with \(B \land C\) occurring in \(A\) as a succedent positive part, consequently, as a positive part of \(G\). It is, further, assumed that the given sequent be reduced in the following way:

\[
\frac{\to G[A_B \land C_+]}{\to G[A_B \land C_+] \quad \|
\to G[A_C_+]}{}. \quad (\to \land_+) 
\]

By H.I., we have \((\text{SN} \vdash \to A[B_+] \text{ or SN} \vdash \to G[\_B_+])\) and \((\text{SN} \vdash \to A[C_+])\) or \(\text{SN} \vdash \to G[\_C_+]\). If \(\text{SN} \vdash \to G[\_C_+]\), we have done it. Otherwise, we have \(\text{SN} \vdash \to A[B_+]\) and \(\text{SN} \vdash \to A[C_+]\) to which \(\text{SN} \vdash \to A[B \land C_+]\) is reduced by way of \((\to \land_+)\).

Case 3: Assume that \(\text{SN} \vdash \to G[A_+]\) is reduced by applying \((\to \lor_-)\) and that the principal formula of the reduction is \(B \lor C\). The case splits up into two subcases in accordance with \(B \lor C\) being or not being contained in \(A\) as its negative part. These are taken care of analogously with the cases 1 and 2 of the induction steps.

Case 4: Let \(G[A_+]\) be \(G_1[A_+\_B \lor C_-]\). Suppose that \(\to G_1[\_+_B \lor C_-]\) is reduced as follows:

\[
\frac{\to G[A_+_B \lor C_-]}{\to G[A_+_B \lor C_-] \quad \|
\to G[A_+_C_-]}{}. \quad (\to \lor_-) 
\]

We, then, have \((\text{SN} \vdash \to A \text{ or SN} \vdash \to G[\_B \lor C_-] \lor B)\) and \((\text{SN} \vdash \to A \text{ or SN} \vdash \to G[\_B \lor C_-] \lor B)\) by H.I.. If \(\to A\) is provable, we are satisfied with it.

In the contrary case, \(\to G[\_B \lor C_-] \lor B\) and \(\to G[\_B \_C_-] \lor B\) are provable in SN, and \(\to G[A_B \lor C_-]\) is reduced to them by \((\to \lor_-)\).

Case 5: Let \(G[A_+]\) be of the form \(G[A_B \lor C_-]\) with \(B \lor C\) occurring in \(A\) as a succedent negative part, consequently, as a negative part of \(G\). It is also assumed that the given sequent be reduced in the following way:

\[
\frac{\to G[A_B \lor C_-]}{\to G[A_B \lor C_-] \quad \|
\to G[A_B \lor C_-]}{}. \quad (\to \lor_-) 
\]

We, then, have \((\text{SN} \vdash \to A \text{ or SN} \vdash \to G[\_B \lor C_-] \lor B)\) and \((\text{SN} \vdash \to A \text{ or SN} \vdash \to G[\_B \lor C_-] \lor B)\) by H.I.. If \(\to A\) is provable, we are satisfied with it.

In the contrary case, \(\to G[\_B \lor C_-] \lor B\) and \(\to G[\_B \lor C_-] \lor B\) are provable in SN, and \(\to G[A_B \lor C_-]\) is reduced to them by \((\to \lor_-)\).
\[
\frac{G[A \supset B \supset C_+]_+}{G[A \supset -]_+ \lor B \lor G[A[C_+]_]_+} \quad (\rightarrow\_+)
\]

By the translation theorem, \(\rightarrow G[A[-]+] \lor B \) gives rise to \( \rightarrow G[A[-]+, \lor B] \), which is proved by a tableau not longer than that for the former. By H.I., we have \((SN \vdash \rightarrow A[-] \lor B \lor G[A[-]+, \lor B]) \lor (SN \vdash \rightarrow G[+]) \lor (SN \vdash \rightarrow A[+])\). If \(SN \vdash \rightarrow G[+], \) we have done it. Otherwise, \( \rightarrow G[A[B \supset C_-]+] \) is reduced to \( \rightarrow A[-] \lor B \) and \( \rightarrow A[C_-] \) by way of \((\rightarrow\_+)\).

Case 6: Let \(G[A_+] \) be \(G_1[A_+, \forall xB(x)_-].\) Suppose that the given sequent is reduced by the following:

\[
\frac{G_1[A_+, \forall xB(x)_-]}{G_1[A_+, \forall xB(x)_-] \lor \sim B(t)} \quad (\rightarrow\_+)
\]

By H.I., we obtain \(SN \vdash A\) or \(SN \vdash G_1[A_+, \forall xB(x)_-] \lor \sim B(t).\) If \(SN \vdash A\) is not provable, then \(SN \vdash G_1[A_+, \forall xB(x)_-] \lor \sim B(t)\) holds. It follows that \(SN \vdash G_1[A_+, \forall xB(x)_-].\) The tableau thereof is of the form:

\[
\frac{G_1[A_+, \forall xB(x)_-]}{G_1[A_+, \forall xB(x)_-] \lor \sim B(t)} \quad (\rightarrow\_+)
\]

Case 7: Let \(G[A_+] \) be \(G_1[\forall xB(x)_-].\) Assume, further, that \(A\) contains \(\forall xB(x)\) as its negative part and that \(G[A_+] \) is reduced by applying \((\rightarrow\_+)\), and the principal formula of the reduction is \(\forall xB(x).\) We can take care of the case as in the case 6. Note that we use the translation theorem as in the case 5 of the induction steps.

Case 8: Let \(G[A_+] \) be \(G_1[\exists xB(x)_+].\) Suppose that \(A\) contains \(\exists xB(x)\) as its positive part and that \(\rightarrow G[A_+] \) is reduced by \((\rightarrow\_+)\) with the principal formula of the reduction being \(\exists xB(x).\) The case is treated analogously in the cases 6 and 7.

**Proof of \(\Leftarrow\) for Theorem 5.7.** If is immediate that \(SN \vdash G_1[A_+] \) holds by means of the thinning theorem and the translation theorem. \(\square\)

**Theorem 5.8** (E-theorem of SN)

\(\text{SN} \vdash \rightarrow G[\exists xA(x)_+] \iff (A \text{ term } t \text{ is found such that } SN \vdash \rightarrow G[A(t)_+] ).\)

**Proof of \(\Rightarrow\) for Theorem 5.8.** We prove the theorem by induction on the length of the given tableau for \( \rightarrow G[\exists xA(x)_+] \) down to the sequents where \( \rightarrow G[\exists xA(x)_+] \) was first introduced. (In what follow the length will be understood in this generalized sense.)

**Basis** Case 1: Assume the given sequent is an axiom. In view of the definition of axioms, \(\exists xA(x)\) is not the principal part of an axiom. Thus, \(\rightarrow G[A(t)_+] \) is also an axiom for any term \(t\) of \(SN\). Case 2: Assume that \(G[\exists xA(x)_+] \) of the given sequent is introduced by thinning as a result of reduction rules \((\rightarrow\_+), (\rightarrow\_+)\) or \((\rightarrow\_+).\) For example, let \(G[\exists xA(x)_+] \) be \(G_1[\exists xA(x)_+, B \supset C_+]\). Suppose the given sequent is subject to the following reduction:
It, then, follows that, for any term $t$, $\rightarrow G_1[A(t)_+, B \supset C_+]$. The sequent is reduced to $B \rightarrow C$ by applying the same rule again. Other cases are similarly dealt with.

\textbf{Induction steps} We shall confine ourselves to three typical cases. The cases for ($\rightarrow \lor -$), ($\rightarrow \lor -$) or ($\rightarrow \forall -$) are proved in the analogous way.

Let $G[\exists xA(x)_+]$ be $G_1[\exists xA(x)_+, B \land C_+]$. Assume the given sequent is reduced in the form:

$$
\frac{\rightarrow G_1[\exists xA(x)_+, B \lor C_+]}{\rightarrow G[\exists xA(x)_+, B \lor C_+]} \quad (\rightarrow \lor +)
$$

By Theorem 5.7, we obtain ($\text{SN} \vdash \exists xA(x)$ or $\text{SN} \vdash G_1[\exists xA(x)_+, B \lor C_+]$) and ($\text{SN} \vdash \exists xA(x)$ or $\text{SN} \vdash G_1[\exists xA(x)_+, B \lor C_+]$). Assume that $\rightarrow \exists xA(x)$ is provable. A term $t$ is, then, found by H.I., such that $\rightarrow H \text{I. such that } B \rightarrow C$. By Theorem 5.7, we obtain

$$
\frac{\rightarrow G_1[\exists xA(x)_+, B \lor C_+]}{\rightarrow G[\exists xA(x)_+, B \lor C_+]} \quad (\rightarrow \lor +)
$$

By Theorem 5.7, we obtain $\text{SN} \vdash \exists xA(x)$ or $\text{SN} \vdash G_1[\exists xA(x)_+, B \lor C_+]$. If $\text{SN} \vdash \exists xA(x)$, then we find a term $s$ such that $\text{SN} \vdash \exists xB(x)_+ \lor B(t)$. Otherwise, there obtains $\text{SN} \vdash G_1[\exists xA(x)_+, B \lor C_+]$. By Theorem 5.7, we obtain $\text{SN} \vdash \exists xA(x)$ or $\text{SN} \vdash G[\exists xA(x)_+] \lor A(t)$. If $\exists xA(x)$ is provable, then we can find a term $t$ as required by H.I.. Otherwise, $\text{SN} \vdash G[A(t)_+]$ is obtained by the translation theorem, and this is the looked-for sequent.

\textbf{Proof of }$\equiv$\textbf{ for Theorem 5.8.} Suppose that a term $t$ is found such that $\text{SN} \vdash \exists xA(x)_+$. By the translation theorem, we have $\text{SN} \vdash G[\exists xA(x)_+] \lor A(t)$. Then applying the thinning theorem to it to have $\text{SN} \vdash G[\exists xA(x)_+] \lor A(t)$, to which $\rightarrow G[\exists xA(x)_+]$ is reduced by ($\rightarrow \lor +$). \ \Box

Theorems 5.7 and 5.8 fully reflect the constructive feature of $\text{SN}$. It is, of course, not satisfied in the classical theory.

\textbf{Theorem 5.9} (Equality theorems)

\begin{enumerate}
\item $\text{SN} \vdash F[s = t^+, A(s)_+] \rightarrow G[A(t)_+]$.
\item $\text{SN} \vdash F[s = t^+, A(t)_+] \rightarrow G[A(s)_+]$.
\item $\text{SN} \vdash F[s = t^+, A(s)_+] \rightarrow G[A(t)_+]$.
\item $\text{SN} \vdash F[s = t^+, A(t)_+] \rightarrow G[A(s)_+]$.
\end{enumerate}
\[(5) \textbf{SN} \vdash F[s = t^+, A(s)^+, A(t)^-] \rightarrow \Delta,\]
\[(6) \textbf{SN} \vdash F[s = t^+, A(t)^+, A(s)^-] \rightarrow \Delta,\]

where \(s\) and \(t\) are terms, and \(A(s)\) and \(A(t)\) are formulas obtained from \(A(x)\) by substituting \(s\) and \(t\), respectively to \(x\).

**Proof.** (1)–(4) are proved simultaneously. For the proof use is made of induction on the length of the formula \(A(s)\).

**Basis Case 1:** Suppose that \(s = t\) are numerical and \(s = t\) is true. In the case, \(A(s)\) and \(A(t)\) are formulas of length 0 equivalent to each other. Consequently, (1)–(4) are respectively provable in view of the axioms 6 and 7.

**Case 2:** Assume that \(s = t\) is numerical and false. Then, (1)–(4) are all the instances of the axiom 2.

**Case 3:** Suppose that \(s = t\) is not numerical. It is immediate that (1)–(4) respectively constitute the instances of the axiom 8.

**Induction steps \(A(s)\)** is one of the forms \(A_1(s) \land A_2(s), A_1(s) \lor A_2(s), A_1(s) \supset A_2(s), \sim A_1(s), \forall x A_1(x, s), \exists x A_1(x, s)\).

Let \(A(s)\) be \(A_1(s) \land A_2(s)\). (1) is reduced as follows:

\[
F[s = t^+, A(s)^+] \rightarrow G[A_1(t) \land A_2(t)^+]
\]

\[
F[s = t^+, A(s)^+] \rightarrow G[A_1(t)] \land [F[s = t^+, A(s)^+] \rightarrow G[A_2(t)^+]].
\]

The formulas under located are both provable in \(\textbf{SN}\) by H.I.. (1) is, hence, provable.

Let \(A(s)\) be \(A_1(s) \lor A_2(s)\). (1) is reduced as shown below:

\[
F[s = t^+, A(s)^+] \rightarrow G[A_1(t)^+],
\]

\[
F[s = t^+, A(s)^+] \rightarrow G[A_2(t)^+].
\]

By H.I., we finish it.

Let \(A(s)\) be \(\sim A_1(s)\). So we have

\[
F[s = t^+, \sim A_1(s)^+] \rightarrow G[\sim A_1(t)^+],
\]

which is of the form:

\[
F[s = t^+, A_1(s)^-] \rightarrow G[A_1(t)^-]. (*)
\]

and (*) is provable by H.I. Therefore, (1) is also provable.

Let \(A(s)\) be \(A_1(s) \supset A_2(s)\). It is suffices to take care of the following reduction:

\[
F[s = t^+, A(s)^+] \rightarrow G[A_1(t) \supset A_2(t)^+]
\]

\[
F[s = t^+, A_1(s) \supset A_2(s)^+] \land A_1(t) \rightarrow A_2(t).
\]

Hence, (1) is provable since the under located sequents are provable by H.I..

Let \(A(s)\) be \(\forall x A_1(x, s)\). (1) is reduced as shown below:

\[
F[s = t^+, \forall x A_1(x, s)^+] \rightarrow G[\forall x A_1(x, t)^+]
\]

\[
F[s = t^+, \forall x A_1(x, s)^+] \rightarrow A_1(b, t)
\]

\[
F[s = t^+, A(s)^+] \land A_1(b, s) \rightarrow A_1(b, t).
\]

17
It is obvious that the sequent at the bottom is provable by H.I. (1), therefore, is provable. The remaining cases are proved similarly.

We next give the proof of (5) and (6). It is remarked that (5) and (6) are proved concurrently like the proof of (1)–(4). (1)–(4) are also required for proving them.

**Basis**

Case 1: Assume that \( s = t \) is numerical and true. \( A(s) \) and \( A(t) \) are, then, equivalent formulas of length 0. Consequently, (5) and (6) are, respectively, the instances of the axiom 5.

Case 2: Suppose that \( s = t \) is numerical and false. Then, (5) and (6) are the instances of the axiom 2 respectively.

Case 3: Assume that \( s = t \) is not numerical. (5) and (6) are, then, the instances of the axiom 8.

**Induction steps**

We shall only prove some representative cases, while the remaining cases will be analogously taken care of as in (1)–(4).

Let \( A(s) \) be \( A_1(s) \supset A_2(s) \). (5) is, then, reduced as follows: using (\( \supset \rightarrow \) +),

\[
F[s = t^+, A_1(s) \supset A_2(s), A(t)^+] \rightarrow \Delta
\]

The left sequent reduced is provable Theorem 5.3.(2). The right sequent reduced is also provable by H.I.. (5) is provable in SN, thus.

Let \( A(s) \) be \( \sim A_a(s) \). (5) then, is thought of as the following sequent:

\[
F^1[s = t^+, A_1(s)^-, A_1(t)^+] \rightarrow \Delta.
\]

(\( \ast \)) is provable by H.I.. So is (5).

**Theorem 5.10**

(1) \( \text{SN} \vdash F[\sim] \rightarrow G[A_+] \Rightarrow \text{SN} \vdash F[A^+] \rightarrow G[\sim] \),

(2) \( \text{SN} \vdash F[+] \rightarrow G[A_-] \Rightarrow \text{SN} \vdash F[A^+] \rightarrow G[-] \),

**Proof.** This is proved by induction on the length of the given tableau. □

Note that the converse does not hold in general. If it did, \( \text{SN} \) would be classical.

**Theorem 5.11**

(1) \( \text{PCN} \vdash F[A^+] \rightarrow \Rightarrow \text{SN} \vdash F[+^+] \rightarrow \sim A \).

(2) \( \text{PCN} \vdash F[A^-] \rightarrow \Rightarrow \text{SN} \vdash F[+^+] \rightarrow A \).

**Proof.** This is proved by induction on the length of the given tableau. □

**Theorem 5.12**

(1) \( \text{FN} \vdash F[P^+] \rightarrow \Rightarrow \text{FN} \vdash F[+] \rightarrow \sim P \),

(2) \( \text{FN} \vdash F[P^-] \rightarrow \Rightarrow \text{FN} \vdash F[-] \rightarrow P \),

where \( P \) is a prime formula.

**Proof.** This is proved by induction on the length of the given tableau. □

We obtain the following important theorem, again, after Schütte [1977] 57.
Theorem 5.13 (Extended axiom theorems)

(1) \( SN \vdash F[A^+] \rightarrow G[B_+] \),
(2) \( SN \vdash F[A^-] \rightarrow G[B_-] \),
(3) \( SN \vdash F[A^+, B^-] \rightarrow \Delta \),

where \( A \) and \( B \) are equivalent formulas.

Proof. (1) and (2) are proved simultaneously before we proceed to the proof of (3). By induction on the length of formula \( A \), (1) and (2) are proved like the proof of Theorem 5.3. (3) is proved similarly with the help of (1) and (2). Note that Theorem 5.13.(1)–(2) could be proved on the basis of Theorem 5.3 and the \( SN \)-cut elimination to be proved in §9 as an analogue of the corresponding one of Schütte’s \( \Delta^1_1 \)-analysis. (For the details, refer to Schütte [1977] [57].)

Lastly, we are presenting the following theorems, which though, trivial, will be made use of in §5.

Theorem 5.14 (Reduction theorem)

\( FN \vdash \Gamma \rightarrow \Rightarrow \quad PCN \vdash \Gamma \rightarrow \).

Proof. Trivial. □

Theorem 5.15

\( FN \vdash F[A \supset B^+] \rightarrow \Rightarrow \quad PCN \vdash F[+] \rightarrow \).

Proof. The theorem is easily proved by induction on the length of the given tableau. □

Lemma 5.16

(1) \( SN \vdash F[A^+] \rightarrow \Delta \iff SN \vdash F[+^] \land A \rightarrow \Delta \),
(2) \( SN \vdash F[A^-] \rightarrow \Delta \iff SN \vdash F[-]^\land \sim A \rightarrow \Delta \),
(3) \( SN \vdash \Gamma \rightarrow G[A_+] \iff SN \vdash \Gamma \rightarrow G[+^] \lor A \),
(4) \( SN \vdash \Gamma \rightarrow G[A_-] \iff SN \vdash \Gamma \rightarrow G[-^] \lor \sim A \).

Proof. The lemma is easily proved by induction on the length of the given tableau. Or just apply Theorem 5.5. If you prefer a semantical proof, then take simultaneous induction on the numbers of procedures which determines the positive and negative parts. □
6 Two cut elimination theorems of SN

There are two cut elimination theorems for the proposed SN. One is a natural orthodox cut elimination theorem. The other is author’s new type cut elimination theorem. First we are presenting the former.

**Theorem 6.1** (SN-cut elimination theorem) (For the simpler predicate logic case, refer to Kanai [1984] [36])

\[(\text{SN} \vdash \Gamma \to G[A\mid_\pm] \quad \text{and} \quad \text{SN} \vdash F[A\mid_\pm] \to \Delta) \Rightarrow \text{SN} \vdash \Gamma \land F[\mid_\pm] \to G[\mid_\pm] \lor \Delta.\]

The proof of Theorem 6.1 will be presented in §9.

For proving the consistency of SN, we need the SN-cut elimination theorem. Assuming SN $\vdash \to A$ and SN $\vdash \to \sim A$ for some formula A of SN, we obtain $\vdash \to$ by the application of the SN-cut elimination theorem. The sequent $\to$, however, is not provable in SN, since we can not reduce it to any other sequent. This leads us to a contradiction. In fact, if SN $\vdash \to$, we have SN $\vdash \to A$ and SN $\vdash \to \sim A$ for some formula A of SN by the thinning theorem. The consistency of SN could be understood more deeply on the basis of the above discussion. In fact, SN is obviously consistent, since SN is formulated as a cut-free tableau system. The sequent $\to 1 = 0$ is not provable in SN.

**Theorem 6.2** (Consistency theorem of SN)

SN is consistent.

*Proof.* Suppose, if possible, SN be inconsistent. Thus, SN $\vdash \to A$ and SN $\vdash \to \sim A$ for some formula A of SN. By the above discussion, $\to$ is provable in SN. This is impossible. □

**Theorem 6.3** The following sentences are equivalent to each other.

1. SN is inconsistent.
2. SN $\vdash \to A$ and SN $\vdash \to \sim A$ for some formula A of SN.
3. SN $\vdash \to$ .
4. SN $\vdash \to 1 = 0$, where 1 is the abbreviation of 0'.
5. SN $\vdash \to A$ for any formula A of SN.

*Proof.* (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) is clear from Theorem 6.2 and the discussion above theorem 6.2 of (3) $\Rightarrow$ (4), (4) is obtained from (3) by Thinning theorem. Conversely, assume (4). Sequent $1 = 0 \to$ is provable in SN, since it is the axiom 2. From (4) and $1 = 0 \to$, we obtain by the SN-cut elimination theorem. Of (5) $\Rightarrow$ (4), (4) is easily obtained from (5). We are taking care of (3) $\Rightarrow$ (5), lastly. For any formula A, $\to A$ is provable, since it is obtained from $\to$ by Thinning theorem. □

Now, another cut elimination theorem will be presented.
Theorem 6.4 (PCN-cut elimination theorem, Inoué [1984] [28])

\[ \text{(PCN} \vdash F_1[A^+] \rightarrow \text{and PCN} \vdash F_2[A^-] \rightarrow \) \Rightarrow \text{ PCN} \vdash F_1[+]+F_2[\neg] \rightarrow . \]

It is again emphasized that the PCN-cut elimination theorem plays an important part of our embedding of CN (classical number theory without complete induction) in SN to be proved in §8. The theorem successfully takes care of the modus ponens of CN within the bound of PCN, since it is a version of the modus ponens in PCN. This is understandable if we remember that PCN is just like the classical system with respect to its axioms and reduction rules. In addition, it is noted, the PCN-cut elimination theorem is by no means generalized to SN as easily shown by a counterexample. (For the details, refer to Inoué [1984] [28]. It was conjectured in the paper that the new cut elimination theorem could be proved in the constructive predicate logic involving strong negation without any restriction.)

Proof of Theorem 6.4. For proving this we use a conventional double induction on the grade and rank of the PCN-cut (i.e., double induction on \( \omega \cdot g + r \) with \( g, r < \omega \).

The formula \( A \) occurring in the premises of the theorem is called cut formula. The grade of a cut formula \( A \) denoted by \( g(A) \), is the length of the formula \( A \).

The left rank of the cut with the cut formula \( A \), denoted by \( rank_l(A) \), is the sum of sequents which contain \( A \), beginning with the left premise up to that in which \( A \) is introduced first. The right rank of the cut with the cut formula \( A \), denoted by \( rank_r(A) \) is similarly defined. The sum of \( rank_l(A) \) and \( rank_r(A) \) is called the rank of the cut with the cut formula \( A \), and denoted by \( rank(A) \). Notice that \( g(A) \geq 0 \) and \( rank(A) \geq 0 \).

For brevity we shall use the following abbreviations, namely, Tr, NTr, Fal and Eq which, respectively, stand for:

\[ Tr(A) \leftrightarrow (A \text{ is a true constant prime formula}), \]
\[ NTr(A) \leftrightarrow (A \text{ is not a true constant prime formula}), \]
\[ Fal(A) \leftrightarrow (A \text{ is a false constant prime formula}), \]
\[ Eq(A, B) \leftrightarrow (A \text{ and } B \text{ are equivalent formulas of length 0}). \]

By Theorem 5.14, it suffices to consider only four axioms (1, 2, 5 and 8) and six reduction rules ((\( \land \rightarrow \)), (\( \lor^+ \rightarrow \)), (\( \lor^- \rightarrow \)), (\( \forall^+ \rightarrow \)), (\( \exists^+ \rightarrow \)) and (\( \exists^- \rightarrow \))) for the construction of the tableau for the premises.

Case 1 (\( g(A) = 0 \) and \( rank(A) = 2 \):

Subcase A: Let \( F_1 \) be \( F_1'[A^+, B^-] \). Assume Eq(A, B) and Tr(A). \( F_1'[A^+, B^-] \land F_1'[\neg, B^-] \land F_2[\neg] \rightarrow \) is the axiom 1 since Tr(B) by the assumption.

Subcase B: Let \( F_1 \) be \( F_1'[A^+, B^-] \) and \( F_2 \) be \( F_2'[A^-, C^+] \). Assume Eq(A, B), Eq(A, C) and NTr(A). \( F_1'[\neg, B^-] \land F_2[\neg, \neg, C^+] \rightarrow \) is, then, the axiom 6 since Eq(B, C) by the assumption.

Subcase C: Let \( F_1 \) be \( F_1'[A^+, B^-] \) and \( F_2 \) be \( F_2'[A^-, C^+] \). Assume Eq(A, B), Eq(A, C) and Tr(C). \( F_1'[\neg, B^-] \land F_2[\neg, \neg, C^+] \rightarrow \) is, then, the axiom 1 by Tr(C).

Subcase D: Let \( F_1 \) be \( F_1'[A^+, B^-] \) and \( F_2 \) be \( F_2'[A^-, C^+] \). Assume Eq(A, B), NTr(A) and Fal(C). \( F_1'[\neg, B^-] \land F_2[\neg, \neg, C^+, D^-] \rightarrow \) is, then, the axiom 2 by Fal(C).
Subcase E: Let $F_1$ be $F'_1[A^+, B^-]$ and $F_2$ be $F'_2[A^+, C^+, D^-]$. Assume $\text{Eq}(A,B)$ and $\text{Eq}(C,D)$. $F'_1[A^+, B^-] \land F'_2[-, C^-] \rightarrow$ is, then, the axiom 5 by $\text{Eq}(C,D)$.

Subcase F: Let $F_1$ be $F'_1[A^+, B^-]$ and $F_2$ be $A(x_1, \ldots, x_n)$ ($n \geq 1$). Assume $\text{Eq}(A,B)$ and $A(x_1, \ldots, x_n)$ satisfies the condition of the axiom 8. The case has already been taken care of by the subcases A–E of the case 1.

Subcase G: Let $F_2$ be $F'_2[A^-, B^+]$. Assume $\text{Fal}(A)$ and $\text{Eq}(A,B)$. $F_1[+ \land F'_2[-, B^+] \rightarrow$ is the axiom 2 since we have $\text{Fal}(B)$ from the assumption.

Subcase H: Let $F_2$ be $F'_2[A^-, B^-]$. Assume $\text{Fal}(A)$ and $\text{Tr}(B)$. $F_1[+ \land F'_2[-, B^-] \rightarrow$ is the axiom 2 by $\text{Fal}(B)$.

Subcase I: Let $F_2$ be $F'_2[A^-, B^+]$. Assume $\text{Fal}(A)$ and $\text{Fal}(B)$. $F_1[+ \land F'_2[\neg, B^+] \rightarrow$ is the axiom 2 by $\text{Fal}(B)$.

Subcase J: Let $F_2$ be $F'_2[A^-, B^+, C^-]$. Assume $\text{Fal}(A)$ and $\text{Eq}(B,C)$. $F_1[+ \land F'_2[\neg, B^+, C^-] \rightarrow$ is, then, the axiom 5 by $\text{Eq}(B,C)$.

Subcase K: Let $F_2$ be $A(x_1, \ldots, x_n)$ ($n \geq 1$). $A(x_1, \ldots, x_n)$ satisfies the condition of the axiom 8. The case has already been considered by the subcases G–K of the case 1.

The case that $F_1[A^+] \rightarrow$ is the the axiom 2 by $\text{Fal}(S)$ has been already taken care of by the subcases G–K of the case 1.

Subcase L: Assume that the cut formula $A$ of $F_1$ is not the principal part of an axiom. From the assumption, $F_1[+ \rightarrow$ is one of the axioms 1, 2, 5 and 8. $F_1[+ \land F'_2[\neg] \rightarrow$ is also an axiom.

Subcase M: Let $F_1$ be $A(x_1, \ldots, x_n)$ ($n \geq 1$). $A(x_1, \ldots, x_n)$ satisfies the condition of the axiom 8. The case has already been considered by the above subcases A–L of the case 1.

Case 2 ($g(A) = 0$, $\text{rank}_k(A) = 1$ and $\text{rank}_r(A) > 1$): By $g(A) = 0$, the cut formula of the premises is not the principal formula of any reduction rules. There are six reduction rules to be taken up, but only two typical ones will be taken care of.

Let $F_2$ be $F'_2[A^-, B \land C^-]$. If $F_2$ is reduced by $(\land \rightarrow)$, we, then, obtain two sequents $F'_2[A^-, B^-] \rightarrow$ and $F'_2[A^-, C^-] \rightarrow$. By H.I.,

$$\text{PCN} \vdash F_1[+ \land F'_2[\neg, B^-] \rightarrow \quad (2–1)$$

$$\text{PCN} \vdash F_1[+ \land F'_2[\neg, C^-] \rightarrow \quad (2–2)$$

By $(2–1)$ and $(2–2)$, $F_1[+ \land F'_2[\neg, B \land C^-] \rightarrow$ is provable in $\text{PCN}$, since it is demonstrable by the following reduction:

$$\frac{F_1[+ \land F'_2[\neg, B \land C^-] \rightarrow}{\frac{F_1[+ \land F'_2[\neg, B^-] \rightarrow \quad | \quad F_1[+ \land F'_2[\neg, C^-] \rightarrow}{(\land \rightarrow)}}$$

Another case we wish to try is the one where $F_2$ is $F'_2[A^-, \forall x B(x)]$. By the inversion theorem $F_2$ is reduced, and we obtain the sequent $F'_2[\neg, B(b)^-] \rightarrow$, where $b$ is a proper (eigen) variable. By H.I.,

$$\text{PCN} \vdash F_1[+ \land F'_2[\neg, B(b)^-] \rightarrow \quad (2–3)$$

Let $c$ be a variable not occurring as a free variable in $F_1[+ \land F'_2[\neg, \forall x B(x)^-] \rightarrow$. Let $x_2, \ldots, x_n$ are pairwise distinct free variables of $F_1[+ \land F'_2[\neg, B(b)^-] \rightarrow$.  

22
Assume further that \(x_2, \ldots, x_n\) are not bound and distinct from \(b\). By Theorem 5.1 and (2–3),

\[
\text{PCN} \vdash F_1[+] \land F'_2[-], B(b)^-][c/b, x_2/x_2, \ldots, x_n/x_n] \rightarrow .
\]

Namely, we have

\[
\text{PCN} \vdash F_1[+] \land F'_2[-], B(b)^-] \rightarrow . \quad (2–5)
\]

From (2–5), we obtain \(\text{PCN} \vdash F_1[+] \land F'_2[-], \forall xB(x)^-] \rightarrow\), since it is reduced as follows:

\[
\frac{F_1[+] \land F'_2[-], \forall xB(x)^-] \rightarrow}{\Gamma \rightarrow}.
\]

Notice that we employ Theorem 5.6 (Interchange theorems) in the case of \((\forall \rightarrow)\) and \((\exists \rightarrow)\).

Case 3 \((g(A) = 0, \text{rank}_l(A) > 1 \text{ and } \text{rank}_r(A) = 1)\): The proof is similar to that for the case 2.

Case 4 \((g(A) = 0, \text{rank}_l(A) > 1 \text{ and } \text{rank}_r(A) > 1)\): The treatment of this case is analogous to that for the case 2.

Case 5 \((g(A) > 0 \text{ and } \text{rank}(A) = 2)\):

Subcase A: Suppose that the cut formula \(A\) is of the form \(A_1 \land A_2\). By the inversion theorem, \(F_2[A_1^-] \rightarrow\) and \(F_2[A_2^-] \rightarrow\) are obtained from \(F_2[A_1 \land A_2^-] \rightarrow\). In what follows, such inferences as:

\[
(\vdash \Gamma_1 \rightarrow \Delta_1 \text{ and } \vdash \Gamma_2 \rightarrow \Delta_2) \Rightarrow \vdash \Gamma_3 \rightarrow \Delta_3,
\]

is described à la Gentzen as:

\[
\frac{\Gamma_1 \rightarrow \Delta_1}{\Gamma_3 \rightarrow \Delta_3}, \quad \frac{\Gamma_2 \rightarrow \Delta_2}{\Gamma_3 \rightarrow \Delta_3},
\]

where \(\Gamma_1 \rightarrow \Delta_1\) and \(\Gamma_2 \rightarrow \Delta_2\) are understood accompanying the respective closed tableau to be obtained by reducing them. (A closed tableau is one of which each branch ends with one of axioms. A sequent which has at least one closed tableau is provable.) We, then, obtain a series of inferences as follows:

\[
\begin{align*}
\text{Interchange} & \quad \frac{F_1[+ \land A_2^+] \rightarrow F_2[A_1^-] \rightarrow \quad F_2[A_1^-]}{F_2[A_2^-] \rightarrow} \quad \text{H.I. on grade} \\
& \quad \frac{F_1[+ \land A_2^+] \rightarrow F_2[A_1^-] \rightarrow}{}
\end{align*}
\]

Contraction

The interchange theorem is called for with a view to making cut more easily applicable.
Subcase B: Assume that the cut formula has the form $A_1 \lor A_2$. By the inversion theorem, we obtain $F_1[A_1^+] \rightarrow$ and $F_1[A_2^+] \rightarrow$ from $F_1[A_1 \lor A_2^+] \rightarrow$. We, then, have:

\[
\frac{F_1[A_1^+] \rightarrow \text{Translation (twice)}}{\text{H.I. on grade}} \quad \frac{F_1[A_1^+] \rightarrow F_2[A_1 \lor A_2^-] \rightarrow (F_1[A_1^+] \land F_2[\neg]) \land \sim A_2 \rightarrow}{\text{H.I. on grade}} \quad \frac{(F_1[A_1^+] \land F_2[\neg]) \land A_2 \rightarrow}{\text{Contraction}}
\]

Subcase C: Suppose that the cut formula $A$ is $A_1 \supset A_2$. By Theorems 5.13 and 5.14, $\text{PCN} \vdash F_1[\vdash] \rightarrow$ is forthcoming from $F_1[A_1 \supset A_2^+] \rightarrow$. Then $\text{PCN} \vdash F_1[\vdash] \land F_2[\neg] \rightarrow$ is obtained by the thinning theorem.

Subcase D: Let $A$ be $\sim A$. $F_1[A_1^+]$ and $F_2[A_1^-]$ are thought of as $F_3[A_1^-]$ and $F_4[A_1^+]$, respectively. We, then, obtain the following series of inferences:

\[
\frac{F_4[A_1^+] \rightarrow F_3[A_1^-] \rightarrow}{\text{H.I. on grade}} \quad \frac{F_4[A_1^+] \rightarrow F_3[A_1^-] \rightarrow}{\text{Interchange}} \quad \frac{F_3[A_1^+] \land F_3[A_1^-] \rightarrow}{\text{Interchange}} \quad \frac{F_3[A_1^+] \land F_3[A_1^-] \rightarrow}{\text{Interchange}}
\]

$F_3[A_1^+] \land F_3[A_1^-]$ is obviously $F_1[A_1^+] \land F_2[A_1^-]$.

Subcase E: Assume that the cut formula $A$ is of the form $\forall x A(x)$. $F_1[\forall x A(x)^+] \rightarrow$ is an axiom by the assumption that $\text{rank}_i(\forall x A(x)) = 1$. $F_1[\vdash] \rightarrow$ is also an axiom since $\forall x A(x)$ is by no means the principal part of an axiom. $F_1[\vdash] \land F_2[\neg] \rightarrow$ remains an axiom, thus.

Subcase F: Let the cut formula $A$ be $\exists x A(x)$. $F_2[\neg] \rightarrow$ is an axiom again by the assumption. It follows that $F_1[\vdash] \land F_2[\neg] \rightarrow$ remains an axiom of the same kind.

Case 6 ($g(A) = 0$, $\text{rank}_i(A) = 1$ and $\text{rank}_c(A) > 1$):

Subcase A: Assume $A$ is of the form $A_1 \land A_2$. By the assumption, $F_2[A_1 \land A_2^-] \rightarrow$ is not an axiom. If $F_2[A_1 \land A_2^-] \rightarrow$ is reduced by $(\land \rightarrow)$, we have:

\[
\frac{F_2[A_1 \land A_2^+, B \land C^-] \rightarrow}{\text{H.I. on grade}} \quad \frac{F_2[A_1 \land A_2^+, B^-] \rightarrow}{\text{H.I. on grade}} \quad \frac{F_2[A_1 \land A_2^+, C^-] \rightarrow}{\text{H.I. on grade}}
\]

$F_2' \rightarrow F_2$. We, then, have:

\[
\frac{F_1[A_1 \land A_2^+] \rightarrow}{\text{H.I. on rank}} \quad \frac{F_1[A_1 \land A_2^+] \rightarrow}{\text{H.I. on rank}}
\]

$F_1[A_1 \land A_2^+] \land F_2'[-, B^-] \rightarrow$, $(*)$

$F_1[A_1 \land A_2^+] \land F_2'[-, C^-] \rightarrow$. $F_1[A_1 \land A_2^+] \land F_2[-, B \land C^-] \rightarrow$ is, thus, reduced to $(*)$ and $(**)$. The case that $(\forall^+ \rightarrow)$, $(\forall^- \rightarrow)$ and $(\forall^- \rightarrow)$ are employed for the reduction are similarly taken care of as in the above case.
Subcase B: Let $A$ be $A_1 \lor A_2$. By $g(A) > 1$, $F_1[+] \rightarrow$ is an axiom. Then, $F_1[+] \land F_2[-] \rightarrow$ also constitutes an axiom.

Subcase C: Assume the cut formula $A$ is $A_1 \supset A_2$. The case is analogously dealt with as in the subcase C of the case 5.

Subcase D: Assume that the cut formula $A$ is one of forms $\forall x A(x)$ and $\exists x A(x)$. In each case, we similarly obtain $PCN \vdash F_1[+] \land F_2[-] \rightarrow$ as in the preceding subcase B of the case 6.

Subcase E: Let $A$ be $\sim A_1$. The case is analogously proved as in the subcase D of the case 4.

Case 7 ($g(A) > 1$, $rank_l(A) > 1$ and $rank_r(A) = 1$): The case is analogously treated as in the case 6.

Case 8 ($g(A) > 1$, $rank_l(A) > 1$ and $rank_r(A) > 1$):

Subcase A: If the cut formula $A$ is one of $A_1 \land A_2$, $A_1 \lor A_2$, $A_1 \supset A_2$ and $\sim A_1$. Then each case is similarly taken care of as in the case 5.

Subcase B: Let $A$ be $\forall x A_1(x)$. Assume that $F_1[\forall x A_1(x)^+] \rightarrow$ is reduced in the following way:

\[
\frac{F_1[\forall x A(x)^+] \land A_1(t) \rightarrow}{F_1[\forall x A(x)^+] \land A_1(t) \rightarrow} \land \rightarrow
\]

We, then, have:

\[
\frac{F_1[\forall x A_1(x)^+] \land A_1(t) \rightarrow}{F_1[\forall x A_1(x)^+] \land A_1(t) \rightarrow} \land \rightarrow
\]

\[
\frac{(F_1[+] \land A_1(t)) \land F_2[-] \rightarrow}{H.I. \text{ on rank}}
\]

Interchange

\[
\frac{(F_1[+] \land A_1(t)) \land A_1(t) \rightarrow}{(F_1[+] \land F_2[-]) \land A_1(t) \rightarrow}
\]

Now, by the inversion theorem there obtains $F_2[A_1(b)^+] \rightarrow$ from $F_2[\forall x A_1(x)^+] \rightarrow$. From this we obtain by substitution $F_2[A_1(t)^+] \rightarrow$. We, then, have:

\[
\frac{(F_1[+] \land F_2[-]) \land A_1(t) \rightarrow}{H.I. \text{ on grade}}
\]

Contraction

\[
\frac{(F_1[+] \land F_2[-]) \land A_1(t) \rightarrow}{(F_1[+] \land F_2[-]) \land A_1(t) \rightarrow}
\]

If the cut formula $A$ is not principal formula of reduction rules, then we prove the cases by H.I. on rank as in the subcase A of the case 6.

Subcase C: Let $A$ be $\exists x A_1(x)$. The treatment is similar to that for the subcase B of the case 8. □

7 Embedding theorem of CN in SN

In this section, we will prove a Kolmogorov-Gödel-type embedding theorem of CN in SN, which is the core for the consistency proof in §8. The following lemmas will be presented first for facilitating the proof.

Lemma 7.1 If a sequent $\Gamma \rightarrow$ is provable in SN and contains no implication sign, then $\Gamma \rightarrow$ is provable in PCN.
Proof. \( \Gamma \rightarrow \) is provable in \( \text{FN} \) by the premise. The sequent is, then, provable in \( \text{PCN} \) by Theorem 5.14. \( \Box \)

**Theorem 7.2**

\begin{align*}
(1) & \quad \text{SN} \vdash A \rightarrow B \iff \text{SNH} \vdash A \supset B, \\
(2) & \quad \text{SN} \vdash A \rightarrow \iff \text{SNH} \vdash A \supset \sim A, \\
(3) & \quad \text{SN} \vdash \rightarrow \iff \text{SNH} \vdash A, \\
(4) & \quad \text{SN} \vdash \rightarrow \iff \text{SNH} \vdash A \text{ for every formula } A \text{ of } \text{SN},
\end{align*}

where \( A \) and \( B \) are formulas of \( \text{SN} \) (or \( \text{SNH} \)), and \( \text{SNH} \) is a Hilbert-type version of \( \text{SN} \).

**Proof.** An easy proof will be omitted. Although the theorem is made use of in the proof of \( \Leftarrow \) in the following embedding theorem, \( \Leftarrow \) in the theorem is dispensed with for the consistency proof of \( \text{CN} \). (cf. Ishimoto [1970] [32] amd Takano [1970] [63]) \( \Box \)

**Theorem 7.3** (Embedding theorem, for a simpler predicate logic part, Ishimoto [197?] [31] (refer to [29], [31]))

\[ \text{CN} \vdash^n A \iff \text{PCN} \vdash \sim TA \rightarrow . \]

where \( A \) is a formula of \( \text{CN} \) and the operator \( T \), which translates a formula of \( \text{CN} \) into its counterpart in \( \text{SN} \), is recursively defined as follows:

\begin{align*}
T A & = A \text{ for every prime formula } A \text{ of } \text{CN}, \\
T \sim A & = \sim TA, \\
T A \land B & = T A \land TB, \\
T A \lor B & = T A \lor TB, \\
T A \supset B & = \sim T A \lor TB, \\
T \forall x A(x) & = \forall x TA(x), \\
T \exists x A(x) & = \exists x TA(x).
\end{align*}

Let us consider the following example of the translation:

\begin{align*}
T \forall y \exists x ((x = 0 \land y = 0') \supset x = y), \\
= \forall y T \exists x ((x = 0 \land y = 0') \supset x = y), \\
= \forall y \sim T \exists x ((x = 0 \land y = 0') \supset x = y), \\
= \forall y \sim \exists x T ((x = 0 \land y = 0') \supset x = y), \\
= \forall y \sim \exists x (\sim (x = 0 \land y = 0') \lor Tx \supset y), \\
= \forall y \sim \exists x (\sim (Tx = 0 \land Ty = 0') \lor x = y), \\
= \forall y \sim \exists x (x = 0 \land y = 0') \lor x = y).
\end{align*}

It is noticed here that the logical symbols in the formula to be translated are different from the original ones which are, of course, interpreted classically. They are constructive as seen from Theorems 5.7 and 5.8. It is remarked \( TA \) has no implication sign as seen from the definition of \( T \) above.
Proof of Theorem 7.3. We are first taking care of \( \Rightarrow \) of the theorem by induction on the length \( n \) of the proof of \( A \).

**Basis (\( n = 0 \))**

Case 1 (\( \text{CN} \vdash^0 A \supset .B \supset A \)): The formula is translated by \( T \) into the following sequent of \( \text{SN} \):

\[
\sim (\sim TA \lor .\sim TB \lor TA) \rightarrow .
\]

By Theorem 5.13, \( \sim (\sim TA \lor .\sim TB \lor TA) \rightarrow \) is thought of as \( F[TA^+, TA^-] \rightarrow . \)

(Two identical formulas are obviously equivalent to each other in our sense. And \( TA \) is equivalent to \( TA_0 \).) \( F[TA^+, TA^-] \rightarrow \), moreover, has no implication sign as remarked above. \( F[TA^+, TA^-] \rightarrow \) is, therefore, provable in \( \text{PCN} \) by Lemma 7.1.

Case 2 (\( \text{CN} \vdash^0 A \supset B \supset .(A \supset .B \supset C) \supset (A \supset C) \)): The sequent obtained by the translation is of the form;

\[
\sim (\sim TA \lor TB) \lor \sim (\sim TA \lor .(\sim TB \lor TC)) \lor (\sim TA \lor TC) \rightarrow .
\]

We, then, have the reduction:

\[
\begin{array}{c}
S_1 \\
\hline
S_3 \quad | \quad S_5 \\
\hline
S_4 \quad | \quad S_6 \\
\end{array}
\]

\[
\begin{array}{c}
S_1 = \sim (\sim TA \lor .\sim TA \lor \sim TB \lor TC) \lor (\sim TA \lor TC) \rightarrow ,
S_2 = \sim (\sim TB \lor .(TA \lor .TB \lor TC) \lor (TA \lor TC)) \rightarrow ,
S_3 = \sim (\sim TB \lor .\sim TA \lor (TA \lor TC)) \rightarrow ,
S_4 = \sim (\sim TB \lor .\sim TA \lor (TA \lor TC)) \rightarrow ,
S_5 = \sim (TA \lor .\sim TB \lor (TA \lor TC)) \rightarrow ,
S_6 = \sim (TA \lor .\sim TB \lor (TA \lor TC)) \rightarrow .
\end{array}
\]

\( S_1, S_3, S_5 \) and \( S_6 \) are, respectively, thought of as:

\[
\begin{align*}
F[TA^+, TA^-] \rightarrow , \\
F[TA^+, TA^-] \rightarrow , \\
F_3[TA^+, TB^-] \rightarrow , \\
F_4[TC^+, TC^-] \rightarrow .
\end{align*}
\]

By Theorem 5.13, \( S_1, S_3, S_5 \) and \( S_6 \) are all provable in \( \text{SN} \). They are also seen provable in \( \text{PCN} \) by taking their structures into consideration as in the case 1. The given sequent obtained by the translation is, thus, reduced to sequents provable in \( \text{PCN} \) in the above reduction, which are easily turned into complete tableaux. Thus,

\[
\text{PCN} \vdash (\sim TA \lor TB) \lor .(\sim TA \lor .TB \lor TC) \lor (\sim TA \lor T) \rightarrow .
\]

For brevity we will mention only the sequents obtained by the translation and corresponding reductions in the cases 3–12 below. The details are similar to those of the cases 1 and 2.
Case 3 ($\text{CN} \vdash A \supset B \supset (A \land B)$): Upon translation we have:

$$\vdash TA \lor . \vdash TB \lor (TA \land TB) \rightarrow .$$

The reduction is then,

$$\frac{\vdash \sim (TA \lor . \vdash TB \lor TA) \rightarrow \quad \vdash \sim (TA \lor . \vdash TB \lor TA) \rightarrow}{S_1 \mid S_2} \quad (\lor \rightarrow)$$

where

$$S_1 = \vdash \sim (TA \lor . \vdash TB \lor TA) \rightarrow,$$
$$S_2 = \vdash \sim (TA \lor . \vdash TB \lor TA) \rightarrow.$$

$S_1$ and $S_2$ are, then, thought of as $F_1[TA^+, TA^-] \rightarrow$ and $F_1[TB^+, TB^-] \rightarrow$, respectively.

Case 4 ($\text{CN} \vdash A \land B \supset A$): The sequent to be proved is in $\text{PCN}$ as $\vdash \sim (TA \land TB) \rightarrow$. The sequent is of the structure $F[TA^+, TA^-] \rightarrow$, and provable in $\text{PCN}$.

Case 5 ($\text{CN} \vdash A \land B \supset A$): The sequent obtained upon translation is

$$\vdash \sim (TA \land TB) \lor TB \rightarrow ,$$

which is of the form $F[TB^+, TB^-] \rightarrow$.

Case 6 ($\text{CN} \vdash B \supset A \lor B$): As a result of translation we have $\vdash \sim (TA \lor . TA \lor TB) \rightarrow$. This is thought of $F[TB^+, TB^-] \rightarrow$.

Case 7 ($\text{CN} \vdash B \supset . A \lor B$): This case is similar to the case 6.

Case 8 ($\text{CN} \vdash A \supset C \supset (B \supset C \supset (A \land B \supset C))$): It is transformed into:

$$\vdash \sim (TA \lor TC) \lor . \vdash \sim TB \lor TC \lor (TA \lor TB) \lor TC) \rightarrow .$$

The reduction proceeds in the following way:

$$\frac{\vdash \sim (TA \lor TC) \lor . \vdash (TA \lor TB) \lor TC) \lor (TA \lor TB) \lor TC) \rightarrow}{S_1 \mid S_4} \quad (\lor \rightarrow)$$

$$\frac{\vdash \sim \vdash \sim \vdash \sim \vdash \sim \vdash \sim}{S_3 \mid S_4} \quad (\lor \rightarrow) \quad \frac{\vdash \sim \vdash \sim \vdash \sim}{S_5 \mid S_6} \quad (\lor \rightarrow)$$

where

$$S_1 = \sim (TA \lor TC) \lor . \sim (TA \lor TC) \lor (TA \lor TC) \rightarrow,$$
$$S_2 = \sim (TA \lor TC) \lor . \sim (TA \lor TC) \lor (TA \lor TC) \rightarrow,$$
$$S_3 = \sim (TA \lor TC) \lor . \sim (TA \lor TC) \lor (TA \lor TC) \rightarrow,$$
$$S_4 = \sim (TC \lor . \sim (TA \lor TC) \lor (TA \lor TC) \rightarrow,$$
$$S_5 = \sim (TA \lor TC) \lor . \sim (TA \lor TC) \lor (TA \lor TC) \rightarrow,$$
$$S_6 = \sim (TA \lor TC) \lor . \sim (TA \lor TC) \lor (TA \lor TC) \rightarrow.$$

$S_3$, $S_4$, $S_5$ and $S_6$ are, respectively, thought of as:

$$F_1[TA^+, TA^-] \rightarrow,$$
$$F_2[TC^+, TC^-] \rightarrow,$$
$$F[TB^+, TB^-] \rightarrow,$$
\( F[TC^+, TC^-] \rightarrow. \)

Case 9 (CN \( \vdash^0 A \supset B. \supset (A \supset \sim B) \supset \sim A \)): The formula is transformed into:

\[
\sim (\sim TA \lor TB) \lor \sim (TA \lor \sim TB) \lor \sim TA.
\]

This is subject to the following series of reductions:

\[
\begin{align*}
S_1 & = \sim (\sim TA \lor \sim (TA \lor \sim TB) \lor \sim TA) \rightarrow, \\
S_2 & = \sim (\sim TA \lor \sim (TA \lor TB) \lor \sim TA) \rightarrow, \\
S_3 & = \sim (\sim TA \lor \sim TA \lor TA) \rightarrow, \\
S_4 & = \sim (\sim TA \lor \sim TA \lor TA) \rightarrow.
\end{align*}
\]

Here, \( S_1, S_3 \) and \( S_4 \) are, respectively, of the forms:

- \( F_1[T A^+, T A^-] \rightarrow, \)
- \( F_2[T A^+, T A^-] \rightarrow, \)
- \( F_3[T B^+, T B^-] \rightarrow. \)

Case 10 (CN \( \vdash^0 \sim A \supset A \)): It is transformed into the sequent \( \sim (\sim TA \lor TA) \rightarrow, \) which is of the form \( F_1[T A^+, T A^-] \rightarrow. \)

Case 11 (CN \( \vdash^0 \forall x(A(x) \supset A(t)) \)): The translation of the formula results in the sequent:

\[
\sim (\sim \forall xTA(x) \lor TA(t)) \rightarrow.
\]

The sequent is subject to the following reduction:

\[
\frac{\sim (\sim \forall xTA(x) \lor TA(t)) \rightarrow}{\sim (\forall xTA(x) \lor TA(t)) \rightarrow} (\forall^+ \rightarrow)
\]

The sequent obtained by the reduction is thought of as \( F[TA(t)^+, TA(t)^-] \rightarrow. \)

Case 12 (CN \( \vdash^0 A(t) \supset \exists xA(x) \)): The sequent to be proved is of the form: the sequent:

\[
\sim (\sim TA(t) \lor \exists xTA(x)) \rightarrow.
\]

By applying \( (\exists^+ \rightarrow) \), the above sequent is reduced to

\[
\sim (\sim TA(t) \lor \exists xTA(x)) \lor TA(t) \rightarrow.
\]

which is regarded as \( F[TA(t)^+, TA(t)^-] \rightarrow. \)

Case 13 (CN \( \vdash^0 a' = b' \lor a = b \)): The sequent to be proved is for the form:

\[
\sim (\sim a' = b' \lor a = b) \rightarrow. \quad (\ast)
\]
If we substitute $0^{(k)}$ and $0^{(l)}$ for $a$ and $b$, respectively in (*), then the sequent:

$$\sim (\sim (0^{(k)})' = (0^{(l)})' \lor 0^{(k)} = 0^{(l)}) \to,$$

is obtained. Assume $k = l$. (+++) is, then, the axiom 1, since $0^{(l)} = 0^{(l)}$ is an antecedent negative part of (**), and a true constant prime formula. Assume $k \neq l$. (***) is, then, the axiom 2, since $(0^{(k)})' = (0^{(l)})'$, i.e., $0^{(k+1)} = 0^{(l+1)}$ is an antecedent positive part of (***) and a false constant prime formula. From the above, (*) is the axiom 8 since $k$ and $l$ are thought of as arbitrary natural numbers.

Case 14 ($\text{CN} \vdash 0 \sim a' = 0$): The sequent in question has the form:

$$\sim \sim a' = 0 \to. \quad (*)$$

If $0^{(k)}$ is substituted for $a$ in (*), then the sequent

$$\sim (0^{(k)})' = 0 \to, \quad (***)$$

is obtained. (*) is, the axiom 8 since (***) is the axiom 2 for every natural number $k$. More specifically, $(0^{(k)})' = 0$ is a false constant prime formula and an antecedent positive part of (***)

Case 15 ($\text{CN} \vdash 0 \sim a = b \supset . a = c \supset b = c$): It is transformed into:

$$\sim (\sim a = b \lor . \sim a = c \lor b = c) \to, \quad (**)$$

Let $A(x)$ be $x = c$. By Theorem 5.9.(1),(*) is provable in SN since the sequent is regarded as $F[a = b^+, A(a)^+, A(b)^-] \to$.

Case 16 ($\text{CN} \vdash 0 \sim a = b \supset a' = b'$): The sequent transformed is, then, the axiom 8 since we can almost similarly prove it as in the case 14 of the vasis case.

Case 17 ($\text{CN} \vdash 0 \sim a + 0 = a$): The sequent to be proved is obviously $\sim a + 0 = a \to$. For every natural number $k$, $\sim 0^{(k)} + 0 = 0^{(k)} \to$ is the axiom 1 since $0^{(k)} + 0 = 0^{(k)}$ is antecedent negative part of the sequent and a true constant prime formula,

Case 18 ($\text{CN} \vdash 0 \sim a + b' = (a + b)'$): The sequent to be proved is

$$\sim a + b' = (a + b)' \to. \quad (*)$$

This satisfies the condition of the axiom 8 since $F[0^{(k)} + (0^{(l)})' = (0^{(k)}) + 0^{(l)}] \to$, i.e., a substitution instance of (*) is the axiom 1. Like the treatment of the case 17, $0^{(k)} + (0^{(l)})' = (0^{(k)}) + 0^{(l)}$ is a true constant prime formula, since $0^{(k)} + (0^{(l)})'$ and $(0^{(k)}) + 0^{(l)}$ have the same value $k + l + 1$.

Case 19 ($\text{CN} \vdash 0 \sim a \cdot 0 = 0$): The translation leads to the sequent of the form $\sim a \cdot 0 = 0 \to$. The sequent is, then, the axiom 8 since $\sim 0^{(k)} \cdot 0 = 0 \to$ is the axiom 1 for every natural number $k$.

Case 20 ($\text{CN} \vdash 0 \sim a \cdot b' = a \cdot b + a$): The formula is transformed into $\sim a \cdot b' = a \cdot b + a \to$, which is the axiom8. In fact, given, particular natural number $k$ and $l$, we can decide that $0^{(k)} \cdot (0^{(l)})'$ and $(0^{(k)} \cdot 0^{(l)}) + 0^{(k)}$ have the same value $k \cdot (l + 1)$.
By the basis, we obtain the proposition

\[ \text{CN} \vdash A \quad \Rightarrow \quad \text{SN} \vdash \sim TA \to . \]

Further, we have:

\[ \text{SN} \vdash \sim TA \to \quad \Rightarrow \quad \text{PCN} \vdash \sim TA \to , \]

since \( TA \) contains no implication sign. This leads to the proposition:

\[ \text{CN} \vdash A \quad \Rightarrow \quad \text{PCN} \vdash \sim TA \to . \]

**Induction steps** \((n \geq 0)\)

Case 1: Assume \( A \) is obtained by the following inference (namely, modus ponens):

\[ \begin{align*}
\text{CN} \vdash n_1 B & \quad \text{CN} \vdash n_2 B \supset A \\
\text{CN} \vdash n A ,
\end{align*} \]

where \( n = \max(n_1, n_2) + 1 \). By H.I. \((n_1 < n \text{ and } n_2 < n)\), we obtain:

\[ \text{PCN} \vdash \sim TB \to \quad (*) \]

and

\[ \text{PCN} \vdash \sim T(B \supset A) \to \]

i.e.,

\[ \text{PCN} \vdash \sim (\sim TB \lor TA) \to \quad (**) \]

Here, use is made of an easy lemma to the effect:

\[ \text{PCN} \vdash A \to \quad \Rightarrow \quad \text{FN} \vdash A \to , \]

by means of which the cut elimination theorem proved earlier turns into its counterpart in \( \text{PCN} \). By the \( \text{PCN} \)-cut elimination theorem (Theorem 6.4), there obtains the looked-for sequent \( \sim TA \to \) from \((*)\) and \((**))

Case 2: Let \( A \) be \( C \supset \forall x A_1(x) \). Assume that the formula is inferred by way of a rule of classical predicate logic in the following way:

\[ \begin{align*}
\text{CN} \vdash n_1 A_1(x) & \quad \text{CN} \vdash n_2 \supset C \\
\text{CN} \vdash n \supset \forall x A_1(x),
\end{align*} \]

By H.I., we obtain \( \text{PCN} \vdash n \sim T(C \supset \forall x A_1(x)) \to \), to which \( \sim TC \lor \forall x TA_1(x) \to \) is reduced by applying \( (\forall \to) \) under the restriction on variables. \( \sim (TC \lor \forall x TA_1(x)) \to \) is obviously the sequent obtain by the translation of \( C \supset \forall x A_1(x) \)

Case 3: Let \( A \) be \( \exists x A_1(x) \supset C \). Suppose that the formulas inferred by way of another rule of predicate logic as follows:

\[ \begin{align*}
\text{CN} \vdash n_1 A_1(x) & \quad \text{CN} \vdash n \exists x A_1(x) \supset C,
\end{align*} \]

31
By H.I., we obtain $\text{PCN} \vdash T(A_1(x) \supset C) \rightarrow$, i.e.,

$$\text{PCN} \vdash \sim (TA_1(x) \lor TC) \rightarrow.$$

The looked-for sequent $\sim (\exists x TA_1(x) \lor TC) \rightarrow$ is reduced to (*) by $(\exists \rightarrow)$ under the restriction on variables.

**Proof of $\Leftarrow$ in the embedding theorem.** Assume $\text{SN} \vdash T A \rightarrow$. By Theorem 7.2, $\text{SNH} \vdash TA \supset \sim TA$. Then, $\text{CN} \vdash TA \supset \sim TA$ since $\text{SNH}$ is a subsystem of $\text{CN}$. $TA$ is thus provable in $\text{CN}$ since $\text{CN} \vdash TA \supset \sim TA$. (Here, $A \equiv B$ stands for $A \equiv B$. $\land B \supset A$.) $A$ is, therefore, provable in $\text{CN}$ since $\text{CN} \vdash TA \equiv TA$. $\square$

8 Consistency of CN

**Theorem 8.1** (Consistency theorem of CN)

$\text{CN}$ is consistent.

**Proof.** Assume, if possible, $\text{CN}$ were inconsistent. Every formula of $\text{CN}$ would, then, be provable in $\text{CN}$. $\text{CN} \vdash 0 = 1$, thus. (We abbreviate $0'$ as 1.) $\text{PCN} \vdash \sim 0 = 1 \rightarrow$ is forthcoming right away by the embedding theorem (Theorem 6.3). It is not provable in $\text{PCN}$, however. In fact, $\sim 0 = 1 \rightarrow$ is not any axiom of $\text{PCN}$ and is by no means reduced by applying any reduction rule of $\text{PCN}$.

Here, we are presenting another consistency proof of theorem. Assume $\text{CN}$, if possible, be inconsistent. Every formula would, then, provable in $\text{CN}$. We, thus, obtain $\text{CN} \vdash 0 = 1$ and $\text{CN} \vdash \sim 0 = 1$. Then, $\sim 0 = 1 \rightarrow$ and $\sim \sim 0 = 1 \rightarrow$ would be both provable in $\text{PCN}$ by the embedding theorem. It follows that $\text{PCN} \vdash \rightarrow 0 = 1$ and $\text{PCN} \vdash \rightarrow \sim 0 = 1$ by Theorem 5.12. This is impossible since $\text{SN}$ is consistent (Theorem 6.2). Another standard argument for the consistency is the following. By the $\text{SN}$-cut elimination theorem, $\rightarrow$ is obtained from $\text{SN} \vdash \rightarrow 0 = 1$ and $\text{SN} \vdash \rightarrow \sim 0 = 1$. But $\rightarrow$ is by no means provable in $\text{SN}$. $\square$

9 The proof of SN-cut elimination theorem and its restricted version

We first recall our cut elimination theorem.

**Theorem 6.4** (SN-cut elimination theorem)

$$(\text{SN} \vdash \Gamma \rightarrow G[A_{\pm}] \text{ and } \text{SN} \vdash F[A_{\pm}] \rightarrow \Delta) \Rightarrow \text{SN} \vdash \Gamma \rightarrow G[\pm] \rightarrow F[\pm] \rightarrow G[\pm] \rightarrow \Delta.$$

**Proof.** A syntactic proof may be taken as an adaptation of Kanai [1984] to our system as that of $\text{PCN}$-cut elimination theorem. A semantical proof is also possible. $\square$

In this section, as a novelty, we shall here give a simple proof of a restricted version of $\text{SN}$-cut elimination theorem as an application of the disjunction property, using $\text{PCN}$-cut elimination theorem.
Theorem 9.1 (A restricted SN-cut elimination theorem) We have:

(1) \((\text{SN} \vdash \Gamma \to G[+^a] \to \Delta) \Rightarrow \text{SN} \vdash \Gamma \land F[+] \to G[+^a] \lor \Delta\),

where
\[
\text{SN} \vdash \Gamma \land \sim A \Rightarrow \quad \text{PCN} \vdash \Gamma \land A \to
\]
and
\[
\text{SN} \vdash F[+^a] \land \sim \Delta \Rightarrow \quad \text{PCN} \vdash F[+^a] \land \sim \Delta \to
\]
hold.

(2) \((\text{SN} \vdash \Gamma \to G[-^a] \to \Delta) \Rightarrow \text{SN} \vdash \Gamma \land F[-^a] \to G[-^a] \lor \Delta\),

where
\[
\text{SN} \vdash \Gamma \land A \Rightarrow \quad \text{PCN} \vdash \Gamma \land A \to
\]
and
\[
\text{SN} \vdash F[-^a] \land \sim \Delta \Rightarrow \quad \text{PCN} \vdash F[-^a] \land \sim \Delta \to
\]
hold.

Proof. First we shall prove (1).

\((\text{SN} \vdash \Gamma \to G[+^a] \text{ and } \text{SN} \vdash F[+^a] \to \Delta) \Rightarrow \text{SN} \vdash \Gamma \land F[+] \to G[+^a] \lor \Delta\).

Suppose \(\text{SN} \vdash \Gamma \to G[+^a]\) and \(\text{SN} \vdash F[+^a] \to \Delta\). By the disjunction property (Theorem 5.7), we have \(\text{SN} \vdash \Gamma \to G[+^a]\) or \(\text{SN} \vdash \Gamma \to A\). Assume \(\text{SN} \vdash \Gamma \to G[+^a]\). Then we immediately obtain \(\text{SN} \vdash \Gamma \land F[+] \to G[+^a] \lor \Delta\) by the thinning theorem. Assume \(\text{SN} \vdash \Gamma \to A\). By Theorem 5.10, we get \(\text{SN} \vdash \Gamma \land \sim A \to\) and \(\text{SN} \vdash F[+^a] \land \sim \Delta \to\). From the assumption of the theorem, we obtain

\[
\text{PCN} \vdash \Gamma \land A \to, \quad (*1)
\]
\[
\text{PCN} \vdash F[+] \land \sim \Delta \to. \quad (*2)
\]
Apply PCN-cut elimination theorem to (*1) and (*2). Then we obtain

\[
\text{PCN} \vdash \Gamma \land F[+] \land \sim \Delta \to. \quad (*3)
\]

From (*3) and Theorem 5.11, we have \(\text{SN} \vdash \Gamma \land F[+] \to \Delta\). By the thinning theorem, the desired sequent

\[
\text{SN} \vdash \Gamma \land F[+] \to G[+^a] \lor \Delta
\]

holds. For the proof of (2), we can take similar arguments. □

The restriction required by Theorem 9.1 is very strong. But some applications of Theorem 9.1 seem to be possible in computer science, when we pursue to constructive nature appeared in the subject.

33
10 On future studies

There may be a plan to extend the results of this paper to the full system with complete induction, a system with bar induction and formal analysis, etc. For those studies, the following papers will be useful: Bezem [1985, 1989] [9, 10], Pohlers [2009] [53], Ferreira [2015] [18], Kahle and Rathjen [2015] [35], Siders [2015] [59], Spector [1962] [61], Tait [2015] [62], etc.

It would also be a way to build the strong negation version of Schütte’s book, Proof Theory (1977 version) [57]. I shall have a plan to do so for the future Part II of this paper.

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To write this paper, I was very much inspired by the papers and the book, Siders [2015] [59], Ferreira [2015] [18] and Odintsov [2008] [50]. So, I would like to thank Prof. Siders, Prof. Ferreira and Prof. Odintsov. I also appreciate some reviews of zbMATH by Prof. M. Yasuhara and Prof. M. J. Beeson.

References

[1] P. H. G. Aczel, H. Simmons and S. S. Wainer, (eds.) , Proof Theory, Cambridge U. P.,1992.

[2] W. Ackermann, Begründung des ‘Tertium non datur’ mittels der Hilbertschen Theorie der Widerspruchsfreiheit, Mathematische Annalen, Vol. 93 (1924), pp. 1–36.

[3] W. Ackermann, Zur Widerspruchsfreiheit der Zahlentheorie, Mathematische Annalen, Vol. 117 (1940), pp. 162–194.

[4] S. Akama, Constructive predicate logic with strong negation and model theory, Notre Dame Journal of Formal Logic, Vol.29 (1988), pp. 18–27.

[5] A. Almukdad and D. Nelson, Constructible falsity and inexact predicates, The Journal of Symbolic Logic, Vol. 49 (1984), pp. 231–233.

[6] T. Arai, Mathematical Logic, (in Japanese), Iwanami Shoten, Tokyo, 2011.

[7] L. Bellotti, Novikov’s cut elimination, Logique et Analyse, Vol.61, No. 242 (2018), pp. 183-199.
[8] P. Bernays, On the original Gentzen consistency proof for number theory, In A. Kino, J. Myhill and R. E. Vesley (eds.), Intuitionism and Proof Theory, pp. 409–417, North-Holland, Amsterdam, 1970.

[9] M. Bezem, Strongly majorizable functionals of finite type: a model for bar recursion containing discontinuous functionals, The Journal of Symbolic Logic, Vol.50 (1985), pp.652–660.

[10] M. Bazem, Compact and majorizable functionals of finite type, The Journal of Symbolic Logic, Vol.54 (1989), pp.271–280.

[11] W. Buchholz, S. Feferman, W. Pohlers and W. Sieg, Iterated Inductive Definitions and Subsystems of Analysis: Recent Proof-Theoretical Studies (Lecture Notes in Mathematics Vol. 897), Springer, 1981.

[12] S. L. Buss, (ed.), Handbook of Proof Theory, Elsevier, Amsterdam, 1998.

[13] S. L. Buss and A. Ignjatović, Unprovability of Consistency Statements in Fragments of Bounded Arithmetic, Annals of Pure and Applied Logic, Vol.74 (1995), pp. 221-244.

[14] J. C. E. Dekker (ed.), Recursive function theory, Proceedings of Symposia in Pure Mathematics, Vol. 5, American Mathematical Society, Providence, Rhode Island, 1962.

[15] S. Feferman, How we got from there to here, In [11], pp. 1–15.

[16] S. Feferman and W. Sieg, Iterated inductive definitions and subsystems of analysis. In [11], pp. 16–77.

[17] S. Feferman and W. Sieg, Proof theoretic equivalences between classical and constructive theories for analysis. In [11], pp. 78–142.

[18] F. Ferreira, Spector’s Proof of the Consistency of Analysis, In [35], pp. 279–300.

[19] R. C. Flagg and H. Friedman, Epistemic and intuitionistic formal systems, Annals of Pure and Applied Logic, Vol. 32 (1986), pp. 53–60.

[20] M. Fitting, Proof Methods for Modal and Intuitionistic Logics, D. Reidel, Dordrecht, 1983.

[21] R. C. Flagg and H. Friedman, A framework for measuring the complexity of mathematical concepts, Advances in Pure and Applied Mathematics, Vol. 11 (1991), pp. 1–34.

[22] G. Gentzen, Untersuchungen über das Logische Schliessen I, Math. Zeitschrift, Vol. 39 (1934), pp. 176–210.

[23] G. Gentzen, Untersuchungen über das Logische Schliessen II, Math. Zeitschrift, Vol. 39 (1934), pp. 405–431.
[24] G. Gentzen, Widerspruchsfreiheit der reinen Zahlenlehre, *Math. Ann.*, Vol. 112 (1936), pp. 493–565.

[25] K. Gödel, Zur intuitionistischen Arithmetik und Zahlentheorie, *Ergebnisse eines mathematischen Kolloquiums*, Vol. 4 (1933), pp. 34–38.

[26] J. Herbrand, Sur la non-contradiction de l’arithmétique, *Journal für reine und angewandte Mathematik*, Vol. 166 (1931), pp. 1–8. English translation: ‘On the consistency of arithmetic’, In J. van Heijenoort (ed.) , *From Frege to Gödel*, Harvard U. P., Cambridge, 1967, pp. 620–628.

[27] D. Hilbert and P. Bernays, *Grundlagen der Mathematik*, Bd. I (1934), Bd. II (1939), Springer, Berlin. (Second edition. 1968 and 1970, respectively)

[28] T. Inoué, New type cut elimination theorem and its application to embedding theorem in S. Unpublished note, 1984.

[29] T. Inoué, On the Accommodation of Prolog to Constructive Logic with Strong Negation S using its Embedding Theorem (in Japanese), In [33], pp. 41–93.

[30] T. Inoué, On the fundamental theorem of Logic programming: the reduction of Prolog to constructive Logic with strong negation, in preparation.

[31] A. Ishimoto, Constructive propositional logic with strong negation and their completeness. Unpublished note, 1977.

[32] A. Ishimoto, A Schütte-type formulation of the intuitionistic functional calculus with strong negation, *Bulletin of the Tokyo Institute of Technology*, Vol. 100 (1970), pp. 161–189.

[33] A. Ishimoto, (ed.), *The progress report III of Ishimoto’s group*, Grant-in-Aid for Scientific Research No.57115012, Ministry of Education, Japan (1984),

[34] A. Ishimoto, (ed.), *The Logic of Natural Language and its Ontology* (in Japanese), Taga Shuppan, Tokyo, 1990.

[35] R. Kahle and M. Rathjen (eds.), *Gentzen’s Centenary, The Quest for Consistency*, Springer, 2015.

[36] N. Kanai, The cut elimination theorem of constructive predicate logic with strong negation (in Japanese). In [34], pp. 94–105.

[37] I. N. Khlodovskii, A new proof of the consistency of arithmetic, (in Russian) *Uspekhi Mat. Nauk*, Vol.14 (1959), pp.105–140. (as I. N. Holdovskii) English translation of the preceding by Moshe Machover. American Mathematical Society translations, ser. 2, Vol. 23 (*Nine Papers on Logic and Quantum Electrodynamics*) (1963), pp. 191-230.
[38] S. C. Kleene, Permutability of inferences in Gentzen’s calculi LK and LJ, in S. C. Kleene, Two Papers on the Predicate Calculus, Memoirs of the American Mathematical Society, No. 10 (1952), The American Mathematical Society, Providence, Rhode Island, pp. 1–26.

[39] S. C. Kleene, Introduction to Metamathematics, North-Holland, Amsterdam, 1952.

[40] S. C. Kleene, Mathematical Logic, J. Wiley and Sons, New York, 1967. (There is a Russian translation by G. E. Mints.)

[41] A. N. Kolmogorov, On the principle of excluded middle (in Russian). In J. van Heijenoort (ed.), From Frege to Gödel, pp. 417-437, Cambridge, Harvard U. P., 1967.

[42] S. Maehara, Mathematical Logic, (in Japanese), Baitukan, Tokyo, 1973.

[43] A. A. Markov, A constructive logic (in Russian), Uspehi Mathematicskikh, Vol. 5 (1950), pp. 187–188.

[44] T. Matsuda and A. Ishimoto, Relationship between logic computer language and constructive predicate logic with strong negation, (in Japanese), In [53], pp. 109–121.

[45] R. Murawski, On proofs of the consistency of arithmetic, Studies in Logic, Grammar and Rhetoric, Vol. 4 (2001), pp.41–50.

[46] D. Nelson, Constructible falsity, The Journal of Symbolic Logic, Vol. 14 (1949), pp. 16–26.

[47] J. von Neumann, Zur Hilbertschen Beweistheorie, Mathematische Zeitschrift, Vol. 26 (1927), pp. 1–46.

[48] D. Monk, Mathematical Logic, Springer-Verlag, New York, 1976.

[49] P. S. Novikov, Elements of Mathematical Logic, (in Russian), Moscow, 1959. (English translation: Elements of Mathematical Logic (translated by L. F. Boron), Oliver & Boyd, Edinburgh and London, 1964.) (German translation: Grundzüge der mathematischen Logik (translated by K. Rosenbaum), Friedr. Vieweg + Sohn, Braunschweig, 1973. There is a Japanese translation by the late Prof. A. Ishimoto, Tokyo Toshio, Tokyo, 1965.

[50] S. P. Odintsov, Constructive Negations and Paraconsistency, Springer, 2008.

[51] K. Ono, Logische Untersuchungen über die Grundlagen der Mathematik, Journal of the Faculty of Science I, Imperial University of Tokyo, Vol. 3 (1938), pp. 329–389.

[52] H. N. Rasiowa, Lattices and constructive logic with strong negation, Fundamenta Mathematicae, Vol.46 (1958), pp.61–80.
[53] W. Pohlers, Proof Theory, The First Step into Impredicativity, Springer, Berlin, 2009.

[54] K. Schütte, Beweistheorie, Springer-Verlag, Berlin, 1960.

[55] K. Schütte, Der Interpolationsatz der intuitionistischen Prädikatenlogik, Mathematische Annalen, Vol. 148 (1962), pp. 192–200.

[56] K. Schütte, Vollständige Systeme Modaler und Intuitionistischer Logik, Springer-Verlag, Berlin, 1968.

[57] K. Schütte, Proof Theory, Springer-Verlag, Berlin, 1977.

[58] S. Shimizu, A study on constructive logic with strong negation - its soundness and completeness, (in Japanese). In [34], pp. 241–267.

[59] A. Siders, A Direct Gentzen-Style Consistency Proof for Heyting Arithmetic. In [35], pp.177–211.

[60] R. M. Smullyan, First-Order Logic, Springer, New-York, 1968. There is a Dover verion of this book with a short comment for literature.

[61] C. Spector, Provably recursive functionals of analysis: a consistency proof of analysis by an extension of principles formulated in current intuitionistic mathematics. In [14], pp. pp. 1-27.

[62] W. W. Tait, Gentzen’s original consistency proof and the Bar theorem, In [35], pp. 213–228.

[63] M. Takano, A formulation of the Fitch functional calculus as a Sequenzenkalkül, Bulletin of the Tokyo Institute of Technology, Vol. 100 (1970), pp. 143–160.

[64] G. Takeuti, Proof Theory, 2nd, North-Holland, Amsterdam, 1987.

[65] S. Toledo, Tableau Systems for First Order Number Theory and Certain Higher Order Theories, (Lecture Notes in Mathematics Vol. 447), Springer, Berlin, 1975.

[66] A. S. Troelstra and H. Schwichtenberg, Basic Proof Theory, 2nd, Cambridge University Press, New York, 2000.

[67] A. S. Troelstra and D. van Dalen, Constructivism in Mathematics, An Introduction, Vol. I, II, North-Holland, Amsterdam, 1988.

[68] N. N. Vorob’ev, A constructive propositional calculus with strong negation (in Russian), Doklady Akademii Nauk SSSR, Vol. 85 (1952), pp. 465–468.

[69] N. N. Vorob’ev, The problem of deducibility in constructive propositional calculus with strong negation (in Russian), Doklady Akademii Nauk SSSR, Vol. 85 (1952), pp. 689–692.

[70] N. N. Vorob’ev, Constructive propositional calculus with strong negation (in Russian), Transactions of Steklov’s Institute, Vol. 72 (1964), pp. 195–227.
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