General paradigm for distilling classical key from quantum states
Karol Horodecki, Michał Horodecki, Paweł Horodecki, Jonathan Oppenheim

Abstract
We develop a formalism for distilling a classical key from a quantum state in a systematic way, expanding on our previous work on secure key from bound entanglement [K. Horodecki et. al., Phys. Rev. Lett. 94 (2005)]. More detailed proofs, discussion and examples are provided of the main results. Namely, we demonstrate that all quantum cryptographic protocols can be recast in a way which looks like entanglement theory, with the only change being that instead of distilling EPR pairs, the parties distill private states. The form of these general private states are given, and we show that there are a number of useful ways of expressing them. Some of the private states can be approximated by certain states which are bound entangled. Thus distillable entanglement is not a requirement for a private key. We find that such bound entangled states are useful for a cryptographic primitive we call a controlled private quantum channel. We also find a general class of states which have negative partial transpose (are NPT), but which appear to be bound entangled. The relative entropy distance is shown to be an upper bound on the rate of key. This allows us to compute the exact value of distillable key for a certain class of private states.

Index Terms
Quantum Information Theory, Classical Key Distillation, Quantum Entanglement
I. Introduction

We often want to communicate with friends or strangers in private. Classically, this is impossible if we wish to communicate over long distances, unless we have met before with our friend and exchanged a secret key which is as long as the message we want to send. On the other hand, quantum cryptography allows two people to communicate privately with only a very short key which is just used to authenticate the message.

Every quantum cryptographic protocol is equivalent to the situation where both parties (Alice and Bob) share some quantum state $\rho_{AB}$, and then perform local operations on that state and engage in public communication (LOPC) to obtain a key which is private from any eavesdropper. Until recently, every quantum protocol was also equivalent to distilling pure entanglement from this shared state. I.e., achieving privacy was equivalent to the two parties converting many copies of the state

$$\rho_{AB}, \quad (1)$$

using local operations and classical communication (LOCC), and then performing a measurement on the EPR pairs in the computational basis. Examples of such protocols include BB84 [7], [49], B92 [5], [51], and of course, E91 [24]. It was thus thought that achieving security is equivalent to distilling pure entanglement, and a number of results pointed in this direction [18], [19], [29], [48], [1], [28], [11].

Recently, however, we have shown that this is not the case – there exist examples of bound entangled states which can be used to obtain a secret key [34]. Bound entangled states [36] are ones which need pure entanglement to create, but no pure entanglement can be distilled from them. This helps explain the properties of bound entangled states. They have entanglement which protects correlations from the environment (or an eavesdropper), but the entanglement is so twisted that it can’t be brought into pure form. This then raised the question of what types of quantum states provide privacy. In [34] we were able to find the general form of private quantum states $\gamma_{ABA'B'}$. This allowed us to recast the theory of privacy (under local operations and public communication – or LOPC) in terms of entanglement theory (local operations and classical communication – or LOCC). In entanglement theory, the basic unit is the EPR pair, while in privacy theory, the only difference is that one replaces the EPR pair with general private states $\gamma_{ABA'B'}$ as the basic units.

In the present article, we review the results of [34] in greater detail, and expand on the proofs and tools. Namely, we study and show that the general form of a private state on a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_{A'} \otimes \mathcal{H}_B \otimes \mathcal{H}_{B'}$ with dimensions $d_A = d_B = d$, $d_{A'}$ and $d_{B'}$, is of the form

$$|\psi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad (1)$$

IX Equality of key rates in LOCC and LOPC paradigms
IX-A Coherent version of LOPC key distillation protocol
IX-B Equivalence of paradigms: The case of exact key
IX-C Distillation of classical key and distillation of pdits - equivalence in general (asymptotically exact) case

X Distilling key from bound entangled states
X-A The new family of PPT states ...
X-B ... can approximate pdits
X-C Distillation of secure key

XI Relative entropy of entanglement as upper bound on distillable key

References
\[ \gamma_{ABA'B'} = U P^{+}_{AB} \otimes \sigma_{A'B'} U^{\dagger} , \]

where \( P^{+}_{AB} \) is a projector onto the maximally entangled state \( \psi_{+} = \sum_{i} \frac{1}{\sqrt{d}} |e_{i}f_{i}\rangle \), and \( U \) is the arbitrary twisting operation

\[ U = \sum_{k,l=0}^{d-1} |e_{k}f_{l}\rangle \langle e_{k}f_{l}| AB \otimes U_{A'B'}^{kl} . \]

The key is obtained after measuring in the \( |e_{i}f_{j}\rangle \) basis. We will henceforth refer to \( \psi_{+} \) as the maximally entangled or EPR state (or Bell state in dimension \( 2 \times 2 \)). We show that the rate of key \( K_{D} \) which can be obtained from a quantum state can be strictly greater than the distillable entanglement, and this even holds if the distillable entanglement is strictly zero. We also show [34] that the size of the private key is generally bounded from above by the regularized relative entropy of entanglement \( E_{r}^{\infty} \) [54]. This will be sufficient to prove that one can have a maximal rate of key strictly less than the entanglement cost (the number of singlets required to prepare a state under LOCC).

In Section II we introduce some of the basic concepts and terminology we will use throughout the paper. This includes the notion of private states, phits which contain one bit of private key, and pdits which have many bits of key. In Section III we show that a state is secure if and only if it is of the form given above. Then we show different useful ways to write the eavesdropper’s density matrix in Section VI. This allows us to interpret previous results in terms of the eavesdropper’s notion of irreducibility which is used to define the basic unit of privacy for private states.

States which have a perfect bit of key must have some distillable entanglement [39]. The case of bound entangled states with secure key is only found in the case of states which are not perfectly secure, although they are arbitrarily secure. This motivates our investigation in Section VII of approximate pbits. We then demonstrate how to rewrite a bipartite state in terms of the eavesdropper’s density matrix in Section VI. This allows us to interpret previous results in terms of the eavesdropper’s states. Then, in Section VII we summarize the previous results in preparation for showing that bound entangled states can have a key. In Sections VIII and IX we review the paradigms of entanglement (LOCC) and privacy theory (LOPC), and show the equivalence of key rates in the two paradigms. We then discuss and compare security criteria in these paradigms in Section XV-C of the Appendix. In Section X we give a number of bound entangled states and show that they can produce a private key. The methods allow one to find a wide class of states which are bound entangled, because the fact that they have key automatically ensures that they are entangled, which is usually the difficult part in showing that a state is bound entangled (the PPT criteria can be quickly checked to see that the states are non-distillable). In Section XI we prove that the relative entropy distance is an upper bound on the rate of key.

In Section XII a class of NPT states are introduced which appear to be bound entangled. They are derived from a class of bound entangled private key states. An additional result discussed in Section XIII which we only mentioned in passing in [34], is that the bound entangled key states can be used as the basis of a cryptographic primitive we call a controlled private quantum channel. We conclude in Section XIV with a few open questions.

## II. Security contained in quantum states

In this section we will introduce the class of states which contain at least one bit (or dit) of perfectly secure key which is directly accessible – these we call private bits (or dits). We discuss the properties of these states and argue the generality of this approach. In particular we introduce the notion of twisting, which is a basic concept in dealing with private states.

A well known state that contains one bit of secure key which is directly accessible is the singlet state. After measuring it in a local basis Alice and Bob obtain bits that are perfectly correlated with each other and completely uncorrelated with the rest of the world including an eavesdropper Eve. This is because the singlet state as a whole is decoupled from the environment, being a pure state. However even if a state is mixed, it can contain secure key. Yet the key must then be located only in a part of it. More formally, we consider a four-partite mixed state \( \rho_{ABA'B'} \) of two systems \( A, A' \) belonging to Alice and \( B, B' \) belonging to Bob. The \( AB \) subsystem of the state will be called the key part of the state – it is the part of the state which produces key upon measurement. The \( A'B' \) subsystem will be called the shield of the state. It is called this, because its presence is what will cause the \( AB \) part of the state to be secure, by shielding information from an eavesdropper.

We assume the worst case scenario – that the state is the reduced density matrix of the pure state \( \psi_{ABA'B'E} \) where we trace out the system \( E \) belonging to eavesdropper Eve. We then distinguish a product basis \( B = \{ e_{i}, f_{j} \} \) in system \( AB \). For our purposes, without loss of generality, we often choose \( B \) to be the standard basis \( \{ \lvert ij \rangle \} \). Distinguishing the basis is connected with the fact that we are dealing with classical security, which finally is realized in some fixed basis. Now, consider the state of systems \( ABE \) after measurement performed in the basis \( B \) by Alice and Bob. This state is of the form

\[ \rho_{ccq} = \sum_{i,j=0}^{d-1} p_{ij}|e_{i}f_{j}\rangle_{AB}\langle e_{i}f_{j}| \otimes \rho_{ij}^{E} . \]
The above form of the state is usually called a ccq state. We will therefore refer to a ccq state associated with state $\rho_{ABA'B'}$, and it is understood that it is also related to chosen basis $\mathcal{B}$. The distribution $p_{ij}$ will sometimes be referred to as the distribution of the ccq state.

We can now distinguish types of states $\rho_{ABA'B'}$ via looking at their ccq states (always assuming that some fixed basis $\mathcal{B}$ was chosen):

**Definition 1:** A state $\rho_{ABA'B'}$ is called secure with respect to a basis $\mathcal{B} \equiv \{|e_i f_j\}_{AB}^d_{i,j=1}$ if the state obtained via measurement on AB subsystem of its purification in basis $\mathcal{B}$ followed by tracing out $A'B'$ subsystem (i.e. its ccq state) is product with Eve’s subsystem:

$$\left( \sum_{i,j=0}^{d-1} p_{ij} |e_i f_j\rangle \langle e_i f_j|_{AB} \right) \otimes \rho_E.$$  

(5)

Such a state $\rho_{ABA'B'}$ will be also called "$\mathcal{B}$ secure". Moreover if the distribution $\{p_{ij}\} = \{\frac{1}{d} \delta_{ij}\}$ so that the ccq state is of the form

$$\left( \sum_{i=0}^{d-1} \frac{1}{d} |e_i f_i\rangle \langle e_i f_i|_{AB} \right) \otimes \rho_E,$$  

(6)

the state $\rho_{ABA'B'}$ is said to have $\mathcal{B}$-key.

One can ask when two states $\rho_{ABA'B'}$ and $\sigma_{ABA'B'}$ are equally secure with respect to a given product basis $\mathcal{B}$. First let us define what does it mean "equally secure". A natural definition would be that when Alice and Bob measure systems $AB$ in the basis, then Eve by any means cannot distinguish between two situations, as far as the outcomes of the measurement are concerned. In particular, the states are definitely equally secure, when their ccq states are equal.

For our purpose we will need to know when for two states the latter relation holds. It is obvious that any unitary transformation applied to systems $A'B'$ of the state $\rho_{ABA'B'}$ will not change the ccq state. (Note that it cannot be just any CP map; for example, partial trace of systems $A'B'$ would mean giving it to Eve, which of course would change the ccq states). As it will be demonstrated in the next section, we can actually do much more without changing the ccq state. Namely we can apply an operation called "twisting". This operation is defined for system $ABA'B'$ and with respect to a product basis $\mathcal{B}$ of $AB$ system as follows:

**Definition 2:** Given product basis $\mathcal{B} = \{e_i, f_j\}_{k,l}$ on systems $AB$, the unitary operation acting on system $ABA'B'$ of the form

$$U = \sum_{k,l=0}^{d-1} |e_k f_l\rangle \langle e_k f_l|_{AB} \otimes U_{A'B'}^{kl},$$  

(7)

is called $\mathcal{B}$-twisting, or shortly twisting.

Finally we define the class of private states. The states from that class are proven [34] to be the only quantum states which after measurement on Alice and Bob subsystems give an ideal key. In other words these are the only states from which Alice and Bob can get an ideal ccq state (6) according to definition 1 of security. For the sake of clarity, we recall this proof with details in Section III.

**Definition 3:** A state $\rho_{ABA'B'}$ of a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_A' \otimes \mathcal{H}_B \otimes \mathcal{H}_B'$ with dimensions $d_A = d_B = d$, $d_{A'}$ and $d_{B'}$, of the form

$$\gamma^{(d)} = \frac{1}{d} \sum_{i,j=0}^{d-1} |e_i f_j\rangle \langle e_j f_j|_{AB} \otimes U_j \sigma_{A'B'} U_j^\dagger,$$  

(8)

where the state $\sigma_{A'B'}$ is an arbitrary state of subsystem $A'B'$, $U_i$'s are arbitrary unitary transformations and $\{\{|e_i\}\}^d_{i=0}, \{\{|f_i\}\}^d_{i=0}$ are local basis on $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, is called private state or pdit. In case of $d = 2$ the state is called pbit.

Note, that maximally entangled states are also private states, which is in case when $d_A' = d_B' = 1$. In general, any pdit can be created out of a maximally entangled state with additional state on $\sigma_{A'B'}$ (which we will call basic pdit) by some twisting.

**Definition 4:** A state $\rho_{ABA'B'}$ of a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_A' \otimes \mathcal{H}_B \otimes \mathcal{H}_B'$ with dimensions $d_A = d_B = d$, $d_{A'}$ and $d_{B'}$, of the form

$$\rho_{ABA'B'} = P^+_{AB} \otimes \sigma_{A'B'},$$  

(9)

is called a basic pdit.

**Remark 1:** Let us note, that one could define states with ideal key also in a different way than in definition 1. Namely, we could say that the state $\rho_{AB}$ has $\mathcal{B}$ key iff it has a subsystem $ab$ such, that the subsystem $abE$ of its purification $|\psi_{ABE}\rangle$ is an ideal ccq state of eq. (6), where $\mathcal{B} = \{|e_i f_i\}_{i=1}^d$. However maximally entangled state is not of this form, but is locally equivalent to a state of this form, i.e. they would be transformable into one another by means of unitary embeddings or partial isometries. In fact the whole class of so defined states with ideally secure key, would be locally equivalent to the one we have introduced in definition 3 and by characterization of the latter, equivalent to the class of private states. Another definition of
Reordering of the block operators

In the above equation, the partial transposition on LHS is with respect to system \( A \), as the partial transposition with respect to system \( B \) resulted already in appropriate reordering of the block operators \( A_{ijkl} \).

In what follows, we will repeatedly use equivalence of the trace norm distance and fidelity proved by Fuchs and van de Graaf [26]:

**Lemma 1:** For any states \( \rho, \rho' \) there holds

\[
1 - F(\rho, \rho') \leq \frac{1}{2} \|\rho - \rho'\| \leq \sqrt{1 - F(\rho, \rho')^2}
\]

(13)

Here \( F(\rho, \rho') = \text{Tr} \sqrt{\sqrt{\rho^2 \rho'}} \) is fidelity;

We also use the Fannes inequality [25] (in the form of [2]):

\[
|S(\rho) - S(\sigma)| \leq 2 \||\rho - \sigma\|| \log d + h(\||\rho - \sigma\||),
\]

(14)

which holds for arbitrary states \( \rho \) and \( \sigma \) satisfying \( \||\rho - \sigma\|| \leq 1 \).

A. Some facts and notations

In what follows, by \( \|\cdot\| \) we mean the trace norm, i.e. the sum of the singular values of an operator. For any bipartite operator \( X \in B(\mathcal{H}_1 \otimes \mathcal{H}_2) \), by \( X^T \) we mean the partial transposition of \( X \) with respect to system \( B \), that is:

\[
(I_1 \otimes T_2)X,
\]

(10)

where \( T_2 \) denotes the matrix transposition over system \( 2 \) of a matrix \( X \). For brevity, we will use the same symbol for any partial transposition. In particular, we deal often with systems of four subsystems \( ABA'B' \), and we will take partial transposition with respect to subsystems \( B \) and \( B' \). Hence \( \rho^\Gamma_{\rho ABA'B'} \) denotes \( (I_A \otimes T_B \otimes I_{A'} \otimes T_{B'})/(\rho_{\rho ABA'B'}) \), with \( T_B \) and \( T_{B'} \) denoting the matrix transposition on systems \( B \) and \( B' \) respectively. To give example of a mixed notation, we consider \( \rho_{\rho ABA'B'} \in B(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^d \otimes \mathbb{C}^d) \). In block matrix form, such a state has a bipartite structure of blocks, so that it reads:

\[
\rho_{\rho ABA'B'} = \begin{bmatrix}
A_{0000} & A_{0001} & A_{0010} & A_{0011} \\
A_{0100} & A_{0101} & A_{0110} & A_{0111} \\
A_{1000} & A_{1001} & A_{1010} & A_{1011} \\
A_{1100} & A_{1101} & A_{1110} & A_{1111}
\end{bmatrix}.
\]

(11)

This state after partial transposition with respect to system \( BB' \) reads:

\[
\rho^\Gamma_{\rho ABA'B'} = \begin{bmatrix}
A^F_{0000} & A^F_{0001} & A^F_{0010} & A^F_{0011} \\
A^F_{0100} & A^F_{0101} & A^F_{0110} & A^F_{0111} \\
A^F_{1000} & A^F_{1001} & A^F_{1010} & A^F_{1011} \\
A^F_{1100} & A^F_{1101} & A^F_{1110} & A^F_{1111}
\end{bmatrix}.
\]

(12)

In the above equation, the partial transposition on LHS is with respect to system \( BB' \) and on the RHS, only with respect to system \( B' \), as the partial transposition with respect to system \( B \), which is a one qubit system, resulted already in appropriate reordering of the block operators \( A_{ijkl} \).

In what follows, we will repeatedly use equivalence of the trace norm distance and fidelity proved by Fuchs and van de Graaf [26]:

**Lemma 1:** For any states \( \rho, \rho' \) there holds

\[
1 - F(\rho, \rho') \leq \frac{1}{2} \|\rho - \rho'\| \leq \sqrt{1 - F(\rho, \rho')^2}
\]

(13)

Here \( F(\rho, \rho') = \text{Tr} \sqrt{\sqrt{\rho^2 \rho'}} \) is fidelity.

We also use the Fannes inequality [25] (in the form of [2]):

\[
|S(\rho) - S(\sigma)| \leq 2 \||\rho - \sigma\|| \log d + h(\||\rho - \sigma\||),
\]

(14)

which holds for arbitrary states \( \rho \) and \( \sigma \) satisfying \( \||\rho - \sigma\|| \leq 1 \).

B. On twisting and privacy squeezing

Here we will show that twisting does not change the ccq state arising from measurement of the key part. Then we will introduce a useful tool by showing that twisting can pump entanglement responsible for security of ccq state into the key part.

We have the following theorem:

**Theorem 1:** For any state \( \rho_{\rho AAB'B'} \) and any \( B \)-twisting operation \( U \), the states \( \rho_{\rho AAB'B'} \) and \( \rho_{\rho AAB'B'} = U_{\rho_{\rho AAB'B'}} U^\dagger \) have the same ccq states w.r.t \( B \), i.e. after measurement in basis \( B \), the corresponding ccq states are equal: \( \tilde{\rho}_{\rho ABE} = \tilde{\sigma}_{\rho ABE} \).

**Proof:** To show that subsystem \( \rho_{\rho ABE} \) is not affected by \( B \) controlled unitary with a target on \( A'B' \) we will consider the whole pure state:

\[
|\psi_{\rho AAB'B'}\rangle = \sum_{ijklm} a_{ijklm} |ijklm\rangle \equiv |\psi\rangle
\]

(15)

(without loss of generality we take \( B \) to be standard basis). After von Neumann measurement on \( B \) and tracing out the \( A'B' \) part the output state is the following:

\[
\tilde{\rho}_{\rho ABE} = \sum_{ijklmn} a_{ijklm} \tilde{a}_{ijklm} |ij\rangle \otimes |m\rangle \langle n|.
\]

(16)
Let us now subject $|\psi\rangle$ to controlled unitary $U_{ABA'B'} \otimes I_E$, 
\begin{equation}
U_{ABA'B'} \otimes I_E|\psi\rangle = \sum_{ijklm} a_{ijklm}|ij\rangle U^{ij}|kl\rangle |m\rangle \equiv |\tilde{\psi}\rangle,
\end{equation}
and then on the output state $|\tilde{\psi}\rangle$ perform a complete measurement on $B$ reading the output:
\begin{equation}
P_{ij}|\tilde{\psi}\rangle \langle \tilde{\psi}| = \sum_{klm,n} a_{ijklm} a_{ijstm}|ij\rangle |kl\rangle \langle st|(U^{ij})^\dagger |AB\rangle \otimes |m\rangle \langle n|_E.
\end{equation}
Performing partial trace and summing over $i,j$ we obtain the same density matrix as in \[16\] which ends the proof.

The above theorem states that two states which differ by some twisting $U$, have the same *ccq* state obtained by measuring their key parts, and tracing out their shields. However, since twisting does not affect only the $ccq$ state, one can be interested in how the whole state changes when subjected to such an operation. We will show now an example of twisting which will be of great importance for further considerations in this paper. Subsequently, we will construct from this twisting an operation called *privacy squeezing* (shortly: *p-squeezing*), which shows the importance of the above theorem. The operation of *p-squeezing* is a kind of primitive in the paradigm which we will present in the paper.

Consider the following technical lemma:

**Lemma 2:** For any state $\sigma_{ABA'B'} \in \mathcal{B}(C^2 \otimes C^2 \otimes C^d \otimes C^d)$ expressed in the form $\sigma_{ABA'B'} = \sum_{ijkl} |ij\rangle \langle kl| \otimes A_{ijkl}$ there exists twisting $U_{ps}$ such that if we apply this to $\sigma_{ABA'B'}$, and trace out $A'B'$ part, the resulting state on $AB$ $\rho_{AB} = \text{Tr}_{A'B'}[U_{ps}\sigma_{ABA'B'}U_{ps}^\dagger]$ will have the form
\begin{equation}
\rho_{AB} = \begin{bmatrix}
\times & \times & \times & |A_{0011}\rangle \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times
\end{bmatrix},
\end{equation}
where we omit non-important elements of $\rho_{AB}$.

**Proof:** Twisting, by its definition (7) is determined by the set of unitary transformations. In the case of pbit which we now consider, there are four unitary transformations which determine it: $\{U_{kl}\}_{k,l=0}^1$. Let us consider singular value decomposition of the operator $A_{0011}$ to be $VRV^\dagger$ with $V,V^\dagger$ unitary transformations, and $R$ - nonnegative diagonal operator. Note, that by unitary invariance of norm, we have that $\|A_{0011}\| = \|R\| = \text{Tr}R$. We then define twisting by choosing $U_{00} = V^\dagger$, $U_{11} = V$, and $U_{01} = U_{10} = I$. The $AB$ subsystem of twisted $\sigma_{ABA'B'}$ state is
\begin{equation}
\rho_{AB} = \sum_{ijkl=0}^1 \text{Tr}(U_{ij}A_{ijkl}U_{kl}^\dagger)|ij\rangle \langle kl|,
\end{equation}
so for such chosen twisting we have indeed, that the element $|00\rangle \langle 11|$ of the matrix of $\rho_{AB}$ is equal to $\text{Tr}U_{00}^\dagger VRV^\dagger U_{11}^\dagger = \text{Tr}R = \|A_{0011}\|$, which proves the assertion.

We will give now the following corollary, which will serve as simple exemplification of this result.

**Corollary 1:** Let the key part be two qubit system. Consider then a state of the form (where blocks are operator acting on $A'B'$ system):
\begin{equation}
\sigma_{ABA'B'} = \begin{bmatrix}
A_{0000} & 0 & 0 & A_{0011} \\
0 & A_{0101} & A_{0110} & 0 \\
0 & A_{1001} & A_{1010} & 0 \\
A_{1100} & 0 & 0 & A_{1111}
\end{bmatrix},
\end{equation}
there exists twisting such that the state after partial trace on $A'B'$ has a form
\begin{equation}
\rho_{AB} = \begin{bmatrix}
\|A_{0000}\| & 0 & 0 & \|A_{0011}\| \\
0 & \|A_{0101}\| & \|A_{0110}\| & 0 \\
0 & \|A_{1001}\| & \|A_{1010}\| & 0 \\
\|A_{1100}\| & 0 & 0 & \|A_{1111}\|
\end{bmatrix}.
\end{equation}

**Proof:** The construction of the twisting is similar as in lemma above. This time one has to consider also the singular value decomposition of the operator $A_{0110} = VSW'$. We can see now, that with any state $\rho_{ABA'B'}$, which has two qubit key part $AB$, we can associate a state obtained in the following way:

1) For state $\rho_{ABA'B'}$ find twisting $U_{ps}$, such that (according to lemma \[2\]) it changes upper-right element of $AB$ subsystem of $\rho_{ABA'B'}$ into $\|A_{0011}\|$.

2) Apply $U_{ps}$ to $\rho_{ABA'B'}$ obtaining $\rho_{ABA'B'}' = U_{ps}\rho_{ABA'B'}U_{ps}^\dagger$. 

3) Trace out the shield ($A'B'$ subsystem) of state $\rho_{ABA'B'}$ obtaining two-qubit state

$$\rho'_{AB} = Tr_{A'B'} \rho_{ABA'B'}.$$  

This operation we will call 
**privacy squeezing**, or shortly **p-squeezing**, and the state $\rho'_{AB}$ which is the output of such operation on the state $\rho_{ABA'B'} \in B(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^d \otimes \mathbb{C}^d)$ the **p-squeezed state of** the state $\rho_{ABA'B'}$. Sometimes we shall use the term privacy squeezing in more informal sense, namely, with the twisting $U_{ps}$ which makes the key part close to maximally entangled state.

Note, that the ccq state of p-squeezed state has no more secret correlations than that of the original state. This is because it emerges from the operation of twisting which preserves security in some sense, i.e. it does not change the ccq state which can be obtained from the original state. The next operation performed in definition of p-squeezed state is tracing out $A'B'$ part which means giving the $A'B'$ subsystem to Eve. Such operation can not increase security of the state in any possible sense.

We will be interested in applying p-squeezing in the case, where the key part of the initial state was weakly entangled, or completely separable. Then the p-squeezing operation will make it entangled.

We can say, that the operation of privacy squeezing pumps the entanglement of the state which is distributed along subsystems $AA'B'B'$ into its key part $AB$. The entanglement once concentrated in the two qubit part, may be much more powerful than the one spread over the whole system. Further in the paper, we will see that from the bound entangled state, the operation of p-squeezing can produce approximately a maximally entangled state of two qubits. Then the analysis of how much key one can draw from the ccq state is much easier in the case of the p-squeezing state.

III. General form of states containing ideal key.

In this section we will provide general form of the states $\rho_{ABA'B'}$ which have $B$ key, i.e. states such that the outcomes of measurement in basis $B$ are both perfectly correlated and perfectly secure. It turns out that this is precisely the class of private states. Hence the definitions I and III are equivalent. We have the following theorem:

**Theorem 2:** Any state $\rho_{ABA'B'}$ of a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_{A'} \otimes \mathcal{H}_B \otimes \mathcal{H}_{B'}$ with dimensions $d_A = d_B = d$, $d_{A'}$ and $d_{B'}$, has $B$-key if and only if it is of the form

$$\rho_{ABA'B'} = \frac{1}{d} \sum_{i,j=0}^{d-1} |e_i f_i⟩⟨e_j f_j|_{AB} \otimes U_i \sigma_{A'B'} U_j^\dagger$$  

where the state $\sigma_{A'B'}$ is an arbitrary state of subsystem $A'B'$, $U_i$’s are arbitrary unitary transformations and $\{e_i \otimes f_j\} = B$.

We can rewrite the state (24) in the following, more appealing form

$$\rho_{ABA'B'} = UP_{AB}^+ \otimes \sigma_{A'B'} U^\dagger,$$  

where $P_{AB}^+$ is a projector onto the maximally entangled state $\psi_+ = \sum_i \frac{1}{\sqrt{d}} |e_i f_i⟩$, and $U$ is arbitrary twisting operation (7). Since the state $P_{AB}^+$ has many matrix elements vanishing, not all unitaries from definition of twisting are actually used here. In fact, unitaries $U_i$ in equation (24) are to be identified with unitaries $U_{kek}$ from equation (7). Note, that we can take $\sigma_{A'B'}$ to be “classically correlated” in the sense that it is diagonal in some product basis. Indeed, twisting can change the state $\sigma_{A'B'}$ into any other state having the same eigenvalues (simply, twisting can incorporate a unitary transformation acting solely on $A'B'$).

Thus we see that the states which have key, are closely connected with the maximally entangled state, which has been so far a "symbol" of quantum security. As we shall see, the maximally entangled state may get twisted so much, that after measurement in many bases of the $AB$ part the outcomes will be correlated with Eve, which is not the case for the maximally entangled state itself. Still, however the basis $B$ will remain secure. Note that here we deal with perfect security. We will later discuss approximate security in Section [V].

**Proof:** ($\Leftarrow$)

This part of the proof is a consequence of the theorem I. Namely a basic pdit (9) is obviously $B$-secure, because it has maximal correlations in this basis, and moreover it is a pure state, hence the one completely decoupled from Eve. More formally, it is evident that the ccq state of basic pdit is of the form (6). Now we can apply theorem I which says that after twisting the ccq state is unchanged. Hence any state of the form (24) has also $B$ key.

**Proof:** ($\Rightarrow$)

In this part we assume, that the state $\rho_{ABA'B'}$ has $B$-key i.e. that after measurement on it’s $AB$ part, one gets a perfectly correlated state (between Alice and Bob) that is uncorrelated with Eve:

$$\left(\sum_{i=0}^{d-1} \frac{1}{d} |e_i f_i⟩⟨e_i f_i|_{AB}\right) \otimes \rho_E.$$  

(26)
Let us consider general pure state for which dimensions of \( A, B \) are \( d, \) dimensions of \( A', B' \) are \( d_A, d_B \) respectively, and dimension of subsystem \( E \) is the smallest one which allows for the whole state being a pure one.

\[
|\psi\rangle = |\psi_{AB\prime A'B'E}\rangle = \sum_{ijklm} a_{ijklm} |e_i f_j klm\rangle.
\]

(27)

one can rewrite it as

\[
|\psi\rangle = \sum_{ij} |e_i f_j\rangle_{AB} \tilde{\psi}^{(ij)}_{A'B'E}.
\]

(28)

with \( |\tilde{\psi}^{(ij)}\rangle_{A'B'E} = \sum_{klm} a_{ijklm} |klm\rangle. \)

It is easy to see that the scalar product \( \langle \tilde{\psi}^{(ij)} | \tilde{\psi}^{(ij)} \rangle \) equals the probability of obtaining the state \(|e_i f_j\rangle_{AB}\) on the system \( AB \) after measurement in basis \( B. \) Now, since the subsystem \( \rho_{AB'E} \) (after measurement in \( B \) on \( AB \)) must be maximally correlated, the vectors \( |\tilde{\psi}^{(ij)}\rangle \) should satisfy \( \langle \tilde{\psi}^{(ij)} | \tilde{\psi}^{(ij)} \rangle = \frac{1}{d} \delta_{ij}. \) We can normalize these states (in case \( i = j \)) to have:

\[
|\psi^{(ii)}\rangle := \frac{|\tilde{\psi}^{(ii)}\rangle}{\sqrt{\langle \tilde{\psi}^{(ii)} | \tilde{\psi}^{(ii)} \rangle}} = \frac{|\tilde{\psi}^{(ii)}\rangle}{\sqrt{d}}
\]

(29)

so that the total state has a form:

\[
|\psi\rangle = \sum_{i=0}^{d-1} 1 \frac{1}{\sqrt{d}} |e_i f_i\rangle_{AB} |\psi^{(ii)}\rangle_{A'B'E}.
\]

(30)

"Cryptographical" interpretation of this state is the following: if Alice and Bob gets \( i \)-th result, then Eve gets subsystem \( \rho_i^E \) of a state \( |\psi^{(ii)}\rangle_{A'B'E}. \) Indeed, the ccq state is then of the form

\[
\rho_{ccq} = \sum_{i=0}^{d-1} \frac{1}{d} |e_i f_i\rangle_{AB} \rho_i^E \otimes \rho_i^E,
\]

(31)

with \( \rho_i = Tr_{A'B'}(|\psi^{(ii)}\rangle_{A'B'E} |\psi^{(ii)}\rangle). \) Now the condition (26) implies that, \( \rho_i^E \) should be all equal to each other. In particular, it follows that rank of Eve’s total density matrix is no greater than dimension of \( A'B' \) system, hence we can assume that \( d_E = d_A d_B = d'. \) It is convenient to rewrite this pure state in a form

\[
|\psi^{(ii)}\rangle_{A'B'E} = \sum_{k=0}^{d-1} |k\rangle_{A'B'} X_i |k\rangle_E,
\]

(32)

where \( \{|k\rangle\} \) is standard basis of \( A'B' \) and of \( E \) system, \( X_i \) is \( d_E \times d_E \) matrix that fully represents this state. It is easy to check, that \( \rho_i^E = X_i X_i^\dagger. \) Consider now singular value decomposition of \( X_i \) given by \( V_i \sqrt{\rho_i} U_i^\dagger \) where \( \rho_i \) is now diagonal in basis \( \{|k\rangle\}. \) One then gets that \( \rho_i^E = V_i \rho_i V_i^\dagger. \) The state (32) may be rewritten

\[
|\psi^{(ii)}\rangle_{A'B'E} = \sum_k X_i^T |k\rangle_{A'B'} |k\rangle_E,
\]

(33)

where \( T \) is transposition in basis \( \{|k\rangle\}. \) Now it is easy to check, that subsystem \( A'B' \) of \( |\psi^{(ii)}\rangle_{A'B'E} \) is in state \( X_i^T (X_i^T)^\dagger, \) so that the whole state \( \rho_{ABAB'} \) is the following:

\[
\rho_{ABAB'} = \frac{1}{d} \sum_{i,j=0}^{d-1} |e_i f_i\rangle \langle e_j f_j|_{AB} \otimes X_i^T (X_i^T)^\dagger.
\]

(34)

We can express this state using states accessible to Eve, namely \( \rho_j^E: \)

\[
\rho_{ABAB'} = \frac{1}{d} \sum_{i,j=0}^{d-1} |e_i f_i\rangle \langle e_j f_j|_{AB} \otimes (U_i^* V_i^T)^T \sqrt{\rho_j^E} V_i T_i^\dagger U_i^* V_i^T \sqrt{\rho_j^E} T_i^\dagger.
\]

(35)

(For expressing state in terms of Eve’s states in more general case, see Section [VI]). Denoting by \( W_i \) the unitary transformation \( U_i^* V_i^T \) one gets:

\[
\rho_{ABAB'} = \frac{1}{d} \sum_{i,j=0}^{d-1} |e_i f_i\rangle \langle e_j f_j|_{AB} \otimes W_i \sqrt{\rho_j^E T_i^\dagger} \cdot \sqrt{\rho_j^E T_j^\dagger} W_j^T.
\]

(36)
However, as mentioned above, Eve’s density matrices are equal to each other, i.e. \( \rho_i^E = \rho_j^E \) for all \( i, j \). We then obtain
\[
\rho_{ABA'B'} = \frac{1}{d} \sum_{i,j=0}^{d-1} |e_i f_i \rangle \langle e_j f_j |_{AB} \otimes W_i \rho W_j^\dagger_{A'B'}.
\] (37)

This completes the proof of theorem[2].

IV. Pdits and their properties

In this section we will present various forms of pdits and pbits. We will first write the pbit in matrix form according to its original definition. We can write it in block form
\[
\gamma_{ABA'B'}^{(2)} = \frac{1}{2} \begin{bmatrix} U_0 T_{ABA} & 0 & 0 & U_0 T_{ABA} U_1^\dagger \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ U_1 T_{ABA} U_1^\dagger & 0 & 0 & U_1 T_{ABA} U_1^\dagger \end{bmatrix},
\] (38)

where \( \sigma_{A'B'} \) is arbitrary state on \( A'B' \) subsystem, and \( U_0 \) and \( U_1 \) are arbitrary unitary transformations which act on \( A'B' \). "Generalized EPR form" of pbit. Since by the theorem of the previous section pdits are the only states that contain \( B \)-key, they could be called generalized EPR states (maximally entangled state). We have already seen that they are "twisted EPR states with operator amplitudes". One can notice an even closer connection. Namely, a pbit can be viewed as an EPR states with operator amplitudes. Indeed, one can rewrite equation (34) in a more appealing form
\[
\gamma_{ABA'B'}^{(d)} = \Psi \Psi^\dagger,
\] (39)

with
\[
\Psi = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} Y_i A'B' \otimes |e_i f_i \rangle_{AB}.
\] (40)

We have written here (unlike in the rest of the paper) first the \( A'B' \) system and then the \( AB \) one, so that this form of pbit would recall a form of pure state. Thus instead of \( c \)-numbers the amplitudes are now \( q \)-numbers, so that states which have key are "second quantized EPR states". In the case of pbits, the matrix form is the following:
\[
\gamma_{ABA'B'}^{(2)} = \frac{1}{2} \begin{bmatrix} Y_0 Y_0^\dagger & 0 & 0 & Y_0 Y_1^\dagger \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ Y_1 Y_0^\dagger & 0 & 0 & Y_1 Y_1^\dagger \end{bmatrix}.
\] (41)

Let us consider the polar decomposition of operators \( Y_i \). From definition of pbit it follows that the only constraint on these operators is the following
\[
\forall i, Y_i = U_i \sqrt{\rho},
\] (42)

where \( U_i \) is unitary transformation and \( \rho \) is a normalized state as so is the \( \sigma_{A'B'} \) state in form (38).

This reflects the fact, that similarly like maximally entangled state which produces correlated outputs has amplitudes with probably different phases, but the same modulus, the "maximally private" state can have "operator amplitudes" which differ by \( U_i \) (a counterpart of the phase) but have the same \( \sqrt{\rho} \) in polar decomposition (which is a counterpart of the modulus).

There is yet another similarity to EPR states, namely the norm of upper-right block \( Y_0 Y_1^\dagger \) is equal to \( \frac{1}{2} \), like the modulus of the coherence of the EPR state. "X-form" of pbit. In special case of pbits one can have representation by just one normalized operator:
\[
\gamma_{ABA'B'}^{(2)} = \frac{1}{2} \begin{bmatrix} X^\dagger & 0 & 0 & X \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ X^\dagger & 0 & 0 & \sqrt{X^\dagger X} \end{bmatrix},
\] (43)

for any operator \( X \) satisfying \( ||X|| = 1 \).

Justification of equivalence of this form and standard form is the following. Consider singular value decomposition of \( X \), \( X = USW \) with \( U \) and \( W \) unitary transformations and \( \sigma \) being diagonal, positive matrix. Since \( X \) has trace norm 1, the same is for \( \sigma \), therefore it can be viewed as \( X = U \rho W \) with \( \rho \) being a legitimate state. Identifying \( U_0 = U \) and \( U_1 = W^\dagger \) we obtain standard form.

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It is important, that in nontrivial cases \( X \) should be a non-positive operator. Otherwise the pbit is equal to basic pbit. Indeed, if it is positive, then since its trace norm is 1, it is itself a legitimate state, call it \( \rho \). Then \( \sqrt{XX^\dagger} = \sqrt{X^\dagger X} = \rho \), so that

\[
\rho_{ABA' B'} = \frac{1}{2} \sum_{i,j=0}^1 |ii\rangle\langle jj| \otimes \rho = |\psi_+\rangle\langle \psi_+| \otimes \rho.
\]

which is the basic pbit [9].

Note, that in higher dimension to have the \( X \)-form we need more than one operator, and the operators depend on each other, which is not as simple representation as in the case of pbit. For example in \( d = 3 \) case we have:

\[
\begin{bmatrix}
\sqrt{XX^\dagger} & 0 & 0 & 0 & X & 0 & 0 & 0 & XY \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
X^\dagger & 0 & 0 & 0 & \sqrt{X^\dagger X} & 0 & 0 & 0 & Y \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(XY)^\dagger & 0 & 0 & 0 & Y^\dagger & 0 & 0 & 0 & \sqrt{Y^\dagger Y} \\
\end{bmatrix},
\]

where the operators \( X \) and \( Y \) satisfy: \( ||X|| = 1 \) and \( X = W Y^\dagger \) for arbitrary unitary transformation \( W \).

"Flags form": special case of \( X \)-form. If the operator \( X \) which represents pbit in its \( X \)-form is additionally hermitian, any such pbit can be seen as a mixture of basic pbit and a variation of basic pbit which has EPR states with different phase:

\[
\gamma^{(2)}_{ABA' B'} = p|\psi_+\rangle\langle \psi_+| \otimes \rho^{+}_{A' B'} + (1 - p)|\psi_-\rangle\langle \psi_-| \otimes \rho^{-}_{A' B'},
\]

where \( |\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \). Derivation of this form is straightforward, if we consider decomposition of \( X \) into positive and negative part [9]:

\[
X = X_+ - X_-,
\]

where \( X_+ \) and \( X_- \) are by definition orthogonal, and positive. Thus denoting \( p = \text{Tr} X_+ \), together with assumption of \( X \)-form that \( ||X|| = \text{Tr}\sqrt{X X^\dagger} = 1 \), we can rewrite \( X \) as

\[
X = pp_+ - (1 - p)p_-,
\]

where \( p_\pm \) are normalized positive and negative parts of \( X \). Moreover, since the states \( \rho_+ \) and \( \rho_- \) are orthogonal: \( \text{Tr} \rho_- \rho_+ = 0 \), we obtain the form [43].

### A. Private bits - examples

We will give now two examples of private bits, and study its entanglement distillation properties.

**Examples of pbit**

1) Let us consider state \( \gamma^V \in B(C^2 \otimes C^2 \otimes C^d \otimes C^d) \) of the following form:

\[
\gamma^V = \frac{1}{2} \begin{bmatrix}
I_d & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{V}{d} \\
0 & 0 & \frac{1}{d} & \frac{V}{d} \\
\end{bmatrix},
\]

where \( V \) is the swap operator which reads: \( V = \sum_{i=0}^{d-1} |ij\rangle\langle ji| \). If we consider positive and negative part of \( V \), which are symmetric and antisymmetric subspace, it is easy to see, that

\[
\gamma^V = p|\psi_+\rangle\langle \psi_+| \otimes \rho_s + (1 - p)|\psi_-\rangle\langle \psi_-| \otimes \rho_a
\]

where

\[
\rho_s = \frac{2}{d^2 + d} P_{\text{sym}} \quad \rho_a = \frac{2}{d^2 + d} P_{\text{asym}}
\]

are symmetric and antisymmetric Werner states, and \( p = \frac{1}{2}(1 + \frac{1}{d}) \). Thus we have obtained, that it is also a pbit with natural "flags form", with flags being (orthogonal) Werner states [58].

2) The second example is the state known as "flower state", which was shown [31] to lock entanglement cost. We have that \( \rho_{\text{flower}} \in B(C^2 \otimes C^2 \otimes C^{d^2} \otimes C^{d^2}) \) is of the form:

\[
\gamma_{\text{flower}} = \frac{1}{2} \begin{bmatrix}
\sigma & 0 & 0 & \frac{1}{d}U^T \\
0 & 0 & 0 & 0 \\
\frac{1}{d}U^* & 0 & 0 & \sigma \\
\end{bmatrix},
\]

where
where $\sigma$ is classical maximally correlated state: $\sigma = \sum_{i=0}^{d-1} \frac{1}{d} |ii\rangle\langle ii|$, and $U$ is the embedding of unitary transformation $W = \sum_{i,j=0}^{d-1} w_{ij} |i\rangle\langle j| = H^{\otimes d}$ with $H$ being Hadamard transform in the following way:

$$U = \sum_{i,j=0}^{d-1} w_{ij} |ii\rangle\langle jj|.$$  

We can check now, that this state is pbit with $X$-form. In this case $X = U^T$. To see this consider unitary transformation $S := U^* + \sum_{i \neq j} |ij\rangle\langle ij|$. Composing $S$ with $U^T$ does not change the norm, which is unitarily invariant, so that

$$||\frac{1}{d} U^T|| = ||\frac{1}{d} U^T S|| = ||\frac{1}{d} \sum_{i=0}^{d-1} |ii\rangle\langle ii|| = 1.$$  

(52)

Thus we see, that $||X|| = 1$. We have also $\sqrt{XX^T} = \sigma$:

$$\sqrt{\frac{1}{d^2} U^T U^*} = [\frac{1}{d^2} \sum_{i=0}^{d-1} |ii\rangle\langle ii|]^{\frac{1}{2}} = \sigma.$$  

(53)

We will show now, that in case of $\gamma^V$ given in eq. (48), the distillable entanglement $E_D$ is strictly smaller then the amount of secure key $K_D$ gained from these states. The formal definition of $K_D$ is given in Section VIII. Here it is enough to base only on its intuitive properties. Namely, any pdit by its very definition has $\gamma$ of secure key.

Lemma 3: For any pbit in $X$-form, if $\sqrt{XX^T}$ and $\sqrt{X^T}X$ are PPT, the log negativity of the pbit in $X$-form reads $E_N = \log(1 + ||X^\Gamma||)$, where $\Gamma$ is transposition performed on the system $B'$.  

The proof of this lemma is given in Sec. X of Appendix. Using this lemma, one can check the negativity of the state $\gamma^V$. We have in this case $X = \frac{1}{\sqrt{d}}$, with $d \geq 2$. Since $V^T = dP_+$, we obtain $E_N(\gamma^V) = \log(1 + \frac{1}{d})$. It implies:

$$E_D(\gamma^V) \leq E_N(\gamma^V) = \log(1 + \frac{1}{d}) < 1 \leq K_D(\gamma^V),$$  

(54)

which demonstrates a desired gap between distillable key and distillable entanglement:

$$E_D(\gamma^V) < K_D(\gamma^V).$$  

(55)

B. Relative entropy of entanglement and pdits

In this section we will consider the entanglement contents of the pbit in terms of a measure of entanglement called relative entropy of entanglement, defined as follows:

$$E_r(\rho) = \inf_{\sigma_{sep} \in SEP} S(\rho|\sigma_{sep})$$  

(56)

where $S(\rho|\sigma) = -S(\rho) - Tr\rho \log \sigma$ is the relative entropy, and $SEP$ is the set of separable states. In the Section XI we will show, that for any state, the relative entropy of entanglement is an upper bound on the key rate, that can be obtained from the state (for generalizations of this result to a wide class of entanglement monotones see [13], [14]). It is then easy to see, that for any pbit $\gamma$, $E_r(\gamma)$ is greater than $\log d$ since $K_D(\gamma) \geq \log d$ by definition of pdits. The question we address here, is the upper bound on the relative entropy of the pdit. We relate its value to the states which appear on the shield of the pdits, when Alice and Bob get key by measuring the key part of the pdit. The theorem below states it formally.

Theorem 3: For any pdit $\gamma_{ABA'B'} \in B(C^d \otimes C^d \otimes C^{d_A} \otimes C^{d_B})$, which is secure in standard basis, let $\rho_{ABA'}^{(i)}$, denote states which appears on shield of the pbit, after obtaining outcome $ii$ in measurement performed in standard basis on its key part. Then we have

$$E_r(\gamma_{ABA'B'}) \leq \log d + \frac{1}{d} \sum_{i=0}^{d-1} E_r(\rho_{ABA'}^{(i)})$$  

(57)

where $AB$ denotes key and $A'B'$ shieldpart of the pbit.

Proof: One can view the quantity $\frac{1}{d} \sum_{i=0}^{d-1} E_r(\rho_{ABA'}^{(i)})$ as the relative entropy of $\gamma_{ABA'B'}$ dephased on $AB$ in computational basis [31]. In case $d = 2^k$, it can be easily done with applying unitary $U_i$ - random sequence of $\sigma_x$ and $I$ unitary transformations. In general case one can use the so called Weyl unitary operators (see e.g. [27]). Such an implementation of dephasing uses $\log d$ bits of randomness.
Following the proof of non-lockability of relative entropy of entanglement [31] (see also [42]), we can write

\[ E_r(\gamma_{AB'A'B'}) - E_r(\sum_i p_i \sigma_i) \leq \log d \] (58)

where \( \sigma_i = U_i \otimes I_{AB'} \gamma_{AB'A'B'} U_i^\dagger \otimes I_{AB'} \) and \( p_i = \frac{1}{d} \). As we have observed above, the relative entropy of dephased state \( \sum_i p_i \sigma_i \) equals \( \frac{1}{d} \sum_{i=0}^{d-1} E_r(\rho_{AB'}^{(i)}) \) which ends the proof.

The above theorem is valid also for regularized relative entropy, defined as [21]

\[ E_r(\rho)^\infty = \lim_{n \to \infty} \frac{1}{n} E_r(\rho^\otimes n). \] (59)

**Theorem 4:** Under the assumptions of theorem 3 there holds:

\[
E_r^\infty(\gamma_{AB'A'B'}) \leq \log d + \frac{1}{d} \sum_{i=0}^{d-1} E_r^\infty(\rho_{AB'}^{(i)}),
\] (60)

For the proof of this theorem, as rather technical, we refer the reader to Appendix XV-F.

**C. Irreducible pdit - a unit of privacy**

In Section XI we have characterized states which contain ideal key i.e. pdits. A pdit has an AB subsystem called here the key part. \( \log d \) bits of key can be obtained from such a pdit by a complete measurement in some basis performed on this key part of pdit. However, as it follows from the characterization given in theorem 2, pdits have also the A'B' subsystem, called here the shield. This part can also serve as a source of key. Indeed there are plenty of such pdits that contain more than \( \log d \) key, due to their shield. Therefore not every pdit can serve as a unit of privacy and we need the following definition:

**Definition 5:** Any pdit \( \gamma \) (with \( d \)-dimensional key part) for which \( K_D(\gamma) = \log d \) is called irreducible.

This definition distinguishes those pdits for which measuring their key part is the optimal protocol for drawing key. They are called irreducible in opposite to those, which can be reduced by distillation protocol to some other pdits which have more than \( \log d \) of key. Irreducible pdits are by definition units of privacy (although they are not generally interconvertible).

Determining the class of irreducible pdits is potentially a difficult task, as it leads to optimisation over protocols of key distillation. However we are able to show a subclass of pdits, which are irreducible. To this end we use a result, which is proven in Section XI, namely that the relative entropy of entanglement is an upper bound on distillable key. Having this we can state the following proposition:

**Proposition 1:** Any pdit \( \gamma \), with \( E_r(\gamma) = \log d \), is irreducible.

**Proof:** By definition of pdit we have \( K_D(\gamma) \geq \log d \) and by theorem 2 from Section XI we have \( K_D(\gamma) \leq E_r(\gamma) \) which is in turn less than \( \log d \) by assumption, and the assertion follows. We can provide now a class of pdits which have \( E_r = \log d \) and by the above proposition are irreducible. These are pdits which have separable states that appear on shield conditionally on outcomes of complete measurement on key part in the computational basis.

**Proposition 2:** For any pdit \( \gamma_{AB'A'B'} \in B(C^d \otimes C^d \otimes C^d \otimes C^d) \), which is secure in standard basis, if \( \rho_{AB'}^{(i)} \) denote states which appear on shield of the pbit, after obtaining outcome \( |i\rangle \langle i| \) in measurement performed in standard basis on its key part are separable states, then pdit \( \gamma_{AB'A'B'} \) is irreducible.

**Proof:** Due to bound on relative entropy of pdit given in theorem 3 we have that \( E_r(\gamma) \) is less then or equal to \( \log d \) since conditional states \( \rho_{AB'}^{(i)} \) are separable and hence have relative entropy of entanglement equal to zero. \( E_r(\gamma) \) is also not less then \( \log d \), since it is greater than the amount of distillable key, which ends the proof.

Note, that examples [45], [47] given in Section XV-A fulfill the assumptions of this theorem, and are therefore irreducible pbits. They are also the first known non trivial states (different than pure state) for which the amount of distillable key has been calculated. Using the bound of relative entropy on distillable key, one can also show, that the class of maximally correlated states has \( K_D = E_D = E_r \), since for the latter \( E_D = E_r \).

**V. APPROXIMATE PBITS**

We present here a special property of states which are close to pbit. We have already seen, that pbits have similar properties to the maximally entangled EPR states. In particular, the norm of the upper-right block in standard form as well as in X-form of pbit is equal to \( \frac{1}{2} \). We will show here, that for general states the norm of that block tells how close the state is to a pbit: any state which is close in trace norm to pbit must have the norm of this block close to \( \frac{1}{2} \) and vice versa.

We will need the following lemma that relates the value of coherence to the distance from the maximally entangled state for two qubit states.

**Lemma 4:** For any bipartite state \( \rho_{AB} \in B(C^2 \otimes C^2) \) expressed on the form \( \rho_{AB} = \sum_{ijkl=0}^{1} a_{ijkl} |ij\rangle \langle kl| \) we have:

\[
\text{Tr} \rho_{AB} P_+ \geq 1 - \epsilon \Rightarrow Re(a_{0011}) > \frac{1}{2} - \epsilon
\] (61)
Re(a_{0011}) > \frac{1}{2} - \epsilon \Rightarrow Tr \rho_{AB} P_+ \geq 1 - 2\epsilon \quad (62)

Proof: For the proof of this lemma, see Appendix XV.G.

We can prove now that approximate pbits have norm of an appropriate block close to \( \frac{1}{2} \).

Proposition 3: If the state \( \sigma_{ABA'B'} \in B(C^2 \otimes C^2 \otimes C^d \otimes C^d) \) written in the form \( \sigma_{ABA'B'} = \sum_{ijkl=0}^1 |ij\rangle \langle kl| \otimes A_{ijkl} \) fulfills

\[ ||\sigma_{ABA'B'} - \gamma_{ABA'B'}|| \leq \epsilon \quad (63) \]

for some pbit \( \gamma \), then for \( 0 < \epsilon < 1 \) there holds \( ||A_{0011}|| \geq \frac{1}{2} - \epsilon \).

Proof: The pbit \( \gamma \) is a twisted EPR state, which means that there exists twisting \( U \) which applied to basic pbit \( P_+ \otimes \rho \) gives \( \gamma \). We apply this \( U \) to both states \( \sigma_{ABA'B'} \) and \( \gamma \) and trace out the \( A'B' \) subsystem of both of them. Since these operations can not increase the norm distance between these states, so that we have for \( \sigma_{AB} = Tr_{A'B'} U \sigma_{ABA'B'} U^\dagger \)

\[ ||\sigma_{AB} - P_+|| \leq \epsilon. \quad (64) \]

It implies, by equivalence of norm and fidelity (11) that

\[ F(\sigma_{AB}, P_+) \geq 1 - \frac{1}{2} \epsilon. \quad (65) \]

We have also that \( F(\sigma_{AB}, P_+)^2 = Tr \sigma_{AB} P_+ \) so that

\[ Tr \sigma_{AB} P_+ > 1 - \epsilon \quad (66) \]

for \( \epsilon < 1 \). Now by lemma (4) this yields \( |a_{0011}| \geq Re(a_{0011}) \geq \frac{1}{2} - \epsilon \), where \( a_{0011} \) is coherence of the state \( \rho_{AB} = \sum_{ijkl=0}^1 a_{ijkl} |ij\rangle \langle kl| \). However, we have

\[ |a_{0011}| = |Tr U_{00} A_{0011} U_{11}^\dagger| \quad (67) \]

where \( U_{00} \) and \( U_{11} \) come from twisting, that we have applied. Using now the fact that ||A|| = sup_U Tr A U, where supremum is taken over unitary transformations we get

\[ ||A_{0011}|| \geq |a_{0011}| \geq 1 - \epsilon. \quad (68) \]

This ends the proof.

Now we will formulate and prove the converse statement, saying that when the norm of the right upper block is close to \( 1/2 \), then the state is close to some pbit.

Proposition 4: If the state \( \sigma_{ABA'B'} \in B(C^2 \otimes C^2 \otimes C^d \otimes C^d) \) with a form \( \sigma_{ABA'B'} = \sum_{ijkl=0}^1 |ij\rangle \langle kl| \otimes A_{ijkl} \) fulfills

\[ ||A_{0011}|| > \frac{1}{2} - \epsilon \]

then for \( 0 < \epsilon < \frac{1}{2} \) there exists pbit \( \gamma \) such that

\[ ||\sigma_{ABA'B'} - \gamma_{ABA'B'}|| \leq \delta(\epsilon) \quad (69) \]

with \( \delta(\epsilon) \) vanishing, when \( \epsilon \) approaches zero. More specifically,

\[ \delta(\epsilon) = 2 \sqrt{8\sqrt{2\epsilon} + h(2\sqrt{2\epsilon}) + 2\sqrt{2\epsilon}} \quad (70) \]

with \( h(.) \) being the binary entropy function \( h(x) = -x \log x - (1-x) \log (1-x) \).

Proof: In this proof by \( \rho_X \) we denote respective reduced density matrix of the state \( \rho_{ABA'B'} \). Let \( \rho_{AB} \) be the privacy-squeezed state of the state \( \sigma_{ABA'B'} \) i.e. \( \rho_{AB} = Tr_{A'B'} \rho_{ABA'B'} \) where \( \rho_{ABA'B'} = U_{ps} \sigma_{ABA'B'} U_{ps}^\dagger \) for certain twisting \( U_{ps} \). By definition the entry \( a_{0011} \) of \( \rho_{AB} \) is equal to \( ||A_{0011}|| \). By assumption we have, \( a_{0011} = ||A_{0011}|| > \frac{1}{2} - \epsilon \). By lemma 4 (equation (62)) we have that

\[ Tr \rho_{AB} P_+ > 1 - 2\epsilon. \quad (71) \]

We have then

\[ F(\rho_{AB}, P_+)^2 = Tr \rho_{AB} P_+ \quad (72) \]

which, by equivalence of norm and fidelity (13) gives

\[ ||\rho_{AB} - P_+|| \leq 2\sqrt{2\epsilon}. \quad (73) \]

Let us now consider the state \( \rho_{ABA'B'} = U_{ps} \sigma_{ABA'B'} U_{ps}^\dagger \) and its purification to Eve’s subsystem \( \psi_{ABA'B'E} \) so that we have:

\[ \rho_{AB} = Tr_{A'B'E}(\psi_{ABA'B'E}) \quad (74) \]

By the Fannes inequality (see eq. (14) in Sec. II.A) we have that

\[ S(\rho_{AB}) = S(\rho_{A'B'E}) \leq 4\sqrt{2\epsilon} \log d_{AB} + h(2\sqrt{2\epsilon}). \quad (75) \]
From this we will get that \( \| \psi_{ABA'B'} - \rho_{AB} \otimes \rho_{A'B'} \| \) is of order of \( \epsilon \). We prove this as follows. Since norm distance is bounded by relative entropy as follows [45]

\[
\frac{1}{2} \| \rho_1 - \rho_2 \|^2 \leq S(\rho_1|\rho_2),
\]

one gets:

\[
\| \psi_{ABA'B'} - \rho_{AB} \otimes \rho_{A'B'} \| \leq \sqrt{2S(\psi_{ABA'B'}|\rho_{AB} \otimes \rho_{A'B'})}. \tag{77}
\]

The relative entropy distance of the state to it’s subsystems is equal to quantum mutual information

\[
I(\psi_{AB'|A'B'}) = S(\rho_{AB}) + S(\rho_{A'B'}) - S(\psi_{ABA'B'}), \tag{78}
\]

We will henceforth use shorthand notation \( I(X : Y), S(X) \), which gives

\[
I(AB : A'B') = 2S(AB) \leq 2(4\sqrt{2}\epsilon \log d_{AB} + h(2\sqrt{2}\epsilon)). \tag{79}
\]

where last inequality comes from Eq. (75). Coming back to inequality (77) we have that

\[
\| \psi_{ABA'B'} - \rho_{AB} \otimes \rho_{A'B'} \| \leq \sqrt{2I(AB : A'B')} \leq 2\sqrt{4\epsilon \log d_{AB} + h(2\sqrt{2}\epsilon)} \tag{80}
\]

If we trace out the subsystem \( E \) the inequality is preserved:

\[
\| \rho_{ABA'B'} - \rho_{AB} \otimes \rho_{A'B'} \| \leq 2\sqrt{8\epsilon + h(2\sqrt{\epsilon)}}, \tag{81}
\]

where we have put \( d_{AB} = 4 \), as we deal with pbts. Now by triangle inequality one has:

\[
\| \rho_{ABA'B'} - \rho_{AB} \otimes \rho_{A'B'} \| \leq \| \rho_{ABA'B'} - P_+ \otimes \rho_{A'B'} \| + \| \rho_{AB} \otimes \rho_{A'B'} - P_+ \otimes \rho_{A'B'} \|. \tag{82}
\]

We can apply now the bounds (73) and (81) to the above inequality obtaining

\[
\| \rho_{ABA'B'} - P_+ \otimes \rho_{A'B'} \| \leq 2\sqrt{8\epsilon + h(2\sqrt{\epsilon}) + 2\sqrt{2\epsilon}}. \tag{83}
\]

Let us now apply the twisting \( U_{ps}^\dagger \) (transformation which is inverse to twisting \( U_{ps} \)) to both states on left-hand-side of the above inequality. Since \( \rho_{ABA'B'} \) is defined as \( U_{ps} \sigma_{ABA'B'} U_{ps}^\dagger \) we get that:

\[
\| \sigma_{ABA'B'} - U_{ps}^\dagger P_+ \otimes \rho_{A'B'} U_{ps} \| \leq 2\sqrt{8\epsilon + h(2\sqrt{\epsilon}) + 2\sqrt{2\epsilon}}, \tag{84}
\]

i.e. our state is close to pbpt \( \gamma = U_{ps}^\dagger P_+ \otimes \rho_{A'B'} U_{ps} \). Then the theorem follows with \( \delta(\epsilon) = 2\sqrt{8\epsilon + h(2\sqrt{\epsilon}) + 2\sqrt{2\epsilon}} \).

Remark 2: The above propositions establish the norm of upper-right block of matrix (written in computational basis according to ABA’ order of subsystems), as a parameter that measures closeness to pbpt, and in this sense it measures security of the bit obtained from the key part. The state of form (21) is close to a pbpt if and only if the norm of this block is close to \( \frac{1}{\epsilon} \). This is the property of approximate pbpts, however it seems not to have an analogue for approximate pdts with \( d \geq 3 \).

VI. EXPRESSING ALICE AND BOB STATES IN TERMS OF EVE’S STATES

In this section we will express the state \( \rho_{ABA'B'} \) in such a way that one explicitly sees Eve’s states in it. We will then interpret the results of the previous sections in terms of such a representation. In particular, we will see that the norm of the upper-right block not only measures closeness to pbpt, but it also measures the security of the bit from the key part directly, in terms of fidelity between corresponding Eve’s states.

A. The case without shield. "Abelian" twisting.

Consider first the easier case of a state without shield i.e.

\[
\rho_{AB} = \sum_{ij, ij'} \rho_{ij|ij'} |ij\rangle \langle ij'|. \tag{85}
\]

Purification of this state is of the following form

\[
\psi_{ABE} = \sum_{ij} \sqrt{\rho_{ij}} |ij\rangle \rho_{AB} |ij\rangle_E \tag{86}
\]

where \( \rho_{ij} = \rho_{ijij} \). We see, that when Alice and Bob measure the state in basis \( |ij\rangle \), Eve’s states corresponding to outcomes \( ij \) are \( \psi_{ij} \), and they occur with probabilities \( p_{ij} \). Performing partial trace over Eve’s system, one obtains

\[
\rho_{AB} = \sum_{ij, i'j'} \sqrt{p_{ij}p_{i'j'}} \langle \psi_E^{ij}\psi_E^{i'j'}|ij\rangle \langle i'j'|. \tag{87}
\]
Thus the matrix elements of $\rho_{AB}$ are inner products of Eve’s states. If we have all inner products between set of states, we have complete knowledge about the set, up to a total unitary rotation, which is irrelevant for security issues (since Eve can perform this herself). Thus density matrix $\rho_{AB}$ can be represented in such a way that all properties of Eve’s states are explicitly displayed. Moreover, moduli of matrix elements are related to fidelity between Eve’s states:

$$|\rho_{ijj'}| = \sqrt{p_{ij}p_{j'i'}} F(\psi_E^{ij},\psi_E^{j'i'})$.\tag{88}$$

a) Two-qubit case.: For example, for two qubits, the density matrix looks as follows (we have not shown all elements)

$$\rho_{AB} = \begin{pmatrix} p_{00} & \times & \times & \sqrt{p_{00}p_{11}}(\psi_E^{11}|\psi_E^{00}\rangle \\
\times & p_{01} & \times & \times \\
\times & \times & p_{10} & \times \\
\times & \times & \times & p_{11} \end{pmatrix}$.\tag{89}$$

Let us now consider the conditions for having one bit of perfect key obtained from the measurement in the two qubit case. They are as follows: (i) $p_{00} = p_{11} = 1/2$ and (ii) $\psi_{00} = \psi_{11}$ up to a phase factor. The latter condition is equivalent to $F(\psi_{00},\psi_{11}) = 1$ (we have dropped here the index $E$). The two conditions can be represented by a single condition:

$$\sqrt{p_{00}p_{11}} F(\psi_{00},\psi_{11}) = \frac{1}{2}$.\tag{90}$$

However, we know from (88) that this means that upper-right matrix element of $\rho_{AB}$ should satisfy $|\rho_{0011}| = 1/2$. Consider now approximate bit of key, so that the conditions are satisfied up to some accuracy. Again we can combine them into single condition

$$\sqrt{p_{00}p_{11}} F(\psi_{00},\psi_{11}) > \frac{1}{2} - \epsilon$.\tag{91}$$

This translates into

$$|\rho_{0011}| \geq \frac{1}{2} - \epsilon$,\tag{92}$$

which gives:

$$F^2 \leq \frac{1}{2}(1 + 2|\rho_{0011}|)$$.\tag{93}

where $\psi_{ME} = \frac{1}{\sqrt{2}}(e^{i\phi_{00}}|00\rangle + e^{i\phi_{11}}|11\rangle)$, and $F^2 = (\psi_{ME}|\rho|\psi_{ME}\rangle$.

Thus, in particular for $F = 1$, we must have $|\rho_{0011}| = 1/2$. The change of phases can be viewed as a unitary operation, where phases are controlled by the basis $|ij\rangle$:

$$U = \sum_{ij} |ij\rangle\langle ij| e^{i\phi_{ij}}$$.\tag{94}

Since we have $\psi_{\text{max}} = U|\psi_+\rangle$, this operation can be called "abelian" twisting. Abelian because only phases are controlled. Thus we can summarize our considerations by the following statement. A two-qubit state has perfectly secure one bit of key with respect to basis $|ij\rangle$, if and only if it is a twisted EPR state (by abelian twisting of Eq. (54)):

$$\rho_{AB} = U|\psi_+\rangle\langle \psi_+|U^\dagger$.\tag{95}

Moreover, if a state satisfies security condition approximately, it must be close in fidelity to some state $U\psi_+$. The quality of the bit of key is given by magnitude of a c-number $|\rho_{0011}|$.

B. The general case.

In this section we will represent in terms of Eve’s states the state which has both key part and shield. We will see then, how the twisting becomes "nonabelian", and the condition of closeness to pure state $U\psi_+$ changes into that of closeness to pbit. If we write state in basis of system $AB$ (key part) we get blocks $A_{ijj'}$ instead of matrix elements

$$\rho_{ABA'B'} = \sum_{ijj'} |ij\rangle_A |i'j'\rangle_B \otimes A_{ijj'}^{i'j'}$.\tag{96}$$

After suitable transformations (see Appendix XV-A for details), we arrive at the following form:

$$\rho_{ABA'B'} = \sum_{ijj'} \sqrt{p_{ij}p_{j'i'}} |ij\rangle_A |i'j'\rangle_B \otimes (U_{ij} \sqrt{\rho_E^{ij}} \sqrt{\rho_E^{j'i'}} U_{ij}^\dagger)^T$,\tag{97}$$

where the operator $U_{ij}^\dagger \equiv U_{ij} W V_{ij}$ maps the space $\mathcal{H}_{A'B'}$ exactly onto a support of $\rho_E^{ij}$ in space $\mathcal{H}_E$, and the dual operator $U_{ij} = V_{ij}^\dagger W^\dagger U_{ij}^\dagger$ maps the support of $\rho_E^{ij}$ back to $\mathcal{H}_{A'B'}$.

Let us note, that in parallel to Eq. (55) we have that the trace norms of the blocks $A$ are connected with fidelities between Eve’s states

$$||A_{ijj'}|| = \sqrt{p_{ij}p_{j'i'}} F(\rho_E^{ij},\rho_E^{j'i'})$.\tag{98}$$
b) The case of two qubit key part: If the key part is two qubit system we get

\[
\rho_{ABA'B'} = \left[ p_{00} |\mathbb{U}_{00} X_0^{00} X_0^{00} |^T \times p_{01} |\mathbb{U}_{01} X_0^{01} X_0^{01} |^T \times \sqrt{p_{00} p_{11}} |\rho_{E}^{00} \rho_{E}^{01} |^T \times p_{10} |\mathbb{U}_{10} X_0^{10} X_0^{10} |^T \times \sqrt{p_{01} p_{10}} |\mathbb{U}_{11} X_0^{11} X_0^{11} |^T \right]
\]

(99)

Let us now discuss conditions for presence of one bit of key. They are again (i) \( p_{00} + p_{11} = \frac{1}{2} \) and (ii) Eve’s states are the same \( \rho_{E}^{00} = \rho_{E}^{11} \). This is equivalent to

\[
\sqrt{p_{00} p_{11}} F(\rho_{E}^{00}, \rho_{E}^{11}) = \frac{1}{2}
\]

(100)

which is nothing but trace norm of upper-right block \( ||A_{0011}|| \). Also conditions for approximate bit of key requires the norm to be close to \( \frac{1}{2} \). Moreover, to see how pbit and the twisting arise, let us put all Eve’s states equal to each other, and probabilities corresponding to perfect correlations. We then obtain

\[
\rho_{ABA'B'} = \frac{1}{d} \sum_{ij} |ii\rangle_{AB} \langle jj| \otimes |\mathbb{U}_{ii} \rho_{E} \mathbb{U}_{jj}^\dagger|^T.
\]

(101)

where \( \rho_{E} \) is one fixed state, that Eve has irrespectively of outcomes. We see here almost the form of pbit. One difference might be is that instead of usual unitaries, we have some embeddings \( \mathbb{U}_{ii} \). However, since now Eve’s space is of the same dimension as \( A'B' \) (because Eve has single state), they are actually usual unitaries. The transposition does not really make a difference, as it can be absorbed both by state, and by unitaries. It is interesting to see here in place of phases from previous section the unitaries appeared, so that abelian twisting changed into nonabelian one. Also the condition for key changed from modulus of c-number - matrix element, to a trace norm of q-number - a block.

VII. OVERVIEW

In this section we will shortly summarize what we have done so far. Then we will describe the goals of the paper, and briefly outline how we will achieve them.

A. Pbits and twisting

We have considered a state shared by Alice and Bob, which was divided into two parts: the key part \( AB \) and the shield \( A'B' \). The key part is measured in a local basis, while the shield is kept. The latter is seen by Eve as an environment that may restrict her knowledge about outcomes of measurement performed on the pdit.

We have shown two important facts. First, we have characterized all the states for which measurement on the key part gives perfect key. The states are called pdits, and they have a very simple form. Moreover, we have shown that twisting does not change the ccq state arising from measurement on the key part. (We should emphasize here, that twisting must be controlled by just the same basis in which the measurement is performed.)

This is an interesting feature, because twisting may be a nonlocal transformation. Thus even though we apply a nonlocal transformation to the state, the quality of the key established by measuring the key part (in the same basis) does not change.

From the exhibited examples of pbits, we have seen that some of them have very small distillable entanglement. Since pbits are EPR states subjected to twisting, we see that in this case the twisting must have been very nonlocal, since it significantly diminished distillable entanglement. Because pbits contain at least one bit of secure key, we have already seen that distillable key can be much larger than distillable entanglement.

However our main goal is to show that there are bound entangled states from which one can draw key. Thus we need distillable entanglement to be strictly zero. Here it is easily seen that any perfect pdit is an NPT state. Even more, one can show that pdits are always distillable [39]. Thus we cannot realize our goals by analysing perfect pdits.

B. Approximating pbits with PPT states

After realising that pbits cannot be bound entangled, one finds that this still does not exclude bound entangled states with private key. Namely, even though bound entangled states cannot contain exact key (as they would be pbits then) they may contain almost exact key. Such states would be in some sense close to pbits. Note that this would be impossible, if the only states containing perfect key were maximally entangled state. Indeed, for \( d \otimes d \) system if only a state has greater overlap than \( 1/d \) with a maximally entangled state we can distill singlets from it [6].

Recall, that for a state with key part being two qubits, the measure of quality of the bit of key coming from measuring the key part is trace norm of upper-right block. The key is perfect if the norm is \( 1/2 \) (we have then pbit) and it is close to perfect, if the norm is close to \( 1/2 \). Thus our first goal will be to find bound entangled states having the trace norm of that block arbitrarily close to \( 1/2 \). We will actually construct such PPT states (hence bound entangled) in Sections X-A X-B. In this way we will show that there exist bound entangled states that contain an arbitrarily (though not perfectly) secure bit of key.
C. Nonzero rate of key from bound entangled states

It is not enough to construct bound entangled states with arbitrary secure single bit of key. The next important step is to show that given many copies of BE states one can draw nonzero asymptotic rate of secure key. To show this we will employ (in Section X-C) the BE states with almost perfect bit of key. Let us outline here the most direct way of proving the claim.

To be more specific, we will consider many copies of states $\rho$, which have upper-right block trace norm equal to $1/2 - \epsilon$. We will argue that one can get key by measuring the key part of each of them, and then process via local classical manipulations and public discussion the outcomes. How to see that one can get nonzero rate in this way?

We will first argue, that the situation is the same, as if the outcomes were obtained from a state which is close to maximally entangled. To this end we will apply the idea of privacy squeezing described in Section [I-B]

First recall, that we have shown that operation of twisting does not change security of ccq state - more precisely, it does not change the state of the Eve’s system and key part of Alice and Bob systems, which would arise, if Alice and Bob measured the key part. Thus whatever twisting we will apply, from cryptographic point of view the situation will not change. The total state will change, yet this can be noticed only by those who have access to the shield of Alice and Bob systems, and Eve does not have such access.

We will choose such a twisting, that will change the upper-right block of $\rho$ into a positive operator. This is exactly the one which realizes privacy squeezing of this state. Now, even though security is not changed, the state is changed in a very favorable way for our purposes. Namely, we can now trace out the shield, and the remaining state of the key part (a p-squeezed state of the initial one) will be close to maximally entangled. Indeed, twisting does not change trace norm of the upper-right block. Because now the block is a positive operator, its trace norm is equal to its trace, and tracing out shield amounts just to evaluating trace of blocks. Since the trace norm was $1/2 - \epsilon$, the upper-right element of the state of key part is is equal to $1/2 - \epsilon$, which means that state is close to maximally entangled (where the corresponding element is equal to $1/2$). One can worry, that it is now not guaranteed that the security is the same, because we have performed not only twisting, but also partial trace over shield. However the latter operation could only make situation worse, since partial trace means giving the traced system to Eve.

Now the only remaining thing is to show that we can draw key from data obtained by measuring many copies of state close to an EPR state, then definitely we can draw key from many copies of more secure ccq state obtained from our $\rho'$. To achieve the goal, we thus need some results about drawing key from ccq state. Let us recall that we work in scenario, where Alice and Bob are promised to share i.i.d. state, so that the ccq state is tensor product of identical copies. In this case, the needed results have been provided in [19]. It follows, that the rate of key is at least $I(A : B) - I(A : E)$, where $I$ is mutual information. If instead of almost-EPR state, we have just an EPR state, the above quantity is equal to 1. Indeed, perfect correlations, and perfect randomness of outcomes gives $I(A : B) = 1$ and purity of the EPR state gives $I(A : E) = 0$. Since we have state close to an EPR state, to continuity of entropies, we will get $I(A : B) \approx 1$, and $I(A : E) \approx 0$. Thus given $n$ copies of states that approximate pbits one can get almost $n$ bits of key in limit of large $n$.

D. Drawing key and transforming into pbits by LOCC

Apart from showing that key can be drawn from BE states, we want to develop the theory of key distillation from quantum states. To this end in Section VIII we recast definition of distilling key in terms of distilling pbits by local operations and classical communication. This is important change of viewpoint: drawing key requires referring to Eve; while distilling pbits by LOCC concerns solely bipartite states shared by Alice and Bob, and never requires explicit referring to Eve’s system. Thus we are able to pass from the game involving three parties: Alice, Bob and Eve to the two players game, involving only Alice and Bob.

We will employ two basic tools: (i) the concept of making a protocol coherent; (ii) the fact (which we will prove) that having almost perfectly secure ccq state is equivalent to having a state close to some pdt. Note that in one direction, the reasoning is very simple: if we can get nonzero rate of asymptotically perfect pbits by LOCC, we can also measure them at the end, and get in this way asymptotically perfect ccq states, which is ensured by item (ii) above. The converse direction is a little bit more involved: we take any protocol that produces key, apply it coherently, and this gives pure final state of Alice, Bob and Eve’s systems. From (ii) it follows, that the total state of Alice and Bob must be close to pdt.

Let us briefly discuss how we will show the fact (ii). The essential observation is that both the ccq state $\tilde{\rho}_{ABE}$ and Alice and Bob total state $\rho_{ABA'B'}$ are reductions of the same pure state $\psi_{ABA'B'}.E$. Here some explanation is needed: in general, since the ccq state is obtained by measurement, it is not reduction of $\psi_{ABA'B'}.E$. However, one can first apply measurement coherently to the state $\rho_{ABA'B'}$. Then the ccq of the new state $\rho'_{ABA'B'}$ is indeed the reduction of $\psi'_{ABA'B'}.E$. In the actual proof we will proceed in a slightly different way.

Now, if we have two nearby ccq states, we can find their purifications that are close to each other. Then also Alice and Bob states arising when we trace out Eve’s system are close to each other (because partial trace can only make states closer). The whole argument is slightly more complicated, but the above reasoning is the main tool.

The equivalence we obtain puts the task of drawing key into the standard picture of state manipulations by means of LOCC. The theory of such manipulations is well developed and, in particular, there are quite general methods of obtaining bounds on
transition rates (in our case the transition rate is just distillable key), see [35]. Indeed, we will be able to show that relative entropy of entanglement is an upper bound for distillable key. The main idea of deriving the bound is similar to the methods from LOCC state manipulations. However significant obstacles arise, to overcome which we have developed essentially new tools.

VIII. TWO DEFINITIONS OF DISTILLABLE KEY: LOCC AND LOPC PARADIGMS

In this section we show that distillable amount of pdits by use of LOCC denoted by $K_D$ is equal to classical secure key distillable by means of local operations and public communication (LOPC).

A. Distillation of pdits

We have established a family of states - pdits - which have the following property: after measurement in some basis $B$ they give a perfect bit of key. In entanglement theory one of the important aims is to distill singlets (maximally entangled states) which leads to operational measure of distillable entanglement. We will pose now an analogous task namely distilling pdits (private states) which are of the form (8). This gives rise to a definition of distillable key i.e. maximal achievable rate of distillation of pdits. Similarly as in the case of distillation of singlet, it is usually not possible to distill exact pdits. Therefore the formal definition of distillable key $K_D$ will be a bit more involved.

Definition 6: For any given state $\rho_{AB} \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ let us consider sequence $P_n$ of LOCC operations such that $P_n(\rho_{AB} \otimes I_B) = \sigma_n$, where $\sigma_n \in B(\mathcal{H}_A^{(n)} \otimes \mathcal{H}_B^{(n)})$. A set of operations $\mathcal{P} = \cup_{n=1}^{\infty} \{P_n\}$ is called pdit distillation protocol of state $\rho_{AB}$ if there holds

$$\lim_{n \to \infty} ||\sigma_n - \gamma_{d_n}|| = 0,$$

where $\gamma_{d_n}$ is a pdit whose key part is of dimension $d_n \times d_n$.

For given protocol $\mathcal{P}$, its rate is given by

$$R(\mathcal{P}) = \lim_{n \to \infty} \sup \frac{\log d_n}{n}$$

The distillable key of state $\rho_{AB}$ is given by

$$K_D(\rho_{AB}) = \sup_{\mathcal{P}} R(\mathcal{P}).$$

In other words, due to this definition, Alice and Bob given $n$ copies of state $\rho_{AB}$ try to get a state which is close to some pdit state with $d = d_n$. Unlike so far in entanglement theory, effect of distillation of quantum key depends not only on the number $n$ of copies of initial state but also on the choice of the output state. This is because private dits appears not to be reversibly transformable with each other by means of LOCC operations, as it is in case of maximally entangled states in LOCC entanglement distillation. Thus the quantity $K_D$ is a rate of distillation to the large class of states. (Of course, since the definition involves optimization, $K_D$ is well defined; in particular the expensive pdits will be suppressed).

One can be interested now if this new parameter of states $K_D(\rho)$ has an operational meaning for quantum cryptography. One connection is obvious: given a quantum state Alice and Bob may try to distill some pdit state, and hence get (according to the above definition) $K_D(\rho_{AB})$ bits of key if such distillation has nonzero rate. However the question arises: is it the best way of extraction of a classical secure key from a quantum state? I.e. given a quantum state is the largest amount of classical key distillable from a state equal to $K_D$. We will give to this question a positive answer now. It means, that distilling private dits i.e. states of the form (24) is the best way of distilling classical key from a quantum state.

B. Distillable classical secure key: LOPC paradigm

The issue of drawing classical secure key from a quantum state is formally quite different from the definition of drawing pdits. However it will turn out that it is essentially the same thing. In the LOCC paradigm, we have an initial state $\rho$ hold by Alice and Bob who apply to it an LOCC map, and obtain a final state $\rho'$. Thus the LOCC paradigm is essentially a bipartite paradigm.

In the paradigm of drawing secure classical key (see e.g. [12], [13]), there are three parties, Alice, Bob and Eve. They start with some joint state $\rho_{ABE}$ where subsystems $A, B, E$ belong to Alice, Bob and Eve respectively. Now, Alice and Bob essentially perform again some LOCC operations. However we have now a tripartite system, and we should know how that operation act on the whole system.

Definition 7: An operation $\Lambda$ belongs to LOPC class if it is composition of

(i) Local Alice (Bob) operations, i.e. operations of the form $\Lambda_A \otimes I_{BE}$ (or $\Lambda_B \otimes I_{AE}$).

(ii) Public communication from Alice to Bob (and from Bob to Alice). E.g. the process of communication from Alice to Bob is described by the following map

$$\Lambda(\rho_{ABE}) = \sum_i P_i \rho_{A|BE} P_i \otimes |i\rangle_b \langle i| \otimes |i\rangle_e \langle i|$$

(105)
where $P_i = I_{ABE} \otimes |i\rangle_a \langle i|$.

Here the subsystem $a$ carries the message to be sent, and the subsystems $b$ and $e$ of Bob and Eve represent the received message.

Let us note, that LOCC operations can be defined in the same way, the only difference is that we drop Eve’s systems (both $E$ and $e$) and corresponding operators.

Now, drawing secure key means obtaining the following state

$$\rho_{\text{ideal}}^{ccq} = \frac{1}{d} \sum_{i=0}^{d-1} |ii\rangle \langle ii|_{AB} \otimes \rho^E$$

(106)

by means of LOPC. Since output states usually can not be exactly $\rho_{\text{ideal}}^{ccq}$ Alice and Bob will get state of the ccq form [4] i.e.

$$\rho_{\text{real}}^{ccq} = \sum_{i,j=1}^{d} p_{ij} |ij\rangle_{AB} \langle ij| \otimes \rho_{ij}^E$$

(107)

There are two issues here: first, Alice and Bob should have almost perfect correlations, second, Eve states should have small correlations with states $|ij\rangle$ of Alice and Bob systems. The first condition refers to uniformity, the second one to security. There are several ways of quantifying these correlations, and some of them are equivalent. To quantify security [3] one can use Holevo function of distilled ccq state, namely:

$$\chi(\rho_{ccq}) \equiv S(\rho_E) - \sum_{i,j=1}^{d} p_{ij} S(\rho_{ij}) \leq \epsilon$$

(108)

where $S(\rho) = \text{Tr} \rho \log \rho$ denotes von Neumann entropy, and

$$\rho_E = \sum_{i,j=1}^{d} p_{ij} \rho_{ij}.$$ 

(109)

Alternatively, one can use similar condition based on norm

$$\sum_{ij} p_{ij} \|\rho_E - \rho_{ij}^E\| \leq \epsilon$$

(110)

The condition of maximal correlations between Alice and Bob (uniformity) can be of the following form

$$\| \sum_{i,j=1}^{d} p_{ij} |ij\rangle \langle ij| - \frac{1}{d} \sum_{i=1}^{d} |ii\rangle \langle ii| \| \leq \epsilon$$

(111)

One can also use again the trace norm between the real state (107) that is obtained and the ideal desired state (106) as done in [19], which includes both maximal correlations condition as well as security condition. The condition says that the state $\rho_{\text{real}}^{ccq}$ obtained by Alice and Bob is closed to some ideal state

$$\| \rho_{\text{real}}^{ccq} - \rho_{\text{ideal}}^{ccq}\| \leq \epsilon$$

(112)

We will discuss relations between this condition, and security criteria (108) and (110) as well as with uniformity criterion (111) in Appendix XV-C.

For the purpose of definition of secret key rate in this paper, we apply the joint criterion (112). Consequently, we adopt the following measure of distillable classical secure key from a quantum tripartite state:

**Definition 8**: For any given state $\rho_{ABE} \in B(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ let us consider sequence $P_n$ of LOPC protocols such that $P_n(\rho_{ABE}^{ccq}) = \beta_n'$, where $\beta_n'$ is ccq state

$$\beta_n' = \sum_{i,j=0}^{d_n-1} p_{ij} |ij\rangle \langle ij|_{AB} \otimes \rho_{ij}^E$$

(113)

from $B(\mathcal{H}^{(n)}) = B(\mathcal{H}_A^{(n)} \otimes \mathcal{H}_B^{(n)} \otimes \mathcal{H}_E^{(n)})$ with $\dim \mathcal{H}_A^{(n)} = \dim \mathcal{H}_B^{(n)} = d_n$. A set of operations $\mathcal{P} \equiv \cup_{n=1}^{\infty} \{P_n\}$ is called classical key distillation protocol of state $\rho_{AB}$ if there holds

$$\lim_{n \to \infty} \| \beta_n' - \beta_{d_n} \| = 0,$$

(114)

where $\beta_{d_n} \in B(\mathcal{H}^{(n)})$ is of the form

$$\frac{1}{d_n} \sum_{i=1}^{d_n} |ii\rangle_{AB} \langle ii| \otimes \rho_{ii}^E,$$

(115)
\(\rho_n^E\) are arbitrary states from \(B(H_E^{(n)})\). The rate of a protocol \(P\) is given by

\[
R(P) = \lim_{n \to \infty} \frac{\log d_n}{n}
\]

Then the distillable classical key of state \(\rho_{ABE}\) is defined as suprema of rates

\[
C_D(\rho_{ABE}) = \sup_P R(P).
\]

The above definition works for any input tripartite state \(\rho_{ABE}\). However in this paper we are only interested in the case where the total state is pure. The latter is determined by state \(\rho_{AB} = \text{Tr}_E \rho_{ABE}\) up to unitary transformations on Eve’s side. Since from the very definition \(C_D\) does not change under such transformations, the latter freedom is not an issue, so that we can say the state \(\rho_{AB}\) completely determines the total state. Thus we get the definition of distillable classical secure key from bipartite state \(\rho_{AB}\).

**Definition 9:** For given bipartite state \(\rho_{AB}\) the distillable classical secure key is given by

\[
C_D(\rho_{AB}) \equiv C_D(\psi_{ABE})
\]

where \(\psi_{ABE}\) purification of \(\rho_{AB}\).

**C. Comparison of paradigms**

Let us compare two definitions \(6\) and \(9\) of distilling cryptographical key. The difference is mostly that the first one deals only with bipartite system, and the goal is to get the desired final state by applying a class of LOCC operations. Within the second paradigm, we have tripartite state and we want to get a wanted state by means of LOPC operations. Thus the first paradigm is much more standard in quantum information theory. The second one comes from classical security theory (see e.g. [43]), where probability distributions of triples of random variables \(P(X, Y, Z)\) are being processed.

In the next section we will see that if the tripartite initial state is pure, the two paradigms are tightly connected. In the case of distillation of exact key, they are almost obviously identical, while in the inexact case, the only issue is to make the asymptotic security requirements equivalent. We will see that an output pdit obtained by LOCC implies some ccq state obtained by LOPC, and vice versa.

**D. Composability issues**

In the present paper we consider the promised scenario, which is the first step to consider in unconditionally secure QKD.\(^1\) The latter security definition is required to be *universally composable*, which means that a QKD protocol can be used as a subroutine of any other cryptographic protocol [4], [3], [47]. To this end, one needs to choose carefully the measure of security. Our starting point is the LOPC paradigm, where the measure of security is the trace norm. This is compatible with [3], where it was shown that such security measure implies indeed composability (see eq. (10) of [3]). In particular, if we concatenate \(n\) QKD protocols with security \(\epsilon\) measured by trace norm between the obtained state and the ideal target, the overall protocol will have security bounded by \(n\epsilon\). Thus trace norm in LOPC paradigm can be called *composable security condition*.

However, we want to recast QKD within the LOCC paradigm, i.e. in terms of distance between Alice and Bob states rather than tripartite states. In this spirit, in [3] it is shown that fidelity with maximally entangled state implies composability, in the sense that if fidelity is \(1 - \epsilon\) then the norm is bounded by \(\sqrt{\epsilon}\) (see eq. (20) of [3]). This may seem a bit uncomfortable, because while composing many protocols we have now to add not epsilons, but rather their square roots. However one can see that it is in general not possible to do better while starting from fidelity. Due to inequality \(1 - F(\sigma, \rho) \leq \frac{2}{\pi} \|\rho - \sigma\|\) [26] we can equally well use the trace norm. Indeed it will give us at worse the \(\sqrt{\epsilon}\) bound for composable security condition in the LOPC paradigm.

In our paper we have a more general situation, i.e. we deal with distance to private states rather than just maximally entangled states. According to the above discussion, we have decided to use trace norm in LOCC scenario. In our techniques, in the mid-steps we shall use fidelity, hence we are again left with \(\sqrt{\epsilon}\). We do not know, whether one can omit fidelity, and get rid of the square root.

Having said all that, we should emphasize that finally the minimal requirement is that security is an exponentially decreasing function of some security parameter, whose role can play e.g. the number of all qubits in the game. This condition is of course not spoiled by square root. (Example of a situation, where this requirement is not met was conjectured in [3] and then proved in [41]). In our paper we shall derive equivalences between various security conditions, and the word “equivalence” will be understood in the sense of not spoiling the proper exponential dependence on total number of qubits. In particular, the security conditions will be called equivalent also when they differ by the factor polynomial in number of qubits (see e.g. [30]).

\(^1\)The next step was subsequently done in papers [33], [32]
IX. EQUALITY OF KEY RATES IN LOCC AND LOPC PARADIGMS

In this section we will show that definitions 6 and 9 give rise to the same quantities. In this way the problem of drawing key within original LOPC paradigm is recast in terms of transition to a desired state by LOCC. First we will describe a coherent version of LOPC protocol. Then we will use it to derive equivalence in exact case (where protocols produce as outputs ideal ccq states or ideal pdits). Subsequently we will turn to the general case where inexact transitions are allowed.

A. Coherent version of LOPC key distillation protocol

The main difference between LOPC and LOCC paradigms is that in the first one we have transformations between tripartite states shared by Alice, Bob and Eve, while in the latter one - between bipartite states shared by Alice and Bob. Thus in LOPC paradigm, the part of the state held by Alice and Bob does not, in general tell us about security. To judge if Alice and Bob have secure key we need the whole \( \rho_{ABE} \) state. Security is assured by the lack of correlations of this state with Eve. Thus if we want to recast the task of drawing key in terms of LOCC paradigm, we need to consider such LOCC protocol which produces output state that assure security of key itself. We will do this by considering coherent version of LOPC key distillation protocols (cf. [40], [19]).

The most important feature of the version will be that given any LOPC protocol, starting with some initial pure state \( \psi_{ABE} \) and ending up with some ccq state \( \rho_{ABE} \), its coherent version will end up with a state \( \psi'_{A'BB'B'} \) such that tracing out \( A'B' \) part will give exactly the ccq state \( \rho_{ABE} \). In this way, the total Alice and Bob state \( \rho_{A'A'B'B'} \) will keep the whole information about Eve (because up to unitary on Eve’s system, purification is unique).

In coherent version of key distillation protocol Alice and Bob perform their local operations in a coherent way i.e. by adding ancillas, performing unitary transformations and putting aside appropriate parts of the system. This additional part of the system is discarded in the usual protocol. However, holding this part allows one to keep the total state of Alice Bob and Eve pure in each step of the protocol. This is because we use pure ancillas, pure initial state and apply only unitary transformations which preserve purity.

Alice and Bob can also perform public communication. Its coherent version is that e.g. Bob and Eve apply C-NOT operations to Alice’s subsystem which holds the result of measurement.

Formally, the coherent version of process of communication from definition 9 is an operation of the form

\[
\Lambda(\rho_{aABE}) = U(\rho_{aABE} \otimes |0\rangle_a' \langle 0| \otimes |0\rangle_b \otimes |0\rangle_e \langle 0|)U^\dagger
\]  

where

\[
U = I_{ABE} \otimes \sum_i |i\rangle_a \langle i| \otimes U_{a'}^{(i)} \otimes U_b^{(i)} \otimes U_e^{(i)}
\]

with unitary transformation \( U_{a'}^{(i)}, U_b^{(i)}, U_e^{(i)} \) satisfying \( U_e^{(i)} |0\rangle_e = |i\rangle_e \) and similarly for \( U_b^{(i)}, U_a^{(i)} \). To finalize the operation, Alice puts aside the system \( a' \).

Such a coherent version of LOPC protocol has the following two features: (i) keeps the state pure. (ii) after tracing out subsystems that are put aside we obtain exactly the same state as in the original protocol.

Now we are in position to construct for a given LOPC protocol a suitable LOCC protocol, which will output pbits when the former protocol will output ideal key. Namely, the local operations from LOPC protocol, we replace with their coherent versions. The public communication we take in original - incoherent - form (of course, without broadcast to Eve since we now deal with bipartite states). One notes that such operation produces the same state of Alice and Bob as the one produced by coherent version of the original LOPC protocol traced out over Eve’s system. This is because, if we trace out Eve, then there is no difference between coherent and incoherent versions of public communication.

Thus from an LOPC protocol we have obtained some LOCC protocol - a special one, where local systems are not traced out. In this way one gets a bridge which joins the two approaches, and shows that different definitions of distillable key are equivalent. In particular, suppose that the LOPC protocol produced the ideal ccq state. Then the output of LOCC protocol obtained as a coherent version of this protocol will produce a state which (due to theorem 2) must be a pdit.

Remark 3: Of course, the notion of coherent version need not concern just some LOPC protocol. Also an LOCC operation, that contained measurements and partial traces can be made coherent, which in view of the above considerations means simply, that the systems are not traced out, but only "kept aside" and measurements are replaced by appropriate local unitaries. Actually, if we include all pure ancillas that will be added in the course of realizing the LOCC operation, the coherent version of the operation is nothing but a closed LOCC operation [37] introduced for sake of counting local resources such as local information. One can think of the shield part, as being the state of all the lab equipment and quantum states, left over from the process of key distillation.
B. Equivalence of paradigms: The case of exact key

Here we will consider the ideal case, where the distillation of the key gives exactly the demanded output state. One can state it formally and observe:

**Proposition 5:** Let $K_D^{exact}$ and $C_D^{exact}$ denote optimal rates achievable by LOPC and LOCC protocols which as outputs have exact ccq states (106) and pdit states (24), respectively. Then for any state $\rho_{AB}$ we have

$$K_D^{exact}(\rho_{AB}) = C_D^{exact}(\rho_{AB})$$

**Proof:** If after LOPC protocol $\mathcal{P}$, Alice and Bob obtained exact $d \times d$ ccq state (106), then the coherent application of $\mathcal{P}$ due to theorem [2] and discussion of section [X-A] will produce pdit of the same dimension. Conversely, if by LOCC Alice and Bob can get a pdit, then after measurement, again by theorem [2] they will obtain exact ccq state (106) of the same dimension.

C. Distillation of classical key and distillation of pdits - equivalence in general (asymptotically exact) case

We will prove here the theorem, which implies, that even in nonexact case, distillation of pdits from initial bipartite state by LOCC is equivalent to distillation of key by LOPC from initial pure state, that is purification of the bipartite state. This in turn means that the rates in both paradigms are equal.

**Theorem 5:** Let Alice and Bob share a state $\rho$ such that Eve has it’s purification. Then the following holds: if Alice and Bob can distill by LOPC operations a state such that with Eve’s subsystem it is ccq state i.e. of the form

$$\rho_{AB} = \sum_{i,j=1}^{d} p_{ij} |ij⟩⟨ij| ⊗ \rho_{ij}^E,$$

with $||\rho_{AB}^{ccq} - \rho_{AB}^{ideal}|| \leq \epsilon$, then they can distill by LOCC operations a state $\rho_{out}$ which is close to some pdit state $\gamma$ in trace norm:

$$||\rho_{out} - \gamma|| \leq \sqrt{2\epsilon},$$

(123)

where the key part of a pdit $\gamma$ is of dimension $d \times d$.

Conversely, if by LOCC they can get state $\rho_{out}$ satisfying $||\rho_{out} - \gamma|| \leq \epsilon$, then by LOPC they can get state $\rho_{ccq}$ satisfying $||\rho_{AB}^{ccq} - \rho_{AB}^{ideal}|| \leq \sqrt{2\epsilon}$.

**Proof:** The "if" part of this theorem is proven as follows. By assumption Alice and Bob are able to get by some LOPC protocol $\mathcal{P}$ a ccq state $\rho_{AB}$ satisfying

$$||\rho_{AB}^{ccq} - \rho_{AB}^{ideal}|| \leq \epsilon.$$  

(124)

Now by equivalence between norm and fidelity (Eq. (13) of Appendix) we can rewrite this inequality as follows

$$F(\rho_{AB}^{ccq},\rho_{ideal}) > 1 - \frac{1}{2}\epsilon.$$  

(125)

By definition of fidelity

$$F(\rho,\sigma) = \max_{\psi,\phi} [⟨\psi|\phi⟩]$$

(126)

where maximum is taken over all purifications $\psi$ and $\phi$ of $\rho$ and $\sigma$, respectively, we can fix one of these purification arbitrarily, and optimise over the other one. Let us then choose such a purification $\psi_{ABA'B'E}$ of $\rho_{AB}^{ccq}$ which is the output of coherent application of the mentioned protocol $\mathcal{P}$. There exists purification $\phi_{ABA'B'E}$ of $\rho_{ideal}$ such that it’s overlap with $\psi$ is greater than $1 - \frac{1}{2}\epsilon$. Since the fidelity can only increase after partial trace applied to both the states, it will be still greater than $1 - \frac{1}{2}\epsilon$ once we trace over Eve’s subsystem. Thus we have

$$F(\rho_{ABA'B'}^{\psi},\sigma_{ABA'B'}^{\phi}) > 1 - \frac{1}{2}\epsilon.$$  

(127)

where $\sigma_{ABA'B'}^{\phi}$ and $\rho_{ABA'B'}^{\psi}$ are partial traces of $\phi$ and $\psi$ respectively. The state $\sigma_{ABA'B'}^{\phi}$ (partial trace of $\phi$) comes from purification of an ideal state, and by the very definition it is some pdit state $\gamma$. At the same time, the state $\rho_{ABA'B'}^{\psi}$ (partial trace of $\psi$) is the one which is the output of coherent application of protocol $\mathcal{P}$. Thus by coherent version of $\mathcal{P}$ Alice and Bob can obtain state close to pdit which proves the "if" part of the theorem.

To obtain equivalence let us prove now the converse implication. The proof is a sort of "symmetric reflection" of the proof of the previous part.

This time we assume that there exists LOCC protocol starting with $\rho$, ending up with final state $\rho_{out}$ with key part od $d \times d$ dimension which is close to some pdit in norm i.e.

$$||\rho_{out} - \gamma|| \leq \epsilon.$$  

(128)
Due to equivalence between fidelity and norm, we have

$$F(\rho_{out}, \gamma) \geq 1 - \epsilon/2$$  \hfill (129)

The total state after protocol is $\psi_{ABA'B'E}$, and if partially traced over Eve it returns $\rho_{out}$. Then we can find such $\phi$, purification of $\gamma$, that $F(\psi, \phi) > 1 - \epsilon$. Now let Alice and Bob measure the key part and trace out the shield. Then out of $\psi$ we get some ccq state $\rho_{out}^{ccq}$. The same operation applied to $\phi$ gives ideal ccq state $\rho_{ideal}^{ccq}$. The operation can only increase the fidelity, so that

$$F(\rho_{out}^{ccq}, \rho_{ideal}^{ccq}) \geq 1 - \epsilon/2$$  \hfill (130)

Returning to norms we get

$$||\rho_{out}^{ccq} - \rho_{ideal}^{ccq}|| \leq \sqrt{2\epsilon}.$$  \hfill (131)

X. DISTILLING KEY FROM BOUND ENTANGLED STATES

In this section we will provide a family of states. Then we will show that for certain regions of parameters they have positive partial transpose (which means that they are non-distillable). Subsequently, we shall show that out of the above PPT states one can produce, by an LOCC operation, states arbitrarily close to pbits (which also implies that they are entangled, hence bound entangled). More precisely, for any $\epsilon$ we will find PPT states, from which by a LOCC protocol, one gets with some probability a state $\epsilon$-close to some pbit. Since LOCC preserves the PPT property, this shows that pbits can be approximated with arbitrary accuracy by PPT states, in sharp contrast with maximally entangled states. We then show how to get from a state sufficiently close to a pbit with non vanishing asymptotic rate of key. We obtain it by reducing the problem to drawing key from states that are close to the maximally entangled state.

A. The new family of PPT states ...

Here we will present a family of states, and will determine the range of parameters for which the states are PPT. The idea of construction of the family is based on the so called hiding states found by Eggeling and Werner in [22]. Let us briefly recall this result. In [52], [20] it was shown that one can hide one bit of information in two states by correlating the bit of information with a pair of states which are almost indistinguishable by use of LOCC operations, yet being almost distinguishable by global operations. The resulting state with the hidden bit is of the form:

$$\rho_{hb} = \frac{1}{2}|0\rangle\langle 0|_{AB} \otimes \rho_{hiding}^{1} + \frac{1}{2}|1\rangle\langle 1| \otimes \rho_{hiding}^{2}$$  \hfill (132)

In [22] it was shown that there are separable states, which can serve as arbitrarily good hiding states. These states are

$$\tau_{1} = \left(\frac{\rho_{s} + \rho_{a}}{2}\right) \otimes k, \quad \tau_{2} = \left(\rho_{s}\right) \otimes k,$$  \hfill (133)

where $\rho_{s}$ and $\rho_{a}$ are symmetric and antisymmetric Werner states $\rho_{W}$. The higher is the parameter $k$, the more indistinguishable by LOCC protocols the states become.

We adopt the idea of hiding bits to hide entanglement. Namely instead of bits one can correlate two orthogonal maximally entangled states with these two hiding states and get the state:

$$\rho_{he} = \frac{1}{2}|\psi_{+}\rangle\langle \psi_{+}|_{AB} \otimes \tau_{1}^{A'B'_{1}} + \frac{1}{2}|\psi_{-}\rangle\langle \psi_{-}|_{AB} \otimes \tau_{2}^{A'B'_{2}}$$  \hfill (134)

Let us recall, that our purpose is to get the family of states which though entangled are not distillable, and can approximate pbit states. Then the choice of $\rho_{he}$ as a starting point has double advantage. First, because $\tau_{1}$ and $\tau_{2}$ are hiding, $\rho_{he}$ will not allow for distillation of entanglement by just distinguishing them. Second, the hiding states are separable, so they do not bring in any entanglement to the state $\rho_{he}$. However the state $\rho_{he}$ is obviously NPT. Indeed, consider partial transposition of $BB'$ system. It is composition of partial transpositions of $B$ and $B'$ subsystems. If one applies it to the state (134), one gets

$$\rho_{ABA'B'}^{\Gamma} = (I_{A} \otimes T_{B} \otimes I_{A'} \otimes T_{B'}) \left(\rho_{ABA'B'}\right) =$$

$$= \begin{bmatrix}
\frac{1}{2}(\tau_{1}^{2} + \tau_{2}^{2}) & 0 & 0 & 0 \\
0 & \frac{1}{2}(\tau_{1} - \tau_{2}) & \frac{1}{2}(\tau_{1} - \tau_{2}) & 0 \\
0 & \frac{1}{2}(\tau_{1} - \tau_{2}) & \frac{1}{2}(\tau_{1} + \tau_{2}) & 0 \\
0 & 0 & 0 & \frac{1}{2}(\tau_{1} + \tau_{2})
\end{bmatrix}$$  \hfill (135)
where $\Gamma$ denotes partial transposition over subsystem $B'$ (as partial transposition over B caused interchange of blocks of matrix of (134)). This matrix is obviously not positive for the lack of middle-diagonal blocks. To prevent this we admix a separable state $\frac{1}{2}(|01\rangle + |10\rangle) \otimes \tau_2$ with a probability $(1 - 2p)$, where $p \in (0, \frac{1}{2})$. It’s matrix reads then

$$
\rho_{(p,d,k)} = \begin{pmatrix}
 p\left(\frac{\tau_1 + \tau_2}{2}\right) & 0 & 0 & p\left(\frac{\tau_1 - \tau_2}{2}\right) \\
 0 & (1 - p)\tau_2 & 0 & 0 \\
 0 & 0 & (1 - p)\tau_2 & 0 \\
p\left(\frac{\tau_1 + \tau_2}{2}\right) & 0 & 0 & p\left(\frac{\tau_1 - \tau_2}{2}\right)
\end{pmatrix},
$$

(136)

In subscript we explicitly write the parameters on which this state depends implicitly: $d = d_A = d_{B'}$ is the dimension of symmetric and antisymmetric Werner states used for hiding states (133) and $k$ is parameter of tensoring in their construction. We shall see, that for some range of $p$, almost every state of this family is a PPT state. We formalise it in the next lemma.

Lemma 5: Let $\rho_a \in B(C^d \otimes C^d)$ and $\rho_b \in B(C^d \otimes C^d)$ be symmetric and antisymmetric Werner states respectively, and let $k$ be such that

$$
\tau_1 = (\rho_a + \rho_b)^{\otimes k}, \quad \tau_2 = (\rho_a)^{\otimes k}
$$

(137)

holds. Then for any $p \in (0, \frac{1}{2}]$ and any $k$ there exists $d$ such that state (136) has positive partial transposition. More specifically, the state (136) is PPT if and only if the following conditions are fulfilled

$$
0 < p \leq \frac{1}{3}, \quad \frac{1 - p}{p} \geq \left(\frac{d}{d - 1}\right)^k
$$

(138)

For the proof of this lemma see Appendix XV-B.

B. ... can approximate pdis

We have just established a family $\rho_{(p,d,k)}$ such that for certain $p$, $k$ and $d$ they are PPT states. We will then show, that by LOCC one can transform some of them to a state close to pbits. More precisely, for any fixed accuracy, we will always find $p$, $d$ and $k$ such that it is possible to reach pbit up to this accuracy, starting from some number of copies of $\rho_{(p,d,k)}$ and applying LOCC operations.

Subsequently, we will show that one can always choose the initial states $\rho_{(p,d,k)}$ to be PPT. Since LOCC operations do not change PPT property, we will in this way show that there are PPT states that approximate pbits to arbitrarily high accuracy.

We will first prove the following theorem.

Theorem 6: For any $\epsilon > 0$ and any $p \in (\frac{1}{4}, 1]$ there exist state $\rho$ from family of state $\{\rho_{(p,d,k)}\}$ (136) such that for some $m$ from $\rho^{\otimes m}$ one can get by LOCC (with nonzero probability of success) a state $\sigma$ satisfying $||\sigma - \gamma|| \leq \epsilon$ for some private bit $\gamma$.

Proof: First of all let us notice that by theorem 4 it is enough to show, that one can transform $\rho^{\otimes m}$ into a state $\rho'$ which has sufficiently large norm of the upper-right block.

Let Alice and Bob share $m$ copies of a state $\rho$ from the family (136). The number $m$ and parameters $(p, k, d)$ of this state will be fixed later.

Now let Alice and Bob apply the well known recurrence protocol - ingredient of protocols of distillation of singlet states [8]. Namely they take one system in state $\rho$ as source system, and iterate the following procedure. In $i$-th step they take one system in state $\rho$, and treat it as a target system. Let us remind that both systems have four subsystems $A$, $B$, $A'$ and $B'$. To distinguish the source and target system, the corresponding subsystems of a target system we call $\tilde{A}, \tilde{B}, \tilde{A}', \tilde{B}'$. On the source and target system they both perform a CNOT gate with a source at the $A(\tilde{A})$ part of a source system and target at $A(\tilde{A})$ part of a target system for Alice (Bob) respectively. Then, they both measure the $\tilde{A}$ and $\tilde{B}$ subsystem of the target system in computational basis respectively, and compare the results. If the results agree, they proceed the protocol, getting rid of the $\tilde{A}\tilde{B}$ subsystem. If they do not agree, they abort the protocol. With nonzero probability of success they can perform this operation $m - 1$ times having each time the same source system, and some fresh target system in state $\rho$. That is they start with $m$ systems in state $\rho$ and in each step (upon success) they use up one system and pass to the next step.

One can easily check, that the submatrices (blocks) of the state $\rho_{(p,d,k)}$ which survives $m - 1$ steps of this recurrence protocol (which clearly happens with nonzero probability) are equal to the $m$-fold tensor power of the elements of initial matrix $\rho_{(p,d,k)}$:

$$
\rho_{(p,d,k)}^{rec} = \frac{1}{N} \begin{pmatrix}
 [p\left(\frac{\tau_1 + \tau_2}{2}\right)]^{\otimes m} & 0 & 0 & [p\left(\frac{\tau_1 - \tau_2}{2}\right)]^{\otimes m} \\
 0 & [\frac{1}{2} - p]\tau_2^{\otimes m} & 0 & 0 \\
 0 & 0 & [\frac{1}{2} - p]\tau_2^{\otimes m} & 0 \\
 [p\left(\frac{\tau_1 + \tau_2}{2}\right)]^{\otimes m} & 0 & 0 & [p\left(\frac{\tau_1 - \tau_2}{2}\right)]^{\otimes m}
\end{pmatrix},
$$

(139)
where the normalisation is given by
\[ N = \text{Tr}[\rho_{ECC}^{cc_p,k}] = 2p^m + 2(\frac{1}{2} - p)^m. \] (140)

Let us consider the upper-right block \( \tilde{A}_{0011} \) of the matrix (139) without normalisation. Norm of this block is equal to
\[ \| \tilde{A}_{0011} \| = \left( \frac{p}{2} \right)^m \| (\rho_k - \rho_s) \otimes k - \rho_s \otimes k \| = \left( \frac{p}{2} \right)^m (2(1 - 2^{-k}))^m = p^m (1 - 2^{-k})^m. \] (141)

where second equality is consequence of the fact, that \( \rho_k \) and \( \rho_s \) have orthogonal supports which gives that \( \rho_s \otimes k \) is orthogonal to any term in expansion of \( (\frac{1}{\sqrt{2}}|0\rangle \langle 0| - \frac{1}{\sqrt{2}}|1\rangle \langle 1|) \otimes k \) but the one \( |0\rangle \langle 0| \otimes k \). Thus the result is equal to norm of \( (|p\rangle \langle 0| - \frac{1}{\sqrt{2}}|s\rangle \langle 1|) \) (which is \( (1 - \frac{1}{2p}) \) plus norm of the difference \( |s\rangle \langle 1| \) which gives the above formula. Thus the norm of the upper-right block \( A_{0011} \) of the state (139) is given by
\[ \| A_{0011} \| = \frac{1}{N} \| \tilde{A}_{0011} \| = \frac{1}{2} (1 - \frac{1}{2k})^m \frac{1}{1 + (\frac{1}{2k})^m}. \] (142)

We want now to see, if we can make the norm to be arbitrary close to 1/2. (then by Lemma 4 the state will be arbitrary close to a pbit). Since \( p > \frac{1}{2} \), we get that \( (\frac{1}{2k})^m \) converges to 0 with \( m \). Although increasing \( m \) diminishes the term \( (1 - \frac{1}{2k})^m \), we can first fix \( k \) large enough, so that the whole expression (142) will be as close to \( \frac{1}{2} \) as it is required.

Now we have the following situation. We know that for \( p \in (\frac{1}{2}, 1) \) if \( \frac{1}{2} (1 - \frac{1}{2k})^m \frac{1}{1 + (\frac{1}{2k})^m} \) is close to \( 1/2 \), then the state (139) is close to a pbit. On the other hand, from lemma 5 it follows that for (i) \( p \in (0, 1/3) \) and (ii) \( \frac{1}{2p} \geq \left( \frac{d-1}{d+1} \right)^k \) the state \( \rho_{(p,d,k)} \) is PPT, hence also the state (139) is PPT (because it was obtained from the former one by LOCC operation). If we now fix \( p \) from interval \( (1/4, 1/3) \), then by choosing high \( m \) and for such \( m \), high enough \( k \), then the state (139) is close to pbit. Now, we can fix also \( m \) and \( k \), and choose \( d \) so large that the condition (ii) is also fulfilled so that the state becomes PPT. This proves the following theorem, which is main result of this section.

Theorem 7: PPT states can be arbitrarily close to pbit in trace norm.

Here might be the appropriate place to note an amusing property of state (134): Namely, Eve knows one bit of information about Alice and Bob’s state – she knows the phase of their Bell state. But she only has one bit of information about their state, thus it cannot be that she also knows the bit of their state, which is the key. In some sense, giving Eve the bit of phase information, means that she cannot know the bit value.

C. Distillation of secure key

In the previous subsection we have shown that private bits can be approximated by PPT states. Now, the question is whether given many copies of one of such PPT states Alice and Bob can get nonzero rate of classical key. Below, we will give the positive answer.

The main idea of the proof is to show that from the PPT state which is close to pbit Alice and Bob by measuring, can obtain ccq state satisfying conditions of protocol (DW) found by Devetak and Winter [19]. Namely, they have shown that for an initial ccq state (state which is classical only on Alice side; this includes ccq state as special case) between Alice, Bob and Eve,
\[ C_D(\rho_{ABE}) \geq I(A : B) - I(A : E) \] (143)

Here \( I(A : B) \) stands for the quantum mutual information of the state \( \rho \) with subsystems \( A \) and \( B \) given by
\[ I(A : B) = S(A) + S(B) - S(AB), \] (144)

where \( S(X) \) stands for the von Neumann entropy of \( X \) (sub)system of the state \( \rho \).

Using the above result we can prove now, that from many copies of states close to a pbit, one can draw nonzero asymptotic rate of key.

Lemma 6: If a state \( \rho \) is close enough to pbit in trace norm, then \( K_D(\rho) > 0 \).

Proof: The idea of the proof is as follows. Suppose that \( \sigma \) is close to pbit \( \gamma \). We then consider twisting that changes \( \gamma \) into basic pbit \( \rho^{AB} \otimes \sigma_{A'B'} \) with some state \( \sigma \) on \( A'B' \). We apply twisting to both states, so that they are still close to each other. Of course, this is only a mathematical tool: Alice and Bob cannot apply twisting, which is usually a nonlocal operation. The main point is that after twisting, according to theorem 1 the ccq state does not change. If we now trace out systems \( A'B' \) the resulting state will be close to maximally entangled, and the resulting ccq state – at most worse from Alice and Bob point of view (because tracing out means giving to Eve). What we have done is just privacy squeezing of the state \( \rho \). Now, the latter ccq state has come from measurement of a state close to the maximally entangled one. Thus the task reduces to estimate quantities \( I(A : B) \) and \( I(A : E) \) for a ccq state obtained from measuring the maximally entangled state. However due to
suitable continuities, first one is close to 1 and second one close to 0. Now by DW protocol, one can draw a pretty high rate of key from such ccq state. Let us now proceed with the formal proof.

We assume that for some pbit $\gamma$ we have

$$\|\rho - \gamma\| \leq \epsilon. \quad (145)$$

Let us consider twisting $U$ which changes pdit $\gamma$ into a basic pdit. Existence of such $U$ is assured by theorem\[2\]. If both states $\gamma$ and $\rho$ are subjected to this transformation, the norm is preserved, so that

$$\|U\rho U^\dagger - U\gamma U^\dagger\| \leq \epsilon. \quad (146)$$

Also due to theorem\[1\] the ccq state obtained by measuring key part of $U\rho U^\dagger$ is the same as that from $\rho$. Now, the amount of key drawn from such ccq state will not increase if we trace out shield. Thus we apply such partial trace to $U\rho U^\dagger$ and to $\gamma$, and by monotonicity of trace norm get

$$\|\tilde{\rho}_{AB} - P_{AB}^+\| \leq \epsilon. \quad (147)$$

It is now enough to show that from ccq state obtained by measuring $\tilde{\rho}_{AB}$ (where Eve holds the rest of its purification) one can get nonzero rate of key.

To this end let us note, that for ccq state obtained from any bipartite state $\rho_{AB}$ by measuring its purification on subsystems A and B in computational basis, we have the following bound for $I(A : E)$:

$$I(A : E)_{ccq} \leq S(\rho_{AB}). \quad (148)$$

Now, since our state $\tilde{\rho}_{AB}$ is close to $P_+$, for which $S(P_+) = 0$ and $I(A : B) = 1$, we can use continuity of entropy, to bound these quantities for the state. From Fannes inequality (see eq. (14) in Sec. II-A), we get

$$I(A : B)_{ccq} \geq 1 - 4\epsilon \log d_{AB} - h(\epsilon), \quad (149)$$

$$I(A : E)_{ccq} \leq S(\tilde{\rho}_{AB}) \leq 8\epsilon - h(\epsilon). \quad (150)$$

To get first estimate, it is enough to note, that due to monotonicity of trace norm, the estimate (147) is also valid if we dephase the state $\tilde{\rho}_{AB}$ and $P_{AB}^+$. Thus we obtain that

$$K_D(\rho) \geq I(A : B) - I(A : E) \geq 1 - 16\epsilon \quad (151)$$

This ends the proof of the lemma. \[\blacksquare\]

Let us note here, that it was not necessary to know, that the state $\rho$ is close to pbit. Rather, it was enough to know that trace norm of upper right block is close to 1/2, as we have proved that it is equivalent to previous condition (see Sec. V). From this it follows, that after twisting, and tracing out $A'B'$ the resulting state $\tilde{\rho}$ is close to the EPR state, which ensures nonzero rate of key (actually the rate is close to 1).

We now can combine the lemma with the fact that we know PPT states that are close to pbit, to obtain that there exist PPT states from which one can draw secure key. The states must be entangled, as from separable states one cannot draw key. Namely separable state can be established by public discussion. If it could then serve as a source of secret key, one could obtain secret key by public discussion which can not be possible. For formal arguments see [17]. Thus our PPT states are entangled. But, since they are PPT, one cannot distill singlets from them [36], hence they are bound entangled. In this way we have obtained the following theorem

*Theorem 8:* There exist bound entangled states with $K_D > 0$.

We have split the way towards bound entangled states with nonzero key into two parts. First, we have shown that from PPT states $\tilde{\rho}_{(p,d,k)}$ by recurrence one can get a state that is close to pbit. Then we have shown, that from a state close to pbit one can draw private key.

Note that we have two quite different steps; recurrence was the quantum operation preformed on quantum Alice and Bob states, while Devetak-Winter protocol in our case, is classical processing of the outputs of measurement. We could unify the picture in two ways. First Alice and Bob could measure the key part of the initial state $\rho_{(p,d,k)}$, and preform recurrence classically (since the quantum recurrence is merely coherent application of classical protocol). Then the whole process of drawing key from $\rho_{(p,d,k)}$ would be classical (of course, taking into account that Eve has quantum states). On the other hand, the DW protocol could be applied coheretly, so that till the very end, we would have quantum state of Alice and Bob.

The result we have obtained allows to distinguish two measures of entanglement

*Corrollary 2:* Distillable entanglement and distillable classical secure key are different measures of entanglement i.e. there are states for which there holds

$$K_D(\rho) > D(\rho) = 0. \quad (152)$$

In further section we will also show that $K_D$ is different than entanglement cost, as it is bounded by relative entropy of entanglement.
XI. RELATIVE ENTROPY OF ENTANGLEMENT AS UPPER BOUND ON DISTILLABLE KEY

In this section we will provide complete proof of the theorem announced [34] which gives general upper bound on distillable key $K_D$. This upper bound is given by regularised relative entropy of entanglement [56]. The relative entropy of entanglement [55], [54] is given by

$$ E_r(\rho) = \inf_{\sigma_{sept}} S(\rho|\sigma_{sept}), \quad (153) $$

where $S(\rho|\sigma) = \text{Tr} \rho \log \rho - \text{Tr} \rho \log \sigma$ is relative entropy, and infimum is taken over all separable states $\sigma_{sept}$. The regularized version of $E_r$ is given by

$$ E_r^\infty(\rho) = \lim_{n} \frac{E_r(\rho^{\otimes n})}{n}. \quad (154) $$

The limit exists, and due to subadditivity of $E_r$, we have

$$ E_r^\infty(\rho) \leq E_r. \quad (155) $$

It follows that also relative entropy of entanglement is upper bound for $K_D$.

We recall now the following lemma obtained in [34], the proof of which we provide in Appendix XV-H.

**Lemma 7:** Consider a set $\mathcal{S}^r := \{ U\rho_{ABA'B'}U^\dagger | \rho_{ABA'B'} \in SEP, \rho_{ABA'B'} \in B(C^d \otimes C^d, C^{d_A} \otimes C^{d_B}) \}$ where $U$ is $B$-twisting with $B$ being a standard product basis in $C^d \otimes C^d$. Let $\sigma_{ABA'B'} \in \mathcal{S}^r$ and $\sigma_{AB} = \text{Tr}_{A'B'}\sigma_{ABA'B'}$. We have then

$$ S(P_+|\sigma_{AB}) \geq \log d, \quad (156) $$

where $P_+ = |\psi_+^A\rangle\langle \psi_+^A|$. We will also need asymptotic continuity of the relative entropy distance from some set of states obtained in [21] in the form of [50].

**Proposition 6:** For any compact, convex set of state $\mathcal{S}$ that contains maximally mixed state, the relative entropy distance from this set given by

$$ E_r^\mathcal{S}(\rho) = \inf_{\sigma \in \mathcal{S}} S(\rho||\sigma), \quad (157) $$

is asymptotically continuous i.e. it satisfies

$$ |E_r^\mathcal{S}(\rho_1) - E_r^\mathcal{S}(\rho_2)| < 4\epsilon \log d + h(\epsilon) \quad (158) $$

for any states $\rho_1$, $\rho_2$ acting on Hilbert space $\mathcal{H}$ of dimension $d$, with $\epsilon = \|\rho_1 - \rho_2\|$ with $\epsilon \leq 1$.

Let us mention, that the original relative entropy of entanglement [54] has in place of $S$ the set of separable states. Another version has been considered in [46], where $S$ was set of PPT states. The latter set has entangled states, but they can be only weakly entangled. In contrast we will have set in which there may be quite strongly entangled states.

We are now in position to formulate and prove the main result of this section.

**Theorem 9:** For any bipartite state $\rho_{AB} \in B(C^{d_A} \otimes C^{d_B})$ there holds

$$ K_D(\rho_{AB}) \leq E_r^\infty(\rho_{AB}), \quad (159) $$

**Proof:** By definition of $K_D(\rho_{AB})$ there exists protocol (i.e. sequence of maps $\Lambda_n$), such that

$$ \Lambda_n(\rho^{\otimes n}) = \gamma_d' \quad (160) $$

where

$$ \lim_n \frac{\log d}{n} = K_D(\rho_{AB}) \quad (161) $$

and

$$ \lim_n ||\gamma_d' - \gamma_d|| = \lim_n \epsilon_n = 0 \quad (162) $$

with $\gamma_d$ being pdit with dimension $d^2$ of the key part.

We will present now the chain of (in)equalities, and comment it below.

$$ S(\rho_{AB}^{\otimes n}||\sigma_{sept}) \geq S(\rho_{AB}^{\otimes n}||\sigma_{sept}) = S(U_{\gamma_{ABA'B'}}U^\dagger[\rho_{ABA'B'}^{\otimes n}U_{\gamma_{ABA'B'}}U^\dagger]) \geq S(\text{Tr}_{A'B'}[U_{\gamma_{ABA'B'}}U^\dagger]|\rho_{ABA'B'}^{\otimes n}U_{\gamma_{ABA'B'}}U^\dagger]) \geq S(P_+|\sigma) \geq \inf_{\tau \in \mathcal{S}} S(P_+|\sigma) := E_r^T(P_+) \geq E_r^T(P_+) - 4\|P_+ - P_+^r\| \log d - h(||P_+ - P_+^r||) \geq (1 - 4\epsilon_n) \log d - h(\epsilon_n) \quad (163) $$
Inequality (163) is due to the fact, that relative entropy does not increase under completely positive maps; in particular it can not increase under LOCC action applied to it’s both arguments (second argument becomes other separable state since LOCC operations can not create entanglement). In the next step, Eq. (164) we perform twisting $U_\gamma$ controlled by the basis in which state $\gamma_m$ is secure (without loss of generality we can assume it is standard basis). The equality follows from the fact that unitary transformation doesn’t change the relative entropy. Next (165) we trace out $A'B'$ subsystem of both states which only decreases the relative entropy. After this operation, the first argument is $P'_+\otimes$, which is a state close to the EPR state $P_+$. ($P'_+$ would be equal to the EPR state if $\gamma_{A'B'}$ were exactly pdit) while second argument becomes some – not necessarily separable – state $\sigma$. The state belongs to the set $T$ constructed as follows. We take set of separable states on system $A'B'$ subject to twisting $U_\gamma$ and subsequently trace out the $A'B'$ subsystem. The inequality (166) holds, because we take infimum over all states from set $T$ of the function $S(P'_+|\sigma)$. This minimised version is named there $E_r^T(P'_+)$ as it is relative entropy distance of $P'_+$ from the set $T$.

Let us check now, that set $T$ fulfills the conditions of proposition 3. Convexity of this set is obvious, since (for fixed unitary $U_\gamma$) by linearity it is due to convexity of the set of separable states. This set contains the identity state, since it contains maximally mixed separable state which is unitarily invariant (i.e. invariant under $U_\gamma$ ) and whose subsystem $AB$ by definition is the maximally mixed state as well. Thus by proposition 3 we have that $E_r^T$ is asymptotically continuous

$$|E_r^T(P'_+) - E_r^T(P'_d)| < \frac{4}{n} \log d + h(||P'_+ - P_+||),$$

where we assume that the EPR state $P_+$ is of local dimension $d$. Since $P'_d$ and $P_+$ come out of $\gamma_{AB}$ and $\gamma_d$ by the same transformation described above (twisting, and partial trace) which doesn’t increase norm distance, by (162) we have that $||P'_+ - P_+|| \leq \epsilon_n$. This, together with asymptotic continuity (170) implies (167). Now by lemma 7 we have

$$E_r^T(P'_+) \geq \log d,$$

which gives the last inequality:

$$E_r^T(P'_+) \geq (1 - 4\epsilon_n) \log d - h(\epsilon_n).$$

Summarizing this chain of inequalities (163)-(168), we have that for any separable state $\sigma_{sep}$:

$$S(\rho_{AB}^{\otimes n}|\sigma_{sep}) \geq (1 - 4\epsilon_n) \log d - h(\epsilon_n)$$

Taking now infimum over all separable states $\sigma_{sep}$ we get

$$E_r(\rho_{AB}^{\otimes n}) \geq (1 - 4\epsilon_n) \log d - h(\epsilon_n).$$

Now we divide both sides by $n$ and take the limit. Then the left-hand-side converges to $E_r^\infty$. Due to (162) $\epsilon_n \to 0$ and due to (161), $\log d/n \to K_D(\rho_{AB})$. Thus due to continuity of $h$ we obtain

$$E_r^\infty \geq K_D$$

As an application of the above upper bound, we consider now the relation between distillable key and entanglement cost. For maximally entangled states these two quantities are of course equal, unlike for general pdits. As an example let us consider again a flower state given in eq. (51). As follows from [31], the flower state has $E_C$ strictly greater than the relative entropy of entanglement. Since by the above theorem we have $K_D(\gamma_{flower}) \leq E_r(\gamma_{flower})$, having $E_r(\gamma_{flower}) < E_C(\gamma_{flower})$, we obtain in this case $K_D(\gamma_{flower}) < E_C(\gamma_{flower})$.

Let us note, that the entanglement monotone approach initiated here was then used in full extent in [13], [14]. It is shown there, that in fact any bipartite monotone $E$, which is continuous and normalized on private states (i.e. $E(\gamma_d) \geq \log d$), is an upper bound on distillable key. In particular it is shown that the squashed entanglement [53], [15] is also an upper bound on distillable key.

XII. A CANDIDATE FOR NPT BOUND ENTANGLEMENT

Thus far, all known bound entangled states have positive partial transpose (are PPT). A long-standing and interesting open question is whether there exist bound entangled states which are also NPT. If such states existed, it would imply that the quantum channel capacity is non-additive. Since any NPT state is distillable with the aid of some PPT state [57], we would have the curious property that one can have two states which are each non-distillable, but if you have both states, then the joint state would be distillable.

We now present a candidate for NPT bound entangled states which are based on the states of equation (134),

$$\rho_{he} = \frac{1}{2} |\psi_+\rangle \langle \psi_+|_{AB} \otimes \tau_1^{A'B'} + \frac{1}{2} |\psi_-\rangle \langle \psi_-|_{AB} \otimes \tau_2^{A'B'}$$

and which intuitively appear to be bound entangled. Globally, the flags $\tau_1^{A'B'}$, are distinguishable, but under LOCC the flags appear almost identical, thus after Alice and Bob attempt to distinguish the flags, the state on $AB$ will be very close to an
equal mixture of \( \psi_+ \) and \( \psi_- \). The equal mixture of only two different EPR states is separable in dimension 2 \( \times \) 2, but it is at the edge of separability. A slight biasing of the mixture, causes the state to be entangled. Thus, if Alice and Bob are able to obtain even a small amount of information about which \( \tau_i \) they have, they will have a distillable state. More explicitly, if Alice and Bob attempt distillation by first guessing which hiding state flag they have, and then grouping the remaining parts of the states into two sets depending on their guess of the hiding state, they will be left with states of the form

\[
\rho_{he} = \frac{1}{2} |\psi_+\rangle \langle \psi_+|_{AB} + \frac{1}{2} |\psi_-\rangle \langle \psi_-|_{AB}.
\]  

(177)

This state is distillable.

But what if we mix in more than two different EPR states? Namely, instead of only considering hiding states (flags) correlated to odd parity Bell states (anti-key states) \( |\psi_+\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \), we also add mix in flags correlated to the even parity Bell states (key type states) \( |\phi_\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \), Consider:

\[
\rho = p_{11} |\phi_+\rangle \langle \phi_+| \otimes \rho_{11} + p_{12} |\phi_-\rangle \langle \phi_-| \otimes \rho_{12} + + p_{21} |\psi_+\rangle \langle \psi_+| \otimes \rho_{21} + p_{22} |\psi_-\rangle \langle \psi_-| \otimes \rho_{22}
\]

(178)

where

\[
\rho_{ij} = \tau_i \otimes \tau_j.
\]

(179)

Let us take for example, all \( p_{ij} = 1/4 \). Then, after attempting to distinguish the hiding states, Alice and Bob will have a state which is very close to the maximally mixed state (i.e. the state will be very close to a mixture of all four Bell states). The maximally mixed state is very far from being entangled, thus even if Alice and Bob’s measurements on the hiding states are able to bias the mixture away from the maximally mixed state, the state will still be separable.

Intuitively, it is thus clear why the state of equation (178) will not be distillable. Any protocol which attempts to first distinguish which Bell state the parties have, will fail. But is the state entangled? Indeed it is, in fact it has negative partial transpose. To see this, we look at the block-matrix form of the state

\[
\rho = \frac{1}{4} \begin{bmatrix}
\tau_1 \otimes (\tau_1 + \tau_2) & 0 & 0 & \tau_1 \otimes (\tau_1 - \tau_2) \\
0 & \tau_2 \otimes (\tau_1 + \tau_2) & \tau_2 \otimes (\tau_1 - \tau_2) & 0 \\
0 & \tau_2 \otimes (\tau_1 - \tau_2) & \tau_2 \otimes (\tau_1 + \tau_2) & 0 \\
\tau_1 \otimes (\tau_1 - \tau_2) & 0 & 0 & \tau_1 \otimes (\tau_1 + \tau_2)
\end{bmatrix}
\]

(180)

If the matrix were PPT we would have in particular

\[
\tau_1^\Gamma \otimes (\tau_1^\Gamma + \tau_2^\Gamma) \geq \tau_2^\Gamma \otimes (\tau_1^\Gamma - \tau_2^\Gamma)
\]

(181)

We will argue that it is not true. Let us recall that

\[
\tau_1^\Gamma = \left(\frac{P_+^\perp}{d^2 - 1}\right) \otimes k
\]

\[
\tau_2^\Gamma = \left(\frac{P_+^\perp}{d^2 + d} + \frac{(1 + d)P_+^\perp}{d^2 + d}\right) \otimes k = \left(\frac{P_+^\perp}{d^2 + d}\right) \otimes k + R
\]

\[
\tau_1^\Gamma - \tau_2^\Gamma = \left[(P_+^\perp)^{\otimes k} \frac{1}{(d^2 - 1)^k} - \frac{1}{(d^2 + d)^k} - R\right]
\]

\[
\tau_1^\Gamma + \tau_2^\Gamma = \left[(P_+^\perp)^{\otimes k} \frac{1}{(d^2 - 1)^k} + \frac{1}{(d^2 + d)^k} + R\right]
\]

(182)

where we use notation from sec. XI. To see that (181) is not satisfied we consider the following projector

\[ Q = I - (P_+^\perp)^{\otimes k} \]

(183)

i.e. \( Q \) is projector onto support of positive operator \( R \). Since \( \text{Tr}(Q \tau_1^\Gamma) = 0 \), we have

\[
\text{Tr}[(Q \otimes (P_+^\perp)^{\otimes k})\tau_1^\Gamma \otimes (\tau_1^\Gamma + \tau_2^\Gamma)] = 0
\]

(184)

Moreover we have

\[
\text{Tr}(Q \otimes (P_+^\perp)^{\otimes k})[\tau_2^\Gamma \otimes (\tau_1^\Gamma - \tau_2^\Gamma)] = \\
= \text{Tr}R \left(\frac{1}{(d^2 - 1)^k} - \frac{1}{(d^2 + d)^k}\right)
\]

(185)

The above quantity is strictly greater than zero for \( d \geq 2 \). Thus inequality (181) is violated on projector \( Q \otimes (P_+^\perp)^{\otimes k} \).
Now, it may be that there is a protocol which succeeds in distilling from the state \( \psi \) which does not rely on first performing a measurement to distinguish the hiding states. However, even taking many copies of the state, produces a state of the form
\[
\rho = \sum_i |\psi_i\rangle \langle \psi_i| \otimes \rho_i \tag{186}
\]
with the \( \rho_i \) being binary strings encoded in hiding states and \( \psi_i \) being the basis of maximally entangled states. Thus the form of the state is invariant under tensoring. There is thus a very strong intuition that these states are NPT bound entangled, and a very good understanding of why they might be so. Effectively, the partial transpose does not feel very strongly the fact that the states \( \rho \) are hiding states, but more strongly feels the fact that they are globally orthogonal.

### XIII. Controlled Private Quantum Channels

Here, we demonstrate a cryptographic application of bound entangled states which have key. A private quantum channel (PQC)\([10],[44]\) allows for the sending of quantum states such that an eavesdropper learns nothing about the sent states. Here, we consider the cryptographic primitive of having the ability to securely send quantum states (a PQC), but that this ability can be turned on and off by a controller. Namely, we consider a three party scenario (Alice, Bob, and the (C)controller) and demand

- Alice and Bob have a private quantum channel, which they can use to send an unknown qubit from one to the other in such a way that they can be sure that no eavesdropper (including the Controller), can gain information about the state being sent.
- the Controller has the ability to determine whether or not Alice and Bob can send the qubit

We now show that this can be done using shared quantum states in such a way that the Controller only needs to send classical communication to one of the parties in order to activate the channel. First, let us note that the standard way of controlling the entanglement of two parties is via the GHZ state
\[
|\psi\rangle_{ABC} = |000\rangle + |111\rangle . \tag{187}
\]
If the Controller, (Claire), measures in the basis \( |0\rangle \pm |1\rangle \), then, depending on the outcome, Alice and Bob will share either the Bell state \( |\psi_+\rangle \) or \( |\psi_-\rangle \). If \( C \) then tells them the result, they will have one unit of entanglement (ebit) which they can then use to teleport quantum states. However, if the Controller wants to give them the ability to send a single qubit securely, then the GHZ state cannot be used for this, because the Controller can trick Alice and Bob into sending part of the quantum state to her. She can claim that she obtained measurement outcome +, when in reality she has not performed a measurement at all. Then, when Alice attempts to teleport a qubit to Bob, she is in fact teleporting to both Bob and the Controller. The controller can then perform a measurement on her qubit to obtain partial information about the sent qubit. Note that here we are concerned with the ability to give single shot access to a quantum channel. If the controller gives Alice and Bob many ebits by performing measurements on many copies of a GHZ state, then Alice and Bob could always perform purity testing to determine that the Controller is honest.

Let us now show that unlike the GHZ, the states of Eq. (134) can be used in such a way that the Controller can give Alice and Bob single shot access to a private quantum channel, in such a way that Alice and Bob are sure that the Controller cannot obtain any information about the sent states even when the Controller cheats. We will then show that we can do the same thing with fully bound entangled states, so that Alice and Bob possess no distillable entanglement unless the Controller gives it to them.

First, we assume the shared state as a trusted resource. I.e. a trusted party gives Alice, Bob and the Controller some state which they use to implement the primitive. This assumption can be removed in the limit of many copies, since if Alice and Bob have many copies of the state, they can perform tomography to ensure that they indeed possess the correct state. The state we initially use is the purification of Eq. (134)
\[
\rho_{he} = \frac{1}{2} |\psi_+\rangle \langle \psi_+|_{AB} \otimes \tau_1^{A'B'} + \frac{1}{2} |\psi_-\rangle \langle \psi_-|_{AB} \otimes \tau_2^{A'B'} \tag{188}
\]
Namely,
\[
|\psi\rangle_{ABC} = |00\rangle_{AB} \otimes |\phi_1\rangle_{A'B'C} + |11\rangle_{AB} \otimes |\phi_2\rangle_{A'B'C} \tag{189}
\]
such that \( \text{Tr}_C(|\phi_1\rangle \langle \phi_1|) = \tau_1 \).

Thus, \( \langle \phi_1| \phi_2 \rangle = 0 \) and since the \( \tau_i \) are orthogonal, the Controller’s states \( \text{Tr}_{A'B'}(|\phi_i\rangle \langle \phi_i|) = \sigma_i^C \) will be orthogonal. The controller can thus give Alice and Bob one ebit by performing a measurement to distinguish the \( \sigma_i^C \). She then tells Alice and Bob the result. Alice and Bob on the other hand, are guaranteed security by the fact that they either possess the state \( |\psi_+\rangle \) or \( |\psi_-\rangle \). I.e. it is an incoherent mixture of the two states, and they either have one of the states or the other, they just don’t know which one they have.

The state of Eq. (134) however, does have an arbitrarily small amount of distillable entanglement. Thus, Alice and Bob will have access to a private quantum channel in the case of having many copies of the state. If we want to give full control to
Claire, we need to ensure that the state held by Alice and Bob in the absence of Claire’s communication is non-distillable. This can be achieved by using the bound entangled states of equation (139) which approximate a pbit. It is not hard to verify, by explicitly writing the state in the Bell basis on $AB$, that the state is arbitrarily close to a state of the same form as equation (139), and thus has the desired properties.

**XIV. Conclusion**

We have seen that one can recast obtaining a private key under LOPC in terms of distilling private states under LOCC. One finds a general class of states which are unconditionally secure. This class includes bound entangled states from which one cannot distill pure entanglement. This then enables one to use tools developed in entanglement theory to tackle privacy theory.

Many open questions remain. The most important problem in this context is whether all entangled states have non-zero private state cost. In particular, the regularized relative entropy of entanglement was found to be an upper bound on the rate of private key. This then enables one to use tools developed in entanglement theory to tackle privacy theory.

Exploring the wide class of private states especially in the context of the well established theory of distillation of entanglement appears to be a necessary step in order to solve the above important problems.

**XV. Appendix**

A. Derivation of formula (96) of Sec. VI

We show that a bipartite state of four subsystems $AB' \ A' B'$ given as

$$\rho_{ABA'B'} = \sum_{ij} |ij\rangle_{AB} \langle i'j'| \otimes A_{ij}^{i'j'}_{AB'},$$

(190)

where $A_{ij}^{i'j'}$ are block matrices can be written as

$$\rho_{ABA'B'} = \sum_{ij} \sqrt{p_{ij} p_{i'j'}} |ij\rangle_{AB} \langle i'j'| \otimes \left[U_{ij} \sqrt{\rho_E^{ij}} \sqrt{\rho_E^{i'j'}} U_{ij}^\dagger \right]^{T},$$

(191)

To see this, we first write down its total purification:

$$\psi_{ABA'B'E} = \sum_{ij} \sqrt{p_{ij}} |ij\rangle_{AB} \langle i'j'| \otimes \left[U_{ij} \sqrt{\rho_E^{ij}} \sqrt{\rho_E^{ij}} U_{ij}^\dagger \right]^{T}.$$

(192)

The states $\psi_{ABA'B'E}^{ij}$ can be written as

$$\psi_{ABA'B'E}^{ij} = \sum_{k=1}^{d_{ABA'B'}} \lambda_{ij}^{ij} V_{ij}^{kA} B' \otimes U_{ij} W |k\rangle_{ABA'B'}.$$  

(193)

Here $U_{ij}$ is unitary transformation acting on Eve’s system, $V_{ij}$ is unitary transformation acting on shield $A'B'$ and $W$ is some fixed embedding of $\mathcal{H}_{ABA'B'}$ into $\mathcal{H}_E$ (this is needed if Eve’s systems are greater than the system $A'B'$):

$$W: \mathcal{H}_{ABA'B'} \rightarrow \mathcal{H}_E, \quad W|k\rangle_{ABA'B'} = |k\rangle_E$$

(194)

where $|k\rangle_{ABA'B'}$, $k = 1, \ldots, d_{ABA'B'}$ is a fixed basis in system $A'B'$, while $|k\rangle_E$, $k = 1, \ldots, d_E$ is a fixed basis in system $E$. We will also need a dual operation, which is fixed projection of space $\mathcal{H}_E$ into $\mathcal{H}_{ABA'B'}$:

$$W^\dagger: \mathcal{H}_E \rightarrow \mathcal{H}_{ABA'B'},$$

(195)

with

$$W^\dagger|k\rangle_E = |k\rangle_{ABA'B'} \quad \text{for} \quad k = 1, \ldots, d_{ABA'B'}$$

(196)

$$W^\dagger|k\rangle_E = 0 \quad \text{for} \quad k > d_{ABA'B'}$$

(197)

One then finds that

$$(A_{ij}^{i'j'})^T = V_{ij}^{ij} W^\dagger U_{ij} \sqrt{\rho_E^{ij}} \sqrt{\rho_E^{i'j'}} U_{ij}^\dagger W V_{ij}^{ij},$$

(198)
where $T$ is matrix transposition. One can find, the operator $U_{j}^{t} = U_{ij}WV_{ij}$ maps the space $H_{A'B'}$ exactly onto a support of $\rho_{E}^{i'j'}$ in space $H_{E}$, and the dual operator $U_{ij}^{t} = V_{ij}^{t}W^{t}U_{ij}^{t}$ maps the support of $\rho_{E}^{ij}$ back to $H_{A'B'}$. Finally, our state is of the form

$$\rho_{ABA'B'} = \sum_{i'j'} \sqrt{p_{ij}p_{i'j'}}|ij\rangle_{AB}|i'j'\rangle_{A'B'} \otimes \|U_{ij}\sqrt{\rho_{E}^{ij}}\sqrt{\rho_{E}^{i'j'}}U_{ij}^{t}\|^{T},$$

(199)

which we aimed to show.

B. The proof of lemma 5 Section XA

We prove now that the states from a family that we have introduced in eq. (136), are indeed PPT for certain range of parameters, as it is stated in lemma 5.

Proof: The matrix of the state (136) after partial transposition has a form

$$\rho_{ABA'B'}^{T} = \begin{bmatrix}
p(\frac{\tau_{1}+\tau_{2}}{2})^{T} & 0 & 0 & 0 \\
0 & \frac{1}{2} - \rho_{e} & p(\frac{\tau_{1}-\tau_{2}}{2})^{T} & 0 \\
0 & p(\frac{\tau_{1}-\tau_{2}}{2})^{T} & \frac{1}{2} - \rho_{e} & 0 \\
0 & 0 & 0 & p(\frac{\tau_{1}+\tau_{2}}{2})^{T}
\end{bmatrix}.$$  

(200)

Since $\tau_{1}$ and $\tau_{2}$ are separable (and hence PPT), so is their mixture. Thus extreme-diagonal blocks of the above matrix are positive. It remains to check positivity of the middle block matrix. Since any block matrix of the form

$$\begin{bmatrix}
A & B \\
B & A
\end{bmatrix},$$

(201)

is positive if there holds $A \geq |B|$ where $A$ and $B$ are arbitrary hermitian matrices, our question of positivity of (200) reads

$$\frac{1}{2} - \rho_{e} \geq p(\frac{\tau_{1}-\tau_{2}}{2})^{T}.$$  

(202)

Having $\rho_{e} = \frac{1}{\tau_{1}+\tau_{2}}(I + V)$ and $\rho_{e} = \frac{1}{\tau_{1}+\tau_{2}}(I - V)$ where $V$ swaps $d$-dimensional spaces and applying $V^{T} = dP_{+}$ one easily gets that

$$\tau_{1}^{T} = \left(\frac{P_{+}}{d^2 - 1}\right)^{\otimes k}$$

(203)

$$\tau_{2}^{T} = \left(\frac{P_{+}}{d^2 - 1} + \frac{(1+d)P_{+}}{d^2 + d}\right)^{\otimes k}$$

(204)

where $P_{+} \equiv I - P_{+}$ is projector onto subspace orthogonal to the projector onto maximally entangled state $P_{+} = |\psi_{+}\rangle\langle\psi_{+}|$.

We check then the inequality

$$\left(\frac{1}{2} - \rho_{e}\right)\left(\frac{P_{+}}{d^2 + d} + \frac{(1+d)P_{+}}{d^2 + d}\right)^{\otimes k} \geq$$

$$\geq \frac{p}{2} \left|\left(\frac{P_{+}}{d^2 - 1}\right)^{\otimes k} - \left(\frac{P_{+}}{d^2 + d} + \frac{(1+d)P_{+}}{d^2 + d}\right)^{\otimes k}\right|.$$  

(205)

To solve this inequality it is useful to represent the term on LHS as a sum:

$$\left(\frac{P_{+}}{d^2 + d} + \frac{(1+d)P_{+}}{d^2 + d}\right)^{\otimes k} = \left(\frac{P_{+}}{d^2 + d}\right)^{\otimes k} + R$$

(206)

where operator $R$ is an unnormalised state which consists of all terms coming out of $k$-fold tensor product of $\left(\frac{P_{+}}{d^2 + d} + \frac{(1+d)P_{+}}{d^2 + d}\right)$ apart from the first term $\left(\frac{P_{+}}{d^2 + d}\right)^{\otimes k}$. It is good to note that $R$ has support on subspace orthogonal to $(P_{+})^{\otimes k}$. This fact allows to omit the modulus and to get

$$\left(\frac{1}{2} - \rho_{e}\right)\left(\frac{P_{+}}{d^2 + d}\right)^{\otimes k} + R \geq$$

$$\geq \frac{p}{2} \left(\left(P_{+}\right)^{\otimes k} \left(\frac{1}{(d^2 - 1)^k} - \frac{1}{(d^2 + d)^k}\right) + R\right)$$

(207)
Since $R$ and $(P^\perp)\otimes^k$ are orthogonal, this inequality is equivalent to the following two inequalities
\[
\begin{align*}
\frac{1}{2} - \frac{3}{2^p} R & \geq 0 \\
\frac{1}{2} - p \left( \frac{P^\perp}{d^2 + d} \right)^{\otimes^k} & \geq \frac{p}{2} (P^\perp)^{\otimes^k} \times \frac{1}{(d^2 - 1)^k} - \frac{1}{(d^2 + d)^k}
\end{align*}
\] (208)

To save first inequality one needs $p \leq \frac{1}{4}$. Preserving the second one requires
\[
\frac{1 - p}{p} \geq \left( \frac{d}{d - 1} \right)^k
\] (210)

This however is fulfilled for any $p \in (0, \frac{1}{4}]$ if $d$ is taken properly large for some fixed $k$. Indeed, the $k$-th root of $\frac{1 - p}{p}$ (which converges to 1 with $k$) can be greater than $\frac{d}{d - 1}$ (which converges to 1 with $d$) for some large $d$. ■

C. Comparison of two criteria for secure key

In this section, we shall compare the joint cryptographic criterion, i.e. the requirement of (112):

\[
\|\rho_{real}^{ccq} - \rho_{ideal}^{ccq}\| \leq \epsilon
\] (211)

which includes both uniformity and security in one formula with the double condition where uniformity and security are treated separately, namely:

\[
\begin{align*}
\chi(\{p_{ij}, \rho_{ij}^E\}) & \leq \epsilon \\
\|\rho_{AB} - \rho_{ideal}^{AB}\| & \leq \epsilon
\end{align*}
\] (212)

The connection between these two criteria for quantum cryptographical security of the state is given in the theorem below.

**Theorem 10:** For any ccq state $\rho_{ABE} = \sum_{i,j=0}^{d-1} p_{ij} |ij\rangle \langle ij| \otimes \rho_{ij}^E$ and $\rho_{ideal} = \sum_{i=0}^{d-1} \frac{1}{d} |ii\rangle \langle ii| \otimes \rho_E$ where $\rho_E = \sum_{ij} p_{ij} \rho_{ij}^E$, the following implications holds:

\[
\begin{align*}
\|\rho_{AB} - \rho_{ideal}^{AB}\| & \leq \epsilon \\
& \Rightarrow \chi(\{p_{ij}, \rho_{ij}^E\}) \leq \epsilon \leq \|\rho_{ABE} - \rho_{ideal}\| \leq \epsilon + \sqrt{\epsilon}
\end{align*}
\] (213)

\[
\begin{align*}
\|\rho_{ABE} - \rho_{ideal}\| & \leq \epsilon \\
& \Rightarrow \left\{ \begin{array}{l}
\chi(\rho_{ABE}) \leq 4\epsilon \log d + h(\epsilon) \\
\|\rho_{ABE} - \rho_{ideal}\| \leq \epsilon.
\end{array} \right.
\end{align*}
\] (214)

where $\rho_{AB} = \text{Tr}_E \rho_{ABE}$, $\rho_{ideal}^{AB} = \text{Tr}_E \rho_{ideal}$.

**Remark 4:** We see that the result (213) is not fully satisfactory due to the term $\log d$. However, one cannot get a better result. Indeed it is easy to construct a state, for which the Holevo function is of $\epsilon \log d$ order, though the state $\rho_{ABE}$ is $\epsilon$ close to some $\rho_{ideal}$ state. As an example may serve an appropriate extension of the isotropic state, measured in computational basis:

\[
\rho_{ABE} = (1 - \epsilon) (\sum_{i=0}^{d-1} \frac{1}{d} |ii\rangle \langle ii|) \otimes (|00\rangle \langle 00|)_E + \epsilon \sum_{i,j \neq 0} \frac{1}{d^2 - d} |ij\rangle \langle ij| \otimes (|ij\rangle \langle ij|)
\] (215)

where $\sigma$ is maximally mixed state. If we consider now $\rho_{ideal} = (\sum_{i=0}^{d-1} \frac{1}{d} |ii\rangle \langle ii|) \otimes (|00\rangle \langle 00|)_E$, it is easy to see that $\|\rho_{ABE} - \rho_{ideal}\| = 2\epsilon$. However the value of the Holevo function equals $h(\epsilon) + \epsilon \log (d^2 - d)$.

**Remark 5:** The main difficulty in the proof of the above theorem is to get the term $\log d$ ($d \times d$ is size of $AB$ system) rather than $\log d_{ABA'B'}$. The latter one would be obtained directly from Fannes type continuities. However to get $\log d$ we have to apply tricks based on twisting. It is quite convenient not to have Eve’s dimension in equivalence formula. This is because Eve’s dimension depends on the protocol that lead to the key (more specifically, it depends on the amount of communication). In contrast, dimension of Alice and Bob system is only the number of bits of obtained key. Thus, our equivalence is independent of the protocol.

**Proof:** For the first part of the theorem [10] we assume that $\chi(\{p_{ij}, \rho_{ij}^E\}) \leq \epsilon$ which by proposition [7] in Sec. XV-D of Appendix means that we have:

\[
\|\sum_{i,j=0}^{d-1} p_{ij} |ij\rangle \langle ij| \otimes \rho_{ij}^E - \sum_{i,j=0}^{d-1} p_{ij} |ij\rangle \langle ij| \otimes \rho_E\| \leq \sqrt{\epsilon},
\] (216)
with \( \rho_E = \sum_{i,j=0}^{d-1} p_{ij} \rho_{ij}^E \). Moreover by second assumption that
\[
\| \sum_{i,j=0}^{d-1} p_{ij} |ij\rangle\langle ij| - \sum_{i=0}^{d-1} \frac{1}{d} |ii\rangle\langle ii| \| \leq \epsilon
\]  
(217)
one gets
\[
\| \sum_{i,j=0}^{d-1} p_{ij} |ij\rangle\langle ij| AB \otimes \rho_E - \sum_{i=0}^{d-1} \frac{1}{d} |ii\rangle\langle ii| AB \otimes \rho_E \| \leq \epsilon.
\]  
(218)
Using triangle inequality, and Eqs. (216) and (218) one obtains the
\[
\|\rho_{ABE} - \rho_{\text{ideal}}\| \leq \epsilon + \sqrt{\epsilon}.
\]  
(219)

The proof of the second part of the theorem [10] is a bit more involved. Of course, it is immediate that due to monotonicity of trace norm under partial trace, from \( \|\rho_{ABE} - \rho_{\text{ideal}}\| \leq \epsilon \) it follows \( \|\rho_{AB} - \rho_{\text{ideal}}^{\text{AB}}\| \leq \epsilon \). The non-obvious task is to bound also \( \chi \). So, we assume that
\[
\|\rho_{ABE} - \rho_{\text{ideal}}\| \leq \epsilon,
\]  
(220)
By equality of norm and fidelity condition [13], there holds
\[
F(\rho_{ABE}; \rho_{\text{ideal}}) \geq 1 - \frac{1}{2} \epsilon
\]  
(221)
By definition of fidelity, there are pure states \( \psi \) and \( \phi \) (purifications of \( \rho_{ABE} \) and \( \rho_{\text{ideal}} \) respectively), such that \( F(\psi, \phi) = F(\rho_{ABE}; \rho_{\text{ideal}}) \). Without loss of generality we can consider the system which purifies both states to be bipartite. We will call it \( \text{A}'B' \). Now let us perform twisting operation on the \( \text{A}'\text{B}' \) parts of the pure states \( \psi \) and \( \phi \), which in the case of state \( \rho_{\text{ideal}} \) transforms \( \text{AB} \) subsystem of into maximally entangled state - \( P^+_d \), (we can choose such twisting because by the theorem [2] purification of an ideal state is some pdit state). I.e. after such twisting, pdit will become a basic pdit (9) which is product with \( \text{A}'B' \) subsystem. Since unitary transformation and tracing out can only increase fidelity, then applying again [13] we have that subsystem \( \text{AB} \) of \( \rho_{ABE} \) is close to a singlet state in norm:
\[
\|\rho_{AB} - P^+_d\| \leq 2\sqrt{\epsilon}.
\]  
(222)
Using Fannes inequality (see eq. [14] in Sec. II) we get
\[
S(\rho_{AB}) \leq 4\sqrt{\epsilon} \log d + h(2\sqrt{\epsilon}).
\]  
(223)
Since the total state of systems \( \text{A}'\text{B}'E \) is pure, we get that \( S(\rho_{\text{A}'\text{B}'E}) = S(\rho_{AB}) \) hence
\[
S(\rho_{\text{A}'\text{B}'E}) \leq 4\sqrt{\epsilon} \log d + h(2\sqrt{\epsilon}).
\]  
(224)
Now, note that the state of the system \( \text{A}'\text{B}'E \) has the form
\[
\rho_{\text{A}'\text{B}'E} = \sum_{i,j=0}^{d-1} p_{ij} \rho_{ij}^{\text{A}'\text{B}'E},
\]  
(225)
where the state \( \rho_{ij}^{\text{A}'\text{B}'E} \) denotes state of \( \text{A}'\text{B}' \) system after twisting and given that \( \text{AB} \) subsystem is in state \( |ij\rangle\langle ij| \) (i.e. if after twisting one measure the system \( \text{AB} \) in basis \( |ij\rangle \) the system \( \text{A}'\text{B}' \) would collapse to \( \rho_{ij}^{\text{A}'\text{B}'E} \)). By definition of Holevo function there holds:
\[
\chi(\{p_{ij}, \rho_{ij}^{\text{A}'\text{B}'E}\}) \leq S(\rho_{\text{A}'\text{B}'E}).
\]  
(226)
The question is how the Holevo function of the \( \{p_{ij}, \rho_{ij}^{\text{A}'\text{B}'E}\} \) ensemble is related to Holevo function of \( \{p_{ij}, \rho_{ij}^E\} \) which we would like to bound from above. It is crucial, that by theorem [1] twisting operation does not affect the ccq state which comes out of the measurement of \( \text{AB} \) in control basis of twisting. In other words, the ensemble \( \{p_i, \rho_{ij}^E\} \) does not change under twisting, so that \( \rho_{ij}^E = \text{Tr}_{A'B'}\rho_{ij}^{\text{A}'\text{B}'E} \). It is easy now to compare the functions \( \chi(\{p_{ij}, \rho_{ij}^E\}) \) and \( \chi(\{p_{ij}, \rho_{ij}^{\text{A}'\text{B}'E}\}) \):
\[
\chi(\{p_{ij}, \rho_{ij}^E\}) \leq \chi(\{p_{ij}, \rho_{ij}^{\text{A}'\text{B}'E}\})
\]  
(227)
This is due to the fact, that each state \( \rho_{ij}^E \) can be obtained from \( \rho_{ij}^{\text{A}'\text{B}'E} \) by tracing out \( \text{A}'\text{B}' \) subsystem. However tracing out can only decrease Holevo function, because this function is equal to the average relative entropy:
\[
\chi(\{p_{ij}, \rho_{ij}^E\}) = \sum_k p_k S(\rho_k) \sum_k p_k \rho_k.
\]  
(228)
Summing up the chain of inequalities (224), (226) and (227) one gets
\[
\chi(\{p_{ij}, \rho_{ij}^E\}) \leq 4\sqrt{\epsilon} \log d + h(2\sqrt{\epsilon}),
\]  
(229)
which is a desired security condition - bound on the Holevo function of the ansamble \( \{p_{ij}, \rho_{ij}^E\} \). ■
D. Useful inequalities relating security conditions

In this section we collect relations between different security conditions for ccq states. Some of these relations have been studied in [3]. Since we will not deal with uniformity, but solely with security, it is convenient to use single index $k$ in place of $ij$. We thus consider ccq state (which could be actually called cq state)

$$\rho = \sum_{k=0}^{d^2-1} p_k |k\rangle \langle k| \otimes \rho_k$$  \hspace{1cm} (230)

Basing on this fact, we can state another lemma establishing some equivalences:

**Lemma 8:** For any state (230) and any positive real $\epsilon \leq \frac{1}{4}$, the following implications hold

1) \[ || \sum_k p_k |k\rangle \langle k| \otimes \rho_k - (\sum_j p_j |j\rangle \langle j|) \otimes \rho_\rangle || \leq \epsilon \]
   \[ \Rightarrow \sum_k p_k F(\rho_k, \rho) \geq 1 - \frac{1}{2\epsilon} \]  \hspace{1cm} (231)

2) \[ \sum_k p_k F(\rho_k, \rho) \geq 1 - \epsilon \Rightarrow \sum_k p_k ||\rho_k - \rho|| \leq 8\epsilon \]  \hspace{1cm} (232)

3) \[ \sum_k p_k ||\rho_k - \rho|| \leq \epsilon \Rightarrow \]
   \[ \Rightarrow || \sum_k p_k |k\rangle \langle k| \otimes \rho_k - (\sum_j p_j |j\rangle \langle j|) \otimes \rho_\rangle || \leq \epsilon. \]  \hspace{1cm} (233)

Here $\rho = \sum_k p_k \rho_k$.

**Proof:** The first thesis follows from the mentioned equivalence of norm and fidelity and definition of fidelity. Namely one can make use of lemma [11] so that if (231) holds, the fidelity $F(\sum_k p_k |k\rangle \langle k| \otimes \rho_k, (\sum_j p_j |j\rangle \langle j|) \otimes \rho)$ is no less than $1 - \frac{1}{2\epsilon}$.

However it is equal to average fidelity $\sum_k p_k F(\rho_k, \rho)$. Indeed,

$$F(\sum_k p_k |k\rangle \langle k| \otimes \rho_k, (\sum_j p_j |j\rangle \langle j|) \otimes \rho) = \text{Tr}\left( (\sum_k p_k |k\rangle \langle k| \otimes \rho_k)^{\frac{1}{2}} (\sum_j p_j |j\rangle \langle j|) \otimes \rho (\sum_l p_l |l\rangle \langle l| \otimes \rho_l)^{\frac{1}{2}} \right)^{\frac{1}{2}}$$  \hspace{1cm} (234)

Now by orthogonality of vectors $|k\rangle$ one has

$$\sqrt{\sum_k p_k |k\rangle \langle k| \otimes \rho_k} = \sum_k \sqrt{p_k} |k\rangle \otimes \sqrt{\rho_k}.$$  \hspace{1cm} (235)

Multiplying now the $(\sum_j p_j |j\rangle \langle j|) \otimes \rho$ matrix by the above from left-hand-side and right-hand-side one gets

$$\sum_k p_k^2 |k\rangle \langle k| \otimes \sqrt{\rho_k} \sqrt{\rho_k}.$$  \hspace{1cm} (236)

This immediately gives the above formula equal to

$$\sum_k p_k \text{Tr} \sqrt{\rho_k} \sqrt{\rho_k}$$  \hspace{1cm} (237)

which is just average fidelity from (232).

The second thesis of this lemma (Eq. (232)) is again a consequence of (13). If applied to each pair $\rho_k, \rho$, and averaged over probabilities of $p_k$ gives that

$$\sum_k p_k \sqrt{1 - \frac{1}{4} ||\rho - \rho_k||^2} \geq 1 - \epsilon$$  \hspace{1cm} (238)

which is equivalent to

$$\sum_k p_k \left( 1 - \sqrt{1 - ||\rho - \rho_k||^2/4} \right) \leq \epsilon.$$  \hspace{1cm} (239)

Now by the fact that

$$1 - \sqrt{1 - \frac{1}{4} ||\rho - \rho_k||^2}$$  \hspace{1cm} (240)
is a convex function of \(|||\rho - \rho_k|||\) on interval \((0, 2)\) we get
\[
1 - \sqrt{1 - \frac{1}{4} \sum_k p_k |||\rho - \rho_k|||}^2 \leq \epsilon
\]
(241)
This however reads for \(0 < \epsilon < 1\)
\[
\sum_k p_k |||\rho - \rho_k||| \leq 8\epsilon.
\]
(242)
Since \(|||\rho - \rho_k||| \leq 2\) one has, that for \(\epsilon \geq 1\) the above inequality is also valid, which completes the proof of the second thesis of lemma 8.

The last implication (Eq. (233)) is a consequence of triangle inequality, which completes the lemma.\(^\blacksquare\)

Let us notice, that this lemma establishes a kind of equivalence of security conditions, namely:
\[
\|\sum_k p_k |k\rangle \langle k| \otimes \rho_k - (\sum_j p_j |j\rangle \langle j|) \otimes \rho\| \leq \epsilon
\]
\[
\Rightarrow \sum_k p_k |||\rho - \rho_k||| \leq 4\epsilon
\]
\[
\Rightarrow \|\sum_k p_k |k\rangle \langle k| \otimes \rho_k - (\sum_j p_j |j\rangle \langle j|) \otimes \rho\| \leq 4\epsilon
\]
(243)

We can show now links between the above conditions on ccq state and Holevo function \(\chi\) of this state, i.e. of an ensemble \(\{p_k, \rho_k\}\) which we shall write \(\chi(\rho_{ccq})\).

Lemma 9: For any ccq state \(\rho_{ccq}\) (240) there holds:
\[
\chi(\rho_{ccq}) \leq \epsilon \Rightarrow \sum_k p_k |||\rho_k - \rho||| \leq \sqrt{2\epsilon}
\]
(244)
\[
\sum_k p_k |||\rho_k - \rho||| \leq \epsilon \Rightarrow \chi(\rho_{ccq}) \leq 8\epsilon \log d + \max(h(\epsilon), 2\epsilon)
\]
where \(\sum_k p_k \rho_k = \rho\), which acts on Hilbert space \(\mathcal{H} = \mathcal{C}^d\), and \(h(\epsilon) = -\epsilon \log \epsilon - (1 - \epsilon) \log (1 - \epsilon)\) is binary entropy.

Proof: For the first statement of this lemma, let us notice that \(\chi(\rho_{ccq}) = S(\rho) - \sum_k p_k S(\rho_k)\) is just equal to average relative entropy distance \(\sum_k p_k S(\rho_k|\rho)\). Thus, by assumption we have
\[
\chi(\rho_{ccq}) = \sum_k p_k S(\rho_k|\rho) \leq \epsilon.
\]
(245)

Now we can make use of the inequality [45]:
\[
\frac{1}{2} |||\rho - \rho_k|||^2 \leq S(\rho_k|\rho)
\]
(246)
which after averaging over probabilities and by concavity of square root gives
\[
\sum_k p_k |||\rho - \rho_k||| \leq \sqrt{2 \sum_k p_k S(\rho_k|\rho)}.
\]
(247)
Applying now bound (245) we obtain
\[
\sum_k p_k |||\rho - \rho_k||| \leq \sqrt{2\epsilon}
\]
(248)
which completes first thesis of this lemma.

To prove the second statement of the lemma we use the Fannes inequality (see eq. (14) in Sec. 11-A). Namely, for \(|||\rho - \rho_k||| \leq 1\) there holds:
\[
|S(\rho) - S(\rho_k)| \leq 2|||\rho - \rho_k||| \log d + h(|||\rho - \rho_k|||).
\]
(249)
Let \(G = \{k : |||\rho - \rho_k||| \leq 1\}\) and \(B = \{k : |||\rho - \rho_k||| > 1\}\), and denote \(\sum_{k \in G} p_k [S(\rho) - S(\rho_k)] = \chi_G(\rho_{ccq})\), and \(\chi_B(\rho_{ccq})\) analogously. We then have, that
\[
\chi(\rho_{ccq}) = \chi_G(\rho_{ccq}) + \chi_B(\rho_{ccq}).
\]
(250)

We will give now the bounds for \(\chi_G(\rho_{ccq})\) and \(\chi_B(\rho_{ccq})\) respectively. The first quantity is directly bounded by the analogous sum over LHS of the Fannes inequality:
\[
\chi_G(\rho_{ccq}) \leq \sum_{k \in G} p_k (2|||\rho - \rho_k||| \log d + h(|||\rho - \rho_k|||)).
\]
(251)
\[ \sum_k p_k \| \rho - \rho_k \| \leq \epsilon \]  

(252)

It follows that \( \sum_{k \in E} p_k \| \rho - \rho_k \| \leq \epsilon \). Using this, and adding non-negative terms \( \sum_{k \in B} p_k h(\| \rho - \rho_k \|) \), we get:

\[ \chi_G(\rho_{ccq}) \leq 2 \epsilon \log d + \sum_k p_k h(\| \rho - \rho_k \|). \]  

(253)

Now by concavity of binary entropy one gets

\[ \chi_G(\rho_{ccq}) \leq 2 \epsilon \log d + h(\sum_k p_k \| \rho - \rho_k \|), \]  

(254)

Were the entropy increasing on \([0, \infty]\) interval, one could use directly the assumption that \( \sum_k p_k \| \rho - \rho_k \| \leq \epsilon \), and bound \( h(\sum_k p_k \| \rho - \rho_k \|) \) by \( h(\epsilon) \). Since it is the case only for \( \epsilon \in [0, \frac{1}{2}] \), we have to end up with more ugly, but nonetheless useful expression. Namely on the interval \((\frac{1}{2}, \infty)\) where the entropy becomes decreasing, it is bounded by 1, and hence not greater than \( 2 \epsilon \) for \( \epsilon \in (\frac{1}{2}, \infty) \). Thus finally one gets

\[ \chi_G(\rho_{ccq}) \leq 2 \epsilon \log d + \max(h(\epsilon), 2 \epsilon). \]  

(255)

We turn now to give the bound for \( \chi_B(\rho_{ccq}) \). For the latter we use the fact, that the Holevo quantity is bounded from above by \( \log d \), which gives:

\[ \chi_B(\rho_{ccq}) \leq \sum_{k \in B} p_k \log d. \]  

(256)

To bound the last inequality we observe, that by definition of the set \( B \) we have \( \sum_{k \in B} p_k \| \rho - \rho_k \| \geq \sum_{k \in B} p_k \). Then, again by assumption (252), we have

\[ \chi_B(\rho_{ccq}) \leq \epsilon \log d. \]  

(257)

Collecting inequality (255) and the above one, we arrive at the formula

\[ \chi(\rho_{ccq}) \leq 3 \epsilon \log d + \max(h(\epsilon), 2 \epsilon), \]  

(258)

which ends the proof of the lemma. ■

The lemmas above allow to prove the following proposition

**Proposition 7:** For any ccq state (230) the following holds:

\[ \| \sum_k p_k |k\rangle \langle k| \otimes \rho_k - \left( \sum_j p_j |j\rangle \langle j| \right) \otimes \rho \| \leq \sqrt{2\epsilon} \]  

(259)

where \( \rho = \sum_k p_k \rho_k \).

**Proof:** Assuming that Holevo function is smaller than \( \epsilon \), we get by lemma 3 that \( \sum_k p_k \| \rho_k - \rho \| \leq \sqrt{2\epsilon} \). This however implies by lemma 8 that \( \| \sum_k p_k |k\rangle \langle k| \otimes \rho_k - \left( \sum_j p_j |j\rangle \langle j| \right) \otimes \rho \| \) is also not greater than \( \sqrt{2\epsilon} \), which completes proof of the proposition. ■

**E. Properties of pbits**

We shall give here detailed proof of the lemma 3 Section [IV.A]

**Proof:** Log-negativity \([56]\) (cf. \([59]\)) is defined as \( E_N(\rho) = \log(\| \rho^\Gamma \|) \). It is easy to see, that after partial transposition on \( BB' \) subsystem, the pbit \( \gamma \) in \( X \)-form changes into

\[ \gamma^\Gamma_{ABA'B'} = \frac{1}{2} \begin{bmatrix} \sqrt{XX^\Gamma} & 0 & 0 & 0 \\ 0 & 0 & X^\Gamma & 0 \\ 0 & (X^\Gamma)^\Gamma & 0 & 0 \\ 0 & 0 & 0 & \sqrt{XX^\Gamma} \end{bmatrix}. \]  

(260)

We have

\[ \| \gamma^\Gamma \| = \frac{1}{2}(\| \sqrt{XX^\Gamma} \| + \| \sqrt{XX^\Gamma} \| + \| A \|), \]  

(261)

where

\[ A = \begin{bmatrix} 0 \\ (X^\Gamma)^\Gamma \\ 0 \\ (X^\Gamma)^\Gamma \end{bmatrix}. \]  

(262)
By assumption, the operators \([XX^\dagger]^\Gamma\) and \([X^\dagger X]^\Gamma\) are positive, so that
\[
||\sqrt{XX^\dagger}||^\Gamma + ||\sqrt{X^\dagger X}||^\Gamma = \text{Tr}(\sqrt{XX^\dagger} + \sqrt{X^\dagger X})^\Gamma = 2\text{Tr}\gamma^\Gamma = 2. \tag{263}
\]
The last equality comes from the fact that \(\Gamma\) preserves trace. To evaluate norm of \(A\), we note that due to unitary invariance of trace norm we have \(||A|| = ||\sigma_x \otimes I_{A'B'}A||\). Consequently
\[
||A|| = ||X^\Gamma|| + ||(X^\dagger)^\Gamma|| = 2||X^\Gamma||. \tag{264}
\]
The last equality follows from the fact that \(\Gamma\) commutes with Hermitian conjugation, and trace norm is invariant under Hermitian conjugation \(||X|| = ||X^\dagger||\). Thus we get
\[
E_N(\gamma) = \log(1 + ||X^\Gamma||), \tag{265}
\]
which proves the lemma.

\[\]

F. Relative entropy of entanglement and pdits

We give now the proof of the theorem 4, Section XV-E.

Proof: If one consider the thesis of theorem 3 for the state \(\rho = \gamma_{ABA'B'}\), it follows that
\[
E_r(\rho) \leq \log d^n + \frac{1}{d^n} \sum_{k=0}^{d^n-1} E_r(\sigma_k), \tag{266}
\]
with \(k\) being the multiindex \(k = (i_1, ..., i_n)\) with \(i_l \in \{0, ..., d-1\}\) for \(l \in \{1, ..., n\}\) and \(\sigma_k = \rho_{i_1} \otimes ... \otimes \rho_{i_n}\). Dividing both sides by \(n\) we obtain
\[
\frac{1}{n} E_r(\gamma_{ABA'B'}^n) \leq \log d + \frac{1}{nd^n} \sum_{k=0}^{d^n-1} E_r(\sigma_k), \tag{267}
\]
The left-hand-side of this inequality approaches \(E_r^\infty(\gamma)\) with \(n\). What has to be shown is that
\[
\lim_{n \to \infty} \frac{1}{nd^n} \sum_{k=0}^{d^n-1} E_r(\sigma_k) \leq \frac{1}{d} \sum_{i=0}^{d-1} E_r^\infty(\rho_i), \tag{268}
\]
with \(\rho_i\) denoting the conditional states on \(A'B'\) subsystem.

Let us first observe that \(E_r(\sigma_k) = E_r(\sigma_{k'})\) for any \(k\) and \(k'\) which are of the same type, i.e. which has the same numbers of occurrence of symbols from set \(\{0, ..., d-1\}\). This is because \(\sigma_k\) and \(\sigma_k'\) differ by local reversible transformation which does not change the entanglement. Moreover, as we will see, one can consider only those \(\sigma_k\) for which \(k\) is \(\delta\)-strongly typical i.e. such, that for some fixed \(\delta > 0\) there holds [16]:
\[
\forall a \in \{0, ..., d-1\} \quad \left| \frac{a(k)}{n} - \frac{1}{d} \right| < \delta, \tag{269}
\]
where \(a(k)\) denotes frequency of symbol \(a\) in sequence \(k\). The set of such \(k\) of length \(n\) we will denote as \(ST_\delta^n\). It is known, that the strongly typical set carries almost whole probability mass for large \(n\), that is for any \(\delta > 0\) and any \(\epsilon > 0\) there exists \(n_0\) such that for all \(n \geq n_0:\n
\[
P^{\otimes n}(ST_\delta^n) > 1 - \epsilon. \tag{270}
\]
Now, since we deal here with homogeneous distribution, we can say, that the probability of the set of events is directly related to the power of this set. Namely we have:
\[
|ST_\delta^n|/d^n > 1 - \epsilon, \tag{271}
\]
which gives \(|ST_\delta^n| > (1 - \epsilon)d^n\).

We can rewrite now the term of the LHS of (268) as follows:
\[
\frac{1}{nd^n} \sum_{k=0}^{d^n-1} E_r(\sigma_k) = \frac{1}{nd^n} \left( \sum_{k \in ST_\delta^n} E_r(\sigma_k) + \sum_{k \notin ST_\delta^n} E_r(\sigma_k) \right). \tag{272}
\]
We can get rid of the second term of the RHS of the above equality, because we can bound from above for each \(k\) the term \(E_r(\sigma_k)\) by \(n \log d\), which gives:
\[
\frac{1}{nd^n} \sum_{k=0}^{d^n-1} E_r(\sigma_k) \leq \frac{1}{nd^n} \left( \sum_{k \in ST_\delta^n} E_r(\sigma_k) + n \log d \sum_{k \notin ST_\delta^n} E_r(\sigma_k) \right) \leq \frac{1}{nd^n} \sum_{k \in ST_\delta^n} E_r(\sigma_k) + \epsilon \log d \tag{273}
\]
Now for each \( k \in ST^a_{\delta} \) we have
\[
E_r(\sigma_k) = E_r(\rho_0^{\otimes \hat{m}_0} \otimes \rho_1^{\otimes \hat{m}_1} \otimes \cdots \otimes \rho_{d-1}^{\otimes \hat{m}_{d-1}})
\] (274)
with \( \hat{m}_i = i(k) \) with \( i \) in place of \( a \) in (269). By subadditivity of \( E_r \) one has
\[
E_r(\sigma_k) \leq \sum_{l=0}^{d-1} E_r(\rho_l^{\otimes \hat{m}_l}).
\]
(275)

Note, that \( \rho_0 \) stands here for the state on shield part of one copy of \( \gamma_{ABA'B'} \). Applying this inequality for each \( k \in ST^a_{\delta} \) and taking maximum of LHS of the above inequality over \( k \), we have a bound:
\[
\frac{1}{nd^n} \sum_{k=0}^{d^n-1} E_r(\sigma_k) \leq \sum_{l=0}^{d-1} \frac{1}{n} E_r(\rho_l^{\otimes m_l}) + \epsilon \log d,
\]
(276)
where \( m_l \) are the coefficients of the decomposition of some \( \sigma_k \) into \( \rho_0^{\otimes m_0} \), which yields the maximal value of \( E_r(\sigma_k) \) over all strongly typical \( k \). Now, we can rewrite the RHS of the above inequality as:
\[
\sum_{l=0}^{d-1} \frac{m_l}{n} \frac{1}{m_l} E_r(\rho_l^{\otimes m_l}) + \epsilon \log d \leq \left( \frac{1}{d} + \delta \right) \sum_{l=0}^{d-1} \frac{1}{m_l} E_r(\rho_l^{\otimes m_l}) + \epsilon \log d
\]
(277)
where the last inequality holds for sufficiently high \( n \) by assumption of strong typicality. Thus, for every \( \delta \) and \( \epsilon \) and sufficiently large \( n \) there holds:
\[
\frac{1}{nd^n} \sum_{k=0}^{d^n-1} E_r(\sigma_k) \leq \left( \frac{1}{d} + \delta \right) \sum_{l=0}^{d-1} \frac{1}{m_l} E_r(\rho_l^{\otimes m_l}) + \epsilon \log d,
\]
(278)
One then sees, that the RHS approaches
\[
\left( \frac{1}{d} + \delta \right) \sum_{l=0}^{d-1} E_r^\infty(\rho_l) + \epsilon \log d
\]
(279)
in limit of large \( n \). Indeed, since for every \( l \) the limit \( E_r^\infty(\rho_l) \) exists, the subsequence \( \frac{1}{n} E_r(\rho_l^{\otimes m_l}) \) approaches this limit. Taking now infimum over \( \delta \) and \( \epsilon \) we prove the inequality (268). This proves the theorem [4] ■

G. Approximate pbits

We give here the proof of lemma [4]

Proof: Assume first, that \( \text{Tr}_{\rho_{AB}} P_+ > 1 - \epsilon \). Since the elements \( a_{ijkl} \) are real, by hermicity of the state we have
\[
\text{Tr}_{\rho_{AB}} P_+ = \frac{1}{2} (a_{0000} + a_{1111} + 2 \text{Re}(a_{0011}))
\]
(280)
This is however less than or equal to \( \frac{1}{2} (1 + 2 a_{0011}) \), which is in turn greater than \( 1 - \epsilon \), and the assertion follows. For the second part of the lemma, assume that \( a_{0011} > \frac{1}{2} - \epsilon \). We then have
\[
\text{Tr}_{\rho_{AB}} P_+ > \frac{1}{2} (a_{0000} + a_{1111} + 1 - 2\epsilon).
\]

We now bound the sum of \( a_{0000} \) and \( a_{1111} \). By positivity of the state, we have that \( \sqrt{a_{0000}a_{1111}} > |a_{0011}| \). Now, by arithmetic-geometric mean inequality, we have that \( a_{0000} + a_{1111} \geq 2 \sqrt{a_{0000}a_{1111}} \) which gives the proof. ■

H. Relative entropy bound

Proof: (of Lemma [7] Section [XI] Let us first show, that
\[
\text{Tr}_{P_d^+} \sigma_{AB} \leq \frac{1}{d}
\]
(281)
for any \( \sigma_{AB} \in T \). We first show this for \( \sigma_{AB} \) “derived” from some pure product states \( |\psi\rangle \langle \psi| \):
\[
\sigma_{AB} = \text{Tr}_{A'B'} U^\dagger |\psi\rangle \langle \psi| U.
\]
(282)
Because \( \psi \) is product, it can be written as
\[
\psi = (\sum a_i |i_A\rangle |\psi_i\rangle) \otimes (\sum b_i |i_B\rangle |\phi_i\rangle)
\]
(283)
with \( a_i, b_i \) normalized and \( |i_A\rangle, |i_B\rangle, |\psi_i\rangle, |\phi_i\rangle \) on subsystem \( A, B, A', B' \) respectively.
Now the condition that the reduced $AB$ state has overlap with $P_d^+$ no greater than $1/d$ is
\[
\sum_{ij} a_i b_j a_i^* b_j^* \langle x_i | x_j \rangle \leq 1
\]  
(284)
where $x_k$ are arbitrary vectors of norm one arising from the action of $U$ on $\psi_i$ and $\phi_i$. Since the $x_k$ are arbitrary they can incorporate the phases of $a_i, b_i$ so that we require now $\sum_{ij} \sqrt{p_ip_j} |x_i\rangle |x_j\rangle \leq 1$, where $p_i$ and $q_i$ are probabilities. Now, the right hand side will not decrease if we assume $\langle x_i | x_j \rangle = 1$ so we require $\sum_{i} \sqrt{p_i q_i} \leq 1$ which is satisfied by any probability distribution, which gives the proof of (281) for special $\sigma_{AB}$.

To show the inequality is true in general we find that
\[
\text{Tr}P_d^+ \text{Tr}_{AB} U^\dagger \sum_k p_k |\psi_k\rangle \langle \psi_k| U = \sum_k p_k \text{Tr}P_d^+ \text{Tr}_{AB} U^\dagger |\psi_k\rangle \langle \psi_k| U.
\]  
(285)
Thus if (281) holds for $\sigma_{AB}$ derived from pure (product) state, by averaging over probabilities, we will have (281) for an arbitrary $\sigma_{AB}$ from the set $T$.

Now by concavity of logarithm, we have for any states $\rho$ and $\sigma$:
\[
S(\rho||\sigma) = -S(\rho) - \text{Tr}(\rho \log \sigma) \geq -S(\rho) - \log(\text{Tr}\rho\sigma)
\]  
(286)
Applying inequality (281) we have that
\[
- \log(\text{Tr}\rho\sigma) \geq \log d.
\]  
(287)
Now by (286) we have that
\[
S(P_d^+ || \sigma_{AB}) \geq \log d,
\]  
(288)
which is a desired bound.

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