A proposal for a first class conversion formalism based on the symmetries of the Wess-Zumino terms

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Abstract

We propose a new procedure to embed second class systems by introducing Wess-Zumino (WZ) fields in order to unveil hidden symmetries existent in the models. This formalism is based on the direct imposition that the new Hamiltonian must be invariant by gauge-symmetry transformations. An interesting feature in this approach is the possibility to fix the WZ fields in a convenient way, which leads to preserve the gauge symmetry in the original phase space. Consequently, the gauge invariant Hamiltonian can be written only in terms of the original phase-space variables. We apply this formalism to important physical models: the reduced-SU(2) Skyrme model, the Chern-Simons-Proca quantum mechanics and the chiral bosons field theory. In all these examples, the gauge-invariant Hamiltonians are derived in a very simple way when compared with the traditional BFFT approach [1].

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I. INTRODUCTION

It is well known that through symmetries, important properties present on physical systems might be investigated in a more general way. In view of this, many works [2] have embedded second class systems into first class ones by enlarging the phase-space with the introduction of WZ variables. The motivation was to reveal symmetries and, subsequently, to cancel anomalies. At the same time, an alternative gauge-invariant approach [3,4] has been proposed, which considers part of the total second class constraints as gauge fixing terms, while the remaining ones form a subset that satisfies a first class algebra. This formalism has an elegant property that does not extend the phase-space with extra variables.

The main feature of this paper is to propose an alternative scheme to embed noninvariant models and, in some cases, to extract hidden symmetries existent in those models. This new approach mixes the WZ and projection concepts idealized in Ref. [5] and Refs. [3,4], respectively. We will extend the initial noninvariant gauge Hamiltonian with the introduction of an arbitrary function \( G \) written in terms of the original phase space and WZ variables, as suggested by Faddeev [5]. Afterward, the extended Hamiltonian is constructed such as it must satisfy the variational condition, \( \delta H = 0 \), i.e., the new Hamiltonian must be invariant by gauge-symmetry transformations. Here, it is opportune to mention that symmetries, obtained in the other constrained conversion formalisms [1,2], appear as a consequence of the first class conversion mechanism. We will see that the possibility of choosing particular symmetries for the WZ terms lead to the considerable simplifications in the determination of the gauge invariant Hamiltonian. Further, we show that the WZ terms can be fixed in some cases. Consequently, the invariant Hamiltonian can be written only as function of the original phase-space variables.

In order to clarify the exposition of the subject, the paper is organized as follows. In Sec. II, the formalism, which it will be called as “variational gauge-invariant formalism”, will be presented in detail. In Sec. III, we will apply this formalism on some important physical systems in order to unveil the hidden symmetry and also to eliminate anomalies that hamper the quantization process of chiral theories. We begin considering to study the SU(2) Skyrme model [6], which is an effective field theory to describe hadrons physics. Nappi et al. [7], using collective coordinates, reduced the SU(2) Skyrme model to a nonlinear quantum mechanical model depending explicitly on the time-dependent collective variables, which satisfies a spherical constraint. Afterward, the Chern-Simons-Proca (CSP) model [8,9]
is considered. It is a quantum mechanical system obtained from the Abelian Chern-Simons theory, which has the gauge field modified by adding a Proca mass term. This modification leads to a significant perturbation of the Chern-Simons action in the infrared limit, which could have physically relevant consequences, for example, in the quantum Hall effect or high-temperature superconductivity. In Ref. [9], the authors investigate the infrared limit of the Abelian Chern-Simons-Proca theory and found that this limit can be described by two a priori different topological quantum mechanical model [8]. To finish, we consider the two dimensional self-dual boson theory proposed by Siegel [10] many years ago. However, this description could not be quantized because it is anomalous at the quantum level. Many authors have attempted to solve this problem following different strategies, however, it was not achieved until the Floreanini and Jackiw’s paper [11], where the self-dual fields were quantized in a consistent way just basing the process on an unconventional Poincaré symmetry. In Sec. IV, the last section, we will discuss our findings together with our final comments and conclusions.

II. GENERAL FORMALISM

In this section, we present a sketch of the variational gauge-invariant formalism. To this end, a general second class constrained mechanical system is considered to study. This system has the dynamics governed by a Lagrangian \( \mathcal{L}(q_i, \dot{q}_i, t) \) (with \( i = 1, 2, \ldots, N \)) with a set of second class \( (T_a(q_i, p_i), \quad a = 1, 2, \ldots, M < N) \), where \( q_i \) and \( \dot{q}_i \) are the space and velocities variables, respectively. Notice that this consideration does not lead to lost generality or physical content. In order to systematize this formalism, we separate the development following two steps. The first one is the computation of the set of second class constraints and consecutive split up it in two subset, where one is chosen to construct the symmetry generator, while the other one is considered as being the gauge-fixing terms, which are discarded by the variational gauge-invariant formalism. In the second step, an arbitrary function \( (G) \) dependent on the original phase space variables \( (q_i, p_i) \) and WZ variable \( (\theta_\alpha) \) is introduced into the model, right on the canonical Hamiltonian. We impose that the new Hamiltonian, \( \tilde{H} \), must be invariant by gauge-symmetry transformation. Consequently, this procedure leads to the determination of the arbitrary function.

Let us to start considering the following set of second class constraints

\[ (T_a(q_i, p_i), \quad a = 1, 2, \ldots, M < N) \]
\[ T_a(q_i, p_i) \approx 0, \quad \text{with} \quad a = 1, 2, \ldots, M, \quad (1) \]

obtained through the iterative Dirac’s procedure.

Afterward, an arbitrary function \( G(q_i, p_i, \theta_\alpha) \), expanded as

\[ G(q_i, p_i, \theta_\alpha) = \sum_{n=0}^{\infty} \sum_{\alpha=1}^{R} G_\alpha^n(q_i, p_i) \theta_\alpha^n, \quad \text{with} \quad R \leq M, \quad (2) \]

which also satisfies the following boundary condition

\[ G(q_i, p_i, \theta_\alpha = 0) = 0, \quad (3) \]

is introduced into the canonical Hamiltonian, namely,

\[ \tilde{H} = H_c + G(q_i, p_i, \theta_\alpha). \quad (4) \]

In order to obtain the arbitrary function, the variational condition \( \delta \tilde{H} = 0 \) must be obeyed by the WZ extended Hamiltonian, given in Eq.(4). The algebraic form of the generator of the symmetry with the WZ term is

\[ \tilde{T}(q_i, p_i, \theta_\alpha, \pi_\theta) = C_\alpha T_\alpha + D_\alpha T^\alpha_\theta, \quad \text{with} \quad \alpha = 1, 2, \ldots R, \quad (5) \]

where \( C_\alpha \) and \( D_\alpha \) are constants to be determined later, while \( T^\alpha_\theta \) is a function of the WZ variable \( (\theta_\alpha) \) and its canonical conjugate momentum \( (\pi_\theta) \). Since \( \tilde{T}(q_i, p_i, \theta_\alpha, \pi_\theta) \) is the symmetry generator, the infinitesimal gauge transformations can be computed as well, namely,

\[ \delta q_i = \varepsilon \{ q_i, \tilde{T} \} = \varepsilon C_\alpha \{ q_i, T_\alpha \}, \]
\[ \delta p_i = \varepsilon \{ p_i, \tilde{T} \} = \varepsilon C_\alpha \{ p_i, T_\alpha \}, \]
\[ \delta \theta_\alpha = \varepsilon \{ \theta_\alpha, \tilde{T} \} = \varepsilon D_\alpha \{ \theta_\alpha, T^\alpha_\theta \}. \quad (6) \]

It is clear that if \( \tilde{T} \) is the symmetry generator, it must to satisfy the first class algebra, given by

\[ \{ \tilde{T}, \tilde{T} \} = 0. \quad (7) \]

At this point, we are ready to compute the corrections terms \( G_\alpha^n(q_i, p_i) \). To this end, the invariant Hamiltonian must obey the variational principle, which generates the following general equation
\[ \delta \tilde{H} = \delta H_c + \sum_{n=1}^{\infty} \sum_{\alpha=1}^{R} \left( \delta G^\alpha_{(n)}(q_i, p_i) \theta_\alpha^n + n G^\alpha_{(n)}(q_i, p_i) \theta_\alpha^{(n-1)} \delta \theta_\alpha \right) = 0, \tag{8} \]

which allows us to compute each correction term \( G^\alpha_{(n)}(q_i, p_i) \). For linear correction term \( (n = 1) \), we have the following relation

\[ \delta H_c + \sum_{\alpha=1}^{R} G^\alpha_{(1)}(q_i, p_i) \delta \theta_\alpha = 0. \tag{9} \]

For the quadratic one \( (n = 2) \), we get \( R \) relations

\[
\begin{align*}
\delta G^1_{(1)}(q_i, p_i) + 2G^2_{(2)}(q_i, p_i) \delta \theta_1 &= 0, \\
\delta G^2_{(1)}(q_i, p_i) + 2G^2_{(2)}(q_i, p_i) \delta \theta_2 &= 0, \\
&\vdots \\
\delta G^R_{(1)}(q_i, p_i) + 2G^R_{(2)}(q_i, p_i) \delta \theta_R &= 0. \tag{10}
\end{align*}
\]

For \( n \geq 1 \) and \( \alpha = 1, 2, \ldots, R \), the general relation is

\[ \delta G^\alpha_{(n)}(q_i, p_i) + (n + 1) G^\alpha_{(n+1)}(q_i, p_i) \delta \theta_\alpha = 0, \tag{11} \]

This iterative process is successively repeated until the recursive relations (11) becomes identically null. It leads to a complete determination of the arbitrary function \( G(q_i, p_i, \theta_\alpha) \) and, consequently, a complete determination of the invariant Hamiltonian \( \tilde{H} \).

In general, we can fix the WZ terms finding a representation for the WZ variable written only in terms of the original phase space variable \( (q_i, p_i) \), i.e., \( \theta_\alpha = f_\alpha(q_i, p_i) \). In order to obtain this function, we impose that it has the same infinitesimal gauge transformation displayed by \( \theta_\alpha \), namely,

\[ \delta \theta_\alpha = \delta f_\alpha(q_i, p_i) = \varepsilon \{ \theta_\alpha, D^\gamma T^\gamma_\theta \}. \tag{12} \]

Thus, it is possible to derive a gauge-invariant Hamiltonian written only as a function of the original phase space variables \( (q_i, p_i) \) satisfying the first class algebra

\[ \{ \tilde{H}, C^\alpha T_\alpha \} = 0. \tag{13} \]

**III. APPLICATIONS OF THE FORMALISM**
A. The reduced-SU(2) Skyrme model

Few decades ago, Skyrme proposed to describe baryons as topological solutions of the SU(2) NLSM with an appropriate stabilizing term. The semi-classical quantization of the model was obtaining in \[7\] separating the collective coordinate. Let us consider the SU(2) Skyrmion Lagrangian

\[
L = \int d^3 x \left\{ \frac{f_\pi^2}{4} Tr \left( \partial_\mu U^\dagger \partial^\mu U \right) + \frac{1}{32e^2} Tr \left[ U^\dagger \partial^\mu U, U^\dagger \partial^\nu U \right]^2 \right\},
\]

where \(f_\pi\) is the pion decay constant and \(e\) is a dimensionless parameter. \(U\) is a SU(2) matrix transforming as \(U \rightarrow AUB^{-1}\) under chiral SU(2) × SU(2), satisfying the boundary condition \(\lim_{r \rightarrow \infty} U = I\) so that the pion field vanishes as \(r\) goes to infinity. There are soliton solutions described by the action (14) whose topological number are identified with the baryon number.

To describe the static soliton we start with the ansatz \(U(r) = \exp\{i\vec{\tau}_a \cdot \hat{x} f(r)\}\) where \(\vec{\tau}_a\) are Pauli matrices, \(\hat{x} = \vec{x}/r\) and \(\lim_{r \rightarrow \infty} f(r) = 0\) and \(f(0) = \pi\). Performing the collective semi-classical expansion in \(14\) \[4\], where \(U(r, t) = A(t)U(r)A^\dagger(t)\) and \(A \in SU(2)\), we obtain after performing the space integral,

\[
L = -M + J Tr \left( \partial_0 A \partial_0 A^{-1} \right).
\]

\(M\) and \(J\) are the soliton mass and the moment of inertia, respectively, which in the hedgehog ansatz are given by

\[
M = 2\pi \int_0^\infty dr r^2 \left[ f_\pi^2 \left( \frac{df}{dr} \right)^2 + 2 \frac{\sin^2 f}{r^2} + \sin^2 f \left( \frac{df}{dr} \right)^2 + \frac{\sin^2 f}{r^2} \right],
\]

and

\[
J = \frac{8\pi}{3} \int_0^\infty dr r^2 \sin^2 f \left[ f_\pi^2 + \frac{1}{e^2} \left( \frac{df}{dr} \right)^2 + \frac{\sin^2 f}{r^2} \right].
\]

The unitary matrix \(A\) may be represented by \(A = a_0 + i \vec{a} \cdot \vec{\tau}\), which satisfies the spherical constraint

\[
a_i a_i - 1 = 0,
\]

since the condition \(AA^\dagger = 1\) must be obeyed. In terms of these variables, the Skyrmion Lagrangian (15) becomes

\[
L = -M + 2J \dot{a}_i \dot{a}_i + \zeta (a_i a_i - 1),
\]
where $\zeta$ is a Lagrange multiplier that enforces the spherical constraint into the model. The corresponding Hamiltonian is

$$H = M + \frac{1}{8I} \pi_i \pi_i - \zeta (a_i a_i - 1), \quad (20)$$

where the canonical momenta conjugated to the collective coordinates $(a_i)$ are

$$\pi_i = 4I \dot{a}_i, \quad (21)$$

while the canonical momentum conjugated to the Lagrange multiplier $\zeta$ is, indeed, a primary constraint, which is read as

$$T_1 = \pi_\zeta. \quad (22)$$

From the temporal stability condition, secondary, tertiary and quaternary constraints are required, namely,

$$T_2 = a_i a_i - 1, \quad T_3 = a_i \pi_i, \quad T_4 = \frac{1}{8I} \pi_i \pi_i + \zeta a_i a_i. \quad (23)$$

Note that the last relation allows to fix the Lagrange multiplier, consequently, no more constraints arise from the iterative Dirac procedure. Due to this, the model has four second class constraints.

At this stage, we are ready to address the question of constraint conversion through the variational gauge-invariant formalism proposed in the last Section. The conversion process starts assuming that one second class constraints must be picked up to construct the gauge symmetry generator. To put our work in perspective with other papers [12,14], we choose the spherical constraint $T_2$, Eq.(23), to forge the first class constraint that will play the role of gauge symmetry generator. To this end, we begin to pick up the gauge symmetry generator as

$$\tilde{T} = -\frac{1}{2} T_2 + \pi_\theta, \quad (24)$$

where we set $C = -\frac{1}{2}$ and $D = 1$. Afterward, the invariant Hamiltonian is written as

$$\tilde{H} = M + \frac{1}{8I} \pi_i \pi_i - \zeta (a_i a_i - 1) + G(a_i, \pi_i, \theta). \quad (25)$$
Recall that $\pi_\theta$ is a WZ variable satisfying the canonical algebra $\{\theta, \pi_\theta\} = 1$, and the arbitrary function $G(a_i, \pi_i, \theta)$ can be written in an expansion form, given in Eq.(8), obeying the boundary condition, given in Eq.(3).

In agreement with the variational gauge-invariant formalism, the Hamiltonian $\tilde{H}$ must obey the variational principle $\delta \tilde{H} = 0$, i.e., this Hamiltonian must be invariant under the infinitesimal gauge transformations generated by symmetry generator $\tilde{T}$, given by

$$\delta a_i = \varepsilon \{a_i, \tilde{T}\} = 0,$$
$$\delta \pi_i = \varepsilon \{\pi_i, \tilde{T}\} = \varepsilon a_i,$$
$$\delta \theta = \varepsilon \{\theta, \tilde{T}\} = \varepsilon,$$
$$\delta \zeta = \varepsilon \{\zeta, \tilde{T}\} = 0,$$

(26)

where $\varepsilon$ is an infinitesimal time-independent parameter.

From the invariance condition $\delta \tilde{H} = 0$, given in Eq.(8), and using the infinitesimal gauge transformations (26), we can compute all correction terms in order of $\theta$. For linear correction term in order of $\theta$, Eq.(9), we get

$$\delta H_c + \varepsilon G_{(1)} = 0,$$
$$\frac{1}{4T} \varepsilon a_i \pi_i + \varepsilon G_{(1)} = 0,$$

(27)

$$G_{(1)} = -\frac{1}{4T} a_i \pi_i.$$

For the quadratic term, Eq.(10), we have

$$\delta G_{(1)} + 2 \varepsilon G_{(2)} = 0,$$
$$-\frac{1}{4T} \varepsilon a_i a_i + 2 \varepsilon G_{(2)} = 0,$$

(28)

$$G_{(2)} = \frac{1}{8T} a_i a_i.$$

For the tertiary term, we obtain $G_{(3)} = 0$, since $\delta G_{(2)} = \{G_{(2)}, \tilde{T}\} = 0$. Due to this, all correction terms $G_{(n)}$ with $n \geq 3$ are null. Therefore, the gauge invariant Hamiltonian is

$$\tilde{H} = M + \frac{1}{8T} \pi_i \pi_i - \frac{1}{4T} (a_i \pi_i) \theta + \frac{1}{8T} a_i a_i \theta^2 - \zeta (a_i a_i - 1),$$

(29)

which was also obtained by us using the symplectic gauge-invariant formalism [14]. This Hamiltonian, by construction, satisfies the gauge invariance property,

$$\{\tilde{H}, \tilde{T}\} = 0.$$

(30)
Note that the invariant Hamiltonian, Eq.(29), can be elegantly written in terms of a gauge field shifted,

\[ \tilde{H} = M + \frac{1}{8I} \tilde{\pi}_i \tilde{\pi}_i - \zeta(a_i a_i - 1), \]  

where

\[ \tilde{\pi}_i = \pi_i - a_i \theta. \]  

(32)

This algebraic expression reminds the field-shifting Stückelberg formalism [15].

The next step is to look for the Lagrangian that leads to this new theory. A consistently way of doing this is by means of the path integral formalism, where the Faddeev procedure [16] has to be used. The general expression for the vacuum functional is

\[ Z = N \int [d\mu] \exp \{ i \int dt [\dot{a}_i \pi_i + \dot{\theta} \pi_\theta + \dot{\zeta} \pi_\zeta - \tilde{H}] \}, \]  

(33)

with the measure \([d\mu] \) given by

\[ [d\mu] = [da_i][d\pi_i][d\theta][d\pi_\theta][d\zeta][d\pi_\zeta][det \{,\}][\delta(a_i a_i - 1 + \pi_\theta)] \prod_\alpha \delta(\tilde{\Lambda}_\alpha), \]  

(34)

where \( \tilde{\Lambda}_\alpha \) are the gauge fixing conditions corresponding to the first class constraints \( \tilde{T}_\alpha \), and the term \( |det \{,\}| \) represents the determinant of all constraints of the theory, including the gauge-fixing ones. The quantity \( N \) that appears in (33) is the usual normalization factor.

Starting from the Hamiltonian (29) and performing the integration over the momenta, we identify the Lagrangian of the new theory, read as

\[ \tilde{L} = -M + 2I \dot{a}_i \dot{a}_i + (\dot{a}_i a_i) \theta + \zeta(a_i a_i - 1), \]  

(35)

which is invariant under the infinitesimal gauge transformations given in Eq.(26).

An alternative way to fix \( \theta \) comes from the infinitesimal transformation (26). The infinitesimal transformation generated by \( \tilde{T} \) (Eq.(24)), \( \delta \theta = \varepsilon \), is obtained since \( \theta \) is fixed as

\[ \theta = \frac{a_i \pi_i}{a_i a_i}. \]  

(36)

Substituting the relation above in the Eq.(29), we get the invariant canonical Hamiltonian written only in terms of the original phase-space variables, given by
\[ \tilde{H} = M + \frac{1}{8\lambda} \left( \pi_i \pi_i - \frac{(a_i \pi_i)^2}{a_j a_j} \right) - \zeta(a_i a_i - 1), \]
\[ = M + \frac{1}{8\lambda} \pi_i M^{ij} \pi_j - \zeta(a_i a_i - 1), \quad (37) \]

where the phase space metric \( M^{ij} \), given by

\[ M^{ij} = \delta^{ij} - \frac{a^i a^j}{a_i^2}, \quad (38) \]

is a singular matrix that has \( a_i \) as an eigenvector with null eigenvalue, namely,

\[ a_i M^{ij} = 0. \quad (39) \]

Due to this, it is easy to show that the Hamiltonian (37) is invariant under the infinitesimal gauge transformations (26). It is important to note that the gauge symmetry is achieved after the elimination of the WZ sector. In view of this, the original second class constraint \( T_2 \) becomes the gauge symmetry generator.

In order to show the equivalence of our first class Hamiltonian, Eq. (37), and the initial second class Skyrme model, we will give an outline of the quantum mechanics treatment using for this the Dirac’s first class procedure. A very detailed description of our procedure can be found in reference [14]. The physical wave functions must be annihilated by the first class operator constraint, reads as

\[ T_2 |\psi\rangle_{phys} = 0. \quad (40) \]

The physical states that satisfy (40) are

\[ |\psi\rangle_{phys} = \frac{1}{V} \delta(a_i a_i - 1) |\text{polynomial} \rangle. \quad (41) \]

where \( V \) is the normalization factor and \( |\text{polynomial} \rangle = \frac{1}{N(0)} (a_1 + ia_2^d \). The corresponding quantum Hamiltonian is

\[ \tilde{H} = M + \frac{1}{8\lambda} \left[ \pi_i \pi_i - \frac{(a_i \pi_i)^2}{a_j a_j} \right] - \zeta(a_i a_i - 1). \quad (42) \]

1 At first, due this property, it is not possible to obtain the first class Skyrmion Lagrangian written only in terms of the original phase-space variables [13].
The spectrum of the theory is determined by taking the scalar product of the symmetrized invariant Hamiltonian, \( \langle \psi | \bar{H} | \psi \rangle \), given by

\[
\langle \text{polynomial} | \frac{1}{V^2} \int da_i \delta(a_i a_i - 1) \bar{H} \delta(a_i a_i - 1) | \text{polynomial} \rangle. \tag{43}
\]

Integrating over \( a_i \), we obtain

\[
\langle \text{polynomial} | M + \frac{1}{8\lambda} \left[ \pi_i \pi_i - (a_i \pi_i)^2 \right] | \text{polynomial} \rangle \\
= M + \frac{1}{8\lambda} \left[ -\partial_j \partial_j + \frac{1}{2} \left( OpOp + 2Op + \frac{5}{4} \right) \right] \\
= M + \frac{1}{8\lambda} \left[ l(l + 2) + \frac{5}{4} \right], \tag{44}
\]

where \( Op \) is defined as \( Op \equiv a_i \partial_i \). It is interesting to point out that the energy levels, formula (44), is the same obtained in a constrained second class treatment of the SU(2) Skyrme model [17]. Thus, this important result indicates that the variational gauge-invariant formalism produces a correct result when compared with the original second class system.

B. Chiral boson quantum mechanics

Chiral bosons, usually called self-dual fields in two space-time dimensions, have received much attention over the last decade because of their significant role played in the understanding of several models with intrinsic chirality, as heterotic strings [18] and quantum Hall effect [19], for example. At the present time, it has experienced a revival since the study of the noncommutativity geometry became a relevant feature in the quantization of the Dp-brane in background \( B_{\mu\nu} \) field [20]. These models are specially interesting since the unique structure (Chern-Simons terms) that are available in three dimensions, give rise to topologically intricate phenomena without even-dimensions analogs. A quantum mechanical version of gauge field theory involving Chern-Simons terms has been proposed by Jackiw et al. [8]. This was done in order to investigate in detail the change in symplectic structure that occurs when the vanishing of a parameter takes a second-order Lagrangian into a first-order one.

The model proposed in Ref. [8] is a quantum mechanical particle of mass \( m \) and charge \( e \) constrained to move on a two-dimensional plane, interacting with a constant magnetic
field (B), which is orthogonal to the plane. This model has its dynamics governed by the following Lagrangian

$$L = \frac{m}{2} q_i^2 + \frac{B}{2} q_i \epsilon_{ij} \dot{q}_j - \frac{k}{2} q_i^2,$$  \hspace{1cm} (45)

where $\epsilon_{ij}$ is an antisymmetric tensor, $(\epsilon^{12} = \epsilon_{12} = 1)$. This model is analogous to the Lagrangian density for the three-dimensional topological massive electrodynamics in the Weyl gauge ($A^0 = 0$), reads as

$$\mathcal{L} = \frac{1}{2} \dot{A}^2 + \frac{\mu}{2} A \times \dot{A} - \frac{1}{2} (\nabla \times A)^2.$$  \hspace{1cm} (46)

In Ref. [8,21] the behavior of the model (45) was investigated in the vanishing mass limit ($m \to 0$) and was also shown to be analogue of a pure Chern-Simons (CS) gauge theory. Rescaling $A \to \sqrt{2/\mu} A$ and setting $\mu \to 0$, the Lagrangian above is reduced to the pure CS theory

$$\mathcal{L}_{CS} = \frac{k}{2} A \times \dot{A}.$$  \hspace{1cm} (47)

Correspondingly, the vanishing mass limit in Eq.(45) produces the following Lagrangian

$$L = \frac{B}{2} q_i \epsilon_{ij} \dot{q}_j - \frac{k}{2} q_i^2,$$  \hspace{1cm} (48)

usually called Chern-Simons-Proca (CSP) quantum mechanical model. Recently, a similar approach was discussed in Ref. [22] in order to investigate the contribution of noncommutative geometry in the quantization of D3-brane in background B-field.

The CSP model, described above, is an example of second class constrained theory since the constraints, given by

$$T_i = p_i + \frac{B}{2} \epsilon_{ij} q_j \approx 0, \hspace{0.5cm} (i = 1, 2)$$  \hspace{1cm} (49)

where $p_i = \frac{\partial L}{\partial \dot{q}_i}$ are the canonical momenta, satisfy the following Poisson algebra

$$\{T_i, T_j\} = B \epsilon_{ij}.$$  \hspace{1cm} (50)

Due to this, the noncommutative nature of the model could be displayed after the computation of the Dirac brackets among the phase space coordinates. It will be done through the symplectic method [11]. As this model is described by a first-order Lagrangian, given in (48), the symplectic variables $\zeta_\alpha$ and the respective one-form canonical momenta $A_{\zeta_\alpha}$ are
\[ \zeta_{q_i} = q_i, \]
\[ A_{q_i} = \frac{B}{2} q_i \epsilon_{ij}, \]
with the following Hamiltonian (symplectic potential)
\[ H = \frac{k}{2} q_i q_i. \] (51)

The corresponding symplectic matrix is
\[ f_{q_i q_j} = \frac{B}{2} \epsilon_{ij}. \] (52)

Since this matrix is nonsingular, it can be inverted to provide the noncommutative Dirac brackets, written as
\[ \{ q_i, q_j \}_{DB} = \frac{2}{B} \epsilon_{ij}. \] (53)

This complete our proposal of this section.

C. The gauge invariant CSP model

In this section, we are involved with the reformulation of CSP model as a gauge invariant theory by using the variational gauge-invariant formalism. The main feature behind of this formalism is the enlargement of the phase space with the introduction of an arbitrary function \( G(q_i, p_i, \theta) \), given in Eq. (2), into the Hamiltonian. In agreement with this formalism, the CSP Hamiltonian, given in Eq. (51), becomes
\[ \tilde{H} = \frac{k}{2} q_i q_i + G(q_i, p_i, \theta). \] (54)

Afterwards, an algebraic form is settle to be the symmetry generator which, for the present problem, we choose
\[ \tilde{T} = T_1 + p_\theta = p_1 + \frac{B}{2} q_2 + p_\theta, \] (55)
where \( p_\theta \) is a WZ variable which obeys the canonical algebra \( \{ \theta, p_\theta \} = 1 \). Due to this, the infinitesimal gauge transformations are
\[ \delta q_1 = \varepsilon \{ q_1, \tilde{T} \} = \varepsilon, \]
\[ \delta p_1 = \varepsilon \{ p_1, \tilde{T} \} = 0, \]
\[ \delta q_2 = \varepsilon \{ q_2, \tilde{T} \} = 0, \]
\[ \delta p_2 = \varepsilon \{ p_2, \tilde{T} \} = -\varepsilon \frac{B}{2}, \]
\[ \delta \theta = \varepsilon \{ \theta, \tilde{T} \} = \varepsilon, \] (56)
where $\varepsilon$ is a time-independent parameter.

Following the variational gauge-invariant formulation sketched in Section II, the variational condition $\delta \tilde{H} = 0$, given in Eq.(3), is

$$
\delta \tilde{H} = \varepsilon \{H_c, \tilde{T}\} + \sum_{n=1}^{\infty} \left( \delta G_{(n)}(q_i, p_i)\theta^n + n G_{(n)}(q_i, p_i)\theta^{(n-1)}\delta \theta \right) = 0.
$$

(57)

From this general equation and using the relations, given in Eq.(56), each correction term in order of $\theta$ can be computed. For the linear term in $\theta$, Eq.(9), we get

$$
\delta H_c + G_{(1)} = 0,
$$

$$
kq_1 + G_{(1)} = 0,
$$

$$
G_{(1)} = -kq_1.
$$

(58)

For the quadratic term, Eq.(10), we have

$$
\delta G_{(1)} + 2G_{(2)}\delta \theta = 0,
$$

$$
G_{(2)} = \frac{k}{2},
$$

(59)

while for the tertiary term, we obtain

$$
\delta G_{(2)} + 3G_{(3)}\delta \theta = 0,
$$

$$
G_{(3)} = 0.
$$

(60)

In view of this, all correction terms $G_{(n)}$ with $n \geq 3$ are null. Thus, the invariant Hamiltonian with the WZ terms reads as

$$
\tilde{H} = \frac{k}{2}q_iq_i - kq_1\theta + \frac{k}{2}\theta^2,
$$

(61)

which by construction satisfies the gauge invariance property,

$$
\{\tilde{H}, \tilde{T}\} = 0.
$$

(62)

Note that the invariant Hamiltonian in Eq.(61) can be elegantly written in terms of a gauge field shifted,

$$
\tilde{H} = \frac{k}{2} \left[ (q_1 - \theta)^2 + q_2q_2 \right]
$$

$$
= \frac{k}{2}\tilde{q}_i\tilde{q}_i,
$$

(63)
where

\[
\tilde{q}_1 = q_1 - \theta, \\
\tilde{q}_2 = q_2.
\]  
(64)

This algebraic expression reminds the field-shifting Stückelberg formalism [15].

The main goal of the variational gauge-invariant formalism consists in to reveal the gauge symmetry existent in the model described by the original phase space fields. To this end, we choose a representation for \( \theta \) which preserves its infinitesimal gauge transformation given in Eq. (56), reads as

\[
\theta = -\frac{1}{B}[p_2 - \frac{B}{2}q_1] = -\frac{1}{B}T_2.
\]  
(65)

Substituting the result (65) in the Hamiltonian, Eq. (63), we get a gauge invariant Hamiltonian written only in terms of the original phase space variables, reads as

\[
\tilde{H} = \frac{k}{2} \left[ \left( \frac{q_1}{2} + \frac{p_2}{B} \right)^2 + q_2\dot{q}_2 \right],
\]  
(66)

with the constraint \( T_1 = p_1 + \frac{B}{2}q_2 \) as the gauge symmetry generator of the infinitesimal gauge transformations given in Eq. (56). It is easy to verify that \( \tilde{H} \), given in Eq. (66), satisfies the first class algebra

\[
\{ \tilde{H}, T_1 \} = 0,
\]  
(67)

and, consequently, the Hamiltonian (66) is invariant under the infinitesimal gauge transformations (56).

The obtainment of the corresponding Lagrangian is just a matter of direct calculation by means of the constrained path integral formalism. The result is

\[
\tilde{L} = \frac{B^2}{2k}q_2^2 - \frac{1}{2}kq_2^2 - \frac{B}{2}\dot{q}_1q_2 - \frac{B}{2}q_1\dot{q}_2,
\]

\[
= \frac{B^2}{2k}q_2^2 - \frac{1}{2}kq_2^2 - \frac{B}{2} \frac{d}{dt}(q_1q_2),
\]  
(68)

which is invariant under the infinitesimal transformations (56),

\[
\delta L = \frac{B}{2} \frac{d}{dt}(\varepsilon q_2).
\]  
(69)

Note that \( q_1 \) variable has not appeared in Eq. (68) except into the total derivative. Furthermore, the Lagrangian except the total derivative is just a usual harmonic oscillator having the frequency \( T = k/B \).
Finally, it is important to mention that the variational gauge-invariant formalism is capable to reveal the gauge symmetry existent on the original phase and configuration spaces, a result that was not yet discussed in the literature.

**D. Chiral-bosons field theory**

Considerable attention has been given to the chiral-bosons field theory. This model is relevant to the comprehension of superstrings, W gravities, and general two-dimensional field theories in the light cone. Its apparent simplicity hides intriguing and interesting points that remain until now.

The Floreanini-Jackiw (FJ) chiral-boson model has its dynamics governed by the following Lagrangian density \[11\]

\[
\mathcal{L} = \dot{\phi} \phi' - \phi'^2, \tag{70}
\]

where dots and primes represent derivatives with respect to time and space coordinates, respectively. Spacetime is assumed to be two dimensional Minkowskian variety. The primary constraint is

\[
T(\phi, \pi) = \pi - \phi', \tag{71}
\]

and the canonical Hamiltonian is

\[
\mathcal{H}_c = \phi'^2. \tag{72}
\]

The time stability condition for the constraint \(T\) does not lead to any new one because it satisfies the following Poisson bracket relation

\[
\{T(x), T(y)\} = -2\delta'(x - y). \tag{73}
\]

At this point, it is important to discuss the degree of freedom of the model. The model has one second class constraint and the phase space sums two dimensions \((\phi, \pi_\phi)\). Since each second class constraint fixes one field, the model has a half independent degree of freedom.

The goal of this section is to open up the possibility to implement a consistent covariant quantization of FJ chiral-boson. To this end, the variational gauge-invariant formalism will be used. This formalism begins introducing an arbitrary function into the Hamiltonian,
\[ \tilde{H} = \mathcal{H}_c + G(\phi, \pi_\phi, \theta) = \phi'^2 + G(\phi, \pi_\phi, \theta), \quad (74) \]

where \( G(\phi, \pi_\phi, \theta) \) is given by Eq.(2).

The generator of gauge symmetry (\( \tilde{T} \)) is chosen as

\[ \tilde{T} = \pi - \phi' + \theta, \quad (75) \]

where the auxiliary field satisfies a non canonical Poisson bracket relation

\[ \{\theta(x), \theta(y)\} = 2\delta'(x - y). \quad (76) \]

Combining Eqs.(73) and (76), we have the first class Poisson bracket

\[ \{\tilde{T}(x), \tilde{T}(y)\} = 0. \quad (77) \]

In order to begin with the variational gauge-invariant formalism, the variational condition (\( \delta \tilde{H} = 0 \)), given in Eq.(8) must be obeyed. The gauge infinitesimal transformations generated by \( \tilde{T} \) are

\[ \begin{align*}
\delta \phi(x) &= \varepsilon \{\phi(x), \tilde{T}(y)\} = \varepsilon \delta(x - y), \\
\delta \pi(x) &= \varepsilon \{\pi(x), \tilde{T}(y)\} = -\varepsilon \delta'(x - y), \\
\delta \mathcal{H}_c(x) &= \varepsilon(y) \{\partial_x \phi(x)^2, \tilde{T}(y)\} = 2 \varepsilon \partial_x \phi(x) \delta'(x - y), \\
\delta \theta(x) &= \varepsilon \{\theta(x), \tilde{T}(y)\} = 2 \varepsilon \delta'(x - y).
\end{align*} \quad (78) \]

Using these relations and following the prescription of the variational gauge-invariant formalism, the linear and quadratic terms are, respectively, obtained as

\[ \begin{align*}
\delta \mathcal{H}_c + G(1) &= 0, \\
G(1) &= -\phi', \\
\delta G(1) + 2G(2)\delta \theta &= 0, \\
G(2) &= \frac{1}{4}.
\end{align*} \quad (79) \]

As the secondary correction term is a scalar, all correction terms \( G(n) \), with \( n \geq 3 \), are null. Therefore, the gauge invariant Hamiltonian density is

\[ \text{It is not difficult to see that the constraint that leads to the brackets(76) is given by } T = \pi' + \frac{1}{2} \theta \approx 0. \]

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\[ \tilde{H} = \phi'^2 - \phi'\theta + \frac{1}{4}\theta^2, \]
\[ = (\phi' - \frac{1}{2}\theta)^2. \]

Using Eqs.(78), it is easy to verify that \( \tilde{H} \), Eq.(80), satisfies a first class algebra, given by
\[ \{ \tilde{H}, \tilde{T} \} = 0, \]
with \( \tilde{T} = \pi - \phi' + \theta. \)

The gauge-invariant Hamiltonian, Eq.(80), is the same obtained by Amorim and Barcelos in [24] via BFFT formalism with the advantage that we have used few algebraic steps. Then, our results indicate the equivalence between the variational gauge-invariant formalism and the BFFT first class conversion method.

The obtainment of the corresponding density Lagrangian is just a matter of direct calculation by means of the constrained path integral formalism. The complete details can be found in [24] and we just mention the result
\[ L = \dot{\phi}\phi' - \phi'^2 + \theta(\phi' - \dot{\phi}) - \frac{1}{4}\theta^2 - \frac{1}{2}\theta \int dy\Theta(x - y)\theta(y), \]
where \( \Theta(x - y) \) is the step function.

It is opportune to comment that in the chiral bosons model, at first, is not possible to fix the WZ field in terms of the original phase space variables. It occurs due to the singular property of the FJ chiral-boson model, whose constraint, Eq.(71), satisfies a second class algebra, given in Eq.(73). Thus, it is necessary, in principle, to adding a WZ variable in the obtainment of the first class algebra, Eq.(77).

**IV. CONCLUSIONS**

In this paper, we have proposed a new approach to reformulate second class systems as gauge invariant theories. This gauge-invariant formalism is based on early conception that invariant models satisfy the variational principle. Following this idea and the Faddeev’s suggestion [3], we apply the variational principle on the WZ extended system in order to unveil symmetries present on the original second class system. One important feature of this formalism is the possibility to choose a convenient gauge symmetry generator which allows to investigate physical properties connected to gauge symmetries. In general, it is possible
to fix the WZ field into the extended Hamiltonian that, subsequently, generates a gauge invariant Hamiltonian written in terms of the original phase-space fields. It is a meaningful characteristic displayed by the variational gauge-invariant formalism. Another point that deserves to mention is the simple algebraic computation of the WZ extended Hamiltonian when compared with other constraint conversion formalisms [2]. The variational gauge-invariant formalism was applied to different physical systems. First, we consider the reduced SU(2) Skyrme model, where we have obtained a first class version for the original second class model. We have also computed the energy spectrum, which reproduces the same results obtained in the literature [12,14]. Second, we reformulate the CSP model as an invariant model which is written only in terms of the original phase space variables. As long we known, it is a new result not yet presents in the literature. Third, the second class nature of the two dimensional self-dual model has been changed to the first one through the variational formalism, which reproduces the results, given in Ref. [24], in an effortless way.

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REFERENCES

[1] I.A.Batalin and I.V.Tyutin, Int. J. Mod. Phys. A6, 3255 (1991).

[2] M. Moshe and Y. Oz, Phys. Lett. B224, 145 (1989); T.Fujiwara, Y. Igarashi and J. Kubo, Nucl. Phys. B341, 695 (1990); C.Wotzasek, Int. J. Mod. Phys. A5, 1123 (1990); C.Neves and C.Wotzasek, J. Math. Phys.34, 1807 (1993); J.Barcelos-Neto, Phys. Rev. D55, 2265 (1997); J. Barcelos-Neto and W. Oliveira, Phys. Rev. D56, 2257 (1997); W. Oliveira and J. Ananias Neto, Int. J. Mod. Phys. A12, 4895 (1997); W. Oliveira and J. Ananias Neto, Nucl. Phys. B533, 611 (1998); S.-T. Hong, Y.-W. Kim and Y.-J. Park, Mod. Phys. Lett. A15, 55 (2000); C. Neves and C. Wotzasek, Phys. Rev. D59, 125018 (1999); C. Neves and C. Wotzasek, Phys. Rev. C62, 025205 (2000); C. Neves and C. Wotzasek, J. Phys. A: Math.Gen.33, 1 (2000).

[3] P.Mitra and R.Rajaraman, Ann. Phys.(N.Y.) 203, 157 (1990).

[4] A.S. Vytheeswaran, Ann. Phys. (N.Y.) 206, 297 (1994); Contribution to Photon and Poincaré Group(New Science Publisher, New York, 1999).

[5] L.Faddeev and S.L.Shatashivilli, Phys. Lett. B167, 225 (1986).

[6] T.H.Skyrme, Proc. R. Soc. London A260, 127 (1961).

[7] G. Adkins, Chiral Solitons, ed. Keh-Fei Liu (World Scientific,Singapore, 1987) p.99; G. S. Adkins, C. R. Nappi and E. Witten, Nucl. Phys. B 228, 552 (1983).

[8] G.V. Dunne, R. Jackiw and C.A. Trungenberger, Phys. Rev. D41, 661 (1990).

[9] Antti J. Niemi (Helsinki U.), V.V. Sreedhar, Phys. Lett. B336, 381 (1994).

[10] W. Siegel, Nucl. Phys. B238, 455 (1987).

[11] R.Floreanini and R.Jackiw, Phys. Rev. Lett.59, 1873 (1987).

[12] S.-T. Hong, Y.-W. Kim and Y.-J. Park, Phys. Rev. D59, 114026 (1999).

[13] T. D. Lee, Particle Physics and Introduction to Field Theory, (Harwood Academic Publishers, 1981)p.480.

[14] J. Ananias Neto, C. Neves and W. Oliveira, Phys. Rev. D 63, 85018 (2001).

[15] E. C. G. Stückelberg, Helv. Phys. Act. 30, 209 (1957).
[16] L.D. Faddeev, Theor. Math. Phys. 1, 1 (1970); P. Senjanovich, Ann. Phys. (N.Y.) 100, 277 (1976).

[17] J. Ananias Neto, J. Phys. G21, 695 (1995).

[18] D. J. Gross, J. A. Harvey, E. Martinec, and R. Rohm, Phys. Rev. Lett. 54, 502 (1985).

[19] M. Stone, Phys. Rev. B41, 212 (1990); X.G. Wen, Phys. Rev. Lett. 64, 2206 (1990).

[20] M.M. Sheikh-Jabari, Phys. Lett. B450, 119 (1999); E. Witten and N. Seiberg, J. High Energy Phys. 09, 32 (1999).

[21] D. Bazeia, Mod. Phys. Lett. A6, 1147 (1991).

[22] D. Bigatti and L. Susskind, Phys. Rev. D62, 066004 (2000).

[23] T. Maskawa and H. Nakajima, Prog. Theor. Phys. 56, 1295 (1976).

[24] R. Amorim and J. Barcelos-Neto, Phys. Rev. D53, 7129 (1996).