Dual pairs of algebras and finite commutative group schemes

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Abstract. We introduce a category of dual pairs of finite locally free algebras over a ring. This gives an efficient way to represent finite locally free commutative group schemes. We give a number of algorithms to compute with dual pairs of algebras, and we apply our results to Galois representations on finite Abelian groups.

1. Introduction

In this article we introduce an efficient way to write down finite locally free commutative group schemes over a ring $R$. Our approach is motivated by Cartier duality and is based on simultaneously representing a group scheme and its Cartier dual. This gives rise to the definition of a category of dual pairs of finite locally free $R$-algebras. On the theoretical side, we prove that this category is equivalent to the category of finite locally free commutative group schemes over $R$, and anti-equivalent to the corresponding category of finite locally free commutative cocommutative Hopf algebras over $R$. On the algorithmic side, we show that this way of representing group schemes leads to transparent algorithms for performing various important operations with group schemes and groups of $S$-valued points for $R$-algebras $S$.

In the case $R = \mathbb{Q}$, dual pairs of $\mathbb{Q}$-algebras give a concise way to write down representations of the absolute Galois group of $\mathbb{Q}$ on finite Abelian groups. An important advantage of our approach is that in practice, the data that one needs to compute with has considerably smaller height than when the computations are done using Hopf algebras. This is especially relevant when computing the data numerically, such as in the explicit computations of Galois representations by Bosman (see [1] and [6, Chapters 6 and 7]), Tian [15], Yin and Zeng [16], Derickx, van Hoeij and Zeng [5], Mascot [9], and the author (unpublished, see [2] and [3]). Until now, the output of these computations has consisted of polynomials whose splitting fields cut out the sought-for Galois representations. When one has computed such a polynomial $F$ but no further data, one has to certify that $F$ has the expected Galois group and ramification properties in order to prove that $F$ indeed cuts out the expected Galois representation. This approach does yield enough information to compute conjugacy classes of Frobenius elements, but does not easily produce data that amounts to specifying the group scheme structure. The approach described here makes it easier to write down such data, namely an object of a category that is equivalent to the category of finite commutative group schemes over $\mathbb{Q}$. In particular, the fact that a dual pair of algebras possesses more structure than just the defining polynomials of the algebras can be exploited to make the certification of the Galois group much more direct.

The outline of the paper is as follows. In §2 we fix the general notation and conventions that we will use. In §3, we define the category of dual pairs of algebras over a ring $R$ and compare it (via the category of Hopf algebras over $R$) to the category of finite locally free commutative group schemes over $R$. In §4, we give two examples (one étale and one non-étale) of dual pairs of algebras representing the 2-torsion subschemes of certain elliptic curves. In §5, we describe algorithms for a number of relevant operations with group schemes and their groups of points in our setting. In §6, we focus on Galois representations of a field $K$, which we can view as finite commutative group schemes over $K$. We relate our description of Galois representations to other descriptions and give an example to show that our description leads to data of particularly small height. In §7, we sketch two directions for future work. In the appendix, we give a generic algorithm to identify a finite Abelian group from a certain type of “pairing matrix” that is used in §5.

Remark. For most of the article, we work with group schemes over a base ring $R$. Whenever we are in this setting, all our constructions are compatible with arbitrary base change and can be extended without difficulties to arbitrary base schemes.

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2. Notation and conventions

All rings (and in particular all algebras) are assumed to be commutative, unless otherwise stated. Similarly, all Hopf algebras are assumed to be commutative and cocommutative.

If \( M \) and \( N \) are modules over a ring \( R \), a bilinear map \( M \times N \to R \) is said to be \textit{perfect} if the induced \( R \)-bilinear maps

\[
\begin{align*}
M & \to \text{Hom}_{\text{Mod}_R}(N, R), \\
N & \to \text{Hom}_{\text{Mod}_R}(M, R)
\end{align*}
\]

are isomorphisms.

Let \( R \) be a ring, and let \( M \) be a finite locally free \( R \)-module. We write \( M^\vee = \text{Hom}_{\text{Mod}_R}(M, R) \) for the \( R \)-linear dual of \( M \). We have a canonical perfect \( R \)-bilinear map

\[
\Phi_M: M \times M^\vee \to R.
\]

Furthermore, the canonical \( R \)-linear map

\[
M \otimes_R M^\vee \to \text{Hom}_{\text{Mod}_R}(M, M)
\]

is an isomorphism, and we write

\[
\theta_M \in M \otimes_R M^\vee
\]

for the unique element mapping to \( \text{id}_M \in \text{Hom}_{\text{Mod}_R}(M, M) \) under the above isomorphism. If \( M \) is free over \( R \), then given an \( R \)-basis \((a_1, \ldots, a_n)\) of \( M \) and the corresponding dual basis \((b_1, \ldots, b_n)\) of \( M^\vee \), we have

\[
\theta_M = \sum_{i=1}^n a_i \otimes b_i.
\]

Let \( M \) and \( N \) be finite free modules over a ring \( R \) with bases \((x_1, \ldots, x_m)\) and \((y_1, \ldots, y_n)\), respectively. We will sometimes identify \( M \otimes_R N \) with the \( R \)-module of \( m \times n \)-matrices with coefficients in \( R \) by representing an element \( \sum_{i=1}^m \sum_{j=1}^n c_{i,j} x_i \otimes y_j \) as the matrix \((c_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}\). Similarly, if \( \Phi: M \times N \to R \) is an \( R \)-bilinear map, we will sometimes represent \( \Phi \) by the \( m \times n \)-matrix \((\Phi(x_i, y_j))_{1 \leq i \leq m, 1 \leq j \leq n}\). Furthermore, we will write \( \Phi^t \) for the bilinear map \( N \times M \to R \) defined by \((n, m) \mapsto \Phi(m, n)\).

Let \( M \) and \( N \) be finite locally free modules over a ring \( R \), and let \( \Phi: M \times N \to R \) be a perfect \( R \)-bilinear map. By assumption, the \( R \)-linear map

\[
N \overset{\sim}{\to} M^\vee \\
n \mapsto (m \mapsto \Phi(m, n))
\]

is an isomorphism. Inverting this, we obtain an isomorphism \( M^\vee \overset{\sim}{\to} N \), which in turn determines an element

\[
\theta_\Phi \in M \otimes_R N
\]

via the isomorphism

\[
M \otimes_R N \overset{\sim}{\to} \text{Hom}(M^\vee, N) \\
m \otimes n \mapsto (\phi \mapsto \phi(m)n).
\]

When \( M \) and \( N \) are free \( R \)-modules with given bases, one checks easily that the matrices of \( \Phi \) and \( \theta_\Phi \) with respect to these bases are inverse transposes of each other.
3. Dual pairs of algebras

3.1. Motivation: Cartier duality for group schemes and Hopf algebras

Let $H$ be a finite locally free commutative group scheme over a ring $R$, and let $A$ be the corresponding finite locally free Hopf algebra over $R$, so $A \cong O_H(H)$ and $H \cong \text{Spec } A$. We denote the structure maps of the Hopf algebra $A$ by

\[
m: A \otimes_R A \to A \quad \text{(multiplication)}, \quad e: R \to A \quad \text{(unit)},
\]

\[
\mu: A \to A \otimes_R A \quad \text{(comultiplication)}, \quad \epsilon: A \to R \quad \text{(counit)}.
\]

Remark. Part of the definition of a Hopf algebra is the existence of an antipode $A \xrightarrow{\sim} A$. However, the antipode itself does not need to be included in the data defining the Hopf algebra structure, since an antipode, if it exists, is unique.

Let $A^\vee$ be the Hopf algebra dual to $A$; as an $R$-module, this is defined by

\[
A^\vee = \text{Hom}_{\text{Mod}_R}(A, R)
\]

and the structure maps of $A^\vee$ are defined by dualising those of $A$. Let $H^*$ be the Cartier dual of $H$, i.e. the spectrum of $A^\vee$; this is again a finite locally free commutative group scheme over $R$. It is well known that $H^*$ represents the sheaf of Abelian groups $\text{Hom}(H, G_{m, R})$ for the $fppf$ topology on $\text{Spec } R$; see for example Oort [10, Theorem 16.1]. We therefore have a canonical morphism of $R$-schemes

\[
H \times H^* \to G_{m, R}
\]

and a corresponding $R$-algebra homomorphism

\[
R[x, x^{-1}] \to A \otimes_R A^\vee.
\]

The element $\theta_A$ defined earlier is the image of $x$ under this homomorphism. Furthermore, a theorem of Deligne stating that a finite locally free commutative group scheme is annihilated by its rank (see Tate and Oort [14, §1]) implies that $\theta_A$ is a (not necessarily primitive) $n$-th root of unity, where $n$ is the rank of $A$.

The basic observation that motivates our definition of dual pairs of algebras in §3.2 below is that if $A$ is a finite locally free Hopf algebra over $R$, then $A$ is determined up to isomorphism by the triple $(A^\vee, (A^\vee)^{al}, \Phi)$, where the subscript “al” indicates the forgetful functor from the category of Hopf algebras over $R$ to the category of algebras over $R$ and $\Phi$ is the canonical $R$-bilinear map $A \times A^\vee \to R$.

3.2. The category of dual pairs of algebras over a ring

We introduce some notation in preparation for our main definition. If $A$ and $B$ are two finite locally free $R$-algebras and $\Phi: A \times B \to R$ is a perfect $R$-bilinear map, then $\Phi$ induces a perfect $R$-bilinear map

\[
\Phi^{(2)}: (A \otimes_R A) \times (B \otimes_R B) \to R
\]

\[
(a \otimes a', b \otimes b') \mapsto \Phi(a, b)\Phi(a', b').
\]

We define

\[
\mu^\Phi_1: A \to A \otimes_R A \quad \text{and} \quad \mu^\Phi_2: B \to B \otimes_R B
\]

as the unique $R$-linear maps making the diagrams commutative, where the top horizontal arrows are induced by $\Phi$, the bottom horizontal arrows are induced by $\Phi^{(2)}$ and the right vertical arrows are the duals of the multiplication maps of $B$ and $A$, respectively. In other words, $\mu^\Phi_1$ and $\mu^\Phi_2$ are uniquely determined by the identities

\[
\Phi(a, bb') = \Phi^{(2)}(\mu^\Phi_1(a), b \otimes b') \quad \text{for all } a \in A, b, b' \in B,
\]

\[
\Phi(aa', b) = \Phi^{(2)}(a \otimes a', \mu^\Phi_2(b)) \quad \text{for all } a, a' \in A, b \in B.
\]
Analogously, we define
\[ \epsilon_1^\Phi : A \to R \quad \text{and} \quad \epsilon_2^\Phi : B \to R \]
as the unique \( R \)-linear maps making the diagrams
\[
\begin{array}{ccc}
A & \xrightarrow{\sim} & B^\vee \\
\downarrow & & \downarrow \\
R & \xrightarrow{id} & R
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
B & \xrightarrow{\sim} & A^\vee \\
\downarrow & & \downarrow \\
R & \xrightarrow{id} & R
\end{array}
\]
commutative, where the right vertical arrows are defined as evaluation in the unit elements of \( B \) and \( A \), respectively. Thus \( \epsilon_1^\Phi \) and \( \epsilon_2^\Phi \) are uniquely determined by the identities
\[
\epsilon_1^\Phi (a) = \Phi(a, 1_B) \quad \text{for all} \quad a \in A,
\]
\[
\epsilon_2^\Phi (b) = \Phi(1_A, b) \quad \text{for all} \quad b \in B.
\]

**Definition.** Let \( R \) be a ring. A dual pair of algebras over \( R \) is a triple \((A, B, \Phi)\), where \( A \) and \( B \) are finite locally free \( R \)-algebras and where
\[
\Phi : A \times B \to R
\]
is a perfect \( R \)-bilinear map such that the following hold:

1. we have \( \Phi(1_A, 1_B) = 1 \);
2. for all \( a, a' \in A \) we have \( \Phi(aa', 1_B) = \Phi(a, 1_B)\Phi(a', 1_B) \);
3. for all \( b, b' \in B \) we have \( \Phi(1_A, bb') = \Phi(1_A, b)\Phi(1_A, b') \);
4. for all \( a, a' \in A \) and \( b, b' \in B \) we have
\[
\Phi^{(2)}(\mu_1^\Phi(a)\mu_1^\Phi(a'), b \otimes b') = \Phi(aa', bb') = \Phi^{(2)}(a \otimes a', \mu_2^\Phi(b)\mu_2^\Phi(b')).
\]

**Remark.** The conditions (1)–(4) are equivalent to saying that the \( R \)-linear maps \( \mu_1^\Phi, \mu_2^\Phi, \epsilon_1^\Phi \) and \( \epsilon_2^\Phi \) are in fact homomorphisms of \( R \)-algebras.

**Definition.** Let \((A, B, \Phi), (A', B', \Phi')\) be two dual pairs of algebras over a ring \( R \). A morphism from \((A, B, \Phi)\) to \((A', B', \Phi')\) is a pair of \( R \)-algebra homomorphisms \((f, g) : A' \to A, g : B \to B'\) satisfying
\[
\Phi(f(a'), b) = \Phi'(a', g(b)) \quad \text{for all} \quad a' \in A' \text{ and } b \in B.
\]

Morphisms can be composed as follows: given a morphism \((f, g)\) from \((A, B, \Phi)\) to \((A', B', \Phi')\) and a morphism \((f', g')\) from \((A', B', \Phi')\) to \((A'', B'', \Phi'')\), we put
\[
(f', g') \circ (f, g) = (f \circ f', g' \circ g);
\]
one checks immediately that this is a morphism from \((A, B, \Phi)\) to \((A'', B'', \Phi'')\).

For every ring \( R \), we denote by \( \text{DP}_R \) the category of dual pairs of \( R \)-algebras with morphisms as defined above.

**Remark.** Alternatively, we could have declared a morphism \((f, g)\) as above to be a “morphism from \((A', B', \Phi')\) to \((A, B, \Phi)\)” instead of the other way around. The reason for our chosen convention is that it gives us an equivalence of categories, rather than an anti-equivalence, in Corollary 3.2.

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3.3. (Anti-)equivalences with Hopf algebras and group schemes

Let $R$ be a ring. We write $\text{HA}_R$ for the category of finite, locally free, commutative and cocommutative Hopf algebras over $R$.

If $(A,B,\Phi)$ is a dual pair of $R$-algebras, we write

$$\text{H}_R(A,B,\Phi) = (A,m_A,e_A,\mu_\Phi,\epsilon_\Phi);$$

see §3.2 for the definition of $\mu_\Phi$ and $\epsilon_\Phi$. The definition of dual pairs implies that $\text{H}_R(A,B,\Phi)$ is an object of $\text{HA}_R$. Furthermore, given a morphism $(f,g):(A,B,\Phi) \rightarrow (A',B',\Phi)$ of dual pairs of $R$-algebras, the morphism $f:A' \rightarrow A$ is a homomorphism of Hopf algebras in the opposite direction. In this way we obtain a functor

$$\text{H}_R: \text{DP}_R \rightarrow \text{HA}^{\text{op}}_R.$$  

Conversely, if $(A,m,e,\mu,\epsilon)$ is a Hopf algebra over $R$, we write

$$\text{D}_R(A,m,e,\mu,\epsilon) = (A,A^\vee,\Phi_A).$$

Here $A$ is equipped with the algebra structure defined by $m$ and $e$, and $A^\vee$ is equipped with the algebra structure defined by $e^\vee$ and $\mu^\vee$. The definition of dual pairs implies that $\text{D}_R(A,m,e,\mu,\epsilon)$ is a dual pair of algebras over $R$. Furthermore, given a homomorphism $f:(A',m',e',\mu',\epsilon') \rightarrow (A,m,e,\mu,\epsilon)$ of Hopf algebras, the pair $(f: A' \rightarrow A, f^\vee: A^\vee \rightarrow A^\vee)$ is a morphism from the dual pair $(A,A^\vee,\Phi_A)$ to the dual pair $(A',A'^\vee,\Phi_{A'})$. In this way we obtain a functor

$$\text{D}_R: \text{HA}^{\text{op}}_R \rightarrow \text{DP}_R.$$  

**Theorem 3.1.** Let $R$ be a ring. The functors $\text{H}_R: \text{DP}_R \rightarrow \text{HA}^{\text{op}}_R$ and $\text{D}_R: \text{HA}^{\text{op}}_R \rightarrow \text{DP}_R$ are anti-equivalences of categories between the category of dual pairs over $R$ and the category of finite locally free commutative cocommutative Hopf algebras over $R$.

**Proof.** It follows immediately from the definitions that the functor $\text{H}_R \circ \text{D}_R$ is the identity on the category $\text{HA}^{\text{op}}_R$. The functor $\text{D}_R \circ \text{H}_R$ sends a dual pair $(A,B,\Phi)$ to the dual pair $(A,A^\vee,\Phi_A)$; there is a natural isomorphism

$$\eta_{(A,B,\Phi)}: (A,B,\Phi) \xrightarrow{\sim} (A,A^\vee,\Phi_A)$$

in $\text{DP}_R$ defined by the pair of isomorphisms $(\text{id}: A \rightarrow A, \phi: B \rightarrow A^\vee)$, where $\phi$ is defined by $\phi(b)(a) = \Phi(a,b)$. The isomorphisms $\eta_{A,B,\Phi}$ define an isomorphism

$$\eta: \text{id}_{\text{DP}_R} \xrightarrow{\sim} \text{D}_R \circ \text{H}_R$$

of functors from the category $\text{DP}_R$ to itself. One checks easily that each of the natural isomorphisms

$$\eta_{D_R}: D_R \xrightarrow{\eta} D_R \circ H_R \circ D_R,$$

$$H_R \eta: H_R \xrightarrow{\eta} H_R \circ D_R \circ H_R$$

is the identity. We conclude that $(H_R, D_R, \eta, \text{id})$ is an adjoint equivalence of categories from $\text{DP}_R$ to $\text{HA}^{\text{op}}_R$ (see Mac Lane [8, §IV.4] for the definition of an adjoint equivalence).

**Remark.** One may wonder why we have introduced the notion of dual pairs when Theorem 3.1 shows that it is essentially the same as that of Hopf algebras. The reason is that for algorithmic purposes, we would like to avoid computing directly with comultiplication maps. The proof of Theorem 3.1 shows that $\text{HA}^{\text{op}}_R$ is canonically embedded into $\text{DP}_R$, but the category $\text{DP}_R$ has more objects because in a dual pair $(A,B,\Phi)$ the $R$-module $B$ is only isomorphic, and not necessarily identical, to $A^\vee$. We can use this extra “degree of freedom” to present $B$ in a computationally efficient way; the multiplication map on $B$ is then used to construct the comultiplication map on $A$, rather than the other way around.
Let $\text{GS}_R$ denote the category of finite locally free commutative group schemes over $R$.

**Corollary 3.2.** Let $R$ be a ring. There is a canonical equivalence of categories

$$G_R: \text{DP}_R \rightarrow \text{GS}_R$$

such that for every dual pair $(A, B, \Phi)$ of $R$-algebras, the underlying $R$-scheme of $G_R(A, B, \Phi)$ equals $\text{Spec } A$.

By construction, the above equivalence is compatible with duality in the following sense: if $(A, B, \Phi)$ is a dual pair of $R$-algebras and $H$ is the group scheme determined by $(A, B, \Phi)$, then the Cartier dual $H^*$ is canonically isomorphic to the group scheme determined by $(B, A, \Phi^t)$. The resulting isomorphisms $B \xrightarrow{\sim} A^\vee$ and $A \xrightarrow{\sim} B^\vee$ of $R$-modules equal those arising from $\Phi$, and the image of $x$ under the resulting $R$-algebra homomorphism

$$R[x, x^{-1}] \rightarrow A \otimes_R B$$

equals the element $\theta \in A \otimes_R B$ defined earlier.

**4. Examples**

4.1. The 2-torsion of an elliptic curve over $\mathbb{Q}$

We consider the elliptic curve $E$ over $\mathbb{Q}$ given by a Weierstraß equation of the form

$$E: y^2 = x^3 - ax$$

with $a \in \mathbb{Q}^\times$.

**Proposition 4.1.** The group scheme $E[2]$ over $\mathbb{Q}$ can be represented by the dual pair $(A, B, \Phi)$ of $\mathbb{Q}$-algebras, where

$$A = B = \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}[t]/(t^2 - a)$$

and the matrix of the $\mathbb{Q}$-bilinear map $\Phi$ with respect to the basis $((1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 0, t))$ is

$$\Phi = \begin{pmatrix} 1/4 & 1/4 & 1/2 & 0 \\ 1/4 & 1/4 & -1/2 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}.$$  

**Proof.** The coordinate ring of $E[2]$ is $\mathbb{Q} \times \text{Spec } \mathbb{Q}[x]/(x^3 - ax)$, where the first factor comes from the origin of $E$ and the second factor from the points of order 2. Using the isomorphism

$$\mathbb{Q}[x]/(x^3 - ax) \xrightarrow{\sim} \mathbb{Q} \times \mathbb{Q}[t]/(t^2 - a)$$

$x \mapsto (0, t)$,

we identify this coordinate ring with $A$. The group scheme $E[2]$ is identified with its own Cartier dual by the Weil pairing $E[2] \times E[2] \rightarrow \mu_2, \mathbb{Q}$.

The splitting field of $E[2]$ is $L = \mathbb{Q}(\sqrt{a})$. The four points of $E[2]$ over this field correspond to the four $\mathbb{Q}$-algebra homomorphisms

$$p_0, p_1, p_2, p_3: A \rightarrow L$$

$$(a, b, c + dt) \mapsto \begin{cases} a & \text{if } i = 0, \\ b & \text{if } i = 1, \\ c + d\sqrt{a} & \text{if } i = 2, \\ c - d\sqrt{a} & \text{if } i = 3. \end{cases}$$

The matrix of the induced $L$-algebra isomorphism $(p_0, p_1, p_2, p_3): A \otimes \mathbb{Q} L \rightarrow L^4$ with respect to the above basis of $A$ and the standard basis of $L^4$ is

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \sqrt{a} \\ 0 & 0 & 1 & -\sqrt{a} \end{pmatrix}.$$
For $i, j \in \{0, 1, 2, 3\}$, we consider the $\mathbb{Q}$-algebra homomorphism

$$p_i \otimes p_j : A \otimes_{\mathbb{Q}} B \longrightarrow L,$$

$$a \otimes b \mapsto p_i(a)p_j(b).$$

The Weil pairing is determined by $\theta \in A \otimes_{\mathbb{Q}} B$ such that

$$(p_i \otimes p_j)(\theta) = \begin{cases} 1 & \text{if } i = 0, j = 0 \text{ or } i = j \\ -1 & \text{otherwise.} \end{cases}$$

Writing $\Theta$ for the $4 \times 4$-matrix of coefficients of $\theta$ with respect to the fixed basis of $A \otimes_{\mathbb{Q}} B$, we see that the above condition is equivalent to the matrix equation

$$P \Theta P^t = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$ 

The unique solution is

$$\Theta = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1/a \end{pmatrix}.$$ 

By our earlier remark that $\Phi = \Theta^t \cdot -1$, the proposition is proved. \[ \square \]

4.2. The 2-torsion of a supersingular elliptic curve in characteristic 2

We now consider a non-étale group scheme, namely the 2-torsion of the supersingular elliptic curve $E$ over $\mathbb{F}_2$ given by the (projective) Weierstraß equation

$$E : y^2z + yz^2 = x^3.$$ 

As an effective Cartier divisor, $E[2]$ equals the point $(0 : 1 : 0)$ with multiplicity 4.

**Proposition 4.2.** The group scheme $E[2]$ over $\mathbb{F}_2$ can be represented by the dual pair $(A, B, \Phi)$ of $\mathbb{F}_2$-algebras, where

$$A = B = \mathbb{F}_2[t]/(t^4)$$

and the matrix of the $\mathbb{F}_2$-bilinear map $\Phi$ with respect to the power basis $(1, t, t^2, t^3)$ is

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. $$

**Proof.** A local parameter of $E$ at $(0 : 1 : 0)$ is $t = x/y$; this identifies $E[2]$ as a scheme with Spec $A$. Let $E_A = E \times_{\text{Spec} \mathbb{F}_2} \text{Spec} A$. For any section $P : A \rightarrow E_A$, the image of $P$ is an relative effective Cartier divisor on $E_A$, which we denote by $[P]$. The “universal 2-torsion point on $E$” is the section

$$P : \text{Spec} A \rightarrow E_A$$

that can be described in the affine patch $\{y = 1\}$ as $P = (t : 1 : t^3)$. Let $D$ be the relative Cartier divisor $[P] = [0]$ on $E_A$. Then $2D$ is a principal divisor. Below we compute a relative Cartier divisor $D'$ on $E_A$ that is linearly equivalent to $D$ but has disjoint support from $D$, as well as rational functions $f$ and $f'$ on $E_A$ (global sections of the sheaf of total quotient rings; see Hartshorne [7, §11.6]) with divisors $D$ and $D'$, respectively. We then pull back $D$ and $f$ to $E_A \otimes_{\mathbb{F}_2} B = E \times_{\text{Spec} \mathbb{F}_2} \text{Spec} B$ via the first projection $\text{Spec} A \times_{\text{Spec} \mathbb{F}_2} \text{Spec} B \rightarrow \text{Spec} A$, and we pull back $D'$ and $f'$ via the second projection $\text{Spec} A \times_{\text{Spec} \mathbb{F}_2} \text{Spec} B \rightarrow \text{Spec} B$, identifying $\text{Spec} B$ with $\text{Spec} A$. Then the Weil pairing is given by the element

$$\theta = f'(D)/f(D') \in \mu_2(A \otimes_{\mathbb{F}_2} B).$$
We will use the following sections Spec $A \to E_A$:

$$O = (0 : 1 : 0), \quad P = (t : 1 : t^3), \quad Q = (0 : 0 : 1), \quad R = (t : t^3 : 1).$$

The lines $x = ty$ and $x = tz$ intersect $E_A$ in the relative effective Cartier divisors $[P] + [Q] + [(t : 1 + t^3 : 1)]$ and $[O] + [R] + [(t : 1 + t^3 : 1)]$, respectively; this shows that $P + Q = R$ and that $D = [P] - [O]$ is linearly equivalent to $D' = [R] - [Q]$. Furthermore, $2D = 2[P] - 2[O]$ is the divisor of the rational function $f = (z - t^2x)/z$, and $2D' = 2[R] - 2[Q]$ is the divisor of the rational function $f' = (y - t^2x)/y$. Pulling the above divisors and functions back to the base ring $A \otimes_{\mathbb{F}_2} B$ as described above and writing this ring as $\mathbb{F}_2[t, t_2]/(t_1^4, t_2^4)$, we obtain divisors and functions

$$D = [(t_1 : 1 : t_2^4)] - [(0 : 1 : 0)], \quad D' = [(t_2 : t_2^2 : 1)] - [(0 : 0 : 1)],$$

$$f = (z - t_2^2x)/z, \quad f' = (y - t_2^2x)/y.$$

Using these data, we compute the Weil pairing on $E[2]$ as

$$\theta = \frac{f'(D)}{f(D')} = \frac{f'(t_1 : 1 : t_2^4)/f(0 : 1 : 0)}{f(t_2 : t_2^2 : 1)/f(0 : 0 : 1)} = \frac{(1 - t_2^2t_1)/1}{(1 - t_2^2t_2)/1} = 1 + t_1t_2 + t_1t_2^2 + t_1^2t_2^3.$$

This means that the matrix $\Theta$ of coefficients of $\theta$ with respect to the power basis $(1, t, t^2, t^3)$ is

$$\Theta = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$

By our earlier remark that $\Phi = \Theta^{t-1}$, the proposition is proved. \hfill \Box

**Remark.** The above result allows us to recover the comultiplication map by computing the group operation as in §5.2 for the “universal pair of points” with values in $A \otimes_{\mathbb{F}_2} A \simeq \mathbb{F}_2[t, t_2]/(t_1^4, t_2^4)$. The result is the $\mathbb{F}_2$-algebra homomorphism

$$\mu_{E[2]} : \mathbb{F}_2[t]/(t^4) \to \mathbb{F}_2[t, t_2]/(t_1^4, t_2^4)$$

$$t \mapsto t_1 + t_2 + t_1^2t_2.$$

As expected, this agrees with the formal group of $E$ (modulo $t^4$) as computed from the Weierstraß equation; see for example Tate [13, §3].

5. Algorithms

In this section we describe a number of algorithms for working with finite locally free commutative group schemes over a ring $R$. We will represent finite group schemes over $R$ as dual pairs of $R$-algebras, and describe our algorithms in terms of these. We also analyse the running time of our algorithms in terms of the required number of operations in $R$, and where applicable also the complexity of factoring univariate polynomials over $R$.

Throughout this section, the base ring $R$ will be assumed to be equipped with an “algorithmic representation”, in the sense that we have a way to write down elements of $R$ (for example as finite bit strings), and that we can perform ring operations and equality testing using this representation. Whenever we refer to an $R$-algebra $S$, we assume $S$ to be represented algorithmically in a similar way as $R$. 

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5.1. Efficiently representing group schemes when the algebras are monogenic

Let \((A, B, \Phi)\) be a dual pair of \(R\)-algebras. We make the assumption that the algebras \(A\) and \(B\) are monogenic over \(R\), and that we have an explicit presentation

\[ A = R[x]/(f), \quad B = R[y]/(g) \]

where \(f \in R[x]\) and \(g \in R[y]\) are monic polynomials, say of degree \(n \geq 0\). This implies that \(A\) and \(B\) are free over \(R\); we fix \(R\)-bases \((a_0, \ldots, a_{n-1})\) and \((b_0, \ldots, b_{n-1})\) of \(A\) and \(B\), respectively, by

\[ a_i = x^i \mod f \quad \text{and} \quad b_i = y^i \mod g. \]

We represent the perfect bilinear map \(\Phi\) by its matrix with respect to these bases. In this way we represent \(A\) and \(B\) by \(n\) elements of \(R\) each, and \(\Phi\) by \(n^2\) elements of \(R\).

Remark. In the important case where \(R = \mathbb{Q}\), the algebras \(A\) and \(B\) are products of number fields. To minimise the height of the entries of the matrix of the bilinear map \(\Phi\), it is useful to construct our chosen \(\mathbb{Q}\)-bases of \(A\) and \(B\) from integral bases of these number fields instead of bases consisting of powers of fixed primitive elements.

Below we will sometimes need to extend the base ring of our dual pairs. If \(S\) is an \(R\)-algebra and \(A = R[x]/(f)\), then we represent the tensor product \(A \otimes_R S\) as

\[ A \otimes_R S = S[x]/(f), \]

where \(f\) is now viewed as an element of \(S[x]\), and similarly for \(B\).

Given monic polynomials \(f \in R[x]\) and \(g \in R[y]\) of degree \(n\) and an \(n \times n\)-matrix \(M\), it is in principle straightforward to check whether these data define a dual pair of \(R\)-algebras by verifying the conditions (1)–(4) in the definition. However, the required number of operations in \(R\) quickly becomes large; we will explain in §§5.7 and 5.8 below how this verification can be done faster in cases where \(R\) is a domain and we can compute or approximate a common splitting field for \(A\) and \(B\).

5.2. Groups of points

Let \(G\) be a finite locally free commutative group scheme over a ring \(R\), represented by a dual pair of \(R\)-algebras \((A, B, \Phi)\). Let \(S\) be an \(R\)-algebra. We will denote the group operation on \(G(S) = \text{Hom}_{\text{Spec} R}(S, G)\) by \((p, q) \mapsto p \ast q\).

Using the maps

\[ G(S) = \text{Hom}_{\text{Spec} R}(S, G) \cong \text{Hom}^\text{Alg}_n(A, S) \subseteq \text{Hom}^\text{Mod}_n(A, S), \]

we can represent elements of the Abelian group \(G(S)\) by \(R\)-linear maps \(A \to S\). The group operation on \(G(S)\) can be computed as follows. Given two elements of \(G(S)\), represented as \(R\)-linear maps

\[ p, q : A \to S, \]

we use \(\Phi\) to convert \(p\) and \(q\) to elements \(\hat{p}\) and \(\hat{q}\) of the \(R\)-algebra \(B \otimes_R S\). We then compute the product \(\hat{p} \hat{q}\) in \(B \otimes_R S\) and convert the result back to an \(R\)-algebra homomorphism \(p \ast q : A \to S\) corresponding to the product of \(p\) and \(q\) under the group operation on \(G(S)\).

Let \(S\) be an \(R\)-algebra. Suppose that given a monic polynomial \(f \in R[x]\) we can compute the set of roots of \(f\) in \(S\). As noted above, the group of points \(G(S)\) is canonically isomorphic to \(\text{Hom}^\text{Alg}_n(A, S)\). If \(A = R[x]/(f)\), then determining \(G(S)\) as a set is therefore equivalent to computing the set of roots of \(f\) in \(S\). In §5.7 below, we will describe how the group structure on \(G(S)\) can be computed in the case where \(K\) is a field of characteristic not dividing the degree \(n\) of \(G\) and \(S\) is a finite extension of \(K\) containing an \(n\)-th root of unity.
5.3. Cartier duality

Let $G$ be a finite locally free commutative group scheme over a ring $R$, represented by a dual pair of $R$-algebras $(A, B, \Phi)$, and let $S$ be an $R$-algebra. Cartier duality gives an isomorphism

$$\tau: G^*(S) \xrightarrow{\sim} \text{Hom}(G_S, G_m)$$

of Abelian groups. Given an element $q \in G^*(S)$, viewed as an $R$-linear map $q: B \to S$, we apply the $R$-linear map

$$\text{id}_A \otimes q: A \otimes_R B \to A \otimes_R S$$

to the element $\theta_R \in A \otimes_R B$. The result is the invertible element in $A \otimes_R S$ corresponding to $\tau(q)$.

We can compute the duality pairing

$$\langle f, g \rangle: G(S) \times G^*(S) \to G_m(S) = S^*$$

in a similar way. Given elements $p \in G(S)$ and $q \in G^*(S)$, represented as $R$-linear maps $p: A \to S$ and $q: B \to S$, we apply the $R$-linear map

$$p \otimes q: A \otimes_R B \to S \otimes_R S$$

to the element $\theta_R \in A \otimes_R B$ and apply the multiplication map $S \otimes_R S$ to the result to obtain $\langle p, q \rangle$. If $A$ and $B$ are free over $R$ and we represent $\theta_R$ by a matrix over $R$ and $p, q$ as row vectors over $S$, then $\langle p, q \rangle$ is nothing but the product $p \theta_R q^t \in S$.

5.4. Groups of morphisms in $\text{DP}_R$

Let $(A_B, \Phi)$ and $(A'_B, \Phi')$ be two dual pairs of algebras over a ring $R$. We will now make the group operation in $\text{Hom}_{\text{DP}_R}((A, B, \Phi), (A', B', \Phi'))$ explicit. Given two morphisms

$$(f, g), (f', g'): (A, B, \Phi) \to (A', B', \Phi'),$$

we denote the sum $(f, g) + (f', g')$ by $(f'', g'')$. We compute $(f'', g'')$ as follows. Writing $G'$ for the group scheme $G_R(A', B', \Phi')$, we view $f, f': A' \to A$ as $A$-valued points of $G'$. We add these as in §5.2 to obtain $f''$. We then compute $g'': B \to B'$ as the unique $R$-linear map satisfying the adjointness property (3.1); this is automatically an $R$-algebra homomorphism.

Moreover, if $R$ and $A$ are such that given a monic polynomial $h \in R[x]$ we can compute the set of roots of $f$ in $A$, then we can compute the set $\text{Hom}_{\text{DP}_R}((A, B, \Phi), (A', B', \Phi'))$ as follows. We first compute the set $\text{Hom}_{\text{Alg}_R}(A', A)$ of $R$-algebra homomorphisms $f: A' \to A$. For each $f \in \text{Hom}_{\text{Alg}_R}(A', A)$, we then compute the unique $R$-linear map $g: B \to B'$ satisfying the adjointness property (3.1). The pair $(f, g)$ is in $\text{Hom}_{\text{DP}_R}((A, B, \Phi), (A', B', \Phi'))$ if and only if $g$ is also an $R$-algebra homomorphism.

Finally, we note that computing $\text{Hom}_{\text{DP}_R}((A, B, \Phi), (A', B', \Phi'))$ allows us in particular to determine whether the two objects $(A, B, \Phi)$ and $(A', B', \Phi')$ of $\text{DP}_R$ are isomorphic.

5.5. Direct sums, kernels and cokernels

The category $\text{GS}_R$ of finite locally free commutative group schemes over a ring is an Abelian category in some cases, for example if $R$ is a field. However, $\text{GS}_R$ fails to be an Abelian category in general, because quotients of finite locally free commutative group schemes are not necessarily locally free. For example, the unique non-trivial homomorphism $\mathbb{Z}/2\mathbb{Z} \to \mu_2$ of group schemes over $\mathbb{Z}$ has trivial kernel and cokernel in $\text{GS}_\mathbb{Z}$, yet it is not an isomorphism. However, $\text{GS}_R$ is always an exact category in the sense of Quillen [12, §2], and in particular is an additive category.

Let $(A, B, \Phi)$ and $(A', B', \Phi')$ be two dual pairs of algebras over a ring $R$. We define a perfect $R$-bilinear map between $A \otimes_R A'$ and $B \otimes_R B'$ by

$$\Phi'': (A \otimes_R A') \times (B \otimes_R B') \to R$$

$$(a \otimes a', b \otimes b') \mapsto \Phi(a, b)\Phi(a', b').$$

Writing $A'' = A \otimes_R A$ and $B'' = B \otimes_R B'$, one can check that $(A'', B'', \Phi'')$ is again a dual pair. It follows from the compatibility of Cartier duality with direct sums that $(A'', B'', \Phi'')$, together
with the “obvious” morphisms to and from the dual pairs \((A, B, \Phi)\) and \((A', B', \Phi')\), is a direct sum (and product) of these two dual pairs.

If \(R\) is a field, then we can compute kernels and cokernels of morphisms in \(\text{DP}_R\). Given a morphism \((f, g)\) from a dual pair \((A, B, \Phi)\) to a dual pair \((A', B', \Phi')\), let
\[
(k, l): (A'', B'', \Phi'') \longrightarrow (A, B, \Phi)
\]
be the kernel of \((f, g)\). Also, let
\[
(f_0, g_0): (A, B, \Phi) \longrightarrow (A', B', \Phi')
\]
be the zero morphism of dual pairs. We can then compute the quotient algebra \(k: A \to A''\) as the coequaliser of \(f, f_0: A' \to A\), and the subalgebra \(l: B'' \to B\) as the equaliser of \(g, g_0: B \to B'\). Then \(B''\) and \(k\) are each other’s orthogonal complements under \(\Phi\), and \(\Phi''\) is the bilinear map induced by \(\Phi\) on \(A'' \otimes_R B''\). Computing the cokernel of \((f, g)\) is entirely analogous.

5.6. Validating input
Let \(R\) be a domain, and let \(K\) be the field of fractions of \(R\). Suppose we are given a triple \((A, B, \Phi)\), where \(A\) and \(B\) are two finite locally free \(R\)-algebras of degree \(n\) such that the characteristic of \(K\) does not divide \(n\), and \(\Phi: A \times B \to R\) is a perfect \(R\)-bilinear map. By a common splitting field for \(A\) and \(B\), we will mean an extension field \(L\) of \(K\) such that the \(L\)-algebras \(A \otimes_R L\) and \(B \otimes_R L\) are both isomorphic to \(L^n\). Under the assumption that we can find such a field \(L\), we will show how to decide whether is \((A, B, \Phi)\) a dual pair of \(R\)-algebras.

We first note that if \(L\) is any extension field of \(K\), then a necessary and sufficient condition for \((A, B, \Phi)\) to be a dual pair of \(R\)-algebras is that the base change \((A_L, B_L, \Phi_L)\) to \(L\) is a dual pair of \(L\)-algebras. The reason for this is that under our assumptions on \((A, B, \Phi)\), the statement that \((A, B, \Phi)\) is a dual pair of \(R\)-algebras is equivalent to a list of identities involving maps between finite locally free \(R\)-modules. These can be checked after base extension to \(L\) because the modules are flat over \(R\) and the homomorphism \(R \to L\) is injective. From now on, we will therefore work over a field \(L\) as above.

We assume furthermore (enlarging \(L\) if necessary) that \(L\) is a common splitting field of \(A\) and \(B\), that \(L\) contains a root of unity of order \(n\) as well as all elements of \(\text{Hom}_{\text{Alg}_L}(A, L)\) and \(\text{Hom}_{\text{Alg}_L}(B, L)\). We have a function of sets
\[
\langle \cdot, \cdot \rangle: \text{Hom}_{\text{Alg}_L}(A, L) \times \text{Hom}_{\text{Alg}_L}(B, L) \longrightarrow L
\]
defined as in §5.3. If \((A, B, \Phi)\) is a dual pair of \(R\)-algebras corresponding to a group scheme \(G\), then \(\langle \cdot, \cdot \rangle\) can be identified with the duality pairing \(G(L) \times G''(L) \longrightarrow L^*\), but the definition makes sense without assuming that \((A, B, \Phi)\) is a dual pair.

We now observe (using the notation and terminology of the appendix) that \((A_L, B_L, \Phi_L)\) is a dual pair of \(L\)-algebras if and only if it is isomorphic to the dual pair \(((H_d)_L, (H^*_d)_L, \Phi_d)\) for some sequence of elementary divisors \(d\). Here \((H_d)_L\) and \((H^*_d)_L\) are the constant group schemes over \(L\) associated with \(H_d\) and \(H^*_d = \text{Hom}(H_d, \mathbb{Q}/\mathbb{Z})\), and \(\Phi_d\) is defined by viewing \((H^*_d)_L\) as the Cartier dual of \((H_d)_L\) via the isomorphism \(\frac{1}{n}\mathbb{Z}/\mathbb{Z} \ightarrow \langle \zeta \rangle\) sending \(1/n\) to \(\zeta\).

**Algorithm 5.1** (Input validation). Given a ring \(R\), two finite locally free algebras \(A\) and \(B\) of degree \(n\) over \(R\), a perfect \(R\)-bilinear map \(\Phi: A \times B \to R\) and a common splitting field \(L\) of \(A\) and \(B\) containing a root of unity of order \(n\), this algorithm outputs “True” if \((A, B, \Phi)\) is a dual pair of \(R\)-algebras, and “False” otherwise.

1. Fix bijections \(\alpha: \{1, \ldots, n\} \longrightarrow \text{Hom}_{\text{Alg}_L}(A, L)\) and \(\beta: \{1, \ldots, n\} \longrightarrow \text{Hom}_{\text{Alg}_L}(B, L)\).
2. Check whether all elements \(\langle \alpha(i), \beta(j) \rangle \in L\) are \(n\)-th roots of unity; if not, output “False” and stop.
3. Using the group isomorphism
\[
\lambda: \langle \zeta \rangle \overset{\sim}{\longrightarrow} \frac{1}{n}\mathbb{Z}/\mathbb{Z}
\]
sending \(\zeta\) to \(1/n\), compute the matrix
\[
T = (\lambda(\langle \alpha(i), \beta(j) \rangle))_{i,j=1}^n
\]
with entries in \( \frac{1}{n} \mathbb{Z}/\mathbb{Z} \). (Note that \( \lambda \) can be computed by listing all powers of \( \zeta \), for example.)

4. Using Algorithm 8.1, check whether \( T \) represents an Abelian group. If so, output “True”, otherwise output “False”.

The above procedure for validating the input relies on explicitly computing a splitting field for \((A, B, \Phi)\). In practice, it is desirable to perform this task without explicitly computing a splitting field. In the case of dual pairs of \( \mathbb{Q} \)-algebras, we will give an alternative approach in §5.8 below.

Remark. The above procedure remains valid when the assumption “\( L \) contains a root of unity of order \( n \)” is weakened to “\( L \) contains a root of unity of order equal to the exponent of \( G(L) \)”, where \( L \) is a separable closure of \( L \). This exponent may be smaller than \( n \), but is in general not known in advance.

5.7. Determining group structures

Let \( K \) be a field, and let \((A, B, \Phi)\) be a dual pair of \( K \)-algebras corresponding to a finite commutative group scheme \( G \) over \( K \). We will now describe how to determine the structure of the Abelian group \( G(K) \). After enlarging \( K \) if necessary, we assume that \( K \) contains a root of unity \( \zeta \) of order \( n \); this is not a real restriction because one can take \( \text{Gal}(\mathbb{K}(\zeta)/K) \)-invariants afterwards if desired.

Viewing the constant group scheme \( G(K)_K \) over \( K \) as a subgroup scheme of \( G \), we write \( G(K)_K \) for the orthogonal complement of \( G(K)_K \) in \( G^* \) under the duality pairing, and we put

\[
G^* \{ K \} = G^*/G(K)_K^*.
\]

Then \( G(K)_K \) corresponds to the largest quotient algebra \( A' \) of \( A \) that is a product of copies of \( K \), and \( G^* \{ K \} \) corresponds to a \( K \)-subalgebra \( B' \subseteq B \). We have now reduced to the case of constant group schemes over \( K \), and we have a perfect pairing

\[
\langle \cdot , \cdot \rangle : G(K) \times G^* \{ K \} \rightarrow K^*.
\]

We assume that we can explicitly determine a root of unity \( \zeta \in K \) of order \( n \) as well as all elements of \( G(K) \cong \text{Hom}_{\text{Alg}_K}(A, K) \simeq \text{Hom}_{\text{Alg}_K}(A', K) \) and \( G^* \{ K \} \cong \text{Hom}_{\text{Alg}_K}(B', K) \).

Algorithm 5.2 (Group structure). Given a field \( K \) containing a root of unity \( \zeta \) of order \( n \) and a dual pair \((A, B, \Phi)\) of \( K \)-algebras of degree \( n \) over \( K \), this algorithm outputs the sequence \( d = (d_1, \ldots, d_r) \) of elementary divisors of \( G(K) \) and a group isomorphism

\[
f: H_d \rightarrow \text{Hom}_{\text{Alg}_K}(A, K).
\]

1. Compute the largest quotient algebra \( A' \) of \( A \) that is a product of copies of \( K \).
2. Compute the subalgebra \( B' \subseteq B \) as the orthogonal complement of the kernel of the quotient map \( A \rightarrow A' \) under \( \Phi \), and \( \Phi \) induces a perfect pairing \( \Phi': A' \times B' \rightarrow K \).
3. Fix bijections \( \alpha : \{ 1, \ldots, m \} \rightarrow \text{Hom}_{\text{Alg}_K}(A', K) \) and \( \beta : \{ 1, \ldots, m \} \rightarrow \text{Hom}_{\text{Alg}_K}(B', K) \).
4. Using the group isomorphism \( \lambda : (\zeta) \sim \rightarrow \frac{1}{n} \mathbb{Z}/\mathbb{Z} \)

\[
\text{sending } \zeta \text{ to } 1/n, \text{ compute the matrix}
\]

\[
T = \left( \lambda((\alpha(i), \beta(j))) \right)_{i,j=1}^m
\]

with entries in \( \frac{1}{n} \mathbb{Z}/\mathbb{Z} \).
5. By applying Algorithm 8.1 to \( T \), determine a sequence \( d \) of elementary divisors and bijections

\[
p : \{ 1, \ldots, m \} \rightarrow H_d \quad \text{and} \quad q : \{ 1, \ldots, m \} \rightarrow H'_d.
\]
6. Output \( d \) and the bijection

\[
\alpha \circ p^{-1} : H_d \rightarrow \text{Hom}_{\text{Alg}_K}(A, L).
\]

Remark. For this algorithm to work, it is in fact only necessary that \( K \) contains a root of unity of order equal to the exponent of \( G(K) \).

Remark. One can also use “black box” algorithms to determine the group structure of \( G(L) \); see for example Buchmann and Schmidt [4]. However, the advantage of our approach is that it only uses the pairing and does not need to perform any group operations.
5.8. Validating input and determining group structures via complex approximation

We now describe a numerical variant of the algorithm from § 5.7 to verify whether a given triple \( (A, B, \Phi) \), where \( A \) and \( B \) are finite \( \mathbb{Q} \)-algebras with distinguished \( \mathbb{Q} \)-bases and \( \Phi \) is the matrix of a perfect \( \mathbb{Q} \)-bilinear map \( A \times B \to \mathbb{Q} \), is a dual pair of \( \mathbb{Q} \)-algebras. In case the answer is yes, this algorithm also determines the group structure of \( G(\overline{\mathbb{Q}}) \), where \( G \) is the finite commutative group scheme corresponding to \( (A, B, \Phi) \). Although our approach uses numerical approximations, it is made rigorous thanks to height bounds. It is important to stress that this approach does not require us to compute any splitting fields. We restrict to the field \( \mathbb{Q} \) and the embedding \( \mathbb{Q} \to \mathbb{C} \) for clarity; the idea can be generalised to other number fields and to ultrametric places instead of Archimedean places.

If \( K \) is a number field, we write \( \Omega_K \) for the set of places of \( K \); for every place \( v \) of \( K \) we denote by \( | \cdot |_v : K \to \mathbb{R} \) the normalised absolute value defined by \( v \). Let \( h : \overline{\mathbb{Q}} \to \mathbb{R} \) denote the absolute logarithmic height function, given by

\[
h(x) = \frac{1}{[K: \mathbb{Q}]} \sum_{v \in \Omega_K} \log \max \{1, |x|_v\}
\]

where \( K \) is a number field with \( x \in K \subseteq \overline{\mathbb{Q}} \); this is independent of the choice of \( K \). It is straightforward to check that for all \( x, y \in \overline{\mathbb{Q}} \) we have

\[
h(xy) \leq h(x) + h(y) \quad \text{and} \quad h(x + y) \leq h(x) + h(y) + \log 2.
\]

**Lemma 5.3.** Let \( \alpha \in C \) be algebraic of degree at most \( d \). If \( \alpha \neq 0 \), then we have

\[\neg \log |\alpha| \leq d h(\alpha).\]

**Proof.** This follows from the identity

\[h(\alpha) = h(1/\alpha),\]

the inequality

\[h(\alpha) \geq d^{-1} \log \max \{1, |\alpha|\} \geq d^{-1} \log |\alpha|\]

and the corresponding inequality for \( 1/\alpha \). \( \square \)

**Lemma 5.4.** Let \( \beta \in C \) be algebraic of degree at most \( d \), and let \( \zeta \in C \) be an \( n \)-th root of unity. Then we have the implication

\[|\beta - \zeta| < \exp\left(-d\phi(n)(h(\beta) + \log 2)\right) \implies \beta = \zeta,\]

where \( \phi \) is Euler’s \( \phi \)-function.

**Proof.** This follows from Lemma 5.3 applied to \( \alpha = \beta - \zeta \); note that \( \alpha \) lies in a number field of degree at most \( d\phi(n) \) and has height at most \( h(\beta) + \log 2 \) since \( h(\zeta) = 0 \). \( \square \)

To motivate the following algorithm, we note that if \( (A, B, \Phi) \) is a dual pair of \( \mathbb{Q} \)-algebras and \( \sigma : A \to C \) and \( \tau : B \to C \) are \( \mathbb{Q} \)-algebra homomorphisms, then \( \sigma \) and \( \tau \) represent \( C \)-valued points of the corresponding pair of Cartier dual group schemes \( G \) and \( G^* \). Furthermore, \( (\sigma \otimes \tau)(\theta_\Phi) \) is the value of the duality pairing between \( \sigma \) and \( \tau \), and in particular is an \( n \)-th root of unity.

**Algorithm 5.5** (Input validation and group structure using approximation). Given two finite \( \mathbb{Q} \)-algebras \( A \) and \( B \) of degree \( n \) with fixed \( \mathbb{Q} \)-bases \( (a_1, \ldots, a_n) \) and \( (b_1, \ldots, b_n) \), respectively, and the matrix of a perfect \( \mathbb{Q} \)-bilinear map \( \Phi : A \times B \to \mathbb{Q} \) with respect to these bases, this algorithm decides whether \( (A, B, \Phi) \) defines a commutative group scheme \( G \) over \( \mathbb{Q} \). If so, it outputs the sequence \( d = (d_1, \ldots, d_r) \) of elementary divisors of \( G(\overline{\mathbb{Q}}) \cong G(C) \) and group isomorphisms

\[
f : H_d \xrightarrow{\sim} \text{Hom}_{\text{Alg}_\mathbb{Q}}(A, C),
\]

\[
g : H_d^* \xrightarrow{\sim} \text{Hom}_{\text{Alg}_\mathbb{Q}}(B, C)
\]

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such that for all $x \in H_d$ and $\xi \in H_d^*$ we have
\[
\exp(2\pi i \xi(x)) = (f(x), g(\xi)),
\]
where the pairing on the right is induced by the canonical isomorphisms $\text{Hom}_{\text{Alg}_\mathbb{Q}}(A, C) \cong G(C)$ and $\text{Hom}_{\text{Alg}_\mathbb{Q}}(B, C) \cong \text{Hom}(G(C), C^*)$. Otherwise, it outputs “False”.

1. Check whether $A$ and $B$ are étale over $\mathbb{Q}$; if not, output “False” and stop.

2. Compute the matrix $\Theta$ of the element $\theta_\Phi \in A \otimes_{\mathbb{Q}} B$ with respect to the fixed bases of $A$ and $B$ as the inverse transpose of the matrix of $\Phi$.

3. Determine a positive integer $d$ such that the element $\theta_\Phi \in A \otimes_{\mathbb{Q}} B$ is a root of a monic polynomial of degree at most $d$ over $\mathbb{Q}$ (for example, take $d = n^2$).

4. Fix a numbering $(\sigma_1, \ldots, \sigma_n)$ of the $\mathbb{Q}$-algebra homomorphisms $A \to C$ and a numbering $(\tau_1, \ldots, \tau_n)$ of the $\mathbb{Q}$-algebra homomorphisms $B \to C$.

5. Compute a numerical approximation to the $n \times n$-matrices
\[
P = (\sigma_i(a_j))_{i,j=1}^n, \quad Q = (\tau_i(b_j))_{i,j=1}^n
\]
over $\mathbb{C}$, with sufficient complex precision to perform the remaining steps.

6. Compute a real number $C \geq 0$ such that for all $\mathbb{Q}$-algebra homomorphisms $\psi: A \otimes_{\mathbb{Q}} B \to \overline{\mathbb{Q}}$ we have $h(\psi(\theta_\Phi)) \leq C$.

7. Compute a numerical approximation to the $n \times n$-matrix
\[
Z = P\Theta Q = ((\sigma_i \otimes \tau_j)(\theta_\Phi))_{i,j=1}^n
\]
over $\mathbb{C}$, with sufficient precision to perform the next step.

8. Use Lemma 5.4 to verify whether the entries of $Z$ are $n$-th roots of unity, and if so, to compute the unique $n \times n$-matrix $T$ with coefficients in $\frac{1}{n} \mathbb{Z}/\mathbb{Z}$ such that $Z$ is obtained from $T$ by pointwise applying the map $t \mapsto \exp(2\pi i t)$. Otherwise, output “False” and stop.

9. Using Algorithm 8.1, determine whether $T$ describes an Abelian group. If so, output $d$ and the bijections $\alpha \circ p^{-1}$ and $\beta \circ q^{-1}$, where $d$, $p$ and $q$ are the output of Algorithm 8.1 and where $\alpha: \{1, \ldots, n\} \rightarrow \text{Hom}_{\text{Alg}_\mathbb{Q}}(A, C)$ and $\beta: \{1, \ldots, n\} \rightarrow \text{Hom}_{\text{Alg}_\mathbb{Q}}(B, C)$ are our fixed numberings of the $\mathbb{Q}$-algebra homomorphisms $A \to C$ and $B \to C$.

Remark. If one is willing to perform $\mathbb{Q}$-algebra operations in $A \otimes_{\mathbb{Q}} B$, then one can first check whether $\theta_\Phi$ is an $n$-th root of unity in $A \otimes_{\mathbb{Q}} B$, which is a necessary condition for $(A, B, \Phi)$ to be a dual pair. If so, one can take $C = 0$ and $d = \phi(n)$. Moreover, instead of applying Lemma 5.2 to find the matrix $T$, it suffices to determine, for each entry of $Z$, the $n$-th root of unity closest to it.

6. Galois representations

Let $K$ be a field, and let $\bar{K}$ be a separable closure of $K$. In this section, by a Galois representation of $K$ we will mean a finite Abelian group $V$ of order not divisible by the characteristic of $K$ equipped with a continuous action of the Galois group $\text{Gal}(\bar{K}/K)$. Given two Galois representations $V$ and $W$ of $K$, we view the Abelian group $\text{Hom}(V, W)$ of all group homomorphisms $V \to W$ as a Galois representation of $K$ by equipping it the with the $\text{Gal}(\bar{K}/K)$-action defined by
\[
(\sigma f)(v) = \sigma(f(\sigma^{-1} v)) \quad \text{for all } v \in V, \, f \in \text{Hom}(V, W) \text{ and } \sigma \in \text{Gal}(\bar{K}/K).
\]
We will apply this in particular to the case $W = \mathbb{G}_m(K)$.

Given a Galois representation $V$ of $K$, let $A_V$ denote the finite étale $K$-algebra of $\text{Gal}(\bar{K}/K)$-equivariant functions $V \to \bar{K}$. Then $A_V$ has a natural structure of Hopf algebra over $K$, and the finite étale $K$-scheme
\[
H_V = \text{Spec } A_V
\]
has a natural structure of finite commutative group scheme over $K$. Conversely, one can reconstruct $V$ from either $A_V$ or $H_V$ by
\[
V = \text{Hom}_{\text{Alg}_K}(A_V, \bar{K}) = H_V(\bar{K}).
\]
We can therefore write down Galois representations in the form of dual pairs of $K$-algebras.
6.1. Computing the matrix of an automorphism under a Galois representation

Let \((A, B, \Phi)\) be a dual pair of \(K\)-algebras of degree \(n\), where \(n\) is not divisible by the characteristic of \(K\), and let \(G\) be the corresponding finite commutative group scheme over \(K\). Let \(L\) be an extension field of \(K\) containing a root of unity \(\zeta\) of order \(n\). In particular, if \(L\) is a Galois extension of \(K\), then \(G(L)\) is a Galois representation of \(K\). We will show how to evaluate the map

\[ \text{Aut}_K L \to \text{Aut}(G(L)). \]

We assume that we have computed the structure of the group \(G(L)\) as in §5.7. In particular, we have a sequence \(d = (d_1, \ldots, d_r)\) of elementary divisors and isomorphisms

\[ H_d \sim \to G(L), \quad H^*_d \sim \to G^*\{L\}. \]

As in §5.7, let \(\lambda: (\zeta) \sim \to \frac{1}{n}\mathbb{Z}/\mathbb{Z}\) be the isomorphism sending \(\zeta\) to \(1/n\). Let \(P_1, \ldots, P_r \in G(L)\) and \(Q_1, \ldots, Q_r \in G^*\{L\}\) be the images of the standard generators of \(H_d\) under these isomorphisms. We also assume that the matrix

\[ U = (\lambda((P_i, Q_j)))_{i,j=1}^r \]

with entries in \(\frac{1}{n}\mathbb{Z}/\mathbb{Z}\) is known. (This matrix can in fact easily be extracted from the data computed while determining the group structure of \(G(L)\).)

We will use the following notation. If \(d = (d_1, \ldots, d_r)\) is a sequence of elementary divisors and \(\text{End} H_d\) is the (not necessarily commutative) endomorphism ring of the Abelian group \(H_d\), then we identify the additive group of \(\text{End} H_d\) with the direct sum

\[ \bigoplus_{i=1}^r \bigoplus_{j=1}^r \text{Hom}(\mathbb{Z}/d_j \mathbb{Z}, \mathbb{Z}/d_i \mathbb{Z}). \]

Under this identification, an element \(f \in \text{End} H_d\) corresponds to a collection of group homomorphisms \(f_{i,j}: \mathbb{Z}/d_j \mathbb{Z} \to \mathbb{Z}/d_i \mathbb{Z}\). We represent \(f\) by the \(r \times r\)-matrix \((f_{i,j}(1))_{i,j=1}^r\); note that \(f_{i,j}(1)\) is an element of \(\mathbb{Z}/d_i \mathbb{Z}\) annihilated by \(d_j\).

Now let \(\sigma\) be an automorphism of \(L\) over \(K\). We compute the element \(M(\sigma) \in \text{Aut} H_d\) (represented by a matrix as above) as follows. We compute the matrix

\[ V(\sigma) = (\lambda(\sigma P_i, Q_j)))_{i,j=1}^r \]

with entries in \(\frac{1}{n}\mathbb{Z}/\mathbb{Z}\). By the definition of \(M(\sigma)\), we have

\[ \sigma P_i = \sum_{k=1}^r M(\sigma)_{k,i} P_k. \]

From the above identities, we deduce the system of equations

\[ V(\sigma)_{i,j} = \sum_{k=1}^r M(\sigma)_{k,i} U_{k,j}, \]

which we can solve for the \(M(\sigma)_{k,i}\).

**Example.** Assume that \(K\) is a number field. Let \(p\) be a finite place of \(K\) such that \((A, B, \Phi)\) has good reduction at \(p\), in the sense that \((A, B, \Phi)\) can be written as the base change of a dual pair \((A_p, B_p, \Phi_p)\) of algebras over the valuation ring \(\mathbb{Z}_{K,p} \subset K\) of \(p\). Then we can compute the matrix of a Frobenius element at \(p\) as follows. Let \((\tilde{A}_p, \tilde{B}_p, \Phi_p)\) be the base change of \((A_p, B_p, \Phi_p)\) to the residue field \(k(p)\) of \(p\). Applying the algorithm described above to the dual pair \((\tilde{A}_p, \tilde{B}_p, \Phi_p)\) of algebras over \(k(p)\) and the Frobenius automorphism of a finite splitting field of this dual pair over \(k(p)\), we obtain the matrix of a Frobenius elements at \(p\) up to conjugacy in \(\text{End} H_d\).
6.2. Computing a dual pair corresponding to a Galois representation

We now describe how to compute a dual pair of $K$-algebras corresponding to a Cartier dual pair of Galois representations. Suppose that we are in the situation where we have two Galois representations $V$ and $V'$ of $K$ together with a perfect pairing

$$\langle \ , \ \rangle : V \times V' \to \mathbb{G}_m(K).$$

that is $\text{Gal}(\bar{K}/K)$-equivariant in the sense that

$$\langle \sigma v, \sigma v' \rangle = \sigma \langle v, v' \rangle$$

for all $v \in V$, $v' \in V'$ and $\sigma \in \text{Gal}(\bar{K}/K)$.

Then the finite commutative group schemes $H_V$ and $H_{V'}$ over $K$ are Cartier dual to each other. We can compute a dual pair of $K$-algebras corresponding to these group schemes as follows. We choose $\text{Gal}(\bar{K}/K)$-equivariant functions

$$\psi : V \to \bar{K}, \quad \psi' : V' \to \bar{K}.$$

Then the monic polynomials

$$f = \prod_{v \in V} (x - \psi(v)) \in \bar{K}[x], \quad g = \prod_{v' \in V'} (y - \psi'(v')) \in \bar{K}[y]$$

have coefficients in $K$. We write

$$A = K[x]/(f), \quad B = K[y]/(g).$$

Furthermore, we define an element

$$\theta \in A \otimes_K B \cong K[x,y]/(f(x), g(y))$$

as the unique element satisfying the interpolation property

$$\theta(\psi(v), \psi'(v')) = \langle v, v' \rangle$$

for all $v \in V$, $v' \in V'$.

We can then compute the matrix of $\Phi$ with respect to the power bases of $A$ and $B$ as the inverse transpose of the matrix of $\theta$.

If $K$ is a number field and $V$ and $V'$ are realised inside some quasi-projective variety over $K$, then except in the smallest examples, the above computations are only feasible in practice using numerical approximation.

**Remark.** One can in fact obtain an explicit upper bound on the height of the matrix of $\Phi$ in terms of the heights of the roots of the polynomials $f$ and $g$. We do not give the details here.

6.3. Explicit computations

The author explicitly computed several dozen dual pairs of $\mathbb{Q}$-algebras corresponding to Galois representations of $\mathbb{Q}$ on groups of the form $\mathbb{Z}/n\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$. The sources of these Galois representations are characters of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, torsion subschemes of elliptic curves over $\mathbb{Q}$, and Hecke eigenforms over finite fields.

In practice, the data defining a dual pair $(A, B, \Phi)$ of $\mathbb{Q}$-algebras often turns out to have smaller height than the data defining the corresponding Hopf algebra, especially if $A$ and $B$ are not isomorphic (i.e. the group scheme is not isomorphic to its own Cartier dual). This is useful in practice because our computations rely on numerical approximation, and the required precision is directly related to the height of the output.
Example. The author computed a dual pair of $\mathbb{Q}$-algebras corresponding to the Cartier dual pair of Galois representations

$$\rho_1, \rho_2 : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{F}_7)$$

attached to the two Hecke eigenforms $f_1$ and $f_2$ for the group $\Gamma_1(13)$ over $\mathbb{F}_7$ with $q$-expansions

$$f_1 = q + q^2 + q^3 + 5q^4 + 5q^5 + q^6 + 2q^7 + 2q^8 + O(q^{10}),$$

$$f_2 = q + 3q^2 + 4q^3 + 3q^4 + 2q^5 + 5q^6 + 5q^8 + 4q^9 + O(q^{10}).$$

These representations occur in the 7-torsion of the Jacobian of the modular curve $X_1(13)$ of genus 2; the duality between them is realized by the Weil pairing. The computations were done using the author’s modgalrep software package [3], written in C and PARI/GP [11]. The height of the resulting polynomials $f \in \mathbb{Q}[x]$ and $g \in \mathbb{Q}[y]$ (i.e. the maximum of the logarithmic heights of the coefficients) is about 88 for $f$ and 93 for $g$. The height of the matrix of $\Phi$ with respect to the power bases of $A = \mathbb{Q}[x]/(f)$ and $B = \mathbb{Q}[y]/(g)$ is about 585. In contrast, the height of the image of $x$ under the comultiplication map

$$\mu : A \rightarrow A \otimes_{\mathbb{Q}} A \cong \mathbb{Q}[x_1, x_2]/(f(x_1), f(x_2))$$

is roughly 1626.

7. Future work

As we have seen, dual pairs of algebras over a ring $R$ are a concise way to write down finite locally commutative group schemes over $R$, and the category of dual pairs of algebras over $R$ is equivalent to the category of finite commutative group schemes over $R$. Moreover, if $R$ is a number field, the resulting objects have comparatively small height. For these reasons, in the near future the author is planning to work on building a database of Galois representations stored as dual pairs of algebras.

Besides representing finite commutative group schemes by a dual pair of algebras over a ring $R$, it is also possible to represent torsors over such group schemes. If $G$ is a finite commutative group scheme represented by the dual pair $(A, B, \Phi)$ over $R$, then as an algebraic structure representing $G$-torsors one can take a triple $(T, U, \Psi)$, where $T$ is a finite locally free $R$-algebra, $U$ is a locally free $B$-module of rank 1, and $\Psi : T \times U \rightarrow R$ is a perfect $R$-bilinear map. Another future direction is to develop algorithms for working with torsors in this representation; this could be useful for computing in Selmer groups.

8. Appendix: identifying a finite Abelian group from a pairing

Let $H$ be a finite Abelian group of order $n$, and let $H^* = \text{Hom}(H, \mathbb{Q}/\mathbb{Z})$ be the dual group. Let $p : \{1, 2, \ldots, n\} \rightarrow H$ and $q : \{1, 2, \ldots, n\} \rightarrow H^*$ be two bijections, viewed as enumerations of the elements of $H$ and $H^*$, respectively. To these data we associate the $n \times n$-matrix $T = (q(j)(p(i)))_{i,j=1}^n$ with entries in the group $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$.

We say that an $n \times n$-matrix $T$ with entries in $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$ describes an Abelian group if it is of the form $(q(j)(p(i)))_{i,j=1}^n$ for some choice of $H$, $p$ and $q$ as above. It is not hard to see that $T$ determines $H$ up to isomorphism.

Algorithm 8.1 below checks whether a given $n \times n$-matrix with entries in $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$ describes an Abelian group $H$, and if so, identifies $H$. For the latter, we will use the following terminology and notation. By a sequence of elementary divisors we mean a sequence $d = (d_1, d_2, \ldots, d_r)$ of integers greater than 1 satisfying $d_r | d_{r-1} | \ldots | d_1$. For such a sequence $d$, we write

$$H_d = \mathbb{Z}/d_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_r \mathbb{Z}.$$

Algorithm 8.1 (Identify a finite Abelian group from a pairing matrix). Given a positive integer $n$ and an $n \times n$-matrix $T$ with entries in $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$, this algorithm decides whether $T$ describes an Abelian group. If so, it outputs a sequence $d$ of elementary divisors and two bijections

$$p : \{1, 2, \ldots, n\} \sim \rightarrow H_d \quad \text{and} \quad q : \{1, 2, \ldots, n\} \sim \rightarrow H_d^*.$$
satisfying $T = (q(j)(p(i)))_{i,j=1}^n$; otherwise, it outputs “False”.

1. If $n = 1$, output the empty sequence $d$ and the unique bijections $p$ and $q$.

2. Let $d_1$ be the maximum of the denominators of the entries of $T$. If $T$ contains the element $1/d_1$, then choose a pair $(i_1, j_1)$ of row and column indices such that $T_{i_1,j_1} = 1/d_1$; otherwise, output “False” and stop.

3. Determine injective functions

$$f_1: \mathbb{Z}/d_1 \mathbb{Z} \to \{1, 2, \ldots, n\}, \quad g_1: \mathbb{Z}/d_1 \mathbb{Z} \to \{1, 2, \ldots, n\}$$

such that for all $x \in \mathbb{Z}/d_1$, the $f_1(x)$-th row of $T$ equals $x$ times the $i_1$-th row and the $g_1(x)$-th column of $T$ equals $x$ times the $j_1$-th column. If these do not exist, output “False” and stop.

4. Write $n' = n/d_1$. Let $T'$ be the submatrix formed by the $T_{i,j}$ where $i$ runs over the row indices such that $T_{i,j_1} = 0$ and $j$ runs over the column indices such that $T_{i_1,j} = 0$. If $T'$ is not an $n' \times n'$-matrix with entries in $\mathbb{Z}/d_1$, output “False” and stop.

5. Apply the algorithm recursively to $T'$ to check whether $T'$ describes an Abelian group, and if so, to find the corresponding sequence $d'$ of elementary divisors and bijections $p'$ and $q'$. Since $T'$ is a submatrix of $T$, the inverses of $p'$ and $q'$ can be viewed as injective functions

$$f': H_{d'} \to \{1, 2, \ldots, n\}, \quad g': H_{d'}^* \to \{1, 2, \ldots, n\}.$$

6. Writing $d' = (d_2, \ldots, d_r)$, let $d = (d_1, d_2, \ldots, d_r)$. Determine whether $d_2$ divides $d_1$, and whether there exist (necessarily unique) bijections

$$f: H_d \to \{1, 2, \ldots, n\}, \quad g: H_d^* \to \{1, 2, \ldots, n\}$$

such that for all $x \in \mathbb{Z}/d_1$, $f(x_1,x')$-th row of $T$ equals the $f_1(x_1)$-th row plus the $f'(x')$-th row, and similarly for $g$ (using the columns of $T$). If so, put $p = f^{-1}$ and $q = g^{-1}$ and output $(d, p, q)$; otherwise, output “False” and stop.

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