Nonabelian mirrors and Gromov-Witten invariants

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Abstract

We propose Picard-Fuchs equations for periods of nonabelian mirrors [11] in this paper. The number of parameters in our Picard-Fuchs equations is the rank of the gauge group of the nonabelian GLSM, which is eventually reduced to the actual number of Kähler parameters. These Picard-Fuchs equations are concise and novel. We justify our proposal by reproducing existing mathematical results, namely Picard-Fuchs equations of Grassmannians and Calabi-Yau manifolds as complete intersections in Grassmannians [2]. Furthermore, our approach can be applied to other nonabelian GLSMs, so we compute Picard-Fuchs equations of some other Fano-spaces, which were not calculated in the literature before. Finally, the cohomology-valued generating functions of mirrors can be read off from our Picard-Fuchs equations. Using these generating functions, we compute Gromov-Witten invariants of various Calabi-Yau manifolds, including complete intersection Calabi-Yau manifolds in Grassmannians and non-complete intersection Calabi-Yau examples such as Pfaffian Calabi-Yau threefold and Gulliksen-Negård Calabi-Yau threefold, and find agreement with existing results in the literature. The generating functions we propose for non-complete intersection Calabi-Yau manifolds are genuinely new.
Mirror symmetry \cite{3-5} is a powerful tool in computing worldsheet instanton corrections to the moduli space of Calabi-Yau threefolds. Because mirror symmetry maps the calculation of quantum corrections to a classical computation in algebraic geometry. It is the work of Candelas et al. \cite{6} that made a tremendous shock in mathematical circles and let mathematicians become interested in mirror symmetry. Candelas et al. in \cite{6} used tools in quantum field theory to solve an open geometric question \cite{7}: counting the numbers of curves in Calabi-Yau spaces of a given degree, which are known as Gromov-Witten invariants. Then, mathematicians \cite{8,9} translated the physical framework of \cite{6} into a precise mathematical conjecture, which has been proved in \cite{10,11} and now is named “Mirror Theorem” in mathematics. However, the technique developed in those papers is only for Calabi-Yau three-folds that have a realization as complete intersections in toric varieties.

During the same period, physicists also studied another important related phenomenon in string compactification: the correspondence between Calabi-Yau manifolds and Landau-Ginzburg (LG) models \cite{12,14}. In \cite{15}, Witten developed a comprehensive framework to
understand the same problem. He constructed gauged linear sigma models (GLSMs) and noted that an NLSM on a Calabi-Yau manifold and an LG model are often two different phases of the same GLSM. Since Witten’s work, there has been a great deal of work on GLSMs. Morrison and Plesser [16] obtained Yukawa-couplings of Calabi-Yau targets by summing instantons in abelian gauge theories, which match results obtained from mirror symmetry. Following [15,17], Hori and Tong [18] studied nonabelian GLSMs for complete intersection Calabi-Yau manifolds in Grassmannians. The Yukawa-couplings of these Calabi-Yau manifolds were obtained in that paper by a physical approach, which agrees with Yukawa-couplings obtained from mirror symmetry studied in [2]. See [19,22] for other works on nonabelian GLSMs. On the one hand, those aspects of GLSMs indicated that if we take into account full quantum corrections in the gauge theory, we can obtain Gromov-Witten invariants without relying on mirrors. Therefore, the authors of [23] proposed that one can extract Gromov-Witten invariants from an exact Kähler potential on the quantum Kähler moduli space of Calabi-Yau threefold, which can be computed exactly from two-sphere partition function by supersymmetric localization [24,25]. This identification is not surprising from $tt^*$ equations [26] and the exact RG-flow in quantum field theory [27]. However, the GLSM Lagrangian on the two-sphere is different from the Lagrangian on the flat worldsheet, thus the authors of [23] made a conjecture. Then, the authors in [28] provided more evidence to support this conjecture. However, a fully satisfactory physical understanding of this conjecture can be found in [29]. Based on this, the authors of [23] can even compute Gromov-Witten invariants of Calabi-Yau manifolds, which are not complete intersections in toric varieties. And these Calabi-Yau manifolds are usually happened to be geometric phases of nonabelian gauged linear sigma models. Finally, we shall briefly mention some other works on the computation of Gromov-Witten invariants based on exact results in GLSMs. The authors of [30] computed Gromov-Witten invariants from vortex partition functions [24]. While the authors of [31] calculated Gromov-Witten invariants from exact results in equivariant A-twisted GLSMs studied in [32]. See also [33,34] for other related works.

On the other hand, as Witten [15] mentioned, mirror symmetry can be interpreted in terms of exchanging chiral multiplets and twisted chiral multiplets between two GLSMs. Following this picture, Morrison and Plesser [35,36] explained mirror symmetry as a duality between pairs of GLSMs for Calabi-Yau manifolds. In 2000, Hori and Vafa [37] applied T-duality [38] to abelian gauged linear sigma models for general targets to obtain mirror Landau-Ginzburg theories. In 2018, one author of the present paper and Sharpe [1] proposed mirror constructions for nonabelian GLSMs. The idea of [1] is simple, where a low energy effective theory of the nonabelian GLSM was considered. It also flows to the same NLSM as the original nonabelian GLSM. This low energy effective theory is an abelian gauge theory with a Weyl-group orbifold, and it is called the associated Cartan theory [39]. Following [37,40], nonabelian mirror constructions were proposed by applying T-duality to associated Cartan theories [1]. However, the exact kinetic term in the abelian-like effective theory under RG-flow is not known due to quantum corrections. Thus the proposal of [1] is not a complete physical derivation in the sense of [37] for nonabelian mirrors. In [1,41,42], many consistency checks for nonabelian mirrors were performed, which
match results obtained from gauge theories. Furthermore, \[13,14\] predicted new results for pure gauge theories from mirrors, which have been checked in \[15\] by a direct computation in gauge theories. Despite these new developments in nonabelian mirrors, one may naturally ask a question: whether the nonabelian mirror construction can be used to compute Gromov-Witten invariants of Calabi-Yau manifolds. This paper is devoted to addressing this question.

To compute Gromov-Witten invariants of Calabi-Yau manifolds from mirrors, one needs to find Picard-Fuchs equations from mirrors \[6\], which periods of mirrors shall satisfy. Not only can Yukawa-couplings be obtained from Picard-Fuchs equations, but also mirror maps. If we have a mirror-geometry, one can define periods directly by integrating the holomorphic top form of the mirror over its A-cycles \[16\]. For Landau-Ginzburg mirrors of abelian GLSMs, the definition of periods has been discussed in \[37,38\]. Because periods do not depend on the Kähler moduli, we can evaluate them in any phase. However, A-branes or A-cycles do depend on the Kähler moduli, which must be defined carefully in our setup. One difficulty is that the definition of A-branes in general orbifolded Landau-Ginzburg models is still an open question. Some primary studies of A-branes in the minimal model can be found in \[49\]. We avoid this difficulty by evaluating periods in the UV where A-branes can be defined explicitly in the geometric phase. If the Landau-Ginzburg is a UV fundamental theory, one can treat the superpotential of LG perturbatively \[50,51\], then A-branes are just Lagrangian-submanifolds of \(\mathbb{C}^n\). Unfortunately, this is not the case for our nonabelian mirror Landau-Ginzburg models. Thus we have to provide a non-perturbative definition of A-branes following \[37\]. Then from the explicit integral formula of the period, we can read off Picard-Fuchs equations from these periods. Our Picard-Fuchs equations are novel as we use as many Kähler parameters as the rank of the gauge group to define differential equations. To justify our proposal, we reproduce existing Picard-Fuchs equations of mirrors of Grassmannians and complete intersection Calabi-Yau manifolds in Grassmannian studied in \[2\] with our approach. As one can apply our framework to any target admitting a GLSM description, we also predict Picard-Fuchs equations of other Fano targets, which have not been explicitly studied in the literature before. Furthermore, one can get generating functions of mirrors of GLSMs from our method. We reproduce known generating functions of complete intersection Calabi-Yau manifolds in Grassmannians and find generating functions of non-complete intersection examples, which have not been studied in the literature before. One can then read off mirror maps and Gromov-Witten invariants from these generating functions of Calabi-Yau manifolds, and all of our results agree with existing results in the literature. However, in principle, our machinery applies to any Calabi-Yau target with a GLSM description.

The outline of this paper is the following: In section \[2\] we review how Candelas et al. \[1\] got Gromov-Witten invariants from Yukawa-couplings and mirror maps. In section \[3\] we review some basics of GLSMs and mirror constructions. This section is serving for the following sections. In section \[4\] we propose Picard-Fuchs equations of mirrors of nonabelian GLSMs. By using our Picard-Fuchs equations, we reproduce existing results of Picard-Fuchs equations in the math literature and predict some new results. In section \[5\] we
compute Gromov-Witten invariants of various Calabi-Yau manifolds and find the agreement with existing results in the literature. Moreover, one can obtain generating functions of non-complete intersection Calabi-Yau manifolds by means of our approach, and they are genuinely new. We end with a summary of our results and future directions.

2 The computation of instanton numbers from mirrors

In this paper, we mainly focus on computing Gromov-Witten invariants of Calabi-Yau manifolds. To compute Gromov-Witten invariants of Calabi-Yau manifolds from mirrors [6], we need Yukawa-couplings and mirror maps. Both of them can be obtained from Picard-Fuchs equations defined globally on the complex-moduli space of mirrors and do not depend on the Kähler moduli. Thus, one can find Picard-Fuchs equations from non-geometric mirrors, for example, Landau-Ginzburg mirrors.

We review some basics of this computational process. We denote the Calabi-Yau manifold as $X$, and the mirror as $\hat{X}$. We have $h^{1,1}(X)$ Kähler parameters denoted by $\tau^a$ that are flat coordinates on the Kähler moduli space of $X$. The mirror $\hat{X}$ has algebraic coordinates $t^a$ on the complex moduli that have the same dimension as the Kähler moduli space of $X$. If the mirror $\hat{X}$ has a geometrical representation, its dimension of the complex-moduli space satisfies $h^{2,1}(\hat{X}) \equiv h^{1,1}(X)$. Mirror symmetry suggests that the $A -$ twisted correlation function of Calabi-Yau $X$ can be obtained from the mirror $B -$ twisted correlation function under the mirror map. More specifically, we have the following equality

$$K_{z^a z^b z^c} = \frac{1}{(\Pi_0(q(z)))^2} K_{q^a q^b q^c} \left( \frac{q^a d z^a}{z^a dq^a} \right) \left( \frac{q^b d z^b}{z^b dq^b} \right) \left( \frac{q^c d z^c}{z^c dq^c} \right),$$  \hspace{0.5cm} (2.1)

where indices are not summed, $z^a = \exp(-\tau^a)$ and $q^a = \exp(-t^a)$. The fundamental period $\Pi_0(q)$ satisfies Picard-Fuchs equations:

$$\mathcal{L}_a \Pi_0(q) = 0, \quad a \in 1, \ldots, h^{1,1}(X).$$  \hspace{0.5cm} (2.2)

For toric varieties (abelian GIT-quotients), one can obtain differential operators $\mathcal{L}_a$ from known mirrors [52]. Then, one can use solutions of Picard-Fuchs equations to define a cohomology-valued generating function, which called I-function [10] on the gauge theory side, or $B$-series [53]

$$\mathcal{L}_a \Pi(\eta; q) = 0,$$  \hspace{0.5cm} (2.3)

where $\Pi(\eta; q)$ can be expanded in terms of $H^{even}(X)$ cohomology classes corresponding to mirror Jacobian rings of $\hat{X}$, or $H^3(\hat{X})$ if the mirror has a geometric representation, thus we have

$$\Pi(\eta; q) = \Pi_0(q) + \eta_a \Pi_1^a(q) + f_2(\eta) \Pi_2^a(q) + \ldots.$$  \hspace{0.5cm} (2.4)

$\eta_a$ in (2.4) are generators of $H^{1,1}(X, \mathbb{Z})$, and $f_i(\eta)$ is a cohomology class in $H^{i,i}(X, \mathbb{Z})$ that can be expressed in terms of a homogenous degree $i$ polynomial of $\eta_a$. One shall be able to read off relations among generators from equ’n (2.3).
For nonabelian GIT-quotient targets and surfaces in those targets, no mirror construction was known until \[1\]; however, that paper did not include the study of Picard-Fuchs equations from mirrors. We fill this gap in this paper. In section 4, we define fundamental periods in nonabelian mirrors \[1\], and then we propose Picard-Fuchs equations for those periods.

If we know periods \(\Pi_0(q)\) and \(\Pi_1(q)\), we can define flat coordinates \(\tau^a\):

\[
  z^a = \exp\left(-\frac{\Pi_1(q)}{\Pi_0(q)}\right) \exp\left(2i\pi t^a_0\right),
\]

where the factor \(\exp\left(2i\pi t^a_0\right)\) in equ’n \((2.5)\) can be fixed by requiring positivity of the first Gromov-Witten invariant \[23\]. Subsequently, Gromov-Witten invariants of the Calabi-Yau 3-folds can be read off from the \(z\)-series in the \(A - twisted\) correlation function:

\[
  K_{zzz} = n_0 + \sum_{\{N_a\}} \frac{n_{\{N_a\}}N_aN_bN_c}{1 - \prod(z^d)^{N_d}},
\]

where \(n_0\) is the classical intersection number, and \(n_{\{N_a\}}\) are Gromov-Witten invariants. Notice \(N_a\) is the degree of the map from the worldsheet \(M\) to the target \(X\) and defined by \(N_a = \int_M \eta_a \in \mathbb{Z}\). For one-dimensional Kähler parameter, the Yukawa-coupling, at the large volume limit, is

\[
  K_{zzz} = n_0 + \sum_{N \geq 1} \frac{n_N N^3 z^N}{1 - z^N}.
\]

This formula was conjectured in \[6\] and then proved in \[56\]. We have recalled the method for computing Gromov-Witten invariants, and this approach requires an explicit mirror theory. Thus we will review mirror constructions in the next section.

3 Gauged linear sigma models and mirror symmetry

For a Fano space or a Calabi-Yau manifold, the worldsheet quantum field theory admits a UV description called the gauged linear sigma model (GLSM) \[15\]. Linear sigma models include both chiral and twisted chiral superfields, which is fact useful in understanding mirror symmetry \[15\]. So after we define GLSMs, we will provide a framework for constructing mirrors from GLSMs.

3.1 GLSMs

We define a GLSM by specifying the following data.

- **Gauge group**: a compact Lie group \(G\) with the associated Lie algebra \(\mathfrak{g}\).  

\[1\]For homogenous space, one can find a mathematical mirror construction in \[57\].
• Chiral matter fields: \( \Phi_{i=1, \ldots, N} \) are in irreducible representations \( R_i \) of \( G \). This vector space is denoted by \( V \equiv \mathbb{C}^{N \cdot \text{rank}(G)} \).

• Adjoint fields: The gauge field \( V \) is in the adjoint representation of \( G \), the field strength is a twisted chiral superfield denoted by \( \Sigma \).

• Superpotential: a holomorphic, \( G \)-invariant polynomial \( W : V \to \mathbb{C} \), namely \( W \in \text{Sym}(V^\ast)^G \).

• Fayet-Iliopoulos (FI) parameters and theta angles: a set of FI-parameters \( r^a \) and periodic theta angles \( \theta^a \in \mathbb{R}/2\pi\mathbb{Z} \) where the index \( a \) runs over the number of \( U(1) \) sectors of \( G \). One can combine them to define \( q^a = \exp(-t^a) \in \mathbb{C}^\ast \), where \( t^a = r^a - i\theta^a \).

• R-symmetry: a vector \( U(1)_V \) and axial \( U(1)_A \) R-symmetries that commute with the action of \( G \) on \( V \). To (classically) preserve the \( U(1)_V \) symmetry, the superpotential must have weight two under it in our convention:

\[
W(\lambda^a \phi) = \lambda^2 W(\phi)
\]

where \( \phi \) denotes the coordinates in \( V \).

For abelian theories, many results are known [15]. While nonabelian gauged linear sigma models are still under active development, see these new works [58–60].

Phases

The FI-theta parameters \( t \) can be shown to belong to \( \mathfrak{z}^\ast/(2\pi i P^W) \) where \( P^W \) is the lattice of Weyl invariant weights and \( \mathfrak{z} = \text{Lie}(Z(G)) \). We denote the real and imaginary part of \( t \) by

\[
t = r - i\theta \in \mathfrak{z}^\ast/2\pi i P^W.
\]

The differential of the representation \( R \) induces a \( \mathbb{R} \)-valued moment map:

\[
\mu : V \to \mathfrak{g}^\ast.
\]

Then we define the space of classical vacua in a phase \( r \)

\[
X_r := (dW^{-1}(0) \cap \mu^{-1}(r))/G.
\]

For geometric phases, \( X_r \) is smooth and compact, and those phases correspond to NLSMs on target \( X_r \) but, in general, we should think of a phase at \( r \) as a hybrid model

\[
(\mu^{-1}(r)/G, W)
\]

where \( W \) is understood as \( W \) restricted to \( \mu^{-1}(r) \).

CY Criteria
For a GLSM with gauge group $G$, and $n$ chirals in representations $R_i$ the condition for the cancellation of $U(1)_A$ anomaly takes the following form:

$$\sum_{i=1}^{n} \text{tr}_{R_i}(t) = 0, \quad t \in g,$$

$$\sum_{i=1}^{n} (1 - q_i) \text{dim}(R_i) - \text{dim}(g) = \hat{c} = 3.$$

For abelian GLSMs, we can T-dualize GLSMs to get mirrors. While for nonabelian GLSMs, we do not have a systematic nonabelian T-dual construction yet [61]. Thus, to get nonabelian mirrors, we study gauge theory on the “generic Coulomb branch.” More specifically, we have an “associated Cartan” gauge theory with the following data [39]:

- **Gauge group:** gauge group $T = U(1)^{\text{rank}(G)} \rtimes S$, $U(1)^{\text{rank}(G)}$ is the maximal torus of gauge group $G$, and $S$ is the Weyl group of $G$.

- **Chiral matter fields:** $\Phi_{i=1,\ldots,N}$ are charged by weights $\rho^a_i$ under gauge group $T$, the field space $\Phi$ is a Weyl-orbifold free subset of $\mathbb{C}^{N\cdot\text{rank}(G)}$ denoted by $V^o$. Additional Weyl orbifold free $\text{dim}(g) - \text{rank}(g)$ vector R-charge 2 with gauge-charges given by the roots $\alpha^a_\mu$ of $G$.

- **Adjoint fields:** $V$ is the adjoint representation of $U(1)^{\text{rank}(G)} \rtimes S$, the field strength is the twisted chiral superfield also denoted as $\Sigma$.

- **Superpotential:** a holomorphic, $T$-invariant polynomial $W : V^o \to \mathbb{C}$, namely $W \in \text{Sym}((V^o)^*)^G$.

- **Fayet-Iliopoulos (FI) parameters and theta angles:** we enlarge the number of FI parameters $t^\mu = r^\mu - i\theta_\mu$, $\mu = 1, \ldots, \text{rank}(G)$, the number of physical FI-parameters are still as before.

- **R-symmetry:** as before

It is clear that “associated Cartan” theories are effective theories of nonabelian gauge-linear sigma models. One may be confused why we have additional chiral superfields as it seems that they are from off-diagonal components of vector multiplets. One can argue this from the physical condition: only chiral superfields (here the $W$-bosons) can be charged under gauge group $T$ on the generic Coulomb branch. It would be interesting to derive this statement from the RG-flow; however, this can not be the full story. The additional chiral superfields must have vector R-charge 2 to match the physics [40]. If the theory flows to an SCFT at the IR, the contribution to the SCFT central charge from gauge fields is negative. This indicates that the additional chiral superfields in the associated Cartan theory must have vector R-charge 2. One can also observe this by studying the exact results in GLSMs for general targets [39].
3.2 Mirror constructions

In the previous section, we have defined gauged linear sigma models. Now, we specify the data of mirrors.

Abelian mirrors

If we consider the gauge theory for a target without a superpotential, then mirrors are:

- **Chiral matter fields**: \( Y_i = g_i + i\vartheta_i \), where \( i = 1, \cdots, N \), \( g_i \in \mathbb{R} \) and \( \vartheta_i \) is the periodic coordinate of \( S^1 \) of period \( 2\pi \), and \( \Sigma_{a=1,\cdots,k} \in \mathbb{C} \). The field space is \((\mathbb{C}^*)^N \times \mathbb{C}^k\).

- **Complex moduli parameters**: \( t^a = r^a - i\theta^a \) same as complexified FI parameters.

- **Superpotential**: a holomorphic function \( W = \Sigma_a (Q_a Y_i - t^a(\mu)) + \mu \sum_i \exp(-Y_i) \), \( \mu \) is the worldsheet physical scale.

One can easily observe that our abelian mirrors are Landau-Ginzburg models (LG). The Kähler potential of a non-compact Calabi-Yau target\(^2\) is an irrelevant term under the RG-flow in the LG theory \([62]\), so we do not specify it and take the flat Kähler metric in the computation.

To derive the mirror construction above, Hori and Vafa considered the mirror of abelian Higgs model first. Then one can obtain the mirror construction of a general toric variety in the weak coupling limit of the extra gauge interactions of a theory with a larger gauge symmetry where \( U(1)^{N-1} \) are gauged. They referred to this procedure as localization; see \([57]\) section 3.2.1] for more details. The mirror of the abelian Higgs model is the target space \( \mathbb{C}^* \times \mathbb{C} \) with the superpotential \( W = \Sigma (Y - t(\mu)) + \mu \exp(-Y) \). One can derive the quadratic term in the superpotential directly from the worldsheet T-duality. To reproduce all contributions from the vortex equation of the abelian Higgs model in the dual variable, we should include \( \mu \exp(-Y) \) in the superpotential \([63]\). Another way to argue the existence of an LG mirror is based on the confinement feature of the abelian Higgs model. The summing of vortex configurations indicates that the abelian Higgs model is confined even in the Higgs phase. Thus, one can construct gauge neutral fields out of the original gauge theory in the Higgs phase, whose formulations agree with those in \([64]\) by applying the T-duality. A similar idea was extensively discussed in 3d mirror symmetry \([65]\).

For GLSMs for target spaces with superpotentials, Hori and Vafa got mirrors indirectly. However, inspired by many exact results obtained in the last decade, we can now directly define mirrors of GLSMs with superpotentials

- **Chiral matter fields**: \( X_i = \exp\left(-\frac{q_i}{2}Y_i\right) \), where \( \sum_{i=1}^N q_i = 2 \), and \( \Sigma_{a=1,\cdots,k} \in \mathbb{C} \). The field space is \( \prod_{i=1}^N \frac{\mathbb{C}}{z_{2/q_i}} \times \mathbb{C}^k \).

- **Complex moduli parameters**: \( t^a = r^a - i\theta^a \).

\(^2\)Do not be confused with the non-compact Calabi-Yau with a nontrivial D-term. A superpotential on that target space is a hybrid model.
• **Superpotential**: a function $W = \sum_a \left( -\frac{2}{n_i} Q_i^a \log (X_i) - t^a (\mu) \right) + \mu \sum_i X_i^\frac{2}{n_i}$, $\mu$ is the worldsheet physical scale.

R-charges of matter fields in gauge theories depends on the phase, although they can mix with gauge charges in the UV. Unlike the mirror construction for GLSM without a superpotential, poles in the superpotential indicate that it is an effective theory, which means the perturbative canonical quantization does not apply to this case. However, one can still define the path integral for an effective theory, and it flows to an orbifold-SCFT as expected. At the large complex-structure limit, one advantage in the UV is that one can define A-branes straightforwardly without worrying about the orbifold structure in the IR. We put the discussion of A-branes in appendix A.

**Nonabelian mirrors**

For a nonabelian GLSM, the mirror is a Weyl-group-orbifold of a Landau-Ginzburg model. The Landau-Ginzburg model has the following matter fields:

- $N$ chiral superfields $X_i = \exp \left( -\frac{n_i}{2} Y_i \right)$, or $Y_i$ if $q_i = 0$, corresponding to matters with arbitrary representations $R_i$ in gauge theory with vector R-charge $q_i$,
- $\text{rank}(g)$ chiral superfields $\Sigma_a$, corresponding to a choice of Cartan subalgebra of the Lie algebra of $G$,
- $\dim(g) - \text{rank}(g)$ chiral superfields $X_{\mu}$,

with superpotential

$$W = \sum_{a=1}^{\text{rank}(g)} \Sigma_a \left( \sum_m \sum_{\rho \in \Lambda_{\text{asym}}} \rho^a Y_\rho^m - \sum_{\mu=1}^{\dim(g)-\text{rank}(g)} \alpha^a_\mu \ln X_\mu - t^a \right)$$

$$+ \sum_{\Lambda_{\text{asym}}} \exp \left( -Y_\rho^m \right) + \sum_{\mu=1}^{\dim(g)-\text{rank}(g)} X_\mu.$$

The Weyl-group-orbifold maps weights to weights

$$Y_i \mapsto Y_j \quad \sum_a \Sigma_a \rho^a_i \mapsto \sum_a \Sigma_a \rho^a_j,$$

roots to roots

$$X_\mu \mapsto X_\nu \quad \sum_a \Sigma_a \alpha^a_\mu \mapsto \sum_a \Sigma_a \alpha^a_\nu,$$

and parameters $t^a$ to parameters $t^b$. One can check that the superpotential is invariant under the Weyl-group action. If we forget about the superpotential, the field space is $(\mathbb{C}^*)^{\text{rank}(g)N} \times \mathbb{C}^{\dim(g)}/S$, which has Weyl-orbifold fixed loci. Therefore, it can not be a dual of the original GLSM in the UV free theory limit. This is not surprising. As we proposed, the mirror construction is dual to an effective theory of a nonabelian GLSM. Thus, one
should expect that the mirror Landau-Ginzburg model is also an effective theory, which can be seen from superpotential: it has poles. Therefore we cannot take the free field limit in the mirror. This is not a problem to define 2d mirror symmetry, as we only need to match the physics of NLSM on the original target. However, it is still important to have nonabelian T-duality, as it could provide a physical derivation of nonabelian mirrors with fundamental theories. Or if we know details of Kähler potential in the associated Cartan theory under the RG-flow, we can also claim a physical derivation of the nonabelian mirror proposed in [1].

3.3 Yukawa couplings

In this paper, we compute Gromov-Witten invariants of Calabi-Yau manifolds following section 2. From the discussion of that section, one can notice that we need Yukawa-couplings. Therefore, this section gives a brief summary of Yukawa-couplings in GLSMs studied in the literature. For abelian theories, we have several methods to get Yukawa couplings of target spaces. For example, one can get Yukawa-couplings from Picard-Fuchs equations, which was initially studied in [6,8]. Furthermore, one can also obtain the Yukawa-coupling by summing instantons in the GLSM setup [16]. In this paper, we cite a formula for Yukawa-couplings in GLSMs, and this formula applies to nonabelian GLSMs. Following exact results studied in higher-dimensional gauge theories [66,67], the authors of [32] computed A-twisted correlation functions of GLSMs by supersymmetric localization, and the formula of A-twisted correlation functions in GLSMs is

$$\langle O(x) \rangle_{g=0} = \frac{1}{|S|} \sum_{k} (q^a)^{k^a} \oint_{JK(\nu \nu)} \left( \prod_{a=1}^{\text{rank}(G)} \frac{d\sigma_a}{2i\pi} \right) Z_{1-\text{loop}}^{1-\text{loop}}(\sigma_a) O(\sigma).$$  (3.2)

The one-loop determinant factor includes the one-loop determinant of chiral multiplets and vector multiplets

$$Z_{1-\text{loop}} = Z_{\text{matter}}^{1-\text{loop}} \cdot Z_{\text{gauge}}^{1-\text{loop}},$$

where

$$Z_{\text{matter}}^{1-\text{loop}} = \prod_{\rho \in \mathbb{R}} \left[ \frac{1}{(\rho(\sigma))^{\rho(q)} - 1} \right], \quad Z_{\text{gauge}}^{1-\text{loop}} = (-1)^{\sum_{\alpha > 0} \alpha(k)} \prod_{\alpha \in G} \alpha(\sigma).$$

Because we have nonabelian Landau-Ginzburg mirrors, which are summarized in section 3.2, thus the same correlators can be computed in mirror B-twisted Landau-Ginzburg models as well. In particular, in the B-twisted LG theory, correlation functions can be written in the form of residues [1][68]

$$\langle O(x) \rangle_{g=0} = \frac{1}{|S|} \oint \prod_{a} \partial_{\sigma_a} W \cdot \prod_{\mu} \partial_{x_{\tilde{\mu}}} W \cdot \prod_{a} \partial_{x_{a}} W \cdot \prod_{p} \partial_{x_{p}} W O(x).$$  (3.3)

3In section 4, we will define Picard-Fuchs equations of nonabelian mirrors, and one should be able to derive Yukawa-couplings from those equations. However, we will not perform such derivation in this paper.
Calabi-Yau complete intersections in Grassmannians. Hori and Tong studied GLSMs for Calabi-Yau complete intersections in Grassmannians [18]. It is a \( U(k) \) gauge theory with \( N \) fundamental fields \( \Phi_i \) and \( m \) chiral superfields \( P^\beta \) transforming in the \( \text{det}^{-Q_\beta} \) representation of the gauge group, with \( Q_\beta > 0 \). The Calabi-Yau condition is \( N = \sum_\beta Q_\beta \). To have a compact Calabi-Yau, we need the superpotential \( W = \sum_{\beta=1}^{m} P^\beta G^\beta(B) \), where \( G^\beta \) are smooth degree \( Q_\beta \) polynomials in terms of baryons \( B^i_1\ldots i_k = \varepsilon_{a_1\ldots a_k} \Phi_i^\alpha_1 \cdots \Phi_i^\alpha_k \).

In the geometric phase, we assign vector R-charge 2 to \( P^\beta \) fields and 0 to \( \Phi_i \). One can then write down the mirror Landau-Ginzburg model from the GLSM data. From equation (3.2) or (3.3), one can derive a clean formula for Yukawa-couplings of Calabi-Yau complete intersections in Grassmannians

\[
\langle \mathcal{O} \rangle = \left( \prod_\beta Q_\beta \right) \text{Res} \frac{\left[ \prod_{\mu<\nu} (z_\mu - z_\nu)^2 \right] \cdot \left( \sum_{a=1}^{k} z_a \right)^m \cdot \mathcal{O}(z)}{\left[ \prod_{a=1}^{k-1} (z_a^N - 1) \right] \left[ 1 + (-1)^{k-N} q \prod_{\beta=1}^{n} Q_\beta^N \left( \sum_{a=1}^{k} z_a \right)^N \right]}, \tag{3.4}
\]

where \( z_a = \sigma_a / \sigma_k \).

Now we list Yukawa-couplings of some CY3 complete intersections in Grassmannians [32] obtained from formula (3.4) by inserting physical observables \( \mathcal{O}(\sigma) \), which are holomorphic functions of gauge-invariant variables \( u_j(\sigma) \equiv \text{Tr}(\sigma^j) \).

- \( X_4 \subset G(2,4) \): the correlators are

\[
\langle u_1(\sigma)^3 \rangle = \frac{8}{1 - 2^{10} q}, \quad \langle u_2(\sigma)u_1(\sigma) \rangle = 0. \tag{3.5}
\]

The singular locus is \( q = 2^{-10} \), which can also be obtained from the twisted effective superpotential [18].

- \( X_{2^{2},1} \subset G(2,5) \): the Yukawa-couplings are

\[
\begin{align*}
\langle u_1(\sigma)^3 \rangle &= \frac{20}{1 + 11(2^4 q) - (2^4 q)^2} \\
\langle u_1(\sigma)u_2(\sigma) \rangle &= \frac{4(1 - 32 q)}{1 + 11(2^4 q) - (2^4 q)^2}
\end{align*} \tag{3.6}
\]

- \( X_{3,12} \subset G(2,5) \): the Yukawa-couplings are

\[
\begin{align*}
\langle u_1(\sigma)^3 \rangle &= \frac{15}{1 + 11(3^3 q) - (3^3 q)^2} \\
\langle u_1(\sigma)u_2(\sigma) \rangle &= \frac{3(1 - 54 q)}{1 + 11(3^3 q) - (3^3 q)^2}
\end{align*} \tag{3.7}
\]
• $X_{2,14} \subset G(2,6)$: the Yukawa-couplings are

$$\langle u_1(\sigma)^3 \rangle = \frac{28}{(1 + 2^2 q)(1 - 27(2^2 q))}$$

$$\langle u_1(\sigma)u_2(\sigma) \rangle = \frac{8(1 + 18q)}{(1 + 2^2 q)(1 - 27(2^2 q))}$$

(3.8)

• $X_{16} \subset G(3,6)$: the Yukawa-couplings are

$$\langle u_1(\sigma)^3 \rangle = \frac{42}{(1 - q)(1 - 64q)}$$

$$\langle u_1(\sigma)u_2(\sigma) \rangle = 0$$

$$\langle u_3(\sigma) \rangle = \frac{-6(1 - 8q)}{(1 - q)(1 - 64q)}$$

(3.9)

• $X_{17} \subset G(2,7)$: the Yukawa-couplings are

$$\langle u_1(\sigma)^3 \rangle = \frac{14(3 + q)}{1 + 57q - 289q^2 - q^3}$$

$$\langle u_1(\sigma)u_2(\sigma) \rangle = \frac{14(1 - 9q)}{1 + 57q - 289q^2 - q^3}$$

(3.10)

The poles in these Yukawa-couplings are conifold points in Kähler moduli spaces. It would be interesting to understand the vanishing points of correlation functions to testing arguments regarding the global properties of SCFT moduli spaces discussed in [29].

4 Picard-Fuchs equations for periods

We have reviewed nonabelian mirror constructions in the previous section. To find Picard-Fuchs equations for periods of mirrors, in this section, we study the open string theory with A-boundary in the Landau-Ginzburg model [47], and these boundary configurations are called A-branes or Lagrangian submanifolds in the context of LG models denoted by $\gamma_i$. If we consider B-twisted bulk with A-boundary, we can then define the pairing between B-model observables $O_a(X)$ and A-boundary

$$\Pi_{a,i} = \langle \gamma_i \mid O_a(X) \rangle.$$  

(4.1)

It has been shown that these periods satisfy the flatness equation in the context of $tt^*$-equations:

$$\nabla_a \Pi_b = (D_a + C_a) \Pi_b = 0 \quad \bar{\nabla}_a \bar{\Pi}_b = (\bar{D}_a + \bar{C}_a) \bar{\Pi}_b = 0,$$
where $C_a$ is the multiplication by the chiral ring-observable corresponding to $O_a(X)$ in the rings and $D_a$ was defined in [26] as a covariant derivative. The B-twisted bulk suggests the path-integral in defining periods depends solely on the constant map from worldsheet with the boundary to the LG model so that the periods can be defined as an ordinary integral

$$\Pi_{a,i} = \int_{\gamma_i} O_a(X) \exp(-W(X)) \, d\Omega, \quad (4.2)$$

where $d\Omega$ is the holomorphic top-form on the non-compact Calabi-Yau field space of the Landau-Ginzburg theory, when considering a trivial identity observable $O_a(X) = 1$ in the pairing, it defines the Landau-Ginzburg analog of the BPS mass for the A-brane $\gamma_i$.

$$\Pi_{1,i} = \int_{\gamma_i} \exp(-W(X)) \, d\Omega. \quad (4.3)$$

If we choose topological coordinates, we have

$$\partial_a \Pi_{1,i} = \Pi_{a,i} = \int_{\gamma_i} O_a(X) \exp(-W(X)) \, d\Omega, \quad (4.4)$$

$$\partial_a \partial_b \Pi_{1,i} = \partial_a \Pi_{b,i} = C_{abc} \Pi_{c,i}. \quad (4.5)$$

One can find periods that carry the full information about the chiral ring structure constants. Before we define the periods, we shall define A-branes explicitly. However, to get Picard-Fuchs equations for periods of mirrors, we do not need the concrete formulation of A-branes. Thus we put the definition and discussion of A-branes in Appendix A.

### 4.1 Picard-Fuchs equations of mirrors of toric varieties

We first consider Picard-Fuchs equations for periods of mirrors of toric varieties. Recall the data of GLSMs for the toric varieties is $k$ gauge fields $\Sigma_a$, and $N$ matter fields $\Phi_i$ with the charge lattice $Q^a_i$. The mirror is a Landau-Ginzburg (LG) theory: the non-compact Calabi-Yau $(\mathbb{C}^*)^N \times \mathbb{C}^k$ with a superpotential $W$:

$$W = \Sigma_a (Q^a_i Y_i - t^a) + \sum_i e^{-Y_i}. \quad (4.1)$$

Following equ’n (4.3), and we have the period

$$\Pi_{1,i} = \int_{\gamma_i} \prod_a d\Sigma_a \prod_i dY_i e^{-\Sigma_a (Q^a_i Y_i - t^a) - \sum_i e^{-Y_i}}. \quad (4.2)$$

Integrating out the $\Sigma$ fields, we have

$$\Pi_{1,i} = \int \prod_i dY_i \prod_a \delta \left( \sum_a Q^a_i Y_i - t^a \right) \exp \left( - \sum_i e^{-Y_i} \right),$$

where $\delta$ is the Kronecker delta.
where we have suppressed the information of A-branes under the RG-flow, we left this discussion in Appendix A. Then, following the trick in [37], we consider instead

\[ \Pi_{1,i}(\mu_i, t^a) = \int \prod_{i,a} dY_i \prod_a \delta \left( \sum Q^a_i Y_i - t^a \right) \exp \left( -\sum_i \mu_i e^{-Y_i} \right). \]

One can show that

\[ \left[ \prod_{Q^a_i > 0} \left( \frac{\partial}{\partial \mu_i} \right)^{Q^a_i} \right] \Pi(\mu_j, t^a) = e^{-t^a} \left[ \prod_{Q^a_i < 0} \left( \frac{\partial}{\partial \mu_i} \right)^{-Q^a_i} \right] \Pi(\mu_j, t^a). \quad (4.6) \]

To reproduce the original period we started with, one can shift \( Y_i \) by \( \log \mu_i \) to get rid of the \( \mu_i \) dependence above, except for a shift in the delta function constraint. Then we have

\[ \Pi(\mu_i, t^a) = \Pi_{0i} \left( 1, t^a - \log \prod_i \mu_i Q^a_i \right), \]

thus equ’n (4.6) can be re-written as differential equations in terms of Kähler moduli parameters \( t^a \)

\[ \left[ \prod_{Q^a_i > 0} \prod_{k^a = 0}^{Q^a_i - 1} (Q^a_i \theta_a + k^a) \right] \Pi(t^a) = e^{-t^a} \left[ \prod_{Q^a_j < 0} \prod_{l^a = 0}^{-Q^a_j - 1} (-Q^a_j \theta_a + l^a) \right] \Pi(t^a). \quad (4.7) \]

We have set \( \mu_j = 1 \) in the above equation. These are Picard-Fuchs equation for periods of mirrors of abelian varieties.

Examples:

\[ \mathbb{CP}^4 \]

The GLSM for \( \mathbb{CP}^4 \) has one twisted chiral superfield \( \Sigma \) and five chiral superfields with gauge charges \( (1, 1, 1, 1, 1) \) and vanishing vector R-charges. Following equ’n (4.6), we have

\[ \prod_{i=1}^5 \frac{\partial}{\partial \mu_i} \Pi = e^{-t} \Pi. \]

Now define \( \theta = -d/dt \), and from the constraint imposed by the delta function in the period, one can find that the period depends on \( \mu_i \) through the combination \( t - \log (\prod_{i=1}^5 \mu_i) \), we then have the Picard-Fuchs equation

\[ \theta^5 \Pi = e^{-t} \Pi. \]

---

4One may understand this trick deeply by considering the T-duality of GLSMs with the omega deformation [69, 70].
The GLSM for $\mathcal{O}(-5)$ over $\mathbb{CP}^4$ has one gauge field strength super-multiplet $\Sigma$ and six chiral superfields with gauge charges $(1, 1, 1, 1, 1, -5)$ and vanishing vector R-charges, then we have

$$\prod_{i=1}^{5} \frac{\partial}{\partial \mu_i} \Pi = e^{-t} \frac{\partial^5}{\partial \mu_6^5} \Pi.$$  

Now define $\theta = -d/dt$, and from the constraint imposed by the delta function in the period, one can find that the period depends on $\mu_i$ through the combination $t - \log (\prod_{i=1}^{5} \mu_i/\mu_6^5)$, we then have the Picard-Fuchs equation

$$\theta^5 \Pi = e^{-t} (5\theta + 4) (5\theta + 3) (5\theta + 2) (5\theta + 1) (5\theta) \Pi.$$  

The GLSM for $\mathcal{O}(-4, -4)$ over $\mathbb{CP}^3 \times \mathbb{CP}^3$ has two twisted chiral superfields $\Sigma_1$ and $\Sigma_2$, and nine chiral superfields with charge matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & -4 \\ 1 & 1 & 1 & 1 & -4 \end{pmatrix},$$

then we have

$$\prod_{i=1}^{4} \frac{\partial}{\partial \mu_{1i}} \Pi = e^{-t^1} \frac{\partial^4}{\partial \mu_{61}^4} \Pi, \quad \prod_{i=1}^{4} \frac{\partial}{\partial \mu_{2i}} \Pi = e^{-t^2} \frac{\partial^4}{\partial \mu_{62}^4} \Pi.$$

Now define $\theta_1 = -d/dt^1$, and $\theta_2 = -d/dt^2$, one can find the period depends on $\mu_{ai}$ through the combination $t^1 - \log (\prod_{i=1}^{4} \mu_{ai}/\mu_{61}^4)$ and $t^2 - \log (\prod_{i=1}^{4} \mu_{2i}/\mu_{62}^4)$, we then have the Picard-Fuchs equations

$$\theta_1^4 \Pi = e^{-t^1} (4(\theta_1 + \theta_2) + 3)(4(\theta_1 + \theta_2) + 2)(4(\theta_1 + \theta_2) + 1)4(\theta_1 + \theta_2)\Pi,$$

$$\theta_2^4 \Pi = e^{-t^2} (4(\theta_1 + \theta_2) + 3)(4(\theta_1 + \theta_2) + 2)(4(\theta_1 + \theta_2) + 1)4(\theta_1 + \theta_2)\Pi.$$  

### 4.2 Picard-Fuchs equations of abelian mirrors of surfaces

As reviewed in section 3.2, the mirrors of GLSMs for surfaces need one more ingredient: the fundamental variable in mirrors is $X_i = \exp (-\frac{1}{2}Y_i)$ for a non-vanishing vector R-charge $q_i$ of the original matter field $\phi_i$. The GLSM for a surface is defined by the following data: $k$ twisted chiral superfields $\Sigma_{a=1,\ldots,k}$, $M$ chiral superfields $\phi_i$ with the positive charge sub-matrix $Q_{i=1,\ldots,M}^a$, and $N - M$ matter superfields $P_j$ with the negative charge sub-matrix...
and a superpotential $W = P_j G_j$. We study the mirror in the geometric phase, and then we shall assign vector R-charge 2 to $P_i$ fields and vanishing R-charge to $\phi_i$. The mirror of a GLSM for a surface is an LG model defined on non-compact Calabi-Yau $(\mathbb{C}^*)^M \times \mathbb{C}^{N-M+k}$ with a superpotential

$$W = \Sigma_a \left( Q_i^a Y_i - \tilde{Q}_j^a \log X_j - t^a \right) + \sum_{i=1}^M \exp (-Y_i) + \sum_{j=N-M+1}^{N-M} X_j. \quad (4.8)$$

The period is defined as

$$\Pi = \int \prod_a d\Sigma_a \prod_i dY_i \prod_j dX_j e^{-\Sigma_a \left( Q_i^a Y_i - \tilde{Q}_j^a \log X_j - t^a \right) - \sum_{i=1}^M \exp (-Y_i) - \sum_{j=N-M+1}^{N-M} X_j}. \quad (4.9)$$

Integrating out $\Sigma_a$ fields, we have

$$\tilde{\Pi} = \int \prod_i dY_i \prod_j dX_j \prod_a \delta \left( \sum_a Q_i^a Y_i - \tilde{Q}_j^a \log X_j - t^a \right) \exp \left( - \sum_{i=1}^M \exp (-Y_i) - \sum_{j=N-M+1}^{N-M} X_j \right). \quad (4.10)$$

then consider instead

$$\tilde{\Pi}(\mu_i, t^a) = \int \prod_{i=1}^M dY_i \prod_{j=N-M+1}^N (\mu_j dX_j) \prod_a \delta \left( \sum_a Q_i^a Y_i - \tilde{Q}_j^a \log X_j - t^a \right) \exp \left( - \sum_{i=1}^M \mu_i e^{-Y_i} - \sum_{j=N-M+1}^N \mu_j X_j \right). \quad (4.11)$$

One can observe that

$$\tilde{\Pi}(\mu_i, t^a) = \prod_{j=N-M+1}^N \left( \mu_j \frac{\partial}{\partial \mu_j} \right) \Pi(\mu_i, t^a) = \prod_{j=N-M+1}^N \left( \tilde{Q}_j^a \theta_a \right) \Pi(\mu_i, t^a). \quad (4.12)$$

The period $\Pi(\mu_i, t^a)$ is the same as we defined before for mirrors of abelian varieties and obeys equ’n \([4.6]\). Thus, the period $\tilde{\Pi}(\mu_i, t^a)$ obeys the following equation

$$\begin{bmatrix} \prod_{i=1}^{N-M} \left( \frac{\partial}{\partial \mu_i} \right) Q_{i\alpha} \end{bmatrix} \prod_{j=N-M+1}^N \left( \tilde{Q}_j^a \theta_a \right)^{-1} \tilde{\Pi}(\mu_j, t^a) = \prod_{j=N-M+1}^N \left( \tilde{Q}_j^a \theta_a \right)^{-1} \tilde{\Pi}(\mu_j, t^a). \quad (4.13)$$

\[5\text{Other phases are hybrid models in general, but they can be closed to geometric phases, see [18, 20].}\]
Then, we have Picard-Fuchs equations for periods of mirrors of surfaces
\[
\left[ \prod_{i=1}^{N-M} \prod_{k^a=0}^{Q^a_i-1} (Q^a_i \theta_a + k^a) \right] \prod_{j=N-M+1}^{N} \left( \tilde{Q}^a_j \theta_a \right)^{-1} \tilde{\Pi}(t^a) = e^{-t^a} \left[ \prod_{j=N-M+1}^{N} \left( -\tilde{Q}^a_j \theta_a + l^a \right) \right] \tilde{\Pi}(t^a).
\]

The appearance of the negative power of differential operators in the above equation makes it look odd. But we will see examples in our paper that these negative powers will be canceled out in the final expression. However, it may not be correct for generic multiple dimensional Kähler parameter spaces, and this is related to the factorization problem in GKZ-equations \[46\]. Another way to represent the differential equation is the following
\[
\left[ \prod_{i=1}^{N-M} \prod_{k^a=0}^{Q^a_i-1} (Q^a_i \theta_a + k^a) \right] \cdot \tilde{\Pi}(t^a) = \prod_{j=N-M+1}^{N} \left( \tilde{Q}^a_j \theta_a \right)^{-1} \prod_{i=1}^{N-M} \prod_{k^a=0}^{Q^a_i-1} (Q^a_i \theta_a + k^a) \cdot \Pi(t^a)
\]

where the index in \((\tilde{Q}^a_j e^{-t^a})\) is not summed. The precise mathematical terminology for equ’n \((4.15)\) in the math literature is named GKZ-equations \[71, 72\]. If GKZ-equations can be factorized into a differential operators times Picard-Fuchs equations, then one can read off Picard-Fuchs equations from the GKZ-equations. The degree of a GKZ-equation is higher than the degree of the Picard-Fuchs equation, but one may expect that the extra-solutions of the GKZ-equation are spurious solutions of the PF-equation \[73\]. We will come back to this point in a concrete example.

Finally, we should comment that we can only prove the statement in equ’n \((4.12)\) at the large volume limit in this paper. Now, we sketch the idea of the proof. Recall the definition of the period is \(\Pi = \langle \gamma_i \mid \mathcal{O}_a(X) \rangle\), and we know a ground state of the mirror of the GLSM for a surface has the following relation with the ground state of the mirror of the GLSM for the ambient abelian variety:
\[
\mid \mathcal{O}_a(X) \rangle_{\text{surface}} \equiv \prod_j (-X_j) \mid \mathcal{O}_a(X) \rangle_{\text{variety}}.
\]

One can find that A-branes of these two Landau-Ginzburg models are almost the same at the large volume limit following the discussion in Appendix \[A\]. So we have proved the
validity of equ’n (4.12) at the large volume limit. However, we expect equ’n (4.12) to hold on the whole Kähler moduli space, and we leave this conjecture to future work.

Quintic

The GLSM for quintic is a $U(1)$ gauge theory with six matter fields $\phi_{i=1,\ldots,5}$ and a $P$-field with charge matrix $(1, 1, 1, 1, 1, -5)$, and it also has a superpotential $W = PG$. The $P$-field has vector R-charge 2, and other fields have vanishing vector R-charge, so $\phi_{i=1,\ldots,5}$ can have nontrivial expectation values corresponding to the geometric configuration. The period from the mirror is

$$\tilde{\Pi} = \int \prod_{i=1}^{5} dY_i dX_p \delta \left( \sum_{i}^{5} Y_i + 5 \log X_p - t \right) \exp \left( - \sum_{i=1}^{5} e^{-Y_i} - X_p \right)$$

One can find $\tilde{\Pi} = 5\theta \Pi$, where $\Pi$ is the period of the mirror of Tot $O_{\mathbb{P}^4} (-5)$. From equ’n (4.14), it is easy to see the fundamental period of mirror quintic obeys the Picard-Fuchs equation

$$\theta^4 \tilde{\Pi} = 5q (5\theta + 4) (5\theta + 3) (5\theta + 2) (5\theta + 1) \tilde{\Pi}.$$  \hspace{1cm} (4.16)

A different representation can be obtained from equ’n (4.15), which reads

$$\theta^5 \tilde{\Pi} = q (5\theta + 5) (5\theta + 4) (5\theta + 3) (5\theta + 2) (5\theta + 1) \tilde{\Pi}.$$  \hspace{1cm} (4.17)

One can easily notice that if we act the operator $\theta$ on equ’n (4.16) will give us equ’n (4.17). One needs to use $\theta \cdot (q \cdots) = (q + \theta) \cdot (q \cdots)$ in the computation.

We can naturally write down an $H^* (\mathbb{CP}^4)$-valued generating function or I-function in the math literature [10, 73], which is

$$\Pi (\Sigma, t) = \Pi_0 (t) + \Sigma \cdot \Pi_1 (t) + \Sigma^2 \cdot \Pi_2 (t) + \Sigma^3 \cdot \Pi_3 (t) + \Sigma^4 \cdot \Pi_4 (t),$$  \hspace{1cm} (4.18)

where $\Pi_0, \ldots, \Pi_4$ are solutions of equ’n (4.17). However, the Picard-Fuchs equation (4.16) only has four solutions. This discrepancy can be resolved by introducing a different generating function, which is:

$$\tilde{\Pi} (\Sigma, t) = 5\Sigma \cdot \Pi (\Sigma, t) = 5\Sigma \cdot \Pi_0 (t) + 5\Sigma^2 \cdot \Pi_1 (t) + 5\Sigma^3 \cdot \Pi_2 (t) + 5\Sigma^4 \cdot \Pi_3 (t)$$

since $\Sigma^5 = 0$. So $\Pi_4 (t)$ is a spurious solution, and only $\Pi_0 (t), \ldots, \Pi_3 (t)$ are solutions of the Picard-Fuchs equation (4.16). More details can be found in [73, section 5.5.2 and 6.3.4]. We will provide a physical explanation in section 4.4, and we will choose the generating function of mirror quintic to be

$$\Pi_{\text{quintic}} (\Sigma, t) = \frac{\tilde{\Pi} (\Sigma, t)}{5\Sigma} = \Pi_0 (t) + \Sigma \cdot \Pi_1 (t) + \Sigma^2 \cdot \Pi_2 (t) + \Sigma^3 \cdot \Pi_3 (t).$$

We have used the relation $\Sigma^4 = 0$ for quintic.
4.3 Picard-Fuchs equations of nonabelian mirrors

Now, we can define Picard-Fuchs equations for periods of nonabelian mirrors. We focus on PF-equations of mirrors of nonabelian GIT-quotient spaces first and then consider mirrors of surfaces in those target spaces.

4.3.1 Nonabelian GIT quotient spaces

The mirror Landau-Ginzburg is defined as before. It is the field space \( ((\mathbb{C}^*)^n \times \mathbb{C}^{r+k}) / S \) with superpotential

\[
W = \sum_{a=1}^{k} \Sigma_a \left( \sum_{i=1}^{n} \rho_i^a Y_i - \sum_{\mu=1}^{r} \alpha^a_\mu \ln X_\mu - t^a \right) + \sum_i \exp (-Y_i) + \sum_{\tilde{\mu}=1}^{\tilde{r}} \mu_{\tilde{\mu}}.
\]

The physical FI-parameters correspond to \( U(1) \)-sectors in \( G \). However, we formally turn on as many Kähler parameters as the rank of \( G \) to define differential equations. We then obtain generating functions from these differential equations and eventually reduce the parameters to the physical Kähler moduli space. Notice that it is a Weyl-orbifold theory, so every step in the calculation shall be gauge invariant.

The period is

\[
\Pi = \int \prod_{a=1}^{k} d\Sigma_a \prod_{i=1}^{n} dY_i \prod_{\tilde{\mu}=1}^{r} dX_{\tilde{\mu}} \exp \left( - \sum_{a=1}^{k} \Sigma_a \left( \sum_{i=1}^{n} \rho_i^a Y_i - \sum_{\mu=1}^{r} \alpha^a_\mu \ln X_\mu - t^a \right) - \sum_i \exp (-Y_i) - \sum_{\tilde{\mu}=1}^{\tilde{r}} \mu_{\tilde{\mu}} \right).
\]

Then use the same trick as before

\[
\Pi(\mu_i, \tilde{\mu}, t^a) = \int \prod_{a=1}^{k} d\Sigma_a \prod_{i=1}^{n} dY_i \prod_{\tilde{\mu}=1}^{r} (\mu_{\tilde{\mu}} dX_{\tilde{\mu}}) \exp \left( - \sum_{a=1}^{k} \Sigma_a \left( \sum_{i=1}^{n} \rho_i^a Y_i - \sum_{\mu=1}^{r} \alpha^a_\mu \ln X_\mu - t^a \right) - \sum_i \mu_i \exp (-Y_i) - \sum_{\tilde{\mu}=1}^{\tilde{r}} \mu_{\tilde{\mu}} X_{\tilde{\mu}} \right).
\]

One then easily observe that

\[
\prod_i \left( \frac{\partial}{\partial \mu_i} \right) \prod_{\alpha^a_\mu > 0} \left( \mu_{\tilde{\mu}} \frac{\partial}{\partial \mu_{\tilde{\mu}}} \right) \Pi(\mu, t^a) = (-1)^{\sum_{\alpha^a_\mu > 0}} \sum_{\alpha^a_\mu > 0} e^{-t^a} \prod_{\alpha^a_\mu > 0} \left( \mu_{\tilde{\mu}} \frac{\partial}{\partial \mu_{\tilde{\mu}}} \right) \prod \Pi(\mu, t^a)
\]

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We have used the fact that the root is always paired with a charge-conjugated root called anti-root. One can then define the period

$$
\Pi(t^a) = \prod_{\alpha_{\tilde{\mu}} > 0} \left( \alpha_{\tilde{\mu}} \theta_a \right) \Pi^{Ab} \left( t^a - i\pi \sum_{\alpha_{\tilde{\mu}} > 0} \alpha_{\tilde{\mu}} \right),
$$

where \(\Pi^{Ab}\) obeys the following Picard-Fuchs equations

$$
\prod_{i} \rho_{a_i}^{-1} \Pi^{Ab} \left( t^a - i\pi \sum_{\alpha_{\tilde{\mu}} > 0} \alpha_{\tilde{\mu}} \right) = (-1)^{\sum_{\alpha_{\tilde{\mu}} > 0} \alpha_{\tilde{\mu}}} e^{-i\pi} \Pi^{Ab} \left( t^a - i\pi \sum_{\alpha_{\tilde{\mu}} > 0} \alpha_{\tilde{\mu}} \right). 
$$

The above equation implies that \(\Pi^{Ab}\) is the period of a mirror of an abelian variety. This claim is valid, at least at the large volume limit, where we define the Gromov-Witten theory. At the large volume limit, the configuration of a Lagrangian submanifold is approximately

$$
\gamma = L \times \mathbb{R}^n \times (\mathbb{R}_{>0})^r.
$$

We have omitted the known information of \(\theta\)-angles as their detailed information is not crucial in our computations. See appendix A for a nonperturbative definition of A-branes of the mirror LG of the GLSM for Grassmannian.

Now we only focus on the \(X_{\tilde{\mu}}\) integral contribution in formula (4.22) that is

$$
\prod_{\alpha_{\tilde{\mu}} > 0} \int_0^\infty dX_{\tilde{\mu}} e^{-\alpha_{\tilde{\mu}} \Sigma_a \ln X_{\tilde{\mu}} - X_{\tilde{\mu}}} \prod_{\alpha_{\tilde{\nu}} < 0} \int_0^\infty dX_{\tilde{\nu}} e^{\alpha_{\tilde{\nu}} \Sigma_a \ln X_{\tilde{\nu}} - X_{\tilde{\nu}}},
$$

where we have used the fact that the positive root \(\tilde{\mu}\) is always paired with the negative root \(-\tilde{\mu}\). Recall the well-known gamma function

$$
\Gamma(z) = \int_0^\infty e^{-t^z} dt, \quad \Re(z) > 0.
$$

Assuming

$$
-1 < \Re \left( \alpha_{\tilde{\mu}} \Sigma_a \right) < 1 \quad \text{for} \quad \alpha_{\tilde{\mu}} > 0,
$$

then the integral equ’n (4.27) reduces to

$$
\prod_{\alpha_{\tilde{\mu}} > 0} \Gamma \left( 1 - \alpha_{\tilde{\mu}} \Sigma_a \right) \Gamma \left( 1 + \alpha_{\tilde{\mu}} \Sigma_a \right).
$$

The above result is a meromorphic function that is holomorphic in field space of \(\Sigma_a\) except the non-positive integers, where the function has simple poles. Thus, we define formula (4.28) as the analytic continuation of the integral (4.27). From the identity

$$
\Gamma \left( 1 - z \right) \Gamma \left( z \right) = \frac{\pi}{\sin \pi z},
$$

20
one can then write (4.28) as

\[
\prod_{\alpha > 0} \frac{\Gamma \left( 1 + \alpha a_{\mu} \sum_a \right)}{\Gamma \left( \alpha a_{\mu} \sum_a \right)} \frac{\pi}{\sin \pi \alpha a_{\mu} \sum_a} = (2i\pi)^{\sum_a} \prod_{\alpha > 0} \left( \alpha a_{\mu} \sum_a \right) e^{i\pi \alpha a_{\mu} \sum_a} e^{2i\pi \alpha a_{\mu} \sum_a} - 1. \tag{4.29}
\]

Now, we focus on the factor

\[
\prod_{\alpha > 0} \left( e^{2i\pi \alpha a_{\mu} \sum_a} - 1 \right)^{-1}
\]

in equ’n (4.29). It can be computed in one renormalization scheme\(^6\). More specifically, if \(| \exp \left( 2i\pi \alpha a_{\mu > 0} \sum_a \right) | < 1\), we have

\[
\prod_{\alpha > 0} \left( e^{2i\pi \alpha a_{\mu} \sum_a} - 1 \right)^{-1} = \prod_{\alpha \neq \alpha_1} \left( e^{2i\pi \alpha a_{\mu} \sum_a} - 1 \right)^{-1} \left( -1 - \sum_{m=1} e^{i2m\pi \alpha a_{\mu} \sum_a} \right),
\]

Notice that we can shift \( t \mapsto t + i \cdot 2m\pi \alpha a_{\mu} \sum_a \) without affecting the physics in Landau-Ginzburg models, so we have

\[
\left( -1 - \sum_{m=1} e^{i2m\pi \alpha a_{\mu} \sum_a} \right) \sim \left( -\sum_{m=0} \right) = \frac{1}{2},
\]

where we have used the Riemann Zeta function regularization such as \( \zeta \left( 0 \right) = 1 + 1 + 1 + \ldots = -\frac{1}{2} \). We will obtain the same result if \(| \exp \left( 2i\pi \alpha a_{\mu > 0} \sum_a \right) | > 1\). Continue this process, we then get

\[
\prod_{\alpha > 0} \left( e^{2i\pi \alpha a_{\mu} \sum_a} - 1 \right)^{-1} \sim \frac{1}{2\pi}.
\]

One can get another non-vanishing value by following a different procedure of renormalization.

So, up to an overall constant, we eventually obtain

\[
\Pi = \int \prod_{a=1}^k d\Sigma_a \prod_{i=1}^n dY_i \prod_{\alpha > 0} \left( \alpha a_{\mu} \sum_a \right) \exp \left( - \sum_{a=1}^k \sum_{i=1}^n \rho_i a_{\mu} Y_i - t^a + i\pi \sum_{\alpha > 0} \alpha a_{\mu} \sum_a \right) - \sum_i \exp \left( -Y_i \right), \tag{4.30}
\]

\(^6\)The period only receives an overall constant difference in a different renormalization scheme, and the physics is still the same.
then we have
\[
\Pi (t^a) = \prod_{\alpha^a_{\mu} > 0} \left( \alpha^a_{\mu} \theta_a \right) \int \prod_{a=1}^k d\Sigma_a \prod_{i=1}^n dY_i \tag{4.31}
\]
\[
\exp \left( - \sum_{a=1}^k \Sigma_a \left( \sum_{i=1}^n \rho^a_i Y_i - t^a + i\pi \sum_{\alpha^a_{\mu} > 0} \alpha^a_{\mu} \right) - \sum_i \exp (-Y_i) \right).
\]

Indeed at the large volume limit, the period is approximate to
\[
\Pi^{Ab}(t^a - i\pi \sum_{\alpha^a_{\mu} > 0} \alpha^a_{\mu}) = \int \prod_{a=1}^k d\Sigma_a \prod_{i=1}^n dY_i \tag{4.32}
\]
\[
\exp \left( - \sum_{a=1}^k \Sigma_a \left( \sum_{i=1}^n \rho^a_i Y_i - t^a + i\pi \sum_{\alpha^a_{\mu} > 0} \alpha^a_{\mu} \right) - \sum_i \exp (-Y_i) \right).
\]

For the case of Grassmannian, equ’n (4.31) agrees with the periods discussed in the appendix of [37].

Grassmannian

The gauged linear sigma model for $Gr(k, N)$ is a $U(k)$ gauge theory with $N$ fundamental chiral superfields. The dual Landau-Ginzburg model is \((\mathbb{C}^*)^{kN} \times \mathbb{C}^{k^2})/S_k\) with superpotential
\[
W = \sum_{a=1}^k \sum_{p=1}^N Y^a_p + \alpha^a_{\mu} \log X_{\mu} - t^a + \sum_{p\alpha} \exp (-Y^a_p) + \sum_{\mu=1}^{k(k-1)} X_{\mu}. \tag{4.33}
\]

The mirror period obeys the following Picard-Fuchs equations
\[
\theta^N \Pi^{Ab} = e^{-t^a} (-1)^{k-1} \Pi^{Ab}, \tag{4.34}
\]

where the period of the mirror of Grassmannian
\[
\Pi = \prod_{1 \leq a < b \leq k} \left( -\theta_a + \theta_b \right) \big|_{t^a = t^b} \Pi^{Ab}. \tag{4.35}
\]

At the large volume limit, $\Pi^{Ab}$ is the period of the mirror of the GLSM for the abelian variety: $\mathbb{P}^{N-1} \times \cdots \times \mathbb{P}^{N-1}$.

4.3.2 Nonabelian Calabi-Yau manifolds

In this paper, we mainly focus on the computation of Gromov-Witten invariants of Calabi-Yau manifolds. However, one can define Picard-Fuchs equations for other surfaces as well.
The discussion is pretty similar to the study of abelian cases.

The mirror is an LG-model, which is the target space \((\mathbb{C}^\star)^n \times \mathbb{C}^m \times \mathbb{C}^{r+k}/S\) with superpotential

\[
W = \sum_{a=1}^{k} \Sigma_a \left( \sum_{i=1}^{n} \rho_i^a Y_i - \sum_{j=1}^{m} \tilde{\rho}_j \log X_j + \sum_{\tilde{\mu}=1}^{r} \alpha_{\tilde{\mu}}^a \log X_{\tilde{\mu}} - t^a \right) + \sum_{i=1}^{n} \exp (-Y_i) + \sum_{j=1}^{m} X_j + \sum_{\tilde{\mu}=1}^{r} X_{\tilde{\mu}}
\]  

(4.36)

The number of physical parameters \(t^a\) is smaller than \(k\), we keep \(k\) parameters in doing the computation and eventually reduce to the right number of physical parameters. One can define the period \(\Pi (t^a)\) as before following section 4.2 and 4.3.1, we then have

\[
\Pi (t^a) = \prod_{\alpha^a_\beta > 0} (\alpha^a_\beta \theta_a) \Pi^{Ab} (t^a),
\]

(4.37)

where \(\Pi^{Ab} (t^a)\) obeys the following Picard-Fuchs equations

\[
\prod_i \prod_{k=0}^{\rho_i^a-1} (\rho_i^a \theta_a + k) (\tilde{\rho}_j \theta_a)^{-1} \Pi^{Ab} (t^a)
\]

(4.38)

\[
= (-1)^{\sum_{\alpha^a_\beta > 0} \alpha^a_\beta} e^{-t^a} \prod_j \prod_{l=0}^{-\tilde{\rho}_j^a-1} (-\tilde{\rho}_j \theta_a + l) (\tilde{\rho}_j \theta_a)^{-1} \Pi^{Ab} (t^a).
\]

So one shall first solve equ’n (4.38) to get \(\Pi^{Ab} (t^a)\), and then obtain the period \(\Pi (t^b)\) of mirrors of Calabi-Yau manifolds by plugging \(\Pi^{Ab} (t^a)\) into equ’n (4.37) and reduce the Kähler parameters \(t^a\) to the actual physical Kähler parameters \(t^b\), where the index \(b\) is from 1 to the dimension of the \(U(1)\) sector of the nonabelian gauge group in the GLSM.

We have defined Picard-Fuchs equations of periods of nonabelian mirrors of general Calabi-Yau manifolds, so let us look at a concrete example.

\(Gr(2, 4)[4]\)

The GLSM for Calabi-Yau in \(Gr(2, 4)\) is a \(U(2)\) gauge theory with four fundamental chiral superfields and one chiral matter field in \(\text{det}^{-4}\) representation of \(U(2)\). The mirror is an LG model: \((\mathbb{C}^\star)^8 \times \mathbb{C}^4 \times \mathbb{C}/S_2\) with the superpotential

\[
W = \Sigma_1 \left( \sum_{i=1}^{4} Y_i^1 + 4 \log X_p + \log X_1 - \log X_2 - t^1 \right)
\]

(4.39)

\[
+ \Sigma_2 \left( \sum_{i=1}^{4} Y_i^2 + 4 \log X_p - \log X_1 + \log X_2 - t^2 \right)
\]

\[
+ \sum_{i=1}^{4} \exp (-Y_i^1) + \sum_{i=1}^{4} \exp (-Y_i^2) + X_p + X_1 + X_2.
\]
The period satisfies the Picard-Fuchs equations

\[
\theta_1^4 \left( \theta_1 + \theta_2 \right)^{-1} \Pi^{Ab} \left( t^1, t^2 \right) = -4q^1 \prod_{k=1}^{3} \left( 4 \left( \theta_1 + \theta_2 \right) + k \right) \Pi^{Ab} \left( t^1, t^2 \right) \quad (4.40)
\]

\[
\theta_2^4 \left( \theta_1 + \theta_2 \right)^{-1} \Pi^{Ab} \left( t^1, t^2 \right) = -4q^2 \prod_{k=1}^{3} \left( 4 \left( \theta_1 + \theta_2 \right) + k \right) \Pi^{Ab} \left( t^1, t^2 \right) \quad (4.41)
\]

One may worry about the \((\theta_1 + \theta_2)^{-1}\) factor in equation (4.40) and (4.41). However, we will see in section 4.5 that this factor can be canceled out after we take \(t_1 = t_2\). Given a solution of the Picard-Fuchs equation (4.40) and (4.41), the period \(\Pi(t)\) of the mirror of \(Gr(2, 4)[4]\) is obtained by

\[
\Pi(t) = \left( \partial t_1 - \partial t_2 \right) \bigg|_{t_1 = t_2 = t} \cdot \Pi^{Ab} \left( t^1, t^2 \right).
\]

### 4.4 Matter fields with vector R-charge 2 and mirrors

In section 3.2, we have mentioned how the matter field’s R-charge affects the mirror’s fundamental variable. We will give a more refined discussion in this section, which will be helpful in our computation of Gromov-Witten invariants in section 5.

**GLSM for the quintic v.s GLSM for \(O(-5)\).** We have seen in section 4.2 that periods in the mirrors of these two GLSMs obey the relation \(\tilde{\Pi} = 5\theta \Pi\). Now, we study more about these two Landau-Ginzburg models.

The mirror construction of GLSM for the quintic is the Landau-Ginzburg model: target space \((\mathbb{C}^*)^5 \times \mathbb{C}^2\) with superpotential

\[
W = \sum \left( \sum_{i=1}^{5} Y_i + 5 \log X_p - t \right) + \sum_{i=1}^{5} e^{-Y_i} + X_p.
\]

The mirror of NLSM on the quintic is the Landau-Ginzburg SCFT: target space \(\left( \mathbb{C}^5 / \mathbb{Z}_5 \right)^5 \) with superpotential

\[
W = X_1^5 + \cdots + X_5^5 + q^{-\frac{1}{2}} X_1 X_2 X_3 X_4 X_5.
\]

This is the low energy effective theory of the mirror Landau-Ginzburg of GLSM for the quintic. One can obtain the ring relations of the untwisted sector by computing

\[
\partial_{X_i} W = 0, \quad i = 1, \cdots, 5,
\]

which gives

\[
5X_i^4 + q^{-\frac{1}{2}} \prod_{j \neq i} X_j = 0, \quad i = 1, \cdots, 5.
\]

So the orbifold invariant Jacobian-rings in the untwisted sector are: 1, \(X_1 \cdots X_5\), \((X_1 \cdots X_5)^2\), \((X_1 \cdots X_5)^3\). These operators are in one to one correspondence with the even cohomology
classes of the quintic, and those are $H_{i,i}^{(\text{quintic})}$ as expected from mirror symmetry. The ground states of this LG model can be represented by

$$(X_1 \cdots X_5)^p \mid \Omega \rangle_c, \quad p = 0, 1, 2, 3. \quad (4.42)$$

The mirror construction of GLSM for $\mathcal{O}(-5)$ is the Landau-Ginzburg model: target space $(\mathbb{C}^*)^6 \times \mathbb{C}$ with superpotential

$$W = \sum_{i=1}^{5} Y_i - 5Y_p - t + \sum_{i=1}^{5} e^{-Y_i} + e^{-Y_p}.$$ 

The mirror of NLSM on $\mathcal{O}(-5)$ is the low energy effective Landau-Ginzburg model: target space $(\mathbb{C}^*)^5$ with superpotential

$$W = X_5^5 + \cdots + X_5^5 + q^{-\frac{1}{5}} X_1 X_2 X_3 X_4 X_5.$$ 

The superpotential is the same as the mirror of the quintic. However, the fundamental variables are $Y_i$ rather than $X_i = e^{-\frac{Y_i}{5}}$ as in the mirror of the quintic. Different choice of fundamental variables will affect generators and relations. More specifically, we compute

$$\partial_{Y_i} W = 0, \quad i = 1, \cdots, 5,$$

then we have

$$5X_i^5 + q^{-\frac{1}{5}} X_1 X_2 X_3 X_4 X_5 = 0, \quad i = 1, \cdots, 5.$$ 

So the generators are $1, X_1 \cdots X_5, (X_1 \cdots X_5)^2, (X_1 \cdots X_5)^3, (X_1 \cdots X_5)^4$. These Jacobian-rings can be mapped to the even cohomology classes of $\mathcal{O}(-5)$, as expected. The ground states of this LG model can be represented by

$$(X_1 \cdots X_5)^p \mid \Omega \rangle_{\text{nc}}, \quad p = 0, 1, 2, 4. \quad (4.43)$$

One can explain the difference of rings of two models on the gauge theory side. Consider the GLSM for the non-compact Calabi-Yau $\mathcal{O}(-5)$. The ground states can be represented by $1, \Sigma, \ldots, \Sigma^4$. These ground states are in one to one correspondence with $\mathcal{O}(-5)$’s cohomology classes by mapping $\Sigma$ to the hyperplane class $H$ of $\mathcal{O}(-5)$. The GLSM for the quintic is a compact theory that can be understood by turning on a perturbative superpotential in GLSM for the non-compact Calabi-Yau $\mathcal{O}(-5)$. Only ground states that correspond to normalized forms in the non-compact theory will flow to the ground states of the compact theory. These states are $\Sigma, \ldots, \Sigma^4$, while $1$ corresponds to the cohomology class dual to the total space of $\mathcal{O}(-5)$, and this is a non-normalized state due to the non-compactness of the space. The analysis of the ground states in the gauge theory suggests the following

$$| \Omega \rangle_{\text{nc}} \mapsto 1, \quad | \Omega \rangle_c \mapsto \Sigma \mid \Omega \rangle_{\text{nc}}. \quad (4.44)$$
One can read off the above relation from the definition of the period as well, and we have already briefly discussed this in section 4.2. We know that A-branes of these two Landau-Ginzburg models are almost the same at the large volume limit, and the relation of the periods for these two models is \( \bar{\Pi} = 5\theta \Pi \) from section 4.2. So we must have

\[
| \Omega_c \rangle = 5\Sigma | \Omega_{nc} \rangle \tag{4.45}
\]

at the large volume limit.

In section 2 we expanded the generating function in terms of the even cohomology basis of the target space, and the first cohomology class is 1. So the generating function we will be using in computing Gromov-Witten invariants of the quintic is

\[
\Pi_{\text{quintic}} (\Sigma; t) = \frac{\theta \Pi_{\mathcal{O}(-5)} (\Sigma; t)}{\Sigma}. \tag{4.46}
\]

Now, as discussed in equ’n (2.31), we can expand \( \Pi_{\text{quintic}} (\Sigma; t) \) as

\[
\Pi_{\text{quintic}} (\Sigma; t) = \tilde{\Pi}_0 (q) + \Sigma \tilde{\Pi}_1 (q) + \Sigma^2 \tilde{\Pi}_2 (q) + \Sigma^3 \tilde{\Pi}_3 (q), \quad \text{mod} \quad \Sigma^4 = 0.
\]

One should notice that we can also choose \( 5\Sigma \cdot \Pi_{\text{quintic}} \) for computing Gromov-Witten invariants. The differences are that the expansion basis starts from \( 5\Sigma \) rather than the identity, and we shall take account of \( \Sigma^5 = 0 \). One can easily extend this proposition to other surfaces as well.

**Roots in mirrors.** In the nonabelian mirror construction, one needs to consider the effect of vector R-charge 2 in the mirror, which means the fundamental variables in the mirror of roots are \( X_{\tilde{\mu}} = \exp (-Z_{\mu}) \). So they look similar to the fundamental variable \( X_p \) for \( p \)-field in the mirror of the GLSM for quintic. However, in the case of the quintic, the vector R-charge 2 of the P-field comes from the superpotential in the GLSM. It has a geometrical meaning in the setup of NLSM on quintic, which has been proved in mathematically [82]. It is called the quantum-Serre duality: the vector R-charge 2 of bundle coordinate \( p \) will induce a degree 5 hypersurface inside \( \mathbb{C}P^4 \). Intuitively, the vector R-charge 2 field corresponds to a normal bundle over quintic in \( \mathbb{C}P^4 \). One may expect that the R-charge 2 chiral superfields from the roots in the nonabelian GLSM correspond to the normal bundle over the nonabelian target in the associated abelian ambient space. We leave a detailed discussion of this point of view to future work.

From the discussion above, we arrive at the following generating function for computing Gromov-Witten invariants:

\[
\tilde{\Pi} (\Sigma_a; t^a) = \frac{\Pi (\Sigma_a; t^a)}{\prod_{\alpha > 0} \alpha_{\mu}^\alpha \Sigma_a^\alpha} = \frac{\Pi_{\alpha > 0} (\alpha_{\mu}^a \theta_a)}{\prod_{\alpha > 0} \alpha_{\mu}^a \Sigma_a^\alpha} \cdot \Pi_{\mathcal{Ab}} \left( \Sigma_a, t^a + i\pi \sum_{\alpha > 0} \alpha^a \right). \tag{4.47}
\]

We shall take the Weyl orbifold invariant part of the above expression at the end of the computation, and then reduce the number of parameters to the actual number of \( U(1) \)-sectors in the nonabelian compact Lie group.
4.4.1 Fano varieties

\( Gr(2,4) \)

Our discussion starts with the simplest nontrivial case \( Gr(2,4) \), and then move on to the general Grassmannians. Following equ’n (4.25), we have the generating function

\[
\Pi_{Gr(2,4)} (\Sigma_1, \Sigma_2; t) = -\partial_{t_1} + \partial_{t_2} \bigg|_{t_1 = t_2 = t} \Pi_{\mathbb{CP}^3 \times \mathbb{CP}^3} (\Sigma_1, \Sigma_2; t^1 + i\pi, t^2 + i\pi)
\]

for the mirror of \( Gr(2,4) \). We denote \( \Pi_{\mathbb{CP}^3 \times \mathbb{CP}^3} \) as the generating function of the mirror of \( \mathbb{CP}^3 \times \mathbb{CP}^3 \) and \( \Pi_{Gr(2,4)} \) as the generating function of the mirror of \( Gr(2,4) \).

The mirror of the GLSM for \( \mathbb{CP}^3 \times \mathbb{CP}^3 \) is the Landau-Ginzburg model \((C^*)^8 \times \mathbb{C}^2\) with the superpotential

\[
W = \Sigma_1 \left( \sum_{i=1}^{4} Y_i^1 - t + i\pi \right) + \Sigma_2 \left( \sum_{i=1}^{4} Y_i^2 - t + i\pi \right) + \sum_{i=1}^{4} \exp (-Y_i^1) + \sum_{i=1}^{4} \exp (-Y_i^1).
\]

The chiral ring relations can be obtained from the vacuum equations

\[
\partial_{Y_i} W = \partial_{Y_i} W = 0, \quad a = 1, 2 \quad \text{and} \quad i = 1, \cdots, 4,
\]

giving rise to

\[
\Sigma_1 = X_i^1 = \exp (-Y_i^1), \quad \Sigma_1^4 = -q^1; \quad \Sigma_2 = X_i^2 = \exp (-Y_i^2), \quad \Sigma_2^4 = -q^2.
\]

A basis of the chiral ring consists of 16 operators:

\[
\Sigma_1^{i_1} \Sigma_2^{i_2}, i_1, i_2 = 0, \cdots, 3.
\]

Following Witten [17], we map \( \Sigma \) to the hyperplane class \( H \) of a projective space. As expected from mirror symmetry, the Jacobian-rings obtained from the Landau-Ginzburg model are in one to one correspondence with the cohomology classes of \( \mathbb{CP}^3 \times \mathbb{CP}^3 \). \( H_1^{1i_1} H_2^{1i_2}, i_1, i_2 = 0, \cdots, 3 \). Now, we expand the generating function in this basis

\[
\Pi_{\mathbb{CP}^3 \times \mathbb{CP}^3} (\Sigma_1, \Sigma_2; t^1 + i\pi, t^2 + i\pi) = \Sigma_1^{i_1} \Sigma_2^{i_2} \Pi_{\mathbb{CP}^3 \times \mathbb{CP}^3, i_1i_2} (t^1 + i\pi, t^2 + i\pi).
\]

It is easy to see that functions symmetric in \( t^1, t^2 \) are annihilated by the operator \((\partial_{t_1} - \partial_{t_2})_{t_1 = t_2 = t}\). There are 10 independent periods symmetric in \( t^1, t^2 \) out of the 16 periods of the mirror of \( \mathbb{CP}^3 \times \mathbb{CP}^3 \), and those are the coefficients in (4.49) of the symmetric polynomials in

\[
\mathbb{C} [\Sigma_1, \Sigma_2] / (\Sigma_1^4, \Sigma_2^4).
\]

The other 6 are anti-symmetric in \( t^1 \) and \( t^2 \), and are coupled to the following anti-symmetric polynomials in \( \Sigma_1 \) and \( \Sigma_2 \):

\[
(\Sigma_1 - \Sigma_2) S_\lambda (\Sigma_1, \Sigma_2),
\]

27
where \( \lambda \) labels Young diagrams in the \( 2 \times 2 \) grid (note that there are 6 of them), and \( S_\lambda \) is the corresponding Schur polynomial. We can list them explicitly

\[
1; \Sigma_1 + \Sigma_2; \Sigma_1^2 + \Sigma_1 \Sigma_2 + \Sigma_2^2, \Sigma_1 \Sigma_2; \Sigma_1 \Sigma_2 (\Sigma_1 + \Sigma_2); \Sigma_1^2 \Sigma_2^2.
\]

Now it is easy to see from (4.48) that

\[
\Pi_{Gr(2,4)} (\Sigma_1, \Sigma_2, t) = \sum_\lambda S_\lambda (\Sigma_1, \Sigma_2) \Pi_\lambda (t),
\]

where \( \Pi_\lambda (t) \) span the image of the operator \((\partial_{t^1} - \partial_{t^2})_{t^1=t^2=t}\), can be written down explicitly

\[
\partial_i \Pi_{\mathbb{C}P^3,j}(t) \Pi_{\mathbb{C}P^3,j}(t) - \Pi_{\mathbb{C}P^3,j}(t) \partial_i \Pi_{\mathbb{C}P^3,j}(t), \quad \text{for} \quad i < j, \quad i, j = 0, \ldots, 3,
\]

where \( \Pi_{\mathbb{C}P^3,j}(t) \) is from the expansion of the generating function of the mirror of \( \mathbb{C}P^3 \),

\[
\Pi_{\mathbb{C}P^3}(\Sigma, t) = \Pi_{\mathbb{C}P^3,0}(t) + \Sigma \Pi_{\mathbb{C}P^3,1}(t) + \Sigma^2 \Pi_{\mathbb{C}P^3,2}(t) + \Sigma^3 \Pi_{\mathbb{C}P^3,3}(t), \quad \text{mod} \quad \Sigma^4.
\]

In section 4.5, we will prove equ’n (4.50) satisfies the one-parameter Picard-Fuchs equation of the mirror of \( Gr(2, 4) \) in the literature.

\( Gr(k, N) \)

The study of the period of the mirror of the general Grassmannian is almost identical to \( Gr(2, 4) \), so we omit some details. Following equ’n (4.47), we have

\[
\Pi_{Gr(k,N)} (\Sigma_1, \ldots, \Sigma_k; t) = \prod_{1 \leq a < b \leq k} \frac{-\partial_{t^a} + \partial_{t^b}}{\Sigma_a - \Sigma_b} |_{t^a=t^b} (\Sigma_1, \ldots, \Sigma_k; t^1 + i(k-1)\pi, \ldots, t^k + i(k-1)\pi).
\]

The rings of the mirror of \( \mathbb{C}P^{N-1} \times \cdots \times \mathbb{C}P^{N-1} \) correspond to the cohomology classes of \( H^* (\mathbb{C}P^{N-1} \times \cdots \times \mathbb{C}P^{N-1}) \) as expected from mirror symmetry. The dimension of the ring is \( N^k \), and it can be represented as

\[
\mathbb{C} \left[ \Sigma_1, \ldots, \Sigma_k \right] / \langle \Sigma_1^N, \ldots, \Sigma_k^N \rangle.
\]

It is not hard to find that only functions \( \Pi (t_{i_a}) \) proportional to \( \varepsilon_{i_1 \cdots i_k} \) are not annihilated by the operator \( \prod_{1 \leq a < b \leq k} (\partial_{t^a} - \partial_{t^b}) \). Then, one can get the conclusion that the ring basis in front of these functions must also be proportional to the Levi-Civita symbol \( \varepsilon_{i_1 \cdots i_k} \). Denote \( (\xi_1, \ldots, \xi_k) \) by \( \xi \) where the entries \( \xi_i \) are nonnegative integers and \( \xi_i > \xi_j \) if \( i < j \). Then ring basis we denote as \( J_\xi \) can be defined as the determinant of the \( k \times k \) matrix with entries \( \Sigma_{\xi_a}^\xi_b \):

\[
J_\xi (\Sigma_1, \ldots, \Sigma_k) = \det (\Sigma_{\xi_a}^\xi_b), \quad 1 \leq a, b \leq k.
\]

The special case with \( \delta = (k-1, k-2, \cdots, 0) \) is the Vandermonde determinant

\[
J_\delta (\Sigma_1, \ldots, \Sigma_k) = \det (\Sigma_{\xi_a}^\xi_b) = \prod_{1 \leq a < b \leq k} (\Sigma_a - \Sigma_b).
\]
We decompose $\zeta = \delta + \lambda$, where $\lambda$ labels Young diagrams in the $k \times (N - k)$ grid, then we have

$$J_\zeta (\Sigma_1, \cdots, \Sigma_k) = J_\delta (\Sigma_1, \cdots, \Sigma_k) S_\lambda (\Sigma_1, \cdots, \Sigma_k).$$

(4.54)

So we find that one can expand the generating function of the mirror of $Gr(k, N)$ in terms of the Schur polynomials in $\Sigma_a$

$$\Pi_{Gr(k,N)} (\Sigma_1, \cdots, \Sigma_k, t) = \sum_{\lambda} S_\lambda (\Sigma_1, \cdots, \Sigma_k) \Pi_\lambda (t),$$

(4.55)

where

$$\Pi_\lambda (t) = \det \left[ \frac{\partial^{b-1} \Pi_{\Sigma} (t^a)}{k!}_{a,b} \right]_{t^a = t}, \quad x_b \in \{0, \cdots, N - 1\}.$$

This is not surprising from mirror symmetry, as we know that the cohomology classes of Grassmannian are Schur polynomials of Chern roots $\Sigma_a$ of the dual tautological bundle. For more general target spaces such as Flag varieties, the cohomology classes are called Schubert classes, it seems that our method can compute them explicitly. We leave the study on Schubert classes of general homogenous spaces as a future work.

**Symplectic Grassmannian.** The GLSM for $SG(k, 2n)$ has been studied in [44, 57], and it is then a $U(k)$ gauge theory with $2n$ chirals $\Phi^a_{\pm i}$ in the fundamental representation $V$ ($a \in \{1, \cdots, k\}, i \in \{1, \cdots, n\}$), and one chiral superfield $q_{ab}$ in the representation $\wedge^2 V^*$, with superpotential

$$W = \sum_{\alpha, \beta} q_{ab} \Phi^a_{\alpha} \Phi^b_{-\beta} \omega^{\alpha \beta} = \sum_{i=1}^n q_{ab} \Phi^a_{\alpha} \Phi^b_{-\alpha}.$$

The superpotential breaks the $U(2n)$-global symmetry to $Sp(2n, \mathbb{C})$. One can find more details about the GLSM for $SG(k, 2n)$ in [44]. The nonabelian mirror Landau-Ginzburg was also proposed in [44] by following [1]. It has chiral superfields $\Sigma_a, Y_{ia}, U_{bc}$ and $X_{\mu \nu}$ where the index $i \in \{\pm 1, \cdots, \pm n\}$ and indices $a, b, c, \mu, \nu \in \{1, \cdots, k\}$. The superpotential is

$$W = \sum_a \sum_{i} \left( \sum_i Y_{ia} - \sum_{b>c} \rho_{bc}^a \ln U_{bc} - \sum_{\mu \neq \nu} \alpha_{\mu \nu}^a \ln X_{\mu \nu} - t^a \right)$$

$$+ \sum_{i,a} \exp (-Y_{ia}) + \sum U_{bc} + \sum X_{\mu \nu},$$

where $\rho_{bc}^a = -\delta^a_b - \delta^a_c$. We consider the general symplectic Grassmannians $SG(k, 2n)$ with $k \leq n$, which contains Lagrangian Grassmannians as a special case when $k = n$.

One can define the period and follow equ’n (4.47) to observe that the generating function
shall look like

\[ \Pi_{SG(k,2n)}(\Sigma_a; t) = \sum_{d_1,\ldots,d_k=0}^{1 \leq a < b \leq k} \left. \prod \frac{-\partial a + \partial b}{\Sigma_a - \Sigma_b} \right|_{t^n=t} \cdot \Pi^{Ab}(\Sigma_a; t^a), \]  
\[ (4.56) \]

\[ \Pi^{Ab}(\Sigma_a; t^a) = \frac{\prod_{1 \leq a < b \leq k} (\Sigma_a + \Sigma_b + j)}{\prod_{a=1}^{k} \prod_{i=1}^{d_i} (\Sigma_a + l)^{2n}} \cdot (-1)^{k(d_1+\cdots+d_k)} \exp\left( -t^1 (d_1 + \Sigma_1) - \cdots - t^k (d_k + \Sigma_k) \right). \]  
\[ (4.57) \]

The generating function of the special case \( LG(n,2n) \) has been studied in the math literature [78].

### 4.4.2 Calabi-Yau manifolds

\( Gr(k, N)[Q_1, \ldots, Q_m] \)

Most examples we will compute here have already been computed in [2] by a different method; however, it is still useful to repeat these computations for two reasons. The first one is that the sign of \( q \) in Yukawa couplings in physics literature could be different from the sign of \( q \) in the math literature [2, 32] for some cases, and we want to make sure that Gromov-Witten invariants of these examples are not affected by this difference. The other reason is that the periods we compute here have different combinatorial expressions that look superficially different from those in [2]. Certain combinatorial identities ensure the consistency between these two methods.

The Calabi-Yau condition is \( \sum_{\beta=1}^{m} Q_{\beta} = N \). By the section 4.3.2, we know the mirror of the GLSM for Calabi-Yau manifolds \( Gr(k, N)[Q_1, \ldots, Q_m] \) in our construction is the Landau-Ginzburg model: target space \((\mathbb{C}^*)^{k:N} / S_k \times \mathbb{C}^{k^2-k} / S_k \times \mathbb{C}^k / S_k \times \mathbb{C}^n\) with the superpotential

\[ W = \sum_{a=1}^{k} \Sigma_a \left( \sum_{i=1}^{N} Y_i^a + \sum_{\beta=1}^{m} Q_{\beta} \log X_{\beta} + \sum_{\mu\nu} \alpha_{\mu\nu}^a X_{\mu\nu} - t^a \right) \]
\[ + \sum_{i=1}^{N} \sum_{a=1}^{k} \exp(-Y_i^a) + \sum_{\mu\neq\nu=1}^{k} X_{\mu\nu} + \sum_{\beta=1}^{m} X_{\beta}, \]

where \( \alpha_{\mu\nu}^a = -\delta_{\mu}^a + \delta_{\nu}^a \). Following equ’n (4.25), we then have

\[ \Pi_{Gr(k,N)[Q_1,\cdots,Q_m]}(\Sigma_1, \cdots, \Sigma_k; t) = \prod_{1 \leq a < b \leq k} \left. \frac{-\partial a + \partial b}{\Sigma_a - \Sigma_b} \right|_{t^n=t}. \]  
\[ (4.58) \]

\[ \Pi_M(\Sigma_1, \cdots, \Sigma_k, t^1 + i(k-1)\pi, \cdots, t^k + i(k-1)\pi), \]
where $\mathcal{M}$ is an abstract notation which does not have a geometrical meaning. However, the period $\Pi_{\mathcal{M}}$ is related to the period of the mirror of an abelian variety

$$\tau = \left( \prod_{\beta=1}^{m} O(-Q_{\beta}, \ldots, -Q_{\beta}) \right) \over \mathbb{CP}^{N-1} \times \cdots \times_{k-1} \mathbb{CP}^{N-1}.$$

More specifically, following the previous discussion on the effect of R-charge 2 matters in the mirror, we expect

$$\Pi_{\mathcal{M}}(\Sigma_1, \ldots, \Sigma_k; t^1 + i(k-1)\pi, \ldots, t^k + i(k-1)\pi) = \prod_{\beta=1}^{n} \left( Q_{\beta} \sum_{a=1}^{k} \theta_a \right) \Pi_{T}(\Sigma_1, \ldots, \Sigma_k, t^1 + i\pi, \ldots, t^k + i\pi).$$

The generating function of the mirror of $\Pi_{T}$ could be derived exactly in physics if we know precisely the Brane factor of A-cycles in Landau-Ginzburg mirrors. Unfortunately, the general definition of the brane factor for A-cycles is still an open question. To avoid this difficulty, one strategy of this paper is to guess the formula of the generating function from the structure of gamma functions in the period integral of mirrors (or the hemisphere partition function in B-twisted Landau-Ginzburg models $[79]$). On the contrary, the brane factor of a B-brane in the gauge theory has been well-studied in $[51]$; see also $[50, 80, 81]$ for some useful information. So one can then obtain the generating function of $\Pi_{T}$ on the gauge theory side as well, see $[51]$ section 8.2.2. Finally, we want to mention that the generating function of a toric variety has been well-studied in the math community $[82, 83]$, so one can find the generating function of $\Pi_{T}$ is

$$\Pi_{T}(\Sigma_1, \ldots, \Sigma_k; t^1 + i(k-1)\pi, \ldots, t^k + i(k-1)\pi) \equiv \sum_{d_1, \ldots, d_k=0}^{\infty} \prod_{l_i=0}^{\sum_{a=1}^{k} Q_{a} d_{a}} \left( Q_{1} \left( -\sum_{a=1}^{k} \Sigma_{a} - l_{i} \right) \times \cdots \times \prod_{l_{m}=0}^{\sum_{a=1}^{k} Q_{m} d_{a}} \left( Q_{m} \left( -\sum_{a=1}^{k} \Sigma_{a} - l_{m} \right) \right) \right) \prod_{l_{1}=1}^{d_{1}} (\Sigma_{1} + l_{1})^{N} \times \cdots \times \prod_{l_{k}=1}^{d_{k}} (\Sigma_{k} + l_{k})^{N} \cdot \exp \left( -\sum_{a=1}^{k} (\Sigma_{a} + d_{a}) (t^{a} + i(k-1)\pi) \right) \mod \Sigma_{a}^{N}. \quad (a \in \{1, \ldots, k\}).$$

Now, we can expand the generating function of the mirror of a Calabi-Yau manifold as

$$\Pi_{Gr(k,N)[Q_{1}, \ldots, Q_{m}]}(\Sigma_1, \ldots, \Sigma_k; t) = \Pi_{Gr(k,N)[Q_{1}, \ldots, Q_{m}], 0}(t) + \left( \sum_{a=1}^{k} \Sigma_{a} \right) \cdot \Pi_{Gr(k,N)[Q_{1}, \ldots, Q_{m}], 1}(t) + \sum_{i=2}^{3} \left( \sum_{a=1}^{k} \Sigma_{a} \right) ^{i} \cdot \Pi_{Gr(k,N)[Q_{1}, \ldots, Q_{m}], i}(t) \mod \left( \sum_{a=1}^{k} \Sigma_{a} \right) ^{4} = 0.$$

$^7$We thank Mauricio Romo for pointing out this to us.

$^8$Recently, the authors of $[84, 85]$ used the hemisphere partition function of the GLSM to obtain the generating function in the LG-model phase. Then one can get FJRW invariants $[86]$ from this generating function.
All components $\Pi_{Gr(k,N)\{Q_1,\cdots,Q_m\},\lambda}(t)$ can be computed explicitly by using the formula (5.40), then we can define a new function from the period, which is called $J$-function in the math literature

$$J_{Gr(k,N)\{Q_1,\cdots,Q_m\}}(\Sigma_1,\cdots,\Sigma_k, t) \equiv \frac{\Pi_{Gr(k,N)\{Q_1,\cdots,Q_m\}}(\Sigma_1,\cdots,\Sigma_k, t)}{\Pi_{Gr(k,N)\{Q_1,\cdots,Q_m\},0}}.$$  \hfill (4.60)

Then the flat coordinate on the Kähler moduli of a Calabi-Yau manifold $Gr(k,N)\{Q_1,\cdots,Q_m\}$ can be defined as

$$\tau \equiv -\frac{\Pi_{Gr(k,N)\{Q_1,\cdots,Q_m\},1}(t)}{\Pi_{Gr(k,N)\{Q_1,\cdots,Q_m\},0}(t)} + 2\pi t_0. \hfill (4.61)$$

One can fix $t_0$ by requiring the first Gromov-Witten invariant to be a positive number. Finally, we want to mention that the study of non-complete intersection Calabi-Yau manifolds is similar. In section 5, we will compute Gromov-Witten invariants of various nonabelian Calabi-Yau manifolds by using the framework we developed in this section.

4.5 Picard-Fuchs equations in physical parameters

In previous sections, we have defined novel Picard-Fuchs equations for any nonabelian mirror. This section will justify our proposal by matching existing results in the literature [2]. Then, we will also predict new results in our framework.

4.5.1 Grassmannians

We check our proposal by starting with the simplest nontrivial examples: $Gr(k,N)$. Our strategy is not to compute all of the examples; instead, we will provide an algorithm to compute them, and this algorithm is different from the one in the math literature [2]. Before talking about the general case, we start with $Gr(2,N)$, which can be easily generalized to $Gr(k,N)$. In our proposal, the periods of the mirror LG of $Gr(2,n)$ is of the form

$$\Pi(t) = (\partial t_1 - \partial t_2)|_{t_1 = t_2 = t} \Pi(t^1 + i\pi, t^2 + i\pi), \hfill (4.62)$$

where $\Pi(t^1,t^2)$ satisfies the Picard-Fuchs equation of the mirror of $\mathbb{P}^{N-1} \times \mathbb{P}^{N-1}$:

$$(-\partial_{t_1})^N \Pi(t^1 + i\pi, t^2 + i\pi) = -q^1 \Pi(t^1 + i\pi, t^2 + i\pi), \hfill (4.63)$$

$$(-\partial_{t_2})^N \Pi(t^1 + i\pi, t^2 + i\pi) = -q^2 \Pi(t^1 + i\pi, t^2 + i\pi).$$

Let us denote by $\Pi_0, \Pi_1, \cdots, \Pi_{N-1}$ a linearly independent set of solutions to the Picard-Fuchs equation of the mirror of $\mathbb{P}^{N-1}$, i.e.

$$(-\partial_{t_a})^N \Pi_i = -q^a \Pi_i \hfill (4.64)$$

for all $i = 0,1,\cdots,N-1$. We do not need the explicit expression of these solutions in our discussion. However, for concreteness, one may notice that $\Pi_i$ can be taken to be the
coefficient of $\Sigma^i$ in the generating function of the mirror of $\mathbb{P}^{N-1}$ or I-function of $\mathbb{P}^{N-1}$. So we have
\[ \Pi_{\mathbb{CP}^{N-1}} (t) = \Pi_0 (t) \cdot 1 + \Pi_1 (t) \cdot \Sigma + \cdots + \Pi_1 (t) \cdot \Sigma^{N-1}, \quad \text{mod} \quad \Sigma^N = 0 \quad (4.65) \]

The general solution to equ’n (4.63) is of the form
\[ \Pi (t^1, t^2) = \sum_{i,j=0}^{N-1} C_{ij} \Pi_i (t^1) \Pi_j (t^2), \quad (4.66) \]
where $C_{ij}$ are constants. This can be proved as follows: from the first equation of (4.63), $\Pi (t^1 + i\pi, t^2 + i\pi)$ can be written as
\[ \Pi (t^1 + i\pi, t^2 + i\pi) = \sum_{i=0}^{N-1} \Pi_i (t^1) g_i (t^2) \]
for some functions $g_i (t)$. The second equation of (4.63) then implies
\[ \sum_{i=0}^{N-1} \Pi_i (t^1) \left( (-\partial_{t^2})^N + q^2 \right) g_i (t^2) = 0. \]
Because $\Pi_i$ are linearly independent, the equation above implies
\[ \left( (-\partial_{t^2})^N + q^2 \right) g_i (t^2) = 0 \]
for all $i$ and $t^2$. Then there exist constants $C_{ij}$ such that
\[ g_i (t^2) = \sum_{j=0}^{N-1} C_{ij} \Pi_j (t^2). \]

We propose that the general solution to (4.62) is of the form
\[ \Pi (t) = \sum_{i,j=0}^{N-1} A_{ij} (\partial_t \Pi_i (t) \Pi_j (t) - \Pi_i (t) \partial_t \Pi_j (t)), \quad (4.67) \]
where the constant matrix $A_{ij}$ satisfy
\[ A_{ij} = -A_{ji}. \]

Now we outline the proof of (4.67). From (4.66), we have
\[ \Pi (t^1, t^2) = \sum_{i,j=0}^{N-1} C_{ij} \Pi_i (t^1) \Pi_j (t^2) \]
for some $N \times N$ constant matrix $C_{ij}$. If we decompose $C_{ij}$ into symmetric and anti-symmetric parts, then

$$\Pi(t^1, t^2) = \Pi_S(t^1, t^2) + \Pi_A(t^1, t^2),$$

where

$$\Pi_S(t^1, t^2) = \sum_{i,j=0}^{N-1} S_{ij} \Pi_i(t^1) \Pi_j(t^2),$$

$$\Pi_A(t^1, t^2) = \sum_{i,j=0}^{N-1} A_{ij} \Pi_i(t^1) \Pi_j(t^2),$$

$$C_{ij} = S_{ij} + A_{ij}, \quad S_{ij} = S_{ji}, \quad A_{ij} = -A_{ji}.$$

Notice that

$$(\partial_{t^1} - \partial_{t^2})|_{t^1 = t^2 = t} \Pi_S(t^1, t^2) = 0,$$

one concludes

$$\Pi(t) = (\partial_{t^1} - \partial_{t^2})|_{t^1 = t^2 = t} \Pi_A(t^1, t^2) = \sum_{i,j=0}^{N-1} A_{ij} (\partial_t \Pi_i(t) \cdot \Pi_j(t) - \Pi_i(t) \cdot \partial_t \Pi_j(t)).$$

If we take a basis for the set of $N \times N$ anti-symmetric matrices, we get a set of functions, which spans the solution space of (4.62). For example, if we define

$$\Pi_{ij}(t) = \partial_t \Pi_i(t) \cdot \Pi_j(t) - \Pi_i(t) \cdot \partial_t \Pi_j(t),$$

for $i < j$, $i, j = 0, \ldots, N - 1$, then any solution to (4.62) can be written as a linear combination of $\Pi_{ij}(t)$.

One can easily extend the study of $Gr(2, N)$ to the general $Gr(k, N)$ case. According to section 4.4, we have equ’n (4.51), which is

$$\Pi_{Gr(k, N)}(\Sigma_1, \ldots, \Sigma_k; t) = \prod_{1 \leq a < b \leq k} \left. \frac{-\partial_{\Sigma_a} + \partial_{\Sigma_b}}{\Sigma_a - \Sigma_b} \right|_{t^a = t^b} \Pi_{CP^{N-1} \times \ldots \times CP^{N-1}}(\Sigma_1, \ldots, \Sigma_k; t^1 + i(k - 1)\pi, \ldots, t^k + i(k - 1)\pi),$$

then from equ’n (4.65), we have

$$\Pi_{CP^{N-1} \times \ldots \times CP^{N-1}}(\Sigma_a; t^a + i(k - 1)\pi) = \sum_{i_1, \ldots, i_k=0}^{N-1} \Sigma_{i_1}^{n_1} \cdots \Sigma_{i_k}^{n_k} \Pi_{i_1}(t^1 + (k - 1)\pi) \cdots \Pi_{i_k}(t^1 + (k - 1)\pi)$$

The differential operator

$$\prod_{1 \leq a < b \leq k} \left. \frac{-\partial_{\Sigma_a} + \partial_{\Sigma_b}}{\Sigma_a - \Sigma_b} \right|_{t^a = t^b}$$

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is anti-symmetric in $t^a$, so we have

$$\Pi_{Gr(k,N)}(S_\lambda; t) = \sum_{i_1, \ldots, i_k=0}^{N-1} \varepsilon_{a_1, \ldots, a_k} \Pi_{i_1}^{(a_1)}(t + (k-1)\pi) \cdot \Pi_{i_2}^{(a_2)}(t + (k-1)\pi) \cdots \Pi_{i_k}^{(a_k)}(t + (k-1)\pi) \cdot S_\lambda,$$

where $S_\lambda$ is the cohomology class of $Gr(k, N)$ discussed in section 4.4, $\varepsilon_{a_1, \ldots, a_k}$ is the Levi-Civita symbol with index $a_l \in \{0, \ldots, k-1\}$, and $\Pi^{(a_l)}(t + (k-1)\pi)$ means $(-\frac{d}{dt})^a \cdot \Pi(t + (k-1)\pi)$. Each $\Pi_i(t + i\pi)$ obeys

$$\left(-\frac{d}{dt}\right)^N \Pi_i = (-1)^{k-1} q \Pi_i.$$  \hfill (4.69)

From equ’n (4.68) and (4.69), we have the following algorithm of deriving the Picard-Fuchs equations of $Gr(k, N)$ in terms of the one-dimensional physical Kähler parameter.

- For each derivative $-\frac{d}{dt}$, we use the chain rule to add the superscripts $\{m_1 \leq \cdots \leq m_k\}$ of

$$\Pi_{i_1}^{(m_1)}(t + i(k-1)\pi) \cdot \Pi_{i_2}^{(m_2)}(t + i(k-1)\pi) \cdots \Pi_{i_k}^{(m_k)}(t + i(k-1)\pi)$$

according to the constraint imposed by the Levi-Civita symbol, $\{m_1, \cdots, m_k\}$ are distinct integers, i.e. $\{m_1 < \cdots < m_k\}$, otherwise the above expression vanishes,

- when $m_k = N$,

$$\Pi_{i_1}^{(m_1)}(t + i(k-1)\pi) \cdots \Pi_{i_k}^{(m_k)}(t + i(k-1)\pi) = q \cdot \Pi_{i_1}^{(0)}(t + i(k-1)\pi) \cdots \Pi_{i_{k-1}}^{(m_{k-1})}(t + i(k-1)\pi),$$

- Differentiate the period repeatedly until the result can be written as a linear combination of the lower order derivatives. Then one can read off the Picard-Fuchs equation $\text{Poly}(-\frac{d}{dt}) \cdot \Pi_{Gr(k,N)}(t) = 0$ from the linear relation. One should notice $-\frac{d}{dt}$ acts nontrivially on $q$ as well.

Now let us look at some concrete examples.

**Gr(2,4).** Consider

$$\Pi \equiv \varepsilon^{m_1, m_2} \Pi_{m_1}^{(1)}(t) \cdot \Pi_{m_2}^{(1)}(t), \quad m \in \{0, \cdots, 3\}$$
and we have \((-\frac{d}{dt})^4 \Pi_t (t) = -q\Pi_t (t)\) then we can compute
\[
\begin{align*}
\Pi^{(1)} &= \varepsilon^{m_1, m_2} \Pi_{m_1} (t) \Pi^{(2)}_{m_2} (t), \\
\Pi^{(2)} &= \varepsilon^{m_1, m_2} (\Pi^{(1)}_{m_1} (t) \Pi^{(2)}_{m_2} (t) + \Pi_{m_1} (t) \Pi^{(3)}_{m_2} (t)), \\
\Pi^{(3)} &= \varepsilon^{m_1, m_2} (2\Pi^{(1)}_{m_1} (t) \Pi^{(3)}_{m_2} (t)), \\
\Pi^{(4)} &= \varepsilon^{m_1, m_2} (2\Pi^{(2)}_{m_1} (t) \Pi^{(3)}_{m_2} (t) + 2q\Pi_{m_1} (t) \Pi^{(1)}_{m_2} (t)), \\
\Pi^{(5)} &= \varepsilon^{m_1, m_2} (4q\Pi_{m_1} (t) \Pi^{(2)}_{m_2} (t) + 2q\Pi_{m_1} (t) \Pi^{(1)}_{m_2} (t))
\end{align*}
\]
thus, we have
\[
\left(-\frac{d}{dt}\right)^5 \Pi = 2q \left(2 \left(-\frac{d}{dt}\right) + 1\right) \Pi.
\] (4.70)
This agrees with the result in [2]. Furthermore, we know \(Gr(2, 4)\) can be realized as a quadric hypersurface in \(\mathbb{P}^5\), and indeed the Picard-Fuchs equation (4.70) is the same as that of the mirror of \(\mathbb{P}^5[2]\).

**Gr(3,4).** We have
\[
\Pi \equiv \varepsilon^{m_1, m_2, m_3} \Pi_{m_1} (t) \Pi^{(1)}_{m_2} (t) \Pi^{(2)}_{m_3} (t), \quad m \in \{0, \cdots, 3\}
\]
and
\[
\left(-\frac{d}{dt}\right)^4 \Pi_j = q \cdot \Pi_j.
\]
We then compute
\[
\begin{align*}
\Pi^{(1)} &= \varepsilon^{m_1, m_2, m_3} \Pi_{m_1} (t) \Pi^{(1)}_{m_2} (t) \Pi^{(3)}_{m_3} (t), \\
\Pi^{(2)} &= \varepsilon^{m_1, m_2, m_3} \Pi_{m_1} (t) \Pi^{(2)}_{m_2} (t) \Pi^{(3)}_{m_3} (t), \\
\Pi^{(3)} &= \varepsilon^{m_1, m_2, m_3} \Pi^{(1)}_{m_1} (t) \Pi^{(2)}_{m_2} (t) \Pi^{(3)}_{m_3} (t), \\
\Pi^{(4)} &= \varepsilon^{m_1, m_2, m_3} q\Pi_{m_1} (t) \Pi^{(1)}_{m_2} (t) \Pi^{(2)}_{m_3} (t),
\end{align*}
\]
thus we get the Picard-Fuchs equation
\[
\left(-\frac{d}{dt}\right)^4 \Pi = q\Pi.
\]
This result is expected from the dual relation between \(Gr(3, 4)\) and \(Gr(1, 4)\).

**Gr(2,5).** We have
\[
\Pi \equiv \varepsilon^{m, n} \Pi_{m} (t) \Pi^{(1)}_{n} (t), \quad m, n \in \{0, \cdots, 4\}
\]
and
\[
\left( -\frac{d}{dt} \right)^5 \Pi_t = -q \cdot \Pi_t.
\]

Then we can compute
\[
\begin{align*}
\Pi^{(1)} &= \varepsilon^{m,n} \Pi_m(t) \Pi_n^{(2)}(t), \\
\Pi^{(2)} &= \varepsilon^{m,n} \left( \Pi_m^{(1)}(t) \Pi_n^{(3)}(t) + \Pi_m(t) \Pi_n^{(3)}(t) \right), \\
\Pi^{(3)} &= \varepsilon^{m,n} \left( 2\Pi_m^{(1)}(t) \Pi_n^{(3)}(t) + \Pi_m(t) \Pi_n^{(4)}(t) \right), \\
\Pi^{(4)} &= \varepsilon^{m,n} \left( 2\Pi_m^{(2)}(t) \Pi_n^{(3)}(t) + 3\Pi_m^{(1)}(t) \Pi_n^{(4)}(t) \right), \\
\Pi^{(5)} &= \varepsilon^{m,n} \left( 5\Pi_m^{(2)}(t) \Pi_n^{(4)}(t) + 3q\Pi_m(t) \Pi_n^{(1)}(t) \right), \\
\Pi^{(6)} &= \varepsilon^{m,n} \left( 5\Pi_m^{(3)}(t) \Pi_n^{(4)}(t) + 3q\Pi_m(t) \Pi_n^{(1)}(t) + 8q\Pi_m(t) \Pi_n^{(2)}(t) \right), \\
\Pi^{(7)} &= \varepsilon^{m,n} \left( 3q\Pi_m(t) \Pi_n^{(1)}(t) + 11q\Pi_m(t) \Pi_n^{(2)}(t) + 8\Pi_m^{(1)}(t) \Pi_n^{(2)}(t) + 13q\Pi_m(t) \Pi_n^{(3)}(t) \right), \\
\Pi^{(8)} &= \varepsilon^{m,n} \left( 3q\Pi_m(t) \Pi_n^{(1)}(t) + 14q\Pi_m(t) \Pi_n^{(2)}(t) + 19q\Pi_m^{(1)}(t) \Pi_n^{(2)}(t) + 24q\Pi_m(t) \Pi_n^{(3)}(t) \\
&\quad + 21q\Pi_m^{(1)}(t) \Pi_n^{(3)}(t) + 13q\Pi_m(t) \Pi_n^{(4)}(t) \right), \\
\Pi^{(9)} &= \varepsilon^{m,n} \left( 3q\Pi_m(t) \Pi_n^{(1)}(t) + 17q\Pi_m(t) \Pi_n^{(2)}(t) + 33q\Pi_m^{(1)}(t) \Pi_n^{(2)}(t) + 38q\Pi_m(t) \Pi_n^{(3)}(t) \\
&\quad + 64q\Pi_m^{(1)}(t) \Pi_n^{(3)}(t) + 37q\Pi_m(t) \Pi_n^{(4)}(t) + 21q\Pi_m^{(2)}(t) \Pi_n^{(3)}(t) + 34q\Pi_m^{(1)}(t) \Pi_n^{(4)}(t) \right), \\
\Pi^{(10)} &= \varepsilon^{m,n} \left( 3q\Pi_m(t) \Pi_n^{(1)}(t) + 20q\Pi_m(t) \Pi_n^{(2)}(t) + 50q\Pi_m^{(1)}(t) \Pi_n^{(2)}(t) + 55q\Pi_m(t) \Pi_n^{(3)}(t) \\
&\quad + 135q\Pi_m^{(1)}(t) \Pi_n^{(3)}(t) + 75q\Pi_m(t) \Pi_n^{(4)}(t) + 85q\Pi_m^{(2)}(t) \Pi_n^{(3)}(t) + 135q\Pi_m^{(1)}(t) \Pi_n^{(4)}(t) \\
&\quad + 55q\Pi_m^{(2)}(t) \Pi_n^{(4)}(t) + 34q^2\Pi_m(t) \Pi_n^{(1)}(t) \right).
\end{align*}
\]

One can then find that \( \Pi^{(10)} \) can be expressed in terms of a linear combination of lower order derivatives, which reads
\[
\Pi^{(10)} = 3\Pi^{(9)} - 3\Pi^{(8)} + \Pi^{(7)} + q \left( 11\Pi^{(5)} + 11\Pi^{(4)} + 3\Pi^{(3)} \right) + q^2\Pi.
\]

So we have the Picard-Fuchs equation of the mirror of \( Gr(2, 5) \)
\[
\left( \left( -\frac{d}{dt} \right)^7 \left( -\frac{d}{dt} - 1 \right)^3 - q \left( 11 \left( -\frac{d}{dt} \right)^5 + 11 \left( -\frac{d}{dt} \right)^4 + 3 \left( -\frac{d}{dt} \right)^3 \right) - q^2 \right) \cdot \Pi(t) = 0.
\]

(4.71)
This agrees with the result in \[2\]. One can also reproduce Picard-Fuchs equations of mirrors of $Gr(2,6)$, $Gr(3,6)$, as well as, $Gr(2,7)$ studied in \[2\]. We will not perform the calculations of all examples here; however, it would be interesting to find a closed formula for Picard-Fuchs equations of mirrors of Grassmannians.

4.5.2 Symplectic Grassmannian

The next simplest nontrivial examples would be $SG(k,2n)$. The generating function of $SG(k,2n)$ has been given in equ’n (4.56), which we rewrite here is

$$
\Pi_{SG(k,2n)} (\Sigma_a; t) = \sum_{d_1, \ldots, d_k=0}^{1 \leq a \leq k} \prod \frac{-\partial_{t_a} + \partial_{t_b}}{\Sigma_a - \Sigma_b} |_{t^a = t} \cdot \Pi^{Ab} (\Sigma_a; t^a),
$$

$$
\Pi^{Ab} (\Sigma_a; t^a) = \prod_{1 \leq b < c \leq k} \prod_{j=1}^{d_b+d_c} (\Sigma_b + \Sigma_c + j) \cdot (-1)^{k(d_1+ \cdots + d_k)} \exp (-t^1 (d_1 + \Sigma_1) - \cdots - t^k (d_k + \Sigma_k)).
$$

One can directly show that $\Pi^{Ab} (\Sigma_a; t^a)$ obeys the following equation

$$
(-\partial_{t_a})^{2n} \Pi^{Ab} = (-1)^{k-1} e^{-t^a} \prod_{b \neq a} (-\partial_{t_a} - \partial_{t_b} + 1) \Pi^{Ab}.
$$

(4.72)

Then from (4.56), we have a similar result as the Grassmannian case, i.e.

$$
\Pi_{SG(k,2n)} (t) = \varepsilon^{m_1, \cdots, m_k} (-\partial t_1)^{0} (-\partial t_2)^{1} \cdots (-\partial t^n)^{n-1} G_{m_1, \cdots, m_k} (t^1, \cdots, t^k)_{t=1} \cdot S_{\lambda},
$$

where $G_{m_1, \cdots, m_k} (t^1, \cdots, t^k)$ satisfies

$$
(-\partial_{t^a})^{2n} G_{m_1, \cdots, m_k} (t^1, \cdots, t^k) = (-1)^{k-1} e^{-t^a} \prod_{b \neq a} (-\partial_{t_a} - \partial_{t_b} + 1) G_{m_1, \cdots, m_k} (t^1, \cdots, t^k).
$$

Now let us consider some concrete examples.

**LG(2,4).** We have

$$
(-\partial_{t_1})^4 G_{m_1 m_2}^{0,1} (t^1, t^2) = -q^1 (-\partial_{t_1} - \partial_{t_2} + 1) G_{m_1 m_2}^{0,1} (t^1, t^2),
$$

$$
(-\partial_{t_2})^4 G_{m_1 m_2}^{0,1} (t^1, t^2) = -q^2 (-\partial_{t_1} - \partial_{t_2} + 1) G_{m_1 m_2}^{0,1} (t^1, t^2),
$$

where

$$
G_{m_1 m_2}^{0,1} (t^1, t^2) = (-\partial_{t_1})^{0} (-\partial_{t_2})^{1} G_{m_1 m_2} (t^1, t^2).
$$
Set $-\partial_t = -\partial_{t_1} - \partial_{t_2}$, then we compute

\[
(-\partial_t) \cdot \varepsilon^{m_1 m_2 G_{m_1 m_2}^{0,1}} (t^1, t^2) = \varepsilon^{m_1 m_2 G_{m_1 m_2}^{0,1}} (t^1, t^2)
\]

\[
(-\partial_t)^2 \cdot \varepsilon^{m_1 m_2 G_{m_1 m_2}^{0,1}} (t^1, t^2) = \varepsilon^{m_1 m_2 (G^{1,2}_{m_1 m_2} (t^1, t^2) + G^{0,2}_{m_1 m_2} (t^1, t^2))}
\]

\[
(-\partial_t)^3 \cdot \varepsilon^{m_1 m_2 G_{m_1 m_2}^{0,1}} (t^1, t^2) = \varepsilon^{m_1 m_2 (2G^{0,3}_{m_1 m_2} (t^1, t^2))}
\]

\[
(-\partial_t)^4 \cdot \varepsilon^{m_1 m_2 G_{m_1 m_2}^{0,1}} (t^1, t^2) = \varepsilon^{m_1 m_2 (2G^{2,3}_{m_1 m_2} (t^1, t^2) + 2q^2 (-\partial_t + 1) G^{0,1}_{m_1 m_2} (t^1, t^2))}
\]

\[
(-\partial_t)^5 \cdot \varepsilon^{m_1 m_2 G_{m_1 m_2}^{0,1}} (t^1, t^2) = \varepsilon^{m_1 m_2 (4q^2 (-\partial_t + 1) G^{0,2}_{m_1 m_2} (t^1, t^2) + 2q^2 (-\partial_t + 1) G^{0,1}_{m_1 m_2} (t^1, t^2))}
\]

One can then obtain the Picard-Fuchs equation of the mirror of $LG(2, 4)$:

\[
\left( -\frac{d}{dt} \right) \left[ \left( -\frac{d}{dt} \right)^4 - 2q \left( -\frac{d}{dt} + 1 \right) \right] \Pi(t) = 0.
\]

This is not surprising, as we know the Picard-Fuchs equation of the mirror of $LG(2, 4)$ is the same Picard-Fuchs equation of the mirror of $\mathbb{P}^4[2]$.

The computations of Picard-Fuchs equations of mirrors of $SG(2, 2n)$ is similar. For example, one can find the Picard-Fuchs operator of the mirror of $LG(2, 6)$ to be

\[
\left( -\frac{d}{dt} \right) 8 \left( -\frac{d}{dt} - 1 \right)^5 - q \left( -\frac{d}{dt} \right)^4 \left( -2 \frac{d}{dt} + 1 \right) \left( 13 \frac{d^2}{dt^2} - 13 \frac{d}{dt} + 4 \right)
\]

\[
- q^2 \left( -3 \frac{d}{dt} + 2 \right) \left( -3 \frac{d}{dt} + 3 \right) \left( -3 \frac{d}{dt} + 4 \right).
\]

We leave the study of general cases to the interested reader.

### 4.5.3 Calabi-Yau three-folds in Grassmannians

The generating functions of mirrors of Calabi-Yau three-folds in Grassmannians, in general, do not share the same factorization property of generating functions of mirrors of Grassmannians. So we have to search for a different algorithm to get Picard-Fuchs equations of mirrors in those cases. Let us start with the simplest case: $Gr(2, 4)[4]$. We denote the period of the mirror of $Gr(2, 4)[4]$ by $\Pi(t)$. Then according to our proposal

\[
\Pi(t) = (-\partial_{t_1} + \partial_{t_2})_{t_1 = t_2 = t} \cdot f(t^1, t^2),
\]

where $f(t^1, t^2)$ satisfies the Picard-Fuchs equations defined in section 4.3.2

\[
(-\partial_{t_1})^4 f(t^1, t^2) = -q^1 D \cdot f(t^1, t^2),
\]

\[
(-\partial_{t_2})^4 f(t^1, t^2) = -q^2 D \cdot f(t^1, t^2),
\]

where $D = \prod_{k=0}^{3}(-4(\partial_{t_1} + \partial_{t_2}) + k)$, $q^a = \exp(-t^a)$. Equ’n (4.73) suggests that we only need to consider functions anti-symmetric in $t^1$ and $t^2$. As an operator acting upon functions anti-symmetric in $t^1$ and $t^2$, $\partial_{t_1} + \partial_{t_2}$ reduces to $\frac{d}{dt}$ in the limit $t^1 \to t, t^2 \to t$. Let us define

\[
h(t^1, t^2) = (-\partial_{t_1} + \partial_{t_2}) \cdot f(t^1, t^2),
\]

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then, from (4.74), one computes
\[(\partial_{t_1} + \partial_{t_2})^3 h(t^1, t^2) = (\partial^3 t_1 + 2 \partial^2 t_1 \partial t_2 - 2 \partial^2 t_1 \partial t_2 - \partial^3 t_2) \cdot f(t^1, t^2) \cdot \] \[-q^1 + q^2 \cdot D \cdot f(t^1, t^2) + 2 \cdot (\partial^2 t_1 \partial t_2 - \partial^2 t_2 \partial t_1) \cdot f(t^1, t^2) \cdot \] \[(\partial_{t_1} + \partial_{t_2})^5 h(t^1, t^2) = (-q^1 + q^2)^5 \left(1 - 2 (\partial_{t_1} + \partial_{t_2}) + (\partial_{t_1} + \partial_{t_2})^2 + 2 (\partial_{t_1} \partial_{t_2}) \right) \cdot f(t^1, t^2) \cdot \] \[+ 2 \cdot (-q^1 \partial_{t_2} + q^2 \partial_{t_1}) \cdot f(t^1, t^2) \cdot \] \[-4 \cdot (q^1 \partial_{t_2}^2 - q^2 \partial_{t_1}^2) \cdot f(t^1, t^2) \cdot \] \end{align*} 

Under the limit \( t^1 \to t, \ t^2 \to t, \ h(t^1, t^2) \to h(t), \) therefore
\[\left( \frac{d}{dt} \right)^5 h(t) = 2q \left( 2 \frac{d}{dt} - 1 \right) D \cdot h(t), \]

where \( D = \prod_{k=0}^{3} \left( 4 (-q^4) + k \right) \). Let \( \theta = -\frac{d}{dt} \), then
\[\theta^5 h(t) = 2q (2\theta + 1) D \cdot h(t). \]

Then as discussed in section 4.4, \( \tilde{h}(t) = \theta h(t) \) is the period of the mirror of Calabi-Yau \( Gr(2,4)[4] \). It obeys
\[(\theta^5 - 16q (2\theta + 1)^2 (4\theta + 1) (4\theta + 3)) \tilde{h}(t) = 0, \]

which is the Picard-Fuchs equation of Calabi-Yau threefold \( Gr(2,4)[4] \). In principle, one can perform a similar computation for the general cases, although they are very lengthy.

However, we will provide a useful trick for getting the Picard-Fuchs equation of the general case quickly. Let us apply this trick to the quintic first, recall the Picard-Fuchs equation of the mirror of \( \mathbb{P}^4 \) is
\[(-\partial_t)^5 \Pi_{\mathbb{P}^4} = q \Pi_{\mathbb{P}^4}. \]

Now, we replace \( q \) by \( \hat{q} \), where \( \hat{D} = (5\partial_t) (5\partial_t - 1) \cdots (5\partial_t - 4) \). Then we have the Picard-Fuchs equation of \( \mathcal{O}(-5) \):
\[(-\partial_t)^5 \Pi_{\mathcal{O}(-5)} = -q (-5\partial_t) \cdots (-5\partial_t + 4) \Pi_{\mathcal{O}(-5)}. \]

From the discussion of section 4.4, we know \( \Pi_{\text{quintic}} = (-\partial_t) \Pi_{\mathcal{O}(-5)} \), then indeed we find the correct Picard-Fuchs equation
\[(-\partial_t)^4 \Pi_{\text{quintic}} = -5q (-5\partial_t + 1) \cdots (-5\partial_t + 4) \Pi_{\text{quintic}}. \]

One can interpret the negative sign in front of \( q \) as a finite shift in \( \theta \) by \( 5\pi \) [16].

Now we apply the same trick to Calabi-Yau manifolds as complete intersections in Grassmannians. Let us consider \( Gr(2,5) \)[1^2, 3] first. The Picard-Fuchs equation of the mirror of \( Gr(2,5) \) has been derived in section 4.5.1
\[\left( -\frac{d}{dt} \right)^7 \left( -\frac{d}{dt} - 1 \right)^3 - q \left( 11 \left( -\frac{d}{dt} \right)^5 + 11 \left( -\frac{d}{dt} \right)^4 + 3 \left( -\frac{d}{dt} \right)^3 \right) - q^2 \cdot \Pi_{Gr(2,5)} = 0. \]
Now we replace $q$ by $q\hat{D}$, where
\[
\hat{D} = \left(-\frac{d}{dt}\right)^2 \cdot \left(3\frac{d}{dt}\right) \left(\frac{3d}{dt} - 1\right) \left(\frac{3d}{dt} - 2\right).
\]
This substitution yields
\[
\left(-\frac{d}{dt}\right)^7 \left(-\frac{d}{dt} - 1\right)^3 \cdot \Pi(t) - q\hat{D} \left(11 \left(-\frac{d}{dt}\right)^5 + 11 \left(-\frac{d}{dt}\right)^4 + 3 \left(-\frac{d}{dt}\right)^3\right) \cdot \Pi(t) \quad (4.75)
\]
\[
- \left(q\hat{D}\right) \left(q\hat{D}\right) \cdot \Pi(t) = 0.
\]
Notice that $q$ does not commute with the operator $\hat{D}$, so we do not have the naive equality $q\hat{D} \cdot q\hat{D} = q^2\hat{D}^2$. We find that equ’n (4.75) is the Picard-Fuchs equation of the corresponding local Calabi-Yau. Recall the discussion in section 4.4, and we define the period of the mirror of $Gr(2,5)[1^2, 3]$ to be $\tilde{\Pi}(t) = \Pi^{(3)}(t)$, then (4.75) reduces to
\[
\left(-\frac{d}{dt}\right)^4 \left(-\frac{d}{dt} - 1\right)^3 \cdot \tilde{\Pi}(t) + 3 \left(q\hat{D}\right) \left(q \left(-3\frac{d}{dt} + 1\right) \left(-3\frac{d}{dt} + 2\right)\right) \cdot \tilde{\Pi}(t)
\]
\[
+ 3q \left(-\frac{d}{dt}\right)^3 \left(-3\frac{d}{dt} + 1\right) \left(-3\frac{d}{dt} + 2\right) \left(11 \left(-\frac{d}{dt}\right)^2 + 11 \left(-\frac{d}{dt}\right) + 3\right) \cdot \tilde{\Pi}(t) = 0.
\]
However, the Picard-Fuchs equation of the mirror of $Gr(2,5)[1^2, 3]$ should be a fourth order differential equation. This suggests that equ’n (4.76) is not irreducible. Indeed, the differential operator can be factorized. Notice that
\[
q \cdot \left(-\frac{d}{dt}\right)^3 \times (\cdots) = \left(-\frac{d}{dt} - 1\right)^3 (q \times \cdots),
\]
\[
\left(-3\frac{d}{dt} + 1\right) \left(-3\frac{d}{dt} + 2\right) \cdot (q \cdots) = q \left(-3\frac{d}{dt} + 4\right) \left(-3\frac{d}{dt} + 5\right) \times (\cdots).
\]
Thus we obtain
\[
(\theta - 1)^3 \cdot \left(\theta^4 + 3q (3\theta + 2) (3\theta + 1) (11\theta^2 + 11\theta + 3)
\]
\[
+ 9q^2 (3\theta + 5) (3\theta + 4) (3\theta + 2) (3\theta + 1)\right) \cdot \tilde{\Pi}(t) = 0.
\]
Indeed, we get the Picard-Fuchs equation of the mirror of $Gr(2,5)[1, 1, 3]$:
\[
\left(\theta^4 + 3q (3\theta + 2) (3\theta + 1) (11\theta^2 + 11\theta + 3)
\]
\[
+ 9q^2 (3\theta + 5) (3\theta + 4) (3\theta + 2) (3\theta + 1)\right) \cdot \tilde{\Pi}(t) = 0.
\]
This agrees with the result in [2], with \( q \mapsto -q \). One may notice that this sign difference in \( q \) between the results in physics literature and math literature has already appeared in Yukawa-couplings discussed in section 3.3. Because Picard-Fuchs equations determine Yukawa-couplings, this sign difference is not a surprise.

One can do the similar computations for cases such as \( Gr(2, 5)[1, 2^2] \), \( Gr(2, 6)[1^4, 2] \), \( Gr(2, 7)[1^7] \) as well as \( Gr(3, 6)[1^6] \). The results agree with those in [2]. For more general Calabi-Yau manifolds [22], a similar procedure is expected to give Picard-Fuchs equations of mirrors.

4.5.4 Flag manifolds

Our method can applied to other manifold as well if a GLSM construction is known. For example, GLSMs for flag manifolds have been studied in [21] [87]. Thus, we can build an algorithm to compute Picard-Fuchs equations of the mirror of any Flag variety. We will illustrate the algorithm by computing \( Fl(1, 2, 3) \) and \( Fl(1, 2, 4) \) explicitly.

\( Fl(1, 2, 3) \). The period of the mirror of \( Fl(1, 2, 3) \) can be expressed as

\[
\Pi(t, \tilde{t}) = (-\partial_{t^1} + \partial_{t^2})_{t^1 = t^2 = \tilde{t}} \cdot f(t; t^1, t^2),
\]

where \( f(t; t^1, t^2) \) obeys the partial differential equations

\[
(-\partial_t + \partial_{t^1}) \cdot (-\partial_t + \partial_{t^2}) f(t; t^1, t^2) = e^{-t} f(t; t^1, t^2),
\]

\[
(-\partial_{t^1})^2 f(t; t^1, t^2) = -\tilde{q}^2 (-\partial_{t^1} + \partial_{t^2}) f(t; t^1, t^2),
\]

\[
(-\partial_{t^2})^2 f(t; t^1, t^2) = -\tilde{q}^2 (-\partial_{t^1} + \partial_{t^2}) f(t; t^1, t^2).
\]

Now, as before let us define \( h(t; t^1, t^2) = (-\partial_{t^1} + \partial_{t^2}) \cdot f(t; t^1, t^2) \), then, from equ’n (4.85), one can compute

\[
(\partial_{t^1} + \partial_{t^2})^2 h(t; t^1, t^2) = (-\partial_{t^1} + \partial_{t^2}) \cdot f(t; t^1, t^2) - \partial_{t^1} \partial_{t^2} (\partial_{t^1} - \partial_{t^2}) \cdot f(t; t^1, t^2),
\]

\[
= (-\tilde{q}^2 (-\partial_{t^1} + \partial_{t^2} + \tilde{q}^2 (-\partial_{t^1} + \partial_{t^2})) \cdot f(t; t^1, t^2) + \partial_{t^1} \partial_{t^2} h(t; t^1, t^2).
\]

Now, we have

\[
(\partial_t)^2 h(t; t^1, t^2)\big|_{t^1 = t^2 = \tilde{t}} = q \cdot \Pi(t; \tilde{t}) + (-\partial_t) (-\partial_{\tilde{t}}) \cdot \Pi(t; \tilde{t}) - (-\partial_t) (-\partial_{\tilde{t}}) \cdot h(t; t^1, t^2)\big|_{t^1 = t^2 = \tilde{t}} = (q + \tilde{q}) \cdot \Pi(t; \tilde{t}) - (-\partial_t)^2 \cdot \Pi(t; \tilde{t}),
\]

where we have used equ’n (4.81) and \( \partial_{\tilde{t}} = \partial_{t^1} + \partial_{t^2} \big|_{t^1 = t^2 = \tilde{t}} \). So we obtain the first Picard-Fuchs equation of \( Fl(1, 2, 3) \)

\[
((-\partial_t)^2 - (-\partial_t) (-\partial_{\tilde{t}}) + (\partial_{\tilde{t}})^2) \cdot \Pi(t; \tilde{t}) = (q + \tilde{q}) \cdot \Pi(t; \tilde{t}).
\]

One can also compute the following

\[
(-\partial_{t^1} - \partial_{t^2})^3 h(t; t^1, t^2)\big|_{t^1 = t^2 = \tilde{t}} = -\tilde{q} (-1 + \partial_t + \partial_{\tilde{t}}) \cdot \Pi(t; \tilde{t}),
\]

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then we obtain the second Picard-Fuchs equation of the mirror of $Fl(1, 2, 3)$
\[ (-\partial t)^3 \Pi (t; \tilde{t}) = -\tilde{q} \left( -1 + \partial_t + \partial_{\tilde{t}} \right) \Pi (t; \tilde{t}). \] (4.83)

Our method of getting Picard-Fuchs equations is not complicated; however, to the best of our knowledge, our computation is a genuinely new result.

$Fl(1, 2, 4)$. The period of the mirror of $Fl(1, 2, 4)$ satisfies
\[ \Pi (t, \tilde{t}) = (-\partial_t + \partial_{\tilde{t}})_{t^1 = t^2 = \tilde{t}} \cdot g \left( t; t^1, t^2 \right), \] (4.84)
where $g (t; t^1, t^2)$ obeys the partial differential equations
\[
\begin{align*}
(-\partial_t + \partial_{\tilde{t}}) \cdot (-\partial_t + \partial_{\tilde{t}}) g (t; t^1, t^2) &= e^{-t} g (t; t^1, t^2), \\
(-\partial_t^4) g (t; t^1, t^2) &= -\tilde{q}^1 \left(-\partial_t + \partial_{\tilde{t}}\right) g (t; t^1, t^2), \\
(-\partial_{\tilde{t}}^4) g (t; t^1, t^2) &= -\tilde{q}^2 \left(-\partial_t + \partial_{\tilde{t}}\right) g (t; t^1, t^2).
\end{align*}
\] (4.85)

Following equ'n (4.85), one can compute the Picard-Fuchs equations of the mirror of $Fl(1, 2, 4)$
\[
\begin{align*}
&\left(2 (-\partial_t)^2 - 2 (-\partial_t) (-\partial_{\tilde{t}})^2 + (-\partial_{\tilde{t}})^3\right) \cdot \Pi (t; \tilde{t}) = (2q (-\partial_t) + \tilde{q}) \cdot \Pi (t; \tilde{t}), \\
&(-\partial_{\tilde{t}})^5 \cdot \Pi (t; \tilde{t}) = \tilde{q} \left( (-\partial_{\tilde{t}} + 1)^2 + (2\partial_t) + 2 (-\partial_t) (-\partial_{\tilde{t}}) + 2 (-\partial_{\tilde{t}})^2 - 2q \right) \cdot \Pi (t; \tilde{t}).
\end{align*}
\] (4.86)

As far as we know, our computation of $Fl(1, 2, 4)$ is a genuinely new result.

In principle, the Picard-Fuchs equations of any Flag manifold can be obtained this way. For example, the mirror of $Fl(k_1, k_2, N)$ has the following structure
\[
\Pi (t; \tilde{t}) = \prod_{1 \leq a < b \leq k_1} (-\partial_{t^a} + \partial_{\tilde{t}^b}) \bigg|_{t^a = t} \cdot \prod_{1 \leq a \leq k_2} (-\partial_{\tilde{t}^a} + \partial_{\tilde{t}^b}) \bigg|_{\tilde{t}^a = \tilde{t}} \cdot f \left( t^1, \ldots, t^{k_1}; \tilde{t}^1, \ldots, \tilde{t}^{k_2} \right),
\] (4.87)

where $f (t; \tilde{t})$ obeys Picard-Fuchs equations
\[
\prod_b (-\partial_{t^b} + \partial_{\tilde{t}^b}) \cdot f (t; \tilde{t}) = (-1)^{k_1 - 1} q f (t; \tilde{t}),
\]
\[
(-\partial_{\tilde{t}^b})^N \cdot f (t; \tilde{t}) = (-1)^{k_2 - 1} \tilde{q} \prod_{a=1}^{k_1} (-\partial_{t^a} + \partial_{\tilde{t}^b}) \cdot f \left( t^1, \ldots, t^{k_1}; \tilde{t}^1, \ldots, \tilde{t}^{k_2} \right).
\]

In order to get Picard-Fuchs equations, one need to perform the computation of
\[
(\partial_{t^1} + \cdots + \partial_{t^{k_1}})^m \cdot (\partial_{\tilde{t}^1} + \cdots + \partial_{\tilde{t}^{k_2}})^n \cdot f \left( t^1, \ldots, t^{k_1}; \tilde{t}^1, \ldots, \tilde{t}^{k_2} \right) \bigg|_{t^a = t, \tilde{t}^a = \tilde{t}} = \cdots .
\]
One can then read off two Picard-Fuchs equations of mirrors of $Fl(k_1, k_2, N)$ from the right choice of values of $m$ and $n$. Finally, we want to mention that it would be interesting to find a closed formula for the Picard-Fuchs equation of any $Fl(k_1, k_2, N)$. The argument for the general Flag manifold $Fl(k_1, \ldots, k_n, N)$ is similar, which we leave to the reader.
4.5.5 Comments on periods and Picard-Fuchs equations

In our study of previous examples, we derived the Picard-Fuchs equations known to mathematicians only from the structure of Picard-Fuchs equations defined in this paper. We did not use any explicit solutions of our Picard-Fuchs equations in the derivation yet, so one may naturally propose another algorithm of getting Picard-Fuchs equations in physical parameters for any targets:

- Solving the first period $\Pi_0(t^a)$ explicitly from our novel Picard-Fuchs equation,
- then find differential equations such that $\text{Poly} (\partial_a t^a; q^a) \cdot \Pi_0(t^a) = 0$.

Here is an example, consider the mirror of $Gr(2, N)$. One can write down the period explicitly from our Picard-Fuchs equation of the mirror of $Gr(2, N)$, which is

$$
\Pi_0(t) = \sum_{d_1, d_2=0} \frac{1}{(d_1!)^N (d_2!)^N} \left( 1 - \frac{N (d_1 - d_2)}{2} (H_{d_1} - H_{d_2}) \right) \exp \left( - (t + i\pi) (d_1 + d_2) \right).
$$

Then the Picard-Fuchs equation can be obtained by solving $\text{Poly} (\partial_a t^a; q^a) \cdot \Pi_0(t^a) = 0$, it would be interesting to find a closed formula of the Picard-Fuchs equations for $Gr(2, N)$ from the explicit expression of $\Pi_0(t)$. The idea can be similarly applied to other examples.

5 Gromov-Witten invariants

This section uses the technique we developed in previous sections to compute Gromov-Witten invariants of various nonabelian Calabi-Yau manifolds.

5.1 Complete intersections in Grassmannians

- $Gr(2, 4)$ [4].

Following section 4.4, we have the generating function of the mirror of $Gr(2, 4)$ [4]:

$$
\Pi_{Gr(2,4)[4]}(\Sigma_1, \Sigma_2; t) = \left. \frac{-\partial_{t_1} + \partial_{t_2}}{\Sigma_1 - \Sigma_2} \right|_{t_1 = t_2 = t} \sum_{d_1, d_2=0}^{\infty} \prod_{i=1}^{d_1 + d_2} (\Sigma_1 + l_1)^4 \cdot \prod_{l_2=1}^{\infty} (\Sigma_2 + l_2)^4
\cdot \exp \left( - (\Sigma_1 + d_1) (t^1 + i\pi) \right) \exp \left( - (\Sigma_2 + d_2) (t^2 + i\pi) \right)
\right) (5.1)
$$

For our purpose of computing the Gromov-Witten invariants, we only need $\Pi_{Gr(2,4)[4],0}$ and $\Pi_{Gr(2,4)[4],1}$, which can be computed directly from equ'n (5.1). Notice that the action of $-\partial_{t_1} + \partial_{t_2}$ on $\exp \left( - (\Sigma_2 + d_2) (t^2 + i\pi) \right)$ will introduce a factor

$$
\frac{d_1 - d_2}{\Sigma_1 - \Sigma_2} + 1,
$$

(5.2)
and the denominator can be expanded in terms of $\Sigma$’s

$$
\prod_{k=1}^{d_1} (\Sigma_1 + k)^{-4} \cdot \prod_{k=1}^{d_2} (\Sigma_2 + k)^{-4} = (d_1!)^{-4} (d_1!)^{-4} \left( 1 - 4\Sigma_1 H_{d_1} - 4\Sigma_2 H_{d_2} \right) (5.3)
$$

$$
+ (\Sigma_1 + \Sigma_2) (\Sigma_1 - \Sigma_2) \left( 4H_{d_1}^2 + \sum_{k=1}^{d_1} \frac{1}{k^2} - 4H_{d_2}^2 - \sum_{k=1}^{d_2} \frac{1}{k^2} \right) + \ldots,
$$

where $H_d$ is the harmonic number $H_d := \sum_{k=1}^{d} \frac{1}{k}$ and $H_0 := 0$. From equ’n (5.2) and (5.3), we then read off

$$
\Pi_0(t) = \sum_{d_1,d_2 \geq 0} \frac{(4d_1 + 4d_2)!}{(d_1!)^4 (d_1!)^4} (1 - 2(d_1 - d_2)(H_{d_1} - H_{d_2})) \exp \left( -(t + i\pi)(d_1 + d_2) \right) (5.4)
$$

$$
\Pi_1(t) = -(t + i\pi) \Pi_0(t)
$$

$$
+ \sum_{d_1,d_2 \geq 0} \frac{(4d_1 + 4d_2)!}{(d_1!)^4 (d_1!)^4} \left( 4H_{4(d_1+d_2)} - 2H_{d_1} - 2H_{d_2} - 8(d_1 - d_2) H_{4(d_1+d_2)} (H_{d_1} - H_{d_2}) \right)
$$

$$
+ (d_1 - d_2) \left( 4H_{d_1}^2 + \sum_{k=1}^{d_1} \frac{1}{k^2} - 4H_{d_2}^2 - \sum_{k=1}^{d_2} \frac{1}{k^2} \right) \exp \left( -(d_1 + d_2)(t + i\pi) \right),
$$

We can compute the first several terms of (5.4) explicitly

$$
\Pi_0(t) = 1 + 48q + 15120q^2 + 7392000q^3 + 444410000q^4 + \mathcal{O}(e^{-5t}),
$$

$$
\Pi_1(t) = -t - i\pi + 16q(16 - 3(t + i\pi)) + 144q^2(634 - 105(t + i\pi))
$$

$$
+ \frac{6400}{3} q^3(21874 - 3465(t + i\pi)) + 98000q^4(290929 - 45045(t + i\pi)) + \mathcal{O}(e^{-5t}).
$$

The flat coordinate $\tau$ is

$$
\tau = -\frac{\Pi_1(t)}{\Pi_0(t)} - 2i\pi t_0 = -\log q - 256q - 79008q^2 - \frac{117004288}{3} q^3 - 23552020432q^4 + \mathcal{O}(e^{-5t}),
$$

(5.5)

where we have chosen $t_0 = 1/2$. Expressing $q$ in terms of $z = e^{-\tau}$, we have

$$
q = z - 256z^2 + 19296z^3 - 2836480z^4 + \mathcal{O}(z^5).
$$

The Yukawa-coupling of $Gr(2,4)[4]$ is given by equ’n (3.5), and the substitution of the above equation into (2.1) yields

$$
8 + 1280z + 739584z^2 + 422690816z^3 + 248570501376z^4 + \mathcal{O}(z^5).
$$

(5.7)

Gromov-Witten invariants $n_a$ of $Gr(2,4)[4]$ can be read off from formula (2.6), which are

$$
1280, \quad 92288, \quad 15655168, \quad 3883902528, \ldots.
$$

(5.8)
These results agree with the computation in [2, 23]. Moreover, because one can treat the nonabelian Calabi-Yau $Gr(2, 4)[4]$ as an abelian Calabi-Yau $P^5[2, 4]$, and compute Gromov-Witten invariants of $P^5[2, 4]$ from toric mirror symmetry, the result serves as a nontrivial consistency check of our framework in computing Gromov-Witten invariants from nonabelian mirrors.

- $Gr(2, 5) [2^2, 1]$. Following section 4.4, the period of the mirror of $Gr(2, 5) [2^2, 1]$ is

$$
\Pi_{Gr(2,5)[2^2,1]}(\Sigma_1, \Sigma_2; t) = \left. \frac{-\partial_{d_1} + \partial_{d_2}}{\Sigma_1 - \Sigma_2} \right|_{t^1=t^2=t, d_1, d_2=0} \sum_{d_1, d_2=0}^{\infty} \prod_{k=1}^{2(d_1+d_2)} (\Sigma_1 + \Sigma_2 + k)^{2(d_1+d_2)} 
\prod_{k=1}^{d_1} (\Sigma_1 + k)^{5} \cdot \prod_{k=1}^{d_2} (\Sigma_2 + k)^{5} \cdot \exp \left( -\left( \begin{array}{c} \Sigma_1 + d_1 \\ 2 \end{array} \right) t^1 \right) \exp \left( -\left( \begin{array}{c} \Sigma_2 + d_2 \\ 2 \end{array} \right) t^2 \right).
$$

(5.9)

One can then compute $\Pi_{Gr(2,5)[2^2,1],0}$ and $\Pi_{Gr(2,5)[2^2,1],1}$ from the above equation, and they are

$$
\Pi_0(t) = \sum_{d_1, d_2=0} \left[ \frac{(d_1 + d_2)!}{(d_1!)^5 (d_2!)^5} \left( 1 - \frac{5}{2} (d_1 - d_2) (H_{d_1} - H_{d_2}) \right) \exp \left( -(d_1 + d_2) t \right),
\right.

(5.10)

$$
\Pi_1(t) = -t \Pi_0 + 
\sum_{d_1, d_2=0} \left[ \frac{(d_1 + d_2)!}{(d_1!)^5 (d_2!)^5} \left( 4H_{2d_1+2d_2} + H_{d_1+d_2} - \frac{5}{2} H_{d_1} - \frac{5}{2} H_{d_2} - (d_1 - d_2) \right) \cdot \left( 10H_{2d_1+2d_2} + \frac{5}{2} H_{d_1+d_2} \right) \cdot (H_{d_1} - H_{d_2}) \cdot \left( 25 \left( H_{d_1}^2 - H_{d_2}^2 \right) + \frac{5}{4} \left( \sum_{k=1}^{d_1} \frac{1}{k^2} - \sum_{k=1}^{d_2} \frac{1}{k^2} \right) \right) \right) \cdot \exp \left( -(d_1 + d_2) t \right).
$$

Series expansion of (5.10) gives

$$
\Pi_0(t) = 1 - 12q + 684q^2 - 58800q^3 + 6129900q^4 + O(q^5),
\Pi_1(t) = -t - 4q(11 - 3t) + 6q^2(491 - 114t) - \frac{280}{3} q^3(2879 - 630t) + 315q^4(91683 - 19460t) + O(e^{-5t}).
$$

The flat coordinate $\tau$ is

$$
\tau = -\frac{\Pi_1(t)}{\Pi_0(t)} - 2i\pi t_0 = -i\pi - log q + 44q - 2418q^2 + \frac{628784}{3} q^3 - 22123897q^4 + O(e^{-t})^5,
$$

(5.11)

where we have chosen $t_0 = 1/2$. In terms of the variable $z = e^{-\tau}$, we have

$$
q = -z + 44z^2 - 486z^3 + 11184z^4 + O(z^5).
$$

(5.12)
Plugging all of the data into the formula \((2.1)\), we then have

\[
K_{zzz} = 20 + 400z + 44720z^2 + 4439200z^3 + 451877040z^4 + \mathcal{O}(z^5).
\]  

(5.13)

It suggests Gromov-Witten invariants of \(Gr(2, 5)[2^2, 1]\) are

\[
400, \quad 5540, \quad 164400, \quad 7059880, \ldots.
\]  

(5.14)

These invariants agree with the result in the literature \([2]\). However, our formula \((5.10)\) of periods looks superficially different from the corresponding formula \([2, \text{ page 31}]\). This suggests a nontrivial combinatorial identity to match two different formulas. In our framework of computing the Gromov-Witten invariants, one can quickly check that \(Gr(3, 5)[2^2, 1]\) shares the same data as \(Gr(2, 5)[2^2, 1]\) by a direct computation.

- \(Gr(2, 5)[1^2, 3]\).

The generating function of the mirror of \(Gr(2, 5)[1^2, 3]\) can be found to be

\[
\Pi_{Gr(2, 5)[1^2, 3]}(\Sigma_1, \Sigma_2; t) = \frac{-\partial_{\Sigma_1} + \partial_{\Sigma_2}}{\Sigma_1 - \Sigma_2} \sum_{d_1, d_2=0}^{\infty} \frac{\prod_{k=1}^{2(\Sigma_1+\Sigma_2)} (3(\Sigma_1 + \Sigma_2) + k) \prod_{k=1}^{\Sigma_1+\Sigma_2} ((\Sigma_1 + \Sigma_2) + k)^2}{\prod_{k=1}^{\Sigma_1} (\Sigma_1 + k)^5 \cdot \prod_{k=1}^{\Sigma_2} (\Sigma_2 + k)^5} \cdot \exp\left(- (\Sigma_1 + d_1) t^1\right) \exp\left(- (\Sigma_2 + d_2) t^2\right).
\]  

One can read off \(\Pi_{Gr(2, 5)[1^2, 3], 0}\) and \(\Pi_{Gr(2, 5)[1^2, 3], 1}\)

\[
\Pi_0(t) = \sum_{d_1, d_2=0} (3d_1 + 3d_2)! \cdot ((d_1 + d_2)!)^2 \cdot \frac{(d_1!)^5 \cdot (d_2!)^5}{(d_1!)^5 \cdot (d_2!)^5} \left(1 - \frac{5}{2} (d_1 - d_2) (H_{d_1} - H_{d_2})\right) \exp\left(- (d_1 + d_2) t\right),
\]  

(5.16)

\[
\Pi_1(t) = -t \cdot \Pi_0 + \sum_{d_1, d_2=0} (3d_1 + 3d_2)! \cdot ((d_1 + d_2)!)^2 \cdot \frac{3H_{3d_1+3d_2} - 2H_{d_1 + d_2} - \frac{5}{2} H_{d_1} - \frac{5}{2} H_{d_2} - (d_1 - d_2)}{(d_1!)^5 \cdot (d_2!)^5} \cdot \left(5H_{d_1+d_2} + \frac{15}{2} H_{3d_1+3d_2}\right) (H_{d_1} - H_{d_2}) + \left(\frac{25}{4} H_{d_1}^2 - H_{d_2}^2\right) + \frac{5}{4} \left(\sum_{k=1}^{d_1} \frac{1}{k^2} - \sum_{k=1}^{d_2} \frac{1}{k^2}\right) \right) \right) \cdot \exp\left(- (d_1 + d_2) t\right),
\]

which have series expansions

\[
\Pi_0(t) = 1 - 18q + 1710q^2 - 246960q^3 + 43347150q^4 + \mathcal{O}(q^5),
\]

\[
\Pi_1(t) = -t - 3q(25 - 6t) + \frac{9}{2} q^2(491 - 114t) - 98q^3(12827 - 2520t) + \frac{135}{4} q^4(6720919 - 1284360t) + \mathcal{O}(e^{-5t}).
\]
So the flat coordinate $\tau$ is
\[
\tau = -\frac{\Pi_1(t)}{\Pi_0(t)} - 2i\pi t_0 = -i\pi - \log q + 75q - \frac{13797}{2}q^2 + 1004623q^3 - \frac{713717469}{4}q^4 + \mathcal{O}(e^{-5t}).
\] (5.17)

We have chosen $t_0 = 1/2$. Then expressing $q$ in terms of $z = e^{-\tau}$, we have
\[
q = -z + 75z^2 - 1539z^3 + 60073z^4 + \mathcal{O}(z^5).
\] (5.18)

Following formula (2.1), one can find $K_{zzz} = 15 + 540z + 100980z^2 + 16776045z^3 + 2873237940z^4 + \mathcal{O}(z^5)$. (5.19)

Consequently, Gromov-Witten invariants of $Gr(2,5)\{1^2,3\}$ are
\[
540, 12555, 621315, 44892765, \ldots .
\] (5.20)

- $Gr(2,6)\{1^4,2\}$.

One can write out the generating function of the mirror of $Gr(2,6)\{1^4,2\}$ as
\[
\Pi_{Gr(2,5)\{1^4,2\}}(\Sigma_1, \Sigma_2; t) = \frac{-\partial_{d_1} + \partial_{d_2}}{\Sigma_1 - \Sigma_2} \bigg|_{t^1=t^2=t}
\] (5.21)
\[
\sum_{d_1,d_2=0}^{\infty} \prod_{k=1}^{2(d_1+d_2)} (2 (\Sigma_1 + \Sigma_2) + k) \prod_{k=1}^{d_1+d_2} ((\Sigma_1 + \Sigma_2) + k)^4
\prod_{k=1}^{d_1} (\Sigma_1 + k)^6 \prod_{k=1}^{d_2} (\Sigma_2 + k)^6
\cdot \exp\left( - (\Sigma_1 + d_1) (t^1 + i\pi) \right) \exp\left( - (\Sigma_2 + d_2) (t^2 + i\pi) \right).
\]
which implies
\[
\Pi_0(t) = \sum_{d_1,d_2=0} \frac{(2d_1 + 2d_2)! \cdot [(d_1 + d_2)!]^4}{(d_1!)^6 \cdot (d_2!)^6} (1 - 3 (d_1 - d_2) (H_{d_1} - H_{d_2})) \cdot \exp\left( - (d_1 + d_2) (t + i\pi) \right),
\] (5.22)
\[
\Pi_1(t) = -(t + i\pi) \Pi_0 + \sum_{d_1,d_2=0} \frac{(2d_1 + 2d_2)! \cdot [(d_1 + d_2)!]^4}{(d_1!)^6 \cdot (d_2!)^6} \left( 2H_{2d_1+2d_2} + 4H_{d_1+d_2} - 3H_{d_1} - 3H_{d_2} - (d_1 - d_2) 
\cdot (6H_{2d_1+2d_2} + 12H_{d_1+d_2}) (H_{d_1} - H_{d_2}) + \left( 9 (H_{d_1}^2 - H_{d_2}^2) + \frac{3}{2} \left( \sum_{k=1}^{d_1} \frac{1}{k^2} - \sum_{k=1}^{d_2} \frac{1}{k^2} \right) \right) \right)
\cdot \exp\left( - (d_1 + d_2) (t + i\pi) \right),
\]
whose series expansions read
\[
\Pi_0(t) = 1 + 8q + 288q^2 + 15200q^3 + 968800q^4 + \mathcal{O}(q^5),
\]
\[
\Pi_1(t) = -t - i\pi + 2q(13 - 4(t + i\pi)) + 3q^2(367 - 96(t + i\pi)) + \frac{20}{3}q^3(9301 - 2280(t + i\pi)) + q^4 \left( \frac{24519335}{6} - 968800(t + i\pi) \right) + \mathcal{O}(e^{-5t}).
\]

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Thus, the flat coordinate $\tau$ is
\[
\tau = -\frac{\Pi_1(t)}{\Pi_0(t)} - 2i\pi t_0 = \log q - 26q - 893q^2 - \frac{142124}{3}q^3 - \frac{6110349}{2}q^4 + \mathcal{O}(e^{-t})^5, \tag{5.23}
\]
where we have have chosen $t_0 = 1/2$. Then we have,
\[
q = z + 26z^2 + 121z^3 + 1372z^4 + \mathcal{O}(z^5). \tag{5.24}
\]
Following formula (2.1), one can get
\[
K_{zzz} = 28 + 280z + 21672z^2 + 1303624z^3 + 81880232z^4 + \mathcal{O}(z^5). \tag{5.25}
\]
Therefore, Gromov-Witten invariants of $Gr(2, 6)[1^4, 2]$ are
\[
280, \quad 2674, \quad 48272, \quad 1279040, \ldots . \tag{5.26}
\]

- $Gr(3, 6) [1^6]$.

The generating function of the mirror of $Gr(3, 6) [1^6]$ is
\[
\Pi_{Gr(3, 6)[1^6]}(\Sigma_1, \Sigma_2; t) = \left(\frac{-\partial_{t_1} + \partial_{t_2}}{\Sigma_1 - \Sigma_2}\right) \left(\frac{-\partial_{t_1} + \partial_{t_3}}{\Sigma_1 - \Sigma_3}\right) \left(\frac{-\partial_{t_2} + \partial_{t_3}}{\Sigma_2 - \Sigma_3}\right) \bigg|_{t^1=t^2=t^3=t} \tag{5.27}
\]
\[
\sum_{d_1, d_2, d_3=0}^\infty \frac{\prod_{k=1}^{d_1+d_2+d_3} ((\Sigma_1 + \Sigma_2) + k)^6}{\prod_{k=1}^{d_1} (\Sigma_1 + k)^6 \cdot \prod_{k=1}^{d_2} (\Sigma_2 + k)^6 \cdot \prod_{k=1}^{d_3} (\Sigma_3 + k)^6} \cdot \exp \left(- (\Sigma_1 + d_1) t^1\right) \exp \left(- (\Sigma_2 + d_2) t^2\right) \exp \left(- (\Sigma_3 + d_3) t^3\right).
\]
Notice the operator $(-\partial_{t_1} + \partial_{t_2})(-\partial_{t_2} + \partial_{t_3})(-\partial_{t_1} + \partial_{t_3})$ introduces the following factor
\[
\frac{(\Sigma_1 - \Sigma_2 + d_1 - d_2) (\Sigma_1 - \Sigma_3 + d_1 - d_3) (\Sigma_2 - \Sigma_3 + d_2 - d_3)}{(\Sigma_1 - \Sigma_2) (\Sigma_1 - \Sigma_3) (\Sigma_2 - \Sigma_3)}. \tag{5.28}
\]
So we should write the variables $\Sigma_1$, $\Sigma_2$ and $\Sigma_3$ as
\[
\Sigma_1 = \frac{\Sigma_1 + \Sigma_2 + \Sigma_3}{3} + \frac{\Sigma_1 - \Sigma_2}{3} + \frac{\Sigma_1 - \Sigma_3}{3},
\]
\[
\Sigma_2 = \frac{\Sigma_1 + \Sigma_2 + \Sigma_3}{3} + \frac{\Sigma_2 - \Sigma_1}{3} + \frac{\Sigma_2 - \Sigma_3}{3},
\]
\[
\Sigma_3 = \frac{\Sigma_1 + \Sigma_2 + \Sigma_3}{3} + \frac{\Sigma_3 - \Sigma_1}{3} + \frac{\Sigma_3 - \Sigma_2}{3}.
\]
One can then find expressions of $\Pi_0(t)$ and $\Pi_1(t)$, respectively. Here, we only list the first several terms of the series instead of the lengthy exact formulas:
\[
\Pi_0(t) = 1 + 6q + 126q^2 + 3948q^3 + 149310q^4 + \mathcal{O}(q^5),
\]
\[
\Pi_1(t) = -t + 2q(8 - 3t) + 2q^2(208 - 63t) + q^3 \left(\frac{42136}{3} - 3948t\right) + 6q^4 (91976 - 24885t) + \mathcal{O}(e^{-5t}).
\]
Then the flat coordinate is
\[ \tau = -\frac{\Pi_1(t)}{\Pi_0(t)} - 2i\pi t_0 = -\log q - 16q - 320q^2 - \frac{30328}{3}q^3 - 387712q^4 + O(e^{-t})^5, \]  
(5.29)
where we have taken \( t_0 = 0 \). Then,
\[ q = z - 16z^2 + 64z^3 - 552z^4 + O(z^5). \]  
(5.30)
Following formula (2.1), one can get
\[ K_{zzz} = 42 + 210z + 9618z^2 + 354018z^3 + 12933522z^4 + O(z^5). \]  
(5.31)
Thus Gromov-Witten invariants of \( Gr(3, 6) \) \cite{6} are
\[ 210, \quad 1176, \quad 13104, \quad 201936, \ldots . \]  
(5.32)
• \( Gr(2, 7) \) \cite{7}.

One can write the generating function of the mirror of \( Gr(2, 7) \) \cite{7} as
\[ \Pi_{Gr(2,7)\{1\}}(\Sigma_1, \Sigma_2; t) = \frac{-\partial_{\Sigma_1} + \partial_{\Sigma_2}}{\Sigma_1 - \Sigma_2} \bigg|_{t_1 = t_2 = t} \sum_{d_1, d_2 = 0}^{\infty} \prod_{k=1}^{d_1+d_2} ((\Sigma_1 + \Sigma_2) + k)^7 \prod_{k=1}^{d_1}(\Sigma_1 + k)^7 \prod_{k=1}^{d_2}(\Sigma_2 + k)^7 \exp \left( - (\Sigma_1 + d_1) t^1 \right) \exp \left( - (\Sigma_2 + d_2) t^2 \right), \]  
(5.33)
which implies
\[ \Pi_0(t) = \sum_{d_1, d_2 = 0} [(d_1 + d_2)!]^7 (1 - \frac{7}{2} (d_1 - d_2) (H_{d_1} - H_{d_2})) \exp (- (d_1 + d_2) t), \]  
(5.34)
\[ \Pi_1(t) = -t \cdot \Pi_0 + \sum_{d_1, d_2 = 0} \frac{[(d_1 + d_2)!]^7}{(d_1)! \cdot (d_2)!^7} \left( 7H_{d_1+d_2} - \frac{7}{2}H_{d_1} - \frac{7}{2}H_{d_2} - (d_1 - d_2) \right) \cdot \left( \frac{49}{2}H_{d_1+d_2} (H_{d_1} - H_{d_2}) - \left( \frac{49}{4} (H_{d_1}^2 - H_{d_2}^2) + \frac{7}{4} \left( \sum_{k=1}^{d_1} \frac{1}{k^2} - \sum_{k=1}^{d_2} \frac{1}{k^2} \right) \right) \right) \cdot \exp (- (d_1 + d_2) t), \]  
whose series expansions read
\[ \Pi_0(t) = 1 - 5q + 109q^2 - 3317q^3 + 121501q^4 + O(q^5), \]  
(5.35)
\[ \Pi_1(t) = -t - q(14 - 5t) + q^2(357 - 109t) - \frac{1}{3} q^3(35105 - 9951t) + q^4 \left( \frac{2669975}{6} - 121501t \right) + O(e^{-5t}). \]
Then we have the flat coordinate $\tau$

$$\tau = -\frac{\Pi_1(t)}{\Pi_0(t)} - 2i\pi t_0 = -i\pi - \log q + 14q - 287q^2 + \frac{26222}{3}q^3 - \frac{647143}{2}q^4 + O(e^{-5t}), \quad (5.36)$$

where we have chosen $t_0 = 1/2$. In terms of $z = e^{-\tau}$, we have

$$q = -z + 14z^2 - 7z^3 - 14z^4 + O(z^5). \quad (5.37)$$

Formula (2.1) results in

$$K_{zzz} = 42 + 196z + 9996z^2 + 344176z^3 + 12685708z^4 + O(z^5). \quad (5.38)$$

Hence Gromov-Witten invariants of $Gr(2, 7)[1^7]$ are

$$196, \quad 1225, \quad 12740, \quad 198058, \ldots . \quad (5.39)$$

### 5.2 The Pfaffian Calabi-Yau threefold

In [54], Rødland conjectured that two Calabi-Yau manifolds $X = Gr(2, 7)[1^7]$ and $Y =$ Pfaffian variety in $\mathbb{CP}^{20} \cap \mathbb{CP}^6$ share the same Kähler moduli space. The first evidence he found was that these two spaces’ cohomology rings are identical, and he then studied the one-parameter Picard-Fuchs equation of the proposed mirrors of $X$ and $Y$ and found that they are the same. In [18], Hori and Tong gave a field-theoretic proof of Rødland’s conjecture, and they concluded that $X$ and $Y$ are two different phases of a GLSM, which has been studied extensively in their paper. More specifically, $X$ is at the large volume limit $r \gg 0$ of the GLSM, and $Y$ is at the Landau-Ginzburg point of the GLSM where $r \ll 0$. In general, the theory at the LG point is a hybrid-model: a base manifold fibered by a Landau-Ginzburg model. For the Pfaffian Calabi-Yau, the superpotential of the Landau-Ginzburg model leads to a massive theory, which has trivial FJRW invariants [86]. That means the invariants we compute at the Landau-Ginzburg point of this model are Gromov-Witten invariants as well. One may naively change the sign of the FI-parameter of the GLSM to indicate that the Pfaffian Calabi-Yau is also at the large volume limit of the GLSM. It seems that Gromov-Witten invariants of this example have not been computed in the physics literature before, so we will perform a detailed computation of this example by proposing a generating function of the Pfaffian Calabi-Yau threefold, which, to our best knowledge, is a new result.

Because the Pfaffian Calabi-Yau shares the same GLSM with $X = Gr(2, 7)[1^7]$, and in the Pfaffian Calabi-Yau phase, we shall assign R-charge 1 to fundamental fields and vanishing R-charges to fields in the determinantal representation, then we can write the
superpotential of the mirror Landau-Ginzburg as

\[ W = \Sigma_1 \left( -2 \sum_{i=1}^{7} \log X_i^1 + \log X_1 - \log X_2 - \sum_{\beta=1}^{7} Y_\beta - t^1 \right) \]

\[ + \Sigma_2 \left( -2 \sum_{i=1}^{7} \log X_i^2 + \log X_2 - \log X_1 - \sum_{\beta=1}^{7} Y_\beta - t^2 \right) \]

\[ + \sum_{i=1}^{7} (X_i^1)^2 + \sum_{i=1}^{7} (X_i^2)^2 + X_1 + X_2 + \sum_{\beta=1}^{7} e^{-Y_\beta}, \]  

where the fundamental variables \( X_i^1 \) and \( X_i^2 \) can be represented as \( X_i^1 = e^{-\frac{Y_i^1}{2}} \) and \( X_i^2 = e^{-\frac{Y_i^2}{2}} \). One can compute the critical loci from

\[ \frac{\partial W}{\partial \Sigma_a} = \frac{\partial W}{\partial X_i^a} = \frac{\partial W}{\partial Y_\beta} = 0, \]

then we have

\[ X_1 = -\Sigma_1 + \Sigma_2 = -X_2 \]

\[ e^{-Y_i^a} = \Sigma_a, \quad e^{-Y_\beta} = -\Sigma_1 - \Sigma_2, \]

and

\[ \Sigma_1^7 = q^1 (\Sigma_1 + \Sigma_2)^7 \]

\[ \Sigma_2^7 = q^2 (\Sigma_1 + \Sigma_2)^7, \quad \Sigma_1 \neq \Sigma_2. \]  

If \( r \to -\infty \), we have \( q \to \infty \), and from the above equation, we have \( \Sigma_1 + \Sigma_2 \to 0 \).

Now we compute the period

\[ \Pi = \int \prod_{i,a} dX_i^a \prod_{\beta=1}^{7} dY_\beta dX_1 dX_2 \cdot e^{-W + \frac{t^1 + t^2}{2}}. \]

The insertion of an extra factor \( e^{\frac{t^1 + t^2}{2}} \) is due to the same reason mentioned in \[54\]. Now, we want to integrate the variables \( X_1 \) and \( X_2 \) out first in this model. The large volume limit of this model looks superficially different from what we usually see in this paper. So we have to check whether we can still use the A-brane we define in appendix A for doing the computation. From equ’n (5.41), we know that

\[ \left( \frac{\Sigma_1}{\Sigma_2} \right)^7 = \frac{q^1}{q^2}, \]

\[ ^{9} \text{In order to get a global section of the canonical bundle near } q = \infty. \]
because the constraint $\Sigma_1 \neq \Sigma_2$, we have

$$\Sigma_1 = \left(\frac{q^1}{q^2}\right) \omega^j \Sigma_2, \quad \omega^7 = 1.$$ 

Although $\Sigma_1$ and $\Sigma_2$ shall take different values, we find they are both close to zero. This is because $\Sigma_1 + \Sigma_2$ is close to zero when $q \to \infty$, which means we can choose the same cycle defined in appendix A for $X_1$ and $X_2$ when $q \to \infty$, and it is: $(0, +\infty)$.

Now, we can perform the same calculation as we did in section 4.3.1 to get the following expression of the period

$$\Pi = \int \prod_{i=1}^{7} \prod_{a=1}^{2} dX^a \prod_{\beta=1}^{7} dY_\beta \cdot (\Sigma_1 - \Sigma_2) e^{-W_{\text{eff}} + \frac{t^1 + t^2}{2}},$$

where the effective superpotential

$$W_{\text{eff}} = \Sigma_1 \left(-2 \sum_{i=1}^{7} \log X^1_i - \sum_{\beta=1}^{7} Y_\beta - t^1 + i\pi\right) + \Sigma_2 \left(-2 \sum_{i=1}^{7} \log X^2_i - \sum_{\beta=1}^{7} Y_\beta - t^2 + i\pi\right) + \sum_{i=1}^{7} \sum_{a=1}^{2} (X^a_i)^2 + \sum_{\beta=1}^{7} e^{-Y_\beta}.$$

Now, we propose that the generating series of the Pfaffian Calabi-Yau is

$$\Pi_{\text{Pfaffian}} = \left. -\frac{\partial}{\partial t^1} + \frac{\partial}{\partial t^2} \right|_{t^1 = t^2 = t} \Pi^{Ab},$$

where

$$\Pi^{Ab} = \int \prod_{i,a} dX^a_i \prod_{\beta=1}^{7} dY_\beta \cdot e^{-W_{\text{eff}} + \frac{t^1 + t^2}{2}}.$$ 

As expected, $\Pi^{Ab}$ obeys the following Picard-Fuchs equations

$$\left(-\frac{\partial}{\partial t^1}\right)^7 \Pi^{Ab} = q^1 \left(-\frac{\partial}{\partial t^1} - \frac{\partial}{\partial t^2} + 1\right)^7 \Pi^{Ab}$$

and

$$\left(-\frac{\partial}{\partial t^2}\right)^7 \Pi^{Ab} = q^2 \left(-\frac{\partial}{\partial t^1} - \frac{\partial}{\partial t^2} + 1\right)^7 \Pi^{Ab}. \quad (5.42)$$

Then, we follow the A-brane we defined in the LG model to get the concrete formula of the period $\Pi^{Ab}$. Integrate out $X^a_i$ and $Y_\beta$, we have

$$\Pi^{Ab} \propto \int d\Sigma_1 d\Sigma_2 \cdot \Gamma^7 \left(\Sigma_1 + \frac{1}{2}\right) \Gamma^7 \left(\Sigma_2 + \frac{1}{2}\right) \Gamma^7 \left(-\Sigma_1 - \Sigma_2\right) \cdot e^{t^1 \Sigma_1 + t^2 \Sigma_2 + \frac{t^1 + t^2}{2} + i\pi (\Sigma_1 + \Sigma_2)}. \quad (5.43)$$
One can easily find that equ’n (5.43) does satisfy equ’n (5.42). We have chosen a cycle in Σ space such that Σ can treated as a holomorphic variable, then we can apply the contour-integral technique to do the computation. Because \( t^1 \to -\infty \) and \( t^2 \to -\infty \), and we require \( \exp(t^1\Sigma_1 + t^2\Sigma_2) \) to vanish at the boundary of the field space, this means \( \Sigma_1 \) and \( \Sigma_2 \) must take positive values, so the contour we choose will encircle poles of the \( \Gamma \) function in the upper half-plane of \( \Sigma_1 \) and \( \Sigma_2 \). Notice that \( \Gamma (-\Sigma_1 - \Sigma_2) \) has poles at \( -\Sigma_1 - \Sigma_2 = -n \) for \( n \geq 0 \). We expand the variable \( \Sigma_1 + \Sigma_2 \) around poles: \( \Sigma_1 + \Sigma_2 = n + \epsilon_1 + \epsilon_2 \), then we have

\[
\Gamma (-\Sigma_1 - \Sigma_2) = \Gamma (-n - \epsilon_1 - \epsilon_2) = \frac{\pi(-1)^{n+1}}{\sin(\pi(\epsilon_1 + \epsilon_2))} \cdot \frac{1}{\Gamma(1 + n + \epsilon_1 + \epsilon_2)}.
\]

Now, we plug \( \Sigma_1 + \Sigma_2 = n + \epsilon_1 + \epsilon_2 \) into \( \Gamma (\Sigma_1) \) or \( \Gamma (\Sigma_2) \) to get

\[
\Gamma^7 \left( \Sigma_1 + \frac{1}{2} \right) \Gamma^7 \left( n + \frac{1}{2} + \epsilon_1 + \epsilon_2 - \Sigma_1 \right),
\]

or

\[
\Gamma^7 \left( \Sigma_2 + \frac{1}{2} \right) \Gamma^7 \left( n + \frac{1}{2} + \epsilon_1 + \epsilon_2 - \Sigma_2 \right).
\]

One can observe that there are additional poles in \( \Gamma^7 \left( n + \frac{1}{2} + \epsilon_1 + \epsilon_2 - \Sigma_1 \right) \) or \( \Gamma^7 \left( n + \frac{1}{2} + \epsilon_1 + \epsilon_2 - \Sigma_2 \right) \). Expansion around these poles has the form

\[
n + \frac{1}{2} + \epsilon_1 + \epsilon_2 - \Sigma_1 = -m + \epsilon_2, \quad \text{or} \quad m + \frac{1}{2} + \epsilon_1 + \epsilon_2 - \Sigma_2 = -n + \epsilon_2,
\]

then

\[
\Sigma_1 = n + m + \frac{1}{2} + \epsilon_1, \quad \Sigma_2 = -m - \frac{1}{2} + \epsilon_2,
\]

or

\[
\Sigma_1 = -m - \frac{1}{2} + \epsilon_1, \quad \Sigma_2 = n + m + \frac{1}{2} + \epsilon_2.
\]

Using

\[
\Gamma (-m + \epsilon_2) = \frac{\pi(-1)^{m+1}}{\sin(\pi \epsilon_2)} \frac{1}{\Gamma(1 + m - \epsilon_2)},
\]

we have

\[
\Pi^{Ab} \propto \sum_{m,n \geq 0} \int \frac{d\epsilon_1 d\epsilon_2 (-1)^n (-1)^m}{\sin^7(\pi(\epsilon_1 + \epsilon_2)) \sin^7(\pi \epsilon_2)} \left[ \Gamma^7(1 + m + n + \epsilon_1) \Gamma^7(1 + n + \epsilon_1 + \epsilon_2) \right] \cdot \exp \left[ (t^1 + i\pi)(-m + \epsilon_1) + (t^2 + i\pi)(n + m + 1 + \epsilon_2) \right] + 1 \Leftrightarrow 2.
\]

The notation \( 1 \Leftrightarrow 2 \) stands for the Weyl-group \( S_2 \) action on variables and parameters. From the structure of the equation above, one can naturally write down the ambient abelian theory’s generating function. It is then straightforward to propose a cohomology-valued
generating function of the mirror of Pfaffian threefold Calabi-Yau manifold, which is
\[
-\frac{\partial_{t_1} + \partial_{t_2}}{\Sigma_1 - \Sigma_2} \Bigg|_{t_1 = t_2 = t} \cdot \left[ \sum_{m,n \geq 0} \frac{\prod_{k=1}^{m+n} (\Sigma_2 + k)^7 (-1)^m}{\prod_{k=1}^{n} (\Sigma_1 + \Sigma_2 + k)^7 \prod_{k=1}^{m} (-\Sigma_1 + k)^7} \right]
\exp \left( \frac{\left( t_1^2 + t_2^2 \right) (\Sigma_1 + \Sigma_2) + \left( \frac{t_1^2 - t_2^2}{2} \right) (\Sigma_1 - \Sigma_2) + t^2(n + 1) + (t^2 - t^4) m}{} \right)
\]
+ 1 \leftrightarrow 2.

To find the mirror map for the case of the Pfaffian Calabi-Yau threefold, we shall expand
\[
\Pi_{\text{Pfaffian}} = \Pi_0(t) + \Pi_1(t) (-\Sigma_1 - \Sigma_2) + \ldots .
\]
Notice \((-\partial_{t_1} + \partial_{t_2}) / (\Sigma_1 - \Sigma_2)\) induces the factor
\[
\frac{d_1 - d_2}{\Sigma_1 - \Sigma_2} + 1,
\]
and we have
\[
\prod_{k=1}^{m} \left( \frac{k + \Sigma_2}{\Sigma_1 - \Sigma_2} \right)^7 = \prod_{k=1}^{m} \left( \frac{1 + \Sigma_2}{1 + \Sigma_2/k} \right) \cdot \left( 1 + \frac{\Sigma_1}{k} + \frac{\Sigma_2}{k^2} + \ldots \right)^7
\]
\[
= \prod_{k=1}^{m} \left( 1 + \frac{\Sigma_1 + \Sigma_2}{k} + \frac{\Sigma_1 \cdot \Sigma_2}{k^2} + \frac{(\Sigma_1 + \Sigma_2)^2}{k^2} + \ldots \right)^7
\]
\[
= 1 + 7 (\Sigma_1 + \Sigma_2) H_m + \frac{7 (\Sigma_1 + \Sigma_2)}{2} (\Sigma_1 - \Sigma_2) \left( \sum_{k=1}^{m} \frac{1}{k^2} \right) + \ldots ,
\]
\[(5.45)\]
\[
\prod_{k=1}^{n} \frac{1}{(k + \Sigma_1 + \Sigma_2)} = \frac{1}{(n!)^7} \left[ \frac{1}{1 + \frac{\Sigma_1 + \Sigma_2}{k}} \right]^7
\]
\[
= \frac{1}{(n!)^7} (1 - 7H_n (\Sigma_1 + \Sigma_2) + \ldots)
\]
and
\[
\prod_{k=m+1}^{m+n} \frac{(k + \Sigma_2)}{m!} = \left( \frac{(m + n)!}{m!} \right)^7 \left( 1 + 7 (H_{m+n} - H_m) \left( \frac{\Sigma_1 + \Sigma_2}{2} - \frac{\Sigma_1 - \Sigma_2}{2} \right) + \ldots \right).
\]

We can then read from (5.44)
\[
\Pi_0(t) = \sum_{n,m=0}^{+\infty} \frac{(-1)^m ((m + n)!)^7}{(n!)^7 (m!)^7} \left( 1 + \frac{7}{2} (2m + n + 1) (H_{m+n} - H_m) \right)
\cdot \left( e^{t_2(n+1)} e^{(t^2 - t^4)} + e^{t_1(n+1)} e^{(t^4 - t^2)} \right) \bigg|_{t_1 = t_2 = t} ,
\]
\[(5.46)\]
and

\[ \Pi_1(t) = -t\Pi_0(t) - \sum_{n,m=0} (-1)^m ((m + n)!)^7 \left( \frac{7H_m - 7H_n + \frac{7(H_{n+m} - H_m)}{2}}{(n!)^7 (m!)^7} \right) \]

\[ - \frac{7}{2} (2m + n + 1) \left( \sum_{k=1}^{m} \frac{1}{k^2} \right) + \frac{49}{2} (2m + n + 1) (H_m - H_n) (H_{m+n} - H_m) \]

\[ \cdot \left( e^{(n+1)t^2} e^{(t^2-t^1)t} + e^{(n+1)t^1} e^{(t^1-t^2)t} \right) \bigg|_{t^1=t^2=t}. \]

One can find that we have to sum over all \( m \) for a fixed degree \( n \) in the calculation, and

several series we may use in the computation, which are

\[ \frac{-e^{\delta}}{1+e^{\delta}} = \sum_{m=1}^{\infty} \left(-\right)^m \exp\left(m\delta\right), \]

\[ \left(\frac{-e^{\delta}}{1+e^{\delta}}\right)^{(k)} = \sum_{m=1}^{\infty} m^k \left(-\right)^m \exp\left(m\delta\right), \]

\[ \frac{d\text{Li}_2(z)}{dz} = -\frac{\ln\left(1-z\right)}{z}, \]

where \( \text{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \) is a special function called polylogarithm.

The first several terms of \( \Pi_0(t) \) and \( \Pi_1(t) \) are

\[ \Pi_0(t) = q^{-1} - 17q^{-2} + 1549q^{-3} - 215585q^{-4} + 36505501q^{-5} + \ldots, \]

\[ \Pi_1(t) = -t\Pi_0 + 70q^{-2} - 7413q^{-3} + \frac{3268573}{3} q^{-4} - \frac{1138372375}{6} q^{-5} + \ldots. \]

The mirror map is

\[ z^{-1} = e^{t+2\pi t_0} = e^{-\frac{\Pi_1(t)}{\Pi_0(t)} - 2\pi t_0}, \]

where we choose \( t_0 = \frac{1}{2} \),

\[ q^{-1} = -z^{-1} + 70z^{-2} - 1127z^{-3} + 47530z^{-4} + O(z^{-5}). \]

Recall the Yukawa-coupling is

\[ \langle (-\Sigma_1 - \Sigma_2)^3 \rangle = \frac{14 (3q^{-1} + 1)}{1 - 289q^{-1} - 57q^{-2} + q^{-3}}, \]

which is an analytic continuation to the Yukawa-coupling of \( Gr(2, 7)[1^7] \) up to an overall sign. By dividing the Yukawa-coupling by \( \Pi_0^2(t) \) and plugging in the mirror map, we then have

\[ K_{\sqrt{-1}z_1 \sqrt{-1}z_2} = 14 + 588z^{-1} + 97412z^{-2} + 15765456z^{-3} + 2647082116z^{-4} + \ldots. \]
Consequently, Gromov-Witten invariants of the Pfaffian Calabi-Yau threefold are

\[588, \quad 12103, \quad 583884, \quad 41359136, \quad \ldots\, .\]

(5.51)

Gromov-Witten invariants we computed agree with the results in [54, 55]. However, the generating function we proposed in equ’n (5.44) is a new result, which is not known both in physics and math literature before, so it would be interesting to understand the generating function from the mathematical point of view.

5.3 The determinantal Gulliksen-Negård Calabi-Yau threefold

Our last example, Gulliksen-Negård Calabi-Yau threefold [88], is realized by a model with two physical parameters. The gauged linear sigma model for this space was studied in [22]. The gauge group is \(U(1) \times U(2)\) with adjoint fields for each group denoted as \(\Sigma_0\) and \(\Sigma\). There are 8 chiral superfields \(\Phi_{\mu}\) of charge +1 under \(U(1)\), 4 chiral superfields \(P_i\) in the bifundamental representation of \(U(1) \times U(2)\), and 4 chiral superfields \(X_i\) in the anti-fundamental representation of \(U(2)\), and a superpotential

\[W = \text{tr}(PA(\Phi)X),\]

where \(A(\Phi) = \sum_{\mu=1}^{8} A^\mu \Phi_{\mu}\) and \(A^\mu\) are 8 constant \(4 \times 4\) matrices. From the superpotential, one may not be surprised that it is called the PAX model [22]. We denote the FI-parameters by \(r^0, r^1\) and \(r^2\) respectively. To recover the physical parameter, we shall set \(r^1 = r^2 = r\). In this paper, we focus on the computation of Gromov-Witten invariants in the geometric phase I of the GLSM, corresponding to the subregion constrained by \(r^0 + r^1 + r^2 > 0\), \(r^1 > 0\) and \(r^2 > 0\) in the FI parameter space.

The mirror Landau-Ginzburg model has the superpotential

\[\begin{align*}
W &= \Sigma_0 \left( \sum_{\mu=1}^{8} Y_{\mu} - 4 \sum_{i=1}^{4} (Y^1_{P_i} + Y^2_{P_i}) - t^0 \right) \\
&\quad + \Sigma_1 \left( \sum_{i=1}^{4} Y^1_{P_i} + \sum_{i=1}^{8} \log X^1_i + \log X_1 - \log X_2 - t^1 \right) \\
&\quad + \Sigma_2 \left( \sum_{i=1}^{4} Y^2_{P_i} + \sum_{i=1}^{8} \log X^2_i + \log X_2 - \log X_1 - t^2 \right) \\
&\quad + \sum_{\mu=1}^{8} e^{-Y_{\mu}} + \sum_{i=1}^{4} \left( e^{-Y^1_{P_i}} + e^{-Y^2_{P_i}} \right) + \sum_{i=1}^{8} (X^1_i + X^2_i) + X_1 + X_2.
\end{align*}\]

(5.52)

Following the same procedure as before, one get the following generating series in the
We then obtain

\[ \Pi_{GN} (\Sigma_1, \Sigma_2, t^0, t) \]

\[ = -\partial_1 + \partial_2 \left| \Sigma_1 - \Sigma_2 \right| \sum_{i_1=t_1=t, d_{1,1}, d_2=0}^{\infty} \cdot \prod_{k=1}^{d_{1}+d_{2}} (\Sigma_1 + k)^4 \prod_{k=1}^{d_{2}+d_{2}} (\Sigma_2 + k)^4 \prod_{k=1}^{d_{1}+d} (\Sigma_1 + k)^4 \prod_{k=1}^{d_{2}+d} (\Sigma_2 + k)^4 \]

\[ \cdot \exp \left( - (\Sigma_0 + d) t^0 - (\Sigma_1 + d_1) (t_1 + i\pi) - (\Sigma_2 + d_2) (t_2 + i\pi) \right). \]

As discussed in [22], a basis of the cohomology subspace \( H^{1,1} \) consists of two divisors represented by \( \Sigma_0 \) and \( \Sigma_1 + \Sigma_2 - 2\Sigma_0 \). If we define the new variables

\[ \tilde{\Sigma}_1 \equiv \Sigma_1 - \Sigma_0, \quad \tilde{\Sigma}_2 \equiv \Sigma_2 - \Sigma_0, \]

then eqn (5.53) can be written as

\[ \Pi_{GN} (\Sigma_1, \Sigma_2, t^0, t) \]

\[ = -\partial_1 + \partial_2 \left| \Sigma_1 - \Sigma_2 \right| \sum_{i_1=t_1=t, d_{1,1}, d_2=0}^{\infty} \cdot \prod_{k=1}^{d_{1}+d_{2}} (\tilde{\Sigma}_1 + \Sigma_0 + k)^4 \prod_{k=1}^{d_{2}+d_{2}} (\tilde{\Sigma}_2 + \Sigma_0 + k)^4 \prod_{k=1}^{d_{1}+d} (\tilde{\Sigma}_1 + k)^4 \prod_{k=1}^{d_{2}+d} (\tilde{\Sigma}_2 + k)^4 \]

\[ \cdot \exp \left( - (\Sigma_0 + d) t^0 - (\tilde{\Sigma}_1 + d_1 - d) (t_1 + i\pi) - (\tilde{\Sigma}_2 + d_2 - d) (t_2 + i\pi) \right). \]

We expand \( \Pi_{GN} (\Sigma_1, \Sigma_2, t^0, t) \) in terms of Schubert classes

\[ \Pi_{GN} (\Sigma_1, \Sigma_2, t^0, t) = \Pi_{GN,0} (t^0, t) + \Pi_{GN,11} (t^0, t) \Sigma + \Pi_{GN,12} (t^0, t) (\Sigma_1 + \Sigma_2 - 2\Sigma) + \cdots. \]

Notice the following expansion,

\[ \prod_{k=1}^{d} (\Sigma_0 + k)^{-8} = 1 - 8\Sigma_0 H_d + \cdots, \]

\[ \prod_{k=1}^{d_{a}} (\tilde{\Sigma}_a + k)^{-4} = 1 - 4\tilde{\Sigma}_a H_{d_{a}} + \tilde{\Sigma}_a^2 \left( 8H_{d_{a}}^2 + 2\sum_{k=1}^{d_{a}} \frac{1}{k^2} \right) \cdots, \quad \text{for} \quad a = 1, 2, \]

\[ \prod_{k=1}^{d_{a}+d} (\tilde{\Sigma}_a + \Sigma_0 + k)^4 = 1 + 4 \left( \tilde{\Sigma}_a + \Sigma_0 \right) H_{d_{a}+d} + \left( \tilde{\Sigma}_a + \Sigma_0 \right)^2 \left( 8H_{d_{a}+d}^2 - 2\sum_{k=1}^{d_{a}+d} \frac{1}{k^2} \right) + \cdots. \]

We then obtain

\[ \Pi_0 = \left( \frac{(d_1 + d)!}{(d_1)!} \right)^4 \left( \frac{(d_2 + d)!}{(d_2)!} \right)^4 \cdot e^{-d_0 \tau_0} e^{-d_0 (t_1 + i\pi)} \cdot (1 + 2 (d_1 - d_2) (H_{d_{1}+d} - H_{d_{2}+d} - H_{d_{1}} + H_{d_{2}})). \]
\[ \Pi_{11} = - (t^0 + t^1 + t^2) \Pi_0 + \frac{((d_1 + d)!)^4 ((d_2 + d)!)^4}{(d!)^8 ((d_1)!)^4 ((d_2)!)^4} \cdot e^{-d \cdot t^0} e^{-(d_1 + d_2)(t + i \pi)} \cdot \left( \sum (4H_{d_1+d} + 4H_{d_2+d} - 8H_d) + \sum \cdot (d_1 - d_2) \left( 8 \left( H_{d_1+d}^2 - H_{d_2+d}^2 \right) - 2 \left( \sum_{k=1}^{d_1+d} \frac{1}{k^2} - \sum_{k=1}^{d_2+d} \frac{1}{k^2} \right) \right) \right) + 16H_d (H_{d_1} - H_{d_2} - H_{d_1+d} + H_{d_2+d}) + 8 (H_{d_1+d} + H_{d_2+d}) \cdot (H_{d_2} - H_{d_1}) , \]

and

\[ \Pi_{12} = - \left( \frac{t^1 + t^2}{2} + i \pi \right) \Pi_0 + \frac{((d_1 + d)!)^4 ((d_2 + d)!)^4}{(d!)^8 ((d_1)!)^4 ((d_2)!)^4} \cdot e^{-d \cdot t^0} e^{-(d_1 + d_2)(t + i \pi)} \cdot \left( (\Sigma_1 + \Sigma_2 - 2\Sigma) (2H_{d_1+d} + 2H_{d_2+d} - 2H_{d_1} - 2H_d) \right) + (\Sigma_1 + \Sigma_2 - 2\Sigma) \cdot (d_1 - d_2) \left( 4 \left( H_{d_1+d}^2 - H_{d_2+d}^2 \right) - \left( \sum_{k=1}^{d_1+d} \frac{1}{k^2} - \sum_{k=1}^{d_2+d} \frac{1}{k^2} \right) \right) \right) + 4H_{d_1}^2 + \sum_{k=1}^{d_1} \frac{1}{k^2} - 4H_{d_2}^2 - \sum_{k=1}^{d_2} \frac{1}{k^2} - 4 (H_{d_1+d} + H_{d_2+d}) (H_{d_1} - H_{d_2}) \left. - 4 (H_{d_1+d} + H_{d_2+d}) (H_{d_1+d} - H_{d_2+d}) \right) . \]

As in [22], we use the notation \( w = -q \) and \( z = q^0 w^2 \). The series expansions of \( \Pi_0, \Pi_{11} \) and \( \Pi_{12} \) read

\[ \Pi_0 (w, z) = 1 + 2w + z + 3w^2 + z^2 + 4w^3 - 14w^2z - 54wz^2 + z^3 + \ldots , \]

\[ \Pi_{11} (w, z) = - (t^0 + t^1 + t^2) \Pi_0 + 4w + 10w^2 + \frac{52}{3} w^3 - 16wz - 110w^2z - 108wz^2 + \ldots , \]

\[ \Pi_{12} (w, z) = - \left( \frac{t^1 + t^2}{2} + i \pi \right) \Pi_0 + 4z + 6z^2 + 24wz + \frac{22}{3} z^3 + 80w^2z - 108wz^2 + \ldots . \]

The mirror maps can be defined as

\[ \tau^0 = - \log z - 4w - 2w^2 - \frac{4}{3} w^3 + 20wz + 72w^2z + 92wz^2 + \ldots , \]

\[ \tau^1 = - \log (-w) - 4z - 2z^2 - \frac{4}{3} z^3 - 16wz - 36w^2z + 128wz^2 + \ldots . \]

One can find that the above two flat coordinates are exactly the same flat coordinates obtained from the two-sphere partition function of the GLSM for Gulliksen-Negård Calabi-Yau in [23]. Because Yukawa-couplings of Gulliksen-Negård Calabi-Yau threefold have been computed in [32], one can then plug the mirror maps into Yukawa-couplings to get Gromov-Witten invariants, and the result should agree with Gromov-Witten invariants computed in [23] with a different method.
6 Conclusions and future directions

In this paper, we proposed Picard-Fuchs equations for periods \[47\] defined in nonabelian mirrors \[1\] of nonabelian GLSMs. Our Picard-Fuchs equations are not traditional because the number of parameters in differential equations is the same as the rank of the gauge group of the GLSM, although eventually, we recover the actual number of Kähler parameters. We verified our proposal by reproducing known Picard-Fuchs equations of Grassmannians as well as complete intersection Calabi-Yau manifolds in Grassmannians studied in \[2\]. Furthermore, one can apply our method to any target admitting a UV GLSM-description. Thus we obtained ordinary Picard-Fuchs equations for cases like \(SG(k, 2n)\) and \(Fl(k_1, \ldots, k_n, N)\) from our Picard-Fuchs equations. To our knowledge, they have not been computed in the literature before. From our Picard-Fuchs equations, we proposed cohomology-valued generating functions for mirrors of Fanos such as \(Gr(k, N)\) and \(SG(k, 2N)\), and found agreement with \[78\]. We also proposed cohomology-valued generating functions for mirrors of various Calabi-Yau manifolds, including complete intersections in Grassmannians as well as non-complete intersection Calabi-Yau manifolds: Pfaffian Calabi-Yau threefold and Gulliksen-Negård Calabi-Yau threefold. Then we found Gromov-Witten invariants of these examples computed by our method match results obtained in \[2, 23, 54, 55\]. To the best of our knowledge, generating functions of non-complete intersection Calabi-Yau manifolds are genuinely new results.

As reviewed in section 3, each nonabelian GLSM has an abelian-like effective theory under the RG-flow. Following the terminology of \[39\], this effective theory is associated Cartan theory. This idea is also called nonabelian/abelian correspondence in the literature, which has been studied comprehensively in several aspects. See \[74\] for the correspondence at the level of cohomology. And the correspondence of genus-zero Gromov-Witten invariants was conjectured in \[78\]. Furthermore, I-functions of nonabelian GIT quotients have been studied in \[89\] following the nonabelian/abelian correspondence. Those works only include Fano-spaces and complete intersections in Fanos. Thus, it would be interesting to translate our results for non-complete intersections into a precise mathematical conjecture, which could be proved as a “nonabelian mirror theorem” in mathematics. It is also worth noting that mathematicians have studied higher-genus GLSMs, see \[91, 93\] and references therein. However, the full study of higher-genus GLSMs is still open in physics. Combined with existing tools \[94\], our approach can be generalized to the study of higher genus invariants. We hope to elucidate these directions in future work.

Our periods are defined in mirror Landau-Ginzburg models \[47\]. Although A-cycles in mirrors do depend on Kähler moduli, the period in our computation may shed light on mirror manifolds of nonabelian Calabi-Yau manifolds because it does not rely on the Kähler moduli. The existence of mirror manifold provides a geometrical understanding of nonabelian mirror symmetry \[38\]. If the mirror Calabi-Yau manifold has a GLSM-description, one can compute exact results from the GLSM for the mirror and probably uncover more dualities between gauge theories.
Finally, we want to mention that physicists are primarily interested in the heterotic Gromov-Witten theory in string compactification \([95]\). There is some progress in understanding heterotic mirror symmetry in the past few years; see \([96,97]\) and \([63,98]\). However, a full understanding of heterotic mirror symmetry is still missing. Furthermore, there is no heterotic analogue of Gromov-Witten invariant that has ever been computed in the literature. We hope our approach would shed some light on this direction.

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### A A-branes in Landau-Ginzburg models

To compute the period in the Landau-Ginzburg model, we need to define the A-brane first. Recall the Landau-Ginzburg model is defined by a non-compact Calabi-Yau manifold and a holomorphic superpotential \(W\), and this non-compact Calabi-Yau manifold does not have a nontrivial D-term. If the superpotential has no poles, the LG model is a fundamental theory, which one can define the A-brane in the UV, where the superpotential can be treated as a perturbation. Therefore, we first define A-branes for free targets \(\mathbb{C}^n\) and \((\mathbb{C}^*)^n\).

**Special Lagrangians.** For a Calabi-Yau target space \(Y\), the special Lagrangian submanifold is an oriented \(10\) Lagrangian submanifold \(\gamma \subset Y\) satisfying

\[
\text{Im} e^{-i\pi \xi(\gamma)} \Omega |_\gamma = 0 \tag{A.1}
\]

for some real constant \(\xi(\gamma)\).

- A-branes in \(\mathbb{C}^n\) are defined by the following submanifolds. Consider a linear subspace \(\mathbb{R}^n \subset \mathbb{C}^n\), and we can specify its embedding in terms of angles \(\theta_i\), as

\[
\gamma^\vartheta = \{ (e^{i\theta_1}x_1, e^{i\theta_2}x_2, \ldots, e^{i\theta_n}x_n) : x_i \in \mathbb{R} \}.
\]

The holomorphic n-form \(\Omega = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n\) can be written as

\[
\Omega = e^{i(\theta_1 + \cdots + \theta_n)} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.
\]

A-branes shall obey equation \((A.1)\). This gives

\[
\xi(\gamma) = \frac{1}{\pi} \sum_{i=1}^{n} \theta_i.
\]

\(10\)It means we have chosen a non-vanishing top form on the submanifold.
We denote by $\gamma^\partial$ the plane with orientation $dx_1 \wedge \cdots \wedge dx_n$, embedding $\theta_1 = \cdots = \theta_n = 0$ and thus $\xi(\gamma) = 0$.

- A-branes of non-compact Calabi-Yau $(\mathbb{C}^*)^n = \prod_{i=1}^n (\rho_i + i\theta_i)$ can be defined straightforwardly
  \[ \gamma^\partial = \{ (\rho_1 + i\theta_1, \rho_2 + i\theta_2, \cdots, \rho_n + i\theta_n) : \rho_i \in \mathbb{R} \} \quad (A.2) \]
  For this target, $\xi(\gamma)$ vanishes.

However, one can not treat the superpotential in the Landau-Ginzburg model as a perturbation if the superpotential has poles. Thus one can not use a UV perturbative theory to define A-branes of the Landau-Ginzburg model. In this situation, one needs to search for a nonperturbative definition of A-branes in the Landau-Ginzburg model. Fortunately, because of the non-renormalization theorem in the LG model, one can define A-branes exactly in Landau-Ginzburg models [37, 47].

- A-branes in an LG model: $((\mathbb{C}^*)^m \times \mathbb{C}^n, W)$. The Lagrangian submanifold whose image in the $W$-plane must be parallel to the real line, or equivalently
  \[ \text{Im}W = \text{constant} \quad (A.3) \]
  If we assume imaginary parts of critical values are all distinct, then there is a canonical choice for A-branes in the $W$-plane, which corresponds to semi-infinite lines starting from critical values and parallel to the real axis. One can find more details on this constraint from the boundary condition of open string in [47]. However, one can relax the constraint (A.3) by modifying the Lagrangian with a suitable boundary interaction, leading to a non-standard one. In this paper, the Landau-Ginzburg model we consider has $\Sigma$ fields with other matter fields. Following [50, 51], we choose a Lagrangian submanifold in $\Sigma$ field space, which reduces the computation of periods to contour-integrals, while the Lagrangian submanifolds in other field spaces still obey the constraint in (A.3). The final result shall not depend on our particular choice.

The mirror of GLSM for $\mathbb{C}\mathbb{P}^{N-1}$. We have the Landau-Ginzburg model whose target space is $(\mathbb{C}^*)^N \times \mathbb{C}$ with superpotential,

\[ W = \Sigma \left( \sum_{i=1}^N Y_i - t \right) + \sum_{i=1}^N e^{-Y_i}. \]

The critical loci can be computed by solving vacuum equations

\[ \partial Y_i W = 0, \]

which gives

\[ \Sigma = X_i = e^{-Y_i}, \quad \Sigma^N = X_i^N = q. \]
Thus Lagrangian submanifolds can be chosen to be
\[
\gamma = \left\{ (L, \rho_1 + i \left( \frac{\theta}{N} + \frac{2\pi k}{N} \right), \ldots, \rho_N + i \left( \frac{\theta}{N} + \frac{2\pi k}{N} \right)) : \rho_i \in (-\infty, \frac{r}{N}) \right\}.
\]

Following [51], we will not specify \(L\) in the field space of \(\Sigma\), but it obeys constraints discussed in [51]. Because of the additional constraint from the superpotential, one can quickly notice that the nonperturbative definition of Lagrangian submanifolds agrees with a subclass of Lagrangian submanifolds in the perturbative theory (A.2) as expected at the large volume limit.

Mirror of GLSM for \(Gr(k, N)\). The mirror of GLSM for \(Gr(k, N)\) is \((\mathbb{C}^*)^{N-k} \times \mathbb{C}^k / S_k\) with superpotential,
\[
W = \sum_a \left( \sum_{i=1}^N Y_i^a + \alpha^a_{\mu\nu} \log X_{\mu\nu} - t \right) + \sum_{i=1}^N \sum_{a=1}^k e^{-Y_i^a} + \sum_{\mu\nu} X_{\mu\nu}.
\]

Solving vacuum equations
\[
\partial_{Y_i^a} W = \partial_{X_{\mu\nu}} W = 0
\]
gives the critical loci
\[
\Sigma_a = X_i^a = e^{-Y_i^a}, \quad X_{\mu\nu} = \Sigma_\mu - \Sigma_\nu \neq 0 \quad \Sigma_a^N = (X_i^a)^N = (-1)^{k-1} q.
\]

It seems that one can naturally choose Lagrangian submanifolds in this LG model as
\[
\gamma = \left\{ (L, \rho_1^a + i \left( \frac{\theta}{N} + \frac{2\pi k^a + \pi (k-1)}{N} \right), \ldots, \rho_N^a + i \left( \frac{\theta}{N} + \frac{2\pi k^a + \pi (k-1)}{N} \right)) : \rho_i^a \in (-\infty, \frac{r}{N}), \quad \rho \in [0, +\infty), \quad k^a \neq k^b \text{ if } a \neq b \right\}.
\]

Again, we will not specify \(L\) in the \(\Sigma_a\)-field space. This is not the end of the story, as our Landau-Ginzburg model is an \(S_k\) orbifold theory, which means Lagrangian submanifolds should be gauge-invariant. From equation (A.1), it looks like we should need the holomorphic top form over the field space to be gauge-invariant. Fortunately, the top-form of the LG model is gauge-invariant under the Weyl-group action only up to a change in orientation
\[
S_k \cdot \Omega = \pm \Omega.
\]

If the orientation is preserved, configurations defined in (A.4) are well-defined physical A-branes. If the orientation is not gauge-invariant, we need to do a little bit more here. We first impose a gauge fixing for the orientation of \(L\)
\[
S_k \cdot L = L.
\]
We also require

\[
S_k \cdot \left( \rho_1^a + i \left( \frac{\theta}{N} + \frac{2\pi k^a + \pi (k - 1)}{N} \right) \right), \cdots , \rho_N^a + i \left( \frac{\theta}{N} + \frac{2\pi k^a + \pi (k - 1)}{N} \right), \cdots , \rho_2^a + i \left( \frac{\theta}{N} + \frac{2\pi k^a + \pi (k - 1)}{N} \right)
\]

which means \( \wedge^a d\Sigma_a \wedge^a dX^a \) is gauge-invariant under the Weyl-group action. We then only need to worry about \( X_{\mu\nu} \) fields

\[
S_k \cdot (\wedge^\mu dX^\mu) = - \wedge^\mu dX^\mu.
\]

Fortunately, the superpotential has no critical loci intersecting the orbifold fixed point. Expressing \( X_{\mu\nu} \) in polar coordinates \( X_{\mu\nu} = \rho_{\mu\nu} e^{i\theta_{\mu\nu}} \), then the orientation reversing of \( \wedge^\mu dX^\mu \) under the Weyl-group action suggests the following identification

\[
\sum_{\mu\nu} \theta_{\mu\nu} \equiv \sum_{\mu\nu} \theta_{\mu\nu} + k\pi, \quad k = 2Z + 1. \tag{A.5}
\]

Langrangian submanifolds, now, can be defined. The difference here is that the fiber over \( \gamma \) is defined by formula (A.5) rather than the usual torus fibration in geometry \[38\]. This is not an issue in our mirror, as \( X_{\mu\nu} \) are heavy matters, which shall be integrated out in the mirror of a geometric target. Finally, one can check that the configuration in (A.4) lies on this novel fiber bundle at the large volume limit.

One may also suspect that integrating out \( \Sigma_a \) fields alone might not be well-defined. Fortunately, this is not the case, as in a physical mirror, one needs to integrate out antiholomorphic fields \( \Sigma_a \) along with holomorphic \( \Sigma_a \) fields altogether, which are gauge-invariant. The effective LG theory may not have a non-vanishing holomorphic top-form on the field space, but the holomorphic top-form is changed only up to an orientation under the Weyl-group action. Thus one can still apply B-twist to the effective LG theory \[99\] and define low energy effective A-branes as we discussed here. Finally, one can easily extend the study of A-branes of nonabelian mirrors to other Fano spaces and Calabi-Yau manifolds.

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