The Orbifold Cohomology Ring of Simplicial Toric Stack Bundles

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Abstract

We introduce extended toric Deligne-Mumford stacks. We use an extended toric Deligne-Mumford stack to get the toric stack bundle and compute its orbifold Chow ring. Finally we generalize one result of Borisov, Chen and Smith so that the orbifold Chow ring of the toric stack bundle and the Chow ring of its crepant resolution are fibres of a flat family.

1 Introduction

The Chen-Ruan orbifold cohomology was constructed by the genus zero and degree zero orbifold Gromov-Witten invariants of Deligne-Mumford stacks, see [5],[6],[1]. In this paper we discuss the case of a toric stack bundle. The orbifold Chow ring of the general toric Deligne-Mumford stack was obtained by Borisov, Chen and Smith [4]. See also [11] for the case of weighted projective space. Let $\Sigma := (N, \Sigma, \beta)$ be a stacky fan. The toric Deligne-Mumford stack $\mathcal{X}(\Sigma) = [Z/G]$ is a quotient stack, where $Z = \mathbb{C}^n - V(J_\Sigma)$ and $J_\Sigma$ is the square-free ideal of $\Sigma$. The action of $G$ on $Z$ is through the map $\alpha : G \to (\mathbb{C}^\times)^n$ determined by the stacky fan. Let $P \to B$ be a principal $(\mathbb{C}^\times)^n$-bundle over a smooth variety $B$, define $P\mathcal{X}(\Sigma)$ to be the quotient stack $[(P \times (\mathbb{C}^\times)^n Z)/G]$, where $G$ acts on $P$ trivially. The stack $[(P \times (\mathbb{C}^\times)^n Z)/G]$ may be written as $P \times (\mathbb{C}^\times)^n [Z/G]$, then $P\mathcal{X}(\Sigma) \to B$ is a toric stack bundle over $B$ with fibre the toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$. We study the orbifold Chow ring of $P\mathcal{X}(\Sigma)$.

Before we go further, let’s consider the case when $N$ is a finite abelian group, then in the stacky fan $\Sigma = (N, \Sigma, \beta)$, $\Sigma = 0$ and $\beta$ is the zero homomorphism $0 \to N$. The toric Deligne-Mumford stack $\mathcal{X}(\Sigma) = [pt/G]$ is the classifying stack $B\underline{G}$, where $G = Hom_\mathbb{Z}(N, \mathbb{C}^\times)$. As a stack $B\underline{G}$ can have different representations. For example, if $N = \mathbb{Z}/3\mathbb{Z}$, $G = \mu_3$ is the cyclic group of order 3, then $B\mu_3$ can also be represented by the quotient stack $[\mathbb{C}^\times/\mathbb{C}^\times]$, where the action is given by $(\cdot)^3$. The stack $[\mathbb{C}^\times/\mathbb{C}^\times]$ is not a toric Deligne-Mumford stack in the sense of [4]. If $B\mu_3 = [\mathbb{C}^\times/\mathbb{C}^\times]$, let $L \to B$ be a line bundle over a smooth variety $B$, then $[(L^\times \times_{\mathbb{C}^\times} \mathbb{C}^\times)/\mathbb{C}^\times]$ is a nontrivial $\mu_3$-gerbe over $B$ if the line bundle is nontrivial. While if $B\mu_3 = [pt/\mu_3]$, we can’t twist it by any line bundle.
In order to make $B_{\mu_3} = [\mathbb{C}^\times/\mathbb{C}^\times]$ a toric Deligne-Mumford stack, we slightly generalize the construction of toric Deligne-Mumford stacks. We introduce \emph{extended stacky fan} $\Sigma^e := (N, \Sigma, \beta^e)$, where $N$ and $\Sigma$ are the same as in a stacky fan $\Sigma = (N, \Sigma, \beta)$, but $\beta^e : \mathbb{Z}^m \to N$ is determined by $b_1, \ldots, b_n, b_{n+1}, \ldots, b_m \in N$ satisfying the conditions that $m \geq n$, $\overline{b}_i$ generates the ray $\rho_i$ for $1 \leq i \leq n$ and all the other data $\{b_{n+1}, \ldots, b_m\}$ belong to $N$, where $\overline{b}_i \in \overline{N}$, and $\overline{N}$ is the lattice in $N_0$ determined by the projection $N \to \overline{N}$. We call $\{b_{n+1}, \ldots, b_m\}$ the extra data in $\Sigma^e$. Associated to an extended stacky fan $\Sigma^e$, we define the extended toric Deligne-Mumford stack $\mathcal{X}(\Sigma^e) := [Z^e/G^e]$ as a quotient stack, where $Z^e = Z \times (\mathbb{C}^\times)^{m-n}$ and $G^e$ acts on $Z^e$ through the homomorphism $\alpha^e : G^e \to (\mathbb{C}^\times)^m$ determined by the extended stacky fan. It is easy to see that every extended stacky fan $\Sigma^e$ naturally determines a stacky fan $\Sigma$. We prove that the extended toric Deligne-Mumford stack $\mathcal{X}(\Sigma^e)$ is isomorphic to the underlying toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$. But we have more freedom to twist as our example above shows.

Given an extended toric Deligne-Mumford stack $\mathcal{X}(\Sigma^e)$, from the extended stacky fan $\Sigma^e$, we have the following exact sequence:

$$1 \to \mu \to G^e \xrightarrow{\alpha^e} (\mathbb{C}^\times)^m \to T \to 1$$

where $T = (\mathbb{C}^\times)^d$. Let $P \to B$ be a principal $(\mathbb{C}^\times)^m$-bundle, let $P, \mathcal{X}(\Sigma^e)$ be the quotient stack $[P \times (\mathbb{C}^\times)^m Z^e/G^e]$, where $G^e$ acts on $P$ trivially and on $(\mathbb{C}^\times)^m$ through the map $\alpha^e$ in the above exact sequence. Then $P, \mathcal{X}(\Sigma^e)$ is a toric stack bundle over $B$ with fibre the extended toric Deligne-Mumford stack $\mathcal{X}(\Sigma^e)$. The extra data $\{b_{n+1}, \ldots, b_m\}$ in $\Sigma^e$ can be put into the $B_{\mu_3}(\Sigma^e)$ which do not influence the structure of the toric stack bundle $P, \mathcal{X}(\Sigma^e)$. The choice of torsion and nontorsion extra data does affect the structure of $P, \mathcal{X}(\Sigma^e)$, but does not affect the orbifold cohomology. To describe the orbifold Chow ring of $P, \mathcal{X}(\Sigma^e)$, we introduce some line bundles over $B$. Let $M = N^*$ be the dual of $N$. For $\theta \in M$, let $\xi_\theta \to B$ be the line bundle coming from the principal $T$ bundle $E \to B$ by "extending" the structure group via $\chi^\theta : T \to \mathbb{C}^\times$, where $E \to B$ is induced from the $(\mathbb{C}^\times)^m$-bundle $P$ in the above exact sequence. Define the deformed ring $A^*(B)[N]\Sigma^e = A^*(B) \otimes \mathbb{Q}[N]\Sigma^e$ where $\mathbb{Q}[N]\Sigma^e := \bigoplus_{e \in N} y^e$, $y$ is the formal variable and $A^*(B)$ is the Chow ring of $B$. The multiplication of $\mathbb{Q}[N]\Sigma^e$ is given by:

$$y^{e_1} y^{e_2} := \begin{cases} y^{e_1+e_2} & \text{if there is a cone } \sigma \in \Sigma \text{ such that } \overline{c}_1 \in \sigma, \overline{c}_2 \in \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Let $I(\Sigma^e)$ be the ideal in $A^*(B)[N]\Sigma^e$ generated by the elements:

$$\left( c_1(\xi_\theta) + \sum_{i=1}^n \theta(b_i) y^{b_i} \right)_{\theta \in M}$$

for $\theta \in M$ and $A^*_\text{orb}(P, \mathcal{X}(\Sigma^e))$ be the orbifold Chow ring of the toric stack bundle. Then we have the following Theorem:
Theorem 1.1 If $P\mathcal{X}(\Sigma^e) \to B$ is a toric stack bundle over a smooth variety $B$ with fibre the extended toric Deligne-Mumford stack $\mathcal{X}(\Sigma^e)$ associated to an extended stacky fan $\Sigma^e$, then we have an isomorphism of $\mathbb{Q}$-graded rings:

$$A^*_{\text{orb}}(P\mathcal{X}(\Sigma^e)) \cong A^*(B)[N]_{\mathcal{I}(\Sigma^e)} \Sigma^e$$

To prove this theorem, first using the similar result in [4] that the components of the inertia stack $\mathcal{I}(\mathcal{X}(\Sigma^e))$ of $\mathcal{X}(\Sigma^e)$ is given by $\text{Box}(\Sigma^e)$ which determines all the elements in the local group of $\mathcal{X}(\Sigma^e)$, we explain that the twist by the $(\mathbb{C}^\times)^m$-bundle $P$ does not twist the components of the inertia stack of the toric stack bundle $P\mathcal{X}(\Sigma^e)$. Then this makes it possible to use the similar methods as in [4] to determine 3-twisted sectors, obstruction bundles of $P\mathcal{X}(\Sigma^e)$ and compute the orbifold Chow ring of $P\mathcal{X}(\Sigma^e)$. As an example, let $N$ be a finite abelian group and $\mathbb{Z} \to N$ be any homomorphism. Then $\Sigma^e = (N, 0, \beta^e)$ is an extended stacky fan, and $\mathcal{X}(\Sigma^e) = B\mu$, where $\mu = Hom(N, \mathbb{C}^\times)$. Twist this extended toric Deligne-Mumford stack by a line bundle $L$ over a smooth variety $B$, we get the $\mu$-gerbe over $B$. We determine its inertia stack and compute its orbifold Chow ring.

The paper is organized as follows. In Section 2 we introduce extended toric Deligne-Mumford stacks. In Section 3 we define the toric stack bundle and discuss its properties. In Section 4 we describe the orbifold Chow ring of the toric stack bundle. In Section 5 we give an interesting example of toric stack bundle, the $\mu$-gerbe $\mathcal{X}$ over $B$ for a finite abelian group $\mu$ and smooth variety $B$. Finally in Section 6 we give some applications of crepant resolutions.

In this paper, we use the rational numbers $\mathbb{Q}$ as coefficients of the Chow ring and orbifold Chow ring. By an orbifold we mean a smooth Deligne-Mumford stack such that at the generic point the automorphism group is trivial. This type of orbifold is sometimes called a reduced orbifold in differential geometry.

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2 The Extended Toric Deligne-Mumford stacks.

In this section we introduce extended stacky fans and construct extended toric Deligne-Mumford stacks. We prove that the extended toric Deligne-Mumford stack is isomorphic to the underlying toric Deligne-Mumford stack.

We refer to [4] the construction and notation of toric Deligne-Mumford stacks. Let $N$ be a finitely generated abelian group of rank $d$. Let $\overline{N}$ be the lattice generated by $N$ in the $d$-dimensional vector space $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$. The natural map $N \to \overline{N}$ is denoted by $b \mapsto \overline{b}$. Let $\Sigma$ be a rational simplicial fan in $N_{\mathbb{Q}}$. Suppose $\rho_1, \ldots, \rho_n$ are the rays in $\Sigma$. We fix $b_i \in N$ for $1 \leq i \leq n$ such that $b_i$ generates the cone $\rho_i$. We choose extra data \{\(b_{n+1}, \ldots, b_m\)\} \(\subset N\)
and consider the homomorphism \( \beta^e : \mathbb{Z}^m \to N \) determined by the elements \( b_1, \ldots, b_m \). We require that \( \beta^e \) has finite cokernel. \( \Sigma^e := (N, \Sigma, \beta^e) \) is called an extended stacky fan.

It is easy to see that any extended stacky fan \( \Sigma^e = (N, \Sigma, \beta^e) \) naturally determines a stacky fan \( \Sigma := (N, \Sigma, \beta) \), where \( \beta : \mathbb{Z}^n \to N \) is given by \( b_1, \ldots, b_n \in N \). Now since \( \beta^e \) has finite cokernel, from Proposition 2.2 in [4], we have exact sequences:

\[
0 \to DG(\beta^e)^* \to \mathbb{Z}^m \xrightarrow{\beta^e} N \to \text{Coker}(\beta^e) \to 0
\]

\[
0 \to N^* \to \mathbb{Z}^m (\beta^e)^\vee \to DG(\beta^e) \to \text{Coker}(\beta^e)^\vee \to 0
\]

where \( (\beta^e)^\vee \) is the Gale dual of \( \beta^e \). As a \( \mathbb{Z} \)-module, \( \mathbb{C}^\times \) is divisible, so it is an injective \( \mathbb{Z} \)-module, from [13], the functor \( \text{Hom}_\mathbb{Z}(-, \mathbb{C}^\times) \) is exact. We get the exact sequence:

\[
1 \to \text{Hom}_\mathbb{Z}(\text{Coker}(\beta^e)^\vee, \mathbb{C}^\times) \to \text{Hom}_\mathbb{Z}(DG(\beta^e), \mathbb{C}^\times) \to \text{Hom}_\mathbb{Z}(\mathbb{Z}^m, \mathbb{C}^\times) \to \text{Hom}_\mathbb{Z}(N^*, \mathbb{C}^\times) \to 1
\]

Let \( \mu := \text{Hom}_\mathbb{Z}(\text{Coker}(\beta^e)^\vee, \mathbb{C}^\times) \), we have the exact sequence:

\[
1 \to \mu \to G^e \xrightarrow{\alpha^e} (\mathbb{C}^\times)^m \to T \to 1 \tag{3}
\]

From [4], the toric Deligne-Mumford stack \( \mathcal{X}(\Sigma) = [Z/G] \) is a quotient stack, where they use the method of quotient construction of toric varieties [7]. Define \( Z^e := Z \times (\mathbb{C}^\times)^{m-n} \), then there exists a natural action of \( (\mathbb{C}^\times)^m \) on \( Z^e \). The group \( G^e \) acts on \( Z^e \) through the map \( \alpha^e \) in \( \mathfrak{3} \). The quotient stack \( [Z^e/G^e] \) is associated to the groupoid \( Z^e \times G^e \rightrightarrows Z^e \). Define the morphism \( \varphi : Z^e \times G^e \to Z^e \times Z^e \) to be \( \varphi(x, g) = (x, g \cdot x) \). Since \( Z^e = Z \times (\mathbb{C}^\times)^{m-n} \), we can mimic the proof the Lemma 3.1 in [4] to get that \( \varphi \) is finite. So the stack \( [Z^e/G^e] \) is a Deligne-Mumford stack.

**Lemma 2.1** The morphism \( \varphi : Z^e \times G^e \to Z^e \times Z^e \) is a finite morphism.

**Definition 2.2** For an extended stacky fan \( \Sigma^e = (N, \Sigma, \beta^e) \), define the extended toric Deligne-Mumford stack \( \mathcal{X}(\Sigma^e) \) to be the quotient stack \( [Z^e/G^e] \).

**Proposition 2.3** For an extended stacky fan \( \Sigma^e = (N, \Sigma, \beta^e) \), the extended toric Deligne-Mumford stack \( \mathcal{X}(\Sigma^e) \) is isomorphic to the underlying toric Deligne-Mumford stack \( \mathcal{X}(\Sigma) \).

**Proof.** From the definitions of extended stacky fan \( \Sigma^e \) and stacky fan \( \Sigma \), we have the following commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \to & \mathbb{Z}^n & \to & \mathbb{Z}^m & \to & \mathbb{Z}^{m-n} & \to & 0 \\
\downarrow{\beta} & & \downarrow{\beta^e} & & \downarrow{\beta} & \ & \\
0 & \to & N & \xrightarrow{id} & N & \to & 0 & \to & 0
\end{array}
\]

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From the definition of Gale dual, we compute $DG(\tilde{\beta}) = Z^{m-n}$ and $\tilde{\beta}^\vee$ is an isomorphism. So from Lemma 2.3 in [4], applying the Gale dual and the $Hom_{Z}(-, \mathbb{C}^\times)$ functor to the above diagram we get:

\[
\begin{array}{c}
1 \longrightarrow G \xrightarrow{\varphi_1} G^e \longrightarrow (\mathbb{C}^\times)^{m-n} \longrightarrow 1 \\
\downarrow \alpha \quad \downarrow \alpha^e \quad \downarrow \tilde{\alpha} \\
1 \longrightarrow (\mathbb{C}^\times)^n \longrightarrow (\mathbb{C}^\times)^m \longrightarrow (\mathbb{C}^\times)^{m-n} \longrightarrow 1
\end{array}
\] (4)

We define the morphism $\varphi_0 : Z \longrightarrow Z^e = Z \times (\mathbb{C}^\times)^{m-n}$ to be the inclusion defined by $z \mapsto (z, 1)$. So $(\varphi_0 \times \varphi_1, \varphi_0) : (Z \times G \rightrightarrows Z) \longrightarrow (Z^e \times G^e \rightrightarrows Z^e)$ defines a morphism between groupoids. Let $\varphi : [Z/G] \longrightarrow [Z^e/G^e]$ be the morphism of stacks induced from $(\varphi_0 \times \varphi_1, \varphi_0)$. From the above commutative diagram we have the following commutative diagram:

\[
\begin{array}{c}
Z \times G \xrightarrow{\varphi_0 \times \varphi_1} Z^e \times G^e \\
\downarrow (s,t) \quad \downarrow (s,t) \\
Z \times Z \xrightarrow{\varphi_0 \times \varphi_0} Z^e \times Z^e
\end{array}
\]

In [4], $\tilde{\alpha}$ is an isomorphism which implies that the left square in (4) is cartesian. So the above commutative diagram is cartesian. Given an element $(z_1, \ldots, z_n, z_{n+1}, \ldots, z_m) \in Z^e$, there exists an element $g^e \in (\mathbb{C}^\times)^{m-n}$ such that $g^e \cdot (z_1, \ldots, z_n, z_{n+1}, \ldots, z_m) = (z_1, \ldots, z_n, 1, \ldots, 1)$. From (4), $g^e$ determine an element in $G^e$, so $\varphi$ is surjective. The stacks $\mathcal{X}(\Sigma^e)$ and $\mathcal{X}(\Sigma)$ are isomorphic. □

Let $X(\Sigma)$ be the simplicial toric variety associated to the extended stacky fan $\Sigma^e$. We have the following corollaries:

**Corollary 2.4** Given an extended stacky fan $\Sigma^e$, then the coarse moduli space of the extended toric Deligne-Mumford stack $\mathcal{X}(\Sigma^e)$ is also the simplicial toric variety $X(\Sigma)$.

**Remark** As in [4], let $\sigma$ be a top dimensional cone in $\Sigma$, denote by $Box(\sigma)$ to be the set of elements $v \in N$ such that $\overline{v} = \sum_{\rho \subseteq \sigma} a_i b_i$ for some $0 \leq a_i < 1$. The set $Box(\sigma)$ is in one-to-one correspondence with the elements in the finite group $N(\sigma) = N/N_{\sigma}$, where $N(\sigma)$ is a local group of the stack $\mathcal{X}(\Sigma^e)$. If $\tau \subseteq \sigma$ is a low dimensional cone, we define $Box(\tau)$ to be the set of elements in $v \in N$ such that $\overline{v} = \sum_{\rho \subseteq \tau} a_i b_i$, where $0 \leq a_i < 1$. It is easy to see that $Box(\tau) \subset Box(\sigma)$. In fact the elements in $Box(\tau)$ generate a subgroup of the local group $N(\sigma)$. Let $Box(\Sigma^e)$ be the union of $Box(\sigma)$ for all $d$-dimensional cones $\sigma \in \Sigma$. For $v_1, \ldots, v_n \in N$, let $\sigma(\overline{v}_1, \ldots, \overline{v}_n)$ be the unique minimal cone in $\Sigma$ containing $\overline{v}_1, \ldots, \overline{v}_n$. 

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3 The Toric Stack Bundle $P \mathcal{X}(\Sigma^e)$.

In this section we introduce the toric stack bundle $P \mathcal{X}(\Sigma^e)$ and determine its twisted sectors. Let $P \rightarrow B$ be a principal $(\mathbb{C}^\times)^m$-bundle over a smooth variety $B$. We give the following definition.

**Definition 3.1** We define the toric stack bundle $P \mathcal{X}(\Sigma^e) \rightarrow B$ to be the quotient stack

$$P \mathcal{X}(\Sigma^e) := [(P \times_{(\mathbb{C}^\times)^m} \mathbb{Z}^e)/G^e]$$

where $G^e$ acts on $P$ trivially.

**Remark** Let $\phi : \mathbb{Z}^m \rightarrow \mathbb{Z}^m$ be the map given by $e_i \mapsto e_i$ for $1 \leq i \leq n$, and $e_j \mapsto e_j + \sum_{i=1}^{n} a_i^j e_i$ for $n + 1 \leq j \leq m$, where $a_i^j \in \mathbb{Z}$. Then consider the following commutative diagram:

$$
\begin{array}{cccccc}
0 & \rightarrow & \mathbb{Z}^m & \xrightarrow{\phi} & \mathbb{Z}^m & \rightarrow & 0 \\
& & \downarrow{\beta^e} & & \downarrow{\beta^e} & & \downarrow{id} \\
0 & \rightarrow & N & \xrightarrow{id} & N & \rightarrow & 0
\end{array}
$$

We obtain a new extended stacky fan $\tilde{\Sigma}^e = (N, \Sigma, \tilde{\beta}^e)$, where the extra data in $\tilde{\Sigma}^e$ are $\tilde{b}_{n+1} = b_{n+1} + \sum_{i=1}^{n} a_i^{n+1} b_i$, $\tilde{h}_m = b_m + \sum_{i=1}^{n} a_i^m b_i$. The map $\phi$ gives a map $\mathbb{C}^n \times (\mathbb{C}^\times)^{m-n} \rightarrow \mathbb{C}^n \times (\mathbb{C}^\times)^{m-n}$ which is the identity on the first factor and given by $\phi$ on the second factor. Since the map in the above diagram doesn’t change the fan in the extended stacky fans, we have a map $\tilde{\phi}_0 : P \times_{(\mathbb{C}^\times)^m} \mathbb{Z}^e \rightarrow P \times_{(\mathbb{C}^\times)^m} \mathbb{Z}^e$, we use the same proof in Proposition 2.3 to prove that $P \mathcal{X}(\Sigma^e) \cong P \mathcal{X}(\tilde{\Sigma}^e)$. So this means that we always can choose the extra data such that $b_j = \sum_{i=1}^{n} a_i b_i$ for $j = n + 1, \cdots, m$ and $0 \leq a_i < 1$. These extra data are actually in the $Box(\Sigma^e)$.

**Example** From the above Remark, the extra data can be put into $Box(\Sigma^e)$. In this example we prove that they can not be put into the torsion subgroup of $N$. Let $N = \mathbb{Z}$ and $b_1 = 2$, $b_2 = -2$. Then $\Sigma = \{b_1, b_2\}$ is a simplicial fan in $NQ$. Let $\Sigma^e = (\Sigma, \Sigma, \tilde{\beta}^e)$, where $\tilde{\beta}^e : \mathbb{Z}^3 \rightarrow \mathbb{Z}$ is determined by $\{b_1, b_2, b_3 = 1\}$, then we compute $DG(\tilde{\beta}^e) = \mathbb{Z}^2$ and the Gale dual $(\tilde{\beta}^e)^\vee : \mathbb{Z}^3 \rightarrow \mathbb{Z}^2$ is given by the matrix $\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$. From Section 2, the extended toric Deligne-Mumford stack $\mathcal{X}(\Sigma^e) = [(\mathbb{C}^2 - \{0\}) \times \mathbb{C}^\times/(\mathbb{C}^\times)^2]$, where the action is given by $(\lambda_1, \lambda_2)(x, y, z) = (\lambda_1 \lambda_2^{-1} x, \lambda_1 y, \lambda_2^{-1} z)$. We get $\mathcal{X}(\Sigma^e) = \mathbb{P}^1 \times [\mathbb{C}^\times/\mathbb{C}^\times] = \mathbb{P}^1 \times B_{\mu_2}$. Now let $\Sigma^e = (\Sigma, \Sigma, \tilde{\beta}^e)$, where $\tilde{\beta}^e : \mathbb{Z}^3 \rightarrow \mathbb{Z}$ is determined by $\{b_1, b_2, b_3 = 0\}$, then we compute $DG(\tilde{\beta}^e) = \mathbb{Z}^2 \oplus \mathbb{Z}_2$ and the Gale dual $(\tilde{\beta}^e)^\vee : \mathbb{Z}^3 \rightarrow \mathbb{Z}^2 \oplus \mathbb{Z}_2$ is given by the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. The extended toric
Deligne-Mumford stack $\mathcal{X}(\Sigma^e) = [(\mathbb{C}^2 - \{0\}) \times \mathbb{C}^\times/(\mathbb{C}^\times)^2]$, where the action is given by $(\lambda_1, \lambda_2, \lambda_3)(x, y, z) = (\lambda_1 \cdot x, \lambda_1 \cdot y, \lambda_2 \cdot z)$. We get $\mathcal{X}(\Sigma^e) = [\mathbb{P}^1/\mu_2] = \mathbb{P}^1 \times B\mu_2$. Let $B = \mathbb{P}^1$ and $P = \mathbb{C}^\times \oplus \mathbb{C}^\times \oplus \mathcal{O}(-1)^\times$, then $\mathcal{X}(\Sigma^e)$ is a nontrivial $\mu_2$-gerbe over $\mathbb{P}^1 \times \mathbb{P}^1$ coming from the line bundle $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, -1)$.

Let $Q = \mathcal{O}(n_1)^\times \oplus \mathcal{O}(n_2)^\times \oplus \mathcal{O}(n_3)^\times$, then $Q\mathcal{X}(\Sigma^e)$ is the trivial $\mu_2$-gerbe over the $\mathbb{P}^1$-bundle $E$ over $\mathbb{P}^1$. So $\mathcal{X}(\Sigma^e)$ is not isomorphic to $Q\mathcal{X}(\Sigma^e)$ for any $Q$.

From Corollary 2.4, $\mathcal{X}(\Sigma^e)$ has the coarse moduli space $X(\Sigma)$ which is the simplicial toric variety associated to the simplicial fan $\Sigma$. From the exact sequence in (4), a $(\mathbb{C}^\times)^m$-bundle over $B$ determine a $T$-bundle over $B$ naturally. Let $E \rightarrow B$ be the principal $T$-bundle induced by $P$, then we have the twists $P\mathcal{X}_{\text{red}}(\Sigma^e) \rightarrow B$ with fibre the toric orbifold $X_{\text{red}}(\Sigma^e)$ and $E X(\Sigma) \rightarrow B$ with fibre the simplicial toric variety $X(\Sigma)$, where $P\mathcal{X}_{\text{red}}(\Sigma^e) := [(P \times (\mathbb{C}^\times)^m Z^e)/\mathcal{G}^e]$ and $E X(\Sigma) := E \times_T X(\Sigma)$, and $\mathcal{G}^e = \text{Im}(\alpha^e)$ in (4), so we obtain the exact sequence:

$$1 \rightarrow \mu \rightarrow G^e \xrightarrow{\alpha^e} \mathcal{G}^e \rightarrow 1 \quad (6)$$

From [8], we have:

**Proposition 3.2** $P\mathcal{X}(\Sigma^e)$ is a $\mu$-gerbe over $P\mathcal{X}_{\text{red}}(\Sigma^e)$ for a finite abelian group $\mu$.

**Remark** In fact, any extended toric Deligne-Mumford stack is $\mu$-gerbe over the underlying toric orbifold for a finite abelian group $\mu$ and some kind of $\mu$-gerbes over toric Deligne-Mumford stacks are again toric Deligne-Mumford stacks, see [3].

Because any toric stack bundle is a $\mu$-gerbe over the corresponding toric orbifold bundle and can be represented as a quotient stack, we have the following propositions:

**Proposition 3.3** The simplicial toric bundle $E X(\Sigma)$ is the coarse moduli space of the toric stack bundle $P\mathcal{X}(\Sigma^e)$ and the toric orbifold bundle $P\mathcal{X}_{\text{red}}(\Sigma^e)$.

**Proof** The toric stack bundle $P\mathcal{X}(\Sigma^e)$ is a $\mu$-gerbe over the simplicial toric orbifold bundle $P\mathcal{X}_{\text{red}}(\Sigma^e)$ for a finite abelian group $\mu$, $P\mathcal{X}(\Sigma^e) = [(P \times (\mathbb{C}^\times)^m Z^e)/\mathcal{G}^e]$ and $P\mathcal{X}_{\text{red}}(\Sigma^e) = [(P \times (\mathbb{C}^\times)^m Z^e)/\mathcal{G}^e]$ are quotient stacks. Take the geometric quotient, we have the coarse moduli space $(P \times (\mathbb{C}^\times)^m Z^e)/\mathcal{G}^e = (P \times Z^e)/((\mathbb{C}^\times)^m \times \mathcal{G}^e)$. From Proposition 2.1 in [3], we have $X(\Sigma) = Z/\mathcal{G}^e = Z^e/\mathcal{G}^e$, so

$$E \times_T (Z^e/\mathcal{G}^e) = (P \times (\mathbb{C}^\times)^m T) \times_T (Z^e/\mathcal{G}^e) = (P \times Z^e)/((\mathbb{C}^\times)^m \times \mathcal{G}^e)$$

From the universal geometric quotients in [14], $E X(\Sigma)$ is the coarse moduli space of $P\mathcal{X}(\Sigma^e)$ and $P\mathcal{X}_{\text{red}}(\Sigma^e)$. □

**Proposition 3.4** The toric stack bundle $P\mathcal{X}(\Sigma^e)$ is a Deligne-Mumford stack.
Proof. From [3], \( P\mathcal{X}(\Sigma^e) = [(P \times_{(\mathbb{C}^*)^m} Z^c)/G^e] \) is a quotient stack, where \( G^e \) acts trivially on \( P \). The action of \( G^e \) on \( Z^c \) has finite, reduced stabilizers because the stack \( [Z^c/G^e] \) is a Deligne-Mumford stack, so the action of \( G^e \) on \( P \times_{(\mathbb{C}^*)^m} Z^c \) also has finite, reduced stabilizers. From Corollary 2.2 of [9], \( P\mathcal{X}(\Sigma^e) \) is a Deligne-Mumford stack. □

For an extended stacky fan \( \Sigma^e \), let \( \sigma \in \Sigma \) be a cone, let \( \text{link}(\tau) := \{ \sigma : \sigma + \tau \in \Sigma, \sigma \cap \tau = 0 \} \), and \( \widetilde{\rho}_1, \ldots, \widetilde{\rho}_l \) be the rays in \( \text{link}(\sigma) \). Then \( \Sigma^e/\sigma = (N(\sigma), \Sigma/\sigma, \beta^e(\sigma)) \) is an extended stacky fan, where \( \beta^e(\sigma) : Z^{l+m-n} \to N(\sigma) \) is given by the images of \( b_1, \ldots, b_{n+1}, \ldots, b_m \) under \( N \to N(\sigma) \). From the construction of extended toric Deligne-Mumford stack, we have \( \mathcal{X}(\Sigma^e/\sigma) := \{ Z^c(\sigma)/G^e(\sigma) \} \), where \( Z^c(\sigma) = (\mathbb{A}^l - V(J_{\Sigma^e/\sigma})) \times (\mathbb{C}^*)^{m-n} = Z(\sigma) \times (\mathbb{C}^*)^{m-n}, G^e(\sigma) = \text{Hom}_Z(DG(\beta^e(\sigma)), \mathbb{C}^\times) \). We have an action of \( (\mathbb{C}^*)^m \) on \( Z^c(\sigma) \) induced by the natural action of \( (\mathbb{C}^*)^{l+m-n} \) on \( Z(\sigma) \) and the projection \( (\mathbb{C}^*)^m \to (\mathbb{C}^*)^{l+m-n} \).

We let

\[
P\mathcal{X}(\Sigma^e/\sigma) = \left[ (P \times_{(\mathbb{C}^*)^m} (\mathbb{C}^*)^{l+m-n} \times_{(\mathbb{C}^*)^{l+m-n}} Z^c(\sigma))/G^e(\sigma) \right]
\]

be the quotient stack. Then we have:

**Proposition 3.5** Let \( \sigma \) be a cone in the extended stacky fan \( \Sigma^e \), then \( P\mathcal{X}(\Sigma^e/\sigma) \) defines a closed substack of \( P\mathcal{X}(\Sigma^e) \).

Proof. Let \( \mathcal{X}(\Sigma^e) = [Z^c/G^e] \), if \( \sigma \) is a cone, let \( W^c(\sigma) \) be the closed subvariety of \( Z^c \) defined by \( J(\sigma) := \{ z_i : \rho_i \subseteq \sigma > \} \) in \( \mathbb{C}[z_1, \ldots, z_n, z_{n+1}, \ldots, z_m] \), then we see that \( W^c(\sigma) = W(\sigma) \times (\mathbb{C}^*)^{m-n} \), where \( W(\sigma) \) is the closed subvariety of \( Z \) defined by \( J(\sigma) := \{ z_i : \rho_i \subseteq \sigma > \} \) in \( \mathbb{C}[z_1, \ldots, z_n] \). From [4], there is a map \( \varphi_0 : W^c(\sigma) \to Z^c(\sigma) \) which is \( (\mathbb{C}^*)^n \)-equivariant, we define the map \( W^c(\sigma) \to Z^c(\sigma) \) by \( \varphi_0 \times 1 \). Twist it by the bundle \( P \), we have a map \( \varphi_0 : P \times_{(\mathbb{C}^*)^m} W^c(\sigma) \to P \times_{(\mathbb{C}^*)^m} Z^c(\sigma) \). From the following diagram:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z}^{n-l} & \longrightarrow & \mathbb{Z}^m & \longrightarrow & \mathbb{Z}^{l+m-n} & \longrightarrow & 0 \\
\downarrow & & \downarrow \varphi & & \downarrow \varphi(\sigma) & & \downarrow & \\
0 & \longrightarrow & N_\sigma & \longrightarrow & N & \longrightarrow & N(\sigma) & \longrightarrow & 0
\end{array}
\]

Applying Gale dual and \( \text{Hom} \) functor we get the commutative diagram:

\[
\begin{array}{ccc}
(\mathbb{C}^*)^m & \longrightarrow & (\mathbb{C}^*)^{l+m-n} \\
\downarrow & & \downarrow \\
G^e & \longrightarrow & G^e(\sigma) \\
\downarrow \alpha^e & & \downarrow \alpha^e(\sigma) \\
(\mathbb{C}^*)^m & \longrightarrow & (\mathbb{C}^*)^{l+m-n}
\end{array}
\]

(7)

So we get a map of groupoids: \( \varphi_0 \times \varphi_1 : P \times_{(\mathbb{C}^*)^m} W^c(\sigma) \times G^e \to P \times_{(\mathbb{C}^*)^m} Z^c(\sigma) \times G^e(\sigma) \) which is Morita equivalent. So we have an isomorphism of stacks.
$[(P \times_{(C^\times)^m} W(\sigma))/G^e] \cong [(P \times_{(C^\times)^m} Z^e(\sigma))/G^e(\sigma)]$. Since $W^e(\sigma)$ is a subvariety of $Z^e$, and $P \times_{(C^\times)^m} W^e(\sigma)$ is a subvariety of $P \times_{(C^\times)^m} Z^e$, so $[(P \times_{(C^\times)^m} W^e(\sigma))/G^e]$ is a substack of $[(P \times_{(C^\times)^m} Z^e)/G^e] = P X(\Sigma^e)$. So $P X(\Sigma^e)/\sigma)$ is a closed substack of $P X(\Sigma^e)$.

**Remark** From [4], $W(\sigma) = Z^{g_1,\ldots,g_r}$ for some group elements in $G$. From Proposition 2.3, the extended Deligne-Mumford stack $[Z^e(\sigma)/G^e(\sigma)]$ is isomorphic to the stack $[Z(\sigma)/G(\sigma)]$. Let $g_1,\ldots,g_r$ still represent the elements in $G$. From (4), we have a map $\varphi_1$ in $[4]$. Then $W^e(\sigma) = (Z^e)^{g_1,\ldots,g_r}$.

**Proposition 3.6** Let $P X(\Sigma^e) \to B$ be a toric stack bundle over a smooth variety $B$ with fibre $X(\Sigma^e)$ the extended toric Deligne-Mumford stack associated to the extended stacky fan $\Sigma^e$, then the $r$-th inertia stack of this toric stack bundle is

$$I_r(P X(\Sigma^e)) = \prod_{(v_1,\ldots,v_r) \in Box(\Sigma^e)^r} P X(\Sigma^e)/\sigma(\tau_1,\ldots,\tau_r)$$

**Proof.** From [4], $P X(\Sigma^e) = [(P \times_{(C^\times)^m} Z^e)/G^e]$ is a quotient stack. Because $G$ is an abelian group and the the action has finite, reduced stabilizers, we have the $r$-th inertia stack:

$$I_r(P X(\Sigma^e)) = \left[ \prod_{(g_1,\ldots,g_r) \in (G^e)^r} (P \times_{(C^\times)^m} Z^e)^H / G^e \right]$$

where $H$ is the subgroup in $G$ generated by the elements $g_1,\ldots,g_r$. From Lemma 4.6 in [4], there is a map from $Box(\Sigma^e)$ to $G$, from the map $\varphi_1$ in [4], we have a map $\rho : Box(\Sigma^e) \to G$ such that $\rho(v) = g(v)$. For a $r$-tuple $(v_1,\ldots,v_r)$ in the $Box(\Sigma^e)$, from Proposition 3.5 and the above Remark, we have: $P X(\Sigma^e)/\sigma(\tau_1,\ldots,\tau_r) \cong [(P \times_{(C^\times)^m} (Z^e)^H / G^e]$. Taking the disjoint union over all $r$-tuples in $Box(\Sigma^e)$ we get a map:

$$\psi : \prod_{(v_1,\ldots,v_r) \in Box(\Sigma^e)^r} P X(\Sigma^e)/\sigma(\tau_1,\ldots,\tau_r) \to I_r(P X(\Sigma^e))$$

The toric stack bundle $P X(\Sigma^e)$ locally like a smooth variety times the extended toric Deligne-Mumford stack $X(\Sigma^e)$. From [4], the map $\psi$ is an isomorphism locally in the Zariski topology of the base $B$, so $\psi$ is an isomorphism globally. We complete the proof of the Proposition. □

**Remark** For any pair $(v_1,v_2) \in Box(\Sigma^e)^2$, there exists a unique element $v_3 \in Box(\Sigma^e)$ such that $v_1 + v_2 + v_3 \equiv 0$. This means that in the local group $N/N_{\sigma(\tau_1,\tau_2)}$, the corresponding group elements $g_1,g_2,g_3$ satisfy $g_1g_2g_3 = 1$. So this implies that $\sigma(\tau_1,\tau_2,\tau_3) = \sigma(\tau_1,\tau_2)$. In fact, the Proposition determines all the 3-twisted sectors of the toric stack bundle $P X(\Sigma^e)$. See also in [17],[11] for the case of toric varieties.
4 The Orbifold Cohomology Ring.

In this section we describe the ring structure of the orbifold cohomology space of the toric stack bundles.

4.1 The Module Structure on $A^{\ast}_{\text{orb}}(P\chi(\Sigma_{e}))$.

Let $P\chi(\Sigma_{e}) \to B$ be a toric stack bundle. Let $E \to B$ be the associated $T$-bundle over $B$ induced from $P$ from the exact sequence $\mathcal{E}$. Let $M := N^\ast$ be the dual of $N$, and let $\theta \in M$, define $\xi_{B} \to B$ to be the line bundle coming from $E \to B$ by “extending” the structure group via $\chi^{\theta} : T \to \mathbb{C}^\times$. We give several definitions:

**Definition 4.1** Let $A^{\ast}(B)$ denote the Chow ring over $\mathbb{Q}$ of the smooth variety $B$. Define the deformed ring $A^{\ast}(B)[N]\Sigma_{e}$ as follows: $A^{\ast}(B)[N]\Sigma_{e} = A^{\ast}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[N]^{\Sigma_{e}}$, where $y$ is a formal variable. Multiplication is given by $(\mathcal{I})$.

The deformed ring $A^{\ast}(B)[N]\Sigma_{e}$ has a $\mathbb{Q}$-grading defined by: if $\mathcal{I} = \sum_{i \in I} a_{i} y^{i}$, then $\deg(y^{i}) = \sum_{i \in I} a_{i} \in \mathbb{Q}$. If $\gamma \in A^{\ast}(B)$, then $\deg(\gamma \cdot y^{i}) = \deg(\gamma) + \deg(y^{i})$. Let $\mathcal{I}(\Sigma_{e})$ be the ideal in $(\mathcal{I})$.

**Definition 4.2** Let $\Sigma_{e} = (N, \Sigma, \beta_{e})$ be an extended stacky fan in $N_{\mathbb{Q}}$. Define ring $S_{\Sigma_{e}} := A^{\ast}(B)[x_{1}, \ldots, x_{n}] / I_{\Sigma_{e}}$, where the ideal $I_{\Sigma_{e}}$ is generated by the square-free monomials $x_{i_{1}} \cdots x_{i_{s}}$ with $i_{1} \cdots + i_{s} \notin \Sigma$.

Note that $S_{\Sigma_{e}}$ is a subring of $A^{\ast}(B)[N]\Sigma_{e}$ given by the map $x_{i} \mapsto y^{n_{i}}$ for $1 \leq i \leq n$. Let $\{\rho_{1}, \ldots, \rho_{n}\}$ be the rays of $\Sigma_{e}$, then each $\rho_{i}$ corresponds to a line bundle $L_{i}$ over the extended toric Deligne-Mumford stack $\mathcal{X}(\Sigma_{e})$. This line bundle can be defined as follow. The line bundle $L_{i}$ on the toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ is given by the trivial line bundle $\mathbb{C} \times Z$ over $Z$ with the $G$ action on $\mathbb{C}$ given by the $i$-th component $\alpha_{i}$ of $\alpha : G \to (\mathbb{C}^\times)^{n}$ in $(\mathcal{I})$ when $\Sigma_{e} = \Sigma$. From $(\mathcal{I})$, we have:

$$
\begin{array}{ccc}
G & \xrightarrow{\varphi_{1}} & G^{e} \\
\downarrow{\alpha} & & \downarrow{\alpha^{e}} \\
(\mathbb{C}^\times)^{n} & \xrightarrow{i} & (\mathbb{C}^{\times})^{m}
\end{array}
$$

(8)

**Definition 4.3** For each $\rho_{i}$, define the line bundle $L_{i}$ over $\mathcal{X}(\Sigma_{e})$ to be the quotient of the trivial line bundle $Z^{e} \times \mathbb{C}$ over $Z^{e}$ under the action of $G^{e}$ on $\mathbb{C}$ through one component of $\alpha^{e}$ such that the pullback component in $\alpha$ through $\mathcal{I}$ is $\alpha_{i}$. Twist it by the principal $(\mathbb{C}^{\times})^{m}$-bundle $P$, we get the line bundle $L_{i}$ over the toric stack bundle $P\mathcal{X}(\Sigma_{e})$.

First we describe the ordinary Chow ring of the toric stack bundle:
Lemma 4.4 Let \( P X(\Sigma^e) \rightarrow B \) be a toric stack bundle over a smooth variety \( B \) with fibre \( X(\Sigma^e) \) the extended toric Deligne-Mumford stack associated to the extended stacky fan \( \Sigma^e \), then there is an isomorphism of \( \mathbb{Q} \)-graded rings:

\[
\frac{S_{\Sigma^e}}{I(\Sigma^e)} \cong A^*((P X(\Sigma^e))
\]

given by \( x_i \mapsto c_1(L_i) \).

\textbf{Proof.} From Corollary 2.4, let \( X(\Sigma) \) be the coarse moduli space of the extended toric Deligne-Mumford stack \( X(\Sigma^e) \). Let \( E \rightarrow B \) be the principal \( T \)-bundle induced from the \((\mathbb{C}^*)^m\)-bundle \( P \), then from Proposition 3.3, \( E X(\Sigma) \) is the coarse moduli space of the toric stack bundle \( P X(\Sigma^e) \). Let \( a_i \) be the first lattice vector in the ray generated by \( b_i \), then \( b_i = l_i a_i \) for some positive integer \( l_i \). The ideal \( I(\Sigma^e) \) in \( \mathbb{Z} \) also define an ideal in \( S_{\Sigma^e} \). From [19], we have

\[
A^*(E X(\Sigma)) \cong \frac{S_{\Sigma^e}}{I(\Sigma^e)}
\]

which is given by \( x_i \mapsto E(V(\rho_i)) \), where \( E(V(\rho_i)) \) is the associated bundle over \( B \) corresponding to the \( T \)-invariant divisor \( V(\rho_i) \). From [2],[20], the Chow ring of the stack \( P X(\Sigma^e) \) is isomorphic to the Chow ring of its coarse moduli space \( E X(\Sigma) \) given by \( c_1(L_i) \mapsto l_i^{-1} \cdot E(V(\rho_i)) \), and \( c_1(\xi) + \sum_{i=1}^{n} \theta(a_i) l_i y_i^b = c_1(\xi) + \sum_{i=1}^{n} \theta(b_i) y_i^b \), so we prove the Lemma. □

Now we talk about the module structure on \( A_{\text{orb}}^*(P X(\Sigma^e)) \). Because \( \Sigma \) is a simplicial fan, we have:

Lemma 4.5 For any \( c \in \mathbb{N} \), let \( \sigma \) be the minimal cone in \( \Sigma \) containing \( \tau \), then there exists a unique expression

\[
c = v + \sum_{\rho_i \subset \sigma} m_i b_i
\]

where \( m_i \in \mathbb{Z}_{\geq 0} \), and \( v \in \text{Box}(\sigma) \). □

Lemma 4.6 If \( \tau \) is a cone in the complete simplicial fan \( \Sigma \), \( \{\rho_1, \ldots, \rho_s\} \subset \text{link}(\tau) \), suppose \( \rho_1, \ldots, \rho_s \) are contained in a cone \( \sigma \subset \Sigma \). Then \( \sigma \cup \tau \) is contained in a cone of \( \Sigma \).

\textbf{Proof.} Using the following result: If \( \rho_1, \ldots, \rho_s \) are rays in the complete simplicial fan \( \Sigma \), if for any \( i, j \), \( \rho_i, \rho_j \) generate a cone, then \( \rho_1, \ldots, \rho_s \) generate a cone, see [10],[15]. The Lemma is proved. □

Proposition 4.7 Let \( P X(\Sigma^e) \rightarrow B \) be a toric stack bundle over a smooth variety \( B \) with fibre \( X(\Sigma^e) \) the extended toric Deligne-Mumford stack associated to the extended stacky fan \( \Sigma^e \), then we have an isomorphism of \( A^*(P X(\Sigma^e)) \)-modules:

\[
\bigoplus_{v \in \text{Box}(\Sigma^e)} A^*(P X(\Sigma^e/\sigma(\tau))) [\deg(g^v)] \cong \frac{A^*(B)[N][\Sigma^e]}{I(\Sigma^e)}
\]
Proof. From the definition of $A^*(B)[N]\Sigma^e$ and Lemma 4.5, we see that $A^*(B)[N]\Sigma^e = \bigoplus_{v \in Box(\Sigma^e)} y^v \cdot S_{\Sigma^e}$. Since $I(\Sigma^e)$ is the ideal in $A^*(B)[N]\Sigma^e$ defined in (3). Then $\bigoplus_{v \in Box(\Sigma^e)} y^v \cdot I(\Sigma^e)$ is the ideal $I(\Sigma^e)$ in $\bigoplus_{v \in Box(\Sigma^e)} y^v \cdot S_{\Sigma^e} = A^*(B)[N]\Sigma^e$. So we obtain the isomorphism of $A^*(\mathcal{L}(\Sigma^e))$-modules:

$$\frac{A^*(B)[N]\Sigma^e}{I(\Sigma^e)} \cong \bigoplus_{v \in Box(\Sigma^e)} \frac{y^v \cdot S_{\Sigma^e}}{y^v \cdot I(\Sigma^e)} \quad (9)$$

For any $v \in Box(\Sigma^e)$, let $\sigma(\Omega)$ be the minimal cone in $\Sigma$ containing $\Omega$. Let $\rho_1, \ldots, \rho_l \in \text{link}(\sigma(\Omega))$, and $\rho_i$ be the image of $\rho_i$ under the natural map $N \rightarrow N(\sigma(\Omega)) = N/N(\sigma(\Omega))$. Then $S_{\Sigma^e/\sigma(\Omega)} \subset A^*(B)[N(\sigma(\Omega))]\Sigma^e/\sigma(\Omega)$ is the subring given by: $\tilde{x}_i \mapsto y^{b_i}$, for $\rho_i \in \text{link}(\sigma(\Omega))$. Consider the morphism: $i : A^*(B)[\tilde{x}_1, \ldots, \tilde{x}_l] \rightarrow A^*(B)[x_1, \ldots, x_n]$ given by $\tilde{x}_i \mapsto x_i$. From Lemma 4.6, it is easy to check that the ideal $I_{\Sigma^e/\sigma(\Omega)}$ goes to the ideal $I_{\Sigma^e}$, so we have a morphism $S_{\Sigma^e/\sigma(\Omega)} \rightarrow S_{\Sigma^e}$. Since $S_{\Sigma^e}$ is a subring of $A^*(B)[N]\Sigma^e$ given by $x_i \mapsto y^{b_i}$, we use the notations $y^{b_i}$. Let $\tilde{\Psi}_v : S_{\Sigma^e/\sigma(\Omega)}[\text{deg}(y^v)] \rightarrow y^v \cdot S_{\Sigma^e}$ be the morphism given by: $y^{b_i} \mapsto y^v \cdot y^{b_i}$. If $\sum_{i=1}^l \tilde{\theta}(b_i)y^{b_i} + c_1(\xi_b)$ belongs to the ideal $I(\Sigma^e)$, then

$$\tilde{\Psi}_v \left( \sum_{i=1}^l \tilde{\theta}(b_i)y^{b_i} + c_1(\xi_b) \right) = y^v \cdot \left( \sum_{i=1}^l \tilde{\theta}(b_i)y^{b_i} + c_1(\xi_b) \right) = y^v \cdot \left( \sum_{i=1}^n \theta(b_i)y^{b_i} + c_1(\xi_b) \right)$$

where $\theta$ is determined by the diagram:

$$\begin{array}{ccc}
N & \xrightarrow{\theta} & \mathbb{Z} \\
\downarrow{\pi} & & \\
N(\sigma(\Omega)) & \xrightarrow{\hat{\sigma}} & \mathbb{Z}
\end{array} \quad (10)$$

So $\theta(b_i) = \tilde{\theta}(b_i)$. From the definition of the line bundle $\xi_b$, we have $\xi_{b_i} \cong \xi_{b_i}$. We obtain that $\tilde{\Psi}_v(\sum_{i=1}^l \tilde{\theta}(b_i)y^{b_i} + c_1(\xi_b)) \in y^v \cdot I(\Sigma^e)$. So $\tilde{\Psi}_v$ induces a morphism $\Psi : \bigoplus_{\sigma(\Omega)} y^v \cdot S_{\Sigma^e} \rightarrow y^v \cdot S_{\Sigma^e}$, such that $\Psi([y^{b_i}]) = [y^v \cdot y^{b_i}]$.

Conversely, for such $v \in Box(\Sigma^e)$ and $\rho_i \subset \sigma(\Omega)$, choose $\theta_i \in Hom(N, \mathbb{Q})$ such that $\theta_i(b_i) = 1$ and $\theta_i(b_j) = 0$ for $b_j \neq b_i \in \sigma(\Omega)$. We consider the following morphism $p : A^*(B)[x_1, \ldots, x_n] \rightarrow A^*(B)[\tilde{x}_1, \ldots, \tilde{x}_l]$, where $p$ is given by:

$$x_i \mapsto \begin{cases} 
\tilde{x}_i & \text{if } \rho_i \subseteq \text{link}(\sigma(\Omega)), \\
-\sum_{j=1}^l \theta_i(b_j)\tilde{x}_j & \text{if } \rho_i \subseteq \sigma(\Omega), \\
0 & \text{if } \rho_i \nsubseteq \sigma(\Omega) \cup \text{link}(\sigma(\Omega)).
\end{cases}$$

For any $x_i \cdots x_i$ in $I_{\Sigma^e}$, also from Lemma 4.6 we prove that $p(x_i \cdots x_i) \in I_{\Sigma^e}$. We also use the notations $y^{b_i}$ to replace $x_i$, then $p$ induces a surjective map: $S_{\Sigma^e} \rightarrow S_{\Sigma^e/\sigma(\Omega)}$ and a surjective map: $\Phi_v : y^v \cdot S_{\Sigma^e} \rightarrow$
For $\theta \in M$, we have $\theta = \theta_v + \theta'_v$, where $\theta_v \in N(\sigma(\mathcal{T}))^\perp = M \cap \sigma(\mathcal{T})^\perp$ and $\theta'_v$ belongs to the orthogonal complement of the subspace $\sigma(\mathcal{T})^\perp$ in $M$. From (10), we have:

$$\tilde{\Phi}_v \left( y^v \cdot \left( \sum_{i=1}^n \theta(b_i) y^{b_i} + c_1(\xi_\theta) \right) \right)$$

$$= \sum_{i=1}^l \theta_v(\tilde{b}_i) y^{\tilde{b}_i} + c_1(\xi_{\theta_v}) + \sum_{\rho_i \in \sigma(\mathcal{T})} \theta'_v(b_i) \left( - \sum_{j=1}^l \theta_i(b_j) y^{b_j} \right) + c_1(\xi_{\theta'_v}) + \sum_{i=1}^l \theta'_v(b_i) y^{b_i}$$

Note that $\left( \sum_{i=1}^l \theta_v(\tilde{b}_i) y^{\tilde{b}_i} + c_1(\xi_{\theta_v}) \right) \in \mathcal{I}(\Sigma^e/\sigma(\mathcal{T}))$. From the definition of $\xi_{\theta}$ over $X(\Sigma^e/\sigma(\mathcal{T}))$, $\xi_{\theta'_v} = 0$. Now let $\theta'_v = \sum_{\rho_i \in \sigma(\mathcal{T})} a_i \theta_i$, where $a_i \in \mathbb{Q}$, then $\sum_{\rho_i \in \sigma(\mathcal{T})} \theta'_v(b_i) = \sum_{\rho_i \in \sigma(\mathcal{T})} a_i \theta_i(b_i)$. We have:

$$\sum_{\rho_i \in \sigma(\mathcal{T})} a_i \theta_i(b_i) \left( - \sum_{j=1}^l \theta_i(b_j) y^{b_j} \right) + \sum_{\rho_i \in \sigma(\mathcal{T})} a_i \theta_i(b_i) y^{b_i} = 0,$$

so we have $\tilde{\Phi}_v \left( y^v \cdot \left( \sum_{i=1}^n \theta(b_i) y^{b_i} + c_1(\xi_\theta) \right) \right) \in \mathcal{I}(\Sigma^e/\sigma(\mathcal{T}))$. So $\tilde{\Phi}_v$ induces a morphism $\Phi : \frac{S_{\Sigma^e/\sigma(\mathcal{T})}}{y^v \cdot \mathcal{I}(\Sigma^e/\sigma(\mathcal{T}))} \to \frac{S_{\Sigma^e/\sigma(\mathcal{T})}}{deg(y^v)}$. Note that $\Phi_v \Psi_v = 1$ is easy to check. For any $[y^v \cdot y^{b_i}] \in \frac{S_{\Sigma^e/\sigma(\mathcal{T})}}{y^v \cdot \mathcal{I}(\Sigma^e/\sigma(\mathcal{T}))}$, since $y^v \cdot \left( - \sum_{j=1}^l \theta_i(b_j) y^{b_j} + \sum_{i=1}^n \theta_i(b_i) y^{b_i} \right) = y^v \cdot y^{b_i}$, we have $[y^v \cdot (- \sum_{j=1}^l \theta_i(b_j) y^{b_j})] = [y^v \cdot y^{b_i}]$. So we check that $\Psi_v \Phi_v = 1$. So $\Phi_v$ is an isomorphism. From Lemma 4.4, for any $v \in \text{Box}(\Sigma^e)$, we have an isomorphism of Chow rings: $\frac{S_{\Sigma^e/\sigma(\mathcal{T})}}{y^v \cdot \mathcal{I}(\Sigma^e/\sigma(\mathcal{T}))} \cong A^*(P^\mathcal{T}(\Sigma^e/\sigma(\mathcal{T})))$. Taking into account all the $v$ in $\text{Box}(\Sigma^e)$ and (9) we have the isomorphism:

$$\bigoplus_{v \in \text{Box}(\Sigma^e)} A^\sigma \left( P^\mathcal{T}(\Sigma^e/\sigma(\mathcal{T})) \right) \cong \bigoplus_{v \in \text{Box}(\Sigma^e)} A^\sigma \left( P^\mathcal{T}(\Sigma^e/\sigma(\mathcal{T})) \right).$$

Note that both sides of (9) are $S_{\Sigma^e/\mathcal{I}(\Sigma^e)} = A^*(P^\mathcal{T}(\Sigma^e))$-modules, we complete the proof. □

**Remark** In Proposition 5.2 of [4], the authors give a proof of the Proposition for toric Deligne-Mumford stacks. We give a more explicit proof of this isomorphism for the toric stack bundle in this Proposition.

### 4.2 The Orbifold Cup Product.

In this section we consider the orbifold cup product on $A_{orb}^\sigma \left( P^\mathcal{T}(\Sigma^e) \right)$. First we determine the 3-twisted sectors of $P^\mathcal{T}(\Sigma^e)$. From the orbifold Gromov-Witten theory, the 3-twisted sectors of $P^\mathcal{T}(\Sigma^e)$ are the components of the double inertia stack $\Delta_3(P^\mathcal{T}(\Sigma^e))$ of $P^\mathcal{T}(\Sigma^e)$, see [6]. So from the remark after Proposition 3.6, we have all the 3-twisted sectors of $P^\mathcal{T}(\Sigma^e)$:

$$\prod_{(g_1, g_2, g_3) \in \text{Box}(\Sigma^e)^3, g_1 g_2 g_3 = 1} P^\mathcal{T}(\Sigma^e/\sigma(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3))$$ (11)

where $\sigma(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ is the minimal cone in $\Sigma$ containing $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$. For any 3-twisted sector $P^\mathcal{T}(\Sigma^e)_{(g_1, g_2, g_3)} = P^\mathcal{T}(\Sigma^e/\sigma(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3))$, we have an inclusion
Let $\Sigma$ sector of the stack $N$.

**Definition 4.8** ([5]) The obstruction bundle $O_{(g_1,g_2,g_3)}$ over $P\mathcal{X}(\Sigma^e/\sigma(\bar{g}_1,\bar{g}_2,\bar{g}_3))$ is defined by the $H$-invariant:

$$(e^*T(P\mathcal{X}(\Sigma^e))) \otimes H^1(C,O_C))^H$$

**Proposition 4.9** Let $P\mathcal{X}(\Sigma^e)(g_1,g_2,g_3) = P\mathcal{X}(\Sigma^e/\sigma(\bar{g}_1,\bar{g}_2,\bar{g}_3))$ be a 3-twisted sector of the stack $P\mathcal{X}(\Sigma^e)$, let $g_1 + g_2 + g_3 = \sum_{\rho_i \subset \sigma(\bar{g}_1,\bar{g}_2,\bar{g}_3)} a_i b_i$, $a_i = 1, 2$, then the Euler class of the obstruction bundle $O_{(g_1,g_2,g_3)}$ over $P\mathcal{X}(\Sigma^e)(g_1,g_2,g_3)$ is:

$$\prod_{a_i=2} c_1(L_i)|_{P\mathcal{X}(\Sigma^e/\sigma(\bar{g}_1,\bar{g}_2,\bar{g}_3))}$$

where $L_i$ is the line bundle over $P\mathcal{X}(\Sigma^e)$ in definition 4.3.

**Proof.** Let $\mathcal{X}(\Sigma^e)$ be the extended toric Deligne-Mumford stack corresponding to the extended stacky fan $\Sigma^e$ in $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$. Let $\sigma(\bar{g}_1,\bar{g}_2,\bar{g}_3)$ be the minimal cone in $\Sigma$ containing $\bar{g}_1,\bar{g}_2,\bar{g}_3$. From Corollary 2.5 and [3] we have the 3-twisted sector $\mathcal{X}(\Sigma^e)(g_1,g_2,g_3) = \mathcal{X}(\Sigma^e/\sigma(\bar{g}_1,\bar{g}_2,\bar{g}_3))$ and $P\mathcal{X}(\Sigma^e)(g_1,g_2,g_3) = P\mathcal{X}(\Sigma^e/\sigma(\bar{g}_1,\bar{g}_2,\bar{g}_3))$. Since $e : \mathcal{X}(\Sigma^e)(g_1,g_2,g_3) \rightarrow \mathcal{X}(\Sigma^e)$ is an inclusion, we have an exact sequence:

$$0 \rightarrow T\mathcal{X}(\Sigma^e/\sigma(\bar{g}_1,\bar{g}_2,\bar{g}_3)) \rightarrow e^*T\mathcal{X}(\Sigma^e) \rightarrow N(\mathcal{X}(\Sigma^e/\sigma(\bar{g}_1,\bar{g}_2,\bar{g}_3))/\mathcal{X}(\Sigma^e)) \rightarrow 0$$

where $N(\mathcal{X}(\Sigma^e/\sigma(\bar{g}_1,\bar{g}_2,\bar{g}_3))/\mathcal{X}(\Sigma^e))$ is the normal bundle of $\mathcal{X}(\Sigma^e/\sigma(\bar{g}_1,\bar{g}_2,\bar{g}_3))$ in $\mathcal{X}(\Sigma^e)$.

Since $\mathcal{X}(\Sigma^e) := [Z^e/G^e]$, the tangent bundle $T(\mathcal{X}(\Sigma^e)) = [T(Z^e)/T(G^e)]$ is a quotient stack. Since $Z^e$ is an open subvariety of $A^n \times (\mathbb{C}^*)^{m-n}$, $T(Z^e) = O_{Z^e}^\times$. Now from the construction of the line bundle $L_k$ over $\mathcal{X}(\Sigma^e)$, we have a canonical map: $\bigoplus_{k=1}^n L_k \rightarrow T(\mathcal{X}(\Sigma^e))$. Since we have a natural map $T(\mathcal{X}(\Sigma^e)) \rightarrow N(\mathcal{X}(\Sigma^e/\sigma(\bar{g}_1,\bar{g}_2,\bar{g}_3))/\mathcal{X}(\Sigma^e))$, we obtain a map of vector bundles over $\mathcal{X}(\Sigma^e/\sigma(\bar{g}_1,\bar{g}_2,\bar{g}_3))$:

$$\varphi : \bigoplus_{\rho_k \subset \sigma(\bar{g}_1,\bar{g}_2,\bar{g}_3)} L_k \rightarrow N(\mathcal{X}(\Sigma^e/\sigma(\bar{g}_1,\bar{g}_2,\bar{g}_3))/\mathcal{X}(\Sigma^e))$$

Then from the definition of the line bundle $L_k$ over $P\mathcal{X}(\Sigma^e)$, we have the map:

$$\bar{\varphi} : \bigoplus_{\rho_k \subset \sigma(\bar{g}_1,\bar{g}_2,\bar{g}_3)} L_k \rightarrow N(P\mathcal{X}(\Sigma^e/\sigma(\bar{g}_1,\bar{g}_2,\bar{g}_3))/P\mathcal{X}(\Sigma^e))$$

where $N(P\mathcal{X}(\Sigma^e/\sigma(\bar{g}_1,\bar{g}_2,\bar{g}_3))/P\mathcal{X}(\Sigma^e))$ is the normal bundle of $P\mathcal{X}(\Sigma^e/\sigma(\bar{g}_1,\bar{g}_2,\bar{g}_3))$ in $P\mathcal{X}(\Sigma^e)$. For any point map:

$$x : \text{Spec} \mathbb{C} \rightarrow \mathcal{X}(\Sigma^e/\sigma(\bar{g}_1,\bar{g}_2,\bar{g}_3)) \rightarrow P\mathcal{X}(\Sigma^e/\sigma(\bar{g}_1,\bar{g}_2,\bar{g}_3))$$
note that $x^+ \varphi$ is an isomorphism, so $\varphi$ is an isomorphism. We have the exact sequence:

$$0 \rightarrow T \left( P^{\mathcal{X}(\Sigma^e/\sigma(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3))} \right) \rightarrow e^* T \left( P^{\mathcal{X}(\Sigma^e)} \right) \rightarrow \bigoplus_{\rho \in \sigma(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3)} L_k \rightarrow 0$$

Now using the result in the proof of Proposition 6.3 in [4], we have

$$(\dim_{\mathbb{C}} L_k \otimes H^1(C, \mathcal{O}_C))^H = 0 \text{ if } a_k = 1, \quad (\dim_{\mathbb{C}} (L_k \otimes H^1(C, \mathcal{O}_C)))^H = 1 \text{ if } a_k = 2$$

So from the definition 4.8, we have:

$$e(O_{(g_1, g_2, g_3)}) \cong \prod_{a_i = 2} c_1(L_i)_{|P^{\mathcal{X}(\Sigma^e/\sigma(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3))}}$$

$\square$

**Proof of Theorem 1.1:**

From the definition of the orbifold cohomology in [5] and Proposition 4.7, we know that $A^*_{orb} \left( P^{\mathcal{X}(\Sigma^e)} \right) = \bigoplus_{g \in Box(\Sigma^e)} A^* \left( P^{\mathcal{X}(\Sigma^e/\sigma(\mathcal{G}))} \right) \left[ \deg(y^g) \right]$, and from Proposition 4.7, we have an isomorphism between $A^*(P^{\mathcal{X}(\Sigma^e)})$-modules:

$$\bigoplus_{g \in Box(\Sigma^e)} A^* \left( P^{\mathcal{X}(\Sigma^e/\sigma(\mathcal{G}))} \right) \left[ \deg(y^g) \right] \cong A^*(B)[N]/\mathcal{I}(\Sigma^e)$$

So we have an isomorphism of $A^*(P^{\mathcal{X}(\Sigma^e)})$-modules: $A^*_{ orb} \left( P^{\mathcal{X}(\Sigma^e)} \right) \cong \frac{A^*(B)[N]}{\mathcal{I}(\Sigma^e)}$. Next we show that the orbifold cup product defined in [5] coincides with the product in ring $A^*(B)[N]/\mathcal{I}(\Sigma^e)$. From the above isomorphisms, it suffices to consider the canonical generators $y^b, y^g$ where $g \in Box(\Sigma^e)$ and $\gamma \in A^*(B)$. Since $b_i \in N$, the twisted sector determined by $b_i$ is the whole toric stack bundle $P^{\mathcal{X}(\Sigma^e)}$, $y^b \cup_{orb} \gamma$ is the usual product $y^b \cdot \gamma$ in the deformed ring because $y^b$ and $\gamma$ belong to the ordinary Chow ring of $P^{\mathcal{X}(\Sigma^e)}$.

For $y^g \cup_{orb} y^h$ and $y^g \cup_{orb} \gamma, g \in Box(\Sigma^e)$, so $g$ determine a twisted sector $P^{\mathcal{X}(\Sigma^e/\sigma(\mathcal{G}))}$. The corresponding twisted sector to $b_i$ and $\gamma$ are the whole toric stack bundle $P^{\mathcal{X}(\Sigma^e)}$. It is easy to see that the 3-twisted sector corresponding to $(g, b_i)$ and $(g, \gamma)$ are $P^{\mathcal{X}(\Sigma^e)}_{(g, 1, g^{-1})} \cong P^{\mathcal{X}(\Sigma^e/\sigma(\mathcal{G}))}$, where $g^{-1}$ is the inverse of $g$ in the local group. From the dimension formula in [5], the obstruction bundle over $P^{\mathcal{X}(\Sigma^e)}_{(g, 1, g^{-1})}$ has dimension zero. So from the definition of orbifold cup product in [5] it is easy to check that $y^g \cup_{orb} y^h = y^g \cdot y^h$, $y^g \cup_{orb} \gamma = y^g \cdot \gamma$.

For the orbifold product $y^{g_1} \cup_{orb} y^{g_2}$, where $g_1, g_2 \in Box(\Sigma^e)$. From (\ref{eq:definition}), we see that if there is no cone in $\Sigma$ containing $\mathcal{G}_1, \mathcal{G}_2$, then there is no 3-twisted sector corresponding to the elements $g_1, g_2$, so the orbifold cup product is zero from the definition. On the other hand from the definition of the group ring $A^*(B)[N]$, $y^{g_1} \cdot y^{g_2} = 0$, so $y^{g_1} \cup_{orb} y^{g_2} = y^{g_1} \cdot y^{g_2}$. If there is a cone $\sigma \in \Sigma$ such that $\mathcal{G}_1, \mathcal{G}_2 \in \sigma$, let $g_3 \in Box(\Sigma^e)$ such that $g_3 \in \sigma(\mathcal{G}_1, \mathcal{G}_2)$ and $g_1 g_2 g_3 = 1$ in the local group. Using the same method in [4], we get: $y^{g_1} \cup_{orb} y^{g_2} = y^{g_1} \cdot y^{g_2}$. The theorem is proved. $\square$
5 The $\mu$-Gerbe.

In this section we talk about the degenerate case of the extended toric Deligne-Mumford stacks. In this case $N$ is a finite abelian group, the simplicial fan $\Sigma$ is 0. The toric stack bundle is a $\mu$-gerbe $\mathcal{X}$ over $B$ for a finite abelian group $\mu$.

Let $N = \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_s^{n_s}}$ be a finite abelian group, where $p_1, \ldots, p_s$ are prime numbers and $n_1, \ldots, n_s > 1$. Let $\beta^e : \mathbb{Z} \to N$ be given by the vector $(1, 1, \ldots, 1)$. Because $N_{\mathbb{Q}} = 0$, so $\Sigma = 0$, then $\Sigma^e = (N, \Sigma, \beta^e)$ is an extended stacky fan from Section 2.1. Let $n = \text{lcm}(p_1^{n_1}, \ldots, p_s^{n_s})$, then $n = p_1^{n_1} \cdots p_s^{n_s}$, where $p_1, \ldots, p_s$ are the distinct prime number which have the highest powers $n_1, \ldots, n_s$. Note that the vector $(1, 1, \ldots, 1)$ generates an order $n$ cyclic subgroup of $N$. We calculate the Gale dual $(\beta^e)^\vee : \mathbb{Z} \to \mathbb{Z} \oplus \bigoplus_{i \in \{i_1, \ldots, i_t\}} \mathbb{Z}_{p_i^{n_i}}$, where $DG(\beta^e) = \mathbb{Z} \oplus \bigoplus_{i \in \{i_1, \ldots, i_t\}} \mathbb{Z}_{p_i^{n_i}}$, so we have the following exact sequence:

$$0 \to \mathbb{Z} \to \mathbb{Z}^{\beta^e} \to N \to \bigoplus_{i \in \{i_1, \ldots, i_t\}} \mathbb{Z}_{p_i^{n_i}}^n \to 0$$

$$0 \to 0 \to \mathbb{Z} \overset{(\beta^e)^\vee}{\longrightarrow} \mathbb{Z} \oplus \bigoplus_{i \in \{i_1, \ldots, i_t\}} \mathbb{Z}_{p_i^{n_i}}^n \to \mathbb{Z}_n \oplus \bigoplus_{i \in \{i_1, \ldots, i_t\}} \mathbb{Z}_{p_i^{n_i}}^n \to 0$$

So we obtain

$$1 \to \mu \to \mathbb{C}^\times \times \prod_{i \notin \{i_1, \ldots, i_t\}} \mu_{p_i^{n_i}}^n \overset{\alpha^e}{\longrightarrow} \mathbb{C}^\times \to 1 \quad (12)$$

where the map $\alpha^e$ in (12) is given by the matrix $\begin{bmatrix} n \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ and $\mu = \mu_n \times \prod_{i \notin \{i_1, \ldots, i_t\}} \mu_{p_i^{n_i}}^n \cong N$. The extended toric Deligne-Mumford stack is $\mathcal{X}(\Sigma^e) = [\mathbb{C}^\times / \mathbb{C}^\times \times \prod_{i \notin \{i_1, \ldots, i_t\}} \mu_{p_i^{n_i}}^n] = \mathcal{B}\mu$, the classifying stack of the group $\mu$. Let $L$ be a line bundle over a smooth variety $B$, let $L^\times$ be the principal $\mathbb{C}^\times$-bundle induced from $L$ removing the zero section. From our twist we have $L^\times \mathcal{X}(\Sigma^e) = L^\times \mathcal{X} \times \mathbb{C}^\times \times \prod_{i \notin \{i_1, \ldots, i_t\}} \mu_{p_i^{n_i}}^n = [L^\times / \mathbb{C}^\times \times \prod_{i \notin \{i_1, \ldots, i_t\}} \mu_{p_i^{n_i}}^n]$, which is exactly a $\mu$-gerbe $\mathcal{X}$ over $B$. The structure of this gerbe is a $\mu$-gerbe coming from the line bundle $L$ plus a trivial $\prod_{i \notin \{i_1, \ldots, i_t\}} \mu_{p_i^{n_i}}$-gerbe over $B$. For this toric stack bundle, we know that $Box(\Sigma^e) = N$, so we have the following Proposition for the inertia stack.

**Proposition 5.1** The inertia stack of this toric stack bundle $\mathcal{X}$ is $p_1^{n_1} \cdots p_s^{n_s}$ copies of the $\mu$-gerbe $\mathcal{X}$.

From our main Theorem, we have:

**Proposition 5.2** The orbifold cohomology ring of the toric stack bundle $\mathcal{X}$ is given by:

$$H^*_{orb}(\mathcal{X}, \mathbb{Q}) \cong H^*(B, \mathbb{Q}) \otimes H^*_{orb}(\mathcal{B}\mu, \mathbb{Q})$$
where \( H^*_{orb}(B; \mathbb{Q}) = \mathbb{Q}[t_1, \cdots, t_s]/(t_0^{n_1} - 1) \).

Let \( N = \mathbb{Z}_r \), and \( \beta : \mathbb{Z} \to \mathbb{Z}_r \) be the natural projection. The toric Deligne-Mumford stack \( \mathcal{X}(\Sigma) = B_\mu_r \). Let \( L \to B \) be a line bundle, then the toric stack bundle \( \mathcal{X} = B_{(L,r)} \) is the \( \mu_r \)-gerbe over \( B \) determined by the line bundle \( L \). We have:

**Corollary 5.3** The orbifold cohomology ring of \( B_{(L,r)} \) is isomorphic to \( H^*(B)[t]/(t^r - 1) \).

If the variety \( B \) is not a toric variety, then the toric stack bundle over \( B \) is not a toric Deligne-Mumford stack. But suppose \( B \) is a smooth toric variety, then a \( \mu_r \)-gerbe \( \mathcal{X} \) can give a toric Deligne-Mumford stack in [4].

**Example** Let \( B = \mathbb{P}^d \) be the \( d \)-dimensional projective space. We give stacky fan \( \Sigma = (N, \Sigma, \beta) \) as follows: let \( N = \mathbb{Z}^d \oplus \mathbb{Z}_r \), \( \beta : \mathbb{Z}^{d+1} \to N \) be determined by the vectors:

\[
\{(1, 0, \ldots, 0, 0), (0, 1, \ldots, 0, 0), \ldots, (0, 0, \ldots, 1, 0), (-1, -1, \ldots, -1, 1)\}
\]

Then \( DG(\beta) = \mathbb{Z} \), and the Gale dual \( \beta^\vee \) is given by the matrix \([r, r, \ldots, r] \). So we have the following exact sequences:

\[
0 \to \mathbb{Z} \xrightarrow{} \mathbb{Z}^{d+1} \xrightarrow{\beta} \mathbb{Z}^d \oplus \mathbb{Z}_r \to 0 \to 0
\]

\[
0 \to \mathbb{Z}^d \xrightarrow{} \mathbb{Z}^{d+1} \xrightarrow{\beta^\vee} \mathbb{Z} \xrightarrow{} \mathbb{Z}_r \to 0
\]

Then we obtain the exact sequence:

\[
1 \to \mu_r \xrightarrow{} \mathbb{C}^\times \xrightarrow{\alpha} (\mathbb{C}^\times)^{d+1} \to (\mathbb{C}^\times)^d \to 1
\]

The toric Deligne-Mumford stack \( \mathcal{X}(\Sigma) := [\mathbb{C}^{d+1} - \{0\}/\mathbb{C}^\times] \) is the canonical \( \mu_r \)-gerbe over the projective space \( \mathbb{P}^d \) coming from the canonical line bundle, where the \( \mathbb{C}^\times \) action is given by \( \lambda \cdot (z_1, \ldots, z_{d+1}) = (\lambda^r \cdot z_1, \ldots, \lambda^r \cdot z_{d+1}) \). Denote this toric Deligne-Mumford stack by \( \mathcal{G}_r = \mathbb{P}(r, \ldots, r) \). If the homomorphism \( \beta : \mathbb{Z}^{d+1} \to N \) is determined by the vectors:

\[
\{(1, 0, \ldots, 0, 0), (0, 1, \ldots, 0, 0), \ldots, (0, 0, \ldots, 1, 0), (-1, -1, \ldots, -1, 0)\}
\]

then \( DG(\beta) = \mathbb{Z} \oplus \mathbb{Z}_r \). Comparing to the former exact sequence, we have the exact sequence:

\[
1 \to \mu_r \to \mathbb{C}^\times \times \mu_r \to (\mathbb{C}^\times)^{d+1} \to (\mathbb{C}^\times)^d \to 1
\]

The corresponding toric Deligne-Mumford stack is the trivial \( \mu_r \)-gerbe \( \mathbb{P}^d \times B_\mu_r \) coming from the trivial line bundle over \( \mathbb{P}^d \). The coarse Moduli spaces of these two stacks are all the projective space \( \mathbb{P}^d \). From the Theorem of this paper or the main Theorem in [4], the orbifold cohomology rings of these two stacks are isomorphic although as stacks, they are different.
Remark Let $H$ represent the hyperplane class of $\mathbb{P}^d$, then $H^*\text{orb}(G_r, \mathbb{Q}) \cong \mathbb{Q}[H]/(H^{d+1} \otimes \mathbb{Q}[t]/(t^r - 1))$. We conjecture that the orbifold quantum cohomology ring of $G_r$ defined in [6] is isomorphic to $\mathbb{Q}[H]/(H^{d+1} - f(H, q)) \otimes \mathbb{Q}[t]/(t^r - 1 - g(t, q))$, where $f, g$ are two relations and $q$ is the quantum parameter. The orbifold quantum cohomology of trivial gerbe case has been computed in [12], where $f(H, q) = q$ and $g(t, q) = 0$.

Remark We conjecture that the small orbifold quantum cohomology ring of the nontrivial $\mu_r$-gerbe and trivial $\mu_r$-gerbe over the projective space $\mathbb{P}^d$ should be different. This means that the orbifold quantum cohomology can classify these two different stacks.

6 Application.

In this section we generalize one result of Borisov, Chen and Smith [4] to the toric stack bundle case.

Let $X(\Sigma)$ be a simplicial toric variety, and let $\mathcal{X}(\Sigma)$ be the associated toric DM stack, where $\Sigma = (N, \Sigma, \beta)$. Let $\Sigma'$ be a subdivision of $\Sigma$ such that $X(\Sigma')$ is a crepant resolution of $X(\Sigma)$. Suppose there are $m$ rays in $\Sigma'$, let $i : (\mathbb{C}^\times)^n \rightarrow (\mathbb{C}^\times)^m$ be the inclusion. From the following commutative diagram:

$$
\begin{array}{ccc}
0 & \rightarrow & \mathbb{Z}^{n-d} \rightarrow \mathbb{Z}^n \rightarrow N \rightarrow 0 \\
\downarrow & & \downarrow i \\
0 & \rightarrow & \mathbb{Z}^{m-d} \rightarrow \mathbb{Z}^m \rightarrow N \rightarrow 0 \\
\end{array}
$$

Taking Gale dual we get:

$$
\begin{array}{ccc}
0 & \rightarrow & N^* \rightarrow (\mathbb{Z}^n)^* \rightarrow N \rightarrow 0 \\
\downarrow id & & \downarrow (\beta')^\vee \\
0 & \rightarrow & (\mathbb{Z}^m)^* \rightarrow N^* \rightarrow N \rightarrow 0 \\
\end{array}
$$

So applying the $\text{Hom}$ functor we have the following diagram:

$$
\begin{array}{ccc}
(\mathbb{C}^\times)^n & \rightarrow & (\mathbb{C}^\times)^m \\
T \downarrow id & & T \downarrow id \\
\end{array}
$$

Let $P \rightarrow B$ be a principal $(\mathbb{C}^\times)^n$-bundle, we still use $P$ to represent the principal $(\mathbb{C}^\times)^m$-bundle induced by $i$, then they induce the same principal $T$ bundle $E$ over $B$. So $E^X(\Sigma') \rightarrow E^X(\Sigma)$ is the crepant resolution. And $E^X(\Sigma)$ is the coarse moduli space of the toric stack bundle $E^X(\Sigma)$ from Proposition 3.3. We have the following result:
Proposition 6.1 If the Chow ring of the smooth variety $B$ is a Cohen-Macaulay ring. Then there is a flat family $\mathcal{S} \rightarrow \mathbb{P}^1$ of schemes such that $\mathcal{S}_0 \cong \text{Spec}(A^\ast_{orb}(\mathcal{P}\chi(\Sigma)))$ and $\mathcal{S}_\infty \cong \text{Spec}(A^\ast(\mathcal{P}\chi(\Sigma)))$.

Proof. We also construct a family of algebras over $\mathcal{S}/I$ system of parameters on $\mathcal{A}$ a smooth variety, $I$ binomial ideal determined by (1). Let $\tilde{\mathcal{I}}$ be the ideal generated by $c_1(\xi_{\theta_i}) + \sum_{i=1}^n \theta_j(b_i)y^{b_i}$ for $1 \leq j \leq d$, where $\theta_1, \ldots, \theta_d$ is a basis of $N^*$. Since $\Sigma'$ is a regular subdivision of $\Sigma$, then there is a $\Sigma'$-linear support function $h : N \rightarrow \mathbb{Z}$ such that $\sum_{i=1}^n \theta_j(b_i)y^{b_i}$ is also a homogeneous system of parameters on $S/I_2$. The Chow ring $A^\ast(B)$ is a Cohen-Macaulay ring, so $S/I_2$ is also Cohen-Macaulay. So the sequence is a regular sequence. Therefore, the Hilbert function of the family $S[t_1/(I_1 + I_2)]$ is constant outside a finite set in $\mathbb{Q}^*$. On the other hand, for the family over $\mathbb{P}^1 - \{0\}$, we use the similar method in [4] to get the family $S[t_2/(I_1 + I_2)]$, where $I_2$ is the binomial ideal in $S[t_2]$ defined in [4]. There exists an automorphism $\varphi$ between these two families so that we construct such a family over $\mathbb{P}^1$. All the left proof is the same as the one in [4]. We omit the details. $\square$

Remark Ruan [18] conjectured that the cohomology ring of crepant resolution is isomorphic to the orbifold Chow ring of the orbifold if we add some quantum corrections on the ordinary cohomology ring of the crepant resolution coming from the exceptional divisors. Let $\mathbb{P}(1,1,2)$ be the weighted projective plane with one orbifold point with local group $\mathbb{Z}_2$, the Hirzburch surface is the crepant resolution of $\mathbb{P}(1,1,2)$. We can compute the quantum correction of the cohomology ring of the Hirzburch surface and check Ruan’s conjecture. This case has been done recently in [16].

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