Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals based on multiple Fourier-Legendre series

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Abstract. The article is devoted to optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 4 based on multiple Fourier-Legendre series. The mentioned stochastic integrals are part of strong numerical methods with convergence orders 1.0, 1.5, and 2.0 for Ito stochastic differential equations with multidimensional non-commutative noise. We show that the lengths of sequences of independent standard Gaussian random variables required for the mean-square approximation of iterated Ito stochastic integrals can be significantly reduced without the loss of the mean-square accuracy of approximation for these stochastic integrals.

1. Strong Taylor-Ito numerical schemes with convergence orders 1.0, 1.5, and 2.0

Today, a large number of mathematical models of dynamical systems under random disturbances are known in the form of Ito stochastic differential equations (SDEs) and systems of Ito SDEs. Among them, we note models in the aerospace industry (helicopter rotor models and satellite dynamics models), hydrology, seismology, geophysics, genetics, electrodynamics, chemical kinetics [1], stochastic financial mathematics [2], biology [3], epidemiology [4], medicine [5], and other fields. Moreover, Ito SDEs arise when solving various mathematical problems such as stochastic stability (for example, satellite orbital stability), signals filtering against the background of random noises, parameter estimation of stochastic systems, stochastic optimal control [1]. The exact solutions to Ito SDEs are known in rare cases. In addition, often, even knowing the exact solution to the Ito SDE, we cannot simulate it without using special numerical methods. The above facts well explain the importance and relevance of the problem of numerical integration of Ito SDEs.

It is well known that one of the promising approaches to the numerical integration of Ito SDEs is an approach based on the stochastic Taylor expansions [1, 6]. The essential feature of such expansions is a presence in them of the so-called iterated stochastic integrals, which play the key role for solving the problem of numerical integration of Ito SDEs.

The article is devoted to the development of an effective method of iterated Ito stochastic integrals approximation based on generalized multiple Fourier series and proposed by the first author of the article in monograph [7]. More precisely, the article solves the optimization problem for the method of...
generalized multiple Fourier series, which is still the most general and effective among the existing methods for the mean-square approximation of iterated Ito stochastic integrals with multiplicities \( k = 3, 4, \ldots \) The detailed comparison of the method based on generalized multiple Fourier series with the existing methods of the mean-square approximation of iterated Ito stochastic integrals is given in monograph [8].

Let \( (\Omega, F, P) \) be a complete probability space, let \( \{F_t, t \in [0, T] \} \) be a nondecreasing right-continuous family of \( \sigma \)-algebras of F and let \( w_t \) be a standard \( m \)-dimensional Wiener process, which is \( F_t \)-measurable for any \( t \in [0, T] \). We assume that the components \( w_i(t) \) of this process are independent. Consider an Ito SDE in the integral form

\[
x_t = x_0 + \int_0^t a(x_s, s) \, ds + \int_0^t \sum_{j=1}^n B_j(x_s, s) \, dw_j(s), \quad x_0 = 0, \quad \omega,
\]

where \( \omega \in \Omega, \ a(x, t) : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n, \ B_j(x, t) \) is the \( j \)th component of \( B(x, t) : \mathbb{R}^n \times [0, T] \to \mathbb{R}^{m \times m}, \ x_0 \) is \( F_0 \)-measurable, \( M|x_0|^2 < \infty \) (\( M \) is an expectation), \( x_0 \) and \( w_t - w_0 \) are independent \( (t > 0) \).

Let us consider the following differential operators

\[
L = \frac{\partial}{\partial t} + \sum_{i=1}^n a^{(i)}(x, t) \frac{\partial}{\partial x^{(i)}} + \frac{1}{2} \sum_{j=1}^n \sum_{i=1}^n B^{(ij)}(x, t)B^{(ij)}(x, t) \frac{\partial^2}{\partial x^{(i)} \partial x^{(j)}},
\]

(2)

\[
G^{(i)}_0 = \sum_{j=1}^n B^{(ij)}(x, t) \frac{\partial}{\partial x^{(j)}}, \quad i = 1, \ldots, m,
\]

(3)

where \( a^{(i)}(x, t) \) is the \( i \)th component of \( a(x, t) \) and \( B^{(ij)}(x, t) \) is the \( ij \)th element of \( B(x, t) \).

Consider the following iterated Ito stochastic integrals

\[
I^{(i_1 \ldots i_k)}_{(t_1 \ldots t_k), x, t} = \int_{t=0}^T (t-t_k)^{i_k} \ldots \int_{t=0}^{t_{i_{k-1}}} (t-t_{i_{k-1}})^{i_{k-1}} dW_{i_1}^{(i_{k-1})} \ldots dW_{i_k}^{(i_{k-1})} (i_1, \ldots, i_k = 1, 2, \ldots \text{ and } i_1, \ldots, i_k = 1, \ldots, m).
\]

(4)

Assume that \( a(x, t) \) and \( B(x, t) \) are enough smooth functions with respect to the variables \( x \) and \( t \). Consider the partition \( \{t_1, t_2, \ldots, t_N \} \) such that \( 0 = t_0 < t_1 < \ldots < t_N = T, \ \Delta_N = \max_{0 \leq i < N} |t_{i+1} - t_i| \).

We will say [1] that a numerical scheme \( \hat{y}_q, q = 0, 1, \ldots, N \) converges strongly with order \( \gamma > 0 \) at time moment \( T \) to the process \( x_t, t \in [0, T] \) if there exist a constant \( C > 0 \), which does not depend on \( \Delta_N \), and a \( \hat{\sigma} > 0 \) such that \( \mathbb{E}[|x_T - \hat{y}_T|^\gamma] \leq C(\Delta_N)^\hat{\sigma} \) for each \( \Delta_N \in (0, \hat{\sigma}) \).

Consider the strong Taylor-Ito scheme with convergence order 2.0 [1, 6, 7]:

\[
y_{q+1} = y_q + \sum_{i=1}^n B_i^{(i)}(0)_{(i)_{0,1}, r_q} + \Delta a + \sum_{i, j=1}^n G_0^{(ij)}B_j^{(i)}(0)_{(0)_{r_q, r_q}, r_q} + \sum_{i=1}^n \left[ G_0^{(i)}(a\Delta t^{(i)}_{(0)_{r_q, r_q}, r_q} + \hat{\Delta t}^{(i)}_{(0)_{r_q, r_q}, r_q}) - LB_0 \hat{\Delta t}^{(i)}_{(0)_{r_q, r_q}, r_q} \right]
\]

\[
+ \sum_{i, j=1}^n G_0^{(ij)}B_j^{(i)}(0)_{(00)_{r_q, r_q}, r_q} + \frac{\Delta^2}{2} \Delta a + \sum_{i=1}^n \left[ G_0^{(i)}(LB_0 \hat{\Delta t}^{(i)}_{(10)_{r_q, r_q}, r_q} - \hat{\Delta t}^{(i)}_{(01)_{r_q, r_q}, r_q}) - LB_0 \hat{\Delta t}^{(i)}_{(10)_{r_q, r_q}, r_q} \right]
\]

\[
+ G_0^{(i)}(a\Delta t^{(i)}_{(10)_{r_q, r_q}, r_q} + \Delta t^{(i)}_{(00)_{r_q, r_q}, r_q})) + \sum_{i=1}^n \left[ G_0^{(i)}(LB_0 \hat{\Delta t}^{(i)}_{(11)_{r_q, r_q}, r_q} - \hat{\Delta t}^{(i)}_{(101)_{r_q, r_q}, r_q}) \right]
\]

\[
+ G_0^{(i)}(a\Delta t^{(i)}_{(11)_{r_q, r_q}, r_q} + \Delta t^{(i)}_{(000)_{r_q, r_q}, r_q}) \right]
\]

(5)

where the functions \( B_i, a, G_0^{(i)}B_j, \ldots \) are calculated at the point \( (y_q, r_q) \), \( y_q \) is an approximation of solution to Ito SDE (1) at moment \( r_q, \ \hat{\Delta t}^{(i)}_{(i, j)_{r_q, r_q}, r_q} \) is an approximation of the iterated Ito stochastic integral \( I^{(i, j)}_{(i, j)_{r_q, r_q}, r_q} \), and \( L, G_0^{(i)}, i = 1, \ldots, m \) are defined by equalities (2) and (3). Note that the first
four terms on the right-hand side of scheme (5) correspond to the Milstein scheme [6] and terms up to $0.5 \cdot \Delta t \cdot \mathbf{a}$ inclusively on the right-hand side of scheme (5) correspond to the strong scheme with convergence order $1.5$. Among the standard conditions ensuring the convergence of the above numerical schemes, we note the condition for approximations of iterated Ito stochastic integrals [1]:

$$M \left( \int I_{(i_1, \ldots, i_k)}^{(j_1, \ldots, j_k)}_{(l_1, \ldots, l_k)_{r_{j_{i_l}}}, r_{j_{l_k}}} - \int I_{(i_1, \ldots, i_k)}^{(j_1, \ldots, j_k)}_{(l_1, \ldots, l_k)_{r_{j_{i_l}}}, r_{j_{l_k}}} \right)^2 \leq C \Delta^{-r},$$

(6)

where $q = 0, 1, \ldots, N - 1$, constant $C$ is independent of $\Delta$, and $0.5 \cdot r = 1.0, 1.5, \text{ and } 2.0$.

2. Method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series

Let us consider the effective approach to expansion and mean-square approximation of iterated Ito stochastic integrals (other approaches can be found in [1-3, 9-19]):

$$J[y^{(k)}]_{r_j} = \int_t^T \psi_k(t) \psi_{l_1}(t_1) \cdots \psi_{l_k}(t_k) d\mathbf{w}_{i_1}^{(l_1)} \cdots d\mathbf{w}_{i_k}^{(l_k)},$$

(7)

where $i_1, \ldots, i_k = 0, 1, \ldots, m$, every $\psi_i(r)$ ($l = 1, \ldots, k$) is a continuous non-random function on $[t, T]$, $w_i^{(l)} (i = 1, \ldots, m)$ are independent standard Wiener processes, and $w_i^{(l)} = \tau$.

Let $K(t_1, \ldots, t_k) = \psi_k(t) \psi_{l_1}(t_1) \cdots \psi_{l_k}(t_k) 1_{[t_1, \ldots, t_k]}$ for $t_1, \ldots, t_k \in [t, T]$ ($k \geq 2$) and $K(t) = \psi_k(t)$ for $t \in [t, T]$, where $1_{A}$ is the indicator of the set $A$. Suppose that $\{\phi_j(x)\}_{j=0}^\infty$ is a complete orthonormal system of functions in $L_2([t, T])$. It is well known that the generalized multiple Fourier series of $K(t_1, \ldots, t_k) \in L_2([t, T]^k)$ is converging to $K(t_1, \ldots, t_k)$ in the mean-square sense, i.e.

$$\lim_{\rho_1, \ldots, \rho_k \to \infty} \left\| K(t_1, \ldots, t_k) - \sum_{j=0}^{\rho_1} \cdots \sum_{j=0}^{\rho_k} C_{l_1, \ldots, l_k} \prod_{j=1}^{k} \phi_j(t_j) dt_1 \cdots dt_k \right\| = 0,$$

(8)

where

$$C_{l_1, \ldots, l_k} = \int_{[t, T]^k} K(t_1, \ldots, t_k) \prod_{j=1}^{k} \phi_j(t_j) dt_1 \cdots dt_k$$

is the Fourier coefficient and $\| \|$ is the $L_2([t, T]^k)$-norm.

Theorem 1 [7]. Suppose that every $\psi_i(r)$ ($l = 1, \ldots, k$) is a continuous non-random function on $[t, T]$ and $\{\phi_j(x)\}_{j=0}^\infty$ is a complete orthonormal system of continuous functions in the space $L_2([t, T])$. Then

$$M \left( J[y^{(k)}]_{r_j} - J[y^{(k)}]_{\rho_1, \ldots, \rho_k} \right)^2 \leq k! \left( I_k - \sum_{j=0}^{\rho_1} \cdots \sum_{j=0}^{\rho_k} C_{l_1, \ldots, l_k}^2 \right) \to 0$$

(9)

if $\rho_1, \ldots, \rho_k \to \infty$, where $I_k^{1/2}$ is the $L_2([t, T]^k)$-norm of $K(t_1, \ldots, t_k)$,

$$J[y^{(k)}]_{\rho_1, \ldots, \rho_k} = \sum_{j=0}^{\rho_1} \cdots \sum_{j=0}^{\rho_k} C_{l_1, \ldots, l_k} \left( \prod_{i=1}^{k} \phi_i(t_i) \Delta w_{i_1}^{(l_1)} \cdots \Delta w_{i_k}^{(l_k)} \right),$$

$$i_1, \ldots, i_k = 1, \ldots, m \text{ for } T - t \in (0, \infty) \text{ and } i_1, \ldots, i_k = 0, 1, \ldots, m \text{ for } T - t \in (0, 1),$$

$\Delta w_{i_1}^{(l_1)}$ is defined by formula (7), i.i.m. is a limit in the mean-square sense, $\zeta_j^{(i)} = \int_t^T \phi_j(s) d\mathbf{w}_{i_1}^{(l_1)} (i = 0, 1, \ldots, m)$ are i.i.d. $N(0, 1)$-r.v.'s for various $i$ or $j$ if $i \neq 0$,

$G_k = H_k \setminus L_k$, $H_k = \{(l_1, \ldots, l_k) : l_1 \leq 1, \ldots, l_k = 0, 1, \ldots, N - 1 \}$,

$L_k = \{(l_1, \ldots, l_k) : l_1, \ldots, l_k = 0, 1, \ldots, N - 1 \}$, $\Delta w_{i_1}^{(l_1)} = \mathbf{w}_{i_1}^{(l_1)} - \mathbf{w}_{i_1}^{(l_1)}$, $\{\tau_q\}_{q=0}^N$ is a
partition of $[t,T]$ such that $t = \tau_0 < \cdots < \tau_N = T$, $\max_{0\leq q\leq N-1} (\tau_{q+1} - \tau_q) \to 0$ if $N \to \infty$, $C_{j_1 \cdots j_k}$ is defined by formula (8),

Let $J[\psi^{(k)}]_{T,j}^p$ be $J[\psi^{(k)}]_{T,j}^{p_1 \cdots p_k}$ for $p_1 = \cdots = p_k = p$ and let $E_k^p = \mathbf{M} \left( J[\psi^{(k)}]_{T,j}^p - J[\psi^{(k)}]_{T,j}^p \right)^2$.

Combining estimates (6) and (9) for $p_1 = \cdots = p_k = p$, we obtain

$$k! \left( I_k - \sum_{j_1,\ldots,j_k = 0} C_{j_1 \cdots j_k}^2 \right) \leq C(T-t)^{r-1}. \tag{10}$$

It is not difficult to see that the multiplier factor $k!$ on the left-hand side of inequality (10) leads to a significant increase of computational costs for approximation of iterated Ito stochastic integrals. The mentioned problem can be overcome if we calculate the mean-square approximation error $E_k^p$ exactly.

Theorem 2 [8]. Suppose that the conditions of theorem 1 are satisfied. Then

$$E_k^p = I_k - \sum_{j_1,\ldots,j_k = 0} C_{j_1 \cdots j_k} \mathbf{M} \left( J[\psi^{(k)}]_{T,j}^p - J[\psi^{(k)}]_{T,j}^p \right), \tag{11}$$

where $i_1,\ldots,i_k = 1,\ldots,m$; expression $\sum_{(j_1,\ldots,j_k)}$ means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. At the same time if $j_r$ swapped with $j_q$ in the permutation $(j_1,\ldots,j_k)$, then $i_r$ swapped with $i_q$ in the permutation $(i_1,\ldots,i_k)$; another notations are the same as in theorem 1.

For the further consideration, we note that

$$\mathbf{M} \left( J[\psi^{(k)}]_{T,j}^p - J[\psi^{(k)}]_{T,j}^p \right) = C_{j_1 \cdots j_k}. \tag{12}$$

3. Approximation of iterated Ito stochastic integrals using Legendre polynomials

Using theorem 1 and the complete orthonormal system of Legendre polynomials in $L_2([t,T])$, we obtain the following formulas for numerical modeling of iterated Ito stochastic integrals from numerical scheme (5) [7]

$$I_{(0)T,j}^{(i_1) q_1} = \frac{T-t}{2} \left( \frac{\rho(i_1)}{\rho(i_1)} + \sum_{j=1}^{q_1} \frac{1}{\sqrt{4j^2-1}} \left( \frac{\rho(i_1)}{\rho(i_1)} - \frac{\rho(i_1)}{\rho(i_1)} \right) - 1 \right),$$

$$I_{(0)T,j}^{(i_1) q_2} = \sum_{j_1,\ldots,j_k = 0} C_{j_1 \cdots j_k} \left( \frac{\rho(i_1)}{\rho(i_1)} \frac{\rho(i_1)}{\rho(i_1)} - 1 \right) \left( \frac{\rho(i_1)}{\rho(i_1)} \frac{\rho(i_1)}{\rho(i_1)} - 1 \right),$$

$$I_{(10)T,j}^{(i_1) q_2} = \sum_{j_1,\ldots,j_k = 0} C_{j_1 \cdots j_k} \left( \frac{\rho(i_1)}{\rho(i_1)} \frac{\rho(i_1)}{\rho(i_1)} - 1 \right) \left( \frac{\rho(i_1)}{\rho(i_1)} \frac{\rho(i_1)}{\rho(i_1)} - 1 \right),$$

$$I_{(000)T,j}^{(i_1) q_2} = \sum_{j_1,\ldots,j_k = 0} C_{j_1 \cdots j_k} \left( \frac{\rho(i_1)}{\rho(i_1)} \frac{\rho(i_1)}{\rho(i_1)} - 1 \right) \left( \frac{\rho(i_1)}{\rho(i_1)} \frac{\rho(i_1)}{\rho(i_1)} - 1 \right),$$

$$I_{(000)T,j}^{(i_1) q_2} = \sum_{j_1,\ldots,j_k = 0} C_{j_1 \cdots j_k} \left( \frac{\rho(i_1)}{\rho(i_1)} \frac{\rho(i_1)}{\rho(i_1)} - 1 \right) \left( \frac{\rho(i_1)}{\rho(i_1)} \frac{\rho(i_1)}{\rho(i_1)} - 1 \right),$$

$$1_A$$ is the indicator of the set $A$.\]
\[ C_{j,h,h}^{00} = \frac{1}{8} L_{j,h,h} (T-t)^{3/2} \mathcal{E}_{j,h,h}^{00}, \quad C_{j,h}^{01} = \frac{1}{8} L_{j,h} (T-t)^{2} \mathcal{C}_{j,h}^{01}, \]
\[ C_{j,j,j}^{10} = \frac{1}{8} L_{j,j,j} (T-t)^{2} \mathcal{C}_{j,j,j}^{10}, \quad C_{j,h,j,j,j}^{00} = \frac{1}{16} L_{j,h,j,j,j} (T-t)^{4} \mathcal{C}_{j,h,j,j,j}^{00}. \]
\[ L_{j,j,j} = \left( \prod_{l=1}^{j} (2j_l+1) \right)^{1/2}, \quad \mathcal{C}_{j,j,j}^{00} = \int_{-j}^{j} \frac{1}{2} \sum_{i=0}^{j} P_j (x_i) \sum_{j=0}^{i} P_j (x_j) \, dx_1 \ldots dx_j, \]
\[ \mathcal{C}_{j,j,j}^{10} = \int_{-j}^{j} \frac{1}{2} \sum_{i=0}^{j} P_j (x_i) \sum_{j=0}^{i} \left( 1+x_i \right) P_j (x_j) \, dx_1 \ldots dx_j, \]
\[ P_j (x) \text{ is the Legendre polynomial; other notations are the same as in theorem 1.} \]

4. Main results
This section is devoted to the optimization of approximations of iterated It\'o stochastic integrals, i.e. we discuss how to essentially minimize the values \( q, q_1, q_2, q_3 \) from the previous section.

Let \( E_{(j_i-j_i)^p}^{q_1,q_2,q_3} \) be \( E_{(j)^p}^{q_1,q_2,q_3} \) for stochastic integral (4). From theorem 2 we obtain (equality (12))
\[ E_{(j)^0}^{(0)} = \frac{(T-t)^2}{2} \left( \frac{1}{2} - \sum_{i=0}^{j} \frac{1}{4i^2-1} \right), \quad i \neq i, \]
\[ E_{(j)^0}^{(1)} = (T-t)^4 \left( \frac{1}{4} - \frac{1}{64} \sum_{j=0}^{p} L_{j,j}^{0} \left( \mathcal{C}_{j,j}^{01} \right)^2 \right), \quad i \neq i, \]
\[ E_{(j)^0}^{(1)} = (T-t)^4 \left( \frac{1}{4} - \frac{1}{64} \sum_{j=0}^{p} L_{j,j}^{0} \left( \mathcal{C}_{j,j}^{10} \right)^2 \right), \quad i = i, \]
\[ E_{(j)^0}^{(0)} = (T-t)^4 \left( \frac{1}{12} - \frac{1}{64} \sum_{j=0}^{p} L_{j,j}^{0} \left( \mathcal{C}_{j,j}^{10} \right)^2 \right), \quad i \neq i, \]
\[ E_{(j)^0}^{(1)} = (T-t)^4 \left( \frac{1}{6} - \frac{1}{64} \sum_{j=0}^{p} L_{j,j}^{0} \left( \mathcal{C}_{j,j}^{01} \right)^2 \right), \quad i \neq i, \]
\[ E_{(j)^0}^{(0)} = (T-t)^4 \left( \frac{1}{6} - \frac{1}{64} \sum_{j=0}^{p} L_{j,j}^{0} \left( \mathcal{C}_{j,j}^{10} \right)^2 \right), \quad i = i, \]
\[ E_{(j)^0}^{(0)} = (T-t)^4 \left( \frac{1}{6} - \frac{1}{64} \sum_{j=0}^{p} L_{j,j}^{0} \left( \mathcal{C}_{j,j}^{10} \right)^2 \right), \quad i \neq i, \]
\[ E_{(j)^0}^{(0)} = (T-t)^4 \left( \frac{1}{6} - \frac{1}{64} \sum_{j=0}^{p} L_{j,j}^{0} \left( \mathcal{C}_{j,j}^{10} \right)^2 \right), \quad i = i, \]
\[ E_{(j)^0}^{(0)} = (T-t)^4 \left( \frac{1}{6} - \frac{1}{64} \sum_{j=0}^{p} L_{j,j}^{0} \left( \mathcal{C}_{j,j}^{10} \right)^2 \right), \quad i = i, \]
\[ E_{(j)^0}^{(0)} = (T-t)^4 \left( \frac{1}{6} - \frac{1}{64} \sum_{j=0}^{p} L_{j,j}^{0} \left( \mathcal{C}_{j,j}^{10} \right)^2 \right), \quad i = i, \]
\[ E_{(j)^0}^{(0)} = (T-t)^4 \left( \frac{1}{6} - \frac{1}{64} \sum_{j=0}^{p} L_{j,j}^{0} \left( \mathcal{C}_{j,j}^{10} \right)^2 \right), \quad \text{if } i_1 \neq i_2 \text{ are pairwise different}, \]
\[ E_{4}^{(0000)}(T-t)^4 \left( \frac{1}{24} - \frac{1}{256} \sum_{j_1, j_2, j_3, j_4} L_{j_1, j_2}^{j_2, j_3, j_4} \sum_{i_1, i_2, i_3, i_4} C_{j_1, j_2; j_3, j_4}^{0000} \right), \quad i_1 = i_3 \neq i_2, i_4, \]

(23)

\[ E_{4}^{(0000)}(T-t)^4 \left( \frac{1}{24} - \frac{1}{256} \sum_{j_1, j_2, j_3, j_4} L_{j_1, j_2}^{j_2, j_3, j_4} \sum_{i_1, i_2, i_3, i_4} C_{j_1, j_2; j_3, j_4}^{0000} \right), \quad i_1 = i_2 \neq i_3, i_4, \]

(24)

\[ E_{4}^{(0000)}(T-t)^4 \left( \frac{1}{24} - \frac{1}{256} \sum_{j_1, j_2, j_3, j_4} L_{j_1, j_2}^{j_2, j_3, j_4} \sum_{i_1, i_2, i_3, i_4} C_{j_1, j_2; j_3, j_4}^{0000} \right), \quad i_2 = i_3 \neq i_1, i_4, \]

(25)

\[ E_{4}^{(0000)}(T-t)^4 \left( \frac{1}{24} - \frac{1}{256} \sum_{j_1, j_2, j_3, j_4} L_{j_1, j_2}^{j_2, j_3, j_4} \sum_{i_1, i_2, i_3, i_4} C_{j_1, j_2; j_3, j_4}^{0000} \right), \quad i_3 = i_4 \neq i_1, i_2, \]

(26)

\[ E_{4}^{(0000)}(T-t)^4 \left( \frac{1}{24} - \frac{1}{256} \sum_{j_1, j_2, j_3, j_4} L_{j_1, j_2}^{j_2, j_3, j_4} \sum_{i_1, i_2, i_3, i_4} C_{j_1, j_2; j_3, j_4}^{0000} \right), \quad i_1 = i_2 = i_3 \neq i_4, \]

(27)

\[ E_{4}^{(0000)}(T-t)^4 \left( \frac{1}{24} - \frac{1}{256} \sum_{j_1, j_2, j_3, j_4} L_{j_1, j_2}^{j_2, j_3, j_4} \sum_{i_1, i_2, i_3, i_4} C_{j_1, j_2; j_3, j_4}^{0000} \right), \quad i_2 = i_3 = i_4 \neq i_1, \]

(28)

\[ E_{4}^{(0000)}(T-t)^4 \left( \frac{1}{24} - \frac{1}{256} \sum_{j_1, j_2, j_3, j_4} L_{j_1, j_2}^{j_2, j_3, j_4} \sum_{i_1, i_2, i_3, i_4} C_{j_1, j_2; j_3, j_4}^{0000} \right), \quad i_3 = i_4 = i_1 \neq i_2, \]

(29)

\[ E_{4}^{(0000)}(T-t)^4 \left( \frac{1}{24} - \frac{1}{256} \sum_{j_1, j_2, j_3, j_4} L_{j_1, j_2}^{j_2, j_3, j_4} \sum_{i_1, i_2, i_3, i_4} C_{j_1, j_2; j_3, j_4}^{0000} \right), \quad i_3 = i_4 = i_2 \neq i_1, \]

(30)

\[ E_{4}^{(0000)}(T-t)^4 \left( \frac{1}{24} - \frac{1}{256} \sum_{j_1, j_2, j_3, j_4} L_{j_1, j_2}^{j_2, j_3, j_4} \sum_{i_1, i_2, i_3, i_4} C_{j_1, j_2; j_3, j_4}^{0000} \right), \quad i_4 = i_1 \neq i_2, i_3, \]

(31)

\[ E_{4}^{(0000)}(T-t)^4 \left( \frac{1}{24} - \frac{1}{256} \sum_{j_1, j_2, j_3, j_4} L_{j_1, j_2}^{j_2, j_3, j_4} \sum_{i_1, i_2, i_3, i_4} C_{j_1, j_2; j_3, j_4}^{0000} \right), \quad i_1 = i_2 = i_3 = i_4, \]

(32)

\[ E_{4}^{(0000)}(T-t)^4 \left( \frac{1}{24} - \frac{1}{256} \sum_{j_1, j_2, j_3, j_4} L_{j_1, j_2}^{j_2, j_3, j_4} \sum_{i_1, i_2, i_3, i_4} C_{j_1, j_2; j_3, j_4}^{0000} \right), \quad i_1 = i_2 \neq i_3 = i_4, \]

(33)

\[ E_{4}^{(0000)}(T-t)^4 \left( \frac{1}{24} - \frac{1}{256} \sum_{j_1, j_2, j_3, j_4} L_{j_1, j_2}^{j_2, j_3, j_4} \sum_{i_1, i_2, i_3, i_4} C_{j_1, j_2; j_3, j_4}^{0000} \right), \quad i_1 = i_3 = i_4 \neq i_2, \]

(34)

\[ E_{4}^{(0000)}(T-t)^4 \left( \frac{1}{24} - \frac{1}{256} \sum_{j_1, j_2, j_3, j_4} L_{j_1, j_2}^{j_2, j_3, j_4} \sum_{i_1, i_2, i_3, i_4} C_{j_1, j_2; j_3, j_4}^{0000} \right), \quad i_1 = i_4 \neq i_2 = i_3, \]

(35)

Obviously, conditions (13)-(35) do not contain the multiplier factors \(2!, 3!, 4!\) in contrast to estimate (9). However, the number of the mentioned conditions is quite large, which is inconvenient for practice. In this paper we propose the hypothesis that all formulas (13)-(35) can be replaced by formulas (13), (14), (16), (18), (22) in which we can suppose that \(i_1, \ldots, i_4 = 1, \ldots, m\). At that we will not have a noticeable loss of the mean-square approximation accuracy of iterated Ito stochastic integrals.

It should be noted that unlike the method based on theorem 1, existing approaches to the mean-square approximation of iterated stochastic integrals (see, for example, [1, 2, 9-17, 19]) do not allow to choose different numbers \(p\) for approximations of different iterated stochastic integrals with multiplicities.
Moreover, the noted approaches exclude the possibility for obtaining of approximate and exact expressions similar to formulas (9), (11). The detailed comparison of theorem 1 with methods from [1, 2, 9-17, 19] is given in chapter 6 of monograph [8].

Let \( q(\alpha) \) be numbers \( p \) from formulas (13)-(35), where \( \alpha \) are numbers of these formulas. For example, \( q(35) \) is the number \( p \) from equality (35). Let

\[
E_2^p \leq (T - t)^\alpha, \quad E_3^p \leq (T - t)^\alpha,
\]

(36)

where \( E_2^p, E_3^p \) mean the left-hand sides of equalities (13) and (18)-(21) respectively. Let

\[
E_2^p \leq (T - t)^\alpha, \quad E_3^p \leq (T - t)^\alpha, \quad E_4^p \leq (T - t)^\alpha,
\]

(37)

where \( E_2^p \) means the left-hand sides of formulas (13)-(17), \( E_3^p \) means the left-hand sides of formulas (18)-(21), and \( E_4^p \) means the left-hand sides of formulas (22)-(35).

### Table 1. Condition (36).

| \( T-t \) | 0.011 | 0.008 | 0.0045 | 0.0035 | 0.0027 | 0.0025 |
|-----------|-------|-------|--------|--------|--------|--------|
| \( q(18) \) | 12    | 16    | 28     | 36     | 47     | 50     |
| \( q(19) \) | 6     | 8     | 14     | 18     | 23     | 25     |
| \( q(20) \) | 6     | 8     | 14     | 18     | 23     | 25     |
| \( q(21) \) | 12    | 16    | 28     | 36     | 47     | 51     |

### Table 2. Condition (37).

| \( T-t \) | 0.011 | 0.008 | 0.0045 | 0.0042 | 0.0040 |
|-----------|-------|-------|--------|--------|--------|
| \( q(22) \) | 6     | 8     | 14     | 15     | 16     |
| \( q(23) \) | 4     | 5     | 10     | 11     | 11     |
| \( q(24) \) | 6     | 8     | 14     | 15     | 16     |
| \( q(25) \) | 6     | 8     | 14     | 15     | 16     |
| \( q(26) \) | 3     | 5     | 9      | 9      | 10     |
| \( q(27) \) | 6     | 8     | 14     | 15     | 16     |
| \( q(28) \) | 4     | 5     | 10     | 11     | 11     |
| \( q(29) \) | 2     | 3     | 4      | 5      | 5      |
| \( q(30) \) | 2     | 3     | 4      | 5      | 5      |
| \( q(31) \) | 4     | 6     | 10     | 11     | 11     |
| \( q(32) \) | 4     | 6     | 10     | 11     | 11     |
| \( q(33) \) | 2     | 3     | 5      | 6      | 6      |
| \( q(34) \) | 6     | 8     | 14     | 15     | 16     |
| \( q(35) \) | 3     | 5     | 9      | 9      | 10     |
Table 3. Condition (37).

| $T-t$ | 0.010 | 0.005 | 0.0025 |
|-------|--------|--------|--------|
| $q(14)$ | 4     | 8     | 16     |
| $q(15)$ | 1     | 1     | 1      |
| $q(16)$ | 4     | 8     | 16     |
| $q(17)$ | 1     | 1     | 1      |

Table 4. Strong scheme with order 1.5. Condition (36).

| $T-t$ | $2^{-1}$ | $2^{-3}$ | $2^{-5}$ | $2^{-8}$ |
|-------|----------|----------|----------|----------|
| $q(13)$ | 1        | 8        | 128      | 8192     |
| $q(18)$ | 0        | 1        | 4        | 32       |
| $q(19)$ | 0        | 0        | 2        | 16       |
| $q(20)$ | 0        | 0        | 2        | 16       |
| $q(21)$ | 0        | 0        | 4        | 33       |

Table 5. Strong scheme with order 2.0. Condition (37).

| $T-t$ | $2^{-1}$ | $2^{-3}$ | $2^{-5}$ | $2^{-8}$ |
|-------|----------|----------|----------|----------|
| $q(13)$ | 1        | 8        | 64       | 512      |
| $q(18)$ | 0        | 2        | 4        | 32       |
| $q(19)$ | 0        | 1        | 4        | 16       |
| $q(20)$ | 0        | 1        | 4        | 16       |
| $q(21)$ | 0        | 2        | 8        | 33       |
| $q(14)$ | 0        | 0        | 1        | 1        |
| $q(15), \ldots, q(17)$ | 0  | 0  | 0  | 0  |
| $q(22), \ldots, q(35)$ | 0  | 0  | 0  | 0  |

Table 6. Values $E_s^{(000)} \cdot (T-t)^{\frac{1}{3}} \defeq E_p$.

| $T-t$ | 0.011 | 0.008 | 0.0045 | 0.0035 | 0.0027 | 0.0025 |
|-------|-------|-------|--------|--------|--------|--------|
| $p=p(18)$ | 12   | 16   | 28     | 36     | 47     | 50     |
| $E_p$   | 0.010154 | 0.007681 | 0.004433 | 0.003456 | 0.002652 | 0.002494 |
| $p=p(19)$ | 12   | 16   | 28     | 36     | 47     | 50     |
| $E_p$   | 0.005077 | 0.003841 | 0.002216 | 0.001728 | 0.001326 | 0.001247 |
| $p=p(20)$ | 12   | 16   | 28     | 36     | 47     | 50     |
| $E_p$   | 0.005077 | 0.003841 | 0.002216 | 0.001728 | 0.001326 | 0.001247 |
| $p=p(21)$ | 12   | 16   | 28     | 36     | 47     | 50     |
| $E_p$   | 0.010308 | 0.007787 | 0.004480 | 0.003488 | 0.002673 | 0.002513 |

Let us show by numerical experiments that in most situations the following inequalities are fulfilled (under conditions (36) and (37)): $q(14) \geq q(15)$; $q(16) \geq q(17)$; $q(18) \geq q(19), \ldots, q(21)$, and
$q(22) \geq q(23), \ldots, q(35)$, where all numbers in these inequalities are minimal natural numbers satisfying to conditions (36) and (37).

In tables 1-8 we can see the results of numerical experiments. In tables 6-8 numbers $p(14), \ldots, p(35)$ means numbers $p$ corresponding to formulas (14)-(35) respectively. These results confirm the hypothesis proposed in this paper.

Table 7. Values $E_4^{(0000)\mu} \cdot (T-t)^{\frac{4\mu}{4}} = E_p$.

| $T-t$ | 0.011 | 0.008 | 0.0045 | 0.0042 |
|-------|-------|-------|-------|-------|
| $p=p(22)$ | 6 | 8 | 14 | 15 |
| $E_p$ | 0.009636 | 0.007425 | 0.004378 | 0.004096 |
| $p=p(23)$ | 6 | 8 | 14 | 15 |
| $E_p$ | 0.006771 | 0.005191 | 0.003041 | 0.002843 |
| $p=p(24)$ | 6 | 8 | 14 | 15 |
| $E_p$ | 0.009722 | 0.007502 | 0.004424 | 0.004139 |
| $p=p(25)$ | 6 | 8 | 14 | 15 |
| $E_p$ | 0.009641 | 0.007427 | 0.004379 | 0.004097 |
| $p=p(26)$ | 6 | 8 | 14 | 15 |
| $E_p$ | 0.005997 | 0.004614 | 0.002720 | 0.002545 |
| $p=p(27)$ | 6 | 8 | 14 | 15 |
| $E_p$ | 0.009722 | 0.007502 | 0.004424 | 0.004139 |
| $p=p(28)$ | 6 | 8 | 14 | 15 |
| $E_p$ | 0.006771 | 0.005191 | 0.003041 | 0.002843 |
| $p=p(29)$ | 6 | 8 | 14 | 15 |
| $E_p$ | 0.003095 | 0.002364 | 0.001379 | 0.001290 |
| $p=p(30)$ | 6 | 8 | 14 | 15 |
| $E_p$ | 0.003095 | 0.002364 | 0.001379 | 0.001290 |
| $p=p(31)$ | 6 | 8 | 14 | 15 |
| $E_p$ | 0.006885 | 0.005282 | 0.003090 | 0.002889 |
| $p=p(32)$ | 6 | 8 | 14 | 15 |
| $E_p$ | 0.006885 | 0.005282 | 0.003090 | 0.002889 |
| $p=p(33)$ | 6 | 8 | 14 | 15 |
| $E_p$ | 0.003690 | 0.002834 | 0.001663 | 0.001555 |
| $p=p(34)$ | 6 | 8 | 14 | 15 |
| $E_p$ | 0.009756 | 0.007545 | 0.004457 | 0.004170 |
| $p=p(35)$ | 6 | 8 | 14 | 15 |
| $E_p$ | 0.006010 | 0.004621 | 0.002722 | 0.002547 |

Let $q_1$ and $q_3$ be minimal natural numbers satisfying to $E_3^{(0000)\mu} \leq (T-t)^4$, $E_4^{(0000)\mu} \leq (T-t)^5$ (the left-hand sides of these inequalities are defined by formulas (18) and (22) respectively). Let $p_1$ and $p_3$
be minimal natural numbers satisfying to $3!E_3^{(000)} \leq (T-t)^4$, $4!E_4^{(000)} \leq (T-t)^5$, where the values $E_3^{(000)}$ and $E_4^{(000)}$ are defined by formulas (18) and (22) respectively. In tables 9, 10 we can see the numerical comparison of the numbers $q_1, q_3$ with the numbers $p_1, p_3$ respectively. Obviously, excluding of the multiplier factors $3!$ and $4!$ essentially (in many times) reduces the calculation costs for the mean-square approximations of iterated Ito stochastic integrals. Note that in this paper we use the exactly calculated Fourier-Legendre coefficients using the Python programming language [20].

| $T-t$  | 0.010 | 0.005 | 0.0025 |
|--------|-------|-------|--------|
| $p=p(14)$ | 4    | 8    | 16    |
| $E_p^{(01)}$ | 0.008950 | 0.004660 | 0.002383 |
| $p=p(15)$ | 4    | 8    | 16    |
| $E_p^{(01)}$ | 0.000042 | 0.000006 | 0.000001 |
| $p=p(16)$ | 4    | 8    | 16    |
| $E_p^{(10)}$ | 0.008950 | 0.004660 | 0.002383 |
| $p=p(17)$ | 4    | 8    | 16    |
| $E_p^{(10)}$ | 0.000042 | 0.000006 | 0.0000001 |

Table 9. Comparison of numbers $q_1$ and $p_1$.

| $T-t$  | $2^{-1}$ | $2^{-2}$ | $2^{-3}$ | $2^{-4}$ | $2^{-5}$ | $2^{-6}$ |
|--------|----------|----------|----------|----------|----------|----------|
| $q_1$  | 0        | 0        | 1        | 2        | 4        | 8        |
| $p_1$  | 1        | 3        | 6        | 12       | 24       | 48       |

Table 10. Comparison of numbers $q_3$ and $p_3$.

| $T-t$  | $2^{-1}$ | $2^{-2}$ | $2^{-3}$ | $2^{-4}$ | $2^{-5}$ | $2^{-6}$ |
|--------|----------|----------|----------|----------|----------|----------|
| $q_3$  | 0        | 0        | 0        | 0        | 0        | 0        |
| $p_3$  | 3        | 4        | 6        | 9        | 12       | 17       |

5. Conclusion
As we mentioned above, existing approaches to the mean-square approximation of iterated stochastic integrals [1, 2, 9-17, 19] do not allow to choose different numbers $p$ (see theorem 2) for approximations of different iterated stochastic integrals with multiplicity $k = 3, 4, ...$ and exclude the possibility for obtaining of approximate and exact expressions similar to formulas (9), (11). This leads to unnecessary terms usage in the expansions of iterated Ito stochastic integrals and, as a consequence, to essential increase of computational costs for the implementation of numerical methods for Ito SDEs. In this article, we have optimized method based on theorems 1, 2, which makes it possible to correctly choose the lengths of sequences of standard Gaussian random variables required for the approximation of iterated Ito stochastic integrals. Thus, the computational costs for the implementation of numerical methods for Ito SDEs are significantly reduced.
On the base of the obtained results we can recommend the following conditions for correct choosing the minimal natural numbers $q, q_1, q_2, q_3$: $E_2^{(00)q} \leq C(T-t)^3$ (for the Milstein scheme), $E_2^{(00)q} \leq C(T-t)^4$, $E_2^{(11)q} \leq C(T-t)^4$ (for the strong scheme with order 1.5), and $E_2^{(00)q} \leq C(T-t)^5$, $E_3^{(00)q} \leq C(T-t)^5$, $E_3^{(11)q} \leq C(T-t)^5$, $E_3^{(01)q} \leq C(T-t)^5$, $E_4^{(000)q} \leq C(T-t)^5$ (for numerical scheme (5)). Here the left-hand sides of the above inequalities are defined by relations (13), (18), (14), (16), and (22), correspondingly. Here $C$ is a constant from the condition (6).

The paper results can be applied, for example, for numerical modeling of helicopter rotors dynamics and satellites orbital stability analysis.

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