Closed-Form Solutions to A Category of Nuclear Norm Minimization Problems

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Abstract

It is an efficient and effective strategy to utilize the nuclear norm approximation to learn low-rank matrices, which arise frequently in machine learning and computer vision. So the exploration of nuclear norm minimization problems is gaining much attention recently. In this paper we shall prove that the following Low-Rank Representation (LRR) [2, 1] problem:

$$\min_Z \|Z\|_*, \quad \text{s.t.,} \quad X = AZ,$$

has a unique and closed-form solution, where $X$ and $A$ are given matrices. The proof is based on proving a lemma that allows us to get closed-form solutions to a category of nuclear norm minimization problems.

1 Introduction

In real applications, our observations are often noisy, or even grossly corrupted, and some observations may be missing. This fact naturally leads to the problem of recovering a low-rank matrix $X^0$ from a corrupted observation matrix $X = X^0 + E^0$ (each column of $X$ is an observation vector), with $E^0$ being the unknown noise. Due to the low-rank property of $X^0$, it is straightforward to consider the following regularized rank minimization problem:

$$\min_{D,E} \text{rank} (D) + \lambda \|E\|_\ell, \quad \text{s.t.} \quad X = D + E,$$

where $\lambda > 0$ is a parameter and $\|\cdot\|_\ell$ is some kind of regularization strategy, such as the $\ell_1$-norm adopted by [3, 4], for characterizing the noise $E^0$. As a common practice in rank minimization problems, one could replace the rank function with the nuclear norm, resulting in the following convex optimization problem:

$$\min_{D,E} \|D\|_* + \lambda \|E\|_\ell, \quad \text{s.t.} \quad X = D + E. \quad (1)$$

The minimizer $D^*$ (with respect to the variable $D$) gives a low-rank recovery to the original data $X^0$. The above formulation is adopted by the recently established Robust PCA (RPCA) method [3, 4], which uses the $\ell_1$-norm to characterize the noise. However, such a formulation implicitly assumes that the underlying data structure is a single low-rank subspace. When the data is drawn from a union of multiple subspaces, denoted as $S_1, S_2, \cdots, S_k$, the PRCA method actually treats the data as being sampled from a single subspace defined by $S = \sum_{i=1}^k S_i$. The specifics of the individual subspaces are not well considered, so the recovery may be inaccurate.

To better handle the mixed data, in [2, 1] we suggest a more general rank minimization problem defined as follows:

$$\min_{Z,E} \text{rank} (Z) + \lambda \|E\|_\ell, \quad \text{s.t.} \quad X = AZ + E,$$

\[\]
where $A$ is a “dictionary” that linearly spans the data space. By replacing the rank function with the nuclear norm, we have the following convex optimization problem:

$$
\min_{Z,E} \|Z\|_* + \lambda \|E\|_\ell, \quad \text{s.t.} \quad X = AZ + E. \tag{2}
$$

After obtaining an optimal solution $(Z^*, E^*)$, we could recover the original data by using $AZ^*$ (or $X - E^*$). Since $\text{rank}(AZ^*) \leq \text{rank}(Z^*)$, $AZ^*$ is also a low-rank recovery to the original data $X^0$. By setting $A = I$, the formulation (2) falls back to (1). So our LRR method could be regarded as a generalization of RPCA that essentially uses the standard basis as the dictionary. By choosing an appropriate dictionary $A$, as shown in [2], the lowest-rank representation also reveals the segmentation of data such that LRR could handle well the data drawn from a mixture of multiple subspaces.

For ease of understanding the LRR method, in this work we consider the “ideal” case that the data is noiseless. That is, we consider the following optimization problem:

$$
\min_{Z} \|Z\|_*, \quad \text{s.t.} \quad X = AZ. \tag{3}
$$

We will show that this optimization problem always has a unique and closed-form minimizer, provided that $X = AZ$ has feasible solutions.

### 2 A Closed-Form Solution to Problem (3)

The nuclear norm is convex, but not strongly convex. So it is possible that problem (3) has multiple optimal solutions. Fortunately, it can be proven that the minimizer to problem (3) is always uniquely defined by a closed form. This is summarized in the following theorem.

**Theorem 2.1 (Uniqueness)** Assume $A \neq 0$ and $X = AZ$ have feasible solutions, i.e., $X \in \text{span}(A)$. Then

$$
Z^* = V_A(V_A^T V_A)^{-1} V_A^T X, \tag{4}
$$

is the unique minimizer to problem (3), where $V_X$ and $V_A$ are calculated as follows: Compute the skinny Singular Value Decomposition (SVD) of $[X, A]$, denoted as $[X, A] = U \Sigma V^T$, and partition $V$ as $V = [V_X; V_A]$ such that $X = U \Sigma V_X^T$ and $A = U \Sigma V_A^T$.

From the above theorem we have the following two corollaries. First, when the matrix $A$ is of full row rank, the closed-form solution defined by (4) can be represented in a simpler form.

**Corollary 2.1** Suppose the matrix $A$ has full row rank, then

$$
Z^* = A^T (AA^T)^{-1} X,
$$

is the unique minimizer to problem (3), where $A^T (AA^T)^{-1}$ is the generalized inverse of $A$.

Second, when the data matrix itself is used as the dictionary, i.e., $A = X$, the solution to problem (3) falls back to the outputs of a factorization based method.

**Corollary 2.2** Assume $X \neq 0$. Then the following optimization problem

$$
\min_{Z} \|Z\|_*, \quad \text{s.t.} \quad X = XZ,
$$

has a unique minimizer

$$
Z^* = \text{SIM}(X), \tag{5}
$$

where $\text{SIM}(X) = V_X V_X^T$ is called the Shape Interaction Matrix (SIM) [5] in computer vision and $X = U \Sigma X V_X^T$ is the skinny SVD of $X$.

The proof of Theorem 2.1 is based on the following three lemmas.
Lemma 2.1 Let $U$, $V$ and $M$ be matrices of compatible dimensions. Suppose both $U$ and $V$ have orthogonal columns, i.e., $U^T U = I$ and $V^T V = I$, then we have

$$\|M\|_* = \|UMV^T\|_*.$$

Proof Let the full SVD of $M$ be $M = U_M \Sigma_M V_M^T$, then $UMV^T = (UU_M) \Sigma_M (V V_M)^T$. As $(UU_M)^T(UU_M) = I$ and $(VV_M)^T(VV_M) = I$, $(UU_M) \Sigma_M (V M)^T$ is actually the SVD of $UMV^T$. By the definition of the nuclear norm, we have $\|M\|_* = \text{tr} (\Sigma_M) = \|UMV^T\|_*$.

Lemma 2.2 For any four matrices $B$, $C$, $D$ and $F$ of compatible dimensions, we have

$$\begin{bmatrix} B & C \\ D & F \end{bmatrix} \geq \begin{bmatrix} B \end{bmatrix},$$

where the equality holds if and only if $C = 0$, $D = 0$ and $F = 0$.

Proof Lemma 10 of [6] directly leads to the above conclusion.

Lemma 2.3 Let $U$, $V$ and $M$ be given matrices of compatible dimensions. Suppose both $U$ and $V$ have orthogonal columns, i.e., $U^T U = I$ and $V^T V = I$, then the following optimization problem

$$\min_Z \|Z\|_*, \text{ s.t. } U^T Z V = M,$$

has a unique minimizer $Z^* = UMV^T$.

Proof First, we prove that $\|M\|_*$ is the minimum objective function value and $Z^* = UMV^T$ is a minimizer. For any feasible solution $Z$, let $Z = U_Z \Sigma_Z V_Z^T$ be its full SVD. Let $B = U^T U_Z$ and $C = V^T V_Z$. Then the constraint $U^T Z V = M$ is equal to

$$B \Sigma_Z C = M.$$  \hspace{1cm} (7)

Since $BB^T = I$ and $C^T C = I$, we can find the orthogonal complements $B_\perp$ and $C_\perp$ such that

$$\begin{bmatrix} B \\ B_\perp \end{bmatrix} \text{ and } \begin{bmatrix} C \\ C_\perp \end{bmatrix}$$

are orthogonal matrices. According to the unitary invariance of the nuclear norm, Lemma 2.2 and (7), we have

$$\|Z\|_* = \|\Sigma_Z\|_* = \\begin{bmatrix} B \\ B_\perp \end{bmatrix} \Sigma_Z \begin{bmatrix} C \\ C_\perp \end{bmatrix} \|_* \geq \|B \Sigma_Z C\|_*= \|M\|_*,$$

Hence, $\|M\|_*$ is the minimum objective function value of problem (6). At the same time, Lemma 2.1 proves that $\|Z^*\|_* = \|UMV^T\|_* = \|M\|_*$. So $Z^* = UMV^T$ is a minimizer to problem (6).

Second, we prove that $Z^* = UMV^T$ is the unique minimizer. Assume that $Z_1 = UMV^T + H$ is another optimal solution. By $U^T Z_1 V = M$, we have

$$U^T H V = 0.$$  \hspace{1cm} (8)

Since $U^T U = I$ and $V^T V = I$, similar to above, we can construct two orthogonal matrices: $[U, U_\perp]$ and $[V, V_\perp]$. By the optimality of $Z_1$, we have

$$\|M\|_* = \|Z_1\|_* = \|UMV^T + H\|_* = \\begin{bmatrix} U^T \\ U_\perp^T \end{bmatrix} (UMV^T + H) [V, V_\perp] \|_* \geq \|M\|_* .$$

According to Lemma 2.2 the above equality can hold if and only if

$$U^T H V_\perp = U_\perp^T H V = U_\perp^T H V_\perp = 0.$$  

Together with (8), we conclude that $H = 0$. So the optimal solution is unique.

\footnote{Note here that $U$ and $V$ may not be orthogonal, namely, $UU^T \neq I$ and $VV^T \neq I$.}

\footnote{When $B$ and/or $C$ are already orthogonal matrices, i.e., $B_\perp = \emptyset$ and/or $C_\perp = \emptyset$, our proof is still valid.}
The above lemma allows us to get closed-form solutions to a class of nuclear norm minimization problems. This leads to a simple proof of Theorem 2.1.

**Proof (of Theorem 2.1)** Since $X \in \text{span}(A)$, we have $\text{rank}(X, A) = \text{rank}(A)$. By the definitions of $V_X$ and $V_A$ (see Theorem 2.1), it can be concluded that the matrix $V_A^T$ has full row rank. That is, if the skinny SVD of $V_A^T$ is $U_1 \Sigma_1 V_1^T$, then $U_1$ is an orthogonal matrix. Through some simple computations, we have

$$V_A(V_A^T V_A)^{-1} = V_1 \Sigma_1^{-1} U_1^T.$$  \hspace{1cm} (9)

Also, it can be calculated that the constraint $X = AZ$ is equal to $V_A^T = V_A^T Z$, which is also equal to $\Sigma_1^{-1} U_1^T V_1^T = V_1^T Z$. So problem (3) is equal to the following optimization problem:

$$\min_Z \|Z\|_*, \quad \text{s.t.} \quad V_1^T Z = \Sigma_1^{-1} U_1^T V_1^T X.$$  

By Lemma 2.3 and (9), problem (3) has a unique minimizer

$$Z^* = V_1 \Sigma_1^{-1} U_1^T V_1^T X = V_A(V_A^T V_A)^{-1} V_1^T X.$$  

3 Conclusion

In this paper, we prove that problem (3) has a unique and closed-form solution. The heart of the proof is Lemma 2.3, which actually allows us to get the closed-form solutions to a category of nuclear norm minimization problems. For example, by following the clues presented in this paper, it is simple for one to get the closed-form solution to the following optimization problem:

$$\min_Z \|Z\|_*, \quad \text{s.t.} \quad X = AZ B,$$

where $X$, $A$, and $B$ are given matrices. Our theorems are useful for studying the LRR problems. For example, based on Lemma 2.3, we have devised a method to recovery the effects of the unobserved, hidden data in LRR [7].

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