Vogel’s notion of regularity for non-coherent rings

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Abstract :
We study the notion of regular ring in the sense of Vogel, which generalizes the classical notion for non-necessarily coherent rings. We build a class of groups $R$ containing Waldhausen’s class $\mathcal{C}$ in $[Wal78]$ and study its stability results. We show that for a regular ring $R$ and a group $G$ in $R$, the group ring $R[G]$ is still regular. Finally, we state Vogel’s excision conjecture generalizing Waldhausen’s results in $[Wal78]$ concerning Whitehead and $K\text{Nil}$ groups.

Keywords :
REGULAR NON-COHERENT RINGS
EXCISION IN WALDHAUSEN ALGEBRAIC K-THEORY

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1 Regular coherent rings

In the classical setting of algebraic geometry, every ring we consider is always noetherian, or at least coherent. In this case, the category \( \text{Mod}^{\text{fp}}(C) \) of finitely presented \( C \)-modules is abelian, thus every finitely presented \( C \)-module \( M \) admits a resolution by finitely generated projective \( P_i \):

\[
\ldots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \ldots \rightarrow P_1 \rightarrow P_0 \rightarrow M
\]

We recall the classical definition of regularity for coherent rings:

**Definition 1.**

Let \( C \) be a coherent ring. The ring \( C \) is called “regular” if every finitely presented \( C \)-module \( M \) admits a finite resolution by finitely generated projective \( P_1 \):  
\[
0 \rightarrow P_n \rightarrow \ldots \rightarrow P_1 \rightarrow P_0 \rightarrow M
\]

For the matter that concerns algebraic K-theorists, this setting is the ground base for Quillen’s d’evissage theorem, and this is the key-tool in Waldhausen’s proof of the vanishing of the spectrum \( K\text{Nil}(C,S) \) that is the obstruction to excision for \( C \) a coherent regular ring, and \( S \) a flat on the left \( C \)-bimodule. With this technical result, Waldhausen proved that \( Wh^R(G) = 0 \) for a group \( G \) in a large class containing the \( \pi_1 \) of all irreducible 3-dimensional Haken manifolds, the \( \pi_1 \) of all submanifolds of the 3-sphere, the \( \pi_1 \) of all surfaces other than the projective plane, all free groups, all free abelian groups, all poly-\( \mathbb{Z} \)-groups, all torsion-free 1-relator groups, \ldots (please refer to [Wal78], th. 17.4 & 17.5 p 250).

2 General Definition

Throughout this section, \( C \) is supposed to be a ring (unitary, associative).

The usual problem for a topologist is that it’s very difficult to prove that a ring is coherent, especially when dealing with group rings, or with the \( \pi_1 \) of CW-complexes. Actually, the gluing process for attaching cells, translates via the Van Kampen theorem into one of the 3 cases studied by Waldhausen in [Wal78]: polynomial extension, free product, or HNN-extension. But none of these operations conserve coherence! This leads us to search for a more flexible notion of “regularity” for non-coherent rings. The notion introduced by Vogel in [Vog90] unpublished, is based on a more categorical approach of modules:

**Definition 2.**

Let \( \mathcal{C} \) be a class of modules in \( \text{Mod}_C \). \( ^{\dagger} \)

The class \( \mathcal{C} \) is called “exact” if

(i) \( \mathcal{C} \) is stable under filtering colimits.

(ii) \( \mathcal{C} \) verifies the ’2/3 axiom’:

let \( M \rightarrow N \rightarrow P \) be a short exact sequence in \( \text{Mod}_C \),

if two of these modules are in \( \mathcal{C} \), so is the third.

\( ^{\dagger} \)For the sake of simplicity, we will work only with the category \( \text{Mod}_C \) of “right” \( C \)-modules. So the notion we define is that of a “right regular ring” \( C \). We will try to write proofs only for right modules (except in certain cases explicitly stated). But every argument can be transported to “left” modules, so that the theory of “left regular ring” is similar. The
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Definition 3.
Let \( C_0 \) be the smallest exact class in \( \text{Mod}_C \) containing the ring \( C \) itself; the modules in \( C_0 \) are called “regular”. The ring \( C \) is called “regular” (in the sense of Vogel) if \( C_0 = \text{Mod}_C \), that means if every \( C \)-module is regular, or equivalently if \( \text{Mod}_C \) is the only exact class containing \( C \).

We’ll prove in chapter 6 that this definition generalizes the classical one for coherent rings and we’ll study the stability results for this notion well-suited for topologists in chapter 3. Moreover, the vanishing results of Waldhausen should be true in this setting: we’ll state Vogel’s conjecture in chapter 4, and some partial results proven by localization and categorical decomposition will be published in the following articles in preparation [Bih02, Bih03].

Let’s now state a technical result, useful hereafter:

Lemma 1.
Suppose that a class \( \mathcal{C} \) in \( \text{Mod}_C \) is stable under filtering colimits. Then this class \( \mathcal{C} \) is also stable under direct summands.

Proof:
Suppose that we have \( M \cong N \oplus P \) with \( M \) in \( \mathcal{C} \). We shall write the following inductive system: for every integer index \( i \), we take \( E_i = M \) and \( f_i : E_i \to E_{i+1} \) defined by the identity \( \text{Id} : N \to N \) and the zero maps anywhere else. We obtain: \( \varinjlim E_i \cong N \), each \( E_i \) being in \( \mathcal{C} \), the filtering colimit \( N \) is thus in \( \mathcal{C} \).

Remark:
Thus, for an exact class \( \mathcal{C} \), containing the base ring \( C \) is equivalent to containing all projective \( C \)-modules, and even all flat \( C \)-modules. Knowing that the notion of “projective” object is stable under Morita-equivalence, we thus see that the notion of “regular” ring is Morita-invariant, and can then be defined in every abelian category.

Examples:
- The axiom (i) tells us that every flat \( C \)-module is in \( \mathcal{C}_0 \). By the axiom (ii), every finite homological dimension \( C \)-module is in \( \mathcal{C}_0 \). Thus, every finite homological dimension ring \( C \) is regular (more precisely: it suffices that every finitely presented \( C \)-module be of finite homological dimension).
- Let \( C = \mathbb{Z}/4\mathbb{Z} \), the class of all free modules is exact, thus it’s the class \( \mathcal{C}_0 \). But it doesn’t contain \( \mathbb{Z}/2\mathbb{Z} \), thus the ring \( C \) is not regular.
- Let \( D = \mathbb{Z}[G] \) the group ring associated to a finite group \( G \). Let \( \mathcal{C} \) the class of modules \( M \) such that all cohomology groups \( H^i(G, M) \) vanish for all \( i > 0 \). It’s an exact class, but it doesn’t contain the trivial \( G \)-module \( \mathbb{Z} \). Thus the ring \( D \) is not regular.

---

\( \text{two notions are a priori independent, but in the case of a group ring } C = \mathbb{Z}[G], \text{ the two categories } \text{Mod}_C \text{ and } C\text{Mod} \text{ are equivalent, so here the notions of “left regular” and “right regular” coincide. In the classical setting of algebraic geometry evoked above, when the ring } C \text{ is not commutative, Waldhausen chooses to work with right modules, but the “left” and “right” notions coincide when applied to his class } \mathcal{C} \text{ of regular groups.} \)

\( \text{cf theorem 1 page 14 by D. Lazard in [Bou80]} \)
3 Stability of the notion of regularity

1. Morphism of rings

**Proposition 1.**
Let \( f : A \to B \) be a ring homomorphism. We have two distinct cases:
(i) We suppose that \( A \) is isomorphic to some direct summand of \( B \), as a \( B \)-bimodule and that \( B \) is flat as a right \( A \)-module. Then if \( B \) is right regular, \( A \) is right regular too.
(ii) We suppose that the canonical morphism of \( B \)-bimodules \( B \otimes_A B \to B \) is split surjective and that \( B \) is flat as a left \( A \)-module. Then conversely, if \( A \) is right regular, \( B \) is right regular too.

**Proof:**
(i) Let \( \mathcal{C}_0 \) be an exact class in \( \text{Mod}_A \) containing \( A \).
The ring homomorphism induces a scalar-restriction functor \( R : \text{Mod}_B \to \text{Mod}_A \) which is exact, and commutes to colimits. We then consider the class \( \mathcal{C} = \{ M \in \text{Mod}_B | R(M) \in \mathcal{C}_0 \} \). As the functor \( R \) is exact, this class \( \mathcal{C} \) verifies the 2/3 axiom. As the functor \( R \) commutes to colimits, the class \( \mathcal{C} \) is stable under filtering colimits. At last, every flat module being a filtering colimit of finitely generated projective modules \( \dagger \), the condition “\( B \) flat as a right \( A \)-module” implies that \( B \) is in \( \mathcal{C} \). Applying the regularity of the ring \( B \), we then deduce \( \mathcal{C} = \text{Mod}_B \).

Let now \( N \) be any right \( A \)-module. By hypothesis, \( B \simeq A \oplus C \), we tensorize to obtain the following decomposition: \( N \otimes_A B \simeq N \oplus (N \otimes_A C) \). The module \( N \) is hence a direct summand of a module \( N \otimes_A B \) already in \( \mathcal{C}_0 \) (because it’s in \( \mathcal{C} \)), thus applying lemma \( \square \) \( N \) is in \( \mathcal{C}_0 \). Therefore \( \mathcal{C}_0 = \text{Mod}_A \) and the ring \( A \) is regular.

(ii) Let \( \mathcal{D}_0 \) be an exact class in \( \text{Mod}_B \) containing \( B \).
The ring homomorphism induces a tensor functor \( \otimes_A B : \text{Mod}_A \to \text{Mod}_B \) which is exact because \( B \) is flat as a left \( A \)-module, and commutes to colimits. We then consider the class \( \mathcal{D} = \{ N \in \text{Mod}_A | N \otimes_A B \in \mathcal{D}_0 \} \). As above, the two conditions on the functor \( \otimes_A B \) imply that \( \mathcal{D} \) is an exact class in \( \text{Mod}_A \). The isomorphism \( A \otimes_A B \simeq B \) tells us that \( A \) is in \( \mathcal{D} \), hence, as the ring \( A \) is regular, we can deduce \( \mathcal{D} = \text{Mod}_A \). Let now \( M \) be any right \( B \)-module. We write \( M \otimes_B B \simeq M \) and \( M \otimes_B (B \otimes_A B) \simeq M \otimes_A B \). The existence of a section implies that \( M \) is a direct summand of \( M \otimes_A B \) in \( \mathcal{D}_0 \), hence by lemma \( \Box \) \( M \) is in \( \mathcal{D}_0 \). Finally, we get \( \mathcal{D}_0 = \text{Mod}_B \) and the ring \( B \) is regular. \( \blacksquare \)

2. Stability under filtering colimits

**Proposition 2.**
Let \( \mathcal{F} \) be a filtering category of index. Let \( (A_i)_{i \in \mathcal{F}} \) be an inductive system of rings. Suppose that for each map \( i \to j \) in \( \mathcal{F} \), the ring \( A_j \) is flat as a left \( A_i \)-module. Suppose that all rings \( A_i \) are right regular. Then the colimit \( A = \varinjlim A_i \) is a right regular ring too.

**Proof:**
Let \( \mathcal{C}_0 \) be the smallest exact class in \( \text{Mod}_A \) containing \( A \). Let \( M \) be a finitely presented right \( A \)-module. We consider the exact sequence \( \xymatrix{ A^p \ar[r]^\alpha & A^q \ar[r] & M \ar[r] & 0 } \). Let \( (e_i) \) be
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a basis of $A^p$. Since the category $\mathcal{F}$ is filtering, there exists an index $i$ such that the ring $A_i$ contains the images $\alpha(e_A)$ of the basis. Hence we get the diagram with exact lines:

\[
\begin{array}{ccc}
A^p & \overset{\alpha}{\rightarrow} & A^q \\
\downarrow & & \downarrow \\
A_i^p & \overset{\alpha}{\rightarrow} & A_i^q
\end{array}
\]

where we keep the same basis ($e_A$) for $A_i^p$. To the lower line we apply the functor $\cdot \otimes_{A_i} A$ to obtain the exact sequence: $A^p \rightarrow A^q \rightarrow M \otimes_{A_i} A \rightarrow 0$. Let $\mathcal{C}_0$ be the class of regular $A$-modules. We then consider the class $\mathcal{C}_1 = \{ M \in \text{Mod}_{A_i} | M \otimes_{A_i} A \in \mathcal{C}_0 \}$. Since $A$ is a right $A_i$-module, the class $\mathcal{C}_1$ is exact and contains $A_i$. Since the ring $A_i$ is regular, we get: $\mathcal{C}_1 = \text{Mod}_{A_i}$ and thus $M$ is in $\mathcal{C}_0$. At last, as every module is the filtering colimit of finitely presented modules, we get: $\mathcal{C}_0 = \text{Mod}_A$ hence the ring $A$ is regular.

3. Stability by product

Proposition 3.
Let $A$ and $B$ be two right regular rings.
Then the product $A \times B$ is a right regular ring too.

Proof:
Let $P$ be a right module on $A \times B$, then we have the decomposition: $P = M \times N$ with $M$ in $\text{Mod}_A$ and $N$ in $\text{Mod}_B$. Let $\mathcal{C}_0$ be the class of regular right $A \times B$-modules. Now the class $\mathcal{C}_1 = \{ M \in \text{Mod}_{A_i} | M \times 0 \in \mathcal{C}_0 \}$ is exact, because the exact functor $\cdot \times 0 : \text{Mod}_A \rightarrow \text{Mod}_{A \times B}$ commutes to colimits. The class $\mathcal{C}_1$ contains $A$, since $A \times 0$ is $A \times B$-projective. Since the ring $A$ is regular, this proves that: $\mathcal{C}_1 = \text{Mod}_A$.
Similarly, the class $\mathcal{C}_2 = \{ N \in \text{Mod}_B | 0 \times N \in \mathcal{C}_0 \}$ is an exact class containing $B$; as the ring $B$ is regular, we get: $\mathcal{C}_2 = \text{Mod}_B$. Finally write the decomposition: $P = M \times N \simeq (M \times 0) \oplus (0 \times N)$ belonging to $\mathcal{C}_0$. Therefore $\mathcal{C}_0 = \text{Mod}_{A \times B}$ and the ring $A \times B$ is regular.

4. Group rings

Proposition 4.
Let $G$ be a group of finite homological dimension, and $A$ a right regular ring. Then the associated group ring $A[G]$ is right regular too.

Proof:
We consider $\mathcal{C}_0$ the class of regular right $A[G]$-modules. If $\text{dim}_A(G) = p$, then the right global dimension and the left global dimension coincide for $\mathcal{Z}(G)$ and the global dimension for bimodules is $\text{dim}_A(G \times G) \leq p \times p$. Let $0 \rightarrow C_n \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow Z \rightarrow 0$ be a resolution of $Z$ by $\mathcal{Z}(G)$-bimodules flat on both sides. Let $M$ be any right $A[G]$-module. We tensorize our exact sequence by $M$ over $Z$ (doted with the diagonal action): $0 \rightarrow M \otimes_Z C_n \rightarrow \cdots \rightarrow M \otimes_Z C_1 \rightarrow M \otimes_Z C_0 \rightarrow M \rightarrow 0$ is also an exact sequence because all the $C_i$ are flat. Let’s describe the structure of $A[G]$-module on $M : A$ acts on $M$ only on the right, and $G$ acts on the $M \otimes_Z C_i$ diagonally. The exact sequence above is hence a resolution of $M$ by right $A[G]$-modules. We can thus reduce the problem to showing that $M \otimes_Z C$ is in the class $\mathcal{C}_0$ for every $C$ right flat, and

\[\text{cf Proposition 7 page 11 in [Bou80]}\]
then $C$ finitely generated projective, and then $C$ finite dimensional free, and finally only for $C = \mathbb{Z}[G]$. Now we shall kill the action of $G$ on $M$ by considering the following isomorphism, where $M_0$ is the module $M$ endowed with its structure of right $A$-module, but where the action of $G$ is trivial: (we write the action of $\gamma \in G$)

$$
M \otimes \mathbb{Z}[G] \cong M_0 \otimes \mathbb{Z}[G]
$$

$$
u \otimes g \quad \mapsto \quad (ug^{-1}) \otimes g
$$

The diagonal action of $G$ on the left side is sent on a trivial action on the right side.

The canonical isomorphism of right $A$-modules is hence also an isomorphism of right $A[G]$-modules. Finally we consider: $\mathcal{G}_1 = \{N \in \text{Mod}_A | N \otimes \mathbb{Z}[G] \in \mathcal{G}_0\}$. It’s an exact class containing $A$, but the ring $A$ is regular, thus we get: $\mathcal{G}_1 = \text{Mod}_A$. Then $(M_0 \otimes \mathbb{Z}[G])$ is in $\mathcal{G}_0$ and our sufficient case is proven. Hence $\mathcal{G}_0 = \text{Mod}_{A[G]}$ and the ring $A[G]$ is regular. 

**Remark**: Let $A$ be a right regular ring. As $\dim_h(\mathbb{Z}[\mathbb{Z}]) = 2$, we can apply inductively Proposition 4 to the group $G = \mathbb{Z}$ to prove that the group rings $A[\mathbb{Z}^n]$ are right regular for every integer $n$.

5. **Stability by Waldhausen’s diagrams**

We shall adopt the definitions and setting used by Friedhelm Waldhausen in his Proposition 4.1 of page 161 in [Waldhausen](#). Every ring here is supposed associative and unitary. Recall that an embedding $\alpha : C \to A$ is said 'pure' if there exists a splitting of $C$-bimodules: $A = \alpha(C) \oplus A'$. We shall always assume that $A'$ is free as a left $C$-module.

**Proposition 5**.

Let the ring $R$ be either :

- 1) the free product in the situation $\alpha : C \to A, \beta : C \to B$ or
- 2) the Laurent extension with respect to $\alpha, \beta : C \to A$ or
- 3) the tensor algebra of the $C$-bimodule $S$.

Assume that the maps $\alpha, \beta$ are pure embeddings, with complements free from the left; likewise $S$ is free from the left.

Suppose that the rings $C, A, B$ be right regular (in the sense of Vogel). Then the ring $R$ is right regular too.

**Proof**:

With the hypothesis above, we can apply Proposition 4.1 of [Waldhausen](#):

Let $M$ be a $R$-module. There exists two $C$-modules $M_C$ and $M'_C$, a $A$-module $M_A$, a $B$-module $M_B$, and a short exact sequence of $R$-modules (respectively for each case):

- 1) $0 \to M_C \otimes_C R \to M_A \otimes_A R \oplus M_B \otimes_B R \to M \to 0$
- 2) $0 \to M_C \otimes_C R \to M_A \otimes_A R \to M \to 0$
- 3) $0 \to M_C \otimes_C R \to M'_C \otimes_C R \to M \to 0$

Let $\mathcal{G}_0$ be the class of regular right $R$-modules. By the '2/3 axiom', the problem is thus reduced to showing that $M_A \otimes_A R$ is in $\mathcal{G}_0$, for $A_1$ one of the three rings $A, B, C$.

Consider then the class $\mathcal{G}_1 = \{N_i \in \text{Mod}_{A_1} | N_i \otimes_{A_1} R \in \mathcal{G}_0\}$, it’s an exact class because $R$ is flat as a left $A_1$-module, it contains $A_1$, but the ring $A_1$ is regular, therefore we get: $\mathcal{G}_1 = \text{Mod}_{A_1}$, and the proof is complete.
Remark: In particular, the case of a polynomial extension $A[t]$ is treated as $A[S]$ with the bimodule $S = A$ itself. By induction, if the ring $A$ is regular, the polynomial rings $A[t_1, \ldots, t_n]$ are regular for every integer $n$.

4 The class $\mathcal{R}$ of Vogel

Let’s introduce a new class $\mathcal{R}$ of groups, first exposed by Pierre VOGEL in [Vog90], larger than the class $\mathcal{C}$ of Friedhelm WALDHAUSEN, exposed in Def.19.2 in [Wal78], that will verify a generalization of Th.17.5 on page 249, due to our more practical notion of regularity for a non-necessarily-coherent ring $R$, and in particular due to the stability properties of the preceding part, applied to group rings $R[G]$. We will also expose Vogel’s conjecture that generalizes Th.19.4 in this setting (cf the forth-coming articles [Bih02] and [Bih03] for two approaches towards a proof).

Definition 4.

Let $\mathcal{R}$ be the smallest class of groups verifying:

1. The trivial group 1 is in $\mathcal{R}$.
2. If $G_0$ and $G_1$ are in $\mathcal{R}$, and $\alpha$, $\beta$ are two injections from $G_0$ to $G_1$, then the HNN-extension

   \[ G_0 \xrightarrow{\alpha} G_1 \xrightarrow{\beta} G_1 \]

   is in $\mathcal{R}$ too.
3. If $G_0$, $G_1$ and $G_2$ are in $\mathcal{R}$, and $\alpha$, $\beta$ are two injections from $G_0$ to $G_1$ and $G_2$, then the amalgamated free product

   \[ G_0 \xrightarrow{\alpha} G_1 \xrightarrow{\beta} G_2 \]

   is in $\mathcal{R}$ too.
4. $\mathcal{R}$ is stable under filtering colimits.

Definition 5.

A group $G$ is called “regular” if and only if for every regular* ring $A$, the associated group ring $A[G]$ is also regular*.

Proposition 6.

Let $G$ be a group of finite homological dimension. Then $G$ is regular.

Proof:

Use Proposition 4 of the preceding part.

Proposition 7.

Let $G$ be a group in the class $\mathcal{R}$ of Vogel. Then $G$ is regular.

Proof:

The objects of the class $\mathcal{R}$ are constructed inductively through the elementary stages corresponding to Proposition 2 and 5 of the preceding part (HNN-extension of groups gives generalized Laurent extension of rings, and amalgamated free product of groups gives generalized free product of rings).

*Choose equivalently “right regular” for both, or “left regular” for both.
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Theorem 1 [Vog90]

(i) \( R \) is stable under taking subgroups.
(ii) \( R \) is stable under taking extensions.
(iii) \( R \) contains all torsion-free abelian groups.
(iv) \( R \) contains all torsion-free one-relation groups.
(v) \( R \) contains the fundamental groups of all irreducible Haken manifolds.
(vi) For each connected CW-complex \( X \), there exists a group \( G \) in \( R \) such that \( X \) is obtained from \( BG \) by Quillen’s plus construction.

Proof:

(i) We consider the class \( \mathcal{C} \) of groups \( B \) such that all their subgroups \( A \) are in \( R \). Then the class \( \mathcal{C} \) contains 1. The class \( \mathcal{C} \) is stable under filtering colimits: let \( B = \lim_{\to} B_i \), with \( B_i \) in \( \mathcal{C} \), and \( A \) any subgroup of \( B \), we consider the pullback:

\[
  \begin{array}{ccc}
    A & \longrightarrow & B_i \\
    \downarrow & & \downarrow \\
    A & \longrightarrow & B \\
  \end{array}
\]

by construction, \( A_i \) is a subgroup of \( B_i \), hence \( A_i \) is in \( R \).

We then apply the functor \( \lim_{\to} \) to the entire diagram:

\[
  \begin{array}{ccc}
    \lim A_i & \longrightarrow & \lim B_i \\
    \downarrow & & \downarrow \\
    A & \longrightarrow & B \\
  \end{array}
\]

which is also a pullback because the colimit is filtering (cf. [Zis67]), hence we obtain the isomorphism: \( A \simeq \lim A_i \) is in \( R \), and finally: \( B \) is in \( \mathcal{C} \). We now need to prove that the class \( \mathcal{C} \) is stable under taking “amalgamated free product” and “HNN-extension”: at this point we’ll have the inclusion \( R \subset \mathcal{C} \), in other words, the class \( R \) is stable under taking subgroups. For this, we need the lemma [1] hereafter.

(ii) Let \( A \) be a fixed group in \( R \). We consider the class \( \mathcal{D} \) of groups \( C \) such that, for every extension \( A \longrightarrow B \longrightarrow C \), the group \( B \) is in \( R \). Then the class \( \mathcal{D} \) contains 1.

The class \( \mathcal{D} \) is stable under filtering colimits: let \( C = \lim_{\to} C_i \), with \( C_i \) in \( \mathcal{C} \), then by pullback along the structural map \( C_i \to C \), we obtain the following commutative diagram:

\[
  \begin{array}{ccc}
    A & \longrightarrow & B_i \longrightarrow & C_i \\
    \downarrow & & \downarrow & & \downarrow \\
    A & \longrightarrow & B & \longrightarrow & C \\
  \end{array}
\]

where the line upside is also an extension, hence \( B_i \) is in \( R \).

We then apply the exact functor \( \lim_{\to} \) to the entire diagram, and we get:

\[
  \begin{array}{ccc}
    A & \longrightarrow & \lim B_i \longrightarrow & \lim C_i \\
    \downarrow & & \downarrow & & \downarrow \\
    A & \longrightarrow & B & \longrightarrow & C \\
  \end{array}
\]

and now by the 5 lemma, we get the isomorphism: \( B \simeq \lim B_i \) is in \( R \), thus \( C \) is in \( \mathcal{D} \).

We now need to prove that the class \( \mathcal{D} \) is stable under taking “amalgamated free product” and “HNN-extension”: at this point, we’ll have the inclusion \( R \subset \mathcal{D} \), in other words, the
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class \( \mathcal{R} \) is stable under taking extensions. For this, we need the lemma hereafter.

(iii) to (v) : The class of groups \( \mathcal{R} \) defined by Vogel is larger than the class \( \mathcal{C} \) defined by Waldhausen in [Wal78] because here we require no more coherence condition on the base ring for an amalgamated sum or an HNN-extension; we recollect here some results of his article.

(vi) We refer the reader to the proof given by Baumslag-Dyer-Heller in their article [Bau80]; it then suffices to verify that the different groups \( G \) that occur always lie in the class \( \mathcal{R} \).

\[ \text{Definition 6.} \]

Let \( \Gamma \) be a graph. We call a "\( \Gamma_\mathcal{R} \)-space" a tuple \((E_\Gamma,(E_x)_{x \in \Gamma_0}, (E_a)_{a \in \Gamma_1}, f)\) where for each vertex \( x \) of the graph \( \Gamma \), the associated topological space \( E_x \) is the disjoint union of Eilenberg-Mac Lane spaces \( \coprod K(G,1) \), with \( G \) a group in the class \( \mathcal{R} \); similarly for each edge \( a \) of the graph \( \Gamma \), the associated topological space \( E_a \) is the disjoint union of Eilenberg-Mac Lane spaces \( \coprod K(G',1) \), with \( G' \) a group in the class \( \mathcal{R} \); for each incidence relation \( x \in a \), we are given an associated map \( i : E_a \to E_x \) injective on the \( \pi_1 \) for every choice of a base-point; the map \( f \) is defined on the vertex \( E_x \to x \) and is locally a trivial fibration \( E_a \times a \to a \) over the edges; finally, the CW-complex \( E_\Gamma \) is obtained by gluing as the following pushout, making \( f \) a cellular map \( f : E_\Gamma \to \Gamma \).

\[ \begin{array}{c}
\coprod_{\sigma \in \Gamma_1} E_{\sigma} \times \partial \sigma \\
\downarrow \\
\coprod_{\sigma \in \Gamma_0} E_{\sigma}
\end{array} \]

\[ \coprod_{\sigma \in \Gamma_1} E_{\sigma} \times \sigma \to E_\Gamma \]

By abuse of notation, we shall talk about the \( \Gamma_\mathcal{R} \)-space \( E_\Gamma \).

\[ \begin{array}{c}
E_a \times a \\
\coprod
\end{array} \]

\[ \begin{array}{c}
E_\Gamma \\
\coprod
\end{array} \]

\[ \begin{array}{c}
\text{CW-complex } E_\Gamma \\
\coprod
\end{array} \]

\[ \begin{array}{c}
\text{Graph } \Gamma
\end{array} \]

\[ \begin{array}{c}
x \\
\coprod
\end{array} \]

\[ \begin{array}{c}
y \\
\coprod
\end{array} \]

\[ \begin{array}{c}
\text{Lemma 2.} \]

Let \( E_\Gamma \) be a \( \Gamma_\mathcal{R} \)-space. Then the underlying space \( E_\Gamma \) obtained by gluing is itself the disjoint union of some Eilenberg Mac-Lane spaces \( \coprod K(\pi,1) \) with groups \( \pi \) in the class \( \mathcal{R} \); and for each vertex \( x \) and each edge \( a \) of the graph \( \Gamma \), the structural maps \( E_x \to E_\Gamma \) and \( E_a \to E_\Gamma \) are injective on the \( \pi_1 \) for every choice of a base-point.
Proof:
a. As the space $E_{\Gamma}$ is obtained by a filtering colimit indexed by all finite subgraphs $\Gamma_0$ in $\Gamma$:
$$E_{\Gamma} = \lim_{\Gamma_0 \subseteq \Gamma} E_{\Gamma_0},$$
we can suppose thereafter that the graph $\Gamma$ is finite.

b. We proceed by induction on the number of cells (in other words the number of vertices and edges): we take as induction hypothesis the conclusion of the geometrical lemma. We'll suppose moreover as a practical hypothesis for work that all spaces $E_x$ and $E_a$ are connected.

If the graph contains no edge, then the space $E_{\Gamma}$ is the disjoint union of the spaces $E_x$ and the lemma is proven. Otherwise, we can choose an edge $a$ in $\Gamma_1$ and we decompose: $\Gamma = \Gamma' \bigcup a$.

Two cases then may arise:

1st case: The edge $a$ links two distinct connected components $\Gamma'$ and $\Gamma''$. In this case, the map $E_a \to E_{\Gamma'}$ is injective on the $\pi_1$ because composed of $E_a \to E_x$ injective on the $\pi_1$ by beginning hypothesis, and $E_x \to E_{\Gamma'}$ injective on the $\pi_1$ by induction hypothesis. The same argument holds for the map $E_a \to E_{\Gamma''}$. But now we're in the case of an amalgamated sum:

$$E_a \times a \to E_{\Gamma'} \to \to E_{\Gamma''} \to \to E$$

By Van-Kampen theorem, the pushout $E$ is then connected, it’s an Eilenberg Mac-Lane space $K(G, 1)$ with $G$ an amalgamated sum of groups already in $\mathcal{R}$, hence $G$ is in the class $\mathcal{R}$.

2nd case: The edge $a$ has its two vertices in the same connected component $\Gamma'_1$. The same argument shows that the maps $E_a \to E_{\Gamma'_1}$ are injective on the $\pi_1$, and we're now in the case of an HNN-extension:

$$E_a \times a \to E_{\Gamma'_1}$$

By Van-Kampen theorem, the gluing $E$ is then connected, it’s an Eilenberg-Mac Lane space $K(G, 1)$ with $G$ an HNN-extension of groups already in $\mathcal{R}$, hence $G$ is in the class $\mathcal{R}$.

c. By induction, the lemma is thus proven when the spaces $E_x$ and $E_a$ are connected. The general case goes back to this one through the following change of variables: we construct another graph $\Gamma'$ with $\Gamma'_0$ the set of couples $(x, u)$ where $x$ is a vertex of $\Gamma$ and $u$ is a connected component in $E_x$. In the same way, we let $\Gamma'_1$ be the set of couples $(a, v)$ where $a$ is an edge of $\Gamma$ and $v$ is a connected component in $E_a$. Then we take $E'_{(x, u)} = u$ and $E'_{(a, v)} = v$. Hence we get $E'_{\Gamma'} = E_{\Gamma}$, and the lemma is proven. ■
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Practical Use:
- For an HNN-extension $C \twoheadrightarrow A$, we shall take for $\Gamma$ one point $x$ doted with a circular line $a$, endowed with the fibres $E_x = K(A, 1)$ and $E_a = K(C, 1)$.
- For an amalgamated free product $C \twoheadrightarrow A \twoheadrightarrow G$, we shall take for $\Gamma$ one line $a$ with distinct vertices $x$ and $y$, endowed with the fibres $E_x = K(A, 1)$, $E_y = K(B, 1)$ and $E_a = K(C, 1)$.

End of Proof for Theorem 1:

Stability under taking subgroups:

Let $\Gamma$ be one of the two graphs above, representing an HNN-extension, or an amalgamated free product of groups. Suppose $G$ obtained by the diagram $\Gamma$ from cells $G_\alpha$ in the class $\mathcal{C}$.

Let $H$ be a subgroup of $G$. We take $X = E_G$ and construct over $X$ the subcovering $\tilde{X} = E_H$ of the universal covering, of fundamental group $\pi_1 \tilde{X} = H$. The pullback is functorial from the category of objects over $X$ towards the category of objects over $Y$, thus we obtain:

\[
\begin{array}{ccc}
\tilde{X}_a \twoheadrightarrow \tilde{X}_x & \twoheadrightarrow & \tilde{X} \\
\downarrow & & \downarrow \\
X_a & \twoheadrightarrow & X
\end{array}
\]

The choice of a base-point in $\tilde{X}_a$ extends in $X_a$, then in $X_x$, hence in a connected component of $\tilde{X}_a$. The composed map $\pi_1 \tilde{X}_a \to \pi_1 \tilde{X}_x \to \pi_1 \tilde{X}$ being injective, we deduce that the map $\pi_1 \tilde{X}_a \to \pi_1 \tilde{X}_x$ is injective, hence the hypothesis of the lemma are verified for the composed map $X \to \tilde{X} \to \Gamma$. The space $E_H$ is hence obtained by gluing spaces $\tilde{X}_a$ of fundamental groups $H_\alpha = \pi_1 \tilde{X}_a$ being subgroups of $G_\alpha = \pi_1 X_\alpha$. But $G_\alpha$ is in $\mathcal{C}$ by beginning hypothesis, hence $H_\alpha$ is in $\mathcal{R}_\alpha$, and by the lemma, $H$ is thus in $\mathcal{R}$; it’s equivalent to say that $G$ is in $\mathcal{C}$.

Stability under taking extensions:

Let $\Gamma$ be one of the two graphs above, representing an HNN-extension, or an amalgamated free product of groups. Suppose $A$ fixed in $\mathcal{R}$ and $C$ obtained by the diagram $\Gamma$ from cells $C_\alpha$ in the class $\mathcal{D}$. We consider an extension $A \twoheadrightarrow B \twoheadrightarrow C$. We want to show that $B$ is in $\mathcal{R}$. We construct over the total space $E_C = K(C, 1)$ a fibration $X \to E_C$ of fixed fiber $F = K(A, 1)$, which gives by pullback some induced fibrations $F \twoheadrightarrow X_\sigma \twoheadrightarrow E_{C_\sigma}$ over the cells. The group $B = \pi_1 X$ is then obtained by gluing cells $B_\sigma = \pi_1 X_\sigma$, obtained by the pullback extension $A \twoheadrightarrow B_\sigma \twoheadrightarrow C_\sigma$. But then each $C_\sigma$ is in the class $\mathcal{D}$ by beginning hypothesis, thus $B_\sigma$ is in $\mathcal{R}$; hence by the lemma (the injectivity hypothesis are verified as above), $B$ is thus in the class $\mathcal{R}$; it’s equivalent to say that $C$ is in the class $\mathcal{D}$.

Remark:
We shall note that this little geometrical lemma develops ideas very near from Waldhausen’s notion of “splitting” of groups exposed in [Wal78], page 249.

We shall now state Vogel’s Conjecture, and establish as a matter of consequence some theorems analog to the famous th.19.4 on page 249 in [Wal78].
Conjecture: \[\text{[Vog90]}\]
Let \( C \) be a regular \(^{11}\) ring (in the sense of Vogel) and let \( S \) be a \( C \)-bimodule, flat on the left. Then the Waldhausen groups \( K_n(\text{Nil}(C, S)) \) vanish for all \( i \geq 0 \).

Remark:
In fact, using the suspension functor \( \Sigma \) introduced by Karoubi, this conjecture implies the vanishing of the groups \( K_n(\text{Nil}(C, S)) \) for all \( i \in \mathbb{Z} \). Hence the canonical inclusion \( C \to C[S] \) induces an isomorphism at the level of the \( K \)-theory non-connective spectra: \( K_*(C[S]) \simeq K_*(C) \).

Let’s now recall the definitions of the various \( \text{Nil} \) used by Waldhausen in the three cases described on page 6. (originally Prop. 4.1 of page 161 in \[\text{Wal78}\].)

Definition 7:

1. In the case of a generalized free product with maps \( \alpha : C \to A \) and \( \beta : C \to B \) pure with complements \( A' \), \( B' \); define \( \text{Nil}(C; A', B') \) to be the category of tuples \( (P, Q, p, q) \) with \( P, Q \) two right projective \( C \)-modules, and \( p : P \to Q \otimes_C A' \) and \( q : Q \to P \otimes_C B' \) two nilpotent maps (i.e. \( (p \circ q)^n \) vanishes for \( n \) large enough).

2. In the case of a Laurent extension of rings, with maps \( \alpha, \beta : C \to A \) pure with complements \( A', A'' \); define \( \text{Nil}(C; A', A'') \) to be the category of tuples \( (P, Q, p, q) \) with \( P, Q \) two right projective \( C \)-modules, provided with \( p : P \to (Q \otimes_C A_{\alpha}') \oplus (P \otimes_C A_{\beta}) \) and \( q : Q \to (P \otimes_C A_{\beta}'' \beta) \oplus (Q \otimes_C A_{\alpha}) \) two nilpotent maps (i.e. \( (p \circ q)^n \) vanishes for \( n \) large enough).

3. Finally, in the case of the tensor algebra of the \( C \)-bimodule \( S \); define \( \text{Nil}(C; S) \) to be the category of pairs \( (P, p) \) with \( P \) a right projective \( C \)-module, dotted with \( p : P \to P \otimes_C S \) a nilpotent map (i.e. \( p^n \) vanishes for \( n \) large enough).

Proposition 8:

1. There exists a \( C \times C \)-bimodule \( X \) and an equivalence of Waldhausen categories: \( \text{Nil}(C; A', B') \simeq \text{Nil}(C \times C; X) \) induced by the following direct sum functor: \( (P, Q, p, q) \mapsto (P \oplus Q, p \oplus q) \). This induces an equivalence at the level of the \( K \)-theory non-connective spectra.

2. There exists a \( C \times C \)-bimodule \( Y \) and an equivalence of Waldhausen categories: \( \text{Nil}(C; A_{\alpha}', \beta A_{\beta}') \simeq \text{Nil}(C \times C; Y) \) induced by the following functor: \( (P, Q, p, q) \mapsto (P \oplus Q, p \oplus q) \). This induces an equivalence at the level of the \( K \)-theory non-connective spectra.

Proof:

1. One takes \( X = A' \oplus B' \) with the action of the first \( C \) on \( A' \) via \( \alpha \) on the right, on \( B' \) via \( \beta \) on the left, and the action of the second \( C \) on \( A' \) via \( \alpha \) on the left, on \( B' \) via \( \beta \) on the right. All other actions are trivial. \( \mathbb{2} \) One takes \( Y = \alpha A_{\alpha}' \oplus \beta A_{\beta}' \oplus \alpha A_{\beta} \oplus \beta A_{\alpha} \) with the action the first \( C \) via \( \alpha \) and the second \( C \) via \( \beta \) (the notation is suggestive). All other actions are trivial. The verifications are obvious.

\(^{11}\) Here it should be important to precise, according to the needs of the proof, if we ask the ring \( C \) to be “right regular” or “left regular”, or even “regular on both sides”.

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Theorem 2. Suppose that the [Conjecture] is true. Let the ring \( R \) be either:
1. the free product in the situation \( \alpha : \mathcal{C} \to A, \beta : \mathcal{C} \to B \) or
2. the Laurent extension with respect to \( \alpha, \beta : \mathcal{C} \to A \) or
3. the tensor algebra of the \( C \)-bimodule \( S \).
Assume that the maps \( \alpha, \beta \) are pure embeddings, with complements flat from the left; likewise \( S \) is flat from the left. Suppose that the ring \( C \) is regular (in the sense of Vogel). Then all the \( KN\text{Nil} \) groups vanish, and we have the corresponding Mayer-Vietoris long exact sequences (for every index \( i \in \mathbb{Z} \)):
1. \[ \ldots \to K_i(C) \to K_i(A) \oplus K_i(B) \to K_i(R) \to K_{i-1}(C) \to \ldots \]
2. \[ \ldots \to K_i(C) \to K_i(A) \to K_i(R) \to K_{i-1}(C) \to \ldots \]
3. \[ K_i(R) \simeq K_i(C) \]

Proof:
- First, let’s recall the construction of Karoubi’s functor \( \Sigma \). We note \( C(\mathbb{Z}) \) the ring of infinite matrices with coefficients in \( \mathbb{Z} \), all zero except finitely many in each line and each column. Let \( M(\mathbb{Z}) \) be the sub-ring of finite matrices, and \( \Sigma(\mathbb{Z}) \) the quotient. The tensor product over \( R \) gives the exact sequence: \( M(\mathbb{R}) \to C(\mathbb{R}) \to \Sigma(\mathbb{R}) \). Karoubi deduces the homotopy fibration: \( K(M(\mathbb{R})) \xrightarrow{\sim} K(C(\mathbb{R})) \to K(\Sigma(\mathbb{R})) \). The left term gives \( K(R) \) by Morita equivalence. The middle term vanishes due to a flasque functor \( F \) such that: \( F_* = F_* + \text{Id}_* \). Finally, \( K(R) \simeq \Omega K(\Sigma(\mathbb{R})) \) allows us to define negative K-theory groups: \( K_{-i}(R) = K_0(\Sigma^i(\mathbb{R})) \).

- By immediate computation: \( K_{i-1}(\text{Nil}(C; S)) = K_0(\Sigma(\mathbb{R})) \).
- We know from the fundamental theorem of Bass-Heller-Swan-Quillen that for every ring \( C \) and every index \( i \in \mathbb{Z} \) the map \( K_{i+1}(C(\mathbb{Z})) \to K_i(C) \) is surjective. Thus we get a surjection: \( K_{i+1}(\text{Nil}(C(\mathbb{Z}); S(\mathbb{Z})) \to K_i(\text{Nil}(C; S)) \).

- Finally, we treat the Laurent extension of rings. By Proposition 5, the obstruction to excision is the Waldhausen non-connective spectrum \( K_*\text{Nil}(C; A, B) \simeq K_*\text{Nil}(C \times C; X) \).

Theorem 3 [Vog90] Suppose that the [Conjecture] is true. Let \( R \) be a regular ring, and \( G \) a group in the class \( \mathfrak{R} \). Then Whitehead’s obstruction non-connective spectrum \( Wh^R(\mathbb{G}) \) is contractible. In other words, the non-connective spectrum of algebraic K-theory \( K(R[G]) \) behaves like the canonical homology theory associated to the \( \Omega \)-spectrum \( BG^+ \wedge K(R) \) with respect to the variable of groups \( G \) in the class \( \mathfrak{R} \); in particular it verifies the excision theorem, and gives Mayer-Vietoris long exact sequences (therein some explicit calculus is possible for \( K(R[G]) \) via spectral sequences).
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Proof:
By definition, we have the homotopy fibration: \( H_*(BG\wedge K(R)) \to K_*(R[G]) \to Wh^R(G) \).
Thus the Whitehead space measures to what extent \( K_*(R[G]) \) differs from a homology theory when \( R \) is fixed and \( G \) varies in the class \( R \). But now, Theorem\[2\] shows that excision holds for the generalized free product, the Laurent extension, and the tensor algebra cases. Hence the proof can be made by induction on \( G \in R \): it’s obvious for \( G = 1 \) and the property is stable under amalgamated free product, HNN-extension, and filtering colimits for both homology theories (via the Mayer-Vietoris long exact sequences above). Thus the result holds for every \( G \in R \). That’s precisely the generalization we wanted for Th.19.4 on page 249 in [Wal78]. ■

More precise results on the structure of Waldhausen \( \tilde{K}Nil \) groups will appear in [Bih02, Bih03], that shall give partial answers to the [Conjecture]: the first based on a powerful localization theorem by Vogel on complexes of diagrams; the second on a careful study of categories and functors involved in the \( \tilde{Nil} \) terms, (constructing a new cyclic functor on the graded categories). Both articles intensively make use of Vogel’s notion of regularity exposed here.

5 Caracterization of regular modules

Fix now a ring \( C \) (unitary, associative). We can simplify the characterization of the class \( C_0 \) of regular \( C \)-modules (hereafter we give the proof from [Vog90]):

Proposition 9.
Let \( C \) be a ring, and \( C_0 \) the class of regular \( C \)-modules. Consider now \( C \) the smallest class containing all free \( C \)-modules, stable under filtering colimits, and verifying for each exact sequence \( 0 \to M \to N \to P \to 0 \), if \( M, N \in C \), then \( P \in C \) [We shall say that \( C \) is 'stable under cokernels of cofibrations']. Then the two classes coincide: \( C_0 = C \).

Proof:
For each ordinal \( \alpha \), we construct inductively a class \( D_\alpha \) in the following way:
\( D_0 \) is the class of all free modules. If \( \alpha \) is a limit-ordinal, a module \( M \) is in \( D_\alpha \) if and only if it’s a filtering colimit of modules in \( \bigcup_{\beta < \alpha} D_\beta \). If \( \alpha = \beta + 1 \), a module \( M \) is in \( D_\alpha \) if and only if it’s the cokernel of a monomorphism \( A \to B \), with \( A \) and \( B \in D_\beta \). We immediately get the inclusion: \( D_\alpha \subset C \). We need then to prove that the union \( C \) of all classes \( D_\alpha \) is exactly the class of regular modules \( C_0 \). The only thing that remains to be proven is that this class \( C \) is stable by the '2/3 axiom'. For this, we verify 3 lemmas:

Lemma 3.
For each ordinal \( \alpha \), the kernel of an epimorphism from a free module towards a module in \( D_\alpha \) is in \( C \).

Proof:
It’s true for \( \alpha = 0 \). Proceeding by induction, we’ll suppose the lemma true for all \( \beta < \alpha \). Let \( f : F \to M \) be an epimorphism from a free module \( F \) towards a module \( M \) in \( D_\alpha \). If

\[\text{\[2\]}\]
\[\text{From now on and till the end of this article, “regular” means “right regular” and “module” means “right module”. But all proofs go through similarly if we take complexes of left modules over left regular rings.}\]
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\[ \alpha = \beta + 1, \text{ we get a short exact sequence } 0 \to M' \to M'' \to M \to 0 \text{ with } M' \text{ and } M'' \text{ in } \mathcal{D}_\beta. \]

We can complete the following diagram:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & M' & \xrightarrow{f'} & M'' & \xrightarrow{f} & M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & E' & \xrightarrow{f'} & E'' & \xrightarrow{f} & E & \rightarrow & 0 \\
\end{array}
\]

where all lines are exact, all vertical maps are surjective, and the modules \( F', E', F \) are free. By the induction hypothesis, \( \text{Ker}(f') \) and \( \text{Ker}(f'') \) are in \( \mathcal{C} \). Thus the kernel \( \text{Ker}(f) = \text{Coker}((\text{Ker}(f') \to \text{Ker}(f''))) \) is in \( \mathcal{C} \) too. If \( \alpha \) is a limit-ordinal, \( M \) is the filtering colimit of a system of modules \( (M_i)_{i \in \mathbb{J}} \) where \( \mathbb{J} \) is a filtering category (in the sense of MacLane in \( \text{[Mac71]} \)) and each \( M_i \) is in \( \mathbb{D}_\beta \) with \( \beta < \alpha \). The difficulty consists in finding an inductive system \( (F_i)_{i \in \mathbb{J}} \) with compatible short exact sequences \( F_i \to F_j \to M_i \) such that \( \lim F_i \) is a free module. One functional way of doing this is the following: note \( M_\bullet \) this system of modules. For each index \( i \), let \( F_\bullet \) be the following system: for each index \( j \), let \( F_{ij} \) be the free \( C \)-module generated by all maps in \( \mathbb{J} \) from \( i \) to \( j \). For each map \( j \to k \) in \( \mathbb{J} \), the induced map \( F_{ij} \to F_{ik} \) is given by composition. Clearly, \( \text{Hom}(F_{ij}, M_\bullet) \) is isomorphic to \( M_k \) [the isomorphism is induced by the image of \( Id : i \to i \) ] and the colimit of the \( F_{ij} \) is isomorphic to \( C \) [the map \( Id : i \to i \) induces all maps \( i \to j \) in the inductive limit]. Let \( \mathbb{J} \) be the set of couples \((i, u)\) with \( i \) an index in \( \mathbb{J} \) and \( u \) a map from \( F_\bullet \) to \( M_\bullet \). Let \( F_\bullet = \bigoplus_{(i,u) \in \mathbb{J}} F_{ij} \). We get a canonical map \( \phi_\bullet : F_\bullet \to M_\bullet \). For each index \( i \), \( \phi_i : F_j \to M_j \) is surjective, its kernel \( K_j \) is in \( \mathcal{C} \) (by induction hypothesis). Pass to the filtering colimit through the index \( j \): it’s an exact functor, hence we get the short exact sequence \( \lim K_j \xrightarrow{\lim F_j} \lim M \) to be compared with the short exact sequence given at the beginning: \( \lim K_j \xrightarrow{\lim F_j} \lim F_j \oplus K \). In the left part, the colimit is in \( \mathcal{C} \) because each \( K_j \) is there. A simple proof by induction then shows that adding in a direct sum a fixed free module \( F \) to any object in \( \mathcal{C} \) gives an object in \( \mathcal{C} \) too. Therefore \( K \) is a direct summand of an object in \( \mathcal{C} \) hence by lemma 1, the module \( K \) is in \( \mathcal{C} \). This ends the case of a limit-ordinal, and thus the proof. ■

\[\text{Lemma 4}.\]
Consider \( 0 \to M \to N \to P \to 0 \) a short exact sequence. If \( M, P \in \mathcal{C} \), then \( N \in \mathcal{C} \) too.

\[\text{Proof}:\]
Let \( f : F \to P \) be an epimorphism from a free module \( F \) to \( P \). Let \( Q \) be the pullback of \( F \) and \( N \) over \( P \). As \( F \) is free, the exact sequence splits, and the module \( Q \) is isomorphic to \( M \oplus F \); hence it’s in \( \mathcal{C} \). By lemma \( \text{[3]} \) \( \text{Ker}(f) \) is in \( \mathcal{C} \). Thus \( N = \text{Coker}(\text{Ker}(f) \to Q) \) is in \( \mathcal{C} \) too. ■

\[\text{Lemma 5}.\]
Consider \( 0 \to M \to N \to P \to 0 \) a short exact sequence. If \( N, P \in \mathcal{C} \), then \( M \in \mathcal{C} \) too.

\[\text{Proof}:\]
Let \( f : F \to N \) be an epimorphism from a free module \( F \) to \( N \). Let’s note \( K \) the kernel of the composed map \( F \to N \to P \). We get a short exact sequence \( 0 \to K \to M \oplus F \to N \to 0 \). By lemma \( \text{[3]} \) \( K \) is in \( \mathcal{C} \). By lemma \( \text{[4]} \) the extension \( M \oplus F \) is in \( \mathcal{C} \) too. But \( M \) is a direct summand of \( M \oplus F \), hence by stability under filtering colimits, \( M \) is in \( \mathcal{C} \). ■

The class \( \mathcal{C} \) thus verifies the three conditions in the ‘2/3 axiom’, hence it’s an exact class, and it’s exactly the class \( \mathcal{C}_0 \) of all regular modules. ■
6 Complexes on a regular ring

Let’s now remind some definitions:

**Definition 8**: Fix $R$ a ring, and $\mathcal{C}_0$ the class of all regular $R$-modules.

- We’ll define a “$R$-complex” $C_*$ to be a complex of projective $R$-modules.
- We’ll say that $C_*$ is “bounded from below” if $C_n$ vanishes for $n$ small enough.
- We’ll say that $C_*$ is “quasi-coherent” if all $C_n$ are finitely generated.
- We’ll say that $C_*$ is “finite” if $\bigoplus_{n \in \mathbb{N}} C_n$ is finitely generated.
- At last, we’ll call a $R$-complex $D_*$ “finite up to homotopy” if there exists a finite $R$-complex $C_*$ and two chain morphisms $f : C_* \to D_*$ and $g : D_* \to C_*$, such that $f \circ g$ is homotopic to the identity of $D_*$, and conversely, $g \circ f$ is homotopic to the identity of $C_*$. With these notations, we can approach the fundamental theorem of regular rings (proven by Pierre Vogel in his unpublished article [Vog90]):

**Lemma 6 [Technical]**. Let $C_*$ be a quasi-coherent $R$-complex, and $M$ be a regular module. Every chain morphism from $C_*$ to $M$ factorizes through a finite $R$-complex.

**Proof**: Here the module $M$ is considered as a graded differential module, concentrated in degree 0 and with a trivial differential. Let $\mathcal{C}$ be the class of $R$-modules $M$ such that, for every quasi-coherent $R$-complex $C_*$, every chain morphism from $C_*$ to $M$ factorizes through a finite $R$-complex.

(i) Let $F$ be a free $R$-module, and $f : C_* \to F$ be a chain morphism. Then $f$ is given by $f_0 : C_0 \to F$ and thus $f$ factorizes through a free finitely generated $R$-module $F'$, contained in the free module $F$. As $F'$ is a finite complex, thus $F$ is in $\mathcal{C}$.

(ii) Let $M = \varinjlim_{i \in J} M_i$, with each $M_i$ in $\mathcal{C}$. Let $f$ be a chain morphism from a quasi-coherent $R$-complex $C_*$ to $M$. As $f$ is defined by a map from the finitely presented module $\text{Coker}(C_1 \to C_0)$ to $M$, $f$ factorizes through one of the $M_i$; but the map $C_* \to M_i$ factorizes through a finite $R$-complex; hence $M$ is in $\mathcal{C}$.

(iii) Let $0 \to M \to N \to P \to 0$ be an exact sequence of $R$-modules. Suppose that $M$ and $N$ are in $\mathcal{C}$. Let $f$ be a chain morphism from a quasi-coherent $R$-complex $C_*$ to $P$. Note $D$ the cone of the identity $\text{Id} : \Sigma^{-1}C \to \Sigma^{-1}C$. The $R$-complex $D$ is contractible, quasi-coherent, and maps surjectively on $C_*$. As $D$ is contractible, there is no obstruction to lift the chain morphism $D \to C \to P$ at the level of $N$. Then we obtain the following diagram:

```
0 ----> M ----> N ----> P ----> 0
       |                        |        |
0 ----> \Sigma^{-1}C ----> D ----> C ----> 0
```

As $M$ is in $\mathcal{C}$ and $\Sigma^{-1}C$ is quasi-coherent, the map $\Sigma^{-1}C \to M$ factorizes through a finite complex $W$. Let $E$ be the pushout of $D$ and $W$ over $\Sigma^{-1}C$. The complex $E$ is quasi-coherent and $N$ is in $\mathcal{C}$. Thus the map $E \to N$ factorizes through a finite complex $X$. Note $F$ the mapping cone of the identity $\text{Id} : W \to W$. As $F$ is contractible, the chain morphism $W \to F$ extends to $E$. Let $L$ be the direct sum $X \oplus F$. The construction above gives us a factorization of $E \to N$ through $L$ and the map $W \to L$ is injective, with projective cokernel we shall note...
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$K$. The complexes $W, L$ and $K$ are finite, and the map $C \to P$ factorizes through $K$. We can overview this construction in the following diagram:

```
0 → M → N → P → 0
0 → W → L → K → 0
0 → Σ⁻¹C → D → C → 0

E
```

Then $\mathcal{C}$ contains all free modules, is stable under filtering colimits, and under cokernels of cofibrations; so by Proposition $\mathcal{C}$ contains the category $\mathcal{C}_0$ of all regular modules. ■

Theorem 4 [Fundamental] [Vog90]
Let $R$ be a regular ring. Let $C_n$ be a quasi-coherent $R$-complex, and $C_n'$ a bounded from below $R$-complex having only finitely many non-trivial homology groups. Then every chain morphism from $C$ to $C'$ factorizes, up to homotopy, through a finite $R$-complex.

Proof:
We proceed by induction on the number of non-trivial homology groups of $C'$. If $C'$ has no homology, $C'$ is contractible (because it’s projective and bounded from below) and every chain morphism from $C$ to $C'$ factorizes up to homotopy, through the zero complex. Let now $C'$ be a $R$-complex with $n$ non-trivial homology groups. We can kill the last non-trivial homology group of $C'$ by adding algebraic cells; this way we obtain new $R$-complexes $C_0'$ and $C_1'$ and a short exact sequence $0 → C_0' → C_1' → 0$ such that $C_1'$ has only one non-trivial homology group, and $C_n'$ has only $(n-1)$ ones. By induction, the map $C \to C' \to C_0'$ factorizes, up to homotopy, through a finite $R$-complex $K_0$. Let $E$ be the mapping cone of the identity $Id : C → C$. The complex $E$ is quasi-coherent, contractible, and contains $C$. The difference of the maps $C → C' → C_0'$ and $C → K_0 → C_1'$ is null-homotopic, hence factorizes through $E$. Thus the composed map $C → C' → C_0'$ factorizes through the complex $K_0' = K_0 \oplus E$ where $K_0'$ is quasi-coherent, and has the homotopy type of a finite complex. Moreover, the map $C → K_0'$ is injective, with projective cokernel we shall note $C_1$. The $R$-complex $C_1$ is quasi-coherent, and we have a chain morphism $g : C_1 → C_1'$. But $C_1'$ has only one non-trivial homology group $M$, in other words: $C_1'$ is a projective resolution of $M$. For every quasi-coherent $R$-complex $L$, the homotopy classes of chain morphisms from $L$ to $C_1'$ are isomorphic to the homotopy classes of chain morphisms from $L$ to $M$. By the technical lemma, the map $g$ factorizes, up to homotopy, through a finite $R$-complex $K_1$. The construction above gives then a quasi-coherent $R$-complex $K_1'$, with the homotopy type of a finite complex, and a factorization of $g$ through $K_1'$. Note $K'$ the homotopy kernel of the chain morphism $K_0' → K_1'$. In other words the desuspension of its mapping cone). By construction, $f$ factorizes through $K'$, this $R$-complex $K'$ has the homotopy type of a finite complex $K$, and $f$ factorizes, up to homotopy, through $K$. Let’s overview all this construction by a diagram:

```
0 → C' → C_0' → C_1' → 0
K' → K_0' → K_1'
0 → C → K_0' → C_1 → 0
```
Corollary 1.
Let $R$ be a regular ring and $C_\ast$ be a quasi-coherent $R$-complex. Then $C_\ast$ is finite up to homotopy, if and only if $C_\ast$ has only a finite number of non-trivial homology groups.

Proof:
The proof needs 2 steps:
1. If $C_\ast$ is quasi-coherent and bounded from below, with only a finite number of non-trivial homology groups, we can apply the theorem above to the map $Id : C_\ast \to C_\ast$. Thus the identity map $Id$ factorizes, up to homotopy, through a finite complex $K$, hence the complex $C_\ast$ is a direct summand of $K$ (up to homotopy), and thus $C_\ast$ is finite up to homotopy.
2. Let now $C_\ast$ be quasi-coherent, no more bounded from below, such that $H_i(C_\ast) \neq 0$ implies $i \in [a, b]$. We consider the dual complex $\hat{C}_\ast$ defined by $\hat{C}_n = \text{Hom}(C_{-n}, R)$, and the class $\mathcal{C} = \{ M \in \text{Mod}_R \mid \forall i > -b, H^i(\hat{C}, M) = 0 \}$. Then this class contains all free modules because the $C_n$ are finitely generated, it is stable under filtering colimits on $M$, at last it is stable under cokernels of cofibrations (look at the cohomology long exact sequence). By proposition 9 and the hypothesis that $R$ is regular, we get $\mathcal{C} = \text{Mod}_R$. This implies in particular the existence of a map $\epsilon_i : \hat{C}_{i-1} \to \hat{C}_i/d\hat{C}_{i+1}$ that splits the canonical surjection. This map forms a homotopy $Id_{\hat{C}_i} \sim 0$ in degree $i > -b$; thus $\hat{C}$ is homotopic to a bounded from above complex. By duality, $\hat{C}$ is hence homotopic to a bounded from below $R$-complex. That leads us back to case (1).

Proposition 10.
Let $R$ be a coherent ring.

Then $R$ is regular (in the classical sense) if and only if $R$ is regular (in the sense of P. Vogel).

Proof:
A ring $R$ is coherent regular in the classical sense if every finitely presented $R$-module admits a resolution: $0 \to C_0 \to \ldots \to C_1 \to C_0 \to M \to 0$ where the $C_i$ are finitely generated projective. (i) $\to$ (ii): As every $R$-module is the filtering colimit of finitely presented modules, we need only showing that every finitely presented module $M$ is in $\mathcal{C}_0$. We split the finite resolution of $M$ in short exact sequences, and we apply the '2/3 axiom'. (ii) $\to$ (i): Suppose the ring $R$ regular (in the sense of Vogel) and coherent. Let $M$ be a finitely presented $R$-module. As $R$ is coherent, $M$ admits a projective resolution $C_\ast$, which is a quasi-coherent bounded from below $R$-complex, with only non-trivial homology group. The corollary above tells us that $C_\ast$ has the homotopy type of a finite $R$-complex. Hence $M$ admits a finite projective resolution, and the ring $R$ is regular (in the classical sense).

Hence the notion of regularity introduced by Mr Vogel really extends the classical notion, pre-existing in algebraic geometry, and the results obtained on the $K\text{Nil}$ in this setup will extend those of Waldhausen in [Wal78].

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