Crystalline gauge fields and discrete geometric response for Abelian topological phases with lattice symmetry

Naren Manjunath and Maissam Barkeshli
Department of Physics, Condensed Matter Theory Center, and Joint Quantum Institute, University of Maryland, College Park, Maryland 20742, USA

Clean isotropic quantum Hall fluids in the continuum possess a host of symmetry-protected quantized invariants, such as the quantized Hall conductivity, shift and Hall viscosity, and fractional quantum numbers of quasiparticles. Here we develop topological field theories using discrete crystalline gauge fields to fully characterize quantized invariants of (2+1)D Abelian topological orders in the presence of symmetry group \( G = U(1) \times \mathbb{Z}_M \), where \( \mathbb{Z}_M \) consists of orientation-preserving space group symmetries on the lattice. Discrete rotational and translational symmetry fractionalization is characterized by a discrete spin vector, a discrete torsion vector which has no analog in the continuum or in the absence of lattice rotation symmetry, and an area vector, which also has no analog in the continuum. In particular, we find a type of crystal momentum fractionalization that is only non-trivial for 2, 3, and 4-fold rotation symmetry. The quantized topological response theory includes a discrete version of the shift which binds fractional charge to disclinations and corners, rotationally symmetric fractional charge polarization, constraints on charge filling and their discrete angular momentum counterparts, momentum bound to dislocations and units of area, and all of their duals.

Introduction. One of the most striking discoveries in physics is the quantized Hall conductivity of integer and fractional quantum Hall (FQH) systems \([1,2]\). The quantized Hall conductivity \([3]\), which requires \(U(1)\) charge conservation to define, is however only one of many symmetry-protected topological invariants. In the continuum, clean isotropic quantum Hall systems possess additional symmetry-protected invariants, such as a quantized Hall viscosity \([4,5]\), the shift and fractional orbital spin of quasiparticles \([6]\), and which correspond to quantized responses to deformations of the spatial geometry \([7,8,9]\). In this paper we generalize this understanding to the case of (2+1)D Abelian topological phases with discrete orientation-preserving space group symmetries of the lattice. We develop a theory of discrete “crystalline gauge fields,” which include gauge fields associated with translation and rotation symmetries, and then we develop an effective topological field theory to study the possible symmetry fractionalization patterns and fractionally quantized responses to lattice defects and magnetic fluxes.

Crystalline gauge fields are discrete analogs of the frame fields and spin connection used for continuum geometry and arise in the theory of elasticity \([10,11]\), although previous treatments have not fully taken into account the non-trivial structure of the space groups involved. Crystalline gauge theory has recently also been described abstractly in terms of fiber bundles \([12]\).

Recently a powerful algebraic theory using \(G\)-crossed braided tensor categories has been developed to comprehensively characterize and classify (2+1)D topologically ordered phases of matter with symmetry \([13]\). In the case of Abelian topological orders with symmetries whose action does not permute distinct quasiparticle types, an alternate approach using topological effective actions, which we develop here, is significantly simpler and yields insight into the physical response.

Crystalline gauge fields. We consider a (2+1)D spacetime manifold \( M = \Sigma^2 \times \mathbb{R} \), where \( \Sigma^2 \) is the space on which the clean lattice system is defined. We fix an arbitrary triangulation of \( M \) and we define on the links a gauge field valued in the symmetry group \( G = U(1) \times \mathbb{Z}_M \). \( \mathbb{Z}_M \) contains translation symmetry and a discrete \( M \)-fold rotation symmetry for \( M = 1, 2, 3, 4, 6 \). Physical results will be independent of triangulation. We define a \( U(1) \) gauge field \( A_{ij} \) on the link \( ij \) of the triangulation, with the link directed towards \( j \) (with \( A_{ij} = -A_{ji} \) and \( A_{ij} \sim A_{ij} + 2\pi \)). Next, we define the crystalline gauge field \( B_{ij} = (\vec{R}_{ij}, C_{ij}) \). Here, \( \vec{R}_{ij} = (X_{ij}, Y_{ij}) = ((R_{ij})_x, (R_{ij})_y) \in 2\pi\mathbb{Z}^2 \) is an integer gauge field corresponding to \( \mathbb{Z}^2 \) translations. The field \( C \) corresponds to point group rotations, where we take \( C_{ij} \in \mathbb{Z}_M \), with \( C_{ij} \sim C_{ij} + 2\pi \). Group multiplication is given by \( (\vec{R}_1, C_1)(\vec{R}_2, C_2) = (\vec{R}_1 + U(C_1)\vec{R}_2, C_1 + C_2) \), where we use addition in place of multiplication when the group is Abelian. \( U(C_1) \) is the \( 2 \times 2 \) rotation matrix corresponding to \( C_1 \). Formally \( \frac{1}{2\pi}B \) is a lift of an element of \( \mathbb{Z}_M \) to \( \mathbb{R} \).

The gauge freedom in \( \vec{R} \) corresponds to the freedom to relabel lattice coordinates. It arises from the well-known ambiguity in elasticity theory that the displacement vector is only meaningful up to an integer lattice vector \([11]\). The gauge freedom in \( C \) corresponds to the freedom in locally orienting the \( x \) and \( y \) axes at every point in space and time. For example, if for \( M = 4 \) we have \( C_{ij} = \pi/2 \) on some link \( ij \), this means the local coordinate axes at \( i \) and \( j \) will be rotated relative to each other by an angle \( \pi/2 \). Under a gauge transformation which places the gauge variable \( (\vec{r}_i, h_i) \) at the vertex \( i \), we have:

\[
B_{ij} \rightarrow (\vec{r}_i, h_i)^{-1}B_{ij}(\vec{r}_j, h_j) = (U(-h_i)(\vec{R}_{ij} + U(C_{ij})\vec{r}_j - \vec{r}_i), -h_i + C_{ij} + h_j) \quad (1)
\]

The underlying lattice of the physical system specifies the gauge invariant quantities of the crystalline gauge...
field. Flux of $C$ corresponds to disclinations: $f_C$ gives the total angle of disclinations within the cycle $\gamma$. If $C$ vanishes everywhere, then $\oint_C f_C$ gives the total Burgers vector of dislocations contained in $\gamma$. If space is a torus and $C$ vanishes everywhere, then $\oint_C f_x Y$ gives the lengths of the torus in the $x$ and $y$ directions, while $\oint_C X$ gives the shear in the $x$ direction upon traversing the $y$ cycle, and similarly for $\oint_C Y$.

When $C$ is non-zero, one needs to take into account the local change of coordinate frame along $\gamma$. We can define a Burgers vector $\vec{R}(\gamma)$, where $\vec{R}_{k,k+1} = U(C_{01} + C_{12} + \cdots + C_{k-1,k}) \vec{R}_{k,k+1}$ for some arbitrary choice of origin 0 and path from 0 to $k$. The extra $C$ factors play a role analogous to the covariant derivative allowing parallel transport of $\vec{R}$ on the lattice. Under a gauge transformation, $\vec{f}_x \vec{R}(\gamma) \rightarrow U(h_0) \vec{f}_x \vec{R}(\gamma)$, corresponding to the fact that the Burgers vector rotates under rotation of the local coordinate system at the origin. To compare Burgers vectors in different regions, it is thus important to choose a common origin 0 is chosen.

$(\vec{R},C)$ thus play a role similar to the frame field and spin connection used in continuum geometry (see Appendix F); it is useful to distinguish them because $(\vec{R},C)$ have discrete gauge transformations, which plays a crucial role in the classification of topological terms. Note that we do not consider the continuous elastic response of the crystal due to stresses and strains, which does not receive any topological, quantized contributions [17,18].

**Symmetry fluxes.** The $U(1)$ gauge flux $dA[012] = A_{01} + A_{12} - A_{02}$ defined on a 2-simplex [012] of the triangulation is gauge-invariant, with $dA \sim dA + 2\pi \chi$. $C$ behaves mathematically like a discrete version of $A$; the flux $\int_D dC$ for any region $D$ is gauge-invariant and gives the total angle of disclinations within $D$.

Naively one may think that $d\vec{R}(0)$ should be the gauge-invariant physical quantity corresponding to the dislocation density. However $d\vec{R}(0)$ depends on a choice of origin together with a choice of local coordinate frame at that origin. Therefore $d\vec{R}(0)$ is both non-local in general and also not gauge invariant. The local quantity $\vec{R}$ differs from $\vec{R}(0)$ at each link by a rotation, implying that $\vec{R}_{ij} = \vec{R}_{ij}^{(0)} + (1 - U(2\pi/M))\vec{F}_{ij}$ where $\frac{1}{2\pi} \vec{F}_{ij} \in \mathbb{Z}^2$ (see Appendix B1d). Here $U(\frac{2\pi}{M})$ is the generator of the $2\pi/M$ rotations (see Appendix B1a). This further implies that $d\vec{R} = d\vec{R}(0) + (1 - U(2\pi/M))d\vec{F}$. Under gauge transformations, we can show that both terms on the right-hand side change by quantities of the form $(1 - U(2\pi/M))d\vec{F}$, where $\frac{1}{2\pi} d\vec{F} \in \mathbb{Z}^2$. It follows that the fractional part of $\frac{1}{2\pi} (1 - U(\frac{2\pi}{M}))^{-1} d\vec{R}$ is gauge invariant (see Appendix B1c). This motivates us to define the local quantity $\vec{R} = (1 - U(\frac{2\pi}{M}))^{-1} \vec{R}$.

The possible holonomies thus fall into different classes based on $\frac{1}{2\pi} \oint_{0D} \vec{R} \mod 1$. To understand this physically, note that to each region we can assign a local Burgers vector with the choice of origin 0 in $D$. Without picking a common origin, the Burgers vector for a region containing two subregions $D$ and $D'$ is thus ambiguous up to separate local rotations of the coordinate axes for the origins 0 in $D$ and origin' 0 in $D'$. The part of the Burgers vector that is gauge invariant can be defined locally defines a finite group grading on Burgers vectors (see Appendix B1d) for further explanation), where we denote $K_M$ as the finite group. The results for various $M$ are given in Table[1]. Another way of understanding $K_M$ is to consider distinct $\mathbb{Z}_M$ rotationally symmetric polarization configurations on a space with boundary (see Appendix B1d).

We can also construct a flux $A_{XY}$, which is quadratic in $\vec{R}$ and corresponds to an area element. We define $A_{XY}[ijk] = \frac{1}{2\pi} \vec{R}_{ij} \times (U(C_{ij})R_{jk})$, where $\times$ is the cross product of vectors. When $C$ vanishes, this gives the usual area element as expected, and it is easy to verify that on a torus $T^2$ whose side lengths are $L_x$ and $L_y$, $\frac{1}{2\pi} \oint_{\partial D} A_{XY} = L_x L_y$. The factor $U(C_{ij})$ keeps track of the relative orientation of the coordinate axes at $i$ and $j$ when $C \neq 0$. In the absence of dislocations, $A_{XY}$ is gauge-invariant up to a boundary term, so that $A_{XY}$ integrated over a closed surface is gauge invariant. To obtain a well-defined area on spaces with boundary, we require the translation gauge transformations to reduce from $\mathbb{Z}_M^2$ to the subgroup of translations preserved by the boundary. Dislocations can be treated as punctures with non-trivial holonomy, which reduces to the case of a manifold with boundary (see Appendix B2).

**Effective actions.** The topological effective Lagrangian is $\mathcal{L} = -\frac{1}{4\pi} a^I \cup K_{1I} da^I + \mathcal{L}_{frac} + \mathcal{L}_{SPT}$, with

\[
\mathcal{L}_{frac} = \frac{1}{2\pi} a^I \cup (qI dA + sj dC + \vec{l}_I \cdot d\vec{R} + m_I A_{XY})
\]

\[
\mathcal{L}_{SPT} = \frac{k_1}{2\pi} A \cup dA + \frac{k_2}{2\pi} A \cup dC + \frac{k_3}{2\pi} C \cup dC + \frac{1}{2\pi} A \cup (\vec{k}_I \cdot d\vec{R}) + \frac{1}{2\pi} C \cup (\vec{k}_5 \cdot d\vec{R}) + \left( \frac{k_6}{2\pi} A + \frac{k_7}{2\pi} C \right) \cup A_{XY}.
\]
We have used the cup product from cohomology: \((A \cup dA)[ijkl] = A_{ij}dA_{jkl}\) for a 3-simplex \([ijkl]\).

The non-degenerate \(D \times D\) symmetric integer matrix \(K\), which couples the dynamical \(U(1)\) gauge field \(a^I\), characterizes the intrinsic topological order \([20]\), \([21]\).

Topologically distinct quasiparticles correspond to integer vectors \(\vec{l} \sim \vec{l} + K\vec{\Lambda}\), where \(\vec{l}, \vec{\Lambda} \in \mathbb{Z}^D\). The quasiparticles form an Abelian group \(A = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_D}\) under fusion, where the \(n_i\) are the diagonal entries in the Smith normal form of \(K\).

\(L_{\text{trac}}\), which contains the coupling between the background gauge fields and the \(a^I\), specifies symmetry fractionalization, i.e. how the anyons carry fractional symmetry quantum numbers. Mathematically this is classified by the second group cohomology \(\mathcal{H}^2(G,A)\) \([16]\), \([22]\). The terms in \(L_{\text{SPT}}\) correspond to Dijkgraaf-Witten (DW) terms, classified by \(\mathcal{H}^3(G, U(1))\) \([24]\). Physically the DW terms can be understood in terms of stacking symmetry-protected topological (SPT) states \([16]\), \([25]\). The terms we have written above are complete for bosonic systems; for fermionic systems, there can be additional terms, e.g. related to group supercohomology \([20]\), \([27]\), which we leave for future work. While we have defined our topological field theory using the framework of discrete gauge theory, we can equivalently use integral, real-valued differential forms as discussed in Appendix A.

Note that the above action is only uniquely defined when the gauge fields are flat: \(da^I, dA, dc, dK, dR \in 2\pi \mathbb{Z}\). When the gauge fields are not flat, the action is not invariant under the shift of \(a, A, \text{or } C\) by \(2\pi\) on a single 1-simplex. More generally, for non-flat gauge fields, one can add additional terms to the action which depend on the field strength and which are not uniquely specified \([28]\). Non-trivial fluxes of \(a, A, C, \text{and } R\) can be included by treating them as punctures in the spatial manifold around which the gauge fields have non-trivial holonomy, such that the gauge fields remain flat. The above also implies the action is invariant under changes of lift \(a_{ij} \rightarrow a_{ij} + 2\pi\) as long as \(\vec{q}, \vec{s}, \vec{l}, \vec{m}\) are integer vectors.

The charge vector \(\vec{q}\) assigns fractional electric charge \(q^I K^{-1} \vec{l}\) to the anyon \(\vec{l}\). Alternatively, this term induces an anyon \(\vec{s}\) under insertion of \(2\pi\) flux. Two charge vectors \(\vec{q}, \vec{q}'\) describe the same anyon if \(\vec{q}' = \vec{q} + K\vec{\Lambda}\) for some \(\vec{\Lambda} \in \mathbb{Z}^D\). Therefore the group of inequivalent choices for \(\vec{q}\) is \(A\). Note that for a fixed state, this equivalence is realized in the effective action by relabelling \(\vec{a} \rightarrow \vec{a} - \vec{\Lambda} A\).

Shifting \(\vec{q}\) thus also changes the values of \(k_1, k_2, k_4\) and \(k_6\), which depend on \(A\). The full equivalence relation is \((\vec{q}; k_1, k_2, k_{4,6}) \sim (\vec{q} + K\vec{\Lambda}; k_1 - \vec{q} \cdot \vec{\Lambda}, k_2 - \vec{q} \cdot \vec{\Lambda}, k_4 = -\vec{q} \cdot \vec{\Lambda}, k_{6} - \vec{q} \cdot \vec{\Lambda})\).

The coupling \(\vec{s}\) is a discrete version of the spin vector \([10]\). This term induces an anyon \(\vec{s}\) under the insertion (fusion) of \(M\) elementary disclinations. A quasiparticle \(\vec{l}\) transported around \(M\) elementary disclinations thus picks up a braiding phase \(e^{2\pi i s^I K^{-1} \vec{l}}\). Alternatively, this term associates fractional orbital angular momentum \(s^I K^{-1} \vec{l}\) to the quasiparticle \(\vec{l}\). Note that taking \(\vec{s} = M\vec{\Lambda}\) for \(\vec{\Lambda} \in \mathbb{Z}^D\) is trivial, since it would simply bind an anyon \(\vec{\Lambda}\) to an elementary disclination, which can always be done by adjusting the local energetics at disclinations. The non-trivial case cannot be captured simply by associating an anyon to an elementary disclination. Therefore we have two equivalence relations: \((\vec{s}, \{k_i\}) \sim (\vec{s} + K\vec{\Lambda}, \{k_i\})\) (by relabelling \(a \rightarrow a - \vec{\Lambda} C\)), and \((\vec{s}, k_5) \sim (\vec{s} + M\vec{\Lambda}, k_5)\). The choices of \(\vec{s}\) inequivalent under both relations constitute the group \(A/M\). For \(A = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_D}, A/M = \mathbb{Z}_{(n,M)}\), where \((n,M) = \gcd(n,M)\). Note that the theory predicts the angular momentum of an anyon \(\vec{l}\) modulo \(M\vec{\Lambda}^T K^{-1} \vec{l}\).

The integer vector \(\vec{t}_i = (t_{x,i}, t_{y,i})^T\), which we refer to as the discrete torsion vector, does not have an analog in the continuum because torsion is not quantized in continuum geometry. Furthermore this term is non-trivial only in the presence of rotational symmetry, with \(1 < M < 6\) (see Appendix B 1 d). This term associates a fractional (linear) momentum \((1 - U(2\pi/M))^{-1} \vec{p}\), for \(p_i = t_i \vec{l}/M \vec{\Lambda}^{-1} \vec{l}\) to the anyon \(\vec{l}\), which is well-defined modulo the equivalence on \(\vec{t} (\vec{t}_i \sim \vec{t}_i + (1 - U(2\pi/M))\mathbb{Z}^2, \vec{t}_i = \vec{t}_i + K\vec{\Lambda}^T \mathbb{Z}^D, i = x, y\). We emphasize that this “crystal momentum fractionalization,” which is only non-trivial for \(M = 2, 3, 4\), is fundamentally distinct from the more familiar notion usually discussed in the context of quantum spin liquids (see e.g. \([22]\), \([29]\)). The latter case is associated with non-commutativity of the translation operator restricted to a given anyon and arises from the existence of an anyon per unit cell (discussed below), which can be non-trivial even in the case \(M = 1\). Furthermore, \(\vec{t}\) associates an anyon \((\vec{t}_x, \vec{t}_y) \cdot (a, b)\) to a region with Burgers vector \((1 - U(2\pi/M)) \cdot (a, b)\). Note that an anyon is attached only for Burgers vectors in the trivial class in \(K_M\); therefore this term cannot arise solely from attaching an anyon to the elementary dislocations. The latter would be topologically trivial as it could arise by adjusting the local energetics of a dislocation. It follows that the topologically distinct values of \((\vec{t}_x, \vec{t}_y)\) correspond to the greatest common subgroup of \(K_M\) and \(A\), which is denoted as \(K_M \otimes A\) (see Appendix D for a definition).

Finally, \(\vec{m} \in \mathbb{Z}^D\), which we refer to as the area vector, also has no analog in the continuum. This associates an anyon \(\vec{m}\) per unit cell, as has been discussed algebraically in previous work \([30]\), \([31]\) and gives rise to certain notions of “crystal momentum fractionalization” discussed previously \([22]\), \([29]\). This means that if a quasiparticle \(\vec{l}\) is taken around a region \(S\) containing \(\text{Num}(S)\) unit cells, the wave function acquires a braiding phase \(e^{2\pi i s^I K^{-1} \vec{m} \text{Num}(S)}\).

Classification. The four generalized charge vectors \(\vec{q}, \vec{s}, \vec{l}, \vec{m}\) described above can all be included independently in the action, as long \(G = U(1) \times G_{\text{space}}\). Therefore the group classification of the generalized charge vectors is \(A \times (A/M) \times (K_M \otimes A) \times A = M^2 (G, A)\). For \(M = 1\) the correct classification is produced by taking \(K_1\) to be trivial. When the magnetic flux per unit cell is not an integer, the group structure becomes a non-trivial central
The first term gives the Hall conductivity, \( \sigma_H = (2k_1 + \hat{q}^T K^{-1} \hat{q}) / 2\pi \). The second and third terms are discrete analogs of the known continuum geometric response of FQH states \([1, 5, 8, 13, 33]\). The third term corresponds to a fractional angular momentum \( \ell_c / M \) for \( \ell_c = (2k_3 + \hat{q}^T K^{-1} \hat{s}) \), attached to the elementary \( 2\pi / M \) disclination. The remaining terms in \( \mathcal{L}_{\text{eff}} \) are intrinsic to the lattice and have no analog in continuum FQH states.

The second term gives a discrete analog of the shift \( \mathcal{S} \), where \( \sigma_H \mathcal{S} = (k_2 + \hat{q}^T K^{-1} \hat{s}) / \pi \). In particular, this term implies that lattice corners and disclinations carry fractional \( U(1) \) charge. Both an elementary \( 2\pi / M \) disclination and a corner of angle \( 2\pi / M \) carry a fractional \( U(1) \) charge of \( \sigma_H \mathcal{S}/M \). For example, if \( M = 4 \) and the system is defined at the surface of a 3D cube, there are effectively 8 disclinations, each one carrying a fractional charge \( \sigma_H \mathcal{S}/4 \); if the system is defined on a square, each corner also has a fractional charge \( \sigma_H \mathcal{S}/4 \). This term therefore implies the system is a fractional “higher order” topological state \([34, 36]\). Note that the response theory only predicts the fractional charge, angular momentum, and linear momentum of the dislocations and disclinations up to those of the elementary anyons, as anyons can always be bound to these defects by adjusting the local energetics.

The term with \( \hat{P}_{c,i} \cdot \hat{b} \) to a dislocation with Burgers vector \( \hat{b} \) (with the caveat discussed above), and a momentum \( 2\pi (\hat{P}_{c,1}, \hat{P}_{c,2}) \) to a \( 2\pi \) instanton (i.e. to \( 2\pi \) flux spread uniformly throughout the system). It associates a fractional charge per unit length \( \hat{P}_c \cdot \hat{e} \) to a boundary along the direction \( \hat{e} \). This corresponds to a fractional polarization \( \hat{P}_c = \hat{P}_c \times \hat{e} \). If the dislocation described by \( \hat{b} \) is connected to an edge of the system, the holonomy at the edge is changed by the amount \( -\hat{b} \). Hence there must be a compensating fractional charge at the edge. In this sense, this term also describes a fractional higher order topological phase. Note that without rotational symmetry, the polarization is a non-quantized topological response \([37]\).

The term with \( \hat{P}_s \) is the rotational analog of \( \hat{P}_c \), where \( \hat{P}_{s,i} = \sum_{j=1}^2 (k_{5;i} + \hat{q}^T K^{-1} \hat{r}_j)(1 - U(2\pi / M))_{ij}^{-1} \). It associates a fractionally quantized angular momentum \( \hat{P}_s \cdot \hat{b} \) to a dislocation with Burgers vector \( \hat{b} \). It is not clear whether the analog of attaching momentum to the \( 2\pi \) instanton makes sense in this context.

The term with \( \pi_{ij} \), where \( \pi_{ij} = (2k_4 + \hat{q}^T K^{-1} \hat{r}) \), associates a fractionally quantized moment to \( k_i = \sum_{j} \pi_{ij} b_j \) to a dislocation with Burgers vector \( b_j \). This is analogous to a “torsional Hall viscosity” as has been discussed for continuum Dirac theories \([38, 39]\), although there the corresponding term is not quantized and is sensitive to the ultraviolet cutoff; the non-trivial quantization only occurs for lattice systems with rotational symmetry, \( 1 < M < 6 \).

The term \( \hat{R} \cup A_{XY} \) associates a momentum of \( \Pi_j = \sum_{i} t_i^T K^{-1} \hat{m}(1 - U(2\pi / M))_{ij}^{-1} \) per unit area of the system. It arises from the fact that there is an anyon \( \hat{m} \) per unit cell, which in turn carries a momentum as specified by the coupling \( \hat{T} \). Remarkably, this implies that the ground state may carry momentum, depending on the area of the system; only for certain commensurate areas is the ground state momentum trivial. Further, this term is also only non-trivial when the system has rotational symmetry, \( M > 1 \).

The term proportional to \( A \cup A_{XY} \) corresponds to a charge of \( \nu_c = k_6 + \hat{q}^T K^{-1} \hat{m} \) per unit area. This gives a generalized Lieb-Schulz-Mattis constraint \([30]\) which imposes constraints on \( \hat{q} \), \( K \), and \( \hat{m} \) in terms of the filling \( \nu_c \). Likewise, the term proportional to \( C \cup A_{XY} \) associates a fractional angular momentum of \( \nu_s = (k_7 + \hat{q}^T K^{-1} \hat{m}) \) to each unit area.
Finally, the term formally written as $A_{XY} \cup d^{-1} A_{XY}$, with $\alpha = m^T K^{-1} m$, corresponds to $A_{XY} \cup C$, where $d c = A_{XY}$. This term arises from the fact that an anyon is associated with each unit cell. However it is not clear how or whether this term can be physically measured as a quantized geometric response.

**Discussion.** The techniques of this paper can be directly applied to continuum systems with symmetry $U(1) \times \mathbb{R}^2$, where $\mathbb{R}^2$ is the two-dimensional Euclidean group. It can be explicitly shown that $C$ and $\tilde{R}$ then correspond to the spin connection $\omega$ and the coframe fields $e^a$, respectively (see Appendix A for further details).

We note that the topological field theory itself does possess a continuous space-time symmetry corresponding to diffeomorphism invariance. In fact, for chiral topological phases, a gravitational CS term, proportional to the chiral central charge $c_-$, for the full spin connection $\omega$ is also required. Mathematically we may consider $\omega$ to be separate from $C$ and $\tilde{R}$. However to be physically meaningful, the space-time manifold $M$ should split into space and time separately, and $\omega$ must reduce to $C$ appropriately (see Appendix A). This will then contribute an additional angular momentum to disclinations proportional to $c_-$.

**Acknowledgements.** We thank Andrey Gromov and Su-Kuan Chu for helpful discussions. This work is supported by NSF CAREER (DMR-1753240), an Alfred P. Sloan Research Fellowship, UMD startup funds, and the NSF Physics Frontier Center at the Joint Quantum Institute at UMD.

[1] S. M. Girvin, “The quantum hall effect: Novel excitations and broken symmetries,” (1999), arXiv:cond-mat/9907002 [cond-mat.mes-hall]
[2] M. O. Goerbig, “Quantum hall effects,” (2009), arXiv:0909.1998 [cond-mat.mes-hall]
[3] R. B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983)
[4] J. E. Avron, R. Seiler, and P. G. Zograf, Phys. Rev. Lett. 75, 697 (1995)
[5] I. V. Tokatly and G. Vignale, Phys. Rev. B 76, 161305 (2007)
[6] N. Read, Phys. Rev. B 79, 045308 (2009)
[7] N. Read and E. H. Rezayi, Phys. Rev. B 84, 085316 (2011)
[8] B. Bradlyn and N. Read, Phys. Rev. B 91, 125303 (2015)
[9] S. Klevtsov and P. Wiegmann, Phys. Rev. Lett. 115, 086801 (2015)
[10] X. G. Wen and A. Zee, Phys. Rev. Lett. 69, 953 (1992)
[11] G. Y. Cho, Y. You, and E. Fradkin, Phys. Rev. B 90, 115139 (2014)
[12] A. G. Abanov and A. Gromov, Phys. Rev. B 90, 014435 (2014)
[13] A. Gromov, G. Y. Cho, Y. You, A. G. Abanov, and E. Fradkin, Phys. Rev. Lett. 114, 016805 (2015)
[14] H. Kleinert, *Gauge Fields in Condensed Matter*, Vol. 3.
[15] R. Thorngren and D. V. Else, Phys. Rev. X 8, 011040 (2018)
[16] M. Barkeshli, P. Bonderson, M. Cheng, and Z. Wang, Phys. Rev. B 100, 115147 (2019)
[17] M. Barkeshli, S. B. Chung, and X.-L. Qi, Phys. Rev. B 85, 245107 (2012)
[18] P. Rao and B. Bradlyn, Phys. Rev. X 10, 021005 (2020)
[19] Mathematically $d_a$ corresponds to the coboundary operation on the triangulation.
[20] X. G. Wen, *Quantum Field Theory of Many-Body Systems – From the Origin of Sound to an Origin of Light and Electrons* (Oxford University Press, 2004).
[21] Formally $d_a$ here are the lifts from $U(1)$ to $\mathbb{R}$.
[22] A. M. Essin and M. Hermele, Phys. Rev. B 87, 104406 (2013)
[23] Note that the symmetry fractionalization anomaly always vanishes here because $\mathcal{H}^1(U(1) \times \Omega_{\text{space}}, U(1))$ is trivial.
Appendix A: Crystalline gauge theory: continuum approach and relation to elasticity theory

In this section we discuss three aspects of crystalline gauge theory. The first is that our method of defining discrete crystalline gauge fields on simplices and using simplicial calculus to evaluate the action was a practical choice to make direct the relation with the group cohomology classifications of symmetry-enriched topological states (SETS)\cite{Ref.10}. However we expect that the same results can also be obtained by working with real-valued differential forms.

The second aspect is that the discrete translation and rotation gauge fields defined in this work are directly related to the continuum geometry and are known to be closely related to elasticity theory.

Finally, we provide some background on the origin of crystalline gauge fields in terms of the gauge theory of elasticity as discussed in Ref.\cite{14}.

1. Crystalline gauge fields as differential forms

In order to construct actions from discrete gauge fields, it was convenient to work in terms of simplicial cohomology and simplicial calculus (see Appendix A of Ref.\cite{28} for a review). There, our translation gauge fields could be viewed as $\mathbb{Z}^2$-valued 1-cochains defined on the triangulated space-time manifold $M$: that is, $X, Y \in C^1(M, \mathbb{Z})$. Similarly, the rotation gauge field can be viewed as a $\mathbb{Z}_M$-valued 1-cochain, $C \in C^1(M, \mathbb{Z}_M)$ (strictly speaking, in the main text $C$ corresponded to a lift of the $\mathbb{Z}_M$ gauge field to $2\pi\mathbb{Z}$). The action is then invariant under changes of lift, e.g. shifting $C_{ij} \rightarrow C_{ij} + 2\pi$ for a single 1-simplex $ij$.

We can consider instead a formulation where we take the gauge fields to be real-valued differential 1-forms. We thus can define

$$a^I, A, X, Y, C \in \Omega^1(M, \mathbb{R}),$$

where $\Omega^k(M, \mathbb{R})$ denotes the space of real-valued differential $k$-forms. $a^I, A, X, Y,$ and $C$ are the internal, $U(1)$, translation, and rotation gauge fields, respectively, now defined as differential 1-forms.

The discreteness of the gauge fields enters through constraints on the holonomies of these gauge fields. Given a cycle $\gamma$, we require

$$\oint_\gamma R \in 2\pi\mathbb{Z}^2, \quad \oint_\gamma C \in \frac{2\pi}{M} \mathbb{Z},$$

with the equivalence

$$\oint_\gamma C \sim \oint_\gamma C + 2\pi, \quad \oint_\gamma A \sim \oint_\gamma A + 2\pi$$

$$\oint_\gamma a^I \sim \oint_\gamma a^I + 2\pi$$

Dislocations and disclinations must therefore correspond to singular sources of flux for $X, Y, C$. Differential forms which are required to integrate to discrete values along cycles are referred to as integral differential forms.

The gauge transformations are also real-valued. In particular large gauge transformations for $a^I, A$ and $C$ must be quantized in units of $2\pi$.

We then write the effective action using the wedge product:

$$\mathcal{L} = -\frac{1}{4\pi} K_{IJK} a^J \wedge da^I + \mathcal{L}_{\text{frac}} + \mathcal{L}_{\text{SPT}}$$

$$\mathcal{L}_{\text{frac}} = \frac{1}{2\pi} a^I \wedge (q_I da + s_J dC + t_I \cdot \tilde{R} + m_1 A_{XY})$$

$$\mathcal{L}_{\text{SPT}} = \frac{k_1}{2\pi} A \wedge dA + \frac{k_2}{2\pi} A \wedge dC + \frac{k_3}{2\pi} C \wedge dC + \frac{1}{2\pi} A \wedge \frac{1}{2\pi} C \wedge \frac{1}{2\pi} C + \frac{1}{2\pi} C \wedge (k_4 \cdot \tilde{R}) + \frac{1}{2\pi} C \wedge (k_5 \cdot \tilde{R}) + \left(\frac{k_6}{2\pi} A + \frac{k_7}{2\pi} C\right) \wedge A_{XY}$$

Here $A_{XY}$ is the continuum analog of the area element we defined in the simplicial formulation. For example, when $C = 0$, $A_{XY} = \frac{1}{2\pi} (X \wedge Y - Y \wedge X)$. Note that the terms in $\mathcal{L}$ aside from those involving $A_{XY}$ and $d\tilde{R}$ are standard. To ensure that the terms involving $d\tilde{R}$ are invariant under large gauge transformations of $a^I, A,$ and $C$, we require $\frac{1}{2\pi} \int_W d\tilde{R} \in \mathbb{Z}^2$ over any closed 2-cycle $W$. When $W$ is the space, for example, this physically corresponds to the fact that the total Burgers vector of the whole closed space is trivial.

While we do not pursue a formal proof here, we expect that the effective action defined using this continuum formulation yields identical physical results as compared with the lattice gauge theory formulation used in the main text.

Given a triangulation of the space-time manifold $M$, we can understand the relation between the discrete formulation and the continuum formulation as follows. Given a link (1-simplex) $ij$ with vertices $i$ and $j$, the discrete gauge fields, $A_{ij}, C_{ij}, X_{ij}$ and $Y_{ij}$ are taken to be the integral from $i$ to $j$ along the 1-simplex $ij$ of their...
continuum counterparts. Note that only those continuum gauge field configurations can be used that give rise to the appropriate discrete values of $C$, $X$, and $Y$. Since the only gauge invariant quantities for $\tilde{R}$ and $C$ are associated with disclinations and dislocations, we expect that such gauge configurations can always be found.

We can see how to specify the action of $C$ on $\tilde{R}$ by noting that in the continuum setting, $\tilde{R}$ and $C$ correspond exactly to the continuum coframe fields $e$ and spin connection $\omega$. In the following section we discuss this correspondence in more detail.

2. Gauge fields for continuous spacetime symmetries: frame field and spin connection

The Euclidean group $\mathbb{E}^2 = \mathbb{R}^2 \times SO(2)$ is a semidirect product of the group of continuous rotations in 2D, $SO(2) = U(1)$ and the group of continuous translations, $\mathbb{R}^2$. In this case we can consider background gauge fields associated with the continuous translation and rotation symmetries.

The translation gauge fields in the continuum setting now correspond to the 1-form coframe fields $e^a$, $a = x, y$ associated with the space $\Sigma^2$. For physically realistic space-time manifolds of the form $M = \Sigma^2 \times \mathbb{R}$, where $\Sigma^2$ is space, we choose $e^x, e^y$ to be of the form $e^a dx = e^x dx + e^y dy$. There is also a fixed time-component of the coframe field, $e^t = dt$. Below we will assume the space $\Sigma^2$ can be curved, but time is separate, as is appropriate for directly describing a condensed matter system. That is, the metric tensor $g = g_{ij} dx^i dx^j + g_{tt} dt^2$.

The coframe fields diagonalize the metric tensor $g_{ij} = e^a_i e^b_j \delta_{ab}$, (A5) where $\delta_{ij}$ (the Kronecker delta) is the flat space metric. In the linearized approximation where $e^a_i = \delta^a_i + \epsilon^a_i$, we have

$$g_{ij} = \delta_{ij} + \epsilon^1_i + \epsilon^1_j,$$

where $\delta^a_i = \delta_{ai}$ is the Kronecker delta.

A translation gauge transformation can be identified as an infinitesimal diffeomorphism:

$$x^i \rightarrow f^i(x) = x^i + \epsilon^i(x),$$

under which

$$e^a_i \rightarrow \partial_i f^a_j = (\delta^a_i + \partial_i \epsilon^a_j) e^a_j = (\delta^a_i + \partial_i \epsilon^a_j) = \delta^a_i + \epsilon^a_i + \partial_i \epsilon^a + \cdots = e^a_i + \partial_i \epsilon^a + \cdots,$$

(A8)

where the $\cdots$ indicate the subleading term which we ignore in the linearized approximation. We see therefore that in the linearized approximation, the gauge transformations of $e^a_i$ are the continuous analog of the discrete translation gauge transformations $\tilde{R}_{ij} \rightarrow \tilde{R}_{ij} + \vec{r}_j - \vec{r}_i$ on the lattice. Note that as in the discrete case, the continuous translation gauge transformations should preserve the gauge-invariant holonomies associated with $e^a$. In particular, the gauge transformations therefore correspond to diffeomorphisms that preserve the lengths along non-contractible cycles.

Physically, the continuous translation gauge fields $e^a_i$ correspond to the plastic distortion tensor discussed in Ref. [14]. The full strain tensor $u$ is the sum of the elastic strain tensor $u^e$ and the plastic strain tensor $u^p$: $u = u^e + u^p$. The gauge-invariant combination is $u^e = u - u^p$ [14].

In addition to the translation gauge transformations, there are also rotation gauge transformations. These correspond to locally rotating the coordinate axes by an element of $SO(2)$, at every point. The gauge field associated with these gauge transformations is the spin connection, which is a 1-form gauge field $\omega$ that corresponds to the continuous spatial rotation symmetry. The spin connection specifies how the frame fields at nearby points are rotated relative to each other. In terms of the full 3D space-time spin connection $\omega_{\mu}^a$, the spin connection associated with spatial rotations corresponds to $\omega_{\mu}^{a \nu}$. In this language, we can explicitly write the correspondence between the continuum and discrete gauge fields as $\tilde{R} \sim (e^x, e^y)$ and $C \sim \omega$. We emphasize that when the continuous $\mathbb{E}^2$ symmetry is broken down to a discrete space group symmetry, there is no distinction between $(\tilde{R}, C)$ and $(\tilde{R}, \omega)$. The gauge-invariant properties associated to $(\tilde{R}, C)$ can equally be calculated using $e^a, \omega$.

To further clarify the correspondence between the discrete translation and rotation gauge fields and the continuum coframe fields and spin connection, we calculate the contribution of $\omega$ to the covariant derivative of $e^a$ using our discrete formulation with certain limiting arguments. At a point $\vec{r} + \delta \vec{r}$, the coframe field (written here using the translation gauge field notation of the main text) is

$$R_j(\delta \vec{r}) \approx R_j(\vec{r}) + \partial_i R_j \cdot \delta \vec{r}_i.$$

(A9)

As stated in the main text, the vector $\tilde{R}(\vec{r})$ parallel transported to $\vec{r} + \delta \vec{r}$ is $U(C_{\vec{r}, \vec{r} + \delta \vec{r}}) \tilde{R}(\vec{r})$, where we have chosen $\vec{r}$ as the origin. In the continuum, we can write $U(C_{\vec{r}, \vec{r} + \delta \vec{r}}) = \theta$ as a rotation matrix $e^{i \theta \sigma}$ (this would not be appropriate on a lattice, where we need to use GL(2, Z) matrices in a lattice basis, but it is not a problem in the continuum). The total rotation applied between $\vec{r}$ and $\vec{r} + d\vec{r}$ is written in terms of $\omega$ as $e^{-i \int_{\vec{r}}^{\vec{r} + d\vec{r}} \omega(\vec{r}') \cdot d\vec{r}' \sigma}$. Here we have written the spin connection as a vector with components $\omega_\mu$. This representation of $C$ shows that it directly corresponds to $\omega$ in the continuum.
To first order in $\delta \vec{r}$, we can approximate
\[
U(C_{\vec{r}, \vec{r} + \delta \vec{r}}) = e^{-\frac{1}{2} \int d\vec{r} \omega(\vec{r}) \, \delta \vec{r}} \approx 1 - i \sigma_y \omega(\vec{r}) \cdot \delta \vec{r}
\]  
(A10)

The covariant derivative of $\vec{R}$ in the direction $x^i$ can then be written as
\[
D_i R_j(\vec{r}) = \lim_{x^i \to 0} \frac{1}{x^i} (R_j(\vec{r} + x^i) - (U(C_{\vec{r}, \vec{r} + x^i}) \vec{R}(\vec{r}))_j)
\]
\[
= \lim_{x^i \to 0} \frac{1}{x^i} (\partial_i R_j + \omega_i(\vec{r}) x^i \times (i \sigma_y R)_j)
\]
\[
= \partial_i R_j + \omega_i(\vec{r}) \epsilon_{jk} R_k
\]
(A12)

(A13)

This is precisely the formula for the covariant derivative $D$ of $e^a$ in terms of $\omega$, which is written in the usual notation as
\[
T^a \equiv D e^a = de^a + \epsilon^a_b \omega^b e^b = de^a + \omega^a_b \wedge e^b
\]
(A15)

Here $\omega^a_b$ is the full spin connection. We have proved this formula using the fact that $\omega^a_b = -\omega^b_a$ is anti-symmetric, so that $\omega^1_2 = \omega^2_1 = 0$ and $\omega^2_2 = \omega$.

$T^a$ is the torsion 2-form, which characterizes how the frame field is rotated along the path traced by a curve in spacetime. The torsion as defined above can be directly related to the dislocation density, i.e. to the holonomy of translation gauge fields after accounting for parallel transport, similar to the quantity $d\vec{R}$ used in our work. Furthermore, the flux associated to rotational symmetry along $(dC$ in the lattice formulation, or $dw$ in the continuum) gives the curvature of the manifold, which is directly related to the disclination density. Therefore couplings involving $dC$ or $dw$ are essentially coupling the system to curvature. Given that torsion is not quantized in the continuum, there cannot be any quantized topological terms formed by coupling anyons or symmetry charges to the torsion (although nonquantized terms which are topological in the sense of being independent of changes in the underlying metric are well-known).

The classification of SET phases with $U(1) \times \mathbb{R}^2$ symmetry is identical to the classification for $U(1) \times U(1)$ symmetry (this can be proved, for example, by computing the relevant cohomology groups). So while the translation group $\mathbb{R}^2$ has associated gauge fields $X$ and $Y$, the Lagrangian does not have any contribution from $X$ and $Y$; the only relevant terms for Euclidean group symmetry fractionalization and for the associated SPT states are given by $\frac{i}{2\pi} a^i \wedge dw$ and $\frac{1}{2\pi} \omega \wedge dw$ respectively.

3. Connection to gauge theories of elasticity

The discrete translation gauge field $\vec{R}$ that we use has previously been discussed in elasticity theory [13]. Here we provide a brief review of how the discrete crystalline gauge fields arise in elasticity theory, following Ch. 9 of Ref. [14].

In elasticity theory, the basic variables are the displacements $u_i(\vec{r})$ of a particle on a lattice whose mean position is $\vec{r}$, along each direction $i$. The elastic energy is a function of the strain tensor components $\partial_i u_j$ and to lowest order has the form
\[
E = \frac{1}{2} \sum_{\vec{r}, \vec{r}'} \lambda_{ijkl}(\partial_i u_j - \frac{a}{2\pi} R_{ij})(\partial_k u_l - \frac{a}{2\pi} R_{kl})
\]
(A16)

where the operator $\partial$ is now interpreted as a discrete gradient. The corresponding classical partition function is given by
\[
Z = \prod_{\vec{r}, i} \left( \int_{-\infty}^{\infty} \frac{du_i(\vec{r})}{a} \right) e^{-\beta E}
\]
(A17)

Demanding that the energy is invariant under rigid rotations leads to the conditions $\lambda_{ijkl} = \lambda_{klji} = \lambda_{jikl}$ among the elastic moduli [17]. This is the most general translation-invariant Lagrangian that can be written at lowest order in derivatives of $u_i$.

At low temperatures and in a classical theory, the displacements $u_i(\vec{r})$ are generally much smaller than the lattice spacing $a$. However, it is possible for thermal or quantum fluctuations to result in particles exchanging their positions over long times. Indeed, the diffusion of particles within the lattice means that it is appropriate to think of $u_i$ as being defined only up to a lattice constant; therefore, our partition function must be invariant under a transformation
\[
u_i(\vec{r}) \rightarrow u_i(\vec{r}) + a N_i(\vec{r}),
\]
(A18)

where $a$ is the lattice spacing and $N_i$ is an integer vector field defined at the discrete positions $F$. The transformation (A18) is a gauge transformation which reflects the physical reality that the coordinates can be relabelled up to integers. To ensure gauge invariance under this transformation, we introduce new integer-valued gauge fields $\frac{1}{2\pi} R_{ij} \in \mathbb{Z}$ and replace
\[
\partial_i u_j(\vec{r}) \rightarrow \partial_i u_j(\vec{r}) - \frac{a}{2\pi} R_{ij}(\vec{r}).
\]
(A19)

(Here $i, j \in \{x, y\}$ and $R_{ij}(\vec{r})$ is a function defined on a lattice; this notation should not be confused with the notation $\vec{R}_{ij}$ in a simplicial formulation, where $ij$ is a 1-simplex on a triangulation.) The partition function then includes a sum over all possible values of $R_{ij}$:
\[
Z = \sum_{\{R_{ij}(\vec{r})\}} \prod_{\vec{r}, i} \left( \int_{-\infty}^{\infty} \frac{du_i(\vec{r})}{a} \right) e^{-\beta \tilde{E}}
\]
(A20)

\[
\tilde{E} = \frac{1}{2} \sum_{\vec{r}} \left( \lambda_{ijkl}(\partial_i u_j - \frac{a}{2\pi} R_{ij})(\partial_k u_l - \frac{a}{2\pi} R_{kl}) \right)
\]
(A21)
The change of variables and subsequent sum over \( R_{ij} \) encode the fact that the quantities \( \partial_i u_j \) can change by any integer values at every lattice point, and that the different particle configurations are all treated equally. As originally desired, \( Z \) is now invariant under the gauge transformation

\[
\begin{align*}
  u_i(\vec{r}) &\to u_i(\vec{r}) + aN_i(\vec{r}) \\
  R_{ij}(\vec{r}) &\to R_{ij}(\vec{r}) + 2\pi \partial_i N_j(\vec{r})
\end{align*}
\] (A22) (A23)

The \( R_{ij} \) are precisely the discrete translation gauge fields suitably defined on a lattice: \( R_{xi} = X_i, R_{yi} = Y_i \). Integrating out the displacements \( u_i \) will result in a pure gauge theory in terms of the gauge fields \( R_{ij} \).

To further understand the fields \( R_{ij} \), we next look at how this gauge theory treats dislocations. A lattice dislocation corresponds to a missing or extra line of atoms such that the number of nearest neighbours at the dislocation point changes. The fields \( R_{ij} \) allow for such configurations, which are deviations from an ideal lattice configuration. These configurations would not be included in the partition function if we restricted ourselves to a change of variable \( u_i(\vec{r}) \to u_i(\vec{r}) + aN_i(\vec{r}) \), as this transformation amounts to a relabelling of coordinates but keeps the particles in an ideal lattice configuration. Another way to say this is that the integral \( \oint_{\gamma} \partial_i N_j d\vec{l} \), where \( d\vec{l} \) is the infinitesimal line element along the loop \( \gamma \), will always be zero and cannot represent a dislocation. A dislocation Burgers vector is obtained from the holonomy \( \frac{1}{2\pi} \oint_{\gamma} \vec{R} \). The symmetrized quantity \( \frac{1}{2}(R_{ij} + R_{ji}) \) is the discontinuous part of the symmetrized strain tensor. A similar procedure can be followed for a continuous elastic medium, where the analog of \( \frac{1}{2\pi} R_{ij} \) is referred to as the plastic strain tensor \( u^{(p)}_{ij} \) and is directly related to the coframe field used in differential geometry, as discussed in Sec. A.2.

We can also introduce disclinations in elasticity theory via a rotation symmetry gauge field. Disclinations, the fluxes of this rotation symmetry field, are related to the antisymmetric component of the strain tensor, which does not enter the action at the usual quadratic order. These effects can be included by adding higher derivative terms to the usual Lagrangian. Conventional elasticity theory does not, however, include translation as well as rotation symmetry via a nonabelian gauge field, as we have done. Instead, it makes certain approximations that allow rotations to be incorporated without dealing with the full space group symmetry. This does not affect the calculations greatly for thermodynamic purposes, but in dealing with topological properties we saw that the nonabelian gauge field led to a situation where only certain properties of dislocations are gauge-invariant. This feature cannot be reproduced by an approximate calculation.

Appendix B: Symmetry fluxes

In this Appendix we provide additional discussion regarding the symmetry fluxes \( d\vec{R} \) and \( \lambda_{XY} \), which are associated with dislocations and the area, respectively.

1. Classifying dislocations with rotation symmetry

a. 2D Point group rotation matrices

An important role in the main text was played by the \( 2 \times 2 \) rotation matrix \( U(2\pi/M) \), associated with the generator of point group rotations. Due to the presence of a lattice, there is a natural basis in which point group rotation matrices \( U(2\pi/M) \) can be defined. We define the \( x \) and \( y \) axes to be the lattice vectors, such that for \( M \)-fold point group rotations, the \( x \) and \( y \) axes subtend an angle \( 2\pi/M \). For \( M = 2 \), \( U(2\pi/2) = -1 \), where \( -1 \) here denotes the \( 2 \times 2 \) identity matrix. For \( M = 3, 4, 6 \) an elementary \( 2\pi/M \) rotation can always be defined to take \( x \to y \). In turn, the existence of a lattice ensures the rotated position of \( y \) can be expressed as a linear combination of the original \( x \) and \( y \). The result for \( U(2\pi/M) \) is given in Table II along with the matrices \((1 - U(2\pi/M))^{-1} \), that also arise frequently.

In our calculations we have assumed that the lengths \( L_x, L_y \) are defined along these possibly nonorthogonal axes. Moreover, integrals \( \int f(x,y)dxdy \) should be carried out with \( x \) and \( y \) defined by this lattice-specific coordinate system. The advantage of using these coordinates is that we always work with integer vectors and matrices, so the coefficients of the theory are always integers or fractions of integers.

| \( M \) | \( U(\frac{2\pi}{M}) \) | \( (1 - U(\frac{2\pi}{M}))^{-1} \) |
|---|---|---|
| 2 | \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} | \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} |
| 3 | \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} | \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} |
| 4 | \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} | \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} |
| 6 | \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} | \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} |

TABLE II. Elementary rotation matrices \( U(2\pi/M) \) for different \( M \).

b. Dislocations are described by \( \vec{R}^{(0)} \)

Consider the product \( B_{01} B_{12} \ldots B_{n-1,n} \), where \( B_{ij} \in G_{\text{space}} \). The part of this corresponding to translations is given by

\[
\int \vec{R}^{(0)} := \sum_{k=0}^{n} U(C_{01} + C_{12} + \cdots + C_{k-1,k}) R_{k,k+1}
\] (B1)

This is the definition of \( \vec{R}^{(0)} \). In the special case of a closed loop in a flat configuration, \( \oint_{C} \vec{R}^{(0)} = 0 \). In gen-
eral the value of this integral around a closed loop $\gamma$ defines the total Burgers vector for any dislocations located inside $\gamma$. The value of this Burgers vector is well-defined up to an overall rotation. Under a gauge transformation, we have

$$
\oint_{\gamma} \vec{R}^{(0)}(\theta) = \sum_{k=0}^{n} U(C_{01} + C_{12} + \cdots + C_{k-1,k}) R_{k,k+1}
$$

(B2)

$$
\rightarrow \sum_{k=0}^{n} (U(-h_0 + C_{01} + h_1 - \cdots - h_{k-1} + C_{k-1,k} + h_k)) \times U(-h_k)(R_{k,k+1} + U(C_{k,k+1})\vec{r}_{k+1} - \vec{r}_k)
$$

(B3)

$$
= \sum_{k=0}^{n} U(-h_0 + C_{01} + C_{12} + \cdots + C_{k-1,k})(R_{k,k+1} + U(C_{k,k+1})\vec{r}_{k+1} - \vec{r}_k)
$$

(B4)

$$
= \sum_{k=0}^{n} U(-h_0 + C_{01} + C_{12} + \cdots + C_{k-1,k}) R_{k,k+1}
$$

(B5)

$$
= U(-h_0) \oint_{\gamma} \vec{R}^{(0)}
$$

(B6)

The contributions from the $\vec{r}_i$ cancel as a telescoping sum (we have used $U(C_1)U(C_2) = U(C_1 + C_2)$ in the second and third lines). Therefore the integral is invariant under the $\vec{r}$-dependent part of the gauge transformation (i.e. the translation gauge transformations), but can be rotated depending on the orientation of the coordinate axes at the origin $0$. Note that in the above calculation we picked the curve $\gamma$ such that the origin $0$ was contained in $\gamma$, however this is not essential.

$\vec{R}^{(0)}$ therefore captures our physical intuition regarding dislocations. However, we cannot use $\vec{R}^{(0)}$ directly in the action because it is not local. Moreover, in the presence of a disclination, the value of $\vec{R}^{(0)}_{ij}$ depends on the precise path chosen between the origin $i$, and is therefore ambiguous up to a rotation by the disclination angle. The solution is to instead use the $\vec{R}$ fields themselves, which are local. But there is considerable ambiguity in $\vec{R}$ under gauge transformations. This fact is of crucial importance.

c. Gauge transformations of $d\vec{R}$

From the definition of $\vec{R}^{(0)}$ we have that

$$
\vec{R}_{ij}^{(0)} = \vec{R}^{(0)}_{ij} + (1 - U(C_{0\to i}))(\vec{R}_{ij})
$$

(B7)

Here we have defined $C_{0\to i} = \int_0^i C$ for a path $\gamma$ from the origin 0 to the point $i$. The last term is of the form $(1 - U(2\pi k/M))\vec{R}_{ij}$. Let $C_{0\to i} = \frac{2\pi k_0}{M}$. Using the fact that $1 - U^k = (1 - U)(1 + U + \cdots + U^{k-1})$, we conclude that

$$
\vec{R}_{ij}^{(0)} = \vec{R}^{(0)}_{ij} + (1 - U(2\pi/M))(1 + U(2\pi/M) + \cdots + U^{k_0,i-1})(2\pi/M))\vec{R}_{ij}
$$

(B8)

$$
:= \vec{R}_{ij}^{(0)} + (1 - U(2\pi/M))\vec{R}_{ij}
$$

(B9)

The last line defines the vector field $\vec{K}$ in terms of $\vec{R}$, with $\frac{1}{2\pi} \vec{K} \in \mathbb{Z}^2$.

A general gauge transformation sends $\vec{R}_{ij} \rightarrow U(-h_i)(\vec{R}_{ij} + U(C_{ij})\vec{r}_i - \vec{r}_j)$. But the above relation will still hold with $\vec{K}$ replaced by some $\vec{K}'$ where $\frac{1}{2\pi} \vec{K}' \in \mathbb{Z}^2$. Now under gauge transformations, assume that the coordinate axes at the origin are rotated by the angle $2\pi m/M$. Then $d\vec{R}$ transforms as

$$
d\vec{R} = d\vec{R}^{(0)} + (1 - U(2\pi/M))d\vec{K}
$$

(B10)

$$
\rightarrow U(2\pi m/M)d\vec{R}^{(0)} + (1 - U(2\pi/M))d\vec{K}'
$$

(B11)

$$
= d\vec{R}^{(0)} + (1 - U(2\pi/M))(d\vec{\Lambda} + d\vec{\bar{K}}')
$$

(B12)

where $\vec{\Lambda} = (1 + U(2\pi/M) + \cdots + U^{m-1}(2\pi/M))\vec{R}^{(0)}$, and also satisfies $\frac{1}{2\pi} \vec{\Lambda} \in \mathbb{Z}^2$.

Therefore gauge transformations preserve the value of
$d\vec{R}$ only up to terms of the form $(1 - U(\frac{2\pi}{M}))d\vec{R}$.

To summarize, the correct definition of a Burgers vector, given by $\vec{R}^{(0)}$, is nonlocal due to the choice of origin 0, and so we are forced to use the field $\vec{R}$ instead in the effective action. $d\vec{R}$ is not gauge-invariant: it is determined only up to terms of the form $(1 - U(\frac{2\pi}{M}))d\vec{R}$. However, $d\vec{R}$ and $d\vec{R}^{(0)}$ are equivalent up to such terms. Therefore the fractional part of $(1 - U(\frac{2\pi}{M}))^{-1}d\vec{R}$ is (i) local, (ii) gauge-invariant, and (iii) equal to the physically meaningful quantity $\frac{1}{2\pi}(1 - U(\frac{2\pi}{M}))^{-1}d\vec{R}^{(0)} \mod 1$. This quantity therefore captures the local, gauge-invariant part of a Burgers vector.

The distinct values taken by $\frac{1}{2\pi}d\vec{R}$ are classified by the group $K_M$. In the following section we provide some intuitive and physical ways to understand the group $K_M$.

d. Understanding the group $K_M$

There are a number of ways to understand $K_M$ more intuitively and physically. Let us consider the most direct way following the mathematical derivation above. Let us first consider the case $M = 2$, and start by considering a small region with a locally defined Burgers vector $(a, b)$ (see Fig. 1). Under a local rotation of the space, this Burgers vector transforms to $(a, b) \rightarrow (-a, -b)$. Thus the Burgers vector $(1, 0) \sim (-1, 0)$ and $(0, 1) \sim (0, -1)$. Now consider two regions, each with a locally defined Burgers vector $(a, b)$ and $(a', b')$. The combined Burgers vector thus would be $(a + a', b + b')$. Upon a $\pi$ rotation of the second region however, $(a', b') \rightarrow (-a', -b')$, so $(a + a', b + b') \rightarrow (a - a', b - b')$. Therefore, when considering the Burgers vector of a large region containing Burgers vectors in smaller regions, $(2, 0) \sim (0, 0)$ and similarly $(0, 2) \sim (0, 0)$. We see that the Burgers vectors form the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, due to the fact that the Burgers vector of a region, when including these local rotations, is only partially well-defined. An equivalent analysis for $M = 3, 4, 6$ gives the groups $\mathbb{Z}_3, \mathbb{Z}_2$, and the trivial group (see Fig. 1). In general, dislocations whose Burgers vectors are of the form $(1 - U(2\pi/M))b$ are equivalent to zero. If we have two neighbouring dislocations with $\vec{b}$ and $-\vec{b}$, the total Burgers vector associated to a loop containing the dislocations is zero. However a local rotation of $-\vec{b}$ by the angle $2\pi/M$ will give a net holonomy equal to $\vec{b} - U(2\pi/M)\vec{b}$ around the same loop. These values of Burgers vectors are therefore considered to be in the trivial equivalence class. This is what we mean by the statement that rotation gauge symmetry induces a finite group grading on Burgers vectors. They are thus classified by elements of $\mathbb{Z}^2$ modulo $(1 - U(2\pi/M))\mathbb{Z}^2$, which can be taken as the mathematical definition of $K_M$.

We can also understand the group $K_M$ by considering a system with fractional charge per unit length along its boundary given by $\vec{P} \cdot \vec{n}$, where $\vec{P}$ is the polarization vector and $\vec{n}$ is the normal to the boundary. An integer value of $\vec{P}$ corresponds to placing an integer charge per unit length on the boundary, which can always be done locally. This is shown pictorially in Fig. 2 where we assign fractional charge per unit length to each boundary segment under one choice of coordinate axes. For example, if we consider a system with $M = 4$, the charge per unit length on the boundaries normal to $\hat{x}, \hat{y}, -\hat{x}, -\hat{y}$ are $(q_1, q_2, q_1, -q_2)$ respectively. Now we can perform rotations of the axes by $2\pi/M$, which will relabel the charge on each segment since the normal vectors $\vec{n}$ get redefined. In this case, the coordinate axes are rotated by an angle $\pi/2$, and the charges on the same boundary segments will now be labelled as $(-q_2, q_1, q_2, -q_1)$ (see Fig. 2). However, the fractional charge on each edge should be the same from either calculation. Therefore we must have $(q_1, q_2, -q_1, -q_2) = (q_2, q_1, q_2, -q_1) \mod 1$. We can see that the only solutions are $(q_1, q_2) = (0, 0)$ or $(1/2, 1/2)$. Therefore the group of distinct assignments of charge at the boundary is $K_4 \cong \mathbb{Z}_2$. One can work out the other cases similarly.

We note that in the first example, the $K_M$ classification arose from general properties of the dislocations that do not depend on a particular Lagrangian, while in the second, it was based on a physical response related to the term $\frac{\vec{P} \cdot A}{2\pi}d\vec{R}$ in the Lagrangian.

Finally, we look at the case with $M = 1$, corresponding to the absence of rotation symmetry. We cannot directly apply the previous reasoning in this case to obtain a useful classification. In a system without rotation symmetry, the Burgers vector of any dislocation is well-defined: the value of $d\vec{R}$ is gauge-invariant. Since there is no gauge transformation relating them, there is no grading of Burgers vectors. In the example of boundary charge, one can now have any assignment of fractional charges per unit length on the boundary of such a system. In either case, the group classifying inequivalent dislocations or fractional boundary charge configurations is not a finite group. However, if we define $K_1$ so that it classifies the quantized fractional charges per unit length that can be assigned to a boundary, the group is trivial. The quantization was a direct result of discrete rotation symmetry, which is broken when $M = 1$.

2. Area flux $A_{XY}$

Here we study the behavior of the area flux $A_{XY}$ under a gauge transformation and discuss its properties in the presence of dislocations and boundaries.
K is given by $M(0)$. (a) For $M=2$, the vectors $(a, b)$ and $(-a, -b)$ are in the same equivalence class. Moreover, the sum of two neighbouring Burgers vectors can be viewed as either $(a, b) + (a', b')$ or $(a, b) - (a', b')$; this gives the relations $(0, 0) \sim (2, 0) \sim (0, 2)$, which reduce the classification to a group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(b) For $M=3$, we see that $(3, 0) \sim (1+U(2\pi/3)+U(4\pi/3))(1, 0)^T = (0, 0)$; in general $(2a+b, b-a) \sim (0, 0)$, so the classification is given by $K_3 \cong \mathbb{Z}_3$. (c) For $M=6$, we can combine the $M=2$ and $M=3$ results to show that $(0, 0) \sim (2, 0) \sim (3, 0)$; thus $(0, 0) \sim (1, 0)$, and similarly $(0, 0) \sim (0, 1)$. Therefore every Burgers vector can be trivialized. Similar reasoning applied to the $M=4$ case gives $K_4 \cong \mathbb{Z}_2$.

Recall that the value of $A_{XY}$ on a 2-simplex $[ijk]$ is

$$A_{XY}[ijk] = \frac{1}{4\pi} \vec{R}_{ij} \times U(C_{ij}) \vec{R}_{jk}$$

(B13)

$$= \frac{1}{4\pi} U(C_{0} \to i) \vec{R}_{ij} \times U(C_{0} \to i) U(C_{ij}) \vec{R}_{jk}$$

(B14)

$$= \frac{1}{4\pi} \vec{R}^{(0)}_{ij} \times \vec{R}^{(0)}_{jk}$$

(B15)

where $\times$ refers to the cross product: $\vec{v} \times \vec{u} = v_x u_y - v_y u_x$.

The second line uses the fact that the cross product is invariant under an equal rotation of both arguments; the symbol $C_{0} \to i$ refers to the sum of $C$'s on any path from the origin 0 to the point i. The last line uses the definition of $\vec{R}^{(0)}$. Note that the cross product of two $\vec{R}^{(0)}$ fields taken in this manner is thus local even though a single such field is not. Since $A_{XY}$ is independent of the choice of origin 0, we drop this superscript and simply write

$$A_{XY} = \frac{1}{4\pi} \vec{R}_{ij} \times \vec{R}_{jk},$$

(B16)

with the understanding that $\vec{R}_{ij}$ is defined with respect to an arbitrary choice of origin 0.

Under a gauge transformation, this equality implies that

$$4\pi A_{XY}[ijk] = \vec{R}_{ij} \times \vec{R}_{jk}$$

(B17)

$$\to (\vec{R}_{ij} + d\vec{R}_{ij}) \times (\vec{R}_{jk} + d\vec{R}_{jk})$$

(B18)

Here we have defined $\tilde{r}_i = U(C_{0} \to i) C_{i}$ (for the same arbitrary choice of origin 0 used to define $\vec{R}$). The difference $\delta A_{XY}$ can be written as

$$4\pi \delta A_{XY}[ijk] = (\vec{R}_{ij}) \times d\vec{R}_{jk} + d\vec{R}_{ij} \times (\vec{R}_{jk} + d\vec{R}_{jk})$$

(B19)

Defining

$$f_{ij} = \vec{R}_{ij} \times \vec{r}_j + \vec{r}_i \times (\vec{R}_{ij} + d\vec{R}_{ij}),$$

(B20)

we see that $\delta A_{XY}$ is a coboundary whenever $d\vec{R} = 0$:

$$4\pi \delta A_{XY}[ijk] = df[ijk]$$

(B21)

Therefore when $A_{XY}$ is integrated over the entire manifold, this property implies that a gauge transformation
will only contribute boundary terms to the integral (assuming $\mathcal{R}$ is flat). Therefore $\frac{1}{2\pi} \int_{\Sigma^2} A_{XY}$ over a closed 2-manifold $\Sigma^2$ is gauge-invariant, which we physically interpret as the area of the space $\Sigma^2$. Note that since the cross-product gives the area of a parallelogram, the integration over the whole space covers the manifold twice, such that $A_{XY}$ is quantized to be an integer multiple of $2\pi$ when integrated over a 2-cycle.

Although we have defined a gauge-invariant area only for closed manifolds, we can also define a gauge-invariant area for manifolds with boundary by restricting the gauge transformations on the boundary. Specifically, we require that the quantity $f$ defined above must vanish for every boundary 1-simplex. For this to occur, it is sufficient that the boundary fields $\mathcal{R}_{ij}$ and the boundary gauge transformation variables $\tilde{r}_j$ be parallel to each other. This requirement can also be viewed as a consequence of the fact that a boundary can be chosen to break exactly one of the two $\mathbb{Z}$ translation symmetries, so that the $\mathcal{R}$ field essentially reduces to a $\mathbb{Z}$ gauge field on the boundary.

For example, suppose the space is a square formed by the region $0 \leq x, y \leq a$ with origin $(0,0)$. For simplicity let $C = 0$ everywhere on the boundary except on links associated with the corners, which have $C = \pi/2$. Let the fields $\mathcal{R}_{ij}$ on the $y = 0$ line have zero $y$-component. Now as we meet the corner $(a,0)$, we meet a 1-simplex with $C = \pi/2$. The above condition on $\mathcal{R}$ now means that on the $x = a$ line, $\mathcal{R}$ has zero $x$-component. In fact, one component of $\mathcal{R}$ is always constrained to vanish on the boundary.

The discussion above has so far required that $\mathcal{R}$ be flat. In particular, when $\mathcal{R}$ is flat, $\mathcal{R}_{ij} \times \mathcal{R}_{jk} = \mathcal{R}_{jk} \times \mathcal{R}_{ki}$, so the definition of $A_{XY}$ does not depend on the ordering of the vertices. But if we assume that the simplex $[ijk]$ contains a dislocation, this equality no longer holds. Instead, we have

$$\mathcal{R}^{(0)}_{ij} + \mathcal{R}^{(0)}_{jk} + \mathcal{R}^{(0)}_{ki} = \mathcal{R}^{(0)}_{ij} \times \mathcal{R}^{(0)}_{jk} \times \mathcal{R}^{(0)}_{ki} = \mathcal{R}^{(0)}_{ij} \times \mathcal{R}^{(0)}_{jk} \neq 0 \quad (B22)$$

This means that the area of a simplex with nonvanishing holonomy of $\mathcal{R}$ is not well defined. This is physically expected: on a lattice with a dislocation, the number of unit cells within a region containing a dislocation cannot be obtained purely from the dimensions of the boundary. In fact, the number of unit cells in a small region containing

![FIG. 2. The $K_M$ classification of rotationally symmetric configurations of boundary charge for (a) $M = 2$, (b) $M = 3$, (c) $M = 4$, and (d) $M = 6$. We choose our coordinate axes to be normal to the boundaries, and place a charge per unit length equal to $(q_1, q_2)^T \hat{n}$ on the boundary with normal vector $\hat{n}$. Thus in (b), for $M = 3$ we have the arrangement $A = (q_1, q_2, -q_1 - q_2)$ as we proceed anticlockwise around the boundary segments. Now under a $2\pi/3$ rotation of axes, the charge per unit length at the same three segments gets redefined as $A' = (-q_1, q_2, q_1 - q_2)$. Since the fractional charge per unit length on each boundary segment remains the same if we only rotate the coordinate axes, we should have $A = A' \mod 1$. This implies that $q_1 = q_2$ and $3q_1 \in \mathbb{Z}$; the three distinct choices of $q_1$ now determine the group $K_3$. We can follow similar reasoning in (a),(c) and (d).]
a dislocation is not well-defined. Moreover, as the dislocation moves, additional unit cells are added or removed. Therefore extensive observables such as the total charge or angular momentum will no longer be gauge-invariant. However, intensive quantities such as the filling or angular momentum per unit cell will still be well-defined, because they are a ratio of two extensive quantities computed with the same triangulation.

A well-defined area can be defined for a given fixed configuration of dislocations by cutting out the regions containing the dislocations. Then the system is viewed as a manifold with boundary, and a gauge-invariant area can be defined as discussed above by restricting the gauge transformations on the boundary. Effectively this approach treats the dislocation a hole in the simplicial approach treats the dislocation as a puncture (for example, a sphere $S^2$ with a puncture would correspond to the plane $\mathbb{R}^2$), but then we cannot describe the open set near the puncture in terms of a finite triangulation.

Appendix C: Classification of SETs and reduction of $\mathcal{H}^3(G,U(1))$

The effective action contains two classes of terms: $\mathcal{L}_{\text{frac}}$ and $\mathcal{L}_{\text{SPT}}$. As was discussed in the main text, $\mathcal{L}_{\text{frac}}$ specifies the symmetry fractionalization class through the choice of the generalized charge vectors, which corresponds to the classification $\mathcal{H}^2(G,A)$. $\mathcal{L}_{\text{SPT}}$ contains additional terms depending only on the background gauge fields, and is classified by $\mathcal{H}^3(G,U(1))$. The choice of $\mathcal{H}^3(G,U(1))$, which corresponds to changing the coefficients $k_i$ in $\mathcal{L}_{\text{SPT}}$, can be understood as stacking $(2+1)$D SPT states. Physically, the effect of changing the action by a choice of $\mathcal{H}^3(G,U(1))$ is to change the braiding and fusion properties of the symmetry defects [16].

Depending on the choice of symmetry fractionalization class and the precise topological order involved, it is possible that changing the action by a non-trivial choice of $\mathcal{H}^3(G,U(1))$ does not yield a distinct phase of matter. Therefore, keeping the symmetry fractionalization choice fixed, the true classification of distinct symmetry-enriched topological states (SETs) is reduced from $\mathcal{H}^3(G,U(1))$ to a smaller group. In the G-crossed tensor category formulation [16], this reduction corresponds to cases where changing the algebraic theory of defects by an element of $\mathcal{H}^3(G,U(1))$ can be completely accounted for by a relabeling of the symmetry defects.

We can also see this reduction from $\mathcal{H}^3(G,U(1))$ in the context of our topological effective action. In this context, we see that field redefinitions can be made to absorb the effect of changing the couplings in $\mathcal{L}_{\text{SPT}}$ by certain amounts. Since this analysis is heavily dependent on the precise topological order (precise choice of $K$ matrix) involved, here we will focus on some simple examples.

To illustrate the main idea, let us begin by considering the case where $G = \mathbb{Z}_M$, with the symmetry fractionalization class specified by the spin vector $\vec{s}$, and the associated defect class given by $k$. The $\mathbb{Z}_M$ gauge field $\mathcal{C}$ couples to $a$ as follows:

$$\mathcal{L} = -\frac{1}{4\pi} K_{IJ} a^I \cup d(a^J + u^J) C + \frac{s_I + K_{IJ} u_J}{2\pi} (a^I + u^I) C \cup dC$$

(C1)

In this case, there are naively $M$ distinct choices of $k$, $k = 0, \cdots, M-1$, corresponding to $\mathcal{H}^3(\mathbb{Z}_M,U(1)) = \mathbb{Z}_M$.

First, we note that the choice of couplings $(\vec{s}, k)$ has the following redundancies:

$$(\vec{s} + M\vec{\Lambda}, k) \sim (\vec{s}, k) \sim (\vec{s}, k + M)$$

(C2)

This first equivalence is because $M\vec{\Lambda} \cdot \vec{a} \cup d\mathcal{C}$ is trivial, as explained in the main text. The second equivalence follows from $\mathcal{H}^3(\mathbb{Z}_M,U(1)) = \mathbb{Z}_M$.

Next, observe that we can rewrite the Lagrangian as

$$\mathcal{L} = -\frac{1}{4\pi} K_{IJ} (a^I + u^I) C \cup d(a^J + u^J) C + \frac{s_I + K_{IJ} u_J}{2\pi} (a^I + u^I) C \cup dC + \frac{2k - \vec{u}^T K \vec{u} - \frac{2s}{2\pi} \cdot \vec{u}}{4\pi} C \cup dC,$$

(C3)

where $\vec{u} \in \mathbb{Z}^D$. Since $a^I$ is dynamical, the shift $a^I \rightarrow a^I + u^I C$ can be trivially absorbed by redefining the integration variables. Note that $a^I + u^I C$ still obeys the flux quantization condition since $d\mathcal{C}$ integrates to $2\pi \mathbb{Z}$ over any 2-cycle.

Therefore, we have the additional equivalence

$$(\vec{s}, k) \sim (\vec{s} + K\vec{u}, k - \frac{\vec{u}^T K \vec{u}}{2} - \frac{s}{2\pi} \cdot \vec{u}).$$

(C4)

Combining the equivalences in (C4) and (C2), we see that whenever $K\vec{u} = M\vec{\Lambda}$, we get

$$(\vec{s}, k) \sim (\vec{s}, k - \frac{\vec{u}^T K \vec{u}}{2} - \frac{s}{2\pi} \cdot \vec{u}).$$

(C5)

For a fixed choice of $\vec{s}$, this corresponds in general to a reduction of $\mathcal{H}^3(\mathbb{Z}_M,U(1))$. Now we can work out some specific examples. Consider the $1/N$ Laughlin state with $N$ even, for which $K = N$, and take $M = 2$. Since $s \sim s + M$, there are two possible spin vector classes, given by $s$ odd or $s$ even. Suppose we
choose \( u = 1 \) and \( \Lambda = N/2 \). Then we have
\[
(s, k) \sim (s, (k - N/2 - s) \mod 2)
\] (C6)

For \( s = 1 \), then this relabelling will take \( k \to (k - 1 - N/2) \mod 2 \). Hence, if \( N \) is a multiple of 4, the SET classes corresponding to \((s, k) = (1, 0)\) and \((1, 1)\) are the same, while the two classes \((s, k) = (0, 0)\) and \((1, 1)\) are distinct. The above result was previously also obtained using the edge physics of Chern-Simons theories in Ref [32]; here we have reproduced their result with the notation of Eq. (2) (main text). In this case, the analog of Table III demonstrates the SET classification for general \( K \)-matrix states, including Table III.

Consider another example with the \( 1/2 \) Laughlin state and \( p4 = \mathbb{Z}^2 \times \mathbb{Z}_4 \) space group symmetry. We will use the notation of Eq. (2) (main text). In this case, the analog of Eq. (C2) is
\[
(\vec{s}, k_3, k_5, k_7) \sim (\vec{s} + M\vec{\Lambda}, k_3, k_5, k_7) \sim (\vec{s}, k_3 + M, k_5, k_7)
\]
\[
\sim (\vec{s}, k_3, k_5 + (1 - U(\pi/2))\vec{u}, k_7) \sim (\vec{s}, k_3, k_5, k_7 + M)
\] (C7)

where the additional relations all arise from the group structure of \( H^4(p4, U(1)) \). On the other hand, the relabelling \( a^I \to a^I + u^I C \) (which is again a physically trivial operation) leads to the following relation:
\[
(\vec{s}, k_3, k_5, k_7)
\]
\[
\sim (\vec{s} + K\vec{u}, k - \frac{a^T K u}{2} - \vec{s}^T \vec{u}, k_{5,i} - \vec{a}^T \vec{t}_i, k_7 - \vec{u}^T \vec{m})
\] (C8)

Thus for \( K\vec{u} = M\vec{\Lambda} \), we can shift \( \vec{s} + K\vec{u} \) to its original value as in the previous example, obtaining
\[
(\vec{s}, k_3, k_5, k_7)
\]
\[
\sim (\vec{s}, k - \frac{a^T K u}{2} - \vec{s}^T \vec{u}, k_{5,i} - \vec{a}^T \vec{t}_i, k_7 - \vec{u}^T \vec{m})
\] (C9)

For the Laughlin state we have \( M = 4, K = N \), and \( s, u \) are integers with \( Nu = 4\Lambda \). The above relation becomes
\[
(\vec{s}, k_3, k_5, k_7)
\]
\[
\sim (\vec{s}, k - \frac{u^2 N}{2} - su, k_{5,i} - ut_i, k_7 - mu)
\] (C10)

With \( N \) a multiple of 4, we can without loss of generality take \( u = 1 \); when \( N \) is of the form \( 4N' + 2 \) we can choose \( u = 2 \). The corresponding SET equivalences are given in Table III. Note that in our examples, it is crucial that \( C \) is discrete, so that we can add trivial terms such as \( \frac{M\vec{a}}{2\Lambda} a^I \) and change the values of \( \vec{s} \) without affecting the value of \( k \). This is not possible for continuous symmetry gauge fields: a term proportional to \( a^I \) cannot be trivial. This means that there is no chain of equivalences relating different elements of \( H^4(U(1), U(1)) \) while keeping the charge vector \( \vec{q} \) fixed. This is consistent with the fact that the different \( U(1) \) SETs with the same charge vector all have different Hall conductivities, and are thus physically distinct states of matter.

In our final example, we will count the number of distinct SETs associated to the \( 1/2 \) Laughlin state with \( U(1) \times G_{\text{space}} \) symmetry. We will only present the results, which can be derived exactly as shown above. Now the parameters \( q, s, t_x, t_y, m \) can correspond to the identity particle \( I \) or to the semion which we will call \( S \) (\( I \) corresponds to an even value of the parameters \( q, s, t_x, t_y, m \) while \( S \) corresponds to an odd value). The parameters \( k_3, k_4 \) and \( k_6 \), which are not associated to rotation symmetry, will not be affected by relabellings and will always contribute a factor of \( \mathbb{Z} \times K_M \) to the overall SET classification; we assume they are also fixed. The remaining SET parameters \( k_j \) for \( j = 2, 3, 5, 7 \) are classified by \( \mathbb{Z}_M \times \mathbb{Z}_M \times K_M \times \mathbb{Z}_M \). The final equivalence relation among them, for fixed \( q, s, t_x, t_y, m \), is
\[
(k_2, k_3, k_5, k_7) = (k_2 - qu, k_3 - u^2 - su, k_5, k_7 - um)
\] (C11)

where \( 2u = M\Lambda \) for \( \lambda \in \mathbb{Z} \).

1. For \( M = 2 \), there are 2 ways to choose each of \( q, s, t_x, t_y, m \). We have \( 2^5 = 32 \) SETs whenever \( q = t_x = m = I \) and \( s = S \) (there are no non-trivial relabellings in this case). Otherwise each relabelling relates exactly 2 states and we get \( 2^4 = 16 \) SETs. Therefore there are \( 31 \times 2^4 = 512 \) SETs obtained by varying over all distinct choices of charge vectors and \( k_2, k_3, k_5, k_7 \).

2. For \( M = 3 \) there are 2 choices each for \( q \) and \( m \), but all possible choices for \( s \) and \( t_x \) are trivial. There are no relabellings, so we get \( 3^4 = 81 \)

| \( N \) | Generalized charge vectors | Relabelled SPT parameters \((k'_3, k'_5, k'_7)\) |
|---|---|---|
| \( 4N' \) | \((s \mod 4, t, m \mod N)\) | \((k_3 + \frac{\pi}{2} - s, k_5, k_7 - t, k_7 - m)\) |
| \( 4N' + 2 \) | \((s \mod 2, t, m \mod N)\) | \((k_3 - 2s, k_5, k_7 - 2t, k_7 - 2m)\) |

TABLE III. The effect of relabellings on the SET classification demonstrated for the \( 1/N \) Laughlin state with \( p4 \) wallpaper group symmetry. The spin and area vectors \( s \) and \( m \) are now integers, while the torsion vector \( \vec{t} \) is valued in \( \mathbb{Z}^2 \) (the symmetry fractionalization class is specified by \( s \mod 4 \), \((t_x + t_y) \mod 2 \) and \( m \mod N \)). Now the fields can be relabelled in a manner that leaves the generalized charge vectors unchanged but changes the SPT parameters \((k_3, k_5, k_7)\) to \((k'_3, k'_5, k'_7)\) (shown in the last column). Therefore stacking the SPT given by their difference will leave the SET invariant. Note that the precise coefficients of this stacked SPT are different for different charge vectors and for different values of \( N \).
distinct SETs for each symmetry fractionalization class, and $4 \times 81 = 324$ SETs in total.

3. For $M = 4$, there are 2 choices each for $q, s, m$ and $t_x + t_y$, giving 16 choices of charge vectors in total. We have $2 \times 4^3 = 128$ SETs whenever $q = s = m = I$ (there are no relabellings); otherwise we have $4^3/2 = 64$ SETs. This gives $2 \times 128 + 14 \times 64 = 1152$ SETs in total.

4. For $M = 6$, there are 2 choices each for $q, s, m$, while $t$ is anyway trivial. We have $6^3 = 216$ SETs whenever $q = M = I$ ans $s = S$ (there are no relabellings); otherwise we have $6^3/2 = 108$ SETs. This gives $1 \times 216 + 7 \times 108 = 972$ SETs in total. The reduction by a factor of 2 in these examples is because we are working with the 1/2 Laughlin state. In general the result can be much more complicated.

We reiterate that in the above counting, we vary over all inequivalent choices of charge vectors and $k_2, k_3, k_5, k_7$. The freedom in choosing $k_4$ gives an extra factor of $|K_M|$, while $k_1$ and $k_6$ each have a $\mathbb{Z}$ worth of choices that become fixed when the charge filling per unit area and the Hall conductivity are fixed. After accounting for the variation of $\mathbb{Z}$, we have $8 \times 16 \times 16 = 256$ SETs for flat $G$ gauge fields fall into equivalence classes determined by the quotient $\mathcal{H}^3(G,U(1)) := \frac{Z^3(G,U(1))}{B^3(G,U(1))}$. It has been shown that this fully characterizes topological gauge theories for gauge group $G$ [24], and also believed to fully characterize $(2+1)$D SPTs [23 42]. It is also known to classify the fusion and braiding properties of symmetry defects in $(2+1)$D SETs once the symmetry fractionalization class has been fixed [10].

Let us now summarize the relationship between $\mathbb{H}^2(G, \mathcal{A})$ and $\mathcal{L}_{\text{frac}}$. Consider the coupling of flat $G$ gauge fields to flat internal gauge fields describing the Abelian topological order (we assume that the symmetry does not permute anyons):

1. Consider a single internal $U(1)$ gauge field $a$. Consider a topological term which is an integer multiple of $\frac{1}{2\pi} a \cup B$, where $B \in 2\pi \mathbb{Z}$ is obtained in terms of the $G$ gauge field and is defined on 2-simplices. Note that $B \in 2\pi \mathbb{Z}$ in order for this term to be invariant under large gauge transformations of $a$.

2. This action can be thought of as an action for $U(1) \times G$ symmetry. Demanding retriangulation invariance implies that $\frac{1}{2\pi} a \cup B$ must be a 3-cocycle: $\frac{1}{2\pi} d(a \cup B) = \frac{1}{2\pi} (da \cup B + a \cup dB) \in 2\pi \mathbb{Z}$. Since $a$ is flat, $da \in 2\pi \mathbb{Z}$, so we find $dB = 0$.

A $G$ gauge transformation which takes $B \rightarrow B + d\Gamma$, where $\frac{1}{2\pi} \Gamma \in \mathbb{Z}$, changes the Lagrangian by a 2-cocoboundary of $G$ with $\mathbb{Z}$ coefficients. Therefore the gauge inequivalent actions fall into equivalence classes determined by the quotient $\mathbb{H}^2(G, \mathbb{Z}) := \frac{Z^2(G, \mathbb{Z})}{B^2(G, \mathbb{Z})}$.

When there are $D$ independent internal gauge fields, the coefficient changes from $\mathbb{Z}$ to $\mathbb{Z}^D$.

3. The $K$-matrix coupling ensures that if $B$ is of the form $K \bar{\Lambda}$ where $\frac{1}{2\pi} \bar{\Lambda}$ is an integer vector, the theory is trivial. This is because the anyon associated to the “symmetry flux” $B$ is trivial. Therefore the correct coefficients which classify physically distinct couplings of the $K$-matrix theory to the background $G$ gauge field are $\mathbb{Z}^D/K\mathbb{Z}^D \cong \mathcal{A}$. This is in fact the definition of $\mathcal{A}$, the group of anyons. Therefore the classification of such actions is given by $\mathbb{H}^2(G, \mathcal{A})$.

Appendix D: Topological terms and group cohomology

The correspondence between the topological effective action and the group cohomology formulation runs deeper than giving the same overall classification. There is a one-to-one correspondence between topological terms in the action involving flat background $G$ gauge fields and cocycles in group cohomology. In this section we will explain this relationship through concrete calculations.

Let us first summarize the relationship between $\mathbb{H}^3(G,U(1))$ and the topological terms in $\mathcal{L}_{\text{SPT}}$, which correspond to topological effective actions for $(2+1)$D SPT states. See Ref. [21] and [25] for a more detailed discussion. For an overview of simplicial calculus, see Ref. [28].

1. A topological Lagrangian for an SPT involving flat $G$ gauge fields (defined on 1-simplices) can be integrated over a 3-simplex of a triangulation, which gives an action $S$ associated to a single 3-simplex. The resulting $e^{iS}$, which depends on the values of the flat gauge field defined on the 1-simplices, is thus a 3-cochain of $G$ valued in $U(1)$, i.e. an element of $C^3(G,U(1))$.

2. In fact $e^{iS}$ is a 3-cocycle of $G$ valued in $U(1)$, i.e. an element of $Z^3(G,U(1))$. The 3-cocycle condition arises by demanding that the theory be independent of the triangulation.
In what follows we describe in more detail the precise relation between the topological terms in the effective action and the group cohomology cocycles for the symmetry group discussed in this paper, \( G = U(1) \times G_{\text{space}} \).

1. Cocycles for \( G = U(1) \) and \( G = \mathbb{Z}_M \)

\[ G = U(1): \text{In this case we have} \]
\[ \mathcal{L}_{\text{frac}} = \frac{1}{2\pi} q_I a_I^I \cup dA \quad (D1) \]
\[ \mathcal{L}_{\text{SPT}} = \frac{k}{2\pi} A \cup dA \quad (D2) \]

where \( q_I, k \in \mathbb{Z} \). Here \( \bar{q} \) is the charge vector. Define a flat, real-valued gauge field \( A \) such that \( A_{12} = a \) and \( A_{23} = b \). Formally \( A \) is the lift of a \( U(1) \) gauge field to \( \mathbb{R} \). A corresponding element of \( U(1) = \mathbb{R}/2\pi \mathbb{Z} \) is written as \( a = a \mod 2\pi \); therefore \( a = \lfloor a \rfloor + 2\pi n_a \) for some \( n_a \in \mathbb{Z} \). Now, for the 3-simplex [0123], \( \mathcal{L}_{\text{frac}} \) becomes

\[ \frac{q_I}{2\pi} a_I^1 dA_{123} = \frac{q_I}{2\pi} a_I^1 ([a] + [b] + [a + b] + d\{a, b\}) \quad (D3) \]

where \( d\{a, b\} = n_a + n_b - n_{a+b} \). The quantity \( \frac{q_I}{2\pi} ([a] + [b] + [a + b] + d\{a, b\}) \) defines an anyon, i.e. an element in \( A \), and can thus be viewed as an \( A \)-valued 2-cocycle, i.e. an element of \( Z^2(U(1), A) \). The quantity \( \frac{q_I}{2\pi} d\{a, b\} \) is an \( A \)-valued 2-coboundary, i.e. an element of \( B^2(U(1), A) \)

In general, coboundaries correspond to changes of lift. Inequivalent choices of \( \bar{q} \) determine inequivalent classes in the cohomology group \( H^2(U(1), A) \). The Lagrangian integrated on a 3-simplex with \( C_{01} = a, C_{12} = b, C_{23} = c \) gives

\[ \frac{k}{2\pi} C \cup dC[0123] = \frac{2\pi k}{M^2} ([a] + M n_a) ([b] + M c - [b + c] + M d\{b, c\}) \quad (D8) \]

When evaluated modulo \( 2\pi \), the rhs is a 3-cocycle which represents a cohomology class in \( H^3(U(1), U(1)) \) identified by \( k \); the terms which explicitly depend on \( n \) arise by choosing alternative lifts. For each choice of charge vector \( \bar{q} \), it is possible to add a \( \mathbb{Z} \) worth of SPT states. This means that for each symmetry fractionalization class, one can obtain a set of topological phases related to each other by stacking \( G \)-SPT states, given by elements of \( H^3(G, U(1)) \).

\[ G = \mathbb{Z}_M: \text{Effective SPT actions for } G = \mathbb{Z}_M \text{ have been related to } \mathbb{Z}_M \text{ group cocycles in previous work} \quad [33]. \text{ The action for } G = \mathbb{Z}_M \text{ is} \]

\[ \mathcal{L}_{\text{frac}} = \frac{s_I}{2\pi} a_I^I \cup dC \quad (D5) \]
\[ \mathcal{L}_{\text{SPT}} = \frac{k}{2\pi} C \cup dC \quad (D6) \]

Define a flat gauge field \( \mathcal{C} \in \mathbb{R}/2\pi \mathbb{Z} \) such that \( C_{12} = 2\pi n_a/M \) and \( C_{23} = 2\pi n_b/M \) where \( a, b \) are integers. Formally \( C \) is a lift from \( \mathbb{Z}_M \) to \( \mathbb{R}/2\pi \mathbb{Z} \). A corresponding element of \( \mathbb{Z}_M \) is written as \( \frac{2\pi a}{M} = \frac{2\pi a}{M} \mod 2\pi \), where we define \( [a]_M = a \mod M \); therefore \( a = [a]_M + M n_a \) for some \( n_a \in \mathbb{Z} \). Now, \( \mathcal{L}_{\text{frac}} \) becomes

\[ \frac{s_I}{2\pi} a_I^1 \cup dC \]
\[ = \frac{s_I}{2\pi} a_I^1 ([a]_M + [b]_M - [a + b]_M + M d\{a, b\}) \quad (D7) \]

The quantity \( \frac{s_I}{M} ([a]_M + [b]_M - [a + b]_M + M d\{a, b\}) \) is a 2-cocycle in the group \( Z^2(\mathbb{Z}_M, A) \). The quantity \( s_I d\{a, b\} \), which is the difference between two different choices of lifts, is a 2-coboundary in the group \( B^2(\mathbb{Z}_M, A) \). Note that the most general coboundary relation implies that shifting \( s_I \) by a multiple of \( M \) corresponds to changing the lift; therefore \( s_I + M \Lambda_1 \) for \( \Lambda_1 \in \mathbb{Z} \) is equivalent to \( s_I \). With these conditions we see that the equivalence classes of \( \bar{s} \) are in bijection with cohomology classes \([w] \in H^2(\mathbb{Z}_M, A) \). When \( A = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r} \), we simply have \( H^2(\mathbb{Z}_M, A) = \mathbb{Z}_{(M, n_1)} \times \cdots \times \mathbb{Z}_{(M, n_r)} = A/M A \) (\( M A \) is defined as \( \{ M \{ a \in A \} \} \)).

Next we analyze \( \mathcal{L}_{\text{SPT}} \). The Lagrangian integrated on a 3-simplex with \( C_{01} = a, C_{12} = b, C_{23} = c \) gives

\[ \frac{k}{2\pi} C \cup dC[0123] = \frac{2\pi k}{M^2} ([a]_M + M n_a) ([b]_M + [c]_M - [b + c]_M + M d\{b, c\}) \quad (D8) \]

Taken modulo \( 2\pi \), this function is a 3-cocycle in \( Z^3(\mathbb{Z}_M, U(1)) \). Choosing \( k \) to be a multiple of \( M \) results in a 3-coboundary; therefore the classification is \( H^3(\mathbb{Z}_M, U(1)) \). Since the 3-cocycles of \( \mathbb{Z}_M \) and \( U(1) \) have a similar form, the resulting SPT terms, which are of the form \( A \cup dA \) and \( C \cup dC \), also have the same Chern-Simons structure.

2. Cocycles for \( H^2(G_{\text{space}}, \mathbb{Z}) \)

The part of the action with terms from the group \( G_{\text{space}} \) is

\[ \mathcal{L}_{\text{frac}} = \frac{s_I}{2\pi} a_I^I \cup dC + \frac{t_I}{2\pi} a_I^I \cup d\bar{R} + \frac{m_I}{2\pi} a_I^I \cup A_{XY} \quad (D9) \]
\[ \mathcal{L}_{\text{SPT}} = \frac{k_3}{2\pi} C \cup dC + \frac{k_7}{2\pi} C \cup d\bar{R} + \frac{k_7}{2\pi} C \cup A_{XY} \quad (D10) \]

Since the group cocycles for \( G_{\text{space}} \) are less common than those of \( U(1) \) or \( \mathbb{Z}_M \), we will first derive them abstractly and then discuss their relationship to the gauge fields \( \bar{R} \) and \( C \). A space group \( G_{\text{space}} \) can always be written as a group extension of a point group \( \bar{H} \) by the group of translations \( \mathbb{Z}^2 \), with some action \( \theta: \bar{H} \rightarrow \text{Aut}(\mathbb{Z}^2) \), as summarized by the short exact sequence

\[ 1 \rightarrow \mathbb{Z}^2 \rightarrow G_{\text{space}} \rightarrow \bar{H} \rightarrow 1 \quad (D11) \]
When $H$ is a rotation point group, the above extension is always a semidirect product, in which case it is possible to use the Kunneth formula to derive the group cohomology of $G$. The Kunneth formula for a semidirect product extension is

$$\mathcal{H}^n(G_{\text{space}}, A) = \prod_{k=0}^{n} \mathcal{H}^k_{\theta_k}(H, \mathcal{H}^{n-k}(Z^2, A)) \quad (D12)$$

Here $A$ is an arbitrary abelian group. The action $\theta_k$ is not on $Z^2$ itself, but on the cohomology group $\mathcal{H}^{n-k}(Z^2, A)$; it is induced by the action $\theta$ of $H$ on $Z^2$, and will be discussed further below. Let us first study symmetry fractionalization. It is easiest to use $Z$ coefficients and then shift to $A$ coefficients. The Kunneth formula gives

$$\mathcal{H}^2(G_{\text{space}}, Z) = \mathcal{H}^2_{\theta_2}(Z_M, \mathcal{H}^0(Z^2, Z))$$

$$\times \mathcal{H}^1_{\theta_1}(Z_M, \mathcal{H}^1(Z^2, Z)) \times \mathcal{H}^0_{\theta_0}(Z_M, \mathcal{H}^2(Z^2, Z)) \quad (D13)$$

$$= \mathcal{H}^2_{\theta_2}(Z_M, Z) \times \mathcal{H}^1_{\theta_1}(Z_M, Z \times Z) \times \mathcal{H}^0_{\theta_0}(Z_M, Z) \quad (D14)$$

The first line is the Kunneth formula. In the second line, we substituted the known cohomology groups $\mathcal{H}^s(Z^2, Z) = Z(\hat{\theta})$. The three terms are further discussed individually. Note that all 2-cocycles must satisfy the condition

$$f_3(g_1, g_2) + f_2(g_1 g_2, g_3) = f_2(g_2, g_3) + f_2(g_1, g_2 g_3) \quad (D15)$$

where, if $g_i = (\hat{r}_i, h_i)$, then $g_1 g_2 = (\hat{r}_1 + U(h_1)\hat{r}_2, h_1 + h_2)$. In what follows, we assume that the translation gauge field $\hat{R}$ is valued in $2\pi Z$, while the $Z^2$ group elements $\hat{r}_i$ are assumed to be integer-valued. Cocycles of $G_{\text{space}}$ of degree $d$ are denoted $f_d$.

a. $\mathcal{H}^2_{\theta_2}(Z_M, \mathcal{H}^0(Z^2, Z))$

The cocycles in the coefficient group $\mathcal{H}^0(Z^2, Z)$ in the first term of Eq. (D13) are constant functions valued in $Z$. The $Z_M$, rotations, which act on $Z^2$, therefore do not change the value of these functions, so that $\theta_2$ is the trivial action. The first term is thus isomorphic to $\mathcal{H}^2(Z_M, Z)$, and the associated cocycle representatives of $G_{\text{space}}$ are $f_2(g_1, g_2) = \frac{s}{M}([h_1]_M + [h_2]_M - [h_1 + h_2]_M)$ with $s \in Z_M$ as discussed previously. The corresponding field-theoretic element is $\frac{s}{2\pi} dC$.

b. $\mathcal{H}^1_{\theta_1}(Z_M, \mathcal{H}^1(Z^2, Z))$

Now we consider the second term of Eq. (D13). The coefficient module $\mathcal{H}^1(Z^2, Z)$ has cocycle representatives of the form $p_{\hat{r}}$, where $p_{\hat{r}}(\hat{r}) = \hat{r} \cdot \hat{r}$ for some $\hat{r} \in Z^2$. Under a rotation $U(h)$, $p_{\hat{r}}$ gets transformed as $\hat{r} \cdot U(h)\hat{r} = (U(-h)\hat{r}) \cdot \hat{r} = p_{U(-h)\hat{r}}(\hat{r})$. This means that the induced action on the coefficients is equivalent to the rotation action $\theta_1 = \theta$ on $Z^2$.

The first observation is that the group $\mathcal{H}^1(\theta_1(Z_M, Z))$ classifies functions $f_1$ taking elements $h$ of $Z_M$ to vectors in $Z^2$. We have $\mathcal{H}^1(Z_M, Z^2) \cong \frac{Z^2}{(I-U(\frac{2\pi}{M}))Z^2} \cong K_M$, using standard results on the cohomology of cyclic groups (see for eg. Ref [25]). A representative cocycle $f_1$ of this group has the form

$$f_1(h) = \frac{1}{1 - U(\frac{2\pi}{M})} f_1(2\pi/M), \quad (D16)$$

where $f_1(2\pi/M) = \tilde{r}$ for some $\tilde{r} \in Z^2$, and the $U$ matrices act on $\tilde{r}$ by rotation.

Next we consider the more detailed decomposition $\mathcal{H}^1_{\theta_1}(Z_M, \mathcal{H}^1(Z^2, Z))$. A cocycle of this group maps an element $h \in Z_M$ to a cohomology class $[p_{f_1(h)}] \in \mathcal{H}^1(Z^2, Z)$ whose representatives are functions $p_{f_1(h)}$.

The desired 2-cocycle of $G_{\text{space}}$ is completely determined in terms of $p_{f_1(h)}$ as follows:

$$f_2(g_1, g_2) = p_{f_1(h_1)}(\tilde{r}_2)$$

$$= \frac{1}{1 - U(\frac{2\pi}{M})} \tilde{r}_2 \quad (D17)$$

This function, whose parameter is $\tilde{r}$, satisfies the 2-cocycle condition for $G_{\text{space}}$. Values of $\tilde{r}$ which are of the form $\tilde{r} = (1 - U(2\pi/M))\hat{r}$ are trivial, as the resulting cocycles are actually 2-coboundaries $db$ of $G_{\text{space}}$, where $b(g) = \hat{r} \cdot \hat{r}$. It is easy to motivate this function by looking at a 2-simplex [012]. If $\frac{1}{2\pi} \tilde{R}_{01} = \hat{r}_1$, $\tilde{R}_{02} = \hat{r}_2$, then from flatness of $(\hat{R}, C)$ we have $\hat{R}_{02} = \hat{r}_1 + U(h_1)\hat{r}_2$. Therefore $\frac{1}{2\pi} d\tilde{R}[012] = \hat{r}_1 + \hat{r}_2 - (\hat{r}_1 + U(h_1)\hat{r}_2) = (1 - U(h_1))\hat{r}_2$. Since $U(h_1)$ is a power of $U(\frac{2\pi}{M})$, this function is always a multiple of $(1 - U(\frac{2\pi}{M})).$ Therefore, $f_2(g_1, g_2) = \tilde{r} \cdot (1 - U(\frac{2\pi}{M}))^{-1} \hat{r}$ is integer valued for all $\tilde{r} \in Z^2$. However, it cannot be generated on a 2-simplex by a 2-coboundary $df(g_1, g_2)$ (the only function that would give $df$ is $f(g) = \hat{r} \cdot (1 - U(\frac{2\pi}{M}))^{-1} \hat{r}$, which is not integer-valued, unless $\tilde{r}$ has the trivial form). The field theory element giving this value is $\frac{\tilde{r}}{2\pi} \cdot d\tilde{R} = \frac{\tilde{r}}{2\pi} \cdot (1 - U(\frac{2\pi}{M}))^{-1} d\tilde{R}$.

c. $\mathcal{H}^0_{\theta_0}(Z_M, \mathcal{H}^2(Z^2, Z))$

Finally, we study the third term of Eq. (D13). The coefficient module $\mathcal{H}^2(Z^2, Z)$ has representatives $w_m$ for $m \in Z$, satisfying $w_m(\hat{r}_1, \hat{r}_2) - w_m(\hat{r}_2, \hat{r}_1) = m \hat{r}_1 \times \hat{r}_2$. Although the rotation action changes the form of $w_m$, the above cross product (and hence the value of $m$) is rotationally invariant, and in this sense $\theta_0$ is trivial. Now the group $\mathcal{H}^0_{\theta_0}(Z_M, Z)$ classifies functions $f_0$ taking each $h \in Z_M$ to some fixed integer $f_0(h) = m \in Z$. Therefore a cocycle in the group $\mathcal{H}^0_{\theta_0}(Z_M, \mathcal{H}^2(Z^2, Z))$ should
take \( h \) to the cohomology class \([w_{f_0(h)}]\) whose representatives \( w_{f_0(h)} \) are such that \( w_{f_0(h)}(\bar{r}_1, \bar{r}_2) - w_{f_0(h)}(\bar{r}_2, \bar{r}_1) \) is rotationally invariant.

It can be verified that the following function is a 2-cocycle of \( G_{\text{space}} \) with these properties:

\[
f_2(g_1, g_2) = w_{f_0(h_1)}(\bar{r}_1, U(h_1)\bar{r}_2) = mr_{1,x}(U(h_1)\bar{r}_2)_y
\]

In this case we have \( w_{f_0(h_1)}(\bar{r}_1, U(h_1)\bar{r}_2) - w_{f_0(h_1)}(U(h_1)\bar{r}_1, \bar{r}_2) = m\bar{r}_1 \times U(h_1)\bar{r}_2 \). The cross product is invariant under rotations and is a measure of area. If operations 2 and 1 are performed successively, the rotation \( h_1 \) changes the relative orientation of axes used to measure the two translations. The vector \( \bar{r}_2 \) is therefore rotated by \( U(h_1) \) so as to meaningfully take a cross product with \( \bar{r}_1 \).

Consider the quantity \( w_{f_0(h_1)}(\bar{r}_1, U(h_1)\bar{r}_2) = \frac{m}{2} \bar{r}_1 \times U(h_1)\bar{r}_2 \). Although it is not an integer-valued cocycle, it satisfies the 2-cocycle condition with \( \frac{1}{2} Z \) coefficients (hence it can be used to obtain a topologically invariant action on 3-simplices). This function satisfies \( w_{f_0(h_1)}(\bar{r}_1, U(h_1)\bar{r}_2) = w_{f_0(h_1)}(U(h_1)\bar{r}_1, \bar{r}_2) = m\bar{r}_1 \times U(h_1)\bar{r}_2 \), i.e. it has the same gauge-invariant property as \( f_2(g_1, g_2) \); moreover, it is already rotationally invariant. We use this \( \frac{1}{2} Z \) valued cocycle in the field theory because it is closely related to the integer-valued space group cocycles, and furthermore an intuitive measure of area. The corresponding field theory object is \( \frac{1}{2} T \langle X_{\text{XY}} \rangle \), where \( A_{\text{XY}}[012] = \frac{1}{2} (\bar{R}_{01}) \times U(C_{01})\bar{R}_{12} \). The gauge transformation behaviour of \( A_{\text{XY}} \) and its physical relationship to the area element were discussed in Appendix B2.

\[d. \text{Classification}\]

The classification \( \mathcal{H}^2(G_{\text{space}}, Z) \) is therefore \( Z \wr K_{M} \times Z \). To obtain cocycles classifying symmetry fractionalization, we use the Universal Coefficient Theorem to write \( \mathcal{H}^2(G_{\text{space}}, A) = \mathcal{H}^2(G_{\text{space}}, Z) \otimes A \) (the usual formula has a second term, which vanishes for our examples). The \( \otimes \) (tensor product) symbol defines the tensor product \( G \otimes H \) of abelian groups \( G \) and \( H \). The group \( G \otimes H \) should be understood as the set of pairs \( g \otimes h \) where \( g \in G, h \in H \), where \( \otimes \) is a bilinear operation. Therefore \( g \otimes h \) is trivial if either \( g \) or \( h \) is trivial. For example if \( n g = 1_G \) (the identity element of \( G \)), \( n (g \otimes h) = (ng) \otimes h = 1_G \otimes h \); this argument runs similarly for \( h \). The group \( G \otimes H \) is completely defined by the following properties:

\[
G \otimes H \cong \mathcal{H} \otimes G \tag{D20}
\]

\[
\prod G_i \otimes \prod H_j \cong \prod (G_i \otimes H_j) \tag{D21}
\]

\[
G \otimes Z \cong G \tag{D22}
\]

\[
Z_m \otimes Z_n \cong Z_d, d = \gcd(m, n) \tag{D23}
\]

When \( G = H^2(G_{\text{space}}, Z) \) and \( H = \mathcal{A} \), the elements of \( H^2(G_{\text{space}}, Z) \otimes A \) are therefore composites of symmetry fluxes from \( H^2(G_{\text{space}}, Z) \) and anyons from \( \mathcal{A} \). The formal effect of the \( \otimes \) symbol is to replace the \( Z \) coefficients by \( A \) coefficients. This means that \( s, \ell, m \) are replaced by \( s, \ell, m \) in \( Z^D \). Moreover, if these parameters take the form \( \tilde{K}, \tilde{A} \), they are trivial. With this change, the above cocycles all become cocycle representatives for \( H^2(G_{\text{space}}, \mathcal{A}) = (A, MA) \times (K_M \otimes A) \times A \). (Note that \( Z_M \otimes A = A, MA \).) Effective actions corresponding to these cocycles are recovered by taking a cup product of the vector \( \alpha_f \) of internal gauge fields with the field theory term corresponding to a representative of \( H^2(G, \mathcal{A}) \).

\[3. \text{Cocycles for } \mathcal{H}^3(G_{\text{space}}, U(1))\]

With our knowledge of \( \mathcal{H}^2(G_{\text{space}}, Z) \), it is easy to understand the group \( \mathcal{H}^3(G_{\text{space}}, U(1)) \). We can derive its cocycle representatives in the following direct way. The \( G_{\text{space}} \) charges are classified by \( \mathcal{H}^3(G_{\text{space}}, U(1)) \cong Z_M \) (corresponding to the charges of \( C \)), whose generator is represented by the cocycle \( f_1(h) = 2\pi |h|_M/M \) mod \( 2\pi \). The associated field theory element is just \( C \). The fluxes are classified by the group \( \mathcal{H}^2(G_{\text{space}}, Z) \). Therefore SPT cocycles, which associate symmetry flux to an elementary symmetry charge, are all of the form \( \nu(g_1, g_2, g_3) = \frac{2\pi |h|_M}{M} \beta(g_2, g_3) \) mod \( 2\pi \), where \( \beta \in \mathcal{H}^2(G_{\text{space}}, Z) \). These functions satisfy the 3-cocycle condition for \( \mathcal{H}^3(G_{\text{space}}, U(1)) \), and correspond to taking the cup product of a cocycle in \( \mathcal{H}^1(Z_M, U(1)) \) with another from \( \mathcal{H}^2(G_{\text{space}}, Z) \). Regarding the SPT classification, consider the three subgroups \( S_1, S_2, S_3 \) of \( \mathcal{H}^2(G_{\text{space}}, Z) \) written previously. From the definition of the tensor product, the classification of SPT terms obtained by associating an elementary \( Z_M \) charge to a flux represented by a cocycle of \( S_1 \) is \( Z_M \otimes S_1 \). The full SPT classification is therefore \( \prod_{i=1}^3 Z_M \otimes S_i = (Z_M \otimes Z_M) \times (Z_M \otimes K_M) \times (Z_M \otimes Z_M) = Z_M \times K_M \times Z_M \). This is the same as Kunneth formula result: \( \mathcal{H}^3(G_{\text{space}}, U(1)) = \mathcal{H}^3(Z_M, U(1)) \times \mathcal{H}^2(Z_M, U(1)) \times \mathcal{H}^1(Z_M, U(1), U(1)) \). The flux-charge construction accounts for all the group cohomology SPTs.

The cocycles for mixed SPTs of \( U(1) \) and \( G_{\text{space}} \) symmetry are obtained by a cup product of a 1-cocycle of \( \mathcal{H}^1(U(1), U(1)) \) (generated by \( f_1(a) = [a] \) mod \( 2\pi \)) and a 2-cocycle of \( \mathcal{H}^2(G_{\text{space}}, Z) \). Finally, the full \( \mathcal{H}^3(G_{\text{space}} \times U(1), U(1)) \) classification can also be obtained from the Kunneth formula: it equals \( Z^2 \times Z_3^M \times K_M^3 \). In this case, the possible charges are classified by the group \( Z \times Z_M \), corresponding to those of \( A \) and \( C \). These charges couple to fluxes, i.e. representatives of the group \( \mathcal{H}^2(G_{\text{space}}, Z) \), to give the full SPT action for the group \( G_{\text{space}} \times U(1) \).