Dark Energy: the equation of state description versus scalar-tensor or modified gravity

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Dark energy dynamics of the universe can be achieved by equivalent mathematical descriptions taking into account generalized fluid equations of state in General Relativity, scalar-tensor theories or modified $F(R)$ gravity in Einstein or Jordan frames. The corresponding technique transforming equation of state description to scalar-tensor or modified gravity is explicitly presented. We show that such equivalent pictures can be discriminated by matching solutions with data capable of selecting the true physical frame.

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1. According to recent astrophysical data, our universe is dominated by a mysterious form of Dark Energy (DE), for a review and a list of refs., see [1], which counts to about 70% of total mass-energy density. As a result, the universe expansion is accelerating. In terms of constant equation of state (EoS) parameterization, observational data indicate that constant EoS parameter is roughly equal to $-1$. In other words, the accelerating universe could be in the cosmological constant ($w = -1$), quintessence ($-1 < w < -1/3$) or phantom era ($w < -1$). Without taking into account the so-called "first and second coincidence problems" (for a recent discussion, see [2]) the fundamental problem is to select the (correct) EoS for the observed universe consistently related to the early epochs. Even considering (perfect/imperfect) ideal fluid description, there are various possibilities: constant EoS, time-dependent EoS, complicated (explicit or implicit) EoS functional dependence of the pressure from energy density (and time), inhomogeneous EoS, etc. The situation is even more complicated since several proposals for DE (from scalars to string-inspired gravity) exist.

In the present Letter, we develop a technique by which it is possible, whatever the ideal fluid EoS description is, to transform such a fluid in a scalar-tensor theory taking into account the "same" FRW scale factor. However, the process can be reversed. Subsequently, the scalar-tensor theory can be represented as a modified gravity theory (without scalar field) with the same scale factor in Jordan or Einstein frames conformally related. Of course, these three descriptions, leading to the same FRW dynamics, differ in various respects (for instance, the Newton law is different, quantum versions of such theories are not equivalent, nucleosynthesis and LSS can be achieved in different ways, etc). The proposal is to discriminate among the three approaches to DE considering observational data: in this sense, the "true" selection of mathematically equivalent descriptions is operated at the solution level spanning as much as possible wide ranges of cosmological parameters like the redshift $z$.

2. Let us start from the following action:

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R - \frac{1}{2} \omega(\phi) \partial_\mu \phi \partial^\mu \phi - V(\phi) \right\} .$$

(1)

Here $\omega(\phi)$ and $V(\phi)$ are functions of the scalar field $\phi$. The function $\omega(\phi)$ actually may be chosen to be equal to 1 or -1 as it is shown below. Its possible role is to realize transitions between deceleration/acceleration phases or non-phantom and phantom phases [3]. Let us now assume a spatially-flat FRW metric:

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^{3} (dx^i)^2 .$$

(2)

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and that the scalar field \( \phi \) only depends on the time coordinate \( t \). Then the FRW equations are given by

\[
\frac{3}{k^{2}}H^{2} = \rho \quad , \quad -\frac{2}{k^{2}}\dot{H} = p + \rho .
\]

(3)

Here the energy density \( \rho \) and the pressure \( p \) are

\[
\rho = \frac{1}{2}\omega(\phi)\dot{\phi}^{2} + V(\phi) \quad , \quad p = \frac{1}{2}\omega(\phi)\dot{\phi}^{2} - V(\phi) .
\]

(4)

Combining \( \ref{1} \) and \( \ref{2} \), one obtains

\[
\omega(\phi)\dot{\phi}^{2} = -\frac{2}{k^{2}}\dot{H} \quad , \quad V(\phi) = \frac{1}{k^{2}}\left(3H^{2} + \dot{H}\right) .
\]

(5)

The interesting case is that of the state (EoS):

\[
\omega = \frac{1}{2} \quad \left( \omega_{0} = \omega \right)
\]

(6)

Since we can always redefine the scalar field \( \phi \rightarrow \phi + \phi_{0} \), \( \omega \) and \( \omega_{0} \) are realized by \( \ref{5} \).

Conversely, if an EoS is given by \( \ref{6} \), since \( \rho \) and \( p \) are given by \( \ref{4} \), the corresponding \( f(\phi) \) can be obtained by solving the following differential equation:

\[
F\left(\frac{3}{k^{2}}f(\phi)^{2}, -\frac{3}{k^{2}}f(\phi)^{2} - \frac{2}{k^{2}}\frac{df(\phi)}{d\phi}\right) = 0 .
\]

(7)

If we define a new field \( \varphi \) as

\[
\varphi = \int d\phi \sqrt{\omega(\phi)} ,
\]

(8)

the action \( \ref{11} \) can be rewritten as

\[
S = \int d^{4}x \sqrt{-g}\left\{ \frac{1}{2\kappa^{2}}R + \frac{1}{2}\partial_{\mu}\varphi\partial^{\mu}\varphi - \tilde{V}(\varphi) \right\} .
\]

(9)

The sign in front of the kinetic term depends on the sign of \( \omega(\phi) \). If the sign of \( \omega \) and then the sign of \( \dot{H} \) is positive (negative), the sign of the kinetic term is \( + \) (\( - \)). Therefore, in the phantom phase, the sign is always + and in the non-phantom phase, always -. One assumes \( \phi \) can be solved with respect to \( \varphi \): \( \phi = \tilde{\phi}(\varphi) \). Then the potential \( \tilde{V}(\varphi) \) is given by \( \tilde{V}(\varphi) = V(\phi(\varphi)) \). Since \( \dot{V}(\varphi) \) could be uniquely determined, there is one to one correspondence between \( H \) and \( \dot{V}(\varphi) \).

By the variation over \( \varphi \) in the action \( \ref{11} \), in the FRW metric, the scalar-field equation follows

\[
0 = \pm\ddot{\varphi} + 3H\dot{\varphi} + \dddot{\varphi} .
\]

(10)

Since the energy density and the pressure is now given by

\[
\rho = \pm \frac{1}{2}\dot{\varphi}^{2} + \ddot{\varphi} , \quad p = \pm \frac{1}{2}\dot{\varphi}^{2} - \ddot{\varphi} ,
\]

(11)

the conservation of the energy can be obtained by using \( \ref{10} \):

\[
\dot{\rho} + 3H(\rho + p) = \pm\dot{\varphi}^{2} + \ddot{\varphi}^{2} + 3H\dot{\varphi}^{2} = \dot{\varphi}^{2} + \ddot{\varphi}^{2} + 3H\dot{\varphi}^{2} + \dddot{\varphi}^{2} = 0 .
\]

(12)

Then one can start either from the EoS ideal fluid description or from the scalar-tensor theory \( \ref{14} \) description: the emerging cosmology is the same.

3. Let us consider several examples. As first example, the simplest case, we take into account a dust model with \( p = 0 \). Since

\[
-\frac{3}{k^{2}}f(\phi)^{2} - \frac{2}{k^{2}}\frac{df(\phi)}{d\phi} = 0 ,
\]

(13)
Hence,

\[ f(\phi) = \frac{2}{3\phi}, \]  

(19)

which gives

\[ \varphi = \frac{2}{\kappa \sqrt{3}} \ln \frac{\phi}{\phi_0}, \]

\[ \tilde{V}(\varphi) = V_0 e^{-\kappa \sqrt{3} \varphi}, \quad V_0 \equiv \frac{2}{3\kappa^2 \phi_0^2}. \]  

(20)

In the case where the EoS parameter \( w = p/\rho \) is constant, this function is

\[ f(\phi) = \frac{2}{3(1 + w)} \phi. \]  

(21)

Here, \( \phi_0 \) is an integration constant. Eq. (21) gives

\[ H = \frac{2}{3(1 + w) |t|}, \]  

(23)

when \( w > -1 \) and by shifting \( t \to t - t_s \) and considering the case \( t < t_s \), we obtain

\[ H = \frac{2}{3(1 + w) (t_s - t)}. \]  

(24)

Eq. (21) expresses the accelerating expansion of the universe. In phantom case \( w < -1 \), the sign in front of the kinetic term in (11) is +.

We have to note, however, that the general solution for (22) is not restricted to the solution corresponding to a constant \( w \) [1, 2]. Let us define new variables \( u, v \) and new time coordinate \( \tau \) by

\[ a = e^{\frac{i\omega}{2}}, \quad \varphi = \frac{2(u - \omega)}{\sqrt{3}}, \]

\[ dt = dt \sqrt{\frac{3V_0}{8}} e^{-\frac{2(u - \omega)}{\sqrt{3} \phi_0}}, \]

\[ \varphi_0 = \frac{2}{\kappa \sqrt{3} |1 + w|}. \]  

(25)

The Hamiltonian constraint and other equation reduce to

\[ \frac{dv}{dt} = a, \]

\[ \frac{du}{dt} = \omega, \]  

\[ \frac{d^2U}{dt^2} = (1 - \alpha^2) \dot{U}, \quad \frac{d^2V}{dt^2} = (1 - \alpha^2) \dot{V}. \]  

(26)
with constant \( h_0 \), which gives
\[
H = h_0 \left( \frac{1}{t} + \frac{1}{t_s - t} \right). \tag{36}
\]
This corresponds to the following EoS:
\[
0 = 4\kappa^2 t_s^2 p^2 - 9\kappa^2 h_0^2 (p + \rho)^2 - 48h_0 t_s \rho. \tag{37}
\]
In this case, one finds
\[
\varphi = \sqrt{2h_0} \int dt \sqrt{\frac{1}{\varphi^2} \beta \alpha - \frac{1}{(t_s - \varphi)^2}}. \tag{38}
\]
By solving (38) with respect to \( \varphi \) as \( \varphi = \phi(\varphi) \), we obtain the corresponding potential:
\[
\ddot{\varphi}(\varphi) = \frac{1}{\kappa^2} \left\{ 3h_0^2 \left( \frac{1}{\phi(\varphi)} + \frac{1}{t_s - \phi(\varphi)} \right)^2 \right\} + h_0 \left( \frac{1}{\phi(\varphi)^2} - \frac{1}{(t_s - \phi(\varphi))^2} \right). \tag{39}
\]
Since
\[
\ddot{H} = \frac{h_0 (2t - t_s)}{t^2 (t_s - t)}, \tag{40}
\]
when \( t < t_s/2 \), the universe is in non-phantom phase \((\ddot{H} < 0)\) but when \( t > t_s/2 \), in phantom phase \((\ddot{H} > 0)\). While using \( \phi \), the transition is smooth although \( \omega(\phi) = 0 \) at the transition point but the definition of \( \varphi_0 \) in (38) has a discontinuity there. Then the EoS (37) admits the phantom phase but, in such a phase, the sign of \( \phi \) kinetic term in (37) becomes +.

Now one may consider
\[
f(\phi) = \left\{ \frac{3}{2} (1 - 2\alpha) \left( \frac{3}{\kappa^2} \right)^{\alpha^{-1}} \right\} \frac{1}{\beta}\phi^{1-\alpha}. \tag{41}
\]
with constants \( f_0 \) and \( \alpha \). The corresponding EoS is
\[
p = -\rho + f_0 \rho^\alpha. \tag{42}
\]
By using (38), we find
\[
\varphi = \varphi_0 \phi^{\frac{1}{1-\alpha}},
\]
\[
\varphi_0 = \sqrt{\frac{2}{|1 - 2\alpha|f_1|}} \frac{1}{|\alpha|\kappa},
\]
\[
f_1 = \left\{ \frac{3}{2} (1 - 2\alpha) \left( \frac{3}{\kappa^2} \right)^{\alpha^{-1}} \right\} \frac{1}{\beta\alpha}. \tag{43}
\]
Then the corresponding potential is given by
\[
\ddot{V}(\varphi) = \frac{1}{\kappa^2} \left\{ f_0^2 \left( \frac{\varphi}{\varphi_0} \right)^{\frac{1}{\beta}} + \frac{f_1}{1 - 2\alpha} \left( \frac{\varphi}{\varphi_0} \right)^{\frac{2}{\beta}} \right\}. \tag{44}
\]
Eq. (41) gives
\[
H = f_t t^{\frac{1}{1-\alpha}}, \quad \dot{H} = \frac{f_t t^{\frac{2}{1-\alpha}}}{1 - 2\alpha}. \tag{45}
\]
Since \( f_1 \) in (43) is positive, if \( \alpha < 1/2 \), the universe is in the phantom phase. In order that \( f_1 \) is real, we also require \( f_0 > 0 \). In the phantom (non-phantom) phase, the sign of the kinetic term of \( \varphi \) in (41) becomes + (−).

As further example, we consider the following EoS:
\[
\rho = -p + \frac{AB\rho^{\alpha + \beta}}{A^{\alpha} + B^\beta}, \tag{46}
\]
with constants, \( A, B, \alpha, \beta \). This EoS has been proposed in [4]. By solving (12), we find that \( f(\phi) \) is the solution of the following algebraic equation:
\[
-\frac{2}{\kappa^2} \left\{ \frac{1}{(1 - \beta \alpha)} B \left( \frac{3}{\kappa^2} \right)^{\beta} f(\phi)^{1 - \frac{\beta}{\alpha}} \right\}
+ \frac{1}{(1 - \frac{\beta}{2}) A} \left( \frac{3}{\kappa^2} \right)^{\frac{\beta}{2}} f(\phi)^{1 - \frac{\beta}{\alpha}} = \phi. \tag{47}
\]
Let the solution of (14) be \( f(\phi) = f_S(\phi) \). Then one may define a new scalar field as
\[
\varphi_S = \frac{2}{\kappa} \int d\phi \sqrt{|f_S(\phi)|}, \tag{48}
\]
which can be solved as \( \phi = \phi(\varphi_S) \). The corresponding potential \( \ddot{V}_S(\varphi_S) \) is given as
\[
\ddot{V}_S(\varphi_S) = \frac{1}{\kappa^2} \left\{ 3f_S(\phi(\varphi_S))^2 + f'(\phi(\varphi_S)) \right\}. \tag{49}
\]
As final example, we consider a fluid with a Van der Waals EoS type [10]
\[
p = -\rho + \frac{\gamma \rho}{1 - \beta \rho} - \alpha \rho^2, \tag{50}
\]
where \( \alpha, \beta, \) and \( \gamma \) are constants. The corresponding \( f(\phi) \) is the solution of the following differential equation:
\[
f'(\phi) = -\frac{2}{1 - \frac{3\gamma}{\kappa^2}} f(\phi)^2 + \frac{9\alpha}{\kappa^2} f(\phi)^4 - \frac{3}{2} f(\phi)^2. \tag{51}
\]
which is not so easy to solve explicitly. For some solution of (51) as \( f(\phi) = f_{V,dW}(\phi) \), a new scalar field can be defined as

\[
\varphi_{V,dW} = \frac{2}{\kappa} \int d\phi \sqrt{|f_{V,dW}(\phi)|} ,
\]

which can be solved as \( \phi = \phi(\varphi_{V,dW}) \). The corresponding potential \( \tilde{V}_{V,dW}(\varphi_{V,dW}) \) is found to be

\[
\tilde{V}_{V,dW}(\varphi_{V,dW}) = \frac{1}{\kappa^2} \left( 3f_{V,dW}(\varphi_{V,dW}) \right)^2 + f'(\phi(\varphi_{V,dW})) .
\]

Similarly, one can construct, in principle, the correspondence between any dark energy EoS and a given scalar-tensor theory. In other words, any explicit (implicit) ideal-fluid EoS of the universe, governed by General Relativity, could be represented as some scalar-tensor theory with specific potential, and vice-versa.

4. Let us now investigate the relations between the scalar-tensor theory (11) and modified \( F(R) \)-gravity, whose Lagrangian density is given by a proper function \( F(R) \) of the scalar curvature \( R \). In such a case, the sign in front of the kinetic term of \( \varphi \) in (13) is \(-\). We can use the conformal transformation

\[ g_{\mu\nu} \rightarrow e^{\pm \kappa \varphi} \sqrt{\bar{g}} g_{\mu\nu} , \]

and make the kinetic term of \( \varphi \) vanish. Hence, one obtains

\[
S = \int d^4x \sqrt{-\bar{g}} \left\{ \frac{e^{\pm \kappa \varphi} \sqrt{\bar{g}}}{2\kappa} R - e^{\pm 2\kappa \varphi} \sqrt{\bar{g}} \bar{V}(\varphi) \right\} .
\]

The action (55) is the so-called "Jordan frame action" while the action (14) is the "Einstein frame action" due to either the non-minimal coupling or the standard coupling in front of the Ricci scalar. Since \( \varphi \) is the auxiliary field, one may cancel out \( \varphi \) by using the equation of motion:

\[
R = e^{\pm \kappa \varphi} \sqrt{\bar{g}} \left( 4\kappa^2 \bar{V}(\varphi) \pm 2\kappa \sqrt{\frac{3}{2}} \bar{V}'(\varphi) \right) ,
\]

which can be solved with respect to \( \varphi \) as \( \varphi = \varphi(R) \). We can rewrite the action (55) in the form of \( F(R) \)-gravity:

\[
S = \int d^4x \sqrt{-F(R)} ,
\]

\[
F(R) \equiv \frac{e^{\pm \kappa \varphi(R)} \sqrt{\bar{g}}}{2\kappa^2} R - e^{\pm 2\kappa \varphi(R)} \sqrt{\bar{g}} \bar{V}(\varphi(R)) .
\]

Note that one can rewrite the scalar-tensor theory (11) or equivalently (14), only when the sign in front of the kinetic term is \(-\) in (13), that is, \( \omega(\phi) \) is positive. In the phantom phase, \( \omega(\phi) \) is negative. In this case, the above procedure to rewrite the phantom scalar-tensor theory as \( F(R) \)-gravity does not work. However, we should note that, for the metric transformed as in (54), even if the universe in the original metric (corresponding to the Einstein frame) is in a phantom phase, the universe in the re-scaled metric (corresponding to the Jordan frame) can be, in general, in a non-phantom phase.

For example, in the case of (21) or (22), after the scale transformation (54) and cancelling out \( \varphi \), one obtains the action corresponding to (22):

\[
S = A \int d^4x \sqrt{-g\bar{R}} .
\]

Here

\[
A \equiv \left( 3 + 3 \left( 1 + \frac{w}{2} \right) \right)^{-\frac{1}{2}} V_0^{-\frac{1}{2}} ,
\]

\[
\alpha \equiv \frac{1}{1 + 3 \left( 1 + \frac{w}{2} \right)} .
\]

We have to note that, if we start from the action (58), due to the conformal transformation (54), the behavior of the universe is different from that given by (21), which is \( a = a_0 \left( \frac{t}{\phi_0} \right)^{\frac{1}{2}} \) with a constant \( a_0 \) (of course, also the Newton law, in the weak energy limit, is modified and interesting results come out as the fact that flat rotation curves of galaxies could be explained without the need of dark matter [12]). By the conformal transformation (54), the FRW metric (2), corresponding to the Einstein frame, is transformed as

\[
\begin{align*}
\bar{ds}^2 & \rightarrow \frac{1}{\phi_0^2} \sqrt{\bar{g}} \bar{ds}^2 = \left( \frac{t}{\phi_0} \right)^{\frac{\mp 2w}{3}} \\
& \times \left\{ -dt^2 + a(t)^2 \sum_{i=1}^{3} (dx_i)^2 \right\} .
\end{align*}
\]

The transformed metric can be regarded as that in the Jordan frame. With a new cosmic time \( \tilde{t} \)

\[
\tilde{t} = \left( \frac{t}{\phi_0} \right)^{\frac{\mp 2w}{3}} dt ,
\]
$ds^2$ can be written in the FRW form as:

$$ds^2 = -dt^2 + \tilde{a}(\tilde{t})^2 \sum_{i=1}^3 (dx^i)^2 .$$

(62)

Hence, $\tilde{a}(\tilde{t})$ behaves as

$$\tilde{a}(\tilde{t}) \sim t^{2 \frac{1}{1 + \frac{1}{3} \sqrt{1 + \frac{2}{1 + w}}}} .$$

(63)

Therefore if we start from the action [58], the behavior of the universe is different from that given by [21]. The exponent in (63) can be defined as $\tilde{h}_0$:

$$\tilde{h}_0 = \frac{2}{3(1 + w)} + \frac{1}{3} \sqrt{\frac{2}{1 + w}} .$$

(64)

In case of -- sign in (64), we find $\tilde{h}_0 < 0$ when

$$-\frac{7}{9} < w < 1 .$$

(65)

Then if we shift $t$ as $t \rightarrow t - t_s$, when $t < t_s$, the universe is in a phantom phase (super-accelerating). In terms of $\alpha$ in the action [58], which is defined in [59], the region in [65] corresponds to

$$\alpha > 0 ,$$

which is consistent with the result in [11]. Hence, even if the universe is not super-accelerating in the original Einstein frame, it is super-accelerating in the shifted Jordan frame in [65]. This demonstrates that being mathematically equivalent, the physics in two such frames may be different.

For the model [42], the equation corresponding to [60] has the following form:

$$R = e^{\kappa \varphi} \sqrt{2} \left[ 4 \left\{ f_1 \left( \frac{\varphi}{\varphi_0} \right)^{\frac{1 + \alpha}{1 - \alpha}} \right. \right.$$  

$$+ \frac{2}{\kappa \varphi_0} \sqrt{3} \left( \frac{2f_1}{1 - \alpha} \left( \frac{\varphi}{\varphi_0} \right)^{\frac{1 + \alpha}{1 - \alpha}} \right. \right. $$

$$+ \frac{2 \alpha f_1}{(1 - 2 \alpha)(1 - \alpha)} \left( \frac{\varphi}{\varphi_0} \right)^{\frac{1 + \alpha}{1 - \alpha}} \right\} ,$$

(67)

which can, in principle, be solved with respect to $\varphi$ as $\varphi = \varphi(R)$. Then the $F(R)$-gravity action corresponding to [67] has the following form:

$$F(R) = \frac{e^{\kappa \varphi(R)} \sqrt{2}}{2 \kappa} R$$

$$- \frac{e^{2\kappa \varphi(R)} \sqrt{2}}{\kappa^2} \left\{ f_1 \left( \frac{\varphi(R)}{\varphi_0} \right)^{\frac{1}{1 - \alpha}} \right.$$  

$$+ \frac{f_1}{1 - 2 \alpha} \left( \frac{\varphi(R)}{\varphi_0} \right)^{\frac{1 + \alpha}{1 - \alpha}} \right\} .$$

(68)

Thus, the explicit examples presented above show that (canonical/quintessence) scalar-tensor theory may be always mapped to modified gravity theory with the same FRW dynamics in one of the frames but the corresponding Newton law is different.

As a generalization of [11], we may consider the following action:

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2 \kappa^2} R - \frac{1}{2} \omega(\phi) \partial \mu \phi \partial^\mu \phi$$

$$+ \eta(\phi) R - V(\phi) \right\} .$$

(69)

Here $\eta(\phi)$ is a function of $\phi$. The late-time, accelerating cosmology for the above theory has been discussed in detail in Ref. [13] and refs. therein. It is worth noting that the stability conditions for the above theory found from perturbation analysis (see last ref. in [13]) confirms such stability conditions, earlier defined in ref. [14] for equivalent modified gravity where quantum considerations have been used.

As before, FRW metric [2] and $\phi$ only depends on $t$. The explicit calculation gives

$$\rho = \frac{1}{2} \omega(\phi) \dot{\phi}^2 + V(\phi) - 6H^2 \eta(\phi) - 6H \eta'(\phi) \dot{\phi} ,$$

$$p = \frac{1}{2} \omega(\phi) \dot{\phi}^2 - V(\phi) + 2 \left( 2 \dot{H} + 3H^2 \right) \eta(\phi)$$

$$+ 2\eta''(\phi) \dot{\phi}^2 + 2\eta'(\phi) \ddot{\phi} + 6H \eta'(\phi) \dot{\phi} .$$

(70)

When $\phi = t$, Eqs. (70) reduce to

$$\rho = \frac{1}{2} \omega(\phi) + V(\phi) - 6H^2 \eta(\phi) - 6H \eta'(\phi) ,$$

$$p = \frac{1}{2} \omega(\phi) - V(\phi) + 2 \left( 2 \dot{H} + 3H^2 \right) \eta(\phi)$$

$$+ 2\eta''(\phi) + 6H \eta'(\phi) .$$

(71)

Deleting $\phi$ from Eqs. (71), we obtain the inhomogeneous EoS description recently introduced in [17]:

$$F(\rho, p, H, \dot{H}) = 0 .$$

(72)
TABLE I: Summary of the three approaches (Eos, Scalar-Tensor and \( F(R) \)) equivalent at level of Lagrangians, field equations and solutions. The solutions are in the Einstein frame for Eos and ST-gravity while they are found in the Jordan frame for \( F(R) \)-gravity.

| \( EoS \) | \( \mathcal{L}_{ST} \) | \( \mathcal{L}_{F(R)} \) |
| --- | --- | --- |
| ↓ | ↓ | ↓ |
| Einstein Eqs. | ST Eqs. | \( F(R) \) Eqs. |
| ↓ | ↓ | ↓ |
| E-frame Sol. | E-frame Sol.+\( \phi \) | J-frame Sol. |

For specific cases, such inhomogeneous Eos description is equivalent to Eos description with time-dependent bulk viscosity. 5.

5. The previous results can be summarized in the Table I, where the mathematical equivalence of the three approaches is shown. In principle, the physical frame (Einstein or Jordan) and the "true" physical approach (Eos, ST or \( F(R) \)) is "selected" by experimental data and observations. The solutions ranging, in principle, from inflation to present accelerated era. Considering the example in previous section, the "mathematical equivalence" between Jordan and Einstein frame of cosmological solutions in scalar-tensor or in \( F(R) \)-gravity can be removed if, in a certain redshift range, data confirm or not super-acceleration: the physical frame (and then the physical theory) is the one in agreement with data. Specifically, the method can be outlined as follows. In order to constrain the parameters characterizing the cosmological solution, we have to maximize the following likelihood function:

\[
\mathcal{L} \propto \exp \left[-\frac{\chi^2(p)}{2}\right] \tag{74}
\]

where \( p \) are the parameters of the cosmological solution and the \( \chi^2 \) merit function is defined as:

\[
\chi^2(p) = \sum_{i=1}^{N} \left[ \frac{y^{th}(z_i, p) - y^{obs}_i}{\sigma_i} \right]^2 + \left[ \frac{\mathcal{R}(p) - 1.716}{0.062} \right]^2 + \left[ \frac{A(p) - 0.469}{0.017} \right]^2 \tag{75}
\]

Terms entering Eq. (75) can be characterized as follows. The dimensionless coordinate distances \( y \) to objects at redshifts \( z \) are considered in the first term. They are defined as:

\[
y(z) = \int_0^z \frac{dz'}{E(z')} \tag{76}
\]

where \( E(z) = H(z)/H_0 \) is a normalized Hubble parameter. This is the quantity which allows to compare the theoretical results with data. The function \( y \) is related to the luminosity distance \( D_L = (1+z)r(z) \) or, equivalently, to the distance modulus \( \mu \). A sample of data on \( y(z) \) for the 157 SNeIa in the Riess et al. \[17\] Gold dataset and 20 radio-galaxies from \[18\] is in \[18\]. These authors fit with good accuracy the linear Hubble law at low redshift (\( z < 0.1 \)) obtaining the Hubble dimensionless parameter \( h = 0.664 \pm 0.008 \). Such a number can be consistently taken into account at low redshift. This value is in agreement with \( H_0 = 72 \pm 8 \) km s\(^{-1}\) Mpc\(^{-1}\) given by the HST Key project \[20\] based on the local distance ladder and estimates coming from time delays in multiply imaged quasars \[21\] and Sunyaev-Zel’dovich effect in X-ray emitting clusters \[22\]. The second term in Eq. (75) allows to extend the \( z \)-range to probe \( y(z) \) up to the last scattering surface \( (z \geq 1000) \). The shift parameter \[23, 24\] can be determined from the CMBR anisotropy spectrum, where \( z_{ls} \) is the redshift of the last scattering surface which can be approximated as \( z_{ls} = 1048 (1 + 0.00124 \omega_b^{0.738}) (1 + g_1 \omega_M^{0.738}) \) with \( \omega_b = \Omega_b h^2 \) (with \( i = b, M \) for baryons and total matter respectively) and \((g_1, g_2)\) given in \[24\]. The parameter \( \omega_b \) is constrained by the baryogenesis calculations contrasted to the observed abundances of
primordial elements. Using this method, the value \( \omega_b = 0.0214 \pm 0.0020 \) is found [20]. In any case, it is worth noting that the exact value of \( z_{ls} \) has a negligible impact on the results and setting \( z_{ls} = 1100 \) does not change constraints and priors on the other parameters of the given model. The third term in the function \( \chi^2 \) takes into account the acoustic peak of the large scale correlation function at 100 \( h^{-1} \) Mpc separation, detected by using 46748 luminous red galaxies (LRG) selected from the SDSS Main Sample [22, 23]. The quantity

\[
A = \sqrt{\Omega_M} \left( \frac{z_{LRG}}{E(z_{LRG})} \right)^{1/3}
\]

is related to the position of acoustic peak where \( z_{LRG} = 0.35 \) is the effective redshift of the above sample. The parameter \( A \) depends on the dimensionless coordinate distance (and thus on the integrated expansion rate), on \( \Omega_M \) and \( E(z) \). This dependence removes some of the degeneracies intrinsic in distance fitting methods. Due to this reason, it is particularly interesting to include \( A \) as a further constraint on the model parameters using its measured value \( A = 0.469 \pm 0.017 \) reported in [23]. Note that, although similar to the usual \( \chi^2 \) introduced in statistics, the reduced \( \chi^2 \) (i.e., the ratio between the \( \chi^2 \) and the number of degrees of freedom) is not forced to be 1 for the best fit model because of the presence of the priors on \( R \) and \( A \) and since the uncertainties \( \sigma_i \) are not Gaussian distributed, but take care of both statistical errors and systematic uncertainties. With the definition [24] of the likelihood function, the best fit model parameters are those that maximize \( \mathcal{L}(p) \).

6. Using the method sketched above, the models studied here can be constrained and selected by observations. From an observational point of view, inhomogeneous EoS with further terms added to \( p = -\rho \) are preferable for the following reasons. Several evidences indicates that \( \Lambda CD M \) \( (p = -\rho) \) is the cosmological scenario able to realistically describe the today observed universe. Any evolutionary model passing from deceleration (dark matter dominance) to acceleration (dark energy dominance) should consistently reproduce, based on the today available observations, such a scenario. Adding terms in Hubble parameter and its derivative to the \( \Lambda \) EoS allows, in any case, a comparison with standard matter parameters (\( \omega_M \) and \( \Omega_M \)) which are directly observable by astrophysical standard methods. Then the number of arbitrary choices (for example, fixing priors) is not so large. On the other hand, implicit EoS, of

| \( \text{par.} \) | \( \text{ACDM} - \log \mathcal{L} = 765.3 \) | \( \text{Exp} - \log \mathcal{L} = 767.3 \) |
|----------|----------------|----------------|
| \( \Omega_b h^2 \) | 0.0226 | 0.0230 | 0.0276 | 0.023 | 0.0213 | 0.0266 |
| \( \Omega_{CDM} h^2 \) | 0.120 | 0.120 | 0.139 | 0.110 | 0.094 | 0.134 |
| \( n_s \) | 0.669 | 0.669 | 0.669 | 0.669 | 0.669 | 0.669 |
| \( \Omega_M \) | 0.298 | 0.298 | 0.379 | 0.298 | 0.298 | 0.383 |
| \( z_{re} \) | 12.4 | 12.4 | 23.4 | 23.4 | 23.4 | 23.4 |
| \( h \) | 0.692 | 0.692 | 0.770 | 0.692 | 0.692 | 0.770 |

TABLE II: Best fit parameters comparing \( \Lambda CD M \) and exponential potential models using the WMAP [29] and CBI [31] data. \( \mathcal{L} \) is the likelihood function defined in the text. The lower and upper limits of the parameter values are the extremal points of the 6-dimensional confidence region. At a first look, the two models seem compatible.

the general form in Eq. (11), needs several arbitrary choices which could result completely inconsistent to further and more refined observations. For example, phantom-like regimes, which could result consistent with observations at large \( z \) (far distances and early universe), could be improperly discarded imposing arbitrary constraints on \( q_0 \) at present epoch. Due to these reasons, from an observational point of view, it is preferable to study models which imply corrections to the \( \Lambda \) EoS rather than giving EoS in implicit form as shown, for example, in Refs. [14, 15]. However, from a theoretical viewpoint, this is not a definitive enough argument in favor of EoS picture.

In order to give a significant example, let us compare, from an observational standpoint, the \( \Lambda CD M \) model with the exponential potential model of scalar-tensor theory. With respect to the arguments presented in this paper, these are paradigmatic examples since they can be always conformally transformed from Jordan to Einstein frame and vice versa, corresponding \( F(R) \)-models can be recovered in any case, and, finally, observational parameters are always referred to the \( \Lambda CD M \) model. The constant \( w \)-case is a particular solution of Eq. (22) as discussed above. As said, we can constrain the cosmological parameters considering the likelihood function [24] and the method sketched above. Using the values of cosmological parameters derived from most popular datasets [29, 30], the two models seem to coincide (Table II) and are practically the same at low redshifts (see Fig. 11). In order to improve the result, we have taken into account also the reionization redshift \( z_{re} \) and the spectral index of scalar fluctuations \( n_s \).

The situation slightly changes if CMBR angular power spectrum is used. In this case, the set of data
FIG. 1: The distance moduli $\mu$ for the $\Lambda$CDM and the exponential potential models, obtained from the best fit parameters of Table II, are compared with SNeIa data in [17]. The two curves practically coincide for $z \leq 2$.

is more complete and refined with respect to the SNeIa ones. From Fig. (2), it is clear that models differ for small $l$.

The differences between the two models are put in evidence if one chooses suitable variables by which representing the $w$ evolution in asymptotic regimes, as shown in Fig. (3). This fact agrees with previous considerations by which inhomogeneous EoS could realistically represent the today observed cosmological scenario allowing also to achieve the early and the late evolution of the universe.

In conclusion, larger and more detailed samples of data than those today available are needed to fit solutions in wide ranges of $z$. Only in this situation, the true physical frame could be univocally selected.

FIG. 2: The CMBR angular power spectrum $Q \equiv l(l+1)C_l/2\pi$ for the two models, obtained with CAMB codes [31] from the best fit parameters of Table II. The two curves do not coincide for small $l$'s, where the exponential potential gives higher values. We have to note that we are exploring a different range of $z$ with respect to that in Fig.1.

FIG. 3: Plot of the scalar-field equation of state versus $\log_{10} a$ with the best fit value of $\Omega_M = 0.298$. The vertical bar marks the today value of scale factor log$_{10} a_0$. Only with this choice of variables, there is evidence of a transition from $w \approx -1$ in the past to $w \approx -0.5$ in the future. This means that other constant values of $w$ can be generically recovered from scalar-tensor theory, also for exponential potentials, so that $\Lambda$CDM does not coincides with exponential models for any $w$ and any $z$. 
