A NOTE ON ORDERABILITY AND DEHN FILLING

CHRISTOPHER HERALD AND XINGRU ZHANG

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ABSTRACT. We improve upon a recent result of Culler and Dunfield on orderability of certain Dehn fillings by removing a difficult condition they required.

If $M$ is the exterior of a knot $K$ in an integral homology 3-sphere, $\mu$ and $\lambda$ will be the canonical meridian and longitude in $\partial M$, i.e., $\mu$ bounds a meridian disk of $K$ and $\lambda$ is null-homologous in $M$. The set of slopes in $\partial M$ will be identified with $\mathbb{Q} \cup \{1/0\}$ with respect to the chosen meridian and longitude in the usual way so that $\mu$ is identified with $1/0$ and $\lambda$ with $0$, and $M(r)$ will denote the Dehn filling of $M$ with slope $r$. Recall that a group is called left-orderable if there is a total ordering on the group which is invariant under left multiplication. The purpose of this note is to update a recent result of Culler and Dunfield [CD] to the following

**Theorem 1.** Let $M$ be the exterior of a knot in an integral homology 3-sphere such that $M$ is irreducible. If the Alexander polynomial $\Delta(t)$ of $M$ has a simple root on the unit circle, then there exists a real number $a > 0$ such that, for every rational slope $r \in (-a, 0) \cup (0, a)$, the Dehn filling $M(r)$ has left-orderable fundamental group.

The above theorem was proved in [CD] under the additional condition that every closed essential surface in $M(0)$ is a fiber in a fibration of $M(0)$ over $S^1$ ([CD, Theorem 1.2]) or under the additional condition that each positive dimensional component of the $PSL_2(\mathbb{C})$ character variety of $M(0)$ consists entirely of characters of reducible representations [CD, Theorem 7.1]. The former condition is very restrictive and the latter one is hard to verify in general. So the updated theorem will be much more applicable.

We also remark that if $M(0)$ is prime (which is always true if $M$ is the exterior of a knot in $S^3$) then $\pi_1(M(0))$ is left-orderable since the first Betti number of $M(0)$ is positive ([BRW, Theorem 1.3]). In this case, we may replace the intervals $(-a, 0) \cup (0, a)$ in Theorem 1 by the interval $(-a, a)$.

The motivation for studying if a Dehn filling has left-orderable fundamental group is its connection to the following now well known

**Conjecture 2.** ([BGW]) For a closed connected orientable prime 3-manifold $W$, the following are equivalent:

1. $W$ has left-orderable fundamental group.
2. $W$ is not an L-space.
3. $W$ admits a co-orientable taut foliation.
Combining Theorem 1 with [R, Theorem 4.7] as well as the fact that an L-space cannot have a co-orientable taut foliation [OS], we can update [CD, Corollary 1.3] to the following

**Corollary 3.** Let $M$ be the exterior of a knot in an integral homology 3-sphere such that $M$ fibers over the circle with pseudo-Anosov monodromy. If the Alexander polynomial of $M$ has a simple root on the unit circle, then there is a real number $a > 0$ such that for every rational slope $r \in (-a,a)$ the Dehn filling $M(r)$ satisfies Conjecture 2.

Similarly combining Theorem 1 with [LR, Theorem 1.1] we have

**Corollary 4.** Let $M$ be the exterior of a nontrivial knot in $S^3$. If the Alexander polynomial of $M$ has a simple root on the unit circle, then there is an $a > 0$ such that for every rational slope $r \in (-a,a)$ the Dehn filling $M(r)$ satisfies Conjecture 2.

Now we proceed to prove Theorem 1. From now on we assume that $M$ is the exterior of a knot in an integral homology 3-sphere such that $M$ is irreducible.

The main new input is a quick application of some results from [H] which we recall now. Let $F$ be a Seifert surface of genus $g$ for $M$ and let $F \times [-1,1]$ be a product neighborhood of $F$ in $M$ so that $F = F \times \{0\}$. If $\{x_i\}_{1 \leq i \leq 2g}$ is a basis for $H_1(F;\mathbb{Z})$, let $x_i^+$ denote the pushoff of $x_i$ in $F \times \{1\}$. Define the linking matrix $V$ by $V_{ij} = lk(x_i,x_j^+)$. The symmetrized Alexander matrix for $M$ is the matrix

$$A(t) = t^{1/2}V - t^{-1/2}V^T$$

and $\Delta(t) = \det A(t)$ is the Alexander polynomial of $M$. Let $B(t) = (t^{-1/2} - t^{1/2})A(t)$. The complex values $t \neq \pm 1$ for which $B(t)$ is singular are exactly the roots of the Alexander polynomial $\Delta(t)$.

Identify $SU(2)$ with the set of unit quaternions and identify $U(1)$ with the unit circle in the complex plane. If $t \in U(1)$, then $B(t)$ is a Hermitian matrix and hence has only real eigenvalues. The equivariant knot signature of $M$, denoted by $\text{Sign}(B(t^2))$, is the function from $U(1)$ to $\mathbb{Z}$ taking $t$ to the number of positive eigenvalues minus the number of negative eigenvalues for $B(t^2)$, counted with multiplicity. This function is independent of the choice of $F$, $\{x_i\}$, and the product neighborhood of $F$. The relationship between $B(t)$ and the Alexander matrix $A(t)$ implies that $\text{Sign}(B(t^2))$ is continuous in $t \in U(1)$ except possibly at square roots of roots of the Alexander polynomial.

For each $0 < \alpha < \pi$, let $\rho_\alpha : \pi_1(M) \rightarrow SU(2)$ be the abelian representation determined by $\rho(\mu) = e^{i\alpha}$. The following results were obtained in [H].

**Theorem 5.** (1) ([H, Theorem 1]) If $e^{i2\alpha}$ is a root of $\Delta(t)$ for which the right and left hand limits

$$\lim_{\beta \rightarrow \alpha^\pm} \text{Sign}(B(e^{i2\beta}))$$

do not agree, then there is a continuous family of irreducible $SU(2)$ representations of $\pi_1(M)$ limiting to $\rho_\alpha$.

(2) ([H, Corollary 2]) If $e^{i2\alpha}$ is a root of $\Delta(t)$ of odd multiplicity, then the condition in part
(1) holds and thus there is a continuous family of irreducible $SU(2)$ representations of $\pi_1(M)$ limiting to $\rho_\alpha$.

(3) ([H, Corollary 3]) Suppose that $e^{i2\alpha}$ is a root of $\Delta(t)$ such that as $t \in U(1)$ moves through the value $e^{i\alpha}$, all eigenvalues of $B(t^2)$ touching zero cross zero transversely, and all do so in the same direction. Then all of the $SU(2)$ irreducible representations $\{\rho_s\}$ near $\rho_\alpha$ (provided by part (1)) send $\lambda$ to $e^{i\sigma(\rho_s)}$ for some small $\sigma(\rho_s) \neq 0$.

What we need in this paper is the following special consequence of Theorem 5.

**Corollary 6.** If $e^{i2\alpha}$ is a simple root of $\Delta(t)$, then there is a continuous family of irreducible $SU(2)$ representations $\{\rho_s\}$ of $\pi_1(M)$ limiting to $\rho_\alpha$. Moreover all of these $\rho_s$ near $\rho_\alpha$ send $\lambda$ to $e^{i\sigma(\rho_s)}$ for some small $\sigma(\rho_s) \neq 0$.

**Proof.** The first assertion is immediate by part (2) of Theorem 5. To get the second assertion, let $t_0 = e^{i\alpha}$ and we have

$$\det B(t^2) = (t^{-1} - t)^{2g} \Delta(t^2) = (t - t_0)f(t)$$

where $f(t)$ is a holomorphic function such that $f(t_0) \neq 0$. The product rule for derivative shows that the derivative of $\det B(t^2)$ at $t_0$ is not zero. As $\det B(t^2)$ is a product of its eigenvalues $\lambda_1(t), \ldots, \lambda_{2g}(t)$ for which we may assume that $\lambda_1(t_0) = 0$ and $\lambda_j(t_0) \neq 0$ for all $1 < j \leq 2g$, applying the product rule for derivative again we see that $\lambda_1(t)$ has nonzero derivative at $t_0$. So $\lambda_1(t)$ cross zero transversely as $t \in U(1)$ moves through the value $e^{i\alpha}$. The second assertion now follows from part (3) of Theorem 5.

Now let $R(M) = \text{Hom}(\pi_1M, SL_2(\mathbb{C}))$ be the $SL_2(\mathbb{C})$ representation variety of $M$ and $X(M)$ the corresponding character variety. Recall that the character $\chi_\rho \in X(M)$ of a representation $\rho \in R(M)$ is the function $\chi_\rho : \pi_1M \to \mathbb{C}$ defined by $\chi_\rho(\gamma) = \text{trace}(\rho(\gamma))$ for $\gamma \in \pi_1M$. Two irreducible representations in $R(M)$ are conjugate if and only if they have the same character.

From now on we consider $SU(2)$ as a subgroup of $SL_2(\mathbb{C})$. So the abelian representation $\rho_\alpha$ sends $\mu$ to $\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}$. The following results were shown in [HPP].

**Theorem 7.** Suppose $e^{i2\alpha}$ is a simple root of the Alexander polynomial $\Delta(t)$ of $M$.

(1) ([HPP, Theorem 1.2]) The character $\chi_{\rho_\alpha}$ of the abelian representation $\rho_\alpha$ is contained in a unique algebraic component $X_0$ of $X(M)$ which contains characters of irreducible representations and is a smooth point of $X_0$.

(2) ([HPP, Theorem 1.1]) The complex dimension of $X_0$ is one.

(3) ([HPP, Corollary 1.4]) The character $\chi_{\rho_\alpha}$ is a middle point of a smooth arc of real valued characters $\{\chi_t: t \in (-\epsilon, \epsilon)\}$ in $X_0$ such that $\chi_0 = \chi_{\rho_\alpha}$, $\chi_t$ is the character of an irreducible representation for $t \neq 0$. Moreover $\chi_t$ is the character of a representation into $SU(2)$ for $t > 0$ and $SU(1,1)$ for $t < 0$.

Now consider a continuous family of irreducible $SU(2)$ representations $\{\rho_s\}$ limiting to the abelian representation $\rho_\alpha$, provided by Corollary 6. Since the character of an irreducible representation cannot be equal to the character of a reducible representation, $\{\chi_{\rho_s}\}$ is a nonconstant
continuous family of characters limiting to $\chi_{\rho_{\alpha}}$. Hence by part (1) of Theorem 7, $\chi_{\rho_{s}} \in X_0$ for all $\chi_{\rho_{s}}$ sufficiently close to $\chi_{\rho_{\alpha}}$.

Let $f_{\lambda} : X_0 \to \mathbb{C}$ be the function defined by $f_{\lambda}(\chi_{\rho}) = \chi_{\rho}(\lambda) - 2$. Then $f_{\lambda}$ is a regular function on the irreducible variety $X_0$. Note that $\rho_{\alpha}(\lambda) = I$ and by Corollary 6 $\rho_{s}(\lambda) \neq I$ for all $\rho_{s}$ sufficiently close to $\rho_{\alpha}$, where $I$ is the identity matrix of $SL_2(\mathbb{C})$. Since $\rho_{s}(\lambda) \in SU(2)$, its trace cannot be 2. It follows that the function $f_{\lambda}$ is non-constant on $X_0$. Since $dim_{\mathbb{C}} X_0 = 1$ by part (1) of Theorem 7, any regular function on $X_0$ is either a constant function or has finitely many zero points. Hence the function $f_{\lambda}$ can have only finitely many zeros in $X_0$. In particular we may assume that $f_{\lambda}$ is never zero valued on the curve $\{\chi_t; t \in (-\epsilon,0)\}$, provided by part (3) of Theorem 7 (by choosing smaller $\epsilon > 0$ if necessary). The proof of [HPP, Corollary 1.4] given in [HPP, Section 5] actually shows that there is a smooth path of irreducible $SU(1,1)$ representations $\{\rho_t; t \in (-\epsilon,0)\}$ of $\pi_1(M)$ limiting to $\rho_{\alpha}$ as $t \to 0$ such that $\chi_t = \chi_{\rho_{t}}$. So for $\rho_t$, $t < 0$, we have $\rho_t(\lambda) \neq I$.

Recall that $SU(1,1)$ is the subgroup of $SL_2(\mathbb{C})$ consisting of matrices $\left\{ \begin{pmatrix} x & y \\ y & x \end{pmatrix} : x - y = 1 \right\}$, which is conjugate to $SL_2(\mathbb{R})$ by the element $\begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$. Hence the path of irreducible $SU(1,1)$ representations $\{\rho_t; t \in (-\epsilon,0)\}$ given above is conjugate to a path of irreducible $SL_2(\mathbb{R})$ representations $\{\rho'_t; t \in (-\epsilon,0)\}$ limiting to the abelian representation $\rho'_\alpha$ as $t \to 0$, where $\rho'_\alpha$ sends $\mu$ to $\begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}$. Of course $\rho'_t(\lambda) \neq I$ for $t \in (-\epsilon,0)$.

Now the argument in [CD, Section 7] shows that the existence of a path of $SL_2(\mathbb{R})$ representations $\rho'_t$ as given in the preceding paragraph will imply the conclusion of Theorem 1. More concretely, this path $\rho'_t$ will lift to a path of $SL_2(\mathbb{R})$ representations $\tilde{\rho}'_t$ of $\pi_1(M)$, where $SL_2(\mathbb{R})$ is the universal covering group of $SL_2(\mathbb{R})$, and moreover there is an $a > 0$ such that for each slope $r \in (-a,0) \cup (0,a)$ some $\tilde{\rho}'_t$ will factor through $\pi_1(M(r))$ yielding a nontrivial representation $\pi_1(M(r)) \to SL_2(\mathbb{R})$. Since $M$ can have at most three Dehn fillings yielding reducible manifolds by [GL], we may assume that $a > 0$ has been chosen so that $M(r)$ is irreducible for each slope $r \in (-a,0) \cup (0,a)$. Hence $\pi_1(M(r))$ is left-orderable, by [BRW, Theorem 1.1].

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Department of Mathematics and Statistics, University of Nevada, Reno, NV 89557

E-mail address: herald@unr.edu

Department of Mathematics, University at Buffalo, Buffalo, NY 14260

E-mail address: xinzhang@buffalo.edu