AN INTEGRAL INEQUALITY FOR CONSTANT SCALAR CURVATURE METRICS ON KähLER MANIFOLDS

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Abstract. We present in this note a lower bound for the Calabi functional in a given Kähler class. This yields an integral inequality for constant scalar curvature metrics, which can be viewed as a refined version of Yau’s Chern number inequality.

1. Introduction and main results

Suppose \((M, J)\) is an \(n\)-dimensional compact complex manifold with a Kähler metric \(g\). This \(g\) determines a positive \((1,1)\)-form, the Kähler form \(\omega(-,\cdot) := \frac{1}{2\pi} g(J\cdot,\cdot)\), and vice versa. Under local coordinates \((z^1,\ldots, z^n)\), we have

\[
g = (g_{i\bar{j}}) := (g(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j})), \quad \omega = \frac{\sqrt{-1}}{2\pi} g_{i\bar{j}} dz^i \wedge d\bar{z}^j.
\]

Here and throughout the paper we use the Einstein convention for various summations. Since Kähler metrics \(g\) are one-to-one correspondence to their Kähler forms \(\omega\), hereafter we don’t distinguish them. Let \([\omega] \in H^{1,1}(M, \mathbb{R})\) be a Kähler class and denote the set of all Kähler forms in \([\omega]\) by \([\omega]^{+}\). In [6], Calabi introduced the following functional, now called the Calabi functional:

\[
[\omega]^{+} \to \mathbb{R}, \quad \omega \mapsto \text{Ca}(\omega) := \int_M s^2(\omega) \omega^n,
\]

where \(s(\omega) := 2g^{\bar{i}j} r_{i\bar{j}}\) is the (Riemannian) scalar curvature of the metric \(\omega\). Here \((g^{\bar{i}j}) := (g_{i\bar{j}})^{-1}\) and \(r_{i\bar{j}}\) are the component functions of

\[
\text{Ric}(\omega) = \frac{\sqrt{-1}}{2\pi} r_{i\bar{j}} dz^i \wedge d\bar{z}^j := -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log(\det(g_{i\bar{j}})),
\]

the Ricci form of \(\omega\) which represents the first Chern class of \(M\).

Calabi proposed finding critical points of this functional as candidates for canonical metrics, which are called extremal Kähler metrics and can be viewed as a generalization of the notion of constant scalar curvature Kähler (“cscK” for short) metrics. It turns out that

\[
\inf_{\omega \in [\omega]} \text{Ca}(\omega),
\]

the greatest lower bound of \(\text{Ca}(\omega)\) in the Kähler class \([\omega]\), is deeply related to the difficult open problem of relating existence of cscK metrics in \([\omega]^{+}\) to algebro-geometric stability ([9], [7]). However, aside from this deep relationship, there is a rather trivial lower bound on this
functional involving only the cohomology class $[\omega]$ and the first Chern class $c_1$ due to the well-known fact that $s(\omega)\omega^n = 2nc_1(\omega) \wedge \omega^{n-1}$:

$$
(1.1) \quad \int_M s^2(\omega)\omega^n \geq \left( \frac{\int_M s(\omega)\omega^n}{\int_M \omega^n} \right)^2 = \frac{(2nc_1(\omega)^{n-1})^2}{[\omega]^{n-1}}.
$$

where the equality holds if and only if $\omega$ is a cscK metric.

Recall that the Kähler curvature tensor $R$ of a Kähler metric $\omega$ has an orthogonal decomposition under the $L^2$ norm, $R = S + P + B$, where $S$ is the scalar part, $P$ is the traceless Ricci part, and $B$ is the Bochner curvature tensor. We call $\omega$ a Bochner-Kähler ("B-K" for short) metric if $B \equiv 0$. The metric $\omega$ is Einstein if and only if $P \equiv 0$. $(M, \omega)$ is a complex space form, i.e., $\omega$ has constant holomorphic sectional curvature, if and only if $P = B \equiv 0$.

It is well-known via the Chern-Weil theory that the two integrals

$$
(1.2) \quad \int_M c_1^2(\omega) \wedge \omega^{n-2} \quad \text{and} \quad \int_M c_2(\omega) \wedge \omega^{n-2},
$$

where $c_1(\omega) (= \text{Ric}(\omega))$ and $c_2(\omega)$ are the first two Chern forms of $\omega$, can be expressed in terms of the $L^2$ norms of the above-mentioned tensors. Apte should be the first one who derived the expression for $c_2(\omega) \wedge \omega^{n-2}$ in [1]. These two expressions have many important related applications. For instance, together with the Aubin-Yau theorem on Kähler-Einstein ("K-E" for short) metrics with negative scalar curvature and the Calabi-Yau theorem on Ricci-flat Kähler metrics, they can be led to Yau's remarkable Chern number inequality ([14]). In [3, p. 80, (2.80), (2.80a)] of Besse's highly influential book, Apte's formula was refined in terms of three terms: the scalar curvature $s(\omega)$ and the squared norms of the traceless tensor $\tilde{\text{Ric}}(\omega) := \text{Ric}(\omega) - \frac{s(\omega)}{2n}\omega$, and the Bochner curvature tensor $B$. In [3, p. 80] these are denoted by the symbols $s$, $\rho_0$ and $B_0$ respectively. Then they use this expression and that of $c_1^2(\omega) \wedge \omega^{n-2}$ to deduce ([3, p. 80, (2.82a)]) the expression of

$$
\int_M (2(n+1)c_2(\omega) - nc_1^2(\omega)) \wedge \omega^{n-2},
$$

which was applied in turn to deduce Yau's Chern number inequality in [3, p. 325].

Here we would like to point out that the coefficient before the traceless part $|\rho_0|^2$ in [3, (2.80), (2.80a)] is incorrect. According to our detailed calculations (see Section 3.1), it should be $-\frac{2m}{m+2}$ rather than $-\frac{2(m-1)}{m}$ (they denote by $m$ the complex dimension of the Kähler manifold). Accordingly, the coefficient before $|\rho_0|^2$ in [3, (2.82a)] should be $-(1 - \frac{2}{m+2})$ rather than $-(1 - \frac{2}{m})$. However, this inaccuracy of the coefficient before $|\rho_0|^2$ does not affect the above-mentioned application as the condition of $\omega$ being Kähler-Einstein requires that $|\rho_0| = 0$. As a byproduct of correcting this coefficient, which will be done in details in Section 3.1, we will have a lower bound on the Calabi functional as follows.

**Proposition 1.1.** Suppose $(M, \omega)$ is an $n$-dimensional compact Kähler manifold with a Kähler metric $\omega$. Then we have

$$
(1.3) \quad \int_M s^2(\omega)\omega^n \geq 8(n+1)(nc_1^2[\omega]^{n-2} - (n+2)c_2[\omega]^{n-2}),
$$

and the equality holds if and only if $\omega$ is a B-K metric.

**Remark 1.2.**
1. As we have commented above, the inaccuracy of the coefficient before $|\rho_0|^2$ in [3, (2.80a)] does not affect its deduction to Yau’s Chern number inequality. But the correctness of this coefficient is crucial to our Proposition 1.1 and the subsequent Theorem 1.3.

2. A direct corollary of Proposition 1.1 is that a B-K metric must be an extremal metric, which was proved by Matsumoto in [12].

3. Clearly the lower bound in (1.3) makes no sense unless

$$nc_1^2[\omega]^{n-2} > (n+2)c_2[\omega]^{n-2}.$$ 

Even if this holds, one may ask that whether in some cases this lower bound is really sharper than the trivial one in (1.1). In Section 2, we will use an example of a Fano manifold, which was first noticed by Batyrev ([2]) to disprove an old conjecture, to illustrate that for some Kähler classes of this manifold the lower bound in (1.3) is strictly larger than that in (1.1) and hence these two lower bounds are actually independent to each other.

Although the appearance of the Bochner tensor $B$ in the decomposition of the Kähler curvature tensor $R$ as an irreducible summand under the unitary group has been known since the 1949 work of Bochner ([4]), B-K metrics did not receive enough attention for a long time until the work of Kamishima and Bryant ([10], [5]). In particular, their uniformization theorem for compact B-K manifolds tells us that the only $n$-dimensional compact B-K manifolds are the compact quotients of the symmetric B-K manifolds $M_p^c \times M_n^c$ (cf. [4, p. 682]), where $M_p^c$ denotes the $p$-dimensional complex space form of constant holomorphic sectional curvature $c$. An important corollary of this remarkable result is that any B-K metric on a compact complex manifold must be a cscK metric. (This conclusion is not valid for non-compact manifolds as Professor Bryant pointed out to me that there are B-K metrics on $\mathbb{C}^n$ that are not cscK).

Now combining this result with (1.1) and Proposition 1.1 leads to the following integral inequality, which provides an obstruction to the existence of cscK metrics in a given Kähler class and is indeed a refinement of Yau’s Chern number inequality (see Corollary 1.5 and its proof).

**Theorem 1.3.** Suppose the Kähler class $[\omega]$ of an $n$-dimensional compact Kähler manifold contains a cscK metric. Then we have

$$n^2(c_1[\omega]^{n-1})^2 \geq 2(n+1) \cdot [\omega]^n \cdot (nc_1^2[\omega]^{n-2} - (n+2)c_2[\omega]^{n-2}),$$

where the equality holds if and only if $[\omega]$ contains a B-K metric.

**Remark 1.4.**

1. Note that in the above conclusion the B-K metric in $[\omega]$ needs not necessarily to be the original cscK metric in it. It is this place that we need the fact that B-K metrics are cscK metrics in the compact case.

2. As we have mentioned in Item (3) of Remark 1.2, for some Kähler classes of the Fano manifold which will be described in details in Section 2, (1.4) does not hold and so these Kähler classes cannot contain cscK metrics. Indeed, it can be shown that the holomorphic automorphism group of this Fano manifold is not reductive and thus our conclusion is consistent with the Matsushima-Lichnerowicz theorem ([11]).
In [13, §2.3] of his famous lecture notes, Tian discussed Yau’s Chern number inequality along the line of the uniformization theorem for constant holomorphic sectional curvature Kähler manifolds. Indeed he has realized that the curvature integral expressions for (1.2) can be used to give an integral inequality for cscK metrics and provided one without a proof in [13, p. 21, Remark 2.15]. In Section 3.2 we will explain how this inequality can be derived.

**Corollary 1.5 (Yau’s Chern number inequality).** Suppose $M$ is an $n$-dimensional compact Kähler manifold.

1. If $c_1 < 0$, then
   
   $$2(n + 1)c_2(-c_1)^{n-2} \geq n(-c_1)^n,$$
   
   where the equality holds if and only if $M$ is covered by the unit ball in $\mathbb{C}^n$.

2. If $c_1 = 0$, then $c_2[\omega]^{n-2} \geq 0$ for any Kähler class $[\omega]$, where the equality holds if and only if $M$ is covered by a complex torus.

**Proof.** If $c_1 < 0$, then $-c_1$ is a Kähler class and contains a K-E metric by Aubin and Yau’s theorem. Replacing the Kähler class $[\omega]$ in (1.4) with $-c_1$ we obtain

$$2(n + 1)c_2(-c_1)^{n-2} \geq n(-c_1)^n,$$

where the equality holds if and only if the Kähler class $-c_1$ contains a B-E metric, say $\omega$. By the above-mentioned Kamishima-Bryant’s result this Kähler metric $\omega$ is a cscK metric and thus a K-E metric ([13, p. 19, Prop. 2.12]). This means the traceless part tensor $P = 0$ and the Bochner tensor $B = 0$. So this metric $\omega$ has negative constant holomorphic sectional curvature and thus is covered by the unit ball in $\mathbb{C}^n$ ([13, Theorem 1.12]).

If $c_1 = 0$, then any Kähler class $[\omega]$ has a Ricci-flat Kähler metric by the Calabi-Yau theorem. So (1.4) tells us that $c_2[\omega]^{n-2} \geq 0$. The equality holds if and only if $[\omega]$ contains a B-K metric, say $\omega$. Similar to the above argument we know that this $\omega$ is a K-E metric and thus is Ricci-flat. This means $M$ is a compact Kähler manifold with vanishing holomorphic sectional curvature and so it is covered by a complex torus. \qed

2. An example

Let $\mathbb{P}^{n-1}$ denote the $(n-1)$-dimensional complex projective space and

$$X := \mathbb{P}(O_{\mathbb{P}^{n-1}} \oplus O_{\mathbb{P}^{n-1}}(n-1)),$$

which is an $n$-dimensional Fano manifold. There was an old conjecture asserting that, among all the $n$-dimensional Fano manifolds $M$, the maximum of the top intersection number $c_1^n(M)$, also called the degree of $M$, can only be attained by $\mathbb{P}^n$. Namely, $c_1^n(M) \leq (n + 1)^n$ and with equality if and only if $M \cong \mathbb{P}^n$. In [2], Batyrev noticed that

$$c_1^n(X) = \frac{(2n - 1)^n - 1}{n - 1} \approx \frac{2^n e^{-3/2}}{n} (n + 1)^n$$

and thus disproved this conjecture. In [8, p. 137-139] Debarre extended the construction of $X$ to a family of Fano manifolds and used them to illustrate that there is indeed no universal polynomial upper bound on $\sqrt[n-1]{c_1^n(M)}$ among all the $n$-dimensional Fano manifolds $M$. 

In this section we will see that this $n$-dimensional Fano manifold $X$ is also an ideal example for our purpose. More precisely, for some Kähler classes of $X$ we can show that they don’t satisfy (1.4) and thus cannot contain cscK metrics. Consequently, for these Kähler classes of $X$ the lower bound in (1.3) is sharper than that in (1.1) and hence clarify the non-triviality of Proposition 1.1.

Let $L$ and $H$ denote the first Chern classes of the line bundle $\mathcal{O}_X(1)$ and the pull back of the hyperplane line bundle $\mathcal{O}_{\mathbb{P}^{n−1}}(1)$ respectively. Then the intersection ring of $X$ is generated by $L$ and $H$ with the relations (cf. [8, p. 138])

\begin{equation}
L^2 = (n-1) LH, \quad LH^{n-1} = 1, \quad H^n = 0.
\end{equation}

Standard calculation tells us that the first two Chern classes of $X$ are as follows.

$c_1 = 2L + H, \quad c_2 = 2nLH - \frac{n(n-1)}{2}H^2$.

With these data in hand, we can easily get the following lemma.

**Lemma 2.1.** Let $\Omega_{\alpha, \beta} := \alpha L + \beta H$ $(\alpha, \beta > 0)$ be a Kähler class of $X$. If we set $t := (n-1)\alpha + \beta$,

then we have

\begin{align*}
\Omega_{\alpha, \beta}^n &= \frac{1}{n-1}(t^n - \beta^n), \\
c_1\Omega_{\alpha, \beta}^{n-1} &= \frac{1}{n-1}[(2n-1)t^{n-1} - \beta^{n-1}], \\
c_1^2\Omega_{\alpha, \beta}^{n-2} &= \frac{1}{n-1}[(2n-1)^2t^{n-2} - \beta^{n-2}], \\
c_2\Omega_{\alpha, \beta}^{n-2} &= \frac{n}{2}[3t^{n-2} + \beta^{n-2}].
\end{align*}

**Proof.** We only treat $\Omega_{\alpha, \beta}^n$ and $c_1^2\Omega_{\alpha, \beta}^{n-2}$ and the other two cases are similar. First note that the relation (2.1) implies

\begin{equation}
L^i H^{n-i} = \begin{cases} 
0, & \text{if } i = 0, \\
(n-1)^{i-1}, & \text{if } 1 \leq i \leq n.
\end{cases}
\end{equation}

Therefore,

\begin{align*}
\Omega_{\alpha, \beta}^n &= (\alpha L + \beta H)^n \\
&= \sum_{i=0}^{n} \binom{n}{i} \alpha^i \beta^{n-i} L^i H^{n-i} \\
&= \sum_{i=1}^{n} \binom{n}{i} \alpha^i \beta^{n-i} (n-1)^{i-1} \quad \text{(by (2.2))} \\
&= \frac{1}{n-1} \left\{ [(n-1)\alpha + \beta]^n - \beta^n \right\} \\
&= \frac{1}{n-1} (t^n - \beta^n).
\end{align*}
\[c_1^n \Omega_{\alpha, \beta}^{n-2} = (2L + H)^2 (\alpha L + \beta H)^{n-2}\]

\[= (4nLH + H^2) \sum_{i=0}^{n-2} \binom{n-2}{i} \alpha^i \beta^{n-2-i} L^i H^{n-2-i}\]

\[= 4n \sum_{i=0}^{n-2} \binom{n-2}{i} \alpha^i \beta^{n-2-i} L^i H^{n-1-i} + \sum_{i=0}^{n-2} \binom{n-2}{i} \alpha^i \beta^{n-2-i} L^i H^{n-i}\]

\[= 4n \sum_{i=0}^{n-2} \binom{n-2}{i} \alpha^i \beta^{n-2-i} (n-1)^i + \sum_{i=1}^{n-2} \binom{n-2}{i} \alpha^i \beta^{n-2-i} (n-1)^{i-1} \quad \text{(by (2.2))}\]

\[= 4nt^{n-2} + \frac{1}{n-1} (t^{n-2} - \beta^{n-2})\]

\[= \frac{1}{n-1} [(2n-1)^2 t^{n-2} - \beta^{n-2}].\]

□

This lemma leads to the following

**Proposition 2.2.** Set

\[f(n, \alpha, \beta) := 2(n+1)\Omega_{\alpha, \beta}^n [n^2 \Omega_{\alpha, \beta}^{n-2} - (n+2)c_2 \Omega_{\alpha, \beta}^{n-2}] - n^2 (c_1 \Omega_{\alpha, \beta}^{n-1})^2.\]

For arbitrary positive numbers \(\alpha, \beta\), there exists a positive integer \(N(\alpha, \beta)\) such that

\[f(n, \alpha, \beta) > 0 \quad \text{whenever} \quad n \geq N(\alpha, \beta).\]

Consequently, when \(n \geq N(\alpha, \beta)\), the lower bound (1.3) for the Calabi functional in the Kähler class \(\Omega_{\alpha, \beta}\) is sharper than that in (1.1). This means the non-triviality of Proposition 1.1 with respect to the trivial one in (1.1). Moreover, these Kähler classes \(\Omega_{\alpha, \beta}\) don’t contain cscK metrics.

**Proof.** Using Lemma 2.1 we have

\[f(n, \alpha, \beta) = \frac{2(n+1)}{n-1} (t^n - \beta^n) \left[ \frac{n(2n-1)^2}{n-1} t^{n-2} - \frac{n}{n-1} \beta^{n-2} - \frac{3n(n+2)}{2} t^{n-2} - \frac{n(n+2)}{2} \beta^{n-2} \right] - \frac{n^2}{(n-1)^2} [(2n-1)t^{n-1} - \beta^{n-1}]^2.\]

Direct computation shows that

\[\frac{(n-1)^2}{n} f(n, \alpha, \beta) = (n^3 - 2n^2 - 4n + 8)t^{2n-2} + (n^3 + 2n^2)\beta^{2n-2} + (4n^2 - 2n)t^{n-1} \beta^{n-1} - n(n+1)^2 t^n \beta^{n-2} - (n+1)(5n^2 - 11n + 8)t^{n-2} \beta^n.\]
Thus
\[
\frac{(n-1)^2}{nt^{n-1} t^n} f(n, \alpha, \beta)
\]
(2.3)

\[
= (n^3 - 2n^2 - 4n + 8) \left( \frac{t}{\beta} \right)^{n-1} + (n^3 + 2n^2) \left( \frac{\beta}{t} \right)^{n-1} + (4n^2 - 2n)
\]

\[
- (n^2 + n) \left( \frac{t}{\beta} \right) - (n + 1)(5n^2 - 11n + 8) \left( \frac{\beta}{t} \right).
\]

The fact that \( \frac{t}{\beta} = (n-1) \frac{\alpha}{\beta} + 1 \) leads to the desired result. \( \square \)

**Remark 2.3.** It can be shown that the holomorphic automorphism group of \( X \) is not reductive (cf. [8, p. 138]). So the nonexistence of cscK metrics in these Kähler classes can also be obtained via the Matsushima-Lichnerowicz theorem ([11]). At the time of writing this note the author is not able to find out a compact Kähler manifold \((M, \omega)\) whose holomorphic automorphism group is reductive such that the Kähler class \([\omega]\) does not satisfy (1.4).

3. PROOF OF PROPOSITION 1.1 AND RELATED REMARKS

3.1. **Proof of Proposition 1.1.** Suppose \( M \) is a compact \( n \)-dimensional Kähler manifold with a Kähler metric \( g \). Under local complex coordinates \((z^1, \ldots, z^n)\), we write the Kähler metric \( g \), its Kähler form \( \omega \), the \((4,0)\)-type Kähler curvature tensor \( R \), the Ricci form \( \text{Ric}(\omega) \), and the (Riemannian) scalar curvature \( s(\omega) \) as follows.

\[
g = (g_{ij}) := (g_i^j \frac{\partial}{\partial z^i} \frac{\partial}{\partial \bar{z}^j}), \quad \omega = \frac{\sqrt{-1}}{2\pi} g_{ij} dz_i \wedge d\bar{z}_j, \quad R_{ijkl} := R(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^k}, \frac{\partial}{\partial \bar{z}^l})
\]

\[
\text{Ric}(\omega) = \frac{\sqrt{-1}}{2\pi} r_{ij} dz_i \wedge d\bar{z}_j = \frac{\sqrt{-1}}{2\pi} g^{kl} R_{ijkl} dz_i \wedge d\bar{z}_j,
\]

\[
s(\omega) := 2g^{ij} r_{ij}, \quad (g^{ij}) := (g_{ij})^{-1}.
\]

The pointwise squared norms of \( R \) and \( \text{Ric}(\omega) \) are defined as follows:

\[
|R|^2 := R_{ijkl} R_{pqrs} g^{ij} g^{pj} g^{kq} g^{rl}, \quad |\text{Ric}(\omega)|^2 := r_{ij} r_{pq} g^{ij} g^{pq}.
\]

It is well-known that these norms are independent of the choices of local coordinates. With these notions and symbols understood, we have the following two well-known facts, which essentially should be due to Apte ([1]). A detailed proof can be found in [15, p. 225-226].

**Lemma 3.1.** (Apte).

(3.1)

\[
\text{Ric}^2(\omega) \wedge \omega^{n-2} = \frac{\frac{s^2(\omega)}{4} - |\text{Ric}(\omega)|^2}{n(n-1)} : \frac{\omega^n}{n(n-1)}
\]

(3.2)

\[
c_2(\omega) \wedge \omega^{n-2} = \frac{\frac{s^2(\omega)}{4} - 2|Ric(\omega)|^2 + |R|^2}{2n(n-1)} : \frac{\omega^n}{2n(n-1)}
\]

**Remark 3.2.**

(1) In [15], the Kähler form is defined to be \( \frac{\sqrt{-1}}{2} g_{ij} dz_i \wedge d\bar{z}_j \) and so there is an extra factor \( \pi^2 \) in the expressions.
(2) Apte only stated (3.2) in [1, p. 150] in a slightly different form. In her notation in [1], “\( R \)” denotes the scalar curvature \( s(\omega) \) and “\( R_{ij} R^{ij} \)” denotes the pointwise squared norm of the Ricci tensor. Namely, if we denote by \( g_{ij} \) and \( r_{ij} \) the Riemannian metric \( g \) and the Ricci tensor under local real coordinates, “\( R_{ij} R^{ij} \)” means \( r_{ij} r_{pq} g^{ip} g^{jq} \), which can be shown to be exactly twice of \( |\text{Ric}(\omega)|^2 \).

Under the local coordinates \((z^1, \ldots, z^n)\), the decomposition \( R = S + P + B \) has the following expression (cf. [4, p. 86]).

\[
R_{i j k l} = S_{i j k l} + P_{i j k l} + B_{i j k l},
\]

where

\[
S_{i j k l} = \frac{1}{2n(n + 1)} (g_{ij} g_{kl} + g_{il} g_{kj}),
\]

\[
P_{i j k l} = \frac{1}{n + 2} (g_{ij} \bar{r}_{kl} + g_{kl} \bar{r}_{ij} + g_{il} \bar{r}_{kj} + g_{kj} \bar{r}_{il}), \quad \bar{r}_{ij} := r_{ij} - \frac{s(\omega)}{2n} g_{ij}.
\]

Clearly \( \omega \) is a Einstein metric if and only if the traceless \((1,1)\)-form

\[
\bar{\text{Ric}}(\omega) := \text{Ric}(\omega) - \frac{s(\omega)}{2n} \omega = \frac{\sqrt{-1}}{2\pi} \bar{r}_{ij} dz_i \wedge d\bar{z}_j \equiv 0.
\]

**Lemma 3.3.** The pointwise squared norms of \( \text{Ric}(\omega) \), \( \bar{\text{Ric}}(\omega) \), \( S \) and \( P \) satisfy

\[
|\text{Ric}(\omega)|^2 = |\bar{\text{Ric}}(\omega)|^2 + \frac{s^2(\omega)}{4n}, \quad |S|^2 = \frac{s^2(\omega)}{2n(n + 1)}, \quad |P|^2 = \frac{4}{n + 2} |\bar{\text{Ric}}(\omega)|^2,
\]

where \( |\bar{\text{Ric}}(\omega)|^2 := \bar{r}_{ij} \bar{r}_{pq} g^{ij} g^{pq} \) and the definitions of \(|S|^2\) and \(|P|^2\) are similar to that of \(|R|^2\).

**Proof.** The first two are well-known to experts. Since the coefficient \( 4/(n + 2) \) in the third one is crucial to our later use, here we give a detailed computation for it. For simplicity we can choose a local unitary frame field and so assume that \( g_{ij} = \delta_{ij} \). Thus

\[
|P|^2 = P_{i j k l} P_{j i k l} = \frac{1}{n + 2} (\delta_{ij} \bar{r}_{k l} + \delta_{kl} \bar{r}_{ij} + \delta_{il} \bar{r}_{kj} + \delta_{kj} \bar{r}_{il}) P_{j i k l}
\]

\[
= \frac{1}{n + 2} (\delta_{ij} \bar{r}_{k l} P_{j i k l} + \text{three other terms}).
\]

Note that under our assumption we have

\[
\sum \bar{r}_{ij} \bar{r}_{ji} = |\bar{\text{Ric}}(\omega)|^2 \quad \text{and} \quad \sum \bar{r}_{ii} = 0.
\]

Therefore,

\[
\delta_{ij} \bar{r}_{k l} \cdot (n + 2) P_{j i k l} = \delta_{ij} \delta_{ji} \bar{r}_{k l} \bar{r}_{ij} + \delta_{ij} \delta_{lk} \bar{r}_{ij} \bar{r}_{kj} + \delta_{ij} \delta_{il} \bar{r}_{k l} \bar{r}_{il} + \delta_{ij} \delta_{kl} \bar{r}_{k l} \bar{r}_{jk}
\]

\[
= n|\bar{\text{Ric}}(\omega)|^2 + 0 + |\bar{\text{Ric}}(\omega)|^2 + |\bar{\text{Ric}}(\omega)|^2
\]

\[
= (n + 2)|\bar{\text{Ric}}(\omega)|^2.
\]

The situations for the “other three terms” in (3.4) are similar. Thus we have

\[
|P|^2 = P_{i j k l} P_{j i k l} = \frac{1}{n + 2} \cdot 4 \cdot |\bar{\text{Ric}}(\omega)|^2 = \frac{4}{n + 2} |\bar{\text{Ric}}(\omega)|^2.
\]
Now we can prove our Proposition 1.1 via the following

**Proposition 3.4.**

\[(3.5)\]

\[c_1^2[\omega]^{n-2} = \int_M \left[ \frac{s^2(\omega)}{4n^2} - \frac{\overline{\text{Ric}}(\omega)^2}{n(n-1)} \right] \cdot \omega^n \]

\[(3.6)\]

\[c_2[\omega]^{n-2} = \int_M \left[ \frac{s^2(\omega)}{8n(n+1)} - \frac{\overline{\text{Ric}}(\omega)^2}{(n+2)(n-1)} + \frac{|B|^2}{2n(n-1)} \right] \cdot \omega^n \]

In particular,

\[(3.7)\]

\[nc_1^2[\omega]^{n-2} - (n+2)c_2[\omega]^{n-2} = \int_M \frac{s^2(\omega)}{8(n+1)} \cdot \omega^n - \int_M \frac{n+2}{2n(n-1)} |B|^2 \cdot \omega^n \]

and thus Proposition 1.1 holds.

**Proof.** Integrating (3.1) and (3.2) over \(M\) and using the relations (3.3) to replace the terms \(|\text{Ric}(\omega)|^2\) and \(|R|^2\) lead to (3.5) and (3.6).

Now we can correct the coefficient in [3, (2.80a)] by rewriting (3.5) and (3.6) as follows.

\[\frac{1}{(n-2)!}c_1^2[\omega]^{n-2} = \int_M \left[ \frac{n-1}{4n} s^2(\omega) - |\overline{\text{Ric}}(\omega)|^2 \right] \cdot \frac{\omega^n}{n!} \]

\[\frac{1}{(n-2)!}c_2[\omega]^{n-2} = \frac{1}{2} \int_M \left[ \frac{n-1}{4(n+1)} s^2(\omega) - \frac{2n}{n+2} |\overline{\text{Ric}}(\omega)|^2 + \frac{|B|^2}{n+2} \right] \cdot \frac{\omega^n}{n!} \]

Note that in [3, p. 80] the Kähler form is defined to be \(2\pi\omega\) in our notation and so there is an additional factor \(4\pi^2\). Clearly the correctness of the coefficient \(-\frac{2n}{n+2}\) before \(|\overline{\text{Ric}}(\omega)|^2\) is crucial to establish (3.7) and Proposition 1.1.

### 3.2. On a remark of Tian.

In Chapter 2 of his lecture notes [13], Tian discussed Yau’s Chern number inequality along the line of the uniformization theorem for Kähler manifolds with constant holomorphic sectional curvature. At the end of chapter 2 ([13, Remark 2.15]), he remarked that, if an \(n\)-dimensional compact Kähler manifold \((M, \omega)\) has constant scalar curvature \(s(\omega)\), then we have the following integral inequality

\[(3.8)\]

\[c_1^2[\omega]^{n-2} - c_2[\omega]^{n-2} \leq \frac{n+2}{8n^2(n+1)} s^2(\omega)[\omega]^n, \]

where the equality holds if and only if \(\omega\) is of constant holomorphic sectional curvature. Using our notation the Kähler form and the scalar curvature in [13] are defined to be \(\pi\omega\) and one half of \(s(\omega)\) respectively.

Indeed, (3.8) can be proved by using (3.5) and (3.6) directly:

\[c_1^2[\omega]^{n-2} - c_2[\omega]^{n-2} = \int_M \left[ \frac{n+2}{8n^2(n+1)} s^2(\omega) - \frac{2}{n(n+2)(n-1)} |\overline{\text{Ric}}(\omega)|^2 - \frac{|B|^2}{2n(n-1)} \right] \omega^n \]

\[\leq \int_M \frac{n+2}{8n^2(n+1)} s^2(\omega) \omega^n \]

\[= \frac{n+2}{8n^2(n+1)} s^2(\omega)[\omega]^n, \quad (\omega \text{ is cscK})\]
where the equality holds if and only if \( P = B = 0 \) and so \( \omega \) has constant holomorphic sectional curvature.

Under the assumption that \( \omega \) be a cscK metric, we can compare (1.4) and (3.8) by rewriting (1.4) as follows.

\[
\frac{c_2^n - c_2^n}{n} \leq \frac{n(c_1^n - 1)^2}{2(n + 1)|\omega|^n} + \frac{2}{n} c_2^n
\]

(3.9)

\[
= \frac{1}{8n(n + 1)} s^2(\omega)|\omega|^n + \frac{2}{n} c_2^n.
\]

Now the difference of the upper bounds in (3.9) and (3.8) is

\[
\frac{2}{n} c_2^n - \frac{2}{8n^2(n + 1)} s^2(\omega)|\omega|^n
\]

\[
= \frac{2}{n} \left( c_2^n - \frac{1}{8n(n + 1)} s^2(\omega)|\omega|^n \right)
\]

\[
= \int_M \left[ - \frac{2|\text{Ric}(\omega)|^2}{n(n + 2)(n - 1)} + \frac{|B|^2}{n^2(n - 1)} \right] \cdot \omega^n \quad \text{(by (3.6))},
\]

whose sign can be either negative or positive.

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