Anomaly analysis of Hawking radiation from Kaluza-Klein black hole with squashed horizon

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Abstract

Considering gravitational and gauge anomalies at the horizon, a new method that to derive Hawking radiations from black holes has been developed by Wilczek et al. In this paper, we apply this method to non-rotating and rotating Kaluza-Klein black holes with squashed horizon, respectively. For the rotating case, we found that, after the dimensional reduction, an effective $U(1)$ gauge field is generated by an angular isometry. The results show that the gauge current and energy-momentum tensor fluxes are exactly equivalent to Hawking radiation from the event horizon.

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I. INTRODUCTION

Hawking [1] has been derived that black hole can radiate from the event horizon like a black body at the temperature \( T = \frac{\hbar}{2\pi} \) using the method of quantum field theory in curved spacetime. This rises the interest of investigation for the Hawking radiation. An elegant derivation is the tunneling method, which based on pair creations of particles and antiparticles near the horizon through calculating WKB amplitudes for classically forbidden trajectories [2, 3]. It is generally believed that the Hawking radiation should be determined only by some universal quantum effects just on the horizon.

Recent proposal of deriving Hawking radiation via gravitational and gauge anomalies proposed by Wilczek and his collaborators [4, 5] has been attracted a lot of interests. This rejuvenates the interest of investigation for Hawking radiation. For various types of black holes, these investigations have been carried out [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. In fact, the anomaly analysis can be traced back to Christensen and Fulling’s early work [21], in which they suggested that there exists a relation between the Hawking radiation and anomalous trace of the field under the condition that the covariant conservation law is valid. Wilczek et al. showed that the Hawking fluxes of the black hole can be accounted by the gauge and gravitational anomalies at the horizon. Their basic idea is that, near the horizon, a quantum field in a black hole background can be effectively described by an infinite collection of (1+1)-dimensional fields on \((t, r)\) space, where \(r\) is the radial direction. In this two-dimensional reduction, because all the ingoing modes can not classically affect physics outside the horizon, the two-dimensional effective action in the exterior region becomes anomalous with respect to gauge or general coordinate symmetries. To cancel the anomaly, they found the Hawking flux is universally determined only by the value of anomalies at the horizon. The anomaly method is believed to be universal and independent of the gravity theory and the dimension of spacetime. On the other hand, Banerjee and Kulkarni [22, 23] pointed that the Hawking radiation can be obtained only use the covariant expression.

In Ref. [11], the original work of [4, 5] was extended to the more general case where the metric determinant \( \sqrt{-g} \neq 1 \), and the case in which \( \sqrt{-g} \) vanishes at the horizon had also been investigated [12] in details. Recently, in [24], a non-spherical topological black hole was considered and the Hawking radiation via gravitational anomalies was derived. The aim of this paper is to generalize the anomaly method to non-rotating and rotating Kaluza-Klein
black holes with squashed horizon. At first, we will perform a dimensional reduction of an action given by a scalar field minimally coupled to gravity in the background of a (4+1)-dimensional Kaluza-Klein black hole. The result shows that the theory is reduced to an effective theory of an infinite collection of (1+1)-dimensional scalar fields near the horizon. Then, through studying the gauge and gravitational anomalies, we obtain the Hawking flux. Especially, for the rotating case, we found that, after the dimensional reduction, an effective $U(1)$ gauge field is generated by an angular isometry. The azimuthal quantum number $\lambda$ serves as the charge of each partial wave, and this result accords with that of 25.

The paper is organized as follows. In section II, we review the basic properties of the (4+1)-dimensional non-rotating Kaluza-Klein black hole with squashed horizon and carry out the dimensional reduction. It’s Hawking radiation is derived via anomalies. The same calculation is applied to the rotating Kaluza-Klein black hole with squashed horizon in section III. Finally, the paper ends with a brief summary.

II. QUANTUM FIELD IN NON-ROTATING KALUZA-KLEIN BLACK HOLE WITH SQUASHED HORIZON

We firstly review some main results of the Kaluza-Klein black hole with squashed horizon. Some relevant discussions can be found in 26, 27, 28, 29. The metric of Kaluza-Klein black hole with squashed horizon is given by

$$ds^2 = \frac{k^2(r)}{f(r)}dt^2 - \frac{r^2}{4}k(r)d\Omega^2 + \frac{r^2}{4}(d\psi + \cos\theta d\phi)^2,$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element on the unit sphere. The coordinate ranges are

$$\theta \in \left[ 0, \pi \right), \phi \in \left[ 0, 2\pi \right), \psi \in \left[ 0, 4\pi \right).$$

The functions $f(r)$ and $k(r)$ take the following forms:

$$f(r) = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^4},$$

$$k(r) = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{(r^2 - r_\infty^2)^2}.$$  

There are three coordinate singularities $r_+$, $r_-$ and $r_\infty$, which satisfy $r_- < r_+ < r_\infty$, $0 < r < r_\infty$. $r = r_+$ and $r = r_-$ are outer and inner event horizons of the black hole. $r = r_\infty$ is
the spatial infinity. They relate to the mass $M$, charge $Q$ by

$$M = \frac{3\pi(r_+^2 + r_-^2 - 2r^2)}{8G\sqrt{(r_+^2 - r_+^2)r_+^2}}, \quad Q = \frac{\sqrt{3\pi r_+}}{2G}.$$

It has been shown that the shape of the horizon is a deformed sphere determined by the factor $k(r_+)$. The non-vanishing gauge potential is given by

$$A_t = \pm \frac{\sqrt{3}}{2} \sqrt{r_+^2 - \frac{1}{r_+^2}}. \quad (4)$$

In this paper, we use the metric

$$ds^2 = -F(\rho) d\tau^2 + \frac{K^2(\rho)}{F(\rho)} d\rho^2 + \rho^2 K^2(\rho) d\Omega^2 + \frac{r_+^2}{4K^2(\rho)} (d\psi + \cos\theta d\phi)^2, \quad (5)$$

which can be deduced from (1) via coordinate transformation

$$\tau = \frac{2\rho_0 t}{r_+}, \quad \rho = \frac{\rho_0 r^2}{(r_+^2 - r^2)}, \quad (6)$$

where $\rho_0^2 = (r_+^2 - r_+^2)(r_+^2 - r_-^2)/(4r_+^2)$. The new functions $F(\rho)$ and $K(\rho)$ are given by

$$F = \left(1 - \frac{\rho_+}{\rho}\right)\left(1 - \frac{\rho_-}{\rho}\right), \quad K^2 = 1 + \frac{\rho_0}{\rho}, \quad (7)$$

and $\rho_0 = \rho_0 r_+^2/(r_+^2 - r_+^2)$. In this coordinate, the non-vanishing gauge potential $A_\tau$, the black hole mass $M$, the charge $Q$ and the surface gravity $\kappa$ of the outer horizon are respectively given by

$$A_\tau = \pm \frac{\sqrt{3}}{2} \sqrt{\rho_+ \rho_-}, \quad M = \frac{3\pi r_+}{4G}(\rho_+ + \rho_-), \quad Q = \pm \frac{\sqrt{3\pi r_+}}{G} \sqrt{\rho_+ \rho_-}, \quad \kappa = \frac{(\rho_+ - \rho_-)}{2\rho_+^2} \sqrt{\rho_+ + \rho_-}. \quad (8)$$

Next, we consider a complex scalar field in the squashed Kaluza-Klein black hole background. The action functional is given by

$$S[\varphi] = -\frac{1}{2} \int d^5 x \sqrt{-g} g^{\mu\nu} (D_\mu \varphi)^* D_\nu \varphi$$

$$= \frac{1}{2} \int d^5 x \varphi^* (\partial_\mu - ieA_\mu) [\sqrt{-g} g^{\mu\nu} (\partial_\nu - ieA_\nu) \varphi]$$

$$= \frac{1}{2} \int d^5 x \rho_+^2 K^2 r_+ \sin \theta \varphi^* \left\{ -\frac{1}{F}(\partial_\tau - ieA_\tau)^2 \varphi + \frac{1}{\rho_+^2 K^2} \partial_\rho \rho_+^2 F \partial_\rho \varphi \right.$$

$$+ \frac{1}{\rho_+^2 K^2} \left[ \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \left( \frac{1}{\sin \theta} \partial_\phi - \frac{\cos \theta}{\sin \theta} \partial_\psi \right)^2 \right] \varphi + \frac{4K^2}{r_+^2} \partial_\psi^2 \varphi \left. \right\}. \quad (9)$$
Performing the partial wave decomposition

$$\varphi = \sum_{lm\lambda} \varphi_{lm\lambda}(\tau, \rho) e^{i\lambda\psi} e^{im\phi} S^\lambda_{lm}(\theta),$$

(10)

where $S(\theta)$ is the so-called spin-weighted spherical function satisfying the equation (recent paper can be found in \[30\])

$$\frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta S) - \frac{(m - \lambda \cos \theta)^2}{\sin^2 \theta} S + [l(l + 1) - \lambda^2] S = 0,$$

(11)

The orthogonality condition reads

$$\int S^\lambda_{lm} \overline{S}^\lambda_{l'm'} d\theta = \delta_{ll'} \delta_{mm'},$$

(12)

where

$$\overline{S}^\lambda_{lm} = (-1)^{\lambda+m} S^{-\lambda}_{l(-m)}.$$

(13)

Define the tortoise coordinate as

$$\frac{d\rho_*}{d\rho} = \frac{K}{F}.$$

(14)

With the tortoise coordinate $\rho_*$, we can write the action (9) as

$$S[\varphi] = -\frac{1}{2} \sum_{lm\lambda} \int d\tau d\rho_* K^2 r_\infty \rho \varphi_{lm\lambda}^* \left[ - (\partial_\tau - i e A_\tau)^2 \varphi_{lm\lambda} + \frac{1}{\rho^2 K} \partial_\rho \rho^2 K \partial_\rho \varphi_{lm\lambda} ight.$$

$$- \left. \frac{F[l(l + 1) - \lambda^2 K^2 + \frac{4\lambda^2 K^2}{\rho^2 K^2} + \frac{4\lambda^2 K^2}{r_\infty^2}]}{\rho^2 K^2} \varphi_{lm\lambda} \right],$$

(15)

where we have integrated the angle coordinate parts. Near the horizon, the factor $F(\rho)$ vanishes. So, the last term can be ignored. Back to the coordinate $\rho$, effective two-dimensional action functional is simplified as

$$S[\varphi] = \frac{1}{2} \sum_{lm\lambda} \int d\tau d\rho K r_\infty \rho^2 \varphi_{lm\lambda}^* \left[ - \frac{K}{F} (\partial_\tau - i e A_\tau)^2 \varphi_{lm\lambda} + \partial_\rho \left( \frac{F}{K} \partial_\rho \varphi_{lm\lambda} \right) \right].$$

(16)

After undergoing the dimensional reduction near the horizon, each partial wave of the scalar field can be effectively described by an infinite collection of complex scalar field in the background of a $(1+1)$-dimensional metric. The dilaton field $\Psi$, and the gauge potential $A_\mu$:

$$ds^2 = -\frac{F}{K} d\tau^2 + \frac{K}{F} d\rho^2,$$

$$\Psi = K \rho^2, \quad A_\tau = \pm \sqrt{3} \frac{\sqrt{\rho^2 - \rho^2}}{\rho}, \quad A_\rho = 0.$$

(17)
For the $U(1)$ gauge current $J_\mu$, the consistent form of $d = 2$ abelian anomaly is considered in [4, 5]

$$\nabla_\mu J^\mu = \pm \frac{e^2}{4\pi} \epsilon^{\mu\nu} \partial_\mu A_\nu, \quad (18)$$

where $+(-)$ respectively corresponds to the left(right)-handed fields and $\epsilon^{\mu\nu}$ is antisymmetric with $\epsilon^{01} = 1$. The current $J^\mu$ is not a covariant current. However, a covariant current can be defined as [5]

$$\tilde{J}^\mu = J^\mu \mp \frac{e^2}{4\pi} A^\lambda \epsilon^{\lambda\mu}, \quad (19)$$

which satisfies

$$\nabla_\mu \tilde{J}^\mu = \pm \frac{e^2}{4\pi} \epsilon_{\mu\nu} F^{\mu\nu}. \quad (20)$$

The consistent current of (18) can be written as

$$J^\mu = J^\mu_{(o)} \Theta_+(\rho) + J^\mu_{(H)} H(\rho), \quad (21)$$

where $\Theta_+(\rho) = \Theta(\rho - \rho_+ - \epsilon)$ and $H(\rho) = 1 - \Theta_+(\rho)$. $J^\mu_{(o)}(\rho)$, the current outside the horizon, is conserved

$$\partial_\rho [J^\rho_{(o)}] = 0. \quad (22)$$

While $J^\mu_{(H)}(\rho)$, the current near the horizon, satisfies the anomalous equation

$$\partial_\rho [J^\rho_{(H)}] = \frac{e^2}{4\pi} \partial_\rho A_\tau. \quad (23)$$

These two equations can be easily integrated as

$$J^\rho_{(o)} = c_o, \quad J^\rho_{(H)} = c_H + \frac{e^2}{4\pi} (A_\tau(\rho) - A_\tau(\rho_+)). \quad (24)$$

where $c_o$ and $c_H$ are integration constants. The variation of quantum effective action $W$ under a gauge transformation with gauge parameter $\zeta$ is given by

$$- \delta W = \int d^2 x \zeta \nabla_\mu J^\mu$$

$$= \int d^2 x \zeta \left[ \partial_\rho \left( \frac{e^2}{4\pi} A_\tau H(\rho) + \delta(\rho - \rho_+ - \epsilon) \left( (J^\rho_{(o)} - J^\rho_{(H)}) + \frac{e^2}{4\pi} A_\tau \right) \right) \right], \quad (25)$$
The coefficient of the delta function in the above equation should vanish

\[ c_o = c_H - \frac{e^2}{4\pi} A_\tau(\rho_+). \] (26)

Imposing the condition that the covariant current \( \tilde{J}^\rho = J^\rho + \frac{e^2}{4\pi} A_\tau(\rho)H(\rho) \) vanishes at the horizon, one can fix the value of the consistent current at the horizon

\[ c_H = -\frac{e^2}{4\pi} A_\tau(\rho_+). \] (27)

The charge flux can be obtained

\[ c_o = -\frac{e^2}{2\pi} A_\tau(\rho_+) = \frac{e^2}{2\pi} V, \] (28)

which agrees with the current flow associated with the Hawking thermal (blackbody) radiation including a chemical potential. \( V = -A_\tau(\rho_+) = \mp \frac{\sqrt{3}}{2} \sqrt{\rho_+} \) is the electrostatic potential of the black hole.

With the presence of gauge fields, the energy-momentum tensor does not preserve the current conservation law and the corresponding anomalous Ward identity is \( \nabla_\mu T^\mu_\nu = F^\mu_\nu J^\mu + A^\nu. \) Considering the outside the horizon and near the horizon, the energy-momentum tensor can be written as

\[ T^\mu_\nu = T^\mu_\nu(\Theta_+ + H(\rho)). \] (29)

Outside the horizon, we have

\[ \partial_\rho [T^\rho_\tau(\Theta_+)] = J^\rho_\tau A_\tau. \] (30)

Near the horizon, we have the anomalous equation

\[ \partial_\rho [T^\rho_\tau(H)] = [J^\rho_\tau A_\tau + A_\tau \partial_\rho J^\rho_\tau] + \partial_\rho N^\rho_\tau, \] (31)

where \( N^\rho_\tau \) is given by

\[ N^\rho_\tau = \frac{1}{96\pi} \epsilon^{\mu\nu} \partial_\nu \Gamma^\rho_{\tau\mu}. \] (32)

These two equations can also be integrated as

\[ T^\rho_\tau(\Theta) = a_o + c_o A_\tau, \]

\[ T^\rho_\tau(H) = a_H + \int_{\rho_+}^\rho d\rho' \partial_\rho \left( c_o A_\tau + \frac{e^2}{4\pi} A^2_\tau + N^\rho_\tau \right). \] (33)
Under the infinitesimal general coordinate transformation \( s \), the effective action changes as

\[
- \delta W = \int d^2x \xi^\tau \nabla_\mu T^\mu_\tau
\]

\[
= \int d^2x \xi^\tau \left\{ c_0 \partial_\mu A_\tau + \partial_\rho \left[ \frac{e^2}{4\pi} A^2_\tau + N^\rho_\tau H \right] \right. \\
+ \left. \delta(\rho - \rho_+ - \epsilon) \left[ (T^\rho_\tau)_H - T^\rho_\tau(o) \right] + N^\rho_\tau + \frac{e^2}{4\pi} A^2_\tau \right\}.
\]

The coefficient of the delta function term should also vanish at the horizon

\[
a_o = a_H + \frac{e^2}{4\pi} A^2_\tau(\rho_+) - N^\rho_\tau(\rho_+).
\]

We impose a vanishing condition for the covariant energy-momentum tensor \( \tilde{T}^\rho_\tau = T^\rho_\tau + \frac{1}{192\pi} ((\frac{F}{K})'' - 2((\frac{F}{K})')^2) \) at the horizon, which gives the equation

\[
a_H = 2N^\rho_\tau(\rho_+).
\]

The total flux of the energy-momentum tensor is given by

\[
a_o = \frac{e^2}{4\pi} A^2_\tau(\rho_+) + N^\rho_\tau(\rho_+),
\]

From Eq. (32), we can calculate

\[
N^\rho_\tau(\rho_+) = \frac{1}{192\pi} \left( \frac{F}{K} \right)^2 \bigg|_{\rho_+} = \frac{\kappa^2}{192\pi}.
\]

So we have

\[
a_o = \frac{e^2}{4\pi} V^2 + \frac{\kappa^2}{48\pi},
\]

where \( V^2 = \frac{3}{4} \frac{\rho_+ - \rho}{\rho_+} \) and \( \kappa \) is the surface gravity.

### III. QUANTUM FIELD IN A ROTATING KALUZA-KLEIN BLACK HOLE WITH SQUASHED HORIZON

The five-dimensional rotating squashed Kaluza-Klein black hole is described by

\[
ds^2 = -dt^2 + \frac{\Sigma_0}{\Delta_0} k(r)^2 dr^2 + \frac{r^2 + a^2}{4} \left[ k(r)(\sigma_1^2 + \sigma_2^2) + \sigma_3^2 \right] + \frac{M}{r^2 + a^2} (dt - \frac{a}{2} \sigma_3)^2,
\]

with

\[
\sigma_1 = -\sin \psi d\theta + \cos \psi \sin \theta d\phi, \\
\sigma_2 = \cos \psi d\theta + \sin \psi \sin \theta d\phi, \\
\sigma_3 = d\psi + \cos \theta d\phi.
\]

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The functions are given by

\[ \Sigma_0 = r^2(r^2 + a^2), \]
\[ \Delta_0 = (r^2 + a^2)^2 - Mr^2, \]
\[ k(r) = \frac{(r_\infty^2 - r_+^2)(r_\infty^2 - r_-^2)}{(r_\infty^2 - r^2)^2}, \]

where \( M \) and \( a \) correspond to mass and angular momenta, respectively. \( r_+ \) and \( r_- \) are outer and inner event horizons of the black hole. \( r_\infty \) is the spatial infinity. The relations between the parameters are \( a^4 = (r_+ r_-)^2, M - 2a^2 = r_+^2 + r_-^2 \). It can be seen that the shape of black hole horizon is also deformed by the parameter \( k(r) \).

The determinant of the metric is

\[ g = -\frac{1}{64} k^4 r^2 (a^2 + r^2)^2 \sin^2 \theta, \]
and non-zero metric coefficients read

\[ g^{00} = -1 - \frac{M(a^2 + r^2)}{\Delta_0}, \]
\[ g^{04} = g^{40} = -\frac{2aM}{\Delta_0}, \]
\[ g^{11} = \frac{\Delta_0}{k^2 \Sigma_0}, \]
\[ g^{22} = \frac{4}{k(a^2 + r^2)}, \]
\[ g^{33} = \frac{4}{k(a^2 + r^2) \sin^2 \theta}, \]
\[ g^{34} = g^{43} = -\frac{4 \cos \theta}{k(a^2 + r^2) \sin^2 \theta}, \]
\[ g^{44} = \frac{4(a^2 - M + r^2)}{\Delta_0} + \frac{4 \cos^2 \theta}{k(a^2 + r^2) \sin^2 \theta}. \]

We note that \( g^{03} \) vanishes, which is different from that in the usual five dimensional Kerr black hole. So, this implies that there exists some special properties in such spacetime.

Now, we consider the complex scalar field in a rotating Kaluza-Klein black hole background. The action functional is given by

\[ S = \frac{1}{2} \int d^5x \sqrt{-g} \Phi^* \nabla^2 \Phi + S_{\text{int}}, \]
where \( S_{\text{int}} \) includes mass and interaction terms. Near the outer horizon, the first term of (45) gives a dominant contribution to the action and thus we can ignore the mass and interaction
terms $S_{int}$. And $\nabla^2$ is the Laplace-Beltrami operator defined by

$$\nabla^2 \equiv \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu). \quad (46)$$

Hence, near the outer horizon, the action becomes

$$S[\Phi] = \frac{1}{2} \int d^5 x \Phi^* \left\{ \sqrt{-g} g^{00} \partial_t^2 + 2\sqrt{-g} g^{04} \partial_t \partial_\psi + \partial_r (\sqrt{-g} g^{11} \partial_r) + \frac{kr \sin \theta}{2} \left[ \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + (\frac{1}{\sin \theta} \partial_\phi - \frac{\cos \theta}{\sin \theta} \partial_\psi) \right]^2 \right. \right.$$  
$$+ \left. \frac{k(a^2 + r^2)(a^2 - M + r^2)}{\Delta_0} \partial_\psi^2 \right\} \Phi. \quad (47)$$

Performing the partial wave decomposition

$$\Phi(t, r, \theta, \phi, \psi) = \sum_{\lambda m} A^\lambda_{lm}(t, r) e^{im\phi + i\lambda \psi} S^\lambda_{lm}(\theta), \quad (48)$$

where function $S(\theta)$ is the spin-weighted spherical function and satisfies equation (11). The physics near the horizon can be effectively described by an infinite collection of massless $(1+1)$-dimensional fields with the following action

$$S[\Phi] = \frac{1}{2} \sum_{\lambda m} \int dt dr (A^\lambda_{lm})^* \left\{ \sqrt{-g} g^{tt} \partial_t + i\lambda A^\lambda_{lm} \right. \right.$$  
$$\left. \left[ \frac{1}{a^4 + r^4 + a^2(M + 2r^2)} \right]^2 \right. \right.$$  
$$+ \left. \partial_r \left( \frac{\sqrt{-g}}{\sin \theta} g^{11} \partial_r \right) - \left[ \frac{kr}{2} E_{\lambda m} + \frac{\lambda^2 k^2 r(a^2 + r^2)^2}{2(a^4 + r^4 + a^2(M + 2r^2))} \right] \right\} A^\lambda_{lm}, \quad (49)$$

where $(A^\lambda_{lm})^* = 2\pi (-1)^{l-m} A^{-\lambda}_{(l-m)}$ and $E_{\lambda m} = -\{l(l+1) - \lambda^2\}$. Define the tortoise coordinate as

$$\frac{dr_*}{dr} = \frac{1}{\Delta_0}. \quad (50)$$

The factor $\Delta_0 = (r^2 + a^2)^2 - Mr^2$ vanishes near the horizon. Carrying out the same method in Section (III), one will see that the last term in the action (49) can be ignored. Then the effective two-dimensional action (49) can be simplified as

$$S[\Phi] = \frac{1}{2} \sum_{\lambda m} \int dt dr \Psi (A^\lambda_{lm})^* \left\{ g^{tt} (\partial_t + i\lambda A_t)^2 + \partial_r (g^{rr} \partial_r) \right\} A^\lambda_{lm}, \quad (51)$$

where

$$g^{tt} = -(g^{rr})^{-1} = -\left[ \frac{(r^2 + a^2)^2 + Ma^2}{(r^2 + a^2)^2 - Mr^2} \right]^{1/2} k, \quad (52)$$

$$\Psi = \frac{1}{8} kr (a^2 + r^2) \left[ \frac{(r^2 + a^2)^2 + Ma^2}{\Sigma_0^{1/2}} \right]^{1/2}, \quad (53)$$

$$A_t = \frac{2aM}{(r^2 + a^2)^2 + Ma^2}. \quad (54)$$
Hence, after undergoing the dimensional reduction near the horizon, each partial wave of the scalar field can be effectively described by an infinite collection of complex scalar field in the background of the dilaton field $\Psi$ (53), the gauge potential $A_\mu = (A_t, 0)$ (54), and a $(1+1)$-dimensional metric described by

$$ds^2 = -\frac{(r^2 + a^2)^2 - M r^2}{[(r^2 + a^2)^2 + M a^2]^{1/2} \Sigma_0^{1/2} k} dt^2 + \frac{[(r^2 + a^2)^2 + M a^2]^{1/2} \Sigma_0^{1/2} k}{(r^2 + a^2)^2 - M r^2} dr^2. \quad (55)$$

Now, the total flux of the energy-momentum tensor for the rotating Kaluza-Klein black hole with squashed horizon is calculated as

$$a_o = \frac{e^2}{4\pi} A_t^2(r_+) + N_\ell^\rho(r_+)$$

$$= \frac{e^2}{4\pi} \left( \frac{2 a M}{a^4 + r_+^4 + a^2(M + 2 r_+^2)} \right)^2 + \frac{\pi}{12\beta^2}, \quad (56)$$

where

$$\beta = \frac{2\pi}{(r_+ + r_-)} \frac{(r_+^2 - r_-^2)}{(r_+ - r_-)} \frac{(r_\infty^2 - r_-^2)}{(r_\infty^2 - r_+^2)}. \quad (57)$$

Note that the flux is proportional to $(T_H)^2$ with $T_H = 1/\beta$ the Hawking temperature of the black hole. The angular velocity on the event horizon is given by

$$\Omega = \frac{2 a M}{a^4 + r_+^4 + a^2(M + 2 r_+^2)}. \quad (58)$$

It is worth to point out that the Hawking temperature $T_H$ here is different from the results in Refs. [27] and [28]. In fact, this is because the different selection of the scale of time. If considering a time coordinate transformation $t = \sqrt{\alpha} \tau$ with $\alpha = \sqrt{\frac{(r_\infty^2 + a^2)^2 + M a^2}{(r_\infty^2 + a^2)^2 - M r_\infty^2}}$, one will obtain the same Hawking temperature with Ref. [27]:

$$T'_H = \frac{1}{\beta} \frac{1}{\alpha} \frac{(r_+ - r_-) \sqrt{(r_\infty^2 + r_+ r_-)^2 + r_+ r_- (r_+ + r_-)^2}}{2\pi r_+ (r_+ + r_-) (r_\infty^2 - r_-^2)} \sqrt{\frac{r_\infty^2 - r_-^2}{r_\infty^2 - r_+^2}}. \quad (59)$$

IV. SUMMARY

Using the method of quantum anomalies, we obtained the Hawking flux from (4+1)-dimensional Kaluza-Klein black hole with squashed horizon. The result shows that there is no difference between the squashed horizon and the normal one, this may be the reason that they are equivalent in topology. The method adopted here only considers the quantum
anomalies near the horizon, so it is generally believed to be universal. We also extended this method to the (4+1)-dimensional rotating Kaluza-Klein black hole with squashed horizon. By integrating the action given by a scalar field minimally coupled to gravity in the background of black hole near the horizon, we obtained a (1+1)-dimensional effect action. That means physics near the horizon can be described with an infinite collection of massless (1+1)-dimensional scalar fields. We also found that, after the dimensional reduction, an effective $U(1)$ gauge field is generated by an angular isometry. The azimuthal quantum number $\lambda$ serves as the charge of each partial wave. The results show that the gauge current and energy-momentum tensor fluxes are exactly equivalent to Hawking radiation from the event horizon for both the non-rotating and rotating Kaluza-Klein black holes with squashed horizon.

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