IDEMPOTENTS IN INTEGRAL RING OF DIHEDRAL QUANDLE

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Abstract. We provide the structure of some idempotents in the integral ring of a dihedral quandles, for odd case. The solution is complete in case $n = 5$. This produces an example of a connected quandle $X$ such that $\text{Aut}(X) \not\cong \text{Aut}(\mathbb{Z}[X])$.

1. Introduction

A quandle is a pair $(A, \cdot)$ such that ‘$\cdot$’ is a binary operation satisfying

1) the map $S_a : A \rightarrow A$, defined as $S_a(b) = b \cdot a$ is an automorphism for all $a \in A$,

2) for all $a \in A$, we have $S_a(a) = a$.

The automorphisms $S_a$ are called the inner automorphisms of the quandle $A$. A quandle is said to be connected if the group of inner automorphisms acts transitively on the quandle. Let $\mathbb{Z}_n$ denote the cyclic group of order $n$. Then defining $a \cdot b = 2b - a$ defines a quandle structure on $A = \mathbb{Z}_n$. This is known as the dihedral quandle, to be denoted by $Q_n$. Any dihedral quandle of odd order is connected. For other examples see [BPS19].

To get a better understanding of the structure, a theory parallel to group rings was introduced by Bardakov, Passi and Singh in [BPS19]. The quandle ring of a quandle $A$ is defined as follows. Let $R$ be a commutative ring. Consider

$$R[A] = \left\{ \sum_i r_i a_i : r_i \in R, a_i \in A \right\}.$$
Then this is an additive group in usual way. Define multiplication as
\[
\left( \sum_i r_i a_i \right) \cdot \left( \sum_j s_j a_j \right) = \sum_{i,j} r_i s_j (a_i \cdot a_j).
\]
The terms idempotents and ring automorphism are of usual meaning, as in ring theory. Since \( \mathbb{Z}[A] \) is an infinite ring, it is usually difficult to conclude about the idempotent structure of the ring. Also since in quandle \( a \ast a = a \) for all \( a \in A \), these are trivial idempotents in the quandle ring \( \mathbb{Z}[A] \).

Here we introduce the notion of an adjacency matrix of a quandle. This helps us in finding some of the idempotents in integral quandle ring of a dihedral quandle of odd order. Using this we further show that \( \mathbb{Z}[Q_5] \) has only trivial idempotents. This further helps in finiding the automorphism group of \( \mathbb{Z}[Q_5] \). This answers a question of Elhamdadi in negation, whether \( \text{Aut}(\mathbb{Z}[X]) \cong \text{Aut}(X) \), for a connected quandle \( X \).

### 2. Adjacency Matrix of a Quandle

The following definition is from [NH05]

**Definition 2.1.** Let \((X, \ast)\) be a quandle of size \( n \). Define the \( n \times n \) matrix \( A = (a_{ij}) \) with \( a_{ij} = x_i \ast x_j \). This matrix will be called as the adjacency matrix of the quandle.

**Example 2.2.** Let \( Q_5 \) be the dihedral quandle of size 5. Then the adjacency matrix of \( Q_5 \) will be given by

\[
\begin{pmatrix}
0 & 2 & 4 & 1 & 3 \\
4 & 1 & 3 & 0 & 2 \\
3 & 0 & 2 & 4 & 1 \\
2 & 4 & 1 & 3 & 0 \\
1 & 3 & 0 & 2 & 4
\end{pmatrix}.
\]

**Lemma 2.3.** Let \( n \) be an odd integer. Then the rows and columns of the adjacency matrix corresponding to a dihedral quandle of order \( n \), is determined by two elements \( \rho, \sigma \in S_n \). Here \( S_n \) denotes the symmetric group on letters \( \{0, 1, \ldots, n-1\} \). Furthermore, the first row is determined by

\[
\rho = (0 \ 2 \ \cdots \ n-1 \ 1 \ 3 \ \cdots \ n-2),
\]

and the first column is determined by

\[
\sigma = (0 \ 1 \ 2 \ \cdots \ n-1)^{-1}.
\]
Proof. We first prove that all the rows of the adjacency matrix is determined by one row. To achieve this let us assume \( x_i * x_j = x_k \) and \( x_i * x_{j+1} = x_l \). We need to show that \( x_m * x_r = x_k \) implies \( x_m * x_{r+1} = x_l \). From the given three conditions we have

\[
2j - i = k, \quad 2j + 2 - i = l, \quad 2r - m = k \pmod{n}.
\]

Hence we have \( 2r + 2 - m = l \pmod{n} \). Now we compute the first row of the matrix. This can be easily shown to be equal to \( \rho \). The comment about \( \sigma \) follows similarly.

\[\square\]

Lemma 2.4. Let \( n \) be an even integer. Then the rows and columns of the adjacency matrix corresponding to a dihedral quandle of order \( n \), is determined by three elements \( \rho_1, \rho_2, \sigma \in S_n \). Here \( S_n \) denotes the symmetric group on letters \( \{0, 1, \ldots, n-1\} \). Furthermore, the row is determined by

\[
\rho_1 = (0 \ 2 \ \cdots \ n-2) \\
\rho_2 = (1 \ 3 \ \cdots \ n-1),
\]

and the column is determined by

\[
\sigma = (0 \ 1 \ 2 \ \cdots \ n-1)^{-1}.
\]

Proof. Same as the last lemma and hence left to the reader.

\[\square\]

Lemma 2.5. Let \( Q_n \) denote the dihedral quandle of odd order \( n \). The \( \sum_{i=0}^{n-1} \alpha_i x_i \in \mathbb{Z}[Q_n] \) is an idempotent if and only if \( (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \) is a solution to the system of equations given by

\[
B \cdot T = T, \quad \sum_{i=0}^{n-1} t_i = 1,
\]

where \( T = (t_0, t_1, \ldots, t_{n-1})^t \) and \( b_{ij} = t_{a_{ij}} \) with \( A = (a_{ij}) \) being the adjacency matrix of \( Q_n \).

Proof. Let \( \sum_{i=0}^{n-1} \alpha_i x_i \in \mathbb{Z}[Q_n] \) is an idempotent element. Then we get that

\[
\left( \sum_{i=0}^{n-1} \alpha_i x_i \right)^2 = \sum_{i=0}^{n-1} \alpha_i x_i
\]

\[
\Rightarrow \sum_{i=0}^{n-1} \alpha_i^2 x_i + \sum_{0 \leq i < j \leq n} \alpha_i \alpha_j (x_i * x_j + x_j * x_i) = \sum_{i=0}^{n-1} \alpha_i x_i.
\]
Now we want to compare coefficient of $x_k$ of both side. Now note that for all $0 \leq i \leq n - 1$, we have the equality

$$(x_k \ast x_i) \ast x_i = x_k.$$ 

Hence comparing the both side we get that

$$\alpha_k = \sum_{i=0}^{n-1} \alpha_{k \ast i} \alpha_i.$$ 

Since we know that $\sum_{i=0}^{n-1} \alpha_i = 1$, we get that $(\alpha_0, \alpha_1, \cdots, \alpha_{n-1})$ should satisfy the given system of equations. A mutatis mutandis produce the other direction for the proof of the other direction. □

**Definition 2.6.** Let $a = \sum \alpha_i x_i \in \mathbb{Z}[X]$. We define the length of $a$ to be $\ell(a) = |\{i : \alpha_i \neq 0\}|$.

**Lemma 2.7.** Let $n$ be odd. Then there is no idempotent $a \in \mathbb{Z}[Q_n]$ satisfying $\ell(a) = 2$.

**Proof.** Suppose on the contrary assume that there exist $i \neq j$ with $\alpha_i, \alpha_j \neq 0$ and $\alpha_k = 0$ for all $k \neq i, j$. Consider the $i$-the column of the adjacency matrix. Then as seen above it will be the permutation $\sigma$. Therefore there exist a $k$ such that $k$-th entry of the column is $\alpha_j$. Moreover $k$ cannot be equal to $i$. If $k$ were equal to $j$, then $x_j \ast x_i = x_k$ since $(x_j \ast x_i) \ast x_i = x_j$. This will force $i = j$, contrary to our assumptions. Hence $k \neq i, j$. So from the $k$-th row we get an equation of the form $\alpha_i, \alpha_j = 0$ or $2\alpha_i, \alpha_j = 0$ again contradicting our assumption $\alpha_i, \alpha_j \neq 0$. □

## 3. Idempotents in $\mathbb{Z}[Q_5]$ 

**Theorem 3.1.** The idempotents of $\mathbb{Z}[Q_5]$ are the trivial idempotents.

**Proof.** We will be using Lemma 2.5. Note that the adjacency matrix is given by

$$
\begin{bmatrix}
0 & 2 & 4 & 1 & 3 \\
4 & 1 & 3 & 0 & 2 \\
3 & 0 & 2 & 4 & 1 \\
2 & 4 & 1 & 3 & 0 \\
1 & 3 & 0 & 2 & 4
\end{bmatrix}
$$
Hence we need to find \( \mathbb{Z} \)-solutions of the system of equations given by
\[
\begin{pmatrix}
t_0 & t_2 & t_4 & t_1 & t_3 \\
t_4 & t_1 & t_3 & t_0 & t_2 \\
t_3 & t_0 & t_2 & t_4 & t_1 \\
t_2 & t_4 & t_1 & t_3 & t_0 \\
t_1 & t_3 & t_0 & t_2 & t_4
\end{pmatrix}
\begin{pmatrix}
t_0 \\
t_1 \\
t_2 \\
t_3 \\
t_4
\end{pmatrix}
=
\begin{pmatrix}
t_0 \\
t_1 \\
t_2 \\
t_3 \\
t_4
\end{pmatrix}
\quad \sum_{i=0}^{4} t_i = 1.
\]

We solve this using the Groebner basis, since a common zero of the above set will be a common zero of the Groebner basis. The Groebner basis of the above equations is given by
\[
\begin{align*}
t_4^3 - t_4^2, t_1^2 - t_1, t_1 t_2 + 3t_4^3 - 3t_4 t_2 - 3t_4^2 - t_2 + 3t_4, t_1 t_3 - t_1^2 + t_1, t_2 t_3,
\end{align*}
\]
\[
\begin{align*}
t_3^2 - 4t_2^2 - t_3 + 4t_2, t_1 t_4 + t_4^2 - t_4, t_2 t_4, t_3 t_4, 5t_4^2 - 5t_4.
\end{align*}
\]

This system can be easily seen to have only solutions \( t_i = 1 \) and \( t_j = 0 \) for all \( i \neq j, i, j \in \{0, 1, 2, 3, 4\} \). Hence the idempotents of \( \mathbb{Z}[Q_5] \) are trivial. \( \Box \)

**Corollary 3.2.** We have that \( \text{Aut}(\mathbb{Z}[Q_5]) \cong S_5 \).

**Proof.** Note that an automorphism \( \varphi \) of \( \mathbb{Z}[Q_5] \) is determined by the images of \( x_i \). Since \( \varphi(x_i) \) is an idempotent and \( \mathbb{Z}[Q_5] \) has only trivial idempotents, it follows that \( \varphi(x_i) = x_j \) for some \( j \). This finishes the proof. \( \Box \)

**References**

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