On D-brane interaction & its related properties

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Abstract

We compute the closed-string cylinder amplitude between one Dp brane and
the other Dp′ brane, placed parallel at a separation, with each carrying a general
worldvolume flux and with \( p - p' = 0, 2, 4, 6 \) and \( p \leq 6 \). For the \( p = p' \), we show
that the main part of the amplitude for \( p = p' < 5 \) is a special case of that for
\( p = p' = 5 \) or 6 case. For all other \( p - p' = 2, 4, 6 \) cases, we show that the amplitude
is just a special case of the corresponding one for \( p = p' \) case. Combined both, we
obtain the general formula for the amplitude, which is valid for each of the cases
considered and for arbitrary worldvolume fluxes. The corresponding general open
string one-loop annulus amplitude is also obtained by a Jacobi transformation of
the general cylinder one. We give also the general open string pair production rate.
We study the properties of the amplitude such as the nature of the interaction, the
open string tachyonic instability, and the possible open string pair production and
its potential enhancement. In particular, in the presence of pure magnetic fluxes
or magnetic-like fluxes, we find that the nature of interaction is correlated with
the existence of potential open string tachyonic instability. When the interaction is
attractive, there always exists an open string tachyonic instability when the brane
separation reaches the minimum determined by the so-called tachyonic shift. When
the interaction is repulsive, there is no such instability for any brane separation
even in the presence of magnetic fluxes. We also find that the enhancement of open
string pair production, in the presence of pure electric fluxes, can occur only for the
\( p - p' = 2 \) case.
1 Introduction

Computing the interaction amplitude between one Dp and the other Dp’, placed parallel at a separation transverse to the Dp brane, with each carrying a general worldvolume flux and with \( p - p' = 0, 2, 4, 6 \) and \( p \leq 6 \), has its own interest by itself. As we will see, the amplitude itself exhibits many interesting properties. For example, the contribution from the so-called NS-NS sector or R-R sector has a nice form, determined by the certain properties of the worldvolume background fluxes relevant to the amplitude, and can be expressed in terms of certain \( \theta \)-functions and the Dedekind \( \eta \)-function. The total amplitude can also be expressed in terms of a certain \( \theta \)-function, usually the \( \theta_1 \)-function, using a special form of the more general identity relating various different \( \theta \)-functions obtained from the contributions of the NS-NS and R-R sectors after the so-called Gliozzi-Scherk-Olive (GSO) projection, and the Dedekind \( \eta \)-function, so exhibiting the expected modular property of the amplitude.

A Dp brane carrying no worldvolume flux is a non-perturbative stable Bogomol’nyi-Prasad-Sommereld (BPS) solitonic extended object in superstring theories (for example, see [2]), preserving one half of the spacetime supersymmetries. It has its tension and carries the so-called RR charge. When we place two such Dp branes parallel at a separation, the net interaction between the two actually vanishes due to the 1/2 BPS nature of this system. We can check this explicitly by computing the lowest order stringy interaction amplitude in terms of the closed string tree-level cylinder diagram. We have here the so-called NS-NS contribution, due to the brane tension, which is attractive, and the so-called R-R contribution, due to the RR charges, which is repulsive. The BPS nature of each Dp brane identifies its tension with its RR charge in certain units and as such the sum of the two gives an expected zero net interaction by making use of the usual ‘abstruse identity’ [3]. This same amplitude can also be computed via the so-called open string one-loop annulus diagram. The same conclusion can be reached.

When one of the above two Dp branes is replaced by a Dp’ with \( p > p' \) and \( p \leq 6 \),

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1 For a system with \( p - p' = \kappa = 0, 2, 4, 6 \), we have as usual \( NN = p' + 1 \) for which the two ends of open string obey the Neumann boundary conditions, \( ND = \kappa \) for which one end of the open string obeys the Dirichlet boundary conditions while the other the Neumann ones, and \( DD = 9 - p' - \kappa \) for which the two ends of open string obey the Dirichlet boundary conditions.

2 In general, placing an infinitely extended Dp in spacetime will cause it to curve. For our purpose, we try to avoid this to happen at least to the probe distance in which we are interested. For this, we need to limit our discussion in this paper to \( p \leq 6 \) cases since these Dp branes have well-behaved supergravity configurations which are all asymptotically flat. Moreover, when the string coupling is small, i.e. \( g_s \ll 1 \), placing one such Dp in spacetime will keep the spacetime flat even for a probe distance to the brane in the substringy scale \( \alpha'^{1/2} \gg r \gg g_s^{1/(7-p)} \alpha'^{1/2} \) as discussed in section 2 of [1].
we have only the NS-NS contribution since different brane RR charge does not interact. For the $p - p' = 2$ case, we have an attractive interaction while for the $p - p' = 6$ case we have a repulsive one. As such, the underlying spacetime supersymmetries are all broken. However, for the $p - p' = 4$ case, the net interaction vanishes and the underlying system is still BPS, preserving 1/4 of spacetime supersymmetries. Each of these, regarding the supersymmetry breaking or preservation, can be checked explicitly following [5,6], for example.

When the brane worldvolume fluxes are present, we now expect in general a non-vanishing interaction. Except for the $(p = 6, p' = 0)$ case mentioned above and the $(p = 6, p' \leq 6)$ cases to be considered later in this paper, the long-range interaction between the Dp and Dp' for other cases is in general attractive when the electric and/or magnetic fluxes on the Dp' are parallel to the corresponding ones on the Dp, respectively. The reason for this is simple since only different constituent branes contribute to this long-range interaction and each contribution is from the respective NS-NS sector and is attractive. For example, if we have both electric and magnetic fluxes present on Dp and Dp', the F-strings (see footnote 4) within Dp' have no interaction with their parallel F-strings within Dp but have a long-range attractive interaction with D(p - 2) branes (see footnote 4) within Dp. However, as indicated above for $p = 6$ and $p' \leq 6$, the long-range interaction can be repulsive in the presence of certain types of fluxes. This has been demonstrated in the simplest possible cases in [18] when $p - p' = 2$. We will spell out the condition in general for this to be true later in this paper.

For certain type of fluxes (to be specified later on), the nature of the interaction at small brane separation (attractive or repulsive) remains unclear if it is computed in terms of the closed string tree-level cylinder amplitude. In general, this implies new physics to appear. The best description is now in terms of the open string one-loop annulus ampli-

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3 For the $p - p' = 2$ case, the long-range interaction is attractive since the contribution from either the exchange of massless dilaton or the exchange of massless graviton is attractive while for the $p - p' = 6$ case the long-range interaction is repulsive since the contribution from the exchange of massless dilaton is repulsive and exceeds the attractive contribution from the exchange of massless graviton. However, for the $p - p' = 4$ case, the repulsive contribution from the exchange of massless dilaton just cancels the attractive one from the exchange of massless graviton and this gives a net vanishing interaction. Each of these can be checked explicitly, for example, see [4]. Each of these remains to be true for any brane separation as we will show later in this paper.

4 The electric flux on a Dp-brane stands for the presence of F-strings, forming the so-called (F, Dp) non-threshold bound state [7,13], while a magnetic flux stands for that of co-dimension 2 D-branes inside the original Dp brane, forming the so-called (D(p-2), Dp) non-threshold bound state [15,17], from the spacetime perspective. These fluxes are in general quantized. We will not discuss their quantizations in the text for simplicity due to their irrelevance for the purpose of this paper.
tude for which many interesting properties such as certain instabilities become manifest.

When only magnetic fluxes are present (or even no fluxes are present with $p - p' = 2$), we find that there is a correlation between the nature of the interaction between the Dp and the Dp' and the potential open string tachyonic instability. If the interaction is attractive, the open string connecting the two D-branes has a tachyonic shift to its spectrum \[19,20\]. We have then the onset of tachyonic instability when the brane separation reaches the minimum determined by the shift. Once this instability develops, the attractive brane interaction diverges. We have then the so-called tachyon condensation and as such a phase transition occurs, releasing the excess energy of this system. For example, for $p = p'$, this process restores not only the gauge symmetry from $U(1) \times U(1) \to U(2)$ but also the supersymmetry from none to half of the spacetime supersymmetries \[21\]. In the so-called weak field limit, the corresponding instability is just the analog of the Nielsen-Olesen one \[40\] of non-abelian gauge theories such as the electroweak one and the gauge symmetry restoration was considered in \[23,24\]. On the other hand, if the interaction is repulsive, we don't have this tachyonic shift and therefore have no tachyonic instability to begin with.

When we have only worldvolume electric fluxes present, the underlying system is in general no-longer 1/2 BPS and breaks all its supersymmetries, therefore unstable. This manifests itself by the appearance of an infinite number of simple poles in the integrand of the integral representation of the open string one-loop annulus amplitude, implying that the interaction amplitude has an imaginary part. Each of these simple poles actually indicates the corresponding open string pair production under the action of the applied electric fluxes \[21,25,26\]. The imaginary amplitude just reflects the decay of the underlying system via the so-called open string pair production, releasing the excess energy of the system until it reaches the corresponding 1/2 BPS stable one. This is the analog of the Schwinger pair production in quantum electrodynamics (QED) \[27\]. For unoriented bosonic string and type I superstring, this was pursued a while ago in \[28,29\]. When the applied electric flux reaches its so-called critical field determined by the fundamental string tension, the open string pairs are produced cascadingly and there is also an instability developed.

When both electric and magnetic fluxes are present in a certain way, the open string pair production has an enhancement, uncovered recently in \[6,18,21,26,30,31\].

As explained in \[21,31\], there is no open string pair production for a single Dp brane in Type II string theories even if we apply an electric flux on the brane, unless it reaches its critical value\[10\]. The simple reason for this is that each of the virtual open strings in

\[\text{[Footnote: When the applied electric field reaches its critical value, it will break the virtual open strings from...]}\]
the pairs from the vacuum has its two ends carrying charge + and −, respectively, giving a net zero-charge of the open string, called the charge-neutral open string, and the net force acting on the string vanishes under the applied constant electric field. So the electric field can only stretch each open string but cannot separate the virtual open string and the virtual anti-open string in each pair to give rise to the pair production. This can also be explained by the fact that a Dp brane carrying a constant worldvolume electric flux is actually a 1/2 BPS non-threshold bound state \[11, 14\], therefore it is stable and cannot decay via the open string pair production.

In order to produce the pair production in Type II string theories, a possible choice is to let the two ends of the open string experience different electric fields since the charge-neutral nature of the open strings cannot be altered. The above mentioned two Dp-brane system is probably the simplest one for this purpose. We compute this pair production rate \[6, 25, 26, 30\] and find it indeed non-vanishing. However, for any realistic electric field applied, the rate is in general vanishingly small and it has no practical use. But when an additional magnetic flux is added in a certain way, the rate can be enhanced enormously \[6, 30\] and the largest possible rate is for the system of two D3 branes when the electric and magnetic fields are along the same or opposite direction \[30, 31\]. This enhanced pair production may have the potential to be detected \[21\].

For this to occur in practice, we need to assume one of the D3 branes to be relevant to our 4-dimensional world and the other D3 to be invisible (hidden or dark) to us. For this simple system, it appears that there is a possibility for the detection of the pair production but there is an issue if one carefully examines the underlying physics as discussed in detail in \[21\]. The mass spectrum of the open string connecting the two D3 at a separation \(y\) has a mass shift \(m = y/(2\pi\alpha')\) at each mass-level. That the corresponding modes at each given mass-level all have this same shift is due to the underlying supersymmetry in the absence of worldvolume fluxes. For example, the lowest-mass eight bosons and eight fermions all have the same mass \(y/(2\pi\alpha')\) which becomes massless at \(y = 0\) and the underlying system is 1/2 BPS in the absence of worldvolume fluxes. In general, the laboratory electric and magnetic fields are much smaller than the string scale and the weak field limit holds. So the contribution to the pair production is due to the above 8 bosonic \((8_B)\) and 8 fermionic \((8_F)\) lowest-mass charged modes of the open string. From the worldvolume viewpoint, these \(8_B + 8_F\) massive charged modes, in the absence of worldvolume fluxes, are due to the symmetry breaking of \(U(2) \rightarrow U(1) \times U(1)\) with one of scalars taking its expectation value \(\sim y\) but the underlying 16 supersymmetries remain
intact, giving rise to the 4-dimensional $N = 4$ massive gauge theory with one massive charged vector (W-boson), 5 massive charged scalars and their corresponding fermionic super partners, all with mass $m = y/(2\pi\alpha')$. In the presence of worldvolume fluxes and if the brane interaction is non-vanishing, the underlying supersymmetries are all broken. The presence of practical magnetic fluxes can also give a tiny mass shift to the massive charged vector \cite{20}.

We therefore naturally expect the mass scale $m = y/(2\pi\alpha')$ no less than that of Standard model W-boson and this requires the electric and magnetic fields 18 orders of magnitude larger than the current laboratory limit to make the detection possible. The other choice is to take the other D3 as a dark one and for this, we don’t have a priori knowledge of the mass scale $m$. If it happens to be no larger than the electron mass, we may have an opportunity to detect the open string rate if the QED Schwinger pair production becomes feasible. Even so, we still have to explain why the other charged fermions other than the one identified with the electron, the charged scalars and the charged vector, all having the same mass $m \sim m_e = 0.51$ MeV, are not the Standard model particles.

All these issues, one way or the other, are due to that the 16 $(8_B + 8_F)$ relevant modes all have the same mass $m = y/(2\pi\alpha')$ from the underlying supersymmetry. In addition, the currently available laboratory electric and magnetic fields are too small. The natural question is: does there exist a possibility that we can get around these issues in practice?

A while ago, one of the present authors along with his collaborator considered a system of one Dp and one Dp', placed parallel at a separation transverse to the Dp brane, with $p - p' = 2$ and with each carrying only one flux \cite{18}, and found that whenever there is an electric flux present along the NN-directions, there is an open string pair production enhancement even in the absence of a magnetic flux. The novel feature found in \cite{18} is that the Dp'-brane plays effectively as a magnetic flux of stringy order (see footnote (4)). In other words, if our D3 brane has a nearby D-string, for example as a cosmic string, this D-string appears effectively as a stringy magnetic field. This field can give rise to the pair production enhancement, which can hardly possible with a laboratory magnetic field, if our D3 carries an electric flux along the D-string direction. In addition, the underlying system breaks all supersymmetries intrinsically. So this consideration may provide a solution to the above question raised. This is the other line motivating us to consider the brane interaction in general between one Dp and the other Dp' with $p - p' = k = 0, 2, 4, 6$ as specified at the outset of this introduction.

Without further ado, we in this paper compute the lowest-order stringy interaction amplitude between one Dp and the other Dp', placed parallel at a separation transverse
to the Dp, with each carrying a general worldvolume flux and with \( p - p' = 0, 2, 4, 6 \) and \( p \leq 6 \). We will show that the key part of the amplitude in terms of the \( \theta_1 \)-functions and the Dedekind \( \eta \)-function for each of the \( p = p' < 5 \) cases is a special case of that for the \( p = p' = 5 \) or 6 case. We further demonstrate that the amplitude for \( p - p' = 2, 4, 6 \) can be obtained, respectively, from the corresponding \( p = p' \) case by choosing specific magnetic fluxes along the 2, 4, 6 ND-directions, a trick greatly simplifying the computations. We compute first the closed-string cylinder amplitude using the closed string boundary state representation of D-brane \([32, 36]\), which has the advantage of holding true for a general worldvolume constant flux \([14]\). By a Jacobi transformation of this, we can obtain the corresponding open-string annulus amplitude. We will also compute the open string pair production rate if any and discuss the relevant analytical structures of the amplitude. We will explore the nature (attractive or repulsive) of the interaction at large brane separation and small brane separation, respectively, and study various instabilities such as the onset of tachyonic one at small brane separation. In particular, we find that there is a correlation between the nature of interaction being attractive and the existence of tachyonic shift, which can give rise to the onset of tachyonic instability when the brane separation reaches the minimum determined by the tachyonic shift. We will determine at which conditions there exists the open string pair production and its possible enhancement. We will also speculate possible applications of the enhanced open string pair production for practical purpose.

This paper is organized as follows. In section 2, we give a brief review of the closed-string boundary state representation of Dp-brane carrying a general worldvolume flux and set up conventions for latter sections. We give also a general discussion on computing the closed-string cylinder amplitude between a Dp and a Dp', placed parallel at a separation transverse to the Dp, with each carrying a general worldvolume flux and with \( p - p' = 0, 2, 4, 6 \) and \( p \leq 6 \). In section 3, we first compute the closed-string cylinder amplitude for each of the \( 0 \leq p = p' \leq 6 \) cases. In this computation, we will make use of certain tricks and simplifications in evaluating this amplitude. Once this is done, we will see the expected nice structure of the amplitude in terms of certain \( \theta \)-functions and the Dedekind \( \eta \)-function. We study the nature of interaction and find that the repulsive interaction can only be possible for \( p = p' = 6 \) and for certain purely magnetic fluxes present. For all other cases, the long-range interaction is attractive. We also find the correlation between the nature of interaction being attractive and the existence of tachyonic shift, which will give rise to the onset of tachyonic instability when the brane separation reaches the minimum determined by the tachyonic shift. We compute the decay rate of the underlying system...

\(^{6}\text{We will also provide the physical rationale for this.}\)
and the corresponding open string pair production rate when they exist and discuss their potential enhancement. In section 4, we move to compute the amplitude for each of $p - p' = 2, 4, 6$ and $p \leq 6$, respectively, using the known $p = p'$ one with a specific choice of the magnetic fluxes on the Dp', along now the 2, 4, 6 ND-directions. Basically, the amplitude for each $p - p' = 2, 4, 6$ and $p \leq 6$ can simply be obtained from the corresponding $p = p'$ one computed in section 3 by a special choice of magnetic fluxes along the 2, 4, 6 ND-directions on the Dp' brane. We also provide the underlying physical reason for this. Similar properties of the amplitude as discussed in section 3 are also given. We discuss and conclude this paper in section 5.

2 The Dp-brane boundary state

In this section, we give a brief review of Dp-brane boundary state carrying a general worldvolume flux, following [36]. We also give a general discussion in computing the closed-string cylinder amplitude between a Dp brane and a Dp' brane, placed parallel at a separation transverse to the Dp, with each carrying a general worldvolume flux and with $p - p' = 0, 2, 4, 6$ and $p \leq 6$.

For such a description of Dp-brane, there are two sectors, namely NS-NS and R-R sectors. In either sector, we have two implementations for the boundary conditions of a Dp brane, giving two boundary states $|B, \eta\rangle$, with $\eta = \pm$. However, only the combinations

$$|B\rangle_{NS} = \frac{1}{2} [ |B, +\rangle_{NS} - |B, -\rangle_{NS} ],$$

$$|B\rangle_{R} = \frac{1}{2} [ |B, +\rangle_{R} + |B, -\rangle_{R} ],$$

(1)

are selected by the Gliozzi-Scherk-Olive (GSO) projection in the NS-NS and R-R sectors, respectively. The boundary state $|B, \eta\rangle$ for a Dp-brane can be expressed as the product of a matter part and a ghost part [34–36], i.e.

$$|B, \eta\rangle = \frac{c_p}{2} |B_{mat}, \eta\rangle |B_{g}, \eta\rangle,$$

(2)

where

$$|B_{mat}, \eta\rangle = |B_X\rangle |B_{\psi}, \eta\rangle, \quad |B_{g}, \eta\rangle = |B_{gh}\rangle |B_{sgh}, \eta\rangle$$

(3)

and the overall normalization $c_p = \sqrt{\pi} \left(2\pi\sqrt{\alpha'}\right)^{3-p}$.

As discussed in [14][36], the operator structure of the boundary state holds true even with a general worldvolume flux and is always of the form

$$|B_X\rangle = \exp(-\sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n} \cdot M \cdot \tilde{\alpha}_{-n}) |B_X\rangle_0,$$

(4)
and
\[ |B_\psi, \eta\rangle_{NS} = -i \exp(i \eta \sum_{m=1/2}^{\infty} \psi_{-m} \cdot M \cdot \tilde{\psi}_{-m}) |0\rangle, \]  
(5)
for the NS-NS sector and
\[ |B_\psi, \eta\rangle_{R} = -\exp(i \sum_{m=1}^{\infty} \psi_{-m} \cdot M \cdot \tilde{\psi}_{-m}) |B, \eta\rangle_{0R}, \]  
(6)
for the R-R sector. The ghost boundary states are the standard ones as given in [34], independent of the fluxes, which we will not present here. The M-matrix\footnote{We have changed the previously often used symbol $S$ to the current $M$ to avoid a possible confusion with the S-matrix in scattering amplitude.}, the zero-modes $|B_X\rangle_0$ and $|B, \eta\rangle_{0R}$ encode all information about the overlap equations that the string coordinates have to satisfy. They can be determined respectively [14, 32, 36] as
\[ M = ([[(\eta - \hat{F})(\eta + \hat{F})^{-1}]_{\alpha\beta}, -\delta_{ij}], \]  
(7)
\[ |B_X\rangle_0 = [-\det(\eta + \hat{F})]^{1/2} \delta^{9-p}(q^i - y^i) \prod_{\mu=0}^9 |k^\mu = 0\rangle, \]  
(8)
for the bosonic sector, and
\[ |B_\psi, \eta\rangle_{0R} = (C \Gamma^0 \Gamma^1 \cdots \Gamma^p \frac{1 + i \eta \Gamma_{11}}{1 + i \eta} U)_{AB} |A\rangle |\tilde{B}\rangle, \]  
(9)
for the R-R sector. In the above, the Greek indices $\alpha, \beta, \cdots$ label the world-volume directions $0, 1, \cdots, p$ along which the Dp brane extends, while the Latin indices $i, j, \cdots$ label the directions transverse to the brane, i.e., $p + 1, \cdots, 9$. We define $\hat{F} = 2\pi \alpha' F$ with $F$ the external worldvolume field. We also have denoted by $y^i$ the positions of the D-brane along the transverse directions, by $C$ the charge conjugation matrix and by $U$ the matrix
\[ U(\hat{F}) = \frac{1}{\sqrt{-\det(\eta + \hat{F})}} \exp \left(-\frac{1}{2} \hat{F}_{\alpha\beta} \Gamma^\alpha \Gamma^\beta \right); \]  
(10)
with the symbol $; ;$ denoting the indices of the $\Gamma$-matrices completely anti-symmetrized in each term of the exponential expansion. $|A\rangle |\tilde{B}\rangle$ stands for the spinor vacuum of the R-R sector. Note that the $\eta$ in the above denotes either sign $\pm$ or the worldvolume Minkowski flat metric and should be clear from the content.

We now come to compute the closed-string tree-level cylinder amplitude between a Dp and a Dp’ as stated earlier via
\[ \Gamma = \langle B_{p'}(\hat{F}') |D| B_p(\hat{F}) \rangle, \]  
(11)
where $D$ is the closed string propagator defined as

$$D = \frac{\alpha'}{4\pi} \int_{|z| \leq 1} \frac{d^2z}{|z|^2} z L_0 \bar{z} L_0. \quad (12)$$

Here $L_0$ and $\bar{L}_0$ are the respective left and right mover total zero-mode Virasoro generators of matter fields, ghosts and superghosts. For example, $L_0 = L_0^X + L_0^\psi + L_0^{gh} + L_0^{sgh}$ where $L_0^X, L_0^\psi, L_0^{gh}$ and $L_0^{sgh}$ are the respective ones from matter fields $X^\mu$, matter fields $\psi^\mu$, ghosts $b$ and $c$, and superghosts $\beta$ and $\gamma$, and their explicit expressions can be found in any standard discussion of superstring theories, for example in [35], therefore will not be presented here. The above total amplitude has contributions from both NS-NS and R-R sectors, respectively, and can be written as $\Gamma_{p,p'} = \Gamma_{\text{NSNS}} + \Gamma_{\text{RR}}$. In calculating either $\Gamma_{\text{NSNS}}$ or $\Gamma_{\text{RR}}$, we need to keep in mind that the boundary state used should be the GSO projected one as given in (11).

For this, we need to calculate first the amplitude $\Gamma(\eta', \eta) = \langle B_{p'}, \eta'|D|B_p, \eta \rangle$ in each sector with $\eta' \eta = +$ or $-$, $B' = B_{p'}(\hat{F}')$ and $B = B_p(\hat{F})$. Actually, $\Gamma(\eta', \eta)$ depends only on the product of $\eta'$ and $\eta$, i.e., $\Gamma(\eta', \eta) = \Gamma(\eta' \eta)$. In the NS-NS sector, this gives $\Gamma_{\text{NSNS}}(\pm) \equiv \Gamma(\eta', \eta)$ when $\eta' \eta = \pm$, respectively. Similarly we have $\Gamma_{\text{RR}}(\pm) \equiv \Gamma(\eta', \eta)$ when $\eta' \eta = \pm$ in the R-R sector. We then have

$$\Gamma_{\text{NSNS}} = \frac{1}{2} [\Gamma_{\text{NSNS}}(+) - \Gamma_{\text{NSNS}}(-)], \quad \Gamma_{\text{RR}} = \frac{1}{2} [\Gamma_{\text{RR}}(+) + \Gamma_{\text{RR}}(-)]. \quad (13)$$

Given the structure of the boundary state, the amplitude $\Gamma(\eta' \eta)$ can be factorized as

$$\Gamma(\eta' \eta) = \frac{c_{p'} c_p \alpha'}{4 \pi} \int_{|z| \leq 1} \frac{d^2z}{|z|^2} A^X A^{bc} A^\psi(\eta' \eta) A^{\beta\gamma}(\eta' \eta). \quad (14)$$

In the above, we have

$$A^X = \langle B'_X |z|^{2L^X_0} |B_X \rangle, \quad A^\psi(\eta' \eta) = \langle B'_\psi, \eta'|z|^{2L^\psi_0} |B_\psi, \eta \rangle, \quad A^{bc} = \langle B_{gh} |z|^{2L^{gh}_0} |B_{gh} \rangle, \quad A^{\beta\gamma}(\eta' \eta) = \langle B_{sgh}, \eta'|z|^{2L^{sgh}_0} |B_{sgh}, \eta \rangle. \quad (15)$$

The above ghost and superghost matrix elements $A^{bc}$ and $A^{\beta\gamma}(\eta' \eta)$, both independent of the fluxes and the dimensionalities of the branes involved, can be calculated to give,

$$A^{bc} = |z|^{-2} \prod_{n=1}^\infty \left(1 - |z|^{2n}\right)^2, \quad (16)$$

and in the NS-NS sector

$$A^{\beta\gamma}_{\text{NSNS}}(\eta' \eta) = |z| \prod_{n=1}^\infty \left(1 + \eta' \eta |z|^{2n-1}\right)^{-2}, \quad (17)$$
while in the R-R sector

\[ A_{RR}^{\psi}(\eta' \eta) = R_0 \langle B_{gh}^\psi, \eta' | B_{gh}^\psi, \eta \rangle_{OR} |z|^{\frac{5}{2}} \prod_{n=1}^{\infty} \left(1 + \eta' \eta |z|^{2n}\right)^{-2}, \]  

(18)

where \( R_0 \langle B_{gh}^\psi, \eta' | B_{gh}^\psi, \eta \rangle_{OR} \) denotes the superghost zero-mode contribution which requires a regularization along with the zero-mode contribution of matter field \( \psi \) in this sector. We will discuss this regularization later on.

With the above preparation, we are ready to compute the closed string tree-level amplitudes for the systems under consideration. We first compute the closed-string tree-level cylinder amplitude for the case of \( p = p' \). This serves as the basis for computing the amplitude for each of the \( p \neq p' \) cases. The general steps follow those given in section 2 of [26] but with a few refinements. Once the closed string tree-level cylinder amplitude is obtained, we use a Jacobi transformation to obtain the corresponding open string one-loop annulus amplitude. We will have these done in the following two sections one by one. We also discuss the properties of the respective amplitude in each case.

3. The amplitude and its properties: the \( p = p' \) case

As indicated already in the previous section, the computations of the amplitude boil down to computing the matrix elements of matter part, i.e., \( A_X \) and \( A_\psi \) given in (15). For this, the following property of the matrix \( M \) given in (7) can be used to simplify their computations greatly,

\[ M_\mu^\rho (MT)_\rho^\nu = (MT)_\mu^\rho M_\rho^\nu = \delta_\mu^\nu, \]

(19)

where \( T \) denotes the transpose of matrix. Following [26], for a system of two Dp branes, placed parallel at a separation \( y \), with one carrying flux \( \hat{F}' \) and the other carrying flux \( \hat{F} \), we can then have,

\[ A_X = V_{p+1} \left( \frac{\det(\eta + \hat{F}')}{\det(\eta + \hat{F})} \right)^{\frac{9-p}{2}} e^{-\frac{\alpha'}{2} \sum_{\nu}^2} \prod_{n=1}^{\infty} \left(1 - \frac{1}{1 - |z|^{2n}} \right)^{9-p} \prod_{\alpha=0}^{p} \frac{1}{1 - \lambda_\alpha |z|^{2n}} \]

(20)

and in the NS-NS sector

\[ A_{NSNS}^{\psi}(\eta' \eta) = \prod_{n=1}^{\infty} \left(1 + \eta' \eta |z|^{2n-1}\right)^{9-p} \prod_{\alpha=0}^{p} \left(1 + \eta' \eta \lambda_\alpha |z|^{2n-1}\right) , \]

(21)

while in the R-R sector

\[ A_{RR}^{\psi}(\eta' \eta) = R_0 \langle B_{\psi}^\psi, \eta' | B_{\psi}^\psi, \eta \rangle_{OR} |z|^{\frac{5}{2}} \prod_{n=1}^{\infty} \left(1 + \eta' \eta |z|^{2n}\right)^{9-p} \prod_{\alpha=0}^{p} \left(1 + \eta' \eta \lambda_\alpha |z|^{2n}\right) , \]

(22)
where \( \langle R'_{\psi}, \eta' | B_{\psi}, \eta \rangle \rangle_{0R} \) denotes the zero-mode contribution in this sector which, when combined with the zero-mode contribution from the superghost, needs a regularization mentioned earlier. We will present the result of this regularization later on. In the above, \( |z| = e^{-\pi t}, V_{p+1} \) denotes the volume of the Dp brane worldvolume, \( \lambda_\alpha \) are the eigenvalues of the matrix \( w_{(1+p) \times (1+p)} \) defined in the following

\[
W = MM^T = \begin{pmatrix}
w_{(1+p) \times (1+p)} & 0 \\
0 & 0
\end{pmatrix},
\]

where the matrix \( M \) or \( M' \) is the one given in (7) when the corresponding flux is \( \hat{F} \) or \( \hat{F}' \), respectively, and \( \mathbb{I} \) stands for the unit matrix. We can also express the matrix \( w \) in terms of matrix \( s \) and \( s' \) as

\[
w_{\alpha}^{\beta} = (ss'^T)_{\alpha}^{\beta} = \left[ (\mathbb{I} - \hat{F})(\mathbb{I} + \hat{F})^{-1}(\mathbb{I} + \hat{F}')(\mathbb{I} - \hat{F}')^{-1} \right]_{\alpha}^{\beta},
\]

where

\[
s_{\alpha}^{\beta} = [ (\mathbb{I} - \hat{F})(\mathbb{I} + \hat{F})^{-1} ]_{\alpha}^{\beta},
\]

and similarly for \( s' \) but with \( \hat{F} \) replaced by \( \hat{F}' \). Note that the two factors \( (\mathbb{I} - \hat{F}) \) and \( (\mathbb{I} + \hat{F})^{-1} \) in \( s \) are inter-changeable and this remains also true for the \( s' \). In the above \( \mathbb{I} \) stands for the \((1+p) \times (1+p)\) unit matrix. For matrix \( s \), we have \( s_\alpha^{\gamma}(s^T)_\gamma^{\beta} = \delta^{\alpha \beta} \). This holds also for the matrix \( s' \) and matrix \( w \). The above orthogonal matrix \( W \), satisfying

\[
WW^T = W^TW = \mathbb{I}_{10 \times 10},
\]

can be obtained from a redefinition of the certain oscillator modes, say \( \tilde{a}_{n\nu} \), which is a trick used in simplifying the evaluation of the matrix elements of matter part from the contribution of oscillator modes. Let us take the following as a simple illustration for obtaining the matrix \( W \). In obtaining \( A^X \), we need to evaluate, for given \( n > 0 \), the following matrix element,

\[
\langle 0 | e^{-\frac{1}{\pi} \alpha_\mu^{n}(M')_\mu^{\nu} \tilde{a}_{n\nu}} | z \rangle 2\alpha_\mu^{n} \alpha_\nu^{n} e^{-\frac{1}{\pi} \alpha_\mu^{n}(M)_\mu^{\nu} \tilde{a}_{n\nu}} e^{-\frac{1}{\pi} \alpha_\mu^{n}(M)_\mu^{\nu} \tilde{a}_{n\nu}} | 0 \rangle = \langle 0 | e^{-\frac{1}{\pi} \alpha_\mu^{n}(M')_\mu^{\nu} \tilde{a}_{n\nu}} e^{-\frac{1}{\pi} \alpha_\mu^{n}(M)_\mu^{\nu} \tilde{a}_{n\nu}} | 0 \rangle,
\]

where \( |0\rangle \) stands for the vacuum. Purely for simplifying the evaluation of the matrix element on the right of the above equality, we first define \( \tilde{\alpha}'_\mu = (M')_\mu^{\rho} \tilde{a}_\rho \) where we have omitted the index \( n \) since this works for both \( n > 0 \) and \( n < 0 \), due to the matrix \( M' \) being real. Note that the commutation relation \( [\tilde{\alpha}'_{n\mu}, \tilde{\alpha}'_{m\nu}] = \eta_{\mu\nu} \delta_{n+m,0} \) continues to hold, using the property of matrix \( M' \) as given in (19). With this property of matrix \( M' \),
we can have $\tilde{\alpha}_\mu = (M^T)_{\mu}^{\nu} \tilde{\alpha}'_{\nu}$. Substituting this into (27) for $n < 0$ and also dropping the prime on $\tilde{\alpha}'$, we have (27) as

$$\langle 0 | e^{-\frac{1}{n} \tilde{\alpha}_n^{\nu} \alpha_n^{\mu} e^{-\frac{|x|^2}{n} \alpha_{-n}^{\rho} W_{\rho}^\sigma \tilde{\alpha}_{-n}^{\sigma} | 0} \rangle,$$

where $W$ is precisely the one given in (23). Since $W$ is an unit matrix in the absence of fluxes, we expect that it can be diagonalized with the deformation of adding fluxes using the following non-singular matrix $V$,

$$V = \begin{pmatrix} v(1+p)^{(1+p)} & 0 \\ I_{(9-p) \times (9-p)} \end{pmatrix},$$

such that

$$W = VW_0V^{-1}.$$

In the above,

$$W_0 = \begin{pmatrix} \lambda_0 & \vdots & \lambda_p \\ 0 & \ddots & 0 \\ 0 & 0 & I_{(9-p) \times (9-p)} \end{pmatrix},$$

and $v$ is a $(1+p) \times (1+p)$ non-singular matrix. We prove (29), (30) and (31) to hold true in general in Appendix A. We now further define, for $n > 0$, $\alpha'_{n\mu} = (V^{-1})_{\mu}^{\nu} \alpha_{n\nu}$ and $\tilde{\alpha}'_{-n\mu} = \alpha'_{-n\nu} V_{\nu}^\mu$, and $\alpha'_{-n\mu} = (V^{-1})_{\mu}^{\nu} \tilde{\alpha}_{-n\nu}$ and $\tilde{\alpha}'_{n\mu} = \tilde{\alpha}_{n\nu} V_{\nu}^\mu$. Note that $\tilde{\alpha}'_{n\mu} \alpha'_{n\mu} = \tilde{\alpha}'_{n\mu} \alpha'_{n\mu}$. The matrix element (28) becomes

$$\langle 0 | e^{-\frac{1}{n} \tilde{\alpha}_n^{\nu} \alpha_n^{\mu} e^{-\frac{|x|^2}{n} \alpha_{-n}^{\rho} \tilde{\alpha}_{-n}^{\sigma} | 0} \rangle.$$

We have now the commutator relations $[\alpha'_{n\mu}, \alpha'_{m\nu}] = n \delta^\nu_{\mu} \delta_{n,m}$ and $[\tilde{\alpha}'_{n\mu}, \tilde{\alpha}'_{m\nu}] = n \delta^\nu_{\mu} \delta_{n,m}$ when $n, m > 0$. We still have $\alpha'_{n\mu} | 0 \rangle = \tilde{\alpha}'_{n\mu} | 0 \rangle = 0$ and $\langle 0 | \alpha'_{n\mu} = \langle 0 | \tilde{\alpha}'_{-n\mu} = 0$. The evaluation of (32) becomes then as easy as the case without the presence of fluxes, giving the results of (20) to (22), respectively.

The $p + 1$ eigenvalues $\lambda_\alpha$ with $\alpha = 0, \ldots, p$ are not all independent and can actually be determined by the given worldvolume fluxes. First from the given property of $w$, we have

$$\det w = 1,$$

which gives

$$\prod_{\alpha=0}^{p} \lambda_\alpha = 1.$$  

---

8This purely serves the purpose of simplifying the evaluation of the matrix element (28). For this, we keep the annihilation operator $\alpha'_{n\mu}$ with a lower Lorentz index $\mu$ while the creation operator $\alpha'_{-n\nu}$ with an upper Lorentz index $\nu$. It will be opposite for the corresponding oscillators with tilde.
The eigenvalue $\lambda$ satisfies the following equation
\[ \det \left( \lambda \delta_{\alpha}^{\beta} - w_{\alpha}^{\beta} \right) = 0, \] (34)
as well as the equation
\[ \det \left( \lambda^{-1} \delta_{\alpha}^{\beta} - (w^{-1})_{\alpha}^{\beta} \right) = 0. \] (35)
The last one can also be written as
\[ \det \left( \lambda^{-1} \delta_{\alpha}^{\beta} - w_{\alpha}^{\beta} \right) = 0, \] (36)
where we have used $(w^{-1})_{\alpha}^{\beta} = (w^T)_{\beta}^{\alpha} = \eta^{\beta\beta'} w_{\beta'}^{\alpha} \eta_{\alpha\alpha'}$. In other words, for every eigenvalue $\lambda$ of $w$, its inverse $\lambda^{-1}$ is also an eigenvalue. So the $p + 1$ eigenvalues $\lambda_\alpha$ are pairwise. When $p = \text{even}$, this must imply that one of the eigenvalues is unity. Given this property of $\lambda_\alpha$, the equation (33) satisfies automatically. For convenience, we now relabel the eigenvalues pairwise as $\lambda_\alpha$ and $\lambda^{-1}_\alpha$ with $\alpha = 0, 1, \cdots, [(p - 1)/2]$ and keep in mind that there is one additional unity eigenvalue, i.e., $\lambda = 1$, when $p = \text{even}$. Here $[(p - 1)/2]$ denotes the corresponding integral part of $(p - 1)/2$. For example, for $p = 6$, it gives an integer 2.

For a general $p \leq 6$, we need at most the following three equations to determine the corresponding eigenvalues $\lambda_\alpha, \lambda^{-1}_\alpha$ with $\alpha = 0, 1, \cdots, [(p - 1)/2]$ plus $\lambda = 1$ if $p = \text{even}$. For $p = \text{even}$, we have $\lambda = 1$ and
\[ 1 + \sum_{\alpha=0}^{[(p-1)/2]} (\lambda_\alpha + \lambda^{-1}_\alpha) = \text{tr} w, \quad 1 + \sum_{\alpha=0}^{[(p-1)/2]} (\lambda_\alpha^2 + \lambda^{-2}_\alpha) = \text{tr} w^2, \quad 1 + \sum_{\alpha=0}^{[(p-1)/2]} (\lambda_\alpha^3 + \lambda^{-3}_\alpha) = \text{tr} w^3, \] (37)
while for $p = \text{odd}$, we have instead
\[ \sum_{\alpha=0}^{[(p-1)/2]} (\lambda_\alpha + \lambda^{-1}_\alpha) = \text{tr} w, \quad \sum_{\alpha=0}^{[(p-1)/2]} (\lambda_\alpha^2 + \lambda^{-2}_\alpha) = \text{tr} w^2, \quad \sum_{\alpha=0}^{[(p-1)/2]} (\lambda_\alpha^3 + \lambda^{-3}_\alpha) = \text{tr} w^3. \] (38)
In the above, the $w$ is given in (24) in terms of fluxes $\hat{F}$ and $\hat{F}'$. Concretely, we list the respective equations needed to determine the corresponding eigenvalues in Table 1 for $p \leq 6$. We actually don’t need to solve the eigenvalues from the equations given in Table 1 for the matrix elements given in (20), (21) and (22), respectively, for each case. Let us use one particular example for $p = 3$ to illustrate this. For example, the following product
Equation(s) for eigenvalue(s)

\[ \lambda_0 + \lambda_0^{-1} = trw \]

\[ \lambda_0 + \lambda_0^{-1} = trw - 1, \ \lambda = 1 \]

\[ \sum_{\alpha=0}^{1}(\lambda_\alpha + \lambda_\alpha^{-1}) = trw, \sum_{\alpha=0}^{1}(\lambda_\alpha^2 + \lambda_\alpha^{-2}) = trw^2 \]

\[ \sum_{\alpha=0}^{1}(\lambda_\alpha + \lambda_\alpha^{-1}) = trw - 1, \sum_{\alpha=0}^{1}(\lambda_\alpha^2 + \lambda_\alpha^{-2}) = trw^2 - 1, \ \lambda = 1 \]

\[ \sum_{\alpha=0}^{2}(\lambda_\alpha + \lambda_\alpha^{-1}) = trw, \sum_{\alpha=0}^{2}(\lambda_\alpha^2 + \lambda_\alpha^{-2}) = trw^2, \sum_{\alpha=0}^{2}(\lambda_\alpha^3 + \lambda_\alpha^{-3}) = trw^3 \]

\[ \sum_{\alpha=0}^{2}(\lambda_\alpha + \lambda_\alpha^{-1}) = trw - 1, \sum_{\alpha=0}^{2}(\lambda_\alpha^2 + \lambda_\alpha^{-2}) = trw^2 - 1, \sum_{\alpha=0}^{2}(\lambda_\alpha^3 + \lambda_\alpha^{-3}) = trw^3 - 1, \ \lambda = 1 \]

Table 1: The equations needed to determine the corresponding eigenvalues for \( p \leq 6 \).

\[
(1 - \lambda_0|z|^{2n}) (1 - \lambda_0^{-1}|z|^{2n}) (1 - \lambda_1|z|^{2n}) (1 - \lambda_1^{-1}|z|^{2n}) \\
= 1 - \sum_{\alpha=0}^{1} (\lambda_\alpha + \lambda_\alpha^{-1}) |z|^{2n} + [2 + (\lambda_0 + \lambda_0^{-1}) (\lambda_1 + \lambda_1^{-1})] |z|^{4n} \\
- \sum_{\alpha=0}^{1} (\lambda_\alpha + \lambda_\alpha^{-1}) |z|^{6n} + |z|^{8n}, \\
= 1 - trw |z|^{2n} + \frac{1}{2} [(trw)^2 - trw^2] |z|^{4n} - trw |z|^{6n} + |z|^{8n}. \tag{39}
\]

We are now ready to express the amplitude (14) given in the previous section in the NS-NS or R-R sector in a more compact form. For the NS-NS sector, using (13), (17), (20) and (21) for the contributions from the ghost \( bc \), superghost \( \beta \gamma \), the matter \( X \) and \( \psi \), respectively, we have the NSNS-amplitude as

\[
\Gamma_{\text{NSNS}}(\eta'\eta) = \frac{V_{p+1}}{\sqrt{(8\pi^2\alpha')^{\frac{d+1}{2}}}} \sqrt{\det(\eta + \hat{F}) \det(\eta + \hat{F})} \int_0^\infty \frac{dt}{t^{d-4}} e^{-\frac{8\pi^2}{2\pi^2\alpha'} |z|^{-1}} \\
\times \prod_{n=1}^{\infty} \left( \frac{1 + \eta'\eta|z|^{2n-1}}{1 - |z|^{2n}} \right)^{7+\delta_{p,\text{even}}-p} \prod_{\alpha=0}^{p+1} \frac{(1 + \eta'\eta\lambda_\alpha|z|^{2n-1}) (1 + \eta'\eta\lambda_\alpha^{-1}|z|^{2n-1})}{(1 - \lambda_\alpha|z|^{2n}) (1 - \lambda_\alpha^{-1}|z|^{2n})}, \tag{40}
\]
and similarly for the R-R sector, using (13), (18), (20) and (22), we have

\[
\Gamma_{RR}(\eta'\eta) = \frac{V_{p+1}}{(8\pi^2\alpha')^{\frac{1+p}{2}}} \frac{\det(\eta + \hat{F}') \det(\eta + \hat{F})}{\det(\eta + \hat{F}) \det(\eta + \hat{F}')} \int_{0}^{\infty} \frac{dt}{t^{\frac{9-p}{2}}} e^{-\frac{\eta^2}{2\pi^2\alpha'^2}} \left[ \prod_{n=1}^{\infty} e^{-\frac{\eta^2}{2\pi^2\alpha'^2}} \right]^{-1} \left[ \prod_{n=1}^{\infty} A_n(+) - \prod_{n=1}^{\infty} A_n(-) \right],
\]

where

\[
\frac{c_p^2}{2\pi^2(2\pi^2\alpha')^{\frac{1+p}{2}}} = \frac{1}{(8\pi^2\alpha')^{\frac{1+p}{2}}}, \quad \int_{0}^{\infty} \frac{d^2z}{|z|^2} = 2\pi \int_{0}^{\infty} dt.
\]

In obtaining the above, we have used the following relations

\[
\Gamma_{RR}(\eta'\eta) \equiv \sigma_{\text{g.s.}}(\eta'|\eta)_{0R} \equiv 0R_{\text{g.s.}}(\eta'|\eta)_{0R} \equiv 0R_{\text{g.s.}}(\eta'|\eta)_{0R} \equiv \Gamma_{NSNS}(\eta'\eta)\eta_{0R} = \Gamma_{NSNS}(\eta'\eta)_{0R} \equiv \Gamma_{RR}(\eta'\eta)_{0R} \equiv \Gamma_{RR}(\eta'\eta)_{0R}.
\]

From (40) and (41) as well as (13) and following the regularization scheme given in [34, 37], we have the GSO projected NSNS-amplitude,

\[
\Gamma_{NSNS} = \frac{1}{2} \left[ \Gamma_{NSNS}(+) - \Gamma_{NSNS}(-) \right],
\]

and the GSO projected RR-amplitude

\[
\Gamma_{RR} = \frac{1}{2} \left[ \Gamma_{RR}(+) + \Gamma_{RR}(-) \right],
\]

as given earlier. The above zero-mode contribution

\[
\langle B', \eta'| B, \eta \rangle_{0R} = \frac{1}{8\pi^2\alpha'} \frac{\det(\eta + \hat{F}) \det(\eta + \hat{F}')}{\det(\eta + \hat{F}) \det(\eta + \hat{F}')}
\]

can be computed for the general fluxes \( \hat{F} \) and \( \hat{F}' \), using the expression for the R-R sector zero-mode (9) along with (11) and following the regularization scheme given in [34, 37], as

\[
\frac{c_p^2}{2\pi^2(\pi^2\alpha')^{\frac{1+p}{2}}} = \frac{1}{(8\pi^2\alpha')^{\frac{1+p}{2}}}, \quad \int_{0}^{\infty} \frac{d^2z}{|z|^2} = 2\pi \int_{0}^{\infty} dt.
\]

where the explicit expression for \( c_p \) as given right after (3) has been used and \( |z| = e^{-\pi t} \) as given earlier. The above zero-mode contribution

\[
\langle B', \eta'| B, \eta \rangle_{0R} = \frac{1}{8\pi^2\alpha'} \frac{\det(\eta + \hat{F}) \det(\eta + \hat{F}')}{\det(\eta + \hat{F}) \det(\eta + \hat{F}')}
\]

as given earlier. The above zero-mode contribution

\[
\langle B', \eta'| B, \eta \rangle_{0R} = \frac{1}{8\pi^2\alpha'} \frac{\det(\eta + \hat{F}) \det(\eta + \hat{F}')}{\det(\eta + \hat{F}) \det(\eta + \hat{F}')}
\]

as given earlier. The above zero-mode contribution

\[
\langle B', \eta'| B, \eta \rangle_{0R} = \frac{1}{8\pi^2\alpha'} \frac{\det(\eta + \hat{F}) \det(\eta + \hat{F}')}{\det(\eta + \hat{F}) \det(\eta + \hat{F}')}
\]

as given earlier. The above zero-mode contribution

\[
\langle B', \eta'| B, \eta \rangle_{0R} = \frac{1}{8\pi^2\alpha'} \frac{\det(\eta + \hat{F}) \det(\eta + \hat{F}')}{\det(\eta + \hat{F}) \det(\eta + \hat{F}')}
\]

as given earlier. The above zero-mode contribution

\[
\langle B', \eta'| B, \eta \rangle_{0R} = \frac{1}{8\pi^2\alpha'} \frac{\det(\eta + \hat{F}) \det(\eta + \hat{F}')}{\det(\eta + \hat{F}) \det(\eta + \hat{F}')}
\]
where

$$A_n(\pm) = \left( \frac{1 + |z|^{2n-1}}{1 - |z|^{2n}} \right)^{7+\delta_p,\text{even}} \prod_{\alpha=0}^{[\nu+1]} \frac{(1 \pm \lambda_\alpha |z|^{2n-1}) (1 \pm \lambda_\alpha^{-1} |z|^{2n-1})}{(1 - \lambda_\alpha |z|^{2n}) (1 - \lambda_\alpha^{-1} |z|^{2n})},$$

$$B_n(+) = \left( \frac{1 + |z|^{2n}}{1 - |z|^{2n}} \right)^{7+\delta_p,\text{even}} \prod_{\alpha=0}^{[\nu+1]} \frac{(1 + \lambda_\alpha |z|^{2n}) (1 + \lambda_\alpha^{-1} |z|^{2n})}{(1 - \lambda_\alpha |z|^{2n}) (1 - \lambda_\alpha^{-1} |z|^{2n})}. \quad (47)$$

In obtaining (46), we have used the property of the zero-mode (44) which has the only contribution from $\eta\eta' = +$.

To proceed, we express the eigenvalues $\lambda_\alpha = e^{2\pi i \nu_\alpha}$ with $\alpha = 0, \cdots, [(p - 1)/2]$. $\nu_\alpha$ takes either real or purely imaginary value. In the former case, we can take $\nu_\alpha \in [0, 1/2]$ since only $\lambda_\alpha + \lambda_\alpha^{-1} = 2 \cos 2\pi \nu_\alpha$ appears in the amplitude and $\nu_\alpha = 0$ corresponds to the absence of fluxes. In the latter case, one can show that at most one of the $\nu_\alpha$'s can take imaginary value (see Appendix A) and we can choose this particular $\nu = \nu_0 = i\tilde{\nu}_0$ with $\tilde{\nu}_0 \in (0, \infty)$ since $\lambda + \lambda^{-1} = 2 \cosh 2\pi \tilde{\nu}_0$. This is actually the consequence of the matrix $w$ (21) being a general Lorentz transformation. It happens whenever the applied electric fluxes cannot be eliminated by a Lorentz transformation.

We can now express the NSNS-amplitude (45) in terms of the $\theta$-functions and the Dedekind $\eta$-function as (see, for example, [38] for their definitions)

$$\Gamma_{\text{NSNS}} = \left[ \theta_3(0|it) \right]^{12 - \frac{[\nu+1]}{2}} \prod_{\alpha=0}^{[\nu+1]} \theta_3(\nu_\alpha|it) - \left[ \theta_4(0|it) \right]^{12 - \frac{[\nu+1]}{2}} \prod_{\alpha=0}^{[\nu+1]} \theta_4(\nu_\alpha|it) \right]. \quad (48)$$

Similarly for the RR-amplitude (46), we have

$$\Gamma_{\text{RR}} = \left[ \theta_2(0|it) \right]^{12 - \frac{[\nu+1]}{2}} \prod_{\alpha=0}^{[\nu+1]} \theta_2(\nu_\alpha|it) \prod_{\alpha=0}^{[\nu+1]} \frac{\sin \pi \nu_\alpha}{\cos \pi \nu_\alpha}. \quad (49)$$

In Appendix B, we will show that the zero-mode contribution (44) can be written in terms of the $\nu_\alpha$ as

$$0R\langle B', \eta'|B, \eta \rangle_{0R} = -2^4 \delta_{\eta\eta'} + \prod_{\alpha=0}^{[\nu+1]} \cos \pi \nu_\alpha. \quad (50)$$
The $\theta$-terms in the square bracket in (51) and their simplification:

| $p$ | \[ \theta_\alpha^\nu(0|it) - \theta_\beta^\nu(0|it) - \theta_\gamma^\nu(0|it) = 2 \theta_1^\nu(0|it) = 0 \] |
|-----|-----------------------------------------------------------------------------------------------------------------|
| 0 or 1 | \[ 2 \theta_3^\nu(0|it) \theta_3(\nu_0|it) - \theta_4^\nu(0|it) \theta_4(\nu_0|it) - \theta_2^\nu(0|it) \theta_2(\nu_0|it) = 2 \theta_1^\nu(0|it) \] |
| 1 or 2 | \[ \theta_3^\nu(0|it) \theta_3(\nu_0|it) \theta_3(\nu_1|it) - \theta_4^\nu(0|it) \theta_4(\nu_0|it) \theta_4(\nu_1|it) - \theta_2^\nu(0|it) \theta_2(\nu_0|it) \theta_2(\nu_1|it) = 2 \theta_1^\nu(0|it) \] |
| 3 or 4 | \[ \theta_3(0|it) \theta_3(\nu_0|it) \theta_3(\nu_1|it) \theta_3(\nu_2|it) - 2 \theta_2(0|it) \theta_2(\nu_0|it) \theta_2(\nu_1|it) \theta_2(\nu_2|it) = 2 \theta_1 \left( \frac{\nu_0 + \nu_1}{2} \right) \right. \] |
| 5 or 6 | \[ \theta_3(0|it) \theta_3(\nu_0|it) \theta_3(\nu_1|it) \theta_3(\nu_2|it) - \theta_4(0|it) \theta_4(\nu_0|it) \theta_4(\nu_1|it) \theta_4(\nu_2|it) = 2 \theta_1 \left( \frac{\nu_0 + \nu_1}{2} \right) \right. \] |

With this, we can have the total interaction amplitude for $0 \leq p = p' \leq 6$ as

\[
\Gamma_{p,p} = \Gamma_{\text{NSNS}} + \Gamma_{RR},
\]

\[
= 2 \left[ \frac{\pi + 1}{\pi} \right] V_{p+1} \sqrt{\det(\eta + \hat{F}') \det(\eta + \hat{F})} \int_0^\infty \frac{dt}{t^{p+1/2}} e^{-\pi \eta \nu \alpha} \sum_{\alpha=0}^{[\pi/2]} \sin \pi \nu \alpha \left[ \theta_1(\nu_0|it) \right]^{[\pi/2]} \prod_{\alpha=0}^{[\pi/2]} \theta_1(\nu_0|it)
\]

\[
= 2 \left[ \frac{\pi + 1}{\pi} \right] V_{p+1} \sqrt{\det(\eta + \hat{F}') \det(\eta + \hat{F})} \int_0^\infty \frac{dt}{t^{p+1/2}} \left[ \theta_1(\nu_0|it) \right]^{[\pi/2]} \prod_{\alpha=0}^{[\pi/2]} \theta_1(\nu_0|it)
\]

\[
= 2 \left[ \frac{\pi + 1}{\pi} \right] V_{p+1} \sqrt{\det(\eta + \hat{F}') \det(\eta + \hat{F})} \int_0^\infty \frac{dt}{t^{p+1/2}} \left[ \theta_1(\nu_0|it) \right]^{[\pi/2]} \prod_{\alpha=0}^{[\pi/2]} \theta_1(\nu_0|it)
\]

We would like to stress that in spite of its appearance, each term in the above square bracket is the product of four theta-functions of the same type. For convenience, we list the three terms in the square bracket for each case in Table 2. In the Table, we also use the following identity for the $\theta$-functions from [39] to simplify the formula

\[
2 \theta_1(w|\tau) \theta_1(x|\tau) \theta_1(y|\tau) \theta_1(z|\tau) = \theta_1(w'|\tau) \theta_1(x'|\tau) \theta_1(y'|\tau) \theta_1(z'|\tau)
\]

\[
+ \theta_2(w'|\tau) \theta_2(x'|\tau) \theta_2(y'|\tau) \theta_2(z'|\tau) - \theta_3(w'|\tau) \theta_3(x'|\tau) \theta_3(y'|\tau) \theta_3(z'|\tau)
\]

\[
+ \theta_4(w'|\tau) \theta_4(x'|\tau) \theta_4(y'|\tau) \theta_4(z'|\tau)
\]

where $w', x', y'$ and $z'$ are related to $w, x, y, z$ as

\[
2w' = -w + x + y + z, \quad 2x' = w - x + y + z,
\]

\[
2y' = w + x - y + z, \quad 2z' = w + x + y - z.
\]
Note that $\theta_1(0|\tau) = 0$. From Table 2, we observe that the $p = 4$ or 3, $p = 2$ or 1 and $p = 0$ cases can be obtained from the $p = 6$ or 5 case by setting $\nu_2 = 0$, $\nu_2 = \nu_1 = 0$ and $\nu_2 = \nu_1 = \nu_0 = 0$, respectively, or by taking the respective limits. This very fact gives us an opportunity to express the amplitude (51) for each of the $p = p' < 5$ as a special case of that for the $p = p' = 5$ or 6 in the following sense, once the aforementioned limit is taken,

$$
\Gamma_{p,p} = \frac{2^3 V_{p+1} \left[ \det(\eta + \hat{F}') \det(\eta + \hat{F}) \right]^{\frac{1}{2}} \prod_{\alpha=0}^{2} \sin \pi \nu_\alpha}{\left(8\pi^2 \alpha' \right)^{\frac{p+1}{2}}} \int_0^\infty \frac{dt}{t^{\frac{p+2}{2}}} e^{-\frac{y^2}{\pi \alpha'}} \left[ \frac{1}{\eta_1(\nu_0 + \nu_1 + \nu_2) - \nu_1 + \nu_2} \right] \\
x \times \frac{1}{\eta_1(\nu_0 + \nu_1 + \nu_2) - \nu_1 + \nu_2} \\
\times \left[ \sum_{\alpha=0}^{2} \cos \pi \nu_\alpha - 2 \prod_{\alpha=0}^{2} \cos \pi \nu_\alpha - 1 \right] \\
\times \int_0^\infty \frac{dt}{t^{\frac{p+2}{2}}} e^{-\frac{y^2}{\pi \alpha'}} \prod_{n=1}^{\infty} C_n, \quad (54)
$$

where

$$
C_n = \frac{\tilde{C}_n}{(1 - |z|^{2n})^2 \prod_{\alpha=0}^{2} \left[ 1 - 2|z|^{2n} \cos 2\pi \nu_\alpha + |z|^{4n} \right]}, \quad (55)
$$

with

$$
\tilde{C}_n = [1 - 2|z|^{2n} \cos \pi (\nu_0 + \nu_1 + \nu_2) + |z|^{4n}] [1 - 2|z|^{2n} \cos \pi (\nu_0 - \nu_1 + \nu_2) + |z|^{4n}] \\
\times [1 - 2|z|^{2n} \cos \pi (\nu_0 + \nu_1 - \nu_2) + |z|^{4n}] [1 - 2|z|^{2n} \cos \pi (\nu_0 - \nu_1 - \nu_2) + |z|^{4n}], \quad (56)
$$

or

$$
[1 - 2|z|^{2n} e^{i\pi \nu_0} \cos \pi (\nu_1 + \nu_2) + e^{2i\pi \nu_0} |z|^{4n}] \\
\times [1 - 2|z|^{2n} e^{i\pi \nu_0} \cos \pi (\nu_1 - \nu_2) + e^{2i\pi \nu_0} |z|^{4n}] \\
\times [1 - 2|z|^{2n} e^{-i\pi \nu_0} \cos \pi (\nu_1 + \nu_2) + e^{-2i\pi \nu_0} |z|^{4n}] \\
\times [1 - 2|z|^{2n} e^{-i\pi \nu_0} \cos \pi (\nu_1 - \nu_2) + e^{-2i\pi \nu_0} |z|^{4n}], \quad (57)
$$

This same amplitude can also be expressed in terms of the open string one-loop annulus...
one via the Jacobi transformation $t \to t' = 1/t$, from the first equality in (58), as

$$\Gamma_{p,p} = -\frac{2^3 i V_{p+1} \left[ \det(\eta + \hat{F}') \det(\eta + \hat{F}) \right]^{1/2} \prod_{\alpha=0}^2 \sin \pi \nu_\alpha}{\left(8\pi^2 \alpha' \right)^{1/2}} \int_0^\infty \frac{dt'}{t'^{1/2} \exp \left(-\frac{3i}{4\pi} \right)} \prod_{\alpha=0}^{2} \sin \pi \nu_\alpha \sinh \pi \nu_\alpha t \sum_{\alpha=0}^{2} \cosh^2 \pi \nu_\alpha t - 2 \prod_{\alpha=0}^{2} \cosh \pi \nu_\alpha t - 1 \right] \prod_{n=1}^\infty Z_n,$$

(58)

where in obtaining the first equality in (58), we have used the following relations for the $\theta_1$-function and the Dedekind $\eta$-function,

$$\eta(\tau) = \frac{1}{(-i\tau)^{1/2}} \eta \left( \frac{-1}{\tau} \right), \quad \theta_1(\nu|\tau) = i e^{-i\pi \nu^2 / \tau} \left( \frac{\nu}{\tau} \right)^{1/2} \theta_1 \left( \frac{\nu}{\tau} \right),$$

(59)

in the second equality we have dropped the prime on $t$, and

$$Z_n = \frac{\tilde{Z}_n}{(1 - |z|^{2n})^2 \prod_{\alpha=0}^2 \left[ 1 - 2 |z|^{2n} \cosh 2 \pi \nu_\alpha t + |z|^{4n} \right]}.$$  

(60)

with

$$\tilde{Z}_n = \left[ 1 - 2 |z|^{2n} \cosh \pi (\nu_0 + \nu_1 + \nu_2) t + |z|^{4n} \right] \left[ 1 - 2 |z|^{2n} \cosh \pi (\nu_0 - \nu_1 + \nu_2) t + |z|^{4n} \right] \left[ 1 - 2 |z|^{2n} \cosh \pi (\nu_0 - \nu_1 - \nu_2) t + |z|^{4n} \right] \left[ 1 - 2 |z|^{2n} \cosh \pi (\nu_0 + \nu_1 - \nu_2) t + |z|^{4n} \right],$$

(61)

or

$$\left[ 1 - 2 |z|^{2n} e^{-\pi \nu_0 t} \cosh \pi (\nu_1 + \nu_2) t + e^{-2\pi \nu_0 t} |z|^{4n} \right] \left[ 1 - 2 |z|^{2n} e^{-\pi \nu_1 t} \cosh \pi (\nu_1 - \nu_2) t + e^{-2\pi \nu_1 t} |z|^{4n} \right] \left[ 1 - 2 |z|^{2n} e^{-\pi \nu_2 t} \cosh \pi (\nu_1 + \nu_2) t + e^{-2\pi \nu_2 t} |z|^{4n} \right] \left[ 1 - 2 |z|^{2n} e^{-\pi \nu_2 t} \cosh \pi (\nu_1 - \nu_2) t + e^{-2\pi \nu_2 t} |z|^{4n} \right].$$

(62)

In what follows, we will discuss each of the four cases: 1) $p = p' = 6$ or 5, 2) $p = p' = 4$ or 3, 3) $p = p' = 2$ or 1, and 4) $p = p' = 0$, separately.
3.1 The $p = p' = 6$ or 5 case

For the respective general worldvolume fluxes $\hat{F}$ and $\hat{F}'$, from the second equality of (54), we have the interaction amplitude for $p = p' = 6$ or 5 as

$$
\Gamma_{p,p} = \frac{2^2 V_{p+1} \left[ \sum_{\alpha=0}^{2} \cos^2 \pi \nu_\alpha - 2 \prod_{\alpha=0}^{2} \cos \pi \nu_\alpha - 1 \right] \sqrt{\det(\eta + \hat{F}') \det(\eta + \hat{F})}}{(8\pi^2 \alpha')^{\frac{p+1}{2}}}
\times \int_0^\infty \frac{dt}{t^{\frac{p}{2}}} e^{-\frac{t}{2\pi \alpha'}} \prod_{n=1}^{\infty} C_n,
$$

(63)

where $C_n$ is given in (55). For large brane separation $y$, the amplitude has its contribution mostly from the large $t$ integration for which $C_n \approx 1$. We have therefore

$$
\frac{\Gamma_{p,p}}{V_{p+1}} \approx \frac{\left[ \sum_{\alpha=0}^{2} \cos^2 \pi \nu_\alpha - 2 \prod_{\alpha=0}^{2} \cos \pi \nu_\alpha - 1 \right] \sqrt{\det(\eta + \hat{F}') \det(\eta + \hat{F})}}{2^{-2(8\pi^2 \alpha')^{\frac{p+1}{2}}}} \times \int_0^\infty \frac{dt}{t^{\frac{p}{2}}} e^{-\frac{t}{2\pi \alpha'}}
\times \frac{\sqrt{\det(\eta + \hat{F}') \det(\eta + \hat{F})}}{2^{p-1} \pi^{\frac{p+1}{2}} (2\pi \alpha')^{p-3} y^{7-p}} \int_0^\infty \frac{dt}{t^{\frac{p}{2}}} e^{-\frac{t}{2}}
\times \frac{\sqrt{\det(\eta + \hat{F}') \det(\eta + \hat{F})}}{2^{p-1} \pi^{\frac{p+1}{2}} (2\pi \alpha')^{p-3} y^{7-p}} \int_0^\infty \frac{dt}{t^{\frac{p}{2}}} e^{-\frac{t}{2}} \Gamma\left(\frac{7-p}{2}\right),
$$

(64)

where in the second equality, we have rescaled the integration variable $t$, and $\Gamma(7 - p/2)$ is the $\Gamma$-function with $\Gamma(1/2) = \sqrt{\pi}$ for $p = 6$ and $\Gamma(1) = 1$ for $p = 5$, respectively. According to our conventions, $\Gamma_{p,p} > 0$ gives an attractive interaction while $\Gamma_{p,p} < 0$ gives a repulsive one. The sign of the factor $F = \sum_{\alpha=0}^{2} \cos^2 \pi \nu_\alpha - 2 \prod_{\alpha=0}^{2} \cos \pi \nu_\alpha - 1$ in the above determines that of the amplitude. When any of $\nu_0, \nu_1$ and $\nu_2$ vanishes, it is non-negative. For example, taking $\nu_0 = 0$, we can write it as $F = (\cos \nu_2 - \cos \nu_1)^2 \geq 0$. When none of them vanishes, we have two cases to consider. Case 1): one of them is imaginary, say, $\nu_0 = im_0$ with $m_0 \in (0, \infty)$, and $\nu_1, \nu_2 \in (0, 1/2]$. We have now

$$
F = \sinh^2 \pi \nu_0 + \cos^2 \pi \nu_1 + \cos^2 \pi \nu_2 - 2 \cosh \pi \nu_0 \cos \pi \nu_1 \cos \pi \nu_2,
$$

$$
= \sinh^2 \pi \nu_0 + \cos^2 \pi \nu_1 + \cos^2 \pi \nu_2 - 2 \left(1 + 2 \sinh^2 \pi \nu_0 \right) \cos \pi \nu_1 \cos \pi \nu_2,
$$

$$
= 4 \left(\cosh^2 \pi \nu_0 - \cos \pi \nu_1 \cos \pi \nu_2\right) \sinh^2 \frac{\pi \nu_0}{2} + (\cos \pi \nu_1 - \cos \pi \nu_2)^2 > 0.
$$

(65)
In other words, the large brane separation is always attractive. Case 2): all \( \nu_0, \nu_1, \nu_2 \in (0, 1/2] \). This can only be possible for \( p = 6 \) since it needs at least six worldvolume spatial directions. For this case, in order to determine the condition for the sign of \( \mathcal{F} \), we rewrite it as

\[
\mathcal{F} = 2^3 \sin \pi \frac{\nu_0 + \nu_1 + \nu_2}{2} \sin \pi \frac{\nu_0 - \nu_1 + \nu_2}{2} \sin \pi \frac{\nu_0 + \nu_1 - \nu_2}{2} \sin \pi \frac{\nu_0 - \nu_1 - \nu_2}{2}. \tag{66}
\]

Note that \( \mathcal{F} \) is symmetric under the exchange of any two of \( \nu \)'s. So without loss of generality, we can assume \( \nu_0 \leq \nu_1 \leq \nu_2 \). Given this and the allowed range for each of the \( \nu \)'s, it is clear that the first two factors \( \sin \pi (\nu_0 + \nu_1 + \nu_2)/2 > 0 \) and \( \sin \pi (\nu_0 - \nu_1 + \nu_2)/2 > 0 \) while the last factor \( \sin \pi (\nu_0 - \nu_1 - \nu_2)/2 < 0 \). So the sign of \( \mathcal{F} \) is determined by that of the third factor \( \sin \pi (\nu_0 + \nu_1 - \nu_2)/2 \). If \( \nu_0 + \nu_1 > \nu_2 \), then \( \mathcal{F} < 0 \) while on the other hand if \( \nu_0 + \nu_1 < \nu_2 \), we have \( \mathcal{F} > 0 \). So in general, if the possible largest one of the three \( \nu \)'s is smaller than the sum of the remaining two, the long-range interaction is repulsive. In other words, the long-range interaction can only be repulsive when the three \( \nu \)'s are non-vanishingly real and when the possible largest one of the three \( \nu \)'s is smaller than the sum of the remaining two. This actually remains true for any brane separation as we will show this later on. When the possible largest one of the three \( \nu \)'s is larger than the sum of the remaining two, the corresponding interaction is still attractive. This also remains true at least until the brane separation reaches the minimum determined by the so-called tachyonic shift. When the possible largest one equals to the sum of the remaining two, the interaction vanishes for any brane separation.

We now come to explain the above. It is well-known that when a D2 or D4 brane is placed parallel to a D6 at a separation transverse to the D6, the interaction between them is zero or attractive while by the same token, the interaction between the D6 and a D0 is repulsive. Each of these cases can be examined easily in the following section when we consider the case of \( p - p' = 2, 4, 6 \) with \( p \leq 6 \). For the above, we need to have \( p = p' = 6 \) along with \( \nu_\alpha \in (0, 1/2] \) for \( \alpha = 0, 1, 2 \). In other words, at least one of the two D6 carries a constant magnetic flux, for example, \( \hat{F} \), with non-vanishing components \( \hat{F}_{12}, \hat{F}_{34} \) and \( \hat{F}_{56} \), which can give rise to the above three \( \nu_\alpha \in (0, 1/2] \). For such a flux, it implies that the D6 has its delocalized D4, D2 and D0 within the D6, which can be easily understood from the following coupling on the D6 worldvolume as

\[
T_6 \int \left( C_{7-2n} \wedge \hat{F}_2^n \right), \tag{67}
\]

where \( T_6 \) is the D6 brane tension, \( C_{p+1} \) is the potential minimally coupled with Dp, and \( \hat{F}_2^n = \hat{F}_2 \wedge \cdots \wedge \hat{F}_2 \) stands for the wedge product of \( n \hat{F}_2 \). So \( n = 0 \) gives the coupling
of D6 with the R-R potential $C_7$, $n = 1$ gives the coupling of D4 with $C_5$, $n = 2$ gives the coupling of D2 with $C_3$ and $n = 3$ gives the coupling of D0 with $C_1$. Given what we have for the non-vanishing components of $\hat{F}$, we can have $n = 0, 1, 2, 3$. So this gives an explanation of the delocalized D0, D2, D4 within D6 mentioned above. So when the possible largest one of the three $\nu$’s is smaller than the sum of the remaining two, the above cylinder amplitude shows that the repulsive interaction between the other D6 and the delocalized D0 on this D6 overtakes the attractive ones between the other D6 and the delocalized D2 or D4. Otherwise, the attractive interaction overtakes the repulsive one. The net interaction vanishes when the two equals. This also explains that when one of the three $\nu$’s is imaginary or vanishes, the net interaction is attractive since we must have one of the $\hat{F}_{12}, \hat{F}_{34}$ and $\hat{F}_{56}$ being zero which implies the absence of D0 branes on the D6.

For small brane separation, the small $t$ integration in (63) becomes important and has to be considered. For small $t$, $C_n$ in (55) can be large. In particular, $C_n$ blows up when $t \to 0$ due to the factor $(1 - \left|z\right|^{2n})^2$ in its denominator. So overall we have a blowing up factor $\prod_{n=1}^{\infty} (1 - \left|z\right|^{2n})^{-2}$ for $t \to 0$ in the integrand of the amplitude (63). Note also that the exponentially suppressing factor $e^{-y^2/(2\pi\alpha')t}$ in the integrand becomes vanishingly small when $t \to 0$. So there is a competition between the two and one expects a potentially interesting physics to occur when $t \to 0$. This will become manifest when we transform the closed string cylinder amplitude to the corresponding open string one-loop annulus one and it is a potential open string tachyonic instability. For now, the nature of $\Gamma_{p,p}$ depends on that of the parameters $\nu_0, \nu_1$ and $\nu_2$.

Following the previous discussion for large $y$, we have that the interaction is attractive whenever the three $\nu$’s are all real and the possible largest one of these is larger than the sum of the remaining two or one of them is imaginary which we will address later on. We now take a close look for a general $y$. Note that the $C_n$ (55) is still positive even for small $t$ and this can be easily checked. Each factor in the numerator $\tilde{C}_n$ (56) is positive, for example, the first factor $[1 - 2|z|^{2n}\cos \pi(\nu_0 + \nu_1 + \nu_2) + |z|^{4n}] > 1 - 2|z|^{2n} + |z|^{4n} = (1 - |z|^{2n})^2 > 0$. By the same token, each factor in the denominator is also positive. In other words, the sign of $\Gamma_{p,p}$ is still determined by that of $F$ given earlier. For this, the attractive interaction acting between the two D6 has a tendency to move the two towards each other and to make the brane separation smaller. Therefore the exponential factor $e^{-y^2/(2\pi\alpha't)}$ will make its suppressing less important and one expects that the diverging effect from $C_n$ at small $t$ will become to dominate at a certain point. So we expect then an instability mentioned above to occur. On the other hand, when the possible largest one of three $\nu$’s is less than the sum of the remaining two, the interaction is repulsive and as such has a tendency to move the two D6 apart further. So this makes the
suppression of the exponential factor $e^{-y^2/(2\pi\alpha' t)}$ in the integration more important and disfavors the instability to occur. So this appears to provide a correlation between the nature of interaction and the existence of potential tachyonic instability. We will show later that this is indeed true.

For small $t$, there appears a new feature when one of three $\nu$’s takes an imaginary value. For example, we choose $\nu_0 = i\bar{\nu}_0$ with $\bar{\nu}_0 \in (0, \infty)$. Now $C_n$ can be negative. By the same token as given in the previous paragraph, each factor in $\tilde{C}_n$ continues to be positive, for example, the third factor $[1 - 2|z|^{2n}e^{\pi\bar{\nu}_0} \cos \pi(\nu_1 + \nu_2) + e^{2\pi\bar{\nu}_0}|z|^{4n}] > 1 - 2|z|^{2n}e^{\pi\bar{\nu}_0} + e^{2\pi\bar{\nu}_0}|z|^{4n} = (1 - e^{\pi\bar{\nu}_0}|z|^{2n})^2 > 0$. However, the factor $[1 - 2|z|^{2n} \cosh 2\pi\bar{\nu}_0 + |z|^{4n}] \approx 2(1 - \cosh 2\pi\bar{\nu}_0) < 0$, in the denominator of $C_n$, for small $t$. Since there are an infinite number of $C_n$ appearing as product in the integrand, the sign of $\Gamma_{p,p}$ will then be indefinite. So for small $y$, the nature of the interaction becomes obscure for the case under consideration and this indicates that there should exist new physical process occurring in addition to the potential tachyonic instability mentioned above for small $t$. This new physics is actually the decay of the underlying system via the so-called open string pair production under the action of applied electric fluxes which makes $\nu_0$ become imaginary. All these will become manifest when the interaction is expressed in terms of the open string variable as the open string one-loop annulus amplitude (58) for $p = 6$ which we turn next. Note that this consideration applies to both $p = p' = 6$ and $p = p' = 5$ cases.

The open string one-loop annulus amplitude for $p = p' = 5, 6$, respectively, can be read from the second equality of (58) as

$$\Gamma_{p,p} = \frac{2^2 V_{p+1} \left[ \det(\eta + \hat{F}^\prime) \det(\eta + \hat{F}) \right]^{\frac{1}{2}} \int_0^\infty \frac{dt}{t^{\frac{p+1}{2}}} e^{-\frac{y^2}{2p\alpha'}} \prod_{\alpha=0}^2 \frac{\sin \pi\nu_\alpha}{\sinh \pi\nu_\alpha t} \cdot \left[ \sum_{\alpha=0}^2 \cosh^2 \pi\nu_\alpha t - 2 \prod_{\alpha=0}^2 \cosh \pi\nu_\alpha t - 1 \right] \prod_{n=1}^\infty Z_n,$$

where $Z_n$ is given in (60). Note that the closed string $t$-variable and the open string $t$-variable are inversely related to each other. So small $t$ in closed string case implies large $t$ in open string one. So the potential open string tachyonic instability should show up for large $t$ in the integrand of the above amplitude if it exists at all. Let us examine this in detail.

For large $t$, note that $n \geq 1$ and so $Z_n \approx 1$ for either all $\nu$’s are real (none of them bigger than $1/2$) or one of them is imaginary and the rest are real and not bigger than $1/2$. When all three $\nu$’s are real (only valid for $p = p' = 6$), we once again assume $\nu_0 \leq \nu_1 \leq \nu_2$ for convenience and without loss of generality since the amplitude is symmetric under
the exchange of any two of the three $\nu_0, \nu_1$ and $\nu_2$. From the discussion in the closed string variable, we know that when $\nu_0 + \nu_1 > \nu_2$, the interaction is repulsive and one expects no tachyonic instability. Let us check this here. From (68), one sees for large $t$ and $\nu_0 + \nu_1 > \nu_2$ that the dominant term in the square bracket is the second one with a minus sign, therefore the integrand is negative and further it behaves like

$$\sim -e^{-\frac{y^2 t}{2\pi \alpha'}} \cosh \pi \nu_0 t \cosh \pi \nu_1 \cosh \pi \nu_2 t \frac{t \to \infty}{\sinh \pi \nu_0 t \sinh \pi \nu_1 \sinh \pi \nu_2 t} \to -e^{-\frac{y^2 t}{2\pi \alpha'}},$$

(69)

which shows no tachyonic shift and therefore no potential tachyonic instability. This is consistent with our anticipation. However, when $\nu_0 + \nu_1 < \nu_2$, we do expect to see the potential tachyonic instability. Now the the term $\cosh^2 \pi \nu_2 t$ in the sum in the square bracket in (68) dominates and the integrand is positive and behaves like

$$\sim e^{-\frac{y^2 t}{2\pi \alpha'}} \cosh^2 \pi \nu_2 t \frac{t \to \infty}{\sinh \pi \nu_0 t \sinh \pi \nu_1 \sinh \pi \nu_2 t} \to e^{-\frac{y^2 t}{2\pi \alpha'}} e^{\pi(\nu_2 - \nu_0 - \nu_1)t},$$

(70)

where we have a so-called tachyonic shift $\pi(\nu_2 - \nu_0 - \nu_1)/2 > 0$ \cite{20}. The effective mass square for the open string is

$$m^2 = \frac{y^2}{(2\pi \alpha')^2} - \frac{\nu_2 - \nu_0 - \nu_1}{2\alpha'},$$

(71)

which becomes tachyonic if $y < \pi \sqrt{2(\nu_2 - \nu_0 - \nu_1)\alpha'}$. Whenever this happens, the integrand blows up for $t \to \infty$ and this reflects the onset of tachyonic instability. Then we will have a phase transition via the so-called tachyon condensation. Once again, this is consistent with our expectation. So this confirms our earlier assertion that there is indeed a correlation between the nature of interaction and the existence of a tachyonic instability.

The tachyonic shift and the appearance of tachyon mode can also be understood from the spectrum of the open string connecting the two D6 carrying magnetic fluxes which give rise to the $\nu_0, \nu_1$ and $\nu_2$ following \cite{19,20}. Let us use an explicit example for $p = p' = 6$ to demonstrate this. For this purpose, we choose the following magnetic flux for $\hat{F}$

$$\hat{F} = \begin{pmatrix} 0 & 0 & -\hat{g}_0 \\ 0 & -\hat{g}_1 & 0 \\ -\hat{g}_0 & 0 & 0 \\ \hat{g}_1 & 0 & -\hat{g}_2 \\ \hat{g}_2 & 0 & 0 \end{pmatrix},$$

(72)
and for $\hat{F}'$ we just replace each $g$’s in $\hat{F}$ with a prime on it. Following the prescription given earlier, we have

$$\lambda_\alpha + \lambda_\alpha^{-1} = \frac{2(1 - \hat{g}_{\alpha}^2)(1 - \hat{g}_{\alpha}'^2) + 4\hat{g}_{\alpha}\hat{g}_{\alpha}''}{(1 + \hat{g}_{\alpha}^2)(1 + \hat{g}_{\alpha}'^2)},$$

(73)

where $\alpha = 0, 1, 2$ and which gives, noting $\lambda_\alpha = e^{2\pi i \nu_\alpha}$ with $\nu_\alpha \in [0, 1/2]$,

$$\tan \pi \nu_\alpha = \left| \frac{\hat{g}_{\alpha} - \hat{g}_{\alpha}'}{1 + \hat{g}_{\alpha}\hat{g}_{\alpha}'} \right|.$$

(74)

We have also now the amplitude [68] with

$$\left[ \det(\eta + \hat{F}') \det(\eta + \hat{F}'') \right]^{1/2} = \prod_{\alpha=0}^{1/2} [(1 + \hat{g}_{\alpha}^2)(1 + \hat{g}_{\alpha}'^2)]^{1/2}.$$

(75)

Type I superstring in a single magnetic background, say, the magnetic field being in 56-directions, has been discussed in [20]. Here we have three magnetic fields, the first in 12-directions, the second in 34-directions and the third in 56-directions. The generalization of the discussion given there to the present case in the R-sector is straightforward and the conclusion remains the same even if we exclude the contribution of $y^2/(2\pi \alpha')^2$ to the energy square. In other words, unlike the case in the NS-sector which we will turn next, there is no possibility for the existence of tachyonic shift in the R-sector. The GSO-projected R-sector ground states (the eight fermions $8_F$) all have the same mass given by $y/(2\pi \alpha')$. In what follows, we focus here only on the generalization to the present NS-sector. As before, without loss of generality, we assume once again $\nu_0 \leq \nu_1 \leq \nu_2$. The energy spectrum is now

$$\alpha' E_{\text{NS}}^2 = \frac{y^2}{(2\pi)^2 \alpha'} + \sum_{\alpha=0}^{2} \left[ (2N_\alpha + 1) \frac{\nu_\alpha}{2} - \nu_\alpha S_\alpha \right] + L_{\text{NS}}^\text{free},$$

(76)

where

$$N_\alpha = b_{\alpha,0}^+ b_{\alpha,0}, \quad S_\alpha = \sum_{n=1}^{\infty} (a_{\alpha,n}^+ a_{\alpha,n} - b_{\alpha,n}^+ b_{\alpha,n}) + \sum_{r=1/2}^{\infty} \left( d_{\alpha,r}^+ d_{\alpha,r} - \tilde{d}_{\alpha,r}^+ \tilde{d}_{\alpha,r} \right),$$

$$L_{\text{NS}}^\text{free} = \sum_{\alpha=0}^{2} \left[ \sum_{n=1}^{\infty} n(a_{\alpha,n}^+ a_{\alpha,n} + b_{\alpha,n}^+ b_{\alpha,n}) + \sum_{r=1/2}^{\infty} r(d_{\alpha,r}^+ d_{\alpha,r} + \tilde{d}_{\alpha,r}^+ \tilde{d}_{\alpha,r}) \right] - \frac{1}{2} + L_{\text{NS}}^\text{free},$$

(77)

where $N_\alpha$ defines the corresponding Landau-level for $\alpha = 0, 1, 2$, respectively, $S_\alpha$ is the spin operator and $L_{\text{NS}}^\text{free}$ is the part contributing to the zero-mode Virasoro generator
from the 0, 7, 8, 9-directions. The possible lowest energy state is from the GSO-projected
ground state, $d_{2,1/2}^+ |0\rangle_{NS}$ and for this we have

$$\alpha' E_{NS}^2 = \frac{y^2}{(2\pi)^2 \alpha'} - \frac{\nu_2 - \nu_0 - \nu_1}{2},$$
$$S_2 = 1, \quad S_0 = S_1 = 0, \quad L_{NS}^\text{free} = 0. \quad (78)$$

Here the first equation is exactly the same as (71). In other words, when $\nu_2 > \nu_0 + \nu_1$ we
have a tachyonic shift and this gives a potential tachyonic instability. Otherwise we don’t.
So the conclusion remains the same as before and we will not repeat it here. It is nice to
see the same from a different approach. From either (71) or (78), we see that the tachyonic
shift would be just $\nu_2/2$, rather than the smaller one $(\nu_2 - \nu_0 - \nu_1)/2$, in the absence of the
other two fluxes. In other words, in order to have the largest tachyonic shift, we need to
choose to apply the largest magnetic one but no more. This largest tachyonic shift is also
responsible for the largest open string pair production enhancement discussed in [21,31].
We will also address this later when we discuss the open string pair production in the
presence of electric fluxes.

We now move to the case when one of three $\nu$’s is imaginary, say, $\nu_0 = i\nu_0$, and both
$\nu_1$ and $\nu_2$ are real. So this applies to both $p = p' = 5$ and $p = p' = 6$. Now the open
string annulus amplitude (68) becomes

$$\Gamma_{p,p} = \frac{2^2 V_{p+1}}{(8\pi^2 \alpha')^{3/2}} \left[ \det(\eta + \hat{F}) \det(\eta + \hat{F}) \right]^{1/2} \int_0^\infty \frac{dt}{i\pi \alpha'} e^{-\frac{x^2 t}{2\pi \alpha'}} \frac{\sinh \pi \nu_0 \sin \pi \nu_1 \sin \pi \nu_2}{\sin \pi \nu_0 \sin \pi \nu_1 \sin \pi \nu_2 t} \times \left[ \left( \cosh \pi \nu_1 t - \cosh \pi \nu_2 t \right)^2 + 4 \sin^2 \frac{\pi \nu_0 t}{2} \left( \cosh \pi \nu_1 t \cosh \pi \nu_2 t - \cos^2 \frac{\pi \nu_0 t}{2} \right) \right] \times \prod_{n=1}^{\infty} Z_n, \quad (79)$$

where $Z_n$, from (60), becomes

$$Z_n = \frac{\tilde{Z}_n}{(1 - |z|^{2n})^2 \left[ 1 - 2|z|^{2n} \cos 2\pi \nu_0 t + |z|^{4n} \right] \prod_{\alpha=1}^{2} \left[ 1 - 2|z|^{2n} \cosh 2\pi \nu_\alpha t + |z|^{4n} \right]}, \quad (80)$$

with $\tilde{Z}_n$, from (62), as

$$\tilde{Z}_n = \left| \left[ 1 - 2|z|^{2n} e^{-i\pi \nu_0 t} \cosh \pi (\nu_1 + \nu_2) t + e^{-2i\pi \nu_0 t} |z|^{4n} \right] \right|^2 \times \left| \left[ 1 - 2|z|^{2n} e^{-i\pi \nu_0 t} \cosh \pi (\nu_1 - \nu_2) t + e^{-2i\pi \nu_0 t} |z|^{4n} \right] \right|^2 > 0. \quad (81)$$

---

$^9$ Both the state $d_{0,1/2}^+ |0\rangle_{NS}$ and $d_{1,1/2}^+ |0\rangle_{NS}$ have their respective energy higher than that of $d_{2,1/2}^+ |0\rangle_{NS}$. 

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Note that in the denominator of $Z_n$, the factor $1 - 2|z|^{2n} \cos \pi \bar{\nu}_0 t + |z|^{4n} > (1 - |z|^{2n})^2 > 0$ and for $\alpha = 1, 2, 1 - 2|z|^{2n} \cosh \pi \nu_0 t + |z|^{4n} = (1 - |z|^{2n} e^{2\pi \nu_0 t})(1 - |z|^{2n} e^{-2\pi \nu_0 t}) > 0$ due to $n \geq 1$ and $\nu_0 \in [0, 1/2]$. So we have $Z_n > 0$. Note also that every other factor in the integrand, except for the $\sin \pi \bar{\nu}_0 t$ in the denominator, is also positive. The interesting physics shows up precisely due to this factor $\sin \pi \bar{\nu}_0 t$. It gives an infinite number of simple poles of the integrand at $t_k = k/\bar{\nu}_0$ with $k = 1, 2, \ldots$ along the positive $t$-axis. This implies that the interaction amplitude has an imaginary part, indicating the decay of the underlying system via the so-called open string pair production. By saying this, we first need to note that the integral has no singularity when we take $t \to 0$. Secondly, we need to have $y > \pi \sqrt{2|\nu_1 - \nu_2| \alpha'}$ to avoid a potential tachyonic instability and to validate our amplitude computations since otherwise the integrand blows up for large $t$ as

$$\sim e^{-\frac{y^2 t}{2\pi \alpha'}} e^{\pi |\nu_1 - \nu_2| t} = e^{-2\pi t} \left[ \frac{2^2}{(2\pi)^{2\alpha'}} \frac{\nu_1 - \nu_2}{2} \right],$$

and as such a phase transition, called tachyon condensation, occurs. The decay rate of the underlying system per unit volume of Dp brane via the open string pair production can be computed, following $\text{[28]}$, as the sum of the residues of the simple poles of the integrand in $\text{(79)}$ times $\pi$ and is given as

$$W_{p,p} = -\frac{2 \text{Im} \Gamma}{V_{p+1}},$$

$$= \frac{2^3}{\bar{\nu}_0 (8\pi^2 \alpha')} \left[ \det(\eta + \bar{F}'') \det(\eta + \bar{F}) \right]^{\frac{1}{2}} \sinh \pi \bar{\nu}_0 \sin \pi \nu_1 \sin \pi \nu_2 \sum_{k=1}^{\infty} (-k+1) \left( \frac{\bar{\nu}_0}{k} \right)^{p-3} \frac{2 \nu_0^2}{\pi \nu_0^2} \left[ \frac{k \nu_1 \nu_2}{\nu_0} - (-k) \cosh \frac{k \nu_1 \nu_2}{\nu_0} \right] Z_k(\nu_0, \nu_1, \nu_2),$$

where

$$Z_k = \prod_{n=1}^{\infty} \frac{[1 - 2(-)^k|z_k|^{2n} \cosh \frac{k \nu_1 + \nu_2}{\nu_0} + |z_k|^{4n}]^2 [1 - 2(-)^k|z_k|^{2n} \cosh \frac{k \nu_1 - \nu_2}{\nu_0} + |z_k|^{4n}]^2}{(1 - |z_k|^{2n})^4 [1 - 2|z_k|^{2n} \cosh \frac{2k \nu_1 \nu_2}{\nu_0} + |z_k|^{4n}] [1 - 2|z_k|^{2n} \cosh \frac{2k \nu_1 \nu_2}{\nu_0} + |z_k|^{4n}]},$$

with $|z_k| = e^{-k \pi / \nu_0}$. Note that when $\bar{\nu}_0 \to \infty$, we have $Z_k \to \infty$ for $k = \text{odd}$ while $Z_k \to 1$ for $k = \text{even}$ due to $|z_k| \to 1$. For the rate, the odd-$k$ terms, each is blowing up and positive, are dominant over the almost vanishing negative even-terms, and so the rate blows up. This gives another singularity. As we will see, this is due to the electric field reaching its critical value. The open strings break under the action of the critical field and their production cascades.
According to [42], the rate (83) should be more properly interpreted as the decay one of the underlying system rather than the pair production one. The pair production rate is just the leading \( k = 1 \) term in the above rate as

\[
W^{(1)}_{p,p} = \frac{2^2 \left[ \det(\eta + \hat{F}') \det(\eta + \hat{F}) \right]^{1/2} \sinh \pi \bar{\nu}_0 \sin \pi \nu_1 \sin \pi \nu_2 \frac{\eta^{-3}}{\bar{\nu}_0^2} e^{-\eta^2/2\pi \bar{\nu}_0} \right. \\
\times \left. \left( \cosh \frac{\nu_1}{\bar{\nu}_0} + \cosh \frac{\nu_2}{\bar{\nu}_0} \right)^2 \sinh \frac{\pi \nu_1}{\bar{\nu}_0} \sinh \frac{\pi \nu_2}{\bar{\nu}_0} \right] \frac{1}{Z_1(\bar{\nu}_0, \nu_1, \nu_2)}. \tag{85}
\]

So the pair production simply cascades when \( \bar{\nu}_0 \to \infty \). Since \( W^{(1)} \) is symmetric to \( \nu_1 \) and \( \nu_2 \), without loss of generality and for convenience, we assume \( \nu_1 \geq \nu_2 \) for the following discussion. Given \( \nu_1 \geq \nu_2 \in (0, 1/2] \), one can check that \( Z_1 \), from (84), increases as \( \bar{\nu}_0 \) increases. When \( \bar{\nu}_0 \gg \nu_1 \), the factor \( \left( \cosh \frac{\nu_1}{\bar{\nu}_0} + \cosh \frac{\nu_2}{\bar{\nu}_0} \right)^2 \sinh \frac{\pi \nu_1}{\bar{\nu}_0} \sinh \frac{\pi \nu_2}{\bar{\nu}_0} \sim 4\bar{\nu}_0^2/(\pi^2 \nu_1 \nu_2) \) increases also when we increase \( \bar{\nu}_0 \). Since \( p \geq 5 \), all other factors have an overall increase when we increase \( \bar{\nu}_0 \). This holds true at least for the most interesting cases with a large enhancement of the rate and is also expected since \( \bar{\nu}_0 \) is related to the applied electric fluxes and increases when any of them increases, which will be explicitly demonstrated in an example given later. When \( \bar{\nu}_0 \sim \nu_1 \geq \nu_2 \), this same factor will not play important role for the rate. The rate is now controlled by the other factors and still increases with \( \bar{\nu}_0 \). If \( \bar{\nu}_0 \ll \nu_2 \leq \nu_1 \), this implies \( \bar{\nu}_0 \ll 1 \). So \( Z_1(\bar{\nu}_0, \nu_1, \nu_2) \approx 1 \). The pair production rate is

\[
W^{(1)}_{p,p} \approx \frac{8\pi \left[ \det(\eta + \hat{F}') \det(\eta + \hat{F}) \right]^{1/2} \sin \pi \nu_1 \sin \pi \nu_2 \frac{\bar{\nu}_0^{-3}}{\nu_0^2} e^{-\eta^2/2\pi \bar{\nu}_0} \right. \\
\times \left. \left( \cosh \frac{\nu_1}{\bar{\nu}_0} + \cosh \frac{\nu_2}{\bar{\nu}_0} \right)^2 \sinh \frac{\pi \nu_1}{\bar{\nu}_0} \sinh \frac{\pi \nu_2}{\bar{\nu}_0} \right] \frac{1}{Z_1(\bar{\nu}_0, \nu_1, \nu_2)}. \tag{86}
\]

where the factor \( e^{\pi(\nu_1-\nu_2)/\nu_0} \gg 1 \), a large enhancement of the rate in the presence of magnetic fluxes. If \( y > \pi \sqrt{2(\nu_1 - \nu_2)\alpha'} \) (for avoiding the tachyonic instability), the rate still increases when we increase \( \bar{\nu}_0 \). For the purpose of illustration, we consider the \( p = p' = 5 \) case and take the following simple flux for \( \hat{F} \) as

\[
\hat{F} = \begin{pmatrix}
0 & -\hat{f} \\
\hat{f} & 0 \\
0 & -\hat{g}_1 \\
\hat{g}_1 & 0 \\
0 & -\hat{g}_2 \\
g_2 & 0
\end{pmatrix}, \tag{87}
\]
where $\hat{f}$ stands for the electric flux along 01-directions while $\hat{g}_1, \hat{g}_2$ are the magnetic ones along 23- and 45-directions, respectively. Similarly for $\hat{F}'$ but denoting every quantity with a prime. We can then determine the eigenvalues as

$$\lambda_0 + \lambda_0^{-1} = 2 \frac{(1 + \hat{f}^2)(1 + \hat{f}'^2) - 4\hat{f}\hat{f}'}{(1 - \hat{f}^2)(1 - \hat{f}'^2)}, \quad \lambda_a + \lambda_a^{-1} = 2 \frac{(1 - \hat{g}_a^2)(1 - \hat{g}'_a^2) + 4\hat{g}_a\hat{g}'_a}{(1 + \hat{g}_a^2)(1 + \hat{g}'_a^2)},$$

where $a = 1, 2$. By setting $\lambda_a = e^{2\pi\nu_a}$ with $\alpha = (0, a)$, we have

$$\tanh \pi \nu_0 = \frac{|\hat{f} - \hat{f}'|}{1 - \hat{f}\hat{f}'}, \quad \tan \pi \nu_a = \frac{|\hat{g}_a - \hat{g}'_a|}{1 + \hat{g}_a\hat{g}'_a}$$

where we have set $\nu_0 = i\nu_0$. Note that $|\hat{f}|, |\hat{f}'| < 1$ while $|\hat{g}_a|, |\hat{g}'_a| < \infty$. As explained earlier, we always have $\nu_0 \in (0, \infty)$ and $\nu_a \in [0, 1/2]$ for the amplitude and the rate. It is clear that $\nu_0$ increases when we increase $\hat{f}$ or $\hat{f}'$ as mentioned earlier. Now the factor $\det(\eta + \hat{F}') \det(\eta + \hat{F}) = (1 - \hat{f}^2)(1 - \hat{f}'^2) \prod_{a=1}^2 (1 + \hat{g}_a^2)(1 + \hat{g}'_a^2)$. The open string pair production rate can now be expressed as

$$\mathcal{W}_{p,p}^{(1)} = \frac{2^3 |\hat{f} - \hat{f}'| |\hat{g}_1 - \hat{g}'_1| |\hat{g}_2 - \hat{g}'_2|}{\nu_0 (8\pi^2 \alpha')^{\frac{8}{11}}} \frac{\pi^3}{\nu_0^2} e^{-\frac{8\pi^2}{24\nu_0 \nu_a}} \left(\frac{\cosh \frac{\pi \nu_0}{\nu_0} + \cosh \frac{\pi \nu_a}{\nu_a}}{\sinh \frac{\pi \nu_0}{\nu_0} \sinh \frac{\pi \nu_a}{\nu_a}}\right)^2 Z_1(\nu_0, \nu_1, \nu_2),$$

where $\nu_0$ and $\nu_a$ with $a = 1, 2$ are given in (89). This rate also applies to the $p = p' = 6$ case. The earlier general discussion on how the pair production rate depends on the applied electric fluxes or $\nu_0$ for fixed $\nu_a$ with $a = 1, 2$ continues to hold and we will not repeat it here. We focus now on how the rate depends on the applied magnetic fluxes or $\nu_a$ for fixed non-vanishing $\nu_0$. The rate for vanishing magnetic fluxes can be obtained from (90) as

$$\mathcal{W}_{p,p}^{(1)}(\hat{g}_a = \hat{g}'_a = 0) = \frac{2^5 |\hat{f} - \hat{f}'|}{(8\pi^2 \alpha')^{\frac{8}{11}}} \frac{\pi^3}{\nu_0^2} e^{-\frac{8\pi^2}{24\nu_0 \nu_a}} Z_1(\nu_0, \nu_a = 0).$$

We have then

$$\frac{\mathcal{W}_{p,p}^{(1)}(\hat{g}_a, \hat{g}'_a \neq 0)}{\mathcal{W}_{p,p}^{(1)}(\hat{g}_a = \hat{g}'_a = 0)} = \frac{|\hat{g}_1 - \hat{g}'_1| |\hat{g}_2 - \hat{g}'_2|}{4\nu_0^2} \left(\frac{\cosh \frac{\pi \nu_0}{\nu_0} + \cosh \frac{\pi \nu_a}{\nu_a}}{\sinh \frac{\pi \nu_0}{\nu_0} \sinh \frac{\pi \nu_a}{\nu_a}}\right)^2 \frac{Z_1(\nu_0, \nu_a \neq 0)}{Z_1(\nu_0, \nu_a = 0)}.$$  

For non-vanishing $\nu_a \in (0, 1/2]$ and $|\hat{g}_a - \hat{g}'_a| \sim \mathcal{O}(1)$, if $\nu_0$ is not too small, the presence of magnetic fluxes will not give a significant enhancement of the rate as can be seen from the above. However, if instead $\nu_a/\nu_0 \gg 1$ and $|\hat{g}_a - \hat{g}'_a| \geq 1$ or $\nu_a/\nu_0 \ll 1$ but all $\hat{g}_a, \hat{g}'_a$ are large, with $a = 1, 2$, the rate has a significant enhancement. Let us consider the
later case for which all $\hat{g}_a$ and $\hat{g}_a'$ are large. Now $\nu_a$ are small. So from (89), we have
\[ |1 + \hat{g}_a\hat{g}_a'|\nu_a = |\hat{g}_a - \hat{g}_a'|. \]
The ratio of (92) becomes
\[ \mathcal{W}_{p,p}^{(1)}(\hat{g}_a, \hat{g}_a' \neq 0) \approx |\hat{g}_a\hat{g}_a'\hat{g}_2\hat{g}_2'| \gg 1, \] (93)
much enhanced. In the above, we have used $Z_1(\bar{\nu}_0, \nu_a) \approx Z_1(\bar{\nu}_0, \nu_a = 0)$.

Unless we consider relevant physics in string scale, the fluxes $\hat{f}$, $\hat{f}'$, $\hat{g}_a$ and $\hat{g}_a'$ are in general small in terms of string scale. In other words, $|\hat{f}| \ll 1, |\hat{f}'| \ll 1, |\hat{g}_a| \ll 1$ and $|\hat{g}_a'| \ll 1$. We then have $\pi\nu_0 = |\hat{f} - \hat{f}'| \ll 1, \pi\nu_a = |\hat{g}_a - \hat{g}_a'| \ll 1$. The rate (90) becomes
\[ \mathcal{W}_{p,p}^{(1)} = \frac{8\pi^3 \nu_1 \nu_2}{(8\pi^2 \alpha')^{p-3}} e^{-\frac{\nu^2}{2\pi\nu_0}} \left( \frac{\cosh \frac{\pi\nu_1}{\nu_0} + \cosh \frac{\pi\nu_2}{\nu_0}}{\sinh \frac{\pi\nu_1}{\nu_0} \sinh \frac{\pi\nu_2}{\nu_0}} \right)^2, \] (94)
where $Z_1(\bar{\nu}_0, \nu_1, \nu_2) \approx 1$. In the weak field limit, we showed in 31 that adding magnetic fluxes $\hat{g}_2$ and $\hat{g}_2'$, assuming $\nu_2 \leq \nu_1$, in general diminishes rather than enhances the rate. This can also be understood via the tachyonic shift discussed earlier. So for the purpose of enhancing the rate via adding magnetic fluxes, we merely need to add only the possible largest fluxes $\hat{g}_1$ and $\hat{g}_1'$. In other words, for given $\bar{\nu}_0 \ll 1$ and the largest possible $\nu_1$, the corresponding largest possible rate is
\[ \mathcal{W}_{p,p}^{(1)} = \frac{8\pi^2 \nu_1 \nu_2}{(8\pi^2 \alpha')^{p-1}} e^{-\frac{\nu^2}{2\pi\nu_0}} \left( \frac{\cosh \frac{\pi\nu_1}{\nu_0} + 1}{\sinh \frac{\pi\nu_1}{\nu_0}} \right)^2. \] (95)
This rate formula is actually valid for $p \geq 3$ 26. For given $\bar{\nu}_0 \ll 1$ and $\nu_1 \ll 1$, it is clear that the smallest $p = 3$ case gives the largest rate 21,31. The enhancement due to the added magnetic fluxes is
\[ \frac{\mathcal{W}_{p,p}^{(1)}(\bar{\nu}_0, \nu_1 \neq 0)}{\mathcal{W}_{p,p}^{(1)}(\bar{\nu}_0, \nu_1 = 0)} = \frac{\pi \nu_1}{\bar{\nu}_0} \left[ 1 + \cosh \frac{\pi\nu_1}{\nu_0} \right]^2 \frac{4 \sinh \frac{\pi\nu_1}{\nu_0}}{\nu_0 \nu_1}, \] (96)
which is always greater than unity for $\pi\nu_1/\bar{\nu}_0 > 0$. One can check this numerically. In particular, when $\nu_1/\bar{\nu}_0 \gg 1$, this ratio becomes
\[ \frac{\mathcal{W}_{p,p}^{(1)}(\bar{\nu}_0, \nu_1 \neq 0)}{\mathcal{W}_{p,p}^{(1)}(\bar{\nu}_0, \nu_1 = 0)} = \frac{\pi \nu_1}{8 \bar{\nu}_0} e^{\frac{\pi\nu_1}{\nu_0}} \gg 1. \] (97)

### 3.2 The $p = p' < 5$ cases

Given the discussion for $p = p' = 5,6$ cases in the previous subsection, the relevant discussion for $p = p' < 5$ is straightforward. We will spell out this in detail in this
subsection. To be concrete, let us explain the rationale behind the integral representation structure of the cylinder amplitude \((54)\) for \(p = p' < 5\).

In using the closed string boundary state representation of D-brane to compute the cylinder interaction amplitude between two parallel placed D-branes of the same or different dimensionality at a separation, we note that the worldvolume dimensionality of the respective D-brane is encoded in its \(M\)-matrix \((7)\), the bosonic zero-mode \((8)\) in the bosonic sector and the fermionic zero-mode \((9)\) in the R-R sector in the matter part. The rest are independent of this dimensionality. Let us first examine carefully the \(M\)-matrix \((7)\) which we rewrite here for convenience,

\[
M = \left[ (\eta - \hat{F})(\eta + \hat{F})^{-1}\right]_{\alpha\beta, -\delta_{ij}},
\]

where \(\alpha, \beta\) are along the brane directions while \(i, j\) are along the directions transverse to the brane. For example, let us first consider the D6 brane. In other words, \(\alpha, \beta = 0, 1, \ldots, 6\) and \(i, j = 7, 8, 9\). For any other \(Dp\) with even \(p < 6\), we denote their \(\alpha', \beta' = 0, 1, \ldots, p\) along its brane directions and \(i', j' = p + 1, \ldots, 9 - p\) as directions transverse to this brane. Its corresponding \(M_p\)-matrix with a general worldvolume flux \((\hat{F}_p)^{\alpha'\beta'}\) can be taken as a special case of the D6 brane, namely \(M_6\), in the following sense. For the \(Dp\) with even \(p < 6\), we have

\[
M_p = \left[ (\eta_p - \hat{F}_p)(\eta_p + \hat{F}_p)^{-1}\right]_{\alpha'\beta', -\delta_{i'j'}}.
\]

We now extend \(\alpha', \beta' = 0, 1, \ldots, p\) to \(\alpha, \beta = 0, 1, \ldots, 6\) and \(\hat{F}_p\) to \(\hat{F}_6\) taking the following special form,

\[
(\hat{F}_6)_{\alpha\beta} = \begin{pmatrix}
(\hat{F}_p)^{\alpha'\beta'} & 0 & \hat{g}_1 & 0 & \cdots

0 & \hat{g}_1 & 0 & \cdots

\vdots & 0 & \hat{g}_{p-2} & \hat{g}_{p-1} & 0

\end{pmatrix}_{7 \times 7}.
\]

With this special choice, the \(M_6\) turns out to give just \(M_p\) when we take the special magnetic fluxes \(\hat{g}_k \to \infty\) with \(k = 1, \cdots, (6 - p)/2\). Let us see this in detail. With the
special flux \([100]\), we have

\[
(M_6)_{\alpha\beta} = \begin{pmatrix}
(M_p)_{\alpha'\beta'} & \frac{1-g^2_2}{1+g^2_2} & \frac{2-g_1}{1+g^2_2} \\
\frac{1+g^2_2}{1+g^2_2} & \frac{1-g^2_2}{1+g^2_2} & \frac{2-g_1}{1+g^2_2} \\
\frac{2-g_1}{1+g^2_2} & \frac{2-g_1}{1+g^2_2} & \frac{1-g^2_2}{1+g^2_2} \\
\vdots & \vdots & \vdots \\
\frac{1-g^2_2}{1+g^2_2} & \frac{2-g_1}{1+g^2_2} & \frac{1-g^2_2}{1+g^2_2} \\
\end{pmatrix},
\]

which becomes \((M_6)_{\alpha\beta} = ((M_p)_{\alpha'\beta'}, -\delta_{\nu'\nu'})\) with \(k', \nu' = p + 1, \ldots, 6\) when we take \(g_k \to \infty\) with \(k = 1, \ldots, (6 - p)/2\). So we have \(M_6 = ((M_p)_{\alpha'\beta'}, -\delta_{\nu'\nu'}) = M_p\) for the above special choice of the flux \(\hat{F}_6(100)\) when we take \(\hat{g}_k \to \infty\) with \(k = 1, \ldots, (6 - p)/2\). In other words, \(M_p\) is just a special case of \(M_6\) when the worldvolume flux of \(D6\) takes a special choice as indicated above. This same discussion applies to the odd \(p < 5\) from \(p = 5\).

This same applies to the R-R zero-mode contribution \([50]\) to the amplitude. We discuss this in great detail in Appendix B and refer there for detail. These two considerations explain the following part of the integrand in the amplitude \([54]\),

\[
\frac{\theta_1 (\nu_0 + \nu_1 + \nu_2) | it \rangle \theta_1 (\nu_0 - \nu_1 - \nu_2) | it \rangle \theta_1 (\nu_0 + \nu_1 - \nu_2) | it \rangle \theta_1 (\nu_0 - \nu_1 + \nu_2) | it \rangle \theta_1 (\nu_0 | it \rangle \theta_1 (\nu_1 | it \rangle \theta_1 (\nu_2 | it \rangle \eta^3 | it \rangle \prod_{a=0}^{2} \sin \pi \nu_a,\]

which is valid in general for \(p = p' = 5\) or 6 but will reduce to the corresponding expected one for \(p = p' < 5\) once the respective special flux such as \([100]\) is chosen and the corresponding limit is taken. However, the story for the bosonic zero-mode \([55]\) is different. Except for the overall factor \([\det(\eta_p + \hat{F}_p)]^{1/2}\), the other part of the zero mode has nothing to do with the applied worldvolume flux and therefore this same trick as used for the \(M\)-matrix and the RR zero-mode does not apply here. This zero-mode contribution to the amplitude gives essentially the other part of the integrand as

\[
\sqrt{\det(\eta_p + \hat{F}_p') \det(\eta_p + \hat{F}_p)} V_{NN} (2\pi^2 \alpha' t)^{-\frac{DD}{2}} e^{-\frac{\nu^2}{2\pi \alpha'}},
\]

where \(V_{NN} = V_{p'+1}\) denotes the volume of the \(Dp'\) worldvolume following the conventions given in footnote \([1]\) and DD denotes the DD-directions. Here DD = 9 − \(p\) with our conventions. It is obvious that the \(t^{-(9-p)/2}\)-factor in the amplitude \([54]\) for \(p = p' < 5\) cannot be obtained from \(p = p' = 5\) or 6 even we choose the respective special fluxes and take the corresponding limits.
We therefore give an explanation to the cylinder amplitude (54) for the case of \( p = p' < 5 \). Given the extended flux (100) for \( \hat{F}_6 \) (or \( \hat{F}_5 \)) and similarly for \( \hat{F}_6' \) (or \( \hat{F}_5' \)) but with now \( \hat{g}'_k \) with a prime, we have now at least

\[
\tan \pi \nu_2 = \left\{ \begin{array}{ll}
\frac{\hat{g}_{a-p} - \hat{g}'_{a-p}}{1 + \hat{g}_{a-p} \hat{g}'_{a-p}} & \text{for even } p < 6,
\frac{\hat{g}_{5-p} - \hat{g}'_{5-p}}{1 + \hat{g}_{5-p} \hat{g}'_{5-p}} & \text{for odd } p < 5,
\end{array} \right.
\tag{104}
\]

which gives \( \nu_2 \to 0 \) when we take the limits \( \hat{g}(6-p)/2 \to \infty \) and \( \hat{g}'(6-p)/2 \to \infty \) (or \( \hat{g}(5-p)/2 \to \infty \) and \( \hat{g}'(5-p)/2 \to \infty \)). So with a vanishing \( \nu_2 = 0 \), we have \( \theta_1(\nu_2|it) \to 2\eta^3(it) \sin \pi \nu_2 \) and the closed string tree-level cylinder amplitude (54) now becomes

\[
\Gamma_{p,p} = \frac{2^2 V_{p+1} \left[ \det(\eta_p + \hat{F}'_p) \det(\eta_p + \hat{F}_p) \right]^{1/2}}{(8\pi^2 \alpha')^{2+1/2}} \int_0^{\infty} \frac{dt}{t^{2-\frac{p}{2}}} \frac{e^{-\frac{y^2}{4\pi \alpha'}}}{\theta_1^{\ast}(\nu_0|it)\theta_1^{\ast}(\nu_1|it)} \theta_1^{\ast}(\nu_0+\nu_1|it) \theta_1^{\ast}(\nu_0-\nu_1|it)
\times \prod_{\alpha=0}^{1} \sin \pi \nu_{\alpha}

= \frac{2^2 V_{p+1} \left[ \det(\eta_p + \hat{F}'_p) \det(\eta_p + \hat{F}_p) \right]^{1/2} (\cos \pi \nu_0 - \cos \pi \nu_1)^2}{(8\pi^2 \alpha')^{2+1/2}} \int_0^{\infty} \frac{dt}{t^{2-\frac{p}{2}}} e^{-\frac{y^2}{2\pi \alpha'}} \prod_{n=1}^{\infty} C_n,
\tag{105}
\]

where \( C_n \), from (55), becomes,

\[
C_n = \frac{\tilde{C}_n}{(1 - |z|^{2n})^4 \prod_{\alpha=0}^{1} \left[ 1 - 2 |z|^{2n} \cos 2\pi \nu_{\alpha} + |z|^{4n} \right]},
\tag{106}
\]

with \( \tilde{C}_n \), from (56) or (57),

\[
\tilde{C}_n = [1 - 2 |z|^{2n} \cos \pi (\nu_0 + \nu_1) + |z|^{4n}]^2 [1 - 2 |z|^{2n} \cos \pi (\nu_0 - \nu_1) + |z|^{4n}]^2,
\tag{107}
\]

\[
\tilde{C}_n \overset{\text{or}}{=} [1 - 2 |z|^{2n} e^{i\pi \nu_0} \cos \pi \nu_1 + e^{2\pi i \nu_0} |z|^{4n}]^2 [1 - 2 |z|^{2n} e^{-i\pi \nu_0} \cos \pi \nu_1 + e^{-2\pi i \nu_0} |z|^{4n}]^2.
\tag{108}
\]
The corresponding open string one-loop annulus amplitude, from (58), is now

\[
\Gamma_{p,p} = -\frac{2^2 V_{p+1} \left[ \det(\eta_p + \tilde{F}_p) \det(\eta_p + \tilde{F}_p') \right]^{\frac{1}{2}} \prod_{\alpha=0}^{1} \sin \pi \nu_{\alpha} }{(8 \pi^2 \alpha')^{\frac{p+1}{2}}} \int_0^{\infty} \frac{dt}{t^{\frac{p+1}{2}}} e^{-\frac{3}{2} \pi t} \eta^p(it) \\
\times \left. \frac{\theta^2_1 \left( \frac{\nu_0+\nu_1}{2} \right) }{\theta_1(\nu_0it|it)\theta_1(\nu_1it|it)} \right|_{t=0}^{\infty}, \\
= 2^2 V_{p+1} \left[ \det(\eta_p + \tilde{F}_p) \det(\eta_p + \tilde{F}_p') \right]^{\frac{1}{2}} \prod_{\alpha=0}^{1} \sin \pi \nu_{\alpha} \int_0^{\infty} \frac{dt}{t^{\frac{p+1}{2}}} e^{-\frac{3}{2} \pi t} \\
\times \left. \frac{(\cosh \pi \nu_0 t - \cosh \pi \nu_1 t)^2}{\sinh \pi \nu_0 t \sinh \pi \nu_1 t} \right|_{t=0}^{\infty} \prod_{n=1}^{\infty} Z_n, \\
\right. \\
\] (109)

where \( Z_n \), from (60), becomes

\[
Z_n = \left. \frac{\tilde{Z}_n}{(1 - |z|^{2n})^4 \prod_{\alpha=0}^{1} [1 - 2|z|^{2n} \cosh 2\pi \nu_{\alpha} t + |z|^{4n}]} \right|_{t=0}^{\infty}, \\
\] (110)

with \( \tilde{Z}_n \), from (61) or (62),

\[
\tilde{Z}_n = \left[ 1 - 2|z|^{2n} \cosh \pi (\nu_0 + \nu_1) t + |z|^{4n} \right]^2 \left[ 1 - 2|z|^{2n} \cosh \pi (\nu_0 - \nu_1) t + |z|^{4n} \right]^2, \\
\] (111)

or

\[
= \left[ 1 - 2|z|^{2n} e^{-\pi \nu_0 t} \cosh \pi \nu_1 t + e^{-2\pi \nu_0 t} |z|^{4n} \right]^2 \left[ 1 - 2|z|^{2n} e^{\pi \nu_0 t} \cosh \pi \nu_1 t + e^{2\pi \nu_0 t} |z|^{4n} \right]^2. \\
\] (112)

As before, the large \( y \) interaction can be obtained from (105) with the large \( t \)-integration as

\[
\frac{\Gamma_{p,p}}{V_{p+1}} \approx \left. \frac{(\cos \pi \nu_0 - \cos \pi \nu_1)^2 \sqrt{\det(\eta_p + \tilde{F}_p) \det(\eta_p + \tilde{F}_p')}}{2^{p-1} \pi^{p+1} (2 \pi \alpha')^{p-3} y^{7-p}} \right|_{t=0}^{\infty} \Gamma \left( \frac{7 - p}{2} \right), \\
\] which is always non-negative, therefore implying an attractive interaction in general. This is consistent with what has been discussed for the \( p = p' = 5 \) or 6 case given in the previous subsection. In other words, whenever \( p = p' < 6 \), the long-range interaction is always non- repulsive. The only possible long-range repulsive interaction for \( p = p' = 6 \) with all three \( \nu_0, \nu_1, \nu_2 \) real and non-vanishing. When \( \nu_0 \) is imaginary, given as \( \nu_0 = i\tilde{\nu}_0 \) with \( \tilde{\nu}_0 \in (0, \infty) \), once again the integrand of the open string one-loop amplitude (109) has an infinite number of simples poles at \( t_k = k/\tilde{\nu}_0 \) with \( k = 1, 2, \cdots \) along the positive \( t \)-axis, indicating the decay of the underlying system via the open string pair production. The decay rate and the corresponding open string pair production rate can be computed
as before, respectively, as

$$W_{p,p} = \frac{2^3}{\bar{v}_0(8\pi^2\alpha')^{\frac{p+1}{2}}} \det(\eta_p + \hat{F}_p) \det(\eta_p + \hat{F}_p) \sqrt{\pi\bar{v}_0 \sin \pi\nu_1} \sum_{k=1}^{\infty} (-)^{k+1} \left( \frac{\bar{v}_0}{k} \right)^{\frac{p-1}{2}} e^{-\frac{k\pi^2 \nu_0}{\bar{v}_0}} \left( \cosh \frac{k\pi\nu_1}{\bar{v}_0} - (-)^k \right)^2 \frac{\sinh \frac{k\pi\nu_1}{\bar{v}_0}}{\sinh \frac{k\pi\nu_1}{\bar{v}_0}} Z_k(\bar{v}_0, \nu_1),$$

(113)

where

$$Z_k(\bar{v}_0, \nu_1) = \prod_{n=1}^{\infty} \left[ 1 - 2(-)^k |z_k|^{2n} \cosh \frac{k\pi\nu_1}{\bar{v}_0} + |z_k|^{4n} \right]^{\frac{1}{4}} \left[ 1 - 2|z_k|^{2n} \cosh \frac{2k\pi\nu_1}{\bar{v}_0} + |z_k|^{4n} \right]^{\frac{1}{2}},$$

(114)

with $|z_k| = e^{-k\pi/\bar{v}_0}$, and

$$W_{p,p}^{(1)} = \frac{2^3}{\bar{v}_0(8\pi^2\alpha')^{\frac{p+1}{2}}} \det(\eta_p + \hat{F}_p) \det(\eta_p + \hat{F}_p) \sqrt{\pi\bar{v}_0 \sin \pi\nu_1} \frac{\bar{v}_0^{\frac{p-1}{2}} e^{-\frac{k\pi^2 \nu_0}{\bar{v}_0}}}{\bar{v}_0^2} \left( \cosh \frac{\pi\nu_1}{\bar{v}_0} + 1 \right)^2 \frac{\sinh \frac{\pi\nu_1}{\bar{v}_0}}{\sinh \frac{\pi\nu_1}{\bar{v}_0}} Z_1(\bar{v}_0, \nu_1),$$

(115)

The above two rates can be obtained from (83) and (85), respectively, by taking the limit $\nu_2 \to 0$. We now discuss the cases for $p = p' < 5$ one by one in what follows.

The $p = p' = 3$ or 4 case: The $p = p' = 3$ can be obtained from the $p = p' = 5$ while $p = p' = 4$ can be obtained from the $p = p' = 6$ in the sense described in the present subsection given above. In either case, the worldvolume flux can be extended the following way,

$$\hat{F}_{\alpha\beta} = \begin{pmatrix} \hat{F}_{\alpha'\beta'} & \hat{g}_2 \\ -\hat{g}_2 & 0 \end{pmatrix}, \quad \hat{F}_{\alpha'\beta'} = \begin{pmatrix} 0 & \hat{g}'_2 \\ -\hat{g}'_2 & 0 \end{pmatrix}. $$

(116)

We then have here

$$\tan \pi\nu_2 = \left| \frac{\hat{g}_2 - \hat{g}'_2}{1 + \hat{g}_2\hat{g}'_2} \right|,$$

(117)

which gives $\nu_2 \to 0$ when we take $\hat{g}_2 \to \infty$ and $\hat{g}'_2 \to \infty$. The general closed string tree-level cylinder amplitude is just given by (105) while the corresponding open string one-loop annulus one is given by (109). The respective physics of these amplitudes such as the nature of the interaction, the relevant instabilities and the potential open string
pair production and its enhancement can be similarly discussed in general following what we have done for the respective $p = p' = 5$ and $p = p' = 6$ cases. So we will not repeat the same discussion here. For example, one typical interesting case is the $p = p' = 3$ one for the following choice of fluxes,

$$
\hat{F}_{\alpha'\beta'} = \begin{pmatrix}
0 & \hat{f} \\
-\hat{f} & 0 \\
0 & \hat{g}_1 \\
-\hat{g}_1 & 0
\end{pmatrix}_{4\times4},
\quad \hat{F}'_{\alpha'\beta'} = \begin{pmatrix}
0 & \hat{f}' \\
-\hat{f}' & 0 \\
0 & \hat{g}'_1 \\
-\hat{g}'_1 & 0
\end{pmatrix}_{4\times4}.
$$

With the above fluxes, we have

$$
\tanh \pi \hat{\nu}_0 = \frac{|\hat{f} - \hat{f}'|}{1 - \hat{f} \hat{f}'};
\quad \tan \pi \nu_1 = \left| \frac{\hat{g}_1 - \hat{g}'_1}{1 + \hat{g}_1 \hat{g}'_1} \right|.
$$

The closed string cylinder amplitude, the open string annulus amplitude, the decay rate and open string pair production rate of this system can be directly read from (105), (109), (113) and (115), respectively, for the present case. Their explicit expressions will not be written down here. Their analysis, in particular the open string pair production and its enhancement along with their potential applications, has been discussed in great detail in [21, 30, 31]. Again we will not repeat it here and refer there for detail. For the $p = p' = 3$ case, the discussion with the most general worldvolume fluxes is given explicitly in a forthcoming paper by one of the present authors [43] and the basic conclusion remains the same. For example, the interaction amplitude can be given in terms of six Lorentz invariants constructed from the fluxes.

The $p = p' = 1$ or 2 case: The $p = p' = 1$ can be obtained from the $p = p' = 5$ while $p = p' = 2$ can be obtained from the $p = p' = 6$ again in the sense described in the present subsection given earlier. In either case, the worldvolume flux can be extended the following way as

$$
\hat{F}_{\alpha\beta} = \begin{pmatrix}
\hat{F}_{\alpha'\beta'} \\
0 & \hat{g}_1 \\
-\hat{g}_1 & 0 \\
0 & \hat{g}_2 \\
-\hat{g}_2 & 0
\end{pmatrix},
\quad \hat{F}'_{\alpha\beta} = \begin{pmatrix}
\hat{F}'_{\alpha'\beta'} \\
0 & \hat{g}'_1 \\
-\hat{g}'_1 & 0 \\
0 & \hat{g}'_2 \\
-\hat{g}'_2 & 0
\end{pmatrix}.
$$

We then have

$$
\tan \pi \nu_1 = \left| \frac{\hat{g}_1 - \hat{g}'_1}{1 + \hat{g}_1 \hat{g}'_1} \right|;
\quad \tan \pi \nu_2 = \left| \frac{\hat{g}_2 - \hat{g}'_2}{1 + \hat{g}_2 \hat{g}'_2} \right|.
$$

37
where both $\nu_1 \to 0$ and $\nu_2 \to 0$ when we take $\hat{g}_k \to \infty, \hat{g}'_k \to \infty$ with $k = 1, 2$. The closed string cylinder amplitude can be obtained from (105) along with (110) and (111) or (112) by taking $\nu_1 \to 0$ limit. The open string one-loop annulus amplitude can be obtained from (109) along with (110) and (111) or (112) also by taking $\nu_1 \to 0$ limit. Since either of these is straightforward, we will not rewrite the corresponding amplitude here. The nature of interaction, the potential instabilities as well as the open string pair production can also be similarly discussed and will not be present here. However, we would like to stress for the present case that we don’t have the same enhancement of the open string pair production as discussed in the $p = p' = 5$ or 6 as well as in the previous work [6, 21, 26, 30], which requires $p = p' \geq 3$ so that the needed magnetic flux can be added. There can be some mild enhancement of open string pair production for the system of $p = p' = 2$ case, which also occurs for $2 \leq p = p' \leq 6$, as discussed in [26] by one of the present authors, when the added fluxes satisfy certain conditions. We refer this to [26] for detail.

The $p = p' = 0$ case: This is a trivial one and can be obtained from the $p = p' = 6$, similarly as above, by setting $\nu_0 \to 0$ and $\nu_1 \to 0$ in the amplitude (105) or (109). As expected, we simply have here $\Gamma_{0,0} = 0$. For this system, there are no fluxes which can be added to the worldvolume and therefore this system remains still as a 1/2 BPS one. The $\Gamma_{0,0} = 0$ is just the usual “no-force” condition.

In summary, in this section, we compute the closed string cylinder as well as the corresponding open string one-loop annulus amplitude for the system of two Dp branes, placed parallel at separation, with $0 \leq p = p' \leq 6$, carrying the most general respective worldvolume fluxes. We use a trick, based on the properties of the matrix $M (7)$ and the various zero-modes in the matter sector of the closed string boundary state representation of Dp-brane, to obtain the respective closed string cylinder amplitude for the lower dimensional D-brane system of $p = p' < 5$ from that of either $p = p' = 5$ or 6 in the sense described earlier. We give a general discussion on the properties of the amplitudes such as the nature of the interaction, the onset of potential tachyonic instability which is associated with the added worldvolume magnetic fluxes and the open string pair production when an electric flux is added. In particular, we find that the interaction can be repulsive only for $p = p' = 6$ and when the added fluxes are all magnetic with the possible largest one of three $\nu_\alpha$’s smaller than the sum of the remaining two. Otherwise, it is attractive. We also find that the nature of the interaction is correlated with the existence of a potential tachyonic instability. When the interaction is repulsive, there is no tachyonic instability. Otherwise, there is a potential one. We give also a detail discussion on the open string
4 Amplitude and its properties: the \( p \neq p' \) case

In this section, we move to compute the closed string tree-level cylinder amplitude between one Dp and the other Dp', placed parallel at a separation transverse to the Dp, with \( p - p' = \kappa = 2, 4, 6 \) and \( p \leq 6 \) and with each brane carrying a general worldvolume flux. Here without loss of generality, we assume \( p > p' \). The discussion given in the previous section for computing the cylinder amplitude for \( p = p' \) case makes it easier to carry out the computations in the present section. Once the cylinder amplitude is obtained, we can again use a Jacobi transformation to obtain the corresponding open string one-loop annulus amplitude.

The trick used in the subsection 3.2 helps us here in obtaining the amplitude for \( p \neq p' \) from that of \( p = p' \) if we extend the general flux \( \hat{F}_{p'} \) on the Dp' to the \( \hat{F}_p \) on a Dp in a similar fashion as we did in extending a Dp brane flux for \( p < 5 \) to the one on D5 or D6 there. In other words, we first have the following extension of the flux \( \hat{F}_{p'} \) on the Dp' as,

\[
\hat{F}_{\alpha\beta}' = \left(\begin{array}{ccc}
0 & \hat{g}_1' & \cdots \\
-\hat{g}_1' & 0 & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
\end{array}\right)_{(p+1) \times (p+1)},
\]

(122)

where \( \alpha, \beta = 0, 1, \cdots, p \) and \( \alpha', \beta' = 0, 1, \cdots, p' \). Here \( \kappa = 2, 4, 6 \). As before, at the end of computations, we need to send \( \hat{g}_k' \to \infty \) with \( k = 1, \cdots, \kappa/2 \). As discussed in the subsection 3.2, this extension will not change anything about the corresponding matrix \( M_{p'} \) given in (7) for the present case and the RR zero-mode contribution in the matter part to the amplitude so long the above limit is taken at the end of computations. Moreover, unlike the extension given there, we have here a bonus for the extension of the bosonic zero-mode contribution to the amplitude so long things are taken care of properly. Let us explain this in detail. If one computes the bosonic zero-mode contribution in the matter part to the amplitude for the present case, as already given in (103), it is

\[
\sqrt{\det(\eta_{p'} + \hat{F}_{p'}) \det(\eta_p + \hat{F}_p)} V_{NN} \left(2 \pi^2 \alpha' t\right)^{-\frac{DD}{2}} e^{-\frac{D^2}{2\pi \alpha'}}. \tag{123}
\]

In the present context, we have \( V_{NN} = V_{p' + 1} \) and \( DD = 9 - p \). If we use the trick mentioned
above, we have then the following
\[
\sqrt{\det(\eta_p + \hat{F}_p^\prime) \det(\eta_p + \hat{F}_p) V_{p+1}} (2\pi^2 \alpha'^t) \frac{2^{p-\mu}}{2} e^{-\frac{\pi^2}{2\pi \alpha'^t}}.
\] (124)
Comparing the two, the nice thing here is that the t-dependent part is the same and their difference occurs only in the t-independent part. From (122), we have \( \det(\eta_p + \hat{F}_p^\prime) = (1 + \hat{g}_1^2) \cdots (1 + \hat{g}_{\mu/2}^2) \det(\eta_p' + \hat{F}_p') \) when we take \( \hat{g}_k' \to \infty \) with \( k = 1, \cdots, \kappa/2 \). So we have
\[
\sqrt{\det(\eta_p + \hat{F}_p^\prime) \det(\eta_p + \hat{F}_p) \cdot \hat{g}_1' \cdots \hat{g}_{\mu/2}'} \sqrt{\det(\eta_p' + \hat{F}_p') \det(\eta_p + \hat{F}_p)}. \] (125)
Note also that \( V_{p+1} = V_{p'+1} \). For a Dp brane with the flux (122), following the discussion of (67), we have the following coupling among others,
\[
T_p \int (C_{p'+1} \wedge \hat{F}' \wedge \hat{F}' \cdots \wedge \hat{F}')_{p+1},
\] (126)
where the number of \( \hat{F}' \)'s is \( (p-p')/2 \) and the \( C_{p'+1} \) is the \( (p'+1) \)-form RR potential which can couple with Dp' brane. It is clear that when we take all \( \hat{g}_k' \to \infty \) with \( k = 1, \cdots, \kappa/2 \), the only dominant coupling is the following one
\[
T_p V_{\kappa} \hat{g}_1' \cdots \hat{g}_{\mu/2}' \int C_{p'+1},
\] (127)
where we have now \( p - p' = \kappa \) and the coefficient in front of the coupling denotes the quantized charge \( N \) of Dp' brane in terms of its tension. In other words, we have
\[
NT_{p'} = T_p V_{\kappa} \hat{g}_1' \cdots \hat{g}_{\mu/2}',
\] (128)
which gives
\[
V_{\kappa} = \frac{N}{\hat{g}_1' \cdots \hat{g}_{\mu/2}' T_p}. \] (129)
With the above considerations, now (124) becomes
\[
\sqrt{\det(\eta_p + \hat{F}_p^\prime) \det(\eta_p + \hat{F}_p) V_{p+1}} (2\pi^2 \alpha'^t) \frac{2^{p-\mu}}{2} e^{-\frac{\pi^2}{2\pi \alpha'^t}} = \hat{g}_1' \cdots \hat{g}_{\mu/2}' \sqrt{\det(\eta_p' + \hat{F}_p') \det(\eta_p + \hat{F}_p) V_{p'+1}} \frac{N}{\hat{g}_1' \cdots \hat{g}_{\mu/2}' T_p} \left(2\pi^2 \alpha'^t\right)^{\frac{2-\mu}{2}} e^{-\frac{\pi^2}{2\pi \alpha'^t}} = N \frac{c_{p'}}{c_p} \sqrt{\det(\eta_p' + \hat{F}_p') \det(\eta_p + \hat{F}_p) V_{p'+1}} (2\pi^2 \alpha'^t)^{\frac{2-\mu}{2}} e^{-\frac{\pi^2}{2\pi \alpha'^t}}, \] (130)
where we have used the relation \( T_{p'}/T_p = c_{p'}/c_p \) with the normalization \( c_p = \sqrt{\pi} (2\pi \sqrt{\alpha'})^{3-p} \) for the boundary state given right after (3). This factor \( c_{p'}/c_p \) is the one just needed to
convert the factor \( c_p^2 \), which is used to compute the cylinder amplitude when \( p = p' \) (for example, see (14)), to \( c_p^2 \times c_{p'}/c_p = c_p c_{p'} \), the correct one for the present amplitude. The large integer \( N \) here implies what has been computed using the trick described is actually between one single \( D_p \) and \( N \) \( D_p' \) branes (not a single \( D_p' \)). To obtain the wanted amplitude with a single \( D_p' \), we need to divide so obtained amplitude by \( N \). Given what has been discussed, the closed string tree-level cylinder amplitude for \( p - p' = \kappa \neq 0 \) can be obtained from (54) as

\[
\Gamma_{p,p'} = \frac{\Gamma_{p,p}}{N},
\]

\[
= \frac{2^3 V_{p+1} \left[ \det(\eta_p + \hat{F}_p') \det(\eta_p + \hat{F}_p) \right]^{\frac{1}{2}} \prod_{\alpha=0}^{2} \sin \pi \nu_{\alpha}}{N(8\pi^2 \alpha')^{\frac{p+1}{2}}} \int_0^\infty \frac{dt}{t^{\frac{9-p}{2}}} \eta^3(it) \times \theta_1 \left( \frac{\nu_0 + \nu_2 + \nu_2}{2} \right) it \theta_1 \left( \frac{\nu_0 - \nu_2 + \nu_2}{2} \right) it \theta_1 \left( \nu_1 | it \right) \theta_1 \left( \nu_2 | it \right)
\]

\[
= c_{p'} \frac{2^3 V_{p'+1} \left[ \det(\eta_{p'} + \hat{F}_{p'}') \det(\eta_p + \hat{F}_p) \right]^{\frac{1}{2}} \prod_{\alpha=0}^{2} \sin \pi \nu_{\alpha}}{(8\pi^2 \alpha')^{\frac{p'+1}{2}}} \int_0^\infty \frac{dt}{t^{\frac{9-p'}{2}}} \eta^3(it) \times \theta_1 \left( \frac{\nu_0 + \nu_2 + \nu_2}{2} \right) it \theta_1 \left( \frac{\nu_0 - \nu_2 + \nu_2}{2} \right) it \theta_1 \left( \nu_1 | it \right) \theta_1 \left( \nu_2 | it \right)
\]

\[
= \frac{2^3 V_{p'+1} \left[ \det(\eta_{p'} + \hat{F}_{p'}') \det(\eta_p + \hat{F}_p) \right]^{\frac{1}{2}} \prod_{\alpha=0}^{2} \sin \pi \nu_{\alpha}}{2^2 (8\pi^2 \alpha')^{\frac{p'+1}{2}}} \int_0^\infty \frac{dt}{t^{\frac{9-p'}{2}}} \eta^3(it) \times \theta_1 \left( \frac{\nu_0 + \nu_2 + \nu_2}{2} \right) it \theta_1 \left( \frac{\nu_0 - \nu_2 + \nu_2}{2} \right) it \theta_1 \left( \nu_1 | it \right) \theta_1 \left( \nu_2 | it \right)
\]

\[
= \frac{2^2 V_{p'+1} \left[ \det(\eta_{p'} + \hat{F}_{p'}') \det(\eta_p + \hat{F}_p) \right]^{\frac{1}{2}} \left[ \sum_{\alpha=0}^{2} \cos^2 \pi \nu_{\alpha} - 2 \prod_{\alpha=0}^{2} \cos \pi \nu_{\alpha} - 1 \right]}{2^2 (8\pi^2 \alpha')^{\frac{p'+1}{2}}} \times \prod_{n=1}^{\infty} \frac{dt}{t^{\frac{9-p}{2}}} \prod_{n=1}^{\infty} C_n,
\]

(131)

where in the first equality the \( \Gamma_{p,p} \) is the cylinder amplitude (54) for the extended flux \( \hat{F}_p' \) given in (122), in the second equality we have used \( V_{p+1} = V_{p'+1} V_n \) and (130), in the third equality we have used the explicit expression for \( c_p = \sqrt{\pi}(2\pi \sqrt{\alpha'})^{3-p} \), and \( C_n \) continues to be given by (55) and the extension trick discussed in section 3 for \( \nu_{\alpha}' \)'s still applies here. It is clear that the basic structure of the above cylinder amplitude is the same as that for the \( p = p' \) case discussed in the previous section. So we expect the same properties of the amplitude as discussed there such as the nature of the interaction and the potential.
instabilities. So we will not repeat this discussion here. Moreover, we also expect some special features to arise here which will be discussed later in this section.

Once we have the closed string tree-level cylinder amplitude (131), the corresponding open string one-loop annulus amplitude can be obtained from the next to the last equality of this amplitude above by a Jacobi transformation following the standard prescription given earlier in the previous section. This open string one-loop annulus amplitude is then

\[
\Gamma_{p,p'} = -\frac{2^3 i V_{p'+1}}{(2\pi^2 \alpha'_p)^{\nu'_p}} \left[ \det(\eta_{p'}) \det(\eta_p) \right]^{1/2} \prod_{\alpha=0}^{2} \sin \pi \nu_\alpha \int_0^\infty \frac{dt}{t^{3/2}} e^{-\frac{y^2 t}{2\pi \alpha'}} \int_0^\infty \frac{dt}{t^{3/2}} e^{-\frac{y'^2 t}{2\pi \alpha'}} \prod_{\alpha=0}^{2} \sin \pi \nu_\alpha \sinh \pi \nu_\alpha t \prod_{n=1}^{\infty} Z_n,
\]

(132)

where \( Z_n \) is still given by (60). The use of this open string one-loop annulus amplitude is for analyzing the small \( y \) behavior such as the onset of tachyonic instability and that when \( \nu_0 \) is imaginary, the underlying system will decay via the so-called open string pair production. So we will give here a general discussion of both.

When all three \( \nu_0, \nu_1 \) and \( \nu_2 \) are real, once again without loss of generality and for convenience, we assume \( \nu_0 \leq \nu_1 \leq \nu_2 \). If \( \nu_0 + \nu_1 \geq \nu_2 \), the interaction (131) is repulsive and there is no potential tachyonic instability which can be checked from the integrand of (132) for large \( t \). On the other hand, if \( \nu_0 + \nu_1 < \nu_2 \), the interaction is attractive and for large \( t \), it behaves as

\[
\sim e^{-\frac{y^2 t}{2\pi \alpha'}} e^{\pi (\nu_2 - \nu_1 - \nu_0) t} = e^{-2\pi t \left( \frac{y^2 (\nu_2 - \nu_1 - \nu_0)}{2} \right)},
\]

(133)

which blows up if \( y < \pi \sqrt{2(\nu_2 - \nu_1 - \nu_0)\alpha'} \), indicating the onset of tachyonic instability. So everything here is consistent with what has been discussed in the previous section for \( p = p' \) case. If \( \nu_0 = i\bar{\nu}_0 \), i.e., imaginary, with \( \bar{\nu}_0 \in (0, \infty) \), so the factor \( \sin \pi \nu_0 / \sinh \pi \nu_0 t \) in the integrand of (132) becomes \( \sinh \pi \bar{\nu}_0 / \sin \pi \bar{\nu}_0 t \), indicating the appearance of an infinite number of simple poles of the integrand at \( t_k = k/\bar{\nu}_0 \) with \( k = 1, 2, \cdots \). So this implies that the amplitude has an imaginary part, indicating the decay of the underlying system via the so-called open string pair production. The decay rate per unit volume of \( Dp' \)
brane worldvolume can be computed as before as
\[ W_{p,p'} = -\frac{2 \text{Im} \Gamma}{\nu_{p'+1}}, \]

\[ = \frac{2^{3-k} \left[ \det(\eta'_{p'} + \hat{F}'_{p'}) \det(\eta_{p} + \hat{F}^\nu_p) \right]^\frac{1}{2} \sinh \pi \nu_{0} \sin \pi \nu_{1} \sin \pi \nu_{2}}{\nu_{0}} \sum_{k=1}^{\infty} (-)^{k+1} \left( \frac{\nu_{0}}{k} \right) \frac{\nu_{p} + 5}{2} e^{-\frac{3y^2}{2 \pi \nu_{0}}} Z_{k}(\nu_{0}, \nu_{1}, \nu_{2}), \]

where
\[ Z_{k} = \prod_{n=1}^{\infty} \left( \frac{1 + |z|^{4n} - 2(-)^{k} |z|^{2n} \cosh \frac{k \nu_{1}}{\nu_{0}} \cosh \frac{k \nu_{2}}{\nu_{0}}}{(1 - |z|^{2n})^2 (1 - 2 |z|^{2n} \cosh \frac{2k \nu_{1}}{\nu_{0}} + |z|^{4n})} \right), \]

with \(|z| = e^{-k \nu_{0}}\). As before, the open string pair production rate is given by the \(k = 1\) term of the above as
\[ W_{p,p'}^{(1)} = \frac{2^{3-k} \left[ \det(\eta'_{p'} + \hat{F}'_{p'}) \det(\eta_{p} + \hat{F}^\nu_p) \right]^\frac{1}{2} \sinh \pi \nu_{0} \sin \pi \nu_{1} \sin \pi \nu_{2}}{\nu_{0}} \frac{\nu_{p} + 5}{2} e^{-\frac{3y^2}{2 \pi \nu_{0}}} Z_{1}(\nu_{0}, \nu_{1}, \nu_{2}). \]

In the above, we assume \(p \geq 5\). The \(p = 3, 4\) amplitude or rate can be obtained by sending \(\nu_{2} \to 0\) and the \(p = 1, 2\) amplitude or rate can further be obtained by sending \(\nu_{1}, \nu_{2} \to 0\). This becomes clear when we discuss the \(p - p' = 2, p - p' = 4\) and \(p - p' = 6\) one by one in the following.

Before we move to that, let us check one thing mentioned in the previous section. It is that the interaction between a Dp and a Dp' for \(6 \geq p > p'\), placed parallel at a separation and without any brane flux present, is attractive when \(p - p' = 2\), vanishes when \(p - p' = 4\) and is repulsive when \(p - p' = 6\) (\(p = 6, p' = 0\)). Let us check each of them explicitly here. For \(p = 6\) and \(p' = 0\), we have here \(\nu_{2} = \nu_{1} = \nu_{0} = 1/2\) and so we have, for example, the cylinder amplitude from the last equality of (131) as
\[ \Gamma_{6,0} = -\frac{V_{1}}{2(8\pi^{2}\alpha')^{\frac{1}{2}}} \int_{0}^{\infty} \frac{dt}{t^{2}} e^{-\frac{3y^{2}}{2 \pi \alpha' t}} \prod_{n=1}^{\infty} \frac{(1 + |z|^{4n})^{4}}{(1 - |z|^{4n})^{2}(1 + |z|^{2n})^{4}}, \]

where we have used (55) for \(C_{n}\). It is indeed repulsive since \(\Gamma_{6,0} < 0\) for any \(y\). For \(p - p' = 2\), we have one of three \(\nu_{0}, \nu_{1}, \nu_{2}\) being half and the remaining two being zero.

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while for $p - p' = 4$, we have two of them being half and the remaining one being zero. For the former case, the cylinder amplitude (131) is now,

$$
\Gamma_{\kappa=2} = \left. \frac{2V_{p'+1}}{(8\pi^2\alpha')^{p'+1/2}} \int_0^\infty \frac{dt}{t^{9/2}} e^{-\frac{y^2}{2\pi\alpha'}} \prod_{n=1}^{\infty} \frac{(1 + |z|^{4n})^2}{(1 - |z|^{2n})^4(1 - |z|^{4n})^2}, \right)
$$

(138)

where we have used [35] for $C_n$. It is indeed attractive since $\Gamma_{\kappa=2} > 0$ for any $y$. For the latter case, the amplitude $\Gamma_{\kappa=4}$ simply vanishes due to the constant factor $\sum_{\alpha=0}^2 \cos^2 \pi\nu_\alpha - 2 \prod_{\alpha=0}^2 \cos \pi\nu_\alpha - 1$ in the amplitude (131) being zero for the present case. All are as expected. We now discuss each separate case mentioned earlier.

### 4.1 The $p - p' = 2$ case

In this subsection, we will focus on the $p - p' = 2$ case, specifically. We will discuss here each of $p = 6$ or 5; $p = 4$ or 3 and $p = 2$, separately. Let us begin with $p = 6$ or 5 case.

**The $p = 6$ or 5 case:** For either of these two cases, we can extend the flux $\hat{F}_{p'}$ to $\hat{F}_p$, prescribed in (122), as

$$
(\hat{F}_p')_{\alpha\beta} = \begin{pmatrix} (\hat{F}_p')_{\alpha'\beta'} & 0 & \hat{g}' \\ -\hat{g}' & 0 \end{pmatrix},
$$

(139)

where we will take $\hat{g}' \to \infty$ at the end of computations. For illustration purpose, we consider the flux $\hat{F}_p$ on the Dp brane the following form

$$
\hat{F}_p = \begin{pmatrix} (\hat{F}_p')_{\alpha'\beta'} & 0 & \hat{g} \\ -\hat{g} & 0 \end{pmatrix},
$$

(140)

where $g$ is finite. We can then determine $\nu_2$ as

$$
\tan \pi\nu_2 = \frac{\hat{g}' - \hat{g}}{1 + \hat{g}'\hat{g}},
$$

(141)

which gives $\tan \pi\nu_2 = 1/|\hat{g}|$ when we take $\hat{g}' \to \infty$ limit. For a given fixed $\hat{g}$, the discussion goes exactly the same as we did for $p = p' = 6$ case given in the previous section. For this reason, we will not repeat it here. We here focus on vanishingly small $\hat{g}$ (in practice $\hat{g} \ll 1$) for which we have $\nu_2 \to 1/2$. From (131), we have the closed string cylinder amplitude for the present case as

$$
\Gamma_{p,p'} = \left. \frac{2V_{p'+1}\left[\det(\eta_{p'} + \hat{F}_p')\det(\eta_p + \hat{F}_p)\right]^{1/2}}{(8\pi^2\alpha')^{p'+1/2}} \left(\cos^2 \pi\nu_0 - \sin^2 \pi\nu_1\right) \int_0^\infty \frac{dt}{t^{9/2}} e^{-\frac{y^2}{2\pi\alpha'}} \prod_{n=1}^{\infty} C_n, \right)
$$

(142)
where $C_n$, from (132), is now

$$C_n = \frac{\tilde{C}_n}{(1 - |z|^{4n})^2 \prod_{\alpha=0}^{1} [1 - 2|z|^{2n} \cos 2\pi\nu_\alpha + |z|^{4n}]}, \quad (143)$$

with

$$\tilde{C}_n = \left[ (1 + |z|^{4n})^2 - 4|z|^{4n}\sin^2 \pi(\nu_0 + \nu_1) \right] \left[ (1 + |z|^{4n})^2 - 4|z|^{4n}\sin^2 \pi(\nu_0 - \nu_1) \right], \quad (144)$$

$$= \left[ (1 + e^{2\pi i\nu_0}|z|^{4n})^2 - 4|z|^{4n}e^{2\pi i\nu_0}\sin^2 \pi\nu_1 \right] \times \left[ (1 + e^{-2\pi i\nu_0}|z|^{4n})^2 - 4|z|^{4n}e^{-2\pi i\nu_0}\sin^2 \pi\nu_1 \right]. \quad (145)$$

When $p = 6$, we have two choices: 1) $\nu_0, \nu_1 \in [0, 1/2]$ and 2) $\nu_0 = i\tilde{\nu}_0$ with $\tilde{\nu}_0 \in (0, \infty)$ and $\nu_1 \in [0, 1/2]$. For the first case, the interaction is attractive if $\cos \pi\nu_0 > \sin \pi\nu_1$. Otherwise, it is repulsive. In the former case, we need $\nu_0 + \nu_1 < 1/2$ while in the latter we have $\nu_0 + \nu_1 > 1/2$. The interaction vanishes if $\cos \pi\nu_0 = \sin \pi\nu_1$ which requires $\nu_0 + \nu_1 = 1/2$. Everything here is consistent with what has been discussed in the $p = p' = 6$ case in the previous section if we take the present $\nu_2 = 1/2$. So here is just a special case with $\nu_2 = 1/2$ of the general discussion of the $p = p' = 6$ in the previous section.

For small $y$, the small $t$ integration becomes important for the amplitude. This gives also a potential singularity of this amplitude since we have two potential sources for this. One is from the $t^{-(9-p)/2}$ factor in the integrand and the other comes from infinite product of $C_n$, each of which has a factor $(1 - |z|^{4n})^2 \sim t^2$ for small $t$ in the denominator of $C_n$. Both of them blow up for small $t$.

For the second case, the sign of the integrand becomes again indefinite for small $t$ and the discussion goes the same as we did for the $p = p'$ case and will not be repeated here.

The underlying physics for either of these two cases will become clear if we examine it from the corresponding open string one-loop annulus amplitude. This annulus amplitude can be read from (132) for the present case as

$$\Gamma_{p,p'} = \frac{2 V_{p+1} \left[ \det(\eta_{p'} + \tilde{F}_{p'}) \det(\eta_0 + \tilde{F}_p) \right]^2}{(8\pi^2\alpha')^{\nu_0' + 1}} \left[ \int_0^\infty \frac{dt}{t^{\nu_0'}} e^{-\frac{\nu_0'^2}{2}} \frac{\sin \pi\tilde{\nu}_0 \sin \pi\nu_1}{\sin \pi\nu_0 t \sinh \pi\nu_1 t \sinh \frac{\pi t}{2}} \right] \times \left[ \left( \cosh \pi\nu_1 t - \cosh \frac{\pi t}{2} \right)^2 + 4 \sin^2 \frac{\pi \tilde{\nu}_0 t}{2} \left( \cosh \pi\nu_1 t \cosh \frac{\pi t}{2} - \cos^2 \frac{\pi \tilde{\nu}_0 t}{2} \right) \right] \prod_{n=1}^{\infty} Z_n, \quad (146)$$

where $Z_n$ can be read from (130) as

$$Z_n = \frac{\left[ 1 - 2|z|^{2n} \cos 2\pi\tilde{\nu}_0 t + |z|^{4n} - 1 \right] \tilde{Z}_n}{(1 - |z|^{2n})^2 \left[ 1 - 2|z|^{2n} \cosh \pi t + |z|^{4n} \right] \left[ 1 - 2|z|^{2n} \cosh 2\pi\nu_1 t + |z|^{4n} \right]}, \quad (147)$$

$$\text{45}$$
with

\[ \tilde{Z}_n = \left[ (1 - 2|z|^{2n} \cos \pi \tilde{\nu}_0 t \cosh \pi \left( \nu_1 + \frac{1}{2} \right) t + |z|^{4n})^2 + 4|z|^{4n} \sin \pi \tilde{\nu}_0 t \sinh^2 \pi \left( \nu_1 + \frac{1}{2} \right) t \right] \left[ (1 - 2|z|^{2n} \cos \pi \tilde{\nu}_0 t \cosh \pi \left( \nu_1 - \frac{1}{2} \right) t + |z|^{4n})^2 + 4|z|^{4n} \sin \pi \tilde{\nu}_0 t \sinh^2 \pi \left( \nu_1 - \frac{1}{2} \right) t \right] > 0. \] (148)

For large \( t \), \( Z_n \approx 1 \) and the integrand is

\[ \sim e^{-\frac{y^2}{2\alpha'}} \pi^{\frac{1}{2} - \nu_1} t = e^{-2\pi\left( \frac{\nu}{2\alpha'} \frac{1}{2} - \frac{1}{2} \nu_1 \right)}, \] (149)

which blows up when \( y < \pi \sqrt{(1 - 2\nu_1)\alpha'} \), indicating the onset of tachyonic instability mentioned above. Once again, the factor \( \sin \pi \tilde{\nu}_0 t \) in the denominator of the integrand of the amplitude gives an infinite number of simple poles along the positive \( t \)-axis (note that the integrand is regular as \( t \to 0 \)) at \( t_k = k/\tilde{\nu}_0 \) with \( k = 1, 2, \ldots \). This implies that the amplitude has an imaginary part, indicating the decay of the underlying system via the so-called open string pair production. The decay rate per unit \( p' \)-brane volume can be computed as before to give

\[ \mathcal{W}_{p,p'} = -2\text{Im}\Gamma \left( \frac{1}{p'} + 1 \right). \]

\[ = 2^2 \left[ \frac{\det(\eta_{p'} + \tilde{F}_{p'}) \det(\eta_p + \tilde{F}_p)^{\frac{1}{2}} \sinh \pi \tilde{\nu}_0 \sin \pi \nu_1}{\tilde{\nu}_0(8\pi^2 \alpha')^{\nu_1 + 1}} \sum_{k=1}^{\infty} (-)^{k+1} \left( \frac{\tilde{\nu}_0}{k} \right)^{\frac{\nu_1}{2}} e^{-\frac{k\pi^2}{2\alpha' \nu_0}} \right] \times \frac{\left( \cosh \frac{k\pi \nu_1}{\nu_0} - (-)^k \cosh \frac{k\pi}{2\nu_0} \right)^2}{\sinh \frac{k\pi \nu_1}{\nu_0} \sinh \frac{k\pi}{2\nu_0}} \tilde{Z}_k(\tilde{\nu}_0, \nu_1), \] (150)

where

\[ Z_k(\tilde{\nu}_0, \nu_1) = \prod_{n=1}^{\infty} \frac{\left[ 1 - 2(-)^k |z_k|^{2n} \cosh \frac{k\pi}{\tilde{\nu}_0} \left( \nu_1 + \frac{1}{2} \right) + |z_k|^{4n} \right]^2}{\left( 1 - |z_k|^{2n} \right)^4 \left[ 1 - 2|z_k|^{2n} \cosh \frac{k\pi}{\tilde{\nu}_0} + |z_k|^{4n} \right] \times \left[ 1 - 2(-)^k |z_k|^{2n} \cosh \frac{2k\pi \nu_1}{\nu_0} \left( \nu_1 - \frac{1}{2} \right) + |z_k|^{4n} \right]^2} , \] (151)

with \( |z_k| = e^{-k\pi/\tilde{\nu}_0} \). The open string pair production rate is the \( k = 1 \) term of the above
and it is

\[ W^{(1)}_{p,p'} = \frac{2^2 \left[ \det(\eta_p + \hat{F}_p) \det(\eta_{p'} + \hat{F}_{p'}) \right]^{\frac{1}{2}} \sinh \pi \nu_0 \sin \pi \nu_1 \nu_0^{\frac{8}{2} - \frac{y_0^2}{2\pi^2 \alpha'}} \left( 8\pi^2 \alpha' \right)^{\frac{y_0^2 + 1}{2}}}{\nu_0^{\frac{8}{2} - \frac{y_0^2}{2\pi^2 \alpha'}} e^{-\frac{y_0^2}{2\pi^2 \alpha'}} \sinh \frac{\pi \nu_1}{\nu_0} \sinh \frac{\pi \nu_0}{2\nu_0}} \times \left( \cosh \frac{\pi \nu_1}{\nu_0} + \cosh \frac{\pi \nu_0}{2\nu_0} \right)^2 \sinh \frac{\pi \nu_0}{2\nu_0} Z_1(\bar{\nu}_0, \nu_1). \]  

(152)

One can check easily that the above decay rate or the open string pair production rate is just the special case of (134) or (136) for \( \nu_2 = 1/2 \) and \( \kappa = 2 \), respectively. There is an interesting enhancement of the pair production rate even in the absence of magnetic flux for which we have \( \nu_1 = 0 \) for small \( \bar{\nu}_0 \). This rate can be obtained from the above by taking \( \nu_1 \to 0 \) limit as

\[ W^{(1)}_{p,p'} = \frac{2^2 \left[ \det(\eta_p + \hat{F}_p) \det(\eta_{p'} + \hat{F}_{p'}) \right]^{\frac{1}{2}} \sinh \pi \nu_0 \sin \pi \nu_1 \nu_0^{\frac{8}{2} - \frac{y_0^2}{2\pi^2 \alpha'}} \left( 8\pi^2 \alpha' \right)^{\frac{y_0^2 + 1}{2}}}{\nu_0^{\frac{8}{2} - \frac{y_0^2}{2\pi^2 \alpha'}} e^{-\frac{y_0^2}{2\pi^2 \alpha'}} \sinh \frac{\pi \nu_0}{2\nu_0} Z_1(\nu_0, 0), \]  

(153)

where

\[ Z_1(\nu_0, 0) = \prod_{n=1}^{\infty} \left[ 1 + 2|z_1|^{2n} \cosh \frac{\pi \nu_0}{2\nu_0} + |z_1|^{4n} \right] \left[ 1 - 2|z_1|^{2n} \cosh \frac{\pi \nu_0}{2\nu_0} + |z_1|^{4n} \right]. \]  

(154)

We would like to remark here that given the form of flux \( \hat{F}_p \) (140), the above decay or pair production rate is also valid for \( p \geq 3 \) when we take \( \nu_1 = 0 \). For illustration, let us have a consideration of the following special choice of fluxes \( \hat{F}_p \) and \( \hat{F}_{p'} \) for \( p = 5 \) as

\[ \hat{F}_5 = \begin{pmatrix} 0 & \hat{f} & 0 & 0 & \cdots & \cdots \\ -\hat{f} & 0 & 0 & \cdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\ \end{pmatrix}_{6 \times 6}, \quad \hat{F}_{p'} = \begin{pmatrix} 0 & \hat{f}' & 0 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \end{pmatrix}_{4 \times 4}, \]  

(155)

where there is no magnetic flux present. This gives the \( \hat{g} = 0 \) in (140) and so we have \( \nu_2 = 1/2 \). With this special choice of fluxes, we have \( \nu_1 = 0 \) and

\[ \tanh \pi \nu_0 = \frac{|\hat{f} - \hat{f}'|}{1 - \hat{f} \hat{f}'}. \]  

(156)
The pair production rate (153) becomes

$$W^{(1)}_{p,p'} = \frac{2^2 |\hat{f} - \hat{f}'|}{(8\pi^2\alpha')^{3/2}} e^{-\frac{\pi^2}{2\alpha'\beta_0}} \left(1 + \cosh\frac{\pi}{2\beta_0}\right)^2 \frac{1}{\sinh\frac{\pi}{2\beta_0}} Z_1(\beta_0, 0),$$

(157)

where $p = 5$ and $p' = 3$. As pointed out above, this rate is also valid for $p = 3$ and $p' = 1$. For small $\beta_0$, $Z_1(\beta_0, 0) \approx 1$ and we have a large enhancement factor of $e^{\pi/(2\alpha')} \gg 1$ which is not seen in the $p = p'$ case. This large enhancement was also considered by one of the present authors in [18] and it is essentially due to the D$p$' brane acting effectively as a stringy magnetic flux.

The $p = 4$ or 3 case: For this case, we need to set $\nu_2 = 0$ from the outset. Now the role of $\nu_2$ in the above $p = 6$ or 5 case is replaced by that of $\nu_1$ in the present one. By the same token, we extend the flux $\hat{F}'_{p'}$ to $\hat{F}'_p$ the following way,

$$(\hat{F}'_p)_{\alpha\beta} = \begin{pmatrix} (\hat{F}'_{p'})_{\alpha'\beta'} & 0 & \hat{g}' \\ -\hat{g}' & 0 \end{pmatrix},$$

(158)

where we will take $\hat{g}' \to \infty$ at the end of computations. For illustration purpose, we consider the flux $\hat{F}_p$ on the D$p$ brane the following form

$$\hat{F}_{\alpha\beta} = \begin{pmatrix} (\hat{F}'_{p'})_{\alpha'\beta'} & 0 & \hat{g} \\ -\hat{g} & 0 \end{pmatrix},$$

(159)

where $\hat{g}$ is finite. We can then determine $\nu_1$ as

$$\tan \pi \nu_1 = \frac{\hat{g}' - \hat{g}}{1 + \hat{g}'\hat{g}},$$

(160)

which gives $\tan \pi \nu_1 = 1/|\hat{g}|$ when we take $\hat{g}' \to \infty$. For a general $\hat{g}$, the present discussion is not different from its correspondence in the $p = p'$ case in the previous section and we will not repeat it here. We here also focus on the small or vanishing $\hat{g}$ for which we have $\nu_1 \to 1/2$. The closed string cylinder amplitude can be read from the last equality in (131) as

$$\Gamma_{p,p'} = \frac{2 V_{p'+1} \left[ \det(\eta'_{p'}) \det(\eta_p + \hat{F}_p) \right]^{1/2} \cos^2 \pi \beta_0}{(8\pi^2\alpha')^{3/2}} \int_0^\infty dt e^{-\pi^2 t} \prod_{n=1}^\infty C_n,$$

(161)
where \( C_n \) can be read from (55) as

\[
C_n = \frac{(1 + |z|^{2n})^2 (1 + 2|z|^{4n} \cos 2\pi \nu_0 + |z|^{8n})^2}{(1 - |z|^{4n})^4 (1 - 2|z|^{2n} \cos 2\pi \nu_0 + |z|^{4n})^2}. \tag{162}
\]

It is clear that this interaction can only be attractive which is consistent with what we have achieved in the previous section. This interaction vanishes if \( \nu_0 = 1/2 \). This can easily be understood as follows. The \( \nu_0 = 1/2 \) can be understood either from that the Dp carries an infinite large magnetic flux and Dp' carries no flux or from that the Dp carries no flux but the Dp' carries such a magnetic flux. In the former case, the contribution to the interaction from the Dp is actually dominated by the infinitely large magnetic flux which gives an infinite many of D(p - 2) branes whose dimensionality is the same as that of the Dp' with \( p' = p - 2 \) in the present case. We know that there is no interaction acting between D branes with the same dimensionality and placed parallel at a separation. So this explains the result. For the latter, by the same token, the Dp' behaves effectively as infinitely many D(p' - 2) branes. So now the interaction is between one Dp and infinitely many D(p - 4)-branes, placed parallel at a separation, which vanishes since there does not exist any interaction between D-branes whose dimensionality differs by 4. Given what has been said, the two cases are still different in that the former case preserves 1/2 spacetime supersymmetries while the later preserves only 1/4.

Again the small \( y \) physics can be best described in terms of the corresponding open string one-loop annulus amplitude (132). For the present case, it is

\[
\Gamma_{p,p'} = \frac{2 V_{p'+1} \left[ \det(\eta_{p'} + \hat{F}_{p'}) \det(\eta_p + \hat{F}_p) \right]^{1/2} \sin \pi \nu_0}{(8\pi^2 \alpha')^{p'+1/2}} \int_0^\infty dt \frac{e^{-y^2 t}}{t^{p'+1} e^{\pi t}} \frac{\cosh \pi \nu_0 t - \cosh \frac{\pi t}{2}}{\sinh \pi \nu_0 t \sinh \frac{\pi t}{2}} \prod_{n=1}^{\infty} Z_n, \tag{163}
\]

where \( Z_n \) can be read from (60) as

\[
Z_n = \frac{\left[ (1 + |z|^{4n} - 2|z|^{2n} \cosh \pi \nu_0 t \cosh \frac{\pi t}{2})^2 - 4|z|^{4n} \sinh^2 \pi \nu_0 t \sinh^2 \frac{\pi t}{2} \right]^2}{(1 - |z|^{2n})^4 (1 - 2|z|^{2n} \cosh \pi t + |z|^{4n})(1 - 2|z|^{2n} \cosh 2\pi \nu_0 t + |z|^{4n})}. \tag{164}
\]

When \( \nu_0 = 1/2 \), once again the amplitude vanishes and we have explained this in the cylinder amplitude. We now assume \( \nu_0 \in [0, 1/2) \). For large \( t \), we have \( Z_n \approx 1 \) and the integrand behaves

\[
\sim e^{-y^2 t \frac{\pi t}{2\pi \alpha} e^{\pi t}} = e^{-2\pi t \left[ \frac{y^2}{(2\pi \alpha')^2} \frac{1 - 2\nu_0}{4} \right]}, \tag{165}
\]
which blows up when \( y < \pi \sqrt{(1 - 2\nu_0)\alpha'} \), indicating again the onset of tachyonic instability. We now consider an imaginary \( \nu_0 = i\tilde{\nu}_0 \) with \( \tilde{\nu}_0 \in (0, \infty) \). We can use the following specific fluxes \( \hat{F}_p \) and \( \hat{F}'_{p'} \) to give a representative discussion,

\[
(\hat{F}_p)_{01} = -(\hat{F}_p)_{10} = \hat{f}, \quad (\hat{F}'_{p'})_{01} = -(\hat{F}'_{p'})_{10} = \hat{f}',
\]

where the rest components of both \( \hat{F}_p \) and \( \hat{F}'_{p'} \) are zero and we have also taken the \( \hat{g} = 0 \) given in (169). With this choice, we have

\[
tanh \pi \tilde{\nu}_0 = \frac{|\hat{f} - \hat{f}'|}{1 - \hat{f}'\hat{f}}.
\]

We have now the amplitude (163) as

\[
\Gamma_{p,p'}' = \frac{2^{\nu_{p+1}}|\hat{f} - \hat{f}'|}{\nu_{p+1}} \int_0^\infty dt \frac{e^{4\pi t} (\cos \pi \tilde{\nu}_0 t - \cosh \frac{\pi t}{2})^2}{\sin \pi \tilde{\nu}_0 t \sinh \frac{\pi t}{2}} \prod_{n=1}^\infty Z_n,
\]

where \( Z_n \) continues to be given by (163) but with \( \nu_0 = i\tilde{\nu}_0 \). This amplitude has now a tachyonic instability when \( y < \pi \sqrt{\alpha'} \). In addition, the \( \sin \pi \tilde{\nu}_0 t \) factor in the denominator of the integrand of the above amplitude gives again an infinite number of simple poles at \( t_k = k/\tilde{\nu}_0 \) with \( k = 1, 2, \ldots \) and therefore the amplitude has an imaginary part, indicating the decay of the underlying system via the so-called open string pair production. The decay rate per unit volume of Dp' brane can be computed to give

\[
\mathcal{W}_{p,p'}' = -\frac{2 \text{Im}\Gamma}{V_{p'+1}},
\]

\[
= \frac{4|\hat{f} - \hat{f}'|}{\nu_0 (8\pi^2 \alpha')} \prod_{k=1}^{\nu_{p+1}} (-k)^{k+1} \left( \frac{\nu_0}{k} \right)^{\nu_{p+1}} e^{-4\pi^2 k\pi^2 \alpha'\nu_0} \left( \frac{\cosh \frac{k\pi}{2\nu_0} - (-k)^2}{\sinh \frac{k\pi}{2\nu_0}} \right) Z_k(\tilde{\nu}_0, \nu_1 = 1/2),
\]

where

\[
Z_k(\tilde{\nu}_0, \nu_1 = 1/2) = \prod_{n=1}^\infty \frac{[1 - 2(-)^k |z_k|^{2n} \cosh \frac{k\pi}{2\nu_0} + |z_k|^{4n}]}{(1 - |z_k|^{2n})^3 (1 - 2|z_k|^{2n} \cosh \frac{k\pi}{2\nu_0} + |z_k|^{4n})},
\]

with \( |z_k| = e^{-k\pi/\nu_0} \). The open string pair production rate is given by the first \( k = 1 \) term of the above and it is

\[
\mathcal{W}_{p,p'}'^{(1)} = \frac{4|\hat{f} - \hat{f}'|}{(8\pi^2 \alpha')} \nu_0^{\nu_{p+1}} e^{-4\pi^2 \nu_0} \left( \frac{\cosh \frac{\pi}{2\nu_0} + 1}{\sinh \frac{\pi}{2\nu_0}} \right)^2 Z_1(\tilde{\nu}_0, \nu_1 = 1/2).
\]
where
\[
Z_1(\tilde{\nu}_0, \nu_1 = 1/2) = \prod_{n=1}^{\infty} \frac{[1 + 2|z_1|^{2n}\cosh \frac{π}{2\nu_0} + |z_1|^{4n}]^4}{(1 - |z_1|^{2n})^6(1 - 2|z_1|^{2n}\cosh \frac{π}{\nu_0} + |z_1|^{4n})^4}.
\] (172)

This pair production rate is, as expected, the same as that given in (157). The same discussion applies here, too. Note also that the decay rate (169) and the pair production rate (171) are just special cases of (134) and (136) when we take \(\nu_2 = 0\) and \(\nu_1 = 1/2\), as expected.

The \(p = 2\) case: This is the last case we will discuss in this subsection. The D0 brane cannot carry any worldvolume flux. However, by the same token as before, we can have the following extension as
\[
\hat{F}_2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \hat{g}' \\
0 & -\hat{g}' & 0
\end{pmatrix},
\] (173)
where we will set \(\hat{g}' \to \infty\) at the end of computations. For this case, we will consider the most general D2 worldvolume flux as an example. This flux can be expressed as
\[
\hat{F}_2 = \begin{pmatrix}
0 & \hat{f}_1 & \hat{f}_2 \\
-\hat{f}_1 & 0 & \hat{g} \\
-\hat{f}_2 & -\hat{g} & 0
\end{pmatrix}.
\] (174)

Using (24), we have
\[
w_{\alpha \beta} = \begin{pmatrix}
\frac{1+\hat{g}^2 + \hat{f}_1^2 + \hat{f}_2^2}{2(\hat{f}_1 - \hat{f}_2)} & -\frac{2(\hat{f}_1 + \hat{f}_2)(1-\hat{g}^2)-4(\hat{f}_2 - \hat{f}_1)\hat{g} \hat{g}'}{(1+\hat{g}^2-\hat{f}_1^2-\hat{f}_2^2)(1+\hat{g}^2)} & -\frac{4(\hat{f}_1 + \hat{f}_2)\hat{g} \hat{g}'+2(\hat{f}_2 - \hat{f}_1)\hat{g} (1-\hat{g}^2)}{(1+\hat{g}^2-\hat{f}_1^2-\hat{f}_2^2)(1+\hat{g}^2)} \\
\frac{1+\hat{g}^2 - \hat{f}_1^2 - \hat{f}_2^2}{2(\hat{f}_2 + \hat{f}_1)} & -\frac{(1-\hat{g}^2+\hat{f}_1^2-\hat{f}_2^2)(1-\hat{g}^2)+4(\hat{g} - \hat{f}_1 \hat{f}_2)\hat{g} \hat{g}'}{(1+\hat{g}^2-\hat{f}_1^2-\hat{f}_2^2)(1+\hat{g}^2)} & \frac{2(1-\hat{g}^2+\hat{f}_1^2-\hat{f}_2^2)\hat{g} - 2(\hat{g}-\hat{f}_1 \hat{f}_2)(1-\hat{g}^2)}{(1+\hat{g}^2-\hat{f}_1^2-\hat{f}_2^2)(1+\hat{g}^2)} \\
\frac{1+\hat{g}^2 - \hat{f}_2^2 - \hat{f}_1^2}{2(\hat{f}_2 + \hat{f}_1)} & -\frac{2(\hat{g} + \hat{f}_1 \hat{f}_2)(1-\hat{g}^2)-2(1-\hat{g}^2-\hat{f}_1^2-\hat{f}_2^2)\hat{g} \hat{g}'}{(1+\hat{g}^2-\hat{f}_1^2-\hat{f}_2^2)(1+\hat{g}^2)} & \frac{(1+\hat{g}^2-\hat{f}_1^2-\hat{f}_2^2)\hat{g} - 2(\hat{g} - \hat{f}_1 \hat{f}_2)(1-\hat{g}^2)}{(1+\hat{g}^2-\hat{f}_1^2-\hat{f}_2^2)(1+\hat{g}^2)}
\end{pmatrix}.
\] (175)

One can check explicitly that the above \(w\) has one eigenvalue unity and the other two \(\lambda_0\) and \(\lambda_0^{-1}\) satisfy
\[
\lambda_0 + \lambda_0^{-1} = \frac{2(\hat{g}^2 + \hat{f}_1^2 + \hat{f}_2^2 - 1)}{1 + \hat{g}^2 - \hat{f}_1^2 - \hat{f}_2^2},
\] (176)
where we have taken \(\hat{g}' \to \infty\). Setting \(\lambda_0 = e^{2\pi i \nu_0}\), we have
\[
\tan \pi \nu_0 = \sqrt{1 - \frac{\hat{f}_1^2 - \hat{f}_2^2}{|\hat{g}|}}.
\] (177)

We have two cases to consider: 1) \(\hat{f}_1^2 + \hat{f}_2^2 < 1\), 2) \(1 < \hat{f}_1^2 + \hat{f}_2^2 < 1 + \hat{g}^2\). For the first case, \(\nu_0 \in [0, 1/2]\). If \(\hat{g}\) is finite, the discussion goes the same as the pure magnetic case of
where we also use (55) for \( C_n \). The integrand of this amplitude has a potential divergence but has no sign ambiguity for small \( t \), indicating a potential tachyonic instability but no open string pair production, even though there exist applied electric fluxes. To see both of these clearly, we need the corresponding open string one-loop annulus amplitude which can be read from (132) as

\[
\Gamma_{2,0} = \frac{2V_1}{(8\pi^2\alpha')^{\frac{3}{2}}} \int_0^\infty dt \frac{e^{-\frac{\nu t^2}{2 \pi \alpha'}}}{t^2} \left[ 1 - |z|^{2n} \cosh \frac{\pi t}{2} \right] \left( \frac{\sinh \frac{\pi t}{2}}{\sinh \frac{\pi t}{2}} \right)
\]

\[
\times \prod_{n=1}^\infty \frac{[1 - |z|^{2n} \cosh \frac{\pi t}{2} + |z|^{4n}]^4}{(1 - |z|^{2n})^6 [1 - |z|^{2n} \cosh \pi t + |z|^{4n}]^4},
\]

where we have used (60) for \( Z_n \). For large \( t \), the integrand of the above behaves like

\[
\sim e^{-\frac{\nu t^2}{2 \pi \alpha'}} e^{\frac{\pi t}{2}} = e^{-2\pi t \left[ \frac{\sqrt{\nu^2}}{\sqrt{\pi^2 \alpha'}} - \frac{1}{4} \right]},
\]

which blows up when \( y < \pi \sqrt{\alpha'} \), indicating the onset of tachyonic instability. The integrand is regular at \( t \to 0 \) and has no simple poles and so as anticipated there is no open string pair production even though there are applied electric fluxes on the D2 brane. The explanation for this is similar to that a single D-brane carrying a constant electric flux cannot give rise to open string pair productions. Here the story is that the two ends of virtual open string and virtual anti open string attracted on the D2 can be pulled away while the other two ends on the D0 cannot. So the electric fields applied can only stretch the virtual open string and the virtual anti open string to certain extend but cannot separate them even if we take \( 1 - \tilde{f}_1^2 - \tilde{f}_2^2 = \epsilon \to 0^+ \). With \( \hat{g} = 0 \), from (176), we always have \( \nu_0 = 1/2 \) and it holds true even in the limit \( 1 - \tilde{f}_1^2 - \tilde{f}_2^2 = \epsilon \to 0^+ \). Due to the tachyonic instability when \( y < \pi \sqrt{\alpha'} \), we need to have \( y > \pi \sqrt{\alpha'} \) to validate the amplitude computations. Once this holds, the effective tension on the virtual open strings is less than the critical one even if we take \( 1 - \tilde{f}_1^2 - \tilde{f}_2^2 \to 0^+ \). So this limiting tension cannot break the open strings and therefore there is no open string pair production.

We now move to the second case for which we cannot set \( \hat{g} \) vanish. So we have \( \hat{g}^2 < \tilde{g}^2 + \tilde{f}_1^2 + \tilde{f}_2^2 - 1 < 2\tilde{g}^2 \) and \( 0 < \hat{g}^2 + 1 - \tilde{f}_1^2 - \tilde{f}_2^2 < 2\hat{g}^2 \). From (176), this must imply
that \( \nu_0 \) is imaginary, i.e., \( \nu_0 = i\nu_0 \) with \( \nu_0 \in (0, \infty) \). This can also be seen directly from (177) and it is now
\[
\tanh \pi \nu_0 = \frac{\sqrt{\hat{f}_1^2 + \hat{f}_2^2 - 1}}{|\hat{g}|}.
\]
The present closed string cylinder amplitude can be read from (131) with \( \nu_2 = \nu_1 = 0 \) and \( \nu_0 = i\nu_0 \) as
\[
\Gamma_{2,0} = \frac{2 V_1 \sqrt{1 + \hat{g}^2 - \hat{f}_1^2 - \hat{f}_2^2} \left( \cosh \pi \nu_0 - 1 \right)^2}{(8\pi^2\alpha')^{3/2}} \int_0^\infty \frac{dt}{t^2} e^{-\frac{\nu_0^2}{2\pi \alpha'}} \prod_{n=1}^\infty \frac{[1 - 2|z|^{2n} \cosh \pi \nu_0 + |z|^{4n}]^4}{(1 - |z|^{2n})^6 (1 - 2|z|^{2n} \cosh 2\pi \nu_0 + |z|^{4n})},
\]
where we have used (65) for \( C_n \). As before, the large separation interaction is obviously attractive but the integrand for small \( t \) has an ambiguity of its sign in addition to a potential singularity. The sign ambiguity implies a decay of the underlying system via the open string pair production while the potential singularity implies a potential tachyonic instability. To check both of these explicitly, we need to examine the corresponding open string one-loop annulus amplitude which can be read from (132) as
\[
\Gamma_{2,0} = \frac{2 V_1 \sqrt{1 + \hat{g}^2 - \hat{f}_1^2 - \hat{f}_2^2} \sinh \pi \nu_0}{(8\pi^2\alpha')^{3/2}} \int_0^\infty \frac{dt}{t^2} e^{-\frac{\nu_0^2}{2\pi \alpha'}} \frac{(1 - \cos \pi \nu_0 t)^2}{\sin \pi \nu_0 t} \prod_{n=1}^\infty \frac{[1 - 2|z|^{2n} \cos \pi \nu_0 t + |z|^{4n}]^4}{(1 - |z|^{2n})^6 (1 - 2|z|^{2n} \cos 2\pi \nu_0 t + |z|^{4n})},
\]
where we have used (69) for \( Z_n \). For large \( t \), the integrand of this annulus amplitude does not have a blowing up behavior and therefore there is no potential tachyonic singularity. However, the integrand does have an infinite number of simple poles at \( t_k = (2k - 1)/\nu_0 \) with \( k = 1, 2, \ldots \), indicating the decay of the system via the open string pair production. The decay rate and the open string pair production rate can be read from (134) and (136), respectively, with \( \nu_2 = \nu_1 = 0 \), as
\[
W_{p,p'} = \frac{16 \hat{\nu}_0 \sqrt{\hat{f}_1^2 + \hat{f}_2^2 - 1}}{(8\pi^2\alpha')^{3/2}} \sum_{k=1}^\infty \left( \frac{\nu_0}{2k - 1} \right)^{\nu_0^2} e^{-\frac{(2k-1)\nu_0^2}{2\pi \alpha' \nu_0}} \prod_{n=1}^\infty \frac{1 - |z_{2k-1}|^{2n}}{1 - |z_{2k-1}|^{2n}}^{8},
\]
where we have used (135) for \( Z_k \) and \( |z_k| = e^{-k\pi/\nu_0} \), and
\[
W_{p,p'}^{(1)} = \frac{16 \sqrt{\hat{f}_1^2 + \hat{f}_2^2 - 1}}{(8\pi^2\alpha')^{3/2}} \nu_0^{-\nu_0^2} e^{-\frac{\nu_0^2}{2\pi \alpha' \nu_0}} \prod_{n=1}^\infty \frac{1 - |z|^{2n}}{1 - |z|^{2n}}^{8},
\]
53
Both of the rates blow up when \( \nu_0 \to \infty \) for which \( \tilde{f}_1^2 + \tilde{f}_2^2 - 1 \to \hat{g}^2 \), the critical limit.

In the above, we have an interesting thing happening. Note that the above discussion for \( \tilde{f}_1^2 + \tilde{f}_2^2 < 1 \) holds true also for \( \hat{g} \neq 0 \). For given \( \hat{g} \neq 0 \), there is a potential open string tachyonic instability but no open string pair production if \( \tilde{f}_1^2 + \tilde{f}_2^2 < 1 \) while there is open string pair production but no open string tachyonic instability if \( \tilde{f}_1^2 + \tilde{f}_2^2 > 1 \). For the former, the electric fluxes representing the respective delocalized fundamental strings (see footnote (4)) have no interaction with the D0 brane. So their presence just gives certain modifications of the pure magnetic case of the underlying system but not its characteristic behavior, as discussed above. So a potential tachyonic instability just gives certain modifications of the pure magnetic case of the underlying system but not its characteristic behavior, as discussed above. So a potential tachyonic instability is expected when the brane separation reaches the distance determined by the tachyonic shift. For the latter, we have to admit that we don’t have a better explanation of it except for the following observation. Before that, we would also like to point out that when \( \tilde{f}_1^2 + \tilde{f}_2^2 = 1 \), the above amplitudes and rates computed all vanish.

For this, let us examine the matrix \( w \) given in (175) while keeping \( \hat{g}' \) large. Note that the tr\( w \) is a D2 worldvolume Lorentz invariant and the eigenvalue equation (176) is now replaced by

\[
\lambda_0 + \lambda_0^{-1} = \text{tr} w - 1 = 2 \frac{(1 + \hat{g} \hat{g}')^2 - (\hat{g} - \hat{g}')^2 + (1 + \hat{g}'^2)(\tilde{f}_1^2 + \tilde{f}_2^2)}{(1 + \hat{g}'^2)(1 + \hat{g}^2 - \tilde{f}_1^2 - \tilde{f}_2^2)},
\]

(186)

which gives (176) if we send \( \hat{g}' \to \infty \). For the present purpose, we keep \( \hat{g}' \) large and take the limit \( \hat{g}' \to \infty \) only at the end of the discussion. If we set \( \lambda_0 = e^{2\pi i \nu_0} \), we have from the above,

\[
\cos \pi \nu_0 = \frac{1 + \hat{g} \hat{g}'}{\sqrt{(1 + \hat{g}'^2)(1 + \hat{g}^2 - \tilde{f}_1^2 - \tilde{f}_2^2)}},
\]

\[
\sin \pi \nu_0 = \frac{\sqrt{(\hat{g} - \hat{g}')^2 - (1 + \hat{g}'^2)(\tilde{f}_1^2 + \tilde{f}_2^2)}}{\sqrt{(1 + \hat{g}'^2)(1 + \hat{g}^2 - \tilde{f}_1^2 - \tilde{f}_2^2)}}.
\]

(187)

Note that we have a few Lorentz invariants of D2 brane worldvolume: \( \hat{F}_{\alpha \beta} \hat{F}^{\alpha \beta} = 2[\hat{g}'^2 - (\tilde{f}_1^2 + \tilde{f}_2^2)] \), \( \hat{F}'_{\alpha \beta} \hat{F}'^{\alpha \beta} = 2 \hat{g}^2 \), \( \hat{F}_{\alpha \beta} \hat{F}'^{\alpha \beta} = 2 \hat{g} \hat{g}' \). Therefore the numerator on the right side of \( \sin \pi \nu_0 \) in (187) is also Lorentz invariant. In other words, the \( \nu_0 \) is a Lorentz invariant. Let us now examine this numerator which can be rewritten as \( [\hat{g}'^2(1 - \tilde{f}_1^2 - \tilde{f}_2^2) + \hat{g}^2 - 2 \hat{g}' \hat{g} - (\tilde{f}_1^2 + \tilde{f}_2^2)]^{1/2} \). Due to the \( \hat{g}' \)-factor in the denominator, we need to have a \( \hat{g}' \)-factor in the numerator to give a non-vanishing \( \nu_0 \) and the numerator becomes \( [(1 - \tilde{f}_1^2 - \tilde{f}_2^2) \hat{g}'^2]^{1/2} \) when we take \( \hat{g}' \to \infty \). Note that \( (1 - \tilde{f}_1^2 - \tilde{f}_2^2) \hat{g}'^2 \) is also a Lorentz invariant of D2 worldvolume.

\[\text{Footnote 10:} \text{ Note that in both cases we need to have } \tilde{f}_1^2 + \tilde{f}_2^2 < 1 + \hat{g}^2 \text{ and for the former case, it satisfies trivially.} \]
since it is related to \((\epsilon_{\alpha'\beta'}\epsilon^{\alpha'\beta'} + \hat{F}_{\alpha_1\beta_1}\epsilon^{\alpha_1\beta_1}F^{\alpha_2\beta_2}g)\hat{F}^{\gamma\tau}\hat{F}_{\gamma\tau} = 4\hat{g}'^2(1 - \hat{f}_1^2 - \hat{f}_2^2)\). So it is clear now that the sign of \(1 - \hat{f}_1^2 - \hat{f}_2^2\) determines the nature of \(\nu_0\), real or imaginary! When \(\hat{f}_1^2 + \hat{f}_2^2 < 1\), \(\nu_0\) is real and the underlying system with \(\hat{g}' \to \infty\) resembles a pure magnetic case. When \(\hat{f}_1^2 + \hat{f}_2^2 = 1\), \(\nu_0\) vanishes. While \(\hat{f}_1^2 + \hat{f}_2^2 > 1\), the \(\nu_0\) is imaginary and the underlying system with \(\hat{g}' \to \infty\) has a long-range attractive interaction but at small brane separation the amplitude has a sign ambiguity, indicating a decay via the open string pair production as described above.

### 4.2 The \(p - p' = 4\) case

For \(p \leq 6\), we have only three cases to consider in this subsection: 1) \(p = 6, p' = 2\); 2) \(p = 5, p' = 1\) and 3) \(p = 4, p' = 0\). The extension of \(\hat{F}'_p\) on the Dp' brane to \(\hat{F}_p\) given in \((122)\) in the present context takes the following form

\[
\hat{F}'_p = \hat{F}'_p + \hat{F}'_{p'} = \begin{pmatrix}
\hat{F}'_{p'} \\
0 \\
-\hat{g}'_1 \\
0 \\
-\hat{g}'_2
\end{pmatrix}
\]

where we need to take both \(\hat{g}'_1 \to \infty\) and \(\hat{g}'_2 \to \infty\) at the end of relevant computations.

**The \(p = 6\) case:** For a general flux \(\hat{F}_6\) on D6, even with the above extension \((188)\) for \(\hat{F}'_2\) on D2, the characteristic behavior of the closed string cylinder amplitude or the corresponding open string one-loop annulus amplitude is similar to that for the \(p = p' = 6\) discussed in the previous section. We here specify the \(\hat{F}_6\) to the following form along with the extension of \(\hat{F}'_2\) as,

\[
\hat{F}_{\alpha\beta} = \begin{pmatrix}
\hat{F}_2 \\
0 \\
-\hat{g}_1 \\
0 \\
-\hat{g}_2
\end{pmatrix}, \quad \hat{F}'_{\alpha\beta} = \begin{pmatrix}
\hat{F}'_2 \\
0 \\
-\hat{g}'_1 \\
0 \\
-\hat{g}'_2
\end{pmatrix}
\]

We have therefore

\[
\tan \pi \nu_1 = \frac{1}{|\hat{g}_1|}, \quad \tan \pi \nu_2 = \frac{1}{|\hat{g}_2|},
\]

where we have taken \(\hat{g}'_1 \to \infty\) and \(\hat{g}'_2 \to \infty\). For general \(\hat{g}_1\) and \(\hat{g}_2\), the discussion continues to be the same as that of the \(p = p' = 6\) case. We further specify to the case of both
\( \hat{g}_1 = 0 \) and \( \hat{g}_2 = 0 \) for which \( \nu_1 = \nu_2 = 1/2 \). Now the closed string cylinder amplitude can be read from (131) with \( \nu_1 = \nu_2 = 1/2 \) as

\[
\Gamma_{6,2} = \frac{V_3 \left[ \text{det}(\eta_2 + \hat{F}_2) \text{det}(\eta_2 + \hat{F}_2') \right]^{\frac{1}{2}} (\cos^2 \pi \nu_0 - 1)}{(8\pi^2 \alpha')^{\frac{1}{2}}} \int_0^\infty \frac{dt}{t^2} e^{-\frac{\nu_0^2}{2t^2}} \times \prod_{n=1}^\infty \frac{\left[ (1 + |z|^{4n})^2 - 4 |z|^{4n} \cos^2 \pi \nu_0 \right]^2}{(1 - |z|^{2n})^2 (1 + |z|^{2n})^4 (1 - 2 |z|^{2n} \cos 2\pi \nu_0 + |z|^{4n})},
\]

where we have used (55) for \( C_n \). The amplitude vanishes for \( \nu_0 = 0 \) when \( \hat{F}_2 = \hat{F}_2' = 0 \). This is consistent with the fact that there is no interaction between a Dp and a Dp', placed parallel at a separation, with \( p - p' = 4 \). When \( \nu_0 \in (0, 1/2] \), the sign of this amplitude is determined by that of the factor \( \cos^2 \pi \nu_0 - 1 \) which is negative since all other factors in the integrand are positive. So the interaction is repulsive. This is also consistent with the conclusion reached for \( p = p' = 6 \) in the previous section since we have here \( \nu_0 + \nu_1 > \nu_2 \), i.e., when the possible largest one among the three \( \nu_0, \nu_1, \nu_2 \) is less than the sum of the remaining two. We therefore don't expect to have a potential open string tachyonic instability.

When \( \nu_0 \) is imaginary, i.e., \( \nu_0 = i\bar{\nu}_0 \) with \( \bar{\nu}_0 \in (0, \infty) \), the large separation interaction becomes attractive. For small \( y \), the small \( t \) becomes important in the integration. But for small \( t \), the factor \( (1 - 2|z|^{2n} \cosh 2\pi \bar{\nu}_0 + |z|^{4n}) \) in the denominator of the infinite product in the integrand can be negative and this gives the ambiguity about the sign of the integrand. As before, we expect the decay of the underlying system via the so-called open string pair production.

Either of the above will become manifest if we look from the corresponding open string one-loop annulus amplitude which can be read from (132) in the present context as

\[
\Gamma_{6,2} = \frac{4V_3 \left[ \text{det}(\eta_2 + \hat{F}_2) \text{det}(\eta_2 + \hat{F}_2') \right]^{\frac{1}{2}} \sin^2 \pi \nu_0 }{(8\pi^2 \alpha')^{\frac{1}{2}}} \int_0^\infty \frac{dt}{t^2} e^{-\frac{\nu_0^2}{2t^2}} \times \frac{\sinh^2 \frac{\pi \nu_0 t}{2} - \cos^2 \frac{\pi t}{2}}{\sinh \pi \nu_0 t \sinh^2 \frac{\pi t}{2}} \prod_{n=1}^\infty \frac{\pi \nu_0 t}{\sinh \pi \nu_0 t \sinh^2 \frac{\pi t}{2}} Z_n, \tag{192}
\]

where \( Z_n \) can be read from (60) as

\[
Z_n = \frac{[1 - 2|z|^{2n} \cosh \pi \nu_0 t + |z|^{4n}]^2}{(1 - |z|^{2n})^2 (1 - 2|z|^{2n} \cosh \pi t + |z|^{4n})^2} \times \left[ (1 + |z|^{4n} - 2|z|^{2n} \cosh \pi \nu_0 t \cosh \pi t)^2 - 4 |z|^{4n} \sinh^2 \pi \nu_0 t \sinh^2 \pi t \right] \left( 1 - 2|z|^{2n} \cosh 2\pi \nu_0 t + |z|^{4n} \right). \tag{193}
\]
For large $t$, $Z_n \approx 1$ and the integrand behaves like
\[ \sim e^{-\frac{\nu y^2}{2a}}, \tag{194} \]
which indicates no possibility for tachyonic instability as expected. When $\nu_0 = i\bar{\nu}_0$ with $\bar{\nu}_0 \in (0, \infty)$, the integrand has an infinite number of simple poles occurring at $t_k = k/\bar{\nu}_0$ with $k = 1, 2, \cdots$, indicating the decay of the underlying system via the so-called open string pair production. The decay rate and the open string pair production rate are given by the (134) and (136), respectively, with $\nu_1 = \nu_2 = 1/2$, and are not given here explicitly. In particular, we would like to point out that there is no open string enhancement here even for small $\bar{\nu}_0$ since it is in general given by $e^{\pi|\nu_1 - \nu_2|/\bar{\nu}_0}$ which is unity here. However, for general non-vanishing fluxes $\hat{g}_1$ and $\hat{g}_2$, this enhancement can still be significant.

**The $p = 5$ case:** Here $p' = 1$. The extension of a general flux $\hat{F}_1'$ on D1 to $\hat{F}_5'$, following (122), is
\[ (\hat{F}_5')_{\alpha\beta} = \begin{pmatrix} 0 & \hat{f}' & 0 \\ -\hat{f}' & 0 & \hat{g}_1' \\ 0 & -\hat{g}_1' & 0 \end{pmatrix}, \tag{195} \]
where as before we take both $\hat{g}_1' \to \infty$ and $\hat{g}_2' \to \infty$ at the end of computations. For a general $\hat{F}_5$ on D5, the discussion goes the same as what has been discussed for the $p = p' = 5$ case in the previous section. An example of the following flux on D5
\[ (\hat{F}_5)_{\alpha\beta} = \begin{pmatrix} 0 & \hat{f} & 0 \\ -\hat{f} & 0 & \hat{g}_1 \\ 0 & -\hat{g}_1 & 0 \end{pmatrix}, \tag{196} \]
along with the extension (195) corresponds just to a special case of what has been discussed in great detail in [31]. So we refer there for detail and will not repeat it here. For this, it is also essentially the same as the $\nu_0$ being the imaginary case of the D6-D2 system discussed above.
The $p = 4$ case: Here $p' = 0$. The extension of no flux on D0 to $\hat{F}'_4$, following (122), as

\[
(\hat{F}'_4)_{\alpha\beta} = \begin{pmatrix}
0 & 0 & \hat{g}'_1 \\
-\hat{g}'_1 & 0 & 0 \\
0 & \hat{g}'_2 & -\hat{g}'_2 \\
\end{pmatrix}, \tag{197}
\]

where once again we take both $\hat{g}'_1 \to \infty$ and $\hat{g}'_2 \to \infty$ at the end of computations. For a general flux on D4, the discussion will go the same as that for the $p = p' = 4$ case given in the previous section. We could give some sample discussion for the present extended flux (197) and some special choice of flux on D4 but this will not give anything new. The closed string cylinder amplitude, the open string annulus one, the potential decay rate and the potential open string pair production rate can all be read from the corresponding from (131), (132), (134) and (136), respectively, for the present consideration. So we omit to write each of them explicitly here.

4.3 The $p - p' = 6$ case

This is the last case to be considered in this section. For $p \leq 6$, we have only one case to consider, namely, $p = 6, p' = 0$. The extension of no flux on D0 to $\hat{F}'_6$, following (122), as

\[
(\hat{F}'_6)_{\alpha\beta} = \begin{pmatrix}
0 & 0 & \hat{g}'_0 \\
-\hat{g}'_0 & 0 & 0 \\
0 & \hat{g}'_1 & -\hat{g}'_1 \\
\end{pmatrix}, \tag{198}
\]

where similarly we need to take $\hat{g}'_0 \to \infty, \hat{g}'_1 \to \infty$ and $\hat{g}'_2 \to \infty$ at the end of computations. As before, for a general flux $\hat{F}_6$ on D6, the relevant discussion goes more or less the same as that for the $p = p' = 6$ case discussed in the previous section and we will not repeat it
here. We could give a sample discussion for the following flux on $D_6$,

$$
(\hat{F}_6)_{\alpha\beta} = \begin{pmatrix}
0 & \hat{f}_1 & \hat{f}_2 \\
-\hat{f}_1 & 0 & \hat{g}_0 \\
-\hat{f}_2 & -\hat{g}_0 & 0 \\
0 & \hat{g}_1 & 0 \\
-\hat{g}_1 & 0 & \hat{g}_2 \\
-\hat{g}_2 & 0 & 0
\end{pmatrix},
$$

(199)

and this is still a rather general case of the more general discussion for the $p = p' = 6$ case mentioned above but now with

$$
\tan \pi \nu_0 = \frac{\sqrt{1 - \hat{f}_1^2 - \hat{f}_2^2}}{|\hat{g}_0|}, \quad \tan \pi \nu_1 = \frac{1}{|\hat{g}_1|}, \quad \tan \pi \nu_2 = \frac{1}{|\hat{g}_2|},
$$

(200)

where we have taken $\hat{g}_0' \to \infty, \hat{g}_1' \to \infty$ and $\hat{g}_2' \to \infty$. If we further set $\hat{g}_1 = \hat{g}_2 = 0$, we have $\nu_1 = \nu_2 = 1/2$. For this special case, the closed string cylinder amplitude can be read from (131) as

$$
\Gamma_{6,0} = \frac{V_1}{2(8\pi^2\alpha')^{1/2}} \int_0^\infty dt \frac{t^3 e^{-\frac{\sqrt{2}t}{\pi\alpha'}}}{(1 - |z|^{2n})^2 (1 + |z|^{2n})^4 (1 - 2|z|^{2n} \cos 2\pi\nu_0 + |z|^{4n})},
$$

(201)

where we have used (55) for $C_n$. Except for the overall constant factor, this amplitude looks essentially the same as the corresponding one for the $p = 6, p' = 2$ case discussed in subsection 4.2. For real and non-vanishing $\nu_0$, which requires $\hat{f}_1^2 + \hat{f}_2^2 < 1$ from (200), the amplitude is negative and therefore the interaction is repulsive again since here $\nu_0 + \nu_1 > \nu_2$, i.e. the sum of $\nu_0$ and $\nu_1$, with each being less than or equal to the $\nu_2$, is larger than the largest $\nu_2$, as discussed for the $p = p' = 6$ case in the previous section. When $\nu_0 = 0$ for which $\hat{g}_0 \neq 0$ and $\hat{f}_1^2 + \hat{f}_2^2 = 1$, the amplitude vanishes but this is different from the $p = 6, p' = 2$ case for which the the fluxes on $D_6$ and $D_2$ all vanish (or the fluxes on $D_6$ are vanishing except for the ones along the $D_2$ directions which are identical to those on the $D_2$). The explanation for the present vanishing interaction goes like this. The interaction between a $D_0$ and a $D_6$ carrying no flux is repulsive. The magnetic flux $\hat{g}_0$ stands for delocalized $D_4$ within $D_6$ which has no interaction with $D_0$ since their dimensionality differs by four. The electric flux $\hat{f}_1, \hat{f}_2$ stand for the delocalized fundamental strings within
D6 which have attractive interaction with the D0. So the vanishing of this amplitude must imply the cancellation of the repulsive interaction between the D6 and the D0 with the attractive one between the D0 and the fundamental F-strings within the D6 when \( \hat{f}_1^2 + \hat{f}_2^2 = 1 \). This is also consistent with the general conclusion reached for the \( p = p' = 6 \) case in the previous section that \( \nu_0 + \nu_1 = \nu_2 \) for which the amplitude vanishes.

For real non-vanishing \( \nu_0 \), given what we learned earlier in this paper, we expect no open string tachyonic instability. Let us check this explicitly by examining the corresponding open string one-loop annulus amplitude which can be read from (132) as

\[
\Gamma_{6,0} = \frac{V_1}{(8\pi^2\alpha')^{1/2}} \int_0^\infty \frac{dt}{t^2} e^{-\frac{\nu_0 t}{2\pi\alpha'}} \frac{\sinh \frac{\nu_0 t}{2}}{\cosh \frac{\nu_0 t}{2} \sinh^2 \frac{\pi t}{2}} \prod_{n=1}^\infty Z_n,
\]

where \( Z_n \), read from (60), is

\[
Z_n = \frac{\left|1 - 2|z|^{2n} \cosh \pi \nu_0 t + |z|^{4n}\right|^2}{(1 - |z|^{2n})^2(1 - 2|z|^{2n} \cosh \pi t + |z|^{4n})^2} \times \left|\frac{(1 - 2|z|^{2n} \cosh \pi \nu_0 t \cosh \pi t + |z|^{4n})^2 - 4|z|^{4n} \sinh^2 \pi \nu_0 t \sinh^2 \pi t}{(1 - 2|z|^{2n} \cosh 2\pi \nu_0 t + |z|^{4n})}\right|.
\]

For large \( t \), \( Z_n \approx 1 \) and the integrand of the above amplitude behaves like

\[
\sim -e^{-\frac{\nu_0 t}{2\pi\alpha'}},
\]

which vanishes for all \( y \neq 0 \), therefore no tachyonic divergence as expected.

Let us consider \( \nu_0 \) to be imaginary which requires \( \hat{g}_0 \neq 0 \) and \( 1 < \hat{f}_1^2 + \hat{f}_2^2 < 1 + \hat{g}_0^2 \). We now set \( \nu_0 = i\bar{\nu}_0 \) with \( \nu_0 \in (0, \infty) \). We have now from (200)

\[
\tanh \pi \bar{\nu}_0 = \frac{\sqrt{\hat{f}_1^2 + \hat{f}_2^2 - 1}}{|\hat{g}_0|},
\]

in addition to \( \nu_1 = \nu_2 = 1/2 \) when we take \( \hat{g}_1 = \hat{g}_2 = 0 \). The open string one-loop annulus amplitude is now, from (202) with \( \nu_0 = i\bar{\nu}_0 \),

\[
\Gamma_{6,0} = \frac{V_1}{(8\pi^2\alpha')^{1/2}} \int_0^\infty \frac{dt}{t^2} e^{-\frac{\nu_0 t}{2\pi\alpha'}} \frac{\sin \frac{\nu_0 t}{2}}{\cos \frac{\nu_0 t}{2} \sinh^2 \frac{\pi t}{2}} \prod_{n=1}^\infty Z_n,
\]

where \( Z_n \) continues to be given by (202) but now with \( \nu_0 = i\bar{\nu}_0 \). Now this amplitude has an infinite number of simples poles of its integrand occurring at \( t_k = (2k - 1)/\bar{\nu}_0 \) with \( k = 1, 2, \ldots \), giving an imaginary part of the amplitude. This further indicates the decay of the underlying system via the so-called open string pair production. The decay rate
and the open string pair production rate are given, respectively, by \(134\) and \(136\) for the present case with \(\nu_1 = \nu_2 = 1/2\) as

\[
\mathcal{W} = \frac{4 \sqrt{\hat{f}_1^2 + \hat{f}_2^2 - 1}}{\tilde{\nu}_0 (8\pi^2 \alpha')^{1/2}} \sum_{k=1}^{\infty} \left( \frac{\tilde{\nu}_0}{2k-1} \right)^3 \cosh^2 \left( \frac{(2k-1)\pi}{2\tilde{\nu}_0} \right) \frac{\sinh^2 \left( \frac{(2k-1)\pi}{2\tilde{\nu}_0} \right)}{\sinh^2 \left( \frac{2\pi\alpha'}{\tilde{\nu}_0} \right)} e^{-\frac{(2k-1)^2\alpha'}{2\tilde{\nu}_0}} Z_{2k-1}(\tilde{\nu}_0, 1/2, 1/2),
\]

where

\[
Z_k(\tilde{\nu}_0, 1/2, 1/2) = \prod_{n=1}^{\infty} \left( \frac{1 + |z_k|^{2n}}{1 - |z_k|^{2n}} \right)^4 \left( 1 + 2|z_k|^{2n} \cosh \frac{k\pi}{\tilde{\nu}_0} + |z_k|^{4n} \right)^2 \cosh \frac{\pi}{\tilde{\nu}_0} \sinh \frac{\pi}{\tilde{\nu}_0} e^{-\frac{\pi^2}{8\pi\alpha' \tilde{\nu}_0}} Z_1(\tilde{\nu}_0, 1/2, 1/2).
\]

We would like to point out that with the choices of fluxes \(198\) and \(199\), the above amplitudes and rates share qualitatively the same properties as the corresponding, respectively, in the case of \(p = 2, p' = 0\) discussed in subsection \(4.1\), even though the details are different. For example, both of the rates blow up when \(\tilde{\nu}_0 \to \infty\) which occurs as \(\hat{f}_1^2 + \hat{f}_2^2 - 1 \to \hat{g}_0^2\), reaching the so-called critical field. For small \(\tilde{\nu}_0 \ll 1\) and the open string pair production rate above looks also like that of the \(p = 2, p' = 0\) case. For a general \(\tilde{\nu}_0\), these two rates are different. Note that we don’t have the exponential enhancement of the rate for small \(\tilde{\nu}_0\), either.

5 Discussion and conclusion

We compute, in this paper, the closed string cylinder amplitude between a Dp and a Dp', placed parallel at a separation along the directions transverse to the Dp, with each carrying their general worldvolume fluxes and with \(p - p' = \kappa = 0, 2, 4, 6\) and \(p \leq 6\). We find that the amplitude for each of the \(p - p' = \kappa \neq 0\) cases can be obtained as just a special case of the corresponding amplitude for the \(p = p'\) case based on the related physical consideration presented in the previous sections. As such, we find a universal
formula for this closed string cylinder amplitude, valid for all cases specified above, as

$$
\Gamma_{p,p'} = \frac{2^3 V_{p'+1} \left[ \det(\eta_{p'} + \hat{F}_{p'}) \det(\eta_p + \hat{F}_p) \right]^{\frac{1}{2}} \prod_{\alpha=0}^2 \sin \pi \nu_{\alpha} \prod_{n=0}^\infty \frac{dt}{t^{\frac{p-1}{2}}} e^{-\frac{\pi^2 t}{2}}}{2^\frac{2}{2} (8\pi^2 \alpha')^{\frac{p+1}{2}}} \times \theta_1 \left( \frac{i\nu_0 + i\nu_2}{2} \right) \theta_1 \left( \frac{i\nu_0 - i\nu_2}{2} \right) \theta_1 \left( \frac{i\nu_0 + i\nu_2}{2} \right) \theta_1 \left( \frac{i\nu_0 - i\nu_2}{2} \right),
$$

$$
\Gamma_{p,p'} = \frac{2^2 V_{p'+1} \left[ \det(\eta_{p'} + \hat{F}_{p'}) \det(\eta_p + \hat{F}_p) \right]^{\frac{1}{2}} \left[ \sum_{\alpha=0}^2 \cos^2 \pi \nu_{\alpha} - 2 \prod_{\alpha=0}^2 \cos \pi \nu_{\alpha} - 1 \right]}{2^\frac{2}{2} (8\pi^2 \alpha')^{\frac{p+1}{2}}} \times \prod_{n=1}^\infty C_n,
$$

(209)

where $C_n$ continues to be given by (59). The amplitude for each given pair of $p$ and $p'$ and the corresponding given worldvolume fluxes can be obtained from the above as a special case as prescribed in the previous two sections. The corresponding open string one-loop annulus universal amplitude can be obtained from the above via the Jacobi transformation $t \to t' = 1/t$ along with the relations for the $\theta_1$-function and the Dedekind $\eta$-function given in (59) as

$$
\Gamma_{p,p'} = \frac{-2^3 i V_{p'+1} \left[ \det(\eta_{p'} + \hat{F}_{p'}) \det(\eta_p + \hat{F}_p) \right]^{\frac{1}{2}} \prod_{\alpha=0}^2 \sin \pi \nu_{\alpha} \prod_{n=0}^\infty \frac{dt}{t^{\frac{p-1}{2}}} e^{-\frac{\pi^2 t}{2}}}{2^\frac{2}{2} (8\pi^2 \alpha')^{\frac{p+1}{2}}} \times \theta_1 \left( \frac{i\nu_0 + i\nu_2}{2} \right) \theta_1 \left( \frac{i\nu_0 - i\nu_2}{2} \right) \theta_1 \left( \frac{i\nu_0 + i\nu_2}{2} \right) \theta_1 \left( \frac{i\nu_0 - i\nu_2}{2} \right),
$$

$$
\Gamma_{p,p'} = \frac{2^2 V_{p'+1} \left[ \det(\eta_{p'} + \hat{F}_{p'}) \det(\eta_p + \hat{F}_p) \right]^{\frac{1}{2}} \left[ \sum_{\alpha=0}^2 \cos^2 \pi \nu_{\alpha} t - 2 \prod_{\alpha=0}^2 \cosh \pi \nu_{\alpha} t - 1 \right] \prod_{n=1}^\infty Z_n}{2^\frac{2}{2} (8\pi^2 \alpha')^{\frac{p+1}{2}}} \times \prod_{\alpha=0}^2 \frac{\sin \pi \nu_{\alpha} t}{\sinh \pi \nu_{\alpha} t},
$$

(210)

where we have dropped the prime on the open string variable $t$ and $Z_n$ continues to be given by (60). If one of three $\nu_{\alpha}$, say $\nu_0$, is imaginary, the underlying system decays via the open string pair production. The general decay rate is

$$
\mathcal{W}_{p,p'} = \frac{2^{3-\frac{p}{2}} \left[ \det(\eta_{p'} + \hat{F}_{p'}) \det(\eta_p + \hat{F}_p) \right]^{\frac{1}{2}} \sinh \pi \nu_0 \sin \pi \nu_1 \sin \pi \nu_2 \sum_{k=1}^\infty (-)^{k+1} \left( \frac{\nu_0}{k} \right)^{\frac{p-3}{2}}}{\nu_0 (8\pi^2 \alpha')^{\frac{p+1}{2}}} \times \left( \cosh \frac{k \pi \nu_1}{\nu_0} - (-)^k \cosh \frac{k \pi \nu_2}{\nu_0} \right)^2 \frac{\cosh k \frac{\pi \nu_1}{\nu_0} \nu_1}{\sinh k \frac{\pi \nu_1}{\nu_0} \sinh k \frac{\pi \nu_2}{\nu_0}} e^{-\frac{\pi^2 \nu_0}{2 \pi \nu_0}} Z_k(\nu_0, \nu_1, \nu_2),
$$

(211)
where $Z_k$ is given by (135). The corresponding open string pair production rate is given by the leading $k = 1$ term of the above as

$$
W^{(1)}_{p,p'} = \frac{2^{3-\frac{p}{2}} \left[ \det(\eta_{p'} + \hat{F}_{p'}) \det(\eta_p + \hat{F}_p) \right]^{\frac{1}{2}} \sinh \pi \nu_0 \sin \pi \nu_1 \sin \pi \nu_2}{\nu_0^2} \cdot \frac{\nu_{p-5}}{e^{-\frac{\nu^2}{2\pi \bar{\nu} \nu_0}}} \cdot \left(8\pi^2 \alpha'\nu_0^2\right)^{\frac{p'+1}{2}} \times \left(\cosh \frac{\pi \nu_1}{\nu_0} + \cosh \frac{\pi \nu_2}{\nu_0}\right)^2 \sinh \frac{\pi \nu_1}{\nu_0} \sinh \frac{\pi \nu_2}{\nu_0} Z_1(\nu_0, \nu_1, \nu_2).
$$

(212)

With the above, we have studied various properties of the amplitudes for each of the systems considered such as the nature of the interaction, the open string tachyonic instability, and the open string pair production if it exists and the associated enhancement. In particular, we find that the interaction can be repulsive for $p' \leq p = 6$ with all three parameters $\nu_0, \nu_1, \nu_2$ being real and non-vanishing. Since the amplitude is symmetric with respect to the three $\nu_0, \nu_1, \nu_2$, we can assume $\nu_0 \leq \nu_1 \leq \nu_2$ without loss of generality. The repulsive interaction occurs indeed when $\nu_0 + \nu_1 > \nu_2$. In other words, whenever the sum of two smaller $\nu$’s (here $\nu_0$ and $\nu_1$) is larger than the largest $\nu$ (here $\nu_2$), the underlying interaction is repulsive. The reason for the above requirements is simple. The repulsive inter-brane interaction occurs, in the absence of the worldvolume fluxes, only for the system of $p = 6$ and $p' = 0$ for which we have $\nu_0 = \nu_1 = \nu_2 = 1/2$ following the description given in the previous two sections (Here $\nu_0 + \nu_1 > \nu_2$ meets the above condition for repulsive interaction). For all other choices of $p$ and $p'$, the inter-brane interaction, in the absence of the worldvolume fluxes, is either attractive or vanishing. So to have a potential repulsive-interaction, we first need to have the presence of D6 and secondly we need to have D0 which can be realized in general for $p' \leq p = 6$ with all three $\nu_0, \nu_1, \nu_2 \in (0, 1/2]$. Note also that when $\nu_0, \nu_1, \nu_2 \in (0, 1/2]$, the worldvolume fluxes give rise to not only D0 but also D2 and D4. The latter branes give instead the attractive inter-brane interaction in addition to the repulsive one between D6 and D0. So whether the net inter-interaction is repulsive, attractive or vanishes depends on the competition between the repulsive component and the attractive one mentioned above. Using the above assumption $\nu_0 \leq \nu_1 \leq \nu_2$, we have shown in section 2 and checked for each case considered later that whenever $\nu_0 + \nu_1 > \nu_2$, the net interaction is repulsive. The interaction vanishes if $\nu_0 + \nu_1 = \nu_2$. The interaction is attractive whenever $\nu_0 + \nu_1 < \nu_2$.

We also find that there is a correlation between the nature of interaction and the existence of the open string tachyonic instability of the underlying system when the brane separation reaches the distance determined by the so-called tachyonic shift. When the interaction is repulsive, there is no open string tachyonic instability, independent of the
brane separation. When the brane separation is attractive, we do have the onset of
tachyonic instability when the brane separation reaches the distance set by the tachyonic
shift. We analyze this from various means and confirm this correlation.

When one of three parameters $\nu_0, \nu_1, \nu_2$ is imaginary, the underlying system is unstable
and decays via the so-called open string pair production. This is reflected in that the open
string one-loop amplitude has an imaginary part. Again without loss of generality, we
choose $\nu_0 = i\bar{\nu}_0$ with $\bar{\nu}_0 \in (0, \infty)$. This is related to the applied electric flux(es). When
the applied electric flux reaches its critical one, we have $\bar{\nu}_0 \rightarrow \infty$, the open string pair
production rate diverges and the pair production cascades, giving rise also to the other
instability of the system. We have also studied the potential enhancement of the pair
production rate in the presence of magnetic fluxes and our findings here are consistent with
our previous studies on this. The enhancement is determined by the so-called tachyonic
shift which can be given in general as $|\nu_2 - \nu_1|/2$ with $\nu_1, \nu_2 \in [0, 1/2]$. In practice, all
$\bar{\nu}_0, \nu_1, \nu_2$ are small. We have that the larger the shift is, the larger the open string pair
production enhancement. For this purpose, we prefer to have the presence of the larger
of $\nu_1$ and $\nu_2$ while turning off the smaller one such that the enhancement is larger. For
example, we keep $\nu_1$ while drop $\nu_2$. So the question is: can we realize the largest shift
which is $\nu_1/2 = 1/4$? This is one of the motivations for this paper as mentioned in the
Introduction. We now have the answer and it can come from the system of $p - p' = 2$
without adding any worldvolume magnetic fluxes given the above consideration. This
is due to that the D$p'$ brane acts effectively as a magnetic field which can give rise to
$\nu_1 = 1/2$ as shown in subsection 4.1. The largest pair production rate for practically given
small $\bar{\nu}_0$ occurs for $p = 3, p' = 1$ with purely added electric fluxes along the D1-directions.
This system has $\nu_1 = 1/2$, giving the possible largest enhancement. This may have a
potential application in practice which we would like to pursue in the near future.

One last thing we have not mentioned so far is the relationship of the present discussion
with that for a system of D$p$ and D$p'$ ($p \geq p'$) with the two branes not at rest but with a
constant relative motion transverse at least to the D$p'$ and/or a rotation between certain
transverse directions and the brane directions. As discussed in [30], a D$p$ brane carrying
a constant electric flux along certain spatial direction is equivalent to a boosted and
delocalized D$(p - 1)$ brane along this direction. They are related by a T-duality along this
direction and the boost velocity is determined by the electric flux. By a similar token,
a D$p$ brane carrying a magnetic strength $F_{ij}$ with $i < j$, for example, is equivalent to
a D$(p - 1)$ brane rotated between the spatial $i$-direction and the spatial $j$-direction and
delocalized along the $j$-direction, for example. Here the D$p$ and D$(p - 1)$ are related by a
T-duality along the $j$-direction and the rotation is determined by the magnetic flux. Since
the resulting D(p - 1) brane in either case is delocalized along certain directions transverse to the brane, it is probably much easier and much more straightforward to compute the same interaction between two such D branes using their equivalent ones carrying fluxes at rest as discussed in this paper even though computations of the interaction for localized such objects are known (for example, see 36[44,45]).

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Appendix A

In this Appendix, we first give a general discussion of the eigenvalues of the matrix $w$ (24). Note that

$$s = (\eta - \tilde{F})(\eta + \tilde{F})^{-1}, \quad s' = (\eta - \tilde{F}')(\eta + \tilde{F}')^{-1}, \quad w = ss'^T, \quad (213)$$

and each of them satisfies the same relation, e.g.,

$$w_\alpha^\gamma(w^T)_\gamma^\beta = \delta_\alpha^\beta. \quad (214)$$

This is actually a relation satisfied by a Lorentz transformation in $(1 + p)$ dimensions. In other words, either $s$ or $s'$ is a general Lorentz transformation since either flux $\tilde{F}$ or $\tilde{F}'$ counts the number of independent parameters of $SO(1,p)$ as $(p + 1)p/2$. This holds also true for $w$ since it is the product of $s$ and $s'$. In addition, we have $\det{s} = \det[(\eta - \tilde{F})(\eta + \tilde{F})^{-1}] = \det(\eta - \tilde{F}) \det(\eta + \tilde{F})^{-1} = \det(\eta - \tilde{F}) \det(\eta - \tilde{F}')^{-1} = 1$ where we have used $(\eta + \tilde{F})^T = (\eta - \tilde{F})$ in the third equality since $\tilde{F}$ is antisymmetric. This same holds for $s'$, too. So we have $\det{w} = \det{s} \det{s'^T} = 1$ as well. Note also here

$$\eta_{\alpha\beta} = (-1, 1, \cdots 1), \quad \alpha, \beta = 0, 1, \cdots p. \quad (215)$$

Given the above, it is clear that a purely electric flux gives a Lorentz boost while a purely magnetic one gives only a rotation of $SO(p)$. When both $\tilde{F}$ and $\tilde{F}'$ are each purely electric, the resulting $w$ is in general not a pure Lorentz boost unless the two electric fluxes are collinear. However, the resulting $w$ is a rotation when both $\tilde{F}$ and $\tilde{F}'$ are each purely magnetic. If $w$ is indeed a pure Lorentz boost, it can always be brought to the following form by a $SO(p)$ rotation $R$,

$$w = r^T \tilde{w} r, \quad (216)$$

\[65\]
where
\[ \tilde{w} = \begin{pmatrix} \gamma & \gamma v \\ \gamma v & \gamma \end{pmatrix}, \quad r = \begin{pmatrix} 1 \\ R_{p \times p} \end{pmatrix}, \]
with \( \gamma = (1 - v^2)^{-1/2} \) and \( v < 1 \). The rotation just brings the velocity along the ‘1’-direction. So it is clear that a general Lorentz boost has only one non-trivial pair real eigenvalues of \( \lambda_{\pm} = \gamma(1 \pm v) \) with \( \lambda_+ \lambda_- = 1 \) and the rest are all unity. If \( w \) is instead a pure SO(p) rotation, e.g. \( w = r \) with \( r \) given above, we have its eigenvalues 1 and the rest in pairs \( \lambda_\alpha, \lambda_-^\alpha \) with \( \lambda_\alpha = e^{2\pi i \nu_\alpha} \) and \( \nu_\alpha = 0, 1, \cdots, (p - 2)/2 \) when \( p \) is even or \( 1, 1, \cdots, (p - 3)/2 \) when \( p \) is odd.

For a general \( w \), we need to put some extra efforts to figure out the nature of its eigenvalues. For this, since \( w \) is a Lorentz matrix with \( \det w = 1 \), we can set
\[ w = e^K, \]
where from \((w^{-1})_{\alpha \beta} = (w^T)_{\alpha \beta}\) we have
\[ K_\beta^\alpha = -K_\alpha^\beta. \]

Now solving the eigenvalue problem of \( w \) is transformed to that of \( K \). In other words, we have
\[ f(\rho) = \det (\rho \delta_\alpha^\beta - K_\alpha^\beta) = 0. \]

Since \( K^T \) has the same eigenvalues as \( K \), we have also
\[ f(\rho) = \det [\rho \delta_\alpha^\beta - (K^T)_\alpha^\beta] = 0, \]
which implies whenever \( \rho \) is an eigenvalue so is \(-\rho\) from \([219]\). In other words, the eigenvalues appear in pairs. When \( p \) is even, we have always one zero-eigenvalue since \( \det K_\alpha^\beta = -\det K_\alpha^\beta = 0 \), giving the unity eigenvalue of \( w \) discussed in Section 3, and the rest are in pairs.

Let us discuss the \( p = \text{even} \) case first. For the zero-eigenvalue, we can choose the corresponding eigenvector as \( e \) such that \( K_\alpha^\beta e_\beta = 0 \). We have two sub-cases to consider: \( e \cdot e = -1 \) and \( e \cdot e = 1 \) where we have normalized each as indicated \([12]\). For the first

\[11\] We choose here the same conventions as used in Section 3.

\[12\] For certain choice of the fluxes, we may have the eigenvector being light-like. This can always be taken as certain limit of the time-like or space-like limit as discussed. When this happens, the corresponding \( K \) matrix can still be diagonalized. Here we use a simple example for \( p = 2 \) to illustrate this. Now the
subcase, we choose \( e^0 = e \) such that \( \{ e^0, e^1, \ldots, e^p \} \) forms a complete normalized basis of the eigenvector space, giving

\[
\eta^{\alpha\beta} e^\alpha e^\beta \beta = \eta^{\alpha\beta} \quad \text{or} \quad \eta_{\alpha\beta} e^\alpha e^\beta \beta = \eta_{\alpha\beta},
\]

(225)

where the \( \alpha, \beta \) indices are raised or lowered using \( \eta^{\alpha\beta} \) or \( \eta_{\alpha\beta} \) and similarly for the \( \bar{\alpha}, \bar{\beta} \) indices. So \( e_\alpha \bar{\alpha} \) or \( e^{\alpha} \bar{\alpha} \) is also a Lorentz transformation. We take \( \alpha = (0, a) \) \((\bar{\alpha} = (0, \bar{a})\) from now on with \( a = 1, 2, \ldots, p \) \((\bar{a} = 1, 2, \ldots, p)\). We have now \( K_\alpha \beta e_\beta 0 = 0 \) and from (219) we have also \( (e^T)^0 \beta K_\alpha = 0 \). With these two, we have, from (225),

\[
\bar{K}_\alpha \bar{\beta} \equiv (e^T)_\bar{\alpha} K_\alpha \beta e_\beta \bar{\beta} = \begin{pmatrix} 0 & 0 \\ 0 & \bar{K}_\alpha \bar{\beta} \end{pmatrix},
\]

(226)

where \( \bar{K}_a \bar{\beta} = \bar{K}_{\bar{a}} \bar{\beta} = -\bar{K}_{\bar{a}} \bar{\beta} \) is real and antisymmetric from the property of \( K \) given in (219). So it can be diagonalized by a unitary matrix \( u \) of the following form with its purely imaginary eigenvalues in pairs as \( (\rho_{\bar{a}}, -\rho_{\bar{a}}) \) with \( \bar{a} = 1, \ldots, p/2 \),

\[
\bar{K} = U \bar{K}_0 U^+ = \begin{pmatrix} 1 & 0 \\ 0 & u_{p\times p} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & (\bar{K}_0)_{p \times p} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u_{p\times p}^+ \end{pmatrix},
\]

(227)

most general \( K \) can be expressed as

\[
K_\alpha \beta = \begin{pmatrix} 0 & f_1 & f_2 \\ f_1 & 0 & g \\ f_2 & -g & 0 \end{pmatrix},
\]

(222)

where we assume \( f_1, f_2, g \) are all non-negative without loss of generality. This matrix has three expected eigenvalues: \( 0, \rho_0, -\rho_0 \) with \( \rho_0 = \sqrt{f_1^2 + f_2^2 + g^2} \). It can be diagonalized as

\[
K = V \bar{K}_0 V^{-1},
\]

(223)

where the diagonal matrix \( (\bar{K}_0)_\alpha \beta = (0, \rho_0, -\rho_0) \) and the non-singular matrix

\[
V = \frac{1}{\rho_0} \begin{pmatrix} g & -f_2 & f_1 \\ \frac{g f_1 - f_2 \rho_0}{g^2 - f_1^2} & \frac{g \rho_0 - f_1 f_2}{g^2 - f_1^2} & 1 \\ -f_1 g - f_2 \rho_0 & g \rho_0 + f_1 f_2 & f_1^2 - g^2 \end{pmatrix},
\]

(224)

with its \( \det V = 2 \). For the 0-eigenvalue, the corresponding eigenvector can be taken as \( e = (g, f_2, -f_1)^T \), giving \( K_\alpha \beta e_\beta = 0 \). Here \( e^2 = \eta^{\alpha\beta} e_\alpha e_\beta = f_1^2 + f_2^2 - g^2 \) which can be either time-like or space-like in general. However, it becomes null when \( f_1^2 + f_2^2 = g^2 \), which can be taken as the corresponding limit of either time-like or space-like case. So long this is taken as a limiting case, we don’t have a problem to diagonalize the matrix \( K \) since \( \det V = 2 \) is non-singular.
exactly the same line as above and end up with now forms a complete normalized basis of the eigenvector space. The following discussion goes same as $K$ where $\bar{f}$ the function and is eigenvalues: one zero and the rest being purely imaginary in pairs as indicated in \((228)\).

With the above, the original $K$ is diagonalized as

$$K = V \bar{K}_0 V^{-1},$$ \hspace{1cm} \text{(229)}

where $V = eU$ with $(e^{-1})^\alpha \beta = (e^T)^\alpha \beta$ and $U^{-1} = U^+$. Here $\bar{K}_0$ gives the expected eigenvalues: one zero and the rest being purely imaginary in pairs as indicated in (228).

For the second subcase, i.e. $e \cdot e = 1$, we take now $e^p = e$ such that $\{e^0, e^1, \cdots, e^p\}$ forms a complete normalized basis of the eigenvector space. The following discussion goes exactly the same line as above and end up with now

$$K_{\alpha \beta} \equiv (e^T)^\alpha K_{\alpha}^\beta e^\beta = \left(\begin{array}{cc}
\bar{K}_{\alpha'}^{\beta'} & 0 \\
0 & 0
\end{array}\right),$$ \hspace{1cm} \text{(230)}

where $\alpha', \beta' = 0, 1, \cdots p - 1$. The diagonalization of the matrix $\bar{K}_{\alpha'}^{\beta'}$ follows exactly the same as $K_{\alpha}^\beta$ for odd $p$. So we turn now to the odd $p$ case.

For this case, since the eigenvalues are in pairs as $(\rho_{\alpha}, -\rho_{\alpha})$ with $\alpha = 0, 1, \cdots (p-1)/2$, the function $f(\rho)$ from \((220)\) must be even in power of $\rho$ when we expand the determinant and is

$$f(\rho) = \rho^{p+1} + c_1 \rho^{p-1} + c_2 \rho^{p-3} + \cdots + c_{p-1} \rho^2 + \det K_{\alpha}^\beta.$$ \hspace{1cm} \text{(231)}

Note that $\det K_{\alpha}^\beta = - det K_{\alpha}^\beta = -(\text{pf}(K_{\alpha}^\beta))^2 < 0$ with $K_{\alpha}^\beta = -K_{\beta}^\alpha$ from (219) and pf($K$) denoting the Pfaffian of antisymmetric matrix $K_{\alpha}^\beta$. So we have $f(0) = det K_{\alpha}^\beta < 0$. For very large $\rho > 0$, the highest power of $\rho$ dominates and so we have $f(\rho) > 0$. Therefore we must have at least one pair $(\rho_0, -\rho_0)$ with $\rho_0$ positive real satisfying the eigenvalue equation $f(\pm \rho_0) = 0$. If det $K_{\alpha}^\beta = 0$, we will have a pair of zero eigenvalues unless the above $c_{p-1} = 0$ and if this is case we can do what we have done for the above even $p$ case.

If we have more zero eigenvalues, we just repeat this process until we have non-zero ones. For now, we assume $\det K_{\alpha}^\beta < 0$, so we have $\rho_0 \neq 0$. The corresponding eigenvectors, denoting as $x^0$ and $x^1$, satisfy their respective equations as

$$K_{\alpha}^\beta x^0_{\alpha} = \rho_0 x^0_{\alpha}, \hspace{1cm} K_{\alpha}^\beta x^1_{\alpha} = -\rho_0 x^1_{\alpha}.$$ \hspace{1cm} \text{(232)}

From the first one, we have $\rho_0 x^0 \cdot x^0 = (x^0)^\alpha K_{\alpha}^\beta x^0_{\beta} = x^0_{\alpha} K_{\alpha}^\beta x^0_{\beta} = 0$ where we have used the property of $K$ from (219). This must imply $x^0 \cdot x^0 = 0$ since $\rho_0 \neq 0$. In other
words, \( x^0 \) is a null vector. By the same token, we have also \( x^1 \) as a null vector. We now show \( x^0 \cdot x^1 \neq 0 \). Since both are null vectors, without loss of generality, we can always choose \( x^0 = (1, 1, 0, \cdots, 0) \) and \( x^1 = (\vec{a}, \vec{a}) \). If \( x^0 \cdot x^1 = 0 \), we must have \( |\vec{a}| = a_1 > 0 \) and \( x^1 = (a_1, a_1, 0, \cdots, 0) = a_1 x^0 \). This contradicts the fact that the two eigenvectors are independent since they correspond to different eigenvalues. Therefore, we must have \( x^0 \cdot x^1 \neq 0 \). For convenience, we choose to have \( x^0 \cdot x^1 = -2 \). With this, we define

\[
e^0 = \frac{1}{2}(x^0 + x^1), \quad e^1 = \frac{1}{2}(x^0 - x^1),
\]

such that \((e^0)^2 = -1\), \((e^1)^2 = 1\) and \(e^0 \cdot e^1 = 0\). We now construct an orthogonal basis \(\{e^0, e^1, \cdots, e^p\}\) satisfying the same relations as those given in (225). Note that

\[
\bar{K}_{\bar{\alpha}}{\bar{\beta}} = (e^T)_{\bar{\alpha}}{\bar{\beta}} K_{\alpha}{\beta} e_{\beta} = \begin{pmatrix} 0 & \rho_0 \\ \rho_0 & 0 \\ & & \bar{K}_{\bar{\alpha}}{\bar{\beta}} \end{pmatrix},
\]

where \(\bar{K}_{\bar{\alpha}}{\bar{\beta}}\) is a \((p-1) \times (p-1)\) antisymmetric matrix \((\bar{K}_{\bar{a}}{\bar{b}} = \bar{K}_{\bar{b}}{\bar{a}} = -\bar{K}_{\bar{a}}{\bar{a}})\) and can be diagonalized, as before, by a \((p-1) \times (p-1)\) unitary matrix \(u\), with its purely imaginary eigenvalues in pairs as \((\rho_{\bar{c}}, -\rho_{\bar{c}})\) with \(\bar{c} = 1, 2, \cdots, (p-1)/2\). Note that the symmetric sub-matrix in \(\bar{K}_{\bar{\alpha}}{\bar{\beta}}\) can be diagonalized by a \(2 \times 2\) matrix \(R\) as

\[
\begin{pmatrix} 0 & \rho_0 \\ \rho_0 & 0 \end{pmatrix} = R \begin{pmatrix} \rho_0 & 0 \\ 0 & -\rho_0 \end{pmatrix} R^{-1},
\]

where specifically

\[
R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad R^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.
\]

In other words, the matrix \(\bar{K}_{\bar{\alpha}}{\bar{\beta}}\) can be diagonalized by the unitary matrix \(rU\) as

\[
\bar{K} = r \begin{pmatrix} \rho_0 & 0 \\ 0 & -\rho_0 \end{pmatrix} \bar{K}_{(p-1) \times (p-1)} r^{-1} = rU \bar{K}_0 U^T = (rU) \bar{K}_0 (rU)^{-1},
\]

where

\[
r = \begin{pmatrix} R & \mathbb{I}_{(p-1) \times (p-1)} \end{pmatrix}, \quad U = \begin{pmatrix} \mathbb{I}_{2 \times 2} & u \end{pmatrix},
\]

\(69\)
and the diagonal matrix $\bar{K}_0 = (\rho_0, -\rho_0, \rho_1, -\rho_1, \cdots, \rho_{(p-1)/2}, -\rho_{(p-1)/2})$. Given the above and from (237), we have

$$K = e\bar{K}e^T = e r U \bar{K}_0 (r U)^+ e^T = (e r U) K_0 (e r U)^{-1},$$

where we have used $e^{-1} = e^T$. So we prove that in general $K$ has a pair of real eigenvalue $(\rho_0, -\rho_0)$ and the rest are all imaginary and given in pairs as $(\rho_c, -\rho_c)$ with $c = 1, 2, \cdots, (p-1)/2$.

In summary, when $p = \text{even}$, we have two cases: 1) one zero eigenvalue and the rest are all imaginary and given in pairs as $(\rho_c, -\rho_c)$ with $c = 1, 2, \cdots, p/2; 2)$ one zero eigenvalue, a pair of real eigenvalues $(\rho_0, \rho_0)$ and the rest are all imaginary and given in pairs as $(\rho_c, -\rho_c)$ with $c = 1, 2, \cdots, (p-2)/2$. For $p = \text{odd}$, we have in general a pair of real eigenvalues $(\rho_0, -\rho_0)$ and the rest are all imaginary and given as $(\rho_c, -\rho_c)$ with $c = 1, 2, \cdots, (p-1)/2$.

If we set the positive real eigenvalue $\rho_0 = 2\pi \nu_0$ and the imaginary eigenvalues $\rho_c = 2\pi i \nu_c$, from (218) we then obtain the same eigenvalues of $\omega$ as discussed in Section 3.

Appendix B

The zero-mode contribution to the amplitude in the RR sector for $p - p' = \nu = 0, 2, 4, 6$ and for $p \leq 6$ can be computed, following the regularization given in [34,37], to give

$$0R \langle B', \eta'|B, \eta\rangle_{0R} \equiv 0R \langle B_{sgh}, \eta'|B_{sgh}, \eta\rangle_{0R} 0R \langle B'_\psi, \eta'|B'_\psi, \eta\rangle_{0R},$$

$$= -\frac{2^4 \delta_{p'p}}{\sqrt{\det(\eta_{\nu} + \hat{F}) \det(\eta_{\nu'} + \hat{F}')}} \sum_{n=0}^{[\nu'+1]} \frac{[2(n + \frac{\nu}{2})]!}{2^{2n+\nu} n!(n + \frac{\nu}{2})!} \hat{F}_{[\alpha_1', \beta_1']\cdots \hat{F}_{[\alpha_n', \beta_n']} \hat{F}^{(p'+1)(p'+2)\cdots(p'+\nu+1)(p'+\nu)} \hat{F}_{[\alpha_1', \beta_1']\cdots \hat{F}_{[\alpha_n', \beta_n']}},$$

(240)

where the indices inside the square bracket denote their antisymmetrization. As indicated already in the previous sections, the above zero-mode matrix element for lower $p$ and $p'$ cases can be obtained from either $p = 6$ or $p = 5$ case depending on $p$ and $p'$ being even or odd if their worldvolume fluxes are extended in a specific way which we turn next. Let us take two explicit examples to demonstrate this. The first one is for $p = 5$ and $p' = 3$ and we will show that the corresponding zero-mode matrix element can be obtained from $p = p' = 5$ case if we extend the $p' = 3$ worldvolume flux $\hat{F}_{\alpha'\beta'}$ to the $p' = p = 5$
worldvolume flux $\hat{F}'_{\alpha\beta}$ the following way,

$$\hat{F}'_{\alpha\beta} = \begin{pmatrix} \hat{F}'_{\alpha'\beta'} & 0 & \hat{g}' \\ 0 & -\hat{g}' & 0 \end{pmatrix},$$

(241)

where we take the magnetic flux $\hat{g}' > 0$ to be infinite at the end of computations. From (240), we have for $p = 5$ and $p' = 3$

$$\delta_{\alpha'\beta'} \frac{2^4 \delta_{\eta'\eta'} + \sqrt{\text{det}(\eta_5 + \hat{F}) \text{det}(\eta_5 + \hat{F}')}}{\sqrt{\text{det}(\eta_5 + \hat{F}) \text{det}(\eta_5 + \hat{F}')}} \left( \hat{F}'_{45} + \frac{3}{2} \hat{F}'_{[\alpha'\beta']} \hat{F}'_{45} \hat{F}'_{\alpha'\beta'} \\ + \frac{15}{8} \hat{F}'_{[\alpha'_1\beta'_1} \hat{F}'_{\alpha'_2\beta'_2]} \hat{F}'_{45} \hat{F}'_{[\alpha'\beta'_3]} \hat{F}'_{\alpha'\beta'_2]} \right).$$

(242)

We now show that this can also be obtained from the $p = p' = 5$ case but with $\hat{F}'_{\alpha\beta}$ given by (241). We have now from (240)

$$\delta_{\alpha'\beta'} \frac{2^4 \delta_{\eta'\eta'} + \sqrt{\text{det}(\eta_5 + \hat{F}) \text{det}(\eta_5 + \hat{F}')}}{\sqrt{\text{det}(\eta_5 + \hat{F}) \text{det}(\eta_5 + \hat{F}')}} \left( \hat{F}'_{45} + \frac{1}{2} \hat{F}'_{\alpha\beta} \hat{F}'_{45} \\ + \left( \frac{1}{2^2 2!} \right)^2 4! \hat{F}'_{[\alpha_1\beta_1} \hat{F}'_{\alpha_2\beta_2]} \hat{F}'_{[\alpha_1\beta_1} \hat{F}'_{\alpha_2\beta_2]} \\ + \left( \frac{1}{2^3 3!} \right)^2 6! \hat{F}'_{[\alpha_1\beta_1} \hat{F}'_{\alpha_2\beta_2} \hat{F}'_{\alpha_3\beta_3]} \hat{F}'_{[\alpha_1\beta_1} \hat{F}'_{\alpha_2\beta_2} \hat{F}'_{\alpha_3\beta_3]} \right).$$

(243)

Note that $\text{det}(\eta_\alpha + \hat{F}'_{\alpha'}) = (1 + \hat{g}'^2) \text{det}(\eta_{\alpha'\beta'} + \hat{F}'_{\alpha'\beta'})$ which gives $\hat{g}'^2 \text{det}(\eta_{\alpha'\beta'} + \hat{F}'_{\alpha'\beta'})$ for $\hat{g}' \to \infty$. To have a finite contribution at $\hat{g}' \to \infty$, we need each term in the bracket to have a factor $\hat{F}'_{45} = \hat{g}'$. For this,

$$\hat{F}'_{\alpha\beta} \hat{F}'_{\alpha\beta} = 2 \hat{F}'_{45} + \cdots,$$

$$\hat{F}'_{[\alpha_1\beta_1} \hat{F}'_{\alpha_2\beta_2]} \hat{F}'_{[\alpha_1\beta_1} \hat{F}'_{\alpha_2\beta_2]} = 4 \hat{g}' \hat{F}'_{[\alpha_1\beta_1} \hat{F}'_{45] \hat{F}'_{\alpha_1\beta_1] + \cdots},}$$

$$\hat{F}'_{[\alpha_1\beta_1} \hat{F}'_{\alpha_2\beta_2} \hat{F}'_{\alpha_3\beta_3]} \hat{F}'_{[\alpha_1\beta_1} \hat{F}'_{\alpha_2\beta_2} \hat{F}'_{\alpha_3\beta_3]} = 6 \hat{g}' \hat{F}'_{[\alpha_1\beta_1} \hat{F}'_{\alpha_2\beta_2} \hat{F}'_{45] \hat{F}'_{\alpha_1\beta_1] + \cdots,}$$

(244)

where the $\cdots$ terms are independent of $\hat{g}'$. With these and taking $\hat{g}' \to \infty$, we can check easily that (243) is exactly the same as (242).

The second example is to take $p = p' = 4$ and we will show that the zero-mode matrix element can be obtained from the $p = p' = 6$ case by taking the worldvolume fluxes, respectively, as

$$\hat{F}'_{\alpha\beta} = \begin{pmatrix} \hat{F}'_{\alpha'\beta'} & 0 & \hat{g}' \\ 0 & -\hat{g}' & 0 \end{pmatrix}, \quad \hat{F}'_{\alpha\beta} = \begin{pmatrix} \hat{F}'_{\alpha'\beta'} & 0 & \hat{g}' \\ 0 & -\hat{g}' & 0 \end{pmatrix},$$

(245)
where $\alpha, \beta = 0, 1, \cdots, 6$ and $\alpha', \beta' = 0, 1, \cdots, 4$. Note that we need to take $\hat{g}, \hat{g}' \to \infty$ at the end of computations. For $p = p' = 4$, the zero-mode matrix element from from (240) is

\[
0_R \langle B', \eta'| B, \eta \rangle_{OR} = -\frac{2^4 \delta_{\eta\eta'}^{+}}{\sqrt{\det(\eta_4 + \hat{F}) \det(\eta_4 + \hat{F}')}} \left(1 + \frac{1}{2} \hat{F}_{\alpha\alpha'} \hat{F}_{\alpha'\beta'}^{\eta} + \frac{3}{8} \hat{F}[\alpha_1 \beta_1 \hat{F}[\alpha_2 \beta_2] \hat{F}[\alpha_3 \beta_3] \hat{F}[\alpha_4 \beta_4] \hat{F}[\alpha_5 \beta_5] \hat{F}[\alpha_6 \beta_6]} \right).
\]  

(246)

For $p = p' = 6$ with the respective worldvolume fluxes given in (245), we have the corresponding zero-mode matrix element from (240) as

\[
0_R \langle B', \eta'| B, \eta \rangle_{OR} = -\frac{2^4 \delta_{\eta\eta'}^{+}}{\sqrt{\det(\eta_6 + \hat{F}) \det(\eta_6 + \hat{F}')}} \left(1 + \frac{1}{2} \hat{F}_{\alpha\alpha'} \hat{F}_{\alpha'\beta'}^{\eta} + \frac{4!}{26} \hat{F}[\alpha_1 \beta_1 \hat{F}[\alpha_2 \beta_2] \hat{F}[\alpha_3 \beta_3] \hat{F}[\alpha_4 \beta_4] \hat{F}[\alpha_5 \beta_5] \hat{F}[\alpha_6 \beta_6]} \right).
\]  

(247)

We will show that the above is actually the same as that given in (246) for $\hat{g}, \hat{g}' \to \infty$. For this, note that

\[
\sqrt{\det(\eta_{\alpha\beta} + \hat{F}_{\alpha\beta}) \det(\eta_{\alpha\beta} + \hat{F}_{\alpha'\beta'})} = \hat{g} \hat{g}' \sqrt{\det(\eta_{\alpha\beta} + \hat{F}_{\alpha\beta}) \det(\eta_{\alpha'\beta'} + \hat{F}_{\alpha'\beta'})},
\]  

(248)

where we have used (245) and taken $\hat{g}, \hat{g}' \to \infty$. To have a finite limit, only those terms in the bracket proportional to $\hat{g} \hat{g}'$ in (247) survive. We have

\[
\hat{F}_{\alpha\beta} \hat{F}_{\alpha'\beta'}^{\eta} = 2 \hat{g} \hat{g}' + \cdots,
\]

\[
\hat{F}[\alpha_1 \beta_1 \hat{F}[\alpha_2 \beta_2] \hat{F}[\alpha_3 \beta_3] \hat{F}[\alpha_4 \beta_4] \hat{F}[\alpha_5 \beta_5] \hat{F}[\alpha_6 \beta_6]} = \frac{2^5}{4!} \hat{g} \hat{g}' \hat{F}_{\alpha\beta} \hat{F}_{\alpha'\beta'}^{\eta} + \cdots,
\]

\[
\hat{F}[\alpha_1 \beta_1 \hat{F}[\alpha_2 \beta_2] \hat{F}[\alpha_3 \beta_3] \hat{F}[\alpha_4 \beta_4] \hat{F}[\alpha_5 \beta_5] \hat{F}[\alpha_6 \beta_6]} = \frac{(3!)^2 4!}{6!} \hat{g} \hat{g}' \hat{F}[\alpha_1 \beta_1 \hat{F}[\alpha_2 \beta_2] \hat{F}[\alpha_3 \beta_3] \hat{F}[\alpha_4 \beta_4] \hat{F}[\alpha_5 \beta_5] \hat{F}[\alpha_6 \beta_6]} + \cdots,
\]  

(249)

where the $\cdots$ terms are independent of $\hat{g} \hat{g}'$. Plugging the above terms to (247) and taking $\hat{g}, \hat{g}' \to \infty$, we get exactly (246).

In summary, for various RR zero-mode contributions given in (240), they each can be obtained from the $p = p' = 6$ or the $p = p' = 5$ case by choosing the corresponding worldvolume fluxes in a way as indicated in the above examples. So we only need to focus on these two cases. They each can be obtained from (240) as

\[
0_R \langle B', \eta'| B, \eta \rangle_{OR} = -\frac{2^4 \delta_{\eta\eta'}^{+}}{\sqrt{\det(\eta_p + \hat{F}) \det(\eta_p + \hat{F}')}} \times \sum_{n=0}^{3} \frac{(2n)!}{2^{2n} (n!)^2} \hat{F}[\alpha_1 \beta_1 \cdots \hat{F}[\alpha_n \beta_n] \hat{F}[\alpha_1 \beta_1 \cdots \hat{F}[\alpha_n \beta_n]}.
\]  

(250)
where \( p = 5 \text{ or } 6 \). For either case, let us write the zero-mode contribution as

\[
0R \langle B', \eta' | B, \eta \rangle_{0R} = -\frac{2^4 \delta_{\eta \eta'}}{\sqrt{\det(\eta + \hat{F}) \det(\eta + \hat{F}')}} S
\]

(251)

where

\[
S = \sum_{n=0}^{3} \frac{(2n)!}{2^{2n} (n!)^2} \hat{F}[\alpha_1 \beta_1; \ldots; \hat{F}'\alpha_n \beta_n] \hat{F}'[\alpha_1 \beta_1; \ldots; \hat{F}'\alpha_n \beta_n].
\]

(252)

In what follows, we will show

\[
(0R \langle B', \eta' | B, \eta \rangle_{0R})^2 = 2^{7-p} \delta_{\eta \eta'} + \det(\mathbb{I} + w),
\]

(253)

where \( w \) is given in (24) and for convenience we rewrite it explicitly here

\[
w = (\mathbb{I} - \hat{F}) (\mathbb{I} + \hat{F})^{-1} (\mathbb{I} + \hat{F}') (\mathbb{I} - \hat{F}')^{-1},
\]

(254)

with \( \mathbb{I} \) the \((p+1) \times (p+1)\) unity matrix. Note that

\[
\det(\mathbb{I} + w) = \frac{\det[(\mathbb{I} + \hat{F}) (\mathbb{I} + w)(\mathbb{I} - \hat{F}')] \det(\mathbb{I} + \hat{F}')} {\det(\mathbb{I} + \hat{F}) \det(\mathbb{I} - \hat{F}')},
\]

(255)

With this, for (253) to hold, we need to show

\[
S^2 = \det(\mathbb{I} - \hat{F} \hat{F}').
\]

(256)

For this, we represent the \( S \) in terms of the following Grassmannian integration,

\[
S = (-)^{p+1} \int \prod_{\gamma=0}^{p} d\theta^\gamma d\theta'_\gamma (1 + \theta^\gamma \theta'_\gamma) e^{-\frac{1}{2} \hat{F}_{\alpha \beta} \theta^\alpha \theta'_\beta} e^{\frac{1}{2} \hat{F}'_{\alpha \beta} \theta'_\alpha \theta'_\beta},
\]

(257)

where \( \theta^\gamma \) and \( \theta'_\gamma \) are all real Grassmannian variables. With this, we have

\[
S^2 = \int \prod_{\gamma=0}^{p} d\theta^\gamma d\theta'_\gamma d\tilde{\theta}^\gamma d\tilde{\theta}'_\gamma (1 + \theta^\gamma \theta'_\gamma)(1 + \tilde{\theta}^\gamma \tilde{\theta}'_\gamma) e^{-\frac{1}{2} \hat{F}_{\alpha \beta} (\theta^\alpha \theta'_\beta + \tilde{\theta}^\gamma \tilde{\theta}'_\gamma) + \frac{1}{2} \hat{F}'_{\alpha \beta} (\theta'_\alpha \theta'_\beta + \tilde{\theta}'_\gamma \tilde{\theta}'_\gamma)},
\]

(258)

Now we change the integration variables as

\[
\theta^\gamma = \frac{\eta^\gamma + \eta'^{\gamma*}}{\sqrt{2}}, \quad \tilde{\theta}^\gamma = \frac{\eta^\gamma - \eta'^{\gamma*}}{i \sqrt{2}}, \quad \theta'_\gamma = \frac{\eta'_\gamma + \eta'^{\gamma*}}{\sqrt{2}}, \quad \tilde{\theta}'_\gamma = \frac{\eta'_\gamma - \eta'^{\gamma*}}{i \sqrt{2}}.
\]

(259)
where * denotes the complex conjugate. In terms of $\eta^\gamma, \eta^{*\gamma}, \eta^\prime_\gamma$ and $\eta^{*\prime}_\gamma$, we have

$$S^2 = \int \prod_{\gamma=0}^{p} d\eta^*_\gamma d\eta^\gamma d\eta^\prime_\gamma d\eta^{*\gamma}(1 + \eta^\gamma \eta^{*\gamma})(1 + \eta^{*\gamma} \eta^\prime_\gamma) e^{-\hat{F}_{\alpha\beta} \eta^\alpha \eta^{*\beta}} e^{-\hat{F}^{\alpha\beta} \eta^\prime_\alpha \eta^{*\prime}_\beta}.$$  \hspace{1cm} (260)

The evaluation of the integral can be simplified if we do the following integration first

$$I = \int \prod_{\gamma=0}^{p} d\eta^\prime_\gamma d\eta^* d\eta^\gamma d\eta^{*\gamma}(1 + \eta^{*\gamma} \eta^\prime_\gamma) e^{-\hat{F}_{\alpha\beta} \eta^\alpha \eta^{*\beta}} e^{-\hat{F}^{\alpha\beta} \eta^\prime_\alpha \eta^{*\prime}_\beta},$$

$$= e^{-(\hat{F}^{\alpha\beta} \eta^{*\alpha} \eta^{\prime*\beta}}.$$ \hspace{1cm} (261)

With this, we have

$$S^2 = \int \prod_{\gamma=0}^{p} d\eta^*_\gamma d\eta^\gamma(1 + \eta^{*\gamma} \eta^\prime_\gamma)I,$n

$$= \int \prod_{\gamma=0}^{p} d\eta^*_\gamma d\eta^\gamma e^{[-(\hat{F}^{\alpha\beta})_{\alpha} \eta^{*\alpha} \eta^{\prime*\beta}},$$

$$= \det(\mathbb{I} - \hat{F} \hat{F}').$$ \hspace{1cm} (262)

In other words, (253) holds indeed. In Appendix A, we have shown

$$w = V w_0 V^{-1}$$

for $w_0$ diagonal with eigenvalue 1 and others in pairs as $(\lambda_\alpha, \lambda_\alpha)$ with $\alpha = 0, 1, 2$ for $p = 6$ or with eigenvalues in pairs as $(\lambda_\alpha, \lambda_\alpha)$ with $\alpha = 0, 1, 2$ for $p = 5$. We then have from (253) for $p = 5$ or 6

$$(0_R \langle B', \eta'| B, \eta \rangle_{0_R})^2 = 2^{7-p} \delta_{\eta^\prime, +} \det(\mathbb{I} + w),$$

$$= 2^{7-p} \delta_{\eta^\prime, +} \det(\mathbb{I} + w_0),$$

$$= 2^{7-p} \delta_{\eta^\prime, +} (1 + \delta_{p,6}) \prod_{\alpha=0}^{2}(1 + \lambda_\alpha)(1 + \lambda^{-1}_\alpha),$$

$$= 2^8 \delta_{\eta^\prime, +} \prod_{\alpha=0}^{2} \cos^2 \pi \nu_\alpha,$$ \hspace{1cm} (263)

where we have used $\lambda_\alpha = e^{2i\pi \nu_\alpha}$. This gives the expected result

$$0_R \langle B', \eta'| B, \eta \rangle_{0_R} = -2^4 \delta_{\eta^\prime, +} \prod_{\alpha=0}^{2} \cos \pi \nu_\alpha,$$ \hspace{1cm} (264)

where we have used the known result $0_R \langle B', \eta'| B, \eta \rangle_{0_R} = -2^4 \delta_{\eta^\prime, +}$ in the absence of fluxes for which $\nu_\alpha = 0$. Given what has been discussed, for a general $p \leq 6$, we have

$$0_R \langle B', \eta'| B, \eta \rangle_{0_R} = -2^4 \delta_{\eta^\prime, +} \prod_{\alpha=0}^{[p-1]} \cos \pi \nu_\alpha.$$ \hspace{1cm} (265)
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