Power law tails and non Markovian dynamics in open quantum systems: An exact solution from Keldysh field theory

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The Born-Markov approximation is widely used to study dynamics of open quantum systems coupled to external baths. Using Keldysh formalism, we show that the dynamics of a system of bosons (fermions) linearly coupled to non-interacting bosonic (fermionic) bath falls outside this paradigm if the bath spectral function has non-analyticities as a function of frequency. In this case, we show that the dissipative and noise kernels governing the dynamics have distinct power law tails. The Green’s functions show a short time “quasi” Markovian exponential decay before crossing over to a power law tail governed by the non-analyticity of the spectral function. We study a system of bosons (fermions) hopping on a one dimensional lattice, where each site is coupled linearly to an independent bath of non-interacting bosons (fermions). We obtain exact expressions for the Green’s functions of this system which show power law decay $\sim |t - t'|^{-3/2}$. We use these to calculate density and current profile, as well as unequal time current-current correlators. While the density and current profiles show interesting quantitative deviations from Markovian results, the current-current correlators show qualitatively distinct long time power law tails $\sim |t - t'|^{-3}$, characteristic of non-Markovian dynamics. We show that the power law decays survive in presence of inter-particle interaction in the system, but the cross-over time scale is shifted to larger values with increasing interaction strength.

I. INTRODUCTION

The dynamics of open quantum systems (OQS)\textsuperscript{1} is the key to some fundamental questions in statistical physics, including issues of emergence of irreversibility and generation of entropy\textsuperscript{2}, generation of quantum entanglement\textsuperscript{3} and approach to thermal equilibrium\textsuperscript{4}. It is also crucial to understand the dynamics of a quantum system coupled to an external bath to design and control possible platforms for creating an architecture of quantum computing, like superconducting qubits\textsuperscript{5}, spin-qubits\textsuperscript{6}, cavity QED\textsuperscript{7}, cavity optomechanics\textsuperscript{8}, quantum dot arrays coupled to cavities\textsuperscript{9}, nanowire junctions\textsuperscript{10}, ultra-cold atomic systems\textsuperscript{11,12} etc. The phenomenal experimental advance over the past decade in implementing and controlling these platforms has reigned the interest in dynamics of OQS. Additionally, theoretical ideas of bath/dissipation engineering\textsuperscript{12} to guide open quantum systems to novel steady states\textsuperscript{13} and using these states as resources in quantum computing\textsuperscript{14} also require a deep understanding of behaviour of OQS.

A widely used paradigm to analyze the dynamics of OQS is the Born-Markov approximation, which assumes that (i) coupling of the quantum system to the bath does not change the dynamics of the bath and (ii) the effective reduced dynamics of the system is local in time. This is often presented in the form of a time local (Markovian) quantum master equation (QME)\textsuperscript{15}, where the positive rates of transition from one configuration of the system to another depends only on the state of the system at that time. Further approximations lead to well known formulations like the Redfield\textsuperscript{16} and Lindblad\textsuperscript{17} master equations. The other related approach is that of stochastic Schrodinger equations\textsuperscript{18,19}, which are generalizations of familiar Langevin equations in classical systems. In systems with short range memory kernels, the Markovian approximation emerges as a coarse grained description of OQS dynamics over a scale $t_{\text{course}}$ in the limit $\tau_s \gg t_{\text{course}} \gg \tau_b$. Here $\tau_b$ and $\tau_s$ are the autocorrelation times in the bath and the system.

Non-Markovian dynamics of OQS has been gaining prominence in recent years\textsuperscript{20}. A broad class of systems, like BEC in trapped ultracold atoms\textsuperscript{21}, quantum dots coupled to superfluid reservoirs\textsuperscript{22}, nanomechanical oscillators coupled to BEC\textsuperscript{23}, atoms/impurities coupled to radiation field in photonic crystals\textsuperscript{24} have displayed non-Markovian dynamics in various forms. Non-Markovian dynamics has been key to recent proposals for bath engineering\textsuperscript{25}, quantum metrology\textsuperscript{26}, and can be used as resources for quantum communication\textsuperscript{27} and quantum memory\textsuperscript{28}. Recent experiments have been successful in tuning the dynamics of an open quantum system from Markovian to non-Markovian by controlling the bath degrees of freedom\textsuperscript{29}. Non-Markovian dynamics\textsuperscript{30} has been traditionally treated using Nakajima-Zwanzig type master equation\textsuperscript{11,31} as well as effective time convolution-less master equation\textsuperscript{32}. In fact, different types of dynamical behaviour or properties of equation of motion is clubbed under the rubric of non-Markovian dynamics with debates over essential definition of “non-Markovian”ness\textsuperscript{33}.

A canonical model of OQS\textsuperscript{35}, consisting of a few bosons (fermions) coupled linearly to non-interacting bath of bosons (fermions), has been studied previously using QME\textsuperscript{36,37} as well as non-equilibrium Greens’ function technique\textsuperscript{38,39}. In this paper, we use Schwinger-Keldysh field theory formalism\textsuperscript{40} to study a system bosons(fermions) hoping on a 1d lattice where each site is connected to a non-interacting bosonic (fermionic) bath kept at fixed temperature and chemical potential. Integrating out the bath degrees of freedom, we obtain a description of the effective steady state dynamics of the OQS. We relate the self-energies induced by the bath to the dissipative and noise kernels in a stochastic Schrodinger equation through a saddle point approximation. We show that any non-analyticity in the bath spectral function leads to temporally long range dissipative and noise kernels with power law tails. The exponent of the power law is determined solely by the nature of the non-analyticity and is independent of other microscopic details. Such non-analytic behaviour can arise
from a variety of sources like band edges, Van-Hove singularities, Kohn anomalies\textsuperscript{41} in phonon spectrum, phase transitions in the bath\textsuperscript{42} and non-Fermi liquid\textsuperscript{43} nature of the bath. The power law tail in the kernels preclude the possibility of coarse graining to obtain a Markovian description for the dynamics of the system. We emphasize that, for bosonic baths kept at fixed chemical potentials, the spectrum must be bounded from below and its spectral function must be non-analytic at the bottom of the band. Hence non-Markovian dynamics will be ubiquitous in such bosonic systems.

The Green’s functions inherit these power law tails along with a short time exponential decay. The crossover from the exponential decay to the power law tail occurs at a time $\tau_0 \sim 1/\epsilon^2$, where $\epsilon$ is the scale of system bath coupling. Thus at very weak system bath coupling the system dynamics appears Markovian for a very long time, and we call this regime “quasi-Markovian”. With increase in system bath coupling, the power law tail dominates the dynamics and the non-Markovian behavior is easier to detect in experiments. We have studied both the density and current profile in the system, as well as the unequal time observables like current-current correlators. While the equal time correlators show important quantitative differences from a Markovian calculation, the unequal time correlators inherit the power law tails and can be used as a direct probe of these singularities.

The features described above are robust, i.e. they only require a non-analytic bath spectral function and a linear system-bath coupling. To illustrate these features, we consider a 1d lattice system of bosonic (fermionic) particles with a nearest hopping, where each site is connected to an independent bosonic (fermionic) bath with its own temperature and chemical potential. We consider a bath whose spectral function has square root derivative singularity at the band edges. After integrating out the bath, we solve this problem analytically and obtain closed form expressions for Greens’ function and the observables mentioned above. We show that the noise and dissipative kernel show a $|t - t'|^{-\frac{3}{2}}$ decay at long times. The Greens’ function and current current correlators have a short time exponential decay followed by a power law tail, where the Greens’ functions $\sim |t - t'|^{-\frac{3}{2}}$ and current-current correlators $\sim |t - t'|^{-3}$. We consider the effect of the inter-particle interactions in the system on the dynamics of OQS within mean field theory. We find that, the power law tail survives, while the crossover scale increases with increasing interaction strength.

We have organized the paper into several sections: In section II, we discuss a model of non-interacting bosons linearly coupled to a bosonic bath. We set up the Keldysh formalism and sketch the steps taken to calculate the Green’s function and observables in the resulting OQS. In section III, we connect the self energies induced by integrating out the bath variables with the dissipative and noise kernels in an equation of motion approach and discuss the power law kernels induced by the non-analyticities in the bath spectral function. In section IV, we solve the Dyson equation and find the analytic Green’s function for the linear chain. We discuss the non-analytic structure of the Green’s function in frequency space, and its behaviour in real time. In section V, we consider equal time and unequal time observables and show how the unequal time correlators show the signature of the power law tails. In section VI, we generalize the results to the case of a fermionic system coupled to a fermionic bath, and show that similar results hold in this case. Finally, in section VII, we consider the effect of inter-particle interactions in the system on the power law tail and the crossover time-scale.

II. KELDYSH THEORY FOR BOSONIC SYSTEM

We consider a system of 1-dimensional lattice of non-interacting bosons (representing arrays of oscillators) hopping between nearest neighbours with an amplitude $g$ (representing coupling between successive oscillators). The fermionic version of this model, which can represent quantum dot arrays, will be taken up in section VI. Each site, $l$, of the lattice is coupled to an independent bosonic bath with a temperature $T_l$ and a chemical potential $\mu_l$. The setup is schematically shown in Fig. 1. We will be interested in the steady state response of the system to different profiles of $\mu_l$ and $T_l$.

We describe each bath by a semi-infinite 1 dimensional lattice of non-interacting bosons with nearest neighbour hopping $t_B$ and assume each system site is coupled locally to the first site of the corresponding bath, as shown in Fig. 1. The system bath coupling , controlled by the scale $\epsilon$, is linear in both the system and bath degrees of freedom. The total Hamiltonian of the system ($H_s$), the baths ($H_b$) and system bath coupling ($H_{sb}$) are then given by,

$$H_s = -g \sum_{l=1}^{N} a_{l}^\dagger a_{l+1} + h.c \quad \text{and} \quad H_{sb} = \epsilon \sum_{l=1}^{N} a_{l}^\dagger b_{l}^{(l)} + h.c$$

$$H_b = -t_B \sum_{l=1}^{N} \sum_{s=1}^{\infty} b_{s}^{(l)} b_{s+1}^{(l)} + h.c. \quad (1)$$
The imaginary part of the noise kernel (Keldysh self energy) $\Sigma^K_{\gamma}(t - t')$ of a site $l$ coupled to a bath with $T_B = 0.625 t_B$ and $\mu_B = -2.25 t_B$ as a function of $|t - t'|$ on a log-log plot. The system bath coupling in this case is $\epsilon = 0.3 t_B$. Note that the absolute value of the self energy is plotted in this case. The solid line is the analytic answer for the envelop of the leading power law $\sim |t - t'|^{-3/2}$ decay of the kernel. We use $t_B = 2$ to set units of $t - t'$ and $\Sigma^K(t - t')$.

where $a_{l}^\dagger$ creates a boson at site $l$ of the system, $b_{s}^{(l)}$ is the bosonic creation operator at site $s$ of $l^{th}$ bath. It is useful to rewrite the bath degrees of freedom in terms of the eigenoperators $B_{\alpha}^{(l)}$, which diagonalize the bath Hamiltonian:

$$H_B = \sum_{l,\alpha} \Omega_\alpha B^\dagger_{\alpha}(l) B_{\alpha}(l) , \quad H_{sb} = \epsilon \sum_{l,\alpha} \kappa_\alpha B^\dagger_{\alpha}(l) a_l + h.c (2)$$

where $\Omega_\alpha$ is the energy of the eigenmode $\alpha$ and $\kappa_\alpha$ is its amplitude on the first site of the bath. We use Schwinger Keldysh functional integral formalism $^{26}$ to study the effective non-unitary dynamics of the QoS in the steady state. The field theoretic technique yields exact results for arbitrary parameter values in non-interacting system, and is not restricted to weak system bath couplings. The Keldysh approach constructs a path integral representation of the dynamics of the density matrix. It requires two copies of fields at each instant of time $t$, namely $\phi_+(t)$ and $\phi_-(t)$, corresponding to the forward and backward evolution inherent in $\rho(t) = U^\dagger(t, -\infty) \rho(-\infty) U(t, -\infty)$. It is customary to work with the symmetric or classical $\phi_{cl} = (\phi_+ + \phi_-)/\sqrt{2}$ and anti-symmetric or quantum $\phi_{q} = (\phi_+ - \phi_-)/\sqrt{2}$ fields. Using $\phi^{(l)}$ and $\psi^{(l)}$ as the fields corresponding to $a_l$ and $B_{\alpha}^{(l)}$, the Keldysh action for the system $S_s$, baths $S_B$, and the system bath couplings $S_{sb}$ is given by

$$S_s = \sum_{l,\nu} \int d\omega \phi_{l}^{\dagger}(\omega) \left[ G_{0}^{-1R}(l, l', \omega) \delta_{l,l'} \right] \phi_l(\omega)$$

$$S_{sb} = -\epsilon \sum_{l,\alpha} \int d\omega \kappa_\alpha \psi_{l\alpha}^\dagger(\omega) \phi_l(\omega) + h.c \quad (3)$$

where $\phi_{l}^{(l)} = [\phi_{cl}^{(l)}, \phi_{q}^{(l)}]$, $\psi_{l\alpha}^{(l)} = [\psi_{cl\alpha}^{(l)}, \psi_{q\alpha}^{(l)}]$ and $G_{0}^{-1R}(l, l', \omega) = (\omega \pm i\eta)\delta_{l,l'} + g \delta_{l,l' \pm 1}$ and $\eta \rightarrow 0^+$.

Since the action (3) is quadratic in $\psi_{l\alpha}^{(l)}$, we can integrate out the bath degrees of freedom to obtain the effective dissipative action for the open quantum system,

$$S_{oqs} = \sum_{l,l'} \int d\omega \hat{\Sigma}_l(\omega) \left[ G_{0}^{-1R}(l, l', \omega) - \Sigma^R(\omega) \delta_{l,l'} \right] \phi_l(\omega), \quad (5)$$

where $\hat{\Sigma}_l(\omega) = \sum_{\alpha} |\kappa_\alpha|^2 e^2 \delta_{1,2} G_{B}(\alpha, l; \omega) \delta_{1}$. The summation over the bath eigenmodes makes the self-energy matrix a function only of the bath spectral function

$$J(\omega) = 2\pi \sum_{\alpha} |\kappa_\alpha|^2 \delta(\omega - \Omega_\alpha), \quad (6)$$

and the distribution function $F_l(\omega)$. All the results of this paper will depend on properties of $J(\omega)$ and not on other microscopic details of the baths. In that sense, we can simply think of different kinds of baths being represented by different $J(\omega)$, and the results obtained here are applicable to a large class of systems. We discuss the cases of some other physically relevant $J(\omega)$ in Appendix A. For our particular case, the bath spectral function $^{37}$,

$$J(\omega) = \Theta(4t_B^2 - \omega^2) \frac{2}{t_B} \sqrt{1 - \frac{\omega^2}{4t_B^2}}, \quad (7)$$

has a square root derivative singularity at the band edges $\omega = \pm 2t_B$. Writing the components of self energy matrix explicitly, we have

$$\Sigma^R(\omega) = -e^2 \int \frac{d\omega'}{2\pi} \frac{J(\omega')}{\omega' - \omega - i\eta}$$

$$\Sigma^K_l(\omega) = -ie^2 J(\omega) \coth \left[ \frac{\omega - \mu_l}{2T_l} \right] \quad (8)$$
Inverting the kernel in the action (5) we obtain exact expressions for the 1-particle Green’s function of the bosons. Using this, we calculate equal time observables (e.g., current, occupation number) as well as unequal time observables (e.g., current current correlation functions) in the steady state. Green’s functions and current-current correlators in the steady state show power law tails indicative of non-Markovian dynamics of the system. In the next section, we work in real time domain and connect the Keldysh formalism with the widely used stochastic Schrodinger equation\( ^{10} \) formalism.

### III. POWER LAW TAIL IN DISSIPATIVE AND NOISE KERNELS

The Keldysh field theory description obtained in the previous section is equivalent to a stochastic Schrodinger equation with non-local memory kernels for dissipation and noise in the system given by the self-energies \( \Sigma^R \) and \( \Sigma^K \). To see this, we write the effective Keldysh action for the QOS in real time,

\[
S = \sum_{l,l'} \int dt dt' \phi_l^\dagger(t) \left[ -\frac{\partial}{\partial t} G_{l,l'}^{-1} R(t, t') \right] \phi_{l'}(t')
\]

with \( G_{l,l'}^{-1} R(t, t') = \delta(t - t') (i\partial_t \delta_{l,l'} + g_0 \delta_{l,l'\pm1}) - \Sigma^R(t - t') \delta_{l,l'} \). The terms quadratic in \( \phi_q \) are first converted into terms in linear in \( \phi_q \) (source terms) by a Hubbard Stratonovich transformation with an auxiliary field \( \zeta(t) \).

The Keldysh partition function is then given by \( Z = \int D[\zeta^\dagger] F(\zeta^\dagger) \int D[\phi^\dagger] e^{18(\phi^\dagger \phi + \zeta^\dagger \zeta)} \) where

\[
S = \int dt \int dt' \sum_{l,l'} [\phi_{q_l}^\dagger(t) G_{l,l'}^{-1} R(t, t') \phi_{q_{l'}}(t') + h.c]
\]

\[
+ \int dt \sum_{l} \left[ \phi_{q_l}^\dagger(t) \zeta_l(t) + \zeta_l^\dagger(t) \phi_{q_l}(t) \right] \text{and}
\]

\[
F(\zeta^\dagger) = e^{-\int dt \int dt' \zeta_l(t) [\Sigma^R_l(t, t')]^{-1} \zeta_l(t')}
\]

The equation of motion, obtained from classical saddle point condition \( \partial S \middle/ \partial \phi \bigg|_{\phi=0} = 0 \), is

\[
i\partial_t \phi_{q_l}(t) - \int dt' \Sigma^R_{l,l'}(t, t') \phi_{q_{l'}}(t') + g_0 \phi_{q_l}^{\dagger\pm1}(t) = \zeta_l(t),
\]

where \( \zeta_l(t) \) is a complex random field with a non-local but Gaussian distribution, \( \langle \zeta_l^\dagger(t) \zeta_l(t') \rangle = -i\delta_{l,l'} \Sigma^K_l(t, t') \). The imaginary part of the retarded self energy is thus the dissipative kernel, while the Keldysh self energy is the noise correlation kernel. The self energies in real time are related to the bath spectral function by

\[
\Sigma^R(t - t') = -i e^2 \Theta(t - t') \int \frac{d\omega}{2\pi} J(\omega) e^{i\omega(t-t')}
\]

\[
\Sigma^K(t - t') = -i e^2 \int \frac{d\omega}{2\pi} J(\omega) \coth \left( \frac{\omega - \mu}{2T_B} \right) e^{i\omega(t-t')}
\]

We note that for any \( J(\omega) \) which is even in \( \omega \), the retarded self energy is purely imaginary and hence dissipative in nature. For the bath spectral function that we consider (eqn. 7),

\[
\Sigma^R(t, t') = -i \Theta(t-t') \frac{e^2}{l_B} \frac{J_1(2t_B|t-t'|)}{|t-t'|},
\]

where \( J_1(x) \) is the Bessel function of first order. It is clear from the above exact expression, that \( \Sigma^R(t - t') \sim -i e^2 (2t_B - t' - t)^{3/2} \cos(2t_B|t-t'|) \) for \( |t-t'| \rightarrow \infty \) i.e the dissipative kernel decays slowly as a power law in the long time limit. Hence, there is no time scale in the system over which one can coarse grain to obtain an effective local description. The dynamics is thus essentially non-Markovian. Although a closed form expression for \( \Sigma^K(t - t') \) cannot be obtained, it can be shown (see Appendix A) that at long times the envelop of \( \Sigma^K(t - t') \sim -e^{2(2\pi)^{1/2}} |t_B(t - t'|-3/2) \coth [(2t_B - \mu)/2T_B] \) where the \((-+\) sign corresponds to its real (imaginary) part. Hence it has the same \( \sim |t-t'|^{-3/2} \) tail as \( \Sigma^R \). In Fig. 2, we plot the imaginary part of the Keldysh self energy of a site coupled to a bath with temperature \( T_B = 0.625 \) and chemical potential \( \mu/T_B = -2.25 \) as a function of \( |t-t'| \) on a log-log plot. The analytic envelop of \( \Sigma^K \) is plotted in the same figure with a solid line. It matches with the numerically obtained results remarkably well. Here, the chemical potential is kept below the band bottom to avoid the pathological case of non-interacting BEC in the bath. We note that in the limit \( \mu \rightarrow -2t_B \), i.e a BEC transition is approached, the \( \Sigma^K(\omega) \) has a square root divergence at \( \omega = \pm 2t_B \) and the power law tail will change to \( |t-t'|^{-1/2} \).

The origin of the power law tail can be traced back to the fact that the bath spectral function \( J(\omega) \) is non-analytic at \( \omega = \pm 2t_B \). This is similar to Friedel oscillations\(^{48}\) or RKKY interaction for impurities in a Fermi gas, where non-analyticity of the polarization function leads to power law decays in space. The relation between power exponents and nature of non-analyticity of \( J(\omega) \) is explored in details in Appendix A for several varieties of non-analyticities. This analysis shows that the decay exponent is independent of all other microscopic details of system and the baths. Power laws tails in dissipative and noise kernels is thus a generic phenomenon for any non-analytic bath spectral function.

For bosons with number conservation (i.e where one can consider chemical potential for baths), the spectrum must be bounded from below and at least one non-analytic point is guaranteed for the bath spectral function. Non-Markovian dynamics is thus ubiquitous for systems linearly coupled to such bosonic baths. Non analytic \( J(\omega) \) can occur due to a variety of reasons: e.g. Van-Hove singularities (both bosons and fermions), Kohn anomalies (phonons)\(^{41}\), non Fermi liquids (fermions)\(^{43}\), critical fluctuations near a quantum phase transitions (bosonic modes)\(^{42}\) etc. The non-Markovian dynamics can then be used to detect the presence and nature of these non-analyticities. If the bath spectral function has sharp but non-singular features, the kernel will be an approximate power law for a large intermediate time period, before show-
This can be inverted. Greens’ functions show an initial exponential decay, followed by a power law tail at long time with a power law due to the non-analytic nature of the system-bath coupling. Even in this case, there is an additional non-analyticity in $G^R(\omega)$ at $\omega = \pm 2\Omega \tau$, inherited from the non-analytic nature of $\Sigma^R(\omega)$. A careful analysis of Appendix A shows that the nature of the leading non-analyticity of $G^R(\omega)$ is same as that of $\Sigma^R(\omega)$. This leads to a power law tail at long time with $G^R(t-t') \sim (t-t')^{-3/2}$. The Keldysh Green’s function also inherits the same power law tail in long time, as can be easily seen from eqn.[16]. In Fig. 3 (a) and (b), we plot respectively Green’s function $G^R_{12}(t-t')$ and the real part of the Keldysh Green’s function $G^R_{12}(t-t')$ for the two site model as a function of time in a log-log plot. The plots are obtained for a system with $\epsilon = 0.3t_B$ and $g = 0.5t_B$ connected to two independent baths of common temperature $T = 0.625t_B$ and chemical potential $\mu_1 = -2.5t_B$ and $\mu_2 = -5t_B$. At short times, the Greens’ functions are dominated by the exponential decay from the poles, while the long time behaviour is governed by the power law due to the non-analytic $J(\omega)$. The power law is clearly visible after the exponential part has decayed, i.e. beyond a time scale $\tau_0 \sim \gamma^{-1} \sim \frac{1}{\Omega}$, which is marked in the figures with an arrow. $\tau_0$ is the time at which the system is in the non-Markovian regime for $t > \tau_0$. However, at strong system-bath coupling, $\tau_0$ is very short and the dynamics is mostly governed by the non-Markovian power law decay in the memory kernels. Thus,

$$M_t = \frac{\sinh[(i+1)\lambda]}{\sinh[\lambda]} \text{ with } \cosh[\lambda] = D/2. \quad (15)$$

The Keldysh Greens’ function is then given by,

$$G^K_{i,j}(\omega) = -ie^2J(\omega) \sum_{l=1}^N G^R_{i,l}(\omega) \coth \left( \frac{\omega - \mu_l}{2T_l} \right) G^R_{l,j}(\omega). \quad (16)$$

To obtain a clear insight about the structure of the Greens’ function, we first consider a simplified toy model consisting of 2-sites connected to two different baths $^{37}$. In this two site model,

$$G^R_{\alpha,\alpha}(\omega) = \frac{D}{g(D^2 - 1)} \quad \text{and} \quad G^R_{\alpha,\beta}(\omega) = -\frac{1}{g(D^2 - 1)}. \quad (17)$$

For the bath spectral function given in eqn.(7), $\Sigma^R(\omega) = e^{\omega/2\Delta_B^2} - i \left( e^{\omega/2\Delta_B^2} - \omega + i\mu \right) e^{\omega/2\Delta_B^2}$, where $\Delta_B^2 = \frac{g^2}{4} - \frac{e^2}{2t_B}$. For $g^2/4 + e^2 > t_B^2$, the pole is on the real axis (outside the bandwidth), leading to an oscillation in the long time limit, similar to the behaviour found by Nori et. al in Ref. 31. In this paper, we will focus on the regime where the presence of the leads to damping in the system. Even in this case, there is an additional non-analyticity in $G^R(\omega)$ at $\omega = \pm 2\tau \Omega$, inherited from the non-analytic nature of $\Sigma^R(\omega)$. A careful analysis (see Appendix A) shows that the nature of the leading non-analyticity of $G^R(\omega)$ is same as that of $\Sigma^R(\omega)$. This leads to a power law tail at long time with $G^R(t-t') \sim (t-t')^{-3/2}$. The Keldysh Green’s function also inherits the same power law tail in long time, as can be easily seen from eqn.[16]. In Fig. 3 (a) and (b), we plot respectively Green’s function $G^R_{1,2}(t-t')$ and the real part of the Keldysh Green’s function $G^R_{1,2}(t-t')$ for the two site model as a function of time in a log-log plot. The plots are obtained for a system with $\epsilon = 0.3t_B$ and $g = 0.5t_B$ connected to two independent baths of common temperature $T = 0.625t_B$ and chemical potential $\mu_1 = -2.5t_B$ and $\mu_2 = -5t_B$. At short times, the Greens’ functions are dominated by the exponential decay from the poles, while the long time behaviour is governed by the power law due to the non-analytic $J(\omega)$. The power law is clearly visible after the exponential part has decayed, i.e. beyond a time scale $\tau_0 \sim \gamma^{-1} \sim \frac{1}{\Omega}$, which is marked in the figures with an arrow. $\tau_0$ is the time at which the system is in the non-Markovian regime for $t > \tau_0$. However, at strong system-bath coupling, $\tau_0$ is very short and the dynamics is mostly governed by the non-Markovian power law decay in the memory kernels. Thus,
it would be easier to detect observable consequences of non-Markovian dynamics at large $\epsilon$. $\tau_0$ has a very weak dependence on temperature and is essentially set by the system-bath coupling. We emphasis that, in spite of the presence of a scale $\tau_0$ in the system dynamics, we can not coarse-grain over this scale and obtain a Markovian description, as we are then left with the power law decay of the Greens' function.

The characteristic features shown by the two-site system is carried over to the solution for the $N$ site system given in eqn.[14]. In this case, the poles are given by sinh$[[N + 1]x] = 0$, where the location of the poles satisfies $z_1 = 1 - \epsilon^2/2\sqrt{2}t_B + i\epsilon^2/t_B\sqrt{1 - \epsilon^2/4t_B^2} = 2g\cos[m\pi/(N + 1)]$, with $m = 1, 2, ..., N$. The pole has a positive imaginary part $\gamma \sim \epsilon^2/t_B$ for $g^2\cos^2[\pi/(N + 1)] + \epsilon^2 < t_B^2$. With this modification, a similar structure of an exponential decay followed by a power law tail is also obtained in this case. We consider a $N = 50$ site chain with $\epsilon = 0.3t_B$ and $g = 0.5t_B$, connected to independent baths with common temperature $T = 0.625t_B$ and a chemical potential profile which varies linearly with the site number of the system going from $\mu_1 = -2.5t_B$ and $\mu_{50} = -5t_B$. In Fig. 3(c) and (d), we respectively plot the retarded Greens' function $G^{\alpha\beta}(t - t')$ and the real part of the Keldysh Green’s function $G^{\alpha\beta}(t - t')$ of the system as a function of time in a log-log plot. We have used $i = 16$ and $j = 18$ to avoid the boundary of the chain. This Green’s functions also show a quasi-Markovian exponential decay followed by a non-Markovian $|t - t'|^{-\frac{5}{2}}$ decay, similar to those found in the two site system.

In the next section, we will construct experimentally accessible quantities which can clearly distinguish between the quasi-Markovian and non-Markovian dynamics and discuss the observable consequences of long range memory kernels.

V. OBSERVABLES IN STEADY STATE

In this section, we focus on experimental observables which can detect non-Markovian behaviour in these open quantum systems. We will divide these observables into two classes: (i) equal time observables like occupation numbers and currents and (ii) unequal time current-current correlators. While the first class of observables show interesting quantitative deviations from Markovian answers, especially in the limit of large system bath couplings, the unequal time correlators show qualitatively different behaviour indicating the presence of non-Markovian power law tails.

A. Equal Time Correlators

We will study two types of equal time correlators: (a) occupation number of sites or eigenmodes and (b) current through the system. We first consider the two site problem. In absence of any coupling to the bath, the system Hamiltonian can be diagonalized to obtain two states at $E = \pm g$ with mode operator $A^\pm = (1/\sqrt{2})(a_1 \pm a_2)$. The first observable we focus on is the occupation number of these modes, $n^\pm$. The change in occupation number of the modes due to coupling to baths, $\delta n^\pm = \frac{1}{4} \sum_\alpha \int \frac{d\omega}{2\pi} [G^{\alpha\alpha}_G(\omega) \pm G^{\alpha\alpha}_K(\omega)]$. (19)

We first consider the case where a system with $g = 0.5t_B$ is coupled to two baths kept at the same temperature and chemical potential. In Fig. 4(a) we plot the occupation of the ground state, $n^+$, in solid (red and green) lines as a function of the system bath coupling $\epsilon$ for two different bath temperatures, $T = 0.5t_B$ and $T = 1.25t_B$. The common chemical potential of the baths is $\mu = -2.25t_B$. We have also plotted the thermal equilibrium distribution, $n^+_eq = (e^{-g\epsilon - 1} - 1)^{-1}$, in dotted lines. The exact steady state distribution approaches the thermal distribution at small $\epsilon$, but starts deviating as $\epsilon$ grows. This deviation increases with the temperature of the bath. As $\epsilon$ increases, the eigenenergies of the system are dressed by the bath to $\pm \tilde{g} = \pm g/[1 - \epsilon^2/(2t_B^2)]$. In Fig. 4(a) we have also plotted the thermal distribution corresponding to the effective spectrum $e^{-\tilde{g}^2 - 1}$ in dashed lines. It is clear that the steady state distribution is different from the equilibrium distribution even with the dressed spectrum showing that the system does not thermalize at large $\epsilon$.

We now consider the steady state where the two baths coupled to the two sites are kept at same $T$, but different $\mu$. In this case, a finite current flows through the system and it is simpler to analyze the system in the site basis. The change in the local occupation numbers, $\delta n_1$ and $\delta n_2$, is plotted with $\epsilon$ in Fig. 4(b) for $T = t_B$ and $g = 0.25t_B$, and $\mu_1 = -2.5t_B$ and $\mu_2 = -5.0t_B$. As $\epsilon$ increases, the influx of particles to site 1 increases, but the probability of particles tunneling from site 1 to 2 saturates when $\epsilon \gtrsim g$. The steady state number density on site 1 then needs to increase to match the outflux with the influx, as seen in Fig. 4(b). The density at site 2 simultaneously shows a decrease with $\epsilon$ in the same regime, since the influx to site 2 has saturated. The number density at site 2 then needs to decrease to to maintain a steady current. Finally, the current in the link between sites $l$ and $l + 1$ is given by, $I_l = g \int \frac{d\omega}{2\pi} Re[G^{\alpha\alpha}_l(\omega)]$. (20)

The current on the link between site 1 and 2 is plotted as a function of $\epsilon$ in Fig. 4(c) for two different values of $g = 0.5t_B$ and $0.25t_B$. The current matches with the Born-Markov answer obtained through the solution of Redfield equations in the small $\epsilon$ limit and deviates from the master equation result for $\epsilon \gtrsim g$. At small $\epsilon$, the current in the system is constrained by exchange of particles between baths and sites of the system. Any particle that has reached site 1 can be thought to be delocalized to site 2 on a time scale $g^{-1}$ with $g^{-1} < < \epsilon^{-1}$ in the Markovian limit. For $\epsilon \gtrsim g$, this is no longer true and steady state current is constrained by the hopping rate $g$ and becomes almost independent of $\epsilon$ for $\epsilon \gg g$. This leads to the saturating trends seen in Fig. 4(c). The increase of $n_1$ and decrease of $n_2$ in Fig. 4(b) is a consequence of this bottleneck between sites 1 and 2.

We now consider a linear chain of $N$ sites where each site...
is connected to an independent bath of temperature $T_l$ and chemical potential $\mu_l$. We focus on the situation where the baths have a fixed temperature $T_l = T$ and a chemical potential linearly varying with space i.e $\mu_l = \mu_1 + \nu(l - 1)$. Specifically we look at a $N = 250$ site system with $g = 0.5t_B$, $\epsilon = 0.25t_B$, $\mu_1 = -2.25t_B$ and $\mu_{250} = -5.0t_B$. The density profile in the system shows an exponential decay with site number on top of a constant value, as seen in a semi-log plot in Fig. 4(d), where the local density is measured with respect to the density at the $N^{th}$ site. We plot the density profile for three different temperatures, $T = 0.33t_B, 0.5t_B, 1t_B$. We find that the decay length scale increases with temperature. The current through the link between sites $l$ and $l + 1$, $I_l$, is plotted as function of $l$ in Fig. 4(e). The current initially increases with distance from the boundary, then settles into an exponential decay over a large range of sites, as seen in the linear graph in the semi-log plot. We plot the current profile for three different temperatures, $T = 0.17t_B, 0.25t_B, 0.5t_B$. Once again we find a decay length scale increasing with temperature. In Fig. 4(f), we plot the decay length from the current profile, $\xi$, as a function of temperature and show that $\xi$ is proportional to $T$. In the extreme classical limit, when $T \rightarrow \infty$, the system shows a constant current independent of the link number. The exponential decay and the variation of the decay length with temperature can be understood from the exact Green’s function for the linear chain, as shown in Appendix B. We have checked that a similar exponential decay is also seen when the temperature of the baths vary linearly in space while the chemical potential is kept fixed.

**B. Unequal Time Correlators**

The density and current profiles in the open quantum system show important quantitative traits as a function of the system bath coupling. However they do not provide a smoking-gun signature of the underlying non-Markovian dynamics. This is provided by the unequal time density-density or current-current correlator, which shows a long time power law decay with an exponent that is twice the exponent of the power law tail in $\Sigma$ and $G$. Here we compute the unequal time current-current correlator $G_{ij}$, which is given by,
\[ C_{kl}(t-t') = \langle I_k(t)I_l(t') \rangle = g^2[G_{t+1,k}(t',t) G_{t+1,l}(t',t') + G_{t,k+1}(t',t) G_{t+1,l}(t',t') - (l \leftrightarrow l+1)] \] (21)

FIG. 5. Unequal time current-current correlator \( C_{kl}(t-t') \) is plotted as a function of \(|t-t'|\) for (a) \( N=2 \) site model and (b) for a \( N=50 \) site chain with \( k=16,18 \). Here each site is connected to a bath. The baths have common temperature \( T = 0.625 t_B \) and linearly varying \( \mu \) where \( \mu_1 = -2.5 t_B, \mu_N = -5.0 t_B, \) and \( g = 0.5 t_B \). In (a) the blue dots correspond to \( \epsilon = 0.1 t_B \) while the orange line corresponds to \( \epsilon = 0.9 t_B \). In (b) the blue dots correspond to \( \epsilon = 0.4 t_B \) while the orange line corresponds to \( \epsilon = 0.9 t_B \). It shows a short time exponential decay followed by a non-Markovian power law tail \( \sim |t-t'|^{-3} \) in the long time limit. The power law tail appears at shorter times as \( \epsilon \) increases.

where \( G^{>}(<) = (G^K \pm (G^R - G^A))/2 \). We first focus on the \( 2 \) site system. In Fig 5(a), we plot \( C_{1,1}(t-t') \), normalized by \( I_1^2 \), as a function of \( t-t' \) on a log-log plot. We plot the correlator for two values of \( \epsilon \): a weak system bath coupling of \( \epsilon = 0.1 t_B \) (blue circles) and a strong system-bath coupling of \( \epsilon = 0.9 t_B \) (orange line). The graphs are obtained for a system with \( g = 0.5 t_B \), coupled to two baths with common \( T_1 = 0.625 t_B \) and chemical potentials \( \mu_1 = -2.5 t_B \) and \( \mu_2 = -5.0 t_B \). In the weak coupling limit (\( \epsilon = 0.1 t_B \)), we see that \( C_{1,1}(t-t') \) decreases exponentially in time before crossing over to a power law \( C_{1,1}(t-t') \sim |t-t'|^{-3} \) at a very large time \( \sim t_B/\epsilon^2 \). However, by this time the correlator has decayed by orders of magnitude and the power law tail may be hard to observe experimentally. Thus, although the behaviour of the system is not Markovian in the long time limit, experiments may see an effectively Markovian dynamics. However, at large \( \epsilon \sim 0.9 t_B \), the initial exponential dynamics is extremely short lived and the power law tail in the auto correlation function should be clearly visible in the experiments. This observable can hence be used to identify non-Markovian dynamics in the system and detect the nature of non-analyticity in the corresponding bath spectral function. Further, one can tune the system dynamics from “quasi” Markovian to non-Markovian by tuning the system-bath coupling in these OQS.

We now consider the current correlations in a linear chain of \( N = 50 \) sites coupled to independent baths of fixed temperature and linearly varying chemical potential. We ignore the boundary regions (where the current is not exponentially decaying) and focus on the middle of the chain. In Fig. 5(b), we plot the current-current correlator \( C_{ij}(t-t') \) as a function of \( t-t' \) by fixing \( i = 16 \) and \( j = 18 \) for a system with \( g = 0.5 t_B \) coupled to baths with \( \mu_1 = -2.5 t_B \), \( \mu_50 = -5.0 t_B \) and \( T = 0.625 t_B \). The current current correlator function should be clearly visible in the experiments. This regime should be easily observed in experiments.

FIG. 6. The current current correlator \( C_{1,1}(t-t') \) (normalized by \( I_1^2 \)) is plotted for a two site fermionic system with \( g = 0.5 t_B \), coupled linearly to two fermionic baths of same temperature \( T = 0.625 t_B \) and chemical potentials \( \mu_1 = -2.5 t_B \) and \( \mu_2 = -5.0 t_B \). It shows a short time exponential decay followed by a non-Markovian power law tail \( \sim |t-t'|^{-3} \) in the long time limit. Power law kernel appears at shorter times as \( \epsilon \) increases.

VI. FERMIONIC SYSTEM COUPLED TO FERMIONIC BATHS

The formalism developed in the previous sections for treating a bosonic open quantum system with non-Markovian dynamics can be easily modified to treat a system of fermions interacting with fermionic baths. The key features, which make it exactly solvable, are non-interacting Hamiltonians for the system and bath and a linear system-bath coupling. As long as these conditions are met, minor changes in the formalism allow us to treat the fermionic system with similar power law tails obtained for self-energies, Greens’ function and unequal time correlators.

For fermions, the Keldysh field theory is set up in terms of doubled Grassmann fields \( \chi_{\pm} \) and their conjugates \( \chi^\dagger_{\pm} \). The only difference from the bosonic theory is that it is more convenient to work in terms of \( \chi_{1(2)} = [\chi_{+} \pm \chi_{-}] / \sqrt{2} \) and \( \chi^\dagger_{1(2)} = [\chi^\dagger_{+} \pm \chi^\dagger_{-}] / \sqrt{2} \), which take care of anti-commutation relations between fermionic fields in a natural way. We consider a fermionic chain, where each site is coupled to an independent fermionic bath with its own temperature and chemical potential. The Hamiltonian of the system is given by equations (1,2), with \( a_i^\dagger \) and \( B_i^\dagger \) now representing fermionic creation operators. The Keldysh action for the system is given...
by,
\[ S_s = \sum_{l,l'} \int d\omega \chi_l^+(\omega) \begin{bmatrix} G_0^{-1R}(l,l',\omega) & G_0^{-1K}(\omega) \delta_{l,l'} \\ 0 & G_0^{-1A}(l,l',\omega) \end{bmatrix} \chi_{l'}(\omega) \]

\[ S_b = \sum_{l,\alpha} \int d\omega \xi^{(l)}_\alpha(\omega) \begin{bmatrix} \omega - \Omega_\alpha + i\eta & 2i\eta F_l(\Omega_\alpha) \\ 0 & \omega - \Omega_\alpha - i\eta \end{bmatrix} \xi^{(l)}(\omega) \]

\[ S_{sb} = -\epsilon \sum_{l,\alpha} \int d\omega \kappa_\alpha \xi^{(l)}_\alpha(\omega) \chi_l(\omega) + h.c. \] (22)

where \( \Sigma^R(\omega) \) is given by equation [8], just as in case of bosons, while the Keldysh self energy \( \Sigma^K(\omega) \) is given by,

\[ \Sigma^K(\omega) = -ie^2 J(\omega) \tanh \left( \frac{\omega - \mu_l}{2T_l} \right) \] (24)

The Keldysh self energy function for the fermionic case can thus be obtained from the bosonic case by replacing \( \cosh \left( \frac{\omega - \mu_l}{2T_l} \right) \) by \( \tanh \left( \frac{\omega - \mu_l}{2T_l} \right) \), as one would expect due to different statistics of particles.

With this replacement, all the expression for the Greens’ functions and observables, obtained in section III, IV and V for bosonic system, can be used for the fermionic system as well. We note that, unlike the bosonic case, there is no reason to place the bath chemical potentials below the band bottom for the fermionic baths. The self-energies in real time show similar power law behaviour \( \Sigma^R(t - t') \sim \Sigma^K_1(t - t') \sim |t - t'|^{-\frac{3}{2}} \) for large time, with the power law originating from the non-analyticities in the bath spectral function given by equation [7]. However, it is harder in this case to give an interpretation to self-energies, since fermions do not have a classical limit and it is impossible to talk about a “classical saddle point” for non-interacting fermionic action. We note that the mere presence of the singularity in the bath spectral function is enough to generate the power law tail, the singularity does not need to be either close to the spectral gap of the system or to the chemical potential of the bath. The Greens’ functions then inherit this power law along with an exponential decay, which also gets reflected in the long time behaviour of density-density auto-correlation function and current-current correlation function, similar to the bosonic system considered earlier. Working with the 2-site fermionic system, we plot \( C_{1,1}(t - t') = \langle \hat{n}_1(t) \hat{n}_1(t') \rangle \), normalized by \( I_2^2 \), as a function of \( t - t' \) on a log-log plot in Fig. 6 for a system with \( g = 0.5t_B \) coupled to baths with \( \mu_1 = -2.5t_B \) and \( \mu_2 = -5.0t_B \) and \( T = 0.625t_B \). The current-current correlator clearly shows an exponential decay followed by a power law tail \( \sim |t - t'|^{-3} \), as seen in the bosonic system. Once again the power law dominates the dynamics and leads to large values of the correlators for long \( t - t' \) at strong system bath coupling.

We have obtained exact solutions for Green’s functions of 1D non-interacting bosonic and fermionic baths coupled to independent baths and demonstrated non-Markovian behaviour in both systems. An obvious question is how the interaction among system degrees of freedom, which is inadvertently present in any realistic system, affects the description we have obtained here. In the next section, we try to answer this question by focusing on the bosonic system and making mean field approximation to the underlying interaction terms.

**VII. EFFECT OF INTERACTION**

We consider the case of an interacting chain of bosons coupled linearly to independent non-interacting bosonic baths. In addition to the hopping \( g \), the system bosons are interacting with each other with a local repulsion \( U \) and nearest neighbour repulsion \( V \), i.e. we add to the Hamiltonian given by equation [1], a term

\[ H_{int} = U \sum_i \hat{n}_i(\hat{n}_i - 1) + V \sum_i \hat{n}_i \hat{n}_{i+1}, \] (25)

where \( \hat{n}_i = a_i^\dagger a_i \) is the density at site \( i \). The bath degrees of freedom can be integrated out as before to obtain the quadratic dissipative action \( S_{obs} \) [ eqn. 5]. This action and its associated Green’s functions can then be used as a free theory around which we consider the effects of interactions. The Keldysh action corresponding to the inter-particle interaction is given by\(^\text{48}\).
$$S_{int} = -\frac{U}{2} \sum_i \int dt \phi_{el}^*(t) \phi_{cl}^*(t) \phi_{cl}^*(t) \phi_{el}^*(t) - \frac{V}{2} \sum_i \int dt \phi_{el}^*(t) \phi_{cl}^*(t) \phi_{cl}^*(t) \phi_{el}^*(t) + h.c + c_l \leftrightarrow q. $$ (26)

We consider the effect of this interaction on the Greens’ functions within mean field theory. This is equivalent to considering the one-loop corrections to the self-energy, shown in Fig. 7 (a) and (b) for the Keldysh and retarded self-energies. Note that the Keldysh self energy correction vanishes because $G^R(t, t) + G^A(t, t) = 0$ 40, while the retarded self-energy due to the interaction is purely real. This is related to the fact that the one loop terms do not lead to redistribution of energy among the modes. The corrections to self-energy, $\Sigma^R$ induced by the interaction term are given by, $\Sigma^R_{\alpha} = n_{\alpha} + (V/2)n_{\alpha \pm 1, \alpha \pm 1} - U$, and $\Sigma^R_{\alpha \pm 1} = (V/2)(n_{\alpha \pm 1, \alpha \pm 1})$, where $n_{\alpha \pm 1} = \int (d\omega / 2\pi) G^K_{\alpha \pm 1}(\omega)$. The retarded self-energy due to inter-particle interaction change the retarded Greens’ functions obtained from $(1 - G^{R} - \Sigma^R_{\alpha \pm 1})^{-1}$ where $G^{-1}R$ already includes self-energies induced by the bath. These retarded Greens’ functions in turn change $G^K$ through $G^K = G^K\Sigma^K G^K$ where $\Sigma^K$ is the Keldysh self-energy induced by the bath (the interaction contribution to it is zero, as shown above). The Keldysh Greens’ function used in calculating the retarded self-energy shown in Fig. 7 (b) is this renormalized $G^K$, obtained self-consistently. Hence, this completes a self-consistency loop. The mean field approximation is then equivalent to resummation of particular set of loop diagrams. From an equation of motion perspective, this is also equivalent to solving stochastic Gross-Pitaevskii equation 49.

A key question we want to answer is how does the interaction affect the non-Markovian dynamics? We have seen earlier that the Greens’ functions and unequal time observables for non-interacting systems show a pattern of exponential decay at short times, which crosses over to power law tail with exponent fixed by the nature of the non-analyticity in the bath spectral function. We can then ask if the power law tail survives in the interacting system, and, if it survives, whether the crossover time scale increases or decreases with the interaction strength.

In order to gain analytic insight into the problem, we once again consider the case of a chain with 2-sites. In this case, $\Sigma^R_{11} = n_{11} + (V/2)n_{22} - \frac{U}{2}$, $\Sigma^R_{12} = n_{22} + (V/2)n_{11} - \frac{U}{2}$, and $\Sigma^R_{12} = (V/2)n_{12}$. The retarded Greens’ function is then given by

$$G^K(\omega) = \frac{1}{Q(\omega)} \left[ \omega - \Sigma^R - \Sigma^R_{12} \right] \left[ -g + \Sigma^R_{12} \right]$$

where $Q(\omega) = \left[ \omega - \Sigma^R - \Sigma^R_{12} \right] \left[ \omega - \Sigma^R - \Sigma^R_{12} \right] - \Sigma^R_{12} - g^2$. Considering the denominator of the equation [27], we find that the poles of the Greens’ function are shifted from $z_0$ in the non-interacting case to $z_\pm$, where $z_\pm$ is obtained from $z_0$ (given in eqn.[18]) by replacing $g$ with $c_\pm = a \pm \sqrt{b^2 + |g|^2}$.

Here $a(b) = (\Sigma^R_{11} \pm \Sigma^R_{12})/2$, and $g = -g + \Sigma^R_{12}$. The interaction induced self-energy splits the single pole of the non-interacting case into two. Using the fact that $a = (U + V/2)(n_1 + n_2) + V/2$, where $n_1(2)$ are the densities at site 1(2), one can show that $a > 0$, and hence $c_+ > g$, for small $\Sigma_{12}$. We have checked that $\Sigma^R_{12}$ is not large enough to change this argument for a wide range of $U$ and $V$ (see Appendix C for details). Comparing $z_+$ and $z_0$, we then see that $z_+$ has a smaller imaginary part compared to $z_0$. This leads to an exponential decay slower than the non-interacting case.

The power law tail survives (in fact it can be shown to survive to all orders in perturbation theory), but the crossover scale is further pushed out because of the slower exponential decay. In Fig. 7(c), we plot the real part of the Keldysh Greens’ function $G^K(t, t')$ for a two site bosonic system coupled to external bath in the mean field approximation. The correction to Keldysh self energy vanishes. The loop propagator in (b) is the self consistent Keldysh Green’s function. (c) The real part of the local Keldysh Green’s function $G^K_{\alpha \beta}(t, t')$ for a two site bosonic system coupled to two baths plotted as a function of $t - t'$ in a semi-log plot. In both cases the bath temperature $T = 0.625t_B$ and chemical potentials $\mu_1 = -2.5t_B$ and $\mu_2 = -5.0t_B$. The systems have a hopping $g = 0.5t_B$ and system-bath coupling $\epsilon = 0.2t_B$. The orange line is for a non-interacting system, while the blue circles are for an interacting system with $U = 0.75t_B$ and $V = 0.6t_B$. The interaction increases the crossover scale, but does not eliminate the power law tail.
The fact that the exponential decay is slowed down in presence of the interaction seems odd, given our expectation of faster decay due to scattering events. This is due to fact that the mean field theory misses processes leading to redistribution of energy, which contribute at two loops or higher order. This is the same reason a prethermalized regime is found in weakly interacting bosonic/fermionic systems under a quantum quench. Thus, although the mean field theory is non-perturbative in interaction strength, we expect that at larger interaction strengths, these additional processes may increase the rate of exponential decay of the Greens’ function, thus pulling back the timescale for crossover from “quasi” Markovian to non-Markovian dynamics in the system.

VIII. CONCLUSIONS

We have used a Schwinger Keldysh field theory to describe the dynamics of a bosonic (fermionic) system coupled linearly to a non-interacting bosonic (fermionic) bath. Integrating out the bath degrees of freedom we obtain the effective dissipative action for the open quantum system. We identify the retarded and Keldysh self energies induced by the bath with the dissipative and noise kernel in a non-local stochastic Schrödinger equation for the open quantum system. We understand this analytically in terms of the structure of the Green’s functions in the system. Increasing interaction strengths, these additional processes may increase the rate of exponential decay of the Greens’ functions, thus pulling back the timescale for crossover from “quasi” Markovian to non-Markovian dynamics in the system.

We finally consider the effect of interactions on the above picture. The power law tails survive to all orders in perturbation theory in the interaction strength. Within a mean field theory for the bosonic two site system, which is equivalent to solving a stochastic Gross Pitaevski equations, we find that the exponential decay slows down and the crossover timescale for observing the power law tails is pushed to larger values. This needs to be further investigated, although our prediction of a larger crossover timescale should hold in the weakly interacting interacting system.

We focus our attention on two classes of observables in the open quantum system: equal time correlators like density and current, and unequal time correlators like current-current correlation functions. We first consider a two site system coupled to two baths of same temperature and chemical potential and show that the steady state mode occupation numbers deviate from the thermal equilibrium answers with increasing strength of system bath couplings which cannot be explained by a simple dressing of the system spectrum. For the same system, if the two baths are kept at different chemical potentials, as the system-bath coupling becomes larger than the hopping within the system, both local densities and current through the system deviates from the answers provided by the quantum master equation approach. The densities show a pileup on one site compared to the other, while the current saturates for \( \epsilon > g \).

We then consider the 1-d chain, with sites coupled to independent baths at the same temperature, but with a chemical potential profile. For a linear variation of \( \mu \) with space, we find an exponential decay of the current with distance. The density also shows an exponential variation in space on the top of a constant value. The length scale obtained from the decay of the current increases with temperature, with a spatially independent current obtained in the limit of infinite temperature. This behaviour of the current is understood analytically from the exact Green’s function obtained for the chain.

We next focus our attention on the unequal time current-current correlators and show that they follow a pattern similar to the Green’s functions: a short time exponential decay, followed by a power law tail \( (t - t')^{-3} \). This occurs both in the two site system and in the linear chain, and is common to both bosonic and fermionic systems. The timescale for the crossover from exponential to power law decay decreases while the value of the correlator in the power law regime increases with system-bath coupling. These power law tails can then be used to detect non-analyticities in the bath spectral functions, and hence possible phase transitions in the baths.

We then consider the effect of interactions on the above picture. The power law tails survive to all orders in perturbation theory in the interaction strength. Within a mean field theory for the bosonic two site system, which is equivalent to solving a stochastic Gross Pitaevski equations, we find that the exponential decay slows down and the crossover timescale for observing the power law tails is pushed to larger values. We understand this analytically in terms of the structure of the Green’s functions in the system. Increasing interaction strength should, in principle lead to a larger scattering rate and hence to faster decays, but such two loop processes redistributing energy is not captured within the mean field theory. This needs to be further investigated, although our prediction of a larger crossover timescale should hold in the weakly interacting interacting system.
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Appendix A: Power Law tails and non analytic bath Spectral Functions

In section (III) and (IV) of the main text, we have claimed that the non-analyticity of the bath spectral function leads to power law tails in self energies and Greens’ functions using Keldysh formalism. We have also claimed that the exponent of the power law depends solely on the nature of the non-analyticity and is independent of its location. In the main text we had shown this with an exact expression for retarded self-energy for the spectral function of semi-infinite bath having
\[ J(\omega) = \Theta(4t_B^2 - \omega^2) \frac{1}{2} \left( 1 - \frac{\omega^2}{4t_B^2} \right)^{-1/2}. \]
In this appendix, we make the connection between non-analyticities of \( J(\omega) \) and power law tails clearer by (a) calculating exact expressions and asymptotic power law tails for retarded self energies with different types of non-analyticities and (b) showing the relation between the non-analyticity and the exponent of the power law for \( \Sigma^K(t-t') \) and \( G^{R/K}(t-t') \) for the semi-infinite bath.

We will first focus on the calculation of retarded self-energy for different \( J(\omega) \) where,
\[ \Sigma^R(t-t') = -ie^2 \Theta(t-t') \int \frac{d\omega}{2\pi} J(\omega) e^{i\omega(t-t')} \] (A1)

In addition to the \( J(\omega) \) considered in the main text, we also consider (i) \( J(\omega) = \Theta(4t_B^2 - \omega^2) \frac{1}{2} \left( 1 - \frac{\omega^2}{4t_B^2} \right)^{-\frac{1}{2}} \), which is the bath spectral function for the infinite 1d lattice where the system couples to a generic site in the middle of the bath. In this case \( J(\omega) \) diverges as \( \omega^{-\frac{1}{2}} \) for \( \omega = \pm 2t_B(1-\delta) \) where \( \delta \rightarrow 0^+ \). Hence \( \Sigma^K(t-t') \) decays as \( |t-t'|^{-\frac{1}{2}} \) for large \( |t-t'| \),

(ii). \( J(\omega) = \Theta(\omega) \frac{\omega^2}{4\pi} \) exp\((-\frac{\omega^2}{4}) \) which has been considered in Ref[31] as a prototype of ohmic,sub-ohmic,supercritical baths for \( x = 1, <1, >1 \) respectively. Here \( J(\omega) \) is non-analytic at \( \omega \rightarrow 0 \) and \( \Sigma^K(t-t') \) falls off as \( |t-t'|^{-(x+1)} \) for long \( |t-t'| \), (iii) \( J(\omega) = K_0(1-\omega^2)\Theta(1-\omega^2) \) where \( K_0 \) is an elliptical function of first kind. This spectral function occurs for a 2d lattice and has logarithmic Van-Hove singularity for \( \omega \rightarrow 0 \). In this case we find that \( \Sigma^K(t-t') \sim |t-t'|^{-1} \) for \( |t-t'| \rightarrow \infty \). These results are obtained from asymptotic expansion of the exact expressions of Fourier transform of various \( J(\omega) \) mentioned above. In Table (I), we tabulated these results, where we have also displayed the exact results from which asymptotic power laws are calculated.

We now shift our attention to calculate the asymptotic power law of \( \Sigma^K(t-t') \) and \( G^{R/K}(t-t') \), where the Fourier transform cannot be obtained in a closed form. To do this, we will first focus on non-analyticity in \( \Sigma^K(\omega) \) for \( J(\omega) = \Theta(4t_B^2 - \omega^2) \frac{1}{2} \left( 1 - \frac{\omega^2}{4t_B^2} \right)^{-\frac{1}{2}} \) and show how this is connected to power law exponent. This connection will help us in understanding the nature of power law for any non-analytic spectral function, if we know its leading order non-analytic piece. We will then use this argument to show that \( \Sigma^K(\omega), G^{R/K}(\omega), G^{K}(\omega) \) all have the same leading singularity and hence the same power law tail appears in their long time profile. The Keldysh self energy is given by,
\[ \Sigma^K(t-t') = -ie^2 \int \frac{d\omega}{2\pi} J(\omega) \frac{\omega - \mu_l}{2T_l} e^{-i\omega(t-t')} \] (A2)

We note that the chemical potential, \( \mu_l < -2t_B \) and hence \( \coth\left(\frac{\omega-\mu_l}{2T_l}\right) \) is a smooth function in the integration range and \( J(\omega) \) and hence \( \Sigma^K(\omega) \) is non-analytic only at \( \omega = \pm 2t_B \). Expanding \( \omega = \pm 2t_B(1-\delta) \) we get \( J(\omega) \sim \delta^\tau \) for \( \delta \rightarrow 0^+ \) and hence the integrand \( I(\omega) = \frac{1}{2\pi} J(\omega) \coth\left(\frac{\omega-\mu_l}{2T_l}\right) \) can be expanded as \( I(\pm 2t_B(1-\delta)) = \sum_{n=0} C_n^+ \delta^n + \frac{1}{2} \) where \( C_n^+ = \frac{\sqrt{\pi}}{2\pi i} \coth\left(\frac{\pm 2t_B - \mu_l}{2T_l}\right) \). Considering the combination of the non-analytic piece, we get,

\[ \Sigma^K(t-t') = -ie^2 \frac{2t_B}{\tau^2} e^{-i\tau} \int_0^\tau dz \sum_{n=0} C_n^+ \frac{\tau^n}{\tau^n} z^{n+\frac{1}{2}} e^{i\frac{1}{2}} + e^{i\tau} \int_0^\tau dz \sum_{n=0} C_n^- \frac{z^{n+\frac{1}{2}} e^{-i\frac{1}{2}}}{\tau^n} \]

(A3)

where \( \tau = 2t_B \) \( (t-t') \), \( z = \delta \tau \) and the subscript \( l \) is dropped here for notational convenience. These integrations can be performed in the form of incomplete Gamma function

\[ \Gamma(\alpha, z), \]

\[ \int_0^\tau dz z^{n+\frac{1}{2}} e^{i\frac{1}{2}} = (-i)^{n-\frac{3}{2}} \left[ \Gamma(n+\frac{3}{2}, -i\tau) - \Gamma(n+\frac{3}{2}, 0) \right] \] (A4)
Having shown the connection between the non-analyticity and power law tails, we now show that for the 2-site model, \( G_{1,2}^R \) also has a \(|t - t'|^{-\frac{3}{2}}\) tail. Similar considerations will apply for other \( G_{i,j}^R \) and \( G_{i,j}^K \), both in the 2 site model and in the chain. Writing \( D = D' + iD'' \) the retarded Greens’ function \( G_{i,j}^R \) is given by,

\[
Re \left[ G_{i,j}^R \right] = -\frac{1}{g} \frac{D^2 - D''^2 - 1}{(D^2 - D''^2 - 1)^2 + 4D'^2D''^2} \quad \text{and} \\
Im \left[ G_{i,j}^R \right] = -\frac{1}{g} \frac{-2D'D''}{(D^2 - D''^2 - 1)^2 + 4D'^2D''^2} \quad \text{A7}
\]

### Table I. Leading power law decay in dissipative kernel (\( \Sigma^R(t - t') \)) at large \( t - t' \) for different non-analytic bath spectral functions \( J(\omega) \).

| Physical model                  | Form of \( J(\omega) \) | \( \Sigma^R(t - t') \) | Asymptotic limit |
|---------------------------------|--------------------------|-------------------------|------------------|
| Ohmic, Sub and Super Ohmic bath | \( \Theta(\omega) \frac{\omega^2}{\omega^2 + \Gamma}\) \( e^{\exp(-\frac{\omega}{\omega_c})} \) | \( \frac{\Gamma(1+x)}{[1 - \omega_c(t-t')]^{x+1}} \) | \(|t - t'|^{-x-1}\) |
| Infinite 1-d bath              | \( \Theta(4t_B^2 - \omega^2) \frac{2}{t_B} \frac{1}{\sqrt{1 - \frac{\omega^2}{4t_B^2}}} \) | \( J_0(2t_B|t - t'|) \) | \(|t - t'|^{-\frac{1}{2}}\) |
| Semi-infinite 1-d bath         | \( \Theta(4t_B^2 - \omega^2) \frac{2}{t_B} \sqrt{1 - \frac{\omega^2}{4t_B^2}} \) | \( J_1(2t_B|t - t'|) \) | \(|t - t'|^{-\frac{3}{2}}\) |
| 2-d square lattice bath       | \( \frac{1}{w_c} \log \left( \frac{|w-w_c|}{w_c} \right) \) | \( -e^{i\omega_c(t-t') \frac{\text{sign}(t)}{t-t'}} \) | \(|t - t'|^{-1}\) |

which has its asymptotic form,

\[
\Gamma(n + \frac{3}{2}, -i\tau) = e^{i\tau}(-i\tau)^{n+\frac{3}{2}} \left[ 1 + \frac{i}{\tau} \left( n + \frac{1}{2} \right) + O\left( \frac{1}{\tau^2} \right) \right]
\]

Using this and substituting the values of \( C_{0,1}^\pm \), we obtain \( \Sigma^K(t - t') \) which is given by equation (A5). It is thus clear that the leading order power law tail in \( \Sigma^K(t - t') \) goes as \(|t - t'|^{-\frac{3}{2}}\).

\[
\Sigma^K(t - t') \sim -te^{2} t_B \frac{\pi}{|2t_B(t-t')|^3} \left( -i \right)^{-\frac{3}{2}} e^{-i2t_B(t-t')} C_{0}^+ + (1) -i e^{-i2t_B(t-t')} C_{0}^+ + O\left( \frac{1}{\tau^2} \right)
\]

\[
\sim -i e^{2} \frac{2}{\pi |2t_B(t-t')|^3} \left( e^{-i[2t_B(t-t') - \frac{3\pi}{4}]} \coth \left( \frac{2t_B - \mu}{2T} \right) + e^{i[2t_B(t-t') - \frac{3\pi}{4}]} \coth \left( \frac{-2t_B - \mu}{2T} \right) \right) + O\left( \frac{1}{\tau^2} \right) \quad \text{A5}
\]

The leading order answer for \( Im[\Sigma^K(t - t')] \) is then,

\[
Im[\Sigma^K(t - t')] \sim -e^{2} \frac{2}{\pi |2t_B(t-t')|^3} \left( \coth \left( \frac{2t_B - \mu}{2T} \right) + \coth \left( \frac{-2t_B - \mu}{2T} \right) \right) \cos \left( 2t_B(t-t') - \frac{3\pi}{4} \right) \quad \text{A6}
\]

The expression, without the oscillation, is plotted in Fig. [1] as a solid line and matches with the numerically obtained plot very well.
where

$$D' = \frac{1}{g} \left[ w \left( 1 - \frac{\epsilon^2}{2t_B} \right) + \frac{\epsilon^2}{t_B} \text{sign}(\omega) \Theta(|\omega| - 2t_B) \sqrt{\frac{\omega^2}{4t_B^2} - 1} \right]$$

$$D'' = \frac{1}{g} \frac{\epsilon^2}{t_B} \Theta(4t_B^2 - \omega^2) \sqrt{1 - \frac{\omega^2}{4t_B^2}}$$

(A8)

We note that $D'' \sim \delta^\frac{1}{2}$ for $\omega = \pm 2t_B(1 - \delta)$ while $D' \sim (1 + C_0 \delta^\frac{1}{2})$ for $\omega = \pm 2t_B(1 + \delta)$ and one needs to consider all these non-analytic pieces to calculate the power law tail. For Fourier transform of $Im [G^{R}_{1,2}]$, the integration goes from $-2t_B < \omega < 2t_B$ and since $D'$ is smooth in this region, only non-analyticity of $D''$ leads to power law. One can then follow an argument similar to that of $\Sigma^R(t - t')$ and show that leading $\delta^\frac{1}{2}$ non-analyticity implies a power law $\sim |t - t'|^{-\frac{3}{2}}$. For $Re [G^{R}_{1,2}]$, the argument is more complicated. The integral is broken into three ranges, (a) from $-\infty$ to $-2t_B$, (b) from $2t_B$ to $\infty$ and (c) from $-2t_B$ to $2t_B$. Note that in the regions (a) and (b) the integrand can be expanded near $\omega = \pm 2t_B(1 + \delta)$ as $\sum_{n=0}^{\infty} C_n^\pm \delta^\frac{n}{2}$ coming from $D''$ while the same in the region (c) goes as $\sum_{n=0}^{\infty} C_n^\pm \delta^\frac{n}{2}$ near $\omega = \pm 2t_B(1 - \delta)$. Then all the three integral in the ranges (a), (b) and (c) have $O(\tau^{-\frac{3}{2}})$ coefficients which contribute to obtain a finite $O\left( \tau^{-\frac{3}{2}} \right)$ coefficient. Thus Fourier transform of $Re [G^{R}_{1,2}]$ also scales as $|t - t'|^{-\frac{3}{2}}$ for $|t - t'| \rightarrow \infty$. We note that similar arguments can also be applied in the analysis of $G^K$ to show its $|t - t'|^{-\frac{3}{2}}$ decay in the long time limit.

Appendix B: Exponential decay of current in a linear chain

In section (IV) of the main text, we have derived the retarded and the Keldysh Greens’ function for a linear chain, where each site is coupled to an independent bath with its own temperature and chemical potential. In section (V), we found that when the baths have a common temperature but chemical potential of the bath varies linearly with lattice site no., the current in the links of the system show an exponential decay in space, away from the edges of the chain. In this appendix we use the analytic forms of the Greens’ function in section (IV) and consider their low temperature form to explain the exponential decay of current. We will also show that the decay length is proportional to the temperature of the baths. To illustrate this, we use eqns. (14) and (16) of the main text to write,

$$G^{K}_{l,l+1} = i \frac{\epsilon^2}{g^2} \frac{J(w)}{|M_0|^2} \left( M_{N-l} M_{N-l-1} \sum_{j=1}^{l} |M_{j-1}|^2 \text{Coth} \left[ \frac{w - \mu_j}{2T} \right] + M_{l-1} M_{l}^* \sum_{j=l+1}^{N} |M_{N-j}|^2 \text{Coth} \left[ \frac{w - \mu_j}{2T} \right] \right)$$

(B1)

where $M_l$ is now written in a form different from eqn (15) of the main text to facilitate the derivation. We now write,

$$M_l = \frac{y^{l+1} - |y|^{l+1}}{2 \sqrt{D^2 - 1}} \text{ with } y = \frac{D}{2} + \sqrt{\frac{D^2}{4} - 1}$$

(B2)

where $D$ is given by eqn (A8) of appendix A. One can easily check that this definition of $M_l$ is equivalent to that in equation (15) of the main text.

$$\sum_{j=1}^{l} |M_{j-1}|^2 \text{Coth} \left[ \frac{w - \mu_j}{2T} \right] = 2 e^{-\left( \omega - \mu_1 + \nu \right)/T} \sum_{j=1}^{l} \left[ r^{2j} + r^{-2j} - 2 \cos(2j \theta) \right] e^{\nu j/T}$$

$$\sum_{j=1}^{l} \left[ M_{N-j} \right]^2 \text{Coth} \left[ \frac{w - \mu_j}{2T} \right] = 2 e^{-\left( \omega - \mu_1 + \nu \right)/T} \sum_{j=1}^{l} \left[ r^{2(N-j+1)} + r^{-2(N-j+1)} - 2 \cos(2(N-j+1) \theta) \right] e^{\nu j/T}$$

Now, we can have either $r > 1$ or $r < 1$. We will first focus on the case $r > 1$ and later show that similar argument works for $r < 1$. Here, we are interested in the region of the chain far away from its boundary so that $r^N >> r^l$. The first series can then be summed (GP) to obtain terms like,
FIG. 8. The difference between the imaginary part of the pole for the non-interacting case ($z''_0$) and those of the interacting case ($z''_i$) is plotted as a function of the interaction strength $U/t_B$ and $V/t_B$ for the 2-site model connected to two independent baths of same temperature $T = 0.625 t_B$ but different chemical potential $\mu_1 = -2.5 t_B$ and $\mu_2 = -5.0 t_B$ where we use $g = 0.5 t_B$, $\epsilon = 0.2 t_B$. It shows that in the entire parameter space of the interaction strength , one of the two poles ($z''_i$) of the interacting model always has smaller imaginary part than the non-interacting one. Hence the cross-over timescale from the exponential decay to power law tail in Greens’ function and consequently in unequal time observables always shifts to larger $t - t'$ values as depicted in Fig. 6(c).

\[
\left[r^2 e^{\nu/T}\right]^{1\pm 1} r^2 e^{\nu/T} / \left(r^2 e^{\nu/T} - 1\right) + r \leftrightarrow e^{\pm i \theta}.
\]

One can then multiply with appropriate factors using equation (B1) to obtain $G_{1,i+1}^0$, whose leading order terms fall off exponentially $\sim e^{\nu 1/T}$ where the other terms are suppressed by factor of $\nu^{2N}$ coming from $\left|M_N\right|^2$. This shows that $\xi = T / \nu$ and explains the linear scaling of $\xi$ with $T$ seen in Fig 3(f). One can similarly work out the other terms and the conclusion obtained above remains robust. If $r < 1$, one can neglect $r^i$ in favour of $r^{-i}$, but the final conclusion $I \sim e^{\nu 1/T}$ works out. Note that $\nu < 0$ in our case, and hence this indicates a decay of current in space. A similar argument can be made for the density profile, except the term with 1 in the expansion of $\coth$ does not sum to zero. So $n(x) \sim n_0 + n_1 e^{\nu 1/T}$, i.e the variation around the constant value decays exponentially.

Appendix C: Effect of interaction on non-Markovian dynamics

In section (VII) , we had shown that, within mean field theory , the exponential decay in the temporal profile of the Greens’ functions of the interacting system is slower than that of the non-interacting system, and hence the time scale for cross-over from the ‘quasi-Markovian’ to non-Markovian dynamics increases. We argued this based on the fact that imaginary part of the one of the poles of the Greens’ function, which controls the decay rate, decreases with interaction. In Fig [8] , we plot the differences in imaginary part of the pole between non-interacting and interacting systems as a function of $U$ and $V$. We have considered a 2-site system with $g = 0.5 t_B$, $\epsilon = 0.2 t_B$, coupled to two baths with common temperature $T = 0.625 t_B$, $\mu_1 = -2.5 t_B$ and $\mu_2 = -5.0 t_B$. We find that the difference $> 0$ for all $U$ and $V$ for one of the poles, i.e, the non-interacting Greens’ function decay faster. For the other pole, the difference is negative for small values of $U$ and $V$ and becomes positive for larger values of $U$ and $V$. We note that the long time decay (before power law tail) is governed by the smallest decay rate , and hence the cross-over time scale increases for all $U$ and $V$.

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