Provability Logic and the Completeness Principle

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Master Thesis

January 9, 2018

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Chapter 1

Introduction

Around 1930, Kurt Gödel proved his celebrated incompleteness theorems. While these results can be seen as the culmination of one era of logical research, they also cleared the way for several new fields within mathematical logic. An example of such a field is provability logic, a topic that still occupies logicians today. Provability logic takes one of the main ingredients of Gödel’s theorems as its starting point. This ingredient is the formalization of the notion ‘formally provable in a certain arithmetical theory $T$’ inside the language of arithmetic itself. Once this step has been taken, one may wonder what a theory $T$ is able to prove about its own notion of provability. This object, i.e. what a theory $T$ can prove about its own notion of provability, is called the provability logic of $T$. Let us write, as we will below, ‘$\vdash_T A$’ for ‘$A$ is formally provable in $T$’, and ‘$\Box_T A$’ for the arithmetical formula expressing that $A$ is formally provable in $T$. Then under some reasonable assumptions, the following turn out to hold:

(i) if $\vdash_T A$, then $\vdash_T \Box_T A$;
(ii) $\vdash_T \Box_T (A \rightarrow B) \rightarrow (\Box_T A \rightarrow \Box_T B)$;
(iii) $\vdash_T \Box_T A \rightarrow \Box_T \Box_T A$.

These are known nowadays as the Hilbert-Bernays-Löb derivability conditions. Using the Diagonalization Lemma, another key idea from Gödel’s theorems, one can derive from these that $\vdash_T \Box_T (\Box_T A \rightarrow A) \rightarrow \Box_T A$, a result known as Löb’s Theorem. In 1976, Robert Solovay proved that for the theory Peano Arithmetic, the schemes (i)-(iii) and the formalized Löb Theorem completely describe the provability logic of Peano Arithmetic.

Provability logics are not monotone in their corresponding theories. That is, if $T$ is a theory extending another theory $U$, then it is not in general true that the provability logic of $T$ extends the provability logic of $U$. In light of this, it is all the more surprising that, in the classical case, provability logics are immensely stable. Solovay’s proof can be modified to show that any $\Sigma_1$-sound theory interpreting Peano Arithmetic has the same provability logic as Peano Arithmetic. We will not explain exactly what this means, but it includes theories as strong as Zermelo-Fraenkel Set Theory (with or without the Axiom of Choice).

Peano Arithmetic is a classical theory, which is why we made the caveat ‘in the classical case’ below. In the intuitionistic case, the situation is completely different. Solovay’s proof simply
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does not work for intuitionistic theories. This shows itself in the fact that the provability logic of Heyting Arithmetic, the intuitionistic counterpart of Peano Arithmetic, contains principles that the provability logic of Peano Arithmetic does not share. These principles are somewhat exotic, and it is unknown what the provability logic of Heyting Arithmetic exactly is. In fact, as far as we are aware, there is presently only one intuitionistic theory for which a nontrivial provability logic is known, a result due to A. Visser (see Remark 4.2.1).

In Solovay’s proof, the semantics of (classical) modal logic play a major role. The larger part of the proof consists of embedding models for modal logic in a certain way into the theory $T$. These models are equipped with an accessibility relation. Solovay uses the predicate $\Box_T$ to represent this relation inside the theory $T$. One may try to give a Solovay-style proof by replacing the models for classical modal logic by models for intuitionistic modal logic. The difficulty about these models, however, is that they also possess an intuitionistic relation, in addition to the accessibility relation. The main question then becomes how we can deal with these two relations.

This question sets the main goal for this thesis: to find an interesting situation where we can give a Solovay-style embedding of a model for intuitionistic modal logic. A. Visser suggested to consider theories that prove their own completeness principle, a principle for which the modal semantics is not too complicated. The availability of the completeness principle is an advantage that is specific to the intuitionistic context, since in the classical context, the completeness principle trivializes questions about provability logic. Furthermore, Visser suggested to use a nonstandard notion of provability to interpret one of the two relations on our models. This approach turned out to be successful, and our Solovay-style embedding is presented in detail below. Our Solovay-style embedding can be used to obtain a variety of results in provability logic. Among these is the determination of the $\Sigma_1$-provability logic of Heyting Arithmetic, an object related to the ordinary provability logic of Heyting Arithmetic. This is not a new result. It was already obtained in 2014 by M. Ardeshir and S. Mojtaba Mojtahedi [1], but the present work arrives at it in a different way. We stress, however, that our proof could not have been devised without the work from the paper [1]. First of all, it is of course easier to determine a provability logic if one already knows what it should be. Moreover, even though our proof is different, we do use some key ingredients from the paper [1], most notably the TNNIL-algorithm.

Let us briefly outline the structure of the thesis. First of all, in Chapter 2, we discuss all the necessary prerequisite knowledge, and fix our notation. This chapter contains no essentially new results, but we do prove some results from the paper [10] under weaker assumptions. For reasons of space (and energy), we will not spell out any specific Gödel numberings or give an explicit definition of the predicate $\Box_T$. Therefore, it will be useful to have some experience with Gödel’s incompleteness theorems and with provability logic (in the classical case) when reading this thesis. A reader that is already familiar with (some of) the concepts discussed in Chapter 2 may want to read (a portion of) this chapter only superficially, and refer back to it if necessary. In Chapter 3, we present our Solovay-style embedding, and formulate a completeness theorem. This theorem will be stated in an abstract way that does not yet mention any specific theories or provability predicates. In Chapter 4, we will present several applications of our completeness theorem, among which the determination of the $\Sigma_1$-provability logic of Heyting Arithmetic.
Chapter 2

Prerequisites

In this chapter, we develop some notation and theory that will be used in the later chapters. First, in Section 2.1, we fix some basic notions about arithmetical theories and provability predicates. Then, in Section 2.2, we discuss the $T$-translation, which will lead to theories that prove their own completeness. In Section 2.3, we turn our attention to two nonstandard notions of provability, called fast and slow provability. Finally, in Section 2.4, we develop some intuitionistic modal (propositional) logic.

2.1 Arithmetic and Provability

All the theories we shall consider will be theories for intuitionistic predicate logic with equality. As our proof system, we pick natural deduction with equality. An axiom will be viewed as a special case of an inference rule, namely as an inference rule whose premiss set is empty. For equality, we have the axiom $x = x$, and an inference rule for substitution. The language in which our theories will be formulated will be the language of arithmetic $\mathcal{L} = \{0, S, +, \times, \exp\}$. Here $0$ is a constant symbol, $S$ is a unary function symbol and $+, \times$ and $\exp$ are binary function symbols. For each $n \in \mathbb{N}$, we can define the $\mathcal{L}$-term $S \ldots S0$, where the $S$ occurs exactly $n$ times. This term is called the numeral of $n$, and we denote it by $\bar{n}$. Usually, however, we omit the bar and just write $n$ for the numeral of $n$. For terms $s$ and $t$, we define $s \leq t$ as $\exists x(s + x = t)$ and $s < t$ as $\exists x(s + x + 1 = t)$. Here $x$ should not occur in $s$ or $t$, of course. We notice that the language $\mathcal{L}$ has a straightforward interpretation in the natural numbers, yielding the standard model $\mathbb{N}$. We introduce two special classes of formulae.

**Definition 2.1.1.** (i) The set of $\Delta_0$-formulae is defined by recursion, as follows:

(a) all atomic $\mathcal{L}$-formulae are $\Delta_0$-formulae;
(b) the set of $\Delta_0$-formulae is closed under conjunction, disjunction and implication;
(c) if $A$ is a $\Delta_0$-formula, and $t$ is an $\mathcal{L}$-term not containing the variable $x$, then the formulae $\exists x(x < t \land A)$ and $\forall x(x < t \rightarrow A)$ are also $\Delta_0$-formulae.

We write $A \in \Delta_0$ if $A$ is a $\Delta_0$-formula.

(ii) The set of $\Sigma_1$-formulae consists of all $\mathcal{L}$-formulae of the form $\exists x.A$, where $A \in \Delta_0$. We write $S \in \Sigma_1$ if $S$ is a $\Sigma_1$-formula.
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To each \( \mathcal{L} \)-expression \( \alpha \) (which can be a term, a formula or a sequence of formulae), we assign a Gödel number \( \lceil \alpha \rceil \) in some reasonable way. More precisely, we require that elementary syntactic operations concerning \( \mathcal{L} \) are primitive recursive in their Gödel numbers.

Definition 2.1.2. A theory \( T \) will be a pair \((\text{Th}(T), \text{Ax}_T)\), where \( \text{Ax}_T \) is a \( \Sigma_1 \)-formula in one free variable, and \( \text{Th}(T) \) is precisely the set of \( \mathcal{L} \)-formulae derivable from the axiom set

\[
\{ A \mid A \text{ an } \mathcal{L} \text{-formula, } N \models \text{Ax}_T(\lceil \alpha \rceil) \}.
\]

In other words, a theory is a set of \( \mathcal{L} \)-formulae that is closed under derivability in intuitionistic predicate logic with equality, together with a \( \Sigma_1 \)-formula that defines an axiom set for the theory in the standard model. Usually, we will define a theory by giving its axioms, understanding that there is some natural \( \Sigma_1 \)-formulation in \( \mathcal{L} \) for axiomhood. For a set of \( \mathcal{L} \)-formulae \( \Gamma \) and an \( \mathcal{L} \)-formula \( A \), we write \( \Gamma \vdash_T A \) to indicate that \( A \) is provable using open assumptions from \( \Gamma \) and the axioms of \( T \). Notice that \( \vdash_T A \) just means \( A \in \text{Th}(T) \).

Now we define three theories that will be of great interest to us.

Definition 2.1.3. (i) The theory \( \text{HA} \), called Heyting arithmetic, has the axioms

\[
\neg (Sx = 0) \quad x \times 0 = 0 \\
Sx = Sy \rightarrow x = y \quad x \times Sy = x \times y + x \\
x + 0 = 0 \quad \exp(x, 0) = 1 \\
x + Sy = S(x + y) \quad \exp(x, Sy) = \exp(x, y) \times x
\]

and, for each \( \mathcal{L} \)-formula \( A \), the induction axiom \( A[0/x] \land \forall x (A \to A[Sx/x]) \to \forall x A \).

(ii) The theory \( \text{EA} \), called elementary arithmetic, has the same axioms as \( \text{HA} \), except that the induction scheme is restricted to formulae \( A \in \Delta_0 \).

(iii) The theory \( \text{PA} \), called Peano arithmetic, has all the axioms of \( \text{HA} \), together with the Law of the Excluded Middle: \( A \lor \neg A \), where \( A \) is an \( \mathcal{L} \)-formula.

Even though the axiom set we presented for \( \text{EA} \) is infinite, the theory \( \text{EA} \) is actually finitely axiomatizable (see e.g. [3], Theorem V.5.6), and there in fact exists a choice for \( \text{Ax}_{\text{EA}} \) such the finite axiomatizability of \( \text{EA} \) can be verified in \( \text{EA} \) itself. It is also well-known that \( \text{EA} \), and hence any theory extending it, is \( \Sigma_1 \)-complete. That is, every \( \Sigma_1 \)-sentence true in the standard model can be proven inside \( \text{EA} \).

Concerning \( \text{HA} \), we have the following well-known result.

Proposition 2.1.1. Let \( F : \mathbb{N}^k \to \mathbb{N} \) be primitive recursive. Then there exists a \( \Sigma_1 \)-formula \( A_F(x, y) \) satisfying:

\[
(i) \vdash_{\text{HA}} A_F(\bar{n}, F(\bar{n})) \text{ for all } \bar{n} \in \mathbb{N}^k; \\
(ii) \vdash_{\text{HA}} \exists y \forall z (A_F(x, z) \leftrightarrow y = z).
\]

Moreover, this formula can be chosen in such a way that the definition of \( F \) as a primitive recursive function is verifiable in \( \text{HA} \).

Notice that we have a primitive recursive function \( \text{Subst} : \mathbb{N}^2 \to \mathbb{N} \) that is defined as follows. If \( a \) is the Gödel number of some formula \( A(v) \) in one free variable \( v \), then \( \text{Subst}(a, b) = \lceil A(b) \rceil \).
otherwise, Subst(a, b) = 0. We can represent this function in HA using Proposition 2.1.1. In fact, for this particular function the clauses (i) and (ii) even hold when HA is replaced by EA. If A(v) is a formula with one free variable, we will write \( \gamma A(\vec{x}) \) for Subst(\( \gamma A(v) \)), which makes sense when working in a theory extending EA. We apply similar conventions for multiple free variables. We will need the following famous result, that we will not prove.

**Theorem 2.1.2** (Diagonalization Lemma). Suppose \( A(\vec{x}, y) \) is an \( \mathcal{L} \)-formula. Then there exists an \( \mathcal{L} \)-formula \( B(\vec{x}) \) such that \( \vdash_{EA} B(\vec{x}) \iff A(\vec{x}, \gamma B(\vec{x})) \).

Concerning the \( \Delta_0 \)- and \( \Sigma_1 \)-formulae, we have the following results.

**Proposition 2.1.3.** (i) If \( A \in \Delta_0 \), then \( \vdash_{HA} A \lor \neg A \).

(ii) If \( S \in \Sigma_1 \), then we have \( \vdash_{HA} S \iff \exists x(a = t) \) for a certain variable \( x \) and certain \( \mathcal{L} \)-terms \( s \) and \( t \).

Now suppose we have a theory \( T \). Using the \( \Sigma_1 \)-formula \( \text{Ax}_T \), we can construct a \( \Sigma_1 \)-formula \( \text{Bew}_T(x) \) that expresses ‘\( x \) is the Gödel number of some formula \( A \) such that \( \vdash_T A \)’ in a natural way. We can write \( \text{Bew}(x) \) as \( \forall y \Prf_T(y, x) \) for some \( \Delta_0 \)-formula \( \Prf_T \). We think of \( \Prf(y, x) \) as expressing the fact that \( y \) codes a \( T \)-proof of the formula that has \( x \) as its Gödel number.

For a formula \( A = A(x_1, \ldots, x_n) \), we write \( \Box_T A \) for \( \text{Bew}_T(\gamma A(x_1, \ldots, x_n)) \). In particular, \( \Box_T A \) has the same free variables as \( A \). Now we can define certain relations between theories.

**Definition 2.1.4.** Let \( U \) and \( T \) be theories. We write:

(i) \( U \subseteq T \) if \( \text{Th}(U) \subseteq \text{Th}(T) \);

(ii) \( U = T \) if \( \text{Th}(U) = \text{Th}(T) \);

(iii) \( U \leq T \) if \( \vdash_{HA} \text{Bew}_U(x) \rightarrow \text{Bew}_T(x) \);

(iv) \( U \equiv T \) if \( \vdash_{HA} \text{Bew}_U(x) \leftrightarrow \text{Bew}_T(x) \).

We emphasize that, then we write \( U = T \), we do not mean an equality of the pairs \( (\text{Th}(U), \text{Ax}_U) \) and \( (\text{Th}(T), \text{Ax}_T) \), but only an equality of the first coordinate. Since HA is sound, we see that \( U \leq T \) implies that \( U \subseteq T \). We also notice that, if \( U \) and \( T \) are theories such that \( \vdash_{HA} \text{Ax}_U(x) \rightarrow \text{Ax}_T(x) \), then \( U \leq T \) clearly holds. However, this requirement is not necessary: it can also be the case that every \( U \)-proof can (verifiably in HA) be transformed into a \( T \)-proof without the one axiom set being contained in the other. Before we can develop more theory, we need to restrict our investigation to theories that, verifiably in HA, can perform a minimal amount of arithmetic.

**Convention 2.1.1.** All the theories \( T \) we shall consider, will satisfy \( \text{EA} \leq T \).

Notice that this clearly holds for the three theories from Definition 2.1.3. With this requirement in place, we can state some basic properties of \( \Box_T \), that we will not prove.

**Proposition 2.1.4.** Let \( T \) be a theory and let \( A, B \) and \( S \) be \( \mathcal{L} \)-formulae. Then we have:

(i) We have \( \vdash_T A \) if and only if \( \mathbb{N} \models \Box_T A \), if and only if \( \vdash_{EA} \Box_T A \);

(ii) \( \vdash_{HA} \Box_T (A \rightarrow B) \rightarrow (\Box_T A \rightarrow \Box_T B) \);

(iii) \( \vdash_{HA} \Box_T A \rightarrow \Box_T \Box_T A \);

(iv) (Formalized \( \Sigma_1 \)-completeness) if \( S \in \Sigma_1 \), then \( \vdash_{HA} S \rightarrow \Box_T S \);
Remark 2.1.1. Our use of the term ‘provability predicate’ differs from its use in other literature. Usually, a provability predicate is defined as a formula that satisfies the Hilbert-Bernays-Löb derivability conditions. Our definition is stronger, since these conditions follow from (i)-(iii) above. Indeed let \( A \) be an \( L \)-sentence. Moreover, (ii), (iii), (iv) and (vi) are verifiable in \( \text{HA} \).

We remark that for (iii)-(vi), we need Convention 2.1.1. In the next section, we will need the following facts.

**Proposition 2.1.5.** Let \( U \) and \( T \) be theories.

(i) If \( U \subseteq T \), then \( \vdash_U A \) implies \( \vdash_T \Box_T A \) for all \( L \)-formulae \( A \).

(ii) If HA \( \subseteq U \subseteq T \), then \( \vdash_{HA} \text{Bew}_U(x) \rightarrow \Box_U \text{Bew}_T(x) \). In particular, \( \vdash_{HA} \Box_U A \rightarrow \Box_U \Box_TA \) for all \( L \)-formulae \( A \).

**Proof.** (i) If \( \vdash_U A \), then also \( \vdash_T A \), so \( \vdash_{EA} \Box_T A \). Since \( EA \subseteq U \), we also get \( \vdash_U \Box_T A \).

(ii) Since \( U \subseteq T \), we have \( \vdash_{HA} \text{Bew}_U(x) \rightarrow \text{Bew}_T(x) \). Since HA \( \subseteq U \), it follows from (i) that \( \vdash_{HA} \Box_U \text{Bew}_U(x) \rightarrow \Box_U \text{Bew}_T(x) \). We also have \( \vdash_{HA} \text{Bew}_U(x) \rightarrow \Box_U \text{Bew}_T(x) \) by formalized \( \Sigma_1 \)-completeness, and now the result follows.

For future use, we state the following definition.

**Definition 2.1.5.** Let \( T \supseteq \text{HA} \) be a theory and let \( S(x) \) be a \( \Sigma_1 \)-formula in one free variable. For an \( L \)-sentence \( A \), we write \( \Box A \) for \( S(\ulcorner A \urcorner) \). We say that \( S \) is a provability predicate for \( T \) if the following hold for all \( L \)-sentences \( A, B \) and \( R \):

(i) if \( \vdash_T A \), then \( \mathbb{N} \models \Box A \);

(ii) \( \vdash_{HA} \Box (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) \);

(iii) if \( R \in \Sigma_1 \), then \( \vdash_{HA} R \rightarrow \Box R \).

Using Proposition 2.1.4, we see that \( \text{Bew}_T \) is always a provability predicate for \( T \).

**Remark 2.1.1.** Our use of the term ‘provability predicate’ differs from its use in other literature. Usually, a provability predicate is defined as a formula that satisfies the Hilbert-Bernays-Löb derivability conditions. Our definition is stronger, since these conditions follow from (i)-(iii) above. Indeed let \( A \) be an \( L \)-sentence. Since \( \text{HA} \) is \( \Sigma_1 \)-complete and \( \Box A \) is a \( \Sigma_1 \)-sentence, we have that \( \mathbb{N} \models \Box A \) implies \( \vdash_{HA} \Box A \). From (iii), it follows that \( \vdash_{HA} \Box A \rightarrow \Box \Box A \), again since \( \Box A \) is a \( \Sigma_1 \)-sentence. In particular, we can derive Löb’s Principle for \( \Box \), using the Diagonalization Lemma. That is, if \( U \) is a theory such that \( \text{HA} \subseteq U \subseteq T \), and \( A \) is an \( L \)-sentence such that \( \vdash_U \Box A \rightarrow A \), then \( \vdash_U A \). We also have Löb’s Theorem for \( \Box \), that is, \( \vdash_{HA} \Box (\Box A \rightarrow A) \rightarrow \Box A \) for all \( L \)-sentences \( A \).

2.2 The Completeness Principle

In this section, we introduce the \( T \)-translation, that will allow us to define theories that prove their own completeness. All results in this section are from Visser’s paper *On the Completeness Principle* [10], but we have formulated some of them under weaker conditions.
Definition 2.2.1. Let $T$ be a theory. We define the $T$-translation $(\cdot)^T$ from the set of $\mathcal{L}$-formulae to itself by recursion. For all $\mathcal{L}$-terms $s$ and $t$ and $\mathcal{L}$-formulae $A$ and $B$, we set:

(i) $(s = t)^T$ is $s = t$ and $\bot^T$ is $\bot$;
(ii) $(A \circ B)^T$ is $A^T \circ B^T$ for $\circ \in \{\land, \lor\}$;
(iii) $(A \rightarrow B)^T$ is $(A^T \rightarrow B^T) \land \square_T(A^T \rightarrow B^T)$;
(iv) $(\exists x A)^T$ is $\exists x A^T$;
(v) $(\forall x A)^T$ is $\forall x A^T \land \square_T(\forall x A^T)$.

Based on the $T$-translation, we can construct new theories out of old ones.

Definition 2.2.2. Let $U$ and $T$ be theories. We define the theory $U^T$ as $\text{HA} + \{A \mid \vdash_U A^T\}$. For a theory $U$, we write $U^*$ for $U^U$.

We make some remarks on how $\text{Ax}_{U,T}$ can be defined. Clearly, the function $(\cdot)^T : \mathbb{N} \rightarrow \mathbb{N}$ that satisfies $x^T = \Gamma A$ is $x$ is the Gödel number of an $\mathcal{L}$-formula $A$, and $x^T = 0$ otherwise, is primitive recursive. So we can represent this function in HA using Proposition 2.1.1. Now we define $\text{Ax}_{U,T}(x)$ as $\text{Ax}_{\text{HA}}(x) \lor (\text{Form}(x) \land \text{Bew}_U(x^T))$, where Form$(x)$ naturally expresses the fact that $x$ is the Gödel number of an $\mathcal{L}$-formula. We study the relation between provability in $U^T$ and provability in $U$ through the following lemmata.

Lemma 2.2.1. For all $\mathcal{L}$-formulae $A$, we have $\vdash_{\text{HA}} A^T \rightarrow \square_T A^T$, and this fact is verifiable in $\text{HA}$.

Proof. We proceed by induction on the complexity of $A$.

At If $A$ is atomic, then $A^T = A$ and the claim follows from Proposition 2.1.4(iv) since $A$ is a $\Sigma_1$-formula.

$\land$ Suppose $A = B \land C$ and the claim holds for $B$ and $C$. Then $A^T$ is $B^T \land C^T$, and we have

$$\vdash_{\text{HA}} B^T \land C^T \rightarrow \square_T B^T \land \square_T C^T \rightarrow \square_T(B^T \land C^T),$$

as desired.

$\lor$ Suppose $A = B \lor C$ and the claim holds for $B$ and $C$. Then $A^T$ is $B^T \lor C^T$, and we have $\vdash_{\text{HA}} B^T \rightarrow \square_T B^T \rightarrow \square_T(B^T \lor C^T)$ and $\vdash_{\text{HA}} C^T \rightarrow \square_T C^T \rightarrow \square_T(B^T \lor C^T)$, which together yield $\vdash_{\text{HA}} B^T \lor C^T \rightarrow \square_T(B^T \lor C^T)$, as desired.

$\rightarrow$ Suppose $A = B \rightarrow C$ and the claim holds for $B$ and $C$. Then the formula $A^T$ is equal to $(B^T \rightarrow C^T) \land \square_T(B^T \rightarrow C^T)$, and we have

$$\vdash_{\text{HA}} (B^T \rightarrow C^T) \land \square_T(B^T \rightarrow C^T) \rightarrow \square_T(B^T \rightarrow C^T) \rightarrow \square_T(B^T \rightarrow C^T) \land \square_T(B^T \rightarrow C^T) \rightarrow \square_T(B^T \rightarrow C^T),$$

as desired.

$\exists$ Suppose $A = \exists x B$ and the claim holds for $B$. Then $A^T$ is $\exists x B^T$. It is provable in intuitionistic predicate logic that $B^T \rightarrow \exists x B^T$, so we also have $\vdash_{\text{HA}} \square_T B^T \rightarrow \square_T(\exists x B^T)$. We get $\vdash_{\text{HA}} \exists x B^T \rightarrow \exists x \square_T B^T \rightarrow \square_T(\exists x B^T)$, as desired.
\( \forall \) Suppose \( A = \forall x B \) and the claim holds for \( B \). Then \( A^T = \forall x B^T \land \Box_T(\forall x B^T) \), and we have

\[
\vdash_{HA} \forall x B^T \land \Box_T(\forall x B^T) \rightarrow \Box_T(\forall x B^T)
\]

\[
\rightarrow \Box_T(\forall x B^T) \land \Box_T(\forall x B^T)
\]

\[
\rightarrow \Box_T(\forall x B^T \land \Box_T(\forall x B^T))
\]

as desired.

For the second statement, we should carry out this induction inside HA. One should notice that now we need that clauses (ii)-(iv) from Proposition 2.1.4 are verifiable in HA. \( \square \)

**Lemma 2.2.2.** Let \( A \) be a formula, let \( x \) be a variable, and let \( s \) be a term.

(i) \( A \) and \( A^T \) have the same free variables;

(ii) \( s \) is free for \( x \) in \( A \) if and only if \( s \) is free for \( x \) in \( A^T \);

(iii) if \( s \) is free for \( x \) in \( A \), then \( \vdash_{HA} (A^T)[s/x] \leftrightarrow (A[s/x]^T) \).

Moreover, these are all verifiable in HA.

**Proof.** All three statements can be proven by an easy induction on the complexity of \( A \). For the induction steps for implication and universal quantification in statement (iii), one should observe that, verifiably in HA, we have \( \vdash_{HA} (\Box_T A)[s/x] \leftrightarrow \Box_T(A[s/x]) \) for all \( L \)-terms \( s \) and \( L \)-formulae \( A \). \( \square \)

Using these two lemmata, we can prove the following crucial result.

**Theorem 2.2.3.** Let \( U \) and \( T \) be a theories such that HA \( \subseteq U \) and: \( \vdash_U B \) implies \( \vdash_U \Box_T B \) for all \( L \)-formulae \( B \). For a set of \( L \)-formulae \( \Gamma \), write \( \Gamma^T = \{ B^T \mid B \in \Gamma \} \). Then for all \( L \)-formulae \( A \), we have \( \Gamma \vdash_U \Gamma^T \) \( A \) if and only if \( \Gamma^T \vdash_U A^T \).

**Remark 2.2.1.** (i) By Proposition 2.1.5(i), the conditions on \( U \) and \( T \) apply in particular when HA \( \subseteq U \subseteq T \). We formulate this theorem (and Corollary 2.2.6 below) in such a strong way in order to obtain Lemma 4.3.1 towards the end of this thesis.

(ii) We warn the reader that, under these conditions, we cannot necessarily verify the result \( \Gamma \vdash_U \Gamma^T \) \( A \) if and only if \( \Gamma^T \vdash_U A^T \) inside HA; see Corollary 2.2.6 below. \( \diamond \)

**Proof of Theorem 2.2.3.** Suppose that \( \Gamma^T \vdash_U A^T \). Then there exist \( n \geq 0 \) and \( C_1, \ldots, C_n \in \Gamma \) such that \( \vdash_U C_0^T \land \ldots \land C_n^T \rightarrow A^T \). Then we also have \( \vdash_U (C_0 \land \ldots \land C_n)^T \rightarrow A^T \), and by our assumption, we also get \( \vdash_U \Box_T((C_0 \land \ldots \land C_n)^T \rightarrow A^T) \). So \( \vdash_U (C_0 \land \ldots \land C_n \rightarrow A)^T \), and therefore we get \( \vdash_{U^T} C_0 \land \ldots \land C_n \rightarrow A \). Finally, this clearly yields that \( \Gamma \vdash_U \Gamma^T \) \( A \).

For the converse direction, we proceed by induction on the proof tree for \( \Gamma \vdash_U \Gamma^T \) \( A \). Before we start, we notice the following: if \( \vdash_U B \rightarrow C \) for certain \( L \)-formulae \( B \) and \( C \), then by our assumption, \( \vdash_U \Box_T(B \rightarrow C) \). Since HA \( \subseteq U \), we also have \( \vdash_U \Box_T(B \rightarrow C) \rightarrow (\Box_T B \rightarrow \Box_T C) \), so in particular, we get \( \vdash_U \Box_T B \rightarrow \Box_T C \). We also note: if \( \vdash_{HA} B \), then \( \vdash_U B \), whence \( \vdash_U \Box_T B \).

First, suppose that \( A \) is an axiom of \( U^T \). That is, we suppose that \( A \) is an axiom of HA or \( \vdash_U A^T \). In the latter case, we are done. So suppose that \( A \) is an axiom of HA. We need to show that \( \vdash_U A^T \). If \( A \) is the axiom \( x = x \), then \( A^T \) is also \( x = x \), which is clearly provable in
2.2. THE COMPLETENESS PRINCIPLE

$U$. If $A$ is the axiom $\neg(\forall x = 0)$, then $A^T$ is $\neg(\forall x = 0) \land \square_T(\neg(\forall x = 0))$. Since $\vdash_{HA} \neg(\forall x = 0)$, we also have $\vdash_U \neg(\forall x = 0)$, and by our assumption, $\vdash_U \square_T(\neg(\forall x = 0))$. So we indeed have $\vdash_U A^T$. If $A$ is another basic axiom of $\text{HA}$, then $A$ is atomic, so $A^T$ is equal to $A$ itself again. So $\vdash_{HA} A^T$, hence also $\vdash_U A^T$. It remains to prove the claim for the case where $A$ is an induction axiom, say $B[0/x] \land \forall x (B \rightarrow B[\{s/x\}]) \rightarrow \forall x B$. First of all, we notice that

$$
\vdash_U (\forall x (B \rightarrow B[\{s/x\}]))^T \leftrightarrow \forall x (B \rightarrow B[\{s/x\}])^T \land \square_T(\forall x (B \rightarrow B[\{s/x\}])^T)
$$

$$
\rightarrow \forall x (B^T \rightarrow (B[\{s/x\}]))^T \land \square_T(\forall x (B \rightarrow B[\{s/x\}])^T)
$$

$$
\leftrightarrow \forall x (B^T \rightarrow (B^T[\{s/x\}])) \land \square_T(\forall x (B^T \rightarrow (B^T[\{s/x\}])).
$$

(2.1)

Furthermore, we know that $\vdash_{HA} (B[0/x])^T \leftrightarrow (B^T)[0/x]$ and that $(\forall x B)^T$ is the formula $\forall x B^T \land \square_T(\forall x B^T)$. Define the formulae

$$
C = (B^T)[0/x] \land \forall x (B^T \rightarrow (B^T[\{s/x\}]) \land \square_T(\forall x (B^T \rightarrow (B^T[\{s/x\}]) \rightarrow \forall x B^T \land \square_T(\forall x B^T),
$$

(2.2)

since the displayed formula is an induction axiom and $\text{HA} \subseteq U$. Now it follows that

$$
\vdash_U \square_T((B^T)[0/x] \land \square_T(\forall x (B^T \rightarrow (B^T[\{s/x\}]) \rightarrow \square_T(\forall x B^T).
$$

(2.3)

Finally, since $\vdash_{HA} (B^T)[0/x] \leftrightarrow (B[0/x])^T$, we can use Lemma 2.2.1 to see that

$$
\vdash_U (B^T)[0/x] \rightarrow (B[0/x])^T \rightarrow \square_T((B[0/x])^T) \rightarrow \square_T((B^T)[0/x]).
$$

(2.4)

From (2.2), (2.3) and (2.4), we may deduce that $C$ is indeed provable in $U$, as desired.

Now we treat the rules of inference. Since the $T$-translation commutes with conjunction, disjunction and existential quantification, the induction steps for rules of inference for these operators are trivial. It remains to check the rules for implication and universal quantification, and the substitution rule.

→E Suppose that $\Gamma^T \vdash_U (B \rightarrow C)^T$ and $\Gamma^T \vdash_U B^T$. We need to show that $\Gamma^T \vdash_U C^T$. But this is obvious since $\vdash_U (B \rightarrow C)^T \rightarrow (B^T \rightarrow C^T).

→I Suppose that $\Gamma^T, B^T \vdash_U C^T$. We need to show that $\Gamma^T \vdash_U (B \rightarrow C)^T$. We certainly have $\Gamma^T \vdash_U B^T \rightarrow C^T$. But then we also have $\square_T(\Gamma^T) \vdash_U \square_T(B^T \rightarrow C^T)$, where $\square_T(\Gamma^T) = \{\square_T(D^T) \mid D \in \Gamma\}$. Since $\text{HA} \subseteq U$ and $\vdash_{HA} D^T \rightarrow \square_T D^T$ for all $D \in \Gamma$, we get $\Gamma^T \vdash_U \square_T(B^T \rightarrow C^T)$. Combining our results, we find

$$
\Gamma^T \vdash_U (B^T \rightarrow C^T) \land \square_T(B^T \rightarrow C^T),
$$

as desired.

∀E Suppose that $\Gamma^T \vdash_U (\forall x B)^T$. We need to show that $\Gamma^T \vdash_U (B[\{s/x\}])^T$. Since $\vdash_U (\forall x B)^T \rightarrow (\forall x B)^T$, we see that $\Gamma^T \vdash_U (B[\{s/x\}])^T$. Since we also know that $\text{HA} \subseteq U$ and $\vdash_{HA} (B^T)[s/x] \leftrightarrow (B[\{s/x\}])^T$, we get $\Gamma^T \vdash_U (B[\{s/x\}])^T$, as desired.
∀I Suppose that \( \Gamma^T \vdash_U B^T \), where the variable \( x \) does not occur anywhere in \( \Gamma \). We need to show that \( \Gamma^T \vdash_U (\forall x)B^T \). First of all, we certainly have \( \Gamma^T \vdash_U \forall x B^T \), since \( x \) does not occur free anywhere in \( \Gamma^T \). By applying the same reasoning as in the \( \rightarrow \)-case, we find \( \Gamma^T \vdash_U \Box_T(\forall x B^T) \). We conclude that \( \Gamma^T \vdash_U \forall x B^T \land \Box_T(\forall x B^T) \), as desired.

Subst Suppose that \( \Gamma^T \vdash_U (B[s/x])^T \) and \( \Gamma^T \vdash_U (s = t)^T \). We need to show that \( \Gamma^T \vdash_U (B^T)[t/x] \). We have \( \Gamma^T \vdash_U s = t \) and by Lemma 2.2.2(iii) and the fact that \( \text{HA} \subseteq U \), we get \( \Gamma^T \vdash_U (B^T)[t/x] \). This yields \( \Gamma^T \vdash_U (B^T)[t/x] \), and thus \( \Gamma^T \vdash_U (B^T)[t/x] \), as desired.

This completes the induction.

From this theorem, we can deduce the following results.

**Corollary 2.2.4.** If \( \text{HA} \subseteq U \subseteq T \), then the theory \( U^T \) is consistent if and only if \( U \) is consistent.

**Corollary 2.2.5.** Suppose that \( \text{HA} \subseteq T \). If \( A \vdash_{\text{HA}} B \), then \( A^T \vdash_{\text{HA}^T} B^T \).

**Proof.** If \( A \vdash_{\text{HA}} B \), then also \( A \vdash_{\text{HA}^T} B \). By applying Theorem 2.2.3 with \( U \equiv \text{HA} \), we find that \( A^T \vdash_{\text{HA}^T} B^T \). \( \Box \)

**Corollary 2.2.6.** Let \( U \) and \( T \) be theories such that \( \text{HA} \leq U \) and \( \vdash_{\text{HA}} \text{Bew}_U(x) \rightarrow \Box_U \text{Bew}_T(x) \). Then \( \vdash_{\text{HA}} \Box_{U^T} A \leftrightarrow \Box_{U^T} A^T \) for all \( L \)-formulae \( A \).

**Remark 2.2.2.** By Proposition 2.1.5(ii), the requirements on \( U \) and \( T \) are satisfied when \( \text{HA} \leq U \leq T \). \( \Box \)

**Proof of Corollary 2.2.6.** The ‘\( \leftarrow \)’-direction is immediate as it follows from the definition of \( U^T \), and it does not need the requirements on \( U \) and \( T \). Concretely, we have

\[ \vdash_{\text{HA}} \text{Form}(x) \land \text{Bew}_U(x^T) \rightarrow \text{Ax}_{U^T}(x) \rightarrow \text{Bew}_{U^T}(x). \]

From this, the desired result follows.

For the ‘\( \rightarrow \)’-direction, we formalize the proof of the left-to-right direction of Theorem 2.2.3 inside HA. We need that the statements of Proposition 2.1.4, Lemma 2.2.1 and Lemma 2.2.2 are verifiable in HA. If we restrict the result to the case where \( \Gamma \) is empty, we get \( \vdash_{\text{HA}} \text{Bew}_{U^T}(x) \rightarrow \text{Bew}_U(x^T) \), from which the desired result will follow. \( \Box \)

Next, we prove an important conservation result for \( U^T \). First, we need the following definition.

**Definition 2.2.3.** The set \( A \) is the smallest set of \( L \)-formulae such that

(i) \( A \) contains all atomic \( L \)-formulae;

(ii) \( A \) is closed under conjunction, disjunction, and both existential and universal quantification;

(iii) if \( S \in \Sigma_1 \) and \( A \in A \), then \( S \rightarrow A \in A \).

**Lemma 2.2.7.** Let \( T \) be a theory such that \( \text{HA} \subseteq T \).

(i) We have \( \vdash_{\text{HA}} S \leftrightarrow S^T \) for all \( S \in \Sigma_1 \).

(ii) We have \( \vdash_{\text{HA}} A^T \rightarrow A \) for all \( A \in A \).
Now suppose that we have another theory $U$ such that $\text{HA} \subseteq U \subseteq T$.

(iii) The theories $U$ and $U^T$ prove the same $\Sigma_1$-formulæ.

(iv) The theory $U^T$ is $A$-conservative over $U$.

Proof. (i) Let $S \in \Sigma_1$. By Proposition 2.1.3(ii), we have that $\vdash_{\text{HA}} S \leftrightarrow \exists x (s = t)$ for certain $L$-terms $s$ and $t$. By Corollary 2.2.5, we also have $\vdash_{\text{HA}} S^T \leftrightarrow (\exists x (s = t))^T$. Since $(\exists x (s = t))^T$ is just $\exists x (s = t)$, the result follows.

(ii) We proceed by induction on the complexity of $A$. Only clause (iii) in the definition of $A$ is nontrivial. Suppose that $A$ is $S \rightarrow B$, where $S \in \Sigma_1$, and that we already know the result for $B$. Then $\vdash_{\text{HA}} S \leftrightarrow S^T$ and $\vdash_{\text{HA}} B^T \rightarrow B$, so $\vdash_{\text{HA}} (S \rightarrow B)^T \rightarrow (S^T \rightarrow B^T) \rightarrow (S \rightarrow B)$, as desired.

(iii) Let $S \in \Sigma_1$. Then by item (i) and Theorem 2.2.3, we have that $\vdash_{U^T} S$ if and only if $\vdash_U S^T$, if and only if $\vdash_U S$.

(iv) Suppose we have $A \in A$ such that $\vdash_{U^T} A$. By Theorem 2.2.3, we have $\vdash_U A^T$, and by item (ii), we get $\vdash_U A$. \qed

At the beginning of this section, we promised to construct theories that prove their own completeness. We now make this precise.

Definition 2.2.4. Let $S(x)$ be a provability predicate for a theory $T$. Again, if $A$ is an $L$-sentence, we write $\square A$ for $S(⌜A⌝)$.

(i) The completeness principle $\text{CP}_{\square}$ is the axiom scheme $A \rightarrow \square A$, where $A$ is an $L$-sentence.

(ii) The strong L"ob principle $\text{SLP}_{\square}$ is the axiom scheme $(\square A \rightarrow A) \rightarrow A$, where $A$ is an $L$-sentence.

We write $\text{CP}_T$ for $\text{CP}_{\square}$, and similarly for $\text{SLP}_T$.

Notice that ‘$\text{CP}_{\square}$’ is a slight abuse of notation, since the $\square$ is just an abbreviation and $\text{CP}_{\square}$ actually depends on $S(x)$.

Lemma 2.2.8. Let $S(x)$ be a provability predicate for a theory $T$. Then the completeness principle and the strong L"ob principle for $S$ are interderivable over HA.

Proof. Define $\square$ as above, and let $A$ be an $L$-sentence. First, we show that $\vdash_{\text{HA+CP}_{\square}} \text{SLP}_{\square}$. Since $S(x)$ is a provability predicate, we have the formalized L"ob Theorem for $\square$, so $\vdash_{\text{HA+CP}_{\square}} (\square A \rightarrow A) \rightarrow \square (\square A \rightarrow A) \rightarrow \square A,$ from which $\vdash_{\text{HA+CP}_{\square}} (\square A \rightarrow A) \rightarrow A$ follows.

Now we show that $\vdash_{\text{HA+SLP}_{\square}} \text{CP}_{\square}$. Clearly, we have $\vdash_{\text{HA}} \square (A \land \square A) \rightarrow A$, so $\vdash_{\text{HA+SLP}_{\square}} A \rightarrow (\square (A \land \square A) \rightarrow A \land \square A)$

$\quad \rightarrow A \land \square A$

$\quad \rightarrow \square A,$

as desired. \qed
Lemma 2.2.9. Let $U$ and $T$ be theories such that $\mathsf{HA} \subseteq U,T$. Then for all $\mathcal{L}$-sentences $A$, we have $\vdash_{U,T} A \rightarrow \square_T A^T$. In particular, $\vdash_{U,T} \mathcal{CP}_{T^*}$.

Proof. Let $A$ be an $\mathcal{L}$-sentence. Since $\mathsf{HA} \subseteq T$ and $\square_T A^T \in \Sigma_1$, we have that $\vdash_{\mathsf{HA}} \square_T A^T \rightarrow (\square_T A^T)^T$. So we get $\vdash_{\mathsf{HA}} A^T \rightarrow \square_T A^T \rightarrow (\square_T A^T)^T$, and since $\mathsf{HA} \subseteq T$, we also get $\vdash_{\mathsf{HA}} \square_T (A^T) \rightarrow (\square_T A^T)^T)$. So we find $\vdash_{\mathsf{HA}} (A \rightarrow \square_T A^T)^T$. This means that $\vdash_U (A \rightarrow \square_T A^T)^T$, since $\mathsf{HA} \subseteq U$. It follows that $\vdash_{U,T} A \rightarrow \square_T A^T \rightarrow \square_{T^*} A$, as desired. \hfill \Box

Remark 2.2.3. Notice that under the present assumptions, $\square_T A^T$ is not necessarily equivalent, over $\mathsf{HA}$, to $\square_{T^*} A$.

Corollary 2.2.10. Let $U$ be a theory such that $\mathsf{HA} \subseteq U$. Then $\vdash_{U^*} \mathcal{CP}_{U^*}$.

Proof. This follows by taking $U \equiv T$ in Lemma 2.2.9.

Remark 2.2.4. We remark that the proof of Lemma 2.2.9 also goes through if we replace the first line with ‘Let $A$ be an $\mathcal{L}$-formula.’ We will not need this greater generality. \hfill \Box

2.3 Fast and Slow Provability

In this section, we introduce two nonstandard notions of provability. The first of these is fast provability, which can be seen as iterated provability. The second is slow provability, a notion of provability that puts a certain size restriction on the axioms that may be used in a proof. For developing the theory of fast provability, the following technique, that is also used in [4], will prove useful.

Lemma 2.3.1 (Reflexive induction). Let $T \supseteq \mathsf{HA}$ be a theory. Suppose $A$ is an $\mathcal{L}$-formula in one free variable such that $\vdash_{\mathsf{HA}} A[0/x]$ and $\vdash_{\mathsf{HA}} \square_T A \rightarrow A[Sx/x]$. Then $\vdash_{\mathsf{HA}} A$.

Proof. It is provable in intuitionistic predicate logic that $\forall x A \rightarrow A$. So from our assumptions, it follows that $\vdash_{\mathsf{HA}} \square_T (\forall x A) \rightarrow \square_T A \rightarrow A[Sx/x]$. Since we also know that $\vdash_{\mathsf{HA}} A[0/x]$, we get $\vdash_{\mathsf{HA}} \square_T (\forall x A) \rightarrow \forall x A$. Using Löb’s Principle, we can conclude that $\vdash_{\mathsf{HA}} \forall x A$, so $\vdash_{\mathsf{HA}} A$. \hfill \Box

Definition 2.3.1. Let $T$ be a theory.

(i) We define $\mathcal{IB}_{T}(u,x)$ as a formula satisfying

$$
\vdash_{\mathsf{EA}} \mathcal{IB}_{T}(u,x) \leftrightarrow ((u = 0 \land \mathcal{IB}_{T}(x)) \lor \exists v (u = S v \land \square_T \mathcal{IB}_{T}(v,x)))
$$

as provided by the Diagonalization Lemma.

(ii) For an $\mathcal{L}$-formula $A = A(x_1,\ldots,x_n)$, we write $\square^{\#^+}_T A$ for $\mathcal{IB}_{T}(u, \Gamma A(x_1,\ldots,x_n) \}^\#_T$.

(iii) We write $\mathcal{IB}_{\mathsf{P}}^f(x)$ for $\exists u \mathcal{IB}_{T}(u,x)$. For an $\mathcal{L}$-formula $A$, we write $\square^{\#^+}_T A$ for $\exists u \square^{\#^+}_T A$.

Remark 2.3.1. As we shall see shortly, $\mathcal{IB}_{T}$ is a provability predicate for $T$. This notion of provability is called fast provability and was introduced by Parikh in [8]. In this paper, fast provability is introduced in a different way, namely by closing the set of theorems of $T$ under Parikh’s rule ‘from $\vdash \square_T A$, infer $\vdash A’$, where $A$ is an $\mathcal{L}$-sentence. This yields, verifiably in $\mathsf{HA}$, the same notion of provability we defined here. If $T$ is $\Sigma_1$-sound, then Parikh’s rule does not lead to any new theorems, so the notions of ordinary provability and fast provability
2.3. FAST AND SLOW PROVABILITY

coincide. However, the use of Parikh’s rule can lead to much shorter proofs, which explains the name ‘fast provability’. Later in this section, we will show that, if \( T \) is consistent, it is never verifiable in \( \text{HA} \) that fast provability coincides with ordinary provability. \( \diamond \)

We notice that \( \text{IB} \text{ew}_T \) is equivalent, over \( \text{EA} \), to a \( \Sigma_1 \)-formula. Informally, \( \text{IB} \text{ew}_T(u, x) \) can be thought of as the formula \( \text{B} \text{ew}_T(\cdots(\text{Bew}_T(x))\cdots) \), where the \( \text{Bew}_T \) occurs \( u + 1 \) times, so we can see \( \text{IB} \text{ew}_T \) as representing ‘iterated provability’. Notice that we write ‘\( u + 1 \)’ in the superscript of \( \square_T \), to indicate that the \( \square_T \) ‘occurs’ \( u + 1 \) times. We prove a number of technical facts about \( \square_T^{u+1} \) and \( \square_T^f \).

**Lemma 2.3.2.** Let \( T \supseteq \text{HA} \) be a theory and let \( A \) and \( B \) be \( L \)-formulae. Then we have:

(i) \( \vdash_{\text{HA}} \square_T \square_T^{u+1} A \leftrightarrow \square_T^{S^{u+1}A} \leftrightarrow \square_T^{u+1} \square_T A \);

(ii) \( \vdash_{\text{HA}} u \leq v \to (\square_T^{u+1} A \to \square_T^{v+1} A) \);

(iii) \( \vdash_{\text{HA}} \square_T^{u+1} (A \to B) \to (\square_T^{u+1} A \to \square_T^{u+1} B) \);

(iv) \( \vdash_{\text{HA}} \square_T A \to \square_T^f A \);

(v) \( \text{Bew}_T^f \) is a provability predicate for \( T \);

(vi) if \( T \) is \( \Sigma_1 \)-sound, then \( \mathbb{N} \models \square_T^f A \) if and only if \( \mathbb{N} \models \square_T A \), if and only if \( \vdash_T A \);

(vii) \( \vdash_{\text{HA}} \square_T \square_T^f A \leftrightarrow \square_T^f A \).

(viii) if \( T \) is consistent, then \( \forall_{\text{HA}} \square_T^f \perp \to \square_T \perp \).

**Proof.** (i) From the definition of \( \text{IB} \text{ew}_T \), it follows that \( \vdash_{\text{HA}} \square_T \text{IB} \text{ew}_T(u, x) \leftrightarrow \text{IB} \text{ew}_T(S^u, x) \), so the first equivalence is immediate. For the second equivalence, we proceed by reflexive induction. First of all, we have \( \vdash_{\text{HA}} \square_T^{S^{0+1}A} \leftrightarrow \square_T \square_T^{0+1} A \leftrightarrow \square_T \square_T^{0+1} A \leftrightarrow \square_T \square_T A \). Furthermore,

\[
\vdash_{\text{HA}} \square_T (\square_T^{S^{u+1}A} \leftrightarrow \square_T^{u+1} \square_T A) \to \\
[\square_T^{S^{u+1}A} \leftrightarrow \square_T \square_T^{u+1} A \\
\to \square_T \square_T^{u+1} \square_T A \\
\to \square_T^{S^{u+1} \square_T A}],
\]

which completes the proof.

(ii) Since \( \square_T^{u+1} A \) is a \( \Sigma_1 \)-formula, we have \( \vdash_{\text{HA}} \square_T^{u+1} A \to \square_T \square_T^{u+1} A \to \square_T^{S^{u+1}A} \). Now the claim follows by induction on \( v \) inside \( \text{HA} \).

(iii) We proceed by reflexive induction. First of all, we have

\[
\vdash_{\text{HA}} \square_T^{0+1} (A \to B) \leftrightarrow \square_T (A \to B) \to (\square_T A \to \square_T B) \leftrightarrow (\square_T^{0+1} A \to \square_T^{0+1} B).
\]

Furthermore,

\[
\vdash_{\text{HA}} \square_T (\square_T^{u+1} A \to B) \to (\square_T^{u+1} A \to \square_T^{u+1} B)) \to \\
[\square_T^{S^{u+1}A} \to \square_T^{u+1} (A \to B) \\
\to \square_T (\square_T^{u+1} A \to \square_T^{u+1} B) \\
\to (\square_T \square_T^{u+1} A \to \square_T \square_T^{u+1} B) \\
\to (\square_T^{S^{u+1}A} \to \square_T^{S^{u+1} B})],
\]
which completes the proof.

(iv) This is immediate as \( \vdash_{\text{HA}} \Box T A \leftrightarrow \Box^{0+1} T A \).

(v) This follows easily from (ii), (iii) and (iv).

(vi) The second statement was already asserted in Proposition 2.1.4(i). So we prove the first statement. The right-to-left direction follows from (iv). For the converse, suppose that \( \mathbb{N} \models \Box^{n+1} T A \) for a certain \( n \in \mathbb{N} \). If \( n = 0 \), then we are done. So suppose that \( n = m + 1 \) for a certain \( m \geq 0 \). Then \( \mathbb{N} \models \Box T \Box^{m+1} T A \), so \( \vdash_{T} \Box^{m+1} T A \). Since \( T \) is \( \Sigma_1 \)-sound, we see that \( \mathbb{N} \models \Box^{m+1} T A \). By repeating this argument, we find \( \mathbb{N} \models \Box^{T} A \), as desired.

(vii) This follows from
\[
\vdash_{\text{HA}} \Box^{f} T \perp \rightarrow \Box^{f} T A
\]
and
\[
\vdash_{\text{HA}} \Box^{f} T A \rightarrow \Box^{S_{u}+1} T A \rightarrow \Box^{f} T \perp
\]
so by Löb’s Theorem, we get \( \vdash_{\text{HA}} \Box T \perp \). We conclude that \( \vdash_{T} \perp \).

Now we prove the analogue of Corollary 2.2.6 for fast provability.

**Lemma 2.3.3.** Suppose \( \text{HA} \leq U \leq T \) are theories and let \( A \) be an \( \mathcal{L} \)-formula. Then we have
\[
\vdash_{\text{HA}} \Box^{u+1} U T A \leftrightarrow \Box^{u+1} U A T
\]
In particular, \( \vdash_{\text{HA}} \Box^{f} U T A \leftrightarrow \Box^{f} U A T \).

**Proof.** We proceed by reflexive induction. First of all, by Corollary 2.2.6, we have
\[
\vdash_{\text{HA}} \Box^{0+1} U T A \leftrightarrow \Box^{0+1} U A T
\]
Furthermore, we have
\[
\vdash_{\text{HA}} \Box^{u+1} U T (A) \rightarrow \Box^{u+1} U T (A) \rightarrow \Box^{S_{u}+1} U T (A)
\]
This completes the proof.

Now we turn to slow provability. We will not give as many details as we did for fast provability, but instead we will refer to the paper [5]. There are two reasons for this. First of all, developing the theory of slow provability is rather involved, so reasons of space do not permit us to provide all the details. The second reason involves our intended usage of fast and slow provability. In Chapter 4, we will obtain results about the provability logic of fast provability. In order to understand and appreciate these results, it is important to know what fast provability is, exactly. Slow provability, on the other hand, will only be used as a tool to obtain results that themselves do not mention slow provability. In order to understand these results, it is not necessary to know all the details about slow provability.

In the paper [5], the authors define a certain ‘fast-growing’ total recursive function \( F : \mathbb{N} \rightarrow \mathbb{N} \). There exists a \( \Sigma_1 \)-formula \( \varphi_F(x, y) \) representing \( F \) in \( \text{HA} \). This means that the definition of \( F \) as a total recursive function is verifiable in \( \text{HA} \), and we have
\[
\vdash_{\text{HA}} \forall y (\varphi_F(n, y) \leftrightarrow y = F(n)) \quad \text{for all } n \in \mathbb{N}.
\]
The $\Sigma_1$-formula $F(x) \downarrow$, which we read as ‘$F(x)$ is defined’, is shorthand for $\exists y \varphi_F(x, y)$. We clearly have that $\vdash_{HA} F(n) \downarrow$ for all $n \in \mathbb{N}$. However, the fast-growing function $F$ is constructed in such a way that $F$ is not provably total. That is, we do not have $\vdash_{HA} F(x) \downarrow$. Now we are ready to define slow provability.

**Definition 2.3.2.** The theory slow Heyting Arithmetic, denoted $sHA$, is given by the axiom formula

$$Ax_{sHA}(x) :\iff Ax_{EA}(x) \lor (Ax_{sHA}(x) \land \exists y \geq x(F(y) \downarrow)).$$

Intuitively, we demand that the axioms we use must not be ‘too large’: they must be not be so large that they are entirely beyond the domain of $F$. Since $F$ is in fact total, we have $\mathbb{N} \models Ax_{sHA}(x) :\iff Ax_{HA}(x)$, which means that $HA = sHA$. We also clearly have that $\vdash_{HA} Ax_{sHA}(x) \rightarrow Ax_{HA}(x)$, so $sHA \leq HA$. However, as we shall show shortly, we do not have $HA \leq sHA$. So from the viewpoint of $HA$, the requirement that the axioms must not be too large is a genuine one.

Even though the base theory used in the paper [5] is the classical theory $PA$, many results carry over to the present case. The most important of these is:

**Proposition 2.3.4.** We have $\vdash_{HA} \text{Bew}_{HA}(x) \rightarrow \Box_{HA}\text{Bew}_{sHA}(x)$, and in particular, we have $\vdash_{HA} \Box_{HA}A \leftrightarrow \Box_{HA}A^{sHA}$ for all $L$-formulae $A$.

**Proof.** The first statement is proven as in [5], Corollary 15, taking $S_n$ to be the theory axiomatized by the axioms of $HA$ having Gödel number at most $n$. The second statement follows from Corollary 2.2.6 with $U \equiv HA$ and $T \equiv sHA$. 

The converse of this result, which is valid for the classical case, does not carry over to the current setting, because the authors use a model theoretic argument to derive this result. However, we will only need a very weak version of this converse, which we can ‘steal’ from the classical case. This proof was suggested by A. Visser.

**Proposition 2.3.5.** (i) For all $\Sigma_1$-sentences $S$, we have $\vdash_{HA} \Box_{HA}\Box_{sHA}S \rightarrow \Box_{HA}S$.

(ii) We have $\forall_{HA} \Box_{HA}\perp \rightarrow \Box_{sHA}\perp$. In particular, $HA \nleq sHA$.

**Proof.** (i) We define the analogue of slow provability for $PA$, e.g. by setting

$$Ax_{sPA}(x) :\iff Ax_{EA}(x) \lor (Ax_{sPA}(x) \land \exists y \geq x(F(y) \downarrow)).$$

Since $\vdash_{HA} Ax_{HA}(x) \rightarrow Ax_{PA}(x)$, it is clear that $HA \leq PA$ and $sHA \leq sPA$. We know from [5], Theorem 4, that $\vdash_{PA} \Box_{PA}\Box_{sPA}S \rightarrow \Box_{PA}S$. So we get

$$\vdash_{PA} \Box_{HA}\Box_{sHA}S \rightarrow \Box_{PA}\Box_{sPA}S \rightarrow \Box_{PA}S \rightarrow \Box_{HA}S,$$

where the final step holds since $PA$ is, verifiably in $HA$, $\Sigma_1$-conservative over $HA$. We notice that $\Box_{HA}\Box_{sHA}S \rightarrow \Box_{HA}S$ is equivalent, over $HA$, to a $\Pi_2$-sentence, that is, a sentence of the form $\forall x R(x)$, where $R \in \Sigma_1$. Since $PA$ is $\Pi_2$-conservative over $HA$, we also find that $\vdash_{HA} \Box_{HA}\Box_{sHA}S \rightarrow \Box_{HA}S$.

(ii) Suppose that $\vdash_{HA} \Box_{HA}\perp \rightarrow \Box_{sHA}\perp$. Since $\perp \in \Sigma_1$, we have

$$\vdash_{HA} \Box_{HA}\Box_{sHA}\perp \rightarrow \Box_{HA}\Box_{sHA}\perp \rightarrow \Box_{HA}\perp,$$

so by Löb’s Theorem, we get $\vdash_{HA} \Box_{HA}\perp$. But then $HA$ is inconsistent, contradiction. 

\[\square\]
2.4 Intuitionistic Modal Logic

In this section, we briefly review intuitionistic modal logic, abbreviated IML, and we define the system of IML that will be relevant to us. The language $L$ of IML has a countable set of propositional constants, the absurdity sign $\bot$, the usual binary connectives $\land$, $\lor$ and $\to$, and the unary sentential operator $\Box$. We shall also use $L$ to denote the set of all $L$-formulae.

As our proof system, we pick a Hilbert-style system that has two inference rules:

\[
\frac{A \to B}{\to \text{E}}\quad \text{and} \quad \frac{A}{\Box A - \text{Nec}}.
\]

**Definition 2.4.1.** (i) The set $iK \subseteq L$ is the smallest set that contains:

(a) all ($L$-substitution instances of) tautologies of intuitionistic propositional logic;
(b) all $L$-sentences of the form $\Box(A \to B) \to (\Box A \to \Box B)$, where $A, B \in L$,

and is closed under $\to \text{E}$ and $\text{Nec}$.

(ii) A *theory for IML* will be a set $T$ that satisfies $iK \subseteq T \subseteq L$ and is closed under $\to \text{E}$ and $\text{Nec}$. If $A \in L$ and $\Gamma \subseteq L$, we write $\Gamma \vdash T A$ if there exists a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\bigwedge \Gamma_0 \to A$ is in $T$.

(iii) The theory $iGL$ is the smallest theory for IML that contains $iK$ and all sentences of the form $\Box(\Box A \to A) \to \Box A$, where $A \in L$.

(iv) The theory $iGLC$ is the smallest theory for IML contains $iGL$ and all sentences of the forms $A \to \Box A$, where $A \in L$.

We now proceed to define the semantics of modal intuitionistic logic.

**Definition 2.4.2.** (i) Consider a triple $\langle W, \leq, \sqsubseteq \rangle$, where $W$ is a nonempty set and $\leq$ and $\sqsubseteq$ are binary relations on $W$. We say that this triple satisfies the *model property* if $\leq \circ \sqsubseteq$ is a subrelation of $\sqsubseteq$. That is, for all $w, v, u \in W$ we should have: if $w \leq v \sqsubseteq u$, then $w \sqsubseteq u$.

(ii) A *frame for IML* is a triple $\langle W, \leq, \sqsubseteq \rangle$, where $W$ is a nonempty set and $\leq$ and $\sqsubseteq$ are binary relations on $W$, such that: $\langle W, \leq \rangle$ is a poset and $\langle W, \leq, \sqsubseteq \rangle$ satisfies the model property.

(iii) A *model for IML* is a quadruple $\langle W, \leq, \sqsubseteq, V \rangle$, where $\langle W, \leq, \sqsubseteq \rangle$ is a frame for IML and $V$ is a relation (called the *valuation*) between $W$ and the proposition letters from $L$ satisfying:

\[
w \leq v \text{ and } wVp \text{ implies } vVp,
\]

for all $w, v \in W$ and proposition letters $p$.

(iv) Let $M = \langle W, \preceq, \sqsubseteq, V \rangle$ be a model for IML, let $w \in W$ and let $A \in L$. We define the forcing relation $M, w \models A$ by recursion on $A$, as follows. For all $B, C \in L$, we set:

(a) $M, w \models p$ if $wVp$ for all proposition letters $p$;
(b) $M, w \models B \land C$ iff $M, w \models B$ and $M, w \models C$;
(c) $M, w \models B \lor C$ iff $M, w \models B$ or $M, w \models C$;
(d) $M, w \models B \to C$ iff for all $v \in W$ such that $w \preceq v$ and $M, v \models B$, we have $M, v \models C$;
(e) $M, w \models \Box B$ iff for all $v \in W$ such that $w \sqsubseteq v$, we have $M, v \models B$.
If $M$ is understood, we just write $w \models A$ instead of $M, w \models A$. We write $M \models A$ if $M, w \models A$ for all $w \in W$, in which case we say that $A$ is valid on $M$. Given a frame $\langle W, \preceq, \sqsubseteq \rangle$ for IML, we say that $A \in L$ is valid on this frame iff for all models $M = \langle W, \preceq, \sqsubseteq, V \rangle$ for IML, we have that $A$ is valid on $M$.

Usually, one writes ‘$\mathcal{R}$’ for the modal relation we call ‘$\sqsubseteq$’ here. Our notation has certain advantages that will become apparent in the next chapter. We impose the model property on our frames because we want the following result:

**Proposition 2.4.1** (Preservativity of Knowledge). Let $M = \langle W, \preceq, \sqsubseteq, V \rangle$ be a model for IML. If we have $w, v \in W$ and $A \in L$ such that $w \models A$ and $w \preceq v$, then $v \models A$.

**Proof.** We proceed by induction on the complexity of $A$. The base case and the induction steps for conjunction, disjunction and implication are trivial. So suppose that $A$ is $\Box B$ and that we have $w, v \in W$ such that $w \preceq v$ and $w \models \Box B$. Consider any $u \in W$ such that $v \sqsubset u$. Then $w \preceq v \sqsubset u$, so since $\langle W, \preceq, \sqsubseteq \rangle$ has the model property, we get $w \models \Box B$. Since $w$ was arbitrary, we can conclude that $v \models \Box B$, as desired. \qed

For our purposes, the relevant frame properties are the following.

**Definition 2.4.3.** Let $\langle W, \preceq, \sqsubseteq \rangle$ be a frame for IML.

(i) We say that this frame is transitive if $\sqsubseteq \circ \sqsubseteq$ is a subrelation of $\sqsubseteq$.

(ii) We say that this frame is semi-transitive if $\sqsubseteq \circ \sqsubseteq$ is a subrelation of $\sqsubseteq \circ \preceq$.

(iii) We say that this frame is realistic if $\sqsubseteq$ is a subrelation of $\preceq$.

(iv) We say that this frame is conversely well-founded if every nonempty subset of $W$ has a maximal element w.r.t. $\sqsubseteq$.

We say that a model for IML has one of the properties mentioned above iff the underlying frame has it.

The terminology from (iii) is not standard and was suggested by R. Iemhoff. The idea behind it is as follows. We can view $\sqsubseteq$ as an accessibility relation that is relative to the various worlds, while $\preceq$ represents the ‘real’ accessibility between worlds. If, in a realistic frame, a world $w$ thinks that some world $v$ is accessible, then $v$ is also really accessible from $w$. We observe that, due to the model property, a realistic frame is automatically transitive. Indeed, suppose that $\langle W, \preceq, \sqsubseteq \rangle$ is a realistic frame for IML and suppose we have $w, v, u \in W$ such that $w \sqsubset v \sqsubset u$. Then we also have $w \preceq v \sqsubset u$, so $w \sqsubset u$ follows, as desired.

Now we relate our frame properties to the axioms of iGLC.

**Proposition 2.4.2.** Let $F = \langle W, \preceq, \sqsubseteq \rangle$ be a frame for intuitionistic modal logic.

(i) The sentence $\Box (\Box p \rightarrow p) \rightarrow \Box p$ is valid on $F$ if and only if $F$ is semi-transitive and conversely well-founded.

(ii) The sentence $p \rightarrow \Box p$ is valid on $F$ if and only if $F$ is realistic.

In particular, all theorems of iGLC are valid on all realistic and conversely well-founded frames.

**Proof.** (i) This result is known from the literature. We refer the reader to the paper [6], Lemma 8.
(ii) First, suppose that $F$ is realistic. Let $V$ be a valuation on $F$, and suppose we have $w \in W$ such that $w \models p$. If $v \in W$ is such that $w \subseteq v$, then also $w \leq v$, so by preservativity of knowledge, we get $v \models p$. We conclude that $w \models \Box p$, and thus that $p \rightarrow \Box p$ is valid on $F$.

Conversely, suppose that $F$ is not realistic. Then there exist $w, v \in K$ such that $w \subseteq v$, but also $w \nsubseteq v$. We define a valuation $V$ on $F$ such that

$$xVp \text{ if and only if } w \leq x.$$ 

Then $wVp$, but since $w \subseteq v$ and $\neg(vVp)$, we also have $w \nsubseteq \Box p$. We conclude that $w \nsubseteq p \rightarrow \Box p$ and thus that $p \rightarrow \Box p$ is not valid on $F$.

The final statement is easily proven by an induction on $\iGLC$-proofs. \qed

In order to get a completeness theorem, we need the following terminology.

**Definition 2.4.4.** Let $T$ be a theory for intuitionistic modal logic.

(i) A set $X \subseteq \mathcal{L}_\Box$ is called adequate if it is closed under taking subformulæ.

(ii) Suppose $X \subseteq \mathcal{L}_\Box$ is adequate. A set $S \subseteq X$ is called $X$-saturated if the following hold:
   
   (a) $S$ is consistent, that is, $S \not\vdash \bot$;
   
   (b) if $A \in X$ and $S \vdash_T A$, then $A \in S$;
   
   (c) if $A \lor B \in S$, then $A \in S$ or $B \in S$.

Notice that the converse of item (b) also holds: if $A \in S$, then clearly $A \in X$ and $S \vdash_T A$. We will need the following result.

**Lemma 2.4.3 (Extension Lemma).** Let $T$ be a theory for intuitionistic modal logic and let $X \subseteq \mathcal{L}_\Box$ be an adequate set. Suppose we have $R \subseteq X$ and $A \in \mathcal{L}_\Box$ such that $R \not\vdash_T A$. Then there exists an $X$-saturated set $S \supseteq R$ such that $S \not\vdash_T A$.

**Proof.** We fix an enumeration $B_0, B_1, B_2, \ldots$ of the formulæ in $X$ such that every element of $X$ occurs infinitely many times in the enumeration. We define the sequence $S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots$ by recursion. First of all, we set $S_0 = R$. Now suppose that $S_n$ has been defined. If $S_n \not\vdash_T B_n$, then $S_{n+1}$ is just $S_n$. If $S_n \vdash_T B_n$, then

$$S_{n+1} = \begin{cases} 
S_n \cup \{B_n\} & \text{if } B_n \text{ is not a disjunction; } \\
S_n \cup \{B_n, C\} & \text{if } B_n \text{ is } C \lor D, \text{ and } S_n \cup \{C\} \not\vdash_T A; \\
S_n \cup \{B_n, D\} & \text{if } B_n \text{ is } C \lor D, \text{ and } S_n \cup \{C\} \vdash_T A;
\end{cases}$$

We define $S$ as $\bigcup_{n \in \mathbb{N}} S_n$. Clearly, we have $S_n \subseteq X$ for all $n \in \mathbb{N}$, so $S \subseteq X$.

Now we use induction on $n$ to prove that $S_n \not\vdash_T A$ for all $n \in \mathbb{N}$. For $n = 0$, this holds by assumption. Now suppose that $S_n \not\vdash_T A$ for a certain $n \in \mathbb{N}$; we need to show that $S_{n+1} \not\vdash_T A$.

If $S_n \not\vdash_T B_n$, then this holds trivially. So suppose that $S_n \vdash_T B_n$. Then we must have that $S_n \cup \{B_n\} \not\vdash_T A$, so if $B_n$ is not a disjunction, then we are also done. So suppose that $B_n$ is $C \lor D$. If $S_n \vdash_T \{C\} \not\vdash_T A$, then we also have $S_{n+1} = S_n \cup \{B_n, C\} \not\vdash_T A$, so we are done.

Finally, suppose that $S_n \cup \{C\} \vdash_T A$. Then we cannot have $S_n \cup \{D\} \vdash_T A$. Indeed, if we have both $S_n \cup \{C\} \vdash_T A$ and $S_n \cup \{D\} \vdash_T A$, then also $S_n \cup \{C \lor D\} \vdash_T A$, which is not the case. So $S_n \cup \{D\} \not\vdash_T A$, and it follows that $S_{n+1} = S_n \cup \{B_n, D\} \not\vdash_T A$, as desired. This completes the induction.
It follows that $S \not\vDash T A$, and in particular, $S$ is consistent. We check that $S$ is $X$-saturated. Now suppose that $C \in X$ and $S \vDash T C$. Then there must be an $n \in \mathbb{N}$ such that $S_n \vDash T C$. Let $m \geq n$ be minimal such that $B_m$ is $C$. Then $S_m \vDash T B_m$, so we get $B_m \in S_{m+1} \subseteq S$, that is $C \in S$. Finally, suppose that $C \lor D \in S$. Then there must be an $n \in \mathbb{N}$ such that $C \lor D \in S_n$. Let $m \geq n$ be minimal such that $B_m$ is $C \lor D$. Then $B_m \in S_n \subseteq S_m$, so we certainly have $S_m \vDash T B_m$. It follows that $C \in S_m \subseteq S$ or $D \in S_m \subseteq S$. This concludes the proof. 

Using the Extension Lemma, we can prove a sound- and completeness theorem for iGLC. This result also appears, in a stronger form, as Theorem 4.25 in [1].

**Theorem 2.4.4.** Let $A$ be an $L_{\Box}$-sentence. Then $\vdash_{iGLC} A$ if and only if $A$ is valid on all finite irreflexive realistic frames.

**Proof.** It is well-known that any finite irreflexive transitive frame is conversely well-founded. So if $\vdash_{iGLC} A$, then $A$ is indeed valid on all finite irreflexive realistic frames, by Proposition 2.4.2. Conversely, suppose that $\not\vdash_{iGLC} A$. Let $X_0$ be the set of subformulae of $A$, and let $X_1 = \{ \Box B \mid B \in X_0 \}$. Then $X := X_0 \cup X_1$ is an adequate set. We let $W$ be the set of all X-saturated sets. Clearly, $W$ is finite, and we have the subset relation $\subseteq$ on $W$. For $w, v \in W$, we write $w \sqsubseteq v$ if:

(i) whenever $B \in L_{\Box}$ and $\Box B \in w$, we have $B \in v$;

(ii) there exists a $C \in L_{\Box}$ such that $\Box C \not\in w$ and $\Box C \in v$.

For $w \in W$ and $p$ a proposition letter, we say that $w Vp$ if and only if $p \in w$. We clearly have: if $w Vp$ and $w \sqsubseteq v$, then $v Vp$. It is also not difficult to check that $(W, \subseteq, \sqsubseteq)$ satisfies the model property. Finally, since $\not\vdash_{iGLC} A$, there exists a $w_0 \in W$ such that $w_0 \not\vDash A$, by the Extension Lemma. In particular, $W$ is nonempty, so $M = \langle W, \subseteq, \sqsubseteq, V \rangle$ is a model for intuitionistic modal logic.

We claim that the frame $\langle W, \subseteq, \sqsubseteq \rangle$ is irreflexive and realistic. Irreflexivity is immediate from the definition. Now suppose we have $w, v \in W$ such that $w \sqsubseteq v$, and $B \in w$. If $B \in X_0$, then $\Box B \in X_1 \subseteq X$ and $w \vdash_{iGLC} \Box B$, so $\Box B \in w$. Since $w \sqsubseteq v$ we get $B \in v$. Now suppose that $B \in X_1$. Then $B$ is $\Box C$ for some $C \in X_0$. Since $w \sqsubseteq v$, we get $C \in v$. This means that $v \vdash_{iGLC} B$, so $B \in v$. In both cases, we get $B \in v$, so we conclude that $w \subseteq v$, as desired.

Now we show that $w \vDash B$ if and only if $B \in w$, for all $w \in W$ and $B \in X$. We proceed by induction on the complexity of $B$.

At For proposition letters, the result holds by the definition of $V$.

$\wedge$ Suppose that $B$ is $C \land D$ and that the result holds for $C$ and $D$. If $w \in W$, then $w \vDash C \land D$ if $w \vDash C$ and $w \vDash D$, iff $C \in w$ and $D \in w$. Now suppose that $C \in w$ and $D \in w$. Then $w \vdash_{iGLC} C \land D$ and $C \land D \in X$, so $C \land D \in w$. Conversely, suppose that $C \land D \in w$. Then $w \vdash_{iGLC} C, D$ and $C, D \in X$, so we get $C \in w$ and $D \in w$.

$\lor$ Suppose that $B$ is $C \lor D$ and that the result holds for $C$ and $D$. If $w \in W$, then $w \vDash C \lor D$ if $w \vDash C$ or $w \vDash D$, iff $C \in w$ or $D \in w$. Suppose that $C \in w$ or $D \in w$. Then in both cases, we have $w \vdash_{iGLC} C \lor D$. Since $C \lor D \in X$, we get $C \lor D \in w$. Conversely, if $C \lor D \in w$, then $C \in w$ or $D \in w$ since $w$ is $X$-saturated.

$\to$ Suppose that $B$ is $C \to D$ and that the result holds for $C$ and $D$. If $w \in W$, then $w \vDash C \to D$ if for all $v \supseteq w$, we have that $v \vDash C$ implies $v \vDash D$. And this holds iff for all $v \supseteq w$, we have that $C \in v$ implies $D \in v$. Now suppose that $C \to D \in w$ and that we have $v \supseteq w$ such that $C \in v$. Then also $C \to D \in v$, so $v \vdash_{iGLC} D$. Since $D \in X$,
we get $D \in v$. Conversely, suppose that $C \rightarrow D \not\in w$. Since $C \rightarrow D \in X$, this means that $w \not\models \text{GLC} C \rightarrow D$, and hence $w \cup \{C\} \not\models \text{GLC} D$. Since $w \cup \{C\} \subseteq X$, we can use the Extension Lemma to find a $v \in W$ such that $w \cup \{C\} \subseteq v$ and $v \not\models \text{GLC} D$. Then $w \subseteq v$, $C \in v$, and $D \not\in v$, so it follows that $w \not\models B \rightarrow C$. 

\[\square\]

Suppose that $B$ is $\Box C$ and that the result holds for $C$. If $w \in W$, then $w \models \Box C$ iff for all $v \supseteq w$, we have $v \models C$. And this holds iff for all $v \supseteq w$, we have $C \in v$. Now suppose that $\Box C \in w$ and that we have $v \supseteq w$. Then by the definition of $\supseteq$, we get $C \in v$. Conversely, suppose that $\Box C \not\in w$. Consider the set $R = \{D \in \mathcal{L}_\Box | \Box D \in w\} \cup \{\Box C\} \subseteq X$. Suppose that $R \not\models \text{GLC} \Box C \rightarrow C$. Then $\{D \in \mathcal{L}_\Box | \Box D \in w\} \not\models \text{GLC} \Box (\Box C \rightarrow C)$, so we also get $\{\Box D \in \mathcal{L}_\Box | \Box D \in w\} \not\models \text{GLC} \Box (\Box C \rightarrow C)$. In particular, $w \not\models \text{GLC} \Box (\Box C \rightarrow C)$, which yields $w \not\models \text{GLC} \Box C$. However, we also have $\Box C \in X$, so we get $\Box C \in w$, contradiction. So $R \not\models \text{GLC} \Box C$. By the Extension Lemma, there exists a $v \in W$ such that $R \subseteq v$ and $v \not\models \text{GLC} \Box C$. We have $\{D \in \mathcal{L}_\Box | \Box D \in w\} \subseteq v$, $\Box C \not\in w$ and $\Box C \in v$, so $w \not\supseteq v$. Furthermore, we have $C \not\in v$, so $w \not\models \Box C$.

This completes the induction. Since $w_0 \not\models \text{GLC} A$, we have $A \not\in w$. Since $A \in X$, we can apply the above result to conclude that $w_0 \not\models A$. So $A$ is not valid on the finite irreflexive realistic frame $(W, \subseteq, \supseteq)$. \[\square\]
Chapter 3

An Abstract Arithmetical Completeness Theorem

In this chapter, we prove a completeness theorem for certain kinds of provability logics. We prove the theorem in a rather abstract form, not yet mentioning any specific provability predicates. In Section 3.1, we introduce the general framework and define the required Solovay function along with the intended realization of the propositional letters of $\mathcal{L}_\square$. Section 3.2 is of a rather technical nature and forms the heart of the proof. Here we show that the realization we defined commutes with the logical operators of $\mathcal{L}_\square$. In Section 3.3, we formulate the completeness theorem and use the preceding material to prove it.

3.1 Definition of the Solovay Function

The general setting of this chapter is given by the following definition.

**Definition 3.1.1.** Let $T \supseteq \text{HA}$ be a theory and let $S(x)$ and $R(x)$ be $\Sigma_1$-formulae in one free variable. If $A$ is an $\mathcal{L}$-sentence, we write $\square A$ for $S(\langle A \rangle)$. We also write $\triangle A$ for $R(\langle A \rangle)$. We say that $(S,R)$ is a **good pair** for $T$ if the following conditions are satisfied:

1. $S$ and $R$ are provability predicates for $T$;
2. if $\mathbb{N} \models \square A$, then $\vdash_T A$, for all $\mathcal{L}$-sentences $A$;
3. $\vdash_T \text{SLP}_\triangle$ (or equivalently, $\vdash_T \text{CP}_\triangle$);
4. $\vdash_{\text{HA}} \square \triangle S \rightarrow \square S$ for all $\Sigma_1$-sentences $S$.

We immediately observe that, if these clauses apply and $S$ is a $\Sigma_1$-sentence, then we also have $\vdash_{\text{HA}} \triangle S \rightarrow \square \triangle S \rightarrow \square S$. We also notice that, since $\text{HA} \subseteq T$, we have that $\vdash_{\text{HA}} A$ implies $\vdash_T A$, which implies $\vdash_{\text{HA}} \square A$ and $\vdash_{\text{HA}} \triangle A$ for all $\mathcal{L}$-sentences $A$.

**Remark 3.1.1.** We remark that the definition of a good pair does not occur anywhere in the literature. This definition is extremely artificial and tailor made to obtain the result of this chapter.
In the remainder of this chapter, we suppose that a theory \( T \) extending \( \text{HA} \) and a good pair \((S,R)\) for \( T \) are given. We also use \( \Box \) and \( \triangle \) as defined above.

Let \( M_0 = \langle W_0, \preceq_0, \sqsubseteq_0, V_0 \rangle \) be a finite irreflexive realistic model for IML such that \( W_0 \) has a least element w.r.t. \( \preceq_0 \). Let \( r > 0 \) be the cardinality of \( W_0 \). We assume that \( W_0 = \{1, \ldots, r\} \) and that the node \( r \) is the least element of \( W_0 \) w.r.t. \( \preceq_0 \). Now we expand \( M_0 \) to a new model \( M = \langle W, \preceq, \sqsubseteq, V \rangle \) for IML. Intuitively, we append a copy of \( 1 + \omega^{op} \) (in the \( \sqsubseteq \)-order relation) to the node \( r \). Formally, we do this as follows. We take \( W = \mathbb{N} \supset W_0 \). The relation \( \preceq \) is defined by:

\[
i \preceq j \iff 1 \leq i, j \leq r \text{ and } i \preceq_0 j,
\text{ or } i > r \text{ and } 1 \leq j \leq i,
\text{ or } i = 0,
\]

for all \( i, j \in \mathbb{N} \). The relation \( \sqsubseteq \) is defined by:

\[
i \sqsubseteq j \iff 1 \leq i, j \leq r \text{ and } i \sqsubseteq_0 j,
\text{ or } i > r \text{ and } 1 \leq j < i,
\text{ or } i = 0 \text{ and } j > 0,
\]

for all \( i, j \in \mathbb{N} \). Finally, \( V \) is defined by:

\[
i V p \iff 1 \leq i \leq r \text{ and } i V_0 p,
\]

for all \( i \in \mathbb{N} \) and proposition letters \( p \).

We can prove that \( M \) is again a realistic irreflexive model for IML; but of course \( M \) is no longer finite. However, \( M \) is conversely well-founded, so \( M \) still validates all theorems of \( \text{IGLC} \). Since \( \preceq_0 \) and \( \sqsubseteq_0 \) are finite relations, we can give \( \Delta_0 \)-definitions of these relations inside \( \text{HA} \). Now we can formalize the definitions of \( \preceq \) and \( \sqsubseteq \) given above in order to obtain \( \Delta_0 \)-definitions of \( \preceq \) and \( \sqsubseteq \) inside \( \text{HA} \). Then \( \text{HA} \) verifies the relevant properties of \( M \): that \( \preceq \) is a poset, that \( \sqsubseteq \) is irreflexive, that \( \langle W, \preceq, \sqsubseteq \rangle \) has the model property, and that this frame is realistic. E.g. by verification of the model property we mean that \( \vdash_{\text{HA}} x \preceq y \wedge y \sqsubseteq z \rightarrow x \sqsubseteq z \).

Since \( \preceq \) is defined by a \( \Delta_0 \)-formula, we have: if \( i \preceq j \), then \( \vdash_{\text{HA}} i \preceq j \), and if \( i \not\preceq j \), then \( \vdash_{\text{HA}} \neg (i \preceq j) \). A similar result holds for \( \sqsubseteq \). Moreover, by Proposition 2.1.3(i), we have safely make case distinctions like \( x \preceq y \vee \neg (x \preceq y) \) inside \( \text{HA} \). Since we assumed that \( \text{HA} \subseteq T \), all these remarks also hold for \( T \) instead of \( \text{HA} \).

For an \( A \in \mathcal{L}_{\Box} \), we define the set \([A]\) as \( \{i \in \mathbb{N} \mid i \models A\} \). The model \( M \) is constructed in such a way that the following result holds.

**Lemma 3.1.1.** If \( A \in \mathcal{L}_{\Box} \), then \([A]\) is finite or \([A] = \mathbb{N} \).

**Proof.** We have to show the following: if \( i \in [A] \) for all \( i > 0 \), then \( 0 \in [A] \). We proceed by induction on the complexity of \( A \). The atomic case clearly holds, and the steps for \( \wedge \) and \( \vee \) are trivial. Now suppose that \( A \) is \( B \rightarrow C \) and that the claim holds for \( B \) and \( C \). Suppose that \( i \in [B \rightarrow C] \) for all \( i > 0 \), and that \( 0 \not\in [B \rightarrow C] \). Then we must have \( 0 \in [B] \) and \( 0 \not\in [C] \) as well. By the induction hypothesis, \( i \not\in C \) for some \( i > 0 \). However, since \( 0 \preceq i \), we also have \( i \in [B] \), so \( i \not\in [B \rightarrow C] \), contradiction. Finally, suppose that \( A = \Box B \) and that
the claim holds for $B$. Suppose that $i \in \Box B$ for all $i > 0$. We should show that $0 \in \Box B$.

By preservativity of knowledge, it suffices to show that $j \in [B]$ for all $j \geq r$. But for such $j$, we have $j + 1 \in [\Box B]$ by assumption, and $j + 1 \equiv j$, so we indeed have $j \in [B]$. $\square$

We now proceed to define the Solovay function. Our models are equipped with two relations, as opposed to just one in the classical case, and we need to find some way to incorporate this into the Solovay function. A. Visser suggested to use two separate provability predicates to take care of the relations $\leq$ and $\equiv$. This is where our good pair comes in. Since $S(x)$ and $R(x)$ are $\Sigma_1$-formulae, we can write $S(x)$ as $\exists y \Prf$\hspace{0.2cm}©$ (y, x)$ and $R(x)$ as $\exists y \Prf$\hspace{0.2cm}△$ (y, x), where $\Prf$\hspace{0.2cm}©$ and $\Prf$\hspace{0.2cm}△$ are $\Delta_0$-formulae.

Let $\langle \cdot, \cdot \rangle : \mathbb{N}^2 \to \mathbb{N}$ be a primitive recursive pairing function that can be formulated inside $\text{HA}$ using a $\Delta_0$-formula. Let $p_0 : \mathbb{N} \to \mathbb{N}$ be the primitive recursive function that gives the projection onto the first coordinate. By replacing $\Prf$\hspace{0.2cm}©$ (y, x)$ with $\exists z \leq y (y = (x, z) \land \Prf$\hspace{0.2cm}©$ (z, x))$, we may assume without loss of generality that

$$\vdash \text{HA} \Prf$\hspace{0.2cm}©$ (y, x) \to x = p_0(y). \quad (3.1)$$

We do the same for $\Prf$\hspace{0.2cm}△$.

In the sequel, we write $x \prec y$ for $x \leq y \land \neg (x = y)$ and $x \sqsubseteq y$ for $x \equiv y \lor x = y$. We define the function $h : \mathbb{N} \to \mathbb{N}$ by $h(0) = 0$ and

$$h(k + 1) = \begin{cases} 
  m & \text{if } h(k) \sqsubseteq m \text{ and } \Prf$\hspace{0.2cm}©$ (k, \vdash \exists x \neg (h(x) \sqsubseteq m)); \\
  n & \text{if } h(k) \prec n \text{ and } \Prf$\hspace{0.2cm}△$ (k, \vdash \exists y \neg (h(y) \leq n)); \\
  h(k) & \text{if neither of these apply.}
\end{cases}$$

Here $x$ and $y$ are two (syntactically) distinct variables, so by our assumption (3.1) above, the first two clauses can never apply simultaneously. Using (3.1) again, we also see that $m$ as in the first clause, if it exists, is unique, and similarly for the second clause. So $h$ is well-defined. Using the Diagonalization Lemma, we can give a $\Sigma_1$-definition of $h$ inside $\text{EA}$. The argument that $h$ is well-defined above can then be formalized inside $\text{HA}$, so $\text{HA}$ proves that $h$ is a function. We also have $\vdash \text{HA} x \leq y \to h(x) \leq h(y)$.

Notice that it is in some sense ‘easier’ to move along $\sqsubseteq$ than it is to move along $\preceq$. We have $\vdash \text{HA} \exists y \neg (h(y) \leq m) \to \exists x \neg (h(x) \sqsubseteq m)$. Since $\text{HA} \subseteq T$ and $R$ is a provability predicate for $T$, we also find that $\vdash \text{HA} \Delta (\exists y \neg (h(y) \leq m)) \to \Delta (\exists x \neg (h(x) \sqsubseteq m))$. We also observe that $\exists x \neg (h(x) \sqsubseteq m)$ is equivalent, over $\text{EA}$, to a $\Sigma_1$-sentence. This means that we also have $\vdash \text{HA} \Delta (\exists x \neg (h(x) \sqsubseteq m)) \to \Box (\exists x \neg (h(x) \sqsubseteq m))$. We conclude that

$$\vdash \text{HA} \Delta (\exists y \neg (h(y) \leq m)) \to \Box (\exists x \neg (h(x) \sqsubseteq m)), \quad (3.2)$$

for any $m \in \mathbb{N}$. We will need this in the sequel. We also need the following observation: if $i \neq 0$ is a natural number, then

$$\vdash \text{HA} \neg (x \leq i) \leftrightarrow \bigvee_{j \in U} x = j, \quad (3.3)$$
where \( U = \{ j \in \mathbb{N} \mid j \neq i \} \) is finite. In other words, if HA knows that \( x \neq i \) for some standard \( i \neq 0 \), then HA knows that \( x \) is some standard number as well. For \( \subseteq \), a similar remark applies.

To close this section, we define, for any \( A \in L_\square \), the \( L \)-sentence

\[
[A] = \begin{cases} 
\bigvee_{i \in [A]} \exists x (h(x) = i) & \text{if } [A] \text{ is finite}; \\
\top & \text{if } [A] = \mathbb{N}.
\end{cases}
\]

We notice that \([A]\) is always a \( \Sigma_1 \)-sentence.

### 3.2 Preservation of the Logical Structure

In this rather technical section, we show that \([\cdot]\) commutes with all the logical operators figuring in \( L_\square \). The proofs in this section will become increasingly difficult. We adopt all the notation introduced in the previous section.

**Lemma 3.2.1.** We have \( \vdash_{HA} [B \lor C] \leftrightarrow [B] \lor [C] \) for \( B, C \in L_\square \).

**Proof.** This is immediate from the definition of \([\cdot]\). \( \square \)

**Lemma 3.2.2.** We have \( \vdash_{HA} [B \land C] \leftrightarrow [B] \land [C] \) for \( B, C \in L_\square \).

**Proof.** If \( [B] = \mathbb{N} \), then \( [B \land C] = [C] \), so we have \( \vdash_{HA} [B \land C] \leftrightarrow [C] \leftrightarrow [B] \land [C] \). Similarly, the result follows if \([C] = \mathbb{N}\). So suppose that \([B]\) and \([C]\) are both finite; then \([B \land C]\) is finite as well.

The ‘\( \lor \)’-statement is immediate in this case. For, the other direction, we should show that

\[ \vdash_{HA} \exists x (h(x) = i) \land \exists y (h(y) = j) \rightarrow [B \land C] \]

whenever \( i \in [B] \) and \( j \in [C] \). First of all, we notice that \( \vdash_{HA} \exists x (h(x) = i) \land \exists y (h(y) = j) \rightarrow i \leq j \lor j \leq i \). Indeed, reason inside HA and suppose we have \( x \) and \( y \) such that \( h(x) = i \) and \( h(y) = j \). Since \( x \leq y \lor y \leq x \), we can conclude that \( i \leq j \lor j \leq i \), as desired.

Now, if \( i \) and \( j \) are incomparable w.r.t. \( \preceq \), then \( \vdash_{HA} \neg(i \leq j \lor j \leq i) \), so by the above we have \( \vdash_{HA} \neg(\exists x (h(x) = i) \land \exists y (h(y) = j)) \), in which case the result is clear. If \( i \) and \( j \) are comparable w.r.t. \( \preceq \), then assume without loss of generality that \( i \leq j \). Then \( j \in [B \land C] \), so

\[ \vdash_{HA} \exists x (h(x) = i) \land \exists y (h(y) = j) \rightarrow \exists y (h(y) = j) \rightarrow [B \land C], \]

as desired. \( \square \)

**Lemma 3.2.3.** We have \( \vdash_T [B \rightarrow C] \leftrightarrow ([B] \rightarrow [C]) \) for \( B, C \in L_\square \).

**Proof.** If \( [B \rightarrow C] = \mathbb{N} \), then \( [B] \subseteq [C] \), so \([B \rightarrow C]\) and \([B] \rightarrow [C]\) are both equivalent to \( \top \), even over HA. Now suppose that \([B \rightarrow C]\) is finite.

We first treat the \( \leftarrow \)-direction. Let \( j_0, \ldots, j_{s-1} \neq 0 \) be the \( \preceq \)-maximal elements \( j \) of \( \mathbb{N} \) such that \( j \notin [B \rightarrow C] \). Then for all \( t < s \), we have \( j_t \in [B] \) and \( j_t \notin [C] \). Using the fact that \( \prec \) is also a conversely well-founded relation, we can show that for all \( i \in \mathbb{N} \), we have \( i \in [B \rightarrow C] \) if and only if \( i \not< j_t \) for all \( t < s \).

Now we reason inside HA. Suppose that \([B] \rightarrow [C]\) and \( \triangle [B \rightarrow C] \). Since \([B \rightarrow C] \rightarrow \exists y \neg(h(y) \leq j_i) \), we have \( \Delta (\exists y \neg(h(y) \leq j_i)) \). Now let \( k_i \) satisfy \( \text{Prf}_\Delta (k_i, \exists y \neg(h(y) \leq j_i)) \).

We distinguish three cases (which is constructively acceptable).
1. Suppose that \( h(k_t) < j_t \) for some \( t < s \). Then by the definition of \( h \), we get \( h(k_t + 1) = j_t \). But \( j_t \in [B] \), so \([B] \) holds, so \([C] \) holds, and therefore \([B \to C] \) also holds.

2. Suppose that \( h(k_t) = j_t \) for some \( t < s \). Then \([B \to C] \) again follows.

3. Suppose that \( \neg(h(k_t) \leq j_t) \) for all \( t < s \). Let \( k = \max_{t < s} k_t \). Then we also know that \( \neg(h(k) \leq j_t) \) for all \( t < s \). Indeed, suppose that \( h(k) \leq j_t \) for some \( t \). Since \( k_t \leq k \), we get \( h(k_t) \leq h(k) \leq j_t \), so, since \( \leq \) is (provably) transitive, \( h(k_t) \leq j_t \), which we already excluded. So we indeed have \( \neg(h(k) \leq j_t) \) for all \( t < s \). But then by (3.3) applied to \( j_0, \ldots, j_{s-1} \), we see that \( \bigvee_{j \in U} h(k) = j \), where \( U = \{ j \in \mathbb{N} \mid j \notin j_t \text{ for all } t < s \} \) is a finite set. We see (outside HA) that \( U \subseteq [B \to C] \), so (inside HA again) we get \([B \to C] \).

We conclude that \( \vdash_{\text{HA}} ([B] \to [C]) \to ([B \to C] \to [B \to C]) \). Since HA \subseteq T and \( \vdash_T \text{SLP}_{\Delta} \), we may conclude that \( \vdash_T ([B] \to [C]) \to [B \to C] \).

The \( \to \)-direction is even provable in HA. Notice that \([B \land (B \to C)] \subseteq [C] \), so by Lemma 3.2.2, we have \( \vdash_{\text{HA}} ([B \land (B \to C)] \to [B \land (B \to C)] \to [C] \), so \( \vdash_{\text{HA}} ([B \to C] \to ([B] \to [C]) \). \( \square \)

The idea of defining \( M \) from \( M_0 \) and the techniques used in the previous three lemmata were already introduced by A. Visser. The proof that \( [\cdot] \) commutes with \( \square \) contains some new ideas. First, we need some auxiliary results.

**Lemma 3.2.4.** Suppose \( i > 0 \) is a natural number. Then

\[
\vdash_{\text{HA}} \exists x(h(x) = i) \to \square(\exists y(i < h(y))).
\]

**Proof.** Before we start proving the displayed sentence inside HA, we need to verify two auxiliary facts inside HA. First of all, we claim that

\[
\vdash_{\text{HA}} (\neg(h(y) \leq i) \land h(x) = i) \to i < h(y).
\]

Reason inside HA and assume the antecedent. If \( y < x \), then \( h(y) \leq h(x) = i \), quod non. So \( x \leq y \), which means that \( i = h(x) \leq h(y) \). But \( h(y) \) cannot be equal to \( i \), so \( i < h(y) \), as desired. Now we also have:

\[
\vdash_{\text{HA}} (\exists y (\neg(h(y) \leq i) \land \exists x(h(x) = i)) \to \exists y(i < h(y)). \quad (3.4)
\]

Secondly, we claim that

\[
\vdash_{\text{HA}} (\neg(h(y) \sqsubseteq i) \land h(x) = i \land h(x-1) \sqsubseteq i) \to i < h(y).
\]

Again, reason inside HA and assume the antecedent. Suppose that \( y < x \). Then \( y \leq x - 1 \), so \( h(y) \leq h(x-1) \sqsubseteq i \). Since our frame (provably) has the model property, we get \( h(y) \sqsubseteq i \), contradiction. So \( y \geq x \). But then \( i = h(x) \leq h(y) \) and \( \neg(i = h(y)) \), so \( i < h(y) \), as desired. We also find:

\[
\vdash_{\text{HA}} \exists y (\neg(h(y) \sqsubseteq i)) \land \exists x(h(x) = i \land h(x-1) \sqsubseteq i) \to \exists y(i < h(y)). \quad (3.5)
\]

Now we start the main part of the proof. Reason inside HA, and suppose that we have an \( x \) such that \( h(x) = i \). Since \( h \) is (provably) a function, we can consider the least \( x \) such that \( h(x) = i \). Then \( x > 0 \), and \( h(x-1) < i \). Again, we make a constructively acceptable case distinction.
1. Suppose that \( \neg(h(x-1) \sqsubseteq i) \). Then \( \triangle(\exists y \neg(h(y) \leq i)) \) (otherwise, we wouldn’t have moved up to \( i \)). Since \( \exists x (h(x) = i) \) is a \( \Sigma_1 \)-sentence, we also know get \( \triangle(\exists x (h(x) = i)) \).

Using (3.4) and the properties of \( \triangle \), we can conclude that \( \triangle(\exists y (i < h(y))) \). Since \( \exists y (i < h(y)) \) is a \( \Sigma_1 \)-sentence, we also get \( \Box(\exists y (i < h(y))) \), as desired.

2. Suppose that \( h(x-1) \sqsubseteq i \). Then, from the fact that we moved up to \( i \), we can deduce that \( \Box(\exists x \neg(h(x) \sqsubseteq i)) \) or \( \Box(\exists y (h(y) \leq i)) \). By (3.2), we can conclude that \( \Box(\exists y (\neg(h(y) \sqsubseteq i)) \) in both cases. At this point, we have \( \exists x (h(x) = i \land h(x-1) \sqsubseteq i) \).

Since this is a \( \Sigma_1 \)-sentence, we also get \( \Box(\exists x h(x) = i \land h(x-1) \sqsubseteq i)) \). Using (3.5) and the properties of \( \Box \), we again find \( \Box(\exists y (i < h(y))) \), as desired.

Lemma 3.2.5. Let \( i, j \) be natural numbers such that \( i < j \) and \( \neg(i \sqsubseteq j) \). Then

\[ \vdash_{HA} \exists x (h(x) = i) \land \exists y (h(y) = j) \rightarrow \triangle(\exists z (j < h(z))) \].

Proof. First of all, we notice that we also know that \( i < j \) and \( \neg(i \sqsubseteq j) \) inside HA. Now reason inside HA, and suppose that \( \exists x (h(x) = i) \) and \( \exists y (h(y) = j) \). Since \( h \) is (provably) a function, we can consider the least \( y \) such that \( h(y) = j \). Then \( y > 0 \), and \( h(y-1) < j \).

Consider an \( x \) such that \( h(x) = i \). Suppose that \( y \leq x \). Then \( j = h(y) < h(x) = i < j \), which is a contradiction since \( \leq \) is (provably) antisymmetric. So \( x < y \), which also means \( x \leq y - 1 \).

Now we get \( i = h(x) \leq h(y-1) \).

If \( h(y-1) \sqsubseteq j \), then \( i \leq h(y-1) \sqsubseteq j \), so \( i \sqsubseteq j \). But we also have \( \neg(i \sqsubseteq j) \), contradiction. So \( \neg(h(y-1) \sqsubseteq j) \). Now we can use the exact same reasoning as in case 1 in the proof of Lemma 3.2.4 (with \( j \) instead of \( i \), and \( y \) instead of \( x \)) to arrive at \( \triangle(\exists z (j < h(z))) \), as desired.

Now we have proven these tedious lemmata, we can derive our crucial result.

Lemma 3.2.6. We have \( \vdash_{HA} \Box[B] \iff \Box[B] \) for all \( B \in \mathcal{L}_{\Box} \).

Proof. If \( [B] = \mathbb{N} \), then \( [\Box B] = \mathbb{N} \) as well, and we see that \( [\Box B] \) and \( [B] \) are both equivalent to \( \top \) over HA. Now suppose that \( [B] \) is finite.

We first treat the \( \leftarrow \)-direction. Let \( j_0, \ldots, j_{s-1} \neq 0 \) be the \( \sqsubseteq \)-maximal elements \( j \) of \( \mathbb{N} \) such that \( j \notin [B] \). Notice that \( j_t \in [\Box B] \) for all \( t < s \). Suppose that we have \( i \in [B] \) and \( t < s \) such that \( i \sqsubseteq j_t \). Since \( M \) is realistic, we get \( i \leq j_t \), so by preservativity of knowledge, \( j_t \in [B] \), contradiction. So if \( i \in [B] \), then \( i \not\sqsubseteq j_t \). In particular, we have \( \vdash_{HA} [B] \rightarrow \exists x \neg(h(x) \sqsubseteq j_t) \) for all \( t < s \). Using the fact that \( \Box \) is a conversely well-founded relation, we can also show: if \( i \not\sqsubseteq j_t \) for all \( t < s \), then \( i \in [B] \).

Now we reason inside HA and suppose that \( [B] \). Then \( \Box(\exists x \neg(h(x) \sqsubseteq j_t)) \) also holds. Let \( k_t \) satisfy \( \text{Prf}_\Box(k_t, \exists x \neg(h(x) \sqsubseteq j_t)) \). We distinguish three cases.

1. Suppose \( h(k_t) \sqsubseteq j_t \) for some \( t < s \). Then by the definition of \( h \), we have \( h(k_t + 1) = j_t \), and \( [B] \) follows.

2. Suppose \( h(k_t) = j_t \) for some \( t < s \). Then \( [B] \) again follows.

3. Suppose that \( \neg(h(k_t) \sqsubseteq j_t) \) for all \( t < s \). Let \( k = \max_{t < s} k_t \). If \( h(k) = j_t \) for some \( t < s \), then \( [B] \) again follows. Suppose \( h(k) \sqsubset j_t \) for some \( t < s \). Since \( h(k_t) \leq h(k) \sqsubset j_t \) and our frame (provably) has the model property, we get \( h(k_t) \sqsubset j_t \), which we already excluded. So we have \( \neg(h(k) \sqsubseteq j_t) \) for all \( t < s \). But then using the \( \sqsubseteq \)-analogue of (3.3) for \( j_0, \ldots, j_{s-1} \), we see that \( \bigcup_{t < s} h(k_t) = j \), where \( U = \{ j \in \mathbb{N} \mid j \not\sqsubseteq j_t \) for all \( t < s \} \) is a finite set.

We see (outside HA) that \( U \subseteq [B] \subseteq [\Box B] \), where the latter inclusion holds.
since $M$ is realistic. So (inside $\text{HA}$ again), we get $\Box B$, as desired.

Now we treat the $\rightarrow$-direction. Consider an $i \in [\Box B]$. Then $i > 0$, since $[B]$ is finite. So by Lemma 3.2.4, we have

$$
\vdash_{\text{HA}} \exists x (h(x) = i) \rightarrow \Box (\exists y (i < h(y))).
$$

(3.6)

Every nonzero node $k$ of $M$ has a finite $\prec$-rank, which is the greatest $n$ such that there exists a sequence $k = k_0 \prec k_1 \prec \cdots \prec k_n$. Let $a \in \mathbb{N}$ be the $\prec$-rank of $i$. For $b \in \mathbb{N}$, we define the finite set

$$
U_b = \{ j \in \mathbb{N} \mid i \prec j, i \not\preceq j \text{ and } \text{rank}(j) < b \}.
$$

We know (inside $\text{HA}$) that $i \prec h(y)$ implies that $h(y)$ is a standard number. Moreover, such a standard number must have rank smaller than $a$, so it is either in $[B]$ (if $i \sqsubset h(y)$) or in $U_a$ (if $i \not\preceq h(y)$). That is, we have

$$
\vdash_{\text{HA}} i \prec h(y) \rightarrow \bigvee_{j \in [B]} \top \lor \bigvee_{j \in U_a} \exists y (h(y) = j).
$$

From this, it follows that

$$
\vdash_{\text{HA}} \exists y (i \prec h(y)) \rightarrow [B] \lor \bigvee_{j \in U_a} \exists y (h(y) = j).
$$

So (3.6) together with the properties of the $\Box$ implies that

$$
\vdash_{\text{HA}} \exists x (h(x) = i) \rightarrow \Box \left( [B] \lor \bigvee_{j \in U_a} \exists y (h(y) = j) \right).
$$

(3.7)

Suppose that $j \in U_b$ for a certain $b \geq 1$. By Lemma 3.2.5, we know that

$$
\vdash_{\text{HA}} \exists x (h(x) = i) \land \exists y (h(y) = j) \rightarrow \bigtriangleup (\exists z (j < h(z))).
$$

(3.8)

Furthermore, if $j \prec h(z)$, then we know (inside $\text{HA}$) that $h(z)$ is some standard number. Moreover, such a standard number must have lower $\prec$-rank than $j$, so it is either in $[B]$ (if $i \sqsubset h(z)$) or in $U_{b-1}$ (if $i \not\preceq h(z)$). That is, we have

$$
\vdash_{\text{HA}} j \prec h(z) \rightarrow \bigvee_{k \in [B]} h(z) = k \lor \bigvee_{k \in U_{b-1}} h(z) = k.
$$

From this, it follows that

$$
\vdash_{\text{HA}} \exists z (j < h(z)) \rightarrow [B] \lor \bigvee_{k \in U_{b-1}} \exists z (h(z) = k).
$$

So using (3.8) and the properties of $\bigtriangleup$, we get

$$
\vdash_{\text{HA}} \exists x (h(x) = i) \land \exists y (h(y) = j) \rightarrow \bigtriangleup \left( [B] \lor \bigvee_{k \in U_{b-1}} \exists z (h(z) = k) \right),
$$

\[\]
This holds for all \( j \in U_b \), so
\[
\vdash_{\text{HA}} \exists x \left( h(x) = i \right) \land \bigvee_{j \in U_b} (\exists y \left( h(y) = j \right)) \rightarrow \Delta \left( [B] \lor \bigvee_{j \in U_{b-1}} \exists y \left( h(y) = j \right) \right).
\]

(We changed some bound variables on the right hand side.) Since \([B]\) is equivalent, over \text{HA}, to a \( \Sigma_1 \)-sentence, we also have
\[
\vdash_{\text{HA}} [B] \rightarrow \Delta [B] \rightarrow \Delta \left( [B] \lor \bigvee_{j \in U_{b-1}} \exists y \left( h(y) = j \right) \right).
\]

So we conclude that
\[
\vdash_{\text{HA}} \exists x \left( h(x) = i \right) \land \left( [B] \lor \bigvee_{j \in U_b} (\exists y \left( h(y) = j \right)) \right) \rightarrow \Delta \left( [B] \lor \bigvee_{j \in U_{b-1}} \exists y \left( h(y) = j \right) \right).
\]

Since \( \exists x \left( h(x) = i \right) \) is equivalent, over \text{HA}, to a \( \Sigma_1 \)-sentence, we have
\[
\vdash_{\text{HA}} \exists x \left( h(x) = i \right) \rightarrow \Box (\exists x \left( h(x) = i \right)).
\]

Now we see:
\[
\vdash_{\text{HA}} \exists x \left( h(x) = i \right) \land \Box \left( [B] \lor \bigvee_{j \in U_b} \exists y \left( h(y) = j \right) \right)
\]
\[
\rightarrow \Box \left( \exists x \left( h(x) = i \right) \land \left( [B] \lor \bigvee_{j \in U_b} (\exists y \left( h(y) = j \right)) \right) \right)
\]
\[
\rightarrow \Box \Delta \left( [B] \lor \bigvee_{j \in U_{b-1}} \exists y \left( h(y) = j \right) \right)
\]
\[
\rightarrow \Box \left( [B] \lor \bigvee_{j \in U_{b-1}} \exists y \left( h(y) = j \right) \right),
\]

where the final step holds since \( [B] \lor \bigvee_{j \in U_{b-1}} \exists y \left( h(y) = j \right) \) is equivalent, over \text{HA}, to a \( \Sigma_1 \)-sentence. Now we can apply this repeatedly to (3.7) in order to obtain
\[
\vdash_{\text{HA}} \exists x \left( h(x) = i \right) \rightarrow \Box \left( [B] \lor \bigvee_{j \in U_0} \exists y \left( h(y) = j \right) \right)
\]
\[
\leftrightarrow \Box ([B] \lor \perp)
\]
\[
\leftrightarrow \Box [B],
\]

where we used that \( U_0 = \emptyset \).

Since this holds for all \( i \in [\Box B] \), we can conclude that \( \vdash_{\text{HA}} [\Box B] \rightarrow \Box [B] \), as desired. \( \Box \)

### 3.3 The Completeness Theorem

In this section, we formulate and prove our completeness theorem in its abstract form. First, we define provability logics.
3.3. THE COMPLETENESS THEOREM

Definition 3.3.1. Let $T$ be a theory and let $S(x)$ be a provability predicate for $T$. If $A$ is an $\mathcal{L}$-sentence, we write $\Box A$ for $S(⌜A⌝)$.

(i) A realization is a function $\sigma$ that assigns, to each proposition letters $p$ in $\mathcal{L}_\Box$, an $\mathcal{L}$-sentence $\sigma(p)$. We call $\sigma$ a $\Sigma_1$-realization if $\sigma(p) \in \Sigma_1$ for every proposition letter $p$.

(ii) Given a realization $\sigma$, we define the function $\sigma_\Box : \mathcal{L}_\Box \to \mathcal{L}$ by:

(a) $\sigma_\Box(\bot) = \bot$ and $\sigma_\Box(p)$ is $\sigma(p)$ for every proposition letter $p$;

(b) $\sigma_\Box(B \circ C) = \sigma_\Box(B) \circ \sigma_\Box(C)$ for all $B,C \in \mathcal{L}_\Box$ and $\circ \in \{\land, \lor, \to\}$.

(c) $\sigma_\Box(\Box B)$ is $\Box(\sigma_\Box(B))$ for all $B \in \mathcal{L}_\Box$.

(As in Definition 2.2.4, this is a slight abuse of notation.) Notice that $\sigma_\Box(B)$ is a sentence for every $A \in \mathcal{L}_\Box$.

(iii) The logic for $\Box$ is defined as the set of all $A \in \mathcal{L}_\Box$ such that $\vdash_T \sigma_\Box(A)$ for every realization $\sigma$. The $\Sigma_1$-logic for $\Box$ is the set of all $A \in \mathcal{L}_\Box$ such that $\vdash_T \sigma_\Box(A)$ for every $\Sigma_1$-realization $\sigma$.

We write $\sigma_T$ for $\sigma_\Box_T$ and $\sigma^f_T$ for $\sigma^f_{\Box_T}$. The $(\Sigma_1)$-logic for $\Box_T$ is called the $(\Sigma_1)$-provability logic of $T$, and the $(\Sigma_1)$-logic for $\Box^f_T$ is called the fast $(\Sigma_1)$-provability logic of $T$.

Now, we again adopt the conventions and notation from Section 3.1. All the work from Section 3.2 now leads to the following result.

Theorem 3.3.1. Define the $\Sigma_1$-realization $\sigma$ by $\sigma(p) = [p]$ for every proposition letter $p$. Then $\vdash_T \sigma_\Box(A) \leftrightarrow [A]$ for all $A \in \mathcal{L}_\Box$.

Proof. This follows by induction on the complexity of $A$ using Lemma 3.2.1, Lemma 3.2.2, Lemma 3.2.3 and Lemma 3.2.6.

The following result tells us what the ‘real’ behaviour of the Solovay function $h$ is, in the case that $T$ is $\Sigma_1$-sound.

Proposition 3.3.2. Suppose that $T$ is $\Sigma_1$-sound. Then $\mathbb{N} \models h(x) = 0$.

Proof. Since $\prec$ is conversely well-founded, we know that $h$ must have a certain limit $i \in \mathbb{N}$. Suppose that $i > 0$. Then $\vdash_{HA} \exists x(h(x) = i) \to \Box(\exists y(i \prec h(y)))$ by Lemma 3.2.4, so since $HA$ is sound, we get $\mathbb{N} \models \exists x(h(x) = i) \to \Box(\exists y(i \prec h(y)))$. By assumption, $\mathbb{N} \models \exists x(h(x) = i)$, so $\mathbb{N} \models \Box(\exists y(i \prec h(y)))$. By requirement (ii) for a good pair, we have $\vdash_T \exists y(i \prec h(y))$. Since $\exists y(i \prec h(y))$ is a $\Sigma_1$-sentence and $T$ is $\Sigma_1$-sound, we get $\mathbb{N} \models \exists y(i \prec h(y))$. However, this is impossible as $i$ is supposed to be the limit of $h$. So $i = 0$, and the result follows.

Now we can finally formulate and prove our main result.

Theorem 3.3.3. Let $T \supseteq HA$ be a $\Sigma_1$-sound theory and suppose we have a good pair $(S,R)$ for $T$. If $A$ is an $\mathcal{L}$-sentence, we write $\Box A$ for $S(⌜A⌝)$.

(i) The $\Sigma_1$-logic for $\Box$ is equal to the set of theorems of iGLC.
(ii) If $\vdash_T CP_\Box$, then the logic for $\Box$ is equal to the set of theorems of iGLC.
Proof. (i) Since \( S(x) \) is a provability predicate for \( T \), we see that the \( \Sigma_1 \)-logic for \( \Box \) contains \( i\text{GLC} \) and is closed under \( \rightarrow \text{E} \) and \( \text{Nec} \). So the logic \( \Sigma_1 \)-logic for \( \Box \) contains all theorems of \( i\text{GLC} \).

Now suppose that we have \( A \in \mathcal{L}_\Box \) such that \( i\text{GLC} \nvdash A \). Then by Theorem 2.4.4, there exists a finite, irreflexive, realistic model \( M_0 = \langle W_0, \preceq_0, \sqsubseteq_0, V_0 \rangle \) in which \( A \) is not valid. We label the nodes of \( M_0 \) as \( W_0 = \{1, \ldots, r\} \) in such a way that \( M_0, r \nvdash A \). By shrinking \( W_0 \) to \( \{i \in W_0 \mid r \preceq_0 i\} \) if necessary, we may assume without loss of generality that \( r \) is the \( \preceq_0 \)-least element of \( W_0 \).

Now define the model \( M \), the Solovay function \( h \), and the \( \Sigma_1 \)-sentences \( [B] \) for \( B \in \mathcal{L}_\Box \) as above. It is easy to show that \( M_0, i \vDash B \) iff \( M, i \vDash B \) for all \( B \in \mathcal{L}_\Box \) and all \( i \) with \( 1 \leq i \leq r \). So we have \( M, r \nvdash A \), that is, \( r \notin [A] \). Now define the \( \Sigma_1 \)-realization \( \sigma \) by \( \sigma(p) = [p] \) for every proposition letter \( p \). By Theorem 3.3.1, we have \( \vdash_T \sigma_\Box(B) \leftrightarrow [B] \) for all \( B \in \mathcal{L}_\Box \).

Now suppose for the sake of contradiction that \( \vdash_T \sigma_\Box(A) \). Then we also get \( \vdash_T [A] \). Since \( [A] \) is (equivalent to) a \( \Sigma_1 \)-sentence and \( T \) is \( \Sigma_1 \)-sound, we see that \( \mathbb{N} \models [A] \). By Proposition 3.3.2, we also know that \( \mathbb{N} \models h(x) = 0 \). This implies that \( 0 \in [A] \). However, we also have \( 0 \preceq_0 r \) and \( r \notin [A] \), which yields a contradiction. We conclude that \( A \) is not in the \( \Sigma_1 \)-logic for \( \Box \), as desired.

(ii) Since \( S(x) \) is a provability predicate for \( T \), we see that the logic for \( \Box \) contains \( i\text{GL} \) and is closed under \( \rightarrow \text{E} \) and \( \text{Nec} \). Since \( \vdash_T \text{CP}_\Box \), we also have that \( A \rightarrow \Box A \) is in the logic for \( \Box \), for every \( A \in \mathcal{L}_\Box \). So the logic for \( \Box \) contains all theorems of \( i\text{GLC} \). Conversely, the logic for \( \Box \) is contained in the \( \Sigma_1 \)-logic for \( \Box \), which is contained in the set of theorems of \( i\text{GLC} \). \( \square \)
Chapter 4

Applications of the Completeness Theorem

In the previous chapter, we proved a completeness theorem in a very abstract form. In this chapter, we provide several applications of this theorem. In particular, we will determine the fast provability logics of the theories $U^*$, for $\Sigma_1$-sound theories $U \supseteq \text{HA}$, and we will determine the fast and ordinary $\Sigma_1$-provability logics of $\text{HA}$. First of all, we lay some further groundwork in Section 4.1. Then, in Section 4.2, we determine the fast provability logics mentioned above. Finally, in Section 4.3, we determine the $\Sigma_1$-provability logic of $\text{HA}$.

4.1 The Sets $\text{NNIL}$ and $\text{TNNIL}$

In the sequel, $\mathcal{L}_p$ is the language of propositional logic, and for $A \in \mathcal{L}_p$, we write ‘$\vdash_{IPC} A$’ to indicate that $A$ is provable in intuitionistic propositional logic. We notice that, if $\sigma$ is a substitution, $A \in \mathcal{L}_p$ and $S(x)$ is a provability predicate for a certain theory $T$, then $\sigma(\Box A)$ does not depend on the provability predicate $S$. So we will just write $\sigma(A)$ instead of $\sigma(\Box A)$. We will also drop the brackets in expressions of the form $\sigma(A)$ and $\sigma_T(A)$.

Like the authors of [1], we introduce the set of $\text{NNIL}$-formulae.

Definition 4.1.1. The set $\text{NNIL} \subseteq \mathcal{L}_p$ (‘no nested implications on the left’) is defined recursively, as follows:

(i) all proposition letters are in $\text{NNIL}$, as is $\perp$;
(ii) if $A, B \in \text{NNIL}$, then $A \land B, A \lor B \in \text{NNIL}$;
(iii) if $A \in \mathcal{L}_p$ contains no implications and $B \in \text{NNIL}$, then $A \rightarrow B \in \text{NNIL}$.

That is, a $\text{NNIL}$-formula is a propositional formulae in which no implication occurs in the antecedent of another implication. In the paper [11], we find the following result, that we will not prove here.
Theorem 4.1.1. There exists a computable function $(\cdot)^* : L_p \to \text{NNIL}$, called the NNIL-algorithm, such that for every $A \in L_p$, the following hold:

(i) $\vdash_{\text{IPC}} A^* \to A$;

(ii) if $B \in \text{NNIL}$ and $\vdash_{\text{IPC}} B \to A$, then $\vdash_{\text{IPC}} B \to A^*$;

(iii) if $\sigma$ is a $\Sigma_1$-realization, then $\vdash_{\text{HA}} \Box_{\text{HA}}(\sigma A) \leftrightarrow \Box_{\text{HA}}(\sigma A^*)$.

Remark 4.1.1. Consider the preorder $(L_p, \leq)$, there $\leq$ is defined by: $A \leq B$ if and only if $\vdash_{\text{IPC}} A \to B$, for $A, B \in L_p$. Consider also the subpreorder $(\text{NNIL}, \leq)$. Then items (i) and (ii) above say that the NNIL-algorithm is left adjoint to the inclusion $\text{NNIL} \to L_p$. $\diamond$

We can get an analogue of (iii) for fast provability.

Corollary 4.1.2. Let $A \in L_p$ and let $\sigma$ be a $\Sigma_1$-realization. Then

$$\vdash_{\text{HA}} \Box_{\text{HA}}^f(\sigma A) \leftrightarrow \Box_{\text{HA}}^f(\sigma A^*).$$

Proof. Since $\text{Bew}_{\text{HA}}^f$ is a provability predicate for $\text{HA}$, we can derive from Theorem 4.1.1(iii) that

$$\vdash_{\text{HA}} \Box_{\text{HA}}^f(\sigma A) \leftrightarrow \Box_{\text{HA}}^f(\sigma A) \leftrightarrow \Box_{\text{HA}}^f(\sigma A^*) \leftrightarrow \Box_{\text{HA}}^f(\sigma A^*),$$

where we also used Lemma 2.3.2(vii). $\square$

The NNIL-algorithm behaves nicely with respect to the theories $U^T$ and $\Sigma_1$-realizations.

Proposition 4.1.3. Suppose that $U$ and $T$ are theories such that $\text{HA} \subseteq U \subseteq U^T$, $\text{HA} \subseteq T$, and $\vdash_{\text{HA}} \text{Bew}_U(x) \to \Box_U \text{Bew}_T(x)$. Then for all $\Sigma_1$-realizations $\sigma$ and $C \in \text{NNIL}$, we have

$$\vdash_{\text{HA}} \Box_U(\sigma C) \leftrightarrow \Box_U(\sigma C).$$

(4.1)

Proof. The $\vdash$-direction holds since $U \subseteq U^T$. For the converse, we notice that, since $\sigma$ is a $\Sigma_1$-realization and $C \in \text{NNIL}$, we have that $\sigma C$ is equivalent, over $\text{HA}$, to $\sigma C$, over $A$, a sentence in $A$. Since $\text{HA} \subseteq T$, we have $\vdash_{\text{HA}} (\sigma C)^T \to \sigma C$, by Corollary 2.2.5 and Lemma 2.2.7(ii). Since $\text{HA} \subseteq U$, we get $\vdash_{\text{HA}} \Box_U(\sigma C)^T \to \Box_U(\sigma C)$. Finally, we notice that the conditions of Corollary 2.2.6 hold, so we get $\vdash_{\text{HA}} \Box_U(\sigma C) \to \Box_U(\sigma C)^T \to \Box_U(\sigma C)$, as desired. $\square$

Corollary 4.1.4. Suppose that $U$ and $T$ are theories such that $\text{HA} \subseteq U \subseteq T$ and $U \subseteq U^T$. Then for all $\Sigma_1$-realizations $\sigma$ and $C \in \text{NNIL}$, we have

$$\vdash_{\text{HA}} \Box_U^f(\sigma C) \leftrightarrow \Box_U^f(\sigma C).$$

(4.2)

Proof. We notice that the conditions from Proposition 4.1.3 apply, so we have

$$\vdash_{\text{HA}} \Box_U(\sigma C) \leftrightarrow \Box_U(\sigma C) \leftrightarrow (\Box_U(\sigma C))^T,$$

where the latter equivalence holds by Lemma 2.2.7(i) and the fact that $\text{HA} \subseteq T$. Since $\text{HA} \subseteq U$, we also get $\vdash_U \Box_U(\sigma C) \leftrightarrow (\Box_U(\sigma C))^T$. We know that $\text{Bew}_U^f$ is a provability
predicate for $U$, so we also get $\vdash_{HA} □_U □_U (σC) \leftrightarrow □_U (□_{UT} (σC))^T$. Using Lemma 2.3.2(vii) and Lemma 2.3.3, we find

$$\vdash_{HA} □_U (σC) \leftrightarrow □_U (□_{UT} (σC))^T \leftrightarrow □_U □_{UT} (σC) \leftrightarrow □_U (σC),$$

as desired. \qed

Following [1], we now extend the notion of ‘no nested implication on the left’ to modal formulae.

**Definition 4.1.2.** The set $\text{TNNIL} \subseteq \mathcal{L}_{□}$ (‘thoroughly no nested implications on the left’) is defined by recursion, as follows:

(i) all proposition letters are in $\text{TNNIL}$;

(ii) if $A, B \in \text{TNNIL}$, then $A \land B, A \lor B, □A \in \text{TNNIL}$;

(iii) if $A, B \in \text{TNNIL}$ and $A$ contains no implications outside a box, then $A \rightarrow B \in \text{TNNIL}$.

We notice that every $A \in \mathcal{L}_{□}$ can, in a unique way, be written as $C(\vec{p}, □B_1, \ldots, □B_k)$, for certain $C(\vec{p}, q_1, \ldots, q_k) \in \mathcal{L}_p$ and distinct $B_1, \ldots, B_k \in \mathcal{L}_{□}$. It is easy to show that, with this notation, we have $A \in \text{TNNIL}$ if and only if $C \in \text{NNIL}$ and $B_i \in \text{TNNIL}$ for $1 \leq i \leq k$. Now we define an operation on modal formulae as in [1].

**Definition 4.1.3.** The $\text{TNNIL}$-algorithm $(\cdot)^+ : \mathcal{L}_{□} \rightarrow \text{TNNIL}$ is defined by recursion, as follows. For $A \in \mathcal{L}_{□}$, write $A = C(\vec{p}, □B_1, \ldots, □B_k)$, where $C(\vec{p}, q_1, \ldots, q_k) \in \mathcal{L}_p$ and $B_1, \ldots, B_k \in \mathcal{L}_{□}$ are distinct. Then

$$A^+ := C^*(\vec{p}, □B_1^+, \ldots, □B_k^+).$$

Notice that, since all the $B_i$ have lower complexity than $A$, the operation $(\cdot)^+$ is well-defined.

We can use our results about $\text{NNIL}$ and $(\cdot)^*$ to obtain the following lemmata about $\text{TNNIL}$ and $(\cdot)^+$. We notice that Lemma 4.1.5(i) also occurs in [1] as Corollary 4.7.1.

**Lemma 4.1.5.** Let $A \in \mathcal{L}_p$ and let $σ$ be a $Σ_1$-realization. Then the following hold:

(i) $\vdash_{HA} □_{HA}(σ_{HA}A) \leftrightarrow □_{HA}(σ_{HA}A^+)$;

(ii) $\vdash_{HA} □_H(σ_{HA}A) \leftrightarrow □_H(σ_{HA}A^+)$.\n
**Proof.** (i) We proceed by strong induction on the boxdepth of $A$. As above, we write $A$ as $C(\vec{p}, □B_1, \ldots, □B_k)$, where $C(\vec{p}, q_1, \ldots, q_k) \in \mathcal{L}_p$ and $B_1, \ldots, B_k \in \mathcal{L}_{□}$ are distinct. Then all the $B_i$ have smaller boxdepth than $A$, so we assume by induction hypothesis that

$$\vdash_{HA} □_{HA}(σ_{HA}B_i) \leftrightarrow □_{HA}(σ_{HA}B_i^+) \text{ for } 1 \leq i \leq k.$$  (4.3)
If $\vec{p} = p_1, \ldots, p_l$, then we write $\sigma \vec{p}$ as a shorthand for $\sigma(p_1), \ldots, \sigma(p_l)$. Now we take a $\Sigma_1$-realization $\tau$ such that $\tau \vec{p} = \sigma \vec{p}$ and $\tau(q_i) = \Box_{\text{HA}}(\sigma_{\text{HA}} B_i)$ for $1 \leq i \leq k$. Now we observe that

$$
\begin{align*}
\sigma_{\text{HA}} A &= C(\sigma \vec{p}, \Box_{\text{HA}}(\sigma_{\text{HA}} B_1), \ldots, \Box_{\text{HA}}(\sigma_{\text{HA}} B_k)) = \tau C, \\
\sigma_{\text{HA}} A^+ &= C^*(\sigma \vec{p}, \Box_{\text{HA}}(\sigma_{\text{HA}} B_1^+), \ldots, \Box_{\text{HA}}(\sigma_{\text{HA}} B_k^+)) \quad \text{and} \\
\tau C^* &= C^*(\sigma \vec{p}, \Box_{\text{HA}}(\sigma_{\text{HA}} B_1), \ldots, \Box_{\text{HA}}(\sigma_{\text{HA}} B_k)).
\end{align*}
$$

So (4.3) gives $\vdash_{\text{HA}} \sigma_{\text{HA}} A^+ \leftrightarrow \tau C^*$. Since $\text{Bew}_{\text{HA}}$ is a provability predicate for HA, we conclude that

$$
\vdash_{\text{HA}} \Box_{\text{HA}}(\sigma_{\text{HA}} A) \leftrightarrow \Box_{\text{HA}}(\tau C) \leftrightarrow \Box_{\text{HA}}(\tau C^*) \leftrightarrow \Box_{\text{HA}}(\sigma_{\text{HA}} A^+),
$$

where we used Theorem 4.1.1(iii). This completes the induction.

(ii) The proof is completely analogous, but with an appeal to Corollary 4.1.2 instead of Theorem 4.1.1(iii).

\begin{lemma}
Let $U$ and $T$ be theories.

(i) If (4.1) holds for all $C \in \text{NNIL}$ and $\Sigma_1$-realizations $\sigma$, then for every $A \in \text{TNNIL}$, we have:

$$
\vdash_{\text{HA}} \Box_U(\sigma U A) \leftrightarrow \Box_{\text{T}}(\sigma_{\text{T}} A)
$$

(ii) If (4.2) holds for all $C \in \text{NNIL}$ and $\Sigma_1$-realizations $\sigma$, then for every $A \in \text{TNNIL}$, we have:

$$
\vdash_{\text{HA}} \Box_U^f(\sigma U A) \leftrightarrow \Box_{\text{T}}^f(\sigma_{\text{T}} A)
$$

\end{lemma}

\begin{proof}
(i) The proof is very similar to the proof of Lemma 4.1.5.

We proceed by strong induction on the boxdepth of $A$. Write $A = C(\vec{p}, \Box B_1, \ldots, \Box B_k)$, where $C(\vec{p}, q_1, \ldots, q_k) \in \text{NNIL}$ and $B_1, \ldots, B_k \in \text{TNNIL}$ are distinct. Then all the $B_i$ have smaller boxdepth than $A$, so we assume by induction hypothesis that

$$
\vdash_{\text{HA}} \Box_U(\sigma_{\text{HA}} B_i) \leftrightarrow \Box_{\text{T}}(\sigma_{\text{T}} B_i) \quad \text{for} \quad 1 \leq i \leq k.
$$

Now we take a $\Sigma_1$-realization $\tau$ such that $\tau \vec{p} = \sigma \vec{p}$ and $\tau(q_i) = \Box_U(\sigma_{\text{HA}} B_i)$ for $1 \leq i \leq k$. Now we observe that

$$
\begin{align*}
\sigma_{\text{HA}} A &= C(\sigma \vec{p}, \Box_U(\sigma_{\text{HA}} B_1), \ldots, \Box_U(\sigma_{\text{HA}} B_k)) = \tau C \quad \text{and} \\
\sigma_{\text{T}} A &= C(\sigma \vec{p}, \Box_{\text{T}}(\sigma_{\text{T}} B_1), \ldots, \Box_{\text{T}}(\sigma_{\text{T}} B_k)).
\end{align*}
$$

So (4.4) gives $\vdash_{\text{HA}} \sigma_{\text{HA}} A \leftrightarrow \tau C$. Since $\text{HA} \subseteq U^T$, we also get $\vdash_{\text{T}} \sigma_{\text{T}} A \leftrightarrow \tau C$. We also know that $\text{Bew}_{\text{T}}$ is a provability predicate for $U^T$, so we also find $\vdash_{\text{HA}} \Box_{\text{T}}(\sigma_{\text{T}} A) \leftrightarrow \Box_{\text{T}}(\tau C)$. Using (4.1), we get,

$$
\vdash_{\text{HA}} \Box_U(\sigma_{\text{HA}} A) \leftrightarrow \Box_U(\tau C) \leftrightarrow \Box_{\text{T}}(\tau C) \leftrightarrow \Box_{\text{T}}(\sigma_{\text{T}} A),
$$

which completes the induction.

(ii) The proof is again completely analogous, but of course with an appeal to (4.2) instead of (4.1).
\end{proof}
4.2 Some Fast ($\Sigma_1$-)Provability Logics

Let $U \supseteq \text{HA}$ be a $\Sigma_1$-sound theory. We consider the theory $U^* \supseteq \text{HA}$. First of all, we prove the following.

**Lemma 4.2.1.** The pair $\left(\text{Bew}_{U^*}(x), \text{Bew}_{U^*}(x)\right)$ is good for $U^*$.

**Proof.** By Lemma 2.3.2(v), we know that $\text{Bew}_{U^*}$ is a provability predicate for $U^*$, and we also know that $\text{Bew}_{U^*}$ is a provability predicate for $U^*$. By Lemma 2.2.7(iii), the theories $U$ and $U^*$ prove the same $\Sigma_1$-formulae, so $U^*$ is $\Sigma_1$-sound as well. By Lemma 2.3.2(vi), we see that $N \models \Box_U^* A$ implies that $N \models \Box_{U^*} A$, which implies $\vdash_{U^*} A$, for all $L$-sentences $A$.

By Corollary 2.2.10, we have $\vdash_{U^*} \text{CP}_{U^*}$. The final requirement for a good pair follows from Lemma 2.3.2(vii).

**Theorem 4.2.2.** Let $U \supseteq \text{HA}$ be a $\Sigma_1$-sound theory. Then the fast ($\Sigma_1$-)provability logic of $U^*$ is equal to the set of theorems of $\text{iGLC}$.

**Proof.** By Corollary 2.2.10, we have $\vdash_{U^*} A \rightarrow \Box_{U^*} A \rightarrow \Box_U^* A$ for all $L$-sentences $A$. Now both statements follow from Theorem 3.3.3 and Lemma 4.2.1.

**Remark 4.2.1.** Presently, the ordinary ($\Sigma_1$-)provability logic of $\text{HA}^*$ is unknown. We conjecture that it is equal to $\text{iGLC}$ as well. For $\text{PA}^*$, more is known. Since $\text{PA}$ is a classical theory, we have $\vdash_{\text{PA}} B \lor (B \rightarrow A)$ for all $L$-formulae $A$ and $B$. This means that we also have

$$\vdash_{\text{PA}} \Box_{\text{PA}}\Box_{\text{PA}} (B \lor (B \rightarrow A))$$

for all $L$-formulae $A$ and $B$. This, in turn, implies that

$$\vdash_{\text{PA}} \Box_{\text{PA}}\Box_{\text{PA}} (B \lor (B \rightarrow A)),$$

for all $L$-formulae $A$ and $B$. So the ($\Sigma_1$-)provability logic of $\text{PA}^*$ contains at least the theorems of $\text{iGLC}$ extended with the axiom scheme $\Box A \rightarrow (B \lor (B \rightarrow A))$. This scheme is called the propositional trace principle, or PTP for short. The theory $\text{iGLC} + \text{PTP}$ for IML is sound and complete with respect to finite frames $\langle W, \preceq, \sqsubset \rangle$, such that $w \sqsubset v$ iff $w \prec v$ for all $w, v \in W$. A. Visser used a proof in the style of Chapter 3 to show that the ($\Sigma_1$-)provability logic of $\text{PA}^*$ contains exactly the theorems of $\text{iGLC} + \text{PTP}$. Since in this case, the model relation $\sqsubset$ can be defined in terms of the intuitionistic relation $\preceq$, the definition of the Solovay function and the induction step for $\Box$ (our Lemma 3.2.6) are somewhat easier. Since $\text{iGLC} + \text{PTP}$ is a proper extension of $\text{iGLC}$ (i.e. has more theorems), we have an example of a theory for which the fast and ordinary provability logics do not coincide.

We now turn our attention to determining the fast $\Sigma_1$-provability logic of $\text{HA}$.

**Theorem 4.2.3.** Let $A \in L_\Box$. Then $A$ is in the fast $\Sigma_1$-provability logic of $\text{HA}$ if and only if $\text{iGLC} \vdash A^+$.

**Remark 4.2.2.** This result gives an ‘indirect’ characterization of the fast $\Sigma_1$-provability logic of $\text{HA}$, since we first have to apply the $\text{TNNI}$-algorithm, and then see whether the result is provable in $\text{iGLC}$. But we can already see that the fast $\Sigma_1$-provability logic of $\text{HA}$ is decidable,
since iGLC is decidable (this follows from the proof of Theorem 2.4.4). In the paper [1], the authors give a direct characterization of the set \{A ∈ L_\square \mid ⊢_{iGLC} A^+\}, by providing an axiomatization for it.

Proof of Theorem 4.2.3. Since HA ≤ HA and HA ≤ HA*, the conditions of Corollary 4.1.4 apply for U ≡ T ≡ HA, so (4.2) holds for U ≡ T ≡ HA. Now let A ∈ L_\square and let σ be a Σ₁-realization. Using Lemma 4.1.5(ii) and Lemma 4.1.6(ii), we find that

\[ ⊢_{HA} \square^f_{HA}(σ_{HA}A) ⇔ \square^f_{HA}(σ_{HA^+}A) ⇔ \square^f_{HA^*}(σ_{HA^*}A^+). \]

Since HA is sound, we see that \( N \models \square^f_{HA}(σ_{HA}A) \) if and only if \( N \models \square^f_{HA^*}(σ_{HA^*}A^+) \). Using Lemma 2.2.7(iii), we see that HA* is Σ₁-sound. Using Lemma 2.3.2(vi), we can now see that

\[ ⊢_{HA} σ_{HA}A \iff N \models \square^f_{HA}(σ_{HA}A) \iff N \models \square^f_{HA^*}(σ_{HA^*}A^+) \iff ⊢_{HA^*} σ_{HA^*}A^+. \]

This means that A is in the fast Σ₁-provability logic of HA if and only if \( A^+ \) is in the fast Σ₁-provability logic of HA*. By Theorem 4.2.2, the latter holds if and only if \( ⊢_{iGLC} A^+ \). ⊓⊔

4.3 The Σ₁-Provability Logic of HA

In this final section, we determine the (ordinary) Σ₁-provability logic of HA. This is also the main result of the paper [1], but the authors arrive at it using different methods.

Recall the theory slow Heyting Arithmetic sHA, that satisfies sHA = HA and sHA ≤ HA, but not HA ≤ sHA. We consider the theory \( \bar{HA} := HA^{sHA} ⊆ HA \). By Proposition 2.3.4, we know that \( ⊢_{HA} \square_{\bar{HA}}A \leftrightarrow \square_{HA}A_{sHA} \) for all \( L \)-formulae A.

Our first goal is to show that the (Σ₁-)provability logic of this theory is equal to the set of theorems of iGLC. In order to do this, we need to find a good pair for \( \bar{HA} \). In the previous section, the role of \( S(x) \) was fulfilled by fast provability. In this section, we put ordinary provability here. We define the Σ₁-formula \( R(x) \) as \( \text{Bew}_{sHA}(x_{sHA}) \). As usual, for an \( L \)-sentence A, we write \( ∆A \) for \( R(⌜A⌝) \). Then we see that \( ∆A \) is equivalent, over EA, to \( □_{sHA}A_{sHA} \).

Lemma 4.3.1. The pair \( (\text{Bew}_{\bar{HA}}(x), R(x)) \) is good for \( \bar{HA} \).

Proof. We already know that \( \text{Bew}_{\bar{HA}} \) is a provability predicate for \( \bar{HA} \). We show that \( R(x) \) is a provability predicate for \( \bar{HA} \) as well. First of all, let A and B be \( L \)-sentences and suppose that \( ⊢_{\bar{HA}} A \). Then by Theorem 2.2.3, we see that \( ⊢_{HA} A_{sHA} \), since HA ⊆ L_\square ⊆ sHA holds. But HA = sHA, so \( ⊢_{sHA} A_{sHA} \) as well, which yields \( N \models □_{sHA}A_{sHA} \). Moreover, since \( \text{Bew}_{sHA} \), we find

\[ ⊢_{HA} □_{sHA}(A \rightarrow B)_{sHA} \rightarrow □_{sHA}(A_{sHA} \rightarrow B_{sHA}) \rightarrow (□_{sHA}A_{sHA} \rightarrow □_{sHA}B_{sHA}). \]

Finally, if S is a Σ₁-sentence, then by Lemma 2.2.7(i), we get \( ⊢_{HA} S \leftrightarrow S_{sHA} \). Since \( \text{Bew}_{sHA} \) is a provability predicate for sHA, hence also for HA, we get

\[ ⊢_{HA} S \rightarrow □_{sHA}S \rightarrow □_{sHA}S_{sHA}, \]
4.3. THE $\Sigma_1$-PROVABILITY LOGIC OF HA

as desired.

Next, let $A$ be an $L$-sentence. We know from Proposition 2.1.4(i) that $N \models \square_{HA} A$ implies $\vdash_{HA} A$.

Moreover, by Lemma 2.2.9 with $U \equiv HA$ and $T \equiv sHA$, we have $\vdash_{HA} A \rightarrow \square_{sHA} A^{sHA}$.

Finally, let $S$ be a $\Sigma_1$-sentence. We recall that $\vdash_{HA} S \leftrightarrow S^{sHA}$.

Now we use Proposition 2.3.5(i) to find that:

\[
\vdash_{HA} \square_{HA} \square_{sHA} S^{sHA} \leftrightarrow \square_{HA} \square_{sHA} S^{sHA} \leftrightarrow \square_{HA} S^{sHA} \leftrightarrow \square_{HA} S,
\]

which finishes the proof.

Now that we have our good pair, we can prove the following.

**Lemma 4.3.2.** The $(\Sigma_1)$-provability logic of $\widehat{HA}$ is exactly the set of theorems of $iGLC$.

**Proof.** Since $sHA \leq HA$, we see that $\vdash_{HA} A \rightarrow \square_{sHA} A^{sHA} \rightarrow \square_{HA} A^{sHA} \rightarrow \square_{HA} A$

for every $L$-sentence $A$. This means that $\vdash_{HA} CP_{\widehat{HA}}$, so both statements follow from Theorem 3.3.3 and Lemma 4.3.1.

Since the theory $\widehat{HA}$ is rather ad hoc, this result is in itself not very interesting. But we can use it to obtain the theorem we were after.

**Theorem 4.3.3.** Let $A \in L_{\square}$. Then $A$ is in the $\Sigma_1$-provability logic of $HA$ if and only if $iGLC \vdash A^+$.

**Proof.** We have $HA \leq HA \leq \widehat{HA}$ and $HA \subseteq sHA$. Moreover, by Proposition 2.3.4, we have $\vdash_{HA} \text{Bew}_{HA}(x) \rightarrow \square_{HA} \text{Bew}_{sHA}(x)$. That is, the conditions of Proposition 4.1.3 apply, so equation (4.1) holds for $U \equiv HA$ and $T \equiv sHA$. Now let $\sigma$ be a $\Sigma_1$-realization. Using Lemma 4.1.5(i) and Lemma 4.1.6(i), we find that

\[
\vdash_{HA} \square_{HA}(\sigma_{HA} A) \leftrightarrow \square_{HA}(\sigma_{HA} A^+) \leftrightarrow \square_{HA}(\sigma_{HA} A^+).
\]

Since $HA$ is sound, we get

\[
\vdash_{HA} \sigma_{HA} A \iff N \models \square_{HA} \sigma_{HA} A \iff N \models \square_{HA}(\sigma_{HA} A^+) \iff \vdash_{HA} \sigma_{HA} A^+.
\]

This means that $A$ is in the $\Sigma_1$-provability logic of $HA$ if and only if $A^+$ is in the $\Sigma_1$-provability logic of $\widehat{HA}$. By Lemma 4.3.2, the latter holds if and only if $\vdash_{iGLC} A^+$.
Chapter 5

Conclusion

In this thesis, our goal was to give a Solovay-style embedding of frames equipped with both an intuitionistic relation $\preceq$ and a modal relation $\Box$. In order to approach this task, we considered theories that prove their own completeness principle. This project has led to the following results and insights.

(i) We were able to give a Solovay-style embedding of finite, irreflexive, realistic frames for IML, in the presence of the completeness principle and the principle $\Box \Delta S \rightarrow \Box S$ for $S \in \Sigma_1$.

(ii) We reproved the result from [1] that the $\Sigma_1$-provability logic of Heyting Arithmetic is equal to the set $\{ A \in \mathcal{L} \mid \vdash_{iGLC} A^+ \}$.

(iii) We showed that the fast $\Sigma_1$-provability logic of HA is also equal to this set.

(iv) We showed that for any $\Sigma_1$-sound theory $U \supseteq HA$, the fast ($\Sigma_1$-)provability logic of $U^*$ is equal to the set of theorems of iGLC.

(v) We found an intuitionistic theory of arithmetic other than PA*, namely the theory $\hat{HA}$, for which we were able to determine the provability logic.

(vi) We discovered that for the theory PA*, the fast provability logic and the ordinary provability logic do not coincide.

Of course, a variety of questions remains open. We mention a few of them, ordered from small to large.

(i) Does the result $\vdash_{HA} \Box_{HA} \Box_{HA} S \rightarrow \Box_{HA} S$ for $S \in \Sigma_1$ (Proposition 2.3.5(i)) also hold for sentences that are not $\Sigma_1$, as it does in the classical case? Or do we even have this principle uniformly over formulae, i.e. $\vdash_{HA} \text{Form}(x) \land \Box_{HA} \text{Bew}_{HA}(x) \rightarrow \text{Bew}_{HA}(x)$?

(ii) What is the ordinary provability logic of HA* and of other theories $U^*$, where $U \supseteq HA$ is $\Sigma_1$-sound? We conjecture that the former is equal to the set of theorems of iGLC. We would like to make two remarks about possible approaches to proving this.

(a) If $T \supseteq HA$ is a consistent theory, then a good pair $(S, R)$ for $T$ can never satisfy $\vdash_T S(x) \leftrightarrow R(x)$. Therefore, Theorem 3.3.3 can only provide us with the logic for $\Box$, and not with the logic for $\Delta$. Since HA* is specifically designed to prove its own
completeness principle $\text{CT}_{\text{HA}^*}$, i.e. $\text{CT}_{\Box_{\text{HA}^*}}$, Theorem 3.3.3 seems to be useless for determining the provability logic of $\text{HA}^*$, i.e. the logic for $\Box_{\text{HA}^*}$.

(b) As we remarked in the Introduction, Solovay’s original proof has an incredible upward stability. That is, we can use almost the very same proof to show that any $\Sigma_1$-sound theory extending $\text{PA}$ has the same provability logic as $\text{PA}$. In the present context, this strength is worrisome. If one wishes to gives a Solovay-style proof, one might aim for a theorem like: ‘if $T \supseteq \text{HA}$ is a $\Sigma_1$-sound theory such that $\vdash_T \text{CT}_T$, then the provability logic of $T$ is equal to the set of theorems of $i\text{GLC}$’. This ‘theorem’ is false, however, since $\text{PA}^*$ is a counterexample. If we want to show that the provability logic of $\text{HA}^*$ is equal to the set of theorems of $i\text{GLC}$, we will need a way to distinguish $\text{HA}^*$ from $\text{PA}^*$ in our proof.

(iii) What is the provability logic of $\text{HA}$? We have not really touched upon this question, and it remains wide open. The $\Sigma_1$-provability logic of $\text{HA}$ at least provides a non-trivial upper bound, but we know that this upper bound cannot be strict, since $A \rightarrow \Box A$, which is in the $\Sigma_1$-provability logic of $\text{HA}$, is definitely not in the provability logic of $\text{HA}$ itself.
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[1] M. Ardeshir and S. Mojtahedi. The $\Sigma_1$ Provability Logic of HA. ArXiv e-prints. Available at arxiv.org/abs/1409.5699v1.

[2] George Boolos. The Logic of Provability. Cambridge University Press, 1993.

[3] Petr Hájek and Pavel Pudlák. Metamathematics of First-Order Arithmetic. Springer-Verlag, 1993.

[4] Paula Henk. Nonstandard Provability for Peano Arithmetic: A Modal Perspective. PhD thesis, Institute for Language, Logic and Computation, University of Amsterdam, 2016.

[5] Paula Henk and Fedor Pakhomov. Slow and Ordinary Provability for Peano Arithmetic. ArXiv e-prints. Available at arxiv.org/abs/1602.01822.

[6] R. Iemhoff. A Modal Analysis of Some Principles of the Provability Logic of Heyting Arithmetic. In Proceedings of AiML’98, volume 2, 2001.

[7] Per Lindström. On Parikh Provability: An Exercise in Modal Logic. In S. Lindström H. Lagerlund and R. Sliwinski, editors, Modality Matters: Twenty-Five Essays in Honour of Krister Segerberg, volume 53 of Uppsala Philosophical Studies. Uppsala Universitet, 2006.

[8] Rohit Parikh. Existence and Feasibility. Journal of Symbolic Logic, 36(3):494–508, 1971.

[9] A.S. Troelstra and D. van Dalen. Contractivism in Mathematics I, volume 121 of Studies in Logic and the Foundations of Mathematics. Elsevier, 1988.

[10] Albert Visser. On the Completeness Principle: A Study of Provability in Heyting’s Arithmetic and Extensions. Annals of Mathematical Logic, 22(3):263–295, 1982.

[11] Albert Visser. Substitutions of $\Sigma^0_1$-sentences: Explorations between Intuitionistic Propositional Logic and Intuitionistic Arithmetic. Annals of Pure and Applied Logic, 114:227–271, 2002.