Logarithmic correction of the BTZ black hole and adaptive model of Graphene

Behnam Pourhassana,∗ Mir Faizalb,c† and S. Ahmad Ketabia,‡

aSchool of Physics, Damghan University, Damghan, 3671641167, Iran
bDepartment of Physics and Astronomy, University of Lethbridge, Lethbridge, AB T1K 3M4, Canada
cIrving K. Barber School of Arts and Sciences, University of British Columbia - Okanagan, Kelowna, BC V1V 1V7, Canada

Abstract

It is known that almost all approaches to quantum gravity produce a logarithmic correction term to the entropy of a black hole, but the exact coefficient of such a term varies between the different approach to quantum gravity. Such logarithmic terms can also occur due to thermal fluctuations in both analogous and real black holes so that we will analyze the effects of logarithmic corrections term with variable coefficient on properties of analogous black hole. As these properties can be experimentally tested, they can be used to obtain the correct coefficient for such terms for an analogous black hole. We will argue that as even the real black holes can be considered as thermodynamical objects in Jacobson formalism, so such analogous black holes can be used to obtain the correct coefficient for the real black holes, and this in turn can be used to select the correct approach to quantum gravity. In that case, we use an adaptive model of graphene, which is still far from real graphene, to investigate some thermodynamics quantities of BTZ black hole.

Keywords: Black Hole, Thermal Fluctuation, Quantum Gravity.

1 Introduction

It is important to understand how gravity is quantized to understand physical systems like the black holes and even the physics at the big bang. The problem with testing quantum
gravity is that the energy scale at which quantum gravity operates is so large that it has not been possible to directly test quantum gravitational effects. Furthermore, there are various approaches to quantum gravity, and they seem to take different approaches to describe the fundamental structure of space-time. Thus, it becomes even more difficult to test quantum gravitational effects. However, there is one result that seems to be almost a universal prediction of all theories of quantum gravity. This prediction is that the leading order term in the entropy of a black hole is proportional to the area of the black hole, \( S_0 \sim A \). There are good reasons for such a prediction, as this result can also be produced from semi-classical approximation. So, it cannot be a purely quantum gravitational effect, but it seems to be an effect that is generated because of quantum field theory in a classical space-time. It may be noted that it is not possible to measure the black hole thermodynamics directly, and so it is difficult to use the black hole thermodynamics to test any quantum gravitational effect. However, it is possible to use analogous black hole like adaptive model of graphene to analyze such systems, as it is possible to construct an effective BTZ-like black hole geometry in graphene [1, 2]. In this effective BTZ-like solution, the velocity of light gets replaced by the Fermi velocity \((v_f \approx 0.003c)\), and other constants are also defined effectively. This is because the effective field theory describing a graphene sheet resembles a massless Dirac equation in three dimensions [3]. A sheet of graphene is made up of carbon atoms arranged in hexagonal structures. In this structure, there exists a triangular lattice with two atoms per unite cell. So, the lattice vector for graphene is given by

\[
\mathbf{a}_1 = \frac{a}{2} \left( 3, \sqrt{3} \right), \quad \mathbf{a}_2 = \frac{a}{2} \left( 3, -\sqrt{3} \right),
\]

The carbon-carbon distance for this structure can be written as

\[a \sim 1.42 \text{Å}.\]

The reciprocal lattice vector for graphene is given by

\[
\mathbf{b}_1 = \frac{2\pi}{3a} \left( 1, \sqrt{3} \right), \quad \mathbf{b}_2 = \frac{2\pi}{3a} \left( 1, -\sqrt{3} \right).
\]

Furthermore, two points at the corners of graphene Brillouin zone are called the Dirac points, and they are located at

\[
K = \left( \frac{2\pi}{3a}, \frac{2\pi}{3\sqrt{3}a} \right), \quad K' = \left( \frac{2\pi}{3a}, -\frac{2\pi}{3\sqrt{3}a} \right).
\]

In this structure, the three nearest neighbors can be represented by

\[
\delta_1 = \frac{a}{2} \left( 1, \sqrt{3} \right), \quad \delta_2 = \frac{a}{2} \left( 1, -\sqrt{3} \right), \quad \delta_3 = -a \left( 1, 0 \right).
\]

The six next to the nearest neighbors can be represented by

\[
\delta_1 = \pm \mathbf{a}_1, \quad \delta_2 = \pm \mathbf{a}_2, \quad \delta_3 = \pm \left( \mathbf{a}_1 - \mathbf{a}_2 \right).
\]
The electrons in this structure are described by the tight-binding approach. So, electrons in graphene can hop to both nearest atoms and next to the nearest atoms and the Hamiltonian for graphene can be written as

\[
H = -t \sum (a_\uparrow \sigma b_\sigma + h.c.) - t' \sum (a_\uparrow \sigma a_\sigma + b_\uparrow \sigma b_\sigma + h.c.),
\]

(7)

where \(\sigma\) denotes the spin. The hopping energy to the nearest neighbor is \(t \sim 2.8 eV\), and the hopping energy to the next to the nearest neighbor is denoted by \(t'\). So, the energy bands for graphene can be written as

\[
E_\pm = \pm t \sqrt{3 + f(k)} - t' f(k),
\]

(8)

and here \(f(k)\) is defined as

\[
f(k) = 2 \cos \sqrt{3}(k_y a) + 4 \cos \left(\frac{\sqrt{3}}{2} k_y a\right) \cos \left(\frac{3}{2} k_x a\right).
\]

(9)

If this energy band is expanded around the Dirac point, then we will obtain

\[
E_\pm \sim \pm v_f |q|,
\]

(10)

where \(k = K + q\), and a similar expression can be obtained for \(K'\). The velocity \(v_f \approx 0.003c\) is the Fermi velocity. Now, to obtain the Hamiltonian of graphene close to the Dirac point, the following approximation can be used

\[
\sum_i e^{\pm K \delta_i} = \sum_i e^{\pm K' \delta_i} = 0.
\]

(11)

It may be noted that near the Dirac point, in the continuum limit, this Hamiltonian for graphene is the resembles the Dirac Hamiltonian \([7]\). The only difference between this Hamiltonian and the usual Dirac Hamiltonian is that the velocity of light is replaced by the Fermi velocity. In fact, it has also been verified experimentally that the graphene is described by a massless Dirac equation in three dimensions \([4, 5]\). So, the effective field theory of a flat sheet of graphene is the Dirac equation in \((2 + 1)d\) space-time, with an effective Lorentz symmetry. However, in this effective field theory, the velocity of light being replaced by Fermi velocity. This is the main difference between the relativistic Dirac equation in three dimensions and this effective field theory describing graphene \([3]\).

It is also known an effective curvature like effects can be induced in a deformed sheet of graphene. A graphene sheet has been used for constructing the surface of revolution with constant negative curvature \([11]\). A gravito-magnetic field can be introduced because of rotation, and its effects on particles resemble that of a magnetic field. So, an externally magnetic field would induce an effective gauge field on the graphene sheet. This would break time reversal invariance but not conformal invariance. Thus, it has been argued that the curved graphene sheet with constant negative curvature in the externally applied electromagnetic field could be modeled by the stationary optical metric of the Zermelo form,
and it is conformal to the BTZ black hole \[1, 2\]. Thus, a deformation of the graphene sheet can produce a BTZ black hole like solution, and in this BTZ black hole like solution, the velocity of light is replaced by the Fermi velocity. So, it is possible to analyze an effective BTZ-like black hole solution in graphene \[1, 14\]. It should have pointed out that it is only suggestive and the real graphene is still far. It may be noted that the horizon of a usual BTZ black hole traps photons, but the effective horizon of the black hole in graphene traps fermions moving with the Fermi velocities. Entropy is associated with all black objects in general relativity \[12, 13\], and so we expect that an entropy should also be associated with the black hole like solutions in graphene. In fact, it has been demonstrated that Hawking radiation is radiated from such solution in graphene \[2, 14, 15\]. It may be noted that the Hawking radiation from such a solution is an effective phenomenon, and it occurs as the effective horizon traps Fermi velocities.

In this paper, we will use the effective black hole thermodynamics in the analogous BTZ like black hole solution in the adaptive model of graphene, which is still far from the real graphene, to gain the correct approach to quantum gravity. This will be done by analyzing the effect of the corrections to the entropy of such an effective BTZ-like black hole solution in the adaptive model of graphene. Such quantum corrections of rotating BTZ black hole recently studied by the Ref. \[16\].

## 2 Logarithmic correction

Almost all approaches to quantum gravity also predict that the leading order correction to the entropy of a black hole would be a logarithmic correction. This logarithmic correction has been obtained using non-perturbative quantum general relativity \[17\]. In this approach, the relation between the density of states of a black hole and the conformal blocks of a well-defined conformal field theory have been used to obtain this logarithmic correction. The Cardy formula has also been used for obtaining such correction terms \[18\]. An exact partition function has been used for analyzing the correction to the entropy of a BTZ black hole, and it has been demonstrated that the leading order corrections are to the entropy of a BTZ black hole are logarithmic corrections \[18\]. The leading order corrections terms for dilatonic black holes have also been demonstrated to be logarithmic corrections \[19\]. String theory has also been used for analyzing the corrections to the entropy of a black hole, and it has been observed that string theoretical effects also produce a logarithmic correction to the entropy of a black hole \[20, 21, 22\]. It has been demonstrated that the generalized uncertainty principle has also generated logarithmic correction terms for the entropy of a black hole \[23, 24\].

Even though the existence of such a logarithmic term is predicted by almost all approaches to quantum gravity, the exact coefficient of such a term is different in different approaches to quantum gravity. Thus, we can write the general form for the leading order correction terms for the entropy of a black hole as \(S = S_0 + S_1\), where \(S_0 \sim A\) and \(S_1 \sim \ln A\). The quantum fluctuations become important when the size of the black hole becomes sufficiently small. However, the temperature of a black hole increases as the size of the black hole
becomes small, and so the effects of thermal fluctuations also increase as the size of the black hole becomes small \cite{25}. Hence, it seems that the thermal fluctuations in the thermodynamics of a black hole can be related to the quantum fluctuations in the geometry of a black hole. This is more obvious in the Jacobson formalism \cite{26,27}. This is because in the Jacobson formalism, that the Einstein’s equations are obtained the thermodynamics by requiring that the Clausius relation holds for all the local Rindler causal horizons through each space-time point \cite{26,27}. So, in the Jacobson formalism, thermal fluctuations in the thermodynamics will generate quantum fluctuations in the geometry of the space-time. Corrections to the thermodynamics of black holes from such thermal fluctuations have been analyzed \cite{28,29}, and it was observed that such corrections are again logarithmic corrections.

In the Ref. \cite{30} the black hole entropy corrections due to thermal fluctuations in the various AdS black hole extensive parameters have been studied. It was found universality in the logarithmic corrections to charged black hole entropy in various dimensions. These corrections are expressed regarding the black hole response coefficients via fluctuation moments. Also, the black hole entropy correction was calculated from the quantum correction point of view using the near horizon conformal algebra \cite{31}.

In fact, the entropy of a very small black hole after thermal fluctuations were taken into account can be written as

$$S = S_0 + S_1 = S_0 - \ln S_0 T^2 / 2.$$ 

As the existence of a logarithmic term is a universal feature of all approaches to quantum gravity, but the exact coefficient depends on the exact approach to quantum gravity chosen, we will not fix the coefficient of the logarithmic term, and write the corrected entropy of the black hole as

$$S = S_0 - \alpha \ln S_0 T^2 / 2,$$

where $\alpha$ is a parameter that depends on the details of the model used. It may be noted that for thermal fluctuations in the Jacobson formalism \cite{26,27}, $\alpha = 1$ holds for very small black holes where the effects of thermal fluctuation have to be taken into consideration, and for very large black hole where the effects of thermal fluctuations can be neglected it is possible to take $\alpha = 0$. The effect of thermal fluctuations on the thermodynamics of black holes in anti-de Sitter space-time has been studied using this formalism \cite{32}. This formalism has also been used for analyzing the corrections to the thermodynamics of a black Saturn \cite{33,34} and modified Hayward black hole \cite{35}. In the Ref. \cite{36} we have been study P-V criticality of logarithm-corrected dyonic charged AdS black holes as well as AdS black holes in massive gravity \cite{37} and investigate thermal fluctuation effects. Also, thermodynamics of an infinitesimal singly spinning Kerr-AdS black hole investigated with mentioned logarithmic correction \cite{38}. Such logarithmic term considered for the STU black hole and found important modification in the thermodynamics and also hydrodynamics \cite{39}. Logarithmic correction as quantum effects considered recently for the Horava-Lifshitz black hole \cite{40}. As logarithmic correction terms occur in all approaches to quantum gravity, we can analyze such effects using an adaptive model of graphene. In that case, already we study that using dumb holes \cite{41}. Furthermore, as such terms can be generated from thermal fluctuations, such terms will occur in the adaptive model of graphene.
3 Thermodynamics

So, now we will analyze the thermodynamics of an effective BTZ-like solution in the adaptive model of graphene, with the velocity of light effectively replaced by the Fermi velocity, and this effective BTZ-like black holes in adaptive model of graphene can be written as [1, 2],

\[ ds^2_{BTZ} = \frac{dr^2}{\Delta} + r^2d\phi^2 + \left(\frac{J^2}{4r^2} - \Delta\right)v_f^2dt^2 - Jv_fdtd\phi, \]  

(12)

where \( \Delta = r^2(l^2)^{-1} - M + J^2(4r^2)^{-1}, M \) is mass and \( J \) is the angular momentum of black hole. The black hole horizons obtained by setting \( \Delta = 0 \). Thus, for a BTZ black hole [42] like solution in graphene [1, 14], we can write

\[ r^2_{\pm} = l(lM \pm \sqrt{(lM)^2 - J^2})/2, \]

(13)

where \( r^2_{\pm} = lJ/2 \). The standard entropy of the black hole is given by

\[ S_{0\pm} = \frac{A_{\pm}v_f^3}{4} = 4\pi r^2_{\pm}v_f^3 = 2\sqrt{2}\pi \sqrt{l(lM \pm \sqrt{(lM)^2 - J^2})}v_f^3, \]

(14)

where \( S_{0\pm} \) is original entropy, which is obtained by neglecting thermal fluctuations. The Hawking temperature of the inner horizon and the outer horizon is given by,

\[ T_{\pm} = \frac{\kappa_{\pm}}{2\pi} = \frac{1}{4\pi} \left(\frac{2r_{\pm} + J^2}{l^2 - 2r_{\pm}^3}\right) \]

(15)

\[ = \frac{\sqrt{2}(lM)^2 \pm lM \sqrt{(lM)^2 - J^2} - J^2}{2\pi l^2}(lM \pm \sqrt{(lM)^2 - J^2})^2. \]

Here the surface gravity of the inner is denoted by \( \kappa_{-} \) and the surface gravity of the outer horizon is denoted by \( \kappa_{+} \). It is possible to express the product of temperatures as

\[ 4\pi^2T_{\pm}T_{\mp} = \frac{16r^4_{\pm} - 4l^2J^2(r^2_{\pm} + r^2_{\mp}) - l^4J^4}{16l^4r^3_{\pm}r^3_{\mp}}, \]

(16)

The entropy and temperature of this black hole can be used to write the Komar energy of the black hole, [43],

\[ E_{0\pm} = 2S_{0\pm}T_{\pm} = \frac{4r^4_{\pm} - l^2J^2r^2_{\pm}}{l^2r^2_{\pm}}v_f^3 = 4v_f^3 \left[ M - \frac{2J^2}{r^2_{\pm}} \right]. \]

(17)

Again, the effects of thermal fluctuations have been neglected. The product of the Komar energies can be expressed as

\[ E_{0+}E_{0-} = 16v_f^6 \left[ M^2 - \frac{2MJ^2}{r^2_{\mp}} - \frac{2MJ^2}{r^2_{\pm}} + \frac{4J^4}{r^2_{\pm}r^2_{\mp}} \right]. \]
It may be noted that the product of the temperatures and Komar energies depends on the black hole mass, while $A_+ A_-$ and $S_0^+, S_0^-$ are independent of the mass of the black hole.

Using the Eq. (13), we can obtain ADM mass of the black hole

$$M_{0_{\text{ADM}}} = \frac{S_0^+}{16\pi^2 l^2} + \frac{4\pi^2 J^2}{S_0^\pm}.$$  \hspace{1cm} (18)

Now, the first law of thermodynamics for this black hole may be

$$dM_{0_{\text{ADM}}} = T_\pm dS_0^\pm + \Omega_0^\pm dJ,$$  \hspace{1cm} (19)

where

$$\Omega_0^\pm = \left(\frac{dM_{0_{\text{ADM}}}}{dJ}\right)_{S_0^\pm} = \frac{8\pi^2 J}{S_0^\pm}.$$  \hspace{1cm} (20)

and

$$T_\pm = \left(\frac{dM_{0_{\text{ADM}}}}{dS_0^\pm}\right)_J.$$  \hspace{1cm} (21)

It may be noted that the first law of thermodynamics satisfied for this black hole solution. We can also calculate the specific heat of this black hole solution as

$$C_{0^\pm} = \frac{dM_{0_{\text{ADM}}}}{dT_\pm} = 4\pi v^2 r_\pm \frac{4r_\pm^4 - l^2 J^2}{4r_\pm^4 + 3l^2 J^2}.$$  \hspace{1cm} (22)

Thus, this black hole is in stable phase ($C_{0^+} \geq 0$) for $lM \geq J$, and $C_{0^-} < 0$ for $lM \geq J$.

4 Corrected thermodynamics

We can now include the thermal fluctuations and analyze the produced effects by them. Thus, we will write the corrected entropy of this black hole as

$$S_\pm = S_0^\pm - \frac{\alpha}{2} \ln S_0^\pm T_\pm^2,$$  \hspace{1cm} (23)

We expected that $C_\pm \geq 0$ for $lM \geq J$, i.e., both $C_+$ and $C_-$ have positive values. In the Fig. 1 we have plotted the value of the specific heat. It may be noted that for $C_-$ to be positive, the logarithmic correction has to be included. In Fig. 1(a), it is demonstrated that $C_+$ without logarithmic correction is negative for any choice of $J$ and $M$. In the presence of logarithmic correction, we have $C_+ \geq 0$ for the massive black hole. Thus, for $C_\pm \geq 0$, with appropriate choice of the black hole mass, the effect of thermal fluctuations has to be considered. In that case, we have $E_\pm = 2S_\pm T_\pm$. It is easy to show that $E_+ \leq E_{0^+}$, and the equality hold for small masses. We also observe that $E_{0^-}$, as well as $E_-$, may change the sign for a massive black hole. It is clear that logarithmic correction increases the value of the energy. This situation is illustrated by the plots of the Fig. 2. Furthermore, $E_{0^+}E_{0^-}$ is negative for a massive black hole, and $E_+E_-$ is positive for a massive black hole. For the
small values $M$, ($Ml \approx 2J$), we have $E_{0+}E_{0-} = E_{+}E_{-}$.

To have a comparison with the adaptive model of graphene, we should represent our results in terms of temperature. In the case of $lJ \ll 1$ one can obtain analytical expressions. In the Fig. (a) we can see temperature dependence specific heat. As before, in the case of $\alpha = 0$ we find negative specific heat, but in the presence of logarithmic correction we see a phase transition. Specific heat is positive for the low temperature, while it is negative for the higher temperature. It means that at the higher temperature, graphene structure is broken and entropy will be negative (as illustrated by the Fig. (b)). Asymptotic behavior of specific heat is corresponding to maximum value of the specific heat of the adaptive model of graphene [44]. We find that value of the corrected entropy does not affected by small $J$. Also, in the Fig. (c) we can see behavior of Helmholtz free energy in terms of temperature...
and see that is increasing function of temperature with positive value for the case of \( \alpha = 0 \) while there is a maximum for the logarithmic corrected case at very low temperature, and general behavior of Helmholtz free energy is increasing function of the temperature with negative value, which is in agreement with behavior of the adaptive model of graphene [44].

Figure 3: (a) Specific heat in term of temperature with \( l = 1, v_f = 0.003, \alpha = 1, J = 0 \) (solid red), \( J = 0.2 \) (dashed blue), \( J = 0.4 \) (dotted green). (b) Entropy in term of temperature with \( l = 1, v_f = 0.003, \alpha = 1 \). (c) Helmholtz free energy in terms of temperature with \( l = 10, v_f = 0.009, \) and \( J = 0.1; \alpha = 0 \) (dashed blue), \( \alpha = 1 \) (solid red).

5 Hawking radiation

Emission rate of Hawking radiation given by [45],

\[
\Gamma = e^{\Delta S_t},
\]  

(24)
which obtained from the action for an outgoing positive-energy particle which crosses the event horizon outwards from

\[ r_{in+} = \sqrt{\frac{l}{2}(lM + \sqrt{(lM)^2 - J^2})}, \]  

(25)

to

\[ r_{out+} = \sqrt{\frac{l}{2}(l(M - \omega) + \sqrt{(l(M - \omega))^2 - J^2})}, \]  

(26)

so \( \Delta S_+ = S(r_{out+}) - S(r_{in+}) \). In the Fig. 4 we can see the effect of thermal fluctuations on the emission rate of Hawking radiation given by the equation (24). It is clear that thermal fluctuation increases the value of the emission rate.

![Figure 4](image-url)

Figure 4: Emission rate of Hawking radiation in terms of the logarithmic correction parameter with \( l = 1, \omega = 1 \) and \( M = 2 \). \( J = 0 \) (blue dot), \( J = 0.2 \) (cyan dash), \( J = 0.4 \) (green dash dot), \( J = 0.6 \) (red long dash), \( J = 0.8 \) (black solid).

We can use \( J = 0 \) limit of BTZ black hole with \((-+,+,+,+)\) signature as ansatz for graphene [46], hence rewrite metric (12) as follow,

\[ ds_{BTZ}^2(J = 0) = -\left(\frac{R^2}{c^2} - M\right)dt^2 + \frac{dR^2}{\frac{R^2}{c^2} - M} - R^2d\chi^2, \]

(27)

where \( c \) and \( M \) are positive constants, and \( \chi \) is the angular variable (\( \phi \)). It can be rewrite as,

\[ ds_{BTZ}^2(J = 0) = -\left(\frac{r^2}{c^2} - M\right)dt^2 + \frac{c^2r^2 - Mc^4}{(r^2 - r^2_\pm)^2}dr^2 + \frac{(r^2 - Mc^2)r^2}{r^2 - r^2_\pm}d\chi^2, \]

(28)
where we used our notation for radial coordinate $R \equiv r$. Also, $r^+ = c\sqrt{M}$ is the black hole event horizon. In that case, the black hole temperature can be written as,

$$T = c_1 r^+, \quad (29)$$

where $c_1 = (2\pi c^2)^{-1}$ is a positive constant. Also, the black hole entropy given by,

$$S_0 = c_2 r^+, \quad (30)$$

where $c_2 = 4\pi c^3$ is a positive constant. Hence, we can write,

$$S_0 = c_3 T, \quad (31)$$

where $c_3 = 8\pi^2 c^5$ is another positive constant. Of course, the ansatz for graphene tell $c = l$. Therefore, we have the following corrected entropy,

$$S = c_3 T - \frac{\alpha}{2} \ln (c_3 T^3). \quad (32)$$

In that case, the specific heat obtained as,

$$C = T \left( \frac{dS}{dT} \right) = c_3 T - \frac{3\alpha}{2}. \quad (33)$$

It is clear that the effect of logarithmic correction is a reduction of specific heat (for positive $\alpha$ as usual). Now we can obtain internal energy as,

$$U = \frac{c_3}{2} T^2 - \frac{3\alpha}{2} T. \quad (34)$$

We can see that the logarithmic correction decreased the value of the internal energy. By using the entropy (32) and internal energy (34) we can obtain Helmholtz free energy as follow,

$$F = E - TS = \frac{c_3}{2} T^2 - (c_3 + \frac{3\alpha}{2}) T + \frac{\alpha}{2} \ln (c_3 T^3). \quad (35)$$

In general, we can say that effect of logarithmic correction is decreasing of thermodynamics potentials like Helmholtz free energy. Without logarithmic correction ($\alpha \neq 0$) there is a minimum for the free energy while in the presence of logarithmic correction there is no minimum point for the Helmholtz free energy instead there is an inflection point.

As before we can investigate emission rate of Hawking radiation which illustrated by the Fig. 5.

### 6 Randers optical metric

The BTZ like metric (12) can conformally transform to the Randers optical metric, and in a sheet of graphene, it can be written as [1]

$$ds^2_R = \frac{d\nu^2}{\Delta \Delta_0} + \frac{1}{\Delta^2_0} \left( \Delta r^2 - \frac{J^2}{4} \right) d\phi^2 - \frac{J^2}{\Delta_0} v_f dt d\phi, \quad (36)$$
where \( \Delta_0 = \frac{r^2}{l^2} - M \), and the value of \( \Delta \) is similar to its value in the original BTZ metric. So, the value of the original entropy, \( S_{0\pm} \), is also similar to the the value obtained in Eq. (13). The Hawking temperature of the inner horizon and the outer horizon is now given by,

\[
T_{\pm} = \frac{8r_{\pm}^6 - 8Ml^2r_{\pm}^4 + MI^4J^2}{8\pi l^4r_{\pm}^3} = \frac{\sqrt{2}J^2 - (lM)^2 \pm lM \sqrt{(lM)^2 - J^2}}{4\pi l^2} \frac{r_{\pm}^6}{(lM \pm \sqrt{(lM)^2 - J^2})^2}. \tag{37}
\]

The product of temperatures on both the horizons can now be written as

\[
4\pi^2 T_+ T_- = \frac{64Ml^2r^4_+ r^4_- (MI^2 - r^2_- - r^2_+ + \frac{r^4_+ r^4_-}{MI^4})}{16l^4 r^3_+ r^3_-} + \frac{8MJ^2l^4 (r^6_+ + r^6_- - MI^2(r^4_+ + r^4_-)) + J^4M^2l^8}{16l^4 r^3_+ r^3_-}. \tag{38}
\]

Now, we can again analyze the effect of thermal fluctuations on the thermodynamics of the black holes. We observe that the energy and specific heat are negative even in the presence of thermal fluctuations. The Zermelo optical metric in a sheet of graphene can be written as

\[
\frac{ds^2}{\Delta^2} = \frac{dr^2}{\Delta^2} + \frac{r^2}{\Delta} d\phi^2 + \left( \frac{J^2}{4\Delta r^2} - 1 \right) \nu_2^2 dt^2 - \frac{J}{\Delta} \nu_2 dt d\phi. \tag{39}
\]
Here the temperature on the horizon is given by $T_\pm = 0$, so specific heat is zero, and in absence of thermal fluctuations the entropy is given by the Eq. (13). Hence, the entropy in presence of thermal fluctuations will be infinite and system will remain in equilibrium. Thus, using Eq. (21), we obtain,

$$\left(\frac{dM_{0\text{ADM}}}{dS_0}\right)_J = \frac{S_0}{8\pi^2 l^2} - \frac{8\pi^2 J^2}{S_0} = 0.$$ (40)

So, in this case we have $lM = J$, and a black hole is in the extremal limit where $r_+ = r_-$. Thus, we can analyze the effect of logarithmic correction of a sheet of graphene. This can be used to obtain the correct value of $\alpha$ comparing the adaptive model of graphene. As even the real black holes can be considered as thermodynamical objects in the Jacobson formalism [26, 27], we will argue that this is the value of $\alpha$ which should occur even in real black holes. Thus, we can select the correct approach to quantum gravity.

7 Summary

We can summarize our argument as follows:

1. It may be noted that in the effective field theory describing graphene, the velocity of light $c$ gets replaced by a much smaller Fermi velocity $v_f$. However, the main symmetries of background geometry describing a flat sheet of graphene is an effective $(2+1)d$ Lorentz symmetry, and any curvature in the sheet of graphene is described as a curved $(2+1)d$ space-time. Thus, it is possible to have an effective horizon in graphene, and in this effective horizon the Fermi velocity effectively acts like the velocity of light for such analogous black holes.

2. It is known that almost all approaches to quantum gravity predict the logarithmic corrections to the entropy of a black hole, but the coefficient of this logarithmic term differs between different approaches to quantum gravity. Hence, this coefficient can be used to select between the different approaches to quantum gravity. So, if we can know what the correct coefficient for the logarithmic term, we can select the correct approach to quantum gravity.

3. Analogous black holes can be used to test the correct coefficient for the logarithmic term, as it has been demonstrated that an analogous BTZ like metric can exist in graphene [1, 2]. So, in this paper, we analyzed the effects of the logarithmic corrections to the entropy of such an analogous black hole in adaptive model of graphene. We keep the coefficient of this BTZ like black hole as a variable. Thus, in this paper, we analyze the effects of the logarithmic corrections to the entropy of a BTZ like black holes in adaptive model of graphene, with a variable coefficient for this logarithmic term.

4. It is possible to calculate the effective Hawking-like radiation in graphene, and also analyze the effect of logarithmic corrections on such effective processes. However, such
effective Hawking radiation can be analyzed in experimentally using a curved sheet of graphene, and thus, the coefficients of such a term can be fixed. Thus, we can use the adaptive model of graphene to fix the coefficient of quantum correction to an analogous effective black hole.

5. It may be noted that there is no fundamental difference between analogous black holes and real black holes in the Jacobson formalism [26, 27]. As in this formalism, even the real black holes (and all of geometry in general relativity) are some thermodynamical objects, just like the analogous black hole. However, we still expect logarithmic corrections even in Jacobson formalism and analogous black holes due to thermal fluctuations [28, 29].

8 Conclusion and discussion

In this paper, we analyzed the black hole like solution that has been that occurs in the effective field theory describing an adaptive model of graphene. Although it is still far from real graphene but it is possible to study from several points of view like superconductivity [47, 48].

It is possible to obtain an effective BTZ-like solution in graphene [1, 2]. Here the velocity of light gets replaced by the Fermi velocity. However, it is possible to have a horizon for such an analogous black hole, and just as a black hole traps photon moving with the velocity of light, this analogous black hole traps fermions moving with the Fermi velocity. Furthermore, the Hawking radiation from analogous back holes in graphene has been studied [1, 2, 14], and so, we analyze the entropy associated with such back holes. In fact, we analyze the corrections to the entropy of such black holes because of thermal fluctuations, and keep the coefficient of such a correction term as a variable. This is because almost all approaches to quantum gravity predict that the corrections to the entropy of a black hole are a logarithmic correction term, but the coefficient of such a term varies between different approaches to quantum gravity. So, if we can experimentally verify the correct coefficient to the logarithmic correction term, we can select the correct approach to quantum gravity. As we cannot test the black hole thermodynamics directly, in this work, we proposed to analyze it using an analogous black hole like solution in the adaptive model of graphene. In fact, there is no fundamental difference between analogous black holes and real black holes in the Jacobson formalism [26, 27], except that the velocity of light gets replaced by the Fermi velocity. This even real black holes can be described as thermodynamical objects in the Jacobson formalism, and the effect of quantum fluctuation can be described in terms of thermal fluctuations [28, 29].

We should notice that the logarithmic corrected entropy used in this paper has similar shape as the entropy studied in a $Co - C$ system [49]. In this system the low entropy in graphene was studied using the $Co - C$ system with activation energy values of order $0.1 - 1eV$, which coincident with the internal energy is given by the Fig. 2 in the presence of logarithmic correction for small black hole masses. The specific heat is of order unity [50], which suggest presence of thermal correction (see Fig. 1). Without thermal correction, specific heat is of order $10^{14}$ with negative value while we find specific heat is about 0.3
in presence of logarithmic correction, and so it is close to experimental results. It is also interesting to compute thermal conductivity of this system and compare it with experimental results [51].

Now, it is possible to test the effect of logarithmic correct on the effective black hole like solution in the adaptive model of graphene, and we have explicitly analyzed its effect on the effective Hawking radiation from such analogous black holes. These results may be tested experimentally and used to determine the correct coefficient for the logarithmic correction term to the entropy of a black hole. This can in turn be used to select the correct approach to quantum gravity, which would produce such a coefficient for the logarithmic correct term of a black hole. Thus, it is possible to analyze quantum gravitational effects using analogous black hole like solution in graphene.

9 Acknowledgments

We would like to thanks Alfredo Iorio for useful suggestions that helped us improve this paper.

References

[1] M. Cvetic and G. W. Gibbons, Annals Phys. 327, 2617 (2012)
[2] A. Iorio and G. Lambiase, Phys. Rev. D 90, 025006 (2014)
[3] K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, Y. Zhang, S. V. Dubonos, I. V. Gregorieva and A. A. Firsov, Science 306, 666 (2004).
[4] A. K. Geim, Science 324, 1530 (2009)
[5] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov and A. K. Geim, Rev. Mod. Phys. 81, 109 (2009)
[6] P. R. Wallace, Phys. Rev. 71, 622 (1947)
[7] G. W. Semenoff, Phys. Rev. Lett. 53, 2449 (1984)
[8] A. Iorio, Ann. Phys. 326, 1334 (2011)
[9] A. Cortijo and M. A. H. Vozmediano, Eur. Phys. J. ST 148, 83 (2007)
[10] A. Iorio, Eur. Phys. J. Plus 127, 156 (2012)
[11] A. Iorio, G. Lambiase, Phys. Lett. B 716, 334 (2012)
[12] J. D. Bekenstein, Phys. Rev. D 9, 3292 (1974)
[13] J. D. Bekenstein, Phys. Rev. D 7, 2333 (1973)
[14] P. Chen and H. C. Rosu, Mod. Phys. Lett. A 27, 1250218 (2012)
[15] A. Capolupo and G. Vitiello, Phys. Rev. D 88, 024027 (2013)
[16] B. Pourhassan, K. Kokabi, Int. J. Theor. Phys. 57, 780 (2018)
[17] A. Ashtekar, Lectures on Non-perturbative Canonical Gravity, World Scientific (1991)
[18] T. R. Govindarajan, R. K. Kaul and V. Suneeta, Class. Quant. Grav. 18, 2877 (2001)
[19] J. Jing and M. L Yan, Phys. Rev. D63, 24003 (2001)
[20] S. N. Solodukhin, Phys. Rev. D57, 2410 (1998)
[21] A. Sen, JHEP 04, 156 (2013)
[22] D. A. Lowe and S. Roy, Phys. Rev. D82, 063508 (2010)
[23] M. Faizal and M. Khalil, Int. J. Mod. Phys. A30, 1550144 (2015)
[24] A. F. Ali, JHEP 1209, 067 (2012)
[25] B. Pourhassan, M. Faizal, S. Upadhyay, L. Al Asfar, Eur. Phys. J. C 77, 555 (2017)
[26] T. Jacobson, Phys. Rev. Lett. 75, 1260 (1995)
[27] R. G. Cai and S. P. Kim, JHEP 0502, 050 (2005)
[28] S. Das, P. Majumdar and R. K. Bhaduri, Class. Quant. Grav. 19, 2355 (2002)
[29] J. Sadeghi, B. Pourhassan, and F. Rahimi, Can. J. Phys. 92, 1638 (2014)
[30] S. Mahapatra, P. Phukon and T. Sarkar, Phys. Rev. D 84, 044041 (2011)
[31] S. Mahapatra, Eur. Phys. J. C 78, 23 (2018)
[32] B. Pourhassan and M. Faizal, Europhys. Lett. 111, 40006 (2015)
[33] M. Faizal and B. Pourhassan, Phys. Lett. B 751, 487 (2015)
[34] B. Pourhassan, M. Faizal, Phys. Lett. B 755, 444 (2016)
[35] B. Pourhassan, M. Faizal, U. Debnath, Eur. Phys. J. C 76, 145 (2016)
[36] J. Sadeghi, B. Pourhassan, M. Rostami, Phys. Rev. D 94, 064006 (2016)
[37] S. Upadhyay, B. Pourhassan, H. Farahani, Phys. Rev. D 95, 106014 (2017)
[38] B. Pourhassan, M. Faizal, Nuclear Physics B 913, 834 (2016)
[39] B. Pourhassan, M. Faizal, Eur. Phys. J. C 77, 96 (2017)
[40] B. Pourhassan, S. Upadhyay, H. Saadat, H. Farahani, Nuclear Physics B 928, 415 (2018)
[41] B. Pourhassan, M. Faizal, S. Capozziello, Annals of Physics 377, 108 (2017)
[42] M. Banados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69, 1849 (1992)
[43] A. Komar, Phys. Rev. 113, 934 (1959)
[44] M. Xia, Y. Song, S. Zhang, Phys. Lett. A, 375, 3726 (2011)
[45] E.C. Vagenas, Mod. Phys. Lett. A17, 609 (2002)
[46] A. Iorio, Int. J. Mod. Phys. D 24, 1530013 (2015)
[47] A. Sepehri, R. Pincak, K. Bamba, S. Capozziello, E. N. Saridakis, Int. J. Mod. Phys. D26 (2017) 1750094
[48] S. Capozziello, R. Pincak, E. N. Saridakis, Annals Phys. 390 (2018) 303-333
[49] G. Amato, F. Beccaria, U. Vignolo, F. Celegato, [arXiv:1604.05928 [cond-mat.mtrl-sci]]
[50] E. Pop, V. Varshney, and A. K. Roy, MRS Bull. 37, 1273 (2012)
[51] A. A. Balandin, S. Ghosh, W. Bao, I. Calizo, D. Teweldebrhan, F. Miao, C. N. Lau, Nano Letters 8, 902 (2008)