Dynamic Axisymmetric Problem of Direct Piezoelectric Effect for A Bimorphic Plate of Stepwise Variable Thickness

Dmitrij Averkievich Shlyahin1, Olesya Viktorovna Ratmanova1
1Samara State Technical University, Academy of Architecture and Civil Engineering, Molodogvardeyskaya st., 194, Samara, 443001, Russia
olesya654@yandex.ru

Abstract. The dynamic axisymmetric problem of direct piezoelectric effect for a circular two-layer transducer consisting of a metal substrate and a piezoceramic plate of smaller diameter is considered. The oscillations of the electroelastic system are carried out due to the action of a mechanical load on its front surface, which is an arbitrary function of time and radial coordinate. The presented mathematical model of calculation allows to take into account the connectedness of longitudinal-transverse oscillations of a structure that is asymmetric in height. A new analytical solution was obtained in the framework of the refined theory of Timoshenko by the method of finite integral transformations. The calculated ratios make it possible to choose the geometric dimensions of the bimorphic plate, which most effectively allow to convert the mechanical load into the generated electrical pulse. The considered designs are widely used as touch switches that regulate the operation of valves for various purposes.

1. Introduction
To evaluate work efficiency and study the stress-strain state of composite electroelastic systems, the classical applied theory for thin plates is used, in which the kinematic hypotheses are supplemented by similar assumptions concerning the nature of electric field intensity change along the axial coordinate [1, 6, 9, 11]. In this formulation, the propagation rate of shear strains is unlimited. To eliminate this drawback in the problems of electroelasticity [8], the refined two-mode theory of Timoshenko is used [4].

In the study of structures that are asymmetric with respect to the median surface, it is necessary to consider the associated membrane and bending vibrations. In such a statement it is possible to note a limited number of the constructed decisions. In particular, in [3] the closed solution is obtained for rectangular anisotropic plates of stepwise variable thickness. The analysis of the work of asymmetric bimorphic transducer of constant thickness in the case of harmonic mechanical load is carried out in [5]. The study [10] is devoted to the analysis of longitudinal-transverse harmonic vibrations of the variable cross-section plate.

The article investigates an asymmetric bimorphic metal piezoceramics plate of stepwise variable thickness and stiffness. The authors propose to construct a new closed solution of the dynamic problem taking into account the coupling of radial and bending deformations by the method of finite integral transformations. The design ratio will allow us to scientifically substantiate the geometrical dimensions of multilayer piezoelectric ceramic transducers of resonant and non-resonant classes.
2. Problem statement

Suppose a round bimorphic plate consist of a metal grounded substrate of radius $r_b = b$ and thickness $h_2$, as well as a piezoceramic axially polarized element with a hexagonal crystal lattice of class 6 mm ($r_a = a$, $a < b$, $h_1$). Axisymmetric bending vibrations are formed as a result of the action on the upper front surface of the structure of the mechanical load $q^*(r, t)$, which is an arbitrary function of time $t$ and radial coordinate $r$ (figure 1). The electrode surfaces connection of piezoceramic plates to the measuring device allows to determine the potential difference $V^*(t)$. The conditions for fixing the cylindrical surface ($r_c = b$) of a bimorphic structure can be arbitrary. For certainty we consider it to be rigidly pinched.

In the cylindrical coordinate system $(r, \vartheta, z)$, the location of the surface $z = 0$ is determined under the condition of taking into account all the inertial characteristics of the structure when constructing a closed solution. The coordinates of the lower plane of the piezoceramic plate, as well as the lower and upper surfaces of the substrate are indicated respectively: $z_1 = h_1 + e$, $z_2 = e$, $z_3 = e - h_2$,

$$e = \frac{\rho^{(2)} h_2^2 - \rho^{(1)} h_1^2}{2 (\rho^{(2)} h_2 + \rho^{(1)} h_1)} \quad (\rho^{(1)}, \rho^{(2)} - \text{the volumetric density of piezoelectric ceramic and metal}).$$

![Figure 1. Model of structure](image)

Differential equations of the radial and axisymmetric vibrations of a bimorph plate in relation to normal $N_r(r, t)$, $N_\vartheta(r, t)$, cross $Q_r(r, t)$ forces, as well as bending moments $M_r(r, t)$, $M_\vartheta(r, t)$ using the following kinematic hypotheses

$$U^*(r, z, t) = U_0^*(r, t) - zw(r, t), W^*(r, z, t) = W^*(r, t)$$

are as follows:

$$\frac{\partial N_r}{\partial r} + \frac{N_r - N_\vartheta}{r} - a_z \frac{\partial^2 U_0^*}{\partial r^2} = 0 \quad (2)$$

$$\frac{\partial Q}{\partial r} + \frac{Q}{r} - a_z \frac{\partial^2 W^*}{\partial r^2} = q^*(r, t)$$

$$\frac{\partial M_r}{\partial r} - Q + \frac{M_r - M_\vartheta}{r} + a_z \frac{\partial^2 \varphi}{\partial r^2} = 0$$

where $U^*, W^*, \varphi - \text{radial, axial movements and the rotation angle of the section in the plane}(r, z)$; $U_0^* - \text{radial displacement of the surface at } z = 0$;

$$a_z = \rho^{(1)} h_1 H(a - r_b) + \left(\rho^{(1)} h_1 + \rho^{(2)} h_2\right) H(r_b - a)$$
33 33 33
21 223 23 12
zz zz zzaH a r H r a
        
(3)

As a result of substitution (4) in (2), taking into account hypotheses (1), (3), we obtain the following system of differential equations in dimensionless form:

\[
p_1 \frac{\partial}{\partial r} \nabla U_0 - p_2 \frac{\partial}{\partial r} \nabla \psi - a_1 \frac{\partial^2 U_0}{\partial t^2} = 0
\]

\[
p_1 \left( \nabla \frac{\partial W}{\partial r} - \nabla \psi \right) - a_1 \frac{\partial^2 W}{\partial t^2} = q
\]

\[
p_4 \frac{\partial}{\partial r} \nabla \psi + p_1 \left( \frac{\partial W}{\partial r} - \psi \right) - p_2 \frac{\partial}{\partial r} \nabla U_0 - a_4 \frac{\partial^2 \psi}{\partial t^2} = 0
\]

where \( \{W, U_0, r, R\} = \{W^*, U_0^*, r^*, a\} / b, t = \frac{r}{b} \sqrt{\frac{C_{11}^{[2]}}{\rho^{[2]}_0}}, \ q = \frac{q^*}{m_i} \),

\[
a_3 = \frac{a_1}{\rho^{[2]}_0 h_2}, \quad a_4 = \frac{a_2}{\rho^{[2]}_0 b^2 h_2},
\]

\[
\begin{bmatrix}
p_1 \\
p_2 \\
p_3 \\
p_4
\end{bmatrix} = 
\begin{bmatrix}
m_1 \\
m_2 \\
m_3
\end{bmatrix} \begin{bmatrix}
m_1 b^{-1} & m_2 b^{-1} & H(r - R) \\
m_0 & m_1 b^{-2} & H(R - r) \\
m_1 b^{-2} & m_3 b^{-2}
\end{bmatrix}
\]

The boundary conditions in a dimensionless form have the form:

\[
r = 1,0 \quad U_0(1, t) = 0, \quad W(1, t) = 0, \quad \psi(1, t) = 0
\]

\[
U_0(0, t) < \infty, \quad W(0, t) < \infty, \quad \psi(0, t) < \infty,
\]

\[
t = 0 \quad U_0(r, 0) = u_0(r), \quad W(r, 0) = W_0(r), \quad \psi(r, 0) = \psi_0(r)
\]
\[
\frac{\partial U_0}{\partial t} \bigg|_{t=0} = \dot{u}_0(r), \quad \frac{\partial W}{\partial t} \bigg|_{t=0} = \dot{W}_0(r), \quad \frac{\partial \psi}{\partial t} \bigg|_{t=0} = \psi(r),
\]

where

\[
m_{(2)} = C_{11(12)}^{(2)} h_2, \quad m_{(4)} = C_{11(12)}^{(2)} m_{17}, \quad m_{(6)} = m_{(2)} + \left( C_{11(12)}^{(4)} + \frac{e_{31}^2}{e_{33}} \right) h_1,
\]

\[
m_{(8)} = m_{(4)} + \left( C_{11(12)}^{(4)} + \frac{e_{31}^2}{e_{33}} \right) m_{15}, \quad m_9 = k_2 C_{55}^{(2)} h_2, \quad m_{10} = k_1 \left( C_{55}^{(2)} h_2 + C_{51}^{(1)} h_1 \right),
\]

\[
m_{1(12)} = C_{11(12)}^{(2)} m_{18}, \quad m_{13(14)} = m_{1(12)} + \left( C_{11(12)}^{(4)} + \frac{e_{31}^2}{e_{33}} \right) m_{16},
\]

\[
m_{15} = \frac{z_1^2 - z_2^2}{2}, \quad m_{16} = \frac{z_1^3 - z_2^3}{3}, \quad m_{17} = \frac{z_2^3 - z_1^3}{2}, \quad m_{18} = \frac{z_3^3 - z_3^3}{3}
\]

\[e_{31}, e_{33} - \text{piezoceramic material constants; } C_{mk}^{(s)} - \text{the modules of elasticity of electroelastic } (s=1) \text{ and elastic } (s=2) \text{ materials} (m,k=1,2,5); \quad W_0, u_0, \psi_0, \dot{W}_0, \dot{u}_0, \dot{\psi}_0 - \text{given at } t=0 \text{ movements and their speeds.}
\]

For this design solution, the electric no load voltage \( V^*(t_*) \) is defined as follows:

\[
V(t_*) = S^{-1} \int_c \phi(r, z, t_*) dS
\]

where \( \phi(r, z, t_*) \) - electric field potential generated on the lower front surface of the piezoceramic plate \( (E_z = -\frac{\partial \phi}{\partial z}) \), \( S \) - the area of the electroelastic plate.

Final function expression \( V(t_*) \) are as follows:

\[
V(t_*) = \frac{e_{31}}{e_{33}} h \left[ 2U_0^*(a, t_*) - (h_1 + 2\epsilon) \psi(a, t_*) \right].
\]

### 3. General solution construction

To solve the initial boundary value problem (5)–(7), we introduce a finite integral transformation (control measuring device) \[7\] with unknown components on the segment \([0,1]\) \( K_i(\lambda, r) \), \( K_2(\lambda, r) \), \( K_3(\lambda, r) \) vector-functions of the transformation kernel and weight functions \( \alpha, \beta, \gamma \):

\[
G(\lambda, t) = \int_0^1 \left[ a U_0(r, t) K_1(\lambda, r) + \beta W(r, t) K_2(\lambda, r) + \gamma \psi(r, t) K_3(\lambda, r) \right] dr
\]

\[
\{ U_0(r, t), W(r, t), \psi(r, t) \} = \sum_{i=1}^{\infty} G(\lambda, t) \{ K_1(\lambda, r), K_2(\lambda, r), K_3(\lambda, r) \} \| K_i \|^2
\]

\[
K_i = \int_0^1 \left[ a K_1^i(\lambda, r) + \beta K_2^i(\lambda, r) + \gamma K_3^i(\lambda, r) \right] dr
\]

where \( \lambda_i \) - dimensionless circular frequencies of axisymmetric oscillations \( (i = 1, \infty) \), associated with dimensional frequencies \( \omega_i \) following dependence:
\[ \omega_i = \frac{\lambda}{b} \sqrt{c_{ij}^{(2)}} \]  

(10)

Introducing piecewise smooth functions \( U_i(\rho, t), W(\rho, t), \psi(\rho, t) \) in the form

\[ \{U_i, W, \psi\} = \{U_i^{(a)}, W^{(a)}, \psi^{(a)}\} H(r - R) + \{U_i^{(b)}, W^{(b)}, \psi^{(b)}\} H(R - r) \]  

(11)

we subject the system of equations (5) and initial conditions (7) to transformations of control measuring device (CMD) in accordance with the structural algorithm [7]. As a result, we obtain the initial problem for the transformant 

\[ G(\lambda, t) \] general integral has the form

\[ G(\lambda, t) = G_0(\lambda, \rho) + G_0(\lambda, \rho) \int_0^t F(\lambda, \rho) \sin(\rho - \tau) d\tau \]  

(12)

And subject to (6), the homogeneous problem for the kernel component of CMD

\[ p_i \frac{d}{dr} \nabla K_i^{(j)} - p_i \frac{d}{dr} \psi K_i^{(j)} + a_i \lambda_i^2 K_i^{(j)} = 0, \quad (j = a, b) \]  

(13)

\[ r = 1,0 \quad K_i^{(a)}(\lambda_1, 1) = K_i^{(b)}(\lambda_1, 1) = K_1^{(a)}(\lambda_1, 1) = 0 \]  

(14)

\[ \{K_i^{(b)}(\lambda_1, 0), K_i^{(b)}(\lambda_1, 0), K_i^{(b)}(\lambda_1, 0)\} < \infty \]

\[ r = R: \quad K_i^{(a)}(\lambda_1, R) = K_i^{(b)}(\lambda_1, R), \quad K_i^{(a)}(\lambda_1, R) = K_i^{(b)}(\lambda_1, R) \]  

(15)

\[ m_0 \left( \frac{dK_i^{(a)}}{dr} - K_i^{(a)} \right)_{r = R} = m_0 \left( \frac{dK_i^{(b)}}{dr} - K_i^{(b)} \right)_{r = R}, \]

\[ \left( m_i \frac{dU_i}{dr} + m_i \frac{dW}{dr} - m_i \psi \frac{d\psi}{dr} \right)_{r = R} = \left( m_i \frac{dU_i}{dr} + m_i \frac{dW}{dr} - m_i \psi \frac{d\psi}{dr} \right)_{r = R}, \]

\[ \left( m_i \frac{dU_i}{dr} + m_i \frac{dW}{dr} - m_i \psi \frac{d\psi}{dr} \right)_{r = R} = \left( m_i \frac{dU_i}{dr} + m_i \frac{dW}{dr} - m_i \psi \frac{d\psi}{dr} \right)_{r = R}, \]

here \( \{K_i, K_2, K_3\} = \{K_i^{(a)}, K_i^{(b)}, K_i^{(a)}\} H(r - R) + \{K_i^{(b)}, K_i^{(b)}, K_i^{(b)}\} H(R - r), \)

\[ F(\lambda, t) = \int_0^t a_s^{-1} q(\rho, t) \beta K_s(\lambda_1, r) dr, \quad \alpha = \beta = 1, \quad \gamma = \frac{a_s}{a_1}, \]

\[ G_0(\lambda_1) = \int_0^t (\alpha u_0 K_1 + \beta W_0 K_1 + \gamma \psi_0 K_1) dr, \]

\[ \tilde{G}_0(\lambda_1) = \int_0^t (\alpha \tilde{u}_0 K_1 + \beta \tilde{W}_0 K_1 + \gamma \tilde{\psi}_0 K_1) dr. \]

Equality (16) determine the conditions of continuity of deformations and forces in the field of irregularity of the construction structure, \( r = R \), which are formed in the implementation of equality:
To solve the boundary value problem (14)–(16) new functions are introduced $R_1^{(j)}(\lambda_m, r)$, $R_2^{(j)}(\lambda_m, r)$, $R_3^{(j)}(\lambda_m, r)$ by formulæ:

$$
R_1^{(j)}(\lambda_m, r) = rK_1^{(j)}(\lambda_m, r), \\
R_2^{(j)}(\lambda_m, r) = \frac{dK_2^{(j)}(\lambda_m, r)}{dr}, \\
R_3^{(j)}(\lambda_m, r) = rK_3^{(j)}(\lambda_m, r),
$$

and then the system (14) is reduced to the following differential equation with respect to $R_3^{(j)}(\lambda_m, r)$:

$$
\left( \nabla^2 \nabla^2 - f_1 \nabla^2 + f_2 \nabla + f_3 \right) R_3^{(j)} = 0,
$$

where $\nabla^2 = \frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr}$, $f_1 = \frac{n_2 + n_4}{n_3} + a_3 \lambda_m^2 n_5$, $f_2 = n_1 n_3 + (n_4 p_3)^{-1} (a_3 n_2 + n_4 p_3)$, $f_3 = (n_4 p_3)^{-1} a_3 \lambda_m^2 n_3$, $n_1 = p_1 p_4$, $n_2 = a_3 \lambda_m^2 - p_3$, $n_3 = a_3 \lambda_m^2 p_3$.

The use of the generating equation

$$
\nabla^2 R_3^{(j)} = -\beta_1^2 R_3^{(j)},
$$

allows to formulate the following bicubic characteristic equation

$$
\left( \beta_1^2 \right)^3 - f_1 \left( \beta_1^2 \right)^2 + f_2 \beta_1^2 - f_3 = 0.
$$

The ratio of the physical characteristics of the materials used in the production of this construction allows to obtain the following equation roots (19): $\beta_{21}, \beta_{2i} -$ the real positive numbers, $\beta_{3i} -$ the imaginary number.

In this case the general integral of the differential equation (18) is written as

$$
R_3^{(j)}(\lambda_m, r) = r \left[ \sum_{\nu=1}^{2} D_{(2\nu)}^{(j)} J_1(\beta_{\nu} r) + D_{(2\nu)}^{(j)} Y_1(\beta_{\nu} r) \right] + D_{(2\nu)}^{(j)} I_1(\beta_{\nu} r) + D_{(2\nu)}^{(j)} K_1(\beta_{\nu} r),
$$

where $J_\mu(...), Y_\mu(...), I_\mu(...), K_\mu(...)$ — ordinary and modified Bessel functions I and II $\mu$ ($\mu = 0, 1$).

Expressions for the components of the kernel of the transformation $K_1^{(j)}(\lambda_m, r), K_2^{(j)}(\lambda_m, r), K_3^{(j)}(\lambda_m, r)$ are determined taking into account (17), and the links obtained in the process of bringing (14) to (18):

$$
R_2^{(j)} = \frac{p_3}{a_3 \lambda_m^2} \left[ n_1 \nabla^2 + (n_2 + n_4) \nabla^2 + n_3 \right] R_3^{(j)},
$$

$$
R_1^{(j)} = p_3 R_2^{(j)} + p_4 \nabla^2 R_3^{(j)} + \left( a_3 \lambda_m^2 - p_3 \right) R_3^{(j)}.
$$
Substitution $K_1^{(l)}, K_2^{(l)}, K_3^{(l)}$ in boundary conditions (15), (16) allows to define constants $D_{a1}^{(a)}...D_{a1}^{(b)}, D_{b1}^{(a)}...D_{b1}^{(b)}$ and eigenvalues $\lambda_i$.

Applying to expression transformants (13) formulas for the treatment of (10), we obtain a general expression for the dimensionless dynamic displacement component $U_0(r,t)$, $W(r,t)$ and projection angle of the section $\psi (r,t)$.

As an example, a bimorphic structure consisting of a metal substrate made of brass is considered ($h_2 = 0.5\times10^{-3}$ m, $b = 3\times10^{-2}$ m) and the piezoceramic plate of the composition of TSTS–19 (zirconium titanium lead). The following physical characteristics of materials are used: brass $E = 9.8\times10^6$ N/m$^2$, $\nu = 0.35$, $\rho = 8600$ kg/m$^3$, $E, \nu, \rho$ – modulus of elasticity, Poisson's ratio and density of isotropic material($C_{11}^{(2)} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}, C_{12}^{(2)} = \frac{E\nu}{(1+\nu)(1-2\nu)}, C_{55}^{(2)} = \frac{E}{2(1+\nu)}$); piezoceramics–$\varepsilon_{33} = 7.26\times10^{-9}$ F/m, $e_{31} = -4.9$ C/m$^2$, $\{C_{11}^{(i)}, C_{12}^{(i)}, C_{55}^{(i)}\} = \{10.9, 9.1, 6.1, 5.4, 2.4\} \times 10^{10}$ N/m$^2$.

Consider the case of the harmonic load: $q(r,t) = q_0 (1-r)\sin \chi t$, where $q_0, \chi$ – the amplitude of the load and the frequency of forced oscillations in a dimensionless form.

Figure 2 presents time changes graphs of vertical displacements of the bimorph system $W(0,t)$ (solid line) and potential differences between the electrode surfaces of the piezoceramic plate $V(t)$ (dashed line), and figures 3, 4 show the dependence change in the amplitude values of the potential difference $V(t)$ for different values of the geometric dimensions of the piezoceramic plate.

Numerical analysis of the results allows us to draw the following conclusions:

1) under high-frequency external effects $\chi = 0.8\lambda_i$ the effect of "beating" observed in the design when calculating the vertical displacements $W(0,t)$, also applies to the nature of the change $V(t)$ in time;

2) analysis of the calculation results obtained for the plate of constant and stepwise variable thickness, shows that reducing the stiffness of the structure at $a < b$, leads to an increase in displacement $W(0,t)$ and potential difference $V(t)$;

3) for the given metal substrate($b = 3\times10^{-2}$ m, $h_2 = 0.5\times10^{-3}$ m), it is possible to choose the geometric dimensions of the piezoceramic plate, allowing the most effective conversion of the mechanical load into an electric pulse, in particular, in this example, the following parameters $R = 0.62$, ($a = 1.86\times10^{-2}$ m), $h_i = 0.9\times10^{-3}$ m. should be used.
8

Figure 2. Change $W(0,t), V(t)$ in time $t$ ($h_1 = 0.5 \times 10^{-3}$ m, $a = 1.8 \times 10^{-2}$ m, $\chi = 0.8 \lambda_i$).

Figure 3. The dependence of the amplitude value $V(t)$ from piezoceramic plate radius $R$ ($\chi = 0.2 \lambda_i$).

Figure 4. The dependence of the amplitude value $V(t)$ from piezoceramic plate thickness $h_1$ ($\chi = 0.2 \lambda_i$).

4. Conclusion
The obtained calculation results show that to describe the work and improve the functionality of the given construction it is necessary to use a mathematical model taking into consideration the dynamic characteristics of the electroelastic system. Therefore, the study of the bimorphic plate, even at harmonic effects on it, due to the imposition of deformation waves, cannot be carried out within the established regime of forced oscillations, usually used in the given problems solving.

References
[1] Vatulyan A.O., Rynkov A.A. A model of Flexural vibrations of piezoelectric bimorphs with split electrodes and its applications. Izv. Russian Academy of Sciences. MTT. 2007; 4: 114–122.
[2] Grinchenko V.T., Ulitko A.F., Shulga N.Ah. Mechanics of related fields in design elements. Kyiv, Sciences Dumka, 1989, 279 p.
[3] Elnicki E.J., Dyachenko P.Y. Application of the method of initial parameters to the solution of unsteady problems of dynamics for a rectangular plate stepped section. Izv. higher educational. Construction. 1997; 11:13–18.

[4] Paymushin V.N. The ratio of the theory of thin shells such as the theory of Tymoshenko with arbitrary displacements and deformations. Applied mechanics and technical physics. 2014(55);5:P.135-149.

[5] Rudnitsky S.I., Sharapov V.M., Shulga N.. Vibrations of disk bimorph transducer of the type metal–piezoceramics. Prikl. mechanics. 1990(26);10:P. 64–71.

[6] Savin V.G., Babaev A.E. The Effect of acoustic pulse with flat elastic and electric system of bimorph Information system mechani Caruana. Kyiv, National Technica UN-t Ukraine. 2009;3:30–39.

[7] Senitskiy Yu.E. Method of finite integral transformations is a generalization of the classical procedure of expansion in eigen vector-functions. Izv. Saratov UN-TA. Series. Mathematics, Mechanics, Informatics. 2011;3:61–89.

[8] Shlyakhin D. A. Forced axisymmetric vibrations of a piezoceramic thin bimorph plate. Izv. Russian Academy of Sciences. MTT. 2013;2:P. 77–85.

[9] I. Yanchevskii V. Control of oscillation of the bending of circular piezoelectric transducer asymmetrical bimorph with split electrodes problems. mechanical engineering, 2012, Vol. 15, № 2. P. 37-43.

[10] Dzyuba V.A., Steblyanko P.O. Use of splines in the calculation of deflections for plates of variable thickness. Science and Education a New Dimension. Natural and Technical Sciences, II(4). 2014; 32:41–47.

[11] Wang Y., Xu R.Q., Ding H.J. Analytical solutions of functionally graded piezoelectric circular plates subjected to axisymmetric loads. Acta Mechanica. 2010(215);1–4:287–305