HELGASON-GABOR FOURIER TRANSFORM AND UNCERTAINTY PRINCIPLES

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Abstract. Windowing a Fourier transform is a useful tool, which gives us the similarity between the signal and time frequency signal, and it allows to get sense when/where certain frequencies occur in the input signal, this method is introduced by Dennis Gabor. In this paper, we generalize the classical Gabor-Fourier transform (GFT) to the Riemannian symmetric space called the Helgason Gabor Fourier transform (HGFT). We continue with proving several important properties of HGFT, like the reconstruction formula, the Plancherel formula, and Parseval formula. Finally we establish some local uncertainty principle such as Benedicks-type uncertainty principle.

Introduction

The Fourier transform has been a useful tool for analyzing frequency properties of a signal, but this transform still insufficient to represent and compute location information for a given signal. To solve this problem, in [5] Gabor formulated a fundamental method by multiplying the function to be transformed by a Gaussian function. This transform becomes a powerful method for determining the sinusoidal frequency and phase content of signal local sections considering its changes over time. In addition, it used for filtering and modifying the signal in the limited region.

In classical case, the Gabor transform is given by ([5])

\[ \mathcal{G}\{f\}(b, \omega) = \int_{-\infty}^{+\infty} f(t) e^{-\pi(t-b)^2} e^{-i\omega t} dt \]

In the general case, we take the windowed function \( \varphi \) as square integral function. The Gabor Fourier transform has other names used in the literature, like as short-time Fourier transform and windowed Fourier transform. Motivated by this concept, in this paper we study a generalization of the classical Gabor Fourier transform to the Riemannian Symmetric spaces see [9, 11], which we call the Helgason Gabor Fourier transform (HGFT). Then, we derive important harmonic analysis properties of HGFT.
This paper is organized as follows, in the first section we remind some results about the classical Helgason-Fourier transform, in the second we define the HGFT, and we establish for it several harmonic analysis properties, such as the inversion formula, Plancherel and Parseval formulas, in the last one we demonstrate some local uncertainty principles for HGFT like Benedick’s theorem.

1 Helgason Transform

1.1 Helgason Transform

In this section we describe the necessary preliminaries regarding semi-simple Lie groups and harmonic analysis on associated Riemannian symmetric spaces.

If $X$ is a Riemannian symmetric space of noncompact type then $X$ can be viewed as a quotient space $G/K$ where $G$ is a connected, noncompact, semi-simple Lie group with finite center and $K$ a maximal compact subgroup of $G$.

Let $G = N A K$ be an Iwasawa decomposition of $G$ and let $a$ be the Lie algebra of $A$. Denoting by $M$ the centralizer of $A$ in $K$ and putting $B = K/M$, we write $A(x, b) = A(k^{-1}g)$. Let $dx$ be a $G$-invariant measure on $X$, and let $db$ and $dk$ be the respective normed $K$-invariant measures on $B$ and $K$.

Let $o = K$ be the origin in $X$ and denote the action of $G$ on $X$ by $(g, x) \mapsto gx$ for $g \in G, x \in X$. The Lie algebras of $G$ and $K$ are respectively denoted by $g$ and $k$.

We denote by $C_\infty^c(X)$ the set of infinity differentiable compactly-supported functions on $X$. Let $dg$ be the element of the Haar measure on $G$.

We assume that the Haar measure on $G$ is normed, so that

$$\int_X f(x) dx = \int_G f(go) dg, \quad f \in C_\infty^c(X)$$

Let $\mathfrak{a}^*$ be the real dual of $\mathfrak{a}$ and $\mathfrak{a}_C^*$ be its complexification; the finite Weyl group $W$ acts on $\mathfrak{a}^*$. Suppose that $\Sigma$ is the set of bounded roots ($\Sigma \subset \mathfrak{a}^*$), $\Sigma^+$ is the set of positive bounded roots, and $\mathfrak{a}^+$ is the positive Weyl chamber so that

$$\mathfrak{a}^+ = \{h \in \mathfrak{a} : \alpha(h) > 0 \text{ for } \alpha \in \Sigma^+\}.$$ 

Denote by $\rho$ the half-sum of the positive bounded roots (counted with their multiplicities); then $\rho \in \mathfrak{a}^*$. Let $\langle , \rangle$ be the Killing form on the Lie algebra $\mathfrak{a}$. For $\lambda \in \mathfrak{a}^*$, let $A_\lambda$ be the vector in $\mathfrak{a}$ such that $\lambda(A) = \langle A_\lambda, A \rangle$ for all $A \in \mathfrak{a}$. Given $\lambda, \mu \in \mathfrak{a}^*$, we set $\langle \lambda, \mu \rangle := \langle A_\lambda, A_\mu \rangle$. 
The correspondence $\lambda \mapsto A_\lambda$ enables us to identify $a^*$ with $a$. Using this identification, we can translate the action of the Weyl group $W$ to $a$. Let

$$a^*_+ = \{ \lambda \in a^* : A_\lambda \in a^* \}$$

The Helgason Fourier transform is a powerful tool in harmonic analysis on noncompact Riemannian symmetric spaces $G/K$ ([11]). This transform associates to any smooth compactly supported right $K$-invariant function $f$ on $G$.

For integrable functions $f$ on $C^\infty_c(X)$, $b \in K$ and $\lambda \in a^*$, Helgason-Fourier transform is defined as in ([9]) by:

$$\hat{f}(\lambda, b) = \int_X f(x)e^{-(i\lambda + \rho)(A(x,b))}dx, \quad \lambda \in a^*, b \in B = K/M$$

We will assume throughout this paper $X$ is of rank 1, and hence $\dim a^* = 1$. In this case we identify $a^*_C$ with $C$ by identifying $\lambda_\rho$ with $\lambda \in C$. Under this identification, $a^* = \mathbb{R}$ by means of the correspondence $\lambda \mapsto \lambda \alpha, \lambda \in \mathbb{R}$.

We norm the measure on $X$ and we conclude this section with the following properties, due to Helgason.

The original function $f \in C^\infty_c(X)$ can then be reconstructed from $\hat{f}$ by means of the inversion formula

$$f(x) = \frac{1}{|W|} \int_{a^* \times B} \hat{f}(\lambda, b)e^{(i\lambda + \rho)(A(x,b))}|c(\lambda)|^{-2}d\lambda db,$$

where $|W|$ is the order of the Weyl group of $G/K$, $d\lambda$ is the element of the Euclidean measure on $a^*$ and $c(\lambda)$ is the Harish-Chandra function.

We also state the Plancherel formula for the Fourier transform:

**Theorem 1.1.** The Fourier transform defined on $C^\infty_c(X)$ by (1) extends to an isometry of $L^2(X)$ onto $L^2(a^* \times B)$ (with the measure $|c(\lambda)|^{-2}d\lambda db$ on $a^* \times B$). Moreover,

$$\int_X f_1(x)f_2(x)dx = \frac{1}{|W|} \int_{a^* \times B} \hat{f}_1(\lambda, b)\hat{f}_2(\lambda, b)e^{(i\lambda + \rho)(A(x,b))}|c(\lambda)|^{-2}d\lambda db,$$

for all $f_1, f_2 \in L^2(X)$.

**Proof.** See [10] Theorem 2, page 227.

It follows from the above arguments that for $f \in L^2(X)$, we have
\[ \int_X |f(x)|^2 dx = \frac{1}{|W|} \int_{a^\ast \times B} |\hat{f}(\lambda, b)|^2 d\mu(\lambda) db = \int_{a^\ast \times B} |\hat{f}(\lambda, b)|^2 d\mu(\lambda) db, \quad (1.5) \]

where \( d\mu(\lambda) := |c(\lambda)|^{-2} d\lambda \).

Given \( h \in G \). For a function \( f \in \mathcal{C}_0(X) \), the translation operator \( T_h \) is given by the formula

\[ (T_h f)(x) := \int_K f(gkh) dk. \]

We remind that a function \( \varphi \in \mathcal{C}_0(X) \) is called a spherical function if \( \varphi \) is \( K \)-invariant, \( \varphi(o) = 1 \), and for each \( D \in \mathcal{C}_c^\infty(X) \), there exists \( \lambda_D \in \mathcal{C} \) such that \( D\varphi = \lambda_D \varphi \).

We now list down some well known properties of the elementary spherical functions on \( X \) based on the Harish-Chandras result [9, Chapitre 4, Theorem 4.3].

First, we give the following lemma proved in [9, Lemma 3].

**Lemma 1.2.** For \( f \in L^2(X) \), we have

\[ (\widehat{T_h(f))}(\lambda, b) = \varphi_\lambda(h) \hat{f}(\lambda, b), \quad h \in G \]

where \( \hat{f}(\lambda, b) \) is the Fourier transform of \( f \)

# 2 Gabor-Helgason transform

The classical Gabor transform of a function \( f \in L^2(\mathbb{R}) \) cannot possess a support of finite Lebesgue measure. In [15] the author showed that the portion of this transform lying outside some set \( M \) of finite Lebesgue measure cannot be arbitrarily small, either. For sufficiently small \( M \), this can be seen immediately by estimating the Hilbert-Schmidt norm of a suitably defined operator. In this section, we try to give some new harmonic analysis results related to Gabor transform in the case of Riemannian symmetric space \( X \).

We define first the Gabor-Helgason transform by:

\[ \mathcal{G}_\varphi \{f\}(\lambda, b, h) = \int_X f(x) T^{-1}_h \varphi(x) e^{(-i\lambda+\rho)(A(x,b))} dx \]

with \( \lambda \in a^\ast \), \( b \in B = K/M \).
2.1 Inversion formula

Before to give the reconstruction formula for the HGFT, we need the following lemma, which proves that, the translation $T_h$ is an isometric operator for the norm of the space $L^2$.

**Lemma 2.1.** For every function $f \in L^2(X)$ and $h \in G$: We have

$$\|T_h f\|_{L^2(X)}^2 = \|f\|_{L^2(X)}^2$$

**Proof.** Applying the relation (1.1), we get,

$$\|T_h f\|_{L^2(X)}^2 = \int_X |T_h f(x)|^2 dx = \int_G |T_h(f(go))|^2 dg = \int_G T_h(f(go))T_h(f(go))dg$$

$$= \int_G \int_K f(gkho)dk \int_K \overline{f(gk'ho)dk'}dg = \int_K \int_K \int_G f(gkho)\overline{f(gk'ho)dg}dkdk'$$

$$= \int_K \int_K \int_G f(go)\overline{f(go)dg}dkdk' = \int_G \|f(go)\|^2 dg = \|f\|_{L^2(X)}^2 \square$$

**Theorem 2.2.** Let $\varphi \in L^2(X)$ be a window function. Then every function $f \in L^2(X)$, can be reconstructed by

$$f(x) = \frac{1}{|W||\varphi|^2} \int_X \int_{a^* \times B} G_{\varphi}^\varphi(\lambda, b, h)e^{(i\lambda + \rho)(A(x, b))}T_{h^{-1}}\varphi(x)c(\lambda)|^{-2} d\lambda db dh$$

**Proof.** We can obtain the inversion formula by using the fact that:

$$G_{\varphi}^\varphi \{ f \} = (T_{h^{-1}}\varphi)(\lambda, b)$$

So,

$$f(x)T_{h^{-1}}\varphi(x) = \frac{1}{|W|} \int_{a^* \times B} G_{\varphi}^\varphi \{ f \} e^{(i\lambda + \rho)(A(x, b))}c(\lambda)|^{-2} d\lambda db, \quad (2.1)$$

We multiply the both sides of (2.1) by $T_{h^{-1}}\varphi$, we obtain

$$f(x)|T_{h^{-1}}\varphi(x)|^2 = \frac{1}{|W|} \int_{a^* \times B} G_{\varphi}^\varphi \{ f \} e^{(i\lambda + \rho)(A(x, b))}c(\lambda)|^{-2} d\lambda db T_{h^{-1}}\varphi(x) \quad (2.2)$$
on integer the inequality \[22\] with respect the measure \(dh\), we get
\[
f(x) \int_G |T_{h^{-1}} \varphi(x)|^2 dh = \frac{1}{|W|} \int_G \int_{a^* \times B} G_\varphi \{f\}(\lambda, b, h) e^{(i \lambda + \rho)(A(x, b))} c(\lambda)^{-2} T_{h^{-1}} \varphi(x) d\lambda db dh
\]
by using the first lemma \[2.1\] we obtain,
\[
f(x) \|\varphi\|^2_{L^2(X)} = \frac{1}{|W|} \int_G \int_{a^* \times B} G_\varphi \{f\}(\lambda, b, h) e^{(i \lambda + \rho)(A(x, b))} T_{h^{-1}} \varphi(x) c(\lambda)^{-2} d\lambda db dh \tag{2.3}
\]
Now, simplifying both sides of \[2.3\] by \(\|\varphi\|^2_{L^2(X)}\), we get our result.
\[
f(x) = \frac{1}{|W| \|\varphi\|^2_{L^2(X)}} \int_G \int_{a^* \times B} G_\varphi \{f\}(\lambda, b, h) e^{(i \lambda + \rho)(A(x, b))} T_{h^{-1}} \varphi(x) c(\lambda)^{-2} d\lambda db dh
\]
\[\square\]

**Theorem 2.3 (Plancherel formula).** For \(f \in L^2(X)\) and \(\varphi \in L^2(X)\) a windowed function, we have
\[
\|G_\varphi \{f\}\|^2_{L^2(a^* \times B \times G)} = \|f\|^2_{L^2(X)} \|\varphi\|^2_{L^2(X)}
\]

**Proof.** We have,
\[
\|G_\varphi \{f\}\|^2_{L^2(a^* \times B \times G)} = \|\widehat{T_{h^{-1}} \varphi f}\|^2_{L^2(a^* \times B \times G)} = \|\widehat{T_{h^{-1}} \varphi f}\|^2_{L^2(X \times G)} = \int_G \int_X |f(x)| T_{h^{-1}} \varphi(x) f(x) \overline{T_{h^{-1}} \varphi(x)} dx dh = \int_G \int_X |f(x)|^2 \overline{T_{h^{-1}} \varphi(x)}^2 dx dh = \int_G \int_X |f(x)|^2 |T_{h^{-1}} \varphi(x)|^2 dx dh
\]
using the equation \[1.1\], lemma \[2.1\] and the Fubini’s theorem, we have,
\[
= \int_G \int_K |f(go)|^2 \int_K \varphi(gkh^{-1}o) dk \overline{\varphi(gkh^{-1}o)} dk' dh dg = \int_G |f(go)|^2 \int_K \int_K \varphi(gkh^{-1}o) \overline{\varphi(gkh^{-1}o)} dk dk' dh dg
\]
using the invariance of the Haar measure \(dg\) by \(K\), we get
\[
= \int_G |f(go)|^2 \int_G \varphi(h^{-1}o) \int_K \overline{\varphi(h^{-1}o)} dk dk' dh dg = \int_G |f(go)|^2 \int_K dk \int_K dk' \int_G |\varphi(h^{-1}o)|^2 dh dg = \int_X |f(x)|^2 dx \int_X |\varphi(y)|^2 dy = \|f\|^2_{L^2(X)} \|\varphi\|^2_{L^2(X)}
\]
Theorem 2.4 (Parseval’s identity). Let \( \varphi \in L^2(X) \) be a window function and \( f, g \in L^2(X) \) arbitrary. Then we have

\[
\int_{G \times \mathbb{R}^+ \times B} G_{\varphi} \{ f(\lambda, b, h) \} \overline{G_{\varphi}(\lambda, b, h)} |c(\lambda)|^{-2} \, d\lambda \, db \, dh = \| \varphi \|_{L^2(X)}^2 \int_X f(x) \overline{g(x)} \, dx
\]

Proof. we have by the lemma 2.1

\[
\int_{G \times \mathbb{R}^+ \times B} G_{\varphi} \{ f(\lambda, b, h) \} \overline{G_{\varphi}(\lambda, b, h)} |c(\lambda)|^{-2} \, d\lambda \, db \, dh = \int_{G \times \mathbb{R}^+ \times B} \int_G \int_X f(x) \overline{g(x)} \overline{T_{h^{-1}} \varphi(.)} (\lambda, b, h) |c(\lambda)|^{-2} \, d\lambda \, db \, dh
\]

\[
= \int_G \int_X f(x) \overline{T_{h^{-1}} \varphi(.)} g(x) T_{h^{-1}} \varphi(.) \, dx \, dh
\]

\[
= \int_G \int_X f(x) \overline{T_{h^{-1}} \varphi(.)} g(x) \overline{T_{h^{-1}} \varphi(.)} \overline{T_{h^{-1}} \varphi(.)} \, dx \, dh
\]

\[
= \int_G \int_X f(x) \overline{g(x)} |T_{h^{-1}} \varphi(.)|^2 \, dx \, dh
\]

\[
= \int_X f(x) \overline{g(x)} \, dx \int_G |T_{h^{-1}} \varphi(.)|^2 \, dh
\]

Such as the proof of the Plancherel’s theorem 2.3, Applying the equation (1.1), lemma 2.1 and the Fubini’s theorem, we get,

\[
= \int_G \int_G f(go) \overline{g(go)} \int_K \varphi(gkh^{-1}h^{-1}o) \, dk \int_K \overline{\varphi(gkh^{-1}h^{-1}o)} \, dk \, dh \, dg
\]

\[
= \int_G \int_K \int_K \varphi(gkh^{-1}h^{-1}o) \varphi(gkh^{-1}h^{-1}o) \, dk \, dk \, dh \, dg
\]

using the invariance of the Haar measure \( dg \) by \( K \), we obtain

\[
= \int_G \int_K \int_K \varphi(h^{-1}o) \, dk \, dk \, dh \, dg
\]

\[
= \int_G \int_K \varphi(h^{-1}o) \, dk \, dh \, dg
\]

\[
= \| \varphi \|_{L^2(X)}^2 \int_X f(x) \overline{g(x)} \, dx
\]
We shall now discuss the validity of some uncertainty principles in the case of Gabor-Helgason transform.

3 Uncertainty principle

In quantum physics, the uncertainty principles state that, we cannot give simultaneously the position and moment time of particle with high precision. The formulation mathematics of this concept is that, the function and its Fourier transform cannot both be sharply localized. Many formulations are given, the first one is proved by Heinseberg in 1927 [8], after, many authors give some generations, such as, Hardy’s theorem [7], Morgan’s theorem [12]. Years after, the locally uncertainty principles arise, those theorems asset that, when the uncertainty of the momentum is small, the probability of being localized at any point is very small [2,3,14].

Our first result will be the following local uncertainty principle,

Lemma 3.1. Let \( \varphi, f \in L^2(X) \), we have

\[
\| \mathcal{G}_\varphi \{ f \}(\omega, y) \|_{L^\infty(a^* \times B \times X)} \leq \| f \|_{L^2(X)} \| \varphi \|_{L^2(X)}
\]

Proof. We have

\[
| \mathcal{G}_\varphi \{ f \}(\lambda, b, h) | = | \int_X f(x) T_{h^{-1}} \varphi(x) e^{-i(x+b)A(x,b)} dx |
\]

Using H"{o}lder inequality we get our result

\[
\| \mathcal{G}_\varphi \{ f \}(\lambda, b, h) \|_{L^\infty(a^* \times B \times X)} \leq \| f \|_{L^2(X)} \| \varphi \|_{L^2(X)}
\]

\[ \square \]

Theorem 3.2. Let \( \varphi \) a windowed function and let \( \Sigma \) a subset of \( a^* \times B \times X \) such that \( 0 < m(\Sigma) < +\infty \), for all \( f \in L^2(X) \) we have,

\[
\| f \|_{L^2(X)} \| \varphi \|_{L^2(X)} \leq \frac{1}{\sqrt{1 - m(\Sigma)^2}} \| \mathcal{G}_\varphi \{ f \} \chi_\Sigma \|_{L^2(a^* \times B \times X)}
\]

Proof. For every \( f \in L^2(X) \); we have

\[
\| \mathcal{G}_\varphi \{ f \} \|_{L^2(X)}^2 = \| \mathcal{G}_\varphi \{ f \} \chi_\Sigma \|_{L^2(a^* \times B \times X)}^2 + \| \mathcal{G}_\varphi \{ f \} \chi_\Sigma^c \|_{L^2(a^* \times B \times X)}^2
\]
Applying the (3.1) and the Plancherel formula 2.3, we get
\[
\|G_\varphi \{f\} \chi \Sigma\|_{L^2(a^* \times B \times X)}^2 \leq m(\Sigma)^2 \|G_\varphi \{f\}\|_{L^\infty(a^* \times B \times X)}^2 \leq m(\Sigma)^2 \|f\|_{L^2(X)} \|G_\varphi \{f\}\|_{L^2(X)},
\]
(3.4)

Thus, by the equation (3.3)
\[
\|G_\varphi \{f\} \chi \Sigma\|_{L^2(a^* \times B \times X)}^2 \geq (1 - m(\Sigma)^2) \|G_\varphi \{f\}\|_{L^2(X)}^2 \|f\|_{L^2(X)}^2
\]
(3.5)

Thus, by the equation (3.3)
\[
\|G_\varphi \{f\} \chi \Sigma\|_{L^2(a^* \times B \times X)}^2 \geq \|G_\varphi \{f\}\|_{L^2(X)}^2 \|f\|_{L^2(X)}^2(1 - m(\Sigma)^2) \|f\|_{L^2(X)}^2
\]
\[
= \|G_\varphi \{f\} \chi \Sigma\|_{L^2(a^* \times B \times X)}^2 \geq \|G_\varphi \{f\}\|_{L^2(X)}^2 \|f\|_{L^2(X)}^2(1 - m(\Sigma)^2) \|f\|_{L^2(X)}^2
\]
(3.6)

**Theorem 3.3** (Concentration of \(G_\varphi \{f\}\) in small sets).

Let \(\varphi\) be a window function and \(\Sigma \subset a^* \times B \times X\) with \(m(\Sigma) < 1\).

Then, for \(f \in L^2(X)\) we have
\[
\|G_\varphi \{f\} - \chi \Sigma G_\varphi \{f\}\|_{L^2(a^* \times B \times X)} \geq \|\varphi\|_{L^2(X)} \|f\|_{L^2(X)} \|G_\varphi \{f\}\|_{L^2(X)}^2(1 - m(\Sigma)) \|f\|_{L^2(X)}^2
\]
(3.6)

**Proof.** We have
\[
\|G_\varphi \{f\} - \chi \Sigma G_\varphi \{f\}\|_{L^2(a^* \times B \times X)} = \|G_\varphi \{f\}(1 - \chi \Sigma)\|_{L^2(a^* \times B \times X)}
\]
From theorem 2.3
\[
\|G_\varphi \{f\} - \chi \Sigma G_\varphi \{f\}\|_{L^2(a^* \times B \times X)} \geq \|G_\varphi \{f\}\|_{L^2(a^* \times B \times X)} \|f\|_{L^2(X)} \|1 - m(\Sigma)\|_{L^2(X)}^2 \|f\|_{L^2(X)}^2
\]
Hence,
\[
\|G_\varphi \{f\} - \chi \Sigma G_\varphi \{f\}\|_{L^2(a^* \times B \times X)} \geq \|\varphi\|_{L^2(X)} \|f\|_{L^2(X)} \|G_\varphi \{f\}\|_{L^2(X)}^2(1 - m(\Sigma)) \|f\|_{L^2(X)}^2
\]
\[
\square
\]

**Theorem 3.4.** Let \(s > 0\). Then there exists a constant \(C_s > 0\) such that, for all \(f, \varphi \in L^2(X)\)
\[
\|f\|_{L^2(X)} \|\varphi\|_{L^2(X)} \leq C_s \left( \int_{a^* \times B \times X} |(\lambda, b, h)|^{2s} |G_\varphi \{f\}(\lambda, b, h)|^2 |c(\lambda)|^{-2} d\lambda db dh \right)^{\frac{1}{2}}
\]
(3.7)
Theorem 3.6 (Benedicks-type uncertainty principle for $G_{\varphi}$).

Let $r, R > 0$. Let $\varphi \in L^2(X) \cap L^\infty(X)$ be a non zero window function such that $\text{supp} \varphi \subset B_r$ and let $\Sigma = S \times B_R \subset a^* \times B \times X$, be a subset of finite measure. Then

$$\text{Im}\{P_{\varphi}\} \cap \text{Im}\{P_\Sigma\} = \{0\}$$
i.e, \( \Sigma \) is weakly annihilating.

Proof. Let \( F \in \text{Im}\{P_\varphi\} \cap \text{Im}\{P_\Sigma\} \), then, there exists a function \( f \in L^2(X) \) such that, \( F = G_\varphi(f) \) and \( \text{supp}\{F\} \subset \Sigma \).

Then for all \( (\lambda, h, b) \in \Sigma \)

\[
F(\lambda, h, b) = (\hat{fT_{h^{-1}}\varphi})(\lambda, b)
\]

Thus \( \text{supp}\{\hat{fT_{h^{-1}}\varphi}\} \subset S \), with \( m(S) < +\infty \). On other hand \( \text{supp}\varphi \subset B_r \), we have \( \text{supp}\{fT_{h^{-1}}\varphi\} \subset B_{r+R} \).

Hence, by the Benedicks theorem of the Helgason transform (see theorem 6.1 in [11]).

We deduce that \( fT_{h^{-1}}\varphi \equiv 0 \) then \( F = 0 \). \( \square \)

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