SPDEs leading to local, relativistic quantum vector fields with indefinite metric and nontrivial S-matrix

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Abstract

In this article we review the construction of local, relativistic quantum vector fields by analytic continuation of Euclidean vector fields obtained as solutions of covariant SPDEs. We revise the formulation of such SPDEs by introducing new Gaussian noise terms – a procedure which avoids the re-definition of the two point functions needed in previous approaches in order to obtain relativistic fields with nontrivial scattering. We describe the construction of asymptotic states and the scattering of the analytically continued solutions of these new SPDEs and we give precise conditions for nontrivial and well-defined scattering.

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1 Introduction

Since the work of Symanzik and Nelson [30, 31, 40], the construction of local, relativistic quantum fields via analytic continuation from Euclidean random
fields has been the most vital and productive paradigm in constructive quantum field theory (QFT), see e.g. the by now classical expositions in [16, 37]. This Euclidean strategy has been completed successfully in $d = 2$ space-time dimensions and partial results have been obtained for $d = 3$ [11, 15]. In the physical space-time dimension $d = 4$ however, the standard approach to the definition of local potentials via renormalization up to now is plagued by seemingly incurable ultra-violet divergences, and no construction of a non-trivial (interacting) quantum field is known within that approach.

In a series of papers beginning with the mid-80ies [6, 7, 8], a different approach was suggested for $d = 4$ dimensions, where Euclidean random fields are being constructed as solutions of the quaternionic Cauchy-Riemann partial differential equation driven by a non Gaussian white noise. For the so-defined models, the analytic continuation to relativistic fields of self-interacting electromagnetic type can be performed [9] and the corresponding relativistic models have non-trivial scattering behavior [3]. The physical interpretation of the given fields, i.e. a solution of the Gauge problem in the sense of Gupta and Bleuler [21, 14] (i.e. the identification of the physical Hilbert space), is still an unsolved problem and therefore these models can not yet be considered as yielding fully satisfactory 4-d relativistic quantum fields. We however note that one can verify the modified Wightman axioms for quantum fields with indefinite metric [29, 39] and the asymptotic incoming and outgoing fields can be gauged as in usual free quantum electrodynamics. Furthermore, gauge invariance of the scattering (S-) matrix has been verified in [4].

In [12, 19, 20] a generalization of this program was suggested by Becker, Gielarak and Lugiewicz and the equation $DA = F$ was studied systematically in arbitrary space-time dimension, where $D$ is a covariant partial differential operator with constant coefficients and $F$ is a generalized (non-Gaussian) white noise vector field. Solution of this SPDE under suitable conditions on $D$ were given in [12], vacuum expectation values of the associated relativistic fields are given in [5, 12, 19, 20] and the scattering behavior has been calculated in [5].

Here we review the construction of relativistic quantum vector fields with indefinite metric via solutions of $DA = F$. Most of the material in this article has been taken from the above references, but we introduce some technical improvements in the construction of the vector noise $F$ by adding new Gaussian noise terms (depending on $D$) which render obsolete the re-definition of relativistic two point functions in [5]. This re-definition in [5] was necessary to obtain a well-defined scattering behavior for a certain class of models, but it unfortunately destroyed the direct connection of quantum fields with the random field models. We here demonstrate that a modification of the SPDE can do the same job and the relativistic and Euclidean side remain directly connected.

The article contains the following materials: In Section 2 we introduce the generalized vector noise fields, we discuss covariance properties of the operator $D$ and we set up and solve a modified SPDE $DA = F + F^\gamma$. We calculate two- and $n$-point Schwinger functions (moments) of the random field $A$ and we show that under suitable conditions on $D$ the two point function of $A$ is a superposition of two point functions of free Euclidean vector fields. Section 3
SPDEs and QFT with indefinite metric

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deals with the analytic continuation of the Schwinger functions to relativistic Wightman functions. Section 4 is devoted to the construction of quantum fields with indefinite metric s.t. the Wightman functions constructed in Section 3 are the vacuum expectation values of these fields. Scattering of the relativistic fields is being discussed in section 5: Under given conditions on $D$ ("no dipole condition", cf. [18] for a physical interpretation), we construct asymptotic states and explicitly calculate the scattering (S-) matrix of the models following [1].

2 Construction of Euclidean random fields via (modified) covariant SPDEs

Let $d$ be the space-time dimension, we then identify the Euclidean space-time with $\mathbb{R}^d$. Let $\tau : \text{SO}(d) \to \text{Gl}(L)$ be a $L$-dimensional (not necessarily irreducible) real representation of the orthogonal group. Since $\text{SO}(d)$ is compact, we may assume without loss of generality that $\tau : \text{SO}(d) \to \text{SO}(L)$. We want to study covariant noise vector fields over $\mathbb{R}^d$, which are defined as follows:

A (tempered) random vector field of dimension $L$ over $\mathbb{R}^d$ is a mapping $\phi$ from the space of $\mathbb{R}^d$-valued Schwarz test functions $\mathcal{S} = \mathcal{S}(\mathbb{R}^d, \mathbb{R}^L)$ to the real valued random variables of some probability space $(\Omega, \mathcal{B}, P)$ such that 1) $\phi$ is linear $P$ a.s. and 2) $f_n \to f$ in $\mathcal{S} \Rightarrow \phi(f_n) \to \phi(f)$ in probability law. $\phi$ by definition transforms covariantly under $\tau$ if $\phi(f) = \phi(f_g)$ in probability law $\forall f \in \mathcal{S}$ where $g \in \text{SO}(d)$ and $f_g(x) = \tau(g)f(g^{-1}x)$. Furthermore, $\phi$ is called translation invariant or stationary if $\phi(f_a) = \phi(f)$ in law $\forall f \in \mathcal{S}$ where $a \in \mathbb{R}^d$ and $f_a(x) = f(x-a)$.

Definition 2.1 A $\tau$-covariant noise field is an $L$-dimensional random vector field over $\mathbb{R}^d$ which transforms covariantly under $\tau$, is stationary and is independent when localized in non intersecting regions (i.e. $\phi(f)$ is independent of $\phi(h)$ if $\text{supp} f \cap \text{supp} h = \emptyset$.)

A characteristic functional is a mapping $\mathcal{C} : \mathcal{S} \to \mathbb{C}$ fulfilling 1) $\mathcal{C}$ is continuous, 2) $\mathcal{C}$ is positive definite and 3) $\mathcal{C}$ is normalized $\mathcal{C}(0) = 1$. By Minlos’ theorem [20, 25] there is a one-to-one (up to equivalence in law) correspondence between (L-dimensional) random fields $\phi$ and characteristic functionals $\mathcal{C}_{\phi}$ given by $\mathcal{C}_{\phi}(f) = E[e^{i\phi(f)}]$. Furthermore, $\phi$ can be realized as coordinate process on the measurable space $(\mathcal{S}', \mathcal{B})$, with $\mathcal{S}'$ the topological dual space of $\mathcal{S}$ and $\mathcal{B}$ the Borel sigma algebra on $\mathcal{S}'$. Explicitly, given a random vector field $\phi$ there exists a unique probability measure $P_{\phi}$ on this measurable space such that $\phi(f) = \langle .., f \rangle \forall f \in \mathcal{S}$ in law where $\langle .., \rangle$ is the dualization of $\mathcal{S}'$ and $\mathcal{S}$. Thus the path space of $\phi$ is always contained in the space of tempered distributions.

As a simple consequence of Minlos’ theorem we get that $F$ is a $\tau$-covariant noise if and only if the associated characteristic functional $\mathcal{C}_F$ fulfills $\mathcal{C}_F(f_a) = \mathcal{C}_F(f)\forall f \in \mathcal{S}, a \in \mathbb{R}^d$ and $\mathcal{C}_F(f + h) = \mathcal{C}_F(f)\mathcal{C}_F(h)\forall f, h \in \mathcal{S}$ s.t. $\text{supp} f \cap \text{supp} h = \emptyset$. To construct noise fields, it is thus sufficient to define a characteristic functional with the above properties.
Let $\psi$ be a $C^\infty$ Lévy characteristic over $\mathbb{R}^L$, i.e.

$$\psi(t) = ia \cdot t - \frac{\sigma t \cdot \sigma t}{2} + \int_{\mathbb{R}^L - \{0\}} \left[e^{i t \cdot s} - 1\right] dM(s)$$  \hspace{1cm} (1)

where $t, a \in \mathbb{R}^L$, $\sigma \in \text{Mat}_{n \times L}(\mathbb{R})$ is symmetric and positive semi definite and the Lévy measure $M$ is a measure on $\mathbb{R}^L - \{0\}$ s.t. $M$ has moments of all orders. $\psi$ is uniquely determined by a (deterministic part), $\sigma$ (Gaussian part) and $M$ (Poisson part). We say that $\psi$ is uniquely determined by $\psi = \tau$, where $t, a \in \mathbb{R}^L$. Obviously, this is the case if and only if $\tau(a, h) = 0$.

**Proposition 2.2** Let $\psi$ be a $C^\infty$ and $\tau$-invariant Lévy characteristic. Then (i)

$$\mathcal{C}_F(f) = \exp \left[\int_{\mathbb{R}^d} \psi(f) \, dx\right], \quad f \in \mathcal{S}$$  \hspace{1cm} (2)

is a characteristic functional. The associated random vector field $F$ is a $\tau$-covariant noise field.

(ii) Let $\Delta$ be the Laplacian on $\mathbb{R}^d$, $p : \mathbb{R} \to \mathbb{R}$ be a polynomial which is positive semi-definite on $[0, \infty)$ and $\bar{\sigma}^2 = -\left(\frac{\partial^2 \psi(t)}{\partial t \partial \tau^\theta} \right)_{t=0}^{\alpha, \beta=1, \ldots, n}$. Then

$$\mathcal{C}_{Fs}(f) = \exp \left[-\int_{\mathbb{R}^d} \bar{\sigma} f \cdot \bar{\sigma} p(-\Delta) f \, dx\right], \quad f \in \mathcal{S}$$  \hspace{1cm} (3)

defines a Gaussian $\tau$-covariant noise field.

**Proof.** (i) Continuity of $\mathcal{C}_F$ follows from $\psi$ being $C^\infty$, normalization is a consequence of $\psi(0) = 0$ and positive definiteness can be derived from the fact that $\psi$ is a conditionally positive definite\footnote{\sum_{l,j=1}^{n} \psi(t_l - t_j) z_l z_j \geq 0 \text{ if } \sum_{l=1}^{n} z_l = 0, \quad t_j \in \mathbb{R}^L, \quad z_j \in \mathbf{I}, \quad j = 1, \ldots, n; \quad n \in \mathbb{N}}$ function in $t$ and thus $\int_{\mathbb{R}^d} \psi(f) \, dx$ is conditionally positive definite in $f$. But the exponential of a conditionally positive definite function is positive definite by Schoenberg’s theorem [13]. $\tau$-invariance of $\mathcal{C}_F$ follows from the $\tau$-invariance of $\psi$ and the invariance of the Lebesgue measure under orthogonal transformations. Likewise, invariance of $\mathcal{C}_F$ under translations follows from translation invariance of $dx$. That $\mathcal{C}_F(f + h)$ factors for $\text{supp} f \cap \text{supp} h = \emptyset$ is implied by $\psi(0) = 0$.

(ii) The proof that $\mathcal{C}_{Fs}$ is the characteristic functional of a Gaussian random field is standard (we note that $\bar{\sigma}$ by definition is positive semi-definite and commutes with the positive operator $p(-\Delta)$.) Invariance follows from the fact that $\bar{\sigma}^2$ commutes with $\tau(g)$ and $p(-\Delta)$ is invariant under orthogonal transformations. Translation invariance follows as in (i). That $\mathcal{C}_{Fs}(f + h)$ factors if $\text{supp} f \cap \text{supp} h = \emptyset$ is a consequence of

$$\int_{\mathbb{R}^d} \bar{\sigma} f \cdot \bar{\sigma} p(-\Delta) h \, dx = 0$$
for \( f, h \) as above. \( \blacksquare \)

Let \( D : S \to S \) be a \( \tau \)-covariant partial differential operator with constant coefficients, i.e. \( Df_g = (Df)_g \) and \( Df_a = (Df)_a \) \( \forall f \in S, g \in SO(d), a \in \mathbb{R}^d \). Assuming that \( D \) is continuously invertible, the following representation has been obtained \([12]\) for the Fourier transform of the Green’s function of \( D \):

\[
\hat{D}^{-1}(k) = \frac{Q_E(k)}{\prod_{i=1}^{N}(|k|^2 + m_i^2)^{\nu_i}}
\]

with \( m_i \in \mathbb{C} - (-\infty, 0], m_j \neq m_i \) for \( l \neq j \) and \( \nu_i \in \mathbb{N} \). \( Q_E(k) \) is an \( L \times L \)-matrix with polynomial entries of order \( \leq \kappa = 2(\sum_{i=1}^{N} \nu_i - 1) \) which fulfills the Euclidean transformation law \( \text{ad}_{e^{i\theta}}(Q_E(g^{-1}k)) = Q_E(k) \forall g \in SO(d) \). Without loss of generality we assume that \( Q_E \) is prime w.r.t the factors \( (|k|^2 + m_l^2) \), i.e. that none of them divides all of the polynomial matrix elements of \( Q_E \) and furthermore we impose a “positive mass spectrum” condition \( m_i \in (0, \infty) \) for \( l = 1, \ldots, N \).

Given this representation of \( \hat{D}^{-1}(k) \) we define

\[
p(t) = p(t, D) = \frac{\prod_{i=1}^{N}(t + m_i^2)^{\nu_i}}{\prod_{i=1}^{N} m_i^{2\nu_i}} - 1
\]

and we note that \( p(t) \geq 0 \) as \( t \geq 0 \).

Let \( \psi \) be a \( \tau \)-invariant Lévy characteristic s.t. \( d\psi(t)/dt \big|_{t=0} = 0 \) and \( D \) as above. We define \( F \) as in Proposition \([22]\) (i) and \( F^g \) as in Proposition \([22]\) (ii) with \( p(t) = p(t, D) \) as in Equation \((5)\), \( F^g \) independent of \( F \) (in the stochastic sense). We set up the crucial SPDE of this work as

\[
DA = F + F^g
\]

which can be solved pathwisely in \( S' \) since the random field \( F + F^g \) by Minlos’ theorem has paths in \( S' \) and \( D \) by our assumptions (and duality) is continuously invertible on that space. We get for the properties of \( A \):

**Theorem 2.3** A obtained as the unique solution of \((6)\) in \( S' \) is a \( \tau \)-covariant random vector field. The Schwinger functions (moments) of \( A \) are given by

\[
S_{n, \alpha_1 \cdots \alpha_n}(x_1, \ldots, x_n) = E[A_{\alpha_1}(x_1) \cdots A_{\alpha_n}(x_n)]
\]

\[
= \sum_{I \in \mathcal{P}^{(n)}} \prod_{I_j \in \mathcal{P}^{(n)}} S_{I_{j_1}, \alpha_{j_1} \cdots \alpha_{j_l}}^T(x_{j_1}, \ldots, x_{j_l})
\]

(7)

where \( \mathcal{P}^{(n)} \) is the collection of all partitions of \( \{1, \ldots, n\} \) into nonempty subsets. For \( n = 2 \)

\[
S_{2, \alpha_1 \alpha_2}^T(x_1, x_2) = \frac{Q_E^T_{2, \alpha_1 \alpha_2}(-\Delta)}{\prod_{i=1}^{N} m_i^{2\nu_i}} \left[ \prod_{i=1}^{N} (-\Delta + m_i^2)^{-\nu_i} \right] (x_1 - x_2)
\]

(8)
and for $n \geq 3$

$$S_{n,\alpha_1,\ldots,\alpha_n}^T(x_1,\ldots,x_n) = Q_{n,\alpha_1,\ldots,\alpha_n}^E(-i\Sigma_n) \times \int_{\mathbb{R}^d} \prod_{j=1}^n \left[ \prod_{l=1}^N (-\Delta + m_l^2)^{-\nu_l} \right] (x_j - x) \, dx \quad (9)$$

where

$$Q_{n,\alpha_1,\ldots,\alpha_n}^E(-i\Sigma_n) = C_{\beta_1,\ldots,\beta_n} \prod_{l=1}^n Q_{E,\beta_l,\alpha_l}^E(-i\frac{\partial}{\partial x_l}) \quad (10)$$

with

$$C_{\beta_1,\ldots,\beta_n} = (-i)^n \left. \frac{\partial^n \psi(t)}{\partial t_{\beta_1} \cdots \partial t_{\beta_n}} \right|_{t=0} \quad (11)$$

and we applied the Einstein convention of summation and upping/lowering of indices on $\mathbb{R}^L$ w.r.t. the invariant inner product $\cdot$. The Schwinger functions fulfill the requirements of $\tau$-covariance, translation invariance, symmetry, clustering and Hermiticity from the Osterwalder-Schrader axioms [32, 33] of Euclidean QFT.

**Proof.** We give a short outline of the proof: $\tau$-covariance of $A$ and stationarity can be deduced from the $\tau$- (and translation-)invariance of the characteristic functional $C_A(f) = C_F(D^{-1}f)C_{F_\tau}(D^{-1}f)$. This property follows from the related property for $C_F$ and $C_{F_\tau}$ and $\tau$-covariance (translation invariance) of $D$. The characteristic functional $C_A$ is given by the following explicit formula

$$C_A(f) = \exp \left[ \int_{\mathbb{R}^d} \psi(D^{-1}f) - \bar{\sigma} D^{-1}f \cdot \bar{\sigma} p(-\Delta) D^{-1}f \, dx \right] \quad f \in S \quad (12)$$

and we can calculate the Schwinger functions (or moments) of $A$ by

$$S_{n,\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n) = (-i)^n \frac{\delta^n}{\delta A_{\alpha_1}(x_1) \cdots \delta A_{\alpha_n}(x_n)} C_A(f) \bigg|_{f=0} \quad (13)$$

Using the linked cluster identity (see e.g. [2]) we get

$$S_{n,\alpha_1,\ldots,\alpha_n}^T(x_1,\ldots,x_n) = (-i)^n \frac{\delta^n}{\delta A_{\alpha_1}(x_1) \cdots \delta A_{\alpha_n}(x_n)} \ln C_A(f) \bigg|_{f=0} \quad (14)$$

By Eqs. (1-3), (12) and (14) one derives Eq. (8) and (9) by explicit calculations.

The $\tau$-covariance / translation invariance properties of the Schwinger functions follow from the related properties of the random vector field $A$. Symmetry and Hermiticity are trivial (note that $A$ is a real vector field). The cluster property can be verified from the fact that the Green’s function

$$\left[ \prod_{l=1}^N (-\Delta + m_l^2)^{-\nu_l} \right] (x) \quad (15)$$
and its derivatives are of exponential decay as $|x| \to \infty$, see [2] for a related situation.

Here we would like to point out that the difference between $A$ as defined here and the corresponding fields defined in [5] is that in that reference

$$S_{201,02}^T(x_1 - x_2) = Q_{201,02}^E(-i\nabla_2) \left[ \prod_{l=1}^N (-\Delta + m_l^2)^{-2\nu_l} \right] (x_1 - x_2)$$

and the effect of the Gaussian noise $F^g$ is to replace (15) by (8). As we shall see in the discussion of relativistic scattering theory below, this correction leads to a well-defined particle-like asymptotics, for $\nu_l = 1, l = 1, \ldots, n$, of the related relativistic quantum field models. Without this correction (or the somewhat ad hoc replacement in [5]) the asymptotics would be dipole-like rather than particle like. Such effects now only occur when some of the $\nu_l$ are strictly larger than one, cf. [18].

**Condition 2.4** We say that the partial differential operator $D$ fulfills the non-dipole condition, if in the representation (4) we have $\nu_l = 1$ for $l = 1, \ldots, N$. We restrict from now on our models to fulfill this condition (cf. [18] for the meaning of this condition).

## 3 Analytic continuation of the Schwinger functions

In this section we discuss the analytic continuation of the truncated Schwinger functions $S_n^T$ to relativistic truncated Wightman functions $W_n^T$. A solution to this problem can be obtained by representing $S_n^T$ as Fourier-Laplace transform, i.e.

$$S_{n,\alpha_1 \ldots \alpha_n}^T(x_1, \ldots, x_n) = (2\pi)^{-dn/2} \int_{\mathbb{R}^{dn}} \exp\left(\sum_{l=1}^n -k_l^0 x_l^0 + ik_l \cdot \vec{x}_l\right) \times W_{n,\alpha_1 \ldots \alpha_n}^T(k_1, \ldots, k_n) \, dk_1 \cdots dk_n,$$

where $x_1^0 < \ldots < x_n^0$ and $W_{n,\alpha_1 \ldots \alpha_n}^T$ (the Fourier transform of $W_n^T$) is a tempered distribution which fulfills the spectral property, i.e. it has support in the cone $\{(k_1, \ldots, k_n) \in \mathbb{R}^{dn} : q_j = \sum_{l=1}^n k_l \in \overline{V_0^j}, j = 1, \ldots, n-1\}$. Here, $\overline{V_0}$ stands for the closed backward lightcone (that we do not use the forward lightcone for the formulation of the spectral condition as done in some other references, is a matter of convention on the Fourier transform). Under this condition the above integral representation exists. From the general theory of quantum fields it follows that $S_n^T$ is the analytic continuation $W_n^T$ from points with purely relativistically real time to the Euclidean points of purely imaginary time. Furthermore, it follows from the symmetry and Euclidean covariance of the $S_n^T$ that $W_n^T$ fulfills the requirements of Poincaré covariance (w.r.t. the analytic continuation of the representation $\tau$) and locality, see e.g. [52].
Following an idea of [12], we expand the denominator of Eq. (4) into partial fractions (we recall that we assume Condition 2.4 to hold)

\[
\frac{1}{\prod_{l=1}^{N}(|k|^2 + m_l^2)} = \sum_{l=1}^{N} \frac{b_l}{(|k|^2 + m_l^2)}
\]

(17)

with \(b_l \in \mathbb{R}\) uniquely determined and \(b_l \neq 0\). Thus, \(D^{-1}(x)\) can be represented as \(Q_E(-i\partial/\partial x)\sum_{l=1}^{N} b_l(-\Delta + m_l^2)^{-1}(x)\) and for \(n \geq 3\)

\[
S^T_{n,\alpha_1 \ldots \alpha_n}(x_1, \ldots, x_n) = Q_E^{(-\sum_{l=1}^{N} b_l(-\Delta + m_l^2)^{-1}(x - x_j) dx}
\]

(18)

Setting

\[
S^T_{n,m_1 \ldots m_n}(x_1, \ldots, x_n) = \int_{\mathbb{R}^d} \prod_{j=1}^{n} (-\Delta + m_j^2)^{-1}(x - x_j) dx
\]

(19)

we note that a Fourier-Laplace representation (16) of \(S^T_{n,m_1 \ldots m_n}(x_1, \ldots, x_n)\) has been calculated in [2] Proposition 7.8. \(\hat{W}^T_{n,m_1 \ldots m_n}(k_1, \ldots, k_n)\) is given by

\[
(2\pi)^{-\left(d(n-2)-2\right)/2} \left\{ \sum_{j=1}^{n-1} \prod_{l=1}^{j-1} \delta_{m_l}(k_l) \prod_{j+1}^{n} \delta_{m_j}(k_j) \right\} \delta(\sum_{l=1}^{n} k_l)
\]

(20)

Here \(\delta_{m_j}(k) = \theta(\pm k^0)\delta(k^2 - m^2)\) where \(\theta\) is the Heaviside step function and \(k^2 = k^0^2 - |\vec{k}|^2\). Furthermore, let

\[
Q_n^M((i k_1^0, \vec{k}_1), \ldots, (i k_n^0, \vec{k}_n)) = Q_n^E((ik_1^0, \vec{k}_1), \ldots, (ik_n^0, \vec{k}_n)).
\]

(21)

We then define

\[
\hat{W}^T_{2,\alpha_1 \alpha_2}(k_1, k_2) = (2\pi)^{(d+1)/2} \frac{Q^M_{2,\alpha_1 \alpha_2}(k_1, k_2)}{\prod_{l=1}^{N} m_l^2} \sum_{l=1}^{N} b_l \delta_{m_l}(k_l) \delta(k_1 + k_2)
\]

(22)

and

\[
\hat{W}^T_{n,\alpha_1 \ldots \alpha_n}(k_1, \ldots, k_n) = Q^M_{n,\alpha_1 \ldots \alpha_n}(k_1, \ldots, k_n)
\]

\[
\times \sum_{l_1, \ldots, l_n=1}^{N} \prod_{j=1}^{n} b_{l_j} \hat{W}^T_{n,m_{l_1} \ldots m_{l_n}}(k_1, \ldots, k_n)
\]

(23)

and we can now put together these pieces in the following theorem:
Theorem 3.1 The truncated Schwinger functions $S^T_n$ have a Fourier-Laplace representation \(^{16}\) with $W^T_n$ defined in Eqs. \(^{22}\) and \(^{23}\). Equivalently, $S^T_n$ is the analytic continuation of $W^T_n$ from purely real relativistic time to purely imaginary Euclidean time. The truncated Wightman functions $W^T_n$ fulfill the requirements of temperedness, relativistic covariance w.r.t. the representation of the orthochronous, proper Lorentz group $\tilde{\tau}: L^\uparrow(d) \to \text{Gl}(L)$, locality, spectral property and cluster property. Here $\tilde{\tau}$ is obtained by analytic continuation of $\tau$ to a representation of the proper complex Lorentz group over $\mathbb{C}^d$ (which contains $SO(d)$ as a real submanifold) and restriction of this representation to the real orthochronous proper Lorentz group.

Proof. That $S^T_n$ has a Fourier-Laplace representation with $W^T_n$ defined as above for the case $n \geq 3$ follows from the related representation for $S^T_{n,m_1,\ldots,m_n}$, the linearity of the Fourier-Laplace transform and the general formula for differentiation of a Fourier-Laplace transform. That $W^T_n$ is tempered can be derived as in \([3]\) or \([1]\). The formula for $n = 2$ can be derived as in the case of the two-point function of the free field. The properties of $W^T_n$ now follow from the general formalism of analytic continuation, \([32]\). □

4 Quantum fields with indefinite metric

In this section we show that the Wightman functions constructed in Theorem 3.1 can be considered as vacuum expectation values of some quantum field theory (QFT) with indefinite metric \([29]\).

We first introduce the concept of a QFT with indefinite metric: Let $H$ be a (separable) Hilbert space and $\mathcal{D} \subseteq H$ a dense domain. Let $\eta$ be a self-adjoint operator on $H$ s.t. $\eta^2 = 1$. $\eta$ is called the metric operator. We define $O_{\eta}(\mathcal{D})$ as the unital, involutive algebra of (unbounded) Hilbert space operators $A : \mathcal{D} \to \mathcal{D}$ s.t. $A^\ast = \eta A^\ast \eta|\mathcal{D} : \mathcal{D} \to \mathcal{D}$ exists. Here $A^\ast$ stands for the Hilbert space adjoint of $(A, \mathcal{D})$. The canonical topology on $O_{\eta}(\mathcal{D})$ is induced by the seminorms $A \to |(\Psi_1, \eta A \Psi_2)|$, $\Psi_1, \Psi_2 \in \mathcal{D}$.

We say that a sequence of Wightman functions $\{W_n\}_{n \in \mathbb{N}_0}$, $W_n \in \mathcal{S}^n$ (where $\mathcal{S}_n = \mathcal{S}^\otimes n$ and $'$ stands for the topological dual - in contrast to the previous sections test functions and distributions from now on are complex valued) fulfills the Hilbert space structure condition (HSSC) if on $\mathcal{S}_n$ there is a Hilbert seminorm $p_n$ s.t.

$$|W_{j+l}(f \otimes h)| \leq p_j(f)p_l(h) \quad \forall f \in \mathcal{S}_j, h \in \mathcal{S}_l, j, l \in \mathbb{N}. \quad (24)$$

We note that the HSSC for $\{W_n\}_{n \in \mathbb{N}_0}$ is implied by the existence of a Schwartz norm $\|\|$ on $\mathcal{S}$ s.t. $W_n^T \in \mathcal{S}_n$ is a continuous distribution w.r.t. $\|\|^\otimes n$ on $\mathcal{S}_n$ (since $\mathcal{S}$ is a nuclear space, the tensor product of norms is well-defined) \([4]\) \([22]\).

Theorem 4.1 Let $\{W_n\}_{n \in \mathbb{N}_0}$ be a sequence of Wightman functions which fulfill the requirements of temperedness, $\tilde{\tau}$-covariance, spectrality, locality and Hermiticity. If furthermore the HSSC \([22]\) holds, then there exists an algebra $O_{\eta}(\mathcal{D})$. 

acting on a separable Hilbert space $\mathcal{H}$ with a distinguished normalized vector $\Psi_0 \in \mathcal{H}$, $\eta \Psi_0 = \Psi_0$, ('vacuum'), an operator valued distribution $\phi : \mathcal{S} \ni f \to \phi(f) \in \mathcal{O}_n(\mathcal{D})$ and a $\eta$-unitary continuous representation of the orthochronous, proper Poincaré group\(^2\) $U : L^1_+(d) \to \mathcal{O}_n(\mathcal{D})$ ($U^* = U^{-1}$) such that

(i) $\mathcal{D}$ is generated by repeated application of operators $\phi(f), f \in \mathcal{S}$ on $\Psi_0$, $\Psi_0$ is invariant under the representation $U$ of $L^1_+(d)$ and $U$ fulfills the spectral condition $\int_{\mathbb{R}^n} (\eta \Psi_1, U(a) \Psi_2) e^{i p \cdot a} da = 0$ if $p$ is not in the forward lightcone (here $\cdot$ is the Minkowski inner product);

(ii) $\phi$ is Hermitean $\phi^\dagger(f) = \phi(f^*)$ where $f^*$ is the (component wise) complex conjugation of $f$; $\phi$ is local ($\phi(f)$ and $\phi(h)$ commute on $\mathcal{D}$ if the support of $f$ and $h$ are space-like separated: $(x-y)^2 < 0$ for $x \in \text{supp} f, y \in \text{supp} h$); $\phi$ transforms $\tau$-covariantly $(U(g) \phi(f) U(g^{-1})) \forall g \in L^1_+$ with $f_g(x) = \tilde{\tau}(g) f(g^{-1} x)$;

(iii) $W_n(f_1 \otimes \cdots \otimes f_n) = (\Psi_0, \phi(f_1) \cdots \phi(f_n) \Psi_0) \forall n \in \mathcal{N}, f_i \in \mathcal{S}$.

For the proof, a kind of GNS-construction on an inner product space is performed, see e.g. [36, 29, 25]. It should be noted that the above assignment of a QFT with indefinite metric to a sequence of Wightman functions in general depends on the Hilbert seminorms $p_n$ in [24] and therefore is not 'intrinsic' for the sequence of Wightman functions. For an example see [10].

Concerning our models in Theorem 4.1 we now get:

**Theorem 4.2** The Wightman functions defined in Section 3 fulfill the HSSC [24]. In particular, there exists a QFT with indefinite metric (cf. Theorem 4.1) s.t. the Wightman functions are given as the vacuum expectation values of that QFT.

**Proof.** By Theorem 4.1 the Wightman functions fulfill all the requirements of Theorem 4.1 except for the HSSC. That also the HSSC holds, can be seen most easily by verifying a uniform continuity property w.r.t. $\| \cdot \|_{\ominus n}$ for the truncated Wightman functions $W^T_{n}$, as explained above. Here $\| \cdot \|$ is some Schwartz norm on $\mathcal{S}$. It has been verified in [11, 21] that there is such a uniform continuity for $W^T_{n,m_1,\ldots,m_n}$ and thus also the linear combinations of these distributions in [22] have this property. But the Fourier transformed Wightman functions of our model are given by the multiplication of the described linear combination by a polynomial $Q^M_n(k_1,\ldots,k_n)$ and it is thus sufficient to verify that the degree of $Q^M_n$ in any variable $k_i$ is bounded independently of $n$, since we then can replace the Schwartz norm $\| \cdot \|$ by the Schwartz norm $\|(1 + |k|^2)^{l/2}\|$ for $l$ larger or equal to this uniform degree. That such a uniform bound of the degree in the $k_i$ exists is a straightforward consequence of the definitions [10] and [21]. □

\(^2\) $P^1_+(d)$ is the semidirect product of $L^1_+(d)$ with the translation group $\mathbb{R}^d$. 
5 On the construction of asymptotic states and the S-matrix

Here we describe the scattering behavior of the QFT models with indefinite metric following [1, 17]. Since the standard axiomatic scattering theory [22, 23, 35, 34] heavily relies on positivity of the Wightman functions, we can not apply these methods here. Let us therefore first consider the general problem of the construction of asymptotic states for quantum fields with indefinite metric following [17]:

Let \( S^{\text{ext}} = \mathcal{S} \otimes \mathbb{R}^3 \) be the "extended test function space". We want to construct an (in general non-local) "extended quantum field" \( \Phi : S^{\text{ext}} \rightarrow \mathcal{O}_q(D) \), where the three components of \( \Phi \) can be interpreted as the incoming- local- and outgoing field \( \Phi = (\phi^{\text{in}}, \phi, \phi^{\text{out}}) \), using the GNS-like construction of Theorem 4.1. We define \( J(f^{\text{in}}, f^{\text{loc}}, f^{\text{out}}) = f^{\text{in}} + f^{\text{loc}} + f^{\text{out}} \) and furthermore \( J^{\text{in/loc/out}} : \mathcal{S} \rightarrow S^{\text{ext}} \) as the injection in the first/second/third component. Let the mass content of the theory be given by the masses \( m_1, \ldots, m_N > 0 \), which in our case are determined by the representation \( \mathcal{H} \) of the partial differential operator \( \mathcal{D} \). Let \( \varphi \in C_0^\infty (\mathbb{R}, \mathbb{R}) \) with support in \((-\epsilon, \epsilon)\) such that \( 0 < \epsilon < \min \{ m_i^2, m_i^2 - m_j^2 \}, i, j = 1, \ldots, N, i \neq j \). We define \( \chi^\pm(k, m) = \theta(\pm k^0) \varphi(k^2 - m^2) \) and we set

\[
\chi_t(a, k) = \begin{cases} 
\sum_{l=1}^N \left[ \chi^+(k, m_l) e^{-i(k^0 - \omega_l)t} + \chi^-(k, m_l) e^{-i(k^0 + \omega_l)t} \right] & \text{for } a = \text{in} \\
1 & \text{for } a = \text{loc} \\
\sum_{l=1}^N \left[ \chi^+(k, m_l) e^{i(k^0 - \omega_l)t} + \chi^-(k, m_l) e^{i(k^0 + \omega_l)t} \right] & \text{for } a = \text{out}
\end{cases}
\]

Here \( \omega_l = (|\vec{k}|^2 + m_l^2)^{1/2} \). We now define \( \Omega_{\text{in}} S^{\text{ext}} \rightarrow \mathcal{S} \) and \( \Omega^{\text{in/out}} : \mathcal{S} \rightarrow \mathcal{S} \) by

\[
\mathcal{F} \Omega^{\text{ext}} \mathcal{F} = \begin{pmatrix}
\chi_t(k, \text{in}) & 0 & 0 \\
0 & \chi_t(k, \text{loc}) & 0 \\
0 & 0 & \chi_t(k, \text{out})
\end{pmatrix}
\]

and \( \Omega_t = J \circ \Omega_{\text{ext}} \), \( \Omega^{\text{in/out}} = \Omega_t \circ J^{\text{in/out}} \). Here \( \mathcal{F} \) (\( \mathcal{F} \)) denotes the (inverse) Fourier transform. For \( f_t \in S^{\text{ext}} \) we set

\[
F_n(f_1 \otimes \cdots \otimes f_n) = \lim_{t_1, \ldots, t_n \rightarrow +\infty} W_n(\Omega_t f_1 \otimes \cdots \otimes \Omega_{t_n} f_n)
\]

provided that the limit exists and \( F_n \) is in \( S^{\text{ext}'} = (S^{\text{ext} \otimes n})' \). Here the limit \( t_1, \ldots, t_n \rightarrow +\infty \) has to be understood in the sense that first one \( t_i \) goes to infinity, then the next etc. and that the limit does not depend on the chosen order. If \( F_n \) exists for any \( n \) then the sequence \( \{ F_n \}_{n \in \mathbb{N}_0} \) is called the form factor functional. Heuristically, \( \{ F_n \}_{n \in \mathbb{N}_0} \) contains the 'mixed' vacuum expectation values of in- loc- and out-fields. If the form factor functional fulfills the HSSC, then one can construct in-, loc- and out-fields acting on one Hilbert space \( \mathcal{H} \) by applying the GNS-like construction of Theorem 4.1 in order to construct a "extended quantum field" \( \Phi \). Given \( \Phi \), we then define \( \phi^{\text{in/loc/out}} = \Phi \circ J^{\text{in/loc/out}} \). The main result for the present section is that.
this whole construction is possible for the models defined in Theorem 4.2. To formulate this, we require some more definitions: We set

$$\Delta_m(a, k) = \begin{cases} -i\pi(\delta_+^m(k) - \delta_-^m(k)) & \text{for } a = \text{in} \\ 1/(k^2 - m^2) & \text{for } a = \text{loc} \\ i\pi(\delta_+^m(k) - \delta_-^m(k)) & \text{for } a = \text{out} \end{cases} \quad (28)$$

and we define the Fourier-transformed truncated form factor functional associated to the Wightman functions $W_{n,m_1,\ldots,m_n}^T$ defined in Section 3 for $n \geq 3$ $a_l = \text{in}/\text{loc}/\text{out}$ via

$$\hat{F}_{n,m_1,\ldots,m_n}^T(a_1,\ldots,a_n)(k_1,\ldots,k_n) = -(2\pi)^{-(d_n-2)/2} \left\{ \sum_{j=1}^{n-1} \prod_{l=1}^{j-1} \delta_{m_1}^j(k_l) \hat{\Delta}_{m_j}(a_j, k_j) \prod_{l=j+1}^{n} \delta_{m_l}^+(k_l) \right\} \delta (\sum_{l=1}^{n} k_l) \quad (29)$$

and we finally define the truncated, Fourier transformed form factor functional for our models as $F_{n,m}^T(a_1,a_2) = W_{n,2}^T$, $a_1,a_2 = \text{in}/\text{loc}/\text{out}$ and for $n \geq 3$ we set in analogy to Eq. (28)

$$\hat{F}_{n,a_1,\ldots,a_n}^T(k_1,\ldots,k_n) = Q_{n,a_1,\ldots,a_n}(k_1,\ldots,k_n) \times \sum_{l_1,\ldots,l_n=1}^{N} b_{l_1} \hat{F}_{n,m_{l_1},\ldots,m_{l_n}}^T(k_1,\ldots,k_n) \quad (30)$$

We define $\{F_n\}_{n \in N_0}$ to be the sequence of distributions associated with the truncated sequence $\{F_n^T\}_{n \in N}$ defined above. We get

**Theorem 5.1** Let $\{W_n\}_{n \in N_0}$ be the Wightman functions of the QFT-models defined in Theorem 4.2. Then the associated form factor functional is given by $\{F_n\}_{n \in N_0}$ and fulfills the HSSC. Thus,

(i) There exists a QFT with indefinite metric (over $S^{\text{out}}$) $(\mathcal{H}, \eta, \Phi_0, \Phi, U)$ fulfilling Hermiticity, spectrality and clustering s.t. the fields $\phi^{\text{in/loc/out}} = \Phi \circ J^{\text{in/loc/out}}$ in addition are $\tau$-covariant and local.

(ii) The asymptotic fields $\phi^{\text{in/out}}$ are given as a sum of independent free $\tau$-vector fields with masses $m_1,\ldots,m_N$.

(iii) $\phi = \phi^{\text{loc}}$ fulfills the LSZ asymptotic condition w.r.t. $\phi^{\text{in/out}}$, i.e.

$$\lim_{t \to +\infty} \phi(\Omega^{\text{in/out}}_t f) = \phi^{\text{in/out}}(f) \quad \forall f \in S \quad (31)$$

where the convergence is in $\mathcal{O}_0(D)$.

**Proof.** That $W_{n,m_1,\ldots,m_n}^T$ and $F_{n,m_1,\ldots,m_n}^T$ fulfill Eq. (27) (here truncation plays no rôle, cf. Proposition 3.4 of [1]), can be seen as follows: The relation has been proven for a single mass $m$ in [1]. The proof can be extended by a simple adaptation of notation for the case where $m_1,\ldots,m_n$ are different masses and
the multipliers $\chi_l(k_i, a)$ depend on only one mass depending on $l$. Given this observation, let us consider our more complicated multipliers in Eq. (25). If a factor $\chi^\pm(k, m_l) e^{\pm i(k^2 \omega \eta t)}$ is multiplied with a factor $\delta^\pm_{m_j}(k)$, then the result is zero by the support properties of $\chi^\pm(k, m_l)$ whenever $j \neq l$. Likewise, if a factor $1/(k_j^2 - m_j^2)$ is being multiplied with such a factor, then this expression vanishes in the limit $t \to \infty$ by the Riemann-Lebesgue lemma [34]. We thus see that only those terms with the "right" masses count in (25) and this establishes the result for $W_{n,m_1,\ldots,m_n}$ and $F_{n,m_1,\ldots,m_n}^T$, $n \geq 3$. The case $n = 2$ is trivial.

The statement that $\{F_n\}$ is the form factor functional associated to $\{W_n\}$ now follows from the fact that multiplication by $Q_n^M$ in energy-momentum space and the limit in (24) can be interchanged.

That the so-defined form factor functional fulfills the HSSC can be proven in a similar way as in Theorem 4.1, cf. [1]. The remaining statements of the theorem then follow by the GNS-like construction as in Theorem 4.1 cf. [1] for the details.

It should be remarked that the metric on the asymptotic Hilbert spaces $H_{\text{in/out}}$ generated by repeated application of the fields $\phi_{\text{in/out}}$ to the vacuum depends on the properties of $C^\beta_1,\beta_2$, $b_1$ and $Q_E$ and gauge principles for the asymptotic fields have to be developed depending on these quantities. For a single, scalar field ($\psi > 0$, $b = 1$, $Q_E = 1$) these spaces carry a positive semi-definite metric [1].

Finally we want to show that the scattering of the fields in Theorem 5.1 is non-trivial. Given the form factor functional, one can define the $S$-matrix of the theory via

$$S_{r,n-r}(f_1 \otimes \cdots \otimes f_r; f_{r+1} \otimes \cdots \otimes f_n) = F_n(J^\text{in} f_1 \otimes \cdots \otimes J^\text{in} f_r \otimes J^\text{out} f_{r+1} \otimes \cdots \otimes J^\text{out} f_n) = \langle \phi^\text{in}(f_1^* \cdots f_r^*) \phi_0^\text{out}(f_{r+1}^* \cdots f_n^*) \rangle$$

where $\langle \cdot, \cdot \rangle = (\cdot, \eta)$ is the indefinite inner product on $H$.

Using the definitions (28)-(30) one can verify by an explicit calculation the following corollary:

**Corollary 5.2** The $S$-matrix of the models in Theorem 5.1 is non-trivial (if $\psi(t)$ has a Poisson part). The Fourier transformed, truncated $S$-matrix is given by $S^T_n(k_1; k_2) = W_{2}^T(k_1, k_2)$ and

$$S^T_{r,n-r,\alpha_1;\cdots;\alpha_r;\alpha_{r+1};\cdots;\alpha_n}(k_1,\ldots,k_r; k_{r+1},\ldots,k_n) = -i(2\pi)^{-(dn-2)/2} Q^{M\alpha_1;\cdots;\alpha_n}(k_1,\ldots,k_n) \sum_{l_1,\ldots,l_n=1}^{N} \prod_{j=1}^{n} b_j$$

$$\times \prod_{j=1}^{r} \delta_{m_j}(k_l) \prod_{l=r+1}^{n} \delta^+_m(k_l) \delta(\sum_{l=1}^{n} k_l)$$

for $n \geq 3$ where $k_1^0,\ldots,k_r^0 < 0$ and $k_{r+1}^0,\ldots,k_n^0 > 0$. 

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We remark that in- and out- fields are free fields and fulfill canonical commutation relations, the relations given in Corollary 5.2 suffice to determine the whole scattering matrix.

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