Poly-Dedekind sums associated with poly-Bernoulli functions

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Abstract
Apostol considered generalized Dedekind sums by replacing the first Bernoulli function appearing in Dedekind sums by any Bernoulli functions and derived a reciprocity relation for them. Recently, poly-Dedekind sums were introduced by replacing the first Bernoulli function appearing in Dedekind sums by any type 2 poly-Bernoulli functions of arbitrary indices and were shown to satisfy a reciprocity relation. In this paper, we consider other poly-Dedekind sums that are obtained by replacing the first Bernoulli function appearing in Dedekind sums by any poly-Bernoulli functions of arbitrary indices. We derive a reciprocity relation for these poly-Dedekind sums.

MSC: 11F20; 11B68; 11B83
Keywords: Poly-Dedekind sum; Polylogarithmic function; Poly-Bernoulli polynomial

1 Introduction
The sawtooth function, denoted by ((x)), is defined by

\[
((x)) = \begin{cases} 
  x - [x] - \frac{1}{2}, & \text{if } x \notin \mathbb{Z}, \\
  0, & \text{if } x \in \mathbb{Z}, 
\end{cases}
\]

(see [1–5]),

where \([x]\) denotes the greatest integer function not exceeding \(x\).

The Dedekind sums are defined by

\[
S(h, m) = \sum_{\mu=1}^{m-1} \left( \mu \cdot \left( \frac{h \mu}{m} \right) \right) \left( \frac{\mu}{m} \right) \left( \frac{h \mu}{m} \right) 
= \sum_{\mu=1}^{m-1} \left( \mu \cdot \left( \frac{h \mu}{m} \right) \right) 
= \sum_{\mu=1}^{m-1} \left( \frac{h \mu}{m} \right),
\]

where \(h\) is any integer and \(m\) is a positive integer (see [9–11, 17, 19, 20]).
It is well known that the Bernoulli polynomials are defined by

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^\infty B_n(x) \frac{t^n}{n!} \quad (|t| < 2\pi),\text{ (see [1–13, 17, 19, 20]).} \quad (3)$$

When \( x = 0 \), \( B_n = B_n(0), (n \geq 0) \) are called the Bernoulli numbers.

From (3), we note that

$$B_n(x) = \sum_{l=0}^n \binom{n}{l} B_{n-l} x^l \quad (n \geq 0),\text{ (see [7–13]).} \quad (4)$$

By (3), we easily get

$$\sum_{l=0}^{n-1} \binom{n}{l} B_{n-l} x^l = \frac{1}{m+1} \left( B_{m+1}(n) - B_{m+1} \right), \quad (n \in \mathbb{N}, m \geq 0),\text{ (see [13])}, \quad (5)$$

and

$$d^{n-1} \sum_{l=0}^{d-1} B_n \left( \frac{x + i}{d} \right) = B_n(x), \quad (n \geq 0, d \in \mathbb{N}),\text{ (see [10, 13]).} \quad (6)$$

The modified Hardy’s polyexponential function of index \( k \) is defined by

$$\text{Ei}_k(x) = \sum_{n=1}^\infty \frac{x^n}{n^k(n-1)!} \quad (k \in \mathbb{Z}),\text{ (see [7]).} \quad (7)$$

Note that \( \text{Ei}_1(x) = e^x - 1 \).

Recently, the type 2 poly-Bernoulli polynomials of index \( k \) are defined by

$$\frac{\text{Ei}_k(\log(1+t))}{e^t - 1} e^{xt} = \sum_{n=0}^\infty B^{(k)}_n(x) \frac{t^n}{n!} \quad (k \in \mathbb{Z}). \quad (8)$$

When \( x = 0 \), \( B^{(k)}_n = B^{(k)}_n(0), (n \geq 0) \) are called the type 2 poly-Bernoulli numbers of index \( k \).

Note that \( B^{(1)}_n(x) = B_n(x), (n \geq 0) \).

It is well known that the polylogarithmic function of index \( k \) is defined by

$$\text{Li}_k(x) = \sum_{n=1}^\infty \frac{x^n}{n^k} \quad (k \in \mathbb{Z}, |x| < 1),\text{ (see [6, 9, 12]).} \quad (9)$$

Note that \( \text{Li}_1(x) = -\log(1-x) \).

In [6, 7, 12], the poly-Bernoulli polynomials of index \( k \) are defined by the generating function

$$\frac{\text{Li}_k(1-e^{-t})}{e^t - 1} e^{xt} = \sum_{n=0}^\infty \beta^{(k)}_n(x) \frac{t^n}{n!}. \quad (10)$$

When \( x = 0 \), \( \beta^{(k)}_n = \beta^{(k)}_n(0) \) are called the poly-Bernoulli numbers of index \( k \).
From (10), we note that
\[
\sum_{i=0}^{n} \binom{n}{i} \beta^{(k)}_{n-i} x^i = \beta^{(k)}_n(x), \quad (n \geq 0), \text{ (see [6, 7, 12])}.
\] (11)

The fractional part of \(x\) is defined by
\[
\langle x \rangle = x - [x].
\]

The Bernoulli functions are defined by
\[
B_n(x) = B_n(\langle x \rangle), \quad (n \geq 0), \text{ (see [1, 2])}.
\]

From (2), we have
\[
S(h, m) = \sum_{\mu=1}^{m-1} \frac{\mu}{m} \left( \frac{h\mu}{m} - \left\lfloor \frac{h\mu}{m} \right\rfloor - \frac{1}{2} \right),
\]
\[
= \sum_{\mu=1}^{m-1} \left( \frac{\mu}{m} - \frac{1}{2} \right) \left( \frac{h\mu}{m} - \left\lfloor \frac{h\mu}{m} \right\rfloor - \frac{1}{2} \right)
\]
\[
= \sum_{\mu=1}^{m-1} B_1 \left( \frac{\mu}{m} \right) B_1 \left( \frac{h\mu}{m} \right),
\] (12)

where \(h, m\) are relatively prime positive integers.

Apostol considered the generalized Dedekind sums, which are given by
\[
S_p(h, m) = \sum_{\mu=1}^{m-1} \frac{\mu}{m} B_p \left( \frac{h\mu}{m} \right),
\] (13)

and showed in [1, 2] that they satisfy the reciprocity relation
\[
(p + 1)(hn^p S_p(h, m) + mh^p S_p(m, h)) = pB_{p+1} + \sum_{s=0}^{p+1} \left( \frac{p+1}{s} \right) (-1)^s B_{s} B_{p+1-s} h^{s} m^{p+1-s}.
\]

As one generalization of Apostol’s generalized Dedekind sums, the poly-Dedekind sums associated with the type 2 poly-Bernoulli functions of index \(k\)
\[
S^{(k)}_p(h, m) = \sum_{\mu=1}^{m-1} \frac{\mu}{m} \beta^{(k)}_p \left( \frac{h\mu}{m} \right),
\] (14)

were recently introduced (see [13]) and, among other things, a reciprocity relation for them was derived.

In this paper, as another generalization of Apostol’s generalized Dedekind sums, we consider the poly-Dedekind sums defined by
\[
T^{(k)}_p(h, m) = \sum_{\mu=1}^{m-1} \frac{\mu}{m} \beta_p^{(k)} \left( \frac{h\mu}{m} \right),
\]
where \( \overline{P}_p^{(k)}(x) = \overline{P}_p^{(k)}(x) \) are the poly-Bernoulli functions of index \( k \) (see (10)). Note here that \( T_{p}^{(1)}(h, m) = S_p(h, m) \). We show the following reciprocity relation for the poly-Dedekind sums given by (see Theorem 7)

\[
hm^{p}T_{p}^{(k)}(h, m) + mh^{p}T_{p}^{(k)}(m, h)
= \sum_{\mu=0}^{m-1} \sum_{j=0}^{p-1} \sum_{\nu=0}^{h-1} \sum_{l=1}^{p-j+1} \frac{(mh)^{p-j-1} \overline{P}_2(p-j+1, l)}{(p-j+1)!} \left( \frac{v}{h} + \frac{\mu}{m} \right).
\]

For \( k = 1 \), this reciprocity relation for the poly-Dedekind sums reduces to that for Apostol’s generalized Dedekind sums given by (see Corollary 8)

\[
hm^{p}S_{p}(h, m) + mh^{p}S_{p}(m, h)
= \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{h-1} (mh)^{p-1} (\mu h + mv) \overline{B}_p \left( \frac{v}{h} + \frac{\mu}{m} \right).
\]

We recommend the readers to look at the articles [15, 16, 18, 21] and the more recent one [14], which are related to the present paper. In Sect. 2, we derive various facts about the poly-Bernoulli polynomials that will be needed in the next section. In Sect. 3, we define the poly-Dedekind sums associated with the poly-Bernoulli functions and demonstrate a reciprocity relation for them.

### 2 Poly-Dedekind sums associated with poly-Bernoulli functions

Let \( n \) be a nonnegative integer. Then the Stirling numbers of the second kind are defined by

\[
x^n = \sum_{k=0}^{n} S_2(n, k) (x)_k, \quad (n \geq 0), \text{ (see [1–14, 17, 19])},
\]

where \( (x)_0 = 1 \), \( (x)_n = x(x-1) \cdots (x-n+1) \), \( (n \geq 1) \).

From (9) and (10), we note that

\[
\frac{\text{Li}_k(1 - e^{-t})}{e^t - 1} = \sum_{n=0}^{\infty} \beta_n^{(k)} \frac{t^n}{n!}.
\]

Thus, by (15), we get

\[
\frac{\text{Li}_k(1 - e^{-t})}{e^t - 1} = \left( \sum_{l=0}^{\infty} \beta_l^{(k)} \frac{t^l}{l!} \right) (e^t - 1)
= \sum_{n=0}^{\infty} (\beta_n^{(k)}(1) - \beta_n^{(k)}) \frac{t^n}{n!}.
\]
On the other hand,

\[
\text{Li}_k(1 - e^{-t}) = \sum_{m=1}^{\infty} \frac{1}{m^k}(1 - e^{-t})^m = \sum_{m=1}^{\infty} \frac{(-1)^m m!}{m^k} (e^{-t} - 1)^m
\]

\[
= \sum_{m=1}^{\infty} \frac{(-1)^m m!}{m^k} \sum_{n=m}^{\infty} S_2(n, m) \frac{(-1)^n t^n}{n!} = \sum_{m=1}^{\infty} \left( \sum_{m=1}^{n} \frac{(-1)^{n-m} m!}{m^k} S_2(n, m) \right) \frac{t^n}{n!}.
\]

Therefore, by (16) and (17), we obtain the following theorem.

**Theorem 1** For \( n \in \mathbb{N} \), we have

\[
\beta_n^{(k)}(1) - \beta_n^{(k)} = \sum_{m=1}^{n} \frac{(-1)^{n-m} m!}{m^k} S_2(n, m).
\]

From Theorem 1, we note that

\[
\beta_0^{(k)} = 1, \quad \beta_1^{(k)} = -1 + \frac{1}{2^k}, \quad \beta_2^{(k)} = 1 - \frac{3}{2^k} + \frac{2}{3^k} \ldots.
\]

Taking \( k = 1 \) in Theorem 1 gives us the following corollary.

**Corollary 2** For \( n \in \mathbb{N} \), we have

\[
\sum_{m=1}^{n} (-1)^{n-m} (m-1)! S_2(n, m) = \delta_{n,1},
\]

where \( \delta_{n,k} \) is the Kronecker symbol.

The three identities in the following lemma can be shown just as in Theorem 3, Corollary 4, and Theorem 5 of [13], and hence their proofs are left to the reader as exercises.

**Lemma 3** For \( s, p \in \mathbb{N} \), we have

\[
\sum_{v=0}^{p} \binom{p}{v} \frac{\beta_v^{(k)}}{p - v + 2} = \frac{(p + 1)}{s} \beta_{p+s+1}^{(k)}(1) + \frac{s - 1}{p + 1} \frac{(p + 2)}{s} \beta_{p+s+2}^{(k)}(1),
\]

\[
\sum_{v=0}^{p-s+1} \binom{p}{v} \frac{\beta_v^{(k)}}{p - v + 2} = \left( \frac{p + 1}{s} \right) \beta_{p+s+1}^{(k)}(1) + \frac{s - 1}{p + 1} \left( \frac{p + 2}{s} \right) \beta_{p+s+2}^{(k)}(1) - \frac{1}{s} \left( \frac{p}{s - 2} \right) \beta_{p+s+2}^{(k)}
\]

and

\[
\sum_{s=0}^{p} \binom{p}{s} \frac{1}{p + 2 - s} = \frac{\beta_{p+1}^{(k)}(1)}{p + 1} - \frac{\beta_{p+2}^{(k)}(1)}{(p+1)(p+2)} + \frac{\beta_{p+2}^{(k)}}{(p+1)(p+2)}.
\]
As a further generalization of Apostol’s Dedekind sums, we study poly-Dedekind sums associated with poly-Bernoulli functions of index $k$, which are given by

$$T_p^{(k)}(h, m) = \sum_{\nu=1}^{m-1} \frac{\mu}{m} \beta_p^{(k)} \left( \frac{h \mu}{m} \right),$$  

where $h, m, p \in \mathbb{N}$, $k \in \mathbb{Z}$, and $\beta_p^{(k)}(x) = \beta_p^{(k)}(\langle x \rangle)$ are the poly-Bernoulli functions of index $k$.

Note that

$$T_p^{(1)}(h, m) = \sum_{\mu=1}^{m-1} \frac{\mu}{m} \beta_p^{(1)} \left( \frac{h \mu}{m} \right) = S_p(h, m).$$

The two identities in Lemma 4 can be proved in the same way as in Proposition 6 and Theorem 7 in [13], while the identity in Lemma 5 can be shown just as in Theorem 8 in [13]. Therefore their proofs are left to the reader.

**Lemma 4** Let $p$ be an odd positive integer $\geq 3$, and $m \in \mathbb{N}$. Then we have

$$m^p T_p^{(k)}(1, m)$$

$$= \sum_{v=0}^{p} \binom{p}{v} \frac{\beta_p^{(k)}}{p+2-v} m^{p+1} + \sum_{i=1}^{p-1} \left[ \sum_{v=0}^{p-i} \binom{p-v}{v} \left( \frac{p+2-v}{i} \right) \right] \frac{\beta_p^{(k)}}{p+2-v} B_i m^{p+1-i} + B_{p+1}$$

and

$$(p+1)m^p T_p^{(k)}(1, m)$$

$$= \sum_{i=0}^{p+1} \binom{p+1}{i} B_i m^{p+1-i} \beta_p^{(k)} \left( \frac{1}{1} \right)$$

$$+ \frac{1}{p+2} \sum_{i=0}^{p+1} \binom{p+2}{i} (i-1) B_i m^{p+1-i} \beta_p^{(k)} \left( \frac{1}{1} \right) - \beta_p^{(k)} \left( \frac{1}{1} \right).$$

**Lemma 5** For $m, n, h \in \mathbb{N}$ with $(h, m) = 1$, and $p$ any positive odd integer $\geq 3$, we have

$$\sum_{s=0}^{p+1} \binom{p+1}{s} B_s \beta_p^{(k)} \left( \frac{1}{1} \right) (mh)^{p+1-s}$$

$$= m^p \sum_{\nu=0}^{m-1} \sum_{s=0}^{p+1} \binom{p+1}{s} h^s \beta_p^{(k)} \left( \frac{\mu}{m} \right) B_{p+1-s} \left( \frac{h - \frac{h \mu}{m}}{m} \right).$$

For $d \in \mathbb{N}$, we observe that

$$\sum_{n=0}^{\infty} \beta_n^{(k)}(x) \frac{t^n}{n!} = \frac{\text{Li}_k(1-e^{-t})}{e^t - 1} e^{xt} = \frac{\text{Li}_k(1-e^{-t})}{e^t - 1} \sum_{i=0}^{d-1} e^{i \mu t}$$

$$= \frac{1}{dt} \text{Li}_k(1-e^{-t}) \sum_{i=0}^{d-1} e^{i \mu t} \sum_{i=0}^{d-1} e^{i \mu t}$$

$$= \frac{1}{dt} \text{Li}_k(1-e^{-t}) \sum_{i=0}^{d-1} e^{i \mu t}$$

(19)
Theorem 6  For \( k \in \mathbb{Z}, d \in \mathbb{N}, \) and \( n \geq 0, \) we have

\[
\beta_n^{(k)}(x) = \sum_{j=0}^{n} \sum_{i=0}^{d+j-1} \binom{n}{j} d^{-1} B_j \left( \frac{x+i}{d} \right) \frac{\Gamma(-1)^{n-j+1-l}}{(n-j+1)l} \frac{S_2(n-j+1,l)}{(n-j+1)!}.
\]

By (18), Lemmas 3–5, and Theorem 6, we get

\[
\begin{align*}
\beta_n^{(k)}(x) & = \sum_{j=0}^{n} \sum_{i=0}^{d+j-1} \binom{n}{j} d^{-1} B_j \left( \frac{x+i}{d} \right) \frac{\Gamma(-1)^{n-j+1-l}}{(n-j+1)l} \frac{S_2(n-j+1,l)}{(n-j+1)!}.
\end{align*}
\]
\[
\sum_{\mu=0}^{m-1} \sum_{j=0}^{p-1} \sum_{\nu=0}^{p-1-j} \sum_{l=1}^{\mu h} \frac{(mh)^{p+j}(p-j+1)!}{(p-j+1)!} \times \binom{p}{j} (-1)^{p-j-1} \left( (\mu h)m^{p-j} + (\nu h)l \right) B_{\mu h} \left( \frac{\nu}{h} + \frac{\mu}{m} \right). 
\]

Therefore, by (20), we obtain the following reciprocity theorem for the poly-Dedekind sums associated with poly-Bernoulli functions with index \( k \).

**Theorem 7** For \( m, h, p \in \mathbb{N} \) and \( k \in \mathbb{Z} \), we have

\[
hm^p T_p^{(k)}(h, m) + mh^p T_p^{(k)}(m, h) = \sum_{\mu=0}^{m-1} \sum_{j=0}^{p-1} \sum_{\nu=0}^{p-1-j} \sum_{l=1}^{\mu h} \frac{(mh)^{p+j}(p-j+1)!}{(p-j+1)!} \times \binom{p}{j} (-1)^{p-j-1} \left( (\mu h)m^{p-j} + (\nu h)l \right) B_{\mu h} \left( \frac{\nu}{h} + \frac{\mu}{m} \right). 
\]

In case of \( k = 1 \), by making use of Corollary 2, we obtain the following reciprocity relation for the generalized Dedekind sums defined by Apostol.

**Corollary 8** For \( m, h, p \in \mathbb{N} \), we have

\[
hm^p T_p^{(1)}(h, m) + mh^p T_p^{(1)}(m, h) = mh^p S_p(h, m) + mh^p S_p(m, h) = \sum_{\mu=0}^{m-1} \sum_{j=0}^{p-1} (\mu h + \nu m) B_{\mu h} \left( \frac{\nu}{h} + \frac{\mu}{m} \right). 
\]

**3 Conclusion**

The quantity called the Dedekind sum,

\[
S(h, m) = \sum_{\mu=1}^{m-1} \frac{\mu}{m} B_1 \left( \frac{h \mu}{m} \right),
\]

occurs in the transformation behavior of the logarithm of the Dedekind eta-function under substitutions from the modular group. It was shown by Dedekind that they satisfy the following reciprocity relation:

\[
S(h, m) + S(m, h) = \frac{1}{12} \left( \frac{h}{m} + \frac{h m}{h} + \frac{m}{h} \right) - \frac{1}{4}
\]

if \( h \) and \( m \) are relatively prime positive integers.

Apostol considered the generalized Dedekind sums

\[
S_p(h, m) = \sum_{\mu=1}^{m-1} \frac{\mu}{m} B_p \left( \frac{h \mu}{m} \right)
\]
and derived a reciprocity relation for them. Recently, as one generalization of the generalize Dedekind sums, the poly-Dedekind sums

\[ S_p^{(k)}(h,m) = \sum_{\mu=1}^{m-1} \frac{\mu}{m} B_p^{(k)} \left( \frac{h\mu}{m} \right), \]

associated with the type 2 poly-Bernoulli functions of arbitrary indices, were introduced and were shown to satisfy a reciprocity relation. In this paper, as another generalization of the generalized Dedekind sums, we considered the poly-Dedekind sums

\[ T_p^{(k)}(h,m) = \sum_{\mu=1}^{m-1} \frac{\mu}{m} B_p^{(k)} \left( \frac{h\mu}{m} \right), \]

associated with the poly-Bernoulli functions of arbitrary indices, and derived a reciprocity relation for these poly-Dedekind sums.

Acknowledgements
The authors would like to thank the reviewers for their suggestions that helped improve the original manuscript in its present form. Also, the authors thank Jangjeon Research Institute for Mathematical Sciences for the support of this research.

Funding
This research was funded by the National Natural Science Foundation of China (No. 11871317, 11926325, 11926321).

Availability of data and materials
Not applicable.

Ethics approval and consent to participate
All authors reveal that there is no ethical problem in the production of this paper.

Competing interests
The authors declare that they have no competing interests.

Consent for publication
All authors want to publish this paper in this journal.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 25 September 2020 Accepted: 11 November 2020 Published online: 23 November 2020

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