CHERN-SIMONS MATRIX MODELS AND STIETJES-WIGERT POLYNOMIALS

YACINE DOLIVET AND MIGUEL TIERZ

Abstract. Employing the random matrix formulation of Chern-Simons theory on Seifert manifolds, we show how the Stieltjes-Wigert orthogonal polynomials are useful in exact computations in Chern-Simons matrix models. We construct a biorthogonal extension of the Stieltjes-Wigert polynomials, not available in the literature, necessary to study Chern-Simons matrix models when the geometry is a lens space. We also discuss several other results based on the properties of the polynomials: the equivalence between the Stieltjes-Wigert matrix model and the discrete model that appears in q-2D Yang-Mills and the relationship with Rogers-Szegő polynomials and the corresponding equivalence with an unitary matrix model. Finally, we also give a detailed proof of a result that relates quantum dimensions with averages of Schur polynomials in the Stieltjes-Wigert ensemble.

1. Introduction

In the late eighties [1], Witten considered a topological gauge theory for a connection on an arbitrary three-manifold $M$, based on the Chern-Simons action:

$$S_{CS}(A) = \frac{k}{4\pi} \int_M \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A),$$

with $k$ an integer number. One of the most important aspects of Chern-Simons theory is that it provides a physical approach to three dimensional topology. In particular, it gives three-manifold invariants and knot invariants. For example, the partition function,

$$Z_k(M) = \int \mathcal{D}A e^{iS_{CS}(A)},$$

delivers a topological invariant of $M$, the so-called Reshetikhin-Turaev-Witten invariant. Recent reviews are [2, 3].

As reviewed in [3], a great deal of interest has focused on the fact that Chern-Simons theory provides large $N$ duals of topological strings. This connection between Chern-Simons theory and topological strings was already pointed out by Witten [5] (see also [6]), and then extended in [7].

Recent progress in Chern-Simons theory includes a description of Chern-Simons theory on certain geometries in terms of models of random matrices. Consider the partition function of Chern-Simons theory on a Seifert space $M = X(p_1, q_1, \ldots, p_n, q_n)$. This is obtained by doing surgery on a link in $S^3$ with $n + 1$ components, out of which $n$ are parallel, unlinked unknots, and one has link number 1 with each of the

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n unknots. The surgery data are \( p_j/q_j \) for the unlinked unknots, \( j = 1, \ldots, n \), and 0 for the last component. The partition function is:

\[
Z_{CS}(M) = \frac{(-1)^{|\Delta_+|}}{|W| (2\pi i)^r} \left( \frac{\text{Vol} \Lambda_w}{\text{Vol} \Lambda_r} \right) \frac{\text{sign}(P)|^{\Delta_+}| e^{\frac{2\pi}{3} \text{sign}(H/P) - \frac{\pi i S}{2\pi} \phi}}{|P|^{r/2}} \times \sum_{t \in \Lambda_r/H\Lambda_r} \int d\beta e^{-\beta^2/2g_s t_{\beta} \beta} \prod_{i=1}^{n} \prod_{k<l} 2 \text{sinh} \frac{\beta_i - \beta_l}{2} \prod_{k<l} \left( 2 \text{sinh} \frac{\beta_i - \beta_l}{2} \right)^{n-2}.
\]

This expression gives the contribution of the reducible flat connections to the partition functions. Recall that for both \( S^3 \) and lens spaces this amounts to the exact partition function. The case \( n = 0 \) corresponds to the three-sphere \( S^3 \) that leads to (1.5).

Thus, for the case of \( U(N) \), and focusing on a particular sector of flat connections, we get the following matrix model:

\[
Z_{CS}(M) = N \prod_{i=1}^{N} \int_{-\infty}^{\infty} dy_i e^{-y_i^2/2g_s t_{y_i} y_i} \prod_{j=1}^{n} \prod_{k<l} 2 \text{sinh} \frac{y_j - y_l}{2p_j} \prod_{k<l} \left( 2 \text{sinh} \frac{y_j - y_l}{2} \right)^{n-2}.
\]

Of course, the simplest case is that of \( S^3 \) with gauge group \( U(N) \), which is given by the partition function of the following random matrix model:

\[
Z = e^{-\frac{N^2-1}{N!}} \int \prod_{i=1}^{N} e^{-u_i^2/2g_s} \prod_{i<j} \left( 2 \text{sinh} \frac{u_i - u_j}{2} \right)^2 du_i / 2\pi.
\]

From the point of view of topological strings, this describes open topological A strings on \( T^*S^3 \) with \( N \) branes wrapping \( S^3 \). This latter case, as shown in [3], can be studied with usual techniques of random matrix theory. More precisely, the Stieltjes-Wigert polynomials, a member of the \( q \)-deformed orthogonal polynomials family [3], allows to compute, in exact fashion, quantities associated to the matrix model. In the computation, the \( q \)-parameter of the polynomials turns out to be naturally identified with the \( q \)-parameter of the quantum group invariants associated to the Chern-Simons theory. This is so because the previous model can be easily mapped into:

\[
Z = \int [dM] e^{-\frac{1}{2} \text{Tr} (\log M)^2}.
\]

named Stieltjes-Wigert ensembles, after the associated orthogonal polynomials.

Chern-Simons matrix models have been further considered in [10] and [11-17] and also play a central role in \( q \)-2D Yang-Mills theory [18, 22]. Most of these works focus on the relevance to topological strings. In [3, 10], the emphasis is on exact solutions and on the special features of the matrix models. The works of Caporaso et al. [20, 19, 23] also make an extensive use of the properties of the Stieltjes-Wigert orthogonal polynomials. We shall be focussing here on aspects of the Chern-Simons matrix models that have to do with the associated system of orthogonal polynomials.

This paper is organized as follows. In the next section we shall construct a biorthogonal extension of the Stieltjes-Wigert polynomials, in order to study the matrix model (1.4) when \( n = 1 \) and \( n = 2 \). These polynomials have not been

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1 See the Appendix C for details on the notation.
discussed in the (vast) orthogonal polynomials literature, so most of our effort is on their derivation and to establish some of its fundamental properties. They are necessary if one wants to obtain full analytic results when the geometry is something more complicated than $S^3$. Note that matrix models in the lens space case have already been studied (with loop equations) \[13\], but if one desires an all order result as in \[8\], the knowledge of orthogonal polynomials is then necessary. After the construction of the biorthogonal Stieltjes-Wigert polynomials in Section 2, we discuss some of their mathematical properties in Section 3. In the last Section, we discuss several aspects of the Chern-Simons matrix models by focussing exclusively on properties of the (ordinary) Stieltjes-Wigert polynomials. In particular, we clearly establish the relationship with the discrete matrix model that also appears in $q$-2D Yang-Mills theory, and also employ the intimate relationship between Stieltjes-Wigert and Rogers-Szegö polynomials to find the exact relation between the Stieltjes-Wigert matrix model and Okuda’s unitary matrix model \[15\]. Finally, we give a detailed proof, employing a mixture of combinatorial and orthogonal polynomials results, of the equality between quantum dimensions and averages of Schur polynomials in the Stieltjes-Wigert ensemble \[2\]. We conclude with a summary and with some avenues for further research, presented in the Conclusions and Outlook.

2. Biorthogonal Stieltjes-Wigert

Let us consider the generic expression \[13\] in the $n = 1$ and $n = 2$ cases, that correspond to the case of lens spaces. We are lead to a biorthogonal extension of the $S^3$ model:

\[
Z = \int \prod_{i=1}^N e^{-u_i^2/2g} \prod_{i<j} \left(2 \sinh \frac{u_i - u_j}{2P} \right) \left(2 \sinh \frac{u_i - u_j}{2Q} \right) \frac{d u_i}{2\pi}.
\]

Recall that a biorthogonal ensemble of random matrices has the probability density \[24\]:

\[
P(x_1, ..., x_N) = C_N \prod_{i=1}^N \omega(x_i) \prod_{i<j} (x_i - x_j) (x_i^k - x_j^k),
\]

where $k$ is a fixed real number. In total analogy with the usual Hermitian case ($k = 1$) one can study \[22\] by considering a pair of biorthogonal polynomials:

\[
\int \omega(x) Y_n(x,k) Z_m(x,k) dx = h_{n,k} \delta_{n,m},
\]

with:

\[
\int Y_n(x,k) x^j \omega(x) dx = \alpha_n^{(k)} \delta_{n,j},
\]

\[
\int Z_n(x,k) x^j \omega(x) dx = \beta_n^{(k)} \delta_{n,j}.
\]

We warn the reader that the term biorthogonal is employed in different contexts in the literature. The classical cases (Hermite, Laguerre and Jacobi) were worked out in \[24\]. Note that \[22\] is exactly the type of ensemble that \[21\] leads us to consider since:
$$Z^{P,Q} = \int \prod_i \frac{du_i}{2\pi} e^{-u_i^2/2gs} \prod_{i<j} (2\sinh(\frac{u_i-u_j}{2P}))(2\sinh(\frac{u_i-u_j}{2Q}))$$

$$= q^{-\sum_{i<j} (2\sinh(\frac{u_i}{P} - \frac{u_j}{Q}))}(2\sinh(\frac{u_i}{P} - \frac{u_j}{Q})),$$

with $u_i = \log e^{\frac{\alpha}{g_s}} y_i$, $\kappa^2 = 1/2g_s$ and $\alpha = -1 - \frac{(N-1)}{2}$. Finally, with $y_i = e^{\frac{\alpha}{P} x_i}$ and some rewriting:

$$Z^{P,Q} = P^N e^{-\frac{N}{2}\kappa^2 \log^2 x_i} \prod_{i<j} (x_i - x_j)(x_i^{P/Q} - x_j^{P/Q}),$$

which is of the form (2.2) with the log-normal (Stieltjes-Wigert) weight function: $\omega(x) = e^{-\kappa^2 \log^2 x_i}$, the $q$-parameter is then $q = e^{-\frac{1}{P} + \frac{1}{Q}}$. Therefore, if we want to go beyond the $S^3$ case one has to construct the biorthogonal Stieltjes-Wigert polynomials, not available in the literature. Thus, this is our main task in what follows. The method we have chosen is based on a simple but fundamental result by Askey, that relates the $q$-Laguerre orthogonal polynomials and the Stieltjes-Wigert polynomials [25]:

$$\lim_{\alpha \to \infty} L_{\alpha}^n (q^{-\alpha}x; q) = S_n (x; q),$$

and then we take into account the biorthogonal construction of the $q$-Laguerre polynomials, carried out by Al-Salam and Verma in the early eighties [24]. The Stieltjes-Wigert polynomials are [27]:

$$S_n (x|q) \equiv \frac{1}{(q; q)_n} \sum_{r=0}^{n} \frac{n!}{r!} (-1)^r q^{r^2} x^r.$$

The limit (2.7) will provide us with a biorthogonal extension of the SW polynomials starting with the $q$-Konhauser polynomials. Therefore, following [26], let us write:

$$Z_n^{(\alpha)}(x, k|q) \equiv \frac{[q^{1+\alpha}]_{nk}}{(q^k; q^k)_n} \sum_{j=0}^{n} \frac{(q^{-nk}; q^k)_j q^{kj(kj-1)+kj(n+\alpha+1)}}{(q^k; q^k)_j [q^{1+\alpha}]_{kj}} x^{kj},$$

and

$$Y_n^{(\alpha)}(x, k|q) \equiv \frac{1}{[q]_n} \sum_{r=0}^{n} x^r q^{\frac{k}{2}(r-1)} b_n^0,$$

with

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2The resulting polynomials were named $q$-Konhauser as they could also be interpreted as a $q$-deformed version of the biorthogonal Laguerre polynomials, worked out by Konhauser.

3In [25] they appear, in Eq. (2.5), slightly reformulated.
(2.11) \[ b_r^\alpha = \sum_{s=0}^r \frac{[q^{-r}]_s}{[q]_s} q^s (q^{1+\alpha+s}; q^k)_n. \]

These polynomials satisfy:

(2.12) \[ < Z_n^{(\alpha)}(x, k|q), Y_n^{(\alpha)}(x, k|q) > = k_n^{(\alpha)} \delta_{n,m} \quad \text{with} \quad k_n^{(\alpha)} = \frac{[q^{1+\alpha}]_n q^{-nk}}{[q]_n}, \]

with respect to the normalized \( q \)-Laguerre measure. We have to study:

(2.13) \[ Z_n(x, k|q) \equiv \lim_{\alpha \to \infty} Z_n^{(\alpha)}(q^{-\alpha} x, k|q), \]

\[ Y_n(x, k|q) \equiv \lim_{\alpha \to \infty} Y_n^{(\alpha)}(q^{-\alpha} x, k|q). \]

In the first case, one readily finds:

(2.14) \[ Z_n(x, k|q) = \frac{1}{(q^k; q^k)_n} \sum_{j=0}^n (q^{-nk}; q^k)_j q^{\frac{1}{2} j(kj-1)+kj(n+1)} x^{kj}, \]

which can be conveniently reexpressed:

(2.15) \[ Z_n(x, k|q) = \frac{1}{(q^k; q^k)_n} \sum_{r=0}^n \binom{n}{r} q^{r^2 k(k+1)} x^r. \]

Regarding \( Y_n(x, k|q) \) we have to find \( b_r \equiv \lim_{\alpha \to \infty} q^{-\alpha} b_r^\alpha \). Employing \( q \)-Taylor one can write:

(2.16) \[ (q^{1+\alpha}; q^k)_n = \sum_{r=0}^n \frac{x^r [1/x]_r}{[q]_r} \sum_{s=0}^r \frac{[q^{-r}]_s}{[q]_s} q^s (q^{1+\alpha+s}; q^k)_n, \]

therefore:

(2.17) \[ (qx; q^k)_n = \sum_{r=0}^n \frac{x^r [q^\alpha/x]_r}{[q]_r} q^{-\alpha r} b_r^\alpha. \]

Taking the \( \alpha \to \infty \) limit and using the finite \( q \)-binomial theorem one can write:

(2.18) \[ (qx; q^k)_n = \sum_{r=0}^n \frac{x^r [q^\alpha/x]_r}{[q]_r} q^{-\alpha r} b_n^{q^\alpha}. \]

so that

(2.19) \[ \frac{b_n^{q^\alpha}}{[q]_r} = (-1)^r \binom{n}{r} q^{\frac{1}{2} kr(r-1)+r}. \]

For later use note that one also have:
\[
\frac{b_{n,r}}{[q]_r} = \frac{1}{r!} \left( \frac{d}{dx} \right)^{(r)} (qx; q^k)_{n|x=0}.
\]

From this one gets:

\[
Y_n(x, k|q) = \frac{1}{[q]_n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} q^{r(r+1)} + \frac{1}{2} kr(r-1) x^r.
\]

For \( k = 1 \), both polynomials reduce to the Stieltjes-Wigert polynomials \( Y_n(x, k|q) \).

Writing \( Y_n(x, k|q) = y_{n,k} x^n + \ldots \) and \( Z_n(x, k|q) = z_{n,k} x^n + \ldots \) one finds:

\[
Z_{n,k} = \frac{(q^{-nk}; q^k)_n}{(q^k; q^k)_n} q^{\frac{1}{2} kn(n+1)} = \frac{(-1)^n q^{\frac{1}{2} n^2 k(k+1)}}{(q^k; q^k)_n},
\]

and

\[
y_{n,k} = \frac{1}{[q]_n} q^{-n(n-1)} \left( \frac{d}{dx} \right)^{(n)} (qx; q^k)_{n|x=0} = \frac{(-1)^n q^{\frac{1}{2} n^2 (k+1) (n(n+1)) + n}}{[q]_n}.
\]

This leads to:

\[
< Y_n(x, k|q), Z_m(x, k|q) > = h_n \delta_{n,m},
\]

with respect to the measure \( \frac{dx}{-x|_\infty} \cdot \frac{dx}{-q^x|_\infty} \), with \( A \) such that \( < 1, 1 > = 1 \)

\[
h_n = \frac{q^{-nk}}{[q]_n}.
\]

Using this, we can find for example:

\[
Z_{P,Q} = N! \frac{(g_s)}{2 \pi i} \frac{N/2}{q} - \frac{N/2}{q} \left[ -1 \right] \left[ 1 \right] \frac{(N-1)}{2} \left[ 1 \right] Z \left( 1 - q^{2N} \right) N - j,
\]

which reduces to the known formula when \( P = Q = 1 \).

3. Mathematical properties of the biorthogonal polynomials

Since the biorthogonal Stieltjes-Wigert polynomials have not been addressed in the literature, we derive here some of its fundamental properties.

3.1. Behavior under dilatation. First, we find some generating functions for \( Z_n(x; k|q) \) and \( Y_n(x; k|q) \) \( (t \neq q^{-k}) \)

\[
\sum_{n \geq 0} Z_n(x; k|q) t^n = \frac{f(tx^k)}{(t; q^k)_{\infty}},
\]

and

\[
\sum_{n \geq 0} \frac{[q]_n}{(q^k; q^k)_n} Y_n(x; k|q) t^n = \frac{g(tx)}{(t; q^k)_{\infty}},
\]
with
\begin{equation}
(3.3)
\end{equation}

\[ f(z) = \sum_{r \geq 0} \frac{q^{r^2} z^r}{(q^k; q^k)_r}(-z)^r \] and
\[ g(z) = \sum_{r \geq 0} \frac{q^{r(r-1)} z^r}{r!} \left( \frac{d}{dx} \right)^r (q x; q^k)_n |_{x=0}. \]

We rely for this on formula (4.2) from [26]. The expression for \( Z \) is essentially property (4.1) in [26]. For (3.2) we use the explicit expression obtained for \( Y_n(x; k|q) \). Let us introduce the moment generating function:
\begin{equation}
(3.4)
\end{equation}

\[ G(t, x) \equiv \sum_{n \geq 0} \frac{[q]_n}{(q^k; q^k)_n} Y_n(x; k|q)t^n, \]

that can be written as:
\begin{equation}
(3.5)
\end{equation}

\[ G(t, x) = \sum_{r \geq 0} \frac{q^{r(r-1)}}{r!} x^r \left( \frac{d}{dx} \right)^r (q x; q^k)_n |_{x=0} \]
\[ = \sum_{r \geq 0} \frac{q^{r(r-1)}}{r!} x^r \left( \frac{d}{dx} \right)^r (q x; q^k)_n |_{x=0} \]
\[ = \sum_{r \geq 0} \frac{q^{r(r-1)}}{r!} x^r \left( \frac{d}{dx} \right)^r (q x; q^k)_n |_{x=0} \]
\[ = \frac{1}{(t; q^k)_\infty} \sum_{r \geq 0} \frac{q^{r(r-1)}}{r!} (x t)^r \left( \frac{d}{dx} \right)^r (q x; q^k)_n |_{x=0} ; \quad t \neq q^{-k} \]

In the second line, the extra piece we add, being a degree \( r - 1 \) polynomial in \( x \) does not contribute due to the derivative. In the third line we use the \( q \)-binomial theorem, and in the fourth one we make the change of variable \( x \rightarrow x t \).

Now, by taking (3.2) with \( x \rightarrow \lambda x \), introducing in the r.h.s the factor \( \frac{(\lambda t q^k)_\infty}{(t q^k)_\infty} \) and matching the coefficients of \( t^n \) on both sides one gets:
\begin{equation}
(3.6)
\end{equation}

\[ Y_n(\lambda x; k|q) = \sum_{j=0}^{n} \gamma_{nj}(\lambda) Y_j(x; k|q), \]

with:
\begin{equation}
(3.7)
\end{equation}

\[ \gamma_{nj}(\lambda) = \frac{[q]_j (q^k; q^k)_n \lambda^j (q^k; q^k)_{n-j}}{[q]_n (q^k; q^k)_j (q^k; q^k)_{n-j}}; \]

and similar steps involving Eq. 3.1 give:
\begin{equation}
(3.8)
\end{equation}

\[ Z_n(\lambda x; k|q) = \sum_{j=0}^{n} \zeta_{nj}(\lambda) Z_j(x; k|q), \]

with
(3.9) \[
\zeta_{nj}(\lambda) = \frac{1}{(q^k; q^k)_{n-j}} \lambda^{kj}(\lambda^k; q^k)_{n-j},
\]

One has \( \zeta_{nj} = \gamma_{nj} \) for \( k = 1 \) as it should. Moreover \( \gamma_{nn} = \zeta_{nn} = \lambda^n \), by matching the dominant coefficients on both sides. Note that equation (4.2) in [26] contains a typo as it does not fulfill this last condition (it would give \( \zeta_{nn} = 1 \)).

And of course one has \( \zeta_{nj}(1) = \gamma_{nj}(1) = \delta_{nj} \).

Even though \( \zeta_{nj} \) and \( \gamma_{nj} \) are defined for \( j \leq n \) we extend for convenience their definition through

(3.10) \[
\zeta_{nj} = \gamma_{nj} = 0 \quad \text{if} \quad j > n.
\]

3.2. Recurrence formulae. The Stieltjes-Wigert polynomials (2.8) satisfy:

(3.11) \[
S_{n-1}(x|q) = (1 - q^n)S_n(x|q) + xq^nS_{n-1}(xq|q),
\]

an identity used by Chihara in [28], to prove that the zeros of the polynomials satisfy:

(3.12) \[
x_{n,m} < x_{n-1,m} < qx_{n,m+1},
\]

where \( n \) denotes the order of the polynomial and \( m \) indexes the zero. Note that the zeros of the SW polynomials are an interesting quantity in the context of topological strings [29]. In what follows, we find the same identities for the biorthogonal polynomials.

3.2.1. Fundamental recurrence relation. Note that for the particular value \( \lambda = q^{-1} \)

(3.13) \[
\zeta_{nj}(q^{-1}) = 0 \quad \text{if} \quad j \leq n - 2.
\]

This implies the following simple recurrence relation for \( \tau_{n}(x,k|q) \):

(3.14) \[
\tau_{n}(x,k|q) - \tau_{n-1}(x,k|q) = q^{kn}\tau_{n}(q^{-1}x,k|q).
\]

Certainly, if one writes the \( \tau_{n}(x,k|q) = \sum_{j=0}^{n} \tau_{n,j}x^{kj} \), it can be checked directly, from the explicit expression in (2.15), that one has:

(3.15) \[
\tau_{n,j} - \tau_{n-1,j} = q^{k(n-j)}\tau_{n,j},
\]

which implies the recurrence relation^4.

For \( k = 1 \), (3.14) reduces to the following relation for the Stieltjes-Wigert polynomials

(3.16) \[
S_n(y) - S_{n-1}(y) = q^nS_n(q^{-1}y).
\]

^4Incidentally, this a check that Eq. (3.9) is correct.
3.2.2. Moment generating recurrences. From (2.19) one has\( (b_{n,0} = 1, b_{n,-1} = 0)\):

\[
\frac{b_{n+1,r}}{[q]_r} = \frac{b_{n,r}}{[q]_r} - q^{nk+1} \frac{b_{n,r-1}}{[q]_{r-1}},
\]

which implies the following recurrence relation for the \(Y_n(x,k|q)\):

\[ (1 - q^{n+1})Y_{n+1}(x,k|q) = Y_n(x,k|q) - q^{nk+1}xY_n(qx,k|q), \]

or, equivalently:

\[
xY_n(x,k|q) = q^{-nk} \left( Y_n(q^{-1}x,k|q) - (1 - q^{n+1})Y_{n+1}(q^{-1}x,k|q) \right).
\]

We proceed in analogous way for \(Z_n(x,k|q)\). For convenience we introduce coefficients \(c_{n,r}\), such that

\[
Z_n(x,k|q) \equiv \frac{1}{(q^k; q^k)_n} \sum_{r=0}^{\infty} x^r q^{\frac{r}{2}(kr-(kr-1))} c_{n,r},
\]

that is:

\[
\frac{c_{n,r}}{(q^k; q^k)_r} = \frac{(q^{-nk}; q^k)_r q^{kr(n+1)}}{(q^k; q^k)_r}.
\]

Then, as in the previous case \(c_{n,0} = 1, c_{n,-1} = 0\), then:

\[
\frac{c_{n+1,r}}{(q^k; q^k)_r} = \frac{c_{n,r}}{(q^k; q^k)_r} - q^{nk+k} \frac{c_{n,r-1}}{(q^k; q^k)_{r-1}},
\]

and one gets the recurrence relation for \(Z_n(x,k|q)\):

\[ (1 - q^{k(n+1)})Z_{n+1}(x,k|q) = Z_n(x,k|q) - q^{nk+k+k+1}x^kZ_n(q^kx,k|q), \]

equivalently:

\[
x^kZ_n(x,k|q) = q^{-nk-k(k+1)} \left( Z_n(q^{-k}x,k|q) - (1 - q^{k(n+1)})Z_{n+1}(q^{-k}x,k|q) \right).
\]

One can easily check that these recurrence relations both reduce, taking \(k = 1\), to (3.11).

To conclude this Section, since we know the explicit behavior of the polynomials under dilatation, we can employ (3.8) and (3.10), and then, (3.19) and (3.24), to obtain an explicit way to compute the moments \(x^lY_n(x,k|q)Z_m(x,k|q)\). For

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\[ \tau_{n,r} = \frac{c_{n,r}}{(q^k; q^k)_r}. \]
instance, one has:

\[
< xY_n(x, k|q)Z_m(x, k|q) > = q^{-nk}k^m(\gamma_{n,m}(q^{-1}) - (1 - q^{n+1})\gamma_{n+1,m}(q^{-1}))
\]

\[
= \frac{q^{-(n+m)k}}{|q|m}(\gamma_{n,m}(q^{-1}) - (1 - q^{n+1})\gamma_{n+1,m}(q^{-1})).
\]

Note that this does not work so well when \(k = 1\) as \(\gamma_{n,m}(q^{-1})\) is not well defined in this case, according to the discussion above.

4. Stieltjes-Wigert, other results

4.1. Moment problem and \(q\)-2D Yang-Mills. The Stieltjes-Wigert matrix model posses distinctive mathematical features, in comparison with other, more usual models in the literature, such as matrix models with Gaussian or polynomial potentials. The log-normal weight function leads to an indeterminate moment problem \([30, 31]\) and consequently, the Stieltjes-Wigert polynomials are not dense in \(L^2(x, w(x))\) (see \([8, 10]\) for details). One of the consequences, discussed in \([10]\), is the discretization of the Chern-Simons matrix model. This opens the possibility for the discrete and continuous versions of the same model to share the same orthogonal polynomials. Note that in \([21]\), we find the suggestion of studying the discrete matrix model (see below) with orthogonal polynomials, as done in \([32]\) for the 2D Yang-Mills theory case, that lead to a discrete Gaussian matrix model. Recall that Gross and Matytsin \([32]\) found a discrete version of the ordinary Gaussian matrix model in their study of the \(1/N\) expansion of the partition function of 2D QCD on the sphere:

\[
Z(A, N) = \sum_{u_1, \ldots, u_N = -\infty}^{+\infty} e^{-\frac{A}{N} \sum_i u_i^2} \prod_{j<k} (u_j - u_k)^2.
\]

In contrast to the continuum case, that is solved with Hermite polynomials, the discrete Gaussian weight does not have a closed system of orthogonal polynomials associated (see \([33]\), for a recent discussion of discrete matrix models). So, they could not rely on known orthogonal polynomials, hence the difficulty of studying the discrete Gaussian model. Actually, the large \(N\) phase transition of the theory is related with the discrepancy between the discrete and continuous orthogonal polynomials.

The orthogonal polynomials for the discrete matrix model in the Chern-Simons case are the Stieltjes-Wigert polynomials, as we show below. Note that the model in Chern-Simons/\(q\)-2D Yang-Mills differs from the Gaussian/2D Yang-Mills model in the exponentiation of the eigenvalue repulsion at large distances, \((u_j - u_k)^2 \rightarrow \sinh^2(u_i - u_j)\). But this exponentiation is precisely the ultimate responsible of this continuum/discrete equivalence (see \([10]\)).

The following detailed computation highlights this special property of the Chern-Simons matrix model. Let us proceed then to show the details of the derivation from Eq. (25) to Eq. (26) in \([10]\).
\[ (4.2) \quad Z_d = \sum_{u_1, \ldots, u_N = -\infty}^{+\infty} e^{-\frac{N}{2} \sum_j u_j^2} \prod_{j<k} \left( \frac{g_s}{2} (u_j - u_k) \right) = \sum_{u_1, \ldots, u_N = -\infty}^{+\infty} e^{-\frac{N}{2} \sum_j u_j^2} e^{(N-1)g_s \sum_j u_j} \prod_{j<k} (e^{-g_s u_j} - e^{-g_s u_k})^2 \]

\[ = \sum_{u_1, \ldots, u_N = -\infty}^{+\infty} q^{\frac{N}{2} \sum_j u_j^2} (cq)^{\sum_j u_j} \prod_{j<k} (q^{u_j} - q^{u_k})^2 \]

\[ = \sum_{u_1, \ldots, u_N = -\infty}^{+\infty} \prod_{i=1}^{N} e^{u_i^2} q^{\frac{N}{2} u_i^2 + u_i} \prod_{j<k} (q^{u_j} - q^{u_k})^2 \]

\[ = c^{N(1-N)} \int_{0}^{+\infty} dx_i \sum_{n=-\infty}^{+\infty} e^n q^{\frac{N}{2} n^2 + n} \delta(x - cq^n) \prod_{j<k} (x_j - x_k)^2 \]

where we have introduced \[ c \equiv e^{N g_s} = q^{-N} \] and we recall that we have as usual \[ q \equiv e^{-g_s} \]. Moreover, one has:

\[ (4.3) \quad w_d(x) \equiv \frac{1}{\sqrt{q M(c)}} \sum_{n=-\infty}^{+\infty} e^n q^{\frac{N}{2} n^2} \delta(x - cq^n), \]

which is the discrete measure equivalent to the continuous distribution \[ w(x) \] in \[ (1.6) \] as far as the integer moments are concerned \[ [28, 34, 35] \]. The normalization is given by:

\[ (4.4) \quad M(c) \equiv (-cq^{3/2}, -c^{-1}q^{-1/2}, q; q)_{\infty} = [-cq^{3/2}]_{\infty}[-c^{-1}q^{-1/2}]_{\infty}[q]_{\infty}, \]

therefore, we can use the equivalence between these two measures to write:

\[ (4.5) \quad Z_d = c^{N(1-N)} q^{\frac{N}{2} M(c)} \int_{0}^{+\infty} \prod_{i=1}^{N} dx_i \prod_{j<k} (x_j - x_k)^2. \]

However recall that, in the simplest case \[ P = Q = 1 \] in \[ (2.6) \], one has:

\[ (4.6) \quad Z^{1,1} = q^{-\frac{N^2}{2}} \left( \frac{g_s}{2\pi} \right) \int_{0}^{+\infty} \prod_{i=1}^{N} dx_i \prod_{j<k} (x_j - x_k)^2. \]

Thus, one has a quite simple relation between the discrete and continuous Stieltjes-Wigert ensembles:

\[ ^6 \text{Note that even though there are infinitely many discrete measures } w_d \text{ equivalent to } w \text{ (the parameter } c \text{ is a real number), in the Chern-Simons case we are discussing, the constant } c \text{ is a function of } g_s. \]
Note the inversion of the coupling constant between the l.h.s. and r.h.s.

4.2. From Stieltjes-Wigert to Rogers-Szegö: unitary matrix model. The Stieltjes-Wigert polynomials turn out to be intimately related to the Rogers-Szegö polynomials [36, 37], that are orthogonal on the unit circle. This is useful to establish in detail the exact relationship between the Stieltjes-Wigert matrix model and the unitary model considered by Okuda [15]. Following [37], we recall the definition and relations between the Rogers-Szegö and Stieltjes-Wigert polynomial. The Rogers-Szegö polynomials are defined as:

\[
H_n(z|q) \equiv \sum_{k=0}^{n} \left[ \frac{n}{k} \right]_q z^k,
\]

and they satisfy an orthogonality relation on the complex unit circle:

\[
\frac{1}{2\pi} \int_{|w|=1} H_m(-q^{-1/2}w|q)H_n(-q^{-1/2}w|q)\Theta_3 \left( \frac{\log w}{2\sqrt{q}} \right) \frac{dw}{w} = \left[ \frac{q}{q^n} \right]_m \delta_{mn},
\]

where \( \Theta_3(z|q) \) is the third Jacobi theta function (see definitions in the Appendix). Note that the orthogonality coefficients \( h_m = \left[ \frac{q}{q^n} \right]_m \) are identical to the ones (Stieltjes-Wigert) that directly give the Chern-Simons partition function in the \( S^3 \times U(N) \) case [3]. This is enough to write down an unitary matrix model for the Chern-Simons partition function. However, let us show this point with detail. The polynomials are also orthogonal with respect to a measure defined on the full real line [37].

\[
\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} H_m(-q^{-1/2}e^{-2i\mu x}|q)H_n(-q^{-1/2}e^{2i\mu x}|q)e^{-x^2} \, dx = \left[ \frac{q}{q^n} \right]_m \delta_{mn},
\]

introducing \( \mu \) through \( q \equiv e^{-2\mu^2} \). Now consider the Stieltjes-Wigert polynomials \( S_n(x) \) [37]:

\[
S_n(x) = \frac{(-1)^n q^{n/2+\frac{1}{4}}}{\sqrt{|q^n|}} \sum_{\nu=0}^{n} \left[ \frac{n}{\nu} \right]_q q^{\nu^2} (-\sqrt{q}x)^\nu
= \frac{(-1)^n q^{n/2+\frac{1}{4}}}{\sqrt{|q^n|}} \hat{S}_n(-\sqrt{q}x|q), \text{ with } \hat{S}_n(z|q) \equiv \sum_{k=0}^{n} \left[ \frac{n}{k} \right]_q q^{k^2} z^k.
\]

These polynomials fulfill the following orthogonality relation on the real line:
Eq. (3.22) in [15], it reads:

\[
Z(4.16) = 00 (4.17) \Theta
\]

where

Then:

\[
(4.15) \quad H_n(x|q^{-1}) = \hat{S}_n(q^{-n}x|q) \quad \text{and} \quad \hat{S}_n(x|q^{-1}) = H_n(q^{-n}x|q).
\]

Then:

\[
(4.14) \quad < p_m, p_m > = \rho_{m,n} \frac{k}{\sqrt{\pi}} \int_0^{\infty} e^{-k^2 \log^2 z} \hat{S}_m(-q^{1/2}z)\hat{S}_n(-q^{1/2}z) \, dz
\]

\[
= \rho_{m,n} q^{-1/2} \frac{k}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-k^2 (x- \frac{\pi}{2})^2} \hat{S}_m(-q^{1/2}e^x)\hat{S}_n(-q^{1/2}e^x) \, dx
\]

\[
= \rho_{m,n} q^{-1/2} \frac{1}{\sqrt{\pi}} \int_{\infty}^{\infty} e^{-x^2} \hat{S}_m(-q^{1/2}e^{x})\hat{S}_n(-q^{1/2}e^{x}) \, dx
\]

\[
= \rho_{m,n} q^{-1/2} \frac{1}{2\pi} \int_0^{2\pi} H_m(-q^{-1/2}\pi | q)H_n(-q^{-1/2} \pi | q)\Theta_3 \left( \frac{\log w}{2i} | q \right) d\theta
\]

where we also have used, between lines 3 and 4, the fact that \( \hat{S}_n(\ae^{-2\kappa x} | q) \) and \( H_n(\ae^{2i\nu y} | q) \) are related by a Fourier transform [37]. We also introduced \( 2\mu = \frac{1}{2} \) and:

\[
(4.15) \quad \rho_{m,n} = \frac{\sqrt{1}}{q^{m+n}} = \frac{\sqrt{1}}{q^{\frac{m+n}{2}}}. \]

The next line is given by the results of the previous section. Now consider Eq. (3.22) in [15], it reads:

\[
Z_{\text{CS}} = \frac{1}{|W|} \int \left( \prod_{i,j=1}^{N} \frac{d\theta_i}{2\pi} \Theta_{00}(e^{i\theta_i} | q) \right) \left( \prod_{i<j} \left( \sin(\theta_i - \theta_j) \right) \right)^2
\]

\[
= (-1)^{N(N-1)} \left( \prod_{i,j=1}^{N} \frac{d\theta_i}{2\pi} \Theta_{00}(e^{i\theta_i} | q) \right) \prod_{i<j} (e^{i\theta_i} - e^{-i\theta_j}) \prod_{i<j} (e^{-i\theta_i} - e^{i\theta_j})
\]

\[
= (-1)^{N(N-1)} \left( \prod_{i,j=1}^{N} \frac{d\theta_i}{2\pi} \Theta_{00}(e^{i\theta_i} | q) \right) \det_{1\leq i,j\leq N} (H_{j-1}(e^{i\theta_i})) \det_{1\leq i,j\leq N} (H_{j-1}(e^{-i\theta_i}))
\]

where

\[
(4.17) \quad \Theta_{00}(e^{i\theta} | q) = \sum_{j \in \mathbb{Z}} q^{\frac{1}{2}j^2} e^{i j \theta}.
\]
Then, considering that [15] and [37] have different conventions for the third Jacobi function one sees that:

\[(4.18) \quad \Theta^{(O)}_{00} (e^{\theta} | q) = \Theta^{(A)}_{3} \left( \frac{\theta}{2} | \sqrt{q} \right). \]

Therefore, one can continue the computation and write:

\[(4.19) \quad \tilde{Z}_{CS} = \left( -1 \right)^{\frac{N(N-1)}{2}} N! \left( \frac{1}{|W|} \right) \left( \det_{1 \leq i, j \leq N} \left( (-1)^{j-i} q^{-\frac{i-j}{2}} | q |_{j-i-1} \mathcal{P}_{j-i-1}(z_i) \right) \right)^2 > w, \]

which then connects with the usual expression of the partition function in terms of the orthogonal polynomials for the measure on the real line.

Incidentally, both Stieltjes-Wigert and Rogers-Szegő can be interpreted as the ground-state wavefunction of a \(q\)-deformed harmonic oscillator [37]. This is an appealing property as it has been recently shown that the Stieltjes-Wigert polynomial describes \(B\)-brane amplitudes on the conifold [29].

4.3. Quantum dimensions as averages of Schur polynomials in the Stieltjes-Wigert ensemble. In this section we prove a formula for the averages of Schur polynomials that appears in [2, 39], without relying on the equivalence with Chern-Simons theory.

Recall that Schur polynomials \(s_{\lambda}[38]\) constitute a basis of symmetric functions in a given set of variables \(x = (x_i)\) and are indexed by Young diagrams \(\lambda\). If the variables \(x\) are seen as eigenvalues of some matrix \(M \in \text{sl}_n\), then \(s_{\lambda}(M) \equiv \text{Tr}_{\lambda}(M)\) is the trace of \(M\) in the representation associated to \(\lambda\). The Schur polynomials may also be more directly defined in terms of the skew-symmetric polynomials \(a_{\mu} = \det(x_{i+j+n-1})\) as:

\[(4.20) \quad s_{\lambda}(x) \equiv \frac{a_{\lambda+\delta(x)}}{a_{\delta}(x)}, \]

where \(a_{\delta}(x)\) is the Vandermonde in the variables \(x\). The result we want to show is the following:

\[(4.21) \quad < s_{\lambda}(M) >_w = q^{-n|\lambda| - \frac{n}{2} C^{U(n)}_{\lambda}} \mathcal{D}_{\lambda}. \]

with

\[(4.22) \quad C^{U(n)}_{\lambda} = (n+1)|\lambda| + \sum_{i} (\lambda_i^2 - 2i \lambda_i), \]

the Casimir of the representation labeled by the Young diagram \(\lambda\) and \(|\lambda|\) its total number of boxes. Background material for this Section is presented in Appendix [13]. Note that this quantity can be rewritten using Eq.(15.6) as:

\[(4.23) \quad C^{U(n)}_{\lambda} = n|\lambda| + 2(n(\lambda') - n(\lambda)), \]
where $\lambda'$ denotes the conjugate partition. The quantum dimension is defined by the $q$-hook formula\footnote{See for instance §4.4 in \cite{40} for a clear discussion of the definition and properties of quantum dimensions.}

\begin{equation}
\mathcal{D}_\lambda \equiv \prod_{x \in \lambda} \frac{[n + c(x)]}{[h(x)]}.
\end{equation}

where for each box $x = (i, j)$ of the diagram $h(x) \equiv \lambda_i + \lambda_j' - i - j + 1$ is the hook-length and $c(x) \equiv j - i$ the content of $x$.

### 4.3.1. Case of 1-column diagrams.

The quantum dimension of the $j$-th fundamental representation of $\text{sl}_n$, which is associated\footnote{We will freely switch notations between Young diagrams and partitions in the following.} to the partition $(1^j)$, or a one-column Young tableau of length $j$, is

\begin{equation}
\mathcal{D}_{(1^j)} = \dim_q \Lambda_{(j)} = \left\lfloor \frac{n}{j} \right\rfloor.
\end{equation}

Moreover, the monic Stieltjes-Wigert polynomials can be written\footnote{The measure being here normalized such that $<1>_w = 1.$}

\begin{equation}
\pi_n(x) = \sum_{j=0}^{n} (-1)^{n-j} q^{(j-n)(j+n+\frac{1}{2})} \left\lfloor \frac{n}{j} \right\rfloor x^j = < \det(x - M) >_w.
\end{equation}

Besides, the following formula holds for the characteristic polynomial:

\begin{equation}
\det(x - M) = \sum_{j=0}^{n} (-1)^{n-j} s_{(1^{n-j})}(M) x^j,
\end{equation}

with $s_\lambda(M)$ the Schur polynomial associated to the partition $\lambda$. Therefore:

\begin{equation}
\sum_{j=0}^{n} (-1)^{n-j} < s_{(1^{n-j})}(M) >_w x^j = \sum_{j=0}^{n} (-1)^{n-j} q^{(j-n)(j+n+\frac{1}{2})} \left\lfloor \frac{n}{j} \right\rfloor x^j,
\end{equation}

from which we extract:

\begin{equation}
< s_{(1^j)}(M) >_w = q^{-j(2n-j+\frac{1}{2})} \left\lfloor \frac{n}{j} \right\rfloor = q^{-\frac{j}{2}(3n-j+1)} \left\lfloor \frac{n}{j} \right\rfloor.
\end{equation}

Using \cite{422}, one sees that Eq. \cite{425} is indeed consistent with Eq. \cite{421}.

### 4.3.2. General case.

To study the case of a general Young diagram we note that as a generalization of Eq. \cite{427} higher powers of the characteristic polynomial are generating functions for diagrams with a higher number of columns. Relying on a formula first computed in \cite{11} we then relate the average of Schur polynomials to some determinant of Stieltjes-Wigert polynomials. From \cite{38} (I.4 Example 5 p.67) we see (taking a slightly more convenient notation)
(4.30) \[ \prod_{i=1}^{k} \prod_{j=1}^{n} (x_i + y_j) = \sum_{\lambda; \lambda_1 \leq k} s_\lambda(y) s_{\lambda'}(x), \]

where \( \lambda = (\lambda_1, \ldots, \lambda_n) \) is a Young diagram with at most \( k \) columns as imposed by the condition \( \lambda_1 \leq k \). The associate diagram \( \lambda' \) is defined as \( (n - \lambda_n, \ldots, n - \lambda_1) \).

Therefore, if one considers \( -y_j \) to be the eigenvalues of \( M \), one immediately gets:

(4.31) \[ \prod_{i=1}^{k} \det(x_i - M) = \sum_{\lambda; \lambda_1 \leq k} (-1)^{|\lambda|} s_\lambda(M) s_{\lambda'}(x). \]

By a standard result on characteristic polynomials [11], we have:

(4.32) \[ < \prod_{i=1}^{k} \det(x_i - M) >_{\omega} = \frac{1}{a_\delta(x)} \begin{vmatrix} \pi_n(x_1) & \cdots & \pi_{n+k-1}(x_1) \\ \vdots & \ddots & \vdots \\ \pi_n(x_k) & \cdots & \pi_{n+k-1}(x_k) \end{vmatrix}, \]

with \( a_\delta(x) \) the Vandermonde determinant of the \( x \) variables.

We now turn to the r.h.s., that we call \( \Delta \) for convenience. From the explicit expression for the Stieltjes-Wigert polynomials one obtains:

\[
\begin{align*}
\text{as}(x)\Delta &= \sum_{i_1, \ldots, i_k} \sum_{\sigma \in S_k} \epsilon(\sigma) (-1)^{i_1 + \cdots + i_k} \prod_{j=1}^{k} q^{-i_j} \prod_{j=1}^{k} q^{-(2n+2j-2-i_j+\frac{j}{2})} \prod_{j=1}^{k} \left[ n + \sigma(j) - 1 \right] x_j^{n+\sigma(j)-1-i_j} \\
&= \sum_{i_1, \ldots, i_k} (-1)^{i_1 + \cdots + i_k + \frac{k(k-1)}{2}} \prod_{j=1}^{k} q^{-i_j} \prod_{j=1}^{k} q^{-(2n-i_j + \frac{j}{2}) - (j-1)(2n+j - \frac{j}{2})} \\
&\quad \times \left( \sum_{\sigma \in S_k} \epsilon(\sigma) \prod_{j=1}^{k} \left[ n + \sigma(j) - 1 \right] x_j^{n-i_j} \right) \\
&= \sum_{i_1, \ldots, i_k} (-1)^{i_1 + \cdots + i_k + \frac{k(k-1)}{2}} \prod_{j=1}^{k} q^{-i_j} \prod_{j=1}^{k} q^{-(2n-i_j + \frac{j}{2}) - (j-1)(2n+j - \frac{j}{2})} \\
&\quad \times \det_{1 \leq a, b \leq k} \left[ \begin{array}{c} n + b - 1 \\ i_a + b - 1 \end{array} \right] \prod_{j=1}^{k} x_j^{n-i_j},
\end{align*}
\]

in the second line we have relabeled \( i_j \rightarrow i_j + \sigma(j) - 1 \). Now we study the determinant in the previous expression and show that if \( i_1 > \ldots > i_n \):

(4.33) \[ \det_{1 \leq a, b \leq k} \left[ \begin{array}{c} n + b - 1 \\
\end{array} \right] = A_q(\lambda) \left[ \begin{array}{c} n \\
\lambda \end{array} \right]. \]
with the constant \( A_q(\lambda) = q^{n(\lambda')} \), as we show in appendix B. Here \( \lambda' \) is the partition conjugate to \( \lambda \) and equal to \((i_1, i_2 + 1, \ldots, i_k + k - 1)\). \([n]_q\) is a notation generalizing the \( q \)-binomial coefficients \([n]_j\) and which is defined by the \( q \)-hook formula:

\[
[n]_\lambda = \prod_{x \in \lambda} \frac{1 - q^{n+c(x)}}{1 - q^{h(x)}},
\]

This is nothing but the analog of quantum dimension Eq. (4.24) where instead of using the \([.]\) version of the \( q \)-integers one rather uses \([.]_q\). Identifying with the l.h.s. of Eq. (4.31) we obtain:

\[
< \mathfrak{s}_\lambda(M) >_w = q^{\sum_j -i_j(2n-i_j+\frac{1}{2})-(j-1)(2n+j-\frac{3}{2})} q^{n(\lambda')} [n]_\lambda.
\]

The last step we need to perform now is to convert \([n]_\lambda\) in terms of \( D_\lambda \) and check that the prefactor is given by Eq. (4.22). To this end we first note that due to Eq. (B.8) and Eq. (B.7) we have:

\[
[n]_\lambda = q^{\frac{1}{2}(n-1)|\lambda|-n(\lambda)} D_\lambda,
\]

and

\[
\sum_j -i_j(2n-i_j+\frac{1}{2})-(j-1)(2n+j-\frac{3}{2}) = -(2n-\frac{3}{2})|\lambda| + \sum_j \lambda_j^2 - 2j\lambda_j.
\]

To rewrite things in terms of the partition itself, rather than its transposed, we use the relationship:

\[
\sum_i \lambda_i^2 - 2i\lambda_i = 2(n(\lambda') - n(\lambda)) - |\lambda|.
\]

Collecting all the prefactors we can eventually write our final result:

\[
< \mathfrak{s}_\lambda > = q^{-\frac{1}{2}((3n+1)|\lambda|+\sum \lambda_i^2 - 2i\lambda_i)} D_\lambda,
\]

which coincides with Eq. (4.21).

5. Conclusions and Outlook

We have constructed the biorhogonal Stieltjes-Wigert polynomials, necessary for computing expressions such as (2.1), that appear in Chern-Simons theory. The polynomials are not discussed in the mathematics literature, so a great deal of the effort has been devoted to the explicit description of their fundamental properties.

The construction of the biorhogonal Stieltjes-Wigert polynomials may very well be a necessary technical step for the computation of knot invariants in exact fashion employing orthogonal polynomials. Note that, so far, the topological invariants computed with orthogonal polynomials are only Chern-Simons partition functions (more precisely, only the case of \( S^3 \) with gauge group \( U(N) \) \[8\]). Indeed, according to Mariño \[42\], the results in \[4\] can be extended in order to obtain random

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10 We warn the reader that for convenience we adopt a slightly different notation for \([n]_q\) compared to \[38\] in the sense that its value for the partition \((1^n)\) is the usual Gaussian polynomial \([n]\) whereas in \[38\], \([n]_q = [n]_q^{(1^n)}\).

11 Recall that \([n] = q^{\frac{1}{2}n(n-1)}\).
matrix descriptions of other Chern-Simons observables. The case of torus knots for example, amounts to:

\[ W_R^{(P,Q)} = e^{-\frac{g_s}{2} \left( P Q (\Lambda^2 - \rho^2) + \frac{\rho^2}{2} \right)} \frac{1}{|PQ|^{\frac{N}{2}}} N! \]

\[ \times \int \prod_{i=1}^{N} \frac{du_i}{2\pi} e^{-\sum_{i} u_i^2/2g_s} \prod_{i<j} \left( \frac{2 \sinh \frac{u_i - u_j}{2P}}{2 \sinh \frac{u_i - u_j}{2Q}} \right) S_\lambda(e^{u_i}), \]

where \( S_\lambda(x_i) \) are Schur polynomials associated to the partition \( \lambda \) (representations of \( U(N) \) are labelled by partitions \( \lambda \)).

That is to say, the (2.1) solved here, but with an insertion of a Schur polynomial. However, an obstacle can be the lack of a computational device for random matrix-like quantities with such a term. Note that the case of an ordinary Gaussian Hermitian matrix model with a Schur polynomial insertion, was solved in \[43\] by purely combinatorial methods, with no use of Hermite polynomials at all. Nevertheless, as we have seen in the last Section, a computation of the Stieltjes-Wigert ensemble with the Schur polynomial can be carried out with a mixture of combinatorial and orthogonal polynomials techniques. Therefore, it turns out that we have studied in detail the two cases comprised in (5.1). The biorthogonal case without the Schur insertion and the average of the Schur polynomial in the orthogonal \((P = Q = 1)\) ensemble.

We have also studied other aspects of the (ordinary) Stieltjes-Wigert polynomials that are of direct relevance for the corresponding matrix model. In particular, we have discussed in detail the equivalence of the Chern-Simons matrix model with its discrete counterpart, of very much interest in \(q\)-\(2D\) Yang-Mills theory, and also discussed the close ties with Rogers-Szegő polynomials, which are defined on the unit circle (both sets of polynomials being an equivalent solution of a \(q\)-deformed harmonic oscillator problem). This relationship allows to clearly establish the relationship with the unitary matrix model discussed in \[15\]. Fundamental properties of the Stieltjes-Wigert polynomials like their asymptotic behavior and the above mentioned \(q\)-deformed harmonic oscillator property, may be of interest in connection with the recently established role of the polynomial in the study of topological strings \[29\]. We hope to address some of these issues in future work.

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Appendix A. Normalizations

To follow standard conventions it is convenient to have the orthogonal polynomials either monic or normalized to unity. Hence, we rewrite the previous ones a little bit (it will also make the link with the usual Stieltjes-Wigert for \( k = 1 \) more transparent).

A.1. Notations, \( k = 1 \). From Szegö [27] we have for the Stieltjes-Wigert polynomials

\[
p_n(x) = \left(\frac{-1}{\sqrt{q}}\right)^{n/2} \sum_{\nu=0}^{\left\lfloor n/2 \right\rfloor} \binom{n}{\nu}_q q^{\nu^2} (-\sqrt{q}x)\nu,
\]

with \( \binom{n}{\nu}_q \) the \( q \)-binomial coefficient:

\[
\binom{n}{\nu} = \frac{[q]^n}{[q]^\nu [q]^{n-\nu}}.
\]

These polynomials are orthonormal for the scalar product \(<,>\) induced by:

\[
w(x) = \kappa \sqrt{\pi} e^{-\kappa^2 \log^2 x},
\]

with \( q = e^{-\frac{1}{2}\kappa^2} \) as usual. Note that one has:

\[
<1,1>_{w} = 1/\sqrt{q}.
\]

The \( S_n \) polynomials in Eq. (2.8) to which Askey refers [25] as the SW polynomials are written in a slightly different form. They satisfy

\[
< S_n, S_m > = \frac{q^{-n}}{[q]^n} \delta_{n,m},
\]

with \(<,>\) the scalar product associated to the measure \( \frac{4dx}{[-x]_{\infty} [-q/x]_{\infty}} \) with \( A \) a normalization constant such that \(<1,1>= 1\).

Then, the polynomials defined by:

\[
\tilde{S}_n(x) \equiv (-1)^n \sqrt{[q]^n q^{n/2}} S_n(x),
\]

are orthonormal for \(<,>\). One then sees that:

\[
p_n(x) = q^{1/4} \tilde{S}_n(\sqrt{q}x).
\]

Since Al-Salam and Verma [26] have the same notations as Askey for the \( q \)-Laguerre polynomials, we will make the same rewriting when using the biorthogonal Stieltjes-Wigert polynomials in the context of Chern-Simons theory computations.

A.2. \( k \) arbitrary. Normalizing and changing variables as in the previous section we define new polynomials

\[
R_n(x,k|q) \equiv \frac{(-1)^n q^{1/4}}{\sqrt{k_n}} Y_n(\sqrt{q}x,k|q) = r_{n,k} x^n + ...
\]

and
\begin{align}
  T_n(x, k|q) &= (-1)^n q^{1/4} \sqrt{k_n} Z_n(\sqrt{q}x, k|q) = t_{n,k} x^{n_k} + ...
\end{align}

Then one has
\begin{align}
  r_{n,k} &= q^{(n+\frac{1}{2})^2} \sqrt{[q]_n} q^{(k-\frac{1}{2})^2},
\end{align}

and
\begin{align}
  t_{n,k} &= \sqrt{[q]_n} q^{(nk+\frac{1}{2})^2 - \frac{1}{2} n^2 k(k-1)},
\end{align}

which reduce to $q^{(n+1)^2/2} \sqrt{[q]_n}$ when $k = 1$ as expected. Then one has
\begin{align}
  < R_n(x, k|q) T_n(x, k|q) >_w &= \delta_{m,n}.
\end{align}

Following Eq. (3.19) and (3.24) the recurrence relations for these orthonormal polynomials read
\begin{align}
  x R_n(x, k|q) &= q^{-nk-1/2} \left( R_n(q^{-1} x, k|q) + q^{-k/2} \sqrt{1 - q^{n+1}} R_{n+1}(q^{-1} x, k|q) \right),
\end{align}

\begin{align}
  x^k T_n(x, k|q) &= q^{-nk+\frac{k(k-2)}{2}} \left( T_n(q^{-k} x, k|q) + q^{-k/2} (1 - q^{k(n+1)}) \sqrt{k_n+1} T_{n+1}(q^{-k} x, k|q) \right).
\end{align}

APPENDIX B. PROOF OF Eq. (4.33)

Eq. (4.33) is not obvious at first sight because when $q = 1$, Giambelli’s formula would lead us to write:
\begin{align}
  \det_{1 \leq a, b \leq k} \left( \begin{array}{c} n \\ i_a + b - 1 \end{array} \right) = \dim \lambda.
\end{align}

However, classically one also has:
\begin{align}
  \det_{1 \leq a, b \leq k} \left( \begin{array}{c} n \\ i_a + b - 1 \end{array} \right) &= \det_{1 \leq a, b \leq k} \left( \begin{array}{c} n + b - 1 \\ i_a + b - 1 \end{array} \right),
\end{align}

which can be seen to hold by using Pascal’s identity. Nevertheless, Pascal’s identity in the quantum case, is slightly more complicated and reads:
\begin{align}
  \begin{bmatrix} n + 1 \\ j + 1 \end{bmatrix} &= \begin{bmatrix} n \\ j + 1 \end{bmatrix} + q^{n-j} \begin{bmatrix} n \\ j \end{bmatrix},
\end{align}

\text{See for instance [44] Eq. (16.114) for a nice presentation.}
and we thus see that in the quantum case the same kind of simplification cannot be shown to hold as simply as in the classical setting.

To prove Eq. (4.33) nevertheless, first recall that in the space of symmetric polynomials the change of basis between the elementary symmetric polynomials \( e_r \) and the Schur polynomials \( s_\lambda \) is given by (38 Eq. (3.5)):

\[
(B.4) \quad s_\lambda = \det(e_{\lambda'_i - i + j}),
\]

where \( \lambda = (\lambda_1 \geq \ldots \geq \lambda_n) \) is a partition and \( \lambda' \) its conjugate partition. This is actually nothing else than Giambelli’s identity, written for the symmetric polynomials and not just for the dimensions of the associated representations. To compute from this, note that if one considers \( x = (1, q, \ldots, q^{n-1}) \), then one has (38 §I.3 Ex. 1. p.44):

\[
(B.5) \quad s_\lambda(x) = q^{n(\lambda)} \left[ \frac{n}{\lambda} \right],
\]

where \( n(\lambda) \) is defined as:

\[
(B.6) \quad n(\lambda) = \sum_{i \geq 1} (i - 1) \lambda_i = \sum_{j \geq 1} \left( \frac{\lambda'_j}{2} \right),
\]

and satisfies the following useful formulae:

\[
(B.7) \quad \sum_{x \in \lambda} c(x) = n(\lambda') - n(\lambda)
\]

for the content (38, §1 Ex.3, p11 ), and another one for the hook-lengths (38, §1 Ex.2, p.11 ):

\[
(B.8) \quad \sum_{x \in \lambda} h(x) = n(\lambda) + n(\lambda') + |\lambda|.
\]

Let us come back to our computation and particularize Eq. (B.4) to \( x = (1, q, \ldots, q^{n-1}) \), that gives:

\[
(B.9) \quad q^{n(\lambda)} \left[ \frac{n}{\lambda} \right] = \det \left( q^{\frac{\lambda'_i - i + j}{2}} \left[ \frac{n}{\lambda'_i - i + j} \right] \right),
\]

or, introducing \( i_a + a - 1 = \lambda'_a \),

\[
(B.10) \quad q^{n(\lambda)} \left[ \frac{n}{\lambda} \right] = \det \left( q^{\frac{i_a + b - 1}{2}} \left[ \frac{n}{i_a + b - 1} \right] \right).
\]

This is still not quite what we want. To proceed further note that according to Eq. (B.3) one has:

\[
(B.11) \quad q^{\frac{\lambda'_i - 1}{2}} \left[ \frac{n + 1}{\lambda'_i + 1} \right] = q^{\frac{\lambda'_i - 1}{2}} \left[ \frac{n}{\lambda'_i + 1} \right] + q^n \left( q^{\frac{\lambda'_i - 1}{2}} \left[ \frac{n}{\lambda'_i + 1} \right] \right).
\]

Therefore, by multiple linear combinations of columns one can write:

\[13\] Elementary symmetric polynomials are special cases of Schur polynomials of 1-column diagrams, or \( e_r = s_{\lambda_r} \) in our notations.
whose first homology group

From which we finally obtain:

can be constructed by performing surgery on a link $L$

understand the origin of other quantities in (1.3), one has to take into account the

in the text, mainly in (1.3). For more information, see [4] and references therein. To

Now, for convenience extract a factor $q^{\frac{n(n-1)}{2}}$ in each line to get:

To proceed further note the following property of the $q$-binomial coefficients:

Therefore, once we write,

it is easy to see that

Collecting everything we thus have

From which we finally obtain:

Appendix C. Notation for Chern-Simons Quantities

We give here some information about the Chern-Simons quantities that appear in the text, mainly in (1.3). For more information, see [4] and references therein. To understand the origin of other quantities in (1.3), one has to take into account the constructions of Seifert homology spheres from surgery. Seifert homology spheres can be constructed by performing surgery on a link $L$ in $S^3$ with $n+1$ components, consisting of $n$ parallel and unlinked unknots together with a single unknot whose linking number with each of the other $n$ unknots is one. The surgery data are $p_j/q_j$ for the unlinked unknots, $j = 1, \cdots, n$, and 0 on the final component. $p_j$ is coprime to $q_j$ for all $j = 1, \cdots, n$, and the $p_j$'s are pairwise coprime. After doing surgery, one obtains the Seifert space $M = X(\frac{p_1}{q_1}, \cdots, \frac{p_n}{q_n})$. This is rational homology sphere whose first homology group $H_1(M, \mathbb{Z})$ has order $|H|$, where

$$H = P \sum_{j=1}^{n} \frac{q_j}{p_j}, \quad \text{and} \quad P = \prod_{j=1}^{n} p_j.$$  

Another topological invariant that will enter the computation is the signature of $L$, which turns out to be:

$$\sigma(L) = \sum_{i=1}^{n} \text{sign}(\frac{q_i}{p_i}) - \text{sign}(\frac{H}{P}).$$
For \( n = 1, 2 \), Seifert homology spheres reduce to lens spaces, and one has that \( L(p, q) = X(q/p) \). For \( n = 3 \), we obtain the Brieskorn homology spheres \( \Sigma(p_1, p_2, p_3) \) (in this case the manifold is independent of \( q_1, q_2, q_3 \)). In particular, \( \Sigma(2, 3, 5) \) is the Poincaré homology sphere. Finally, the Seifert manifold \( X(2 - 1, m(m+1)/2, t - m) \), with \( m \) odd, can be obtained by integer surgery on a \((2, m)\) torus knot with framing \( t \). Note that in \( \Sigma \) the weight and root lattices of \( G \) are denoted by \( \Lambda_w \) and \( \Lambda_r \), respectively.

Finally, there is a phase factor in \( \Sigma \) that comes from the framing correction, that guarantees that the resulting invariant is in the canonical framing for the three-manifold \( M \). Its explicit expression is:

\[
\phi = 3 \text{sign} \left( \frac{H}{P} \right) + \sum_{i=1}^{n} 12 s \left( q_i, p_i \right) - q_i p_i,
\]

where \( \sigma(L) \) is again the signature of the linking matrix of \( L \) and \( s(p, q) \) is the Dedekind sum:

\[
s(p, q) = \frac{1}{4q} \sum_{n=1}^{q-1} \cot \left( \frac{\pi n}{q} \right) \cot \left( \frac{\pi n p}{q} \right).
\]

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Laboratoire de Physique Théorique de l’École Normale Supérieure. 24 rue L’homond 75231, Paris Cedex 05, France.
E-mail address: dolivet@lpt.ens.fr

Institut d’Estudis Espacials de Catalunya (IEEC/CSIC). Campus UAB, Facultat de Ciències, Torre C5-Parell-2a planta. E-08193 Bellaterra (Barcelona) Spain.
E-mail address: tierz@ieec.fcr.es