PROJEC TIVE CRYSTALLINE REPRESENTATIONS OF ÉTALE FUNDAMENTAL GROUPS AND TWISTED PERIODIC HIGGS-DE RHAM FLOW

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ABSTRACT. We introduce new notions of projective crystalline representations and twisted periodic Higgs-de Rham flows. These new notions generalize crystalline representations of étale fundamental groups introduced in [4, 6] and periodic Higgs-de Rham flows introduced in [12]. We show that the categories of projective crystalline representations and twisted periodic Higgs-de Rham flows are equivalent via the category of twisted Fontaine-Faltings module which is also introduced in this paper. As an application of our results, we use stable graded Higgs bundle to construct infinitely many irreducible PGL($\mathbb{Z}_p^u$)-crystalline representations of the étale fundamental group of the projective line minus at least four marked points, and of smooth projective curves of genus $g \geq 2$.

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This work was supported by SFB/Transregio 45 Periods, Moduli Spaces and Arithmetic of Algebraic Varieties of the DFG (Deutsche Forschungsgemeinschaft) and also supported by National Key Basic Research Program of China (Grant No. 2013CB834202).
0. Introduction

The nonabelian Hodge theory established by Hitchin and Simpson associates representations of the topological fundamental group of an algebraic variety $X$ over $\mathbb{C}$ to a holomorphic object on $X$ named Higgs bundle. Later, Ogus and Vologodsky established the nonabelian Hodge theory in positive characteristic in their spectacular work [19]. They constructed the Cartier functor and the inverse Cartier functor, which give an equivalence of categories between the category of nilpotent Higgs modules and the category of nilpotent flat modules over a smooth proper and $W_2(k)$-liftable variety. This equivalence generalizes the classical Cartier descent theorem. Moreover, it is the starting point of the theory of Higgs-de Rham flows in [12].

To attach representations of the fundamental group to these $\mathcal{O}_X$-linear objects on $X$, one still needs an analogue of the classical Riemann-Hilbert correspondence. Unfortunately, there is no direct generalization of Riemann-Hilbert correspondence in the characteristic $p$ case. However, in the $p$-adic case, a good $p$-adic analogue of the category of polarized complex variations of Hodge structures, the category $\mathcal{MF}_{[a,b]}^\nabla(\mathcal{X}/W)$, is introduced by Fontaine and Laffaille [9] for $\mathcal{X} = \text{Spec} W(k)$ in their study of $p$-adic Hodge theory. The theory of Fontaine and Laffaille was later generalized by Faltings [4] to the general geometric base case. The objects in this theory are called Fontaine modules and consists of a quadruple $(V, \nabla, \text{Fil}, \varphi)$, where $(V, \nabla, \text{Fil})$ is a filtered de Rham bundle over $\mathcal{X}$ and $\varphi$ is a relative Frobenius which is horizontal with respect to $\nabla$ and satisfies the strong $p$-divisibility condition. The latter condition is a $p$-adic analogue of the Riemann-Hodge bilinear relations. Then the Fontaine-Laffaille-Faltings correspondence gives a fully faithful functor from $\mathcal{MF}_{[0,w]}^\nabla(\mathcal{X}/W)$ ($w \leq p - 2$) to the category of
crystalline representations of \( \pi_1(X_K) \), where \( X_K \) is the generic fiber of \( X \). This can be regarded as a \( p \)-adic version of the Riemann-Hilbert correspondence.

Faltings has established an equivalence of categories between the category of generalized representations of the geometric fundamental group and the category of Higgs bundles over a \( p \)-adic curve, which has generalized the earlier work of Deninger-Werner on a partial \( p \)-adic analogue of Narasimhan-Seshadri theory.

In order to establish a \( p \)-adic analogue of the Hitchin-Simpson correspondence between the category of representations with small coefficients, namely \( \text{GL}_r(W_n(\mathbb{F}_q)) \)-representations, and the category of Higgs bundles, Lan, Sheng and Zuo introduced the notion of Higgs-de Rham flows, which can be considered as an analogue of the Yang-Mills-Higgs flow attached to a Higgs bundle in the complex analytic situation. The latter is used to solve the Yang-Mills-Higgs equation. The stability of the Higgs bundles guarantees the existence of solutions, the so-called Yang-Mills-Higgs connections. Roughly speaking, a Higgs-de Rham flow is a sequence of graded Higgs bundles and filtered de Rham bundles, connected by the inverse Cartier transform defined by the fundamental work of Ogus and Vologodsky and the grading functor by the attached Hodge filtrations on the de Rham bundles (for details see Section 3 in [12] or Section 3.1 in this paper). The following diagram presents a Higgs-de Rham flow over \( X_1 \) (\( X_1 \) is the special fiber of \( X \)):

A Higgs-de Rham flow is said to be periodic (of period \( f \in \mathbb{N} \)) if \( f \) is the smallest integer such that there exists an isomorphism of Higgs bundles \( \phi : (E_f, \theta_f) \cong (E_0, \theta_0) \). For Higgs bundles over \( X_n/W_n(k) \), the key point of defining the Higgs-de Rham flow is to find the suitable lifting \( C_n^{-1} \) of \( C_1^{-1} \) over the truncated Witt ring \( W_n(k) \). This was done by Lan-Sheng-Zuo in section 4 of [12]. Because of the importance of it in lifting the Higgs-de Rham flow to \( X/W(k) \), we recall the construction of functor \( C_n^{-1} \) briefly in Section 2.

**Theorem 0.1** (Theorem 1.4 in [12]). Let \( X \) be a smooth proper scheme over \( W \). For each integer \( 0 \leq w \leq p-2 \) and each \( f \in \mathbb{N} \), there is an equivalence of categories between the category of \( p \)-torsion free Fontaine-Faltings modules over \( X \) of Hodge-Tate weight \( \leq m \) with endomorphism structure \( W(\mathbb{F}_{p^f}) \) and the category of periodic Higgs-de Rham flows over \( X \) of level \( \leq w \) and whose periods are \( f \).

**Remark.** It is straightforward to generalize Theorem 0.1 to the logarithmic setting. To be more precise, let \( X \) be a smooth proper scheme over \( W \) and
let $D \subset X$ be a simple normal crossings divisor relative to $W$. Then, for each positive integer $f$, there is an equivalence of categories between the category of strict $p^n$-torsion logarithmic Fontaine modules (with logarithmic structure along $D \times W_n \subset X \times W_n$) with endomorphism structure of $W_n(F_{p^f})$ whose Hodge-Tate weights $\leq p - 2$ and the category of periodic logarithmic Higgs-de Rham flows over $X \times W_n$ (with logarithmic structure along $D \times W_n \subset X \times W_n$) whose periods are factors of $f$ and nilpotent exponents are $\leq p - 2$.

By Theorem 6.6 in [12], a periodic Higgs bundle must have trivial Chern classes. This fact limits the application of the $p$-adic Hitchin-Simpson correspondence. For instance, Simpson constructed a canonical Hodge bundle $\Omega^1_X \oplus O_X$ on $X$ in his proof of the Miyaoka-Yau inequality (Proposition 9.8 and Proposition 9.9 in [21]), which has nontrivial Chern classes in general. In fact, the classical nonabelian Hodge theorem tells us that the Yang-Mills-Higgs equation is still solvable for a polystable Higgs bundle with nontrivial Chern classes. Instead of getting a flat connection, one can get a projective flat connection in this case, whose monodromy gives a $\text{PGL}_r$-representation of the fundamental group. This motivates us to find a $p$-adic Hitchin-Simpson correspondence for graded Higgs bundles with nontrivial Chern classes.

A projective flat connection $\nabla$ on a bundle $V$ over $\mathbb{C}$ is a (usual) connection whose curvature has the special form $\Theta = \omega \otimes \text{Id}_V$, where $\omega$ is a rational closed $(1,1)$-form representing $\frac{1}{\text{rk}(V)}c_1(V)$. Note that, if $[\omega] \in H^2(X, \mathbb{Z})$, then by the Lefschetz theorem on $(1,1)$-classes one can actually find a line bundle $L$ with a metric connection $\nabla_L$ such that $(V, \nabla) \otimes (L, \nabla_L)$ becomes a flat bundle. Inspired by this we first introduce the 1-periodic twisted Higgs-de Rham flow over $X_1$ as follows

$$
\begin{array}{ccc}
(V, \nabla, \text{Fil})_0 & \xrightarrow{C_1^{-1}} & (E, \theta)_0 \sim \phi_L (E, \theta) \otimes (L, 0) \\
\downarrow \text{Gr}(\cdot) \otimes (L, 0) & & (E, \theta)_1 \otimes (L, 0)
\end{array}
$$

Here $L$ is called a twisting line bundle on $X_1$, and $\phi_L : (E_1, \theta_1) \otimes (L, 0) \cong (E_0, \theta_0)$ is called the twisted $\phi$-structure.

On the Fontaine module side, we also introduce the twisted Fontaine-Faltings module over $X_1$. The latter consists of the following data: a filtered de Rham bundle $(V, \nabla, \text{Fil})$ together with an isomorphism between de Rham bundles:

$$
\varphi_L : (C_1^{-1} \circ \text{GrFil}(V, \nabla)) \otimes (L^\otimes p, \nabla_{\text{can}}) \cong (V, \nabla).
$$

We will refer to the isomorphism $\varphi_L$ as the twisted $\varphi$-structure. The general construction of twisted Fontaine-Faltings modules and twisted periodic Higgs-de Rham flows are given in Section 1.5 and Section 3.2 (over $X_n/W_n(k)$, and multi-periodic case).
Theorem 0.2 (Theorem 3.3). Let $\mathcal{X}$ be a smooth proper scheme over $W$. For each integer $0 \leq a \leq p - 2$ and each $f \in \mathbb{N}$, there is an equivalence of categories between the category of all twisted $f$-periodic Higgs-de Rham flows over $X_n$ of level $\leq a$ and the category of strict $p^n$-torsion twisted Fontaine-Faltings modules over $X_n$ of Hodge-Tate weight $\leq a$ with an endomorphism structure of $W_n(F_{p^f})$.

Theorem 0.2 can be generalized to the logarithmic case. The precise statement is as follows.

Theorem 0.3 (Theorem 3.4). Let $\mathcal{X}$ be a smooth proper scheme over $W$ with a simple normal crossing divisor $D \subset X$ relative to $W$. Then for each natural number $f \in \mathbb{N}$, there is an equivalence of categories between the category of strict $p^n$-torsion twisted logarithmic Fontaine-Faltings modules (with pole along $D \times W_n \subset X \times W_n$) with endomorphism structure of $W_n(F_{p^f})$ whose Hodge-Tate weight $\leq p - 2$ and the category of twisted $f$-periodic logarithmic Higgs-de Rham flows (with pole along $D \times W_n \subset X \times W_n$) over $X \times W_n$ whose nilpotent exponents are $\leq p - 2$.

One of our goals is to associate a PGL$_n$-representation of $\pi_1$ to a twisted (logarithmic) Fontaine-Faltings module. To do so, we will need to generalize Faltings’s work. Following Faltings [4], we construct a functor $\mathbb{D}^P$ in section 2.5 which associates to a twisted (logarithmic) Fontaine-Faltings module a PGL$_n$ representation of the étale fundamental group.

Theorem 0.4 (Theorem 2.10). Let $\mathcal{X}$ be a smooth proper geometrically connected scheme over $W$ with a simple normal crossing divisor $D \subset X$ relative to $W$. Suppose $F_{p^f} \subset k$. Let $M$ be a twisted logarithmic Fontaine-Faltings module over $X$ (with pole along $D$) with endomorphism structure of $W(F_{p^f})$. Applying $\mathbb{D}^P$-functor, one gets a projective representation

$$\rho : \pi_1(X^o_K) \to \text{PGL}(\mathbb{D}^P(M)),$$

where $X^o_K$ is the generic fiber of $X^o = X \setminus D$.

In Section 3.4, we study several properties of this functor $\mathbb{D}^P$. For instance, we prove that a projective sub-representation of $\mathbb{D}^P(M)$ corresponds to a sub-object $N \subset M$ such that $\mathbb{D}^P(M/N)$ is isomorphic to this subrepresentation. Combining this with Theorem 3.3, we infer that a projective representation coming from a stable twisted periodic Higgs bundle $(E, \theta)$ with $(\text{rank}(E), \deg_H(E)) = 1$ must be irreducible.

The next theorem gives a $p$-adic analogue of the existence of projective flat Yang-Mills-Higgs connection in terms of semistability of Higgs bundles and triviality of discriminant.

Theorem 0.5 (Theorem 3.10). A semistable Higgs bundle over $X_1$ initials a twisted preperiodic Higgs-de Rham flow if and only if it is semistable and has trivial discriminant.
Consequently we obtain the following theorem on the existence of non-trivial representations of étale fundamental group in terms of the existence of semistable graded Higgs bundles.

**Theorem 0.6** (Theorem 3.14). Let \( k \) be a finite field of characteristic \( p \). Let \( \mathcal{X} \) be a smooth proper geometrically connected scheme over \( W(k) \) together with a smooth log structure \( \mathcal{D}/W(k) \). Assume that there exists a semistable graded logarithmic Higgs bundle \( (E, \theta)/(\mathcal{X}, \mathcal{D})_1 \) with \( r := \text{rank}(E) \leq p - 1 \), discriminant \( \Delta_H(E) = 0 \), \( r \) and \( \text{deg}_H(E) \) are coprime. Let \( \mathcal{X}^o = \mathcal{X} \setminus \mathcal{D} \) and \( K' = W(k \cdot \mathbb{F}_{p^f})[1/p] \). Then, there exist a positive integer \( f \) and a \( \text{PGL}_r(\mathbb{F}_{p^f}) \)-crystalline representation \( \rho \) of \( \pi_1(\mathcal{X}^o_{K'}) \), which is irreducible in \( \text{PGL}_r(\mathbb{F}_p) \).

Finally we give two applications in Section 4 to show how the general machinery developed in the previous sections works in some concrete situations. Taking the moduli space \( M \) of graded stable Higgs bundles of rank-2 and degree 1 over \( \mathbb{P}^1 \) with logarithmic structure on \( m (> 3) \) marked points we show that the self map induced by Higgs-de Rham flow stabilizes the component \( M(1,0) \) of \( M \) of maximal dimension (\( \dim = m - 3 \)) as a rational and dominant map. Hence by Hrushovski’s theorem [8] the subset of periodic Higgs bundles is Zariski dense in \( M(1,0) \). In this way we produce infinitely many \( \text{PGL}_2(\mathbb{F}_{p^f}) \)-crystalline representations, which are irreducible in \( \text{PGL}_2(\mathbb{F}_p) \). By Theorem 3.14 all these representations lift to \( \text{PGL}_2(\mathbb{Z}_{p^r}) \)-crystalline representations.

For the case of 4 marked points \( \{0, 1, \infty, \lambda\} \) we state an explicit formula for the self map and use it to study the dynamic of Higgs-de Rham flows for \( p = 3 \) and several values of \( \lambda \).

In the last subsection 4.3 we consider a smooth projective curve \( \mathcal{X} \) over \( W(k) \) of genus \( g \geq 2 \). In the Appendix of [20], de Jong and Osserman have shown that the subset of twisted periodic vector bundles over \( X_1 \) in the moduli space of semistable vector bundles over \( X_1 \) of any rank and any degree is always Zariski dense. By applying our main theorem for twisted periodic Higgs de Rham flows with zero Higgs fields, which should be regarded as projective étale trivializable vector bundles in the projective version of Lange-Stuhe’s theorem (see [14]), they all correspond to \( \text{PGL}_r(\mathbb{F}_{p^f}) \)-representations of \( \pi_1(X_1) \). Once again we show that they all lift to \( \text{PGL}_r(\mathbb{Z}_{p^r}) \) of \( \pi_1(X_1) \). It should be very interesting to make a comparison between the lifting theorem obtained here lifting \( \text{GL}_r(\mathbb{F}_{p^f}) \)-representations of \( \pi_1(X_1) \) to \( \text{GL}_r(\mathbb{Z}_{p^r}) \)-representation of \( \pi_1(X_1 \mathbb{F}_p) \) and the lifting theorem developed by Deninger-Werner [2]. In their paper, they have shown that any vector bundle over \( \mathcal{X}/W \) which is étale trivializable over \( X_1 \) lifts to a \( \text{GL}_r(\mathbb{C}_p) \)-representation of \( \pi_1(\mathcal{X}/W) \).
In this section, we will recall the definition of Fontaine-Faltings modules in [4] and generalize it to the twisted version.

1.1. Fontaine-Faltings modules. Let $X_n$ be a smooth and proper variety over $W_n(k)$. And $(V, \nabla)$ is a de Rham sheaf (i.e. a sheaf with an integrable connection) over $X_n$. In this paper, a filtration Fil on $(V, \nabla)$ will be called a Hodge filtration of level in $[a, b]$ if the following conditions hold:

- Fil$^i V$’s are locally split sub-sheaves of $V$, with $V = \text{Fil}^a V \supset \text{Fil}^{a+1} V \supset \cdots \supset \text{Fil}^b V \supset \text{Fil}^{b+1} V = 0$,

and locally on all open subsets $U \subset X_n$, the graded factor Fil$^i V(U)/\text{Fil}^{i+1} V(U)$ are finite direct sums of $O_{X_n(U)}$-modules of form $O_{X_n(U)}/p^e$.

- Fil satisfies Griffiths transversality with respect to the connection $\nabla$.

In this case, the triple $(V, \nabla, \text{Fil})$ is called a filtered de Rham sheaf. One similarly gives the conceptions of (filtered) de Rham modules over a $W$-algebra.

1.1.1. Fontaine-Faltings modules over a small affine base. Let $\mathcal{U} = \text{Spec} R$ be a small affine scheme (which means there exist an étale map $W_n[T_1^{\pm 1}, T_2^{\pm 1}, \cdots, T_d^{\pm 1}] \to O_{X_n(U)}$, see [4]) over $W$ and $\Phi : \hat{R} \to \hat{R}$ be a lifting of the absolute Frobenius on $R/pR$, where $\hat{R}$ is the $p$-adic completion of $R$. A Fontaine-Faltings module over $\mathcal{U}$ of Hodge-Tate weight in $[a, b]$ is a quadruple $(V, \nabla, \text{Fil}, \varphi)$, where

- $(V, \nabla)$ is a de Rham $R$-module;
- Fil is a Hodge filtration on $(V, \nabla)$ of level in $[a, b]$;
- $\varphi$ is an $R$-linear isomorphism

$$\varphi : F^*_{\mathcal{U}, \Phi} \tilde{V} = \tilde{V} \otimes_{\hat{R}} \tilde{R} \to V,$$

where $F^*_{\mathcal{U}, \Phi} = \text{Spec}(\Phi)$, $\tilde{V}$ is the quotient $\bigoplus_{i=a}^{b} \text{Fil}^i V / \sim$ with $x \sim py$ for any $x \in \text{Fil}^i V$ and $y$ is the image of $x$ under the natural inclusion $\text{Fil}^i V \hookrightarrow \text{Fil}^{i-1} V$.

- The relative Frobenius $\varphi$ is horizontal with respect to the connections $F^*_{\mathcal{U}, \Phi} \nabla$ on $F^*_{\mathcal{U}, \Phi} \tilde{V}$ and $\nabla$ on $V$, i.e. the following diagram commutes:

$$\begin{align*}
F^*_{\mathcal{U}, \Phi} \tilde{V} & \xrightarrow{\varphi} V \\
\downarrow F^*_{\mathcal{U}, \Phi} \nabla & \quad \downarrow \nabla \\
F^*_{\mathcal{U}, \Phi} \tilde{V} \otimes \Omega^1_{\mathcal{U}/W} & \xrightarrow{\varphi \otimes \text{id}} V \otimes \Omega^1_{\mathcal{U}/W}
\end{align*}$$

In this case, the triple $(V, \nabla, \text{Fil})$ is called a filtered de Rham sheaf. One similarly gives the conceptions of (filtered) de Rham modules over a $W$-algebra.
Let $M_1 = (V_1, \nabla_1, \text{Fil}_1, \varphi_1)$ and $M_2 = (V_2, \nabla_2, \text{Fil}_2, \varphi_2)$ be two Fontaine-Faltings modules over $\mathcal{U}$ of Hodge-Tate weight in $[a, b]$. The homomorphism set between $M_1$ and $M_2$ constitutes by those morphism $f : V_1 \to V_2$ of $R$-modules, satisfying:

- $f$ is for the filtrations. i.e. $f^{-1}(\text{Fil}_2 V_2) = \text{Fil}_1 V_1$.
- $f$ is a morphism of de Rham modules. i.e. $(f \otimes \text{id}) \circ \nabla_1 = \nabla_2 \circ f$.
- $f$ commutes with the $\varphi$-structures. i.e. $(\tilde{f} \otimes \text{id}) \circ \varphi_1 = \varphi_2 \circ f$, where $\tilde{f}$ is the image of $f$ under Faltings’ tilde functor.

Denote by $\mathcal{MF}^{\nabla, \Phi}_{[a, b]}(U/W)$ the category of all Fontaine-Faltings modules over $U$ of Hodge-Tate weight in $[a, b]$.

The gluing functor. In the following, we recall the gluing functor of Faltings. In other words, up to a canonical equivalence of categories, the category $\mathcal{MF}^{\nabla, \Phi}_{[a, b]}(U/W)$ does not depend on the choice of $\Phi$. More explicitly, the equivalent functor is given as follows.

Let $\Psi$ be another lifting of the absolute Frobenius. For any filtered de Rham module $(V, \nabla, \text{Fil})$, Faltings shows that there is a canonical isomorphism by Taylor formula

$$\alpha_{\Phi, \Psi} : F^n_{\mathcal{U}, \Phi} \tilde{V} \simeq F^n_{\mathcal{U}, \Psi} \tilde{V},$$

which is parallel with respect to the connection, satisfies the cocycle conditions and induces an equivalent functor of categories

$$\begin{align*}
\mathcal{MF}^{\nabla, \Phi}_{[a, b]}(U/W) & \longrightarrow \mathcal{MF}^{\nabla, \Phi}_{[a, b]}(U/W) \\
(V, \nabla, \text{Fil}, \varphi) & \longrightarrow (V, \nabla, \text{Fil}, \varphi \circ \alpha_{\Phi, \Psi})
\end{align*} \tag{1.1}$$

1.1.2. Fontaine-Faltings modules over global base. Let $I$ be the index set of all pairs $(\mathcal{U}_i, \Phi_i)$. The $\mathcal{U}_i$ is a small affine open subset of $X$, and $\Phi_i$ is a lift of the absolute Frobenius on $O_X(\mathcal{U}_i) \otimes_W k$. Recall that the category $\mathcal{MF}^{\nabla, \Phi}_{[a, b]}(X/W)$ is constructed by gluing those categories $\mathcal{MF}^{\nabla, \Phi_i}_{[a, b]}(\mathcal{U}_i/W)$.

Actually $\mathcal{MF}^{\nabla, \Phi_i}_{[a, b]}(\mathcal{U}_i/W)$ can be described more precisely as below.

A Fontaine-Faltings module over $X$ of Hodge-Tate weight in $[a, b]$ is a tuple $(V, \nabla, \text{Fil}, \{\varphi_i\}_{i \in I})$ over $X$, i.e. a filtered de Rham sheaf $(V, \nabla, \text{Fil})$ together with $\varphi_i : \tilde{V}(\mathcal{U}_i) \otimes_{\Phi_i} \tilde{O}_X(\mathcal{U}_i) \to V(\mathcal{U}_i)$ such that

- $M_i := (V(\mathcal{U}_i), \nabla, \text{Fil}, \varphi_i) \in \mathcal{MF}^{\nabla, \Phi_i}_{[a, b]}(\mathcal{U}_i/W)$.
- For all $i, j \in I$, on the overlap open set $\mathcal{U}_i \cap \mathcal{U}_j$, local Fontaine-Faltings modules $M_i |_{\mathcal{U}_i \cap \mathcal{U}_j}$ and $M_j |_{\mathcal{U}_i \cap \mathcal{U}_j}$ are associated to each other by the equivalent functor respecting these two liftings $\Phi_i$ and $\Phi_j$. In other
words, the following diagram commutes
\[
\begin{array}{ccc}
\tilde{V}(U_{ij}) \otimes \Phi_i & \xrightarrow{\alpha_{i,j} \Phi_j} & \tilde{V}(U_{ij}) \otimes \Phi_j \\
\phi_i & \downarrow & \phi_j \\
V(U_{ij}) & \xrightarrow{id} & V(U_{ij})
\end{array}
\]  
\text{(1.2)}

Morphisms between Fontaine-Faltings modules are those between sheaves and locally they are morphisms between local Fontaine-Faltings modules. More precisely, for a morphism \( f \) of the underlying sheaves of two Fontaine-Faltings modules over \( X \), the map \( f \) is called a morphism of Fontaine-Faltings modules if and only if \( f(U_i) \in \text{Mor} \left( \mathcal{MF}_{[a,b]}(U_i/W) \right) \), for all \( i \in I \).

Denote by \( \mathcal{MF}_{[a,b]}(X/W) \) the category of all Fontaine-Faltings modules over \( X \) of Hodge-Tate weight in \([a,b]\). And denote by \( \mathcal{MF}_{[a,b]}(X/W) \) the sub-category of \( \mathcal{MF}_{[a,b]}(X/W) \) consisted of strict \( p^n \)-torsion Fontaine-Faltings modules over \( X \) of Hodge-Tate weight in \([a,b]\).

1.2. Inverse Cartier functor. For a Fontaine-Faltings module \((V, \nabla, \Fil, \{\phi_i\}_{i \in I})\), we call \( \{\phi_i\}_i \) the \( \phi \)-structure of the Fontaine-Faltings module. In this section, we first recall a global description of the \( \phi \)-structure via the inverse Cartier functor over truncated Witt rings constructed by Lan, Sheng and Zuo [12].

Note that the inverse Cartier functor \( C_1^{-1} \) (the characteristic \( p \) case) is introduced in the seminal work of Ogus-Vologodsky [19]. Here we sketch an explicit construction of \( C_1^{-1} \) presented in [12]. Let \((E, \theta)\) be a nilpotent Higgs bundle over \( X_1 \). Locally we have

- \( V_i = F_{U_i}^* (E \mid_{U_i}) \),
- \( \nabla_i = d + \frac{dF_{U_i}}{[p]} (F_{U_i}^* \theta \mid_{U_i}) : V_i \to V_i \otimes \Omega^1_{U_i} \),
- \( G_{ij} = \exp(h_{ij} (F_{U_i}^* \theta \mid_{U_i})) : V_i \mid_{U_{ij}} \to V_j \mid_{U_{ij}} \),

where \( F_{U_i} \) is the absolute Frobenius on \( U_i \) and \( h_{ij} : F_{U_i}^* \Omega^1_{U_i} \to \mathcal{O}_{U_{ij}} \) is the homomorphism given by the Deligne-Illusie’s Lemma [1]. Those local data \((V_i, \nabla_i)\)’s can be glued into a global sheave \( H \) with integrable connection \( \nabla \) via the transition maps \( \{G_{ij}\} \) (Theorem 3 in [13]). The inverse Cartier functor on \((E, \theta)\) is

\[
C_1^{-1}(E, \theta) := (V, \nabla).
\]

Remark. Note that the inverse Cartier transform \( C_1^{-1} \) also has the logarithmic version. When the log structure is given by a simple normal crossing divisor, an explicit construction of the log inverse Cartier functor is given in the Appendix of [11].

As mentioned in the introduction, we need to generalize \( C_1^{-1} \) to the inverse Cartier transform over the truncated Witt ring for Higgs bundles over \( X_n/W_m(k) \). We briefly recall the construction in section 4 of [12].
1.2.1. *Inverse Cartier functor over truncated Witt ring.* Let $S = \text{Spec}(W(k))$ and $F_S$ be the Frobenius map on $S$. Let $X_{n+1} \supset X_n$ be a $W_{n+1}$-lifting of smooth proper varieties. Recall that the functor $C_{n}^{-1}$ is defined as the composition of $C_n^{-1}$ and the base change $F_S: X'_n = X_n \times_{F_S} S \to X_n$ (by abusing notation, we still denote it by $F_S$). The functor $C_{n}^{-1}$ is defined as the composition of two functors $\mathcal{T}_n$ and $\mathcal{F}_n$. In general, we have the following diagram and its commutativity follows easily from the construction of those functors.

These categories appeared in the diagram are explained as following:

- **MCF$_a(X_n)$** is the category of filtered de Rham sheaves over $X_n$ of level in $[0,a]$.
- **$\mathcal{H}(X_n)$** (resp. **$\mathcal{H}(X'_n)$**) is the category of tuples $(E, \theta, \nabla, Fil, \psi)$, where
  - $(E, \theta)$ is a graded Higgs module over $X_n$ (resp. $X'_n = X_n \otimes_\sigma W$) of exponent $\leq p - 2$;
  - $(\nabla, Fil)$ is a filtered de Rham sheaf over $X_{n-1}$ (resp. over $X'_{n-1}$);
  - and $\psi: Gr_{Fil}(\nabla) \simeq (E, \theta) \otimes \mathbb{Z}/p^n-1\mathbb{Z}$ is an isomorphism of Higgs sheaves over $X_n$ (resp. $X'_n$).
- **$\widetilde{\text{MIC}}(X_n)$** (resp. **$\widetilde{\text{MIC}}(X'_n)$**) is the category of sheaves over $X_n$ (resp. $X'_n$) with integrable $p$-connection.
- **$\text{MIC}(X_n)$** (resp. **$\text{MIC}(X'_n)$**) is the category of de Rham sheaves over $X_n$ (resp. $X'_n$).

*Functor $\overline{\text{Gr.}}$* For an object $(V, \nabla, Fil)$ in $\text{MCF}_{p-2}(X_n)$, the functor $\overline{\text{Gr}}$ is given by

$$\overline{\text{Gr}}(V, \nabla, Fil) = (E, \theta, \nabla, Fil, \psi),$$

where $(E, \theta) = \text{Gr}(V, \nabla, Fil)$ is the graded sheaf with Higgs field, $(\nabla, Fil)$ is the modulo $p^{n-1}$-reduction of $(V, \nabla, Fil)$ and $\psi$ is the identifying map $\text{Gr}(\nabla) \cong E \otimes \mathbb{Z}/p^{n-1}\mathbb{Z}$.

*Faltings tilde functor $\widetilde{\cdot}$.* For an object $(V, \nabla, Fil)$ in $\text{MCF}_{p-2}(X_n)$, the $(\nabla, Fil)$ will be denoted as the quotient $\bigoplus_{i=0}^{p-2} Fil^i/ \sim$ with $x \sim py$ for any $x \in Fil^iV$ and $y$ the image of $x$ under the natural inclusion $Fil^iV \hookrightarrow Fil^{i-1}V$. 
The construction of functor \( T_n \). Let \((E, \theta, \bar{V}, \bar{\nabla}, \bar{Fil}, \psi)\) be an object in \( \mathcal{H}(X_n) \) (resp. \( \mathcal{H}(X_n') \)). Locally on an affine open subset \( U \subset X \) (resp. \( U \subset X' \)), there exists \((V_U, \nabla_U, \Fil_U)\) (Lemma 4.6 in \cite{12}), a filtered de Rham sheaf, such that
\[
- (\bar{V}, \bar{\nabla}, \bar{Fil})|_U \cong (V_U, \nabla_U, \Fil_U) \otimes \mathbb{Z}/p^{n-1}\mathbb{Z};
- (E, \theta)|_U \cong \Gr(V_U, \nabla_U, \Fil_U).
\]

The tilde functor associates \((V_U, \nabla_U, \Fil_U)\) to a sheaf with \( p \)-connection over \( U \). By gluing those sheaves with \( p \)-connections over all \( U \)'s (Lemma 4.10 in \cite{12}), one gets a global sheaf with \( p \)-connection over \( X_n \) (resp. \( X'_n \)). Denote it by \( T_n(E, \theta, \bar{V}, \bar{\nabla}, \Fil, \psi) \).

For small affine open subset \( U \) of \( X \), there exists endomorphism \( F_U \) on \( U \) which lifts the absolute Frobenius on \( U_k \) and is compatible with the Frobenius map \( F_S \) on \( S = \Spec(W(k)) \). Thus there is a map \( F_{U/S} : U \to U' = U \times_{F_S} S \) satisfying \( F_U = F_S \circ F_{U/S} \).

Let \((\bar{V}', \bar{\nabla}')\) be an object in \( \widehat{MIC}(X'_n) \). Locally on \( U \), applying functor \( F^{*}_{U/S} \), we get a de Rham sheaf over \( U \)
\[
F^{*}_{U/S}(\bar{V}'|_U, \bar{\nabla}'|_U).
\]

By Taylor formula, up to a canonical isomorphism, it does not depends on the choice of \( F_U \). In particular, on the overlap of two small affine open subsets, there is an canonical isomorphism of two de Rham sheaves. By gluing those isomorphisms, one gets a de Rham sheaf over \( X_n \), we denote it by \( \mathcal{T}_n(E, \theta, \bar{V}, \bar{\nabla}, \Fil, \psi) \).

1.3. Global description of the \( \varphi \)-structure in Fontaine-Faltings modules (via the inverse Cartier functor). Let \((V, \nabla, \Fil) \in \MFC_{p-2}(X_n) \) be a filtered de Rham sheaf over \( X_n \) of level in \([0, p-2]\). From the commutativity of diagram (1.3), for any \( i \in I \), one has
\[
C^{-1}_n \circ \Gr(V, \nabla, \Fil)|_{U_i} = \mathcal{T}_n \circ F^{*}_S \circ \Gr(V, \nabla, \Fil)|_{U_i} = F_n \circ F^{*}_S(V, \nabla)|_{U_i} \approx F^{*}_{U_i}(\bar{V}, \bar{\nabla})|_{U_i}.
\]
Here \( F_{i,k} = \text{Spec}(\Phi_i) \colon \mathcal{U}_i \to \mathcal{U}_i \) is the lifting of the absolute Frobenius on \( \mathcal{U}_{i,k} \). As the \( \mathcal{F}_n \) is glued by using the Taylor formula, for any \( i,j \in I \), one has the following commutative diagram

\[
\begin{array}{ccc}
F_{i,k}^* (\tilde{V} |_{\mathcal{U}_i}, \nabla |_{\mathcal{U}_i}) & \xrightarrow{\alpha_{n,i},\psi_j} & C_n^{-1} \circ \text{Gr}(V, \nabla, \Fil) |_{\mathcal{U}_i} \\
\end{array}
\]

(1.6)

To give a system of compatible \( \varphi \)-structures (for all \( i \in I \))

\[
\varphi_i : F_{i,k}^* (\tilde{V} |_{\mathcal{U}_i}, \nabla |_{\mathcal{U}_i}) \to (V |_{\mathcal{U}_i}, \nabla |_{\mathcal{U}_i}),
\]

it is equivalent to give an isomorphism

\[
\varphi : C_n^{-1} \circ \text{Gr}(V, \nabla, \Fil) \to (V, \nabla).
\]

In particular, we have the following results

**Lemma 1.1** (Lemma 5.6 in \cite{12}). To give a Fontaine-Faltings module in \( \mathcal{MF}_{[0,p-2]}(X/W) \), it is equivalent to give a tuple \((V, \nabla, \Fil, \varphi)\), where

- \((V, \nabla, \Fil) \in \text{MCF}_{p-2}(X_n)\) is a filtered de Rham sheaf over \( X_n \) of level in \([0,p-2]\), for some positive integer \( n \);
- \( \varphi : C_n^{-1} \circ \text{Gr}(V, \nabla, \Fil) \to (V, \nabla) \) is an isomorphism of de Rham sheaves.

**1.4. Fontaine-Faltings modules with endomorphism structure.** Let \( f \) be a positive integer. We call \((V, \nabla, \Fil, \varphi, \iota)\) a Fontaine-Faltings module over \( \mathcal{X} \) with endomorphism structure of \( W(\mathbb{F}_p) \) whose Hodge-Tate weights lie in \([a,b]\), if \((V, \nabla, \Fil, \varphi)\) is an object in \( \mathcal{MF}_{[a,b]}^{\nabla}(\mathcal{X}/W) \) and

\[
\iota : W(\mathbb{F}_p) \to \text{End}_{\mathcal{MF}}(V, \nabla, \Fil, \varphi)
\]

is a continuous ring homomorphism. We call \( \iota \) an endomorphism structure of \( W(\mathbb{F}_p) \) on \((V, \nabla, \Fil, \varphi)\). Let’s denote by \( \mathcal{MF}_{[a,b],f}^{\nabla}(\mathcal{X}/W) \) the category of Fontaine-Faltings modules with endomorphism structure of \( W(\mathbb{F}_p) \) whose Hodge-Tate weights lie in \([a,b]\). And denote by \( \mathcal{MF}_{[0,p-2],f}^{\nabla}(X_{n+1}/W_{n+1}) \) the subcategory of \( \mathcal{MF}_{[0,p-2],f}^{\nabla}(\mathcal{X}/W) \) consisted by strict \( p^n \)-torsion objects.

**Lemma 1.2.** Assume \( f \) is a positive integer with \( \mathbb{F}_p \subset k \). Then giving an object in \( \mathcal{MF}_{[0,p-2],f}^{\nabla}(\mathcal{X}/W) \) is equivalent to give \( f \)-ordered objects

\[(V_i, \nabla_i, \Fil_i) \in \text{MCF}_{p-2}(X_n), \quad i = 0, 1, \cdots, f - 1 \]

(for some \( n \in \mathbb{N} \)) together with isomorphisms of de Rham sheaves

\[
\varphi_i : C_n^{-1} \circ \text{Gr}(V_i, \nabla_i, \Fil_i) \to (V_{i+1}, \nabla_{i+1}), \quad \text{for } 0 \leq i \leq f - 2,
\]

and

\[
\varphi_{f-1} : C_n^{-1} \circ \text{Gr}(V_{f-1}, \nabla_{f-1}, \Fil_{f-1}) \to (V_0, \nabla_0).
\]
Proof. Let \((V, \nabla, \Fil, \varphi, \iota)\) be an object in \(\mathcal{M}F_{\smallsetimes, 0, n-2, f}(X/W)\). Let \(\sigma\) be the Frobenius map on \(W(p^f)\) and let \(\xi\) be a generator of \(W(p^f)\) as a \(\mathbb{Z}_p\)-algebra. Then \(\iota(\xi)\) is an endomorphism of the Fontaine-Faltings module \((V, \nabla, \Fil, \varphi)\). Since \(\mathbb{F}_p^f \subset k\), all conjugate elements of \(\xi\) are of form \(\sigma^i(\xi)\), which are contained in \(w = w(K)\). The filtered de Rham sheaf \((V, \nabla, \Fil)\) can be decomposed into eigenspaces
\[
(V, \nabla, \Fil) = \bigoplus_{i=0}^{f-1} (V_i, \nabla_i, \Fil_i),
\]
where \((V_i, \nabla_i, \Fil_i) = (V, \nabla, \Fil)^{\iota(\xi) = \sigma^i(\xi)}\) is the \(\sigma^i(\xi)\)-eigenspace of \(\iota(\xi)\). Applying \(C_{n-1} \circ \overline{\text{Gr}}\) on both side, we get
\[
C_{n-1} \circ \overline{\text{Gr}}(V, \nabla, \Fil) = \bigoplus_{i=0}^{f-1} C_{n-1} \circ \overline{\text{Gr}}(V_i, \nabla_i, \Fil_i).
\]
Comparing \(\sigma^{i+1}(\xi)\)-eigenspaces of \(\iota(\xi)\) on both side of
\[
\varphi : C_{n-1} \circ \overline{\text{Gr}}(V, \nabla, \Fil) \simeq (V, \nabla),
\]
one gets the restrictive isomorphisms
\[
\varphi_i : C_{n-1} \circ \overline{\text{Gr}}(V_i, \nabla_i, \Fil_i) \rightarrow (V_{i+1}, \nabla_{i+1}), \text{ for all } 0 \leq i \leq f - 2,
\]
and
\[
\varphi_{f-1} : C_{n-1} \circ \overline{\text{Gr}}(V_{f-1}, \nabla_{f-1}, \Fil_{f-1}) \rightarrow (V_0, \nabla_0).
\]
Conversely, we can construct the Fontaine-Faltings module with endomorphism structure in an obvious way. \(\square\)

1.5. Twisted Fontaine-Faltings modules with endomorphism structure. Let \(L_n\) be a line bundle over \(X_n\). Then there is a natural connection \(\nabla_{\text{can}}\) on \(L_n^{p^n}\) by 5.1.1 in [10]. Tensoring with \((L_n^{p^n}, \nabla_{\text{can}})\) induces a self equivalence functor on the category of de Rham bundles over \(X_n\).

**Definition 1.3.** An \(L_n\)-twisted Fontaine-Faltings module over \(X_n\) with endomorphism structure of \(W_n(p^f)\) whose Hodge-Tate weights lie in \([a, b]\) is a tuple consisting the following data:

- for \(0 \leq i \leq f-1\), a filtered de Rham bundle \((V_i, \nabla_i, \Fil_i)\) over \(X_n\) of level in \([a, b]\);
- for \(0 \leq i \leq f-2\), an isomorphism of de Rham sheaves
  \[
  \varphi_i : C_{n-1} \circ \overline{\text{Gr}}(V_i, \nabla_i, \Fil_i) \rightarrow (V_{i+1}, \nabla_{i+1});
  \]
- an isomorphism of de Rham sheaves
  \[
  \varphi_{f-1} : (C_{n-1} \circ \overline{\text{Gr}}(V_{f-1}, \nabla_{f-1}, \Fil_{f-1})) \otimes (L_n^{p^n}, \nabla_{\text{can}}) \rightarrow (V_0, \nabla_0).
  \]

We use \((V_i, \nabla_i, \Fil_i, \varphi_i)_{0 \leq i \leq f}\) to denote the \(L_n\)-twisted Fontaine-Faltings module and use \(\mathcal{T}MF_{a, b, f}(X_{n+1}/W_{n+1})\) to denote the category of all twisted Fontaine-Faltings modules over \(X_n\) with endomorphism structure of \(W_n(p^f)\) whose Hodge-Tate weights lie in \([a, b]\).
A morphism between two objects \((V_i, \nabla_i, \Fil_i, \varphi_i)_{0 \leq i < f}\) and \((V_i', \nabla_i', \Fil_i', \varphi_i')_{0 \leq i < f}\) is an \(f\)-tuple \((g_0, g_1, \cdots, g_{f-1})\) of morphisms of filtered de Rham sheaves 

\[ g_i : (V_i, \nabla_i, \Fil_i, \varphi_i) \to (V_i', \nabla_i', \Fil_i'), \quad i = 0, 1, \cdots, f - 1 \]
satisfying 

\[ g_{i+1} \circ \varphi_i = \varphi_i' \circ (C_n^{-1} \circ \Gr(g_i)) , \quad \text{for } 0 \leq i \leq f - 2, \]

and 

\[ \left( g_0 \otimes L_n^\alpha \right) \circ \varphi_{f-1} = \varphi_{f-1} \circ (C_n^{-1} \circ \Gr(g_{f-1})) . \]

Remark. i). By Lemma 1.1 to give an object in \(\cT MF^V_{[a,b],1}(X_n/W_n)\) with \(L_n = \O_{X_n}\) is equivalent to give a strict \(p^n\)-torsion Fontaine-Faltings module over \(X_n\) whose Hodge-Tate weights lie in \([a,b]\).

ii). Suppose \(\mathbb{F}_{p^n} \subset k\). By Lemma 1.2 to give an object in \(\cT MF^V_{[a,b],f}(X_n/W_n)\) with \(L_n = \O_{X_n}\) is equivalent to give a strict \(p^n\)-torsion Fontaine-Faltings module over \(X_n\) with endomorphism structure of \(W_n(\mathbb{F}_{p^n})\) and whose Hodge-Tate weight in \([a,b]\).

Local trivialization. Let \(j \in I\). Locally on the small open affine set \(U_j\) \((R_j = \O_X(U_j))\), we choose and fix a lifting \(\Phi_j : \hat{R}_j \to \hat{R}_j\) and a trivialization of the line bundle \(L_n\)

\[ \tau_j : \O_{X_n}(U_j) \simeq L_n(U_j). \]

It induces a trivialization of flat bundle \(\tau_j^{p^n} : (\O_{X_n}(U), \nabla) \simeq (L_n^{p^n}(U), \nabla_{\text{can}})\)

Let \(M = (V_i, \nabla_i, \Fil_i, \varphi_i)_{0 \leq i < f} \in \cT MF^V_{[a,b],f}(X_{n+1}/W_{n+1})\) be an \(L_n\)-twisted Fontaine-Faltings module over \(X_n\) with endomorphism structure of \(W_n(\mathbb{F}_{p^n})\) whose Hodge-Tate weights lie in \([a,b]\). Then one gets a local Fontaine-Faltings module over \(R_j\) with endomorphism structure of \(W_n(\mathbb{F}_{p^n})\) whose Hodge-Tate weights lie in \([a,b]\]

\[ M(\tau_j) = \left( \oplus V_i(U_j), \oplus \nabla_i, \oplus \Fil_i, \bigoplus_{i=0}^{f-2} \varphi_i + \varphi_{f-1} \circ (\id \otimes \tau_j^{p^n}) \right) . \]

We call \(M(\tau_j)\) the trivialization of \(M\) on \(U_j\) via \(\tau_j\).

Logarithmic version. Finally, let us mention that everything in this section extends to the logarithmic context. Let \(\mathcal{X}\) be a smooth and proper scheme over \(W\) and \(\mathcal{X}^o\) is the complement of a simple normal crossing divisor \(\mathcal{D} \subset \mathcal{X}\) relative to \(W\). Similarly, one constructs the category \(\cT MF^V_{[a,b],f}(X_{n+1}/W_{n+1})\) of strict \(p^n\)-torsion twisted logarithmic Fontaine modules (with pole along \(\mathcal{D} \times W_n \subset \mathcal{X} \times W_n\)) with endomorphism structure of \(W_n(\mathbb{F}_{p^n})\) whose Hodge-Tate weights lie in \([a,b]\).

2. Projective Fontaine-Laffie-Faltings functor

2.1. The Fontaine-Laffie-Faltings’ \(\mathbb{D}\)-functor.
The functor $\mathbb{D}_\Phi$. Let $R$ be a small affine algebra over $W = W(k)$ with a $\sigma$-linear map $\Phi : \hat{R} \to \hat{R}$ which lifts the absolute Frobenius of $R/pR$. If $\Phi$ happens to be étale in characteristic 0, Faltings (page 36 of [4]) constructed a map $\kappa_\Phi : \hat{R} \to B^+(\hat{R})$ which respects Frobenius-lifts. Thus the following diagram commutes

$$\begin{array}{c}
\hat{R} \\
\Phi \\
\hat{R}
\end{array} \xrightarrow{\kappa_\Phi} \begin{array}{c}
B^+(\hat{R}) \\
\Phi_B \\
B^+(\hat{R}).
\end{array}$$

(2.1)

Here $\Phi_B$ is the Frobenius on $B^+(\hat{R})$. Denote $D = B^+(\hat{R})[1/p]/B^+(\hat{R})$, which is equipped with the natural $\varphi$-structure and filtration.

Let $M = (V, \nabla, \text{Fil}, \varphi)$ be an object in $\mathcal{MF}_{[a,b]}(\mathcal{U}/W)$. Faltings constructed an operator $\mathbb{D}_\Phi$ by

$$\mathbb{D}_\Phi(M) = \text{Hom}(V \otimes_{\kappa_\Phi} B^+(\hat{R}), D),$$

where the homomorphisms are $B^+(\hat{R})$-linear and respect filtrations and the $\varphi$-structure. The action of $\text{Gal}(\hat{R}/\hat{R})$ on the tensor product $V \otimes_{\kappa_\Phi} B^+(\hat{R})$ is defined via the connection on $V$, which commutes with the $\varphi$'s and hence induces an action of $\text{Gal}(\hat{R}/\hat{R})$ on $\mathbb{D}_\Phi(M)$. Since $V$ is a $p$-power torsion finitely generated $R$-module, $\mathbb{D}_\Phi(M)$ is a finite $\mathbb{Z}_p$-module. Faltings shows that the functor $\mathbb{D}_\Phi$ from $\mathcal{MF}_{[a,b]}(\mathcal{U}/W)$ to $\text{Rep}_{\mathbb{Z}_p}^\text{finite}(\text{Gal}(\hat{R}/\hat{R}))$, the category of finite $\mathbb{Z}_p$-representation of $\text{Gal}(\hat{R}/\hat{R})$, is fully faithful and its image is closed under subobjects and quotients.

The functor $\mathbb{D}$. Recall that $I$ is the index set of all pair $(\mathcal{U}_i, \Phi_i)$ of small affine open subset $\mathcal{U}_i$ of $\mathcal{X}$ and lifting $\Phi_i$ of the absolute Frobenius on $\mathcal{O}_{\mathcal{X}}(\mathcal{U}_i) \otimes Wk$. For each $i \in I$, the functor $\mathbb{D}_\Phi$, associates to any Fontaine-Faltings module over $\mathcal{X}$ a compatible system of étale sheaves on $\hat{U}_{i,K}$ (the generic fiber of $\hat{U}_i$). By gluing and using the results in EGA3, one obtains a locally constant sheaf on $\mathcal{X}_K$ and a globally defined functor $\mathbb{D}$.

In the following, we give a slightly different way to construct the functor $\mathbb{D}$. Let $J$ be a finite subset of the index set $I$, such that $\{\mathcal{U}_j\}_{j \in J}$ forms a covering of $\mathcal{X}$. Denote $U_j = (\mathcal{U}_j)_K$ and choose $\pi$ a geometric point of $\mathcal{X}_K$ contained in $\bigcap_{j \in J} U_j$.

Let $(V, \nabla, \text{Fil}, \{\varphi_i\}_{i \in I})$ be a Fontaine-Faltings module over $\mathcal{X}$. For each $j \in J$, the functor $\mathbb{D}_\Phi$ gives us a finite $\mathbb{Z}_p$-representation of $\pi^1(\hat{U}_j, \pi)$. Recall that the functor $\mathbb{D}_\Phi$ does not depends on the choice of $\Phi$, up to a canonical isomorphism. In particular, for all $j_1, j_2 \in J$, there is a natural isomorphism of $\pi^1(\hat{U}_{j_1} \cap \hat{U}_{j_2}, s)$-representations

$$\mathbb{D}(V(\mathcal{U}_{j_1} \cap \mathcal{U}_{j_2}), \nabla, \text{Fil}, \varphi_{j_1}) \simeq \mathbb{D}(V(\mathcal{U}_{j_1} \cap \mathcal{U}_{j_2}), \nabla, \text{Fil}, \varphi_{j_2}).$$
By Theorem 2.6 all representations \( \mathbb{D}(V(U_j), \nabla, \Fil, \varphi_j) \)'s descend to a \( \mathbb{Z}_p \)-representations of \( \pi^1(X_K, \overline{\mathbb{F}}) \). Up to a canonical isomorphism, this representation does not depend on the choice of \( J \) and \( s \). This representation is just \( \mathbb{D}(V, \nabla, \Fil, \{ \varphi_i \}_{i \in I}) \) and we construct the Fontaine-Laffaille-Faltings’ \( \mathbb{D} \)-functor in this way.

**Theorem 2.1** (Faltings). The functor \( \mathbb{D} \) induces an equivalence of the category \( \mathcal{M}_R^{[0, p-2]}(X/W) \) with the full subcategory of finite \( \mathbb{Z}_p[\pi^1(X_K)] \)-modules whose objects are dual-crystalline representations. This subcategory is closed under sub-objects and quotients.

The extra \( W(\mathbb{F}_{p^f}) \)-structure. Suppose \( \mathbb{F}_{p^f} \subset k \). Let \( (V, \nabla, \Fil, \varphi, \iota) \) be an object in \( \mathcal{M}_R^{[0, p-2]}(X/W) \). Since the functor \( \mathbb{D} \) is fully faithful, we get an extra \( W(\mathbb{F}_{p^f}) \)-structure on \( \mathbb{D}(V, \nabla, \Fil, \varphi) \), via the composition

\[
W(\mathbb{F}_{p^f}) \xrightarrow{\sim} \End_{\mathcal{M}_F}(V, \nabla, \Fil, \varphi) \xrightarrow{\sim} \End(\mathbb{D}(V, \nabla, \Fil, \varphi)).
\]

Since \( V \) is strictly \( p^f \)-torsion, the \( W_n(\mathbb{F}_{p^f}) \)-module \( \mathbb{D}(V, \nabla, \Fil, \varphi) \) is free with a linear action of \( \pi^1(X_K) \). We write this \( W_n(\mathbb{F}_{p^f}) \)-representation as

\[
\mathbb{D}(V, \nabla, \Fil, \varphi, \iota).
\]

### 2.2. The category of projective representations.

The categories \( \Rep_O(G) \) and \( \Rep_O^{\text{free}}(G) \). Let \( O \) be a commutative topological ring with identity and let \( G \) be a topological group. Note that all morphisms of topological groups and all actions of groups are continuous in this section. Denote by \( \Rep_O(G) \) the category of all finitely generated \( O \)-modules with an action of \( G \) and denote by \( \Rep_O^{\text{free}}(G) \) the subcategory of all free \( O \)-modules of finite rank with an action of \( G \).

The categories \( \PRep_O(G) \) and \( \PRep_O^{\text{free}}(G) \). For a finitely generated \( O \)-module \( \mathbb{V} \), we denote by \( \text{PGL}_O(\mathbb{V}) \) the quotient group \( \text{GL}(\mathbb{V})/\mathbb{O}^\times \). If \( \rho : G \to \text{PGL}_O(\mathbb{V}) \) is a group morphism, then there exists a group action of \( G \) on the quotient set \( \mathbb{V}/\mathbb{O}^\times \) defined by \( g([v]) := [\rho(g)v] \) for any \( g \in G \) and \( v \in \mathbb{V} \).

In this case, we call the pair \( (\mathbb{V}, \rho) \) a projective \( O \)-representation of \( G \). A morphism of projective \( O \)-representations from \( (\mathbb{V}_1, \rho_1) \) to \( (\mathbb{V}_2, \rho_2) \) is an \( O \)-linear morphism \( f : \mathbb{V}_1 \to \mathbb{V}_2 \) such that the quotient map from \( \mathbb{V}_1/\mathbb{O}^\times \) to \( \mathbb{V}_2/\mathbb{O}^\times \) induced by \( f \) is a morphism of \( G \)-sets. Denote \( \PRep_O(G) \) the category of finite projective \( O \)-representations of \( G \). Denote by \( \PRep_O^{\text{free}}(G) \) the subcategory with \( \mathbb{V} \) being a free \( O \)-module.

### 2.3. Gluing representations and projective representations.

Let \( S \) be an irreducible scheme. We fix a geometric point \( s \) of \( S \). In this section, \( U \) is an open subset of \( S \) containing \( s \).

**Proposition 2.2** (SGA1 [3], see also Proposition 5.5.4 in [23]). The open immersion \( U \to S \) induces a surjective morphism of fundamental groups

\[
\rho^S_U : \pi_1(U, s) \to \pi_1(S, s).
\]
Thus, there is a natural restriction functor \( \text{res} \) from the category of \( \pi_1(S, s) \)-sets to the category of \( \pi_1(U, s) \)-sets, which is given by
\[
\text{res}(\rho) = \rho \circ \rho^S_U.
\]

**Corollary 2.3.** The restriction functor \( \text{res} \) is fully faithful.

The proof of this corollary directly follows from the surjectivity proved in Proposition 2.2 and Lemma 52.4.1 in [22, Tag 0BN6].

Let \( \tilde{S} \) be a finite étale covering of \( S \). Then there is a natural action of \( \pi_1(S, s) \) on the fiber \( F_s(\tilde{S}) \).

**Proposition 2.4.** i). The fiber functor \( F_s \) induces an equivalence from the category of finite étale coverings of \( S \) to the category of \( \pi_1(S, s) \)-sets.

ii). The functor \( F_s \) is compatible with the restrictions of covering to open set \( U \subset S \) and restrictions of \( \pi_1(S, s) \)-sets to \( \pi_1(U, s) \)-sets by \( \rho^S_U \).

See Proposition 52.3.10 in [22, Tag 0BN6] for a proof of the first statement. The second one follows the very definition, one can find the proof in 5.1 of [18].

As a consequence, one has the following result, which should be well-known for the experts. We still give a proof for the reader’s convenience.

**Corollary 2.5 (Rigid).** The restriction functor \( \cdot|_U \) from the category of finite étale coverings of \( S \) to the category of finite étale coverings of \( U \) is fully faithful. Suppose that there is an isomorphism \( f_U : \tilde{S}'|_U \to \tilde{S}|_U \) of finite étale coverings of \( U \), for some finite étale coverings \( \tilde{S} \) and \( \tilde{S}' \) of \( S \). Then there is a unique isomorphism \( f_S : \tilde{S}' \to \tilde{S} \) of finite étale coverings of \( S \), such that \( f_U = f_S|_U \).

**Proof.** According to Proposition 2.4, we have the following commutative diagram, with two bijective horizontal maps \( F_s \).

\[
\begin{array}{ccc}
\text{Hom}_S(\tilde{S}', \tilde{S}) & \overset{F_s}{\longrightarrow} & \text{Hom}_{\pi_1(S, s)}(\Sigma', \Sigma) \\
\downarrow{\cdot|_U} & & \downarrow{\text{res}} \\
\text{Hom}_U(\tilde{S}'|_U, \tilde{S}|_U) & \overset{F_s}{\longrightarrow} & \text{Hom}_{\pi_1(U, s)}(\Sigma', \Sigma)
\end{array}
\]

(2.2)

The fully faithful of the restriction functor follows from Corollary 2.3. Under a fully faithful functor, a morphism is an isomorphism if and only if its image under this functor is an isomorphism. So the corollary 2.5 follows.

In the following, we fix a finite index set \( J \) and an open covering \( \{U_j\}_{j \in J} \) of \( S \) with \( s \in \bigcap_j U_j \). Then for any \( j \in J \), the inclusion map \( U_j \to S \) induces a surjective group morphism of fundamental groups
\[
\tau_j : \pi_1(U_j, s) \twoheadrightarrow \pi_1(S, s).
\]
Denote \( U_{j_1} := U_{j_1,j_2} \cdots := U_{j_1} \cap U_{j_2} \cap \cdots \cap U_{j_r} \) for any \( J_1 = \{ j_1, j_2, \ldots, j_r \} \subset J \). Similarly, for any \( J_1 \subset J_2 \subset J \), we have a surjective group morphism of fundamental groups

\[
\tau_{J_2}^{J_1} : \pi_1(U_{J_2}, s) \twoheadrightarrow \pi_1(U_{J_1}, s).
\]

Now we can view every \( \pi_1(U_{J_1}, s) \)-set as a \( \pi_1(U_{J_2}, s) \)-set through this group morphism.

**Theorem 2.6.** Let \((\Sigma_j, \rho_j)\) be a finite \( \pi_1(U_j, s) \)-set for each \( j \in J \). Suppose for each pair \( i, j \in J \), there exists an isomorphism of \( \pi_1(U_{ij}, s) \)-sets \( \eta_{ij} : \Sigma_i \simeq \Sigma_j \). Then every \( \Sigma_j \) descends to a \( \pi_1(S, s) \)-set \((\Sigma_j, \tilde{\rho}_j)\) uniquely. Moreover, the image of \( \rho_j \) equals that of \( \tilde{\rho}_j \).

**Proof.** Fix \( j_0 \in J \). One has an isomorphism of \( \pi_1(U_{j_0}, s) \)-sets \( \eta_{jj_0} : \Sigma_j \simeq \Sigma_{j_0} \). As \( F_s \) is an equivalent functor, there is a covering of \( \tilde{U}_j \) of \( U_j \) with isomorphism \( \eta_j : F_s(\tilde{U}_j) \to \Sigma_j \) of \( \pi_1(U_j, s) \)-sets for each \( j \in J \). Denote

\[
\overline{\tau}_j = \eta_{j_0}^{-1} \circ \eta_{jj_0} \circ \eta_j.
\]

The \( \eta_j \) and \( \eta_{jj_0} \) are \( \pi_1(U_j, s) \)-isomorphisms, so do \( \overline{\tau}_j \). Denote

\[
\overline{\tau}_{j_1, j_2} = \overline{\tau}_{j_2} \circ \overline{\tau}_{j_1},
\]

which is also a \( \pi_1(U_j, s) \)-isomorphisms. One has

\[
\overline{\tau}_{j_0, j_1} \circ \overline{\tau}_{j_2, j_3} \circ \overline{\tau}_{j_1, j_2} = \text{id}_{F_s(\tilde{U}_{j_1})}.
\]

The equivalence of \( F_s \) over \( U_j \) induces the following commutative diagram of finite étale coverings of \( U_j \).
In particular, for all \( j_1, j_2, j_3 \in J \),
\[
F_s^{-1}(\mathcal{F}_{j_3,j_1}) \circ F_s^{-1}(\mathcal{F}_{j_2,j_3}) \circ F_s^{-1}(\mathcal{F}_{j_1,j_2}) = \text{id}_{U_{j_1} \mid U_{j_2}}.
\]  
(2.3)

By Corollary 2.5, there is a unique isomorphism of finite étale coverings of \( U_{j_1,j_2} \)
\[
f_{j_1,j_2} : \tilde{U}_{j_1} \mid U_{j_1,j_2} \to \tilde{U}_{j_1} \mid U_{j_1,j_2},
\]
such that \( f_{j_1,j_2} \mid U_{j} = F_s^{-1}(\mathcal{F}_{j_1,j_2}) \). By (2.3), we have
\[
(f_{j_3,j_1} \mid U_{j_1,j_2,j_3}) \circ (f_{j_2,j_3} \mid U_{j_1,j_2,j_3}) \circ (f_{j_1,j_2} \mid U_{j_1,j_2,j_3}) \mid U_{j} = \text{id} \left( (\tilde{\mathcal{U}}_{j_1})_{U_{j_1,j_2,j_3}} \right)_{U_{j}}.
\]

Using Corollary 2.5 once again, one has
\[
f_{j_3,j_1} \mid U_{j_1,j_2,j_3} \circ f_{j_2,j_3} \mid U_{j_1,j_2,j_3} \circ f_{j_1,j_2} \mid U_{j_1,j_2,j_3} = \text{id} \left( (\tilde{\mathcal{U}}_{j_1})_{U_{j_1,j_2,j_3}} \right)_{U_{j}}.
\]

So one can glue \( \{ \tilde{U}_j \}_{j \in J} \) by isomorphisms \( \{ f_{ij} \}_{i,j \in J} \) into a finite étale covering \( \tilde{S} \) of \( S \). Applying the fiber functor \( F_s \) on the structure isomorphisms \( f_j : \tilde{S} \mid U_j \cong \tilde{U}_j \), one gets \( \pi_1(U_j, s) \)-isomorphisms \( F_s(f_j) : F_s(\tilde{S}) \cong F_s(\tilde{U}_j) \). The bijections \( F_s(f_j) \) and \( \eta_j \) give us isomorphisms of permutation groups
\[
\text{Aut}(F_s(\tilde{S})) \cong \text{Aut}(F_s(\tilde{U}_j)) \cong \text{Aut}(\Sigma_j).
\]

Since the \( F_s(f_j) \) and \( \eta_j \) are isomorphisms of \( \pi_1(U_j, s) \)-sets, the following diagram commutes
\[
\begin{array}{ccc}
\pi_1(U_j, s) & \xrightarrow{\tau_j} & \pi_1(U_j, s) \\
\downarrow \rho_j & & \downarrow \rho_j \\
\pi_1(S, s) & \xrightarrow{\rho_j} & \pi_1(S, s) \\
\end{array}
\]
(2.4)

Let \( \hat{\rho}_j \) denote the composition
\[
\pi_1(S, s) \to \text{Aut}(F_s(\tilde{S})) \cong \text{Aut}(F_s(\tilde{U}_j)) \cong \text{Aut}(\Sigma_j).
\]

The commutativity of diagram (2.4) means that \( \rho_i \) descends to \( \hat{\rho}_j \). Other statements can be easily deduced from the surjectivity of \( \tau_j \) and \( \tau_{ij} \). \( \square \)

2.4. Comparing representations associated to local Fontaine-Faltings modules underlying isomorphic filtered de Rham sheaves.

In this section we compare several representations associated to local Fontaine-Faltings modules underlying isomorphic filtered de Rham sheaves. To do so, we first introduce a local Fontaine-Faltings module, which corresponds to a \( W_{\mu}(\mathbb{F}_p) \)-character of the local fundamental group. We will then use this character to measure the difference of the associated representations.
Let $R$ be a small affine algebra over $W(k)$ and denote $R_n = R/p^nR$ for all $n \geq 1$. Fix a lifting $\Phi : \hat{R} \to \hat{R}$ of the absolute Frobenius on $R/pR$. Recall that $\kappa_\Phi : \hat{R} \to B^+(\hat{R})$ is the lifting of $B^+(\hat{R})/F^1B^+(\hat{R}) \simeq \hat{R}$ with respect to the $\Phi$. Under such a lifting, the Frobenius $\Phi_B$ on $B^+(\hat{R})$ extends to $\Phi$ on $\hat{R}$.

Element $a_{n,r}$. Let $f$ be a positive integer. For any $r \in \hat{R}^\times$, we construct a Fontaine-Faltings module of rank $f$ as following. Let $V = R_n e_0 \oplus R_n e_1 \oplus \cdots \oplus R_n e_{f-1}$ be a free $R_n$-module of rank $f$. The integrable connection $\nabla$ on $V$ is defined by formula

$$\nabla(e_i) = 0,$$

and the filtration Fil on $V$ is the trivial one. Applying the tilde functor and twisting by the map $\Phi$, one gets

$$\tilde{V} \otimes \Phi \hat{R} = R_n \cdot (\tilde{e}_0 \otimes \Phi 1) \oplus R_n \cdot (\tilde{e}_1 \otimes \Phi 1) \oplus \cdots \oplus R_n \cdot (\tilde{e}_{f-1} \otimes \Phi 1),$$

where the connection on $\tilde{V} \otimes \Phi \hat{R}$ is determined by

$$\nabla(\tilde{e}_i \otimes \Phi 1) = 0.$$

Denote by $\varphi$ the $R_n$-linear map from $(\tilde{V} \otimes \Phi \hat{R}, \nabla)$ to $(V, \nabla)$

$$\varphi(\tilde{e}_0 \otimes 1, \tilde{e}_1 \otimes 1, \cdots, \tilde{e}_{f-1} \otimes 1) = (e_0, e_1, \cdots, e_{f-1}) \begin{pmatrix} 0 & & & \cr & 1 & 0 & \cr & & 1 & 0 \cr \cdots & \cdots & \cdots & \cdots \cr & & & 1 \end{pmatrix}.$$  

The $\varphi$ is parallel due to $d(r^{p^n}) \equiv 0 \pmod{p^n}$. By lemma 1.1, the tuple $(V, \nabla, \text{Fil}, \varphi)$ forms a Fontaine-Faltings module. Applying Fontaine-Laffaille-Faltings’ functor $\mathbb{D}_\Phi$, one gets a finite $\mathbb{Z}_p$-representation of $\text{Gal}(\hat{R}/\hat{R})$, which is a free $\mathbb{Z}/p^n\mathbb{Z}$-module of rank $f$.

**Lemma 2.7.** Let $n$ and $f$ be two positive integers and let $r$ be an invertible element in $R$. Then there exists an $a_{n,r} \in B^+(\hat{R})^\times$ such that

$$\Phi_B^f(a_{n,r}) \equiv \kappa_\Phi(r)^{p^n} \cdot a_{n,r} \pmod{p^n}. \quad (2.5)$$

**Proof.** Since $\mathbb{D}_\Phi(V, \nabla, \text{Fil}, \varphi)$ is free over $\mathbb{Z}/p^n\mathbb{Z}$ of rank $f$, one can find an element $g$ with order $p^n$. Recall that $\mathbb{D}_\Phi(V, \nabla, \text{Fil}, \varphi)$ is the sub-$\mathbb{Z}_p$-module of $\text{Hom}_{B^+(\hat{R})}(V \otimes \kappa_\Phi B^+(\hat{R}), D)$ consisted by elements respecting the filtration.
and \( \varphi \). In particular, the following diagram commutes

\[
\begin{array}{ccc}
\left( \mathcal{V} \otimes_{\kappa_{\Phi}} B^+(\widehat{R}) \right) \otimes_{\Phi} B^+(\widehat{R}) & \xrightarrow{g \otimes \text{id}} & D \otimes_{\Phi} B^+(\widehat{R}) \\
\downarrow \varphi \otimes \text{id} & & \downarrow \approx \\
V \otimes_{\kappa_{\Phi}} B^+(\widehat{R}) & \xrightarrow{g} & D
\end{array}
\]  

Comparing images of \((e_i \otimes_{\kappa_{\Phi}} 1) \otimes_{\Phi} 1\) under the diagram, we have

\[\Phi(g(v_i \otimes_{\kappa_{\Phi}} 1)) = g(v_{i+1} \otimes_{\kappa_{\Phi}} 1), \quad \text{for all } 0 \leq i \leq f - 2;\]

and

\[\Phi(g(v_{f-1} \otimes_{\kappa_{\Phi}} 1)) = \kappa_{\Phi}(r) p^n \cdot g(v_0 \otimes_{\kappa_{\Phi}} 1).\]

So we have

\[\Phi^f(g(v_0 \otimes_{\kappa_{\Phi}} 1)) = \kappa_{\Phi}(r) p^n \cdot g(v_0 \otimes_{\kappa_{\Phi}} 1).\]

Since the image of \(g\) is \(p^n\)-torsion, \(\text{Im}(g)\) is contained in \(D[p^n] = \frac{1}{p^n} B^+(\widehat{R})/B^+(\widehat{R})\), the \(p^n\)-torsion part of \(D\). Choose a lifting \(a_{n,r}\) of \(g(e_0 \otimes_{\kappa_{\Phi}} 1)\) under the surjective map \(B^+(\widehat{R}) \xrightarrow{1} D[p^n]\). Then the equation \((2.5)\) follows. Similarly, one can define \(a_{n,r-1}\) for \(r^{-1}\). By equation \((2.5)\), we have

\[\Phi^f(a_{n,r} \cdot a_{n,r-1}) = a_{n,r} \cdot a_{n,r-1}.\]

Thus \(a_{n,r} \cdot a_{n,r-1} \in W(\mathbb{F}_p)\). Since both \(a_{n,r}\) and \(a_{n,r-1}\) are not divided by \(p\) (by the choice of \(g\)), we know that \(a_{n,r} \cdot a_{n,r-1} \in W(\mathbb{F}_p)^{\times}\). The invertibility of \(a_{n,r}\) follows. \(\square\)

Comparing representations. Let \(n\) and \(f\) be two positive integers. For all \(0 \leq i \leq f\), let \((V_i, \nabla_i, \text{Fil}_i)\) be filtered de Rham \(R_n = R/p^n R\)-modules of level \(a\) \((a \leq p - 1)\). We write \(V = \bigoplus V_i, \nabla = \bigoplus \nabla_i\) and \(\text{Fil} = \bigoplus \text{Fil}_i\) for short. Let

\[\begin{align*}
\varphi_i : C^{-1} & \circ \text{Gr}(V_i, \nabla_i, \text{Fil}_i) \simeq (V_{i+1}, \nabla_{i+1}), & 0 \leq i \leq f - 2 \\
\varphi_{f-1} : C^{-1} & \circ \text{Gr}(V_{f-1}, \nabla_{f-1}, \text{Fil}_{f-1}) \simeq (V_0, \nabla_0)
\end{align*}\]

be isomorphisms of de Rham \(R\)-modules. Let \(r\) be an element in \(R^\times\). Since \(d(rp^n) = 0 \pmod {p^n}\), the map \(r p^n \varphi_{f-1}\) is also an isomorphism of de Rham \(R_n\)-modules. Thus

\[M = (V, \nabla, \text{Fil}, \varphi)\]

and \(M' = (V, \nabla, \text{Fil}, \varphi')\) are Fontaine-Faltings modules over \(R_n\), where \(\varphi = \sum_{i=0}^{f-1} \varphi_i\) and \(\varphi' = \sum_{i=0}^{f-2} \varphi_i + r p^n \varphi_{f-1}\).
Proposition 2.8.  

(i) There are $W_n(\mathbb{F}_{p'})$-module structures on $\mathbb{D}_\Phi(M)$ and $\mathbb{D}_\Phi(M')$. And the actions of $\text{Gal}(\widehat{\mathbb{R}}/\widehat{\mathbb{R}})$ are semi-linear.

(ii) The multiplication of $a_{n,r}$ on $\text{Hom}_{B^+(\widehat{\mathbb{R}})}(V \otimes_{\kappa_{\Phi}} B^+(\widehat{\mathbb{R}}), D)$ induces a $W_n(\mathbb{F}_{p'})$-linear map between these two submodules.

Proof.  

(i). We only give the $W_n(\mathbb{F}_{p'})$-linear structure on $\mathbb{D}_\Phi(M)$. Let $g: \bigoplus_{i=0}^{f-1} V_i \otimes_{\kappa_{\Phi}} B^+(\widehat{\mathbb{R}}) \to D$ be an element in $\mathbb{D}_\Phi(M)$. For all $a \in W_n(\mathbb{F}_{p'})$, define

$$a \cdot g = \sum_{i=0}^{f-1} \sigma^i(a) g_i,$$

where $g_i$ is the restriction of $g$ on the $i$-th component $V_i(U_j) \otimes_{\kappa_{\Phi}} B^+(\widehat{\mathbb{R}}_j)$. One checks that $a \cdot g$ is also contained in $\mathbb{D}_\Phi_j(M(\tau_j))$. Let $\delta$ be an element in $\text{Gal}(\widehat{\mathbb{R}}/\widehat{\mathbb{R}})$. Then

$$\delta(a \cdot g) = \delta \circ \left( \sum_{i=0}^{f-1} \sigma^i(a) g_i \right) \circ \delta^{-1}$$

$$= \sum_{i=0}^{f-1} \sigma^i(\delta(a)) \delta \circ g_i \circ \delta^{-1}$$

$$= \delta(a) \cdot \delta(g) \quad (2.8)$$

In this way, $\mathbb{D}_\Phi_j(M(\tau_j))$ forms a $W_n(\mathbb{F}_{p'})$-module with a continuous semi-linear action of $\pi_1(U_K)$.

(ii). Recall that $\mathbb{D}_\Phi(M)$ (resp. $\mathbb{D}_\Phi(M')$) is defined to be the set of all morphisms in $\text{Hom}_{B^+(\widehat{\mathbb{R}})}(V \otimes_{\kappa_{\Phi}} B^+(\widehat{\mathbb{R}}), D)$ compatible with the filtration and $\varphi$ (resp. $\varphi'$). Comparing the rank of $\mathbb{D}_\Phi(M)$ and $\mathbb{D}_\Phi(M')$, we only need to show that $a_{n,r} \cdot f \in \mathbb{D}_\Phi(M')$ for all $f \in \mathbb{D}_\Phi(M)$. Suppose $f: V \otimes_{\kappa_{\Phi}} B^+(\widehat{\mathbb{R}}) \to D$ is an element in $\mathbb{D}_\Phi(M)$, which means that $f$ satisfies the following two conditions:

1. $f$ is strict for the filtrations. i.e.

$$\sum_{\ell_1 + \ell_2 = \ell} \text{Fil}^{\ell_1} V \otimes_{\kappa_{\Phi}} \text{Fil}^{\ell_2} B^+(\widehat{\mathbb{R}}) = f^{-1}(\text{Fil}^\ell D).$$
2). $f \otimes \Phi \text{id} = f \circ (\varphi \otimes \kappa_{\varphi} \text{id})$. i.e. the following diagram commutes

\[
\begin{array}{ccc}
W \otimes_{\kappa_{\varphi}} B^+ (\hat{R}) & \otimes_{\Phi} B^+ (\hat{R}) & D \otimes_{\Phi} B^+ (\hat{R}) \\
\downarrow & \downarrow & \downarrow \\
W \otimes_{\Phi} \hat{R} & \otimes_{\kappa_{\varphi}} B^+ (\hat{R}) & \approx \\
\varphi_{n,r} \otimes \text{id} & & \\
V \otimes_{\kappa_{\varphi}} B^+ (\hat{R}) & f & D
\end{array}
\]

Since $a_{n,r} \in B^+(\hat{R}) \subset \text{Fil}^0 B^+(\hat{R}) \setminus \text{Fil}^1 B^+(\hat{R})$, we have $a_{n,r} \cdot \text{Fil}^\ell D = \text{Fil}^\ell D$, and thus

\[
\sum_{\ell_1 + \ell_2 = \ell} \text{Fil}^{\ell_1} W \otimes_{\kappa_{\varphi}} \text{Fil}^{\ell_2} B^+ (\hat{R}) = f^{-1} (\text{Fil}^\ell D) = (a_{n,r} \cdot f)^{-1} (\text{Fil}^\ell D).
\]

Simultaneously, we have

\[
(a_{n,r} \cdot f) \otimes \Phi \text{id} = f \otimes \Phi (a_{n,r}) \cdot \text{id}
\]

\[
= f \otimes a_{n,r} \cdot \kappa_{\varphi}(r)^{p^n} \cdot \text{id} = (a_{n,r} \cdot \kappa_{\varphi}(r)^{p^n}) \cdot (f \otimes \Phi \text{id})
\]

\[
= a_{n,r} \cdot \kappa_{\varphi}(r)^{p^n} \cdot (f \circ (\varphi \otimes \kappa_{\varphi} \text{id}))
\]

\[
= (a_{n,r} \cdot f) \circ (r^{p^n} \varphi \otimes \kappa_{\varphi} \text{id})
\]

So by definition $a_{n,r} \cdot f \in \mathcal{D}_\Phi(M')$.

**Corollary 2.9.** Suppose that $\mathbb{F}_{p^f} \subset k$. The map from $\mathcal{D}_\Phi(M)$ to $\mathcal{D}_\Phi(M')$ is an isomorphism of projective $W_n(\mathbb{F}_{p^f})$-representations of $\text{Gal}(\overline{R}/\hat{R})$. In particular, we have an bijection of $\text{Gal}(\overline{R}/\hat{R})$-sets

\[
\mathcal{D}_\Phi(M)/W_n(\mathbb{F}_{p^f})^\times \to \mathcal{D}_\Phi(M'/M)/W_n(\mathbb{F}_{p^f})^\times.
\]

**2.5. The functor $\mathcal{D}^p$.** In this section, we assume $f$ to be a positive integer with $\mathbb{F}_{p^f} \subset k$. Let $\{U_j\}_{j \in J}$ be a finite small affine open covering of $X$. Let $U_j = (U_j)_K$. For every $j \in J$, fix $\Phi_j$ as a lifting of the absolute Frobenius on $U_j \otimes_k \mathbb{F}_{p^f}$. Fix $\overline{x}$ as a geometric point in $U_j = \bigcap_{j \in J} U_j$ and fix $j_0$ an element in $J$.

Let $(V, \nabla, \text{Fil}, \varphi, t)$ be a Fontaine-Faltings module over $X_n$ with an endomorphism structure of $W(\mathbb{F}_{p^f})$ whose Hodge-Tate weights lie in $[0, p-2]$. Locally, Applying Fontaine-Laffaille-Faltings’ functor $\mathcal{D}_{\Phi,j}$, one gets a finite $W_n(\mathbb{F}_{p^f})$-representation $\vartheta_j$ of $\pi_1(U_j, \overline{x})$. Faltings shows that there is an isomorphism $\vartheta_{j_1} \simeq \vartheta_{j_2}$ of $\mathbb{Z}/p^n\mathbb{Z}$-representations of $\pi_1(U_{j_1,j_2}, \overline{x})$. By Lan-Sheng-Zuo [12], this isomorphism is $W_n(\mathbb{F}_{p^f})$-linear. By Theorem [2.6], these $\vartheta_j$‘s uniquely descend to a $W_n(\mathbb{F}_{p^f})$-representation of $\pi_1(X_K, \overline{x})$. Thus one reconstructs the $W_n(\mathbb{F}_{p^f})$-representation $\mathcal{D}(V, \nabla, \text{Fil}, \varphi, t)$ in this way.
Now we construct functor $\mathbb{D}^P$ for twisted Fontaine-Faltings modules, in a similar way. Let $(V_i, \nabla_i, \Fil_i, \varphi_i)_{0 \leq i < f} \in \mathcal{T}\mathcal{M}\mathcal{F}^\nabla_{[0, p-2]}(X_{n+1}/W_{n+1})$ be an $L_n$-twisted Fontaine-Faltings module over $X_n$ with endomorphism structure of $W_n(\mathbb{F}_{p^f})$ whose Hodge-Tate weights lie in $[0, p-2]$. For each $j \in J$, choosing a trivialization $M(\tau_j)$ and applying Fontaine-Laffaille-Faltings’ functor $\mathbb{D}_\Phi$, we get a $W_n(\mathbb{F}_{p^f})$-module together with a linear action of $\pi_1(U_j, \varpi)$. Denote its projectification by $\rho_j$. By Corollary 2.9 there is an isomorphism $\rho_{j_1} \simeq \rho_{j_2}$ as projective $W_n(\mathbb{F}_{p^f})$-representations of $\pi_1(U_{j_1,j_2}, \varpi)$. In what follows, we will show that these $\rho_j$’s uniquely descend to a projective $W_n(\mathbb{F}_{p^f})$-representation of $\pi_1(\mathcal{X}_K, \varpi)$ by using Theorem 2.6. In order to use Theorem 2.6, set $\Sigma_j$ to be the quotient $\pi_1(U_j, \varpi)$-set $\mathbb{D}_\Phi(\tau_j)/W_n(\mathbb{F}_{p^f})^\times$.

Obviously the kernel of the canonical group morphism

$$GL(\mathbb{D}_\Phi(\tau_j))/W_n(\mathbb{F}_{p^f})^\times \rightarrow Aut(\Sigma_j)$$

is just $W_n(\mathbb{F}_{p^f})^\times$, we identify the image of this morphism with

$$PGL(\mathbb{D}_\Phi(\tau_j))/W_n(\mathbb{F}_{p^f})^\times.$$ 

Let’s denote by $\rho_j$ the composition of $\rho_j$ and $GL(\mathbb{D}_\Phi(\tau_j))/W_n(\mathbb{F}_{p^f})^\times \rightarrow Aut(\Sigma_j)$ for all $j \in J$.

By Corollary 2.9 the restrictions of $(\Sigma_{j_1}, \rho_{j_1})$ and $(\Sigma_{j_2}, \rho_{j_2})$ on $\pi_1(U_{j_1,j_2}, \varpi)$ are isomorphic for all $j_1, j_2 \in J$. Hence by Theorem 2.6 the map $\rho_{j_0}$ descends to some $\tilde{\rho}_{j_0}$ and the image of $\tilde{\rho}_{j_0}$ is contained in $PGL(\mathbb{D}_\Phi(\tau_{j_0}))/\pi_1(U_{j_0}, \varpi)$. So the projective $W_n(\mathbb{F}_{p^f})$-representation $(\mathbb{D}_\Phi(\tau_{j_0}), \rho_{j_0})$ of $\pi_1(U_{j_0}, \varpi)$ descends to projective representation $(\mathbb{D}_\Phi(\tau_{j_0}), \tilde{\rho}_{j_0})$ of $\pi_1(\mathcal{X}_K, \varpi)$. Up to a canonical isomorphism, this projective representation does not depend on the choices of the covering $\{U_j\}_{j \in J}$, the liftings $\Phi_j$’s and $j_0$. And we denote this projective $W_n(\mathbb{F}_{p^f})$-representation of $\pi_1(\mathcal{X}_K, \varpi)$ by

$$\mathbb{D}^P\left((V_i, \nabla_i, \Fil_i, \varphi_i)_{0 \leq i < f}\right).$$

Similarly as Faltings’s functor $\mathbb{D}$ in [4], our construction of the $\mathbb{D}^P$ functor can also be extended to the logarithmic version. More precisely, let $\mathcal{X}$ be a smooth and proper scheme over $W$ and let $\mathcal{X}^{\circ}$ be the complement of a simple normal crossing divisor $\mathcal{D} \subset \mathcal{X}$ relative to $W$. Similarly, by replacing
\( X_K \) and \( U_j \) with \( X_K^0 = X_K \) and \( U_j^0 \), we construct the functor

\[
D^P : \mathcal{T}MF_{\{0, p-2\}, f}(X_{n+1}/W_{n+1}) \to \text{PRep}_{\text{free}}^{W_n(\mathbb{F}_{p^f})}(\pi_1(X_K^0)) \tag{2.10}
\]

from the category of strict \( p^n \)-torsion twisted logarithmic Fontaine modules (with pole along \( D \times W_n \subset X \times W_n \)) with endomorphism structure of \( W_n(\mathbb{F}_{p^f}) \) whose Hodge-Tate weights lie in \([0, p-2]\) to the category of free \( W_n(\mathbb{F}_{p^f}) \)-modules with projective actions of \( \pi_1(X_K^0) \).

Summarizing this section, we get the following result.

**Theorem 2.10.** Let \( M \) be a twisted logarithmic Fontaine-Faltings module over \( X \) (with pole along \( D \)) with endomorphism structure of \( W(\mathbb{F}_{p^f}) \). The \( D^P \)-functor associates to \( M \) and its endomorphism structure a projective representation

\[ \rho : \pi_1(X_K^0) \to \text{PGL}(D^P(M)), \]

where \( X_K^0 \) is the generic fiber of \( X^0 = X \setminus D \).

### 3. Twisted periodic Higgs-de Rham flows

In this section, we will recall the definition of periodic Higgs-de Rham flows and generalize it to the twisted version.

**3.1. Higgs-de Rham flow over \( X_n \subset X_{n+1} \).** Recall [12] that a Higgs-de Rham flow over \( X_n \subset X_{n+1} \) is a sequence consisting of infinitely many alternating terms of filtered de Rham bundles and Higgs bundles

\[
\left\{ (V, \nabla, \text{Fil})_{-1}^{(n-1)}, (E, \theta)^{(n)}_0, (V, \nabla, \text{Fil})_{0}^{(n)}, (E, \theta)^{(n)}_1, (V, \nabla, \text{Fil})_{1}^{(n)}, \cdots \right\},
\]

which are related to each other by the following diagram inductively

where
- \((V, \nabla, \text{Fil})_{-1}^{(n-1)}\) is a filtered de Rham bundle over \( X_{n-1} \) of level in \([0, p-2]\);
- \((E, \theta)^{(n)}_0\) is a lifting of the graded Higgs bundle \( \text{Gr} \left( (V, \nabla, \text{Fil})_{-1}^{(n-1)} \right) \) over \( X_n \), \((V, \nabla)_0^{(n)} := C_{n-1}((E, \theta)^{(n)}_0, (V, \nabla, \text{Fil})_{-1}^{(n-1)}, \psi) \) and \( \text{Fil}_0^{(n)} \) is a Hodge filtration on \((V, \nabla)_0^{(n)}\) of level in \([0, p-2]\);
Let $\text{Definition 3.1}$. And for any $\text{Remark.}$ In case $n = 1$, the data of $(V, \nabla, \text{Fil})_{(n-1)}$ is empty. The Higgs-de Rham flow can be rewritten in the following form
\[ \left\{ (E, \theta)^{(1)}_0, (V, \nabla, \text{Fil})_0^{(1)}, (E, \theta)^{(1)}_1, (V, \nabla, \text{Fil})_1^{(1)}, \ldots \right\}. \]
In this way, the diagram becomes

\[ \begin{array}{ccc}
(E, \theta)_0^{(1)} & \xrightarrow{c_1^{-1}} & (V, \nabla, \text{Fil})_0^{(1)} \\
\downarrow{\phi} & & \downarrow{G_r} \\
(E, \theta)_1^{(1)} & \xrightarrow{c_1^{-1}} & (V, \nabla, \text{Fil})_1^{(1)}
\end{array} \]

In the rest of this section, we will give the definition of twisted periodic Higgs-de Rham flow and equivalent categories. Let $L_\ell$ be a line bundle over $X_n$. For all $1 \leq \ell < n$, denote $L_\ell = L_n \otimes O_{X_\ell}$. In this subsection, let $a \leq p - 2$ be a positive integer. We will give the definition of $L_n$-twisted Higgs-de Rham flow of level in $[0, a]$. 3.2. Twisted periodic Higgs-de Rham flow and equivalent categories. Let $L_n$ be a line bundle over $X_n$. For all $1 \leq \ell < n$, denote $L_\ell = L_n \otimes O_{X_\ell}$ the reduction of $L_n$ on $X_\ell$. In this subsection, let $a \leq p - 2$ be a positive integer. We will give the definition of $L_n$-twisted Higgs-de Rham flow of level in $[0, a]$. 3.2.1. Twisted periodic Higgs-de Rham flow over $X_1$.

**Definition 3.1.** Let $f$ be a positive integer. An $f$-periodic $L_1$-twisted Higgs-de Rham flow over $X_1 \subset X_2$ of level in $[0, a]$, is a Higgs-de Rham flow over $X_1$

\[ \left\{ (E, \theta)^{(1)}_0, (V, \nabla, \text{Fil})_0^{(1)}, (E, \theta)^{(1)}_1, (V, \nabla, \text{Fil})_1^{(1)}, \ldots \right\} \]

together with isomorphisms $\phi^{(1)}_{f+i} : (E, \theta)^{(1)}_f \otimes (L_1^p, 0) \rightarrow (E, \theta)^{(1)}_i$ of Higgs bundles for all $i \geq 0$.

\[ \begin{array}{ccc}
(E, \theta)_0^{(1)} & \xrightarrow{c_1^{-1}} & (V, \nabla, \text{Fil})_0^{(1)} \\
\downarrow{\phi} & & \downarrow{G_r} \\
(E, \theta)_1^{(1)} & \xrightarrow{c_1^{-1}} & (V, \nabla, \text{Fil})_1^{(1)}
\end{array} \]

And for any $i \geq 0$ the isomorphism

\[ C_1^{-1}(\phi^{(1)}_{f+i}) : (V, \nabla)^{(1)}_f \otimes (L_1^p, \nabla_{\text{can}}) \rightarrow (V, \nabla)^{(1)}_i, \]
strictly respects filtrations Fil_{f,i}^{(1)} and Fil_{f+i}^{(1)}. Those \( \phi_{f+i}^{(1)} \)'s are relative to each other by formula

\[
\phi_{f+i+1}^{(1)} = \text{Gr} \circ C_{f+1}^{-1}(\phi_{f+i}^{(1)}).
\]

Denote the category of all twisted \( f \)-periodic Higgs-de Rham flow over \( X_1 \) of level in \([0, a]\) by \( \mathcal{HDF}_{a,f}(X_2/W_2) \).

### 3.2.2. Twisted periodic Higgs-de Rham flow

Let \( n \geq 2 \) be an integer and \( f \) be a positive integer. And \( L_n \) is a line bundle over \( X_n \). Denote by \( L_\ell \) the reduction of \( L_n \) modulo \( p^\ell \). We define the category \( \mathcal{HDF}_{a,f}(X_{n+1}/W_{n+1}) \) of all \( f \)-periodic twisted Higgs-de Rham flow over \( X_n \subset X_{n+1} \) of level in \([0, a]\) in the following inductive way.

**Definition 3.2.** An \( L_n \)-twisted \( f \)-periodic Higgs-de Rham flow over \( X_n \subset X_{n+1} \) is a Higgs-de Rham flow

\[
\left\{(V, \nabla, \text{Fil})^{(n-1)}_{n-2}, (E, \theta)^{(n)}_{n-1}, (V, \nabla, \text{Fil})^{(n)}_{n-1}, (E, \theta)^{(n)}_{n}, \ldots \right\}_{X_n \subset X_{n+1}}
\]

which is a lifting of an \( L_{n-1} \)-twisted \( f \)-periodic Higgs-de Rham flow

\[
\left\{(E, \theta)^{(1)}_{0}, \ldots, (E, \theta)^{(1)}_{1}, \ldots, \phi_1^{(1)} \right\}_{X_1 \subset X_2}
\]

It is constructed by the following diagram for \( 2 \leq \ell \leq n \), inductively

![Diagram](image)

Here
- \( (E, \theta)^{(1)}_{\ell-1}/X_\ell \) is a lifting of \( (E, \theta)^{(\ell-1)}_{\ell-1}/X_{\ell-1} \), which implies automatically \((V, \nabla)^{(1)}_{\ell-1} := C_{\ell-1}^{-1} \left((E, \theta)^{(1)}_{\ell-1}, (V, \nabla, \text{Fil})^{(\ell-1)}_{\ell-2}, \text{id}\right)\) is a lifting of \((V, \nabla)^{(\ell-1)}_{\ell-1}\) since \( C_{\ell-1}^{-1} \) is a lifting of \( C_{\ell-1}^{-1} \).
- \( \text{Fil}_{\ell-1}^{(1)} \subset (V, \nabla)^{(1)}_{\ell-1} \) is a lifting of the Hodge filtration \( \text{Fil}_{\ell-1}^{(1)} \subset (V, \nabla)^{(\ell-1)}_{\ell-1} \), which implies that \( (E, \theta)^{(1)}_{\ell} = \text{Gr} \left((V, \nabla, \text{Fil})^{(1)}_{\ell-1}\right)/X_\ell \) is a lifting of \( (E, \theta)^{(\ell-1)}_{\ell-1}/X_{\ell-1} \).
and \((V, \nabla)^{(\ell)}_k := C_{\ell-1}^{-1}((E, \theta)^{(\ell)}_k, (V, \nabla, \text{Fil})^{(\ell-1)}_k, \text{id})\).

- Repeating the process above, one gets the data \(\text{Fil}^{(\ell)}_i, (E, \theta)^{(\ell)}_{i+1}\) and \((V, \nabla)^{(\ell)}_{i+1}\) for all \(i \geq \ell\).
- Finally, for all \(i \geq \ell - 1\), \(\phi_{i+f}^{(\ell)} : (E, \theta)^{(\ell)}_{i+f} \to (E, \theta)^{(\ell)}_i\) is a lifting of \(\phi_{i+f}^{(\ell-1)}\). And these morphisms are related to each other by formula \(\phi_{i+f+1} = \text{Gr} \circ C_{\ell-1}^{-1}(\phi_{i+f}^{(\ell)})\). Denote the twisted periodic Higgs-de Rham flow by

\[
\{(V, \nabla, \text{Fil})^{(n-1)}_{n-2}, (E, \theta)^{(n)}_{n-1}, (V, \nabla, \text{Fil})^{(n)}_{n-1}, (E, \theta)^{(n)}_n, \ldots ; \phi^{(n)}\}_{/X_n \subset X_{n+1}}
\]

The category of all periodic twisted Higgs-de Rham flow over \(X_n \subset X_{n+1}\) of level in \([0, a]\) is denoted by \(\mathcal{T}HDF_{a,f}(X_{n+1}/W_{n+1})\).

**Remark.** For the trivial line bundle \(L_n\), the definition above is equivalent to the original definition of periodic Higgs-de Rham flow in [12] by using the identification \(\phi : (E, \theta)_0 = (E, \theta)_f\).

Note that we can also define the logarithmic version of the twisted periodic Higgs-de Rham flow, since we already have the log version of inverse Cartier transform. \(X\) is a smooth proper scheme over \(W\) and \(\mathcal{X}\) is the complement of a simple normal crossing divisor \(\mathcal{D} \subset \mathcal{X}\) relative to \(W\). Similarly, one constructs the category \(\mathcal{T}HDF_{a,f}(X_{n+1}/W_{n+1})\) of twisted \(f\)-periodic logarithmic Higgs-de Rham flows (with pole along \(D \times W_n \subset X \times W_n\)) over \(X \times W_n\) whose nilpotent exponents are \(\leq p - 2\).

### 3.2.3. Equivalence of categories.
We establish an equivalence of categories between \(\mathcal{T}HDF_{a,f}(X_{n+1}/W_{n+1})\) and \(\mathcal{T}MF_{[0,a],f}(X_{n+1}/W_{n+1})\).

**Theorem 3.3.** Let \(a \leq p - 1\) be a natural number and \(f\) be a positive integer. Then there is an equivalence of categories between \(\mathcal{T}HDF_{a,f}(X_{n+1}/W_{n+1})\) and \(\mathcal{T}MF_{[0,a],f}(X_{n+1}/W_{n+1})\).

**Proof.** Let

\[
\mathcal{E} = \left\{(V, \nabla, \text{Fil})^{(n-1)}_{n-2}, (E, \theta)^{(n)}_{n-1}, (V, \nabla, \text{Fil})^{(n)}_{n-1}, (E, \theta)^{(n)}_n, \ldots ; \phi^{(n)}\right\}_{/X_n \subset X_{n+1}}
\]

be an \(f\)-periodic \(L_n\)-twisted Higgs-de Rham flow over \(X_n\) with level in \([0, a]\). Taking out \(f\) terms of filtered de Rham bundles

\[
(V, \nabla, \text{Fil})^{(n)}_0, (V, \nabla, \text{Fil})^{(n)}_1, \ldots, (V, \nabla, \text{Fil})^{(n)}_{f-1}
\]

together with \(f - 1\) terms of identities maps

\[
\varphi_i : C_n^{-1} \circ \text{Gr} \left( (V, \nabla, \text{Fil})^{(n)}_i \right) = (V, \nabla)^{(n)}_{i+1}, \quad i = 0, 1, \ldots, f - 2,
\]

and \(\varphi_{f-1} := C_n^{-1}\left(\phi^{(n)}_f\right)\), one gets a tuple

\[
\mathcal{IC}(\mathcal{E}) := \left\{(V^{(n)}_i, \nabla^{(n)}_i, \text{Fil}^{(n)}_i, \varphi_i)_{0 \leq i < f}\right\},
\]
This tuple forms an $L_n$-twisted Fontaine-Faltings module by definition. It gives us the functor $\mathcal{F}$ from $\mathcal{T}HF_{0,f}(X_{n+1}/W_{n+1})$ to $\mathcal{T}MF_{[0,n]}^\nabla(X_{n+1}/W_{n+1})$.

Conversely, let $(V_i, \nabla_i, \Fil_i, \varphi_i)_{0 \leq i < f}$ be an $L_n$-twisted Fontaine-Faltings module. For $0 \leq i \leq f - 2$, we identify $(V_{i+1}, \nabla_{i+1})$ with $C^{-1}_n \circ \Gr(V_i, \nabla_i, \Fil_i)$ via $\varphi_i$. We construct the corresponding flow by induction on $n$.

In case $n = 1$, we already have following diagram

\[
\begin{array}{c}
\xymatrix{ (V, \nabla, \Fil)_{0} & (V, \nabla, \Fil)_{2} & \cdots & (V, \nabla, \Fil)_{f-1} & (V, \nabla)_{f} \\
(E, \theta)_{0} & (E, \theta)_{2} & \cdots & (E, \theta)_{f-1} & (E, \theta)_{f} }
\end{array}
\]

(3.1)

Denote $(E, \theta)_0 = (E, \theta)_f \otimes (L_1, 0)$. Then

\[
C^{-1}_1(E_0, \theta_0) \simeq (V_f, \nabla_f) \otimes (L^p_1, \nabla_{\text{can}}) \simeq (V_0, \nabla_0).
\]

By this isomorphism, we identify $(V_0, \nabla_0)$ with $C^{-1}_1(E_0, \theta_0)$. Under this isomorphism, the Hodge filtration $\Fil_0$ induces a Hodge filtration $\Fil_f$ on $(V_f, \nabla_f)$.

Take Grading and denote

\[
(E_{f+1}, \theta_{f+1}) := \Gr(V_f, \nabla_f, \Fil_f).
\]

Inductively, for $i > f$, we denote $(V_i, \nabla_i) = C^{-1}_1(E_i, \theta_i)$. By the isomorphism

\[
(C^{-1}_1 \circ \Gr)^{-f} (\varphi_{f-1}) : (V_i, \nabla_i) \otimes (L_{i+1}^{p-1}, \nabla_{\text{can}}) \to (V_{i-f}, \nabla_{i-f}),
\]

the Hodge filtration $\Fil_{i-f}$ induces a Hodge filtration $\Fil_i$ on $(V_i, \nabla_i)$. Denote

\[
(E_{i+1}, \nabla_{i+1}) := \Gr(V_i, \nabla_i, \Fil_i).
\]

Then we extend above diagram into the following twisted periodic Higgs-de Rham flow over $X_1$

\[
\begin{array}{c}
\xymatrix{ (V, \nabla, \Fil)_{0} & (V, \nabla, \Fil)_{1} & \cdots & (V, \nabla)_{i} & (V, \nabla)_{i+1} \\
(E, \theta)_0 & (E, \theta)_1 & \cdots & (E, \theta)_i & (E, \theta)_{i+1} }
\end{array}
\]

(3.2)

For $n \geq 2$, denote

\[
(\nabla_{-1}, \nabla_{-1}, \Fil_{-1}) := (\nabla_{f-1} \otimes L^{p-1}_{n-1}, \nabla_{f-1} \otimes \nabla_{\text{can}}, \Fil_{f-1} \otimes \Fil_{f-1}),
\]

where $(\nabla_{f-1}, \nabla_{f-1}, \Fil_{f-1})$ is the modulo $p^{n-1}$ reduction of $(V_{f-1}, \nabla_{f-1}, \Fil_{f-1})$. Those $\varphi_i$ reduce to a $\varphi$-structure on $(\nabla_i, \nabla_i, \Fil_i)_{1 \leq i < f-1}$. This gives us a $L_{n-1}$-twisted Fontaine-Faltings module over $X_{n-1}$

\[
(\nabla_i, \nabla_i, \Fil_i, \varphi_i)_{1 \leq i < f-1}
\]
By induction, we have a twisted periodic Higgs-de Rham flow over $X_{n-1}$

$$
\begin{array}{cccc}
\tau_1 & \cdots & \tau_{f-1} \\
\tau_1 & \cdots & \tau_{f-1} \\
\tau_1 & \cdots & \tau_{f-1} \\
\end{array}
$$

where the first $f$-terms of filtered de Rham bundles over $X_{n-1}$ are those appeared in the twisted Fontaine-Faltings module over $X_{n-1}$.

Based on this flow over $X_{n-1}$, we extend the diagram similarly as the $n = 1$ case,

$$
\begin{array}{cccc}
(V, \nabla, \Fil)_0 & \cdots & (V, \nabla, \Fil)_{f-1} & (V, \nabla) \\
(E, \theta) & \cdots & (E, \theta)_{f-1} & (E, \theta) \\
\end{array}
$$

Now it is a twisted periodic Higgs-de Rham flow over $X_n$. Denote this flow by

$$\mathcal{GR}((V_i, \nabla_i, \Fil_i, \varphi_i)_{0 \leq i < f}).$$

It is straightforward to verify $\mathcal{GR} \circ \mathcal{IC} \simeq \text{id}$ and $\mathcal{IC} \circ \mathcal{GR} \simeq \text{id}$. □

This Theorem can be straightforwardly generalized to the logarithmic case and the proof is similar as that of Theorem 3.3.

**Theorem 3.4.** Let $X$ be a smooth proper scheme over $W$ with a simple normal crossings divisor $D \subset X$ relative to $W$. Then for each natural number $f \in \mathbb{N}$, there is an equivalence of categories between $\mathcal{T}HF_{a.f}(X^0_{n+1}/W_{n+1})$ and $\mathcal{T}MF_{[0,f]}(X^0_{n+1}/W_{n+1})$.

3.2.4. A sufficient condition for lifting the twisted periodic Higgs-de Rham flow. We suppose that the field $k$ is finite in this section. Let $X$ be a smooth proper variety over $W(k)$ and denote $X_n = X \times_{W(k)} W_n(k)$. Let $D_1 \subset X_1$ be a $W(k)$-liftable normal crossing divisor over $k$. Let $D \subset X$ be a lifting of $D_1$.

**Proposition 3.5.** Let $n$ be an positive integer and let $L_{n+1}$ be a line bundle over $X_{n+1}$. Denote by $L_\ell$ the reduction of $L_{n+1}$ on $X_\ell$. Let

$$\{(V, \nabla, \Fil)^{(n-1)}_{n-2}, (E, \theta)^{(n)}_{n-1}, (V, \nabla, \Fil)^{(n)}_{n-1}, (E, \theta)^{(n)}_n, \cdots, \phi^{(n)}_n\} /_{X_n \subset X_{n+1}}$$

be an $L_n$-twisted periodic Higgs-de Rham flow over $X_n \subset X_{n+1}$. Suppose

- Lifting of the graded Higgs bundle $(E, \theta)^{(n)}_i$ is unobstructed. i.e. there exist a logarithmic graded Higgs bundle $(E, \theta)^{(n+1)}_i$ over $X_{n+1}$, whose reduction on $X_n$ is isomorphic to $(E, \theta)^{(n)}_i$. 

Lifting of the Hodge filtration $\text{Fil}_i^{(n)}$ is unobstructed, i.e., for any lifting $(V, \nabla)_i^{(n+1)}$ of $(V, \nabla)_i^{(n)}$ over $X_{n+1}$, there exists a Hodge filtration $\text{Fil}_i^{(n+1)}$ on $(V, \nabla)_i^{(n+1)}$, whose reduction on $X_n$ is $\text{Fil}_i^{(n)}$.

Then every twisted periodic Higgs-de Rham flow over $X_n$ can be lifted to a twisted periodic Higgs-de Rham flow over $X_{n+1}$.

Proof. By assumption, we choose $(E', \theta')_n^{(n+1)}$ a lifting of $(E', \theta')_n^{(n)}$. Inductively, for all $i \geq n$, we construct $(V', \nabla', \text{Fil}'_i^{(n+1)})$ and $(E', \theta')_i^{(n+1)}$ as follows. Denote

$$(V', \nabla')_i^{(n+1)} = C_{n+1}^{-1} \left((E', \theta')_i^{(n+1)}\right),$$

which is a lifting of $(V, \nabla)_i^{(n)}$. By assumption, we choose a lifting $\text{Fil}'_i^{(n+1)}$ on $(V', \nabla')_i^{(n+1)}$ of the Hodge filtration $\text{Fil}_i^{(n)}$ and denote

$$(E', \theta')_i^{(n+1)} = \text{Gr}(V', \nabla', \text{Fil}'_i^{(n+1)}),$$

which is a lifting of $(E, \theta)_i^{(n)}$.

From the $\phi$-structure of the Higgs-de Rham flow, for all $m \geq 0$ there is an isomorphism

$$(E, \theta)_n^{(n)} \simeq (E, \theta)_n^{(n+1)} \otimes L_{n+1}^{p_{n+1} + p_{n+2} + \cdots + p_{n+m-2}}.$$

Twisting $(E', \theta')_n^{(n+1)}$ with $L_{n+1}^{p_{n+1} + p_{n+2} + \cdots + p_{n+m-2}}$, one gets a lifting of $(E, \theta)_n^{(n)}$.

By deformation theory, the lifting space of $(E, \theta)_n^{(n)}$ is a torsor space modeled by $H^1_{\text{Hig}}(X_1, \text{End}((E, \theta)_1^{(1)}))$. Therefore, the torsor space of lifting $(E, \theta)_n^{(n)}$ as a graded Higgs bundle should be modeled by a subspace of $H^1_{\text{Hig}}$. We give a description of this subspace as follows. For simplicity of notations, we shall replace $(E, \theta)_n^{(1)}$ by $(E, \theta)$ in this paragraph. The decomposition of $E = \bigoplus_{p+q=n} E^{p,q}$ induces a decomposition of $\text{End}(E)$:

$$(\text{End}(E))^{k, -k} := \bigoplus_{p+q=n} (E^{p,q})^\vee \otimes E^{p+q, q-k}$$

Furthermore, it also induces a decomposition of the Higgs complex $\text{End}(E, \theta)$. One can prove that the hypercohomology of the following Higgs subcomplex

$$H^1((\text{End}(E))^{0,0} \xrightarrow{\theta_{\text{End}}} (\text{End}(E))^{1,1} \otimes \Omega^1 \xrightarrow{\theta_{\text{End}}} \cdots) \quad (3.4)$$

gives the subspace corresponding to the lifting space of graded Higgs bundles.

Thus by the finiteness of the torsor space, there are two integers $m > m' \geq 0$, such that

$$(E', \theta')_{n+m}^{(n+1)} \otimes L_{n+1}^{p_{n+1} + p_{n+2} + \cdots + p_{n+m-2}} \simeq (E', \theta')_{n+m'}^{(n+1)} \otimes L_{n+1}^{p_{n+1} + p_{n+2} + \cdots + p_{n+m'-2}}.$$

(3.5)
By twisting suitable power of the line bundle $L_{n+1}$ we may assume $m' = 0$. By replacing the period $f$ with $mf$, we may assume $m = 1$. For integer $i \in [n, n + f - 1]$ we denote

$$(E, \theta, V, \nabla, \Fil)^{(n+1)}_i := (E', \theta', V', \nabla', \Fil')^{(n+1)}_i.$$  

Then (3.5) can be rewritten as

$$(3.5) \text{ can be rewritten as}$$  

$$(E, \theta, V, \nabla, \Fil)^{(n+1)}_i := (E', \theta', V', \nabla', \Fil')^{(n+1)}_i.$$  

Then (3.6) can be rewritten as

$$(3.6) \text{ can be rewritten as}$$  

$$\phi^{(n+1)}_{n+f} : (E, \theta)^{(n+1)}_{n+f} \otimes L_{n+1}^{p_{n+1} - p + p^{n+f-2}} \rightarrow (E, \theta)^{(n+1)}_n$$  

where $(E, \theta)^{(n+1)}_{n+f} = (E', \theta')^{(n+1)}_{n+f} = \Gr \left( (V, \nabla, \Fil)^{(n+1)}_{n+f-1} \right)$. Inductively, for all $i \geq n + f$, we construct $(V, \nabla, \Fil)^{(n+1)}_i, (E, \theta)^{(n+1)}_{i+1}$ and $\phi^{(n+1)}_{i+1}$ as follows. Denote

$$(V, \nabla)^{(n+1)}_i = C^{-1}_{n+1} \left( (E, \theta)^{(n+1)}_i \right).$$  

According to the isomorphism

$$(3.7) \text{ the Hodge filtration Fil}_{i-f}^{(n+1)} on (V, \nabla)^{(n+1)}_i \text{ induces a Hodge filtration Fil}_{i-f}^{(n+1)} \text{ on } (V, \nabla)^{(n+1)}_i.$$

Taking grading on equation (3.7), one gets a lifting of $\phi^{(n+1)}_{i+1}$

$$\phi^{(n+1)}_{i+1} : (E, \theta)^{(n+1)}_{i+1} \otimes L_{i+1}^{p_{i+1} - p + p^{i+1}} \rightarrow (E, \theta)^{(n+1)}_{i+1-f}$$  

and a twisted Higgs-de Rham flow over $X_{n+1} \subset X_{n+2}$

$$\left\{ (V, \nabla, \Fil)^{(n)}_{n-1}, (E, \theta)^{(n)}_n, (V, \nabla, \Fil)^{(n+1)}_n, (E, \theta)^{(n+1)}_n, \ldots ; \phi^{(n+1)} \right\} / X_{n+1} \subset X_{n+2}$$

which lifts the given twisted periodic flow over $X_n \subset X_{n+1}$. \hfill \Box

Remark. In the proof we see that one needs to enlarge the period for lifting the twisted periodic Higgs-de Rham flow.

3.3. The choice of the twisting line bundle and semi-stable Higgs bundle with trivial discriminant. Let $X_1$ be a smooth proper $W_2$-liftable variety over $k$, with dim $X_1 = n$. Let $H$ be a polarization of $X_1$. Let $r < p$ be a positive integer and $(E, \theta)_0$ be a nilpotent semistable Higgs bundle over $X_1$ of rank $r$. Recall the main result in the Appendix of [12]:

**Theorem 3.6.** There is a Higgs-de Rham flow over $X_1$ with initial term $(E, \theta)_0$.

In the construction of the Higgs-de Rham flow given by Theorem 3.6, the key step is to prove the existence of Simpson’s graded semistable Hodge filtration Fil (Theorem A.4 in [12] and Theorem 5.12 in [16]), which is the most coarse Griffiths transverse filtration on a semi-stable de Rham module such that the associated graded Higgs module is still semi-stable. Denote
\[(V, \nabla)_0 := C^{-1}_1(E_0, \theta_0)\] and \(\text{Fil}_0\) the Simpson’s graded semistable Hodge filtration on \((V, \nabla)_0\). Denote \((V, \nabla)_1 := C^{-1}_1(E_1, \theta_1)\) and \(\text{Fil}_1\) the Simpson’s graded semistable Hodge filtration on \((V, \nabla)_1\). Repeating this process, we construct a Higgs-de Rham flow over \(X_1\) with initial term \((E, \theta)\)

\[
\begin{array}{ccccccc}
(E, \theta)_0 & \xrightarrow{c^{-1}_1} & (V, \nabla, \text{Fil}_0) & \xrightarrow{c^{-1}_1} & \cdots & \xrightarrow{c^{-1}_r} & (E, \theta)_r \\
\end{array}
\]

Since the Simpson’s graded semistable Hodge filtration is unique, this flow is also uniquely determined by \((E, \theta)_0\). The purpose of this subsection is to find a canonical choice of the twisting line bundle \(L\) such that this Higgs-de Rham flow is twisted preperiodic.

Firstly, we want to find a positive integer \(f_1\) and a suitable twisting line bundle \(L_1\) such that \((E'_{f_1}, \theta'_{f_1}) := (E_{f_1}, \theta_{f_1}) \otimes (L_1, 0)\) satisfies the following conditions

- (i) \(c_1(E'_{f_1}) = c_1(E_0)\)
- (ii) \(c_2(E'_{f_1}) \cdot [H]^{n-2} = c_2(E_0) \cdot [H]^{n-2}\).

Under these two condition, both \((E, \theta)_0\) and \((E, \theta)_{f_1}\) are contained in the moduli scheme \(M^{ss}_{Higgs}(X_1/k, r, a_1, a_2)\), which is constructed by Langer in [15] and classifies all the semistable Higgs bundles over \(X_1\) with some fixed topological invariants (which will be explained later). Following [15], we introduce \(S'_{X_1/k}(d; r, a_1, a_2, \mu_{\text{max}})\) the family of Higgs sheaves over \(X_1\) such that \((E, \theta)\) is a member of the family if \(E\) is reflexive of dimension \(d\), \(\mu_{\text{max}}(E, \theta) \leq \mu_{\text{max}}(E_0) = r, a_1(E) = a_1\) and \(a_2(E) \geq a_2\). Here \(\mu_{\text{max}}(E, \theta)\) is the slope of the maximal destabilizing sub sheaf of \((E, \theta)\), and \(a_i(E)\) are defined by

\[
\chi(X_1/k, E(m)) = \sum_{i=0}^{d} a_i(E) \binom{m + d - i}{d - i}.
\]

By the results of Langer, the family \(S'_{X_1/k}(d; r, a_1, a_2, \mu_{\text{max}})\) is bounded (see Theorem 4.4 of [15]). So \(M^{ss}_{Higgs}(X_1/k, r, a_1, a_2)\) is the moduli scheme which corepresents this family. Note that \(a_1(E) = \chi(E|_{\text{Fil}_i \leq H_j})\) where \(H_1, \ldots, H_d \in |\mathcal{O}(H)|\) is an \(E\)-regular sequence (see [9]). Using Hirzebruch-Riemann-Roch theorem, one finds that \(a_1(E), a_2(E)\) will be fixed if \(c_1(E)\) and \(c_2(E) \cdot [H]^{n-2}\) are fixed.

**Proposition 3.7.** Assume discriminant of \(E_0\) (with respect to the polarization \(H)\) \(\Delta(E_0) := (c_2(E_0) - \frac{1}{2} c_1(E_0)^2) \cdot [H]^{n-2}\) equals to zero. Let \(f_1\) be the minimal positive integer with \(r \mid p^{f_1} - 1\), and let \(L_1 = \text{det}(E_0)^{1-p^{f_1}}\). Then the two conditions above are satisfied.

**Proof.** Since \(c_1(C^{-1}_1(E_0, \theta_0)) = pc_1(E_0)\) and \(c_1(L_1) = \frac{1-p^{f_1}}{r} \cdot c_1(E_0)\), we have

\[
c_1(E'_{f_1}) = r c_1(L_1) + c_1 \left((Gr \circ C^{-1}_1)(E_0, \theta_0)\right) = \left(r \cdot \frac{1-p^{f_1}}{r} + p^{f_1}\right) c_1(E_0).
\]
One gets Condition (i). Note that the discriminant $\Delta$ is invariant under twisting line bundles, and $\Delta(C_1^{-1}(E_0, \theta_0)) = p^2 \Delta(E_0)$, one gets

$$\Delta(E'_{f_1}) = \Delta(Gr \circ C_1^{-1}(E_0, \theta_0)) = p^{2f_2} \Delta(E_0) = 0.$$  

So we have $c_2(E_0) \cdot [H]^{n-2} = c_1(E_0)^2 \cdot [H]^{n-2}$ and $c_2(E'_{f_1}) \cdot [H]^{n-2} = c_1(E'_{f_1})^2$. Since $c_1(E'_{f_1}) = c_1(E_0)$, we already get Condition (ii).

**Corollary-Definition 3.8.** There is a self map $\Upsilon$ on $M_{\text{Higgs}}^{ss}(X_1/k, r, a_1, a_2)$ by sending $(E, \theta)_0$ to $(E, \theta)_f \otimes (\det(E_0)^{\frac{1-pf_1}{r}}, 0)$, where $f_1$ is the minimal positive integer with $r | p^{f_1} - 1$.

**Proposition 3.9.** Suppose that discriminant of $E_0$ equals to zero and there exists $f_2$ a positive integer with $\Upsilon^{f_2}(E_0, \theta) \simeq (E_0, \theta_0)$. Then the Higgs-de Rham flow (3.9) is $\det(E_0)^{\frac{pf_1-1}{2}}$-twisted $f$-periodic, where $f = f_1f_2$.

**Proof.** Inductively, one shows that

$$\Upsilon^m(E_0, \theta_0) = (E, \theta)_{mf_1} \otimes (\det(E_0)^{\frac{1-pmf_1}{r}}, 0).$$

(3.10)

Since $\Upsilon^{f_2}(E_0, \theta_0) \simeq (E_0, \theta_0)$, there is an isomorphism of Higgs bundles

$$\phi_f : (E_f, \theta_f) \otimes (\det(E_0)^{\frac{pf_1-1}{2}}, 0) \to (E_0, \theta_0).$$

By formula $\phi_i = (Gr \circ C_1^{-1})^{i-f}(\phi_f)$ for all $i \geq f$, we construct the twisted $\phi$-structure. Under this $\phi$-structure the Higgs-de Rham flow is $\det(E_0)^{\frac{pf_1-1}{2}}$-twisted $f$-periodic. □

**Theorem 3.10.** A semistable Higgs bundle over $X_1$ with trivial discriminant is preperiodic after twisting. Conversely, a twisted preperiodic Higgs bundle is semistable with trivial discriminant.

**Proof.** For a Higgs bundle $(E, \theta)$ in $M_{\text{Higgs}}^{ss}(X_1/k, r, a_1, a_2)$, we consider the iteration of the self-map $\Upsilon$. Since $M_{\text{Higgs}}^{ss}(X_1/k, r, a_1, a_2)$ is of finite type over $k$ and has only finitely many $k$-points, there must exist a pair of integers $(e, f_2)$ such that $\Upsilon^e(E, \theta) \cong \Upsilon^{e+f_2}(E, \theta)$. By Proposition 3.9 we know that $(E, \theta)$ is preperiodic after twisting.

Conversely, let $(E, \theta)$ be the initial term of a twisted $f$-preperiodic Higgs-de Rham flows. We show that it is semistable. Let $(F, \theta) \subset (E, \theta)$ be a proper sub bundle. Denote $(F_{i}^{1}, \theta_{i}^{1})$ and $(E_{i}^{1}, \theta_{i}^{1})$ are the terms appearing in the Higgs-de Rham flows. By the preperiodicity, there exists a line bundle $L$ and an isomorphism $\phi : (E_{i}, \theta_{i}) \cong (E_{i+f}, \theta_{i+f}) \otimes (L, 0)$. Calculating the slope on both side, one get $\mu(L) = (1 - p^f) \mu(E_i)$. Iterating $m$ times of this isomorphism $\phi$, one get

$$\phi^m : (E_{e}, \theta_{e}) \cong (E_{e+mf}, \theta_{e+mf}) \otimes (L^{1+p^f+\cdots+p^{(m-1)f}}, 0).$$
So \((\phi^m)^{-1}\left(F_{e+mf} \otimes L^{1+p^f+\cdots+p^{m-1}f}\right)\) forms a sub sheaf of \(E_e\) of slope

\[p^m \mu(F_e) + (1 + p^f + \cdots + p^{(m-1)f})\mu(L) = p^m \mu(F_e) - \mu(E_e) + \mu(E_e)\].

So \(\mu(F_e) \leq \mu(E_e)\) (otherwise there are subsheaves of \(E_e\) with unbounded slopes, but this is impossible). So we have

\[\mu(F) = \frac{1}{p^f} \mu(F_e) \leq \frac{1}{p^f} \mu(E_e) = \mu(E)\].

This shows that \((E, \theta)\) is semistable. The discriminant equals zero follows from the fact that \(\Delta(C_1^{-1}(E, \theta)) = p^2 \Delta(E)\). \(\square\)

**Corollary 3.11.** Let \((E, \theta) \supset (F, \theta)\) be the initial terms of a twisted periodic Higgs-de Rham flow and a sub twisted periodic Higgs-de Rham flow. Then

\[\mu(F) = \mu(E)\].

### 3.4. Sub-representations and sub periodic Higgs-de Rham flows.
In this section, we assume \(F_{p^f}\) is contained in \(k\). Recall that the functor \(D^P\) is contravariant and sends quotient object to subobject, i.e. for any sub twisted Fontaine-Faltings module \(N \subset M\) with endomorphism structure, the projective representation \(D^P(M/N)\) is a sub-projective representation of \(D^P(M)\). Conversely, we will show that every sub-projective representation comes from this way. By the equivalence of the category of twisted Fontaine-Faltings modules and the category of twisted periodic Higgs-de Rham flows, we construct a twisted periodic sub Higgs-de Rham flow for each sub-projective representation.

Let \(\mathcal{X}\) be a smooth proper \(W(k)\)-variety. Denote by \(X_n\) the reduction of \(\mathcal{X}\) on \(W_n(k)\). Let \(\{U_i\}_{i \in I}\) be a finite covering of small affine open subsets and we choose a geometric point \(x\) in \(\bigcap_{i \in I} U_i, \mathfrak{X}\).

**Proposition 3.12.** Let \(M\) be an object in \(\mathcal{T}_M F^\nabla_{[a,b],f}(X_2/W_2)\). Suppose we have a sub-projective \(F_{p^f}\)-representation of \(\pi^1(\mathcal{X}_K) \forall \subset D^P(M)\), then there exists a subobject \(N\) of \(M\) such that \(\forall\) equals to \(D^P(M/N)\).

**Proof.** Recall that the functor \(D^P\) is defined by gluing representations of \(\Delta_i = \pi^1(U_i, K, x)\) into a projective representation of \(\Delta = \pi^1(\mathcal{X}_K, x)\). Firstly, we show that the sub-projective representation \(\forall\) is actually corresponding to some local sub-representations. Secondly, since the Fontaine-Laffaille-Faltings’ functor \(D\) is fully faithful, there exists local Fontaine-Faltings modules corresponding to those sub-representations. Thirdly, we glue those local Fontaine-Faltings modules into a global twisted Fontaine-Faltings module. For \(i \in I\), we choose a trivialization \(M_i = \mathcal{M}(\tau_i)\) of \(M\) on \(U_i\), which gives a local Fontaine-Faltings module with endomorphism structure on \(U_i\). By definition of \(D^P\), those representations \(D_{U_i}(M_i)\) of \(\Delta_i\) are glued into the projective representation \(D^P(M)\). In other words, we have the following
commutative diagram of $\Delta_{ij} = \pi^1(U_{i,K} \cap U_{j,K}, x)$-sets

\[
\begin{array}{ccc}
\mathbb{D}_{\mathcal{U}_i}(M_i)/\mathbb{F}^\times_{p^f} & \xrightarrow{a_{1,r}} & \mathbb{D}_{\mathcal{U}_i}(M_j)/\mathbb{F}^\times_{p^f} \\
\mathbb{D}^P(M)/\mathbb{F}^\times_{p^f} & \xrightarrow{a_{1,r}} & \mathbb{D}_{\mathcal{U}_j}(M_j)/\mathbb{F}^\times_{p^f}
\end{array}
\]  

(3.11)

Here $r$ is the difference of the trivializations of the twisting line bundle on $\mathcal{U}_i$ and $\mathcal{U}_j$. And $a_{1,r}$ is the elements given in Lemma 2.7.

Assume that $\mathbb{V}$ is a sub-projective $\mathbb{F}^\times_{p^f}$-representation of $\mathbb{D}^P(M)$ of $\pi^1(X_K, x)$, i.e. $\mathbb{V}/\mathbb{F}^\times_{p^f}$ is a sub $\pi^1(X_K)$-set of $\mathbb{D}^P(M)/\mathbb{F}^\times_{p^f}$. Then $\mathbb{V}_i$, the image of $\mathbb{V}$ under the map $\mathbb{D}^P(M) \to \mathbb{D}_{\mathcal{U}_i}(M_i)$, is a sub projective $\mathbb{F}^\times_{p^f}$-representation of $\mathbb{D}_{\mathcal{U}_i}(M_i)$. So we have the following commutative diagram of $\Delta_{ij}$-sets

\[
\begin{array}{ccc}
\mathbb{V}_i/\mathbb{F}^\times_{p^f} & \xrightarrow{a_{1,r}} & \mathbb{D}_{\mathcal{U}_i}(M_i)/\mathbb{F}^\times_{p^f} \\
\mathbb{V}/\mathbb{F}^\times_{p^f} & \xrightarrow{a_{1,r}} & \mathbb{D}^P(M)/\mathbb{F}^\times_{p^f} \\
\mathbb{V}_j/\mathbb{F}^\times_{p^f} & \xrightarrow{a_{1,r}} & \mathbb{D}_{\mathcal{U}_j}(M_j)/\mathbb{F}^\times_{p^f}
\end{array}
\]  

(3.12)

Notice that $\mathbb{D}_{\mathcal{U}_i}(M_i)/\mathbb{F}^\times_{p^f}$ is the projectification of the $\mathbb{F}^\times_{p^f}$-representation $\mathbb{D}_{\mathcal{U}_i}(M_i)$ of $\Delta_i$. So $\mathbb{V}_i \subset \mathbb{D}_{\mathcal{U}_i}(M_i)$ is actually a sub $\mathbb{F}^\times_{p^f}$-representation of $\Delta_i$.

Since the image of the contravariant functor $\mathbb{D}_{\mathcal{U}_i}$ is closed under subobjects, there exists $N_i \subset M_i$ as a sub-Fontaine-Faltings module with endomorphism structure of $\mathbb{F}^\times_{p^f}$, such that

$$\mathbb{V}_i = \mathbb{D}_{\mathcal{U}_i}(M_i/N_i).$$

On the overlap $\mathcal{U}_i \cap \mathcal{U}_j$, those two Fontaine-Faltings module $M_i$ and $M_j$ have the same underlying filtered de Rham sheaf. We can twist the $\varphi$-structure of $M_i$ to get $M_j$ by the element $r$. Doing the same twisting on $N_i$, we get a sub-Fontaine-Faltings module $N_i'$ of $M_j$. By the functoriality of $\mathbb{D}$, one has the following commutative diagram

\[
\begin{array}{ccc}
\mathbb{D}(M_i/N_i) & \xrightarrow{a_{1,r}} & \mathbb{D}(N_i) \\
\mathbb{D}(M_j/N_i') & \xrightarrow{a_{1,r}} & \mathbb{D}(M_j/N_j') \\
\mathbb{D}(M_j/N_j) & \xrightarrow{a_{1,r}} & \mathbb{D}(N_j')
\end{array}
\]  

(3.13)

So we have $\mathbb{D}(M_j/N_i') = a_{1,r}\mathbb{D}(M_i/N_i) = a_{1,r}\mathbb{V}_i$. On the other hand, one has $\mathbb{D}(M_j/N_j) = \mathbb{V}_j = a_{1,r}\mathbb{V}_i$ by diagram (3.12). Thus one has $\mathbb{D}(M_j/N_i') = \mathbb{D}(M_j/N_j)$. Since $\mathbb{D}$ is fully faithful and contravariant, $N_i' = N_j$. In particular, on the overlap $\mathcal{U}_i \cap \mathcal{U}_j$ the local Fontaine-Faltings modules $N_i$ and
Let $E$ be a twisted $f$-periodic Higgs-de Rham flow. Denote by $M = IC(E)$ the Fontaine module with the endomorphism structure corresponding to $E$. By the equivalence of the category of twisted Fontaine-Faltings modules and the category of periodic Higgs-de Rham flow, one gets the following result.

**Corollary 3.13.** Suppose $V \subset \mathbb{D}^P(M)$ is a non-trivial sub-projective $\mathbb{F}_p f$-representation. Then there exists a non-trivial sub-twisted periodic Higgs-de Rham flow of $E$ which corresponds to $\mathbb{D}^P(M)/V$.

Finally we arrive at the main theorem of our paper:

**Theorem 3.14.** Let $k$ be a finite field of characteristic $p$. Let $X$ be a smooth proper scheme over $W(k)$ together with a smooth log structure $\mathcal{D}/W(k)$. Assume that there exists a semistable graded logarithmic Higgs bundle $(E, \theta)/(X, \mathcal{D})_1$ with discriminant $\Delta_H(E) = 0$, rank($E$) < $p$ and $(\text{rank}(E), \deg_H(E)) = 1$. Then there exists a positive integer $f$ and an absolutely irreducible projective $\mathbb{F}_p f$-representation $\rho$ of $\pi_1(X^o_{K'})$, where $X^o = X \setminus \mathcal{D}$ and $K' = W(k \cdot \mathbb{F}_p)[1/p]$.

**Proof.** We only show the result for $\mathcal{D} = \emptyset$, as the proof of the general case is similar. By Theorem 3.10, there is a twisted preperiodic Higgs-de Rham flow with initial term $(E, \theta)$. Removing finitely many terms if necessary, we may assume that it is twisted $f$-periodic, for some positive integer $f$. By using Theorem 3.3 and applying functor $\mathbb{D}^P$, one gets a $\text{PGL}_{\text{rank}(E)}/(\mathbb{F}_p f)$-representation $\rho$ of $\pi_1(X^o_{K'})$.

Since $(\text{rank}(E), \deg_H(E)) = 1$, the semi-stable bundle $E$ is actually stable. According to Corollary 3.11, there is no non-trivial sub twisted periodic Higgs-de Rham flow. By Corollary 3.13, there is no non-trivial sub projective representation of $\rho$, so that $\rho$ is irreducible. □

**Remark.** For simplicity, we only consider results on $X_1$. Actually, all results in this section can be extended to truncated level.

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**4. Constructing crystalline representations of étale fundamental groups of $p$-adic curves via Higgs bundles**

As an application of the main theorem (Theorem 3.14), we construct irreducible $\text{PGL}_2$ crystalline representations of $\pi_1$ of the projective line removing $m$ ($m \geq 4$) marked points. Let $M$ be the moduli space of semistable graded Higgs bundles of rank 2 degree 1 over $\mathbb{P}^1/W(k)$, with logarithmic Higgs fields which have $m$ poles $\{x_1, x_2, \ldots, x_m\}$ (actually stable, since the rank and degree are coprime to each other). The main object of this section is to study the self map $\Upsilon$ (Corollary-Definition 3.8) on $M$. In section 4.1, we decompose $M$ into connected components. In section 4.2, we show that the
self-map is rational and dominant on the component of \( M \) with maximal dimension. In section 4.3, we give the explicit formula in case of \( m = 4 \).

### 4.1. Connected components of the moduli space \( M \)

First, let’s investigate the geometry of \( M \). For any \( [(E, \theta)] \in M \), \( E \cong \mathcal{O}(d_2) \oplus \mathcal{O}(d_1) \) with \( d_1 + d_2 = 1 \) and \( d_1 < d_2 \). And the graded semi-stable Higgs bundle with nilpotent non-zero Higgs field

\[
\theta : \mathcal{O}(d_1) \longrightarrow \mathcal{O}(d_2) \otimes \Omega^1_{\mathbb{P}^1}(m)
\]

By the condition \( \theta \neq 0 \) in \( \text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}(d_1), \mathcal{O}(d_2 + m - 2)) \), we have \( d_1 \leq d_2 + m - 2 \). Combining with the assumption \( d_1 + d_2 = 1 \), one get

\[
(d_1, d_2) = (1, 0), (2, -1), \cdots, \text{or } \left( \frac{m - 1}{2}, \frac{4 - m}{2} \right),
\]

where \([·]\) is the greatest integer function. Therefore, \( M \) admits a decomposition

\[
M = \coprod_{(d_2, d_1)} M(d_2, d_1)
\]

where \( M(d_2, d_1) \) is isomorphic to

\[
\mathbb{P}(\text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}(d_1), \mathcal{O}(d_2) \otimes \Omega^1_{\mathbb{P}^1}(\log D))) \cong \mathbb{P} \left( H^0(\mathbb{P}^1, \mathcal{O}(d_2 - d_1 + m - 2)) \right)
\]

(note that in this case two Higgs bundles are isomorphic if the Higgs fields differ by a scalar). For \( m = 3, 4 \), the decomposition is trivial because \( (d_2, d_1) = (0, 1) \) is the only choice. But for \( m \geq 5 \), there are more choices. The following table presents the information of \( M(d_2, d_1) \):

| \( (d_1, d_2) \) | \( m \)   | 3 | 4 | 5 | 6 | 7 | 8 | 9 | \cdots |
|------------------|--------|---|---|---|---|---|---|---|-------|
| (1, 0)           | \( \mathbb{P}^0 \) | \( \mathbb{P}^1 \) | \( \mathbb{P}^2 \) | \( \mathbb{P}^3 \) | \( \mathbb{P}^4 \) | \( \mathbb{P}^5 \) | \( \mathbb{P}^6 \) | \( \cdots \) |
| (2, -1)          | \( \mathbb{P}^0 \) | \( \mathbb{P}^1 \) | \( \mathbb{P}^2 \) | \( \mathbb{P}^3 \) | \( \mathbb{P}^4 \) | \( \cdots \) |
| (3, -2)          | \( \cdots \) | \( \cdots \) | \( \cdots \) | \( \cdots \) | \( \cdots \) | \( \cdots \) |
| \vdots           | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots |

### 4.2. Self-maps on moduli spaces of Higgs bundles on \( \mathbb{P}^1 \) with marked points

Let \( p \) be an odd prime number. Since the rank \( r = 2 \) for any element in \( M \), by Corollary-Definition 3.8 we know that \( f_1 = 1 \) and \( L_1 = \mathcal{O}_{\mathbb{P}^1}(\frac{1-p}{2}) \). In other words, the self-map is given by

\[
\Upsilon : (E, \theta) \mapsto (\text{Gr} \circ C^{-1}_{1}(E, \theta)) \otimes \mathcal{O}_{\mathbb{P}^1}(\frac{1-p}{2}),
\]

where the filtration on \( C^{-1}_{1}(E, \theta) \) is the Simpson’s graded semistable Hodge filtration. Let’s denote \( (V, \nabla) = C^{-1}_{1}(E, \theta) \), which is a rank 2 degree \( p \) stable de Rham bundle over \( \mathbb{P}^1 \). Using Grothendieck’s theorem, one gets \( V \cong \mathcal{O}(l_1) \oplus \mathcal{O}(l_2) \) with \( l_1 + l_2 = p \) (assume \( l_1 < l_2 \)). In this case, the
Simpson’s graded semistable Hodge filtration is just the natural filtration \((\mathcal{O}(l_2) \subset V)\).

Since \((V, \nabla)\) is stable, \(\mathcal{O}(l_2)\) cannot be \(\nabla\)-invariant, which means the Higgs field

\[
\text{Gr}_\nabla : \mathcal{O}(l_2) \to \mathcal{O}(l_1) \otimes \Omega^1_{\mathbb{P}^1}(m) \cong \mathcal{O}(l_1 + m - 2)
\]

is nontrivial. Thus, \(l_2 \leq l_1 + m - 2\). Combining with the fact \(l_1 + l_2 = p\) and \(\ell_1 < \ell_2\), one gets

\[
(l_1, l_2) = \left(\frac{p-1}{2}, \frac{p+1}{2}\right), \left(\frac{p-3}{2}, \frac{p+3}{2}\right), \ldots, \left(\frac{p-m+3}{2}, \frac{p+m-2}{2}\right).
\]

For \(m \geq 5\), the jumping phenomena appears, i.e. there exists \([(E, \theta)] \in M(d_2, d_1)\) such that the type of \((\text{Gr} \circ C^{-1}_1(E, \theta)) \otimes \mathcal{O}(\frac{1-p}{2})\) is different from \((d_2, d_1)\).

Next we shall characterize the jumping locus on \(M(d_2, d_1)\). Define a \(\mathbb{Z}\)-valued function \(l\) on \(M(d_2, d_1)\): for each \([E, \theta] \in M(d_2, d_1)\), let \(l([(E, \theta)]) = l([\theta]) := l_2\).

**Lemma 4.1.** The function \(l\) on \(M(d_2, d_1)\) is upper semicontinuous.

**Proof.** Define \(U_n := \{[\theta] \in \mathbb{P}^H(\mathcal{O}(d_2 - d_1 + m - 2)) \mid l([\theta]) \leq n\}\). One only need to prove that \(U_n\) is Zariski open in \(\mathbb{P}^{d_2-d_1+m-2}\) for all \(n \in \mathbb{Z}\). Recall the proof of Grothendieck’s theorem, for \((V_0, \nabla) := C^{-1}_1(\mathcal{O}(d_2) \oplus \mathcal{O}(d_1), \theta)\) one defines

\[
m := \min \{\lambda \in \mathbb{Z} \mid H^0(\mathbb{P}^1, V_\theta(\lambda)) \neq 0\}
\]

and gets the splitting \(V_\theta \cong \mathcal{O}(-m) \oplus \mathcal{O}(p + m) (p + m \leq -m)\). Therefore, \(l([\theta]) = -m\). Since \([\theta] \in U_n\), one gets \(-m \leq n\). But this means \(-n - 1 < -n \leq m\). Thus \(H^0(\mathbb{P}^1, V_\theta(-n-1)) = 0\).

By the semicontinuity of the rank of the direct image sheaf, we know that \(H^0(\mathbb{P}^1, V_\theta(-n-1)) = 0\) for \(\theta'\) in a neighborhood of \(\theta\). This means \(l([\theta']) \leq n\) in a neighborhood. Therefore, \(U_n\) is Zariski open for each \(n \in \mathbb{Z}\). \(\square\)

Now consider the first component of moduli scheme \(M(\frac{p-1}{2}, \frac{p+1}{2})\). By the lemma above, \(U_{\frac{p+1}{2}}\) is a Zariski open set of \(M(\frac{p-1}{2}, \frac{p+1}{2})\). All Higgs bundles in \(U_{\frac{p+1}{2}}\) will be sent back to \(M\) by applying the inverse Cartier transform, taking the quotient of \(\mathcal{O}(\frac{p+1}{2})\) and tensoring with \(\mathcal{O}(\frac{1-p}{2})\). This process actually gives us a functor, which we denote as \(\text{Gr}_{\frac{p+1}{2}} \circ C^{-1}_1(\cdot) \otimes \mathcal{O}(\frac{1-p}{2})\).

**Lemma 4.2.** The functor \(\text{Gr}_{\frac{p+1}{2}} \circ C^{-1}_1(\cdot) \otimes \mathcal{O}(\frac{1-p}{2})\) induces a rational map

\[
\varphi : M(1, 0) \to M(1, 0).
\]

**Proof.** Let \(M(1, 0)\) denote the moduli functor which is corepresented by the moduli scheme \(M(1, 0)\). And \(U_{\frac{p+1}{2}}\) denotes the subfunctor corresponding to \(U_{\frac{p+1}{2}}\).

Note that the functor \(\text{Gr}_{\frac{p+1}{2}} \circ C^{-1}_1(\cdot) \otimes \mathcal{O}(\frac{1-p}{2})\) gives a natural transform
between these two moduli functors $U_{p+1}$ and $M(1,0)$. Since $M(1,0)$ is corepresented by $M(1,0)$, one gets the following diagram

$$
\begin{array}{ccc}
U_{p+1} & \to & M(1,0) \\
\downarrow & & \downarrow \\
\to & & \to \\
\end{array}
$$

By the universal property of the coarse moduli scheme, one gets a natural transform

$$
\text{Hom}_k(\cdot, U_{p+1}) \to \text{Hom}_k(\cdot, M(1,0))
$$

Take $Id \in \text{Hom}_k(U_{p+1}, U_{p+1})$, the natural transform will give the $k$-morphism

$$
U_{p+1} \to M(1,0)
$$

One can easily check that this map is induced by the self-map.

Now we want to prove:

**Lemma 4.3.** The rational map $\varphi$ is dominant.

**Proof.** We prove this lemma by induction on the number $m$ of the marked points. For $m = 3$, the lemma trivially holds since $M$ is just a point. Now suppose the statement is true for the case of $m - 1$ marked points. We want to prove $\varphi$ is dominant for the case of $m$ marked points. Set $Z := \overline{\text{Im}(\varphi)} \subset M(1,0)$ and we want to prove $Z = M(1,0)$. Suppose $Z$ is a proper subscheme of $M(1,0) \cong \mathbb{P}^{m-3}$. Then $\dim Z \leq m - 4$. Denote $M(\hat{x}_i)$ to be the moduli space of semistable graded Higgs bundles of rank 2 degree 1 over $\mathbb{P}^1$, with nilpotent logarithmic Higgs fields which have $m - 1$ poles \{\(x_1, \ldots, \hat{x}_i, \ldots, x_m\)\}. Then one can define a natural embedding $M(\hat{x}_i) \hookrightarrow M$ by forgetting one marked point $x_i$. Therefore,

$$
\bigcup_i \varphi(M(\hat{x}_i; 1,0)) \subset Z
$$

where $M(\hat{x}_i; 1,0)$ is the component of $M(\hat{x}_i)$ with maximal dimension. Then we know that $M(\hat{x}_i; 1,0) \cong \mathbb{P}^{m-4}$. So $\dim Z = m - 4$ by the assumption that $\varphi$ is dominant for $m - 1$ case. And $Z$ has more than one irreducible component. But this is impossible since $Z$ is the Zariski closure of $\varphi(M(1,0)) \cong \varphi(\mathbb{P}^{m-3})$, which is irreducible. □

Now we can state and prove the main result of this section:

**Theorem 4.4.** The set of periodic points of $\varphi$ is Zariski dense in $M(1,0)$.

Combining this with Proposition 3.9, one gets infinitely many irreducible crystalline projective representations of the fundamental group.

To prove this we need a theorem of Hrushovski:
**Theorem 4.5** (Hrushovski [8], see also Theorem 3.7 in [3]). Let $Y$ be an affine variety over $\mathbb{F}_q$, and let $\Gamma \subset (Y \times_{\mathbb{F}_q} Y) \otimes_{\mathbb{F}_q} \mathbb{F}_q$ be an irreducible sub variety over $\mathbb{F}_q$. Assume the two projections $\Gamma \to Y$ are dominant. Then, for any closed sub variety $W \subseteq Y$, there exists $x \in Y(\mathbb{F}_q)$ such that $(x, x^{q^m}) \in \Gamma$ and $x \notin W$ for large enough natural number $m$.

**proof of Theorem 4.4.** For each Zariski open subset $U \subset M(\frac{p-1}{2}, \frac{p+1}{2})$, we need to find a periodic point $x$ of $\varphi$ such that $x \in U$. We take $Y$ to be an affine neighborhood of $M(\frac{p-1}{2}, \frac{p+1}{2}+1)$. And $\Gamma$ is the intersection of $\varphi \otimes_{\mathbb{F}_q} \mathbb{F}_q$ and $(Y \times_{\mathbb{F}_q} Y) \otimes_{\mathbb{F}_q} \mathbb{F}_q$. $W$ is defined to be the union of $(M(\frac{p-1}{2}, \frac{p+1}{2}+1) \backslash U) \cap Y$ and the indeterminacy of $\varphi$. By Lemma 4.3 the projections $\Gamma \to Y$ are dominant. So we can apply Theorem 4.5 and find a point $x \in Y(\mathbb{F}_q)$ such that $(x, x^{q^m}) \in \Gamma$ and $x \notin W$ for some $m$. Therefore, $x \in U$, $\varphi$ is well-defined at $x$ and $\varphi(x) = x^{q^m} (Y \subset A^r$, so $x$ can be written as $(x_1, \ldots, x_r) \in A^r(\mathbb{F}_q)$ and $x^{q^m} := (x_1^{q^m}, \ldots, x_r^{q^m})$). The rational map $\varphi$ is well-defined at $x$ means that $\varphi$ is also well-defined at $x^{q^N}$ for any $N \in \mathbb{N}$. Then we have

$$\varphi(\varphi(x)) = \varphi(x^{q^m}) = \varphi(x)^{q^m} = x^{q^{2m}}$$

Thus $\varphi^N(x) = x^{q^{Nm}} = x$ for $N$ large enough. That means, $x$ is a periodic point of $\varphi$. □

### 4.3. An explicit formula of the self map in the case of four marked points.

In this section, we give an explicit formula of the self map in case of $m = 4$ marked points. Using Möbius transformation on $\mathbb{P}^1$, we may assume these 4 points are of form $\{0, 1, \infty, \lambda\}$. By section 4.1 the moduli space $M$ is connected and isomorphic to $\mathbb{P}^1$, where the isomorphism is given by sending $(E, \theta)$ to the zero locus $(\theta)_0 \in \mathbb{P}^1$. To emphasize the dependence of the self map on $\lambda$ and $p$, we rewrite the self map by $\varphi_{\lambda, p}$. By calculation we get

$$\varphi_{\lambda, p}(z) = \frac{z^p}{\lambda^p-1} \cdot \left( \frac{f_{\lambda}(z^p)}{g_{\lambda}(z^p)} \right)^2,$$

where $f_{\lambda}(z^p)$ is the determinant of matrix

$$
\begin{pmatrix}
\frac{\lambda^p(1-z^p)-\lambda^p(\lambda^p-z^p)}{(p+1)/2} & \cdots & \lambda^p(1-z^p)-\lambda^p(\lambda^p-z^p) \\
\frac{\lambda^p(1-z^p)-\lambda^p(\lambda^p-z^p)}{2} & \cdots & \lambda^p(1-z^p)-\lambda^p(\lambda^p-z^p) \\
\cdots & \cdots & \cdots \\
\frac{\lambda^p(1-z^p)-\lambda^p(\lambda^p-z^p)}{(p-1)/2} & \cdots & \lambda^p(1-z^p)-\lambda^p(\lambda^p-z^p)
\end{pmatrix}
$$

and $g_{\lambda}(z^p)$ is the determinant of matrix

$$
\begin{pmatrix}
\frac{\lambda^p(1-z^p)-\lambda^p(\lambda^p-z^p)}{2} & \cdots & \lambda^p(1-z^p)-\lambda^p(\lambda^p-z^p) \\
\frac{\lambda^p(1-z^p)-\lambda^p(\lambda^p-z^p)}{2} & \cdots & \lambda^p(1-z^p)-\lambda^p(\lambda^p-z^p) \\
\cdots & \cdots & \cdots \\
\frac{\lambda^p(1-z^p)-\lambda^p(\lambda^p-z^p)}{(p-1)/2} & \cdots & \lambda^p(1-z^p)-\lambda^p(\lambda^p-z^p)
\end{pmatrix}
$$
By calculation, for $p = 3$ one has
\[
\varphi_{\lambda,3}(z) = z^3 \left( \frac{z^3 + \lambda(\lambda + 1)}{(\lambda + 1)z^3 + \lambda^2} \right)^2
\]
and $\varphi_{\lambda,3}(z) = z^{3^2}$ if and only if $\lambda = -1$; for $p = 5$, one has
\[
\varphi_{\lambda,5}(z) = z^5 \left( \frac{z^{10} - \lambda(\lambda + 1)(\lambda^2 - \lambda + 1)z^5 + \lambda^4(\lambda^2 - \lambda + 1)}{(\lambda^2 - \lambda + 1)z^{10} - \lambda^2(\lambda + 1)(\lambda^2 - \lambda + 1)z^5 + \lambda^6} \right)^2,
\]
and $\varphi_{\lambda,5}(z) = z^{5^2}$ if and only if $\lambda$ is a 6-th primitive root of unit; for $p = 7$ one has
\[
\varphi_{\lambda,7}(z) = z^7 \left( \frac{z^{21} + 14\lambda(\lambda + 1)(\lambda^2 + \lambda + 1)(\lambda^2 + 3\lambda + 1)z^{14} + \lambda^6(\lambda^2 + \lambda + 1)}{(\lambda^2 + \lambda + 1)z^{21} + \lambda^2(\lambda^2 + \lambda + 1)(\lambda^2 + 3\lambda + 1)z^{14} + 2\lambda^6(\lambda^2 + \lambda + 1)(\lambda^2 + 3\lambda + 1)z^7 + \lambda^{12}} \right)^2
\]
and $\varphi_{\lambda,7}(z) = z^{7^2}$ if and only if $(\lambda + 1)(\lambda^2 + \lambda + 1) = 0$.

We regard $\varphi_{\lambda,p}$ as a self map on $\mathbb{P}^1$, which is rational and of degree $p^2 \neq 1$. Thus it has $p^2 + 1$ fixed points defined over $\overline{k}$. Let $(E, \theta)/\mathbb{P}^1_{k'}$ be a fixed point of $\varphi_{\lambda,p}$ defined over some extension field $k'$ of $k$. Then in the language of Higgs-de Rham flow, $(E, \theta)$ is the initial term of a twisted 1-periodic Higgs-de Rham flow over $\mathbb{P}^1_{k'}$.

To lifting of twisted periodic Higgs-de Rham flow. Here we just consider 1-periodic case, for higher-periodic case the treatment is similar. Since those two conditions in Proposition 3.5 are trivially hold, one lifts $(E, \theta)$ to a twist periodic Higgs bundle over $\mathbb{P}^1_{W_2}$. Recall the proof of Proposition 3.5, one constructs a self map on the torsor space of all liftings of $(E, \theta)$, and the fixed points of this self map correspond to those liftings of the twisted 1-periodic Higgs-de Rham flow. By explicit calculate, we identify the torsor space in $\mathbb{P}^1$ with $k$ and the self map on this torsor space is of form
\[
z \mapsto az^p + b,
\]
where $a, b \in k$.

**Case 1:** $a = 0$. Then $z = b$ is the unique fixed point of the self map. In other words there is a unique twisted periodic lifting of the given twisted 1-periodic Higgs-de Rham flow over $\mathbb{P}^1_{W_2(k)}$.

**Case 2:** $a \neq 0$. Let $z_0 \in \overline{k}$ be a solution of $z = az^p + b$. Then $\Sigma = \left\{ i \cdot a^{- \frac{1}{p-1}} + z_0 \mid i \in F_p \right\}$ is the set of all solutions. If $a \neq 0$ is not a $(p - 1)$-th power of any element in $k^\times$, then $\#(\Sigma \cap k) \leq 1$. In other words there is at most one twisted 1-periodic lifting over $\mathbb{P}^1_{W_2(k)}$ of the given twisted 1-periodic Higgs-de Rham flow. If $a \neq 0$ is a $(p - 1)$-th power of some element in $k^\times$, then $\#(\Sigma \cap k) = 0$ or $p$. In other words, if the twisted 1-periodic Higgs-de Rham flow is liftable then there are exactly $p$ liftings over $\mathbb{P}^1_{W_2(k)}$. 
If we consider the lifting problem over an extension $k'$ of $k$, which contains $\Sigma$, then there are exactly $p$ liftings of the twisted 1-periodic Higgs-de Rham flow over $\mathbb{P}^1_{W_2(k')}$. 

4.4. **Examples of dynamic of Higgs-de Rham flow on $\mathbb{P}^1$ with four-marked points.** In the following, we give some examples in case $k = \mathbb{F}_{3^4}$. For any $\lambda \in k \setminus \{0, 1\}$, the map $\varphi_{\lambda, 3}$ is a self $k$-morphism on $\mathbb{P}^1_k$. So it can be restricted as a self map on the set of all $k$-points 

$$\varphi_{\lambda, 3} : k \cup \{\infty\} \rightarrow k \cup \{\infty\}.$$ 

Since $\alpha = \sqrt{1 + \sqrt{-1}}$ is a generator of $k = \mathbb{F}_{3^4}$ over $\mathbb{F}_3$, every elements in $k$ can be uniquely expressed in form $a_3\alpha^3 + a_2\alpha^2 + a_1\alpha + a_0$, where $a_3, a_2, a_1, a_0 \in \{0, 1, 2\}$. We use the integer $27a_3 + 9a_2 + 3a_1 + a_0 \in [0, 80]$ stand for the element $a_3\alpha^3 + a_2\alpha^2 + a_1\alpha + a_0$. By identifying the set $k \cup \infty$ with $\{0, 1, 2, \cdots, 80, \infty\}$ in this way, we get a self map on $\{0, 1, 2, \cdots, 80, \infty\}$ for all $\lambda \in k$

$$\varphi_{\lambda, 3} : \{0, 1, 2, \cdots, 80, \infty\} \rightarrow \{0, 1, 2, \cdots, 80, \infty\}.$$ 

In the following diagrams, the arrow $\circlearrowleft \rightarrow \circlearrowright$ means $\gamma = \varphi_{\lambda, 3}(\beta)$. And an $m$-length loop in the following diagrams just stands for a twisted $m$-periodic Higgs-de Rham flow, which corresponds to $\text{PGL}_2(\mathbb{F}_{3^m})$-representation by Theorem 3.14 and Theorem 3.14.

- For $\lambda = 2\sqrt{1 + \sqrt{-1}}$, we have

![Diagram](image-url)
The 1-length loops \( \circ \) and \( \circ \) in the diagrams above correspond to projective representations of form

\[
\pi_1 \left( \mathbb{P}^1_{W(F_{34})[1/3]} \setminus \left\{ 0, 1, \infty, 2\sqrt{1 + \sqrt{-1}} \right\} \right) \to \text{PGL}_2(F_3),
\]

here \( W(F_{34})[1/3] \) is the unique unramified extension of \( \mathbb{Q}_3 \) of degree 4.

- For \( \lambda = \sqrt{-1} \), we have

The 2-length loop \( \circ \) corresponds to a projective representation of form

\[
\pi_1 \left( \mathbb{P}^1_{W(F_{34})[1/3]} \setminus \left\{ 0, 1, \infty, \sqrt{-1} \right\} \right) \to \text{PGL}_2(F_{32}).
\]

We also have diagram

which is an 8-length loop and corresponds to a projective representation of form

\[
\pi_1 \left( \mathbb{P}^1_{W(F_{38})[1/3]} \setminus \left\{ 0, 1, \infty, \sqrt{-1} \right\} \right) \to \text{PGL}_2(F_{38}).
\]

- For \( \lambda = 2 + \sqrt{1 + \sqrt{-1}} \), one has
and the 3-length loop in this diagram corresponds to a projective representation of form
\[
\pi_1 \left( \mathbb{P}^1_{W(F_{3^4}[1/3]} \setminus \left\{ 0, 1, \infty, 2 + \sqrt{1 + \sqrt{-1}} \right\} \right) \to \text{PGL}_2(\mathbb{F}_{3^4}).
\]

We also have
\[
\text{which is a 4-length loop and corresponds to a projective representation of form}
\pi_1 \left( \mathbb{P}^1_{W(F_{3^4}[1/3]} \setminus \left\{ 0, 1, \infty, 2 + \sqrt{1 + \sqrt{-1}} \right\} \right) \to \text{PGL}_2(\mathbb{F}_{3^4}).
\]

4.5. Projective $F$-units crystalline on smooth projective curves.
Let $X$ be a smooth proper scheme over $W(k)$. In [12] an equivalence between the category of $f$-periodic vector bundles $(E, 0)$ of rank-$r$ over $X$ (i.e. $(E, 0)$ initials an $f$-periodic Higgs-de Rham flow with zero Higgs fields in all Higgs terms) and the category of $\text{GL}_r(W_n(F_{p_f}))$-representations of $\pi_1(X)$ has been established. This result generalizes Katz’s original theorem for $X$ being an affine variety. As an application of our main theorem, we show that

**Theorem 4.6.** The $\mathbb{D}^P$ functor is faithful from the category of rank-$r$ twisted $f$-periodic vector bundles $(E, 0)$ over $X$ to the category of projective $W_n(F_{p_f})$-representations of $\pi_1(X_{1,k'})$ of rank $r$, where $k'$ is the minimal extension of $k$ containing $F_{p_f}$. 

Remark. For $n = 1$ the above theorem is just a projective version of Lange-Stuhne’s theorem.

**Theorem 4.7** (lifting twisted periodic vector bundles). Let $(E, 0)/X_1$ be an $f$-periodic vector bundle after twisting line bundle. Assume $H^2(X_1, \text{End}(E)) = 0$. Then for any $n \in \mathbb{N}$ there exists some positive integer $f_n$ with $f | f_n$ such that $(E, 0)$ lifts to a twisted $f_n$-periodic vector bundle over $X_n$.

Translate the above theorem in of representations:

**Theorem 4.8** (lifting projective representations of $\pi_1(X_1)$). Let $\rho$ be a projective $\mathbb{F}_{p^f}$-representation of $\pi_1(X_1)$. Assume $H^2(X_1, \text{End}(\rho)) = 0$, then there exist an positive integer $f_n$ divded by $f$ such that $\rho$ lifts to a projective $W_n(\mathbb{F}_{p^f_n})$-representation of $\pi_1(X_1,k')$ for any $n \in \mathbb{N}$, where $k'$ is the minimal extension of $k$ containing $\mathbb{F}_{p^f_n}$.

Assume $\mathcal{X}$ is a smooth proper curve over $W(k)$, de Jong and Osserman (see Appendix A in [20]) have shown that the subset of periodic vector bundles over $X_1\mathcal{X}$ is Zariski dense in the moduli space of semistable vector bundles over $X_1$ (Laszlo and Pauly have also studied some special case, see [17]). Hence by Lange-Stuhne’s theorem (see [14]) every periodic vector bundle corresponds to a $(P)GL_r(\mathbb{F}_{p^f'})$-representations of $\pi_1(X_1,k')$, where $f$ is the period and $k'$ is a definition field of the periodic vector bundle containing $\mathbb{F}_{p^f}$.

**Corollary 4.9.** Every $(P)GL_r(\mathbb{F}_{p^f'})$-representation of $\pi_1(X_1,k')$ lifts to a $(P)GL_r(W_n(\mathbb{F}_{p^f_n}))$-representation of $\pi_1(X_1,k')$ for some positive integer $f_n$ divided by $f$, where $k''$ is a definition field of the periodic vector bundle containing $\mathbb{F}_{p^f_n}$.

Remark. It shall be very interesting to compare this result with Deninger-Werner’s theorem (see [2]), they have shown that any vector bundle over $\mathcal{X}$, which is preperiodic over $X_1$, lifts to a $GL_r(\mathbb{C})$-representation of $\pi_1(\mathcal{X}_K)$.

**Acknowledgement.** We would like to thank Adrian Langer, Mao Sheng and Carlos Simpson for their interests in this paper. We thank warmly Christian Pauly for pointing out a paper of de Jong and Osserman. We are grateful to Xiaotao Sun, Deqi Zhang, Shouwu Zhang and Hang Xue for the discussion on dynamic properties of self maps in characteristic $p$. The discussion with Ariyan Javanpeykar about the self map on the moduli space of Higgs bundles over the projective line with logarithmic structure on marked points is very helpful for us, and he also read the preliminary version of this paper carefully and suggested a lot for the improvement. We thank him heavily.

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