REMARKS ON NUMERICAL RELATIVITY, 
GEODESIC MOTIONS, 
BINARY NEUTRON STAR EVOLUTION

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Abstract. The computations of numerical relativity make use of (3 + 1)--decompositions of Einstein field equations. We examine the conceptual characteristics of this method; instances of compact-star binaries are considered. The preeminent role of the geodesic motions is emphasized.

PACS 04.20 – General relativity.

1. Introduction. In sect. 2 we recall properties and geodesics of the Gaussian-normal coordinate system. In sect. 3 we emphasize that the physical results derived from any solution of Einstein field equations must be independent of the adopted reference frame. In sects. 4 and 5 we give a résumé of the properties of the (3 + 1)-decompositions of Einstein equations, and we point out that: i) the results of the approximate computations obtained with the employment of the gravitational energy-pseudotensor have an illusive value; ii) coordinate-system independence of the results is not proved; iii) no mathematical theorem of existence supports the approximate computations of numerical relativity. Sect. 5bis: the particles of a discrete “cloud of dust” describe geodesic lines – and therefore no GW is emitted by them: an emblematic instance. Sects. 6 and 7 contain some comments on the numerical computations concerning the binaries composed of two “Schwarschilidian mass-points” and of two neutron stars; an analysis is given of the notions of ADM-mass. Appendix A: The Einstein equations in a Gaussian-normal frame. Appendix B: The equations of the standard (3 + 1)-decomposition. Appendix C: The four-dimensional world as “a mass of plasticine” (Weyl), and some interesting consequences.

2. – The concept of Gaussian-normal coordinate system is due to Hilbert [1]. Landau and Lifshitz called it synchronous [2]. Let us consider in the four-dimensional world a three-dimensional space $S_3$ such that every line in it is space-like; $(x^1, x^2, x^3)$ be the point-coordinates in $S_3$. Starting from each point $(x^1, x^2, x^3)$, we trace the geodesic lines which are orthogonal to $S_3$. They will become particular time-like lines if we report on them the time $x^0 = ct$ as the proper time $\tau$. It follows easily that
(1) \[ ds^2 = -(dx^0)^2 + g_{\alpha\beta} \, dx^\alpha \, dx^\beta, \, (\alpha, \beta = 1, 2, 3) \].

By virtue of the very construction of the three-dimensional space \( x^0 = 0 \), the quadratic form \( g_{\alpha\beta} \, dx^\alpha \, dx^\beta \) is necessarily a definite positive one.

It is interesting to write the differential equations of all the geodesic lines of the metric (1). The Lagrangian \( \mathcal{L} \):

(2) \[ \mathcal{L} := -\dot{x}^0 + g_{\alpha\beta}(x^0, x^1, x^2, x^3) \, \dot{x}^\alpha \, \dot{x}^\beta = A \epsilon^2, \]

where the overdot denotes a derivative with respect to an affine parameter \( \sigma \), and \( A \) is a constant, gives the Lagrangian equations

(3) \[ \ddot{x}^0 + \frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial x^0} \, \dot{x}^\alpha \, \dot{x}^\beta = 0 ; \]

(3') \[ g_{\alpha\gamma} \ddot{x}^\alpha + \frac{\partial g_{\alpha\gamma}}{\partial x^0} \, \dot{x}^\alpha \, \ddot{x}^0 + \frac{\partial g_{\alpha\gamma}}{\partial x^0} \, \dot{x}^\gamma \, \dot{x}^0 - \frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial x^0} \, \dot{x}^\alpha \, \dot{x}^\beta = 0 , \, (\gamma, \delta = 1, 2, 3). \]

\( \mathcal{L} \) is a first integral of eqs.(3–3'): \( A \) is negative, zero, positive for time-like, null, space-like geodesics, respectively. If \( \sigma = \tau \), we have \( A = -1 \).

The geodesics orthogonal to \( x^0 = 0 \) are characterized by \( 0 = \dot{x}^\alpha = \ddot{x}^\alpha \), and \( \sigma = \tau \).

The solutions of eqs.(3–3'), are commonly interpreted, if \( \sigma = \tau \), as the trajectories of test-particles. However, they represent also the geodesic trajectories of the material elements of a “cloud”, which generates the Gaussian field.

There are infinite expressions for the Gaussian-normal \( ds^2 \), which are obtained by one of them with: \( i) \) any Lorentz transformation of the spacetime coordinates, \( ii) \) any transformation of the space coordinates.

Landau and Lifshitz [2] write the Einstein field equations in a Gaussian-normal frame; then, they separate space – and time – derivatives, thus obtaining a particular \( (3 + 1) \)-formalism. All the operations of raising and lowering of the indices and the covariant derivatives are performed with respect to the space metric \( g_{\alpha\beta}(x^0, x^1, x^2, x^3) \). They put \( \kappa_{\alpha\beta} := \partial g_{\alpha\beta}/\partial t \), and in their equations (99,10), (99,11), (99,12) (see App. A) the symbol \( P_{\alpha\beta} \) denotes the Ricci tensor with respect to the three-dimensional space. It is immediate to divide eqs. (99,10), (99,11), (99,12) in constraint equations, containing \( g_{\alpha\beta} \) and \( \partial g_{\alpha\beta}/\partial t \), and in evolution equations, containing \( \partial^2 g_{\alpha\beta}/\partial t^2 \). It can be proved [3] that

(4) \[ \frac{\partial^2 g_{\alpha\beta}}{\partial t^2} = \frac{F_{\alpha\beta}}{g} , \]

where \( g \) is the determinant of the Gaussian metric matrix, and \( F_{\alpha\beta} \) is an entire rational function of the \( g_{\alpha\beta} \)’s, of the \( \partial g_{\alpha\beta}/\partial t \)'s, of the second spatial and spatiotemporal derivatives of \( g_{\alpha\beta} \). (Of course, \( F_{\alpha\beta} \) depends also on the matter tensor).
For the solution of the Cauchy problem it is necessary to give the values of $g_{\alpha\beta}$ and $\partial g_{\alpha\beta}/\partial t$ at all the space points $(x^1, x^2, x^3)$ at an initial instant $t = t_0$.

3. – In principle, the Cauchy problem could be solved for any physical system in any system of coordinates, thus giving origin to an implicit, particular $(3+1)$–formulation of the problem.

As it was emphasized by Hilbert [1], as a rule any solution of Einstein field equations must satisfy the following criterion: all the physical results which we deduce from it must be independent of the coordinate system in which the solution is expressed. Of course, this does not exclude the existence of interesting physical phenomena – as, e.g., the gravitational redshift of the spectral lines –, whose entity depends on the chosen reference system. An emblematic instance of independence of the coordinate system is represented by the geodesic motions of the elements of a continuous “cloud of dust”.

4. – The numerical relativity makes use of two $(3+1)$-decompositions of the Einstein field equations [4a, b]). The standard decomposition is (see, e.g., p.41 of [4a]):

$$ds^2 = -\alpha^2 dt^2 + \gamma_{jk} (dx^j + \beta^j dt) (dx^k + \beta^k dt),$$

where $(j, k) = (1, 2, 3)$, and $c = G = 1$. The matrix of the components of metric (5) is:

$$
\begin{pmatrix}
-\alpha^2 + \gamma_{jk} \beta^j \beta^k & \gamma_{jk} \beta^j \\
\gamma_{jk} \beta^k & \gamma_{jk}
\end{pmatrix}.
$$

In Appendix B we report the field equations of the standard decomposition.

If $\alpha = 1$ and the three-vector $\beta^j$ is zero, eq. (5) gives a Gaussian-normal $ds^2$. At p.37 of [4a]) we read: “The freedom to choose these four gauge functions $\alpha, \beta^1, \beta^2, \beta^3$ completely arbitrary embodies the fourfold coordinate degrees of freedom inherent in general relativity ...” The lapse function $\alpha$ reflects the freedom to choose the sequence of the space-like hypersurfaces, and the shift vector $\beta^j$ reflects the freedom to relabel the spatial coordinates on each of the above hypersurfaces in an arbitrary way. This means that the covariance of a $(3+1)$-decomposition is restricted to the continuous transformations of of the space coordinates $x^1, x^2, x^3$, and to the freedom to choose the initial $t$-hypersurfaces. Consequently, the $(3+1)$-decompositions are not equivalent to the Einstein field equations, which possess the general covariance with respect to all the continuous transformations of the space-time coordinates $x^0, x^1, x^2, x^3$. The $(3+1)$-decompositions are contrary to the spirit of GR. And in them a vast class of coordinate systems is not employable.
5. – In the applications to various problems of the formalisms of the \((3+1)\)-decompositions of Einstein equations the authors choose suitable particularizations of the metric \((5)–(6)\). In these numerical computations an important role is commonly played by the gravitational energy-pseudotensor, which is a tensor only with respect to linear coordinate transformations, and therefore its intervention renders illusive the physical value of any result. Indeed, it is not sufficient to remark that the total pseudoenergy and the total pseudomomentum of the system form a four-dimensional vector in the very distant Minkowskian spacetime. Further, the authors omit to prove that the obtained results are independent of the adopted reference system. No mathematical theorem of existence is given: the numerical solutions presuppose that “suitable boundary conditions and initial data are chosen so that these solutions do indeed exist” ([4a], p.21).

5bis. – Articles and treatises of numerical relativity and \((3+1)\)-decompositions of the Einstein equations contain many considerations and computations concerning the properties and the generation of GWs. We have given several demonstrations that GR, if properly understood, excludes the physical existence of GWs [5] – and the experience corroborates our proofs. Therefore, we do not discuss here the numerical-relativity methods regarding the GWs. We give instead the simple proof that also the motions of the particles of a discrete “cloud of dust” are geodesic. First of all, according to a method by Infeld [6] we write the Einstein equations, in a generic coordinate system \(x^0, x^1, x^2, x^3\), using tensor densities:

\[
\sqrt{-g} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \sqrt{-g} R = -8\pi \sqrt{-g} T_{\mu\nu} \quad (\mu, \nu = 0, 1, 2, 3) , \quad (c = G = 1).
\]

If \(\delta(x - \xi) := \delta(x_1 - \xi_1) \delta(x_2 - \xi_2) \delta(x_3 - \xi_3)\) is the three-dimensional Dirac’s distribution, and \(\xi_1, \xi_2, \xi_3\) are the coordinate of a particle, we put – if the “cloud” is composed of \(s\) particles:

\[
\sqrt{-g} \ T^{\mu\nu} = \sum_{p=1}^{s} \sqrt{-g} \ T_{p}^{\mu\nu} ,
\]

\[
\sqrt{-g} \ T_{p}^{\mu\nu} = \bar{m} (t) \frac{d \xi^\mu}{ds} \frac{d \xi^\nu}{ds} \delta \ (x - \xi) ;
\]

now, if the world-lines of the particles never intersect, it is not difficult to verify that the equations of motion of the particles:

\[
\sum_{p=1}^{s} (\sqrt{-g} \ T_{p}^{\mu\nu})_{,\nu} = 0 ,
\]

where the semicolon denotes a covariant derivative, are the differential equations of geodesic lines. Consequently, no GW is generated by our “dust”; a result very general, because there exists no limitation for the
values of the particle kinematical-elements (velocities, accelerations, time derivative of the accelerations, etc.).

We see that the gravitational self-force theory, which is based on a wrong analogy with the electromagnetic self-force of a charge, does not make any physical sense. And the post-Newtonian approximations which concern a discrete “cloud of dust” cannot give GWs.

6. – With regard to the binary stars composed of two Schwarzschildian mass-points (see, e.g., Chapt. 13 – “Binary Black Hole Evolution” – of [4a]), we observe that nobody has succeeded in proving the theoretical existence of two Schwarzschildian mass-points, and therefore the approximate solution of Einstein equations obtained by numerical computations, which would describe this system, has a very doubtful value. Further, we remark that the black-hole interpretation of the Schwarzschildian point-mass solution of Einstein equations has a mathematically unfounded basis (see, e.g., sects. 3b, 3c of [5]). At any rate, the curvature (“hard”) singularity at \( r = 0 \), which characterizes the gravitational field of a point-mass in a current consideration of the standard (Hilbert-Droste-Weyl) form of the metric, can be simply removed if one adopts the original Schwarzschild’s coordinate system, or Brillouin’s system, for which the metric has only a “soft” singularity at \( r = 0 \). Numerical stratagems, as the BH-excision, the moving puncture method, etc., are superfluous.

7. – The notion of ADM-mass (ADM: Arnowitt-Deser-Misner) is currently employed in the numerical computations [4a]). Now, the definition of the ADM-mass is founded on the properties of the gravitational energy-pseudotensor (see sect. 5), and therefore the physical meaning of an ADM-mass is rather uncertain.

Chapt. 16 of [4a] regards the “Binary Neutron Star Evolution”. The approximate computations make use of the above mentioned notion of ADM-mass. We limit ourselves to emphasize that the time evaluations of the inspiral and merger phases of the considered binaries and of the evolutions of the amplitudes \( h_{+} \) and \( h_{\times} \) of hypothesized GWs (computed with the disputable quadrupole formula) do not possess a physical value. Indeed, these time intervals would make sense only if we could prove their independence of the adopted coordinate system.

We conclude that the approximate computations of numerical relativity are essentially self-referential, because no experimental, or observational, proof and no theorem of mathematical existence supports them. Further, they do not even satisfy the criterion according to which the physical value of the results depends on their independence of the reference frame.
APPENDIX A

The Einstein field equations written in a Gaussian-normal coordinate system (see sect. 2 and Landau and Lifshitz [2]) are \((\alpha, \beta = 1, 2, 3)\):

\[(A1) \quad R^0_0 = \frac{1}{2c} \frac{\partial \kappa_{\alpha}^\alpha}{\partial t} + \frac{1}{4} \kappa_{\alpha}^\alpha \kappa_\beta^\beta = \frac{8\pi G}{c^4} \left( T^0_0 - \frac{1}{2} T \right) ,\]

\[(A2) \quad R^0_\alpha = \frac{1}{2} (\kappa_\beta^\beta - \kappa_\alpha^\alpha) = \frac{8\pi G}{c^4} T^0_\alpha ,\]

\[(A3) \quad R^\beta_\alpha = P^\beta_\alpha + \frac{1}{2c\sqrt{-g}} \frac{\partial}{\partial t} (\sqrt{-g} \delta_\alpha^\beta) = \frac{8\pi G}{c^4} \left( T^a_\alpha - \frac{1}{2} \delta_\alpha^\beta T \right) ;\]

here: \(\kappa_{\alpha\beta} := \frac{\partial g_{\alpha\beta}}{\partial t}\); \(P^\beta_\alpha\) is the three-dimensional Ricci tensor; the semicolon denotes a covariant derivative with respect to the three-dimensional metric \(g_{\alpha\beta}\), \((\alpha, \beta = 1, 2, 3)\).

APPENDIX B

We give the basic formulae of the standard \((3+1)\)-decomposition of the Einstein field equations [7].

Index notations: \((i, j, \ldots) = (1, 2, 3); (a, b, \ldots = 0, 1, 2, 3)\). The extrinsic curvature \(K_{ij} = \sqrt{-g^{\alpha\beta}} T^0_\iota^\alpha g_{i\iota}^\beta\) of a hypersurface measures how much normal vectors to the hypersurface differ at neighboring points; \(K = \gamma_{ij} K^{ij} = K^{ij}_j\). The symbol \(D_j\) denotes the covariant derivative with respect to \(\gamma^{ij}\).

Normal vector: \(n_a = (-\alpha, 0, 0, 0); n^a = (\alpha^{-1}, -\alpha^{-1} \beta^1, -\alpha^{-1} \beta^2, -\alpha^{-1} \beta^3)\).

The matter tensor \(T_{ab}\) is the fluidodynamical energy-tensor, with various polytropic equations of state (EOSs); the matter source terms are:

\(\varrho = n_a n_b T^{ab}; \quad S^a = -\gamma^{ij} n^a T_{aj}; \quad S_{ij} = \gamma_{ia} \gamma_{jb} T^{ab}; \quad S = \gamma^{ij} S_{ij}\).

Metric \((c = G = 1)\):

\[(B1) \quad ds^2 = -\alpha^2 dt^2 + \gamma_{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt) .\]

Two constraint equations \((R\) is the Ricci scalar):

\[(B2) \quad R + K^2 - K_{ij} K^{ij} = 16\pi \varrho ;\]

\[(B3) \quad D_j (K^{ij} - \gamma^{ij} K) = 8\pi S^i .\]

Evolution equation for the space metric \(\gamma_{ij}\):
(B4) \[ \frac{\partial \gamma_{ij}}{\partial t} = -2 \alpha K_{ij} + D_t \beta_j + D_j \beta_t . \]

Evolution equation for the extrinsic curvature \( K_{ij} \):

(B5) \[ \frac{\partial K_{ij}}{\partial t} = \alpha \left( K_{ij} - 2K_{ik} K^k_j + K K_{ij} \right) - D_i D_j \alpha - 8\pi \alpha \left[ S_{ij} - \frac{1}{2} \gamma_{ij} (S - \rho) \right] + \beta^k \frac{\partial K_{ij}}{\partial x^k} + K_{ik} \frac{\partial \beta^k}{\partial x^j} + K_{kj} \frac{\partial \beta^k}{\partial x^i} . \]

**APPENDIX C**

We read in Weyl [8] a penetrant analysis of a fundamental property of the general-relativistic coordinate systems. He wrote [4b]: "... the concept of relative motion of several bodies has, as the postulate of general relativity shows, no more foundation than the concept of absolute motion of a single body. Let us imagine the four-dimensional world as a mass of plasticine traversed by individual fibers, the world lines of the material particles. Except for the condition that no two world lines intersect, their pattern may be arbitrarily given. The plasticine can then be continuously deformed so that not only one but all fibers become vertical straight lines." It is clear that this consideration implies that there exists always a coordinate transformation, which allows us to pass from any coordinate system for which some bodies are in motion to a co-moving coordinate system for which all these bodies are at rest.

Now, no class of privileged coordinate systems exists in GR, and any physical effect must be frame independent [1]. Consequently, the fact that bodies at rest cannot generate gravitational waves has a general significance: no coordinate system exists for which the motions of the bodies generate gravitational waves.

Remark that these considerations hold for the general case in which both gravitational and non-gravitational forces are present. For a different proof, founded on the Einstein field equations, that all the general-relativistic motions can be geodesically described, see our paper of ref. [9].

An immediate corollary: the gravitational field of a body, whose motion is geodesic, moves en bloc with the body, and is propagated instantaneously. In special relativity we have a partial analogue: the static-electromagnetic fields created by an electric charge in a rectilinear and uniform motion (a Minkowskian geodesic motion) move along with the charge, and are propagated instantaneously [10]. (This corresponds perfectly to the results found by an observer in a rectilinear and uniform motion, who travels with respect to the charge at rest).
This fact has been considered a paradox by some physicists, who have tried to get rid of it with the gratuitous surmise that it is hold only for infinite spatio-temporal motions of the electric charge.

An instantaneous propagation of a field is not always in contradiction with the theory of relativity.

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