HIGH DEGREE DIOPHANTINE EQUATION \( c^q = a^p + b^p \)

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Abstract. The main idea of this article is simply calculating integer functions in module (Modulated Function and digital function). This article studies power and exponent functions, logarithm function between integer modules, module of complex number, the analytic method of digit by digit, and modular integration and differentials in discrete subspace. Finally prove a condition of non-solution of Diophantine Equation \( a^p + b^p = c^q \): \( a, b > 0, (a, b) = (b, c) = 1, p, q \geq 41, p \) is prime.

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Truth is ordinary.

1. INTRODUCTION

When talking about high degree diophantine equation the most famous result is Fermat’s last theorem. This article applies purely algebraic method to discuss unequal logarithms of finite integers under module, and get a nice result on equation \( c^q = a^p + b^p \).

2. MODULATED FUNCTION

In this section \( p \) is a prime greater than 2 unless further indication.

Definition 2.1. Function of \( x \in \mathbb{Z} \): \( c + \sum_{i=1}^{m} c_i x^i \) is power-analytic (i.e power series). Function of \( x \): \( c + \sum_{i=1}^{m} c_i e^{ix} \) is linear exponent-analytic of bottom \( e \).

\( e, c, c_i, i \) are constant integers. \( m \) is finite positive integer.

Date: May 4, 2010.

2000 Mathematics Subject Classification. Primary 11D41, Secondary 11T06, 11C08, 11C20, 13F20.

Key words and phrases. High degree Diophantine equation, Modulated function, Modulated logarithm, Digital Analytic, Discrete geometry, Fermat’s Last Theorem.
**Theorem 2.2.** Power-analytic functions modulo \( p \) are all the functions from mod \( p \) to mod \( p \), if \( p \) is a prime. And \((1, x^i), (0 < i \leq p - 1)\) are linear independent group. (for convenience always write \( 1 \) as \( x^0 \), and \( x^{p-1} \) is different from \( x^0 \))

**Proof.** Make \( n \)-th order matrix \( X \):

\[ X_{i,1} = 1, X_{ij} = i^j - 1 \quad (1 \leq i \leq p, 2 \leq j \leq p) \]

The columnar vector of this matrix is the values of \( x^i \). This matrix is Vandermonde’s matrix and its determinant is not zero modulo \( p \). The number of functions in mod \( p \) and the number of the linear combinations of the columnar vectors are the same \( p^p \). So the theorem is valid. \( \square \)

A proportion of the row vector is values of exponent function modulo \( p \).

**Theorem 2.3.** Exponent-analytic functions modulo \( p \) and of a certain bottom are all the functions from mod \( p - 1 \) to mod \( p \), if \( p \) is a prime.

**Proof.** From theorem 2.2, \( p - 1 \) is the least positive number \( a \) for:

\[ \forall x \neq 0 \text{ mod } p \left( x^a = 1 \text{ mod } p \right) \]

or exists two unequal number \( c, b \) mod \( p - 1 \) such that functions \( x^c, x^b : x^c = x^b \) mod \( p \). Hence exists \( e \) whose exponent can be any member in mod \( p \) except 0. Because the part of row vector in matrix \( X \) (as in the previous theorems) is values of exponent function, so this theorem is valid. \( \square \)

**Theorem 2.4.** \( p \) is a prime. The members except zero factors in mod \( p^n \) is a multiple group that is generated by single element \( e \) (here called generating element of mod \( p^n \)).

**Definition 2.5.** (Modulated Logarithm modulo \( p^{m} \)) \( p \) is a prime, \( e \) is the generating element as in the last theorem:

\[ lme(x) : \ x \in \mathbb{Z}(x, p) = 1 \rightarrow \text{ mod } p^{m-1}(p - 1) : \ e^{lme(x)} = x \text{ mod } p^{m} \]

Similarly it’s written as that \( y = lmb(x) \text{ mod } p^{m-1}, b = e^{p-1} \text{ mod } p \). Because for \( x \) such that \( x = 1 \text{ mod } p \) there is only one \( y \text{ mod } p^{m-1} \) meeting \( b^y = x \text{ mod } p^m \). \( p \) is prime.

**Lemma 2.6.**

\[ lme(-1) = p^{m-1}(p - 1)/2 \text{ mod } p^{m-1}(p - 1) \]

\( p \) is a prime. \( e \) is defined in mod \( p^m \).

**Lemma 2.7.** The power series expansions of \( \log(1 + x), (|x| < 1) \) (real natural logarithm), \( exp(x) \) (real natural exponent), and the series for \( exp(\log(1 + x)), (|x| < 1) \) with the previous two substituted into are absolutely convergent.

**Definition 2.8.** Because:

\[ a \text{ mod } p^m = kp^n \leftrightarrow a = 0 \text{ mod } p^{m+n} \]

\( a, k \in \mathbb{Z} \), it’s valid to make the rational number set modulo integers if it applies to equations (written as \( a/p^m = 0 \text{ mod } p^n \)).

**Definition 2.9.** \( p^i||a \) means \( p^i|a \) and not \( p^j|a, j > i \).
Theorem 2.10. \( p \) is a prime greater than 2. Defining
\[
E = \sum_{i=0}^{n} \frac{p^i}{i!} \mod p^m
\]
n is sufficiently great and dependent on \( m \). \( e^{1-p^m} = E \mod p^m \), \( e \) is the generating element (Here the logarithm: \( \text{Im}_c(x) \) is written as \( \text{Im}(x) \)). Then for \( x \in \mathbb{Z} ::
\[
E^x = \sum_{i=0}^{n} \frac{p^i}{i!} x^i \mod p^m
\]
\[
\text{Im}_E(px + 1) = \sum_{i=1}^{n} \frac{(-1)^{i+1}p^{i-1}}{i} x^i =: f(x) \mod p^{m-1}
\]
\[
\text{Im}_E(x^{1-p^m}) = \text{Im}(x^{1-p^m})/\text{Im}(E) = \text{Im}(x^{1-p^m}) = \text{Im}(x) \mod p^{m-1}.
\]
In fact \( m \) is free to choose. And \( E \) is nearly \( \exp(p) \).
If \( 2 \mid x \) this theorem is also valid for \( p = 2 \).

Proof. To prove the theorem, contrast the coefficients of \( E^x \) and \( E^f(x) \) to those of \( \exp(px) \) and \( \exp(\log(px + 1)) \). \( \square \)

Theorem 2.11. Set \( d_m : p^{dm} \mod p^m/m! \). It’s valid \( d_m(>p^n) > d_p^n \).

Theorem 2.12. (Modulated Derivation) \( p \) is a prime greater than 2. \( f(x) \) is a certain power-analytic form \( \mod p^m \), \( f^{(i)}(x) \) is the \( i \)-th order real derivation (hence called modulated derivation relative to the special difference by \( zp \) as this theorem): \( (n \) is sufficiently great)
\[
f(x + zp) = \sum_{i=0}^{n} \frac{p^i}{i!} z^i f^{(i)}(x) \mod p^m
\]
If \( 2 \mid z \) this theorem is also valid for \( p = 2 \).

Definition 2.13. (Example of Modulated Function) Besides taking functions as integer function, some functions can be defined (for increasingly positive integer \( m \)) by equations \( \mod p^m \), even though with irrational value as real function in form sometimes. This kinds of function is called Modulated Function. For example:
\[
(1 + p^2 x)^{\frac{r}{p}} \mod p^m
\]
is as the unique solution of the equation for \( y ::
\[
1 + p^2 x = y^p \mod p^{m+1}
\]
By calculation to verify:
\[
plm_E(y) = \text{Im}_E(1 + p^2 x) = \sum_{i=1}^{n} (-1)^{i+1}p^{i-1}(px)^i \mod p^{m+1}
\]
Lemma 2.14.
\[
(E^x)' = pE^x \mod p^m
\]
This modulated derivation is not necessary to relate to difference by \( zp \), it’s valid for difference by 1.
Lemma 2.15. The derivation of \((1 + x)^{1/p} \mod p^{m+2}\) at the points \(x : p^2 \mid x\) is:
\[
(1 + x)^{1/p}' = \left(\frac{\text{E}_{l,m E}(1+x)}{p}\right)' \mod p^m
\]
\[
= \frac{1}{p} \left(1 + x\right)^{1/p} \frac{1}{1 + x} \mod p^m
\]

Theorem 2.16. Because
\[
1 - x^{p^{n-1}(p-1)} = \begin{cases} 
0, & (x \neq 0 \mod p) \\
1, & (x = 0 \mod p) 
\end{cases} \mod p^n
\]
and in \(x = 0 \mod p\), any power-analytic function is of the form:
\[
\sum_{i=0}^{n-1} a_i x^i
\]
hence the power-analytic function is of the form:
\[
\sum_{i=0}^{n-1} (1 - (x - i)^{p^{n-1}(p-1)}) \left(\sum_{k=0}^{n-1} a_{ki}(x - i)^k\right) \mod p^n
\]

Theorem 2.17. Modulated Derivation of power-analytic and modulated function \(f(x) \mod p^{2m}\) can be calculated as
\[
f'(x) = (f(x + p^m) - f(x))/p^m \mod p^m
\]
The modulated derivations of equal power analytic functions \(\mod p^{2m}\) are equal \(\mod p^m\).

Theorem 2.18. Modulated \(plm(x)\) is power-analytic modulo \(p^m\).

3. Some Definitions

In this section \(p, p_i\) is prime. \(m, m'\) are sufficiently great.

Definition 3.1. \(x \rightarrow a\) means the variable \(x\) is set value \(a\).

Definition 3.2. \(a, b, c, d, k, p, q\) are integers, \((p, q) = 1:\)
\[
[a]_p = \text{[a + kp]}_p \\
[a]_p + [b]_p = [a + b]_p
\]
\([a = b]_p\) means \([a]_p = [b]_p\).
\[
[a]_p [b]_q = \{x : [x = b]_p, [x = b]_q\}_{pq} \\
[a]_p \cdot [b]_p = [ab]_p
\]
Easy to verify:
\[
[a + c]_p [b + d]_q = [a]_p [b]_q + [c]_p [d]_q
\]
\[
[ka]_p [kb]_q = k[a]_p [b]_q
\]
\[
[a^k]_p [b^k]_q = ([a]_p [b]_q)^k
\]

Definition 3.3. \(\sigma(x)\) is the Euler’s character number as the least positive integer \(s\) meeting
\[
\forall y((y, x) = 1 \rightarrow [y^s = 1]_{x})
\]
Definition 3.4. The complete logarithm on composite modules is complicated. But it can be easily defined

\[ [\text{l}(x)]_{p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}} := [\text{l}(x)]_{p_1^1} [\text{l}(x)]_{p_2^2} \cdots [\text{l}(x)]_{p_m^{n_m}} \]

\( p_i \) is distinct primes. This definition will be used without detailed indication.

Definition 3.5. \( x = q[a] : [a = x]_q, 0 \leq x < q \)

Definition 3.6. For module \( p^i : F_{p^i}(x)(:= p^a) \) means \( F_{p^i}(x) \mid x \); For composite module \( Q_1 Q_2 \) meeting \( (Q_1, Q_2) = 1 \) : \( F_{Q_1 Q_2}(x) := F_{Q_1}(x) F_{Q_2}(x) \).

Definition 3.7. \( P(q) \) is the multiple of all the distinct prime factors of \( q \).

Definition 3.8. \( Q(x) = \prod_i [p_i]_{p_i^{m_i}}, p_i \) is all the prime factors of \( x \). \( m \) is sufficiently great.

Theorem 3.9. \( 2|q \rightarrow 2|x: \)

\[ [Q(q)\text{l}(1 + xq) = \sum_{i=1}^{[a]} (xq)^i (-1)^{i+1}/i]_q \]

The method of proof is getting result in powered prime module and synthesizing them in composite module.

Definition 3.10. \[ [a^{1/2} := e^{p-1}[\text{l}(a)]/2]_p \]

It can be can proven that \[ [a^{1/2}(1/a)^{1/2} = -1]_p \]

Definition 3.11. \[ [\text{l}(pk) = p\text{l}(k)]_{p^m} \]

\( p \) is a prime.

Theorem 3.12. \[ p\text{l}(x) = (x^{p^m(1-p^m)} - 1)/p^m, [x \neq 0]_p \]

\[ p\text{l}(x) = \sum_{i=1}^{m'} (-1)^{i+1}((x - x^{p^m})/x^{p^m})^i/i, [x \neq 0]_p \]

Hence \[ [p\text{l}'(x) = 1/x]_{p^m}, [x \neq 0]_p \]

Definition 3.13. \( [y = x^{1/a}]_p \) is the solutions of equation \[ |y^a = x|_p \]. When \( (a, p - 1) \neq 1 \), \( x^{1/a} \) is multi-valued function or empty at all.
4. The Modulus Of Prime $p = 4n - 1$ On Complex Numbers

In this section $p$ is prime other than 2.

For prime $p = 4n - 1$ the equation $[i^2 = -1]_p$ has no solution, then it's suitable to extend the module to complex numbers.

**Definition 4.1.**

$$\mathbb{PZ} = \{x + yi : x, y \in \mathbb{Z}\}$$

For $p = 4n - 1$, define

$$[x : x \in \mathbb{PZ}]_{p^n} = [x + tp^n : t \in \mathbb{PZ}]_{p^n}$$

This definition is sound and good because there is no zero factor other than $p^j$.

**Definition 4.2.** For prime $p = 4n - 1$. $a, b \in \mathbb{Z}$. Define $e^i$ in $\mathbb{PZ}$, for any $j \in \mathbb{Z}$ and some $a, b$:

$$[e^{ji}] = \frac{2ab}{a^2 + b^2} + i \frac{a^2 - b^2}{a^2 + b^2}$$

$$[e^{(1-p^m)i}] = \sum_{j=0}^{n} \frac{p^j}{j!} =: E^i_p$$

($n$ is sufficiently great and dependent on $m$). Analyzing the group formed by all solutions of $[z^*z = 1]_p$ in mod $p$ (count $p+1$ and $[z^p = z^*]_p$) can find this definition is all right.

Define $[e^{a+bi} = e^a e^{bi}]_{p^n}$. It can be found the results on exponent’s and logarithm’s expansion are valid in $\mathbb{PZ}$ similar to the form as in $\mathbb{Z}$.

**Definition 4.3.** $(q_1, q_2) = 1$, $a, b, a', b' \in \mathbb{Z}$:

$$[a + bi]_{q_1} [a' + b'i]_{q_2} = [a]_{q_1} [a']_{q_2} + [i(b)]_{q_1} [i(b')]_{q_2}$$

Also define the triangular functions by $e^z$.

**Definition 4.4.** For mod $p^m, p = 4n + 1$, $\omega : [\omega^2 = -1]_p$ was chosen as pseudo-imaginary unit $i$ (and treat $\omega, i$ differently), define for all equations that is formed by functions of arguments $z_1 + z_2, z_1, z_2 \in \mathbb{Z}$, with property pseudo-conjugation:

$$[z = a + bi = 0]_{p^m} \rightarrow [z^* = a - bi = 0]_{p^m}$$

$a, b \in \mathbb{Z}$. Then the $i$ has the similar property like the true imaginary unit because from the above condition, it’s implied:

$$[a + bi = 0]_{p^m} \rightarrow [a = 0, b = 0]_{p^m}$$

Elements like $a, b$ are called pseudo-real.

Notice that the original value of $i$: $\omega$ should be treated as pseudo-real.

For $i$ (pseudo-imaginary unit for $p = 4n + 1$) actually real, Strengthen $i$ as $[1]_{p-1}\iota_{p^m}$ (but there will be trouble in composite modulus regarding exponent) and set pseudo-conjugation to all the equations involved. Defining $e^i$ for pseudo-real $a, b$ meeting

$$[e^i = a + bi]_{p^m}, [a, b \neq 0]_p$$

$$[e^{i(1-p^m)}] = E^i_p$$

$$([e^i]^*) = e^{-i} = a - bi]_{p^m}$$
it means to meet the harmony in real and complex senses. Think about the group formed by $[z : zz^* = 1]_p$, whose elements count $p - 1$ ie. all non-zero $[n]_p$, it can be found the above identities are possible.

Complete logarithm is complicated, but logarithm mod $p^n$ is easy, it will be used without detailed indication.

Pseudo-conjugation will be used without detailed indication.

5. Digital Analytic

In this section $p$ is prime unless further indication. $m, m'$ are sufficiently great.

Definition 5.1.

\[ T(q', x) = y : [x = y]_{q'}, 2|q' : -q'/2 < y < q'/2 + 1 \text{ otherwise } -q'/2 < y < q'/2 \]

Digit can be express as

\[ D_{q'}(x) := (T(q^n, x) - T(q^{n-1}, x))/q^{n-1} \]

Digital function is digit in digits’s power analytic function. Also define

\[ D_q(x) := D_p((x - T(q, x))/q) \]

Definition 5.2. Independent Digital variables (Functions) is the digits can not be constrained in root set of a nonzero digital function.

Theorem 5.3. Resolve function digit by digit. For an integer function $f(x) \mod p^m$, whose value can be express by its digits. The digit of the function is determined by its arguments’s digits, then the digit can be express by Digital function

\[ D_{p^k}(f(x)) = \sum_j a_j \prod_{i=1}^n D_{p^j}(x) \]

$0 \leq j_i \leq p - 1$. With this method of Digit by Digit the whole function can be resolved in the similar form for each digits of the function.

Digital functional resolution has some important properties, it can express arbitrary map $f(x)$ between the same module.

Definition 5.4. Digital functions group

\[ [s_i = f_i(x, j), i, j = 0, \cdot \cdot, n] \]

called square group or square function.

Theorem 5.5. Functional independent square group is invertible.

Proof. Independence means the function value travel all, it’s one to one map and invertible:

\[ [x_i = g_i(s, j), i, j = 0, \cdot \cdot, n] \]
6. Modular Integration

In this section \( p \) is prime unless further indication. \( m, m' \) are sufficiently great.

**Definition 6.1.** The size of a set is called the freedom of the set.

**Definition 6.2.** With consideration of mod \( p \):

\[
\delta(x_i - C_i) := \begin{cases} 
1 & [(x_i, C_i)] \mod p \\
0 & \text{otherwise}
\end{cases}
\]

**Definition 6.3.** The algebraic derivation of the shortest expression (clean expression) of digital function is called clean derivation. The clean derivation of \([f(x)]_p\) is denoted as \( f^D(x), Df(x)/Dx \) formally. This definition will be used without detailed indication. The real algebraic derivation, or modulated derivation is denoted as \( f'(x), df(x)/dx \) for \( f(x) \) or \( f(x)^{p^n} \).

**Theorem 6.4.** The clean derivation expressed in algebraic derivation is

\[
[f^D(x) = d_k f(x)]_p
\]

with \( k \) sufficiently great, and \( d_k \) is:

\[
\begin{align*}
d_0 & := d/dx, d_1 := d_0 + (d^p/d^px)/p! \\
d_n & := d_{n-1} + (d^{n(p-1)+1}/d^{n(p-1)+1}x)/(n(p-1) + 1)!
\end{align*}
\]

**Definition 6.5.** For convenience it’s taken that

\[
[1/x := x^{p(p-1)-1}]_p^2
\]

when calculate digital functions.

**Definition 6.6.**

\[
[f^D(x) = -\sum_{t=0}^{p-1} f(t)(t - x)^{p-2}]_p
\]

or concisely

\[
[= -\sum_t f(t)(t - x)^{p-2}]_p
\]

Prove this by the formula of power sum and bernoulli number.

**Definition 6.7.** The reduced function is clean and without a term that has a factor of the highest degree on single argument.

**Theorem 6.8.**

\[
[I^t(x) := -\sum_{i=0}^{p-2} x^{p-1-t}i^{i+1}/(i + 1)]_p
\]

\[
[I'_{t_0}(x) := I^t(x) - I^{t_0}(x)]_p
\]

then

\[
\int_0^t f(x)dx = \sum_x f(x)I'(x)]_p
\]

\( f \) is reduced and clean.

**Theorem 6.9.** The \( I^t(C) \) has \( p - 1 \) distinct values and two zero values if \([C \neq 0]_p\).
**Proof.** Take the clean function as vector with units $x^n$. If the equation $I^t(x) = C$ has roots, observe the freedom of the set generated by transform

$$f(x) \to \sum_x f(x)(I^t(x) - C)$$

\[\square\]

**Theorem 6.10.**

$[I^t(x) \neq -t]_p, [t \neq 0]_p$

$I^t(x) = -I^t(t)$

**Definition 6.11.**

$[f(x) \cdot I^t(x) := \sum_x f(x)I^t(x)]_p$

$[f(x_i) \cdot \prod_i I^t_i(x_i) := \sum_x f(x_i) \prod_i I^t_i(x_i)]_p$

**Definition 6.12.**

$[Dt := I^t(t) - I^{t-1}(t) - I^t(1)]_p$

$[f(t) \cdot Dt := f(t) \cdot I^t(t) - f(t) \cdot I^{t-1}(t) - f(t) \cdot I^t(1)]_p$

$[f(t) \cdot I^t(1) := \sum_x f(x)I^t(1)]_p$

Define the *modular integration* in an area $A$:

$$\int_A f(x, x_i) \prod_i^D x_i := \sum_{(x, i)} \delta_A(x, x_i)(f(x, x_i) \cdot \prod_i^D x_i)]_p$$

$$[\delta_A(x, x_i) = \sum_{(x, i) \in A} \delta(x, x_i)]_p$$

$$\int_a^b f(x) \cdot D x := \sum_{x \in (a, b)} f(x) \cdot D x]_p$$

Obviously

$$\int_0^x \delta(x) \cdot D x = \int_0^x \delta(x)D(x + C)]_p$$

$$\int_0^x f(x) \cdot D x = \int_0^x f(x)D(x + C)]_p$$

$$[\delta(x) \cdot D x = -I^x(1)]_p$$

$$\forall x[f(x) \cdot D x = 0]_p \leftrightarrow \forall x[f(x) = 0]_p$$

**Definition 6.13.** Define

$[f^t(x) := f(x) \cdot D x]_p$

$[f^t(t) := \sum_x f^t(x)]_p$

$[f^\Delta(x) : f^\Delta(x) - f^\Delta(x - 1) = f^t(x), f^\Delta(0) = 0]_p$

$$\int f(x) \cdot D x := f^\Delta(x) + C$$
\(f^{\Delta}(x)\) (is called \textit{original function}) is defined by \(f^{I}(x)\) uniquely except for a constant difference. For example

\[
\left[\int\delta(x)\,Dx = -(x^p - x)/p + C = x \text{lm}(x)|_{x \neq 0} + C = \sum_{x=0}^{x} I^1(z) + C\right]_p
\]

Note that the function \(x \text{lm}(x)\) is not digital function, in fact it’s defined in mod \(p^2\), this means the integration is depend on integral track, especially as the track \((a, b)\) crosses zero mod \(p\).

It’s obvious that

\[
[DF^{I}(x)/Dx = f(x) - f(x - 1)]_p
\]

The definition is extended to multi-arguments function as

\[
[f^{I}(x, x_i) := f(x, x_i) \cdot \prod Dx_i]_p
\]

\[
[f^{\Sigma}(x, x_i) := (\prod_{i} f^{I}(x, x_i))]_p
\]

\[
[f^{\Delta}(x, x_i, x_i) : f^{\Delta}(x_0, x_i, x_i) - f^{\Delta}(x_0 - 1, x_i - 1, x_i) = f^{I}(x, x_i), f^{\Delta}(x, x_i - 1, 0, x_i + 1) = 0]_p
\]

\[
(\prod_i \int f(x, x_i) \prod Dx_i := f^{\Delta}(x, x_i) + C(x, x_i))
\]

**Definition 6.14.** Define \textit{modular derivation} of digital function formally as

\[
[f^{D}(x) := \sum_{t} -f(x^p + t^p)/t^p]_p
\]

It’s the inverse of the modular integration.

**Theorem 6.15.**

\[
(\prod_{x}^{b_i} f(x, x_i) \prod Dx_i = (\prod_{x}^{b_i} \Delta_{x_i} f^{\Delta}(x, x_i))]_p
\]

\(a_i, b_i\) are constants.

**7. Discrete Geometry and Subspace**

In this section \(m\) is sufficiently great. \(p\) is prime.

**Definition 7.1.** Define \textit{modular differential} as the inversion of \textit{square (modular) linear integration}:

\[
(\prod_{x}^{b_i} f(x, x_i) \prod Dx_i = (\prod_{x}^{b_i} \Delta_{x_i} f^{\Delta}(x, x_i))]_p
\]

\(a_i, b_i\) are constants.
Definition 7.2. The so-called *discrete geometry* always discuss the square box
\[(x_1, x_2) = ([a, b], [c, d]), (x_i,) = ([a_i, b_i],)\]
And take these square box of different dimensions into consideration of real geometry. Obviously it can be find the result is similar to that in real differential geometry
\[\int_D D^\wedge F = \int_{\partial D} F|_p\]
\(F\) is *antisymmetrical modular differential tensor*.

Definition 7.3.
\[D(G + G') = DG + DG', D^\wedge (G + G') = D^\wedge G + D^\wedge G'\]
\(G, G'\) is differential tensor.
\[\left[D(K(X) \bigwedge_i Dx_{\sigma(i)}) = DK(X) \bigwedge_i Dx_{\sigma(i)}\right]_p\]
\[\left[D^\wedge(K(X) \bigwedge_i Dx_{\sigma(i)}) = DK(X) \bigwedge_i Dx_{\sigma(i)}\right]_p\]
\(\sigma\) is a map in \(N\).

Definition 7.4. Use \(\Delta_{k,x}\) and \(D_{k,x}\) to express difference and differential, and for example, \(D_{k,x}^2\) means \(D_{k,x} \cdot D_{k,x}\), \(D_{2,x}^2 D_{1,x}\) means \((D_{2,x} \cdot D_{2,x}) \otimes D_{1,x}\).

Definition 7.5. For the discrete space
\[F(X) = (f_0(x_i),, f_i(x_j),, f_{n-1}(x_j),)]_p\]
\((F(X)\) is square invertible), the *subspace* in digital functions set
\[\text{sub } f_{k \in A}(X)\]
is the module of the ideal generated from \(f_{k \in A}(X)\).

Definition 7.6. *Span function* of arguments \((x_i,)\) is digital function
\[\left[F(\cdot, x_i, \Delta_{k,x_i},)\right]_p\]

Definition 7.7. The difference of span function is defined by
\[\left[\Delta' \Delta_{X_i} = 0\right]_p\]
For all \(i\).

Theorem 7.8. *The difference can be calculate by operator*
\[\left[\Delta = \sum_{n=1}^{m} \left(\sum_{i} \Delta_{x_i} \frac{D}{Dx_i} \right)^n / n\right]_p\]

Theorem 7.9.
\[\Delta (f(x)g(x)) = g(x)\Delta f(x) + f(x)\Delta g(x) + \Delta f(x)\Delta g(x)\]

Definition 7.10. The *Correspondence* between clean span function \(S\) and tensor \(T\) is substitution:
\[S \rightarrow T = TC(S) : \Delta_{k,x_i} \rightarrow D_{k,x_i}, T \rightarrow S = SC(T) : D_{k,x_i} \rightarrow \Delta_{k,x_i}\]

Definition 7.11. The sum of all terms of the lowest degree of the difference of original arguments in the span function \(f\) is denoted as \(LD(f)\)
Definition 7.12. The differential tensor in subspace \( f_{i \in A}(x, j) \) is defined as module of all
\[
\prod_{i} D_{\sigma(i)} \left( \prod_{k \in A} f_{i}^{k}(X) \right)_{p}
\]
\( \sigma \) is arbitrary map in \( N \).

Definition 7.13. The span function in subspace \( f_{i \in A}(x, j) \) is defined as module of all
\[
\prod_{i=1}^{n} \Delta_{\sigma(i)} \left( \prod_{k \in A} f_{i}^{k}(X) \right)_{p}, n \geq 0
\]

Theorem 7.14. \( F = 0 \rightarrow \Delta F = 0 \) \( \text{sub} f_{i \in A}(x, i) \)
\( F \) is a span function.

Theorem 7.15. \( G = 0 \leftrightarrow SC(G) = 0 \) \( \text{sub} f_{i \in A}(x, i) \)
\( G \) is is differential tensor.

Theorem 7.16. \( T = 0 \rightarrow DT = 0 \) \( \text{sub} f_{i \in A}(x, i) \)
\( T \) is a differential tensor.

Theorem 7.17. If a function in subspace \( g(x, j) = 0 \) \( \text{sub} f_{i \in A}(x, j) \)
Then
\( g(x, i) = C \) \( \text{sub} f_{i \in A}(x, j) \)

Proof. The span function in sub \( f_{i \in A}(x, j) \) is in fact the substitution
\( f_{i \in A}(x, j) \rightarrow 0 \)
and taking the arguments of \( f_{i}(x, j) \) to express the functions. \( \square \)

Definition 7.18. Derivation of the group of functions \( (., f_{i}(.,.), 0 \leq i < n) \)
\( \mathcal{G}_{(., f_{i}(.,.),)} := \left[ \frac{\partial(., f_{i}(.,.),)}{\partial(., f_{i}(.,.),)} \right] \)
is called geometry derivation. If the derivation is operated on clean functions group in a prime module then it’s called clean geometry derivation.

Definition 7.19. For convenience for the latter define
\( \delta(x) := 1 - x^{p-1} \)

Definition 7.20. Express a square digital function as sum of delta branches
\( f_{j}(x_{0}, x_{i}, x_{n}) = a_{j}k_{0}, k_{i}, k_{n}, \delta(x_{0} - k_{0}, x_{i} - k_{i}, x_{n} - k_{n}) \) \( p \)
The clean geometry derivation in \( (., x_{i}) = (., k_{i}) \) is only depend on the delta branches of
\( \left[ (a_{i} \delta(., k_{i}, k_{i}/k))_{i} \right]_{p} \)
\( \delta(., x_{i} - k_{i}) := \delta(., k_{i}, k_{i-1}, k_{i+1}) \) \( (k_{i}/k) := (k_{i-1}, k, k_{i+1}, k) \)
The points as in the supporting set of these delta branches are called relative chain of the point \( (., k_{i}) \). The points in relative chain of point \( P \) is denoted as \( RC(P) \).
Theorem 7.21. For a square digital function, the relative chain of the point of \( P \) are restricted, and the functional values of members of \( R \) are distinct. Then construct like this: Alter the function’s value by adding on delta branches but do not alter that of the chain \( R \), hence to form a invertible function with the clean geometry derivations are unchanged at the point \( P \), and the determinant of the square sub-matrix of the partial derivation matrix (of dimensions at least 2) of the square digital functions, is also unchanged.

8. Diophantine Equation \( a^p + b^p = c^q \)

\( m \) are sufficiently great.

Definition 8.1. For real number \( a \)

\[ [a] = \max \{ x \in \mathbb{Z} : x \leq a \} \]

Theorem 8.2. \( 0 < b < a < q/P^3(q), P^{11}(q) \, | \, q, (a, q) = (b, q) = 1. \) For any prime \( p/q \) meeting \( [a^2 \neq b^2]_p \). Then

\[ [\text{ln}(a) \neq \text{ln}(b)]_q \]

Proof. \( r = P(q) = \prod_i p_i \). If \( [\text{ln}(a) = \text{ln}(b)]_q \).

Considering module \( q^2 r \). Make

\[ f(x, y) = \prod_i \left[ T((q^2, p^m_i), (a + rx)^{p_i-1} + (b + ry)^{p_i-1} - a^{p_i-1} - b^{p_i-1}) + (q^2, p^m_i) s \right] \]

\[ x = \sum_i q_i D_i, q_i = q/(q, p^m_i) \]

\[ y = \sum_i q_i D'_i \]

set

\[ D_i = T((q, p^m_i), D_i), D'_i = T((q, p^m_i), D'_i), [x = D_i]_{(q^2, p^m_i)}, [y = D'_i]_{(q^2, p^m_i)} \]

The function \( f(x, y) \) can be expressed digit by digit in the square function

\[ [F_{ji}, (D_{li}(x), D_{li+1,i}(x), D_{li}(y), D_{li+1,i}(y),)] = D_{pj}([f(x, y)]_{(p^m_i, p_i)}, p_i, p_i)_{(q^2 r)} \]

\[ D_{li}(x) := D_{pj}^{\prime}(D_i), p_i, q \]

\( F_{ji} \) depends on \( D_k, D'_k \). The derivation on \( D_{li}(x), D_{li}(y) \) of the square group \( F_{ji} \)

\[ \text{is derived from modulated derivation by noticing the special power of the function } f(x, y): \]

\[ [df(x, y)]_{(p^m_i, q^2 r)} = \sum_j p_j^j dF_{ji}([p^m_i, q^2 r]) \]

\( dF_{ji} \) is in the form of the real algebraical differential on variable forms of \( D_{li}(x), D_{li}(y) \).

It can be observed that replace the differential unit \( dD_{li}(x), dD_{li}(y) \) with free digits, then the according replacements of \( df(x, y) \) is generated at sub-module of mod \( (q^2 r, p^m_i) \). If the derivation on \( D_{li}(x), D_{li}(y) \) between several \( |F_{ji}|_{p_i} \), are independent of each others then the according digits of the replacements of these \( df(x, y) \) are free of each others. Calculate the case of the derivation at \( (x, y) \to (0, 0), s \to 0 \) and that \( \Delta x, \Delta y \) replace \( dx, dy \). The distinct generated function values mod \( q^2 r \) count less than

\[ rab(q/a + q/b) \]
Make
\[ \text{rab}(q/a + q/b) < q^2/r \]
to formulate for some \( p_i \)
\[ \lfloor \prod_{t > 1, p_i | q^2} DF_{ki} \rfloor_{(x,y)=(0,0)} = 0 \] _p_i.

Further more for some \( k \) at \((x, y) = (0, 0)\)
\[ [DF_{ki} = 0]_p, \text{ mod } [DF_{ji}]_p, p_i^j | q^2, j \geq k \]
This results is proved by reasoning in that: the replacements of \( df \) is
\[ CDr := Cr \sum q_i (D_i/a + D_i'/b) =: Cr(q_i A(= D_i/a + D_i'/b)) + (p_i^m, q)B \]
Then varying \( A \) (or \( B \)) is to draw a straight line in 2-dimensional space
\[(\text{mod}(p_i^m, q^2), \text{mod} q^2/(p_i^m, q^2))\]
like
\[ ([D]_{(p_i^m, q^2)}, [D]_q^2/(p_i^m, q^2)) \]
Presume doesn’t exist \( D_1, D_2 : 0 < |D_1|, |D_2| < p_i ;\)
\[ [D_1/a = D_2/b]_{(q/r,p_i^m)} \]
Consider points \((x, y) = (0, 0)\) and its relative chain in mod \((q^2r, p_i^m)\). Operate on the domain of \( x, y \) to set, except the first digit invariable, \( f(x, y) \) all distinct mod \((p_i^m, q^2r)\), like the theorem [7,21] with functional values of the point \((x, y) = (0, 0)\) and its relative chain unchanged, and also set
\[ [f(RC(0, q_i s)) \to f(RC(0, 0)) + p_i^{k-1}s]_{(q^2r,p_i^m)}, s = T(p_i, s) \]
\([s = s'], p_i , s' \) is choose to not disturb the digits of \( f(RC(0, 0)) \) except the \( k \)-th digit and the \( RC(0, q_i s) \) and RC(0, 0) are created by adding the same modular value on \((0, q_i s), (0, 0) \) respectively.
After these it’s valid that for the new digits \( F_{ji} \)
\[ 0 = \delta(F_{ki} - F_{vi}) \]
\[ DF_{ki} = Ds = DF_{ki} | p \text{ sub } F_{ji}, F_{vi} - F_{ki}, p_i^j | q^2, p_i^r = (p_i^m, q^2r), j \neq k \]
\[ [F_{ki} = C]_p \text{ sub } F_{ji}, F_{vi} - F_{ki}, p_i^j | q^2, p_i^r = (p_i^m, q^2r), j \neq k \]
it’s obviously not valid.
If exists \( D_1, D_2 : 0 < |D_1|, |D_2| < p_i ;\)
\[ [D_1/a = D_2/b]_{(q/r,p_i^m)}, (D_1, D_2) = 1 \]
discussion on
\[ [f(x, y) = T(p_i^{10}, (d_1 + p_i x)p_i^{d_1 - 1} + (d_2 + p_i y)p_i^{d_2 - 1} - d_i^{p_i - 1} - d_i^{p_i - 1} + p_i^{10}s_{p_i 1}) \]
\[ p_i < \max(|d_1|, |d_2|) < p_i^2, d_1 = D_1^a, d_2 = D_2^a \]
This case can be proved impossible unless \([a = \pm b]_{(q^2r,p_i^m)} \).

**Theorem 8.3.** \( 0 < b < a < q/P^3(q), (a, q) = (b, q) = 1. P^{11}(q)|q. \) Then
\[ [\text{lm}(a) \neq \text{lm}(b)]_{(q/r,p_i^m)} \]
Proof. If not, then:
If the product of all prime sub-modules in which
\[ a^2 = b^2 \]
is greater than \( q^2 / P^2(q) \), then
\[ a^2 = b^2 \]
\( \square \)

Theorem 8.4. For prime \( p \) and positive integer \( q \) the equation
\[ a^p + b^p = c^q \]
has no integer solution \((a, b, c)\) such that \( a, b > 0, (a, b) = (b, c) = (a, c) = 1 \) if \( p, q \geq 41 \).
Proof. The method is to make logarithm in mod \( c^q \). It’s a condition sufficient for controversy:
\[ \frac{q}{p} \leq 6[(q - 2)/39] \]
\( \square \)

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