A REMARK ON RENORMALIZATION GROUP THEORETICAL PERTURBATION IN A CLASS OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

We revisit the renormalization group (RG) theoretical perturbation theory on oscillator type second order ordinary differential equations. For a class of potentials, we show a simple functional relation among secular coefficients of the harmonics in the naive perturbation series. It leads to an inversion formula between bare and renormalized amplitudes and an elementary proof of the absence of secular terms in all order of the RG series. The result covers non-autonomous as well as autonomous cases and refines earlier studies including the classic examples as Van der Pol, Mathieu, Duffing and Rayleigh equations.

1. Introduction

In the second order ordinary differential equations such as

Van der Pol: \[ \frac{d^2 y}{dt^2} + y + \varepsilon(y^2 - 1)\frac{dy}{dt} = 0, \]
Mathieu: \[ \frac{d^2 y}{dt^2} + y + \varepsilon(g + 2\cos t)y = 0, \]
Duffing: \[ \frac{d^2 y}{dt^2} + y + \varepsilon\left(\frac{dy}{dt} + gy^3\right) = 0, \]
Rayleigh: \[ \frac{d^2 y}{dt^2} + y + \varepsilon\frac{dy}{dt}\left(\frac{1}{3}\left(\frac{dy}{dt}\right)^2 - 1\right) = 0, \]

naive perturbation around \( \varepsilon = 0 \) leads to a series of the form

\[ y = \sum_{k \geq 0} \sum_{n \in \mathbb{Z}} (\text{polynomial in } t)\varepsilon^ke^{nit}. \]

The polynomiality of the coefficients is called secular and invalidates the effective description beyond the time scale typically like \( \mathcal{O}(\varepsilon^{-1}) \). The renormalization group (RG) theoretical approach is a successful example of singular perturbation (cf. [1, 5, 7]) which circumvents the difficulty and offers an effective resummation of the divergent series. The basic strategy is to absorb the secular \( t \)-dependence of (5) into renormalized amplitudes and describe the slow dynamics of the latter by the so called RG equation. The method has rich background and perspectives. See for example [2, 8, 10, 11] and the references therein.

This short note is an elementary and modest addition to the well-developed machinery. We focus on the equations of the form

\[ \frac{d^2 y}{dt^2} + y = \varepsilon V, \quad V = \text{arbitrary polynomial in } \varepsilon, e^{\pm it}, y \text{ and } \frac{dy}{dt}, \]

which are bit specific but well cover (1)–(4). The linear and \( \frac{dy}{dt} \)-free case like Mathieu equation (2) may also be viewed as a stationary Schrödinger equation in one dimensional periodic potentials expressible with finitely many Fourier components.

Let \( \sum_{n \in \mathbb{Z}} P_n(\varepsilon, t, A, B)e^{nit} \) be the naive perturbation series (5) that reduces to \( Ae^{it} + Be^{-it} \) at \( \varepsilon = 0 \). Here \( P_n(\varepsilon, t, A, B) \) is a power series in \( \varepsilon \) which we call secular coefficient. Especially \( P_{\pm 1}(\varepsilon, t, A, B) \) are important in that the relevant \( e^{\pm it} \) are the resonant harmonics. Our main finding is the functional relation (Corollary 5)

\[ P_n(\varepsilon, t, A, B) = P_n(\varepsilon, t - s, P_t(\varepsilon, s, A, B), P_{-1}(\varepsilon, s, A, B)) \quad (\forall n \in \mathbb{Z}), \]
and the resulting refinements in the conventional RG approach. For example, absence of secular terms in all order of the RG series\(^1\) follows immediately, and the manifest bijection (30) between the bare and renormalized amplitudes makes it unnecessary to introduce the so called renormalization constants and to resort to the implicit function theorem.

Our proof of (7) is elementary and elucidates why it works naturally for \(V\) of the form (6).\(^2\) It would be interesting to explore a geometric and/or holographic interpretation of it (cf. [3, 9]), and to seek a similar structure in a wider class of differential equations.

In Section 2 a precise definition of \(P_\star(\varepsilon, t, A, B)\) is given. In Section 3 the main result (7) is proved. In Section 4 renormalized amplitude is introduced and the RG equation is derived in one line calculation. In Section 5 the classical examples (1)–(4) are treated along the scheme of the paper. Similar analyses are available in many literatures and we have no intention to claim the originality, not to mention the basic idea and flow of the RG analysis. The last subsection 5.5 includes an exercise on a nonlinear and non-autonomous example.

### 2. Naïve perturbation

We study the second order ordinary differential equation for \(y = y(t)\) of the form

\[
\frac{d^2y}{dt^2} + y = \varepsilon V(\varepsilon, e^{it}, e^{-it}, y, \frac{dy}{dt}).
\]

(8)

Here \(\varepsilon\) is a parameter with respect to which the perturbation series is to be constructed. The function \(V = V(\varepsilon, e^{it}, e^{-it}, y, \frac{dy}{dt})\), which we call the potential, is a polynomial in the indicated five variables. Namely, we assume that \(V\) has the form

\[
V = \sum_{k \in \mathbb{Z}, l, m, n \in \mathbb{Z}_{\geq 0}} C_{klmn} \varepsilon^n e^{klt} y^l \left(\frac{dy}{dt}\right)^m,
\]

(9)

where \(C_{klmn}\) is the coefficient independent of \(\varepsilon, t\) and is nonzero only for finitely many quartets \((k, l, m, n)\). We work in the generic complex domain, so \(C_{klmn} = C_{-klmn}\) need not be imposed. Thus for example, \(V = \varepsilon^2 t^2 y^2 + (4 + \varepsilon e^{-5it})y(\frac{dy}{dt})^3 + 7e^{-3it}\) is covered but \(V = ty\) is not. We would like to remove the trivial linear case \(V = C_1y + C_2 \frac{dy}{dt} + \sum_k D_k e^{klt}\). So the existence of nonzero \(C_{klmn}\) with \(l + m \geq 1\) and \(|k| \geq \max(2 - l - m, 0)\) is assumed.

In naïve perturbation we will be concerned with the solutions of the form

\[
y(\varepsilon, t) = \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}_{\geq 0}} \varepsilon^k f_{n,k}(t)e^{nit}, \quad f_{n,k}(t) = \text{polynomial in } t.
\]

(10)

It is a formal power series in \(\varepsilon\) and is also a formal Laurent series in \(e^{it}\). As we will see, our construction always leads to \(f_{n,k}(t) = 0\) for sufficiently large \(|n|\) for each fixed \(k\). Therefore any product, say \(y^5 \left(\frac{dy}{dt}\right)^3\), makes sense as a formal Laurent series in \(e^{it}\).

Consider the formal power series expansion

\[
y(\varepsilon, t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots,
\]

(11)

which corresponds to setting \(y_{k}(t) = \sum_{n \in \mathbb{Z}} f_{n,k}(t)e^{nit}\) in (10). Substituting (11) into (8), we get an equation for each power of \(\varepsilon:\)

\[
\frac{d^2y_0}{dt^2} + y_0 = 0,
\]

(12)

\[
\frac{d^2y_1}{dt^2} + y_1 = V(0, e^{it}, e^{-it}, y_0, \frac{dy_0}{dt}),
\]

(13)

\[
\ldots \ldots
\]

\[
\frac{d^2y_k}{dt^2} + y_k = \left[V(\varepsilon, e^{it}, e^{-it}, \sum_{j=0}^{k-1} \varepsilon^j y_j, \sum_{j=0}^{k-1} \varepsilon^j \frac{dy_j}{dt})\right]_{\varepsilon=0} (k \geq 2).
\]

(14)

(15)

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\(^1\)General proof of this basic fact seems lacking in the literature.

\(^2\)The form of \(V\) in (9) is a sufficient but not the necessary condition for the approach in this paper to work. See the remark after the proof of Lemma 3.
At (1) Example 1. Consider Van der Pol equation (1), which corresponds to taking the potential as

\[ V \text{ potential} \]

In contrast to (19), we demand

\[ f_{\pm1,k}(t) = 0 \quad (\forall k \geq 1). \]

This completely fixes \( \alpha_k, \beta_k \), again successively, for \( k = 1, 2, 3, \ldots \). From the construction, it is easy to see the property \( f_{n,k}(t) = 0 \) \((|n| \gg 1, k \text{ fixed})\) mentioned after (10).

We define \( Y(\varepsilon, t, A, B) \) to be the resulting formal solution \( y(\varepsilon, t) \) (11). It can also depend on the other parameters in the potential \( V \) like \( g \) in (2) and (3). This last class of parameters inherent in \( V \) will be suppressed in the notation below. By using the \( Y(\varepsilon, t, A, B) \), define the quantity \( P_n(\varepsilon, t, A, B) \) to be the coefficients occurring in the expansion into harmonics:

\[ Y(\varepsilon, t, A, B) = \sum_{n \in \mathbb{Z}} P_n(\varepsilon, t, A, B)e^{int}. \]

In terms of (10), this means setting \( P_n(\varepsilon, t, A, B) = \sum_{k \geq 0} \varepsilon^k f_{n,k}(t) \). We call \( A, B \) the bare amplitudes and \( P_n(\varepsilon, t, A, B) \) the secular coefficient of the harmonics \( e^{int} \). The special case \( P_{\pm1}(\varepsilon, t, A, B) \), which is called the resonant secular coefficient (cf. [10]), will play a key role in what follows. By the definition they satisfy

\[ P_{\pm1}(0, 0, A, B) = \frac{A}{B}, \]

\[ P_n(0, 0, A, B) = A\delta_{n,1} + B\delta_{n,-1}. \]

In contrast to (19), \( P_n(\varepsilon, 0, 0, A, B) \) with \( n \neq \pm1 \) is nontrivial in general, and will take part in the renormalized expansion. (See (33)). The range of \( n \)-sum in (18) can be some subset of \( \mathbb{Z} \) depending on the potential \( V \). For instance in the list (1) – (4), the sums are actually \( \sum_{n \in 2\mathbb{Z}+1} \) except (2). In another example \( V = ye^{int} \), the sum reduces to \( \sum_{n \in 2\mathbb{Z}} +1 \).

Example 1. Consider Van der Pol equation (1), which corresponds to taking the potential as \( V = (1 - y^2)^{\frac{3}{4}} \) in (8). Then the above definition leads to \( P_n(\varepsilon, t, A, B) = 0 \) for \( n \) even as mentioned. For odd \( n \) we have the following table, where the \( k(\geq 0) \) th row and the \( n(\geq 1) \) th column from the NW corner shows the polynomial \( f_{k,2n-1}(t) \) in (10) corresponding to \( Y(\varepsilon, t, A, B) \). We have set \( C = AB \) to save the space.

\[
\begin{array}{ccccc}
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
y_0 & e^{it} & e^{3it} & e^{5it} & e^{7it} \\
y_1 & \frac{4A}{15}(1 - C) & \frac{1A^3}{4} & 0 & 0 \\
y_2 & \frac{4At}{15}(-2i + 8C^2 - 7it + 6C^2t) & -\frac{4At}{15}(-2i - 12t + 12Ct) & 0 & \frac{4A^5}{192} + t \\
y_3 & -\frac{4At}{15}(96C - 210C^2 + 111C^4 + 24it - 216Ct + 444C^2t^2 - 252C^3t) & \frac{4At}{15}(4i - 42C + 29C^2t) & \frac{4A^5}{4096}(-14i - 3iC - 60t + 60Ct) & -\frac{7A^7}{1152} \quad \text{t} \\
& -8t^2 + 104Ct^2 + 216C^2t^2 + 120C^3t^2 & +72i^2t^2 - 192Ct^2 + 120C^2t^2 & \quad \\
& \vdots & \vdots & \vdots & \vdots \\
P_1(\varepsilon, t, A, B) & P_3(\varepsilon, t, A, B) & P_5(\varepsilon, t, A, B) & P_7(\varepsilon, t, A, B) \\
\end{array}
\]

For instance one has

\[ P_5(\varepsilon, t, A, B) = -\frac{5A^5\varepsilon^2}{192} + \frac{5A^5\varepsilon^3}{4608}(-14i - 3iC - 60t + 60Ct) + \mathcal{O}(\varepsilon^4). \]

In this example, \( P_n \) with negative \( n \) is obtained by \( P_{-n}(\varepsilon, t, A, B) = P_n(\varepsilon, t, B, A) |_{t \to -t} \). Such a relation is not valid in general for the potentials that are not \( e^{it} \leftrightarrow e^{-it} \) symmetric like \( V = ye^{3it} + 2 \frac{d^2}{dt^2} e^{-it} \).

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\(^3\)There is no loss of generality compared with setting \( f_{\pm1,k}(t = t_0) = 0 \) with another parameter \( t_0 \). This degree of freedom is essentially incorporated into the forthcoming (26). The present convention was employed implicitly in [8].
As this example demonstrates, the secular coefficients are formal power series in \( \varepsilon \) whose leading order behaves as

\[
P_n(\varepsilon, t, A, B) \sim \mathcal{O}(\varepsilon^{d_n}), \quad d_n \to \infty \text{ as } |n| \to \infty.
\]  

(22)

3. Properties of secular coefficients

Lemma 2. Let \( s \) be an arbitrary parameter. The formal series \( y = y(\varepsilon, t) \) of the form (10) that satisfies the differential equation (8) and the conditions

\[\begin{align*}
(i) \quad y(0, t) &= Ae^{it} + Be^{-it}, & (ii) \quad [y(\varepsilon, t)]_{\varepsilon = s} = P_{\pm 1}(\varepsilon, s, A, B)
\end{align*}\]

is unique and given by \( y(\varepsilon, t) = Y(\varepsilon, t, A, B) \) in (18). (The LHS of (ii) means the value of the resonant secular coefficients evaluated at \( t = s \)).

Proof. From the construction in the previous section, we know that the solution exists uniquely by choosing the parameters \( \alpha_k, \beta_k \) appropriately so as to fit (ii) in each order of \( \varepsilon \). It is obvious that a solution \( y(\varepsilon, t) = Y(\varepsilon, t, A, B) \) fulfills the both (i) and (ii). \( \square \)

The next lemma is the point where the specific form (9) of the potential \( V \) is used.

Lemma 3. For arbitrary parameters \( s, C \) and \( D \), the formal series

\[\sum_{n \in \mathbb{Z}} P_n(\varepsilon, t - s, C, D)e^{nit}\]

(24)
is also a solution to the differential equation (8).

Proof. The only nontrivial claim is that shifting \( t \) to \( t - s \) in \( P_n \) without changing \( e^{nit} \) keeps it to be a solution. To show this, regard (18) as a formal Laurent series in \( e^{it} \). Then the original equation (8) is equivalent to the family of equations for the coefficient of each harmonics \( e^{nit} \):

\[\frac{\partial^2 P_m(\varepsilon, t, A, B)}{\partial t^2} + 2im \frac{\partial P_m(\varepsilon, t, A, B)}{\partial t} + (1 - m^2)P_m(\varepsilon, t, A, B)
\]

(25)

\[= \varepsilon V(\varepsilon, e^{it}, e^{-it}, \sum_{n \in \mathbb{Z}} P_n(\varepsilon, t, A, B)e^{nit}, \sum_{n \in \mathbb{Z}} \left( \frac{\partial P_n(\varepsilon, t, A, B)}{\partial t} + in \right)e^{nit}) \quad (\forall m \in \mathbb{Z}).
\]

In general, the RHS contains infinite sums like \( \sum_{n_1+n_2+n_3=m+5} P_{n_1}P_{n_2} \frac{\partial P_{n_3}}{\partial t} \) when \( V = e^{-5it}y^2e^{-it} \) for example. However, thanks to (22), they are actually convergent and make sense as formal power series in \( \varepsilon \). The point here is that (25) is totally an autonomous equation, i.e., all the \( t \)-dependence is via \( P_n \) and its derivatives. Therefore the shifted equation \( t \rightarrow t - s \) is equally valid. \( \square \)

The crux of the above argument is that (25) is well-defined and autonomous. It clarifies why the non-autonomous part of \( V \) has to be Laurent polynomials as in (9). Non \( e^{2it} \) type dependence like \( V = (2te^{it} + 3t^2)y^2e^{it} \) spoils the autonomous nature of the RHS of (25), hence invalidates the proof and the statement. We leave the consideration on a wider class of potentials like \( V = e^{y+y^2-2it} \) as a future problem.

The main result of this paper is the following.

Theorem 4. For any \( s, t, A \) and \( B \), the following identity between the formal series is valid:

\[\sum_{n \in \mathbb{Z}} P_n(\varepsilon, t, A, B)e^{nit} = \sum_{n \in \mathbb{Z}} P_n(\varepsilon, t - s, P_1(\varepsilon, s, A, B), P_{-1}(\varepsilon, s, A, B))e^{nit}.
\]

(26)

Proof. From Lemma 2, it suffices to verify that the RHS of (26) is a solution to the equation (8) and satisfies the conditions (i) and (ii) in (23). The fact that it is a solution is assured by Lemma 3. To check (23) is straightforward by using (20). \( \square \)

Theorem 4 tells that the RHS of (26) is independent of \( s \). In the RG context, it implies the independence of the choice of the initial time by a suitable renormalization of the amplitudes. The novelty of (26) is that the required normalization is exactly achieved by \( P_{\pm 1} \) itself.

Corollary 5. The secular coefficients satisfy the functional relation:

\[P_n(\varepsilon, t, A, B) = P_n(\varepsilon, t - s, P_1(\varepsilon, s, A, B), P_{-1}(\varepsilon, s, A, B)) \quad (\forall n \in \mathbb{Z}).
\]

(27)
Further specializing (27)|_{n=\pm 1} to \ t = 0 and applying (19) (with the reset \ s \to -t), we obtain an “inversion” formula:
\[
P_{\pm 1}(\varepsilon, t, P_1(\varepsilon, -t, A, B), P_{-1}(\varepsilon, -t, A, B)) = \left( \begin{array}{c} A \\ B \end{array} \right). 
\] (28)

Although our derivation of Theorem 4 and Corollary 5 has been quite elementary, their consequences are rather nontrivial. For instance, the first nontrivial assertion of (27) about (21) is
\[
P_3(\varepsilon, t, A, B) = P_3(\varepsilon, t - s, A + \frac{1}{2} \varepsilon A s (1 - C), B + \frac{1}{2} \varepsilon B s (1 - C)) \mod \mathcal{O}(\varepsilon^4). \] (29)

4. Renormalized amplitude and RG equation

We introduce \ A_r(t) \ and \ B_r(t) \ by either one of the following two sets of relations:
\[
\left( \begin{array}{c} A_r(t) \\ B_r(t) \end{array} \right) = P_{\pm 1}(\varepsilon, t, A, B) \iff \left( \begin{array}{c} A \\ B \end{array} \right) = P_{\pm 1}(\varepsilon, -t, A_r(t), B_r(t)). \] (30)

Their equivalence is assured by the inversion relation (28). Now Corollary 5 is stated as the identity:
\[
P_n(\varepsilon, t, A, B) = P_n(\varepsilon, t - s, A_r(s), B_r(s)). \] (31)

In particular the case \ s = t \ reads
\[
P_n(\varepsilon, t, A, B) = P_n(\varepsilon, 0, A_r(t), B_r(t)). \] (32)

This relation proves that the secular \ t-dependence in the LHS can be eliminated totally by switching from the bare amplitude \ A, B \ to the new ones \ A_r(t), B_r(t). In this sense the variables \ A_r(t), B_r(t) \ are called the renormalized amplitudes [2, 10, 8]. They allow us to rewrite the naive perturbation series (18) as
\[
Y(\varepsilon, t, A, B) = \sum_{n \in \mathbb{Z}} P_n(\varepsilon, 0, A_r(t), B_r(t)) e^{nit}
= A_r(t) e^{it} + B_r(t) e^{-it} + \sum_{n \in \mathbb{Z} \setminus \{\pm 1\}} P_n(\varepsilon, 0, A_r(t), B_r(t)) e^{nit}. \] (33)

By the construction, the RG series (33) is free of secular terms to all order of \ \varepsilon. 

The remaining task is to describe the dynamics or “modulation” of the renormalized amplitudes \ A_r(t), B_r(t) \ entering (33). The left relation in (30) is certainly an answer, but there is no point in substituting it into (33) since it just brings us back to the original expansion (18) which is full of secular terms. So one should devise an alternative maneuver which suppresses the secular (non-autonomous) \ t-dependence totally and \ t \ always “stays within the house \ A_r(t), B_r(t)”.\footnote{At the time of writing this, the number of COVID-19 infected in the world is 32356828.} Now with the exact renormalization (32) at hand, this can be done in a single line:
\[
\frac{d}{dt} \left( \begin{array}{c} A_r(t) \\ B_r(t) \end{array} \right) = (30) \frac{\partial P_{\pm 1}}{\partial t}(\varepsilon, t, A, B) = (31) \frac{\partial P_{\pm 1}}{\partial t}(\varepsilon, t - s, A_r(s), B_r(s)) = (31) \frac{\partial P_{\pm 1}}{\partial t}(\varepsilon, 0, A_r(t), B_r(t)). \] (34)

In the last step we have changed \ s \ to \ t. This is allowed by the \ s \-independence due to \ \frac{\partial}{\partial s} (\frac{\partial P_{\pm 1}}{\partial t}(\varepsilon, t - s, A_r(s), B_r(s))) = 0. From this maneuver it is clear that the \ t \-derivative in the last expression of (34) does not touch \ A_r(t), B_r(t). The differential equation (34) is called the RG or amplitude equation. One sees that the dynamics of the renormalized amplitude is governed by the resonant secular coefficients \ P_{\pm 1} \ to all order of \ \varepsilon. 

Let us isolate the top term of the power series \ P_{\pm 1}(\varepsilon, t, A, B) \ and name the other part as \ Q_{\pm 1}(\varepsilon, t, A, B):
\[
P_{\pm 1}(\varepsilon, t, A, B) = \left( \begin{array}{c} A \\ B \end{array} \right) + \varepsilon Q_{\pm 1}(\varepsilon, t, A, B). \] (35)

Then \ Q_{\pm 1}(\varepsilon, t, A, B) = \sum_{k \geq 1} \varepsilon^{k-1} f_{\pm 1,k}(t) \ is still a power series in \ \varepsilon \ such that
\[
Q_{\pm 1}(\varepsilon, 0, A, B) = 0 \] (36)

because of (19). Now the RG equation (34) is simplified slightly as
\[
\frac{d}{dt} \left( \begin{array}{c} A_r(t) \\ B_r(t) \end{array} \right) = \varepsilon \frac{\partial Q_{\pm 1}}{\partial t}(\varepsilon, 0, A_r(t), B_r(t)). \] (37)
where, as in (34), the \( t \)-derivative in the RHS does not concern \( A_r(t), B_r(t) \). This representation indicates that the RG dynamics is indeed “slow” in the sense that the RHS is at last of order \( \mathcal{O}(\varepsilon) \).

In the earlier works [10, 8], the right relation in (30) was conventionally formulated as

\[
A = A_r(t)Z_A(\varepsilon, t, A_r(t), B_r(t)), \quad B = B_r(t)Z_b(\varepsilon, t, A_r(t), B_r(t))
\]
by further introducing the so-called the renormalization constants \( Z_A, Z_b \). Moreover, reversing these relations had to be attributed to the implicit function theorem. One of the main achievements in this paper is the manifest bijection (30) between the bare and the renormalized amplitudes that untangles these issues and to have elucidated its elegant origin in the functional relation (27). The renormalization constants mentioned in the above, although they can now be dispensed with, acquire a “closed formula” as

\[
Z_A(\varepsilon, t, A_r(t), B_r(t)) = A_r(t)^{-1}P_1(\varepsilon, -t, A_r(t), B_r(t)) = 1 + \varepsilon A_r(t)^{-1}Q_1(\varepsilon, -t, A_r(t), B_r(t)),
\]

\[
Z_b(\varepsilon, t, A_r(t), B_r(t)) = B_r(t)^{-1}P_{-1}(\varepsilon, -t, A_r(t), B_r(t)) = 1 + \varepsilon B_r(t)^{-1}Q_{-1}(\varepsilon, -t, A_r(t), B_r(t)).
\]

5. Examples

5.1. Van der Pol equation. We consider (1), which was also treated in Example 1. Introduce the variables \( R = R(t) \) and \( \theta = \theta(t) \) connected to the renormalized amplitudes\(^5\)

\[
A_r(t) = R(t)e^{\theta(t)}, \quad B_r(t) = R(t)e^{-\theta(t)}.
\]

Set \( \tau = t + \theta(t) \). Then the renormalized expansion (33) reads

\[
y = 2R \cos \tau - \frac{\varepsilon R^3}{4}\sin 3\tau - \frac{\varepsilon^2 R^3}{96}(6\cos 3\tau + R^2(3\cos 3\tau + 5\cos 5\tau))
\]

\[
- \frac{\varepsilon^3 R^3}{2304}(6\sin 3\tau - 14R^2(27\sin 3\tau + 5\sin 5\tau) + R^4(261\sin 3\tau - 15\sin 5\tau - 28\sin 7\tau)) + \mathcal{O}(\varepsilon^4).
\]

The RG equation (37) is given by

\[
\frac{d\log R}{dt} = \frac{\varepsilon(1 - R^2)}{2} - \frac{\varepsilon^3 R^2(32 - 70R^2 + 37R^4)}{128} + \frac{\varepsilon^5 R^4(-1980 + 8154R^2 - 10757R^4 + 4589R^6)}{36864}
\]

\[
- \frac{\varepsilon^7 R^4(2950992 - 16173432R^2 + 28047688R^4 - 14916436R^6 - 4396557R^8 + 4493323R^{10})}{21233664} + \mathcal{O}(\varepsilon^9),
\]

\[
\frac{d\theta}{dt} = \frac{\varepsilon^2(-2 + 8R^2 - 7R^4)}{16} + \frac{\varepsilon^4(-24 - 192R^2 + 1020R^4 - 1266R^6 + 497R^8)}{3072}
\]

\[
+ \frac{\varepsilon^6(-1728 - 6912R^2 + 181872R^4 - 455608R^6 + 121432R^8 + 417540R^{10} - 266949R^{12})}{1769472} + \mathcal{O}(\varepsilon^8).
\]

By postulating \( \frac{d\log R}{dt} = 0 \), one finds the values on the limit cycle:

\[
2R_c = 2 + \frac{\varepsilon^2}{64} - \frac{23\varepsilon^4}{49152} - \frac{51619\varepsilon^6}{169869312} + \mathcal{O}(\varepsilon^8),
\]

\[
\left(\frac{d\theta}{dt}\right)_c = -\frac{\varepsilon^2}{16} + \frac{17\varepsilon^4}{3072} + \frac{35\varepsilon^6}{884736} + \mathcal{O}(\varepsilon^8).
\]

The approximate leading value \( 2R_c = 2 \) is well-known from the energy balance argument that the total work by the friction term during a period should be zero, i.e., by requiring \( \int_0^{2\pi} (y^2 - 1)(\frac{dy}{dt})^2 \) \( dt \) = 0 for \( y = 2R_c \cos t \).

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\(^5\)This change of variables is optional. The original \( A_r(t) \) and \( B_r(t) \) equally suit the numerical work. The same feature applies to the other equations in this section.
5.2. Mathieu equation. We consider (2) with $g$ dependent on $\varepsilon$ as $g = g_1 + g_2 \varepsilon + g_3 \varepsilon^2 + \cdots$. Then $Q_1(\varepsilon, t, A, B)$ defined in (35) is given by

$$Q_1(\varepsilon, t, A, B) = \frac{i Ag_1 t}{2} - \frac{\varepsilon t}{24} (i(8A + 12B + 3Ag_1^2 - 12Ag_2) + 3Ag_1^2 t) + \frac{\varepsilon^3 t}{144} (i(88Ag_1 + 72Bg_1 + 9Ag_1^3 - 36Ag_1g_2 + 72Ag_3) + 3Ag_1(8 + 3g_1^2 - 12g_2)t - 3iAg_1^3 t^2) + O(\varepsilon^4).$$

(47)

The other one is obtained by $Q_{-1}(\varepsilon, t, A, B) = Q_1(\varepsilon, t, B, A)\big|_{t \to -t}$. This example is exceptional among (1)-(4) in that it is the only equation which is linear and moreover non-autonomous. Reflecting the former feature, the RG equation also becomes linear. In fact, differentiation of (37) can be combined and split into the two identical equations

$$\frac{d^2 A_r(t)}{dt^2} = -\omega^2 A_r(t), \quad \frac{d^2 B_r(t)}{dt^2} = -\omega^2 B_r(t),$$

where the constant $\omega^2$ is given by

$$\omega^2 = \frac{\varepsilon^2}{4} g_1^2 - \frac{\varepsilon^3}{24} g_1 (8 + 3g_1^2 - 12g_2) - \frac{\varepsilon^4}{576} (80 - 400g_1^2 - 45g_1^4 + 192g_2 + 216g_1^2g_2 - 144g_2^2 - 288g_1g_3)$$

$$- \frac{\varepsilon^5}{3456} (-840g_4 + 3920g_1^3 + 189g_5^2 - 4800g_1g_2 - 108g_1^3g_2 + 1296g_1g_2^2 + 1152g_3)$$

$$+ 1296g_1^2g_3 - 1728g_2g_3 - 1728g_1g_4) + O(\varepsilon^6).$$

(49)

The stable region $\omega^2 > 0$ and the unstable region $\omega^2 < 0$ of $A_r(t), B_r(t)$ are separated by the curve $\omega^2 = 0$. Solving it order by order in $\varepsilon$ with respect to $g_1, g_2, g_3, \ldots$ yields two branches. Let us present them for the combination $a := 1 + \varepsilon g$ which is the usual coupling constant in the conventional setting of Mathieu equation (2) as $\frac{d^2 y}{dt^2} + (a + 2\varepsilon \cos t)y = 0$. Then the branches are $a = a^\pm$, where

$$a^- = 1 - \frac{\varepsilon^2}{3} + \frac{5\varepsilon^4}{216} + \cdots, \quad a^+ = 1 + \frac{5\varepsilon^2}{3} - \frac{763\varepsilon^4}{216} + \cdots.$$

(50)

These curves in the $(a, \varepsilon)$ plane specify the boundaries of the unstable region $a^- < a < a^+$ and the stable region in the vicinity of the so-called second resonance point $(a, \varepsilon) = (1, 0)$. (The $n$th resonance is $(a, \varepsilon) = (\frac{n\pi}{4}, 0)$ [1, sec.11.4].) The result $a^\pm$ (50) agrees with the zeros of the determinants $\Delta^\pm(\varepsilon, a) = 0$ of the semi-infinite Jacobi matrices\(^6\) around the resonance [6, sec.7-1], where

$$\Delta^-(\varepsilon, a) = \begin{vmatrix} a - 1^2 & \varepsilon & \varepsilon \\ \varepsilon & a - 2^2 & \varepsilon \\ \varepsilon & \varepsilon & a - 3^2 \end{vmatrix}, \quad \Delta^+(\varepsilon, a) = \begin{vmatrix} \varepsilon & a - 1^2 & \varepsilon \\ \varepsilon & a - 2^2 & \varepsilon \\ \varepsilon & \varepsilon & a - 3^2 \end{vmatrix}.$$

It will be an interesting exercise to see how the RG series fits the exact solution [4].

5.3. Duffing equation. Consider (3) with $g = 1$ which can be attained by $y \to y/\sqrt{g}$. We have included a $\frac{dy}{d\tau}$ term since otherwise the equation is integrable by an elliptic function. Introduce the variables $R = R(t)$ and $\theta = \theta(t)$ connected to the renormalized amplitudes by (41) and set $\tau = t + \theta(t)$. Then the renormalized expansion (33) reads

$$y = 2R \cos \tau + \frac{\varepsilon^3 R^3}{4} \cos 3\tau + \frac{\varepsilon^2 R^3}{32} (6 \sin 3\tau + R^2(\cos 5\tau - 21 \cos 3\tau))$$

$$- \frac{\varepsilon^3 R^3}{768} (6 \cos 3\tau - 2R^2(567 \sin 3\tau - 19 \sin 5\tau) + 3R^4(417 \cos 3\tau - 43 \cos 5\tau + \cos 7\tau)) + O(\varepsilon^4).$$

(51)

\(^6\)Symmetric, tri-diagonal matrices with positive off-diagonal elements. We imagine $\varepsilon$ is positive to reply on this nomenclature. $\Delta^-(\varepsilon, a)$ and $\Delta^+(\varepsilon, a)$ correspond to $\text{Se}(x)$ and $\text{Ce}(x)$ on [6, p176], respectively.
The RG equation (37) is given by
\[
\frac{d \log R}{dt} = \frac{\varepsilon}{2} + \frac{3\varepsilon R^2}{16} - \frac{195\varepsilon^3 R^4}{64} + \frac{5931\varepsilon^4 R^6}{512} - \frac{5\varepsilon^5 R^8 (16092 - 172027 R^2)}{4096} + O(\varepsilon^6),
\]
(52)
\[
\frac{d\theta}{dt} = \frac{3\varepsilon R^2}{2} - \frac{\varepsilon^2 (2 + 15 R^4)}{16} - \frac{3\varepsilon^3 R^2 (8 - 41 R^4)}{128} + \frac{\varepsilon^4 (-8 + 4116 R^4 - 921 R^8)}{1024} - \frac{3\varepsilon^5 R^2 (8 + 21305 R^4 - 193 R^8)}{2048} + O(\varepsilon^6).
\]
(53)

5.4. **Rayleigh equation.** We consider (4). Introduce the variables $R = R(t)$ and $\theta = \theta(t)$ connected to the renormalized amplitudes by (41) and set $\tau = t + \theta(t)$. Then the renormalized expansion (33) reads

\[
y = 2R \cos \tau + \frac{\varepsilon R^3}{12} \sin 3\tau + \frac{\varepsilon^2 R^3}{96} (-6 \cos 3\tau + R^2 (9 \cos 3\tau - \cos 5\tau)) + \frac{\varepsilon^3 R^3}{2304} (-36 \sin 3\tau - 2R^2 (63 \sin 3\tau + 17 \sin 5\tau) + R^4 (111 \sin 3\tau + 51 \sin 5\tau - 4 \sin 7\tau)) + O(\varepsilon^4).
\]
(54)

The RG equation (37) is given by
\[
\frac{d \log R}{dt} = \frac{\varepsilon (1 - R^2)}{2} + \frac{\varepsilon^3 R^2 (22 - 13 R^2)}{128} - \frac{\varepsilon^5 R^4 (2268 - 1026 R^2 - 2683 R^4 + 1603 R^6)}{36864} + O(\varepsilon^7),
\]
(55)
\[
\frac{d\theta}{dt} = \frac{\varepsilon^2 (R^4 - 2)}{16} + \frac{\varepsilon^4 (-24 + 156 R^4 - 234 R^6 + 65 R^8)}{3072} + \frac{\varepsilon^6 (-1728 - 98064 R^4 + 305208 R^6 - 210728 R^8 - 71388 R^{10} + 84627 R^{12})}{1769472} + O(\varepsilon^8).
\]
(56)

5.5. **A non-linear and non-autonomous example.** Finally we consider a nonlinear and non-autonomous example:

\[
\frac{d^2 y}{dt^2} + y = 2\varepsilon \frac{dy}{dt} y \cos t.
\]
(57)

Introduce the variables $R = R(t)$ and $\theta = \theta(t)$ connected to the renormalized amplitudes by (41). Then the renormalized expansion (33) reads

\[
y = 2R \cos (\theta + t) + \frac{1}{4} \varepsilon R^2 \sin (2\theta + 3t) - \frac{\varepsilon^2 R^3}{24} (3 \cos \theta \cos (2\theta + 3t) + \cos (3\theta + 5t)) + \frac{\varepsilon^3 R^4}{4608} (-24 \sin (2\theta + 3t) - 33 \sin (4\theta + 7t) + 14 \sin \theta \cos (3\theta + 5t) + 72 \sin 2\theta \cos (2\theta + 3t) + 288 \cos 2\theta \sin (2\theta + 3t) - 146 \cos \theta \sin (3\theta + 5t)) + O(\varepsilon^4).
\]
(58)

The RG equation (37) is given by
\[
\frac{d \log R}{dt} = \frac{1}{2} \varepsilon R \cos \theta - \frac{1}{4} \varepsilon^2 R^2 \sin 2\theta + \frac{5}{16} \varepsilon^3 R^3 \cos \theta - \varepsilon^4 R^4 \left( \frac{21}{64} \sin 2\theta + \frac{1}{32} \sin 4\theta \right) + O(\varepsilon^5),
\]
(59)
\[
\frac{d\theta}{dt} = \frac{1}{2} \varepsilon R \sin \theta - \varepsilon^2 R^2 \left( \frac{1}{4} \cos 2\theta + \frac{3}{8} \right) + \frac{1}{8} \varepsilon^3 R^3 \sin \theta + \frac{1}{128} \varepsilon^4 R^4 (9 \cos 2\theta - 4 \cos 4\theta - 3) + O(\varepsilon^5).
\]
(60)

Unlike Van de Pol, Duffing and Rayleigh equations, one has the essential mixture of $R$ and $\theta$ in the RHS of the RG equation reflecting the non-autonomous and nonlinear nature of (57). As exemplified in the following plot, there seems only one peak in the envelop $R(t)$ in a certain parameter range.
Figure 1. Plot of $y(t)$ vs $t$ by the direct numerical solution of (57) (blue) and the RG expansion (red) started from the same initial condition $R(0) = 0.2, \theta(0) = -0.1$ with $\varepsilon = 0.25$. We have kept only $\varepsilon^0, \varepsilon^1$ terms in (58) and the leading $\varepsilon^1$ term in (59), (60). Taking $\varepsilon^2$ term in (59) and (60) into account already makes it too difficult to observe the discrepancy.

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