NISNEVICH TOPOLOGY WITH MODULUS
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Abstract. In Voevodsky’s theory of motives, the Nisnevich topology on smooth schemes is used as an important building block. In this paper, we introduce a Grothendieck topology on proper modulus pairs, and prove its fundamental properties. Proper modulus pairs are introduced in [2], as an ingredient to develop a non-homotopy invariant generalization of Voevodsky’s theory.

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1. Introduction

In the theory of motives à la Voevodsky in [9], the Nisnevich topology on the category of smooth schemes over a field $k$ plays a fundamental role. A Nisnevich cover $f : Y \to X$ is an étale cover such that any point $x \in X$ admits a point $y \in Y$ with $f(y) = x$ and $k(y) = k(x)$. Therefore, the Nisnevich topology is finer than the Zariski topology and is coarser than the étale topology. Voevodsky defined the category of effective motives $\text{DM}^\text{eff}$ as the derived category of the abelian category of Nisnevich sheaves with transfers $\text{NST}$, modulo $\mathbb{A}^1$-homotopy invariance:

$$\text{DM}^\text{eff} := \frac{\text{D} (\text{NST})}{(\mathbb{A}^1\text{-homotopy invariance})}.$$
We briefly recall the definition of $\text{NST}$. Let $\text{PST}$ be the category of additive abelian presheaves on the category of finite correspondences $\text{Cor}$. Then we have a natural functor $\text{Sm} \to \text{Cor}$, where $\text{Sm}$ denotes the category of smooth schemes over $k$. Then $\text{NST}$ is defined to be the full subcategory of $\text{PST}$ which consists of $F \in \text{PST}$ such that the restriction $F|_{\text{Sm}}$ is a Nisnevich sheaf on $\text{Sm}$.

The definition of $\text{NST}$ is simple, but it is non-trivial that $\text{NST}$ is an abelian category. It follows from the existence of a left adjoint to the inclusion functor $\text{NST} \to \text{PST}$. A key ingredient of the proof of its existence is the following fact: for any Nisnevich cover $U \to X$, the following Čech complex is exact as a complex of Nisnevich sheaves:

$$
\cdots \to Z_{tr}(U \times_X U) \to Z_{tr}(U) \to Z_{tr}(X) \to 0,
$$

where $Z_{tr}(-) : \text{Cor} \to \text{PST}$ denotes the Yoneda embedding (see for example [7, Prop. 6.12]). Moreover, the Nisnevich topology is subcanonical, i.e., every representable presheaf in $\text{Sm}$ is a sheaf.

The category of motives $\text{DM}_{\text{eff}}$ has provided vast applications to the study of arithmetic geometry, but on the other hand, it has a fundamental constraint that it cannot capture non-$A_1$-homotopy invariant phenomena, e.g., wild ramification. Indeed, the arithmetic fundamental group $\pi_1(X)$, which captures the information of ramifications, is not $A_1$-homotopy invariant.

An attempt to develop a theory of motives which captures non-$A_1$-homotopy invariant phenomena started in [5]. The strategy is to extend Voevodsky’s theory to modulus pairs. A modulus pair is a pair $\mathcal{M} = (\overline{M}, M^\infty)$ of a scheme $\overline{M}$ and an effective Cartier divisor $M^\infty$ on $\overline{M}$ such that the interior $\mathcal{M}^\circ := \overline{M} - M^\infty$ is smooth over $k$. We can define a reasonable notion of morphisms between modulus pairs, and we obtain a category of modulus pairs $\text{MSm}$. A modulus pair $\mathcal{M}$ is proper if $\overline{M}$ is proper over $k$, and we denote by $\text{MSm}$ the full subcategory of $\text{MSm}$ consisting of proper modulus pairs (see Definition 2.1.1 for details).

These categories embed in categories of “modulus correspondences” $\text{MCor}$ and $\text{MCor}$, just as $\text{Sm}$ embeds in $\text{Cor}$ (see Definition 2.3.2). In [5], categories of “modulus sheaves with transfers” $\text{MNST}$ (relative to $\text{MCor}$) and $\text{MNST}$ (relative to $\text{MCor}$) were introduced, in order to parallel the definition of (1.1.1). However, the proof that these categories are abelian was found to contain a gap. This gap was filled in [2] for $\text{MNST}$, by showing that its objects are indeed the sheaves with transfers for a suitable Grothendieck topology on $\text{MSm}$.

In this paper, we construct a Grothendieck topology on $\text{MSm}$ with nice properties. It will be shown in [3], using [1], that the objects of $\text{MNST}$ are the sheaves (with transfers) for this topology and that this
category is abelian. Thus the present paper contains the tools to finish filling the gap of [5]. Moreover, we prove an important exactness result.

Our guide is the following characterization of the Nisnevich topology on $\text{Sm}$: the Nisnevich topology is generated by coverings $U \sqcup V \to X$ associated with some commutative square $S$ in $\text{Sm}$ of the form

\[
\begin{array}{ccc}
W & \to & V \\
\downarrow & & \downarrow \\
U & \to & X
\end{array}
\]

which satisfies the following properties:

1. $S$ is a cartesian square,
2. the horizontal morphisms are open immersions,
3. the vertical morphisms are étale, and
4. the morphism $(V - W)_{\text{red}} \to (X - U)_{\text{red}}$ is an isomorphism.

Such squares are called \textit{elementary Nisnevich squares}. Elementary Nisnevich squares form a cd-structure on $\text{Sm}$ in the sense of [10]. A remarkable property of the Nisnevich cd-structure is the following fact: a presheaf of sets $F$ on $\text{Sm}$ is a Nisnevich sheaf if and only if $F(\emptyset) = \{\ast\}$ and for any elementary Nisnevich square as above, the square

\[
\begin{array}{ccc}
F(M) & \to & F(U) \\
\downarrow & & \downarrow \\
F(V) & \to & F(W)
\end{array}
\]

is cartesian. This equivalence holds for any cd-structure which is complete and regular (see [10, Def. 2.3, 2.10, Cor. 2.17]).

In [2], a cd-structure on $\text{MSm}$ is introduced. It is denoted $P_{\text{MV}}$, and satisfies properties similar to elementary Nisnevich squares. Its definition will be recalled in §4.1. For short, we call the topology on $\text{MSm}$ associated with $P_{\text{MV}}$ the \textbf{MV-topology}.

Our main result is the following.

\textbf{Theorem 1.} \textit{The category of proper modulus pairs $\text{MSm}$ admits a cd-structure $P_{\text{MV}}$ such that the following assertions hold: For short, we call the topology associated with $P_{\text{MV}}$ the MV-topology.}

1. (see Theorem 4.3.1, 4.4.1, 4.4.2) The cd-structure $P_{\text{MV}}$ is complete and regular. In particular, a presheaf of sets $F$ on $\text{MSm}$ is a sheaf for the MV-topology if and only if $F(\emptyset) = \{\ast\}$ and
for any square $T \in P_{\text{MV}}$ of the form

$$
\begin{array}{ccc}
W & \longrightarrow & V \\
\downarrow & & \downarrow \\
U & \longrightarrow & M,
\end{array}
$$

the square

$$
\begin{array}{ccc}
F(M) & \longrightarrow & F(U) \\
\downarrow & & \downarrow \\
F(V) & \longrightarrow & F(W)
\end{array}
$$

is cartesian.

(2) (see Theorem 4.5.1) The $\text{MV}$-topology and the $\text{MV}$-topology are subcanonical.

(3) (see Corollary 5.2.4) For any $M \in \text{MSm}$, consider the preshaf $\mathbb{Z}_{\text{tr}}(M)$ on $\text{MCor}$ represented by $M$, which is a sheaf for the $\text{MV}$-topology by [2, Th. 2 (2)]. Then, for any square as above, the following complex of sheaves for the $\text{MV}$-topology is exact:

$$0 \to \mathbb{Z}_{\text{tr}}(W) \to \mathbb{Z}_{\text{tr}}(U) \oplus \mathbb{Z}_{\text{tr}}(V) \to \mathbb{Z}_{\text{tr}}(M) \to 0.$$

The organization of the paper is as follows. In §2, we recall basic definitions and results on modulus pairs from [2]. In §3, we introduce “the off-diagonal functor”, which is a key ingredient to define the cd-structure on the category of proper modulus pairs. In §4, we define the cd-structure on the category of proper modulus pairs, and prove that it satisfies completeness and regularity. Finally, in §5, we prove the exactness of the Mayer-Vietoris sequences associated with the distinguished squares with respect to the cd-structure.

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Notation and convention. Throughout the paper, we fix a base field $k$. Let $\text{Sm}$ be the category of separated smooth schemes of finite type over $k$, and let $\text{Sch}$ be the category of separated schemes of finite type over $k$. For any scheme $X$ and for any closed subscheme $F \subset X$, we denote by $\text{Bl}_F(X)$ the blow-up of $X$ along $F$. 
2. Basics on modulus pairs

In this section, we introduce basic notions which we use throughout the paper.

2.1. Category of modulus pairs. We recall basic definitions on modulus pairs, which is introduced in [2]. We will also introduce new notations. Especially, the canonical model of fiber product is important (see Definition 2.2.2). Though our main interest in this paper is on proper modulus pairs, we introduce the general definition of modulus pairs for later use.

Definition 2.1.1.

1. A modulus pair is a pair \( (\overline{M}, M^\infty) \) consists of a scheme \( \overline{M} \in \text{Sch} \) and an effective Cartier divisor \( M^\infty \) on \( \overline{M} \) such that
   - the ambient space \( \overline{M} \in \text{Sch} \), and
   - the modulus divisor \( M^\infty \), i.e., an effective Cartier divisor on \( \overline{M} \)

   such that
   - the interior \( M^o := \overline{M} \setminus |M^\infty| \) belongs to \( \text{Sm} \), where \( |M^\infty| \) denotes the support of \( M^\infty \).

   Note that \( M^o \) is a dense open subset of \( \overline{M} \). Moreover, we can prove that \( \overline{M} \) must be a reduced scheme by using the smoothness of \( M^o \) and the assumption that \( M^\infty \) is an effective Cartier divisor.

2. A modulus pair \( M \) is called proper if the ambient space \( \overline{M} \) is proper over \( k \).

3. An admissible morphism \( f : M \to N \) of modulus pairs is a morphism between the interiors \( f^o : M^o \to N^o \) in \( \text{Sm} \) which satisfies the properness condition:
   - Let \( \Gamma \) be the graph of the rational map \( \overline{f} : \overline{M} \dashrightarrow \overline{N} \) which is induced by \( f^o \). Then the natural morphism \( \Gamma \to \overline{M} \) is proper.

   and the modulus condition:
   - Let \( \Gamma^N \) be the normalization of \( \Gamma \). Then we have the following inequality
     \[ M^\infty|_{\Gamma^N} \geq N^\infty|_{\Gamma^N}, \]
     of effective Cartier divisors on \( \Gamma^N \), where \( M^\infty|_{\Gamma^N} \) and \( N^\infty|_{\Gamma^N} \) denote the pullbacks \( M^\infty \) and \( N^\infty \) along the natural morphisms \( \Gamma^N \to \overline{M} \) and \( \Gamma^N \to \overline{N} \). Note that the pullbacks are defined since the rational map \( \overline{f} \) restricts to a morphism \( f^o \), and since \( M^o \) is dense in \( \overline{M} \).
If \(f : M \to N\) and \(g : N \to L\) are admissible morphisms, then the composite \(g \circ f^o : M^o \to L^o\) defines an admissible morphism \(M \to N\) (cf. [2]). If \(N\) is proper, then the properness condition above is always satisfied.

(4) We let \(\text{MSm}\) denote the category whose objects are modulus pairs and whose morphisms are admissible morphisms. The full subcategory of \(\text{MSm}\) consisting of proper modulus pairs is denoted by \(\text{MSm}\).

(5) A morphism \(f : M \to N\) in \(\text{MSm}\) is called ambient if \(f^o : M^o \to N^o\) extends to a morphism \(\overline{M} \to \overline{N}\) in \(\text{Sch}\). Such an extension is unique since \(\overline{M}\) is reduced, \(M^o\) is dense in \(\overline{M}\), and \(\overline{N}\) is separated. We let \(\text{MSm}^\text{fin}\) (resp. \(\text{MSm}^\text{fin}\)) denote the (non-full) subcategory of \(\text{MSm}\) (resp. \(\text{MSm}\)) whose objects are modulus pairs (resp. proper modulus pairs) and whose morphisms are ambient morphisms.

(6) A morphism \(f : M \to N\) in \(\text{MSm}\) is called minimal if \(f\) is ambient and satisfies \(M^\infty = f^* N^\infty\).

(7) We let \(\Sigma^\text{fin}\) denote the subcategory of \(\text{MSm}\) whose objects are the same as \(\text{MSm}\) and whose morphisms are those morphisms \(f : M \to N\) in \(\text{MSm}^\text{fin}\) such that \(f\) is minimal, \(\overline{f} : \overline{M} \to \overline{N}\) is proper and \(f^o : M^o \to N^o\) is an isomorphism in \(\text{Sm}\). Then the canonical functor \(\text{MSm}^\text{fin} \to \text{MSm}\) induces an equivalence of categories \(\Sigma^\text{fin} \text{MSm}^\text{fin} \cong \text{MSm}^\text{fin} [2, \text{Prop. 1.9.2}]\).

(8) Let \(\text{Sq}\) be the product category \([0] \times [0]\), where \([0] = \{0 \to 1\}\). For any category \(\mathcal{C}\), we define \(\mathcal{C}^\text{Sq}\) to be the category of functors from \(\text{Sq}\) to \(\mathcal{C}\). An object \(T\) of \(\mathcal{C}^\text{Sq}\) is given by a commutative diagram

\[
\begin{array}{ccc}
T(00) & \longrightarrow & T(01) \\
\downarrow & & \downarrow \\
T(10) & \longrightarrow & T(11).
\end{array}
\]

in \(\mathcal{C}\), and a morphism \(T_1 \to T_2\) in \(\mathcal{C}^\text{Sq}\) is given by a set of morphisms \(T_1(ij) \to T_2(ij)\), \(i, j = 0, 1\), which are compatible with all the edges of the squares.

(9) A morphism \(T_1 \to T_2\) in \(\text{MSm}^\text{Sq}\) is called ambient if for any \(i, j = 0, 1\), the morphisms \(\overline{T_1(ij)} \to \overline{T_2(ij)}\) in \(\text{MSm}\) are ambient. A square \(T \in \text{MSm}^\text{Sq}\) is called ambient if it is contained in \((\text{MSm}^\text{fin})^\text{Sq} \subset \text{MSm}^\text{Sq}\).

The following lemma is often useful.
Lemma 2.1.2. For any square $T \in \mathbf{M}_{\text{Sm}}^{\text{Sm}}$, there exists an ambient square $T'$ which admits an ambient morphism $T' \to T$ which is an isomorphism in $\mathbf{M}_{\text{Sm}}^{\text{Sm}}$.

Proof. This is just a consequence of a repeated use of the graph trick ([2, Lemma 1.3.6]). Or the reader can consult the calculus of fractions in [2, Prop. 1.9.2]). The details are left to the reader. \qed

2.2. Fiber products. We discuss fiber products in $\mathbf{M}_{\text{Sm}}$ and $\mathbf{MSm}$.

Lemma 2.2.1. Let $X$ be a scheme, and let $D_1$ and $D_2$ be effective Cartier divisors on $X$. Assume that the scheme-theoretic intersection $\text{inf}(D_1, D_2) := D_1 \times_X D_2$ is also an effective Cartier divisor on $X$. Set $X^\infty := D_1 + D_2 - \text{inf}(D_1, D_2)$.

Then, for any morphism $f : Y \to X$ in $\mathbf{Sch}$ such that $Y$ is normal and the image of any irreducible component of $Y$ is not contained in $|X^\infty| = |D_1| \cup |D_2|$, we have

$$f^* X^\infty = \sup(f^* D_1, f^* D_2),$$

where $\sup$ is the supremum of Weil divisors on the normal scheme $Y$.

Proof. See [2, Lem. 1.10.1, Def. 1.10.2 Rem. 1.10.3]. \qed

Definition 2.2.2. Let $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ be morphisms in $\mathbf{MSm}^{\text{fin}}$, and assume that the fiber product $P^0 := M_1^\circ \times_N M_2^\circ$ exists in $\mathbf{Sm}$. We define a modulus pair $P$ as follows. Let $\overline{P}_0$ be the scheme-theoretic closure of $P^0$ in $\overline{M} \times_N \overline{M}$, and let $\overline{P}_{0,i} : \overline{P}_0 \to \overline{M}_1 \times_N \overline{M}_2 \twoheadrightarrow \overline{M}_i$ be the composite of the closed immersion followed by the $i$-th projection for $i = 1, 2$. Let

$$\overline{P} := \text{Bl}_{\overline{P}_{0,1}(M_1^\infty) \times_N \overline{P}_{0,2}(M_2^\infty)}(\overline{P}_0)^N \xrightarrow{\pi_P} \overline{P}_0$$

be the normalized blow-up of $\overline{P}_0$ along the closed subscheme $(\overline{P}_{0,1} M_1^\infty) \times_N (\overline{P}_{0,2} M_2^\infty)$. Set

$$P^\infty := \pi_P^* \overline{P}_{0,1} M_1^\infty + \pi_P^* \overline{P}_{0,2} M_2^\infty - E,$$

where $E := \pi_P^{-1}((\overline{P}_{0,1} M_1^\infty) \times_N (\overline{P}_{0,2} M_2^\infty))$ denotes the exceptional divisor. Then we have $\overline{P} - |P^\infty| = P^0 \in \mathbf{Sm}$ by construction, and we obtain a modulus pair $P = (\overline{P}, P^\infty)$.

We call $P$ the canonical model of fiber product of $f_1$ and $f_2$, and we often write

$$M_1 \times_N M_2 := P.$$
By construction, we have a commutative diagram

\[
\begin{array}{c}
M_1 \times^c_N M_2 \xrightarrow{p_2} M_2 \\
p_1 \quad \downarrow \quad \downarrow f_2 \\
M_1 \xrightarrow{f_1} N
\end{array}
\]

in \( \text{MSm}^{\text{fin}} \). Moreover, we have \((M_1 \times^c_N M_2)^o \cong M_1^o \times_{N^o} M_2^o \).

**Theorem 2.2.3.** Let \( f_1 : M \to N \) and \( f_2 : M_2 \to N \) be morphisms in \( \text{MSm}^{\text{fin}} \). Assume that the fiber product \( M_1^o \times_{N^o} M_2^o \) exists in \( \text{Sm} \). Then the canonical model of fiber product \( M_1 \times^c_N M_2 \) represents the fiber product \( M_1 \times_N M_2 \) in \( \text{MSm} \). Moreover, if \( M_1, M_2, N \) are proper, then \( M_1 \times^c_N M_2 \) (hence \( M_1 \times_N M_2 \)) is proper.

**Remark 2.2.4.** \( M_1 \times^c_N M_2 \) does not necessarily represent a fiber product in \( \text{MSm}^{\text{fin}} \), and it is not functorial in \( \text{MSm}^{\text{fin}} \). However, under some minimality conditions, they behave nicely in \( \text{MSm}^{\text{fin}} \).

**Proof.** We prove that \( P := M_1 \times^c_N M_2 \) satisfies the universal property of fiber product in \( \text{MSm} \). Let \( g_1 : L \to M_1 \) and \( g_2 : L \to M_2 \) be morphisms in \( \text{MSm} \) which coincide at \( N \). Since \( \text{MSm} \cong \Sigma_{\text{fin}}^{-1} \text{MSm}^{\text{fin}} \), we can find morphisms \( L_1 \to L \) in \( \Sigma_{\text{fin}} \) such that the composite morphisms \( L_1 \to L \to M_i \) are ambient for \( i = 1, 2 \), and such that \( L_1 \) is normal. Since \( L_1 \to L \) is an isomorphism in \( \text{MSm} \), we replace \( L \) with \( L_1 \) and assume that \( L_1 \) is normal, and that \( g_1 \) and \( g_2 \) are ambient. Let \( p_1 : P \to M_1 \) and \( p_2 : P \to M_2 \) be the ambient morphisms as in Def. 2.2.2.

There exists a unique morphism \( g^o : L^o \to P^o = M_1^o \times_{N^o} M_2^o \) in \( \text{Sm} \) which is compatible with \( g_1^o, g_2^o, p_1^o \) and \( p_2^o \). It suffices to prove that \( g^o \) defines a morphism \( L \to P \) in \( \text{MSm} \). Let \( \Gamma \subset \overline{L} \times \overline{P} \) be the closure of the graph of \( g^o \), and let \( \Gamma^N \) be the normalization of \( \Gamma \). Let \( s : \Gamma^N \to \overline{L} \) and \( t : \Gamma^N \to \overline{P} \) be the natural projections.

Then, for \( i = 1, 2 \), we obtain a commutative diagram

\[
\begin{array}{c}
\Gamma^N \xrightarrow{t} \overline{P} \\
\downarrow \quad \quad \downarrow g^o \\
\overline{L} \xrightarrow{g^o} \overline{M}_i
\end{array}
\]

where the commutativity follows from the fact that \( \overline{p}_i t \) and \( \overline{g}_i s \) coincide on the dense open subset \( s^{-1}(L^o) \subset \Gamma^N \).
By the construction of $P$ and by Lemma 2.2.1, we have
\[ t^* P^\infty = \sup(t^* p_1^* M_1^\infty, t^* p_2^* M_2^\infty) \]
\[ = \sup(s^* \overline{g}_1^* M_1^\infty, s^* \overline{g}_2^* M_2^\infty), \]
where the second equality follows from the commutativity of the above diagram. Since $g_1$ and $g_2$ are ambient and $\overline{L}$ is normal, we have $\overline{g}_i^* M_i^\infty \leq \overline{L}^\infty$. Therefore, we obtain
\[ t^* P^\infty \leq s^* L^\infty, \]
which shows that $g^c$ defines a morphism $g : L \to P$. This proves the first assertion. The last assertion is obvious by construction. This finishes the proof. □

**Corollary 2.2.5.** Let $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ be morphisms in $\textbf{MSm}$. Assume that the fiber product $M_1^\infty \times_N M_2^\infty$ exists in $\textbf{Sm}$. Then there exists a fiber product $M_1 \times_N M_2$ in $\textbf{MSm}$. Moreover, if $M_1$, $M_2$, and $N$ are proper, then $M_1 \times_N M_2$ is proper.

**Proof.** By [2, Lemma 1.3.6], for each $i = 1, 2$, there exists a morphism $M_i^c \to M_i$ in $\textbf{MSm}^{\text{fin}}$ which is invertible in $\textbf{MSm}$ and such that the composite $M_i^c \to M_i \to N$ is ambient. Theorem 2.2.3 shows that the fiber product $M_1^c \times_N M_2^c$ exists in $\textbf{MSm}$. This also represents a fiber product $M_1 \times_N M_2$, proving the first assertion. The second assertion follows from the construction of the canonical model of fiber product. This finishes the proof. □

**Remark 2.2.6.** The inclusion functor $\tau_s : \textbf{MSm} \to \textbf{MSm}$ preserves fiber products by construction.

Given some minimality assumptions, we can say more about the canonical model of fiber product. We will need this in this paper, but it will be used in the other papers, including [1]. Recall from [2, Definition 1.8.1] the definition of $\textbf{Comp}$.

**Proposition 2.2.7.**

1. Let $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ be morphisms in $\textbf{MSm}^{\text{fin}}$, and assume that $f_1$ is minimal. Then we have
   \[ M_1^c \times_N M_2 = ((M_1^c \times_N M_2)^N, \pi^*(M_1^c \times_N M_2^\infty)), \]
   where $\pi : (M_1^c \times_N M_2)^N \to M_1^c \times_N M_2$ is the normalization.

2. Consider the following commutative diagram
   
   \[
   \begin{array}{ccc}
   U_1 & \to & V \\
   j_1 & & j \\
   M_1 & \to & N \end{array}
   \]

   \[
   \begin{array}{ccc}
   j_2 \\
   \end{array}
   \]

   \[
   \begin{array}{ccc}
   \end{array}
   \]

   \[
   \begin{array}{ccc}
   U_2 \\
   \end{array}
   \]

   \[
   \begin{array}{ccc}
   \end{array}
   \]

   \[
   \begin{array}{ccc}
   M_2 \\
   \end{array}
   \]

   where $j_1$, $j_2$ are ambient and $\overline{L}$ is normal, we have $\overline{g}_i^* M_i^\infty \leq \overline{L}^\infty$. Therefore, we obtain

   \[ t^* P^\infty \leq s^* L^\infty, \]

   which shows that $g^c$ defines a morphism $g : L \to P$. This proves the first assertion. The last assertion is obvious by construction. This finishes the proof. □
in $\mathbf{MSm}^{\text{fin}}$, such that $j_1$ and $j_2$ are minimal. Then the morphism

$$j_1 \times j_2 : U_1 \times_{V_1} U_2 \to M_1 \times_N M_2$$

in $\mathbf{MSm}$, induced by the universal property of fiber product, belongs to $\mathbf{MSm}^{\text{fin}}$ and is minimal.

(3) In the situation of (2), if $j_1, j_2$ are open immersions, and if $U_1 \to V$ is minimal and if $U_1 \times_{V_1} U_2$ is normal, then

$$j_1 \times j_2 : U_1 \times_{V_1} U_2 = U_1 \times_{V_1} U_2 \to M_1 \times_N M_2$$

is an open immersion, where the equality follows by (1).

Proof. (1): This follows from the construction of canonical model of fiber product (see also [2, Corollary 1.10.7]).

(2): Let $\overline{P}$ (resp. $\overline{Q}$) be the closure of $M_1^0 \times_{N_1} M_2^0$ (resp. $U_1^0 \times_{V_1} U_2^0$) in $\overline{M_1 \times_N M_2}$ (resp. $\overline{U_1 \times_N U_2}$). Then the morphisms $j_1$ and $j_2$ induce a morphism

$$\overline{J} : \overline{Q} \to \overline{P}.$$ Then we obtain the following commutative diagrams:

in $\text{Sch}$ for $i = 1, 2$, where $p_i$ and $q_i$ are the natural $i$-th projections. Set $F := p_1^* M_1^\infty \times_{\overline{P}} p_2^* M_2^\infty \subset \overline{P}$ and $G := q_1^* U_1^\infty \times_{\overline{Q}} q_2^* U_2^\infty \subset \overline{Q}$. Then the commutativity of the diagrams shows

$$\overline{J}^{-1} F := F \times_{\overline{P}} \overline{Q} = (q_1^* J_1^* M_1^\infty) \times_{\overline{Q}} (q_2^* J_2^* M_2^\infty) = q_1^* U_1^\infty \times_{\overline{Q}} q_2^* U_2^\infty = G,$$

where the equality in the second line follows from the minimality of $j_1$ and $j_2$. Let $\pi_P : \text{Bl}_F(\overline{P})^N \to \overline{P}$ and $\pi_Q : \text{Bl}_G(\overline{Q})^N \to \overline{Q}$ be the normalized blow-ups. Therefore, by the universal property of blow-up and normalization, $\overline{J}$ lifts to a morphism

$$\overline{J}_1 : \overline{U_1 \times_{V_1} U_2} = \text{Bl}_G(\overline{Q})^N \to \text{Bl}_F(\overline{P})^N = M_1 \times_N M_2,$$
which makes the diagram

\[
\begin{array}{ccc}
\text{Bl}_G(Q)^N & \xrightarrow{\mathcal{J}_1} & \text{Bl}_F(P)^N \\
\pi_Q & & \pi_P \\
Q & \xrightarrow{\mathcal{J}} & P
\end{array}
\]

commute. Moreover, letting \( F' := \pi_P^{-1}(F) \), \( G' := \pi_Q^{-1}(G) \) be the exceptional divisors, the commutativity of the two diagrams as above shows

\[
\mathcal{J}_1^*(M_1^c \times M_2) = \mathcal{J}_1^*(\pi_P^*p_1^*M_1^\infty + \pi_P^*p_1^*M_1^\infty - F')
\]

\[
= \pi_Q^*\mathcal{J}_1^*p_1^*M_1^\infty + \pi_Q^*\mathcal{J}_1^*\pi_2^*M_2^\infty - G'
\]

\[
= \pi_Q^*\mathcal{J}_1^*q_1^*U_1^\infty + \pi_Q^*\mathcal{J}_1^*q_2^*U_2^\infty - G'
\]

\[
= (U_1^c \times U_2^c)^\infty
\]

where the equality in the fourth line follows from the minimality of \( j_1 \) and \( j_2 \). Therefore, the morphism \( \mathcal{J}_1 \) defines a minimal morphism \( U_1 \times U_2 \to M_1 \times M_2 \), as desired.

(3): We take the notation as above. Then \( \overline{U}_1 \times \overline{U}_2 \) is an open subset of \( \overline{P} \). Since \( \mathcal{J}^*F = G \), the minimality of \( U_1 \to V \) shows \( F \cap \overline{U}_1 \times \overline{U}_2 = \overline{U}_1 \times \overline{U}_2 \), where the right hand side is an effective Cartier divisor on \( \overline{U}_1 \times \overline{U}_2 \). Therefore, the normalized blow-up \( \pi_\mathcal{P} \) is an isomorphism over \( \overline{U}_1 \times \overline{U}_2 \), and the open immersion \( \overline{U}_1 \times \overline{U}_2 \to \text{Bl}_F(P) \) uniquely lifts to an open immersion \( \overline{U}_1 \times \overline{U}_2 \to \text{Bl}_F(P) \). This finishes the proof. \( \square \)

### 2.3. A remark on elementary correspondences

In this subsection, we will observe a relationship between cartesian squares and elementary correspondences. First we provide some definitions.

**Definition 2.3.1.** For any \( M_1, M_2 \in \text{MSm} \), we define \( \text{MCor}^{el} \) to be the set of elementary finite correspondence \( V : M_1^\circ \to M_2^\circ \) which satisfies the following admissibility conditions: let \( \overline{V} \) be the closure of \( V \) in \( \overline{M}_1 \times \overline{M}_2 \), and let \( \overline{\mathcal{V}}^N \to \overline{\mathcal{V}} \) be the normalization of \( \overline{\mathcal{V}} \). Let \( \text{pr}_i : \overline{\mathcal{V}}^N \to \overline{M}_i \) be the \( i \)-th projections.

1. \( pr_1 \) is proper.
2. \( pr_1^*M_1^\infty \geq pr_2^*M_2^\infty \).

**Definition 2.3.2 ([2, Def. 1.1.1, 1.3.3]).** A category \( \text{MCor} \) is defined as follows: the objects are the same as \( \text{MSm} \), and and for \( M, N \in \text{MSm} \),
$\mathbf{MCor}$, the set of morphisms is defined as the free abelian group generated on $\mathbf{MCor}^\mathrm{el}(M,N)$. Note that $\mathbf{MCor}(M,N) \subset \mathbf{Cor}(M^o,N^o)$ by definition. The composition is given by the composition of finite correspondences. Define $\mathbf{MCor}$ as the full subcategory of $\mathbf{MCor}$ whose objects are proper modulus pairs.

**Proposition 2.3.3.** For any modulus pair $M$, for any $f : N \to L$ in $\mathbf{MSm}$ and for any $V \in \mathbf{MCor}^\mathrm{el}(M,N)$, the image

$$ f_+(V) := (\text{Id}_{M^o} \times f^o)(V) \subset M^o \times L^o $$

is an irreducible closed subset, and we have $f_+(V) \in \mathbf{MCor}^\mathrm{el}(M,L)$.

Thus, any modulus pair $M$ is associated a covariant functor

$$ \mathbf{MCor}^\mathrm{el}(M,-) : \mathbf{MSm} \to \mathbf{Set}. $$

*Proof.* By [2, Prop. 1.2.3], the composition of finite correspondences $W := \Gamma_f \circ V$ belongs to $\mathbf{MCor}(M,L)$, where $\Gamma_f$ denotes the graph of $f^o : M^o \to N^o$. By the definition of composition, we can verify that $|W| = f(V)$. This implies that $f(V)$ is a component of $W$. Therefore, we have $W \in \mathbf{MCor}(M,L)$, as desired. $\square$

**Proposition 2.3.4.** Let $T$ be a pull-back square in $\mathbf{MSm}$ of the form

$$
\begin{array}{ccc}
T(00) & \xrightarrow{v_T} & T(01) \\
\downarrow q_T & & \downarrow p_T \\
T(10) & \xrightarrow{u_T} & T(11)
\end{array}
$$

(2.3.1)

and let $M$ be a modulus pair. Then the associated commutative diagram of sets

$$
\begin{array}{ccc}
\mathbf{MCor}^\mathrm{el}(M,T(00)) & \xrightarrow{v_T^+} & \mathbf{MCor}^\mathrm{el}(M,T(01)) \\
\downarrow q_T^+ & & \downarrow p_T^+ \\
\mathbf{MCor}^\mathrm{el}(M,T(10)) & \xrightarrow{u_T^+} & \mathbf{MCor}^\mathrm{el}(M,T(11))
\end{array}
$$

is cartesian in $\mathbf{Set}$.

**Remark 2.3.5.** We can formulate another statement by replacing $\mathbf{MCor}^\mathrm{el}$ with $\mathbf{MCor}$ and $(-)_+$ with $(-)_+$, but it will be false. Indeed, if $\alpha_1$ and $\alpha_2$ are distinct elementary correspondences which have the same image $\beta$ under $p_{T^*}$, then the image of the (non-elementary) finite correspondence $\alpha := \alpha_1 - \alpha_2$ is zero, which is trivially contained in the image of $u_{T^*}$. But there is no reason why $\alpha$ is contained in the image of $v_{T^*}$. 

Proof. Take any $\alpha_1 \in \text{MCor}^\text{el}(M, T(10))$ and $\alpha_2 \in \text{MCor}^\text{el}(M, T(01))$, and assume $\beta := u_{T^+}(\alpha_1) = p_{T^+}(\alpha_2)$. Let $\xi_i$ be the generic point of $\alpha_i$ for $i = 1, 2$.

We need to prove that there exists a unique $\gamma \in \text{MCor}^\text{el}(M, T(00))$ which maps to $\alpha_1$ and $\alpha_2$. The uniqueness is easy. Indeed, suppose given such $\gamma$, and let $\zeta$ be the generic point of $\gamma$. Then, since $\zeta$ lies over $\xi_1$ and $\xi_2$ and since $T^0$ is a pull-back diagram in $\text{Sm}$, the point $\zeta$ must be unique.

We prove the existence of $\gamma$. Let $\zeta \in (M^o \times T(10)^o) \times_{M^o \times T^0(11)} (M^o \times T(01)^o) \cong M^o \times T(10)^o \times_{T^0(11)} T(01)^o \cong M^o \times T(00)^o$ be the unique point which lies over $\xi_1$ and $\xi_2$. Let $\gamma := \overline{\{\zeta\}}$ be the closure of $\zeta$ in $M^o \times T(00)^o$, endowed with the reduced scheme structure.

Claim 2.3.6. $\gamma$ is an elementary correspondence from $M^o$ to $T(00)^o$.

Proof. We have to prove that $\gamma$ is finite and surjective over a component of $M^o$. Since $\zeta = (\xi_1, \xi_2) \in \alpha_1 \times_{M^o} \alpha_2$, the scheme $\gamma$ is naturally a closed subscheme of $\alpha_1 \times_{M^o} \alpha_2$. Moreover, since $\zeta$ maps to $\xi_i$ via the projection $\text{pr}_i : \alpha_1 \times_{M^o} \alpha_2 \to \alpha_i$ for each $i = 1, 2$, we obtain dominant maps $\gamma \to \alpha_i$. These maps are finite (hence surjective) since each $\alpha_i$ is finite over $M^o$. Since the natural map $\gamma \to M^o$ factors as $\gamma \to \alpha_1 \to M^o$, and since $\alpha_1$ is finite and surjective over a component, we obtain the claim.

Claim 2.3.7. $\gamma \in \text{MCor}^\text{el}(M, T(00))$.

Proof. We make a preliminary reduction as follows: since the assertion depends only on the isomorphism class of $T$ in $\text{MSm}^{\text{Sq}}$, we may assume that $T$ is ambient by Lemma 2.1.2. Moreover, since $T$ is a pull-back diagram, we have $T(00) \cong T(10) \times_{T(11)} T(01)$, where the right hand side is the canonical model of fiber product in Def. 2.2.2. Therefore, by replacing $T(00)$ with $T(10) \times_{T(11)} T(01)$ (this preserves the condition that $T$ is ambient by the construction of canonical model), we may assume that $\overline{q^*_T} T(10)^\infty$ and $\overline{v^*_T} T^*(01)$ have a universal supremum in the sense of [2, Def. 1.10.2] and that $T(00)^\infty = \sup(\overline{q^*_T} T(10)^\infty, \overline{v^*_T} T(01)^\infty)$.

Let $\overline{\gamma}$ be the closure of $\gamma$ in $\overline{M} \times T(00)$. First we check that $\overline{\gamma}$ is proper over $\overline{M}$. Note that the natural map $\overline{\gamma} \to \overline{M}$ factors as $\overline{\gamma} \to \overline{\alpha_1 \times_{\overline{\pi}} \overline{\alpha_2}} \to \overline{M}$. The first map is proper since the natural map $\overline{T}(00) \to \overline{T}(10) \times_{T(11)} \overline{T}(01)$ is proper by construction of the canonical model of
fiber product, and the latter map is proper since $\pi_i$ are proper over $\overline{M}$ by assumption. This shows that $\gamma \to \overline{M}$ is proper, as desired.

Next we check the modulus condition. Let $\alpha_i$ (resp. $\alpha_2$) be the closure of $\alpha_1$ (resp. $\alpha_2$) in $\overline{M} \times \overline{T}(10)$ (resp. $\overline{M} \times \overline{T}(01))$, and $\pi_i^N$ the normalization of $\alpha_i$. By assumption, we have $\alpha_1 \in \text{MCor}^e(M, T(10))$ and $\alpha_2 \in \text{MCor}^e(M, T(01))$, which means $M^\infty|_{\pi_1^N} \geq T(10)^\infty|_{\pi_1^N}$ and $M^\infty|_{\pi_2^N} \geq T(01)^\infty|_{\pi_2^N}$. Since $\gamma \to \alpha_i$ are dominant for $i = 1, 2$, we obtain morphisms $\gamma^N \to \overline{\alpha_i}^N$ by the universal property of normalization. Therefore, the above inequalities imply

$$M^\infty|_{\pi_i^N} \geq \overline{q_i}^*T(10)^\infty|_{\pi_i^N}, \quad M^\infty|_{\pi_i^N} \geq \overline{v_i}^*T(01)^\infty|_{\pi_i^N}.$$

Thus, since $\overline{q_i}^*T(10)^\infty$ and $\overline{v_i}^*T(01)^\infty$ have a universal supremum and since $T(00)^\infty = \sup(\overline{q_i}^*T(10)^\infty, \overline{v_i}^*T(01)^\infty)$ by assumption, we obtain

$$M^\infty|_{\pi_i^N} \geq \sup(\overline{q_i}^*T(10)^\infty|_{\pi_i^N}, \overline{v_i}^*T(01)^\infty|_{\pi_i^N})$$

$$= \sup(\overline{q_i}^*T(10)^\infty, \overline{v_i}^*T(01)^\infty)|_{\gamma^N}$$

$$= T(00)^\infty|_{\gamma^N}.$$

by [2, Remark 1.10.3 (3)]. This finishes the proof of the claim.

By construction, we have $\alpha_1 = q_T(\gamma)$ and $\alpha_2 = v_T(\gamma)$. This finishes the proof of Proposition 2.3.4.\Box

3. Off-diagonal functor

We introduce the “off-diagonal” functor, which is a key notion used in the definition of the cd-structure on $\text{MSm}$.

**Definition 3.1.1.** Define $\text{MEt}$ as a category such that

1. objects are those morphisms $f : M \to N$ in $\text{MSm}$ such that $f^o : M^o \to N^o$ is étale, and
2. morphisms of $f : M_1 \to N_1$ and $g : M_2 \to N_2$ are those pairs of morphisms $(s : M_1 \to M_2, t : N_1 \to N_2)$ which are compatible with $f, g$ such that $s^o$ and $t^o$ are open immersions.

Define $\overline{\text{MEt}}$ as the full subcategory of $\text{MEt}$ consisting of those $f : M \to N$ such that $M, N \in \text{MSm}$.

**Definition 3.1.2.** For modulus pairs $M$ and $N$, we define the disjoint union of $M$ and $N$ by

$$M \sqcup N := (\overline{M} \sqcup \overline{N}, M^\infty \sqcup N^\infty).$$

We have $(M \sqcup N)^o = M^o \sqcup N^o$, and $M \sqcup N$ represents a coproduct of $M$ and $N$ in the category $\overline{\text{MSm}}$. 
Theorem 3.1.3. There exists a functor
\[ \text{OD} : \text{M} \text{Et} \to \text{MSm} \]
such that for any \( f : M \to N \), there exists a functorial decomposition
\[ M \times_N M \cong M \sqcup \text{OD}(f). \]
Moreover, we have \( \text{OD}(f)^o = M^o \times_{N^o} M^o \setminus \Delta(M^o) \), where \( \Delta : M^o \to M^o \times_{N^o} M^o \) is the diagonal morphism. In particular, if \( f^o \) is an open immersion, then \( \text{OD}(f)^o = \emptyset \), hence \( \text{OD}(f) = \emptyset \). Moreover, the functor \( \text{OD} \) restricts to a functor
\[ \text{OD} : \text{MEt} \to \text{MSm}. \]
We call the functors the off-diagonal functors.

Proof. First, we prove that for any \( f : M \to N \) in \( \text{MEt} \), there exists a morphism \( i : X \to M \times_N M \) such that the induced morphism
\[ M \sqcup X \xrightarrow{\Delta \sqcup i} M \times_N M \]
is an isomorphism in \( \text{MSm} \). Take any object \( f : M \to N \) in \( \text{MEt} \). Since \( f^o \) is étale and separated by the assumption, the diagonal morphism \( \Delta : M^o \to M^o \times_{N^o} M^o \) is an open and closed immersion. Therefore, we obtain a decomposition into two connected components:
\[ M^o \times_{N^o} M^o = \Delta(M^o) \sqcup (M^o \times_{N^o} M^o - \Delta(M^o)). \]

Let \( P \) denote the canonical model of fiber product \( M \times_N M \) as in Def. 2.2.2. Note that \( P^o = M^o \times_{N^o} M^o \).

Define a closed immersion \( \overline{i}_\Delta : \overline{\Delta(f)} \to \overline{P} \) as the scheme-theoretic closure of the open immersion \( \Delta(M^o) \to P^o \to \overline{P} \). Set \( \Delta(f)^\infty := \overline{i}_\Delta^o P^\infty \) and \( \Delta(f) := (\overline{\Delta(f)}, \Delta(f)^\infty) \). Then \( \overline{i}_\Delta \) induces a minimal morphism \( i_\Delta : \Delta(f) \to P \). Moreover, we have \( \Delta(f)^o = \Delta(M^o) \).

Similarly, define a closed immersion \( \overline{i}_{\text{OD}} : \overline{\text{OD}(f)} \to \overline{P} \) as the scheme-theoretic closure of the open immersion \( M^o \times_{N^o} M^o - \Delta(M^o) \to P^o \to \overline{P} \). Set \( \text{OD}(f)^\infty := \overline{i}_{\text{OD}}^o P^\infty \) and \( \text{OD}(f) := (\overline{\text{OD}(f)}, \text{OD}(f)^\infty) \). Then \( \overline{i}_{\text{OD}} \) induces a minimal morphism \( i_{\text{OD}} : \text{OD}(f) \to P \). Moreover, we have \( \text{OD}(f)^o = M^o \times_{N^o} M^o - \Delta(M^o) \).

The morphisms \( i_\Delta \) and \( i_{\text{OD}} \) induce a minimal morphism in \( \text{MSm}^{\text{fin}} \):
\[ i_\Delta \sqcup i_{\text{OD}} : \Delta(f) \sqcup \text{OD}(f) \to P. \]

By (7) in Definition, 2.1.1, this morphism is an isomorphism in \( \text{MSm} \) (not in \( \text{MSm}^{\text{fin}} \)) since \( (i_\Delta \sqcup i_{\text{OD}})^o = \overline{i}_\Delta^o \sqcup \overline{i}_{\text{OD}}^o : \Delta(f)^o \sqcup \text{OD}(f)^o \to P^o \cong M^o \times_{N^o} M^o \) is an isomorphism in \( \text{Sm} \), and since \( \overline{i}_\Delta^o \sqcup \overline{i}_{\text{OD}}^o : \overline{\Delta(f)} \sqcup \overline{\text{OD}(f)} \to \overline{P} \) is proper by construction.
We claim $\Delta(f) \cong M$. Let $\Delta : M \to P(\cong M \times_N M)$ be the diagonal morphism. Then, the composite $M \xrightarrow{\Delta} P \cong \Delta(f) \sqcup \text{OD}(f)$ factors through $\Delta(f)$. The inverse morphism is given by $\Delta(f) \to P \xrightarrow{pr_1} M$, where $pr_1$ denotes the first projection $P \cong M \times_N M \to M$.

Thus, for any $f : M \to N$ in $\text{MfEt}$, we have obtained a decomposition $M \times_N M \cong M \sqcup \text{OD}(f)$.

Next we check the functoriality of $\text{OD}(f)$. Let $(f_1 : M_1 \to N_1) \to (f_2 : M_2 \to N_2)$ be a morphism in $\text{MfEt}$, i.e., a commutative diagram

$$
\begin{array}{ccc}
M_1 & \xrightarrow{s} & M_2 \\
\downarrow f_1 & & \downarrow f_2 \\
N_1 & \xrightarrow{t} & N_2
\end{array}
$$

where $f_1$, $f_2$, $s$ and $t$ are morphisms in $\text{MSm}$ such that $f_1^\circ$ and $f_2^\circ$ are étale and $s^\circ$ and $t^\circ$ are open immersions.

We claim that there exists a unique morphism $\text{OD}(f_1) \to \text{OD}(f_2)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
M_1 \times_{N_1} M_1 & \xrightarrow{s} & M_2 \times_{N_2} M_2 \\
\cong & & \cong \\
M_1 \sqcup \text{OD}(f_1) & \xrightarrow{=} & M_2 \sqcup \text{OD}(f_2).
\end{array}
$$

The uniqueness is obvious by the commutativity of the above diagram. For the existence, we need to show that the composite

$$
\text{OD}(f_1) \to M_1 \times_{N_1} M_1 \to M_2 \times_{N_2} M_2 \cong M_2 \sqcup \text{OD}(f_2)
$$

factors through $\text{OD}(f_2)$. To see this, it suffices to prove that the image of the morphism

$$
M_1^\circ \times_{N_1^\circ} M_1^\circ \setminus \Delta(M_1^\circ) \to M_1^\circ \times_{N_1^\circ} M_1^\circ \xrightarrow{s^\circ \times s^\circ} M_2^\circ \times_{N_2^\circ} M_2^\circ
$$

lands in $M_2^\circ \times_{N_2^\circ} M_2^\circ \setminus \Delta(M_2^\circ)$, which easily follows from the injectivity of the open immersion $s^\circ$. This finishes the proof. \hfill \Box

The off-diagonal functor is compatible with base change.

**Proposition 3.1.4.** Let $f : M \to N$ be an object of $\text{MfEt}$, and $N' \to N$ any morphism in $\text{MSm}$. Then the base change $g := f \times_N N'$ belongs to $\text{MfEt}$, and we have a natural isomorphism $\text{OD}(g) \cong \text{OD}(f) \times_N N'$. 

**Proof.** The first assertion holds since $g^\circ = f^\circ \times_{N'} N'^\circ$ is étale as a base change of an étale morphism. We prove the second assertion. Note
\[(M \times N M) \times N N' \cong M' \times N' M', \] where \(M' := M \times N N'.\) Consider the following diagram in \(\underline{MSm}^1:\)

\[
\begin{array}{ccc}
(M \times_N M) \times_N N' & \cong & (M \sqcup \text{OD}(f)) \times_N N' \\
\downarrow & & \downarrow h \\
M' \times_N N' M' & \cong & M' \sqcup \text{OD}(g)
\end{array}
\]

where all the arrows, except for \(h\), are natural isomorphisms in \(\underline{MSm}^1\), and \(h\) is defined to be the composite. By diagram chase, \(h\) restricts to the identity map on \(M'\) and an isomorphism \(\text{OD}(f) \times_N N' \to \text{OD}(g)\). This finishes the proof. □

In the following, we describe \(\text{OD}\) for special cases.

**Lemma 3.1.5.** Let \(f : U \to M\) be a minimal morphism such that \(\overline{f} : \overline{U} \to \overline{M}\) is étale. Then we have

\[
\text{OD}(f) = (\overline{U} \times_{\overline{M}} \overline{U} - \Delta(\overline{U}))^N, \\
\text{OD}(f)^\infty = \pi^* M^\infty,
\]

where \(\Delta : \overline{U} \to \overline{U} \times_{\overline{M}} \overline{U}\) is the diagonal, \((-)^N\) denotes the normalization, and \(\pi : (\overline{U} \times_{\overline{M}} \overline{U})^N \to \overline{M}\) is the natural morphism.

**Proof.** Since \(U^o \times_{M^o} U^o - \Delta(U^o)\) is dense in \(\overline{U} \times_{\overline{M}} \overline{U} - \Delta(\overline{U})\) (as a complement of the divisor \(U^\infty \times_{\overline{M}} \overline{U} \setminus \Delta(\overline{U})\)), and since \(U^\infty \times_{\overline{M}} \overline{U} = \overline{U} \times_{\overline{M}} U^\infty = \pi^* M^\infty\), the assertion follows from the construction of \(\text{OD}(f)\). This finishes the proof. □

**Proposition 3.1.6.** Let \(S\) be an \(\underline{MV}^1\)-square of the form

\[
\begin{array}{ccc}
S(00) & \xrightarrow{u_S} & S(01) \\
\downarrow q_S & & \downarrow p_S \\
S(10) & \xrightarrow{u_S} & S(11).
\end{array}
\]

Then the morphism \(\text{OD}(q_S) \to \text{OD}(p_S)\) is an isomorphism in \(\underline{MSm}^1\).

**Proof.** Let \(S\) be an \(\underline{MV}^1\)-square. Then, since \(\overline{S}\) is an elementary Nisnevich square, we have a natural isomorphism

\[
\overline{S}(00) \times_{\overline{S}(10)} \overline{S}(00) - \Delta_0(\overline{S}(00)) \cong \overline{S}(01) \times_{\overline{S}(11)} \overline{S}(01) - \Delta_1(\overline{S}(01)),
\]

where \(\Delta_i : \overline{S}(0i) \to \overline{S}(0i) \times_{\overline{S}(1i)} \overline{S}(0i)\) is the diagonal for each \(i = 0, 1\). Then, in view of Lemma 3.1.5, the minimality of \(u_S, p_S, q_S\) shows that the isomorphism as above induces an isomorphism \(\text{OD}(q_S) \to \text{OD}(p_S)\) in \(\underline{MSm}^1\). This finishes the proof. □
Corollary 3.1.7. Let $S$ be an MV-square. Then the natural morphism $\text{OD}(q_S) \to \text{OD}(p_S)$ is an isomorphism in $\mathbf{MSm}$. 

Proof. By definition of MV-square, there exists an MV$^{\text{fin}}$-square $S'$ which is isomorphic to $S$. Then, noting that there are natural isomorphisms $\text{OD}(q_S) \cong \text{OD}(q_{S'})$ and $\text{OD}(p_S) \cong \text{OD}(p_{S'})$ in $\mathbf{MSm}$, the assertion follows from Proposition 3.1.6. □

4. The cd-structure

In this section, we introduce a cd-structure on $\mathbf{MSm}$, and prove its fundamental properties.

4.1. MV-squares. First, we recall from [2] the cd-structure on $\mathbf{MSm}$.

Definition 4.1.1.

(1) An MV$^{\text{fin}}$-square is a square $S \in (\mathbf{MSm}^{\text{fin}})^{\text{Sq}}$ such that the morphisms in $S$ are minimal, and such that the resulting square

\[
\begin{array}{ccc}
S(00) & \longrightarrow & S(01) \\
\downarrow & & \downarrow \\
S(10) & \longrightarrow & S(11)
\end{array}
\]

is an elementary Nisnevich square (on $\text{Sch}$).

(2) An MV-square is a square $S \in \mathbf{MSm}^{\text{Sq}}$ which belongs to the essential image of the inclusion functor $(\mathbf{MSm}^{\text{fin}})^{\text{Sq}} \to \mathbf{MSm}^{\text{Sq}}$.

Proposition 4.1.2 ([2, Prop. 3.2.2]). The MV-squares form a complete and regular cd-structure $P_{\text{MV}}$ on $\mathbf{MSm}$. □

Definition 4.1.3. The topology on $\mathbf{MSm}$ associated with the cd-structure $P_{\text{MV}}$ is called the MV-topology.

4.2. MV-squares.

Definition 4.2.1. Let $T$ be an object of $\mathbf{MSm}^{\text{Sq}}$ of the form (2.3.1). Then $T$ is called an MV-square if the following conditions hold:

(1) $T$ is a pull-back square in $\mathbf{MSm}$.

(2) There exist an MV-square $S$ such that $S(11) \in \mathbf{MSm}$ and a morphism $S \to T$ in $\mathbf{MSm}^{\text{Sq}}$ such that the induced morphism $S^o \to T^o$ is an isomorphism in $\mathbf{Sm}^{\text{Sq}}$ and $S(11) \to T(11)$ is an isomorphism in $\mathbf{MSm}$. In particular, $T^o$ is an elementary Nisnevich square.

(3) $\text{OD}(q_T) \to \text{OD}(p_T)$ is an isomorphism in $\mathbf{MSm}$. 

We let $P_{MV}$ be the cd-structure on $\text{MSm}$ consisting of MV-squares. The topology on $\text{MSm}$ associated with the cd-structure $P_{MV}$ is called the MV-topology for short.

Remark 4.2.2.

(1) For any $T \in \text{MSm}^{\text{Sq}}$ with $T^o$ an elementary Nisnevich square, the induced morphism $\text{OD}(p_T)^o \to \text{OD}(q_T)^o$ between interiors is an isomorphism in $\text{Sm}$. This follows easily from the definition of elementary Nisnevich squares.

(2) If $p_T^o$ and $q_T^o$ are open immersions, then $\text{OD}(q_T) = \text{OD}(p_T) = \emptyset$. In particular, we have $\text{OD}(q_T) \cong \text{OD}(p_T)$.

Proposition 4.2.3. Let $T$ be a square in $\text{MSm}^{\text{Sq}}$ which satisfies Condition (1) (resp. (2), resp. (3)). Then, for any morphism $M \to T(11)$ in $\text{MSm}$, the base change square $T_M := T \times_{T(11)} M$ satisfies (1) (resp. (2), resp. (3)).

Proof. Since base change of a pull-back diagram is a pull-back diagram, Condition (1) is preserved by base change. Prop. 3.1.4 shows that (3) is preserved by the base change.

Finally, we prove that Condition (2) is preserved by base change. Let $S \to T$ be a morphism as in (2), and let $M \to T(11)$ be any morphism in $\text{MSm}$. Then we obtain a morphism $S_M \to T_M$, where $S_M := S \times_{S(11)} M$ and $T_M := T \times_{T(11)} M$. Since $S(11) \cong T(11)$, we obtain $S_M(11) \cong T_M(11)$. Moreover, $S_M$ is a MV-square as the base change of an MV-square (see [2, Theorem 4.1.2]), and we have $S_M^o \cong T_M^o$. Therefore, the morphism $S_M \to T_M$ satisfies the requirement in (2). This finishes the proof. 

4.3. Completeness.

Theorem 4.3.1. The cd-structure $P_{MV}$ is complete.

Proof. By [10, Lemma 2.5], it suffices to prove the following assertions:

(1) Any morphism with values in $\emptyset = (\emptyset, \emptyset)$ is an isomorphism.

(2) For any $T \in P_{MV}$ and for any $M \to T(11)$ in $\text{MSm}$, the square $T_M := T \times_{T(11)} M$, which is obtained by base change, belongs to $P_{MV}$.

(1) is obvious, and (2) is a direct consequence of Proposition 4.2.3.

4.4. Regularity.

Theorem 4.4.1. The cd-structure $P_{MV}$ is regular.

Proof. By [10, Lemma 2.11], it suffices to prove that for any $T \in P_{MV}$, the following assertions hold:
(1) $T$ is a pull-back square in $\text{MSm}$.
(2) $u_T : T(10) \to T(11)$ is a monomorphism.
(3) The fiber products $T(01) \times_{T(11)} T(01)$ and $T(00) \times_{T(10)} T(00)$ exist in $\text{MSm}$, and the derived square

\[
\begin{array}{ccc}
T(00) & \to & T(01) \\
\downarrow & & \downarrow \\
T(00) \times_{T(10)} T(00) & \to & T(01) \times_{T(11)} T(01)
\end{array}
\]

which we denote by $d(T)$, belongs to $P_{\text{MV}}$.

(1) is by the definition of MV-squares. (2) holds since $u_T^o : T^o(10) \to T^o(11)$ is an open immersion. We prove (3): we check the conditions in Def. 4.2.1 for $d(T)$.

Since $\Delta_{p_T}$ and $\Delta_{q_T}$ are open immersions, we have $\text{OD}(\Delta_{q_T}) \cong \emptyset \cong \text{OD}(\Delta_{p_T})$ by Theorem 3.1.3. Hence $d(T)$ satisfies (3) in Def. 4.2.1.

Note that $d(T)$ is isomorphic in $\text{MSm}^{\text{sq}}$ to the following diagram:

\[
\begin{array}{ccc}
T(00) & \to & T(01) \\
\downarrow & & \downarrow \\
T(00) \sqcup \text{OD}(q_T) & \to & T(01) \sqcup \text{OD}(p_T)
\end{array}
\]

where the vertical maps are the canonical inclusions, and the horizontal maps are induced by $v_T$. It is easy to see that this diagram is a pull-back diagram, i.e., $d(T)$ satisfies (1) in Def. 4.2.1. Indeed, suppose that we are given a pair of morphisms $f : M \to T(01)$ and $g : M \to T(00) \sqcup \text{OD}(q_T)$ which coincide at $T(01) \sqcup \text{OD}(p_T)$. Then, one sees that $g^o : M^o \to T(00)^o \sqcup \text{OD}(q_T)^o$ factors through $T(00)^o$, which implies that $g$ factors through $T(00)$.

We are reduced to checking the condition (2) for $d(T)$. Consider the following diagram in $\text{MSm}$:

\[
\begin{array}{ccc}
(T(00)^o, \emptyset) & \to & T(01) \\
\downarrow & & \downarrow \\
(T(00)^o, \emptyset) \sqcup \text{OD}(q_T) & \to & T(01) \sqcup \text{OD}(p_T)
\end{array}
\]

which we denote by $d(T)_0$, where the vertical maps are the canonical inclusions. Then $d(T)_0$ is an $\text{MV}$-square since $\text{OD}(q_T) \cong \text{OD}(p_T)$, and there exists a natural morphism $d(T)_0 \to d(T)$. It induces an isomorphism $d(T)^o_0 \cong d(T)^o$, and we have $d(T)_0(11) \cong d(T)(11)$. Therefore, $d(T)$ satisfies (2) in Def. 4.2.1. This finishes the proof. $\Box$
Theorem 4.4.2. Let $F$ be a presheaf with values in $\text{Sets}$ on $\text{MSm}$. Then $F$ is a sheaf with respect to the $\text{MV}$-topology if and only if $F(\emptyset) = 0$ and for any $\text{MV}$-square $T \in P_{\text{MV}}$, the square

$$
\begin{array}{c}
F(T(11)) \xrightarrow{u_T} F(T(10)) \\
\downarrow \quad \downarrow \\
F(T(01)) \xrightarrow{v_T} F(T(00))
\end{array}
$$

is cartesian.

Proof. This follows from [10, Corollary 2.17], Theorem 4.3.1 and Theorem 4.4.1. □

4.5. Subcanonicity. In this subsection, we prove the following result. Recall that a Grothendieck topology is subcanonical if every representable presheaf is a sheaf.

Theorem 4.5.1. The $\text{MV}$-topology and the $\text{MV}$-topology are subcanonical.

We need the following elementary observation.

Lemma 4.5.2. Let $P$ be a complete and regular cd-structure on a category $\mathcal{C}$. Then the topology associated with $P$ is subcanonical if and only if every square $T \in P$ is cocartesian in $\mathcal{C}$.

Proof. Let $\mathcal{Y}$ denote the Yoneda embedding of $\mathcal{C}$ into the category of presheaves on $\mathcal{C}$. All squares $T \in P$ are cocartesian in $\mathcal{C}$ if and only if for any $T \in P$ and for any $X \in \mathcal{C}$, the square

$$
\begin{array}{c}
\mathcal{Y}(X)(T(11)) \xrightarrow{u'_T} \mathcal{Y}(X)(T(10)) \\
\downarrow \quad \downarrow \\
\mathcal{Y}(X)(T(01)) \xrightarrow{v'_T} \mathcal{Y}(X)(T(00))
\end{array}
$$

is cartesian in $\mathcal{C}$. The latter condition is equivalent to that for any $X \in \mathcal{C}$, the representable presheaf $\mathcal{Y}(X)$ is a sheaf for the topology associated with $P$ by [10, Cor. 2.17]. This finishes the proof. □

We also need the following results:

Lemma 4.5.3 ([6, Lem. 2.2]). Let $f : X \to Y$ be a surjective morphism of normal integral schemes, and let $D, D'$ be two Cartier divisors on $Y$. If $f^* D' \leq f^* D$, then $D' \leq D$. □

Proposition 4.5.4.

(1) Any $\text{MV}$-square is cocartesian in $\text{MSm}$. 


Any MV-square is cocartesian in in $\textbf{MSm}$, hence in $\textbf{MSm}$.

Proof. (1): Let $S$ be an MV-square. We may assume that $S$ is an $\textbf{MV}^\text{fin}$-square since cocartesian-ness is stable under isomorphisms. Let $S(10) \to M$ and $S(01) \to M$ be morphisms in $\textbf{MSm}$ which coincide after restricted to $S(00)$. Since $S^o$ is an elementary Nisnevich square, it is cocartesian in $\textbf{Sm}$. Therefore, the morphisms $S(10)^o \to M^o$ and $S(01)^o \to M^o$ induce a unique morphism $h^o : S(11)^o \to M^o$. It suffices to check that $h^o$ induces a morphism $S(11) \to M$ in $\textbf{MSm}$.

Let $\Gamma$ be the graph of the rational map $\overline{S}(11) \dashrightarrow \overline{M}$, and let $\Gamma^N \to \Gamma$ be the normalization. For any $(ij) \in \textbf{Sq}$, set

$$S_1(ij) := (\overline{S}(ij) \times_{\overline{S}(11)} \Gamma^N, S^\infty \times_{\overline{S}(11)} \Gamma^N).$$

Then minimal morphisms $S_1(ij) \to S_1(kl)$ are induced by $S(ij) \to S(kl)$ for all $(ij) \to (kl)$ in $\textbf{Sq}$, and they form an $\textbf{MV}^\text{fin}$-square $S_1$. Moreover, $S_1(ij)$ are normal for all $(ij) \in \textbf{Sq}$, and the composites

$$\overline{h}_{ij} : \overline{S}_1(ij) \to \overline{S}(11) \dashrightarrow \overline{M}$$

are morphism of schemes for all $(ij) \in \textbf{Sq}$ by construction. Moreover, the morphisms $\overline{S}_1(ij) \to \overline{S}(ij)$ are proper (by the properness of $\Gamma$ over $\overline{S}(11)$). Therefore, by the minimality of $S_1(ij) \to S(ij)$, the morphism $S_1 \to S$ is an isomorphism in $\textbf{MSm}^{\text{Sq}}$.

Claim 4.5.5. $S_1^\infty \geq \overline{h}_{11} M^\infty$.

Proof. The admissibilities of $S(10) \to M$ and $S(01) \to M$ implies those of $S_1(10) \to M$ and $S_1(01) \to M$. Since $\overline{S}(10)$ and $\overline{S}(01)$ are normal, we have

$$S_1^\infty(ij)^\infty \geq \overline{h}_{ij} M^\infty$$

for $(ij) = (10), (01)$. Since $\overline{S}_1(10) \sqcup \overline{S}_1(01) \to \overline{S}_1(11)$ is a surjection between normal schemes and since $S_1^\infty(10) \to S_1(11)$ and $S_1(01) \to S_1(11)$ are minimal, Lemma 4.5.3 implies

$$S_1^\infty(11)^\infty \geq \overline{h}_{11} M^\infty.$$ 

This finishes the proof. \qed

By Claim 4.5.5, we have a morphism $S_1(11) \to M$ in $\textbf{MSm}^{\text{fin}}$. The composite $S(11) \dashrightarrow S_1(11) \to M$ gives the desired morphism. The uniqueness of the morphism follows from the fact that the elementary Nisnevich square $S^o$ is cocartesian in $\textbf{Sm}$. This finishes the proof of (1).

(2): Let $T$ be an MV-square. Then Condition (2) of Definition 2.3.1 shows that there exists an MV-square $S$ and a morphism $S \to T$ in $\textbf{MSm}^{\text{Sq}}$ such that $S(11) \cong T(11)$. Let $f : T(10) \to M$ and $g :
T(01) → M be morphisms in \( \text{MSm} \) which coincide after restricted to T(00). Then the composites
\[ f_S : S(10) → T(10) → T(11), \quad g_S : S(01) → T(01) → T(11) \]
coincide after restricted to S(00). Then \( f_S \) and \( g_S \) induce a unique morphism \( h : T(11) \cong S(11) → M \) since S is cocartesian in \( \text{MSm} \) by (1). Since \( S^o \cong T^o \), we have \( h \circ u_T = f \) and \( h \circ p_T = g \). This finishes the proof of Proposition 4.5.4.

Proof of Theorem 4.5.1. This follows from Lemma 4.5.2 and Proposition 4.5.4 (1) and (2). This finishes the proof.

5. Mayer-Vietoris sequence

5.1. Easy Mayer-Vietoris.

Definition 5.1.1. For any sheaf \( F \) on a site \( \mathcal{C} \), we denote by \( ZF \) the sheaf associated with the presheaf \( \mathcal{C} \ni X \mapsto Z(F(X)) \), where for any set \( S \), we denote by \( ZS \) the free abelian group generated on \( S \).

For any \( M \in \text{MSm} \) (resp. \( \text{MSm} \)), we set \( Z(M) := Z\mathcal{Y}(M) \), where \( \mathcal{Y}(M) \) denotes the presheaf of sets represented by \( M \).

Theorem 5.1.2. Let \( T \) be an MV-square. Then the complex of sheaves on \( \text{MSm} \)
\[ 0 → Z(T(00)) → Z(T(10)) ⊕ Z(T(01)) → Z(T(11)) → 0 \]
is exact.

Proof. This follows from [10, Lemma 2.18], Theorem 4.4.1 and Theorem 4.5.1.

5.2. Mayer-Vietoris with transfers.

Theorem 5.2.1. Let \( T \in \text{MSm}^{\text{Sq}} \). Assume that \( T^o \) is an elementary Nisnevich square, and that \( T \) satisfies Conditions (1) and (3) in Def. 4.2.1. Recall the following notation from Def. 4.2.1:

\[ T(00) \xrightarrow{v_T} T(01) \]
\[ \xrightarrow{q_T} \]
\[ T(10) \xrightarrow{u_T} T(11). \]

Then, for any \( M \in \text{MSm} \), the following complex of abelian groups is exact:
\[ 0 → Z_{tr}(T(00))(M) \xrightarrow{(q_T^*, v_T^*)} Z_{tr}(T(10))(M) ⊕ Z_{tr}(T(01))(M) \xrightarrow{p_T^* - u_T^*} Z_{tr}(T(11))(M), \]
Proof. The injectivity of \((q_{T^*}, v_{T^*})\) is obvious since \(v_{T^*} : T(00)^o \to T(01)^o\) is an open immersion. Therefore, it suffices to prove the exactness at \(\mathbb{Z}_{tr}(T(10))(M) \oplus \mathbb{Z}_{tr}(T(01))(M)\). This is equivalent to proving that the commutative square

\[
\begin{array}{ccc}
\text{MCor}(M, T(00)) & \xrightarrow{v_{T^*}} & \text{MCor}(M, T(01)) \\
q_{T^*} \downarrow & & \downarrow p_{T^*} \\
\text{MCor}(M, T(10)) & \xrightarrow{v_{T^*}} & \text{MCor}(M, T(11))
\end{array}
\]

is cartesian. Note that the horizontal maps are injective.

A key observation is the following lemma. Recall the notation from Proposition 2.3.3.

**Lemma 5.2.2.** Let \(\alpha_1, \alpha_2 \in \text{MCor}^{el}(M, T(01))\) be elementary correspondences with \(\alpha_1 \neq \alpha_2\). Assume that \(p_{T^+}(\alpha_1) = p_{T^+}(\alpha_2)\) holds in \(\text{MCor}^{el}(M, T(11))\). Then \(\alpha_1\) and \(\alpha_2\) belong to the image of \(v_{T^*}\).

**Proof.** Set \(P := T(01) \times_{T(11)} T(01)\), and consider the commutative diagram

\[
\begin{array}{ccc}
\text{MCor}^{el}(M, P) & \xrightarrow{pr_{1^+}} & \text{MCor}^{el}(M, T(01)) \\
pr_{2^+} \downarrow & & \downarrow p_{T^*} \\
\text{MCor}^{el}(M, T(01)) & \longrightarrow & \text{MCor}^{el}(M, T(11))
\end{array}
\]

in \(\text{Set}\). By Proposition 2.3.4, there exists a unique \(\gamma \in \text{MCor}^{el}(M, P)\) such that \(pr_{1^+}(\gamma) = \alpha_1\) and \(pr_{2^+}(\gamma) = \alpha_2\).

We have a canonical identification

\[
\text{MCor}^{el}(M, P) \cong \text{MCor}^{el}(M, T(01)) \sqcup \text{MCor}^{el}(M, \text{OD}(p_{T^*}))
\]

induced by \(P \cong T(01) \sqcup \text{OD}(p_{T^*})\). Through this identification, we regard \(\text{MCor}^{el}(M, \text{OD}(p_{T^*}))\) as a subset of \(\text{MCor}^{el}(M, P)\).

**Claim 5.2.3.** \(\gamma \in \text{MCor}^{el}(M, \text{OD}(p_{T^*}))\).

**Proof.** Let \(\xi_1, \xi_2\) and \(\zeta\) be the generic points of \(\alpha_1, \alpha_2\) and \(\gamma\). Then \(\zeta\) lies over \(\xi_1\) and \(\xi_2\). Since \(\xi_1 \neq \xi_2\) by the assumption that \(\alpha_1 \neq \alpha_2\), we have \(\xi \notin M^o \times \Delta(T(01)^o)\), where \(\Delta(T(01)^o)\) denotes the image of \(\Delta : T(01)^o \to T(01)^o \times_{T(11)^o} T(01)^o\). This implies that \(\xi \in M^o \times \text{OD}(p_{T^*})\). Therefore, we have

\[
\gamma \in \text{Cor}(M^o, \text{OD}(p_{T^*})) \cap \text{MCor}^{el}(M, P) = \text{MCor}^{el}(M, \text{OD}(p_{T^*})).
\]

This finishes the proof of the claim. \(\square\)
By construction, we have \( \alpha_i = \text{pr}_i(\gamma) = |(\text{pr}_i)_*(\gamma)| \), where \( \text{pr}_i : T(01)^{\circ} \times_{T(11)^{\circ}} T(01)^{\circ} \to T(01)^{\circ} \), \( i = 1, 2 \), are the projections. Therefore, in order to prove \( \alpha_i \in \text{MCor}(M, T(00)) \) for \( i = 1, 2 \), it suffices to prove that \( \alpha \in \text{MCor}(M, T(00) \times_{T(10)} T(00)) \). Since \( \alpha \in \text{MCor}(M, \text{OD}(p_T)) \) by the above claim, and since \( \text{OD}(q_T) \cong \text{OD}(p_T) \) by Condition (3) of Definition 4.2.1, we have \( \gamma \in \text{MCor}(M, \text{OD}(q_T)) \subseteq \text{MCor}(M, T(00) \times_{T(10)} T(00)) \). This finishes the proof of Lemma 5.2.2.

Now we are ready to prove that the above square is cartesian. Let \( \alpha \in \text{MCor}(M, T(01)) \) and assume \( p_T \circ \alpha \in \text{MCor}(M, T(10)) \). Write \( \alpha = \sum_{i \in I} m_i \alpha_i \), where \( I \) is a finite set, \( m_i \in \mathbb{Z} - \{0\} \) and \( \alpha_i \) are elementary correspondences which are distinct from each other. Then we have \( \alpha_i \in \text{MCor}(M, T(01)) \) for all \( i \in I \). Set

\[
J := \{ i \in I \mid \exists j \in I - \{i\}, |p_T \circ \alpha_i| = |p_T \circ \alpha_j| \}.
\]

Then, by Lemma 5.2.2, we have \( \alpha_i \in \text{MCor}(M, T(00)) \) for all \( i \in J \). Let \( i \in I - J \), and set \( \beta := |p_T \circ \alpha_i| \). Then the coefficient of \( \beta \) in \( p_T \circ \alpha \) is non-zero, and therefore \( \beta \in \text{MCor}^{el}(M, T(10)) \). By Proposition 2.3.4, there exists a unique \( \gamma \in \text{MCor}^{el}(M, T(00)) \) such that \( \nu_{T+}(\gamma) = \alpha_i \) and \( q_{T+}(\gamma) = \beta \). Since \( T(00)^{\circ} \to T(01)^{\circ} \) is an open immersion, this implies that \( \alpha_i = \gamma \in \text{MCor}^{el}(M, T(00)) \). This finishes the proof of the exactness of (5.2.2).

Recall from [2, Th. 2 (2)] that for any \( M \in \text{MSm} \), the presheaf \( Z_{tr}(M) \) on \( \text{MSm} \) is a sheaf for the \( \text{MV} \)-topology.

**Corollary 5.2.4.** Let \( T \) be an MV-square. Then the following complex of sheaves on \( \text{MSm} \) for the \( \text{MV} \)-topology is exact:

\[
0 \to Z_{tr}(T(00)) \xrightarrow{(q_T, \nu_{T+})} Z_{tr}(T(10)) \oplus Z_{tr}(T(01)) \xrightarrow{p_T - u_T} Z_{tr}(T(11)) \to 0.
\]

**Proof.** By Theorem 5.2.1, it suffices to prove the surjectivity of the last maps of the complexes. Take a morphism \( S \to T \) in \( \text{MSm}^{\text{sq}} \) as in (2) of Definition 4.2.1. Then the map

\[
Z_{tr}(S(10)) \oplus Z_{tr}(S(01)) \to Z_{tr}(S(11)) = Z_{tr}(T(11))
\]

is epi in \( \text{MNST} \) by [2, Theorem 4.5.7]. Since the map factors through

\[
Z_{tr}(T(10)) \oplus Z_{tr}(T(01)),
\]

we are done. \( \square \)
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