Two-dimensional steady-state heat conduction problem for heat networks

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Abstract. The method of determining the level of heat losses through structures of heat networks and the two-dimensional steady-state heat conduction problem for multilayer structures of heat networks have been considered, given that each of the layers has its own physical properties and heat transfer coefficients depend on $r$ and $z$ coordinates. The definition of heat losses is an important practical task, since they are one of the main indicators of energy efficiency in the operating heat networks and are included in heat tariffs.

1. Introduction

Heat network is a set of pipelines and units supplying heat from a source to consumers by means of a heat carrier medium (hot water or steam). Heat network pipelines are multilayer cylinder-shaped non-uniform constructions.

In district heating systems, heat supply from the heat source to consumers is connected with heat losses also through thermal insulation of heat networks [1]. Thermal insulation is the construction element reducing the heat-transfer process and acting as the main thermal resistance in the construction [1]. Thermal insulation serves to reduce heat losses and to secure the temperature tolerance of the insulated surface. Depending on the heat carrier temperature, mineral wool or PU foam is used as the thermal insulation.

Based on energy studies, the rate of heat loss during transfer of heat carrier in existing heat networks may be estimated as 15-30 %, depending on the season [2]. Heat losses are one of the main indicators of energy efficiency in the operating heat networks and are included in heat tariffs. The definition of heat losses is an important practical task.

Heat losses are interpreted as the amount of heat transferred from the heat carrier to the environment through the structures of heat networks.

According to Fourier’s law, the total amount of heat transferred through the pipeline surface is calculated by formula [3]:

$$Q = -\sum_{j=1}^{m} \int_{\sigma_j} \text{div} [\lambda_j \text{grad} T_j] d\sigma, \quad (j = 1, m)$$

(1)
where \( \text{grad} T_j \) is the temperature gradient in layer \( j \); \( \lambda_j \) is the heat transfer coefficient of the non-uniform layer \( j \); \( \sigma_j \) is the surface of layer \( j \) in structures of the heat networks; and \( m \) is the number of layers.

Conduction heat transfer in cylindrical coordinates is defined by formula [4]:

\[
\text{div} \left[ \lambda_j \text{grad} T_j \right] = \frac{\partial \lambda_j}{\partial r} \frac{\partial T_j}{\partial r} + \frac{1}{r^2} \frac{\partial }{\partial \theta} \left( \frac{\partial \lambda_j}{\partial \theta} \frac{\partial T_j}{\partial \theta} \right) + \lambda_j \left( \frac{1}{r^2} \frac{\partial }{\partial r} \left( r \frac{\partial T_j}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T_j}{\partial \theta^2} + \frac{\partial^2 T_j}{\partial z^2} \right),
\]

\( (j=1,m) \)

where \( \frac{\partial T_j}{\partial r}, \frac{\partial T_j}{\partial \theta}, \frac{\partial T_j}{\partial z} \) is the temperature pattern along coordinates \( r, \theta, z \) of layer \( j \); \( \frac{\partial \lambda_j}{\partial r}, \frac{\partial \lambda_j}{\partial \theta}, \frac{\partial \lambda_j}{\partial z} \) is the heat transfer coefficient of the non-uniform layer \( j \) along coordinates \( r, \theta, z \).

To determine the rate of heat losses at the pipeline section in operation [5], a concept of an ideal pipe has been introduced. An ideal pipe is interpreted as a pipe having no roughness on the inner surface, and the pipeline walls being heat-proof. Consequently, no heat loss in such a pipe will take place.

The level of heat losses at the pipeline section of heat networks can be determined as the difference between the quantity of heat in ideal conditions and that in view of heat losses through heat network pipeline layers.

To determine heat losses as per formula (1), it is required to specify the temperature pattern. Research paper [6] considers an approximate solution of the steady-state heat conduction problem at a section of a multilayer heat network pipeline. In the considered task, heat transfer coefficient is a constant value along coordinates \( r \) and \( z \).

Heat conduction of uniform cylindrical constructions is widely considered in works [3], [4], [7] – [9]. Heat transfer coefficient in the solutions of these works is also treated as a constant value. If non-uniform layers of heat network constructions are to be considered, then the heat transfer coefficient is generally dependent on coordinates. One-dimensional steady-state problems of heat conduction have been considered in research [10].

Within the scope of this work, let us consider a two-dimensional steady-state heat conduction problem for a pipeline section of a heat network of finite size, with the heat carrier medium flowing along \( z \)-coordinate, given that each of the layers has its own physical properties and heat transfer coefficients depend on \( r \) and \( z \) coordinates (Ill. 1).

![Figure 1. Multilayer pipeline section of a heat network.](image-url)
2. Methods

Heat conduction equation in cylindrical coordinates takes the form [11]:

\[ \frac{\partial \lambda_j}{\partial r} \frac{\partial T_j}{\partial r} + \lambda_j \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T_j}{\partial r} \right) + \frac{\partial^2 T_j}{\partial z^2} \right) + \frac{\partial \lambda_j}{\partial z} \frac{\partial T_j}{\partial z} = \beta_j \omega_j \frac{\partial T_j}{\partial z}, \]  \tag{2}

where \( \omega_j \) is the rate of heat carrier medium along \( z \) -coordinate, and \( \beta_j \) is the coefficient dependent on the aggregation state of the substance.

To solve an equation (2), let us establish its boundary conditions [12]. We will set the temperature of the environment for the first and last layers, as well as the law of convection heat transfer law between the body surface and the medium. Hence, we obtain:

\[ \frac{\partial T_j(r, z)}{\partial r} \bigg|_{r=R_j} - H_{tr} \left[ T_j(r, z) - T_{\text{men.,}r} \right]_{r=R_i} = 0 \]  \tag{3}

\[ T_j(r, z) \bigg|_{r=R_j} = T_{j+1}(r, z) \bigg|_{r=R_j} \]  \tag{4}

\[ \lambda_j(r) \frac{\partial T_j(r, z)}{\partial r} \bigg|_{r=R_j} = \lambda_{j+1}(r) \frac{\partial T_{j+1}(r, z)}{\partial r} \bigg|_{r=R_j} \]  \tag{5}

\[ \frac{\partial T_m(r, z)}{\partial r} \bigg|_{r=R_m} + H_{mv} \left[ T_m(r, z) - T_{\text{wep.,a}} \right]_{r=R_m} = 0 \]  \tag{6}

where \( H_{tr} = \frac{\alpha_j}{\lambda_j(r)} \); \( H_{mv} = \frac{\alpha_m}{\lambda_m(r)} \); \( \alpha_j \) and \( \alpha_m \) are the thermal transmittance values (U-value) of the heat carrier medium and layer \( m \) of the pipeline, respectively; \( T_{\text{men.,}r} \) is the temperature of the heat carrier medium at the pipeline inlet; and \( T_{\text{wep.,a}} \) is the temperature of the ambient air.

At the boundary \( z = 0 \) for the first layer, let us set the known temperature \( T_{\text{men.,}r} \), and for the end faces of subsequent layers of the multilayer pipeline we will establish such conditions that the heat flow through the end faces would be missing:

\[ T_j(r, z) \bigg|_{z=0} = T_{\text{men.,}r} \]  \tag{7}

\[ \frac{\partial T_{j+1}(r, z)}{\partial z} \bigg|_{z=0} = 0 \]  \tag{8}

At the boundary \( z = l \), we shall set such conditions that the heat flow through the end faces would be missing:

\[ \frac{\partial T_j(r, z)}{\partial z} \bigg|_{z=l} = 0 \]  \tag{9}
3. Results

Let us submit Equation (2) in a form:

$$
\left[ \frac{\partial \lambda_j}{\partial r} \frac{\partial T_j}{\partial r} + \lambda_j \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T_j}{\partial r} \right) \right] + \left[ \frac{\partial \lambda_j}{\partial z} \frac{\partial T_j}{\partial z} + \lambda_j \frac{\partial^2 T_j}{\partial z^2} - \beta_j \omega_j \frac{\partial T_j}{\partial z} \right] = 0
$$

(10)

where \( T_j = T_j(r,z) \) is the temperature pattern of layer \( j \).

\( T_j(r,z) \) will be presented as a product of two functions, one of which is \( r \) -coordinate dependent, and the other one depends on \( z \) -coordinate.

Thus, we get two equations:

$$
\frac{\partial \lambda_j}{\partial r} \varphi_j + \lambda_j \frac{1}{r} \frac{\partial}{\partial r} \left( r \varphi_j \right) = 0
$$

(12)

$$
\frac{\partial \lambda_j}{\partial z} \phi_j + \lambda_j \frac{\partial^2 \phi_j}{\partial z^2} - \beta_j \omega_j \frac{\partial T_j}{\partial z} = 0
$$

(13)

Solving equation (12), we obtain:

$$
\lambda_j'(r) \psi_j'(r) + \lambda_j(r) \left( r \psi_j'(r) \right)' = 0
$$

After a substitution \( r \psi_j'(r) = U_j(r) \):

$$
\lambda_j'(r)U_j(r) + \lambda_j(r)U_j'(r) = 0
$$

$$
\left( \lambda_j(r)U_j(r) \right)' = 0
$$

Upon integrating over the range of \( r_{j-1} \) to \( r \):

$$
\lambda_j(r) \left. \frac{d\psi_j(r)}{dr} \right|_{r_{j-1}} = 0
$$

$$
\lambda_j(r) \frac{d\psi_j(r)}{dr} = \lambda_j(r_{j-1}) \frac{d\psi_j(r_{j-1})}{dr}
$$

$$
\frac{d\psi_j(r)}{dr} = \frac{\lambda_j(r_{j-1}) r_{j-1} F_{j-1}}{\lambda_j(r)}
$$

where

$$
F_{j-1} = \frac{d\psi_j(r_{j-1})}{dr}
$$

$$
\psi_j(r) = \lambda_j(r_{j-1}) r_{j-1} F_{j-1} \int_{r_{j-1}}^{r} \frac{dr}{\lambda_j(r)} + D_{j-1}
$$

(14)

where \( D_{j-1} = \psi_j(r_{j-1}) \).

In equation (14), the constants \( C_{j-1} \) and \( D_{j-1} \) are determined from the boundary conditions.

Basing on equation (13), we get:

$$
\lambda_j(z) \phi_j'(z) + \phi_j'(z) \left( \lambda_j'(z) - \omega_j \right) = 0
$$

Having introduced the notations \( A_j = \lambda_j(z) \), \( B_j = \left( \lambda_j'(z) - \omega_j \right) \), we obtain:
\[ A_1 \phi_j^*(z) + B_j \phi_j(z) = 0 \]  

(15)

Solving the equation (15), we get:

\[ \phi_j(z) = C_1 + C_2 e^{-\frac{B_j}{\lambda_j} z} \]  

(16)

where \( C_1 \) and \( C_2 \) are the unknown coefficients, determined from the boundary conditions.

Application of the obtained solutions (14) and (16) to (11) will provide:

\[ T_j(r,z) = \left[ \lambda_j(r_{j-1})r_{j-1} \int_{r_{j-1}}^{r_j} \frac{d}{\lambda_j(r_j)} r + D_{j-1} \right] \left( C_1 + C_2 e^{-\frac{B_j}{\lambda_j} z} \right) \]  

(17)

Having determined the temperature pattern (17), we further calculate the rate of heat losses through constructions of heat networks by formula (1) as the difference between the amount of heat under ideal conditions and the amount of heat in view of heat losses through constructions of heat networks.

**Conclusions**

A method of calculating heat losses through constructions of heat networks has been defined. A solution of the two-dimensional steady-state heat conduction problem for multilayer structures of heat networks, with any number of layers and any non-uniformity of thermal-physical properties, has been obtained. Using simple transformations, solutions for various boundary conditions can be obtained. The provided solution is common for any non-uniform heat transfer coefficients. The provided solution is convenient, since it is possible to obtain a set of solutions for any type of materials, through setting individual curves of heat transfer coefficients, which is significant for non-uniform materials.

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