Commutativity of quantization with conic reduction for torus actions on compact CR manifolds

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Abstract

We define conic reductions $X^{\text{red}}_\nu$ for torus actions on the boundary $X$ of a strictly pseudo-convex domain and for a given weight $\nu$ labeling a unitary irreducible representation. There is a natural residual circle action on $X^{\text{red}}_\nu$. We have two natural decompositions of the corresponding Hardy spaces $H(X)$ and $H(X^{\text{red}})_\nu$. The first one is given by the ladder of isotypes $H(X)_{k\nu}$, $k \in \mathbb{Z}$, the second one is given by the $k$-th Fourier components $H(X^{\text{red}})^k$ induced by the residual circle action. The aim of this paper is to prove that they are isomorphic for $k$ sufficiently large. The result is given for spaces of $(0, q)$-forms with $L^2$-coefficient when $X$ is a CR manifold with non-degenerate Levi form.

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1 Introduction

Let $X$ be the boundary of a strictly pseudo-convex domain $D$ in $\mathbb{C}^{n+1}$. Then $(X, T^{1,0}X)$ is a contact manifold of dimension $2n + 1$, $n \geq 1$, where $T^{1,0}X$ is the sub-bundle of $TX \otimes \mathbb{C}$ defining the CR structure. We denote by $\omega_0 \in C^\infty(X, T^*X)$ the contact 1-form whose kernel is the horizontal bundle $HX \subset TX$, we refer to Section 2.1 for definitions. Associated with this data we can define the Hardy space $H(X)$, it is the space of boundary values of holomorphic functions in $D$ which lie in $L^2(X)$, the Hilbert space of square
integrable functions on $X$. Suppose a contact and CR action of a $t$-dimensional torus $T$ is given; we denote by $\mu : X \to t^*$ the associated CR moment map. Fix a weight $i\nu$ in the lattice $i\mathbb{Z}^l \subset t^*$, if $0 \in t^*$ does not lie in the image of the moment map the isotypes

$$H(X)_{i\nu} = \{ f \in H(X) : (e^{i\theta} : f)(x) = e^{ik\langle \nu, \theta \rangle} f(x), \theta \in \mathbb{R}^l \}, \quad k \in \mathbb{Z},$$

are finite dimensional.

Suppose that the ray $i\mathbb{R}_+ \cdot \nu \in t^*$ is transversal to $\mu$, then $X_\nu := \mu^{-1}(i\mathbb{R}_+ \cdot \nu)$ is a sub-manifold of $X$ of codimension $l - 1$. There is a well-defined locally free action of $T_{\nu}^{-1} \:=\! \exp_{\nu}(i \ker \nu)$ on $X_\nu$, the resulting orbifold $X_\nu^{\text{red}}$ is called conic reduction of $X$ with respect to the weight $\nu$. Let $\varphi$ be an Euclidean product on $t$, we shall also use the symbol $\langle \cdot, \cdot \rangle$, and denote by $\lambda^\varphi \in t$ be uniquely determined by $\lambda = \varphi(\lambda^\varphi, \cdot)$ and $\|\lambda\|$ the corresponding norm. By abuse of notation we write $\lambda$ for $\lambda^\varphi$ and we identify $t \cong i\mathbb{R}^t$ with its dual. We set

$$\ker \nu = \nu^\perp := \{ \lambda \in t : \langle \nu, \lambda \rangle = 0 \}.$$

The locus $X_\nu$ is $T$-invariant, we will always assume that the action of $T$ on $X_\nu$ is locally free. After replacing $T$ with its quotient by a finite subgroup, we may and will assume without loss of generality that the action is generically free. In Section 2.1, we show that $X_\nu^{\text{red}}$ is CR manifold with positive definite Levi form of dimension $2n - 2t + 3$.

Let us define $T_{\nu}^{1} \:=\! \exp_T(i \nu)$, if $\nu$ is coprime, we have a Lie group isomorphism

$$\kappa_\nu : S^1 \to T_{\nu}^{1}, \quad e^{i\theta} \mapsto e^{i\theta \nu}$$

between $T_{\nu}^{1} \:=\! \exp_T(i \nu)$ and the circle $S^1$. Let us denote by

$$\overline{T_{\nu}^{1}} := T/T_{\nu}^{-1} \cong T_{\nu}^{1}/(T_{\nu}^{1} \cap T_{\nu}^{-1}),$$

then the character $\chi_\nu : T \to S^1, \chi_\nu(e^{i\theta}) := e^{ik\langle \nu, \theta \rangle}$, being trivial on $T_{\nu}^{-1}$, descends to a character $\chi_\nu' : \overline{T_{\nu}^{1}} \to S^1$ which is a Lie group isomorphism, see [23, Lemma 10]. Thus, we have a locally free circle action of $\overline{T_{\nu}^{1}}$ on $X_\nu^{\text{red}}$, which induces an action on the Hardy space $H(X_\nu^{\text{red}})$. Suppose that the action of $\overline{T_{\nu}^{1}}$ on $X$ is transversal to the CR structure. We denote by $H(X_\nu^{\text{red}})_k$ the corresponding $k$-th Fourier component and we call the action of $\overline{T_{\nu}^{1}}$ on $X_\nu^{\text{red}}$ residual circle action. The aim of this paper is to prove that $H(X)_k$ and $H(X_\nu^{\text{red}})_k$ are isomorphic for $k$ sufficiently large.

We prove the aforementioned result in the more general setting of CR manifolds for spaces of $(0, q)$-forms when $k$ is large, more precisely we consider $(0, q)$ forms with $L^2$ coefficients and the corresponding projector $S^{(q)}$ onto the kernel of the Kohn Laplacian $H^q(X)$. Now, we make more precise the assumptions on the CR manifold $X$ and on the group action.

**Assumption 1.1.** Let $(X, T^{1,0}X)$ be a compact connected orientable CR manifold of dimension $2n + 1$, $n \geq 1$, and let $\omega$ be the associated contact 1-form. The Levi form $L$ is non-degenerate of constant signature $(n_-, n_+)$ on $X$. That is, the Levi form has exactly $n_-$ negative and $n_+$ positive eigenvalues at each point of $X$, where $n_- + n_+ = n$.

Concerning the group action, we always assume

**Assumption 1.2.** The action of $T$ preserves the contact form $\omega_0$ and the complex structure $J$. That is, $g^*\omega_0 = \omega_0$ on $X$ and $g_*J = Jg_*$ on the horizontal bundle $HX$ for every $g \in T$, where $g^*$ and $g_*$ denote the pull-back map and push-forward map of $T$, respectively.

Let $X_{\nu}^{\text{red}} := X_\nu/T_{\nu}^{-1}$, defined in the same way as before, more precisely we shall assume
The moment map $\mu$ is transverse to the ray $i \mathbb{R}_+ \cdot \nu \in \mathfrak{t}^*$, the action of $\mathbb{T}$ on $X_\nu$ is locally free and for every $x \in X_\nu$

$$\text{val}_x(\nu^+) \cap \text{val}_x(\nu^+) = \{0\}$$

where $b$ is a bilinear form on $H_x X$ such that

$$b(\cdot, \cdot) = d\omega_b(\cdot, J \cdot)$$

and it is non-degenerate.

We note that $b(U, V) = 2L(U, V)$ for every $U, V \in HX$. By Assumptions above, we will show that $X^\text{red}_\nu$ is a CR manifold with natural CR structure induced by $T^{1,0}X$ of dimension $2n - 2(t - 1) + 1$. Let $L_{X^\text{red}_\nu}$ be the Levi form on $X^\text{red}_\nu$ induced naturally from the Levi form $L$ on $X$. For a given subspace $s$ of $t$, we denote $s_X$ the subspace of infinitesimal vector fields on $X$. Let us consider

$$B = \ker \nu_X \oplus J \ker \nu_X.$$ 

Hence, $b$ has constant signature on $B \times B$, suppose $b$ has $r$ negative eigenvalues on $B \times B$ where $r \leq n_-$ since $L$ and $b$ have the same number of negative eigenvalues on $HX$. Fix $q = n_-$, hence by Lemma 2.1, $L_{X^\text{red}_\nu}$ has $q - r$ negative eigenvalues at each point of $X^\text{red}_\nu$. We refer to Section 2.1 for definitions, we have:

**Theorem 1.1.** Suppose that $\Box^\nu_b$ has $L^2$ closed range. Fix a maximal coprime weight $\nu \neq 0$ in the lattice inside $\mathfrak{t}^*$ and assume that the circle action $\mathbb{T}^\nu$ is a transversal CR action. Fix $q = n_-$, under the assumptions above, $X^\text{red}_\nu$ is a compact CR manifold with non-degenerate Levi form having $q - r$-negative eigenvalues. There is a natural isomorphism of vector spaces $\sigma_k : H^q(X)_{k\nu} \rightarrow H^{q-r}(X^\text{red}_\nu)_{k\nu}$ for $k$ sufficiently large.

For strictly pseudoconvex domain we have $q = n_+ = r = 0$ and thus we have quantization commutes with reduction for spaces of functions for $k$ large. We give a proof in Section 3 which is inspired from [HH] (see [HMM] for the full extension of [HH]) and it is a consequence of the microlocal properties of the projector $S^{(q)}_{k\nu}$ described in Section 2.2 and calculus of Fourier integral operators of complex type, see [MS]. Furthermore, we recall that the way to establish the isometry from kernel expansion for $k$ large comes from [MZ].

The conic reductions defined above appear naturally in geometric quantization. In fact, given a Hamiltonian and holomorphic action with moment map $\Phi$ of a compact Lie Group $G$ on a Hodge manifold $(M, \omega)$ with quantizing circle bundle $\pi : X \rightarrow M$, one can always define an infinitesimal action of the Lie algebra $\mathfrak{g}$ on $X$. If it can be integrated to an action of the whole group $G$, then one has a representation of $G$ on $H(X)$. In [GS2] it was observed that associated to group actions one can define reductions $M^\text{red}_\Theta$ by pulling back a $G$-invariant and proper sub-manifold $\Theta$ of $\mathfrak{g}^*$ via the moment map $\Phi$. When $\Theta$ is chosen to be a cone through a co-adjoint orbit $C(\mathcal{O}_\nu)$ one has associated reduction whose Hardy space of its quantization is

$$H(X^\text{red}_\nu) = \bigoplus_{k \in \mathbb{Z}} H(X^\text{red}_\nu)_k,$$

where $k$ labels an irreducible representation of the residual circle action described above. Now, it is natural to ask if this spaces are related with the decomposition induced by $G$ on $H(X)$. When $G = \mathbb{T}$ Theorem 1.1 states that they are isomorphic to $H(X)_{k\nu}$; the
it is proved that \( \delta_k \) is an isomorphism for \( k \) large enough in the setting when \( X \) is the circle bundle of a polarized Hodge manifold whose Grauert tube is \( D \). Thus, Theorem 1.1 generalizes the main theorem in [P3] to compact “quantizable” pseudo-Kähler manifolds. We also refer to [P2] for examples and the explicit expression of the leading term of the asymptotic expansion of \( \dim H(X)_{k\nu} \) as \( k \) goes to infinity. Along this line of research, Toeplitz operators were studied for circle action, in [G] the study of asymptotics of compositions of Toeplitz operators with quantomopomorphism is addressed for torus actions.

2 Preliminaries

2.1 Geometric setting

We recall some notations concerning CR and contact geometry. Let \( (X, T^{1,0}X) \) be a compact, connected and orientable CR manifold of dimension \( 2n + 1 \), \( n \geq 1 \), where \( T^{1,0}X \) is a CR structure of \( X \). There is a unique sub-bundle \( HX \) of \( TX \) such that \( HX \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X \). Let \( J : HX \rightarrow HX \) be the complex structure map given by \( J(u + \bar{v}) = iu - \bar{v} \), for every \( u \in T^{1,0}X \). By complex linear extension of \( J \) to \( TX \otimes \mathbb{C} \), the \( i \)-eigenspace of \( J \) is \( T^{1,0}X \). We shall also write \( (X, HX, J) \) to denote a CR manifold.

Since \( X \) is orientable, there always exists a real non-vanishing 1-form \( \omega_0 \in \mathcal{C}^\infty(X, T^*X) \) so that \( \langle \omega_0(x), u \rangle = 0 \), for every \( u \in H_xX \), for every \( x \in X \); \( \omega_0 \) is called contact form and it naturally defines a volume form on \( X \). For each \( x \in X \), we define a quadratic form on \( HX \) by

\[
L_x(U, V) = \frac{1}{2} d\omega_0(JU, V), \quad \forall U, V \in H_xX.
\]

Then, we extend \( L \) to \( HX \otimes \mathbb{C} \) by complex linear extension; for \( U, V \in T^{1,0}X \),

\[
L_x(U, \overline{V}) = \frac{1}{2} \overline{d\omega_0(JU, \overline{V})} = -\frac{1}{2i} d\omega_0(U, \overline{V}).
\]

The Hermitian quadratic form \( L_x \) on \( T^{1,0}X \) is called Levi form at \( x \). In the case when \( X \) is the circle bundle of an Hodge manifold \( (M, \omega) \), the positivity of \( \omega \) implies that the number of negative eigenvalues of the Levi form is equal to \( n \). The Reeb vector field \( R \in \mathcal{C}\mathcal{C}^\infty(X, TX) \) is defined to be the non-vanishing vector field determined by

\[
\omega_0(R) = 1, \quad d\omega_0(R, \cdot) = 0 \quad \text{on} \quad TX.
\]

Fix a smooth Hermitian metric \( \langle \cdot | \cdot \rangle \) on \( CTX \) so that \( T^{1,0}X \) is orthogonal to \( T^{0,1}X \), \( \langle u | v \rangle \) is real if \( u, v \) are real tangent vectors, \( \langle R | R \rangle = 1 \) and \( R \) is orthogonal to \( T^{1,0}X \oplus T^{0,1}X \). For \( u \in CTX \), we write \( |u|^2 := \langle u | u \rangle \). Denote by \( T^{*1,0}X \) and \( T^{*0,1}X \) the dual bundles of \( T^{1,0}X \) and \( T^{0,1}X \), respectively. They can be identified with sub-bundles of the complexified cotangent bundle \( CT^*X \).

Assume that \( X \) admits an action of \( t \)-dimensional torus \( T \). In this work, we assume that the \( T \)-action preserves \( \omega_0 \) and \( J \); that is, \( t^*\omega_0 = \omega_0 \) on \( X \) and \( t_*J = Jt_* \) on \( HX \). Let \( \mathfrak{t} \) denote the Lie algebra of \( T \), we identify \( \mathfrak{t} \) with its dual \( \mathfrak{t}^* \) by means of the scalar product \( \langle \cdot, \cdot \rangle \). For any \( \xi \in \mathfrak{t} \), we write \( \xi_X \) to denote the vector field on \( X \) induced by
The moment map associated to the form \( \omega_0 \) is the map \( \mu : X \to \mathfrak{t}^* \) such that, for all \( x \in X \) and \( \xi \in \mathfrak{t} \), we have
\[
\langle \mu(x), \xi \rangle = \omega_0(\xi_X(x)).
\] (2)

Fix a maximal weight \( \nu \neq 0 \) in the lattice inside \( \mathfrak{t}^* \). Suppose that \( i\mathbb{R}_+ \cdot \nu \) is transversal to \( \mu \), so we have
\[
\mathfrak{t}^* = i\mathbb{R}_+ \cdot \nu \oplus d\mu(T_pX)
\] (3)
and \( X_\nu \) is a sub-manifold of \( X \) of codimension \( 2n + 2 - t \). We claim that the action of \( \mathbb{T}^{t-1}_\nu \) on \( X_\nu \) is locally free. In fact, by the contrary suppose that there exists \( \xi \in \ker \nu \) such that \( \xi_X(x) = 0 \) on \( T_xX_\nu, x \in X_\nu \). For each \( v \in T_xX_\nu \), we have
\[
(d_x\mu(v))(\xi) = d_x\omega_0(\xi_X, v) = 0
\]
which contradicts (3).

The action of \( \mathbb{T} \) restricts to an action of \( \mathbb{T}^{t-1} \) whose moment map is given by
\[
\mu|_{\mathbb{T}^{t-1}} = p_\nu \circ \mu, \quad \text{where} \quad p_\nu : \mathfrak{t}^* \to \mathfrak{t}^*_\nu
\]
is the canonical projection onto the \( t-1 \)-dimensional subspace \( \nu_\perp \) in \( \mathfrak{t}^* \). The transversality condition in [12] implies that \( 0 \in \mathfrak{t}^*_\nu^{-1} \) is a regular value for the moment \( \mu|_{\mathbb{T}^{t-1}} \). Thus, we have
\[
X_\nu/\mathbb{T}^{t-1}_\nu = \mu|_{\mathbb{T}^{t-1}}^{-1}(0)/\mathbb{T}^{t-1}_\nu
\]
and by assumptions [12] [13] and [HH Section 2.5] (we shall also refer to [GH2] for definitions concerning CR structures on orbifolds) we have

**Lemma 2.1.** The space of orbits \( X_\nu/\mathbb{T}^{t-1}_\nu \) is a CR orbifold. Let us denote by \( \pi : X_\nu \to X_\nu^{\text{red}} \) and \( i : X_\nu \hookrightarrow X \) the natural projection and inclusion, respectively, then there is a unique induced contact form \( \omega_0^{\text{red}} \) on \( X_\nu/\mathbb{T}^{t-1}_\nu \) such that
\[
\pi^*\omega_0^{\text{red}} = i^*\omega_0.
\]
In particular, set \( HX_\nu = TX_\nu \cap HX \), we have
\[
HX = HX_\nu \oplus J\nu_X^\perp \quad \text{and} \quad HX_\nu = i\nu_X^\perp \oplus d\pi^*HX_\nu^{\text{red}}.
\]

We will also assume that \( \mathbb{T}^{t-1}_\nu \)-action is transversal CR, that is, the infinitesimal vector field
\[
(\nu_X u)(x) = \frac{\partial}{\partial t}(u(\exp(it \nu) \circ x))|_{t=0}, \quad \text{for any} \ u \in C^\infty(X),
\]
preserves the CR structure \( T^{1,0}X \), so that \( \nu_X \) and \( T^{1,0}X \oplus T^{0,1}X \) generate the complex tangent bundle to \( X \),
\[
\mathbb{C}T_xX = \mathbb{C}\nu_X(x) \oplus \mathbb{C}T^{1,0}_xX \oplus \mathbb{C}T^{0,1}_xX \quad (x \in X).
\]

We define local coordinates that will be useful later. Recall that \( X \) admits a CR and transversal \( \mathbb{T}^{t-1}_\nu \)-action which is locally free on \( X_\nu \), \( T \in C^\infty(X, TX) \) denotes the global real vector field given by this infinitesimal circle action. We will take \( T \) to be our Reeb vector field \( R \). In a similar way as in Theorem 3.6 in [HH], there exist local coordinates \( v = (v_1, \ldots, v_{t-1}) \) of \( \mathbb{T}^{t-1}_\nu \) in a small neighborhood \( V_0 \) of the identity \( \mathbb{e} \) with \( v(\mathbb{e}) = (0, \ldots, 0) \), local coordinates \( x = (x_1, \ldots, x_{2n+1}) \) defined in a neighborhood \( U_1 \times U_2 \) of \( p \in X_\nu \), where \( U_1 \subseteq \mathbb{R}^{t-1} \) (resp. \( U_2 \subseteq \mathbb{R}^{2n+2-t} \)) is an open set of \( 0 \in \mathbb{R}^{t-1} \) (resp. \( 0 \in \mathbb{R}^{2n+2-t} \)) and
$p \equiv 0 \in \mathbb{R}^{2n+1}$, and a smooth function $\gamma = (\gamma_1, \ldots, \gamma_{t-1}) \in \mathcal{C}^\infty(U_2, U_1)$ with $\gamma(0) = 0$ such that

$$(v_1, \ldots, v_{t-1}) \circ (\gamma(x_t, \ldots, x_{2n+1}), x_t, \ldots, x_{2n+1})$$

$$= (v_1 + \gamma_1(x_t, \ldots, x_{2n+1}), \ldots, v_{t-1} + \gamma_{t-1}(x_t, \ldots, x_{2n+1}), x_t, \ldots, x_{2n+1})$$

for each $(v_1, \ldots, v_{t-1}) \in V_0$ and $(x_t, \ldots, x_{2n+1}) \in U_2$. Furthermore, we have

$$t = \text{span}\{\partial_{x_j} \mid j = 1, \ldots, t-1\}, \quad \mu^{-1}(i \mathbb{R} \nu \cdot (\nu)) \cap U = \{x_{2d-t+1} = \ldots = x_{2d} = 0\},$$

on $\mu^{-1}(i \mathbb{R} \nu \cdot (\nu)) \cap U$ there exist smooth functions $a_j$'s with $a_j(0) = 0$ for every $0 \leq j \leq t-1$ and independent on $x_1, \ldots, x_{2(t-1)}, x_{2n+1}$ such that

$$J(\partial_{x_j}) = \partial_{x_{t-1}+j} + a_j(x) \partial_{x_{2n+1}} \quad j = 1, \ldots, t-1,$$

the Levi form $L_p$, the Hermitian metric $(\cdot \cdot)$ and the 1-form $\omega_0$ can be written

$$L_p(Z_j, Z_k) = \mu_j \delta_{j,k}, \quad \langle Z_j, Z_k \rangle = \delta_{j,k} \quad (1 \leq j, k \leq n),$$

and

$$\omega_0(x) = (1 + O(|x|))dx_{2n+1} + \sum_{j=1}^{t-1} 4\mu_j x_{t-1+j} dx_j + \sum_{j=t}^{n} 2\mu_j x_{2j} d x_{2j-1}$$

$$- \sum_{j=t}^{2n} 2\mu_j x_{2j-1} dx_j + \sum_{j=r}^{2n} b_j x_{2n+1} dx_j + O(|x|^2)$$

where $b_r, \ldots, b_{2n} \in \mathbb{R}$,

$$T_p^{1,0} X = \text{span}\{Z_1, \ldots, Z_n\}$$

and

$$Z_j = \frac{1}{2} (\partial_{x_j} - i \partial_{x_{t-1}+j})(p) \quad (j = 1, \ldots, t-1),$$

$$Z_j = \frac{1}{2} (\partial_{x_{2j-1}} - i \partial_{x_{2j}})(p) \quad (j = t, \ldots, n).$$

We need to define in local coordinates we just introduced the phase function of the $T_p^{1,0}$-invariant Szegö kernel $\Phi_-(x, y) \in \mathcal{C}^\infty(U \times U)$ which is independent of $(x_1, \ldots, x_{t-1})$ and $(y_1, \ldots, y_{t-1})$. Hence, we write $\Phi_-(x, y) = \Phi_-((0, x''), (0, y'')) := \Phi_-((x'', y'')$ and $\hat{x}'' := (x_1, \ldots, x_{2n}), \hat{y}'' := (y_1, \ldots, y_{2n})$. Moreover, there is a constant $c > 0$ such that

$$\text{Im} \Phi_-(x'', y'') \geq c \left(|\hat{x}''|^2 + |\hat{y}''|^2 + |\hat{x}'' - \hat{y}''|^2\right), \quad \text{for all } ((0, x''), (0, y'')) \in U \times U. \quad (4)$$

Furthermore,

$$\Phi_-(x'', y'') = - x_{2n+1} + y_{2n+1} + 2i \sum_{j=1}^{t-1} |\mu_j| y_{t-1+j}^2 + 2i \sum_{j=1}^{t-1} |\mu_j| x_{t-1+j}^2 + i \sum_{j=t}^{n} |\mu_j| |z_j - w_j|^2$$

$$+ \sum_{j=t}^{n} i\mu_j (z_j w_j - z_j \bar{w}_j) + \sum_{j=1}^{d} (-b_{t-1+j} x_{d+j} x_{2n+1} + b_{t-1+j} y_{t-1+j} y_{2n+1})$$

$$+ \sum_{j=t}^{n} \frac{1}{2} (b_{2j-1} - i b_{2j}) (-z_j x_{2n+1} + w_j y_{2n+1}) + \sum_{j=t}^{n} \frac{1}{2} (b_{2j-1} + i b_{2j}) (\bar{w}_j x_{2n+1} + \bar{z}_j y_{2n+1})$$

$$+ (x_{2n+1} - y_{2n+1}) f(x, y) + O(|(x, y)|^3),$$

$$0 \equiv f \equiv 0 \in \mathcal{C}^\infty(U \times U)$$

and $f$ is independent on $x_1, \ldots, x_{t-1}$.
where \( z_j = x_{2j-1} + ix_{2j}, w_j = y_{2j-1} + iy_{2j}, j = t, \ldots, n, \mu_j, j = 1, \ldots, n, f \) is smooth and satisfies \( f(0, 0) = 0, f(x, y) = \overline{f(y, x)} \).

We now consider the circle action on \( X \). Let \( p \in \mu^{-1}(i\mathbb{R}_+ \cdot v) \), there exist local coordinates \( v = (v_1, \ldots, v_{q-1}) \) of \( \mathbb{T}^{q-1} \), \( q \) is even, in a small neighborhood \( V_0 \) of \( e \) with \( v(e) = (0, \ldots, 0) \), local coordinates \( x = (x_1, \ldots, x_{2n+1}) \) defined in a neighborhood \( U_1 \times U_2 \) of \( p \), where \( U_1 \subset \mathbb{R}^{q-1} \) (resp. \( U_{q-1} \subset \mathbb{R}^{2n+q-1} \)) is an open set of \( 0 \in \mathbb{R}^{q-1} \) (resp. \( 0 \in \mathbb{R}^{2n+q-1} \)) and a smooth function \( \gamma = (\gamma_1, \ldots, \gamma_{q-1}) \in C^\infty(U_2, U_1) \) with \( \gamma(0) = 0 \) such that \( T = -\frac{\partial}{\partial x_{2n+1}} \) and all the properties for the local coordinates defined before hold. The phase function \( \Psi \) satisfies \( \Psi(x, y) = -x_{2n+1} + y_{2n+1} + \hat{\Psi}(\hat{x}''', \hat{y}'') \), where \( \hat{\Psi}(\hat{x}''', \hat{y}'') \in C^\infty(U \times U) \) and \( \Psi \) satisfies \( \quad \).

### 2.2 Hardy spaces

We denote by \( L^2_{(0,q)}(X) \), \( q = 0, 1, \ldots, n \), the completion of \( \Omega^{0,q}(X) \) with respect to \( (\cdot, \cdot) \). We extend \( (\cdot, \cdot) \) to \( L^2_{(0,q)}(X) \) by

\[
\overline{\partial}_b : \text{Dom} \overline{\partial}_b \subset L^2_{(0,q)}(X) \rightarrow L^2_{(0,q+1)}(X),
\]

where \( \text{Dom} \overline{\partial}_b := \{ u \in L^2_{(0,q)}(X); \overline{\partial}_b u \in L^2_{(0,q+1)}(X) \} \) and, for any \( u \in L^2_{(0,q)}(X) \), \( \overline{\partial}_b u \) is defined in the sense of distributions. We also write

\[
\overline{\partial}_b^* : \text{Dom} \overline{\partial}_b^* \subset L^2_{(0,q+1)}(X) \rightarrow L^2_{(0,q)}(X)
\]

to denote the Hilbert space adjoint of \( \overline{\partial}_b \) in the \( L^2 \) space with respect to \( (\cdot, \cdot) \). There is a well-defined orthogonal projection

\[
S^{(q)} : L^2_{(0,q)}(X) \rightarrow \text{Ker} \square^q_b
\]

with respect to the \( L^2 \) inner product \( (\cdot, \cdot) \) and let

\[
S^{(q)}(x, y) \in \mathcal{D}'(X \times X, T^{0,q}X \boxtimes (T^{0,q}X)^*)
\]

denote the distribution kernel of \( S^{(q)} \). We will write \( H^q(X) \) for \( \text{Ker} \square^q_b \) and for functions we simply write \( H(X) \) to mean \( H^0(X) \). The distributional kernel \( S^{(q)}(x, y) \) was studied in [15], before recalling an explicit description describing the oscillatory integral defining the distribution \( S^{(q)}(x, y) \) we need to fix some notation. To begin, let us review the concept of the Hörmander symbol space. Let \( D \subset X \) be a local coordinate patch with local coordinates \( x = (x_1, \ldots, x_{2n+1}) \).

**Definition 2.1.** For every \( m \in \mathbb{R} \), we denote with

\[
S^m_{1,0}(D \times D \times \mathbb{R}_+, T^{0,q}X \boxtimes (T^{0,q}X)^*) \subset C^\infty(D \times D \times \mathbb{R}_+, T^{0,q}X \boxtimes (T^{0,q}X)^*)
\]

the space of all \( a \) such that, for all compact \( K \subset D \times D \) and all \( \alpha, \beta \in \mathbb{N}_0^{2n+1}, \gamma \in \mathbb{N}_0 \), there is a constant \( C_{\alpha,\beta,\gamma} > 0 \) such that

\[
|\partial^{\alpha}_{x'} \partial^{\beta}_{x''} \partial^{\gamma}_{t} a(x, y, t)| \leq C_{\alpha,\beta,\gamma}(1 + |t|)^{m-\gamma}, \quad \text{for every } (x, y, t) \in K \times \mathbb{R}_+, t \geq 1.
\]

For simplicity we denote with \( S^m_{1,0} \) the spaces defined in \( (\text{7}) \); furthermore, we write

\[
S^{-\infty}(D \times D \times \mathbb{R}_+, T^{0,q}X \boxtimes (T^{0,q}X)^*) := \bigcap_{m \in \mathbb{R}} S^m_{1,0}(D \times D \times \mathbb{R}_+, T^{0,q}X \boxtimes (T^{0,q}X)^*).
\]
Let $a_j \in S^m_{1,0}$, $j = 0, 1, 2, \ldots$ with $m_j \to -\infty$, as $j \to \infty$. Then there exists $a \in S^m_{1,0}$
unique modulo $S^{-\infty}$, such that

$$a - \sum_{j=0}^{k-1} a_j \in S^m_{1,0}(D \times D \times \mathbb{R}^+; T^{a,q} X \otimes (T^{a,q} X)^*)$$

for $k = 0, 1, 2, \ldots$. If $a$ and $a_j$ have the properties above, we write $a \sim \sum_{j=0}^{\infty} a_j$ in $S^m_{1,0}$.

It is known that the characteristic set of $\square^q_b$ is given by

$$\Sigma = \Sigma^- \cup \Sigma^+, \quad \Sigma^- = \{(x, \lambda \omega_0(x)) \in T^* X; \lambda < 0\},$$

and $\Sigma^+$ is defined similarly for $\lambda > 0$. We recall the following theorem (see [HS, Theorem 1.2]).

**Theorem 2.1.** Suppose that the Levi form is non-degenerate and $\square^q_b$ has $L^2$ closed range. Then, there exist continuous operators $S_-, S_+: L^2(\mathbb{R}^q_0) \to \text{Ker } \square^q_b$ such that

$$S^{(q)} = S_- + S_+, \quad S_+ \equiv 0 \text{ if } q \neq n_+$$

and

$$\text{WF}'(S_-) = \text{diag}(\Sigma^- \times \Sigma^-), \quad \text{WF}'(S_+) = \text{diag}(\Sigma^+ \times \Sigma^+) \text{ if } q = n_+ = n_+,$$

where $\text{WF}'(S_-) = \{(x, \xi, y, \eta) \in T^* X \times T^* X; (x, \xi, y, -\eta) \in \text{WF}(S_-), \text{WF}(S_-) \}$, $\text{WF}(S_-)$ is the wave front set of $S_-$ in the sense of Hörmander.

Moreover, consider any small local coordinate patch $D \subset X$ with local coordinates $x = (x_1, \ldots, x_{2n+1})$, then $S_-(x, y)$, $S_+(x, y)$ satisfy

$$S_\pm(x, y) \equiv \int_0^\infty e^{i\varphi_\pm(x, y)t} s_\pm(x, y, t) dt \text{ on } D,$$

with

$$s_\pm(x, y, t) \sim \sum_{j=0}^\infty s_\pm^j(x, y)t^{n-j} \text{ in } S^n_{1,0}(D \times D \times \mathbb{R}^+; T^{a,q} X \otimes (T^{a,q} X)^*),$$

$s_\pm(x, y, t) = 0$ if $q \neq n_+$ and $s_\pm^0(x, y) \neq 0$ for all $x \in D$. The phase functions $\varphi_-$, $\varphi_+$ satisfy

$$\varphi_-, \varphi_+ \in C^\infty(D \times D), \quad \text{Im } \varphi_\pm(x, y) \geq 0, \quad \varphi_-(x, x) = 0, \quad \varphi_-(x, y) \neq 0 \text{ if } x \neq y,$$

and

$$d_x \varphi_-(x, y) |_{x=y} = -\omega_0(x), \quad d_y \varphi_-(x, y) |_{x=y} = \omega_0(x), \quad -\varphi_+(x, y) = \varphi_-(x, y).$$

**Remark 2.1.** Kohn [Kohn] proved that if $q = n_- = n_+$ or $|n_- - n_+| > 1$ then $\square^q_b$ has $L^2$
closed range. For a description of the phase function in local coordinates see chapter 8 of part I in [HS].

Now we focus on the decomposition induced by the group action on $H^q(X)$. Fix $t \in \mathbb{T}$ and let $t^*: \Lambda^J_x(\mathbb{C}^* X) \to \Lambda^J_{t^{-1} ox}(\mathbb{C}^* X)$ be the pull-back map. Since $\mathbb{T}$ preserves $J$, we have $t^*: T^{a,q}_x X \to T^{a,q}_{t^{-1} ox} X$, for all $x \in X$. Thus, for $u \in \Omega^{0,q}(X)$, we have $t^* u \in \Omega^{0,q}(X)$. Put

$$\Omega^{0,q}(X)_{kr} := \{u \in \Omega^{0,q}(X); (e^{i\theta})^* u = e^{ik(\nu, \theta)} u, \forall \theta \in \mathbb{R}^+\}. $$
Since the Hermitian metric \( \langle \cdot | \cdot \rangle \) on \( \mathcal{C}T^* X \) is \( T \)-invariant, the \( L^2 \) inner product \( \langle \cdot | \cdot \rangle \) on \( \Omega^{0,q}(X) \) induced by \( \langle \cdot | \cdot \rangle \) is \( T \)-invariant. Let \( u \in L^2_{(0,q)}(X) \) and \( t \in T \), we can also define \( t^* u \) in the standard way. We introduce the following notation

\[
L^2_{(0,q)}(X)_{kJ} := \{ u \in L^2_{(0,q)}(X); (e^{i\theta})^* u = e^{i(k(\nu,\theta))} u, \ \forall \theta \in \mathbb{R}^r \},
\]

and put

\[
(\text{Ker } \Box^q)_kJ := \text{Ker } \Box^q_b \cap L^2_{(0,q)}(X)_{kJ}.
\]

The equivariant Szegö projection is the orthogonal projection

\[
S^q_{kJ} : L^2_{(0,q)}(X) \to (\text{Ker } \Box^q)_kJ
\]

with respect to \( \langle \cdot | \cdot \rangle \). Let \( S^q_{k\nu}(x, y) \in \mathcal{D}'(X \times X, T^{*,0,q} X \boxtimes (T^{*,0,q} X)^*) \) be the distribution kernel of \( S^q_{kJ} \). The asymptotic expansion for the distributional kernel of the projector \( S^q_{k\nu} \) was studied in [H] when \( X \) is the quantizing circle bundle of a given Hodge manifold in Heisenberg local coordinates. Using similar ideas as in [HH] one can generalize the results for \( (0,q) \)-forms in the following theorem, we give a sketch of the proof, which is similar to the proof of [HH] Theorem 1.8], here the group \( G \) in [HH] is \( \mathbb{T}^{-1} \) and the circle action in [HH] is given by the action of \( \mathbb{T}^1 \).

Since the action is locally free we need to recall some notations from [GHI]. Since the action is locally free \( \mu^{-1}(0)/\mathbb{T}^1 \) is an orbifold, let us denote with \( \pi : \mu^{-1}(0) \rightarrow \mu^{-1}(0)/\mathbb{T}^1 \) the projection. Furthermore the action of \( \mathbb{T}^1 \) commutes with the one of \( \mathbb{T}^{-1} \), then we have a smooth locally free action of \( \mathbb{T}^{-1} \) on \( \mu^{-1}(0)/\mathbb{T}^1 \). Given \( y \in X \) and \( g \) in the stabilizer \( \mu^{-1}(0)/\mathbb{T}^1 \), there exist \(|(\mathbb{T}^1)^{-1}_y|\) elements \( e^{i\theta_{\nu,j}} \in \mathbb{T}^1 \) \((j = 1, \ldots, |(\mathbb{T}^1)^{-1}_y|)\) such that

\[
g \circ y = e^{-i\theta_{\nu,j}} \circ y.
\]

Thus, all the elements \( (e^{i\theta_{\nu,j}}, g) \in \mathbb{T}^1 \times \mathbb{T}^{-1} = \mathbb{T} \) satisfying

\[
e^{i\theta} \cdot g \circ y = y,
\]

are of the form \( (e^{i\theta_{\nu,j}}, g) \) for each \( g \in (\mathbb{T}^{-1})_{\pi(y)} \).

**Theorem 2.2.** Suppose that \( \Box^q_b \) has \( L^2 \) closed range. Then, there exist continuous operators \( S^q_{kJ}, S^+_kJ : L^2_{(0,q)}(X) \to (\text{Ker } \Box^q)_kJ \) such that

\[
S^q_{kJ} = S^-_{kJ} + S^+_{kJ}, \quad S^+_{kJ} \equiv 0 \quad \text{if} \ q \neq n+
\]

Suppose for simplicity that \( q \neq n_+ \). If \( q \neq n_- \), then \( S^q_{kJ} \equiv O(k^{-\infty}) \) on \( X \).

Suppose \( q = n_- \) and let \( D \) be an open set in \( X \) such that the intersection \( \mu^{-1}(i \mathbb{R}_+ \cdot \nu) \cap D = \emptyset \). Then \( S^q_{kJ} \equiv O(k^{-\infty}) \) on \( D \).

Let \( p \in \mu^{-1}(i \mathbb{R}_+ \cdot \nu) \) and let \( U \) a local neighborhood of \( p \) with local coordinates \((x_1, \ldots, x_{2n+1})\). Then, if \( q = n_- \), for every \( y \in U \), we consider \( S^q_{kJ}(x, y) \) as a \( k \)-dependent smooth function in \( x \), then

\[
S^q_{kJ}(x, y) = \sum_{h \in G(x,y)} \sum_{j=1}^{|S^j|} e^{i\theta_{h,j}} e^{i\|\nu\| \Psi(x,y)} b(x, y, k\|\nu\|) + O(k^{-\infty}).
\]

for every \( x \in U_y \), where \( U_y \) is a small open neighborhood of \( y \). The phase function \( \Psi(x, y) \) is defined in local coordinates in the end of Section [2.7], the symbol satisfies

\[
b(x, y, k\|\nu\|) \in S^{n+(1-t)/2}_{loc}(1, U \times U, T^{*,(0,q)} X \boxtimes (T^{*,(0,q)})^*)
\]

and the leading term of \( b \) is non-zero.
Proof. Suppose $q = n_-$, on small local neighborhood $D$ of a point $p \in X_\nu$ we have

$$S_{k\nu}^{(q)}(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} e^{ik \theta} S(q)(x, e^{i\theta} y) \, d\theta$$

where $\theta = (\theta_1, \ldots, \theta_t)$. It is easy to prove that the oscillatory integral has a rapidly decreasing asymptotic as $k \to +\infty$ far away from a local neighborhood of those elements $(e^{i\theta}, g) \in T^*_\nu \times T^{t-1}_\nu = \mathbb{T}$ such that

$$e^{i\theta} \circ y = y.$$

We shall consider the case of a local neighborhood of the identity in $\mathbb{T}$, we set $\theta_1$ the variable for circle action of $T^*_\nu$ and $(\theta_1, \ldots, \theta_t)$ the variables for $T^{t-1}_\nu$. We can then use local coordinates defined in [2.1] we have

$$S_{k\nu}^{(q)}(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} S(q)(x, e^{i\theta} y) \, d\theta_1 \cdots d\theta_t \tag{8}$$

where

$$S_{k\nu}^{(q)}(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} S^{(q)}(x, e^{i\theta} y) \, d\theta_1 \cdots d\theta_t \tag{9}$$

and $\hat{x} = (x_1, \ldots, x_n, 0)$ and $\hat{y} = (y_1, \ldots, y_n, 0)$. The action of $\mathbb{T}$ restricts to an action of $T^{t-1}_\nu$ whose moment map is given by

$$\mu_{T^{t-1}_\nu} = p_\nu \circ \mu, \quad \text{where} \ p_\nu : t^* \to t^*$$

is the canonical projection onto the $t - 1$-dimensional subspace $\nu^\perp$ in $t^*$. Since we are assuming $X_\nu \neq \emptyset$ we have that $0 \in t^*_{\nu} \neq 0$ does lie in the image of the moment map $\mu_{T^{t-1}_\nu}$. The proof follows in a similar way as in Theorem 1.8 in [H] where here we have the action of $G = T^{t-1}_\nu$ with moment map $\mu_{T^{t-1}_\nu}$ and the $k\nu$ Fourier components are the ones induced by the transversal $T^*_\nu$-action; notice that [H] Assumption 1.7] is satisfied. We shall also refer to [G] Theorem 5.4] for the locally free action case.

For ease of notation we introduce a definition to say that an operator behaves micro-locally as equivariant Szegő projector we just studied. For simplicity we assume $q = n_-$.

**Definition 2.2** (Equivariant Szegő type operator). Suppose that $q = n_-$ and consider $H : \Omega^{0,q}(X) \to \Omega^{0,q}(X)$ be a continuous operator with distribution kernel

$$H(x, y) \in \mathcal{D}'(X \times X, T^{q,0,q}X \otimes (T^{q,0,q}X)^*) .$$

We say that $H$ is a complex Fourier integral operator of equivariant Szegő type of order $n + (1 - t)/2 \in \mathbb{Z}$ if $H$ is smoothing away $\mu^{-1}(i \mathbb{R}_+ \cdot \nu)$ and for given $p \in \mu^{-1}(i \mathbb{R}_+ \cdot \nu)$ let $D$ a local neighborhood of $p$ with local coordinates $(x_1, \ldots, x_{2n+1})$. Then, the distributional kernel of $H$ satisfies

$$H_k(x, y) = e^{ik \Psi(x, y)} a(x, y, k) + O(k^{-\infty}) \quad \text{on} \ D$$

where $a \in S_{1,0}^{n + (1 - t)/2}(D \times D \times \mathbb{R}_+, T^{q,0,q}X \otimes (T^{q,0,q}X)^*)$, and $\Psi$ is as in the end of Section [2.1].
For $k = 0$, $H^{(q)}(X)_0$ is the space of $\mathbb{T}$-fixed vectors, by Theorem 1.5 in [1111] we have \( \dim H^{(q)}(X)_0 < +\infty \) since when $0 \notin \mu(X)$ the projector $S^{(q)}_{\nu}$ is smoothing. Now, given $k_1, k_2 \in \mathbb{Z}$ with $k_1, k_2 \neq 0$, consider $f_1 \in H^{(q)}(X)_{k_1 \nu}$ and $f_2 \in H^{(q)}(X)_{k_2 \nu}$, we have $f_1 \cdot f_2 \in H^{(q)}(X)_{(k_1 + k_2) \nu}$; we can study $\dim H^{(q)}(X)_{k \nu}$ for $k$ large, we have

$$\dim H^{(q)}(X)_{k \nu} = \int_X S^{(q)}_{k \nu}(x, x) \, dV_X(x).$$

These dimensions can be studied as $k \to +\infty$ by using the microlocal properties of the Szegő kernel and Stationary Phase Lemma in a similar way as in Corollary 1.3 in [11] we have $\dim H^{(q)}(X)_{k \nu} = O(k^{d+1-t})$. So, we have the following generalization of Theorem A.3 in [11], where it is proved for spaces of functions $H^{(q)}(X)_{k \nu}$.

**Lemma 2.2.** If $0 \notin \mu(X)$, then $H^{(q)}(X)_{k \nu}$ are finite dimensional.

### 3 Proof of Theorem 1.1

In this section we shall explain how to prove asymptotic commutativity for quantization and reduction for spaces of $(0, q)$-forms. We recall that $X^\nu_{\text{red}}$ is a CR orbifold whose Levi form is non-degenerate, with $n_- - r$ negative eigenvalues; let us denote with $S^{(q)}_{\text{red}}(\nu)$ the corresponding Szegő kernel for $(0, q)$-forms whose $k$-th Fourier components are the one induced by the $T^\nu_{\nu}$-action on $X^\nu_{\text{red}}$. We shall recall briefly its microlocal expression by [11][12]. Let, now, us denote by $e^{i\theta}$ the transversal and CR locally free $T^\nu_{\nu}$-action and we take the Reeb vector field $R$ to be the vector field on $X$ induced by it.

Let $q = n_- - r$, and consider an open set $\hat{U} \subset X$, $p \in \hat{U}$, and an orbifold chart $(\hat{U}, G_U) \to U$, we denote by $\tilde{x}$ the coordinates on $\hat{U}$. For every $\ell \in \mathbb{N}$, put

$$X_\ell := \{x \in X; e^{i\theta} x \neq x, \theta \in [0, 2\pi/\ell], e^{i\theta/\ell} x = x\}.$$ 

With the assumptions and notations used above, assume that $p \in X_\ell$, for some $\ell \in \mathbb{N}$. We have as $k \to +\infty$,

$$S^{(q)}_{\text{red}, k}(x, y) = \sum_{j=0}^{\ell-1} \sum_{g \in G_U} e^{\frac{2\pi i}{\ell} k} e^{i k \Psi(\tilde{x}, e^{i\theta/\ell} \tilde{y})} b(\tilde{x}, e^{i\theta/\ell} \cdot g \cdot \tilde{y}, k) + O(k^{-\infty})$$

where the phase function

$$\Psi \in C^\infty(\hat{U} \times \hat{U}), \quad \Psi(\tilde{x}, \tilde{x}) = 0, \quad \text{for all } \tilde{x} \in \hat{U},$$

$$\inf_{e^{i\theta} \in S^1} \left\{ \text{dist}^2(\tilde{x}, e^{i\theta} \tilde{y}) \right\} / C \leq \Im \Psi(\tilde{x}, \tilde{y}) \leq C \inf_{e^{i\theta} \in S^1} \left\{ \text{dist}^2(\tilde{x}, e^{i\theta} \tilde{y}) \right\}$$

for each $(\tilde{x}, \tilde{y}) \in \hat{U} \times \hat{U}$, $C > 1$ is a constant; and the symbol satisfies

$$b(\tilde{x}, \tilde{y}, k) \sim \sum_{j=0}^{+\infty} b_j(\tilde{x}, \tilde{y}) k^{n-j} \text{ in } S^{n-(t-1)}(1; \hat{U} \times \hat{U}, T^{0,q}X \boxtimes (T^{0,q}X)^*)$$

and $b_0(\tilde{x}, \tilde{y})$ is non zero.

Let us recall briefly why we have a local chart $(\hat{U}, G_U) \to U$, for ease of notation let us put $G = T^\nu_{\nu}$. We recall that, for every $x \in X_k$, by the slice Theorem a neighborhood of any orbit $T^\nu_{\nu} \cdot x = x_0$ is equivariantly diffeomorphic to a neighborhood of the zero section of the associated principal bundle

$$G \times_{G_x} N_x,$$
where \( N_x \) is the normal space to \( G : x \) in \( X_\nu \) and \( G_x \) is the stabilizer of \( x \) for the action of \( G \), which is finite. Therefore, for some \( \epsilon > 0 \) and for an open ball \( B_{2\epsilon}(x) \subseteq N_x \), one has a homeomorphism \( B_{2\epsilon}(x)/G_x \cong \mathcal{U} \) onto some neighborhood of \( x_0 \) in \( X_\nu^{red} \).

Since \( \nu^\perp_x := \text{val}(\nu^\perp) \) is orthogonal to

\[
H_x X_\nu \cap J_x H_x X_\nu \quad \text{and} \quad H_x X_\nu \cap J_x H_x X_\nu \subset (\nu^\perp_x)^{-1} h
\]

for every \( x \in X_\nu \), we can find a \( \mathbb{T} \)-invariant orthonormal basis \( \{ Z_1, \ldots, Z_n \} \) of \( T^*_{1,0} X \) on \( X_\nu \) such that for each \( j, k = 1, \ldots, n \),

\[
L_x(Z_j(x), \overline{Z}_j(x)) = \delta_{jk} \lambda_j(x),
\]

where

\[
Z_j(x) \in (\nu^\perp_{X,x} + iJ \nu^\perp_{X,x}) \quad \text{for each} \quad j = 1, \ldots, t - 1
\]

and

\[
Z_j(x) \in H_x X_\nu \cap J_x (H_x X_\nu) \quad \text{for each} \quad j = r, \ldots, n .
\]

Let \( \{ Z_1^*, \ldots, Z_n^* \} \) denote the orthonormal basis of \( T^{*0,1} X \) on \( X_\nu \), dual to \( Z_1, \ldots, Z_n \). Fix \( s = 0, 1, 2, \ldots, n - r + 1 \). For \( x \in X_\nu \), put

\[
B_x^{s,0} X = \left\{ \sum_{r \leq j_1 < \cdots < j_s \leq n} a_{j_1, \ldots, j_s} Z_{j_1}^*, \ldots, Z_{j_s}^* ; a_{j_1, \ldots, j_s} \in \mathbb{C} \right\}
\]

and let \( B_x^{s,0} X \) be the vector bundle of \( X_\nu \) with fibre \( B_x^{s,0} \subseteq X_\nu \). Let \( C^\infty(X_\nu, B_x^{s,0} X)^\mathbb{T} \) denote the set of all \( \mathbb{T} \)-invariant sections of \( X_\nu \) with values in \( B_x^{s,0} X \); let

\[
t_{red} : C^\infty(X_\nu, B_x^{s,0} X)^\mathbb{T} \to \Omega^{0,s}(X_\nu^{red})
\]

be the natural identification. Before defining the map between \( H^q(X)_k \) and \( H^{q-r}(X_\nu^{red})_k \) we need one more piece of notation. We can assume that \( \lambda_1 < 0, \ldots, \lambda_r < 0 \) and also \( \lambda_1, \ldots, \lambda_{n-r+t-1} < 0 \). For \( x \in X_\nu \), set

\[
\mathcal{N}(x, n_n) = \{ c Z_t^* \wedge \cdots \wedge Z_{n-r+t-1}^* ; c \in \mathbb{C} \}
\]

and let \( \tilde{\phi} : \mathcal{N}(x, n_n) \to \mathcal{N}(x, n_n) \) to be

\[
\tilde{\phi}(c Z_t^* \wedge \cdots \wedge Z_r^* \wedge Z_t^* \wedge \cdots \wedge Z_{n-r+t-1}^*) = c Z_t^* \wedge \cdots \wedge Z_{n-r+t-1}^*
\]

where \( c \) is a complex number. Put \( t_\nu : X_\nu \to X \) be the natural inclusion and let \( t_\nu^* : \Omega^{0,q}(X) \to \Omega^{0,q}(X_\nu) \) be the pull-back of \( t_\nu \).

Let \( q = n_n \); now, inspired by [11], we define the map

\[
\sigma_{k\nu} : H^q(X)_k \to H^{q-r}(X_\nu^{red})_k
\]

given by

\[
\sigma_{k\nu}(u) := (k\nu)^{(r-1)/4} S^{(q-r)}_{k\nu} \circ t_{red} \circ t_{\nu}^* \circ \mathcal{E} \circ t_{\nu}^* \circ S^{(q)}_{k\nu}(u),
\]

where, \( c(x) \) is a \( \mathbb{T} \)-invariant smooth function on \( X_\nu \) that can be found explicitly. Notice that operators \( S^{(q-r)}_{red,k} \) and \( S^{(q)}_{k\nu} \) appearing in [11] are known explicitly. \( S^{(q-r)}_{red,k} \) is the \( k \)-th Fourier component of the standard Szegő kernel for the CR manifold \( X_\nu^{red} \), on the other hand \( S^{(q)}_{k\nu} \) is described in Section 2.2.

Before stating the next theorem we shall specialize the local coordinates defined in Section 2.1. Let us consider \( p \in X_\nu \) and let \( x = (x_1, \ldots, x_{2n+1}) \) be the local coordinates in an open neighborhood \( U \) of \( p \) defined in Section 2.1. We may assume that \( U = \)
$U_1 \times U_2 \times U_3 \times U_4$, where $U_1 \subset \mathbb{R}^{l-1}$, $U_2 \subset \mathbb{R}^{l-1}$ are open sets of $0 \in \mathbb{R}^{l-1}$, $U_3 \subset \mathbb{R}^{2n-2(l-1)}$ is an open set of $0 \in \mathbb{R}^{2n-2(l-1)}$ and $U_4$ is an open set of $0 \in \mathbb{R}$. From now on, we can identify $U_2$ with

$$\{(0, \ldots, 0, x_t, \ldots, x_{2(l-1)}, 0, \ldots, 0) \in U : (x_t, \ldots, x_{2(l-1)}) \in U_2\}$$

and $U_3$ with

$$\{(0, \ldots, 0, x_{2l-1}, \ldots, x_{2n}, 0) \in U : (x_t, \ldots, x_{2n}) \in U_3\}.$$

For a given orbifold chart $(\tilde{U}, G_U) \to U$ we write $\tilde{U}_i$ for the corresponding open sets in $\tilde{U}$. Eventually we recall that $\tilde{x}'' := (x_{2l-1}, \ldots, x_{2n+1})$.

**Theorem 3.1.** Under the assumptions above, if $x \not\in X_\nu$, then for every sufficiently small open set $D$ of $x$ with $\overline{D} \cap X_\nu = \emptyset$, we have $\sigma_{k\nu} = O(k^{-\infty})$ on $X_\nu^{red} \times D$.

Let $\pi : X_\nu \to X_\nu^{red}$ the projection. If $x, y \in X_\nu$ and $\pi(x) \neq \pi(e^{i\theta} \cdot y)$ for every $e^{i\theta} \in \mathbb{T}_\nu$, then there exist open set $U$ of $\pi(x)$ in $X_\nu^{red}$ and $V$ of $y$ in $X$ such that $\sigma_{k\nu} = O(k^{-\infty})$ on $U \times V$.

Eventually, let $p \in X_\nu \cap X_t$, using the local coordinates defined above, we have

$$\sigma_{k\nu}^{\ell}(\tilde{x}''', \tilde{y}''') = \sum_{j=0}^{n-4} e^{2\pi i r_j} S^{(q-r)}_{\nu, j}(\tilde{x}, \tilde{y}) \alpha(\tilde{x}, e^{2\pi i} \cdot g \cdot \tilde{y}, k) + O(k^{-\infty}) \quad \text{on } (\tilde{U}_3 \times \tilde{U}_4) \times \tilde{U}$$

where

$$\alpha(\tilde{x}'', \tilde{y}'', k) \in S^{-\frac{4}{3}(l-1)}(1; (\tilde{U}_3 \times \tilde{U}_4) \times \tilde{U}, T^{\nu}(0, q-r)X_\nu^{red} \boxtimes (T^{\nu}(0, q)X)^*)$$

and the leading term $\alpha_0$ in the expansion of $\alpha$ can be computed explicitly along the diagonal.

**Proof.** Since, by Theorem 2.2, $S_{k\nu}^{(q)}$ has rapidly decreasing asymptotics as $k$ goes to infinity away $X_\nu$, we obtain that $\sigma_k = O(k^{-\infty})$ on $X_\nu^{red} \times D$.

Now, let us prove the second statement. If $x, y \in X_\nu$ and $\pi(x) \not= \pi(e^{i\theta} \cdot y)$ for every $e^{i\theta} \in \mathbb{T}_\nu$, since $S_{k\nu, j}^{(q-r)}$ is smoothing away from diagonal, then there exist open set $U$ of $\pi(x)$ in $X_\nu^{red}$ and $V$ of $y$ in $X$ such that $\chi S_{k\nu, j}^{(q-r)} \eta = O(k^{-\infty})$ on $X_\nu^{red}$, where $\chi \in C_0(\mathbb{T} \cdot \tilde{U}_3)$ and $\eta \in C_0(\mathbb{T} \cdot V)$. Furthermore by (3), we see that, for sufficiently small neighborhoods of $x$ and $y$, we can integrate by parts in $d\theta_j$ and see that $S_{k\nu}^{(q)}$ has rapidly decreasing asymptotic. The second statements follows.

Eventually, let $p \in X_\nu$ and consider a small open neighborhood $U$ with coordinates defined as above. Let $\chi \in C_0(\mathbb{T} \cdot \tilde{U}_3)$ be a $G_U$ invariant bump function and assume $\chi = 1$ on some neighborhood of $p$, it extends naturally to a function on $\mathbb{T} \cdot \tilde{U}_3$. Let us put $U^\sharp = \{\pi(x) : x \in U\}$. Let $\eta \in C_0(X_\nu^{red})$ such that $\eta = 1$ on some neighborhood of $U^\sharp$ and

$$\text{supp}(\eta) \subset \{\pi(x) \in X_\nu^{red} : x \in X_\nu, \chi(x) = 1\}.$$

Thus, we get

$$\eta \cdot \sigma_{k\nu} \sim (k\nu)^{(l-1)/4} \eta S_{k\nu, j}^{(q-r)} \circ \iota_{red} \circ \hat{\circ} \circ \tau_{x,n} \circ e \circ \iota_{\nu} \circ \chi S_{k\nu}^{(q)}.$$ 

Now, we can compose the operators and we can use the complex stationary phase formula of [MS], we get the theorem.
In fact, $\sigma_{k\nu}$ is an isomorphism for $k$ large if we can prove that
\[
\sigma_{k\nu} : H^q(X) \to H^{q-r}(X^r\nu) \quad \text{and} \quad \sigma_{k\nu}^* : H^{q-r}(X^r\nu) \to H^q(X)
\]
are injective for $k$ large. Notice that, by Theorem 3.1, we have
\[
\sigma_{k\nu}^*(x'', y'') = \sum_{j=0}^{l-1} \sum_{g \in G_U} e^{\frac{2\pi i}{r} \cdot g \cdot y''} \beta(x'', e^{\frac{2\pi i}{r} \cdot g \cdot y''}, k) + O(k^{-\infty})
\]
on $\tilde{U} \times (\tilde{U}_3 \times \tilde{U}_4)$ where we can check $\beta_0(\tilde{x}'', \tilde{y}'') = \alpha_0(\tilde{x}'', \tilde{y}'')$.

The injectivity of the map $\sigma_{k\nu}$ is a consequence of the following theorem which is an adaptation of proof of the main theorem in [HH].

**Theorem 3.2.** There exists a Fourier integral operator $R_k$ of equivariant Szegő type of degree $n - (t - 1)/2$ such that
\[
\sigma_{k\nu}^* \sigma_{k\nu} \equiv c_0 (1 + R_k) S_{k\nu}^{(q)}
\]
where $c_0$ is a positive constant and $1 + R_k : \Omega^q(X) \to \Omega^q(X)$ is an injective Fourier integral operator of equivariant Szegő type.

**Proof.** By Theorem 3.1 and equation (12) we can compose $\sigma_{k\nu}^*$ and $\sigma_{k\nu}$ using the complex stationary phase formula of [MS]. We can pick $e$ in the definition of $\sigma_{k\nu}$ so that the leading term of the symbol of $\sigma_{k\nu}^* \sigma_{k\nu}$ agrees with the one of $S_{k\nu}^{(q)}$ along the diagonal. Thus we can find an operator
\[
R_k(x, y) = \sum_{j=0}^{l-1} \sum_{g \in G_U} e^{\frac{2\pi i}{r} \cdot g \cdot y''} r(x'', e^{\frac{2\pi i}{r} \cdot g \cdot y''}, k) + O(k^{-\infty})
\]
on $\tilde{U} \times \tilde{U}$ such that
\[
r \in S^{n-(t-1)/2}(1; \tilde{U} \times \tilde{U}, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)
\]
and
\[
|r_0(x, y)| \leq C |(x, y) - (x_0, x_0)|
\]
for all $x_0 \in X^r \nu \cap \tilde{U}$, where $C > 0$ is a constant.

The injectivity of the operator $1 + R_k$ follows from direct computations of the leading symbol of $R_k$ which vanishes at diag($X^r \nu \times X^r \nu$). In fact, as a consequence of this, by Lemma 6.7 and Lemma 6.8 in [HH] we have that $\|R_k u\| \leq \epsilon_k \|u\|$ for all $u \in \Omega^0,q(X)$, for all $k \in \mathbb{N}$ where $\epsilon_k$ is a sequence with $\lim_{k \to +\infty} \epsilon_k = 0$; this in turn implies, if $k$ is large enough, that the map $1 + R_k$ is injective.

Analogously, the injectivity of the map $\sigma_{k\nu}^*$ follows by studying $\sigma_{k\nu} \sigma_{k\nu}^*$. We don’t repeat the proof here since it is an application of the stationary phase formula for Fourier integral operators of complex type, see [MS], and it follows in a similar way as above.

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