Affine su(2) fusion rules from gerbe 2-isomorphisms

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Abstract

We give a geometric description of the fusion rules of the affine Lie algebra \( \hat{\mathfrak{su}}(2)_k \) at a positive integer level \( k \) in terms of the \( k \)-th power of the basic gerbe over the Lie group SU(2). The gerbe can be trivialised over conjugacy classes corresponding to dominant weights of \( \hat{\mathfrak{su}}(2)_k \) via a 1-isomorphism. The fusion-rule coefficients are related to the existence of a 2-isomorphism between pullbacks of these 1-isomorphisms to a submanifold of \( SU(2) \times SU(2) \) determined by the corresponding three conjugacy classes. This construction is motivated by its application in the description of junctions of maximally symmetric defect lines in the Wess–Zumino–Witten model.

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1 Introduction

Consider the affine Lie algebra \( \hat{\mathfrak{su}}(2)_k \) at a positive integer level \( k \). The integrable highest-weight representations of \( \hat{\mathfrak{su}}(2)_k \) are labelled by elements of the set of dominant affine weights which we identify with the subset \( P^+_k = \{0, 1, \ldots, k\} \) of the integers. The fusion ring of these representations is given by

\[
[\lambda] \ast [\mu] = \sum_{\nu = |\lambda - \mu|}^{\min(\lambda + \mu, 2k - \lambda - \mu)} [\nu] \quad \text{for } \lambda, \mu \in P^+_k,
\]

where the superscript \((+2)\) means that the sum is carried out in steps of two. We can collect the fusion rules in a discrete subset of \([0, k]^3\),

\[
V = \{ (\lambda, \mu, \nu) \in (P^+_k)^3 \mid [\nu] \text{ appears in } [\lambda] \ast [\mu] \}.
\]

We should, in principle, introduce a conjugation in this definition, but for \( \hat{\mathfrak{su}}(2)_k \) this does not make a difference.

The fusion rules appear in numerous places. The most interesting one for us in the present context is the application in two-dimensional conformal quantum field theory. There, they can be used to compute the dimension of the spaces of conformal blocks in the quantum Wess–Zumino–Witten (WZW) model at level \( k \), cf., e.g., [Be], and they are famously related to the modular properties of affine characters by the Verlinde formula [Ve].

The application we have in mind is the study of the WZW model on surfaces with defect lines and defect junctions. Without going into any detail, we merely mention that the elementary maximally symmetric defects in the quantum WZW model are labelled by weights in \( P^+_k \). Defect lines can meet at points on the world-sheet which we call defect junctions. A nonzero defect junction (of conformal dimension zero) of three defect lines exists if and only if the fusion rule for the three weights labelling the defect lines is non-zero [Fr].

Defect lines and defect junctions also have a description in classical \( \sigma \)-models [FSW, RS1]. The analysis of maximally symmetric defect lines and junctions in the WZW model motivates the geometric construction of the present paper. The relation to the WZW model will be elaborated in [RS2].
We shall identify the fundamental affine Weyl alcove of SU(2) at level \( k \) with the interval \([0, k]\) on the real axis, and assign to \( \lambda \in [0, k] \) the conjugacy class
\[
\mathcal{C}_\lambda = \{ g \cdot \left( \begin{array}{cc} e^{\pi i \lambda / k} & 0 \\ 0 & e^{-\pi i \lambda / k} \end{array} \right) \cdot g^{-1} \mid g \in \text{SU}(2) \}.
\] (1.3)

We introduce the factor \( k \) at this stage in order to reduce the number of its appearances later on. Define three maps \( p_1, p_2, m : \text{SU}(2) \times \text{SU}(2) \to \text{SU}(2) \) via
\[
p_1(g, h) = g, \quad p_2(g, h) = h, \quad m(g, h) = g \cdot h.
\] (1.4)

We shall be interested in the following submanifolds of \( \text{SU}(2) \times \text{SU}(2) \):
\[
\mathcal{T}_{\lambda, \mu}^\nu = p_1^{-1}(\mathcal{C}_\lambda) \cap p_2^{-1}(\mathcal{C}_\mu) \cap m^{-1}(\mathcal{C}_\nu), \quad \text{where} \quad \lambda, \mu, \nu \in [0, k].
\] (1.5)

It is easy to see that \( \mathcal{T}_{\lambda, \mu}^\nu \neq \emptyset \) if and only if \( (\mathcal{C}_\lambda \cdot \mathcal{C}_\mu) \cap \mathcal{C}_\nu \neq \emptyset \). This defines the fusion polytope of \( \text{SU}(2) \),
\[
\mathcal{F} = \{ (\lambda, \mu, \nu) \in [0, k]^3 \mid \mathcal{T}_{\lambda, \mu}^\nu \neq \emptyset \},
\] (1.6)

see Fig. 1. One finds by inspection (see Section 4 below) that if \( [\nu] \) appears in the fusion product \( [\lambda] * [\mu] \), for \( \lambda, \mu \in \mathcal{P}_k^+ \), then also \( (\lambda, \mu, \nu) \in \mathcal{F} \), i.e. \( V \subset \mathcal{F} \). In fact, this relation between intersections of conjugacy classes and affine fusion rules holds more generally [Ha, TW], but we shall only consider \( \text{SU}(2) \) here.

One can now ask about the converse, namely if one can determine \( V \) starting from \( \mathcal{F} \). The simplest idea would be to just intersect \( \mathcal{F} \) with the set of dominant affine weights \( (\mathcal{P}_k^+)^3 \),
but this does not respect the parity-conservation rule by which the sum (1.1) has to be carried out in steps of two. Obtaining this rule from geometric considerations is the main aim of this paper. To this end, we shall need to introduce some additional structures on $SU(2)$, to wit, a certain gerbe $G$ and related 1- and 2-isomorphisms. The construction is summarised below, and then detailed in Sections 2–5.

The Lie group $SU(2)$ is equipped with the family of Cartan 3-forms

$$H = \frac{r}{12\pi} \, \text{tr}(\theta_L \wedge \theta_L \wedge \theta_L), \quad r \in \mathbb{R},$$

(1.7)
defined in terms of the standard left-invariant Maurer–Cartan 1-forms $\theta_L(g) = g^{-1} dg$ on $SU(2)$. There is a gerbe on $SU(2)$ (or on any compact simple connected and simply connected Lie group, for that matter) with curvature $H$ if and only if $r \in \mathbb{Z}$, and this gerbe is unique up to a 1-isomorphism $\{Ga1, Me\}$. We shall set $r = k$ and denote the corresponding gerbe $G$. We review the definition of bundle gerbes and the detailed construction of $G$ in Section 2 below.

Gerbes have an important application in physics: they describe the topological term in the $\sigma$-model action functional which is necessary to preserve the conformal symmetry of the WZW model upon quantisation $\{Wi1, Al, Ga1, FGK\}$.

If we restrict the 3-form $H$ to a conjugacy class $C_\lambda \subset SU(2)$ for $\lambda \in [0, k]$, it becomes exact, $H_{|C_\lambda} = d \omega_\lambda$ for $\omega_\lambda \in \Omega^2(C_\lambda)$. One can then enquire whether also the gerbe $G$ can be trivialised when restricted to $C_\lambda$, i.e. whether there is a 1-isomorphism

$$\Phi_\lambda : G_{|C_\lambda} \rightarrow \mathcal{I}(\omega_\lambda)$$

(1.8)
between $G_{|C_\lambda}$ and the trivial gerbe with curving $\omega_\lambda$ (consult Sections 2 and 3 for definitions). This turns out to be possible if and only if $\lambda \in P^+_k$, and $\Phi_\lambda$ is unique up to a 2-isomorphism in this case $\{GR1, Ga2\}$ (again, a similar statement holds for other compact simple connected and simply connected Lie groups). We review the construction of $\Phi_\lambda$ in Section 3.

The trivialisation of the gerbe over $C_\lambda$ finds its application in the description of the WZW model on surfaces with a non-empty boundary. There, it describes a maximally symmetric boundary condition for the $\sigma$-model fields $\{GR1\}$, and, more relevant to our present concerns, it also describes the WZW model on surfaces with defect lines $\{FSW, RS1\}$, where it defines a maximally symmetric defect gluing condition $\{Wa2, RS2\}$.

Each of the three maps $p_1, p_2$ and $m$ defined in (1.4) can be used to pull back the gerbe $G$ from $SU(2)$ to $SU(2) \times SU(2)$. There is a unique 1-isomorphism $\mathcal{M} : p_1^* G \ast p_2^* G \rightarrow m^* G \ast I(\rho)$, where $\ast$ is a product of gerbes (described in Section 2) and $\rho$ is a 2-form defined globally on $SU(2) \times SU(2)$. The 1-isomorphism $\mathcal{M}$ (together with a preferred 2-isomorphism subject to certain conditions) is called a multiplicative structure on $G$ $\{Ca, Wa2\}$. It enters the final step of our construction: Given $\lambda, \mu, \nu \in P^+_k$, we use the maps $p_1, p_2$ and $m$ to pull back the 1-isomorphisms $\Phi_\lambda, \Phi_\mu$ and $\Phi_\nu$ from their respective conjugacy classes to $T_{\lambda, \mu}$. We then ask if there exists a 2-isomorphism

$$\varphi_{\lambda, \mu}^\nu : p_1^* \Phi_\lambda \ast p_2^* \Phi_\mu \rightarrow (m^* \Phi_\nu \ast \text{id}_{I(\rho)}) \circ \mathcal{M}$$

(1.9)
between the two 1-isomorphisms trivialising \( p_1^*G \ast p_2^*G \) over \( \mathcal{T}_{\lambda,\mu}^\nu \), as detailed in Section 3. The existence of the 2-isomorphism is obstructed in general, and we find by inspection that the obstruction vanishes if and only if \([\nu]\) appears in the fusion product \([\lambda] \ast [\mu]\). Thus, if we define

\[
V_G = \{ (\lambda, \mu, \nu) \in (P_k)^3 \mid \varphi_{\lambda,\mu}^{\nu} \text{ exists} \},
\]

(1.10)

we can state our main result as

**Theorem 1.1.** \( V = V_G \), where \( V \) is given in (1.2) and \( V_G \) in (1.10).

The definition (1.9) and – in particular – the appearance of the multiplicative structure may seem ad hoc, but they will turn out to be very natural from the point of view of the WZW model in the presence of defect lines. In this context, the 2-isomorphism (1.9) is used to define a junction of maximally symmetric defect lines [RS1], as we shall describe in detail in [RS2].

At the moment, we have not much to say about how Theorem 1.1 generalises to other Lie groups, in particular if or how fusion-rule multiplicities could appear in the context of gerbes. Also, Theorem 1.1 is proved by explicitly computing \( V_G \) and comparing to the known answer for \( V \), and it would be desirable to find a direct relation between affine fusion rules and the 2-isomorphisms discussed here. We hope to return to these points in the future.

This paper is organised as follows: In Sections 2 and 3, we review the construction of the gerbe \( G \) and of the 1-isomorphisms trivialising \( G \) upon restriction to conjugacy classes, following [GR1]. Our first new result is the explicit construction of the 1-isomorphism \( M \) restricted to \( \mathcal{T}_{\lambda,\mu}^\nu \) in Section 4; previously, only the existence of \( M \) had been proved. In Section 5, we analyse when \( \varphi_{\lambda,\mu}^{\nu} \) exists and prove Theorem 1.1.

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## 2 The basic gerbe over SU(2)

Abelian gerbes [G] over a given manifold \( M \) provide a geometric realisation of elements of the integral cohomology group \( H^3(M, \mathbb{Z}) \). They admit a number of descriptions, such as the local description [Ga1, Br] in terms of Deligne hypercohomology, and the geometric one [Mu, MS] formulated in terms of complex line bundles over a surjective submersion over \( M \). In the context of two-dimensional non-linear \( \sigma \)-models, the former description was the first one to be used as it is more intuitive and can be abstracted from the construction of the action functional [Ga1]. Nonetheless, the geometric description is more convenient for our explicit calculations, and we shall use it throughout the main text.

Let us begin our lightning review of the necessary elements of the geometric description with the concept of an abelian bundle gerbe with curving, connection and fixed curvature, as
represented concisely by the following diagram

\[(L, \nabla_L, \mu_L)\]

\[
\begin{array}{c}
Y^{[2]} M \\
\downarrow \pi_{Y M}
\end{array} \xrightarrow{\text{pr}_1} \begin{array}{c}
YM, B \end{array} \xrightarrow{\pi_{YM}} \begin{array}{c}(M, H)\end{array}
\]

(2.1)

Here, \(Y^{[2]} M\) is the fibred product

\[Y^{[2]} M \equiv YM \times_M YM = \{ (y_1, y_2) \in YM \times YM \mid \pi_{YM}(y_1) = \pi_{YM}(y_2) \}, \]

(2.2)

with its obvious generalisations \(Y^{[n]} M\) for \(n > 2\). We shall also need the various canonical projections

\[
\text{pr}_{i_1, \ldots, i_n} : Y^{[m]} M \to Y^{[n]} M : (y_1, \ldots, y_m) \mapsto (y_{i_1}, \ldots, y_{i_n}).
\]

(2.3)

For example, \(\text{pr}_i : Y^{[2]} M \to YM\) maps \((y_1, y_2)\) to \(y_i\). The notation does not keep track of the index \(m\) or of the manifold \(M\), and so we shall mention the source and the target explicitly if it is not clear from the context.

**Definition 2.1.** [Mu] A hermitean abelian bundle gerbe with curving and connection (or gerbe for short) of curvature \(H \in \Omega^3(M)\) over a smooth base \(M\) is a quadruple \(G = (YM, B, L, \mu_L)\) with the following entries:

- a surjective submersion \(YM \xrightarrow{\pi_{YM}} M\), together with a global 2-form \(B \in \Omega^2(YM)\) on \(YM\), termed the curving of the gerbe, satisfying the relation
  \[
  \pi_{YM}^* H = dB;
  \]

(2.4)

- a hermitean line bundle \(L \xrightarrow{\pi_L} Y^{[2]} M\) with unitary connection \(\nabla_L\) of curvature
  \[
  \text{curv}(\nabla_L) = \text{pr}_2^* B - \text{pr}_1^* B,
  \]

(2.5)

determined by the pullbacks \(\text{pr}_i^* B\) of the curving along \(\text{pr}_i : Y^{[2]} M \to YM\);

- a groupoid structure \(\mu_L\) on \(L \xrightarrow{\pi_L} Y^{[2]} M\), i.e. a unitary connection-preserving bundle isomorphism
  \[
  \mu_L : \text{pr}_{1,2}^* L \otimes \text{pr}_{2,3}^* L \xrightarrow{\sim} \text{pr}_{1,3}^* L
  \]

(2.6)

\(^1\)A differentiable map \(f : X \to Y\) between a pair of smooth manifolds \(X, Y\) is called a surjective submersion if both \(f\) and its tangent map \(f_* : TX \to TY\) are surjective, and so \(f\) admits smooth local sections.
between pullbacks of $L$ to $Y^{[3]}M$ along $\text{pr}_{i,j} : Y^{[3]}M \to Y^{[2]}M$, which is associative in the sense specified by the commutativity of the diagram

\[
\begin{array}{c}
pr_{1,2}^*L \otimes pr_{2,3}^*L \otimes pr_{3,4}^*L \\
\downarrow \downarrow \downarrow id_{pr_{1,2}^*L \otimes pr_{2,3,4}^*L} \\
pr_{1,3}^*L \otimes pr_{3,4}^*L \\
\downarrow \downarrow pr_{1,3,4}^*L \\
pr_{1,4}^*L
\end{array} \xrightarrow{pr_{1,2}^*L \otimes pr_{2,4}^*L} \pr_{1,2}^*L \otimes pr_{2,4}^*L
\]

(2.7)

of bundle isomorphisms over $Y^{[4]}M$, the latter coming with the canonical projections $pr_{i,j} : Y^{[4]}M \to Y^{[2]}M$ and $pr_{i,j,k} : Y^{[4]}M \to Y^{[3]}M$.

Amidst all gerbes over a given base $M$, there is a distinguished class of trivial gerbes, characterised by the existence of a globally defined curving $\omega \in \Omega^2(M)$. They are represented by quadruples

\[
\mathcal{I}(\omega) = (M, \omega, 1_M, \mu_{1_M}),
\]

(2.8)

with the surjective submersion $id_M : M \to M$, the trivial bundle $1_M = M \times \mathbb{C} \to M$ with the trivial connection $\nabla_{1_M} = d$, and the canonical groupoid structure

\[
\mu_{1_M} : 1_M \otimes 1_M \to 1_M : (x, z) \otimes (x, z') \mapsto (x, z \cdot z').
\]

(2.9)

The following two natural operations defined on gerbes will be useful for our purposes.

**Pullback:** Given a smooth map $f : N \to M$ from a manifold $N$ to the base $M$ of the gerbe $G$, and a surjective submersion $\pi : YN \to N$ together with a map $\tilde{f} : YN \to YM$ that covers $f$ in the sense that it renders the diagram

\[
\begin{array}{c}
YN \\
\pi \downarrow \downarrow \downarrow \tilde{f} \\
N \\
\downarrow \downarrow \downarrow f \\
M
\end{array} \xrightarrow{\pi \downarrow \downarrow \downarrow \pi} \begin{array}{c}
YM \\
\pi \downarrow \downarrow \downarrow \pi \\
N \\
\downarrow \downarrow \downarrow f \\
M
\end{array}
\]

(2.10)

commutative (and so it also induces maps $\tilde{f}^{[n]} : Y^{[n]}N \to Y^{[n]}M$), we define the pullback of $G$ to $N$ along $f$ as the gerbe

\[
f^*G = (YN, \tilde{f}^*B, \tilde{f}^{[2]*}L, \tilde{f}^{[3]*}\mu_L).
\]

(2.11)

**Product:** Consider a pair of gerbes $G_i = (Y_iM, B_i, L_i, \mu_{L_i}), i \in \{1, 2\}$ over a common base $M$. We define their product as the gerbe

\[
G_1 \ast G_2 = (Y_{1,2}M, pr_{1,2}^*B_1 + pr_{2,1}^*B_2, pr_{1,3}^*L_1 \otimes pr_{2,4}^*L_2, pr_{1,3,5}^*\mu_{L_1} \otimes pr_{2,4,6}^*\mu_{L_2})
\]

(2.12)
with the surjective submersion
\[ \pi_{Y,M} \circ \text{pr}_i : Y_{1,2}M = Y_1M \times_M Y_2M \rightarrow M \] (2.13)
expressed in terms of either of the canonical projections
\[ \text{pr}_i : Y_{1,2}M \rightarrow Y_iM : (y_1, y_2) \mapsto y_i \] (2.14)
and their obvious generalisations to higher fibred products \( Y_{1,2}^\llbracket n \rrbracket M \).

The basic gerbe \( \mathcal{G}_1 \) over the group manifold of SU(2) is the unique (up to a 1-isomorphism, see Section 3) gerbe with the curvature given by the Cartan 3-form
\[ \text{curv}(\mathcal{G}_1) = \frac{1}{12\pi} \text{tr}(\theta_L \wedge \theta_L \wedge \theta_L), \quad \theta_L(g) = g^{-1} dg, \quad g \in \text{SU}(2). \] (2.15)
The class of the rescaled 3-form \( \frac{1}{2\pi} \text{curv}(\mathcal{G}_1) \) is a generator of the integral cohomology group \( H^3(\text{SU}(2), \mathbb{Z}) \cong \mathbb{Z} \) embedded in \( \tilde{H}^3(\text{SU}(2), \mathbb{R}) \).

Let us fix a positive integer \( k \) for the rest of the paper. We shall be concerned with a gerbe \( \mathcal{G} \) of \( k \) times the curvature of \( \mathcal{G}_1 \), which is otherwise constructed in the same manner as \( \mathcal{G}_1 \). We shall refer to \( \mathcal{G} \) as the canonical gerbe. An explicit construction of the canonical gerbe over SU(2) in terms of its local data (i.e. in the language of the Deligne hypercohomology) was originally given in [Ga1]. In the remainder of this section, we explain in some detail the geometric construction along the lines of [GR1], providing the level of detail that we need for Sections 3–5.

We shall first set up our conventions and assemble the necessary algebraic objects. The standard Pauli matrices are
\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \] (2.16)
In the complexified Lie algebra \( \text{su}(2)^\mathbb{C} \), we choose the Cartan subalgebra \( t = \mathbb{C} \sigma_3 \) and fix the ad-invariant bilinear (Killing) form given by the trace as
\[ K(X,Y) = \text{tr}(X \cdot Y), \quad X,Y \in \text{su}(2)^\mathbb{C}. \] (2.17)
We subsequently use the latter to identify the Lie algebra with its dual. Under this identification, the weight lattice of the algebra becomes \( \Lambda = \frac{1}{2} \mathbb{Z} \sigma_3 \), with the single fundamental weight \( \Lambda = \frac{1}{2} \sigma_3 \), and the root lattice takes the form \( \tau = \mathbb{Z} \sigma_3 \), with the single simple root \( \alpha = \sigma_3 \).

The fundamental affine Weyl alcove at level \( k \) is the 1-simplex
\[ \mathcal{A}_W^k = \{ \lambda \Lambda \mid \lambda \in [0,k] \}, \] (2.18)
whence also its identification with the closed segment \([0,k]\) mentioned in the introduction and used throughout the paper. Upon exponentiation
\[ e^{2\pi i \lambda \Lambda/k} = \begin{pmatrix} e^{\pi i \lambda/k} & 0 \\ 0 & e^{-\pi i \lambda/k} \end{pmatrix} =: t_\lambda, \] (2.19)
it produces an interval in the Cartan subgroup \( T = U(1) \) which intersects each Ad-orbit at exactly one point.

8
Parameterisations of SU(2)

In what follows, we shall make use of two particularly convenient parameterisations of the group manifold. The first of them is a redundant parameterisation given by the surjective map

$$ c : \tilde{\text{SU}}(2) = [0, k] \times \text{SU}(2) \to \text{SU}(2) : (\lambda, h) \mapsto \text{Ad}_h(t_\lambda) . $$

(2.20)

Since $c(\lambda, h) = c(\lambda', h')$ implies $\lambda = \lambda'$, we can consider the isotropy group for each value of $\lambda$, separately,

$$ \text{SU}(2)_{t_\lambda} = \{ g \in \text{SU}(2) \mid \text{Ad}_g(t_\lambda) = t_\lambda \} . $$

(2.21)

It acts on $\text{SU}(2)$ by right regular translations

$$ \text{SU}(2) \times \text{SU}(2)_{t_\lambda} \to \text{SU}(2) : (h, t) \mapsto h \cdot t , $$

(2.22)

so that we have $c(\lambda, h \cdot t) = c(\lambda, h)$. Many of the expressions below will simplify once pulled back to $\tilde{\text{SU}}(2)$. However, to ensure that functions, forms and bundles on $\tilde{\text{SU}}(2)$ arise as pullbacks from $\text{SU}(2)$, one has to impose appropriate equivariance conditions on these objects over each of the submanifolds $\{\lambda\} \times \text{SU}(2)$ with respect to the isotropy groups $\text{SU}(2)_{t_\lambda}$. We shall denote objects defined on $\tilde{\text{SU}}(2)$ with a tilde $\tilde{\cdot}$.

The other useful parameterisation is in terms of the Euler angles $\varphi \in [0, \pi]$, $\theta \in [0, \pi/2]$, $\psi \in [0, 2\pi]$. Here, an arbitrary group element $g \in \text{SU}(2)$ is expressed as

$$ g(\varphi, \theta, \psi) = e^{i\varphi \sigma_3} \cdot e^{i\theta \sigma_2} \cdot e^{i\psi \sigma_3} . $$

(2.23)

Surjective submersion and curvings

The construction of the canonical gerbe $\mathcal{G}$ begins with the choice of a cover of the group manifold. We take a pair of contractible open subsets ($e$ is the group unit)

$$ \mathcal{O}_0 \cong \text{SU}(2) \setminus \{ -e \} , \quad \mathcal{O}_1 \cong \text{SU}(2) \setminus \{ e \} $$

(2.24)

within $\text{SU}(2)$. The surjective submersion is defined as the disjoint union of the two elements of the open cover,

$$ \text{YSU}(2) = \mathcal{O}_0 \sqcup \mathcal{O}_1 \overset{\pi_{\text{YSU}(2)}}{\longrightarrow} \text{SU}(2) . $$

(2.25)

This choice is related to the shape of the fundamental affine Weyl alcove via the pullback to $\tilde{\text{SU}}(2)$. Namely, $\mathcal{O}_0$ and $\mathcal{O}_1$ are associated with the respective vertices

$$ \lambda_0 = 0 , \quad \lambda_1 = k $$

(2.26)

of $[0, k] \cong \mathcal{A}_W^k$ as per

$$ \tilde{\mathcal{O}_i} := c^{-1}(\mathcal{O}_i) = ([0, k] \setminus \{ \lambda_{1-i} \}) \times \text{SU}(2) . $$

(2.27)
Next, we need to find a global primitive for the curvature

\[ H = \text{curv}(G) = \frac{k}{12\pi} \text{tr}(\theta_L \wedge \theta_L \wedge \theta_L) \quad (2.28) \]

of \( G \) when pulled back to \( YSU(2) \), i.e. we need 2-forms \( B_i \) on \( \mathcal{O}_i \) such that \( dB_i = H|_{\mathcal{O}_i} \). We shall construct these by giving a primitive of the pullback of \( H \) by \( c \) and then checking the necessary equivariance conditions.

Using the Maurer–Cartan equation \( d\theta_L + \theta_L \wedge \theta_L = 0 \), the pullback \( c^* H \) is readily verified to trivialise globally on \( \tilde{\SU}(2) \) as

\[ c^* H(\lambda, h) = d(\tilde{Q}(\lambda, h) + \tilde{F}(\lambda - \lambda_c, h)), \quad (2.29) \]

with

\[ \tilde{Q}(\lambda, h) = \frac{k}{4\pi} \text{tr}(\theta_L(h) \wedge \text{Ad}_{\lambda}\theta_L(h)), \quad \tilde{F}(\lambda, h) = -i \lambda \text{tr}(\Lambda d\theta_L(h)), \quad (2.30) \]

and \( \lambda_c \) an arbitrary constant. For a form \( \eta \) on \( \tilde{\SU}(2) \) to be the pullback of a form on \( SU(2) \), it needs to be basic with respect to the action of the local isotropy groups \( SU(2)_{t\lambda} \). This means that it has to be horizontal and invariant for each \( \lambda \in [0, k] \), that is, for all vector fields on \( \{\lambda\} \times SU(2) \) of the form

\[ X(h) = X^A L_A(h), \quad X^A i \sigma_A \in \text{LieSU}(2)_{t\lambda}, \quad (2.31) \]

it has to satisfy

\[ X \lrcorner \eta = 0 \quad \text{and} \quad \mathcal{L}_X \eta = 0. \quad (2.32) \]

Here, \( L_A \) are the standard left-invariant vector fields on \( SU(2) \) dual to the Maurer–Cartan 1-forms,

\[ L_A \lrcorner \theta_L = i \sigma_A, \quad (2.33) \]

and \( \mathcal{L}_X \) is the Lie derivative along \( X \). The vector fields in (2.31) generate the action (2.22) of the isotropy groups \( SU(2)_{t\lambda} \). For \( \lambda \in ]0, k[ \), the Lie algebra \( \text{LieSU}(2)_{t\lambda} \) is spanned by \( i \sigma_3 \) and we find that

\[ L_3(h) \lrcorner \tilde{Q}(\lambda, h) = \frac{k}{4\pi} \text{tr}(i \sigma_3 \text{Ad}_{\lambda}\theta_L(h)) - \frac{k}{4\pi} \text{tr}(\theta_L(h) \text{Ad}_{\lambda}i \sigma_3) = 0, \quad (2.34) \]

and, similarly, \( L_3(h) \lrcorner \tilde{F}(\lambda, h) = 0 \). Evaluating the Lie derivatives as \( \mathcal{L}_X \eta = X \lrcorner d\eta + d(X \lrcorner \eta) \), we find

\[ \mathcal{L}_{L_3}(\tilde{Q} + \tilde{F})(\lambda, h) = L_3(h) \lrcorner c^* H(\lambda, h) = 0, \quad (2.35) \]

and hence, altogether, \( \tilde{Q}(\lambda, h) + \tilde{F}(\lambda, h) \) is basic on \( ]0, k[ \times SU(2) \).

Furthermore, \( d\tilde{F}(\lambda, h) = -i d\lambda \wedge \text{tr}(\Lambda d\theta_L(h)) \) gives \( \mathcal{L}_{L_3} \tilde{F}(\lambda, h) = 0 \). This implies that also \( \mathcal{L}_{L_3} \tilde{Q}(\lambda, h) = 0 \). Since \( \tilde{Q}(\lambda_i, h) = 0 \) for \( i \in \{0, 1\} \), it is automatically basic for these two values of \( \lambda \), and thus comes from a global 2-form on \( SU(2) \), which we shall denote by \( Q \). On the
other hand, \( \tilde{F}(\lambda_i - \lambda_c, h) \) is horizontal and invariant with respect to \( SU(2)_{\lambda_i} = SU(2) \) only for \( \lambda_c = \lambda_i \) (in which case \( \tilde{F}(\lambda_i - \lambda_c, h) = 0 \)). This prompts us to define

\[
\tilde{B}_i(\lambda, h) = \tilde{Q}(\lambda, h) + \tilde{F}(\lambda - \lambda_i, h),
\]

such that \( \tilde{B}_i(\lambda, h) \) is basic on \( \tilde{O}_i \) and satisfies \( \tilde{B}_i(\lambda_i, h) = 0 \). The \( \tilde{B}_i \) are hence pullbacks to

\[
Y\tilde{SU}(2) = \tilde{O}_0 \sqcup \tilde{O}_1
\]

of the sought-after curvings \( B_i \) on \( O_i \) along the map

\[
\hat{c} : Y\tilde{SU}(2) \to YSU(2) : (i, \lambda, h) \mapsto (i, Ad_h(t_\lambda))
\]

that covers \( c \).

**Kirillov–Kostant–Souriau bundles**

The line bundle over \( Y^{[2]} SU(2) \) will be constructed in terms of the Kirillov–Kostant–Souriau (KKS) bundles over co-adjoint orbits of \( SU(2) \). Denote by

\[
SU(2)_\lambda = \{ g_\lambda \in SU(2) \mid Ad_{g_\lambda}(\lambda \Lambda) = \lambda \Lambda \}
\]

the isotropy group of the point \( \lambda \Lambda \) with respect to the co-adjoint action of \( SU(2) \). The isotropy group acts on \( SU(2) \) through right regular translations,

\[
SU(2) \times SU(2)_\lambda \to SU(2) : (h, t) \mapsto h \cdot t.
\]

The corresponding coadjoint orbit \( SU(2)/SU(2)_\lambda \) is endowed with a canonical symplectic structure, and that structure defines a canonical line bundle \( K_\lambda \to SU(2)/SU(2)_\lambda \), the KKS bundle \([Ko, So, Ki]\), which we proceed to describe. We start from the trivial line bundle

\[
pr_1 : \overline{K}_\lambda = SU(2) \times \mathbb{C} \to SU(2)
\]

with connection

\[
\nabla = d + \lambda \text{tr}(\Lambda d\theta_L)
\]

of curvature

\[
\text{curv}(\nabla) = i\lambda \text{tr}(\Lambda d\theta_L).
\]

For \( \lambda \in \mathbb{Z} \), we can lift the action (2.40) to the bundle \( \overline{K}_\lambda \) as

\[
\overline{K}_\lambda \times SU(2)_\lambda \to \overline{K}_\lambda : (h, z, t) \mapsto (h \cdot t, \chi_\lambda(t) \cdot z)
\]

by means of the characters

\[
\chi_\lambda : SU(2)_\lambda \to \mathbb{C}^\times : e^{i\sigma_3} \mapsto e^{-i\lambda \varphi} \quad \text{for} \ \lambda \neq 0.
\]
For $\lambda = 0$, we have $\text{SU}(2)_\lambda = \text{SU}(2)$ and it is convenient to set $\chi_0(g) = 1$ for all $g \in \text{SU}(2)$.

For $\lambda \neq 0$, this lift of the action to the bundle is fixed uniquely by the demand that the connection form (cf. [Br, Def. 2.2.4] or [Wo, App. A.3]),

$$\widehat{A}(h, z) = \frac{i \varepsilon}{z} + i \lambda \text{tr} (\Lambda \theta_L(h)),$$

induced by $\nabla$ on the complement of the zero section in the total space $K_\lambda$ be annihilated by the vector field

$$\widehat{L}_3(h, z) = \frac{d}{dz} \varepsilon|_{\varepsilon=0} \chi_\lambda(e^{i \varepsilon \sigma_3}) \cdot z \frac{\partial}{\partial z}$$

(2.47)

on $\text{SU}(2) \times \mathbb{C}^\times$ that generates the lifted action. It is also straightforward to check that $\widehat{A}$ is invariant since $\text{SU}(2)_\lambda$ is just the maximal torus. In the case $\lambda = 0$, horizontality and invariance with respect to the action of $\text{SU}(2)_\lambda$ hold trivially.

Horizontality and invariance of $\nabla$ with respect to the action of $\text{SU}(2)_\lambda$ on $K_\lambda$ show that we can pass to the quotient line bundle $K_\lambda = K_\lambda/\text{SU}(2)_\lambda$ with base $\text{SU}(2)/\text{SU}(2)_\lambda$. For $\lambda = 0$, we obtain the trivial line bundle over a point.

**Line bundle with connection over $Y^{[2]}SU(2)$**

For the surjective submersion $Y\text{SU}(2)$ of (2.25), the fibred product is given by

$$Y^{[2]}SU(2) = \bigsqcup_{i,j \in \{0, 1\}} O_{i,j}, \quad O_{i,j} = O_i \cap O_j.$$  

(2.48)

We need to specify a hermitean line bundle

$$\pi_L : L \to Y^{[2]}SU(2)$$

equipped with a unitary connection $\nabla_L$ of curvature

$$\text{curv}(\nabla_L)|_{O_{i,j}} = (B_j - B_i)|_{O_{i,j}}.$$  

(2.50)

We shall give this bundle in terms of an equivariant line bundle $\tilde{L}_{i,j}$ over the base $\tilde{O}_{i,j} = \tilde{O}_i \cap \tilde{O}_j$. The unitary connection $\nabla_{\tilde{L}_{i,j}}$ on $\tilde{L}_{i,j}$ must have the curvature

$$\text{curv}(\nabla_{\tilde{L}_{i,j}})(\lambda, h) = (\tilde{B}_j - \tilde{B}_i)(\lambda, h) = \tilde{F}(\lambda_i - \lambda_j, h) = i \lambda_{i,j} \text{tr}(\Lambda d\theta_L(h)),$$

where we have abbreviated $\lambda_{i,j} = \lambda_j - \lambda_i$. A quick glance at (2.43) shows that the KKS bundle (or rather the equivariant bundle $\overline{K}_{\lambda_{i,j}}$ on $\text{SU}(2)$) has the desired curvature. We therefore define

$$\tilde{L}_{i,j} = \pi_{\text{SU}(2)}^* \overline{K}_{\lambda_{i,j}}, \quad \text{where} \quad \pi_{\text{SU}(2)} : \tilde{O}_{i,j} \to \text{SU}(2) : (\lambda, h) \mapsto h.$$  

(2.52)

The result is simply the trivial bundle $\tilde{L}_{i,j} = \tilde{O}_{i,j} \times \mathbb{C} \to \tilde{O}_{i,j}$ with connection $\nabla_{\tilde{L}_{i,j}} = d + \lambda_{i,j} \text{tr}(\Lambda \theta_L)$. For $i \neq j$, it inherits equivariance with respect to the action of $\text{SU}(2)_{\lambda_{i,j}} = \mathbb{U}(1)$ from $\overline{K}_{\lambda_{i,j}}$ and thus yields a well-defined quotient bundle over $O_{i,j}$. It should be stressed
that there are two distinct isotropy groups entering the above construction, to wit, \( SU(2)_{t,\lambda} \) whose action has to be divided out (over each point \((\lambda, h) \in \tilde{O}_{i,j}\)) when passing from \( \tilde{O}_{i,j} \) to \( O_{i,j} \), and \( SU(2)_{\lambda, i,j} \) whose action on \( \tilde{L}_{i,j} \) is inherited from the KKS bundle. It is the relation \( SU(2)_{t,\lambda} \subset SU(2)_{\lambda, i,j} \), valid for all \((\lambda, h) \in \tilde{O}_{i,j}\), that enables us to descend the \( SU(2)_{\lambda, i,j} \)-equivariant bundle \( \tilde{L}_{i,j} \) to the \( SU(2)_{t,\lambda} \)-quotient \( O_{i,j} \) of \( \tilde{O}_{i,j} \). For \( i = j \), both the connection and the action of the isotropy groups on the fibre are trivial, and so \( \tilde{L}_{i,i} \) induces the trivial bundle with trivial connection over \( \tilde{O}_{i,i} \).

This completes the definition of the line bundle \( L \to Y^{[2]}SU(2) \) in terms of appropriate equivariant bundles \( \tilde{L}_{i,j} \to \tilde{O}_{i,j} \).

**Groupoid structure**

The construction of \( G \) is completed by specifying the groupoid structure on the fibres of \( L \) pulled back to

\[
Y^{[3]}SU(2) = \bigsqcup_{i,j,k \in \{0,1\}} O_{i,j,k} , \quad O_{i,j,k} = O_i \cap O_j \cap O_k . \tag{2.53}
\]

The map \( \mu : \text{pr}_{1,2}^* L \otimes \text{pr}_{2,3}^* L \to \text{pr}_{1,3}^* L \) will be described in terms of an equivariant map \( \bar{\mu} \) on \( \tilde{O}_{i,j,k} = \tilde{O}_i \cap \tilde{O}_j \cap \tilde{O}_k \). Namely, we set

\[
\bar{\mu}|_{\tilde{O}_{i,j,k}} : \tilde{L}_{i,j} \otimes \tilde{L}_{j,k} |_{\tilde{O}_{i,j,k}} \to \tilde{L}_{i,k} |_{\tilde{O}_{i,j,k}} , \quad \bar{\mu}(i, j, k, \lambda, h, z \otimes z') = (i, j, k, \lambda, h, z \cdot z') . \tag{2.54}
\]

This map is clearly unitary and it is easy to see that it is compatible with the connections. Equivariance with respect to the action of the maximal torus, common to all isotropy groups, amounts to the statement that for all \( t \in U(1) \),

\[
\bar{\mu}(i, j, k, \lambda, h \cdot t, \chi_{\lambda, i,j}(t) \cdot z \otimes \chi_{\lambda, j,k}(t) \cdot z') = (i, j, k, \lambda, h \cdot t, \chi_{\lambda, i,k}(t) \cdot z \cdot z') . \tag{2.55}
\]

This holds true by virtue of the equality

\[
\chi_{\lambda, i,j}(t) \cdot \chi_{\lambda, j,k}(t) = \chi_{\lambda, i,k}(t) , \tag{2.56}
\]

readily verified by direct inspection. Once again, we are using the equivariant structures with respect to the action the isotropy groups \( SU(2)_{\lambda, i,j} \), \( SU(2)_{\lambda, j,k} \) and \( SU(2)_{\lambda, i,k} \), defining the three bundles involved, to divide out the action of the isotropy group \( SU(2)_{t,\lambda} \) acting on their base. In so doing, we exploit the relation \( SU(2)_{t,\lambda} \subset SU(2)_{\lambda, i,j} \cap SU(2)_{\lambda, j,k} \cap SU(2)_{\lambda, i,k} \), valid for all \((\lambda, h) \in \tilde{O}_{i,j,k} \). For the non-generic isotropy groups over \( \lambda = 0 \) (in which case \( i = j = k = 0 \)) and \( \lambda = k \) (in which case \( i = j = k = 1 \)), the argument is exactly the same, provided that we allow \( t \in SU(2) \) and recall that we defined \( \chi_0(t) = 1 \) in this case.

Finally, it is clear that \( \mu \) satisfies the associativity condition.

### 3 Trivialisation over conjugacy classes

The next piece of the general theory that we shall need in the subsequent discussion is the definitions of a 1-isomorphism and of a stable isomorphism,
Definition 3.1. [MS, Wa1] Let \( G_i = (Y_iM, B_i, L_i, \mu_{L_i}) \), \( i \in \{1, 2\} \) be a pair of gerbes of the same curvature \( H \) over a common base \( M \).

(i) A 1-isomorphism \( \Phi_{1,2} \) between \( G_1 \) and \( G_2 \), denoted as

\[
\Phi_{1,2} : G_1 \sim \sim G_2,
\]

is a triple \( \Phi_{1,2} = (YY_{1,2}M, E_{1,2}, \alpha_{1,2}) \) which consists of

- a surjective submersion \( YY_{1,2}M \xrightarrow{\pi Y_{1,2}M} Y_{1,2}M = Y_1M \times_M Y_2M \) over the fibred product

\[
Y_1M \times_M Y_2M = \{ (y_1, y_2) \in Y_1M \times Y_2M \mid \pi_{Y_1M}(y_1) = \pi_{Y_2M}(y_2) \};
\]

- a hermitian line bundle \( E_{1,2} \xrightarrow{\pi_{E_{1,2}}} YY_{1,2}M \) with connection \( \nabla_{E_{1,2}} \) of curvature

\[
\text{curv}(\nabla_{E_{1,2}}) = \pi^*_1B_2 - \pi^*_1B_1
\]

fixed by the pullbacks of the two curvings along the maps \( \pi_i = \text{pr}_i \circ \pi_{YY_{1,2}M} \) written in terms of the canonical projections \( \text{pr}_i : Y_1M \times_M Y_2M \to Y_iM : (y_1, y_2) \mapsto y_i \);

- a unitary connection-preserving isomorphism

\[
\alpha_{1,2} : \pi^*_1L_1 \otimes p^*_2E_{1,2} \sim \sim p^*_1E_{1,2} \otimes \pi^*_2L_2,
\]

of line bundles over \( Y_{[2]}Y_{1,2}M = YY_{1,2}M \times_M YY_{1,2}M \), defined in terms of the obvious maps \( \pi_{i,i+2} = \text{pr}_{i,i+2} \circ (\pi_{YY_{1,2}M} \times_M \pi_{YY_{1,2}M}) : YY_{1,2}M \times_M YY_{1,2}M \to Y_{i,[2]}M \) and \( p_i : YY_{1,2}M \times_M YY_{1,2}M \to YY_{1,2}M \); the isomorphism must be compatible with the two groupoid structures \( \mu_{L_i} \) in the sense specified by the commutativity of the diagram

\[
\begin{array}{ccc}
\pi^*_1L_1 \otimes \pi^*_2L_2 \otimes p^*_2E_{1,2} & \sim \sim & \pi^*_1L_1 \otimes p^*_1E_{1,2} \otimes \pi^*_2L_2 \\
\pi^*_1L_1 \otimes p^*_2E_{1,2} \otimes \pi^*_2L_2 & \sim \sim & \pi^*_1L_1 \otimes \pi^*_2L_2 \otimes p^*_1E_{1,2} \\
\end{array}
\]

\[
\begin{array}{ccc}
\pi^*_1L_1 \otimes \pi^*_2L_2 \otimes p^*_2E_{1,2} & \sim \sim & \pi^*_1L_1 \otimes \pi^*_2L_2 \otimes p^*_1E_{1,2} \\
\pi^*_1L_1 \otimes p^*_2E_{1,2} \otimes \pi^*_2L_2 & \sim \sim & \pi^*_1L_1 \otimes \pi^*_2L_2 \otimes p^*_1E_{1,2} \\
\end{array}
\]

of bundle isomorphisms over \( Y_{[3]}Y_{1,2}M = YY_{1,2}M \times_M YY_{1,2}M \times_M YY_{1,2}M \), the latter being endowed with the canonical projections \( p_i : Y_{[3]}Y_{1,2}M \to YY_{1,2}M \) and \( p_{i,j} : Y_{[3]}Y_{1,2}M \to Y_{[2]}Y_{1,2}M \), as well as with the maps \( \pi_{i,j} \) and \( \pi_{i,j,k,l} \) given by the corresponding canonical projections \( \text{pr}_{i,j} \) and \( \text{pr}_{i,j,k,l} \) precomposed with \( (\pi_{YY_{1,2}M})^3 \).
(ii) A stable isomorphism is a 1-isomorphism whose surjective submersion is given by \( YY_{1,2}M = Y_{1,2}M \) with \( \pi_{YY_{1,2}M} = \text{id}_{Y_{1,2}M} \).

The more general surjective submersion allowed in the definition of a 1-isomorphism is necessary when formulating composition below. However, the notions of 1-isomorphism classes and stable-isomorphism classes of gerbes are the same,

**Proposition 3.2.** [Wa1] Let \( \mathcal{G}_i, i \in \{1, 2\} \) be two gerbes. There exists a stable isomorphism between \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) if and only if there exists a 1-isomorphism between these gerbes.

Trivialisations compose a distinguished class of stable isomorphisms. Given a gerbe \( \mathcal{G} \) over base \( M \), they take the form

\[
\Phi : \mathcal{G} \xrightarrow{\sim} \mathcal{I}(\omega),
\]

for some \( \omega \in \Omega^2(M) \). These are special examples of a larger family of bundle-gerbe modules [Bo] whose definition generalises that of trivialisations in that it replaces the notion of a stable isomorphism between a given bundle gerbe and a trivial one with the notion of a 1-morphism, where a higher-rank vector bundle is allowed instead of a line bundle in Definition 3.1 [Wa1].

Just as for gerbes, there are a number of natural operations on 1-isomorphisms. In the present paper, we shall only need the pullback of a 1-isomorphism, the product and composition of 1-isomorphisms, and so we confine our presentation to these particular operations.

**Pullback:** Let \( \mathcal{G}_i = (Y_iM, B_i, L_i, \mu_{L_i}), \ i \in \{1, 2\} \) be a pair of gerbes over a common base \( M \) and let \( \Phi_{1,2} : \mathcal{G}_1 \xrightarrow{\sim} \mathcal{G}_2 \) be a 1-isomorphism with data \( \Phi_{1,2} = (YY_{1,2}M, E_{1,2}, \alpha_{1,2}) \). Furthermore, let \( N \) be a smooth manifold with surjective submersions \( \pi_{YY_{1,2}N} : Y_{1,2}N \to N \), and suppose that we are given a smooth map \( f : N \to M \) together with maps \( \hat{f}_i : Y_iN \to Y_iM \) that cover \( f \), so that we obtain the pullback gerbes \( f^*\mathcal{G}_i \). To define the pullback of \( \Phi_{1,2} \), we need, in addition, a surjective submersion \( \pi_{YY_{1,2}N} : YY_{1,2}N \to YY_{1,2}M \equiv Y_1N \times_N Y_2N \) together with a map \( f_{1,2} : YY_{1,2}N \to YY_{1,2}M \) that covers \( \hat{f}_{1,2} = \hat{f}_1 \times \hat{f}_2 \), as expressed by the commutative diagram

\[
\begin{array}{ccc}
YY_{1,2}N & \xrightarrow{f_{1,2}} & YY_{1,2}M \\
\downarrow{\pi_{YY_{1,2}N}} & & \downarrow{\pi_{YY_{1,2}M}} \\
Y_{1,2}N & \xrightarrow{\hat{f}_{1,2}} & Y_{1,2}M
\end{array}
\]

Given these, the **pullback of** \( \Phi_{1,2} **to N along f** is the 1-isomorphism

\[
f^*\Phi_{1,2} = (YY_{1,2}N, f_{1,2}^*E_{1,2}, f_{1,2}^*[\alpha_{1,2}]), \quad f^*\Phi_{1,2} : f^*\mathcal{G}_1 \xrightarrow{\sim} f^*\mathcal{G}_2,
\]

where we use the maps \( f_{1,2}^{[n]} : Y^{[n]}Y_{1,2}N \to Y^{[n]}Y_{1,2}M \) induced by \( f_{1,2} \).

**Product:** Consider a quadruple of gerbes \( \mathcal{G}_i = (Y_iM, B_i, L_i, \mu_{L_i}), \ i \in \{1, 2, 3, 4\} \) over a common base \( M \), with products \( \mathcal{G}_1 \star \mathcal{G}_2 \) and \( \mathcal{G}_2 \star \mathcal{G}_4 \) as in (2.12), and a pair of 1-isomorphisms
\( \Phi_{1,2} : G_1 \xrightarrow{\sim} G_2, \Phi_{3,4} : G_3 \xrightarrow{\sim} G_4 \) with data \( \Phi_{i,j} = (YY_{i,j}M, E_{i,j}, \alpha_{i,j}) \). The latter permit to define a product 1-isomorphism \( \Phi_{1,2} \ast \Phi_{3,4} : G_1 \ast G_3 \xrightarrow{\sim} G_2 \ast G_4 \) as

\[
\Phi_{1,2} \ast \Phi_{3,4} = (YY_{1,2}M \times_M YY_{3,4}M, pr^*_1E_{1,2} \otimes pr^*_2E_{3,4}, pr^*_{1,3}(\alpha_{1,2} \otimes pr^*_{2,4}\alpha_{3,4})) ,
\]

where the surjective submersion \( \pi_{YY_{1,2}M \times_M YY_{3,4}M} : Y(Y_{1,2}M \times_M YY_{3,4}M) \rightarrow Y_{1,2}M \times_M YY_{3,4}M \) is given by the map \( \pi_{YY_{1,2}M \times_M YY_{3,4}M} = \pi_{2,3} \circ (\pi_{YY_{1,2}M} \times \pi_{YY_{3,4}M}) \) expressed in terms of the transposition map \( \pi_{2,3} : Y_1M \times_M Y_2M \times_M Y_3M \times_M Y_4M \rightarrow Y_1M \times_M Y_2M \times_M Y_3M \times_M Y_4M \) is trivial, this proposition shows

\[
\Phi_{2,3} \circ \Phi_{1,2} = (YY_{1,3}M, E_{1,2,3}, \alpha_{1,2,3}) : G_1 \xrightarrow{\sim} G_3 \]

with the surjective submersion

\[
(pr_1, pr_3) \circ (\pi_{YY_{1,2}M} \times \pi_{YY_{3,4}M}) : YY_{1,3}M = YY_{1,2}M \times_M YY_{2,3}M \rightarrow Y_1M \times_M YY_{2,3}M ,
\]

written in terms of the canonical projections \( p_i : Y_1M \times_M Y_2M \times_M Y_3M \rightarrow Y_iM \), with the hermitean line bundle

\[
E_{1,2,3} = pr_1^*E_{1,2} \otimes pr_2^*E_{2,3} \rightarrow YY_{1,3}M
\]

with the connection

\[
\nabla_{E_{1,2,3}} = pr_1^*\nabla_{E_{1,2}} + pr_2^*\nabla_{E_{2,3}} ,
\]

both written in terms of the canonical projections \( p_i : YY_{1,3}M \rightarrow YY_{i,i+1}M \), and with the composite bundle isomorphism

\[
\alpha_{1,2,3} := (id_{pr_1^*E_{1,2}} \otimes pr_{2,4}^*\alpha_{2,3}) \circ (p_{1,3}^*\alpha_{1,2} \otimes id_{pr_2^*E_{2,3}}) ,
\]

defined in terms of the canonical projections \( p_{i,i+2} : Y[2]YY_{1,3}M \rightarrow Y[2]YY_{i,i+1}M \) and \( p_1 = pr_1 \circ p_{1,3}, p_4 = pr_2 \circ p_{2,4} \), where \( pr_i : Y[2]YY_{i,i+1}M \rightarrow YY_{i,i+1}M \) are, again, the canonical projections.

A useful description of gerbes with a given curvature is provided by the following statement [Ga1], which derives from the relation between bundle gerbes and the Deligne hypercohomology, and from the relation of the latter to sheaf cohomology [Bi1], cf., e.g., [GR1], Sect. 2.3] and [Go, Prop. 2.2].

**Proposition 3.3.** The set of stable-isomorphism classes of gerbes with a given curvature over base \( M \) is a torsor under the action of the cohomology group \( H^2(M, U(1)) \).

In particular, since the cohomology group \( H^2(SU(2), U(1)) \) is trivial, this proposition shows that the canonical gerbe is unique up to a 1-isomorphism.
Remark 3.4. Stable isomorphisms play a central rˆole in the construction of two-dimensional non-linear $\sigma$-models. The topological term of the action functional is given as (the logarithm of) the surface holonomy of a trivialisation of the pullback, along the embedding map $X : \Sigma \to M$, of the gerbe over the target space of the $\sigma$-model to the two-dimensional world-sheet $\Sigma$. From the point of view of this paper, it is also important that a stable isomorphism together with a definition of the world-volume and of the curvature 2-form of the so-called bi-brane describe conformal defects separating phases of the two-dimensional field theory [FSW, RS1, SS] (cf. also [CJM, GR1] for the boundary case).

A particular instantiation of this latter fact occurs on world-sheets with a boundary, capturing the dynamics of the open string. Here, the $\sigma$-model action functional is defined for a submanifold $\mathcal{D}$ of $M$, embedded in the target space as $\iota : \mathcal{D} \to M$ and termed the D-brane (or $G$-brane) world-volume, alongside a trivialisation of the bulk 2-gerbe over $\mathcal{D}$, $\Phi : \iota^*\mathcal{G} \sim \mathcal{I}(\omega)$. The trivialisation is defined in terms of a trivial gerbe $\mathcal{I}(\omega)$ with a globally defined curving $\omega \in \Omega^2(\mathcal{D})$ satisfying $\iota^*\text{curv}(\mathcal{G}) = d\omega$ and called the D-brane curvature.

In the WZW model, there is a distinguished class of D-branes – the so-called (untwisted) maximally symmetric D-branes – whose world-volume is given by conjugacy classes. Below, we review the construction of the associated 1-isomorphism in the case of $SU(2)$.

We shall denote the conjugacy class of the element $t_\lambda$ of the maximal torus as

$$C_\lambda = \{ \text{Ad}_g(t_\lambda) \mid g \in SU(2) \}.$$  \hspace{1cm} (3.15)

The restriction of the Cartan 3-form to $C_\lambda$ admits a global primitive $H|_{C_\lambda} = d\omega_\lambda$, which is simply given by restricting the 2-form $Q$ on $SU(2)$ to $C_\lambda$. Equivalently, we can use the equivariant formulation on $[0,k] \times SU(2)$, where

$$\tilde{C}_\lambda = c^{-1}(C_\lambda) = \{ \lambda \} \times SU(2), \quad \tilde{\omega}(h) = \tilde{Q}(\lambda, h)|_{\tilde{C}_\lambda}.$$ \hspace{1cm} (3.16)

In the remainder of this section, we describe – after [GR1] – the conditions under which there exists a trivialisation of the restricted gerbe

$$\Phi_\lambda : \mathcal{G}|_{C_\lambda} \sim \mathcal{I}(\omega_\lambda)$$ \hspace{1cm} (3.17)

and give the details of its construction.

**Surjective submersion and curvature 2-forms**

We need to give a surjective submersion $YY_{1,2}C_\lambda \xrightarrow{\pi_{Y1,2}C_\lambda} Y_1C_\lambda \times_M Y_2C_\lambda$, where $Y_1C_\lambda$ and $Y_2C_\lambda$ are the surjective submersions of the gerbes $\mathcal{G}|_{C_\lambda}$ and $\mathcal{I}(\omega_\lambda)$, respectively.

For the restricted canonical gerbe $\mathcal{G}|_{C_\lambda}$, we have the surjective submersion

$$YC_\lambda = C_{\lambda,0} \sqcup C_{\lambda,1} \xrightarrow{\pi_{YC_\lambda}} C_\lambda \quad \text{with} \quad C_{\lambda,t} = C_\lambda \cap \mathcal{O}_t.$$ \hspace{1cm} (3.18)

\footnote{In the context of two-dimensional non-linear $\sigma$-models, the term ‘bulk’ refers to the conformally invariant field theory defined away from the boundary of the world-sheet, describing the dynamics of the closed string.}
Thus, \( C_{\lambda,0} = C_{\lambda} \) except for the special cases \( C_{0,1} = \emptyset = C_{k,0} \), which are simply dropped, leaving us with \( YC_0 = C_{0,0} = \{e\} \) and \( YC_k = C_{k,1} = \{-e\} \). For the trivial gerbe \( I(\omega_\lambda) \), we have the trivial surjective submersion \( \text{id}_{C_\lambda} : C_\lambda \to C_\lambda \). The product of the two surjective submersions fibred over the common base \( C_\lambda \) of the two gerbes is given by

\[
YC_\lambda \times_{C_\lambda} C_\lambda = YC_\lambda.
\]  

Finally, for \( YY_{1,2}C_\lambda \to YC_\lambda \), we choose the trivial surjective submersion of \( YC_\lambda \), i.e. \( YY_{1,2}C_\lambda = YC_\lambda \) and \( \pi_{YY_{1,2}C_\lambda} = \text{id}_{YC_\lambda} \).

In order to give the line bundle over \( YC_\lambda \), we shall, again, work in the equivariant formulation via the pullback by \( c \). We therefore define

\[
YC_\lambda = \tilde{C}_{\lambda,0} \sqcup \tilde{C}_{\lambda,1}, \quad \text{with} \quad \tilde{C}_{\lambda,l} = \tilde{C}_\lambda \cap \tilde{O}_l.
\]  

The difference of the pullback curvings in (3.3) on \( \tilde{C}_{\lambda,l} \subset YC_\lambda \) then reads

\[
\bar{\omega}_\lambda(h) - \tilde{B}_l(\lambda, h) = -\tilde{F}(\lambda - \lambda_l, h) = i(\lambda - \lambda_l) \text{tr}(\Lambda d\theta_L(h)).
\]  

The latter 2-form can be the curvature 2-form of a bundle only if its periods over 2-cycles of \( \tilde{C}_{\lambda,l} \) take values in \( 2\pi \mathbb{Z} \). For \( \lambda = 0 \) or \( \lambda = k \), the manifold \( YC_\lambda \) is simply a point, and so the line bundle exists – it is the trivial bundle. For \( \lambda \notin \{0,k\} \), the conjugacy classes \( C_{\lambda,l} \) with \( l = 0,1 \) provide a choice of representatives of the generators of \( H_2(YC_\lambda) \). The conjugacy classes can be parameterised by mapping \( [0, \pi] \times [0, \pi/2] \to \tilde{C}_{\lambda,l} \to C_{\lambda,l} \) as \( (\varphi, \theta) \mapsto (\lambda, h(\varphi, \theta)) \mapsto h(\varphi, \theta) \cdot t_\lambda \cdot h(\varphi, \theta)^{-1} \), where \( h(\varphi, \theta) = e^{i\varphi\sigma_3} \cdot e^{i\theta\sigma_2} \). The pullback of \( -\tilde{F} \) is then given by

\[
-\tilde{F}(\varphi, \theta) = -2(\lambda - \lambda_l) \sin 2\theta \, d\varphi \wedge d\theta,
\]  

so that

\[
\int_{\tilde{C}_{\lambda,l}} (-\tilde{F}) = 2(\lambda_l - \lambda) \int_0^\pi d\varphi \int_0^{\pi/2} d\theta \sin 2\theta = 2\pi (\lambda_l - \lambda).
\]  

The result lies in \( 2\pi \mathbb{Z} \) if and only if \( \lambda \in \mathbb{Z} \), or – equivalently – if and only if \( \lambda \Lambda \in \mathcal{A}_W^k \) is an integral weight.

**Line bundle over \( YC_\lambda \)**

The calculation of periods shows that the line bundle over \( YC_\lambda \) with the curvature (3.21) exists if and only if \( \lambda \in P_+^k \) (so that \( \lambda \Lambda \in \mathcal{A}_W^k \)). In order to construct this bundle – or rather the equivariant line bundle \( \tilde{E}_\lambda \to YC_\lambda \) – we can use the KKS bundle again. Namely, we set

\[
\tilde{E}_{\lambda,l} = \pi^*_{SU(2)} K_{\lambda-\lambda_l}, \quad \text{where} \quad \pi_{SU(2)} : \tilde{C}_{\lambda,l} \to SU(2) : (\lambda, h) \mapsto h.
\]  

For \( \lambda \notin \{0,k\} \), it inherits equivariance with respect to the action of \( SU(2)_{\lambda-\lambda_l} = U(1) \) from \( K_{\lambda-\lambda_l} \), and hence – as \( SU(2)_{t_\lambda} = SU(2)_{\lambda-\lambda_l} \) – it descends to a well-defined quotient bundle \( E_{\lambda,l} \to C_{\lambda,l} \). For \( \lambda = 0 \), we have \( l = 0 \) and both the connection and the action of the isotropy group are trivial, and so \( \tilde{E}_{0,0} \) induces the trivial bundle over \( C_{0,0} = \{e\} \). Similarly for \( \lambda = k \), we get the trivial bundle over \( C_{k,1} = \{-e\} \). Altogether, the quotient bundles \( E_{\lambda,l} \to C_{\lambda,l} \) compose a line bundle \( E_\lambda \to YC_\lambda \).
Isomorphism of pullback bundles

The last piece of data that we need is the isomorphism (3.4) of the pullback bundles. The bundles are pulled back to $Y Y_{1,2} C_\lambda \times C_\lambda Y Y_{1,2} C_\lambda = Y C_\lambda \times C_\lambda Y C_\lambda = Y^{[2]} C_\lambda$, where

$$Y^{[2]} C_\lambda = \bigsqcup_{i,j \in \{0,1\}} C_{\lambda, i,j}, \quad C_{\lambda, i,j} = C_\lambda \cap O_{i,j}. \quad (3.25)$$

As before, we denote by $p_{r_i} : Y^{[2]} C_\lambda \to Y C_\lambda$ the projections onto the two factors. In the present case, the isomorphism (3.4) boils down to

$$\alpha_\lambda : L|_{Y^{[2]} C_\lambda} \otimes \text{pr}_{r_2}^* E_\lambda \to \text{pr}_{r_1}^* E_\lambda. \quad (3.26)$$

In the equivariant formulation, we work with the subsets $\tilde{C}_{\lambda, i,j} = \tilde{C}_\lambda \cap \tilde{O}_{i,j}$. We denote points in $\tilde{C}_{\lambda, i,j}$ as $(i, j, \lambda, h)$, with $h \in SU(2)$. The isomorphism takes the simple form

$$\tilde{\alpha}_\lambda|_{\tilde{C}_{\lambda, i,j}} : \tilde{L}_{i,j} \otimes \tilde{E}_{\lambda,j}|_{\tilde{C}_{\lambda, i,j}} \to \tilde{E}_{\lambda,i}|_{\tilde{C}_{\lambda, i,j}} : (i, j, \lambda, h, z \otimes z') \mapsto (i, j, \lambda, h, z \cdot z'). \quad (3.27)$$

This map is clearly unitary and it is easy to see that it is compatible with the connections. Equivariance with respect to the maximal torus, common to all isotropy groups, amounts to the statement that, for all $s \in U(1)$,

$$\tilde{\alpha}_\lambda(i, j, \lambda, h \cdot s, \chi_{\lambda, i,j}(s) \cdot z \otimes \chi_{\lambda-\lambda_i}(s) \cdot z') = (i, j, \lambda, h \cdot s, \chi_{\lambda-\lambda_i}(s) \cdot z \cdot z'). \quad (3.28)$$

This holds true because of the equality

$$\chi_{\lambda, i,j}(s) \cdot \chi_{\lambda-\lambda_i}(s) = \chi_{\lambda-\lambda_i}(s). \quad (3.29)$$

For the non-generic isotropy groups over $\lambda = 0$ (in which case $i = j = 0$) and $\lambda = k$ (in which case $i = j = 1$), the argument is the same, only that now $s \in SU(2)$ and $\chi_0(s) = 1$. We conclude that $\tilde{\alpha}_\lambda$ yields a bundle isomorphism $\alpha_\lambda$.

Finally, it is clear that $\alpha_\lambda$ is also compatible with the groupoid structures on $G|_{C_\lambda}$ and $I(\omega_\lambda)$.

4 Elements of the multiplicative structure

A multiplicative structure is a 1-isomorphism $\mathcal{M}$ between gerbes on $G \times G$, together with a 2-isomorphism subject to some coherence properties $[Ca, Wa]$. Here, $G$ is a Lie group, and the gerbes in question are obtained by pulling back a power of the basic gerbe on $G$ along the projection maps to each factor, and along the multiplication map, respectively. If $G$ is compact simple and connected, a multiplicative structure exists under certain conditions on the level $[GW]$, and it exists for all levels if $G$ is simply connected $[Wa]$. Whenever a multiplicative structure exists, it is unique up to an isomorphism $[Wa, GW]$. In the present paper, we shall only need the 1-isomorphism $\mathcal{M}$ restricted to certain submanifolds of $SU(2) \times SU(2)$.
2-isomorphisms

The last element of the general theory necessary to fully understand the subsequent discussion is the notion of a 2-isomorphism between 1-isomorphisms. We begin with

**Definition 4.1.** [St, Wa1] Let \( \mathcal{G}_i = (Y_i M, B_i, L_i, \mu_{L_i}), \ i \in \{1, 2\} \) be a pair of gerbes over a common base \( M \) and of the same curvature, and let \( \Phi^A_{1,2} = (Y^A Y_{1,2} M, E^A_{1,2}, \alpha^A_{1,2}) \), \( A \in \{1, 2\} \) be a pair of 1-isomorphisms between \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \),

\[
\Phi^A_{1,2} : \mathcal{G}_1 \sim \sim \mathcal{G}_2. \tag{4.1}
\]

A 2-isomorphism \( \varphi \) between \( \Phi^1_{1,2} \) and \( \Phi^2_{1,2} \), denoted as

\[
\varphi : \Phi^1_{1,2} \Rightarrow \Phi^2_{1,2}, \tag{4.2}
\]

is a pair \((YY^{1,2} Y_{1,2} M, \varphi)\) determined by a choice of a surjective submersion

\[
YY^{1,2} Y_{1,2} M \xrightarrow{\pi_{YY^{1,2} Y_{1,2} M}} Y^{1,2} Y_{1,2} M, \quad Y^{1,2} Y_{1,2} M = Y^1 Y_{1,2} M \times_{Y_{1,2} M} Y^2 Y_{1,2} M, \tag{4.3}
\]

together with a unitary connection-preserving isomorphism

\[
\varphi : p^*_1 E^1_{1,2} \sim \sim p^*_2 E^2_{1,2} \tag{4.4}
\]

between the two line bundles over \( YY^{1,2} Y_{1,2} M \), the latter coming with the maps \( p_A = \text{pr}_A \circ \pi_{YY^{1,2} Y_{1,2} M} : YY^{1,2} Y_{1,2} M \to Y^A Y_{1,2} M \) written in terms of the canonical projections \( \text{pr}_A : Y^{1,2} Y_{1,2} M \to Y^A Y_{1,2} M \). The isomorphism must be compatible with the two isomorphisms \( \alpha^A_{1,2} \) in the sense specified by the commutativity of the diagram

\[
\begin{array}{ccc}
p^*_1 L_1 \otimes \pi_3^* E^1_{1,2} & \xrightarrow{\pi^*_{1,3} \alpha^1_{1,2}} & \pi_1^* E^1_{1,2} \otimes p^*_2 L_2 \\
\downarrow \text{id}_{p^*_1 L_1} \otimes \text{pr}^*_2 \varphi & & \downarrow \text{pr}^*_1 \varphi \otimes \text{id}_{p^*_2 L_2} \\
p^*_1 L_1 \otimes \pi_4^* E^2_{1,2} & \xrightarrow{\pi^*_{1,4} \alpha^2_{1,2}} & \pi_2^* E^2_{1,2} \otimes p^*_2 L_2
\end{array} \tag{4.5}
\]

of bundle isomorphisms over \( Y^{[2]} Y^{1,2} Y_{1,2} M = YY^{1,2} Y_{1,2} M \times_M YY^{1,2} Y_{1,2} M \), the latter coming with the following maps:

- the canonical projections \( \text{pr}_i : Y^{[2]} Y^{1,2} Y_{1,2} M \to YY^{1,2} Y_{1,2} M \);

- the natural maps \( p_{i,i+2} = \text{pr}_{i,i+2} \circ (\pi_{Y Y^{1,2} Y_{1,2} M} \times \pi_{Y Y^{1,2} Y_{1,2} M}) \circ (p_A \times p_A) \), independent of \( A \) and defined in terms of \( p_A \) as above, alongside the canonical projections \( \text{pr}_{i,i+2} : Y_{1,2} M \times_M Y_{1,2} M \to Y_i^{[2]} M \);

- the natural maps \( \pi_{2i-1} : Y^{[2]} Y^{1,2} Y_{1,2} M \xrightarrow{\text{pr}_i} YY^{1,2} Y_{1,2} M \xrightarrow{\pi_1} Y^1 Y_{1,2} M \) and \( \pi_{2i} : Y^{[2]} Y^{1,2} Y_{1,2} M \xrightarrow{\text{pr}_i} YY^{1,2} Y_{1,2} M \xrightarrow{\pi_2} Y^2 Y_{1,2} M \), alongside \( \pi_{1,3} : Y^{[2]} Y^{1,2} Y_{1,2} M \to Y^1 Y_{1,2} M \times_M Y^1 Y_{1,2} M \) and \( \pi_{2,4} : Y^{[2]} Y^{1,2} Y_{1,2} M \to Y^2 Y_{1,2} M \times_M Y^2 Y_{1,2} M \).
The notion of equivalence of 2-isomorphisms is specified in the following definition.

**Definition 4.2.** \([\text{Wa1}]\) Let \((Y, Y^{1,2}Y_{1,2}M, \varphi_i), i \in \{1, 2\}\) be two 2-isomorphisms between a pair \(\Phi^A = (Y^A, Y^{1,2}Y_{1,2}M, E^A, \alpha^A), A \in \{1, 2\}\) of 1-isomorphisms between two gerbes \(G_i = (Y_i, B_i, L_i, \mu_{L_i}), i \in \{1, 2\}\) over a common base \(M\) and of the same curvature. We call the 2-isomorphisms *equivalent* if there exists a smooth space \(Y^{1,2}Y_{1,2}M\) and a pair of surjective submersions \(\pi^i_{Y^{1,2}Y_{1,2}M} : Y^{1,2}Y_{1,2}M \rightarrow Y^{1,2}Y_{1,2}M\) satisfying the identity

\[
\pi^1_{Y^{1,2}Y_{1,2}M} \circ \pi^1_{Y^{1,2}Y_{1,2}M} = \pi^2_{Y^{1,2}Y_{1,2}M} \circ \pi^2_{Y^{1,2}Y_{1,2}M}
\]

and such that

\[
\pi^1_{Y^{1,2}Y_{1,2}M} \varphi_1 = \pi^2_{Y^{1,2}Y_{1,2}M} \varphi_2.
\]

We have

**Proposition 4.3.** Every 2-isomorphism between a pair \(\Phi^A = (Y^A, Y^{1,2}Y_{1,2}M, E^A, \alpha^A), A \in \{1, 2\}\) of 1-isomorphisms between two gerbes \(G_i = (Y_i, B_i, L_i, \mu_{L_i}), i \in \{1, 2\}\) over a common base \(M\) and of the same curvature is equivalent to a 2-isomorphism between these 1-isomorphisms with the trivial surjective submersion \(\text{id}_{Y^{1,2}Y_{1,2}M}\).

The proposition follows from the proof of Theorem 1 in \([\text{Wa1}]\) (there, the statement is made for 2-isomorphisms between stable isomorphisms but it can be generalised).

**Remark 4.4.** Gerbes over a given base \(M\), together with 1-isomorphisms and 2-isomorphisms, form a 2-category, introduced in \([\text{St}]\) and further explored in \([\text{Wa1}]\). In this sense, 2-isomorphisms are a logical completion of the family of structures discussed in the preceding sections. They are essential constituents of (twisted) equivariant structures on bundle gerbes \([\text{GR1}, \text{GR2}, \text{SSW}, \text{GSW1}, \text{GSW2}]\) and play a fundamental rôle in the construction of \(\sigma\)-models on world-sheets with (intersecting) defect lines \([\text{RS1}]\). They are also part of the definition of the multiplicative structure on gerbes over Lie groups \([\text{Ca}, \text{Wa2}]\), which we shall explain next to the extent strictly necessary for understanding subsequent considerations.

There is a convenient cohomological classification of stable isomorphisms between a given pair of gerbes; it can be proved similarly to Proposition 3.3.

**Proposition 4.5.** The set of 2-isomorphism classes of stable isomorphisms between two given gerbes over a common base \(M\) is a torsor under the cohomology group \(H^1(M, U(1))\).

**The submanifolds \(T_{\lambda, \mu}^\nu\)**

Consider the triple of smooth maps

\[
\begin{align*}
p_i &: \text{SU}(2) \times \text{SU}(2) \rightarrow \text{SU}(2) &: (g_1, g_2) \mapsto g_i, \quad i \in \{1, 2\}, \\
m &: \text{SU}(2) \times \text{SU}(2) \rightarrow \text{SU}(2) &: (g_1, g_2) \mapsto g_1 \cdot g_2.
\end{align*}
\]
The submanifolds
\[ \mathcal{T}_{\lambda,\mu}^\nu = p_1^{-1}(\mathcal{C}_\lambda) \cap p_2^{-1}(\mathcal{C}_\mu) \cap \mathfrak{m}^{-1}(\mathcal{C}_\nu) \subset \text{SU}(2) \times \text{SU}(2) \] (4.9)
will play a central rôle in the following. The subset of the parameter space for which these submanifolds are non-empty is called the fusion polytope of SU(2),
\[ \mathcal{F} = \{ (\lambda, \mu, \nu) \in [0, k] \times 3 \mid \mathcal{T}_{\lambda,\mu}^\nu \neq \emptyset \}. \] (4.10)

**Remark 4.6.** Using the three maps \( p_1, p_2, m : \text{SU}(2) \times \text{SU}(2) \rightarrow \text{SU}(2) \), we may pull the various geometric objects introduced previously, such as the canonical gerbe \( \mathcal{G} \) and its trivialisations \( \Phi_\lambda, \Phi_\mu \) and \( \Phi_\nu \), back to \( \mathcal{T}_{\lambda,\mu}^\nu \); we may also restrict the multiplicative structure on \( \text{SU}(2) \times \text{SU}(2) \) to it. The motivation to do so ultimately derives from the study of the maximally symmetric bi-brane of the WZW model for SU(2) [FSW] and will be expounded in full detail in the companion paper [RS2]. As already pointed out in [FSW], the space \( \mathcal{T}_{\lambda,\mu}^\nu \) is closely related to the moduli space of flat connections on a principal SU(2)-bundle over a Riemann sphere with three punctures, with the holonomy around the punctures constrained to take values in the three conjugacy classes \( \mathcal{C}_\lambda, \mathcal{C}_\mu \) and \( \mathcal{C}_\nu \), as reconstructed in [AM]. The moduli space, in turn, reproduces, upon quantisation [W12], the space of conformal blocks of the WZW model for SU(2) with insertions of primary fields from the corresponding representations of highest weights \( \lambda, \mu \) and \( \nu \) of the affine Lie algebra \( \hat{\mathfrak{su}}(2)_k \). Our point of view is slightly different in that we have so far only been considering the fusion rules from the existence of three-fold junctions of defect lines in the classical WZW model [RST] rather than from the quantisation of a moduli space. In the quantum WZW model, the two approaches to fusion rules are equivalent, as such defect junctions exist if and only if the corresponding space of conformal three-point blocks is nonzero [Fr, RST].

An equivalent way of writing the submanifolds \( \mathcal{T}_{\lambda,\mu}^\nu \) is
\[ \mathcal{T}_{\lambda,\mu}^\nu = \{ (\text{Ad}_x(t_\lambda), \text{Ad}_a(t_\mu)) \mid x, a \in \text{SU}(2) \text{ and } t_\lambda \cdot \text{Ad}_a(t_\mu) \in \mathcal{C}_\nu \}. \] (4.11)

Here, \( a \) is taken to run over all of SU(2), but it is easy to see that it suffices to pick a representative \( a \) from each of the equivalence classes
\[ [a] \in \text{SU}(2) \backslash \text{SU}(2) / \text{SU}(2)_{t_\nu} . \] (4.12)
Recalling the parameterisation (2.23) by the Euler angles, we infer that we can choose this representative in the form \( a = e^{i\sigma_2} \) and let \( \theta \) vary over those values in \([0, \pi/2] \) which are allowed by the condition \( t_\lambda \cdot \text{Ad}_a(t_\mu) \in \mathcal{C}_\nu \). In fact, the set of such \( \theta \) is either empty or contains a single element. In order to see this, note, first, that
\[ e^{i\theta_2} \cdot t_\mu \cdot e^{-i\theta_2} = \begin{pmatrix} \cos \frac{\mu \theta}{k} + i \sin \frac{\mu \theta}{k} \cos 2\theta & -i \sin \frac{\mu \theta}{k} \sin 2\theta \\ -i \sin \frac{\mu \theta}{k} \sin 2\theta & \cos \frac{\mu \theta}{k} - i \sin \frac{\mu \theta}{k} \cos 2\theta \end{pmatrix} . \] (4.13)
For SU(2), the condition \( t_\lambda \cdot \text{Ad}_a(t_\mu) \in \mathcal{C}_\nu \) is equivalent to \( \text{tr}(t_\lambda \cdot \text{Ad}_a(t_\mu)) = \text{tr}(t_\nu) \). Explicitly, this condition reads
\[ \cos \left( \frac{\pi (\lambda + \mu)}{k} \right) + (\sin \theta)^2 \left[ \cos \left( \frac{\pi (\lambda - \mu)}{k} \right) - \cos \left( \frac{\pi (\lambda + \mu)}{k} \right) \right] = \cos \left( \frac{\pi \nu}{k} \right) \] (4.14)
and has a solution for \( \theta \) if and only if \( |\lambda - \mu| \leq \nu \leq \min(\lambda + \mu, 2k - \lambda - \mu) \). Altogether, we have proved (cf., e.g., [JW], Prop. 3.1):
Lemma 4.7. The fusion polytope of $SU(2)$ is given by

$$\mathcal{F} = \{ (\lambda, \mu, \nu) \in [0, k]^3 \mid |\lambda - \mu| \leq \nu \leq \min(\lambda + \mu, 2k - \lambda - \mu) \}.$$  \hspace{1cm} (4.15)

For later use, we also define the following subsets of the fusion polytope: the boundary $\partial \mathcal{F}$, the corners $\partial_0 \mathcal{F}$, the boundary less the corners $\partial_{12} \mathcal{F}$, the interior $\mathcal{F}$, the fusion polytope less its corners and edges $\mathcal{F}$. Altogether,

$$\partial \mathcal{F} = \{ (\lambda, \mu, \nu) \in [0, k]^3 \mid |\nu| = |\lambda - \mu| \text{ or } \nu = \min(\lambda + \mu, 2k - \lambda - \mu) \},$$

$$\partial_0 \mathcal{F} = \{ (\lambda, \mu, \nu) \in \mathcal{F} \mid \lambda, \mu, \nu \in \{0, k\} \},$$

$$\partial_{12} \mathcal{F} = \partial \mathcal{F} \setminus \partial_0 \mathcal{F},$$

$$\Hat{\mathcal{F}} = \mathcal{F} \setminus \partial \mathcal{F},$$

$$\mathcal{F} = \mathcal{F} \cap [0, k]^3.$$  \hspace{1cm} (4.16)

For $(\lambda, \mu, \nu) \in \mathcal{F}$ and $\lambda, \mu \not\in \{0, k\}$, equation (4.14) has a unique solution for $\theta$ in the range $[0, \frac{\pi}{2}]$, and we use this to define

$$a_{\lambda, \mu} = e^{i\theta}, \quad \text{with} \quad \theta \in [0, \frac{\pi}{2}] \quad \text{such that} \quad (\sin \theta)^2 = \frac{\cos \frac{\pi \nu}{k} - \cos \frac{\pi(\lambda + \mu)}{k}}{2 \sin \frac{\pi \lambda}{k} \sin \frac{\pi \mu}{k}}. \hspace{1cm} (4.17)$$

By construction, the $a_{\lambda, \mu}$ satisfy $t_\lambda \cdot \text{Ad}_{a_{\lambda, \mu}}(t_\mu) = \text{Ad}_b(t_\nu)$ for some $b \in SU(2)$. Clearly, only the class of $b$ in the coset $[b] \in SU(2)/SU(2)_{t_\nu}$ is relevant, hence we are free to choose $b$ in the form $b = e^{i\varphi_2} \cdot e^{i\theta_2}$. In order to fix $\theta'$, we take the trace of $\text{Ad}_{a_{\lambda, \mu}}(t_\mu) = t_{-\lambda} \cdot \text{Ad}_b(t_\nu)$, whereby $a_{\lambda, \mu}$ and $e^{i\varphi_3}$ drop out. The value of $\varphi'$ can then be determined by matching the matrix entries of $t_\lambda \cdot \text{Ad}_{a_{\lambda, \mu}}(t_\mu)$ with those of $\text{Ad}_b(t_\nu)$. One finds that both $\theta'$ and $\varphi'$ are defined uniquely for $(\lambda, \mu, \nu) \in \mathcal{F}$ with $\lambda \not\in \{0, k\}$ and $\nu \not\in \{|\lambda - \mu|, \min(\lambda + \mu, 2k - \lambda - \mu)\}$. In fact, $\theta'$ is unique in the larger range, namely, for all $(\lambda, \mu, \nu) \in \mathcal{F}$ such that $\lambda, \nu \not\in \{0, k\}$, and so we may use the arbitrariness of $\varphi'$ on the difference of the two ranges to extend the formula for $\varphi'$ to it, whereby we obtain

$$b_{\lambda, \mu} = e^{i\varphi_3} \cdot e^{i\theta_2}, \quad \text{with} \quad \varphi' = \frac{\pi \lambda}{2k} \quad \text{and} \quad \theta' \in [0, \frac{\pi}{2}] \quad \text{such that} \quad (\cos \theta')^2 = \frac{\cos \frac{\pi \nu}{k} - \cos \frac{\pi(\lambda + \mu)}{k}}{2 \sin \frac{\pi \lambda}{k} \sin \frac{\pi \mu}{k}}. \hspace{1cm} (4.18)$$

for all $(\lambda, \mu, \nu) \in \mathcal{F}$ such that $\lambda, \nu \not\in \{0, k\}$. In their respective ranges, $a_{\lambda, \mu}$ and $b_{\lambda, \mu}$ are smooth functions of $(\lambda, \mu, \nu)$. In particular, both are smooth functions on the subset $\Hat{\mathcal{F}}$ of $\mathcal{F}$.

It will be convenient to extend the range of $a_{\lambda, \mu}$ and $b_{\lambda, \mu}$ (non-continuously) to include all triples from $\mathcal{F}$ as

$$a_{0, \lambda} = a_{\lambda, 0} = a_{k, \lambda} = a_{k, \lambda} = e \quad \text{and} \quad b_{0, \mu, \nu} = b_{\mu, \nu} = e = b_{0, \lambda}, \quad b_{k, \lambda} = i \sigma_2 \hspace{1cm} (4.19)$$

for arbitrary $\lambda \in [0, k]$ and $\mu \in \{0, k\}$. With these assignments, we have

$$t_\lambda \cdot \text{Ad}_{a_{\lambda, \mu}}(t_\mu) = \text{Ad}_{b_{\lambda, \mu}}(t_\nu) \quad \text{for all} \quad (\lambda, \mu, \nu) \in \mathcal{F}. \hspace{1cm} (4.20)$$
With the help of \( a_{\lambda,\mu}^{\nu} \), we obtain a surjective map
\[
\tau : \mathcal{F} \times \text{SU}(2) \rightarrow \text{SU}(2) \times \text{SU}(2) : (\lambda, \mu, \nu, h) \mapsto (\text{Ad}_h(t_\lambda), \text{Ad}_h a_{\lambda,\mu}^{\nu}(t_\mu)) .
\]

(4.21)

The map \( \tau \) is not smooth on all of \( \mathcal{F} \times \text{SU}(2) \) but it is smooth on the subset \( \mathcal{F} \times \text{SU}(2) \). Furthermore, for any fixed choice \( (\lambda, \mu, \nu) \in \mathcal{F} \), the map \( \tau(\lambda, \mu, \nu, -) \) is a smooth surjection from \( \text{SU}(2) \) to \( T_{\lambda,\mu}^{\nu} \), and provides a convenient (redundant) parameterisation of the submanifolds \( T_{\lambda,\mu}^{\nu} \), i.e. we have
\[
T_{\lambda,\mu}^{\nu} = \left\{ \tau(\lambda, \mu, \nu, h) \mid h \in \text{SU}(2) \right\} .
\]

(4.22)

**Remark 4.8.** The map \( \tau \) intertwines the left action \( \text{SU}(2) \times (\mathcal{F} \times \text{SU}(2)) \rightarrow \mathcal{F} \times \text{SU}(2) \), given by \( g.(\lambda, \mu, \nu, h) = (\lambda, \mu, \nu, g \cdot h) \), with the diagonal adjoint action on \( \text{SU}(2) \). It follows that \( T_{\lambda,\mu}^{\nu} \) is a single orbit under the diagonal Ad-action of \( \text{SU}(2) \) on \( \text{SU}(2) \). This is peculiar to \( \text{SU}(2) \); for other Lie groups, the corresponding intersection of conjugacy classes as in (4.9) will typically decompose into a continuum of diagonal Ad-orbits.

Below, we shall also need lifts of the maps \( p_1, p_2 \) and \( m \) of (4.8) to maps \( q_1, q_2, q_m : \mathcal{F} \times \text{SU}(2) \rightarrow ]0, k[ \times \text{SU}(2) \) satisfying
\[
p_1 \circ \tau = c \circ q_1 , \quad p_2 \circ \tau = c \circ q_2 , \quad m \circ \tau = c \circ q_m \quad (4.23)
\]
on \( \mathcal{F} \times \text{SU}(2) \). Their definition uses both \( a_{\lambda,\mu}^{\nu} \) and \( b_{\lambda,\mu}^{\nu} \),
\[
q_1(\lambda, \mu, \nu, h) = (\lambda, h) , \quad q_2(\lambda, \mu, \nu, h) = (\mu, h \cdot a_{\lambda,\mu}^{\nu}) , \quad q_m(\lambda, \mu, \nu, h) = (\nu, h \cdot b_{\lambda,\mu}^{\nu}) . \quad (4.24)
\]

Due to the restriction to \( \mathcal{F} \), these maps are smooth, and the property (4.23) follows from (4.20). This allows to express the pullback-gerbe data on \( \text{SU}(2) \times \text{SU}(2) \) in terms of the corresponding equivariant data on \( \mathcal{F} \times \text{SU}(2) \), obtained – in turn – via pullback of equivariant data from \( ]0, k[ \times \text{SU}(2) \). For fixed values of \( \lambda, \mu \) and \( \nu \), when smoothness in the weight variables ceases to play a rôle, we shall also consider maps \( q_{1,\lambda,\mu}^{\nu} \) and \( q_{2,\lambda,\mu}^{\nu} \) defined just like \( q_1, q_2 \) and \( q_m \), respectively, but for an arbitrary fixed triple \( (\lambda, \mu, \nu) \in \mathcal{F} \). These will be useful when pulling back objects from fixed conjugacy classes in \( \text{SU}(2) \) to the manifolds \( \{(\lambda, \mu, \nu)\} \times \text{SU}(2) \).

The adjoint action being transitive on \( T_{\lambda,\mu}^{\nu} \), we can write
\[
T_{\lambda,\mu}^{\nu} \cong \text{SU}(2)/S_{\lambda,\mu}^{\nu} , \quad S_{\lambda,\mu}^{\nu} = \text{SU}(2)_{t_\lambda} \cap \text{Ad}_{a_{\lambda,\mu}^{\nu}} \text{SU}(2)_{t_\mu} \quad (4.25)
\]
for \( S_{\lambda,\mu}^{\nu} \subset \text{SU}(2) \) the isotropy subgroup with respect to the diagonal adjoint action. As indicated by (4.22), whenever we describe gerbe data on \( \text{SU}(2) \times \text{SU}(2) \) in terms of its pullback to \( \mathcal{F} \times \text{SU}(2) \), we have to ensure equivariance with respect to the right action of \( \text{SU}(2) \) restricted to the isotropy subgroups \( S_{\lambda,\mu}^{\nu} \). One verifies that the latter are given by
\[
S_{\lambda,\mu}^{\nu} = \begin{cases} 
\text{SU}(2) & \text{if } (\lambda, \mu, \nu) \in \partial_0 \mathcal{F} \\
\text{U}(1) & \text{if } (\lambda, \mu, \nu) \in \partial_1 \mathcal{F} \\
\mathbb{Z}_2 & \text{if } (\lambda, \mu, \nu) \in \mathcal{F}
\end{cases} . \quad (4.26)
\]
Thus, with the above three possibilities for $S_{\nu}^{\lambda,\mu}$, the manifold $T_{\nu}^{\lambda,\mu}$ is isomorphic to a point, to a 2-sphere, and to $SO(3)$, respectively. The resulting first cohomology groups are

$$H^1(T_{\lambda,\mu}, U(1)) = \begin{cases} 1 & \text{if } (\lambda, \mu, \nu) \in \partial \mathcal{F} \\ \mathbb{Z}_2 & \text{if } (\lambda, \mu, \nu) \in \hat{\mathcal{F}} \end{cases}. \tag{4.27}$$

It now follows from Proposition 4.5 that in the interior of $\mathcal{F}$, there appears an obstruction to the existence of the fusion 2-isomorphism to be constructed in Section 5, which will ultimately lead to the parity-conservation rule that is the main point of this paper.

Finally, note that since $\tau(\lambda, \mu, \nu, h) = \tau(\lambda, \mu, \nu, \zeta \cdot h)$ for $\zeta = \pm e$, the map $\tau$ factors through $SO(3)$.

$$\mathcal{F} \times SU(2) \xrightarrow{\tau} SU(2) \times SU(2) = \mathcal{F} \times SU(2) \xrightarrow{\pi} \mathcal{F} \times SO(3) \xrightarrow{\bar{\tau}} SU(2) \times SU(2). \tag{4.28}$$

Here, $\pi$ is the projection from $SU(2)$ to $SO(3) = SU(2)/\{\pm e\}$ and $\bar{\tau}$ is the induced map on the quotient, which is smooth when restricted to $\mathcal{F} \times SU(2)$.

### Pullback gerbes over $T_{\lambda,\mu}^{\nu}$

Our next goal is to pull the canonical gerbe $G$ back to the manifolds $T_{\lambda,\mu}^{\nu}$ along each of the three maps $p_1, p_2, m$ of (4.8). To this end, we should first specify a surjective submersion over each $T_{\lambda,\mu}^{\nu}$, which we take in the form

$$YT_{\lambda,\mu}^{\nu} := \bigsqcup_{\tilde{k} = (l, m) \in \{0, 1\}^2} T_{\lambda,\mu;\tilde{k}}^{\nu}, \quad T_{\lambda,\mu;\tilde{k}}^{\nu} = p_1^{-1}(C_{\lambda;l}) \cap p_2^{-1}(C_{\mu;m}) \cap m^{-1}(C_{\nu;l+m}), \tag{4.29}$$

where $l + m = \min(l + m, 2 - l - m) = l + m \pmod{2}$. Recall that either $C_{\lambda;l} = C_{\lambda}$ or $C_{\lambda;l} = \emptyset$, etc., and so either $T_{\lambda,\mu;\tilde{k}}^{\nu} = T_{\lambda,\mu}^{\nu}$ or $T_{\lambda,\mu;\tilde{k}}^{\nu} = \emptyset$. In the above definition, it is understood that we keep those index pairs $\tilde{k} = (l, m)$ only for which $T_{\lambda,\mu;\tilde{k}}^{\nu}$ is non-empty. The advantage of working with $YT_{\lambda,\mu}^{\nu}$ is that it comes with a natural choice of maps $\tilde{p}_i, i \in \{1, 2\}$ and $\tilde{m}$ that cover the respective maps on its base $T_{\lambda,\mu}$, cf. Diag. (2.10), to wit

$$\tilde{p}_1(\tilde{k}, g, h) = (l, g), \quad \tilde{m}(\tilde{k}, g, h) = (l + m, g \cdot h),$$

and similarly for $\tilde{p}_2(\tilde{k}, g, h) = (m, h)$, alongside

$$\tilde{m}(\tilde{k}, g, h) = (l + m, g \cdot h),$$

$$\tilde{p}_1(\tilde{k}, g, h) = (l, g).$$
The definition of \( \hat{m} \) ensures that its image is non-empty for every triple \((\lambda, \mu, \nu) \in \mathcal{F}\). Furthermore, maps \( \hat{p}_i, \hat{m} \) canonically induce the corresponding maps on the fibred square \( Y^{[2]} T_{\lambda, \mu}^{\nu} \) that we shall also need later. These are, for \( \bar{k}_i = (l_i, m_i), \) \( i \in \{1, 2\} \), given by the formulæ
\[
\hat{p}_1^{[2]}(\bar{k}_1, \bar{k}_2, g, h) = (l_1, l_2, g), \quad \hat{p}_2^{[2]}(\bar{k}_1, \bar{k}_2, g, h) = (m_1, m_2, h),
\]
\[
\hat{m}^{[2]}(\bar{k}_1, \bar{k}_2, g, h) = (l_1 + m_1, l_2 + m_2, g \cdot h).
\]
(4.32)

With these choices, we may pull the canonical gerbe \( \mathcal{G} \) back to \( T_{\lambda, \mu}^{\nu} \) along the three maps \( p_1, p_2 \) and \( m \). As usual, it proves more convenient to work in the equivariant formalism. For that purpose, we introduce the subsets
\[
\hat{T}_{\lambda, \mu}^{\nu} = \{ (\lambda, \mu, \nu) \} \times \text{SU}(2) \subset \mathcal{F} \times \text{SU}(2).
\]
(4.33)

Over these sets, we need surjective submersions \( \pi_{Y T_{\lambda, \mu}^{\nu}} : Y \hat{T}_{\lambda, \mu}^{\nu} \to \hat{T}_{\lambda, \mu}^{\nu} \) together with a map \( \hat{\tau} \) that covers \( \tau \) restricted to \( \hat{T}_{\lambda, \mu}^{\nu} \). In analogy with (2.27), we define the sets
\[
\hat{O}_{i,j} = ([0, k] \setminus \{\lambda_{i-j}\}) \times ([0, k] \setminus \{\lambda_{1-j}\}) \times ([0, k] \setminus \{\lambda_{1-k}\}) \times \text{SU}(2).
\]
(4.34)

We now have the natural choices
\[
Y \hat{T}_{\lambda, \mu}^{\nu} = \bigsqcup_{k=(l,m) \in \{0,1\}^2} \hat{T}_{\lambda, \mu; k}^{\nu}, \quad \hat{T}_{\lambda, \mu; k}^{\nu} = \hat{T}_{\lambda, \mu}^{\nu} \cap \hat{O}_{l,m}^{1+m},
\]
\[
Y^{[2]} \hat{T}_{\lambda, \mu}^{\nu} = \bigsqcup_{\bar{k}_1, \bar{k}_2 \in \{0,1\}^2} \hat{T}_{\lambda, \mu; k_1, k_2}^{\nu}, \quad \hat{T}_{\lambda, \mu; k_1, k_2}^{\nu} = \hat{T}_{\lambda, \mu}^{\nu} \cap \hat{O}_{l_1,m_1}^{1+m_1} \cap \hat{O}_{l_2,m_2}^{1+m_2},
\]
(4.35)
alongside
\[
\hat{\tau}(\bar{k}, \lambda, \mu, \nu, h) = (\bar{k}, \tau(\lambda, \mu, \nu, h)), \quad \hat{\tau}^{[2]}(\bar{k}_1, \bar{k}_2, \lambda, \mu, \nu, h) = (\bar{k}_1, \bar{k}_2, \tau(\lambda, \mu, \nu, h)).
\]
(4.36)

We want to lift both sides of (4.23) to the respective surjective submersions. To this end, we use the map \( \hat{c} \) from (2.38) and introduce the maps
\[
\hat{q}_{(\lambda), \mu}^{\nu}(\bar{k}, \lambda, \mu, \nu, h) = (l, \lambda, h), \quad \hat{q}_{\lambda, (\mu)}^{\nu}(\bar{k}, \lambda, \mu, \nu, h) = (m, \mu, h \cdot a_{\lambda, \mu}^{\nu}),
\]
\[
\hat{q}_{\lambda, \mu}^{(\nu)}(\bar{k}, \lambda, \mu, \nu, h) = (l + m, \nu, h \cdot b_{\lambda, \mu}^{\nu}),
\]
(4.37)
as well as their counterparts for fibred squares of the surjective submersions,
\[
\hat{c}^{[2]}(i, j, \lambda, h) = (i, j, \text{Ad}_h(t_{\lambda}))
\]
(4.38)

and
\[
\hat{q}_{(\lambda), \mu}^{\nu}[2](\bar{k}_1, \bar{k}_2, \lambda, \mu, \nu, h) = (l_1, l_2, \lambda, h), \quad \hat{q}_{\lambda, (\mu)}^{\nu}[2](\bar{k}_1, \bar{k}_2, \lambda, \mu, \nu, h) = (m_1, m_2, \mu, h \cdot a_{\lambda, \mu}^{\nu}),
\]
\[
\hat{q}_{\lambda, \mu}^{(\nu)}[2](\bar{k}_1, \bar{k}_2, \lambda, \mu, \nu, h) = (l_1 + m_1, l_2 + m_2, \nu, h \cdot b_{\lambda, \mu}^{\nu}).
\]
(4.39)
We may next pull back restrictions of the bundle $L$ along the maps $p_1^{[2]} \circ \tau^{[2]}, p_2^{[2]} \circ \tau^{[2]}$ and $m^{[2]} \circ \tau^{[2]}$ to the common base $Y^{[2]} T_{\lambda, \mu}^\nu$. By construction, we have that $\tilde{L}_{l_1,l_2} |_{\tilde{c}_{\lambda_{l_1,l_2}}} \simeq \tau^{[2]}_* L |_{c_{\lambda_{l_1,l_2}}}$ as equivariant line bundles. Thus, since $p_1^{[2]} \circ \tau^{[2]} = \tau^{[2]} \circ q_{(\lambda, \mu)}^{\nu}$, we may equivalently pull back the corresponding equivariant bundles

$$(p_1^{[2]} \circ \tau^{[2]})^* L |_{c_{\lambda_{l_1,l_2}}} \simeq q_{(\lambda, \mu)}^{\nu}[2]^* \tilde{L}_{l_1,l_2} |_{\tilde{c}_{\lambda_{l_1,l_2}}},$$

and similarly for $\tilde{L}_{m_1,m_2}$ and $\tilde{L}_{l_1+l_m+1, l_2 + m_2}$. For the reader’s convenience, and for later reference, we write out the action of the isotropy group $S_{\lambda, \mu}^\nu$ on the pullback bundles that enables us to descend them to the quotient $T_{\lambda, \mu}^\nu$. The actions take the form (with $g \in S_{\lambda, \mu}^\nu$ and $z \in \mathbb{C}$)

$$(\bar{k}_1, \bar{k}_2, \lambda, \mu, \nu, h, z).g = (\bar{k}_1, \bar{k}_2, \lambda, \mu, \nu, h \cdot g, \chi_{l_1,l_2}(g) \cdot z)$$

for $q_{(\lambda, \mu)}^{\nu}[2]^* \tilde{L}_{l_1,l_2} |_{\tilde{c}_{\lambda_{l_1,l_2}}},$

and

$$(\bar{k}_1, \bar{k}_2, \lambda, \mu, \nu, h, z).g = (\bar{k}_1, \bar{k}_2, \lambda, \mu, \nu, h \cdot g, \chi_{l_1+l_2+m_2}(\text{Ad}_{b_{\lambda,\mu}^{\nu}}^{-1}(g)) \cdot z)$$

for $q_{(\lambda, \mu)}^{\nu}[2]^* \tilde{L}_{l_1+l_1+l_2+m_2} |_{\tilde{c}_{\lambda_{l_1+l_2+m_2}}},$ and

$$(\bar{k}_1, \bar{k}_2, \lambda, \mu, \nu, h, z).g = (\bar{k}_1, \bar{k}_2, \lambda, \mu, \nu, h \cdot g, \chi_{l_1+l_2+m_2}(\text{Ad}_{b_{\lambda,\mu}^{\nu}}^{-1}(g)) \cdot z)$$

for $q_{(\lambda, \mu)}^{\nu}[2]^* \tilde{L}_{l_1+1, l_2+1+m_2} |_{\tilde{c}_{\lambda_{l_1+l_2+m_2}}}.$

Define the map $\varepsilon : SU(2) \rightarrow \{-1, 1\}$ as

$$\varepsilon(g) = \begin{cases} 
-1 & \text{if } g \in U(1) \cdot i \sigma_2 \\
1 & \text{otherwise}
\end{cases}.$$

Using $\varepsilon$, we may rewrite the action of the isotropy subgroup on the fibre of $q_{(\lambda, \mu)}^{\nu}[2]^* \tilde{L}_{l_1+l_1+l_2+m_2} |_{\tilde{c}_{\lambda_{l_1+l_2+m_2}}}$ in the useful form

$$z \mapsto \chi_{l_1+l_2+m_2}(g) \varepsilon(a_{\lambda,\mu}^{\nu}) \cdot z,$$

$$z \mapsto \chi_{l_1+l_2+m_2}(g) \varepsilon(b_{\lambda,\mu}^{\nu}) \cdot z,$$

respectively. To see this, we distinguish and check three cases:

- $(\lambda, \mu, \nu) \in \partial_0 \mathcal{P}$: The characters entering the above expressions are the trivial ones, $\chi_0 = 1$, and so there remains nothing to demonstrate.

- $(\lambda, \mu, \nu) \in \partial_2 \mathcal{P}$: Here, $S_{\lambda, \mu}^\nu = U(1)$, and both $a_{\lambda,\mu}^{\nu}$ and $b_{\lambda,\mu}^{\nu}$ take values in the set $\{e\} \cup U(1) \cdot i \sigma_2$, and so (4.43) follows from the identities $\text{Ad}_{a_{\lambda,\mu}^{\nu}}^{-1}(\sigma_3) = \varepsilon(x) \sigma_3$ for $x \in \{e, i \sigma_2\}$.

- $(\lambda, \mu, \nu) \in \mathcal{P}$: In this case, $S_{\lambda, \mu}^\nu = \mathbb{Z}_2$, and neither $a_{\lambda,\mu}^{\nu}$ nor $b_{\lambda,\mu}^{\nu}$ belongs to $U(1) \cdot i \sigma_2$. The resulting equalities $\varepsilon(a_{\lambda,\mu}^{\nu}) = 1 = \varepsilon(b_{\lambda,\mu}^{\nu})$ are consistent with $\text{Ad}_{a_{\lambda,\mu}^{\nu}}^{-1}(g) = g = \text{Ad}_{b_{\lambda,\mu}^{\nu}}^{-1}(g)$ implied by $g \in \mathbb{Z}_2$.  

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**Proposition 4.10.**

Let $M$ be a manifold with the total space of the bundle with the properties listed, e.g., in [Br, Def. 2.2.4]. Furthermore, a track of the invariant 1-form written in terms of the function $\hat{\varepsilon}$.

We check that the $\rho$-diffeomorphisms $\rho_\zeta : M \to M$ for $\zeta \in \mathbb{Z}_2$ such that $\rho_\zeta \circ \rho_\xi = \rho_{\zeta \xi}$ for all $\zeta, \xi \in \mathbb{Z}_2$.

**Definition 4.9.** Let $M$ be a manifold with $\mathbb{Z}_2$-action $\rho$. 

(i) A $\mathbb{Z}_2$-equivariant line bundle over $M$ is a pair $(L, \hat{\rho})$, where $\pi : L \to M$ is a line bundle, and $\hat{\rho} : \mathbb{Z}_2 \times L \to L$ is an action of $\mathbb{Z}_2$ by diffeomorphisms, i.e. $\hat{\rho}_\zeta \circ \hat{\rho}_\xi = \hat{\rho}_{\zeta \xi}$ for all $\zeta, \xi \in \mathbb{Z}_2$. The action preserves fibres, is linear and unitary on the fibres, and preserves the connection form $\hat{A} \in \Omega^1(L)$ in the sense that $\hat{\rho}^*_\zeta \hat{A} = \hat{A}$ for all $\zeta \in \mathbb{Z}_2$.

(ii) Let $(L, \hat{\rho})$ and $(L', \hat{\rho}')$ be two $\mathbb{Z}_2$-equivariant line bundles over $M$. A $\mathbb{Z}_2$-equivariant isomorphism $L \to L'$ is an isomorphism $\phi : L \to L'$ of hermitean line bundles with unitary connection that intertwines the $\mathbb{Z}_2$-action, i.e. $\hat{\rho}_\zeta \circ \phi = \phi \circ \hat{\rho}_\zeta$ for all $\zeta \in \mathbb{Z}_2$.

Altogether, given a manifold with $\mathbb{Z}_2$-action, we obtain the category (or groupoid) of $\mathbb{Z}_2$-equivariant line bundles over $M$, with objects and morphisms as described in the above definition. If the $\mathbb{Z}_2$-action is free, the quotient space $M/\mathbb{Z}_2$ is again a manifold, and the category of line bundles over $M/\mathbb{Z}_2$ is equivalent to the category of $\mathbb{Z}_2$-equivariant line bundles over $M$.

Let $M$ be a manifold with $\mathbb{Z}_2$-action $\rho$, and let $A$ be a 1-form on $M$ that is invariant under $\rho$, i.e. $\rho_\zeta A = A$ for all $\zeta \in \mathbb{Z}_2$. Consider the trivial bundle $M \times \mathbb{C}$ with connection $\nabla = d + \frac{1}{2} A$ and the maps $\hat{\rho}_\pm : \mathbb{Z}_2 \times M \to M$ given by

$$\hat{\rho}_\pm(m, z) = (\rho_\zeta(m), (\pm 1)^{\zeta} \cdot z),$$

written in terms of the function $\varepsilon : \mathbb{Z}_2 \to \{0, 1\}$ with values

$$\varepsilon_\zeta = \begin{cases} 0 & \text{if } \zeta = e \\ 1 & \text{if } \zeta = -e \end{cases}.$$

One checks that the $\hat{\rho}_\pm$ each turn $M \times \mathbb{C}$ into a $\mathbb{Z}_2$-equivariant line bundle. In order to keep track of the invariant 1-form $A$, we denote them as $(M \times \mathbb{C}, A, \hat{\rho}_\pm)$. We have

**Proposition 4.10.** Let $M$ be a 2-connected manifold with $\mathbb{Z}_2$-action. Every $\mathbb{Z}_2$-equivariant line bundle over $M$ is $\mathbb{Z}_2$-equivariantly isomorphic to one of the form $(M \times \mathbb{C}, A, \hat{\rho}_\pm)$.

**Proof.** Since $M$ is 2-connected, the line bundle $L$ with connection $\nabla_L$ is isomorphic to the trivial bundle $M \times \mathbb{C} \to M$ with connection $\nabla' = d + \frac{1}{4} A'$ for some 1-form $A'$ on $M$. To obtain a $\mathbb{Z}_2$-invariant connection, we average the 1-form with respect to the $\mathbb{Z}_2$-action, i.e. we define $\nabla = d + \frac{1}{2} A$, where $A = \frac{1}{2} (A' + \rho_\zeta^* A')$. Since the curvature of $L$ is $\mathbb{Z}_2$-invariant, we have $dA = dA'$. The set of isomorphism classes of line bundles with a given curvature is
isomorphic to $H^1(M, U(1))$, which is trivial as $M$ is 1-connected. Thus, also $M \times C \to M$ with connection $\nabla$ is isomorphic to $L$. Let $f : M \times C \to L$ be this isomorphism. Denote by $\hat{A}_L$ the connection form of $\nabla_L$, and by $\hat{A}$ that of $\nabla$, so that $\hat{A} = f^*\hat{A}_L$.

Define a $\mathbb{Z}_2$-action $\sigma$ on $M \times C \to M$ as $\sigma_\zeta = f^{-1} \circ \tilde{\rho}_\zeta \circ f$. Since $\hat{A}_L$ obeys $\tilde{\rho}_\zeta \hat{A}_L = \hat{A}_L$, we also have $\sigma_\zeta \hat{A} = \hat{A}$. Let us write the $\mathbb{Z}_2$-action $\sigma$ as $\sigma_\zeta(m, z) = (\rho_\zeta(m), s_\zeta(m) \cdot z)$ for some $U(1)$-valued map $s_\zeta$. For a trivial bundle, the connection form is given by (cf. [33 Sect. 2.2]) $\hat{A}(m, z) = iz^{-1}dz + A(m)$, and so the condition $\sigma_\zeta \hat{A} = \hat{A}$ implies $\rho_\zeta A = -i s^{-1}_\zeta ds_\zeta + A$, i.e. $ds_\zeta = 0$. Thus, $s_\zeta$ is locally constant, and since $M$ is connected, it is globally constant. The relation $\sigma_\zeta \circ \sigma_\zeta = \sigma_{\zeta \zeta}$ implies that $s_\zeta$ is a $\mathbb{Z}_2$-character. Finally, the isomorphism $f$ is $\mathbb{Z}_2$-equivariant by construction.

**Stable isomorphisms between pullback gerbes on $T_{\lambda,\mu}^\nu$**

The pullback gerbes $p_1^*G, p_2^*G_2$ and $m^*G$ over the product manifold $SU(2) \times SU(2)$ are closely related, as indicated by the Polyakov–Wiegmann identity [PW]

$$p_1^*H + p_2^*H = m^*H + d\rho, \quad \rho = \frac{k}{4\pi} \text{tr}(p_1^* \theta_L \wedge p_2^* \theta_R), \quad \theta_R = \text{Ad}_L \theta_L. \quad (4.48)$$

In fact, we have

**Proposition 4.11.** There exists a stable isomorphism

$$\mathcal{M} : p_1^*G \star p_2^*G \sim m^*G \star \mathcal{I}(\rho) \quad (4.49)$$

between pull-back gerbes on $SU(2) \times SU(2)$, and this stable isomorphism is unique up to a 2-isomorphism.

**Proof.** By Proposition 3.3, stable-isomorphism classes of gerbes with a given curvature are in a one-to-one correspondence with elements of $H^2(SU(2) \times SU(2), U(1))$, which is trivial. Thus, (4.48) implies that $\mathcal{M}$ exists. By Proposition 4.3, 2-isomorphism classes of stable isomorphisms, in turn, form a torsor over $H^1(SU(2) \times SU(2), U(1))$, which is also trivial. Hence, $\mathcal{M}$ is unique up to a 2-isomorphism.

The stable isomorphism $\mathcal{M}$ constitutes part of the data of a multiplicative structure on $G$ [Ca, Wa]; here, we shall only require $\mathcal{M}$. An explicit expression for $\mathcal{M}$ is currently not known. In what follows, we shall give such an expression for the restriction of $\mathcal{M}$ to each of the subsets $T_{\lambda,\mu}^\nu$ of $SU(2) \times SU(2)$. We shall do so in two steps: First, we determine all 2-isomorphism classes of stable isomorphisms $p_1^*G \star p_2^*G \sim m^*G \star \mathcal{I}(\rho)$ restricted to $T_{\lambda,\mu}^\nu$ (Lemmas 4.13 and 4.14), and, then, we identify those which arise as a restriction of $\mathcal{M}$ (Lemmas 4.15 and 4.16).

In virtue of Proposition 4.3, taken in conjunction with (4.27), there are two 2-isomorphism classes of stable isomorphisms $p_1^*G \star p_2^*G \sim m^*G \star \mathcal{I}(\rho)$ over $T_{\lambda,\mu}^\nu$ for any triple $(\lambda, \mu, \nu) \in \mathcal{F}$, and a unique class for $(\lambda, \mu, \nu) \in \partial \mathcal{F}$. Below, we give a representative of each of these classes in terms of equivariant bundles over the simply connected covers $\hat{T}_{\lambda,\mu}^\nu$ of $T_{\lambda,\mu}^\nu$.

The surjective submersion of the gerbe $(p_1^*G \star p_2^*G)|_{T_{\lambda,\mu}^\nu}$ is given by $YT_{\lambda,\mu}^\nu \times T_{\lambda,\mu}^\nu Y T_{\lambda,\mu}^\nu = Y^2 T_{\lambda,\mu}^\nu$, and that of $(m^*G \star \mathcal{I}(\rho))|_{T_{\lambda,\mu}^\nu}$ by $YT_{\lambda,\mu}^\nu \times T_{\lambda,\mu}^\nu T_{\lambda,\mu}^\nu = YT_{\lambda,\mu}^\nu$. The surjective submersion
of the stable isomorphism is therefore $Y^{[2]}T_{\lambda,\mu} \times T_{\lambda,\mu} \times Y T_{\lambda,\mu} = Y^{[3]}T_{\lambda,\mu}$. In the equivariant formulation, the line bundle on $Y^{[3]}T_{\lambda,\mu}$ and the isomorphism of pullback line bundles on $Y^{[6]}T_{\lambda,\mu}$ are described as follows: Introduce the canonical projections $\text{pr}_i : Y^{[3]}T_{\lambda,\mu} \to Y T_{\lambda,\mu}$, $i \in \{1, 2, 3\}$ and $\pi_{\text{SU}(2)} : Y^{[3]}T_{\lambda,\mu} \to \text{SU}(2)$. The first datum that we need to give is a $S_{\lambda,\mu}$-equivariant line bundle $\tilde{E}_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3}$ with connection $\nabla_{\tilde{E}_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3}}$ of curvature

$$\text{curv} (\nabla_{\tilde{E}_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3}}) = \text{pr}_3^* \tilde{q}_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3} \tilde{B} |_{\tilde{\xi}_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3}} + \text{pr}_3^* \pi_*^* \nabla^* \tau^* \rho |_{\tilde{E}_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3}} - \text{pr}_1^* \tilde{q}_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3} \tilde{B} |_{\tilde{\xi}_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3}} - \text{pr}_2^* \tilde{q}_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3} \tilde{B} |_{\tilde{\xi}_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3}}. \quad (4.50)$$

Here,

$$\tilde{\tilde{\gamma}}_{\lambda,\mu} = q_{\lambda,\mu}^*(\lambda) \tilde{Q} |_{\tilde{\xi}_{\lambda,\mu}} + \tau^* \rho |_{\tilde{E}_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3}} - q_{\lambda,\mu}^* \tilde{Q} |_{\tilde{\xi}_{\lambda,\mu}} - q_{\lambda,\mu}^* \tilde{Q} |_{\tilde{\xi}_{\lambda,\mu}}, \quad (4.51)$$

and

$$\tilde{\tilde{\gamma}}_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3} (h) = i \text{tr} \left[ (\Lambda - \lambda_1) \Lambda + (\mu - \lambda m_2) \text{Ad}_{\lambda,\mu}(A) - (\nu - \lambda_3 + m_3) \text{Ad}_{\lambda,\mu}(A) \right] \theta_L (h). \quad (4.52)$$

In order to proceed further, we need

**Lemma 4.12.**

$$\tilde{\tilde{\gamma}}_{\lambda,\mu} \equiv 0. \quad (4.53)$$

**Proof.** The tangent space of $T_{\lambda,\mu}$ at a point $(\lambda, \mu, \nu, h)$ is spanned by vectors $\mathcal{V}_A(\lambda, \mu, \nu, h) = R_A(h)$, $A \in \{1, 2, 3\}$, where $R_A$ are the standard right-invariant vector fields on $\text{SU}(2)$, dual to the right-invariant Maurer–Cartan 1-forms, $R_A \cdot \theta_R = i \sigma_A$. The vanishing of $\tilde{\tilde{\gamma}}_{\lambda,\mu} \equiv 0$ can be rephrased equivalently as the identity

$$\mathcal{V}_A \cdot \tilde{\tilde{\gamma}}_{\lambda,\mu}(\lambda, \mu, \nu, h) = 0 \quad \text{for all } A \in \{1, 2, 3\}. \quad (4.54)$$

Using (3.16) and the shorthand notation $(g_1, g_2) = \tau(\lambda, \mu, \nu, h)$, we readily compute

$$\mathcal{V}_A \cdot q_{\lambda,\mu}^* \tilde{Q} = \frac{i}{4 \pi} \text{tr} (\sigma_A (\text{Ad}_{g_1} - \text{Ad}_{g_1^{-1}}) \theta_R (h)), \quad (4.55)$$

The 2-form $\rho$ pulls back to $T_{\lambda,\mu}$ as

$$\tau^* \rho(\lambda, \mu, \nu, h) = - \frac{i}{4 \pi} \text{tr} (\theta_R (h) \wedge (\text{id}_{\text{su}(2)} - \text{Ad}_{g_1}) \circ (\text{id}_{\text{su}(2)} - \text{Ad}_{g_2}) \theta_R (h)), \quad (4.56)$$

and so we also have

$$\mathcal{V}_A \cdot \tau^* \rho(\lambda, \mu, \nu, h) = \frac{i}{4 \pi} \text{tr} (\sigma_A (\text{Ad}_{g_1} - \text{Ad}_{g_1^{-1}} + \text{Ad}_{g_2} - \text{Ad}_{g_2^{-1}} - \text{Ad}_{g_1 g_2} + \text{Ad}_{(g_1 g_2)^{-1}}) \theta_R (h)). \quad (4.57)$$

Putting all the above formulæ together, we obtain the desired result, (4.53), which concludes the proof.  

\[ \square \]
By the above lemma, the curvature has a global primitive, and since $\widetilde{F}^\nu_{\lambda,\mu;k_1,k_2,k_3}$ are simply connected, the line bundle under consideration is trivial,
\[ E^\nu_{\lambda,\mu;k_1,k_2,k_3} = \widetilde{F}^\nu_{\lambda,\mu;k_1,k_2,k_3} \times \mathbb{C}. \] (4.58)
The connection is given by
\[ \nabla E^\nu_{\lambda,\mu;k_1,k_2,k_3} = d + \frac{1}{\pi^*} \pi^* \tau^\nu_{\lambda,\mu;k_1,k_2,k_3}(h). \] (4.59)

The second piece of data of a stable isomorphism between the pullback gerbes is – in the equivariant formulation – a family of $S_{\lambda,\mu}$-equivariant isomorphisms
\[ \alpha^\nu_{\lambda,\mu;k_1,k_2,k_3,k_4,k_5,k_6} : \text{pr}_{1,4}^* \widetilde{q}^\nu_{\lambda,\mu}(\alpha)^{\nu} \otimes \text{pr}_{2,5}^* \widetilde{q}^\nu_{\lambda,\mu}(\beta) \otimes \text{pr}_{4,5,6}^* \widetilde{E}^\nu_{\lambda,\mu;k_4,k_5,k_6} \sim \text{pr}_{1,2,3}^* \widetilde{E}^\nu_{\lambda,\mu;k_1,k_2,k_3} \otimes \text{pr}_{3,4,6}^* \widetilde{q}^\nu_{\lambda,\mu}(\gamma) \] (4.60)
of line bundles over $\widetilde{F}^\nu_{\lambda,\mu;k_1,k_2,k_3,k_4,k_5,k_6}$, the latter space being equipped with the canonical projections $\text{pr}_{i,j}(k_1, \ldots, k_6, \lambda, \mu, \nu, h) = (k_1, k_j, \lambda, \mu, \nu, h)$ and $\text{pr}_{i,j,k}(k_1, \ldots, k_6, \lambda, \mu, \nu, h) = (k_1, k_j, k_k, \lambda, \mu, \nu, h)$. As demonstrated below, the $\alpha$ are of the form
\[ \alpha^\nu_{\lambda,\mu;k_1,k_2,k_3,k_4,k_5,k_6}(k_1, k_2, k_3, k_4, k_5, k_6, \lambda, \mu, \nu, h, z \otimes z' \otimes z'') = (k_1, k_2, k_3, k_4, k_5, k_6, \lambda, \mu, \nu, h, 1 \otimes z \cdot z' \cdot z''). \] (4.61)
It is manifest that the maps $\alpha$ are unitary, associative and compatible with the groupoid structures of the gerbes involved. It is also easy to see that they preserve the bundle connections.

It remains to give the $S_{\lambda,\mu}$-action on $\widetilde{E}^\nu_{\lambda,\mu;k_1,k_2,k_3}$ and verify equivariance of $\nabla \widetilde{E}^\nu_{\lambda,\mu;k_1,k_2,k_3}$ and $\alpha^\nu_{\lambda,\mu;k_1,k_2,k_3,k_4,k_5,k_6}$. Once this is done, we can descend the data to $T^\nu_{\lambda,\mu}$. We shall denote the resulting stable isomorphism as
\[ \Phi^\nu_{\lambda,\mu}(\widetilde{E}^\nu_{\lambda,\mu;k_1,k_2,k_3}) : (p_1^* G \ast p_2^* G)|_{T^\nu_{\lambda,\mu}} \sim (m^* G \ast I^\nu(\rho))|_{T^\nu_{\lambda,\mu}}. \] (4.62)

For $(\lambda, \mu, \nu) \in \partial \mathcal{F}$, one verifies that
\[ \text{Ad}_{a^\nu_{\lambda,\mu}}(\sigma_3) = \varepsilon(a^\nu_{\lambda,\mu}) \sigma_3 \quad \text{and} \quad \text{Ad}_{b^\nu_{\lambda,\mu}}(\sigma_3) = \varepsilon(b^\nu_{\lambda,\mu}) \sigma_3, \] (4.63)
with $\varepsilon(\cdot)$ as defined in (4.44). Comparing the exponents in (4.20), we thus obtain the relation
\[ \lambda + \varepsilon(a^\nu_{\lambda,\mu}) \mu - \varepsilon(b^\nu_{\lambda,\mu}) \nu = 2k n^\nu_{\lambda,\mu}, \quad n^\nu_{\lambda,\mu} \in \mathbb{Z}. \] (4.64)
The definition of $n^\nu_{\lambda,\mu}$ enters the formulation of the following lemma.

**Lemma 4.13.** For $(\lambda, \mu, \nu) \in \partial \mathcal{F}$, any 1-isomorphism $(p_1^* G \ast p_2^* G)|_{T^\nu_{\lambda,\mu}} \sim (m^* G \ast I^\nu(\rho))|_{T^\nu_{\lambda,\mu}}$ is 2-isomorphic to $\Phi^\nu_{\lambda,\mu}(\widetilde{E}^\nu_{\lambda,\mu;k_1,k_2,k_3})$ with
\[ \widetilde{E}^\nu_{\lambda,\mu;k_1,k_2,k_3} = \pi^* \widetilde{K}^{\nu_{\lambda,\mu} - \lambda_1 - \varepsilon(a^\nu_{\lambda,\mu}) \lambda_2 + \varepsilon(b^\nu_{\lambda,\mu}) \lambda_3 + m_3}, \] (4.65)
and with the $S_{\lambda,\mu}$-equivariant structure inherited from $\widetilde{K}^{\nu_{\lambda,\mu} - \lambda_1 - \varepsilon(a^\nu_{\lambda,\mu}) \lambda_2 + \varepsilon(b^\nu_{\lambda,\mu}) \lambda_3 + m_3}$, as defined in (2.44).
Proof. Let us denote $\xi_{k_1,k_2,k_3} = 2k n_{\lambda,\mu} - \lambda_1 - \epsilon(a_{\lambda,\mu}^\nu) \lambda_{m_2} + \epsilon(b_{\lambda,\mu}^\nu) \lambda_{l_3+m_3}$. When $(\lambda, \mu, \nu) \in \partial \mathcal{F}$, the expression (4.62) for the 1-form defining the connection $\nabla_{\xi_{k_1,k_2,k_3}} = d + \frac{1}{2} \pi^*_{\text{SU}(2)} \tilde{A}_{\lambda,\mu}^\nu \xi_{k_1,k_2,k_3}(h)$ on $\tilde{E}_{\lambda,\mu}^\nu_{k_1,k_2,k_3}$ simplifies due to the relations (4.66) and (4.64), giving

$$\tilde{A}_{\lambda,\mu}^\nu_{k_1,k_2,k_3} = i \xi_{k_1,k_2,k_3} \text{tr}(\Lambda \theta_L(h)).$$

As $\tilde{E}_{\lambda,\mu}^\nu_{k_1,k_2,k_3}$ is a trivial bundle, we infer, by comparing the respective connection 1-forms, that it can be identified with the pullback of the equivariant bundle $\pi\xi_{k_1,k_2,k_3} \to \text{SU}(2)$ along the (bijective) projection $\pi_{\text{SU}(2)} : \tilde{\mathcal{T}}_{\lambda,\mu}^\nu \to \text{SU}(2)$. The $\mathcal{S}_{\lambda,\mu}^\nu$-equivariant structure on $\tilde{E}_{\lambda,\mu}^\nu_{k_1,k_2,k_3}$ is obtained by restricting that on $\pi\xi_{k_1,k_2,k_3}$ to $\mathcal{S}_{\lambda,\mu}^\nu$. This is possible because the isotropy subgroup $\text{SU}(2)\xi_{k_1,k_2,k_3}$ of $\pi\xi_{k_1,k_2,k_3}$ contains $\mathcal{S}_{\lambda,\mu}^\nu$ as it always contains $\text{U}(1)$, and equals $\text{SU}(2)$ for $\lambda, \mu \in \{0, k\}$. According to (4.41)–(4.43), the $\mathcal{S}_{\lambda,\mu}^\nu$-equivariance of $\tilde{\alpha}_{\lambda,\mu}^\nu_{k_1,k_2,k_3,k_4,k_5,k_6}$ amounts to

$$\tilde{\alpha}_{\lambda,\mu}^\nu_{k_1,k_2,k_3,k_4,k_5,k_6} \left( h \cdot s, \chi_{\lambda_1,t_4}(s) \cdot z \otimes \chi_{\lambda_2,m_0}(s) \epsilon(a_{\lambda,\mu}^\nu) \cdot z' \otimes \chi_{\lambda_3,k_5}(s) \cdot z'' \right)$$

$$= (h \cdot s, \chi_{\xi_{k_1,k_2,k_3}}(s) \otimes \chi_{\lambda_2,m_0}(s) \epsilon(b_{\lambda,\mu}^\nu) z \cdot z' \cdot z'')$$

for $s \in \mathcal{S}_{\lambda,\mu}^\nu$. The above identity follows immediately from

$$\lambda_1,t_4 + \epsilon(a_{\lambda,\mu}^\nu) \lambda_{m_2,m_0} + 2k n_{\lambda,\mu} - \lambda_1 - \epsilon(a_{\lambda,\mu}^\nu) \lambda_{m_0} + \epsilon(b_{\lambda,\mu}^\nu) \lambda_{l_3+m_3}$$

$$= 2k n_{\lambda,\mu} - \lambda_1 - \epsilon(a_{\lambda,\mu}^\nu) \lambda_{m_2} + \epsilon(b_{\lambda,\mu}^\nu) \lambda_{l_3+m_3} + \epsilon(b_{\lambda,\mu}^\nu) \lambda_{l_3+m_3,m_3}.$$

Lemma 4.14. For $(\lambda, \mu, \nu) \in \mathcal{F}$, any 1-isomorphism $(p_1^* \mathcal{G} \ast p_2^* \mathcal{G})|_{\mathcal{T}_{\lambda,\mu}^\nu} \sim\sim (m^* \mathcal{G} \ast \mathcal{I}(\rho))|_{\mathcal{T}_{\lambda,\mu}^\nu}$ is 2-isomorphic to $\Phi_{\lambda,\mu}^\nu_{k_1,k_2,k_3}$ with

$$\tilde{E}_{\lambda,\mu}^\nu_{k_1,k_2,k_3} = \pi^*_{\text{SU}(2)}(\text{SU}(2) \times \mathbb{C})$$

the trivial line bundle with the connection 1-form given by (4.62), and a $\mathbb{Z}_2$-equivariant structure inherited from that on $\text{SU}(2) \times \mathbb{C} \to \text{SU}(2)$ which lifts the action of $\mathcal{S}_{\lambda,\mu}^\nu \cong \mathbb{Z}_2$ on the base $\text{SU}(2)$ to the total space as

$$(\text{SU}(2) \times \mathbb{C}) \times \mathbb{Z}_2 \to \text{SU}(2) \times \mathbb{C} : (h, z, \zeta) \mapsto (h \cdot \zeta, \chi_{\lambda,\mu}^\nu_{k_1,k_2,k_3}(\zeta) \cdot z),$$

$$\chi_{\lambda,\mu}^\nu_{k_1,k_2,k_3} : \mathbb{Z}_2 \to \text{U}(1) : \zeta \mapsto (-1)^{\epsilon(\lambda_3+m_3,\lambda_3-m_2,\epsilon_{\lambda,\mu}^\nu)}$$

and where either $\epsilon_{\lambda,\mu}^\nu = 0$ or $\epsilon_{\lambda,\mu}^\nu = 1$. 32
Proof. For notational brevity, we shall, in this proof, omit the isomorphism \( \pi_{SU(2)} : \tilde{T}^\nu_{\lambda,\mu;k_1,k_2,k_3} = \{(k_1, k_2, k_3, \lambda, \mu, \nu)\} \times SU(2) \sim SU(2) \) and identify \( \tilde{T}^\nu_{\lambda,\mu;k_1,k_2,k_3} \) with \( SU(2) \). Each of the spaces \( \tilde{T}^\nu_{\lambda,\mu;k_1,k_2,k_3} \) carries a \( \mathbb{Z}_2 \)-action given by multiplication by elements of the centre of \( SU(2) \). The 1-forms \( \tilde{A}^\nu_{\lambda,\mu;k_1,k_2,k_3} \) are \( \mathbb{Z}_2 \)-invariant.

We need to give a \( \mathbb{Z}_2 \)-equivariant line bundle on each of the spaces \( \tilde{T}^\nu_{\lambda,\mu;k_1,k_2,k_3} \), with curvatures \( d\tilde{A}^\nu_{\lambda,\mu;k_1,k_2,k_3} \). Since \( \tilde{T}^\nu_{\lambda,\mu;k_1,k_2,k_3} \) is 2-connected, each such bundle is, by Proposition \([1,10]\), \( \mathbb{Z}_2 \)-equivariantly isomorphic to a trivial bundle with \( \mathbb{Z}_2 \)-action as given in the proposition. Specifically, up to a \( \mathbb{Z}_2 \)-equivariant isomorphism, the most general \( \mathbb{Z}_2 \)-equivariant line bundle is

\[
\tilde{E}^\nu_{\lambda,\mu;k_1,k_2,k_3} = SU(2) \times \mathbb{C},
\]

and the \( \mathbb{Z}_2 \)-action, given by

\[
(\zeta, (h, z)) \mapsto (h \cdot \zeta, \chi_{\lambda,\mu}^\nu_{k_1,k_2,k_3}(\zeta) \cdot z),
\]

is determined by a family of \( \mathbb{Z}_2 \)-characters \( \chi_{\lambda,\mu}^\nu_{k_1,k_2,k_3} \). These characters are constrained by the requirement that \( \tilde{\alpha}^\nu_{\lambda,\mu;k_1,k_2,k_3} \) be a \( \mathbb{Z}_2 \)-equivariant bundle isomorphism. As a bundle isomorphism, it is necessarily of the form

\[
\tilde{\alpha}^\nu_{\lambda,\mu;k_1,k_2,k_3,k_4,k_5,k_6}(h, z \otimes z' \otimes z'') = (h, 1 \otimes \psi_{\lambda,\mu}^\nu_{k_1,k_2,k_3,k_4,k_5,k_6} \cdot z \cdot z' \cdot z'')
\]

for some constant phases \( \psi_{\lambda,\mu}^\nu_{k_1,k_2,k_3,k_4,k_5,k_6} \in U(1) \). This follows from the comparison of the respective 1-forms on both sides of \((4.60)\) (implying local constancy of the phases), in conjunction with connectedness of their common base (implying their global constancy). Equivariance of \( \tilde{\alpha}^\nu_{\lambda,\mu;k_1,k_2,k_3,k_4,k_5,k_6} \) with respect to the action of the isotropy subgroup \( \mathcal{S}_{\lambda,\mu} \) on the bundles involved amounts to the requirement that

\[
\tilde{\alpha}^\nu_{\lambda,\mu;k_1,k_2,k_3,k_4,k_5,k_6}(h \cdot \zeta, \chi_{\lambda,\mu}^\nu_{k_1,k_2,k_3,k_4,k_5,k_6}(\zeta) \cdot z \cdot z' \cdot z'') \cdot \chi_{\lambda,\mu}^\nu_{k_1,k_2,k_3,k_4,k_5,k_6}(\zeta) \cdot z'' \cdot z' \cdot z''
\]

hold true for arbitrary \( \zeta \in \mathbb{Z}_2 \). This is equivalent to the equality

\[
\chi_{\lambda,\mu}^\nu_{k_1,k_2,k_3}(\zeta) = \chi_{\lambda,\mu}^\nu_{k_1,k_2,k_3,k_4,k_5,k_6}(\zeta) \cdot (-1)^{\varepsilon_{\lambda,\mu}}
\]

of functions on \( \mathbb{Z}_2 \). It follows that both sides have to be independent of \( k_1, k_2, k_3 \), so that

\[
\chi_{\lambda,\mu}^\nu_{k_1,k_2,k_3}(\zeta) = \chi_{\lambda,\mu}^\nu_{k_1,k_2,k_3,k_4,k_5,k_6}(\zeta) \cdot (-1)^{\varepsilon_{\lambda,\mu}}
\]

for some constants \( \varepsilon_{\lambda,\mu} \in \{0, 1\} \).

We may now proceed to constrain \( \psi_{\lambda,\mu}^\nu_{k_1,k_2,k_3,k_4,k_5,k_6} \) by demanding compatibility of the isomorphism \( \tilde{\alpha}^\nu_{\lambda,\mu;k_1,k_2,k_3,k_4,k_5,k_6} \) with the (trivial) groupoid structures on \( p^*_1\mathcal{G} \star p^*_2\mathcal{G} \) and \( m^*\mathcal{G} \star \mathcal{I}(\rho) \). When rewritten in terms of the data \( \psi_{\lambda,\mu}^\nu_{k_1,k_2,k_3,k_4,k_5,k_6} \), Diagram \((3.5)\) yields the relation

\[
\psi_{\lambda,\mu}^\nu_{k_1,k_2,k_3,k_4,k_5,k_6} = \psi_{\lambda,\mu}^\nu_{k_1,k_2,k_3,k_4,k_5,k_6} \cdot \psi_{\lambda,\mu}^\nu_{k_1,k_2,k_3,k_4,k_5,k_6}.
\]
Setting all \( \vec{k}_i \) equal to \( \vec{0} \) shows that \( \psi_{\lambda,\mu,\vec{k}_1,\vec{k}_2,\vec{k}_3}^{\nu} = 1 \); setting \( \vec{k}_1 = \vec{k}_2 = \vec{k}_3 = \vec{k}_4 = \vec{k}_5 = \vec{0} \) gives \( \psi_{\lambda,\mu,\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4,\vec{k}_5}^{\nu} = \psi_{\lambda,\mu,\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4,\vec{k}_5,\vec{0},\vec{0}}^{\nu} \); and, finally, setting \( \vec{k}_1 = \vec{k}_5 = \vec{k}_6 = \vec{0} \) implies, in conjunction with the above results, that

\[
\psi_{\lambda,\mu,\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4,\vec{k}_5,\vec{0},\vec{0}}^{\nu} = \psi_{\lambda,\mu,\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4,\vec{k}_5,\vec{0},\vec{0}}^{\nu}.
\]

(4.78)

The line bundles \( \tilde{E}_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3}^{\nu} \) together with the isomorphisms \( \tilde{\alpha}_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4,\vec{k}_5,\vec{k}_6}^{\nu} \) provide the \( \mathbb{Z}_2 \)-equivariant formulation of the data of a stable isomorphism \( (p_1^*G \ast p_2^*G)|_{T_{\lambda,\mu}^{\nu}} \sim (m^*G \ast I(\rho))|_{T_{\lambda,\mu}^{\nu}} \). The latter is 2-isomorphic to the stable isomorphism obtained by replacing

\[
\psi_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4,\vec{k}_5,\vec{k}_6}^{\nu} \sim \psi_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4,\vec{k}_5,\vec{k}_6}^{\nu} \cdot \frac{\gamma_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3}^{\nu}}{\gamma_{\lambda,\mu;\vec{k}_4,\vec{k}_5,\vec{k}_6}^{\nu}}
\]

(4.79)

for some \( U(1) \)-valued constants \( \gamma_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3} \). In order to see this, just take the trivial submersion in Definition 4.1 and the locally constant maps \( (\vec{k}_1, \vec{k}_2, \vec{k}_3, h, z) \mapsto (\vec{k}_1, \vec{k}_2, \vec{k}_3, h, \gamma_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3}^{\nu}, z) \) as bundle isomorphisms. This allows to choose \( \psi_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{0},\vec{0},\vec{0}}^{\nu} = 1 \), and hence also \( \psi_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{0},\vec{0},\vec{0}}^{\nu} = 1 \). Thus, \( \tilde{\alpha}_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4,\vec{k}_5,\vec{k}_6}^{\nu} \) are of the form claimed in (4.61).

The stable isomorphism of the multiplicative structure on \( T_{\lambda,\mu}^{\nu} \)

By Proposition 4.49, there is a unique stable isomorphism \( M \) between the pullback gerbes \( p_1^*G \ast p_2^*G \) and \( m^*G \ast I(\rho) \) on \( SU(2) \times SU(2) \). As proved in Lemma 4.13, the restriction of \( M \) to \( T_{\lambda,\mu}^{\nu} \) with \((\lambda,\mu,\nu) \in \partial \mathcal{F} \) has to be 2-isomorphic to the stable isomorphism described there. On the other hand, according to Lemma 4.14, there are two choices of the stable isomorphism for \((\lambda,\mu,\nu) \in \mathcal{F} \). In this section, we describe which one of these is 2-isomorphic to the restriction of \( M \) to \( T_{\lambda,\mu}^{\nu} \).

Let us commence our study by fixing indices \( (\vec{k}_1, \vec{k}_2, \vec{k}_3) \) and defining the 2-connected space

\[
\tilde{T}_{\vec{k}_1,\vec{k}_2,\vec{k}_3} = \{(\vec{k}_1, \vec{k}_2, \vec{k}_3)\} \times \mathcal{F} \times SU(2).
\]

(4.80)

Note, in particular, that the factor \( \mathcal{F} \) contains points from \( \partial_2 \mathcal{F} \). This will be of prime significance to our subsequent considerations. As discussed on p. 24, the space \( \tilde{T}_{\vec{k}_1,\vec{k}_2,\vec{k}_3} \) parameterises a connected submanifold \( \tau(\mathcal{F} \times SU(2)) \subset SU(2) \times SU(2) \), and the mapping \( \tau \) factorises through the \( \mathbb{Z}_2 \)-orbifold \( \tilde{T}_{\vec{k}_1,\vec{k}_2,\vec{k}_3} \times SU(2)/\mathbb{Z}_2 \). Hence, \( M \) defines a \( \mathbb{Z}_2 \)-equivariant line bundle

\[
\tilde{M}_{\vec{k}_1,\vec{k}_2,\vec{k}_3} \rightarrow \tilde{T}_{\vec{k}_1,\vec{k}_2,\vec{k}_3}.
\]

(4.81)

As \( \tilde{T}_{\vec{k}_1,\vec{k}_2,\vec{k}_3} \) is 2-connected, \( \tilde{M}_{\vec{k}_1,\vec{k}_2,\vec{k}_3} \) is, by Proposition 4.10, \( \mathbb{Z}_2 \)-equivariantly isomorphic to the trivial bundle \( \tilde{T}_{\vec{k}_1,\vec{k}_2,\vec{k}_3} \times \mathbb{C} \) with a connection determined by a \( \mathbb{Z}_2 \)-invariant 1-form \( \tilde{P}_{\vec{k}_1,\vec{k}_2,\vec{k}_3} \) on \( \tilde{T}_{\vec{k}_1,\vec{k}_2,\vec{k}_3} \). To avoid introducing yet another symbol, we denote this trivial (and explicitly
trivialised) bundle by \( \widetilde{\mathcal{M}}_{\tilde{k}_1, \tilde{k}_2, \tilde{k}_3} \) from now onwards. The \( \mathbb{Z}_2 \)-equivariance is implemented on \( \widetilde{\mathcal{M}}_{\tilde{k}_1, \tilde{k}_2, \tilde{k}_3} \) as

\[
\mathbb{Z}_2 \times \widetilde{\mathcal{M}}_{\tilde{k}_1, \tilde{k}_2, \tilde{k}_3} \to \widetilde{\mathcal{M}}_{\tilde{k}_1, \tilde{k}_2, \tilde{k}_3}
\]

\[
: (\zeta, (\tilde{k}_1, \tilde{k}_2, \tilde{k}_3, \lambda, \mu, \nu, h, z)) \mapsto (\tilde{k}_1, \tilde{k}_2, \tilde{k}_3, \lambda, \mu, \nu, \zeta \cdot h, \eta_{\tilde{k}_1, \tilde{k}_2, \tilde{k}_3}(\zeta) \cdot z) \quad (4.82)
\]

by characters \( \eta_{\tilde{k}_1, \tilde{k}_2, \tilde{k}_3} : \mathbb{Z}_2 \to U(1) \). In particular, the \( \eta_{\tilde{k}_1, \tilde{k}_2, \tilde{k}_3} \) are independent of \( \lambda, \mu, \nu \) and \( h \).

As already mentioned above, the stable isomorphism \( \mathcal{M} \) restricted to \( T_{\lambda, \mu}^\nu \) is 2-isomorphic to one of the stable isomorphisms constructed in Lemmas 4.13 and 4.14. By Proposition 4.3, the surjective submersion of the 2-isomorphism can be chosen trivial, and so \( \eta_{\tilde{k}_1, \tilde{k}_2, \tilde{k}_3} \) necessarily has a non-zero winding number, as illustrated by the simple calculation

\[
\mathbf{S}^{\nu}_{\lambda, \mu} \text{-equivariant structure on that bundle } \widetilde{E}_{\lambda, \mu, \tilde{k}_1, \tilde{k}_2, \tilde{k}_3} \text{ which is } \mathbb{Z}_2 \text{-equivariantly isomorphic to } \mathcal{M}_{\tilde{k}_1, \tilde{k}_2, \tilde{k}_3} \bigg|_{\mathcal{T}_{\lambda, \mu, \tilde{k}_1, \tilde{k}_2, \tilde{k}_3}}. \quad (4.83)
\]

\[
\frac{X_{\lambda, \mu; \tilde{k}_1, \tilde{k}_2, \tilde{k}_3}^{\nu}(\zeta)}{\eta_{\tilde{k}_1, \tilde{k}_2, \tilde{k}_3}^{\nu}(\zeta)} = \frac{b(\zeta \cdot h)}{b(h)} \quad \text{for all } h \in SU(2) \text{ and } \zeta \in \{ e, -e \}. \quad (4.84)
\]

Its left-hand side takes values in the set \( \{ -1, +1 \} \). Suppose that \( b(-h) = -b(h) \) for all \( h \in SU(2) \). Restricting \( b \) to the maximal torus \( U(1) \subset SU(2) \), we obtain a map \( U(1) \to U(1) \). This map necessarily has a non-zero winding number, as illustrated by the simple calculation

\[
\frac{1}{2\pi i} \int_{U(1)} d\log b = \frac{1}{2\pi i} \int_0^{2\pi} d\phi \frac{d}{d\phi} \log b(e^{i\phi}) = \frac{1}{2\pi i} \int_0^{\pi} d\phi \frac{d}{d\phi} \log b(e^{i\phi}) = \frac{1}{2\pi i} \log \frac{b(-e)}{b(e)} \in 2\mathbb{Z} + 1,
\]

where we used \( \frac{d}{d\phi} \log b(e^{i\phi}) = \frac{d}{d\phi} \log b(e^{i\phi}) \). However, the maximal torus in \( SU(2) \) is the boundary of a disc, and \( b \) restricts to a smooth map on this disc, and so cannot have non-zero winding. Hence, \( b(-h)/b(h) = 1 \), which proves the lemma.

Taking into account the last lemma, we can now use the known form of \( X_{\lambda, \mu; \tilde{k}_1, \tilde{k}_2, \tilde{k}_3}^{\nu} \) for \( (\lambda, \mu, \nu) \) from the boundary \( \mathcal{F} \) to determine \( \eta_{\tilde{k}_1, \tilde{k}_2, \tilde{k}_3} \), and subsequently employ this result to deduce the characters \( \chi_{\lambda, \mu; \tilde{k}_1, \tilde{k}_2, \tilde{k}_3}^{\nu} \) for \( (\lambda, \mu, \nu) \) from the interior of \( \mathcal{F} \). This is done in the following lemma.
Lemma 4.16. Let \((\lambda, \mu, \nu) \in \mathcal{F}\) and take an arbitrary element \(\zeta = (-1)^{e} e \in \mathbb{Z}_2\). Then,

\[
\chi_{\lambda,\mu}^{\nu} (\zeta) = (-1)^{e} \lambda_{1}^{3} - \lambda_{1}^{2} - \lambda_{1}^{1}.
\]  

(4.85)

Proof. By virtue of Lemma \ref{lem:1.13}, it suffices to verify the thesis for \((\lambda, \mu, \nu) \in \partial_{12} \mathcal{F}\). Lemma \ref{lem:1.13} shows that the relevant \(\mathbb{Z}_2\)-equivariant structure is simply a restriction of the \(U(1)\)-equivariant structure of the KKS bundle \(\mathcal{K}_{k_{1}^{\nu}-A_{k_{1}^{\nu}}^{\lambda} - B_{k_{1}^{\nu}}^{\mu}}\). Upon recalling (2.45) and using integrality of \(n_{\lambda,\mu}^{\nu}\) alongside (4.44), we obtain the desired result. \(\square\)

5 The fusion 2-isomorphism

The preceding sections have equipped us with all the tools necessary to address the issue of existence of the fusion 2-isomorphism \(\varphi_{\lambda,\mu}^{\nu}\) and – in so doing – prove the main result of our paper, as expressed in Theorem 1.1.

We begin by noting that, for any triple \((\lambda, \mu, \nu)\) of weights from the discrete subset

\[
F_{\mathbb{Z}} = \mathcal{F} \cap (P_{k}^{+})^{3}
\]

of the fusion polytope, the product gerbe \(p_{1}^{*} \mathcal{G} \star p_{2}^{*} \mathcal{G}\) discussed previously admits two different trivialisations over the corresponding manifold \(T_{\lambda,\mu}^{\nu}\), namely,

\[
(m^{*} \Phi_{\mu} \star \text{id}_{\mathcal{I}(\rho)}) \circ \mathcal{M} : (p_{1}^{*} \mathcal{G} \star p_{2}^{*} \mathcal{G}) \mid_{T_{\lambda,\mu}^{\nu}} \xrightarrow{\sim} \mathcal{I}(m^{*} \omega_{\nu} + \rho) \mid_{T_{\lambda,\mu}^{\nu}}.
\]

(5.3)

Upon invoking Lemma \ref{lem:1.12} in conjunction with (3.16), we conclude that both trivialisations yield the same trivial gerbe over \(T_{\lambda,\mu}^{\nu}\),

\[
\mathcal{I}(p_{1}^{*} \omega_{\lambda} + p_{2}^{*} \omega_{\mu}) \equiv \mathcal{I}(m^{*} \omega_{\nu} + \rho),
\]

(5.4)

and so it is natural to enquire when there exists a 2-isomorphism \(\varphi_{\lambda,\mu}^{\nu}\) between the 1-isomorphisms \(p_{1}^{*} \Phi_{\lambda} \star p_{2}^{*} \Phi_{\mu}\) and \((m^{*} \Phi_{\nu} \star \text{id}_{\mathcal{I}(\rho)}) \circ \mathcal{M}\), as depicted in the standard 2-categorial diagram

\[
\begin{align*}
(p_{1}^{*} \mathcal{G} \star p_{2}^{*} \mathcal{G}) \mid_{T_{\lambda,\mu}^{\nu}} \xrightarrow{\mathcal{M} \mid_{T_{\lambda,\mu}^{\nu}}} (m^{*} \mathcal{G} \star \mathcal{I}(\rho)) \mid_{T_{\lambda,\mu}^{\nu}} \\
\xrightarrow{\varphi_{\lambda,\mu}^{\nu}} (m^{*} \Phi_{\nu} \star \text{id}_{\mathcal{I}(\rho)}) \mid_{T_{\lambda,\mu}^{\nu}}.
\end{align*}
\]

(5.5)

This 2-isomorphism was dubbed the fusion 2-isomorphism in the introduction, with reference to its underlying physical interpretation detailed in \cite{RS2}.
Lemma 5.1. A 2-isomorphism $\varphi_{\lambda,\mu}^\nu$ exists if and only if $\lambda + \mu - \nu \in 2\mathbb{Z}$, in which case it is unique up to a globally defined $U(1)$-valued constant.

Proof. Let us give the fusion 2-isomorphism in full detail by specialising Definition 3.1 to the setting at hand. To this end, we first write out the line bundles and the bundle isomorphisms of the two 1-isomorphisms that appear in its definition, in keeping with Definition 3.1. Starting with $p_1^*\Phi_\lambda \star p_2^*\Phi_\mu$, we find – for a fixed pair $(\bar{k}_1, \bar{k}_2) = (l_1, m_1, l_2, m_2) \in \{0, 1\}^4$ – the surjective submersion

$$\left( T_{\lambda,\mu;\bar{k}_1} \times T_{\lambda,\mu} \right) \times T_{\lambda,\mu} \left( T_{\lambda,\mu;\bar{k}_2} \times T_{\lambda,\mu} \right) \cong T_{\lambda,\mu;\bar{k}_1,\bar{k}_2}$$

as the base of the product line bundle

$$E_{\lambda \otimes \mu}|_{T_{\lambda,\mu;\bar{k}_1,\bar{k}_2}} = \text{pr}_{1}^* p_1^* E_{\lambda;\bar{l}_1} \otimes \text{pr}_2^* p_2^* E_{\mu;\bar{m}_2} \to T_{\lambda,\mu;\bar{k}_1,\bar{k}_2},$$

written in terms of the canonical projections $\text{pr}_i : T_{\lambda,\mu;\bar{k}_1,\bar{k}_2} \to T_{\lambda,\mu;\bar{k}_1,\bar{k}_2}$, alongside the product bundle isomorphism

$$\alpha_{\lambda \otimes \mu} = \text{pr}_{1,3}^* \widehat{\text{id}}_{\lambda;\bar{l}_1} \otimes \text{pr}_{2,4}^* \widehat{\text{id}}_{\mu;\bar{m}_2} \alpha_{\mu} : \text{pr}_{1,3}^* \widehat{\text{id}}_{\lambda;\bar{l}_1} \otimes \text{pr}_{2,4}^* \widehat{\text{id}}_{\mu;\bar{m}_2} E_{\lambda \otimes \mu} \sim \text{pr}_{1,2}^* E_{\lambda \otimes \mu},$$

of line bundles over a disjoint union of spaces $T_{\lambda,\mu;\bar{k}_1,\bar{k}_2,\bar{k}_3,\bar{k}_4}$, which equiped with the canonical projections $\text{pr}_{i,j} : T_{\lambda,\mu;\bar{k}_1,\bar{k}_2,\bar{k}_3,\bar{k}_4} \to T_{\lambda,\mu;\bar{k}_i,\bar{k}_j}$. In the case of the composite stable isomorphism $(\text{m}^* \Phi_\nu \star \text{id}_{\lambda}) \circ \mathcal{M}$, we obtain – for fixed $\bar{k}_i \in \{0, 1\}^2$ – the surjective submersion

$$T_{\lambda,\mu;\bar{k}_1,\bar{k}_2,\bar{k}_3} \times T_{\lambda,\mu;\bar{k}_3,\bar{k}_4} \left( T_{\lambda,\mu;\bar{k}_2,\bar{k}_3} \times T_{\lambda,\mu;\bar{k}_4,\bar{k}_5} \right) \cong T_{\lambda,\mu;\bar{k}_1,\bar{k}_2,\bar{k}_3}$$

as the base of the product line bundle

$$E_{\nu(\lambda,\mu)}|_{T_{\lambda,\mu;\bar{k}_1,\bar{k}_2,\bar{k}_3}} = E_{\lambda,\mu;\bar{k}_1,\bar{k}_2,\bar{k}_3} \otimes \text{pr}_3^* \widehat{\text{id}}_{\nu;\bar{l}_3 + \bar{m}_3} E_{\nu;\bar{l}_3 + \bar{m}_3} \to T_{\lambda,\mu;\bar{k}_1,\bar{k}_2,\bar{k}_3},$$

written in terms of the canonical projections $\text{pr}_i : T_{\lambda,\mu;\bar{k}_1,\bar{k}_2,\bar{k}_3} \to T_{\lambda,\mu;\bar{k}_i,\bar{k}_j}$, together with the bundle isomorphism

$$\alpha_{\nu(\lambda,\mu)}|_{T_{\lambda,\mu;\bar{k}_1,\bar{k}_2,\bar{k}_3,\bar{k}_4,\bar{k}_5,\bar{l}_6}} = \left( \text{id} \text{pr}_{1,3}^* E_{\lambda,\mu;\bar{k}_1,\bar{k}_2,\bar{k}_3} \otimes \text{pr}_3^* \widehat{\text{id}}_{\nu;\bar{l}_3 + \bar{m}_3} \right) \alpha_{\nu}|_{\nu;\bar{l}_3 + \bar{m}_3 + \bar{l}_6 + \bar{m}_6}$$

$$\circ \left( \alpha_{\lambda,\mu;\bar{k}_1,\bar{k}_2,\bar{k}_3,\bar{k}_4,\bar{k}_5,\bar{l}_6} \otimes \text{id} \text{pr}_5^* \widehat{\text{id}}_{\nu;\bar{l}_6 + \bar{m}_6} \right),$$

$$\alpha_{\nu(\lambda,\mu)} : \text{pr}_1^* \widehat{\text{id}}_{\lambda;\bar{l}_1} \otimes \text{pr}_2^* \widehat{\text{id}}_{\mu;\bar{m}_2} L \otimes \text{pr}_3^* \widehat{\text{id}}_{\nu;\bar{l}_3 + \bar{m}_3 + \bar{l}_5 + \bar{m}_6} E_{\nu(\lambda,\mu)} \sim \text{pr}_{1,2,3}^* E_{\nu(\lambda,\mu)},$$

of line bundles over a disjoint union of spaces $T_{\lambda,\mu;\bar{k}_1,\bar{k}_2,\bar{k}_3,\bar{k}_4,\bar{k}_5,\bar{k}_6}$, which equiped with the canonical projections $\text{pr}_i : T_{\lambda,\mu;\bar{k}_1,\bar{k}_2,\bar{k}_3,\bar{k}_4,\bar{k}_5,\bar{k}_6} \to T_{\lambda,\mu;\bar{k}_i,\bar{k}_j}$ and $\text{pr}_{i,j} : T_{\lambda,\mu;\bar{k}_1,\bar{k}_2,\bar{k}_3,\bar{k}_4,\bar{k}_5,\bar{k}_6} \to T_{\lambda,\mu;\bar{k}_i,\bar{k}_j,\bar{k}_k}$. We may now write the fusion 2-isomorphism as an isomorphism

$$\varphi_{\lambda,\mu}^\nu : \text{pr}_{1,2}^* E_{\lambda \otimes \mu} \sim E_{\nu(\lambda,\mu)}.$$
of hermitean line bundles with connection over a disjoint union of bases $\mathcal{T}_{\nu} \times \mathcal{T}_{\lambda}$, the latter coming with the canonical projection $pr_{1,2} : \mathcal{T}_{\nu} \to \mathcal{T}_{\lambda}$. Note that we have implicitly chosen the trivial surjective submersion $\text{id}_{\mathcal{T}_{\nu}}$ for the fusion 2-isomorphism, which can be done by virtue of Proposition [4,3]. Imposing the requirement that $\varphi_{\lambda,\mu}$ be compatible with $\alpha_{\nu(\lambda,\mu)}$ and $\alpha_{\lambda\otimes\mu}$ becomes equivalent to demanding the commutativity of the diagram

$$
\begin{array}{ccc}
pr_{1,4}^* \tilde{\mathcal{P}}_{1,2}^{[2]*}L \otimes pr_{2,5}^* \tilde{\mathcal{P}}_{2}^{[2]*}L \otimes pr_{4,5}^* \mathcal{T}_{\lambda\otimes\mu} & \xrightarrow{pr_{1,2,4,5}^* \alpha_{\lambda\otimes\mu}} & pr_{1,2}^* \mathcal{T}_{\nu,\lambda\otimes\mu} \\
\text{id}_{pr_{1,4}^* \tilde{\mathcal{P}}_{1,2}^{[2]*}L \otimes pr_{2,5}^* \tilde{\mathcal{P}}_{2}^{[2]*}L \otimes pr_{4,5}^* \mathcal{T}_{\lambda\otimes\mu}} & & pr_{1,2,3}^* \varphi_{\nu,\lambda,\mu} \\
pr_{1,4}^* \tilde{\mathcal{P}}_{1,2}^{[2]*}L \otimes pr_{2,5}^* \tilde{\mathcal{P}}_{2}^{[2]*}L \otimes pr_{4,5}^* \mathcal{T}_{\lambda\otimes\mu} & \xrightarrow{pr_{1,2,4,5}^* \alpha_{\lambda\otimes\mu}} & pr_{1,2}^* \mathcal{T}_{\nu,\lambda\otimes\mu} \end{array}
$$

(5.13)

of isomorphisms of hermitean line bundles with connection over a disjoint union of bases $\mathcal{T}_{\nu} \otimes \mathcal{T}_{\lambda,\mu}$, the latter taken with canonical projections $pr_{i,j}$, $pr_{i,j,k}$ as above and with $pr_{i,j,k,l} : \mathcal{T}_{\nu} \otimes \mathcal{T}_{\lambda,\mu}$ as above and with $pr_{i,j,k,l} : \mathcal{T}_{\nu} \otimes \mathcal{T}_{\lambda,\mu} \to \mathcal{T}_{\nu} \otimes \mathcal{T}_{\lambda,\mu}$.

In what follows, we work with the $S_{\lambda,\mu}$-equivariant counterparts $\tilde{\mathcal{E}}_{\lambda\otimes\mu}, \widetilde{\mathcal{E}}_{\nu(\lambda,\mu)}, \tilde{\mathcal{E}}_{\nu(\lambda,\mu)}$ and $\tilde{\varphi}_{\lambda,\mu}^{\nu}$ of the untilded objects, defined on the respective surjective submersions over $\mathcal{T}_{\lambda,\mu}$. The above abstract definition of the fusion 2-isomorphism then yields, on $\mathcal{T}_{\lambda,\mu}$, $\tilde{\varphi}_{\lambda,\mu}^{\nu}(k_1, k_2, k_3, \lambda, \mu, \nu, h, z \otimes z') = (\tilde{k}_1, \tilde{k}_2, \tilde{k}_3, \lambda, \mu, \nu, h, \tilde{f}_{\lambda,\mu}(k_1, k_2, k_3) \cdot z \otimes z')$ (5.14) for some constant phases $\tilde{f}_{\lambda,\mu}(k_1, k_2, k_3) \in U(1)$. Constancy of $\tilde{f}_{\lambda,\mu}(k_1, k_2, k_3)$ is an immediate consequence of the equality of the connection 1-forms of the line bundles $\tilde{\mathcal{E}}_{\nu(\lambda,\mu)} | \mathcal{T}_{\lambda,\mu}$ and $\tilde{\mathcal{E}}_{\nu(\lambda,\mu)} | \mathcal{T}_{\lambda,\mu}$, and of connectedness of $\tilde{\mathcal{T}}_{\lambda,\mu}^{\nu}(k_1, k_2, k_3)$. The compatibility condition encoded in Diagram 5.13 rewrites as

$$
f_{\lambda,\mu}(k_1, k_2, k_3) = \tilde{f}_{\lambda,\mu}(k_1, k_2, k_3)
$$

(5.15)

and implies independence of $f_{\lambda,\mu}(k_1, k_2, k_3)$ of the indices $k_1, k_2, k_3$,

$$
f_{\lambda,\mu}(k_1, k_2, k_3) = f_{\lambda,\mu} \in U(1)
$$

(5.16)

The remaining freedom in the definition of $\tilde{\varphi}_{\lambda,\mu}^{\nu}$ is hence a $U(1)$-valued constant. Clearly, for the fusion 2-isomorphism to be well-defined, (5.14) has to be consistent with the $S_{\lambda,\mu}$-equivalances entering the definitions of the bundles involved, as expressed by the identity

$$
\tilde{\varphi}_{\lambda,\mu}^{\nu}(k_1, k_2, k_3, \lambda, \mu, \nu, h \cdot g, \chi_{\lambda-\lambda_1}(g) \cdot z \otimes \chi_{\mu-\lambda_2}(g) \cdot z') = (k_1, k_2, k_3, \lambda, \mu, h \cdot g, \tilde{\varphi}_{\lambda,\mu}^{\nu}(g \cdot f_{\lambda,\mu}(k_1, k_2, k_3)) \cdot z \otimes \chi_{\nu-\lambda_3}(g) \cdot z'),
$$

(5.17)
to be imposed for an arbitrary element $g \in S_{\lambda,\mu}^{\nu}$, cf. (4.41)–(4.45). Thus, we obtain the algebraic condition

$$
\Delta_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3}^{\nu}(g) = \sum_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3}^{\nu} (g) \cdot \chi_{\nu-\lambda_{i_3+m}}(g)^{\varepsilon(b_{\lambda,\mu}^\nu)} \cdot \chi_{\lambda_{-\lambda_{i_1}}(g)^{-1}} \cdot \chi_{\lambda_{-\lambda_{i_2}}(g)^{-\varepsilon(a_{\lambda,\mu}^\nu)}} = 1. \quad (5.18)
$$

The fusion 2-isomorphism $\varphi_{\lambda,\mu}^\nu$ exists if and only if this condition is satisfied for all $\vec{k}_1, \vec{k}_2, \vec{k}_3 \in \{0, 1\}$ such that $T_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3} \neq \emptyset$ and for all $g \in S_{\lambda,\mu}^{\nu}$. We shall solve this condition by distinguishing two cases:

- $(\lambda, \mu, \nu) \in \partial F \cap (P_k^+)^3$: For $(\lambda, \mu, \nu) \in \partial_0 F \cap (P_k^+)^3$, the identity $\Delta_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3}^{\nu}(g) = 1$ is automatically implied by triviality of the characters involved. For $(\lambda, \mu, \nu) \in \partial_1 F \cap (P_k^+)^3$, on the other hand, it is readily verified by inspection, using Lemma 4.13 and (2.45).

- $(\lambda, \mu, \nu) \in \mathcal{F} \cap (P_k^+)^3$: In this case, $\Delta_{\lambda,\mu;\vec{k}_1,\vec{k}_2,\vec{k}_3}^{\nu}(\zeta) = (-1)^{\varepsilon(\lambda+\mu-\nu)}$ for all $\zeta \in S_{\lambda,\mu}^{\nu} \cong \mathbb{Z}_2$, as follows directly from Lemma 4.16 taken in conjunction with the identities $\varepsilon(a_{\lambda,\mu}^\nu) = 1 = \varepsilon(b_{\lambda,\mu}^\nu)$ and (2.43).

$\square$

Lemma 5.1 now implies the main result of the paper,

**Theorem 1.1.** $V = V_G$, where $V$ is given in (1.2) and $V_G$ in (1.10).

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