Time Encoding via Unlimited Sampling: Theory, Algorithms and Hardware Validation

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Abstract—An alternative to conventional uniform sampling is that of time encoding, which converts continuous-time signals into streams of trigger times. This gives rise to Event-Driven Sampling (EDS) models. The data-driven nature of EDS acquisition is advantageous in terms of power consumption and time resolution and is inspired by the information representation in biological nervous systems. If an analog signal is outside a predefined dynamic range, then EDS generates a low density of trigger times, which in turn leads to recovery distortion due to aliasing. In this paper, inspired by the Unlimited Sensing Framework (USF), we propose a new EDS architecture that incorporates a modulo nonlinearity prior to acquisition that we refer to as the modulo EDS or MEDS. In MEDS, the modulo nonlinearity folds high dynamic range inputs into low dynamic range amplitudes, thus avoiding recovery distortion. In particular, we consider the asynchronous sigma-delta modulator (ASDM), previously used for low power analog-to-digital conversion. This novel MEDS based acquisition is enabled by a recent generalization of the modulo nonlinearity called modulo-hysteresis. We design a mathematically guaranteed recovery algorithm for bandlimited inputs based on a sampling rate criterion and provide reconstruction error bounds. We go beyond numerical experiments and also provide a first hardware validation of our approach, thus bridging the gap between theory and practice, while corroborating the conceptual underpinnings of our work.

Index Terms—Event-driven, nonuniform sampling, modulo sampling, analog-to-digital conversion (ADC).

I. INTRODUCTION

In Shannon’s sampling paradigm, acquisition is performed by recording the amplitude of a signal at predefined, uniform time locations. Alternatively, in time encoding, the signal is converted into a nonuniform sequence of time events, leading to the acquisition paradigm called Event-Driven Sampling (EDS) [1], [2], [3], [4].

Such time events are induced by significant changes in the input signal values, thus enabling a data-driven approach to sampling. EDS has been adopted in engineering fields such as control engineering and signal processing [5] and has a wide range of applications, including neuromorphic vision [6], machine learning [7], [8] and brain-machine interfaces (BMIs) [9].

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In biology, EDS is used to model the signal transmission in the nervous system of vertebrates [10]. When compared to uniform sampling, event-driven sampling schemes are not controlled by an external clock signal and are characterized by low power consumption [11], [12] and increased time resolution [6].

Bottleneck in the Current EDS Acquisition: In EDS, the input is encoded into an asynchronous sequence of time events or trigger times. The information content in such a sequence can be characterised by the sampling density, a positive quantity measuring the number of trigger instants per unit time. This leads to an encoding barrier: no information is being transmitted for inputs whose amplitudes are outside the range where the sampling density is positive. The barrier is illustrated for the particular case of the asynchronous sigma-delta modulator based EDS in Fig. 1. Therefore, the accurate recovery of the input is affected, which requires a minimal sampling density to prevent aliasing [13]. Thus, in EDS, the input is typically constrained to a predefined dynamic range.

The problem of limited dynamic range is common in traditional encoding schemes such as analog-to-digital converters (ADCs), which clip or saturate the input signals exceeding a predefined threshold. To overcome this fundamental bottleneck, the Unlimited Sensing Framework (USF) [14], [15], [16], [17], [18] was introduced and recently validated in hardware via modulo ADCs [18]—it was shown that, in practice, signals up to 24 times...
larger than ADC’s threshold can be recovered using modulo ADCs [18], despite non-idealities and quantization effects. In USF, the introduction of modulo nonlinearity converts a signal from high dynamic range (HDR) to low dynamic range. This is done by folding the HDR, continuous-time signal back into the dynamic range of the ADC, whenever the input amplitude is outside a predefined range [14], [15], [16], [17], [18]. The follow-up work on USF includes,

- Different signal classes, such as sparse signals [19], [20], sinusoidal mixtures [21] and spline spaces [22].
- New sampling architectures [23], [24], [25], [26].
- Application frontiers such as array signal processing [27], [28] and computational imaging [22], [29], [30].

A more general model for modulo nonlinearity was introduced recently, namely, modulo-hysteresis [23], [25], which, compared to ideal modulo, comprises two additional parameters that can tackle non-idealities present in experimental observations. Moreover, modulo-hysteresis provided guarantees for a new class of reconstruction methods based on thresholding.

To address the dynamic range limitation in event-driven sampling schemes, we propose a new encoding model consisting of a modulo-hysteresis nonlinearity in series with an EDS model, which we call modulo EDS or MEDS. In our setting the EDS model is fixed, and the EDS input (modulo output) is guaranteed to satisfy the EDS dynamic range by selecting an appropriate modulo nonlinearity (Fig. 1(a)). None of the existing recovery methods for EDS models can address the reconstruction of the modulo output, which is a folded signal. Therefore, the use of advanced recovery algorithms enables high dynamic range acquisition via MEDS that is conventionally not possible.

The EDS model in this paper is an asynchronous sigma-delta modulator (ASDM) [1]. Thanks to its low power consumption [31] and modular design [32], the ASDM was first implemented as an alternative to conventional ADCs in [33] and subsequently included in applications such as brain-machine interfaces [9]. Assuming that the input amplitude is in a predefined range of values, the ASDM transmits trigger times measuring changes in the input integral [1]. For inputs of amplitude outside this range, the ASDM transmits no information, as shown in Fig. 1(b).

**Related Work:** The ASDM dynamic range can be extended by changing the model parameters or architecture. This includes increasing the sampling density by adjusting the triggering threshold [1], or introducing an adaptation mechanism via a new ASDM architecture [34]. Another popular EDS model is the integrate-and-fire (IF) neuron, inspired from the biological neuron, for which input reconstruction was studied for a wide range of input classes [3], [35], [36] including recovery methods for EDS leveraging uniform sampling theory [37]. The recovery methods for IF also assume that the input is in a predefined dynamic range. It is possible to encode high dynamic range signals with an IF by generating bidirectional event-driven samples [35], [38]. In this case, however, recovery guarantees do not exist for low amplitude inputs, for which the model does not generate output. Event based cameras, which use models closely related to the IF or ASDM, capture high dynamic range data with logarithmic pixels. This, in turn, decreases the pixel resolution for high intensities [6]. None of the methods above can recover modulo folded inputs from the output of an EDS. A MEDS architecture that extends the IF dynamic range was first numerically validated in [25].

**Contributions:** Our contributions are as follows,

- We introduce a new event-driven modulo sampling scheme, comprising a modulo-hysteresis model in conjunction with an ASDM, which addresses the dynamic range restriction for an ASDM model without any parameter or architecture alteration. The new scheme is fully compatible with the existing ASDM methodology, but can also accommodate inputs that do not satisfy the dynamic range requirement.
- By exploiting the modulo-hysteresis architecture, we provide theoretical conditions for which the input can be recovered from the output trigger times.
- Extensive numerical experiments and a validation of the MEDS-based hardware prototype (cf. Fig. 5) show the validity of our approach.

**Notation:** We use $\mathbb{Z}$ and $\mathbb{R}$ to denote the set of integers and real numbers, respectively, and $\mathbb{N}$ to denote the set of positive integers. Continuous functions are denoted as $f(t)$ and discrete sequences as $f[k]$. The Hilbert space of square-integrable functions is denoted as $L^2(\mathbb{R})$. The norm in any space $H$ is denoted as $\|f\|_H$. The derivative of order $N$ is denoted as $f^{(N)}(t)$ and, for sequences, the finite difference of order $N$ is $\Delta^N f[k]$, which is computed by applying recursively $\Delta^N f[k] = \Delta^N f[k+1] - \Delta^N f[k]$, where $\Delta^0 f = f$. The space of square integrable functions bandlimited to $\Omega$ is the Paley-Wiener space denoted $PW_{\Omega}$. The indicator function $\mathbb{1}_{S}(t)$ is 1 for $t \in S$ and 0 otherwise. The floor and ceiling functions are $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$, respectively, and $\lfloor x \rfloor = x - \lfloor x \rfloor$ denotes the fractional part of $x$. Furthermore, let $\text{supp}(f)$ denote the support of sequence $f[k]$, let $|S|$ denote the cardinality of a set $S$, and let $\emptyset$ denote the empty set. By $f_{\infty}$ or $\|f\|_{\infty}$ we denote the absolute sequence norm $\|f\|_{\ell^\infty}$ or the absolute function norm $\|f\|_{L^\infty}$, respectively. Let $\text{sinc}_{\Omega}$ be the sinc function defined here as $\text{sinc}_{\Omega}(t) = \frac{\sin(\pi t)}{\pi t}$.

**Scope and Organization:** The ASDM model and associated reconstruction methods are summarised in Section II. The MEDS architecture is introduced in Section III. The new reconstruction method from MEDS samples and associated theoretical guarantees are given in Section IV. The numerical demonstration and MEDS hardware prototype are in Section V. Section VI comprises proofs for some of the theoretical results in Section IV, and the conclusions are in Section VII.

II. THE ASYNCHRONOUS SIGMA DELTA MODULATOR

The ASDM is an event-driven sampling model consisting of a feedback loop containing an adder, an integrator, and a non-inverting Schmitt trigger, as illustrated in Fig. 2. We introduce the formal definition of the ASDM model in Section II-A and summarise the recovery method for bandlimited inputs and its shortcomings in Section II-B.
Can be derived directly from (2). This creates a strong transform. \( \pi \) is the Paley-Wiener space of bandlimited functions. As in [19], the modulo nonlinearity is the t-transform \[ (2) \]

\[ \int_{t_k}^{t_{k+1}} \left( g(s) + (-1)^k b \right) ds = (-1)^k 2\delta, \quad k \in \mathbb{Z}_+^* \]

We model the link between the trigger times and input via the operator \( \{ t_k \} = \text{ASDM}_{\delta,b}(g) \). The t-transform was also studied in the context of other EDS models such as the IF neuron \[ [2] \]. By defining \( G(t) \equiv \int_0^t g(s) ds \), it can be shown via induction that \( G(t_k) \) can be derived directly from (2). This creates a strong link between ASDM sampling and nonuniform sampling, and allows using similar analysis and recovery approaches \[ [13] \]. A hardware realisation of an ASDM is in Fig. 5(b).

### B. Reconstruction From Asynchronous Sigma-Delta Samples

The ASDM input \( g(t) \) can be perfectly reconstructed \[ [2] \] if it is bandlimited \( g \in \text{PW}_{\Omega} \) and the maximum distance between the ASDM trigger times satisfies the Nyquist rate condition, i.e., \( |t_{k+1} - t_k| < \frac{\pi}{\Omega} \). This condition is sufficient to guarantee recovery from nonuniform samples \( G(t_k) \) \[ [13] \].

In the case of the ASDM, if \( |g(t_k)| > b \) the encoder does not generate any output, meaning that \( |t_{k+1} - t_k| < \frac{\pi}{\Omega} \) does not always hold. To overcome this problem it is assumed that \( g_\infty < b \), which, via (2), guarantees that \( t_{k+1} - t_k < \frac{2\delta}{b-g_\infty} \). Therefore, the Nyquist rate condition for recovery is guaranteed if

\[ g_\infty < b - \frac{2\delta}{\Omega}, \quad (3) \]

The input is recovered as \[ [1] \]

\[ g(t) = \sum_{m \in \mathbb{Z}} c_m \text{sinc}_\Omega (t - \bar{t}_m), \quad (4) \]

where \( \bar{t}_m = (t_m + t_{m+1})/2 \) are the midpoints of the intervals between the trigger times. The coefficients \( c_m \) are computed via least squares from \( \int_{t_k}^{t_{k+1}} g(s) ds = (-1)^k 2\delta - (-1)^k b(t_{k+1} - t_k) \) \[ (2) \]. In the context of nonuniform sampling, this procedure is also known as reconstruction from local averages \[ [13] \].

Intuitively, whenever \( |g(t_k)| > b \), the input does not trigger the ASDM (Fig. 1(b)). This is because the integrator output \( y(t) \) is no longer monotonic and might not reach the threshold \( \pm b \). This locks in the ASDM output until the threshold is reached. This effect is also observed in hardware, as illustrated in Fig. 3. As will be shown in Section V, the ASDM saturation leads to unstable reconstructions.

### III. THE PROPOSED MODULO EVENT-DRIVEN ENCODER

Here we introduce a new event-driven modulo encoder model, comprising a modulo encoder with hysteresis and an ASDM model, depicted in Fig. 4. We first present the ideal modulo encoder in III-A and explain some of its limitations. The modulo-hysteresis is then described in III-B. The proposed event-driven modulo is introduced in III-C.

#### A. The Ideal Modulo

Let \( g \in \text{PW}_{\Omega} \) where \( \text{PW}_{\Omega} \) is the Paley-Wiener space of \( \Omega \)-bandlimited functions. As in [19], the modulo nonlinearity is
defined by function $\mathcal{M}_\lambda : PW_\Omega \rightarrow L^2(\mathbb{R})$

$$\mathcal{M}_\lambda(g(t)) = 2\lambda \left(\left\lfloor \frac{g(t)}{2\lambda} + \frac{1}{2}\right\rfloor - \frac{1}{2}\right)$$ (5)

where $\lfloor x \rfloor = x - \lfloor x \rfloor$ is the fractional part of $x$, and $L^2(\mathbb{R})$ denotes the space of functions with finite energy. The instantaneous times when $\lfloor x \rfloor$ switches from 1 to 0 are called folding times and denoted $\tau_r, r \in \mathbb{Z}$. The output of the model $x(t) = \mathcal{M}_\lambda(g(t))$ can be expanded as $x(t) = g(t) - \varepsilon_g(t)$, where $\varepsilon_g(t) = 2\lambda \sum_{r \in \mathbb{Z}} s_r 1_{[\tau_r, \infty)}(t), t \in \mathbb{R}, s_r = \pm 1$. In Unlimited Sampling the data is sampled uniformly with period $T$, which yields

$$x(kT) = g(kT) - \varepsilon_g(kT), \quad k \in \mathbb{Z}.$$ (6)

The input $g(kT)$ is recovered by applying a finite difference filter $\Delta^N$ to (6), which annihilates $\Delta^N \varepsilon_g(kT)$ given $\Delta^N x(kT)$. This is possible because $\Delta^N$ vanishes gradually $\Delta^N g(kT)$ but always leaves $\Delta^N \varepsilon_g(kT)$ on an equally spaced grid that is then annihilated via a modulo operation. However this approach does not work for experimental modulo data due to various non-idealities [18], [23]. This approach also would not work when sampling the output with an ASDM, which integrates the input on intervals of various durations. (2).

A different reconstruction approach from uniform samples for the ideal modulo consists of filtering the modulo output with a kernel that amplifies the folding times, which are subsequently detected via thresholding [24], [40]. The thresholding approach is particularly desirable when the data is affected by non-idealities, such as transient folding samples, where the Unlimited Sampling approach fails [23]. However, the input reconstruction is not guaranteed when the folding times are very close, which can happen for the ideal modulo, as depicted in Fig. 6(a). This problem can be solved using the modulo-hysteresis model [23], presented in the next subsection, which will be used in the present work.

**B. The Modulo-Hysteresis Model**

The phenomenon of hysteresis, present in ADC circuits [41], results from a difference between the quantization threshold values when the value is approached from above or below [42]. This effect is also commonly exploited in circuits such as the Schmitt trigger, which avoids noise induced spurious triggering during ADC conversion. Recently, that effect was exploited in a modulo-hysteresis model and associated hardware implementation [23] (see Fig. 5(a)), which limits the minimum distance between two consecutive folding times. We give below the formal definition of modulo-hysteresis.

**Definition 1 (Modulo-Hysteresis):** The modulo-hysteresis with threshold $\lambda$ and hysteresis parameter $h \in [0, \lambda)$, denoted $\mathcal{M}_H = [\lambda, h]$, generates an nonuniform sequence $\{\tau_r\}$ and an analog function $x(t) = \mathcal{M}_H g(t), t \geq \tau_0$ in response to input

![Fig. 5. Hardware experiment. (a) The US-ADC hardware prototype with variable parameters controlling the threshold $\lambda$ and hysteresis $h$. (b) The ASDM hardware realisation. (c) Oscilloscope screenshot showing the input and outputs of the modulo-hysteresis and ASDM hardware prototypes.](image-url)
where

\[ g \in \mathcal{PW}_T. \]

The sequence \( \{ \tau_r \} \) is computed recursively as

\[
\tau_1 = \min \{ t > \tau_0 | \mathcal{M}_h(g(t) + \lambda) = 0 \}
\]

\[
\tau_{r+1} = \min \{ t > \tau_r | \mathcal{M}_h(g(t) - g(\tau_r) + h s_r) = 0 \},
\]

where \( s_r = \text{sign}(g(\tau_r) - g(\tau_{r-1})) \), \( n \geq 1 \). The function \( x(t) \) returned by the encoder is then given by

\[
x(t) = g(t) - \varepsilon_g(t),
\]

where \( \varepsilon_g(t) = 2\lambda h \sum_{r \in \mathbb{Z}} s_r \mathbb{1}_{[\tau_r, \infty)}(t) \) and \( \lambda_h = \lambda - h/2 \).

Modulo-hysteresis has the following important property.

**Lemma 1 (Minimum Separation between Folds):** Let \( \{ \tau_r \}_{r \in \mathbb{Z}} \) be the sequence of folding times corresponding to encoder \( \mathcal{M}_h \), \( H = [\lambda, h] \), for an input \( g \in \mathcal{PW}_T \). Then

\[ \tau_{r+1} - \tau_r \geq h^* \quad \text{and} \quad h^* = \min \{ h, 2\lambda h \}. \]

**Proof:** From Bernštejn’s inequality \( \| g^{(n)} \|_p \leq \Omega^n \| g \|_p \) (cf. pg. 116, [43]) and Definition 1, for \( p = \infty, n = 1 \) and \( r \geq 1 \),

\[ \tau_{r+1} - \tau_r \geq \frac{|g(\tau_{r+1}) - g(\tau_r)|}{\max |g(t)|} \geq \frac{\min \{ h, 2\lambda - h \}}{\Omega \max |g(t)|} = \frac{h^*}{\Omega g_\infty}. \]

When the data is sampled with period \( T \), it was shown that here \( g(kT) \) can still be recovered from \( x(kT) = \mathcal{M}_h g(kT) \) with \( \mathcal{USF} \), \( g(kT) = 0 \). This property no longer holds when the data is sampled with an asynchronous sigma-delta modulator, as will be explained next.

**C. The Proposed Modulo Event-Driven Sampling Model**

The proposed model consists of a modulo-hysteresis in series with an ASDM model. Then, for \( k \geq 0 \), the trigger times \( \{ t_k \} \) generated by the model satisfy (2)

\[
\int_{t_k}^{t_{k+1}} \mathcal{M}_h g(s) ds = q_k,
\]

\[
q_k \triangleq (-1)^k 2 \delta - (-1)^k b (t_{k+1} - t_k).
\]

We assume \( g \in \mathcal{PW}_T \) and \( \lambda \leq b - \frac{24\delta}{n} \) such that if \( |g(t)| < \lambda \) then (3) is satisfied, and thus \( g \) is left unchanged by the modulo nonlinearity. This ensures that (9) is equivalent to (2), and thus the proposed encoder is backwards compatible with the existing results on event-driven sampling. When \( |g(t)| > b - \frac{24\delta}{n} \), the asynchronous sigma-delta modulator leads to information loss, which is prevented by the proposed circuit by folding the input back into the range \( [-\lambda, \lambda] \). According to Definition 1, \( x(t) = \mathcal{M}_h g(t) \) satisfies

\[
x(t) = g(t) - 2\lambda h \sum_{r \in \mathbb{Z}} s_r \mathbb{1}_{[\tau_r, \infty)}(t).
\]

Let \( X(t) = \int_0^t x(s) ds \). Then \( X(t_0) = X(0) = 0 \) and (9) implies that

\[ X(t_k) = \sum_{m=0}^{k-1} q_m, \quad k \geq 1. \]

Furthermore, let \( E_g(t) = \int_0^t g(s) ds \) and \( G(t) = \int_0^t g(s) ds \), and thus, by integrating (10) we get

\[
X(t_k) = G(t_k) - E_g(t_k).
\]

We can then compute samples \( X(t_k) \) using just the model parameters and output trigger times via (11). Further on, the input and residue will be separated using (12). Equation (12) has a similar form as (6), but here \( E_g(t_k) \) no longer lie on an uniform grid as it involves integration on intervals of variable length. Therefore, a different recovery strategy than in the case of \( \mathcal{USF} \) is required, which is explained in the following.

**IV. RECOVERY FROM MODULO EVENT-DRIVEN DATA**

Our goal is to recover \( g(t) \) from (9) given time instances \( \{ t_k \}_{k \in \mathbb{Z}} \) and design parameters \( \{\lambda, h, b, \delta\} \). The proposed encoding circuit guarantees the asynchronous sigma-delta modulator input is in a predefined range. Generally, the EDS input \( x(t) \) is not bandlimited, and does not even belong to a class of shift-invariant spaces, given that \( \tau_r \geq 0 \) do not lie on a uniform grid. Thus \( x(t) \) cannot be recovered with the traditional methods for \( \mathcal{USF} \) models, which assume that the input is bandlimited, or at least smooth [2], [3], [23], [44].

Given samples \( X(t_k) = G(t_k) - E_g(t_k) \), the main challenge is to recover both \( G(t_k) \) and \( E_g(t_k) \). By applying a generic filter \( \mathcal{D} \) to the data \( X(t_k) \) for \( k \geq 2 \) we get (7, 11)

\[
\mathcal{D} X(t_k) = \mathcal{D} G(t_k) - \mathcal{D} E_g(t_k) = \mathcal{D} \left( \sum_{m=0}^{k-1} q_m \right).
\]

The input reconstruction is organised in two stages. In the first stage, \( \mathcal{D} G(t_k) \) is interpreted as noise and an estimation \( \mathcal{E}_g(t_k) \) of \( E_g(t_k) \) is computed by thresholding \( \mathcal{D} X(t_k) \). This approach was used before to reconstruct modulo folded data in the context of uniform samples [23], [24], [40]. In the second recovery stage, an estimation \( \hat{G}(t_k) \) of \( G(t_k) \) is computed from the integral form of the modulo-hysteresis input \( X(t_k) \) and \( \mathcal{E}_g(t_k) \) through (12), and the respective estimate for the continuous input \( \hat{g}(t) \) is reconstructed from the estimated samples \( \hat{G}(t_k) \), such that the error norm \( \| \hat{g} - \hat{g} \|_2 \) is bounded. The section is organised as follows. To accommodate nonuniform samples, we define the nonuniform difference operator \( \mathcal{D}^N \) in Section IV-A. We then describe the first recovery stage, and provide guarantees for computing \( \mathcal{E}_g(t_k) \) in the particular case of one folding time in Section IV-B, and extend it to multiple folding times in Section IV-C. Section IV-D describes the second recovery stage and gives the input reconstruction algorithm and associated error bound.

**A. The Nonuniform Difference Operator**

Here we formulate a thresholding based recovery for the case of the ASDM model. Given samples \( X(t_k) = G(t_k) - E(t_k) \), the main challenge is to recover both \( G(t_k) \) and \( E(t_k) \). We threshold samples \( \mathcal{D} X(t_k) \) to recover the residual \( E_g(t_k) \), which requires identifying the folding times \( \tau_r \) and signs \( s_r \). Operator \( \mathcal{D} \) is designed to enhance the effect of \( E_g \) in (13). In [23] the data was processed with the uniform finite difference.
filter $\Delta^N$. Here we recover the input from nonuniform samples and, to this end, we choose $D = D^N$ defined recursively the nonuniform difference operator of order $N$,

$$D^N f[k] = \frac{D^{N-1} f[k+1] - D^{N-1} f[k]}{t_{k+N} - t_k}, \quad N > 0,$$

(14)

where $D f[k] = f[k]$ and $f[k]$ is a generic sequence. Here, $D^N$ is an extension of the difference operator $\Delta^N$ used for uniform samples. In (13), the values $D^N f[k] = \sum_{m=1}^{g_m} \delta_{m} f[k]$, where $g_m$ can be computed directly from the measurements. As is the case for USF, the measurements lead to a mixture of the samples $D^{N} G(t_k)$ and $D^N E_g(t_k)$. However, unlike the uniform sampling scenario, values $D^N E_g(t_k)$ do not lie on an uniform grid. We will show that the separation of $D^N G(t_k)$ and $D^N E_g(t_k)$ is still possible via a thresholding approach. To this end, we provide the following bound, which will be used in our recovery approach (proof in Section VI).

**Proposition 1:** Let $D^N$ be the nonuniform finite difference operator defined in (14). Then

$$|D^N G(t_k)| \leq \frac{1}{\Omega!} \left( \frac{T_{\max}}{T_{\min}} \right)^N g_{\infty},$$

(15)

where $T_{\min}, T_{\max} > 0$ are two constants satisfying $T_{\min} \leq t_{k+1} - t_k \leq T_{\max}$.

This bound is an extension to nonuniform samples of the bound derived in [14] for $|\Delta^N g(kT)|$. Specifically, we note that in the limit case where $T_{\min} = T_{\max} = T$, and thus $t_k = kT$, we have that $D^N G(t_k) = \Delta^N G(kT)$. According to Bernstein's theorem, we have $||G||_{\infty} \leq \Omega G_{\infty} \Rightarrow g_{\infty} \leq \Omega G_{\infty}$. Then, in this particular case, (15) becomes $|\Delta^N G(kT)| \leq (T_{\max})^N G_{\infty}$, which is in accordance to the bound derived in [14], [16].

**B. The Case of a Single Fold**

For simplicity, let us first assume we have only one folding time $\tau_1$. Let $K_1$ be the index of the trigger time preceding the folding time, such that $\tau_1 \in [K_1, K_{1} + 1)$. The following result calculates the expression of $D^N E_g(t_k)$.

**Proposition 2:** For all $k \in \mathbb{Z}$, sequence $D^N E_g(t_k)$ satisfies

$$D^N E_g(t_k) = \frac{p}{s_1(2\lambda - h)} \begin{cases} \mu_{K_1}^N [k] p + \beta_{K_1}^N [k] (1 - p) & \forall N \geq 2, \end{cases}$$

(16)

where $p = \frac{t_{K_1 + 1 - \tau_1}}{t_{K_1 + 1 - t_1}}, 0 \leq t_{k+N} - t_k$ \in \{0, 1\}$ and $\mu_{K_1}^N, \beta_{K_1}^N$ satisfy

$$\begin{align*}
\mu_{K_1+1}^N & = \mu_{K_1}^N + \frac{1}{t_{K_1+1} - t_k}, \\
\beta_{K_1+1}^N & = \mu_{K_1}^N + \frac{1}{t_{K_1+1} - t_k}.
\end{align*}$$

(17)

for $N \geq 2$, where $\mu_{K_1}^N = \frac{1}{t_{K_1+1} - t_k}, \beta_{K_1}^N = \frac{1}{t_{K_1+1} - t_k}$.

**Proof:** We compute $E_g(t)$ by integrating $e_g(t)$ defined in (7)

$$E_g(t) = \frac{D^N E_g(t_k)}{s_1(2\lambda - h)} = \frac{p}{s_1(2\lambda - h)} \begin{cases} \mu_{K_1}^N [k] + \beta_{K_1}^N [k] (1 - p) & \forall N \geq 2, \end{cases}$$

(18)

We evaluate $D^N E_g(t_k)$ for $N = 1, 2$ as below

$$\begin{align*}
D^N E_g(t_k) & = \frac{p}{s_1(2\lambda - h)} \begin{cases} \mu_{K_1}^N [k] + \beta_{K_1}^N [k] (1 - p), & \forall N = 1, 2, \end{cases}
\end{align*}$$

(19)

and the result follows via (14) and (17).

Using (16), the non-zero values of $D^N E_g(t_k)$ therefore satisfy

$$\text{supp}(D^N E_g(t_k)) \subseteq \mathbb{S}_N,$$

(20)

where $\mathbb{S}_N = \{K_1 - N + 1, \ldots, K_1\}$. We only have access to $D^N X(t_k)$, which also includes the effect of $D^N G(t_k)$. In the first recovery stage, we compute the support of $D^N E_g(t_k)$ via thresholding. In the case of uniform sampling, a similar strategy was proposed with a fixed threshold $\Psi_N = \frac{\lambda}{2\Omega}$. In the present case of nonuniform sampling, we can achieve input reconstruction in a more general scenario by allowing a time-varying threshold $\Psi_N[t]$, which depends on the nonuniform trigger times $\{t_k\}$. In order to constrain the effect of $D^N G(t_k)$ on the measurements $D^N X(t_k)$, the following lemma (proof in Section VI) derives a bound for $D^N G(t_k)$ and subsequently computes support of $D^N E_g(t_k)$.

**Lemma 2:** Assume that

$$|D^N G(t_k)| < \Psi_N[t],$$

(21)

where $\Psi_N[t]$ is defined as

$$\Psi_N[t] = \mu_{K_1}^N [k] + \beta_{K_1}^N [k],$$

and sequences $\phi_{i,t}, i = 0, 1, l \in \mathbb{N}$, satisfy

$$\begin{align*}
\phi_{0,t} & = \frac{|\mu_{K_1}^N [t] - N + 1|}{|\mu_{K_1}^N [t] - N + 1| + |\beta_{K_1}^N [t] - N + 2|}, \\
\phi_{1,t} & = \frac{|\mu_{K_1}^N [t] - 1|}{|\mu_{K_1}^N [t] - 1| + |\beta_{K_1}^N [t] - 1|},
\end{align*}$$

(22)

where $\mu_{K_1}^N [k]$ and $\beta_{K_1}^N [k]$ are given in (17). Furthermore, let $K_N = \{k_m, k_M\}$ be such that

$$k_m = \min M_{\mathbb{N}}, k_M = \max M_{\mathbb{N}},$$

$$M_{\mathbb{N}} = \{k \in \mathbb{Z} | |D^N X(t_k)| \geq \Psi_N[t]\}.$$
3) If \( k_M - k_m = N - 3 \), then \( \hat{\tau}_1 = \frac{t_{k_M} + t_{k_m}}{2} \) and (24) holds.

Given that \( t_{k+1} - t_k \leq \frac{2\delta}{\lambda - \delta} \), \( \forall k \in \mathbb{Z} \), by choosing \( T_{\text{max}} = \frac{2\delta}{\lambda - \delta} \), then \( \tau_1 \) can be recovered with any accuracy for a small enough \( \delta \) given by (24), (25). We note that in USF the folding times \( \{\tau_r\} \) are estimated as \( \tau_r \approx K_r T \), leading to a similar error as here, bounded by the sampling period \( T \).

\( \text{C. The Case of Multiple Folds} \)

In the general case there are \( R \) folding times \( \{\tau_r\}_{r=1}^R \), and each has an associated set \( \mathcal{S}_r^T = \{t_{K_r - N + 1}, \ldots, t_{K_r}\} \), such that the non-zero values of \( \mathcal{D}^N E_g(t_k) \) satisfy (26)

\[
\text{supp}(\mathcal{D}^N E_g(t_k)) \subseteq \bigcup_{r=1, \ldots, R} \mathcal{S}_r^T.
\]

Here the results in Lemma 2 and Theorem 1 can be applied sequentially provided that \( \mathcal{S}_1^T \cap \mathcal{S}_2^T = \emptyset, \forall r_1, r_2 = 1, \ldots, R, r_1 \neq r_2 \). In other words, this means there are at least \( N \) trigger times between each two consecutive folds \( \tau_r, \tau_{r+1} \), which is guaranteed if \( \tau_{r+1} - \tau_r \geq NT_{\text{max}} \). A sufficient condition can be obtained via Lemma 1

\[
\frac{1}{\Omega g_\infty} \geq NT_{\text{max}}.
\]

Then if (20), (27) are true then \( \{\tau_r\}_{r=1}^R \) and \( \{s_r\}_{r=1}^R \) can be computed as follows. Let \( \mathcal{K}_N = \bigcup_{r \in \{1, R\}} \{k_{r_m}^r, k_{r_M}^r\} \), where \( k_{r_m}^r, k_{r_M}^r \) are computed as below. For \( r = 1 \),

\[
k_{1_m}^r = \min \left\{ k \mid |X_k^N| \geq \Psi_N^r[k] \right\},
\]

\[
k_{M}^r = \max \left\{ k \leq k_{1_m}^r + N - 1 \mid |X_k^N| \geq \Psi_N^r[k] \right\}.
\]

For \( r = 2, \ldots, R \)

\[
k_{1_m}^r = \min \left\{ k \mid k > k_{M}^{r-1} \mid |X_k^N| \geq \Psi_N^r[k] \right\},
\]

\[
k_{M}^r = \max \left\{ k \leq k_{1_m}^r + N - 1 \mid |X_k^N| \geq \Psi_N^r[k] \right\},
\]

where \( X_0^N \triangleq \mathcal{D}^N X(t_k) \). Then \( s_r, \tau_r \), \( r = 1, \ldots, R \) can be computed sequentially with Theorem 1 from \( \mathcal{K}_N \). To better explain the proposed technique, we illustrated the thresholding strategy for multiple folds in Fig. 7. The diagram shows the role of \( h \) in separating the folds which, in turn, enables an independent detection of the folding times. The following lemma provides sufficient recovery conditions.

\( \text{Lemma 3 (Sufficient Recovery Conditions):} \) The recovery is possible when either of the conditions 1) and 2) below is true.

1) The values \( s_r, \tau_r \), \( r = 1, \ldots, R \) can be recovered from \( \mathcal{D}^N X(t_k) \) with an error given by (24), (25) if

\begin{align*}
&\text{S1)} \quad \frac{C(\lambda - \delta)}{\lambda - \delta} \Omega e N^{-1} g_\infty < \frac{\lambda}{\lambda g_\infty}.
&\text{S2)} \quad \frac{2h_\infty}{b - \lambda} \leq h^*,
\end{align*}

where \( h^* = \min\{h, 2\lambda - h\} \) and \( C = \frac{b + \lambda}{b - \lambda} \).

2) Furthermore, S1) and S2) are satisfied if

\[
\delta < \frac{(b - \lambda) h^*}{2N \Omega g_\infty} \quad \text{where} \quad \kappa = \min\left\{1, \frac{\lambda}{e^{2h_\infty} C^{2 + \pi}}\right\}.
\]

\( \text{Proof:} \)

1) We use the upper bound for \( |\mathcal{D}^N G(t_k)| \) in Proposition 1 to derive a sufficient condition for (20) as

\[
\left(\frac{T_{\max}}{T_{\min}}\right)^N \frac{\Omega e}{g_\infty} < \frac{\lambda}{\lambda g_\infty} \frac{\lambda}{\lambda T_{\max}}.
\]

To prove (31), note that, in the limit case, \( T_{\max} = T_{\min} = T \) is always satisfied for large enough \( N \) if \( \frac{T_{\max}}{T_{\min}} = T \Omega e < 1 \). The latter is the sampling rate condition for uniform sampling [14]. Then S1) follows from (31) and S2) from (27) where \( T_{\min} \) and \( T_{\max} \) are selected as the following bounds for the intervals between the trigger times [1]

\[
T_{\min} = \frac{2\delta}{b + \lambda} \leq |t_{k+1} - t_k| \leq \frac{2\delta}{b - \lambda} = T_{\max}.
\]

2) From S1) and S2) it follows that \( 2\delta < \min\{B_1(N), B_2(N)\} \), where

\[
B_1(N) = \frac{b - \lambda}{C} \left( \frac{\lambda}{g_\infty N \Omega e} \right) \quad \text{and} \quad B_2(N) = \frac{(b - \lambda) h^*}{N \Omega g_\infty}.
\]

A direct calculation yields \( B_1(N) \geq B_2(N) \), which completes the proof.

If \( g_\infty \) is unknown yet an upper bound is known, the results in Lemma 3 still hold true by replacing \( g_\infty \) with the upper bound. Furthermore, we note that the conditions in Lemma 3 are only sufficient, and can be conservative. Therefore, in practice, one can achieve good reconstructions with higher values of \( N \), some of which that might not satisfy the lemma.
\[ E_g(t_k) = 2 \lambda_h \sum_{r=1}^{R} s_r \cdot 1_{[\tau_r, \infty)}(t_k - \tau_r) \cdot (t_k - \tau_r). \]  

Finally, \( g(t) \) is recovered recursively using reconstruction from local averages: 

\[ \Delta G(t_k) = g(t_{k+1}) - g(t_k) = \int_{t_k}^{t_{k+1}} g(s) ds \]  

where \( G(t_k) = X(t_k) + E_g(t_k) \). This recovery approach was proposed in the context of irregular sampling in [13] and adapted for event-driven ASDM sampling in [1]. In our case, sequence \( \bar{g}_n(t) \) is computed as 

\[ g_0(t) = \mathcal{S}_2 \left[ \Delta G(t) \right]_n(t), \]  
\[ g_{n+1}(t) = \bar{g}_n(t) + \bar{g}_0(t) - \mathcal{S}_2 \left[ L \bar{g}_n(t) \right], \]  

where \( \mathcal{S}_2 : \ell^2 \to L^2(\mathbb{R}), \mathcal{S}_2[a_k] = \sum_{k \in \mathbb{Z}} a_k \cdot \text{sinc}_2(t - s_k), \) \( s_k = \frac{t_k + t_{k+1}}{2}, \) and \( L : L^2(\mathbb{R}) \to \ell^2, (Lf)_k = \int_{t_k}^{t_{k+1}} f(s) ds \). 

We outline the recovery procedure in Algorithm 1.

The following result proven in Section VI gives a bound for the input reconstruction error.

**Proposition 3:** Assuming that the conditions in Lemma 3 are satisfied and that the data consists of \( R \) folding times, then the input can be recovered recursively as \( \bar{g}_n(t) \), where \( n \) is the iteration number. The reconstruction error satisfies: 

\[ \|g - \bar{g}_n\|_{L^2} \leq \frac{4\lambda_h 2\delta R \sqrt{2\pi}}{\pi (b - \lambda) - 2\delta \Omega} + \left( \frac{2\delta \Omega}{\pi (b - \lambda)} \right)^{n+1} \|g\|_{L^2}. \]  

We note that the second term in the error bound (35) is guaranteed to vanish for \( n \to \infty \) if \( \frac{2\delta \Omega}{\pi (b - \lambda)} < 1 \), which is satisfied if condition \( S_2 \) in Lemma 3 is true. Furthermore, the error in (35) can be made arbitrarily small for a small enough ASDM threshold \( \delta \). Furthermore, we note that Algorithm 1 has an intrinsic method to indicate that the data does not verify the recovery conditions, if \( k^*_{M} - k^*_{m} \notin \{N - 3, N - 2, N - 1\} \) for any \( r \in \{1, \ldots, R\} \). Although theoretically there could be situations where the algorithm runs with no error for an input not satisfying the recovery conditions, we found this to be very unlikely in practice.

**V. NUMERICAL AND HARDWARE EXPERIMENTS**

We validate the proposed event-driven model using both synthetic and real data measurements. We generate a bandlimited function \( g \in \mathbb{W} \), which is encoded directly using the stand-alone ASDM into trigger times \( t_k^{\text{ASDM}} \). The same input is then processed with the proposed encoding model, which triggers times \( t_k^{\text{MEDS}} \). The direct reconstruction from ASDM samples is denoted \( g_{\text{ASDM}}(t) \), and the proposed reconstruction from the MEDS samples is denoted \( g_{\text{MEDS}}(t) \). We measure the relative
recovery error via the ratio $\text{Err}(g, \hat{g})$, defined as
\[
\text{Err}(g, \hat{g}) = 100 \times \frac{\|g - \hat{g}\|_2\|g\|_2}{\|g\|_2^2} \text{ (\%)}.
\] (36)

We denote the recovery errors with each method by $\text{Err}_{\text{ASDM}} = \text{Err}(g_{\text{ASDM}}, g)$ and $\text{Err}_{\text{MEDS}} = \text{Err}(g_{\text{MEDS}}, g)$, respectively.

The recovery results from synthetic data are in V-A, and the ones from experimental data are presented in V-B.

### A. Reconstruction Using Synthetic Data

Here we compare the proposed model with the ASDM encoder. The input is $g(t) = Ag_0(t)$, where
\[
g_0(t) = \sum_{n \in \mathbb{Z}} c_n \text{sinc}_\Omega(t - n\pi/\Omega),
\]
where $\Omega = 150 \text{ rad/s}$ and $c_n$ are drawn from the uniform distribution $U([-1, 1])$. The signal is truncated such that $t \in [0, 0.13 \text{ s}]$, and normalized such that $\max_{t} |g_0(t)| = 1$. The ASDM parameters are $\delta = 2.5 \times 10^{-3}$, $b = 9$. The Nyquist rate condition is $|g(t)| < g_{\text{MAX}}$, where $g_{\text{MAX}}$ denotes the ASDM dynamic range $g_{\text{MAX}} = \frac{\pi b - 2\delta}{\pi} = 8.76$.

The proposed encoder consists of a modulo model with threshold $\lambda = g_{\text{MAX}}/2$ and hysteresis parameter $h = \lambda/2$ in series with an ASDM whose parameters are the same as the standalone ASDM encoder. Therefore the models are equivalent for $|g(t)| < \lambda$. For $A = 34.6$, the ASDM generates trigger times $\{\tau_{k_{\text{ASDM}}}\}_{k=1}^{14}$ in response to input $g(t)$ when its amplitude is within the ASDM dynamic range $[-8.76, 8.76]$, and sends no information when the input amplitude is outside the range. However, the proposed model generates trigger times $\{\tau_{k_{\text{MEDS}}}\}_{k=1}^{220}$, which cover the whole duration of $g(t)$, as depicted in Fig. 8(a). The input is first recovered from the ASDM samples $\{\tau_{k_{\text{ASDM}}}\}_{k=1}^{14}$, which leads to a high reconstruction error especially where the input has a large amplitude, as depicted in Fig. 8(d). For the proposed model, we used the $\mathcal{D}^3$ to identify correctly the 21 folding times. The filtered data $\mathcal{D}^3 X(t_{k_{\text{MEDS}}})$, the threshold $\Psi_{\text{MEDS}}[k]$ and the estimated folding times are in Fig. 8(b), (c). We then recovered $\hat{g}_{\text{MEDS}}(t)$ using Algorithm 1. The resulted errors for the two methods are $\text{Err}_{\text{ASDM}} = 6.11 \times 10^{-3}\%$ and $\text{Err}_{\text{MEDS}} = 0.25\%$.

To see the effect of changing the ASDM threshold $\delta$ on the recovery performance, we repeated the recovery for 10 uniformly spaced values of $\delta$ in interval $[10^{-3}, 3 \times 10^{-3}]$. We evaluated the input error as before, the number of spikes generated, and the folding time error evaluated as $\text{Err}_{\tau} = 100 \times \frac{\|\tau - \hat{\tau}\|_2\|\tau\|_2}{\|\tau\|_2^2}$ (\%).

The results, depicted in Fig. 9, show that when $\delta > 2.6 \times 10^{-3}$ the recovery algorithm does not identify the folds correctly leading to a large error. Choosing $\delta < 1.8$ leads to an exponential increase in sample size without a significant decrease in error. Moreover, we remark that, in this example, the input error is around 6 times larger than the folding time error. This is in line with the theoretical derivation that the error for computing $\Delta G(t_k)$ is $2\lambda_{\text{h}} = 6.57$ times larger than the folding time error as shown in (57).
B. Reconstruction Using Experimental Data

In this example, the input is $g(t) = 4.51 \cdot \sin((\Omega(t - \tau_0))$ where $\Omega = 125 \text{ rad/s}$ and $\tau_0 = 1.6 \cdot 10^{-2}$. The output data was generated using a MEDS hardware prototype shown in Fig. 5 that is coarsely calibrated such that the modulo output is within the ASDM dynamic range. The corresponding acquisition pipeline is shown in Fig. 10. The MEDS parameters, estimated using line search based optimization [39], are $\lambda = 1.53$, $h = 1.51$, $\delta = 2.07 \times 10^{-4}$, and $b = 2.22$. Therefore the ASDM dynamic range is $g_{\text{MAX}} = \frac{\frac{\varepsilon h}{2} - 2 b \Omega}{\pi} = 2.206$. The output of the prototype in response to $g(t)$ is $\{\tau^{\text{MEDS}}_{k \in 1} \}$ depicted in Fig. 11(a).

We used the $G$ to identify correctly the 12 folding times. The filtered data $G^2 X(\text{MEDS})$, the threshold $\Psi_N[k]$ and the estimated folding times are in Fig. 11(b), (c). The input is recovered with Algorithm 1. The resulted error is $\text{Err}_{\text{MEDS}} = 0.68\%$.

VI. PROOFS

Here we prove Proposition 1, Lemma 2, Theorem 1, and Proposition 3.

Proof of Proposition 1: We fix an arbitrary value $k \in \mathbb{Z}$ and expand $G(t_{k+l})$, $l \in \{0, \ldots, N\}$ in Taylor series around $\tau = \frac{t_{k+N} + t_k}{2}$

$$G(t_{k+l}) = G(\tau) + \sum_{n=1}^{N-1} (t_{k+l} - \tau)^n \frac{g^{(n-1)}(\tau)}{n!}$$

Further, we note that operator $G$ cancels out polynomials of degree up to $N - 1$, and thus

$$|G^N G(t_k)| = |G^N G(r_k)| = \left| G^{N-1}(k+1) - G^{N-1}(r_k) \right|$$

Similarly, via induction, it can be shown that

$$|G^N G(t_k)| \leq \max_{l=0, \ldots, N} \left| G^{N-1}(r(k+l)) \right|$$

To prove Lemma 2, we first provide a few properties of $\mu^N_N[k]$ and $\beta^N_N[k]$ in the following proposition.

Proposition 4 (Properties of $\mu^N_N$ and $\beta^N_N$): The following hold,

$$\text{supp} \mu^N_N[k] = \{l - N + 1, \ldots, l - 1\},$$

$$\text{supp} \beta^N_N[k] = \{l - N + 2, \ldots, l\}.$$ (38)

For all $k \in \mathbb{Z}, \forall N \geq 2$, it follows that

$$\text{sign} \{\mu^N_N[k]\} \cdot \text{sign} \{\mu^N_N[k + 1]\} \leq 0,$$

$$\text{sign} \{\beta^N_N[k]\} \cdot \text{sign} \{\beta^N_N[k + 1]\} \leq 0,$$

$$\text{sign} \{\mu^N_N[k]\} \cdot \text{sign} \{\beta^N_N[k]\} \leq 0.$$ (39)

The following bounds hold true.

$$\frac{1}{N!T_{\text{max}}} \leq \left| \beta^N_N[k] \right| \leq \frac{1}{N!T_{\text{max}}},$$

$$\frac{1}{N!T_{\text{min}}} \leq \left| \mu^N_N[k] \right| \leq \frac{1}{N!T_{\text{min}}},$$

$$\frac{1}{N!T_{\text{min}}} \leq \left| \beta^N_N[l - N + 2] \right| \leq \frac{N - 2}{N! T_{\text{min}} - 1},$$

$$\frac{1}{N!T_{\text{min}}} \leq \left| \beta^N_N[l - 1] \right| \leq \frac{N - 2}{N! T_{\text{min}} - 1}.$$ (42)

Proof: Equation (38) is derived directly from the definitions of $\mu^N_N$ and $\beta^N_N$ in Proposition 2. Moreover, (39) can be shown directly via induction for $N \geq 2$. Bounds (40) are shown recursively by using $n T_{\text{min}} \leq t_{k+l} - t_k \leq n T_{\text{max}}$.

$$m_{N+1} = \frac{\mu^N_N[l - N + 2] - \mu^N_N[l - N + 3]}{t_{k+2} - t_{k + N + 1}} \leq \frac{m_{N+1} + |\mu^N_N[l - N + 3]|}{(N + 1)!T_{\text{max}}}$$. (43)

Similarly, via induction, it can be shown that

$$|G^N G(t_k)| \leq \max_{l=0, \ldots, N} \left| G^{N-1}(r(k+l)) \right|$$

To derive (42), we may use the following inequalities, respectively.

$$|G^N G(t_k)| \leq \left( \frac{t_{\text{max}}}{T_{\text{min}}} \right)^N \frac{|g^{(N-1)}|}{N!}$$

$$\leq \frac{1}{N!} \left( \frac{t_{\text{max}}}{T_{\text{min}}} \right)^N \frac{\|g\|_{\infty}}{N!}.$$
To show (46), we denote by $f_1, f_2 : [0, 1] \to \mathbb{R}$ two functions

$$f_1(x) = \left| \tilde{\mu}_1 x + \tilde{\beta}_1 (1 - x) \right|, \quad \text{(38)}$$

$$f_2(x) = \left| \tilde{\mu}_2 x + \tilde{\beta}_2 (1 - x) \right|, \quad \text{(39)}$$

where $\tilde{\mu}_1 = \mu^{N}_{K_1}[K_1 - N + i], \tilde{\beta}_1 = \beta^{N}_{K_1}[K_1 - N + i], i = 1, 2$. Given that $E_{K_{N+1}}^N f_1 = f_1(p)$ and $E_{K_{N+2}}^N f_2 = f_2(p)$ (16), which gives

$$\max \{E_{K_{N+1}}^N, E_{K_{N+2}}^N\} \geq \min_{x \in [0, 1]} \left( \max_{i=1,2} f_i(x) \right). \quad \text{(48)}$$

Function $f_1(x)$ is strictly increasing, whereas $f_2(x)$ has one zero $f(x^*) = 0$, where $x^* = \frac{\tilde{\beta}_2}{|\tilde{\mu}_1| + |\tilde{\beta}_2|}$. Then $f_2(x)$ is strictly decreasing for $x \in [0, x^*)$, and thus, given that $f_1(0) = 0$ and $f_2(0) > 0$, it follows that there exists a unique point $x^{**} \in (0, x^*)$ such that $f_1(x^{**}) = f_2(x^{**})$. Indeed, when solving $f_1(x) = f_2(x)$ for $x$, there are two solutions $x_k = \frac{|\tilde{\mu}_1| + |\tilde{\beta}_2|}{|\tilde{\beta}_1| + |\tilde{\mu}_2| + |\tilde{\beta}_2|}, k = 1, 2$, such that $x_2 < x^* < x_1$, and thus $x^* = \frac{|\tilde{\beta}_1|}{|\tilde{\mu}_1| + |\tilde{\beta}_1| + |\beta|}. \bar{\mu}(1)$.

It follows that $\max_{i=1,2} f_i(x) = f_2(x), x \in [0, x^*)$ and $\max_{i=1,2} f_i(x) = f_1(x), x \in [x^*, x^*]$. Given that for $x > x^*$, $f_2(x)$ is strictly increasing, then $f_2(x) > f_2(x^*), \forall x > x^*$, and thus

$$\max_{i=1,2} f_i(x) \geq \max_{i=1,2} f_i(x^*) = \left| \bar{\mu}_1 x^* \right| = \phi_{0, K_1},$$

which yields (46) via (48).

The bound (47) is proven following the same steps as for (46). Specifically, we denote by $f_3, f_4 : [0, 1] \to \mathbb{R}$

$$f_3(x) = \left| \tilde{\mu}_3 x + \tilde{\beta}_3 (1 - x) \right| = \left| \bar{\mu}_3 x - \bar{\beta}_3 (1 - x) \right|, \quad \text{(39)}$$

$$f_4(x) = \left| \tilde{\mu}_4 x + \tilde{\beta}_4 (1 - x) \right| = \left| \bar{\mu}_4 x - \bar{\beta}_4 (1 - x) \right|, \quad \text{(38)}$$

where $\tilde{\mu}_4 = \mu^{N}_{K_1}[K_1 - 4 + i], \tilde{\beta}_4 = \beta^{N}_{K_1}[K_1 - 4 + i], i = 3, 4$. Using $y = 1 - x$ we get the same problem as before, which yields (47) via direct calculation. Finally, (46), (47), together with (44) imply that for $m \leq K_1 - N + 2$ and $m \geq K_1 - 1$. This completes the proof of the lemma via $\mathcal{L}_N \subseteq \mathcal{S}_N$. □

Proof of Theorem 1: To show that $\mathcal{S}_1 = \bar{s}_1$, we observe that $\mathcal{D}^N \mathcal{G}(t_n) < \Psi^N[k_m]$ and $\mathcal{D}^N \mathcal{Z}(t_{k_m}) \geq \Psi^N[k_m]$ which implies that sign $[\mathcal{D}^N \mathcal{Z}(t_{k_m})] = \text{sign}[\mathcal{D}^N \mathcal{G}(t_{k_m})]$ and thus (16)

$$\bar{s}_1 = \text{sign} \left[ \bar{\mu}_1 \mathcal{G}_m + \beta^{K_1}[K_1 - N + 1] \right] \cdot s_1. \quad \text{(49)}$$

1) If $k_N - k_m = N - 1$, then, according to Lemma 2, we know that $k_N = K_1 - N + 1, K_1 = K_1$. Then, due to (38), $\mu^{N}_{K_1}[K_1 - N + 1, K_1] = \beta^{K_1}[K_1 - N + 1] = 1, V > 2$, and thus $\bar{s}_1 = s_1$ via (49). Moreover, $\bar{\tau}_1 = \frac{K_1 - 1}{2}$ which satisfies (24) given that $\bar{\tau}_1 \in [0, 1].$ Note that $\bar{\tau}_1 \in [0, 1].$

2) If $k_N - k_m = N - 2$, via Lemma 2 there are two options

a) $k_N = K_1 - N + 1, K_1 = K_1 - 1$. Here (25) (based on the same reasoning as in 1).

b) $k_N = K_1 - N + 2, K_1 = K_1$. Here we have that

$$\bar{s}_1 = \text{sign} \left( f(p) \right) \cdot s_1, \quad \text{(50)}$$

where $f(p) = \tilde{\mu}_2 p + \tilde{\beta}_2 (1 - p), \mu^{N}_{K_1}[K_1 - N + 2], \beta^{N}_{K_1}[K_1 - N + 2]$. We know that $\mathcal{D}^N \mathcal{Z}(t_{K_1-N+1}) \leq \Psi^N[K_1 - N + 1]$, which implies that $\mathcal{D}^N \mathcal{E}_g(t_{K_1-N+1}) < 2 \Psi^N[K_1 - N + 1]$ via (44). This can be re-written as

$$\frac{\mathcal{D}^N \mathcal{E}_g(t_{K_1-N+1})}{2\lambda_h} \leq \left| \bar{\mu}_1 \cdot p \right| < \Psi^N[K_1 - N + 1] \lambda_h$$

3) If $k_m - k_N = N - 3$, Lemma 2 implies $k_m = K_1 - N + 3, k_N = K_1 - 1$. Then $\bar{s}_1 = s_1$ via the same derivation as in 2 b). Moreover, $\bar{\tau}_1 = \frac{k_0 + k_m + 1}{4}$ which satisfies (24).

Proof of Proposition 3: Assuming the values $\Delta G(t_h) = (Lg)_k$ known, $g$ can be recovered recursively as [45], [46]

$$g_0(t) = L_0 \left[ \Delta G(t_k) \right], \quad \text{(51)}$$

Then, provided that $T_{\text{max}} < \frac{N}{\lambda}$ the following hold [37]

$$\| g - g_n \|_{L_2} \leq \left( \frac{T_{\text{max}} \Omega}{\lambda} \right)^{n-1} \| g \|_{L_2}, \quad \text{(52)}$$

$$\| I - L_0 \|_{L_2} \leq \left( \frac{T_{\text{max}} \Omega}{\lambda} \right), \quad \text{(53)}$$

where $I$ is the identity operator, and thus $\lim_{n \to \infty} g_n = g$.

When comparing (51) with (34), the main difference is in the initial condition $g_0$, which is determined by samples $\Delta G(t_k)$. In the following, we evaluate the error in computing these samples.

We begin by making two observations.

Observation 1: As derived at the end of the Proof for Theorem 1, $\bar{\tau}_r \in \left[ t_{k_h}, t_{K_1} \right]$, this, and via the definition of $K_r$, we get $\tau_r, \bar{\tau}_r \in \left[ t_{K_1}, t_{K+1} \right]$, i.e., there are no spike times in between the true and estimated fold.

Observation 2: Given $\frac{N_{200g}}{\lambda} \leq h^*$ (condition S2 in Lemma 3) we are guaranteed to have at least two spike times between each two consecutive folding times $\tau_r, \tau_{r+1}$.

Using (12) and that $X(t_h)$ is known, we get that $|\Delta G(t_h) - \Delta G(t_k)| = |\Delta E_g(t_h) - \Delta E_g(t_k)|$. We will derive a bound for the latter by considering two cases.

1) $k \in \mathbb{Z}_+ \setminus \left\{ K_1, \ldots, K_H \right\}$. We first consider the trivial case where $k < K_1 \Rightarrow E_g(t_k) = E_0(t_k) = 0$ (Observation 1). If $k > K_1$ then $\exists r \in \left\{ 1, \ldots, R \right\}$ s.t. $k \in \left\{ K_{r_0} + 1, \ldots, K_{r_0+1} - 1 \right\}$ (Observation 2). Using the expression of $E_g$ we get that

$$\Delta E_g(t_k) = \sum_{r=1}^{r_0} 2\lambda_h s_r \Delta(I_{\tau_r+\infty}(t_k) - \tau_r) \quad \text{(54)}$$
\[
\rho = \sum_{r=1}^{R} 2\lambda bs_r (t_{k+1} - t_k).
\]

The equalities use that there are no folds in \([t_k, t_{k+1}]\) given that \(k \neq K r_o\). Using the same derivation for \(\Delta E_g\) and that \(s_r = s_r, \forall r \in \{1, \ldots, R\}\), we get that \(\Delta E_g(t_k) = \Delta E_g(t_k)\).

2) \(k \in \{K, \ldots, K_g\}\). Let \(k = K r_o, r_o \in \{1, \ldots, R\}\). Due to Observation 1 we have that \(\tau_{r_o} - \tau_{r_o} \in [K r_o, t_{K r_o+1}]\). Then, using that \(\tau_1 = t\), we get

\[
\Delta E_g(t_k) = \sum_{r=1}^{R} 2\lambda bs_r (t_{k+1} - t_k) + 2\lambda b s_{r_1} (t_{k+1} - \tau_{r_o}).
\]

Therefore \(|\Delta E_g(t_k) - \Delta E_g(t_k)| = 2\lambda b \tau_{r_0} - \tau_{r_o} |\).

By combining 1) and 2), we get:

\[
|\Delta G(t_k) - \Delta G(t_k)| \leq 2\lambda \max_r |r - \tau_r|.
\]

(56)

To bound theestimation of \(e_n\), we define \(e_n = \bar{g}_n - g_n \Rightarrow e_0 = \mathcal{S}_\Omega[\Delta G(t_k) - \Delta G(t_k)]\) and thus

\[
e_0(t) = \sum_{k \in \mathbb{R}_n} [\Delta G(t_k) - \Delta G(t_k)] \cdot \text{sinc}_\Omega(t - s_k).
\]

(57)

Given that \(|\mathcal{K}_r| \leq 2R\), via the triangle inequality and (57) we have that

\[
\|e_0\|_{L^2} \leq 2R \max_k |\Delta G(t_k) - \Delta G(t_k)| \cdot \|\text{sinc}_\Omega\|_{L^2} \\
\leq 4\lambda b \sqrt{R} \cdot T_{\text{max}} \sqrt{\frac{\Omega}{\pi}}.
\]

(59)

Furthermore, we have that (34, 51)

\[
\|e_n\|_{L^2} = \|e_{n-1} + e_0 - \mathcal{S}_\Omega e_{n-1}\|_{L^2} \leq \|e_0\|_{L^2} + \|I - \mathcal{S}_\Omega\| \cdot \|e_{n-1}\|_{L^2}
\]

(60)

Via induction, given that \(T_{\text{max}} < \frac{\pi}{\Omega}\), we get

\[
\|e_n\|_{L^2} \leq \|e_0\|_{L^2} \sum_{l=0}^{\infty} \left(\frac{\max\Omega}{\pi}\right)^l = \|e_0\|_{L^2} \left(1 - \frac{\max\Omega}{\pi}\right)^{n+1} \leq \|e_0\|_{L^2} \left(1 - \frac{\max\Omega}{\pi}\right)^n.
\]

(61)

Finally, we bound the reconstruction error as

\[
\|\bar{g}_n - g\|_{L^2} \leq \|\bar{g}_n - g_n\|_{L^2} + \|g - g_n\|_{L^2},
\]

(62)

which yields the desired result via (59), (63), (52), where \(T_{\text{max}}\) satisfies (32).

\[ \square \]

VII. CONCLUSIONS AND FUTURE WORK

Summary of Results: We proposed a new encoding model based on the modulo event-driven sampling (MEDS) pipeline consisting of a modulo-hysteresis nonlinearity in series with an asynchronous sigma-delta modulator (ASDM) model. Our result contributes to the existing efforts of alleviating the dynamic range restrictions of existing EDS sensors. We provide mathematical guarantees for the input reconstruction, and validate the MEDS model with synthetic and hardware experiments. Numerical simulations show that the proposed method can recover inputs with a dynamic range up to 20 times the modulo threshold.

Future Work: We plan on following up the results in several directions,

1) The method can be extended to modulo models in series with other event-driven encoders, such as level-crossing encoders and biphase integrate-and-fire neurons.
2) The nonuniform finite difference could be replaced by a more general operator for achieving a better recovery.
3) The current methodology can be extended to inputs in multiple dimensions.

New exciting hardware applications such as event-driven cameras show the promise that this encoding scheme holds in the field of signal processing. Despite this, the sensor dynamic range restriction remains a fundamental bottleneck in the field. Recently, it was shown that a high dynamic range image can be recovered from uniform samples in a single capture [22]. Given that event-driven signals have an inherently low dynamic range, this line of research represents a natural step in the field that could lead to promising new encoding strategies.

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