Abstract

We describe a scheme of constructing classical integrable models in 2 + 1-dimensional discrete space-time, based on the functional tetrahedron equation—equation that makes manifest the symmetries of a model in local form. We construct a very general “block-matrix model” together with its algebro-geometric solutions, study its various particular cases, and also present a remarkably simple scheme of quantization for one of those cases.

Introduction

Consider two electric devices, each made of three resistors and with three outer contacts, and whose diagrams are triangle and star, respectively. It is well known that they are equivalent iff the following relations for resistances hold:

$$r_j r'_j = r'_1 r'_2 + r'_2 r'_3 + r'_3 r'_1 = \frac{r_1 r_2 r_3}{r_1 + r_2 + r_3}, \quad j = 1, 2, 3,$$

where resistances $r_j$ apply to the triangle, and resistances $r'_j$—to the star. These relations enable us to find $r'_j$ for any given triple of $r_j$, and vice versa.
We will represent such electric devices diagrammatically in a simplified manner (in comparison with usual electric diagrams): resistors will be represented as edges of a graph, outer contacts—as blank circles, and inner contacts—as filled circles. In this way, the transformation (1) is represented in Figure 1.

\[ r_j \leftrightarrow \frac{1}{r_j}. \]

This leads to the fact that the “star–triangle” and “triangle–star” transformations, after a simple change of variables, can be expressed as the same transformation. The exact formulation of this will be given below in subsection 3.3.

Consider now the diagrams in Figure 2. We can transform diagram (a) into diagram (b) in four steps, each being a star–triangle or triangle–star transformation for some three resistors. What is important, we can do it in two ways, represented in Figure 3 (where
Figure 3

the obvious arrangement of blank and filled circles is not shown).

It can be seen from “electrical” argument that the two ways in Figure 3 must lead to the same result. Mathematically, this means that the transformation (1) is very specific: it satisfies (when written as in subsection 3.5 below) the \textit{functional tetrahedron equation} (FTE).

In this paper, we will try to show that FTE constitutes the basis of 2 + 1-dimensional integrability. We will explain how to generalize the simple “electrical” construction for its solutions in such a way that it will give not only a huge amount of new integrable classical systems, but even integrable quantum systems. The contents of the rest of this paper is as follows. In section 1 we consider the most general abstract model where the FTE naturally arises. In section 2 we present a concrete, but still very general incarnation of the abstract model, together with the algebro-geometric method of constructing its solutions. In section 3 we give a list of solutions representing the simplest “one-parametric reductions” of the general model. In section 4 we show that the “electric” model described above is connected with the well-known bilinear Miwa equation. In section 5 we present an amazingly direct quantization for one of our “one-parametric” models, using, essentially, the same FTE. Finally, we discuss our results in section 6.

1 Classical straight-string model and functional tetrahedron equation

Let there be several straight strings, i.e. oriented straight lines (with the “positive direction” indicated in each line), in a plane. Let there be among those strings (i) no coinciding ones, (ii) no three of them intersecting in one point, and (iii) let the orientations of strings be \textit{consistent} in the following sense: moving from any point along the strings in positive directions, one cannot return in the same point. Let us attribute to each point of intersec-
tion of two strings, e.g. strings $a$ and $b$, some object $X_{ab}$. Now let us allow each string to move, remaining parallel to its initial position, and pass through points of intersection of other strings, but so that (i) two parallel strings never go through each other and (ii) at no moment there should be four strings intersecting in one point.

Let us assume that an object $X_{ab}$ changes only if some string $c$ goes through the intersection point of $a$ and $b$. To be exact, in such case the triple $(X_{ab}, X_{ac}, X_{bc})$ is transformed into a new triple $(X'_{ab}, X'_{ac}, X'_{bc})$ according to some fixed rule. When we consider such a transformation, we think of both those triples as ordered in the following way: one can get from $X_{ab}$ to $X_{ac}$ and then to $X_{bc}$ moving along the strings in positive directions, while from $X'_{ab}$ to $X'_{ac}$ and then to $X'_{bc}$—in negative directions. This can be represented graphically as turning the triangle inside out, see Figure 4.

![Figure 4](image)

We will say that the operator $R$ acts on the triples of objects:

$$R : \ (X_{ab}, X_{ac}, X_{bc}) \rightarrow (X'_{ab}, X'_{ac}, X'_{bc}).$$

(2)

Note that in Figure 4 the point of intersection of strings $a$ and $b$ goes from the left part of the plane, with respect to orientation of $c$, to the right part. If we make in Figure 4 the changes $c \leftrightarrow a$, $X_{ab} \leftrightarrow X'_{ab}$, LHS $\leftrightarrow$ RHS, we will see that for the case when the point of intersection of strings $a$ and $b$ goes from the right part of the plane, with respect to orientation of $c$, to the left part, $R$ should be replaced by $P_{13}R^{-1}P_{13}$, where $P_{13}$ stands for the interchange of the first and the third objects in a triple. In other respects, we assume $R$ be the same for all triples of strings, i.e. not depending, say, on the angles between them or whatever. At the same time, we will not exclude the situation where $R$ is not single-valued—in such case we will say that the “primed triple” is determined up to a “gauge equivalence”.
If there are many enough non-parallel strings in our model, there exist many possibilities for their movement. It is natural to regard as “integrable” such a model where to a passage from one string configuration to another there will correspond a transformation of objects $X_{ab}$ not depending on the details of that passage, e.g. which string was the first to pass through the intersection point of two others and so on. For this to hold, it is enough to require that the operation of turning a triangle inside out, as in Figure 4, commute with passing a fourth string across the whole triangle. In other words, the two ways of transforming the left side of Figure 5 into its right side, one of which starts with turning inside out triangle 356 and the other one—triangle 123, must lead to equal results. This condition is the functional tetrahedron equation (FTE) to which this paper is devoted:

$$R_{123} \circ R_{145} \circ R_{246} \circ R_{356} = R_{356} \circ R_{246} \circ R_{145} \circ R_{123}. \quad (3)$$

Here, of course, $R_{123}$ acts only on the triple

$$(X_1, X_2, X_3) \overset{\text{def}}{=} (X_{ab}, X_{ac}, X_{bc})$$

and so on.

The diagrammatic representation of equation (3) itself is given in Figure 6.

To conclude this section, note that we could associate $X$-like objects not to the intersection points of strings, but to the segments in which the strings divide each other, or to the domains of the plane in which it is divided by the strings. It would lead us to other versions of the functional tetrahedron equation, in complete analogy with the quantum case [5, 6]. It is natural to call equation (3) the vertex type FTE. In this paper we will be dealing only with this type of equations.
2 The block-matrix model and its algebro-geometric solutions

2.1 Formulation of the model

Let now to each of the strings correspond a finite-dimensional complex linear space, e.g. to the string $a$—the space $V_a$. We will also associate with a set of strings the direct sum of their linear spaces (and not the tensor product, in contrast with the usual theory of Yang–Baxter equation). The objects $X_{ab}$ will be linear operators in $V_a \oplus V_b$. Fixing once and for ever the bases in all $V_a$, we will identify those operators with block matrices of the type

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A$ acts within $V_a$, $B$—from $V_b$ to $V_a$, etc. The “primed” operators—the result of the action of operator $R(2)$—will be determined from the equation

$$X_{ab}X_{ac}X_{bc} = X_{bc}'X_{ac}'X_{ab}',$$

which in expanded form is written as

$$\begin{pmatrix} A_1 & B_1 & 0 \\ C_1 & D_1 & 0 \end{pmatrix} \begin{pmatrix} A_2 & 0 & B_2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & A_3 & B_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & A_1' & B_1' \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & A_2' & B_2' \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & A_3' & B_3' \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & A_1'' & B_1'' \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & A_2'' & B_2'' \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & A_3'' & B_3'' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A_1' & B_1' \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & A_2' & B_2' \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & A_3' & B_3' \end{pmatrix}.$$

Figure 6
i.e. we identify each of the operators

\[ X_{ab} = X_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \ldots, X_{bc}' = X_3' = \begin{pmatrix} A_3' & B_3' \\ C_3' & D_3' \end{pmatrix} \]

with the direct sum of itself and the unity operator in the lacking space.

**Remark.** If we now want to consider Figure 4 as the illustration to formula (4), we have to take into account that the operators in (4) act against the arrows in Figure 4. On the other hand, in Figure 6 the arrows point in the same direction where the (nonlinear) operators \( R \) act.

In case if the operators in the LHS of equation (4) are generic enough, that equation is solvable with respect to the “primed” operators, and those latter are determined uniquely up to the following obvious block-diagonal “gauge transformations”:

\[
\begin{pmatrix} A_3' & B_3' \\ C_3' & D_3' \end{pmatrix} \to \begin{pmatrix} A_3' & B_3' \\ C_3' & D_3' \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & M \end{pmatrix},
\]

\[
\begin{pmatrix} A_2' & B_2' \\ C_2' & D_2' \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & M^{-1} \end{pmatrix} \begin{pmatrix} A_2' & B_2' \\ C_2' & D_2' \end{pmatrix} \begin{pmatrix} K & 0 \\ 0 & 1 \end{pmatrix},
\]

\[
\begin{pmatrix} A_1' & B_1' \\ C_1' & D_1' \end{pmatrix} \to \begin{pmatrix} K^{-1} & 0 \\ 0 & L^{-1} \end{pmatrix} \begin{pmatrix} A_1' & B_1' \\ C_1' & D_1' \end{pmatrix}.
\]

The details can be found in [4], chapter 2.

As for the functional tetrahedron equation, it follows in our block-matrix model from the fact that the product of six \( X \)-matrices corresponding to the RHS of Figure 5 determines those matrices as well (again up to obvious block-diagonal gauge transformations). The proof of this statement is a routine exercise in matrix algebra (to begin, one can unite, for a while, the strings \( c \) and \( d \) in the RHS of Figure 5 in one “thick” string, and then apply several times the uniqueness, up to gauge transformations, of factorizing the product of three matrices).

### 2.2 Algebro-geometric solutions

Let us take some (smooth irreducible) algebraic curve \( \Gamma \) of genus \( g \). Let us attach to each string \( a \) two effective divisors (i.e. finite sets of points of \( \Gamma \) with positive multiplicities) \( D'_a \) and \( D''_a \) of the same degree, equal to \( \dim V_a \) (the dimension of linear space associated with \( a \)). For two non-parallel strings \( a \) and \( b \), we will require that no point enter at the same time in both \( D'_a \) and \( D''_b \) (as well as in both \( D''_a \) and \( D'_b \)). The meaning of this will be clear later.

Next, let us put in correspondence to each point \( A \) of the plane where the strings live a divisor \( D^A \) according to the following rules. First, for each string \( a \), if \( A \) lies to the left of \( a \) (with respect to its orientation), take divisor \( D'_a \), if \( A \) lies to the right of \( a \), take divisor \( D''_a \), if \( A \) lies in \( a \), take the zero divisor, and denote that as \( D^A_a \). Then set \( D^A = \sum_{a \in A} D^A_a \).

We will only be interested in divisors corresponding this way to (inner points of) string segments limited by string intersection points, and to those intersection points themselves.
For example, if there are only two strings, to their halves and to the intersection point will correspond divisors as in Figure 7.

\[ \text{Figure 7} \]

Take now an arbitrary (generic) divisor \( D \) of degree

\[ \deg D = \sum_{\text{over all } a} \dim V_a + g - 1. \]

If to a given segment \( l \) of a string \( a \) corresponds a divisor \( D_l \), let us also put in correspondence to that segment the linear space of meromorphic functions \( f \) on \( \Gamma \) whose divisor \((f)\) (the difference between the zero divisor and the pole divisor) obeys the condition

\[ (f) + D - D_l \geq 0 \]

(this means that the singularities of \( f \) are controlled by divisor \( D \) and, moreover, \( f \) must have zeroes in the points of divisor \( D_l \)). It is easy to see that the space of such functions is isomorphic to (i.e. has the same dimension as) \( V_a \), so we will identify it with \( V_a \), using for that an arbitrary isomorphism.

We will need the following explicit description of isomorphisms of this kind. The space \( V_a \) itself, whose basis, as we have agreed in the beginning of subsection 2.1, is fixed, can be imagined as comprised of column vectors (that we will write sometimes as transposed row vector, using the symbol \( T \) of matrix transposing) of height \( n_a = \dim V_a \)

Whenever \( V_a \) should be identified with some space of meromorphic functions, we should choose in the latter a basis \((f_1, \ldots, f_{n_a})\), and after that put in correspondence to a column vector \((\alpha_1, \ldots, \alpha_{n_a})^T \in V_a\) the function \( \alpha_1 f_1 + \ldots + \alpha_{n_a} f_{n_a} \).

Thus, in the different segments of string \( a \), the space \( V_a \) is identified with different spaces of meromorphic functions by arbitrary but fixed isomorphisms. Every time as, following the movement of the strings, the limiting points of a segment pass through each other, the isomorphism for the "turned inside out" segment is chosen, of course, anew.
Let strings $a$ and $b$ intersect in a vertex $A$. Let us denote $D_A$ the divisor corresponding to $A$ according to the rules described above. The vertex $A$ has two incoming edges (string segments preceding $A$, according to the given positive directions in the strings) and two outgoing ones. It is easily seen that the sum of linear spaces of meromorphic functions corresponding to the incoming edges consists of such functions $f$ that

$$(f) + D - D_A \geq 0$$

and is direct (the particular case shown in Figure 7 can be useful to quickly understand this. Recall the requirement that no point enter in both $D'_a$ and $D'_b$). Exactly the same applies to the outgoing edges, hence the sums of “incoming” and “outgoing” spaces coincide.

The operator $X_{ab}$ is now constructed as follows. Take some vector $\Phi \in V_a \oplus V_b$ and represent it as a sum

$$\Phi = \Phi_a + \Phi_b, \quad \Phi_a \in V_a, \quad \Phi_b \in V_b.$$ 

Identify vectors $\Phi_a$ and $\Phi_b$ with meromorphic functions $\varphi_a$ and $\varphi_b$ using the discussed above isomorphisms for the outgoing edges. Thus, vector $\Phi$ is identified with function

$$\varphi = \varphi_a + \varphi_b.$$ 

Now decompose this function as

$$\varphi = \psi_a + \psi_b,$$

where $\psi_a$ and $\psi_b$ belong to the spaces corresponding to incoming edges, identify $\psi_a$ and $\psi_b$ with vectors $\Psi_a \in V_a$ and $\Psi_b \in V_b$ respectively using the isomorphisms for incoming edges, and set

$$\Psi = \Psi_a + \Psi_b.$$ 

The operator $X_{ab}$ is defined by the requirement that

$$X_{ab}\Phi = \Psi$$

for every vector $\Phi$ (recall that $X_{ab}$ acts “against the arrows”).

The equation (4) for $X$-operators constructed this way follows from the fact that both LHS and RHS of (4) act on each “outgoing” vector $\Phi$, where now $\Phi \in V_a \oplus V_b \oplus V_c$, as follows. First, $\Phi$ is identified with a meromorphic function $\varphi$ using isomorphisms corresponding to three outgoing edges. Then $\varphi$ is identified with some vector $\Psi \in V_a \oplus V_b \oplus V_c$ using three “incoming” isomorphisms. So, both LHS and RHS transform $\Phi$ into $\Psi$, although using different ways of doing this.

The explicit formulae determining $X_{ab}$ are as follows. Recall that the isomorphisms between the spaces $V_a$ of column vectors and the spaces of meromorphic functions are provided by fixing bases in the latter spaces. Let $(f_1, \ldots, f_{n_a})$ and $(g_1, \ldots, g_{n_b})$ be bases corresponding to incoming edges for some $X_{ab}$, while $(\tilde{f}_1, \ldots, \tilde{f}_{n_a})$ and $(\tilde{g}_1, \ldots, \tilde{g}_{n_b})$ be bases corresponding to the outgoing edges. Then it is not hard to verify that $X_{ab}$ is determined by the formula

$$\left(f_1(z), \ldots, f_{n_a}(z), g_1(z), \ldots, g_{n_b}(z)\right) X_{ab} = \left(\tilde{f}_1(z), \ldots, \tilde{f}_{n_a}(z), \tilde{g}_1(z), \ldots, \tilde{g}_{n_b}(z)\right)$$
that must hold for all \( z \in \Gamma \).

**Remark.** Our model, being composed of a finite number of strings moving in the plane, has the obvious "global" integral of motion (or rather all admissible motions)—the product of all \( X \)-operators (complemented by unities in the lacking spaces) taken in the order corresponding to moving along the strings in positive directions. If, however, we want to ensure the possibility of passing to the infinite string number limit, we should prefer more "local" constructions. For example, using the presented method of constructing algebro-geometric solutions, one can add a new string—let us denote it \( h \)—so that it don’t intersect some chosen domain \( \Omega \) of the plane, adding at the same time to divisor \( \mathcal{D} \) one of the divisors \( \mathcal{D}_h \) or \( \mathcal{D}_h' \), depending on whether \( \Omega \) lies to the left or to the right of \( h \). The result will be no change to the \( X \)-matrices (as well as divisors corresponding to string segments) in \( \Omega \). The reader can find some formulae for the infinite kagome lattice in [4], chapter 3.

### 3 One-parametric solutions to the FTE

The simplest case of the "re-factorizing equation" (5) is when the blocks in matrices are just numbers. For numbers, we will use small letters, i.e. let

\[
X_{12} = \begin{pmatrix} a_1 & b_1 & 0 \\ c_1 & d_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad X'_{12} = \begin{pmatrix} a'_1 & b'_1 & 0 \\ c'_1 & d'_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]
\[
X_{13} = \begin{pmatrix} a_2 & 0 & b_2 \\ 0 & 1 & 0 \\ c_2 & 0 & d_2 \end{pmatrix}, \quad X'_{13} = \begin{pmatrix} a'_2 & 0 & b'_2 \\ 0 & 1 & 0 \\ c'_2 & 0 & d'_2 \end{pmatrix},
\]
\[
X_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_3 & b_3 \\ 0 & c_3 & d_3 \end{pmatrix}, \quad X'_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a'_3 & b'_3 \\ 0 & c'_3 & d'_3 \end{pmatrix},
\]

and let us search for solutions of equation

\[
X_{12}X_{13}X_{23} = X'_{23}X'_{13}X'_{12}. \tag{6}
\]

The (nontrivial) one-parametric solutions of (5) were classified in [2]. The word 'one-parametric' means here that \( a_k, b_k, c_k \) and \( d_k \), as well as \( a'_k, b'_k, c'_k \) and \( d'_k \), are supposed to be algebraic functions of one complex parameter \( x \) in such way that

\[
a_k = a(x_k), \quad b_k = b(x_k), \quad c_k = c(x_k), \quad d_k = d(x_k),
\]
\[
a'_k = a(x'_k), \quad b'_k = b(x'_k), \quad c'_k = c(x'_k), \quad d'_k = d(x'_k), \quad k = 1, 2, 3,
\]

and the operator \( R \) can be regarded as a transformation

\[
R : \ (x_1, x_2, x_3) \to (x'_1, x'_2, x'_3).
\]
It turns out that there are, essentially, six different cases for the matrix

\[ X(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}. \]

### 3.1 Case (α)

\[ X(x) = \begin{pmatrix} 1 & x \\ 0 & k \end{pmatrix}, \]

\(k\) being a constant; this gives

\[ R: \quad x_1, x_2, x_3 \rightarrow x_1, kx_2 + x_1x_3, x_3. \]

Inverse map:

\[ R^{-1}: \quad x_1, x_2, x_3 \rightarrow x_1, \frac{x_2 - x_1x_3}{k}, x_3. \]

### 3.2 Case (β)

\[ X(x) = \begin{pmatrix} 1 & x \\ k/x & 0 \end{pmatrix}, \]

this gives

\[ R: \quad x_1, x_2, x_3 \rightarrow \frac{kx_2 + x_1x_3}{x_3}, x_1x_3, \frac{kx_2x_3}{kx_2 + x_1x_3}. \]

Inverse map:

\[ R^{-1}: \quad x_1, x_2, x_3 \rightarrow \frac{x_1x_2}{x_2 + x_1x_3}, \frac{x_1x_3}{k}, \frac{x_2 + x_1x_3}{x_1}. \]

### 3.3 Case (γ)

\[ X(x) = \begin{pmatrix} x & 0 \\ 1 - x & 1 \end{pmatrix}, \]

then

\[ R: \quad x_1, x_2, x_3 \rightarrow \frac{x_3 - x_2 + x_1x_2}{x_3}, \frac{x_1x_2x_3}{x_3 - x_2 + x_1x_2}, x_3. \]

Inverse map:

\[ R^{-1}: \quad x_1, x_2, x_3 \rightarrow \frac{x_1x_2}{x_3 - x_1x_3 + x_1x_2}, x_3 - x_1x_3 + x_1x_2, x_3. \]
3.4 Case (δ)

\[ X(x) = \begin{pmatrix} x & 1 \\ 1-x & 0 \end{pmatrix}. \]

Then

\[ R : \quad x_1, x_2, x_3 \rightarrow \frac{x_1 x_2}{x_1 + x_3 - x_1 x_3}, \quad x_1 + x_3 - x_1 x_3, \quad \frac{(1 - x_1) x_2 x_3}{x_1 + x_3 - x_1 x_2 - x_1 x_3}. \]

Here

\[ R^2 = 1. \]

This transformation is connected with the pentagon equation and was described in [3].

3.5 Case (ε)

\[ X(x) = \begin{pmatrix} x & 1 + ix \\ 1 - ix & x \end{pmatrix}, \]

then

\[ R : \quad x_1, x_2, x_3 \rightarrow \frac{x_1 x_2}{x_1 + x_3 + x_1 x_2 x_3}, \quad x_1 + x_3 + x_1 x_2 x_3, \quad \frac{x_2 x_3}{x_1 + x_3 + x_1 x_2 x_3}, \]

again with

\[ R^2 = 1. \]

This is the electric network transformation, considered in [4]. The variables \( x_j \) and \( x'_j \) are connected with the resistances \( r_j \) and \( r'_j \) in the Introduction as follows:

\[ x_1 = \frac{1}{r_1}, \quad x_2 = r_2, \quad x_3 = \frac{1}{r_3}, \quad x'_1 = r'_1, \quad x'_2 = \frac{1}{r'_2}, \quad x'_3 = r'_3. \]

The reader can verify that the equalness of results of two transformation sequences in Figure 3 means exactly the same that the FTE for \( x \)'s, if all the resistances attached to the edges of graphs in Figure 3 are replaced by \( x_j^{\pm 1} \), where the proper choice of signs is an easy exercise.

Note also that in [4] this transformation was realized in terms of a “free-bosonic local Yang–Baxter equation”, while our case here is a fermionic one, as explained in [4] [2].

3.6 Case (ζ)

\[ X(x) = \begin{pmatrix} x & -s(x) \\ s(x) & x \end{pmatrix}, \]

where \( s(x)^2 = 1 - x^2 \). This case is equivalent to Onsager star–triangle transform, and can also be interpreted as the Euler decomposition of an element of the group \( SO(3) \) into a product of two-dimensional rotations, if we put \( x = \cos \phi, \quad s(x) = \sin \phi \). This case was also considered in [4].
For $R$, this gives

$$R : \begin{pmatrix} x_1, x_2, x_3 \end{pmatrix} \rightarrow \begin{pmatrix} x_1 x_2 \over F(x_1, x_2, x_3), \\ F(x_1, x_2, x_3), \\ x_2 x_3 \over F(x_1, x_2, x_3) \end{pmatrix},$$

where $F$ can be found from

$$s(F) = s(x_2) x_1 x_3 - s(x_1) s(x_3),$$

and so this is a two-foiled transformation.

Inverse map:

$$R^{-1} = IRI,$$

where $I$ transforms a pair $(x, s(x))$ into $(x, -s(x))$.

4 Electric network transformation and Miwa model

It has been shown in paper [1] that the “electric” model described in the Introduction and in subsection 3.5 is connected with the well known integrable Miwa model [7]. In terms of the present paper, this connection reads as follows.

Let us consider, in the three-dimensional space $\mathbb{R}^3 \ni (x, y, z)$, the cubic lattice planes whose equations are $x = l$, $y = m$, and $z = n$, where $l, m, n$ take all integer values. Consider also a “moving” plane with the equation

$$x + y + z = t,$$

where $t$ can be thought of as “time”. With a generic $t$, the intersection of the cubic lattice planes with the plane (8) yields in this latter the regular (infinite) kagome lattice. Let us choose some positive direction for every straight line forming that lattice in such manner that all parallel lines be directed the same way and the orientation of all the lines be consistent in the sense of section 1.

Let us now attach to each kagome lattice vertex a variable $x_k$, and require that the strings move with time $t$ according to equation (8) and the variables $x_k$ change according to (7) whenever a vertex passes through a line. In such way, we will obtain a straight-string model of the kind described in section 1.

Let us introduce unit vectors $f_1, f_2, f_3$ pointing in the directions of axes $x, y, z$ of the space $\mathbb{R}^3$ respectively. Note that each kagome lattice vertex sweeps between its two collisions with strings exactly an edge of the integer cubic lattice in $\mathbb{R}^3$. Thus, we can consider values $x_k$ as corresponding to edges of the cubic lattice. For convenience, and also in order to link our presentation to the paper [1], we will think of $x_k$’s also as attached to vertices of the cubic lattice as follows: let the radius vector of some vertex be $n$, then we will denote as $x_1(n)$ the value $x_1$ corresponding to the edge linking the vertices with radius vectors $n$ and $n - f_1$, as $x_2(n)$—the value $x_2$ corresponding to the edge linking the vertices $n + f_2$ and $n$, and as $x_3(n)$—the value $x_3$ corresponding to the edge linking the vertices $n$ and $n - f_3$. 

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Let us attach signs $\epsilon(n) = \pm 1$ to the cubic lattice vertices in a checkerboard order. Then let us introduce (again in order to link our paper to the work [1]) the vectors $e_1, e_2, e_3$ according to

$$
e_1 = \frac{f_3 - f_2 - f_1}{2}, \quad e_2 = \frac{f_1 + f_2 + f_3}{2}, \quad e_3 = \frac{f_1 - f_2 - f_3}{2}.
$$

Finally, fix some non-zero constants $\alpha_1, \alpha_2$ and $\alpha_3$ and make a substitution

$$x_j(n) = \epsilon(n) \alpha_j \frac{\tau(n + e_j) \tau(n - e_j)}{\tau(n + e_k) \tau(n + e_l)}, \quad \{j, k, l\} = \{1, 2, 3\},
$$

where $\tau$ is the new unknown function, defined, as is seen from (9), in the centers of cubes of the lattice.

As is shown in [1], function $\tau$ satisfies the Miwa equation

$$\sum_{j=1}^{4} \alpha_j \tau(n + e_j) \tau(n - e_j) = 0, \quad e_4 = -e_1 - e_2 - e_3, \quad \alpha_4 = \alpha_1 \alpha_2 \alpha_3.
$$

Note that the variables $a_j(n)$ in [1] are connected with our $x_j(n)$ by

$$a_j(n) = \epsilon(n)x_j(n)^{-1}.
$$

## 5 Quantization of the $(\beta)$ model

The model corresponding to the case $(\beta)$ (subsection 3.2) admits an amazingly straightforward quantization. According to subsection 3.2, the corresponding transformation $(x_1, x_2, x_3)$ to $(x'_1, x'_2, x'_3)$ arises from the relation

$$
\begin{pmatrix}
1 & x_1 & 0 \\
k/x_1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & x_2 \\
0 & 1 & 0 \\
k/x_2 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & x_3 \\
k/x_3 & 0 & 0
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & x_3' \\
k/x_3' & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & x_2' \\
0 & 1 & 0 \\
k/x_2' & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & x_1' & 0 \\
0 & 1 & 0 \\
k/x_1' & 0 & 0
\end{pmatrix}.
$$

To quantize this model, let us take the same relation (10), but with $x_1, x_2$ and $x_3$ belonging to some associative algebra and satisfying the commutation relations

$$x_1x_2 = \omega^{-1}x_2x_1, \quad x_1x_3 = \omega x_3x_1, \quad x_2x_3 = \omega^{-1}x_3x_2,
$$

where $\omega$ is a scalar (and $k$ remains a scalar, too). It can be verified that in this case (10) still determines the “primed” variables without contradiction, namely

$$x'_1 = x_1 + kx_2x_3^{-1}, \quad x'_2 = x_1x_3, \quad x'_3 = k(x_1 + kx_2x_3^{-1})^{-1}x_2.
$$
and the following relations hold:

\[ x'_1 x'_2 = \omega x'_2 x'_1, \quad x'_1 x'_3 = \omega^{-1} x'_3 x'_1, \quad x'_2 x'_3 = \omega x'_3 x'_2. \] (13)

Let us consider now a straight-string model, as in section 1, and define the commutation relations for \( x_k \) in all vertices. We will demand that

- if two \( x_k \) don’t lie in the same string, they commute;
- if the vertices where two \( x_k \) are situated belong to a triangle of the form as in Figure 8 (a) or (b), the commutation relations (11) hold (where we temporarily change the actual \( k \)'s to 1, 2 and 3). Note that there may exist other strings, not depicted in Figure 8, between the vertices 1, 2 and 3;
- if the vertices where two \( x_k \) are situated arise as intersection points of a string with two parallel strings, the commutation relation between them can be obtained if we let one of the vertices in Figure 8 tend to infinity. Here the figures (a) and (b) lead to the same result, however the orientation of strings does play the rôle.

It can be verified that the stated commutation relations are conserved under the transformations as in Figure 4.

6 Discussion

In this paper, we have considered, first of all, classical integrable dynamical systems in 2+1-dimensional discrete space-time. We proposed a very general scheme for constructing such systems together with their solutions. We showed that our scheme includes Miwa bilinear equation as just one of particular cases.
The integrability in our general scheme is ensured by a rich set of symmetries that are expressed in the local form as functional tetrahedron equation (FTE). The form of FTE suggests at once that the corresponding model need not be associated with a regular (e.g. cubic) lattice, but can be naturally defined e.g. for any configuration of intersecting planes (“world sheets” of strings in section [1]) in a three-dimensional space. The algebro-geometric solution for such system is given in terms of an algebraic curve and some constant divisors assigned to the planes. In a sense, the equations of motion, when expressed in terms of those divisors, “disappear”, and this calls to mind the Penrose twistor theory, where field equations “disappear” as well.

We have also shown that, at least in one particular case, FTE can successfully replace the usual (quantum) tetrahedron equation in constructing a quantum integrable model.

Finally, we would like to note that a generalization onto $3 + 1$ dimensions of our scheme for constructing classical models and their algebro-geometric solutions has been found recently in work [8].

References

[1] R.M. Kashaev, *On discrete three-dimensional equations associated with the local Yang–Baxter relation*, Lett. Math. Phys. 35, 389–397 (1996), also solv-int/9512005.

[2] S.M. Sergeev, *Solutions of the functional tetrahedron equation connected with the local Yang–Baxter equation for the ferro-electric*, solv-int/9709006.

[3] R.M. Kashaev and S.M. Sergeev, *On pentagon, ten-term, and tetrahedron relations*, preprint ENSLAPP-L-611/96, also q-alg/9607032, to appear in Commun. Math. Phys.

[4] I.G. Korepanov, *Algebraic integrable dynamical systems, 2 + 1-dimensional models in wholly discrete space-time, and inhomogeneous models in 2-dimensional statistical physics*, solv-int/9506003 (1995).

[5] V.V. Bazhanov and Yu.G. Stroganov, *Uslovija perestanov卵巢nosti matrits perekhoda na mnogomernikh reshjotkah*, Teor. Mat. Fiz. 52, 105–113 (1982) [English trans.: Theor. Math. Phys. 52, 685 (1982)]; M.T. Jaekel and J.-M. Maillard, *Symmetry relations in exactly solvable models*, J. Phys. A15, 1309–1325 (1982).

[6] J. Hietarinta, *Labeling schemes for tetrahedron equations and dualities between them*, J. Phys. A27, 5727–5748 (1994), also hep-th/9402139.

[7] T. Miwa, *On Hirota’s difference equations*, Proc. Japan Acad. 58 ser. A, 9–12 (1982).

[8] I.G. Korepanov, *Integrability in 3 + 1 dimensions: relaxing a tetrahedron relation*, solv-int/9712006.