THE INTERIOR $C^2$ ESTIMATE FOR MONGE-AMPÈRE EQUATION IN DIMENSION $n=2$

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Abstract. In this paper, we introduce a new auxiliary function, and establish the interior $C^2$ estimate for Monge-Ampère equation in dimension $n=2$, which was firstly proved by Heinz [5] by a geometric method.

1. Introduction

In this paper, we consider the convex solution of Monge-Ampère equation

$$\det D^2u = f(x), \quad \text{in } B_R(0) \subset \mathbb{R}^2.$$  \tag{1.1}

When the solution $u$ is convex, the equation (1.1) is elliptic. It is well-known that the interior $C^2$ estimate is an important problem for elliptic equations. For Monge-Ampère equation in dimension $n=2$, the corresponding interior $C^2$ estimate was established by Heinz [5], and for higher dimension $n \geq 3$, Pogorelov [7] constructed his famous counterexample, namely irregular solutions to Monge-Ampère equations.

Later, Urbas [9] generalized the counterexample for $\sigma_k$ Hessian equations with $k \geq 3$. So the interior $C^2$ estimate of $\sigma_2$ Hessian equation

$$\sigma_2(D^2u) = f, \quad \text{in } B_R(0) \subset \mathbb{R}^n,$$  \tag{1.2}

is an interesting problem, where $\sigma_2(D^2u) = \sigma_2(\lambda(D^2u)) = \sum_{1 \leq i_1 < i_2 \leq n} \lambda_{i_1} \lambda_{i_2}$, $\lambda(D^2u) = (\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of $D^2u$, and $f > 0$. For $n = 2$, (1.2) is the Monge-Ampère equation (1.1). For $n = 3$ and $f \equiv 1$, (1.2) can be viewed as a special
Lagrangian equation, and Warren-Yuan obtained the corresponding interior $C^2$ estimate in the celebrated paper [10]. Moreover, the problem is still open for general $f$ with $n \geq 4$ and nonconstant $f$ with $n = 3$.

Moreover, Pogorelov type estimates for the Monge-Ampère equations and $\sigma_k$ Hessian equation ($k \geq 2$) were derived by Pogorelov [7] and Chou-Wang [3] respectively, and see [4] and [6] for some generalizations.

In this paper, we introduce a new auxiliary function, and establish the interior $C^2$ estimate as follows

**Theorem 1.1.** Suppose $u \in C^4(B_R(0))$ be a convex solution of Monge-Ampère equation (1.1) in dimension $n = 2$, where $0 < m \leq f \leq M$ in $B_R(0)$. Then

\[
|D^2u(0)| \leq C_1e^{C_2 \sup_{B_R(0)} |Du|^2},
\]

where $C_1$ is a positive constant depending only on $m$, $M$, $R \sup |\nabla f|$, $R^2 \sup |\nabla^2 f|$, and $C_2$ is a positive constant depending only on $m$ and $M$.

**Remark 1.2.** By Trudinger’s gradient estimates (see [8]), we can bound $|D^2u(0)|$ in terms of $u$. In fact, we can get from the convexity of $u$

\[
\sup_{B_{\frac{R}{2}}(0)} |Du| \leq \frac{\text{osc}_{B_{\frac{R}{2}}(0)} u}{\frac{R}{2}} \leq \frac{4 \sup_{B_R(0)} |u|}{R},
\]

and

\[
|D^2u(0)| \leq C_1e^{C_2 \sup_{B_R(0)} |Du|^2 \left(\frac{R}{2}\right)^2} \leq C_1e^{16C_2 \sup_{B_R(0)} |u|^2 \frac{R}{4}}.
\]

**Remark 1.3.** The result was firstly proved by Heinz [5]. In fact, Heinz’s proof depends on the strict convexity of solutions and the geometry of convex hypersurface in dimension two. Our proof, which is based on a suitable choice of auxiliary functions, is elementary and avoids geometric computations on the graph of solutions.

**Remark 1.4.** The interior $C^2$ estimate of $\sigma_2$ Hessian equation (1.2) in higher dimensions is a longstanding problem. As we all know, it is hard to find a corresponding geometry in higher dimensions, so we can not generalize Heinz’s proof or Warren-Yuan’s proof to higher dimensions. But the method in this paper and the optimal concavity in [2] is helpful for this problem.
The rest of the paper is organized as follows. In Section 2, we give the calculations of the derivatives of eigenvalues and eigenvectors with respect to the matrix. In Section 3, we introduce a new auxiliary function, and prove Theorem 1.1.

2. Derivatives of eigenvalues and eigenvectors

In this section, we give the calculations of the derivatives of eigenvalues and eigenvectors with respect to the matrix. We think the following result is known for many people, for example see [1] for a similar result. For completeness, we give the result and a detailed proof.

Proposition 2.1. Let \( W = \{W_{ij}\} \) is an \( n \times n \) symmetric matrix and \( \lambda(W) = (\lambda_1, \lambda_2, \cdots, \lambda_n) \) are the eigenvalues of the symmetric matrix \( W \), and the corresponding continuous eigenvector field is \( \tau^i = (\tau^i_1, \cdots, \tau^i_n) \in \mathbb{S}^{n-1} \). Suppose that \( W = \{W_{ij}\} \) is diagonal, \( \lambda_i = W_{ii} \) and the corresponding eigenvector \( \tau^i = (0, \cdots, 0, 1, 0, \cdots, 0) \in \mathbb{S}^{n-1} \) at the diagonal matrix \( W \). If \( \lambda_k \) is distinct with other eigenvalues, then we have at the diagonal matrix \( W \)

\[
\frac{\partial \tau^k}{\partial W_{pq}} = 0, \quad \forall \ p, q; \quad \frac{\partial \tau^k_i}{\partial W_{ik}} = \frac{1}{\lambda_k - \lambda_i}, \quad i \neq k; \quad \frac{\partial \tau^k_i}{\partial W_{pq}} = 0, \text{ otherwise.} \tag{2.1}
\]

\[
\frac{\partial^2 \tau^k}{\partial W_{pk} \partial W_{pq}} = -\frac{1}{(\lambda_k - \lambda_p)^2}, \quad p \neq k; \tag{2.2}
\]

\[
\frac{\partial^2 \tau^k_i}{\partial W_{ik} \partial W_{ii}} = \frac{1}{(\lambda_k - \lambda_i)^2}, \quad i \neq k; \quad \frac{\partial^2 \tau^k_i}{\partial W_{ik} \partial W_{kk}} = -\frac{1}{(\lambda_k - \lambda_i)^2}, \quad i \neq k; \tag{2.3}
\]

\[
\frac{\partial^2 \tau^k_i}{\partial W_{iq} \partial W_{qk}} = \frac{1}{\lambda_k - \lambda_i} \frac{1}{\lambda_k - \lambda_q}, \quad i \neq k, i \neq q, q \neq k; \tag{2.4}
\]

\[
\frac{\partial^2 \tau^k_i}{\partial W_{pq} \partial W_{rs}} = 0, \text{ otherwise.} \tag{2.5}
\]

Proof. From the definition of eigenvalue and eigenvector of matrix \( W \), we have

\[(W - \lambda_k I)\tau^k \equiv 0,\]

where \( \tau^k \) is the eigenvector of \( W \) corresponding to the eigenvalue \( \lambda_k \). That is, for \( i = 1, \cdots, n \), it holds

\[
[W_{ii} - \lambda_k] \tau^k_i + \sum_{j \neq i} W_{ij} \tau^k_j = 0. \tag{2.6}
\]
When $W = \{W_{ij}\}$ is diagonal and $\lambda_k$ is distinct with other eigenvalues, $\lambda_k$ and $\tau^k$ are $C^2$ at the matrix $W$. In fact,
\[
\tau^k,k = 1, \quad \tau^k,i = 0, \quad i \neq k, \quad \text{at } W.
\] (2.7)

Taking the first derivative of (2.6), we have
\[
\left[ \frac{\partial W_{ii}}{\partial W_{pq}} - \frac{\partial \lambda_k}{\partial W_{pq}} \right] \tau^k,i + \left[ W_{ii} - \lambda_k \right] \frac{\partial \tau^k,i}{\partial W_{pq}} + \sum_{j \neq i} \left[ \frac{\partial W_{ij}}{\partial W_{pq}} \tau^k,j + W_{ij} \frac{\partial \tau^k,j}{\partial W_{pq}} \right] = 0.
\]

Hence for $i = k$, we get from (2.7)
\[
\frac{\partial \lambda_k}{\partial W_{pq}} = \left\{ \begin{array}{ll}
1, & p = k, q = k; \\
0, & \text{otherwise}.
\end{array} \right.
\] (2.8)

And for $i \neq k$,
\[
\left[ W_{ii} - \lambda_k \right] \frac{\partial \tau^k,i}{\partial W_{pq}} + \sum_{j \neq i} \frac{\partial W_{ij}}{\partial W_{pq}} \tau^k,j = 0,
\]
then
\[
\frac{\partial \tau^k,i}{\partial W_{pq}} = \left\{ \begin{array}{ll}
\frac{1}{\lambda_k - \lambda_i}, & p = i, q = k; \\
0, & \text{otherwise}.
\end{array} \right.
\] (2.9)

Since $\tau^k \in S^{n-1}$, we have
\[
1 = |\tau^k|^2 = (\tau^k,1)^2 + \cdots + (\tau^k,k)^2 + \cdots + (\tau^k,n)^2.
\] (2.10)

Taking the first derivative of (2.10), and using (2.7), it holds
\[
\frac{\partial \tau^k,k}{\partial W_{pq}} = 0, \quad \forall (p, q).
\] (2.11)

For $i = k$, taking the second derivative of (2.6), and using (2.7), it holds
\[
\left[ \frac{\partial^2 W_{kk}}{\partial W_{pq} \partial W_{rs}} - \frac{\partial^2 \lambda_k}{\partial W_{pq} \partial W_{rs}} \right] \tau^k,k + \sum_{j \neq k} \left[ \frac{\partial W_{kj}}{\partial W_{pq} \partial W_{rs}} \frac{\partial \tau^k,j}{\partial W_{rs}} + \frac{\partial W_{kj}}{\partial W_{rs} \partial W_{pq}} \frac{\partial \tau^k,j}{\partial W_{pq}} \right] = 0,
\]
hence
\[
\frac{\partial^2 \lambda_k}{\partial W_{pq} \partial W_{rs}} = \sum_{j \neq k} \left[ \frac{\partial W_{kj}}{\partial W_{pq} \partial W_{rs}} \frac{\partial \tau^k,j}{\partial W_{rs}} + \frac{\partial W_{kj}}{\partial W_{rs} \partial W_{pq}} \frac{\partial \tau^k,j}{\partial W_{pq}} \right]
\]
\[
= \left\{ \begin{array}{ll}
\frac{1}{\lambda_k - \lambda_q}, & p = k, q \neq k, r = q, s = k; \\
\frac{1}{\lambda_k - \lambda_s}, & r = k, s \neq k, p = s, q = k; \\
0, & \text{otherwise}.
\end{array} \right.
\] (2.12)
For $i \neq k$, it holds
\[
\left[ \frac{\partial W_{ii}}{\partial W_{pq}} - \frac{\partial \lambda_k}{\partial W_{pq}} \right] \frac{\partial \tau^{k,i}}{\partial W_{rs}} + \left[ \frac{\partial W_{ii}}{\partial W_{rs}} - \frac{\partial \lambda_k}{\partial W_{rs}} \right] \frac{\partial \tau^{k,i}}{\partial W_{pq}} + \left[ W_{ii} - \lambda_k \right] \frac{\partial^2 \tau^{k,i}}{\partial W_{pq} \partial W_{rs}}
\]
\[+ \sum_{j \neq i} \left[ \frac{\partial W_{ij}}{\partial W_{pq}} \frac{\partial \tau^{k,j}}{\partial W_{rs}} + \frac{\partial W_{ij}}{\partial W_{rs}} \frac{\partial \tau^{k,j}}{\partial W_{pq}} \right] = 0,
\]
then
\begin{align}
\frac{\partial^2 \tau^{k,i}}{\partial W_{ik} \partial W_{ii}} &= \frac{1}{\lambda_k - \lambda_i} \frac{\partial \tau^{k,i}}{\partial W_{ik}} = \frac{1}{\lambda_k - \lambda_i} \frac{1}{\lambda_k - \lambda_i}, \quad i \neq k; \\
\frac{\partial^2 \tau^{k,i}}{\partial W_{iq} \partial W_{qk}} &= \frac{1}{\lambda_k - \lambda_i} \frac{\partial \tau^{k,i}}{\partial W_{iq}} \frac{\partial \tau^{k,q}}{\partial W_{qk}} = \frac{1}{\lambda_k - \lambda_i} \frac{1}{\lambda_k - \lambda_i - \lambda_q}, \quad i \neq k, i \neq q, q \neq k; \\
\frac{\partial^2 \tau^{k,i}}{\partial W_{ik} \partial W_{kk}} &= \frac{1}{\lambda_i - \lambda_k} \left[ \frac{\partial \lambda_k}{\partial W_{pq}} \frac{\partial \tau^{k,i}}{\partial W_{pq}} \right] = -\frac{1}{\lambda_k - \lambda_i} \frac{1}{\lambda_k - \lambda_i}, \quad i \neq k; \\
\frac{\partial^2 \tau^{k,i}}{\partial W_{pq} \partial W_{rs}} &= 0, \quad \text{otherwise}.
\end{align}

From (2.10), we have
\[
2\tau^{k,i} \frac{\partial^2 \tau^{k,k}}{\partial W_{pq} \partial W_{rs}} + \sum_{i \neq k} \frac{\partial \tau^{k,i}}{\partial W_{pq}} \frac{\partial \tau^{k,i}}{\partial W_{rs}} = 0,
\]
then
\[
\frac{\partial^2 \tau^{k,k}}{\partial W_{pq} \partial W_{rs}} = -\sum_{i \neq k} \frac{\partial \tau^{k,i}}{\partial W_{pq}} \frac{\partial \tau^{k,i}}{\partial W_{rs}} = \begin{cases} \frac{1}{\lambda_k - \lambda_p} \frac{1}{\lambda_k - \lambda_p}, & p \neq k, q = k, r = p, s = q; \\
0, & \text{otherwise}. \end{cases}
\]

The proof of Proposition 2.1 is finished. \( \square \)

**Example 2.2.** When $n = 2$, the matrix \( \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \) has two eigenvalues
\[
\lambda_1 = \frac{(u_{11} + u_{22}) + \sqrt{(u_{11} - u_{22})^2 + 4u_{12}u_{21}}}{2},
\]
\[
\lambda_2 = \frac{(u_{11} + u_{22}) - \sqrt{(u_{11} - u_{22})^2 + 4u_{12}u_{21}}}{2},
\]
with $\lambda_1 \geq \lambda_2$. If $\lambda_1 > \lambda_2$, \[
\left[ \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} - \lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0,
\]
we can get \[
\xi_1 = \frac{(u_{22} - u_{11}) - \sqrt{(u_{11} - u_{22})^2 + 4u_{12}u_{21}}}{2}; \quad \xi_2 = -u_{21}.
\]
Then the eigenvector $\tau$ corresponding to $\lambda_1$ is

$$
\tau = -\frac{(\xi_1, \xi_2)}{\sqrt{\xi_1^2 + \xi_2^2}}.
$$

We can verify Proposition 2.1.

3. Proof of Theorem 1.1

Now we start to prove Theorem 1.1.

Let $\tau(x) = \tau(D^2u(x)) = (\tau_1, \tau_2) \in S^1$ be the continuous eigenvector field of $D^2u(x)$ corresponding to the largest eigenvalue. Denote

$$
\Sigma =: \{ x \in B_R(0) : r^2 - |x|^2 + \langle x, \tau(x) \rangle^2 > 0, r^2 - \langle x, \tau(x) \rangle^2 > 0 \},
$$

where $r = \frac{1}{\sqrt{2}}R$. It is easy to know, $\Sigma$ is an open set and $B_r(0) \subset \Sigma \subset B_R(0)$. We introduce a new auxiliary function in $\Sigma$ as follows

$$
(3.2) \quad \phi(x) = \eta(x)^\beta g(\frac{1}{2}|Du|^2)u_{\tau\tau}
$$

where $\eta(x) = (r^2 - |x|^2 + \langle x, \tau(x) \rangle^2)(r^2 - \langle x, \tau(x) \rangle^2)$ with $\beta = 4$ and $g(t) = e^{\frac{ct}{r^2}}$ with $c_0 = \frac{32}{m}$. In fact, $\langle x, \tau(x) \rangle$ is invariant under rotations of the coordinates, so is $\eta(x)$.

From the definition of $\Sigma$, we know $\eta(x) > 0$ in $\Sigma$, and $\eta = 0$ on $\partial\Sigma$. Assume the maximum of $\phi(x)$ in $\Sigma$ is attained at $x_0 \in \Sigma$. By rotating the coordinates, we can assume $D^2u(x_0)$ is diagonal. In the following, we denote $\lambda_i = u_{ii}(x_0)$, $\lambda = (\lambda_1, \lambda_2)$. Without loss of generality, we can assume $\lambda_1 \geq \lambda_2$, and $\tau(x_0) = (1, 0)$.

If $\eta\lambda_1 \leq 10^3(1 + M + r \sup |\nabla f| + \frac{M \sup |Du|}{r})r^4$. Then we can easily get

$$
|u_{\xi}(0)| \leq u_{\tau(0)}(0) \leq 10^3(1 + M + r \sup |\nabla f| + \frac{M \sup |Du|}{r})e^{\frac{c_0 \sup |Du|^2}{r^2}}.
$$

Hence we get

$$
|u_{\xi}(0)| \leq u_{\tau(0)}(0) \leq 10^3(1 + M + r \sup |\nabla f|)e^{(c_0 + \frac{2M}{m} \sup |Du|^2)}r^4, \quad \forall \xi \in S^1.
$$

Then we prove Theorem 1.1 under the condition $\eta\lambda_1 \leq 10^3(1 + M + r \sup |\nabla f| + \frac{M \sup |Du|}{r})r^4$. 

Now, we assume $\eta \lambda_1 \geq 10^3 (1 + M + r \sup |\nabla f| + \frac{M \sup |Du|}{r} r^4)$. Then we have

$$\lambda_1 = \frac{\eta \lambda_1}{\eta} \geq 10^3 (1 + M + r \sup |\nabla f| + \frac{M \sup |Du|}{m} r^4).$$

From the equation (1.1), we have

$$\lambda_2 = \frac{f}{\lambda_1} \leq \frac{M}{\lambda_1} < \lambda_1.$$

Hence $\lambda_1$ is distinct with the other eigenvalue, and $\tau(x)$ is $C^2$ at $x_0$. Moreover, the test function

$$\varphi = \beta \log \eta + \log g(\frac{1}{2} |Du|^2) + \log u_{11}$$

attains the local maximum at $x_0$. In the following, all the calculations are at $x_0$.

Then, we can get

$$0 = \varphi_i = \beta \frac{\eta_i}{\eta} + \frac{g'}{g} \sum_k u_k u_{ki} + \frac{u_{11i}}{u_{11}},$$

so we have

$$\frac{u_{11i}}{u_{11}} = -\beta \frac{\eta_i}{\eta} - \frac{g'}{g} u_i u_{ii}, \quad i = 1, 2.$$

At $x_0$, we also have

$$0 \geq \varphi_{ii} = \beta \left[ \frac{\eta_{ii}}{\eta} - \frac{\eta^2}{\eta^2} \right] + \frac{g''g - g'^2}{g^2} \sum_k u_k u_{ki} \sum_i u_{11i} u_{11i}$$

$$+ \frac{g'}{g} \sum_k (u_{ki} u_{ki} + u_k u_{kii}) + \frac{u_{11i}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2} - u_{11i}^2 u_{11i} - u_{11i} u_{11} - \frac{u_{11i}^2}{u_{11}^2},$$

since $g''g - g'^2 = 0$. Let

$$F^{11} = \frac{\partial \det D^2 u}{\partial u_{11}} = \lambda_2, \quad F^{22} = \frac{\partial \det D^2 u}{\partial u_{22}} = \lambda_1,$$

$$F^{12} = \frac{\partial \det D^2 u}{\partial u_{12}} = 0, \quad F^{21} = \frac{\partial \det D^2 u}{\partial u_{21}} = 0.$$
Differentiating (1.1) once, we can get

\[ F^{11}u_{11i} + F^{22}u_{22i} = f_i, \]

then

\[ u_{22i} = \frac{1}{F_{22}} [f_i - F^{11}u_{11i}] = \frac{f_i}{\lambda_1} - \frac{f}{\lambda_1 u_{11i}}. \]  \hfill (3.7)

Differentiating (1.1) twice, we can get

\[ F^{11}u_{1111} + F^{22}u_{2211} = f_{11} - \frac{\partial^2 \det D^2u}{\partial u_{11}\partial u_{22}}u_{111u_{221}} - \frac{\partial^2 \det D^2u}{\partial u_{12}\partial u_{21}}u_{112u_{211}} = f_{11} - 2u_{111}u_{221} + 2u_{112} \]

\[ = f_{11} + 2u_{112} - 2u_{111}\left[ \frac{f_1}{\lambda_1} - \frac{f}{\lambda_1 u_{11}} \right] \]

\[ = f_{11} + 2u_{112} - 2f_1 \frac{u_{111}}{u_{11}} + 2f \left( \frac{u_{111}}{u_{11}} \right)^2, \]  \hfill (3.8)

and

\[ F^{11}u_{1112} + F^{22}u_{2212} = f_{12} - \frac{\partial^2 \det D^2u}{\partial u_{11}\partial u_{22}}u_{111u_{222}} - \frac{\partial^2 \det D^2u}{\partial u_{22}\partial u_{11}}u_{221u_{112}} - 2\frac{\partial^2 \det D^2u}{\partial u_{12}\partial u_{21}}u_{112u_{121}} = f_{12} - u_{111}u_{222} - u_{112}u_{221} + 2u_{112}u_{221} = f_{12} - u_{111}\left[ \frac{f_2}{\lambda_1} - \frac{f}{\lambda_1 u_{11}} \right] + u_{112}\left[ \frac{f_1}{\lambda_1} - \frac{f}{\lambda_1 u_{11}} \right] \]

\[ = f_{12} + f_1 \frac{u_{112}}{u_{11}} - f_2 \frac{u_{111}}{u_{11}}. \]  \hfill (3.9)
Hence

\[ 0 \geq \sum_{i=1}^{2} F^{ii} \varphi_{ii} \]

\[ = \beta \sum_{i} F^{ii} \left( \frac{\eta_{ii} - \eta_{i}^{2}}{\eta_{i}^{2}} \right) + g' g \sum_{i} F^{ii} u_{i}^{2} + \frac{g' g}{g} \sum_{k} u_{k} f_{k} \]

\[ + \frac{1}{u_{11}} \sum_{i} F^{ii} u_{11i} - \sum_{i} F^{ii} \left( \frac{u_{11i}^{2}}{u_{11}} \right) \]

\[ = \beta \lambda_{2} \left[ \frac{\eta_{11} - \eta_{1}^{2}}{\eta_{1}^{2}} \right] + \beta \lambda_{1} \left[ \frac{\eta_{22} - \eta_{2}^{2}}{\eta_{2}^{2}} \right] + \frac{g' g}{g} [\lambda_{1} + \lambda_{2} f + \frac{g' g}{u_{1} f_{1} + u_{2} f_{2}}] \]

\[ + \frac{1}{u_{11}} \left[ f_{11} + 2u_{112} - 2f_{1} u_{111} + 2f_{11} \left( \frac{u_{111}}{u_{11}} \right)^{2} \right] - \lambda_{2} \left[ \frac{u_{111}^{2}}{u_{11}} \right] - \lambda_{1} \left[ \frac{u_{112}^{2}}{u_{11}} \right] \]

\[ \geq \beta \left[ \frac{f_{11} \eta_{11}}{\lambda_{1} \eta} + \lambda_{1} \frac{\eta_{22}}{\eta} \right] - \beta \left[ \frac{f_{11} \eta_{1}^{2}}{\lambda_{1} \eta_{1}^{2}} - \beta \lambda_{1} \frac{\eta_{2}^{2}}{\eta_{2}^{2}} \right] \]

\[ + \frac{f_{11}}{\lambda_{1}} \left[ \frac{u_{111}^{2}}{u_{11}} \right] - \frac{2}{u_{11}} \frac{f_{11} u_{111}}{u_{11}} + \frac{\lambda_{1}}{2} \left[ \frac{u_{112}^{2}}{u_{11}} \right] + \frac{\lambda_{1}}{2} \left[ \frac{1}{\lambda_{1}} \frac{g' g}{g} u_{112} \right]^{-2} \]

\[ + \frac{g' f_{1}}{g} \lambda_{1} - \frac{g' g}{g} \left| \nabla u \right| \left| \nabla f \right| - \frac{f_{11}}{\lambda_{1}} \]

\[ \geq \beta \left[ \frac{f_{11} \eta_{11}}{\lambda_{1} \eta} + \lambda_{1} \frac{\eta_{22}}{\eta} \right] - \beta \left[ \frac{f_{11} \eta_{1}^{2}}{\lambda_{1} \eta_{1}^{2}} - \beta \lambda_{1} \frac{\eta_{2}^{2}}{\eta_{2}^{2}} \right] \]

\[ + \frac{2}{\lambda_{1}} \left[ \frac{u_{111}^{2}}{u_{11}} \right] + \frac{\lambda_{1}^{2}}{2} \left[ \frac{u_{112}^{2}}{u_{11}} \right] + \frac{\beta^{2}}{2} \lambda_{1} \frac{\eta_{2}^{2}}{\eta_{2}^{2}} + \beta f_{11} \frac{g' g}{g} \eta_{2} \]

\[ + \frac{g' f_{1}}{g} \lambda_{1} - \frac{g' g}{g} \left| \nabla u \right| \left| \nabla f \right| - \frac{f_{11}}{\lambda_{1}} - 2 \frac{f_{11}^{2}}{f_{1} \lambda_{1}} \]

(3.10)

Lemma 3.1. Under the condition \( \eta \lambda_{1} \geq 10^{3} (1 + M + r \sup |\nabla f| + M_{\sup} \sup |Du|) r^{-4} \), we have at \( x_{0} \)

\[ \beta \frac{f_{11} \eta_{1}^{2}}{\lambda_{1} \eta} \leq \frac{8 \beta f_{11} \lambda_{1}}{\eta_{1}^{2}} + \frac{\lambda_{1}}{4} \frac{(u_{112})^{2}}{u_{11}}; \]

(3.11)

\[ \beta f_{11} \frac{g' g}{g} \eta_{2} \geq -4 \beta f_{11} \frac{g' r^{4}}{g} \frac{|u_{2}|}{r}; \]

(3.12)
Proof. At \( x_0, \tau = (\tau_1, \tau_2) = (1, 0) \). Then from Proposition 2.1 we get

\[
\langle x, \partial_1 \tau \rangle = \sum_{m=1}^{2} x_m \frac{\partial \tau_m}{\partial x_i} = \sum_{m=1}^{2} x_m \frac{\partial \tau_m}{\partial u_{pq}} u_{pq} = x_2 \frac{\partial \tau_2}{\partial u_{pq}} u_{pq}
\]

(3.14)

\[
x_2 \frac{u_{12i}}{\lambda_1 - \lambda_2}, \quad i = 1, 2.
\]

From the definition of \( \eta \), then we have at \( x_0 \)

\[
\eta = [r^2 - |x|^2 + \langle x, \tau \rangle^2][r^2 - \langle x, \tau \rangle^2] = (r^2 - x_2^2)(r^2 - x_1^2).
\]

(3.15)

Taking first derivative of \( \eta \), we can get

\[
\eta_i = [-2x_i + 2 \langle x, \tau \rangle \langle x, \tau \rangle_i][r^2 - \langle x, \tau \rangle^2]
\]

\[
+ [r^2 - |x|^2 + \langle x, \tau \rangle^2][-2 \langle x, \tau \rangle \langle x, \tau \rangle_i]
\]

\[
= [-2x_i + 2x_1 (\delta_{i1} + \langle x, \partial_1 \tau \rangle)][r^2 - x_1^2] + (r^2 - x_2^2)[-2x_1 (\delta_{i1} + \langle x, \partial_1 \tau \rangle)]
\]

\[
= \begin{cases} 
-2x_1 (r^2 - x_2^2) + 2x_1 \langle x, \partial_1 \tau \rangle (x_1^2 - x_2^2), \quad i = 1; \\
-2x_2 (r^2 - x_1^2) + 2x_1 \langle x, \partial_2 \tau \rangle (x_2^2 - x_1^2), \quad i = 2.
\end{cases}
\]

Hence

\[
\beta \frac{f}{\lambda_1} \frac{\eta_i^2}{\eta^2} \leq \beta \frac{f}{\lambda_1} \frac{[-2x_1 (r^2 - x_2^2)]}{\eta} + 2x_1 x_2 \frac{x_2^2 - x_1^2}{\eta} \frac{u_{112}}{\lambda_1 - \lambda_2} \]

\[
\leq \beta \frac{f}{\lambda_1} \frac{8r^6}{\eta^2} + \frac{8r^6}{\eta^2} \frac{u_{112}^2}{u_{11}}^2
\]

(3.16)

\[
\leq \frac{8\beta f r^6}{\eta^2 \lambda_1} + \frac{\lambda_1}{4} \frac{(u_{112})^2}{u_{11}}.
\]
Also we have

\[
\frac{\eta_2}{\eta} = \frac{-2x_2(r^2 - x_1^2)}{\eta} + 2x_1x_2 \frac{x_2^2 - x_1^2}{\eta} \frac{u_{221}}{\lambda_1 - \lambda_2} \\
= \frac{-2x_2(r^2 - x_1^2)}{\eta} + 2x_1x_2 \frac{x_2^2 - x_1^2}{\eta} \frac{1}{\lambda_1 - \lambda_2} \left( \frac{f_1}{\lambda_1} - \frac{f}{\lambda_1 u_{111}} \right) \\
= \frac{-2x_2(r^2 - x_1^2)}{\eta} \left[ 1 - x_1 \frac{(x_2^2 - x_1^2)(r^2 - x_2^3)}{\eta} \frac{1}{\lambda_1 - \lambda_2} f_1 \right] \\
+ 2x_1x_2 \frac{x_2^2 - x_1^2}{\eta} \frac{1}{\lambda_1 - \lambda_2 \lambda_1} \frac{f}{\eta} \beta \left[ \frac{\eta_1}{\eta} + \frac{g'}{u_{111}} \right] \\
= \frac{-2x_2(r^2 - x_1^2)}{\eta} \left[ 1 - x_1 \frac{(x_2^2 - x_1^2)(r^2 - x_2^3)}{\eta} \frac{1}{\lambda_1 - \lambda_2} f_1 \left( \frac{f_1}{\lambda_1} + \frac{g'}{u_{111}} \right) \right] \\
+ 2x_1x_2 \frac{x_2^2 - x_1^2}{\eta} \frac{1}{\lambda_1 - \lambda_2 \lambda_1} \frac{f}{\eta} \beta \left[ \frac{2x_1(r^2 - x_2^3)}{\eta} \right] \\
+ [2x_1x_2 \frac{x_2^2 - x_1^2}{\eta} \lambda_1 - \lambda_2 \lambda_1 \beta \frac{\eta_1}{\eta} - \frac{g'}{g} u_{222}],
\]

then we can get

\[
[1 + \beta^2 (2x_1x_2 \frac{x_2^2 - x_1^2}{\eta})^2 (\lambda_1 - \lambda_2)^2 \eta] \frac{\eta_2}{\eta} \\
= \frac{-2x_2(r^2 - x_1^2)}{\eta} \left[ 1 - x_1 \frac{(x_2^2 - x_1^2)(r^2 - x_2^3)}{\eta} \frac{1}{\lambda_1 - \lambda_2} f_1 \left( \frac{f_1}{\lambda_1} + \frac{g'}{g} u_{111} \right) \right] \\
+ 2x_1x_2 \frac{x_2^2 - x_1^2}{\eta} \frac{2x_1^2x_2(x_2^2 - x_1^2)^2(r^2 - x_2^3)}{\eta^2} \frac{f}{\lambda_1 - \lambda_2} \frac{g'}{g} \frac{u_{22}}{\lambda_1}.
\]

Hence

\[
\beta f \frac{g'}{g} \frac{\eta_2}{\eta} u_2 \geq - \beta f \frac{g'}{g} \frac{2x_1^3}{\eta} \left[ 1 + 2x_2^5 \left( \frac{f_1}{\lambda_1} + \frac{g'}{g} u_1 \right) f + \frac{2x_2^3 r^3}{\eta^2} \frac{u_{111}}{\lambda_1} \right] u_2 \\
\geq - \beta f \frac{g'}{g} \frac{2x_1^3}{\eta} \left[ 1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \right] u_2 \\
= -4 \beta f \frac{g'}{g} \frac{r^4}{\eta} u_2.
\]
In fact, \( \frac{\eta}{\eta^2} \approx \frac{-2x_2(r^2 - x_1^2)}{\eta} \) if \( \eta \lambda_1 \) is big enough. And we can get from (3.17)

\[
\frac{\eta^2}{\eta^2} \geq \left\{ 1 + \beta^2(2x_1x_2 \frac{x_2^2 - x_1^2}{\eta})^2, \frac{f}{(\lambda_1 - \lambda_2)^2} \right\}^2 \left[ 1 - \beta^2(2x_1x_2 \frac{x_2^2 - x_1^2}{\eta})^2, \frac{f}{(\lambda_1 - \lambda_2)^2} \right]^2
\]

\[
\geq \left[ \frac{-2x_2(r^2 - x_1^2)}{\eta} \right]^2 \left[ 1 - \frac{2x_5}{\eta \lambda_1} \left( \frac{f_1}{\lambda_1} + \frac{g'}{g} |u_1| f + \frac{2\beta^2 f r^3}{\eta^2 \lambda_1^2} g |u_2| \right)^2 [1 - \beta^2 \frac{16r^8 f^2}{\lambda_1^2}] \right]^2
\]

\[
\geq \left[ \frac{-2x_2(r^2 - x_1^2)}{\eta} \right]^2 \left[ 1 - \frac{1}{10^3} - \frac{64}{10^3} - \frac{1}{10} - \frac{1}{10^3} \right]^2 [1 - \frac{1}{10^3}]^2
\]

Taking second derivatives of \( \eta \), we can get

\[
\eta_{ii} = [-2 + 2(x, \tau) \langle x, \tau, \rangle_{ii} + 2(x, \tau) \langle x, \tau, \rangle_i]\left[ r^2 - (x, \tau)^2 \right] + 2[-x_i + 2(x, \tau) \langle x, \tau, \rangle_i][-2(x, \tau) \langle x, \tau, \rangle_i] + (r^2 - x_i^2)\left[ -2x_i \langle x, \tau, \rangle_{ii} - 2\langle x, \tau, \rangle_i \right]^{(2)} \left[ 2 - 2x_i \langle x, \tau, \rangle_i \right] + 2(x, \tau, \tau) \langle x, \tau, \rangle_{ii} + 2(x, \tau, \tau) \langle x, \tau, \rangle_i]
\]

so

\[
\eta_{11} = -2(r^2 - x_1^2) - 2x_1(x_1^2 - x_2^2) \langle x, \tau, \rangle_{11}
\]

\[
+ (4x_2^2 - 12x_1^2) \langle x, \tau, \rangle + (2x_2^2 - 10x_1^2) \langle x, \tau, \rangle^2
\]

\[
\eta_{22} = -2(r^2 - x_1^2) - 2x_1(x_2^2 - x_2^2) \langle x, \tau, \rangle_{22}
\]

\[
+ 8x_1x_2 \langle x, \tau, \rangle + (2x_2^2 - 10x_1^2) \langle x, \tau, \rangle^2.
\]

Hence

\[
\beta \left[ \frac{\eta_{11}}{\lambda_1} + \lambda_1 \frac{\eta_{22}}{\eta} \right] = -2\beta \left[ \frac{f}{\lambda_1} \left( \frac{r^2 - x_2^2}{\eta} + \lambda_1 \frac{r^2 - x_1^2}{\eta} \right) \right]
\]

\[
- 2\beta x_1 \left( \frac{x_1^2 - x_2^2}{\eta} \right) \langle x, \tau, \rangle_{11} + \lambda_1 \langle x, \tau, \rangle_{22}
\]

\[
+ \beta \left[ \frac{x_2(4x_2^2 - 12x_1^2)}{\eta} \lambda_1 - \lambda_2 + \frac{x_2^2(2x_2^2 - 10x_1^2)}{\eta} \right] \langle u_{112} \rangle_{11} \left( \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2} \right)^2
\]

\[
+ \beta \lambda_1 \frac{8x_1x_2^2}{\eta} \langle u_{221} \rangle + \frac{x_2^2(2x_2^2 - 10x_1^2)}{\eta} \langle u_{221} \rangle \left( \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2} \right)^2.
\]
Direct calculations yield

\[
\langle x, \tau \rangle_{11} = 2 \frac{\partial \tau_1}{\partial x_1} + \sum_{m=1}^{2} x_m \frac{\partial^2 \tau_m}{\partial x_1^2} = 2 \frac{\partial \tau_1}{\partial x_1} + \sum_{m=1}^{2} x_m \frac{\partial^2 \tau_m}{\partial x_1^2} \]

\[
= 2 \frac{\partial \tau_1}{\partial u_{pq}} u_{pq11} + \sum_{m=1}^{2} x_m \left[ \frac{\partial \tau_m}{\partial u_{pq}} u_{pq11} + \frac{\partial^2 \tau_m}{\partial u_{pq} \partial u_{rs}} u_{pq1u_{rs}} \right] \]

\[
= 0 + x_1 \frac{\partial^2 \tau_1}{\partial u_{pq} \partial u_{rs}} u_{pq1u_{rs}} + x_2 \left[ \frac{\partial \tau_2}{\partial u_{pq}} u_{pq11} + \frac{\partial^2 \tau_2}{\partial u_{pq} \partial u_{rs}} u_{pq1u_{rs}} \right] \]

\[
= -x_1 \left[ \frac{u_{112}}{\lambda_1 - \lambda_2} \right]^2 + x_2 \left[ \frac{1}{\lambda_1 - \lambda_2} \right] u_{1211} + 2x_2 \left[ -\frac{u_{112} u_{111}}{(\lambda_1 - \lambda_2)^2} + \frac{u_{112} u_{221}}{(\lambda_1 - \lambda_2)^2} \right],
\]

Similarly, we have

\[
\langle x, \tau \rangle_{22} = 2 \frac{\partial \tau_2}{\partial x_2} + \sum_{m=1}^{2} x_m \frac{\partial^2 \tau_m}{\partial x_2^2} = 2 \frac{\partial \tau_2}{\partial x_2} + \sum_{m=1}^{2} x_m \frac{\partial^2 \tau_m}{\partial x_2^2} \]

\[
= 2 \frac{\partial \tau_2}{\partial u_{pq}} u_{pq22} + \sum_{m=1}^{2} x_m \left[ \frac{\partial \tau_m}{\partial u_{pq}} u_{pq22} + \frac{\partial^2 \tau_m}{\partial u_{pq} \partial u_{rs}} u_{pq2u_{rs}} \right] \]

\[
= 2 \left[ \frac{1}{\lambda_1 - \lambda_2} \right] u_{221} - x_1 \left[ \frac{u_{221}}{\lambda_1 - \lambda_2} \right]^2 \]

\[
+ x_2 \left[ \frac{1}{\lambda_1 - \lambda_2} \right] u_{1222} + 2x_2 \left[ -\frac{u_{112} u_{221}}{(\lambda_1 - \lambda_2)^2} + \frac{u_{222} u_{221}}{(\lambda_1 - \lambda_2)^2} \right],
\]

then

\[
\frac{1}{\lambda_1} \langle x, \tau \rangle_{11} + \lambda_1 \langle x, \tau \rangle_{22} = -x_1 \frac{1}{\lambda_1} \left[ \frac{u_{112}}{\lambda_1 - \lambda_2} \right]^2 + 2x_2 \left[ \frac{u_{221}}{\lambda_1 - \lambda_2} \right] - x_1 \lambda_1 \left[ \frac{u_{221}}{\lambda_1 - \lambda_2} \right]^2 \]

\[
+ x_2 \left[ \frac{1}{\lambda_1 - \lambda_2} \right] \left[ f_{12} + f_1 \frac{u_{112}}{u_{111}} - f_2 \left( \frac{u_{111}}{u_{111}} \right) \right] \]

\[
+ 2x_2 \left[ -\frac{u_{112}}{(\lambda_1 - \lambda_2)^2} f_1 + \frac{u_{221}}{(\lambda_1 - \lambda_2)^2} f_2 \right].
\]
From (3.21) and (3.22), we can get
\[
\beta \left[ \frac{\eta_1}{\lambda_1} \eta + \lambda_1 \eta_2 \right]
\geq -2\beta \left[ \frac{\eta}{\lambda_1} \eta + \lambda_1 \eta_2 \right] - 2\beta \frac{x_1^2 - x_2^2}{\eta} - 2\beta \frac{x_1 x_2 (x_1^2 - x_2^2)}{\eta} \left[ \frac{1}{\lambda_1 - \lambda_2} \right] \left[ f_{12} - f_2 \left( \frac{u_{111}}{u_{11}} \right) \right]
\]
\[
+ \left( \frac{u_{112}}{\lambda_1 - \lambda_2} \right)^2 \left[ 2\beta f \frac{x_1^2 (x_1^2 - x_2^2)}{\eta} + \beta f \frac{x_2 (2x_2^2 - 10x_1^2)}{\eta} \right]
\]
\[
+ \frac{u_{112}}{\lambda_1 - \lambda_2} \left[ -2\beta x_1 (x_1^2 - x_2^2) (x_2 f_1 - 2x_2 f_1) + \beta f x_2 (4x_2^2 - 12x_1^2) \right]
\]
\[
+ \left( \frac{u_{221}}{\lambda_1 - \lambda_2} \right) \left[ 2\beta x_1 (x_1^2 - x_2^2) + \beta x_2 (2x_2^2 - 10x_1^2) \right]
\]
\[
+ \frac{u_{221}}{\lambda_1 - \lambda_2} \left[ -2\beta x_1 (x_1^2 - x_2^2) (2\lambda_1 + 2x_2 \frac{f_2}{\lambda_1 - \lambda_2}) + \beta \lambda_1 \frac{8x_1 x_2^2}{\eta} \right]
\]
\[
\geq -2\beta \frac{r_1^2 - r_2^2}{\eta} - 2\beta f \frac{r_2^2}{\eta \lambda_1} - 2\beta \frac{f_{12}}{\eta (\lambda_1 - \lambda_2)} - 2\beta \left( \frac{r_1^4}{\eta \lambda_1} \right) - 2\beta \left( \frac{f_2 r_2^4}{\eta \lambda_1} \right)
\]
\[
- \left( \frac{u_{112}}{\lambda_1 - \lambda_2} \right)^2 \beta f \frac{8r^4}{\eta \lambda_1} - \left( \frac{u_{112}}{\lambda_1 - \lambda_2} \right) \left[ 6\beta \frac{|f_1| r^4}{\eta \lambda_1} + \beta f \frac{12r^3}{\eta \lambda_1} \right]
\]
\[
- \frac{1}{\lambda_1 - \lambda_2} \left[ u_{221} \right] \left[ \left( \frac{u_{221}}{\lambda_1 - \lambda_2} \right)^2 \beta x_1 \frac{8r^4}{\eta} \right] - \frac{1}{\lambda_1 - \lambda_2} \left[ u_{221} \right] \left[ \left( \frac{u_{221}}{\lambda_1 - \lambda_2} \right)^2 \beta x_1 \frac{8r^4}{\eta} \right]
\]
\[
\geq -2\beta \frac{r_1^2 - r_2^2}{\eta} - 2\beta f \frac{r_2^2}{\eta \lambda_1} - 4\beta \frac{f_{12}}{\eta \lambda_1} - 4\beta \frac{f_2 r_1^4}{\eta \lambda_1}
\]
\[
- \left( \frac{u_{112}}{u_{11}} \right)^2 \beta f \frac{16r^4}{\eta \lambda_1} - \left( \frac{u_{112}}{u_{11}} \right) \left[ 12\beta \frac{|f_1| r^4}{\eta \lambda_1} + \beta f \frac{24r^3}{\eta \lambda_1} \right]
\]
\[
- 2\left( \frac{r_1^2}{\lambda_1} + \left( \frac{u_{111}}{u_{11}} \right)^2 \beta x_1 \frac{16r^4}{\eta} \right) - \left( \frac{|f_1|}{\lambda_1} + \left( \frac{u_{111}}{u_{11}} \right) \right) \left[ 8\beta \frac{r^4}{\eta \lambda_1} + \beta \lambda_1 \frac{24r^3}{\eta \lambda_1} \right]
\]
\[
\geq -2\beta \frac{r_1^2 - r_2^2}{\eta} - 2\beta f \frac{r_2^2}{\eta \lambda_1} - 4\beta |f_{12}| \frac{r_2^4}{\eta \lambda_1} - 32\beta \frac{f_2 r_2^4}{\eta \lambda_1} - 8\beta \frac{f_{12} f_2 r_4^4}{\eta \lambda_1} - 24\beta \frac{|f_1| r^3}{\eta \lambda_1}
\]
\[
- \frac{\lambda_1}{2} \left( \frac{u_{112}}{u_{11}} \right)^2 \left[ 32\beta f r_4^4 + \frac{12\beta r^4}{\eta \lambda_1} + 24\beta f r_4^4 + \frac{12\beta |f_1|^2 r_4^4}{\eta \lambda_1} + 24\beta f r_4^4 \right]
\]
\[
- f \left( \frac{u_{111}}{u_{11}} \right)^2 \left[ \frac{64\beta f r^4}{\eta \lambda_1} + \frac{16\beta r^4}{\eta \lambda_1} + \frac{1}{4} \right] - \frac{f}{2\lambda_1} \left( \frac{u_{111}}{u_{11}} \right)^2 \left[ 16\beta f r_4^2 \right] + \left( \frac{8\beta r^4}{\eta \lambda_1} \right)^2 + \left( \frac{8\beta r^4}{\eta \lambda_1} \right)^2
\]
\[
\geq -2\beta \frac{r_1^2 - r_2^2}{\eta} - 2\beta f \frac{r_2^2}{\eta \lambda_1} - 4\beta |f_{12}| \frac{r_2^4}{\eta \lambda_1} - 32\beta \frac{f_2 r_2^4}{\eta \lambda_1} - 8\beta \frac{f_{12} f_2 r_4^4}{\eta \lambda_1} - 24\beta \frac{|f_1| r^3}{\eta \lambda_1}
\]
\[
- \frac{\lambda_1}{4} \left( \frac{u_{112}}{u_{11}} \right)^2 \left[ 6\beta |f_1|^2 r^4 + 12\beta f r_2^4 \right]
\]
\[
- f \left( \frac{u_{111}}{u_{11}} \right)^2 \left[ \frac{8\beta f r_4^4}{\eta \lambda_1} + \frac{48\beta f r^4}{\eta \lambda_1} + \frac{32\beta |f_2|^2 r^8}{\eta \lambda_1} \right].
\]
Now we just need to estimate $-2\beta\lambda_1 \frac{r^2-x_2^2}{\eta}$. If $x_2^2 \leq \frac{r^2}{2}$, we can get

$$-2\beta\lambda_1 \frac{r^2-x_2^2}{\eta} = -\frac{8}{r^2-x_2^2} \lambda_1 \geq -\frac{16}{r^2} \lambda_1 \geq -\frac{1}{2} \frac{c_0}{r^2} f\lambda_1 = -\frac{1}{2} \frac{g'}{g} f\lambda_1.$$ 

If $x_2^2 \geq \frac{r^2}{2}$, we can get

$$-2\beta\lambda_1 \frac{r^2-x_2^2}{\eta} = -\frac{8}{r^2-x_2^2} \lambda_1 \geq -\frac{x_2^2}{r^2-x_2^2} \frac{8}{r^2-x_2^2} \lambda_1 = -\beta\lambda_1 \left[ \frac{1}{2} \frac{2x_2^2}{r^2-x_2^2} \right]^2 \geq -\lambda_1 \left[ \frac{\eta_2}{\eta} \right]^2.$$ 

Hence

(3.23) $$-2\beta\lambda_1 \frac{r^2-x_2^2}{\eta} \geq -\frac{1}{2} \frac{g'}{g} f\lambda_1 - \beta\lambda_1 \left[ \frac{\eta_2}{\eta} \right]^2.$$ 

and

$$\beta \left[ \frac{f}{\lambda_1} \frac{\eta_1}{\eta} + \lambda_1 \frac{\eta_2}{\eta} \right] \geq \frac{1}{2} \frac{g'}{g} f\lambda_1 - \beta\lambda_1 \left[ \frac{\eta_2}{\eta} \right]^2 - \frac{f}{2\lambda_1} \left( \frac{u_{111}}{u_{11}} \right)^2 - \frac{\lambda_1}{4} \left( \frac{u_{112}}{u_{11}} \right)^2$$

$$- 2\beta f \frac{r^2}{\eta\lambda_1} - 4\beta \frac{f_{12}}{\eta\lambda_1} \frac{r^4}{\eta\lambda_1} - 32\beta \frac{f_{1}^2 r^4}{\eta\lambda_1^2} - 8\beta \frac{f_1 f_2 |r|^4}{\eta\lambda_1^2} - 24\beta \frac{f_1 |r|^3}{\eta\lambda_1}$$

$$- \left[ \frac{6\beta |f_1|^2 r^4}{\eta\lambda_1^2} + \frac{12\beta f r^2}{\eta\lambda_1} \right] - \left[ \frac{8\beta f r^4 |f_2|^2}{\eta^2 \lambda_1} + \frac{(48\beta)^2 f_{1}^2 |r|^6}{\eta^2 \lambda_1^2} + \frac{32\beta^2 |f_2|^2 r^8}{\eta^2 \lambda_1 f} \right].$$

(3.24)
Now we continue to prove Theorem 1.1. From (3.10) and Lemma 3.1, we can get

\[ 0 \geq \sum_{i=1}^{2} F^{ii} \varphi_{ii} \]

\[ \geq \frac{1}{2} g^i_r f \lambda_1 - 2 \beta f \frac{r^2}{\eta \lambda_1} - 4 \beta |f_{12}| \frac{r^4}{\eta \lambda_1} - 32 \beta \frac{f_1 f_2 r^4}{\eta \lambda_1^3} - 8 \beta \frac{|f_1 f_2 r^4|}{\eta \lambda_1^3} - 24 \beta \frac{|f_1 r^3|}{\eta \lambda_1} - \frac{6 \beta f_1 f_2 r^4}{\eta \lambda_1} - \frac{12 \beta f r^2}{\eta \lambda_1} \]

\[ - \frac{8 \beta f r^2 |f_2|^2}{\eta \lambda_1^3} + \frac{(48 \beta)^2 f r^6}{\eta^2 \lambda_1} + \frac{32 \beta^2 |f_2|^2 r^8}{\eta^2 \lambda_1} \]

\[ - 4 \beta f \frac{g^i_r r^4 |u_2|}{g} - \frac{g^i_r |\nabla u|| \nabla f|}{\lambda_1} - \frac{|f_{11}|}{\lambda_1} - 2 \frac{f r^2}{f \lambda_1} - \frac{8 \beta f r^6}{\eta^2 \lambda_1} \]

\[ \geq \frac{c_0 m}{2r^2} \lambda_1 - \frac{8r^2 \cdot M}{\eta \lambda_1} - \frac{32r^2 \cdot r^2 |f_{12}|}{\eta \lambda_1} - \frac{128r^2 \cdot r^2 f_1^2}{\eta \lambda_1} - \frac{32r^2 \cdot r^2 |f_1 f_2|}{\eta \lambda_1} - \frac{96r^2 \cdot r |f_1|}{\eta \lambda_1} \]

\[ - \frac{24r^2 \cdot r^2 |f_1|^2}{\eta \lambda_1} - \frac{48r^2 \cdot M}{\eta \lambda_1} - \frac{32r^2 \cdot M \cdot r^2 |f_1|^2}{\eta \lambda_1} - \frac{192r^2 \cdot M \cdot r^6}{\eta \lambda_1} - \frac{512r^6 \cdot |f_2|^2}{\eta \lambda_1} \]

\[ - \frac{16r^2 \cdot c_0 m |u_2|}{r} - \frac{c_0 r^2 \cdot r |\nabla f|}{r} \cdot \frac{|\nabla u|}{r} - \frac{|f_{11}|}{\lambda_1} - \frac{2}{m \lambda_1} - \frac{32 M r^6}{\eta^2 \lambda_1}. \]

So we can get

(3.25) \[ \eta \lambda_1 \leq C \left( 1 + \frac{|\nabla u|}{r} \right) r^4. \]

where \( C \) is a positive constant depending only on \( c_0, m, M, r |\nabla f|, \) and \( r^2 |\nabla^2 f| \). So we can easily get

\[ u_r(0) = \frac{1}{r} \frac{1}{r^{4 \beta}} \phi(0) \leq \frac{1}{r} \frac{1}{r^{4 \beta}} \phi(x_0) \leq C \left( 1 + \frac{\sup |Du|}{r} \right) e^{c_0 \frac{\sup |Du|}{r^2}} \]

\[ \leq C e^{(c_0 + 2) \frac{\sup |Du|^2}{r^2}}, \]

and

(3.26) \[ |u_{\xi\xi}(0)| \leq u_r(0) \tau(0)(0) \leq C e^{(c_0 + 2) \frac{\sup |Du|^2}{r^2}}, \quad \forall \xi \in S^1. \]

Then we prove Theorem 1.1 under the condition \( \eta \lambda_1 \geq 10^3 (1 + M + r \sup |\nabla f| + \frac{M \sup |Du|}{r}) r^4. \) Hence Theorem 1.1 holds.

**Remark 3.2.** The eigenvector field \( \tau \) is important. In fact, it is well-defined when the largest eigenvalue is distinct with others, and \( \tau \) depends only on the adjoint matrix. For the Monge-Ampère equation in dimension \( n \geq 3 \), we do not know whether the largest eigenvalue is distinct with others. So our method is not suitable for this case.
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