On von Neumann’s Examples of Types

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Abstract

The factorization properties of operator algebras in separable Hilbert spaces was developed by John von Neumann in collaboration with F.J. Murray in four outstanding papers published from 1936 to 1943. Unfortunately, probably because of conceptual difficulties with physical interpretations, some relevant results presented in those papers, in particular the remarkable examples of factors, have not been adequately considered so far. The paper here presented aims to introduce the subject in a new although maybe unusual form pursuing three main goals: speculating about the physical reasons and motivations that are likely to have been at the origin of von Neumann’s investigation; describing the examples of factors provided by von Neumann and Murray with the purpose of clarifying the general concepts standing at the base of the classification of algebraic factors into three general types; outlining the perspective of extending the theory to non–separable Hilbert spaces with the purpose of suggesting a novel approach to the representation of infinite systems controlled by external gauge fields.

Introduction

According to the correspondence principle stated by Niels Bohr (1923), the behavior of a quantum mechanical system approaches that of a classical system in the limit of large quantum numbers. As shown by Dirac in 1926, this principle can be mathematically formulated by saying that Poisson’s brackets are the classical limits of Heisenberg’s commutators divided by $i\hbar$. Correspondingly, the transformations $q' = T_c q$, $p' = T_c p$ of the classical phase–space variables $\{q, p\}$ are found to be the classical limits of the algebraic automorphisms $Q' = TQ T^{-1}$, $P' = TP T^{-1}$ acting on the operators $\{Q, P\}$, which play the role of $\{q, p\}$ in the quantum–mechanical representation.

In other words, the Abelian groups of classical transformations appear to be isomorphic to the Abelian groups of quantum–mechanical automorphisms, rather than to the group of transformations $|\phi'\rangle = T|\phi\rangle$ acting on the unit vectors $|\phi\rangle$ (rays), which represent the states of the quantum–mechanical system. The latter, indeed, are non–Abelian projective extensions of the Abelian groups of phase–space transformations $T_c$.

So, while in classical mechanics the algebraic variables that represent the physical quantities differ qualitatively from the differential operators that represent the generators of physical transformations, in quantum mechanics, by contrast, the Hermitian operators that represent the physical quantities represent also the generators of physical transformations; as if observing and operating were two complementary aspects of physical phenomenology.

A few years later, in The theory of groups and quantum mechanics, §14, [12] (translation of the German edition published in 1929), Hermann Weyl declared to feel certain that the kinematical structure of a physical system is expressed by an irreducible Abelian group of unitary ray rotations in system space (which is a different name for “irreducible non–Abelian projective extension of an Abelian group”). This structure leads automatically to the decomposition of a family of independent observables into pairs of

\[\boxed{\text{1This is in contrast with the attempts to unify the theory of the classical and the quantum–mechanical systems basing on the correspondence between physical states rather than between classical transformations and observable automorphisms.}}\]
conjugated quantities. For quantum–mechanical systems with classical analog, the group is generated by the observables $Q$ and $P$. However, as proved by Weyl himself, also the commutative and anti–commutative algebras generated by the creation and destruction operators of bosonic and fermionic fields can be derived from Abelian groups of automorphisms.

In this view, the general structure of quantum–mechanical systems emerges as a maximal commutative algebra $\mathcal{P}$ of projectors $P, P', \ldots$, representing a complete set of compatible observables, which are equipped with an Abelian group $\mathcal{U}$ of automorphisms $P' = UPU^{-1}$ induced by a group of unitary operators $U \in \mathcal{U}$, which can be interpreted as the carriers of canonical–coordinate transformations. It is worth noticing, however, that after the discovery of the internal–symmetry groups of elementary particles there is no reason to require that the automorphism groups devised by Weyl be Abelian.

The fact that the transformations acting on the states of a classical system are one–to–one with the algebraic automorphisms acting on the observables of a corresponding quantum–mechanical system admits the following physical interpretation: both groups represent the operational possibilities of an ideal macroscopic observer interacting with the system. It is therefore quite natural to assume that, since these possibilities are given in the macroscopic world, they can be represented as a set of mutually exclusive events, as they were the possible results of macroscopic measurements. It is therefore reasonable to assume that a quantum–mechanical representation of this set of operational possibilities be implemented by labeling the vectors of a Hilbert–space basis by the elements of a suitable group $G$. In this way, the observed–observer system could find a suitable representation in the direct product of the Hilbert space formed by the state vectors of the observable system with that spanned by a basis of vectors labeled by the elements of $G$.

An evident disadvantage of this description is that, if we require that the Hilbert space of operational possibilities is separable, then only discrete groups of transformations can be admitted. *En passant*, this leads to hypothesize that the description of a more realistic set of operational possibilities, possibly parameterized by external gauge fields, could be given only in non–separable Hilbert spaces.

By the arguments that we are just on the point to introduce, we want to prove that the direct product of two Hilbert spaces of the type described above produces a hybrid space, in which the algebraic factorization of the entire system in two independent, although mutually correlated parts, has its most natural representation.

Such a factorization is not arbitrary but conditioned by the algebraic structure of the observables of the composed system. Throughout this study, we will elucidate also the problem of classifying and representing the ways in which the parts of a quantum–mechanical system interact with each other. The relevance of this fact lies in that this problem does not find a satisfactory solution if the state space of the composed system is naively assumed as the direct product of the state spaces of the parts.

For instance, if the state of a system formed by two parts $A, A'$ is represented as a unit vector $|\Phi\rangle$ belonging to the direct product $\mathcal{H}$ of the Hilbert spaces $\mathcal{H}_A, \mathcal{H}_{A'}$ of the parts, we have

$$|\Phi\rangle = \sum_{ij} c_{ij} |\alpha_i\rangle \otimes |\alpha'_j\rangle,$$

where $c_{ij}$ are complex constants and $|\alpha_i\rangle, |\alpha'_j\rangle$ are the basis vectors respectively of $\mathcal{H}_A$ and $\mathcal{H}_{A'}$.

In this way, the composition of the two parts produces the entanglement of their respective states. This has no analog in classical mechanics. Now, it is known that, by keeping $|\Phi\rangle$ fixed, the bases of $\mathcal{H}_A, \mathcal{H}_{A'}$ can be rotated so as to obtain the one–index summation

$$|\Phi\rangle = \sum_i \sqrt{w_i} |\alpha_i\rangle \otimes |\alpha'_i\rangle,$$

where the positive constants $w_i$ can be interpreted as the probabilities that a suitable observation of $A$, or of $A'$, detects that the state of the system is $|\alpha_i\rangle \otimes |\alpha'_i\rangle$. We immediately see how the entanglement of part states results into a discrete pairing of part states. Of course, if $|\Phi\rangle$ varies in $\mathcal{H}$, also the states of each pairing and the values of $w_i$ in general change.

Compositions and measurements of this sort explain how, for instance, the observation that a measurement device $A'$ is in a state $|\alpha'_i\rangle$ determines the simultaneous projection of the state of the measured system $A$ into the state $|\alpha_i\rangle$ paired with $|\alpha'_i\rangle$ in $|\Phi\rangle$.  

2
Clearly, these sorts of pairings are possible only if the quantities to be measured possess discrete sets of eigenvalues. But how then could we deal with quantities whose eigenvalue–spectra are with continuous or with infinite collections of quantities?

The problem of measurement and observation of quantum–mechanical systems was carefully analyzed by von Neumann. In his famous book of 1932 on the Mathematical Foundations of Quantum Mechanics [7], after discussing the process of measurement as outlined above, the great author posed also the problem of how, starting from an initial state of the form \( |\Phi_0\rangle = |\phi_0\rangle \otimes |\phi'_0\rangle \), i.e., a state formed by two non–entangled states \( |\phi_0\rangle \) and \( |\phi'_0\rangle \), the entangled state \( |\Phi\rangle = \sum_i \sqrt{w_i} |\alpha_i\rangle \otimes |\alpha'_i\rangle \) can be generated.

To capture the problem in its very essence, von Neumann assumed that the states of the measurement device, and consequently also those of the observed system, be indexed by the elements of a discrete group, precisely the Abelian group of integers \( \mathbb{Z} \).

In particular, he assumed that the initial resting–state of the measurement device is \( |\phi_0\rangle = |\alpha'_0\rangle \) and that the initial state of the system under observation is

\[
|\phi_0\rangle = \sum_{n \in \mathbb{Z}} \sqrt{w_n} |\alpha_n\rangle;
\]

that is, a superposition of the eigenstates of the quantity under observation, whose probability amplitudes \( w_n \) can be related to the statistical frequencies of results obtained in a large number of measurements repeated in identical conditions. The question is then to explain how the transition

\[
|\Phi_0\rangle = \left( \sum_{n \in \mathbb{Z}} \sqrt{w_n} |\alpha_n\rangle \right) \otimes |\alpha'_0\rangle \quad \Rightarrow \quad |\Phi\rangle = \sum_{n \in \mathbb{Z}} \sqrt{w_n} \left( |\alpha_n\rangle \otimes |\alpha'_i\rangle \right),
\]

from the initially disentangled state \( |\Phi_0\rangle \) to the entangled state \( |\Phi\rangle \) can occur before the observation of the measurement device disentangles the pair \( |\alpha_n\rangle \otimes |\alpha'_i\rangle \) with probability \( w_n \).

Von Neumann’s answer was that the transition \( |\Phi_0\rangle \Rightarrow |\Phi\rangle \) can be interpreted as a temporal evolution \( |\Phi\rangle = U |\Phi_0\rangle \) performed by the unitary operator defined by the equations

\[
U |\alpha_n\rangle \otimes |\alpha'_m\rangle = |\alpha_n\rangle \otimes |\alpha'_{n+m}\rangle.
\]

This operator, which unfortunately cannot be expressed as a continuous function of time because of the discreteness of state entanglement, has the form

\[
U = \sum_{n \in \mathbb{Z}} P_n \otimes T^n, \quad P_n = |\alpha_n\rangle \langle \alpha_n|, \quad T = \sum_{n \in \mathbb{Z}} |\alpha'_{n+1}\rangle \langle \alpha'_n|.
\]

Thus, ultimately, the problem of how state entanglement may take place during observation processes leads naturally to represent the observed–observer system in the framework of a hybrid space \( \mathcal{H} = \mathcal{H}_X \otimes \mathcal{H}_Z \), where \( X \) is the index set of the state–projectors \( P_n \) family of the observed system and \( Z \) the translation group of the observing–device pointer.

The celebrated papers of von Neumann (and Murray) on the rings of operators, [5] [6] [9] [10], seem to be directed to exhaust the analysis of the measurement problem, which was introduced only schematically and as an example in the quoted book on Foundations. Those papers seem to indicate that, at the maximum level of generality, the solution to the problem can be reached by pursuing the goal of understanding, not just how the states of the parts can pair with each other, but, more basically, how the operators that represents the observables of the parts can correlate with each other in a measurement process.

In the following, we will see how the operator algebras that can be formed quite naturally in the hybrid spaces of the sort described above, afford a wonderful territory in which the investigation of this subject can be carried out in the most fruitful way.

We will limit ourselves to the case, completely solved by von Neumann, in which the Hilbert space of operational possibilities is separable. That is, the case in which the operations form a non–Abelian discrete group \( G \). In the most natural way, and with maximal generality, the structure of the observed system will be represented by an algebra of projectors indexed by the measurable subsets (in the most general sense) of a set \( X \), in which \( G \) acts by point transformations. For the sake of language simplicity, systems
of this sort will be called discrete. For reasons that will be clear in a second moment, we will assume that $G$ acts on $X$ freely, i.e., with no fixed subsets of non–zero measure, and ergodically, i.e., transitively for non–zero–measure subsets of $X$.

This treatment differs somewhat from that of the great Austro–Hungarian mathematician and of his coworker. In certain sense, it is a considerable simplification of their original treatment (besides, of course, an incomplete and lacunal exposition of it). Nevertheless, we hope that by pursuing the goal of evidencing the physical meanings of the construction and by using a formalism based on the definition of certain operators in the space of operational possibilities, we will be able (hopefully) to make the subject simpler and more transparent.

Before entering the subject, we will introduce without any pretensions of completeness and precision some basic concepts regarding the algebras of observables.

1 The theory of factorization

We will summarize here the most important results obtained by von Neumann on the algebras of bounded operators in Hilbert spaces (the algebras of unbounded operators exhibit incurable pathologies). In the following, the term algebra will indicate a $C^*$–algebra, i.e., an algebra containing the adjoint $A^\dagger$ of every operator $A$.

1.1 Von Neumann algebras

**Definition 1.1** Let $A$ be an algebra of bounded operators of the Hilbert space $\mathcal{H}$. Precisely, a subalgebra of the algebra $I$ formed by all the bounded operators of $\mathcal{H}$. Call the commutant of $A$ the algebra $A'$ formed by all bounded operators of $\mathcal{H}$ that commute with all the elements of $A$. Clearly, $A \subset A''$ and $A' = A'''$. If $A = A''$, we will say that $A$ is a von Neumann’s algebra.

The construction of an algebra by means of finite numbers of algebraic operations on a basic set of bounded operators and its subsequent topological closure by suitable limiting procedures requires a certain care in order to avoid ambiguities that may rise from the possible non–commutativity of limiting procedures. Among various possible kinds of topological closure, the most convenient and interesting is the closure in the weak topology. It consists, in practice, in taking the limit of a sequence of operators by evaluating the limits of its matrix elements. In the following, we will assume that the algebras generated by a set of bounded operators are closed in the weak topology. The well–known theorem called of the bicommutant then holds [8].

**Theorem 1.1** If $A$ contains the unit operator $I$ and is closed in the weak topology then $A'' = A$.

Let us state without proof the following proposition.

**Theorem 1.2** $A = A''$ can be generated from all of its projectors as well as from all of its unitary operators.

Therefore, $A'$ is determined by the condition of commuting with all the projectors of $A$ or with all the unitary operators of $A$.

Let us now study how the weak–closure property permits the conversion of the partial ordering of the subalgebras of $I$ in a complete lattice of subalgebras.

**Definition 1.2** Let $A \subset I$ and $B \subset I$ two von Neumann’s algebras. Let us indicate by $A \vee B$ the smallest von Neumann’s algebra that contains $A$ and $B$ as subalgebras, and by $A \wedge B$ the greatest von Neumann’s algebra contained in $A$ and $B$.

It can be easily proved that $A \wedge B = A \cap B$, where $\cap$ is the set–theoretic intersection.

Since both $A$ and $B$ contain the set $\{\alpha I\}$ formed by the multiples of the unit operator $I$, we have $\{\alpha I\} \subset A \wedge B \subset A \vee B \subset I$. 

4
Note that \( \{\alpha I\}' = \mathcal{I} \setminus \mathcal{I}' = \{\alpha I\} \), and that in the partially ordered set of subalgebras the apex exchanges the adjective containing with the adjective contained and greater with smaller; in particular, it exchanges \( A \subset B \) with \( B' \subset A' \). We arrive thereby to establish the following theorem

**Theorem 1.3** If \( I \in A = A'' \) and \( I \in B = B'' \), then the following lattice-theoretic- ordering relations hold

\[
\{\alpha I\} \subset A \wedge B \subset A \vee B \subset \mathcal{I}; \quad \{\alpha I\} \subset (A \vee B)' \subset (A \wedge B)' \subset \mathcal{I};
\]

and the duality relations \((A \wedge B)' = A' \vee B', \ (A \vee B)' = A' \wedge B'\).

In short, the apex works as an operator of relative orthocomplementation.

Let us now introduce the notions of factor and factorization.

**Definition 1.3** An algebra \( A \) of bounded operators containing \( I \), but not necessarily a von Neumann’s algebra, is called a factor if \( A \wedge A' = \{\alpha I\} \) and \( A \vee A' = \mathcal{I} \). If \( \mathcal{A} \) is a von Neumann’s algebra, the condition \( A \wedge A' = \{\alpha I\} \) is sufficient, as the second equality can be derived from the first by duality. In this case \( A \) and \( A' \) are called coupled factors.

A trivial example of coupled factors is provided by \( A = \mathcal{I} \) and \( A' = \{\alpha I\} \). An elementary example is that formed by the direct products of matrices \( A \otimes I_N \) and \( I_N \otimes A' \), where \( A \) and \( A' \) are square matrices and \( I_N \) are the unit matrices, respectively of dimensions \( N \) and \( N' \). \( N \) and \( N' \) can be finite or infinite. The most interesting cases, however, are those in which the operators of each factor cannot be represented as direct products of the form indicated above. The existence and the properties of these factors are just the subject of our investigation.

### 1.2 Isometric projectors

Here, we introduce briefly a few important notions on partially isometric operators \[5\]. Let \( \mathcal{K} \) be a subspace of \( \mathcal{H} \). In the following, the subspace orthogonal to \( \mathcal{K} \) in \( \mathcal{H} \) will be denoted by \( \mathcal{H} - \mathcal{K} \).

**Definition 1.4** An operator \( \hat{U} \) is called partially isometric if it maps a subspace \( \mathcal{K}_1 \subset \mathcal{H} \) onto a subspace \( \mathcal{K}_2 \subset \mathcal{H} \) leaving vector–norms unchanged and annihilating the subspace \( \mathcal{H} - \mathcal{K}_2 \). That is, if \( f \in \mathcal{K}_1 \) implies \( \hat{U} f \in \mathcal{K}_2 \), with \( (\hat{U} f, \hat{U} f) = (f, f) \), while \( f \in \mathcal{H} - \mathcal{K}_2 \) implies \( \hat{U} f = 0 \).

From the symmetry of the definition it follows that the adjoint operator \( \hat{U}^\dagger \) maps isometrically \( \mathcal{K}_2 \) into \( \mathcal{K}_1 \) annihilating the orthogonal subspace \( \mathcal{H} - \mathcal{K}_2 \). We can easily verify that \( \hat{U}^\dagger \hat{U} \) and \( \hat{U} \hat{U}^\dagger \) project \( \mathcal{H} \) respectively onto \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \). This fact leads naturally to the following definition.

**Definition 1.5** Two projectors \( P_1, P_2 \) will be called isometric, and we will write \( P_1 \sim P_2 \), if there exists a partially isometric operator \( \hat{U} \) such that \( \hat{U}^\dagger \hat{U} = P_1 \), \( \hat{U} \hat{U}^\dagger = P_2 \). The following equalities can be easily verified \( \hat{U} P_1 \hat{U}^\dagger = P_2 \), \( \hat{U}^\dagger P_2 \hat{U} = P_1 \).

In the next, we will use the following definition and soon after a theorem based on the considerations reported along with the definition.

**Definition 1.6** Two partially isometric operators \( \hat{U}^{(1)} \) and \( \hat{U}^{(2)} \) are called orthogonal if \( \hat{U}^{(1)} \hat{U}^{(2)} = \hat{U}^{(2)} \hat{U}^{(1)} = 0 \).

Define the projectors

\[
P^{(1)}_1 = \hat{U}^{(1)} \hat{U}^{(1)} \dagger, \quad P^{(1)}_2 = \hat{U}^{(1)} \hat{U}^{(1)} \dagger, \quad P^{(2)}_1 = \hat{U}^{(2)} \hat{U}^{(2)} \dagger, \quad P^{(2)}_2 = \hat{U}^{(2)} \hat{U}^{(2)} \dagger.
\]

If \( \hat{U}^{(1)} \) and \( \hat{U}^{(2)} \) are orthogonal, then the equations

\[
(\hat{U}^{(1)} + \hat{U}^{(2)}) (\hat{U}^{(1)} + \hat{U}^{(2)}) = P^{(1)}_1 + P^{(2)}_1, \quad (\hat{U}^{(1)} + \hat{U}^{(2)}) (\hat{U}^{(1)} + \hat{U}^{(2)}) \dagger = P^{(1)}_2 + P^{(2)}_2
\]

clearly hold. Moreover \( P^{(1)}_1 P^{(2)}_2 = 0 \), \( P^{(2)}_2 P^{(1)}_1 = 0 \). In short, \( \hat{U}^{(1)} + \hat{U}^{(2)} \) is an isometric operator and \( P^{(1)}_1 + P^{(2)}_1, P^{(1)}_2 + P^{(2)}_2 \) are projectors. Also the converse can be easily proved: if \( P^{(1)}_2 P^{(2)}_2 = 0 \) and \( P^{(1)}_1 P^{(2)}_1 = 0 \), then \( \hat{U}^{(1)} \) e \( \hat{U}^{(2)} \) are orthogonal. These results lead directly to the following conclusion.
Theorem 1.4 If for the projectors $P_1^{(1)}$, $P_2^{(1)}$, $P_1^{(2)}$, $P_2^{(2)}$ the properties
\[ P_1^{(1)}P_2^{(1)} = 0, \quad P_2^{(2)}P_1^{(2)} = 0, \quad P_1^{(1)} \sim P_2^{(1)}, \quad P_1^{(2)} \sim P_2^{(2)} \]
hold, then $P_1^{(1)} + P_2^{(2)}$, $P_2^{(1)} + P_1^{(2)}$ are projectors and $P_1^{(1)} + P_1^{(2)} \sim P_2^{(1)} + P_2^{(2)}$.

1.3 Remarkable relationships between $A$ and $A'$

Let us now evidence a few operator properties that are naturally related to the notions introduced in the previous section. The theorems that we are on the point to present belong to the repertoire of standard notions of the theory of operators [11]. The first two of them are well known, and we wish to recall them only for their importance in the theory of factorization that we want to describe.

Theorem 1.5 (of the root operator) If $B$ is a non-negative definite and bounded operator, then the operator $B^{1/2}$ belongs to the algebra generated by $I$ e $B$.

We can indeed easily prove the convergence of the binomial series
\[ B^{1/2} = [I + (B - I)]^{1/2} = \sum_{n=0}^{\infty} \frac{1}{2} \frac{(\frac{1}{2} - 1) \cdots (\frac{1}{2} - n + 1)}{n!} (B - I)^n. \]

Theorem 1.6 (of the polar decomposition) Every bounded operator $A$ can be written as $A = \hat{U}|A|$, where $|A| = (A^*A)^{1/2}$ is non-negative definite and $\hat{U}$ is partially isometric.

Consequently, also the projectors $P_A = \hat{U}^*\hat{U}$ and $P_{A^*} = \hat{U}\hat{U}^*$ are determined. Let $D(A)$ and $C(A)$ be respectively the domain and the range of $A$. Denote by $[D(A)]$ the closure of $D(A)$, that is, the smallest subspace of $\mathcal{H}$ that contains $D(A)$, and with $[C(A)]$ the closure of $C(A)$. Then $[D(A)] = P_A\mathcal{H}$ and $[C(A)] = P_{A^*}\mathcal{H}$.

Theorem 1.7 (of the commuting projector) Let $A \in \mathcal{A}$ and $\hat{P}$ a projector of $\mathcal{I}$. A necessary and sufficient condition that the equality $[A, P] = 0$ holds is $C(\hat{A}\hat{P}) \subset C(\hat{P})$ and $C(\hat{A}^*\hat{P}) \subset C(\hat{P})$.

The necessity of the condition is evident. As for the sufficiency, it is clear that the inclusive relations between the ranges are equivalent to $PA\hat{P} = \hat{A}P$ and $PA^*\hat{P} = \hat{A}^*P$. The comparison of the first equation with the adjoint of the second yields $[P, A] = 0$.

Theorem 1.8 (of the crossed projector) Let $P$ be a projector of $\mathcal{A}$ and $P'$ a projector of $\mathcal{A}'$. Then $PP' = 0$ implies $P = 0$ or $P' = 0$. In other words, if both $P$ and $P'$ differ from zero then $PP' \neq 0$.

Proof: Consider the subspace $\hat{K}$ formed by all vectors $\hat{f} \in \mathcal{H}$ such that $PX\hat{f} = 0$ for all the operators $X \in \mathcal{A}$. In symbols, $\hat{K} = \{ \hat{f} : PX\hat{f} = 0, X \in \mathcal{A} \}$. Let $\hat{P}$ be the projector defined by $P\mathcal{H} = \hat{K}$. If $A \in \mathcal{A}$, the relation $A\hat{f} \in \hat{K}$ holds as $XA \in \mathcal{A}$; if $A' \in \mathcal{A}'$, $A'\hat{f} \in \hat{K}$ holds as $PXA'\hat{f} = A'PX\hat{f} = 0$. In both cases, we have the inclusive relations
\[ C(A\hat{P}) \subset C(\hat{P}), \quad C(A^*\hat{P}) \subset C(\hat{P}), \quad C(A'\hat{P}) \subset C(\hat{P}), \quad C(A'^*\hat{P}) \subset C(\hat{P}), \]
which enable the application of theorem [11]. We have then $[\hat{P}, A] = [\hat{P}, A'] = 0$. Since $\mathcal{A} \in \mathcal{A}'$ have in common only multiples of the unit operator we obtain $\hat{P} = \alpha I$, with $\alpha = 1$ or $\alpha = 0$, in other terms, $\hat{K} = \mathcal{H}$ or $\hat{K} = \emptyset$. Assume now $PP' = 0$. This means that for every $f \in \mathcal{H}$ we have $P'f \in \hat{K}$ and $Pf \in \mathcal{H} - \hat{K}$. Since $\hat{K} = \mathcal{H}$ or $\hat{K} = \emptyset$, we deduce that at least one of the two projectors is zero.

Theorem 1.9 (of local comparability) Two non-zero projectors $P_1$ and $P_2$ of $\mathcal{A}$ possess isometric segments. That is, there exist a projector $\Delta P_1 \subset P_1$ and a projector $\Delta P_2 \subset P_2$ such that $\Delta P_1 \sim \Delta P_2$.

\[^2\text{ }P_1 \subset P_2 \text{ means } P_1P_2 = P_1.\]
It is sufficient to prove that there exists a non-zero partially isometric projector $\hat{U}_{12}$ such that $\hat{U}_{12}\hat{U}_{12} \subset P_1$ and $\hat{U}_{12}\hat{U}_{12} \subset P_2$.

Assume $f \in P_1\mathcal{H} \equiv \mathcal{K}_1$ and normalize $f$ so that $(f,f) = 1$. Form the space $\mathcal{K}'_1$ that comprises all vectors that can be obtained by applying all the operators of $\mathcal{A}$ to $f$, in symbols $\mathcal{K}'_1 = [A f]$, and denote by $P'_f$ the projector that sends $\mathcal{H}$ into $\mathcal{K}'_1$. Clearly, if $A \in \mathcal{A}$ then $\mathcal{C}(A P'_f) \subset \mathcal{C}(P'_f)$ and moreover $\mathcal{C}(A^1 P'_f) \subset \mathcal{C}(P'_f)$, since also $A^1 \in \mathcal{A}$. Then, from Theor. 1.7 we obtain $[P'_f, A] = 0$ for every $A \in \mathcal{A}$, and therefore $P'_f \in \mathcal{A}'$. Consequently, we obtain not only $f \in P_1 P'_f \neq 0$ by construction, but also $P_2 P'_f \neq 0$ by theorem 1.8.

Let now be $g \in (P_2 P'_f)\mathcal{H} = (P'_f P_2)\mathcal{H}$ with $(g,g) = 1$. This means that for at least one operator $\hat{A} \in \mathcal{A}$ the inequality

$$
\| P_2 \hat{A} f - g \| \equiv \| P_2 \hat{A} P_1 f - g \| < 1
$$

holds, which implies $P_2 \hat{A} P_1 \neq 0$. Pose $A_{12} = P_2 \hat{A} P_1$ and form the polar decomposition $A_{12} = \hat{U}_{12} |A_{12}|$. It can then be easily proved that $\hat{U}_{12}$ has the expected properties.

### 1.4 The first theorem of comparability

Basing on Theor. 1.9 we can carry out the following construction. Assume $P_1, P_2 \in \mathcal{A}$, $\Delta P_1 \subset P_1$, $\Delta P_2 \subset P_2$ and $\Delta P_1 \sim \Delta P_2$. Define $\Delta^{(1)} P_1 = \Delta P_1$ and $\Delta^{(1)} P_2 = \Delta P_2$, with the purpose of starting the enumeration of subsequent applications of the theorem. By subtracting the isometric segments $\Delta P_1$ and $\Delta P_2$ respectively from $P_1, P_2$ we obtain two new projectors $P_1 - \Delta P_1, P_2 - \Delta P_2$. If none of these is zero we can apply again theorem 1.9 thus finding two new isometric segments $\Delta'^1P_1 \sim \Delta'^2P_2$ respectively orthogonal to the previous ones. Pose $\Delta^{(2)} P_1 = \Delta^{(1)} P_1 + \Delta'^1 P_1$, $\Delta^{(2)} P_2 = \Delta^{(1)} P_2 + \Delta'^2 P_2$. Then, from theorem 1.11 we obtain $\Delta^{(2)} P_1 \sim \Delta^{(2)} P_2$. We can iterate this procedure possibly until one of the two remainders becomes zero. Alternatively, the procedure continues indefinitely. Even in this case, we can exhaust one of the remainders by invoking the principle of Transfinite Induction.

Pose $\hat{P}_1 = \sum_n \Delta^{(n)} P_1$, $\hat{P}_2 = \sum_n \Delta^{(n)} P_2$. Since all segments of each projector are mutually orthogonal, we have $\hat{P}_1 \sim \hat{P}_2$. We arrive in this way to state the following comparability theorem 8.

**Theorem 1.10** If $P_1, P_2$ are two projectors of a same factor $\mathcal{A}$, then either there exists a $\hat{P}_1$ such that $P_1 \sim \hat{P}_1 \subset P_2$ or a $\hat{P}_2$ such that $P_2 \sim \hat{P}_2 \subset P_1$. In the first case we will write $P_1 \preceq P_2$, in the second $P_1 \succeq P_2$.

Clearly, $P_1 \preceq P_2 \Leftrightarrow P_2 \preceq P_1$ imply $P_1 \sim P_2$.

In conclusion, the theorem asserts the possibility of mapping one into the other all the projectors of a same factor, which implies an unexpected homogeneity of the algebraic structure of the factor, in particular, of the spectral densities of the projectors. The reason of this homogeneity lies in that the factorization determines the splitting of the algebra $\mathcal{I}$ in two subalgebras, which, although independent of each other, remain mutually correlated by a common dimensional constraint. Note indeed that to prove this comparability theorem we had to involve the projectors of the commuting algebra. This correlation is for certain aspects similar to that determined by a nonholonomic constraint between two parts of a classical system. As the state of the system changes, the two parts roll and crawl onto each other in such a way that each of them can reach any desired configuration independently of the other, leaving however unchanged the dimension of the contact.

In certain aspects, this theorem is the analog of Cantor–Bernstein’s theorem on comparability of sets, which stands at the basis, firstly, of the notion of cardinality, secondly, of that of measurability. If we consider that the projectors of quantum–mechanical systems are the analog of the sets of states of classical systems, the question arises quite naturally of what may be, in the world of projectors, the analog of the measure of a set.

The existence of a relation of total ordering in the isometry class of projectors, together with the additive property of orthogonal projectors, suggests the introduction of an additive measure $D(P)$, to be called the relative dimension over the set of the projectors $P$ of a same factor. To be consistent with the measure analog we impose the following conditions: (i) $D(0) = 0$; (ii) if $P_1 P_2 = 0$ then $D(P_1 + P_2) = D(P_2) + D(P_1)$; (iii) $P_1 \preceq P_2$ is equivalent to $D(P_1) \leq D(P_2)$. 

7
Using the perseverance of additive property across isometric maps, carried out in all possible (finitary or infinitary) ways, among the projectors $P \in \mathcal{A}$, we arrive to the following important result:

**Theorem 1.11** The relative dimensions are exhausted by the following three types of possibilities:

1. $D(P) = 0, 1, \ldots, n$ (type $I_n$); or $D(P) = 0, 1, \ldots, \infty$ (type $I_{\infty}$).
2. $D(P)$ takes a value in the interval $[0, c]$, where $c$ is a positive real number (type $II_c$); or in the interval $[0, \infty]$ (type $II_{\infty}$).
3. $D(P) = 0, \infty$ (type $III$).

Since in case $II_c$ the dimensions are defined up to an arbitrary scale factor, we can pose $c = 1$ and denote type $II_c$ as $H_1$.

As an example of type $I_{\infty}$, we can indicate the algebra $I$ itself. In this case, all projectors of finite dimension have a discrete spectrum. This means that the projectors whose spectrum is continuous or partially continuous have infinite dimension. By contrast, in factors $II_1$ and $II_{\infty}$, the sole projector with a discrete spectrum is zero, the spectra of all others being continuous. This is just the reason why the dimensions of these types of projectors have finite ratios.

### 1.5 The second theorem of comparability

We will now establish a one–to–one mapping between a class of projectors of $\mathcal{A}$ and a class of projectors of $\mathcal{A}'$ by the following construction.

Assume $f \in \mathcal{H}$. Denote by $[Af]$ the smallest subspace of $\mathcal{H}$ that contains the vectors that can be obtained by the application of all the operators $A \in \mathcal{A}$ to $f$, and by $[A'f]$ that obtained by the application of all $A' \in \mathcal{A}'$ to $f$. Denote by $P_f'$ the projector that sends $\mathcal{H}$ onto $[Af]$ and with $P_f$ that sends $\mathcal{H}$ onto $[A'f]$. We wish to prove that $P_f \in \mathcal{A}$ and $P_f' \in \mathcal{A}'$.

In fact, all the unitary operators $U \in \mathcal{A}$ leave invariant $[Af]$ as, if $A \in \mathcal{A}$, also $UA \in \mathcal{A}$ and consequently $UP_fU^\dagger = P_f'$. In the same way, all the unitary operators $U' \in \mathcal{A}'$ leave invariant $[A'f]$ and consequently $U'P_fU'^\dagger = P_f$. Since $\mathcal{A}$ and $\mathcal{A}'$ can be generated from their unitary operators (theorem 1.2), the commutation relations $[P_f', A] = 0$ and $[P_f, A'] = 0$ do hold. We can then say that every vector $f \in \mathcal{H}$ determines a pair of projectors: $P_f \in \mathcal{A}$ and $P_f' \in \mathcal{A}'$.

Let now $P_{f_1}, P_{f_2}, P'_{f_1}, P'_{f_2}$ be the projectors determined by the vectors $f_1, f_2 \in \mathcal{H}$ as indicated above. Then, because of the first comparability theorem, $P_{f_1}$ and $P_{f_2}$ are comparable and such are also $P'_{f_1}$ and $P'_{f_2}$. It is not said, however, that the first ones be compatible with the second ones. The following theorem, which we limit ourselves to state because its proof is rather cumbersome, however holds

**Theorem 1.12** If $P_{f_1} \preceq P_{f_2}$, then $P'_{f_1} \preceq P'_{f_2}$. Moreover, for every $P \in \mathcal{A}$ there exists a vector $f$ for which $P_f \sim P$ and for every $P' \in \mathcal{A}'$ there exists a vector $f$ for which $P_f' \sim P'$.

Clearly, the first part of the theorem implies the correspondence

$$D(P_{f_1}) \preceq D(P_{f_2}) \iff D'(P'_{f_1}) \preceq D'(P'_{f_2}),$$

where $D'(P')$ is the relative dimension of $P' \in \mathcal{A}'$. This means that the range of $D(P_f)$, $f \in \mathcal{H}$, can be mapped onto that of $D'(P'_f)$, $f \in \mathcal{H}$, and/or vice versa. But then, from the second part of the theorem, also the range of $D(P)$, $P \in \mathcal{A}$, can be mapped onto that of $D'(P')$, $P' \in \mathcal{A}'$, and/or vice versa. We deduce that $\mathcal{A}$ and $\mathcal{A}'$ belong to the same type and that the dimensions of their projectors exhaust the following possibilities:

1. $D(P) = 0, 1, 2, \ldots, n \leq \infty$; $D'(P') = 0, 1, 2, \ldots, n' \leq \infty$.
2. $0 \leq D(P) \leq c \leq \infty$; $0 \leq D'(P') \leq c' \leq \infty$.
3. $D(P), D'(P') = 0, \infty$.

If $n = n'$ and $c = c'$, we will say that the factors are **symmetrically coupled**.
2 Discrete systems

The class of systems that we will now describe provides some examples of *symmetrically coupled factors* for all types indicated in the previous section. The class is a rather wide. It is based on the construction of a Hilbert space of measurable functions in which a maximal algebra of projectors is automorphically transformed by a *discrete* group of unitary operators. Because of this, the systems will be defined *discrete*. More general classes of systems could be obtained by introducing finite or infinite continuous groups. The reason of the restriction to discrete systems, however, will be soon apparent.

2.1 Functional Hilbert spaces of maximum generality

Let \( \mathcal{H}_X \) be the Hilbert space of measurable functions \( f(x) \), with \( x \) running over a set of points \( X \). To extend the notions of *measurable set* and *measurable function* to the case in which \( X \) is not a topological set, we introduce the following notion of measure.

For every subset \( S \subset X \) we define a real function \( \mu(S) \) characterized by the following axioms:

(i) **Positivity:** \( 0 \leq \mu(S) \leq \infty \).

(ii) **Ordering:** \( S \subset S' \) implies \( \mu(S) \leq \mu(S') \).

(iii) **Additivity:** For every finite or infinite collection \( S_1, S_2, \ldots \),

\[
\mu(S_1 \cup S_2 \cup \ldots) \leq \mu(S_1) + \mu(S_2) + \ldots
\]

We adopt now the following definition⁴.

**Definition 2.1** Misurability according to Carathéodory:

*The subset \( S \) is called measurable if and only if for every subset \( S' \subset X \) the equality*

\[
\mu(S') = \mu(S' \cap S') + \mu(S' \cap S)
\]

*holds.*

Let us now continue the list of axioms:

(iv) **Covering:** If \( \mu(S_0) < \alpha \) then there exists a measurable set \( S \supseteq S_0 \) con \( \mu(S) < \alpha \).

(v) **Separability:** There exists a countable collection of measurable subsets \( S^{(1)}, S^{(2)}, \ldots \) of finite measure and \( \bigcup_i S^{(i)} = X \), such that if \( x, y \in S^{(i)} \) or \( x, y \notin S^{(i)} \) for all \( i \), then \( x = y \). The latter axiom serves to replace the (absent or ignored) notion of *topological separability*. In case \( X \) is an Euclidean space of finite dimension, we can take as \( S^{(i)} \) all the spheres of rational coordinates and radii.

**Definition 2.2** Generalized notion of Lebesgue measurability:

A complex function \( f(x) \) defined at all points \( x \in X \) is called measurable if for every real number \( \alpha \) the sets

\[
S_\alpha = \{ x ; \Re[f(x)] > \alpha \}, \quad S'_\alpha = \{ x ; \Im[f(x)] > \alpha \}
\]

are measurable.

---

³ In the following, the complementary set of \( S \) in \( X \) will be denoted by \( \bar{S} \) and the symmetric difference \((S \cup S') \cap (S \cup S')\) with \( S - S' \) (then \( \bar{S} \equiv X - S ) \).

⁴ The original formulation reads: If \( x \in S^{(i)} \) is equivalent to \( y \in S^{(i)} \) for all \( i \) then \( x = y \). The condition \( \bigcup_i S^{(i)} = X \) is superfluous. Indeed, if it is not satisfied and the measure of \( S_0 \equiv X - \bigcup_i S^{(i)} \) is finite, we can add \( S_0 \) to the collection so as to have \( \bigcup_i S^{(i)} = X \). If on the contrary \( \mu(S_0) = \infty \) then \( S_0 = \{ x_0 \} \) (one-point set). But in this case every measurable function \( f(x) \) must satisfy the condition \( f(x_0) = 0 \). Therefore, the point \( x_0 \) can be omitted from \( X \).
On this basis, we can introduce a generalized notion of Lebesgue integral. Accordingly, we will define in \( \mathcal{H}_X \) the scalar products

\[
(f_1, f_2) = \int_X f_1^*(x) f_2(x) \, d\mu(x).
\]

It is understood that two functions \( f(x) \) and \( f'(x) \) with square–integrable moduli differing only over a zero–measure set represent the same vector of \( \mathcal{H}_X \). In this case we will write \( f(x) = f'(x) \) a.e. (almost everywhere).

An important class of measurable functions is that of box functions: \( \chi_S(x) = 1 \) for \( x \in S \) and \( \chi_S(x) = 0 \) for \( x \not\in S \), where \( S \) is a measurable subset of \( X \). We will then have

\[
\int_X \chi_S(x) \, d\mu(x) = \int_S \, d\mu(x) = \mu(S).
\]

It is known from Lebesgue’s theory of integration that box–function set is dense in the space of measurable functions. This means that every measurable \( f(x) \) is a.e. equal to a limit of linear combinations of box functions. Shortly, we will say that every \( f(x) \) can be a.e. generated by box functions.

### 2.2 Discrete groups that are free and ergodic

Let \( G \) be a finite or infinite discrete group, whose elements \( g \) operate on \( X \) by point transformations \( x' = gx \). These transform a subset \( S \) of \( X \) to a subset \( S' = gS = \{gx : x \in S\} \).

Let add now the following axioms (Murray and von Neumann, 1936; von Neumann, 1940).

(vi) **Closure of measurability with respect to \( G \):** If \( S \) is measurable, also \( gS \) is measurable.

(vii) **Freedom:** For every \( g \in G \), the set \( S_g = \{ x : gx = x \} \) has measure zero. This notion generalizes the condition that there are no fixed points in case \( X \) is discrete.

(viii) **Ergodicity:** From \( g \neq 1 \) and \( \mu(gS - S) = 0 \), where \( S - S' \) denotes the symmetric difference \( S \cup S' - S \cap S' \), it follows either \( \mu(S) = 0 \) or \( \mu(S) = \mu(X) \).

The axiom just listed characterize the collection of measurable subsets of \( X \) as a Borelian system that is separating and closed with respect to \( G \).

Finally, let us add, only temporarily however, an axiom whose elimination will lead, as we will see, to interesting consequences.

(ix) **Measure invariance:** The measure \( \mu \) is invariant with respect to \( G \), that is, \( \mu(S) = \mu(gS) \).

Therefore, in this case, for every measurable \( f(x) \) and every \( g \in G \) we have the equations

\[
\int_X f(x) \, d\mu(gx) = \int_X f(g^{-1}x) \, d\mu(x) = \int_X f(x) \, d\mu(x).
\]

### 2.3 The algebras of discrete systems

We pass now to describe the algebraic structure of the system that implements in the most natural way the axioms listed above.

**Definition 2.3** Define in \( \mathcal{H}_X \) the operators

\[
L_{\phi(x)} f(x) = \phi(x) f(x),
\]

where \( \phi(x) \) is a bounded and measurable complex function of \( x \).
Clearly, these operators are bounded and if \( \phi(x) = \phi'(x) \) a.e., \( L\phi(x) \) and \( L\phi'(x) \) define the same operator. They form a commutative algebra that will be denoted by \( \mathcal{L} \).

Note that the operators \( P_S \equiv L\chi_S(x) \), where \( \chi_S(x) \) are box functions, form a commutative family of projectors, which in the following will be denoted by \( \mathcal{P} \). These satisfy the relations \( P_S + P_{S'} = P_{S \cup S'} + P_{S \cap S'} \) for all pairs of measurable sets \( S, S' \). In particular, \( P(S) + P(S) = I \) (the unit operator).

Note moreover that the operators \( V_{\lambda(x)} = L\exp i\lambda(x) \), where \( \lambda(x) \) is a real bounded and measurable real function of \( x \), generate an Abelian group, which will be denoted by \( \mathcal{V} \).

Note finally that, since every measurable function \( f(x) \) can be a.e. generated by box functions, both \( \mathcal{L} \) and \( \mathcal{V} \) can be generated by \( \mathcal{P} \).

**Definition 2.4** A commutative algebra \( \mathcal{L} \) is called maximal if it satisfies the equation \( \mathcal{L} = \mathcal{L}' \).

**Definition 2.5** A family of projectors \( \mathcal{P} \) is called maximal if from \( [P, P_S] = 0 \) for any \( P_S \in \mathcal{P} \) it follows \( P \in \mathcal{P} \).

Basing on these definitions we prove the following theorem:

**Theorem 2.1** The family of projectors \( \mathcal{P} \) is maximal. Consequently, \( \mathcal{L} \) is a maximal commutative algebra.

Proof: Firstly, note that, since \( \mathcal{L} \) can be generated by \( \mathcal{P} \), if \( P \) commutes with every \( P_S \in \mathcal{P} \), it commutes also with every \( L\phi(x) \in \mathcal{L} \). We have therefore \( PL\phi(x)f(x) = L\phi(x)Pf(x) \) and consequently

\[
P[\phi(x)f(x)] = \phi(x)[Pf(x)]. \tag{1}
\]

Secondly, note that, since \( P \) is a bounded operator, the functions \( Pf(x) \) are measurable. Now, from the collection \( S^{(1)}, S^{(2)}, \ldots \) introduced with axiom (v) we can form the collection

\[
T^{(i)} = S^{(i)} - \bigcup_{j=1}^{i-1} S^{(j)}.
\]

It can be easily proved that \( \mu(T^{(i)}) < \infty, T^{(i)} \cap T^{(j)} = \emptyset \) (the empty set) and \( \bigcup_i T^{(i)} = X \).

Let us then define the pairwise box-functions \( e_i(x) = 1 \) if \( x \in T^{(i)} \) and \( e_i(x) = 0 \) if \( x \in T^{(i)} \). It is evident that \( e_i(x)^2 = e_i(x) \), \( \sum_i e_i(x) = 1 \) and that the projectors \( P_i \equiv P_{T^{(i)}} \), clearly belonging to \( \mathcal{P} \), operate on the functions \( f(x) \in \mathcal{H}_X \) as follows:

\[
P_i f(x) = e_i(x)f(x).
\]

Therefore, from \( [PP_i - PP_i] e_i(x) = 0 \) we obtain \( Pe_i(x) = e_i(x)Pe_i(x) \). Consequently, since for \( i \neq j \) we have \( e_i(x)e_j(x) = 0 \), there is no loss of generality in posing \( Pe_i(x) = \phi(x)e_i(x) \) for all \( i \). Moreover, since \( P^2 = P \), the equation \( \phi(x) = \phi(x)^2 \) holds. This means that the possible values of \( \phi(x) \) are 0 and 1.

Now let \( S' \) be the set on which \( \phi(x) = 1 \) and write \( \phi(x) = \chi_{S'}(x) \). Then, we can also write \( P e_i(x) = \chi_{S'}(x)e_i(x) \). Using (1) we obtain

\[
P[\phi(x)e_i(x)] = \phi(x)[Pe_i(x)] = \phi(x)\chi_{S'}(x)e_i(x),
\]

from which, summing over \( i \) and posing \( \phi(x) = f(x) \in \mathcal{H}_X \), we obtain \( Pf(x) = \chi_{S'}(x)f(x) \). Since \( Pf(x) \) and \( f(x) \) are measurable and \( \chi_{S'}(x) \) is bounded, also this function is measurable. Therefore \( S' \) is measurable and then \( P \equiv P_{S'} \in \mathcal{P} \).

**Definition 2.6** Now define the operators

\[
U_g f(x) = f(gx).
\]
Note that their product follows the rule $U_gU_g^* = U_gg$. These form a group of unitary operators that will be indicated by $\mathcal{U}$. Indeed, from $d\mu(g^{-1}x) = d\mu(x)$ we obtain

$$(U_gf_1, U_gf_2) = \int_X f_1^*(gx)f_2(gx)\,d\mu(x) = \int_X f_1^*(x)f_2(x)\,d\mu(g^{-1}x) = (f_1, f_2).$$

Also note that

$$U_gL_{\phi(x)}U_g^{-1}f(x) = U_gL_{\phi(x)}f(g^{-1}x) = U_g\phi(x)f(g^{-1}x) = \phi(gx)f(x) = L_{\phi(gx)}f(x).$$

Consequently, because of the arbitrariness and completeness of $f(x)$, we can also write

$$U_gL_{\phi(x)}U_g^{-1} = L_{\phi(gx)}.$$ 

In other terms, the unitary operators $U_g$ induce a group of automorphisms on $\mathcal{L}$ and then also on $\mathcal{P} \in \mathcal{V}$. In particular, we have

$$U_gP_\mathcal{S}U_g^{-1} = P_{\mathcal{S}g}. \tag{2}$$

Let us now prove two important theorems that will be used in the following section.

**Theorem 2.2** The equation $L_{\phi(x)} = L_{\psi(x)}U_g$ is possible only if either $g = 1$ and $\phi(x) = \psi(x) \neq 0$ hold a.e., or if $g \neq 1$ e $\phi(x), \psi(x) = 0$ a.e.

Proof: This theorem is a consequence of group–freedom axiom (vii). To prove this we start from noting that the hypothesized equation is equivalent to $\phi(x)f(x) = \psi(x)f(gx)$ a.e. for every $f(x) \in \mathcal{H}_X$.

Using the sets $T^{(i)}$ and the measurable functions $e_i(x)$ introduced in the proof of theorem 2.1, we obtain the equations $\phi(x)e_i(x) = \psi(x)e_i(gx)$ a.e., then $\phi(x) = \psi(x)$ a.e. in $T^{(i)} \cap gT^{(i)}$. Hence, summing over $i$ and noting that $\bigcup_i T^{(i)} = \bigcup_i gT^{(i)} = X$, we deduce $\phi(x) = \psi(x)$ a.e., $x \in X$. Therefore the hypothesized equality translates to the equation

$$\int_X |\phi(x)|^2|f(x) - f(gx)|^2d\mu(x) = 0 \tag{3}$$

with $g \neq 1$.

Using the sets $S^{(i)}$ introduced in axiom (v), take the box functions $f(x)$ defined by $\hat{e}_i(x) = 1$ if $x \in S^{(i)}$ and $\hat{e}_i(x) = 0$ if $x \in S^{(i)}$. Then in place of (3) we have

$$\int_X |\phi(x)|^2[\hat{e}_i(x) - \hat{e}_i(gx)]d\mu(x) = \int_{S^{(i)} - g^{-1}S^{(i)}} |\phi(x)|^2d\mu(x) = 0.$$

Once posed $X_0 = \bigcup_i (S^{(i)} - g^{-1}S^{(i)})$, these imply

$$\int_{X_0} |\phi(x)|^2d\mu(x) \leq \sum_i \int_{S^{(i)} - g^{-1}S^{(i)}} |\phi(x)|^2d\mu(x) = 0.$$

This means that $\phi(x)$ does not vanish a.e. only in $X_0 = \bigcap_i (S^{(i)} - g^{-1}S^{(i)})$, that is either for both $x$ and $g^{-1}x$ belonging to $S^{(i)}$ (and to $g^{-1}S^{(i)}$) or both belonging to $S^{(i)}$ (and to $g^{-1}S^{(i)}$). But then, the separability axiom (v) ensures that $x = gx$ for every $x \in X_0$ and $g \neq 1$. Moreover, the group–freedom axiom (vii) ensures $\mu(X_0) = 0$ and consequently $\phi(x) = 0$ a.e.

Note that, by posing $\phi(x) = 1$ in Eq. (3), the same line of reasoning leads to the following result

**Corollary 2.3** The equations $f(x) = f(gx)$ hold a.e. for all elements $g \in G$ if and only if $f(x)$ is a.e. constant.

The following theorem is a consequence of the ergodicity axiom (viii):

**Theorem 2.4** Let $\mathcal{U}'$ be the commutant of $\mathcal{U}$. Then $\mathcal{L} \cap \mathcal{U}' = \alpha I$ (multiples of the unit operator).
Proof: Since \( \mathcal{L} \) can be generated by \( P \), it suffices to prove that \( \mathcal{P} \cap \mathcal{U}' = 1 \). Clearly, \( I \) belongs to both \( \mathcal{U}' \) and \( P \); in the first case because \( I \) commutes with every \( U_g \), in the second case because \( I = P(X) \). Assume absurdly that \( \mathcal{P} \cap \mathcal{U}' \) contains a \( P_S \neq 0, I \). Consequently \( U_g P_S U_g^{-1} = P_S \) and, since \( P_S \in \mathcal{U}' \), the equality \( P_S = P_{gS} \) must hold for every \( g \in G \). This means \( S = gS \) a.e., then \( \mu(S - gS) = 0 \) for every \( g \). From the ergodicity axiom (viii), this is possible only if either \( \mu(S) = 0 \) or \( \mu(X - S) = 0 \), that is, if either \( P_S = P_{\emptyset} = 0 \) or \( P_S = P_X = I \). But this means that

**Corollary 2.5** The representation of the algebra generated by the operators \( P_S \) and \( U_g \) is irreducible.

### 3 The space of operational states

In the previous section, it was built in a Hilbert space \( \mathcal{H}_X \) a maximal family \( \mathcal{P} \) of commuting projectors \( P_S \in \mathcal{P} \) indexed by the measurable subsets \( S \) of a set \( X \). This family can be physically interpreted as a complete set of compatible observables. This projector algebra is equipped with a group of automorphisms \( g \in G \) that operate freely and ergodically on the subsets of \( X \). The unitary operators \( U_g \) that implement the automorphism \( U_g P_S U_g^{-1} = P_{gS} \) can be interpreted as the representations of the operations \( g \) that an external observer can perform on the states of the system.

If these operations are carried out in the macroscopic world, we must assume that they behave as a set of classical possibilities, i.e., like a set of mutually exclusive results of a measurement. From the quantum mechanical point of view, this means that they must be represented by a set of pairwise orthogonal vectors \( |g_1\rangle, |g_2\rangle, \ldots \) of a Hilbert space \( \mathcal{H}_G \) indexed by the elements \( g_i \in G \). We can normalize these states so as to have \( \langle g_i|g_j \rangle = \delta_{ij} \).

The operator algebra of this space exhibits some interesting aspects. The inversion operator \( Q \), defined by the equation \( Q|g\rangle = |g^{-1}\rangle \), implies the existence of two groups of operators: \( \mathcal{G}_L \) (the left group) and \( \mathcal{G}_R \) (the right group), whose elements \( L_g \in \mathcal{G}_L \) and \( R_g \in \mathcal{G}_R \) operate on the states of \( \mathcal{H}_G \) as follows:

\[
L_g|g_i\rangle = |gg_i\rangle, \hspace{1cm} R_g|g_i\rangle = |g_i\rangle.
\]

Indeed, from

\[
QL_g|g_i\rangle = Q|gg_i\rangle = |g_i^{-1}g^{-1}\rangle = R_g^{-1}|g_i^{-1}\rangle = R_g^{-1}Q|g_i\rangle
\]

and from \( Q^2 = 1 \) we obtain \( L_g = Q R_{g^{-1}} Q \). It can be easily proved that \( [\mathcal{G}_L, \mathcal{G}_R] = 0 \) and that the operators \( R_g, L_g \) and \( Q \) are unitary.

It is useful to define also the following operators:

\[
P_g = |g\rangle\langle g|, \hspace{1cm} Q_g = |g^{-1}\rangle\langle g|.
\]

Clearly, \( P_g \) is a projector and \( Q_g \) can be defined an *inversor*. The following equations can be easily proved

\[
Q_g^\dagger = Q_{g^{-1}}, \hspace{1cm} Q_g Q_{g'} = P_{g^{-1}g'g}, \hspace{1cm} Q = \sum_i Q_{g_i}.
\]

### 4 Hybrid spaces

The arguments introduced in the previous section ultimately suggest that the quantum–mechanical representation of an observable system that can be automorphically transformed by an external agent, combined with the representation of the actions that can be performed by the agent, takes the form of the direct product

\[
\mathcal{H} = \mathcal{H}_X \otimes \mathcal{H}_G,
\]

which in the following will be called a *hybrid space*. The vectors \( |\Phi\rangle \in \mathcal{H} \) can then be physically interpreted as the states of the observed–observer system. They can be written as

\[
|\Phi\rangle = \sum_i f(g_i, x) \otimes |g_i\rangle,
\]
so that their scalar products are defined by the relationships

\[ \langle \Phi \mid \Phi_2 \rangle = \sum_i \int_X f_1^*(g_i, x) f_2(g_i, x) \, d\mu(x). \]

**Note:** The elements \( g_i \in G \), on which the functions \( f(g_i, x) \) appear to depend, could be safely replaced by the indices \( i \). We prefer, however, to exhibit the dependence on \( g_i \) as this will allow us to exploit identities of the sort

\[ \sum_i a_{g_i} b_{g_i} \equiv \sum_i a_{g_i} b_{g_{g_i}} \equiv \sum_i a_{g_i} b_{g_{g_{g_i}}} \equiv \sum_i a_{g_i} b_{g_{g_{g_{g_i}}}} \], etc.

Let us now introduce a different but particularly useful way to represent the states of \( \mathcal{H} \). Once the basis \( \mathcal{H}_G \) is chosen, any generic vector \( |\Phi\rangle \in \mathcal{H} \) is one–to–one to the ordered collection \( f(g_1, x), f(g_2, x), \ldots, g_i \in G \), of the vectors of \( \mathcal{H}_X \). We can therefore indicate this equivalence by writing

\[ |\Phi\rangle \sim \|f(g_1, x), f(g_2, x), \ldots\| \equiv \|f(g_i, x)\|. \]

The following theorem indicates how this sort of representation can be extended to the operators of \( \mathcal{H} \).

**Theorem 4.1** The basis \( |g_i\rangle \) being fixed, every bounded operator \( A \) of \( \mathcal{H} \) is one–to–one with a matrix \( A^g_{ji} \) of operators of \( \mathcal{H}_X \) satisfying the equations

\[ A|\Phi\rangle = |\Phi'\rangle \sim \|f'(g_i, x)\| = \| \sum_j A^g_{ji} f(g_j, x) \|. \]

In other terms, the operator \( A \) is one–to–one with the doubly ordered collection of the operators \( A^g_{ji} \) of \( \mathcal{H} \). We can therefore indicate this fact by writing

\[ A \sim \| A^g_{ji} \|. \]

and abridge the action of \( A \) on \( |\Phi\rangle \) by the equation

\[ Af(g_i, x) = f'(g_i, x) = \sum_j A^g_{ji} f(g_j, x), \quad g_i, g_j \in G. \]

The proof of this equation can be easily obtained by using the projectors \( \bar{P}_{g_i} \) of \( \mathcal{H} \) defined by the equations \( \bar{P}_{g_i} |\Phi\rangle = f(g_i, x) \otimes |g_i\rangle \), that is, the operators \( \bar{P}_{g_i} \equiv 1 \otimes P_{g_i} \), then by realizing that the operators \( A^g_{ji} \) of \( \mathcal{H}_X \) defined by the equations

\[ [A^g_{ji} f(g_j, x)] \otimes |g_i\rangle = \bar{P}_{g_i} A \bar{P}_{g_j} |\Phi\rangle \]

satisfy the required properties. Also the relationships

\[ \lambda A \sim \| \lambda A^g_{ji} \|; \quad A^\dagger \sim \| \langle A^g_{ji} \rangle \|; \quad A + B \sim \| A^g_{ji} + B^g_{ji} \|; \quad AB \sim \| \sum_k A^g_{ki} B^g_{ji} \|; \tag{4} \]

can be easily proved. In the following, \( \| f(g_i, x) \| \) and \( \| A^g_{ji} \| \) will be respectively called the projections of \( |\Phi\rangle \) and \( A \) in \( \mathcal{H}_X \).

### 4.1 The algebra of a hybrid space

**Definition 4.1** Let us build in \( \mathcal{H} = \mathcal{H}_X \otimes \mathcal{H}_G \) the following operators

\[ \bar{L}_{\phi(x)} = L_{\phi(x)} \otimes 1; \quad \bar{U}_g = U_g \otimes L_{g^{-1}}. \tag{5} \]

They act on \( |\Phi\rangle = \sum_i f(g_i, x) \otimes |g_i\rangle = \sum_i f(gg_i, x) \otimes |gg_i\rangle \) in the following way

\[ \bar{L}_{\phi(x)} |\Phi\rangle = \sum_i \left[ L_{\phi(x)} \otimes 1 \right] f(g_i, x) \otimes |g_i\rangle = \sum_i \phi(x) f(g_i, x) \otimes |g_i\rangle; \]

\[ \bar{U}_g |\Phi\rangle = \sum_i \left[ U_g \otimes L_{g^{-1}} \right] f(gg_i, x) \otimes |gg_i\rangle = \sum_i f(gg_i, gx) \otimes |g_i\rangle. \]
In the abridged form, we can write
\[ \tilde{L}_{\phi(x)} f(g_i, x) = \phi(x) f(g_i, x), \quad \tilde{U}_g f(g_i, x) = f(gg_i, gx), \quad g_i \in G. \]

The operators \( \tilde{L}_{\phi(x)} \) are isomorphic to the operators \( L_{\phi(x)} \) introduced in \( \mathcal{H}_X \) and for them the already established theorems hold. As for the operators \( \tilde{U}_g \), they are unitary because they are direct products of unitary operators. Note that the automorphisms
\[ \tilde{U}_g \tilde{L}_{\phi(x)} \tilde{U}_g^{-1} = \tilde{L}_{\phi(gx)}, \quad \tilde{U}_g \tilde{P}_S \tilde{U}_g^{-1} = \tilde{P}_{gS} \]  

have the same structure of the analogous operators defined in \( \mathcal{H}_X \).

In the following, the commutative algebra formed by \( \tilde{L}_{\phi(x)} \) will be indicated by \( \tilde{\mathcal{L}} \), the group of operators \( \tilde{U}_g \) will be indicated by \( \tilde{\mathcal{U}} \) and the family of projectors \( \tilde{P}_S = \tilde{L}_{x_{e}(x)} \) will be indicated by \( \tilde{\mathcal{P}} \).

**Definition 4.2** Now define the operators
\[ L'_{\phi(x)} = \sum_i L_{\phi(g_i^{-1}x)} \otimes P_{g_i}; \quad \tilde{U}_g = 1 \otimes R_g. \]  

It is evident that they commute with all the operators \( \tilde{U}_g, \tilde{L}_{\phi(x)} \) defined by \( \tilde{\mathcal{L}}, \) than also with all \( \tilde{P}_S \), and that the operators \( \tilde{U}_g \) are unitary. They act on \( |\Phi\rangle = \sum_i f(g_i, x) \otimes |g_i\rangle = \sum_i f(g_i g^{-1}, x) \otimes |g_i g^{-1}\rangle \) in the following way
\[ \tilde{L}'_{\phi(x)} |\Phi\rangle = \sum_{ij} \left[ L_{\phi(g^{-1}_ix)} \otimes P_{g_i} \right] f(g_i, x) \otimes |g_i\rangle = \sum_i \phi(g^{-1}_ix) f(g_i, x) \otimes |g_i\rangle; \]
\[ \tilde{U}_g |\Phi\rangle = \sum_i \left[ 1 \otimes R_g \right] f(g_i g^{-1}, x) \otimes |g_i g^{-1}\rangle = \sum_i f(g_i g^{-1}, x) \otimes |g_i\rangle; \]
or, in the abridged form,
\[ L'_{\phi(x)} f(g_i, x) = \phi(g_i^{-1}x) f(g_i, x), \quad \tilde{U}_g f(g_i, x) = f(g_i g^{-1}, x). \]

**Definition 4.3** Finally, let us define the operator
\[ \tilde{Q} = \sum_i U_{g_i} \otimes Q_{g_i}, \]  

which acts on \( |\Phi\rangle = \sum_i f(g_i, x) \otimes |g_i\rangle = \sum_i f(g_i^{-1}, x) \otimes |g_i^{-1}\rangle \) as follows
\[ \tilde{Q} |\Phi\rangle = \sum_{ij} \left[ U_{g_i} \otimes Q_{g_i} \right] f(g_i^{-1}, x) \otimes |g_i^{-1}\rangle = \sum_i f(g_i^{-1}, g_i^{-1}x) \otimes |g_i\rangle; \]
or, in the abridged form,
\[ \tilde{Q} f(g_i, x) = f(g_i^{-1}, g_i^{-1}x) = U_{g_i^{-1}} f(g_i^{-1}, x), \quad g_i \in G. \]

\( \tilde{Q} \) is unitary and involutory as \( \tilde{Q}^2 = 1 \). The second property is evident. The first follows from the second and from the fact that
\[ \tilde{Q}^\dagger = \sum_i U_{g_i}^\dagger \otimes Q_{g_i}^\dagger = \sum_i U_{g_i^{-1}} \otimes Q_{g_i^{-1}} = \sum_i U_{g_i} \otimes Q_{g_i} = \tilde{Q}. \]

The equations
\[ \tilde{U}_g = \tilde{Q} \tilde{U}_g \tilde{Q}, \quad \tilde{L}'_{\phi(x)} = \tilde{Q} \tilde{L}_{\phi(x)} \tilde{Q}, \quad \tilde{P}'_S = \tilde{Q} \tilde{P}_S \tilde{Q} \]
can be easily verified. Eq.s (6) can be isomorphically translated into
\[ \tilde{U}_g L'_{\phi(x)} \tilde{U}_g^{-1} = L_{\phi(gx)}, \quad \tilde{U}_g P'_S \tilde{U}_g^{-1} = P'_{gS}. \]  

We can then introduce the commutative algebra \( \tilde{\mathcal{L}}^Q = \tilde{Q} \tilde{\mathcal{L}} \tilde{Q} \), the group \( \tilde{\mathcal{U}}^Q = \tilde{Q} \tilde{\mathcal{U}} \tilde{Q} \), the projector family \( \tilde{\mathcal{P}}^Q = \tilde{Q} \tilde{\mathcal{P}} \tilde{Q} \) and summarize the results in the theorem.
Theorem 4.2 The algebra \( \mathcal{F} = \mathcal{U} \cup \mathcal{L} \) generated by \( \mathcal{U} \) and \( \mathcal{L} \) commutes with the algebra \( \mathcal{F}^Q = \mathcal{U}^Q \cup \mathcal{L}^Q \) generated by \( \mathcal{U}^Q = Q\mathcal{U}Q \) and \( \mathcal{L}^Q = Q\mathcal{L}Q \).

From the algebraic isomorphism between Eq.s 6 and Eq.s 9, and from the fact that the property of being maximal is algebraic and not spatial, it follows that

Theorem 4.3 The commutative algebras \( \mathcal{L} \) and \( \mathcal{L}^Q \), formed respectively by \( L_{\phi(x)} \) and \( L'_{\phi(x)} \), are maximal. The same properties hold for the projector families \( \mathcal{P} \) and \( \mathcal{P}^Q \), which are formed respectively by \( P_S = L_{\chi_S(x)} \) and \( P_S' = L'_{\chi_S(x)} \).

Now, let \( \mathcal{U} \cup \mathcal{L} \) be the algebra of von Neumann generated by \( \mathcal{U} \) and \( \mathcal{L} \), and \( \mathcal{U}^Q \cup \mathcal{L}^Q \) that generated by \( \mathcal{U}^Q = Q\mathcal{U}Q \) and \( \mathcal{L}^Q = Q\mathcal{L}Q \). We have

Theorem 4.4 The algebra \( \mathcal{F} = \mathcal{U} \cup \mathcal{L} \) commutes with \( \mathcal{F}^Q = \mathcal{U}^Q \cup \mathcal{L}^Q \) and, since both of them contain 1, we have \( \mathcal{F} = \mathcal{F}'' \), \( \mathcal{F}^Q = (\mathcal{F}^Q)' = \mathcal{F}' \).

The first part of the theorem follows from the fact that \( \mathcal{F} \) and \( \mathcal{F}^Q \) are respectively generated by commuting elements. The second follows from the bicommutant theorem [1] and from the fact that the commutant of an algebra of von Neumann is unique.

Since every element \( A \in \mathcal{F} \) is one-to-one with \( Q\bar{A}Q \in \mathcal{F}^Q \), and vice versa, we can state the following theorem

Theorem 4.5 If \( A \in \mathcal{F} \), then \( Q\bar{A}Q \equiv \bar{A}' \in \mathcal{F}' \) and if \( A' \in \mathcal{F}' \), then \( Q\bar{A}'Q \equiv \bar{A} \in \mathcal{F} \).

The question then arises of how \( \mathcal{F} \) and \( \mathcal{F}' \) relate to the algebra of all bounded operators of \( \mathcal{H} \); in particular, of whether \( \mathcal{F} \cup \mathcal{F}' \) exhausts the totality of bounded operators of \( \mathcal{H} \). To answer this, we need to verify whether \( \mathcal{F} \) is a factor or if \( \mathcal{F}, \mathcal{F}' \) are coupled factors.

### 4.2 The coupled factors \( \mathcal{F}, \mathcal{F}' \)

From the equations in the abridged form

\[
[L_{\phi(x)}, A]f(g_i, x) = \sum_j [L_{\phi(x)}, A_{g_j}] f(g_j, x);
\]

\[
\bar{U}_g A \bar{U}_g^{-1} f(g_i, x) = \bar{U}_g A \sum_j A_{g_j} U_{g^{-1}} f(g^{-1} g_j, x) = \sum_j A_{g_j} U_{g^{-1}} U_{g^{-1}} f(g_j, x);
\]

\[
\bar{Q} A \bar{Q} f(g_i, x) = \bar{Q} A \bar{U}_g^{-1} f(g_i^{-1}, x) = \bar{Q} \sum_j A_{g_j} U_{g^{-1}} f(g_j^{-1}, x) = \sum_j A_{g_j} U_{g^{-1}} U_{g^{-1}} f(g_j, x);
\]

we obtain the equivalence relations

\[
[L_{\phi(x)}, A] \sim \| [L_{\phi(x)}, A_{g_j}] \|, \quad \bar{U}_g A \bar{U}_g^{-1} \sim \| U_g A^{g_{g_j}} U_g^{-1} \|, \tag{10}
\]

\[
\bar{Q} A \bar{Q} \sim \| U_{g^{-1}} A_{g_j} U_{g_j} \|, \tag{11}
\]

Let us pose now the problem of determining the form of operators \( A \) in case that the equality \( A = \bar{A}' \in \mathcal{F}' \) holds.

Clearly, since every operator of this sort commutes with all operators of \( \mathcal{F} \), the equations \( [L_{\phi(x)}, \bar{A}'] = 0 \) and \( \bar{U}_g A \bar{U}_g^{-1} = \bar{A}' \) must hold and consequently, because of Eq.s (10), also the equations

\[
[L_{\phi(x)}, A_{g_j}] = 0, \quad A_{g_j} = U_g A^{g_{g_j}} U_g^{-1}.
\]
From the first of these, since \( L \) is maximal, we deduce \( A_{gi}^g = L_{\alpha(g_i, g_j; x)} \) for some collection of bounded measurable functions \( \alpha(g_i, g_j; x) \). From the second, we deduce that for every \( g \in G \) we must have
\[
L_{\alpha(g_i, g_j; x)} = U_g L_{\alpha(g_i, g_j; x)} U_g^{-1} = L_{\alpha(g_i, g_j; gx)}.
\]
This means that \( \alpha(g_i, g_j; gx) \) does not actually depend on \( g \). We can then pose \( g = g_i^{-1} \) in each of these functions and replace \( \alpha(1, g_i^{-1} g_j; g_i^{-1} x) \) with \( \alpha(g_i^{-1} g_j; g_i^{-1} x) \). Thus, we can write
\[
A_{gi}^g = L_{\alpha(g_i^{-1} g_j; g_i^{-1} x)}.
\]
Hence, the general form of the operators of \( \mathcal{F}' \) is
\[
\hat{A}' \sim \| L_{\alpha(g_i^{-1} g_j; g_i^{-1} x)} \|,
\]
and its action on a state \( |\Phi\rangle \sim \| f(g_i, x) \| \) is
\[
\hat{A}'|\Phi\rangle \sim \| \sum_j \alpha(g_i^{-1} g_j; g_i^{-1} x) f(g_j, x) \|.
\]

Let us pose also the problem of determining the form of the operators \( A \) when \( A = \hat{A} \in \mathcal{F} \). In this case, we can exploit the fact that \( QAQ = \hat{A}' \in \mathcal{F}' \). It will be therefore sufficient to find the abridged form of \( QAQ \).

From Eq.s \( 11 \) and \( 12 \) we obtain
\[
\hat{Q}\hat{A}'\hat{Q} \sim \| U_{g_i^{-1}} L_{\alpha(g_i, g^{-1}_j; x)} U_{g_j} \| = \| L_{\alpha(g_i, g^{-1}_j; x)} U_{g_i^{-1}} U_{g_j} \|.
\]
Hence, since \( U_g U_g = U_{g'g} \), we have
\[
\hat{A} \sim \| L_{\alpha(g_i, g^{-1}_j; x)} U_{g_j} \|,
\]
which acts on \( \langle\Phi\rangle \) as follows
\[
\hat{A}|\Phi\rangle \sim \| \sum_j \alpha(g_i, g^{-1}_j; x) f(g_j, g_j^{-1} x) \|.
\]

We therefore see that a same bounded measurable function \( \alpha(g_i; x) \) is one–to–one with both the bounded operator \( \hat{A} \in \mathcal{F} \) and the bounded operator \( \hat{A}' \in \mathcal{F}' \). Since the totality of such functions is dense in \( \mathcal{H} \), we can state the following theorem

**Theorem 4.6** The dense set of vectors \( \langle\Phi\rangle \sim \| \alpha(g_i; x) \| \in \mathcal{H} \) is one–to–one with the dense set of operators \( \hat{A} \sim \| L_{\alpha(g_i, g^{-1}_j; x)} U_{g_j} \| \) on one side and the dense set operators \( \hat{A}' \sim \| L_{\alpha(g_i, g^{-1}_j; x)} \| \in \mathcal{F}' \) on the other side.

Let us prove now the following structural theorems

**Theorem 4.7** From the conditions that \( G \) be free and ergodic on \( X \), follow the equalities
\[
\mathcal{F} \wedge \mathcal{F}' = \{ \alpha I \}, \quad \mathcal{F} \vee \mathcal{F}' = \mathcal{I},
\]
where \( \mathcal{I} \) is the set of all bounded operators of \( \mathcal{H} \).

Were it not so, there would be an operator
\[
\hat{A} \sim \| L_{\alpha(g_i, g^{-1}_j; x)} U_{g_j} \| \equiv \| L_{\alpha(g_i^{-1} g_j; g_j^{-1} x)} \|.
\]
Because of theorem \( 2.3 \) this is possible only if either \( g_j \neq g_i \) and
\[
\alpha(g_i, g_j^{-1}; x) = \alpha(g_i^{-1} g_j; g_j^{-1} x) = 0 \text{ a.e.}
\]
or \( g_j = g_i \) and \( \alpha(1; x) = \alpha(1; g_j x) \), a.e. The first case implies \( A = I \), the second, because of corollary \( 2.3 \) it implies \( \alpha(1; x) = \alpha (\text{costante}) \) a.e., hence \( A = \alpha I \), because of the ergodicity theorem \( 2.4 \). In this way, we find the first inequality stated by the theorem. The second follows immediately as \( \{ \alpha I \}' = \mathcal{I} \).

In other terms, \( \mathcal{F} \) and \( \mathcal{F}' \) are coupled factors. They can be interpreted as the operator algebras of two systems, which are physically independent, non spatially isomorphic but quantum–mechanically coupled by the algebraic isomorphism \( Q \mathcal{F} Q = \mathcal{F}' \).
4.3 The classification of factors

Studying the possible coupling of two parts of a physical system is ultimately equivalent to studying the spectral properties of the operators of $\mathcal{F}$ and $\mathcal{F}'$, in particular of their projectors.

We can take advantage of the correspondence between states and operators established in the previous subsection to study the algebraic properties of the operators $\bar{A}$, $\bar{A}'$, as respectively defined by Eq.s (14) and (10). To express the fact that both of them correspond to the same collection of functions $\alpha(g_i; x)$ we write

$$\bar{A} \approx \bar{A}' \approx \langle \alpha(g_i; x) \rangle.$$  

(15)

By substituting in Eq.s (4) the expression of $\bar{A}$ in the form (14) and that of $\bar{B}$ in a similar form with $\beta(g_i; x)$ in place of $\alpha(g_i; x)$, we can immediately verify the following correspondences

$$\lambda \bar{A} \approx \langle \lambda \alpha(g_i; x) \rangle;$$
$$\bar{A}^\dagger \approx \langle \alpha^*(g_i^{-1}; g_i^{-1}x) \rangle;$$
$$\bar{A} + \bar{B} \approx \langle \alpha(g_i; x) + \beta(g_i; x) \rangle;$$
$$\bar{A} \bar{B} \approx \langle \sum_j \alpha(g_j; x) \beta(g_j^{-1}g_i; g_j^{-1}x) \rangle.$$  

(16)

In particular, we have

$$\bar{A}^\dagger \bar{A} \approx \langle \sum_j \alpha^*(g_j; g_jx) \alpha(g_jg_i; g_jx) \rangle;$$
$$\bar{A} \bar{A}^\dagger \approx \langle \sum_j \alpha(g_j; x) \alpha^*(g_j^{-1}g_i; g_j^{-1}x) \rangle.$$  

(17)

(18)

If $\bar{A}$ is Hermitian, the equalities $\alpha(g_i; x) = \alpha^*(g_i^{-1}; g_i^{-1}x)$ must hold; in particular, $\alpha(1; x)$ must be real.

If $\bar{P} \approx \langle \chi(g_i; x) \rangle$ is a projector, from $\bar{P} = \bar{P}^\dagger e \bar{P}^2 = \bar{P}$ follow the identities

$$\sum_j \chi^*(g_j; g_jx) \chi(g_jg_i; g_jx) = \chi(g_i; x); \quad \chi(g_i; x) = \chi^*(g_i^{-1}; g_i^{-1}x).$$

For $g_i = 1$ we have

$$\sum_j \chi(g_j; g_jx)^2 = \sum_j \chi(g_j; x)^2 = \chi(1; x).$$

Therefore, if $\chi(1; x) = 0$ a.e., $\bar{P} = 0$. From the equality $\chi(1; x) = \chi(1; x)^2$ we deduce that $\chi(1; x)$ is a box function. There exists therefore a measurable set $S_P$, which will be called a spectral set of $\bar{P}$, on which $\chi(1; x) = 1$ if $x \in S_P$, otherwise $\chi(1; x) = 0$. Consequently, we have

$$\int_X \chi(1; x) d\mu(x) = \int_{S_P} d\mu(x) = \mu(S_P),$$

(19)

which vanishes if and only if $\bar{P} = 0$.

If $\bar{U} \approx \langle \nu(g_i; x) \rangle$ is partially isometric, then $\bar{U}^\dagger \bar{U} = \bar{P}_1 e \bar{U} \bar{U}^\dagger = \bar{P}_2$ are projectors and from Eq.s (17), (18), with $g_i = 1$, we obtain

$$\sum_j |\nu(g_j; g_jx)|^2 = \chi_1(1; x); \quad \sum_i |\nu(g_i; x)|^2 = \chi_2(1; x).$$

(20)

By integrating both members of these and using the equality $d\mu(g_i x) = d\mu(x)$ we obtain

$$\mu(S_{\bar{P}_1}) = \int_X \chi_1(1; x) d\mu(x) = \int_X \chi_2(1; x) d\mu(x) = \mu(S_{\bar{P}_2}),$$

where $S_{\bar{P}_1}$ and $S_{\bar{P}_2}$ are the spectral sets of $\bar{P}_1$ and $\bar{P}_2$.

Let us now focus on the cases in which the projectors belong to $\bar{P}$; that is, the cases in which $\chi(g_i; x) = \chi_S(x) \delta_{g_i,1}$ a.e. If $\bar{P}_S \bar{P}_S \in \bar{P} e \bar{P}_S \bar{P}_S = 0$ we will have $\bar{P}_S \bar{P}_S = \bar{P}_S \bar{P}_S = 0$ and, correspondingly, $\chi_S(x)\chi_S(x) = 0, \chi_S(x) + \chi_S(x) = \chi_S(x)\chi_S(x)$ and $\chi_S(x)\chi_S(x) = \chi_S(x)\chi_S(x)$. We will obtain therefore

$$\int_X \chi_S(x) d\mu(x) = \mu(S_A) + \mu(S_B).$$
This additive property of measures transfers immediately to all pairs of projectors of \( \bar{P}_1, \bar{P}_2 \in \bar{F} \) for which the relations \( \bar{P}_1 \bar{P}_2 = 0, \bar{P}_1 \sim \bar{P}_{S_{A}}, \bar{P}_2 \sim \bar{P}_{S_B} \) hold.

By noting that, \( \bar{P} \) being maximal, every projector \( \bar{P} \in \bar{F} \) is isometric to a projector \( \bar{P}_S \in \bar{P} \) at least, we deduce the additive property
\[
\mu(S_{\bar{P}_1} + S_{\bar{P}_2}) = \mu(S_{\bar{P}_1}) + \mu(S_{\bar{P}_2}), \quad \bar{P}_1 \bar{P}_2 = 0
\]
does hold.

The same results can be obtained, of course, also if \( \bar{P}' \in \bar{F}' \) for all projectors. It is then evident that \( \mu(S_{\bar{P}}) \) has all the properties of a relative dimension \( D(\bar{P}) \). We can then establish a proportionality relation
\[
D(\bar{P}) = c \mu(S_{\bar{P}}),
\]
where \( c \) is a real positive normalization constant. We can therefore state

**Theorem 4.8** Every projector of the sort \( \bar{P} \approx \bar{P}' \approx \langle \chi(g_i; x) \rangle \) has dimension
\[
D(\bar{P}) = D(\bar{P}') = c \int_X \chi(1; x) d\mu(x) ,
\]
where \( c \) is a suitable normalization constant.

We can establish in this way a precise correspondence between the dimensions of the projectors and the measures of their spectral sets in all cases in which these measures are invariant with respect to \( G \). That is, for all factors of types I and II.

Here are some examples of discrete groups that act on \( X \) so as to generate the described types:

\( I_n \). The additive group over the integers modulo \( n \) \( Z_n \) on \( X = |Z_n| \), the set of \( Z_n \) elements.

\( I_\infty \). The additive group \( Z \) over the integers on \( X = |Z| \), the set of \( Z \) elements.

\( II_1 \). The additive group \( Q_1 \) modulo 1 generated by the rational numbers \( g_n = 1/n, n = 1, 2, \ldots \), which acts on the real numbers \( x \in [0, 1] \), or on the set \( |Q_1| \) itself, according to the law \( x' = x + g \).

\( II_\infty \). The additive group of rational numbers \( Q \) that acts on the real numbers \( x \in [-\infty, +\infty] \), or on the set \( |Q| \) itself, according to the law \( x' = x + g \).

### 4.4 The class Trace

If the measure of \( X \) is finite, we can normalize it so as to have \( \mu(X) = 1 \). In this case, \( \bar{F} \) and \( \bar{F}' \) belong to types \( I_n \) or \( II_1 \). The type \( I_n \) occurs when \( X \) is a set of \( n \) points, in which case the measure of a point is \( \mu(x) = 1/n \). The type \( II_1 \) occurs when the projectors \( \bar{P} \) have a continuous spectrum of finite measure. Correspondingly, the vector \( |\Omega\rangle = 1 \otimes |1\rangle \) has the finite norm
\[
\langle \Omega|\Omega\rangle = \int_X d\mu(x) = 1 .
\]

We can therefore establish the equations
\[
\bar{A}^\dagger \bar{A}|\Omega\rangle = \sum_j |\alpha^*(g_j; g_j x)|^2 \otimes |g_j\rangle ;
\]
\[
\bar{A} \bar{A}^\dagger |\Omega\rangle = \sum_j |\alpha^*(g_j; g_j x)|^2 \otimes |g_j\rangle ;
\]
\[
\bar{P}|\Omega\rangle = \sum_j \chi(g_j; g_j x) \otimes |g_j\rangle ;
\]
from which we obtain
\[
\langle \Omega|\bar{A} \bar{A}^\dagger |\Omega\rangle = \langle \Omega|\bar{A}^\dagger \bar{A}|\Omega\rangle = \sum_j \int_X |\alpha(g_j; x)|^2 d\mu(x) .
\]
\[
\langle \Omega|\bar{P}|\Omega\rangle = \int_X \chi(1; x) \mu(x) = D(\bar{P}) \leq 1 .
\]
If $\bar{U} \in \bar{F}$, is unitary, we have $\bar{P} \sim \bar{U} \bar{P} \bar{U}^\dagger$ and, in agreement with the note following Def. 1.5, we have also $\langle \Omega | \bar{P} | \Omega \rangle = \langle \Omega | \bar{U} \bar{P} \bar{U}^\dagger | \Omega \rangle$.

Let us prove that also every self–adjoint operator $\bar{A} \in \bar{F}$ satisfies the equation

$$\langle \Omega | \bar{A} | \Omega \rangle = \langle \Omega | \bar{U} \bar{A} \bar{U}^\dagger | \Omega \rangle.$$ 

It is indeed sufficient to represent $A$ in the form

$$\bar{A} = \int_0^1 \alpha \, d\bar{E}(\alpha),$$

where $\bar{E}(\alpha) \in \bar{F}$ is a suitable matrioska of projectors depending on the real parameter $\alpha$, $0 \leq \alpha \leq 1$. That is, a family of projectors such that $\bar{E}(\alpha_1) \bar{E}(\alpha_2) = \bar{E}(\alpha_1)$ so $\alpha_1 \leq \alpha_2$. Clearly, if $\bar{U} \in \bar{F}$ is unitary, we have $\bar{E}(\alpha) \sim \bar{U} \bar{E}(\alpha) \bar{U}^\dagger$ for every value of $\alpha$. Consequently,

$$\langle \Omega | \bar{E}(\alpha) | \Omega \rangle = \langle \Omega | \bar{U} \bar{E}(\alpha) \bar{U}^\dagger | \Omega \rangle = \int_0^1 d\bar{E}(\alpha) = \mu(\alpha),$$

where $\mu(\alpha)$ is the dimension of the projector $\bar{E}(\alpha)$. But then, the mean value can be expressed as

$$\langle \Omega | \bar{U} \bar{A} \bar{U}^\dagger | \Omega \rangle = \int_0^1 \alpha \, d\mu(\alpha),$$

which is clearly independent of $\bar{U}$. From all of this, it follows that determining the mean value in $| \Omega \rangle$ is equivalent to computing the trace:

$$\langle \Omega | \bar{A} \bar{B} \ldots | \Omega \rangle \equiv \text{Tr}[\bar{A} \bar{B} \ldots].$$

### 4.5 The density matrix

All projectors of a factor of type $I_\infty$, and therefore all of its self–adjoint projectors, have an infinite discrete spectrum and all those of type $II_\infty$ have an infinite continuous spectrum. These are just the cases of interest for quantum mechanics. It is clear, however, that the considerations carried out in the previous subsection do not apply to these types.

The difficulty lies in the fact that the mean values of operators do not exist. Otherwise, the norm of any hypothetical state $| \Omega \rangle = 1 \otimes | 1 \rangle$ would be infinite.

From a physical standpoint, the non–existence of the trace is related to the fact that the probability distributions of the eigenstates of an observable with an infinite spectrum cannot be uniform, otherwise we should allow a sense to infinite sets of vanishing probabilities. Types $I_\infty$ and $II_\infty$ can host the representations of physical systems for which the mean values of observable quantities, no matter whether discrete or continuous, are evaluated over non uniform probability distributions (provided that the eigenvalue spectra do not have infinite degeneration). Of this sort are for instance finite quantum–mechanical systems in thermodynamic equilibrium. In this case, the non–uniformity of the probability distributions is a consequence of the exponentially decreasing profile of Gibbs’ distribution function.

With maximum generality, the means of physical quantities can be calculated as mean values of operators over states represented by vectors of the sort

$$| \Omega \rangle = \sum_i f(g_i, x) \otimes | g_i \rangle, \quad \langle \Omega | \Omega \rangle = \sum_i \int_X |f(g_i, x)|^2 \, d\mu(x) = 1.$$

By setting $w_{g_i}^0(x) = f^*(g_i; g_i; g_i) f(g_j; g_j; g_j)$, $w(x) = \sum_i w_{g_i}^0(x) = \sum_x |f(g_i, x)|^2$, we obtain for $\bar{L}_{\phi(x)} \in \bar{E}$

$$\langle \Omega | \bar{L}_{\phi(x)} | \Omega \rangle = \int_X w(x) \phi(x) \, d\mu(x).$$
This expression is equivalent to evaluating the mean value of \( \phi(x) \) over a probability density \( w(x) \). For a generic bounded operator \( \hat{A} \sim || L_\alpha(g_i g_j^{-1} : x U_{g_i, g_j^{-1}}) || \in \mathcal{F} \) we obtain

\[
\langle \Omega | \hat{A} | \Omega \rangle = \sum_{i,j} \int_X w_{g_i}(x) \alpha(g_i g_j^{-1}; g_i x) \, d\mu(x),
\]

We can interpret \( w_{g_i}(x) \) as the matrix elements of a state–density operator \( W \) of \( \mathcal{H} \).

In the case of type \( I_\infty \), the space \( X \) contains a countable infinity of points \( x_1, x_2, \ldots \) with same \( \mu(x_k) \). Since \( \mu(X) = \infty \), it is not restrictive to assume \( \mu(x_k) = 1 \). In this way, the integration becomes a summation over the indices \( k \) and the averages indicated above became respectively

\[
\langle \Omega | \hat{L}_{\phi(x)} | \Omega \rangle = \sum_{k=1}^\infty w(x_k) \phi(x_k),
\]

\[
\langle \Omega | \hat{A} | \Omega \rangle = \sum_{i,j,k} w_{g_i}(x_k) \alpha(g_i g_j^{-1}; g_i x_k).
\]

5 Factors of Type III

In the previous section, we have proved that for factors of types \( I \) and \( II \) there is a substantial equivalence between the dimensions of the projectors and the measures of their spectral sets.

In the proof, the invariance axiom \( (x) \) introduced in subsection \( 2 \) was used. This ensures that the equality \( \mu(gS) = \mu(S) \) holds for all measurable sets \( S \subset X \) and all elements \( g \in G \). This means that, in the framework of discrete systems here considered, possible examples of factors of type \( III \) must be searched for under the condition that axiom \( (x) \) does not hold. That is, when the inequality \( \mu(gS) \neq \mu(S) \) holds for some \( S \subset X \) and some \( g \in G \). Let us here provide two examples of such groups, the first of which has \( \mu(X) = \infty \) and the second \( \mu(X) < \infty \).

5.1 Examples of groups with non–invariant measures

1. Let \( X \) be the set of real numbers equipped with standard Lebesgue measures. Hence we have \( \mu(X) = \infty \). Let \( \tilde{G} \) be the group of affine transformations defined by the equations

\[
x' = g_{\rho, \sigma} x = \rho x + \sigma, \quad x \in X,
\]

and \( G \) the subgroup of \( \tilde{G} \) with rational \( \rho \) and \( \sigma \). Clearly, \( G \) is discrete free and ergodic and, if \( S \subset X \) is measurable, we have

\[
\mu(g_{\rho, \sigma} S) = \rho \mu(S).
\]

Therefore the measure is not invariant.

2. Let \( X \) be the set of complex numbers \( z \) with \( |z| = 1 \). Assume as measurable sets those formed by the arcs of the unitary circle \( |z| = 1 \). Then \( \mu(X) = 2\pi \).

Let now \( \tilde{G} \) be the group of conformal maps

\[
z' = g_{\rho, \sigma} z = e^{i 2\pi \rho} \frac{z + \sigma}{1 + z \sigma}, \quad \rho \in [0, 1); \quad |\sigma| < 1.
\]

We can easily verify that \( |z'| = 1 \). In other terms \( \tilde{G} \) maps the unitary circle on itself. Let \( G \) be the subgroup of \( \tilde{G} \) generated by \( g_{0, 1/2} \) and \( g_{\rho, 0} \) with rational \( \rho \). Clearly, \( G \) is infinite and discrete. We can immediately verify that it is free and ergodic: Indeed, ergodic is already the subgroup \( G_0 \) formed by the sole elements of \( g_{\rho, 0} \). Now, indicating by \( S_{0, \pi/2} \) the arc delimited by the complex vectors \( z = 1 \) and \( z = i \), we have \( \mu(S_{0, \pi/2}) = \pi/2 \). But we have \( \mu(g_{0, 1/2} S_{0, \pi/2}) = \arctan 3/4 \) as \( g_{0, 1/2} \) sends \( 1 \) into \( 1 \) but \( i \) into \( (4 + 3i)/5 \). Thus, in general, the measures of the arcs are not invariant under \( G \).
5.2 The theorem of Radon-Nikodym

We can define unitary transformations that do not leave invariant the spectral sets of projectors by using a theorem of Lebesgue–Radon–Nikodym adapted by von Neumann to deal with sets that are measurable in the generalized sense defined in subsection 2. The theorem, which we report without proof [9], states the following:

**Theorem 5.1** Let \( \mu_1(S) \) and \( \mu_2(S) \) different measures of a measurable subset \( S \subset X \). Then, there exists a positive measurable function \( \kappa(x) \), which is in general defined up to a set of measure zero, such that for all measurable functions \( f(x) \) the following equations

\[
\int_X f(x) d\mu_2(x) = \int_X f(x) \kappa(x) d\mu_1(x)
\]

hold. We can therefore represent this function as

\[
\kappa(x) = \frac{d\mu_2(x)}{d\mu_1(x)}.
\]

We can apply this theorem to the case in which the diversity of the measure depends upon the non–invariance under \( G \), so as to give a sense to the expression

\[
\frac{d\mu(g_2 x)}{d\mu(g_1 x)} = \frac{d\mu(g_1^{-1} x)}{d\mu(g_2^{-1} x)}.
\]

5.3 The algebras of systems with non–invariant measures

The condition that the operators \( U_g \), which represent the elements \( g \in G \), be unitary imposes the following generalization of the operators introduced in Subsec. 2.3

\[
L_\phi(x) f(x) = \phi(x) f(x), \quad U_g f(x) = f(g x) \left[ \frac{d\mu(g x)}{d\mu(x)} \right]^{1/2}.
\]

We can immediately verify that \( U_g \) is unitary. In a similar way, for the operators \( \mathcal{H} = \mathcal{H}_X \otimes \mathcal{H}_G \), we can define

\[
\tilde{L}_\phi(x) f(g_i x) = \phi(x) f(g_i x); \quad \tilde{U}_g f(g_i x) = f(g g_i x) \left[ \frac{d\mu(g x)}{d\mu(x)} \right]^{1/2};
\]

\[
\tilde{Q} f(g_i x) = f(g_i^{-1} x) \left[ \frac{d\mu(g_i^{-1} x)}{d\mu(x)} \right]^{1/2};
\]

\[
\tilde{L}_\phi'(x) f(g_i x) = \phi(g_i^{-1} x) f(g_i x); \quad \tilde{U}_g' f(g_i x) = f(g g_i^{-1} x).
\]

It is easy to verify that \( \tilde{P}_S = \tilde{L}_{\chi_S(x)} \) and \( \tilde{P}_g = \tilde{L}_{\chi_S(x)} \) are projectors, that \( \tilde{U}_g, \tilde{Q} \) and \( \tilde{U}_g' \) are unitary and that the equalities

\[
Q^2 = Q, \quad \tilde{Q} \tilde{L}_\phi(x) \tilde{Q} = \tilde{L}_\phi'(x), \quad \tilde{Q} \tilde{U}_g \tilde{Q} = \tilde{U}_g'
\]

hold.

We can therefore proceed to the construction of all the representations of operators \( \tilde{A} \in \mathcal{F} \) and \( \tilde{A}' \in \mathcal{F}' \) exactly as they were in the case of invariant–measure systems. We proceed up to the point at which, for the spectral sets \( S_P \) and \( S_{P_g} \) of two projectors \( P, P_g \) linked by the equation \( \tilde{P}_g = \tilde{U}_g \tilde{P} \tilde{U}_g^\dagger \), we find \( S_{P_g} = g S_P \) a.e., and finally \( D(\tilde{P}) = c \mu(S_{\tilde{P}}) \). Since \( \tilde{P} \) and \( \tilde{P}_g \) are isometric, the equation \( D(\tilde{P}) = D(\tilde{P}_g) \) must hold.

If we pose the condition that the equality holds for all \( g \in G \) and all measurable sets \( S \subset X \) with \( c \neq 0, \infty \), then for all projectors of finite spectral measure the equations \( D(\tilde{P}) = c \mu(S_{\tilde{P}}) < \infty \) must hold. In this case, we can assume \( \mu(S) = D(\tilde{P}_S) \) and use the dimensions of projectors as invariant measures of their spectral sets. On this basis, we can establish a direct relation between the measurability properties of spectral sets and the dimensional properties of types I and II.
However, if at least for one element \( g \in G \) and one subset \( S \subset X \) we have \( \mu(S_P) \neq \mu(S_{P'}) \), we must conclude that the relation between dimensions and measures is possible only if either \( c = 0 \) or \( c = \infty \). This means that the factor is of type III.

We can summarize these results in the theorem:

**Theorem 5.2** The classification of factor types is substantially based on the proportionality relation \( D(\bar{P}) = \mu(S_P) \) between the dimensions of projectors \( \bar{P} \) and the measures of spectral sets \( S_P \) (as established by theorem 4.8), and on the invariance of this relation under the action of unitary automorphisms \( U_g P U_{g'}^* \) internal to the factor. If the measure of a spectral set is not invariant for one of these automorphisms, the equality is possible only for \( c = 0 \) or for \( c = \infty \), and therefore the factor is of type III.

This theorem does not state that the spectral sets of the projectors of type III are not measurable, but that the dependence of the measure from certain unitary or isometric transformations makes it impossible to assign a same finite dimension to isometric projectors. From a physical standpoint, we can say that type III is characteristic of physical systems in which some observables can undergo scale transformation under the action of certain physical operations.

### 6 Mean values and measurement processes

In all generality, the mean values of bounded operators of the three types can be evaluated over normalized vectors

\[
|\Omega\rangle = \sum_i f(g_i, x) \otimes |g_i\rangle, \quad \langle \Omega | \Omega \rangle = \sum_i \int_X |f(g_i, x)|^2 d\mu(x) = 1.
\]

For a generic bounded operator \( \bar{A} = \| A_{g_i}^g \| \) of \( \mathcal{H} \) we have

\[
\langle \Omega | \bar{A} | \Omega \rangle = \sum_{i,j} \int_X f^*(g_i; x) f(g_j; x) A_{g_i}^g A_{g_j}^g d\mu(x).
\]

For an operator \( \bar{A} \sim L_{\alpha(g; g^{-1})} U_{g, g^{-1}} \) we find

\[
\langle \Omega | \bar{A} | \Omega \rangle = \sum_{i,j} \int_X f^*(g_i; x) f(g_j; g_i g_i^{-1} x) \alpha(g_i g_i^{-1}; x) \left[ \frac{d\mu(g_i g_i^{-1} x)}{d\mu(x)} \right]^{1/2} d\mu(x),
\]

and for \( \bar{A}' = \bar{Q} \bar{A} \bar{Q} \sim L_{\alpha(g; g^{-1})} \) we find

\[
\langle \Omega | \bar{A}' | \Omega \rangle = \sum_{i,j} \int_X f^*(g_i; x) f(g_j; x) \alpha(g_i^{-1} g_i g_i^{-1}; x) d\mu(x).
\]

Therefore, in general, we have \( \langle \Omega | A | \Omega \rangle \neq \langle \Omega | A' | \Omega \rangle \). It is therefore evident that \( \bar{A} \) and \( \bar{A}' \), although algebraically isomorphic, are spatially different.

However, if \( \alpha(g; x) = \alpha(1; x) \delta_{g, 1} \), the expectation values of \( \bar{A} \) and \( \bar{A}' \) coincide over all states. In this case, we have

\[
\langle \Omega | A | \Omega \rangle = \langle \Omega | A' | \Omega \rangle = \sum_i \int_X |f(g_i; x)|^2 \alpha(1; x) d\mu(x).
\]

By interpreting \( \sum_i |f(g_i; x)|^2 = w(x) \) as a probability density and \( \alpha(x) = \alpha(1; x) \) as an eigenvalue spectrum, we conclude that \( \bar{A} \) and \( \bar{A}' \) possess the same mean value and the same standard deviation. In correspondence with different functions \( \alpha(1; x) \), we will find different pairs of operators, which can be interpreted as a correlation of observables between the two parts of a system.

These are just the conditions that are expected in a measurement process. Conditions in which, for instance, \( \bar{A} \) represents the observed quantity and \( \bar{A}' \) the measuring device. It is worth noticing that the algebraic factorization allow us to deal equally well with the measures of observables with discrete and with continuous spectra. We answer in this way a question that was posed at the beginning of the paper: how
to represent the correlation between the states of an observed and an observing system when the vectors of the pair of correlated systems cannot be represented as vectors of a direct–product space.

Actually, the subject pertains more in general to the way in which two parts of a system correlate with each other during their interaction. Indeed, we know that during the evolution of an isolated system, it occurs in general the exchange of conservative quantities, which are observable quantities common to the different parts of the system. The representation of interactions in direct–product spaces can only account for exchanges of quantities possessing discrete spectra of eigenvalues (quanta). The factorization of the algebra of observables into commuting subalgebras of different types, we arrive to represent such exchanges at the level of maximum generality.

At this point, however, we should point out that the decomposition of an algebra into a pair of coupled factors is possible only if the system is decomposed into two algebraically isomorphic parts, which excludes the case of asymmetric partitions. To treat the general case, we should study the asymmetric decompositions of the bounded operator algebra into coupled factors. This can be done by suitably selecting, out of one coupled factor, a subalgebra as an asymmetric factor. We will not try, however, to expand our analysis in this direction.

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At this point, we should be able to describe the unitary entanglement–operators capable of producing the correlation of the states of initially uncorrelated observables. We leave the solution of this problem as an exercise for the reader.