Znajek-Damour Horizon Boundary Conditions with
Born-Infeld Electrodynamics

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Abstract

In this work, the interaction of electromagnetic fields with a rotating (Kerr) black hole is explored in the context of Born-Infeld (BI) theory of electromagnetism instead of standard Maxwell theory and particularly BI theory versions of the four horizon boundary conditions of Znajek and Damour are derived. Naturally, an issue to be addressed is then whether they would change from the ones given in the Maxwell theory context and if they would, how. Interestingly enough, as long as one employs the same local null tetrad frame as the one adopted in the works by Damour and by Znajek to read out physical values of electromagnetic fields and fictitious surface charge and currents on the horizon, it turns out that one ends up with exactly the same four horizon boundary conditions despite the shift of the electrodynamics theory from a linear Maxwell one to a highly non-linear BI one. Close inspection reveals that this curious and unexpected result can be attributed to the fact that the

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concrete structure of BI equations happens to be such that it is indistinguishable at the horizon to a local observer, say, in Damour’s local tetrad frame from that of standard Maxwell theory.
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I. Introduction

The idea of using rotating black holes as energy sources has a long history. To our knowledge, Salpeter [1] and Zel’dovich [1] were the first to point out that gigantic black holes might serve as power engines for quasars or radio galaxies. Realistic theoretical models to realize this type of energy extraction from rotating black holes also appeared afterwards and they are due to Penrose [2], Press and Teukolsky [2], Ruffini and Wilson [3], Damour [3], and Blandford and Znajek [4]. Among these models, that of Blandford and Znajek is particularly interesting in its formulation and looks quite plausible in its operational mechanism. At first, puzzling over the possible explanation for the observed twin jets pointing oppositely out of a black hole-accretion disk system, Blandford and Znajek conceived of a particular process in which the power going into the jets comes from the hole’s enormous rotational energy. Schematically, their mechanism works as follows; suppose that the rotating hole is threaded by magnetic field lines. As the hole spins, it drags the field lines around, causing them to fling surrounding plasma upward and downward to form two jets. Then the jets shoot out along the hole’s spin axis and their direction is firmly fixed to the hole’s axis of rotation. The magnetic field lines, of course, come from the accretion disk around the hole. Namely, it is the magnetic fields that extract the rotational energy of a black hole and then act to power the jets. According to their careful analysis, on the other hand, as the energy is extracted, electric currents flow into the horizon near the hole’s poles (in the form of positively-charged particles falling inward), and currents flow out of the horizon near the equator (in the form of negatively-charged particles falling inward). It was as though the hole were a voltage generator of an electric circuit driving current out of the horizon’s equator, then up magnetic field lines to a large distance, then through “plasma load” to other field lines near the hole’s spin axis, then down those field lines and into the horizon. Namely, the magnetic field were the wires of the electric circuits, the plasma was the load that exerts power from the circuit. And the two pictures, one schematic and the other analytic, are just two different ways of describing the same phenomenon. This
electric circuit description was totally unexpected and thus curious enough although it was resulted from a careful general relativistic treatment of the problem. Right after the post of this new mechanism, Znajek [5] and, independently, Damour [6] succeeded in translating the careful general relativistic formulation into a surprisingly simple non-relativistic, flat spacetime electrodynamics language, the celebrated four horizon boundary conditions. And the assumption of central importance in this new picture is to endow the horizon with some fictitious surface charge and current as those previously imagined by Hanni and Ruffini [7]. It is really amusing that one now has an option to view the situation in terms of flat spacetime electrodynamics alone at least for rough understanding.

Speaking of the theory that governs the electromagnetism, however, it is interesting to note that historically, there has been another classical theory that can be thought of as a larger class of theory involving the standard Maxwell theory just as its limiting case. It is the theory proposed in the 1930’s by Born and Infeld [9]. In spite of its long history, the Born-Infeld (BI) theory of electrodynamics has remained almost unnoticed and hence nearly uncovered in full detail. This theory may be thought of as a highly nonlinear generalization of or a non-trivial alternative to the standard Maxwell theory of electromagnetism. It is known that Born and Infeld had been led, when they first constructed this theory, by the considerations such as finiteness of the energy in electrodynamics, natural recovery of the usual Maxwell theory as a linear approximation and the hope to find soliton-like solutions representing point-like charged particles. In the present work we would like to explore the interaction of electromagnetic fields with a rotating (Kerr) black hole but in the context of BI theory of electromagnetism instead of Maxwell theory. And our particular concern is to derive BI theory versions of the four horizon boundary conditions to see how they would change from the ones derived originally by Znajek and by Damour in the context of Maxwell theory. Now the motivation for shifting the theory of electromagnetism from that of standard Maxwell to that of BI to study the physics of interaction between “test” electromagnetic field and “background” rotating black hole geometry can be stated as follows. The BI theory, although appeared as a “classical” theory long before the advent of Quantum Electrodynamics (QED)
theory, may be viewed as some kind of an effective low-energy theory of QED in that its highly non-linear structure plays the role of eliminating the short-distance divergences. Normally, the strong magnetic field, believed to be anchored in the central black holes of typical gamma-ray bursters, is regarded as being originated, say, from that of neutron stars that has collapsed to form the black hole. A number of various observations indicate that in young neutron stars, the surface magnetic field strengths are of order $10^{11} \sim 10^{13} (\text{Gauss})$ and in some extreme cases such as magnetars, magnetic field strengths are estimated to be as large as $\geq 5 \times 10^{14} (\text{Gauss})$ [8]. Then the magnetic field of this ultra strength, in turn, stimulates our curiosity and leads us to ask questions such as ; what would happen if we choose to employ the BI theory that, as stated, can be thought of as an effective theory of QED, instead of linear Maxwell theory, to study the physics in the vicinity of rotating hole’s horizon ? And in doing so, we anticipate that perhaps the highly non-linear nature of the BI theory may serve to uncover some hidden interplay between the strong electromagnetic field and ultra strong gravity near the hole’s horizon. Since our main concern is the derivation of the four horizon boundary conditions in BI theory, we now recall some of the basic ingredients of these boundary conditions obtained in the conventional Maxwell theory.

The four “horizon boundary conditions” first derived in the works of Znajek [5] and of Damour [6] and reformulated later in the literature can be briefly described as follows. They may be called radiative ingoing boundary condition, Ohm’s law, Gauss’ law and Ampere’s law, respectively. And in order to represent each boundary condition properly, we need to introduce in advance some quantities that will be derived carefully in the text shortly. They are electric and magnetic fields at the horizon $(\vec{E}_H, \vec{B}_H)$ as seen by a local observer in a null tetrad frame which has been made to be well-behaved at the horizon by the amount of boost that becomes suitably infinite at the horizon and the fictitious charge and current densities $(\sigma, \kappa)$ that have been assigned at the horizon in such a way that the sum of real current 4-vector outside the horizon and this fictitious current 4-vector on the horizon together is conserved. Firstly, then, the radiative ingoing boundary condition first derived by Znajek [5] takes the form $\vec{B}_H = \vec{E}_H \times \hat{n}$ with $\hat{n}$ being the outer unit normal to the horizon.
Evidently, it states that the electric and magnetic fields tangential to the horizon are equal in magnitude and perpendicular in direction and hence their Poynting energy flux is into the hole. Secondly, the Ohm’s law reads $\vec{E}_H = 4\pi \vec{\kappa}$. It has been derived rigorously in the work by Damour [6] and pointed out in the work by Znajek [5]. Clearly, this relation takes on the form of a non-relativistic Ohm’s law for a conductor and hence implies that if we endow the horizon with some charge and current densities (which are to be determined by the surrounding external electromagnetic field $F_{\mu\nu}$ as we shall see in the text), then the horizon behaves as if it is a conductor with finite surface resistivity of $\rho = 4\pi \simeq 377$ (ohms). Actually these two relations are the ones that have been explicitly derived in the works by Znajek and by Damour and play the central role in justifying that the introduction of fictitious charge and current densities on the horizon indeed provides a self-consistent picture. That is, one might wonder what would happen to the Joule heat generated when those surface currents work against the surface resistance and how it would be related to the electromagnetic energy going down the hole through the horizon. In their works, Znajek and Damour provided a simple and natural answer to this question. Namely, they showed in an elegant manner that the total electromagnetic energy flux (i.e., the Poynting flux) into the rotating Kerr hole through the horizon is indeed precisely the same as the amount of Joule heat (Ohmic dissipation) produced by the surface currents when they work against the surface resistivity of $4\pi$. As a result, one may think of the rotating hole as a conducting sphere that absorbs the incident electromagnetic energy flux as a form of Joule heat that the surface current (driven by the electromagnetic fields) generates when it interferes with the surface resistivity. This is indeed an interesting and quite convincing alternative picture of viewing the interaction of external electromagnetic fields with a rotating black hole. Damour [6] also remarked that this result provides a clear confirmation of Carter’s assertion [10] that a black hole is analogous to an ordinary object having finite viscosity and electrical conductivity. Thirdly, if one follows the formulation of Damour but in a slightly different way in taking the local tetrad frame and projecting the Maxwell field tensor and the surface current 4-vector onto that chosen tetrad frame, one also gets the relation $E_{\tilde{r}} = 4\pi \sigma$ which may be identified
with the surface version of Gauss’ law. It says that the fictitious surface charge density we assumed on the horizon plays the role of terminating the normal components of all electric fields that pierce the horizon. Lastly, if we combine the radiative ingoing boundary condition at the horizon that we obtained earlier, $\vec{B}_H = \vec{E}_H \times \hat{n}$ with the Ohm’s law $\vec{E}_H = 4\pi \kappa$, we end up with the fourth relation $\vec{B}_H = 4\pi (\kappa \times \hat{n})$ which can be viewed as the surface version of Ampere’s law. Again, consistently with our motivation for introducing fictitious current density on the horizon, this relation indicates that the current density we assumed plays the role of terminating any tangential components of all magnetic fields penetrating the horizon. And actually these four horizon boundary conditions later on provided a strong motivation for the proposal of so-called “membrane paradigm [11]” of black holes by Thorne and his collaborators. As we already mentioned, in the present work we would like to particularly derive BI theory versions of these four horizon boundary conditions to see if they would change from the ones given above and if they would, how. Interestingly enough, as far as we employ the same local null tetrad frame as the one adopted in the works by Damour and by Znajek, it turns out that we end up with exactly the same four horizon boundary conditions despite the shift of the electrodynamics theory from a linear Maxwell one to a highly non-linear BI one. As we shall see shortly in the text, this curious and unexpected result can be attributed to the fact that the nature of the BI theory or more precisely, the concrete structure of BI equations happens to be such that it is indistinguishable at the horizon to a local observer, say, in Damour’s local tetrad frame from that of standard Maxwell theory. We find this point indeed quite amusing on theoretical side.

II. Choice of coordinate system and tetrad frame

The relevant choice of coordinate system and the proper choice and treatment of the associated tetrad frame for the background Kerr black hole spacetime is of primary importance to discuss electrodynamics on this geometry in terms of physical field values (here, the meaning of “physical” values will be unambiguously defined shortly). Thus in this section, we shall carefully choose the coordinate system and perform a proper treatment of the as-
associated tetrad frame to derive the four horizon boundary conditions based on these choices later on. Generally speaking, in order to represent the background Kerr geometry, we need to choose a coordinate system in which the metric is to be given and in order to obtain physical components of a tensor (such as the electric and magnetic field values), we need to select a tetrad frame (in a given coordinate system) to which the tensor components are to be projected. It has been known for some time that there are three important coordinate systems for Kerr spacetime; ingoing/outgoing Kerr coordinates, Kerr-Schild coordinates, and Boyer-Lindquist coordinates. First, the Kerr-Schild coordinates are quasi-Cartesian coordinates and the well-known ring structure of curvature singularity can only be uncovered in this coordinate system. Next, the Boyer-Lindquist coordinates can be viewed as the generalization of Schwarzschild coordinates to the stationary, axisymmetric case. Lastly, the Kerr coordinates can be thought of as the axisymmetric generalization of Eddington-Finkelstein null coordinates and hence are free of coordinate singularities. In particular, the ingoing (advanced) null coordinates represent a reference frame of “freely-falling” photons. As such, these ingoing Kerr coordinates are well-behaved on the event horizon and thus meet our purpose to explore the electrodynamics near the horizon. Turning to the choice of tetrad frame, there are largely two types of tetrad frames; orthonormal tetrad and null tetrad. As is well-known, the orthonormal tetrad is a set of four mutually orthogonal unit vectors at each point in a given spacetime which give the directions of the four axes of locally-Minkowskian coordinate system

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{AB} e^A e^B \]

\[ = -(e^0)^2 + (e^1)^2 + (e^2)^2 + (e^3)^2 \]  

(1)

where \( e^A = e^A_\mu dx^\mu \). Namely, every physical observer with 4-velocity \( u^\mu \) has associated with him an orthonormal frame in which the basis vectors are the (reciprocal of) orthonormal tetrad \( e_A = \{e_0 = u, e_1, e_2, e_3\} \). And corresponding to this is a null tetrad \( Z_A = \{l, n, m, \bar{m}\} \) defined by
\[ e_0 = \frac{1}{\sqrt{2}} (l + n), \quad e_1 = \frac{1}{\sqrt{2}} (l - n), \tag{2} \]
\[ e_2 = \frac{1}{\sqrt{2}} (m + \bar{m}), \quad e_3 = \frac{1}{\sqrt{2i}} (m - \bar{m}) \]
satisfying the orthogonality relation
\[ -l^\mu n_\mu = 1 = \bar{m}^\mu \bar{m}_\mu \tag{3} \]
with all other contractions being zero and
\[ g^{\mu \nu} = -l^\mu n^\nu - n^\mu l^\nu + m^\mu \bar{m}^\nu + \bar{m}^\mu m^\nu. \tag{4} \]
Conversely, given a non-singular null tetrad, there is a corresponding physical observer. The tetrad vectors then can be used to obtain, from tensors in arbitrary coordinate system, their physical (i.e., finite and non-zero) components measured by an observer in this locally-flat tetrad frame. And the rules for calculating the physical components of a tensor, say, \( T_{\mu \nu} \) in the orthonormal frame and in the null frame are given respectively by
\[ T_{AB} = T_{\mu \nu} (e^\mu_A e^\nu_B), \quad T_{lm} = T_{\mu \nu} (l^\mu m^\nu), \quad \text{etc.} \tag{5} \]
where \( e^\mu_A \) is the inverse of the tetrad vectors \( e_\mu^A \) in that \( e_\mu^A e^\alpha_A = \delta^\alpha_\mu \) and \( e_\mu^A e_\nu^B = \delta_A^B \).

1. Hawking-Hartle (or Teukolsky) tetrad

As we stated earlier in the introduction, we would like to derive Znajek-Damour-type boundary conditions at the horizon of Kerr black hole in the context of BI theory of electromagnetism. Generally speaking, all that is required of the “correct” boundary conditions for electric and magnetic fields at the horizon can be stated as follows. The physical field’s components in the neighborhood of an event horizon should have “nonspecial” values. Or put another way, a physically well-behaved observer at the horizon should see the fields as having finite and non-zero values. And this can be achieved only when one works in the coordinates having non-singular behavior at the horizon with the choice of a null tetrad frame. Such a choice of well-behaved null tetrad frame has been provided long ago by Hawking and Hartle [12] and also by Teukolsky [13]. Thus in order to briefly review the derivation of
their tetrad, we start with Kinnersley’s null tetrad [14] given originally in Boyer-Lindquist coordinates $x^\mu = (t, r, \theta, \tilde{\phi})$

$$l^\mu = \left( \frac{r^2 + a^2}{\Delta}, \ 1, \ 0, \ \frac{a}{\Delta} \right),$$

$$n^\mu = \left( \frac{r^2 + a^2}{2\Sigma}, \ -\frac{\Delta}{2\Sigma}, \ 0, \ \frac{a}{2\Sigma} \right),$$

$$m^\mu = \frac{1}{\sqrt{2}(r + ia\cos\theta)} \left( ia\sin\theta, \ 0, \ 1, \ \frac{i}{\sin\theta} \right).$$

(6)

where $\Sigma = r^2 + a^2 \cos^2 \theta$ and $\Delta = r^2 + a^2 - 2Mr$ with $M$ and $a$ being the ADM mass and the angular momentum per unit mass of the hole respectively. As is well-known, this tetrad is not well-behaved on the horizon where $\Delta = 0$ since it is given in Boyer-Lindquist coordinates (which themselves are singular on the horizon) and hence cannot be of any practical use. Thus we transform them to the ingoing Kerr coordinates $x'\mu = (v, r, \theta, \phi)$ via the coordinate transformation law

$$dv = dt + \frac{(r^2 + a^2)}{\Delta} dr, \quad d\phi = d\tilde{\phi} + \frac{a}{\Delta} dr$$

(7)

to obtain, after the standard procedure,

$$Z'\mu_A = \left( \frac{\partial x'^\mu}{\partial x^\nu} \right) Z^\nu_A \quad \text{where} \quad Z^\mu_A = (l^\mu, n^\mu, m^\mu, \bar{m}^\mu),$$

$$l^\mu = \left( \frac{2(r^2 + a^2)}{\Delta}, \ 1, \ 0, \ \frac{2a}{\Delta} \right),$$

$$n^\mu = \left( 0, \ -\frac{\Delta}{2\Sigma}, \ 0, \ 0 \right),$$

$$m^\mu = \frac{1}{\sqrt{2}(r + ia\cos\theta)} \left( ia\sin\theta, \ 0, \ 1, \ \frac{i}{\sin\theta} \right).$$

(8)

Although it is expressed in these well-behaved ingoing Kerr coordinates, this null tetrad is still singular at the horizon where $\Delta = 0$. At this point, notice that we can get around this difficulty using the tetrad transformations. Namely, recall that the orthogonality relations for null tetrad given in eq.(3) remain invariant under the 6-parameter group of homogeneous Lorentz transformations at each point of spacetime. And this Lorentz group can be decomposed into 3-Abelian subgroups ;
\[(I) \quad l \to l, \quad m \to m + al, \quad n \to n + am + \bar{a}m + a\bar{a}l,\]
\[(II) \quad n \to n, \quad m \to m + bn, \quad l \to l + b\bar{m} + \bar{b}m + b\bar{b}n,\]
\[(III) \quad l \to \Lambda l, \quad n \to \Lambda^{-1} n, \quad m \to e^{i\theta} m,\]

where \(a\) and \(b\) are complex numbers and \(\Lambda\) and \(\theta\) are real. Each of these group transformations are called a “null rotation” and here we particularly consider the null rotation (III). Under this null rotation (III), the corresponding orthonormal tetrad \(e_A\) is boosted in the \(e_1 = e_r\) direction with 3-velocity \((\Lambda^2 - 1)/(\Lambda^2 + 1)\) and spatially rotated about \(e_1 = e_r\) through the angle \(\theta\). Indeed this action is precisely what we need. Namely, in order to get a null tetrad well-behaved at the horizon, we need to boost it by an amount that becomes suitably infinite on the horizon. Thus we perform the null rotation (III) on the Kinnersley’s null tetrad given in ingoing null tetrad given above with \(\Lambda = \Delta / 2(r^2 + a^2)\) and \(e^{i\theta} = \Sigma^{1/2} / (r - ia \cos \theta)\) to obtain the following non-singular null tetrad on the horizon:

\[
I^\mu = \begin{pmatrix}
1, & \frac{\Delta}{2(r^2 + a^2)}, & 0, & \frac{a}{(r^2 + a^2)}
\end{pmatrix},
\]
\[
N^\mu = \begin{pmatrix}
0, & -\frac{(r^2 + a^2)}{\Sigma}, & 0, & 0
\end{pmatrix},
\]
\[
M^\mu = \frac{1}{\sqrt{2\Sigma^{1/2}}} \begin{pmatrix}
ia \sin \theta, & 0, & 1, & \frac{i}{\sin \theta}
\end{pmatrix}.
\]

This is the Hawking-Hartle (or Teukolsky) null tetrad and the associated covariant components are given by

\[
l_\mu = \begin{pmatrix}
-\frac{\Delta}{2(r^2 + a^2)}, & \frac{\Sigma}{(r^2 + a^2)}, & 0, & \frac{\Delta}{2(r^2 + a^2)} a \sin^2 \theta
\end{pmatrix},
\]
\[
n_\mu = \begin{pmatrix}
-\frac{(r^2 + a^2)}{\Sigma}, & 0, & 0, & \frac{(r^2 + a^2)}{\Sigma} a \sin^2 \theta
\end{pmatrix},
\]
\[
m_\mu = \frac{1}{\sqrt{2\Sigma^{1/2}}} \begin{pmatrix}
-ia \sin \theta, & 0, & \Sigma, & i(r^2 + a^2) \sin \theta
\end{pmatrix}.
\]

2. Damour’s quasi-orthonormal tetrad

It is interesting to note that generally one can “mix” half of the null tetrad \(Z_A = (l, n, m, \bar{m})\) and half of the orthonormal tetrad \(e_A = (e_0, e_1, e_2, e_3)\) to form a “quasi-orthonormal” or “mixed” tetrad
\{l^\mu, -n^\mu, e_2^\mu, e_3^\mu\}, \quad \{-n_\mu, l_\mu, e_2^2_\mu, e_3^3_\mu\}. \quad (12)

And if we construct this half-null, half-orthonormal, mixed tetrad from the previous Hawking-Hartle null tetrad, it becomes Damour’s quasi-orthonormal tetrad as we can see shortly. Before we proceed, let us elaborate on the general construction of this mixed tetrad. Using the relations between the orthonormal tetrad \(e_A\) and null tetrad \(Z_A\) given in eq.(2),

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{AB} e^A_\mu e^B_\nu dx^\mu dx^\nu
\]

\[
= (-l_\mu n_\nu - n_\mu l_\nu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu) dx^\mu dx^\nu \quad (13)
\]

and hence \(g_{\mu\nu} = -l_\mu n_\nu - n_\mu l_\nu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu\). However, since the pair \((e_0, e_1)\) is related only to \((l, n)\) while the pair \((e_2, e_3)\) is related only to \((m, \bar{m})\), one can write, using \(m_\mu \bar{m}_\nu = e_2^2 e_2^2 + e_3^3 e_3^3\),

\[
g_{\mu\nu} = -l_\mu n_\nu - n_\mu l_\nu + e_2^2 e_2^2 + e_3^3 e_3^3 \quad \text{or} \quad g^{\mu\nu} = -l^\mu n^\nu - n^\mu l^\nu + e_2^2 e_2^2 + e_3^3 e_3^3. \quad (14)
\]

This obviously implies that one may mix half of null tetrad and half of orthonormal tetrad to form a mixed tetrad as given in eq.(12). Therefore, we now construct this mixed tetrad from the previous Hawking-Hartle tetrad as

\[
e_0^\mu \equiv l^\mu = \left(1, \frac{\Delta}{2(r^2 + a^2)}, 0, \frac{a}{(r^2 + a^2)}\right),
\]

\[
e_1^\mu \equiv -n^\mu = \left(0, \frac{(r^2 + a^2)}{\Sigma}, 0, 0\right),
\]

\[
e_2^\mu = \frac{1}{\sqrt{2}} (m^\mu + \bar{m}^\mu) = \left(0, 0, \frac{1}{\Sigma^{1/2}}, 0\right)
\]

\[
e_3^\mu = \frac{1}{\sqrt{2}i} (m^\mu - \bar{m}^\mu) = \left(\frac{a \sin \theta}{\Sigma^{1/2}}, 0, 0, \frac{1}{\Sigma^{1/2} \sin \theta}\right)
\]

and its dual is

\[
e_0^0 \equiv -n_\mu = \left(\frac{(r^2 + a^2)}{\Sigma}, 0, 0, \frac{-(r^2 + a^2)}{\Sigma} a \sin \theta\right),
\]

\[
e_1^1 \equiv l_\mu = \left(\frac{\Delta}{2(r^2 + a^2)}, \frac{\Sigma}{(r^2 + a^2)}, 0, \frac{\Delta}{2(r^2 + a^2) a \sin^2 \theta}\right), \quad (16)
\]
\[ e^2_\mu = \frac{1}{\sqrt{2}}(m_\mu + \bar{m}_\mu) = (0, 0, \Sigma^{1/2}, 0) \]
\[ e^3_\mu = \frac{1}{\sqrt{2}i}(m_\mu - \bar{m}_\mu) = \left( -a \sin \theta \frac{r^2 + a^2}{\Sigma^{1/2}}, 0, 0, \frac{r^2 + a^2}{\Sigma^{1/2}} \sin \theta \right) \]

where we renamed as

\[ l^\mu \rightarrow e^0_\mu, \quad n^\mu \rightarrow -e^1_\mu, \]
\[ l^\mu \rightarrow e^1_\mu, \quad n^\mu \rightarrow -e^0_\mu, \]

to go from the null tetrad’s orthogonality relations \(-l^\mu n_\mu = 1 = m^\mu \bar{m}_\mu\) to the usual orthonormality condition \(e^\mu_A e^B_\mu = \delta^B_A, \quad e^\mu_A e^A_\nu = \delta^\mu_\nu\). Note that this mixed tetrad precisely coincides with Damour’s choice of quasi-orthonormal tetrad [6]. And the tetrad metric \(\epsilon_{AB} = \epsilon^{AB}\) in

\[ ds^2 = \epsilon_{AB} e^A e^B \]

can be identified with

\[ \epsilon_{AB} = \epsilon^{AB} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}. \]

Particularly, for later use, we explicitly write down the dual basis of vectors as \(e_A = (e_0 = \dot{e}, \quad e_1 = e_r, \quad e_2 = e_\theta, \quad e_3 = e_\phi)\),

\begin{align*}
e_0 &= \partial_v + \Delta \frac{2(r^2 + a^2)}{(r^2 + a^2)^2 + a^2} \partial_\theta, \\
e_1 &= \frac{\Delta}{\Sigma} \partial_r, \\
e_2 &= \frac{1}{\Sigma^{1/2}} \partial_\theta, \\
e_3 &= \frac{1}{\Sigma^{1/2}} \left[ a \sin \theta \partial_v + \frac{1}{\sin \theta} \partial_\phi \right].
\end{align*}

(17)

Note that in all the calculations involved in this work to read off physical components of tensors such as Maxwell field tensor and current 4-vector, we shall strictly use this quasi-orthonormal tetrad given in eqs.(15) and (16) and nothing else. In this sense, our choice
of local tetrad frame is slightly different from that in the original work of Damour [6] in which he introduced, particularly on the 2-dimensional, \( v = \text{const.} \) section of the event horizon, some other orthonormal basis (slightly different from \( \{ e_2^\mu, e_3^\mu \} \) given above) specially adapted to the “intrinsic geometry” of the \( v = \text{const.} \) section of the horizon and used them to project out physical components of tensors.

Before we proceed, we momentarily recall the “Zero-Angular-Momentum-Observer (ZAMO)” tetrad in Boyer-Lindquist coordinates for the sake of comparison with this quasi-orthonormal tetrad in ingoing Kerr coordinates. The dual of ZAMO tetrad is given by

\[
\tilde{e}_A = \left( \tilde{e}_0 = \tilde{e}_0(t), \tilde{e}_1 = \tilde{e}_1(r), \tilde{e}_2 = \tilde{e}_2(\theta), \tilde{e}_3 = \tilde{e}_3(\tilde{\phi}) \right),
\]

\[
\tilde{e}_0 = \frac{1}{\alpha} \left[ \partial_t + \Omega \partial_{\tilde{\phi}} \right],
\]

\[
\tilde{e}_1 = \left( \frac{\Delta}{\Sigma} \right)^{1/2} \partial_r,
\]

\[
\tilde{e}_2 = \frac{1}{\Sigma^{1/2}} \partial_\theta,
\]

\[
\tilde{e}_3 = \left( \frac{\Sigma}{A} \right)^{1/2} \frac{1}{\sin \theta} \partial_{\tilde{\phi}}
\]

(18)

where \( \alpha^2 = (\Sigma\Delta/A), A = [(r^2 + a^2) - \Delta a^2 \sin^2 \theta], \Omega = -g_{t\tilde{\phi}}/g_{\tilde{\phi}\tilde{\phi}} = 2Ma/A \) ith \( \alpha \) being the lapse function. As is well-known, this ZAMO tetrad, particularly \( \tilde{e}_0(t) \) exhibits pathological behavior as the horizon is approached, i.e., in the limit, \( \Delta \to 0 \) or \( \alpha \to 0 \) and it can be attributed to the Boyer-Lindquist coordinate system itself as it is ill-defined at the horizon. As a matter of fact, this is precisely the reason why we (and originally Damour) choose to work in ingoing Kerr coordinates and further employ half-null, half-orthogonal tetrad instead despite its seemingly complex structure. ZAMO is a “FIDO” (fiducial observer) and \( \tilde{e}_0(t) = u^\mu \) (in \( \tilde{e}_0(t) = \tilde{e}_0^\mu(t) \partial_\mu \)) is its 4-velocity whose pathological behavior near the horizon needs to be regularized, for instance, by \( \tilde{e}_0(t) \to \alpha \tilde{e}_0(t) = (\partial/\partial t)^\mu + \Omega(\partial/\partial \tilde{\phi})^\mu \). Obviously, this regularized 4-velocity of ZAMO becomes, at the horizon, Killing vector normal to the horizon, \( \tilde{\chi}^\mu = (\partial/\partial t)^\mu + \Omega_H(\partial/\partial \tilde{\phi})^\mu \) (with \( \Omega_H = a/(r_+^2 + a^2) \) being the angular velocity of the horizon) and hence is null. Only after this regularization, the (dual) of ZAMO tetrad is now made to be well-defined at the future horizon and then can be used, say, to read out
physical components of a given tensor via eq.(5).

We now consider this standard procedure toward the study of electrodynamics in the background of Kerr black hole geometry by employing, instead, Damour’s quasi-orthonormal tetrad in ingoing Kerr coordinates. The first thing that can be noticed is the fact that the 4-velocity of a local observer in this quasi-orthonormal frame, \( e_\hat{\nu}^\mu \) (in \( e_\hat{\nu} = e_\hat{\nu}^\mu \partial_\mu \)) becomes, at the horizon where \( \Delta = 0 \), at once, the usual Killing vector normal to the horizon, \( \chi^\mu = (\partial/\partial v)^\mu + \Omega_H (\partial/\partial \phi)^\mu \) which has no pathological behavior whatsoever. Thus we do not need any \textit{ad hoc} regularization prescription. As we have carefully discussed earlier in the derivation of Hawking-Hartle (or Teukolsky) tetrad, it is interesting to note that this regular behavior of the 4-velocity at the horizon is due to neither the choice of ingoing Kerr coordinates (which is known to be well-behaved at the horizon) nor the employment of the (half) null nature of the tetrad but really due to the action of “null rotation III” in which particularly the associated orthonormal tetrad is boosted in the \( e_\hat{r} \)-direction with 3-velocity \( (\Lambda^2 - 1)/(\Lambda^2 + 1) \) with \( \Lambda = \Delta/2(r^2 + a^2) \). Namely the lesson there was that simply taking null tetrad is not enough and in order to get a well-behaved null tetrad at the horizon, one needs to boost it by an amount that becomes suitably infinite on the horizon. Having a well-behaved tetrad (at the horizon) in our possession, we now can proceed and calculate “physical” components of any given tensor by projecting its components onto this quasi-orthonormal tetrad frame. To summarize, in this comparison between the choice of ZAMO in Boyer-Lindquist coordinates and that of Damour’s quasi-orthonormal tetrad in ingoing Kerr coordinates, it appears that the latter is physically more relevant in that it has been constructed in a more natural manner than the former which, for regularity at the horizon, involves somewhat \textit{ad hoc} prescription. Thus in the present work, we choose to work with Damour’s quasi-orthonormal tetrad in ingoing Kerr coordinates and try to read out physical components of all tensors involved. For example, physical components of Maxwell field strength and electric current 4-vector will be identified with \( F_{AB} = F_{\mu \nu} (e_A^\mu e_B^\nu) \), \( J^A = J^\mu e^A_\mu \) respectively.
III. Identification of electric and magnetic fields on the horizon

As stated, our major concern in the present work is the derivation of curious boundary conditions for electromagnetic fields at the horizon but in the context of non-linear BI electrodynamics. As we shall see in a moment, the highly non-linear BI equations can be made to take on a seemingly linear structure similar to that of Maxwell equations. And to this end, we need to introduce two species of field strength tensors: the new one $G_{\mu\nu}$ for the inhomogeneous BI field equations and the usual one $F_{\mu\nu}$ for the homogeneous Bianchi identity. Despite this added technical complexity, however, the basic field quantities, namely the physical (finite and non-zero) components of electric field and magnetic induction can still be extracted from the standard field strength $F_{\mu\nu}$. Thus before we go on, it might be relevant to remind two alternative typical procedures by which one can read off physical components of electric field and magnetic induction from $F_{\mu\nu}$, generally. Obviously, the first procedure involves the projection of components of $F_{\mu\nu}$ onto the orthonormal tetrad frame chosen, $F_{AB} = F_{\mu\nu} (e^\mu_A e^\nu_B)$ as given above. Since $A,B$ are now tangent space indices in this locally-flat tetrad frame, the physical electric and magnetic field components then can be read off in a standard manner as

$$F_{AB} = \{F_{i0}, F_{ij}\}$$

where

$$E_i = F_{i0}, \quad B_i = \frac{1}{2} \epsilon_{ijk} F^{jk} = \frac{1}{2} \epsilon_{0ijk} F^{jk} = \tilde{F}_{0i} = -\tilde{F}_{i0}.$$  \hspace{1cm} (19)

The second alternative but equivalent procedure can be described as follows. Consider a family of fiducial observers (FIDOs) whose worldlines are a congruence of timelike curves orthogonal to spacelike hypersurfaces. Let $u^\mu$ be the 4-velocity of a FIDO normalized as $u^\alpha u_\alpha = -1$. Since, by definition, all the physically meaningful measurements should be made by these FIDOs, one can expect that the local values of the electric and magnetic fields measured by a FIDO with 4-velocity would be given by
\[ E^\alpha = F^{\alpha \beta} u_\beta, \quad \text{(20)} \]
\[ B^\alpha = -\frac{1}{2} \epsilon^{\alpha \beta \lambda \sigma} u_\beta F_{\lambda \sigma} = -\tilde{F}^{\alpha \beta} u_\beta \]

or

\[ F^{\alpha \beta} = u^\alpha E^\beta - E^\alpha u^\beta + \epsilon^{\alpha \beta \lambda \sigma} u_\lambda B_\sigma. \]

In addition, since both the electric and the magnetic fields are purely spatial vectors, one may expect

\[ u_\alpha E^\alpha = 0 = u_\alpha B^\alpha. \quad \text{(21)} \]

Finally, we project \( E^\mu = (0, E^i) \) and \( B^\mu = (0, B^i) \) onto the orthonormal tetrad frame associated with this FIDO to get their physical (finite and non-zero) components

\[ E^A = \epsilon^A_\mu E^\mu, \quad B^A = \epsilon^A_\mu B^\mu \quad \text{(22)} \]

then \( E_i = E^i, B_i = B^i \) where \( A = (0, i) \). Thus in order to evaluate physical components of electric field and magnetic induction \((E_i, B_i)\), one may choose between these two procedures and in the present work, where the quasi-orthonormal tetrad of Damour is already available, we shall employ the first procedure for actual calculations.

1. Brief review of BI electrodynamics in curved spacetimes

Eventually for the exploration of boundary conditions for BI electromagnetic fields at the horizon of Kerr hole, we now briefly describe general formulation of BI theory in a given curved spacetime. Since this BI theory of electromagnetism is, despite its long history and physically interesting motivations behind it, not well-known and hence might be rather unfamiliar to relativists and workers in theoretical astrophysics community, we provide an introductory review of BI theory in flat spacetime in Appendix. For a recent study of this flat spacetime BI theory particularly in modern field theory perspective, we also refer the reader to [15]. In our discussion below, however, we are implicitly aimed at adapting the theory to the formulation of electrodynamics in a rotating uncharged black hole spacetime. Also at this point in seems worthy of mention that throughout, we will be assuming the “weak
field limit”. To be a little more concrete, we consider the dynamics of electromagnetic field governed by the BI theory in the background of uncharged Kerr black hole spacetime. And we assume that the strength of this external electromagnetic field is small enough not to have any sizable backreaction to the background geometry. Then this means we are not considering a phenomena described by solutions in coupled full Einstein-BI theory but an environment where the test electromagnetic field possesses dynamics governed by the BI theory rather than by the Maxwell theory. Also note that this assumption can be further justified as long as we confine our concern to the electrodynamics around the “uncharged” Kerr black hole. If, instead, one is interested in the same physics but in charged rotating black holes (which, however, is rather uninteresting since it is less likely to happen in realistic astrophysical environments where the black hole charge, if any, gets quickly neutralized by the surrounding plasma), one would have to deal with the full Einstein-BI theory in which, unfortunately, the charged rotating black hole solution is not available. 

Thus we consider here the action of (4-dimensional) BI theory in a fixed background spacetime with metric $g_{\mu\nu}$. And to do so, some explanatory comments might be relevant. Coupling the BI gauge theory to gravity is not so familiar and hence we start first with the BI theory action in 4-dim. flat spacetime.

$$S = \int d^4x \frac{1}{4\pi} \beta^2 \left[ 1 - \sqrt{-\det \left( \eta_{\mu\nu} + \frac{1}{\beta} F_{\mu\nu} \right)} \right]$$

and then elevate it to its curved spacetime version by employing the minimal coupling scheme. This is really the conventional procedure and the result is

$$S = \int d^4x \sqrt{g} \left\{ \frac{1}{4\pi} \beta^2 \left[ 1 - \sqrt{1 + \frac{1}{2\beta^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{16\beta^4} \left( F_{\mu\nu} F^{\mu\nu} \right)^2} \right] + J^\mu A_\mu \right\} \quad (23)$$

where $J^\mu = \rho u^\mu + j_e^\mu$ is the electric source current for the vector potential $A_\mu$. Here, the generic parameter of the theory “$\beta$” having the canonical dimension $\text{dim}[\beta] = \text{dim}[F_{\mu\nu}] = +2$, probes the degree of deviation of BI theory from the standard Maxwell theory as the
limit $\beta \to \infty$ obviously corresponds to the Maxwell theory action. Now extremizing this action with respect to $A_\mu$ yields the dynamical BI field equation

$$\nabla_\nu \left[ \frac{F^{\mu\nu} - \frac{1}{4\beta^2} (F_{\alpha\beta} \tilde{F}^{\alpha\beta}) \tilde{F}^{\mu\nu}}{\sqrt{1 + \frac{1}{2\beta^2} (F_{\alpha\beta} F^{\alpha\beta}) - \frac{1}{16\beta^4} (F_{\alpha\beta} F^{\alpha\beta})^2}} \right] = 4\pi J^\mu \tag{24}$$

while the Bianchi identity, which is a supplementary equation to this field equation is given by

$$\nabla_\nu \tilde{F}^{\mu\nu} = \frac{1}{\sqrt{g}} \partial_\nu \left[ \sqrt{g} \tilde{F}^{\mu\nu} \right] = 0 \tag{25}$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$ is the Hodge dual of $F_{\mu\nu}$. Note that this Bianchi identity is just a geometrical equation independent of the detailed nature of a gauge theory action. Thus it remains the same as that in Maxwell theory. For later use, we also provide the energy-momentum tensor of this BI theory,

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}$$

$$= \frac{1}{4\pi} \left\{ \beta^2 (1 - R) g_{\mu\nu} + \frac{1}{R} \left[ F_{\mu\alpha} F_\nu^\alpha - \frac{1}{4\beta^2} (F_{\alpha\beta} \tilde{F}^{\alpha\beta}) F_{\mu\alpha} \tilde{F}_\nu^\alpha \right] \right\} \tag{26}$$

where $R \equiv \left[ 1 + \frac{1}{2\beta^2} (F_{\alpha\beta} F^{\alpha\beta}) - \frac{1}{16\beta^4} (F_{\alpha\beta} F^{\alpha\beta})^2 \right]^{1/2}$. Now the first thing that one can readily notice in this rather unfamiliar BI theory of electrodynamics might be the fact that even in the absence of the source current, the dynamical BI field equation and the geometrical Bianchi identity clearly are not dual to each other under $F_{\mu\nu} \to \tilde{F}_{\mu\nu}$ and $\tilde{F}_{\mu\nu} \to -F_{\mu\nu}$. Obviously, this is in contrast to what happens in the standard Maxwell theory and can be attributed to the fact that when passing from the Maxwell to this highly non-linear BI theory, only the dynamical field equation undergoes non-trivial change (“non-linearization”) and the geometrical Bianchi identity, as pointed out above, remains unchanged. Therefore in order to deal with this added complexity properly and formulate the BI theory in curved background spacetime in a manner parallel to that for the standard Maxwell theory, we find it relevant to introduce another field strength $G_{\mu\nu}$ which, however, is made up of $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$. To be more precise, consider introducing, for the inhomogeneous BI field equation,
\[ G_{\mu\nu} = \frac{1}{R} \left[ F_{\mu\nu} - \frac{1}{4\beta^2} (F_{\alpha\beta} \tilde{F}^{\alpha\beta}) \tilde{F}_{\mu\nu} \right] \] (27)

and defining the associated fields on each spacelike hypersurfaces, \((D^\alpha, H^\alpha)\) as

\[ D^\alpha = G^{\alpha\beta} u_\beta, \] (28)
\[ H^\alpha = -\frac{1}{2} \epsilon^{\alpha\beta\lambda\sigma} u_\beta G_{\lambda\sigma} = -\tilde{G}^{\alpha\beta} u_\beta \]

which also implies their purely spatial nature

\[ u_\alpha D^\alpha = 0 = u_\alpha H^\alpha. \] (29)

As before, \(u^\mu\) here is the 4-velocity of FIDO (or more precisely ZAMO for rotating Kerr geometry) having a timelike geodesic orthogonal to spacelike hypersurfaces. Then the inhomogeneous BI field equation now takes the form

\[ \nabla_\nu G^{\mu\nu} = 4\pi J^\mu \] (30)

which relates the fields \((D^\mu, H^\mu)\) as defined above to “free” charge and current \(J^\mu = \rho_e u^\mu + j^\mu_e\). Despite this extra elaboration, the fundamental field quantities, namely the electric field and the magnetic induction still can be identified with

\[ E^\alpha = F^{\alpha\beta} u_\beta, \] (31)
\[ B^\alpha = -\frac{1}{2} \epsilon^{\alpha\beta\lambda\sigma} u_\beta F_{\lambda\sigma} = -\tilde{F}^{\alpha\beta} u_\beta \]

which, as before, implies \(u_\alpha E^\alpha = 0 = u_\alpha B^\alpha\). Thus the homogeneous Bianchi identity equation

\[ \nabla_\nu \tilde{F}^{\mu\nu} = 0 \] (32)

is expressible in terms of usual \((E^\mu, B^\mu)\) fields. Then in this new representation of a set of BI equations, we now imagine their space-plus-time decomposition. Obviously, the dynamical BI field equation would split up into two inhomogeneous equations involving \((D^\mu, H^\mu)\) and the “free” source charge and current \(J^\mu = \rho_e u^\mu + j^\mu_e\) whereas the geometrical Bianchi identity
equation decomposes into two homogeneous equations involving \((E^\mu, B^\mu)\). Incidentally, one can then realize that this indeed is reminiscent of Maxwell equations in a “medium”. Namely, in this new representation, the BI theory of electrodynamics can be thought of as taking on the structure of ordinary Maxwell electrodynamics in a medium with non-trivial electric susceptibility and magnetic permeability. In this interpretation of the new representation of the BI theory, then, it is evident that the system is of course not linear in that \((D^\mu, H^\mu)\) and \((E^\mu, B^\mu)\) are related by

\[
D^\mu = \frac{1}{R} \left[ E^\mu + \frac{1}{\beta^2} (E_\alpha B^\alpha) B^\mu \right],
\]

\[
H^\mu = \frac{1}{R} \left[ B^\mu - \frac{1}{\beta^2} (E_\alpha B^\alpha) E^\mu \right],
\]

or inversely

\[
E^\mu = \frac{1}{R} \left[ D^\mu - \frac{1}{\beta^2} (D_\alpha H^\alpha) H^\mu \right],
\]

\[
B^\mu = \frac{1}{R} \left[ H^\mu + \frac{1}{\beta^2} (D_\alpha H^\alpha) D^\mu \right],
\]

with

\[
R = \left[ 1 - \frac{1}{\beta^2} (E_\alpha E^\alpha - B_\alpha B^\alpha) - \frac{1}{\beta^4} (E_\alpha B^\alpha)^2 \right]^{1/2},
\]

or

\[
R = \left[ 1 - \frac{1}{\beta^2} (H_\alpha H^\alpha - D_\alpha D^\alpha) - \frac{1}{\beta^4} (D_\alpha H^\alpha)^2 \right]^{1/2},
\]

where we used eqs.\((27),(28)\) and \((31)\) and \(u^\alpha u_\alpha = -1\), \(F_{\alpha\beta} \tilde{F}^{\alpha\beta} = -2(E_\alpha E^\alpha - B_\alpha B^\alpha)\) and \(F_{\alpha\beta} \tilde{F}^{\alpha\beta} = 4E_\alpha B^\alpha\). It is also noteworthy from above expressions that

\[
E_\alpha B^\alpha = D_\alpha H^\alpha.
\]

Thus from now on, we may call \(D^\mu = (0, D^i)\) as the “electric displacement” 4-vector and \(H^\mu = (0, H^i)\) as the “magnetic field strength” 4-vector.

2. Electric field and magnetic induction on the horizon

Earlier, we mentioned that we shall employ, between the two alternative procedures to evaluate “physical” components of electric field and magnetic induction, the first one. In the context of BI theory of electrodynamics, however, there are a set of fields \(D^\mu = (0, D^i = D_i)\),

\( \mu^\mu = (0, H^i = H_i) \) in addition to \( E^\mu = (0, E^i = E_i) \), \( B^\mu = (0, B^i = B_i) \). Then we shall first evaluate \((D_i, H_i)\) on the horizon and then from them identify \((E_i, B_i)\) afterwards.

With respect to Damour’s quasi-orthonormal tetrad, the physical components of electric displacement \( D_i \) and magnetic field strength \( H_i \) can be read off as

\[
G_{AB} = G_{\mu\nu}(\epsilon_A^\mu \epsilon_B^\nu) \quad \text{and} \quad D_i = G_{i0}, \quad H_i = \frac{1}{2} \epsilon_{ijk} G^{jk} = -\tilde{G}_{i0}.
\]  

(36)

More concretely, since we are working in ingoing Kerr coordinates \((v, r, \theta, \phi)\), the components of electric displacement on the horizon can be read off as

\[
D_r = D_1 = G_{10} = G_{\mu\nu}(\epsilon_1^\mu \epsilon_0^\nu)|_{r_+} = \left( \frac{r_+^2 + a^2}{\Sigma_+} \right) \left[ G_{rv} + \frac{a}{(r_+^2 + a^2)} G_{r\phi} \right],
\]

\[
D_\theta = D_2 = G_{20} = G_{\mu\nu}(\epsilon_2^\mu \epsilon_0^\nu)|_{r_+} = \frac{1}{\Sigma_+^{1/2}} \left[ G_{\theta v} + \frac{a}{(r_+^2 + a^2)} G_{\theta \phi} \right],
\]

\[
D_\phi = D_3 = G_{30} = G_{\mu\nu}(\epsilon_3^\mu \epsilon_0^\nu)|_{r_+} = \frac{\Sigma_+^{1/2}}{(r_+^2 + a^2) \sin \theta} G_{\phi v}
\]

(37)

where \( \Sigma_+ \equiv (r_+^2 + a^2 \cos^2 \theta) \). Next, the components of magnetic field strength again on the horizon can be read off as

\[
H_r = H_1 = -\tilde{G}_{10} = G^{23} = G_{23}
\]

\[
= G_{\mu\nu}(\epsilon_2^\mu \epsilon_3^\nu)|_{r_+} = \frac{1}{\Sigma_+ \sin \theta} [a \sin^2 \theta G_{\theta v} + G_{\theta \phi}],
\]

\[
H_\theta = H_2 = -\tilde{G}_{20} = G^{31} = G_{30}
\]

\[
= D_\phi = \frac{\Sigma_+^{1/2}}{(r_+^2 + a^2) \sin \theta} G_{\phi v},
\]

\[
H_\phi = H_3 = -\tilde{G}_{30} = G^{12} = G_{02}
\]

\[
= -D_\theta = -\frac{1}{\Sigma_+^{1/2}} \left[ G_{\theta v} + \frac{a}{(r_+^2 + a^2)} G_{\theta \phi} \right]
\]

(38)

where we used the Damour’s quasi-orthonormal tetrad metric

\[
ds^2 = 2e^0 e^1 + e^2 e^2 + e^3 e^3 = \epsilon_{AB} e^A e^B
\]
to deduce
\[ G^{23} = \epsilon^2 A \epsilon^3 B G_{AB} = G_{23}, \]
\[ G^{31} = \epsilon^3 A \epsilon^1 B G_{AB} = G_{30}, \]
\[ G^{12} = \epsilon^1 A \epsilon^2 B G_{AB} = G_{02}. \]

Thus it is interesting to note that on the horizon \( H = D \hat{\theta} \) and \( H = -D \hat{\phi} \) or in a vector notation in a tangent space to the horizon,
\[ \vec{H} = \vec{D} \times \hat{n} \quad (39) \]

where \( \hat{n} = \hat{r} \) is the vector (outer) normal to the horizon. This relation indicates that \( \{ \vec{H}, \vec{D}, \hat{n} \} \) form a “triad” on the horizon and hence constitutes the so-called “radiative ingoing (or, inward Poynting flux)” boundary condition at horizon as seen by a local observer at rest in the quasi-orthonormal tetrad frame. Here, however, it seems worthy of note that although this relation is one of the horizon boundary conditions eventually we are after, it has not been obtained essentially from the horizon specifics. As a matter of fact, it holds for any \( r = \text{const.} \) sections and indeed its emergence can be attributed to the “half-null” \( (e_0^\mu = l^\mu, \ e_1^\mu = -n^\mu) \) structure of Damour’s quasi-orthonormal tetrad. Given the observation that the same type of relation as this “radiative ingoing boundary condition” actually holds for any null surface, one might wonder what then would be the distinctive nature of the event horizon (among null surfaces) that actually endows this boundary condition with real physical meaning. Znajek [5] provided one possible answer to this question and it is: the special feature of the event horizon over all other null surfaces is that it is a “stationary” null surface and there is a natural class of time coordinates associated with the frame at infinity in which the black hole is at rest. And the physical components of electric and magnetic fields should be evaluated, in a unique way, in a frame at rest on the horizon. At this point, we remark on another crucial thing happening at the horizon. Namely we note that at the horizon,
\[ D_\alpha H^\alpha = g_{\alpha \beta} D^\alpha H^\beta = (\epsilon_{AB} e_\alpha^A e_\beta^B) D^\alpha H^\beta \]
\[ \epsilon_{AB} D^A H^B = D^0 H^1 + D^1 H^0 + D^2 H^2 + D^3 H^3 = D^2 D^3 - D^3 D^2 = 0, \]

\[ D_\alpha D^\alpha = g_{\alpha\beta} D^\alpha D^\beta = (\epsilon_{AB} \epsilon^A_C \epsilon^B_D) D^\alpha D^\beta \]

\[ = \epsilon_{AB} D^A D^B = D^0 D^1 + D^1 D^0 + D^2 D^2 + D^3 D^3 = D^2 D^2 + D^3 D^3 \]

\[ = H^3 H^3 + H^2 H^2 = \epsilon_{AB} H^A H^B = g_{\alpha\beta} H^\alpha H^\beta = H_\alpha H^\alpha. \]

These relations also hold not only at the horizon but on any \( r = \text{const.} \) sections and again can be attributed to the half-null nature of Damour’s quasi-orthonormal tetrad. One immediate consequence of these relations \( D_\alpha H^\alpha = 0 \) and \( D_\alpha D^\alpha = H_\alpha H^\alpha \) everywhere is that practically \( E^\mu = D^\mu \) and \( B^\mu = H^\mu \) everywhere (due to eqs. (33) and (34)) as seen by a local observer at rest in the quasi-orthonormal tetrad frame. In fact, the interpretation of this is straightforward. Since Damour’s quasi-orthonormal tetrad is half-null in \((v - r)\) sector, an observer in this tetrad frame is actually a null observer who, as a result of his motion, would see the electromagnetic field around him as a “radiation field” all the way which, in turn, turns the BI theory of electrodynamics effectively into the Maxwell theory. What is particularly remarkable concerning this study of electrodynamics in the background of Kerr black hole in the context of BI theory is that the nature of the theory or the concrete structure of BI equations happens to be such that it is indistinguishable to a local observer in Damour’s quasi-orthonormal tetrad frame (indeed to any null observers) from that of standard Maxwell theory. This point is indeed quite amusing on theoretical side. From now on, then, whenever we deal with quantities involving physical components of fields as seen by this observer in Damour’s tetrad frame, we can freely replace \((D^\mu(E^\mu), H^\mu(B^\mu))\) by \((E^\mu(D^\mu), B^\mu(H^\mu))\). Thus the radiative ingoing boundary condition at the horizon obtained above can be given in terms of electric field and magnetic induction as

\[ \bar{B}_H = \bar{E}_H \times \hat{n}. \]

As pointed out earlier, this relation states that the electric and magnetic fields tangential to the horizon are equal in magnitude and perpendicular in direction and hence their Poynting energy flux is into the hole. This boundary condition as seen by a local observer again in a
null tetrad frame (which has been made to be well-behaved at the horizon by the amount of boost that becomes suitably infinite at the horizon) has been derived first by Znajek [5] in the context of standard Maxwell theory and here we just witnessed that precisely the same radiative ingoing boundary condition holds in the BI theory context as well.

IV. (Fictitious) Charge and current on the horizon

It is well appreciated that in any attempt to have an intuitive picture of Blandford-Znajek process for the rotational energy extraction from rotating black holes, the introduction of surface charge and current density on the (stretched) horizon proves to be quite convenient. For instance, the circuit analysis in the membrane paradigm [11] cannot do without the notion of the horizon surface charge and current density. If one follows the original argument of Damour [6], one can justify their introduction as follows. Suppose the existence of a 4-current $J^\mu(v, r, \theta, \phi)$ which is defined and conserved all over the spacetime. Let $r = r_+$ be the location of an event horizon, then obviously some charge and current can plunge into the hole and disappear from the region $r > r_+$. Nevertheless, imagine that we do not want to consider what happens inside the black hole ($r < r_+$) and just wish to keep the charge and current conserved in the region $r > r_+$. Then we would have to endow the surface $r = r_+$ with charge and current densities in such a way that the real current outside the horizon and this fictitious current on the horizon together can complete the circuit. Then the task of constructing the horizon surface current can be described as a mathematical problem as follows: “Given the bulk current $J^\mu(v, r, \theta, \phi)$ such that $\nabla_\mu J^\mu = 0$, find a complementary boundary (surface) current $j^\mu$ on the surface $r = r_+$ such that $I^\mu \equiv [J^\mu Y(r - r_+) + j^\mu]$ (where $Y(r)$ is the Heaviside function defined by $dY(r) = \delta(r)dr$) is conserved.” And in this problem, a crucial point to be noted is that the conservation of the bulk current $J^\mu$ is ensured by the field equation. Obviously then, what changes from the ordinary Maxwell theory case to the present BI theory case is that now the conservation of $J^\mu$ is secured by the inhomogeneous BI field equation instead of the Maxwell equation, i.e.,

$$\nabla_\nu G^{\mu\nu} = 4\pi J^\mu$$

(42)
implies $\nabla_\mu J^\mu = \nabla_\mu \nabla_\nu G^{\mu \nu}/4\pi = 0$ outside the horizon. Then the condition for the conservation of the total current $I^\mu$ reads

$$0 = \nabla_\mu I^\mu = \nabla_\mu [J^\mu Y(r - r_+) + j^\mu]$$

$$= \frac{1}{4\pi} (\nabla_\nu G^{\mu \nu}) \left( \frac{x_\mu}{r} \right) \delta(r - r_+) + \nabla_\mu j^\mu$$ (43)

where we used $\nabla_\nu G^{\mu \nu} = 4\pi J^\mu$, $\nabla_\mu J^\mu = 0$ and $\partial_\mu Y = (x_\mu/r)\delta(r - r_+)$. Obviously, this equation is solved by the complementary surface current given as

$$j^\mu = \frac{1}{4\pi} G^{\mu r}(\partial_r r)\delta(r - r_+)$$

$$\equiv \frac{1}{4\pi} G^{\mu r}\delta(r - r_+).$$ (44)

Further, it is convenient to introduce a “Dirac distribution” $\delta_H$ on the horizon normalized with respect to the time at infinity $v$ and the local proper area $dA$ such that

$$\int d^4x \sqrt{g} \delta_H \delta(v - v_0) f(v, r, \theta, \phi) = \int_{H} dA f(v_0, r_+, \theta, \phi)$$ (45)

which, then, yields

$$\delta_H = \frac{(r_+^2 + a^2)}{\Sigma_+} \delta(r - r_+)$$ (46)

where we used $\sqrt{g} = \Sigma \sin \theta$ and $dA = (r_+^2 + a^2) \sin \theta d\theta d\phi$. Finally, then, the complementary surface current 4-vector on the horizon can be written as $j^\mu = \kappa^\mu \delta_H$ with

$$\kappa^\mu = \frac{1}{4\pi} \frac{\Sigma_+}{(r_+^2 + a^2)} G^{\mu r}_+.$$ (47)

As usual, what matters is the identification of “physical” (i.e., finite and non-zero) components of this current 4-vector (i.e., the horizon charge and current density) as seen by an observer in our quasi-orthonormal tetrad frame. And they can be computed, using the dual of Damour’s mixed tetrad given in eq.(16), in a straightforward manner as

$$\sigma = \kappa^0 = \kappa^\mu e_{\mu}^0|_{r_+} = \frac{1}{4\pi} [G^{\mu r}_+ - a \sin^2 \theta G^{\mu \phi}_+]$$

$$= \frac{1}{4\pi} \left[ \frac{(r_+^2 + a^2)}{\Sigma_+} G_{r^0} + \frac{a}{\Sigma_+} G_{r^\phi} \right] = \frac{1}{4\pi} D_r,$$
\[ \kappa^\hat{r} = \kappa^1 = \kappa^\mu e^1_\mu |_{r_+} = 0, \]

\[ \kappa^\hat{\theta} = \kappa^2 = \kappa^\mu e^2_\mu |_{r_+} = \frac{1}{4\pi} \frac{\Sigma^{3/2}_+}{(r_+^2 + a^2)} G^{\theta r}_+ \]

\[ = \frac{1}{4\pi} \left[ \frac{1}{\Sigma^{1/2}_+} G_{\theta v} + \frac{a}{\Sigma^{1/2}_+ (r_+^2 + a^2)} G_{\theta \phi} \right] = \frac{1}{4\pi} D_\hat{\theta}, \]

\[ \kappa^\hat{\phi} = \kappa^3 = \kappa^\mu e^3_\mu |_{r_+} = \frac{1}{4\pi} \frac{\Sigma^{1/2}_+}{(r_+^2 + a^2)} \left[-a \sin \theta G^{\phi r}_+ + (r_+^2 + a^2) \sin \theta G^{\phi v}_+ \right] \]

\[ = \frac{1}{4\pi} \frac{\Sigma^{1/2}_+}{(r_+^2 + a^2) \sin \theta} G_{\phi v} = \frac{1}{4\pi} D_\hat{\phi} \]

where the subscript “+” denotes the value at the horizon \( r = r_+ \) and we compared these equations with eq.(37) to relate the surface charge and current densities to the components of electric displacement on the horizon.

**V. Ohm’s law, Gauss’ law, and Ampere’s law**

We now are in the position to demonstrate that, as results of central significance, a set of three relations, at the horizon, between the fields \( (D_i = E_i, H_i = B_i) \) and the surface charge and current densities \( (\sigma = \kappa^0, \kappa^i) \) that can be thought of as Ohm’s law, Gauss’ law and Ampere’s law valid at the horizon of a rotating Kerr black hole. First, notice that

\[ D_\hat{\theta} = 4\pi \kappa^\hat{\theta}, \quad D_\hat{\phi} = 4\pi \kappa^\hat{\phi}. \]

These relations can be rewritten in a vector notation in a tangent space to the horizon as

\[ \vec{D}_H = 4\pi \vec{\kappa} \quad \text{or} \quad \vec{E}_H = 4\pi \vec{\kappa} \]

and hence can be interpreted as the “Ohm’s law”. Namely, this relation precisely takes on the form of a non-relativistic Ohm’s law for a conductor and hence implies that if we endow the horizon with some charge and current densities which are to be determined by the surrounding external electromagnetic field \( F_{\mu \nu} \), then the horizon behaves as if it is a conductor with finite surface resistivity of

\[ \rho = 4\pi \simeq 377(\text{ohms}). \]
The derivation of Ohm’s law and this value of surface resistivity has been performed first by Damour [6] and by Znajek [5] independently in the context of standard Maxwell theory. Thus what is indeed remarkable here is that the Ohm’s law above and the value of horizon’s surface resistivity \((4\pi)\) remain unchanged even when we replace the Maxwell theory by the BI theory of electrodynamics. This result cannot be naturally anticipated but close inspection reveals that it can be attributed to the peculiar structure of highly non-linear inhomogeneous BI field equation given in eqs.(30) and (27) which, at the horizon, shows some magic such that there the \((\vec{D}, \vec{H})\) fields become exactly the same as \((\vec{E}, \vec{B})\) as seen by a local observer in Damour’s tetrad frame respectively as can be checked from eqs.(34) and (38) (or (40)). As Damour [6] pointed out, this result constitutes a clear confirmation of Carter’s assertion [10] that a black hole is analogous to an ordinary object having finite viscosity and electrical conductivity. Next, we also notice that

\[ D_\hat{r} = 4\pi\sigma \quad \text{or} \quad E_\hat{r} = 4\pi\sigma \] (52)

which evidently can be identified with the surface version of Gauss’ law. It says that the fictitious surface charge density we assumed on the horizon plays the role of terminating the normal components of all electric fields that pierce the horizon just as we want it to. Lastly, if we combine the radiative ingoing boundary condition at the horizon that we obtained earlier, \(\vec{H}_H = \vec{D}_H \times \hat{n} \) (or \(\vec{B}_H = \vec{E}_H \times \hat{n}\)) and the Ohm’s law above, \(\vec{D}_H = 4\pi\vec{\kappa}\) (or \(\vec{E}_H = 4\pi\vec{\kappa}\)), we end up with the third relation

\[ \vec{H}_H = 4\pi(\vec{\kappa} \times \hat{n}) \quad \text{or} \quad \vec{B}_H = 4\pi(\vec{\kappa} \times \hat{n}) \] (53)

which may be viewed as the surface version of Ampere’s law. Again, consistently with our motivation for introducing fictitious current density on the horizon, this relation indicates that the current density we assumed plays the role of terminating any tangential components of all magnetic fields penetrating the horizon. To summarize, for the reason stated earlier, even the highly non-linear BI theory of electrodynamics leads to the same horizon boundary conditions eqs.(41), (50), (52), and (53) as those in the standard Maxwell theory and indeed
this set of four curious boundary conditions on the horizon actually have provided the motivation for the proposal of membrane paradigm [11] of black holes later on.

VI. Joule’s law or Ohmic dissipation at the horizon

Perhaps one of the most intriguing consequences of assuming the existence of fictitious charge and current densities on the horizon would be that if we choose to do so, the horizon behaves as if it is a conductor with finite conductivity as we stressed in the previous section. Since it is the surrounding external electromagnetic field that drives the surface currents on the horizon, one might naturally wonder what would happen to the Joule heat generated when those currents work against the surface resistance and how it would be related to the electromagnetic energy going down the hole through the horizon. Znajek and Damour also provided a simple and natural answer to this question. Namely, they showed in a consistent and elegant manner that the total electromagnetic energy flux (i.e., the Poynting flux) into the rotating Kerr hole through the horizon is indeed precisely the same as the amount of Joule heat produced by the surface currents when they work against the surface resistivity of $4\pi$. In the following, we shall demonstrate again along the same line of formulation as Damour that indeed the same is true even in the context of BI theory of electrodynamics. It is well-known that for a stationary, axisymmetric black hole spacetime with the horizon-orthogonal Killing field

$$\chi^\mu = (\partial/\partial v)^\mu + \Omega_H(\partial/\partial \phi)^\mu \equiv \xi^\mu + \Omega_H \psi^\mu$$

the mass-energy and the angular momentum flux into the hole through the horizon are given respectively by

$$\frac{dM}{dv} = \int_H dA T^\mu_\nu \xi^\nu \chi_\mu = \int_H dA T^\mu_v \chi_\mu$$

$$\frac{dJ_z}{dv} = -\int_H dA T^\mu_\nu \psi^\nu \chi_\mu = -\int_H dA T^\mu_\phi \chi_\mu$$

where $dA = (r_+^2 + a^2) \sin \theta d\theta d\phi$ is again the area element on the horizon and $T^\mu_\nu$ is the matter energy-momentum tensor at the horizon. Now, combining these with the 1st law of black hole thermodynamics [16]
\[ dQ = \frac{1}{8\pi} \kappa_H dA = dM - \Omega_H dJ_z \]  
(56)

where \( dQ \) denotes the heat dissipated in the hole not charge (recall that we only consider here uncharged Kerr black hole) with \( \kappa_H \) being the surface gravity [16] of the hole, one gets

\[ \frac{dQ}{dv} = \frac{dM}{dv} - \Omega_H \frac{dJ_z}{dv} \]
(57)

\[ = \int_H dA \, T^\mu_\nu (\xi^\nu + \Omega_H \psi^\nu) \chi_\mu = \int_H dA \, T^\mu_\nu \chi^\nu \chi_\mu. \]

Perhaps a word of caution might be relevant here. As we mentioned earlier, we are only interested in the “test” electromagnetic field whose dynamics is governed particularly by the BI theory in the “background” of uncharged Kerr black hole spacetime which is a solution to the vacuum Einstein equation. Therefore, as long as we confine our concern to the case with uncharged Kerr black hole physics, the 1st law of black hole thermodynamics given above still remains to be valid. If, instead, one is interested in the case with charged, rotating black hole physics, one would have to deal with the full, coupled Einstein-BI theory context and then there the associated 1st law should get modified to the extended version like the one given by Rasheed [17] recently. Now, since the “matter” for the case at hand is the BI electromagnetic field, we have at the horizon

\[ T^\mu_\nu \chi^\nu \chi_\mu |_{r_+} = \frac{1}{4\pi} \left\{ \beta^2 (1 - R) \chi^\alpha \chi_\alpha + \frac{1}{R} \left[ F^\alpha_\mu F^\alpha_\nu - \frac{1}{4\beta^2} (F^\alpha_\beta \tilde{F}^\alpha_\beta) F^\alpha_\mu \tilde{F}^\alpha_\nu \chi^\nu \chi_\mu \right] \right\} |_{r_+} \]
(58)

where \( R \) is as defined earlier and in the second line we used that at the horizon where \( g^\alpha_\beta \chi^\alpha \chi^\beta = \chi^\alpha \chi_\alpha = 0, \)

\[ F^\alpha_\beta F^\alpha_\beta = -2(E^\alpha E^\alpha - B^\alpha B^\alpha) = -2(D^\alpha D^\alpha - H^\alpha H^\alpha) = 0, \]

\[ F^\alpha_\beta \tilde{F}^\alpha_\beta = 4E^\alpha B^\alpha = 4D^\alpha H^\alpha = 0, \]

and hence

\[ R = \left[ 1 + \frac{1}{2\beta^2} (F^\alpha_\beta F^\alpha_\beta) - \frac{1}{16\beta^4} (F^\alpha_\beta \tilde{F}^\alpha_\beta)^2 \right]^{1/2} = 1 \]

which also yields, at the horizon, \( G^\mu_\nu = F^\mu_\nu. \) Recall that in the standard Maxwell theory,
\[ T_{\mu \nu} = \frac{1}{4\pi} [F_{\mu \alpha} F^\alpha_\nu - \frac{1}{4} g_{\mu \nu} (F_{\alpha \beta} F^{\alpha \beta})] \] (59)

and thus at the horizon, \( T_{\mu \nu} \chi^\mu \chi^\nu |_{r_+} = \frac{1}{4\pi} (F_{\mu \alpha} F^\alpha_\nu) \chi^\mu \chi^\nu |_{r_+} \), which is the same as its counterpart in BI theory obtained above. This means that, at the horizon, the amount of total electromagnetic energy flux into the hole turns out to be the same and hence indistinguishable between Maxwell theory and BI theory. Further,

\[
T_{\mu \nu} \chi^\mu \chi^\nu |_{r_+} = \frac{1}{4\pi} (F_{\mu \alpha} F^\alpha_\nu) \chi^\mu \chi^\nu |_{r_+} \]
(60)
\[
= \frac{1}{4\pi} \left\{ \left[ \frac{\Sigma_{+}^{1/2}}{r_+^2 + a^2} \sin \theta F_{\phi v} \right]^2 + \left[ \frac{1}{\Sigma_{+}^{1/2}} G_{\theta v} + \frac{a}{\Sigma_{+}^{1/2}(r_+^2 + a^2)} G_{\theta \phi} \right]^2 \right\} 
\]
\[
= 4\pi \left[ (\dot{r}^2) + (\dot{\theta})^2 \right] = 4\pi (\vec{\kappa})^2
\]

where we used \( G_{\mu \nu} = F_{\mu \nu} \) and \( \kappa^2 = 0 \) at the horizon. Thus, finally we end up with

\[
\frac{dQ}{dv} = \int_H dA \ T_{\mu \nu} \chi^\mu \chi^\nu 
= 4\pi \int_H dA \ (\vec{\kappa})^2 = \int_H dA \ (\vec{E}_H \cdot \vec{\kappa})
\]

where we used the Ohm’s law \( \vec{E}_H = 4\pi \vec{\kappa} \), we obtained earlier. As we promised to demonstrate, clearly this is the Joule’s law which is again precisely the same as its Maxwell theory counterpart originally obtained by Znajek [5] and by Damour [6] and implies that the absorption of electromagnetic energy by Kerr holes through the horizon can be translated into an equivalent picture in which the holes gain energy by absorbing Joule heat (or Ohmic dissipation) generated when the surface current \( \vec{\kappa} \) driven by the electric field \( \vec{E}_H \) works against the surface resistivity of \( 4\pi \). And as before, what is remarkable is the fact that even if we replace the Maxwell theory by the highly non-linear BI electrodynamics, the physics of the horizon such as this horizon thermodynamics as well as the horizon boundary conditions remain unchanged. And as we pointed out earlier, this has much to do with the nature of Damour’s quasi-orthonormal tetrad frame (i.e., its half-null structure) in ingoing Kerr coordinates.

VII. Concluding Remarks
In the present work, we have explored the interaction of electromagnetic fields with a rotating (Kerr) black hole in the context of Born-Infeld (BI) theory of electromagnetism and particularly we have derived BI theory versions of the four horizon boundary conditions of Znajek and Damour. Interestingly enough, as far as we employ the same local null tetrad frame as the one adopted in the works by Damour and by Znajek, we ended up with exactly the same four horizon boundary conditions despite the shift of the electrodynamics theory from a linear Maxwell one to a highly non-linear BI one. As we have seen in the text, this curious and unexpected result could be attributed to the fact that the concrete structure of BI equations happens to be such that it is indistinguishable at the horizon to a local observer, say, in Damour’s local tetrad frame from that of standard Maxwell theory. Finally, we have a word of caution to avoid a possible confusion the potential readers might have. Namely, again we point out that in all the calculations involved in this work to read off physical components of tensors such as Maxwell field tensor and current 4-vector, we strictly used the quasi-orthonormal tetrad given in eqs.(15) and (16) and nothing else. In this sense, our choice of local tetrad frame was slightly different from that in the original work of Damour [6] in which he introduced, particularly on the 2-dimensional, $v = \text{const.}$ section of the event horizon, some other orthonormal basis (slightly different from \{e_2^\mu, e_3^\mu\} given in eq.(15)) specially adapted to the “intrinsic geometry” of the $v = \text{const.}$ section of the horizon and used them to project out physical components of tensors. As such, any deviation of the results one may find in the expressions for the electric field, magnetic field and surface charge and current densities appeared in the text of the present work from their counterparts in the original work of Damour can be attributed to this slightly different choices of the local tetrad vectors. This discrepancy, however, is insensitive to the physical nature of this study of the horizon boundary conditions that we try to deliver in the present work.

Acknowledgements
Appendix : An Introduction to Born-Infeld Electrodynamics

The Born-Infeld (BI) theory may be thought of as a highly nonlinear generalization of or a non-trivial alternative to the standard Maxwell theory of electromagnetism. Here, we would like to present an introductory review of BI theory of electrodynamics particularly in modern field theory perspective. As usual, we begin with the action for this BI theory which is given, in 4-dimensions, by (in MKS unit)

\[
S = \int d^4x \left\{ \beta^2 \left[ 1 - \sqrt{\det(\eta_{\mu\nu} + \frac{1}{\beta^2} F_{\mu\nu})} + j^{\mu} A_{\mu} \right] \right\}
\]

where \( \beta \) is a generic parameter of the theory having the dimension \( \text{dim}[\beta] = \text{dim}[F_{\mu\nu}] = +2 \). It probes the degree of deviation of BI gauge theory from the standard Maxwell theory and obviously \( \beta \to \infty \) limit corresponds to the standard Maxwell theory. Again, extremizing this action with respect to \( A_{\mu} \) yields the dynamical BI field equation

\[
\partial_{\mu} \left[ \frac{F_{\mu\nu} - \frac{1}{4\beta^2} (F_{\alpha\beta} F^{\alpha\beta}) \tilde{F}_{\mu\nu}}{\sqrt{1 + \frac{1}{2\beta^2} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{16\beta^4} (F_{\alpha\beta} F^{\alpha\beta})^2}} \right] = -j^{\nu}.
\]

In addition to this, there is a supplementary equation coming from an identity satisfied by the abelian gauge field strength tensor, \( \partial_{\lambda} F_{\mu\nu} + \partial_{\mu} F_{\nu\lambda} + \partial_{\nu} F_{\lambda\mu} = 0 \). This is the Bianchi identity which is just a geometrical equation and in terms of the Hodge dual field strength, \( \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta} \), it can be written as

\[
\partial_{\mu} \tilde{F}_{\mu\nu} = 0
\]
And it seems noteworthy that the field equation for $A_\mu$ in eq.(63) is the dynamical field equation which gets determined by the concrete nature of the gauge theory action such as the one in eq.(62). The Bianchi identity in eq.(64), on the other hand, is simply a geometrical identity and is completely independent of the choice of the context of the gauge theory. Further, if one wishes to decompose these covariant equations, use $\partial^\mu = (-\partial/\partial t, \nabla_i)$, $\partial_\mu = \eta_{\mu\nu} \partial^\nu = (\partial/\partial t, \nabla_i)$ (namely, we use the sign convention, $\eta_{\mu\nu} = diag(-1,1,1,1)$), $A^\mu = (\phi, A^i)$ and the field identification, $E_i = F^i_0$, $B_i = \frac{1}{2} \epsilon_{ijk} F^{jk}$ or $F^{ij} = \epsilon_{ijk} B^k$, and write them in terms of $\vec{E}$ and $\vec{B}$ fields, to get

$$\nabla \cdot \left[ \frac{1}{R} (\vec{E} + \frac{1}{\beta^2} (\vec{E} \cdot \vec{B}) \vec{B}) \right] = \rho_e,$$

$$\nabla \times \left[ \frac{1}{R} (\vec{B} - \frac{1}{\beta^2} (\vec{E} \cdot \vec{B}) \vec{E}) \right] - \frac{\partial}{\partial t} \left[ \frac{1}{R} (\vec{E} + \frac{1}{\beta^2} (\vec{E} \cdot \vec{B}) \vec{B}) \right] = \vec{j}_e$$

where $R \equiv \sqrt{1 + \frac{1}{2\pi} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{16\pi^2} (F_{\alpha\beta} F^{\alpha\beta})^2} = \sqrt{1 - \frac{1}{\beta^2} (\vec{E}^2 - \vec{B}^2)} - \frac{1}{\beta^2} (\vec{E} \cdot \vec{B})^2$ for the dynamical BI field equation and

$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

for the geometrical Bianchi identity and where we used $F_{\mu\nu} F^{\mu\nu} = -2(\vec{E}^2 - \vec{B}^2)$ and $F_{\mu\nu} F^{\mu\nu} = 4\vec{E} \cdot \vec{B}$. We now start with some electrostatics described by this BI theory. As a simplest exercise, we look for the solution to these BI equations that represents a static electric monopole field. Next, the static electric monopole. It can be obtained from one of the dynamical field equations $\nabla \cdot \left[ \{ \vec{E} + (\vec{E} \cdot \vec{B}) \vec{B} / \beta^2 \} / R \right] = e \delta^3(\vec{r})$ with $R \equiv \left\{ 1 - (\vec{E}^2 - \vec{B}^2) / \beta^2 - (\vec{E} \cdot \vec{B})^2 / \beta^4 \right\}^{1/2}$. Again for $\vec{r} \neq 0$, and in spherical-polar coordinates, this equation becomes $[\partial_r (r^2 \sin\theta \hat{E}_r) + \partial_\theta (r \sin\theta \hat{E}_\theta) + \partial_\phi (r \hat{E}_\phi)] / r^2 \sin\theta = 0$ with $\hat{E}_i \equiv [E_i + (\vec{E} \cdot \vec{B}) B_i / \beta^2] / R$. In the absence of the magnetic field, $\hat{E}_i = E_i / \sqrt{1 - \vec{E}^2 / \beta^2}$ and then the above equation is solved by [15]

$$E_r = \frac{e}{4\pi \sqrt{r^4 + \left( \frac{e}{4\pi \beta} \right)^2}}, \quad E_\theta = E_\phi = 0.$$  

(67)

Since the static electric monopole field is not singular as $r \to 0$, the energy stored in the field of electric point charge could be finite and this point seems to be consistent with the
consideration of finiteness of energy, which is one of the motivations to propose this BI electrodynamics when it was first devised. Thus to see if this is indeed the case, consider the energy-momentum tensor of this BI theory

\[ T_{\mu\nu} = \beta^2 (1 - R) \eta_{\mu\nu} + \frac{1}{R} \left[ F_{\mu\alpha} F^\alpha_\nu - \frac{1}{4\beta^2} (F_{\alpha\beta} F^{\alpha\beta}) F_{\mu\alpha} \tilde{F}^\alpha_\nu \right] \]  

(68)

with \( R \) as given earlier. The energy density stored in the electromagnetic field can then be read off as

\[ T_{00} = \beta^2 \left[ \frac{1 + \frac{1}{\beta^2} B^2}{\sqrt{1 - \frac{1}{\beta^2} (E^2 - B^2) - \frac{1}{\beta^2} (E \cdot B)^2}} - 1 \right] \]  

(69)

which does reduce to its Maxwell theory’s counterpart \((E^2 + B^2)/2\) in the limit \( \beta \to \infty \) as it should. We are ready to calculate the energy density stored in the electric field generated by the electric charge \( e \). Using \( \vec{E} = \{ e/4\pi \sqrt{r^4 + (e/4\pi \beta)^2} \} \hat{r} \), one gets

\[ T_{00}^E = \beta^2 \left[ \frac{1}{\sqrt{1 - \frac{1}{\beta^2} E^2}} - 1 \right] = \beta^2 \left[ \sqrt{1 + \frac{e^2}{(4\pi \beta)^2} r^4} - 1 \right]. \]  

(70)

Then the electric monopole energy can be evaluated in a concrete manner as [15]

\[ E = \int d^3x T_{00}^E = \int_0^\infty dr \beta^2 \left[ \sqrt{(4\pi r^2)^2 + \frac{e^2}{\beta^2}} - 4\pi r^2 \right] 
\]

\[ = \frac{\beta e^3}{4\pi} \int_0^\infty dy \left( \sqrt{y^4 + 1} - y^2 \right) \]

\[ = \sqrt{\frac{\beta e^3}{4\pi}} \frac{\pi^{3/2}}{3\Gamma(\frac{3}{2})^2} = 1.23604978 \sqrt{\frac{\beta e^3}{4\pi}} \]  

(71)

where \( y^2 = (4\pi \beta/e^2) r^2 \) and in the \( y \)-integral, integration by part and the elliptic integral have been used. Remarkably, this energy is indeed finite as Born and Infeld hoped when they constructed this theory and if one takes the Maxwell theory limit \( \beta \to \infty \), one recovers divergent energy for a point electric charge as expected. Lastly we turn to some electrodynamics governed by this BI theory. In the dynamical BI field equations given earlier in eq.(69), we define, for the sake of convenience of the formulation, the “electric displacement” \( \vec{D} \) and the “magnetic field” \( \vec{H} \) in terms of the fundamental fields \( \vec{E} \) and \( \vec{B} \) as
\[
\vec{D} = \frac{1}{R} \left\{ \vec{E} + \frac{1}{\beta^2} (\vec{E} \cdot \vec{B}) \vec{B} \right\}, \quad \vec{H} = \frac{1}{R} \left\{ \vec{B} - \frac{1}{\beta^2} (\vec{E} \cdot \vec{B}) \vec{E} \right\}
\]

where \( R = \sqrt{1 - \frac{1}{\beta^2} (\vec{E}^2 - \vec{B}^2) - \frac{1}{\beta^4} (\vec{E} \cdot \vec{B})^2} \) is as defined earlier. Then the BI equations take the form

\[
\begin{align*}
\nabla \cdot \vec{D} &= \rho_e, \\
\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} &= \vec{j}_e \\
\nabla \cdot \vec{B} &= 0, \\
\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0.
\end{align*}
\]

Now \( \vec{E} \cdot (\text{Ampere's law eq.}) - \vec{H} \cdot (\text{Faraday's induction law eq.}) \) yields

\[
\vec{H} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{H}) = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} - \vec{j}_e \cdot \vec{E}.
\]

Further using

\[
\vec{H} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{H}) = \nabla \cdot (\vec{E} \times \vec{H}),
\]

\[
-\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = -\frac{\partial}{\partial t} T_{00}
\]

where \( T_{00} \) is the energy density stored in the electromagnetic field in BI theory given in eq.(69), one arrives at the familiar local energy conservation equation

\[
\nabla \cdot \vec{S} + \frac{\partial u}{\partial t} = -\vec{j}_e \cdot \vec{E}
\]

where \( u = T_{00} \) is the energy density, \( \vec{S} = \vec{E} \times \vec{H} \) is the “Poynting vector” representing the local energy flow per unit time per unit area and \(-\vec{j}_e \cdot \vec{E} \) on the right hand side is the power dissipation per unit volume. In particular for \( \vec{j}_e \cdot \vec{E} = 0 \), one gets

\[
\nabla \cdot \vec{S} + \frac{\partial u}{\partial t} = 0
\]

which is the equation of continuity for electromagnetic energy density with the BI theory version of the Poynting vector given by [15]

\[
\vec{S} = \vec{E} \times \vec{H} = \frac{\vec{E} \times \vec{B}}{\sqrt{1 - \frac{1}{\beta^2} (\vec{E}^2 - \vec{B}^2) - \frac{1}{\beta^4} (\vec{E} \cdot \vec{B})^2}}
\]

which obviously reduces to its Maxwell theory counterpart \( \vec{S} = \vec{E} \times \vec{B} \) in the limit \( \beta \to \infty \).
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