On the Minimum Energy Configuration of a Rotating Barotropic Fluid

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1. Introduction

In a recent paper (astro-ph/0207561, posted on July 25, 2002), Fromang & Balbus (2002, hereafter FB02) claim that the minimum energy configuration of a rotating fluid depends critically on whether or not any part of the fluid rotates supersonically. They state that “a uniformly rotating barotropic fluid in an external potential attains a true energy minimum if and only if the rotation profile is everywhere subsonic. If regions of supersonic rotation are present, fluid variations exist that could take the system to states of lower energy.”

We present here a simple counter-example which demonstrates that the stability criterion presented in FB02 is incorrect.

2. Minimum Energy Equilibria

We first repeat part of their analysis. Following FB02, we consider a barotropic fluid in which the pressure is a unique function of the density. To further simplify matters, and to be consistent with the analysis in the next section, we consider a polytropic equation of state in which the pressure $P$ and the density $\rho$ are related as

$$P = K\rho^\gamma.$$  \hspace{1cm} (1)

Here $K$ is a constant throughout the fluid and is not allowed to vary in any perturbation. Thus we exclude thermal effects. Furthermore, in line with FB02, we consider only perturbations that are adiabatic, which excludes convective instabilities.

For such a fluid, the internal energy per unit volume $\epsilon$, the enthalpy per unit mass $H$, and the adiabatic sound speed $a$, are given by

$$\epsilon = \frac{P}{\gamma - 1}, \quad H = \frac{\gamma}{\gamma - 1} \frac{P}{\rho}, \quad a^2 = \frac{\gamma P}{\rho}.$$ \hspace{1cm} (2)

We assume that the mass $M$ and the total angular momentum $J$ of the system are held fixed:

$$M = \int \rho dV = \text{constant},$$ \hspace{1cm} (3)

$$J = \int \rho R v dV = \text{constant},$$ \hspace{1cm} (4)

where $R$ is the cylindrical radius as measured perpendicular to the rotation axis and $v(R)$ is the azimuthal velocity of the gas at radius $R$. Note that, as do FB02, we take account throughout only of the azimuthal component of the velocity. The implication of this is that we restrict stability considerations to axi-symmetric perturbations alone.
We are interested in minimizing the energy, \( E \), of the system,

\[
E = \int \left( \frac{1}{2} \rho v^2 + \epsilon + \rho \phi \right) dV,
\]

where \( \phi \) is the (time-independent) external potential in which the fluid is confined. (Following FB02, we neglect self-gravity of the fluid.) The minimization of \( E \) subject to the constraints (3) and (4) is a standard problem that can be solved by variational methods. We minimize

\[
I = E - \zeta M - \Omega J = \int \left( \frac{1}{2} \rho v^2 + \epsilon + \rho \phi - \zeta \rho - \Omega \rho R v \right) dV,
\]

where \( \zeta \) and \( \Omega \) are Lagrange multipliers. To find an extremum of the energy, we set the first order variation of \( I \) with respect to changes in \( \rho(R) \) and \( v(R) \) to zero:

\[
\delta I = \int \left[ \left( \frac{1}{2} v^2 - \Omega R v + H + \phi - \zeta \right) \delta \rho + \rho(v - \Omega R) \delta v \right] dV = 0,
\]

where we have used equation (1) to set \( d\epsilon/d\rho = H \). Since \( \delta I \) should vanish for arbitrary choices of \( \delta \rho(R) \) and \( \delta v(R) \), the coefficients of \( \delta \rho(R) \) and \( \delta v(R) \) in the integrand of (7) should individually vanish at each \( R \). This gives the following two conditions:

\[
v = \Omega R, \quad \tag{8}
\]

\[
-\frac{1}{2} \Omega^2 R^2 + H + \phi = \zeta = \text{constant}. \quad \tag{9}
\]

The first condition states that a polytropic fluid in an energy-extremum configuration must be in a state of uniform rotation, a well-known result (e.g. Landau & Lifshitz). The second condition is nothing other than the condition of hydrostatic balance, as can be verified by taking the gradient of equation (9). We thus find that a polytropic fluid with fixed \( M \) and \( J \) and with an extremum in \( E \) has uniform rotation and is in hydrostatic equilibrium.

The analysis is identical to that in FB02, with the sole difference that we have specialized to the case of a polytropic equation of state. Following the above steps, FB02 proceed to take second differences of the energy, on the basis of which they reach the conclusions stated above in \( \S 1 \). We do not repeat their analysis since their final conclusions are incorrect, as we now demonstrate by means of a counter-example.

3. A Simple Counter-Example

We consider a polytropic fluid rotating around the \( z \) axis in a quadratic potential

\[
\phi(R, z) = \frac{1}{2} \Omega^2 K(R^2 + z^2). \quad \tag{10}
\]

We assume that the fluid is in an energy extremum state which, by the analysis of the previous section, means that the fluid rotates with a constant angular velocity \( \Omega \) and is
in hydrostatic equilibrium, satisfying equation (9). Let us assume that the fluid extends from \( R = 0 \) to a maximum radius \( R_0 \) on the equatorial plane. (Such a maximum radius always exists for \( \gamma > 1 \).) At the point \((R, z) = (R_0, 0)\), the enthalpy of the gas vanishes. This enables us to determine the constant \( \zeta \) in equation (9), using which we can show that

\[
H(R, z) = \frac{1}{2}(\Omega_K^2 - \Omega^2) \left[ R_0^2 - R^2 - \frac{\Omega_K^2}{(\Omega_K^2 - \Omega^2)} z^2 \right].
\]  

Not surprisingly, the fluid takes the form of a flattened spheroid, with equatorial radius \( R_0 \) and “vertical” radius \( Z_0 = \left( \frac{\Omega_K^2 - \Omega^2}{\Omega_K^2} \right)^{1/2} R_0 \).

From the expression for \( H \), we may calculate the density \( \rho \) of the fluid,

\[
\rho(R, z) = \left[ \frac{(\gamma - 1)H}{\gamma K} \right]^n,
\]

where we have defined the polytropic index \( n \) in the usual way:

\[
n = \frac{1}{\gamma - 1}.
\]

The vertically integrated surface density of the fluid is then

\[
\Sigma(R) = C(R_0^2 - R^2)^{n+1/2},
\]

\[
C = \frac{\Gamma(1/2)\Gamma(n + 1)}{\Gamma(n + 3/2)} \left( \frac{\gamma - 1}{2\gamma K} \right)^n \left( \frac{\Omega_K^2 - \Omega^2}{\Omega_K} \right)^{n+1/2}.
\]

Through equation (16), the angular velocity \( \Omega \) defines the parameter \( C \). Thus this simple model has two free parameters, \( R_0 \) and \( \Omega \). For a given value of \( \Omega \) we may determine the radius \( R_0 \) by applying the mass constraint

\[
M = \int \Sigma(R) 2\pi RdR = \frac{\pi}{(n + 3/2)} CR_0^{2n+3}.
\]

We then determine \( \Omega \) by applying the angular momentum constraint,

\[
J = \int \Sigma(R) \Omega R^2 2\pi RdR = \frac{1}{(n + 5/2)} M\Omega R_0^2.
\]

From equations (16) and (17) we see that for a given mass, \( R_0^2 \) increases with increasing \( \Omega \), and thus from equation (18) it is clear that \( J \) is also a monotonically increasing function of \( \Omega \). We conclude that for given values of \( M \) and \( J \), there is one and only one energy extremum configuration.
All possible fluid configurations in the model have a positive energy, since all the terms in equation (5) are positive. Therefore, we are guaranteed that the system has a well-defined global energy minimum. Since we have shown that for given values of mass $M$ and angular momentum $J$, there is only one configuration that is an energy extremum, that particular configuration must correspond to the absolute energy minimum state and must be stable.

In this simple analytical model, the orbital velocity $v(R)$ increases monotonically with radius $R$, from $v = 0$ at $R = 0$ to a maximum at $R = R_0$, while the sound speed $a(R, z)$ decreases from a maximum at $R = 0, z = 0$ to $a = 0$ all over the surface (and specifically at the outer edge $R = R_0, z = 0$):

$$v(R) = \Omega R, \quad a^2(R, z) = (\gamma - 1)H(R, z) \propto \left(1 - \frac{R^2}{R_0^2} - \frac{z^2}{Z_0^2}\right). \quad (19)$$

It is evident that the model includes both subsonic and supersonic zones, and should therefore be unstable according to the criterion given by FB02. And yet, as proved above, the fluid is in an energy minimum state and is stable. The model is thus an explicit counter-example to the result claimed in FB02 — supersonic rotation does not imply that the fluid has neighboring states with a lower energy.

4. A Final Comment

In astrophysics, one is generally interested in fluids with free surfaces at zero pressure rather than fluids confined between walls. Since FB02 claim that the only stable barotropic configurations are those that rotate uniformly and are subsonic at all radii, it is interesting to ask whether such fluid configurations are at all possible in the first place. The answer is that, if we leave out the strictly non-rotating case ($\Omega = 0$) and if we restrict ourselves to fluids with polytropic indices $n > 0$, there are no isolated configurations that meet FB02’s requirements. (We thank Martin Rees for pointing this out.)

For instance, if the fluid extends to an infinite radius (as can happen for an isothermal gas), then the rotation velocity $v(R) \to \infty$ at large $R$ whereas the sound speed remains finite. The fluid at large radius is thus guaranteed to be supersonic. On the other hand, if the fluid has an edge at a finite radius, as in the simple model described in §3, then $v$ is finite at the edge whereas the sound speed goes to zero. Once again, the fluid near the outer edge rotates supersonically.

Thus, all uniformly rotating configurations for a wide class of fluids have supersonic motions. Since, surely, all of these fluids must possess stable minimum energy equilibria, the result claimed by FB02 must be incorrect. The example discussed in §3 is a specific example that explicitly demonstrates this.