Behaviour of critical exponent in product of hyperbolic space.

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Abstract

In this article we study the Manhattan curve as defined by M. Burger and different invariant including critical exponent. The principal result is an isolation theorem, precising the sharpness of the rigidity criteria of C. Bishop and S. Steger, for two representations of surface group in $\text{PSL}(2,\mathbb{R})$ to be conjugated. We exhibit some examples lightening the behavior of the critical exponent over the Teichmüller space. In Appendix we add a generalization of a theorem of G. Knieper \cite{Knieper} for two factors comparing critical exponent and the exponential growth of the number of closed geodesics.

1 Introduction

Let $S$ be a compact surface of genus $g \geq 2$, $\Gamma := \pi_1(S)$ the fundamental group of $S$, and $\rho_1, \rho_2$, two faithful and discrete representations of $\Gamma$ in $\text{PSL}(2,\mathbb{R})$. We will denote by $\Gamma_i := \rho_i(\Gamma)$ the corresponding subgroups of $\text{PSL}(2,\mathbb{R})$, and $S_i := \mathbb{H}^2/\Gamma_i$ the hyperbolic surfaces homeomorphic to $S$. We will often forget the $\rho_i$ and denote $\rho_i(\gamma)$ by $\gamma_i$. The group $\Gamma$ acts diagonally on $\mathbb{H}^2 \times \mathbb{H}^2$, i.e. $\gamma(x, y) = (\rho_1(\gamma)x, \rho_2(\gamma)y)$. On the product $\mathbb{H}^2 \times \mathbb{H}^2$ we can put various metrics, and the ones we will be interested in are the different Manhattan metrics, $d_{a_1, a_2}^M$ defined by the weighted sum of the two hyperbolic metrics on each factor, $d_{a_1, a_2}^M := a_1d_1 + a_2d_2$. Let $o = (o_1, o_2) \in \mathbb{H}^2 \times \mathbb{H}^2$ be a point fixed once for all. The Poincaré series is then defined by

$$P_M[\rho_1, \rho_2, a_1, a_2](s) := \sum_{\gamma \in \Gamma} e^{-sd_{a_1, a_2}^M(\gamma o, o)}.$$ 

We define the critical exponent of $\rho_1$ and $\rho_2$ by

$$\delta(\rho_1, \rho_2) := \inf\{s > 0 | P_M[\rho_1, \rho_2, 1, 1](s) < \infty\}.$$ 

By triangular inequality $\delta(\rho_1, \rho_2)$ depends neither on $o \in \mathbb{H}^2 \times \mathbb{H}^2$, nor on the conjugation class of $\rho_1$ and $\rho_2$, hence it defines a function on $\text{Teich}(S) \times \text{Teich}(S)$, still denoted by $\delta$. The primary aim of this article is the study of the function $\delta$ on $\text{Teich}(S) \times \text{Teich}(S)$.

Let $CS$ be the set of free homotopy classes of closed curves on $S$. If $S$ is endowed with a hyperbolic metric, for all $c \in CS$ there is a unique geodesic in the class of $c$. In this article the metric on $S$ will be specified by a subscript and we will note $\ell_i(c)$ for the length of the geodesic representative of $c$ on $S_i$. We endow the Teichmüller space of $S$ with the Thurston distance.

**Definition 1.1.** $\mathcal{R}$. Let $(S_1, S_2)$ be two hyperbolic surfaces. The Thurston distance is defined by

$$d(S_1, S_2) = \log \sup_c \max \left( \frac{\ell_2(c)}{\ell_1(c)}, \frac{\ell_1(c)}{\ell_2(c)} \right).$$
where the supremum is taken over $C_S$.

We will extend a theorem of G. Knieper, saying that the critical exponent can be defined by the geodesic Poincaré series, namely, for two hyperbolic surfaces $S_i := \mathbb{H}^2/\rho_i(\Gamma)$, $i \in \{1, 2\}$,

$$P_M^G[S_1, S_2, a_1, a_2](s) := \sum_{c \in C_S} e^{-s(a_1 \ell_1(c) + a_2 \ell_2(c))}.$$

We will show in Theorem 5.2 that the critical exponent satisfies

$$\delta(\rho_1, \rho_2) = \inf \{ s > 0 \mid P_M^G[S_1, S_2, 1, 1](s) < \infty \}.$$

The proof goes almost along the same lines as Knieper’s one and we add it in the Appendix. A simple consequence of this theorem is the continuity of $\delta$ with respect to the Thurston distance. Since Teichmüller space is also the set of hyperbolic metrics, Theorem 5.2 allows us to have a geometric intuition on the behavior of $\delta$, we will think of $\delta$ as a function on pair of hyperbolic surfaces, and if we want to insist on the geometric meaning of the critical exponent, we will write $\delta(S_1, S_2)$ instead of $\delta(\rho_1, \rho_2)$.

The study of $\delta(S_1, S_2)$ is motivated by the following rigidity result of C. Bishop and T. Steger [? ]. Let $S_1$ and $S_2$ be two hyperbolic surfaces, then

$$\delta(S_1, S_2) \leq \frac{1}{2},$$

with equality if and only if $S_1 = S_2$. In this article, this will be given as a consequence of a theorem of M. Burger about $C_M$[? ], Theorem 2.1. The main result of this paper is the following isolation theorem

**Theorem 1.2.** Let $S_0$ be a fixed hyperbolic surface and $S_n$ be a sequence of hyperbolic surfaces. Then $\lim_{n \to \infty} \delta(S_0, S_n) = \frac{1}{2}$ if and only if $\lim_{n \to \infty} d(S_n, S_0) = 0$.

In Theorem 1.2, fixing $S_0$ is necessary.

**Theorem 1.3.** There exists two sequences of surfaces $S_n$ and $S'_n$ such that $d(S_n, S'_n) \geq 1$ and $\lim_{n \to \infty} \delta(S_0, S_n) = \frac{1}{2}$.

Recall that the thick part of $\text{Teich}(S)$ is the set of hyperbolic surfaces with injectivity radius bounded away from 0. By a simple application of Mumford’s compactness theorem if $S_0$ stays in the thick part of $\text{Teich}(S)$ Theorem 1.2 remains true

**Theorem 1.4.** Let $S_n$ and $S'_n$ be two sequences of hyperbolic surfaces. Suppose that at least one of the sequences stays in the thick part of $\text{Teich}(S)$. Then $\lim_{n \to \infty} \delta(S_n, S'_n) = \frac{1}{2}$ if and only if $\lim_{n \to \infty} d(S_n, S'_n) = 0$.

The study of $\delta$ is closely related to the *Manhattan curve* defined by M. Burger. This curve is a subset of $\mathbb{R}^2$ defined by the points $(a_1, a_2)$ for which the critical exponent of the Poincaré series associated to $d_M^{a_1, a_2}$ is equal to 1.

**Definition 1.5.** The Manhattan curve is

$$C_M := \{(a_1, a_2) \in \mathbb{R}^2 \mid P_M^G[S_1, S_2, a_1, a_2](s) \text{ has critical exponent equal to } 1\}.$$
In addition to studying the behavior of $\delta$, we will review in Section 2 some properties of $C_M$. We will introduce another invariant associated to $S_1$ and $S_2$, namely the correlation number with slope. This invariant is a generalization of the one introduces by R. Schwarz and R. Sharp in [?] and [?], where they call correlation number the real defined by

$$C(S_1, S_2, 1) := \lim_{T \to \infty} \frac{1}{T} \log \text{Card}\{c \in CS | \ell_1(c) \in [T, T+1) \text{ and } \ell_2(c) \in [T, T+1)\}.$$ 

The length of the interval is arbitrary. We would obtain the same invariant by considering the cardinal of the set $\{c \in CS | \ell_1(c) \in [T, T+\epsilon) \text{ and } \ell_2(c) \in [T, T+\epsilon)\}$ for any $\epsilon > 0$. For convenience we decided to set $\epsilon = 1$ in all definitions and theorems concerning correlation numbers. For example, in Theorem 1.7 the dependence on $\epsilon$ would affect only the constant $K$.

Let $\text{dil}^- := \inf_{c \in CS} \frac{\ell_2(c)}{\ell_1(c)}$ and $\text{dil}^+ := \sup_{c \in CS} \frac{\ell_2(c)}{\ell_1(c)}$ be the minimum and the maximum dilatation of the geodesic lengths between $S_1$ and $S_2$. By analogy we set the following definition

**Definition 1.6.** For $\lambda \in (\text{dil}^-, \text{dil}^+)$, the correlation number with slope $\lambda$ is

$$C(S_1, S_2, \lambda) := \lim_{T \to \infty} \frac{1}{T} \log \text{Card}\{c \in CS, \ell_1(c) \in [T, T+1) \text{ and } \ell_2(c) \in [\lambda T, \lambda T+1)\}.$$ 

Let $S_0$ and $S_1$ be two hyperbolic surfaces. In his paper R. Sharp [? , Theorem 1] proved that there exists $K > 0$ and $C > 0$ such that

$$\text{Card}\{c \in CS | \ell_1(c) \in [T, T+1) \text{ and } \ell_2(c) \in [T, T+1)\} \sim_{T \to \infty} K e^{CT} T^{3/2}.$$ 

Moreover $C = C(S_0, S_1, 1) = a_1 + b_1$, where $(a_1, b_1)$ is the point on $C_M$ such that the slope of the normal is 1.

We will establish a generalization of this result, in Section 2.1, namely,

**Theorem 1.7.** Let $S_1$ and $S_2$ be two hyperbolic surfaces and $\lambda \in (\text{dil}^-, \text{dil}^+)$. There exists $K > 0$ and $C > 0$ such that

$$\text{Card}\{c \in CS | \ell_1(c) \in [T, T+1) \text{ and } \ell_2(c) \in [\lambda T, \lambda T+1)\} \sim_{T \to \infty} K e^{CT} T^{3/2}.$$ 

Moreover $C = C(S_1, S_2, \lambda) = a_1(\lambda) + \lambda a_2(\lambda)$, where $(a_1(\lambda), a_2(\lambda))$ is the unique point on $C_M$ where a normal vector to $C_M$ has slope $\lambda$.

Our proof follows the article of [?]. It is based on a theorem of S. Lalley concerning the distribution of orbits of the geodesic flow on a hyperbolic surface.

R. Schwartz and R. Sharp asked in their paper [?] what is the behavior of $C(S_1, S_2, 1)$, more precisely they asked if $C(S_1, S_2, 1)$ is bounded away from 0 as $S_1$ and $S_2$ range over Teich($S$). We answer by the negative to this question and throughout this article we give a precise description of $C(S_1, S_2, 1)$ since it is closely related to $\delta(S_1, S_2)$. This last statement will be made clear in Corollaries 2.18 and 2.19.

In Section 2 we will also study a more common invariant, the directional critical exponent. For $\lambda \in (\text{dil}^-, \text{dil}^+)$ and $\epsilon > 0$, let

$$CS(\lambda, \epsilon) := \left\{ c \in CS \mid \frac{\ell_2(c)}{\ell_1(c)} - \lambda \leq \epsilon \right\}.$$ 

To this set is naturally associated a critical exponent, namely

$$\delta(S_1, S_2, \lambda, \epsilon) := \inf \left\{ s > 0 \mid \sum_{c \in CS(\lambda, \epsilon)} e^{-s(\ell_1(c) + \ell_2(c))} < +\infty \right\}.$$ 

3
Definition 1.8. The critical exponent with slope $\lambda$, is defined by
\[
\delta(S_1, S_2, \lambda) := \lim_{\epsilon \to 0} \delta(S_1, S_2, \lambda, \epsilon).
\]

We obtain the following Corollary 2.16 of Theorem 1.7

**Theorem 1.9.** Let $S_1, S_2$ be two hyperbolic surfaces and $\lambda \in (\text{dil}^-, \text{dil}^+)$, we then have
\[
\frac{\delta(S_1, S_2, \lambda)}{1 + \lambda} = \delta(S_1, S_2, \lambda) = \frac{a_1(\lambda) + \lambda a_2(\lambda)}{1 + \lambda},
\]
where $(a_1(\lambda), a_2(\lambda))$ is the unique point on $C_M$ where a normal vector to $C_M$ has slope $\lambda$.

By derivation of these formulas we obtain that the directional critical exponent is maximal for a unique $\lambda$.

**Corollary 1.10.** Let $S_1, S_2$ be two hyperbolic surfaces and $\lambda \in (\text{dil}^-, \text{dil}^+)$, we then have
\[
\delta(S_1, S_2, \lambda) \leq \delta(S_1, S_2),
\]
with equality for a unique $\lambda \in (\text{dil}^-, \text{dil}^+)$.  

This last corollary implicitly belongs to the work of G. Link [? , Theorem 3.12, Theorem 5.1]. However, G. Link considers $\mathbb{H}^2 \times \mathbb{H}^2$ endowed with the Riemannian metric, hence for the Manhattan metric this was not known.

In Theorem 1.2, one way is just the continuity of critical exponent and the difficult part is the converse. Let’s explain the scheme of the proof for the converse. Let $S_0$ be a fixed surface and $S_n$ be a sequence satisfying $\lim \delta(S_0, S_n) = \frac{1}{2}$, we want to prove that $\lim d(S_n, S_0) = 0$. First, remark that if the sequence $S_n$ stays in a compact set of $\text{Teich}(S)$ we can then suppose that $S_n$ converges and we conclude by continuity and rigidity theorem of C. Bishop and S. Steger. Hence we have to prove that if a sequence $S_n$ goes to the boundary of $\text{Teich}(S)$ then $\delta(S_0, S_n)$ does not tend to $\frac{1}{2}$. The following basic example gives the rough picture of how we manage to prove this. Let $c$ be a simple closed curve on $S$ and take $S_n = \tau^{2n} S_0$ where $\tau$ is the Dehn twist around $c$, then a simple argument shows using invariance of $\delta(S_0, S_n)$ by the mapping class group, shows that the sequence $(\delta(S_0, S_n))_{n \in \mathbb{N}}$ is decreasing, hence cannot tend to $\frac{1}{2}$.

Now the idea is to replace Dehn twists by *earthquakes*, which are a continuous generalization of the latter. A well-known theorem of W. Thurston, [? , Theorem 3.1] says that we can join $S_0$ to every point in the Teichmüller space by an "earthquake" and we manage to show that along earthquake paths the critical exponent does not tends to $\frac{1}{2}$, which proves Theorem 1.2.

The point to prove that $\delta$ cannot tend to $\frac{1}{2}$ along earthquakes is to show that "most" curves on $S_0$ are increasing along earthquake paths. Explaining precisely what we mean by "most" would take us too far in this introduction and this is the purpose of Section 3.5. Roughly we see closed geodesics as probability measures on $T^1 S_0$ and we use a large deviation theorem of Y. Kifer about geodesic flow [? ] saying that the typical geodesics behave like the Liouville measure. So this "most" refers to the Liouville measure, therefore depends only on the metric on $S_0$ and doesn’t take into account the metric on $S_n$. The last difficulty is to relate the metric on $S_0$ to the sum of the metric on $S_0$ and $S_n$. This is solved by the use of directional critical exponent. Recall that Theorem 1.10 says that the critical exponent $\delta(S_0, S_n)$ is equal to $\delta(S_0, S_n, \lambda_n)$ for some $\lambda_n$, hence the length of geodesics on $S_n$ is almost equal to $\lambda_n \ell_0(c)$ and so $\ell_0(c) + \ell_n(c) \approx (1 + \lambda_n) \ell_0(c)$. We don’t know that $\lambda_n$ is uniformly bounded away of 1 hence this is not sufficient to prove Theorem 1.2. In fact we prove that if $\lim_{n \to \infty} \delta(S_0, S_n) = \frac{1}{2}$, then $\lim_{n \to \infty} \lambda_n = 1$, Theorem 2.4.
The end of the proof is by contradiction, suppose that \( \lim_{n \to \infty} \delta(S_0, S_n) = \frac{1}{2} \). The Poincaré sum over the geodesics which length is increasing along earthquake has a critical exponent decreasing to 0, we can then consider only the other geodesics. Using the previous estimate we have \( \ell_0(c) + \ell_n(c) \approx (1 + \lambda_n)\ell_0(c) \approx 2\ell_0(c) \), since \( \lambda_n \to 1 \) by Theorem 2.4. Finally, there are "few" geodesics for the metric \( S_0 \) which length is not increasing under the earthquake path, hence the Poincaré series over those curves has a critical exponent strictly less than \( 1/2 \) which is the required contradiction.

Our work in Section 4 uses the machinery of earthquakes and geodesic currents and unlike the rigidity result of M. Burger and the work of P. Albuquerque, G. Link and J-F. Quint, does not rely on Patterson-Sullivan theory. As a consequence this cannot be easily generalized for example to surfaces of non constant negative curvature, or to larger dimensions by considering two different representations of convex cocompact group of non finite volume. However we want to stress that our theorems are quantitative, they give the value or the limit of the critical exponent. They are of a different essence than those of J-F. Quint [? ] or A. Sambarino [? , Theorem A] which give the existence of an exponential growth rate of discrete group.

Our results have to be compared to the examples given in [? ]. Indeed, in this paper, C. McMullen calculate the critical exponent (actually the Hausdorff dimension of the limit set) of quasifuchsian manifolds, which are parametrized by a pair of Teichmüller representations. It is a surprising fact that the function \( \delta \) in our context behave almost in a opposite manner as the equivalent function for quasifuchsian representation. More precisely, for \( S_1, S_2 \) two points in the Teichmüller space of \( S \), let \( \delta_{QF}(S_1, S_2) \) be the critical exponent of the quasifuchsian manifolds associated by the Bers representation to \( (S_1, S_2) \). Then

- R. Bowen showed in [? ] that \( \delta_{QF}(S_1, S_2) \geq 1 \) with equality if and only if \( S_1 = S_2 \) whereas, as we already said, \( \delta(S_1, S_2) \leq 1/2 \) with equality if and only if \( S_1 = S_2 \).
- C. McMullen showed that for a pseudo-Anosov diffeomorphism \( \phi \) and for the sequence \( S_n = \phi^n(S_0) \) we have \( \lim_{n \to \infty} \delta_{QF}(S_0, S_n) = 2 \) whereas we show in Section 3.4 that \( \lim_{n \to \infty} \delta(S_0, S_n) = 0 \).
- C. McMullen showed that for \( S \), the surface obtained by pinching a simple closed curve on \( S_0 \) we have \( \lim_{t \to \infty} \delta_{QF}(S_0, S_I) < 2 \), whereas we prove that \( \liminf_{t \to \infty} \delta(S_0, S_I) > 0 \).

We don’t know if the analogy can go further. For example, a natural question we would like to have answered but couldn’t, was wether or not the critical exponent was decreasing along earthquake paths. Namely, let \( t \to S_t \) be an earthquake path in Teich\((S)\), is \( t \to \delta(S_0, S_t) \) decreasing ? In comparison C. McMullen showed in [? , Theorem 9.6] that if \( S_I \) is obtained by an earthquake along a simple closed curve, there exists a periodic function \( D(t) \) such that \( \lim_{t \to \infty} \delta_{QF}(S_0, S_I) - D(t) = 0 \). He added a remark saying that \( D \) is probably not constant and that \( \delta_{QF}(S_0, S_I) \) should oscillate along this path. In our case, it seems rather convincing that \( t \to \delta(S_0, S_I) \) admits a limit when \( t \to +\infty \). The reason for this belief is that we have a natural candidate for the latter limit. Let \( CS^+ \) be the set of free homotopy classes of closed curves having positive intersection with the lamination along we do the earthquake to get \( S_t \). Remark that for \( c \in CS^+ \) we have \( \lim_{t \to \infty} \ell_t(c) = +\infty \) and for \( c \not\in CS^+ \) we have \( \ell_t(c) = \ell_0(c) \). Hence the candidate to be the limit of \( t \to \delta(S_0, S_I) \) is the critical exponent of \( \sum_{c \not\in CS^+} e^{-\ell_0(c) - \ell_t(c)} \). Unfortunately the limit \( \lim_{t \to \infty} \ell_t(c) = +\infty \) is not uniform on \( CS^+ \) and this is why we couldn’t answer the question.

The plan of the paper is the following. In Section 2, we introduce the Manhattan curve as defined by M. Burger and make a careful study of the different invariant that can be read out
of this curve: in particular, many inequalities between these invariant are deduced from the convexity of the Manhattan curve. At the end of Section 2, we will prove Theorem 1.7. The third section is a description of geometric results about currents, lamination and earthquakes. We will follow the presentation of F. Bonahon \[?\] in order to get a clear picture of the results we need, namely, Corollary 3.20. This section contains also examples where we can calculate either the limit of the critical exponent of some sequences, or at least get some bounds. One of them is the example proving Theorem 1.3 which shows the sharpness of Theorem 1.2. The fourth section is devoted to the proof of Theorems 1.2 and 1.4. Finally we included an appendix to show the generalization of the theorem of G. Knieper \[?\] previously mentioned.

2 Manhattan curve

We begin this section by some inequalities between critical exponent and intersection number then we study the correlation number with slope. The intersection between two hyperbolic surfaces will be defined in 3.10. In this section the precise definition does not matter, we will just prove some inequalities coming from the theorem of M. Burger \[?\] about the Manhattan curve. For all this section we fix $S_1$ and $S_2$ two hyperbolic surfaces and we call $I$ the function on $\text{Teich}(S) \times \text{Teich}(S)$ defined by, $I : (S_1, S_2) \rightarrow \frac{i(S_1,S_2)}{i(S_1,S_1)}$, where $i(S_i, S_j)$ is the intersection number between $S_i$ and $S_j$. Finally let $\lambda(x)$ be the slope of a normal vector to $C_M$ at the abscissa $x$.

**Theorem 2.1.** \[?\] The Manhattan curve is the straight line between $(1,0)$ and $(0,1)$ if and only if $S_1$ and $S_2$ are equal in $\text{Teich}(S)$.

- $\lambda(1) = I(S_1,S_2)$ and $\lim_{x \to +\infty} \lambda(x) = \text{dil}^+$, $\lim_{x \to -\infty} \lambda(x) = \text{dil}^-$

The other result we will need is the

**Theorem 2.2.** \[?\] The Manhattan curve is real analytic.

For concision we write $\delta$ for $\delta(S_1, S_2)$ and $I$ for $I(S_1, S_2)$.

We first begin by recalling some basic facts about $\delta$ and $C_M$.

**Proposition 2.3.**

1. The points $(1,0)$ and $(0,1)$ are on $C_M$.
2. The intersection point between $C_M$ and the line $y = x$ has coordinates $(\delta, \delta)$.
3. The Manhattan curve is convex and strictly convex if $S_1 \neq S_2$.
4. If $S_1 \neq S_2$ then $\lambda : \mathbb{R} \rightarrow (\text{dil}^-, \text{dil}^+)$ is one-to-one. The inverse image of $\lambda \in (\text{dil}^-, \text{dil}^+)$, will be noted $(a_1(\lambda), a_2(\lambda))$

**Proof.**

1. Follows from the compacity of $S$. Indeed the critical exponent of $\sum_{c \in CS} e^{-s\lambda(c)}$ is 1 for a compact surface.
2. The intersection point has coordinates of the form $(x, x)$ since it is on the line $y = x$. Since $(x, x)$ is on $C_M$, $\sum_{c \in CS} e^{-s(t_1(c)+t_2(c))}$ has critical exponent equal to 1, this exactly means that $x = \delta$
3. Follows from the convexity of the exponential map. More precisely let $(a_1, a_2)$, $(b_1, b_2)$ be two points of $C_M$ then by Hölder’s inequality

$$P[S_1, S_2, ta_1 + (1-t)b_1, ta_2 + (1-t)b_2](s) \leq (P[S_1, S_2, a_1, a_2](s))^t(P[S_1, S_2, b_1, b_2](s))^{1-t}.$$ 

By definition, both series of the right hand side have critical exponent equal to 1, hence $P[S_1, S_2, ta_1 + (1-t)b_1, ta_2 + (1-t)b_2](s)$ has critical less than 1, which exactly means
that \((ta_1 + (1 - t)b_1, ta_2 + (1 - t)b_2)\) is above \(C_M\). The strict convexity follows from the analiticity of \(C_M\).

4. By strict convexity, \(\lambda\) is strictly increasing and it is continuous hence one-to-one.

The convexity of \(C_M\) and Theorem 2.1 imply the rigidity theorem of C. Bishop and S. Steger we mentioned in the introduction, \(\delta \leq \frac{1}{2}\) with equality if and only if \(S_1 = S_2\).

By definition and by item 2 in Proposition 2.3, \(\lambda(\delta)\) is the slope of a normal vector at the intersection point between \(C_M\) and the line \(y = x\), hence by convexity \(C_M\) is above the line \(y = \frac{1}{\lambda(\delta)}(x - \delta) + \delta\). Since \((0, 1) \in C_M\) and \((1, 0) \in C_M\), we obtain the two following inequalities:

\[
\frac{\delta}{1 - \delta} \leq \lambda(\delta) \leq \frac{1 - \delta}{\delta}.
\]

We will use the following corollary of these inequalities.

**Corollary 2.4.** Let \(S_n\) and \(S'_n\) be two sequences of hyperbolic surfaces. If \(\lim_{n \to \infty} \delta(S_n, S'_n) = \frac{1}{2}\) then \(\lim_{n \to \infty} \lambda(\delta(S_n, S'_n)) = 1\).

By convexity again, the line \(y = \frac{1}{\lambda(\delta)}(x - 1)\) is below \(C_M\). Hence taking the intersection with \(y = x\) we get \(\delta \geq \frac{1}{\lambda(\delta)}(\delta - 1)\), which is equivalent to

\[
\delta \geq \frac{1}{1 + I}.
\]

This gives the following.

**Corollary 2.5.** Let \(S_n\) and \(S'_n\) be two sequences of hyperbolic surfaces. If \(\lim_{n \to \infty} I(S_n, S'_n) = 1\), then \(\lim_{n \to \infty} \delta(S_n, S'_n) = \frac{1}{2}\).

Moreover, \(\delta \geq \frac{1}{1 + I}\) gives, \(I \geq \frac{1}{\delta} - 1\), hence we have the following corollary which is in the paper of M. Burger.

**Corollary 2.6.** [Corollary 1] We have \(I(S_1, S_2) \geq 1\), with equality if and only if \(S_1 = S_2\).

A central idea in our work is to study the proportionality factors between the two lengths of the geodesic corresponding to a closed curve on \(S_1\) and \(S_2\).

The principal result in this section is a formula for the critical exponent with slope \(\lambda\) in terms of the Manhattan curve.

**Theorem 2.7.** Let \((a_1, a_2) \in C_M\), and \(\lambda \in (\text{dil}^-, \text{dil}^+)\). We have the following inequality

\[
\delta(\lambda) \leq \frac{a_1 + \lambda a_2}{1 + \lambda},
\]

equality occurs if and only if \((a_1, a_2) = (a_1(\lambda), a_2(\lambda))\).

First we are going to prove the inequality, which is a simple algebraic manipulation. This implicitly appears, but for Riemannian metric, in the work of G. Link [?]. We propose a proof since our context is a little bit different and since recognizing that [? , Lemma 3.7] and 2.8 are similar is not obvious.

**Lemma 2.8.** Compare to [? , Lemma 3.7]. For every \((a_1, a_2) \in C_M\) we have the inequality

\[
\delta(\lambda) \leq \frac{a_1 + \lambda a_2}{1 + \lambda}.
\]
Proof. Let \((a_1, a_2) \in \mathcal{C}_M, \lambda \in (dil^-, dil^+)\), and \(c \in \mathcal{C}S(\lambda, \epsilon)\). First we suppose that \(a_1 > 0\) and \(a_2 > 0\). Then

\[
a_1 \ell_1(c) + a_2 \ell_2(c) = \left(\frac{a_1}{1 + \ell_2(c)/\ell_1(c)} + a_2 \frac{\ell_2(c)/\ell_1(c)}{1 + \ell_2(c)/\ell_1(c)}\right) (\ell_1(c) + \ell_2(c)) \tag{2}
\]

This implies for the Poincaré series that:

\[
\sum_{c \in \mathcal{C}S} e^{-s(a_1 \ell_1(c) + a_2 \ell_2(c))} \geq \sum_{c \in \mathcal{C}S(\lambda, \epsilon)} e^{-s(a_1 \ell_1(c) + a_2 \ell_2(c))} \tag{4}
\]

Since \((a_1, a_2) \in \mathcal{C}_M\) the critical exponent of the left hand side is equal to 1, therefore

\[
1 \geq \delta(\lambda, \epsilon) \left(\frac{1 + \lambda - \epsilon}{a_1 + (\lambda + \epsilon)a_2}\right) \tag{6}
\]

\[
\delta(\lambda, \epsilon) \leq \frac{a_1 + (\lambda + \epsilon)a_2}{1 + \lambda - \epsilon} \tag{7}
\]

Passing to the limit gives,

\[
\delta(\lambda) \leq \frac{a_1 + \lambda a_2}{1 + \lambda} \tag{8}
\]

This end the proof for \(a_1 > 0\) and \(a_2 > 0\).

If \(a_1 < 0\) and \(a_2 > 0\), the inequality 3 would become

\[
a_1 \ell_1(c) + a_2 \ell_2(c) \leq \left(\frac{a_1}{1 + \lambda + \epsilon} + \frac{(\lambda + \epsilon)a_2}{1 + \lambda - \epsilon}\right) (\ell_1(c) + \ell_2(c))
\]

and then the inequality 6 would become

\[
1 \geq \delta(\lambda, \epsilon) \left(\frac{a_1}{1 + \lambda + \epsilon} + \frac{(\lambda + \epsilon)a_2}{1 + \lambda - \epsilon}\right).
\]

Passing to the limit as in 8 ends the proof in the case \(a_1 < 0\) and \(a_2 > 0\).

The case \(a_1 > 0\) and \(a_2 < 0\) can be treated similarly. \(\square\)

Remark. For \(N\) a norm on \(\mathbb{R}^2\), let us define the \(N\)-Manhattan curve \(\mathcal{C}_M^N\) as the set of \((a_1, a_2) \in \mathbb{R}^2\) such that \(\sum_{c \in \mathcal{C}S} e^{-sN(a_1 \ell_1(c), a_2 \ell_2(c))}\) has critical exponent equal to 1. We can define similarly a \(N\)-directional exponent \(\delta^N(\lambda)\), by restricting the sum on the elements such that \(|\ell_2(c)/\ell_1(c) - \lambda| \leq \epsilon\) and taking the limit \(\epsilon \to 0\). The inequality would become for all \((a_1, a_2) \in \mathcal{C}_M^N\):

\[
\delta^N(\lambda) \leq \frac{N(a_1, \lambda a_2)}{N(1, \lambda)}.
\]

Nonetheless, the conjectural equality:

\[
\delta^N(\lambda) = \frac{N(a_1(\lambda), \lambda a_2(\lambda))}{N(1, \lambda)}
\]

cannot be easily deduced. Indeed our proof uses the correlation number, the inequality between the correlation number and the critical exponent with slope \(\lambda\) and finally the formula \(C(\lambda) = a_1(\lambda) + \lambda a_2(\lambda)\). We would have to generalize this latter equality to \(C(\lambda) = N(a_1(\lambda), \lambda a_2(\lambda))\), which is not clear at all using thermodynamic formalism as we will see thereafter.

The equality case in Theorem 2.16 will be our goal for the last part of this section.
2.1 Correlation number with slope

We are actually going to show the equality case using an extension of a formula due to R. Sharp, [? ] about the correlation number. This result of R. Sharp uses thermodynamic formalism for the geodesic flow. We will make a brief survey of results on thermodynamic formalism and geodesic flow which ends with Theorems 2.11 and 2.13. Finally, we will prove Theorem 2.15 extending the theorem of R. Sharp. The proof is very similar to the original one and we include it for the sake of completeness.

Here again $S_1$ and $S_2$ will be fixed hyperbolic surfaces, hence we will note $C(\lambda)$ instead of $C(S_1, S_2, \lambda)$ for the correlation number of slope $\lambda$, cf Definition 1.6 and $\delta(\lambda)$ instead of $\delta(S_1, S_2, \lambda)$ for the directional critical exponent, cf Definition 1.8. We first begin to prove an inequality :

**Lemma 2.9.** $C(\lambda) \leq \delta(\lambda)(1 + \lambda)$.

**Proof.** Let $CC(T, \lambda) := \{ c \in CS, \ell_1(c) \in [T, T + 1) \text{ and } \ell_2(c) \in [\lambda T, \lambda T + 1) \}$ the set of closed and "correlated" curves. Let $\epsilon > 0$ and $c \in CC(T, \lambda)$ then, for $T > \frac{\max(1, \lambda)}{\epsilon}, \left| \frac{\ell_2(c)}{\ell_1(c)} - \lambda \right| \leq \epsilon$, that is to say $c \in CS(\lambda, \epsilon)$.

$$\sum_{k \geq \max(1, \lambda)/\epsilon} \sum_{c \in CC(k, \lambda)} e^{-s(\ell_1(c)+\ell_2(c))} \leq \sum_{c \in CS(\lambda, \epsilon)} e^{-s(\ell_1(c)+\ell_2(c))}.$$  

The right hand side has critical exponent equal to $\delta(\lambda, \epsilon)$ and for $c \in CC(T, \lambda)$ we have $\ell_2(c) \leq \lambda T + 1 \leq \lambda \ell_1(c) + 1$, hence the left hand side satisfies

$$\sum_{k \geq \max(1, \lambda)/\epsilon} e^{-s(\ell_1(c)+\ell_2(c))} \geq \sum_{k \geq \max(1, \lambda)/\epsilon} \sum_{c \in CC(k, \lambda)} e^{-s\ell_1(c)(1+\lambda)-s}.$$  

Since the growth of $\text{Card}CC(k, \lambda)$ is larger than $e^{C(\lambda) - \eta}k$ for all $\eta > 0$, there is $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$

$$\text{Card}CC(k, \lambda) \geq e^{C(\lambda) - \eta}k.$$  

Set $k_1 := \max(k_0, \max(1, \lambda)/\epsilon)$,

$$\sum_{k \geq \max(1, \lambda)/\epsilon} \sum_{c \in CC(k, \lambda)} e^{-s(\ell_1(c)+\ell_2(c))} \geq \sum_{k \geq \max(1, \lambda)/\epsilon} \sum_{c \in CC(k, \lambda)} e^{-s\ell_1(c)(1+\lambda)-s} \geq \sum_{k \geq k_1} \sum_{c \in CC(k, \lambda)} e^{-s(k+1)(1+\lambda)-s} \geq e^{-s(1+\lambda)-s} \sum_{k \geq k_1} e^{C(\lambda) - \eta}k e^{-sk(1+\lambda)}.$$  

In fine, the critical exponent of $\sum_{k \geq \max(1, \lambda)/\epsilon} \sum_{c \in CC(k, \lambda)} e^{-s(\ell_1(c)+\ell_2(c))}$ is larger than $\frac{C(\lambda) - \eta}{1+\lambda}$, therefore we have for all $\epsilon > 0$ and $\eta > 0$, that $C(\lambda) - \eta \leq \delta(\lambda, \epsilon)(1 + \lambda)$. We conclude since $\eta$ and $\epsilon$ are arbitrary.

Our proof of equality case use thermodynamic formalism that we survey in the next paragraph.
Reparametrization of geodesic flow.

**Definition 2.10.** Let \( \varphi_t \) be the geodesic flow on \( S_1 \) and \( \tau_1 \) be a periodic orbit for \( \varphi_t \). Let also \( \psi : T^1S_1 \to \mathbb{R} \) be any Hölder continuous function. We note \( \omega(\psi, \tau_1) := \int_{\tau_1} \psi \) the integral of \( \psi \) with respect to the arc length along the geodesic associated to \( \tau_1 \).

For example if \( \psi = 1 \), we have \( \omega(\psi, \tau_1) = \ell_1(c) \), where \( c \) is the closed geodesic on \( S_1 \) whose support is \( \tau_1 \). We are going to construct a function \( \psi \) such that \( \omega(\psi, \tau_1) = \ell_2(c) \). This reparametrization is classic for example R. Schwartz and R. Sharp propose a method to construct such a function. Let’s describe this construction. The classical references for what we are going to introduce are [? | ?] and [?].

A Hölder cocycle is a map \( u : \Gamma \times \partial \Gamma \to \mathbb{R} \) satisfying:

\[
u(\gamma', \xi) = u(\gamma, \xi) + u(\gamma', \gamma)\]

for every pair \( \gamma, \gamma' \in \Gamma \) and \( \xi \in \partial \Gamma \), which is a Hölder continuous map in the variable \( \xi \). Since the surface \( S \) is compact the boundary of \( \Gamma \) can be identified with \( S^1 \) that is the boundary of \( \mathbb{H}^2 \). The period of a cocycle \( u \), \( \ell_u(\gamma) \) is by definition \( u(\gamma, \gamma^+) \) where \( \gamma^+ \) is the attractive fixed point of \( \gamma \). From [? , Corollary 1 p.106] for every Hölder cocycle \( c \) there exists a function \( \psi \) such that \( \omega(\psi, \tau_1) = \ell_2(c) \) and \( \ell_u(\gamma) = \ell_2(\gamma) \), where \( \ell_2(\gamma) \) is the displacement of the element \( p_2(\gamma) \).

The Busemann cocycle defined by \( \beta_\xi(o, p_2(\gamma)o) := \lim_{x \to \xi} d(o, x) - d(p_2(\gamma)o, x) \), where \( o \) is any point of \( \mathbb{H}^2 \), satisfies this property. This defined a Hölder continuous function \( \psi \) with the desired property.

\[
\sum_{c \in C} e^{-a_1 \ell_1(c) - a_2 \ell_2(c)} = \sum_{\tau} e^{\int_{\tau} -a_1 - a_2 \psi},
\]

where the sum of the right hand side is taken over all closed orbit of \( \varphi_t \).

Given a Hölder continuous function \( f : T^1S_1 \to \mathbb{R} \), we define the pressure of \( f \) to be

\[
P(f) = \sup_{\nu} \left\{ h(\nu) + \int_{T^1S_1} f d\nu \right\},
\]

where the supremum is taken over all \( (\varphi_t) \)-invariant probability measures, and \( h(\nu) \) is the entropy of the geodesic flow with respect to \( \nu \). This supremum is achieved by a unique such measure \( \mu_f \), called the equilibrium state for \( f \). In our setting of a surface of constant curvature, the equilibrium state for \( f \equiv 0 \) is the Liouville measure, which is the local product of the Lebesgue measure for the metric associated to \( S_1 \) and the arc length along the fibre.

We say that a function \( v : T^1S_1 \to \mathbb{R} \) is continuously differentiable with respect to \( \varphi_t \) if the limit

\[
v'(y) := \lim_{t \to 0} \frac{v(\varphi_t(y)) - v(y)}{t}
\]

exists everywhere, and is continuous. Two Hölder functions \( f \) and \( g \) are said to be **cohomologous** if \( f - g = v' \) for some continuously differentiable function \( v \).

W. Parry and M. Pollicott showed in their book [? , Proposition 4.8], that if \( \psi \) is a Hölder continuous function which is not cohomologous to a constant, then the function \( t \to P(t \psi) \) is analytic, strictly convex and furthermore, [? , Proposition 4.12]

\[
P'(t \psi) := \frac{d}{dt} P(t \psi) = \int_{T^1S_1} \psi d\mu_{tf}
\]

holds for each value of \( t \in \mathbb{R} \). A version of this theorem more adapted to our notations can be found in [? , p. 429].
The following is a classical theorem in thermodynamic theory, and has been proved by P. Walters in [7, Theorem 4.1]; here again, the following version of this theorem in [7, p. 106 eq. 10] is more adapted:

**Theorem 2.11.** Let \( f : T^1S_1 \to \mathbb{R} \) be a Hölder continuous function, then

\[
\lim_{T \to +\infty} \frac{1}{T} \log \sum_{\tau, \omega(1,T) \leq T} e^{tf} = P(f).
\]

By the previous discussion the Manhattan curve is the set of \((a_1, a_2) \in \mathbb{R}^2\) such that \(\sum_{s} e^{-s f_{a_1 + a_2}}\) has critical exponent equal to 1. Applying [7, Lemma 1, p106] and the remark after the lemma, to the function \( f := a_1 + a_2 \psi \), we see that critical exponent is equal to 1 if and only if \( P(-f) = 0 \) that is \( P(-a_1 - a_2 \psi) = 0 \). Hence

\[
C_M = \{(a_1, a_2), P(-a_1 - a_2 \psi) = 0\} = \{(a_1, a_2), P(-a_2 \psi) = a_1\}.
\]

The independence lemma of [7] shows that \( \psi \) is not cohomologous to a constant as soon as \( \rho_1 \) and \( \rho_2 \) are not conjugated. Hence we will thereafter suppose that \( \rho_1 \) and \( \rho_2 \) are not conjugated. Moreover, the function \( \psi \) satisfies the property that along any geodesic segment \( s \) on \( S_1 \), \( \int_s \psi = \ell_2(s) \), hence the function \( \psi \) is strictly positive. Taking the derivative of \( t \to P(-t \psi) \) gives \( \frac{dP(-t \psi)}{dt} = -\int_{T^1S_1} \psi d\mu_{t \psi} \neq 0 \) and finally the implicit function theorem assures the existence of a analytic function \( q(t) \) defined by \( P(-q(t) \psi) = t \). By definition \( C_M \) is the graph of \( q \).

We can then calculate the derivative of \( q \):

\[
\frac{d}{dt} P(-q(t) \psi) = -\int \psi d\mu_{-q(t) \psi}
\]

and we obtain

\[
\frac{dq}{dt} = \frac{-1}{\int \psi d\mu_{-q(t) \psi}}
\]

**Definition 2.12.** We set \( J(\psi) := \{P(q\psi), q \in \mathbb{R}\} \). By strict convexity, if \( \lambda \in J(\psi) \), there is a unique real noted \( q_\lambda \) such that \( P(q_\lambda \psi) = \lambda \).

This next theorem is due to S.P Lalley [7].

**Theorem 2.13.** [7, Theorem 1][7, proposition p.429] Let \( \psi : T^1S_1 \to \mathbb{R} \) be a Hölder continuous function, which is not cohomologous to a constant. Let \( \lambda \in J(\psi) \), there exists \( K > 0 \) such that,

\[
\text{Card}\{\tau, \omega(1,\tau) \in [T, T + 1), \omega(\psi, \tau) \in [\lambda T, \lambda T + 1)\} \sim_{T \to \infty} K \frac{e^{h(\mu_{\lambda \psi})T}}{T^{3/2}}.
\]

Applying this theorem to the function \( \psi \) we defined, which is not cohomologous to a constant, gives for every \( \lambda \in J(\psi) \), the existence of \( K > 0 \) such that,

\[
\text{Card} CC(T, \lambda) \sim_{T \to \infty} K \frac{e^{h(\mu_{\lambda \psi})T}}{T^{3/2}},
\]

hence if \( \lambda \in J(\psi) \), we have \( C(\lambda) = h(\mu_{\lambda \psi}) \).

This next proposition is not in the original paper of R. Sharp but the proof is essentially the same as in the remarks at the end of [7]
Proposition 2.14. We have the following inclusion \((\text{dil}^-, \text{dil}^+) \subset J(\psi)\)

Proof. Let \(\lambda \in (\text{dil}^-, \text{dil}^+)\). By definition there exists closed geodesics \(c\) and \(c'\) such that \(\ell_2(c) < \lambda \ell_1(c)\) and \(\ell_2(c') > \lambda \ell_1(c')\). They correspond to closed orbits of \(\varphi_t\), \(\tau\) and \(\tau'\), hence \(\omega(\psi, \tau)/\lambda(1, \tau) < \lambda \text{ and } \omega(\psi, \tau')/\omega(1, \tau') > \lambda\).

Let \(I(\psi)\) denote the set of values \(\int \psi d\nu\) where \(\nu\) ranges over invariant probability measures. Clearly \(I(\psi)\) is a closed interval. If we take \(\nu\) to be the probability measure whose support either \(c\) or \(c'\) we see that \(\omega(\psi, \tau)/\omega(1, \tau) \in I(\psi)\), and \(\omega(\psi, \tau')/\omega(1, \tau') \in I(\psi)\). Hence \(|\lambda - 2\epsilon, \lambda + 2\epsilon| \in I(\psi)\) for some \(\epsilon > 0\).

Since \(\lambda \pm \epsilon \in I(\psi)\), we have,

\[
\forall t \in \mathbb{R}, \quad \lambda t + |t| \epsilon \in t I(\psi).
\]

And by the definition of pressure if \(y \in I(\psi)\) then \(P(t \psi) \geq ty\). Hence

\[
\forall t \in \mathbb{R}, \quad P(t \psi) \geq \sup t I(\psi).
\]

Combining the last two inequalities gives

\[
\forall t \in \mathbb{R}, \quad P(t \psi) - \lambda t \geq |t| \epsilon.
\]

Consider the function \(Q(t) := \frac{P(t \psi) - \lambda t}{t}\); we have \(Q(0) = P(0) = 1\) and we just proved that for all \(|t| > 1/\epsilon, Q(t) > 1\). Therefore \(Q\) has a minimum \(q_\lambda \in [-1/\epsilon, 1/\epsilon]\), where \(Q'(q_\lambda) = 0\) which is to say \(P(q_\lambda \psi) = \lambda\) and \(\lambda \in J(f)\).

Now we are going to prove the formula that extends the result of R. Sharp [?].

Theorem 2.15. Let \(\lambda \in (\text{dil}^-, \text{dil}^+)\). There exists \(K > 0\) and \(C(\lambda)\) such that

\[
\text{Card} CC(T, \lambda) \sim K e^{C(\lambda)T/3/2}.
\]

Moreover \(C(\lambda) = a_1(\lambda) + \lambda a_2(\lambda)\).

Proof. Let \(\lambda \in (\text{dil}^+, \text{dil}^-)\), by Proposition 2.14, we know that \(\lambda \in J(\psi)\). The remark after the Theorem 2.13 says that \(C(\lambda) = h(\mu_{q_\lambda}, \psi)\). We have to show that \(h(\mu_{q_\lambda}, \psi) = a_1(\lambda) + \lambda a_2(\lambda)\). By definition of \(h(\mu_{q_\lambda}, \psi)\) we have that

\[
C(\lambda) = h(\mu_{q_\lambda}, \psi) = P(q_\lambda \psi) - \int q_\lambda \psi d\mu_{q_\lambda},
\]

Set \(a\) such that \(q_{\lambda} = -q(a)\), this gives,

\[
C(\lambda) = P(-q(a) \psi) + q(a) \int \psi d\mu_{-q(a)\psi} = a + q(a) \int \psi d\mu_{-q(a)\psi}.
\]

But \(q_{\lambda}\) is such that \(\int \psi d\mu_{q_\lambda} = P(q_\lambda \psi) = \lambda\), hence

\[
C(\lambda) = a + q(a) \lambda.
\]

Moreover,

\[
\frac{dq}{dt}(a) = \frac{-1}{\psi d\mu_{q(a)\psi}} = \frac{-1}{\lambda}.
\]

This is exactly where the slope of a normal vector at \(q\) is equal to \(\lambda\), that is \(a = a_1(\lambda)\), and

\[
C(\lambda) = a_1(\lambda) + \lambda q(a_1(\lambda)) = a_1(\lambda) + \lambda a_2(\lambda).
\]
We finally conclude the proof of the equality case in Theorem 2.7.

**Corollary 2.16.** Let \( \lambda \in (\text{dil}^-, \text{dil}^+) \), we have
\[
C(\lambda) = \frac{\delta(\lambda) = a_1(\lambda) + \lambda a_2(\lambda)}{1 + \lambda}.
\]

**Proof of Corollary 2.16 and Theorem 2.7.** From Lemma 2.8 and Lemma 2.9 we have for all \((a_1, a_2) \in C_M\) and all \(\lambda \in (\text{dil}^-, \text{dil}^+)\) that \(C(\lambda) \leq \delta(\lambda)(1 + \lambda) \leq a_1 + \lambda a_2\) and with Theorem 2.15, equality occurs for \((a_1, a_2)\).

We now show that the inequality \(\delta(\lambda) \leq \frac{a_1 + \lambda a_2}{1 + \lambda}\) is strict for \((a_1, a_2) \neq (a_1(\lambda), a_2(\lambda))\). This is a simple application of the strict convexity of the Manhattan curve. Let \(a \to q(a)\) be the function which graph is the Manhattan curve. Fix \(\lambda\) and define
\[
g(a) := \frac{a + \lambda q(a)}{1 + \lambda}.
\]

The derivative of \(g\) is \(g'(a) = \frac{1 + \lambda q'(a)}{1 + \lambda}\). By Definition \(\lambda = -1/q'(a)\), hence \(g'(a) = \frac{q'(a)}{q'(a) + 1}\). By strict convexity of \(q\), \(g'(a) = 0\) if and only if \(a = a_1(\lambda)\), which finally implies that the inequality is strict for the others values of \((a_1, a_2) \in C_M\).

Here again, the following corollary implicitly belongs to the work of G. Link. However, G. Link considers \(H^2 \times H^2\) endowed with the Riemannian metric, hence for the Manhattan metric this corollary was not known.

**Corollary 2.17.** [? , compare to Theorem 3.12, Theorem 5.1] Let \(\lambda \in (\text{dil}^-, \text{dil}^+)\) then
\[
\delta(S_1, S_2, \lambda) \leq \delta(S_1, S_2),
\]
the equality occurs for a unique \(\lambda\).

**Proof.** The inequality is obvious by definition. For the strict inequality we are going to use a similar method as the previous corollary. We use the formula from Corollary 2.16, \(\delta(\lambda) = \frac{a_1(\lambda) + \lambda a_2(\lambda)}{1 + \lambda}\). Consider \(\lambda(\lambda)\) as the slope of a normal vector at \((a, q(a)) \in C_M\). Hence \(\delta(\lambda(a)) = \frac{a + \lambda q(a)}{1 + \lambda}\). Denoted by \(h\) the function
\[
h(a) = \frac{a q'(a) - q(a)}{q'(a) - 1},
\]

The derivative of \(h\) is \(h'(a) = \frac{q'(a) (a + q(a))}{q'(a) - 1}\), which has the same sign as \(-a + q(a)\). By Proposition 2.3, \(\delta\) is the intersection of \(C_M\) and the line \(y = x\), this implies that the unique solution of \(-a + q(a) = 0\) is \(a = q(a) = \delta\), and \(h'(a) < 0\) for all \(a > \delta\) and \(h'(a) > 0\) for all \(a < \delta\). Hence for all \(a \neq \delta\) we have \(\delta(\lambda(a)) = h(a) < h(\delta) = \delta\).

**Corollary 2.18.** Let \(S_n\) and \(S'_n\) be two sequences in the Teichmüller space of \(S\). Then the following is equivalent:

- \(\lim_{n \to \infty} \delta(S_n, S'_n) = 0\)
- \(\lim_{n \to \infty} \delta(S_n, S'_n, 1) = 0\)
- \(\lim_{n \to \infty} C(S_n, S'_n, 1) = 0\)

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Proof. From Corollary 2.16, \( \delta(S_n, S'_n, 1) = \frac{C(S_n, S'_n, 1)}{\delta(S_n, S'_n, 1)} \) hence the second and third statements are clearly equivalent. We then show the equivalence of the first and the second ones.

By definition \( 0 \leq \delta(S_n, S'_n, 1) \leq \delta(S_n, S'_n) \), hence \( \delta(S_n, S'_n) \) tends to zero, implies that \( \delta(S_n, S'_n, 1) \) tends also to zero.

Conversely, by Corollary 2.16 \( \delta(S_n, S'_n, 1) = \frac{a^1_n(1) + a^2_n(1)}{2} \), where \( (a^1_n(1), a^2_n(1)) \) is the point on the Manhattan curve \( C^M_n \) for the surfaces \( S_n \) and \( S'_n \), where the slope of one normal vector is 1. From the convexity of \( C^M_n \) we have that \( a^1_n(1) > 0 \) and \( a^2_n(1) > 0 \). Hence if \( \lim_{n \to \infty} \delta(S_n, S'_n, 1) = 0 \) it follows that \( \lim_{n \to \infty} a^1_n(1) = 0 \) and \( \lim_{n \to \infty} a^2_n(1) = 0 \). By continuity of \( C^M_n \), this implies that there are points in \( c^M_n \) which are as close as we want from the origin, and since the critical exponent is equal to the abscissa of the intersection of \( C^M_n \) and the line \( y = x \), this finally implies that the critical exponent can be made as small as we want.

Corollary 2.19. Let \( S_n \) and \( S'_n \) be two sequences in the Teichmüller space of \( S \). Then the following is equivalent:

- \( \lim_{n \to \infty} \delta(S_n, S'_n) = 1/2 \)
- \( \lim_{n \to \infty} \delta(S_n, S'_n, 1) = 1/2 \)
- \( \lim_{n \to \infty} C(S_n, S'_n, 1) = 1 \)

Proof. Here again the second and the third statement are equivalent from the formula obtained in Corollary 2.16.

We now show the equivalence of the first two statements. By definition \( \delta(S_n, S'_n, 1) \leq \delta(S_n, S'_n) \leq 1/2 \) hence the second statement implies the first.

Conversely, if \( \lim \delta(S_n, S'_n, 1) = 1/2 \) by Corollary 2.4, \( \lim \lambda(\delta(S_n, S'_n, 1)) = 1 \) since the the Manhattan curve is \( C^1 \), this implies that \( \lim (a^1_n(1), a^2_n(1)) = \lim (\delta(S_n, S'_n), \delta(S_n, S'_n)) = (1/2, 1/2) \). Hence \( \lim_{n \to \infty} \delta(S_n, S'_n, 1) = \lim_{n \to \infty} \frac{a^1_n(1) + a^2_n(1)}{2} = \lim_{n \to \infty} \frac{\delta(S_n, S'_n) + \delta(S_n, S'_n)}{2} = 1/2. \)

3 Geometric background

3.1 Geodesic currents

The notion of geodesic currents has been introduced by F. Bonahon in [? ?] in order to get, in Bonhanon’s words "a better understanding of homotopy classes of unoriented closed curves on \( S^n \). But it is also a beautiful tool to understand Thurston compactification of Teichmüller space by measured geodesic laminations, [? ?], or the ends of three manifolds as in [? ?]. Geodesic currents are an extension of closed geodesics where the set of geodesic measured laminations as well as Teichmüller space \( \text{Teich}(S) \) have natural embeddings. All the material of this section is well known and can be found in [? ?], [? ?], [? ?], [? ?], [? ?]. We survey this material in order to introduce notation. We will follow the lines of [? ?].

Let \( S \) be a surface of genus \( g \geq 2 \), that we endow with a hyperbolic metric, and \( CS \) the set of closed unoriented geodesics on \( S \). In this section, geodesics will be primitive elements, ie represented by a indivisible element of \( \Gamma \). There is a bijective correspondence between the set of homotopy classes of unoriented closed curves and the set of unoriented closed geodesics with multiplicity. We didn’t make this distinction in the previous section since according to Knieper’s theorem, the exponential growth of the number of primitive geodesics and non-primitive geodesics are the same, hence the resulting Manhattan curve and critical exponent are the same.
Consider \( \tilde{S} \) the universal covering of \( S \), and \( G\tilde{S} \) the set of geodesics of \( \tilde{S} \). Any closed geodesic lifts to a \( \Gamma \)-invariant set of \( G\tilde{S} \). We can take the multiplicity into account if we identify the lifts with a Dirac measure on this discrete subset, where the Dirac measures are multiplied by the multiplicity of the geodesic. This is equivalent to put some weight on geodesics. Hence a weighted sum of geodesics of \( S \) is naturally a \( \Gamma \)-invariant measure of \( G\tilde{S} \).

There is a parametrization of unoriented geodesics by the boundary at infinity of \( \tilde{S} \), say \( \partial \tilde{S} \), that is \( \partial \tilde{S} \simeq \partial S \times \partial \tilde{S} - \Delta/\mathbb{Z}_2 \), where \( \Delta \) is the diagonal and \( \mathbb{Z}_2 \) acts by exchanging the two factors. Giving a hyperbolic metric on \( S \) gives an identification between \( \tilde{S} \) and \( \mathbb{H}^2 \), whose boundary is canonically \( S^1 \). Hence \( G\tilde{S} \) is naturally identified with \( S^1 \times S^1 - \Delta/\mathbb{Z}_2 \).

**Definition 3.1.** A geodesic current is a \( \Gamma \)-invariant borelian measure on \( G\tilde{S} \). The set \( GC \) of geodesic currents is endowed with the weak* topology defined by the family of semi-distances \( d_f \) where \( f \) ranges over all continuous functions \( f : G\tilde{S} \to \mathbb{R} \) with compact support and where \( d_f(\alpha, \beta) = |\alpha(f) - \beta(f)| \).

By the previous discussion the set of homotopy classes of closed curves on \( S \) is embedded in \( GC \).

**Proposition 3.2.** \[ , proposition 2 \] The space \( GC \) is complete and the real multiples of homotopy classes of closed curves are dense in it.

**Definition 3.3.** If \([\alpha] \) and \([\beta] \) are two homotopy classes of closed curves on \( S \), their geometric intersection is the infimum of the cardinality of \( \alpha' \cap \beta' \) for all \( \alpha' \in [\alpha] \) and \( \beta' \in [\beta] \).

This infimum is realized for \( \alpha' \) and \( \beta' \) the geodesic representatives, \[ , Ch 3, proposition 10 \] .

The crucial fact is that the geometric intersection can be extended to geodesic currents.

**Theorem 3.4.** \[ , Paragraph 4.2 \] There is a continuous symmetric bilinear function \( i : GC \times GC \to \mathbb{R}^+ \) such that, for any two homotopy classes of closed curves \( \alpha, \beta \in GC \), \( i(\alpha, \beta) \) is the above geometric intersection.

If we fix a current \( \alpha \) whose support is sufficiently large (if it fills \( S \)), then the set of currents whose intersections with \( \alpha \) is bounded, is compact. More precisely we have the following.

**Theorem 3.5.** \[ , Proposition 4 \] Let \( \alpha \) be a geodesic current with the property that every geodesics of \( \tilde{S} \) transversely meets some geodesic which is in the support of \( \alpha \). Then the set \( \{ \beta \in GC , i(\alpha, \beta) \leq 1 \} \) is compact in \( GC \).

Taking a current which fills \( S \) gives the following,

**Corollary 3.6.** \[ , Corollary 5 \] The space \( PGC \) of projective geodesic currents is compact.

We are not going to use exactly this fact but a related one : the set of geodesic currents of length less or equal to one is compact. This leads us to define a length for any geodesic current. Let \( F \) be the foliation of \( T^1 S \) by the orbit of geodesic flow. For each \( \varphi_t \), invariant finite measure \( \mu \) there exists an associated transverse measure to \( F \), \( \tilde{\mu} \), normalized by the requirement that it is locally a product : \( \mu = \tilde{\mu} \times dt \), where \( dt \) is the Lebesgue measure along orbits of \( \varphi_t \) in \( F \). By definition \( \tilde{\mu} \) is a geodesic current. Hence there is a bijective correspondence between the set of invariant measures by the geodesic flow and the set of currents. In particular the Liouville measure gives rise to a geodesic current, \( L \). This current depends on the hyperbolic metric and we will write the metric as a subscript if there is an ambiguity on the metric we consider on \( S \). For example \( \ell_m \) will be the Liouville geodesic current on \((S, m)\).
Theorem 3.7. Let $(S,m)$ be a hyperbolic surface. For every closed curve $c \in GC$, we have $i(\ell_m, c) = \ell_m(c)$.

This remarkable property allows us to define the length of any geodesic current once we have set a hyperbolic metric on $S$, by the extension of $i$ to any geodesic current. Let $m$ be a hyperbolic metric, for every $\beta \in GC$, by definition $\ell_m(\beta) := i(\ell_m, \beta)$.

Since the Liouville current has the property to intersect transversely every geodesics of $\tilde{S}$, applying theorem 3.5 we get,

Theorem 3.8. For any hyperbolic metric $m$, the set of geodesic currents of $m$-length equal to one is compact. That is the set $\{ \beta \in GC, i(\beta, \ell_m) = 1 \}$ is compact.

The Liouville currents are particularly interesting since they allow to embed Teichmüller space into the set of geodesic currents.

Theorem 3.9. The map $m \rightarrow \ell_m$ is a proper embedding of Teichmüller space into the space of geodesic currents.

Definition 3.10. Let $S_1 = (S, m_1)$ and $S_2 = (S, m_2)$ be two surfaces endowed with hyperbolic metric. Then the intersection between $S_1$ and $S_2$ is defined by

$$i(S_1, S_2) := i(\ell_{m_1}, \ell_{m_2}).$$

For all $m \in \text{Teich}(S)$ we have that $i(\ell_m, \ell_m) = \pi^2|\chi(S)|$ [? , Proposition 15] hence if $\ell_m = k\ell_{m'}$ then $k = 1$ and $m = m'$.

Corollary 3.11. The composition $\text{Teich}(S) \rightarrow GC \rightarrow PGC$ is an embedding.

Finally here is the corollary we are going to use and which is clearly equivalent to the previous one, since there is a homeomorphism between projective current and the current of length 1.

Corollary 3.12. Let $(S_0, m_0)$ be a fixed hyperbolic surface, then the map $m \rightarrow \frac{\ell_m}{i(m_0, \ell_m)}$ is an embedding of Teichmüller space into the space of geodesic currents of $S_0$ length 1.

3.2 Geodesic laminations

Let us introduce geodesic laminations in the context of geodesic currents. The simplest definition in this setting is,

Definition 3.13. A measured geodesic lamination $L$ is a geodesic current whose self-intersection $i(L, L)$ is equal to 0. The set of measured geodesic laminations will be denoted by $ML$.

The original definition of measured laminations due to W. Thurston [? ] has been shown to be equivalent to the above by F. Bonahon. Recall that according to Thurston, a measured geodesic lamination is a closed subset $\Lambda$ which is the union of simple disjoint geodesics endowed with a transverse measure, i.e. a measure on every arc transverse to $\Lambda$ invariant by holonomy. The Thurston’s topology on the set of measured lamination is defined by the semi-distances $d_\gamma(\alpha, \beta) = |i(\alpha, \gamma) - i(\beta, \gamma)|$ where $\gamma$ ranges over all simple closed curves on $S$.

Theorem 3.14. The set $\{ \alpha \in GC, i(\alpha, \alpha) = 0 \}$ has a natural homeomorphic identification with Thurston’s space of measured laminations.
Let $\mathcal{PML}$ be the set of projective measured laminations, that is the quotient of measured lamination by $\mathbb{R}^*_+$, where the equivalence is $(L, \mu) \sim (L, x\mu)$ for $x \in \mathbb{R}^*_+$. The topology of Thurston on $\text{Teich}(S) \cup \mathcal{PML}$ is defined as follows: $\text{Teich}(S)$ is open in $\text{Teich}(S) \cup \mathcal{PML}$ and a sequence $m_j \in \text{Teich}(S)$ converges to $L$ if and only if the ratio $\frac{\ell_{m_j}(\alpha)}{\ell_{m_j}(\beta)}$ converges to $\frac{\ell(L, \alpha)}{\ell(L, \beta)}$ for every pair of simple closed curves $\alpha, \beta$ with $i(\alpha, \beta) \neq 0$.

**Theorem 3.15.** [??, Theorem 18] The topology of $\text{Teich}(S) \cup \mathcal{PML}$ seen as a subspace of $\mathcal{PGL}$ is the same as the Thurston’s topology.

By Corollary 3.12 if we fix a metric $m$, the set of geodesic laminations of $m$-length 1 is compact.

The simplest example of measured geodesic lamination is, of course, a weighted simple geodesic, or a disjoint union of such measures, but in fact any geodesic lamination is a limit of such examples.

**Theorem 3.16.** [??, proposition 4.9] $\mathcal{ML} / \mathcal{PML}$ is the closure of the linear combinations of closed disjoint geodesics.

### 3.3 Earthquakes

We introduced geodesic laminations in order to present a central tool in Teichmüller theory, the earthquake map. They are an extension of the notion of Fenchel-Nielsen twists. Let $c$ be a closed simple geodesic on $(S, m)$, take a tubular neighborhood $U \simeq \mathbb{S}^1 \times [-\epsilon, \epsilon]$ of $c$, such that $\mathbb{S}^1 \times \{0\}$ is isometrically sent to $c$, and every $z \times [-\epsilon, \epsilon]$ corresponds to a geodesic arc orthogonal to $c$. Consider a smooth function $\xi : [0, \epsilon] \to \mathbb{R}$ equal to 0 on a neighborhood of $\epsilon$ and equal to 1 on a neighborhood of 0. Then, for $t \in \mathbb{R}$ consider the map $\phi_t : S \to S$ which is the identity on $S - U$ and on $\mathbb{S}^1 \times [-\epsilon, 0]$ and which is defined by $\phi_t(e^{i\theta}, u) = (e^{i\theta - it\xi(u)}, u)$ on $\mathbb{S}^1 \times [0, \epsilon] \subset U$. The map $\phi_t$ is a diffeomorphism on $S - c$, discontinuous along $c$. The metric $\phi_t^*(m)$ on $S - c$ coincides with the original metric $m$ if we are sufficiently close to $c$, and it extends to a hyperbolic metric over all $S$. We denote the new metric by $\mathcal{E}_t^1(m)$, which is, in Thurston terminology, the metric obtained from $m$ by a left earthquake of amplitude $t$ along the curve $c$. When $c$ is a simple closed curve, this transformation is more classically called Fenchel-Nielsen twist, the extension to any geodesic lamination has been studied by W. Thurston.

**Theorem 3.17.** [??, Proposition 1] There is a continuous function $\mathcal{E} : \mathcal{ML} \times \mathbb{R} \times \text{Teich}(S) \to \text{Teich}(S)$, associating to $m \in \text{Teich}(S)$ an element $\mathcal{E}_t^\lambda(m) \in \text{Teich}(S)$ such that $\mathcal{E}_t^\lambda(m) = \mathcal{E}_t^1(m)$, for all $\lambda > 0$ and all $L \in \mathcal{ML}$, and coincide with a Fenchel-Nielsen twist when $L$ is a closed geodesic.

Moreover S. Kerckhoff showed that the length function is convex along earthquake.

**Theorem 3.18.** [??, p253, Theorem 1] Let $m$ be a hyperbolic metric, $L$ be a measured geodesic lamination, and $c$ be a closed curve. Then the function $t \to \ell(\mathcal{E}_t^1(m)c) = i(\mathcal{E}_t^1(m), c)$ is convex.

The last tool we are going to use is the "geology" theorem of Thurston saying that any two hyperbolic metrics are linked by an earthquake.

**Theorem 3.19.** [??, Appendix] Let $(S_1, m_1)$ and $(S_2, m_2)$ be two hyperbolic metrics on $S$, then there exists a unique measured lamination $L$ such that

$$\mathcal{E}_1^{1}(m_1) = m_2.$$
We are going to use the equivalent following corollary, which is just a renormalization of the measured lamination seen as current.

**Corollary 3.20.** Let $(S_1, m_1)$ and $(S_2, m_2)$ be two hyperbolic metrics on $S$, then there exists a unique measured lamination $\mathcal{L}$ of $m_1$-length 1 and a $T \in \mathbb{R}$ such that:

$$E_T^\mathcal{L}(m_1) = m_2.$$  

### 3.4 Examples

In the paper [? ], R. Sharp is asking about the behavior of the correlation number $C(S_1, S_2, 1)$ as $S_1$ and $S_2$ range over $\text{Teich}(S)$. By Corollaries 2.18 and 2.19, the asymptotic behavior of $C(S_1, S_2, 1)$, $\delta(S_1, S_2)$ and $\delta(S_1, S_2, 1)$ are the same.

- Let $S_0$ be a hyperbolic surface and $A$ be a pseudo-Anosov diffeomorphism, with measured foliations associated to it, $\mathcal{F}^\pm$ and dilatations $k$. Consider the sequences $S_n := A^n S_0$ and $S'_n := A^{-n} S_0$. The following can be found in [? , Ch9, proposition 19]. Let $c \in \mathcal{C}_S$, for all $n \in \mathbb{N}$ we have

$$\ell_n(c) = \ell_0(A^n c) \geq k^n i(\mathcal{F}^+, c)$$

$$\ell'_n(c) = \ell_0(A^{-n} c) \geq k^n i(\mathcal{F}^-, c).$$

Where $i(\mathcal{F}^\pm, c)$ is the integral along $c$ of the transverse measure on $\mathcal{F}^\pm$. Hence

$$\ell_n(c) + \ell'_n(c) \geq k^n i(\mathcal{F}^+ \cup \mathcal{F}^-, c).$$

On the other hand, there exists $K_0$ such that for all $c \in \mathcal{C}_S$, $i(\mathcal{F}^+ \cup \mathcal{F}^-, c) > K_0 \ell_0(c)$, see [? , Ch9, lemma 20]. We then have

$$\ell_n(c) + \ell'_n(c) \geq k^n K_0 \ell_0(c).$$

This finally gives

$$\sum_{c \in \mathcal{C}_S} e^{-s(\ell_n(c) + \ell'_n(c))} \leq e^{-sk^n K_0 \ell_0(c)}$$

and $\delta(S_n, S'_n) \leq \frac{1}{k^n K_0}$ hence goes to zero.

- Let $S_0$ be a hyperbolic surface. Take two pants decomposition of $S_0$, $\mathcal{P}$ and $\mathcal{P}'$, such that $\mathcal{P} \cup \mathcal{P}'$ fills up $S_0$ (ie the complement of $\mathcal{P} \cup \mathcal{P}'$ consists of topological disks). Then we consider the surfaces $S_n$ and $S'_n$, defined by shrinking the geodesics of $\mathcal{P}$ respectively $\mathcal{P}'$ by a factor $e^{-n}$. By the [? , collar lemma], there exists $M$ such that for every $c \in \mathcal{C}_S$, $\ell_n(c) \geq C |\log(e^{-n})| i(c, \mathcal{P}) = C i(c, \mathcal{P}) n$, since the length of the geodesics in $\mathcal{P}$ are less than $e^{-n}$. We also have $\ell'_n(c) \geq M |\log(e^{-n})| i(c, \mathcal{P}') = n M i(c, \mathcal{P}')$. These two inequalities give,

$$\ell_n(c) + \ell'_n(c) \geq n M i(c, \mathcal{P} \cup \mathcal{P}'),$$

By [? , Ch 4, lemma 3], since $\mathcal{P} \cup \mathcal{P}'$ fills up $S_0$, there exists $K_0 > 0$ such that for all $c \in \mathcal{C}_S$ we have $i(c, \mathcal{P} \cup \mathcal{P}') \geq K_0 \ell_0(c)$, hence

$$\ell_n(c) + \ell'_n(c) \geq n M K_0 \ell_0(c).$$

Again $\delta(S_n, S'_n) \leq \frac{1}{n M K_0}$ goes to zero.
• The next example contains the basic idea for the proof of Theorem 1.2: we show that along a sequence of Dehn twists, the critical exponent is decreasing. Let \( S_0 := (S, m_0) \) be a hyperbolic surface and \( \alpha \) a simple closed curve on \( S \). Let \( t_\alpha \) be the Dehn twist along \( \alpha \) and define \( S_n \) (respectively \( S'_n \)) by \( S_n := t^n_\alpha S_0 \) (respectively \( S'_n := t^{-n}_\alpha S_0 \)). By definition we have \( S_n = \mathcal{E}_{G_\alpha}(S_0) \). We denote for \( t \in \mathbb{R} \) the surface \( S_t := \mathcal{E}_{G_\alpha}(S_0) \). Let \( c \) be a closed curve on \( S \) and \( f \) be the function \( f : t \to \ell_t(c) \), that is the length of the geodesic representative of \( c \) on \( S_t \). By theorem 3.18 the function \( f \) is convex. Finally let \( g(t) = f(t) + f(-t) \) and remark that

\[
g(n) = f(n) + f(-n) = \ell_n(c) + \ell'_n(c)
\]

(9)

(10)

\( g \) is convex and \( g'(0) = 0 \) hence, \( g \) is an increasing function on \( \mathbb{R}^+ \). Hence \( g(n) \geq g(n-1) \) or equivalently,

\[
\ell_n(c) + \ell'_n(c) \geq \ell_{n-1}(c) + \ell'_{n-1}(c)
\]

This implies that \( n \to \delta(S_n, S'_n) \) is decreasing. Moreover by Theorem 2.1 \( \delta(S_1, S'_1) < 1/2 \) and we can then conclude

\[
\lim \delta(S_n, S'_n) < 1/2.
\]

The limit exists since \( \delta(S_n, S'_n) \) is decreasing.

• Let \( c \) be a simple closed curve on \( S \). Choose \( \alpha \) and \( \beta \) two closed curves, non homotopic to \( c \) nor 0 (on a connected component ) of \( S \setminus c \), such that \( \gamma_\alpha \) and \( \gamma_\beta \), elements in \( \Gamma \) which project on \( \alpha \) and \( \beta \), generates a free group of rank 2. This can be done since (every connected component of ) \( S \setminus c \) has negative Euler characteristic. Now let \( S_t \) be the hyperbolic surface obtained by shrinking \( c \) by a factor \( e^{-t} \) and let \( \rho_t \) be the representation of \( \Gamma \) into \( \text{PSL}_2(\mathbb{R}) \), such that \( S_t \simeq \mathbb{H}^2/\rho_t(\Gamma) \). Since \( \alpha \) and \( \beta \) are not homotopic to \( c \) their lengths on \( S_t \) are bounded away from 0. Hence the group \( \rho_\alpha(\gamma_\alpha), \rho_\beta(\gamma_\beta) \) is quasi-isometric to \( \langle \gamma_\alpha, \gamma_\beta \rangle \), with a constant of quasi-isometry independent of \( t \). Since the critical exponent of the Poincaré series, \( \sum_{w \in \langle \gamma_\alpha, \gamma_\beta \rangle} e^{-sd(w,0,0)} \), is strictly positive, the one of \( \sum_{w \in \langle \gamma_\alpha, \gamma_\beta \rangle} e^{-sd(w,0,0)+d(\rho_t(w),0,0)} \), is also bounded away from 0 independently of \( t \). Since the Poincaré series of the whole group is larger than the one restricted to \( \langle \gamma_\alpha, \gamma_\beta \rangle \), this finally implies that \( \liminf \delta(S_0, S_t) > 0 \). This example as to be compared to the equivalent of [?] p.3 Example 3.

• Finally we give an example of a family of surfaces \( (S_t, S_t') \) for which the Thurston distance between the two representations is bounded below (or even tends to infinity), but the critical exponent tends to 1/2. Let \( S_0 \) be a hyperbolic surface, and \( c \) be a simple closed curve. Let \( S_t \) the hyperbolic surface obtained by shrinking \( c \) by a factor \( e^{-t} \). The Thurston distance between \( S_t \) and \( S_{t+1} \) is bounded below since it is bigger than the ratio \( \log \left( \frac{\ell_t(c)}{\ell_{t+1}(c)} \right) = \log \left( \frac{e^{-t}}{e^{-t-1}} \right) \) = \( \log e = 1 \). Hence \( d_T(S_t, S_{t+1}) \geq 1 \). The next theorem is Theorem 1.3.

**Theorem 3.21.** For the family \( S_t \), we have \( \lim_{t \to \infty} \delta(S_t, S_{t+1}) = 1/2 \).

**Proof.** This is due to two facts. The first one is that the Weil-Petersson distance of the path \( p : \mathbb{R}^+ \to \text{Teich}(S) \), \( p(t) = S_t \) is finite, this is proved by Wolpert in [?] . The second fact is the link between the intersection and Weil-Petersson metric. Bonahon showed in [?] , Theorem 19] that \( i(m, m') = i(m, m) + o(d_W(p(m, m'))) \).
Theorem 4.1. Let \( h \) surface groups acting on \( \mathbb{H}^1 \). We are now ready to enter into the proof of the theorems of isolation of critical exponent for 4 Isolation theorem intersection of a random geodesic. His method, inspired the proof of our main result. Moreover \( \eta \) Then \( c \) geodesic \( \lim \). Therefore, \( \therefore \)

Moreover, \( \eta > 0 \) is a consequence that \( \hat{L} \) is the only measure of maximal entropy and \( h(\hat{L}) = 1 \). This theorem has been used by Y. Herrera in his thesis \([?]\) to estimate the number of self intersection of a random geodesic. His method, inspired the proof of our main result.

3.5 Large deviation theorem

Finally we will use a theorem of large deviation for orbits of geodesic flow due to Y. Kifer \([?]\) (in a much more general context). Let \( \mathcal{P} \) be the set of \( \varphi_t \)-invariant probability on \( T^1S \). Every closed geodesic \( c \) can be considered as an element of \( \mathcal{P} \), if we see it as \( \frac{\nu}{\eta_T} \), where \( \hat{c} \) is the Dirac measure along \( c \). By the discussion in section section 3.1, the set of invariant probability measures is in bijection with the set of geodesic currents of length equal to 1. Let \( \hat{L} \) be the Liouville measure on \( T^1S \) (which corresponds to the Liouville current \( L \)). We give a name for the set of geodesics of \( S \) of length less than \( T \):

\[
CS(T) := \{ c \in CS \mid \ell(c) \leq T \}.
\]

Theorem 3.22. For any open neighborhood \( \mathcal{U} \) of \( \hat{L} \) in \( \mathcal{P} \), there exists \( \eta > 0 \) such that

\[
\frac{1}{\text{Card} CS(T)} \left\{ \gamma \in CS(T), \frac{\hat{c}}{\ell(c)} \notin \mathcal{U} \right\} = O(e^{-\eta T}).
\]

Moreover \( \eta := \inf_{\nu \in \mathcal{U}} \{ 1 - h(\nu) \} \) where \( h(\nu) \) is the entropy of \( \varphi_T \) with respect to \( \nu \).

The fact that \( \eta > 0 \) is a consequence that \( \hat{L} \) is the only measure of maximal entropy and \( h(\hat{L}) = 1 \).

This theorem has been used by Y. Herrera in his thesis \([?]\) to estimate the number of self intersection of a random geodesic. His method, inspired the proof of our main result.

4 Isolation theorem

We are now ready to enter into the proof of the theorems of isolation of critical exponent for surface groups acting on \( \mathbb{H}^2 \times \mathbb{H}^2 \) by diagonal action.

Theorem 4.1. Let \( S_0 \) be a fixed hyperbolic surface and \( S_n \) be a sequence of hyperbolic surfaces. Then \( \lim_{n \to \infty} \delta(S_0, S_n) = 1/2 \) if and only if \( \lim_{n \to \infty} d_T(S_0, S_n) = 0 \).

Proof. One way is just the continuity of critical exponent at \( S_0 \). Let’s prove it briefly. If \( d_T(S_n, S_0) \to 0 \), for all \( \epsilon > 0 \), there is a \( n_0 \) such that for all \( c \in CS \), and all \( n \geq n_0 \),

\[
1 - \epsilon < \frac{\ell_n(c)}{\ell_0(c)} < 1 + \epsilon.
\]

Hence \( \sum_{c \in CS} e^{-s(\ell_0(c) + \ell_n(c))} > \sum_{c \in CS} e^{-s(\ell_0(c) + (1+\epsilon)\ell_n(c))} \) This implies that for any \( \epsilon > 0 \), and for \( n \geq n_0 \) we have

\[
\delta(S_0, S_n) \geq \frac{1}{2 + \epsilon}.
\]

Therefore, \( \lim \delta(S_0, S_n) \geq \frac{1}{2} \). Recalling that \( 1/2 \geq \delta(S_0, S_n) \) for any surfaces gives the result.

Let’s show the converse. Suppose by absurd that \( d_T(S_0, S_n) \) doesn’t tend to 0. If \( S_n \) stays in a compact subset of \( \text{Teich}(S) \), it admits a converging subsequence, that we still denote by \( S_n \). Denote by \( S_\infty \) its limit which by hypothesis is different of \( S_0 \), then \( \delta(S_0, S_\infty) < 1/2 \) by rigidity Theorem 2.1. By continuity of critical exponent for the Thurston metric, \( \lim \delta(S_0, S_n) = \delta(S_0, S_\infty) < 1/2 \).
sequences over these curves has a decreasing critical exponent. Consider the following function, 
\[ f : \mathcal{ML}_1(S_0) \times \mathcal{GC} \to \mathbb{R} \]
\[ (\mathcal{L}, \nu) \mapsto \frac{i(E^1_{\nu}(m_0), \nu)}{i(\mathcal{L}, \nu)} \]
which is continuous, since earthquakes and intersection are continuous, Theorems 3.17 and 3.4.

And where \( m_0 \) is the metric on \( S_0 \), \( \mathcal{ML}_1(S_0) \) is the set of laminations of \( m_0 \)-length 1 and \( \ell_0 \) is the Liouville current associated to \( m_0 \). The set \( \mathcal{ML}_1(S_0) \) is compact hence \( g(\nu) := \min_{\mathcal{L}} f(\mathcal{L}, \nu) \) is well defined and continuous.

The compacity of \( \mathcal{ML}_1(S_0) \) implies also the existence of \( \mathcal{L}_0 \in \mathcal{ML}_1(S_0) \) such that 
\[ g(\ell_0) = \min_{\mathcal{L}} f(\mathcal{L}, \ell_0) = \frac{i(E^1_{\nu}(m_0), \mathcal{L}_0)}{i(\mathcal{L}, \ell_0)}. \]
Remark that \( E^1_{\nu}(m_0) \neq m_0 \) since \( \mathcal{L}_0 \) is not the trivial lamination, hence by Corollary 2.6 it follows that \( g(\ell_0) > 1 \).

Let \( \mathcal{U} \) be the neighborhood of \( \hat{\ell}_0 \) in \( \mathcal{P} \) defined by \( \mathcal{U} := \{ \hat{\mu} \in \mathcal{P} \mid |g(\mu) - g(\ell_0)| < \epsilon/2 \} \), and consider the set
\[ CSU := \left\{ c \in CS \mid \frac{c}{\ell_0(c)} \in U \right\}. \]

By definition, if \( c \in CS \mathcal{U} \), it satisfies, 
\[ g(c) - g(\ell_0) > -\epsilon/2, \]
that is \( \min_{\mathcal{L}} \frac{i(E^1_{\nu}(m_0), c)}{i(\mathcal{L}, c)} > 1 + \epsilon/2. \) Equivalently, we have for any \( \mathcal{L} \in \mathcal{ML}_1(S_0) \)
\[ \frac{i(E^1_{\nu}(m_0)(c))}{i(\mathcal{L}, c)} > 1 + \epsilon/2. \]

By convexity of length along earthquakes theorem 3.18, we have for all \( \mathcal{L} \in \mathcal{ML}_1(S_0) \), all \( t > 1 \) and all \( c \in CS \mathcal{U} \),
\[ \ell(E^1_{\mathcal{L}}(m_0)(c)) \geq (1 + \epsilon t/2)\ell_0(c). \]
This last inequality is what we mathematically meant, by saying that the length of "most" curves are increasing. Indeed Kifer’s result, theorem 3.22, says that \( \text{Card} CS \mathcal{U} \cap CS(k) \) has smaller exponential growth than \( \text{Card} CS(k) \).

Let’s look to the Poincaré series associated to \( (S_0, S_n) \),
\[ P_{n,t}(s) := \sum_{c \in CS} e^{-s(\ell_0(c) + \ell_n(c))}. \]

By Corollary 3.20, there exists \( \mathcal{L}_n \in \mathcal{ML}_1(S_0) \) and \( t_n \) such that \( E^1_{\mathcal{L}_n}(m_0) = m_n \), hence
\[ P_{n,t}(s) = \sum_{c \in CS} e^{-s(\ell_0(c) + i(E^1_{\mathcal{L}_n}(m_0), c))}. \]

Now we divide this sum into two parts, the curves which are in \( CS \mathcal{U} \) and the others.

Since \( S_n \) goes out of every compacts of \( \text{Teich}(S) \) and by continuity of \( t, \mathcal{L} \mapsto E^1_{\mathcal{L}}(m_0) \), the sequences \( t_n \) must tends to infinity. Hence for \( n \) sufficiently large \( t_n > \frac{4}{\epsilon} \). Moreover
\[ \sum_{c \in CS \mathcal{U}} e^{-s(\ell_0(c) + i(E^1_{\mathcal{L}_n}(m_0), c))} < \sum_{c \in CS \mathcal{U}} e^{-s(\ell_0(c)(1 + \epsilon t_n/2))} \]
Hence the series \( \sum_{c \in \mathcal{C}S_{U}'} e^{-s(\ell_0(c) + f_0(c))} \) has critical exponent strictly less than 1/2. In fact we see that this series has critical exponent tending to 0 when \( n \to \infty \).

In the second step of the proof we get an upper bound on the critical exponent of the Poincaré sum over \( \mathcal{C}S_{U}' \). This relies on Kifer’s theorem and the directional critical exponent since we want that the length of the geodesics on the second factor to be almost proportional to the length on the first.

Let \( \lambda_n \) be the slope for which the directional critical exponent \( \delta(\lambda_n) \) between \( S_0 \) and \( S_n \) is maximal. Let \( u > 0 \) and \( A_n(u) := \left\{ c \in \mathcal{C}S ; \left| \frac{\ell_n(c)}{\ell_0(c)} - \lambda_n \right| < u \right\} \). By theorem 2.7 and Corollary 2.17 for any \( u > 0 \) the critical exponent of \( \sum_{c \in A_n(u)} e^{-s(\ell_0(c) + f_0(c))} \) is equal to the critical exponent of the whole Poincaré series \( P_{0,n} \), that is to say \( \delta(S_0, S_n) \). Hence \( \delta(S_0, S_n) \) is equal to the maximum of the critical exponent of the two following series:

\[
\sum_{c \in \mathcal{C}S_{U} \cap A_n(u)} e^{-s(\ell_0(c) + f_0(c))}
\]

We previously saw that the critical exponent of the first one goes to \( 0 \) hence is strictly less than \( \delta(S_0, S_n) \) for \( n \) sufficiently large, since we suppose that \( \delta(S_0, S_n) \to 1/2 \). Hence \( \delta(S_0, S_n) \) is equal to the critical exponent of \( \sum_{c \in \mathcal{C}S_{U} \cap A_n(u)} e^{-s(\ell_0(c) + f_0(c))} \). The end of the proof consists to show that this exponent cannot tend to 1/2.

Hence, for \( n \) sufficiently large, \( \delta(S_0, S_n) \) is equal to the critical exponent of

\[
\sum_{c \in \mathcal{C}S_{U} \cap A_n(u)} e^{-s(\ell_0(c) + f_0(c))}
\]

For \( c \in A_n(u), \ell_n(c) \geq \ell_0(c)(\lambda_n - u) \), we then have

\[
\sum_{c \in \mathcal{C}S_{U} \cap A_n(u)} e^{-s(\ell_0(c) + f_0(c))} \leq \sum_{c \in \mathcal{C}S_{U} \cap A_n(u)} e^{-sf_0(c)(1+\lambda_n-u)}
\]

By theorem 3.22, the complementary set of \( \mathcal{C}S_{U} \) is "small" for the metric \( m_0 \), that is there exists \( \eta > 0 \) and \( M > 0 \) such that \( \text{Card}(\mathcal{C}S_{U}' \cap \mathcal{C}S(T)) \leq M \text{Card}(\mathcal{C}S(T)) e^{-\eta T} \leq M'e^{(1-\eta)T} \). Let \( \mathcal{C}S_{U}'(k) = \{ c \in \mathcal{C}S_{U}' \} \text{ and } l(c) \in [k, k+1] \} \).

\[
\sum_{c \in \mathcal{C}S_{U}' \cap A_n(u)} e^{-sf_0(c)(1+\lambda_n-u)} \leq \sum_{k \in \mathbb{N}} \sum_{c \in \mathcal{C}S_{U}'(k)} e^{-sk(1+\lambda_n-u)} \leq \sum_{k \in \mathbb{N}} \sum_{c \in \mathcal{C}S_{U}'(k)} e^{-sk(1+\lambda_n-u)}.
\]

And since \( \mathcal{C}S_{U}'(k) \subset \mathcal{C}S_{U}' \cap \mathcal{C}S(k) \),

\[
\sum_{c \in \mathcal{C}S_{U}' \cap A_n(u)} e^{-sf_0(c)(1+\lambda_n-u)} \leq \sum_{k \in \mathbb{N}} \sum_{c \in \mathcal{C}S_{U}' \cap \mathcal{C}S(k)} e^{-sk(1+\lambda_n-u)} \leq \sum_{k \in \mathbb{N}} \text{Card}(\mathcal{C}S_{U}' \cap \mathcal{C}S(k)) e^{-sk(1+\lambda_n-u)} \leq \sum_{k \in \mathbb{N}} M'e^{(1-\eta)k} e^{-sk(1+\lambda_n-u)} \leq \sum_{k \in \mathbb{N}} M'e^{k(1-\eta-s(1+\lambda_n-u))}.
\]
This finally implies that $\sum_{c \in S_n \cap \Delta_n(u)} e^{-\sigma_\delta(c)(1+\lambda_n-u)}$ has critical exponent less than $\frac{1-\eta}{1+u}$.

Combined to the fact that the critical exponent of $P_{0,n}$ is less or equal to this last one, and taking the limit in $u \to 0$, we get:

$$\delta(S_0, S_n) \leq \frac{1-\eta}{1+\lambda_n}$$

Suppose that $\delta(S_0, S_n) \to 1/2$, then by Corollary 2.4 we deduce $\lambda_n \to 1$. Taking the limit $n \to \infty$, we get $1/2 \leq \frac{1-\eta}{2}$ which is absurd. This concludes the proof.

Next we look at what happens if both surfaces change. We call $\text{Teich}(S)$ the thick part of Teichmüller space, that is the surfaces for which no closed geodesic has length less than $\epsilon$. The mapping class group preserves the length spectrum, hence acts on the thick part of Teichmüller.

Recall the Mumford compactness theorem.

**Theorem 4.2.** If for any $\epsilon > 0$, $\text{Teich}(S)/\text{MCG}$ is compact.

So if a surface stays in $\text{Teich}(S)$, we can send it in a fixed compact set by the mapping class group. This remark with the previous theorem allows us to show:

**Corollary 4.3.** Let $S_n$ and $S'_n$ be two sequences of hyperbolic surfaces. Suppose that at least one the sequences stays in $\text{Teich}(S)$ for some $\epsilon$. Then $\lim_{n \to \infty} \delta(S_n, S'_n) = 1/2$ if and only if $\lim_{n \to \infty} d(S_n, S'_n) = 0$.

**Proof.** We are going to prove that if $\lim_{n \to \infty} \delta(S_n, S'_n) = 1/2$ then $\lim_{n \to \infty} d(S_n, S'_n) = 0$, the other implication is again a consequence of the continuity of the critical exponent for the Thurston metric. By hypothesis we can suppose that $S_n$ stays in $\text{Teich}(S)$. Hence there exists a compact $K$ in $\text{Teich}(S)$ and $\phi_n$ in the mapping class group, such that $\phi_n(S_n) \in K$ for every $n$. Hence we can suppose that $\phi_n(S_n)$ converges to $S_\infty \in K$.

Let $u > 0$ for every $n$ sufficiently large,

$$1 - u \leq \frac{\ell_{\phi_n(S_n)}(c)}{\ell_\infty(c)} \leq 1 + u$$

Hence the critical exponents satisfies:

$$(1 - u)\delta_{S_\infty, \phi(S'_n)} \leq \delta_{\phi_n(S_n), \phi_n(S'_n)} \leq (1 + u)\delta_{S_\infty, \phi(S'_n)}.$$

Now since the mapping class group doesn’t change the length spectrum, the critical exponent of $(S_n, S'_n)$ is equal to the critical exponent of $(\phi_n(S_n), \phi_n(S'_n))$. If $\lim_{n \to \infty} \delta(S_n, S'_n) = 1/2$ then $\lim_{n \to \infty} \delta(\phi_n(S_n), \phi_n(S'_n)) = 1/2$ and by the previous inequalities it follows that $\lim_{n \to \infty} \delta(S_\infty, \phi(S'_n)) = 1/2$ since $u$ is arbitrary small. By Theorem 4.1 this implies that $\lim_{n \to \infty} d(\phi(S'_n), S_\infty) = 0$, which finally implies that $\lim_{n \to \infty} d(S_n, S'_n) = 0$.

5 **Appendix**

5.1 **Proof of Knieper’s theorem**

The aim of this section is to extend the following theorem of Knieper to the case of a group acting on two factors.

**Theorem 5.1.** Let $\Gamma$ be a discrete, without torsion, subgroup of $\text{Isom}(\mathbb{H}^2)$. Suppose that $\Gamma$ is cocompact. Then the critical exponent of

$$\sum_{\gamma \in \Gamma} e^{-\sigma_\delta(c)(1+\lambda_n)}$$
is the same as
\[ \sum_{c \in CS} e^{-l(c)}, \]
where \( o \) is any point of \( \mathbb{H}^n \), \( CS \) is the set of homotopy classes of closed curves on \( \mathbb{H}^n/\Gamma \) and \( l(c) \) is the length of the geodesic in the homotopy class of \( c \in CS \).

In [?], Knieper showed this theorem for cocompact group acting on manifolds with bounded negative curvature.

In the context of section 1, we want to show that the Manhattan curve and the geodesic Manhattan curve coincide. The proof follows the same line as Knieper’s one. The modifications we make are essentially solved by the existence of an equivariant homeomorphism between the universal covers of \( S_1 \) and \( S_2 \). By uniformisation theorem these universal covers are isometric to \( \mathbb{H}^2 \) and we will distinguished them by indexing, that is \( \mathbb{H}^2_i \) will correspond to the universal cover of \( S_i, \ i \in \{1,2\} \).

Recall that we defined a priori two different Manhattan curves, the "classical" one:

\[ C_M := \left\{ (a, b) \in \mathbb{R}^2 \left| \sum_{\gamma \in \Gamma} e^{-s(ad_{1}(\gamma, o_1)+bd_{2}(\gamma, o_2))} \right. \right\} \]

and the geodesic one:

\[ C_M^G := \left\{ (a, b) \in \mathbb{R}^2 \left| \sum_{c \in CS} e^{-s(a\ell_1(c)+b\ell_2(c))} \right. \right\}, \]

where \( CS \) is the set of homotopy classes of non trivial closed curves on \( S \).

**Theorem 5.2.** \( C_M = C_M^G \).

Let’s define the two following sets:

\[ \Gamma^{a,b}(t) := \left\{ \gamma \in \Gamma \left| ad_{1}(\gamma, o_1) + bd_{2}(\gamma, o_2) \leq t \right. \right\}. \]

and

\[ CS^{a,b}(t) := \left\{ c \in CS \left| a\ell_1(c) + b\ell_2(c) \leq t \right. \right\} \]

Recall that the critical exponent \( \delta_{a,b} \) of the Poincaré serie \( \sum_{\gamma \in \Gamma} e^{-s(ad_{1}(\gamma, o_1)+bd_{2}(\gamma, o_2))} \) is equal to the exponential growth of \( \text{Card} \Gamma^{a,b}(t) \), that is:

\[ \delta_{a,b} = \lim \sup \frac{\log \text{Card}(\Gamma^{a,b}(t))}{t}. \]

As well the critical exponent of the geodesic Poincaré serie is

\[ \delta_{G}^{a,b} = \lim \sup \frac{\log \text{Card}(CS^{a,b}(t))}{t}. \]

The proof consists in showing that \( \delta_{a,b} = \delta_{G}^{a,b} \), which will of course imply the theorem 5.2.

We begin by a geometric lemma for \( \mathbb{H}^2 \) that we will use for the inequality \( \delta_{G}^{a,b} \leq \delta_{a,b} \). For points \( \xi, \eta \in \mathbb{H}^2 \) we note \((\xi, \eta)\) the geodesic between them. The distance on the boundary is taken to be the Gromov distance pointed on the origin of \( \mathbb{H}^2 \). One way to define the Gromov distance is \( d(\xi, \eta) := e^{-d_{\mathbb{H}^2}(o,(\xi, \eta))} \). The exponent \( d_{\mathbb{H}^2}(o,(\xi, \eta)) \) is the distance between the origin and the geodesic \((\xi, \eta)\) in \( \mathbb{H}^2 \).
Lemma 5.3. \(\bullet\) For every \(x \in \mathbb{H}^2\), for every \(R > 0\) there exists \(\epsilon > 0\) such that for every \(\xi, \eta \in \partial \mathbb{H}^2\), we have:
\[
(\xi, \eta) \cap B(x, R) \neq \emptyset \implies d(\xi, \eta) > \epsilon
\]
\(\bullet\) For every \(x \in \mathbb{H}^2\), for every \(\epsilon > 0\), there is \(R\) such that for every \(\xi, \eta \in \partial \mathbb{H}^2\), we have:
\[
d(\xi, \eta) > \epsilon \implies (\xi, \eta) \cap B(x, R) \neq \emptyset.
\]
where \((\xi, \eta)\) is the geodesic from \(\xi\) to \(\eta\).

Proof. \(\bullet\) Suppose first that \(x = o\) is the origin of \(\mathbb{H}^2\) in the disk model. Then by definition of the Gromov metric on the boundary, \(d(\xi, \eta) = e^{-d(\xi, \eta, o)}\). Hence if \((\xi, \eta) \cap B(o, R) \neq \emptyset\), we have \(d((\xi, \eta), o) \leq R\) and finally \(d((\xi, \eta), o) > e^{-R} = \epsilon\). Now if \(x\) is not the origin, there is \(R'\) such that \(B(x, R) \subset B(o, R')\). If \((\xi, \eta) \cap B(x, R) \neq \emptyset\) then \((\xi, \eta) \cap B(o, R') \neq \emptyset\), hence there is \(\epsilon'\) with the desired property.

\(\bullet\) Here again we suppose first that \(x = o\) the origin of \(\partial \mathbb{H}^2\). In this case we choose \(R\) equal to \(-\log(\epsilon)\). If \(d(\xi, \eta) > \epsilon\) we then have \(e^{-d((\xi, \eta), o)} > \epsilon\) hence, \(d((\xi, \eta), o) < R\) which proves the lemma. If \(x\) is not the origin, then choose, \(R = -\log(\epsilon) + d(o, x)\) and apply what we just did. \(\square\)

Corollary 5.4. For all \(a_1, b_2 \in \mathbb{H}^2_1\), \(a_2, b_2 \in \mathbb{H}^2_2\) and all \(R_1 > 0\) there exists \(R_2 > 0\) such that for all \(\gamma = (\gamma_1, \gamma_2) \in \Gamma\) we have:
\[
(\gamma_1^+, \gamma_2^+) \cap B(a_1, R_1) \neq \emptyset \implies (\gamma_2^+, \gamma_2^+) \cap B(a_2, R_2) \neq \emptyset.
\]

Proof. From \(R_1 > 0\) and \(a_1 \in \mathbb{H}^2_1\) the first part of the previous lemma gives a \(\epsilon_1\). Let \(d\) be the Hölder exponent of the boundary map between \(\partial \mathbb{H}^2_1\) and \(\partial \mathbb{H}^2_2\), which conjugate the action of \(\Gamma_1\) and \(\Gamma_2\). Let \(R_2 > 0\) be the real associated to \(e_1\) by the second part of the previous lemma. Now if \((\gamma_1, \gamma_2^+) \cap B(a_1, R_1) \neq \emptyset\) then \(d(\gamma_1^+, \gamma_2^+) > \epsilon_1\). By Hölder continuity of the boundary application, \(d(\gamma_2^+, \gamma_2^+) > e_1\). Hence \((\gamma_2^+, \gamma_2^+) \cap B(a_2, R_2) \neq \emptyset\). \(\square\)

The inequality \(\delta_G^{a,b} \leq \delta^{a,b}\) is easy to get, this is the aim of the next lemma.

Lemma 5.5. There is a \(k > 0\) such that
\[
\text{card}(CS^{a,b}(t)) \leq \text{card}(\Gamma^{a,b}(t + k)).
\]
Hence \(\delta_G^{a,b} \leq \delta^{a,b}\).

To avoid the multiplication of cases we are going to suppose that \(a, b > 0\), if \(a\) or \(b\) is negative then some inequalities becomes trivial. As we said, we will index the objects which are in the "left" \(\mathbb{H}^2\) by 1, and by 2 the objects which are in the "right" one. We fix two compact fundamental domains \(N_1\) and \(N_2\), for the action of \(\Gamma_1\), respectively \(\Gamma_2\), on \(\mathbb{H}^2_1\), respectively \(\mathbb{H}^2_2\), such that \(a_i \in N_i\) for \(i \in \{1, 2\}\). We call \(D_i\) their diameters. For any \(c \in CS\) or \(c \in CS^{a,b}(t)\) there is a geodesic representant in \(S := \mathbb{H}^2\), that we call \(c_1\).

Proof. Let \(c \in CS^{a,b}(t)\) and \(C_1\) be a geodesic of \(\mathbb{H}^2_1\) which projects to \(c_1 \in \mathbb{H}^2_1/\Gamma_1\) and such that \(C_1 \cap N_1 \neq \emptyset\). There is \(\gamma_1 \in \Gamma_1\) whose axis is \(C_1\). Take \(p_1 \in C_1 \cap N_1\), then
\[
d(\gamma_1 p_1, p_1) \leq d(\gamma_1 p_1, p_1) + 2d(o_1, p_1) \leq \ell_1(c) + 2D_1.
\]
On the other hand, from Corollary 5.4 there is a \(R_2 > 0\) such that \(C_2 \cap B(o_2, R_2) \neq \emptyset\), where \(C_2\) is the geodesic axis of \(\gamma_2\). Take \(p_2 \in C_2 \cap B(o_2, R_2)\),
\[
d(\gamma_2 o_2, p_2) \leq d(\gamma_2 p_2, p_2) + 2d(o_2, p_2) \leq \ell_2(c) + 2R_2.
\]
Hence \(ad_1(\gamma_1 o_1, o_1) + ad_2(\gamma_2 o_2, o_2) \leq a\ell_1(c) + 2aD_1 + b\ell_2(c) + 2bR_2\) and \(\gamma \in \Gamma(t + 2aD_1 + 2bR_2)\).

This gives an injection of \(CS^{a,b}(t)\) into \(\Gamma^{a,b}(t + 2aD_1 + 2bR_2)\) which proves the lemma. \(\square\)
We are now going to show that $\delta_G^{a,b} \leq \delta_G^{a,b}$ by a series of lemmas which allows us to bound by above the quantity $\text{Card}(CS^{a,b}(t))$. As before we begin with a topological lemma about geodesics on $\mathbb{H}^2$, this lemma is also in Knieper’s proof.

**Lemma 5.6.** [*Satz 1.4*] Let $\gamma \in \Gamma$,

1. For every neighbourhood $U^- \subset \mathbb{H}^2$ of $\gamma^-$ and $U^+ \subset \mathbb{H}^2$ of $\gamma^+$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\gamma^n(U^-) \subset U^+$ and $\gamma^n(U^+) \subset U^-$.

2. For every point $o \in \mathbb{H}^2$, there exists neighbourhoods $U^- \subset \mathbb{H}^2$ of $\gamma^-$ and $U^+ \subset \mathbb{H}^2$ of $\gamma^+$, and a constant $K_0 > 0$ such that for all $(\xi, \eta) \in U^- \times U^+$, we have

   $$d((\xi, \eta), o) \leq K_0,$$

where $(\xi, \eta)$ is the geodesic joining $\xi$ to $\eta$.

Let $f : \overline{H}_1^2 \to \overline{H}_2^2$ be a $(\Gamma_1, \Gamma_2)$-equivariant homeomorphism, sending $o_1$ to $o_2$. Let $U^-_1$ and $U^+_1$ be like in Lemma 5.6 for $\gamma_1 \in \Gamma_1$ and set $U^-_2 = f(U^-_1)$ and $U^+_2 = f(U^+_1)$. Then $U^-_2$, $U^+_2$ and $\gamma_2 \in \Gamma_2$ satisfy condition 1 of Lemma 5.6, there is an integer $n_2$

$$\gamma_2^{n_2}(\overline{H}_2^2 \setminus U^-_2) \subset U^+_2$$

and

$$\gamma_2^{-n_2}(\overline{H}_2^2 \setminus U^+_2) \subset U^-_2.$$

They also satisfy condition 2. of Lemma 5.6, there exists $K_0 > 0$ such that for all $(\xi_2, \eta_2) \in U^-_2 \times U^+_2$, we have

$$d((\xi_2, \eta_2), o_2) \leq K_0.$$

For open sets $U^-_1$, $U^+_1 \subset \overline{H}_1^2$ we define

$$A^{a,b}(U^-_1, U^+_1, t) := \{ g \in \Gamma^{a,b}(t) \mid g_1 U^-_1 \cap U^+_1 = \emptyset \}.$$

From the equivariance of $f$ we have :

$$A^{a,b}(U^-_1, U^+_1, t) = \{ g \in \Gamma^{a,b}(t) \mid g_1 U^-_1 \cap U^+_1 = \emptyset \text{ and } g_2 U^-_2 \cap U^+_2 = \emptyset \}.$$

Thanks to the previous lemma, we are going to show that we can associated to any elements $g \in A(U^-_1, U^+_1, t)$ an element $g'$ such that the Manhattan length of $g'$ is less than the Manhattan length of $g$ up to a fixed constant and such that the endpoints of $g'$ are in $U^-_1$ and $U^+_1$. The purpose of this is to restrict our attention on elements satisfying the latter conditions, that is, the elements of $A(U^-_1, U^+_1, t)$. We are then going to show that $\text{Card}(A(U^-_1, U^+_1, t))$ has the same exponential growth as the whole $\text{Card}(\Gamma^{a,b}(t))$. The end of the proof will consist to show that there is a constant $K$ such that $\text{Card}(A(U^-_1, U^+_1, t)) \leq Kt \text{Card}(CS^{a,b}(t))$ which will end the proof.

**Lemma 5.7.** Let $\gamma \in \Gamma$, such that the axes of $\gamma_i$, meet $N_i$. There exists neighborhoods $U^-_i$, $U^+_i$ of $\gamma^+_i$ for $i \in \{1, 2\}$, an integer $n \in \mathbb{N}$ and a constant $K_1 > 0$ such that for all $g \in A^{a,b}(U^-_i, U^+_i, t)$ we have :

$$\begin{align*}
(\gamma^n g \gamma^n)^- & \subset U^-_i, \\
(\gamma^n g \gamma^n)^+ & \subset U^+_i,
\end{align*}$$

for $i \in \{1, 2\}$ and

$$al(\gamma_1^n g_1 \gamma_1^n) + bl(\gamma_2^n g_2 \gamma_2^n) \leq t + K_1.$$
Proof. Let \( p_i \in \text{Axis}(\gamma_i) \cap N_i \) and let \( U_i^-, U_i^+ \) be neighborhoods chosen like in Lemma 5.6 and the remark following Lemma 5.6. There exists \( K_0 > 0 \) such that for all \((x, y) \in U_i^- \times U_i^+, d((x, y), p_i) \leq K_0 \). There is also an integer \( n = \max(n_1, n_2) \), satisfying
\[
\gamma_0^a(H^2 \setminus U_i^+) \subset U_i^-
\]
and
\[
\gamma_0^{-a}(H^2 \setminus U_i^-) \subset U_i^+.
\]
Up to shrinking \( U_i^- \) and \( U_i^+ \) we can suppose that \( U_i^- \subset H^2 \setminus U_i^+ \) and \( U_i^+ \subset H^2 \setminus U_i^- \).

Since \( g_i U_i^- \cap U_i^+ = \emptyset \), we have \( \gamma_0^a g_i \gamma_0^{-a}(U_i^-) \subset U_i^- \) and \( \gamma_0^{-a} g_i \gamma_0^a(U_i^+) \subset U_i^+ \). This implies that the fixed points of \( \gamma_0^a g_i \gamma_0^{-a} \) are in \( U_i^- \) and \( U_i^+ \). Moreover we have
\[
\begin{align*}
I(\gamma_0^a g_i \gamma_0^{-a}) & \leq d(\gamma_0^a p_i \gamma_0 a p_i, p_i) \\
& \leq d(g_i p_i, p_i) + 2n \ell(\gamma_i) \\
& \leq d(g_i o_i, o_i) + 2d(o_i, p_i) + 2n \ell(\gamma_i) \\
& \leq d(g_i o_i, o_i) + 2D_i + 2n \ell(\gamma_i).
\end{align*}
\]

The lemma follows with \( K_1 = a(2D_1 + 2n \ell(\gamma_1)) + b(2D_2 + 2n \ell(\gamma_2)). \)

Hence we have associated to every element of \( g \in A(U_1^-, U_1^+, t) \) an element of \( \Gamma \) whose displacement is close to the one of \( g \) and whose axe is by 5.6 (2), close to the origin of \( H^2 \).

The next lemma formalize with constants the fact that the distance between two distincts geodesics tends to infinity.

**Lemma 5.8.** Let \( U^-, U^+ \) and \( V^-, V^+ \) be four open disjoint subsets of \( H^2 \). Let \( o \in H^2 \) and \((\xi, \eta, \xi', \eta') \in U^- \times U^+ \times V^- \times V^+, \) points in \( \partial H^2 \) and let \( A \in (\xi, \eta) \) and \( A' \in (\xi', \eta') \). For all \( R > 0 \) there exists \( K_2 \) such that if \( d(A, A') \leq R \) then \( d(A, o) \leq K_2 \) and \( d(A', o) \leq K_2 \).

The following is the technical lemma which shows that number of elements of \( A(U_1^-, U_1^+, t) \) has the same exponential growth as the whole group \( \Gamma(t) \). The rough idea is to show that the elements of \( \Gamma(t) \) which send \( U_1^- \) on \( U_1^+ \) move \( o \) from a bounded quantity independent of \( t \). This will be shown by considering a stronger condition, we will consider elements \( g \in \Gamma \) which send \( U_1^- \) on \( U_1^+ \) but also \( U_1^+ \) on \( U_1^- \) and \( V_1^+ \) on \( V_1^- \). For some \( V_1^\pm \), by Lemma 5.8 this will force \( g \) to move \( o \) by a bounded quantity.

**Lemma 5.9.** Let \( \gamma \in \Gamma \). There exists neighborhoods \( (U_1^-, U_1^+) \) of \( \gamma_1^+ \) in \( H^2 \) and constants \( K_3 > 0 \) and \( K_4 > 0 \) such that :
\[
\text{Card}(A^n(b(U_1^-, U_1^+, t + K_3))) \geq \frac{1}{4}[\text{Card}(\Gamma^n(b(t))) - \text{Card}(\Gamma^n(a(K_4)))].
\]

**Proof.** As we said before, if we choose \( U_2^- = f(U_1^-) \) and \( U_2^+ = f(U_1^+) \), for every elements \( g \in \Gamma \), \( g_1 U_1^- \cap U_1^+ = \emptyset \) if and only if \( g_2 U_2^- \cap U_2^+ = \emptyset \). So we choose \( U_1^- \) and \( U_1^+ \) as in the previous lemma.

Let \( \beta \in \Gamma \) be an element which has no common endpoints with \( \gamma \). We set \( V_1^- := \beta(U_1^-) \) and \( V_1^+ := \beta(U_1^+) \). Up to shrinking, we can suppose those four open sets are pairwise disjoint and by Lemma 5.6 that there exists \( K_0 \) such that
\[
\begin{align*}
\forall (x, y) \in U_1^- \times U_1^+, \quad d((x, y), o_1) \leq K_0 \\
\forall (x, y) \in V_1^- \times V_1^+, \quad d((x, y), o_1) \leq K_0.
\end{align*}
\]
We define
\[ A^{a,b}(t) := A^{a,b}(U_1^-, U_1^+, t) \cup A^{a,b}(U_1^+, U_1^-, t) \cup A^{a,b}(V_1^-, V_1^+, t) \cup A^{a,b}(V_1^+, V_1^-, t). \]

Clearly, \( \text{Card}(A^{a,b}(U_1^-, U_1^+, t)) = \text{Card}(A^{a,b}(U_1^+, U_1^-, t)) \), a simple bijection between these sets being \( g \mapsto g^{-1} \). Moreover if \( g' \in A^{a,b}(V_1^-, V_1^+, t) \), then \( g'_1 \beta_1 U_1^- \cap \beta_1 U_1^+ = \emptyset \) and hence \( \beta^{-1} g' \beta \in A^{a,b}(U_1^-, U_1^+, t') \) for \( t' = 2(\text{ad}(\beta_1 o_1) + bd(\beta_2 o_2, o_2)) \) since
\[ d (\beta^{-1} g'_1 \beta_1 o_1, o_1) \leq d(g'_1 o_1, o_1) + 2d(\beta_1 o_1, o_1) \]
Finally \( \text{Card}(A^{a,b}(t)) = 4 \text{Card}(A^{a,b}(U_1^-, U_1^+, t + ar_1 + br_2)) \), where \( r_i = 2d(\beta_i o_1, o_1) \).

Set \( K_3 = 2(\text{ad}(\beta_1 o_1) + bd(\beta_2 o_2, o_2)) \). Then in order to prove the lemma, it is enough to show that there exists \( K_4 > 0 \), such that for all \( t > 0 \)
\[ \Gamma^{a,b}(t) - A^{a,b}(t) \subset \Gamma^{a,b}(K_4). \]

Let \( g \in \Gamma^{a,b}(t) \setminus A^{a,b}(t) \), there exists \((\xi, \eta, \xi', \eta') \in U_1^- \times U_1^+ \times V_1^- \times V_1^+ \), such that \( g_1(\xi) \in U_1^+ \), \( g_1(\eta) \in U_1^- \), \( g_1(\xi') \in V_1^+ \) et \( g_1(\eta') \in V_1^- \). This means that \( g \) makes a permutation between four points in the open sets \( U_1^- \), \( U_1^+ \) and \( V_1^- \), \( V_1^+ \). Roughly, this implies that \( g \) does not move a lot the intersection of \( (\xi, \eta) \) and \( (\xi', \eta') \), and hence that the distance \( d(\beta_1 o_1, o_1) \) is bounded.

More precisely, by Lemma 5.6 there exists \( A \in (\xi, \eta) \) such that \( d(A, o_1) \leq K_0 \) and \( B \in (g_1(\xi), g_1(\eta)) \) such that \( d(B, o_1) \leq K_0 \). We then have
\[ d(g_1 A, B) \geq d(g_1 A, A) - d(A, B) \]
\[ \geq d(g_1 o_1, o_1) - 2d(A, o_1) - d(A, B) \]
\[ \geq d(g_1 o_1, o_1) - 4K_0 \]
Similarly, using Lemma 5.6 there exist \( K'_0 \) and \( A' \in (\xi', \eta') \) and \( B' \in (g_1(\xi'), g_1(\eta')) \) such that \( d(A', o_1) \leq K'_0 \) and \( d(B', o_1) \leq K'_0 \) we then have
\[ d(g_1 A', B') \geq d(g_1 o_1, o_1) - 4K'_0. \]
The constant \( K'_0 \) depends on \( U_1^- \), \( U_1^+ \) and \( \beta \) but not on \( g \).

We have \( d(g_1 A, g_1 A') = d(A, A') \leq K_0 + K'_0 \). Moreover \( \gamma_1 A \in \gamma_1(\xi, \eta) \) and \( g_1 A' \in g_1(\xi', \eta') \) and the geodesics \( \gamma_1(\xi, \eta) \) and \( g_1(\xi', \eta') \) are distinct. Hence applying Lemma 5.8, to \( R = K_0 + K'_0 \), there exists \( K_2 \) such that \( d(g_1 A, o_1) \leq K_2 \). This implies that \( d(g_1 A, B) \leq K_2 + K_0 \). Hence
\[ d(g_1 o_1, o_1) \leq 5K_0 + K_2. \]
The same calculations works for \( g_2 \) and we have
\[ d(g_2 o_2, o_2) \leq 5K_0 + K_2. \]
This ends the lemma with \( K_4 = (a + b)(5K_0 + K_2) \). \( \square \)

The previous lemma implies that
\[ \lim_{t \to \infty} \frac{\log \text{Card}(A^{a,b}(U_1^-, U_1^+, t))}{t} = \lim_{t \to \infty} \frac{\log \text{Card}(\Gamma^{a,b}(t))}{t} = \delta^{a,b}. \]
We are now proving that \( \text{Card}(A^{a,b}(U_1^-, U_1^+, t)) \) can be bounded above by \( K_0 t C^{a,b}(t) \).
We fix $\gamma \in \Gamma$ and $U_1^-, U_1^+$ neighbourhoods of the fixed points of $\gamma_1$ as in the two previous lemmas and set
\[
B_{K_0}^b(t) := \{g \in \Gamma | al(g_1) + bl(g_2) \leq t \text{ and } d(Axis(g_1), o_1) \leq K_0\}.
\]

Let $g \in A^{a,b}(U_1^-, U_1^+, t + K_3)$. From Lemma 5.7 there exist $n \in \mathbb{N}$ such that $al(\gamma_1^n g_1 \gamma_1^{-n}) + bl(\gamma_2^n g_2 \gamma_2^{-n}) \leq t + K_3 + K_1$ and $(\gamma_1^n g_1 \gamma_1^{-n})^{-} \in U_1^-$ and $(\gamma_2^n g_2 \gamma_2^{-n})^{-} \in U_1^+$. The latter conditions with Lemma 5.6, imply that $d(Axis(\gamma_1^n g_1 \gamma_1^{-n}), o_1) \leq K_0$, hence $\gamma_1^n g_1^n \in B_{K_0}^b(t + K_3 + K_1)$. We then have
\[
\text{Card } A^{a,b}(U_1^-, U_1^+, t + K_3) \leq \text{Card } B_{K_0}^b(t + K_3 + K_1).
\]

Recall that an element of $\mathcal{C}S$ can be seen as a conjugacy class of an element in $\Gamma$. To make this distinction here, we will note $[c] \in \mathcal{C}S$, the conjugacy class of any representative $g \in \Gamma$ of $c$.

\[
\text{Card}(B_{K_0}^{a_1,a_2}(t)) = \text{Card}\{g, [c]) \in \Gamma \times \mathcal{C}S | g \in [c], \ell_{a_1^{a_2}}(g) \leq t, \text{ and } d(Axis(g_1), o_1) \leq K_0\}
\]

\[
\leq \text{Card } \mathcal{C}S^{a_1,a_2}(t) \ast \text{Card}\{g | g \in [c], \ell_{a_1^{a_2}}(g) \leq t, \text{ and } d(Axis(g_1), o_1) \leq K_0\}
\]

\[
\text{Card}(B_{K_0}^{a_1,a_2}(t)) = \text{Card}\{g, [c]) \in \Gamma \times \mathcal{C}S | g \in [c], a_1 l(g_1) + a_2 l(g_2) \leq t, \text{ and } d(Axis(g_1), o_1) \leq K_0\}
\]

(16)

To end the proof, it is sufficient to show that there exists $K_7$ such that $\text{Card}\{g | g \in [c], al(g_1) + bl(g_2) \leq t, d(Axis(g_1), o_1) \leq K_0\} \leq K_7 t$.

**Lemma 5.10.** Let $s_i$ be the length of the systole in $\mathbb{H}_i/\Gamma_i$ and let $r_i = s_i/4$. There exists $K_6 > 0$ such that, for all $[c] \in \mathcal{C}S^{a,b}(t)$, and for all $p_i \in \mathbb{H}_i$
\[
\text{Card}\{g \in \Gamma | g \in [c], \text{Axis}(g_1) \cap B(p_1, r_1) \neq \emptyset, \text{Axis}(g_2) \cap B(p_2, r_2) \neq \emptyset\} \leq K_6 t
\]

**Proof.** Let $c \in \Gamma$ be a fixed representative of $[c]$. The elements of $[c]$ are then of the form $\alpha c^{-1}$ for $\alpha \in \Gamma$ and their axes are $\text{Axis}(\alpha c^{-1}) = \alpha \text{Axis}(c)$. Hence we are looking at the number of different $\alpha \in \Gamma$ such that
\[
\alpha_1 \text{Axis}(c_1) \cap B(p_1, r_1) \neq \emptyset,
\]
\[
\alpha_2 \text{Axis}(c_2) \cap B(p_2, r_2) \neq \emptyset.
\]

Let $I_i$ be a geodesic segment of length $\ell_i(c)$ on $\text{Axis}(c_i)$, which is a fundamental domain for the action of $c_i$ on $\text{Axis}(c_i)$. For every $\alpha$ such that $\alpha c^{-1} \in \{g \in \Gamma | g \in [c], \text{Axis}(g_1) \cap B(p_1, r_1) \neq \emptyset, \text{Axis}(g_2) \cap B(p_2, r_2) \neq \emptyset\}$ we can choose points $q_i$ on $I_i$ such that $\alpha g_i^n(q_i) \in B(p_i, r_i)$ for some $n_i \in \mathbb{N}$ since $I_i$ is a fundamental domain for $c_i$. We will then show that for different $\alpha$, the points $q_i$ have to be "far" from each other, this will bound the number of $q_i$ that can be associated to a $\alpha$ and finally bound the cardinal of $\{g \in \Gamma | g \in [c], \text{Axis}(g_1) \cap B(p_1, r_1) \neq \emptyset, \text{Axis}(g_2) \cap B(p_2, r_2) \neq \emptyset\}$ Choose two different elements $\alpha$ and $\alpha'$ and choose points $q_i$ and $q_i'$ on $I_i$ as before, that is:
\[
d(q_i, q_i') \leq \ell_i(c)
\]
and
\[
\alpha g_i^{n_i}(q_i) \in B(p_i, r_i) \text{ and } \alpha' g_i^{n_i'}(q_i') \in B(p_i, r_i).
\]

Define by $e_i := \alpha g_i^{n_i}$ et $e_i' := \alpha' g_i^{n_i'}$. By definition of $s_i$ we have that
\[
s_i \leq d(e_i^{-1} e_i q_i, q_i') = d(e_i q_i, e_i' q_i')
\]
\[
\leq d(e_i q_i, e_i' q_i') + d(e_i' q_i, e_i' q_i')
\]
\[
\leq 2r_i + d(q_i, q_i') = s_i/2 + d(q_i, q_i')
\]

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Hence $d(q_i, q'_i) > s_i/2$, and since $q_i, q'_i \in I_i$ there exists at most $K_6 := \min \left( \frac{2s_1}{2} \frac{2s_2}{2} \right)$ points $q_i$ which can satisfy this condition, and therefore as many different $\alpha$, this implies the lemma.  

From Corollary 5.4, we have

$$\{ g \mid d(\text{Axis}(g_1), o_1) \leq K_0 \} \subset \{ g \mid d(\text{Axis}(g_1), o_1) \leq K_0 \text{ and } d(\text{Axis}(g_2), o_2) \leq K_8 \}.$$ 

Since $B(o_1, K_0) \times B(o_2, K_8)$ is compact, there exists a finite cover with balls of radius $r_i$ applying Lemma 5.10 to every ball of the cover gives a constant $K_7$ such that for $c \in CS^{a,b}(t)$

$$\text{Card}\{g \in \Gamma \mid g \in [c], \text{Axis}(g_1) \cap B(p_1, K_0) \neq \emptyset, \text{Axis}(g_2) \cap B(p_2, K_7) \neq \emptyset\} \leq K_7 t$$

Since $\text{Axis}(g_1) \cap B(p_1, K_0) \neq \emptyset$ implies $\text{Axis}(g_2) \cap B(p_2, K_7) \neq \emptyset$ it follows that

$$\text{Card}\{g \mid g \in [c], al(g_1) + bl(g_2) \leq t, d(\text{Axis}(g_1), o_1) \leq K_0\} \leq K_7 t$$

which ends the proof.

\[30\]