GLOBAL TRANSFORMATIONS PRESERVING
STURM-LIOUVILLE SPECTRAL DATA

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ABSTRACT. We show the existence of a real analytic isomorphism between a space of impedance function $\rho$ of the Sturm-Liouville problem $-\rho^{-2}(\rho^2 f')' + uf$ on $(0,1)$, where $u$ is a function of $\rho, \rho', \rho''$, and that of potential $p$ of the Schrödinger equation $-y'' + py$ on $(0,1)$, keeping their boundary conditions and spectral data. This mapping is associated with the classical Liouville transformation $f \rightarrow \rho f$, and yields a global isomorphism between solutions to inverse problems for the Sturm-Liouville equations of the impedance form and those to the Schrödinger equations.

1. INTRODUCTION AND MAIN RESULTS

1.1. Liouville transformation. It is well-known that the Liouville transformation $f \rightarrow \rho f$ maps the solution to the Sturm-Liouville equation

\begin{equation}
-\rho^{-2}(\rho^2 f')' + uf = \lambda f,
\end{equation}

called the equation of impedance form, to that for the Schrödinger equation

\begin{equation}
-y'' + py = \lambda y,
\end{equation}

and reduces the issues for the former to those for the latter. A characteristic feature of the 1-dimensional inverse problem for the Schrödinger equation is that we know the global structure of solutions, i.e. the existence of a real analytic isomorphism between certain Hilbert space of potentials and that for the spectral data. Therefore, one may well think of transforming back this isomorphism by the inverse Liouville transformation to solve the inverse problem for the Sturm-Liouville equation of impedance form. However, it is by no means obvious to find a proper form of the associated transformation between two inverse problems, in particular, the function spaces for the impedance functions and the potentials. Thus there has been no general result so far except for some restricted cases. In this paper, we show the existence of an analytic isomorphism between a space of impedance functions $\rho$ and that for the potentials $p$, which also preserves the boundary conditions and the spectral data, i.e. eigenvalues and norming constants. Therefore, it gives an isomorphism between the solutions of inverse problems for these equations. Since the inverse problem for (1.2) has already been solved, we can thus solve the inverse problem for (1.1), in particular, the characterization problem. This result can be applied to the Laplacian on the surface of revolution [15] and to the propagation of wave in the media periodic in radius [41], [42]. We shall discuss these problems elsewhere [19].

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Suppose we are given a Sturm-Liouville operator $-\Delta_{q,u}$ defined in $L^2((0,1); g^2 dx)$, where $g = g(x) > 0$, having the form
\begin{equation}
-\Delta_{q,u} f = -\frac{1}{g^2}(g^2 f')' + uf, \quad q = \frac{g'}{g},
\end{equation}
equipped with the boundary condition
\begin{equation}
f'(0) - af(0) = 0, \quad f'(1) + bf(1) = 0, \quad a, b \in \mathbb{R} \cup \{\infty\}.
\end{equation}
In view of (1.3), we take $g(x)$ as follows
\begin{equation}
g(x) = e^{\int_0^x q(t) dt}.
\end{equation}
Using the unitary transformation $\mathcal{U}$ defined by
\begin{equation}
\mathcal{U} : L^2((0,1); g^2 dx) \to L^2((0,1); dx),
\end{equation}
\begin{equation}
f \mapsto \mathcal{U} f = \tilde{f}, \quad g = e^{\int_0^x q(t) dt},
\end{equation}
we can transform the operator $-\Delta_{q,u}$ into the Schrödinger operator on $L^2((0,1); dx)$:
\begin{equation}
\mathcal{U}(-\Delta_{q,u})\mathcal{U}^{-1} = -g^{-1} \frac{d}{dx} \left(g^2 \frac{d}{dx} g^{-1}\right) + u = -\frac{d^2}{dx^2} + q' + q^2 + u.
\end{equation}
We now put
\begin{equation}
S_p = -\frac{d^2}{dx^2} + p,
\end{equation}
\begin{equation}
\begin{cases}
p = q' + q^2 + u - c_0, \\
c_0 = \int_0^1 (q^2 + u) dt,
\end{cases}
\end{equation}
and denote $L^2((0,1); dx)$ by $L^2(0,1)$.

1.2. First main result - Analytic isomorphism. Let $H^m$ be the Sobolev space of order $m$ on $(0,1)$:
\begin{equation}
H^m = \left\{ q \in L^2(0,1) ; q^{(k)} \in L^2(0,1), \ 0 \leq k \leq m \right\},
\end{equation}
and introduce the following space of real functions
\begin{equation}
\mathcal{H}_0^1 = \left\{ q \in H^1 ; q(0) = q(1) = 0 \right\},
\end{equation}
\begin{equation}
\mathcal{H}_0 = \left\{ q \in L^2(0,1) ; \int_0^1 q(x) dx = 0 \right\},
\end{equation}
\begin{equation}
\mathcal{H}_\alpha = \mathcal{H}_0 \cap H^\alpha, \quad \alpha \geq 0,
\end{equation}
equipped with the norms
\begin{equation}
\| q \|_{\mathcal{H}_0^1}^2 = \| q' \|^2 = \int_0^1 |q'(x)|^2 dx, \quad \| q \|_{\mathcal{H}_\alpha}^2 = \| q^{(\alpha)} \|^2 = \int_0^1 |q^{(\alpha)}(x)|^2 dx.
\end{equation}
Here and what follows, $\| \cdot \|$ denotes the norm of $L^2(0,1)$. To show that they define the norms, we have only to pass to the Fourier series. Define the space $L^2_{even}(0,1)$
of even functions and the space $L^2_{\text{odd}}(0,1)$ of odd functions by
\begin{align}
L^2_{\text{odd}}(0,1) = \left\{ q \in L^2(0,1) : q(x) = -q(1-x), \quad \forall \, x \in (0,1) \right\}, \\
L^2_{\text{even}}(0,1) = \left\{ q \in L^2(0,1) : q(x) = q(1-x), \quad \forall \, x \in (0,1) \right\},
\end{align}
and for $\omega = \text{even}$ or $\omega = \text{odd}$ we put
\begin{equation}
\mathcal{H}^0_{\alpha,\omega} = \mathcal{H}^0 \cap L^2_{\omega}(0,1), \quad \mathcal{H}^\alpha = \mathcal{H} \cap L^2_{\omega}(0,1), \quad \alpha \geq 0.
\end{equation}
We also introduce the space $\ell^2_\alpha$ of real sequences $h = (h_n)_1^\infty$, equipped with the norm
\begin{equation}
\|h\|_{\alpha}^2 = 2 \sum_{n \geq 1} (2\pi n)^{2\alpha} |h_n|^2, \quad \alpha \in \mathbb{R},
\end{equation}
and let $\ell^2 = \ell^2_0$.

We assume that the potential $u$ is related to $q$ in the following way.

**Condition U.** The potential $u$ has the form
\begin{equation}
u = u_1(q) + u_2(Q),
\end{equation}
$Q$ being defined in (1.5), with the following properties.

1. Each $u_j : \mathbb{R} \to \mathbb{R}, \quad j = 1,2$, is real analytic and satisfies
\begin{equation}
u_j''(t) \leq 0, \quad \forall \, t \in \mathbb{R}.
\end{equation}
2. There exist nondecreasing functions $F_j : [0,\infty) \to [0,\infty), \quad j = 1,2$, such that
\begin{equation}
\|u_1(q)\| \leq F_1(\|q\|), \quad \|u_2(Q)\| \leq F_2(\|q\|), \quad q \in \mathcal{H}^0_1.
\end{equation}

With Condition (U) in mind, we write $\Delta_q$ instead of $\Delta_{q,u}$. Our first main theorem is the following.

**Theorem 1.1.** The mapping $P : \mathcal{H}^0_1 \ni q \to p = P(q) \in \mathcal{H}_0$ given by (1.9) is a real analytic isomorphism between $\mathcal{H}^0_1$ and $\mathcal{H}_0$. In particular, the operator $\frac{\partial P}{\partial q}$ has a bounded inverse for each $q \in \mathcal{H}^0_1$. Moreover, it has the following properties.

1. The following inequalities hold:
\begin{equation}
\|q'\| \leq \|p\| \leq \|q''\| + \|q''\| F_1(\|q'\|) + \|q\| F_2(\|q\|) + \|q''\|^{\frac{1}{2}} F_2(\|q\|)^{\frac{1}{2}}.
\end{equation}
2. The mapping $P(q) - q' : \mathcal{H}^0_1 \to \mathcal{H}_0$ is compact.
3. The mapping $\mathcal{H}^0_{1,\text{odd}} \ni q \to p = P(q)$ is a real analytic isomorphism between $\mathcal{H}^0_{1,\text{odd}}$ and $\mathcal{H}^0_{\text{even}}$.
4. The mapping $\mathcal{H}^\text{odd} \ni q \to p = P(q)$ is a real analytic isomorphism between $\mathcal{H}^\text{odd}$ and $\mathcal{H}^\text{even}$.

**Remark.** (1) The mapping $q \to p = q'' + q^2 + u - c_0 : \mathcal{H}_1 \to \mathcal{H}_0$ was considered in [22]. In some cases the mapping $\mathcal{H}_0$ into $\mathcal{H}_{-1}$ is also useful (see [23], [2]).
(2) For the surface of revolution (1.9), we need to study the case $u = E q^{-\frac{1}{2}}$. Here $d + 1 \geq 2$ is the dimension of the surface of revolution and $E \geq 0$ is a constant.
1.3. Second main result - Inverse problems. We consider the eigenvalue problems for $-\Delta_q$ and $S_p$ on $(0,1)$ subject to the boundary condition
\begin{equation}
\label{1.18}
f'(0) - af(0) = 0, \quad f'(1) + bf(1) = 0, \quad a, b \in \mathbb{R} \cup \{\infty\}.
\end{equation}

Our second main theorem asserts that the mapping in Theorem 1.1 preserves the boundary conditions and spectral data.

**Theorem 1.2.** Let $p = P(q)$, $q \in \mathcal{H}_0^1$, be defined by (1.26). Then the operators $S_p$ and $-\Delta_q$ are unitarily equivalent. In particular, they have the same boundary conditions, eigenvalues and the norming constants.

Therefore, the inverse problem for $-\Delta_q$ is solvable if and only if so is for $S_p$. Let us consider the following three cases separately.

1.4. Inverse problem for the Dirichlet boundary condition : $a = b = \infty$. Denote by $\mu_n = \mu_n(q)$, $n \geq 1$, the eigenvalues of $-\Delta_q$ subject to the boundary condition (1.18) for the case $a = b = \infty$. It is well-known that all $\mu_n$ are simple and satisfy
\begin{equation}
\label{1.19}
\mu_n = \lambda_n^0 + c_0 + \overline{\mu}_n, \quad \text{where} \quad (\overline{\mu}_n)^+ \in \ell^2, \quad c_0 = \int_0^1 (q^2 + u) dt,
\end{equation}
and $\lambda_n^0 = (\pi n)^2$, $n \geq 1$, denote the unperturbed eigenvalues. The norming constants are defined by
\begin{equation}
\label{1.20}
\chi_n(q) = \log \left| \frac{q(1)f_n'(1, q)}{f_n'(0, q)} \right|, \quad n \geq 1,
\end{equation}
where $f_n$ is the $n$-th eigenfunction satisfying $f_n'(0) \neq 0$ and $f_n'(1) \neq 0$. Applying Theorems 1.1 and 1.2 and the result of the inverse problem for $S_p$ in [39], we have the following theorem.

**Theorem 1.3.** Let $a = b = \infty$ and let $\mathcal{H} = \mathcal{H}_0^1$ or $\mathcal{H} = \mathcal{H}_1$. Then the mapping
\[ \Psi : q \mapsto ((\overline{\mu}_n(q))_n^{\infty}; (\chi_n(q))_n^{\infty}) \]
is a real-analytic isomorphism between $\mathcal{H}$ and $\mathcal{M}_1 \times \ell^2$, where
\begin{equation}
\label{1.21}
\mathcal{M}_1 = \{ (h_n)_{n=1}^{\infty} \in \ell^2; \mu_1^0 + h_1 < \mu_2^0 + h_2 < \cdots \} \subset \ell^2.
\end{equation}

In particular, in the anti-symmetric case, i.e., $q$ is odd, the spectral mapping
\begin{equation}
\label{1.22}
\tilde{\sigma} : \mathcal{H}_{0, \text{odd}} \rightarrow \mathcal{M}_1, \quad \text{given by} \quad p \mapsto \tilde{\sigma} \quad p \mapsto \sigma
\end{equation}
is a real analytic isomorphism between $\mathcal{H}_{0, \text{odd}}$ and $\mathcal{M}_1$, where $\mathcal{H}_{0, \text{odd}} = \mathcal{H}_{0}^1, \mathcal{H}_{0, \text{odd}}^1 = \mathcal{H}_{0, \text{odd}}^1$.

1.5. Inverse problem for the mixed boundary condition : $a = \infty$, $b \in \mathbb{R}$. Let $\lambda_n = \lambda_n(q,b)$, $n \geq 0$, be the eigenvalues of $-\Delta_q$ subject to the boundary condition (1.18) for the case $a = \infty$, $b \in \mathbb{R}$. We then have
\[ \lambda_n = \lambda_n^0 + c_0 + 2b + \overline{\lambda}_n(q,b), \quad \text{where} \quad (\overline{\lambda}_n)_{0}^{\infty} \in \ell^2, \quad c_0 = \int_0^1 (q^2 + u) dt, \]
and $\lambda_n^0 = \pi^2(n + \frac{1}{2})^2$, $n \geq 0$, denote the unperturbed eigenvalues. The norming constants are defined by
\begin{equation}
\label{1.23}
\chi_n(q,b) = \log \left| \frac{q(1)f_n'(1, q,b)}{f_n'(0, q,b)} \right|, \quad n \geq 0,
\end{equation}
where $f_n'$ and $f_n''$ are the $n$-th eigenfunction and its derivative, respectively, satisfying $f_n'(0) \neq 0$ and $f_n''(1) \neq 0$. Applying Theorem 1.1 and the result of the inverse problem for $S_p$ in [39], we have the following theorem.

**Theorem 1.4.** Let $a = \infty$ and let $\mathcal{H} = \mathcal{H}_0^1$ or $\mathcal{H} = \mathcal{H}_1$. Then the mapping
\[ \Psi : q \mapsto ((\overline{\mu}_n(q))_n^{\infty}; (\chi_n(q,b))_n^{\infty}) \]
is a real-analytic isomorphism between $\mathcal{H}$ and $\mathcal{M}_1 \times \ell^2$, where
\begin{equation}
\label{1.24}
\mathcal{M}_1 = \{ (h_n)_{n=1}^{\infty} \in \ell^2; \mu_1^0 + h_1 < \mu_2^0 + h_2 < \cdots \} \subset \ell^2.
\end{equation}

In particular, in the anti-symmetric case, i.e., $q$ is odd, the spectral mapping
\begin{equation}
\label{1.25}
\tilde{\sigma} : \mathcal{H}_{0, \text{odd}} \rightarrow \mathcal{M}_1, \quad \text{given by} \quad p \mapsto \tilde{\sigma}
\end{equation}
is a real analytic isomorphism between $\mathcal{H}_{0, \text{odd}}$ and $\mathcal{M}_1$, where $\mathcal{H}_{0, \text{odd}} = \mathcal{H}_{0}^1, \mathcal{H}_{0, \text{odd}}^1 = \mathcal{H}_{0, \text{odd}}^1$. 
where $f_n$ is the $n$-th eigenfunction satisfying $f_n'(0, q, b) \neq 0$ and $f_n(1, q, b) \neq 0$. A simple calculation gives
\[
\chi_n^0 = \chi_n(0, 0) = -\log \pi(n + \frac{1}{2}), \quad \text{where} \quad \sqrt{\lambda_n^0} = \pi(n + \frac{1}{2}).
\]

Denote by $\varphi(x) = \varphi(x, \lambda, q, b)$ and $\xi(x) = \xi(x, \lambda, q, b)$ the fundamental solutions of the equation
\[
(1.23) \quad -\frac{1}{\varphi}(\varphi^2f')' + uf = \lambda f
\]
such that
\[
(1.24) \quad \varphi(0) = 0, \quad \varphi'(0) = 1, \quad \xi(1) = -1, \quad \xi'(1) = b.
\]
We define the Wronskian $w(\lambda, q, b)$ for the equation (1.23) by
\[
(1.25) \quad w = \{\varphi, \xi\}_q = \varphi(1)(\varphi'(1, \lambda, q, b) + b\varphi(1, \lambda, q, b)),
\]
where $\{f, g\}_q = f(qg') - (qf')g$. Note that $\lambda_n(q, b)$ is a simple root of $w(\lambda, q, b)$, and is given by
\[
(1.26) \quad w(\lambda, q, b) = \cos \sqrt{\lambda} \prod_{n=0}^{+\infty} \frac{\lambda - \lambda_n(q, b)}{\lambda - \lambda_n^0}, \quad \lambda \in \mathbb{C}.
\]

The inverse problem for $S_p$ with $a = \infty, b \in \mathbb{R}$ was solved in [28]. Therefore, applying Theorems [11 12] and the result of the inverse problem for $S_p$ in [28], we have the following theorem.

**Theorem 1.4.** (1) For each $b \in \mathbb{R}$ the mapping
\[
\Psi : q \mapsto \left( (\lambda_n(q, b))_{n=1}^\infty : (\chi_n(q, b) - \chi_n^0)_{n=1}^\infty \right)
\]
is a real-analytic isomorphism between $\mathcal{H}_1^0$ and $\Lambda_1 \times \ell^2$, where $\Lambda_1$ is given by
\[
(1.27) \quad \Lambda_1 = \{(h_n)_{n=1}^\infty \in \ell^2 : \lambda_0^0 + h_1 < \lambda_2^0 + h_2 < \ldots \} \subset \ell^2.
\]
(2) For each $(q, b) \in \mathcal{H}_1^0 \times \mathbb{R}$ the following identity holds true:
\[
(1.28) \quad b = \sum_{n=0}^{\infty} \left( 2 - \frac{e^{\chi_n(q, b)}}{\varphi_{\lambda_n}(\lambda_n, q, b)} \right),
\]
where the series converges uniformly.

**1.6. Inverse problem for the generic boundary condition:** $a, b \in \mathbb{R}$. Let $\mu_n = \mu_n(q, a, b), \ n \geq 0$, be the eigenvalues of $-\Delta_q$ subject to the boundary condition (1.18) for the case $a, b \in \mathbb{R}$. Then we have

\[
\mu_n = \mu_n^0 + c_0 + 2(a + b) + \bar{\mu}_n(q, a, b), \quad \text{where} \quad (\bar{\mu}_n)_{n=1}^\infty \in \ell^2, \quad c_0 = \int_0^1 (q^2 + u) dt,
\]
and $\mu_n^0 = (\pi n)^2, \ n \geq 1$, denote the unperturbed eigenvalues. We introduce the norming constants
\[
(1.29) \quad \phi_n(q, a, b) = \log \frac{|\varphi(1)f_n(1, a, q, b)|}{|\varphi(0)f_n(1, a, q, b)|}, \quad n \geq 0,
\]
where $f_n$ is the $n$-th normalized eigenfunction satisfying $f_n(1, a, q, b) \neq 0$ and $f_n(0, a, q, b) \neq 0$. The inverse problem for $S_p$ with generic boundary condition was solved in [28]. Therefore, by Theorems [11 12] and the results in [28], we have the following theorem.
Theorem 1.5. For any $a, b \in \mathbb{R}$, the mapping
\[
\Psi_{a,b} : q \mapsto ((\bar{\mu}_n(q,a,b))_{n=1}^{\infty}; (\phi_n(q,a,b))_{n=1}^{\infty})
\]
is a real-analytic isomorphism between $\mathscr{H}_1^0$ and $\mathcal{M}_1 \times \ell_1^2$, where $\mathcal{M}_1$ is given by Theorem 1.5.

When $a, b$ are not fixed, the inverse problem for $S_p$ was solved in [18]. Therefore, Theorems 1.1, 1.2 and [18] prove the following theorem.

Theorem 1.6. (1) The mapping
\[
\Psi : (q, a, b) \mapsto ((\bar{\mu}_{n-1}(q,a,b))_{n=1}^{\infty}; (\phi_{n-1}(q,a,b))_{n=1}^{\infty})
\]
is a real-analytic isomorphism between $\mathbb{R}^2 \times \mathscr{H}_1^0$ and $\mathcal{M}_0 \times \ell_1^2$, where
\[
(1.30) \quad \mathcal{M}_0 = \{(h_{n-1})_{n=1}^{\infty} \in \ell^2 : \mu_0^1 + h_0 < \mu_1^0 + h_1 < \ldots \} \subset \ell^2.
\]
(2) For each $(q; a; b) \in \mathcal{W}_n^1 \times \mathbb{R}^2$ the following identities hold true:
\[
(1.31) \quad b = -1 + \sum_{n=0}^{+\infty} \left( 2 - \frac{e^{\phi_n}}{|\partial_\mu \lambda (\mu_n)|} \right), \quad a = -1 + \sum_{n=0}^{+\infty} \left( 2 - \frac{e^{-\phi_n}}{|\partial_\mu \lambda (\mu_n)|} \right),
\]
where $w_2(\lambda) = -\sqrt{\lambda} \sin \sqrt{\lambda} \cdot \prod_{n=1}^{\infty} \frac{\lambda - \mu_n}{\lambda - \mu_n^0}$, $\mu_n = \mu_n(q,a,b)$, $\phi_n = \phi_n(q,a,b)$.

1.7. Plan of the paper. Theorem 1.1 is proved in Section 2. Proof of Theorems 1.2, 1.3 will be given in Section 3. In Section 4, we shall review the results for the 1-dimensional inverse problems for the Schrödinger operator $S_p y = -y'' + py$ on the finite interval.

2. Non-linear mappings

2.1. Unitary transformations. We define a unitary transformation $\mathcal{U}$ by
\[
\mathcal{U} : L^2([0, 1], dx) \ni f \mapsto \rho f \in L^2([0, 1], dx), \quad \rho = e^{\int_0^x q(t) dt},
\]
which transforms the operator $-\Delta_q$ into the Schrödinger operator $S_p$ as follows
\[
(2.1) \quad \mathcal{U}(-\Delta_q) \mathcal{U}^{-1} = -q^{-1} \partial_x \rho^2 \partial_x r^{-1} + u = \mathcal{D}^* \mathcal{D} + u = S_p + c_0.
\]
The operator $S_p$ acts in $L^2([0, 1], dx)$ and is given by
\[
(2.2) \quad S_p = -\partial_x^2 + p, \quad c_0 = \int_0^1 (q^2 + u) dx, \quad p = q' + q^2 + u - c_0, \quad \rho = e^\varrho.
\]
Here, using $\varrho = e^{\int_0^x q(t) dt}$ and letting $A^*$ denote the formal adjoint of $A$, we have computed
\[
(2.3) \quad \mathcal{D} = \rho \partial_x \rho^{-1} = \partial_x - q, \quad \mathcal{D}^* = (\rho \partial_x \rho^{-1})^* = -\partial_x - q, \quad \mathcal{D}^* \mathcal{D} = -(\partial_x + q)(\partial_x - q) = -\partial_x^2 + q' + q^2.
\]
In this section we consider the mapping $P : \mathcal{W}_1^0 \rightarrow \mathcal{H}_0$ defined by
\[
(2.4) \quad \begin{cases} 
P(q) = q' + q^2 + u - c_0, \\
u = u_1(q) + u_2(q), \quad Q = \int_0^x q(t) dt, \quad c_0 = \int_0^1 (q^2 + u) dt\end{cases}
\]
where $u$ satisfies Condition U.
2.2. Estimates. We first derive preliminary estimates on the non-linear mapping. Recall that
\[ \|f\|^2 = (f, f), \quad (f, g) = \int_0^1 fg \, dx. \]

It is easy to show the following inequalities for \( q \in \mathcal{W}^1 \):
\[ \|q\|_{L^\infty(0,1)} \leq \|q\|, \quad \|q\| \leq \|q\|, \quad \|q\| \leq \|q\|. \]

(2.6)
\[ \|Q\|_{L^\infty(0,1)} \leq \|q\|. \]

We need the estimates of \( P(q) \) from above and below.

**Lemma 2.1.** Let \( p = P(q) \), \( q \in \mathcal{W}^1 \) be given by (2.4) and let \( h = q^2 + u - c_0 \). Then the following estimates hold true:
\[ \|q\|^2 + \|h\|^2 \leq \|p\|^2 = \|q\|^2 + \|h\|^2 - (q^2, u'(Q)), \quad \] (2.7)
\[ \|u_2(Q) - u_2(0)\| \leq \|Q\|_{L^\infty(0,1)} F_2(\|q\|), \quad \] (2.8)
\[ \|u_1(q) - u_1(0)\| \leq \|q\|_{L^\infty(0,1)} F_1(\|q\|), \quad \]
(2.9)
\[ \|u - \int_0^1 u \, dx\| \leq \|u - u(0)\| \leq F(\|q\|, \|q\|), \quad \] (2.10)
\[ \|p\| \leq \|q\|^{\frac{3}{2}} + \|h\|^{\frac{3}{2}} + \|q\| F_1(\|q\|), \quad \]
where \( F(\|q\|, \|q\|) = \|q\| F_1(\|q\|) + \|q\| F_2(\|q\|) \).

In particular, if \( u_1 = u_2 = 0 \) and \( c_0 = \|q\|^2 \), we have
\[ \|q\|^2 \leq \|p\|^2 = \|q\|^2 + \|q^2 + u - c_0\|^2 = \|q\|^2 + \|q^2\|^2 - c_0^2 \leq \|q\|^2 + \|q^2\|^2. \]
(2.11)

**Proof.** Let \( h = q^2 + u - c_0 \), where \( u = u_1(q) + u_2(Q) \). We have
\[ \|p\|^2 = \|q^2 + \|h\|^2 + 2(q^2 + u - c_0) = \|q^2 + \|h\|^2 + 2(q^2, u), \quad \]
(2.12)
since \( (q^2, 1) = 0, \quad (q', q^2) = 0 \).

Letting \( U(x) = \int_0^x u_1(t) \, dt \), we have by integration by parts and the Condition \( U \)
\[ (q', u) = \int_0^1 q' u_1(q) \, dx + \int_0^1 q' u_2(Q) \, dx \]
\[ = U_1(q(x)) \bigg|_0^1 - \int_0^1 q^2 u'_2(Q) \, dx = -(q^2, u'_2(Q)) \geq 0, \quad \]
which together with (2.12) yields (2.7).

We have the identity
\[ u_2(Q(x)) - u_2(0) = \int_0^1 Q(x) u'_2(tQ(x)) \, dt. \]
The inequality (2.8) then follows from the computation
\[
\|u_2(Q) - u_2(0)\|^2 = \int_0^1 (u_2(Q(x)) - u_2(0))^2 dx = \int_0^1 Q^2(x)dx \int_0^1 u_2'(tQ(x))dt^2
\]
\[
\leq \int_0^1 Q^2(x)dx \int_0^1 u_2'(tQ(x))^2 dt \leq \|Q\|_{L^\infty(0,1)}^2 \int_0^1 dt\|u_2'(tQ)\|^2
\]
\[
\leq \|Q\|_{L^\infty(0,1)}^2 \int_0^1 F_2^2(t\|q\|)dt \leq \|Q\|_{L^\infty(0,1)}^2 F_2^2(\|q\|),
\]
where we have used (1.16) and the monotonicity of $F_2$.

The identity
\[
u_1(q(x)) - u_1(0) = \int_0^1 q(x)u_1'(tq(x))dt.
\]
and similar arguments (1.16) and the monotonicity of $F_2$) yield
\[
\|u_1(q) - u_1(0)\|^2 = \int_0^1 (u_1(q(x)) - u_1(0))^2 dx = \int_0^1 q^2(x)dx \int_0^1 u_1'(tq(x))dt^2
\]
\[
\leq \int_0^1 q^2(x)dx \int_0^1 u_1'(tq(x))^2 dt \leq \|q\|_{L^\infty(0,1)}^2 \int_0^1 dt\|u_1'(tq)\|^2
\]
\[
\leq \|q\|_{L^\infty(0,1)}^2 \int_0^1 F_1^2(t\|q'\|)dt \leq \|q\|_{L^\infty(0,1)}^2 F_1^2(\|q'\|),
\]
which gives (2.8).

Using (2.8) and (2.6), we obtain
\[
\|u - u(0)\| \leq \|u_1(q) - u_1(0)\| + \|u_2(q) - u_2(0)\| \leq \|q'\|F_1(\|q'\|) + \|q\|F_2(\|q\|).
\]

This estimate and a simple bound for $v \in L^2(0,1)$
\[
||v - \int_0^1 vdx|| \leq \|v\|
\]
give
\[
\|u - \int_0^1 udx\| = \|u - u(0) - \int_0^1 (u - u(0))dx\| \leq \|u - u(0)\| \leq F(\|q'\|, \|q\|),
\]
and
\[
h = \|(q^2 - \|q\|^2) + u - u(0) - \int_0^1 (u - u(0))dx\|
\]
\[
\leq \|(q^2 - \|q\|^2)\| + \|u - u(0)\| \leq \|q^2\|^2 + F(\|q'\|, \|q\|)
\]
which yields (2.9).

Using (1.10), (2.9) and (2.7), we obtain (2.11). If $u_1 = u_2 = 0$, the inequality
(2.11) follows from (2.7)–(2.10).

2.3. Analyticity and invertibility. We show that that mapping $P : \mathcal{H}_0^1 \rightarrow \mathcal{H}_0$ is real analytic.

Lemma 2.2. (1) The map $P : \mathcal{H}_0^1 \rightarrow \mathcal{H}_0$ is real analytic and its gradient is given by
\[
dP(q)\frac{dq}{dq}f = f' + 2qf + u_1'(q)f + u_2'(Q)Jf - \frac{dc_0(q)}{dq}f, \quad \forall q, f \in \mathcal{H}_0^1,
\]
(2.15) \[ \frac{dc_0}{dq} f = \int_0^1 \left( 2qf + u'_1(q)f + u'_2(Q)Jf \right) dx , \]
where \( Jf = \int_0^x f dx . \)

(2) The operator \( \frac{dP(q)}{dq} \) is invertible for all \( q \in \mathcal{W}_1^0 . \)

**Proof.** The proof of (1) is standard. See [39].

We show (1) by a contradiction. Due to (2.14), the linear operator \( \frac{dP(q)}{dq} \) : \( \mathcal{W}_0^1 \rightarrow \mathcal{H}_0^1 \) is a sum of \( \frac{d}{dx} \) and a compact operator for all \( q \in \mathcal{W}_0^1 \). Thus it is a Fredholm operator. We prove that \( \frac{dP(q)}{dq} \) is invertible. Let \( q \in \mathcal{W}_0^1 \) and \( f \in \mathcal{W}_0^1 \) be a solution of the equation

\[ \frac{dP(q)}{dq} f = 0 , \]
which is rewritten as

\[ f' + 2qf + u'_1(q)f + u'_2(Q)Jf - \frac{dc_0(q)}{dq} f = 0 . \]

Setting \( y = Jf \), we obtain the equation for \( y \) given by

\[ -y'' - 2wy' + Vy = C , \quad C = - \int_0^1 \left( 2qf + u'_1(q)f + u'_2(Q)Jf \right) dx , \]

\[ w = q + \frac{1}{2} u'_1(q) , \quad V = -u'_2(Q) \geq 0 , \quad y'(0) = y'(1) = y(0) = 0 . \]

We rewrite (2.18) in the following form

\[ -\frac{1}{r^2}(r^2y')' + Vy = C , \quad r(x) = e^{\int_0^x w dt} , \]

\[ y'(0) = y'(1) = y(0) = 0 . \]

If \( C = 0 \), by multiplying (2.19) by \( r^2y \), we have

\[ 0 = \int_0^1 \left( - (r^2y')y + r^2Vy^2 \right) dx = \int_0^1 \left( (ry')^2 + r^2Vy^2 \right) dx , \]
which implies \( y = 0 \), hence \( f = 0 . \)

If \( C \neq 0 \), we can assume without loss of generality that \( C = 1 \) in (2.19). The solution of (2.20) with \( C = 1 \) has the form

\[ y(x) = \int_0^1 R(x,t)r^2(t)dt \geq 0 , \]

where \( R(x,t) \) is the Green function for the problem

\[ -\frac{1}{r^2}(r^2\psi')' + V\psi = f , \quad 0 < x < 1 , \quad y(0) = y'(1) = 0 . \]

Note that \( R(x,t) \geq 0 \) for any \( x, t \in [0, 1] \), since the first eigenvalue of the Sturm-Liouville problem equation

\[ -\frac{1}{r^2}(r^2\psi')' + V\psi = \lambda \psi , \quad \psi(0) = \psi'(1) = 0 , \]

is positive. Let \( \varphi(x,t) \) be the solution of the equation with parameter \( t \in [0, 1] : \)

\[ -\varphi'' - 2w(x+t)\varphi' + V(x+t)\varphi = 0 , \quad \varphi(0, t) = 0 , \quad \varphi'(0, t) = 1 . \]
A direct calculation shows that \( z(x) \) defined by
\[
z(x) = -\int_0^x \varphi(x-t,t)dt
\]
satisfies
\[-z'' - 2w(x)z' + V(x)z = 1, \quad z(0) = z'(0) = 0,
\]
hence \( z(x) = y(x) \). Note that the eigenvalues \( \lambda \) of the Sturm-Liouville problem (2.22) with \( r(x), V(x) \) replaced by \( r(x+t), V(x+t) \) are positive. Then by the well-known comparison principle, \( \varphi(x,t) \) is positive for \( x, t \in [0,1] \). Therefore, \( y(x) < 0 \) on \([0,1] \). In view of (2.20), we have arrived at a contradiction. □

2.4. Bijectivity of \( P \). The proof of Theorem 1.4 is done by the “direct approach” in [21] based on nonlinear functional analysis, the main theorem of which is the following one.

**Theorem 2.3.** Let \( H, H_1 \) be real separable Hilbert spaces equipped with norms \( || \cdot ||_1, || \cdot ||_1 \). Suppose that the map \( f : H \to H_1 \) satisfies the following conditions:

(i) \( f \) is real analytic and the operator \( df/dq \) has a bounded inverse for all \( q \in H \),

(ii) there is a nondecreasing function \( \eta : [0, \infty) \to [0, \infty) \) such that \( \eta(q) = 0 \) and \( ||q|| \leq \eta(||f(q)||_1) \) for all \( q \in H \),

(iii) there exists a linear isomorphism \( f_0 : H \to H_1 \) such that the map \( f - f_0 : H \to H_1 \) is compact.

Then \( f \) is a real analytic isomorphism between \( H \) and \( H_1 \).

We now prove the main results of this section.

**Proof of Theorem 1.4** We check all conditions in Theorem 2.3 for the mapping \( P(q), q \in \mathcal{H}_1^0 \). The condition (i) is proved in Lemma 2.2. The two-sided estimates (1.17) are proved in Lemma 2.3. Let \( q'' \to q \) weakly in \( \mathcal{H}_1^0 \) as \( \nu \to \infty \). Then we have that \( q'' \to q \) strongly in \( \mathcal{H}_0^0 \) as \( \nu \to \infty \), since the imbedding \( \mathcal{H}_1^0 \to \mathcal{H}_0^0 \) is compact. Hence the mapping \( q \to P(q) - q' \) is compact, and the condition (iii) is satisfied. Therefore, by Theorem 2.3, \( P : \mathcal{H}_1^0 \to \mathcal{H}_0^0 \) is a real analytic isomorphism between the Hilbert spaces \( \mathcal{H}_1^0 \) and \( \mathcal{H}_0^0 \).

If \( q \in \mathcal{H}_1^{0,\text{odd}} \), then \( p = P(q) \in \mathcal{H}_0^{\text{even}} \), by the definition of \( P \). Repeating the arguments for \( P(q) \) when \( q \in \mathcal{H}_1^{0,\text{odd}} \), we deduce that the mapping \( P : \mathcal{H}_1^{0,\text{odd}} \to \mathcal{H}_0^{\text{even}} \) is a real analytic isomorphism between the Hilbert spaces \( \mathcal{H}_1^{0,\text{odd}} \) and \( \mathcal{H}_0^{\text{even}} \).

The proof for the case \( P : \mathcal{H}_0^{\text{odd}} \to \mathcal{H}_0^{\text{even}} \) is similar. □

3. Proof of Theorems 1.2–1.5

Letting \( y = \mathcal{U} f = \varrho f \) for \( q \in \mathcal{H}_1^0 \), we obtain
\[
y(0) = f(0), \quad y'(0) = f'(0), \quad y(1) = \varrho(1)f(1), \quad y'(0) = \varrho(1)f'(1).
\]
Thus the function \( f \) satisfies the boundary condition
\[
f'(0) - af(0) = 0, \quad f'(1) + bf(1) = 0, \quad a, b \in \mathbb{R} \cup \{ \infty \},
\]
if and only if \( y = \mathcal{U} f = \varrho f \) satisfies the same boundary condition
\[
y'(0) - ay(0) = 0, \quad y'(1) + by(1) = 0, \quad a, b \in \mathbb{R} \cup \{ \infty \},
\]
since \( q \in \mathcal{H}_1^0 \) and \( q'(0) = q'(1) = 0 \).
3.1. **Proof of Theorem 1.2.** Let \( p = P(q), q \in W_2^0 \), be defined by (1.9). Then under the transformation \( y = \Psi f = g f \) the operators \( S_p \) and \(-\Delta_y\) are unitarily equivalent. Moreover, due to (3.1) the operators \( S_p \) and \(-\Delta_y\) have the same the boundary conditions. The norming constants are the same by (3.1).

Assume that the mapping \( p \to \) "eigenvalues + norming constants for the operator \( S_p \)" gives the solution of the inverse problem for the operator \( S_p \). Then, since the mapping \( p \to q \) is an analytic isomorphism, we obtain the solution of the inverse problem for \(-\Delta_y\) from that of \( S_p \), and vice versa.

In the proof of Theorems 1.3-1.5 we give a more detailed explanation.

3.2. **Proof of Theorems 1.3-1.5.** Recall that in (2.1) the operator \(-\Delta_y\) acting in \( L^2([0,1],g^2 dx) \) is shown to be unitarily equivalent to

\[
\Psi (-\Delta_y) \Psi^{-1} = S_p + \sigma_n,
\]

acting in \( L^2([0,1], dx) \). This representation is more convenient for us.

**Proof of Theorem 1.3.** Let \( q \in \mathcal{H}_0^1 \) and \( a, b \in \mathbb{R} \). We consider the Sturm-Liouville problem with the Dirichlet boundary conditions:

\[
-\frac{1}{q^2} (g^2 f')' + uf = \lambda f, \quad f(0) = f(1) = 0.
\]

Let \( \mu_n = \mu_n(q), n = 1, 2, 3, \ldots \) be the eigenvalues. It is well-known that

\[
\mu_n = (\pi n)^2 + c_0 + \bar{\mu}_n(q), \quad \text{where } (\bar{\mu}_n)_{n=1}^{\infty} \in \ell^2, \quad c_0 = \int_0^1 (q^2 + u) dt.
\]

Following [39], we introduce the norming constants

\[
(3.5) \quad \phi_n(q) = \log \left| \frac{\phi_n(1, q)}{\phi_n(0, q)} \right|, \quad n \geq 1,
\]

where \( \phi_n \) is the \( n \)-th eigenfunction. Note that \( \phi_n'(0) \neq 0 \). We then have the mapping

\[
\Psi : q \mapsto \Psi(q) = ((\bar{\mu}_n(q))_{n=1}^{\infty} ; (\phi_n(q))_{n=1}^{\infty})
\]

We let \( p = P(q), q \in \mathcal{H}_0^1 \), and apply Theorem 1.2. Consider the Sturm-Liouville problem with the Dirichlet boundary conditions:

\[
S_p y = -y'' + p(x)y, \quad y(0) = y(1) = 0.
\]

Denote by \( \sigma_n = \sigma_n(p), n \geq 1 \) the eigenvalues of \( S_p \) and let \( \varphi_n(p) \) be the corresponding norming constants given by

\[
(3.6) \quad \varphi_n(p) = \log \left| \frac{\varphi_n(1, p)}{\varphi_n(0, p)} \right|, \quad n \geq 0.
\]

Recall that due to [39] (see Theorem 4.1) the mapping

\[
\Phi : p \mapsto \Phi(p) = ((\sigma_n(p))_{n=1}^{\infty} ; (\varphi_n(p))_{n=1}^{\infty})
\]

is a real-analytic isomorphism between \( \mathcal{H}_0^1 \) and \( \mathcal{M}_1 \times \ell_1^2 \), where \( \mathcal{M}_1 \) is defined by (1.20). Due to Theorem 1.2 we obtain the identity

\[
\Phi(P(q)) = \Psi(q), \quad \forall \ q \in \mathcal{H}_0^1.
\]

This is a composition of two mappings \( \Phi \) and \( P \), each of which is an analytic isomorphism (see Theorems 1.1 and 1.1). Then the mapping

\[
\Psi : q \mapsto ((\bar{\mu}_n(q))_{n=1}^{\infty} ; (\phi_n(q))_{n=1}^{\infty})
\]
is a real-analytic isomorphism between $\mathcal{H}_0^1$ and $\mathcal{M}_1 \times \ell_1^2$.

Consider the the spectral mapping in the symmetric case
\[ \overline{\mu} : \mathcal{H}_1^{0,\text{odd}} \to \mathcal{M}_1, \]
given by
\[ p \mapsto \overline{\mu}. \]
Recall that due to [39] (see Theorem 1.4) the mapping
\[ \sigma : p \mapsto \sigma(p) = (\sigma_n(p))_{n=1}^\infty \]
is a real-analytic isomorphism between $\mathcal{H}_0^{\text{even}}$ and $\mathcal{M}_1$, where $\mathcal{M}_1$ is defined by (1.20). Again in view of Theorem 1.2, we obtain the identity
\[ \sigma(P(q)) = \overline{\mu}(q), \quad \forall q \in \mathcal{H}_1^{0,\text{odd}}. \]
This is a composition of two mappings $\sigma$ and $P$, each of which being analytic isomorphism (see Theorem 4.1 and 1.1). Then so is the mapping
\[ \overline{\mu} : q \mapsto (\overline{\mu}_n(q))_{n=1}^\infty : \mathcal{H}_1^{0,\text{odd}} \to \mathcal{M}_1. \]

The proof for the case $q \in \mathcal{H}_1^{0,\text{odd}}$ is similar. ■

**Proof of Theorem 1.4** Let $q \in \mathcal{H}_0^1$ and $b \in \mathbb{R}$. We consider the Sturm-Liouville problem with the mixed boundary condition,
\[ -\frac{1}{\rho^2}(q^2 f')' + uf = \lambda f, \quad f(0) = 0, \quad f'(1) + bf(1) = 0. \]
Let $\lambda_n = \lambda_n(q,b), n = 0, 1, 2, \ldots$ be the associated eigenvalues. Then,
\[ \lambda_n = \pi^2(n+\frac{1}{2})^2 + c_0 + 2b + \lambda_0(q,b), \quad \text{where} \quad (\lambda_n)_{n=1}^\infty \in \ell^2, \quad c_0 = \int_0^1 (q^2 + u)dt. \]
As in [28], the norming constants are defined by
\[ \chi_n(q,b) = \log \left| \frac{\rho(1)f_n(1,q,b)}{f_n'(0,q,b)} \right|, \quad n \geq 0, \]
where $f_n$ is the $n$-th normalized eigenfunction such that $f_n'(0,q,b) > 0$. A simple calculation gives
\[ \chi_n^0 = \chi_n(0,0) = -\log \pi(n+\frac{1}{2}), \quad \text{where} \quad \sqrt{\chi_n^0} = \pi(n+\frac{1}{2}). \]
Thus for fixed $b \in \mathbb{R}$ we have a mapping
\[ \Psi_b : q \mapsto \Psi_b(q) = \left( (\lambda_n(q,b))_{n=1}^\infty : (\chi_{n-1}(q,b) - \chi_n^0)_{n=1}^\infty \right). \]

Consider the Sturm-Liouville problem
\[ S_p y = -y'' + p(x)y, \quad y(0) = 0, \quad y'(1) + by(1) = 0, \quad b \in \mathbb{R}. \]
Denote by $\tau_n = \tau_n(p,b), n \geq 0$ the eigenvalues of $S_p$ and let $\nu_n(p,b)$ be the corresponding norming constants given by
\[ \nu_n(p,b) = \log \left| \frac{y_n(1,p,b)}{y_n'(0,p,b)} \right|, \quad n \geq 0. \]
Recall that due to [28] (see Theorem 1.2), for each $b \in \mathbb{R}$ the mapping
\[ \Phi_b : p \mapsto \Phi_b(p) = \left( (\tau_n(p))_{n=1}^\infty : (\nu_{n-1}(p) - \nu_n^0)_{n=1}^\infty \right) \]
is a real-analytic isomorphism between $\mathcal{H}_0$ and $\Lambda_1 \times \ell_1^2$, where $\Lambda_1$ is defined by (1.27). By Theorem 1.2 we have an identity
\[ \Phi_b(P(q)) = \Psi_b(q), \quad \forall q \in \mathcal{H}_0^1, \]
where \( \Phi_b \) and \( \Psi \) are analytic isomorphisms (see Theorems 1.2 and 1.1). Then for each \( b \in \mathbb{R} \) the mapping
\[
\Psi_b : q \mapsto \left( (\widetilde{\lambda}_n(q,b))_{n=1}^\infty : (\chi_{n-1}(q,b) - \chi_{n-1})_{n=1}^\infty \right)
\]
is a real-analytic isomorphism between \( \mathcal{H}_0^1 \) and \( \Lambda_1 \times \ell_1^2 \).

We consider the mapping \( \Psi_b \), when \( b \) is not fixed.

**Theorem 3.1.** (i) The mapping
\[
\Psi : (q,b) \mapsto \left( (\widetilde{\lambda}_n(q,b))_{n=1}^\infty : (\chi_{n-1}(q,b) - \chi_{n-1})_{n=1}^\infty \right)
\]
is a real-analytic isomorphism between \( \mathcal{H}_0^1 \times \mathbb{R} \) and \( \mathbb{R} \times \Lambda_0 \times \ell_1^2 \), where \( \Lambda_0 \) is given by
\[
(3.9) \quad \Lambda_0 = \{ (h_{n-1})_{n=1}^\infty : \lambda_0^0 + h_0 < \lambda_0^1 + h_1 < \lambda_0^2 + h_2 < \ldots \} \subset \ell^2.
\]

(ii) For each \( (q,b) \in \mathcal{H}_0^1 \times \mathbb{R} \) the following identity holds true:
\[
(3.10) \quad b = \sum_{n=0}^{+\infty} \left( 2 + \frac{e^{\chi_n(q,b)}}{|\dot{w}(\lambda_n, q, b)|} \right).
\]

**Proof.** The proof is similar to the case of fixed \( b \in \mathbb{R} \), and is based on Theorems 1.2 and 1.1.

**Proof of Theorem 1.5.** Let \( q \in \mathcal{H}_0^1 \) and \( a,b \in \mathbb{R} \). We consider the Sturm-Liouville problem with the generic boundary condition,
\[
-\frac{1}{q^2}(\ddot{q}^2 f')' + uf = \lambda f, \quad f'(0) - af(0) = 0, \quad f'(1) + bf(1) = 0.
\]
Let \( \mu_n = \mu_n(q,a,b) \), \( n = 0, 1, 2, \ldots \) be the associated eigenvalues. We then have
\[
\mu_n = (\pi n)^2 + c_0 + 2(a + b) + \bar{\mu}_n(q,b), \quad \text{where} \quad (\bar{\mu}_n)_{n}^\infty \in \ell^2, \quad c_0 = \int_0^1 (q^2 + u)dt.
\]
As in [28], the norming constants are defined by
\[
(3.11) \quad \phi_n(q,a,b) = \log \left| \frac{\varphi(1)f_n(1,q,a,b)}{f_n(0,q,a,b)} \right|, \quad n \geq 0,
\]
where \( f_n \) is the \( n \)-th normalized eigenfunction such that \( f'_n(0) > 0 \). Thus for fixed \( a,b \in \mathbb{R} \) we have the mapping
\[
\Psi_{a,b} : q \mapsto \Psi_{a,b}(q) = ((\bar{\mu}_n(q,a,b))_{n=1}^\infty ; (\phi_n(q,a,b))_{n=1}^\infty)
\]
As above, we consider the Sturm-Liouville problem
\[
S_p y = -y'' + p(x)y, \quad y'(0) - ay(0) = 0, \quad y'(1) + by(1) = 0, \quad a,b \in \mathbb{R}.
\]
Denote by \( \sigma_n = \sigma_n(p) \), \( n \geq 0 \) the eigenvalues of \( S_p \) and let \( \chi_n(p) \) be the corresponding norming constants given by
\[
(3.12) \quad \chi_n(p) = \log \left| \frac{y_n(1,p,b)}{y'_n(0,p,b)} \right|, \quad n \geq 0.
\]
By [28] (see Theorem 1.5) for each \( a,b \in \mathbb{R} \) the mapping
\[
\Phi_{a,b} : p \mapsto \Phi_{a,b}(p) = ((\bar{\sigma}_n(p))_{n=1}^\infty ; (\chi_n(p))_{n=1}^\infty)
\]
is a real-analytic isomorphism between $\mathcal{H}_0$ and $\mathcal{M}_1 \times \ell^2_1$, where $\mathcal{M}_1$ is defined by (1.20). Theorem 1.2 gives the following identity
\[ \Phi_{a,b}(P(q)) = \Psi_{a,b}(q), \quad \forall \ q \in \mathcal{H}_0. \]

By the same arguments as above, using Theorem 4.5 and 1.1, for each $a,b \in \mathbb{R}$ the mapping
\[ \Psi_{a,b} : q \mapsto ((\tilde{\mu}_n(q,a,b)_{n=1}^{\infty}; (\phi_n(q,a,b))_{n=1}^{\infty}) \]
is a real-analytic isomorphism between $\mathcal{H}_0$ and $\mathcal{M}_1 \times \ell^2_1$.

4. Review of 1-dimensional inverse problems

4.1. Results of Gel’fand-Levitan-Marchenko-Ostrovski-Trubowitz. Let us briefly review the inverse spectral theory for Sturm-Liouville operators on a finite interval. We recall only some important steps mostly focusing on the characterization problem, i.e., the complete description of spectral data that correspond to some fixed class of potentials. More information about different approaches to inverse spectral problems can be found in the monographs [36], [33], [39] and references therein.

The inverse spectral theory goes back to the seminal paper [3] (see also a simpler proof [32]). Borg showed that spectra of two Sturm-Liouville problems $-y'' + p(x)y = \lambda y, x \in [0,1]$, with the same boundary conditions at $x = 1$ but different boundary conditions at $x = 0$, determine the potential $p(x)$ and the boundary conditions uniquely. The next step was done by Marchenko [35]. He proved that the so-called spectral function (or, equivalently, the Weyl-Titchmarsh function) determines the potential uniquely. Note that the spectral function is piecewise-linear outside the spectrum $\{\lambda_n\}_{n=1}^{\infty}$ and its jump at $\lambda_n$ is equal to the so-called normalizing constant $1/\alpha_n(p)$ given (4.6). At the same time, a different approach to this problem was developed by Krein [29], [30], [31].

An important result was obtained by Gel’fand and Levitan [14]. They gave an effective method to reconstruct the potential $p(x)$ from its spectral function. More precisely, they derived an integral equation and expressed $p(x)$ explicitly in terms of the solution of this equation. At that time, there was some gap between necessary and sufficient conditions for the spectral functions corresponding to fixed classes of $p(x)$.

Some characterization of spectral data for $q$ such that $p^{(m)} \in L^1(0,1)$ was derived by Levitan and Gasymov [34] for all $m = 0, 1, 2, \ldots$. Also, they formulated the solution of the characterization problem in the case $p'' \in L^2(0,1)$ (without proof). Marchenko and Ostrovski [37] obtained a sharpening of this result. Namely, for all $m = 0, 1, 2, \ldots$ they gave the complete solution of the inverse problem in terms of two spectra, if $p^{(m)} \in L^2(0,1)$.

So, the inverse problem in the case of Dirichlet boundary conditions (1.2) a sharp characterization of all spectral data $(\sigma_n(p), \alpha_n(p))_{n=1}^{\infty}$ that correspond to potentials $p \in \mathcal{H}_0$ is available due to [37]. Namely, the necessary and sufficient conditions are
\[ \sigma_1 < \sigma_2 < \sigma_3 < \ldots, \quad (\sigma_n - \pi^2 n^2)_{n=1}^{\infty} \in \ell^2 \]
and
\[ (2\pi^2 n^2 \alpha_n(p) - 1)_{n=1}^{\infty} \in \ell^2_1. \]

Trubowitz and co-authors (Isaacson [18], Isaacson-McKean [17], Dahlberg [9], Pöschel [39]) suggested another approach to the inverse Sturm-Liouville problem...
on the finite interval with separated boundary conditions. It is based on the analytic properties of the mapping \{potentials\} \to \{spectral data\} and the explicit transforms corresponding to the change of only a finite number of spectral parameters \((\sigma_n(p), \phi_n(p))\)\(^{n=1}\). Their norming constants \(\phi_n(p)\) (defined in (4.4)) differ slightly from Marchenko’s normalizing constants \(\alpha_n(p)\), see (4.6), but the characterizations are equivalent (see Appendix in [7]). Also, this approach was applied to other scalar inverse problems with purely discrete spectrum:

1) Schrödinger operators on the circle (periodic boundary conditions): Garnett-Trubowitz [11], [12], Kargaev-Korotyaev [21] for even potentials and Korotyaev [27], [26] for general potentials.

2) Impedance equations on the unit interval Coleman-McLaughlin [8], and an impedance equation on the circle [25].

3) Singular Sturm-Liouville operators on \([0, 1]\); [13], [4], [40].

4) Perturbed harmonic oscillators: [38], [5], [6].

5) Vector-valued Sturm-Liouville operators on the unit interval with the Dirichlet boundary conditions [7].

There are results, based on other methods, see e.g., [10], [1] for the operator \(-\Delta_q\) and [16] for the operator \(S_p\) and references therein. Since we use the analytic method of Trubowitz, we mention only the papers using this approach.

Thus, the inverse spectral theory for the Sturm-Liouville operators is now well understood. We summarize below the characterization theorems for the case of the Schrödinger operator \(-\frac{d^2}{dx^2} + p(x)\) on the finite interval.

4.2. Inverse problem for Dirichlet boundary conditions : \(a = b = \infty\). We consider the Sturm-Liouville problem with the Dirichlet boundary conditions:

\[
S_p y = -y''(x) + p(x)y(x), \quad y(0) = y(1) = 0,
\]

where a potential \(p\) belongs to the real space \(\mathcal{H}_0\). For further information see [39]. The spectrum of this operator \(S_p\) is discrete and consists of simple eigenvalues \(\sigma_n, n = 1, 2, \ldots\), which satisfy

\[
\sigma_1 < \sigma_2 < \sigma_3 < \ldots < \sigma_n < \sigma_{n+1} < \ldots
\]

\[
\sigma_n = (\pi n)^2 + \bar{\sigma}_n, \quad n \geq 1, \quad \bar{\sigma} = (\bar{\sigma}_n) \in l^2.
\]

Let \(y_n(x) = y_n(x, p)\) be the corresponding real eigenfunctions, which satisfy \(y_n(0) = y_n(1) = 0\) with the condition \(y_n'(0) = 1\). Introduce the norming constants \(\phi_n\) by

\[
\phi_n = \log |y_n'(1)|, \quad n \geq 1.
\]

It is well known that

\[
\phi = (\phi_n) \in l^2_1.
\]

The monotonicity [13] gives that, even if \(p\) varies over the whole space \(\mathcal{H}_0\), the image of \((\sigma_n) \in l^2\) does not coincide with \(l^2\). Note that Marchenko [33] solved the inverse problem in terms of eigenvalues and normalizing constant \(\alpha_n(p)\) given

\[
\alpha_n(p) = \int_0^1 y_n^2(x, p)dx.
\]

The results of Marchenko-Ostrovski-Trubowitz are formulated in the following theorem.
Theorem 4.1. The mapping $\Phi : \mathcal{H}_0 \to \mathcal{M}_1 \times \ell^2_1$ given by
\begin{equation} \label{4.7} p \to \Phi = (\tilde{\sigma}, \phi) \end{equation}
is a real real analytic isomorphism between the Hilbert space $\mathcal{H}_0$ and the set $\mathcal{M}_1 \times \ell^2_1$.

In particular, in the case of even potentials $p$, the spectral mapping
\begin{equation} \label{4.8} \tilde{\sigma} : \mathcal{H}_0^{\text{even}} \to \mathcal{M}_1, \quad \text{given by} \quad p \to \tilde{\sigma} \end{equation}
is a real real analytic isomorphism between the Hilbert space $\mathcal{H}_0^{\text{even}}$ and $\mathcal{M}_1$.

4.3. Inverse problem for the mixed boundary conditions: $a = \infty, b \in \mathbb{R}$. Below we describe the results of [28] for the case $a = \infty, b \in \mathbb{R}$. As far as we know, this is the most detailed characterization of spectral data in this case available in the literature. Let
\begin{equation} S_p y = -y'' + p(x)y, \quad y(0) = 0, \quad y'(1) + by(1) = 0, \quad b \in \mathbb{R}. \end{equation}
Denote by $\tau_n = \tau_n(p, b), n \geq 0$ the eigenvalues of $S_p$. It is well known that all $\tau_n$ are simple and
\begin{equation} \tau_n(p, b) = \pi^2(n + \frac{1}{2})^2 + 2b + \tau_n(p, b), \quad \text{where} \quad (\tau_n(p, b))_{n=1}^{+\infty} \in \ell^2. \end{equation}
Here $\pi^2(n + \frac{1}{2})^2, n \geq 0$, denote the unperturbed eigenvalues.

Let $Y_1(x) = Y_1(x, \lambda, p, b), Y_2(x) = Y_2(x, \lambda, p, b)$ be the solutions of $-y'' + p(x)y = \lambda y$ such that
\begin{equation} Y_1(0) = 0, \quad Y'_1(0) = 1 \quad \text{and} \quad Y_2(1) = -1, \quad Y'_2(1) = b. \end{equation}
Here and below $(') = \frac{d}{dx}$ and $(\cdot) = \frac{d}{d\lambda}$. Note that $\tau_n(p, b)$ are the roots of the Wronskian
\begin{equation} w_1(\lambda) = w_1(\lambda, p, b) \equiv \{Y_1, Y_2\}(\lambda, p, b) \equiv Y'_1(1, \lambda, p, b) + bY_1(1, \lambda, p, b), \quad \lambda \in \mathbb{C}, \end{equation}
where $\{Y_2, Y_1\} = Y_2Y'_1 - Y'_2Y_1$. The Hadamard Factorization Theorem implies
\begin{equation} \label{4.9} w_1(\lambda, p, b) \equiv \cos \sqrt{\lambda} \cdot \prod_{n=0}^{+\infty} \frac{\lambda - \tau_n(p, b)}{\lambda - \tau_n^0}, \quad \lambda \in \mathbb{C}. \end{equation}
Let $y_n(x) = y_n(x, p, b)$ be the $n$-th eigenfunction of $S_p$. Note that $y'_n(0) \neq 0$. We introduce the norming constants as
\begin{equation} \label{4.10} \nu_n(p, b) = \log \left| \frac{y_n(1, p, b)}{y'_n(0, p, b)} \right|, \quad n \geq 0. \end{equation}
A simple calculation gives
\begin{equation} \nu_n^0 = \nu_n(0, 0) = -\log \pi(n + \frac{1}{2}), \quad \text{where} \quad \sqrt{\sigma_n^0} = \pi(n + \frac{1}{2}). \end{equation}

Let the boundary parameter $b \in \mathbb{R}$ be fixed (e.g., $b = 0$ corresponds to the boundary conditions $y(0) = 0, y'(1) = 0$). In this case spectral data $(\tau_n)_{n=0}^{+\infty}, (\nu_n)_{n=0}^{+\infty}$ are not independent, since they satisfy the nonlinear equation (4.11). It turns out that the first eigenvalue $\tau_0(p, b)$ can be uniquely reconstructed from the other spectral data $(\tau_n)_{n=1}^{+\infty}$ and $(\nu_n)_{n=0}^{+\infty}$.

We formulate the result of [28].
Theorem 4.2. Let \( b \in \mathbb{R} \) be fixed. Then the mapping
\[
\Phi_b : q \mapsto ((\tilde{r}_n(p))_{n=1}^\infty; (\nu_{n-1}(p) - \nu_{n-1}^0)_{n=1}^\infty)
\]
is a real-analytic isomorphism between \( \mathcal{H}_0 \) and \( \Lambda_1 \times \ell_1^2 \), where \( \Lambda_1 \) is defined by (1.27). For each \((p;b) \in \mathcal{H}_0 \times \mathbb{R}\) the following identity holds true:
\[
(4.11) \quad b = \sum_{n=0}^\infty \left( 2 - \frac{e^{\nu_n(p,b)}}{|w(\tau_n,p,b)|} \right).
\]

It is possible to “exclude” from the spectral data not the first eigenvalue \( \lambda_0 \) but an arbitrary norming constant \( \mu_m \) (thus, obtaining the parametrization of iso-spectral manifolds). For any \( \{\lambda_n^*\}_{n=0}^\infty \) such that \( \lambda_n^* = \lambda_n^0 + c^* + \mu_n^* \), where \( \{c^*; \{\mu_n^*\}_{n=0}^\infty \} \in \mathbb{R} \times \mathcal{M} \), denote
\[
(4.12) \quad w^*(\lambda) := \cos \sqrt{\lambda} \cdot \prod_{n=0}^{\infty} \frac{\lambda - \lambda_n^*}{\lambda - \lambda_n^0}, \quad \lambda \in \mathbb{C}.
\]
Recall that \( w(\lambda, q, b) \equiv w^*(\lambda) \) for all \( q \in \text{Iso}_b[\{\lambda_n^*\}_{n=0}^\infty] \).

Corollary 4.3. Let \( b \in \mathbb{R} \) and \( (c^*; \{\mu_n^*\}_{n=0}^\infty) \in \mathbb{R} \times \mathcal{M} \). Then for each \( m \geq 0 \) the mapping
\[
q \mapsto \{\nu_n(q,b) - \nu_n^0\}_{n=0,n\neq m}^\infty
\]
is a real-analytic isomorphism between the isospectral set \( \text{Iso}_b[\{\lambda_n^*\}_{n=0}^\infty] \) (which is a real-analytic submanifold of \( L^2(0,1) \)) and the open set
\[
(4.12) \quad \mathcal{N}_m^b = \left\{ (\nu_n - \nu_n^0)_{n=0,n\neq m}^\infty \in \ell_2^1 : \sum_{n=0,n\neq m}^{\infty} \left( 2 - \frac{e^{\nu_n}}{|w^*(\lambda_n^*)|} \right) > b - 2 \right\} \subset \ell_2^1.
\]

We consider the case when \( b \) is not fixed.

Theorem 4.4. The mapping
\[
\Phi : (p;b) \mapsto ((\tilde{r}_{n-1}(q,b))_{n=1}^\infty; (\nu_{n-1}(p,b) - \nu_{n-1}^0)_{n=1}^\infty)
\]
is a real-analytic isomorphism between \( \mathcal{H}_0 \times \mathbb{R} \) and \( \mathcal{M}_0 \times \ell_1^2 \), where \( \mathcal{M}_0 \) is given by (1.27).

4.4. Inverse problem for generic boundary conditions: \( a, b \in \mathbb{R} \). Let \( \sigma_n = \sigma_n(p,a,b) \), \( n = 0, 1, 2, ... \) be the eigenvalues of the Sturm-Liouville problem
\[
-y'' + p(x)y = \lambda y, \quad y'(0) - ay(0) = 0, \quad y'(1) + by(1) = 0,
\]
where \( p \in \mathcal{H}_0 \) and \( a, b \in \mathbb{R} \). It is well known that
\[
\sigma_n = \pi^2n^2 + 2(a + b) + \sigma_n(p,a,b), \quad \text{where} \quad (\sigma_n)_{n=0}^\infty \in \ell^2,
\]
where \( \pi^2n^2, \ n \geq 0 \), denote the unperturbed eigenvalues. Note that \( \sigma_n \) are the (simple) roots of the Wronskian
\[
\omega(\lambda, p, a, b) \equiv (\varphi' + a\varphi + b(\varphi + a\varphi))(1, \lambda, p), \quad \lambda \in \mathbb{C}.
\]
Following [18], we introduce the norming constants
\[
(4.13) \quad \kappa_n(p,a,b) = \log \left| \frac{y_n(1,p,a,b)}{y_n(0,p,a,b)} \right|, \quad n \geq 0,
\]
where \( y_n \) is the \( n \)-th normalized eigenfunction such that \( y_n(0) > 0 \).

The case for fixed \( a, b \) is important. We recall the results of [28].
**Theorem 4.5.** For any \(a, b \in \mathbb{R}\) the mapping
\[
\Psi_{a,b}: p \mapsto (\tilde{\sigma}_n(p, a, b))_{n=1}^{\infty}; (\kappa_n(p, a, b))_{n=1}^{\infty}
\]
is a real-analytic isomorphism between \(\mathcal{H}_0\) and \(\mathcal{M}_1 \times \ell^2_1\), where \(\mathcal{M}_1\) is given by (1.20).

For the case when \(a, b\) are not fixed, we refer the result of [18].

**Theorem 4.6.** The mapping
\[
\Psi: (p; a; b) \mapsto ((\tilde{\sigma}_{n-1}(p, a, b))_{n=1}^{\infty}; (\kappa_{n-1}(p, a, b))_{n=1}^{\infty})
\]
is a real-analytic isomorphism between \(\mathcal{H}_0 \times \mathbb{R}^2\) and \(\mathcal{M}_0 \times \ell^2_1\), where
\[
\mathcal{M}_0 = \{(h_{n-1})_{n=1}^{\infty} \in \ell^2: \mu_0 + h_0 < \mu_1 + h_1 < \ldots \} \subset \ell^2.
\]

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GLOBAL TRANSFORMATIONS PRESERVING SPECTRAL DATA

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