Canonical single field slow-roll inflation with a non-monotonic tensor-to-scalar ratio

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Abstract. We take a pragmatic, model independent approach to single field slow-roll canonical inflation by imposing conditions, not on the potential, but on the slow-roll parameter \( \epsilon(\phi) \) and its derivatives \( \epsilon'(\phi) \) and \( \epsilon''(\phi) \), thereby extracting general conditions on the tensor-to-scalar ratio \( r \) and the running \( n_{sk} \) where the perturbations are produced, some 50–60 \( e \)-folds before the end of inflation. We find quite generally that for models where \( \epsilon(\phi) \) develops a maximum, a relatively large \( r \) is most likely accompanied by a positive running while a negligible tensor-to-scalar ratio implies negative running. The definitive answer, however, is given in terms of the slow-roll parameter \( \xi_2(\phi) \). To accommodate a large tensor-to-scalar ratio that meets the limiting values allowed by the Planck data, we study a non-monotonic \( \epsilon(\phi) \) decreasing during most part of inflation. Since at \( \phi_H \) the slow-roll parameter \( \epsilon(\phi) \) is increasing, we thus require that \( \epsilon(\phi) \) develops a maximum for \( \phi > \phi_H \) after which \( \epsilon(\phi) \) decrease

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to small values where most $e$-folds are produced. The end of inflation might occur through a hybrid mechanism and a small field excursion $\Delta \phi_e \equiv |\phi_H - \phi_e|$ is obtained with a sufficiently thin profile for $\epsilon(\phi)$ which, however, should not conflict with the second slow-roll parameter $\eta(\phi)$. As a consequence of this analysis we find bounds for $\Delta \phi_e$, $r_H$ and for the scalar spectral index $n_{sH}$. Finally we provide examples where these considerations are explicitly realised.

**Keywords:** inflation, physics of the early universe, primordial gravitational waves (theory)

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1 Introduction

Inflation [1–3]\(^1\) has proved to be very useful in explaining not only the homogeneity of the universe on very large scales but also in providing a theory of structure formation. Typically, slow-roll models of inflation are specified by a formula for the potential which in the single field case generically predicts Gaussian, adiabatic and nearly scale-invariant primordial fluctuations. Here, instead of testing the observables for a specific potential we study general characteristics of the inflationary paradigm by looking at properties of the slow-roll parameter \(\epsilon(\phi)\) and its derivatives with respect to \(\phi\) denoted by \(\epsilon(\phi)'\) and \(\epsilon(\phi)''\). In what follows we concentrate on single field slow-roll canonical inflation (for noncanonical kinetic studies see [7]).

Observable scales of primordial perturbations were produced some 50–60 e-folds before the end of inflation. We denote quantities at this scale with the subscript \(H\). When the tensor-to-scalar ratio \(r\) is large, \(r_H \approx 0.12\) (taking the upper limit of the Planck [8] or Planck-Keck-BICEP2 [9] analysis) a slightly modified Lyth bound [10] implies a relatively large range of the inflaton excursion \(\Delta\phi_H \equiv |\phi_H - \phi_8|\) for the observable cosmological scales \(\Delta N \approx 8\) and for an increasing \(\epsilon(\phi)\) during the first few e-folds of observable inflation, \(\epsilon(\phi) \geq r_H/16\), thus \(\Delta\phi_H \approx \Delta N_8 \sqrt{2\epsilon(\phi)} M_{Pl} \geq 8\sqrt{r_H/8} M_{Pl} \approx 0.98 M_{Pl}\). Here \(M_{Pl}\) is the reduced Planck mass \(M_{Pl} = 2.44 \times 10^{18}\) GeV which we set \(M_{Pl} = 1\) in what follows. In the Boubekeur-Lyth bound [11], a stronger result follows when \(\epsilon(\phi)\) does not decrease during inflation. To have a small \(\Delta\phi_e \equiv |\phi_H - \phi_e|\) with a relatively large tensor-to-scalar ratio it seems that we are invited to consider a decreasing \(\epsilon(\phi)\) during most part of inflation with a large number of e-folds generated not around \(\phi_H\) but close to the end of inflation at \(\phi_e\) (for work in this direction see e.g., [12–15]). The present paper explores the possibility of a maximum in the evolution of \(\epsilon(\phi)^2\) with particular attention to the consequent values for \(\Delta\phi_e\) and \(r_H\). It is clear for instance that \(\epsilon(\phi)\) should not decrease too much because then the small-scale power spectrum becomes so large that primordial black holes are overproduced [16, 17]. Moreover, the end of inflation in this case should be achieved not by the inflaton-field itself, but by some other mechanism e.g., an hybrid field, although all of inflation is driven by a single field.

\(^1\)For reviews see e.g. [4], [5] and [6].

\(^2\)This idea has been suggested by [23] where a non-monotonic tensor-to-scalar ratio with a maximum occurs in a very natural way.
This article is organised as follows: in section 2 we discuss general consequences of a non-monotonic tensor-to-scalar ratio during observable inflation, in particular for the running $n_{sk}$ defined by eq. (2.3) below. Section 3 contains a discussion of bounds for $\Delta \phi_e$ and $r_H$ while in section 4 we provide two examples of well motivated models one with a monotonic and another with a non-monotonic tensor-to-scalar ratio. Finally in section 5 one can find a summary of our results and concluding remarks.

2 The scalar spectral index and the running

Here we study consequences of a non-monotonic tensor-to-scalar ratio for the spectral index and for the running. The slow-roll parameters [18] which involve the potential and its derivatives are defined by

$$\epsilon \equiv \frac{1}{2} \left( \frac{V'}{V} \right)^2, \quad \eta \equiv \frac{V''}{V}, \quad \xi_2 \equiv \frac{V'V'''}{V^2},$$

where primes denote derivatives with respect to $\phi$. In the slow-roll approximation the scalar spectral index and the running are given in terms of the usual slow-roll parameters [18] as follows

$$n_s = 1 + 2\eta - 6\epsilon,$$

$$n_{sk} = \frac{dn_s}{d\ln k} = 16\epsilon\eta - 24\epsilon^2 - 2\xi_2.$$  

There is evidence that the power-spectrum over the range of observable scales is decreasing in amplitude as the scales decrease which means that, while this range of scales were leaving the horizon during inflation, $\epsilon$ was increasing. For a non-monotonic $\epsilon$, a simple but interesting possibility is $\epsilon$ developing a maximum at $\phi_{max}$ during inflation (see figure 1). First we realize that $\phi_H$ cannot be located at $\phi_{max}$ because this would be inconsistent with the Planck data [8]: $\delta_{ns} > 0.0307$ together with the result $r = 8\delta_{ns}$ at the maximum, would imply $r > 0.246$. Thus, $\phi_H < \phi_{max}$ and $\epsilon_H < \epsilon_{max}$, which is consistent with an increasing $\epsilon$ during observable inflation. The derivative of the slow-roll parameter $\epsilon$ at its maximum is

$$\epsilon' \equiv \frac{d\epsilon}{d\phi} = \frac{V'}{V} (\eta - 2\epsilon) = 0, \quad \iff \eta_{max} = 2\epsilon_{max}. \quad (2.4)$$

As usual, the second derivative of $\epsilon$ at $\eta = 2\epsilon$ characterises this critical point

$$\epsilon'' \equiv \left. \frac{d^2\epsilon}{d\phi^2} \right|_{\eta=2\epsilon} = \left. \left[ \xi_2 - 2\eta\epsilon - 4\epsilon (\eta - 2\epsilon) + (\eta - 2\epsilon)^2 \right] \right|_{\eta=2\epsilon} = \xi_2 - 4\epsilon^2, \quad (2.5)$$

which for a maximum implies that $\xi_{2,max} < 4\epsilon_{max}^2$.

Let us concentrate for the moment in the expression where $V' < 0$ with the potential $V$ a monotonically decreasing function of $\phi$ during inflation. In this case $\phi$ is evolving away from the origin, thus the derivative of $\epsilon$ is

$$\epsilon' = -\sqrt{2\epsilon} (\eta - 2\epsilon). \quad (2.6)$$

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3From now on we drop the $\phi$-dependence but keep the $H$-subindex label to emphasize that quantities are evaluated at the scale $H$. 

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The maximum is required because, for observable inflation, \( \epsilon(\phi) \) is increasing at \( \phi < \phi_{\text{max}} \). At least 8 \( e \)-folds of inflation should occur in the interval \( \phi_{H} < \phi < \phi_{\text{max}} \). For \( \phi > \phi_{\text{max}} \), \( \epsilon(\phi) \) can decrease practically generating all of inflation. The maximum of \( \epsilon(\phi) \) can not occur at \( \phi_{\text{max}} = \phi_{H} \) because the value of the tensor-to-scalar ratio \( r \) would violate observational bounds. The end of inflation for vanishing \( \epsilon(\phi_{s}) \) is triggered by a hybrid field and a small \( \Delta \phi_{s} \) is obtained when the profile depicted for \( \epsilon(\phi) \) is sufficiently thin which, however, should not conflict with the curvature of the potential, controlled by the slow-roll parameter \( \eta \).

The case \( V' > 0 \) would correspond to \( \phi \) evolving towards the origin and can be analysed in a similar way. In what follows we parameterise the deviation from the Harrison-Zeldovich spectrum with \( \delta_{ns} \) defined by \( \delta_{ns} \equiv 1 - n_{s} \). We note that eq. (2.6) together with eq. (2.2) can be thus written as

\[
\epsilon' = \frac{1}{2} \sqrt{2\epsilon (\delta_{ns} - 2\epsilon)},
\]

(2.7)

Requiring a non-decreasing \( \epsilon \) during observable inflation means that there should be at least 8 \( e \)-folds of inflation between \( \phi_{H} \) and the maximum of \( \epsilon \) at \( \phi_{\text{max}} \). This implies \( \epsilon' > 0 \) at least during this range, or

\[
\delta_{ns} > 2\epsilon > 2\epsilon_{H} > 0,
\]

(2.8)

which means that during this window of observable inflation the spectral index is bounded by

\[
n_{s} < 1 - 2\epsilon_{H} = 1 - r_{H}/8.
\]

(2.9)

In the interval \( \phi_{H} < \phi < \phi_{\text{max}} \), \( \epsilon \) grows from \( \epsilon_{H} \) to \( \epsilon_{\text{max}} \) but its derivative decreases vanishing at \( \phi_{\text{max}} \) thus \( \epsilon_{H} < \epsilon \) and \( \epsilon' > \ell' \). From eq. (2.7) we get \( 1 - n_{s} - 2\epsilon < \frac{\sqrt{\epsilon\ell}}{\sqrt{\epsilon}} (1 - n_{sH} - 2\epsilon_{H}) < 1 - n_{sH} - 2\epsilon_{H} \), or

\[
n_{sH} < n_{s} + 2(\epsilon - \epsilon_{H}),
\]

(2.10)

where \( \epsilon - \epsilon_{H} > 0 \). Thus, although \( \epsilon \) is larger than \( \epsilon_{H} \), \( n_{s} \) is not constrained to be smaller than \( n_{sH} \) due to the contribution of the \( \eta \)-term present in \( n_{s} \) (\( \eta_{H} \)-term present in \( n_{sH} \)).
To find possible consequences for the running let us now consider the second derivative of $\epsilon$. Together with eq. (2.2), we have

$$
\epsilon'' = \xi_2 - 2\eta \epsilon - 4\epsilon (\eta - 2\epsilon) + (\eta - 2\epsilon)^2 = -9\epsilon^2 + 2\epsilon \delta_{ns} + \frac{1}{4} \delta_{ns}^2 + \xi_2
$$

(2.11)

In a similar way, we write the running as

$$
n_{sk} = 16\epsilon \eta - 24\epsilon^2 - 2\xi_2 = 24\epsilon^2 - 8\epsilon \delta_{ns} - 2\xi_2 = \frac{r}{2} \left( \frac{3}{16} r - \delta_{ns} \right) - 2\xi_2.
$$

(2.12)

According to the Planck data (last column of table 4 of [8]), at $\phi_H$, $r_H < 0.15$, $n_{sk,H} = 0.9644 \pm 0.0049$, and $n_{st,H} = -0.0085 \pm 0.0076$. Thus, $\frac{3}{16} r_H - \delta_{ns,H} < 0$. The running $n_{sk,H}$ will be negative if and only if

$$
\xi_{2H} > \frac{r_H}{4} \left( \frac{3}{16} r_H - \delta_{ns,H} \right) > -5.5 \times 10^{-4}.
$$

(2.13)

In particular, a positive $\xi_{2H}$ will always give a negative running and from eq. (2.11), since $-\frac{9}{16} r_H^2 + r_H \delta_{ns,H} + 2\delta_{ns,H}^2 > 0$, a positive $\epsilon''_{H}$. Furthermore, substituting $\xi_2$ from eq. (2.11) into eq. (2.12)

$$
n_{sk} = -2\epsilon'' + \frac{1}{128} (r - 8\delta_{ns}) (3r - 8\delta_{ns}).
$$

(2.14)

Thus, we conclude that, in general, $n_{sk}$ will be negative if and only if

$$
\epsilon'' > \frac{1}{256} (r - 8\delta_{ns}) (3r - 8\delta_{ns}).
$$

(2.15)

The last inequality implies mostly positive values for $\epsilon''_{H}$ except in the region where $\delta_{ns,H}/6 < \epsilon_H < \delta_{ns,H}/2$, equivalently $8 \delta_{ns,H}/3 < r_H < 8 \delta_{ns,H}$ or $0.082 < r_H < 0.246$, which is still within the bound $r_H < 0.15$ when running is allowed (see table 4 of [8]), but interestingly enough, not a zero value either.

When approaching the maximum of $\epsilon$ its derivative $\epsilon'$ tends to zero with negative $\epsilon''$. A positive value for $\epsilon''$ can occur only if in addition $\epsilon(\phi)$ has an inflection point at $\phi_I < \phi_{\text{max}}$, such that $\epsilon'(\phi_I)$ is a maximum and $\epsilon''(\phi_I) = 0$, (see figure 2). In this situation $\phi_H$ can be smaller or larger than $\phi_I$. Thus, for models where $\epsilon$ presents a maximum with an inflection point as in figure 2, a relatively large $r_H$ is most likely accompanied by a negative $\epsilon''$ with positive running, while a negligible tensor-to-scalar ratio implies positive $\epsilon''$ and negative running. The definitive answer, however, is to be found in the inequality given by eq. (2.13) which imposes precise constraints on the parameters of an specific model.

3 Bounds on $\Delta \phi$ and $r$

While observable scales were leaving the horizon during inflation, $\epsilon$ was an increasing function of $\phi$ and the Lyth bound is regarded as an inevitable consequence. However, if for subsequent $\phi$, we study a decreasing $\epsilon$, then $\epsilon$ has to go through a maximum at $\phi_{\text{max}} > \phi_H$ before starting to decrease as in figure 1.
Figure 2. Plot of the slow roll parameter $\epsilon(\phi)$ and its derivatives $\epsilon'(\phi)$ and $\epsilon''(\phi)$ with respect to $\phi$, as functions of the field, for a model where $\epsilon(\phi)$ is a non-monotonic function of $\phi$. The labels $\phi_{\text{max}}$ and $\phi_I$ refer to the values of $\phi$ for which the maximum and the inflection points of $\epsilon(\phi)$ occur, respectively.

The value $\phi_H$ at which $n_{sH} \approx 0.968$ lies to the left of $\phi_{\text{max}}$ so that $\epsilon'_H$ is positive although small. We observe that an $\epsilon$ with the behaviour shown in figure 1 has the potential to generate relatively large values of $r_H$ while sufficient inflation is produced away from $\phi_H$.

Defining $\Delta \phi_{\text{max}} \equiv |\phi_H - \phi_{\text{max}}|$ and $\Delta N_{\text{max}}$ as the corresponding number of e-folds generated during the field excursion of width $\Delta \phi_{\text{max}}$ the usual expression for the number of e-folds gives

$$\Delta N_{\text{max}} = \int_{\phi_H}^{\phi_{\text{max}}} \frac{d\phi}{\sqrt{2\epsilon}} > \Delta \phi_{\text{max}} \min\left\{ \frac{1}{\sqrt{2\epsilon}} \right\} = \Delta \phi_{\text{max}} \frac{1}{\sqrt{2 \max\{\epsilon\}}} = \frac{\Delta \phi_{\text{max}}}{\sqrt{2 \epsilon_{\text{max}}}} \quad (3.1)$$

where $\min\{\ldots\}$ and $\max\{\ldots\}$ denote the minimum/maximum numerical value of the corresponding quantity in the interval $\Delta \phi_{\text{max}} \equiv |\phi_H - \phi_{\text{max}}|$. However from eq. (2.7) we have that $2\epsilon_{\text{max}} = \delta_{n_s,\text{max}}$, thus, eq. (3.1) becomes

$$\Delta \phi_{\text{max}} < \Delta N_{\text{max}} \sqrt{\delta_{n_s,\text{max}}} \quad (3.2)$$

In a similar way

$$\Delta N_{\text{max}} = \int_{\phi_H}^{\phi_{\text{max}}} \frac{d\phi}{\sqrt{2\epsilon}} < \Delta \phi_{\text{max}} \max\left\{ \frac{1}{\sqrt{2\epsilon}} \right\} = \Delta \phi_{\text{max}} \left( \frac{1}{\sqrt{2 \min\{\epsilon\}}} \right) = \frac{\Delta \phi_{\text{max}}}{\sqrt{2 \epsilon_H}} \quad (3.3)$$

Consequently, the (slightly modified) Lyth bound for the interval $\Delta \phi_{\text{max}}$ reads

$$\Delta N_{\text{max}} \sqrt{\frac{T_H}{8}} < \Delta \phi_{\text{max}} \quad (3.4)$$

Combining eqs. (3.2) and (3.4) we find a bounded excursion:

$$\Delta N_{\text{max}} \sqrt{\frac{T_H}{8}} < \Delta \phi_{\text{max}} < \Delta N_{\text{max}} \sqrt{\delta_{n_s,\text{max}}} \quad (3.5)$$

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One can also write just the first inequality as a bound for $r_H$ which, for the desired upper bound $\Delta \phi_e < 1$ together with $\Delta \phi_{\text{max}} < \Delta \phi_e < 1$ and $\Delta N_{\text{max}} \approx 8$ $e$-folds of observable inflation yields

$$r_H \leq 8 \left( \frac{\Delta \phi_{\text{max}}}{\Delta N_{\text{max}}} \right)^2 < 8 \left( \frac{\Delta \phi_e}{\Delta N_{\text{max}}} \right)^2 < \frac{1}{8} \approx 0.12.$$  

(3.6)

This coincides very closely with Planck’s bound. Thus, allowing for an increasing $\epsilon$ during observable inflation with a subsequent decrease, one should be able to construct models where $\Delta \phi_e < 1$ and with $r_H$ saturating the constraints imposed by Planck or Planck-Keck-BICEP2 data.

After the maximum at $\phi_{\text{max}} > \phi_H$ has been reached, $\epsilon$ starts decreasing up to a point where $1/\sqrt{2\epsilon}$ has generated sufficient $e$-folds and inflation is terminated by a waterfall field. The profile of $\epsilon$ in figure 1 implies that close to the end of inflation, at $\phi_e$, the potential becomes very flat and a hybrid mechanism should terminate inflation. It is also evident that a small $\Delta \phi_e$ is obtained with a sufficiently steep drop in the value of $\epsilon$, thus a feature in $\epsilon$ should not only be high with a low end value but also sharp. We would expect that a sharp feature in $\epsilon$ would conflict with the slow-roll parameter $\eta$ which accounts for the curvature of the potential. In order to control large values of $\eta$ after $\phi$ has reached the maximum, let us rewrite eq. (2.6) in a more convenient form as

$$\eta = 2\epsilon - \frac{\epsilon'}{\sqrt{2\epsilon}}, \quad \phi > \phi_{\text{max}}.$$  

(3.7)

After the maximum $\phi_{\text{max}}$, $\epsilon'$ becomes negative. Consequently, both terms $2\epsilon$ and $-\frac{\epsilon'}{\sqrt{2\epsilon}}$ in eq. (3.7) are positive and thus $0 < \eta < 1$ during inflation with a decreasing $\epsilon$. The first term is negligible w.r.t. the second because we want a large $\epsilon'$, thus for $\phi > \phi_{\text{max}}$, $2\epsilon$ decreases while $-\frac{\epsilon'}{\sqrt{2\epsilon}}$ grows large

$$\eta = 2\epsilon - \frac{\epsilon'}{\sqrt{2\epsilon}} \approx -\frac{\epsilon'}{\sqrt{2\epsilon}} \approx \frac{\Delta \epsilon_c}{\Delta \phi_c \sqrt{2\epsilon}}.$$  

(3.8)

where $\Delta \epsilon_c \equiv |\epsilon_{\text{max}} - \epsilon_c|$ and $\Delta \phi_c \equiv |\phi_{\text{max}} - \phi_c|$ denote quantities in the complementary range of inflation. Thus, demanding that inflation is sustained for an interval $\Delta \phi_c$ (i.e. $\eta < 1$), sets a lower bound for

$$\Delta \phi_c \approx \frac{\Delta \epsilon_c}{\sqrt{2\epsilon} \eta} > \frac{\Delta \epsilon_c}{\sqrt{2\epsilon}} > \frac{\epsilon_{\text{max}} - \epsilon_c}{\sqrt{2\epsilon \epsilon_H}} \approx \frac{\epsilon_{\text{max}}}{\sqrt{2\epsilon \epsilon_H}} \approx \left( \frac{\epsilon_H}{2} \right)^{1/2}. \quad (3.9)$$

In the equation above $\epsilon < \epsilon_{\text{max}}$, thus $1/\sqrt{\epsilon} > 1/\sqrt{\epsilon_{\text{max}}} \approx 1/\sqrt{\epsilon_H}$.\footnote{The assumption $\epsilon_H \approx \epsilon_{\text{max}}$ is justified because we expect most $e$-folds to occur for small $\epsilon$ or, equivalently, large $1/\sqrt{2\epsilon}$. While closely after $\phi_{\text{max}}$, one could have $\epsilon_{\text{max}} > \epsilon(\phi) > \epsilon_H$, the required quick drop in $\epsilon$ after $\phi_{\text{max}}$ means that for most of the excursion $\Delta \phi_c$ the value of $\epsilon$ lies below $\epsilon_H$.} Using the bound of eq. (3.6) we get $\Delta \phi_c > 0.06$, for $\phi > \phi_{\text{max}}$ and $\epsilon$ decreasing.

4 Examples of models with monotonic and non-monotonic tensors

We illustrate some of the previous discussion with two well motivated models of inflation: natural Inflation (NI) [19–22] and Hybrid Natural Inflation (HNI) [23–26].
Figure 3. The potential of Natural Inflation with its corresponding tensor-to-scalar ratio (right). Because the minimum of the potential occurs with vanishing energy the (monotonic) tensor-to-scalar ratio diverges at that point. In NI inflation is terminated by the saturation of the condition $\epsilon = 1$ with the inflaton also in charge of ending inflation.

4.1 A model with a monotonic tensor-to-scalar ratio: Natural Inflation

The NI potential is given by

$$V_{NI} = V_0 \left[ 1 + \cos \left( \frac{\phi}{f} \right) \right],$$

and from eq. (2.1) it follows that the tensor-to-scalar ratio $r = 16 \epsilon$ is given by

$$r = 8 \frac{f^2}{a^2} \sin^2 \left( \frac{\phi}{f} \right),$$

quite clearly $r$ grows without bound diverging for $\phi/f = \pi$, as illustrated in figure 3. Typically inflation is terminated when $\epsilon = 1$. In NI not only $\epsilon$ but also $\epsilon'$ and $\epsilon''$ are monotonically increasing functions of $\phi$.

4.2 A model with a non-monotonic tensor-to-scalar ratio: Hybrid Natural Inflation

In HNI [23–26] all of inflation is driven by the single-field $\phi$ although the end of inflation is triggered by a second, waterfall-field. The inflationary sector of HNI is

$$V_{HNI} = V_0 \left[ 1 + a \cos \left( \frac{\phi}{f} \right) \right],$$

where $0 \leq a < 1$. The tensor-to-scalar ratio is given by

$$r = 8 \frac{a^2 \sin^2 \left( \frac{\phi}{f} \right)}{f^2 \left[ 1 + a \cos \left( \frac{\phi}{f} \right) \right]^2}.$$  

We see that $r = 0$ at $\phi/f = 0, \pi$ developing a maximum located at $\cos(\phi_{\text{max}}/f) = -a$ (see figure 4). The maximum value $r$ can take (away from $r_H$) is therefore $r_{\text{max}} = \frac{8a^2}{f^2(1-a^2)}$. In both cases the running can be written as [26]

$$n_{sk} = \frac{r}{32} \left( 3r - 16\delta_{ns} + \frac{8}{f^2} \right),$$

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Figure 4. In Hybrid Natural Inflation there is a nearly flat plateau followed by a smooth fast transition to a lower flat constant plateau where most of inflation can occur and where inflation is terminated by a second waterfall-field. After the inflaton $\phi$ has carried out all of inflation the waterfall-field fast-roll towards the global minimum with vanishing energy (not shown). As a consequence of this the inflaton minimum has a non-vanishing energy (left) allowing for the possibility of a non-monotonic tensor-to-scalar ratio (right).

without $a$ appearing explicitly. Thus, the running $n_{skH}$ will be negative if and only if

$$f > \frac{1}{\sqrt{2\delta_{ns} - \frac{3}{8}r}}.$$  (4.6)

According to the values of table 4 of Planck data [8] we get $f > 3.5$, in Planck units. In NI $f$ is strictly super-Planckian satisfying the previous bound with a negative running while in HNI $f$ can also be sub-Planckian with positive running allowing for the possibility of primordial black hole production during inflation [16, 26].

5 Conclusions

We developed a model-independent study of single field slow-roll canonical inflation by imposing conditions on the slow-roll parameter $\epsilon(\phi)$ and its derivatives, $\epsilon'(\phi)$ and $\epsilon''(\phi)$, to extract general conditions on the tensor-to-scalar ratio $r$ and the running $n_{sk}$. For models where $\epsilon(\phi)$ presents a maximum, a relatively large $r_H$ is most likely accompanied by a positive running, while a negligible tensor-to-scalar ratio typically implies negative running. The definitive answer, however, is given by the condition on the slow-roll parameter $\xi_2$, eq. (2.13). We have also shown that by imposing conditions to the slow-roll parameter $\epsilon(\phi)$ and its derivatives $\epsilon'(\phi)$ and $\epsilon''(\phi)$ we can accommodate sufficient inflation with a relatively large $r_H$ but still satisfying the Planck [8] or Planck-Keck-BICEP2 [9] constraints. The excursion of the field $\Delta \phi_e \equiv |\phi_H - \phi_e|$ will be no larger than one if the function $\epsilon(\phi)$ has a maximum in a thin hill-shaped feature, decreasing to small values close to the end of inflation, at $\phi_e$. The maximum is required because observations indicate that $\epsilon_H$ is increasing in the observables scales, at $\phi_H$. Then $\epsilon(\phi)$ should decrease for $\phi > \phi_{\text{max}}$ for an effective way of generating the majority of $e$-folds of inflation. The contribution to the number of $e$-folds when $\epsilon(\phi)$ is growing can be $8$ $e$-folds for $\Delta \phi_e$ less than one and $r_H$ close to the upper limit $r_H = 0.12$. The end of inflation for vanishing $\epsilon(\phi)$ can be triggered by a hybrid field and a small $\Delta \phi_e \equiv |\phi_{\text{max}} - \phi_e|$ is obtained when $\epsilon(\phi)$ is sufficiently thin which, however, should not conflict with the other slow-roll parameter $\eta(\phi)$. Under these circumstances $\Delta \phi_e$ is restricted to a narrow windows of values.
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