The heat kernel expansion for the electromagnetic field in a cavity

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Abstract

We derive the first six coefficients of the heat kernel expansion for the electromagnetic field in a cavity by relating it to the expansion for the Laplace operator acting on forms. As an application we verify that the electromagnetic Casimir energy is finite.

1 Introduction

The modes of an electromagnetic field in a cavity, taken together with their unphysical, longitudinal counterparts, can be mapped onto the eigenstates of the Laplacian acting on the de Rham complex of a 3-manifold with boundary. The electric and magnetic fields are thereby associated to forms of degree $p = 1$ and $p = 2$ respectively. In this correspondence transverse modes are associated with coexact, resp. exact forms, which permits to further map longitudinal modes to forms of degree $p = 0$ and $p = 3$. We will use this observation, which is explained in detail in Sect. 2 below, to compute the first six coefficients of the heat kernel expansion for the electromagnetic field in a cavity. The result is used to show in a simple way that the Casimir energy in an arbitrary cavity with smooth boundaries is finite, a conclusion which has been reached previously \[3\]. In an appendix the derivation of the numerical coefficients of the expansion is presented.

We shall present a Hilbert space formulation of the classical Maxwell equations in a cavity $\Omega \subset \mathbb{R}^3$. In a preliminary Hilbert space $L^2(\Omega, \mathbb{R}^3)$ we define the dense subspaces

$$\mathcal{R} = \{ V \in L^2(\Omega, \mathbb{R}^3) \mid \text{rot} \, V \in L^2(\Omega, \mathbb{R}^3) \} ,$$

$$\mathcal{R}_0 = \{ V \in \mathcal{R} \mid \langle U, \text{rot} \, V \rangle = \langle \text{rot} \, U, V \rangle, \forall U \in \mathcal{R} \}$$
and the (closed) operator

\[ R = \text{rot} \quad \text{with domain} \quad \mathcal{D}(R) = \mathcal{R}_0. \]

Its adjoint is then given as \( R^* = \text{rot} \) with \( \mathcal{D}(R^*) = \mathcal{R} \). We remark that \( R \), resp. \( R^* \), is also the closure of \( \text{rot} \) defined on smooth vector fields \( \mathbf{V} \) with boundary condition \( \mathbf{V}_\parallel = 0 \) on the smooth boundary \( \partial \Omega \), resp. without boundary conditions. This is what is meant when we later simply say that a differential operator is defined with (or without) a certain boundary condition.

The subspace

\[ \mathcal{H} = \{ \mathbf{V} \in L^2(\Omega, \mathbb{R}^3) \mid \text{div} \mathbf{V} = 0 \} \quad (1) \]

and its orthogonal complement in \( L^2(\Omega, \mathbb{R}^3) \) are preserved by \( R \) and, therefore, by \( R^* \). We will thus view them as operators on the physical Hilbert space \( \mathcal{H} \). The Maxwell equations with boundary condition \( E_\parallel = 0 \) on the ideally conducting shell \( \partial \Omega \) can now be written as

\[ i \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} = M \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} \quad (2) \]

with

\[ M = \begin{pmatrix} 0 & iR^* \\ -iR & 0 \end{pmatrix} = M^* \quad \text{on} \quad \mathcal{H} \oplus \mathcal{H}, \]

cf. [12]. Since no boundary condition has been imposed on \( \mathbf{B} \), we have \( M(0, \mathbf{B}) = 0 \) for all \( \mathbf{B} = \nabla \psi \) with \( \psi \) harmonic, and hence

\[ \dim \text{Ker} \ M = \infty. \quad (3) \]

We shall compute the heat kernel trace

\[ \text{Tr}'_{\mathcal{H} \oplus \mathcal{H}}(e^{-tM^2}) = \sum_k' e^{-t\omega_k^2}, \]

where \( ' \) means that the contributions of zero-modes, i.e., of eigenvalues \( \omega_k = 0 \) of \( M \), have been omitted. This is necessary in view of [3], but a more physical justification, tied to the application to the Casimir effect to be discussed later, is that zero-modes are not subject to quantization.

The square of \( M \) is

\[ M^2 = \begin{pmatrix} R^* R & 0 \\ 0 & RR^* \end{pmatrix} = \begin{pmatrix} -\Delta_E & 0 \\ 0 & -\Delta_B \end{pmatrix}, \quad (4) \]

where \( \Delta_E \), resp. \( \Delta_B \), is the Laplacian on \( \mathcal{H} \) with boundary conditions

\[ E_\parallel = 0, \quad \text{resp.} \quad (\text{rot} \mathbf{B})_\parallel = 0. \quad (5) \]

The operators \( RR^* \) and \( R^* R \) have the same spectrum, including multiplicity, except for zero-modes. Incidentally, we note that eigenfunctions \( (\mathbf{E}, \mathbf{B}) \) corresponding to \( \omega_k \neq 0 \)
satisfy \( \mathbf{B} = -i\omega_k^{-1}\text{rot} \mathbf{E} \) and hence, by Stokes’ theorem, the boundary condition \( \mathbf{B}_1 = 0 \), which we did not impose, but which is usually also associated with ideally conducting shells. Since \( \partial_t^2 + M^2 = (i\partial_t - M)(-i\partial_t - M) \), each pair of non-zero eigenvalues of \( R^* R \) and \( RR^* \) corresponds to a single oscillator mode for \( [2] \). We will thus discuss the heat kernel asymptotics for

\[
\frac{1}{2} \text{Tr}_{\mathcal{H} \oplus \mathcal{H}}(e^{-tM^2}) = \begin{cases} 
\text{Tr}_{\mathcal{H}} e^{i\Delta_E} \\
\text{Tr}_{\mathcal{H}} e^{t\Delta_B}
\end{cases}
\]

\[
= \sum_{n=0}^{\infty} a_n t^{\frac{n-3}{2}}, \quad (t \downarrow 0).
\]

The coefficients \( a_n \) are known, see e.g. [5], for general operators of Laplace type. The direct application of such results is prevented by the divergence constraint in \( \mathcal{H} \), see (1). In the next section we indicate how to remove it. First however we present the main result.

Let

\[
L_{ab} = (\nabla e_a e_b, n), \quad (a, b = 1, 2),
\]

be the second fundamental form on the boundary \( \partial \Omega \) with inward normal \( n \) and local orthonormal frame \( \{e_1, e_2, n\} \). We denote by \( |\Omega| \) the volume of \( \Omega \) and set

\[
f[\partial \Omega] = \int_{\partial \Omega} f(y) dy,
\]

where \( dy \) is the (induced) Euclidean surface element on \( \partial \Omega \). The corresponding Laplacian on \( \partial \Omega \) is denoted by \( \nabla^2 \).

**Theorem 1** Let \( \Omega \subset \mathbb{R}^3 \) be a compact, connected domain with smooth boundary \( \partial \Omega \) consisting of \( n \) components of genera \( g_1, g_2, \ldots, g_n \). Then

\[
a_0 = 2(4\pi)^{-\frac{3}{2}} |\Omega|,
\]

\[
a_1 = 0,
\]

\[
a_2 = -\frac{4}{3}(4\pi)^{-\frac{3}{2}} (\text{tr } L)[\partial \Omega],
\]

\[
a_3 = \frac{1}{64}(4\pi)^{-1} \left(3(\text{tr } L)^2 - 4 \text{ det } L\right)[\partial \Omega] - \frac{1}{2} \sum_{i=1}^{n} (1 + g_i) + 1,
\]

\[
a_4 = \frac{16}{315}(4\pi)^{-\frac{3}{2}} (2(\text{tr } L)^3 - 9 \text{ tr } L \cdot \text{det } L)[\partial \Omega],
\]

\[
a_5 = \frac{1}{122880}(4\pi)^{-1} \left(2295(\text{tr } L)^4 - 12440(\text{tr } L)^2 \text{ det } L + 13424(\text{det } L)^2 + 1200 \text{ tr } L \cdot \nabla^2 \text{tr } L\right)[\partial \Omega].
\]

We will give two partially independent proofs, based on \([6]\), resp. \([7]\). Their agreement is related to the index theorem, as it may be seen from \([11]\). A further, partial check of these coefficients has been made on the basis of general cylindrical domains and of the sphere, where a separation into TE and TM modes is possible.
The coefficient $a_0$ was computed in [13] (except for the factor 2 replaced by 3, as the divergence condition [11] was ignored), $a_1, a_2$ in [1]. The coefficient $a_3$ is closely related to a result of [3], as discussed in Sect. 3.

2 Proofs

We consider the space of (square integrable) forms, $\Lambda(\Omega) = \bigoplus_{p=0}^{n} \Lambda_p(\Omega)$, on the manifold $\Omega$ with boundary, together with the exterior derivative $d_{p+1} : \Lambda_p(\Omega) \rightarrow \Lambda_{p+1}(\Omega)$ defined with relative boundary condition ([11], Sect. 2.7.1)

$$\omega|_{\partial\Omega} = 0,$$

as a form $\omega|_{\partial\Omega} \in \Lambda_p(\partial\Omega)$. For later use we recall that by the de Rham theorem for manifolds with boundary ([9] or [11], Thm. 2.7.3) we have

$$H^p_r(\Omega) \cong H_{n-p}(\Omega) \cong H_p(\partial\Omega),$$

(10)

where $H^p_r(\Omega) = \text{Ker} d_{p+1}/\text{Im} d_p$ is the $p$-th relative cohomology group, $H_p(\Omega)$ is the $p$-th homology group, and $H_p(\partial\Omega)$ is the $p$-th relative homology group, i.e., the homology based on chains mod $\partial\Omega$.

We shall henceforth restrict to $\Omega \subset \mathbb{R}^3$ as in Theorem 1. Using either homology (10), the dimension of $H^p_r(\Omega)$ is seen to be

$$\begin{align*}
0 & \quad (p = 0), \\
 n-1 & \quad (p = 1), \\
\sum_{i=1}^{n} g_i & \quad (p = 2), \\
 1 & \quad (p = 3).
\end{align*}$$

(11)

These are also the dimensions of the spaces of harmonic $p$-forms.

The space $\Lambda(\Omega) = \bigoplus_{p=0}^{3} \Lambda_p(\Omega)$ may be identified as

$$\Lambda(\Omega) = L^2(\Omega) \oplus L^2(\Omega, \mathbb{R}^3) \oplus L^2(\Omega, \mathbb{R}^3) \oplus L^2(\Omega) \ni (\phi, \mathbf{E}, \mathbf{B}, \psi),$$

where $d : \Lambda(\Omega) \rightarrow \Lambda(\Omega)$ acts as

$$d : L^2(\Omega) \xrightarrow{\text{grad}} L^2(\Omega, \mathbb{R}^3) \xrightarrow{\text{rot}} L^2(\Omega, \mathbb{R}^3) \xrightarrow{\text{div}} L^2(\Omega) \rightarrow 0$$

with boundary conditions $\phi = 0$, $\mathbf{E} \parallel = 0$, $\mathbf{B} \perp = 0$ on $\partial\Omega$. Then

$$d^* : 0 \leftarrow L^2(\Omega) \xleftarrow{\text{-div}} L^2(\Omega, \mathbb{R}^3) \xleftarrow{\text{-rot}} L^2(\Omega, \mathbb{R}^3) \xleftarrow{\text{-grad}} L^2(\Omega)$$
without any boundary conditions. The Laplace-Beltrami operator on forms,
\[-\Delta = \bigoplus_{p=0}^{3}(-\Delta_p) = dd^* + d^*d,\]
is seen to correspond to the Euclidean Laplacian with boundary conditions
\[
\begin{align*}
\phi &= 0 & (p = 0), \\
\mathbf{E}_\parallel &= 0, \quad \text{div} \mathbf{E} = 0 & (p = 1), \\
\mathbf{B}_\perp &= 0, \quad (\text{rot} \mathbf{B})_\parallel &= 0 & (p = 2), \\
(\text{grad} \psi)_\perp &= 0 & (p = 3).
\end{align*}
\]
Each of the four problems admits a heat kernel expansion,
\[
\text{Tr}_{\Lambda_p(\Omega)} e^{\Delta_p t} \simeq \sum_{n=0}^{\infty} a_n^{(p)} t^{n-3/2},
\]
whose coefficients have been computed \((n = 0, \ldots, 3)\) \[\|\] or can be computed using existing results \((n = 4, 5)\) \[\|\]. To this end we note that the boundary conditions for \(p = 1, 2\) can be formulated equivalently as
\[
\begin{align*}
\mathbf{E}_\parallel &= 0, \quad \frac{\partial \mathbf{E}_\perp}{\partial n} - (\text{tr} L) \mathbf{E}_\perp &= 0 & (p = 1), \\
\mathbf{B}_\perp &= 0, \quad \frac{\partial \mathbf{B}_\parallel}{\partial n} - L \mathbf{B}_\parallel &= 0 & (p = 2).
\end{align*}
\]
**First approach.** We will compute \((\|)\). We observe that \(\Delta_{\mathbf{E}}\) is just the restriction of \(\Delta_1\) to its invariant subspace
\[
\mathcal{H} = \left\{ \mathbf{E} \in L^2(\Omega, \mathbb{R}^3) \mid \text{div} \mathbf{E} = 0 \right\} = \text{Ker} \, d^*.
\]
Hence
\[
\text{Tr}_{\mathcal{H}} e^{t \Delta_{\mathbf{E}}} = \text{Tr}_{L^2(\Omega, \mathbb{R}^3)} e^{t \Delta_1} - \text{Tr}_{L^2(\Omega, \mathbb{R}^3)} e^{t \Delta_0},
\]
where the orthogonal complement of \(\mathcal{H}\) in \(L^2(\Omega, \mathbb{R}^3)\) is
\[
\mathcal{H}^\perp = \text{Ran} \, d_1 = \text{Ran} \, d_1 = \left\{ \nabla \phi \in L^2(\Omega, \mathbb{R}^3) \mid \phi = 0 \text{ on } \partial \Omega \right\},
\]
(Ran \(d\) is closed by the Hodge decomposition, see e.g. \[\|\] \[\|\].) By \(d\Delta = \Delta d\), the operators \((\Delta_1)\rfloor_{\mathcal{H}^\perp}\) and \(\Delta_0\) have the same spectrum (in fact \(\nabla \phi = 0\) implies \(\phi = 0\) by the boundary condition). Thus, using also \((\|)\), we find
\[
\text{Tr}_{\mathcal{H}} e^{t \Delta_{\mathbf{E}}} = \text{Tr}_{L^2(\Omega, \mathbb{R}^3)} e^{t \Delta_1} - \text{Tr}_{L^2(\Omega)} e^{t \Delta_0} = \text{Tr}_{L^2(\Omega, \mathbb{R}^3)} e^{t \Delta_1} - \text{Tr}_{L^2(\Omega)} e^{t \Delta_0} - (n-1),
\]
i.e.,
\[
\begin{align*}
a_k &= a_k^{(1)} - a_k^{(0)}, & (k \neq 3), \\
a_3 &= a_3^{(1)} - a_3^{(0)} - n + 1.
\end{align*}
\]
These relations, together with the values of $a_k^{(p)}$ computed in the Appendix, yield the values of the coefficients stated in the Theorem. In particular, we will obtain

$$a_3^{(1)} - a_3^{(0)} = \frac{1}{64}(4\pi)^{-1}\left(3(\text{tr} L)^2 + 28 \det L\right)[\partial \Omega].$$

This matches the stated value of $a_3$ because of

$$n = \frac{1}{2} \sum_{i=1}^{n} (1 + g_i) + \frac{1}{2} \sum_{i=1}^{n} (1 - g_i)$$

and of the Gauss-Bonnet theorem,

$$\frac{1}{2} \sum_{i=1}^{n} (1 - g_i) = \frac{1}{2}(4\pi)^{-1}(\det L)[\partial \Omega]. \quad (15)$$

**Second approach.** We now compute (7). As has been noted in the Introduction, eigenmodes of $-\Delta_B$, except for zero-modes, satisfy the boundary condition $B_\perp = 0$, and are thus eigenmodes of $-\Delta_2$ belonging to its invariant subspace $\mathcal{H}$, cf. (5, 12). The converse is obvious. We conclude that

$$\text{Tr}_{\mathcal{H}} e^{t\Delta_B} = \text{Tr}_{L^2(\Omega, \mathbb{R}^3)} e^{t\Delta_2} - \text{Tr}_{\mathcal{H}} e^{t\Delta_3}.$$

Since

$$\mathcal{H} = \{B \in L^2(\Omega, \mathbb{R}^3) \mid \text{div} B = 0\} = \text{Ker} d_3,$$

we have

$$\mathcal{H}^\perp = \overline{\text{Ran} d_3^*} = \text{Ran} d_3^* = \{-\nabla \psi \in L^2(\Omega, \mathbb{R}^3) \mid \psi \in L^2(\Omega)\}.$$

Using $d^* \Delta = \Delta d^*$, we see that $(-\Delta_2)[\mathcal{H}^\perp]$ and $-\Delta_3$ have the same spectrum, except for a single zero-mode (in fact, $-\nabla \psi = 0$ implies $\psi = \text{const}$). We thus find, using (11),

$$\text{Tr}_{\mathcal{H}} e^{t\Delta_B} = \text{Tr}_{L^2(\Omega, \mathbb{R}^3)} e^{t\Delta_2} - \text{Tr}_{L^2(\Omega)} e^{t\Delta_3} = \text{Tr}_{L^2(\Omega, \mathbb{R}^3)} e^{t\Delta_2} - \text{Tr}_{L^2(\Omega)} e^{t\Delta_3} - \left(\sum_{i=1}^{n} g_i - 1\right),$$

i.e.,

$$a_k = a_k^{(2)} - a_k^{(3)}, \quad (k \neq 3),$$

$$a_3 = a_3^{(2)} - a_3^{(3)} - \sum_{i=1}^{n} g_i + 1.$$

From these relations and from the results of the Appendix we again recover Theorem. In particular,

$$a_3^{(2)} - a_3^{(3)} = \frac{1}{64}(4\pi)^{-1}\left(3(\text{tr} L)^2 - 36 \det L\right)[\partial \Omega]$$

leads to the claim for $a_3$, because of

$$\sum_{i=1}^{n} g_i = \frac{1}{2} \sum_{i=1}^{n} (1 + g_i) - \frac{1}{2} \sum_{i=1}^{n} (1 - g_i)$$

and of (15).
3 Application to the Casimir effect

For the purpose of this discussion we simply define the Casimir energy by the mode summation method, see e.g. [3]. In particular, we do not address the issue [6] of whether it is the most appropriate physically. We shall however observe that the Casimir energy is finite – a conclusion obtained in [3], but questioned in [10].

Consider the cavity $\Omega \subset \mathbb{R}^3$ enclosed in a large ball $\Omega_0$. As usual we compare the vacuum energy of the electromagnetic field in the domains $\Omega \cup (\Omega_0 \setminus \Omega)$ with that of the reference domain $\Omega_0$. Each eigenmode of either domain contributes a zero-point energy $\omega_k^2 / 2$, resp. $\omega_0^2 / 2$. As a regulator for the eigenfrequencies $\omega_k = \lambda_k^{1/2}$, we choose $e^{-\gamma \lambda_k}$, $(\gamma > 0)$. The corresponding definition of the Casimir energy is

$$E_C = \frac{1}{2} \lim_{\Omega_0 \to \infty} \lim_{\gamma \downarrow 0} \left( \sum_k \lambda_k^{1/2} e^{-\gamma \lambda_k} - \sum_k (\lambda_k^0)^{1/2} e^{-\gamma \lambda_k^0} \right).$$

We shall prove that the limit $\gamma \downarrow 0$ is finite. It will also be clear that the subsequent limit $\Omega_0 \to \infty$ exists, though we shall not make the effort to prove that (see however e.g. [8], Section 12.7 for the necessary tools). Using $\lambda_k^{1/2} = -\frac{1}{\sqrt{\pi}} \int_0^\infty dt \ t^{-\frac{1}{2}} \frac{d}{dt} e^{-t \lambda_k}$ and [8] we find for the regularized sum of the eigenfrequencies

$$\sum_k \lambda_k^{1/2} e^{-\gamma \lambda_k} \approx -\sum_n \frac{n - 3}{2 \sqrt{\pi}} a_n \int_0^\delta dt \ t^{-\frac{1}{2}} (t + \gamma)^{\frac{n-5}{2}}$$

as $\gamma \downarrow 0$. Here $\delta > 0$ is arbitrary, but fixed, and “≈” means up to terms $O(1)$. Using

$$\int_0^\delta dt \ t^{-\frac{1}{2}} (t + \gamma)^{\frac{n-5}{2}} \approx \begin{cases} \frac{4}{3} \gamma^{-2} & (n = 0), \\ \frac{3}{2} \gamma^{-\frac{3}{2}} & (n = 1), \\ 2 \gamma^{-1} & (n = 2), \\ \pi \gamma^{-\frac{1}{2}} & (n = 3), \\ -\log \gamma & (n = 4), \end{cases}$$

we find

$$\sum_k \lambda_k^{1/2} e^{-\gamma \lambda_k} \approx \frac{2}{\sqrt{\pi}} a_0 \gamma^{-2} + \frac{\sqrt{\pi}}{2} a_1 \gamma^{-\frac{3}{2}} + \frac{1}{\sqrt{\pi}} a_2 \gamma^{-1} + 0 \cdot a_3 \gamma^{-\frac{1}{2}} + \frac{1}{2 \sqrt{\pi}} a_4 \log \gamma.$$

Hence a finite Casimir energy requires (cf. [7]) that $a_0, a_1, a_2, a_4$ (but not necessarily $a_3$!) agree for $\Omega \cup (\Omega_0 \setminus \Omega)$ and for the reference domain $\Omega_0$. This is indeed so for $a_0 = 2(4\pi)^{-\frac{3}{2}} |\Omega_0|$ and for $a_1 = 0$, but also for $a_2, a_4$ as the contribution from the two...
sides of $\partial \Omega$ cancel. The same conclusion is obtained if the regulator $e^{-\gamma \lambda_k}$ is replaced by $e^{-(\gamma \lambda_k)^{1/2}}$ (see [7], Eq. (27)):

$$\sum_k \lambda_k^2 e^{-(\gamma \lambda_k)^{1/2}} \approx \frac{24}{\sqrt{\pi}} a_0 \gamma^{-2} + 4 a_1 \gamma^{-\frac{3}{2}} + \frac{2}{\sqrt{\pi}} a_2 \gamma^{-1} + 0 \cdot a_3 \gamma^{-\frac{5}{2}} + \frac{1}{\sqrt{\pi}} a_4 \log \gamma .$$

Since no renormalization is necessary, the value of $E_C$ agrees with that obtained by means of the zeta function.

In the rest of this section we compare our results with those of [2, 3]. To the extent the comparison is done we will find agreement. An important tool there is the mode generating function, Eq. (4.5) in [2],

$$\Phi(k) = \frac{1}{2} \text{Tr} \left( \frac{-\Delta_E}{-\Delta_E - k^2} + \frac{-\Delta_B}{-\Delta_B - k^2} \right) = \frac{k^2}{2} \text{Tr}' \left( (-\Delta_E - k^2)^{-1} + (-\Delta_B - k^2)^{-1} \right), \quad (k \in \mathbb{C} \setminus \mathbb{R}) ,$$

where “$\doteq$” means equality “within addition of some polynomial in $k^2$”. Since the resolvents in (16) are not trace class, but their squares are, we first consider that replacement. Using $(A + \mu)^{-2} = \int_0^\infty dt \; t \; e^{-t(A+\mu)}$ we obtain, as $\mu \to \infty$,

$$\frac{1}{2} \text{Tr}' \left( (-\Delta_E + \mu)^{-2} + (-\Delta_B + \mu)^{-2} \right) \doteq \sum_{n=0}^\infty a_n \int_0^\infty dt \; t^{\frac{n-3}{2}} e^{-t \mu} = \sum_{n=0}^\infty \Gamma(\frac{n+1}{2}) a_n \mu^{-\frac{n+1}{2}}$$

with coefficients $a_n$ given in Theorem I. Integrating w.r.t. $\mu$ we find

$$\frac{1}{2} \text{Tr}' \left( (-\Delta_E + \mu)^{-1} + (-\Delta_B + \mu)^{-1} \right) \doteq \sum_{n=0}^\infty \Gamma(\frac{n-1}{2}) a_n \mu^{-\frac{n-1}{2}} - a_1 \log \mu$$

and hence, with $\mu^{1/2} = -ik$,

$$\Phi(k) = 2 \sqrt{\pi} a_0 i k^3 - \sqrt{\pi} a_1 k^2 \ln(-k^2) + i \sqrt{\pi} a_2 k - a_3 + O(k^{-1}) .$$

Upon insertion of the mentioned values for $a_0, \ldots, a_3$ this agrees with Eq. (4.40) in [2], except for $a_3$ which is there replaced by its local part, see [3],

$$a_3 = \frac{1}{64} (4\pi)^{-1} (3(\text{tr} L)^2 - 4 \det L) [\partial \Omega] = \frac{1}{64} \int_{\partial \Omega} \text{d}\sigma \left( \frac{3}{4} (\kappa_1^2 + \kappa_2^2) - \kappa_1 \kappa_2 \right) ,$$

where $\kappa_1, \kappa_2$ are the principal curvatures. Note however that this discrepancy is implicit in the definition of “$\doteq$”. It is resolved in [3] by first considering $\delta \Phi(k)$, i.e., the difference of the mode generating functions corresponding to the configurations $\Omega \cup (\Omega_0 \setminus \Omega)$ and $\Omega_0$. Thus

$$\delta \Phi(k) = -2 a_3 + O(k^{-1}) ,$$

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since the contributions to $a_0$, $a_2$ cancel, and those to $\tilde{a}_3$ double the value. Not ambiguous then is “the number of additional modes of finite frequency created by introducing the conducting surface $\partial \Omega$”:

$$ C = \psi(0+) - \psi(\infty) , $$

where $\psi(y) = \delta \Phi(iy)$. For a connected boundary $\partial \Omega$ of genus $g$ the value of $\psi(0+)$ has been established as $\psi(0+) = -g$ (see [3], Eq. (5.8)), resulting in

$$ C = 2\tilde{a}_3 - g . \quad (17) $$

This result agrees with Theorem 1; the non-local terms in (9) take the values $-\frac{1}{2}(g - 1), -\frac{1}{2}g, \frac{1}{2}$ for $\Omega, \Omega_0 \setminus \overline{\Omega}$ and $\Omega_0$ respectively. Thus,

$$ \delta a_3 = 2\tilde{a}_3 - g , $$

in agreement with (17).

## A Appendix

In this appendix we compute the heat kernel coefficients in (13) for $p = 0, \ldots, 3$ and $n = 0, \ldots, 5$ on the basis of Theorems 1 and 4 in [3]. We use the same notation, together with $P = n \otimes n$ denoting the normal projection at the boundary. The vector bundle is $V = \Omega \times \mathbb{R}$ for $p = 0, 3$, resp. $V = T\bar{\Omega}$ for $p = 1, 2$, equipped with the Euclidean connection. The decompositions of $V|_{\partial \Omega} = V_N \oplus V_D \ni (\phi^N, \phi^D)$ (with projections $\Pi_+$, resp. $\Pi_-$) and boundary conditions $\phi^N + S\phi^N = 0$, resp. $\phi^D = 0$, are specified as follows, cf. (14) and [3]:

\begin{align*}
  p = 0 : & \begin{cases} 
    \Pi_+ = 0 , \\
    \Pi_- = 1 , 
  \end{cases} \\
  p = 1 : & \begin{cases} 
    \Pi_+ = P , \\
    \Pi_- = 1 - P , 
  \end{cases} \\
  S = -L_{aa} P , \\
  p = 2 : & \begin{cases} 
    \Pi_+ = 1 - P , \\
    \Pi_- = P , 
  \end{cases} \\
  S = -L , \\
  p = 3 : & \begin{cases} 
    \Pi_+ = 1 , \\
    \Pi_- = 0 . 
  \end{cases} \\
\end{align*} \quad (18)
The result is

\[ a_0^{(p)} = (4\pi)^{-\frac{3}{2}} c_0^{(p)}|\Omega|, \]

\[ a_1^{(p)} = \frac{1}{4} (4\pi)^{-1} c_1^{(p)}|\partial\Omega|, \]

\[ a_2^{(p)} = \frac{1}{3} (4\pi)^{-\frac{3}{2}} c_2^{(p)}(\text{tr} L)[\partial\Omega], \]

\[ a_3^{(p)} = \frac{1}{384} (4\pi)^{-1} \left( c_{31}^{(p)}(\text{tr} L)^2 + c_{32}^{(p)}(\det L) \right)[\partial\Omega], \]

\[ a_4^{(p)} = \frac{1}{315} (4\pi)^{-\frac{3}{2}} \left( c_{41}^{(p)}(\text{tr} L)^3 + c_{42}^{(p)} \text{tr} L \cdot \det L \right)[\partial\Omega], \]

\[ a_5^{(p)} = \frac{1}{245760} (4\pi)^{-1} \left( c_{51}^{(p)}(\text{tr} L)^4 + c_{52}^{(p)}(\text{tr} L)^2 \det L + c_{53}^{(p)}(\det L)^2 + c_{54}^{(p)} \text{tr} L \cdot \nabla^2 \text{tr} L \right)[\partial\Omega] \]

with coefficients given by

|       | \( p = 0 \) | \( p = 1 \) | \( p = 2 \) | \( p = 3 \) |
|-------|-------------|-------------|-------------|-------------|
| \( c_0^{(p)} \) | 1           | 3           | 3           | 1           |
| \( c_1^{(p)} \) | -1          | -1          | 1           | 1           |
| \( c_2^{(p)} \) | 1           | -3          | -3          | 1           |
| \( c_{31}^{(p)} \) | 3           | 21          | 33          | 15          |
| \( c_{32}^{(p)} \) | -20         | 148         | -220        | -4          |
| \( c_{41}^{(p)} \) | 4           | 36          | 60          | 28          |
| \( c_{42}^{(p)} \) | -18         | -162        | -186        | -42         |
| \( c_{51}^{(p)} \) | 555         | 5145        | 8625        | 4035        |
| \( c_{52}^{(p)} \) | -2840       | -27720      | -35720      | -10840      |
| \( c_{53}^{(p)} \) | 2224        | 29072       | 29712       | 2864        |
| \( c_{54}^{(p)} \) | 120         | 2520        | 4680        | 2280        |

These values imply Theorem 1 as explained in its proof.

The computation of the table is based on the general result of [5], which has been
applied to (18) using the following identities:

\[
\begin{align*}
\text{Tr}(P_{:a}P_{:b}) &= 2(L^2)_{ab}, \\
\text{Tr}(P_{:a}P_{:a}P_{:b}P_{:b}) &= (L^4)_{aa} + (L^2)_{aa}(L^2)_{bb}, \\
\text{Tr}(P_{:a}P_{:b}P_{:a}P_{:b}) &= 2(L^4)_{aa}, \\
\text{Tr}(P_{:a}P_{:b}P_{:a}P_{:b}) &= 2L_{acb}L_{bca} + 4(L^4)_{aa} + 4(L^2)_{aa}(L^2)_{bb}, \\
\text{Tr}(P_{:a}P_{:b}P_{:a}P_{:b}) &= 2L_{abc}L_{abc} + 6(L^4)_{aa} + 2(L^2)_{aa}(L^2)_{bb}.
\end{align*}
\]

They can be derived by using \( \nabla_{e_a} n = -L_{ab} e_b \), so that

\[
P_{:a} = -L_{ac} (e_c \otimes n + n \otimes e_c),
\]

and by assuming without loss that \( \nabla_{e_a} e_b \) has no component parallel to \( T_p \partial \Omega \) at the point \( p \) of evaluation, i.e., \( \nabla_{e_a} e_b = L_{ab} n \). Then

\[
P_{:ab} = -L_{acb} (e_c \otimes n + n \otimes e_c) - 2(L^2)_{ab} P + (L_{ac}L_{bd} + L_{ad}L_{bc}) e_c \otimes e_d,
\]

from which the above traces follow. In turn they allow the computation of similar traces with \( P \) replaced by \( \chi = \Pi_+ - \Pi_- \), i.e., by \( \chi = \pm (2P - 1) \) in the cases \( p = 1, 2 \). In these two cases we also have

\[
\begin{align*}
\text{Tr} S_{:a} &= -L_{bca}, \\
\text{Tr} S_{:ab} &= -L_{cc:ab},
\end{align*}
\]

and, moreover, for \( p = 1 \),

\[
\begin{align*}
\text{Tr}(S_{:a}S_{:a}) &= L_{bca}L_{cc:a} + 2L_{bb}L_{cc}(L^2)_{aa}, \\
\text{Tr}(P_{:a}S_{:b}) &= -2(L^2)_{ab}L_{cc}, \\
\text{Tr}(PS_{:a}S_{:a}) &= L_{bca}L_{cc:a} + L_{bb}L_{cc}(L^2)_{aa},
\end{align*}
\]

resp. for \( p = 2 \),

\[
\begin{align*}
\text{Tr}(S_{:a}S_{:a}) &= L_{abc}L_{abc} + 2(L^2)_{aa}, \\
\text{Tr}(P_{:a}S_{:a}) &= 2(L^3)_{aa}, \\
\text{Tr}(PS_{:a}S_{:a}) &= (L^4)_{aa}.
\end{align*}
\]

Furthermore, traces of \( L^k \), \( (k \geq 2) \), were reduced to \( \text{tr} L, \det L \) by means of \( L^2 - (\text{tr} L)L + \det L = 0 \). Finally, we used the Codazzi equation, \( L_{abc} = L_{ac:b} \), as well as

\[
L_{abc:a} - L_{abc:a} = L_{aa}(L^2)_{bc} - (L^2)_{aa}L_{bc},
\]

which follows from the Gauss equation.

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