A stationary bulk planar ideal flow solution for the double shearing model

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Abstract. This paper provides a general ideal flow solution for the double shearing model of pressure-dependent plasticity. This new solution is restricted to a special class of stationary planar flows. A distinguished feature of this class of solutions is that one family of characteristic lines is straight. The solution is analytic. The mapping between Cartesian and principal lines based coordinate systems is given in parametric form with characteristic coordinates being the parameters. A simple relation that connects the scale factor for one family of coordinate curves of the principal lines based coordinate system and the magnitude of velocity is derived. The original ideal flow theory is widely used as the basis for inverse methods for the preliminary design of metal forming processes driven by minimum plastic work. The new theory extends this area of application to granular materials.

1. Introduction

The theory of stationary and non-stationary bulk ideal flow has been mainly developed for rigid perfectly plastic solids satisfying Tresca’s yield condition and its associated flow rule. The first ideal flow solution has been derived in [1]. In this paper, a die profile for maximum efficiency in plane strain strip drawing has been determined. The existence of stationary three-dimensional ideal flows in Tresca’s solids has been demonstrated in [2]. This result has been extended to non-stationary flows in [3]. A comprehensive overview on the ideal flow theory has been provided in [4]. This theory has been used as the basis for inverse methods for the preliminary design of bulk metal forming processes driven by minimum plastic work [1, 5, 6]. It has been shown in [7] that elasticity can be incorporated in the theory for stationary planar flow. A ductile fracture criterion has been incorporated in the ideal flow theory in [8]. The existence of stationary bulk planar ideal flows for anisotropic rigid plastic materials obeying the model proposed in [9] has been proven in [10]. All available results on the ideal flow theory have been obtained for pressure-independent materials. In the present paper, a general stationary ideal flow solution is given for the double shearing model proposed in [11]. This model is based on the Coulomb – Mohr yield criterion.
2. System of equations and its properties

The constitutive equations of the double shearing model under plane strain conditions have been given in [11]. These equations comprise the Mohr-Coulomb yield criterion and the flow rule. Let \((\mu, \nu)\) be an arbitrary orthogonal curvilinear coordinate system. Then, the Mohr-Coulomb yield criterion and the flow rule read

\[
(\sigma_{\mu\mu} + \sigma_{\nu\nu}) \sin \varphi + \sqrt{(\sigma_{\mu\mu} - \sigma_{\nu\nu})^2 + 4\sigma_{\mu\nu}^2} = 2k \cos \varphi, \tag{1}
\]

\[
\varepsilon_{\mu\mu} + \varepsilon_{\nu\nu} = 0, \quad \sin(2\psi)(\varepsilon_{\mu\mu} - \varepsilon_{\nu\nu}) - 2\cos(2\psi)\varepsilon_{\mu\nu} - 2\sin \varphi\left(\omega_{\mu\nu} + \frac{d\psi}{dt}\right) = 0, \tag{2}
\]

respectively. In equations (1) and (2), \(\sigma_{\mu\mu}, \sigma_{\nu\nu},\) and \(\sigma_{\mu\nu}\) are the components of the stress tensor, \(\varepsilon_{\mu\mu}, \varepsilon_{\nu\nu},\) and \(\varepsilon_{\mu\nu}\) are the components of the strain rate tensor and \(\omega_{\mu\nu}\) is the only non-zero spin (vorticity) component in the \((\mu, \nu)\) coordinate system. Also, \(\psi\) is the orientation of the major principal stress axis relative to the \(\mu\)-direction, measured anti-clockwise positive, \(k\) is the cohesion, and \(\varphi\) is the angle of internal friction. Moreover, \(d\psi/dt\) denotes the convected derivative. Equations (1) and (2) should be supplemented with the equilibrium equations.

Let us introduce a principal lines based coordinate system \((\xi, \eta)\) where \(\xi\)-lines are trajectories of the major principal stress \(\sigma_\xi\) and \(\eta\)-lines are trajectories of the other principal stress in planes of flow \(\sigma_\eta\). The system of stress equations comprising the Mohr-Coulomb yield criterion and the equilibrium equations is hyperbolic and the angle between each of the characteristic directions and the principal axis corresponding to the stress \(\sigma_\xi\) is equal to \(\pi/4 + \varphi/2\). The \((\mu, \nu)\) coordinate system can be chosen as a Cartesian coordinate system \((x, y)\). The orientation of the principal lines based coordinate system and the characteristic coordinate system \((\alpha, \beta)\) relative to the Cartesian coordinate system is shown in figure 1.

It has been shown in [12] that the scale factors for the \(\xi\)- and \(\eta\)-coordinate lines, \(h_\xi\) and \(h_\eta\), satisfy the relation

\[
h_\xi h_\eta^b = 1, \tag{3}
\]

where \(b = (1 + \sin \varphi)/(1 - \sin \varphi)\).
3. Mapping between the Cartesian, principal lines based and characteristic coordinate systems

It follows from the geometry of figure 1 that

\[
\frac{\partial x}{h_\xi \partial \xi} = \cos \psi, \quad \frac{\partial x}{h_\eta \partial \eta} = -\sin \psi, \quad \frac{\partial y}{h_\xi \partial \xi} = \sin \psi, \quad \frac{\partial y}{h_\eta \partial \eta} = \cos \psi.
\]

Using (3) these equations can be transformed to

\[
\frac{\partial x}{\partial \xi} = h^{-b} \cos \psi, \quad \frac{\partial x}{\partial \eta} = -h \sin \psi, \quad \frac{\partial y}{\partial \xi} = h^{-b} \sin \psi, \quad \frac{\partial y}{\partial \eta} = h \cos \psi
\]

(4)

where \( h \equiv h_\eta \). Substituting (4) into the compatibility equations

\[
\frac{\partial^2 x}{\partial \xi \partial \eta} = \frac{\partial^2 x}{\partial \eta \partial \xi}, \quad \frac{\partial^2 y}{\partial \xi \partial \eta} = \frac{\partial^2 y}{\partial \eta \partial \xi}
\]

gives

\[
-bh^{-b-1} \cos \psi \frac{\partial h}{\partial \eta} - h^{-b} \sin \psi \frac{\partial \psi}{\partial \eta} + \sin \psi \frac{\partial h}{\partial \xi} + h \cos \psi \frac{\partial \psi}{\partial \xi} = 0,
\]

\[
-bh^{-b-1} \sin \psi \frac{\partial h}{\partial \eta} + h^{-b} \cos \psi \frac{\partial \psi}{\partial \eta} - \cos \psi \frac{\partial h}{\partial \xi} + h \sin \psi \frac{\partial \psi}{\partial \xi} = 0.
\]

(5)

Rotating the Cartesian coordinate system so that its \( x \)-axis coincides with principal axis corresponding to the stress \( \sigma_\xi \) at a given point transforms (5) to the following system of equations for \( h \) and \( \psi \) in the \((\xi, \eta)\) coordinate system

\[
bh^{-b-1} \frac{\partial h}{\partial \eta} - h \frac{\partial \psi}{\partial \xi} = 0, \quad h^{-b} \frac{\partial \psi}{\partial \eta} - \frac{\partial h}{\partial \xi} = 0.
\]

(6)

Using a standard technique it is possible to show that this system of equations is hyperbolic. In particular, the characteristic curves are given by

\[
\frac{d\eta}{d\xi} = -\sqrt{b}h^{-b-1}, \quad \frac{d\eta}{d\xi} = \sqrt{b}h^{-b-1}.
\]

(7)

Here the first equation determines the \( \alpha \)-lines and the second the \( \beta \)-lines. The characteristic relations along these lines are

\[
\sqrt{b}d \ln h + d\psi = 0 \quad \text{on } \alpha \text{-line}, \quad -\sqrt{b}d \ln h + d\psi = 0 \quad \text{on } \beta \text{-line.}
\]

(8)

In what follows, it is assumed that \( \beta \)-lines are straight. In this case \( \psi \) is independent of \( \beta \). Then, it is seen from the second equation in (8) that \( h \) is independent of \( \beta \). The equations in (8) can be immediately integrated. Following [13] it is convenient to write the solution of these equations as

\[
\sqrt{b} \ln \frac{h}{h_0} + \psi - \psi_0 = 0, \quad -\sqrt{b} \ln \frac{h}{h_0} + \psi - \psi_0 = 2\alpha \cos \varphi.
\]

(9)

Here \( h_0 \) and \( \psi_0 \) are constant. Solving (9) for \( h \) and \( \psi \) yields

\[
\sqrt{b} \ln \frac{h}{h_0} = -\alpha \cos \varphi, \quad \psi - \psi_0 = \alpha \cos \varphi.
\]

(10)
Using (10) equations (7) can be rewritten as
\[
\frac{\partial \eta}{\partial \alpha} = -\sqrt{b h_0^{-b-1}} \exp(2\alpha) \frac{\partial \xi}{\partial \alpha}, \quad \frac{\partial \eta}{\partial \beta} = \sqrt{b h_0^{-b-1}} \exp(2\alpha) \frac{\partial \xi}{\partial \beta}.
\] (11)

It is convenient to introduce the new variable \( \omega \) by
\[
\xi = \omega \exp(-2\alpha).
\] (12)

Eliminating \( \xi \) in (11) by means of this equation yields
\[
\frac{\partial \eta}{\partial \alpha} = -\sqrt{b h_0^{-b-1}} \left( \frac{\partial \omega}{\partial \alpha} - 2\omega \right), \quad \frac{\partial \eta}{\partial \beta} = \sqrt{b h_0^{-b-1}} \frac{\partial \omega}{\partial \beta}.
\] (13)

The second equation can be immediately integrated to give
\[
\eta = \sqrt{b h_0^{-b-1}} \omega + 2\sqrt{b h_0^{-b-1}} \Phi_1(\alpha),
\] (14)

where \( \Phi_1(\alpha) \) is an arbitrary function of \( \alpha \). Eliminating \( \eta \) in the first equation in (13) by means of (14) results in
\[
\frac{\partial \omega}{\partial \alpha} - \omega = -\frac{d\Phi_1}{d\alpha}.
\] (15)

The general solution of this equation is
\[
\omega = -\Phi_1(\alpha) - e^{\alpha} \int \Phi_1(\alpha) e^{-\alpha} d\alpha + e^{\alpha} \Phi_2(\beta),
\] (15)

where \( \Phi_2(\beta) \) is an arbitrary function of \( \beta \). Eliminating \( \omega \) in (12) and (14) by means of (15) yields
\[
\frac{\eta h_0^{1+b}}{\sqrt{b}} = -e^{\alpha} \int \Phi_1(\alpha) e^{-\alpha} d\alpha + e^{\alpha} \Phi_2(\beta) + \Phi_1(\alpha).
\] (16)

These equations supply the mapping between the principal line based and characteristic coordinates. The infinitesimal arc lengths of the \( \alpha \)- and \( \beta \)-characteristic curves are
\[
ds_\alpha = \sqrt{\left( \frac{\partial x}{\partial \alpha} \right)^2 + \left( \frac{\partial y}{\partial \alpha} \right)^2} d\alpha, \quad ds_\beta = \sqrt{\left( \frac{\partial x}{\partial \beta} \right)^2 + \left( \frac{\partial y}{\partial \beta} \right)^2} d\beta,
\]
respectively. Using the chain rule these relations can be transformed to
\[
ds_\alpha = \sqrt{\left( \frac{\partial x}{\partial \alpha} \frac{\partial \xi}{\partial \alpha} + \frac{\partial x}{\partial \eta} \frac{\partial \eta}{\partial \alpha} \right)^2 + \left( \frac{\partial y}{\partial \alpha} \frac{\partial \xi}{\partial \alpha} + \frac{\partial y}{\partial \eta} \frac{\partial \eta}{\partial \alpha} \right)^2} d\alpha,
\]
\[
ds_\beta = \sqrt{\left( \frac{\partial x}{\partial \beta} \frac{\partial \xi}{\partial \beta} + \frac{\partial x}{\partial \eta} \frac{\partial \eta}{\partial \beta} \right)^2 + \left( \frac{\partial y}{\partial \beta} \frac{\partial \xi}{\partial \beta} + \frac{\partial y}{\partial \eta} \frac{\partial \eta}{\partial \beta} \right)^2} d\beta.
\]
(17)

Substituting (4), (10), and (11) into (17) gives
\[
ds_\alpha = -\sqrt{1 + bh_0^{-b}} \exp[-\alpha(1 - \sin \varphi)] \left( \frac{d\Phi_1}{d\alpha} + \omega \right) d\alpha,
\]
\[
ds_\beta = \sqrt{1 + bh_0^{-b}} \exp(\alpha \sin \varphi) \frac{d\Phi_2}{d\beta} d\beta.
\]
(18)
It is seen from this equation that it is possible to put with no loss of generality that $\Phi_2(\beta) = \beta$. Then, (16) and (18) become

$$\xi = -\Phi_1(\alpha)e^{-2\alpha} - \Phi_1(\alpha)e^{-\alpha} \int \Phi_1(\alpha)e^{-\alpha} d\alpha + \beta e^{-\alpha},$$

$$\eta \frac{h_0^{1+b}}{b} = -e^\alpha \int \Phi_1(\alpha)e^{-\alpha} d\alpha + \beta e^\alpha + \Phi_1(\alpha),$$

$$d\alpha = -\sqrt{1 + bh_0^{-b}} \exp[-\alpha(1 - \sin \varphi)] \left[ \frac{d\Phi_1}{d\alpha} - \Phi_1(\alpha) - e^\alpha \int \Phi_1(\alpha)e^{-\alpha} d\alpha + e^\alpha \beta \right] d\alpha,$$

$$d\beta = \sqrt{1 + bh_0^{-b}} \exp(\alpha \sin \varphi) d\beta.$$

It follows from the geometry of figure 1 that

$$\frac{\partial x}{\partial s_\alpha} = \cos \left( \psi - \frac{\pi}{4} - \frac{\varphi}{2} \right), \quad \frac{\partial x}{\partial s_\beta} = \cos \left( \psi + \frac{\pi}{4} + \frac{\varphi}{2} \right),$$

$$\frac{\partial y}{\partial s_\alpha} = -\cos \left( \psi + \frac{\pi}{4} - \frac{\varphi}{2} \right), \quad \frac{\partial y}{\partial s_\beta} = \cos \left( \psi - \frac{\pi}{4} + \frac{\varphi}{2} \right).$$

Equations (20) and (21) combine to give

$$\frac{\partial x}{\partial \alpha} = -\sqrt{1 + bh_0^{-b}} \exp[-\alpha(1 - \sin \varphi)] \left[ \frac{d\Phi_1}{d\alpha} - \Phi_1(\alpha) - e^\alpha \int \Phi_1(\alpha)e^{-\alpha} d\alpha + e^\alpha \beta \right] \cos \left( \psi - \frac{\pi}{4} - \frac{\varphi}{2} \right),$$

$$\frac{\partial x}{\partial \beta} = \sqrt{1 + bh_0^{-b}} \exp(\alpha \sin \varphi) \cos \left( \psi + \frac{\pi}{4} + \frac{\varphi}{2} \right),$$

$$\frac{\partial y}{\partial \alpha} = \sqrt{1 + bh_0^{-b}} \exp[-\alpha(1 - \sin \varphi)] \left( \frac{d\Phi_1}{d\alpha} - \Phi_1(\alpha) - e^\alpha \int \Phi_1(\alpha)e^{-\alpha} d\alpha + e^\alpha \beta \right) \cos \left( \psi + \frac{\pi}{4} + \frac{\varphi}{2} \right),$$

$$\frac{\partial y}{\partial \beta} = \sqrt{1 + bh_0^{-b}} \exp(\alpha \sin \varphi) \cos \left( \psi - \frac{\pi}{4} + \frac{\varphi}{2} \right).$$

Integrating the second equation yields

$$x = \sqrt{1 + bh_0^{-b}} \exp(\alpha \sin \varphi) \cos \left( \psi + \frac{\pi}{4} + \frac{\varphi}{2} \right) \beta + \sqrt{1 + bh_0^{-b}} X(\alpha).$$

Here $X(\alpha)$ is an arbitrary function of $\alpha$. Substituting this solution into the first equation in (22) results in

$$\frac{dX}{d\alpha} = - \left( \frac{d\Phi_1}{d\alpha} - \Phi_1(\alpha) - e^\alpha \int \Phi_1(\alpha)e^{-\alpha} d\alpha \right) \exp[\alpha(\sin \varphi - 1)] \cos \left( \psi - \frac{\pi}{4} - \frac{\varphi}{2} \right).$$

Analogously, integrating the forth equation in (22) yields

$$y = \sqrt{1 + bh_0^{-b}} \exp(\alpha \sin \varphi) \cos \left( \psi - \frac{\pi}{4} + \frac{\varphi}{2} \right) \beta + \sqrt{1 + bh_0^{-b}} Y(\alpha).$$

Here $Y(\alpha)$ is an arbitrary function of $\alpha$. Substituting this solution into the third equation in (22) results in

$$\frac{dY}{d\alpha} = \left[ \frac{d\Phi_1}{d\alpha} - \Phi_1(\alpha) - e^\alpha \int \Phi_1(\alpha)e^{-\alpha} d\alpha \right] \exp[\alpha(\sin \varphi - 1)] \cos \left( \psi + \frac{\pi}{4} - \frac{\varphi}{2} \right).$$

Eliminating $\psi$ in (24) and (26) by means of (10) and integrating determine the functions $X(\alpha)$ and $Y(\alpha)$. Thus equations (19), (23), and (25) supply the mapping between the Cartesian and principal lines based coordinate systems in parametric form with $\alpha$ and $\beta$ being the parameters.
4. Ideal flow solution

The solutions (19), (23), and (25) satisfy the equilibrium equations and Coulomb-Mohr yield criterion. In the case of ideal flow, the $\xi$-coordinate lines are streamlines. In order to prove the existence of ideal flow, it is necessary to show that this condition is compatible with the velocity equations of the double shearing model. Let us choose the $(\mu, \nu)$-coordinate system such that the $\mu$-coordinate lines are the $\xi$-coordinate lines. In this case the angle $\psi$ involved in (2) is the angle between the $\xi$-coordinate lines and the principal stress direction corresponding to $\sigma_1$. Therefore, $\psi = 0$ everywhere in the plastic region. Then, equation (2) becomes

$$\varepsilon_{\xi\xi} + \varepsilon_{\eta\eta} = 0, \quad \varepsilon_{\xi\eta} + \sin \varphi \omega_{\xi\eta} = 0.$$  \hfill (27)

It has been taken into account here that $\varepsilon_{\xi\xi} \equiv \varepsilon_{\mu\mu}$, $\varepsilon_{\eta\eta} \equiv \varepsilon_{\nu\nu}$, $\varepsilon_{\xi\eta} \equiv \varepsilon_{\mu\nu}$, and $\omega_{\xi\eta} \equiv \omega_{\mu\nu}$. Since the $\xi$-coordinate lines are streamlines, the velocity vector has one non-zero component $u_\xi = u$.

Using (3) these relations can be transformed to

$$\varepsilon_{\xi\xi} = \frac{1}{h}\frac{\partial u}{\partial \xi}, \quad \varepsilon_{\eta\eta} = \frac{u}{h^2} \frac{\partial h}{\partial \xi}, \quad 2\varepsilon_{\xi\eta} = 1 \frac{\partial u}{h} \frac{\partial h}{\partial \eta} - \frac{u}{h} \frac{\partial h}{\partial \eta}, \quad 2\omega_{\xi\eta} = 1 \frac{\partial u}{h} \frac{\partial h}{\partial \eta} + \frac{u}{h} \frac{\partial h}{\partial \eta}. \hfill (28)$$

Substituting (28) into (27) yields

$$\frac{\partial u}{\partial \xi} + \frac{u}{h} \frac{\partial h}{\partial \xi} = 0, \quad \frac{\partial u}{\partial \eta} + \frac{u}{h} \frac{\partial h}{\partial \eta} = 0.$$

These equations can be immediately integrated to give

$$u = \frac{C}{h}. \hfill (29)$$

Here $C$ is constant. Therefore, once any solution satisfying equations (19), (23), and (25) has been found, the magnitude of the velocity vector is determined from (29). The direction of this vector is tangent to the $\xi$-coordinate lines.

Conclusions

It has been shown that non-trivial stationary planar ideal flow solutions exist in pressure-dependent plasticity if the double shearing model based of the Coulomb–Mohr yield criterion is adopted. The solution found is based on the assumption that one family of characteristic lines is straight. In this case, the solution is practically analytic. In particular, the mapping between the principal lines based and characteristic coordinate systems is given by equation (19) and the mapping between the Cartesian and characteristic coordinate systems by equation (23) and (25). Thus the mapping between the principal lines based and Cartesian coordinate systems is given in parametric form. The function $\Phi_1(\alpha)$ involved in equations (19), (24), and (26) should be determined from boundary conditions. Once a solution for the mappings between different coordinate systems has been found, the magnitude of the velocity vector is determined from (29). The direction of this vector is tangent to the $\xi$-coordinate lines.

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