The Theory of Stochastic Pseudo-differential Operators and Its Applications, I∗

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Abstract

The purpose of this paper is to establish the theory of stochastic pseudo-differential operators and give its applications in stochastic partial differential equations. First, we introduce some concepts on stochastic pseudo-differential operators and prove their fundamental properties. Also, we present the boundedness theory, invertibility of stochastic elliptic operators and the Gårding inequality. Moreover, as an application of the theory of stochastic pseudo-differential operators, we give a Calderón-type uniqueness theorem on the Cauchy problem of stochastic partial differential equations. The proof of the uniqueness theorem is based on a new Carleman-type estimate, which is adapted to the stochastic setting.

1 Introduction

During the past seventy years, more and more studies of stochastic phenomena have appeared. Research in this area is stimulated by the need to take account of random effects in the engineering and physical systems. For such systems, stochastic processes give a natural replacement for deterministic functions as mathematical descriptions. Since K. Itô introduced the stochastic integral, one of the main topics in Probability Theory and Stochastic Process has been the issue on stochastic differential equations. Compared to deterministic differential equations, stochastic differential equations are much more complicated. Indeed, one has to distinguish forward stochastic differential equations, backward stochastic differential equations and forward-backward stochastic differential equations in most of the interesting cases. Now, stochastic ordinary differential equations have been well-developed. However, stochastic partial differential equations (SPDEs for short) make slow progress. By now, the main tool in this field is of functional analysis nature. As far as we know, the real PDE-based approach is not well-developed. Meanwhile, we notice that the theory

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of pseudo-differential operators has been a powerful tool in the study of general partial differential equations, since it was established in 1960’s. It plays a crucial role in the studies of existence, uniqueness and propagation of singularities for the solutions of partial differential equations. Therefore, we would like to introduce such a theory to the stochastic setting and regard it as a tool to solve the problems related to SPDEs.

For this purpose, we establish the theory of stochastic pseudo-differential operators (SPDOs for short) and give some applications to SPDEs. First of all, we introduce some basic notions, which are adapted to the stochastic setting, including symbol, amplitude, SPDO, kernel and uniformly properly supported SPDO. Since a stochastic process has the variables on time and sample point, we add these two variables to all notions and endow them with suitable integrability. Also, in order for the symbol calculus, we study asymptotic expansions of a symbol, and eventually establish an algebra and generalized module of SPDOs. On the other hand, we establish the \( L^p \)-boundedness theory, invertibility of stochastic elliptic operators and the Gårding inequality, which are some fundamental results related to the energy estimates. Moreover, as an application of theory of SPDOs, we present a Calderón-type uniqueness theorem on the Cauchy problem of SPDEs. In his remarkable paper [2], A.-P. Calderón established a fundamental result on the uniqueness of the non-characteristic Cauchy problem for general partial differential equations. One of the main tools introduced in [2] is a preliminary version of the symbol calculation technique. Later, Calderón’s uniqueness theorem was extended to the operators with characteristics of high multiplicity. We refer to [14] and the references cited therein for some deeper results in this topic. However, as far as we know, there is no work addressing the uniqueness on the Cauchy problem for general SPDEs. In this paper, we give a Calderón-type uniqueness result in the stochastic setting, by virtue of the theory of SPDOs. In order to present the key idea in the simplest way, we do not pursue the full technical generality. More precisely, we focus mainly on the Cauchy problem for SPDEs in the case of at most double characteristics.

It is a little surprising that the theory of SPDOs was not available in the previous literatures although a related but clearly different theory for random pseudo-differential operators was introduced in [4]. It deserves to point out that the study of SPDOs seems to be of independent interest. We divide our results on SPDOs into two parts. In this paper, we present the first part and its applications. We will establish the stochastic micro-local analysis and develop singularity propagation theory for stochastic hyperbolic equations of second order in the forthcoming paper.

The rest of this paper is organized as follows. Section 2 is devoted to the basic concepts and properties of SPDOs. In Section 3 we establish the boundedness theory. In Section 4 we give invertibility of stochastic elliptic operators and the Gårding inequality. As an application of the theory of SPDOs, a Calderón-type uniqueness theorem is presented in Section 5.

2 Calculus of stochastic pseudo-differential operators

Pseudo-differential operators developed from the theory of singular integral operators, which were essentially pseudo-differential operators with homogeneous symbol of order 0. The appearance of
both Calderón’s uniqueness theorem ([2]) and the index theorem for elliptic operators by M. F. Atiyah and I. M. Singer ([1]) showed the importance of the theory of singular integral operators. Shortly afterwards, J. J. Kohn and L. Nirenberg ([9]) removed the restriction to order 0 and generalized the notions of pseudo-differential operators to the case of general polyhomogeneous symbols. Later, L. Hörmander ([6]) introduced pseudo-differential operators with the symbols of type $(\rho, \delta)$, by the need to incorporate fundamental solutions of hypoelliptic operators of constant strength. Since the theory of pseudo-differential operators was established in 1960’s, it has been an important mathematical branch ([5], [13]). It plays an important role in many fields, such as partial differential equations, harmonic analysis and differential geometry etc.

In order to introduce the theory of pseudo-differential operators to the stochastic setting, in this section, we shall present some basic concepts and properties of SPDOs. First, we introduce some locally convex topological vector spaces, which will be used later. Then, we give the notions of symbol, amplitude, SPDO, kernel and uniformly properly supported SPDO in sequence. Moreover, we give asymptotic expansions of a symbol and establish an algebra and generalized module of SPDOs.

To begin with, we give some usual notations. Throughout this paper, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ is a complete filtered probability space, on which a one dimensional standard Brownian motion $\{w(t)\}_{t \geq 0}$ is defined. Fix $T > 0$, $n \in \mathbb{N} \setminus \{0\}$, $m \in \mathbb{N}$, $\ell \in \mathbb{R}$, $p$, $q \in [1, \infty]$ and a domain $G$ of $\mathbb{R}^n$. $i$ denotes the imaginary unit. Let $H$ be a Banach space. We denote by $L^p_G(0, T; H)$ the set of all $H$-valued $\{\mathcal{F}_t\}_{t \geq 0}$-adapted process $X(\cdot)$ such that $\mathbb{E} \int_0^T |X(t)|^p_G dt < \infty$; by $L^p_G(0, T; H)$ the Banach space consisting of all $H$-valued $\{\mathcal{F}_t\}_{t \geq 0}$-adapted bounded processes; and by $L^p_G(\Omega; C^m([0, T]; H))$ the Banach space consisting of all $H$-valued $\{\mathcal{F}_t\}_{t \geq 0}$-adapted $m$-th order continuously differentiable processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^p_{C^m([0, T]; H)}) < \infty$. Moreover, we simply write $L^p_G(0, T; \mathbb{R})$ as $L^p_G(0, T)$, and have the similar notations for $L^p_G(0, T; \mathbb{R})$ and $L^p_G(\Omega; C^m([0, T]))$. Furthermore, we denote by $C(\cdot)$ a generic constant, which may be different from one place to another.

### 2.1 Basic function spaces

In this subsection, we introduce some locally convex topological vector spaces, which will be used as the domain or range of a SPDO later. To begin with, we denote by $\mathcal{D}(G)$ the topological space (with the usual inductive topology, see Page 54 in [12]) of infinitely differentiable functions supported by $G$; by $\mathcal{E}(G)$ the topological space of infinitely differentiable functions defined on $G$; and by $\mathcal{S}$ the topological space of rapidly decreasing functions. Let $\{K_j\}_{j \in \mathbb{N}}$ stand for a sequence of compact sets satisfying $K_0 \subseteq K_1 \subseteq \cdots$ and $\bigcup_{j \in \mathbb{N}} K_j = G$. Then, we write

$$|v|_{j,k,1} = \sup_{x \in K_j, \ |\alpha| \leq k} |\partial_x^\alpha v(x)|, \quad (j, \ k \in \mathbb{N});$$

$$|u|_{p,j,k,1} = \left\{ \sup_{x \in K_j, \ |\alpha| \leq k} |\partial_x^\alpha u(\cdot, x)| \right\}_{L^p_G(0, T)}, \quad (j, \ k \in \mathbb{N});$$
\[ |u|_{p,j,k,2} = \sup_{x \in \mathbb{R}^n, x, \alpha, \beta} |x^\alpha \partial^\beta_x u(\cdot, \cdot, x)| \quad (j, k \in \mathbb{N}), \]

where \( \alpha \) and \( \beta \) are multi-indices. On the other hand, by Theorem 1.36 and Theorem 1.35 in [11], if \( B \) is a 0-neighborhood base for the inductive topology on \( \mathcal{D}(G) \), then for any \( \gamma \in B \),

\[ \gamma = \{ v \in \mathcal{D}(G); \mu_\gamma(v) < 1 \}, \]

where \( \mu_\gamma \) is the Minkowski functional of \( \gamma \). Also, \( \{\mu_\gamma\}_{\gamma \in B} \) is a family of generating semi-norms on \( \mathcal{D}(G) \). Set

\[ |u|_{p,\gamma} = |(\mu_\gamma(u))(\cdot, \cdot)|_{L^p_x(0,T)}, \quad (\gamma \in B). \]

Next, we define the following locally convex spaces:

\[ L^p_x(0,T; \mathcal{E}(G)) = \{ u \mid u(t, \omega, \cdot) \in \mathcal{E}(G), \text{ a.e. } (t, \omega) \in (0, T) \times \Omega; u(\cdot, \cdot, x) \text{ is } \{\mathcal{F}_t\}_{t \geq 0}-\text{adapted}, \]

\[ \text{for any } x \in G; \text{ and } |u|_{p,j,k,1} < \infty, \quad j, k = 0, 1, \cdots \}, \]

which is a Fréchet space, generated by a sequence of semi-norms \( \{ | \cdot |_{p,j,k,1} \}_{j,k \in \mathbb{N}} \); \n
\[ L^p_x(0,T; \mathcal{S}) = \{ u \mid u(t, \omega, \cdot) \in \mathcal{S}, \text{ a.e. } (t, \omega) \in (0, T) \times \Omega; u(\cdot, \cdot, x) \text{ is } \{\mathcal{F}_t\}_{t \geq 0}-\text{adapted}, \]

\[ \text{for any } x \in \mathbb{R}^n; \text{ and } |u|_{p,j,k,2} < \infty, \quad j, k = 0, 1, \cdots \}, \]

which is a Fréchet space, generated by a sequence of semi-norms \( \{ | \cdot |_{p,j,k,2} \}_{j,k \in \mathbb{N}} \); \n
\[ L^p_x(0,T; \mathcal{D}(G)) = \{ u \mid u(t, \omega, \cdot) \in \mathcal{D}(G), \text{ a.e. } (t, \omega) \in (0, T) \times \Omega; u(\cdot, \cdot, x) \text{ is } \{\mathcal{F}_t\}_{t \geq 0}-\text{adapted}, \]

\[ \text{for any } x \in G; \text{ and } |u|_{p,\gamma} < \infty, \text{ for each } \gamma \in B \}, \]

which is generated by a family of semi-norms \( \{ | \cdot |_{p,\gamma} \}_{\gamma \in B} \).

In addition, for any compact set \( K \subseteq G \) and \( j \in \mathbb{N} \), we write

\[ |v|_{K,j} = \sup_{x \in K, |\alpha| \leq j} |\partial^\alpha_x v(x)|, \quad |u|_{p,K,j} = |(u(\cdot, \cdot, \cdot))_{j} |_{L^p_x(0,T)}. \]

Then, \( \mathcal{D}_K = \{ v \in C^\infty(G) ; \text{ supp } v \subseteq K \} \) is a locally convex topological vector space, endowed with a sequence of semi-norms \( \{ | \cdot |_{K,j} \}_{j \in \mathbb{N}} \). Also, we introduce the following locally convex spaces:

\[ L^p_x(0,T; \mathcal{D}_K) = \{ u \mid u(t, \omega, \cdot) \in \mathcal{D}_K, \text{ a.e. } (t, \omega) \in (0, T) \times \Omega; u(\cdot, \cdot, x) \text{ is } \{\mathcal{F}_t\}_{t \geq 0}-\text{adapted}, \]

\[ \text{for any } x \in G; \text{ and } |u|_{p,K,j} < \infty, \quad j = 0, 1, \cdots \}, \]

which is generated by a sequence of semi-norms \( \{ | \cdot |_{p,K,j} \}_{j \in \mathbb{N}} \);

\[ L^p_x(0,T; \mathcal{D}_G) = \bigcup_{K \subseteq G \text{ is compact}} L^p_x(0,T; \mathcal{D}_K), \]

which is endowed with the inductive topology.

In order to characterize the topology and convergence in \( L^p_x(0,T; \mathcal{D}_G) \), we first recall the following known result.
Lemma 2.1 ([12, Page 54]) Let $\mathcal{Z}$ and $\mathcal{Z}_K$ ($K \in \Theta$, $\Theta$ is an index set) be vector spaces, let $g_K$ be a linear mapping of $\mathcal{Z}_K$ into $\mathcal{Z}$, and let $\Gamma_K$ be a locally convex topology on $\mathcal{Z}_K$. If we denote by $\Gamma$ the inductive topology on $\mathcal{Z}$ with respect to the family $\{(\mathcal{Z}_K, \Gamma_K, g_K); K \in \Theta\}$, then a 0-neighborhood base for $\Gamma$ is given by the family $\{U\}$ of all balanced, convex, absorbing subsets of $\mathcal{Z}$, such that for each $K \in \Theta$, $g_K^{-1}(U)$ is a 0-neighborhood in $(\mathcal{Z}_K, \Gamma_K)$.

For our problem, for any compact set $K \subseteq G$, we take $\mathcal{Z} = L^p_F(0, T; \mathcal{D}_G)$ and $\mathcal{Z}_K = L^p_F(0, T; \mathcal{D}_K)$. $g_K : \mathcal{Z}_K \rightarrow \mathcal{Z}$ is canonical embedding and the topology $\Gamma_K$ is generated by a sequence of seminorms $\{\| \cdot |_{p,K,j}\}_{j \in \mathbb{N}}$. Then for the inductive topology $\Gamma$ on $L^p_F(0, T; \mathcal{D}_G)$, a 0-neighborhood base $\mathcal{B}$ is given by the family $\{U\}$ of all balanced, convex, absorbing subsets of $L^p_F(0, T; \mathcal{D}_G)$, such that for any compact set $K \subseteq G$, $g_K^{-1}(U)$ is a 0-neighborhood in $(L^p_F(0, T; \mathcal{D}_K), \Gamma_K)$.

Moreover, we recall another known result on locally convex spaces.

Lemma 2.2 ([11, Theorem 1.37]) Suppose that $\mathcal{P}$ is a separating family of semi-norms on a vector space $\mathcal{Z}$. For each $\phi \in \mathcal{P}$ and positive integer $k$, set
\[
V(\phi, k) = \left\{ u \in \mathcal{Z}; \phi(u) < \frac{1}{k}\right\}.
\]

Let $\mathcal{B}$ be the collection of all finite intersections of the sets $V(\phi, k)$. Then $\mathcal{B}$ is a balanced convex absorbing local base, which turns $\mathcal{Z}$ into a locally convex space.

For any compact set $K$, by Lemma 2.2 and the definition of semi-norms $\{\| \cdot |_{p,K,j}\}_{j \in \mathbb{N}}$ on $L^p_F(0, T; \mathcal{D}_K)$, it is easy to show that for any $\{u_k\}_{k \in \mathbb{N}} \subseteq L^p_F(0, T; \mathcal{D}_K)$, $\lim_{k \to \infty} u_k = 0$ in $L^p_F(0, T; \mathcal{D}_K)$ if and only if for any $j \in \mathbb{N}$, $\lim_{k \to \infty} \mathbb{E} \int_0^T \sup_{x \in K, |\alpha| \leq j} |\partial_{\alpha}^j u_k(t, \omega, x)|^p dt = 0$.

In the remainder of this subsection, we give the following result on the convergence in $L^p_F(0, T; \mathcal{D}_G)$.

Proposition 2.1 For any $\{u_j\}_{j \in \mathbb{N}} \subseteq L^p_F(0, T; \mathcal{D}_G)$, $\lim_{j \to \infty} u_j = 0$ in $L^p_F(0, T; \mathcal{D}_G)$ if and only if the following two conditions hold:

1. there exists a compact set $K_*$ such that $\text{supp } u_j(t, \omega, \cdot) \subseteq K_*$ for a.e. $(t, \omega) \in (0, T) \times \Omega$ and any $j \in \mathbb{N}$;
2. for any $k \in \mathbb{N}$, $\lim_{j \to \infty} \mathbb{E} \int_0^T \sup_{x \in K_+, |\alpha| \leq k} |\partial_{\alpha}^j u_j(t, \omega, x)|^p dt = 0$.

Proof. By Lemma 2.1 and Lemma 2.2, we have only to prove (1). Assume the contrary. Then there exist a subsequence of $\{u_{j_k}\}_{k \in \mathbb{N}}$ of $\{u_j\}_{j \in \mathbb{N}}$ and a sequence of compact sets $\{K_{k,*}\}_{k \in \mathbb{N}}$ satisfying $K_{0,*} = \emptyset$, $K_{1,*} \subseteq K_{2,*} \subseteq \cdots$ and $\bigcup_{k \in \mathbb{N}} K_{k,*} = G$, such that
\[
|u_{j_k}(t, \omega, x_k)| \geq \varepsilon_k, \quad k = 1, 2, \cdots, \tag{2.1}
\]
for $(t, \omega) \in T_k \times \Omega_k$ and a positive constant $\varepsilon_k$, where $x_k \in K_{k,*}\setminus K_{k-1,*}$, $(T_k)_{k=1,2,\ldots}$ and $(\Omega_k)_{k=1,2,\ldots}$ are two sequences of measurable sets (with positive measures) of $(0, T)$ and $\Omega$, respectively. Then, we define a semi-norm $\| \cdot |_{p,*}$ on $L^p_F(0, T; \mathcal{D}_G)$ ($p \geq 1$) as follows:
\[
|u|^p_{p,*} = \sum_{k=1}^{\infty} \frac{1}{P(\Omega_k)|T_k|} \int_{\Omega_k} \int_{T_k} \sup_{x \in K_{k,*}\setminus K_{k-1,*}} |\frac{u(t, \omega, x)}{u_{j_k}(t, \omega, x_k)}|\big| dt dP,
\]
where $|T_k|$ denotes the Lebesgue measure of $T_k$. Notice that for any $u \in L^p_T(0,T;D_G)$, the right side of the above equality is indeed a finite sum. Since $x_k \in K_{k,*} \setminus K_{k-1,*}$, it is easy to see that $|u_{j_k,p,*}| \geq 1$ for any $k = 1, 2, \cdots$. Therefore, if we write $U_\ast = \{ u \in L^p_T(0,T;D_G) \mid |u|_{p,*} < 1 \}$, then any $u_{j_k}$ $(k = 1, 2, \cdots)$ does not belong to $U_\ast$.

On the other hand, $U_\ast$ is a 0-neighborhood in $L^p_T(0,T;D_G)$. In fact, by Lemma 2.1, it remains to prove that for any compact set $K$, $V_K = U_\ast \cap L^p_T(0,T;D_K)$ is a 0-neighborhood in $L^p_T(0,T;D_K)$. By the definition of the semi-norm $| \cdot |_{p,*}$, it follows that

$$V_K = \left\{ u \in L^p_T(0,T;D_K) \mid \sum_{k=1}^\infty \frac{1}{P(\Omega_k)} P(\Omega_k) \int_0^1 \int_{T_k} \sup_{x \in (K_{k,*} \setminus K_{k-1,*}) \cap K} \left| u(t,\omega,x) \right|^p dt dP < 1 \right\}.$$  

We suppose that $K$ has nonempty intersections with the sets $K_{k,*} \setminus K_{k-1,*}$ $(k = 1, 2, \cdots, i_*)$, where $i_* = i_*(K)$ is a positive integer. Then, for any $u \in L^p_T(0,T;D_K)$,

$$\sum_{k=1}^\infty \frac{1}{P(\Omega_k)} P(\Omega_k) \int_0^1 \int_{T_k} \sup_{x \in (K_{k,*} \setminus K_{k-1,*}) \cap K} \left| u(t,\omega,x) \right|^p dt dP \leq \sum_{k=1}^{i_*} \frac{1}{P(\Omega_k)} P(\Omega_k) \int_0^T \sup_{x \in K} |u(t,\omega,x)|^p dt.  \tag{2.2}$$

If we take $N_* = 1 + \left\lfloor \sum_{k=1}^{i_*} \frac{1}{P(\Omega_k)} P(\Omega_k) \right\rfloor$, then by (2.2),

$$\left\{ u \in L^p_T(0,T;D_K) \mid \mathbb{E} \int_0^T \sup_{x \in K} |u(t,\omega,x)|^p dt < \frac{1}{N_*} \right\} \subseteq V_K,$$

where $\lfloor \ell \rfloor$ denotes the integral part of a real number $\ell$. Therefore, $V_K$ is a 0-neighborhood of $L^p_T(0,T;D_K)$. This implies that $\{u_j\}_{j \in \mathbb{N}}$ cannot be a sequence converging to 0 in $L^p_T(0,T;D_G)$. This contradiction proves that (1) must be true. Similarly, we can get the desired result for $p = \infty$. The proof is completed.

### 2.2 Symbol and stochastic pseudo-differential operators

In this subsection, we use the Fourier integral representation to define SPDOs. For this purpose, we first introduce the notion of symbols. Compared to the classical one in the deterministic case, we add two variables $t$ and $\omega$, and endow them with the integrability.

**Definition 2.1** A complex-valued function $a$ is called a symbol of order $(\ell,p)$ if it satisfies the following conditions:

1. $a(t,\omega,\cdot,\cdot) \in C^\infty(G \times \mathbb{R}^n)$, a.e. $(t,\omega) \in (0,T) \times \Omega$;
2. $a(\cdot,\cdot,x,\xi)$ is $\{\mathcal{F}_t\}_{t \geq 0}$-adapted, $\forall (x,\xi) \in G \times \mathbb{R}^n$;
(3) for any two multi-indices $\alpha$ and $\beta$, and any compact set $K \subseteq G$, there exists a nonnegative function $M_{\alpha,\beta,K}(\cdot,\cdot) \in L^p_F(0,T)$ such that

$$\left| \partial_{\xi}^\alpha \partial_x^\beta a(t,\omega, x, \xi) \right| \leq M_{\alpha,\beta,K}(t,\omega)(1 + |\xi|^{\ell-|\alpha|},$$

for a.e. $(t,\omega) \in (0,T) \times \Omega$ and any $(x,\xi) \in K \times \mathbb{R}^n$. We write $a \in S^\ell_p(G \times \mathbb{R}^n)$ for short.

**Remark 2.1** In Definition 2.1, if $G = \mathbb{R}^n$ and $M_{\alpha,\beta,K}(\cdot,\cdot) \equiv M_{\alpha,\beta}(\cdot,\cdot)$, which is independent of the compact set $K$, we set $a \in S^\ell_p$.

By Definition 2.1, it is easy to show that for any $\ell_1$, $\ell_2 \in \mathbb{R}$, if $\ell_1 \leq \ell_2$, then $S^{\ell_1}_p(G \times \mathbb{R}^n) \subseteq S^{\ell_2}_p(G \times \mathbb{R}^n)$. Therefore, we write

$$S^\infty_p(G \times \mathbb{R}^n) = \bigcup_{\ell \in \mathbb{R}} S^\ell_p(G \times \mathbb{R}^n), \quad S^{-\infty}_p(G \times \mathbb{R}^n) = \bigcap_{\ell \in \mathbb{R}} S^\ell_p(G \times \mathbb{R}^n).$$

Moreover, if $a_1 \in S^{\ell_1}_p(G \times \mathbb{R}^n)$ and $a_2 \in S^{\ell_2}_q(G \times \mathbb{R}^n)$, then for any multi-indices $\alpha$ and $\beta$,

$$\partial_{\xi}^\alpha \partial_x^\beta a_1 \in S^{\ell_1-|\alpha|}_p(G \times \mathbb{R}^n), \quad a_1 + a_2 \in S^{\max\{\ell_1,\ell_2\}}_p(G \times \mathbb{R}^n), \quad a_1 a_2 \in S^{\ell_1+\ell_2}_q(G \times \mathbb{R}^n),$$

here and hereafter $q^*$ denotes a constant defined as follows: $q^* = \frac{pq}{p+q}$, for $p, q \geq 1$, $pq \geq p+q$; $q^* = p$, for $p \geq 1$, $q = \infty$; $q^* = q$, for $q \geq 1$, $p = \infty$; $q^* = \infty$, for $p = q = \infty$.

Now, we introduce the notion of SPDOs.

**Definition 2.2** A linear operator $A$ is called a SPDO of order $(\ell, p)$ if $a \in S^\ell_p(G \times \mathbb{R}^n)$ and for any $u \in L^p_F(0,T; \mathcal{D}(G))$,

$$(Au)(t,\omega, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(t,\omega, x, \xi) \hat{u}(t,\omega, \xi) d\xi,$$

where $\hat{u}(t,\omega, \xi) = \int_G e^{-ix\cdot\xi} u(t,\omega, x) dx$. We write $A \in \mathcal{L}^\ell_p(G)$. Moreover, if $a \in S^\ell_p$, we set $A \in \mathcal{L}^\ell_p$.

Notice that for a.e. $(t,\omega) \in (0,T) \times \Omega$, the SPDO $A$ in Definition 2.2 is indeed a usual pseudo-differential operator of order $\ell$ in the deterministic case. For simplicity of notation, we write it simply $A$ when no confusion can arise in this paper.

In the following, we will investigate the domain and range of SPDOs. Before that, we present a useful lemma. Write $|v|_\alpha = \sup_{x \in \mathbb{R}^n} \left| (1 + |x|)^{1+n} \partial_x^\beta v(x) \right|$, for any $v \in C^\infty_0(G)$. Then, we have the following result.

**Lemma 2.3** $| \cdot |_\alpha$ is a generating semi-norm on $\mathcal{D}(G)$.

**Proof.** By Lemma 1.34 in [11], the set $U^* = \{ v \in C^\infty_0(G); \ |v|_\alpha < 1 \}$ is balanced, convex, absorbing, and $| \cdot |_\alpha$ is the Minkowski functional of $U^*$. Therefore, it remains to prove that $U^*$ is the member of a 0-neighborhood base for the inductive topology on $\mathcal{D}(G)$. By Lemma 2.1, it suffices
to show that for any compact set $K \subseteq G$, $V^* = \{ v \in \mathcal{D}_K; |v|_\alpha < 1 \}$ is a 0-neighborhood in $\mathcal{D}_K$.
By the definition of $| \cdot |_\alpha$, for any $v \in \mathcal{D}_K$,
\[
\sup_{x \in \mathbb{R}^n; |\beta| \leq 1} |(1 + |x|)^{1+n} \partial_x^{\beta} v(x)| \leq N^* \sup_{x \in K; |\beta| \leq 2(|\alpha| + \ell + 1 + n)} |\partial_x^{\beta} v(x)|,
\]
where $N^* = 1 + (1 + \sup_{x \in K} |x|)^{1+n}$ and $\lfloor \ell \rfloor$ denotes the integral part of a real number $\ell$. This implies that
\[
\left\{ v \in \mathcal{D}_K \mid |v|_{K,2(|\alpha|+\ell+1+n)} = \sup_{x \in K; |\beta| \leq 2(|\alpha|+\ell+1+n)} |\partial_x^{\beta} v(x)| < \frac{1}{N^*} \right\} \subseteq V^*.
\]
Therefore, $V^*$ is a 0-neighborhood in $\mathcal{D}_K$. This finishes the proof.

Based on Lemma 2.3, we get the following result.

**Theorem 2.1** Suppose that $A$ is a SPDO determined by a symbol $a$.
(1) If $a \in S^p_p(G \times \mathbb{R}^n)$, $A : L^q_p(0, T; \mathcal{D}(G)) \to L^q_p(0, T; \mathcal{E}(G))$ is continuous;
(2) If $a \in S^\ell_p$, $A : L^q_p(0, T; \mathcal{S}) \to L^q_p(0, T; \mathcal{S})$ is continuous.

**Proof.** For any $u \in L^q_p(0, T; \mathcal{D}(G))$, multi-index $\alpha$ and any compact set $K \subseteq G$, by Definition 2.1, we have that
\[
|\partial_x^\alpha [e^{ix \xi} a(t, \omega, x, \xi) \hat{u}(t, \omega, \xi)]| = \left| \sum_{\alpha_1 + \alpha_2 = \alpha} \frac{\alpha!}{\alpha_1! \alpha_2!} i^{\alpha_1} \xi^{\alpha_1} e^{ix \xi} \partial_x^{\alpha_2} a(t, \omega, x, \xi) \hat{u}(t, \omega, \xi) \right|
\]
\[
= \left| \sum_{\alpha_1 + \alpha_2 = \alpha} \frac{\alpha!}{\alpha_1! \alpha_2!} i^{\alpha_1} \xi^{\alpha_1} e^{ix \xi} \partial_x^{\alpha_2} a(t, \omega, x, \xi) \int_G e^{-ix \xi} u(t, \omega, x) dx \right|
\]
\[
\leq C(n, \alpha) M_{a, K}(t, \omega) (1 + |\xi|)^{|\alpha| + \ell} \int_G e^{-ix \xi} u(t, \omega, x) dx
\]
\[
\leq C(n, \alpha, \ell) M_{a, K}(t, \omega) (1 + |\xi|)^{-n-1} \int_G (1 + |\xi|^2)^{|\alpha| + \ell + 1 + n} e^{-ix \xi} u(t, \omega, x) dx
\]
\[
= C(n, \alpha, \ell) M_{a, K}(t, \omega) (1 + |\xi|)^{-n-1} \int_G (1 - \Delta_x)^{|\alpha| + \ell + 1 + n} e^{-ix \xi} u(t, \omega, x) dx
\]
\[
\leq C(n, \alpha, \ell) M_{a, K}(t, \omega) (1 + |\xi|)^{-n-1} \sup_{x \in \mathbb{R}^n} (1 + |x|)^{1+n} (1 - \Delta_x)^{|\alpha| + \ell + 1 + n} u(t, \omega, x),
\]
for a.e. $(t, \omega) \in (0, T) \times \Omega$ and any $(x, \xi) \in K \times \mathbb{R}^n$, where $\Delta_x$ denotes the Laplacian operator with respect to $x$. It follows that for a.e. $(t, \omega) \in (0, T) \times \Omega$, $(Au)(t, \omega, \cdot) \in C^\infty(G)$. Moreover,
\[
\sup_{x \in \mathbb{R}^n} |(\partial_x^\alpha (Au))(t, \omega, x)| \leq C(n, \alpha, \ell) M_{a, K}(t, \omega) |u(t, \omega, \cdot)|_\alpha.
\]
Hence,
\[
\sup_{x \in \mathbb{R}^n} |(\partial_x^\alpha (Au))(\cdot, \cdot, x)|_{L^q_p(0, T)} \leq C(n, \alpha, \ell) M_{a, K}(\cdot, \cdot)_{L^q_p(0, T)} |u(\cdot, \cdot, \cdot)|_\alpha_{L^q_p(0, T)}.
\]
By Lemma 2.3, this implies the desired continuity of $A$. Also, notice that both the limit of a sequence of measurable functions and the sum of a finite number of measurable functions are measurable. Therefore, for any $x \in G$, $(Au)(\cdot, x)$ is $\{F_t\}$-adapted.

(2) in Theorem 2.1 can be derived in the same way. The proof is completed. 

**Remark 2.2** It is easy to check that if $a \in S^\ell_p(G \times \mathbb{R}^n)$, then the associated SPDO $A$ satisfies the following conditions:

1. for a.e. $(t, \omega) \in (0, T) \times \Omega$, $A : D(G) \to \mathcal{E}(G)$ is continuous;
2. $A : L^q_p(0, T; D_G) \to L^q_p(0, T; \mathcal{E}(G))$ is continuous.

**Remark 2.3** We can endow $S^\ell_p(G \times \mathbb{R}^n)$ and $\mathcal{L}^\ell_p(G)$ with suitable topological structures, respectively, such that any SPDO of order $(\ell, p)$ has structural stability with respect to its amplitude. Indeed, let $\{K_j\}_{j \in \mathbb{N}}$ be a sequence of compact sets satisfying $K_0 \subseteq K_1 \subseteq \cdots$ and $\bigcup_{j \in \mathbb{N}} K_j = G$. For any $k, N, j \in \mathbb{N}$, set

$$|a|_{k,N,j,\ell,p} = \sup_{x \in K_j, \xi \in \mathbb{R}^n, \beta \leq k, |\beta| \leq N, \ell \geq 0} \frac{|\partial^\beta_x \partial^\beta_{\xi} a(\cdot, x, \xi)|}{(1 + |\xi|)^{\ell - |\beta|}} \in L^q_p(0, T)$$

Then $S^\ell_p(G \times \mathbb{R}^n)$ is a Fréchet space, generated by a sequence of semi-norms $\{| a|_{k,N,j,\ell,p}\}_{k,N,j \in \mathbb{N}}$. On the other hand, by the proof of Theorem 2.1, we see that for any SPDO $A$ determined by a symbol $a$, any multi-index $\alpha$, $j \in \mathbb{N}$ and $u \in L^q_p(0, T; D(G))$,

$$|Au|_{q^*,j,|\alpha|,1} \leq C(n, \alpha, \ell) |a|_{0,|\alpha|,j,\ell,p} \|u(\cdot, \cdot, \cdot)|_{0,|\alpha|,L^q_p(0, T)}$$

where $| \cdot |_{q^*,j,|\alpha|,1}$ and $\| \cdot \|_{L^q_p(0, T)}$ are one of the generating semi-norms of $L^q_p(0, T; D(G))$ and $L^q_p(0, T; D(G))$, respectively. Therefore, if we write

$$|A|_{\alpha,j,\ell,p} = \sup_{u \in L^q_p(0, T; D(G))} \frac{|Au|_{q^*,j,|\alpha|,1}}{\|u(\cdot, \cdot, \cdot)|_{0,|\alpha|,L^q_p(0, T)}}, \quad \forall A \in \mathcal{L}^\ell_p(G),$$

then $\mathcal{L}^\ell_p(G)$ is a Fréchet space, generated by a sequence of semi-norms $\{| a|_{\alpha,j,\ell,p}\}_{j \in \mathbb{N}, |\alpha| \geq 0}$. Moreover, $|A|_{\alpha,j,\ell,p} \leq C(n, \alpha, \ell) |a|_{0,|\alpha|,j,\ell,p}$. This implies that the mapping $S^\ell_p(G \times \mathbb{R}^n) \to \mathcal{L}^\ell_p(G)$, $a \mapsto A_a$ is continuous, where $A_a$ denotes the SPDO determined by a symbol $a$ of order $(\ell, p)$.

### 2.3 Amplitude and stochastic pseudo-differential operator

In this subsection, we introduce a class of SPDOs, which is apparently more general than the class of SPDOs defined in the last subsection. However, we shall show that these two classes in fact coincide under some circumstances later. First of all, we give the notion of amplitudes.

**Definition 2.3** A complex-valued function $a$ is called an amplitude of order $(\ell, p)$ if it satisfies the following conditions:
we first introduce the definition of transpose operators.

Remark 2.5 It is easy to check that for any

\[ A(2) \]

\[ A(1) \]

\[ A \]

Remark 2.4 For a.e. \( t,\omega \) \in (0,T) \times \Omega \), the integral in Definition 2.4 is understood as follows:

\[ (Au)(t,\omega,x) = (2\pi)^{-n} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \int_{G} \chi(\varepsilon \xi) e^{i(x-y)\cdot \xi} a(t,\omega,x,y,\xi) u(t,\omega,y) dy d\xi, \]

where \( \chi \in C_0^\infty(\mathbb{R}^n) \) and \( \chi = 1 \) in a neighborhood of the origin. Similar to the deterministic case, it is easy to show that Definition 2.4 is well posed.

Also, by the same method as that used in the proof of Theorem 2.1, we obtain the following result for the SPDOs determined by amplitudes.

\[ \text{Definition 2.4} \quad \text{The linear operator } A \text{ is called a SPDO of order } (\ell,p) \text{ if } a \in S_p^\ell(G \times G \times \mathbb{R}^n) \]  

for any \( u \in L_p^\ell(0,T;\mathcal{D}(G)) \),

\[ (Au)(t,\omega,x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{G} e^{i(x-y)\cdot \xi} a(t,\omega,x,y,\xi) u(t,\omega,y) dy d\xi. \]

We write \( A \in L_p^\ell(G \times G) \).

Remark 2.4 For a.e. \( t,\omega \) \in (0,T) \times \Omega \) and any \( x \in G \), the integral in Definition 2.4 is understood as follows:

\[ (Au)(t,\omega,x) = (2\pi)^{-n} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \int_{G} \chi(\varepsilon \xi) e^{i(x-y)\cdot \xi} a(t,\omega,x,y,\xi) u(t,\omega,y) dy d\xi, \]

where \( \chi \in C_0^\infty(\mathbb{R}^n) \) and \( \chi = 1 \) in a neighborhood of the origin. Similar to the deterministic case, it is easy to show that Definition 2.4 is well posed.

Also, by the same method as that used in the proof of Theorem 2.1, we obtain the following result for the SPDOs determined by amplitudes.

\[ \text{Theorem 2.2} \quad \text{If } A \in L_p^\ell(G \times G), \text{ then the following assertions hold:} \]

\[ \begin{align*}
(1) \quad & A : L_p^\ell(0,T;\mathcal{D}(G)) \to L_p^\ell(0,T;\mathcal{E}(G)) \text{ is continuous;} \\
(2) \quad & A : L_p^\ell(0,T;\mathcal{D}(G)) \to L_p^\ell(0,T;\mathcal{E}(G)) \text{ is continuous.}
\end{align*} \]

In the following, we extend the domain of SPDOs to a space of distributions. For this purpose, we first introduce the definition of transpose operators.

\[ \text{Definition 2.5} \quad \text{Suppose that } A \text{ is a SPDO of order } (\ell,p) \text{ and } a \text{ is its amplitude. Then an operator } \quad \]

\[ (\ell A) \]

\[ (\ell A)(t,\omega,x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{G} e^{i(x-y)\cdot \xi} a(t,\omega,x,y,-\xi) u(t,\omega,y) dy d\xi. \]

Remark 2.5 It is easy to check that for any \( A \in L_p^\ell(G \times G) \), \( \ell A \in L_p^\ell(G \times G) \). Therefore, Theorem 2.2 holds for \( \ell A \).
Denote by \((L^p_{\mathcal{F}}(0,T;\mathcal{D}(G)))'\) the dual space of the locally convex space \(L^p_{\mathcal{F}}(0,T;\mathcal{D}(G))\); and by \((L^q_{\mathcal{F}}(0,T;\mathcal{E}(G)))'\) the dual space of the locally convex space \(L^q_{\mathcal{F}}(0,T;\mathcal{E}(G))\). Let \(q_s\) be a constant defined as follows: \(q_s = \frac{p}{p - q}\), for \(p,q \geq 1\), \(p > q\); \(q_s = q\), for \(q \geq 1\), \(p = \infty\); \(q_s = \infty\), for \(p = q\). Next, we present the following result on an extension of the domain of SPDOs.

**Theorem 2.3** Suppose that \(A \in \mathcal{L}^q_p(G \times G)\). Then \(A : (L^q_{\mathcal{F}}(0,T;\mathcal{E}(G)))' \rightarrow (L^q_{\mathcal{F}}(0,T;\mathcal{D}(G)))'\) is continuous.

**Proof.** For any \(u \in (L^q_{\mathcal{F}}(0,T;\mathcal{E}(G)))'\) and \(v \in L^q_{\mathcal{F}}(0,T;\mathcal{D}(G))\), define

\[
\langle Au, v \rangle_{(L^q_{\mathcal{F}}(0,T;\mathcal{D}(G)))', L^q_{\mathcal{F}}(0,T;\mathcal{D}(G))} = \langle u, Av \rangle_{(L^q_{\mathcal{F}}(0,T;\mathcal{E}(G)))', L^q_{\mathcal{F}}(0,T;\mathcal{E}(G))}.
\]

Since \(u \in (L^q_{\mathcal{F}}(0,T;\mathcal{E}(G)))'\), there exists a semi-norm \(|\cdot|_{q,j_0,k_0,1}\) defined on \(L^q_{\mathcal{F}}(0,T;\mathcal{E}(G))\) such that

\[
|\langle u, Av \rangle_{(L^q_{\mathcal{F}}(0,T;\mathcal{E}(G)))', L^q_{\mathcal{F}}(0,T;\mathcal{E}(G))}| \leq C(u) |A|_{q,j_0,k_0,1}.
\]

By the above result and Remark 2.5, we can find a semi-norm \(|\cdot|_{q,s,\alpha_0}\) defined on \(L^q_{\mathcal{F}}(0,T;\mathcal{D}(G))\) for a multi-index \(\alpha_0\), such that

\[
|\langle u, Av \rangle_{(L^q_{\mathcal{F}}(0,T;\mathcal{E}(G)))', L^q_{\mathcal{F}}(0,T;\mathcal{E}(G))}| \leq C(u, v) |v|_{q,s,\alpha_0},
\]

which implies that \(Au \in (L^q_{\mathcal{F}}(0,T;\mathcal{D}(G)))'\). Moreover, the continuity of \(A\) is clear from (2.3). The proof of Theorem 2.3 is completed. \(\square\)

**Remark 2.6** It is regrettable that we fail to give a characterization of \((L^q_{\mathcal{F}}(0,T;\mathcal{E}(G)))'\) clearly. Here we only present two classes of function spaces, which are contained in \((L^q_{\mathcal{F}}(0,T;\mathcal{E}(G)))'\).

Let \(\mathcal{E}'(G)\) and \(\mathcal{D}'(G)\) denote the dual spaces of \(\mathcal{E}(G)\) and \(\mathcal{D}(G)\), respectively. Then we see that \(\mathcal{E}'(G) \subseteq (L^q_{\mathcal{F}}(0,T;\mathcal{E}(G)))'\). Indeed, for any \(u \in \mathcal{E}'(G)\) and \(v \in L^q_{\mathcal{F}}(0,T;\mathcal{E}(G))\), define

\[
\langle u, v \rangle_{(L^q_{\mathcal{F}}(0,T;\mathcal{E}(G)))', L^q_{\mathcal{F}}(0,T;\mathcal{E}(G))} = E \int_0^T \langle u(\cdot), v(t,\omega, \cdot) \rangle_{\mathcal{E}'(G), \mathcal{E}(G)} dt.
\]

Then there exist two nonnegative integers \(j_1\) and \(k_1\), such that

\[
|\langle u, v \rangle_{(L^q_{\mathcal{F}}(0,T;\mathcal{E}(G)))', L^q_{\mathcal{F}}(0,T;\mathcal{E}(G))}| \leq E \int_0^T C(u)|v(t,\omega, \cdot)|_{j_1,k_1,1} dt
\]

\[
\leq C(u, T, q) |v(\cdot, \cdot, \cdot)|_{j_1,k_1,1} L^q_{\mathcal{F}}(0,T) = C(u, T, q) |v|_{q,j_1,k_1,1}.
\]

This implies that \(u \in (L^q_{\mathcal{F}}(0,T;\mathcal{E}(G)))'\).

On the other hand, we define a constant \(q'\) as follows: \(q' = \frac{q}{q - 1}\) for \(q > 1\); \(q' = 1\) for \(q = \infty\); \(q' = \infty\) for \(q = 1\). Then, for any given compact set \(K\), it is also easy to prove that

\[
\{ u \in L^{q'}_{\mathcal{F}}(0,T; L^1(G)) \mid \text{supp } u(t,\omega, \cdot) \subseteq K \} \text{ for a.e. } (t,\omega) \in (0,T) \times \Omega \subseteq (L^q_{\mathcal{F}}(0,T;\mathcal{E}(G)))'.
\]
Remark 2.7 For any $A \in \mathcal{L}_p'(G \times G)$, it is easy to show that for a.e. $(t, \omega) \in (0, T) \times \Omega$, $A : \mathcal{E}'(G) \to \mathcal{D}'(G)$ is continuous. Moreover, if we let

\[
\mathcal{D}_s(G) = \{ u | \text{ for a.e. } (t, \omega) \in (0, T) \times \Omega, \text{ } u(t, \omega, \cdot) \in \mathcal{D}(G); \text{ and for any } x \in G,
\]

\[
u(\cdot, \cdot, x) \text{ is } \{ \mathcal{F}_t \}_{t \geq 0-\text{adapted}};\]

\[
\mathcal{E}_s(G) = \{ u | \text{ for a.e. } (t, \omega) \in (0, T) \times \Omega, \text{ } u(t, \omega, \cdot) \in \mathcal{E}(G); \text{ and for any } x \in G,
\]

\[
u(\cdot, \cdot, x) \text{ is } \{ \mathcal{F}_t \}_{t \geq 0-\text{adapted}};\]

\[
\mathcal{D}'_s(G) = \{ u | \text{ for a.e. } (t, \omega) \in (0, T) \times \Omega, \text{ } u(t, \omega, \cdot) \in \mathcal{D}'(G); \text{ and for any } v \in \mathcal{D}_s(G),
\]

\[
\langle u(\cdot, \cdot, \cdot), v(\cdot, \cdot, \cdot) \rangle_{\mathcal{D}(G), \mathcal{D}'(G)} \text{ is } \{ \mathcal{F}_t \}_{t \geq 0-\text{adapted}};\]

\[
\mathcal{E}'_s(G) = \{ u | \text{ for a.e. } (t, \omega) \in (0, T) \times \Omega, \text{ } u(t, \omega, \cdot) \in \mathcal{E}'(G); \text{ and for any } v \in \mathcal{E}_s(G),
\]

\[
\langle u(\cdot, \cdot, \cdot), v(\cdot, \cdot, \cdot) \rangle_{\mathcal{E}(G), \mathcal{E}'(G)} \text{ is } \{ \mathcal{F}_t \}_{t \geq 0-\text{adapted}};\]

then for any $u \in \mathcal{E}'_s(G)$, $Au \in \mathcal{D}'_s(G)$.

2.4 Kernel and pseudo-local property

In this subsection, we introduce the definition of kernels, and then prove the pseudo-local property of SPDOs (see Theorem 2.4). First of all, we give the notion of kernels.

Definition 2.6 Suppose that $a \in S_p^f(G \times G \times \mathbb{R}^n)$ and $A$ is the associated SPDO. Then $K_A \in \mathcal{D}'(G \times G)$ is called a kernel of $A$ if for a.e. $(t, \omega) \in (0, T) \times \Omega$,

\[
(K_A(t, \omega, \cdot, \cdot), v)_{\mathcal{D}'(G \times G), \mathcal{D}(G \times G)} = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{G} e^{i(x-y) \cdot \xi} a(t, \omega, x, y, \xi) v(x, y) dx dy d\xi,
\]

for any $v \in C_0^\infty(G \times G)$.

Notice that for a.e. $(t, \omega) \in (0, T) \times \Omega$, the kernel in Definition 2.6 is indeed the one in the deterministic case. Therefore, the following assertions hold:

Proposition 2.2 Suppose that $A$ is a SPDO and $a$ is its amplitude. Then,

1. if $a \in S_p^f(G \times G \times \mathbb{R}^n)$, $K_A$ is $C^\infty$ off the diagonal in $G \times G$ for a.e. $(t, \omega) \in (0, T) \times \Omega$. Moreover, for any $(x, y) \in G \times G$ with $x \neq y$, $K_A(\cdot, \cdot, x, y)$ is $\{ \mathcal{F}_t \}_{t \geq 0-\text{adapted}}$, and for any compact set $K \subseteq (G \times G) \setminus \{x = y\}$, \[\sup_{(x, y) \in K} |K_A(\cdot, \cdot, x, y)| \in L_p^2(0, T);\]

2. if $a \in S_p^{-\infty}(G \times G \times \mathbb{R}^n)$, $K_A \in C^\infty(G \times G)$ for a.e. $(t, \omega) \in (0, T) \times \Omega$. Moreover, for any $(x, y) \in G \times G$, $K_A(\cdot, \cdot, x, y)$ is $\{ \mathcal{F}_t \}_{t \geq 0-\text{adapted}}$, and for any compact set $K \subseteq G \times G$, \[\sup_{(x, y) \in K} |K_A(\cdot, \cdot, x, y)| \in L_p^-^2(0, T).\]

Sketch of the proof. First, for any $a \in S_p^f(G \times G \times \mathbb{R}^n)$, it is easy to show that for a.e. $(t, \omega) \in (0, T) \times \Omega$ and any $(x, y) \in G \times G$ with $x \neq y$, the integral \[\int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} a(t, \omega, x, y, \xi) d\xi\] can
be understood in the following two senses equivalently:

\[ \int_{\mathbb{R}^n} e^{i(x-y)\xi} a(t,\omega,x,y,\xi)d\xi = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \chi(\varepsilon \xi) e^{i(x-y)\xi} a(t,\omega,x,y,\xi)d\xi, \]

\[ = \int_{\mathbb{R}^n} e^{i(x-y)\xi} (-1)^k |x-y|^{-2k} \Delta^k a(t,\omega,x,y,\xi)d\xi, \]

where \( \chi \in C^\infty_0(\mathbb{R}^n) \), \( \chi = 1 \) in a neighborhood of the origin and \( k \in \mathbb{N} \) with \( \ell - 2k < -n \).

Also, by the meaning of the above integral, for a.e. \( (t,\omega) \in (0,T) \times \Omega \) and any compact set \( K \subseteq (G \times G) \setminus \{ x = y \} \), we have that \( \int_{\mathbb{R}^n} e^{i(x-y)\xi} a(t,\omega,x,y,\xi)d\xi \in C^\infty(K) \).

Moreover, by the Lebesgue dominated convergence theorem, it is easy to check that \( K_A(t,\omega,x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\xi} a(t,\omega,x,y,\xi)d\xi \), for a.e. \( (t,\omega) \in (0,T) \times \Omega \) and any \( (x,y) \in G \times G \) with \( x \neq y \). Therefore, \( K_A(t,\omega,\cdot,\cdot) \in C^\infty((G \times G)\setminus\{ x = y \}) \) for a.e. \( (t,\omega) \in (0,T) \times \Omega \).

Furthermore, by the properties of measurable functions, we see that \( K_A(\cdot,\cdot,\cdot,\cdot) \) is \( \{ F_t \}_{t \geq 0} \)-adapted, and by the definition of amplitudes, \( \sup_{(x,y) \in K} |K_A(\cdot,\cdot,x,y)| \in L^p_F(0,T) \) for any compact set \( K \subseteq (G \times G) \setminus \{ x = y \} \).

Similar to the above procedure, we can also get the desired result (2), if \( a \in S_p^{-\infty}(G \times G \times \mathbb{R}^n) \).

**Remark 2.8** For a SPDO \( A \), we call \( A \) a smoothing operator of order \( p \) if \( K_A \) satisfies the following conditions:

1. for a.e. \( (t,\omega) \in (0,T) \times \Omega \), \( K_A(t,\omega,\cdot,\cdot) \in C^\infty(G \times G) \);
2. for any \( (x,y) \in G \times G \), \( K_A(\cdot,\cdot,x,y) \) is \( \{ F_t \}_{t \geq 0} \)-adapted;
3. for any compact set \( K \subseteq G \times G \), \( \sup_{(x,y) \in K} |K_A(\cdot,\cdot,x,y)| \in L^p_F(0,T) \).

It is easy to check that if a function \( \widetilde{K} \) satisfies the above conditions (1)-(3), then an operator \( A \) defined as follows: \( (Au)(t,\omega,x) = \int_G \widetilde{K}(t,\omega,x,y)u(t,\omega,y)dy \) is a SPDO and its amplitude \( a \in S_p^{-\infty}(G \times G \times \mathbb{R}^n) \). Combining Proposition 2.2 with the above fact, we obtain that \( a \in S_p^{-\infty}(G \times G \times \mathbb{R}^n) \) if and only if \( A \) is a smoothing operator of order \( p \).

Next, we recall the notion of singular supports. For a distribution \( u \), the singular support of \( u \) is the complement of the open set on which \( u \) is smooth and we write it \( \text{sing supp } u \).

In the following, we present the pseudo-local property for SPDOs.

**Theorem 2.4** Suppose that \( A \) is a SPDO. Then for a.e. \( (t,\omega) \in (0,T) \times \Omega \) and any \( u \in \mathcal{E}_s'(G) \),

\[ \text{sing supp } (Au)(t,\omega,\cdot) \subseteq \text{sing supp } u(t,\omega,\cdot). \]

The proof of Theorem 2.4 follows by the similar method as that used in the deterministic case. Therefore, we omit it here.
2.5 Uniformly properly supported stochastic pseudo-differential operator

In order to present the composition of two SPDOs, in this subsection, we introduce uniformly properly supported SPDOs with respect to \((t,\omega)\). First of all, we recall the notion of proper sets. A set \(E \subseteq G \times G\) is called a proper set if \(E\) has compact intersection with \(K \times G\) and with \(G \times K\), for any compact set \(K \subseteq G\). Also, we give some relevant definitions.

**Definition 2.7** A function \(\widetilde{K} \in \mathcal{D}'_c(G \times G)\) is said to be uniformly properly supported with respect to \((t,\omega)\) if there exists a proper set \(E\) such that \(\text{supp} \widetilde{K}(t,\omega,\cdot,\cdot) \subseteq E\) for a.e. \((t,\omega) \in (0,T) \times \Omega\).

**Definition 2.8** We call \(A\) a uniformly properly supported SPDO with respect to \((t,\omega)\) if \(A\) is a SPDO and its kernel \(K_A\) is uniformly properly supported with respect to \((t,\omega)\).

**Definition 2.9** Suppose that \(a \in S^p_0(G \times G \times \mathbb{R}^n)\). \(a\) is said to have uniformly proper support with respect to \((t,\omega,\xi)\) if there exists a proper set \(E \subseteq G \times G\) such that \(\text{supp} a(t,\omega,\cdot,\cdot,\xi) \subseteq E\) for a.e. \((t,\omega) \in (0,T) \times \Omega\) and any \(\xi \in \mathbb{R}^n\).

Next, we give a characterization of amplitudes for uniformly properly supported SPDOs with respect to \((t,\omega)\).

**Proposition 2.3** For a SPDO \(A\), if its amplitude \(a\) has uniformly proper support with respect to \((t,\omega,\xi)\), then \(A\) is a uniformly properly supported SPDO with respect to \((t,\omega)\); Conversely, if \(A\) is a uniformly properly supported SPDO with respect to \((t,\omega)\), then its amplitude \(a\) can be replaced by another one, which has uniformly proper support with respect to \((t,\omega,\xi)\).

**Sketch of the proof.** First, it is easy to show the fact: if \(E\) is a proper set, then there exists a function \(\psi \in C^\infty(G \times G)\) such that \(\psi = 1\) in a neighborhood of \(E\) and \(\text{supp} \psi\) is proper.

Next, if \(A \in L^p_0(G \times G)\) is a uniformly properly supported SPDO with respect to \((t,\omega),\) by the above fact, there exist a function \(\psi_1 \in C^\infty(G \times G)\) and a proper set \(E_1\), such that \(\psi_1 = 1\) in a neighborhood of \(E_1\), \(\text{supp} \psi_1\) is proper and \(K_A(t,\omega,\cdot,\cdot) \subseteq E_1\) for a.e. \((t,\omega) \in (0,T) \times \Omega\). This implies that for any \(u,v \in C_0^\infty(G)\),

\[
\langle (Au)(t,\omega,\cdot,v)_{\mathcal{D}'(G)},{\mathcal{D}(G)} \rangle = \langle K_A(t,\omega,\cdot,\cdot),uv \rangle_{\mathcal{D}'(G \times G),\mathcal{D}(G \times G)}
\]

\[
= \psi_1 K_A(t,\omega,\cdot,\cdot,uv)_{\mathcal{D}'(G \times G),\mathcal{D}(G \times G)} + ((1-\psi_1)K_A(t,\omega,\cdot,\cdot,uv)_{\mathcal{D}'(G \times G),\mathcal{D}(G \times G)}
\]

\[
= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\xi} a(t,\omega,x,y,\xi) \psi_1(x,y)u(y)v(x)dx dy d\xi.
\]

Set \(a_*(t,\omega,x,y,\xi) = a(t,\omega,x,y,\xi)\psi_1(x,y)\) and denote by \(A_*\) its associated SPDO. Then \(a_* \in S^p_0(G \times G \times \mathbb{R}^n)\) and \(A = A_*\). Moreover, since \(\text{supp} a_*(t,\omega,\cdot,\cdot,\xi) \subseteq \text{supp} \psi_1\) for a.e. \((t,\omega) \in (0,T) \times \Omega\) and any \(\xi \in \mathbb{R}^n\), the amplitude \(a_*\) has uniformly proper support with respect to \((t,\omega,\xi)\).

On the other hand, if \(a\) has uniformly proper support with respect to \((t,\omega,\xi)\), then there exists a proper set \(E_2\), such that \(\text{supp} a(t,\omega,\cdot,\cdot,\xi) \subseteq E_2\) for a.e. \((t,\omega) \in (0,T) \times \Omega\) and any \(\xi \in \mathbb{R}^n\). This leads to that for any \(\varphi \in C_0^\infty(\overline{E_2})\),

\[
\langle K_A(t,\omega,\cdot,\cdot),\varphi \rangle_{\mathcal{D}'(G \times G),\mathcal{D}(G \times G)} = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\xi} a(t,\omega,x,y,\xi)\varphi(x,y)dx dy d\xi = 0.
\]
Therefore, \( \text{supp} \, K_A(t, \omega, \cdot, \cdot) \subseteq \overline{E_2} \) for a.e. \( (t, \omega) \in (0, T) \times \Omega \), which implies that \( A \) is a uniformly properly supported SPDO with respect to \( (t, \omega) \).

Based on Proposition 2.3, we get that a uniformly properly supported SPDO with respect to \( (t, \omega) \) actually has better properties than the usual operators.

**Theorem 2.5** Suppose that \( A \) is a uniformly properly supported SPDO of order \((\ell, p)\) with respect to \((t, \omega)\). Then

1. \( A : L^q_{\mathcal{F}}(0, T; D_G) \rightarrow L^q_{\mathcal{F}}(0, T; D_G) \) is continuous;
2. the domain of \( A \) can be extended to be \( L^q_{\mathcal{F}}(0, T; \mathcal{E}(G)) \). Moreover, \( A : L^q_{\mathcal{F}}(0, T; \mathcal{E}(G)) \rightarrow L^q_{\mathcal{F}}(0, T; \mathcal{E}(G)) \) is continuous.

**Proof.** Denote by \( a \) the amplitude of \( A \). Since \( A \) is a uniformly properly supported SPDO with respect to \((t, \omega)\), by Proposition 2.3, without loss of generality, we suppose that there is a proper set \( E_3 \), such that for a.e. \((t, \omega) \in (0, T) \times \Omega \) and any \( \xi \in \mathbb{R}^n \), \( \text{supp} \, a(t, \omega, \cdot, \cdot, \xi) \subseteq E_3 \). Also, for any \( u \in L^q_{\mathcal{F}}(0, T; D_G) \), there exists a compact set \( K^0 \) such that \( \text{supp} \, u(t, \omega, \cdot) \subseteq K^0 \) for a.e. \((t, \omega) \in (0, T) \times \Omega \). Write \( K^1 = \{ x \in G \mid \text{there exists a } y \in K^0 \text{ such that } (x, y) \in E_3 \} \). Then \( K^1 \) is a compact set, and if \( x \) does not belong to \( K^1 \),

\[
(Au)(t, \omega, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{G} e^{i(x-y) \cdot \xi} a(t, \omega, x, y, \xi) u(t, \omega, y) dy \, d\xi = 0.
\]

This means that \( \text{supp} \, Au(t, \omega, \cdot) \subseteq K^1 \) for a.e. \((t, \omega) \in (0, T) \times \Omega \). By Theorem 2.2, \( Au \in L^q_{\mathcal{F}}(0, T; \mathcal{E}(G)) \). Therefore, \( Au \in L^q_{\mathcal{F}}(0, T; D_G) \).

Moreover, if \( \lim_{j \to \infty} u_j = 0 \) in \( L^q_{\mathcal{F}}(0, T; D_G) \), then by Proposition 2.1, there exists a compact set \( K^2 \subseteq G \), such that \( \bigcup_{j \in \mathbb{N}} \text{supp} \, u_j(t, \omega, \cdot) \subseteq K^2 \), for a.e. \((t, \omega) \in (0, T) \times \Omega \). Write \( K^3 = \{ x \in G \mid \text{there exists a } y \in K^2 \text{ such that } (x, y) \in E_3 \} \). Then \( K^3 \) is a compact set and \( \text{supp} \, (Au_j)(t, \omega, \cdot) \subseteq K^3 \), for a.e. \((t, \omega) \in (0, T) \times \Omega \) and any \( j \in \mathbb{N} \). Since \( \lim_{j \to \infty} Au_j = 0 \) in \( L^q_{\mathcal{F}}(0, T; \mathcal{E}(G)) \), then by Proposition 2.1 again, \( \lim_{j \to \infty} Au_j = 0 \) in \( L^q_{\mathcal{F}}(0, T; D_G) \).

On the other hand, for any open set \( U \subseteq G \) with \( \overline{U} \) being compact, we write \( K^4 = \{ y \in G \mid \text{there exists an } x \in \overline{U} \text{ such that } (x, y) \in E_3 \} \), and then \( K^4 \) is a compact set. Choose a function \( \psi_2 \), such that \( \psi_2 \in C_0^\infty(G) \) and \( \psi_2 = 1 \) in \( K^4 \). We extend the domain of a uniformly properly supported SPDO with respect to \((t, \omega)\) as follows: \((Au)(t, \omega, x) = (A(\psi_2 u))(t, \omega, x)\), for any \( u \in L^q_{\mathcal{F}}(0, T; \mathcal{E}(G)) \), a.e. \((t, \omega) \in (0, T) \times \Omega \) and any \( x \in U \). It is easy to check that the above definition of \( A \) is well posed, namely, the definition of \( A \) is independent of \( \psi_2 \) and \( U \).

Furthermore, for any compact set \( K \), there exists an open set \( U_1 \subseteq G \) such that \( K \subseteq U_1 \) and \( \overline{U_1} \) is compact. Write \( K^5 = \{ y \in G \mid \text{there exists an } x \in \overline{U_1} \text{ such that } (x, y) \in E_3 \} \), take a function \( \psi_3 \) such that \( \psi_3 \in C_0^\infty(G) \) and \( \psi_3 = 1 \) in \( K^5 \), and then set \( K^6 = \text{supp} \, \psi_3 \). Then for any \( u \in L^q_{\mathcal{F}}(0, T; \mathcal{E}(G)) \) and multi-index \( \alpha \),

\[
|\partial^\alpha_x (Au)(t, \omega, x)| = \left| (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{G} \partial^\alpha_x [e^{i(x-y) \cdot \xi} a(t, \omega, x, y, \xi) u(t, \omega, y) \psi_3(y)] dy \, d\xi \right|
\]
This implies that

\[ (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{G} \frac{\alpha!}{\alpha_1 \alpha_2!} |\alpha_1|^{|\alpha_1|} |\alpha_2|^{|\alpha_2|} (1 + |\xi|^2)^{-|\alpha|-\ell-n-1} \int_{G} (1 - \Delta_y)^{|\alpha|+\ell+n+1} e^{i(x-y)\cdot \xi} \]

\[ \cdot \partial_x^{\alpha_2} a(t, \omega, x, y, \xi) u(t, \omega, y) \psi_3(y) dy d\xi \]

\[ \leq C(n, \ell, \alpha, K) M_{n, \ell, \alpha, K}(t, \omega) \sup_{y \in K^6} |\partial_y^3 u(t, \omega, y)|, \]

for a function \( M_{n, \ell, \alpha, K} \in L^p_F(0, T) \), a.e. \((t, \omega) \in (0, T) \times \Omega \) and any \( x \in K \). It follows that

\[ \left| \sup_{x \in K} |(\partial_x^n (Au)) (\cdot, \cdot, x)| \right|_{L^p_F(0, T)} \]

\[ \leq C(n, \ell, \alpha, K) |M_{n, \ell, \alpha, K}(\cdot, \cdot)|_{L^p_F(0, T)} \sup_{y \in K^6} \left| \partial_y^3 u(\cdot, \cdot, y) \right|_{L^p_F(0, T)}. \]

This implies that \( A : L^p_F(0, T; \mathcal{E}(G)) \rightarrow L^q_F(0, T; \mathcal{E}(G)) \) is continuous. The proof is completed. \( \square \)

By Theorem 2.5, for a.e. \((t, \omega) \in (0, T) \times \Omega \), a finite number of uniformly properly supported SPDOs with respect to \((t, \omega)\) can be composed. Moreover, for \( \ell_1, \ell_2 \in \mathbb{R} \), suppose that \( A \in \mathcal{L}^\ell_1(G) \) and \( B \in \mathcal{L}^\ell_2(G) \). If \( pq \tilde{q} \geq pq + \tilde{q}(p + q) \), then \( B \circ A : L^q_F(0, T; \mathcal{D}_G) \rightarrow L^q_F(0, T; \mathcal{D}_G) \) and \( B \circ A : L^q_F(0, T; \mathcal{E}(G)) \rightarrow L^q_F(0, T; \mathcal{E}(G)) \) are continuous, where \( \tilde{q} = \frac{pq}{pq + \tilde{q}(p + q)} \). In addition, since a kernel is smooth off the diagonal in \( G \times G \), we may write a SPDO as the sum of a uniformly properly supported operator and a smoothing operator.

**Theorem 2.6** If \( A \) is a SPDO of order \((\ell, p)\), then \( A = A^0 + A^1 \), where \( A^0 \) is a uniformly properly supported SPDO with respect to \((t, \omega)\) and \( A^1 \) is a smoothing operator.

**Sketch of the proof.** Suppose that \( a \) is the amplitude of \( A \). Choose a function \( \psi_4 \in C^\infty(G \times G) \) such that \( \psi_4 = 1 \) in a neighborhood of the set \( \{(x, y) \in G \times G \mid x = y\} \) and \( \text{supp} \psi_4 \) is proper. If we write \( a^0(t, \omega, x, y, \xi) = a(t, \omega, x, y, \xi) \psi_4(x, y) \), then for a.e. \((t, \omega) \in (0, T) \times \Omega \) and any \( \xi \in \mathbb{R}^n \), \( \text{supp} a^0(t, \omega, x, y, \xi) \subseteq \text{supp} \psi_4 \). It follows that the SPDO \( A^0 \) determined by \( a^0 \) is a uniformly properly supported SPDO with respect to \((t, \omega)\). On the other hand, let \( K_A \) be the kernel of \( A \), then \((1 - \psi_4)K_A \in \mathcal{E}_s(G \times G) \). If we denote by \( A^1 \) the operator, whose kernel is \((1 - \psi_4)K_A \), then its amplitude \( a^1 \) turns out to be \((1 - \psi_4)a \). It is easy to check that \((1 - \psi_4)K_A \) satisfies the conditions (1)-(3) mentioned in Remark 2.8. Therefore, \( a^1 \in S_p^{-\infty}(G \times G) \), \( A^1 \) is a smoothing operator and \( A = A^0 + A^1 \). \( \square \)
For a uniformly properly supported SPDO with respect to \((t, \omega)\) determined by an amplitude, one can reduce it to the form represented by a symbol.

**Theorem 2.7** Suppose that \(A\) is a uniformly properly supported SPDO with respect to \((t, \omega)\). Then for any \(u \in L^q_{L^p}(0, T; D(G))\),

\[
(Au)(t, \omega, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma_A(t, \omega, x, \xi) \hat{u}(t, \omega, \xi) d\xi,
\]

where \(\sigma_A(t, \omega, x, \xi) = e^{-ix \cdot \xi} (Ae^{ix \cdot \xi})(t, \omega, x)\).

The proof of Theorem 2.7 is similar to that in the deterministic case. Here we omit it.

For any SPDO \(A\), by Theorem 2.6, \(A\) can be rewritten as a sum of a uniformly properly supported SPDO \(A^0\) with respect to \((t, \omega)\) and a smoothing operator \(A^1\). On the other hand, by Theorem 2.7, \(A^0\) can be represented by a symbol \(\sigma_{A^0}\). We call \(\sigma_{A^0}\) a principal symbol of \(A\) and denote it by \(\sigma_A\) simply.

In the remainder of this subsection, we give an equivalent characterization of SPDOs and omit the proof, since it is similar to that in the deterministic case.

**Corollary 2.1** (1) \(A \in \mathcal{L}_p^l(G \times G)\) if and only if for a.e. \((t, \omega) \in (0, T) \times \Omega\), \(A : C_0^\infty(G) \to C_0^\infty(G)\) is a continuous linear operator, and for any \(\varphi \in C_0^\infty(G)\), \(e^{-ix \cdot \xi} A(\varphi e^{ix \cdot \xi}) \in \mathcal{S}_p^l(G \times \mathbb{R}^n)\);

(2) \(A \in \mathcal{L}_p^l(G \times G)\) is uniformly properly supported with respect to \((t, \omega)\) if and only if for a.e. \((t, \omega) \in (0, T) \times \Omega\), \(A : C_0^\infty(G) \to C_0^\infty(G)\) is a continuous linear operator, its distribution kernel \(K_A\) is uniformly properly supported with respect to \((t, \omega)\), and \(e^{-ix \cdot \xi} A(e^{ix \cdot \xi}) \in \mathcal{S}_p^l(G \times \mathbb{R}^n)\).

### 2.6 Asymptotic expansions of a symbol

In this subsection, we give the notion of asymptotic expansions of a symbol and present some usual results on asymptotic expansions. First, we have the following definition.

**Definition 2.10** Suppose that \(\{\ell_j\}_{j \in \mathbb{N}}\) is a monotone decreasing sequence satisfying \(\ell_j \to -\infty\) \((j \to \infty)\) and \(a_j \in \mathcal{S}_p^{\ell_j}(G \times \mathbb{R}^n)\). Then \(\sum_{j=0}^\infty a_j\) is called asymptotic expansions of a symbol \(a \in \mathcal{S}_p^{\ell_0}(G \times \mathbb{R}^n)\) if

\[
a = \sum_{j=0}^{k-1} a_j + \sum_{j=k}^\infty a_j \in \mathcal{S}_p^{\ell_k}(G \times \mathbb{R}^n), \quad \forall \ k \in \mathbb{N},
\]

and we write \(a \sim \sum_{j=0}^\infty a_j\).

Next, we give a useful lemma.

**Lemma 2.4** Suppose that \(\{\ell_j\}_{j \in \mathbb{N}}\) is a monotone decreasing sequence satisfying \(\ell_j \to -\infty\) \((j \to \infty)\) and \(a_j \in \mathcal{S}_p^{\ell_j}(G \times \mathbb{R}^n)\). Then there exists a symbol \(a \in \mathcal{S}_p^{\ell_0}(G \times \mathbb{R}^n)\) such that \(a \sim \sum_{j=0}^\infty a_j\).
Proof. Pick a sequence of compact sets \( \{K_j\}_{j \in \mathbb{N}} \) such that \( K_0 \subseteq K_1 \subseteq \cdots \) and \( \bigcup_{j \in \mathbb{N}} K_j = G \), and choose a function \( \psi_5 \in C^\infty(\mathbb{R}^n) \) satisfying \( \psi_5(\xi) = 0 \) for \( |\xi| \leq \frac{1}{2} \) and \( \psi_5(\xi) = 1 \) for \( |\xi| \geq 1 \). We construct a function \( a \) of the form
\[
a(t,\omega,x,\xi) = \sum_{j=0}^{\infty} \psi_5(\varepsilon_j \xi) a_j(t,\omega,x,\xi),
\]
where \( \varepsilon_j \) are small constants, which will be specified later. It is easy to verify that \( a \in \mathcal{E}_s(G \times \mathbb{R}^n) \).

On the other hand, since for any \( k \in \mathbb{N} \),
\[
a(t,\omega,x,\xi) - \sum_{j=0}^{k-1} a_j(t,\omega,x,\xi) = \sum_{j=0}^{k-1} [\psi_5(\varepsilon_j \xi) - 1] a_j(t,\omega,x,\xi) + \sum_{j=k}^{\infty} \psi_5(\varepsilon_j \xi) a_j(t,\omega,x,\xi),
\]
and \( \sum_{j=0}^{k-1} [\psi_5(\varepsilon_j \xi) - 1] a_j(t,\omega,x,\xi) \in S_p^{-\infty}(G \times \mathbb{R}^n) \), it remains to prove that \( \sum_{j=k}^{\infty} \psi_5(\varepsilon_j \xi) a_j(t,\omega,x,\xi) \in S_p^{\ell_k}(G \times \mathbb{R}^n) \). Indeed, for any compact set \( K \subseteq G \), there exists an \( i^* \in \mathbb{N} \) such that \( K \subseteq K_{i^*} \). For any two multi-indices \( \alpha \) and \( \beta \), write \( k^* = \max\{k,|\alpha| + |\beta| + i^*\} \). Then
\[
\sum_{j=k}^{\infty} \psi_5(\varepsilon_j \xi) a_j(t,\omega,x,\xi) = \sum_{j=k}^{k^*} \psi_5(\varepsilon_j \xi) a_j(t,\omega,x,\xi) + \sum_{j=k^*+1}^{\infty} \psi_5(\varepsilon_j \xi) a_j(t,\omega,x,\xi).
\]
For a.e. \( (t,\omega) \in (0,T) \times \Omega \) and any \( (x,\xi) \in K \times \mathbb{R}^n \),
\[
\left| \partial_\xi^\alpha \partial_x^\beta \left[ \sum_{j=k}^{k^*} \psi_5(\varepsilon_j \xi) a_j(t,\omega,x,\xi) \right] \right| \leq C(\alpha,\beta,k,K) M_{1,\alpha,\beta,K}^{t,\omega,x,\xi}(1 + |\xi|)^{\ell_k - |\alpha|},
\]
for a function \( M_{1,\alpha,\beta,K}^{t,\omega,x,\xi} \in L_p^p(0,T) \). Also, for any \( j \geq k^* + 1 \),
\[
\left| \partial_\xi^\alpha \partial_x^\beta [\psi_5(\varepsilon_j \xi) a_j(t,\omega,x,\xi)] \right| \leq \sum_{\alpha_1 + \alpha_2 = \alpha} \frac{\alpha_1! \alpha_2!}{\alpha_1! \alpha_2!} \left| \partial_\xi^{\alpha_1} \psi_5(\varepsilon_j \xi) \partial_\xi^{\alpha_2} \partial_x^\beta a_j(t,\omega,x,\xi) \right|
\]
\[
\leq C(j,\alpha) M_{2,\beta}^j(t,\omega) \left( 1 + \frac{1}{2\varepsilon_j} \right)^{\ell_j - |\alpha|},
\]
for a function \( M_{2,\beta}^j(\cdot,\cdot) \in L_p^p(0,T) \). We choose \( \varepsilon_j (j \geq k^* + 1) \) sufficiently small such that
\[
\left( 1 + \frac{1}{2\varepsilon_j} \right)^{\ell_j - |\alpha|} \leq \frac{2^{-j}}{C(j,\alpha) M_{2,\beta}^j(\cdot,\cdot) \|M_{2,\beta}^j(\cdot,\cdot)\|_{L_p^p(0,T) + 1}}.
\]
Then,
\[
\left\{ \sum_{j=k^*+1}^{N} C(j,\alpha) M_{2,\beta}^j(\cdot,\cdot) \left( 1 + \frac{1}{2\varepsilon_j} \right)^{\ell_j - |\alpha|} \right\}_{N \in \mathbb{N}}
\]
is a Cauchy sequence in \( L_p^p(0,T) \). Therefore, we can find a nonnegative function \( M^2(\cdot,\cdot) \in L_p^p(0,T) \) such that
\[
\sum_{j=k^*+1}^{\infty} C(j,\alpha) M_j(t,\omega) \left( 1 + \frac{1}{2\varepsilon_j} \right)^{\ell_j - |\alpha|} = M^2(t,\omega) \quad \text{for a.e. } (t,\omega) \in (0,T) \times \Omega.
\]
This implies that

\[ \left| \partial_\xi^j \partial_\tau^k \sum_{j=0}^{\infty} \psi_5(\epsilon_j, \xi) a_j(t, \omega, x, \xi) \right| \leq M^2(t, \omega)(1 + |\xi|)^{\ell_k - |\alpha|}. \tag{2.8} \]

Hence, by (2.6)-(2.8), \( a \sim \sum_{j=0}^{\infty} a_j \). Since \( a - a_0 \in S_p^{\ell_j}(G \times \mathbb{R}^n) \) and \( a_0 \in S_p^{\ell_j}(G \times \mathbb{R}^n) \), we get that \( a \in S_p^{\ell_j}(G \times \mathbb{R}^n) \).

Lemma 2.4 leads to the following theorem, which shows that the asymptotic relation \( a \sim \sum_{j=0}^{\infty} a_j \) is valid if an apparently weaker condition than (2.4) is assumed to hold.

**Theorem 2.8** Suppose that \( \{\ell_j\}_{j \in \mathbb{N}} \) is a monotone decreasing sequence satisfying \( \ell_j \to -\infty \ (j \to \infty) \) and \( a_j \in S_p^{\ell_j}(G \times \mathbb{R}^n) \). If \( a \in \mathcal{E}_s(G \times \mathbb{R}^n) \) satisfies the following conditions:

1. for any compact set \( K \subseteq G \), and multi-indices \( \alpha \) and \( \beta \), there exist a function \( M_{\alpha,\beta,K}(\cdot) \in L^p_T(0, T) \) and a constant \( \rho = \rho(\alpha, \beta, K) \in \mathbb{R} \) such that
   \[ \left| \partial_\xi^j \partial_\tau^k a(t, \omega, x, \xi) \right| \leq M_{\alpha,\beta,K}(t, \omega)(1 + |\xi|)^{\rho}, \tag{2.9} \]
   for a.e. \( (t, \omega) \in (0, T) \times \Omega \) and any \( (x, \xi) \in K \times \mathbb{R}^n \);

2. there exist a sequence of real numbers \( \{\rho_j\}_{j \in \mathbb{N}} \) satisfying \( \rho_j \to -\infty \ (j \to \infty) \) and a sequence of functions \( \{d_j(\cdot, \cdot)\}_{j \in \mathbb{N}} \subseteq L^p_T(0, T) \), such that for any \( j \in \mathbb{N} \),
   \[ \left| a(t, \omega, x, \xi) - \sum_{k=0}^{j-1} a_k(t, \omega, x, \xi) \right| \leq d_j(t, \omega)(1 + |\xi|)^{\rho_j}, \tag{2.10} \]
   for a.e. \( (t, \omega) \in (0, T) \times \Omega \) and any \( (x, \xi) \in K \times \mathbb{R}^n \), then \( a \in S_p^{\ell_j}(G \times \mathbb{R}^n) \) and \( a \sim \sum_{j=0}^{\infty} a_j \).

Based on Theorem 2.8, we can get some results on asymptotic expansions. The first one is the following asymptotic expansions of a principal symbol.

**Proposition 2.4** Suppose that \( A \) is a SPDO and \( a \) is its amplitude. Then its principal symbol \( \sigma_A \) satisfies that

\[ \sigma_A(t, \omega, x, \xi) \sim \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} \xi^\alpha \partial_\tau^\alpha a(t, \omega, x, y, \xi)|_{y=x}. \]

The next result is the asymptotic expansions for a principal symbol of transpose operators.

**Proposition 2.5** Suppose that \( A \) is a SPDO. Then a principal symbol \( \sigma_A \) of its transpose operator has the following asymptotic expansions:

\[ \sigma_A(t, \omega, x, \xi) \sim \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} \partial_\xi^\alpha \partial_\tau^\alpha \sigma_A(t, \omega, x, -\xi). \]
Finally, for two uniformly properly supported SPDOs with respect to \((t, \omega)\), we present the asymptotic expansions for a principal symbol of their composition operator.

**Proposition 2.6** Let \(\ell_1, \ell_2 \in \mathbb{R}\). Suppose that \(A \in \mathcal{L}^\ell_1(G \times G)\) and \(B \in \mathcal{L}^\ell_2(G \times G)\) are uniformly properly supported SPDOs with respect to \((t, \omega)\). Then the composition operator \(B \circ A\) is a SPDO of order \((\ell_1 + \ell_2, q^*)\). Moreover, its principal symbol \(\sigma_{B \circ A}\) has the following asymptotic expansions:

\[
\sigma_{B \circ A}(t, \omega, x, \xi) \sim \sum_{|\alpha| = 0}^{\infty} \frac{1}{\alpha! |\alpha|} \partial^\alpha_x \sigma_B(t, \omega, x, \xi) \cdot \partial^\alpha_x \sigma_A(t, \omega, x, \xi).
\]

The proofs of Theorem 2.8 and Propositions 2.4-2.6 are similar to those for the usual pseudo-differential operators in the deterministic case. Here we omit them.

### 2.7 Algebra and generalized module of stochastic pseudo-differential operators

In this subsection, we establish an algebra and generalized module of SPDOs. For this purpose, first of all, we give the following result on the composition of two uniformly properly supported SPDOs with respect to \((t, \omega)\).

**Theorem 2.9** Suppose that \(A\) and \(B\) are uniformly properly supported SPDOs with respect to \((t, \omega)\). Then \(B \circ A\) is a uniformly properly supported SPDO with respect to \((t, \omega)\).

**Proof.** Denote by \(a\) and \(b\) the amplitudes of \(A\) and \(B\), respectively. Then for any \(u \in \mathcal{D}_s(G)\),

\[
(B \circ A)u(t, \omega, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} b(t, \omega, x, y, \xi)(Au)(t, \omega, y) dy d\xi
\]

\[
= (2\pi)^{-2n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} b(t, \omega, x, y, \xi) \left[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(y-z) \cdot \varsigma} a(t, \omega, y, z, \varsigma) u(t, \omega, z) dz d\varsigma \right] dy d\xi
\]

\[
= (2\pi)^{-2n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-z) \cdot \xi} \left[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(y-z) \cdot (\xi - \varsigma)} b(t, \omega, y, x, y, \xi) a(t, \omega, y, z, \varsigma) dz d\varsigma \right] u(t, \omega, z) dz d\xi.
\]

Set

\[
c(t, \omega, x, z, \xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(z-y) \cdot (\xi - \varsigma)} b(t, \omega, x, y, \xi) a(t, \omega, y, z, \varsigma) dz d\varsigma.
\]

\[
= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(z-y) \cdot \eta} b(t, \omega, x, y, \xi) a(t, \omega, y, z, \xi - \eta) dy d\eta.
\]

Since \(A\) and \(B\) are two uniformly properly supported SPDOs with respect to \((t, \omega)\), there exist two proper sets \(G_1\) and \(G_2\), such that \(\text{supp } a(t, \omega, \cdot, \cdot, \xi) \subseteq G_1\) and \(\text{supp } b(t, \omega, \cdot, \cdot, \xi) \subseteq G_2\) for a.e. \((t, \omega) \in (0, T) \times \Omega\) and any \(\xi \in \mathbb{R}^n\). If we write

\[
G^* = \{(x, z) \in G \times G \mid \text{there exists a } y \in G, \text{ such that } (x, y) \in G_2, (y, z) \in G_1\},
\]

then it is easy to see that \(G^*\) is proper and \(\text{supp } c(t, \omega, \cdot, \cdot, \xi) \subseteq G^*\) for a.e. \((t, \omega) \in (0, T) \times \Omega\) and any \(\xi \in \mathbb{R}^n\). This finishes the proof. \(\Box\)
In the following, we establish an algebra and generalized module of SPDOs. Notice that a generalized module means a usual module, which does not satisfy the associative law. For any given \( p \in [1, \infty) \), \( \bigcup_{\ell \in \mathbb{R}} \mathcal{L}_p^\ell(G \times G) \) is a linear space. Also, for any \( A, B \in \bigcup_{\ell \in \mathbb{R}} \mathcal{L}_p^\ell(G \times G) \), denote by \( a \) and \( b \) their amplitudes respectively and define an equivalent relation \( \sim \):

\[
A \sim B \iff a - b \in \mathcal{S}^{-\infty}_p(G \times G) \iff A - B \in \bigcap_{\ell \in \mathbb{R}} \mathcal{L}_p^\ell(G \times G).
\]

In every equivalent class \([\cdot]\), there exists a uniformly properly supported SPDO with respect to \((t, \omega)\). Therefore, for any two equivalent classes \( A \) and \( B \), take \( A \in A \) and \( B \in B \) such that \( A \) and \( B \) are two uniformly properly supported SPDOs with respect to \((t, \omega)\). We define the composition of \( A \) and \( B \) as follows:

\[
B \circ A = [B \circ A].
\]

It is easy to check that the definition is well posed. Moreover, \( \bigcup_{\ell \in \mathbb{R}} \mathcal{L}_p^\ell(G \times G) \) constructs an algebra in the above sense, and \( \bigcup_{\ell \in \mathbb{R}} \mathcal{L}_p^\ell(G \times G) \) \( (p \geq 1) \) is a generalized module over \( \bigcup_{\ell \in \mathbb{R}} \mathcal{L}_\infty^\ell(G \times G) \).

**Remark 2.9** For any \( A, B \in \bigcup_{\ell \in \mathbb{R}} \mathcal{L}_\infty^\ell(G \times G) \), define \([A, B] = A \circ B - B \circ A\). Then we conjecture that \([\cdot, \cdot]\) is a Poisson bracket defined on \( \bigcup_{\ell \in \mathbb{R}} \mathcal{L}_\infty^\ell(G \times G) \). However, we do not give a precise proof at this moment.

### 3 Boundedness of stochastic pseudo-differential operators

Many papers have been devoted to a study of the continuity of pseudo-differential operators in \( L^p \) spaces. In this section, we shall establish the \( L^p \)-estimates of SPDOs. The main idea borrows from that used in [7] and [8]. However, different from the deterministic case, results here involve integrability with respect to the variables of time and sample point, and we have to deal with the problem, generated by global estimates. For simplicity of notation, for any bounded linear operator \( L \), we denote by \(|L|\) its operator norm.

#### 3.1 \( L^2 \)-boundedness

In this subsection, we first present the \( L^2 \)-estimates for SPDOs. To begin with, we prove the following basic estimate.

**Lemma 3.1** Suppose that \( \widetilde{K} \) is a function defined on \((0, T) \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n\) such that

\[
\sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} |\widetilde{K}(t, \omega, x, y)| \, dx \leq M(t, \omega), \quad \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |\widetilde{K}(t, \omega, x, y)| \, dy \leq M(t, \omega),
\]

for a nonnegative function \( M(\cdot, \cdot) \) defined on \((0, T) \times \Omega\). Then, for the linear operator \( L \) defined as follows: \((Lu)(t, \omega, x) = \int_{\mathbb{R}^n} \widetilde{K}(t, \omega, x, y)u(t, \omega, y) \, dy\), the following conclusions hold:
(1) for a.e. \((t, \omega) \in (0, T) \times \Omega\), \(L : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)\) is bounded and \(|L| \leq M(t, \omega);\)

(2) if \(M(\cdot, \cdot) \in L^q_{\mathcal{F}}(0, T)\), then \(L : L^q(0, T; L^p(\mathbb{R}^n)) \to L^q(0, T; L^p(\mathbb{R}^n))\) is bounded and \(|L| \leq \|M\|_{L^q_{\mathcal{F}}(0, T)};\)

(3) if \(M(\cdot, \cdot) \in L^q_{\mathcal{F}}(0, T)\) \((\tilde{q} \in [1, \infty))\), then \(L : L^\tilde{q}(0, T; L^p(\mathbb{R}^n)) \to L^\tilde{q}(0, T; L^p(\mathbb{R}^n))\) is bounded and \(|L| \leq \|M\|_{L^\tilde{q}_{\mathcal{F}}(0, T)}\), where \(\tilde{q} = \frac{q \cdot \tilde{q}}{q + \tilde{q}}\) for \(\tilde{q} \geq 1\), \(q \tilde{q} \geq q + \tilde{q}\); and \(q = \tilde{q}\) for \(\tilde{q} \geq 1\), \(q = \infty).\)

**Proof.** For any \(p > 1\), a.e. \((t, \omega) \in (0, T) \times \Omega\) and any \(u \in L^p(\mathbb{R}^n)\), by Hölder’s inequality, we have that

\[
|\langle Lu \rangle(t, \omega, x)^p \leq \int_{\mathbb{R}^n} |K(t, \omega, x, y)||u(y)|^p \, dy \cdot \left( \int_{\mathbb{R}^n} |K(t, \omega, x, y)| \, dy \right)^{p'},
\]

\[
\leq M^{\tilde{p}'}(t, \omega) \int_{\mathbb{R}^n} |K(t, \omega, x, y)||u(y)|^p \, dy,
\]

here and hereafter \(p'\) denotes a constant satisfying \(\frac{1}{p} + \frac{1}{p'} = 1\). Integrating on \(\mathbb{R}^n\) with respect to the variable \(x\), by (3.1), we get that

\[
\int_{\mathbb{R}^n} |\langle Lu \rangle(t, \omega, x)^p \, dx \leq M^{1+\tilde{p}'}(t, \omega) \int_{\mathbb{R}^n} |u(y)|^p \, dy.
\]

This means that for a.e. \((t, \omega) \in (0, T) \times \Omega\), \(L\) is a bounded operator from \(L^p(\mathbb{R}^n)\) to \(L^p(\mathbb{R}^n)\) and \(|L| \leq M(t, \omega).\) If \(M \in L^q_{\mathcal{F}}(0, T)\), then by (3.2), for any \(u \in L^q_{\mathcal{F}}(0, T; L^p(\mathbb{R}^n))\) and a.e. \((t, \omega) \in (0, T) \times \Omega,\)

\[
|\langle Lu \rangle(t, \omega, \cdot)|_{L^p(\mathbb{R}^n)} \leq \|M\|_{L^q_{\mathcal{F}}(0, T)} |u(t, \omega, \cdot)|_{L^p(\mathbb{R}^n)}.
\]

It follows that

\[
\mathbb{E} \int_0^T |\langle Lu \rangle(t, \omega, \cdot)|_{L^p(\mathbb{R}^n)}^q \, dt \leq \|M\|_{L^q_{\mathcal{F}}(0, T)}^q \mathbb{E} \int_0^T |u(t, \omega, \cdot)|_{L^p(\mathbb{R}^n)}^q \, dt, \text{ for } q \geq 1;
\]

\[
|\langle Lu \rangle|_{L^q(0, T; L^p(\mathbb{R}^n))} \leq \|M\|_{L^q_{\mathcal{F}}(0, T)} |u|_{L^q(0, T; L^p(\mathbb{R}^n))}, \text{ for } q = \infty,
\]

which implies that \(L : L^q_{\mathcal{F}}(0, T; L^p(\mathbb{R}^n)) \to L^q(0, T; L^p(\mathbb{R}^n))\) is bounded and \(|L| \leq \|M\|_{L^q_{\mathcal{F}}(0, T)}\). Furthermore, if \(M \in L^\tilde{q}_{\mathcal{F}}(0, T)\) \((\tilde{q} \geq 1)\), by (3.2), for any \(u \in L^\tilde{q}_{\mathcal{F}}(0, T; L^p(\mathbb{R}^n))\), we obtain that

\[
|\langle Lu \rangle(t, \omega, \cdot)|_{L^\tilde{q}(\mathbb{R}^n)} \leq M^{\tilde{q}}(t, \omega) |u(t, \omega, \cdot)|_{L^\tilde{q}(\mathbb{R}^n)}.
\]

Hence,

\[
\mathbb{E} \int_0^T |\langle Lu \rangle(t, \omega, \cdot)|_{L^\tilde{q}(\mathbb{R}^n)}^q \, dt \leq \left[ \mathbb{E} \int_0^T M^{\tilde{q}}(t, \omega) \, dt \right]^{\frac{q}{\tilde{q}}} \cdot \left[ \mathbb{E} \int_0^T |u(t, \omega, \cdot)|_{L^\tilde{q}(\mathbb{R}^n)}^{q} \, dt \right]^{\frac{q}{\tilde{q}}}, \text{ for } q \geq 1;
\]

\[
\mathbb{E} \int_0^T |\langle Lu \rangle(t, \omega, \cdot)|_{L^\tilde{q}(\mathbb{R}^n)}^q \, dt \leq \mathbb{E} \int_0^T M^{\tilde{q}}(t, \omega) \, dt \cdot |u|_{L^\tilde{q}(0, T; L^\tilde{q}(\mathbb{R}^n))}^q, \text{ for } q = \infty.
\]

This implies that \(L : L^\tilde{q}_{\mathcal{F}}(0, T; L^p(\mathbb{R}^n)) \to L^\tilde{q}(0, T; L^p(\mathbb{R}^n))\) is bounded and \(|L| \leq \|M\|_{L^\tilde{q}_{\mathcal{F}}(0, T)}\). Results for the cases of \(p = 1\) and \(p = \infty\) can be derived in the same way. \(\square\)

Next, we give the notion of adjoint operators.
Definition 3.1 Suppose that $A$ is a SPDO of order $(t, p)$ and $a$ is its amplitude. A linear operator $A^*$ is called the adjoint operator of $A$ if for any $u \in L^q_T(0, T; \mathcal{D}(G))$,

$$ (A^*u)(t, \omega, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot \xi} a(t, \omega, y, x, \xi) u(t, \omega, y) dy \, d\xi, $$

where $\overline{z}$ denotes the conjugate of a complex number $z$.

Remark 3.1 It is easy to show that for any SPDO $A \in L^q_p(G \times G)$, $A^* \in L^q_p(G \times G)$. Moreover, for a.e. $(t, \omega) \in (0, T) \times \Omega$ and any $u, v \in C^\infty_0(G)$, $((A^*u)(t, \omega, \cdot), v(\cdot))_{L^2(G)} = (u(\cdot), (Av)(t, \omega, \cdot))_{L^2(G)}$.

Remark 3.2 Similar to Proposition 2.5, for any SPDO $A$, it is easy to show that a principal symbol of its adjoint operator $A^*$ has the following asymptotic expansions:

$$ \sigma_{A^*}(t, \omega, x, \xi) \sim \sum_{|\alpha| = 0}^{\infty} \frac{1}{\alpha ! |\alpha|} \partial^\alpha_\xi \partial^\alpha_x \sigma_A(t, \omega, x, \xi). $$

Now, we give the $L^2$-estimates for a class of the SPDOs of order $(0, \infty)$.

Theorem 3.1 Suppose that $A$ is a SPDO and $a$ is its symbol. If $a \in S^0_p$, $A : L^q_T(0, T; L^2(\mathbb{R}^n)) \to L^q_T(0, T; L^2(\mathbb{R}^n))$ is bounded.

Proof. Step 1. First, we prove that $A : L^q_T(0, T; L^2(\mathbb{R}^n)) \to L^q_T(0, T; L^2(\mathbb{R}^n))$ is bounded, if $a \in S^{n-1}_p$. By the definition of a kernel $K_A$, we have that

$$ |K_A(t, \omega, x, y)| = (2\pi)^{-n} \left| \int_{\mathbb{R}^n} e^{i(x-y)\cdot \xi} a(t, \omega, x, \xi) \, d\xi \right| \leq (2\pi)^{-n} \int_{\mathbb{R}^n} |a(t, \omega, x, \xi)| \, d\xi \leq C(n) M_0(t, \omega), \tag{3.3} $$

where $M_0(\cdot, \cdot) \in L^p_T(0, T)$ is a nonnegative function. On the other hand, for any multi-index $\alpha$,

$$ |(x-y)^\alpha K_A(t, \omega, x, y)| = (2\pi)^{-n} \left| \int_{\mathbb{R}^n} \partial^\alpha_\xi e^{i(x-y)\cdot \xi} a(t, \omega, x, \xi) \, d\xi \right| = (2\pi)^{-n} \left| \int_{\mathbb{R}^n} e^{i(x-y)\cdot \xi} \partial^\alpha_\xi a(t, \omega, x, \xi) \, d\xi \right| \leq C(n) M_\alpha(t, \omega) \int_{\mathbb{R}^n} (1 + |\xi|)^{-1-n-|\alpha|} \, d\xi \leq C(n) M_\alpha(t, \omega), \tag{3.4} $$

where $M_\alpha(\cdot, \cdot) \in L^p_T(0, T)$ is a nonnegative function. Then, by (3.3) and (3.4), it follows that for a nonnegative function $M_1(\cdot, \cdot) \in L^p_T(0, T)$,

$$ |K_A(t, \omega, x, y)| \leq \frac{C(n) M_1(t, \omega)}{(1 + |x-y|)^{1+n}}, $$

which implies that

$$ \sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} |K_A(t, \omega, x, y)| \, dx \leq C(n) M_1(t, \omega), \quad \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |K_A(t, \omega, x, y)| \, dy \leq C(n) M_1(t, \omega). $$

By (2)-(3) in Lemma 3.1, we get the desired result.
Step 2. Suppose that $a \in S^\ell_{\infty}\ (\ell < 0)$. By Remark 3.1, for any SPDO $A \in \mathcal{L}^{-\frac{n-1}{2\ell}}_\infty$, $A^* \circ A \in \mathcal{L}^{-\frac{n-1}{2\ell}}_\infty$. Then, by the proof in Step 1 and (3.2), for a.e. $(t, \omega) \in (0, T) \times \Omega$,

$$
\|(Au)(t, \omega, \cdot)\|^2_{L^2(\mathbb{R}^n)} = \|(Au)(t, \omega, \cdot), (Au)(t, \omega, \cdot)\|_{L^2(\mathbb{R}^n)}
$$

$$
= \|((A^* \circ A)u)(t, \omega, \cdot), u(t, \omega, \cdot)\|_{L^2(\mathbb{R}^n)} \leq M_2'(t, \omega)|u(t, \omega, \cdot)|^2_{L^2(\mathbb{R}^n)},
$$

for a nonnegative function $M_2'(\cdot, \cdot) \in L^\infty_\mathcal{F}(0, T)$. Therefore,

$$
\|((Au)(\cdot, \cdot, \cdot)|_{L^2(\mathbb{R}^n)}\|_{L^\infty_\mathcal{F}(0, T)} \leq |M_2'(\cdot, \cdot)|_{L^\infty_\mathcal{F}(0, T)} \|u(\cdot, \cdot, \cdot)|_{L^2(\mathbb{R}^n)}\|_{L^\infty_\mathcal{F}(0, T)}.
$$

This implies that for any $k \in \mathbb{N}$, $A : L^\infty_k(0, T; L^2(\mathbb{R}^n)) \rightarrow L^\infty_k(0, T; L^2(\mathbb{R}^n))$ is bounded, if $a \in S^\ell_{\infty\ell}$. For any $\ell < 0$, we take a positive integer $k_2$, such that $\ell < -\frac{n-1}{2\ell_2}$. Since $S^\ell_{\infty\ell} \subseteq S^\ell_{\infty\ell}$, we get the desired result of Theorem 3.1.

Step 3. If $a \in S^0_{\infty\ell}$, there exists a constant $C_0 > 0$ such that $|a(t, \omega, x, \xi)| \leq C_0$ for a.e. $(t, \omega) \in (0, T) \times \Omega$ and any $(x, \xi) \in \mathbb{R}^{2n}$. Take $C_1 > 2C_0^2 + 1$ and define $b(t, \omega, x, \xi) = (C_1 - |a(t, \omega, x, \xi)|^2)^{1/2}$. Then $b \in S^0_{\infty\ell}$ and $b\overline{b} + a\overline{a} = C_1$. If we denote by $B$ the SPDO determined by $b$, then $\sigma_{B^* \circ B} + \sigma_{A^* \circ A} - C_1 \in S^{-1}_{\infty\ell}$. Hence, we can find a SPDO $R \in \mathcal{L}^{-\frac{n-1}{2\ell_2}}_\infty$ such that for a.e. $(t, \omega) \in (0, T) \times \Omega$,

$$
|(Au)(t, \omega, \cdot)|^2_{L^2(\mathbb{R}^n)} + |(Bu)(t, \omega, \cdot)|^2_{L^2(\mathbb{R}^n)} = C_1|u(t, \omega, \cdot)|^2_{L^2(\mathbb{R}^n)} + ((Ru)(t, \omega, \cdot), u(t, \omega, \cdot))_{L^2(\mathbb{R}^n)}.
$$

Combining the above equality with the proof in Step 2, we see that

$$
|(Au)(t, \omega, \cdot)|^2_{L^2(\mathbb{R}^n)} \leq C(C_1, R)|u(t, \omega, \cdot)|^2_{L^2(\mathbb{R}^n)}.
$$

This implies that

$$
\|((Au)(\cdot, \cdot, \cdot)|_{L^2(\mathbb{R}^n)}\|_{L^\infty_\mathcal{F}(0, T)} \leq C(C_1, a) \|u(\cdot, \cdot, \cdot)|_{L^2(\mathbb{R}^n)}\|_{L^\infty_\mathcal{F}(0, T)}.
$$

Therefore, the proof is completed.

In the following, we give two corollaries. The first one generalizes the result of Theorem 3.1 to the space $L^q_k(0, T; H^\delta(\mathbb{R}^n))$ for $\delta \in \mathbb{R}$.

Corollary 3.1 Suppose that $A$ is a SPDO and $a$ is its symbol. If $a \in S^\ell_{\infty\ell}$, then for any $\delta \in \mathbb{R}$, $A : L^q_k(0, T; H^\delta(\mathbb{R}^n)) \rightarrow L^q_k(0, T; H^{\delta-\ell}(\mathbb{R}^n))$ is bounded.

Sketch of the proof. First, we recall that for the pseudo-differential operator $\Lambda^\delta$, whose symbol is $(1 + |\xi|^2)^{\frac{\delta}{2}}$, $|v|_{H^\delta(\mathbb{R}^n)} = |\Lambda^\delta v|_{L^2(\mathbb{R}^n)}$ for any $v \in H^\delta(\mathbb{R}^n)$. Therefore, for a.e. $(t, \omega) \in (0, T) \times \Omega$ and any $u \in L^q_k(0, T; H^\delta(\mathbb{R}^n))$, if we write $v = \Lambda^\delta u$, then

$$
|(Au)(t, \omega, \cdot)|_{H^{\delta-\ell}(\mathbb{R}^n)} = |(\Lambda^{\delta-\ell}(Au))(t, \omega, \cdot)|_{L^2(\mathbb{R}^n)} = |\Lambda^{\delta-\ell} A(\Lambda^{-\delta} v)(t, \omega, \cdot)|_{L^2(\mathbb{R}^n)}.
$$

Noticing that $\Lambda^{\delta-\ell} A \Lambda^{-\delta} \in S^0_{\infty\ell}$, by the result of Theorem 3.1, we have that

$$
\mathbb{E} \int_0^T |(Au)(t, \omega, \cdot)|^q_{H^{\delta-\ell}(\mathbb{R}^n)} dt = \mathbb{E} \int_0^T |A^{\delta-\ell} A(\Lambda^{-\delta} v)(t, \omega, \cdot)|^q_{L^2(\mathbb{R}^n)} dt
$$

$$
\leq C(a, q, \delta, \ell) \mathbb{E} \int_0^T |v(t, \omega, \cdot)|^q_{L^2(\mathbb{R}^n)} dt = C(a, q, \delta, \ell) \mathbb{E} \int_0^T |u(t, \omega, \cdot)|^q_{H^\delta(\mathbb{R}^n)} dt.
$$

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This finishes the proof.

The second corollary involves the $L^2$-estimates of the SPDOs defined on a local domain of $\mathbb{R}^n$. For this purpose, we introduce some locally convex topological vector spaces:

$$L^q_{\mathcal{F}}(0, T; H^\delta_{\text{loc}}(G)) = \{ u \mid \text{for any } \psi \in C^\infty_0(G) \text{ and a.e. } (t, \omega) \in (0, T) \times \Omega, \quad (\psi u)(t, \omega, \cdot) \in H^\delta(\mathbb{R}^n); \text{ and } |u|_{q, \delta, \psi} = |\psi u|_{L^q_{\mathcal{F}}(0, T; H^\delta_{\text{loc}}(G))} < \infty \},$$

which is generated by a family of semi-norms $\{ |\cdot|_{q, \delta, \psi} \}_{\psi \in C^\infty_0(G)}$.

$$L^q_{\mathcal{F}}(0, T; H^\delta_K(G)) = \{ u \mid \text{for a.e. } (t, \omega) \in (0, T) \times \Omega, \ u(t, \omega, \cdot) \in H^\delta(\mathbb{R}^n) \text{ and supp } u(t, \omega, \cdot) \subseteq K \},$$

which is generated by the norm $|\cdot|_{L^q_{\mathcal{F}}(0, T; H^\delta_{\text{loc}}(G))}$ and $K$ is any given compact set;

$$L^q_{\mathcal{F}}(0, T; H^\delta_{\text{comp}}(G)) = \bigcup_{K \subseteq G} L^q_{\mathcal{F}}(0, T; H^\delta_K),$$

which is endowed with the inductive topology.

Then, similar to Proposition 2.1, it is easy to show the following conclusions:

1. the locally convex space $L^q_{\mathcal{F}}(0, T; H^\delta_{\text{comp}}(G))$ satisfies the first countability axiom;
2. the locally convex space $L^q_{\mathcal{F}}(0, T; H^\delta_{\text{loc}}(G))$ is a Fréchet space;
3. $\lim_{j \to \infty} u_j = 0$ in $L^q_{\mathcal{F}}(0, T; H^\delta_{\text{comp}}(G))$, if and only if there exists a compact set $K^*$ such that
   $$\bigcup_{j \in \mathbb{N}} \text{supp } u_j(t, \omega, \cdot) \subseteq K^* \text{ for a.e. } (t, \omega) \in (0, T) \times \Omega \text{ and } \lim_{j \to \infty} |u_j|_{L^q_{\mathcal{F}}(0, T; H^\delta_{\text{loc}}(G))} = 0;$$
4. $\lim_{j \to \infty} u_j = 0$ in $L^q_{\mathcal{F}}(0, T; H^\delta_{\text{loc}}(G))$, if and only if for any $\psi \in C^\infty_0(G), \quad \lim_{j \to \infty} |\psi u_j|_{L^q_{\mathcal{F}}(0, T; H^\delta_{\text{loc}}(G))} = 0.$

**Corollary 3.2** Suppose that $A$ is a SPDO and $a$ is its amplitude.

1. If $a \in S^r_\infty(G \times G \times \mathbb{R}^n)$, then for any $\delta \in \mathbb{R}$, $A : L^q_{\mathcal{F}}(0, T; H^\delta_{\text{comp}}(G)) \to L^q_{\mathcal{F}}(0, T; H^{\delta-\ell}_{\text{loc}}(G))$ is continuous;
2. If $A$ is a uniformly properly supported SPDO with respect to $(t, \omega)$ and $a \in S^r_\infty(G \times G \times \mathbb{R}^n)$, then for any $\delta \in \mathbb{R}$, both $A : L^q_{\mathcal{F}}(0, T; H^\delta_{\text{comp}}(G)) \to L^q_{\mathcal{F}}(0, T; H^{\delta-\ell}_{\text{comp}}(G))$ and $A : L^q_{\mathcal{F}}(0, T; H^\delta_{\text{loc}}(G)) \to L^q_{\mathcal{F}}(0, T; H^{\delta-\ell}_{\text{loc}}(G))$ are continuous.

**Proof.** For any $u \in L^q_{\mathcal{F}}(0, T; H^\delta_{\text{comp}}(G))$, by the definition of the space $L^q_{\mathcal{F}}(0, T; H^\delta_{\text{comp}}(G))$, there exists a compact set $K^7$ such that $u \in L^q_{\mathcal{F}}(0, T; H^\delta_{K^7})$. Take a function $\varphi_1 \in C^\infty_0(G)$ such that $\varphi_1 = 1$ in $K^7$. Then for any function $\psi \in C^\infty_0(G),$

$$(\psi(Au))(t, \omega, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{G} e^{i(x-y) \cdot \xi} a(t, \omega, x, y, \xi) \psi(y) \varphi_1(y) u(t, \omega, y) dy d\xi.$$ 

Write $\tilde{a}(t, \omega, x, y, \xi) = a(t, \omega, x, y, \xi) \psi(y) \varphi_1(y)$ and denote by $\tilde{A}$ the SPDO determined by $\tilde{a}$. Then $\tilde{a} \in S^r_\infty$ and $\tilde{A}u = \psi(Au)$. By Corollary 3.1, it follows that $\psi(Au) \in L^q_{\mathcal{F}}(0, T; H^{\delta-\ell}_{\text{comp}}(G))$. This implies that $Au \in L^q_{\mathcal{F}}(0, T; H^{\delta-\ell}_{\text{loc}}(G))$. 

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On the other hand, if \( \lim_{j \to \infty} u_j = 0 \) in \( L^q_{\mathcal{F}}(0, T; H^\delta_{\text{comp}}(G)) \), then there exists a compact set \( K^8 \) such that \( \bigcup_{j \in \mathbb{N}} \text{supp } u_j(t, \omega, \cdot) \subseteq K^8 \) for a.e. \( (t, \omega) \in (0, T) \times \Omega \) and \( \lim_{j \to \infty} |u_j|_{L^p_{\mathcal{F}}(0, T; H^p(\mathbb{R}^n))} = 0 \).

Similarly, take a function \( \varphi_2 \in C_0^\infty(G) \) such that \( \varphi_2 = 1 \) in \( K^8 \). Then for any function \( \psi \in C_0^\infty(G) \),
\[
\psi(Au) = \tilde{A}u, \quad \text{where the amplitude of } \tilde{A} \text{ is } a\psi_2 \text{ and } a\psi_2 \in S^\ell_{\text{loc}}. \quad \text{Therefore, by Corollary 3.1,}
\]
\[
\lim_{j \to \infty} \psi(Au_j) = 0 \text{ in } L^q_{\mathcal{F}}(0, T; H^{3-\ell}_{\text{loc}}(\mathbb{R}^n)). \quad \text{This implies that } \lim_{j \to \infty} Au_j = 0 \text{ in } L^q_{\mathcal{F}}(0, T; H^{3-\ell}_{\text{loc}}(G)).
\]
Hence, we get the desired result (1).

Similarly, we can show that (2) also holds. \( \square \)

### 3.2 \( L^p \)-boundedness

In this subsection, we present the \( L^p \)-estimates \((p > 1, p \neq 2)\) of SPDOs. To begin with, we give the following known lemma, which will be used later.

**Lemma 3.2** ([13, Page 272]) There exist two functions \( \psi*(\cdot), \varphi*(\cdot) \in C_0^\infty(\mathbb{R}^n) \) satisfying \( 0 \leq \psi*(\xi), \varphi*(\xi) \leq 1 \), and

\[
(1) \quad \text{supp } \psi*(\cdot) \subseteq B_1, \quad \text{supp } \varphi*(\cdot) \subseteq G_0; \quad (2) \quad \psi*(\xi) + \sum_{j=0}^{\infty} \varphi*(2^{-j}\xi) = 1, \forall \xi \in \mathbb{R}^n,
\]
where \( B_1 = \{ \xi \in \mathbb{R}^n; \ |\xi| = 1 \} \) and \( G_0 = \{ \xi \in \mathbb{R}^n; \ k_s^{-1} < |\xi| < 2k_s \} \) for a constant \( k_s > 1 \).

Also, we give a lemma, which is a reformation of the known Calderón-Zygmund decomposition and is adapted to the stochastic case.

**Lemma 3.3** Suppose that \( u \in L^1(\mathbb{R}^n; L^p_{\mathcal{F}}(0, T)) \). Then for any \( r > 0 \), there exist the functions \( v(\cdot), w_k(\cdot) \in L^1(\mathbb{R}^n; L^p_{\mathcal{F}}(0, T)) \) \((k = 1, 2, \cdots)\) such that \( u(t, \omega, x) = v(t, \omega, x) + \sum_{k=1}^{\infty} w_k(t, \omega, x) \), where \( v(\cdot) \) and \( w_k(\cdot) \) satisfy the following conditions:

\[
\text{supp } w_k(t, \omega, \cdot) \subseteq \overline{T_k}, \text{ for a.e. } (t, \omega) \in (0, T) \times \Omega, \text{ where } I_k \text{ are disjoint cubes in } \mathbb{R}^n;
\]
\[
\begin{align*}
& r \sum_{k=1}^{\infty} |I_k| \leq |u(\cdot)|_{L^1(\mathbb{R}^n; L^p_{\mathcal{F}}(0, T))}; \\
& \int_{\mathbb{R}^n} w_k(t, \omega, x) dx = 0, \quad |v(\cdot, \cdot, x)|_{L^p_{\mathcal{F}}(0, T)} \leq 2^n r, \text{ for a.e. } (t, \omega, x) \in (0, T) \times \Omega \times \mathbb{R}^n; \\
& |v(t, \omega, \cdot)|_{L^1(\mathbb{R}^n)} + \sum_{k=1}^{\infty} |w_k(t, \omega, \cdot)|_{L^1(\mathbb{R}^n)} \leq 3|u(t, \omega, \cdot)|_{L^1(\mathbb{R}^n)}, \text{ for a.e. } (t, \omega) \in (0, T) \times \Omega; \\
& |v(\cdot, \cdot, x)|_{L^p_{\mathcal{F}}(0, T)} = |u(\cdot, \cdot, x)|_{L^p_{\mathcal{F}}(0, T)}, \text{ for a.e. } x \in \left( \bigcup_{k \in \mathbb{N}} I_k \right)^c; \\
& \int_{I_k} |v(\cdot, \cdot, x)|_{L^p_{\mathcal{F}}(0, T)} dx \leq \int_{I_k} |u(\cdot, \cdot, x)|_{L^p_{\mathcal{F}}(0, T)} dx, \text{ for any } k \in \mathbb{N}.
\end{align*}
\]

**Proof.** First, we divide \( \mathbb{R}^n \) into the cubes, whose volumes are greater than \( r^{-1}|u(\cdot)|_{L^1(\mathbb{R}^n; L^p_{\mathcal{F}}(0, T))} \).

Then for every such cube \( M \),
\[
\frac{|u(\cdot)|_{L^1(M; L^p_{\mathcal{F}}(0, T))}}{|M|} \leq \frac{|u(\cdot)|_{L^1(\mathbb{R}^n; L^p_{\mathcal{F}}(0, T))}}{|M|} < r. \tag{3.5}
\]
Again, we divide every cube $M$ into $2^n$ cubes equally and denote by $I_{1k}$ ($k = 1, 2, \cdots$) those small cubes satisfying $\frac{|u(\cdot)|_{L^1(I_{1k}; L^r_p(\Omega))}}{|I_{1k}|} \geq r$. By (3.5), we obtain that $|u(\cdot)|_{L^1(I_{1k}; L^r_p(\Omega))} < 2^n r |I_{1k}|$. Next, we divide the cubes, which are not $I_{1k}$, into $2^n$ small cubes equally. Denote by $I_{2k}$ ($k = 1, 2, \cdots$) those small cubes satisfying $\frac{|u(\cdot)|_{L^1(I_{2k}; L^r_p(\Omega))}}{|I_{2k}|} \geq r$. Similarly, we get that $|u(\cdot)|_{L^1(I_{2k}; L^r_p(\Omega))} < 2^n r |I_{2k}|$.

We proceed as the above steps, and then get a sequence of cubes (denoted by $\{I_k\}_{k \in \mathbb{N}}$), such that the average $\frac{|u(\cdot)|_{L^1(I_k; L^r_p(\Omega))}}{|I_k|}$ of $|u|$ on $I_k$ is greater than $r$ for all $k \in \mathbb{N}$. Moreover,

$$|u(\cdot)|_{L^1(I_k; L^r_p(\Omega))} < 2^n r |I_k|.$$

(3.6)

Define

$$v(t, \omega, x) = \begin{cases} \int_{I_k} \frac{u(t, \omega, x)dx}{|I_k|} & x \in I_k, \\ u(t, \omega, x) & x \in (\bigcup_{k \in \mathbb{N}} I_k)^c; \end{cases}$$

and

$$w_k(t, \omega, x) = \begin{cases} u(t, \omega, x) - v(t, \omega, x) & x \in I_k, \\ 0 & x \in (I_k)^c. \end{cases}$$

Then, if $x \in I_k$, by (3.6), $|v(\cdot, \cdot, x)|_{L^r_p(\Omega)} = \frac{\left| \int_{I_k} u(\cdot, \cdot, x)dx \right|}{|I_k|} \leq \frac{\int_{I_k} u(\cdot, \cdot, x)dx}{|I_k|} < 2^n r$, for $k \in \mathbb{N}$. If $x \in (\bigcup_{k \in \mathbb{N}} I_k)^c$, there exists a sequence of cubes $\{I_k^*\}_{k \in \mathbb{N}}$ satisfying $x \in \bigcap_{k \in \mathbb{N}} I_k^*$ and $|I_k^*|$ tends to 0 ($k \to \infty$), such that $\frac{|u(\cdot)|_{L^1(I_k^*; L^r_p(\Omega))}}{|I_k^*|} < r$. Therefore, we get that $|v(\cdot, \cdot, x)|_{L^r_p(\Omega)} \leq r$ for a.e. $x \in (\bigcup_{k \in \mathbb{N}} I_k)^c$. This implies that $|v(\cdot, \cdot, x)|_{L^r_p(\Omega)} \leq 2^n r$ for a.e. $x \in \mathbb{R}^n$. Furthermore, it is easy to check other conclusions in Lemma 3.3. The proof is completed.

**Remark 3.3** Similar to Lemma 3.3, we can get the following result: if $u \in L^\infty_{x, t}(0, T; L^1(\mathbb{R}^n))$, then for any $r > 0$, there exist the functions $v(\cdot)$, $w_k(\cdot) \in L^\infty_{x, t}(0, T; L^1(\mathbb{R}^n))$ ($k = 1, 2, \cdots$) such that $u(t, \omega, x) = v(t, \omega, x) + \sum_{k=1}^{\infty} w_k(t, \omega, x)$, where $v(\cdot)$ and $w_k(\cdot)$ satisfy the following conditions:

$$\text{supp } w_k(t, \omega, \cdot) \subseteq \overline{I_k}, \text{ for a.e. } (t, \omega) \in (0, T) \times \Omega, \text{ where } I_k \text{ are disjoint cubes in } \mathbb{R}^n;$$

$$r \sum_{k=1}^{\infty} |I_k| \leq |u(\cdot)|_{L^\infty_{x, t}(0, T; L^1(\mathbb{R}^n))};$$

$$\int_{\mathbb{R}^n} w_k(t, \omega, x)dx = 0, \quad |v(t, \omega, x)| \leq 2^n r, \text{ for a.e. } (t, \omega, x) \in (0, T) \times \Omega \times \mathbb{R}^n;$$

$$|v(t, \omega, \cdot)|_{L^1(\mathbb{R}^n)} + \sum_{k=1}^{\infty} |w_k(t, \omega, \cdot)|_{L^1(\mathbb{R}^n)} \leq 3 |u(t, \omega, \cdot)|_{L^1(\mathbb{R}^n)}.$$
Based on the above lemmas, first of all, we give a boundedness result for a class of SPDOs, whose symbol \( a = a(t, \omega, \xi) \) is independent of the variable \( x \). For the functions \( \psi^* \) and \( \varphi^* \) given in Lemma 3.2, write \( a_{-1,*}(t, \omega, \xi) = \psi^*(\xi)a(t, \omega, \xi) \) and \( a_{j,*}(t, \omega, \xi) = \varphi^*(2^{-j}\xi)a(t, \omega, \xi) \) \( (j \in \mathbb{N}) \). Then we have the following result.

**Lemma 3.4** Suppose that \( a = a(t, \omega, \xi) \in S^0_0 \). Then there exists the functions \( M_n(\cdot, \cdot) \in L^p_\psi(0, T) \) such that for any \( j \in \mathbb{N} \cup \{-1\} \),

\[
\int_{\mathbb{R}^n} |\hat{a}_{j,*}(t, \omega, x)|dx \leq M_n(t, \omega), \text{ a.e. } (t, \omega) \in (0, T) \times \Omega,
\]

where \( \hat{a}_{j,*}(t, \omega, x) \) denotes the Fourier inversion transform of \( a_{j,*}(t, \omega, \xi) \) with respect to the variable \( \xi \). Moreover, for the associated SPDO \( A_{j,*} \) determined by the symbol \( a_{j,*} \) and any \( p, \in [1, \infty] \), \( A_{j,*} : L^p_\psi(0, T; L^p(\mathbb{R}^n)) \to L^q_\psi(0, T; L^p(\mathbb{R}^n)) \) is bounded.

The proof of Lemma 3.4 is similar to that in the deterministic case. Therefore, we omit it.

Next, we present another useful lemma. Notice that by Lemma 3.3, for any \( u \in L^1(\mathbb{R}^n; L^p_\psi(0, T)) \), there exist the functions \( v \) and \( w_k \) \( (k \in \mathbb{N}) \) satisfying all the conditions mentioned in Lemma 3.3. Actually, for any \( u \in L^1(\mathbb{R}^n; L^p_\psi(0, T)) \) and the associated \( v \), the following conclusion also holds.

**Lemma 3.5** Suppose that \( a = a(t, \omega, \xi) \in S^0_0 \), \( u \in L^1(\mathbb{R}^n; L^p_\psi(0, T)) \) and \( u(t, \omega, \cdot) \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \), for a.e. \((t, \omega) \in (0, T) \times \Omega \). Then for any \( r > 0 \) and a.e. \((t, \omega) \in (0, T) \times \Omega \),

\[
|\{ x \in \mathbb{R}^n; |(Au)(t, \omega, x)| > r \}| \leq C(n, a) \left( |u|_{L^1(\mathbb{R}^n; L^p_\psi(0, T))} + |u(t, \omega, \cdot)|_{L^1(\mathbb{R}^n)} + r^{-1} |v(t, \omega, \cdot)|_{L^2(\mathbb{R}^n)}^2 \right),
\]

where \( v \) is the function associated to \( u \) in Lemma 3.3.

**Proof.** For any \( u \in L^1(\mathbb{R}^n; L^p_\psi(0, T)) \) satisfying \( u(t, \omega, \cdot) \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \), for a.e. \((t, \omega) \in (0, T) \times \Omega \), suppose that \( v \) and \( w_k \) \( (k \in \mathbb{N}) \) are the functions mentioned in Lemma 3.3. Then, for any \( r > 0 \), it follows that

\[
|\{ x \in \mathbb{R}^n; |(Av)(t, \omega, x)| > r \}| \leq \left| \{ x \in \mathbb{R}^n; |(Av)(t, \omega, x)| > \frac{r}{2} \} \right| + \left| \{ x \in \mathbb{R}^n; |(A \sum_{k=1}^{\infty} w_k)(t, \omega, x)| > \frac{r}{2} \} \right|. \tag{3.7}
\]

In the following, we estimate two terms in the right side of (3.7), respectively. By Theorem 3.1,

\[
\frac{r^2}{4} \left| \{ x \in \mathbb{R}^n; |(Av)(t, \omega, x)| > \frac{r}{2} \} \right| \leq \int_{\mathbb{R}^n} |(Av)(t, \omega, x)|^2 dx \leq C(a) \int_{\mathbb{R}^n} |v(t, \omega, x)|^2 dx. \tag{3.8}
\]

On the other hand, denote by \( I_k \), the cube, with the same center as \( I_k \) and the length of side twice than \( I_k \). Let \( I^* = \bigcup_{k \in \mathbb{N}} I_k \). Then, for any \( k \in \mathbb{N} \), if we can prove that

\[
\int_{I_k} |(Av_k)(t, \omega, x)|dx \leq C \int_{\mathbb{R}^n} |w_k(t, \omega, x)|dx, \tag{3.9}
\]
it follows that

\[
\frac{r}{2} \left| \{ x \in (I^*)^c; | \sum_{k=1}^{\infty} (Aw_k)(t,\omega,x) | > \frac{r}{2} \} \right| \leq \int_{(I^*)^c} \left| \sum_{k=1}^{\infty} (Aw_k)(t,\omega,x) \right| dx \\
\leq \sum_{k=1}^{\infty} \int_{(I_k,s)^c} |(Aw_k)(t,\omega,x)| dx \leq C \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |w_k(t,\omega,x)| dx \leq C|u(t,\omega,\cdot)|_{L^1(\mathbb{R}^n)}; \tag{3.10}
\]

By (3.7), (3.8) and (3.10), we obtain the desired result. The remainder is devoted to the proof of (3.9). We denote by \( \ell_s \) the length of side of \( I_k \) and write \( h_j(t,\omega,x) = \tilde{a}_{j,s}(t,\omega,\xi) \), where \( \tilde{a}_{j,s} \) are the functions mentioned in Lemma 3.4. Then we see that

\[
\int_{(I_k,s)^c} |(Aw_k)(t,\omega,x)| dx \leq \sum_{j=1}^{\infty} \int_{(I_k,s)^c} \int_{\mathbb{R}^n} h_j(t,\omega,x-y)w_k(t,\omega,y)dy \, dx \\
\leq \sum_{2\ell_s \geq 1} \int_{(I_k,s)^c} \left| \int_{\mathbb{R}^n} h_j(t,\omega,x-y)w_k(t,\omega,y)dy \right| \, dx \\
+ \sum_{2\ell_s < 1} \int_{(I_k,s)^c} \left| \int_{\mathbb{R}^n} [h_j(t,\omega,x-y) - h_j(t,\omega,x)]w_k(t,\omega,y)dy \right| \, dx. \tag{3.11}
\]

Similar to the deterministic case, we estimate two terms in the right side of (3.11), respectively, and then get (3.9). This finishes the proof. \( \square \)

**Remark 3.4** Similar to Lemma 3.5, we can prove the following result: suppose that \( a = a(t,\omega,\xi) \in S_{\infty}^0 \) and \( u \in L_{x}^{\infty}(0,T;L^1(\mathbb{R}^n)) \). Then for any \( r > 0 \) and a.e. \( (t,\omega) \in (0,T) \times \Omega \),

\[
r \cdot |\{ x \in \mathbb{R}^n; |(Au)(t,\omega,x)| > r \}| \leq C(n,a)|u|_{L_{x}^{\infty}(0,T;L^1(\mathbb{R}^n))}.
\]

Now, we give the \( L^p \)-estimates \( (p > 1, p \neq 2) \) of a class of the SPDOs of order \( (0,\infty) \).

**Theorem 3.2** Suppose that \( a = a(t,\omega,\xi) \in S_{\infty}^0 \) and \( p > 1 \). Then for the associated SPDO \( A \), the following conclusions hold:

1. if \( 1 < p < 2 \), \( A : L^p(\mathbb{R}^n;L^p_{x}(0,T)) \to L^p(\mathbb{R}^n;L^p_{x}(0,T)) \) is bounded;
2. if \( p > 2 \), \( A : L^p(\mathbb{R}^n;L^p_{x}(0,T)) \to L^p(\mathbb{R}^n;L^p_{x}(0,T)) \) is bounded.

**Proof.** For any \( r > 0 \), define

\[
u_r(t,\omega,x) = \begin{cases} 
\frac{u(t,\omega,x)}{|u(\cdot,\cdot,x)|_{L^p_x(0,T)}} & |u(\cdot,\cdot,x)|_{L^p_x(0,T)} \geq r, \\
0 & |u(\cdot,\cdot,x)|_{L^p_x(0,T)} < r,
\end{cases}
\]

\[
u^*(t,\omega,x) = \begin{cases} 
0 & |u(\cdot,\cdot,x)|_{L^p_x(0,T)} \geq r, \\
u(t,\omega,x) & |u(\cdot,\cdot,x)|_{L^p_x(0,T)} < r.
\end{cases}
\]

Then, for \( 1 < p < 2 \), by Lemma 3.5, we have that

\[
|\{Au_r\}(t,\omega,\cdot)|_{L^p(\mathbb{R}^n)} = \int_0^{\infty} pr^{p-1} |\{ x \in \mathbb{R}^n; |(Au_r)(t,\omega,x)| > r \}| dr \\
\leq C(n,a,p) \int_0^{\infty} r^{p-2} \left[ |u_r|_{L^1(\mathbb{R}^n;L^p_x(0,T))} + |u_r(t,\omega,\cdot)|_{L^1(\mathbb{R}^n)} + r^{-1} |v_r(t,\omega,\cdot)|_{L^2(\mathbb{R}^n)} \right] dr, \tag{3.12}
\]
where \( v_r \) is the function associated to \( u_r \) in Lemma 3.5. Next, we estimate every term in the right side of (3.12) respectively.

\[
\begin{align*}
\mathbb{E} \int_0^T \int_0^\infty r^{p-2}|u_r(t, \cdot, \cdot)|_{L^p(\mathbb{R}^n)}^p \, dr \, dt &= \int_0^T \int_0^\infty r^{p-2}|u_r(\cdot, \cdot, x)|_{L^p_x(0,T)}^p \, dx \, dr \\
&\leq C(T) \int_0^T \int_0^\infty r^{p-2}|u_r(\cdot, \cdot, x)|_{L^p_x(0,T)}^p \, dx \, dr \\
&\leq C(T,p) \int \int |u_r(\cdot, \cdot, x)|_{L^p_x(0,T)}^p \, dx.
\end{align*}
\]

(3.13)

\[
\begin{align*}
\mathbb{E} \int_0^T \int_0^\infty r^{p-2}|u_r(t, \omega, \cdot)|_{L^1(\mathbb{R}^n)} \, dr \, dt &= \int_0^T \int_0^\infty r^{p-2}|u_r(t, \omega, x)|_{L^p_x(0,T)}^p \, dx \, dr \\
&\leq C(T,p) \int \int |u_r(t, \omega, x)|_{L^p_x(0,T)}^p \, dx.
\end{align*}
\]

(3.14)

\[
\begin{align*}
\mathbb{E} \int_0^T \int_0^\infty r^{p-3}|v_r(t, \omega, \cdot)|_{L^2(\mathbb{R}^n)}^2 \, dr \, dt &= \int_0^T \int_0^\infty r^{p-3}|v_r(t, \omega, x)|_{L^p(0,T)}^2 \, dx \, dr \\
&\leq C(n) \int_0^\infty \int_0^\infty r^{p-3}|v_r(\cdot, \cdot, x)|_{L^p_x(0,T)}^2 \, dx \, dr \\
&\leq C(n) \int_0^\infty \int_0^\infty r^{p-2}|v_r(\cdot, \cdot, x)|_{L^p_x(0,T)}^2 \, dx \, dr.
\end{align*}
\]

(3.15)

By (3.12)-(3.15), we see that

\[
\mathbb{E} \int_0^T |(Au_r)(t, \omega, \cdot)|_{L^p(\mathbb{R}^n)}^p \, dt \leq C(n, a, p, T) \int \int |u(\cdot, \cdot, x)|_{L^p_x(0,T)}^p \, dx.
\]

(3.16)

On the other hand,

\[
\begin{align*}
|(Au)^r(t, \omega, \cdot)|_{L^p(\mathbb{R}^n)}^p &= \int_0^\infty r^{p-1}|x \in \mathbb{R}^n; |(Au)^r(t, \omega, x)| > r| \, dr \\
&\leq \int_0^\infty pr^{p-3} \int |(Au)^r(t, \omega, x)|^2 \, dx \, dr \\
&\leq C(a) \int_0^\infty pr^{p-3} |u^r(t, \omega, x)|^2 \, dx \, dr \\
&\leq C(a) \int_0^\infty pr^{p-3} |u(t, \omega, x)|^2 \, dx.
\end{align*}
\]
Therefore, we obtain that
\[
E \int_0^T |(Au^\nu)(t,\omega,\cdot)|_{L^p}\ dt \leq C(a,p)E \int_0^T \int_{\mathbb{R}^n} |u(\cdot,\cdot,x)|_{L^{p-2}}^p |u(t,\omega,x)|^2 dx dt
\]
\[
= C(a,p) \int_{\mathbb{R}^n} |u(\cdot,\cdot,x)|_{L^{p-2}}^p \left( E \int_0^T |u(t,\omega,x)|^2 dt \right) dx
\]
\[
\leq C(a,p) \int_{\mathbb{R}^n} |u(\cdot,\cdot,x)|_{L^{p-1}}^{p-1} |u(\cdot,\cdot,x)|_{L^p}^p dx \leq C(a,p,T) \int_{\mathbb{R}^n} |u(\cdot,\cdot,x)|_{L^{p'}}^p dx.
\]
(3.16) and (3.17) imply the desired result (1). By Lemma 2.1 in [10] and a duality argument, we can get the result (2) for \( p > 2 \).

\[\square\]

**Remark 3.5** Similar to the proof of Theorem 3.2, by Remark 3.3 and Remark 3.4, we can prove the following result: if \( a = a(t,\omega,\xi) \in S^0_{\infty} \), then for \( 1 < p < 2 \), the associated SPDO \( A : L^p(\mathbb{R}^n; L^p(0,T)) \to L^p(\mathbb{R}^n; L^p(0,T)) \) is bounded. However, for \( p > 2 \), we do not establish the corresponding boundedness result, because we fail to give a characterization for the dual spaces of \( L^p(\mathbb{R}^n; L^p(0,T)) \) and \( L^p(\mathbb{R}^n; L^p(0,T)) \) at this moment.

In the remainder of this subsection, we establish the \( L^p \)-estimates of the SPDOs defined on a local domain. First of all, we give a preliminary.

**Corollary 3.3** Suppose that \( a \in S^0_{\infty}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n) \) and \( A \) is the associated SPDO. If there exists a bounded set \( B_0 \subseteq \mathbb{R}^{2n} \), such that \( \text{supp} \ a(t,\omega,\cdot,\cdot,\xi) \subseteq B_0 \) for a.e. \( (t,\omega) \in (0,T) \times \Omega \) and any \( \xi \in \mathbb{R}^n \), then the conclusions (1)-(2) mentioned in Theorem 3.2 hold.

**Sketch of the proof.** The method of the proof is similar to that used in the deterministic case. First, we construct a family of amplitudes \( \{a_{\eta,\zeta}\}_{(\eta,\zeta) \in \mathbb{R}^{2n}} \), whose associated SPDOs \( \{A_{\eta,\zeta}\}_{(\eta,\zeta) \in \mathbb{R}^{2n}} \) have good estimates with respect to parameters \( (\eta,\zeta) \). Indeed, write
\[
\hat{a}(t,\omega,\eta,\zeta,\xi) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i(x-y)\cdot\eta} a(t,\omega,x,y,\xi) dx dy.
\]
Then for any multi-indices \( \alpha_1, \alpha_2 \) and \( \beta \), it follows that
\[
\eta^{\alpha_1} \zeta^{\alpha_2} \hat{a}(t,\omega,\eta,\zeta,\xi) = \eta^{\alpha_1} \zeta^{\alpha_2} \hat{a}(t,\omega,x,y,\xi) dx dy
\]
\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{(\xi^{[1]})^{[1]} + |\alpha_2|} \hat{a}(t,\omega,x,y,\xi) dx dy
\]
\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{|\alpha_1| + |\alpha_2|} e^{-i(x-y)\cdot\eta} \hat{a}(t,\omega,x,y,\xi) dx dy.
\]
This implies that for any \( k \in \mathbb{N} \), a.e. \( (t,\omega) \in (0,T) \times \Omega \) and any \( (\eta,\zeta,\xi) \in \mathbb{R}^{3n} \),
\[
|\hat{a}(t,\omega,\eta,\zeta,\xi)| \leq C(k,\beta,B_0)(1 + |\xi|)^{-|\beta|} (1 + |\eta| + |\zeta|)^{-k}.
\]
Next, denote by $A_{\eta, \xi}$ the SPDO, whose amplitude is $\hat{a}$. Based on the result of Theorem 3.2, we obtain that

$$|A_{\eta, \xi}|_{L^p_\chi (R^n; L^p_p(0, T))} \leq C(n, k, T, a, B_0, p)(1 + |\eta| + |\xi|)^{-k}, \text{ for } 1 < p < 2;$$

$$|A_{\eta, \xi}|_{L^\infty_\chi (R^n; L^p_p(0, T))} \leq C(n, k, T, a, B_0, p)(1 + |\eta| + |\xi|)^{-k}, \text{ for } p > 2.$$ 

Moreover, notice that

$$(A u)(t, \omega; x) = (2\pi)^{-n} \int_{R^n} \int_{R^n} e^{i(x-y)\cdot\xi} a(t, \omega, x, \xi) u(t, \omega, y) dy$$

$$= (2\pi)^{-3n} \int_{R^n} \int_{R^n} e^{i(x-y)\cdot\xi} \left[ \int_{R^n} \int_{R^n} e^{i(x-\eta+ys)\cdot\xi} \hat{a}(t, \omega, \eta, \xi) d\eta d\xi \right] u(t, \omega, y) dy$$

$$= (2\pi)^{-2n} \int_{R^n} \int_{R^n} e^{i(x-\eta+ys)\cdot\xi} (A_{\eta, \xi} u)(t, \omega, x) d\eta d\xi.$$ 

Therefore, for $1 < p < 2$,

$$|A u|_{L^p_\chi (R^n; L^p_p(0, T))} \leq C(n) \int_{R^n} \int_{R^n} |A_{\eta, \xi} u|_{L^p_\chi (R^n; L^p_p(0, T))} d\eta d\xi$$

$$\leq C(n, T, a, B_0, p) \int_{R^n} \int_{R^n} (1 + |\eta| + |\xi|)^{-2n-2} |u|_{L^p_\chi (R^n; L^p_p(0, T))} d\eta d\xi$$

$$\leq C(n, T, a, B_0, p) |u|_{L^p_\chi (R^n; L^p_p(0, T))}.$$ 

We can get the result in the case of $p > 2$ in the same way. 

Next, we introduce some locally convex topological spaces. For any $p, q > 1$ and any compact set $K \subseteq R^n$,

$$L^p(K; L^q_p(0, T)) = \{ u \in L^p(G; L^q_p(0, T)) \mid \text{supp } u(t, \omega, \cdot) \subseteq K, \text{ for a.e. } (t, \omega) \in (0, T) \times \Omega \},$$

which is generated by the norm $\| \cdot \|_{L^p(G; L^q_p(0, T))}$;

$$L^p_{\text{comp}}(G; L^q_p(0, T)) = \bigcup_{K \subseteq G \text{ is compact}} L^p(K; L^q_p(0, T)),$$

which is endowed with the inductive topology;

$$L^p_{\text{loc}}(G; L^q_p(0, T)) = \left\{ u \mid |u|_{p, q, \psi} = \| \psi u \|_{L^p(G; L^q_p(0, T))} < \infty, \text{ for any } \psi \in C^\infty_0(G) \right\},$$

which is generated by a family of semi-norms $\{ | \cdot |_{p, q, \psi} \}_{\psi \in C^\infty_0(G)}$.

Then, based on Corollary 3.3, by the method similar to that used in the proof of Corollary 3.2, we get the following boundedness result.

**Corollary 3.4** Suppose that $A$ is a SPDO and $a$ is its amplitude.

1. If $a \in S^0_\infty(G \times G \times R^n)$, then for $1 < p < 2$, $A : L^p_{\text{comp}}(G; L^p_p(0, T)) \to L^p_{\text{loc}}(G; L^p_p(0, T))$ is continuous;

2. If $a \in S^0_\infty(G \times G \times R^n)$, then for $p > 2$, $A : L^p_{\text{comp}}(G; L^p_p(0, T)) \to L^p_{\text{loc}}(G; L^p_p(0, T))$ is
continuous;

(3) If $A$ is a uniformly properly supported SPDO with respect to $(t, \omega)$ and $a \in S^0_\infty(G \times G \times \mathbb{R}^n)$, then for $1 < p < 2$, both $A : L^p_{\text{comp}}(G; L^p_\mathcal{F}(0, T)) \to L^p_{\text{comp}}(G; L^p_\mathcal{F}(0, T))$ and $A : L^p_{\text{loc}}(G; L^p_\mathcal{F}(0, T)) \to L^p_{\text{loc}}(G; L^p_\mathcal{F}(0, T))$ are continuous;

(4) If $A$ is a uniformly properly supported SPDO with respect to $(t, \omega)$ and $a \in S^0_\infty(G \times G \times \mathbb{R}^n)$, then for $p > 2$, both $A : L^p_{\text{comp}}(G; L^p_\mathcal{F}(0, T)) \to L^p_{\text{comp}}(G; L^p_\mathcal{F}(0, T))$ and $A : L^p_{\text{loc}}(G; L^p_\mathcal{F}(0, T)) \to L^p_{\text{loc}}(G; L^p_\mathcal{F}(0, T))$ are continuous.

4 Elliptic stochastic pseudo-differential operators

In this subsection, we introduce the notion of elliptic SPDOs and point out their invertibility. Also, we give the Gårding inequality.

First of all, we define an elliptic operator and its parametrix.

**Definition 4.1** Suppose that $A$ is a SPDO of order $(\ell, p)$ and $a$ is its symbol. $A$ is called an elliptic SPDO if for any compact set $K$, there exist two positive constants $C_K$ and $R_K$, such that

$$|a(t, \omega, x, \xi)| \geq C_K(1 + |\xi|^\ell), \text{ for a.e. } (t, \omega) \in (0, T) \times \Omega \text{ and any } (x, \xi) \in K \times \{ \xi \in \mathbb{R}^n; |\xi| \geq R_K \}.$$

**Definition 4.2** A left parametrix $Q_1$ (or a right parametrix $Q_2$) for a SPDO $A$ of order $(\ell, p)$ is a SPDO, which is a single-sided inverse for $A$ modulo smoothing operators satisfying

$$Q_1A - I \in \mathcal{L}^{-\ell}_\infty \quad (\text{or } AQ_2 - I \in \mathcal{L}^{-\ell}_\infty).$$

Next, we present invertibility of elliptic SPDOs.

**Theorem 4.1** Suppose that $A$ is an elliptic SPDO of order $(\ell, \infty)$. Then there exist two SPDOs $Q_1, Q_2 \in \mathcal{L}^{-\ell}_\infty(G)$, which are the left parametrix and right parametrix for $A$, respectively.

Also, we give a lemma, which is a preliminary for the proof of the Gårding inequality.

**Lemma 4.1** Suppose that $A$ is a SPDO of order $(0, \infty)$ and $a$ is its symbol. If for a.e. $(t, \omega) \in (0, T) \times \Omega$ and any $(x, \xi) \in \mathbb{R}^{2n}$, $\text{Re } a(t, \omega, x, \xi) \geq C^*$ for a positive constant $C^*$, then there exists a SPDO $B \in \mathcal{L}^0_\infty$, such that

$$\frac{A + A^*}{2} - B^*B \in \mathcal{L}^{-\ell}_\infty.$$

The proofs of Theorem 4.1 and Lemma 4.1 are similar to those in the deterministic case. Here we omit them.

Now, we prove the following Gårding inequality.
Let $5.1$ Statement of the main result

SPDO. Then, we see that $\text{Re } a(t,\omega,\cdot)$, an estimate for the SPDOs of order $(1,2)$ of at most double characteristics. The key point of the proof is to establish a new Carleman-type theorem on the Cauchy problem of SPDEs. In order to present the main idea, we focus on the case $S_{\ell}^2(U; H^{k_1}([0, T]; C^k(U)))$.

In this section, as an application of the theory of SPDOs, we establish a Calderón-type uniqueness theorem on the Cauchy problem of SPDEs. In order to present the main idea, we focus on the case of at most double characteristics. The key point of the proof is to establish a new Carleman-type estimate for the SPDOs of order $(1,\infty)$.

**Theorem 4.2** Suppose that $A$ is a SPDO of order $(\ell, \infty)$ and $a$ is its symbol. If there exist two positive constants $\delta^*$ and $R^*$, such that $\text{Re } a(t,\omega, x, \xi) \geq \delta^* |\xi|^\ell$ for a.e. $(t,\omega) \in (0,T) \times \Omega$ and any $(x,\xi) \in \mathbb{R}^n \times \{ \xi \in \mathbb{R}^n; \ |\xi| \geq R^* \}$, then for any $\varepsilon > 0$, $r \in \mathbb{R}$ and $u \in H^2_r(\mathbb{R}^n)$, we have

$$\mathbb{E} \int_0^T \text{Re } (Au(t,\omega,\cdot), u(t,\omega,\cdot))_{L^2(\mathbb{R}^n)} dt \geq (\delta^* - \varepsilon) \mathbb{E} \int_0^T |u(t,\omega,\cdot)|^2_{H^2_r(\mathbb{R}^n)} dt - C\mathbb{E} \int_0^T |u(t,\omega,\cdot)|^2_{H^2_r(\mathbb{R}^n)} dt.$$

**Sketch of the proof.** If $\ell = 0$, without of generality, we assume that $\text{Re } a(t,\omega, x, \xi) \geq \delta^* |\xi|^\ell$ for a.e. $(t,\omega) \in (0,T) \times \Omega$ and any $(x,\xi) \in \mathbb{R}^n$. Write $a_\varepsilon = a - (\delta^* - \varepsilon)$ and let $A_\varepsilon$ be the associated SPDO. Then, we see that $\text{Re } a_\varepsilon \geq \varepsilon$, and by Lemma 4.1, there exists a SPDO $B \in \mathcal{L}_0^0$ such that $A_\varepsilon + A_\varepsilon^* - B^*B = R_1 \in \mathcal{L}^{-\infty}_\infty$. Therefore, $\frac{A + A^*}{2} - (\delta^* - \varepsilon)I - B^*B = R_1$. By the $L^2$-estimates, this leads to that

$$\mathbb{E} \int_0^T \text{Re } ((Au(t,\omega,\cdot), u(t,\omega,\cdot))_{L^2(\mathbb{R}^n)} dt$$

$$= \mathbb{E} \int_0^T \text{Re } ((\delta^* - \varepsilon)u(t,\omega,\cdot), u(t,\omega,\cdot))_{L^2(\mathbb{R}^n)} dt + \mathbb{E} \int_0^T \text{Re } ((Bu(t,\omega,\cdot), Bu(t,\omega,\cdot))_{L^2(\mathbb{R}^n)} dt$$

$$+ \mathbb{E} \int_0^T \text{Re } ((R_1 u(t,\omega,\cdot), u(t,\omega,\cdot))_{L^2(\mathbb{R}^n)} dt$$

$$\geq (\delta^* - 2\varepsilon) \mathbb{E} \int_0^T |u(t,\omega,\cdot)|^2_{L^2(\mathbb{R}^n)} dt - C(\varepsilon) \mathbb{E} \int_0^T |u(t,\omega,\cdot)|^2_{H^2_r(\mathbb{R}^n)} dt.$$

We can also get the desired results for $a \in S_{\ell}^\infty (\ell \neq 0)$ by the technique used in Corollary 3.1. This finishes the proof.  

5 Calderón-type theorem of stochastic pseudo-differential operators

In this section, as an application of the theory of SPDOs, we establish a Calderón-type uniqueness theorem on the Cauchy problem of SPDEs. In order to present the main idea, we focus on the case of at most double characteristics. The key point of the proof is to establish a new Carleman-type estimate for the SPDOs of order $(1,\infty)$.

5.1 Statement of the main result

Let $U$ be a neighborhood of the origin in $\mathbb{R}^n$. Set $X_\infty = \bigcap_{j \in \mathbb{N}} L^2_x(\Omega; C^1([0, T]; C^j(U)))$ and $X_m = \bigcap_{k_1 + k_2 = m, \ k_1 \leq k_2} L^2_x(\Omega; C^{k_1}([0, T]; H^{k_2}(U)))$. Consider the Cauchy problem for the following linear SPDE...
of order $m$:

$$
\begin{aligned}
\frac{1}{t} d D_t^{m-1} u &= \sum_{k=0}^{m-1} \sum_{|\alpha| = m-k} a_{\alpha}(t, \omega, x) D_{x}^{\alpha} u dt \\
&\quad + \sum_{|\beta| < m} \left[ b_{\beta}(t, \omega, x) D_{t,x}^{\beta} u dt + c_{\beta}(t, \omega, x) D_{t,x}^{\beta} w(t) \right] \
&\quad \text{in } (0, T) \times \Omega \times U,
\end{aligned}
$$

(5.1)

where $D_t = \frac{1}{t} \partial_t$, $D_{x_k} = \frac{1}{t} \partial_{x_k}$, $\alpha$ and $\beta$ denote two multi-indices, and $a_{\alpha}, b_{\beta}, c_{\beta} \in X_{\infty}$.

Write $p_m(t, \omega, x, \lambda, \xi) = \lambda^m - \sum_{k=0}^{m-1} \sum_{|\alpha| = m-k} a_{\alpha}(t, \omega, x) \xi^\alpha \lambda^k$ and denote by $\{\lambda_k(t, \omega, x, \xi); k = 1, \cdots, m\}$ the characteristic roots of $p_m(t, \omega, x, \lambda, \xi)$ for a.e. $(t, \omega) \in (0, T) \times \Omega$ and any $(x, \xi) \in U \times \mathbb{R}^n$, i.e., $p_m(t, \omega, x, \lambda_k(t, \omega, x, \xi), \xi) = 0$. Also, for a.e. $(t, \omega) \in (0, T) \times \Omega$ and any $(x, \xi) \in U \times \mathbb{R}^n$ satisfying $|\xi| = 1$, we introduce the following hypotheses:

(H1) all real roots are simple and the multiplicity of all complex roots is at most two;

(H1') all roots $\lambda_k(t, \omega, x, \xi)$ $(k = 1, \cdots, m)$ are simple;

(H2) there exists a positive constant $\varepsilon$, (which is independent of $t, \omega$, $x$ and $\xi$) such that for any complex root $\lambda_k(t, \omega, x, \xi)$, $|\Im \lambda_k(t, \omega, x, \xi)| \geq \varepsilon$;

(H3) If a root $\lambda_k(t, \omega, x, \xi)$ is real (complex) at a point, it remains real (complex) at every point;

(H4) The algebraic multiplicity of all complex roots is constant with respect to every variable, and the geometric multiplicity of all complex roots is constant with respect to $\omega$.

Then, the main results in this section are stated as follows. The first one is a uniqueness result for equation (5.1) in the case of single characteristics.

**Theorem 5.1** Suppose that the hypotheses (H1'), (H2) and (H3) hold. If $u \in X_m$ is a strong solution of equation (5.1). Then there exist a neighborhood $V (\subset U)$ of the origin in $\mathbb{R}^n$ and a sufficiently small $T' > 0$ such that $u$ vanishes in $(0, T') \times \Omega \times V$.

The other one generalizes the result of Theorem 5.1 to the case of at most double characteristics.

**Theorem 5.2** Suppose that the hypotheses (H1), (H2), (H3) and (H4) hold. If $u \in X_m$ is a strong solution of equation (5.1). Then there exist a neighborhood $V (\subset U)$ of the origin in $\mathbb{R}^n$ and a sufficiently small $T' > 0$ such that $u$ vanishes in $(0, T') \times \Omega \times V$.

We shall give the proofs of Theorem 5.1 and Theorem 5.2 in the next two subsections respectively.
5.2 Proof of Theorem 5.1

In order to get a Calderón-type uniqueness result, we first point out that it suffices to establish a suitable estimate for a strong solution of a SPDE of order \( m \). Next, by introducing a pseudo-differential operator, we reduce the desired estimate to a new Carleman-type estimate for a SPDE of order \((1, \infty)\). Finally, we give the proof of the Carleman estimate (see Lemma 5.1), based on the theory of SPDOs mentioned in Sections 2-4.

To begin with, we introduce a smooth real function \( \zeta \) satisfying that \( \zeta(t) = 1 \) in \([0, 2T/3]\) and \( \zeta(t) = 0 \) in \([T, \infty)\). Write \( B_r = \{ x \in \mathbb{R}^n; |x| < r \} \). Then, we have the following result.

**Proposition 5.1** Under the assumptions \((H1')\), \((H2)\) and \((H3)\), if \( u \in \mathcal{X}_m \) is a strong solution of equation (5.1) satisfying that \( \text{supp}\, u(t, \omega, \cdot) \subseteq B_r \) for a.e. \((t, \omega) \in (0, T) \times \Omega\), then there exists a constant \( C \), (which is independent of \( u \) and \( \mu \)) such that for sufficiently small positive constants \( r, T \) and \( \mu^{-1} \),

\[
\mathbb{E} \int_0^T e^{\mu(t-T)^2} \sum_{|\alpha| \leq m} |D^a_{t,x}(\zeta u(t))|_{L^2(B_r)}^2 dt \\
\leq C(T + \mu^{-1}) \mathbb{E} \int_0^T e^{\mu(t-T)^2} \sum_{|\beta| < m} f_\beta(t, \omega, \cdot) D^\beta_{t,x} u(t)|_{L^2(B_r)}^2 dt,
\]

where \( f_\beta \in \mathcal{X}_\infty \) for \( 0 \leq |\beta| < m \) depends only on \( \{a_\alpha\}_{1 \leq |\alpha| \leq m}, \{b_\beta\}_{|\beta| < m} \) and \( \{c_\beta\}_{|\beta| < m} \), and \( f_\beta(t, \omega, x) = 0 \) for \((t, w, x) \in [0, 2T/3] \times \Omega \times U\).

**Remark 5.1** Let us show how to deduce Theorem 5.1 from Proposition 5.1. In fact, let \( u \) be a strong solution of equation (5.1). Without loss of generality, we suppose that \( \text{supp}\, u \subseteq \mathbb{R}^+ \times \Omega \times \mathbb{R}^n \) by the given initial condition. In order to construct a function with compact support with respect to the variable \( x \), we make the following Holmgren transformation:

\[
(t, \omega, x) \to (t', \omega', x'), \quad x' = x, \quad \omega' = \omega, \quad t' = t + \delta|x|^2
\]

where \( \delta \) is a sufficiently small positive constant. Then after the coordinate transformation, all conditions in Theorem 5.1 still hold, and for sufficiently small positive constants \( T \) and \( r \), \( \text{supp}\, u(t, \omega, \cdot) \subseteq B_r \) for a.e. \((t, \omega) \in (0, T) \times \Omega\). Therefore, by (5.2), it follows that

\[
e^{\frac{\mu T^2}{2}} \mathbb{E} \int_0^T |u(t)|^2_{L^2(B_r)} dt \leq \mathbb{E} \int_0^T e^{\mu(t-T)^2} |u(t)|^2_{L^2(B_r)} dt \\
\leq C(T, u, r, f_\beta, m) \mu^{-1} \int_0^T e^{\mu(t-T)^2} dt \leq C(T, u, r, f_\beta, m) \mu^{-1} e^{\frac{\mu T^2}{2}}.
\]

Letting \( \mu \to \infty \) in the above inequality, we obtain that \( u = 0 \) in \((0, T/2) \times \Omega \times B_r\), which implies Theorem 5.1.

In the following, we divide the proof of Proposition 5.1 into four parts.

**Step 1** We transform a SPDE of order \( m \) to a stochastic pseudo-differential system of order 1. Write

\[A_k(t, \omega, x, D) = \sum_{|\alpha| = m-k} a_\alpha(t, \omega, x) D_x^\alpha,\]

and then the symbol \( a_k(t, \omega, x, \xi) \) of \( A_k \) is

\[\sum_{|\alpha| = m-k} a_\alpha(t, \omega, x) \xi^\alpha.\]
Therefore, if we denote by \( u \) a strong solution of equation (5.1) and let
\[
Y = (\Lambda^{m-1}(\zeta u), D_t \Lambda^{m-2}(\zeta u), \ldots, D_t^{m-1}(\zeta u))^\top,
\]
it is easy to see that
\[
\frac{1}{i} dY = AY dt + f dt + F dw(t),
\]
where
\[
A = \begin{pmatrix}
0 & \Lambda & 0 & \cdots & 0 \\
0 & 0 & \Lambda & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \Lambda \\
A_0 \Lambda^{1-m} & A_1 \Lambda^{2-m} & A_2 \Lambda^{3-m} & \cdots & A_{m-1}
\end{pmatrix},
\]
\[
f = (0, 0, \ldots, \sum_{|\beta| < m} f_1^\beta(t,\omega,x) D_{t,x}^\beta u + \sum_{|\beta| < m} b_\beta(t,\omega,x) D_{t,x}^\beta(\zeta u))^\top,
\]
\[
F = (0, 0, \ldots, \sum_{|\beta| < m} f_2^\beta(t,\omega,x) D_{t,x}^\beta u + \sum_{|\beta| < m} c_\beta(t,\omega,x) D_{t,x}^\beta(\zeta u))^\top,
\]
and \( f_1^\beta, f_2^\beta \in X_\infty(|\beta| < m) \) depend only on \( m, \{a_\alpha\}_{1 \leq |\alpha| \leq m}, \{b_\beta\}_{|\beta| < m}, \{c_\beta\}_{|\beta| < m} \) and \( \zeta \). Moreover, \( f_1^\beta(t,w,x) = 0 \) for \((t, w, x) \in [0, 2T/3] \times \Omega \times U\) \((j = 1, 2)\). Furthermore, if we denote by \( A_0 \) a SPDO, with the symbol
\[
\sigma(A_0) = \begin{pmatrix}
0 & |\xi| & 0 & \cdots & 0 \\
0 & 0 & |\xi| & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & |\xi| \\
a_0 |\xi|^{1-m} & a_1 |\xi|^{2-m} & a_2 |\xi|^{3-m} & \cdots & a_{m-1}
\end{pmatrix},
\]
then \( A = A_0 + B \), where \( B \in \mathcal{L}_c^0(U) \).

**Step 2** We make the diagonalization for operator \( A_0 \). By the assumption \((H1')\), for a.e. \((t,\omega) \in (0,T) \times \Omega\) and any \((x,\xi) \in U \times \mathbb{R}^n\) satisfying \(|\xi| = 1\), there exists an invertible matrix \( r^*(t,\omega,x,\xi) \) such that \( j^* = r^* \cdot \sigma(A_0) \cdot r^{*-1} \) is a diagonal matrix. Since \( \sigma(A_0) \) is homogeneous of order 1 with respect to \( \xi \), \( j^* \) is still a diagonal matrix after \( r^* \) is extended homogeneously of order 0 to \( \mathbb{R}^n \setminus \{0\} \) with respect to \( \xi \). Denote by \( R, S \) and \( J \) the SPDOs, whose symbols are \( r^*, r^{*-1} \) and \( j^* \), respectively. Then \( J \in \mathcal{L}_c^0(U) \) is diagonal. Let \( Z = RY \). By the assumption \((H2)\), if an element of the diagonal of \( J \) is \( A_1 + iB_1 \), then either \( B_1 = 0 \) or \( B_1 \) is an elliptic SPDO of order \((1,\infty)\) with a real symbol.

**Step 3** We give a new Carleman-type estimate for a SPDO of order \((1,\infty)\).

**Lemma 5.1** Suppose that \( A_1 \) and \( B_1 \) are two SPDOs of order \((1,\infty)\) and their symbols are real. If \( B_1 = 0 \) or \( B_1 \) is elliptic, and \( z \in L_2^2(\Omega; C([0,T]; H^1(\mathbb{R}^n))) \) is an \( H^1(\mathbb{R}^n) \)-valued semimartingale
satisfying $z(0) = z(T) = 0$ a.s., then for sufficiently small $\mu^{-1}$ and $T$, it holds

$$
\mathbb{E} \int_0^T e^{\mu(t-T)^2} |z|^2_{L^2(R^n)} dt + \frac{1}{\mu} \mathbb{E} \int_0^T e^{\mu(t-T)^2} |\mu(t-T)z - B_1(t)z|_{L^2(R^n)}^2 dt
\leq \frac{4}{\mu} \text{Re} \mathbb{E} \int_0^T \int_{R^n} e^{\mu(t-T)^2} \left[ \frac{1}{i} dz - A_1(t)zdt - iB_1(t)zdt \right] \cdot \left[ i\mu(t-T)z(t) - iB_1(t)z(t) \right] dx
$$

\begin{equation}
-2 \text{Re} \mathbb{E} \int_0^T \int_{R^n} e^{\mu(t-T)^2} \left[ \frac{1}{i} dz - A_1(t)zdt - iB_1(t)zdt \right] \cdot (B_1(t) - B_1^*(t))z(t) dx
\end{equation}

\begin{equation}
-2 \mathbb{E} \int_0^T \int_{R^n} (t-T)e^{\mu(t-T)^2} |z|^2 dx - \frac{2}{\mu} \text{Re} \mathbb{E} \int_0^T e^{\mu(t-T)^2} (dz, B_1(t)(dz))_{L^2(R^n)}.
\end{equation}

where $B_1^*$ denotes the conjugate operator of $B_1$.

**Proof.** Set $\theta = e^{\mu(t-T)^2}$ and $\varphi = \theta z$. Then it is easy to show that

$$
\theta \left[ \frac{1}{i} dz - A_1(t)zdt - iB_1(t)zdt \right] = \frac{1}{i} d\varphi - A_1(t)\varphi dt + i\mu(t-T)\varphi dt - iB_1(t)\varphi dt.
$$

Multiplying both sides of (5.4) by $i\mu(t-T)\varphi - iB_1(t)\varphi$, and integrating on $(0, T) \times \Omega \times \mathbb{R}^n$, we obtain that

$$
\text{Re} \mathbb{E} \int_0^T \int_{\mathbb{R}^n} \theta \left[ \frac{1}{i} dz - A_1(t)zdt - iB_1(t)zdt \right] \cdot i\mu(t-T)\varphi - iB_1(t)\varphi dx
$$

\begin{equation}
= \text{Re} \mathbb{E} \int_0^T \int_{\mathbb{R}^n} \left[ \mu(t-T)\varphi - B_1(t)\varphi \right]^2 dt - \mu(t-T)\varphi d\varphi
$$

\begin{equation}
+i\mu(t-T)A_1(t)\varphi \cdot \varphi dt + B_1(t)\varphi d\varphi - iA_1(t)\varphi \cdot B_1(t)\varphi d\varphi \right] dx.
\end{equation}

We estimate one by one the last four terms in (5.5). First, by Itô’s formula, it follows that

\begin{equation}
-\mu \text{Re} \mathbb{E} \int_0^T \int_{\mathbb{R}^n} (t-T)\varphi d\varphi dx = \mu \frac{2}{2i} \mathbb{E} \left[ \int_0^T |\varphi|^2_{L^2(R^n)} dt + \int_0^T \int_{\mathbb{R}^n} (t-T)\theta^2 |dz|^2 dx \right].
\end{equation}

Also, by the $L^2$-estimates of SPDOs, we see that

\begin{equation}
\text{Re} \mathbb{E} \int_0^T \int_{\mathbb{R}^n} i\mu(t-T)A_1(t)\varphi \cdot \varphi dt = \mathbb{E} \int_0^T \frac{\mu(t-T)}{2i} ((A_1(t) - A_1^*(t))\varphi, \varphi) dt
$$

\begin{equation}
\geq -C\mu T \mathbb{E} \int_0^T |\varphi|^2_{L^2(R^n)} dt,
\end{equation}

here and hereafter, we denote by $(\cdot, \cdot)$ the inner product in $L^2(R^n)$. Next, we notice that

\begin{equation}
\text{Re} \mathbb{E} \int_0^T \int_{\mathbb{R}^n} B_1(t)\varphi d\varphi dx
$$

\begin{equation}
= \frac{1}{2} \text{Re} \mathbb{E} \int_0^T \int_{\mathbb{R}^n} ((d\varphi, B_1(t)\varphi) - (\varphi, B_1(t)\varphi)dt - (\varphi, B_1(t)(d\varphi)) - (d\varphi, B_1(t)(d\varphi))]
$$

\begin{equation}
= -\frac{1}{2} \text{Re} \mathbb{E} \int_0^T (\varphi, B_1(t)\varphi) dt
$$

\begin{equation}
+ \frac{1}{2} \text{Re} \mathbb{E} \int_0^T (d\varphi, (B_1(t) - B_1^*(t))\varphi) - \frac{1}{2} \text{Re} \mathbb{E} \int_0^T (d\varphi, B_1(t)(d\varphi)).
\end{equation}

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Therefore, for sufficiently small $\varepsilon > 0$, by Hölder’s inequality and the $L^2$-boundedness of SPDOs, it follows that
\[
\text{Re} \ E \int_0^T \int_{\mathbb{R}^n} \overline{B_1(t)} \phi \overline{\phi} \, dx \, dt - \frac{1}{2} \text{Im} \ E \int_0^T \| \phi \|^2_{L^2(\mathbb{R}^n)} \, dt - \frac{1}{2} \text{Re} \ E \int_0^T (d\phi, B_1(t)(d\phi)) = -C E \int_0^T \| \phi(t) \|^2_{L^2(\mathbb{R}^n)} \, dt
\]

Moreover, for sufficiently small $\varepsilon > 0$, it holds
\[
-C E \int_0^T \| \phi(t) \|^2_{L^2(\mathbb{R}^n)} \, dt - C(\varepsilon) E \int_0^T |\phi|^2_{L^2(\mathbb{R}^n)} \, dt - \frac{1}{2} \text{Re} \ E \int_0^T (d\phi, B_1(t)(d\phi))
\]

By (5.5)–(5.9), we end up with
\[
\begin{align*}
\text{Re} \ E \int_0^T \int_{\mathbb{R}^n} \theta \left[ \frac{1}{i} d\phi - A_1(t) d\phi \right] \overline{\phi} \, dx \\
\text{Im} \ E \int_0^T \| \phi \|^2_{L^2(\mathbb{R}^n)} \, dt - \frac{1}{2} \text{Re} \ E \int_0^T \| \phi \|^2_{L^2(\mathbb{R}^n)} \, dt - C(\varepsilon) E \int_0^T |\phi|^2_{L^2(\mathbb{R}^n)} \, dt
\end{align*}
\]

(5.10)
If $B_1$ is an elliptic SPDO, by Theorem 4.1, it follows that

$$
\mathbb{E} \int_0^T |\varphi|_{H^1(\mathbb{R}^n)}^2 dt \leq C \mathbb{E} \int_0^T |B_1(t)\varphi|_{L^2(\mathbb{R}^n)}^2 dt + C \mathbb{E} \int_0^T |\varphi|_{L^2(\mathbb{R}^n)}^2 dt
$$

(5.11)

$$
\leq C \mathbb{E} \int_0^T |\mu(t-T)\varphi - B_1(t)\varphi|_{L^2(\mathbb{R}^n)}^2 dt + C(1 + T\mu) \mathbb{E} \int_0^T |\varphi|_{L^2(\mathbb{R}^n)}^2 dt.
$$

By (5.10) and (5.11), if we take $\mu^{-1}$ and $T$ sufficiently small, then

$$
\Re \mathbb{E} \int_0^T \int_{\mathbb{R}^n} \theta \left[ \frac{1}{i} d\xi - A_1(t)z dt - iB_1(t)z dt \right] \overline{[\mu(t-T)\varphi - iB_1(t)\varphi]} dx
$$

$$
\geq \frac{\mu}{4} \mathbb{E} \int_0^T |\varphi|_{L^2(\mathbb{R}^n)}^2 dt + \frac{1}{4} \mathbb{E} \int_0^T \int_{\mathbb{R}^n} |\mu(t-T)\varphi - B_1(t)\varphi|^2 dx dt
$$

$$
+ \frac{1}{2} \Im \mathbb{E} \int_0^T \theta^2 \left( \frac{1}{i} d\xi - A_1(t)z dt - iB_1(t)z dt, (B_1(t) - B_1^*(t))(\xi) \right)
$$

$$
+ \frac{\mu}{2} \mathbb{E} \int_0^T \int_{\mathbb{R}^n} (t-T)\theta^2|\varphi|^2 dx - \frac{1}{2} \Re \mathbb{E} \int_0^T \theta^2(d\xi, B_1(t)(d\xi)),
$$

which implies the result of Lemma 5.1.

On the other hand, the case of $B_1 = 0$ can be treated in the same way.

By Lemma 5.1, (5.3) holds for every component of $Z$. Therefore, we obtain the following result.

**Proposition 5.2** For sufficiently small $\mu^{-1}$ and $T$, the following inequality holds:

$$
\mathbb{E} \int_0^T e^{\mu(t-T)^2} |Z(t)|_{L^2(\mathbb{R}^n)}^2 dt + \frac{1}{\mu} \mathbb{E} \int_0^T e^{\mu(t-T)^2} |\mu(t-T)Z(t) - B_1(t)Z(t)|_{L^2(\mathbb{R}^n)}^2 dt
$$

$$
\leq \frac{4}{\mu} \Re \mathbb{E} \int_0^T \int_{\mathbb{R}^n} e^{\mu(t-T)^2} \left( \frac{1}{i} dZ - J(t)Z(t) dt \right) \cdot \overline{[\mu(t-T)Z(t) - iB_1(t)Z(t)]} dx
$$

$$
- \frac{2}{\mu} \Im \mathbb{E} \int_0^T \int_{\mathbb{R}^n} e^{\mu(t-T)^2} \left( \frac{1}{i} dZ - J(t)Z(t) dt \right) \cdot (B_1(t) - B_1^*(t))Z(t) dx
$$

$$
- 2\mathbb{E} \int_0^T \int_{\mathbb{R}^n} (t-T)e^{\mu(t-T)^2} |dZ|^2 dx - \frac{2}{\mu} \Re \mathbb{E} \int_0^T e^{\mu(t-T)^2} (dZ, B_1(t)(dZ)).
$$

**Step 4** By the result of Proposition 5.2, we give an estimate for the vector $Y$. First of all, we notice that there exist a constant $C > 0$ and a SPDO $T_0^* \in L^0_{\infty}(U)$, such that

$$
\left\{
\begin{array}{l}
\mathbb{E} \int_0^T \theta^2|Y(t)|_{L^2(\mathbb{R}^n)}^2 dt \leq C \mathbb{E} \int_0^T \theta^2(|Z(t)|_{L^2(\mathbb{R}^n)}^2 + |Y(t)|_{H^{-1}(\mathbb{R}^n)}) dt; \\
R \left( \frac{1}{i} dY(t) - AY(t) dt \right) = \frac{1}{i} dZ(t) - JZ(t) dt + T_0^*Y dt.
\end{array}
\right.
$$

(5.14)

In fact, by $I_{m \times m} = r^* \cdot r^{-1}$, there exists a SPDO $T_{-1}^* \in L^{-1}_{\infty}(U)$ such that $I = SR + T_{-1}^*$. Hence,

$$
Y = SRY + T_{-1}^*Y = SZ + T_{-1}^*Y.
$$

By the $L^2$-boundedness of SPDOs, the first inequality of (5.14) holds. On the other hand,

$$
R \left( \frac{1}{i} dY - AY dt \right) = \frac{1}{i} dZ + \frac{1}{i} R_Y Y dt - RA(SZ + T_{-1}^*Y) dt
$$

$$
= \frac{1}{i} dZ - JZ dt + JZ dt + \frac{1}{i} R_Y Y dt - RA(SZ + T_{-1}^*Y) dt
$$

$$
= \frac{1}{i} dZ - JZ dt + (D_Y - RAT_{-1}^* + JR - RASR) Y dt,
$$

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where $T_0^* = D, R - RAT_{-1} + JR - RASR$ is a SPDO of order $(0, \infty)$.

Next, by (5.14), Hölder’s inequality and Proposition 5.2, we get that

\[
\mathbb{E} \int_0^T e^{\mu(t-T)^2} |Y|^2_{L^2(\mathbb{R}^n)} dt + \frac{1}{\mu} \mathbb{E} \int_0^T e^{\mu(t-T)^2} |\mu(t-T)RY - B_1(t)RY|^2_{L^2(\mathbb{R}^n)} dt
\]

\[
\leq \frac{4}{\mu} \text{Re} \mathbb{E} \int_0^T \int_{\mathbb{R}^n} e^{\mu(t-T)^2} \left[ R \left( \frac{1}{i}dY - AY dt \right) - T_0^* Y dt \right] \cdot [i\mu(t-T)RY - iB_1(t)RY] dx
\]

\[
- \frac{2}{\mu} \text{Im} \mathbb{E} \int_0^T \int_{\mathbb{R}^n} e^{\mu(t-T)^2} \left[ R \left( \frac{1}{i}dY - AY dt \right) - T_0^* Y dt \right] \cdot (B_1(t) - B_1^*(t))RY dx
\]

\[
+ \text{C} \mathbb{E} \int_0^T e^{\mu(t-T)^2} |Y|^2_{H^{-1}(\mathbb{R}^n)} dt
\]

\[
- 2\mathbb{E} \int_0^T \int_{\mathbb{R}^n} (t - T)\theta^2 |RdY|^2 dx - \frac{2}{\mu} \text{Re} \mathbb{E} \int_0^T \theta^2 (RdY, B_1(t)(RdY))
\]

\[
\leq \frac{4}{\mu} \text{Re} \mathbb{E} \int_0^T \int_{\mathbb{R}^n} e^{\mu(t-T)^2} R \left( \frac{1}{i}dY - AY dt \right) \cdot [i\mu(t-T)RY - iB_1(t)RY] dx
\]

\[
- \frac{2}{\mu} \text{Im} \mathbb{E} \int_0^T \int_{\mathbb{R}^n} e^{\mu(t-T)^2} R \left( \frac{1}{i}dY - AY dt \right) \cdot (B_1(t) - B_1^*(t))RY dx
\]

\[
+ \frac{1}{2\mu} \mathbb{E} \int_0^T e^{\mu(t-T)^2} |\mu(t-T)RY - B_1(t)RY|^2_{L^2(\mathbb{R}^n)} dt
\]

\[
+ \frac{C}{\mu} \mathbb{E} \int_0^T e^{\mu(t-T)^2} |Y|^2_{L^2(\mathbb{R}^n)} dt + \text{C} \mathbb{E} \int_0^T e^{\mu(t-T)^2} |Y|^2_{H^{-1}(\mathbb{R}^n)} dt
\]

\[
+ \text{C} \mathbb{E} \int_0^T \int_{\mathbb{R}^n} \theta^2 |RF|^2 dx dt - \frac{2}{\mu} \text{Re} \mathbb{E} \int_0^T \theta^2 (RF, B_1(t)RF) dt
\]

Since $B_1$ is an elliptic SPDO and its symbol is real, by Theorem 4.2, we see that

\[
- \frac{2}{\mu} \text{Re} \mathbb{E} \int_0^T \theta^2 (RF, B_1(t)RF) dt \leq \frac{C}{\mu} \mathbb{E} \int_0^T \theta^2 |F|^2_{L^2(\mathbb{R}^n)} dt.
\]

It follows that

\[
\mathbb{E} \int_0^T e^{\mu(t-T)^2} |Y|^2_{L^2(\mathbb{R}^n)} dt + \frac{1}{\mu} \mathbb{E} \int_0^T e^{\mu(t-T)^2} |\mu(t-T)RY - B_1(t)RY|^2_{L^2(\mathbb{R}^n)} dt
\]

\[
\leq \frac{4}{\mu} \text{Re} \mathbb{E} \int_0^T \int_{\mathbb{R}^n} e^{\mu(t-T)^2} R f \cdot [i\mu(t-T)RY - iB_1(t)RY] dtdx
\]

\[
- \frac{2}{\mu} \text{Im} \mathbb{E} \int_0^T \int_{\mathbb{R}^n} e^{\mu(t-T)^2} R f \cdot (B_1(t) - B_1^*(t))RY dtdx
\]

\[
+ \frac{1}{2\mu} \mathbb{E} \int_0^T e^{\mu(t-T)^2} |\mu(t-T)RY - B_1(t)RY|^2_{L^2(\mathbb{R}^n)} dt
\]

\[
+ \frac{C}{\mu} \mathbb{E} \int_0^T e^{\mu(t-T)^2} |Y|^2_{L^2(\mathbb{R}^n)} dt + \text{C} \mathbb{E} \int_0^T e^{\mu(t-T)^2} |Y|^2_{H^{-1}(\mathbb{R}^n)} dt
\]

\[
+ \text{C} (T + \frac{1}{\mu}) \mathbb{E} \int_0^T e^{\mu(t-T)^2} |F|^2_{L^2(\mathbb{R}^n)} dt.
\]
Therefore, by Hölder’s inequality, we have that

\[
E \int_0^T e^{\mu(t-T)^2} |Y|_{L^2(\mathbb{R}^n)}^2 dt + \frac{1}{\mu} E \int_0^T e^{\mu(t-T)^2} |\mu(t-T) RY - B_1(t) RY|_{L^2(\mathbb{R}^n)}^2 dt
\]

\[
\leq \frac{3}{4\mu} E \int_0^T e^{\mu(t-T)^2} |i\mu(t-T) RY - iB_1(t) RY|_{L^2(\mathbb{R}^n)}^2 dt + \frac{C}{\mu} E \int_0^T e^{\mu(t-T)^2} |f|_{L^2(\mathbb{R}^n)}^2 dt
\]

\[
+ C(T + \frac{1}{\mu}) E \int_0^T e^{\mu(t-T)^2} |F|_{L^2(\mathbb{R}^n)}^2 dt.
\]

Then, for sufficiently small \( r, T \) and \( \mu^{-1} \), we obtain that

\[
E \int_0^T e^{\mu(t-T)^2} |Y|_{L^2(\mathbb{R}^n)}^2 dt + \frac{1}{\mu} E \int_0^T e^{\mu(t-T)^2} |\mu(t-T) RY - B_1(t) RY|_{L^2(\mathbb{R}^n)}^2 dt
\]

\[
\leq \frac{C}{\mu} E \int_0^T e^{\mu(t-T)^2} |f|_{L^2(\mathbb{R}^n)}^2 dt + C(T + \frac{1}{\mu}) E \int_0^T e^{\mu(t-T)^2} |F|_{L^2(\mathbb{R}^n)}^2 dt.
\]

(5.16)

By the definition of \( Y \), \( \sum_{|\alpha| < m} |(D^{\alpha}_{t,x}(\zeta u))(t)|_{L^2(\mathbb{R}^n)}^2 \leq C |(Y(t))_{L^2(\mathbb{R}^n)}. \) Notice that (5.16) implies (5.2).

### 5.3 Proof of Theorem 5.2

The idea of the proof of Theorem 5.2 is similar to that used in Theorem 5.1. We only need to prove Proposition 5.1 for a strong solution \( u \) of equation (5.1). Different from Theorem 5.1, since the multiplicity of a complex root may be two, the matrix which makes the principal symbol \( \sigma(A_0) \) become a Jordan canonical form is only defined locally. Therefore, by a localization technique, we get the desired estimate. In the following, we give a sketch of the proof of Theorem 5.2.

**Step 1** First, under the conditions (H1), (H3) and (H4), there exist a finite covering \( \{G_k\}_{k=1}^{N_1} \) \((N_1 \in \mathbb{N})\) of \( \{ \xi \in \mathbb{R}^n; |\xi| = 1 \} \), and sufficiently small \( r \) and \( T \) such that

\[
r_k^* \cdot \sigma(A_0) \cdot r_k^{-1} = j_k^*, \quad \text{for a.e.} \ (t, \omega) \in (0, T) \times \Omega \text{ and any } (x, \xi) \in B_r \times G_k,
\]

where \( r_k^*, r_k^{-1} \in \bigcap_{j \in \mathbb{N}} L^\infty_j(0, T; C^1(B_r \times G_k)) \) and \( j_k^* \) is a Jordan canonical form of \( \sigma(A_0) \). Denote by \( \{ \varphi_k\}_{k=1}^{N_1} \) a finite number of smooth functions such that \( \{ \varphi_k\}_{k=1}^{N_1} \) is the partition of unity for \( \{G_k\}_{k=1}^{N_1} \) and \( \sum_{k=1}^{N_1} \varphi_k^2 = 1 \) on \( \{ \xi \in \mathbb{R}^n; |\xi| = 1 \} \). After \( \varphi_k \) is extended homogeneously of order 0 to \( \mathbb{R}^n \setminus \{0\} \) with respect to \( \xi \), we denote by \( \Phi_k \) the associated pseudo-differential operator determined by symbol \( \varphi_k \) and write \( Y_k = \Phi_k Y \). Also, we choose smooth functions \( \tilde{\psi}_k : \{ \xi \in \mathbb{R}^n; |\xi| = 1 \} \rightarrow G_k \) satisfying that \( \tilde{\psi}_k(\xi) = \xi \) on supp \( \varphi_k \) and set

\[
a_k^0(t, \omega, x, \xi) = \sigma(A_0)(t, \omega, x, \tilde{\psi}_k(\xi)), \quad r_k(t, \omega, x, \xi) = r_k^*(t, \omega, x, \tilde{\psi}_k(\xi)).
\]

Then, after we extend \( a_k^0 \) (resp. \( r_k \) and \( r_k^{-1} \)) homogeneously of order 1 (resp. order 0) with respect to \( \xi \), we get that \( r_k \cdot a_k^0 \cdot r_k^{-1} = j_k \) for a.e. \( (t, \omega) \in (0, T) \times \Omega \) and any \( (x, \xi) \in B_r \times \mathbb{R}^n \setminus \{0\}, \)
where $j_k$ is a Jordan canonical form of $a_k^0$. Denote by $R_k$, $A_k$, $S_k$ and $J_k$ the SPDOs determined by $r_k, a_k, r_k^{-1}$ and $j_k$, respectively.

**Step 2** Set $Z_k = R_k Y_k$. If the geometric multiplicity and the algebraic multiplicity for every root equal, then $j_k$ is a diagonal matrix and we can derive the desired result in the same way as that in the last subsection. Otherwise, by Lemma 5.1 and invertibility of elliptic SPDOs, it is easy to see that the following result holds.

**Lemma 5.2** *Suppose that $A_1$ and $B_1$ are two SPDOs of order $(1, \infty)$ and their symbols are real. If $B_1 = 0$ or $B_1$ is elliptic, and $z_1, z_2 \in L^2_z(\Omega; C([0, T]; H^1(\mathbb{R}^n)))$ is an $H^1(\mathbb{R}^n)$-valued semimartingale satisfying $z_j(0) = z_j(T) = 0$ ($j = 1, 2$) a.s., then there exists a constant $C = C(B_1, n)$ such that for sufficiently small $\mu^{-1}$ and $T$, it holds*

\[
\begin{align*}
\mathbb{E} \int_0^T e^{\mu(t-T)^2} |x_1|^2_{L^2(\mathbb{R}^n)} dt + \left(1 + \mathbb{E} \int_0^T e^{\mu(t-T)^2} \right) |x_1|^2_{L^2(\mathbb{R}^n)} dt \\
+ \mathbb{E} \int_0^T e^{\mu(t-T)^2} |x_2|^2_{L^2(\mathbb{R}^n)} dt + \left(1 + \mathbb{E} \int_0^T e^{\mu(t-T)^2} \right) |x_2|^2_{L^2(\mathbb{R}^n)} dt \\
\leq \frac{8}{\mu} \text{Re} \mathbb{E} \int_0^T \int_{\mathbb{R}^n} e^{\mu(t-T)^2} \left[ \frac{1}{i} d z_1 - A_1(t) z_1 dt - i B_1(t) z_1 dt + \Lambda z_2 dt \right] \cdot \left( 3 |\mu(t-T)| z_1 - i B_1(t) z_1 \right) dx \\
- \frac{4}{\mu} \text{Im} \mathbb{E} \int_0^T \int_{\mathbb{R}^n} e^{\mu(t-T)^2} \left[ \frac{1}{i} d z_2 - A_1(t) z_2 dt - i B_1(t) z_2 dt + \Lambda z_2 dt \right] \cdot \left( B_1(t) - B_1^*(t) \right) z_1 dx \\
- \frac{4}{\mu} \text{Re} \mathbb{E} \int_0^T \int_{\mathbb{R}^n} (t-T) e^{\mu(t-T)^2} |z_1|^2 dx - \frac{4}{\mu} \text{Re} \mathbb{E} \int_0^T e^{\mu(t-T)^2} (z_1, B_1(t)(z_1))_{L^2(\mathbb{R}^n)} \\
+ \frac{8C}{\mu} \text{Re} \mathbb{E} \int_0^T \int_{\mathbb{R}^n} e^{\mu(t-T)^2} \left[ \frac{1}{i} d z_2 - A_1(t) z_2 dt - i B_1(t) z_2 dt \right] \cdot \left( i |\mu(t-T)| z_2 - i B_1(t) z_2 \right) dx \\
- \frac{4C}{\mu} \text{Im} \mathbb{E} \int_0^T \int_{\mathbb{R}^n} e^{\mu(t-T)^2} \left[ \frac{1}{i} d z_2 - A_1(t) z_2 dt - i B_1(t) z_2 dt \right] \cdot \left( B_1(t) - B_1^*(t) \right) z_2 dx \\
- 4C \mathbb{E} \int_0^T \int_{\mathbb{R}^n} (t-T) e^{\mu(t-T)^2} |z_2|^2 dx - \frac{4C}{\mu} \text{Re} \mathbb{E} \int_0^T e^{\mu(t-T)^2} (z_2, B_1(t)(z_2))_{L^2(\mathbb{R}^n)}. 
\end{align*}
\]

**Sketch of the proof.** By Lemma 5.1, it remains to estimate the following two terms:

\[
\mathcal{I} = -\frac{4}{\mu} \text{Re} \mathbb{E} \int_0^T \int_{\mathbb{R}^n} e^{\mu(t-T)^2} \Lambda z_2 dt \cdot \left( i |\mu(t-T)| z_1 - i B_1(t) z_1 \right) dx \\
+ \frac{2}{\mu} \text{Im} \mathbb{E} \int_0^T \int_{\mathbb{R}^n} e^{\mu(t-T)^2} \Lambda z_2 dt \cdot \left( B_1(t) - B_1^*(t) \right) z_1 dx.
\]

It is easy to check that

\[
\mathcal{I} \leq \frac{C}{\mu} \mathbb{E} \int_0^T \theta^2 \left( |z_1|_{H^1(\mathbb{R}^n)} |\mu(t-T)| z_1 - B_1(t) z_1 \right)_{L^2(\mathbb{R}^n)} + |z_2|_{H^1(\mathbb{R}^n)} |z_1|_{L^2(\mathbb{R}^n)} \right) dt.
\]

Also, similar to (5.11), we see that

\[
\mathbb{E} \int_0^T \theta^2 |z_2|^2_{H^1(\mathbb{R}^n)} dt \leq C \mathbb{E} \int_0^T \theta^2 \left[ |\mu(t-T)| z_2 - B_1(t) z_2 \right]_{L^2(\mathbb{R}^n)} + (1 + T\mu) |z_2|^2_{L^2(\mathbb{R}^n)} \right) dt.
\]
It follows that

\[ I \leq \frac{1}{2\mu} \mathbb{E} \int_0^T \theta^2 |\mu(t-T)z_1 - B_1(t)z_1|^2_{L^2({\mathbb R}^n)} dt + \frac{C}{\mu} \mathbb{E} \int_0^T \theta^2 |\mu(t-T)z_2 - B_1(t)z_2|^2_{L^2({\mathbb R}^n)} dt \]
\[ + \frac{1}{2\mu} \mathbb{E} \int_0^T \theta^2 |\mu(t-T)z_2 - B_1(t)z_2|^2_{L^2({\mathbb R}^n)} dt + \frac{C}{\mu} (1 + T\mu) \mathbb{E} \int_0^T \theta^2 |z_2|^2_{L^2({\mathbb R}^n)} dt \]
\[ + \frac{C}{\mu} \mathbb{E} \int_0^T \theta^2 |z_1|^2_{L^2({\mathbb R}^n)} dt + \frac{C}{\mu} (1 + T\mu) \mathbb{E} \int_0^T \theta^2 (|z_1|^2_{L^2({\mathbb R}^n)} + |z_2|^2_{L^2({\mathbb R}^n)}) dt. \] (5.17)

Applying the result of Lemma 5.1 to \( z_1 \) and \( z_2 \), respectively, by (5.17), we obtain the desired result.

Lemma 5.2 implies the following result.

**Proposition 5.3** There exists a constant \( C > 0 \), such that for sufficiently small \( \mu^{-1} \) and \( T \),

\[ \mathbb{E} \int_0^T e^{\mu(t-T)^2} |Z_k(t)|^2_{L^2({\mathbb R}^n)} dt + \frac{1}{\mu} \mathbb{E} \int_0^T e^{\mu(t-T)^2} |\mu(t-T)Z_k(t) - B_1(t)Z_k(t)|^2_{L^2({\mathbb R}^n)} dt \]
\[ \leq \frac{C}{\mu} \left| \mathbb{E} \int_0^T \int_{\mathbb R^n} e^{\mu(t-T)^2} \left( \frac{1}{i} dZ_k - J_k(t)Z_k(t) dt \right) \cdot \frac{|i\mu(t-T)Z_k(t) - iB_1(t)Z_k(t)|}{dx} \right| \] (5.18)
\[ + \frac{C}{\mu} \left| \mathbb{E} \int_0^T \int_{\mathbb R^n} e^{\mu(t-T)^2} \left( \frac{1}{i} dZ_k - J_k(t)Z_k(t) dt \right) \cdot \frac{(B_1(t) - B_1^*(t))Z_k(t)}{dx} \right| \]
\[ - C \mathbb{E} \int_0^T \int_{\mathbb R^n} (t-T)e^{\mu(t-T)^2} |dZ_k|^2 dx - \frac{C}{\mu} \text{Re} \mathbb{E} \int_0^T e^{\mu(t-T)^2} (dZ_k, B_1(t)(dZ_k)). \]

**Step 3.** Finally, we give an estimate of \( Y \), which implies Proposition 5.1. Similar to (5.14), we can show that there exist a constant \( C > 0 \) and a SPDO \( T_{0k}^* \in \mathcal{L}^0_\infty(U) \), such that

\[ \begin{cases} 
\mathbb{E} \int_0^T \theta^2 |Y_k(t)|^2_{L^2({\mathbb R}^n)} dt \leq C \mathbb{E} \int_0^T \theta^2 (|Z_k(t)|^2_{L^2({\mathbb R}^n)} + |Y_k(t)|_{H^{-1}({\mathbb R}^n)}^2) dt; \\
R_k \left( \frac{1}{i} dY_k(t) - AY_k(t) dt \right) = \frac{1}{i} dZ_k(t) - J_k Z_k(t) dt + T_{0k}^* Y_k(t). 
\end{cases} \]

Therefore, similar to (5.15), we see that

\[ \mathbb{E} \int_0^T e^{\mu(t-T)^2} |Y_k|^2_{L^2({\mathbb R}^n)} dt + \frac{1}{2\mu} \mathbb{E} \int_0^T e^{\mu(t-T)^2} |\mu(t-T)R_k Y_k - B_1(t) R_k Y_k|^2_{L^2({\mathbb R}^n)} dt \]
\[ \leq \frac{C}{\mu} \left| \mathbb{E} \int_0^T \int_{\mathbb R^n} e^{\mu(t-T)^2} R_k \left( \frac{1}{i} dY_k - AY_k dt \right) \cdot \frac{|i\mu(t-T)R_k Y_k - iB_1(t) R_k Y_k|}{dx} \right| \]
\[ + \frac{C}{\mu} \left| \mathbb{E} \int_0^T \int_{\mathbb R^n} e^{\mu(t-T)^2} R_k \left( \frac{1}{i} dY_k - AY_k dt \right) \cdot \frac{(B_1(t) - B_1^*) R_k Y_k}{dx} \right| \]
\[ + \frac{C}{\mu} \mathbb{E} \int_0^T \int_{\mathbb R^n} e^{\mu(t-T)^2} |Y_k|^2_{L^2({\mathbb R}^n)} dt + C \mathbb{E} \int_0^T e^{\mu(t-T)^2} |Y_k|^2_{H^{-1}({\mathbb R}^n)} dt \]
\[ - \frac{C}{\mu} \mathbb{E} \int_0^T \int_{\mathbb R^n} e^{\mu(t-T)^2} (t-T)|R_k dY_k|^2 dx - \frac{C}{\mu} \text{Re} \mathbb{E} \int_0^T \theta^2 (R_k dY_k, B_1(t) R_k dY_k). \]

By the definition of \( Y_k \), it is easy to show that \( |Y|_{H^s({\mathbb R}^n)} = \sum_{k=1}^{N_1} |Y_k|_{H^s({\mathbb R}^n)} \) for any \( s \in \mathbb{R} \) and
\( A\Phi_k Y - \Phi_k AY = T_{0k} Y \), where \( T_{0k} \in \mathcal{L}^0_\infty(U) \). Hence, it follows that

\[
\mathbb{E} \int_0^T e^{\mu(t-T)^2} |Y|_{L^2(R^n)}^2 \, dt \\
\leq C \mathbb{E} \int_0^T e^{\mu(t-T)^2} |f|_{L^2(R^n)}^2 \, dt + \frac{C}{\mu} \mathbb{E} \int_0^T e^{\mu(t-T)^2} |Y|_{L^2(R^n)}^2 \, dt + C\nu^2 \mathbb{E} \int_0^T e^{\mu(t-T)^2} |Y|_{L^2(R^n)}^2 \, dt \\
+C(T + \frac{1}{\mu}) \mathbb{E} \int_0^T e^{\mu(t-T)^2} |F|_{L^2(R^n)}^2 \, dt.
\]

If we take \( r \) and \( \frac{1}{\mu} \) sufficiently small, then the above inequality implies Proposition 5.1.

**Remark 5.2** In [14], for deterministic partial differential equations of order \( m \), principal symbols are required to have \( C^\infty \) coefficients. However, by the proofs of Theorem 5.1 and Theorem 5.2, we notice that the coefficients of equation (5.1) should only belong to the space \( X_\infty \). This is because we actually consider \( t \) and \( \omega \) as two parameters. In fact, we find that in the deterministic case, all pseudo-differential operators appeared in the proof of uniqueness theorem may regard the variable \( t \) as a parameter. Therefore, in the classical Calderón uniqueness theorem, it is sufficient that the coefficients of principal symbols are required to be \( C^1 \) with respect to \( t \).

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