POLYNOMIAL TREewidth forces
A LARGE GRID-LIKE-MINOR

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Abstract. Robertson and Seymour proved that every graph with sufficiently large treewidth contains a large grid minor. However, the best known bound on the treewidth that forces an $\ell \times \ell$ grid minor is exponential in $\ell$. It is unknown whether polynomial treewidth suffices. We prove a result in this direction. A grid-like-minor of order $\ell$ in a graph $G$ is a set of paths in $G$ whose intersection graph is bipartite and contains a $K_\ell$-minor. For example, the rows and columns of the $\ell \times \ell$ grid are a grid-like-minor of order $\ell + 1$. We prove that polynomial treewidth forces a large grid-like-minor. In particular, every graph with treewidth at least $c\ell^4\sqrt{\log \ell}$ has a grid-like-minor of order $\ell$.

As an application of this result, we prove that the cartesian product $G \Box K_2$ contains a $K_\ell$-minor whenever $G$ has treewidth at least $c\ell^4\sqrt{\log \ell}$.

1. Introduction

A central theorem in Robertson and Seymour’s theory of graph minors states that the grid is a canonical witness for a graph to have large treewidth, in the sense that the $\ell \times \ell$ grid has treewidth $\ell$, and every graph with sufficiently large treewidth contains an $\ell \times \ell$ grid minor [13]. See [7, 11, 12] for alternative proofs. The following theorem is the best-known explicit bound. See [4, 5] for better bounds under additional assumptions.

**Theorem 1.1 (Robertson, Seymour, and Thomas [12]).** Every graph with treewidth at least $20^{2\ell^5}$ contains an $\ell \times \ell$ grid minor.

Robertson et al. [12] also proved that certain random graphs have treewidth proportional to $\ell^2\log \ell$, yet do not contain an $\ell \times \ell$ grid minor. This is the best known lower bound on the function in Theorem 1.1. Thus it is open whether polynomial treewidth forces a large grid minor. This question is not only of theoretic interest—for example, it has direct bearing on certain algorithmic questions [3]. In this paper we prove that polynomial treewidth forces a large ‘grid-like-minor’.

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A grid-like-minor of order \( \ell \) in a graph \( G \) is a set \( \mathcal{P} \) of paths in \( G \), such that the intersection graph of \( \mathcal{P} \) is bipartite and contains a \( K_{\ell} \)-minor. Observe that the intersection graph of the rows and columns of the \( \ell \times \ell \) grid is the complete bipartite graph \( K_{\ell, \ell} \), which contains a \( K_{\ell+1} \)-minor (formed by contracting a matching of \( \ell - 1 \) edges). Hence, the \( \ell \times \ell \) grid contains a grid-like-minor of order \( \ell + 1 \). The following is our main result.

**Theorem 1.2.** Every graph with treewidth at least \( c\ell^{4} \sqrt{\log \ell} \) contains a grid-like-minor of order \( \ell \), for some constant \( c \). Conversely, every graph that contains a grid-like-minor of order \( \ell \) has treewidth at least \( \lceil \frac{\ell}{2} \rceil - 1 \).

Theorem 1.2 proves that grid-like-minors serve as a canonical witness for a graph to have large treewidth, just like grid minors. The advantage of grid-like-minors is that a polynomial bound on treewidth suffices. The disadvantage of grid-like-minors is that they are a broader structure than grid minors (but not as broad as brambles; see Section 2).

Theorem 1.2 has an interesting corollary concerning the cartesian product \( G \square K_2 \). This graph consists of two copies of \( G \) with an edge between corresponding vertices in the two copies. Motivated by Hadwiger’s Conjecture for cartesian products, the second author [17] showed that the maximum order of a complete minor in \( G \square K_2 \) is tied to the treewidth of \( G \). In particular, if \( G \) has treewidth at most \( \ell \), then \( G \square K_2 \) has treewidth at most \( 2\ell + 1 \) and thus contains no \( K_{2\ell+3} \)-minor. Conversely, if \( G \) has treewidth at least \( 2^{4\ell^4} \), then \( G \square K_2 \) contains a \( K_{\ell} \)-minor. The proof of the latter result is based on the version of Theorem 1.1 due to Diestel, Jensen, Gorbunov, and Thomassen [7]. The following theorem is a significant improvement.

**Theorem 1.3.** If a graph \( G \) has treewidth at least \( c\ell^{4} \sqrt{\log \ell} \), then \( G \square K_2 \) contains a \( K_{\ell} \)-minor, for some constant \( c \).

2. Background

All graphs considered in this paper are undirected, simple, and finite. For undefined terminology, see [6]. A graph \( H \) is a minor of a graph \( G \) if a graph isomorphic to \( H \) can be obtained from a subgraph of \( G \) by contracting edges. A graph \( G \) is \( d \)-degenerate if every subgraph of \( G \) has a vertex of degree at most \( d \). Mader [10] proved that every graph with no \( K_{\ell} \)-minor is \( 2^{\ell-2} \)-degenerate. Let \( d(\ell) \) be the minimum integer such that every graph with no \( K_{\ell} \)-minor is \( d(\ell) \)-degenerate. Kostochka [9] and Thomason [15, 16] independently proved that \( d(\ell) \in \Theta(\ell \sqrt{\log \ell}) \).
Theorem 2.1 (Kostochka [9], Thomason [15, 16]). Every graph with no $K_\ell$-minor is $d(\ell)$-degenerate, where $d(\ell) \leq c\ell\sqrt{\log \ell}$ for some constant $c$.

Let $G$ be a graph. Two subgraphs $X$ and $Y$ of $G$ touch if $X \cap Y \neq \emptyset$ or there is an edge of $G$ between $X$ and $Y$. A bramble in $G$ is a set of pairwise touching connected subgraphs. The subgraphs are called bramble elements. A set $S$ of vertices in $G$ is a hitting set of a bramble $\mathcal{B}$ if $S$ intersects every element of $\mathcal{B}$. The order of $\mathcal{B}$ is the minimum size of a hitting set. The canonical example of a bramble of order $\ell$ is the set of crosses (union of a row and column) in the $\ell \times \ell$ grid. The following ‘Treewidth Duality Theorem’ shows the intimate relationship between treewidth and brambles.

Theorem 2.2 (Seymour and Thomas [14]). A graph $G$ has treewidth at least $\ell$ if and only if $G$ contains a bramble of order at least $\ell + 1$.

See [1] for an alternative proof of Theorem 2.2. In light of Theorem 2.2, Theorem 1.1 says that every bramble of large order contains a large grid minor, and Theorem 1.2 says that every bramble of polynomial order contains a large grid-like-minor.

3. Main Proofs

In this section we prove Theorems 1.2 and 1.3. Let $e := 2.718\ldots$ and $[n] := \{1, 2, \ldots, n\}$. The following lemma is by Birmelé, Bondy, and Reed [2]; we include the proof for completeness.

Lemma 3.1 (Birmelé et al. [2]). Let $\mathcal{B}$ be a bramble in a graph $G$. Then $G$ contains a path that intersects every element of $\mathcal{B}$.

Proof. Let $P$ be a path in $G$ that (1) intersects as many elements of $\mathcal{B}$ as possible, and (2) is as short as possible. Let $v$ be an endpoint of $P$. There is a bramble element $X$ that only intersects $P$ at $v$, as otherwise we could delete $v$ from $P$. Suppose on the contrary that $P$ does not intersect some bramble element $Z$. Since $X$ and $Z$ touch, there is a path $Q$ starting at $v$ through $X$ to some vertex in $Z$, and $Q \cap P = \{v\}$. Thus $P \cup Q$ is a path that also hits $Z$. This contradiction proves that $P$ intersects every element of $\mathcal{B}$. $\blacksquare$

Lemma 3.2. Let $G$ be a graph containing a bramble $\mathcal{B}$ of order at least $k\ell$ for some integers $k, \ell \geq 1$. Then $G$ contains $\ell$ disjoint paths $P_1, \ldots, P_\ell$, and for distinct $i, j \in [\ell]$, $G$ contains $k$ disjoint paths between $P_i$ and $P_j$. 
Proof. By Lemma 3.1, there is a path $P = (v_1, \ldots, v_n)$ in $G$ that intersects every element of $B$. For $1 \leq i \leq j \leq n$, let $P(i,j)$ be the sub-path of $P$ induced by $\{v_i, \ldots, v_j\}$. Let $t_1$ be the minimum integer such that the sub-bramble

$$B_1 := \{X \in B : X \cap P(1,t_1) \neq \emptyset\}$$

has order $k$. Now let $t_2$ be the minimum integer such that the sub-bramble

$$B_2 := \{X \in B : X \cap P(1,t_1 + 1,t_2) \neq \emptyset, X \cap P(1,t_1) = \emptyset\}$$

has order $k$. Continuing in this way, since $B$ has order at least $k\ell$, we obtain integers $t_1 < t_2 < \cdots < t_{\ell} \leq n$, such that for each $i \in [\ell]$, the sub-bramble

$$B_i := \{X \in B : X \cap P(t_{i-1} + 1,t_i) \neq \emptyset, X \cap P(1,t_{i-1}) = \emptyset\}$$

has order $k$, where $t_0 := 0$. Let $P_i := P(t_{i-1} + 1,t_i)$ for $i \in [\ell]$. Thus $P_1, \ldots, P_{\ell}$ are disjoint paths in $G$.

Suppose that there is a set $S \subseteq V(G)$ separating some distinct pair of paths $P_i$ and $P_j$, where $|S| \leq k - 1$. Thus $S$ is not a hitting set of $B_i$, since $B_i$ has order $k$. Hence some element $X \in B_i$ does not intersect $S$. Similarly, some element $Y \in B_j$ does not intersect $S$. Thus $S$ separates $X$ from $Y$, and hence $X$ and $Y$ do not touch. This contradiction proves that every set of vertices separating $P_i$ and $P_j$ has at least $k$ vertices. By Menger’s Theorem, there are $k$ disjoint paths between $P_i$ and $P_j$, as desired. \qed

We now prove the main result.

Proof of the first part of Theorem 1.2. Let $k := \lceil 2e(2\ell^3 - 3) d(\ell) \rceil$. Let $G$ be a graph with treewidth at least $c\ell^4 \log \ell$, which is at least $k\ell - 1$ for an appropriate value of $c$. By Theorem 2.2, $G$ has a bramble of order at least $k\ell$. By Lemma 3.2, $G$ contains $\ell$ disjoint paths $P_1, \ldots, P_{\ell}$, and for distinct $i,j \in [\ell]$, $G$ contains a set $Q_{i,j}$ of $k$ disjoint paths between $P_i$ and $P_j$.

For distinct $i,j \in [\ell]$ and distinct $a,b \in [\ell]$ with $\{i,j\} \neq \{a,b\}$, let $H_{i,j,a,b}$ be the intersection graph of $Q_{i,j} \cup Q_{a,b}$. Since $H_{i,j,a,b}$ is bipartite, if $K_\ell$ is a minor of $H_{i,j,a,b}$, then $Q_{i,j} \cup Q_{a,b}$ is a grid-like-minor of order $\ell$. Now assume that $K_\ell$ is not a minor of $H_{i,j,a,b}$. By Theorem 2.1, $H_{i,j,a,b}$ is $d(\ell)$-degenerate.

Let $H$ be the intersection graph of $\cup\{Q_{i,j} : 1 \leq i < j \leq \ell\}$; that is, $H$ is the union of the $H_{i,j,a,b}$. Then $H$ is $\binom{\ell}{2}$-colourable, where each colour class is some $Q_{i,j}$. Each colour class of $H$ has $k$ vertices, and each pair of colour classes in $H$ induce a $d(\ell)$-degenerate subgraph. By Lemma 4.2 (in the following section) with $n = k$ and $r = \binom{\ell}{2}$ and $d = d(\ell)$, $H$ has an independent set with one vertex
from each colour class. That is, in each set \( Q_{i,j} \) there is one path \( Q_{i,j} \) such that \( Q_{i,j} \cap Q_{a,b} = \emptyset \) for distinct pairs \( i, j \) and \( a, b \). Consider the set of paths

\[
\mathcal{P} := \{ P_i : i \in [\ell] \} \cup \{ Q_{i,j} : 1 \leq i < j \leq \ell \}.
\]

The intersection graph of \( \mathcal{P} \) is the 1-subdivision of \( K_\ell \), which is bipartite and contains a \( K_\ell \)-minor. Therefore \( \mathcal{P} \) is a grid-like-minor of order \( \ell \) in \( G \).

The next lemma with \( r = 2 \) implies that if a graph \( G \) contains a grid-like-minor of order \( \ell \), then the treewidth of \( G \) is at least \( \left\lceil \frac{\ell}{2} \right\rceil - 1 \), which is the second part of Theorem 1.2.

**Lemma 3.3.** Let \( H \) be the intersection graph of a set \( \mathcal{X} \) of connected subgraphs in a graph \( G \). If \( H \) contains a \( K_\ell \)-minor, and \( H \) contains no \( K_{r+1} \)-subgraph, then the treewidth of \( G \) is at least \( \left\lceil \frac{\ell}{r} \right\rceil - 1 \).

**Proof.** Let \( H_1, \ldots, H_\ell \) be the branch sets of a \( K_\ell \)-minor in \( H \). Each \( H_i \) corresponds to a subset \( \mathcal{X}_i \subseteq \mathcal{X} \), such that \( \mathcal{X}_i \cap \mathcal{X}_j = \emptyset \) for distinct \( i, j \in [\ell] \). Let \( G_i \) be the subgraph of \( G \) formed by the union of the subgraphs in \( \mathcal{X}_i \). Since \( H_i \) is connected and each subgraph in \( \mathcal{X}_i \) is connected, \( G_i \) is connected. For distinct \( i, j \in [\ell] \), some vertex in \( H_i \) is adjacent to some vertex in \( H_j \). That is, some subgraph in \( \mathcal{X}_i \) intersects some subgraph in \( \mathcal{X}_j \). Hence \( G_i \) and \( G_j \) share a vertex in common, and \( \mathcal{B} := \{ G_1, \ldots, G_\ell \} \) is a bramble in \( G \). Since \( H \) has no \( K_{r+1} \)-subgraph, every vertex of \( G \) is in at most \( r \) bramble elements of \( \mathcal{B} \). Thus every hitting set of \( \mathcal{B} \) has at least \( \left\lceil \frac{\ell}{r} \right\rceil \) vertices. Thus \( \mathcal{B} \) has order at least \( \left\lceil \frac{\ell}{r} \right\rceil \). By Theorem 2.2, \( G \) has treewidth at least \( \left\lceil \frac{\ell}{2} \right\rceil - 1 \).

Theorem 1.3 follows from Theorem 1.2 and the next lemma.

**Lemma 3.4.** Let \( \mathcal{P} \) be a grid-like-minor in a graph \( G \). Then the intersection graph \( H \) of \( \mathcal{P} \) is a minor of \( G \square K_2 \).

**Proof.** Let \( \mathcal{A} \cup \mathcal{B} \) be a bipartition of \( V(H) \). If \( XY \in E(H) \) for some \( X, Y \in \mathcal{P} \), then \( X \in \mathcal{A} \) and \( Y \in \mathcal{B} \), and some vertex \( v \) of \( G \) is in \( X \cap Y \). Thus in \( G \square K_2 \), the copy of \( v \) in the first copy of \( G \) is adjacent to the copy of \( v \) in the second copy of \( G \). Thus \( H \) is obtained by contracting each path in \( \mathcal{A} \) in the first copy of \( G \), and by contracting each path in \( \mathcal{B} \) in the second copy of \( G \), as illustrated in Figure 1.

Note that Lemma 3.4 generalises as follows: If \( H \) is the intersection graph of a set of connected subgraphs of a graph \( G \), then \( H \) is a minor of \( G \square K_{\chi(H)} \).
4. Independent Sets in Coloured Graphs

The proof of Theorem 1.2 depends on the following sufficient condition for a coloured graph to have an independent set with one vertex from each colour class.

The proof is based on the Lovász Local Lemma.

Lemma 4.1 (Erdős and Lovász [8]). Let $X$ be a set of events, such that each event in $X$ has probability at most $p$ and is mutually independent of all but $D$ other events in $X$. If $ep(D + 1) \leq 1$ then with positive probability no event in $X$ occurs.

Lemma 4.2. For some $r \geq 2$, let $V_1, \ldots, V_r$ be the colour classes in an $r$-colouring of a graph $H$. Suppose that $|V_i| \geq n := 2e(2r - 3)d$ for all $i \in [r]$, and $H[V_i \cup V_j]$ is $d$-degenerate for distinct $i, j \in [r]$. Then there exists an independent set $\{x_1, \ldots, x_r\}$ of $H$ such that each $x_i \in V_i$.

Proof. Let $n := [2e(2r - 3)d]$. For each $i \in [r]$, we can assume that $|V_i| = n$ (since deleting vertices from $V_i$ does not change the degeneracy assumption). For each $i \in [r]$, independently and randomly choose one vertex $x_i \in V_i$. Each vertex in $V_i$ is chosen with probability $\frac{1}{n}$. For each edge $vw$ of $G$, let $X_{vw}$ be the event that both $v$ and $w$ are chosen. The probability of $X_{vw}$ equals $\frac{1}{n^2}$.

Consider an event $X_{vw}$, where $v \in V_i$ and $w \in V_j$. Observe that $X_{vw}$ is mutually independent of every event $X_{xy}$ where $\{x, y\} \cap (V_i \cup V_j) = \emptyset$. There are $2r - 3$ pairs of colour classes that include $V_i$ or $V_j$. Between each pair of colour classes there are at most $2dn$ edges (since a $d$-degenerate graph with $N$ vertices has at most...
Thus $X_{vw}$ is mutually independent of all but at most $2dn(2r - 3) - 1$ other events.

By assumption, $e \cdot \frac{1}{n^2} \cdot 2dn(2r - 3) \leq 1$. Thus by Lemma 4.1 with positive probability no event $X_{vw}$ occurs. Hence there exists $x_1, \ldots, x_r$ such that no event $X_{vw}$ occurs. That is, $\{x_1, \ldots, x_r\}$ is the desired independent set. \hfill \square

We now give an example that shows that the lower bound on $|V_i|$ in Lemma 4.2 is best possible up to a constant factor. Say $V_1$ has $d(r - 1)$ vertices. Partition $V_1$ into sets $W_2, \ldots, W_r$ each of size $d$. Connect every vertex in $W_i$ to every vertex in $V_i$ by an edge. Each bichromatic subgraph (ignoring isolated vertices) is the complete bipartite graph $K_{d,n}$ (for some $n$), which is $d$-degenerate. However, since every vertex in $V_1$ dominates some colour class, no independent set has one vertex from each colour class.

It is interesting to determine the best possible lower bound on the size of each colour class in Lemma 4.2. It is possible that $|V_i| \geq d(r - 1) + c$ suffices. This is challenging even for $d = 1$. It would also be of interest to find an algorithmic proof of Lemma 4.2. It is easy to see that if each colour class has at least $r(r - 1)d + 1$ vertices, then a minimum-degree-greedy algorithm works.

Finally, note that Lemma 4.2 is generalised as follows.

**Lemma 4.3.** Let $V_1, \ldots, V_r$ be the colour classes in an $r$-colouring of a graph $H$. For $i \in [r]$, let $n_i := |V_i|$, and let $m_i$ be the number of edges with one endpoint in $V_i$. Suppose that $n_i \geq 2et$ and $m_i \leq tn_i$ for some $t > 0$ and for all $i \in [r]$. Then there exists an independent set $\{x_1, \ldots, x_r\}$ of $H$ such that each $x_i \in V_i$.

**Proof.** Let $n := \lceil 2et \rceil$. Suppose that $n_i > n$ for some $i \in [r]$. Some vertex $v \in V_i$ has degree at least $\frac{m_i}{n_i}$. Thus $\frac{m_i - \deg(v)}{n_i - 1} \leq \frac{m_i}{n_i} \leq t$. Hence $H - v$ satisfies the assumptions. By induction, $H - v$ contains the desired independent set. Now assume that $n_i = n$ for all $i \in [r]$.

For each $i \in [r]$, independently and randomly choose one vertex $x_i \in V_i$. Each vertex in $V_i$ is chosen with probability $\frac{1}{n}$. Consider an edge $vw$, where $v \in V_i$ and $w \in V_j$. Let $X_{vw}$ be the event that both $v$ and $w$ are chosen. Thus $X_{vw}$ has probability $p := \frac{1}{n^2}$. Observe that $X_{vw}$ is mutually independent of every event $X_{xy}$ where $x \notin V_i \cup V_j$ and $y \notin V_i \cup V_j$. Thus $X_{vw}$ is mutually independent of all but at most $D := d_i + m_j - 1$ other events.

Now $2em_i \leq 2etn \leq n^2$ and $2em_i \leq 2etn \leq n^2$. Thus $e(m_i + m_j) \leq n^2$. That is, $ep(D + 1) \leq 1$. By Lemma 4.1 with positive probability no event $X_{vw}$ occurs. Hence there exists $x_1, \ldots, x_r$ such that no event $X_{vw}$ occurs. That is, $\{x_1, \ldots, x_r\}$ is the desired independent set. \hfill \square
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