ON SUBDIRECT FACTORS OF A PROJECTIVE MODULE
AND APPLICATIONS TO SYSTEM THEORY

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ABSTRACT. We extend a result of Napp Avelli, Van der Put, and Rocha with a system-theoretic interpretation to the noncommutative case: Let $P$ be a f.g. projective module over a two-sided Noetherian domain. If $P$ admits a subdirect product structure of the form $P \cong M \times_T N$ over a factor module $T$ of grade at least 2 then the torsion-free factor of $M$ (resp. $N$) is projective.

1. INTRODUCTION

This paper provides two homologically motivated generalizations of a module-theoretic result proved by Napp Avelli, Van der Put, and Rocha. This result was expressed in [NAR10] using the dual system-theoretic language and applied to behavioral control. Their algebraic proof covers at least the polynomial ring $R = k[x_1, \ldots, x_n]$ and the Laurent polynomial ring $R = k[x_1^\pm, \ldots, x_n^\pm]$ over a field $k$. The corresponding module-theoretic statement is the following:

**Theorem 1.1.** Let $R$ be one of the above rings and $M = R^q/A$ a finitely generated torsion-free module. If there exists a submodule $B \leq R^q$ such that $A \cap B = 0$ and $T := R^q/(A + B)$ is of codimension at least 2 then $M$ is free.

In fact, they prove a more general statement of which the previous is obviously a special case. However, the special statement implies the more general one.

**Theorem 1.2 ([NAR10, Theorem 18]).** Let $R$ be one of the above rings and $M = R^q/A$ a finitely generated module. If there exists a submodule $B \leq R^q$ such that $A \cap B = 0$ and $T := R^q/(A + B)$ is of codimension at least 2 then the torsion-free factor $M/\tau(M)$ of $M$ is free.

In the proposed module-theoretic generalization of Theorem 1.1 the notions “torsion-free”, “codimension” and “free” are replaced by the more homological notions “torsionless”, “grade”, and “projective”, respectively.

We start by describing the very basics of the duality between linear systems and modules in Section 2. The two notions “torsionless” and “grade” are briefly recalled in Section 3. In Section 4 a module-theoretic generalization of Theorem 1.1 is stated and proved. The proof relies on an Abelian generalization which is treated in Section 5. Since torsion-freeness admits a system-theoretic interpretation we need to discuss the relation between being torsion-free and being torsionless to justify the word “generalization”. Indeed, torsionless modules are torsion-free but the converse is generally false (cf. Remark 3.4 for a precise statement). Section 6 describes a fairly general setup in which the converse does hold. And only when it holds are we able to prove the corresponding generalization of Theorem 1.2. This is done in Section 7. Finally, Appendix A contains a converse to the key Lemma of this paper.

**Convention:** Unless stated otherwise $R$ will always denote a not necessarily commutative unital ring. The term “domain” will not imply commutativity.

Everything below is valid for left and for right $R$-modules.
2. Duality between Linear System Theory and Module Theory

For an $R$-module $F$ we define the category of $F$-behaviors as the image of the contravariant Hom-functor $\text{Hom}_R(-, F) : R-\text{Mod} \to \text{Mod}-C$, where $C$ is the center of $R$ (or the endomorphism ring of $F$ or any unital subring of thereof). $R$ is called the ring of functional operators and $F$ a signal module or signal space.

An $R$-module $M$ is said to be cogenerated by $F$ if $M$ can be embedded into a direct power $F^I$ for some index set $I$. $F$ is a called a cogenerator if it cogenerates any $R$-module $M$, or, equivalently, if the duality functor $\text{Hom}_R(-, F)$ is faithful. In particular, a cogenerator is a faithful $R$-module. The duality functor $\text{Hom}_R(-, F)$ is exact if and only if $F$ is injective. An injective $F$ is a cogenerator if and only if the solution space $\text{Hom}_R(M, F) \neq 0$ for each $M \neq 0$. In particular, all simples can be embedded into an injective cogenerator. Summing up, $\text{Hom}_R(-, F)$ is exact and reflects exactness (and hence faithful) if and only if $F$ is an injective cogenerator. In this case the $\text{Hom}$-duality between $R$-modules and $F$-behaviors is perfect. The above statements are true in any ABELian category with products [Ste75, § IV.6].

The ABELian group $\mathbb{Q}/\mathbb{Z}$ of characters of $Z$ is an injective cogenerator in the category of ABELian groups. Likewise, the $R$-module $\text{Hom}_Z(R, \mathbb{Q}/\mathbb{Z})$ is called the module of characters of $R$ and is an injective cogenerator in $R-\text{Mod}$. This follows from the adjunction between $\text{Hom}$ and the tensor product functor [Ste75, Proposition I.9.3]. The $k$-dual $R^k := \text{Hom}_k(R, k)$ is an injective cogenerator for each $k$-algebra $R$ over a field $k$. This classical result was already used in [Obe90, Corollary 3.12, Remark 3.13]. PLESKEN and ROBERTZ gave a constructive proof for the injectivity of the $k$-dual $R^k$ when $R$ is a multiple ORE extension over a computable field $k$ admitting a JANET basis notion (cf. [Rob06, Corollary 4.3.7, Theorem 4.4.7]). Furthermore, a minimal injective cogenerator always exists [Lam06, Proposition 19.13] (see [Lam06, Subsection 19A] for more details on injective cogenerators).

However, only those injective cogenerators which can be interpreted as a space of “generalized functions” (like distributions, hyperfunctions, microfunctions) are of direct significance for system theory in the engineering sense. OBERST considers in [Obe90] injective cogenerators $F$ over commutative NOETHERian rings which are large, i.e., satisfying $\text{Ass}(F) = \text{Spec}(R)$. FRÖHLER and OBERST prove in [FO98] that the space of Sfto hyperfunctions on an open interval $\Omega \subset \mathbb{R}$ is an injective cogenerator for the noncommutative ring $R := A[I]_{[d]}$ where $A := \left\{ \frac{f}{g} \mid f, g \in C[t], \forall \lambda \in \Omega : g(\lambda) \neq 0 \right\}$. ZERZ shows in [Zer06] that the space of $\mathbb{R}$-valued functions on $\mathbb{R}$ which are smooth except at finitely many points is an injective cogenerator for the rational WEYL algebra $B_1(\mathbb{R}) = \mathbb{R}(t)[\frac{d}{dt}]$.

From now on let $F$ be an injective cogenerator with system-theoretic relevance. Restricting to factor modules $M = R^d/A$ of a fixed free module $R^d$ yields a (non-intrinsic) GALOIS duality between the submodules $A$ of $R^d$, the so-called equations submodules, and $F$-behaviors $\mathcal{M} = \text{Hom}_R(R^d/A, F)$.

In system-theoretic terms a factor module of the module $M$ corresponds to a subbehavior of $\mathcal{M} = \text{Hom}_R(M, F)$, and the torsion-free factor to the largest controllable subbehavior. All degrees of torsion-freeness (including reflexivity and projectivity) are related to successive

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1One can take the index set to be the solution space $I := \text{Hom}_R(M, F)$ and require in the definition that the evaluation map from $M$ to the direct power $F^I$, sending $m \in M$ to the map $I \to F, \varphi \mapsto \varphi(m)$, is an embedding.
2Cyclic or even simple $R$-modules suffice (cf. [Ish64, Theorem 3.1]).
3This follows easily from the fact that an exact faithful functor of ABELian categories is conservative, i.e., reflects isomorphisms (see, e.g., [BLH13, Lemma A.1]). An exact functor of ABELian categories which also reflects exactness is called “faithfully exact” in [Ish64, Definition 1]. Injective cogenerators are called “faithfully injective” in [Ish64, Definition 3].
4As a referee remarked, this restriction rules out singularities which solutions of ODEs with varying coefficients might generally exhibit.
parametrizability of multi-dimensional systems in [PQ99, CQR05]. Freeness of modules corresponds to flatness of linear systems [Fl90]. A common factor module $T$ of $M = R^I/A$ and $N = R^J/B$ corresponds to the so-called interconnection, i.e., the intersection $\mathcal{M} \cap \mathcal{N}$ of the two behaviors $\mathcal{M}, \mathcal{N}$ corresponding to $M, N$. The interconnection is called regular when $A \cap B = 0$. Finally, the codimension of a module corresponds to the degree of autonomy of the corresponding behavior. This paper suggests, in particular, the use of grade as a substitute for codimension to define the degree of autonomy in the noncommutative setting.

3. TORSIONLESS MODULES AND GRADE

We will use the notion of a torsionless module, due to H. Bass, to provide a natural module-theoretic generalization of Theorem 1.1.

**Definition 3.1.** An $R$-module $M$ is called torsionless if it is cogenerated by the free module $R$, i.e., if it can be embedded into a direct power $R^I := \prod_{i \in I} R$, for some index set $I$.

**Remark 3.2.** From the definition we conclude that:

1. Any submodule of a torsionless module is torsionless and any direct product (and hence sum) of torsionless modules is torsionless.
2. Since direct sums embed in direct products any submodule of a free module is torsionless. Thus, projective modules and left and right ideals are torsionless.

We denote by $M^* := \text{Hom}_R(M, R)$ the $R$-dual of an $R$-module $M$. It is easy to see that $M$ is torsionless iif\footnote{The “only if”-part follows by setting $\lambda$ to be the composition of the embedding $j : M \hookrightarrow R^I$ and the projection $\pi_i : R^I \to R$ such that $\pi_i(j(m)) \neq 0$. The “if”-part follows by setting $I = M^*$ and $j$ to be the evaluation map $\varepsilon_M : M \to M^{**}, m \mapsto (\lambda \mapsto \lambda(m))$ considered as a map to $R^I \supset M^{**}$.} for any $m \in M \setminus \{0\}$ there exists a functional $\lambda \in M^*$ such that $\lambda(m) \neq 0$. Hence, $M$ is torsionless iff the natural evaluation map

\[ \varepsilon_M : M \to M^{**}, m \mapsto (\lambda \mapsto \lambda(m)) \]

is a monomorphism.\footnote{Recall, $M$ is called reflexive if $\varepsilon_M$ is an isomorphism.} The dualized evaluation map $\varepsilon_M^* : M^{***} \to M^*$ is a post-inverse of the evaluation map of the dual module $\varepsilon_M^* : M^* \to M^{***}$, i.e., the latter is a split monomorphism (cf. [Lam99, Remark (4.65),(f)]). In particular, the dual $M^*$ and the double-dual $M^{**} = (M^*)^*$ are torsionless modules. This gives rise to the following definition.

**Definition 3.3.** The torsionless factor of an $R$-module $M$ is the coimage $M/\ker \varepsilon_M$ of the evaluation map.

**Remark 3.4.** If $R$ is a domain then any torsionless module is torsion-free. The converse is false: The infinitely generated $\mathbb{Z}$-module $\mathbb{Q}$ is torsion-free with a zero evaluation map, i.e., the “opposite” of being torsionless. Finitely generated modules behave better in this respect (cf. Theorem 6.1). While submodules of torsion modules are torsion, the torsionless $\mathbb{Z}$-submodule $\mathbb{Z} \leq \mathbb{Q}$ shows that having a zero evaluation map is not stable under passing to submodules. Still, the factor module $\mathbb{Q}/\mathbb{Z}$ has a zero evaluation map.

Recall, an $R$-module $T$ is is said to have grade at least $c$ if $\text{Ext}^i(T, R) = 0$ for all $i < c$. The grade of the associated cyclic module $R/\text{Ann}(T)$ coincides with the grade of $T$.

**Remark 3.5.** Let $R$ be a commutative NOETHERIAN ring. The grade of an $R$-module $T$ coincides, by a theorem of REES, with $\text{depth} \text{Ann}(T) := \text{depth}(\text{Ann}(T), R)$ [Eis95, Proposition 18.4], [BH93, Theorem 1.2.5]. $R$ is called COHEN-MACAULAY if the notions of codimension and grade coincide, i.e., if $\text{codim} T := \text{codim} \text{Ann}(T)$ coincides with $\text{grade} T = \text{depth} \text{Ann}(T)$ for all modules $T$ [Eis95, Introductions to Chapters 9 and 18]. The reader is referred to [BH93, Part II] for large classes of COHEN-MACAULAY rings.

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\[^5\]The “only if”-part follows by setting $\lambda$ to be the composition of the embedding $j : M \hookrightarrow R^I$ and the projection $\pi_i : R^I \to R$ such that $\pi_i(j(m)) \neq 0$. The “if”-part follows by setting $I = M^*$ and $j$ to be the evaluation map $\varepsilon_M : M \to M^{**}, m \mapsto (\lambda \mapsto \lambda(m))$ considered as a map to $R^I \supset M^{**}$.

\[^6\]Recall, $M$ is called reflexive if $\varepsilon_M$ is an isomorphism.

\[^7\]This is a more convenient than defining the grade by an equality.
The following definition is used to formulate the module-theoretic generalization of Theorem 1.1.

**Definition 3.6.** We say that an $R$-module $M$ is **projective up to grade** $c$ if there exists a projective module $P$ and an epimorphism $P \xrightarrow{\pi} M$ such that $A := \ker \pi \leq P$ admits a complement up to grade $c$ in $P$, i.e., if there exists a submodule $B \leq P$ with $A \cap B = 0$ and $T := P/(A + B)$ has grade at least $c$. If $M$ is finitely generated then we insist that $P$ is finitely generated.

4. **A module-theoretic generalization of Theorem 1.1**

Projective modules are torsionless (Remark 3.2.(2)) and obviously projective up to grade $c$, for any $c$. The converse is true for finitely generated modules and $c \geq 2$, yielding a module-theoretic generalization of Theorem 1.1:

**Theorem 4.1.** Let $R$ be a ring and $M$ a finitely generated $R$-module. If $M$ is torsionless and projective up to grade 2 then $M$ is projective.

**Proof.** Let $P \twoheadrightarrow M$ be the f.g. projective module of Definition 3.6, $A := \ker \pi \leq P$, $B$ be a complement up to the grade 2 of $A$ in $P$, and $T := P/(A + B)$ the factor module of grade $\geq 2$, i.e., $\Hom(T, R) = 0 = \Ext^1(T, R)$. The assertion will follow from Theorem 5.1 as soon as we have shown that $\Hom(T, M) = 0 = \Ext^1(T, P)$ which we will do now:

Since $M$ is torsionless there exists an embedding $\Phi: M \hookrightarrow R^I$ in a direct product for some index set $I$. As the left exact covariant $\Hom$-functor commutes with direct products [HS97, Proposition I.3.5] it follows that

$$\Hom(T, M) \cong \Hom(T, \Phi(M)) \leq \Hom(T, R^I) \cong \Hom(T, R)^I = 0.$$ 

And since $P$ is finitely generated projective it is a direct summand of a free module $R^P \cong P \oplus P'$ of finite rank $p$. Finally, the additivity of $\Ext^1(T, \Phi)$ yields

$$\Ext^1(T, P) \leq \Ext^1(T, P) \oplus \Ext^1(T, P') \cong \Ext^1(T, R^P) = \Ext^1(T, R)^P = 0. \quad \square$$

5. **An Abelian generalization of Theorem 1.1**

Let $\mathcal{A}$ be an **Abelian** category and $P \cong M \times_T N \in \mathcal{A}$ a subdirect product\(^9\) of two objects $M$ and $N$ over a common factor object $T$, i.e., $M \leftarrow P \rightarrow N$ is the pullback of the two epis $M \twoheadrightarrow T \twoheadleftarrow N$.

**Theorem 5.1.** If $\Hom(T, M) = 0 = \Ext^1(T, P)$ then the epi $P \twoheadrightarrow M$ is split and $M$ is isomorphic to a direct summand of $P$. If furthermore $P$ is projective then so is $M$.

The following simple lemma is the essence of the short proof of Theorem 5.1. We keep the above notation and set $A := \ker (P \twoheadrightarrow M)$ and $B := \ker (P \twoheadrightarrow N)$.

**Lemma 5.2.** If $\Ext^1(T, A) = 0$ then $A$ has a complement $B' \cong M$ in $P$ which contains $B$. In particular, $M$ is isomorphic to a direct summand of $P$.

**Proof.** Set $S := A + B \leq P$, the direct sum of $A$ and $B$. The assumption $\Ext^1(T, A) = 0$ and the natural isomorphism $S/B \cong A$ imply that the short exact sequence $0 \rightarrow S/B \rightarrow N \rightarrow T \rightarrow 0$ splits. In other words, there exists a subobject $B'$ of $P$ with $B' \geq B$ such that $B'/B$ is a complement of $S/B \cong A$ in $P/B$. Since $B' \cap (A + B) = B$ it follow that $B'$ is a complement of $A$ in $P$, canonically isomorphic to $M$. \(\square\)

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\(^8\)Proposition 6.3 provides an alternative embedding into a free module of finite rank.

\(^9\)Also called fiber product.
Proof of Theorem 5.1. To apply Lemma 5.2 we need to show that $\text{Ext}^1(T, A) = 0$. Indeed, $\underline{\text{Hom}}(T, M) \to \text{Ext}^1(T, A) \to \underline{\text{Ext}}^1(T, P)$ is part of the long exact $\text{Ext}(T, -)$-sequence with respect to the short exact sequence $0 \to A \to P \to M \to 0$. Hence, $\text{Ext}^1(T, A) = 0$. □

Lemma 5.2 has an interesting converse which we did not need here. It is treated in Appendix A.

6. WHEN DOES TORSION-FREE IMPLY TORSIONLESS?

In this section we assume $R$ to be two-sided NOETHERIAN\textsuperscript{10}. A finite projective presentation of a f.g. $R$-module $M$ is an exact sequence $M \leftarrow P_0 \leftarrow P_1$ with f.g. projective $R$-modules $P_0$ and $P_1$. For such a module define the AUSLANDER dual $A(M)$ to be the cokernel of the dual (or pullback) map $\partial^*: P_0^* \to P_1^*$. Like the syzygy modules of $M$, the AUSLANDER dual is well-defined up to projective equivalence. In particular $\text{Ext}^i(A(M), R)$ does not depend on the finite projective presentation for $i > 0$. Furthermore, if $M$ is projective then $A(M) = 0$ (up to projective equivalence) and $\text{Ext}^i(A(M), R) = 0$ for all $i > 0$ (for a converse statement cf. [CQR05, Theorem 7]).

The kernel and cokernel of the evaluation map $\varepsilon: M \to M^{**}$ were characterized by AUSLANDER, where $M$ is assumed to have a finite projective presentation. As one of many applications of his theory of coherent functors [Aus66] he proved the existence of a natural monomorphism $\tau: \text{Ext}^1(A(M), R) \hookrightarrow M$ and a natural epimorphism $\rho: M^{**} \twoheadrightarrow \text{Ext}^2(A(M), R)$ such that

$$
(\varepsilon) \quad 0 \to \text{Ext}^1(A(M), R) \xrightarrow{\tau} M \xrightarrow{\rho} M^{**} \xrightarrow{\rho} \text{Ext}^2(A(M), R) \to 0
$$

is an exact sequence. In particular, $M$ is torsionless iff $\text{Ext}^1(A(M), R) = 0$ and reflexive iff $\text{Ext}^i(A(M), R) = 0$ for $i = 1, 2$. A short elegant proof of $(\varepsilon)$ can be found in [CQR05, Theorem 6] and a generalization in [AB69, Chapter 2, (2.1)] (see also [HS97, Exer. IV.7.3]).

A left (resp. right) NOETHERIAN domain $R$ satisfies the left (resp. right) ORE condition and the set of torsion elements $\tau(M)$ of an $R$-module $M$ form an $R$-submodule. The following theorem states that the two notions “torsion-free” and “torsionless” coincide for finitely generated modules.

**Theorem 6.1** ([CQR05, Theorem 5]). Let $R$ be a two-sided NOETHERIAN domain and $M$ a f.g. $R$-module. Then the image of the natural monomorphism $\tau: \text{Ext}^1(A(M), R) \hookrightarrow M$ is the torsion submodule $\tau(M)$ yielding a canonical isomorphism $\text{Ext}^1(A(M), R) \cong \tau(M)$. In particular, the torsion-free factor and the torsionless factor of $M$ coincide and $M$ is torsion-free iff $M$ is torsionless.

**Corollary 6.2.** Let $R$ be a two-sided NOETHERIAN domain. A finitely generated $R$-module of grade at least 1 is torsion.

**Proof.** Let $T$ be a such a module. By Theorem 6.1 the torsion-free factor coincides with the torsionless factor. The latter is trivial since $\text{Hom}_R(T, R) = 0$ and the evaluation map $T \to T^{**}$ vanishes. Hence $T$ is a torsion module. □

Any finitely generated torsion-free module over a commutative domain can be embedded into a free module of finite rank. This can be easily seen by passing to the quotient field (cf. [Lam99, the paragraph preceding (2.31)]). The exact sequence $(\varepsilon)$ yields a generalization to the noncommutative case. The following proposition is part of [CQR05, Theorem 8].

**Proposition 6.3.** Let $R$ be a two-sided NOETHERIAN domain. A finitely generated torsionless (=torsion-free) $R$-module can be embedded in a free module of finite rank.

\textsuperscript{10}R two-sided coherent is, as usual, enough but we stick to two-sided NOETHERIAN for lack of references.
Proof. The two-sided coherence of $R$ assures the existence of finite rank free resolutions for \(R\)-modules. Let \(M \hookrightarrow F_0 \overset{\partial}{\to} F_1\) be a finite free presentation of \(M\), i.e., with free modules \(F_0\) and \(F_1\) of finite rank. Dualizing we obtain a finite free presentation of \(A(M)\) which we can resolve one step further and obtain \(F_{n-1}^* \to F_0^* \overset{\partial^*}{\to} F_1^* \to A(M)\) with \(F_{n-1}^*\) free of finite rank. Dualizing again yields an exact complex: The defect of exactness at \(F_0^{**}\) is \(\operatorname{Ext}^1(A(M), R)\) which vanishes since \(M\) is torsionless. Using the reflexiveness of free modules of finite rank it follows that \(M\) (as the cokernel of \(\partial\) or \(\partial^{**}\)) embeds into the finite rank free module \(F_{n-1}^*\). 

The above Proposition is implemented for computable rings in \textsc{OreModules} [CQR07] and \textsc{homalg} [Thpa13, BLH11].

7. A module-theoretic generalization of Theorem 1.2

Theorem 7.1. Let \(R\) be a two-sided Noetherian domain and \(M\) a finitely generated \(R\)-module. If \(M\) is projective up to grade 2 then the torsion-free factor \(M/\operatorname{t}(M)\) is projective.

Proof. Let \(P \overset{\pi}{\longrightarrow} M\) be the f.g. projective module of Definition 3.6, \(A := \ker \pi \leq P\), \(B\) be a complement up to the grade 2 of \(A\) in \(P\), and \(T := P/(A + B)\) the factor module of grade \(\geq 2\). Let \(A'\) denote the preimage of \(t(M)\) in \(P\), so \(A'/A \cong t(M)\). The intersection \(A' \cap (A + B) = 0\) since \(A' \cap (A + B) = A\). The latter can be seen as follows: Otherwise \((A' \cap (A + B))/A \leq A'/A \cong t(M)\) would be a nontrivial torsion\(^{11}\) submodule of the torsion-free factor \((A + B)/A \cong B\). The next proposition guarantees that the epimorphic image \(T'' = P/(A' + B)\) of \(T\) is again of grade at least 2. It remains to apply Theorem 4.1 to \(M/\operatorname{t}(M) \cong P/A'\) with \(B\) now a complement of \(A'\) in \(P\) up to grade at least 2. 

Proposition 7.2. Let \(T\) be a torsion module over a domain. If \(T\) has grade at least 2 then any of its factor modules has grade at least 2.

Proof. The grade condition for \(T\) means that \(\operatorname{Hom}(T, R) = 0 = \operatorname{Ext}^1(T, R)\). Let \(T'' = T/T'\) be a factor of \(T\). Any morphism from a torsion module over a domain into a torsion-free module is zero. An since the submodule \(T'\) is again torsion\(^{12}\) it follows that \(\operatorname{Hom}(T', R) = 0\). The long exact \(\operatorname{Ext}(\ldots, R)\)-sequence (w.r.t. \(0 \to T' \to T \to T'' \to 0\))

\[
0 \to \operatorname{Hom}(T'', R) \to \underline{\operatorname{Hom}(T, R)} \to \underline{\operatorname{Hom}(T', R)} \to \underline{\operatorname{Ext}^1(T', R)} \to \underline{\operatorname{Ext}^1(T, R)}
\]

implies that \(\operatorname{Hom}(T'', R) = 0 = \operatorname{Ext}^1(T'', R)\). \hfill \square

We end this section by describing a context in which the original formulation can be retained. If \(M\) has a finite free resolution, e.g., if \(R\) is an FFR ring\(^{13}\), then, by a remark of Serre, \(M\) projective implies \(M\) stably free (cf. [Eis95, Proposition 19.16]). If, additionally, \(R\) is HERMITE then \(M\) projective already implies \(M\) free. If \(R\) is commutative COHEN-MACaulay then the notions of grade and codimension coincide (cf. Remark 3.5). The rings mentioned in the Introduction are FFR, HERMITE, and commutative COHEN-MACaulay (even regular) domains.

Remark 7.3. It should be noted that this paper is less of computational interest as non of the results suggests an algorithm to decide the projectivity of the torsion-free factor of a given finitely presented module. For an overview on algorithms to test projectivity, stably freeness, and freeness see [BLH11, Subsection 3.4] and the references therein. However, given \(M = R^q/A\) and \(B \leq R^q\) over a computable ring \(R\) it can be algorithmically decided whether \(A \cap B = 0\) and grade \(T \geq 2\) for \(T = R^q/(A + B)\). For the definition of a computable ring see [BLH11, ]

\(^{11}\)Here we need that \(\operatorname{t}(M)\) is torsion and not merely having a zero evaluation map.

\(^{12}\)Here we need that \(T\) is torsion and not merely having a zero evaluation map.

\(^{13}\)Finite free resolution ring.
Definition 3.2. The torsion-free factor over finitely presented modules over such rings can be computed, e.g., as the coimage of the evaluation map.

**Appendix A. A converse of Lemma 5.2**

Let $\mathcal{A}$ be an Abelian category and $P \cong M \times_T N \in \mathcal{A}$ a fiber product of two objects $M$ and $N$ over a common factor object $T$. Again we set $A := \ker(P \to M)$, $B := \ker(P \to N)$, and $S := A + B$.

The four factors

$$P/S \cong (P/B)/(S/B)$$

in the first isomorphism theorem applied to $B \leq S \leq P$ can be expressed by four commuting short exact sequences yielding the diagram on the right.

We now formulate the converse of Lemma 5.2 under the assumption that $\text{Ext}^1(T, P) = 0$.

**Proposition A.1.** Under the assumption that $\text{Ext}^1(T, P) = 0$ the following two conditions become equivalent:

1. The extension $0 \to A \to N \to T \to 0$ is trivial.
2. $\text{Ext}^1(T, A) = 0$.

**Proof.** For the nontrivial implication $(1) \implies (2)$ consider the braid diagram below. Condition (1) implies that the connecting homomorphism $\text{Hom}(T, T) \to \text{Ext}^1(T, A)$ is zero, i.e., that $\text{Ext}^1(T, A)$ embeds into $\text{Ext}^1(T, N)$. The homomorphism $\varphi : \text{Ext}^1(T, S) \to \text{Ext}^1(T, N)$ can be written as the composition

$$\text{Ext}^1(T, S) = \text{Ext}^1(T, A + B) \cong \text{Ext}^1(T, A) + \text{Ext}^1(T, B) \to \text{Ext}^1(T, A) \to \text{Ext}^1(T, N),$$

showing that the image of $\varphi$ is isomorphic to $\text{Ext}^1(T, A)$. But $\varphi$ factors through $\text{Ext}^1(T, P) = 0$ and is hence zero, together with its image $\text{Ext}^1(T, A)$.

□
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