Hamiltonization and separation of variables for Chaplygin ball on a rotating plane

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Abstract

We discuss a non-Hamiltonian vector field appearing in consideration of a partial motion of the Chaplygin ball rolling on a horizontal plane which rotates with constant angular velocity. In two partial cases this vector field is expressed via Hamiltonian vector fields using a non-algebraic deformation of the canonical Poisson bivector on $e^*(3)$. For the symmetric ball we also calculate variables of separation, compatible Poisson brackets, algebra of Haantjes operators and $2 \times 2$ Lax matrices.

1 Introduction

Theory of integrable systems appeared as a family of mathematical methods which can be applied to find exact solutions of dynamical systems. The main motivation was to determine the scope of mathematical models of real physical processes. Explicit solutions allow us to test analytical and numerical schemes applied to a given mathematical model and to choose a reasonable approximation to solutions of the model.

In classical mechanics change of time is a standard tool for construction of explicit solutions of equations of motion according to Kepler [24], Jacobi, Maupertuis [27], Weierstrass [47], see discussion in [30, 31] and references within. In nonholonomic mechanics Appel [1] and Chaplygin [11, 12] also used change of time for integrating certain nonholonomic systems with two degrees of freedom. In modern nonholonomic mechanics we have many equations of motion taking a Hamiltonian form after suitable symmetry reduction and time reparametrisation, see [2, 4, 5, 7, 8, 10, 23, 15, 16, 17, 18, 19, 28] and references within. Unfortunately, in most of these publications authors discuss only the form of equations of motion instead of exact solutions of these equations.

The main aim of this note is to compare Hamiltonization and the modern separation of variables method embedding elements of artificial intelligence such as machine learning and deep learning. Harnessing of modern computational abilities for studying integrable systems is naturally placed as a prominent avenue in contemporary classical and quantum mechanics. For instance, this allows us to automate validation of mathematical models of real physical process, see one of the collection of papers about machine learning in physics [13]. In classical mechanics computer modeling currently consists not only of approximate numerical calculations and visualization, but also of algorithmic reduction to quadratures [20, 21, 33, 46].

In this note we take a non-Hamiltonian vector from the recent paper by Borisov, Mamaev and Bizyaev [6] and obtain Poisson bivectors, variables of separation and quadratures using only modern computer software. Our main motivation is to enlarge known collection of deformations of canonical Poisson bivectors appearing both in Hamiltonian and non-Hamiltonian dynamics [3, 10, 20, 21], because sufficiently large datasets are an integral part of the field of machine learning.
1.1 Hamiltonization

In 1903 Chaplygin found quadratures in the mathematical model of inhomogeneous balanced ball rolling without slipping on a horizontal plane [11]. These quadratures for non-Hamiltonian model can be resolved after change of time similar to the well-known Weierstrass change of time in Hamiltonian mechanics [47]. Equations of motion were written in Hamiltonian form only in 2001 [9].

In 1911 Chaplygin introduced the reducing multiplier method and applied this method to integrate what would later become known as the Chaplygin sleigh [12]. He also remarked that his general procedure (using the reducing multiplier) for integrating certain nonholonomic systems with two degrees of freedom was "interesting from a theoretical standpoint as a direct extension of the Jacobi method to simple nonholonomic systems."

The first part of Chaplygin’s theorem states that in case of nonholonomic systems in two generalized coordinates \((q_1, q_2)\) possessing an invariant measure with density \(N(q_1, q_2)\) equations of motion may be written in Hamiltonian form after the time reparameterization \(d\tau = N dt\). The second part of this theorem says that if a nonholonomic system can be written in Hamiltonian form after time reparameterization \(d\tau = f(q_1, q_2) dt\), then the original system has an invariant measure. Both functions \(N\) and \(f\) are known as the reducing multiplier, or simply the multiplier, see historic remarks and discussion in [18, 28]. The reduced phase space of Chaplygin’s system is isomorphic to the cotangent bundle \(T^*Q\) where reduced equations may be formulated as

\[
\frac{dz}{dt} = N(q_1, q_2) P_f dH, \tag{1.1}
\]

where \(z = (q_1, q_2, p_1, p_2)\). Roughly speaking Chaplygin considered conformally Hamiltonian vector fields \(Z\) associated with Turiel type deformations \(P_f\) of the canonical Poisson bivector on \(T^*Q\) [25, 44]. Discussion of symplectic and non-symplectic diffeomorphisms associated with a conformally Hamiltonian vector field in Hamiltonian mechanics can be found in [26].

After introduction of Chaplygin’s theorem, subsequent research on the theorem resulted in, among other things, an extension to the quasi coordinate context, a study of the geometry behind the theorem, discoveries of isomorphisms between nonholonomic systems, an example of a system in higher dimensions which was Hamiltonizable through a similar time reparameterization, determination of necessary conditions for Hamiltonization, study deformations of Poisson structures in nonholonomic systems, etc. More detailed discussions of various modern methods of the Hamiltonization may be found in [2, 4, 5, 8, 10, 15, 16, 17, 18, 19, 28] and the references therein.

The main advantage of any type of Hamiltonization is that we identify phase space with a Poisson or symplectic manifold. It allows us to study non-Hamiltonian systems using standard machinery of symplectic geometry.

The main disadvantage of any type of Hamiltonization is that Hamiltonization only works for a narrow class of nonholonomic systems; even if it works, the reduction to quadratures is not transparent. Another disadvantage is that we do not have an algorithmic procedure for constructing of cotangent bundle \(T^*Q\) with generalized coordinates or quasi-coordinates starting with original physical variables.

1.2 Separation of variables

In [12] Chaplygin discussed a direct extension of the Jacobi method for simple nonholonomic systems. In fact, we do not need any extensions because the original geometric version of the Jacobi methods is independent of time and, therefore, it is directly applicable both for Hamiltonian and non-Hamiltonian systems.

In 1837 Jacobi proved that \(m\) solutions \(h_1 = H_1(x, y), \ldots, h_m = H_m(x, y)\) of \(m\) separation relations

\[
\Phi_j(x_j, y_j, h_1, \ldots, h_m) = 0, \quad j = 1, \ldots, m, \quad \det \left[ \frac{\partial \Phi_j}{\partial h_k} \right] \neq 0, \tag{1.2}
\]
are in involution with respect to the Poisson bracket
\[ \{H_i, H_j\} = 0, \quad i, j = 1, \ldots, m, \]
defined by the Poisson bivector
\[ \mathcal{P}_f = \sum_{j=1}^{m} f_j(x_j, y_j) \left( \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial y_j} - \frac{\partial}{\partial y_j} \wedge \frac{\partial}{\partial x_j} \right), \tag{1.3} \]
where \( f_j(x_j, y_j) \) are arbitrary functions. Equations (1.2) define curves \( X_1, \ldots, X_m \) on a projective plane depending on \( m \) parameters \( h_1, \ldots, h_m \) so that common level surface of functions \( H_i(x, y) = h_i \) is a product of these plane curves
\[ \mathcal{M} : \quad X_1 \times X_2 \times \cdots \times X_m. \]
If \( \mathcal{M} \) is a regular Lagrangian submanifold on phase space, we have a completely integrable system, but in generic case \( \mathcal{M} \) is a product of plane curves only.

Realisations of the common level set \( \mathcal{M} \) as a product of curves is independent from parameterization of trajectories living on \( \mathcal{M} \), i.e. independent of time and of the form of equations of motion. It is pure geometric fact. We can find this realisation without reduction of equations of motion to the Hamiltonian form.

Compatible Poisson bivectors \( \mathcal{P}_f \) and \( \mathcal{P}_g \) associated with two sets of functions \( f_1, \ldots, f_m \) and \( g_1, \ldots, g_m \) are related to each other
\[ \mathcal{P}_g = N \mathcal{P}_f \]
by the formal recursion operator
\[ N = \mathcal{P}_g \mathcal{P}_f^{-1} = \sum_{j=1}^{m} g_j f_j^{-1} L_j, \tag{1.4} \]
where
\[ L_j = \frac{\partial}{\partial x_j} \otimes dx_j + \frac{\partial}{\partial y_j} \otimes dy_j \]
form the so-called algebra of Haantjes operators with vanishing Nijenhuis torsion.

Functions \( H_1, \ldots, H_m \) and compatible Poisson bivectors \( \mathcal{P}_f \) and \( \mathcal{P}_g \) satisfy the equation
\[ \mathcal{P}_g \mathbf{dH} = \mathcal{P}_f \mathbf{F}_{fg} \mathbf{dH}, \quad \mathbf{dH} = (dH_1, dH_2, \ldots, dH_m), \tag{1.5} \]
where \( \mathbf{F}_{fg} \) is the so-called control matrix. Eigenvalues of the control matrix are functions on variables of separation, i.e.
\[ \lambda_j = \lambda_j(x_i, y_i). \]
If one of the compatible Poisson bivectors \( P_f \) or \( P_g \) is non-degenerate, we can calculate variables of separation using the recursion operator \( N \). If both bivectors are degenerate, we can calculate variables of separation using the control matrix \( F \).

Now we are ready to discuss application of the geometric Jacobi method to integration of the equations of motion
\[ \dot{z}_i = Z_i(z_1, \ldots, z_n), \quad i = 1, \ldots, n \tag{1.6} \]
which determine a specific mathematical model of some real physical process, which means that the number of equations is not very big. First integrals of vector field \( Z \) could be obtained by brute force method, i.e. by solving equation
\[ \dot{H}(z) = 0, \tag{1.7} \]
using some anzats for \( H(z) \). If we find some first integrals \( H_1, \ldots, H_m \), we can try to solve algebraic equations
\[
\sum_{ij} P_{f_{ij}} \frac{\partial H_k}{\partial z_i} \frac{\partial H_\ell}{\partial z_j} = 0, \quad k, \ell = 1, \ldots, m
\]
(1.8)
with respect to the Poisson bivector \( P_f \). Because a’priory these equations have infinitely many solutions of the form (1.3) we have to restrict the space of solutions, i.e. use a suitable anzats in order to get a partial solution. Instead of (1.8) we can solve equation
\[
Z = f_1(z) P_f \, dH_1 + \cdots + f_m(z) P_f \, dH_m.
\]
(1.9)
It is easy to see that conformally Hamiltonian vector fields (1.1) belong to a very restricted subspace of solutions for this equation.
If we suppose that equations of motion are reducible to quadratures (completely or partially), then there is also another decomposition
\[
Z = g_1(z) P_g \, dH_1 + \cdots + g_m(z) P_g \, dH_m,
\]
(1.10)
where \( P_g \) is a Poisson bivector compatible with \( P_f \). A pair of compatible Poisson bivectors determines variables of separation, which allows us to reduce the equations of motion to quadratures. In generic case it could be a complete or partial separation of variables.
Thus, if equations of motion (1.1) can be reduced to quadratures in the framework of the Jacobi method, we have an algorithm of reduction:
- solve equation (1.7) with respect to functionally independent first integrals \( H_1, \ldots, H_m \);
- solve equations (1.9,1.10) with respect to compatible Poisson bivectors \( P_f \) and \( P_g \);
- find eigenvalues of the corresponding matrix \( F_{fg} \);
- calculate quadratures associated with variables of separation.
Some results of application of this algorithm in Hamiltonian and non-Hamiltonian mechanics are given in [20, 21, 22, 23, 24, 25, 26, 27].
The main advantage of the Jacobi method is that using variables of separation, we obtain not only quadratures, but also families of compatible Poisson brackets, recursion operators, algebras of Haantjes operators, master symmetries, Lax matrices, new integrable systems and exact discretization of the original equations of motion [11, 12, 13].
The main disadvantage of the Jacobi method is that we can solve equations (1.7) and (1.9,1.10) only using ansatz for a solution. We hope that selection of the suitable ansatz can be automated using elements of artificial intelligence.

1.3 Description of the model

Let us consider the following equations of motion
\[
\dot{\gamma} = \gamma \times \omega, \quad \dot{K} = \Omega \gamma \times K + (K - d\Omega\gamma) \times \omega.
\]
(1.11)
Here vectors \( \gamma = (\gamma_1, \gamma_2, \gamma_3) \) and \( K = (K_1, K_2, K_3) \) are variables on the phase space, \( x \times y \) is a vector product in \( \mathbb{R}^3 \), \( d \) and \( \Omega \) are some parameters, \( A = diag(a_1, a_2, a_3) \) is a diagonal matrix, and
\[
\omega = AK + \frac{(\gamma, AK)}{d^{-1} - (\gamma, A\gamma)} A\gamma.
\]
These equations describe a partial case of motion of the inhomogeneous balanced ball rolling without slipping on a horizontal plane rotating with constant angular velocity \( \Omega \), see equations (22) in [6]. We use the same notations as in [6], where the reader can find a complete description of variables \( z = (\gamma_1, \gamma_2, \gamma_3, K_1, K_2, K_3) \), definitions of parameters and a list of the necessary references.
According to [6] equations of motion (1.11) possess two geometric integrals of motion
\[ C_1 = \gamma^2 = 1, \quad C_2 = (K, \gamma), \] (1.12)
an integral of motion similar to the Jacobi integral
\[ H = \frac{1}{2}(K, A K) - \frac{K^2}{2d} + \frac{d}{2(1 - d(\gamma, A \gamma))}(A K, \gamma)^2, \] (1.13)
and an invariant measure
\[ \mu = \rho dK d\gamma, \quad \text{where} \quad \rho = \left(1 - d(\gamma, A \gamma)\right)^{-\frac{1}{2}}. \] (1.14)

In Section 2 we present Poisson bivector \( P \) which allows us to rewrite vector field Z (1.11) in conformally Hamiltonian form
\[ Z = f(z) P d H(z) \]
at \( C_2 = 0 \). This bivector \( P \) is a linear deformation of the standard Lie-Poisson bivector on algebra \( e^*(3) \) involving non-algebraic functions, i.e. the so-called Turiel type deformation [44].

In Section 3 we discuss a counterpart of the heavy symmetric top at \( a_1 = a_2 \) in (1.11). In this case we have one more first integral [6]:
\[ H_2 = \rho K_3 - \frac{d a_1 \rho (K, \gamma) \gamma_3}{da_1} + \frac{da_1 - 1}{a_1 \sqrt{d(a_1 - a_3)}} \Omega \ln \left(\sqrt{d(a_1 - a_3)} \gamma_3 + \rho^{-1}\right). \]

Using this non-algebraic first integral we can decompose vector field (1.11) into Hamiltonian vector fields
\[ Z = f_1(z) \mathcal{P}' d H(z) + f_2(z) \mathcal{P}' d H_2(z) \]
and find second Poisson bivectors \( \mathcal{P}'' \) compatible with \( \mathcal{P}' \) so that
\[ Z = g_1(z) \mathcal{P}'' d H(z) + g_2(z) \mathcal{P}'' d H_2(z) \]
It allows us to calculate variables of separation for the equations of motion (1.11) on a computer.

The same variables of separation may be obtained more easily using a counterpart of the Lagrange calculations for the symmetric heavy top. In the framework of the Jacobi method these variables of separation determine compatible Poisson brackets, recursion operators, algebra of Haantjes operators and \( 2 \times 2 \) Lax matrices for the vector field (1.11).

2 Conformally Hamiltonian vector field at \( (\gamma, K) = 0 \)

Let us substitute vector field \( Z \) (1.11), its geometric integrals \( C_{1,2} \) (1.12) and the Jacobi integral \( H \) (1.13) into the following system of algebraic equations
\[ Z = f(z) P d H \quad \text{and} \quad P d C_{1,2} = 0. \] (2.1)
Desired Poisson bivector \( P \) shall also satisfy to the Jacobi identity, i.e. system of differential equations
\[ [P, P] = 0 \] (2.2)
coded in a short form using the Schouten bracket
\[ [A, B]_{ijk} = \sum_{m=1}^{\text{dim } M} \left( B_{mk} \frac{\partial A_{ij}}{\partial z_m} + A_{mk} \frac{\partial B_{ij}}{\partial z_m} + \text{cycle}(i, j, k) \right). \]
In our case \( z = (\gamma, K) \) and \( \text{dim } M = 6. \)
Substituting linear anazts for entries of the Poisson bivector

\[ \mathcal{P}_{ij} = \sum_{m=1}^{3} u_{ij}^{m}(\gamma) K_{m} + v_{ij}(\gamma) \]

and function \( f(z) = f(\gamma) \) into (2.1) one gets an inconsistent system of algebraic equations, which has solution only at \( C_{2} = (\gamma, K) = 0 \). This solution of algebraic equations depends on arbitrary function \( f(\gamma) \) on variables \( \gamma_{1}, \gamma_{2} \) and \( \gamma_{3} \).

Substituting this partial solution into the Jacobi identity and solving the resulting differential equations we obtain the desired Poisson bivector. It took us only a few seconds to solve both algebraic and differential equations using one of the modern computer algebra systems.

**Proposition 1** At \( \gamma^{2} = 1 \) and \( (\gamma, K) = 0 \) vector field \( Z \) is a conformally Hamiltonian vector field

\[ Z = f(z) \mathcal{P} dH. \]

with conformal factor depending only on variables \( \gamma_{1}, \gamma_{2} \) and \( \gamma_{3} \)

\[ f(z) = 2 \rho \delta, \]

which is a product of functions \( \rho \) from (1.14) and

\[ \delta = (1 - da_{1})a_{2}a_{3}\gamma_{1}^{2} + (1 - da_{2})a_{1}a_{3}\gamma_{2}^{2} + (1 - da_{3})a_{1}a_{2}\gamma_{3}^{2}. \]

The proof consists of straightforward verification of algebraic and differential equations (2.1-2.2) by using an explicit form of Poisson bivector \( \mathcal{P} \).

In variables \( z = (\gamma, K) \) bivector \( \mathcal{P} \) is equal to

\[ \mathcal{P} = \frac{d}{2\delta \mu} \left( \begin{array}{ccc} 0 & \mathbf{Y} & 0 \\ -\mathbf{Y}^{\top} & \Omega & 0 \\ 0 & 0 & \Pi \end{array} \right) + \frac{d\mu}{2\delta} \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & \Pi \end{array} \right), \]

(2.3)

where \( \mathbf{Y} \) is the following \( 3 \times 3 \) matrix

\[ \mathbf{Y} = \Gamma_{3} - \Gamma A(dA - \text{Id})^{-1}, \]

matrix \( \Gamma \) is equal to

\[ \Gamma = \left( \begin{array}{ccc} 0 & \gamma_{3} & -\gamma_{2} \\ -\gamma_{3} & 0 & \gamma_{1} \\ \gamma_{2} & -\gamma_{1} & 0 \end{array} \right), \]

and

\[ \Gamma_{3} = \left( \begin{array}{ccc} \gamma_{1}\gamma_{2}\gamma_{3}(a_{3}-a_{2}) & \gamma_{2}\gamma_{3}(a_{3}-a_{2}) & \gamma_{2}\gamma_{3}(a_{3}-a_{2}) \\ \gamma_{2}\gamma_{3}(a_{3}-a_{2}) & \gamma_{1}\gamma_{2}\gamma_{3}(a_{3}-a_{2}) & \gamma_{2}\gamma_{3}(a_{3}-a_{2}) \\ \gamma_{2}\gamma_{3}(a_{3}-a_{2}) & \gamma_{2}\gamma_{3}(a_{3}-a_{2}) & \gamma_{1}\gamma_{2}\gamma_{3}(a_{3}-a_{2}) \end{array} \right), \]

so that

\[ \Gamma_{3} \Gamma = 0. \]

Skew-symmetric \( 3 \times 3 \) matrix \( \Pi \) has a more cumbersome form

\[ \Pi = \frac{1}{\det(dA - \text{Id})} \left( \begin{array}{ccc} 0 & a_{1}\gamma_{1}\gamma_{2}\gamma_{3}(\gamma \times K)_{3} + \beta_{3}K_{3} & a_{2}\gamma_{1}\gamma_{2}\gamma_{3}(\gamma \times K)_{2} + \beta_{2}K_{2} \\ * & 0 & a_{1}\gamma_{1}\gamma_{2}\gamma_{3}(\gamma \times K)_{3} + \beta_{3}K_{3} \\ * & * & 0 \end{array} \right), \]
where

\[
\begin{align*}
\alpha_3 &= (a_2 - a_1)(d(a_2 - a_3)(a_3 - a_1)\gamma_3^2 + a_3(1 - da_2)(1 - da_1)) , \\
\alpha_2 &= (a_1 - a_3)(d(a_2 - a_3)(a_1 - a_2)\gamma_2^2 + a_2(1 - da_3)(1 - da_1)) , \\
\alpha_1 &= (a_2 - a_3)(d(a_1 - a_3)(a_2 - a_1)\gamma_1^2 + a_1(1 - da_3)(1 - da_2)) ,
\end{align*}
\]

and

\[
\begin{align*}
\beta_1 &= d(a_1-a_2)^2(1-d_{a_3})\gamma_3^2 + d(a_1-a_3)^2(1-d_{a_2})\gamma_2^2 \\
&- d(a_1-a_2)(d_{a_1}a_2+2da_1a_3-2da_2a_3-a_{3-2a_1+2a_2})\gamma_1^2 \\
&- d(a_1-a_3)(2da_1a_2+da_2a_3-2da_3-a_{3-2a_1+2a_3})\gamma_2^2 \\
&+ (a_{1-2a_2})(2d_{a_1}a_{2-3d}a_3-a_{3-2a_1+2a_2+1})\gamma_2^2 \\
&+ (a_{1-2a_3})(2d_{a_1}a_{2-3d}a_3-a_{3-2a_1+2a_3+1})\gamma_2^2 \\
&+ (2a_{1-2a_2})(2d_{a_1}a_{2-3d}a_3-a_{3-2a_1+2a_2+1})\gamma_2^2 \\
&+ (1-a_{1-2a_1})(2d_{a_1}a_{2-3d}a_3-a_{3-2a_1+2a_3+1})\gamma_2^2 ,
\end{align*}
\]

\[
\begin{align*}
\beta_2 &= d(a_1-a_2)^2(1-d_{a_3})\gamma_3^2 + d(a_2-a_3)^2(1-d_{a_1})\gamma_3^2 \\
&- d(a_1-a_2)(d_{a_1}a_2+2da_1a_3-2da_2a_3-a_{3-2a_1+2a_2})\gamma_1^2 \\
&- d(a_2-a_3)(d_{a_2}a_3-2da_2a_3-a_{3-2a_1+2a_3})\gamma_1^2 \\
&+ (a_{1-2a_2})(2d_{a_1}a_{2-3d}a_3-a_{3-2a_1-2a_3})\gamma_1^2 \\
&+ (a_{2-2a_3})(2d_{a_2}a_{3-3d}a_3-a_{3-2a_1-2a_3+1})\gamma_1^2 \\
&+ (a_{1-2a_3})(2d_{a_1}a_{2-3d}a_3-a_{3-2a_1-2a_3+1})\gamma_1^2 \\
&+ (2a_{1-2a_2})(2d_{a_1}a_{2-3d}a_3-a_{3-2a_1-2a_3+1})\gamma_1^2 ,
\end{align*}
\]

\[
\begin{align*}
\beta_3 &= d(a_1-a_3)^2(1-d_{a_2})\gamma_2^2 + d(a_2-a_3)^2(1-d_{a_1})\gamma_2^2 \\
&- d(a_1-a_3)(d_{a_1}a_3-2da_2a_3-a_{3-2a_1-2a_3})\gamma_3^2 \\
&- d(a_3-a_2)(d_{a_3}a_2-2da_2a_3-a_{3-2a_1-2a_3})\gamma_3^2 \\
&+ (a_{1-2a_3})(2d_{a_1}a_{2-3d}a_3-a_{3-2a_1-2a_3+1})\gamma_3^2 \\
&+ (a_{2-2a_1})(2d_{a_2}a_{3-3d}a_3-a_{3-2a_1-2a_3+1})\gamma_3^2 \\
&+ (a_{1-2a_3})(2d_{a_1}a_{2-3d}a_3-a_{3-2a_1-2a_3+1})\gamma_3^2 \\
&+ (2a_{1-2a_2})(2d_{a_1}a_{2-3d}a_3-a_{3-2a_1-2a_3+1})\gamma_3^2 ,
\end{align*}
\]

It’s easier to get this solution on a computer than to write it out.

At \( \gamma^2 = 1 \) and \((\gamma, K) = 0\) we can simplify these expressions by using algebraic transformation variables \( K_i \rightarrow L_i \) defined by equations of the form

\[
\begin{align*}
L_1 &= 2\rho(1-d_{a_2}) \left( \frac{a_{2a_3}(\gamma_1^2+\gamma_3^2)d-a_3\gamma_3^2-a_2\gamma_1^2}{d(\gamma_2^2+\gamma_3^2)} \right) K_1 - (a_3-a_2)\gamma_1\gamma_2\gamma_3 \left( \gamma \times K \right)_1 , \\
L_2 &= 2\rho(1-d_{a_3}) \left( \frac{a_{a_3}(\gamma_1^2+\gamma_3^2)d-a_3\gamma_3^2-a_1\gamma_3^2}{d(\gamma_1^2+\gamma_3^2)} \right) K_2 - (a_3-a_1)\gamma_1\gamma_2\gamma_3 \left( \gamma \times K \right)_2 , \\
L_3 &= 2\rho(1-d_{a_2}) \left( \frac{d_{a_2}(\gamma_1^2+\gamma_2^2)-\gamma_3^2a_2-a_1\gamma_2^2}{d(\gamma_1^2+\gamma_2^2)} \right) K_3 - (a_2-a_1)\gamma_1\gamma_2\gamma_3 \left( \gamma \times K \right)_3 .
\end{align*}
\]

Here \( \rho(1,1,4) \) is an algebraic function of variables \( \gamma_1, \gamma_2 \) and \( \gamma_3 \).
In variables \( z = (\gamma, L) \) Poisson bivector \( \mathcal{P} \) \( (2.3) \) becomes a quite visible object

\[
\mathcal{P} = \begin{pmatrix}
0 & 0 & 0 & 0 & \gamma_3 & -\gamma_2 \\
0 & 0 & 0 & -\gamma_3 & 0 & \gamma_1 \\
0 & 0 & 0 & \gamma_2 & -\gamma_1 & 0 \\
-\gamma_3 & 0 & \gamma_1 & -L_3 & 0 & L_1 \\
\gamma_2 & -\gamma_1 & 0 & L_2 & -L_1 & 0
\end{pmatrix}
\]  
(2.5)

\[
\mathcal{P} = \frac{2p(da_1 - 1)(da_2 - 1)(da_3 - 1)\Omega}{d} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \gamma_3 & -\gamma_2 & 0 \\
0 & \gamma_1 & -\gamma_3 & 0 & \gamma_1 & 0 \\
0 & 0 & \gamma_2 & -\gamma_1 & 0 & 0
\end{pmatrix}.
\]

Below we prove that this bivector is a non algebraic deformation of the Lie-Poisson bivector on \( e^*(3) \)

\[
\mathcal{P}_0 = \begin{pmatrix}
0 \\
-\Gamma^\top \\
M
\end{pmatrix} 
\]  
(2.6)

where

\[
\Gamma = \begin{pmatrix}
0 & \gamma_3 & -\gamma_2 \\
-\gamma_3 & 0 & \gamma_1 \\
\gamma_2 & -\gamma_1 & 0
\end{pmatrix}, \quad M = \begin{pmatrix}
0 & M_3 & -M_2 \\
-M_3 & 0 & M_1 \\
M_2 & -M_1 & 0
\end{pmatrix}.
\]

It means that we cannot reduce bivector \( \mathcal{P} \) \( (2.3) \) to the Lie-Poisson bivector \( \mathcal{P}_0 \) \( (2.0) \) using only algebraic transformations of variables.

### 2.1 Deformation of the canonical Poisson bivector linear in momenta

Let \( Q \) be an \( n \)-dimensional smooth manifold endowed with \((1,1)\) tensor field \( \Lambda(q) \) with vanishing Nijenhuis torsion and with vector field \( \nu(q) \). Canonical Poisson bracket on its cotangent bundle \( T^*Q \)

\[
\{q_i, q_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}, \quad \{p_i, p_j\} = 0,
\]

where \( q_i, p_i \) are fibered coordinates, after linear transformation of momenta

\[
p_j \to \sum_{i=1}^n A^i_j(q)p_i + \nu_j(q)
\]

looks like

\[
\{q_i, q_j\}' = 0, \quad \{q_i, p_j\}' = A^i_j, \quad \{p_i, p_j\}' = \sum_{k=1}^n \left( \frac{\partial A_k^i}{\partial q_j} - \frac{\partial A_k^j}{\partial q_i} \right) p_k + \frac{\partial \nu_j}{\partial q_i} - \frac{\partial \nu_i}{\partial q_j}.
\]

It is the so-called Turiel type deformation of the canonical Poisson bracket on \( T^*Q \), see \[44, 35\] and references within.

Bivector \( \mathcal{P} \) \( (2.3) \) determines standard Poisson brackets \( \{\gamma_i, \gamma_j\} = 0 \) and \( \{\gamma_i, L_j\} = \varepsilon_{ijk} \gamma_k \) and non-standard brackets \( \{L_i, L_j\} \) between momenta. Let us consider a change of variables

\[
L_i \to M_i = M_i + \eta_i(\gamma),
\]

which reduces these non-standard Poisson brackets to canonical form

\[
\{M_1, M_2\} = M_3, \quad \{M_2, M_3\} = M_1, \quad \{M_3, M_1\} = M_2
\]

(2.7)

at

\[
(\gamma, M) = \eta_1 \gamma_1 + \eta_2 \gamma_2 + \eta_3 \gamma_3 = 0 \quad \text{and} \quad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1.
\]

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These algebraic equations hold if we use the following anzats
\[ \eta_1 = \gamma_2 \eta_4 - \frac{\gamma_1}{\gamma_1 + \gamma_2} \gamma_3 \eta_3, \quad \eta_2 = -\gamma_1 \eta_4 - \frac{\gamma_2}{\gamma_1 + \gamma_2} \gamma_3 \eta_3, \]
for functions \( \eta_{1,2} \). We substitute this anzats into (2.8) and solve the resulting differential equations on a computer.

First partial solution of the equations (2.8) reads as \( \eta_3 = 0 \) and
\[ \eta_4 = \frac{2(1 - da_3)(1 - da_2)(1 - da_1)\Omega}{d^{3/2} \kappa} F \left( \sqrt{d(a_2 - a_1) - d(a_1 - a_2) - da_1}, \sqrt{d(a_1 - a_2) + da_1 + 1} \right), \]
where \( F \) is an incomplete elliptic integral of the first kind.

Second partial solution of the equations (2.8) looks like \( \eta_4 = 0 \) and
\[ \eta_3 = \frac{2(1 - da_3)(1 - da_2)(1 - da_1)\Omega}{d^{3/2} \kappa} \ln \left( \sqrt{d \kappa + 1} \right), \]
where function \( \rho \) defines the invariant measure (1.14) and
\[ \kappa = \sqrt{(a_1 - a_3)\gamma_1^2 + (a_2 - a_3)\gamma_2^2 \over \gamma_1^2 + \gamma_2^2}. \]

Thus, in variables \( z = (\gamma, M) \) vector field \( Z (1.11) \) becomes a conformally Hamiltonian vector field
\[ Z = f(\gamma) \mathcal{P} dH \]
defined by Hamiltonian \( H (1.13) \) which is a non-algebraic function in variables \( M (2.7) \) on the cotangent bundle of two-dimensional sphere \( T^*\mathbb{S}^2 \).

This result was obtained by directly solving algebraic and differential equations on a computer. Now we can apply this result in order to prove that bivector \( \mathcal{P} \) is the Turiel type deformation of canonical Poisson brackets. As we identify phase space with \( T^*\mathbb{S}^2 \) we can introduce spherical coordinates and momenta
\[ \phi = \arctan \gamma_1 / \gamma_2, \quad p_\phi = -K_3, \quad \theta = \arccos \gamma_3, \quad p_\theta = -\gamma_2 K_1 - \gamma_1 K_2 \sqrt{\gamma_1^2 + \gamma_2^2}. \]

**Proposition 2** Transformation of momenta
\[ p_\phi = \Lambda_1(\phi, \theta)p_\phi + \Lambda_1^2(\phi, \theta)p_\phi + \nu_1(\phi, \theta), \]
\[ p_\theta = \Lambda_1^2(\phi, \theta)p_\phi + \Lambda_2^2(\phi, \theta)p_\phi + \nu_2(\phi, \theta) \]
reduces the Poisson bracket associated with bivector \( \mathcal{P} (2.8) \) to a canonical Poisson bracket on the cotangent bundle of two-dimensional sphere
\[ \{ \phi, \theta \} = 0, \quad \{ \phi, p_\phi \} = \{ \theta, p_\theta \} = 1, \quad \{ p_\phi, p_\theta \} = 0, \]
if
\[ \Lambda = \frac{2(da_3 - 1)\rho}{d} \left( \frac{(a_1 - a_2)\cos^2 \phi - da_1 a_2 + a_2}{(a_1 - a_2)\sin 2\phi \cos \theta} \right) \]
\[ \quad \times \left( \frac{(a_1 - a_2)\sin 2\phi \cos \theta}{2 \sin \theta} \right) \times \frac{\lambda}{da_1 - 1}, \]
where
\[ \lambda = -(da_3 - 1)(a_1 - a_2)\cos^2 \theta \cos^2 \phi + (da_2 - 1)(a_1 - a_3)\cos^2 \theta - (da_1 - 1)(da_2 - 1)a_3 \]
and functions \( \nu_{1,2}(\phi, \theta) \) satisfy differential equation
\[ \frac{\partial \nu_1}{\partial \theta} - \frac{\partial \nu_2}{\partial \phi} = 2\Omega(1 - da_1)(1 - da_2)(1 - da_3)\rho \sin \theta. \]

Partial solutions of this equation are incomplete elliptic integral of the first kind and logarithmic function, which have been obtained by brute force method before.
It is easy to prove that (1,1) tensor field $\Lambda(q)$ has zero Nijenhuis torsion and, therefore, bivector $\mathcal{P}$ is a Turiel type deformation of the canonical Poisson bivector.

In nonholonomic mechanics functions $L_j(q)$ and $\nu_j(q)$ are usually algebraic functions on configuration space $Q$. For instance, see linear deformations appearing for:

- reduced motion of the Chaplygin ball on the plane \[34\];
- reduced motion of the Chaplygin ball on the sphere \[36\];
- nonholonomic Veselova system and its generalizations \[37\];
- reduced motion of the Routh ball on the plane \[3\];
- nonholonomic motion of a body of revolution on the plane \[40\];
- nonholonomic motion of a homogeneous ball on the surface of revolution \[40\];
- nonholonomic Heisenberg type systems \[22\];
- other non-Hamiltonian systems associated with Turiel type deformations \[35\].

In contrast with these systems with two degrees of freedom one gets non-algebraic deformations involving elliptic integrals or logarithms for reduced motion of the Chaplygin ball on the rotating plane \[1.11\].

In fact, it is the main result because it allows us to essentially enlarge the list of possible Turiel type deformations appearing in mathematical description of real physical systems. We suppose that similar non-algebraic deformations of the canonical Poisson brackets appear also in other models of rigid body motion on rotating surfaces \[13\], \[25\], \[29\], \[45\], \[48\].

Non-algebraic deformations for the non-Hamiltonian systems with three degrees of freedom are also discussed in \[8\], \[39\].

### 3 Sum of the Hamiltonian vector fields

At $a_1 = a_2$ vector field $Z$ \[(1.11)\] has a formal integral of motion

$$H_2 = \rho K_3 - \frac{da_1 \rho (\gamma, K) \gamma_3}{da_1 - 1} + \frac{da_1 - 1}{a_1 \sqrt{d(a_1 - a_3)}} \Omega \ln\left(\sqrt{d(a_1 - a_3) \gamma_3 + \rho^{-1}}\right),$$

with a logarithmic term, see discussion in \[6\].

**Proposition 3** At $a_1 = a_2$ vector field $Z$ \[(1.11)\] is a sum of two Hamiltonian vector fields

$$Z = f_1(z) \mathcal{P}' dH + f_2(z) \mathcal{P}' dH_2$$

with coefficients

$$f_1 = 2\rho \delta, \quad \text{and} \quad f_2 = \frac{2a_1(a_1 - a_3) \gamma_3 \cdot (\gamma, K)}{d}.$$ 

Here $H$ is a Jacobi integral and Poisson bivector $\mathcal{P}'$ is equal to

$$\mathcal{P}' = \mathcal{P} + \frac{d^2 \rho \cdot (\gamma, K)}{2((a_3 - a_1) \gamma_3^2 + (da_1 - 1)a_3)(da_1 - 1)} \left( \begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\sigma_3 & 0 & \sigma_1 \\
0 & 0 & 0 & -\sigma_3 & -\sigma_1 & 0 \\
0 & 0 & 0 & -\sigma_3 & -\sigma_1 & 0 \\
\end{array} \right),$$

where

$$\sigma_1 = \left( a_1 - \frac{(a_1 - a_3) \gamma_3^2}{1 - \gamma_1^2} \right) \gamma_1, \quad \sigma_2 = \left( a_1 - \frac{(a_1 - a_3) \gamma_3^2}{1 - \gamma_2^2} \right) \gamma_2,$$

and

$$\sigma_3 = \left( a_1a_3(da_1 - 1) - (a_1 - a_3)^2 \gamma_3^2 + \frac{a_3 - a_1}{d} \right) \frac{\gamma_3}{da_3 - 1}.$$
The proof is a straightforward calculation.

In this case we cannot use any type of Hamiltonization for reduction of equations of motion to a Hamiltonian form. Nevertheless, we can calculate variables of separation for equations of motion (1.11) directly solving equation (1.10) in the framework of the Jacobi method.

Because we know the solution (3.2) of equation (1.9) we have to solve (1.10) together with equation
\[ [\mathcal{P}', \mathcal{P}''] = 0, \]
which guarantees compatibility of bivectors \( \mathcal{P}' \) and \( \mathcal{P}'' \). Solving these equations on a computer we use the following ansatzs:

- entries of \( \mathcal{P}'' \) are linear functions on momenta;
- first coefficient \( g_1 \) is a function on \( \gamma_3 \) similar to (3.2);
- second coefficient \( g_2 \) is a linear function in momenta similar to (3.2).

After a few seconds one gets the following answer
\[
\mathcal{P}'' = \begin{pmatrix} 0 & \gamma'' \\ -\gamma''^\top & \Pi'' \end{pmatrix},
\]
where
\[
\gamma'' = \begin{pmatrix} -\gamma_3 \gamma_2 \left( \frac{1}{\rho} - \frac{1}{a_1(\gamma_1 + \gamma_2)^2} \right) - \gamma_3 \left( \frac{\gamma_2^2}{\rho^2} + \frac{\gamma_2^2}{a_1(\gamma_1 + \gamma_2)^2} \right) ; \\
-\frac{\gamma_3}{\gamma_1 + \gamma_2} \left( \frac{\gamma_2^2}{\rho^2} + \frac{\gamma_2^2}{a_1(\gamma_1 + \gamma_2)^2} \right) - \frac{\gamma_3}{\gamma_1 + \gamma_2} \left( \frac{1}{\rho} - \frac{1}{a_1(\gamma_1 + \gamma_2)^2} \right) ; \\
-\frac{\gamma_3}{a_1(\gamma_1 + \gamma_2)^2} & 0 \end{pmatrix}
\]
and entries of the skew symmetric matrix \( \Pi'' \) are equal to
\[
\Pi''_{12} = -\frac{\gamma_3}{\gamma_1 + \gamma_2} \left( -\frac{(\gamma_1 K_1 + \gamma_2 K_2)}{\rho} + \frac{(da_1 - 1)\Omega - a_1 K_3}{a_1^2} + \frac{\rho^2(d a_3 \gamma_3 K_3 - \frac{(a_1 + x_2 K_2)(a_3 - 1)}{\gamma_1 + \gamma_2})}{a_1} \right),
\]
\[
\Pi''_{13} = -\frac{K_3}{\rho} - \frac{(da_1 - 1)\Omega}{a_1} - \frac{d(a_1(\gamma_1 + \gamma_2) + a_3 \gamma_3 K_3)}{a_1} \frac{\gamma_2}{a_1(\gamma_1 + \gamma_2)^2},
\]
\[
\Pi''_{23} = -\frac{K_3}{\rho} + \frac{(da_1 - 1)\Omega}{a_1} + \frac{d(a_1(\gamma_1 + x_2 K_2) + a_3 \gamma_3 K_3)}{a_1} \frac{\gamma_2}{a_1(\gamma_1 + \gamma_2)^2}.
\]

In this case coefficients in decomposition (1.4) are equal to
\[
g_1(z) = \frac{a_1 d(1 - \gamma_2^2)}{2(1 - da_1)}
\]
and
\[
g_2(z) = \left( \frac{a_2(\gamma_1^2 + \gamma_2^2)}{(a_1 - 1)\rho} + da_2 a_3(\gamma_1^2 + \gamma_2^2) - a_3(da_1 - 1)\rho^2 \right) K_3
\]
\[
- a_1 \gamma_3 \left( \frac{a_1}{da_1 - 1}\rho + \frac{da_2(a_3 - 1)\rho}{da_1 - 1} - \frac{(da_3 - 1)\rho^3}{\gamma_1^2 + \gamma_2^2} \right) (\gamma_1 K_1 + \gamma_2 K_2).
\]

Both bivectors \( \mathcal{P}' \) and \( \mathcal{P}'' \) are degenerate and, therefore, we cannot determine recursion operator (1.4), but we can easily calculate control matrix \( \mathbf{F} \) (1.5) and its eigenvalues, which are desired variables of separation. In our case we have to solve algebraic equations
\[
\mathcal{P}'dH = \mathbf{F}_{11}\mathcal{P}'dH + \mathbf{F}_{12}\mathcal{P}''dH_2,
\]
\[
\mathcal{P}'dH_2 = \mathbf{F}_{21}\mathcal{P}'dH + \mathbf{F}_{22}\mathcal{P}''dH_2.
\]
with respect to four entries of matrix \( F \). In our case \( F_{21} = 0 \) and we can easily calculate eigenvalues of \( F \)

\[
\lambda_1 = F_{21} = \frac{d}{2a_1(da_3 - 1)(da_1 - 1)} \quad \text{and} \quad \lambda_2 = \frac{d(1 - \gamma_3^2)a_1\sqrt{d(a_1 - a_3)\gamma_3^2 - a_1d + 1}}{2(da_1 - 1)(da_1a_3 - (a_1 - a_3)\gamma_3^2 - a_3)}.
\]

There is only one nontrivial eigenvalue \( \lambda_2 \) which is the desired variable of separation. Now we have to validate that our algorithm yields quadrature

\[
\frac{d\lambda_2}{dt} = F(\lambda_2),
\]

i.e. an equation which includes only variable \( \lambda_2 \) and its differential. This quadrature will be discussed in the following subsection.

Of course, this variable of separation may be obtained by directly using symmetries of the ball similar to the Lagrange approach to symmetric heavy top. More complicated, but algorithmic computer calculations of variables of separation were given in order to underline the importance of studying equations (1.9-1.10) in nonholonomic mechanics.

### 3.1 Separation of variables

It is easy to prove that equation of motion

\[
\dot{\gamma}_3^2 = a_1^2(\gamma_2 M_1 - \gamma_1 M_2)^2 = F(\gamma_3)
\]

on common level surface of the first integrals

\[
C_1 = 1, \quad C_2 = \ell, \quad H_2 = k, \quad 2H = h
\]

includes only one variable with its own differential. Here

\[
F(x) = \frac{a_1^2d(x^2 - 1)}{da_1 - 1} + \frac{a_1^2(x^2 - 1)(dx^2(a_1 - a_3) + 2d^2a_1a_3 - 2da_1 - da_3 + 1)}{(da_3 - 1)(da_1 - 1)^2} \ell^2
\]

\[
- \left( \frac{a_1x\sqrt{dx^2(a_1 - a_3) - da_1 + 1}}{(da_1 - 1)\sqrt{da_3 - 1}} \ell - a_1\sqrt{da_3 - 1}k \right)
\]

\[
+ \frac{(da_1 - 1)\sqrt{da_3 - 1} \ln \left(\sqrt{d(a_1 - a_3)x + \sqrt{dx^2(a_1 - a_3) - da_1 + 1}}\right)}{\sqrt{d(a_1 - a_3)}}
\]

Thus, at \( a_1 = a_2 \) solution of the six equations of motion (1.11) is reduced to one nontrivial quadrature

\[
\int_{\gamma_3}^{\gamma_3} \frac{dx}{\sqrt{F(x)}} = t
\]

involving non-algebraic function \( F(x) \) [3.6]. We hope that study of such non-Abelian quadratures can be carried out by means other than numerical simulations.

Following Lagrange and Jacobi we can introduce two pairs of independent variables of separation

\[
x_1 = \gamma_3, \quad y_1 = a_1(\gamma_1 K_2 - \gamma_2 K_1), \quad \text{and} \quad x_2 = \phi(\gamma_1/\gamma_2), \quad y_2 = H_2.
\]

so that

\[
\{x_1, x_2\}' = \{y_1, y_2\}' = \{y_1, x_2\}' = \{x_1, y_2\}' = 0
\]
and
\[
\{x_1, y_1\}' = f_1(x_1) = \frac{a_1 \sqrt{d(a_1 - a_3)x_1^2 - a_1d + 1(x_1^2 - 1)}}{d(a_1 - a_3)(a_3(da_1 - 1) - (a_1 - a_3)x_1^2)},
\]
(3.8)
\[
\{x_2, y_2\}' = f_2(x_2) = \frac{\psi(x_2)}{da_1(a_1 - a_3)(a_3 - 1)}, \quad \psi = \frac{d(\gamma_1/\gamma_2)}{d(\gamma_1/\gamma_2)}.
\]

Here \(\phi(\gamma_1/\gamma_2)\) is an arbitrary function and \(\{., .\}'\) is the Poisson bracket associated with the Poisson bivector \(\mathcal{P}'\).

In the framework of the Jacobi method we identify these variables of separation with affine coordinates of divisors. First divisor \(P_1 = (x_1, y_2)\) belongs to plane curve \(X'_1\) defined by non-algebraic equation of the form
\[
X'_1: \quad y^2 - F(x) = 0.
\]
(3.9)
Second divisor \(P_2 = (x_2, y_2)\) lies on horizontal line \(X'_2\) defined by the equation
\[
X'_2: \quad y - k = 0.
\]
(3.10)

Thus, we can formulate the following proposition.

**Proposition 4** At \(a_1 = a_2\) common level surface of the first integrals (3.8) of non-Hamiltonian vector field \(Z\) (1.11) is a product of two plane curves \(X'_1 \times X'_2\) (3.9, 3.10).

The proof is a straightforward calculation.

Divisors \(P_1\) and \(P_2\) determine compatible Poisson brackets, recursion operators, algebra of Haantjes operators with vanishing Nijenhuis torsion and \(2 \times 2\) Lax matrices. For instance, we can easily recover Poisson bivector \(\mathcal{P}'\) (3.8) obtained by brute force method in the previous Section
\[
\mathcal{P}' = f_1(x_1) \left( \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial x_1} \right) + f_2(x_2) \left( \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_2} \wedge \frac{\partial}{\partial x_2} \right),
\]
where \(f_{1,2}\) are given by (3.8).

Constructions of the Poisson brackets (1.3), recursion operators (1.4) and algebra of Haantjes operators
\[
L_0 = Id, \quad L_i = \frac{\partial}{\partial x_i} \otimes dx_i + \frac{\partial}{\partial y_i} \otimes dy_i, \quad i = 1, 2,
\]
are independent from the form of plane curves \(X'_1\) and \(X'_2\) and time variable, in contrast with construction of the \(2 \times 2\) Lax matrices
\[
\mathcal{L} = \begin{pmatrix} V & U \\ W & -V \end{pmatrix} \quad \text{and} \quad \mathcal{A} = \begin{pmatrix} 0 & 1 \\ S & 0 \end{pmatrix}, \quad \text{so that} \quad \dot{\mathcal{L}} = [\mathcal{L}, \mathcal{A}],
\]
here \(x\) is a spectral parameter, polynomials in \(x\)
\[
U(x) = x_2(x - x_1), \quad V(x) = \frac{1}{2} \dot{U}(x),
\]
are the standard Jacobi polynomials on a product of plane curves \(X'_1\) and \(X'_2\) and
\[
W(x) = \frac{F(x) - V^2(x)}{U(x)}, \quad S(x) = -\frac{\dot{W}(x)}{2V(x)},
\]
are functions on \(x\).

Unfortunately, notion of the compatible Poisson brackets, recursion operators, algebra of Haantjes operators with vanishing Nijenhuis torsion and Lax matrices can not help us in the search of real trajectories of motion.
4 Conclusion

In this note we discuss a non-Hamiltonian vector field appearing in consideration of a motion of the Chaplygin ball rolling on a horizontal plane which rotates with constant angular velocity \[6\]. In two partial cases we present division of this vector field by Hamiltonian vector fields using brute force computer calculations. In the first case one reduces equations of motion to Hamiltonian form which can also be done in the framework of Hamiltonization theory. In the second case vector field is a sum of two Hamiltonian vector fields which cannot be obtained by using any type of Hamiltonization. In both cases we obtain Turin type deformations of canonical Poisson brackets on the cotangent bundle to the sphere.

Calculation of at least two different representations of a given vector field via Hamiltonian vector fields is a crucial part of finding variables of separation in the Jacobi method. In any existing theory of Hamiltonization the main aim is to obtain one very special representation, which may not exist for the given mathematical model of a physical process, and, therefore, these theories are not applicable to algorithmic search of variables of separation. We prefer to develop a computer version of the Jacobi method which could be applicable both Hamiltonian and non-Hamiltonian vector fields with relatively low numbers of equations of motion. Of course, these computer algorithms are not applicable to abstract nonholonomic systems with non-fixed arbitrary numbers of degrees of freedom.

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