3-String Junction and BPS Saturated Solutions in $SU(3)$ Supersymmetric Yang-Mills Theory

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Abstract

We construct BPS saturated regular configurations of $\mathcal{N} = 4$ $SU(3)$ supersymmetric Yang-Mills theory carrying non-parallel electric and magnetic charges. These field theory BPS states correspond to the string theory BPS states of 3-string junctions connecting three different D3-branes by regarding the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory as an effective field theory on parallel D3-branes.

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1 Introduction

The recent developments in non-perturbative string theory have provided new tools to investigate non-perturbative features of field theory. Many supersymmetric gauge theories can be studied as effective field theories on branes. In this picture, the non-perturbative features of the supersymmetric gauge theories are tightly related to those of string theory. A BPS state of string theory has a correspondence to a BPS state of the effective field theory on the brane. The 3+1 dimensional $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang-Mills (SYM) theory broken spontaneously to $U(1)^{N-1}$ can be studied as an effective field theory on $N$ parallel D3-branes [1, 2]. Since a D3-brane is an invariant object under the $SL(2, Z)$ duality transformation of IIB string theory, the $SL(2, Z)$ duality implies the $SL(2, Z)$ duality symmetry [3] of the $\mathcal{N} = 4$ SYM theory. A fundamental string between different D3-branes corresponds to a W-boson of the SYM theory, which is a BPS state preserving $\frac{1}{2}$ of the supersymmetries on the D3-branes. By performing the $SL(2, Z)$ duality transformation on the configurations, a $(p, q)$ string should correspond to a BPS state of the field theory with electric charge $p$ and magnetic charge $q$, where $p$ and $q$ are relatively prime integers. The state with $p = 0$ and $q = 1$ is a monopole, while states with $p \neq 0$ and $q = 1$ are dyons. Their field configurations are explicitly known in the form of the Prasad-Sommerfield solution [4]. It is a difficult question whether the other dyon states exist as well in the field theory. In the simplest case of $SU(2)$ with $q = 2$, the existence was shown by the quantization of the collective modes of the two monopole solutions [5].

Recently, BPS configurations of IIB strings with 3-string junctions have been attracting attentions. It was conjectured originally in [6] that a 3-string junction would exist under both the conditions that the forces from the three strings should balance and that the $(p, q)$ charges of the three strings should conserve at the junction. The BPS feature of the 3-string junction has been shown recently by several authors in both the string picture [7, 8] and the M-theory picture [9]. The relations to the BPS states of field theory have been discussed in several contexts [10]-[13]. The U-dual of the Hanany-Witten effect [14] implies the existence of a certain series of 3-string junctions, and they were used in the description of the exceptional Lie groups [15]. On the other hand, requiring only the above two conditions of 3-string junction, more BPS string states were observed than those expected in field theory [12]. The selection rule of proper BPS string states has not been established yet.

Thus it would be interesting to show explicitly the existence of the field theory BPS states
corresponding to some BPS string states with 3-string junctions in a simple context. Since a IIB string with any \((p, q)\) charge can end on a D3-brane, there would be a BPS state of \(\mathcal{N} = 4\) SYM theory corresponding to a 3-string junction connecting three different D3-branes. These BPS states preserve only \(\frac{1}{4}\) of the D3-brane world volume supersymmetry, and hence these states are different from the BPS states mentioned above. An evidence for the existence of such BPS states in \(\mathcal{N} = 4\) SYM theory was recently obtained in [11]. The author argued that such a BPS state has non-parallel electric and magnetic charges in the world volume theory, and showed that, under the assumption of BPS saturation, the mass of such a state agrees with the mass obtained from the IIB string picture.

In this paper, we find spherically symmetric regular solutions to the Bogomol’nyi equations in the cases of non-parallel electric and magnetic charges. Our general solution corresponds to the three-pronged strings carrying charges \((a, 2)\), \((b, 0)\) and \((-a - b, -2)\), where \(a\) and \(b\) take arbitrary real values.

2 Bogomol’nyi equations and the asymptotic behavior

Since we are interested in the 3-string junction, let us consider the simplest case of three D3-brane system, and hence the 3+1 dimensional \(\mathcal{N} = 4\) SYM system with gauge group \(SU(3)\). Among the six scalar fields describing the transverse coordinates of the D3-branes, we keep only the two, \(X\) and \(Y\), corresponding to the two dimensional plane on which the three-pronged strings lie. Therefore, the bosonic part of the Hamiltonian of the system reads

\[
H = \int d^3x \frac{1}{2} \text{Tr} \left\{ (B_i)^2 + (E_i)^2 + (D_iX)^2 + (D_iY)^2 + (D_0X)^2 + (D_0Y)^2 - [X, Y]^2 \right\},
\]

where \(B_i = \frac{1}{2} \epsilon_{ijk} F_{jk}\) and \(E_i = F_{0i}\) are the magnetic and electric fields, respectively, and the covariant derivative is defined by \(D_\mu X = \partial_\mu X - i [A_\mu, X]\). We have put the Yang-Mills coupling constant equal to one, and shall consider the case of vanishing vacuum theta angle. The BPS saturation condition is derived by introducing an angle \(\theta\) as [13]

\[
H = \int d^3x \frac{1}{2} \text{Tr} \left\{ (E_i \cos \theta - B_i \sin \theta - D_iX)^2 + (B_i \cos \theta + E_i \sin \theta - D_iY)^2 + (D_0X)^2 + (D_0Y)^2 - [X, Y]^2 \right\} + (Q_X + M_Y) \cos \theta + (Q_Y - M_X) \sin \theta
\geq \sqrt{(Q_X + M_Y)^2 + (Q_Y - M_X)^2},
\]

(2.2)
where $M_{X,Y}$ and $Q_{X,Y}$ are defined by

$$
M_{X} = \int d^3x \text{Tr} (B_i D_i X) = \int \limits_{r \rightarrow \infty} dS_i \text{Tr} (B_i X), \quad Q_{X} = \int d^3x \text{Tr} (E_i D_i X) = \int \limits_{r \rightarrow \infty} dS_i \text{Tr} (E_i X).
$$

(2.3)

The lower bound in (2.2) is saturated when the following conditions hold:

$$
D_i X = \epsilon_i \cos \theta - B_i \sin \theta, \quad (2.4)
$$

$$
D_i Y = B_i \cos \theta + \epsilon_i \sin \theta, \quad (2.5)
$$

$$
D_0 X = D_0 Y = 0, \quad (2.6)
$$

$$
[X, Y] = 0, \quad (2.7)
$$

and the corresponding angle $\theta$ is given by

$$
\tan \theta = (Q_Y - M_X) / (Q_X + M_Y). \quad (2.8)
$$

Besides the four equations (2.4)–(2.7), we have to impose the Gauss law,

$$
D_i \epsilon_i = 0, \quad (2.9)
$$

since we used it in converting the volume integration of $Q_{X,Y}$ (2.3) into the surface one. Note that eq. (2.8) is an automatic consequence of the two equations (2.4) and (2.5) and need not be imposed separately.

Suppose that we have a static solution $(A_\mu(r), X(r), Y(r))$ to the equations (2.4)–(2.7) and (2.9), and that their asymptotic ($r \rightarrow \infty$) forms are, after a suitable gauge transformation, given (locally) as follows:

$$
X \sim \text{diag}(x_1, x_2, x_3) + \frac{1}{2r} \text{diag}(u_1, u_2, u_3), \quad (2.10)
$$

$$
Y \sim \text{diag}(y_1, y_2, y_3) + \frac{1}{2r} \text{diag}(v_1, v_2, v_3), \quad (2.11)
$$

$$
\epsilon_i \sim \hat{r}_i \frac{1}{2r^2} \text{diag}(e_1, e_2, e_3), \quad (2.12)
$$

$$
B_i \sim \hat{r}_i \frac{1}{2} \text{diag}(g_1, g_2, g_3), \quad (2.13)
$$

with $\hat{r} \equiv r/r$. This solution represents a configuration of three D3-branes $a = 1, 2, 3$ at transverse coordinates $(x_a, y_a)$ from which a string with magnetic and electric charges $(e_a, g_a)$ are emerging in the direction $(u_a, v_a)$. This is because the eigenvalues of the scalars $(X, Y)$ are interpreted as the transverse coordinates of the D3-branes and the “tube-like” part of the
D3-brane surface (corresponding to small \( r \)) can be regarded as a string \([16]\) (see fig. 1). Of course, \((u_a, v_a)\) are not arbitrary but should be determined in terms of other parameters of the solution. In fact, this relation can be obtained by the analysis in the asymptotic region without concretely solving the differential equations: Multiplying eqs. (2.4) and (2.5) with either \(E_i\) or \(B_i\), we obtain a gauge invariant equation

\[
\partial_i \text{Tr} (E_i X) = \text{Tr} (E_i)^2 \cos \theta - \text{Tr} (E_i B_i) \sin \theta,
\]

and three others. Plugging the asymptotic expressions \((2.10)–(2.13)\) into these equations and comparing the \(O(1/r^4)\) terms, we get the known relation between the directions of the strings and their charges \([8]\):

\[
\begin{pmatrix} u_a \\ v_a \end{pmatrix} = - \begin{pmatrix} \cos \theta & - \sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} e_a \\ g_a \end{pmatrix}.
\]

(2.15)

The above observations in the asymptotic region lead to the standard picture of the 3-string junction (see fig. 2): We can show that the three straight lines starting at \((x_a, y_a)\) in the direction \((u_a, v_a)\) meet at a common point. Hence, these straight lines can be regarded as IIB strings forming a junction. In fact, the force balance relation, \(\sum_a T_a(u_a, v_a)(u_a^2 + v_a^2)^{-1/2} = 0\), holds by taking \(T_a = (e_a^2 + g_a^2)^{1/2}\) as the string tension. Furthermore, the sum of the masses of the three strings, \(\sum_i T_i \ell_i\), with \(\ell_i\) being the length of the \(a\)-th straight line, coincides with the BPS bound \((2.2)\) of the SYM hamiltonian \([1]\). However, we shall see that the configurations of the 3-string junctions obtained from the eigenvalues of the scalars \((X, Y)\) of the classical

\[\text{Figure 1: “Tube-like” configuration of D3-brane surface representing a string.}\]

\[\text{Figure 2: 3-string junction connecting three D3-branes.}\]

\[\text{Here, we have to assume that the } O(1/r^3) \text{ terms are missing from } B_i \text{ and } E_i. \text{ This assumption will prove valid in the concrete solutions given later.}\]
solutions of SYM constructed below have more complicated structures than the above string picture.

3 Solutions

Since the angle $\theta$ can be absorbed by the rotation in the $(X, Y)$ plane, we restrict ourselves to the case with $\theta = 0$. Then, equations to be solved are

\begin{align*}
    D_i X &= \mathcal{E}_i, \quad (3.1) \\
    D_i Y &= B_i, \quad (3.2)
\end{align*}

as well as (2.6), (2.7) and (2.9). Our strategy for the construction of the solutions is as follows. First, we prepare a (monopole) solution $(A_i(r), Y(r))$ to eq. (3.2). Then, eq. (3.1) is automatically satisfied by putting $A_0(r) = -X(r)$, while eq. (2.6) holds due to eq. (2.7) and the time-independence of our solution. Therefore, we have only to solve eq. (2.9), i.e.,

\begin{equation}
    D_i D_i X = 0, \quad (3.3)
\end{equation}

under the D-flatness condition (2.7).

The monopole solutions to eq. (3.2) for a gauge group $SU(N)$ were discussed by many people. Among them, we adopt the solutions given by refs. [17, 18] constructed on the basis of the general formalism of spherically symmetric solutions of ref. [19]. Let us first recapitulate the elements of ref. [18] necessary for our construction.

Our solutions $(A_i, X, Y)$ are assumed to be spherically symmetric with respect to $SO(3)$ generator $J$, i.e., they satisfy

\begin{equation}
    [J_i, A_j] = i\epsilon_{ijk}A_k, \quad [J_i, X] = [J_i, Y] = 0, \quad (3.4)
\end{equation}

and the present $J$ is given as

\begin{equation}
    J = L + T, \quad (3.5)
\end{equation}

where $L = -ir \times \nabla$ is the generator of the space rotation and $T$ is the maximal $SU(2)$ embedding in $SU(3)$ with $T_3 = \text{diag}(1, 0, -1)$. The monopole solution of [19] takes the following form for the vector potential:

\begin{equation}
    A(r) = (M(r, \hat{r}) - T) \times \hat{r}/r, \quad (3.6)
\end{equation}
where the Lie algebra valued function $M_i$ should satisfy the spherical symmetry condition,

$$[J_i, M_j] = i\epsilon_{ijk}M_k.$$  \hspace{1cm} (3.7)

Various formulas are derived by using the expression $\nabla = \hat{r}\partial/\partial r - (i/r)\hat{r} \times \mathbf{L}$ for the space derivative as well as the spherical symmetry properties, eqs. (3.4) and (3.7). We list the three necessary for our purpose:

\begin{align*}
DY &= \hat{r}Y' + \frac{i}{r} \hat{r} \times [M, Y], \\
B_i &= -\frac{i}{r^2} \hat{r}_i \hat{r}_j \left( \frac{1}{2} \epsilon_{jkl} [M_k, M_l] - iT_j \right) - \frac{1}{r} \left( \hat{r} \times (\hat{r} \times M') \right)_i, \\
D_iD_iX &= X'' + \frac{2}{r} X' - \frac{1}{r^2} \left( \delta_{ij} - \hat{r}_i \hat{r}_j \right) [M_i, [M_j, X]],
\end{align*}

where the prime denotes the differentiation $\partial/\partial r$.

Due to spherical symmetry we have only to construct solutions on the positive $z$-axis. The monopole equation (3.2) on the $z$-axis is reduced to

\begin{align*}
\frac{r^2}{2} Y'' &= \frac{1}{2} [M_+, M_-] - T_3, \\
M'_\pm &= \mp [M_\pm, Y],
\end{align*}

with $M_\pm \equiv M_1 \pm iM_2$. The solution to eqs. (3.11) and (3.12) given in ref. [18] for the $SU(3)$ case is as follows. The matrices $X$ and $M_\pm$ are expressed as

\begin{align*}
Y &= \text{diag} \left(Y_a\right) = \frac{1}{2} \text{diag} \left(\phi_1, \phi_2 - \phi_1, -\phi_2\right), \\
M_+ &= \begin{pmatrix} 0 & a_1 & 0 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_- = (M_+)^T,
\end{align*}

in terms of $a_m$ and $\phi_m$, which are further given in terms of two functions $(Q_1, Q_2)$ as

\begin{align*}
a_m &= \frac{r}{Q_m} (2Q_{3-m})^{1/2}, \\
\phi_m &= -\frac{d\ln Q_m}{dr} + \frac{2}{r}.
\end{align*}

Eq. (3.12) is an automatic consequence of the expressions (3.13), while eq. (3.11) now becomes

$$(Q'_m)^2 - Q_mQ''_m = Q_{3-m}.$$  \hspace{1cm} (3.16)

In ref. [18] they found the following solution to (3.16):

\begin{align*}
Q_1(r) &= \frac{1}{2} \sum_{a=1,2,3} \exp \left(-2y_ar\right) \prod_{b(\ne a)} \left(y_a - y_b\right)^{-1}, \quad Q_2(r) = Q_1(-r).
\end{align*}

(3.17)
Here, $y_a (\sum_a y_a = 0)$ are the constants characterizing the solution, and they are nothing but $y_a$ in the asymptotic expression (2.11) of $Y$ if the condition $y_1 < y_2 < y_3$ is satisfied. The magnetic charges $g_a$ of the present solution are

$$(g_1, g_2, g_3) = (-2, 0, 2). \quad (3.18)$$

Having prepared the monopole solution $(A_i, Y)$, our task is to solve eq. (3.3) for the scalar $X$, which, using eq. (3.10), becomes on the $z$-axis

$$X'' + \frac{2}{r} X' - \frac{1}{2r^2} ([M_+, [M_-, X]] + [M_-, [M_+, X]]) = 0. \quad (3.19)$$

Since we have adopted a diagonal form (3.13) for $Y$, eq. (2.7) implies that $X$ is also diagonal:

$$X = \text{diag} (X_a) = \frac{1}{4r} \text{diag} (\varphi_+ + \varphi_- - 2\varphi_-, -2\varphi_-, -\varphi_+ + \varphi_-). \quad (3.20)$$

Then, eq. (3.19) for $\varphi_\pm$ reads

$$\varphi''_\pm - \left(\frac{Q_2}{Q_1^2} \pm \frac{Q_1}{Q_2^2}\right) \varphi_+ - 3 \left(\frac{Q_2}{Q_1^2} \pm \frac{Q_1}{Q_2^2}\right) \varphi_- = 0 \quad (3.21)$$

The differential equation (3.21) can be solved both numerically and analytically (we shall present exact solutions in Sec. 3.2). However, let us first carry out the analysis of the solutions near the origin and infinity. This is useful for the understanding of the solutions given later.

### 3.1 Behavior of the solution near $r = 0$ and $\infty$

First, we shall consider (3.21) near the origin $r = 0$. Since $Q_m(r)$ is Taylor-expanded around $r = 0$ as

$$Q_m = r^2 - \frac{1}{3} \sum_{a>b} y_a y_b r^4 + (-)^m \frac{2}{15} y_1 y_2 y_3 r^5 + O(r^6), \quad (3.22)$$

eq. (3.21) is approximated near the origin as

$$\varphi''_+(r) - \left(\frac{2}{r^2} + O(1)\right) \varphi_+(r) = 0, \quad \varphi''_-(r) - \left(\frac{6}{r^2} + O(1)\right) \varphi_-(r) = 0. \quad (3.23)$$

Each of the differential equations (3.23) has two independent solutions; one is regular and the other is singular at $r = 0$. From the physical requirement, we of course have to choose regular ones which behave as

$$\varphi_+ = \tilde{\alpha} r^2 + O(r^4), \quad \varphi_- = \tilde{\beta} r^3 + O(r^5). \quad (3.24)$$
where \((\tilde{\alpha}, \tilde{\beta})\) are arbitrary real constants. Once the parameter \((\tilde{\alpha}, \tilde{\beta})\) is fixed, we can successively determine the coefficients of any higher powers of \(r\) by iterative use of eq. (3.21). Therefore, taking also into account the parameter \(y_a\) of the monopole solution, we see that the number of the freedom of our solutions \((A_\mu, X, Y)\) is four; \((y_1, y_2, \tilde{\alpha}, \tilde{\beta})\).

From eqs. (3.24), (3.15) and (3.22), we obtain the following behavior of the eigenvalues \((X_a(r), Y_a(r))\) near \(r = 0:\)

\[
(X_1, Y_1) \sim -(X_3, Y_3) \sim \left(\frac{\tilde{\alpha}}{4}, \frac{1}{3} \sum_{a>b} y_ay_b\right) r, \quad (X_2, Y_2) \sim -\left(\frac{\tilde{\beta}}{2}, \frac{2}{3} y_1y_2y_3\right) r^2.
\]

(3.25)

Eq. (3.25) implies the followings: First, since both the scalars \(X\) and \(Y\) representing the D3-brane transverse coordinates vanish at \(r = 0\), the three D3-brane surface meet at the origin \((X, Y) = (0, 0)\) of the transverse plane. Second, among the three D3-branes, the branes 1 and 3 are “smoothly” connected at the origin (namely, they have a common tangent at the junction), while the brane 2 meets with the other two with an angle.

Next let us consider eq. (3.21) near \(r = \infty\). Since both the quantities \(Q_2/Q_1^2\) and \(Q_1/Q_2^2\) which appear in eq. (3.21) decay exponentially as \(r \to \infty\), the leading behavior of \(X\) is in fact given by (2.10), and the next order terms are exponentially dumping ones. The other scalar \(Y\) behaves in a similar manner at infinity. This fact, together with the formulas \(E_z = X'\) and \(B_z = Y'\) on the \(z\)-axis, which are consequence of eqs. (3.1), (3.2) and (3.8), implies the validity of the assumption we made in deriving eq. (2.15): the \(O(1/r^3)\) terms are missing from \(E\) and \(B\).

### 3.2 Exact solutions for \(X\)

Though it seems difficult to solve analytically the differential equation (3.21) for arbitrary \((y_a)\), exact solutions are obtained for the following special values of \((y_a)\):

\[
(y_1, y_2, y_3) = (-C, 0, C),
\]

(3.26)

with \(C\) being a real and positive constant. In fact, for this \((y_a)\) we have

\[
Q_1 = Q_2 = \left(\frac{\sinh Cr}{C}\right)^2,
\]

(3.27)

and hence the differential equations (3.21) become separated ones:

\[
\varphi''_+(r) - \frac{2C^2}{\sinh^2 Cr} \varphi_+(r) = 0, \quad \varphi''_-\!(r) - \frac{6C^2}{\sinh^2 Cr} \varphi_-(r) = 0.
\]

(3.28)
Eqs. (3.28) have the following solutions regular at $r = 0$:

$$
\varphi_+ (r) = \alpha (Cr \coth Cr - 1), \quad \varphi_- (r) = \beta \left( 3 \coth Cr - Cr \frac{2 \cosh^2 Cr + 1}{\sinh^2 Cr} \right). \tag{3.29}
$$

where $\alpha$ and $\beta$ are arbitrary constants. These solutions are consistent with (3.24) by the identifications $\tilde{\alpha} = \alpha C^2/3$ and $\tilde{\beta} = -4\beta C^3/15$.

From eq. (3.29), we can read off the following values for the locations of the D3-branes and the electric and magnetic charges of the strings:

$$
(x_a, y_a) = \left\{ C \left( \frac{\alpha - 2\beta}{4}, -1 \right), \quad C \left( \beta, 0 \right), \quad C \left( -\frac{\alpha + 2\beta}{4}, 1 \right) \right\}, \tag{3.30}
$$

$$
(e_a, g_a) = \left\{ \left( \frac{\alpha - 3\beta}{2}, -2 \right), \quad \left( 3\beta, 0 \right), \quad \left( -\frac{\alpha + 3\beta}{2}, 2 \right) \right\}. \tag{3.31}
$$

In particular, for $(\alpha, \beta) = (-1, 1/3)$, the three charges are $(-1, -2)$, $(1, 0)$ and $(0, 2)$. In fig. 3, we plot the trajectories of the D3-brane coordinates in course of changing $r$ by the solid lines. In course of decreasing $r$, the D3-branes approach the origin of $(X, Y)$ and meet there at $r = 0$, where the gauge symmetry is restored. The trajectory of the brane $D_2$ is just a straight line. This is a general feature of our exact solutions and comes from the boundary condition $y_2 = 0$. We obtain a bending trajectory of $D_2$ for a general case of $y_2 \neq 0$, in which we solved the equations only numerically.

The branes $D_1$ and $D_3$ connect smoothly to each other at the origin, as discussed in Sec. 3.1. Noticing the fact that the brane $D_2$ has no magnetic charge\footnote{We mean the charge $(e_a, g_a)$ by the electric and magnetic charges of the brane $D_a.$} while the branes $D_1$ and
$D_3$ have non-zero magnetic charges, this might come from our technical preference that we describe the BPS states by the classical treatment of the SYM theory, i.e. electrically. Then the interpretation of the trajectories might be that the branes $D_1$ and $D_3$ are the two parts of one very heavy smooth magnetic object pulled and bent by the light electric brane $D_2$.

In fig. 3, we also draw three dashed straight lines tangent to the D3-brane trajectories at $r = \infty$. They meet at one point. As discussed in Sec. 2, this configuration agrees with the three-pronged strings in the IIB string picture.

4 Summary and Discussions

We have obtained BPS saturated spherically symmetric regular configurations in $\mathcal{N} = 4 SU(3)$ SYM theory carrying non-parallel electric and magnetic charges $(a, 2), (b, 0)$ and $(-a-b, -2)$, where $a$ and $b$ take arbitrary real values. Regarding the $\mathcal{N} = 4$ SYM theory as an effective field theory on parallel D3-branes, the solutions correspond to 3-string junctions connecting three different D3-branes. Assuming the quantization of the electric charges $a$ and $b$ and the $SL(2, Z)$ duality symmetry, our solutions imply, in general, the existence of the junctions of the three IIB strings carrying the two-form charges $(p, q)$, $(lr, ls)$ and $(-p - lr, -q - ls)$, respectively, where $l, p, q, r, s$ are integers satisfying $ps - qr = 2$.

As discussed in Sec. 2, the asymptotic behavior at $r \sim \infty$ of our solutions leads to the configuration of three-pronged strings in the IIB string picture, i.e. three straight strings meet at one point and their forces balance. This would be a non-trivial consistency check of our approach. On the other hand, the behavior of our solution at finite $r$ is quite different from the above IIB string picture. The trajectories of the D3-branes bend non-trivially, and the two magnetically charged D3-branes connect smoothly to each other. As discussed in Sec. 3.2, this might be due to the fact that we did not treat the problem in an $SL(2, Z)$ invariant way, but obtained the BPS configurations by the classical treatment of the SYM theory, i.e. electrically.

Another difference between our solutions and the IIB string picture is the number of degrees of freedom. Now let us count the number of degrees of freedom of a 3-string junction in the string picture. As for the charges, the three magnetic charges are fixed to (3.18) from the beginning, but we have the freedom of two electric charges (the other electric charge is determined by the charge conservation). Then, since the string tensions are determined by

\footnote{See the previous footnote.}
the charges, the relative directions of the three strings are determined by the force balance condition. We have the freedom to choose the lengths of each string. We should not count the freedom of rotating or shifting parallelly the 3-string junction in the two-dimensional plane, because we fixed these degrees of freedom in solving the equations. We absorbed the angle $\theta$ in Sec. 3, and the center of the three D3-branes are fixed by the tracelessness of the adjoint scalar fields. Thus we have in total five degrees of freedom in the string picture. This does not agree with four, the number of degrees of freedom of our solutions obtained in Sec. 3.1.

This difference might again come from the limitation of the electric description of the 3-string junction. Since electrically charged objects are the fundamental degrees of freedom themselves in the electric description, the introduction of a bare electric charge would necessarily cause the problem of singularities in the solutions. Nevertheless, in our solutions, we have one D3-brane ($D_2$ in fig. 3) which is charged only electrically. However, we may have the possibility that, if we had the freedom to introduce a bare electric charge, the number of the degrees of freedom would increase from four to five.

In fact, the difficulty of a bare electric charge appears in a different way in our construction of the solutions. We took the maximal embedding of $SU(2)$ to $SU(3)$ in this paper. Because of this, the magnetic charge is 2 and not 1. Taking the minimal embedding of $SU(2)$ does not work. It turns out that, to obtain non-parallel electric and magnetic charges, we have a singularity at the origin, which is the bare source of the electric field.

We tried another way to introduce a unit magnetic charge. In the degenerate case of $y_1 = y_2$, the magnetic charges of the D3-branes become $(1, 1, -2)$ [17, 18]. But in this case the asymptotic behavior at $r \sim \infty$ of eq. (3.21) changes from the non-degenerate cases, and the solutions regular at $r = 0$ diverge at $r = \infty$ ($\lim_{r \to \infty} \phi_1(r) = \infty$) except the case of parallel electric and magnetic charges. Thus we could not introduce one unit of magnetic charge in our solutions.

Since the monopole solutions for the general $SU(N)$ case are also known, it would be an interesting extension to find BPS saturated solutions carrying non-parallel electric and magnetic charges for general $SU(N)$ and compare them with the string picture. This work is now in progress.

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