Abstract. It is well known for experts that resonances in nonlinear systems lead to new invariant objects that lead to new behaviors.

The goal of this paper is to study the invariant sets generated by resonances under foliation preserving torus maps. That is torus which preserve a foliation of irrational lines \( L_{\theta_0} = \{ \theta_0 + \Omega t \mid t \in \mathbb{R} \} \subseteq \mathbb{T}^d \).

Foliation preserving maps appear naturally as reparametrization of linear flows in the torus and also play an important role in several applications involving coupled oscillators, delay equations, resonators with moving walls, etc. The invariant objects we find here, lead to predictions on the behavior of these models.

Since the results of this paper are meant to be applied for other problems, we have developed very quantitative results giving very explicit descriptions of the phenomena and the invariant objects that control them.

The structure of the phase locking regions for foliation preserving maps is very different than for generic maps of the torus. Indeed, for the sake of completeness, we have developed similar analysis for the case of generic maps of the torus and shown that the objects that appear in foliation preserving maps are quantitatively and qualitatively different from those of generic torus maps. This has consequences in applications.

1. Introduction

The influential book [Poi99] by Poincaré considered the study of nearly integrable dynamical systems as the main problem of mechanics.

One importance of this study is that qualitatively new phenomena may appear under some circumstances. For example, when we consider perturbations of several oscillators, if the frequencies are rationally independent, the perturbations average out and the system resembles for a long time the unperturbed system. On the other hand, if the system
presents a resonance (i.e. some of the frequencies is a combination of others), the resonance can lead to genuinely new phenomena not present in the original system. For nonlinear systems, one needs to do several orders of perturbation theory and new phenomena can happen at each order of perturbation theory and there are phenomena that happen beyond all orders of perturbation theory.

In Hamiltonian systems, the geometry of the resonances is extremely important for phenomena such as Arnold diffusion [DH09, DdlLS06, Cor08] that show that perturbation theory has limits.

In the very non-resonant regions, one can continue averaging to all orders and indeed the motion is similar to a rotation for all time. In particular cases, one can apply KAM theory and obtain that indeed the system remains a rotation (see Section 3).

The goal of this paper is to study rather quantitatively, the phenomena induced by resonances in some class of torus maps with a special structure (they preserve an irrational foliation). See Subsection 1.1 below for a precise definition. We also present KAM results that show that away from resonances, indeed the system remains qualitatively the same.

The foliation preserving maps appear in several applications. In pure mathematics, they appear as time reparametrizations of constant vector fields in the torus. In more applied contexts, they have appeared in the study of cavities with moving boundaries (and, in general in quasi-periodic systems). For us, a motivating application is the study of state dependent delay equations [HdlL22].

With a view to applications, we will present the results in a very constructive way. The new objects generated (or the KAM theory concluding that the system remains a rotation) are obtained through solutions of a functional equation. The perturbative analysis provides with approximate solutions of the functional equation. The mathematical theory includes also an a-posteriori result that shows that the existence of approximate solutions implies the existence of true solutions.

Note that the usefulness of the a-posteriori result goes beyond the perturbation regime since the approximate solutions of the functional equation can be produced by methods not based on perturbation theory (e.g. by numerical calculations). With a point of view to developing numerical methods, the a-posteriori theorems are proved by establishing the convergence of an iterative procedure. Implementing the iterative procedure provides an algorithm for the computation.

1.1. Foliation preserving mappings. Let $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$. We say that $\Omega \in \mathbb{R}^d$ is an irrational frequency vector when $\Omega \cdot k \neq 0$ for all $k \in \mathbb{Z}^d \setminus \{0\}$.

If $\Omega$ is an irrational frequency vector, the sets

$$L_{\theta_0} = \{\theta_0 + \Omega t \in \mathbb{T}^d : t \in \mathbb{R}\}$$
define a foliation of the torus \( \mathbb{T}^d \), which we will denote by \( \mathcal{F}_\Omega \). Note that \( L_{\theta_0} \) are the equivalent classes of the equivalence relation
\[
\theta \sim \tilde{\theta} \iff \exists t \in \mathbb{R}, \ s.t. \ \theta - \tilde{\theta} = \Omega t \mod 1.
\]
We are specially interested in maps \( f \) of the torus which preserve one of the above foliations (i.e., a leaf gets mapped into another leaf). More precisely, we are interested in maps of the form:
\[
(1.2) \quad T_\varphi(\theta) = \theta + \varphi(\theta)\Omega,
\]
where \( \varphi : \mathbb{T}^d \to \mathbb{R} \) is a differentiable function. To save notations, we write also \( T_\varphi \) for the associated lift on \( \mathbb{R}^d \).

With the notation (1.1), we have that \( T_\varphi(L_{\theta_0}) \subseteq L_{\theta_0} \). Indeed, for any \( \theta = \theta_0 + \Omega t \) on \( L_{\theta_0} \), there is
\[
T_\varphi(\theta) = \theta_0 + \Omega(t + \varphi(\theta_0 + \Omega t)) \in L_{\theta_0}.
\]
Furthermore, let \( \text{Diff}(\mathbb{T}^d) \) be the set of diffeomorphism mapping on the torus and let \( \Xi \) be the subset of \( \text{Diff}(\mathbb{T}^d) \), in which the element \( T_\varphi \) has the form of \( T_\varphi = \text{Id} + \varphi\Omega \). Then one easily verifies that \( \Xi \) is a subgroup of \( \text{Diff}(\mathbb{T}^d) \) under the composition \( T_f \circ T_g = T_{g \circ f \circ T_g} \) and the inverse of \( T_f \) is given by \( T_{-f \circ T_f} \).

Maps of the form (1.2) appear naturally in the study in resonant cavities affected by quasi-periodic perturbations [PdlLV03], in the study of equilibria in quasi-periodic media [dlLSZ16, dlLSZ17] or in the study of state-dependent delay equations [HdlL17, HdlL16]. In these applications, the objects we describe have direct consequences: in the problems of cavities they lead to exponential growth of energy [PdlLV03] and in state dependent delay equations, they lead to lack of analyticity. On the purely mathematical side, they appear as reparametrization of linear flows of the torus [FKW01, Fay02, Cor02].

In contrast with the generic torus maps, the special structure of (1.2) leads to rather different dynamical consequences. For instance, there exists at most one non-vanishing Lyapunov exponent for the foliation preserving torus maps. More precisely, we have the following result, whose proof can be found in [PdlLV03, Proposition 3.1].

**Proposition 1.1.** Let \( T_\varphi \) be a \( C^1 \) torus map of the form (1.2). Then for every \( \theta \in \mathbb{T}^d \), \( d - 1 \) of the Lyapunov exponents of \( T_\varphi \) are zero. Besides these \( d - 1 \) trivial exponents, for almost every \( \theta \) (in the sense of any \( T_\varphi \)-invariant measure), there is one Lyapunov exponent corresponding to the direction of \( \Omega \).

Furthermore, if the function \( \varphi \) in (1.2) is bounded away from zero, then there is no periodic points for \( T_\varphi \). Indeed, since the line \( L_\theta = \{ \theta + \Omega t : t \in \mathbb{R} \} \) in the cover is mapped to itself, the motion on the line is increasing, so that no orbit can come back to itself. So, there are no periodic points in the line. From the irrationality of \( \Omega \), we obtain that two different points on the line are also two different points in \( \mathbb{T}^d \), so that all points in the orbit are different.
Periodic points in maps of the form (1.2) can exist but only when \( \varphi \) has different signs. Clearly if \( \varphi(p) = 0 \), the point \( p \) is a fixed point.

1.2. **Resonant frequency.** We say that \( \Omega \in \mathbb{R}^d \) is resonant if there exist \( k \in \mathbb{Z}^d \setminus \{0\} \) and \( n \in \mathbb{Z} \) such that \( \Omega \cdot k - n = 0 \). The frequency \( \Omega \in \mathbb{R}^d \) is non-resonant if for any \( k \in \mathbb{Z}^d \) and \( n \in \mathbb{Z} \), the relationship \( k \cdot \Omega - n = 0 \) implies \( k = 0 \) and \( n = 0 \).

Note that non-resonant, implies irrational because 0 is a particular case of an integer. However, it could happen that for an irrational \( \Omega \) we have that for some \( k \in \mathbb{Z}^d \setminus \{0\} \) there is \( k \cdot \Omega = n \), where \( n \) is non-zero integer. In such a case, the \( \Omega \) could be irrational but resonant.

Denoting

\[
\Gamma_{\Omega}(\mathbb{Z}) = \{ k \in \mathbb{Z}^d | k \cdot \Omega \in \mathbb{Z} \},
\]

we see that if \( k, \tilde{k} \in \Gamma_{\Omega}(\mathbb{Z}) \) so is \( k + \tilde{k} \). In more algebraic language, we note that \( \Gamma_{\Omega}(\mathbb{Z}) \) is a \( \mathbb{Z} \)-module and called the **resonance module** of \( \Omega \).

Assume \( \Omega \) is resonant (i.e. \( \Gamma_{\Omega}(\mathbb{Z}) \neq \{0\} \)), then there exist \( k_1, \cdots, k_{d-r} \in \mathbb{Z}^d \setminus \{0\} \) linearly independent over \( \mathbb{R} \) such that \( \{k_j\}_{j=1}^{d-r} \) form a basis of \( \Gamma_{\Omega}(\mathbb{Z}) \), which means

\[
\Gamma_{\Omega}(\mathbb{Z}) = \left\{ z \in \mathbb{Z}^d : z = \sum_{j=1}^{d-r} t_j k_j \text{ with } t_j \in \mathbb{Z} \text{ uniquely determined} \right\}.
\]

Moreover, we can also find a matrix \( \mathfrak{A} \in \text{SL}(d, \mathbb{Z}) \), \( \omega \in \mathbb{R}^r \) and \( L \in \mathbb{Z}^d \) in such a way that

\[
\mathfrak{A} \Omega = \begin{pmatrix} \omega \\ 0 \end{pmatrix} + L
\]

with \( \omega \cdot m \neq 0 \) for \( m \in \mathbb{Z}^r \setminus \{0\} \).

We refer to \( \omega \) in (1.4) as the **intrinsic frequency** of \( \Omega \). It is essentially unique, i.e., unique up to change of basis in \( \mathbb{R}^r \) given by a matrix in \( \text{SL}(r, \mathbb{Z}) \).

In our case, it is important to realize that, given an irrational frequency \( \Omega' \), it is possible that there exists a real number \( \alpha \) such that \( \alpha \Omega' \) is resonant. However, in this case, the resonance module of \( \alpha \Omega' \) is only one dimension (see also [dLSZ16]).

**Proposition 1.2.** Let \( \Omega' \in \mathbb{R}^d \) be irrational and \( 0 \neq \alpha \in \mathbb{R} \). If \( \alpha \Omega' \) is resonant, then the dimension of \( \Gamma_{\alpha \Omega'}(\mathbb{Z}) \) is exactly one.

**Proof.** For \( k_1 \cdot \alpha \Omega' - n_1 = 0 \) and \( k_2 \cdot \alpha \Omega' - n_2 = 0 \), we see that

\[
\alpha = \frac{n_1}{k_1 \cdot \Omega'} = \frac{n_2}{k_2 \cdot \Omega'}.
\]

The irrationality of \( \Omega' \) implies that \( n_1k_1 - n_2k_2 = 0 \), which proves the proposition. \( \square \)

Note that for frequencies \( \Omega \) that have a one-dimensional resonance module, the intrinsic frequency has one dimension less.
Note also that if $\Omega'$ is irrational and $\alpha$ real, we have that $\alpha \Omega'$ is irrational. However if we pick $k \in \mathbb{Z}^d \setminus \{0\}$ and $n \in \mathbb{Z}$, if we take $\alpha = n/(k \cdot \Omega')$, we have $k \cdot \alpha \Omega' = n$. Therefore the set of $\alpha$ such that $\alpha \Omega'$ is resonant is dense in $\mathbb{R}$.

An important type of non-resonant frequency that will be used in this manuscript is the following.

**Definition 1.1.** We say that $\omega \in \mathbb{R}^s$ satisfies the Diophantine condition of type $(\nu, \tau)$ if
\begin{equation} (1.5) \quad |k \cdot \omega - n| \geq \nu |k|^{-\tau} \end{equation}
for all $k \in \mathbb{Z}^s \setminus \{0\}$ and $n \in \mathbb{Z}$.

We denote $\mathcal{D}_s(\nu, \tau)$ the set of vectors in $\mathbb{R}^s$ satisfying the Diophantine conditions of type $(\nu, \tau)$. It is well known that the set $\mathcal{D}_s(\tau) = \bigcup_{\nu > 0} \mathcal{D}_s(\nu, \tau)$ occupies full Lebesgue measure in any open sets of $\mathbb{R}^s$ for $\tau > s$. See [dlL01].

1.3. **Preliminaries.** In this paper, we deal with analytic functions and finitely differentiable functions defined on the torus. We collect here some standard notations and results which appear frequently later.

For the finite dimension space $\mathbb{R}^s$ we adopt its supremum norm. We denote by $\mathcal{A}_\rho = \mathcal{A}_\rho(\mathbb{T}^d, \mathbb{R}^s)$ the set of real analytic periodic functions which are analytic in the complex neighborhood $\mathbb{T}^d_\rho$ of $\mathbb{T}^d$ in the complex space. Here $\mathbb{T}^d_\rho = \{x \in \mathbb{C}^d / \mathbb{Z}^d : |\text{Im} x| \leq \rho\}$. We also endow $\mathcal{A}_\rho(\mathbb{T}^d, \mathbb{R}^s)$ the supremum norm defined by
\begin{equation} (1.6) \quad \|w\|_{\rho} = \sup \{|w(x)| : x \in \mathbb{T}^d_\rho\}. \end{equation}

Then by the Cauchy estimate, it is readily seen that, if $w \in \mathcal{A}_\rho$, the partial derivative with respect to its $j$-th argument $x_j$ satisfies
\begin{equation} \|w_{x_j}\|_{\rho - \sigma} \leq \sigma^{-1}\|w\|_{\rho} \end{equation}
for all $0 < \sigma < \rho$ and $1 \leq j \leq d$. Furthermore, $w \in \mathcal{A}_\rho$ can be expanded into Fourier series
\begin{equation} w(x) = \sum_{k \in \mathbb{Z}^d} \widehat{w}(k)e^{2\pi i \langle k, x \rangle}, \end{equation}
whose Fourier coefficients $\widehat{w}(k)$ satisfies
\begin{equation} (1.7) \quad |\widehat{w}(k)| \leq \|w\|_{\rho}e^{-2\pi |k|\rho}. \end{equation}

Next, we denote by $\mathcal{C}^n(\mathbb{T}^d, \mathbb{R}^s)$ the Banach space of $n$ times continuously differentiable periodic functions, whose norm is given by
\begin{equation} (1.8) \quad \|w\|_{\mathcal{C}^n} = \max_{0 \leq j \leq n} \|D^j w\|_{\mathcal{C}^0}, \quad \|w\|_{\mathcal{C}^0} = \sup_{x \in \mathbb{T}^d} |w(x)|. \end{equation}

If $w \in \mathcal{C}^n(\mathbb{T}^d, \mathbb{R}^s)$, we have that
\begin{equation} (1.9) \quad \sup_{k \in \mathbb{Z}^d} (|\widehat{w}(k)| \cdot |k|^n) \leq C_n \|w\|_{\mathcal{C}^n}, \end{equation}
where $C_n$ is the constant that depends only on $n$. For more details on the regularities of functions characterized by the decay rate of Fourier coefficients, we refer to [dIL01, Gra14].

Finally, we collect the following lemma on the cohomology equation (1.10) frequently used in KAM theory, whose proof can be found in [Rüs75, dIL01]. We will encounter equation (1.10) when developing the resonant normal form, formulating Lindstedt series near resonance and proving the KAM theorem for the foliation preserving torus map.

**Lemma 1.1.** Assume that $\omega$ satisfies the Diophantine condition (1.5). Let $Q \in A_\rho$ be periodic function with zero average, i.e., $\int_{\mathbb{T}} Q \, d\theta = 0$. Then there is a unique solution $W$ of

$$W(\theta) - W(\theta + \omega) = Q(\theta)$$

such that $W$ has zero average. Moreover, we have for all $0 < \sigma < \rho$,

$$\|W\|_{\rho-\sigma} \leq C\nu^{1-\sigma}\|Q\|_{\rho},$$

where the constant $C$ depends only on the Diophantine exponent $\tau$ and the dimension of the space.

1.4. **Organization of this paper.** The left of the paper is organized as follows. In Section 2 we study the resonance and phase locking phenomena for the foliation preserving torus maps (FPTM). We first establish a resonant normal form for FPTM, based on which an invariant surface is constructed under some non-degeneracy conditions (see Theorem 2.2). Next we compute the Lindstedt series for the invariant surface, and show the relationship between the Lindstedt series and the true embedding of the invariant surface (see Theorem 2.3). For the two dimensional FPTM, we study more closely the dynamics around the invariant circle. For instance, we establish an extension of the Sternberg linearization theorem for the invariant circle (See Theorem 2.4 and 2.5), which reduces the dynamics of the invariant circle in the normal direction to the linear flow. Moreover, we study the structural stability of FPTM in two dimension. To illustrate our theoretical results, we present a simple example at the end of Section 2.

In Section 3 we establish an a-posteriori KAM theorem for the foliation preserving maps, which is devoted to the conjugation problem of FPTM to the rigid rotation. In Appendix A we study the phase locking phenomena for the general torus maps in the perturbative setting, which provides a comparison to those special maps preserving the foliation and might be of interest itself.

2. **Resonance and phase locking phenomena**

In this section, we study the phase locking phenomena for a family of foliation preserving torus maps close to rotations on the torus. More precisely, we study the following
families of torus maps
\begin{equation}
F_{\alpha, \varepsilon}(x) = x + \alpha \Omega + \varepsilon f_\varepsilon(x) \Omega,
\end{equation}
where \( x \in \mathbb{T}^d, \alpha \in \mathbb{R}, \Omega \in \mathbb{R}^d \) is irrational and \( f_\varepsilon \) is defined on the torus for every \( \varepsilon \). We assume some regularity of \( f \) with respect to \( \varepsilon \) such that the power series expansion in \( \varepsilon \) is valid.

We regard the set \( \mathbb{R} \times \mathbb{R}^+ \) of points \((\alpha, \varepsilon)\) as the parameter space. Typically, when \( \varepsilon = 0 \) and \( \alpha_0 \Omega \) is resonant (see the definition in Subsection 1.2), the torus decomposes into a family of co-dimension one tori. Each orbit is dense on such a lower dimension tori, but not in \( \mathbb{T}^d \).

We are interested in the dynamical behavior of \( F_{\alpha, \varepsilon} \) for those parameters \( \alpha \) close to \( \alpha_0 \) (such that \( \alpha_0 \Omega \) is resonant) and small \( \varepsilon \). To this end, we develop a resonant normal form for the foliation preserving torus maps. Assuming some non-degeneracy condition, we show that the invariant surface for the resonant normal form persists under perturbation. See Theorem 2.2.

The fact that the resonance module of \( F_{\alpha, \varepsilon} \) is one-dimensional (or the natural frequency is zero) leads to the fact that the dynamics of such maps is very different from that of generic maps of the torus close to rotations. As we will see, when the natural frequency is not zero, the structures that are generated by resonances are codimension one manifolds which obstruct the foliation. For generic maps this consequence is not true and one can get many more objects generated by resonances. As we will see, this difference appears already in the study of formal power series solutions.

Furthermore, due to preserving the \( \Omega \)-direction, \( F_{\alpha, \varepsilon} \) can also be described as a skew product flow in the neighborhood of the invariant surface generated by resonance. Then, the evolution on the normal coordinate can be linearized under some hyperbolic hypothesis. See also [KP90]. Moreover, if the evolution on the invariant circle can be conjugated to a rigid rotation, then the skew product flow can be further reduced to linear system with constant coefficients. See Theorem 2.4 and Theorem 2.5.

Finally, we give a simple example to show our theoretical results.

2.1. **Invariant surface generated by resonance.** In the following, we assume that \( \alpha_0 \Omega \) is resonant. Then Proposition 1.2 implies that there exists a matrix \( \mathfrak{A} \in \text{SL}(d, \mathbb{Z}), \omega \in \mathbb{R}^{d-1} \) and \( L \in \mathbb{Z}^d \) such that
\begin{equation}
\mathfrak{A} \alpha_0 \Omega = \begin{pmatrix} \omega \\ 0 \end{pmatrix} + L
\end{equation}
with \( \omega \cdot m \neq 0 \) for any \( m \in \mathbb{Z}^{d-1} \setminus \{0\} \). Furthermore, we assume the intrinsic frequency \( \omega \) satisfies Diophantine condition (1.5)
\[|k \cdot \omega - n| > \nu |k|^{-r}\]
for all \( k \in \mathbb{Z}^{d-1} \setminus \{0\} \) and \( n \in \mathbb{Z} \).
2.1.1. Resonant normal form. We apply the averaging method to obtain a resonant normal form for (2.1) in two steps. We first make some heuristic calculations to formulate the resonant normal form. Then we give a detailed analysis on the convergence of Fourier series which requires solving the cohomology equation (1.10).

To this end, making change of variables by \( H_\varepsilon = Id + \varepsilon h_\varepsilon \Omega \), we have

\[
H_\varepsilon^{-1} \circ F_{\alpha,\varepsilon} \circ H_\varepsilon - T_{\alpha,\Omega} = \varepsilon \left(h^0 - h^0 \circ T_{\alpha,\Omega} + f^0\right) \Omega
\]

\[
+ \varepsilon^2 \left(h^1 - h^1 \circ T_{\alpha,\Omega} + f^1 + Df^0 \cdot \Omega \cdot h^0
\]
\[
+ Dh^0 \circ T_{\alpha,\Omega} \cdot \Omega \cdot h^0 \circ T_{\alpha,\Omega} - Dh^0 \circ T_{\alpha,\Omega} \cdot \Omega \cdot (h^0 + f^0)\right) \Omega
\]
\[
+ \cdots
\]

where \( T_{\alpha,\Omega}(x) = x + \alpha \Omega \) for \( x \in \mathbb{T}^d \), \( f_\varepsilon \) and \( h_\varepsilon \) are expanded into power series in \( \varepsilon \) as

\[
f_\varepsilon = f^0 + \varepsilon f^1 + \cdots, \quad h_\varepsilon = h^0 + \varepsilon h^1 + \varepsilon^2 h^2 + \cdots.
\]

For those \( \alpha \) close enough to \( \alpha_0 \), we denote

\[
\Delta \alpha = \alpha - \alpha_0,
\]

\[
\tilde{\Delta}(\alpha) = H_\varepsilon^{-1} \circ F_{\alpha,\varepsilon} \circ H_\varepsilon - H_\varepsilon^{-1} \circ F_{\alpha_0,\varepsilon} \circ H_\varepsilon.
\]

From the facts that \( F_{\alpha,\varepsilon} - F_{\alpha_0,\varepsilon} = \Omega \Delta \alpha \) and \( D^p(F_{\alpha,\varepsilon} \circ H_\varepsilon) \) is independent of \( \alpha \), we have

\[
\|\tilde{\Delta}(\alpha)\|_{\mathcal{E}^0} \leq \|D(H_\varepsilon^{-1})\|_{\mathcal{E}^0} \cdot |\Omega| \cdot |\Delta \alpha|
\]

and

\[
\|D^p(\tilde{\Delta}(\alpha))\|_{\mathcal{E}^0} \leq C_n \|H_\varepsilon^{-1}\|_{\mathcal{E}^{n-1}} \cdot \|F_{\alpha,\varepsilon} \circ H_\varepsilon\|_{\mathcal{E}^n}
\]

\[
\times (1 + \|F_{\alpha,\varepsilon} \circ H_\varepsilon\|_{\mathcal{E}^{n-1}}) \cdot |\Omega| \cdot |\Delta \alpha|
\]

for any \( n \geq 1 \), where \( \|\cdot\|_{\mathcal{E}^n} \) is defined in (1.8).

The estimates in (2.5) and (2.6) are standard and can be found in [dlLO99] in a uniform way. Nevertheless, (2.5) shows that it suffices to find a normal form for \( F_{\alpha_0,\varepsilon} \) when \( \Delta \alpha \) is sufficiently small.

In order to simplify (2.3) with \( \alpha = \alpha_0 \), we solve the following cohomology equations up to some resonant terms

\[
h^0 \circ T_{\alpha_0,\Omega} - h^0 = f^0,
\]

\[
h^1 \circ T_{\alpha_0,\Omega} - h^1 = f^1 + [Df^0 \cdot h^0 + Dh^0 \circ T_{\alpha_0,\Omega} \cdot h^0 \circ T_{\alpha_0,\Omega}
\]
\[
- Dh^0 \circ T_{\alpha_0,\Omega} \cdot (h^0 + f^0)] \Omega,
\]
\[
\cdots
\]
It is worth noticing that the L.H.S. of equations in (2.7) (for $h^0, h^1, \cdots$) has the same form while the R.H.S. are known depending on lower order terms which can be obtained step by step.

When expanding both sides of (2.7) into Fourier series directly and comparing them, it turns out that, for those $k$'s not lying on the resonance module, the $k$-th Fourier coefficient of the unknowns $h^i$ can be solved explicitly.

To be precise, we make further linear transformation on equation (2.7) such that the intrinsic frequency $\omega$ in (1.4) plays a role. When composing with an appropriate linear transformation $\mathcal{A}^{-1}$ on the right, the first equation in (2.7) reads

$$h^0 \circ T_{\alpha_0 \Omega} \circ \mathcal{A}^{-1} - h^0 \circ \mathcal{A}^{-1} = (h^0 \circ \mathcal{A}^{-1}) \circ T_{\alpha_0 \Omega} - h^0 \circ \mathcal{A}^{-1} = f^0 \circ \mathcal{A}^{-1}.$$  

Then expanding $g^0 = h^0 \circ \mathcal{A}^{-1}$ and $\rho^0 = f^0 \circ \mathcal{A}^{-1}$ into Fourier series, we have

$$g^0(k) = \frac{\rho^0(k)}{e^{2\pi i (k, \omega_0)} - 1}.$$  

For $m = 0$, we choose $g^0(k) = 0$ and leave equation (2.8) unsolved. Combining the choice of the Fourier coefficients $\hat{g}^0(k)$, we see that

$$h^0 \circ \mathcal{A}^{-1} - h^0 \circ T_{\alpha_0 \Omega} \circ \mathcal{A}^{-1} + f^0 \circ \mathcal{A}^{-1} = \sum_{l \in \mathbb{Z}} \rho^0((0, l)) e^{2\pi i (0, l, x)}$$  

and

$$h^0 = g^0 \circ \mathcal{A} = \sum_{k=(m, l) \in \mathbb{Z}^{d-1} \times \mathbb{Z}, m \neq 0} \hat{g}^0(k) e^{2\pi i (k, \mathcal{A})}.$$  

When $F_{\alpha, \varepsilon}$ is analytic such that $f^0 \circ \mathcal{A}^{-1} \in \mathcal{A}_{\rho}$, the analyticity of $h^0$ follows from the Diophantine condition (1.5) and the exponential decay estimate (1.7) for the Fourier coefficient. More precisely, for any $x \in \mathbb{C}^d$ with $|\text{Im}| \chi < \rho - \delta$, we have

$$|g^0(x)| = \sum_{0 \neq m \in \mathbb{Z}^{d-1}} \sum_{l \in \mathbb{Z}} \hat{g}^0(k) e^{2\pi i (m, l, x)}$$

$$\leq \sum_{0 \neq m \in \mathbb{Z}^{d-1}} \sum_{l \in \mathbb{Z}} \frac{\rho^0(k)}{e^{2\pi i (m, \omega_0)} - 1} e^{2\pi |m| \rho - 2\pi |m| \varepsilon}$$

$$\leq |f^0 \circ \mathcal{A}^{-1}|_\rho \sum_{0 \neq m \in \mathbb{Z}^{d-1}} \frac{\pi}{2} |m| e^{-\pi |m| \rho} \sum_{l \in \mathbb{Z}} e^{-2\pi \delta |l|}$$

$$\leq \nu^{-1} C_{d, \varepsilon} \delta^{-(\tau + d)} \|f^0 \circ \mathcal{A}^{-1}\|_\rho,$$

where $\| \cdot \|_\rho$ is given by (1.6).
Similarly, when $F_{a,e}$ belongs to $\mathcal{G}^n$, we have, by (1.9), that
\[
\|g^0\|_{\mathcal{G}^n} \leq (2\pi)^{-n} \sum_{0 \neq m \in \mathbb{Z}^{d-1}} \sum_{l \in \mathbb{Z}} (|m| + |l|)^{n-2} \frac{|p^0(m, l)|}{|e^{2\pi i (m, l)} - 1|}
\]
(2.10)
\[
\leq n^{-1} C_n \sum_{0 \neq m \in \mathbb{Z}^{d-1}} \sum_{l \in \mathbb{Z}} (|m| + |l|)^{n-2} \cdot |m|^r \cdot \|f^0 \circ \mathfrak{A}^{-1}\|_{\mathcal{G}^n}
\]
\[
\leq n^{-1} C_{a,d} \sum_{\mu \geq 0} \mu^{-r-s+1} \cdot \|f^0 \circ \mathfrak{A}^{-1}\|_{\mathcal{G}^n},
\]
where $\| \cdot \|_{\mathcal{G}^n}$ is defined in (1.8). The sum in the R.H.S. converges provided $s > \tau + d$.

Likewise, denoting
\[
p^1 \circ \mathfrak{A} = f^1 + Df^0 \cdot \Omega \cdot h^0 + (Dh^0 \cdot \Omega \cdot h^0) \circ T_{a_0\Omega} - Dh^0 \circ T_{a_0\Omega} \cdot \Omega \cdot (h^0 + f^0),
\]
we obtain
\[
h^1 \circ \mathfrak{A}^{-1} = h^1 \circ T_{a_0\Omega} \circ \mathfrak{A}^{-1} + p^1 = \sum_{l \in \mathbb{Z}} \tilde{p}^1((0, l)) e^{2\pi i ((0, l) \cdot \cdot \cdot )},
\]
where $h^1$ can be written explicitly as $h^0$.

By induction, one can conduct the same averaging process to $h^{j+1}$ in (2.7) when $h^0, \ldots, h^j$ are already known but losing some regularities at each step. Above all, under the averaging procedure and the linear transformation, one actually obtain the following normal form for $F_{\Omega, e}$, say, to the order of $N$
\[
\mathfrak{A} \circ (H^N)^{-1} \circ F_{a_0\Omega, e} \circ H^N \circ \mathfrak{A}^{-1}(x_1, x_2)
\]
(2.11)
\[
= \left( x_1 \right) + \omega \cdot \left( \frac{0}{0} \right) + \alpha_0^{-1} \left( \begin{array}{c} \omega + L_1 \\ L_2 \end{array} \right) \sum_{j=1}^{N} \varepsilon^j \sum_{l \in \mathbb{Z}} \tilde{p}^j((0, l)) e^{2\pi i ((0, l) \cdot \cdot \cdot )}
\]
\[
+ \varepsilon^{N+1} r(x_1, x_2; \varepsilon),
\]
where $(x_1, x_2) \in \mathbb{T}^{d-1} \times \mathbb{T}$, $L = (L_1, L_2) \in \mathbb{Z}^d$ and $\varepsilon^{N+1} r(x_1, x_2; \varepsilon)$ is the Taylor remainder of order $\varepsilon^{N+1}$. Thus $r(x_1, x_2; \varepsilon)$ is bounded for sufficient small $\varepsilon$. Furthermore, the invertible conjugacy function $H^N$ takes the form of
\[
H^N = Id + \varepsilon h^0 \Omega + \varepsilon^2 h^1 \Omega + \cdots \varepsilon^{N} h^{N-1} \Omega.
\]

For future application, we introduce the following notations. Typically, we decompose $r$ into
\[
r(x_1, x_2; \varepsilon) = \left( \begin{array}{c} r_1(x_1, x_2; \varepsilon) \\ r_2(x_1, x_2; \varepsilon) \end{array} \right) \in \mathbb{T}^{d-1} \times \mathbb{T}.
\]
(2.12)
Let
\[
n = \min \left\{ s \geq 1 : \sum_{l \in \mathbb{Z}} \hat{p}^{j}((0, l)) e^{2\pi i (\omega \beta + l \gamma)} \equiv 0, \forall 1 \leq j \leq s, \right. \]
\[
(2.13) \left. \sum_{l \in \mathbb{Z}} \hat{p}^{j}((0, l)) e^{2\pi i (\omega \beta + l \gamma)} \neq 0 \right\}
\]
and we denote
\[
(\beta(x_2, \varepsilon), \eta(x_2, \varepsilon)) = \alpha_0^{-1} \left( \frac{\omega + L_1}{L_2} \right) \sum_{n \leq j \leq n + m - 1} \varepsilon^{j-n} \sum_{l \in \mathbb{Z}} \hat{p}^{j}((0, l)) e^{2\pi i (\beta x_2 + \eta x_2)}
\]
for some integer \( m \geq 1. \) By an abuse of notation, we also denote \( \mathcal{H} \circ \Delta(\alpha) \circ \mathcal{H}^{-1} \) by \( \Delta(\alpha) \) and decompose it into
\[
(2.15) \quad \Delta(\alpha) = \left( \frac{\Delta_1(\alpha)}{\Delta_2(\alpha)} \right) \in \mathbb{T}^{d-1} \times \mathbb{T}.
\]
Taking \( N = n + m - 1, \) we summarize the averaging results in the following theorem.

**Theorem 2.1.** Given \( p, q \in \mathbb{N} \) with \( p > q \geq (n + m - 1)(d + \tau). \) Assume \( \alpha_0 \Omega \) is resonant and the intrinsic frequency \( \omega \) satisfies Diophantine condition (1.5). Assume further that \( F_{\alpha, \varepsilon}(x) \) is real analytic (or \( \mathcal{C}^p \)) in \( x \) and \( \mathcal{C}^{n+m} \) in \( \varepsilon. \) Then, for sufficiently small \( \varepsilon, \) the torus map \( F_{\alpha, \varepsilon} \) can be conjugated by a real analytic (or \( \mathcal{C}^{p-q}, \) respectively) invertible function \( H^{n+m-1}_\varepsilon \) and a linear transformation \( \mathcal{H} \in \text{SL}(d, \mathbb{Z}) \) to the following resonant normal form \( \tilde{F}_{\alpha, \varepsilon} : \mathbb{T}^{d-1} \times \mathbb{T} \to \mathbb{T}^{d-1} \times \mathbb{T} \) defined by
\[
(2.16) \quad \tilde{F}_{\alpha, \varepsilon}(x_1, x_2) = \left( x_1 + \omega + \varepsilon^p \beta(x_2, \varepsilon) + \varepsilon^{n+m} r_1(x_1, x_2; \varepsilon) + \Delta_1(\alpha), x_2 + \varepsilon^p \eta(x_2, \varepsilon) + \varepsilon^{n+m} r_2(x_1, x_2; \varepsilon) + \Delta_2(\alpha) \right),
\]
where \( n, m \) and \( \beta, \eta, r_1, r_2, \Delta_1(\alpha), \Delta_2(\alpha) \) are given in (2.12)-(2.15).

Furthermore, \( r_1 \) and \( r_2 \) are real analytic (or \( \mathcal{C}^{p-q}, \) respectively) periodic functions defined on \( \mathbb{T}^d, \beta \) and \( \eta \) are real analytic periodic (or \( \mathcal{C}^{p-q}, \) respectively) functions on \( \mathbb{T}, \) and \( \Delta(\alpha) \) satisfies (2.5).

**2.1.2. Persistence of invariant surface.** By an abuse of notation, we still denote \( F_{\alpha, \varepsilon} \) by the resonant normal form developed in Theorem 2.1. In this section, we assume that there exists \( x^*_2 \in \mathbb{T} \) such that
\[
(\text{H1}) \quad \eta(x^*_2, 0) = 0 \quad \text{and} \quad D_1 \eta(x^*_2, 0) \neq 0.
\]
By the implicit function theorem, we obtain \( x^*_2(\varepsilon) \) such that \( \eta(x^*_2(\varepsilon), \varepsilon) \equiv \eta(x^*_2, 0) \) and \( x^*_2(0) = x^*_2. \) Then it follows that \( \Gamma = \{(x_1, x^*_2(\varepsilon)) \mid x_1 \in \mathbb{T}^{d-1}\} \) is an invariant surface for the torus map \( (x_1, x_2) \mapsto (x_1 + \omega + \varepsilon^p \beta(x_2, \varepsilon), x_2 + \varepsilon^p \eta(x_2, \varepsilon)) \) on which the motion is a rotation. Without loss of generality, we assume \( x^*_2(\varepsilon) = 0. \)
In what follows, we show the existence of invariant surface for the resonant normal form developed in Theorem 2.1 which is close to $\Gamma$. In the perturbative setting, we see that the invariant manifold of $F_{\alpha,\varepsilon}$ can be represented by a graph of $w : \mathbb{T}^{d-1} \to \mathbb{R}$, whose invariance determines a functional $\mathcal{F}$ defined on some function space. Then by the contraction mapping arguments, we can prove the existence of a fixed point for $\mathcal{F}$, which corresponds to the desired invariant manifold of $F_{\alpha,\varepsilon}$.

**Theorem 2.2.** Let $F_{\alpha,\varepsilon}$ be a family of real analytic ($C^\infty$ or finitely differentiable) torus maps given by (2.1). Assume $\alpha_0\Omega$ is resonant with intrinsic frequency $\omega$ satisfying Diophantine condition (1.5).

Then if the resonant normal form (2.16) satisfies non-degeneracy condition (H1), there exists a $(d-1)$-dimensional finitely differentiable invariant torus of $F_{\alpha,\varepsilon}$ for those parameters $(\alpha, \varepsilon)$ in some neighborhood of $(\alpha_0, 0)$.

If the map is finitely differentiable (but with enough regularity) we have an analogue result but the regularities of the low dimensional invariant torus may be less than those of the map.

In Appendix A, we prove a theorem on the existence of invariant surface for a family of generic torus maps generated by the resonances, from which Theorem 2.2 is an immediate result. Moreover, in the proof of Theorem 2.2 we will employ a particular contraction mapping theorem (see [HdlL17, Lemma 2.4]), which guarantees the existence of the fixed point of the operator $\mathcal{F}$ by verifying that $\mathcal{F}$ maps a closed subset in the high regularity space to itself, and is a contraction in the low regularity space. Similar ideas also appear in [Lan73].

**Remark 2.1.** In Theorem 2.2 we only obtain a finitely differentiable invariant surface even though the map $F_{\alpha,\varepsilon}$ is $C^\infty$ or analytic.

This depends a lot on the dynamics of the map in the invariant manifold. For instance, in the case that the manifold is a one-dimensional circle and the dynamics on the circle have an attractive periodic point, it is shown in [dlL97] that the invariant circles may be finitely differentiable and the regularity is determined by the exponents of the derivative at the periodic points. On the other hand, in the case that the maps inside of the invariant circle have a Diophantine rotation number, the circles are analytic when the map is analytic.

In the case of higher dimensional invariant manifolds, there are similar obstructions to regularity depending on what is the dynamics inside of the manifold (it could have invariant manifolds of further codimension). This requires further study.

### 2.2. Computation of Lindstedt series

We formulate the Lindstedt series for the invariant surface of the foliation preserving torus map generated by resonance, which provides a formal power series of the invariant surface.

Consider the following foliation-preserving torus map

$$F_\varepsilon(\theta) = \theta + \alpha_0\Omega + \varepsilon g(\theta)\Omega,$$

(2.17)
where $\Omega$ is irrational and $\epsilon > 0$. Assume also $\alpha_0 \Omega$ is resonant. Then from (2.2) we have

$$\mathcal{A} \circ F_\epsilon \circ \mathcal{A}^{-1}(\theta) = \theta + \left[ \frac{\omega}{0} + L \right] \cdot \left[ 1 + \epsilon \alpha^{-1} g(\mathcal{A}^{-1} \theta) \right],$$

and by an abuse of notation, we can write

$$F_\epsilon \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} x \\ y \end{array} \right) + \left( \begin{array}{c} \omega \\ 0 \end{array} \right) + \left( \begin{array}{c} \epsilon (\omega + L_1) g(x, y) \\ \epsilon L_2 g(x, y) \end{array} \right).$$

We look for $l_\epsilon = (l^x_\epsilon, l^y_\epsilon) : \mathbb{T}^{d-1} \to \mathbb{T}^d$ and $u_\epsilon : \mathbb{T}^{d-1} \to \mathbb{T}^{d-1}$ such that

(2.18)

$$F_\epsilon \circ l_\epsilon = l_\epsilon \circ u_\epsilon,$$

by which the graph of $l_\epsilon$ is the desired invariant surface. To formulate the Lindstedt series, we expand $l_\epsilon$ and $u_\epsilon$ into power series in $\epsilon$ as

$$l_\epsilon(\sigma) = \sum_{j=0}^{\infty} l_j(\sigma) \epsilon^j, \quad u_\epsilon(\sigma) = \sum_{j=0}^{\infty} u_j(\sigma) \epsilon^j,$$

and then take formal calculations.

By matching the coefficients of the $\epsilon^0$-terms in (2.18), we obtain $F_0 \circ l_0 = l_0 \circ u_0$ or equivalently

$$\begin{cases}
l^x_0(\sigma) + \omega = l^x_0(u_0(\sigma)), \\
l^y_0(\sigma) = l^y_0(u_0(\sigma)),
\end{cases}$$

which can be solved by choosing

$$l^x_0(\sigma) = \sigma, \quad l^y_0(\sigma) = y_0 = \text{Constant}, \quad u_0(\sigma) = \sigma + \omega.$$

For those $\epsilon^1$-terms, we get

$$F_1 \circ l_0(\sigma) + DF_0 \circ l_0(\sigma) \cdot l_1(\sigma) = l_1 \circ u_0(\sigma) + DL_0 \circ u_0(\sigma) \cdot u_1(\sigma),$$

or equivalently

(2.19)

$$\begin{cases}
l^x_1(\sigma + \omega) - l^x_1(\sigma) = (\omega + L_1) g(\sigma, y_0) - u_1(\sigma), \\
l^y_1(\sigma + \omega) - l^y_1(\sigma) = L_2 g(\sigma, y_0).
\end{cases}$$

It suffices to choose $y_0$ such that

(2.20)

$$\int_{\mathbb{T}^{d-1}} g(\sigma, y_0) \, d\sigma = 0,$$

and $u_1(\sigma) \equiv 0$. Then $l_1 = (l^x_1, l^y_1)$ can be solved from (2.19) by using Lemma 1.1 but with the average

$$\langle l_1 \rangle = \int_{\mathbb{T}^{d-1}} l_1(\sigma) \, d\sigma$$

to be specified.
To clarify the induction, we proceed to compute the equation for those $\varepsilon^2$-terms, which reads
\[ F_2 \circ l_0 + DF_1 \circ l_0 \cdot l_1 + DF_0 \circ l_0 \cdot l_2 = l_2 \circ u_0 + Dl_1 \circ u_0 \cdot u_1 + Dl_0 \circ u_0 \cdot u_2, \]
or equivalently
\[ l_2(\sigma + \omega) - l_2(\sigma) = \left( \frac{(\omega + L_1)Dg(\sigma, y_0)l_1(\sigma) - u_2(\sigma)}{L_2Dg(\sigma, y_0)} \right). \]
Let $l_1(\sigma) = \langle l_1 \rangle + \tilde{l}_1(\sigma)$, in which $\tilde{l}_1(\sigma)$ is uniquely determined by (2.19). Then the R.H.S. of (2.21) reads
\[
\left( \frac{\omega + L_1}{L_2} \right) \cdot Dg(\sigma, y_0) \tilde{l}_1(\sigma) + \left( \frac{\omega + L_1}{L_2} \right) Dg(\sigma, y_0) \langle l_1 \rangle - \left( \frac{u_2(\sigma)}{0} \right).
\]
In order the average of R.H.S of (2.21) to vanish, we need to choose parameters $\langle l_1 \rangle = (\langle l_1^1 \rangle, \langle l_1^2 \rangle)$ and $u_2(\sigma)$. More precisely, we have
\[
\langle D_2g(\cdot, y_0) \cdot \langle l_1^1 \rangle \rangle = \langle Dg(\cdot, y_0) \tilde{l}_1(\cdot) \rangle,
\]
\[
(\omega + L_1)\langle D_2g(\cdot, y_0) \cdot \langle l_1^2 \rangle - \langle u_2 \rangle = (\omega + L_1)\langle Dg(\cdot, y_0) \tilde{l}_1(\cdot) \rangle,
\]
since $\langle D_1g(\cdot, y_0) \rangle = 0$ by (2.20). Then if
\[ \langle D_2g(\cdot, y_0) \rangle \neq 0, \]
we just choose $u_2 \equiv 0$ and take
\[
\langle l_1^1 \rangle = \langle D_2g(\cdot, y_0) \rangle^{-1} \cdot \langle Dg(\cdot, y_0) \tilde{l}_1(\cdot) \rangle.
\]
By induction, we assume that, under assumption (2.20) and (2.22), we can choose $u_j \equiv 0$ for all $1 \leq j \leq n - 1$ and always find $\langle l_j^r \rangle$ for $1 \leq j \leq n - 2$ such that (2.18) holds up to $O(\varepsilon^d)$. Comparing the coefficients of $\varepsilon^d$ in (2.18), we obtain
\[
\langle D_2g(\cdot, y_0) \cdot \langle l_{n-1}^r \rangle = \langle G_n[l_0, \cdots, l_{n-1}]; g \rangle \rangle,
\]
\[
(\omega + L_1)\langle D_2g(\cdot, y_0) \cdot \langle l_{n-1}^r \rangle - \langle u_{n}(\sigma) \rangle = (\omega + L_1)\langle G_n[l_0, \cdots, l_{n-1}]; g \rangle, \]
where $G_n$ can be computed explicitly by the known functions $g$ and $l_0, \cdots, l_{n-1}$ from the induction procedure. Then it suffices to choose $u_n(\sigma) \equiv 0$ and solve $\langle l_{n-1}^r \rangle$ from the second equation above. This completes the induction.

We conclude the above results in the following proposition.

**Proposition 2.3.** Let $\alpha_0 \Omega$ be resonant with an intrinsic frequency $\omega \in \mathbb{R}^{d-1}$ defined in (2.2). Assume there exists $y_0 \in \mathbb{R}$ such that (2.20) and (2.22) hold. Then we can find a formal power series $l_\varepsilon = \sum_{j=0}^{\infty} l_j \varepsilon^j$ in $\varepsilon$ such that
\[ F_\varepsilon \circ l_\varepsilon(\sigma) = l_\varepsilon(\sigma + \omega). \]
Combining Theorem 2.2 and Proposition 2.3, we have the following result for the foliation preserving torus map $F_\varepsilon$ in (2.17).

**Theorem 2.3.** Let $p, q, N \in \mathbb{N}$ with $p - 1 > q \geq N(d + \tau)$. Suppose $F_\varepsilon(\theta)$ is $C^p$ in $\theta$ and is smooth in $\varepsilon$. Under the assumptions of Proposition 2.3 there exist two $C^{p-q-1}$ maps $l_* : \mathbb{T}^{d-1} \to \mathbb{T}$ and $u_* : \mathbb{T}^{d-1} \to \mathbb{T}^{d-1}$ such that

$$F_\varepsilon \circ l_* = l_* \circ u_*.$$ 

Let $I^{\varepsilon N} = \sum_{j=0}^{N} l_j(\sigma)\varepsilon^j$ and $u_0(\sigma) = \sigma + \omega$ be the truncated Lindstedt series obtained in Proposition 2.3 satisfying

$$\|F_\varepsilon \circ I^{\varepsilon N} - I^{\varepsilon N} \circ u_0\|_{C^{p-q}} < C \varepsilon^{N+1}.$$ 

Then we have

$$\|I^{\varepsilon N} - l_*\|_{C^{p-q-1}} \leq C' \varepsilon^{N+1}, \quad \|u_0 - u_*\|_{C^{p-q-1}} \leq C' \varepsilon^{N+1},$$

where the constant $C' > 0$ depends on $p, q, N, d, \tau$ and $g$.

**Proof.** Since the conditions (2.20) and (2.22) verify (H1), we obtain the invariant torus $\Gamma_\varepsilon$ of $F_\varepsilon$, which is constructed by the graph of the map $l_* : \mathbb{T}^{d-1} \to \mathbb{T}$ in the proof of Theorem 2.2. We denote by $u_* : \mathbb{T}^{d-1} \to \mathbb{T}^{d-1}$ the evolution on the invariant surface $\Gamma_\varepsilon$. Moreover, we have that $l_*$ and $u_*$ are $C^{p-q-1}$ smooth, in contrast with the $C^{p-q}$ resonant normal form of $F_\varepsilon$. The remaining estimates follow directly from the construction of the Lindstedt series.

**Remark 2.2.** For a family of generic torus maps

$$F_{\Omega,\varepsilon}(x) = x + \Omega + \varepsilon f(x),$$

the Lindstedt series for the resonant invariant surface is more involved than the Lindstedt series we obtained for foliation preserving maps. It is shown in Proposition 1.4 in the Appendix that there exists $l_\varepsilon$ such that

$$F_{\Omega,\varepsilon} \circ l_\varepsilon(\sigma) = l_\varepsilon(\sigma + \omega + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots),$$

where $\omega$ is the intrinsic frequency of $\Omega$ and $\{u_j\}_{j \geq 1}$ is a sequence of constant vectors independent of $\sigma$.

Note that in the case of general maps of the torus, the frequency on the torus depends on the perturbation. However, as we shall see in Subsection 2.4, a $C^1$ perturbation of the foliation preserving torus map does not change the rotational number of its uniformly attracting (or repelling) invariant circle.
2.3. **Sternberg linearization around the invariant circle.** In this section, we consider the particular case of $d = 2$ for the analytic foliation preserving torus map (2.1) near resonance. Then under the assumption of Theorem 2.2 there exists an invariant circle for all $(\alpha, \varepsilon)$ close to $(\alpha_0, 0)$.

Now we study the local dynamics of $F_{\alpha, \varepsilon}$ around the invariant circle. Up to the coordinate transformation, the restriction of $F_{\alpha, \varepsilon}$ on a small neighborhood of the invariant circle can be characterized by a skew product map

$$\varphi: T \times \mathbb{R} \ni (\sigma, \rho) \mapsto (u(\sigma), \Gamma_\sigma(\rho)) \in T \times \mathbb{R},$$

where $\rho$ describes the normal coordinate and $\Gamma_\sigma(0) = 0$. As a result, the invariant circle in the $(\sigma, \rho)$-coordinate is characterized by $T \times \{\rho = 0\}$.

We recall that it could happen that the invariant circle is finitely differentiable only. On the other hand, due to the preservation of the foliation, we have a set of normal fibers that are sent into each other by the map. The motion from a fiber to its image is analytic. In mathematical language this is described as a *skew-product map*. The dynamics in the base may be finitely differentiable but the dynamics on the fiber is analytic. Hence, for a description of the result, it is important to consider functions that have different regularity along different directions.

Assume the dynamics $u$ on the invariant circle is invertible and denote

$$w(\sigma) = u^{-1}(\sigma).$$

Moreover, we assume the invariant circle is uniformly repelling. More precisely, assume that in the normal coordinate, there exists a constant $\lambda > 1$ such that for sufficiently small $\delta > 0$, we can always find a $\gamma > 0$ such that

$$|\Gamma_\sigma'(\rho) - \lambda| < \delta$$

for all $\rho \in B_\gamma(0)$ and $\sigma \in T$, where $B_\gamma(0)$ is the closed ball centered at zero with radius $\gamma$ in the complex plane.

**Theorem 2.4.** (Sternberg Linearization Theorem) Consider the skew product map $\varphi(\sigma, \rho) = (u(\sigma), \Gamma_\sigma(\rho))$ defined in (2.23). Assume $\Gamma_\sigma(\rho)$ is analytic in $\rho$ and $\Gamma_\sigma(0) = 0$. Furthermore, $u(\sigma)$ and $\Gamma_\sigma(\rho)$ are continuous in $\sigma$ satisfying (2.24) and (2.25). Then there exists an invertible coordinate transformation $H: T \times \mathbb{R} \to T \times \mathbb{R}$ such that

$$H^{-1} \circ \varphi \circ H(\sigma, \rho) = (u(\sigma), A_{\sigma, \rho}),$$

where $A_{\sigma, \rho} = \Gamma_\sigma'(0)$. Moreover, the transformation $H$ is analytic in $\rho$ and continuous in $\sigma$.

**Proof.** We denote

$$h_N^\sigma = A_{\sigma} \cdot A_{w(\sigma)} \cdots A_{w^N(\sigma)} \circ \Gamma_{w^N(\sigma)}^{-1} \circ \cdots \circ \Gamma_{\sigma}^{-1},$$

and it follows that

$$h_N^{w(\sigma)} \circ \Gamma_{\sigma}^{-1} = A_{\sigma}^{-1} \cdot h_N^{N+1}.$$
If \( \lim_{N \to \infty} h^N_{\sigma} \) exists and denote
\[
h_{\sigma} = \lim_{N \to \infty} h^N_{\sigma},
\]
we then have
\[
h_{w(\sigma)} \circ \Gamma_{\sigma}^{-1} = A_{\sigma}^{-1} \cdot h_{\sigma}
\]
and equivalently
\[
h_{\sigma} \circ \Gamma_{\sigma} \circ h^{-1}_{w(\sigma)} = A_{\sigma}.
\]

Let
\[H : \mathbb{T} \times \mathbb{R} \ni (\sigma, \rho) \mapsto (\sigma, h^{-1}_{w(\sigma)}(\rho)) \in \mathbb{T} \times \mathbb{R}.\]
We immediately have
\[H^{-1} \circ \varphi \circ H(\rho, \sigma) = (u(\sigma), A_{\sigma}\rho).\]

The only thing left is to show the existence and analyticity of the limit. Observing the fact that \( A_{\sigma} \) and \( \Gamma_{\sigma}(\rho) \) are tangent, we have:
\[
\|h^{N+1}_{\sigma} - h^N_{\sigma}\|_{B^2_{\gamma}(0)} = \left\|A_{\sigma} \cdot A_{w(\sigma)} \cdots A_{w^{N}(\sigma)} \left[A_{w^{N+1}(\sigma)} \circ \Gamma_{w^{N+1}(\sigma)}^{-1} - \text{Id}\right] \circ \Gamma_{w^{N}(\sigma)}^{-1} \circ \cdots \circ \Gamma_{\sigma}^{-1}\right\|_{B^2_{\gamma}(0)} \leq (\lambda + \delta)^N \cdot C \|\Gamma_{w^{N}(\sigma)}^{-1} \circ \cdots \circ \Gamma_{\sigma}^{-1}\|_{B^2_{\gamma}(0)}^2.
\]
Moreover, since \( \Gamma_{\sigma}(0) = 0 \), the mean value theorem implies
\[
\|h^{N+1}_{\sigma} - h^N_{\sigma}\|_{B^2_{\gamma}(0)} \leq C \left[\frac{\lambda + \delta}{(\lambda - \delta)^2}\right]^N.
\]
Therefore, the sequence \( \{h^N_{\sigma}\} \) converges uniformly to an analytic function \( h_{\sigma}(\rho) \) (analytic in \( \rho \)) provided \( \delta \) is small such that
\[
\lambda + \delta < (\lambda - \delta)^2.
\]
This completes the proof of Theorem 2.4. \(\square\)

In the case that the dynamics \( u \) on the invariant circle can be conjugated to a Diophantine rotation, the non-autonomous Sternberg Linearization Theorem 2.4 can also be strengthened in such a way that, in the normal coordinate, the effect of \( \varphi \) is simply to expand by a constant factor.

More precisely, we give a general result Theorem 2.5 which may be of interest itself. This theorem tells that, when the map in the base is conjugate to a Diophantine rotation we can get the normal contraction to be constant (by a smooth change of variables).

**Theorem 2.5.** Let \( \psi : \mathbb{T}^{d-1} \times \mathbb{R} \to \mathbb{T}^{d-1} \times \mathbb{R} \) be of the form
\[
\psi(\sigma, t) = (v(\sigma), a(\sigma)t)
\]
with \( v \in \mathcal{C}^r(T^{d-1},T^{d-1}) \) a diffeomorphism and the function \( a(\sigma) \) belonging to \( \mathcal{C}^r(T^{d-1}, \mathbb{R} \setminus \{0\}) \). Assume that there exists a diffeomorphism \( h \in \mathcal{C}^r(T^{d-1}, T^{d-1}) \) and a vector \( \Omega \in D_{d-1}(\nu, \tau) \) such that

\[
(2.26) \quad h^{-1} \circ v \circ h(\sigma) = \sigma + \Omega.
\]

Assume further that \( \tau < r \).

Then, there exists a function \( b \in \mathcal{C}^{r-\tau}(T^{d-1}, \mathbb{R} \setminus \{0\}) \) and a constant \( \kappa \in \mathbb{R} \setminus \{0\} \) such that the diffeomorphism

\[
H(\sigma, t) = (h(\sigma), b(\sigma) t)
\]

satisfies

\[
(2.27) \quad \psi \circ H(\sigma, t) = H(\sigma + \Omega, \kappa t).
\]

**Proof.** Expanding (2.27), we see that the first component of (2.27) is just our assumption (2.26). The second component of (2.27) is

\[
a \circ h(\sigma) \cdot b(\sigma) = b(\sigma + \Omega) \cdot \kappa.
\]

Noticing that \( a(\sigma) \neq 0 \), hence, it has the same sign for all \( \sigma \in T^{d-1} \). In the case that \( a(\sigma) \) is always positive, we see that

\[
(2.28) \quad (\log b)(\sigma + \Omega) - (\log b)(\sigma) = \log(a \circ h)(\sigma) - \log \kappa,
\]

which is a very standard cohomology equation in KAM theory for \( \log b \) (see [dlL01, Lemma 2.1.8]).

Then we just take

\[
\kappa = \exp \left( \int_{T^{d-1}} \log(a \circ h)(\sigma) \, d\sigma \right)
\]

and solve (2.28) for \( \log b \) (see [R65] for estimates and obtain \( b \) by exponentiating this result).

In the case that \( a(\sigma) \) is negative, we have

\[
(\log b)(\sigma + \Omega) - (\log b)(\sigma) = [\log(-a \circ h)](\sigma) - \log(-\kappa),
\]

and once again, we can repeat the analysis above. This completes the proof of Theorem 2.5. \( \square \)

The meaning of (2.27) in this paper is that, performing change of variable after Theorem 2.4, we reduce the mapping \( \phi \) in the neighborhood of invariant circle to the constant coefficient map \( (\sigma, \rho) \to (\sigma + \Omega, \kappa \rho) \).

**Remark 2.3.** It is worth noticing that if we consider a family of problems, we expect that the hypothesis of existence of function \( h \) conjugating \( v \) to a rotation will be satisfied by a positive measure set of parameters.
Remark 2.4. For general maps of the torus, it is expected that in the vicinities of higher multiplicity resonance, one could find higher co-dimension torus and develop analogues of Sternberg theorems \([\text{KP90}]\). In the case of foliation preserving torus maps, as indicated in \([\text{SdlL12}]\), only resonance of multiplicity one happen.

2.4. Structural stability. In Subsection 2.3, we show the reducibility of the skew product flow (2.23) around the repulsive invariant circle to a linear flow provided that the rotation number of the evolution on the invariant circle is Diophantine. In this part, we study the structural stability of the foliation preserving torus maps in \(\mathbb{T}^2\).

Let \(T_f : \mathbb{T}^2 \to \mathbb{T}^2\) be a \(C^1\) map of form (1.2) with \(f > 0\). Assume that \(T_f\) has a uniformly attracting invariant circle \(\Gamma_f \subset \mathbb{T}^2\). Recall that the only direction with possible non-vanishing Lyapunov exponent is the direction of \(\omega\). Hence, the invariant circle \(\Gamma_f\) is transversal to \(\omega\). Moreover, since \(T_f\) has no periodic points, we see that the rotation number of \(T_f|_{\Gamma_f}\) has to be irrational.

Let \(g : \mathbb{T}^2 \to \mathbb{R}\) be sufficiently \(C^1\) close to \(f\) and remain positive. Consider the map \(T_g\) given by (1.2), by the theory of normally hyperbolic invariant manifold \([\text{Fen74}, \text{Fen77}]\), it follows that \(T_g\) also admits a uniformly attracting invariant circle \(\Gamma_g\) close to \(\Gamma_f\).

Since \(T_g|_{\Gamma_g}\) depends continuously on \(g\), if the rotation number of \(T_g|_{\Gamma_g}\) is different from that of \(T_f|_{\Gamma_f}\), we can find a map \(h\) in the homotopic functions \(\{sf + (1 - s)g : 0 \leq s \leq 1\}\) connecting \(f\) and \(g\) such that \(T_h|_{\Gamma_h}\) has rational rotation number, which is impossible since \(T_h\) also has no periodic points. This shows that small \(C^1\) perturbation of the foliation preserving torus map does not change the rotation number of its uniformly attracting (or repelling) invariant circle.

2.5. Example. In this part, we study a simple example of the foliation preserving torus map to illustrate our theoretical results in this section. Consider

\[(2.29) \quad F_\varepsilon(\theta) = \theta + \Omega + \varepsilon\Omega + \varepsilon\Omega + \varepsilon g(\theta), \quad \theta \in \mathbb{T}^2,\]

where \(\Omega = (\omega, 1)^T \in \mathbb{R}^2\), \(\varepsilon > 0\) and \(g : \mathbb{T}^2 \to \mathbb{R}\) is given by

\[g(\theta) = a + \delta_1 \sin(2\pi x) + \delta_2 \sin(2\pi y), \quad \theta = (x, y).\]

Obviously, \(\Omega\) is resonant and can be written as \(\Omega = (\omega, 0)^T + (0, 1)^T\), where \(\omega\) is the intrinsic frequency of \(\Omega\). Then we have

\[F_\varepsilon\left(\begin{array}{c} x \\ y \end{array}\right) = \left(\begin{array}{c} x \\ y \end{array}\right) + \left(\begin{array}{c} \omega \\ 0 \end{array}\right) + \left(\begin{array}{c} \varepsilon\omega g(x, y) \\ \varepsilon g(x, y) \end{array}\right).\]

Assuming \(\omega\) is Diophantine, we obtain from Theorem 2.1 the resonant normal form

\[F_\varepsilon(x, y) = \left(\begin{array}{c} x + \omega + \varepsilon\beta(y) + \varepsilon^2 r_1(x, y; \varepsilon) \\ y + \varepsilon\eta(y) + \varepsilon^2 r_2(x, y; \varepsilon) \end{array}\right),\]

where

\[\eta(y) = a + \delta_2 \sin(2\pi y), \quad \beta(y) = \omega(a + \delta_2 \sin(2\pi y)).\]
If $|a| < |\delta_2|$, there exists a number $y_0 \in [0, 1]$ such that
\begin{equation}
(2.30) \quad \eta(y_0) = 0
\end{equation}
and $\eta'(y_0) = 2\pi \delta_2 \cos(2\pi y_0) \neq 0$, which verifies the non-degeneracy condition $(H1)$ in Theorem 2.2. It then follows that $F_\epsilon$ has a one-dimensional invariant circle for sufficiently small $\epsilon$.

**Remark 2.5.** Note that, in general, we will get two numbers $y_0$ solving $\eta(y_0) = 0$: one of them with $\eta'(y_0) > 0$ and another one with $\eta'(y_0) < 0$. We will continue the analysis for one of them but the calculations will apply to the other one just as well.

Next, we employ the Lindstedt series to study the invariant circle generated by the resonance. Obviously, conditions (2.20) and (2.22) are exactly the non-degeneracy condition $(H1)$. From Proposition 2.3 we have

$$F_\epsilon \circ l_0(\sigma) = l_0(\sigma + \omega),$$

where

$$l_0(\sigma) = \left( \frac{t_\epsilon(\sigma)}{l_\epsilon(\sigma)} \right) = \left( \sigma + \epsilon^1 l_1(\sigma) + \epsilon^2 l_2(\sigma) + \cdots \right).$$

It follows that the graph $\mathcal{C}$ of $l_0$ is the invariant circle of $F_\epsilon$, whose existence has already been established. Introduce the local coordinate $(\rho, \sigma)$ around the invariant circle $\mathcal{C}$ by defining

$$\Phi : \mathbb{R} \times \mathbb{T} \ni (\rho, \sigma) \mapsto (l_\epsilon(\sigma) + \omega \rho, l_\epsilon(\sigma) + \rho) = (x, y) \in \mathbb{T}^2.$$  

Obviously, $\det \frac{\partial (x, y)}{\partial (\rho, \sigma)} = -1 + O(\epsilon) \neq 0$. Let

$$\Phi^{-1}(x, y) = \left( \frac{y - y_0}{x - \omega(y - y_0)} \right) + \epsilon \left( \begin{array}{c} \varphi(x, y) \\ \psi(x, y) \end{array} \right) + O(\epsilon^2).$$

It follows from $\Phi^{-1} \circ \Phi = Id$ that

$$\varphi(\sigma + \omega \rho, y + \rho) = t_1^\epsilon(\sigma),$$

which is independent of $\rho$. One easily solves from $\omega \cdot \partial_x \varphi + \partial_y \varphi = 0$ that

$$\varphi(x, y) = t_1^\epsilon(x - \omega(y - y_0)).$$

Considering $\tilde{\varphi} = \varphi \circ F_\epsilon \circ \Phi$, we compute

$$\frac{d\tilde{\varphi}}{d\rho} = (\omega + O(\epsilon))\partial_x \varphi + (1 + O(\epsilon))\partial_y \varphi = O(\epsilon).$$

As a result, under the new coordinate, we have

$$\Phi^{-1} \circ F_\epsilon \circ \Phi(\rho, \sigma) = (\Gamma_{\sigma}(\rho), u(\sigma)),$$

where $\Gamma_{\sigma}(0) = 0$ and

$$\Gamma_{\sigma}(\rho) = \rho + t_1^\epsilon(\sigma) + \epsilon g \circ \Phi - y + \epsilon \tilde{\varphi}(\rho, \sigma) + O(\epsilon^2).$$
Compute
\[
\left. \frac{d}{d \rho} \right|_{\rho=0} (g \circ \Phi) = 2\pi [\delta_1 \omega \cos(2\pi l_1'(\sigma)) + \delta_2 \cos(2\pi l_2'(\sigma))].
\]
Recalling that \( l_1'(\sigma) = y_* + O(\varepsilon) \) and \( \cos(2\pi y_*) = \pm \sqrt{1 - (a/\delta_2)^2} \), we can always choose \( \delta_1 \) and \( \delta_2 \) with \( 2\omega |\delta_1| < |\delta_2| \sqrt{1 - (a/\delta_2)^2} = \lambda \) and \( \delta_2 \cos(2\pi y_*) > 0 \) such that \( \left. \frac{d}{d \rho} \right|_{\rho=0} (g \circ \Phi) = \lambda \pi + O(\varepsilon) \neq 0 \). As a result, for \( 0 < \varepsilon \ll 1 \), we have
\[
\Gamma_{\sigma}'(0) > 1 + \frac{\lambda \pi}{4\varepsilon} > 1,
\]
which implies the invariant circle \( \mathcal{C} \) is uniformly repelling.

As indicated in Remark 2.5, there are other solutions of the equation (2.30). In the preceding we have presented the calculations for one of them with \( \eta'(y_*) > 0 \) – this is the solution that plays a more important role in the applications to delay equations in [HDL22]. Proceeding in the same way with the other solution, we obtain another invariant circle which is attractive.

For a family of foliation preserving torus maps \( F_{\alpha,\varepsilon}(\theta) = \theta + \alpha \Omega + \varepsilon \Omega g(\theta) \) close to \( F_\varepsilon \) given by (2.29) (i.e., \(|\alpha - 1| \ll 1\)), there also exists a uniformly repelling invariant circle \( \mathcal{C}_\alpha \) close to \( \mathcal{C} \) by using Theorem 2.2 when \( 0 < \varepsilon \ll 1 \).

Since the rotational number of \( \mathcal{C}_\alpha \) is continuous in \( \alpha \), we see that, for for many points \( \alpha \) close to one, Theorem 2.5 holds and the map \( F_{\alpha,\varepsilon} \) in the neighborhood of \( \mathcal{C}_\alpha \) can be reduced to an autonomous linear map.

A more delicate version of the argument – which we do not carry out in detail – shows that the set of parameters where the rotation number is Diophantine is of positive measure.

3. KAM theory for foliation preserving torus maps

In this section, we consider the conjugation problem of the foliation preserving torus maps to the rigid rotation by the KAM techniques. Instead of treating the nonlinearity as a perturbation, we study a family of non-perturbation foliation preserving torus maps.

The KAM theorem we present is in an a-posteriori format. That is, given an approximate solution satisfying some non-degeneracy condition, there is a true solution nearby.

More precisely, we study the following foliation preserving torus maps
\begin{equation}
F(x) = x + \Omega f(x)
\end{equation}
defined on \( \mathbb{T}^d \) with non-resonant \( \Omega \). Assume \( \alpha \Omega \) satisfy the Diophantine condition (1.5) and \( f \) is real analytic.

Following the extra parameter method of [Mos67], we find an extra parameter \( \lambda \) and a foliation preserving map in such a way that the mapping \( F + \lambda \Omega \) is conjugate to the rotation by \( \alpha \).
Remark 3.1. We recall how the result thus obtained translates into results for families.

From the a-posteriori format of the theorem, it follows that $\lambda(\alpha)$ is a Lipschitz function for $\alpha$ defined in the set of parameters. If we consider a family of parameters indexed by $\varepsilon$, we obtain $\lambda(\varepsilon, \alpha)$ is a Lipschitz function.

If we study the equation $\lambda(\varepsilon, \alpha) = 0$ using the Lipschitz implicit function theorem (under the assumption that $\partial_{\varepsilon} \lambda \neq 0$), we obtain that for a positive set of parameters $\varepsilon$, the map can be conjugated to a rotation. A numerical study of this problem for the families of maps that appear in the study of cavities with moving boundaries appears in [PdlLV03].

More concretely, we are looking for a real analytic periodic function $h$ and parameter $\lambda \in \mathbb{R}$ in such a way that

$$F \circ H = H \circ T_{\alpha \Omega} + \lambda \Omega,$$

where $H = I + h \Omega$ and $T_{\alpha \Omega}(x) = x + \alpha \Omega$.

Denoting the functional $\mathcal{F}$ by

$$\mathcal{F}[h, \lambda] \cdot \Omega = F \circ H - H \circ T_{\alpha \Omega} - \lambda \Omega$$

$$= [h - h \circ T_{\alpha \Omega} - (\alpha + \lambda) + f(\text{Id} + h \Omega)] \cdot \Omega,$$

the conjugation problem of the foliation preserving torus maps is transformed into finding the solution of the functional equation

$$\mathcal{F}[h, \lambda] = 0.$$ 

Following the conventional notations in the KAM theory, we denote in the sequel various constants by the letter $C$ with some subscripts indicating the dependence on the given quantities, which might be different from line to line. These constants could be made explicit from the context, but need not be.

**Theorem 3.1.** Let $\alpha \Omega$ satisfy Diophantine condition (1.5) and $f$ belong to the analytic function space $A_\rho$. Assume that there is an approximate solution $(h_0, \lambda_0)$ satisfying

(i) $h_0 \in A_\rho$;

(ii) $DH_0 = I + Dh_0 \cdot \Omega$ is invertible with $(DH_0)^{-1} \in A_\rho$;

(iii) $\det(I + Dh_0 \cdot \Omega) \neq 0$ with $\langle \cdot \rangle$ being the average of periodic functions.

Denoting the initial error by $e_0$, i.e.,

$$e_0 = \mathcal{F}[h_0, \lambda_0],$$

then if $\|e_0\|_\rho$ is sufficiently small, there is a true solution $(h, \lambda)$ of (3.4) satisfying $h \in A_{\rho/2}$ and

$$|\lambda - \lambda_0|, \|h - h_0\|_{\rho/2} \leq C(4/\rho)^2 \|e_0\|_\rho,$$

where the constant $C$ depends only on the given quantities $h_0, \Omega, d$ and $\nu$ in (1.5).
We apply the Nash-Moser method to prove Theorem 3.1. In subsection 3.1, we analyze the Newton equation for the functional equation (3.4) in which the small divisor problem is overcome by the classical cohomology equation (1.10). Then we prove the convergence of the Newton iteration in subsection 3.2.

3.1. Newton equation. The Newton equation of (3.4) is
\[ \Delta h - \Delta h \circ T_{a\Omega} - \Delta \lambda + Df \circ (Id + h\Omega) \cdot \Omega \Delta h = -e. \]
Differentiating both sides of \( \mathcal{F}[h, \lambda] = e \), we obtain
\[ Dh - Dh \circ T_{a\Omega} + Df \circ (Id + h\Omega)(I + Dh \cdot \Omega) = De. \]

Denoting
\[ \Delta V = (I + Dh \cdot \Omega)^{-1} \cdot \Omega \cdot \Delta h \]
and substituting (3.7)-(3.8) into (3.6), we have
\[
\begin{align*}
\Delta V \circ T_{a\Omega} - \Delta V = & \quad \left( (DH)^{-1} \circ T_{a\Omega} \cdot e \cdot \Omega - (DH)^{-1} \circ T_{a\Omega} \cdot \Delta \lambda \cdot \Omega \right) \\
& + (DH)^{-1} \circ T_{a\Omega} \cdot \Omega \cdot De \cdot \Delta V.
\end{align*}
\]
Since \( De \cdot \Delta V \) is quadratic in error, we ignore it for the moment and consider the modified Newton equation
\[ \Delta V \circ T_{a\Omega} - \Delta V = \left( (DH)^{-1} \circ T_{a\Omega} \cdot e - (DH)^{-1} \circ T_{a\Omega} \cdot \Delta \lambda \right) \cdot \Omega. \]

Then, by Lemma 1.1, we can solve (3.9) uniquely (in the sense of vanishing average) provided
\[ \det \langle I + \Omega Dh \rangle \neq 0, \]
where \( \langle \cdot \rangle \) denotes the average of periodic functions. Furthermore, we also have
\[ \| \Delta V \|_{\rho - \delta} \leq C_{\nu_d} \cdot \| (DH)^{-1} \|_{\rho} \cdot \delta^{-\tau} \cdot \| e \|_{\rho}, \]
and
\[ |\Delta \lambda| = \left| \langle (DH)^{-1} \cdot (DH)^{-1} \circ R_{a\Omega} \cdot e \rangle \right| \leq C_{(DH)} \cdot \| e \|_{\rho}. \]

Hence, it follows from (3.8) that
\[ \| \Omega \Delta h \|_{\rho - \delta} \leq C_{\nu_d} \| DH \|_{\rho} \cdot \| (DH)^{-1} \|_{\rho} \cdot \delta^{-\tau} \cdot \| e \|_{\rho}. \]

Now we give the estimates for the new error
\[ e^+ = \mathcal{F}[h^+, \lambda^+] \]
with
\[ h^+ = h + \Delta h, \quad \lambda^+ = \lambda + \Delta \lambda. \]
Since
\[ F[h^+, \lambda^+] = F[h, \lambda] + D F[h, \lambda] (\Delta h, \Delta \lambda) + \int_0^1 s \int_0^1 D^2 F[h + st \Delta h, \lambda, st \Delta \lambda] (\Delta h, \Delta \lambda) \, ds \, dt \]
\[ = (DH)^{-1} \circ R_\sigma \cdot \sigma \cdot De \cdot \Delta V + \int_0^1 s \int_0^1 D^2 f \circ (Id + \Omega h + \Omega st \Delta h) \cdot (\Omega \Delta h)^{o2} \, ds \, dt, \]
we see that
\[ \|F[h^+, \lambda^+]\|_{\rho-\delta} \leq C_{v_d} \cdot \|(DH)^{-1}\|_{\rho}^2 \cdot \|\Omega\| \cdot \delta^{-(\tau+1)} \cdot \|e\|_{\rho}^2 \]
\[ + C_{v_d}^2 \|D^2 f\|_{\rho} \cdot \|DH\|_{\rho}^2 \cdot \|(DH)^{-1}\|_{\rho}^2 \cdot \delta^{-2\tau} \cdot \|e\|_{\rho}^2, \]
\[ \leq C_{v_d, h, \Omega} \cdot \delta^{-2\tau} \cdot \|e\|_{\rho}^2. \]

We conclude the above analysis in the following iterative lemma.

**Lemma 3.1.** Given the approximate solution \((h, \lambda) \in \mathcal{A}_\rho \times \mathbb{R}\) with the error \(e = F[h, \lambda]\). Assume \(DH\) is invertible and (3.10) holds. Then, for any \(0 < \delta < \rho\), there exists \((\Delta h, \Delta \lambda) \in \mathcal{A}_{\rho-\delta} \times \mathbb{R}\) such that
\[ \|e^+\|_{\rho-\delta} = \|F[h^+, \lambda^+]\|_{\rho-\delta} \leq C_{v_d, h, \Omega} \cdot \delta^{-2\tau} \cdot \|e\|_{\rho}^2, \]
where \(h^+ = h + \Delta h\) and \(\lambda^+ = \lambda + \Delta \lambda\).

Note that the openness of the invertibility and non-degeneracy condition (3.10) in Lemma 3.1 enables us to iterate the Newton steps.

Now we start proving the convergence of the iteration sequences as well as verifying the conditions in Lemma 3.1 at each step. Another key point to ensure the convergence is to prove the uniform boundedness of \(DH_n\) and \((DH_n)^{-1}\) in the common analyticity domain developed below. All the arguments are very standard in the KAM theory.

### 3.2. Proof of the convergence

From the standard techniques in KAM theory, we use the subscript \(n\) to denote the \(n\)-th step for the Newton iterations. More precisely, we choose the loss of the analyticity domain \(\sigma_n\) as \(\sigma_n = 2^{-(n-1)} \sigma\) and \(\sigma = \rho/4\). Let \(\rho_{n+1} = \rho_n - \sigma_{n+1}\) and \(\rho_0 = \rho\). Inductively, we assume the errors \(e_n = F[h_n, \lambda_n]\) satisfy \(\|e_n\|_{\rho_n} \leq e_n\), where \(h_0, \lambda_0\) are given in (3.5) and \(\varepsilon_0 = \|e\|_{\rho_0}\). Noted that \(h_n\) and \(\lambda_n\) are inductively defined by \(h_n = h_{n-1} + \Delta h_{n-1}\) and \(\lambda_n = \lambda_{n-1} + \Delta \lambda_{n-1}\). Furthermore, we also assume that
\[ e_n = C \sigma_n^{-2\tau} e_{n-1}. \]
Generally, if (3.13) holds for all \(n\), it is easy to show that \(e_n\) approaches zero when \(e_0\) is small enough. Indeed, denoting \(\bar{e}_n = C \sigma_n^{-2\tau} 2^{2(n+1)} e_n\), from (3.13) one has \(\bar{e}_{n+1} = \bar{e}_n^2\), which implies
\[ \bar{e}_n = [C (1/\sigma)^{2\tau} e_0]^n. \]
Then if
\[ (3.15) \quad C(1/\sigma)^{2r\varepsilon_0} < 1, \]
\(\varepsilon_n\) obviously approaches zero and satisfies
\[ \sum_{n=1}^{\infty} \tilde{\varepsilon}_n \leq \sum_{n=1}^{\infty} [C(1/\sigma)^{2r\varepsilon_0}]^n \leq C(1/\sigma)^{2r\varepsilon_0}. \]

To prove the \((n + 1)\)-th step, it suffices to verify the conditions in Lemma 3.1. All together, we are led to showing the difference \(\Omega h_n - \Omega h\) is small enough so that the non-degeneracy and invertibility conditions hold. Furthermore, we also need to show \(\text{DH}_n\) and \((\text{DH}_n)^{-1}\) are uniformly bounded along all the iterations. These follow from (3.11) and (3.12) that
\[ |\lambda_n - \lambda_0|, ||\Omega h_n - \Omega h_0||_{\rho_n} \leq \sum_{j=1}^{n} \max \{ |\Delta \lambda_{j-1}|, ||\Omega \Delta h_{j-1}||_{\rho_j} \} \]
\[ \leq \sum_{j=1}^{\infty} C \frac{\tilde{\varepsilon}_{j-1}}{\sigma^j} \leq \sum_{j=1}^{\infty} \tilde{\varepsilon}_j, \]
provided that \(\text{DH}_0\) is invertible and satisfies the non-degeneracy condition (3.10) and \(||e||_{\rho}\) is sufficiently small.

Since \(\rho_n\) decreases to \(\rho/2\), for the convergence of \(h_n\) and \(\lambda_n\), it is sufficient to apply the same estimates in (3.16) to show that \(\{\Omega h_n\}_{n=0}^{\infty}\) is Cauchy on the uniform analyticity domain \(\{ x \in T^d_C = C^d/\mathbb{Z}^d : |\text{Im} x| \leq \rho/2 \}\), which is an immediate result of the convergence of \(\sum_{j=1}^{\infty} \tilde{\varepsilon}_j\).

This completes the proof of the convergence of the Newton iteration. \(\square\)

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Appendix A. Phase locking phenomena for general torus maps

In the Appendix, we summarize results for families of maps close to rotations without the assumption of preservation of a foliation. We hope that this can help understand the effect of the foliation preserving assumption and see the effect in the applications where it appears.

There exist a lot of results on the dynamical properties of torus maps both in mathematical and physical literature. The systems with more than two frequencies have much more complicated dynamics than the single frequency case. For instance, diffeomorphism on the torus can have multiple attractors as well as chaotic trajectories.
In a generic family of torus maps close to a family of rotations, the classical KAM theory asserts that the set of the parameter value for which the diffeomorphism can not be smoothly conjugated to irrational rotations occupies small Lebesgue measure in the parameter space.

There are also resonant regions in parameter space in which periodic orbits with given rotation vector exist. Roughly speaking, resonance is a generalization of phase locking encountered in circle maps with rational rotation number when the circle map has an attracting or repulsive periodic point \cite{Arn88}. The resonant regions and phase locking phenomenon of dynamical systems on the torus have been studied in \cite{KMG89, BGKM91, Gal89, Gal94, PdlLV03}.

In this Appendix, we consider the phase locking phenomenon for a family of torus maps close to a family of rotations on the torus. More precisely, we study the following families of torus map

\[ F_{\Omega, \varepsilon}(x) = x + \Omega + \varepsilon f_{\varepsilon}(x), \]

where \( x \in T^d \) and \((\Omega, \varepsilon)\) is regarded as the parameter. We will study both the analytic case and finitely differentiable case for the torus map.

In contrast with \cite{Gal89, Gal94}, we make further assumptions on the resonant frequency instead of restricting on the Mathieu type perturbations, which enables us to obtain the convergence of the formal Fourier series. Using averaging method and perturbation theory, we obtain a resonant normal form for the torus maps and resonant regions in the parameter space.

A.1. **Resonant normal form.** Consider the following torus map

\[(A.1) \quad F_{\Omega, \varepsilon}(x) = x + \Omega + \varepsilon f_{\varepsilon}(x), \]

where \( x \in T^d \) and \( f_{\varepsilon} \) is an analytic function defined on the torus for every \( \varepsilon \). We also assume some regularity of \( f \) with respect to \( \varepsilon \) such that the power series expansion in \( \varepsilon \) is valid.

We regard \( \mathbb{R}^d \times \mathbb{R}^+ \) of points \((\Omega, \varepsilon)\) as the parameter space. Typically, when \( \varepsilon = 0 \) and \( \Omega \) is resonant, the torus decomposes into a family of lower dimension torus. Each orbit is dense on such a lower dimension torus, but not in \( T^d \). We are interested in the dynamical behavior of \( F_{\Omega, \varepsilon} \) for those parameter \( \Omega \) close to a resonant frequency and small \( \varepsilon \).

In this section, we develop a resonant normal form for \( F_{\Omega, \varepsilon} \) when \( \Omega \) is close to the resonant frequency \( \Omega_0 \) whose resonance module is \((d - r)\)-dimension. From Subsection \ref{Subsec1.2} we know that there exist a matrix \( \mathfrak{H} \in \text{SL}(d, \mathbb{Z}) \), a non-resonant intrinsic frequency \( \omega \in \mathbb{R}^r \) and an integer vector \( L \in \mathbb{Z}^d \) such that

\[(A.2) \quad \mathfrak{H}\Omega_0 = \begin{pmatrix} \omega \\ 0 \end{pmatrix} + L. \]
Repeating the calculations in Subsection 2.1.1, we are able to obtain the resonant normal form of \( F_{\Omega,e} \). The main difference with the foliation preserving torus map is that the coordinate transformation \( H_\varepsilon \) takes the form of \( H_\varepsilon = Id + \varepsilon h_\varepsilon \).

**Theorem 1.1.** Given \( p, q \in \mathbb{N} \) with \( p > q \geq (n + m - 1)(d + \tau) \). Assume \( \Omega_0 \) is resonant and the intrinsic frequency \( \omega \) satisfies Diophantine condition (1.5). If \( F_{\Omega,e}(x) \) is real analytic (or \( \mathcal{C}^p \)) in \( x \) and \( \mathcal{C}^{n+m} \) in \( \varepsilon \), then for sufficiently small \( \varepsilon \), the torus map \( F_{\Omega,e} \) is conjugated by a real analytic (or \( \mathcal{C}^{p-q} \), respectively) invertible function

\[
H_\varepsilon^{n+m-1} = Id + \varepsilon h^0 + \varepsilon^2 h^1 + \cdots + \varepsilon^{n+m-1} h^{n+m-2}
\]

and a linear transformation \( \Phi \in SL(d, \mathbb{Z}) \) to the following resonant normal form \( F_{\Omega,e} : T^r \times T^{d-r} \to T^r \times T^{d-r} \) defined by

\[
(F.3) \quad \tilde{F}_{\Omega,e}(x_1, x_2) = \left( x_1 + \omega + \varepsilon^n \beta(x_2, \varepsilon) + \varepsilon^{n+m} r_1(x_1, x_2; \varepsilon) + \Delta_1(\Omega),
\right. \nonumber \quad
\left. x_2 + \varepsilon^\eta \eta(x_2, \varepsilon) + \varepsilon^{n+m} r_2(x_1, x_2; \varepsilon) + \Delta_2(\Omega) \right),
\]

where \( n, m, h^j \) and \( \beta, \eta, r_1, r_2, \Delta_1(\Omega), \Delta_2(\Omega) \) can be computed explicitly from the averaging procedure.

Furthermore, \( r_1 \) and \( r_2 \) are real analytic (or \( \mathcal{C}^{p-q} \), respectively) periodic functions defined on \( T^d \), \( \beta \) and \( \eta \) are real analytic periodic (or \( \mathcal{C}^{p-q} \), respectively) functions on \( T^{d-r} \), and \( \Delta_j(\Omega) = O(|\Omega - \Omega_0|), j = 1, 2 \).

**A.2. Persistence of the invariant surface.** By an abuse of notation, we still denote by \( F_{\Omega,e} \) the resonant normal form developed in Theorem 1.1. In this section, we assume that there exists \( x_2^* \in T^{d-r} \) such that

\[
\eta(x_2^*, 0) = 0 \quad \text{and} \quad D_1\eta(x_2^*, 0) \quad \text{is non-singular.}
\]

By the implicit function theorem, we obtain \( x_1^*(\varepsilon) \) such that \( \eta(x_1^*(\varepsilon), \varepsilon) \equiv 0 \) and \( x_2^*(0) = x_2^* \). Then it follows that \( \Gamma = \{(x_1, x_2^*(\varepsilon)) \mid x_1 \in T^r\} \) is an invariant surface for the torus map \( (x_1, x_2) \mapsto (x_1 + \omega + \varepsilon^n \beta(x_2, \varepsilon), x_2 + \varepsilon^\eta \eta(x_2, \varepsilon)) \) on which the motion is a rotation. Without loss of generality, we assume \( x_2^*(\varepsilon) = 0 \).

In what follows, we show the existence of invariant surface for the resonant normal form developed in Theorem 1.1 which is close to \( \Gamma \).

**Theorem 1.2.** Let \( F_{\Omega,e} \) be a family of real analytic (\( \mathcal{C}^\infty \) or finitely differentiable) torus maps given by (A.3). Assume \( \Omega_0 \) is resonant with intrinsic frequency \( \omega \) satisfying Diophantine condition (1.5). Then if the resonant normal form (A.3) satisfies non-degeneracy condition (H1), there exists a \( r \)-dimensional, finitely differentiable invariant torus of \( F_{\Omega,e} \) for those parameters \( (\Omega, \varepsilon) \) in some neighborhood of \( (\Omega_0, 0) \).

In the finitely differentiable case (but with enough regularity), the regularities of the low dimensional invariant tori are less than those of \( F_{\Omega,e} \).
In the perturbative setting, we see that the invariant manifold of $F_{\Omega, \varepsilon}$ can be represented by a graph of $w : T^r \to \mathbb{R}^{d-r}$, whose invariance determines a functional $\mathcal{F}$ defined on some function space. Then by the contraction mapping arguments, we prove the existence of a fixed point for $\mathcal{F}$, which corresponds to the desired invariant manifold of $F_{\Omega, \varepsilon}$. We delay the proof of Theorem 1.2 to the end of this section.

A.2.1. Formulation of the functional $\mathcal{F}$. For the resonant normal form for $F_{\Omega, \varepsilon}$, we further assume the hyperbolicity that

\begin{equation}
\text{Spec}(I + \varepsilon^n D_1 \eta(0, \varepsilon)) \cap \{z \in \mathbb{Z} : |z| = 1\} = \emptyset
\end{equation}

holds for small $\varepsilon > 0$.

By the hyperbolicity assumption (A.4), we decompose the space $\mathbb{R}^{d-r}$ into $\mathbb{R}^{d-r} = E_\varepsilon^s \oplus E_\varepsilon^u$ such that

$$I + \varepsilon^n D_1 \eta(0, \varepsilon) = \begin{pmatrix} \Lambda_\varepsilon^s & 0 \\ 0 & \Lambda_\varepsilon^u \end{pmatrix},$$

where the spectral radius of $\Lambda_\varepsilon^s$ is less than one and $\Lambda_\varepsilon^u$ is invertible with spectral radius greater than one.

Assume that there exist positive constants $C_s, C_u$ and adapted norms for $E_\varepsilon^s$ and $E_\varepsilon^u$ such that

\begin{equation}
\text{H2} \quad \|\Lambda_\varepsilon^s\| \leq 1 - C_s\varepsilon^n \quad \text{and} \quad \|\Lambda_\varepsilon^u\|^{-1} \leq 1 - C_u\varepsilon^n.
\end{equation}

Remark 1.1. We know that, for the $\Omega$-foliation preserving torus maps (see (2.1) and Theorem 2.1), $\eta(x_2, \varepsilon)$ is a scalar function. As a result, assumption (H1) implies (H2) for small $\varepsilon$. More precisely, we can always choose constants $C_s$ and $C_u$ such that

$$1 + \varepsilon^n D_1 \eta(0, \varepsilon) \leq 1 - C_s\varepsilon^n \quad \text{if} \quad D_1 \eta(0, 0) < 0,$$

and

$$(1 + \varepsilon^n D_1 \eta(0, \varepsilon))^{-1} \leq 1 - C_u\varepsilon^n \quad \text{if} \quad D_1 \eta(0, 0) > 0,$$

for sufficiently small $\varepsilon$. In this case, it suffices to consider two different situations rather than to decompose the space.

In the perturbative setting, we assume that the invariant manifold of $F_{\Omega, \varepsilon}$, which is close to $\Gamma$, can be represented by a graph of $w : T^r \to \mathbb{R}^{d-r}$. Then by the invariance of the graph, we are led to solving the following functional equation

\begin{equation}
[I + \varepsilon^n D_1 \eta(0, \varepsilon)]w(x_1) + \varepsilon^{n+m}r_2(x_1, w(x_1)) + \varepsilon^n R(w(x_1), \varepsilon) + \tilde{\Delta}_2(\Omega) = w\left(x_1 + \omega + \varepsilon^n\beta(w(x_1), \varepsilon) + \varepsilon^{n+m}r_1(x_1, w(x_1)) + \tilde{\Delta}_1(\Omega)\right)
\end{equation}
for every $x_1 \in \mathbb{T}^r$, where
\[
R(x_2, \varepsilon) = \eta(x_2, \varepsilon) - D_1 \eta(0, \varepsilon) x_2 = \int_0^1 \int_0^1 sD_1 \eta(stx_2, \varepsilon) dsdr \cdot x_2^{\otimes 2}.
\]

Projecting (A.5) onto $E^s_\varepsilon$ and $E^\mu_\varepsilon$ respectively, we obtain that (A.5) is equivalent to
\[
(A.6) \quad \Lambda^s_\varepsilon w^s + \varepsilon^n r^s_2 + \varepsilon^\mu R^s + \tilde{\Delta}^s_2(\Omega) = w^s \circ \mathcal{P}
\]
and
\[
(A.7) \quad \Lambda^\mu_\varepsilon w^\mu + \varepsilon^n r^\mu_2 + \varepsilon^\mu R^\mu + \tilde{\Delta}^\mu_2(\Omega) = w^\mu \circ \mathcal{P},
\]
where
\[
w = (w^s, w^\mu), \quad r_2 = (r_2^s, r_2^\mu), \quad R = (R^s, R^\mu), \quad \tilde{\Delta}_2(\Omega) = (\tilde{\Delta}^s_2(\Omega), \tilde{\Delta}^\mu_2(\Omega)),
\]
and
\[
(r_2^s) = r_2^s(x_1, w^s(x_1), w^\mu(x_1)), \quad r_2^\mu = r_2^\mu(x_1, w^s(x_1), w^\mu(x_1)),
\]
\[
R^s = R^s(w^s(x_1), w^\mu(x_1), \varepsilon), \quad R^\mu = R^\mu(w^s(x_1), w^\mu(x_1), \varepsilon),
\]
\[
\mathcal{P}(x_1) = x_1 + \omega + \varepsilon^\mu \beta(w(x_1), \varepsilon) + \varepsilon^n r_1(x_1, w(x_1)) + \tilde{\Delta}_1(\Omega).
\]

Now we are looking for a functional $\mathcal{F}$ such that its fixed point is a solution to (A.6) and (A.7). Before this, for (A.6), we are led to solving $x_1$ from
\[
(A.8) \quad y_1 = \mathcal{P}(x_1) = x_1 + \omega + \varepsilon^\mu \beta(w(x_1), \varepsilon) + \varepsilon^n r_1(x_1, w(x_1)) + \tilde{\Delta}_1(\Omega).
\]

Similar arguments also appear in [Lan73].

Rewrite (A.8) as
\[
x_1 = y_1 - \omega - \varepsilon^\mu \beta(w^s(x_1), w^\mu(x_1), \varepsilon) - \varepsilon^n r_1(x_1, w^s(x_1), w^\mu(x_1)) - \tilde{\Delta}_1(\Omega)
\]
\[
\equiv \mathcal{G}(x_1, y_1, w^s, w^\mu),
\]
and consider the fixed point problem of operator $\mathcal{G}$ with parameter $(y_1, w)$ belonging to $\mathbb{T}^r \times \mathcal{U}$, where
\[
\mathcal{U} = \{ w^s \in C^1(\mathbb{T}^r, E^s) : ||w^s||_1 \leq 1 \} \times \{ w^\mu \in C^1(\mathbb{T}^r, E^\mu) : ||w^\mu||_1 \leq 1 \}.
\]

Noticing that
\[
|\mathcal{G}(x_1^{(1)}, y_1, w^s, w^\mu) - \mathcal{G}(x_1^{(2)}, y_1, w^s, w^\mu)|
\]
\[
\leq \varepsilon^n \left( ||D_\beta||_0 + 2 \varepsilon^\mu ||Dr_1||_0 + C \frac{||\Delta \Omega||}{\varepsilon^\mu} \right) \cdot |x^{(1)}_1 - x^{(2)}_1|,
\]
we have for
\[
(A.9) \quad ||\Delta \Omega|| \leq C\varepsilon^{n+1}
\]
and sufficiently small $\varepsilon$, the operator is a uniform contraction mapping on $\mathbb{T}^r \times \mathcal{U}$. 
By the uniform contraction mapping theorem (see [Chi06]), we obtain the solution of (A.8) denoting by
\[ x_1 = v(y_1, w^s, w^u). \]
Furthermore, since \( \mathcal{G} : T^r \times U \rightarrow T^r \) is \( \mathcal{C}^1 \), we have the fixed point function \( v : T^r \times U \rightarrow T^r \) is also \( \mathcal{C}^1 \).

Then we rewrite (A.6) as
\[
\begin{align*}
    w^s(y_1) &= \Lambda^s \varepsilon w^s(v(y_1, w^s, w^u)) + \varepsilon^{n+m} R^s(v, w^s(v), w^u(v)) \\
    &\quad + \varepsilon^n R(v, w^s(v), w^u(v), \varepsilon) + \tilde{\Delta}_s^2(\Omega) \\
    &\equiv \mathcal{F}^s[w^s, w^u](y_1).
\end{align*}
\]

For (A.7), it would be much easier to find a functional \( \mathcal{F}^u \) whose fixed point together with \( \mathcal{F}^s \) would be a solution of (A.6) and (A.7). Actually, we have
\[
\begin{align*}
    w^u(x_1) &= (\Lambda^u \varepsilon)^{-1} w^u(\mathcal{P}(x_1)) - \varepsilon^{n+m} (\Lambda^u \varepsilon)^{-1} r^u(x_1, w^s(x_1), w^u(x_1)) \\
    &\quad - \varepsilon^n (\Lambda^u)^{-1} R^u(w^s(x_1), w^u(x_1), \varepsilon) - (\Lambda^u)^{-1} \tilde{\Delta}_s^2(\Omega) \\
    &\equiv \mathcal{F}^u[w^s, w^u](x_1).
\end{align*}
\]

All together, we define
\[
(A.10) \quad \mathcal{F}[w^s, w^u] = (\mathcal{F}^s[w^s, w^u], \mathcal{F}^u[w^s, w^u]).
\]

from \( \mathcal{C}^L(T^r, E^s \varepsilon) \times \mathcal{C}^L(T^r, E^u \varepsilon) \) to itself.

It is readily seen that, given an integer \( L > 0 \) and a real number \( 0 < l \leq n \), there exists a \( \varepsilon_0 > 0 \) such that for all \( 0 < \varepsilon < \varepsilon_0 \), we have
\[
(A.11) \quad \|\Lambda^s \varepsilon\|_0 (1 - \varepsilon^{n+j} \|D\beta\|_0)^{-j} < 1
\]

and
\[
(A.12) \quad \|(\Lambda^u \varepsilon)^{-1}\|(1 + \varepsilon^{n+j} \|D\beta\|_0)^{j} < 1
\]

for any \( 0 \leq j \leq L \).

**Remark 1.2.** The adapted norms of \( \Lambda^s \) and \( (\Lambda^u)^{-1} \) depend also on the perturbation. We will see more clearly about the smallness of \( \varepsilon \) in the one dimensional resonance case. For instance, in one dimensional resonance case and \( D_1 \eta(0, 0) > 0 \), condition (A.12) reduces to
\[
\frac{1}{1 + \varepsilon^{n+j} D_1 \eta(0, 0)} (1 + \varepsilon^{n+j} \|D\beta\|_0)^L < 1.
\]

Then, for sufficiently large \( L \), we see that \( \varepsilon \leq C \sqrt{\frac{1}{L}} \).
A.2.2. Invariant closed subset of the functional. Now we are trying to find a closed subset $\mathcal{L} \subset C^L(\mathbb{T}^r, E^s) \times C^L(\mathbb{T}^r, E^u)$ such that $\mathcal{F}(\mathcal{L}) \subseteq \mathcal{L}$. In this paper, we would like to find the subset $\mathcal{L}$ with different bounds for different orders of derivatives. More precisely, we assume

$$\mathcal{L} = \left\{(w^s, w^u) \in C^L(\mathbb{T}^r, E^s) \times C^L(\mathbb{T}^r, E^u) : \right\}$$

$$\left\|D^j w^u\right\|_0 \leq \delta_j, \left\|D^j w^u\right\|_0 \leq \delta_j \text{ for all } 0 \leq j \leq L \right\}. \quad (A.13)$$

In what follows, we show how to find $\delta_j$'s such that $\mathcal{F}(\mathcal{L}) \subseteq \mathcal{L}$. For $\left\|w^s\right\|_0 \leq \delta_0 < 1$ and $\left\|w^u\right\|_0 \leq \delta_0 < 1$, we have

$$\left\|\mathcal{F}^s[w^s, w^u]\right\|_0 \leq \left\|A^s\delta_0 + \varepsilon^{n+m}\right\|_0 + \varepsilon^n\|\eta(\cdot, 0)\|_2 \delta_0^2 + \varepsilon^{n+1}\|D_2\eta(\cdot, 0)\|_2 + C\|\Delta \Omega\|$$

$$\leq (1 - C_\delta \varepsilon^n)\delta_0 + \varepsilon^{n+1}\|D_2\eta(\cdot, 0)\|_2 + \varepsilon^{n+m}\|\eta(\cdot, 0)\|_2 \delta_0^2 + C\|\Delta \Omega\|.$$

To ensure $\left\|\mathcal{F}^s[w^s, w^u]\right\|_0 \leq \delta_0$, it suffices

$$\left\|\eta\right\|_2 \delta_0^2 - C_\delta \delta_0 + \varepsilon \left(\left\|\eta\right\|_1 + \varepsilon^m\|\eta\|_2 + C\|\Delta \Omega\|\varepsilon^{-n}\right) \leq 0,$$

which implies

$$O(\varepsilon) = C_\delta \left\{\frac{1 - \sqrt{1 - 4\varepsilon\|\eta\|_2 q}}{2\|\eta\|_2} \right\} \leq \delta_0 \leq C_\delta \left\{\frac{1 + \sqrt{1 - 4\varepsilon\|\eta\|_2 q}}{2\|\eta\|_2} \right\} = O(1) \quad (A.14)$$

with $q = \left\|\eta\right\|_1 + \varepsilon^m\|\eta\|_2 + C\|\Delta \Omega\|\varepsilon^{-n}$. Then we see that, when $0 < \alpha_0 < 1$, $\delta_0 = \varepsilon^{\alpha_0}$ satisfies (A.14) for sufficiently small $\varepsilon$.

For $\mathcal{F}^u[w^s, w^u]$, we also have

$$\left\|\mathcal{F}^u[w^s, w^u]\right\|_0 \leq \left\|\left(A^u\right)^{-1}\right\|_0 \left\|\delta_0 + \varepsilon^{n+m}\right\|_0 + \varepsilon^n\|\eta\|_2 \delta_0^2 + \varepsilon^{n+1}\|\eta\|_1 + C\|\Delta \Omega\|$$

$$\leq (1 - C_\delta \varepsilon^n)\delta_0 + \varepsilon^{n+1}\|\eta\|_1 + \varepsilon^{n+m}\|\eta\|_2 \delta_0^2 + C\|\Delta \Omega\|.$$

Similarly, it suffices to choose $\delta_0 = \varepsilon^{\alpha_0}$ with $0 < \alpha_0 < 1$ such that

$$\left\|\mathcal{F}^u[w^s, w^u]\right\|_0 \leq \delta_0.$$

In what follows, we also need to estimate the derivatives of $\mathcal{F}[w^s, w^u]$ up to the order of $L$. We will show that the bound for the first derivative of $\mathcal{F}$ plays an important role in the analysis of other higher order derivatives.
The derivative of $\tilde{F}^u[\omega]$ is
\[
D(\tilde{F}^u[\omega^s, \omega^u])(x_1) = (\Lambda^{\mu}_0)\cdot \left\{ I + \epsilon^{\mu}D_1\beta(\cdot) \right\}
\]
\[
+ \epsilon^{\mu+m}D_1r_1(\cdot) + \epsilon^{\mu+m}D_2r_2(\cdot) \left[ Dw^s(x_1), Dw^u(x_1) \right] + D_{x_1}[\Delta(\Omega)]
\]
\[
- \epsilon^{\mu+m}(\Lambda^{\mu}_0)^{-1}D_1r_2(\cdot) - \epsilon^{\mu+m}(\Lambda^{\mu}_0)^{-1}D_2r_2(\cdot) \left[ Dw^s(x_1), Dw^u(x_1) \right]
\]
\[
- \epsilon^{\mu}(\Lambda^{\mu}_0)^{-1}D_1\epsilon^s(\cdot) \left[ Dw^s(x_1), Dw^u(x_1) \right] - (\Lambda^{\mu}_0)^{-1}D_{x_1}[\Delta(\Omega)],
\]
where $(\cdot)$ denotes the argument omitted in a function.

Similarly, the derivative of $\tilde{F}^s[\omega^s, \omega^u]$ reads
\[
D(\tilde{F}^s[\omega^s, \omega^u])(y_1) = \Lambda^{\epsilon}_s Dw^s(v)D_1\epsilon^s(\cdot) + \epsilon^{\mu+m}D_1r_2(\cdot)
\]
\[
+ \epsilon^{\mu}D_1r_1(\cdot) \left[ Dw^s(v), Dw^u(v) \right] \cdot D_1\epsilon^s(\cdot) + D_{y_1}[\Delta(\Omega)].
\]

However, the derivative of $\tilde{F}^s$ is more involved. We also need to give the derivative of $v(y_1, \omega^s, \omega^u)$ with respect to $y_1$. From $l, m, n$ we obtain
\[
D_1v(y_1, \omega^s, \omega^u) = \left\{ I + \epsilon^{\mu}D_1\beta(\cdot) \right\} \left[ Dw^s(v), Dw^u(v) \right] + \epsilon^{\mu+m}D_1r_1(\cdot)
\]
\[
+ \epsilon^{\mu+m}D_2r_2(\cdot) \left[ Dw^s(v), Dw^u(v) \right] + D_{x_1}[\Delta(\Omega)]
\]
\[
\cdot D_1\epsilon^s(\cdot) + D_{y_1}[\Delta(\Omega)].
\]

For $\|Dw^s\|_0 \leq \delta_1$ and $\|Dw^u\|_0 \leq \delta_1$, we have
\[
\|D, \tilde{F}^u[\omega^s, \omega^u]\|_0 \leq (1 - C_\Omega \epsilon^s) \left\{ \delta_1 \left[ 1 + \epsilon^{\mu}\|D\beta\|_0 \delta_1 + \epsilon^{\mu+m}\|r_1\|_1 \right] \right\}
\]
\[
+ \epsilon^{\mu+m}\|r_2\|_1 \left[ 1 + \delta_1 \right] + \epsilon^{\mu + \alpha_0}\|\eta\|_1 \delta_1 + \epsilon^{\mu}\|\eta\|_2 \delta_0 \delta_1 + C|\Delta\Omega|
\]
\[
\cdot \|Dv(\cdot, \omega^s, \omega^u)\|_0 \leq (1 - C_\Omega \epsilon^s) \left\{ \delta_1 \left[ 1 + \epsilon^{\mu}\|D\beta\|_0 \delta_1 + \epsilon^{\mu+m}\|r_1\|_1 \right] \right\}
\]
\[
+ \epsilon^{\mu+m}\|r_2\|_1 \left[ 1 + \delta_1 \right] + \epsilon^{\mu + \alpha_0}\|\eta\|_1 \delta_1 + \epsilon^{\mu}\|\eta\|_2 \delta_0 \delta_1 + C|\Delta\Omega| e^{-(n+1)}
\]
\[
+ 2\epsilon^{\mu+m}\left( \|r_1\|_1 + \|r_2\|_1 \right).
\]

Then it suffices to choose $\delta_1 = \epsilon^{\alpha_1}$ with $l < \alpha_1 < n + \alpha_0$ such that $\|D, \tilde{F}^u[\omega^s, \omega^u]\|_0 \leq \delta_1$.

To estimate the derivative of $\tilde{F}^s[\omega^s, \omega^u]$, we see that
\[
\|D_1v(\cdot, \omega^s, \omega^u)\|_0 \leq \left( 1 - \epsilon^{\mu}\|D\beta\|_0 \delta_1 \right)^{-1} + 2\|r_1\|_1 \epsilon^{\mu+m} + C|\Delta\Omega|
\]
for sufficiently small $\varepsilon$.

Consequently, we have

$$\|D^j \mathcal{F}[w^s, w^u]\|_0 \leq (1 - C_x \varepsilon^n)(1 - \varepsilon^n\|D\beta\|_0)^{-1}\delta_1 + 4\varepsilon^{n+i}\|\eta(\cdot, 0)\|_2$$

$$+ 4\varepsilon^{n+1}\left(\|D_2\eta(\cdot, 0)\|_1 + C|\Delta\Omega|\varepsilon^{-(n+1)}\right) + 4\varepsilon^{n+m}\|\rho_1\|_1 + \|\rho_2\|_1,$$

which implies that the choice of $\delta_1 = \varepsilon^{n_1}$ with $l < \alpha_1 < n + \alpha_0$ is sufficient to keep $\|D^j \mathcal{F}[w^s, w^u]\|_0 \leq \delta_1$.

For higher derivatives, the Faa-di-Bruno’s formula (see [AR67, p.3]) gives

$$D^j \mathcal{F}[w^s, w^u](x_1)$$

$$= (\Lambda^n)^{-1} D^i w^u(\mathcal{P}(x_1)) \cdot [D\mathcal{P}]^l$$

$$+ (\Lambda^n)^{-1} \sum_{1 \leq q < i} \sum_{k_1 + \cdots + k_q = i} \sigma_i D^{i_q} w^u \circ \mathcal{P} \left[ D^{k_1} \mathcal{P}, \ldots, D^{k_q} \mathcal{P} \right]$$

(A.15)

$$- \varepsilon^{n+m}(\Lambda^n)^{-1} D^i(r_2(x_1, w(x_1))) - \varepsilon^n(\Lambda^n)^{-1}$$

$$\times \sum_{1 \leq q < i} \sum_{k_1 + \cdots + k_q = i} \sigma_i D^{i_q} \eta^u \circ w \left[ (D^{k_1} w^s, D^{k_2} w^u), \ldots, (D^{k_1} w^s, D^{k_2} w^u) \right],$$

where

$$D\mathcal{P} = 1 + \varepsilon^n D_1 \beta(\cdot) \cdot [Dw^s(x_1), Dw^u(x_1)] + \varepsilon^{n+m} D_1 r_1(\cdot)$$

$$+ \varepsilon^{n+m} D_2 r_1(\cdot)[Dw^s(x_1), Dw^u(x_1)] + D_{x_1}[\Lambda_1(\Omega)],$$

$k_1, \cdots, k_q$ are positive integers and $\sigma_{i+1} = \sigma_i(k_1, \cdots, k_q)$ is of integer value and can be calculated explicitly.

Noticing that when $1 \leq q < i$, there always exists a $k_j$ with $1 \leq j \leq q$ such that $k_j \geq 2$. Thus $D^i/\mathcal{P}(x_1)$ will be of order $\varepsilon^n$.

By induction, we assume that for $2 \leq j \leq i - 1$, $\delta_j = \varepsilon^{n_j}$ with $0 < \alpha_j < n$. Then, we see from (A.15) that

(A.16)

$$\|D^j \mathcal{F}[w^s, w^u]\|_0 \leq (1 - C_x \varepsilon^n)(1 - \varepsilon^{n+i}\|D\beta\|_0)^L\delta_i$$

$$+ \varepsilon^n R^u_{i-1}(\delta_0, \cdots, \delta_{i-1}; \varepsilon, \eta, r_1, r_2, L) + C|\Delta\Omega|,$$

where $R^u_i$ is a polynomial of $\delta_0, \cdots, \delta_{i-1}$ whose coefficients depend on $\eta, r_1, r_2, L$ and $\varepsilon$. By assumption (A.12), we have $\delta_i = \varepsilon^{n_i}$ with $0 < \alpha_i < n$ is sufficient to ensure $\|D^i \mathcal{F}[w^s, w^u]\|_0 \leq \delta_i$ when $\varepsilon$ is small. This completes the induction argument.

For the high derivatives of $\mathcal{F}[w^s, w^u]$, one also have the similar estimate to (A.16) as follows

$$\|D^j \mathcal{F}[w^s, w^u]\|_0 \leq (1 - C_x \varepsilon^n)(1 - \varepsilon^{n+i}\|D\beta\|_0)^L\delta_i$$

$$+ \varepsilon^n R^u_i(\delta_0, \cdots, \delta_{i-1}; \varepsilon, \eta, r_1, r_2, L) + C|\Delta\Omega|,$$

which implies that the $\alpha_j$’s obtained keeps $\|D^i \mathcal{F}[w^s, w^u]\|_0 \leq \delta_j$. 

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We conclude the above arguments in the following lemma.

**Lemma 1.1.** Let \( L \in \mathbb{N} \) be fixed and real number \( l \leq n \). Let \( F_{0,\varepsilon} \in \mathcal{C}^p(\mathbb{T}^l, \mathbb{T}^d) \) be the torus map in the resonant normal form (2.16) with \( L + 2 < p \in \mathbb{N} \). Assume \((H1)\) and \((A.4)\) hold. Then for any given \( \alpha_j \) satisfying \( 0 < \alpha_0 < 1, l < \alpha_1 < n + \alpha_0 \) and \( 0 < \alpha_1 \leq n \) for \( 2 \leq i \leq L \), there exists a \( \varepsilon_0 = \varepsilon_0(L, l, \alpha_0, \cdots, \alpha_L; \eta, r_1, r_2) \) such that for any \( 0 < \varepsilon < \varepsilon_0 \) and \( |\Omega - \Omega_0| \leq \varepsilon^{\delta + 1} \), we have \( \mathcal{F}(\mathcal{L}) \subseteq \mathcal{L} \), where \( \mathcal{L} \) is given by \((A.13)\) with \( \delta_j = \varepsilon^{\alpha_j} \).

**Remark 1.3.** Since \( L \) is given and fixed, one should keep in mind that there are only finitely many conditions on the smallness of \( \varepsilon \) which appear in determining \( \delta_j \), for \( 0 \leq j \leq L \).

### A.2.3. Contractility of the functional.

In this subsection, we show that \( \mathcal{F} \) is a contraction on \( \mathcal{L} \) in the \( \mathcal{C}^0 \)-norm.

For \( (w^i_1, w^i_2) \) and \( (w^i_2, w^i_2) \) in \( \mathcal{L} \), we have
\[
\| \mathcal{F}^u[w^i_2, w^i_2] - \mathcal{F}^u[w^i_1, w^i_1] \|_0 \leq \|(\Lambda)^{-1}\| \left\{ \|w^i_2 - w^i_1\|_0 + \|Dw_1\|_0 \cdot [\varepsilon^l \|D\|_0 \cdot \|(w^i_2 - w^i_1, w^i_2 - w^i_1)\|_0 \\
+ \varepsilon^{n+m} \|D_r\|_0 \cdot \|(w^i_2 - w^i_1, w^i_2 - w^i_1)\|_0 \right\} \times \|(w^i_2 - w^i_1, w^i_2 - w^i_1)\|_0 + \varepsilon^l\|\eta\| \cdot \|w^i_1\|_0 \cdot \|w^i_2\|_0 + \|\eta\|_1 \cdot \|\eta\|_2 \cdot \|w^i_1\|_0 \cdot \|w^i_2\|_0 \\
+ \varepsilon^{\delta + 1} \left( \|\eta\|_1 + C|\Delta\| \varepsilon^{-(\delta + 1)} \right) \cdot \|w^i_2 - w^i_1, w^i_2 - w^i_1\|_0 \right\} \leq \left\{ (1 - C_\varepsilon \varepsilon^l) (1 + \varepsilon^{n+\alpha_1} \|D\|_0) + \varepsilon^{n+m+\alpha_1} \|D_r\|_0 \right. \\
+ 2\varepsilon^{n+m} \|\eta\|_2 + \varepsilon^{n+2\alpha_0} + \varepsilon^{\delta + 1} \left( \|\eta\|_1 + C\|\Delta\| \varepsilon^{\delta + 1} \right) \right\} \cdot \|w^i_2 - w^i_1, w^i_2 - w^i_1\|_0,
\]

which enables us to choose sufficiently small \( \varepsilon \) such that \( \mathcal{F}^u \) is a contraction in the \( \mathcal{C}^0 \)-norm.

To show a similar estimate for \( \mathcal{F}^x \), we give the Lipschitz property of \( x_1 = v(x_1, w^i, w^i) \) with respect to \( (w^i, w^i) \) on \( \mathcal{U} \). Obviously, from \((A.8)\), we see that
\[
-(v(x_1, w_2) - v(x_1, w_1)) = \varepsilon^l \left[ \beta(w_2(v(x_1, w_2))) - \beta(v(x_1, w_1)) \right] \\
+ \varepsilon^l \left[ \beta(w_1(v(x_1, w_2))) - \beta(v(x_1, w_1)) \right] \\
+ \varepsilon^{n+m} \left[ r_1(v(x_1, w_2), w_2(v(x_1, w_2))) - r_1(v(x_1, w_1), w_2(v(x_1, w_2))) \right],
\]
Lemma 2.4] the following lemma on the existence and regularity of the fixed point of

Under the assumptions of Lemma 1.1, there exists a fixed point \( w \) of \( \text{Lemma 1.2} \).

Proof of Theorem 1.2. For \( w^1, w^2 \in \mathcal{L} \), one readily sees

\[
|v(y_1, w_2) - v(y_1, w_1)| \leq \frac{\varepsilon^0||D\beta||_0 + \varepsilon^{n+m}||Dr_1||_0 + C|\Delta\Omega|}{1 - \varepsilon^0||D\beta||_0 \delta_1 - \varepsilon^{n+m}||Dr_1||_0 + C|\Delta\Omega|} ||w_2 - w_1||_0,
\]

\[(\text{i.e. Lip}_2 v \leq \frac{\varepsilon^0||D\beta||_0 + \varepsilon^{n+m}||Dr_1||_0 + C|\Delta\Omega|}{1 - \varepsilon^0||D\beta||_0 \delta_1 - \varepsilon^{n+m}||Dr_1||_0 + C|\Delta\Omega|})\]

and

\[
|w_2(v(y_1, w_2)) - w_1(v(y_1, w_1))| \leq ||w_2 - w_1||_0 + ||Drw_1||_0 \cdot |v(y_1, w_2) - v(y_1, w_1)| \leq (1 + \delta_1 \text{Lip}_2 v) \cdot ||w_2 - w_1||_0.
\]

Therefore, one easily gets

\[
||F^1[w^2, w^2] - F^1[w^1, w^1]||_0 \leq \left\{ (1 - C \varepsilon^0)(1 - \varepsilon^0||D\beta||_0 \delta_1)^{-1} + \varepsilon^{n+m}||Dr_2||_0 \cdot \left[ \text{Lip}_2 v + (1 + \delta_1 \text{Lip}_2 v) \right] \right. \\
+ 2\varepsilon^0\delta_0||\eta||_2 + \varepsilon^{n+1} \left( ||\eta||_1 + C|\Delta\Omega| \frac{\varepsilon^{n+1}}{\varepsilon^{n+1}} \right) \cdot ||(w^2_2 - w^2_1, w^2_1 - w^1_1)||_0,
\]

which is also a contraction on \( \mathcal{L} \) if \( \varepsilon \) is chosen sufficiently small.

Together with Lemma [1.1] and the contractility of \( \mathcal{F} \) on \( \mathcal{L} \), we obtain from [HdlL17] Lemma 2.4] the following lemma on the existence and regularity of the fixed point of \( \mathcal{F} \).

Lemma 1.2. Under the assumptions of Lemma 1.1, there exists a fixed point \( w^* \in C^{L-1}(\mathbb{T}^r, \mathbb{R}^{d-r}) \) of \( \mathcal{F} \), which is unique in the closure of \( \mathcal{L} \) in \( C^0(\mathbb{T}^r, \mathbb{R}^{d-r}) \).

Proof of Theorem 1.2 By the formulation of the functional \( \mathcal{F} \), Theorem 1.2 is an immediate result of Theorem 1.1 and Lemma 1.2. The regularity \( L - 1 \) of the invariant surface (given by the graph of \( w^* \)) is less than that of the resonant normal form. \( \square \)

A.3. Lindstedt series for generic torus maps close to rotation. Consider the following torus map

\[
(A.17) \quad F_\varepsilon \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \omega + \varepsilon g(x, y) \\ y + \varepsilon h(x, y) \end{pmatrix} \in \mathbb{T}^r \times \mathbb{T}^{d-r},
\]

where \( \omega \in \mathbb{R}^r \) satisfies the Diophantine condition.
We look for \( l_\varepsilon = (l^x_\varepsilon, l^y_\varepsilon) : \mathbb{T}^r \to \mathbb{T}^d \) and \( u_\varepsilon : \mathbb{T}^r \to \mathbb{T}^r \) such that
\[
F_\varepsilon \circ l_\varepsilon = l_\varepsilon \circ u_\varepsilon,
\]
by which the graph of \( l_\varepsilon \) is the desired invariant surface. To formulate the Lindstedt series, we expand \( l_\varepsilon \) and \( u_\varepsilon \) into power series in \( \varepsilon \) as
\[
l_\varepsilon(\sigma) = \sum_{j \geq 0} \left( \frac{f^x_j(\sigma)}{l^y_j(\sigma)} \right) \varepsilon^j, \quad u_\varepsilon(\sigma) = \sum_{j \geq 0} u_j(\sigma) \varepsilon^j,
\]
and then take formal calculations.

By matching the coefficients of \( \varepsilon^0 \)-terms, we obtain
\[
F_0 \circ l_0 = l_0 \circ u_0, \quad \text{which can be solved by choosing}
\]
\[
l^x_0(\sigma) = \sigma, \quad l^y_0(\sigma) = y_0 = \text{Constant}, \quad u_0(\sigma) = \sigma + \omega.
\]
For those \( \varepsilon^1 \)-terms, we get
\[
F_1 \circ l_0(\sigma) + DF_0 \circ l_0(\sigma) \cdot l_1(\sigma) = l_1 \circ u_0(\sigma) + Dl_0 \circ u_0(\sigma) \cdot u_1(\sigma).
\]
It follows that
\[
\begin{cases}
l^x_1(\sigma + \omega) - l^x_1(\sigma) = g(\sigma, y_0) - u_1(\sigma), \\
l^y_1(\sigma + \omega) - l^y_1(\sigma) = h(\sigma, y_0).
\end{cases}
\]
Assuming that there exists \( y_0 \in \mathbb{T}^{d-r} \) such that
\[
\langle h(\cdot, y_0) \rangle = \frac{1}{\mathbb{T}^{d-r}} \int_{\mathbb{T}^{d-r}} h(\sigma, y_0) \, d\sigma = 0,
\]
then we can choose
\[
u_1(\sigma) = \langle g(\cdot, y_0) \rangle
\]
and solve (A.19) for \( (l^x_1, l^y_1) \) by Lemma 1.1 in KAM theory, leaving the average
\[
\langle l_1 \rangle = (\langle l^x_1 \rangle, \langle l^y_1 \rangle) = \int_{\mathbb{T}^r} l_1(\sigma) \, d\sigma
\]
to be specified.

To clarify the induction, we proceed to compute the equation for those \( \varepsilon^2 \)-terms, which reads
\[
F_2 \circ l_0 + DF_1 \circ l_0 \cdot l_1 + DF_0 \circ l_0 \cdot l_2 = l_2 \circ u_0 + Dl_1 \circ u_0 \cdot u_1 + Dl_0 \circ u_0 \cdot u_2.
\]
It follows that
\[
l^x_2(\sigma + \omega) - l^x_2(\sigma) = \begin{pmatrix}
Dg(\sigma, y_0) l_1(\sigma) - Dl^x_1(\sigma + \omega) u_1 - u_2(\sigma) \\
Dh(\sigma, y_0) l_1(\sigma) - Dl^y_1(\sigma + \omega) u_1
\end{pmatrix}.
\]
Let \( l_1(\sigma) = \langle l_1 \rangle + \tilde{l}_1(\sigma) \) and recall that \( \tilde{l}_1(\sigma) \) is uniquely determined by \( (A.19) \). Then the R.H.S. of \( (A.21) \) reads

\[
\left( \frac{Dg(\sigma, y_0)}{Dh(\sigma, y_0)} \right) \cdot \tilde{l}_1(\sigma) + \left( \frac{Dg(\sigma, y_0)}{Dh(\sigma, y_0)} \right) \cdot \langle l_1 \rangle - \left( \frac{Dl_1^x(\sigma)u_1 + u_2(\sigma)}{0} \right).
\]

In order the average of R.H.S of \( (A.21) \) to vanish, we need to choose parameters \( \langle l_1 \rangle = (\langle l_1^x \rangle, \langle l_1^y \rangle) \) and \( u_2(\sigma) \). More precisely, we have

\[
\langle D_2h(\cdot, y_0) \rangle \cdot \langle l_1^x \rangle = \langle Dh(\cdot, y_0)l_1 \rangle,
\]

\[
\langle D_2g(\cdot, y_0) \rangle \cdot \langle l_1^y \rangle - \langle u_2 \rangle = \langle Dg(\cdot, y_0)l_1 \rangle,
\]

since \( \langle D_1g(\cdot, y_0) \rangle = \langle D_1h(\cdot, y_0) \rangle = \langle Dl_1^x \rangle = 0 \).

Then if \( (A.22) \),

\[
\det(D_2h(\cdot, y_0)) \neq 0,
\]

we just take

\[
\langle l_1^x \rangle = \langle D_2h(\cdot, y_0) \rangle^{-1} \cdot \langle Dh(\cdot, y_0)l_1 \rangle,
\]

and

\[
u_2(\sigma) \equiv \langle D_2g(\cdot, y_0) \rangle \cdot \langle D_2h(\cdot, y_0) \rangle^{-1} \cdot \langle Dh(\cdot, y_0)l_1 \rangle - \langle Dg(\cdot, y_0)l_1 \rangle.
\]

Now the induction is clear. By similar arguments in Section 2.2, we obtain the following result.

**Proposition 1.4.** Let \( \omega \in \mathbb{R}^r \) satisfy Diophantine condition. Assume there exists \( y_0 \in \mathbb{R}^{d-r} \) such that \( (A.20) \) and \( (A.22) \) hold. Then we can find two formal power series

\[
l_\varepsilon(\sigma) = \sum_{j \geq 0} l_j(\sigma)\varepsilon^j \quad \text{and} \quad u_\varepsilon(\sigma) = \sigma + \omega + \sum_{j \geq 1} u_j\varepsilon^j
\]

such that \( (A.18) \) holds, where \( \{u_j\}_{j \geq 1} \) are a sequence of constant vectors.

We are also able to establish the relationship between the Lindstedt series and the true embedding of the invariant surface as that in Theorem 2.3.

**References**

[AR67] Ralph Abraham and Joel Robbin. *Transversal mappings and flows*. An appendix by Al Kelley. W. A. Benjamin, Inc., New York-Amsterdam, 1967.

[Arn88] V. I. Arnold. *Geometrical methods in the theory of ordinary differential equations*, volume 250 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, New York, second edition, 1988. Translated from the Russian by Joseph Szücs [József M. Szűcs].

[BGKM91] Claude Baesens, John Guckenheimer, Seung-hwan Kim, and Robert MacKay. Simple resonance regions of torus diffeomorphisms. In *Patterns and dynamics in reactive media (Minneapolis, MN, 1989)*, volume 37 of IMA Vol. Math. Appl., pages 1–9. Springer, New York, 1991.
[Chi06] Carmen Chicone. *Ordinary differential equations with applications*, volume 34 of *Texts in Applied Mathematics*. Springer, New York, second edition, 2006.

[Cor02] Albert Joseph Cortez. *Dynamics of diffeomorphisms of the torus*. ProQuest LLC, Ann Arbor, MI, 2002. Thesis (Ph.D.)–University of California, Los Angeles.

[Cor08] Bruno Cordani. Frequency modulation indicator, Arnold’s web and diffusion in the Stark-Quadratic-Zeeman problem. *Phys. D.*, 237(21):2797–2815, 2008.

[DdlLS06] Amadeu Delshams, Rafael de la Llave, and Tere M. Seara. A geometric mechanism for diffusion in Hamiltonian systems overcoming the large gap problem: heuristics and rigorous verification on a model. *Mem. Amer. Math. Soc.*, 179(844):viii+141, 2006.

[DH09] Amadeu Delshams and Gemma Huguet. Geography of resonances and Arnold diffusion in a priori unstable Hamiltonian systems. *Nonlinearity*, 22(8):1997–2077, 2009.

[dlL97] Rafael de la Llave. Invariant manifolds associated to nonresonant spectral subspaces. *J. Statist. Phys.*, 87(1-2):211–249, 1997.

[dlL01] Rafael de la Llave. A tutorial on KAM theory. In *Smooth ergodic theory and its applications* (Seattle, WA, 1999), volume 69 of *Proc. Sympos. Pure Math.*, pages 175–292. Amer. Math. Soc., Providence, RI, 2001.

[dlLO99] Rafael de la Llave and Rafael Obaya. Regularity of the composition operator in spaces of Hölder functions. *Discrete Contin. Dynam. Systems*, 5(1):157–184, 1999.

[dLSZ16] Rafael de la Llave, Xifeng Su, and Lei Zhang. Resonant equilibrium configurations in quasiperiodic media: perturbative expansions. *J. Stat. Phys.*, 162(6):1522–1538, 2016.

[dLSZ17] Rafael de la Llave, Xifeng Su, and Lei Zhang. Resonant equilibrium configurations in quasiperiodic media: KAM theory. *SIAM J. Math. Anal.*, 49(1):597–625, 2017.

[Fay02] Bassam R. Fayad. Weak mixing for reparameterized linear flows on the torus. *Ergodic Theory Dynam. Systems*, 22(1):187–201, 2002.

[Fen77] Neil Fenichel. Asymptotic stability with rate conditions. II. *Indiana Univ. Math. J.*, 26(1):81–93, 1977.

[Fen74] Neil Fenichel. Asymptotic stability with rate conditions. *Indiana Univ. Math. J.*, 23:1109–1137, 1973/74.

[FKW01] Bassam Fayad, Anatole Katok, and Alistar Windsor. Mixed spectrum reparameterizations of linear flows on $T^2$. *Mosc. Math. J.*, 1(4):521–537, 644, 2001. Dedicated to the memory of I. G. Petrovskii on the occasion of his 100th anniversary.

[Gal89] O. G. Galkin. Resonance regions for Mathieu type dynamical systems on a torus. *Phys. D.*, 39(2-3):287–298, 1989.

[Gal94] O. G. Galkin. Phase-locking for dynamical systems on the torus and perturbation theory for Mathieu-type problems. *J. Nonlinear Sci.*, 4(2):127–156, 1994.

[Gra14] Loukas Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2014.

[HdlL16] Xiaolong He and Rafael de la Llave. Construction of quasi-periodic solutions of state-dependent delay differential equations by the parameterization method II: Analytic case. *J. Differential Equations*, 261(3):2068–2108, 2016.

[HdlL17] Xiaolong He and Rafael de la Llave. Construction of quasi-periodic solutions of state-dependent delay differential equations by the parameterization method I: Finitely differentiable, hyperbolic case. *Journal of Dynamics and Differential Equations*, 29:1503–1517, 2017.

[HdlL22] Xiaolong He and Rafael de la Llave. Non-analyticity of quasi-periodic solutions for state-dependent delay differential equations. *preprint*, 2022.
[KMG89] Seung-hwan Kim, R. S. MacKay, and J. Guckenheimer. Resonance regions for families of torus maps. *Nonlinearity*, 2(3):391–404, 1989.

[KP90] U. Kirchgraber and K. J. Palmer. *Geometry in the neighborhood of invariant manifolds of maps and flows and linearization*, volume 233 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1990.

[Lan73] III Lanford, Oscar E. Bifurcation of periodic solutions into invariant tori: the work of ruelle and takens. In *Nonlinear Problems in the Physical Science and Biology*, volume 322 of *Lecture Notes in Mathematics*, pages 159–192. Springer Berlin Heidelberg, 1973.

[Mos67] Jürgen Moser. Convergent series expansions for quasi-periodic motions. *Math. Ann.*, 169:136–176, 1967.

[PdLV03] Nikola P. Petrov, Rafael de la Llave, and John A. Vano. Torus maps and the problem of a one-dimensional optical resonator with a quasiperiodically moving wall. *Phys. D*, 180(3-4):140–184, 2003.

[Poi99] H. Poincaré. *Les méthodes nouvelles de la mécanique céleste*. Les Grands Classiques Gauthier-Villars. Paris, 1899.

[Rü75] Helmut Rüssmann. On optimal estimates for the solutions of linear partial differential equations of first order with constant coefficients on the torus. In *Dynamical systems, theory and applications (Rencontres, Battelle Res. Inst., Seattle, Wash., 1974)*, pages 598–624. Lecture Notes in Phys., Vol. 38. Springer, Berlin, 1975.

[SdL12] Xifeng Su and Rafael de la Llave. KAM theory for quasi-periodic equilibria in one-dimensional quasi-periodic media. *SIAM J. Math. Anal.*, 44(6):3901–3927, 2012.

[Van02] John Vano. *A Whitney-Zehnder implicit function theorem*. PhD thesis, University of Texas at Austin, 2002. [http://www.ma.utexas.edu/mp_arc02–276](http://www.ma.utexas.edu/mp_arc02–276).

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