Solving protoplanetary structure equations using Adomian decomposition method

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ABSTRACT

In this paper, the distribution of thermodynamic variables in the protoplanets formed by gravitational instability in the mass range 0.3 – 10 M J (1 M J = 1 Jupiter mass = 1.8966 × 10 30 gm) is investigated in their initial state by solving the structure equations via the Adomian decomposition method. Concerning the heat transfer in the protoplanets, the mode of convection is taken into account. The outcomes indicate that there is a reasonably good agreement between the Adomian semi-analytical solution containing only first 8 terms and the numerical results.

1. Introduction

Since the dawn of civilization, human beings are trying to understand the formation mechanism of the planetary system. Scientific theory in this regard, however, is the vortex theory due to Descartes [1] who suggested that the gas in the universe developed a series of vortices with each vortex producing a star surrounded by a body of gas having minor vortices; the planetary system condensed from these swirling eddies [2]. Since then, there have been a great many advances in understanding planetary formation mechanisms through investigations of many authors. With the confirmation of the first exoplanet or extrasolar planet, 51 Pegasi b (discovered and confirmed in 1995), interests in this regard have been rekindled and a large volume of work has already been done on the physical conditions prevailing in the interior of such planets including those of our solar system [3, 4]. Though the planets are now believed to have formed from high specific angular momentum material leftover from the star’s birth, many of the details regarding the process of their formation are still questioned [5]. One of the standard models of planetary formation suggests that the formation of planets has taken place from a placental cloud of dust and gas. The slowly rotating cloud was initially cold and collapsed due to local gravitational instability (GI). Such a GI can lead to dense and self-gravitating clumps, which might collapse and contract to form gas giant protoplanets [6, 7]. The process of planetary formation through the GI has also been discussed by various other authors [8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. Although some questions arise as to whether stable protoplanets could be formed or not by the GI, the mechanism is believed to be a promising route to the rapid formation of gas giants [5, 18, 19, 20, 21, 22, 23]. However, the primordial structures of the protoplanets, the precursor to planet formation, formed via GI are still unknown, and diverse numerical models report different primordial structures [5, 13, 15, 16, 17, 24, 25, 26, 27]. For example, simulations due to Boss [28] foresee less dense and colder objects than those of Mayer et al. [9, 19] and the simulations carried out by the mentioned investigators reported denser and warmer initial profiles over the ones used in the simulations due to DeCampli and Cameron [29] and Bodenheimer et al. [30], whereas Paul et al. [5, 17] found denser protoplanets than that of Helled and Schubert [13], but having similar central temperature profiles. However, no evidence reveals the existence of such a protoplanet with a definite structure. In this connection, it is worth noting that the protoplanets can give birth to Earth-mass or more massive cores by grain growth and sedimentation, and be tidally disrupted, potentially providing a ‘new’ pathway to forming all kinds of planets [31]. The grain growth and sedimentation depend, among others, on the primordial internal structure of a
protoplanet where the material is found to be distributed due to gravitational stratification of other parameters [16, 27]. Then some mechanisms are needed for the planetary core formation [27]. The obvious mechanism for sedimentation is that the dust grains present in the protoplanets, being heavier than the gas, will grow up through hit-and-stick or other mechanisms and will settle down towards the centre of the protoplanet under its gravitational field [14, 15, 16, 17, 27, 32]. Therefore, the primordial internal structure of a protoplanet can be an important part of planetary evolution. However, Boss [7], in his investigation of planetary formation, assumed that the protoplanet under its gravitational field [14, 15, 16, 17, 27, 32] considered completely convective except the thin outer radiative zones. Such an assumption is consistent with that of Bodenheimer [33] found them to be completely convective except the topplanets in radiative equilibrium. Helled et al. [12], and Helled and Bodenheimer [33] found them to be completely convective except the thin outer radiative zones. Such an assumption is consistent with that of Bodenheimer et al. [30] and Wuchterl et al. [34], whereas Paul et al. [5, 15, 27] considered fully convective initial protoplanets, such an assumption is supported by Helled et al. [35]. On the other hand, Paul et al. [24, 36] estimated the primordial internal structures of the protoplanets considering the conductive-radiative and convective modes of heat transfer to make a comparison.

It is to be noted here that most of the processes mentioned above have been carried out numerically, but as far as the authors’ knowledge, no endeavor has been taken into account to have semi-analytical and analytical or closed-form solutions to the problem without specification of an arbitrary density distribution model. It has been established that the Adomian decomposition method (ADM) is an effective semi-analytical technique for computing an analytic approximate solution to a non-linear equation as an infinite series that usually converges very fast to the exact solution [37]. The ADM can be utilized to solve a large variety of ordinary and partial differential equations (PDEs) as well as several kinds of integral and integro-differential equations arising in linear and non-linear problems related to different fields of science and engineering [38]. To date, a large volume of research articles has been published to demonstrate the practicability of the technique [38]. The main benefit of the ADM is that one can employ it directly to differential and integral equations with constant or variable coefficients, which may be either linear or non-linear as well as either homogeneous or nonhomogeneous [39, 40]. Another important benefit of the method is that it may reduce the size of calculation significantly while still preserving reasonably accurate numerical results over the traditional techniques [41]. A brief note on the ADM is presented in subsection 3.1.

Owing to the ability of providing analytic approximate solutions with faster convergence rate and higher accuracy of the ADM, in this study, we attempt to solve the equations of protoplanetary structure by the mentioned method to have the primordial internal structures of the protoplanets having masses 0.3 – 10 M_\odot assuming the gas blob of each of them to be convective. We intend to show that if the masses and radii for the assumed protoplanets are provided, then the configurations of temperature, pressure, and density can uniquely be determined, suggesting that the GI hypothesis is a reasonable one.

2. The model

Our model assumes isolated spherical gaseous protoplanets with the solar composition of gas formed via GI having the mass in the range 0.3 – 10 M_\odot [42]. It is relevant to note here that the mass range covers the majority of the detected exoplanets [13]. The protoplanets, during their initial phase, contract quasi-statically [17, 30, 42]. In this phase, following the mentioned three groups of investigators, we also assume that such a protoplanet is in a steady state in which the Clausius-Clapeyron equation of state holds well, and the source of energy is only the gravitational contraction of gases. We consider the mode of convection relating to the heat transfer in the protoplanets. Then the structure of such a protoplanet for the period of its pre-collapse phase can be presented by solving the set of equations appended below [5]:

The equation of hydrostatic equilibrium,

\[ \frac{dP_r}{dr} = -\frac{GM_r}{r^2}\rho. \]  (1)

The conservation of mass equation,

\[ \frac{dM_r}{dr} = 4\pi r^2 \rho. \]  (2)

The equation for convection heat transfer,

\[ \frac{dT_r}{dr} = \left(1 - \frac{1}{\gamma} \right)\frac{T_r}{P_r} \frac{dP_r}{dr}. \]  (3)

The Clausius-Clapeyron equation of state,

\[ P_r = \frac{k}{\mu \rho} \theta T_r. \]  (4)

In Eqs. (1), (2), (3), and (4), P_r, T_r, and \rho, symbolize the pressure, temperature, and density, respectively, at distance r measuring from the centre of the protoplanet; M_r represents its mass interior to r; \mu is the mean molecular weight of the gas composed of H_2, He, and heavy elements (solar composition); \theta denotes the mass of a hydrogen atom; k designates the Boltzmann constant; \gamma is the ratio of specific heats; G is the universal gravitational constant.

2.1. Boundary conditions

Taking into consideration a sphere of infinitesimal radius r at the centre of a protoplanet, we see that

\[ M_r = \frac{4}{3} \pi r^3 \rho. \]  (5)

as \rho, can be treated reasonably as a constant there. Therefore, M_r → 0 as r → 0. Eq. (5) also clarifies the fact that at r = R, i.e., the mass within the radius, R is M ( = M_\odot say). Furthermore, we can develop suitable conditions for temperature and pressure at the surface of the protoplanet. An initial protoplanet having a cold origin definitely will have a low surface temperature [14, 43]. Thus, in the first approximation, the surface temperature can be set to zero [5, 15, 16]. Further, the protoplanetary atmospheric mass is just an infinitesimal fraction of its total mass [44]. Therefore, the surface pressure can approximately be taken as zero. The approximate boundary conditions (BCs) for solving the structure equations can thus be given by [5]:

\[ T_r = 0, \quad P_r = 0 \quad \text{at} \quad r = R \quad \text{(surface)}, \]

\[ M_r = M \quad \text{at} \quad r = R \quad \text{(surface)}, \]

\[ M_r = 0 \quad \text{at} \quad r = 0 \quad \text{(centre)}. \]  (6)

3. Solution of the model equations

First, the structure equations, as well as the BCs, were made dimensionless by means of the Schwarzschild transformations as follows [44]:

\[ r = x R, \quad P_r = \frac{GM_r}{4\pi R^2} \rho(x), \quad T_r = \frac{dHGM_r}{dr} \theta(x), \quad M_r = M_\odot \rho(x). \]

Now, making use of the above transformations and Eq. (4) with \gamma = \frac{5}{3} (for a monoatomic gas), Eqs. (1), (2), and (3) are transformed, respectively, into the following equations:

\[ \frac{dp}{dx} = -\frac{pq}{\rho x^2} \]  (7)

\[ \frac{dq}{dx} = \frac{px^2}{\rho} \]  (8)

and

\[ \frac{dp}{dx} = -\frac{2q}{5x^2} \]  (9)

while the BCs described by Eq. (6) take the following form:
\[ \theta(x) = 0, \ p(x) = 0 \text{ at } x = 1 \text{ (surface)}, \]
\[ q(x) = 1 \text{ at } x = 1 \text{ (surface)}, \]
\[ q(x) = 0 \text{ at } x = 0 \text{ (centre)}. \]  

\[ \text{(10)} \]

Eqs. (7) and (8) produce the following equation:
\[ p = E_p \theta^{1/2}, \]

where \( E_p \) is the constant of integration.

One can now recast Eq. (4) in the following form:
\[ \rho(x) = \frac{M}{4\pi R^3} \frac{p}{\theta} \]

Furthermore, differentiating Eq. (9) with respect to \( x \) and then using Eqs. (8) and (11), it is easy to see that
\[ \frac{1}{x^2} \frac{d}{dx} \left( x \frac{d\theta}{dx} \right) = -\left( \frac{E_p}{2.5} \right) \theta^1. \]

\[ \text{(13)} \]

It is obvious from Eq. (13) that if it could be solved analytically for \( \theta \) in terms of \( x \) and hence for \( \frac{d\theta}{dx} \), through Eq. (11), \( p \) could be obtained in terms of \( x \) and hence with the help of Eq. (12), \( \rho \) could be determined. However, a closed-form solution or an analytical solution of Eq. (13), as far the best of authors’ knowledge, is not possible. Thus, it can be solved semi-analytically or numerically. It is to be noted here that the numerical solution of Eq. (13) is detailed in Senthilkumar and Paul [45] for having structures of initially formed protoplanets. To attain its semi-analytical solution, we have employed the ADM. A brief note on the ADM, the application of the method in solving an ordinary differential equation (ODE), and details of the solution procedure of the equation of interest are presented in the following subsections.

### 3.1. A brief note on the ADM

The scientific nonlinear problems describing physical phenomena do not often possess a precise analytical solution [46]. To solve such kinds of nonlinear problems, a powerful method providing efficient algorithms for analytic approximate solutions and numeric simulations for real-world applications in science and engineering was devised in 1981 by G. Adomian (1923–1996) [38], and ever since many mathematicians and scientists have modified the ADM in an endeavor to enhance its accuracy and/or to broaden the applications of the original method [47]. Some examples are as follows: reviews of the application of the ADM to differential and integral equations are discussed in [48, 49, 50], Wazwaz [51] established a modification in the standard ADM that demonstrates a faster rate of convergence of the series solution over the standard ADM. Duan et al. [52] reviewed the ADM and its modifications including different modified and parameterized recursion schemes, the multistage ADM for initial and boundary value problems (BVPs), new developments of the method and its applications to linear or nonlinear, and ODEs or PDEs, including fractional differential equations. The authors concluded that the ADM provides us with an easily computable, readily verifiable, and fast convergent sequence of analytic approximate functions for the solution [52]. Babolian et al. [53] compared the performance of the ADM and homotopy perturbation method (HPM). They showed that the ADM is equivalent to the powerful HPM with a specific convex homotopy for nonlinear differential equations. Harko et al. [54] studied the dynamics of vortices with arbitrary topological charges in weakly interacting Bose-Einstein condensates by solving the nonlinear Gross-Pitaevskii equation in polar coordinates with the use of the ADM. In ref. [54], the authors obtained a series representation of the solution of the Susceptible, Infectious, and/or Recovered (SIR) model by the Laplace-ADM (LADM) to solve the basic evolution equation of the model. Appadu and Kelil [55] solved linearized dispersive KdV equations with homogeneous and non-homogeneous source terms by the LADM, LADM based on Bernstein polynomials (BLADM), HPM, and reduced differential transform method. They found that the BALDM is entirely a comprehensible method, as it diminishes the large volume of calculations and its iteration steps towards an exact solution are direct. Mak et al. [47] presented some applications of the ADM to the Fisher Kolmogorov second order PDE, which was used to describe many physical processes and three important applications in astronomy and astrophysics including the determination of the solutions of the Lane-Emden equation (LEE), Kepler equation, and the general relativistic equation describing the motion of massive particles in the spherically symmetric and static Schwarzschild geometry. They also showed the effectiveness of the ADM in solving the linear, Bernoulli, Riccati, and Abel differential equations.

The ADM is a powerful, accurate, and convenient technique for attaining analytical solutions not only for weakly nonlinear equations but also for strongly ones [56]. The method allows one to solve nonlinear initial-BVPs without immaterial obstructive speculation, viz. required by linearization, perturbation, ad hoc assumptions, guessing the initial term or a set of basis functions, and so forth [52]. The main steps of the ADM can be summarized as follows: the ADM consists of partitioning the equation of interest into linear and nonlinear portions, inverting the derivative operator of the highest-order contained in the linear operator, identifying the initial and/or boundary conditions, and the terms involving the independent variable alone as an initial approximation, decomposing the nonlinear function in terms of special polynomials called Adomian’s polynomials (ADPs) and finding the successive terms of the series solution by a recurrent relation using the ADPs [37, 52, 53, 55]. In this connection, it is worth noting that as the ADM produces a convergent series solution of the differential or integral equation [57] that converges very quickly to the exact solution [58], it saves significant amounts of computing time. Further, in the case of the ADM, there is no need to linearize or discretize the differential and integral equations.

### 3.2. Application of the ADM to an ODE

Following Adomian [49, 59] and Duan et al. [52] depending on our problem, the method for solving an ODE is presented here for a better perspective. Consider the ordinary differential operator (ODO) equation as
\[ Ru = f, \]

where \( f \) is a source term, which is known, and \( F \) is a general nonlinear ODO involving both linear and nonlinear terms. Suppose that \( N \) be the nonlinear part of \( F \) and its linear part can be decomposed into \( L + R \). \( L \) is the highest-order derivative operator, assumed to be invertible, and \( R \) is a linear differential operator having order less than \( L \). Then Eq. (14) is reframed as \( Lu + Ru + Nu = f \), which on solving for \( Lu \) yields
\[ Lu = f - Ru - Nu. \]

\[ \text{(15)} \]

Therefore, applying the inverse operator \( L^{-1} \) to Eq. (15), as \( L \) is an invertible operator, one obtains
\[ u = g - L^{-1}Ru - L^{-1}Nu, \]

where \( g \) represents the terms arising from integrating the source term, \( f \) and from using the conditions that are assumed to be prescribed.

Now the remaining work is to decompose the nonlinear term, \( Nu \). In order to make sure of this, Adomian [49] decomposed the solution as \( u = \sum_{n=0}^{\infty} x^n u_n \), where \( i \) is a decomposition parameter. Adomian [49] assumed that the nonlinear term, \( N(u) \) can be expressed by an infinite series of \( A_n \) given as \( N(u) = \sum_{n=0}^{\infty} x^n A_n(u_1, u_2, u_3, \ldots, u_n) \), where \( A_n \) are termed as ADPs, which can be evaluated for all forms of nonlinearity. The ADPs for \( N(u) \) can be evaluated using the following expression [37]:
\[ A_n = \frac{1}{n!} \frac{d^n}{dx^n} \left( N \sum_{i=0}^{\infty} x^i u_i \right) \bigg|_{x=0}, \quad n = 0, 1, 2, 3, \ldots \text{, which can be simplified as} \]

\[ A_0 = N(u_0), \quad A_1 = u_1 N'(u_0), \quad A_2 = u_2 N'(u_0) + \frac{x^2}{2} N''(u_0), \]
\[ A_3 = u_3 N'(u_0) + u_1 u_2 N''(u_0) + \frac{x^3}{3} N'''(u_0), \ldots \]

Now, we parameterize Eq. (16) in the form given below:

\[ u = g - L^{-1}Ru - L^{-N}Nu, \]

where the decomposition parameter \( \lambda \) is an identifier. The parameter is only introduced for collecting terms in a suitable way such that \( u_0 \) depends on \( u_0, u_1, \ldots, u_{N-1} \), and later it is set as \( \lambda = 1 \). Thus, with \( u = \sum_{n=0}^{\infty} x^n u_n \) and \( N(u) = \sum_{n=0}^{\infty} x^n A_n(u_0, u_2, u_3, \ldots, u_n) \), mentioned above, Eq. (17) can be written as

\[ \sum_{n=0}^{\infty} x^n u_n = g - L^{-1} R u_0 - L^{-N} N u_n, \]

Equating the coefficients of each power of \( \lambda \) on both sides of Eq. (18), we obtain

\[ u_0 = g, \quad u_1 = -L^{-1} R u_0 - L^{-1} A_0, \]
\[ u_2 = -L^{-1} R u_1 - L^{-1} A_1, \ldots \] and in general,
\[ u_n = -L^{-1} R u_{n-1} - L^{-1} A_1, \quad n \geq 1. \]

Finally, an \( N \)-term approximate series solution is sought in the following form:

\[ \psi(x) = \sum_{n=0}^{N-1} u_n(x), \quad n \geq 1, \] and the exact solution is \( u(x) = \lim_{N \to \infty} \psi_n(x) \).

3.3. Solution technique of structure equations

Let \( \theta = T_\phi \phi \), where \( T_\phi \) is a factor chosen in such a way that \( \phi \in [0, 1] \).

Eq. (13) then can be put in the following form:

\[ \frac{d^2 \phi}{dx^2} + \frac{2}{x} \frac{d \phi}{dx} + \frac{E_\phi T_\phi}{2.5} \phi^{1.5} = 0. \]

As in Paul et al. [43], the non-dimensional form of the LE equation, a non-dimensional form of Poisson's equation for the gravitational potential of a Newtonian self-gravitating, spherically symmetric, polytropic fluid [43], with polytropic index \( n = 1.5 \) can be set as

\[ \frac{d^2 \phi}{dx^2} + \frac{2}{x} \frac{d \phi}{dx} + \xi_1 \phi^{1.5} = 0, \]

where \( \xi_1 \) being the first zero of \( \phi \) and \( \xi_1 = 3.65375 \) for \( n = 1.5 \).

It is known that for convective energy transport, \( n = 1.5 \) [43]. Thus, Eq. (20) represents the LE equation with polytropic index \( n = 1.5 \), and hence \( \xi_1 = 3.65375 \). In a series of studies due to Paul et al. [5, 15, 16, 27, 36], the value of \( E_\phi \) was attained to be 45.4. Thus, with this value of \( E_\phi \), the value of \( T_\phi \) is obtained as 0.5404. The conditions then for solving Eq. (20) satisfying Eq. (10) take the following form [14]:

\[ \begin{align*}
\phi &= 1, \quad \frac{d \phi}{dx} = 0 \quad \text{at} \quad x = 0. \\
\end{align*} \]

Eq. (20) in Adomian operator form is given by

\[ L \phi = -C \phi^{1.5}, \]

where \( L = x^{-2} \frac{d}{dx} x^2 \frac{d}{dx} \) (.), and \( C = \frac{E_\phi \sqrt{\xi_1}}{2.5} \).

Operating \( L^{-1} = \int \frac{x}{0} x^2 \frac{d}{dx} \)(.) dx on both sides of Eq. (23), one can see

\[ \begin{align*}
\phi(x) &= \phi(0) - \psi(0)x - CL^{-1}(\phi^{1.5}), \quad \text{which yields} \\
\phi(x) &= 1 - CL^{-1}(\phi^{1.5}). \\
\end{align*} \]

Let \( \psi(x) = \sum_{p=0}^{\infty} \phi_p \psi_p \), be a solution of Eq. (24) and \( \phi^{1.5} = \sum_{p=0}^{\infty} \phi_p \).

Then Eq. (24) yields,

\[ \sum_{p=0}^{\infty} \phi_p = 1 - CL^{-1} \sum_{p=0}^{\infty} \phi_p. \]

Equating the coefficients of each power of \( \lambda \) on both sides of Eq. (25), we get

\[ \psi_0 = 1, \quad \psi_{p+1} = -CL^{-1}(A_p), \quad p = 0, 1, 2, 3, \ldots \]

The so-called ADPs are then given by

\[ \begin{align*}
A_0 &= 0.15^{0.5}, \quad A_1 = 1.5\phi_0^{0.5} \phi_1, \quad A_2 = \frac{0.375}{\phi_0^{0.5}} \phi_2^{2} + 1.5\phi_0^{0.5} \phi_2, \\
A_3 &= 0.625 \phi_0^{0.5} \phi_3^{0.75} + 0.75 \phi_0^{0.5} \phi_0 \phi_2 + 1.5\phi_0^{0.5} \phi_3, \ldots \]
\end{align*} \]

Then the first term is \( \psi_0 = -CL^{-1}(A_0) = -CL^{-1}(\phi_0^{1.5}) = -CL^{-1}(1) \).

Applying \( CL^{-1} = \int \frac{x}{0} x^2 \frac{d}{dx} \)(.) dx, one gets

\[ \phi_1 = 0.16666666666666666666666666666666 \]

Proceeding in a similar manner, we obtain

\[ \begin{align*}
\phi_2 &= 0.01250000000000C_4 \phi_1, \quad \phi_3 = -0.0006944444443C_3 \phi_1, \\
\phi_4 &= 0.0003215020575C_4 \phi_1, \ldots \]
\end{align*} \]

Here the terms are computed using the symbolic computation software, Maple 18. But the calculation of all the \( \phi_n \) is almost impossible. It was found that the calculation of each term after the 12th requires a longer period of time even for Maple 18, and the solution of Eq. (13) is thus approximated by a series given by

\[ \psi(x) = \sum_{n=0}^{N-1} \phi_n(x), \quad n \geq 1, \]

where the exact solution is given by

\[ \psi(x) = \lim_{n \to \infty} \phi_n(x). \]

We attained a sequence of approximated results up to \( N = 12 \) since the Adomian semi-analytical solution containing only first 8 terms was in a reasonable agreement with the numerical results, as we will see later. But for checking the convenience of the series (27), one or two more terms can be generated.

With \( N = 12 \), the approximated power series solution is given by

\[ \begin{align*}
\phi_{12} &= 1 - 0.166766666666666666666666666666666 + 0.01250000000000 \cdot 0.694 \cdot 10^{-3} \cdot C_4 \phi_1 + 0.3215020575 \cdot 10^{-4} \cdot C_4 \phi_1 - 0.1332772165 \cdot 10^{-5} \cdot C_5 \phi_1 \\
+ 5.094196351 \cdot 10^{-3} \cdot C_7 \phi_1 + 1.839741902 \cdot 10^{-9} \cdot C_7 \phi_1 + 6.341223094 \cdot 10^{-11} \cdot C_7 \phi_1 + 2.113552252 \cdot 10^{-12} \cdot C_7 \phi_1 \\
+ 6.825130240 \cdot 10^{-14} \cdot C_7 \phi_1 + 2.160163439 \cdot 10^{-15} \cdot C_7 \phi_1 + 6.660069067 \cdot 10^{-17} \cdot C_7 \phi_1. \\
\end{align*} \]
For testing the convergency of the series given by Eq. (27), we plot different orders of the ADM approximate solutions of Eq. (20) and its numerical solution in Figure 1. From Figure 1, it is seen that the ADM solution converges to the numerical solution after only seven iterations. The radius of convergence of the Lane-Emden function with series approximation given by Eq. (27) is the first zero of \( \varphi(x) \), which is defined as the smallest positive value \( x_0 \) for which \( \varphi(x_0) = 0 \). From Figure 1, it is seen that \( x = 1 \) is the smallest positive value for which \( \varphi(x) = 0 \). Thus, the radius of convergence is 1, i.e., the solution converges throughout the protoplanet.

Our results were estimated with the power series given by Eq. (28). Then with \( \theta = T_K \rho \), the non-dimensional temperature, \( \theta \) can be attained. Subsequently, with the aid of Eq. (11), \( \rho \) can be determined and hence \( \rho \) can be obtained with the help of (12).

We have justified our results through numerical outputs. To find numerical outputs for varying \( x \), it should be specified. However, \( x \) cannot be started up right from the centre, for \( \varphi(x) \) (see Eq. (9)) is undefined at the centre, i.e., at \( x = 0 \) and hence \( x \in (0, 1) \). Now to obtain, the distribution of \( \varphi \) for varying \( x \), the value of \( C \) is needed. With the prescribed value of \( E_p \) along with the value of \( T_k \), \( C \) is obtained as 13.34. Then with the distribution of \( \varphi \), and thereby the distribution of the non-dimensional temperature, \( \theta \) will be attained with \( \theta = T_k \rho \), and hence the non-dimensional pressure distribution will be attained from Eq. (11). Eq. (12) will then yield the non-dimensional distribution of density for varying \( x \) with the obtained distributions of \( p \) and \( \theta \). Then the intended distribution of thermodynamic variables in the assumed protoplanets are obtained with the above stated Schwarzschild transformations.

In this investigation, the mass and radius of each of the assumed protoplanets are taken from the study due to Paul et al. [42], which are similar to those found in [13]. Also, we have used \( \mu = 2.2 \), which is suitable for molecular hydrogen (\( H_2 \)); \( H = 6.63 \times 10^{-27} \) ergs; \( G = 6.67 \times 10^{-8} \) cm\(^3\)kg\(^{-1}\)s\(^{-2}\); \( k = 1.38 \times 10^{-24} \) cm\(^2\)g s\(^{-2}\)K\(^{-1}\).

4. Results, discussion, and conclusion

Following stellar evolution codes, we have investigated the primordial internal structures of some protoplanets formed by the GI having masses 0.3, 1, 3, 5, 7, and 10 \( M_j \) by solving protoplanetary structure equations through the ADM. Each of the protoplanets is considered to be a gaseous sphere of solar composition, which is in a steady state of quasi-static equilibrium. The gas blob of such a protoplanet is presumed to be fully convective and the source of protoplanetary energy is only the gravitational contraction of gases. For estimating the protoplanetary structure, the constant of integration \( E_p \) was needed to be supplied. However, the best solution satisfying the BCs (surface and central) obtained for \( E_p \) due to Paul et al. [5, 15, 16, 27, 36] was 45.4, which was the adopted value of the parameter for the present study. It is worth noting that this value of \( E_p \) obtained by Osterbrock [60] in determining the structure of a convective star was 13.6. Further, the distribution of thermodynamic variables is dependent on the specification of the number of terms in the series given by Eq. (26). We calculated our results with \( N = 12 \) as \( N = 10, 11, \) and 12 yielded almost the same result (see Table 1).

Figure 2 illustrates the internal temperature distribution of the assumed protoplanets in their initial state. From the figure, it is inferred that as massive as is the protoplanet considered, so warmer the interior is obtained. In our calculated profiles of temperature, the consequence is found to be in good agreement with those presented in refs. [5, 15, 27], and the surface as well as central temperatures are in reasonable agreement with those reported in refs. [5, 13, 27, 42, 61].

Figure 3 shows the estimated pressure profiles of the protoplanets mentioned above. It is seen from Figure 3 that the pressures at a height from the centres of the protoplanets increase with their increasing mass except for the one having mass 10 \( M_j \). We find a reasonably good agreement between the distributions of pressure of the protoplanets that came out through our calculation and the corresponding outcomes conferred in refs. [5, 24, 25, 36, 45], but the attained central pressures are found to be conflicted with the ones presented in ref. [13]. The reason behind the fact may be that the authors, in ref. [13], assumed the protoplanets formed via the GI to be convective with a thin outer radiative zone, as is to be expected [30]. But in our study, the protoplanets are assumed to be fully convective, which is consistent with that of Helled et al. [35].

The model computed density distributions for the assumed initial protoplanetary masses are depicted in Figure 4. The figure (Figure 4) demonstrates that the protoplanets are centrally condensed in the order of increasing masses except for the protoplanets with masses 1, 3, and 10 \( M_j \). The protoplanets of masses 1 and 3 \( M_j \) are found to be less centrally
condensed over the one having mass 0.3 $M_J$, whereas the protoplanet of mass 10 $M_J$ is found to be rarer over that of 5 $M_J$. The protoplanet of mass 7 $M_J$ resulted from our calculation is found to be the most centrally condensed, whereas the protoplanet of mass 3 $M_J$ shows to be less dense.

Our computed density distributions of the protoplanets are found to be consistent with the ones presented in refs. [5, 15, 16, 24, 25, 27, 36, 42, 45]. But Helled and Schubert [13] documented in their study that the massive protoplanets are more centrally condensed. In fact, the initial profiles of the protoplanets are still unidentified, and a variety of models report diverse initial configurations [35], as stated above. To compare our calculated findings (for $N = 12$) for varying $x$ directly, they are presented with those obtained in Senthilkumar and Paul [45] and Paul et al. [5] in Figure 5. It is of interest to mention here that Senthilkumar and Paul [45] solved Eq. (13), the equation of our interest, with the embedded RKA-HeM(4,4) method, and Paul et al. [5] solved Eqs. (1), (2), (3), and (4), yielding a protoplanetary structure, with the embedded RKARMS(4,4) technique in order to have optimal solutions. It is further worth noting that Eq. (13) is obtained from Eqs. (1), (2), and (3). It can be seen from Figure 5 that the results obtained in the study match well with those presented in [5, 45]. We have evaluated root mean square errors (RMSEs) between our obtained results with those presented in Senthilkumar and Paul [45] and Paul et al. [5]. The respective RMSE values in attaining temperature, pressure, and density of 1 $M_J$ were found to be 0.53 K, 0.64 dynes cm$^{-2}$, 2.62 $\times$ 10$^{-9}$ gm cm$^{-3}$ and 0.45 K, 0.37 dynes cm$^{-2}$, 3.34 $\times$ 10$^{-9}$ gm cm$^{-3}$, respectively. Thus, there is a reasonable agreement between our computed results and the results presented in the mentioned investigations with respect to the RMSE values.

However, our estimated results with $8 \leq N \leq 12$ are found to be almost the same. Thus, the result with $N = 12$ is justifiable, and therefore for large $N$, approximate solutions may get closer to the exact solution. The obtained outcomes reveal that the ADM is an efficient and powerful approach to obtain an analytic approximate solution to a nonlinear

### Table 1. Distribution of thermodynamic variables inside the protoplanet with mass 1 $M_J$.

| $r/R$ | Temperature, $T$ (K) | Pressure, $P$ (dynes cm$^{-2}$) | Density, $\rho$ (gm cm$^{-3}$) |
|-------|----------------------|-------------------------------|-----------------------------|
|       | $N = 10$             | $N = 11$                      | $N = 12$                    | $N = 10$             | $N = 11$                      | $N = 12$                    |
| 0.99  | 2.1765               | 2.7542                        | 2.5229                      | 0.0008               | 0.0014                        | 0.0011                      | 9.19E-12                     | 1.31E-11                     | 1.15E-11                     |
| 0.9   | 28.2719              | 28.3306                       | 28.3112                     | 0.4561               | 0.4585                        | 0.4577                      | 4.30E-10                     | 4.32E-10                     | 4.31E-10                     |
| 0.8   | 62.4004              | 62.4039                       | 62.4030                     | 3.3011               | 3.3015                        | 3.3014                      | 1.41E-09                     | 1.41E-09                     | 1.41E-09                     |
| 0.7   | 102.1179             | 102.1181                      | 102.1180                    | 11.3095              | 11.3095                       | 11.3095                     | 2.95E-09                     | 2.95E-09                     | 2.95E-09                     |
| 0.6   | 146.3991             | 146.3991                      | 146.3991                    | 27.8313              | 27.8313                       | 27.8313                     | 5.07E-09                     | 5.07E-09                     | 5.07E-09                     |
| 0.5   | 193.1773             | 193.1773                      | 193.1773                    | 55.6644              | 55.6644                       | 55.6644                     | 7.68E-09                     | 7.68E-09                     | 7.68E-09                     |
| 0.4   | 239.4180             | 239.4180                      | 239.4180                    | 95.1874              | 95.1874                       | 95.1874                     | 1.06E-08                     | 1.06E-08                     | 1.06E-08                     |
| 0.3   | 281.3752             | 281.3752                      | 281.3752                    | 142.5287             | 142.5287                      | 142.5287                    | 1.35E-08                     | 1.35E-08                     | 1.35E-08                     |
| 0.2   | 315.0740             | 315.0740                      | 315.0740                    | 189.1119             | 189.1119                      | 189.1119                    | 1.60E-08                     | 1.60E-08                     | 1.60E-08                     |
| 0.1   | 336.9558             | 336.9558                      | 336.9559                    | 223.6763             | 223.6763                      | 223.6763                    | 1.77E-08                     | 1.77E-08                     | 1.77E-08                     |
| 0.01  | 344.4691             | 344.4691                      | 344.4691                    | 236.3541             | 236.3541                      | 236.3541                    | 1.83E-08                     | 1.83E-08                     | 1.83E-08                     |

*Figure 2. Temperature profiles inside the assumed protoplanets in their initial stage formed via GI.*
Figure 3. Primordial internal pressure profiles of the assumed protoplanets formed via GI.

Figure 4. Density distribution inside some initial protoplanets formed via GI.
equation that converges to the actual solution very quickly (converges after only seven iterations, in the case of the present work, see Figure 1). Nevertheless, the system yields a unique solution, which leads to the fact that the planetary formation through the GI is a feasible mechanism. The outcomes emanated in this investigation can be significant in the study of planetary evolution in our solar system and elsewhere. The findings can be compared to the results obtained with other possible semi-analytical methods, namely the HPM, the LADM, the BLADM, etc. These works are in progress and would be addressed in a future paper.

Declarations

Author contribution statement

Gour Chandra Paul: Conceived and designed the experiments; Contributed reagents, materials, analysis tools or data; Analyzed and interpreted the data; Wrote the paper.
Shahinur Khatun: Analyzed and interpreted the data; Wrote the paper.
Md. Nuruzzaman: Conceived and designed the experiments; Wrote the paper.
Dipankar Kumar: Conceived and designed the experiments; Analyzed and interpreted the data; Wrote the paper.
Md. Emran Ali: Analyzed and interpreted the data; Wrote the paper.
Farjana Bilkis: Analyzed and interpreted the data; Wrote the paper.
Mrinal Chandra Barman: Analyzed and interpreted the data; Wrote the paper.

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The authors declare no conflict of interest.

Additional information

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