GLOBAL ATTRACTORS OF TWO LAYER BAROCLINIC QUASI-GEOSTROPHIC MODEL

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Abstract. We study the dynamics of a two-layer baroclinic quasi-geostrophic model. We prove that the semigroup \( \{S(t)\}_{t \geq 0} \) associated with the solutions of the model has a global attractor in both \( \dot{H}^1_0(\Omega) \) and \( \dot{H}^2_0(\Omega) \). Also we show that for any viscosity \( \mu > 0 \), there is an open and dense set of forcing \( G \subset \dot{H}^0_0(\Omega) \) such that for each \( G = (g_1, g_2) \in G \), the set \( S(G, \mu) \subset \dot{H}^4_0(\Omega) \) of the steady state problem is non-empty and finite.

1. Introduction and some preliminaries. Baroclinic instability is one of the most important geophysical fluid dynamical instability, and plays a crucial role in understanding the dominant mechanism shaping the cyclones and anticyclones that dominate weather in mid-latitudes. A two layer baroclinic quasi-geostrophic model was first introduced by J. Pedlosky [6] to study baroclinic instability, and is widely used in geophysical fluid dynamics. In its non-dimensional form, the model is given as follows; see among others [1,2,5] and the references therein:

\[
\begin{align*}
-\frac{\partial}{\partial t} + U_1 - \frac{\partial}{\partial x} + \frac{\partial \varphi_1}{\partial x} \frac{\partial}{\partial y} - \frac{\partial \varphi_1}{\partial y} \frac{\partial}{\partial x} &= \mu \Delta \varphi_1 + F(\varphi_2 - \varphi_1) + (U_1 - U_2)y + \beta y, \\
-\frac{\partial}{\partial t} + U_2 - \frac{\partial}{\partial x} + \frac{\partial \varphi_2}{\partial x} \frac{\partial}{\partial y} - \frac{\partial \varphi_2}{\partial y} \frac{\partial}{\partial x} &= \mu \Delta \varphi_2 + F(\varphi_1 - \varphi_2) + (U_2 - U_1)y + \beta y,
\end{align*}
\]

where \((x, y) \in (0, \frac{2\pi}{\gamma}) \times (0, \pi) = \Omega, \gamma \) is the horizontal aspect ratio, we denote \( \hat{\Omega} = \mathcal{D} \), and \((g_1, g_2)\) is external forcing.

The above problem is supplemented with the periodic boundary conditions

\[
\varphi_k(0, y, t) = \varphi_k\left(\frac{2\pi}{\gamma}, y, t\right), \quad \varphi_k(x, 0, t) = \varphi_k(x, \pi, t), \quad k = 1, 2.
\]

To simplify the form of the equations and for the convenience of computation or estimation, we rewrite (1.1)-(1.2) as the following and still use the same label...
numbers
\[-\frac{\partial}{\partial t}[\Delta \varphi_1 + F(\varphi_2 - \varphi_1)] + \mu \Delta [\Delta \varphi_1 + F(\varphi_2 - \varphi_1)] = J(\Delta \varphi_1 + F(\varphi_2 - \varphi_1), U_{1y}) + J(\varphi_1, \Delta \varphi_1 + F(\varphi_2 - \varphi_1) + \beta y) + g_1,\]
\[-\frac{\partial}{\partial t}[\Delta \varphi_2 + F(\varphi_1 - \varphi_2)] + \mu \Delta [\Delta \varphi_2 + F(\varphi_1 - \varphi_2)] = J(\Delta \varphi_2 + F(\varphi_1 - \varphi_2), U_{2y}) + J(\varphi_2, \Delta \varphi_2 + F(\varphi_1 - \varphi_2) + \beta y) + g_2,\]

where
\[J(f, g) = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}\]
is the Jacobian of two functions $f$ and $g$.

Denote $|\psi|^2_k = \int_D |\nabla^k \psi|^2$, $k = 1, 2, 3$ and $|\psi|^2_0 = \int_D |\psi|^2$.

Let $u = (\varphi_1, \varphi_2)$, we require that
\[\int_D \varphi_1 = \int_D \varphi_2 = 0, \quad (1.4)\]
so that the norm $|\varphi|^k$ is an equivalent norm of $\varphi_k$ in $H^k(\Omega)$, so is $|u|^k_k$, and $|u|^2_k = |\varphi_1|^2_k + |\varphi_2|^2_k$, $k = 1, 2, 3, 4$. Hereafter we shall always use
\[\hat{H}^k_p(\Omega) \quad (1.5)\]
to denote the $H^k$ Sobolev space on $\Omega$ with periodic boundary condition (1.3) and with mean zero assumption (1.4).

We also give the Poincaré inequalities as below, which will be frequently used in the estimates later:
\[C_k|\psi|_k \leq |\psi|_{k+1}, \quad k = 0, 1, 2, 3.\]

Given $u_0 \in \hat{H}^1_p(\Omega)$ or $\hat{H}^2_p(\Omega)$ respectively, applying a standard Faedo-Galerkin method, one can obtain the existence of the solutions to the problem (1.1)-(1.4) with initial data $u_0 \in \hat{H}^1_p(\Omega)$ or $\hat{H}^2_p(\Omega)$ respectively. Furthermore, a slight modification of the a priori estimates, derived in the next section for the proving the existence of absorbing sets, ensures the existence of global in time solutions of the problem. For simplicity, we omit the details.

In view of the existence theorem just mentioned, for every $u_0 \in \hat{H}^1_p(\Omega)$ (or $\hat{H}^2_p(\Omega)$ respectively), let $u(t)$ be the solution of the problem with initial data $u_0$, and set
\[S(t)u_0 = u(t), \quad \forall t \geq 0.\]

Then $\{S(t)\}_{t \geq 0}$ forms a $C_0$ semigroup.

The main objectives of this paper are
- to demonstrate that the system (1.1)-(1.4) gives rise to an infinite dynamical system, which possesses a global attractor in $\hat{H}^1_p(\Omega)$ and $\hat{H}^2_p(\Omega)$ respectively, and
- to show genericity of the steady state solutions of the problem.

2. Estimate of bounded absorbing sets in $\hat{H}^1_p(\Omega)$ and $\hat{H}^2_p(\Omega)$. One main result of this section is the following existence theorem of bounded absorbing sets of the semigroup associate with the solutions of problem (1.1)-(1.4) in $\hat{H}^1_p(\Omega)$ and $\hat{H}^2_p(\Omega)$.
Theorem 2.1. Let $G = (g_1, g_2)^t \in \dot{H}^0_p(\Omega)$. Assume that $\int_\Omega \varphi_1 = \int_\Omega \varphi_2 = 0$, and that $|U_2 - U_1| < \frac{C_0C_2^2\nu}{4F}$, where $C_0$, $C_1$ are the constants in Poincaré inequalities. Then the semigroup $\{S(t)\}_{t \geq 0}$ associated with the solutions of (1.1)-(1.4) has a bounded absorbing set in $\dot{H}^1_p(\Omega)$.

Proof of Theorem 2.1. Multiply (1.1) and (1.2) by $\varphi_1$ and $\varphi_2$ respectively and integrate on $D$, we have

$$
\frac{d}{dt}|\varphi_1|^2 + \frac{d}{dt}|\varphi_2|^2 + \int_D F(\varphi_2 - \varphi_1)\varphi_1 + \mu|\varphi_1|^2 + \mu F(\varphi_2 - \varphi_1)^2 = U_1 \int_D \frac{\partial \varphi_2}{\partial x} \varphi_1 + \int_D g_1 \varphi_1,
$$

$$
\frac{d}{dt}|\varphi_2|^2 + \frac{d}{dt}|\varphi_1|^2 + \int_D F(\varphi_1 - \varphi_2)\varphi_2 + \mu|\varphi_2|^2 + \mu F(\varphi_1 - \varphi_2)^2 = U_2 \int_D \frac{\partial \varphi_1}{\partial x} \varphi_2 + \int_D g_2 \varphi_2.
$$

Combining the above two equalities, we get

$$
\frac{d}{dt}|\varphi_1|^2 + F|\varphi_2 - \varphi_1|^2 + \mu|\varphi_1|^2 + \mu F|\varphi_2 - \varphi_1|^2 = (U_2 - U_1) F \int_D \frac{\partial \varphi_1}{\partial x} \varphi_2 + \int_D g_1 \varphi_1 + \int_D g_2 \varphi_2 \\
\leq F|U_2 - U_1||\varphi_1||\varphi_2| + 2|g_1||\varphi_1| + 2|g_2||\varphi_2|
$$

(2.1)

The Poincaré inequalities, Young inequality and the assumption $|U_2 - U_1| < \frac{C_0C_2^2\mu}{4F}$ implies that

$$
\frac{d}{dt}|\varphi_1|^2 + F|\varphi_2 - \varphi_1|^2 + \mu C_1^2|\varphi_1|^2 + 2\mu C_3^2 F|\varphi_2 - \varphi_1|^2 \\
\leq \frac{F|U_2 - U_1||\varphi_1||\varphi_2|}{C_0} + 2\frac{|g_1|^2}{C_0 F|U_2 - U_1|} \\
\leq \frac{\mu C_1^2 |\varphi_1|^2}{2} + \frac{|g_1|^2}{C_0 F|U_2 - U_1|}
$$

and then

$$
\frac{d}{dt}|\varphi_1|^2 + F|\varphi_2 - \varphi_1|^2 + \mu C_1^2|\varphi_1|^2 + 2\mu C_3^2 F|\varphi_2 - \varphi_1|^2 \\
\leq \frac{2|g_1|^2}{C_0 F|U_2 - U_1|}.
$$

Let $\nu_1 = \min\{\mu C_1^2, 2\mu C_3^2\}$ and $\frac{2|g_1|^2}{C_0 F|U_2 - U_1|} = K$, we have

$$
\frac{d}{dt}|\varphi_1|^2 + F|\varphi_2 - \varphi_1|^2 + \nu_1(|\varphi_1|^2 + F|\varphi_2 - \varphi_1|^2) \leq K.
$$

By Gronwall inequality, we get for any $t \geq 0$,

$$
|u(t)|^2 + F|\varphi_2(t) - \varphi_1(t)|^2 \\
\leq (|u(0)|^2 + F|\varphi_2(0) - \varphi_1(0)|^2)e^{-\nu_1 t} + \frac{K}{\nu_1} (1 - e^{-\nu_1 t}) \\
\leq (|u(0)|^2 + F(|\varphi_2(0)| + |\varphi_1(0)|)^2)e^{-\nu_1 t} + \frac{K}{\nu_1} \\
\leq (|u(0)|^2 + 4F|u(0)|^2)e^{-\nu_1 t} + \frac{K}{\nu_1} \\
\leq (1 + 4F|u(0)|^2)e^{-\nu_1 t} + \frac{K}{\nu_1}
$$

(2.2)
So, for any bounded set \( B_1 \subseteq \dot{H}_p^1(\Omega) \), there is a constant \( M_1 > 0 \), such that 
\[
|u(0)|_1 \leq M_1 \text{ for any } u_0 = u(0) \in B_1.
\]
Then 
\[
|u(t)|_1^2 + F|\varphi_2(t) - \varphi_1(t)|_0^2 \leq (1 + \frac{4F}{C_0^2})M_1^2 + \frac{K}{\nu_1} \triangleq \rho_1^2.
\]

This indicates that the semigroup \( \{S(t)\}_{t \geq 0} \) has a bounded absorbing set \( B(0; \rho_1) \), the ball centered at 0 of radius \( \rho_1 \) in \( \dot{H}_p^1(\Omega) \).

The proof is complete. \( \square \)

**Theorem 2.2.** Let \( G = (g_1, g_2) \in \dot{H}_p^0(\Omega) \). Assume that \( \int_D \varphi_1 = \int_D \varphi_2 = 0 \) and that \( |U_2 - U_1| < \frac{C_1 C_2^2 \mu}{4F} \), where \( C_1, C_2 \) are the constants in Poincaré inequalities. Then the semigroup \( \{S(t)\}_{t \geq 0} \) associated with the solutions of (1.1)-(1.4) has a bounded absorbing set in \( \dot{H}_p^2(\Omega) \).

**Proof of Theorem 2.2.** Multiply (1.1) and (1.2) by \( -\Delta \varphi_1 \) and \( -\Delta \varphi_2 \) respectively and integrate on \( \mathcal{D} \), we have
\[
\frac{1}{2} \frac{d}{dt} |\varphi_1|^2 + \frac{1}{2} F \frac{d}{dt} \int_D (\varphi_2 - \varphi_1) \Delta \varphi_1 + \mu |\varphi_1|^2 \leq \int_D \Delta (\varphi_2 - \varphi_1)(-\Delta \varphi_1)
\]
\[
= -U_1F \int_D \frac{\partial \varphi_2}{\partial x} \Delta \varphi_1 + F \int_D J(\varphi_1, \varphi_2) \Delta \varphi_1 + \int_D g_1(-\Delta \varphi_1),
\]
\[
\int_D \frac{d}{dt} |\varphi_2|^2 + \frac{1}{2} F \int_D (\varphi_2 - \varphi_1) \Delta \varphi_2 + \mu |\varphi_2|^2 \leq \int_D \Delta (\varphi_1 - \varphi_2)(-\Delta \varphi_2)
\]
\[
= -U_2F \int_D \frac{\partial \varphi_1}{\partial x} \Delta \varphi_2 + F \int_D J(\varphi_2, \varphi_1) \Delta \varphi_2 + \int_D g_2(-\Delta \varphi_2).
\]
From the above two equalities we get
\[
\frac{1}{2} \frac{d}{dt} (|u_2|^2 + F|\varphi_2 - \varphi_1|^2) + \mu |u_2|^2 \leq 2\mu C_2^2 F|\varphi_2 - \varphi_1|^2
\]
\[
= (U_2 - U_1)F \int_D \nabla \varphi_1 \cdot \nabla \varphi_2 + F \int_D J(\varphi_1, \varphi_2) \Delta (\varphi_2 - \varphi_1) + \int_D g_1(-\Delta \varphi_1) + \int_D g_2(-\Delta \varphi_2)
\]
\[
\leq F|U_2 - U_1||u_1|^2|u_1|_1 + 4F|u_1|^2|u_2|_2 + 2|g_1|u_2|_2.
\]

For any bounded set \( B_2 \subseteq \dot{H}_p^2(\Omega) \), \( \exists M_2 > 0 \) such that \( \forall u_0 = u(0) \in B_2, |u_0|_2 \leq M_2 \). Since the upper bounded of \( |u_0|_1 \) in the proof of Theorem 2.1 can be chosen large enough, we pick one such that 
\[
|u_0|_1 \leq \frac{1}{C_1} |u_0|_2 \leq \frac{M_2}{C_1} \leq M_1,
\]
so that there is a corresponding \( \rho_1, |u_1|_1 \leq \rho_1 \), also use Poincaré inequalities and the assumption of the theorem, we then have by Young inequality,
\[
\frac{d}{dt} (|u_2|^2 + F|\varphi_2 - \varphi_1|^2) + \mu C_2^2 |u_2|^2 + 2\mu C_2^2 F|\varphi_2 - \varphi_1|^2 \leq \frac{F|U_2 - U_1||u_1|^2 + C_1(2F|u_1|^2 + |g_1|_1^2)}{F|U_2 - U_1|}
\]
\[
\leq \frac{\mu}{2} C_2^2 |u_2|^2 + \frac{C_1(2F\rho_1^2 + |g_1|_1^2)}{F|U_2 - U_1|}
\]
Let \( \nu_2 = \min \{\mu C_2^2, 2\mu C_2^2\} \) and \( \bar{K} \triangleq \frac{C_1(2F\rho_1^2 + |g_1|_1^2)}{F|U_2 - U_1|} \), we arrive
\[
\frac{d}{dt} (|u_2|^2 + F|\varphi_2 - \varphi_1|^2) + \nu_2 (|u_2|^2 + F|\varphi_2 - \varphi_1|^2) \leq \bar{K}.
\]
By Gronwall inequality, we get for any $t \geq 0$,
\[
|u(t)|^2 + F|\varphi_2(t) - \varphi_1(t)|^2 \\
\leq (|u(0)|^2 + F|\varphi_2(0) - \varphi_1(0)|^2)e^{-\nu_2 t} + \tilde{K} \\
\leq (M_2^2 + 4FM_1^2)e^{-\nu_2 t} + \tilde{K} \\
\leq (C_1^2 + 4F)M_1^2e^{-\nu_2 t} + \tilde{K},
\]
and
\[
|u(t)|^2 + F|\varphi_2(t) - \varphi_1(t)|^2 \leq 1 + \tilde{K} \triangleq \rho_2^2,
\]
provided $t \geq t_2 = \frac{1}{\nu_2} \ln(C_1^2 + 4F)M_1^2$.

Thus, the ball of $B(0; \rho_2)$, centered at 0 of radius $\rho_2$ in $\dot{H}_p^2(\Omega)$ is an absorbing set of $\{S(t)\}_{t \geq 0}$ in $\dot{H}_p^2(\Omega)$. □

3. Existence of global attractors in $\dot{H}_p^1(\Omega)$ and $\dot{H}_p^2(\Omega)$.

**Theorem 3.1.** Let $G = (g_1, g_2)^t \in \dot{H}_p^0(\Omega)$. Assume that $\int_D \varphi_1 = \int_D \varphi_2 = 0$ and that $|U_2 - U_1| < \min\left(\frac{C_0 C_1^2 \mu}{4\nu}, \frac{C_1 C_2^2 \mu}{4\nu}\right)$, where $C_0$, $C_1$, $C_2$ are the constants in Poincaré inequalities. Then the semigroup $\{S(t)\}_{t \geq 0}$ associate with the solutions of (1.1)-(1.4) has a global attractor in $\dot{H}_p^1(\Omega)$ and $\dot{H}_p^2(\Omega)$ respectively.

This theorem is proved using the following abstract existence theorem of global attractors.

**Theorem 3.2** ([8]). Assume that $X$ is a metric space and $\{S(t)\}_{t \geq 0}$ are $C^0$ semigroup. We also assume that the following two are satisfied:

1. $\{S(t)\}_{t \geq 0}$ has a bounded absorbing set,
2. $\{S(t)\}_{t \geq 0}$ are uniformly compact for $t$ large, i.e. for every bounded set $\mathcal{B}$ there exists some $t_0$ which may depend on $\mathcal{B}$, such that $\bigcup_{t \geq t_0} S(t)\mathcal{B}$ is relatively compact in $X$.

Then there exists a global attractor for $\{S(t)\}_{t \geq 0}$ in $X$.

**Proof of Theorem 3.1.** By Theorem 2.2 and the compact imbedding from $\dot{H}_p^2(\Omega)$ to $\dot{H}_p^1(\Omega)$, we get that $\{S(t)\}_{t \geq 0}$ are uniformly compact in $\dot{H}_p^1(\Omega)$ for $t$ large. Combine the result of Theorem 2.1, we obtain the existence of a global attractor in $\dot{H}_p^1(\Omega)$ by Theorem 3.2.

We now go to verify the uniform compactness of the semigroup $\{S(t)\}_{t \geq 0}$ in $\dot{H}_p^2(\Omega)$, i.e. for every bounded set $\mathcal{B}$ there exists some $t_0$ which may depend on $\mathcal{B}$, such that $\bigcup_{t \geq t_0} S(t)\mathcal{B}$ is relatively compact in $\dot{H}_p^2(\Omega)$. Recall that $S(t)u_0 = u(t)$ and notice the fact that the imbedding from $\dot{H}_p^3(\Omega)$ into $\dot{H}_p^2(\Omega)$ is compact, we verify by showing that $u(t)$ is bounded in $\dot{H}_p^3(\Omega)$ for $t$ large.
Multiply (1.1) and (1.2) by $\Delta^2 \varphi_1$ and $\Delta^2 \varphi_2$ respectively and integrate on $D$, we have
\[
\frac{1}{2} \frac{d}{dt} |\varphi_1|^2 + \frac{1}{2} F \frac{d}{dt} \int_D \Delta (\varphi_2 - \varphi_1) \Delta \varphi_1 + \mu |\varphi_1|^2 + \mu F \int_D \Delta (\varphi_2 - \varphi_1) \Delta^2 \varphi_1
\]
\[
= U_1 F \int_D \frac{\partial \Delta \varphi_2}{\partial x} \Delta \varphi_1 + \int_D J(\varphi_1, \Delta \varphi_1) \Delta^2 \varphi_1 + F \int_D J(\varphi_1, \varphi_2) \Delta^2 \varphi_1 + \int_D g_1 \Delta^2 \varphi_1,
\]
\[
\frac{1}{2} \frac{d}{dt} |\varphi_2|^2 + \frac{1}{2} F \frac{d}{dt} \int_D \Delta (\varphi_1 - \varphi_2) \Delta \varphi_2 + \mu |\varphi_2|^2 + \mu F \int_D \Delta (\varphi_1 - \varphi_2) \Delta^2 \varphi_2
\]
\[
= U_2 F \int_D \frac{\partial \Delta \varphi_1}{\partial x} \Delta \varphi_2 + \int_D J(\varphi_2, \Delta \varphi_2) \Delta^2 \varphi_2 + F \int_D J(\varphi_2, \varphi_1) \Delta^2 \varphi_2 + \int_D g_2 \Delta^2 \varphi_2,
\]
\[
From the above two equalities we have
\[
\frac{1}{2} \frac{d}{dt} (|u_3|^2 + F |\varphi_2 - \varphi_1|^2) + \mu |u_4|^2 + \mu F |\varphi_2 - \varphi_1|^2
\]
\[
= F(U_2 - U_1) \int_D \frac{\partial \Delta \varphi_2}{\partial x} \Delta \varphi_2 + \int_D J(\varphi_1, \Delta \varphi_1) \Delta^2 \varphi_1 + \int_D J(\varphi_2, \Delta \varphi_2) \Delta^2 \varphi_2
\]
\[
+ F \int_D J(\varphi_2, \varphi_1) \Delta^2 (\varphi_2 - \varphi_1) + \int_D g_1 \Delta^2 \varphi_1 + \int_D g_2 \Delta^2 \varphi_2,
\]
\[
\leq F|U_2 - U_1||u_3|^2|u_4|^2 + 4 |u_1|^2 |u_4|^2 + \mu |u_1|^2 |u_3|^2 + \mu F |\varphi_2 - \varphi_1|^2 + \frac{F}{\mu} |u_3|^2 + \frac{\mu}{2} |u_4|^2 + \frac{|g_0|^2}{\mu},
\]
and
\[
\frac{d}{dt} (|u_3|^2 + F |\varphi_2 - \varphi_1|^2)
\]
\[
\leq \left( \frac{2F|U_2 - U_1|}{C_2} + \frac{16}{\mu} |u_3|^2 |u_4|^2 + \frac{2F}{\mu} |u_3|^2 + \frac{2|g_0|^2}{\mu} \right),
\]

Thanks to the existence of absorbing sets we have proved before, for $t \geq t_2$, we have
\[
\frac{d}{dt} (|u_3|^2 + F |\varphi_2 - \varphi_1|^2) \leq \left( \frac{2F|U_2 - U_1|}{C_2} + \frac{16}{\mu} \rho_1^2 |u_3|^2 + \frac{2F}{\mu} \rho_1^2 + \frac{2|g_0|^2}{\mu} \right).
\]

Notice that $|u_3|^2 + F |\varphi_2 - \varphi_1|^2$, $\left( \frac{2F|U_2 - U_1|}{C_2} + \frac{16}{\mu} \rho_1^2 \right)$, $\frac{2F}{\mu} \rho_1^2 + \frac{2|g_0|^2}{\mu}$ are all nonnegative functions, and for $t \geq t_2$,
\[
\int_t^{t+1} \left( \frac{2F|U_2 - U_1|}{C_2} + \frac{16}{\mu} \rho_1^2 \right) ds = \left( \frac{2F|U_2 - U_1|}{C_2} + \frac{16}{\mu} \rho_1^2 \right) := a_1,
\]
\[
\int_t^{t+1} \left( \frac{2F}{\mu} \rho_2^2 + \frac{2|g_0|^2}{\mu} \right) ds = \left( \frac{2F\rho_2^2 + |g_0|^2}{\mu} \right) := a_2,
\]
To estimate $\int_t^{t+1} (|\varphi_1|^2 + F |\varphi_2 - \varphi_1|^2) ds$, we refer to (2.3), (2.4) and (2.5) can obtain
\[
\frac{d}{dt} (|u_3|^2 + F |\varphi_2 - \varphi_1|^2) + 2 \mu (|u_3|^2 + F |\varphi_2 - \varphi_1|^2)
\]
\[
\leq 2(|U_2 - U_1| \rho_1 \rho_2 + 4 \rho_1^2 \rho_2 + |g_0| \rho_2),
\]
integrate this inequality from $t$ to $t + 1$ we get
\[
|u(t + 1)|_2^2 + F|\varphi_2(t + 1) - \varphi_1(t + 1)|_2^2 - (|u(t)|_2^2 + F|\varphi_2(t) - \varphi_1(t)|_2^2) + 2\mu \int_t^{t+1} (|u(s)|_2^2 + F|\varphi_2(s) - \varphi_1(s)|_2^2)ds \leq 2(|U_2 - U_1|_2 \rho_1 \rho_2 + 4\rho_1^2 \rho_2 + \rho_2^2 + 2F\rho_1^2 + |g|_2\rho_2),
\]
and then
\[
\int_t^{t+1} (|u(s)|_2^2 + F|\varphi_2(s) - \varphi_1(s)|_2^2)ds \leq \frac{1}{\mu}(|U_2 - U_1|_2 \rho_1 \rho_2 + 4\rho_1^2 \rho_2 + \rho_2^2 + 2F\rho_1^2 + |g|_2\rho_2) := a_3.
\]

Thus by the uniform Gronwall inequality, when $t \geq t_2$,
\[
|u(t + 1)|_2^2 + F|\varphi_2(t + 1) - \varphi_1(t + 1)|_2^2 \leq (a_3 + a_2)e^{a_1}.
\]
Set $t_0 = t_2 + 1$, we then have
\[
|u(t)|_2^2 + F|\varphi_2(t) - \varphi_1(t)|_2^2 \leq ((a_3 + a_2)e^{a_1},
\]
provided $t \geq t_0$.

We have proved that \{S(t)\}_{t \geq 0} is uniformly compact in $\dot{H}_p^2(\Omega)$ for $t$ large. The conditions of Theorem 2.2 are satisfied. So \{S(t)\}_{t \geq 0} has a global attractor in $\dot{H}_p^2(\Omega)$. \hfill $\square$

4. **Genericity of Steady-State solutions.** In this section, we consider the steady state problem of (1.1)–(1.4) as follows:
\[
\mu \Delta [\Delta \varphi_1 + F(\varphi_2 - \varphi_1)] = J(\Delta \varphi_1 + F(\varphi_2 - \varphi_1), U_1y) + J(\varphi_1, \Delta \varphi_1 + F(\varphi_2 - \varphi_1) + \beta y) + g_1, \quad (4.1)
\]
\[
\mu \Delta [\Delta \varphi_2 + F(\varphi_1 - \varphi_2)] = J(\Delta \varphi_2 + F(\varphi_1 - \varphi_2), U_2y) + J(\varphi_2, \Delta \varphi_2 + F(\varphi_1 - \varphi_2) + \beta y) + g_2, \quad (4.2)
\]
where $g_1$ and $g_2$ are external forcing.

The main theorem in this section is

**Theorem 4.1.** For any $\mu > 0$, there is an open and dense set $G \subset \dot{H}_p^0(\Omega)$ such that for each $G = (g_1, g_2) \in G$, the set $S(G, \mu) \subset \dot{H}_p^2(\Omega)$ of the steady state problem (4.1) and (4.2) with (1.4) is non-empty and finite.

The proof of Theorem 4.1 relies on the Sard’s theorem on Banach spaces due to S. Smale, which we recall as follows.

Let $E_1$ and $E_2$ be two Banach spaces. A map $G : E_1 \to E_2$ is called a completely continuous field if $G = L + H$, where $L : E_1 \to E_2$ is a linear isomorphism and $H : E_1 \to E_2$ is a compact operator. We note that a $C^1$ completely continuous field $G : E_1 \to E_2$ must be a Fredholm map of index zero.

Let $G : E_1 \to E_2$ be a $C^1$ completely continuous field. A point $u \in E_1$ is called a regular point of $G$ if $G'(u) : E_1 \to E_2$ is an isomorphism, and is singular point if it is not a regular point. The image of a singular point under $G$ is called a singular value of $G$, and the complement of all singular points are called regular values of $G$. Notice that if $f \in E_2$ is not in the image $G(E_1)$, then $f$ is automatically a regular value of $G$. 

Theorem 4.2. (Smale [7]). Let $E_1$ and $E_2$ be two Banach spaces and $G : E_1 \to E_2$ be a $C^1$ completely continuous field. Then the set of all regular values of $G$ is dense in $E_2$. Moreover, if $f \in E_2$ is a regular value of $G$, then $G^{-1}(f)$ is discrete.

Proof of Theorem 4.1. We proceed in two steps as follows.

Step 1. Let $A = L + H$ be defined by
\[
Lu = \left( \begin{array}{c}
\mu \Delta^2 \varphi_1 \\
\mu \Delta^2 \varphi_2
\end{array} \right) \\
Hu = \left( \begin{array}{c}
\mu F \Delta (\varphi_1 - \varphi_2) - J(\Delta \varphi_1 + F(\varphi_2 - \varphi_1), U_{1y}) - J(\varphi_1, \Delta \varphi_1 + F(\varphi_2 - \varphi_1) + \beta y) \\
\mu F \Delta (\varphi_1 - \varphi_2) - J(\Delta \varphi_2 + F(\varphi_1 - \varphi_2), U_{2y}) - J(\varphi_1, \Delta \varphi_2 + F(\varphi_1 - \varphi_2) + \beta y)
\end{array} \right)
\]

Then the steady state equations (4.1) and (4.2) are given by
\[
Lu + Hu = G
\]  
(4.3)

where $G = (g_1, g_2)^t$.

Observe that $L : \dot{H}^4_p(\Omega) \to \dot{H}^0_p(\Omega)$ is an isomorphism. Note that $\dot{H}^0_p(\Omega)$ contains $L^2(\Omega)$ functions with $\int_\Omega u = 0$.

Lemma 4.3. The operator $A : \dot{H}^4_p(\Omega) \to \dot{H}^0_p(\Omega)$ is a completely continuous field. Moreover, $A$ is a surjective map, i.e. $A : \dot{H}^4_p(\Omega) = \dot{H}^0_p(\Omega)$.

Proof. First we know that $L$ is an isomorphism. Second, it is clear that $H$ is compact nonlinear operator from $\dot{H}^4_p(\Omega)$ to $\dot{H}^0_p(\Omega)$. Therefore $A$ is a completely continuous field. The surjectivity of $A$ follows from the existence and regularity of steady-state solutions to (4.1), (4.2) and (1.4).

Step 2. The theorem follows from the following lemma, which is similar to Theorem 4.1.11 in [4].

Lemma 4.4. i) The set of all regular values of $A$ is open and dense in $\dot{H}^0_p(\Omega)$.

ii) If $G \in \dot{H}^0_p(\Omega)$ is a regular value of $A$, then $A^{-1}$ is nonempty and finite.

Proof. By the Sard theorem, Theorem 4.2, and, by Lemma 4.3, the set of all regular values of $A$, denoted by $\mathcal{R} \subset \dot{H}^0_p(\Omega)$, is dense in $\dot{H}^0_p(\Omega)$. Since $A$ is surjective, for any $G \in \mathcal{R}$, $A^{-1}$ is nonempty and finite.

Now, for $G \in \mathcal{R}$, let $A^{-1}(G) = \{u_1, \cdots, u_n\}$. By the implicit function theorem, for each $u_i \in A^{-1}(G), i = 1, \cdots, n$, there exists a neighborhood $U_i \subset \dot{H}^4_p(\Omega)$ of $u_i$ and a neighborhood of $\mathcal{F}_i \subset \dot{H}^0_p(\Omega)$ of $G$, such that
\[
A : U_i \to \mathcal{F}_i
\]
is a homeomorphism.

Therefore, there exists an open set $\mathcal{F} \subset \mathcal{F}_1 \cap \cdots \cap \mathcal{F}_n$ with $G \in \mathcal{F}$ such that
\[
A^{-1}(\mathcal{F}) = V_1 + \cdots + V_n,
\]
\[
V_i \cap V_j = \emptyset, i \neq j,
\]
\[
u_i \in V_i, i = 1, \cdots, n.
\]

For each $u \in V_1 + \cdots + V_n$, $A'(u)$ is an isomorphism. Hence $\mathcal{F} \subset \mathcal{R}$. Therefore $\mathcal{R}$ is open.

Then we have the conclusion of Theorem 4.1. The proof is completed.

\[\square\]
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