Risk Premium Impact in the Perturbative Black Scholes Model

Luca Regis* and Simone Scotti†

Abstract

We study the risk premium impact in the Perturbative Black Scholes model. The Perturbative Black Scholes model, developed by Scotti, is a subjective volatility model based on the classical Black Scholes one, where the volatility used by the trader is an estimation of the market one and contains measurement errors. In this article we analyze the correction to the pricing formulas due to the presence of an underlying drift different from the risk free return. We prove that, under some hypothesis on the parameters, if the asset price is a sub-martingale under historical probability, then the implied volatility presents a skewed structure, and the position of the minimum depends on the risk premium $\lambda$.

Key Words: Dynamic Hedging, Risk Premium, Error Theory using Dirichlet Forms, Bias

1 Introduction

It is now common knowledge that the Black and Scholes model, which worked well before the 1987 crash, is nowadays unable to price options correctly. As can be deduced by comparing the two papers by Rubinstein [23] and Jackwerth and Rubinstein [18], something has evidently changed in the market for options after that event. The most shared explanation for the failure in which BS model incurs today is usually thought to reside in the fact that the constant volatility parameter it proposes is not a good representation of reality anymore. Empirical evidence shows that the underlying stock volatility, for example, is not time invariant during the life of an option.

Moreover, while before 1987 the lognormal distribution of stock prices implied by the Black and Scholes model seemed to be a good approximation of the real one and volatility observed across strike prices had a moderately pronounced smile, from that date onwards the implied volatility curve appears to be steeper and generally skewed to the left. Jackwerth and Rubinstein [18], recovering stock price distribution from observed prices, empirically find a “fatter” left tail phenomenon. Constantinides et al. [8], in the context of an equilibrium model, find stochastic dominance violations on both tails of the implied volatility curve. Christensens and Prabahla [7], for example, suggest that a regime switch has occurred after the crash.

The most natural solution to the problem of pricing options more correctly seems then to let volatility change with time. Many pricing models have then been proposed, with different formulations for the stochastic process driving volatility, e.g. local volatility models, see Dupire [14], or stochastic volatility models. Then, an option pricing model is characterized by a system of differential equations, since two different processes are specified, one for the stock price and one for its underlying volatility.

The first authors to solve the problem of pricing options with stochastic volatility were Hull and White [17]. However, they were able to obtain closed form solutions only for the case of uncorrelated volatilities and stock prices, while Heston [16], using a new technique, managed to find exact prices also for the correlated case. In this stream of literature, one of the most used model in practice is probably the SABR one, introduced in Hagan et al. [14], which provides excellent fitting for interest rates derivatives. More sophisticated models include for example the possibility of jumps in the stock price evolution, e.g. see Brigo et al. [6], or directly in the process for volatility, see Eraker et al. [10].

*University of Torino
address: Via Real Collegio 30, 10024 Moncalieri (TO) Italy
email: luca.regis@unito.it

†University of Torino and Ecole Nationale des Ponts et Chaussées - CERMICS
address: Via Real Collegio 30, 10024 Moncalieri (TO) Italy
email: simone.scotti@unito.it
However, stochastic volatility models like the ones described above are usually complex and characterized by a large number of parameters and, unless in special cases (SABR model, Heston [10]), they do not provide closed form solutions for vanilla options prices.

The PBS model (Scotti [24]) introduced a new category of stochastic volatility pricing models, being founded on the notion of subjective volatility. Using error theory through Dirichlet forms, the PBS model generalizes the standard Black and Scholes one, imposing an error structure on volatility. Thus, rather than specifying a possible pattern of evolution for volatility through time, the perturbative approach deals with the concept of measurement errors present in the estimation procedure performed by the trader.

One of the most important issues of this model lies in the possibility of obtaining closed forms solutions for European vanilla option prices and for each kind of derivative which has a closed form solution using the classical Black and Scholes model such as Asian and barrier options. This important framework, joint with the flexibility of the model, permits us to calibrate it to different markets and fit them, even if they imply opposite behaviours of the implied volatility curve. PBS can reproduce a right-tailed or a left-tailed skewness effect, as well as sharper or flatter slopes, obviously depending on the calibration of parameters. In Scotti [24], sufficient conditions for the presence of a smile are derived in the case with no drift term in the stock price dynamics.

This article studies a natural implication of the PBS model: the dependence of option prices on risk premia. Dependence of option prices on the expected excess return on the stock is ruled out in the classical Black and Scholes model, as a result of the lognormality assumption. Lo and Wang [20], starting from the evidence that the predictability naturally captured by the expected return on stocks is affects option prices, assume an Orstein-Uhlenback process for stock prices and show that the impact of a drift term is not negligible anymore.

In the PBS model we maintain the assumption of lognormality of stock prices. However, we show that the presence of measurement errors in the estimation of parameters induces market incompleteness and lets the hedging position of a trader not invariant to the expected excess return on stock. We analyze the implications of this fact and we show that taking into account the impact of a risk premium on stocks it is possible to reproduce the most commonly observed behaviours of the market for options in terms of implied volatility. The PBS model is able to generate the usual smile and skew effects pointed out by the empirical literature we addressed above.

Section 2 briefly summarizes the basic concepts of error theory using Dirichlet forms, section 3 recalls the most important features of the PBS model, section 4 studies the impact of the drift term on the general profit and loss function and in the particular case of the price of a European call option, section 5 analyzes the sensitivity of the volatility implied by the model to some parameters and above all to risk premium. Finally, the appendix provides technical computations.

2 Preliminaries

This section resumes the notations used throughout the paper and briefly surveys the key notions of error theory using Dirichlet forms.

2.1 Notation

We use the following notation:

- \((\Omega, \mathcal{F}, \mathbb{P})\) is the historical probability space, sometimes simply denoted with \(\Omega\).
- \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\) is a filtration defined on the probability space.
- \(\{W_t\}_{0 \leq t \leq T}\) is the associated Brownian motion, i.e. a Brownian motion adapted to the filtration \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\).
- \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) is another probability space, used to represent the uncertainty on the volatility parameter, denoted simply with \(\tilde{\Omega}\).
- \(\mathbb{E}[\cdot] \) and \(\mathbb{E}[\cdot|\mathcal{F}_t]\) denote, respectively, the expectation and the conditional expectation under the probability measure \(\mathbb{P}\), while \(\tilde{\mathbb{E}}[\cdot]\) denotes the expectation under \(\tilde{\mathbb{P}}\).

For a complete and exhaustive treatment of error theory and Dirichlet forms, see Bouleau [2], [3], [5] and Fukushima [13].
• \((P_t)_{t \geq 0}\) denotes a strongly continuous contraction semi-group, \(A\) its generator, with domain \(D_A\), and \(\Gamma\) the "carré du champ" operator associated with the Dirichlet form of the semi-group, with domain \(\mathcal{D}\).

• \(T\) denotes the maturity, a fixed positive number.

• we suppose that there are two traded assets in the market: a stock, with price \(S_t\) at time \(t\), and a risk free money account. For the sake of simplicity we assume that the risk free interest rate is zero.

• \(\sigma_0\) is the stock market volatility, supposed to be constant, while \(\sigma\) is the volatility estimated by the trader which we assume to be stochastic.

• \(\mu\) is the the expected rate of return on the stock, \(\lambda\) represents the risk premium and \(L\) the cumulated risk premium, i.e. \(L = \lambda \sqrt{T}\).

2.2 Error Theory using Dirichlet Forms

Asset pricing models are characterized by a set of parameters that permits to reproduce the real world features with some degree of freedom. These parameters have to be estimated in order to be used in practice. The classical way to find these estimations is to use market data to make the model fit the prices observed on the market. Hence, the final result of this procedure is that we obtain a mean value for each parameter along with some uncertainty due to estimation errors.

What is the impact of such an uncertainty? Generally, models’ users consider the mean values of the estimations of their parameters and implicitly consider them as deterministic, since their variance is generally small with respect to their mean. However, when non linear functions are then used in pricing procedures, the impact of this uncertainty, even if small, could be important, since the moments of random variables are distorted. What is the amplitude of the bias induced by forgetting the probabilistic nature of an estimated parameter?

Giving an answer to this question is not an easy task. If we consider a non linear function \(F\) and a random variable \(\sigma\) it is well know by Jensen’s inequality that the expected value of \(F(\sigma)\) is different from the function \(F\) evaluated at the expected value of \(\sigma\). A straightforward computation of this difference, however, is usually impossible, since the functional forms involved are often difficult to treat. This is the reason why the probabilistic uncertainty on the estimation of parameters, whose presence was already known by Gauss, is frequently neglected.

If a direct approach to solve this problem is unsuccessful, we can try to find a way to reach a solution addressing the question from another perspective. We can use the fact that the variance of an estimated parameter is very small compared to its mean in order to justify a Taylor expansion; studying the bias and the variance of \(F(\sigma)\) we find:

\[
\begin{align*}
\hat{E}[F(\sigma) - F(\sigma_0)] &= \epsilon \left\{ F'(\sigma_0) \text{ Bias } [\sigma] + \frac{1}{2} F''(\sigma_0) \text{ Variance } [\sigma] \right\} + o(\epsilon) \\
\hat{E}[\left( F(\sigma) - F(\sigma_0) \right)^2] &= \epsilon \left( F'(\sigma_0) \right)^2 \text{ Variance } [\sigma] + o(\epsilon)
\end{align*}
\]

where \(\sigma_0\) is the estimated value of the random variable \(\sigma\), \(\epsilon \text{ Bias } [\sigma]\) is its bias, i.e. the difference between \(\sigma_0\) and \(\hat{E}[\sigma]\), and \(\epsilon \text{ Variance } [\sigma]\) is its variance.

Remark 2.1 The first relation in (2.1) summarizes the key difference between a deterministic uncertainty and a probabilistic one, since the mean of a function evaluated on a random variable is distorted if it is non linear.

The starting point of error theory using Dirichlet forms consists in considering a very small \(\epsilon\) and to stop the Taylor expansion at the first order. The theory involves thus the use of two operators, a bias operator \(A\) and a variance-covariance one \(\Gamma\), linked together by the chain rule in equation (2.1). Such an environment can be studied through semi groups theory. The operator \(A\) is the generator of the semi-group and \(\Gamma\) is the "carré du champ" associated with the Dirichlet form of the semi-group.

The most important framework of error theory is the error structure. We recall its definition:

Definition 2.1 (Error structure) An error structure is a term

\[
\left( \Omega, \mathcal{F}, \mathbb{P}, \mathcal{D}, \Gamma \right)
\]

where
1. \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})\) is a probability space;

2. \(\mathbb{D}\) is a dense sub-vector space of \(L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})\);

3. \(\Gamma\) is a positive symmetric bilinear application from \(\mathbb{D} \times \mathbb{D}\) into \(L^1(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})\) satisfying the functional calculus requirements on the class \(C^1 \cap \text{Lip}\). This means that, if \(F\) and \(G\) are two functions of class \(C^1\) and Lipschitzian, \(u\) and \(v\) \(\in \mathbb{D}\), then \(F(u)\) and \(G(v)\) \(\in \mathbb{D}\) and

\[
\Gamma [F(u), G(v)] = F'(u)G'(v) \Gamma [u, v] \text{ a.s.;}
\]

4. the bilinear form \(E[u, v] = \frac{1}{2} \mathbb{E} [\Gamma[u, v]]\) is closed;

5. The constant function 1 belongs to \(\mathbb{D}\), i.e. the error structure is Markovian.

Hypotheses 2, 3 and 4 together ensure that \(\mathcal{E}\) is a Dirichlet form, with \(\Gamma\) as carré du champ operator.

We use the simplified notation \(\Gamma[u] = \Gamma[u, u]\) to indicate that the operator \(\Gamma\) is applied twice on the same argument. The couple \((\Gamma, \tilde{P})\) defines a unique semi group \((P_t)_{t \geq 0}\) and its generator \(A\) thanks to the Hille-Yosida theorem. Hypothesis 5 provides that the semi-group \((P_t)_{t \geq 0}\) is Markovian. Therefore, we defined two operators \(\Gamma\) and \(A\) that satisfy the chain rule (2.1).

It is now useful to conclude this section providing an example of an error structure:

**Example 2.1 (Orstein-Uhlenbeck structure)**

\[
\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \mathbb{D}, \Gamma\right) = \left(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu, H^1(\mu), \Gamma[u, u] = \{u'\}^2\right)
\]

where \(\mathcal{B}(\mathbb{R})\) is the Borel \(\sigma\)-field of \(\mathbb{R}\), \(\mu\) is a gaussian measure and \(H^1(\mu)\) is the first Sobolev space with respect to the measure \(\mu\), i.e. \(u \in H^1(\mu)\) if \(u \in L^2(\mu)\) and \(u'\) belongs to \(L^2(\mu)\) in distribution sense.

The associated generator has the following domain:

\[
\mathcal{D}[A] = \{u \in L^2(\mu) : u'' - x f' \text{ belongs to } L^2(\mu)\} \text{ in distribution sense}
\]

and the generator operator is

\[
A[u] = \frac{1}{2} u'' - \frac{1}{2} I \cdot u'
\]

where \(I\) is the identity map on \(\mathbb{R}\).

This example gives the basic idea of an error structure on a parameter. Moreover, as shown in Hamza [15], it is important since every Dirichlet form on \(\mathbb{R}\) has a characterization.

### 3 Perturbative Black Scholes model

In this section, we recall the key features of the Perturbative Black Scholes model introduced by Scotti [24].

The PBS model is based on the classical Black Scholes model, (Black and Scholes [1]). If we assume that the interest rate is worth zero or, from an economic point of view, that all assets are priced in terms of the money market, then the underlying stock price follows the SDE

\[
(3.1) \quad dS_t = \mu S_t \, dt + \sigma_0 S_t \, dW_t
\]

where \(\mu\) is the return on the stock, \(\sigma_0\) is the volatility and \(W_t\) is a Brownian motion.

2 see Fukushima [13] for a complete proof
In the BS model pricing formulas depend on the diffusion term only and not on \( \mu \); we find closed forms expressions for the premium and the greeks of vanilla options. In contrast with its simplicity, unluckily the BS model cannot reproduce the so called smile effect: the volatility implied by the BS model is constant across strike prices, while the observed one is usually u-shaped.

We introduce the notation of risk premium \( \lambda \) as the ratio between the expected return on the stock and market volatility.

\[
\lambda = \frac{\mu}{\sigma_0}
\]

The PBS model lies on three main hypotheses:

1. the stock price follows a geometrical brownian motion with fixed and non perturbed volatility \( \sigma_0 \);
2. the trader has to estimate the volatility parameter, and the value of his own estimation contains intrinsic inaccuracies. The model reproduces this fact through an error structure; nonetheless we assume that the stock price \( S_t \) is not erroneous. We evaluate the impact of the perturbation generated by those measurement errors on the profit and loss process used by trader to hedge a position on a vanilla option;
3. the trader knows the existence of the perturbation described above and wants to modify his own offered prices in order to take into account the bias present on volatility and, as a consequence, on the hedged position.

Summarizing, all traders use a geometric Brownian motion to model the stock price process and they hold some positions involving vanilla options; they use observed market prices to determine the values of parameters by inversion of pricing formulas. Thus, they find an observed volatility process \( \varsigma_t \), usually known as implied volatility, they take it as a forecast for future volatility and hedge their portfolio accordingly.

Since we made as an assumption that the trader knows the existence of errors in his estimation procedure, volatility is incorporated into the model in two different ways. It has a “market” value, the classical parameter used to set up standard pricing formulas, and a subjective one. The former is denoted with \( \sigma_0 \) and it is supposed to be a constant parameter as in the Black Scholes model. The “subjective” volatility notion comes from the intuition that in the real world, when an operator deals with the problem of option pricing, he does not know the precise value volatility will assume during its life. Hence, he has to estimate it from market observations. The value he gets from this procedure, as pointed out above, will obviously be subject to measurement errors captured in the PBS model by the error structure form. We assume that the stock volatility “market” value is also the mean value of volatility in the erroneous estimation procedure performed by the trader. The volatility estimated by the trader is then a random variable, and is ”subjective”, since it can assume a different value in the expectation of each operator.

The profit and loss process of a trader has a key role in the PBS model. The value of this process at maturity is given by:

\[
P&L = F(\varsigma_0, S_0, 0) + \int_0^T \frac{\partial F}{\partial x}(\varsigma_t, S_t, t)dS_t - \Phi(S_T)
\]

where \( F(\varsigma_0, S_0, 0) \) is the security premium, the integral term represents the hedging strategy, \( \Phi(S_T) \) is the Payoff and \( S_t \) follows Black Scholes SDE (3.1).

4 Impact of the drift term in security pricing

In this section, we study the impact of a non zero drift term in the diffusive process assumed for stock prices on prices determined with the PBS model.

As we have shown previously, the expected profit and loss function from the hedging position is then in turn a random variable, characterized by a bias and a variance term, which make it different from the one implied by the BS model. We make an important remark:

\footnote{Measurement errors arise from the uncertainty expected using the central limit theorem.}
Remark 4.1 (Drift impact) In the PBS model, the profit and loss process defined in equation (3.2) depends crucially on the drift rate \( \mu \), which is the expected excess return on the stock. As a matter of fact, the integral term in (3.2) depends on the diffusive process described in (3.1) where \( \mu \) plays a role.

In the Black Scholes model, instead, the price of an option does not depend on the drift term. In that case, in fact, the \( P\&L \) process is worth zero almost surely; as a consequence, we can change the probability measure without altering the result. If, as in the PBS model, we assume that the volatility \( \varsigma_t \) used by the trader is \( \sigma \) and not \( \sigma_0 \), the profit and loss process is not worth zero a.s.; on the contrary, it becomes a stochastic process characterized by two random sources:

- the Brownian motion which describes the evolution of the stock price and
- the process \( \varsigma_t \), the trader’s volatility, which depends on an independent probability space.

As a consequence, we cannot change the probability measure without changing the value of the profit and loss process at maturity.

We suppose that the trader’s volatility \( \varsigma_t \) is the time independent random variable \( \sigma \) we defined in the previous section. Using the language of Dirichlet forms, we derive the following expansion for the volatility estimation:

\[
\sigma_0 \to \sigma_0 + \epsilon A[\sigma](\sigma_0) + \sqrt{\epsilon} \Gamma[\sigma](\sigma_0) \tilde{N}
\]

where \( \tilde{N} \) is a standard Gaussian random variable. Moreover, we assume that this error structure admits a sharp operator.

We estimate the variance and bias of the error on \( \mathbb{E}[P\&L] \). In the computation we assume that \( \sigma = \sigma_0 \) is the right value of the random variable, in the sense that if \( \varsigma_t = \sigma_0 \), then \( P\&L(\sigma_0) = 0 \) almost surely. Notice that, however, this does not mean that the trader believes the BS model to be correct.

Then we can prove, see Scotti [24], that we have the following bias and variance terms:

\[
\mathcal{A}[\mathbb{E}[P\&L]] = \left\{ \frac{\partial F}{\partial \sigma}(\sigma_0, x, 0) + \mathbb{E} \left[ \int_0^T \frac{\partial^2 F}{\partial \sigma \partial x}(\sigma_0, S_s, s) \, dS_s \right] \right\} A[\sigma](\sigma_0)
\]

\[
+ \frac{1}{2} \left\{ \frac{\partial^2 F}{\partial \sigma^2}(\sigma_0, x, 0) + \mathbb{E} \left[ \int_0^T \frac{\partial^3 F}{\partial \sigma^2 \partial x}(\sigma_0, S_s, s) \, dS_s \right] \right\} \Gamma[\sigma](\sigma_0)
\]

\[
\Gamma[\mathbb{E}[P\&L]] = \left\{ \frac{\partial F}{\partial \sigma}(\sigma_0, x, 0) + \mathbb{E} \left[ \int_0^T \frac{\partial^2 F}{\partial \sigma \partial x}(\sigma_0, S_s, s) \, dS_s \right] \right\}^2 \Gamma[\sigma](\sigma_0)
\]

These values represent the inaccuracies that the trader knows to be present in his estimates.

We can give the following interpretation to this error structure:

- the bias in the \( P\&L \) process represents a deviation in security prices asked by the trader to the buyer.
- the variance of the \( P\&L \) process naturally generates a bid/ask spread on security prices. The width of the bid-ask spread depends both on the traders’ risk aversion and on the perceived uncertainty on volatility.

As a consequence of the presence of the error structure, the price of a security is thus not unique, but it can be represented, at each instant in time, as a distribution, whose characteristics depend on the parameters which characterize the error structure.

Therefore, we have shown that the trader must modify his prices in order to take into account the two previous effects, namely the variance and the bias on his expected profit and loss process. Thus, he fixes a supportable risk probability \( \alpha < 0.5 \) and accepts to buy the option at a certain price

\[\text{4When, as in the classical B-S formulation, prices are unique, risk-neutral arguments can be formulated in order to solve the partial differential equations which rule the pricing of assets. In this sense, at each instant in time, price can be represented through a Dirac distribution.}\]

\[\text{5As pointed out in Scotti (2007), PBS model induces market incompleteness.}\]
(Bid Premium) = (BS Premium) + \epsilon A [E[P\&L]] + \sqrt{\epsilon} \Gamma [E[P\&L]] N_\alpha

where \( N_\alpha \) is the \( \alpha \)-quantile of the reduced normal law. Analogously, the trader accepts to sell the option at the price

(Ask Premium) = (BS Premium) + \epsilon A [E[P\&L]] + \sqrt{\epsilon} \Gamma [E[P\&L]] N_{1-\alpha}

Since \( N_\alpha + N_{1-\alpha} = 0 \); the mid-premium is

\[(4.2)\]  
(Mid Premium) = (BS Premium) + \epsilon A [E[P\&L]]

and the bid-ask spread is

\[(4.3)\]  
Bid-Ask spread = 2\sqrt{\epsilon} \Gamma [E[P\&L]] N_\alpha

4.1 European Call options

We now focus our attention on European call options and we study the bias and its derivatives in order to derive some sufficient conditions for the presence of a smiled behaviour on implied volatility. We know the premium of a call option (see Lamberton et al. [19]) with strike \( K \), spot price \( x \), volatility \( \sigma_0 \) and maturity \( T \), and we know its hedging strategy in the usual Black Scholes setting:

\[ F(\sigma_0, x, 0) = xN(d_1) - KN(d_2) \]

\[ \text{Delta} = \frac{\partial F}{\partial x}(\sigma_0, x, 0) = N(d_1) \]

where

\[ d_1 = \ln \frac{x - \ln K + \frac{\sigma^2}{2}(T - s)}{\sigma_0 \sqrt{T - s}} \]

and

\[ d_2 = d_1 - \sigma_0 \sqrt{T} \]

The following results are classical (see [19]):

\[ \frac{\partial^2 F}{\partial \sigma^2}(\sigma_0, x, 0) = \frac{x \sqrt{T} e^{-\frac{1}{2}d_2^2}}{\sqrt{2\pi}} \]

\[ \frac{\partial^3 F}{\partial \sigma^3}(\sigma_0, x, 0) = \frac{x \sqrt{T} e^{-\frac{1}{2}d_2^2} d_2}{\sigma_0 \sqrt{2\pi}} d_1 \]

\[ \frac{\partial^3 F}{\partial \sigma^2 \partial x}(\sigma_0, s, s) = \frac{d_1(s, s) + d_2(s, s) - d_1(s, s) d_2(s, s)}{\sqrt{2\pi} \sigma^2_0} e^{-\frac{1}{2}d_1^2(s, s)} \]

and we can easily prove that:

\[ \frac{\partial^2 F}{\partial \sigma \partial x}(\sigma_0, s, s) = -\frac{1}{\sqrt{2\pi} \sigma_0} d_2(s, s) e^{-\frac{1}{2}d_1^2(s, s)} \]

\[(4.5)\]  
We apply the Perturbative Black Scholes model to find the corrections it imposes on the expected profit and loss process for a trader who is hedging a short position on a plain vanilla European call option. Then the bias on
the call premium is given by two terms. The first one is the bias when \( \mu = 0 \). This case is accurately studied in [24].

\[
A_{\mu=0}[C]\big|_{\sigma=\sigma_0} = x \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}} \left\{ A\left[ \sigma \sqrt{T} \right] \big|_{\sigma=\sigma_0} + \frac{d_1 d_2}{2\sigma_0 \sqrt{T}} \Gamma \left[ \sigma \sqrt{T} \big|_{\sigma=\sigma_0} \right] \right\}
\]

Now, if we assume that the drift term of the stock price process is non zero, this correction is not sufficient in order to hedge the position correctly. We have to study another term, which is the correction when \( \mu \neq 0 \).

While when \( \mu = 0 \) the stochastic integrals in equation (4.1) are martingales, if \( \mu > 0 \) we have to evaluate their expectations:

\[
A_{\text{correction}}[C]\big|_{\sigma=\sigma_0} = E \left[ \int_0^T \frac{\partial^2 F}{\partial \sigma \partial x}(\sigma_0, S_s, s) \ dS_s \right] A[\sigma](\sigma_0) + \frac{1}{2} E \left[ \int_0^T \frac{\partial^3 F}{\partial \sigma^2 \partial x}(\sigma_0, S_s, s) \ dS_s \right] \Gamma[\sigma](\sigma_0)
\]

In appendix A we compute the two integrals and we find

\[
A_{\text{correction}}[C]\big|_{\sigma=\sigma_0} = K \left\{ \frac{\mathcal{N}(d_2 + \mathcal{L}) - \mathcal{N}(d_2)}{\mathcal{L}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \right\} A\left[ \sigma \sqrt{T} \right] \big|_{\sigma=\sigma_0}
\]

\[
- \frac{K}{\sigma_0 \sqrt{T}} \left\{ \sigma_0 \sqrt{T} \left[ \frac{3}{\mathcal{L}^2} + \left( 1 + \frac{d_2}{\mathcal{L}} \right) \right] - \frac{8}{\mathcal{L}} - \frac{d_2}{\mathcal{L}^2} \right\} \mathcal{N}(d_2 + \mathcal{L}) - \mathcal{N}(d_2)
\]

\[
+ \frac{1}{\sqrt{2\pi}} \left[ d_2^2 + 4 - 2 \frac{d_2}{\mathcal{L}} + 8 \frac{\mathcal{L}}{\mathcal{L}^2} + \sigma_0 \sqrt{T} \left( d_2 - \frac{4}{\mathcal{L}} - \frac{d_2}{\mathcal{L}^2} \right) \right] e^{-\frac{1}{2}d_2^2}
\]

\[
+ \frac{1}{\sqrt{2\pi}} \left[ \sigma_0 \sqrt{T} \left( 1 + \frac{d_2}{\mathcal{L}} \right) - \frac{8}{\mathcal{L}^2} \right] e^{-\frac{1}{2}(d_2 + \mathcal{L})^2} \right\} \Gamma \left[ \sigma \sqrt{T} \big|_{\sigma=\sigma_0} \right]
\]

where \( \mathcal{L} \) is the cumulated risk premium:

\[
\mathcal{L} = \lambda \sqrt{T} = \frac{\mu T}{\sigma_0 \sqrt{T}}
\]

**Remark 4.2** We remark that the first correction (4.6) derives from the bias of the option price. It is then an uncertainty coming from the error on estimating the value of volatility.

The second correction, (4.7) is a consequence of the presence of a bias on the strategy, which introduces uncertainty on the hedging procedure also.

It is easy to compute the value of the variance term of the error structure for a call option:

\[
\Gamma[\text{Call}] = \left\{ \frac{x}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} + K \left[ \frac{\mathcal{N}(d_2 + \mathcal{L}) - \mathcal{N}(d_2)}{\mathcal{L}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \right] \right\}^2 \Gamma[\sigma \sqrt{T}]
\]

We assume that the ask price is then simply the mid price increased by a standard deviation, symmetrically the bid price is the mid price decreased by a standard deviation. The spread is simply given by

\[
\text{Bid-Ask spread}[\text{Call}] = 2 \sqrt{\epsilon \Gamma[\text{Call}]}
\]
5 Numerical Analysis

In this section we explore the sensitivity of the PBS model to some parameters and, through numerical analysis, we give evidence of the fact that the model is able to reproduce all the observed behaviours of the implied volatility curve. The existence of closed form solutions to the pricing formulas allows us to make some comparative static exercises in order to analyze the dependence of the implied volatility curve on time horizon and, above all, on the drift term.

5.1 Parameter sensitivity

First of all, we point out that the corrections obtained with respect to Black and Scholes prices depend on the choice of the parameters $A$ and $\Gamma$ and on the magnitude of the $\epsilon$ term. $A$ captures the bias introduced on the profit and loss function, while $\Gamma$ is a variance term. $\epsilon$ is just a scale factor. It must be small enough to make the higher order expansion terms be negligible. Let us analyze the sensitivity of the volatility implied by the PBS model to the choice of these parameters. Thus, we fix a value for $\sigma$ and the other real world parameters and we let the coefficients of the error structure vary.

Figure 1 shows the effect of an increase in the absolute value of the coefficient of the bias term. As we will explain below, there are good reasons for considering a negative bias. Then, the lower the coefficient, the more the curve shifts downwards and the point of minimum variance to the right. Increasing the value of the coefficient of the variance term, instead, clearly “opens” up the smile, which becomes more pronounced. Moreover, as can be seen in figure 2, a higher coefficient is associated with a more pronounced skew effect which makes the implied volatility higher for out-of-the-money options, compared to in-the-money ones.

A change in epsilon, instead, combines the two effects described above. Notice that this parameter must be small enough to justify expansion in our case. A higher epsilon, thus, produces both a downward shift of the curve and more pronounced smile and skew effects, see figure 3.

From now on, we fix the values of these three parameters to make some comparative static analysis of the parameters that capture the real world features. We set epsilon to 0.02 for convenience.

The coefficient on the variance term is set, by a normalization argument, to

\begin{equation}
\Gamma[\sigma]_{\sigma_0} = \sigma_0^2
\end{equation}

implying that

\begin{equation}
Variance = \epsilon \sigma_0^2
\end{equation}

The bias coefficient is instead set to

\begin{equation}
A[\sigma]_{\sigma_0} = -5\sigma_0
\end{equation}

leading to

\begin{equation}
Bias = -5\epsilon \sigma_0
\end{equation}

This last choice is made in order to reproduce a precautionary effect. The hypothesis we make is that for some reason, the trader believes he overestimated volatility in his procedure. This feeling can be justified by two reasons, one mathematical and one economic.

---

6 Since we have three parameters, a possible way to calibrate the model to the market behaviour is by using instruments which price variance (e.g. variances swaps). Such derivatives permit to find an implicit link between the bias and the variance term; by fixing an epsilon, it is possible then to calibrate the model on just one parameter, see Scotti [25]. Another possibility is instead to fix an arbitrary epsilon small and use the implied spread to calibrate the two coefficients $A$ and $\Gamma$.

7 Obviously, this coefficient can not be negative. As shown in Figure 4, higher values of this coefficient lead to more pronounced smiles.
Figure 1: Implied volatilities curves depending on $A[\sigma]$.

Figure 2: Implied volatilities curves depending on $\Gamma[\sigma]$. 
The mathematical explanation lies in the analysis of the usual formula used to estimate historical volatility under the hypothesis of lognormality of stock prices. Since it is a concave function, it is more likely that the approximated value found by the estimation procedure is an overestimation of the true one.

The economic explanation lies in the way volatility is usually described in models. Markets are opened for 8 hours a day only. However, the flow of information does not stop when markets are closed: variability accumulates even if securities are not traded. Then, in almost every pricing model, volatility is described as a continuous process. This is of course a simplification, but seems nevertheless reasonable. However, it has been shown by some authors, see Stoll and Whaley [26] that overnight volatility is consistently lower than intra day one. Hence, it is straightforward to believe that usual models overestimate volatility.

Let us consider the PBS model prediction on a one-month European call option with the parameters we set above.

First, we keep the risk premium measure fixed and we analyze how implied volatility changes with different maturities. Our finding is that we obtain curves which are flatter as long as the option time horizon becomes longer. This behaviour is consistent with empirical evidence on almost every derivative market, see Hagan et al. [14]. Figure 4 shows that the implied volatility curve is skewed to the left for each option; the point of minimum variability shifts towards higher strikes for longer time horizons.

As we showed in the previous section, the use of our perturbative approach implies that option prices and, thus, implied volatilities are affected by changes in the risk premium. If we let \( \lambda \), the risk premium we defined previously, change and we fix the parameters that characterize the error structure, we can observe and analyze this sensitivity.

Figures 5 and 6 show the behavior of the implied volatility curve on a 1 month European call option for an expected excess return on stock term that ranges from 0 to 0.2. The curve evidently shifts to the right side of the graph as the risk premium term increases. With almost every value up to 0.2, i.e \( \lambda = 1 \), there is a skew effect towards lower maturities. The lower the risk premium, the higher is the value of implied volatility for deep in the money options and the lower for options which are far out of the money. As the value of \( \mu \) increases the curve appears to become steeper on the right side. In particular, for this parameter choice, for a very high risk premium, there is a slight tendency to change the skew direction. Curves cross approximately at the money, around 102.

These findings are consistent with those obtained by other stochastic volatility models, such as the SABR one (Hagan et al. [14]). The authors of that model, fitting it on prices of Eurodollar options, obtained those behaviours of the implied volatility curve, under the hypothesis that asset prices and volatilities are correlated.

5.2 Spread Analysis

Up to now, we have considered the mid price only. As pointed out before, the perturbative approach used by the PBS model can naturally generate a spread on prices and volatilities.

The spread on implied volatility is then obtained by inversion of the pricing formula.

For the same set of parameters described above, we can thus analyze the effect of changing the drift term on a theoretical bid-ask spread. Figures 7 and 8 give an example of price and volatility spread behavior for a chosen value of \( \mu \). It is straightforward to notice that higher prices imply higher volatility. Figure 8 shows the magnitude of the spread on implied volatility for three different values of \( \mu \). For low strikes, the spread is higher when there is no risk premium; it reaches a minimum around the money, then it starts increasing. For out of the money options, the behavior is reversed: the spread is higher the higher the risk premium. The main difference we find with the standard \( \mu = 0 \) setting is that the spread has no longer its point of minimum variance around the money. The variance spread becomes indeed wider as the strike increases. As shown in figure 10, the relative spread on prices (spread-mid price) is almost zero for in deep in the money options, then increases sharply with both strike and risk premium for out of the money calls.

Let us finally consider directly the effect of the addition of the correction in equation (4.7) to the PBS model implied volatility curve. Figure 11 clearly shows that the presence of a risk premium skews the implied volatility curve toward higher strikes.

---

8 For \( \mu = 0.2 \) the implied volatility at moneyness 1.15 is slightly higher than at 0.85. Unreported simulations show that this behavior is common to every choice of parameter. This could suggest that in periods of high risk premia, volatility should tend to be higher for out of the money options.

9 Notice that the analysis of the relative spread leads to the same conclusions.
Figure 3: Implied volatilities curves depending on $\epsilon$.

Figure 4: Skewed implied volatility depending on maturity.
Figure 5: Implied volatility depending on $\lambda$.

Figure 6: Implied volatility depending on $\lambda$. 
Figure 7: Price spread.

Figure 8: Percentual price spread.
Figure 9: Volatility spread depending on $\lambda$.

Figure 10: Implied volatility in PBS model with and without drift.
6 Conclusion

In this article we have studied the Perturbative Black Scholes model, introduced in [24], when we drop the hypothesis that the underlying is a martingale under the historical probability. Without imposing any behaviour of volatility through time, we showed that the hedging procedure of a trader who estimates it depends on the expected excess return on stocks.

We then introduced a correction with respect to Scotti [24] when the drift term of the diffusive process for the stock price is different from the risk free rate. We found a closed form solution for the pricing of a European vanilla call option. This formula depends on the same parameters of the classical Black Scholes model, i.e. the volatility $\sigma_0$ and the parameter $d_2$, on the two parameters of the PBS model, the variance $\Gamma[\sigma]$ and the bias $A[\sigma]$, which characterize the error structure of the volatility estimated by the trader. Since the PBS model induces market incompleteness, pricing formulas depend also on the cumulated risk premium $\mathcal{L}$, as shown by equation (4.7).

We analyzed how a simple risk aversion argument forces the underlying price to be a sub-martingale and we studied the dependence of implied volatility on the parameters of the model. We numerically studied the most case in which the volatility used by traders is an overestimation of the true value and we showed that higher risk premia tend to increase the skewness and the smile of the implied volatility curve, since the distribution of stock prices at maturity is shifted towards higher values.

We finally found out that the Perturbative Black Scholes model with drift can reproduce the behaviour of the implied volatility curve after the 1987 crash.

Figure 11: Bid, ask, mid implied volatilities in PBS model.
A Computation

We have to compute

\[ \mathbb{E} \left[ \int_0^T \frac{\partial^2 F}{\partial \sigma \partial x} (\sigma_0, S_s, s) \, dS_s \right] \]  

where \( S_t \) follows the Black Scholes diffusion (3.1) and \( F(\sigma_0, S_s, s) \) is the price of a call option with strike \( K \), starting at time \( s \), when the spot value is \( S_s \) and the volatility is \( \sigma_0 \).

\[
\begin{align*}
\mathbb{E} \left[ \int_0^T \frac{\partial^2 F}{\partial \sigma \partial x} (\sigma_0, S_s, s) \, dS_s \right] &= -\frac{\mu}{\sqrt{2 \pi} \sigma_0} \int_0^T \mathbb{E} \left[ d_2(S_s, s) \, e^{-\frac{1}{2} \frac{d_2(S_s, s)^2}{\sigma^2}} S_s \right] \, ds \\
&= -\frac{\mu S_0}{\sqrt{2 \pi} \sigma_0} \int_0^T e^{(\mu-\frac{1}{2}\sigma^2)s} \int_{y} \frac{\ln \frac{S_0}{K} + \mu s - \frac{1}{2} \sigma_0^2 s + \sigma_0 W_s - \frac{\sigma_0^2 (T-s)}{2}}{\sigma_0 \sqrt{T-s}} \\
&\quad \times e^{-\frac{1}{2} \frac{(\ln \frac{S_0}{K} + \mu s - \frac{1}{2} \sigma_0^2 s + \sigma_0 W_s + \frac{\sigma_0^2 (T-s)}{2})^2}{\sigma_0^2}} e^{\sigma_0 W_s} \, ds \\
&= -\frac{\mu K}{\sqrt{2 \pi} \sigma_0} \int_0^T \frac{T-s}{T} \left[ \frac{\ln \frac{S_0}{K} + \mu s}{\sigma_0 \sqrt{T}} \right] e^{-\frac{1}{2} \left( \frac{\ln \frac{S_0}{K} + \mu s}{\sigma_0 \sqrt{T}} \right)^2} ds \\
&= -\frac{K}{\sqrt{2 \pi} \sigma_0} \int_0^1 (1-u) \left[ \frac{\mu T}{\sigma_0 \sqrt{T}} u + d_2 \right] e^{-\frac{1}{2} \left( \frac{\mu T}{\sigma_0 \sqrt{T}} u + d_2 \right)^2} ds
\end{align*}
\]

We integrate by part and we find

\[ \mathbb{E} \left[ \int_0^T \frac{\partial^2 F}{\partial \sigma \partial x} (\sigma_0, S_s, s) \, dS_s \right] = -\frac{K \sqrt{T}}{\sqrt{2 \pi}} e^{-\frac{1}{2} d_2^2} + \frac{K \sqrt{T}}{L} \left[ \mathcal{N} (d_2 + L) - \mathcal{N} (d_2) \right] \]

where \( \mathcal{N} \) is the cumulated distribution function of a reduced gaussian random variable.

The second term that we have to compute in equation (4.7) is

\[ \mathbb{E} \left[ \int_0^T \frac{\partial^3 F}{\partial \sigma^2 \partial x} (\sigma_0, S_s, s) \, dS_s \right] \]

We can compute this term following the same steps we used for the first term (A.1).
where

\[ \Theta = \frac{\ln \frac{\tau_0}{K} + \mu s - \frac{1}{2} \sigma_0^2 s}{\sigma_0 \sqrt{T-s}} \]

\[ \Lambda = \frac{1}{2} \sigma_0 \sqrt{T-s} \]

\[ \sqrt{\frac{T-s}{T}} (\Theta - \Lambda) = \frac{\ln \frac{\tau_0}{K} + \mu s - \frac{1}{2} \sigma_0^2 s}{\sigma_0 \sqrt{T-s}} = d_2 + \frac{\mu}{\sigma_0 \sqrt{T}} s \]

Finally, we integrate by parts three times and we find

\[ \text{(A.4)} \]

\[ \mathbb{E} \left[ \int_0^T \frac{\partial^3 F}{\partial \sigma^2 \partial x}(\sigma_0, S_s, s) dS_s \right] = - \frac{K \sqrt{T}}{\sigma_0 \sqrt{2 \pi}} \left( \frac{\sigma_0 \sqrt{T}}{L} \left( 1 + \frac{d_2}{L} \right) - \frac{8}{L^2} \right) e^{-\frac{1}{2} (d_2 + \zeta)^2} \]

\[ - \frac{K \sqrt{T}}{\sigma_0 \sqrt{2 \pi}} \left( d_2^4 + 4 - 2 \frac{d_2}{L} + \frac{8}{L^2} + \sigma_0 \sqrt{T} \left( d_2 - 4 \frac{d_2}{L} - \frac{d_2}{L^2} \right) \right) e^{-\frac{1}{2} d_2^2} \]

\[ - \frac{K \sqrt{T}}{\sigma_0} \left( \sigma_0 \sqrt{T} \left[ 3 \frac{d_2^2}{L^2} + \left( 1 + \frac{d_2}{L} \right)^2 \right] \right) - \frac{8}{L} \left( d_2 + \frac{d_2}{L} \right) \frac{d_2}{L^2} \right) \left[ N(d_2 + L) - N(d_2) \right] \]
References

[1] Black, F.; Scholes M. (1973): *The Pricing of Options and Corporate Liabilities*, Journal of Political Economy, 81, 637-659.

[2] Bouleau, N.; Hirsch, F. (1991): *Dirichlet Forms and Analysis on Wiener space*, De Gruyter, Berlin.

[3] Bouleau, N. (2003): *Error Calculus for Finance and Physics*, De Gruyter, Berlin.

[4] Bouleau, N. (2003): *Error Calculus and path sensitivity in financial models*, Mathematical Finance, 13, 1, 115-134.

[5] Bouleau, N. (2006): *When and how an error yields a Dirichlet form*, Journal of Functional Analysis, 240, 2, 445-494.

[6] Brigo, D., Mercurio, F., Rapisarda, F. (200x): *Lognormal-mixture dynamics and calibration to market volatility smiles*, Banca IMI.

[7] Christensen, B., Prabhala, N. (1998): *Relation between implied and realized volatility*, Journal of Financial Economics, 50, 125-150.

[8] Constantinides, G., Jackwerth, J., Perrakis, S. (2008): *Mispricing of S&P 500 Index Options*, Review of Financial Studies, forthcoming.

[9] Derman, E.; Kamal, M.; Kani, I. and Zou, J. (1996) *Valuing Contracts with Payoffs based on Realized Volatility*, Global Derivatives Quarterly Review, Goldman Sachs.

[10] Eraker, B., Johannes, M., Polson, N. (2003): *The Impact of Jumps in Volatility and Returns*, Journal of Finance, 58, 3, 1269-1300.

[11] Dupire, B. (1994): *Pricing with Smile*, RISK, January 1994.

[12] Fouque, J. P., Papanicolaou, G. and Sircar, R. (2000): *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University Press, Cambridge.

[13] Fukushima, M.; Oshima, Y.; Takeda, M. (1994): *Dirichlet Forms and Markov Process*, De Gruyter, Berlin.

[14] Hagan, P., Kumar, D., Lesniewski, A., Woodward, D., : *Managing Smile Risk*, Willmot Magazine.

[15] Hamza, M. M. (1975): *Détermination des formes de Dirichlet sur \( \mathbb{R}^n \)*, PhD thesis, Université d’Orsay, Paris.

[16] Heston, S.(1993): *A Closed-form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options*, Review of Financial Studies, 6, 2, 327-343.

[17] Hull, J., White, A. (1987): *The Pricing of Options on Assets with Stochastic Volatility*, Journal of Finance, 42, 2, 281-300.

[18] Jackwerth, J., Rubinstein, M. (1996): *Recovering Probability Distributions from Option Prices*, Journal of Finance, 51, 5, 1611-1631.

[19] Lamberton, D.; Lapeyre, B. (1995): *Introduction to Stochastic Calculus Applied to Finance*, Chapman & Hall, London.

[20] Lo, A., Wang, J. (1995): *Implementing Option Pricing Models When Asset Returns are Predictable*.

[21] Perignon, C.; Villa, C. (2002): *Extracting Information from Options Markets: Smiles, State-Price Densities and risk Aversion*, European Financial Management, 8, 4, 495-513.

[22] Renault, E.; Touzi, N. (1996): *Option Hedging and Implied Volatilities in a Stochastic Volatility Model*, Mathematical Finance, 6, 3, 279-302.
[23] Rubinstein, M. (1985): Non Parametric Test of Alternative Option Pricing Models using all Reported Trades and Quotes on the 30 most active CBOE Option classes from August 23, 1976 through August 31, 1978, Journal of Finance, 40, 455-480.

[24] Scotti, S. (2007): Perturbative Approach on Financial Markets, Preprint Arxiv 0806.0287.

[25] Scotti, S. (2007): Calibration of Perturbative Black Scholes model with Variance Swaps, Report No.2, 2007/2008, fall, Institut Mittag-Leffler, the Royal Swedish Academy of Sciences.

[26] Stoll, H., Whaley, R.: Stock Market Structure and Volatility, Review of Financial Studies, 3, 1, National Bureau of Economic Research Conference: Stock Market Volatility and the Crash, Dorado Beach, March 16-18, 1989, 37-71.