TWISTED SUMS AND A PROBLEM OF KLEE

N. T. PECK

To Victor Klee

In [5], Klee asked whether every vector topology \( \tau \) on a real vector space \( X \) is the supremum of a nearly convex topology \( \tau_1 \) and a trivial dual topology \( \tau_2 \). Recall that a vector topology \( \tau_1 \) on \( X \) is nearly convex if for every \( x \) not in the \( \tau_1 \)-closure of \( \{0\} \) there is \( f \) in \( (X, \tau_1)^* \) with \( f(x) \neq 0 \); \( \tau_2 \) is trivial dual if \( (X, \tau_2)^* = \{0\} \). We do not require that \( \tau_1 \) or \( \tau_2 \) be Hausdorff, even if \( \tau \) itself is Hausdorff. The topology \( \tau \) is the supremum of \( \tau_1 \) and \( \tau_2 \) if \( \tau_1 \) and \( \tau_2 \) are weaker than \( \tau \), and if for every \( \tau \)-neighborhood \( U \) of the origin \( 0 \) there are a \( \tau_1 \)-neighborhood \( V \) of \( 0 \) and a \( \tau_2 \)-neighborhood \( W \) of \( 0 \) such that \( U \supset V \cap W \).

In [5], Klee proved that the usual topology on \( \ell_p \), \( 0 < p < 1 \), is not the supremum of a locally convex topology and a trivial dual topology; this and other examples make the question at the beginning of this paper a natural one. Some related questions on suprema of linear topologies were studied in [7].

Given any vector topology \( \tau \) on \( X \), let \( K(\tau) = \cap \{ f^{-1}(0) : f \in (X, \tau)^* \} \). It is trivial to answer Klee’s question affirmatively in the case that \( K(\tau) \) is complemented. For in this case, \( K(\tau) \) must be a trivial dual space in the relative topology; and if \( L \) is a complement to \( K(\tau) \) in \( X \), the relative topology on \( L \) is nearly convex. Now simply let \( \tau_1 \) be the product of the trivial topology on \( K(\tau) \) and the relative
topology on $L$; and let $\tau_2$ be the product of the relative topology on $K(\tau)$ and the trivial topology on $L$. Then $\tau = \sup(\tau_1, \tau_2)$.

So the interesting case is when $K(\tau)$ is uncomplemented. We study the problem when $(X, \tau)$ is the twisted sum of a separable normed space and the real line. Recall that a real function $F$ on a normed space $E$ is quasi–linear if

1. $F(rx) = rF(x)$ for all scalars $r$ and all $x$ in $E$;
2. $|F(x+y) - F(x) - F(y)| \leq C(\|x\| + \|y\|)$ for all $x, y$ in $E$ and some constant $C$.

Now define the twisted sum of the real line and $E$ (with respect to $F$) as the vector space $X_F = \mathbb{R} \times E$ equipped with quasi–norm $\|(r, x)\| = |r - F(x)| + \|x\|$. It is easy to verify that

$$\|(r_1 + r_2, x_1 + x_2)\| \leq (C + 1)[\|(r_1, x_1)\| + \|(r_2, x_2)\|].$$

The space $E$ is said to be a $K$–space if the subspace $\mathbb{R} \times \{0\}$ is complemented in $X_F$ for every quasi–linear map $F$ on $E$. (This is a slight abuse of terminology; strictly speaking, it is the completion of $E$ that is the $K$–space.) So we are interested in Klee’s question for the non–$K$ spaces. The only known non–$K$ spaces are $\ell_1$–like.

The Ribe function is defined on $\ell_1^0$, the space of finitely supported elements of $\ell_1$, by

$$F_0(x) = \sum_i x_i \ell n|x_i| - \left(\sum_i x_i\right) \ell n \sum_i |x_i|$$

with the convention that $0\ell n 0 = 0$. Ribe [8] proved that $F_0$ is quasi–linear on $\ell_1^0$ and used $F_0$ to show that $\ell_1$ is not a $K$–space. Closely related functions were used by Kalton [2] and Roberts [9] to prove the same result. The reflexive space $\ell_2(\ell_1^0)$ is not a $K$–space, and the $B$–convex spaces are $K$–spaces [2]. Kalton and Roberts
[4] showed that $c_0$ and $\ell_\infty$ are $K$–spaces. It is not known whether the James space is a $K$–space. We are studying Klee's problem for spaces $E$ and quasi-linear maps $F$ on $E$ such that $\mathbb{R} \times \{0\}$ is not complemented in $X_F$. By Theorem 2.5 of [3], there is no linear map $T$ on $E$ such that $|T(x) - F(x)| \leq C\|x\|$ for all $x$ in $E$ (i.e. $F$ does not split on $E$). The corollary to our main theorem implies that none of the spaces above can be a counterexample for Klee’s question, since the $F$ concerned does split on an infinite-dimensional subspace.

We now state our main result:

**Main Theorem.** Let $E$ be an $\aleph_0$–dimensional normed space. Assume $F$ is a quasi–linear function on $E$ for which there are a linearly independent sequence $(x_i)$ in $E$ and a linear map $T$ on $\text{span}(x_i)$ such that

\begin{equation}
|T(x) - F(x)| \leq C\|x\| \text{ for all } x \text{ in } \text{span}(x_i) \text{ and some constant } C.
\end{equation}

Then there are a trivial dual topology $\tau_2$ on $\mathbb{R} \times E$, weaker than the quasi–norm topology, and a $\tau_2$–neighborhood $U$ of 0 such that if $(r, x) \in U$ and $\|x\| \leq 1$, then $\| (r, x) \| < C$ for some constant $C$.

Before we prove the theorem, we set the framework for the construction with some auxiliary results. We begin with:

**Definition.** Suppose $(G_i)$ is a finite or infinite sequence of subsets of $E$, and $(n_i)$ is a sequence of positive integers (of the same length as $(G_i)$). The $(n_i)$–sum of $(G_i)$ is the set of all finite sums

$$z = r_1z_1 + r_2z_2 + r_3z_3 \ldots$$

where $|r_i| \leq 1$ for all $i$ and $z_1, \ldots, z_{n_1}$ are in $G_1$, $z_{n_1+1}, \ldots, z_{n_1+n_2}$ are in $G_2$, $z_{n_1+n_2+1}, \ldots, z_{n_1+n_2+n_3}$ are in $G_3$, etc. Note that if $|r| \leq 1$, $rz$ is also in the $(n_i)$ sum.
Lemma 1. Let $X$ be a vector space and let $(U_n)$ be a neighborhood base at 0 for a pseudo-metrizable vector topology on $X$, chosen so that $U_{n+1} + U_{n+1} \subseteq U_n$ for all $n$ and $[-1,1]U_n \subseteq U_n$ for all $n$. Let $(F_n)$ be a sequence of subsets of $X$, chosen so that $[-1,1]F_n \subseteq F_n$ and $F_{n+1} + F_{n+1} \subseteq F_n$, for all $n$. Then the sequence $(U_n + F_n)$ is a neighborhood base at 0 for a pseudo-metrizable vector topology on $X$ which is weaker than the original topology.

Proof. Immediate. □

In the next lemma, we specify $F_n$ more closely.

Lemma 2. Let $X$ and $(U_n)$ be as in Lemma 1. Let $(G_n)$ be a sequence of subsets of $X$. Define subsets $F_n$ of $X$ as follows: for each $n$ in $N$, $F_n$ is the $(2^{i-n})$–sum of the $G_i$’s for $i \geq n$. Then $(F_n)$ satisfies the hypotheses of Lemma 1.

Proof. $[-1,1]F_n \subseteq F_n$ as remarked already. For a typical sum in $F_{n+1} + F_{n+1}$, at most $2 \cdot 2^{i-(n+1)} = 2^{i-n}$ of the $z_i$’s are in $G_i$ for $i \geq n+1$, so $F_{n+1} + F_{n+1} \subseteq F_n$. □

Remark. Note that there is an apriori bound on the number of elements of $G_i$ appearing in a sum in $F_n$, for any $n$: the bound is $2^{i-1}$; we use the looser bound $2^i$.

In our construction, $(U_n)$ is a neighborhood base at 0 for the twisted sum topology. The $G_i$’s of Lemma 2 will be chosen so that $(U_n + F_n)$ is a neighborhood base at 0 for a trivial dual topology $\tau_2$; they will also have to be chosen so that $\tau$ is the supremum of $\tau_1$ and $\tau_2$. The next lemma identifies the topology $\tau_1$:

Lemma 3. Let $F$ be a quasi-linear map on a normed space $E$ and let $X_F = \mathbb{R} \times E$ with the quasi-norm $||(r,x)|| = |r - F(x)| + \|x\|$. Assume $\mathbb{R} \times \{0\}$ is not complemented in $X_F$. Then the strongest nearly convex topology on $\mathbb{R} \times E$ which

is weaker than the quasi-norm topology has a neighborhood base at 0 of sets of the form \( \{(r, x) : \|x\| < \eta\} \).

**Proof.** Sets of the above type are a neighborhood base at 0 for a nearly convex topology weaker than \( \tau \), the quasi-norm topology. The closure of \( \{0\} \) for this weaker topology is \( \mathbb{R} \times \{0\} \); and if \((r, x)\) is in \( X_F \) and \( x \neq 0 \), there is \( f \) in \( E^* \) with \( f(x) \neq 0 \). Then \( f(\pi(r, x)) \neq 0 \), where \( \pi \) is the quotient map of \( X_F \) onto \( E \).

Now suppose \( \nu \) is a nearly convex topology on \( \mathbb{R} \times E \), weaker than the quasi-norm topology. Since \( \mathbb{R} \times \{0\} = K(\tau) \) is not complemented, the \( \nu \)-closure of \( \{0\} \) must contain \( \mathbb{R} \times \{0\} \). Let \( U \) be \( \nu \)-open containing 0. Choose \( V \) \( \nu \)-open containing 0, with \( V + V \subset U \). Choose \( \epsilon > 0 \) so that if \( \|(r, x)\| < \epsilon \), then \( (r, x) \in V \).

Now, \( \|(F(x), x)\| = \|x\| \), so if \( \|x\| < \epsilon \), then \( (F(x), x) \in V \). Also, \( (r - F(x), 0) \) is in \( V \) since it is in the \( \nu \)-closure of 0, and so \( (r, x) \) is in \( U \). \( \Box \)

**Notation.** Let \( Z \) be a Banach space with a basis \((v_i)\). Let \((v_i^*)\) be the coordinate functionals on \( Z \). For \( n \) a positive integer and \( x \) in \( Z \), set \( x|_{[1, n]} = \sum_{i=1}^{n} v_i^*(x)v_i \), and \( x|_{(n, \infty)} = \sum_{i=n+1}^{\infty} v_i^*(x)v_i \). Say that \( x \) is to the right of \( n \) if \( v_i^*(x) = 0 \) for \( i \leq n \).

We need two more preliminary results before proving our main theorem:

**Lemma 4.** Let \( Z \) be a Banach space with a monotone basis \((v_i)\), let \( K \) be a compact subset of \( Z \), and let \( \epsilon > 0 \). Then there is \( n \) so that if \( y \) is to the right of \( n \) and \( x \in K \), then \( \|x\| < \|x + y\| + \epsilon \).

**Proof.** Choose \( n \) so that \( \|x|_{(n, \infty)}\| < \epsilon \) for every \( x \) in \( K \). Now if \( y \) is to the right of \( n \), then \( \|x|_{[1, n]}\| = \|(x + y)|_{[1, n]}\| \leq \|(x + y)\| \), since the basis is monotone, so \( \|x\| < \|x + y\| + \epsilon \). \( \Box \)
Lemma 5. Let \( y_i, 1 \leq i \leq k \), be linearly independent elements of a normed space \( E \). Define \( y_{k+1} = -\sum_{i=1}^{k} y_i \), and let \( \eta > 0 \). Set \( z_i = my_i, 1 \leq i \leq k+1 \), for some \( m > 0 \). Then we can choose \( m \) so large that the following condition is satisfied: if at most \( k \) of the \( r_i \)'s are non-zero, and if \( \| \sum_{i=1}^{k+1} r_i z_i \| < 3 \), then \( \sum_{i=1}^{k+1} |r_i| < \eta \).

Proof. Choose \( M > 0 \) so that \( \sum_{i=1}^{k} |\alpha_i| \leq M \| \sum_{i=1}^{k} \alpha_i y_i \| \) for all \( k \)-tuples \( (\alpha_i) \). Now suppose \( \| \sum_{i=1}^{k+1} r_i z_i \| < 3 \), with at most \( k \) \( r_i \)'s non-zero. If \( r_{k+1} = 0 \), then
\[
\sum_{i=1}^{k} |r_i| \leq \frac{3M}{m} < \eta
\]
for \( m > 3M/\eta \).

If \( r_{k+1} \neq 0 \), then some other \( r_i \) is 0, \( r_1 \), say; now,
\[
\| \sum_{i=1}^{k+1} r_i z_i \| = \| -mr_{k+1} y_1 + \sum_{i=2}^{k} m(r_i - r_{k+1}) y_i \| < 3.
\]
Therefore
\[
|r_{k+1}| + \sum_{i=2}^{k+1} |r_i - r_{k+1}| < \frac{3M}{m}, \quad \text{so}
\]
\[
\sum_{i=2}^{k+1} |r_i| \leq |r_{k+1}| + \sum_{i=2}^{k+1} |r_i - r_{k+1}| + k|r_{k+1}|
\]
\[
< \frac{3(k+1)M}{m} < \eta
\]
for \( m > 3(k+1)M/\eta \). \( \square \)

Proof of Main Theorem. We will construct inductively the sets \( G_n \) used in Lemma 2. That lemma will give us the sets \( F_n \) and then Lemma 1 will provide the topology.

We may assume that for \( x_i \) and \( T \) in the theorem, \( T(x_i) = 0 \) for each \( i \). This is possible since for each \( i \), there is a scalar \( \alpha_i \) so that \( T(x_{2i-1} + \alpha_i x_{2i}) = 0 \); now the sequence \( x_i' = x_{2i-1} + \alpha_i x_{2i} \) also satisfies condition 1) of the Theorem.

We can regard \( E \) as a subspace of the Banach space \( Z = C[0,1] \), which has a monotone basis. Any positive scalar multiple of the quasi-norm yields the same...
topology as the quasi–norm, so we can and do assume that the constant $C$ in 0(ii) above is 1. This can be done by multiplying $F$ by a suitable positive constant.

Finally, we use $\| \|$ to refer to the norm on $Z$ and on $E$. We only calculate norms of elements of $E$, but we do use the monotonicity of $(v_i)$ in $Z$.

Now we begin the construction of $(G_i)$.

Choose $0 < c_n \leq 2^{-(n+3)}$ (and thus $\sum_{n=1}^{\infty} c_n < \frac{1}{4}$). Let $(d_j)$ be any sequence whose linear span is $E$, and let $(e_i)$ be an indexing of $(d_j)$ such that each $d_j$ occurs infinitely often in $(e_i)$. We can assume that $\|d_j\| \leq 1$ and that $|F(d_j)| \leq 1$ for each $j$, by multiplying $d_j$ by a positive constant.

Assume that finite sets $G_0, G_1 \cdots G_{n-1}$ have been constructed, with $G_0 = \{0\}$, satisfying the following conditions:

(2) for each $1 \leq i \leq n-1$, $G_i$ is a finite set $(w_{i,j} : 1 \leq j \leq 2^{i+1})$, with

$$w_{i,j} = e_i + m_i x_{\ell(i,j)}, \quad j \leq 2^i,$$

$$w_{i,2^i+1} = e_i - \sum_{j=1}^{2^i} m_i x_{\ell(i,j)};$$

here, $(x_i)$ is the sequence in the statement of the theorem.

(3) Set $z_{i,j} = w_{i,j} - e_i$. Then if $\left\| \sum_{j=1}^{2^{i+1}} r_j z_{i,j} \right\| < 3$ with at most $2^i r_j$'s non–zero,

$$\sum_{j=1}^{2^{i+1}} |r_j| < c_i.$$

To define $G_n$, let $K_n'$ be the $(2^i)$–sum of $G_i$ for $i \leq n-1$, (so $K_1' = \{0\}$) and let $K_n = K_n' + [-2^n, 2^n]e_n$. Then $K_n$ is a compact subset of $E \subset Z$. By Lemma 4, there is an integer $s_n$ such that if $y$ is to the right of $s_n$ then $\|x\| < \|x+y\| + c_n$ for all $x$ in $K_n$. By the linear independence of the sequence $(x_i)$, we can choose $x_{\ell(n,1)}, \ldots, x_{\ell(n,2^n)}$, all to the right of $s_n$, with $\ell(n,j) < \ell(n,j')$ if $j < j'$. For ease of notation, put $x_{n,i} = x_{\ell(n,i)}, 1 \leq i \leq 2^n$, and put $x_{n,2^n+1} = -\sum_{n,i} x_{n,i}$. 

By Lemma 5, we can choose $m_n$ so large that if

$$\left\| \sum_{j=1}^{2^n+1} r_{n,j} m_n x_{n,j} \right\| < 3,$$

with at most $2^n$ of the $r_{n,j}$ non-zero, then

$$\sum_{j=1}^{2^n+1} |r_{n,j}| < c_n. \tag{4}$$

Finally, for $1 \leq i \leq 2^n + 1$, put

$$w_{n,i} = e_n + m_n x_{n,i}$$

and let

$$G_n = (w_{n,i}), \quad 1 \leq i \leq 2^n + 1.$$ 

Note that since $\sum_{i=1}^{2^n+1} x_{n,i} = 0$, $e_n \in \text{co}G_n$. (We denote the convex hull of $A$ by $\text{co}A$.)

This finishes the construction of $(G_i)$.

Now let $(F_n)$ be the subsets of $E$ used in Lemma 1: $F_n$ is the $(2^{i-n})$ sum of $(G_i)$ for $i \geq n$. Let $(U_n)$ be a neighborhood base at 0 for the quasi–norm topology on $\mathbb{R} \times E$, with $U_{n+1} + U_{n+1} \subseteq U_n$ and $[-1,1]U_n \subseteq U_n$, for all $n$; also assume that $\|w\| < 1$ if $w \in U_1$. Let $\tau_2$ be the topology yielded by Lemma 1.

We claim that $\tau_2$ is trivial dual. To see this, note that for $m \geq n$, $e_m \in \text{co}(w_{m,i}) \subseteq \text{co}F_n \subseteq \text{co}(F_n + U_n)$; since each $d_j$ occurs infinitely often in the sequence $(e_m)$, $K(\tau_2)$ contains every $d_j$ and therefore contains $\{0\} \times E$. Also, $(1,0) \in \text{co}U_n \subseteq \text{co}(U_n + F_n)$ for every $n$, so $K(\tau_2)$ contains $\mathbb{R} \times \{0\}$. This proves the claim.

Now suppose that

$$x = \sum_{i=1}^{n} \sum_{j=1}^{2^i+1} r_{i,j} (e_i + m_i x_{i,j})$$

is in $E$, and that $\|x\| < 1$. We will first prove that $|F(x)| < 0$. 

Toward that end: since the \( x_{n,j} \) are to the right of \( s_n \), the construction of \( G_n \) implies that

\[
\| \sum_{i=1}^{n-1} \sum_{j=1}^{2^i+1} r_{i,j} (e_i + m_i x_{i,j}) + \sum_{j=1}^{2^n+1} r_{n,j} e_n \| < 1 + c_n \tag{5}
\]

from which, since \( \| x \| < 1 \),

\[
\| \sum_{j=1}^{2^n+1} r_{n,j} m_n x_{n,j} \| < 2 + c_n < 3. \tag{6}
\]

Now from (4) and (6), we have

\[
\sum_{j=1}^{2^n+1} |r_{n,j}| < c_n; \tag{7}
\]

combining this with (5), we have

\[
\| \sum_{i=1}^{n-1} \sum_{j=1}^{2^i+1} r_{i,j} (e_i + m_i x_{i,j}) \| < 1 + 2c_n. \tag{8}
\]

For the induction step, assume that for some \( \ell \),

\[
\| \sum_{i=1}^{\ell-1} \sum_{j=1}^{2^i+1} r_{i,j} (e_i + m_i x_{i,j}) \| < 1 + 2c_n \cdots + 2c_{\ell+1}. \tag{9}
\]

Since the \( x_{\ell,j} \) are to the right of \( s_\ell \), the construction of \( G_\ell \) implies that

\[
\| \sum_{i=1}^{\ell-1} \sum_{j=1}^{2^i+1} r_{i,j} (e_i + m_i x_{i,j}) + \sum_{j=1}^{2^\ell+1} r_{\ell,j} e_\ell \| < 1 + 2c_n \cdots + 2c_{\ell+1} + c_\ell, \tag{10}
\]

from which

\[
\| \sum_{j=1}^{2^\ell+1} r_{\ell,j} m_\ell x_{\ell,j} \| < 2 + 4c_n \cdots + 4c_{\ell+1} + c_\ell < 3. \tag{11}
\]

Now from (4) and (11), we have

\[
\sum_{j=1}^{2^\ell+1} |r_{\ell,j}| < c_\ell; \tag{12}
\]
combining this with (10), we obtain
\[
\left\| \sum_{i=1}^{\ell-1} \sum_{j=1}^{2^i+1} r_{i,j} (e_i + m_i x_{i,j}) \right\| < 1 + 2c_n \cdots + 2c_\ell, \tag{13}
\]
recalling that \(\|e_i\| \leq 1\). This finishes the induction step.

The above argument has yielded that
\[
\left\| \sum_{j=1}^{2^i+1} r_{i,j} e_i \right\| < c_i \tag{14}
\]
for each \(i\); from this and \(\|x\| < 1\), we have
\[
\left\| \sum_{i=1}^{n} \sum_{j=1}^{2^i+1} r_{i,j} m_i x_{i,j} \right\| < 1 + \sum_{n=1}^{\infty} c_n < 2. \tag{15}
\]

From (15) and (1), recalling \(T(x_{i,j}) = 0\),
\[
|F\left( \sum_{i=1}^{n} \sum_{j=1}^{2^i+1} r_{i,j} m_i x_{i,j} \right)| < 2.
\]

To estimate
\[
F\left( \sum_{i=1}^{n} \sum_{j=1}^{2^i+1} r_{i,j} e_i \right),
\]
recall that \(|F(e_i)| \leq 1\) for each \(i\), so
\[
|F\left( \sum_{j=1}^{2^i+1} r_{i,j} e_i \right)| < 2^{-i}.
\]

Therefore
\[
|F\left( \sum_{i=1}^{n} \left( \sum_{j=1}^{2^i+1} r_{i,j} e_i \right) \right)| \leq \sum_{i=1}^{n} 2^{-i} + \sum_{i=1}^{n} i \left\| \sum_{j=1}^{2^i+1} r_{i,j} e_i \right\|
\]
\[
< 1 + \sum_{i=1}^{n} i \cdot 2^{-i} < 4
\]
(using \(|F(\sum u_i)| \leq \sum |F(u_i)| + \sum i \|u_i\|\)). Finally,
\[
|F(x)| \leq 2 + 4 + \left\| \sum_{i=1}^{n} \sum_{j=1}^{2^i+1} r_{i,j} e_i \right\|
\]
\[
+ \left\| \sum_{i=1}^{n} \sum_{j=1}^{2^i+1} r_{i,j} m_i x_{i,j} \right\|
\]
\[
< 2 + 4 + 1 + 2 = 9.
\]
To complete the proof of the theorem, suppose \((r, x) \in U_1 + F_1\) and \(\|x\| \leq 1\). Write \((r, x) = (r, y) + (0, z)\), with \((r, y) \in U_1\) and \(z \in F_1\). Then \(|r - F(y)| + \|y\| \leq 1\); from this and \(\|x\| \leq 1\) follows \(\|z\| \leq 2\). Now since \(z \in F_1\), the preceding paragraph implies \(|F(z)| < 18\). At last,

\[
|r - F(x)| \leq |r - F(y)| + |F(y) - F(x)| \\
\leq 1 + |F(y) - F(x)| \\
\leq 1 + |F(z)| + \|z\| + \|x\| \\
< 22,
\]

so \(||(r, x)|| < 23\). The proof is complete. □

**Corollary.** Let \(E\) be a separable normed space and let \(E_0\) be an \(\aleph_0\)-dimensional subspace of \(E\) which is dense in \(E\). Assume that there is a quasi-linear map \(F\) on \(E_0\) which splits on an infinite-dimensional subspace of \(E_0\). Then the twisted sum topology on \(\mathbb{R} \otimes_F E\) is the supremum of a trivial dual topology and a nearly exotic topology.

**Proof.** Let \(q\) denote the quotient map of \(\mathbb{R} \otimes_F \tilde{E}\) onto \(\tilde{E}\), where \(\tilde{E}\) is the completion of \(E\). (For \(x \in E_0, q(r, x) = x\.) The subspace \(E_0\) satisfies the hypotheses of the main theorem. Therefore there are a trivial dual topology \(\tau_2\) on \(\mathbb{R} \times E_0\), weaker than the twisted sum topology; a \(\tau_2\)-neighborhood \(V\) of 0; and a constant \(C\) so that if \(x \in E_0, (r, x) \in V\), and \(\|x\| < 1\), then \(||(r, x)|| < C\).

We can assume that \(V\) contains a \(\tau_2\)-neighborhood \(U\) of 0 of the form \(B_\alpha + F_n\), where \(F_n \subset E_0\) is as constructed as in the proof of the main theorem, and for any \(\beta > 0\),

\[
B_\beta = \{(r, x) \in \mathbb{R} \times E_0 : ||(r, x)|| < \beta\}.
\]
Sets of the form $B_\beta + q^{-1}(F_m)$, where

$$B_\beta = \{ w \in \mathbb{R} \otimes_F E : \|w\| < \beta \},$$

obviously form a neighborhood base at the origin for a vector topology $\tau_2$ on $\mathbb{R} \otimes_F E$, weaker than the twisted sum topology. The topology $\tau_2$ is trivial dual since its restriction to the dense subspace $\mathbb{R} \times E_0$ is trivial dual.

Now choose $0 < \gamma < 1/2$ so that $B_\gamma \subset B_\alpha + q^{-1}(F_n)$ and $\|q(w)\| < 1/2$. Choose $w_0 \in \mathbb{R} \times E_0$ so that $\|w - w_0\| < \gamma$. Then $\|q(w) - q(w_0)\| < \gamma$, so $\|q(w_0)\| < \gamma + 1/2 < 1$. Clearly, $w_0 \in B_\alpha + q^{-1}(F_n)$, and so from our assumption, $\|w_0\| < C$. Now, $\|w\| < (\alpha/\gamma)(C + 1)$, and the proof is complete. □

The theorem and corollary apply to several spaces which are either not $K$–spaces or for which it is not known whether they are $K$–spaces:

**Theorem.** For the following pairs of normed spaces $E$ and quasi–linear maps $F$ on $E$, the twisted sum topology on $X_F = \mathbb{R} \times E$ is the supremum of a nearly convex topology and a trivial dual topology:

(a) $E$ is any infinite–dimensional subspace of $\ell^0_1$ (whether or not it is a $K$–space), $F$ is the Ribe function $F_0$;

(b) $E$ is the linear span of the usual unit vector basis for the James space, under the James norm; $F$ is any quasi–linear function on $E$;

(c) $E$ is the span of the usual unit vector basis in $\ell_p(\ell^1_n)$, for $1 < p < \infty$ (this is a reflexive non–$K$ space); $F$ will be described below.

**Proof.** For (a), let $H = \{ x \in E : \sum x_i = 0 \}$. Note that if $x, y \in H$ and $x$ and $y$ have disjoint supports, $F(x + y) = F(x) + F(y)$. Since $H$ has codimension at
most 1 in \( E \) and \( E \) is infinite dimensional, there is a sequence of non–zero elements \((x_i)\) in \( H \) satisfying \( \sup (\text{support } x_i) < \inf (\text{support } x_{i+1}) \) for all \( i \).

As remarked above, \( F_0 \) is linear on \( \text{span}(x_i) \), so if we define \( T(x_i) = F(x_i) \), the linear function \( T \) certainly satisfies hypothesis (1) of the theorem. Therefore the theorem applies to \( E \).

For (b), it is known that the even unit vectors \( e_{2n} \) span a pre–Hilbert subspace of the James space (see [1]). The \( B \)-convexity of \( \text{span}(e_{2n}) \) and Theorems 2.6 of [2] and 2.5 of [3] imply that there is a linear map \( T \) on \( \text{span}(e_{2n}) \) such that \(|T(x) - F(x)| \leq C\|x\|\) for all \( x \) in \( \text{span}(e_{2n}) \). Therefore the theorem applies.

(c) For each \( n \) let \((e_{i,n})\) be the usual unit vector basis of \( \ell^1\), and let \( E \) be the span of the \( e_{i,n} \) in \( \ell_p(\ell^1) \). Let \((c_n)\) be any sequence in \( \ell_q, \frac{1}{p} + \frac{1}{q} = 1 \). Let \( F_0 \) be the Ribe function and define \( F \) on \( E \) by

\[
F((x_n)) = \sum_n c_n F_0(x_n).
\]

We claim that \( F \) is quasi–linear. For this, if \((x_n)\) and \((y_n)\) are in \( E \), the sequences \((\|x_n\|_1)\) and \((\|y_n\|_1)\) are \( \ell_p \) sequences, and for each \( n \),

\[
|c_n F_0(x_n + y_n) - c_n F_0(x_n) - c_n F_0(y_n)| \\
\leq c_n (\|x_n\|_1 + \|y_n\|_1).
\]

From Hölder’s inequality,

\[
|F((x_n + y_n)) - F((x_n)) - F((y_n))| \\
\leq \|(c_n)\|_q (\|x_n\|_p + \|y_n\|_p).
\]

Theorem 4.7 of [2] gives that \( E \) is not a \( K \)–space. The \( F \) just defined proves this directly, for suppose there is a linear \( T \) on \( E \) with \(|T(x) - F(x)| \leq C\|x\|\) for
all $x$ in $E$. Then since $F(e_{i,n}) = 0$ for all $i$, $n$, $|T(e_{i,n})| \leq C$ for all $i$, $n$. But

$$F\left(\frac{1}{n} \sum_{i=1}^{n} e_{i,n}\right) = -c_n \log n,$$

a contradiction if we choose $c_n$ so that $(c_n \log n)$ is unbounded.

Finally, our theorem applies in this situation. To show this, for each $n$ pick a unit vector $x_n$ in $\ell_1^n$. The sequence $(x_n)$ is equivalent to the usual basis of $\ell_p$, which is $B$–convex; the results already mentioned imply that there is a linear $T$ on $\text{span}(x_n)$ such that $|T(x) - F(x)| \leq C\|x\|$ for all $x$ in $\text{span}(x_n)$. This finishes the proof. □

Note that, because of the separability, the corollary applies to the completions of the twisted sums in (a)–(c) above.

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