Tightness and weak convergence of probabilities on the Skorokhod space on the dual of a nuclear space and applications

by

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Abstract. Let $\Phi'$ denote the strong dual of a nuclear space $\Phi$ and let $D_T(\Phi'_\beta)$ be the Skorokhod space of right-continuous with left limits (càdlàg) functions from $[0,T]$ into $\Phi'_\beta$. We introduce the concepts of cylindrical random variables and cylindrical measures on $D_T(\Phi'_\beta)$, and prove analogues of the regularization theorem and Minlos theorem for extensions of these objects to bona fide random variables and probability measures on $D_T(\Phi'_\beta)$. Further, we establish analogues of Lévy’s continuity theorem to provide necessary and sufficient conditions for tightness of a family of probability measures on $D_T(\Phi'_\beta)$ and sufficient conditions for weak convergence of a sequence of probability measures on $D_T(\Phi'_\beta)$. Extensions of the above results to the space $D_\infty(\Phi'_\beta)$ of càdlàg functions from $[0,\infty)$ into $\Phi'_\beta$ are also given. Next, we apply our results to the study of weak convergence of $\Phi'_\beta$-valued càdlàg processes and in particular to Lévy processes. Finally, we apply our theory to the study of tightness and weak convergence of probability measures on the Skorokhod space $D_\infty(H)$ where $H$ is a Hilbert space.

1. Introduction. Let $E$ be a topological space and let $D_T(E)$ denote the collection of all right-continuous with left limits (càdlàg) maps $x : [0,T] \to E$. For the case of $E$ being a separable metric space, Skorokhod [37] introduced four topologies on the space $D_T(E)$, with $J1$ being the topology most widely used.

Under the assumption that $\Phi$ is a Fréchet nuclear space with strong dual $\Phi'_\beta$, Mitoma [29] introduced the Skorokhod $J1$ topology on $D_T(\Phi'_\beta)$ and provided characterizations for compact subsets of it. He also introduced sufficient conditions for uniform tightness and weak convergence of sequences of probability measures on $D_T(\Phi'_\beta)$ in terms of uniform tightness and weak convergence of their finite dimensional projections. The work of Mitoma was

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later extended by Fouque \cite{17} to the cases when $\Phi$ is either a countable inductive limit of Fréchet nuclear spaces or the strong dual of a Fréchet nuclear space.

A further extension of the work of Mitoma to $D_T(E)$, where $E$ is a completely regular space, was carried out by Jakubowski \cite{20}. Jakubowski assumes that $E$ has metrizable compacts and that $\{\mu_i\}$ is a family of probability measures on $D_T(E)$ satisfying the compact containment condition and such that $\mu_i \circ f^{-1}$ is uniformly tight on $D_1(\mathbb{R})$ for a set $F$ of continuous functions $f : D_1(E) \to D_1(\mathbb{R})$ that satisfy certain conditions. In a recent work and under the same assumptions on $E$, Kouritzin \cite{27} introduces characterizations of uniform tightness under the compact containment condition and under (several equivalent) modulus of continuity conditions.

The main objective of this article is to provide sufficient and necessary conditions for tightness and weak convergence of random objects on $D_T(\Phi'_\beta)$, where $\Phi'_\beta$ is the strong dual of a general nuclear space $\Phi$, or more generally when $\Phi$ is a Hausdorff locally convex space. We do this by studying properties of the Fourier transforms of these random objects and by proving analogues of the Minlos theorem and Lévy’s continuity theorem on $D_T(\Phi'_\beta)$.

Our motivation is twofold. First, since the pioneering work of Mitoma many applications have emerged, for example [6, 7, 11, 18, 25, 32, 34], to cite but a few. Since we are considering general nuclear spaces, we hope that with our work more applications will appear, especially for modelling of random phenomena taking values on other examples of nuclear spaces not covered by the works of Mitoma and Fouque (see Sect. 9). Second, in [14] a new theory of stochastic integration and stochastic PDE’s in $\Phi'_\beta$ driven by Lévy noise has been introduced. Much of the work in this article goes towards showing convergence of solutions of these stochastic PDE’s. The results will appear elsewhere.

We now give a description of our work. Our first task is to characterize the compact subsets of $D_T(\Phi'_\beta)$. We show that if $\Phi$ is a barrelled nuclear space, then for a set $A \subseteq D_T(\Phi'_\beta)$ compactness of finite-dimensional projections of $A$ implies compactness of $A$ in $D_T(\Phi'_\beta)$ (Theorem 3.5). This extends previous results obtained by Mitoma \cite{29}.

Then, we introduce the concepts of cylindrical measures and cylindrical random variables on $D_T(\Phi'_\beta)$ by considering the space-time algebra of cylindrical subsets of $D_T(\Phi'_\beta)$. We stress that our definitions are not a particular case of the usual theory of cylindrical measures and cylindrical random variables on locally convex spaces, as it is well-known that $D_T(\Phi'_\beta)$ is not a topological vector space. Here, we show an extended version of the regularization theorem given in [13] that says that if $\{X_t\}_{t \in [0,T]}$ is a cylindrical process in $\Phi'$ where the maps $X_t : \Phi \to L^0(\Omega, \mathcal{F}, \mathbb{P})$ are equicontinuous at
the origin, then this cylindrical process has an extension to a $D_T(\Phi'_\beta)$-valued random variable with a Radon probability distribution (Theorem 4.7). We also show an extension of the Minlos theorem (Theorem 4.9) that states that a cylindrical measure on $D_T(\Phi'_\beta)$ that has equicontinuous Fourier transforms for its time projections has a Radon measure extension on $D_T(\Phi'_\beta)$.

Next, we move to the core of this article that consists in establishing necessary and sufficient conditions for a family of probability measures $\{\mu_\alpha : \alpha \in A\}$ on $D_T(\Phi'_\beta)$ to be uniformly tight (Theorem 5.2). In particular, we show that if the finite-dimensional projections of the measures are uniformly tight and if the Fourier transforms of the time projections of the measures are equicontinuous at the origin, then $\{\mu_\alpha : \alpha \in A\}$ is uniformly tight on $D_T(\Phi'_\beta)$. Observe that in contrast to [20, 27] we do not assume that compact subsets of $\Phi'_\beta$ are metrizable, nor that the compact containment condition holds. Furthermore, we show that if $\Phi$ is also ultrabornological then only the uniform tightness of finite-dimensional projections has to be assumed. We extend our results to the space $D_\infty(\Phi'_\beta)$ of càdlàg mappings from $[0, \infty)$ into $\Phi'_\beta$. Our results generalize those obtained by Mitoma [29] and Fouque [17].

At the center of our arguments is the idea of using equicontinuity of Fourier transforms of the time projections of the measures $\{\mu_\alpha : \alpha \in A\}$ on $D_T(\Phi'_\beta)$ to set the problem on the space $D_T((\widetilde{\Phi}_\theta)'_\beta)$ equipped with its Skorokhod topology, where $\widetilde{\Phi}_\theta$ denotes the completion of $\Phi$ equipped with a weaker (compared to nuclear) countably Hilbertian topology $\theta$. The advantage of using this methodology is that $\widetilde{\Phi}_\theta$ is a complete, separable, pseudometrizable space (not necessarily nuclear), hence linear operators and measures defined on $\widetilde{\Phi}_\theta$ have better properties than on $\Phi$. Previously, we have used this tool in [13] to prove existence of continuous or càdlàg versions of cylindrical processes in $\Phi'$.

Our next goal is to provide sufficient conditions for weak convergence of probability measures on $D_\infty(\Phi'_\beta)$ and of $\Phi'_\beta$-valued càdlàg processes (Theorems 6.2 and 6.5). Again our results generalize those in [17, 29], and furthermore we consider the completely new case of convergence of cylindrical processes in $\Phi'$. Applications are then given to weak convergence in $D_\infty(\Phi'_\beta)$ for a sequence of $\Phi'_\beta$-valued Lévy processes in terms of properties of the characteristics of their Lévy-Khinchin formula (Theorem 7.2).

Finally, under the assumption that $\Phi$ is a (Hausdorff) locally convex space and by considering its Sazonov topology, we indicate how our methods for the nuclear space setting extend to provide sufficient conditions for uniform tightness and weak convergence of probability measures on $D_\infty(\Phi'_\beta)$ (Theorems 8.2 and 8.3). A particular case of great importance is when $H$ is a Hilbert space because in that case our result represents an extension
of Sazonov’s theorem and Lévy’s continuity theorem to the space $D_\infty(H)$ (Theorems 8.5 and 8.6). We hope that these results could generate new applications, especially for the Hilbert space setting.

The organization of the paper is the following. In Sect. 2 we list some important notions concerning nuclear spaces and their duals, and also properties of cylindrical measures and cylindrical processes in duals of nuclear spaces. The Skorokhod topology on $D_T(\Phi'_\beta)$ is introduced in Sect. 3 and characterizations for its compact subsets are given. In Sect. 4 we introduce the concepts of cylindrical measures and cylindrical random variables in $D_T(\Phi'_\beta)$ and show the regularization and Minlos theorems in $D_T(\Phi'_\beta)$. Later, in Sect. 5 we study the uniform tightness of probability measures on $D_T(\Phi'_\beta)$ and on $D_\infty(\Phi'_\beta)$. In Sect. 6 we prove a Lévy continuity theorem for weak convergence of probability measures and stochastic processes in $D_\infty(\Phi'_\beta)$. Then, in Sect. 7 we apply our results to characterizing weak convergence in $D_\infty(\Phi'_\beta)$ of a sequence of Lévy processes. In Sect. 8 we show how our results for the dual of a nuclear space setting extend to the case when $\Phi$ is locally convex. Finally, in Sect. 9 we consider concrete examples of nuclear spaces, give some remarks, and compare our results with those in the literature.

2. Preliminaries

2.1. Nuclear spaces and their strong duals. In this section we introduce our notation and review some of the key concepts on nuclear spaces and their dual spaces that we will need throughout this paper. For more information see [35, 39]. Only vector spaces over $\mathbb{R}$ will be considered.

A locally convex space is called quasi-complete if each of its bounded and closed subsets is complete. A barrelled space is a locally convex space on which every lower semicontinuous seminorm is continuous. A locally convex space that is the inductive limit of a family of normed (respectively Banach) spaces is called bornological (respectively ultrabornological).

Let $\Phi$ be a locally convex space. If $p$ is a continuous seminorm on $\Phi$ and $r > 0$, the closed ball $B_p(r) = \{\phi \in \Phi : p(\phi) \leq r\}$ is a closed, convex, balanced neighborhood of zero in $\Phi$. A continuous seminorm (respectively a norm) $p$ on $\Phi$ is called Hilbertian if $p(\phi)^2 = Q(\phi, \phi)$ for all $\phi \in \Phi$, where $Q$ is a symmetric, non-negative bilinear form (respectively inner product) on $\Phi \times \Phi$. Let $\Phi_p$ be the Hilbert space that corresponds to the completion of the pre-Hilbert space $(\Phi/\ker(p), \tilde{p})$, where $\tilde{p}(\phi + \ker(p)) = p(\phi)$ for each $\phi \in \Phi$. The quotient map $\Phi \rightarrow \Phi/\ker(p)$ has a unique continuous linear extension $i_p : \Phi \rightarrow \Phi_p$.

Let $q$ be another continuous Hilbertian seminorm on $\Phi$ for which $p \leq q$. In this case, $\ker(q) \subseteq \ker(p)$. Moreover, the inclusion map from $\Phi/\ker(q)$ into
\( \Phi / \ker(p) \) is linear and continuous, and therefore it has a unique continuous extension \( i_{p,q} : {\Phi}_q \to {\Phi}_p \). Furthermore, \( i_p = i_{p,q} \circ i_q \).

We denote by \( \Phi' \) the topological dual of \( \Phi \) and by \( f[\phi] \) the canonical pairing of \( f \in \Phi' \) and \( \phi \in \Phi \). We denote by \( \Phi'_\beta \) the dual space \( \Phi' \) equipped with its strong topology \( \beta \), i.e. \( \beta \) is the topology on \( \Phi' \) generated by the family of seminorms \( \{ \eta_B \} \), where for each \( B \subseteq \Phi \) bounded we have \( \eta_B(f) = \sup \{|f[\phi]| : \phi \in B\} \) for all \( f \in \Phi' \). If \( p \) is a continuous Hilbertian seminorm on \( \Phi \), then we denote by \( \Phi'_{p,\beta} \) the Hilbert space dual to \( \Phi_{p,\beta} \). The dual norm \( p' \) on \( \Phi'_{p,\beta} \) is given by \( p'(f) = \sup \{|f[\phi]| : \phi \in B_p(1)\} \) for all \( f \in \Phi'_{p,\beta} \). Moreover, the dual operator \( i'_p \) corresponds to the canonical inclusion from \( \Phi'_{p,\beta} \) into \( \Phi'_{p,\beta} \) and it is linear and continuous.

Let \( p \) and \( q \) be continuous Hilbertian seminorms on \( \Phi \) with \( p \leq q \). The space of continuous linear operators (respectively Hilbert–Schmidt operators) from \( \Phi_q \) into \( \Phi_p \) is denoted by \( \mathcal{L}(\Phi_q, \Phi_p) \) (respectively \( \mathcal{L}_2(\Phi_q, \Phi_p) \)) and the operator norm (respectively Hilbert–Schmidt norm) is denoted by \( \| \cdot \|_{\mathcal{L}(\Phi_q, \Phi_p)} \) (respectively \( \| \cdot \|_{\mathcal{L}_2(\Phi_q, \Phi_p)} \)). We employ an analogous notation for operators between the dual spaces \( \Phi'_{p,\beta} \) and \( \Phi'_{q,\beta} \).

Let us recall that a (Hausdorff) locally convex space \((\Phi, \mathcal{T})\) is called nuclear if its topology \( \mathcal{T} \) is generated by a family \( P \) of Hilbertian seminorms such that for each \( p \in P \) there exists \( q \in P \) with \( p \leq q \) such that the canonical inclusion \( i_{p,q} : {\Phi}_q \to {\Phi}_p \) is Hilbert–Schmidt. Other equivalent definitions of nuclear spaces can be found in [33, 39].

Let \( \Phi \) be a nuclear space. If \( p \) is a continuous Hilbertian seminorm on \( \Phi \), then the Hilbert space \( \Phi_p \) is separable (see [33, Proposition 4.4.9 and Theorem 4.4.10, p. 82]). Now, let \( \{p_n\}_{n \in \mathbb{N}} \) be an increasing sequence of continuous Hilbertian seminorms on \( \Phi \). We denote by \( \theta \) the locally convex topology on \( \Phi \) generated by the family \( \{p_n\}_{n \in \mathbb{N}} \). The topology \( \theta \) is weaker than the nuclear topology on \( \Phi \). We will call \( \theta \) a weaker countably Hilbertian topology on \( \Phi \) and we denote by \( \Phi_{\theta} \) the space \((\Phi, \theta)\) and by \( \Phi_{\widetilde{\theta}} \) its completion. The space \( \Phi_{\widetilde{\theta}} \) is a separable, complete, pseudo-metrizable (hence Baire) locally convex space (see [13, Proposition 2.4]). Moreover, \( \Phi_{\theta} \) is ultrabornological because it is bornological and complete (see [30, Example 13.2.8(b) and Theorem 13.2.12, pp. 445, 449]).

2.2. Cylindrical and stochastic processes. Let \( E \) be a topological space and denote by \( \mathcal{B}(E) \) its Borel \( \sigma \)-algebra. Recall that a Borel measure \( \mu \) on \( E \) is called a Radon measure if for every \( \Gamma \in \mathcal{B}(E) \) and \( \epsilon > 0 \), there exists a compact set \( K \subseteq \Gamma \) such that \( \mu(\Gamma \setminus K) < \epsilon \). In general, not every Borel measure on \( E \) is Radon. We denote by \( \mathcal{M}_R^0(E) \) and by \( \mathcal{M}_R^1(E) \) the spaces of all bounded Radon measures and of all Radon probability measures on \( E \). A subset \( M \subseteq \mathcal{M}_R^0(E) \) is called uniformly tight if (i) \( \sup \{ \mu(E) : \mu \in M \} < \infty \), and (ii) for every \( \epsilon > 0 \) there exists a compact set \( K \subseteq E \) such that \( \mu(K^c) < \epsilon \).
for all \( \mu \in M \). A sequence \( (\mu_n : n \in \mathbb{N}) \subseteq \mathfrak{M}^1_R(E) \) converges weakly to \( \mu \in \mathfrak{M}^1_R(E) \) if \( \int_E f d\mu_n \to \int_E f d\mu \) for every \( f \in C_b(E) \); we then write \( \mu_n \Rightarrow \mu \).

Let \( \Phi \) be a locally convex space. Given \( M \subseteq \Phi \), the cylindrical algebra on \( \Phi' \) based on \( M \) is the collection \( \mathcal{Z}(\Phi', M) \) of all the cylindrical sets of the form \( \mathcal{Z}(\phi_1, \ldots, \phi_n; A) = \{ f \in \Phi' : (f[\phi_1], \ldots, f[\phi_n]) \in A \} \) where \( n \in \mathbb{N} \), \( \phi_1, \ldots, \phi_n \in M \) and \( A \in \mathcal{B}(\mathbb{R}^n) \). The \( \sigma \)-algebra generated by \( \mathcal{Z}(\Phi', M) \) is denoted by \( \mathcal{C}(\Phi', M) \). If \( M \) is finite then \( \mathcal{C}(\Phi', M) = \mathcal{Z}(\Phi', M) \). Moreover, we always have \( \mathcal{C}(\Phi') := \mathcal{C}(\Phi', \Phi) \subseteq \mathcal{B}(\Phi'_\beta) \), but equality does not hold in general. A function \( \mu : \mathcal{Z}(\Phi', \Phi) \to [0, \infty] \) is called a cylindrical measure on \( \Phi' \) if for each finite subset \( M \subseteq \Phi \) the restriction of \( \mu \) to \( \mathcal{C}(\Phi', M) \) is a measure. A cylindrical measure \( \mu \) is said to be finite if \( \mu(\Phi') < \infty \), and a cylindrical probability measure if \( \mu(\Phi') = 1 \). The Fourier transform of \( \mu \) is the function \( \hat{\mu} : \Phi \to \mathbb{C} \) defined by

\[
\hat{\mu}(\phi) = \int_{\Phi'} e^{if[\phi]} \mu(df), \quad \forall \phi \in \Phi.
\]

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a (complete) probability space. Denote by \( L^0(\Omega, \mathcal{F}, \mathbb{P}) \) the space of equivalence classes of real-valued random variables defined on \( (\Omega, \mathcal{F}, \mathbb{P}) \). We always consider \( L^0(\Omega, \mathcal{F}, \mathbb{P}) \) to be equipped with the topology of convergence in probability, and in this case it is a complete, metrizable, topological vector space.

A cylindrical random variable in \( \Phi' \) is a linear map \( X : \Phi \to L^0(\Omega, \mathcal{F}, \mathbb{P}) \). If \( Z = \mathcal{Z}(\phi_1, \ldots, \phi_n; A) \) is a cylindrical set, then for \( \phi_1, \ldots, \phi_n \in \Phi \) and \( A \in \mathcal{B}(\mathbb{R}^n) \) let

\[
\mu_X(Z) := \mathbb{P}(X(\phi_1), \ldots, X(\phi_n)) \in A).
\]

The map \( \mu_X \) is a cylindrical probability measure on \( \Phi' \), called the cylindrical distribution of \( X \). The Fourier transform of \( X \) is defined to be the Fourier transform \( \hat{\mu}_X : \Phi \to \mathbb{C} \) of its cylindrical distribution \( \mu_X \).

Let \( X \) be a \( \Phi'_\beta \)-valued random variable, i.e. \( X : \Omega \to \Phi'_\beta \) is a \( \mathcal{F}/\mathcal{B}(\Phi'_\beta) \)-measurable map. We denote by \( \mu_X \) the probability distribution of \( X \), i.e. \( \mu_X(\Gamma) = \mathbb{P}(X \in \Gamma) \) for all \( \Gamma \in \mathcal{B}(\Phi'_\beta) \); it is a Borel probability measure on \( \Phi'_\beta \). For each \( \phi \in \Phi \) we denote by \( X[\phi] \) the real-valued random variable defined by \( X[\phi](\omega) := X(\omega)[\phi] \) for all \( \omega \in \Omega \). It is clear that the mapping \( \phi \mapsto X[\phi] \) defines a cylindrical random variable.

If \( X \) is a cylindrical random variable in \( \Phi' \), a \( \Phi'_\beta \)-valued random variable \( Y \) is a called a version of \( X \) if for every \( \phi \in \Phi \), \( X(\phi) = Y[\phi] \) \( \mathbb{P} \)-a.e. A \( \Phi'_\beta \)-valued random variable \( X \) is called regular if there exists a weaker countably Hilbertian topology \( \theta \) on \( \Phi \) such that \( \mathbb{P}(X \in (\Phi_\theta)') = 1 \).

Let \( J = [0, \infty) \) or \( J = [0, T] \) for some \( T > 0 \). We say that \( X = \{X_t\}_{t \in J} \) is a cylindrical process in \( \Phi' \) if \( X_t \) is a cylindrical random variable for each
Then there exist a weaker countably Hilbertian topology on $\Phi$ for every $\phi \in \Phi$. We will say that it is the cylindrical process determined/induced by $X$. A $\Phi'_\beta$-valued processes $Y = \{Y_t\}_{t \in J}$ is said to be a $\Phi'_\beta$-valued version of the cylindrical process $X = \{X_t\}_{t \in J}$ on $\Phi'$ if for each $t \in J$, $Y_t$ is a $\Phi'_\beta$-valued version of $X_t$.

Let $X = \{X_t\}_{t \in J}$ be a $\Phi'_\beta$-valued process. We say that $X$ is continuous (respectively càdlàg) if for $\mathbb{P}$-a.e. $\omega \in \Omega$, the sample paths $t \mapsto X_t(w) \in \Phi'$ of $X$ are continuous (respectively right-continuous with left limits). We say that the process $X$ is regular if for every $t \in J$, $X_t$ is a regular random variable.

A result of fundamental importance in this work is the following:

**Theorem 2.1 (Regularization Theorem; [13, Theorem 3.2]).** Let $\Phi$ be a nuclear space. Let $X = \{X_t\}_{t \geq 0}$ be a cylindrical process in $\Phi'$ satisfying:

1. For each $\phi \in \Phi$, the real-valued process $X(\phi) = \{X_t(\phi)\}_{t \geq 0}$ has a continuous (respectively càdlàg) version.
2. For every $T > 0$, the family $\{X_t : t \in [0, T]\}$ of linear maps from $\Phi$ into $L^0(\Omega, \mathcal{F}, \mathbb{P})$ is equicontinuous.

Then there exist a weaker countably Hilbertian topology $\theta$ on $\Phi$ and a $(\Phi'_{\theta})'_\beta$-valued continuous (respectively càdlàg) process $Y = \{Y_t\}_{t \geq 0}$ such that for every $\phi \in \Phi$, $Y[\phi] = \{Y_t(\phi)\}_{t \geq 0}$ is a version of $X(\phi) = \{X_t(\phi)\}_{t \geq 0}$. Moreover, $Y$ is a $\Phi'_\beta$-valued, regular, continuous (respectively càdlàg) version of $X$ that is unique up to indistinguishable versions.

### 3. The Skorokhod topology in $D_T(\Phi'_{\beta})$.

Let $\Phi$ be a (Hausdorff) locally convex space and let $\{q_\gamma(\cdot) : \gamma \in \Gamma\}$ be a family of seminorms generating the strong topology $\beta$ on $\Phi'$. Fix $T > 0$ and denote by $D_T(\Phi'_{\beta})$ the collection of all càdlàg (i.e. right-continuous with left limits) maps from $[0, T]$ into $\Phi'_{\beta}$.

Following [20] (see also [29]), for a given $\gamma \in \Gamma$ we consider the pseudometric $d_\gamma$ on $D_T(\Phi'_{\beta})$ given by

$$d_\gamma(x, y) = \inf_{\lambda \in \Lambda} \left\{ \sup_{t \in [0, T]} q_\gamma(x(t) - y(t)) + \sup_{0 \leq s < t \leq T} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right| \right\}$$

for all $x, y \in D_T(\Phi'_{\beta})$, where $\Lambda$ denotes the set of all strictly increasing continuous maps $\lambda$ from $[0, T]$ onto itself.

The family $\{d_\gamma : \gamma \in \Gamma\}$ of seminorms generates a completely regular topology on $D_T(\Phi'_{\beta})$ that is known as the Skorokhod topology (also known as the J1 topology). This topology does not depend on the particular choice of seminorms $\{q_\gamma(\cdot) : \gamma \in \Gamma\}$ on $\Phi'_{\beta}$ (see [20, Theorem 1.3]).
Let $\Phi$ be a nuclear space and let $q$ be a continuous seminorm on $\Phi$. Very important for our further developments is the space $D_T(\Phi'_q)$. Observe that because $\Phi'_q$ is a separable Banach space, $D_T(\Phi'_q)$ is complete, separable and metrizable (see [10, 23]).

The next result characterizes the compact subsets of $D_T(\Phi'_q)$. For its statement we will need the following moduli of continuity:

1. if $x \in D_T(\Phi'_q)$, $\delta > 0$, let
   \[ w'_x(\delta, q) = \inf_{\{t_i\}} \max_{1 \leq i \leq n} \sup_{s, t \in [t_{i-1}, t_i]} q'(x(t) - x(s)) : s, t \in [t_{i-1}, t_i], \]
2. if $x \in D_T(\Phi'_q)$, $\phi \in \Phi'_q$, $\delta > 0$, let
   \[ w'_x(\delta, \phi) = \inf_{\{t_i\}} \max_{1 \leq i \leq n} \sup_{s, t \in [t_{i-1}, t_i]} |x(t)\phi - x(s)\phi| : s, t \in [t_{i-1}, t_i], \]
3. if $x \in D_T(\mathbb{R})$, $\delta > 0$, let
   \[ w'_x(\delta) = \inf_{\{t_i\}} \max_{1 \leq i \leq n} \sup_{s, t \in [t_{i-1}, t_i]} |x(t) - x(s)| : s, t \in [t_{i-1}, t_i], \]

where the infimum is taken over all finite partitions $0 = t_0 < t_1 < \cdots < t_n = T$, $t_i - t_{i-1} > \delta$, $i = 1, \ldots, n$.

**Proposition 3.1 ([23, Theorem 2.4.3]).** Let $q$ be a continuous seminorm on $\Phi$. Then $A \subseteq D_T(\Phi'_q)$ is compact if and only if

1. there exists $K \subseteq \Phi'_q$ compact such that $x(t) \in K$ for all $t \in [0, T], x \in A$,
2. $\lim_{\delta \to 0^+} \sup_{x \in A} w'_x(\delta, q) = 0$.

Now let $\theta$ be a weaker countably Hilbertian topology on the nuclear space $\Phi$. We proceed to study some properties of the space $D_T((\tilde{\Phi}_\theta)'_\beta)$ equipped with its Skorokhod topology, which we will denote temporarily by $D_{s,T}((\tilde{\Phi}_\theta)'_\beta)$ to distinguish it from the inductive limit topology that we introduce below.

Let $(p_n : n \in \mathbb{N})$ be an increasing sequence of continuous Hilbertian seminorms on $\Phi$ generating the topology $\theta$. Because (see [13, Proposition 2.4])

\[ (\tilde{\Phi}_\theta)'_\beta = \bigcup_{n \in \mathbb{N}} \Phi'_{p_n}, \]

the Banach–Steinhaus theorem implies that (see e.g. [20, Proposition 5.3])

\[ D_T((\tilde{\Phi}_\theta)'_\beta) = \bigcup_{n \in \mathbb{N}} D_T(\Phi'_{p_n}). \]

Moreover because the canonical inclusion from $\Phi'_{p_n}$ into $(\tilde{\Phi}_\theta)'_\beta$ is continuous, for each $n \in \mathbb{N}$ the inclusion from $D_T(\Phi'_{p_n})$ into $D_{s,T}((\tilde{\Phi}_\theta)'_\beta)$ is continuous (see [20, Lemma 1.5]).

In view of (3.2) and following an idea of Pérez-Abreu and Tudor [32], we can also consider on $D_T((\tilde{\Phi}_\theta)'_\beta)$ the inductive limit topology with respect
to the spaces \((D_T(\Phi'_{p_n}) : n \in \mathbb{N})\), i.e. the finest topology for which the
inclusions from \(D_T(\Phi'_{p_n})\) into \(D_T((\tilde{\Phi}_p)'_\beta)\) are continuous. We denote the space
\(D_T((\tilde{\Phi}_p)'_\beta)\) equipped with this topology by \(D_{i,T}((\tilde{\Phi}_p)'_\beta)\). We summarize the
properties of \(D_{i,T}((\tilde{\Phi}_p)'_\beta)\) and \(D_{s,T}((\tilde{\Phi}_p)'_\beta)\) in the following result.

**Proposition 3.2.** Let \(\theta\) be a weaker countably Hilbertian topology on the
nuclear space \(\Phi\).

1. The spaces \(D_{i,T}((\tilde{\Phi}_p)'_\beta)\) and \(D_{s,T}((\tilde{\Phi}_p)'_\beta)\) are Suslin.
2. The compact subsets of \(D_{s,T}((\tilde{\Phi}_p)'_\beta)\) are metrizable.
3. The canonical inclusion from \(D_{s,T}((\tilde{\Phi}_p)'_\beta)\) (and hence from \(D_{i,T}((\tilde{\Phi}_p)'_\beta)\))
   into \(D_T(\Phi'_\beta)\) is continuous.

**Proof.** (1) The space \(D_{i,T}((\tilde{\Phi}_p)'_\beta)\), being the inductive limit of the Suslin
spaces \(D_T(\Phi'_{p_n})\), is again a Suslin space (see [39, Proposition A.4(c), p. 551]).
Now, because the canonical inclusion from \(D_{i,T}((\tilde{\Phi}_p)'_\beta)\) into \(D_{s,T}((\tilde{\Phi}_p)'_\beta)\) is
continuous, it follows that \(D_{s,T}((\tilde{\Phi}_p)'_\beta)\) is also Suslin.

(2) Because \(\tilde{\Phi}_\theta\) is ultrabornological, hence barrelled, if \(K \subseteq (\tilde{\Phi}_p)'_\beta\) is
compact then it is equicontinuous (see [35, Theorem IV.5.2, p. 141]). Therefore,
there exists a continuous Hilbertian seminorm \(p\) on \(\tilde{\Phi}_\theta\) such that \(K \subseteq B_{p'}(1)\).
Consequently, the set \(K\) is metrizable. Then, because each compact subset
of \((\tilde{\Phi}_p)'_\beta\) is metrizable, the space \(D_{s,T}((\tilde{\Phi}_p)'_\beta)\) inherits the same property (see
[20, Proposition 1.6(vii)]).

(3) The conclusion follows from the fact that the topology on \((\tilde{\Phi}_p)'_\beta\) is
finer than the induced topology from \(\Phi'_\beta\) and from [20, Lemma 1.5].

**Remark 3.3.** If \(\Phi\) is a Fréchet nuclear space and \(\theta\) coincides with the
nuclear topology on \(\Phi\) (hence \(\tilde{\Phi}_\theta = \Phi\)), it is shown in [32, Proposition 3.1]
that \(\mathcal{B}(D_{i,T}(\Phi'_\beta)) = \mathcal{B}(D_{s,T}(\Phi'_\beta))\).

**Note 3.4.** From now on, unless otherwise specified we will always assume
that \(D_T((\tilde{\Phi}_p)'_\beta)\) and \(D_T(\Phi'_\beta)\) are equipped with their Skorokhod topologies.

The next result gives characterizations for compact subsets of \(D_T(\Phi'_\beta)\)
when \(\Phi\) is a barrelled nuclear space. We will need the following definition:
for each \(\phi \in \Phi\), let \(\Pi_\phi : D_T(\Phi'_\beta) \to D_T(\mathbb{R})\) be the space projection
given by \(x \mapsto x[\phi] = \{x(t)[\phi]\}_{t \in [0,T]}\).

**Theorem 3.5.** Let \(\Phi\) be a nuclear space and let \(A \subseteq D_T(\Phi'_\beta)\). Consider
the following statements:

1. \(A\) is compact in \(D_T(\Phi'_\beta)\).
2. For any \(\phi \in \Phi\), the set \(\Pi_\phi(A) = \{x[\phi] : x \in A\}\) is compact in \(D_T(\mathbb{R})\).
There exists a continuous Hilbertian seminorm \( q \) on \( \Phi \) such that \( A \) is compact in \( D_T(\Phi^\prime_q) \).

Then \((1) \Rightarrow (2)\), \((3) \Rightarrow (1)\), and if \( \Phi \) is a barrelled nuclear space, then also \((2) \Rightarrow (3)\).

Proof. First observe from Proposition 3.2(3) that the canonical inclusion \( D_T(\Phi^\prime_q) \rightarrow D_T(\Phi^\prime_{q'}) \) is continuous, therefore if \( A \) is compact in \( D_T(\Phi^\prime_q) \), it is so in \( D_T(\Phi^\prime_{q'}) \). This shows \((3) \Rightarrow (1)\). Similarly, \((1) \Rightarrow (2)\) is a direct consequence of the continuity of the space projection \( \Pi_\phi \) for each \( \phi \in \Phi \).

Now assume that \( \Phi \) is a barrelled nuclear space. We are going to show that \((2) \Rightarrow (3)\). Let \( K = \bigcup_{x \in A}{\{x(t) : t \in [0, T]\}} \). Clearly, \( A \subseteq D_T((K, \beta \cap K)) \), where \((K, \beta \cap K)\) denotes the subspace \( K \subseteq \Phi^\prime_{q'} \) equipped with the subspace topology induced on \( K \) by the strong topology \( \beta \) on \( \Phi^\prime \). Moreover, because for each \( \phi \in \Phi \) the set \( \Pi_\phi(A) \) is compact, we have

\[
\sup_{x \in A} \sup_{t \in [0, T]} |x(t)|_{\phi} < \infty, \quad \forall \phi \in \Phi.
\]

Therefore the set \( K \subseteq \Phi^\prime_{q'} \) is weakly bounded and because \( \Phi \) is barrelled, this implies that \( K \) is strongly bounded and equicontinuous (see [35, Theorem IV.5.2, p. 141]). Then its polar \( K^0 \) is a neighborhood of zero in \( \Phi \). But because \( \Phi \) is nuclear, there exists a continuous Hilbertian seminorm \( p \) on \( \Phi \) such that \( B_p(1) \subseteq K^0 \). If \( p_{K^0} \) is the continuous seminorm on \( \Phi \) with unit ball \( K^0 \) (the Minkowski functional of \( K^0 \)), then the inclusion \( i_{p_{K^0}, p} : \Phi_p \rightarrow \Phi_{p_{K^0}} \) is continuous, and hence its dual \( i_{p_{K^0}, p}^\prime : \Phi_{\Phi_{K^0}}^\prime \rightarrow \Phi_p \) is also continuous. Let \( q \) be a continuous Hilbertian seminorm on \( \Phi \) such that \( p \leq q \) and \( i_{p, q} : \Phi_q \rightarrow \Phi_p \) is Hilbert–Schmidt. Then \( i_{p, q}^\prime : \Phi_p^\prime \rightarrow \Phi_q^\prime \) is Hilbert–Schmidt. But because \( K \) is the unit ball in \( \Phi_{\Phi_{K^0}}^\prime \), and the map \( i_{p_{K^0}, q} = i_{p, q} \circ i_{p_{K^0}, p}^\prime \) is Hilbert–Schmidt, hence compact, the image of \( K \) under \( i_{p_{K^0}, q}^\prime \) is relatively compact in \( \Phi_{q'}^\prime \).

Thus if \( K \) denotes the closure of \( K \) in \( \Phi_{q'}^\prime \), then \( K \) is compact in \( \Phi_{q'}^\prime \) and \( A \subseteq D_T((K, \beta \cap K)) \subseteq D_T(\Phi_{p_{K^0}}^\prime) \subseteq D_T(\Phi_{q'}^\prime) \) (because the topology on \( \Phi_{p_{K^0}}^\prime \) is finer than that induced by \( \Phi_{q'}^\prime \); see [20, Lemma 1.5]). Hence \( A \) is a subset of \( D_T(\Phi_{q'}^\prime) \) that satisfies the first condition in Proposition 3.1.

We next show that \( A \) also satisfies the second condition there. We will follow some ideas from [23, proof of Theorem 2.4.4].

Let \( (\phi_{q,j}^\prime)_{j \in \mathbb{N}} \subseteq \Phi \) be a complete orthonormal system in \( \Phi_q \). Observe that for each \( j \in \mathbb{N} \), the map \( \Pi_{\phi_{q,j}^\prime} : D_T(\Phi^\prime_{q'}) \rightarrow D_T(\mathbb{R}) \) given by \( x \mapsto x[\phi_{q,j}^\prime] = \{x(t)[\phi_{q,j}^\prime] : t \in [0, T]\} \) is continuous. Then for every \( j \in \mathbb{N} \) the set \( B_j := \Pi_{\phi_{q,j}^\prime}(A) = \{x[\phi_{q,j}^\prime] : x \in A\} \) is compact in \( D_T(\mathbb{R}) \). Therefore

\[
\lim_{\delta \to 0+} \sup_{x \in A} w_{x}(\delta, \phi_{q,j}^\prime) \leq \lim_{\delta \to 0+} \sup_{y \in B_j} w_{y}(\delta) = 0.
\]

Now, recall from our previous arguments that \( x(t) \in K \subseteq B_{p'}(1) \) for all
Let $t \in [0,T]$ and $x \in A$. Then
\[
\sup_{x \in A} w_x'(\delta, \phi_j)^2 \leq \sup_{x \in A} \sup_{t \in [0,T]} 4|x(t)[\phi_j^q]|^2 \leq 4p(\phi_j^q)^2,
\]
but because $i_{p,q}$ is Hilbert–Schmidt we have $\sum_{j=1}^{\infty} p(\phi_j^q)^2 < \infty$. Therefore from the dominated convergence theorem,
\[
\lim_{\delta \to 0^+} \sup_{x \in K} w_x'\delta, \phi_j^q)^2 \leq \sum_{j=1}^{\infty} \lim_{\delta \to 0^+} \sup_{x \in K} w_x'(\delta, \phi_j^q)^2 = 0.
\]
Thus Proposition 3.1 shows that $A$ is compact in $D_T(\Phi'_q)$.

Remark 3.6. If $\Phi$ is a barrelled nuclear space and $(q_i : i \in I)$ is a family of continuous Hilbertian seminorms generating the nuclear topology on $\Phi$, then much as for (3.2) the Banach–Steinhaus theorem shows that $D_T(\Phi'_\beta) = \bigcup_{i \in I} D_T(\Phi'_{q_i})$, and hence we can also define the inductive limit topology on $D_T(\Phi'_\beta)$ with respect to the family of spaces $(D_T(\Phi'_{q_i}) : i \in I)$. Using Theorem 3.5 and similar arguments to those in [32], one can show that the compact subsets of $D_T(\Phi'_\beta)$ coincide under the inductive limit topology and under the Skorokhod topology (see [32, Lemma 3.2] for the details).

4. Measures and random variables in $D_T(\Phi'_\beta)$

Assumption 4.1. Unless otherwise indicated, in this section $\Phi$ will denote a (Hausdorff) locally convex space.

4.1. Cylindrical measures and cylindrical random variables. In this section we introduce the concepts of cylindrical measures and cylindrical random variables in $D_T(\Phi'_\beta)$. We stress that the standard definitions for these objects cannot be applied because $D_T(\Phi'_\beta)$ is not a topological vector space since addition is not continuous. The main motivation for introducing these concepts is that they provide an alternative approach to the usual measurability problems in $D_T(\Phi'_\beta)$ (see [20]).

We start by introducing the class of cylindrical sets. Let $\phi_1, \ldots, \phi_m \in \Phi$, $t_1, \ldots, t_m \in [0,T]$, $m \in \mathbb{N}$. We define the space-time projection map $\Pi_{t_1, \ldots, t_m} : D_T(\Phi'_\beta) \to \mathbb{R}^m$ by
\[
\Pi_{t_1, \ldots, t_m}(x) = (x(t_1)[\phi_1], \ldots, x(t_m)[\phi_m]), \quad \forall x \in D_T(\Phi'_\beta).
\]
If $M = \{\phi_1, \ldots, \phi_m\} \subseteq \Phi$, $I = \{t_1, \ldots, t_m\} \subseteq [0,T]$ and $B \in \mathcal{B}(\mathbb{R}^m)$,
then the set
\[ Z(M, I, B) := (I_{t_1}, \ldots, t_m)^{-1}(B) \]
\[ = \{ x \in D_T(\Phi'_\beta) : (x(t_1)[\phi_1], \ldots, x(t_m)[\phi_m]) \in B \} \]
is called a cylinder set in \( D_T(\Phi'_\beta) \) based on \( (M, I) \). Moreover the collection
\[ \mathcal{C}(D_T(\Phi'_\beta); M, I) = \{ Z(M, I, B) : B \in \mathcal{B}(\mathbb{R}^m) \} \]
is a \( \sigma \)-algebra, called the cylindrical \( \sigma \)-algebra in \( D_T(\Phi'_\beta) \) based on \( (M, I) \). Furthermore, we denote the collection of all cylinder sets in \( D_T(\Phi'_\beta) \) by \( Z(D_T(\Phi'_\beta)) \), and the \( \sigma \)-algebra they generate by \( \mathcal{C}(D_T(\Phi'_\beta)) \). We call \( \mathcal{C}(D_T(\Phi'_\beta)) \) the cylindrical \( \sigma \)-algebra in \( D_T(\Phi'_\beta) \).

One can easily check that \( \mathcal{C}(D_T(\Phi'_\beta)) \subseteq \mathcal{B}(D_T(\Phi'_\beta)) \), but the opposite inclusion is not true in general. Nevertheless, the following result shows that if we consider \( \Phi \) equipped with a weaker countably Hilbertian topology then equality does hold.

**Lemma 4.2.** If \( \theta \) is a weaker countably Hilbertian topology on \( \Phi \), then \( \mathcal{C}(D_T((\Phi_\theta)_\beta)) = \mathcal{B}(D_T((\Phi_\theta)_\beta)) \).

**Proof.** Let \( (p_n : n \in \mathbb{N}) \) be an increasing sequence of continuous Hilbertian seminorms on \( \Phi \) that generates the topology \( \theta \) on \( \Phi \). Then the conclusion follows from (3.2), because for each \( n \in \mathbb{N} \), \( \mathcal{C}(D_T(\Phi'_p)) = \mathcal{B}(D_T(\Phi'_p)) \) (a consequence of each \( \Phi'_p \) being separable and metric; see [20 Corollary 2.4]).

**Definition 4.3.** A cylindrical (probability) measure on \( D_T(\Phi'_\beta) \) is a map
\[ \mu : Z(D_T(\Phi'_\beta)) \to [0, +\infty] \]
such that for each finite \( M \subseteq \Phi \) and \( I \subseteq [0, T] \), the restriction of \( \mu \) to \( \mathcal{C}(D_T(\Phi'_\beta); M, I) \) is a (probability) measure.

Let \( \mu \) be a cylindrical probability measure on \( D_T(\Phi'_\beta) \). For \( t \in [0, T] \), consider the time projection \( \Pi_t : D_T(\Phi'_\beta) \to \Phi'_\beta \) given by \( x \mapsto x(t) \). We define the Fourier transform \( \widehat{\mu}_t \) of the measure \( \mu \) at time \( t \) as the Fourier transform of \( \mu_t := \mu \circ \Pi_t^{-1} \) on \( \Phi'_\beta \), i.e. the function \( \widehat{\mu}_t : \Phi \to \mathbb{C} \) is given by
\[ \widehat{\mu}_t(\phi) = \int_{D_T(\Phi'_\beta)} e^{ix(t)[\phi]} d\mu, \quad \forall \phi \in \Phi. \]

Now, as \( \mathcal{C}(D_T(\Phi'_\beta)) \subseteq \mathcal{B}(D_T(\Phi'_\beta)) \), a Borel probability measure on \( D_T(\Phi'_\beta) \) clearly defines a cylindrical probability measure on \( D_T(\Phi'_\beta) \). When \( \Phi \) is a nuclear space, sufficient conditions for a cylindrical probability measure on \( D_T(\Phi'_\beta) \) to extend to a Borel probability measure on \( D_T(\Phi'_\beta) \) in terms of its Fourier transforms will be given in Theorem 4.9.

**Definition 4.4.** A cylindrical random variable in \( D_T(\Phi'_\beta) \) (defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \)) is a linear map \( X : \Phi \to L^0(\Omega, \mathcal{F}, \mathbb{P}; D_T(\mathbb{R})) \),
where $L^0(\Omega, \mathcal{F}, \mathbb{P}; D_T(\mathbb{R}))$ is the set of $D_T(\mathbb{R})$-valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

If $\mathcal{Z}(M, I, B)$ is a cylinder set in $D_T(\Phi'_\beta)$ with $M = \{\phi_1, \ldots, \phi_m\} \subseteq \Phi$, $I = \{t_1, \ldots, t_m\} \subseteq [0, T]$ and $B \in \mathcal{B}(\mathbb{R}^m)$, let

$$
\mu_X(\mathcal{Z}(M, I, B)) := \mathbb{P}\left((X(\phi_1)(t_1), \ldots, X(\phi_m)(t_m)) \in B \right) = \mathbb{P} \circ X^{-1} \circ (\Pi_{t_1, \ldots, t_m}^{\phi_1, \ldots, \phi_m})^{-1}(B).
$$

The map $\mu_X$ is called the cylindrical distribution of $X$ and it is a cylindrical probability measure on $D_T(\Phi'_\beta)$. The Fourier transform $\hat{\mu}_{X,t}$ at time $t$ of the cylindrical random variable $X$ in $D_T(\Phi'_\beta)$ is the Fourier transform of its cylindrical measure $\mu_X$ at time $t$.

As the next results shows, to every cylindrical probability measure on $D_T(\Phi'_\beta)$ there corresponds a canonical cylindrical random variable in $D_T(\Phi'_\beta)$.

**Theorem 4.5.** Let $\mu$ be a cylindrical probability measure on $D_T(\Phi'_\beta)$. Then there exists a cylindrical random variable $X$ in $D_T(\Phi'_\beta)$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ whose cylindrical distribution is $\mu$.

**Proof.** For the proof we will construct a compatible family of measures that satisfies the assumptions of the Kolmogorov extension theorem.

For each $\phi \in \Phi$, let $(\Omega_{\phi}, \mathcal{F}_{\phi})$ be the Borel space $(D_T(\mathbb{R}), \mathcal{B}(D_T(\mathbb{R})))$. For any $F \subseteq \Phi$, we write $\Omega^F = \times_{\phi \in F} \Omega_{\phi}$ and $\mathcal{F}^F = \bigotimes_{\phi \in F} \mathcal{F}_{\phi}$. Furthermore, we define $\pi_F : D_T(\Phi'_\beta) \to \Omega^F$ by $\pi_F(x) = (x[\phi])_{\phi \in F}$ for all $x \in D_T(\Phi'_\beta)$. For finite $F = \{\phi_1, \ldots, \phi_m\} \subseteq \Phi$, we also use the notation $\pi_{\phi_1, \ldots, \phi_m}$ for $\pi_F$. If $G \subseteq F \subseteq \Phi$, we denote by $\pi_{F,G}$ the map that takes $y \in \Omega^F$ to its restriction to $G$. Clearly, the maps $\pi_F$ and $\pi_{F,G}$ are measurable.

Fix $F = \{\phi_1, \ldots, \phi_m\} \subseteq \Phi$. For any $t_1, \ldots, t_m \in [0, T]$, define $\pi_{t_1, \ldots, t_m} : \Omega^F \to \mathbb{R}^m$ by $\pi_{t_1, \ldots, t_m}(y) = (y_1(t_1), \ldots, y_m(t_m))$ for $y = (y_1, \ldots, y_m) \in \Omega^F$. It is well-known that $\mathcal{B}(D_T(\mathbb{R})) = \sigma(\{y \in D_T(\mathbb{R}) : y(t) \in B\} : t \in [0, T], B \in \mathcal{B}(\mathbb{R}))$, hence it is clear that the family $\{(\pi_{t_1, \ldots, t_m})^{-1}(B_1 \times \cdots \times B_m) : t_1, \ldots, t_m \in [0, T], B_1, \ldots, B_m \in \mathcal{B}(\mathbb{R})\}$ of cylinder sets in $\Omega^F$ generates $\mathcal{F}^F$.

Now, define $\mu_F$ on $\Omega^F$ as follows: first, for $I = \{t_1, \ldots, t_m\} \subseteq [0, T]$ and $B_1, \ldots, B_m \in \mathcal{B}(\mathbb{R})$, let

$$
\mu_F((\pi_{t_1, \ldots, t_m})^{-1}(B_1 \times \cdots \times B_m)) = \mu(\mathcal{Z}(F, I, B_1 \times \cdots \times B_m)).
$$

As the cylinder sets in $\Omega^F$ generate $\mathcal{F}^F$, the above-defined $\mu_F$ extends to a probability measure (denoted again by $\mu_F$) on $(\Omega^F, \mathcal{F}^F)$.

Now, we will show that the measures $(\mu_F : F \subseteq \Phi, F \text{ finite}$ satisfy the consistency condition. We start by showing it for the cylinder sets. Let $G = \{\phi_1, \ldots, \phi_n\} \subseteq F = \{\phi_1, \ldots, \phi_m\} \subseteq \Phi$, and consider $I = \{t_1, \ldots, t_n\} \subseteq J = \{s_1, \ldots, s_m\} \subseteq [0, T]$ and $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R})$. For $i = 1, \ldots, n$, let $s_{j_1}, \ldots, s_{j_n}$ be given by $s_{j_i} = t_i$, and for $j = 1, \ldots, m$, let $A_j = \mathbb{R}$ if $s_j \notin \{s_{j_1}, \ldots, s_{j_n}\}$,
and $A_j = B_j$ if $s_j \in \{s_{j_1}, \ldots, s_{j_n}\}$. Then

$$
\mu_G((\pi_{s_{j_1},\ldots,s_{j_n}})^{-1}(B_1 \times \cdots \times B_n)) \\
= \mu(Z(G, I, B_1 \times \cdots \times B_n)) = \mu(Z(F, J, A_1 \times \cdots \times A_m)) \\
= \mu_F((\pi_{s_{j_1},\ldots,s_{j_n}})^{-1}(A_1 \times \cdots \times A_m)) \\
= \mu_F(\pi_{F,G}^{-1}((\pi_{s_{j_1},\ldots,s_{j_n}})^{-1}(B_1 \times \cdots \times B_n))).
$$

Now, because the above equality holds for any cylinder set, it also holds for any set in $\mathcal{F}^G$, showing that the consistency condition $\mu_G = \mu_F \circ \pi^{-1}_{F,G}$ is satisfied for $G \subseteq F \subseteq \Phi$ with $G$ and $F$ finite.

Hence, by Kolmogorov’s extension theorem (see [31, Theorem 5.1, p. 144]), there exists a unique probability measure $\mathbb{P}$ on $(\Omega^\Phi, \mathcal{F}^\Phi) := (\times_{\phi \in \Phi} D_T(\mathbb{R}), \otimes_{\phi \in \Phi} \mathcal{B}(D_T(\mathbb{R})))$ such that for each finite $F \subseteq \Phi$, \begin{equation}
\mu_F(\Gamma) = \mathbb{P}(\pi_F^{-1}(\Gamma)), \quad \forall \Gamma \in \mathcal{F}^F. \tag{4.2}
\end{equation}

In particular, for every $F = \{\phi_1, \ldots, \phi_n\} \subseteq \Phi$, $I = \{t_1, \ldots, t_n\} \subseteq [0, T]$ and $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R})$, from the above equality and (4.1) we have

$$
\mu(Z(F, I, B_1 \times \cdots \times B_n)) = \mu_F(\pi_{t_1,\ldots,t_n}^{-1}(B_1 \times \cdots \times B_n)) \\
= \mathbb{P}(\pi_{\phi_1,\ldots,\phi_n}^{-1}(\pi_{t_1,\ldots,t_n}^{-1}(B_1 \times \cdots \times B_n))) \\
= \mathbb{P}(\Pi_{t_1,\ldots,t_n}^{-1}(B_1 \times \cdots \times B_n)) \tag{4.3}
$$

Our next step is to define a cylindrical random variable in $D_T(\Phi')$ whose cylindrical distribution is $\mu$. Let $X : \Phi \to L^0(\Omega^\Phi, \mathcal{F}^\Phi, \mathbb{P}; D_T(\mathbb{R}))$ be defined in the following way: for every $\phi \in \Phi$, let $X(\phi) := \pi_\phi$, i.e. $X(\phi)(y) = y(\phi) \in D_T(\mathbb{R})$ for each $y = (y(\phi))_{\phi \in \Phi} \in \Omega^\Phi$. Clearly, each $X(\phi)$ is $\mathcal{F}^\Phi/\mathcal{B}(D_T(\mathbb{R}))$-measurable and therefore $X$ is well-defined. Moreover, $\mu$ is the cylindrical distribution of $X$ as a direct consequence of (4.3).

Now we show that $X$ is linear. Let $\psi_1, \psi_2 \in \Phi$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. Consider the subset $F = \{\lambda_1 \psi_1, \lambda_2 \psi_2, \lambda_1 \psi_1 + \lambda_2 \psi_2\}$ of $\Phi$. Then $\Omega^F = D_T(\mathbb{R})^3$ and $\pi_F : D_T(\Phi_\beta') \to D_T(\mathbb{R})^3$ is given by $x \mapsto (\lambda_1 x[\psi_1], \lambda_2 x[\psi_2], x[\lambda_1 \psi_1 + \lambda_2 \psi_2])$. If $\sigma : D_T(\mathbb{R})^3 \to D_T(\mathbb{R})$ is defined as $(u, v, w) \mapsto u + v - w$, then $\sigma$ is continuous and also $\sigma \circ \pi_F = 0 \in D_T(\mathbb{R})$. Thus for $A \in \mathcal{B}(D_T(\mathbb{R}))$, $\mu_F(\sigma^{-1}(A)) = \mu \circ \pi_F^{-1}(\sigma^{-1}(A))$ takes value 0 if $0 \notin A$ and takes value 1 if $0 \in A$. Hence $\mu_F$ is supported by the plane $\sigma^{-1}(\{0\}) = \{(u, v, w) : u + v - w = 0\}$ of $\Omega^F = D_T(\mathbb{R})^3$. But then from (4.2) we have

$$
\mathbb{P}(\lambda_1 X(\psi_1) + \lambda_2 X(\psi_2) - X(\lambda_1 \psi_1 + \lambda_2 \psi_2) = 0) \\
= \mathbb{P}\left((\lambda_1 X(\psi_1), \lambda_2 X(\psi_2), X(\lambda_1 \psi_1 + \lambda_2 \psi_2)) \in \sigma^{-1}(\{0\})\right) = \mu_F(\sigma^{-1}(\{0\})) = 1.
$$

This proves that $X$ is a cylindrical random variable in $D_T(\Phi')$ defined on $(\Omega^\Phi, \mathcal{F}^\Phi, \mathbb{P})$ with cylindrical measure $\mu$. ■
4.2. Measurability of random elements in $D_T(\Phi'_\beta)$

**Assumption 4.6.** In this section, unless otherwise specified, all the random elements will be defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We start by listing the relationships between the different types of random elements in $D_T(\Phi'_\beta)$.

1. Let $X$ be a $D_T(\Phi'_\beta)$-valued random variable. Clearly $X$ determines a $\Phi'_\beta$-valued càdlàg process $\{X_t\}_{t \in [0,T]}$ given by $X_t(\omega) := X(\omega)(t)$ for all $t \in [0,T]$ and $\omega \in \Omega$.

2. If $X$ is a cylindrical random variable in $D_T(\Phi'_\beta)$, then the linear map $\phi \mapsto \{X(\phi)(t)\}_{t \in [0,T]}$ is a cylindrical process in $\Phi'$. Conversely, if $X = \{X_t\}_{t \in [0,T]}$ is a cylindrical process such that for each $\phi \in \Phi$ the real-valued process $X(\phi) = \{X_t(\phi)\}_{t \in [0,T]}$ is càdlàg, then $X$ is a cylindrical random variable in $D_T(\Phi'_\beta)$.

3. Let $X = \{X_t\}_{t \in [0,T]}$ be a $\Phi'_\beta$-valued càdlàg process. In this case $X$ defines two objects. First, for every $\phi \in \Phi$, the space projection $\Pi_\phi : D_T(\Phi'_\beta) \to D_T(\mathbb{R})$ maps $X$ to $X[\phi] = \Pi_\phi(X) := \{X_t[\phi]\}_{t \in [0,T]}$. This way, $X$ defines a cylindrical random variable in $D_T(\Phi'_\beta)$. Second, $X$ defines a map $X : \Omega \to D_T(\Phi'_\beta)$ by means of its paths $\omega \mapsto (t \mapsto X(\omega)(t) := X_t(\omega))$. This map is $\mathcal{F}/\mathcal{C}(D_T(\Phi'_\beta))$-measurable. However, because the inclusion $\mathcal{C}(D_T(\Phi'_\beta)) \subseteq \mathcal{B}(D_T(\Phi'_\beta))$ might be strict, $X$ is not necessarily a $D_T(\Phi'_\beta)$-valued random variable.

Given a cylindrical process $X$ in $\Phi'$, the next result gives sufficient conditions for the existence of a version that is a $D_T(\Phi'_\beta)$-valued random variable.

**Theorem 4.7** (Regularization theorem on Skorokhod space). Let $\Phi$ be a nuclear space. Let $X = \{X_t\}_{t \in [0,T]}$ be a cylindrical process in $\Phi'$ (e.g. a $\Phi'_\beta$-valued process) such that:

1. For each $\phi \in \Phi$, the real-valued process $X(\phi) = \{X_t(\phi)\}_{t \in [0,T]}$ has a càdlàg version.

2. The family $\{X_t : t \in [0,T]\}$ of linear maps from $\Phi$ into $L^0(\Omega, \mathcal{F}, \mathbb{P})$ is equicontinuous.

Then there exist a weaker countably Hilbertian topology $\theta$ on $\Phi$ and a $D_T((\widetilde{\Phi}_\theta)'\beta)$-valued random variable $Y$ such that for each $\phi \in \Phi$ the real-valued càdlàg processes $X(\phi)$ and $Y[\phi]$ are indistinguishable. In particular, $Y$ is a $D_T(\Phi'_\beta)$-valued random variable whose probability distribution is a Radon measure on $D_T(\Phi'_\beta)$.

**Proof.** From (1) and (2) and Theorem 2.1 there exist a countably Hilbertian topology $\theta$ on $\Phi$ and a $(\widetilde{\Phi}_\theta)'\beta$-valued càdlàg process $Y = \{Y_t\}_{t \geq 0}$ that is a version of $X$ (unique up to indistinguishable versions).
The mapping \( \omega \mapsto Y_t(\omega) \) from \( \Omega \) into \( D_T((\tilde{\Phi}_\theta)'_\beta) \) is \( F/\mathcal{C}(D_T((\tilde{\Phi}_\theta)'_\beta)) \)-measurable. But from Lemma 4.2, it is also \( F/\mathcal{B}(D_T((\tilde{\Phi}_\theta)'_\beta)) \)-measurable. Thus \( Y_t \) defines a \( D_T((\tilde{\Phi}_\theta)'_\beta) \)-valued random variable and its probability distribution on \( D_T((\tilde{\Phi}_\theta)'_\beta) \) is Radon because this space is Suslin (Proposition 3.2(1)). Finally, because the inclusion map from \( D_T((\tilde{\Phi}_\theta)'_\beta) \) into \( D_T(\Phi'_\beta) \) is continuous (Proposition 3.2(3)), \( Y_t \) is also a \( D_T(\Phi'_\beta) \)-valued random variable whose probability distribution on \( D_T(\Phi'_\beta) \) is Radon. 

As the following result shows, we can relax some of the conditions in Theorem 4.7 when the space \( \Phi \) is ultrabornological.

**Corollary 4.8.** Assume that \( \Phi \) is an ultrabornological nuclear space. Let \( X = \{X_t\}_{t \in [0,T]} \) be a \( \Phi'_\beta \)-valued process such that:

1. For each \( \phi \in \Phi \), the real-valued process \( X[\phi] = \{X_t[\phi]\}_{t \geq 0} \) has a càdlàg version.
2. For each \( t \in [0,T] \), the probability distribution \( \mu_t \) of \( X_t \) is a Radon measure on \( \Phi'_\beta \).

Then there exist a weaker countably Hilbertian topology \( \theta \) on \( \Phi \) and a \( (\tilde{\Phi}_\theta)'_\beta \)-valued càdlàg process \( Y = \{Y_t\}_{t \geq 0} \) that is a version of \( X \) and is such that \( Y \) is also a \( D_T((\tilde{\Phi}_\theta)'_\beta) \)-valued random variable. In particular, \( Y \) is a \( D_T(\Phi'_\beta) \)-valued random variable whose probability distribution is a Radon measure on \( D_T(\Phi'_\beta) \).

**Proof.** First, since each \( \mu_t \) is a Radon measure and the space \( \Phi \) is barrelled (because it is ultrabornological, see [30, p. 449]), it follows from [13, Theorem 2.9] that each map \( X_t \) from \( \Phi \) into \( L^0(\Omega, \mathcal{F}, \mathbb{P}) \) is continuous. Moreover, because \( \Phi \) is ultrabornological, the above property together with (1) implies that the linear mapping from \( \Phi \) into \( D_T(\mathbb{R}) \) (equipped with the supremum norm) given by \( \psi \mapsto \{X_t[\psi]\}_{t \in [0,T]} \) is continuous (see [13, Proposition 3.10]). This in particular shows that the family \( \{X_t : t \in [0,T]\} \) of linear maps from \( \Phi \) into \( L^0(\Omega, \mathcal{F}, \mathbb{P}) \) is equicontinuous. The result then follows from Theorem 4.7. 

As an important consequence of Theorem 4.7, we get the following interesting result concerning sufficient conditions for Radon extensions of cylindrical measures on \( D_T(\Phi'_\beta) \).

**Theorem 4.9 (Minlos theorem on Skorokhod space).** Let \( \Phi \) be a nuclear space and let \( \mu \) be a cylindrical probability measure on \( D_T(\Phi'_\beta) \). Suppose that the family \( \{\tilde{\mu}_t : t \in [0,T]\} \) of its Fourier transforms is equicontinuous at zero. Then there exists a Radon probability measure \( \nu \) on \( D_T(\Phi'_\beta) \) that is an extension of \( \mu \). Moreover, there exists a weaker countably Hilbertian topology \( \theta \) on \( \Phi \) such that \( \nu \) is a Radon measure on \( D_T((\tilde{\Phi}_\theta)'_\beta) \).
Proof. First, from Theorem 4.5 there exists a cylindrical random variable \( X \) in \( D_T(\Phi_\beta') \) defined on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\) whose cylindrical distribution is \( \mu \). In particular, \( \phi \mapsto \{X(\phi)(t)\}_{t \in [0,T]} \) is a cylindrical process in \( \Phi' \) satisfying condition (1) in Theorem 4.7.

To check the second condition in Theorem 4.7 observe that from the inequality (see [22, Lemma 5.1, p. 85])

\[
\mathbb{P}(\{X(\phi)(t)\geq \epsilon\} \leq \frac{e}{2} \int_{-2/\epsilon}^{2/\epsilon} (1 - \mathbb{E} e^{isX(\phi)(t)}) ds = \frac{e}{2} \int_{-2/\epsilon}^{2/\epsilon} (1 - \hat{\mu}_t(s\phi)) ds,
\]

valid for every \( \epsilon > 0 \), \( t \in [0,T] \), and \( \phi \in \Phi \), it follows that the equicontinuity of \((\hat{\mu}_t : t \in [0,T])\) at zero implies that of the family of linear maps \( \phi \mapsto X(\phi)(t), t \in [0,T] \), from \( \Phi \) into \( L^0(\Omega, \mathcal{F}, \mathbb{P}) \). Hence by Theorem 4.7 there exist a weaker countably Hilbertian topology \( \theta \) on \( \Phi \) and a \( D_T((\Phi_\theta)_\beta') \)-valued random variable \( Y \) such that \( Y[\phi] = X(\phi) \) \( \mathbb{P} \)-a.e.

Let \( \nu \) be the probability distribution of \( Y \) on \( D_T(\Phi_\beta') \). We know from Theorem 4.7 that \( \nu \) is a Radon measure on \( D_T((\Phi_\theta)_\beta') \), and hence a Radon measure on \( D_T(\Phi_\beta') \). Moreover, for all \( m \in \mathbb{N}, t_1, \ldots, t_m \in [0,T] \), \( \phi_1, \ldots, \phi_m \in \Phi \) and \( A \in \mathcal{B}(\mathbb{R}^m) \), we have

\[
\nu((\Pi_{t_1, \ldots, t_m}^{\phi_1, \ldots, \phi_m})^{-1}(A)) = \mathbb{P}(Y \in (\Pi_{t_1, \ldots, t_m}^{\phi_1, \ldots, \phi_m})^{-1}(A))
\]
\[
= \mathbb{P}((Y(t_1)[\phi_1], \ldots, Y(t_m)[\phi_m]) \in A)
\]
\[
= \mathbb{P}((X(\phi_1)(t_1), \ldots, X(\phi_m)(t_m)) \in A)
\]
\[
= \mu((\Pi_{t_1, \ldots, t_m}^{\phi_1, \ldots, \phi_m})^{-1}(A)).
\]

Thus \( \nu \) is an extension of \( \mu \) as both measures agree on the cylindrical \( \sigma \)-algebra \( \mathcal{C}(D_T(\Phi_\beta')) \) of \( D_T(\Phi_\beta') \).

As the next results shows, to every probability measure on \( D_T(\Phi_\beta') \) with equicontinuous Fourier transforms there corresponds a canonical random variable in \( D_T(\Phi_\beta') \). This is a direct consequence of the proof of Theorem 4.9.

**Corollary 4.10.** Let \( \Phi \) be a nuclear space. Suppose that \( \mu \) is a probability measure on \( D_T(\Phi_\beta') \) for which the family \((\hat{\mu}_t : t \in [0,T])\) is equicontinuous at zero. Then there exists a weaker countably Hilbertian topology \( \theta \) on \( \Phi \) and a \( D_T((\Phi_\theta)_\beta') \)-valued random variable \( Y \) whose probability distribution is \( \mu \).

## 5. Tightness of probability measures on the Skorokhod space

**Assumption 5.1.** Unless otherwise indicated, in this section we will always assume that \( \Phi \) is a nuclear space.

### 5.1. Uniform tightness on \( D_T(\Phi_\beta') \)

The main result of this section is the following theorem that provides necessary and sufficient conditions for
a family \((\mu_\alpha : \alpha \in A)\) of probability measures on \(D_T(\Phi'_\beta)\) to be uniformly tight.

**Theorem 5.2.** Let \((\mu_\alpha : \alpha \in A)\) be a family of probability measures on 
\(D_T(\Phi'_\beta)\) such that:

1. The family \((\hat{\mu}_{\alpha,t} : t \in [0,T], \alpha \in A)\) is equicontinuous at zero.
2. For each \(\phi \in \Phi\), the family \((\mu_\alpha \circ \Pi_\phi^{-1} : \alpha \in A)\) of probability measures 
on \(D_T(\mathbb{R})\) is uniformly tight.

Then there exists a weaker countably Hilbertian topology \(\theta\) on \(\Phi\) such that 
\((\mu_\alpha : \alpha \in A)\) is uniformly tight on \(D_T((\Phi_0)'_\beta)\), and in particular on \(D_T(\Phi'_\beta)\).

Conversely, if \(\Phi\) is a barrelled nuclear space and the family \((\mu_\alpha : \alpha \in A)\) 
is uniformly tight on \(D_T(\Phi'_\beta)\), then conditions (1) and (2) are satisfied.

We proceed to prove Theorem 5.2. The first step is the following.

**Proposition 5.3.** Let \((\mu_\alpha : \alpha \in A)\) be a family of probability measures on 
\(D_T(\Phi'_\beta)\) satisfying condition (1) in Theorem 5.2. Then there exists a weaker countably Hilbertian topology \(\theta\) on \(\Phi\) such that for each \(\alpha \in A\), \(\mu_\alpha\) is a Radon probability measure on \(D_T((\Phi_0)'_\beta)\).

**Proof.** For each \(\alpha \in A\), denote by \(X^\alpha\) the canonical cylindrical random variable in \(D_T(\Phi'_\beta)\) associated to \(\mu_\alpha\) by Theorem 4.5. Without loss of generality we can assume that the \(X^\alpha\)'s are defined on the same probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

The same arguments in the proof of Theorem 4.9 and the equicontinuity of \((\hat{\mu}_{\alpha,t} : t \in [0,T], \alpha \in A)\) at zero show that we can choose a weaker countably Hilbertian topology \(\theta\) on \(\Phi\) (independently of \(\alpha\)) and for each \(\alpha \in A\) a \(D_T((\Phi_0)'_\beta)\)-valued random variable \(Y^\alpha\) such that \(Y^\alpha(t)[\phi] = X^\alpha(\phi)(t)\) \(\mathbb{P}\)-a.e. Hence, following the same arguments as in the last part of the proof of Theorem 4.9 it can be shown that \(\mu_\alpha\) is a Radon probability measure on \(D_T((\Phi_0)'_\beta)\) for each \(\alpha \in A\).

The importance of Proposition 5.3 is that it settles our problem in the context of measures on \(D_T((\Phi_0)'_\beta)\). This fact is to be used in combination with the following result whose proof will be given at the end of this section.

**Proposition 5.4.** Let \((\mu_\alpha : \alpha \in A)\) be a family of probability measures on 
\(D_T(\Psi'_\beta)\) where \(\Psi\) is an ultrabornological space, and suppose that for each \(\phi \in \Psi\) the family \((\mu_\alpha \circ \Pi_\phi^{-1} : \alpha \in A)\) is uniformly tight on \(D_T(\mathbb{R})\). Then for each \(\epsilon > 0\) there exists a continuous seminorm \(p\) on \(\Psi\) such that 
\[
\sup_{\alpha \in A} \int_{D_T(\Psi'_\beta)} |1 - e^{ix(t)[\phi]}| d\mu_\alpha \leq \epsilon, \quad \forall \phi \in B_p(1).
\]

In particular, the family \((\hat{\mu}_{\alpha,t} : \alpha \in A, t \in [0,T])\) is equicontinuous at zero.
Proof of Theorem 5.2. From Proposition 5.3 there exists a weaker countably Hilbertian topology \( \theta \) on \( \Phi \) such that \( \mu_\alpha \) is a Radon probability measure on \( D_T((\tilde{\Phi}_\theta)_\beta') \) for each \( \alpha \in A \).

Let \( \epsilon > 0 \). Because \( \tilde{\Phi}_\theta \) is an ultrabornological space, it follows from Proposition 5.4 that there exists a continuous Hilbertian seminorm \( p \) on \( \tilde{\Phi}_\theta \) (therefore continuous on \( \Phi \)) such that

\[
\operatorname{sup}_{\alpha \in A} \int_{D_T((\tilde{\Phi}_\theta)_\beta')} \sup_{t \in [0, T]} |1 - e^{ix(t)[\phi]}| \, d\mu_\alpha \leq \frac{\epsilon}{12}, \quad \forall \phi \in B_p(1).
\]

Now, because \( |e^{ix(t)[\phi]}| \leq 1 \) for all \( \phi \in \Phi \) and \( x \in D_T((\tilde{\Phi}_\theta)_\beta') \), it follows that

\[
\sup_{\alpha \in A} \int_{D_T((\tilde{\Phi}_\theta)_\beta')} \sup_{t \in [0, T]} |1 - e^{ix(t)[\phi]}| \, d\mu_\alpha \leq \frac{\epsilon}{12} + 2p(\phi)^2, \quad \forall \phi \in \Phi.
\]

Let \( q \) be a continuous Hilbertian seminorm on \( \Phi \) such that \( p \leq q \) and \( i_{p,q} \) is Hilbert–Schmidt. Let \( (\phi_k^q)_{k \in \mathbb{N}} \subseteq \Phi \) be a complete orthonormal system in \( \Phi_q \).

By similar arguments to those in [13] proof of Lemma 3.8 (see also [29] Lemma 3.2) it follows from (5.1) that for all \( C > 0 \) and \( \alpha \in A \),

\[
\mu_\alpha \left( x \in D_T((\tilde{\Phi}_\theta)_\beta') : \sup_{t \in [0, T]} \sum_{k=1}^\infty |x(t)[\phi_k^q]|^2 > C^2 \right)
\]

\[
\leq \lim_{m \to \infty} \frac{\sqrt{e}}{\sqrt{e-1}} \int_{D_T((\tilde{\Phi}_\theta)_\beta')} \sup_{t \in [0, T]} \left( 1 - \exp \frac{-1}{2C^2} \sum_{k=1}^m |x(t)[\phi_k^q]|^2 \right) \, d\mu_\alpha
\]

\[
\leq \lim_{m \to \infty} \frac{\sqrt{e}}{\sqrt{e-1}} \int_{\mathbb{R}^m} \int_{D_T((\tilde{\Phi}_\theta)_\beta')} \sup_{t \in [0, T]} \left( 1 - \exp \frac{i}{2C^2} \sum_{k=1}^m z_k x(t)[\phi_k^q] \right) \, d\mu_\alpha \frac{e^{-|z|^2/2}}{(2\pi)^{m/2}} \, dz
\]

\[
\leq \lim_{m \to \infty} \frac{\sqrt{e}}{\sqrt{e}-1} \left( \frac{\epsilon}{12} + \frac{2}{C^2} \sum_{k=1}^m p(\phi_k^q)^2 \right) = \frac{\sqrt{e}}{\sqrt{e}-1} \left( \frac{\epsilon}{12} + \frac{2}{C^2} \|i_{p,q}\|_{L_2(\Phi_q, \Phi_\theta)}^2 \right).
\]

Then choosing \( C \) such that \( \frac{2}{C^2} \|i_{p,q}\|_{L_2(\Phi_q, \Phi_\theta)}^2 < \frac{\epsilon}{12} \) and considering the probability of the complement, we get

\[
\inf_{\alpha \in A} \mu_\alpha \left( x \in D_T(\Phi_q') : \sup_{t \in [0, T]} q'(x(t)) \leq C \right)
\]

\[
\geq \inf_{\alpha \in A} \mu_\alpha \left( x \in D_T((\tilde{\Phi}_\theta)_\beta') : \sup_{t \in [0, T]} \sum_{k=1}^\infty |x(t)[\phi_k^q]|^2 \leq C^2 \right) \geq 1 - \frac{\epsilon}{2}.
\]

Let \( q \) be a continuous Hilbertian seminorm on \( \Phi \) such that \( q \leq \varrho \) and \( i_{q,\varrho} \) is compact (e.g. Hilbert–Schmidt). Then \( i_{q,\varrho}' \) is also a compact operator. As \( F = \{ f \in \Phi_q' : q'(f) \leq C \} \) is a neighborhood of zero in \( \Phi_q' \), its image under
\( i'_{q,\varrho} \) is a relatively compact subset of \( \Phi'_\varrho \). Let \( K \) be its closure. Then \( K \) is a compact subset of \( \Phi'_\varrho \), and regarding both \( F \) and \( K \) as subsets of \( \Phi'_\varrho \) we have \( F \subseteq K \). Then it follows from (5.2) that

\[
\inf_{\alpha \in A} \mu_\alpha(x \in D_T(\Phi'_\varrho) : x(t) \in K \text{ for all } t \in [0,T]) \geq \inf_{\alpha \in A} \mu_\alpha\left(x \in D_T(\Phi'_q) : \sup_{t \in [0,T]} q'(x(t)) \leq C\right) \geq 1 - \frac{\epsilon}{2}.
\]

Now, since \( \Phi'_\varrho \) is a separable Hilbert space, we can choose a sequence \( (\varrho_j)_{j \in \mathbb{N}} \subseteq \Phi \) that separates the points in \( \Phi'_\varrho \) (see [4, Proposition 6.5.4, p. 17]). For each \( j \in \mathbb{N} \), from our assumption of tightness of \( (\mu_\alpha \circ \Pi^{-1}_{\varrho_j} : \alpha \in A) \), there exists a compact subset \( B_j \) of \( D_T(\mathbb{R}) \) such that

\[
\inf_{\alpha \in A} \mu_\alpha \circ \Pi^{-1}_{\varrho_j}(B_j) > 1 - \frac{\epsilon}{2j+1}.
\]

Let

\[
\Gamma = \left( \bigcap_{j=1}^{\infty} \Pi^{-1}_{\varrho_j}(B_n) \right) \cap \{ x \in D_T(\Phi'_\varrho) : x(t) \in K \text{ for all } t \in [0,T] \}.
\]

Then

1. \( \Gamma \subseteq D_T(K) \) is closed, where \( K \subseteq \Phi'_\varrho \) is equipped with the subspace topology,
2. for each \( j \in \mathbb{N} \), \( \Pi^{-1}_{\varrho_j}(\Gamma) \) is compact in \( D_T(\mathbb{R}) \).

Then [20, Lemma 3.3] shows that \( \Gamma \) is compact in \( D_T(\Phi'_\varrho) \). Moreover, from (5.3) and (5.4),

\[
\sup_{\alpha \in A} \mu_\alpha(\Gamma) \leq \frac{\epsilon}{2} + \sum_{j=1}^{\infty} \frac{\epsilon}{2j+1} = \epsilon.
\]

Now consider a decreasing sequence \( (\epsilon_n : n \in \mathbb{N}) \) of positive numbers converging to zero, and for each \( n \in \mathbb{N} \) choose a continuous Hilbertian seminorm \( \varrho_n \) on \( \Phi \) and a compact subset \( \Gamma_n \) in \( D_T(\Phi'_\varrho) \) such that

\[
\inf_{\alpha \in A} \mu_\alpha(\Gamma_n) \geq 1 - \epsilon_n.
\]

Then it is clear from (3.2) that \( (\mu_\alpha : \alpha \in A) \) is uniformly tight on \( D_T((\Phi_\varrho)'_\beta) \), where \( \varrho \) is the weaker countably Hilbertian topology on \( \Phi \) generated by the seminorms \( (\varrho_n) \). Here it is important that by construction the topology \( \varrho \) is finer than the topology \( \theta \) defined at the beginning of the proof and hence each \( \mu_\alpha \) is a Radon measure on \( D_T((\Phi_\varrho)'_\beta) \). Now, because the inclusion from \( D_T((\Phi_\varrho)'_\beta) \) into \( D_T(\Phi'_\varrho) \) is continuous (Proposition 3.2(3)), we find that \( (\mu_\alpha : \alpha \in A) \) is uniformly tight on \( D_T(\Phi'_\varrho) \).
To prove the converse, assume that \( \Phi \) is a barrelled nuclear space and the family \( (\mu_\alpha : \alpha \in A) \) is uniformly tight on \( D_T(\Phi'_{\beta}) \). Then \( 20 \) Proposition 1.6(vi) shows that the family \( (\mu_\alpha \circ \Pi_t^{-1} : \alpha \in A, t \in [0, T]) \) of probability measures is uniformly tight on \( \Phi'_{\beta} \). But because \( \Phi \) is barrelled and nuclear, the family \( (\hat{\mu}_{\alpha, t} : t \in [0, T], \alpha \in A) \) of its Fourier transforms is equicontinuous at zero (see \( 5 \) Theorem III.2.7, p. 104)). Finally, for each \( \phi \in \Phi \) the continuity of the space projection map \( \Pi_\phi \) and the uniform tightness of \( (\mu_\alpha : \alpha \in A) \) on \( D_T(\Phi'_{\beta}) \) imply that \( (\mu_\alpha \circ \Pi_{\phi}^{-1} : \alpha \in A) \) is uniformly tight on \( D_T(R) \).

If \( \Phi \) is an ultrabornological nuclear space then by Proposition 5.4, condition (2) in Theorem 5.2 implies condition (1). From this we obtain the following important result.

**Theorem 5.5.** Let \( (\mu_\alpha : \alpha \in A) \) be a family of probability measures on \( D_T(\Phi'_{\beta}) \) where \( \Phi \) is an ultrabornological nuclear space. Then the family \( (\mu_\alpha : \alpha \in A) \) is uniformly tight on \( D_T(\Phi'_{\beta}) \) if and only if for each \( \phi \in \Phi \) the family \( (\mu_\alpha \circ \Pi_{\phi}^{-1} : \alpha \in A) \) is uniformly tight on \( D_T(R) \). Moreover, there exists a weaker countably Hilbertian topology \( \theta \) on \( \Phi \) such that \( (\mu_\alpha : \alpha \in A) \) is uniformly tight on \( D_T((\Phi'_{\theta})'_{\beta}) \).

To prove Proposition 5.4, we need the following preliminary results on pseudo-seminorms on vector spaces.

**Definition 5.6.** A function \( x \mapsto |x| \) defined on a vector space \( L \) over \( \mathbb{R} \) is called a pseudo-seminorm on \( L \) if

1. \( |x + y| \leq |x| + |y|, \forall x, y \in L, \)
2. for \( \lambda \in \mathbb{R}, |\lambda| \leq 1 \) implies \( |\lambda x| \leq |x|, \forall x \in L, \)
3. if \( \lambda_n \to 0 \), then \( |\lambda_n x| \to 0, \forall x \in L. \)
4. \( |x_n| \to 0 \) implies \( |\lambda x_n| \to 0, \forall \lambda \in \mathbb{R}. \)

Every pseudo-seminorm defines a pseudo-metric \( d(x, y) = |x - y| \) on \( L \) that generates a metrizable linear topology on \( L \) (see \( 35 \) Section I.6).

The following facts are relevant to our study of pseudo-seminorms on ultrabornological spaces:

1. A topological vector space \( E \) is sequential if every sequentially closed subset is closed. Hence every sequentially lower semicontinuous function on \( E \) is lower semicontinuous.
2. \( E \) is sequential if and only if every sequentially continuous pseudo-seminorm on \( E \) is continuous if and only if every sequentially continuous linear map to an arbitrary Fréchet space is continuous (see \( 26 \) Proposition 2.6]). Hence every bornological space is sequential (see \( 28 \) Theorem 28.3(4), p. 383))
(3) A topological vector space $E$ is $S$-barrelled if every lower semicontinuous pseudo-seminorm on $E$ is sequentially continuous (see [26, Proposition 5.2]) if and only if every pointwise bounded family of continuous linear maps from $E$ into an arbitrary Fréchet space is sequentially equicontinuous (see [26, Proposition 5.5]).

From facts (1)–(3) above and because every ultrabornological space is bornological and barrelled, we have the following:

**Proposition 5.7.** If $E$ is an ultrabornological space, then every sequentially lower semicontinuous pseudo-seminorm on $E$ is continuous.

The main step in the proof of Proposition 5.4 is the following.

**Lemma 5.8.** Let $(\mu_\alpha : \alpha \in A)$ be a family of probability measures on $D_T(\Psi_\beta)$ where $\Psi$ is an ultrabornological space, and suppose that for each $\phi \in \Psi$ the family $(\mu_\alpha \circ \Pi^{-1}_\phi : \alpha \in A)$ is uniformly tight on $D_T(\mathbb{R})$. Let $V : \Psi \to [0, +\infty)$ be given by

$$V(\phi) = \sup_{\alpha \in A} \int_{D_T(\Psi_\beta')} \frac{\sup_{t \in [0,T]} |x(t)[\phi]|}{1 + \sup_{t \in [0,T]} |x(t)[\phi]|} \, d\mu_\alpha, \quad \forall \phi \in \Psi.$$ 

Then $V$ is a continuous pseudo-seminorm on $\Psi$.

**Proof.** We first show that $V$ is a pseudo-seminorm.

1. If $\phi_1, \phi_2 \in \Psi$, then because $x \mapsto \frac{x}{1+x}$ is a subadditive function it is clear that $V(\phi_1 + \phi_2) \leq V(\phi_1) + V(\phi_2)$.
2. Let $\phi \in \Psi$ and $\lambda \in \mathbb{R}$, $|\lambda| \leq 1$. Because the function $x \mapsto \frac{x}{1+x}$ is increasing, $V(\lambda \phi) \leq V(\phi)$.
3. We follow some ideas from [29, proof of Lemma 3.3]. Let $\lambda_m \to 0$, $\epsilon > 0$, and $\phi \in \Psi$. Because the family $(\mu_\alpha \circ \Pi^{-1}_\phi : \alpha \in A)$ is uniformly tight on $D_T(\mathbb{R})$, it follows from [23, Theorem 2.4.3] that there exists $r(\epsilon)$ such that for all $\alpha \in A$,

$$\mu_\alpha \left( x \in D_T(\Psi_\beta') : \sup_{t \in [0,T]} |x(t)[\phi]| > r(\epsilon) \right) = \mu_\alpha \circ \Pi^{-1}_\phi \left( y \in D_T(\mathbb{R}) : \sup_{t \in [0,T]} |y(t)| > r(\epsilon) \right) < \epsilon.$$ 

Let $N \in \mathbb{N}$ be such that $\lambda_m r(\epsilon) < \epsilon$ for all $m \geq N$. Let

$$\Gamma = \left\{ x \in D_T(\Psi_\beta') : \sup_{t \in [0,T]} |x(t)[\phi]| > r(\epsilon) \right\}.$$
Then for all \( m \geq N \) we have
\[
V(\lambda_m \phi) = \sup_{\alpha \in A} \int_{D_T(\Psi'_\beta)} \frac{\sup_{t \in [0,T]} |x(t)[\lambda_m \phi]|}{1 + \sup_{t \in [0,T]} |x(t)[\lambda_m \phi]|} \, d\mu_{\alpha}
\]
\[
\leq \sup_{\alpha \in A} \mu_{\alpha}(\Gamma) + \frac{\epsilon}{1 + \epsilon} < 2\epsilon.
\]

Therefore, \( \lim_{m \to \infty} V(\lambda_m \phi) = 0 \).

(4) Let \( \lambda \in \mathbb{R} \) and let \( (\phi_m)_{m \in \mathbb{N}} \subseteq \Psi \) be such that \( \lim_{m \to \infty} V(\phi_m) = 0 \). Consider any subsequence \( (\phi_{m_k})_{k \in \mathbb{N}} \) of \( (\phi_m)_{m \in \mathbb{N}} \). Then \( \lim_{k \to \infty} V(\phi_{m_k}) = 0 \).

Hence, for each \( r \in \mathbb{N} \) there exists \( \phi_{m_k(r)} \) such that
\[
V(\phi_{m_k(r)}) = \sup_{\alpha \in A} \int_{D_T(\Psi'_\beta)} \frac{\sup_{t \in [0,T]} |x(t)[\phi_{m_k(r)}]|}{1 + \sup_{t \in [0,T]} |x(t)[\phi_{m_k(r)}]|} \, d\mu_{\alpha} \leq \frac{1}{2^{r+2}}.
\]

Therefore,
\[
\sup_{\alpha \in A} \left( \sup_{x \in D_T(\Psi'_\beta)} \sup_{t \in [0,T]} |x(t)[\phi_{m_k(r)}]| > 2^{-r} \right)
\]
\[
= \sup_{\alpha \in A} \left( \sup_{x \in D_T(\Psi'_\beta)} \sup_{t \in [0,T]} \frac{|x(t)[\phi_{m_k(r)}]|}{1 + \sup_{t \in [0,T]} |x(t)[\phi_{m_k(r)}]|} > \frac{2^{-r}}{1 + 2^{-r}} \right)
\]
\[
\leq \frac{1 + 2^{-r}}{2^{-r}} \sup_{\alpha \in A} \int_{D_T(\Psi'_\beta)} \frac{\sup_{t \in [0,T]} |x(t)[\phi_{m_k(r)}]|}{1 + \sup_{t \in [0,T]} |x(t)[\phi_{m_k(r)}]|} \, d\mu_{\alpha}
\]
\[
\leq 3 \cdot \frac{1}{2^{r+2}} \leq \frac{1}{2^r}
\]

Then for every \( r \in \mathbb{N} \) we have
\[
V(\lambda \phi_{m_k(r)}) \leq \sup_{\alpha \in A} \left( \sup_{x \in D_T(\Psi'_\beta)} \sup_{t \in [0,T]} |x(t)[\lambda \phi_{m_k(r)}]| > |\lambda|2^{-r} \right)
\]
\[
+ \frac{|\lambda|2^{-r}}{1 + |\lambda|2^{-r}}
\]
\[
< 2^{-r}(1 + |\lambda|).
\]

So, we conclude that \( \lim_{r \to \infty} V(\lambda \phi_{m_k(r)}) = 0 \). Then, as each subsequence of \( (V(\lambda \phi_m) : m \in \mathbb{N}) \) has a further subsequence that converges to 0, it follows that \( \lim_{m \to \infty} V(\lambda \phi_m) = 0 \).

Thus we have shown that \( V \) is a pseudo-seminorm on \( \Psi \). Our next objective is to show that \( V \) is sequentially lower semicontinuous.

Let \( (\phi_m : m \in \mathbb{N}) \) be a sequence in \( \Psi \) converging to \( \phi \in \Psi \). Because for each \( x \in D_T(\Psi'_\beta) \) the map \( \varphi \mapsto \sup_{t \in [0,T]} |x(t)[\varphi]| \) is lower semicontinuous,
it follows from Fatou’s lemma that

\[ V(\phi) \leq \sup_{\alpha \in A} \int_{D_T(\Psi')} \liminf_{m \to \infty} \frac{\sup_{t \in [0,T]} |x(t)[\phi_m]|}{1 + \sup_{t \in [0,T]} |x(t)[\phi_m]|} \, d\mu_\alpha \]

\[ \leq \sup_{\alpha \in A} \liminf_{m \to \infty} \int_{D_T(\Psi')} \frac{\sup_{t \in [0,T]} |x(t)[\phi_m]|}{1 + \sup_{t \in [0,T]} |x(t)[\phi_m]|} \, d\mu_\alpha \]

\[ \leq \liminf_{m \to \infty} V(\phi_m). \]

Hence \( V \) is a sequentially lower semicontinuous pseudo-seminorm on \( \Psi \) and because this space is ultrabornological, Proposition 5.7 shows that \( V \) is continuous on \( \Psi \).

**Proof of Proposition 5.4.** Let \( \epsilon > 0 \). From the continuity of the exponential function there exists \( \delta_1 > 0 \) such that \( |1 - e^{ir}| \leq \epsilon/2 \) whenever \( |r| < \delta_1 \). Now, by Lemma 5.8 there exists a continuous seminorm \( p \) on \( \Psi \) such that \( V(\phi) \leq (\delta_2)^2 \) for all \( \phi \in B_p(1) \), where \( \delta_2 = \min\{\delta_1, (-1 + \sqrt{1 + \epsilon})/2\} \). For given \( \phi \in \Psi \), let \( \Gamma_\phi = \{ x \in D_T(\Psi') : \sup_{t \in [0,T]} |x(t)[\phi]| \leq \delta_2 \} \).

Then for all \( \phi \in B_p(1) \) we have

\[
\sup_{\alpha \in A} \int_{D_T(\Psi')} \sup_{t \in [0,T]} |1 - e^{ix(t)[\phi]}| \, d\mu_\alpha \leq \sup_{\alpha \in A} \int_{\Gamma_\phi} \sup_{t \in [0,T]} |1 - e^{ix(t)[\phi]}| \, d\mu_\alpha \\
+ 2 \sup_{\alpha \in A} \mu_\alpha(\Gamma_\phi^{c}) \\
\leq \frac{\epsilon}{2} + 2 \frac{1 + \delta_2}{\delta_2} V(\phi) \leq \frac{\epsilon}{2} + 2 \frac{\epsilon}{4} = \epsilon.
\]

Finally, the equicontinuity at zero of \( (\widehat{\mu}_{\alpha,t} : \alpha \in A, t \in [0,T]) \) is just a consequence of the above result and of the inequality

\[
\sup_{\alpha \in A} \sup_{t \in [0,T]} |1 - \widehat{\mu}_{\alpha,t}(\phi)| \leq \sup_{\alpha \in A} \int_{D_T(\Psi')} \sup_{t \in [0,T]} |1 - e^{ix(t)[\phi]}| \, d\mu_\alpha, \quad \forall \phi \in \Psi. \]

**5.2. Uniform tightness on \( D_\infty(\Phi'). \)** Let \( D_\infty(\Phi') \) denote the space of càdlàg mappings \( x : [0, \infty) \to \Phi' \). For every \( s \geq 0 \), let \( r_s : D(\Phi') \to D_{s+1}(\Phi') \) be given by

\[
r_s(x)(t) = \begin{cases} 
    x(t) & \text{if } t \in [0, s], \\
    (s + 1 - t)x(t) & \text{if } t \in [s, s + 1].
\end{cases}
\]

For every \( \gamma \in \Gamma \) let

\[
d_\gamma(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} (1 \wedge d_\gamma^n(r_n(x), r_n(y))).
\]
where for each $n \in \mathbb{N}$, $d_{\gamma}^n$ is the pseudometric defined in (3.1) for $T = n$. It is not hard to check that each $d_{\gamma}^\infty$ is a pseudometric in $D_\infty(\Phi^\prime_\beta)$. The Skorokhod topology in $D_\infty(\Phi^\prime_\beta)$ (see [20, 29]) is the completely regular topology generated by the family $(d_{\gamma}^\infty : \gamma \in \Gamma)$. An equivalent topology is obtained if we consider a family of seminorms other than $(q_\gamma : \gamma \in \Gamma)$ that generates the strong topology $\beta$ on $\Phi^\prime$ (see [20, Theorem 4.3]).

An interesting fact on the topology introduced above is that if for some $T > 0$ we have $C(D_T(\Phi^\prime_\beta)) = B(D_T(\Phi^\prime_\beta))$, then also $C(D_\infty(\Phi^\prime_\beta)) = B(D_\infty(\Phi^\prime_\beta))$ (see [20, Proposition 4.4] and [27, Lemma 9]). Hence, if $\theta$ is a weaker countably Hilbertian topology on $\Phi$, it follows from Lemma 4.2 that $C(D_\infty((\tilde{\Phi}_\theta)^\prime_\beta)) = B(D_\infty((\tilde{\Phi}_\theta)^\prime_\beta))$. Moreover, since for each $T > 0$ compact subsets of $D_T((\tilde{\Phi}_\theta)^\prime_\beta)$ are metrizable (see Proposition 3.2(2)), and the canonical inclusion from $D_\infty((\tilde{\Phi}_\theta)^\prime_\beta)$ into $D_T((\tilde{\Phi}_\theta)^\prime_\beta)$ is continuous, compact subsets of $D_\infty((\tilde{\Phi}_\theta)^\prime_\beta)$ are also metrizable.

It is clear from the above arguments that all the concepts of measures and random elements introduced in Sect. 4 and the results proved there are also valid for the space $D_\infty(\Phi^\prime_\beta)$. We leave to the reader the task of completing the details.

The following theorem provides necessary and sufficient conditions for uniform tightness for probability measures on $D_\infty(\Phi^\prime_\beta)$.

**Theorem 5.9.** Let $(\mu_\alpha : \alpha \in A)$ be a family of probability measures on $D_\infty(\Phi^\prime_\beta)$ such that:

1. For all $T > 0$, the family $(\hat{\mu}_{\alpha,t} : t \in [0, T], \alpha \in A)$ of Fourier transforms is equicontinuous at zero.
2. For each $\phi \in \Phi$, the family $(\mu_\alpha \circ \Pi_\phi^{-1} : \alpha \in A)$ of probability measures on $D_\infty(\mathbb{R})$ is uniformly tight.

Then there exists a weaker countably Hilbertian topology $\theta$ on $\Phi$ such that $(\mu_\alpha : \alpha \in A)$ is uniformly tight on $D_\infty((\tilde{\Phi}_\theta)^\prime_\beta)$. In particular, $(\mu_\alpha : \alpha \in A)$ is uniformly tight on $D_\infty(\Phi^\prime_\beta)$.

Conversely, if $\Phi$ is a barrelled nuclear space and the family $(\mu_\alpha : \alpha \in A)$ is uniformly tight on $D_\infty(\Phi^\prime_\beta)$, then conditions (1) and (2) are satisfied.

**Proof.** We start by showing the following:

**Claim.** Given $\epsilon > 0$, there is a weaker countably Hilbertian topology $\theta_\epsilon$ on $\Phi$ and a compact subset $K_\epsilon$ of $D_\infty((\tilde{\Phi}_{\theta_\epsilon})^\prime_\beta)$ such that $\sup_{\alpha \in A} \mu_\alpha(K_\epsilon^c) < \epsilon$.

Let $\epsilon > 0$. By Theorem 5.2 for each $n \in \mathbb{N}$ there exists a continuous Hilbertian seminorm $q_n$ on $\Phi$ and $K_n \subseteq D_n(\Phi_{q_n})$ compact such that $\sup_{\alpha \in A} \mu_\alpha \circ r_n^{-1}(K_n^c) < \epsilon/2^n$. Then [20, Proposition 1.6(iv)] shows that for each $n \in \mathbb{N}$ there exists a compact $K_n \subseteq \Phi_{q_n}$ such that $K_n \subseteq D_n(K_n)$. 


Let $K = \bigcap_{n=1}^{\infty} r_n^{-1}(K_n) \subseteq D_\infty((\Phi_\theta)'_\beta)$, where $\theta$ is the weaker countably Hilbertian topology on $\Phi$ generated by $(q_n : n \in \mathbb{N})$. As $r_n^{-1}(K_n) \subseteq D_\infty(K_n)$ for all $n \in \mathbb{N}$, the set $K = \bigcap_{n=1}^{\infty} K_n$ is a compact subset in $(\Phi_\theta)'_\beta$ and moreover $K \subseteq D_\infty(K)$ is closed. Hence because $\tilde{\Phi}_\theta$ is separable and metrizable, and $\Pi_\phi(K)$ is compact in $D_\infty(\mathbb{R})$ for all $\phi \in \Phi$, [20, Lemma 3.3] shows that $K$ is compact in $D_\infty((\tilde{\Phi}_\theta)'_\beta)$. Moreover,

$$\sup_{x \in A} \mu_\alpha(K^c) \leq \sup_{x \in A} \sum_{n=1}^{\infty} \mu_\alpha \circ r_n^{-1}(K_n^c) < \epsilon.$$ 

So we have proved our claim.

Now, if $(\epsilon_m : m \in \mathbb{N})$ is a decreasing sequence of positive numbers converging to 0, then for each $m \in \mathbb{N}$ there exist $\theta_m$ and $K_m$ satisfying the properties of the claim. But then, if $\theta$ is the weaker countably topology on $\Phi$ generated by the Hilbertian seminorms generating the topologies $\theta_m$ for $m \in \mathbb{N}$, then each $K_m$ is compact in $D_\infty((\Phi_\theta)'_\beta)$ and therefore the family $(\mu_\alpha : \alpha \in A)$ is uniformly tight on $D_\infty((\Phi_\theta)'_\beta)$, and hence on $D_\infty(\Phi'_\beta)$.

For the converse, if $\Phi$ is barrelled and $(\mu_\alpha : \alpha \in A)$ is tight on $D_\infty(\Phi'_\beta)$, then for each $T > 0$ the family $(\mu_\alpha \circ r_T^{-1} : \alpha \in A)$ is tight on $DT((\Phi_\theta)'_\beta)$, and the result follows from Theorem 5.2. □

If in the above proof we use Theorem 5.5 instead of Theorem 5.2 we get the following result for ultrabornological nuclear spaces.

**Theorem 5.10.** Let $(\mu_\alpha : \alpha \in A)$ be a family of probability measures on $D_\infty(\Phi'_\beta)$ where $\Phi$ is an ultrabornological nuclear space. Then the family $(\mu_\alpha : \alpha \in A)$ is uniformly tight on $D_\infty(\Phi'_\beta)$ if and only if the family $(\mu_\alpha \circ \Pi_\phi^{-1} : \alpha \in A)$ is uniformly tight on $D_\infty(\mathbb{R})$ for each $\phi \in \Phi$. Moreover there exists a weaker countably Hilbertian topology $\theta$ on $\Phi$ such that $(\mu_\alpha : \alpha \in A)$ is uniformly tight on $D_\infty((\Phi_\theta)'_\beta)$.

**6. Weak convergence on the Skorokhod space**

**Assumption 6.1.** Unless otherwise indicated, in this section we will always assume that $\Phi$ is a nuclear space.

**6.1. Weak convergence of probability measures.** The following result shows how the theory of the previous sections can be used to prove the weak convergence of a sequence of probability measures on $D_\infty(\Phi'_\beta)$.

**Theorem 6.2** (Lévy’s continuity theorem on $D_\infty(\Phi'_\beta)$). Let $(\mu_n : n \in \mathbb{N})$ be a sequence of probability measures on $D_\infty(\Phi'_\beta)$ such that:
For each $T > 0$, the family $(\hat{\mu}_{n,t} : t \in [0,T], n \in \mathbb{N})$ is equicontinuous at zero.

(2) For each $\phi \in \Phi$, the family $(\mu_n \circ \Pi^{-1}_\phi : n \in \mathbb{N})$ of probability measures on $D_\infty(\mathbb{R})$ is uniformly tight.

(3) For all $m \in \mathbb{N}$, $\phi_1, \ldots, \phi_m \in \Phi$, $t_1, \ldots, t_m \geq 0$,

$$\mu_n \circ (\Pi_{t_1,\ldots,t_m}^{\phi_1,\ldots,\phi_m})^{-1} \Rightarrow \nu_{t_1,\ldots,t_m}^{\phi_1,\ldots,\phi_m},$$

where $\nu_{t_1,\ldots,t_m}^{\phi_1,\ldots,\phi_m}$ is a Borel probability measure on $\mathbb{R}^m$.

Then there exist a weaker countably Hilbertian topology $\theta$ on $\Phi$ and a probability measure $\mu$ on $D_\infty((\widetilde{\Phi}_\theta)'_\beta)$ such that $\mu_n \Rightarrow \mu$ in $\mathcal{M}^1(D_\infty((\widetilde{\Phi}_\theta)'_\beta))$. Moreover $\mu$ is the unique (up to equivalence) probability measure on $D_\infty(\Phi'_\beta)$ such that $\mu_n \Rightarrow \mu$ in $\mathcal{M}^1(D_\infty(\Phi'_\beta))$.

Proof. First by (1), (2) and Theorem 5.9 there exists a weaker countably Hilbertian topology $\theta$ on $\Phi$ such that $(\mu_n : n \in \mathbb{N})$ is uniformly tight on $D_\infty((\widetilde{\Phi}_\theta)'_\beta)$. As $D_\infty((\widetilde{\Phi}_\theta)'_\beta)$ is a completely regular topological space whose compact subsets are metrizable (see Sect. 5.2), the fact that $(\mu_n : n \in \mathbb{N})$ is uniformly tight on $D_\infty((\widetilde{\Phi}_\theta)'_\beta)$ implies that every subsequence of $(\mu_n : n \in \mathbb{N})$ contains a further weakly convergent subsequence (see [4], Theorem 8.6.7, p. 206).

Let $(\mu_n^1 : n \in \mathbb{N})$ and $(\mu_n^2 : n \in \mathbb{N})$ be two subsequences of $(\mu_n : n \in \mathbb{N})$. Then $(\mu_n^1 : n \in \mathbb{N})$ has a subsequence $(\mu_{n_k}^1 : k \in \mathbb{N})$ that converges weakly to $\nu^1$, and $(\mu_n^2 : n \in \mathbb{N})$ has a subsequence $(\mu_{n_k}^2 : k \in \mathbb{N})$ that converges weakly to $\nu^2$. The hypothesis (3) shows that

$$\nu_1 \circ (\Pi_{t_1,\ldots,t_m}^{\phi_1,\ldots,\phi_m})^{-1} = \nu_2 \circ (\Pi_{t_1,\ldots,t_m}^{\phi_1,\ldots,\phi_m})^{-1}$$

for all $m \in \mathbb{N}$, $\phi_1, \ldots, \phi_m \in \Phi$ and $t_1, \ldots, t_m \geq 0$. Therefore, $\nu_1$ and $\nu_2$ coincide on all cylinder sets and hence on $\mathcal{C}(D_\infty((\widetilde{\Phi}_\theta)'_\beta))$. But then $\nu_1 = \nu_2$ because $\mathcal{C}(D_\infty((\widetilde{\Phi}_\theta)'_\beta)) = \mathcal{B}(D_\infty((\widetilde{\Phi}_\theta)'_\beta))$ (see Sect. 5.2). We have shown that every subsequence of $(\mu_n : n \in \mathbb{N})$ contains a further subsequence that converges weakly to the same limit $\mu$ in $\mathcal{M}^1(D_T((\widetilde{\Phi}_\theta)'_\beta))$. Hence, [2], Theorem 2.6] shows that $\mu_n \Rightarrow \mu$ in $\mathcal{M}^1((\widetilde{\Phi}_\theta)'_\beta)$.

Finally, since the inclusion $j_\theta$ from $(\widetilde{\Phi}_\theta)'_\beta$ into $\Phi'_\beta$ is linear and continuous, we have $f \circ j_\theta \in C_b(\mathcal{M}^1((\widetilde{\Phi}_\theta)'_\beta))$ for all $f \in C_b(\Phi'_\beta)$. Therefore, $\mu_n \Rightarrow \mu$ in $\mathcal{M}^1(D_\infty((\widetilde{\Phi}_\theta)'_\beta))$ implies $\mu_n \Rightarrow \mu$ in $\mathcal{M}^1(D_\infty(\Phi'_\beta))$. This also shows the uniqueness of $\mu$. ■

When $\Phi$ is an ultrabornological nuclear space, if in the proof of Theorem 6.2 we use Theorem 5.10 instead of Theorem 5.9 we obtain the following result.
Theorem 6.3. Let $\Phi$ be an ultrabornological nuclear space and let $(\mu_n : n \in \mathbb{N})$ be a sequence of Borel probability measures on $D_\infty(\Phi')$ such that for each $\phi \in \Phi$, the family $(\mu_n \circ \Pi^{-1}_\phi : n \in \mathbb{N})$ is tight on $D_\infty(\mathbb{R})$. Assume further that for any $m \in \mathbb{N}$, $\phi_1, \ldots, \phi_m \in \Phi$, $t_1, \ldots, t_m \geq 0$, there exists a probability measure $\nu_{t_1, \ldots, t_m}^{\phi_1, \ldots, \phi_m}$ on $\mathbb{R}^m$ such that

$$\mu_n \circ (\Pi_{t_1, \ldots, t_m}^{\phi_1, \ldots, \phi_m})^{-1} \Rightarrow \nu_{t_1, \ldots, t_m}^{\phi_1, \ldots, \phi_m}.$$ 

Then there exist a weaker countably Hilbertian topology $\theta$ on $\Phi$ and a probability measure $\mu$ on $D_\infty((\widetilde{\Phi}_\theta')')$ such that $\mu_n \Rightarrow \mu$ in $\mathcal{M}^1(D_\infty((\widetilde{\Phi}_\theta')'))$. Moreover, $\mu$ is the unique (up to equivalence) probability measure on $D_\infty(\Phi')$ such that $\mu_n \Rightarrow \mu$ in $\mathcal{M}^1(D_\infty(\Phi'))$.

Remark 6.4. Clearly, Theorems 6.2 and 6.3 can also be formulated for measures on $D_T(\Phi')$. We leave to the reader the task of stating and proving them by using Theorems 5.2 and 5.5.

6.2. Weak convergence of (cylindrical) processes in the Skorokhod space. In this section we apply our results to provide sufficient conditions for the weak convergence in $D_\infty(\Phi')$ of a sequence of càdlàg processes. This is done in the following result formulated in the more general setting of cylindrical processes:

Theorem 6.5. For each $n \in \mathbb{N}$, let $X^n = \{X^n_t \geq 0\}$ be a cylindrical process in $\Phi'$ (e.g. a $\Phi'_\beta$-valued process) such that:

1. For each $\phi \in \Phi$ and each $n \in \mathbb{N}$, the real-valued process $X^n(\phi) = \{X_t^n(\phi) \geq 0\}$ is càdlàg.
2. For every $T > 0$, the family $\{X^n_t : t \in [0, T], n \in \mathbb{N}\}$ of linear maps from $\Phi$ into $L^0(\Omega, \mathcal{F}, \mathbb{P})$ is equicontinuous at zero.
3. For each $\phi \in \Phi$, the sequence of distributions of $X^n(\phi)$ is uniformly tight on $D_\infty(\mathbb{R})$.
4. For all $m \in \mathbb{N}$, $\phi_1, \ldots, \phi_m \in \Phi$, and $t_1, \ldots, t_m \geq 0$, the probability distribution of $(X^n_t(\phi_1), \ldots, X^n_t(\phi_m))$ converges in distribution to some probability measure on $\mathbb{R}^m$.

Then there exist a weaker countably Hilbertian topology $\theta$ on $\Phi$ and some $D_\infty((\widetilde{\Phi}_\theta')')$-valued random variables $Y$ and $Y^n$, $n \in \mathbb{N}$, such that:

a. For all $\phi \in \Phi$ and $n \in \mathbb{N}$, the real-valued càdlàg processes $X^n(\phi)$ and $Y^n[\phi]$ are indistinguishable.

b. The sequence $(Y^n : n \in \mathbb{N})$ is tight on $D_\infty((\widetilde{\Phi}_\theta')')$.

c. $Y^n \Rightarrow Y$ in $D_\infty((\widetilde{\Phi}_\theta')')$.

Moreover (b) and (c) are also satisfied for $Y$ and $(Y^n : n \in \mathbb{N})$ as $D_\infty(\Phi')$-valued random variables.
Proof. First for each \( n \in \mathbb{N} \), from (1), (2) and Theorem \( \text{4.7} \) there exists a \( D_\infty(\Phi'_\beta) \)-valued random variable \( Y^n \) such that for each \( \phi \in \Phi \), \( X^n(\phi) \) and \( Y^n[\phi] \) are indistinguishable.

For each \( n \in \mathbb{N} \), let \( \mu_n \) denote the probability distribution of \( Y^n \) on \( D_\infty(\Phi'_\beta) \). Then, for all \( m \in \mathbb{N} \), \( \phi_1, \ldots, \phi_m \in \Phi \), and \( t_1, \ldots, t_m \geq 0 \), it is clear that \( \mu_n \circ (\Pi_{t_1, \ldots, t_m}^\phi)^{-1} \) is the probability distribution of \( (X^n_{t_1}(\phi_1), \ldots, X^n_{t_m}(\phi_m)) \). In particular, for each \( n \in \mathbb{N} \), the Fourier transform \( \hat{\mu}_{n,t} \) is that of \( X^n \) as a cylindrical random variable in \( D_\infty(\Phi'_\beta) \). Therefore, conditions (2)−(4) imply that \( (\mu_n : n \in \mathbb{N}) \) satisfies the conditions of Theorem \( \text{6.2} \). This shows the existence of a weaker countably Hilbertian topology \( \theta \) on \( \Phi \) and a probability measure \( \mu \) on \( D_\infty((\Phi_\theta)'_\beta) \) such that \( \mu_n \Rightarrow \mu \) in \( M_1(D_\infty((\Phi_\theta)'_\beta)) \). Hence each \( Y^n \) is a \( D_\infty((\Phi_\theta)'_\beta) \)-valued random variable and \( Y \) is a \( D_\infty((\Phi_\theta)'_\beta) \)-valued random variable whose probability distribution is \( \mu \) (this is a consequence of Lévy’s theorem and Corollary \( \text{4.10} \)). Therefore (a)−(c) are clearly satisfied. ■

Much as for weak convergence of probability measures, under the assumption that \( \Phi \) is ultrabornological and nuclear we can obtain a version of the above theorem with weaker assumptions.

**Theorem 6.6.** Let \( \Phi \) be an ultrabornological nuclear space. For each \( n \in \mathbb{N} \), let \( X^n = \{X^n_t\}_{t \geq 0} \) be a \( \Phi'_\beta \)-valued càdlàg process such that for each \( t \geq 0 \) the distribution of \( X^n_t \) is a Radon measure on \( \Phi'_\beta \). Suppose moreover that the sequence \( (X^n : n \in \mathbb{N}) \) satisfies (3) and (4) of Theorem \( \text{6.5} \). Then there exists a weaker countably Hilbertian topology \( \theta \) on \( \Phi \) and \( (\Phi_\theta)'_\beta \)-valued càdlàg processes \( Y = \{Y_t\}_{t \geq 0} \) and \( Y^n = \{Y^n_t\}_{t \geq 0} \), for all \( n \in \mathbb{N} \), such that:

(a) For all \( n \in \mathbb{N} \), \( X^n \) and \( Y^n \) are indistinguishable.
(b) \( Y \) and each \( Y^n \) are \( D_\infty((\Phi_\theta)'_\beta) \)-valued random variables.
(c) \( (Y^n : n \in \mathbb{N}) \) is tight on \( D_\infty((\Phi_\theta)'_\beta) \).
(d) \( Y^n \Rightarrow Y \) in \( D_\infty((\Phi_\theta)'_\beta) \).

Moreover, (c) and (d) are also satisfied for \( Y \) and \( (Y^n : n \in \mathbb{N}) \) as \( D_\infty((\Phi'_\beta) \)-valued random variables.

**Proof.** The arguments are similar to those used to deduce Theorem \( \text{6.5} \) from Corollary \( \text{4.8} \) and Theorem \( \text{6.3} \). ■

**7. Weak convergence of Lévy processes in Skorokhod space**

**Assumption 7.1.** In this section, \( \Phi \) always denotes a barrelled nuclear space.

In this section we will provide sufficient conditions for a sequence of \( \Phi'_\beta \)-valued Lévy processes to converge in \( D_\infty((\Phi'_\beta) \). We start by recalling some
basic properties of Lévy processes taking values in \( \Phi'_{\beta} \). For further details see \[15\].

A \( \Phi'_{\beta} \)-valued process \( L = \{L_t\}_{t \geq 0} \) is called a Lévy process if (i) \( L_0 = 0 \) a.s., (ii) \( L \) has independent increments, i.e. for any \( n \in \mathbb{N} \) and \( 0 \leq t_1 < \cdots < t_n < \infty \) the \( \Phi'_{\beta} \)-valued random variables \( L_{t_1}, L_{t_2} - L_{t_1}, \ldots, L_{t_n} - L_{t_{n-1}} \) are independent, (iii) \( L \) has stationary increments, i.e. for any \( 0 \leq s \leq t, L_t - L_s \) and \( L_{t-s} \) are identically distributed, and (iv) for every \( t \geq 0 \) the distribution \( \mu_t \) of \( L_t \) is a Radon measure and the mapping \( t \mapsto \mu_t \) from \( \mathbb{R}_+ \) into the space \( \mathcal{M}_{1R}(\Phi'_{\beta}) \) of Radon probability measures on \( \Phi'_{\beta} \) is continuous at 0 when \( \mathcal{M}_{1R}(\Phi'_{\beta}) \) is equipped with the weak topology.

Every \( \Phi'_{\beta} \)-valued Lévy process \( L = \{L_t\}_{t \geq 0} \) has a regular, càdlàg version \( \hat{L} = \{\hat{L}_t\}_{t \geq 0} \) that is also a Lévy process. Moreover, there exists a weaker countably Hilbertian topology \( \vartheta_L \) on \( \Phi \) such that \( \hat{L} \) is a \( (\Phi_{\vartheta_L})'_{\beta} \)-valued càdlàg process (see \[15\] Corollary 3.11). Therefore, \( L \) can be identified with a \( D_{\infty}(\Phi'_{\beta}) \)-valued random variable whose probability distribution is a Radon measure on \( D_{\infty}(\Phi'_{\beta}) \) (see the proof of Theorem 4.7).

Recall that a Borel measure \( \nu \) on \( \Phi'_{\beta} \) is a Lévy measure (see \[15\]) if:

1. \( \nu(\{0\}) = 0 \).
2. For each neighborhood of zero \( U \subseteq \Phi'_{\beta} \), the restriction \( \nu|_{U^c} \) belongs to the space \( \mathcal{M}_{1R}(\Phi'_{\beta}) \) of bounded Radon measures on \( \Phi'_{\beta} \).
3. There exists a continuous Hilbertian seminorm \( \rho \) on \( \Phi \) such that

\[
(7.1) \quad \int_{B_{\rho'}(1)} \rho'(f)^2 \nu(df) < \infty \quad \text{and} \quad \nu|_{B_{\rho'}(1)} \in \mathcal{M}_{bR}(\Phi'_{\beta}).
\]

One of the most important properties of a \( \Phi'_{\beta} \)-valued Lévy process \( L = \{L_t\}_{t \geq 0} \) is the Lévy–Khinchin formula for its Fourier transform \[15\] Theorem 4.18]: for all \( t \geq 0 \) and \( \phi \in \Phi \),

\[
(7.2) \quad \mathbb{E}(e^{iL_t[\phi]}) = e^{t\eta(\phi)} \quad \text{with}
\]

\[
\eta(\phi) = \text{im}[\phi] - \frac{1}{2} Q(\phi)^2 + \int_{\Phi'_{\beta}} (e^{if[\phi]} - 1 - if[\phi]1_{B_{\rho'}(1)}(f)) \nu(df),
\]

where \( \text{m} \in \Phi'_{\beta} \), \( Q \) is a continuous Hilbertian seminorm on \( \Phi \), \( \nu \) is a Lévy measure on \( \Phi'_{\beta} \) and \( \rho \) is a continuous Hilbertian seminorm on \( \Phi \) for which \( \nu \) satisfies (7.1).

Our main result on convergence of Lévy processes is the following:

**Theorem 7.2.** For every \( n \in \mathbb{N} \), let \( L^n = \{L^n_t\}_{t \geq 0} \) be a \( \Phi'_{\beta} \)-valued càdlàg Lévy process where \( (\text{m}_n, Q_n, \nu_n, \rho_n) \) are as in (7.2). Assume that there exists a continuous Hilbertian seminorm \( q \) on \( \Phi \) with \( Q_n \leq q \) and \( \rho_n \leq q \) for all \( n \in \mathbb{N} \), and such that:
(1) \((m_n : n \in \mathbb{N})\) is relatively compact in \(\Phi'\).
(2) \(\sup_{n \in \mathbb{N}} \| i_{Q_n,q} \|_{L^2(\Phi_q, \Phi_{Q_n})} < \infty\).
(3) \(\sup_{n \in \mathbb{N}} \int_{\Phi'_\beta} (q'(f)^2 \wedge 1) \nu_n(df) < \infty\).

Suppose moreover that for all \(m \in \mathbb{N}, \phi_1, \ldots, \phi_m \in \Phi, \) and \(t_1, \ldots, t_m \geq 0,\) the sequence of distributions of \((L^n_{t_1}(\phi_1), \ldots, L^n_{t_m}(\phi_m))\) converges in distribution to some probability measure on \(\mathbb{R}^m\). Then the sequence \((L^n : n \in \mathbb{N})\) is uniformly tight on \(D_\infty(\Phi'_\beta)\) and there exists a \(\Phi'_\beta\)-valued Lévy process \(L = \{L_t\}_{t \geq 0}\) such that \(L^n \Rightarrow L\) in \(D_\infty(\Phi'_\beta)\).

**Proof.** For each \(n \in \mathbb{N}, \) let \(\mu_n\) be the distribution of \(L^n\) as a random variable in \(D_\infty(\Phi'_{\beta})\). Then for each \(t \geq 0,\) the Fourier transform \(\hat{\mu}_{n,t}\) of \(\mu_n \circ \Pi_t^{-1}\) is precisely the Fourier transform of \(L^n_t\). Hence if for each \(n \in \mathbb{N}, \) \(\eta_n\) is defined by (7.2), then \(\hat{\mu}_{n,t}(\phi) = e^{i\eta_n(\phi)}\) for each \(t \geq 0\) and \(\phi \in \Phi.\) Thus, in order to show that for every \(T > 0,\) \((\hat{\mu}_{n,t} : t \in [0, T], n \in \mathbb{N})\) is equicontinuous at zero, it is enough to show that \((\hat{\mu}_{n,1} : n \in \mathbb{N})\) is equicontinuous at zero.

Now, because the family \((\mu_n \circ \Pi_t^{-1} : n \in \mathbb{N})\) of measures is infinitely divisible (they correspond to the distributions of the sequence \((L^n_{t_1}(\phi_1), \ldots, L^n_{t_m}(\phi_m))\); see [15, Theorem 3.5]), it follows from [8, Satz 2.8] that conditions (1)–(3) imply that \((\mu_n \circ \Pi_1^{-1} : n \in \mathbb{N})\) is uniformly tight on \(\Phi'_{\beta}\). But as \(\Phi\) is a barrelled space, the above implies that \((\hat{\mu}_{n,1} : n \in \mathbb{N})\) is equicontinuous at zero (see [3, Theorem III.2.7, p. 104]). Hence, \((\hat{\mu}_{n,t} : t \in [0, T], n \in \mathbb{N})\) is equicontinuous at zero for each \(T > 0.\)

Furthermore for every \(t \geq 0\) and \(\phi \in \Phi\) the sequence \((L^n_t[\phi])\) converges weakly and because \(L^n[\phi] = (L^n_t[\phi])_{t \geq 0}\) is a real-valued càdlàg Lévy process for each \(n \in \mathbb{N},\) the sequence \((L^n[\phi] : n \in \mathbb{N})\) converges weakly in \(D_\infty(\mathbb{R})\) (see [1, Proposition 12.4]) for each \(\phi \in \Phi.\) Therefore by the Prokhorov theorem, \((L^n[\phi] : n \in \mathbb{N})\) is uniformly tight on \(D_\infty(\mathbb{R}),\) and hence \((\mu_n \circ \Pi_\phi^{-1} : n \in \mathbb{N})\) is uniformly tight on \(D_\infty(\mathbb{R}).\)

Thus all the conditions in Theorem 6.5 are satisfied, and so \((L^n : n \in \mathbb{N})\) is uniformly tight on \(D_\infty(\Phi'_{\beta})\) and also there exists a \(\Phi'_{\beta}\)-valued càdlàg process \(L = \{L_t\}_{t \geq 0}\) such that \(L^n \Rightarrow L\) in \(D_\infty(\Phi'_{\beta}).\) Finally, because for all \(m \in \mathbb{N}, \phi_1, \ldots, \phi_m \in \Phi, t_1, \ldots, t_m \in [0, T],\) \((L^n_{t_1}(\phi_1), \ldots, L^n_{t_m}(\phi_m))\) converges in distribution to \((L_{t_1}(\phi_1), \ldots, L_{t_m}(\phi_m))\), it follows that \(L\) is a cylindrical Lévy process. But then \(L\) is a \(\Phi'_{\beta}\)-valued Lévy process by [15, Theorem 3.8]. ■

8. Tightness on the Skorokhod space of a locally convex space

**Assumption 8.1.** In this section, \((\Phi, \tau)\) always denotes a (Hausdorff) locally convex space.

In this section we will show how the machinery developed in the previous sections for the case of the dual of a nuclear space can be applied to study uniform tightness of probability measures on \(D_\infty(\Phi'_{\beta}).\) This can be done
through the use of the Sazonov topology whose definition will be recalled for the convenience of the reader. For further details, see [4, 36, 38].

Let \( P(\Phi, \tau) \) denote the following collection of seminorms: \( p \in P(\Phi, \tau) \) if and only if \( p \) is a continuous Hilbertian seminorm on \( (\Phi, \tau) \) for which there exists a continuous Hilbertian seminorm \( q \) on \( \Phi \) such that \( p \leq q \), \( \Phi_q \) is separable, and the canonical inclusion \( i_{p, q}: \Phi_q \to \Phi_p \) is Hilbert–Schmidt. The collection \( P(\Phi, \tau) \) is not empty as every seminorm on \( \Phi \) continuous with respect to the weak topology \( \sigma \) is in \( P(\Phi, \tau) \).

The locally convex topology on \( \Phi \) generated by \( P(\Phi, \tau) \) is called the Sazonov topology with respect to \( \tau \) and is denoted by \( \tau_S \). Considering finite-dimensional subspaces of \( \Phi \) as Hilbert spaces, it is clear that \( \sigma \) is weaker than \( \tau_S \). On the other hand, each \( p \in P(\Phi, \tau) \) is a continuous Hilbertian seminorm on \( \Phi \), and therefore \( \tau_S \) is weaker than \( \tau \). Moreover, \( \tau_S = \tau \) if and only if \( (\Phi, \tau) \) is a nuclear space.

**Theorem 8.2.** Let \( (\mu_\alpha : \alpha \in A) \) be a family of probability measures on \( D_\infty(\Phi'_\beta) \) that satisfies the following conditions:

1. For all \( T > 0 \), the family \( \mu_{\alpha, t} : t \in [0, T], \alpha \in A \) is equicontinuous at zero on \( (\Phi, \tau_S) \).
2. For each \( \phi \in \Phi \), the family \( \mu_\alpha \circ \Pi_{\phi}^{-1} : \alpha \in A \) of probability measures on \( D_\infty(\mathbb{R}) \) is uniformly tight.

Then there exists a weaker countably Hilbertian topology \( \theta \) on \( \Phi \) such that \( (\mu_\alpha : \alpha \in A) \) is uniformly tight on \( D_\infty((\tilde{\Phi}_\theta)'_\beta) \). In particular, \( (\mu_\alpha : \alpha \in A) \) is uniformly tight on \( D_\infty(\Phi'_\beta) \).

**Proof.** The proof is a modification of the arguments used in the proofs of Theorems 5.2 and 5.9. For the benefit of the reader we will sketch the main steps.

First, the regularization theorem (Theorem 2.1) remains valid if we assume equicontinuity with respect to the Sazonov topology \( \tau_S \) (see [16]). Then the regularization theorem and the Minlos theorem on the Skorokhod space (Theorems 4.7 and 4.9) (and therefore Proposition 5.3) remain valid if we assume equicontinuity with respect to \( \tau_S \). In a similar way, Theorem 5.2 can be proved if we assume equicontinuity of the Fourier transforms with respect to \( \tau_S \). In fact, our assumptions imply that in (5.1) we can choose the continuous Hilbertian seminorm \( p \) to be \( \tau_S \)-continuous. Therefore, from the definition of \( \tau_S \) and since every Hilbert–Schmidt operator can be factored into the composition of a Hilbert–Schmidt operator and a compact operator (see [36, Proposition II.3.6, p. 217]), for the \( \tau_S \)-continuous seminorm \( p \) one can find two continuous Hilbertian seminorms \( q \) and \( q \) on \( \Phi \) such that the canonical inclusions \( i_{p, q}: \Phi_q \to \Phi_p \) and \( i_{q, q}: \Phi_q \to \Phi_q \) are respectively
Hilbert–Schmidt and compact. The proof of Theorem 5.2 can then be replicated with almost no changes.

Finally, in the proof of Theorem 5.9 we only used the corresponding result from Theorem 5.2 but since the conclusions of the latter remain valid under the assumption of equicontinuity with respect to \( \tau_S \), the conclusions of Theorem 5.9 are valid as well. In the above result we deduce the tightness of \((\mu_{\alpha}: \alpha \in A)\) on \( D_{\infty}(\Phi, \tau_S)'_\beta \), but since the inclusion from \( D_{\infty}(\Phi, \tau_S)'_\beta \) into \( D_{\infty}(\Phi)'_\beta \) is continuous (because the topology on \( (\Phi, \tau_S)'_\beta \) is finer than the topology induced from \( \Phi' \)), we infer the tightness of \((\mu_{\alpha}: \alpha \in A)\) on \( D_{\infty}(\Phi)'_\beta \).

In a similar way, modifying the arguments in the proof of Theorem 6.2 by using Theorem 8.2 we can prove the following:

**Theorem 8.3.** Let \((\mu_n: n \in \mathbb{N})\) be a sequence of probability measures on \( D_{\infty}(\Phi)'_\beta \) that satisfies (1) and (2) of Theorem 8.2 and such that for all \( m \in \mathbb{N}, \phi_1, \ldots, \phi_m \in \Phi, t_1, \ldots, t_m \geq 0 \), the sequence \( \mu_n \circ (\Pi_{t_1, \ldots, t_m}^{\phi_1, \ldots, \phi_m})^{-1} \) converges weakly on \( \mathbb{R}^m \). Then \( \mu_n \Rightarrow \mu \) in \( \mathcal{M}^1(D_{\infty}(\Phi)'_\beta) \).

**Remark 8.4.** It should be clear that we can also prove a version of Theorem 6.5 in the context of locally convex spaces provided that we assume that the linear maps \( \{X^n_t: t \in [0,T], n \in \mathbb{N}\} \) from \( \Phi \) into \( L^0(\Omega, \mathcal{F}, \mathbb{P}) \) are equicontinuous at zero on \( (\Phi, \tau_S) \). We leave the details to the reader.

Recall that if \( H \) is a separable Hilbert space, the Sazonov topology \( \tau_S \) on \( H \) is generated by the seminorms on \( H \) of the form \( p_S(\phi) = \|S\phi\|_H \) for \( \phi \in H \), where \( S \) runs over the Hilbert–Schmidt operators on \( H \). This definition is equivalent to the one introduced above for general locally convex spaces (see [38]). Since the conclusions of Theorem 4.9 remain valid if we assume equicontinuity with respect to the Sazonov topology, we obtain the following generalization of Sazonov’s theorem on the Skorokhod space:

**Theorem 8.5.** Let \( H \) be a separable Hilbert space and let \( \mu \) be a cylindrical probability measure on \( D_T(H) \). Suppose that the family \((\hat{\mu}_t: t \in [0,T])\) is equicontinuous at zero on \( (H, \tau_S) \). Then there exists a Radon probability measure \( \nu \) on \( D_T(H) \) that is an extension of \( \mu \).

Now, if we use Theorems 8.2 and 8.3 we obtain the following generalization of Lévy’s continuity theorem on the Skorokhod space:

**Theorem 8.6.** Let \( H \) be a separable Hilbert space. For a sequence \((\mu_n: n \in \mathbb{N})\) of probability measures on \( D_{\infty}(H) \) to be uniformly tight it is sufficient that the following conditions are satisfied:

1. For all \( T > 0 \), the family \((\hat{\mu}_{n,t}: t \in [0,T], n \in \mathbb{N})\) of Fourier transforms is equicontinuous at zero on \( (H, \tau_S) \).
(2) For each \( h \in H \), the sequence \((\mu_n \circ \Pi^{-1}_h : n \in \mathbb{N})\) of probability measures on \( D_\infty(\mathbb{R}) \) is uniformly tight.

If moreover for all \( m \in \mathbb{N} \), \( h_1, \ldots, h_m \in H \), and \( t_1, \ldots, t_m \geq 0 \), the sequence \( \mu_n \circ (\Pi_{t_1, \ldots, t_m}^{h_1, \ldots, h_m})^{-1} \) converges weakly on \( \mathbb{R}^m \), then \( \mu_n \Rightarrow \mu \) in \( \mathcal{M}_1(D_\infty(H)) \).

We hope that Theorems 8.5 and 8.6 can serve as useful sufficient conditions for tightness and weak convergence on the Skorokhod space of a Hilbert space; this is a contribution to the literature on the Skorokhod space of a metric space (see e.g. [2, 10, 37]).

9. Remarks and examples. Throughout this paper we have considered random variables and probability measures on the dual of a nuclear space. Most of our results have been formulated for a general nuclear space \( \Phi \), but occasionally we have assumed some additional structure on \( \Phi \), for example that \( \Phi \) is barrelled or ultrabornological. The purpose of this section is to provide concrete examples of nuclear spaces satisfying these conditions and to attribute to each of them the properties used throughout the paper and thus the results valid for them. Some additional remarks are given, together with comparison of our results with those obtained by other authors.

9.1. The case of ultrabornological and barrelled nuclear spaces

Examples. There are many examples of spaces of functions widely used in analysis that are nuclear. For example, it is known (see e.g. [33, 35, 39]) that the function spaces \( \mathcal{E}_K := C_\infty(K) \) \((K \text{ a compact subset of } \mathbb{R}^d)\), \( \mathcal{E} := C_\infty(\mathbb{R}^d) \), the rapidly decreasing functions \( \mathcal{S}(\mathbb{R}^d) \), and the space \( \mathcal{H}(U) \) of harmonic functions \((U \text{ an open subset of } \mathbb{R}^d)\) are all examples of Fréchet nuclear spaces. Their (strong) duals \( \mathcal{E}_K', \mathcal{E}', \mathcal{S}'(\mathbb{R}^d), \mathcal{H}'(U) \) are also nuclear. On the other hand, the space \( \mathcal{D}(U) := C_\infty(U) \) of test functions \((U \text{ an open subset of } \mathbb{R}^d)\), the space \( \mathcal{P}_n \) of polynomials in \( n \) variables, and the space \( \mathbb{R}^N \) of real-valued sequences (with direct sum topology) are strict inductive limits of Fréchet nuclear spaces (hence they are also nuclear). The space \( \mathcal{D}'(U) \) of distributions \((U \text{ an open subset of } \mathbb{R}^d)\) is also nuclear. All the above are examples of (complete) ultrabornological nuclear spaces.

Compactness on Skorokhod space. In Theorem 3.5 we have proved that for a barrelled nuclear space \( \Phi \) and \( A \subseteq DT(\Phi') \), compactness of finite-dimensional projections of \( A \) implies compactness. The same characterization but for \( \Phi \) being a Fréchet nuclear space was proved by Mitoma [29]. We are not aware of any further extension of the result of Mitoma to any other classes of nuclear spaces. Moreover, since any ultrabornological space is also barrelled, the examples given in the previous paragraph are all examples of barrelled nuclear spaces for which Theorem 3.5 is valid.
Now, it is important to mention that there are examples of (complete) nuclear spaces that are not barrelled and for which compactness of finite-dimensional projections does not imply compactness (and hence the characterization in Theorem 3.5 fails). To provide an example, let $E$ be an infinite-dimensional Banach space. Then it is known that $E$ is the strong dual of some complete nuclear space $\Phi$ (see [19, Corollary 1 of Theorem IV.4.3.3]). Note that $\Phi$ cannot be barrelled because if it were, it would be reflexive and then $\Phi = E'_{\beta}$ (see [35, Theorems III.7.2 and IV.5.6]). But this equality is impossible because in that case $\Phi$ would be both nuclear and Banach, and this is only possible if $\Phi$ is finite-dimensional (see [33, Theorem 4.4.14]).

Now, let $B$ denote the closed unit ball in $E$ and $\delta > 0$. Let $A = A(B, \delta)$ denote the collection of all $x \in D_T(E) = D_T(\Phi'_{\beta})$ that are of the form $x(t) = f_j$ for $t \in [t_{j-1}, t_j)$, $j = 1, \ldots, m$, where $t_j - t_{j-1} > \delta$, $f_j \in B$, $t_0 = 0$, $t_m = T$. For each $\phi \in \Phi$, the set $\Pi_{\phi}(A) = \{x[\phi] : x \in A\}$ is relatively compact in $D_T(\mathbb{R})$: this is a consequence of the fact that $B[\phi] = \{f[\phi] : f \in B\}$ is relatively compact in $\mathbb{R}$ and of [23, Lemma 2.4.1]. However, $A$ cannot be relatively compact in $D_T(E)$ because in that case the closure of $\{x(t) : t \in [0, T], x \in A\}$, which is equal to $B$, must be compact (see [20, proof of Proposition 1.6(vi)]); but this is impossible as $E$ is infinite-dimensional.

**Tightness and weak convergence on Skorokhod space.** In Theorems 5.5 and 5.10 we have proved that if $\Phi$ is an ultrabornological space, tightness of a family of probability measures on the Skorokhod space in $\Phi'_{\beta}$ is equivalent to tightness of its one-dimensional projections on the Skorokhod space in $\mathbb{R}$. Under the same hypothesis on $\Phi$, in Theorem 6.3 we proved that for weak convergence of a sequence of probability measures on the Skorokhod space in $\Phi'_{\beta}$ it is sufficient to have tightness of one-dimensional projections on $D_{\infty}(\mathbb{R})$ and weak convergence of time-space finite-dimensional projections. The analogous result for weak convergence of a sequence of $\Phi'_{\beta}$-valued càdlàg processes is given in Theorem 6.6.

The above results were first proved by Mitoma [29] for $\Phi$ being a Fréchet nuclear space. They were later extended by Fouque [17] to the case when $\Phi$ is a countable inductive limit of nuclear Fréchet spaces. However, if $\Phi$ is a Fréchet space or the countable inductive limit of Fréchet spaces, then it is ultrabornological (see [21, Corollaries 4 and 5, Section 13.1, p. 273]). Hence, Theorems 5.5, 5.10, 6.3 and 6.6 generalize the results obtained by Mitoma and Fouque under the same hypothesis. Moreover, our results work for classes of ultrabornological nuclear spaces that are not covered by Mitoma and Fouque’s assumptions, for example the space $\mathcal{A}(V)$ of real-analytic functions ($V$ a closed subset of $\mathbb{R}^d$, see [19]), or the space $\mathbb{R}^N$ equipped with the product topology (where $N$ denotes the cardinality of the continuum, see [24]).
9.2. The case of general nuclear spaces

Examples. There are interesting examples of nuclear spaces that are not (or might not be) ultrabornological or barrelled. As examples we have the space $\mathbb{R}^D$ equipped with its product topology ($D$ an arbitrary set, see [39]) and nuclear Köthe sequence spaces (see [19, 21]).

Many more examples can be generated if one considers spaces of functions defined on nuclear spaces or spaces whose strong dual is nuclear (also called dual nuclear spaces). One has for example the space of holomorphic functions defined on a (quasi-)complete dual nuclear space (e.g. $H(D',\mathcal{R}^d)$, see [9]), the space of continuous linear operators from a semi-reflexive dual nuclear space into a nuclear space (e.g. the space $D'(\mathcal{U};\mathcal{S}(\mathcal{R}^d)) := L(\mathcal{C}_c(\mathcal{U}),\mathcal{R}^N)$ of distributions with values in $\mathcal{R}^N$, $\mathcal{U} \subseteq \mathcal{R}^n$ open), and tensor products of arbitrary nuclear spaces (e.g. the space $H(U;D'(\mathcal{R}^d)) \cong H(U) \otimes D'(\mathcal{R}^d)$ of holomorphic functions with values in the space of distributions, with $\mathcal{U} \subseteq \mathcal{R}^n$ open); for references see [35, 39]. A particular example of a non-barrelled nuclear space that has been used in the study of weak convergence of sequences of $\mathcal{S}(\mathcal{R}^d)$-valued càdlàg processes is the space $D(\mathcal{R}) \otimes \mathcal{S}(\mathcal{R}^d)$ (see [3] for details).

Tightness and weak convergence on Skorokhod space. Apart from studying tightness and weak convergence of probability measures and random variables in the Skorokhod space $D_T(\Phi'_\beta)$, in this article we have also introduced the more general concepts of cylindrical measures and cylindrical random variables in $D_T(\Phi'_\beta)$. We are not aware of any other work that makes a systematic study of these objects. In particular, for $\Phi$ a nuclear space we have shown that there is an analogue of the regularization theorem (Theorem 4.7) and of the Minlos theorem (Theorem 4.9) for cylindrical random variables and cylindrical measures in $D_T(\Phi'_\beta)$.

Now, if $\Phi$ is a general nuclear space, in Theorems 5.2 and 5.9 we have shown that for a family of probability measures on $D_T(\Phi'_\beta)$ (or on $D_{\infty}(\Phi'_\beta)$), the equicontinuity of Fourier transforms and tightness of its one-dimensional projections on the Skorokhod space in $\mathbb{R}$ is sufficient for tightness on $D_T(\Phi'_\beta)$ (or on $D_{\infty}(\Phi'_\beta)$), and necessary if $\Phi$ is barrelled. There are two important comments we want to make on this result.

First, the methodology used in our proofs shows that the use of the Baire category argument, which was a common element in the arguments of Mitoma and Fouque (see the proofs of [29, Lemma 3.3] and [17, Lemme II.1 and Lemme IV.1]), can be replaced in a very efficient way by the use of a weaker countably Hilbertian topology. This was a fundamental step in our arguments to extend the results of Mitoma to general nuclear spaces. Observe that equicontinuity of Fourier transforms played a fundamental role in showing the existence of the weaker countably Hilbertian topology.
Second, our condition of equicontinuity of Fourier transforms seems to be less demanding than the compact containment condition introduced by Jakubowski [20]. The latter condition, together with weak tightness with respect to some separating family, constitutes the main characterization for tightness on the Skorokhod space for a completely regular space whose compact subsets are metrizable. The more recent work of Kouritzin [27] replaces the weak tightness assumption by some modulus of continuity conditions, but the compact containment condition is still present. Apart from the argument given above, observe that unlike the results in [20] and [27], in Theorems 5.2 and 8.2 we do not need to assume that compact subsets of $\Phi'_\beta$ are metrizable. This is of importance because not for every locally convex space are its compact subsets metrizable. Necessary and sufficient conditions for this to be true are given in [12].

In Theorem 6.2 we introduced an analogue of Lévy’s continuity theorem for weak convergence of probability measures on $D_\infty(\Phi'_\beta)$. The corresponding result for weak convergence of (cylindrical) processes in $\Phi'_\beta$ is given in Theorem 6.5. We have illustrated the usefulness of these results in Theorem 7.2 where we provide sufficient conditions for the weak convergence of a sequence of Lévy processes in $D_\infty(\Phi'_\beta)$. We hope our results can be applied to the study of weak convergence of SPDEs taking values in the dual of a nuclear space defined in [14]; the same is done for example in [11, 32] for SPDEs in the dual of a Fréchet nuclear space.

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