On a magnetic characterization of spectral minimal partitions.

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Abstract

Given a bounded open set Ω in \( \mathbb{R}^n \) (or in a Riemannian manifold) and a partition of Ω by \( k \) open sets \( D_j \), we consider the quantity \( \max_j \lambda(D_j) \) where \( \lambda(D_j) \) is the ground state energy of the Dirichlet realization of the Laplacian in \( D_j \). If we denote by \( \Sigma_k(\Omega) \) the infimum over all the \( k \)-partitions of \( \max_j \lambda(D_j) \), a minimal \( k \)-partition is then a partition which realizes the infimum. When \( k = 2 \), we find the two nodal domains of a second eigenfunction, but the analysis of higher \( k \)'s is non trivial and quite interesting. In this paper, we give the proof of one conjecture formulated in [5] and [18] about a magnetic characterization of the minimal partitions when \( n = 2 \).

Keywords: minimal partitions, nodal sets, Aharonov-Bohm Hamiltonians, Courant’s nodal theorem.

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1 Introduction

1.1 Main definitions

We consider mainly the Dirichlet Laplacian in a bounded domain \( \Omega \subset \mathbb{R}^2 \). We would like to analyze the relations between the nodal domains of the real-valued eigenfunctions of this Laplacian and the partitions of \( \Omega \) by \( k \) open sets \( D_i \) which are minimal in the sense that the maximum over the \( D_i \)'s of the ground state energy\(^1\) of the Dirichlet realization of the Laplacian \( H(D_i) \) in \( D_i \) is minimal. In the case of a Riemannian compact manifold, the natural extension is to consider the Laplace Beltrami operator. We denote

\(^1\)The ground state energy is the smallest eigenvalue.
by $\lambda_j(\Omega)$ the increasing sequence of its eigenvalues and by $u_j$ some associated orthonormal basis of real-valued eigenfunctions. The ground state $u_1$ can be chosen to be strictly positive in $\Omega$, but the other eigenfunctions $u_k$ must have zerosets. For any real-valued $u \in C^0_0(\Omega)$, we define the zero set as

$$N(u) = \{x \in \Omega \mid u(x) = 0\}$$

and call the components of $\Omega \setminus N(u)$ the nodal domains of $u$. The number of nodal domains of $u$ is called $\mu(u)$. These $\mu(u)$ nodal domains define a $k$-partition of $\Omega$, with $k = \mu(u)$.

We recall that the Courant nodal theorem says that, for $k \geq 1$, and if $\lambda_k$ denotes the $k$-th eigenvalue and $E(\lambda_k)$ the eigenspace of $H(\Omega)$ associated with $\lambda_k$, then, for all real-valued $u \in E(\lambda_k) \setminus \{0\}$, $\mu(u) \leq k$.

In dimension 1 the Sturm-Liouville theory says that we have always equality (for Dirichlet in a bounded interval) in the previous theorem (this is what we will call later a Courant-sharp situation). A theorem due to Pleijel [27] in 1956 says that this cannot be true when the dimension (here we consider the 2D-case) is larger than one.

We now introduce for $k \in \mathbb{N}$ ($k \geq 1$), the notion of $k$-partition. We will call $k$-partition of $\Omega$ a family $D = \{D_i\}_{i=1}^k$ of mutually disjoint sets in $\Omega$. We call it open if the $D_i$ are open sets of $\Omega$, connected if the $D_i$ are connected. We denote by $\mathcal{O}_k(\Omega)$ the set of open connected partitions of $\Omega$. We now introduce the notion of spectral minimal partition sequence.

**Definition 1.1**

For any integer $k \geq 1$, and for $D$ in $\mathcal{O}_k(\Omega)$, we introduce

$$\Lambda(D) = \max_i \lambda(D_i).$$

Then we define

$$\mathcal{L}_k(\Omega) = \inf_{D \in \mathcal{O}_k} \Lambda(D).$$

and call $D \in \mathcal{O}_k$ a minimal $k$-partition if $\mathcal{L}_k = \Lambda(D)$.

If $k = 2$, it is rather well known (see [19] or [15]) that $\mathcal{L}_2 = \lambda_2$ and that the associated minimal 2-partition is a nodal partition, i.e. a partition whose elements are the nodal domains of some eigenfunction corresponding to $\lambda_2$. 

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A partition $\mathcal{D} = \{D_i\}_{i=1}^k$ of $\Omega$ in $\mathcal{O}_k$ is called **strong** if

$$\text{Int} \left( \bigcup_i D_i \right) \setminus \partial \Omega = \Omega,$$

where, for a set $A \subset \mathbb{R}^2$, Int $(A)$ means the interior of $A$.

Attached to a strong partition, we associate a closed set in $\overline{\Omega}$, which is called the **boundary set** of the partition:

$$N(\mathcal{D}) = \bigcup_i (\partial D_i \cap \Omega).$$

$N(\mathcal{D})$ plays the role of the nodal set (in the case of a nodal partition).

This suggests the following definition:

**Definition 1.2**

We call a partition $\mathcal{D}$ regular if its associated boundary set $N(\mathcal{D})$, has the following properties:

(i) Except for finitely many distinct $x_i \in \Omega \cap N$ in the neighborhood of which $N$ is the union of $\nu_i = \nu(x_i)$ smooth curves ($\nu_i \geq 3$) with one end at $x_i$, $N$ is locally diffeomorphic to a regular curve.

(ii) $\partial \Omega \cap N$ consists of a (possibly empty) finite set of points $z_i$. Moreover $N$ is near $z_i$ the union of $\rho_i$ distinct smooth half-curves which hit $z_i$.

(iii) $N$ has the **equal angle meeting property**

The $x_i$ are called the critical points and define the set $X(N)$. Similarly we denote by $Y(N)$ the set of the boundary points $z_i$. By equal angle meeting property, we mean that the half curves meet with equal angle at each critical point of $N$ and also at the boundary together with the tangent to the boundary.

We say that $D_i, D_j$ are **neighbors** or $D_i \sim D_j$, if $D_{ij} := \text{Int} \left( D_i \cup D_j \right) \setminus \partial \Omega$ is connected. We associate with each $\mathcal{D}$ a graph $G(\mathcal{D})$ by associating with each $D_i$ a vertex and to each pair $D_i \sim D_j$ an edge. We will say that the graph is **bipartite** if it can be colored by two colors (two neighbours having two different colors). We recall that the graph associated with a collection of nodal domains of an eigenfunction is always bipartite.

### 1.2 Motivation and outlook

Before we state some results on spectral minimal partitions, discuss their properties and finally formulate and prove the central result of the present
paper, we give an informal outlook on our results. The main result is a new characterization of minimal partitions via specific magnetic Hamiltonians, see Section 4 for the necessary definitions and explanations of those operators.

In [23] we have characterized via minimal partitions the case of equality in Courant’s nodal theorem, see Theorem 2.3 below. Roughly speaking, see Theorem 2.2 if a minimal partition could in principle stem from an eigenfunction it must be already be produced by the nodal domains of an eigenfunction and this can only happen if there is equality in (7). Pleijel’s result, [27], implies, roughly speaking, that eigenfunctions associated to higher eigenvalues cannot lead to equality in (7).

In Section 3 we give a few pictures of non-nodal minimal partitions, or more precisely natural candidates, since it is notoriously hard to work out explicit examples for such partitions. A first glance shows that there are points where an odd number of nodal arcs meet.

More than 10 years ago together with Maria Hoffmann-Ostenhof and Mark Owen we investigated some special magnetic Schrödinger operators, called Aharonov Bohm Hamiltonians, i.e. Hamiltonians with zero magnetic field but with singular magnetic vector potential and with half integer circulation around holes in [21, 22], see Section 4. This investigation was motivated by the at this time surprising result of Berger and Rubinstein, [3], about the zeroset of a groundstate for such a problem with one hole. For more than one hole similar results were obtained on zerosets: each hole was hit by an odd number of nodal arcs.

The findings in [21, 22] motivated the conjecture in [5] and [18] and is reformulated in the present paper. The result says roughly that spectral minimal partitions are obtained by minimizing a certain eigenvalue of a Aharonov Bohm Hamiltonian with respect to the number and the position of poles if we assume that \( \Omega \) is simply connected. See Theorem 5.1 for the full result.

This new approach to spectral minimal partitions sheds new light on those spectral minimal partitions. While in in original formulation, [23], say for a fixed \( \Omega \) the \( \mathcal{L}_k(\Omega) \) and the associated minimal partitions as defined by Definition 1.1 require the calculation of \( \Lambda(\mathcal{D}) \) for k-partitions, the new formulation can be considered as an, admittedly involved, eigenvalue minimization.

Acknowledgments
When writing this paper we benefitted from useful discussion with V. Bonnaillie-
Theorem 2.1
For any $k$, there exists a minimal regular $k$-partition. Moreover any minimal $k$-partition has a regular representative$^3$.

Other proofs of a somewhat weaker version of this statement have been given by Bucur-Buttazzo-Henrot [10], Caffarelli-F.H. Lin [12].

A natural question is whether a minimal partition of $\Omega$ is a nodal partition, i.e. the family of nodal domains of an eigenfunction of $H(\Omega)$. We have first the following converse theorem ([19], [23]):

Theorem 2.2
If the graph of a minimal partition is bipartite, then this partition is nodal.

A natural question is now to determine how general the previous situation is. Surprisingly this only occurs in the so called Courant-sharp situation. We say that $u$ is **Courant-sharp** if

$$u \in E(\lambda_k) \setminus \{0\} \quad \text{and} \quad \mu(u) = k .$$

For any integer $k \geq 1$, we denote by $L_k(\Omega)$ the smallest eigenvalue of $H(\Omega)$, whose eigenspace contains an eigenfunction with $k$ nodal domains. We set $L_k(\Omega) = \infty$, if there are no eigenfunction with $k$ nodal domains. In general, one can show that

$$\lambda_k(\Omega) \leq \mathcal{L}_k(\Omega) \leq L_k(\Omega) .$$

(6)

The last result gives the full picture of the equality cases:

Theorem 2.3
Suppose $\Omega \subset \mathbb{R}^2$ is regular. If $\mathcal{L}_k(\Omega) = L_k(\Omega)$ or $\mathcal{L}_k(\Omega) = \lambda_k(\Omega)$ then

$$\lambda_k(\Omega) = \mathcal{L}_k(\Omega) = L_k(\Omega) .$$

(7)

In addition, one can find in $E(\lambda_k)$ a Courant-sharp eigenfunction.

$^3$Modulo sets of capacity 0.
Remark 2.4

Very recently spectral partitions for discrete problems, namely quantum graphs, have been investigated in [2].

3 Examples of minimal $k$-partitions for special domains

Using Theorem 2.3, it is now easier to analyze the situation for the disk or for rectangles (at least in the irrational case), since we have just to check for which eigenvalues one can find associated Courant-sharp eigenfunctions.

The possible topological types of a minimal partition $\mathcal{D}$ rely essentially on Euler’s formula and the fact that the $D_i$’s have to be nice, that means

$$\text{Int} (\overline{D_i}) \cap \Omega = D_i.$$  \hspace{1cm} (8)

Figures 2 and 3 illustrate possible situations.

Proposition 3.1

Let $U$ be an open set in $\mathbb{R}^2$ with piecewise-$C^1$ boundary and let $N$ a closed set such that $U \setminus N$ has $k$ components and such that $N$ satisfies the properties of Definition 1.2. Let $b_0$ be the number of components of $\partial U$ and $b_1$ be the number of components of $N \cup \partial U$. Denote by $\nu(x_i)$ and $\rho(z_i)$ the numbers of arcs associated with the $x_i \in X(N)$, respectively $z_i \in Y(N)$. Then

$$k = b_1 - b_0 + \sum_{x_i \in X(N)} \left( \frac{\nu(x_i)}{2} - 1 \right) + \frac{1}{2} \sum_{z_i \in Y(N)} \rho(z_i) + 1.$$  \hspace{1cm} (9)

This allows us to analyze minimal partitions of a specific topological type. If in addition the domain has some symmetries and we assume that a minimal partition keeps some of these symmetries, then we find natural candidates for minimal partitions.

Minimal 3-partitions

In the case of the disk (see [20]), we have no proof that the minimal 3-partition is the “Mercedes star” or Y-partition, i.e. the partition created by
three straight rays meeting at the center with equal angle. But if we assume that the minimal 3-partition has a unique singular point at the center then one can show that is indeed the $Y$-partition. This point of view is explored numerically by Bonnaillie-Helffer [5] (using some method equivalent to the Aharonov-Bohm approach and playing with the location of the critical point). There is also an interesting theoretical analysis by Noris-Terracini [25].

We have no example of minimal 3-partitions with two critical points. For the disk and the square the minimal 4-partitions are nodal.

**Minimal 5-partitions**

Using the covering approach, we were able (with V. Bonnaillie) in [5] to produce numerically the following candidate $D_1$ for a minimal 5-partition assuming a specific topological type.

![Figure 1: Candidate $D_1$ for the 5-partition of the square.](image)

It is interesting to compare with other possible topological types of minimal 5-partitions. They can be classified by using Euler’s formula (see formula [9]). Inspired by numerical computations in [16], one looks for a configuration which has the symmetries of the square and four critical points. We get two types of models that we can reduce to a Dirichlet-Neumann problem on a triangle corresponding to the eighth of the square. Moving the Neumann boundary on one side like in [7] leads us to two candidates $D_2$ and $D_3$. One has a lower energy $\Lambda(D)$ and one recovers the pictures in [16].

Note that in the case of the disk a similar analysis leads to a different answer. The partition of the disk by five half-rays with equal angles has a lower energy than the minimal 5-partition with four singular points.
4 The Aharonov-Bohm approach

Let us recall some definitions and results about the Aharonov-Bohm Hamiltonian (for short ABX-Hamiltonian) defined in an open set $\Omega$ which can be simply connected or not. These results were initially motivated by the work of Berger-Rubinstein [3], and further developed in [1 21 22 6 5].

Simply connected case : one pole

We first consider the case when one pole, denoted by $X = (x_0, y_0)$, is chosen in $\Omega$ and introduce the magnetic potential :

$$A^X(x, y) = (A_1^X(x, y), A_2^X(x, y)) = \frac{\Phi}{2\pi} \left(-\frac{y - y_0}{r^2}, \frac{x - x_0}{r^2}\right). \quad (10)$$

We know that in this case the magnetic field vanishes identically in $\hat{\Omega}_X$, where

$$\hat{\Omega}_X = \Omega \setminus \{X\}. \quad (11)$$
The $ABX$-Hamiltonian is defined by considering the Friedrichs extension starting from $C_0^\infty(\hat{\Omega}_X)$ and the associated differential operator is

$$-\Delta_{A^X} := (D_x - A_1^X)^2 + (D_y - A_2^X)^2 \text{ with } D_x = -i\partial_x \text{ and } D_y = -i\partial_y.$$  

We will consider in the sequel the very special case when the flux $\Phi$ created at $X = (x_0, y_0)$, which can be computed by considering the circulation of $A^X$ along a simple closed path turning once anti-clockwise around $X$, satisfies:

$$\frac{\Phi}{2\pi} = \frac{1}{2}.$$  

Under assumption (13), let $K_X$ be the anti-linear operator

$$K_X = e^{i\theta_X} \Gamma,$$

with $(x-x_0) + i(y-y_0) = \sqrt{|x-x_0|^2 + |y-y_0|^2} e^{i\theta_X}$, where $\Gamma$ is the complex conjugation operator

$$\Gamma u = \bar{u}$$

and

$$\nabla\theta_X = 2A^X,$$

which can also be rewritten in the form

$$-A^X = A^X - \nabla\theta_X.$$

The flux condition (13) shows that one can find a solution $\theta_X$ of (14) (a priori multi-valued) such that $e^{i\theta_X}$ is uni-valued and $C^\infty$. Hence $-\Delta_{A^X}$ and $-\Delta_{-A^X}$ are intertwined by the gauge transformation associated with $e^{i\theta_X}$. Then we have

$$K_X \Delta_{A^X} = \Delta_{A^X} K_X.$$  

We say that a function $u$ is $K_X$-real, if it satisfies $K_X u = u$. Then the operator $-\Delta_{A^X}$ is preserving the $K_X$-real functions. In the same way one proves that the usual Dirichlet Laplacian admits an orthonormal basis of real valued eigenfunctions or one restricts this Laplacian to the vector space over $\mathbb{R}$ of the real-valued $L^2$ functions, one can construct for $-\Delta_{A^X}$ a basis of $K_X$-real eigenfunctions or, alternately, consider the restriction of the $ABX$-Hamiltonian to the vector space over $\mathbb{R}$

$$L^2_{K_X}(\hat{\Omega}_X) = \{ u \in L^2(\hat{\Omega}_X) \ , \ K_X u = u \}.$$
Non simply connected case
In this situation, magnetic potentials in $\Omega$ with zero magnetic field can be different from gradients if some fluxes around some holes are not in $(2\pi)\mathbb{Z}$. In this situation we will be interested in potentials where the created flux by some hole is $\pi$. This will be realized in this article by introducing a pole in the hole. Except that $\hat{\Omega}_\mathcal{X} = \Omega$ (there are no singularity in $\Omega$) all what has been defined before goes through and this is actually the initial case treated in the pioneering work by [3].

Poles and holes
We can extend our construction of an Aharonov-Bohm Hamiltonian in the case of a configuration with $\ell$ distinct points $X_1, \ldots, X_\ell$ (putting a flux $\pi$ at each of these points). These points can be chosen in $\Omega$ or in the holes. They are distinct and each hole contains at most one $X_k$. We can just take as magnetic potential

$$A^X = \sum_{j=1}^{\ell} A^{X_j},$$

where $X = (X_1, \ldots, X_\ell)$. Our Hamiltonian will be defined in $\hat{\Omega}_\mathcal{X} = \Omega \setminus X$. We can also construct (see [21, 22]) the anti-linear operator $K^{\mathcal{X}}$, where $\theta^{\mathcal{X}}$ is replaced by a multivalued function $\phi^{\mathcal{X}}$ such that $\nabla \phi^{\mathcal{X}} = 2A^X$ and $e^{i\phi^{\mathcal{X}}}$ is uni-valued and $C^\infty$. We can then consider the real subspace of the $K^{\mathcal{X}}$-real functions in $L^2_{K^{\mathcal{X}}} (\hat{\Omega}_\mathcal{X})$ and our operator as an unbounded selfadjoint operator on $L^2_{K^{\mathcal{X}}} (\hat{\Omega}_\mathcal{X})$.

It was shown in [21, 22] for the case with holes and in [1] for the case with poles that the nodal set of such a $K^{\mathcal{X}}$-real eigenfunction has the same structure as the nodal set of a real-valued eigenfunction of the Laplacian except that an odd number of half-lines meet at each pole and at the boundary of each hole containing some $X_k$. In the case of one hole, this fact was first observed by Berger-Rubinstein [3] for a first eigenfunction (assuming that the first eigenvalue is simple). We denote by $L_k(\hat{\Omega}_\mathcal{X})$ the lowest eigenvalue, if it exists, such that there exists a $K^{\mathcal{X}}$-real eigenfunction with $k$ nodal domains and we set $L_k(\hat{\Omega}_\mathcal{X}) = +\infty$ if there is no such eigenvalue.
5 The magnetic characterization of a minimal partition

We now prove the following conjecture presented (in the simply-connected case) in [5] and [18].

**Theorem 5.1**

Suppose $\Omega$ is a bounded, not necessarily simply connected, domain with $m$ disjoint closed holes $B_i\ (i = 1, \ldots, m)$ with non empty interiors. Again we assume that $\partial \Omega$ is piecewise $C^1$. Then

$$\mathfrak{L}_k(\Omega) = \inf_{\ell \in \mathbb{N}} \inf X_1, \ldots, X_\ell L_k(\hat{\Omega}_X)$$

where in the infimum each $X_j = (x_j, y_j)$ is either in Int $(B_i)$ or in $\Omega$. In each $B_i$ there is either one or no $X_i$. The $X_i \in \Omega$ are distinct points.

Let us first give the proof in the simply connected case.

**Step 1** : $\inf_{\ell \in \mathbb{N}} \inf X_1, \ldots, X_\ell L_k(\hat{\Omega}_X) \leq \mathfrak{L}_k(\Omega)$

Considering a minimal $k$-partition $D = (D_1, \ldots, D_k)$, we know that it has a regular representative and we denote by $X^{\text{odd}}(D) := (X_1, \ldots, X_\ell)$ the critical points of the boundary set of the partition for which an odd number of half-curves meet.

For proving Step 1, we have indeed just to prove that, for this family of points $X = X^{\text{odd}}(D)$, $\mathfrak{L}_k(\Omega)$ is an eigenvalue of the Aharonov-Bohm Hamiltonian associated with $\hat{\Omega}_X$ and to explicitly construct the corresponding eigenfunction with $k$ nodal domains described by the $D_i$’s.

For this, we recall that we have proven in [23] the existence of a family $(u_i)_{i=1, \ldots, k}$ such that $u_i$ is a ground state of $H(D_i)$ and $u_i - u_j$ is a second eigenfunction of $H(D_{ij})$ when $D_i \sim D_j$. The claim is that one can find a sequence $\epsilon_i(x)$ of $S^1$-valued functions, where $\epsilon_i$ is a suitable $4$ square root of $e^{i\phi}$ in $D_i$, such that $\sum_i \epsilon_i(x) u_i(x)$ is an eigenfunction of the ABX-Hamiltonian associated with the eigenvalue $\mathfrak{L}_k$.

More explicitly, let us describe how we can construct $\epsilon_i(x)$. We start from some $i_0$ and define $\epsilon_{i_0}(x) = e^{i\phi}$ in $D_i$. According to the footnote $\epsilon_{i_0}(x)$ is

Note that by construction the $D_i$’s never contain any point of $X$. Hence the ground state energy of the Hamiltonian $H(D_I)$ is the same as the ground state energy of $H_{ABX}(D_I)$. 

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a well defined $C^\infty$ function. Let $D_i$ a nearest neighbor of $D_{i_0}$ then we define $\epsilon_i(x) = -e^{\frac{1}{2} \phi x}$. Then we can extend the definition by considering the neighbors of the neighbors. Now we have to check that the construction is consistent. The problem can be reduced to the following question. Consider a closed simple path $\gamma$ in $\hat{\Omega}_X$ transversal to $N(D)$ (and avoiding the critical points). Take some origin $x_0$ on $\gamma \cap D_{i_1}$. We start from $\epsilon(x) = e^{\frac{1}{2} \phi x(x)}$ in $D_{i_1}$ and, choosing the positive orientation, multiply by $-1$ each time that we cross an arc of $N(D)$. It is then a consequence of Euler’s formula that the number of crossings along $\gamma$ is odd if and only if there is an odd number of points of $X$ inside $\gamma$ (apply Euler’s formula (9) with $U$ being the open set delimited by $\gamma$). It is then clear that $\epsilon(x)$ is well defined along $\gamma$.

**Step 2**: \( \inf_{\ell \in \mathbb{N}} \inf_{X_1, \ldots, X_{\ell}} L_k(\hat{\Omega}_X) \geq \mathcal{L}_k(\Omega) \)

Conversely, given $\ell$ distinct points $X_i$ in $\Omega$, any family of nodal domains of a $K_X$-real eigenfunction of the Aharonov-Bohm operator on $\hat{\Omega}_X$ corresponding to $L_k$ gives a $k$-partition. Using the results of [21] and [1], we immediately see that the $X_i$’s corresponding to the ”odd” singular points of the partitions. In each of these nodal domains $D_i$, $L_k$ is an eigenvalue of the Dirichlet realization of the Schrödinger operator with magnetic potential $A^X$, which is by the diamagnetic inequality higher as the ground state energy of the Dirichlet Laplacian in $D_i$ without magnetic field. Hence the energy $\Lambda_k(D)$ of this partition is indeed less than $L_k(\hat{\Omega}_X)$.

**Step 3**: Proof in the non simply connected case

The main change is in step 1. In the non simply connected case, the set $X$ consists of the singular points of the boundary set inside $\Omega$ where an odd number of half-lines arrive together with those points in the holes whose boundary is hit by an odd number of half-curves.

**Examples**

Let us present a few examples illustrating the theorem in the case of a simply connected domain. When $k = 2$, there is no need to consider punctured $\Omega$’s. The infimum is obtained for $\ell = 0$. When $k = 3$, it is possible to show (see Remark 5.3 below) that it is enough to minimize over $\ell = 0$, $\ell = 1$ and $\ell = 2$. In the case of the disk and the square, it is proven that the infimum cannot be for $\ell = 0$ and we conjecture that the infimum is for $\ell = 1$ and attained for
the punctured domain at the center. For $k=5$, it seems that the infimum is for $\ell=4$ in the case of the square (see Figure 2) and for $\ell=1$ in the case of the disk (see Figure 3).

**Remark 5.2**

If $D$ is a regular representative of a minimal $k$-partition and if $\hat{\Omega}_X$ is constructed like in Step 1 of the proof of the previous theorem, then $\Sigma_k(\Omega) = \lambda_k(\hat{\Omega}_X)$ (Courant sharp situation). Coming back indeed to this step, one can follow the proof of Theorem 1.13 (Section 6) in [23].

**Remark 5.3**

Euler’s formula (9), implies that for a minimal $k$-partition $D$ of a simply connected domain $\Omega$ the cardinality of $X^{\text{odd}}(D)$ satisfies

$$\#X^{\text{odd}}(D) \leq 2k - 3.$$  \hfill (17)

Note that if $b_1 = b_0$, we necessarily have a singular point in the boundary. The argument depends only on Euler’s formula. If we implement the additional property that the open sets $D_i$’s of a minimal partition are nice (see (8)), we can exclude the case when there is only one point on the boundary. We emphasize that this was not a priori excluded from the results of [21, 1]. Hence, we obtain

$$b_1 - b_0 + \frac{1}{2} \sum \rho(y_i) \geq 1,$$

which implies the inequality

$$\#X^{\text{odd}}(D) \leq 2k - 4.$$  \hfill (18)

This estimate seems optimal for a general geometry although all the known candidates for minimal partitions for $k=3$ and 5 have a lower cardinality of odd critical points.

**Remark 5.4**

The argument around (8) shows that a nodal set of a $K_X$-real eigenfunction that corresponds to a minimal partition cannot have a critical point that is met only by one nodal arc. Actually that can happen for ground states of Aharonov-Bohm Hamiltonians, see [27] which of course do not correspond to minimal partitions.
Remark 5.5
It would be interesting to look at the case of the sphere (already considered in [24]) and the first problem in this case is to define the suitable magnetic Laplacian. We refer to [28] for one of the first papers on this question. More specifically, we would like to construct in our case an Aharonov-Bohm Hamiltonian. Note for example that we can not have such an operator with one pole and a flux $\pi$ around this pole. Fortunately there are no minimal $k$-partition whose boundary set consists of one "odd" critical point on the sphere, as can be seen by Euler’s formula for the sphere (see in [24], Remark 4.2). We indeed know that the cardinality of "odd" critical points is even. This is actually a standard result from graph theory that the number of vertices with odd degree is even. (See for example Corollary 1.2 in [4]). This suggests that instead of putting the flux $\pi$ around each pole, we take alternately $\pi$ and $-\pi$ for the fluxes in order to get a total flux equal to 0. In other words, we should probably describe $X_{\text{odd}}(\mathcal{D})$ as a union of dipoles.

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