Sharp Estimates of the Generalized Euler-Mascheroni Constant

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Abstract: Let \(a \in (0, \infty)\), \(\gamma(a)\) be the Generalized Euler-Mascheroni Constant, and let

\[
\begin{align*}
x_n &= \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \ln \frac{a+n}{a}, \\
y_n &= \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \ln \frac{a+n-1}{a}.
\end{align*}
\]

In this paper, we determine the best possible constants \(\alpha_i, \beta_i (i = 1, 2, 3, 4)\) such that the following inequalities are valid for all integers \(n \geq 1\).

\[
\begin{align*}
\frac{1}{2(n+a) - \alpha_1} &\leq \gamma(a) - x_n < \frac{1}{2(n+a) - \beta_1}, \\
\frac{1}{2(n+a) - \alpha_2} &\leq y_n - \gamma(a) < \frac{1}{2(n+a) - \beta_2}, \\
\frac{1}{2(n+a) + \alpha_3} + \frac{\alpha_3}{(n+a)^2} &\leq \gamma(a) - x_n < \frac{1}{2(n+a) + \beta_3} + \beta_3, \\
\frac{1}{2(n+a - 1) + \alpha_4} + \frac{\alpha_4}{(n+a - 1)^2} &\leq y_n - \gamma(a) < \frac{1}{2(n+a - 1) + \beta_4} + \beta_4.
\end{align*}
\]

Key Words: Generalized Euler-Mascheroni Constant, Inequalities, Psi function.

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1 Introduction

One of the most important sequences in analysis and number theory of the form

\[
\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n,
\]

considered by Leonhard Euler in 1735, is known to converge towards the limit

\[
\gamma = 0.57721566490115328\cdots,
\]

which is now called the Euler-Mascheroni Constant. For \(\gamma_n - \gamma\), many lower and upper estimates have been given in the literature\[2, 4, 7, 8, 10\]. We remind some of them:
In [2], Alzer proved that for \( n \geq 1, \)
\[
\frac{1}{2n+1} \leq \gamma_n - \gamma \leq \frac{1}{2n}.
\]
Tóth [10] proved that for \( n \geq 1, \)
\[
\frac{1}{2n + \frac{1}{5}} < \gamma_n - \gamma \leq \frac{1}{2n + \frac{1}{3}}.
\]  
(1.3)
In [4], Chen proved that for \( n \geq 1, \)
\[
\frac{1}{2n + \alpha} \leq \gamma_n - \gamma < \frac{1}{2n + \beta},
\]  
(1.4)
where the constants \( \alpha = \frac{(2\gamma - 1)}{(1 - \gamma)} \) and \( \beta = 1/3 \) are the best possible.
Qiu and Vuorinen [8] showed the double inequality
\[
\frac{1}{2n} - \frac{\alpha}{n^2} \leq \gamma_n - \gamma \leq \frac{1}{2n} - \frac{\beta}{n^2}, \quad n \geq 1,
\]  
(1.5)
where the constants \( \alpha = 1/12 \) and \( \beta = \gamma - 1/2 \) are the best possible.
For every \( a > 0 \), the numbers of the form
\[
\gamma(a) = \lim_{n \to \infty} \left( \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \ln \frac{a+n-1}{a} \right)
\]  
(1.6)
were introduced in the monograph by Knopp [6]. There are known now as the generalized Euler-Mascheroni constant, since \( \gamma(1) = \gamma \). Recently, the generalized Euler-Mascheroni constant \( \gamma(a) \) has been the subject of intensive research [3, 7, 9], similar to \( \gamma \).
In [9], Sintămărian consider the following sequences
\[
x_n = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \ln \frac{a+n}{a},
\]  
(1.7)
\[
y_n = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \ln \frac{a+n-1}{a}.
\]  
(1.8)
and proved that for \( n \geq 1, \) the following inequalities hold,
\[
\frac{1}{2(n+a)} \leq \gamma(a) - x_n \leq \frac{1}{2(n+a-1)}, \quad \frac{1}{2(n+a)} \leq y_n - \gamma(a) \leq \frac{1}{2(n+a-1)}.
\]  
(1.9)
Hence, we can easily know that \( x_n, y_n \) converge to \( \gamma(a) \) like \( n^{-1} \).
In [3], Berinde and Mortici gave better bounds for \( \gamma(a) - x_n, y_n - \gamma(a) \), showing the following

**Theorem 1.1.** For each \( a > 0, n \geq 2, \) then
\[
\frac{1}{2(n+a) - \frac{1}{4}} < \gamma(a) - x_n < \frac{1}{2(n+a) - \frac{1}{3}}, \quad \frac{1}{2(n+a) - \frac{4}{3}} < y_n - \gamma(a) < \frac{1}{2(n+a) - \frac{5}{3}}.
\]  
(1.10)

In the same paper, Berinde and Mortici obtained the following theorem.

**Theorem 1.2.** a) For each \( a \geq \frac{13}{30} \) and each integer \( n \geq 1, \) then
\[
\frac{1}{2(n+a) - \frac{1}{3} + \frac{1}{18n}} \leq \gamma(a) - x_n.
\]  
(1.11)
b) For each \( a \geq \frac{17}{30} \) and each integer \( n \geq 1, \) then
\[
\frac{1}{2(n+a) - \frac{5}{3} + \frac{1}{18n}} \leq y_n - \gamma(a).
\]  
(1.12)
It is natural to extend the above inequalities (1.4) and (1.5) in terms of generalized Euler-Mascheroni constant $\gamma(a)$. In this paper, we will consider the two sequences $\gamma - x_n, y_n - \gamma$ where $x_n, y_n$ are defined by (2.3) and (2.4) respectively.

The main results are stated as follows.

**Theorem 1.3.** For each $a > 0$ and integer $n \geq 1$, let $\gamma(a)$ be generalized Euler-Mascheroni constant.

1. Let the sequences $x_n$ be defined by (2.3), then

$$\frac{1}{2(n+a) - \alpha_1} \leq \gamma(a) - x_n < \frac{1}{2(n+a) - \beta_1},$$

(1.13)

with the best possible constants

$$\alpha_1 = 2(1 + a) - \frac{1}{\psi(1 + a) - \ln(1 + a)}, \quad \beta_1 = \frac{1}{3}. \quad (1.14)$$

2. Let the sequences $y_n$ be defined by (2.4), then

$$\frac{1}{2(n+a) - \alpha_2} \leq y_n - \gamma(a) < \frac{1}{2(n+a) - \beta_2},$$

(1.15)

with the best possible constants

$$\alpha_2 = 2(1 - d), \quad \beta_2 = \frac{5}{3}. \quad (1.16)$$

where

$$d = \max\{\tilde{f}_2(a), \tilde{f}_2(1 + a), \tilde{f}_2(2 + a)\}, \quad \tilde{f}_2(x) = \frac{1}{2(\psi(x + 1) - \ln(x))} - x.$$

**Theorem 1.4.** For each $a > 0$, and integer $n \geq 1$, let the sequences $x_n, y_n$ be defined by (2.3), (2.4) and $\gamma(a)$ be generalized Euler-Mascheroni constant, then

$$\frac{1}{2(n+a)} + \frac{\alpha_3}{(n+a)^2} \leq \gamma(a) - x_n < \frac{1}{2(n+a)} + \frac{\beta_3}{(n+a)^2},$$

(1.17)

$$\frac{1}{2(n+a-1)} + \frac{\alpha_4}{(n+a-1)^2} < y_n - \gamma(a) \leq \frac{1}{2(n+a-1)} + \frac{\beta_4}{(n+a-1)^2},$$

(1.18)

with the best possible constants

$$\alpha_3 = (1 + a)^2[\ln(1 + a) - \psi(1 + a)] - \frac{1 + a}{2}, \quad \beta_3 = \frac{1}{12}, \quad (1.19)$$

$$\alpha_4 = -\frac{1}{12}, \quad \beta_4 = a^2[\psi(a) - \ln(a)] + \frac{a}{2}. \quad (1.20)$$

### 2 Preliminaries

In this section, we give out several formulas and lemmas in order to establish our main results stated in section 1. Firstly, let us recall some known results for the psi (or digamma) function $\psi(x)$.

For real numbers $x, y > 0$, the gamma and psi functions are defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

respectively.
The psi function $\psi(x)$ has the following Recurrence Formulas [1]

$$
\psi(n + z) = \frac{1}{(n - 1) + z} + \frac{1}{(n - 2) + z} + \cdots + \frac{1}{2 + z} + \frac{1}{1 + z} + \frac{1}{z} + \psi(z). \quad (2.1)
$$

and the Asymptotic Formulas (11),

$$
\psi(z) \sim \ln(z) - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots \quad (z \to \infty \text{ in } |\arg z| < \pi), \quad (2.2)
$$

According to (2.1) and the definition of $x_n, y_n$, we have

$$
x_n = \psi(n + a) - \psi(a) - \ln \frac{n + a}{a}, \quad (2.3)
$$

$$
y_n = \psi(n + a) - \psi(a) - \ln \frac{n + a - 1}{a}. \quad (2.4)
$$

By the definition of $\gamma(a)$ (1.6) and the Asymptotic Formulas (2.2), then

$$
\gamma(a) = \lim_{n \to \infty} y_n = \lim_{n \to \infty} (\psi(n + a) - \ln(n + a) + \ln(n) - \psi(a)) = \ln(n) - \psi(a). \quad (2.5)
$$

Hence

$$
\gamma(a) - x_n = \ln(n + a) - \psi(n + a), \quad (2.6)
$$

$$
y_n - \gamma(a) = \psi(n + a) - \ln(n + a - 1). \quad (2.7)
$$

**Lemma 2.1.** (1) The function

$$
f_1(x) = \frac{1}{\ln(x) - \psi(x)} - 2x \quad (2.8)
$$

is strictly decreasing from $(1, \infty)$ onto $(-1/3, 1/\gamma - 2)$.

(2) The function

$$
f_2(x) = \frac{1}{\psi(x + 1) - \ln(x)} - 2x \quad (2.9)
$$

is strictly decreasing from $[2, \infty)$ onto $(1/3, f_2(2)]$.

**Proof.** (1) Differentiation gives

$$
(\ln(x) - \psi(x))^2 f_1'(x) = \psi'(x) - \frac{1}{x} - 2(\ln(x) - \psi(x))^2,
$$

Using the inequalities [5]

$$
\psi'(x) - \frac{1}{x} < \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7}, \quad \ln(x) - \psi(x) > \frac{1}{2x} + \frac{1}{12x^2} - \frac{1}{120x^4},
$$

We have

$$
(\ln(x) - \psi(x))^2 f_1'(x) < \frac{1}{50400x^5} F_1(x), \quad (2.10)
$$

where

$$
F_1(x) = -207 - 3840(x - 1) - 6580(x - 1)^2 - 3640(x - 1)^3 - 700(x - 1)^4,
$$

we have $F(x) < 0, f_1'(x) < 0$ for $x \geq 1$, and the monotonicity of $f_1(x)$ follows.

Clearly, $f_1(1) = 1/\gamma - 2$. The limiting value $\lim_{x \to \infty} f_1(x) = -1/3$ follows from the Asymptotic Formulas (2.2).
Differentiation yields
\[ 2 \left( \psi(x + 1) - \ln(x) \right)^2 f_2'(x) = \frac{1}{x} + \frac{1}{x^2} - \psi'(x) - 2\psi(x) + \frac{1}{x} - \ln(x)^2. \]

Using the inequalities \[5\], for \( x > 0, \)
\[ \frac{1}{x} + \frac{1}{x^2} - \psi'(x) < \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^6}, \quad \psi(x) + \frac{1}{x} - \ln(x) > \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^5} - \frac{1}{252x^6}, \]
we obtain
\[ 2 \left( \psi(x + 1) - \ln(x) \right)^2 f_2'(x) < -\frac{F_2(x)}{3175200x^{12}} \]

where
\[ F_2(x) = 3217636 + 17887632(x - 2) + 39443124(x - 2)^2 + 47009928(x - 2)^3 + 33797841(x - 2)^4 \]
\[ + 15180480(x - 2)^5 + 4189500(x - 2)^6 + 652680(x - 2)^7 + 44100(x - 2)^8 \]
\[ > 0 \quad (x \geq 2). \]

Hence \( f_2(x) \) is a decreasing function on \([2, \infty)\). The limiting value \( \lim_{x \to \infty} f_2(x) = 1/6 \) follows from the Asymptotic Formulas \((2.2)\). \( \square \)

The following Lemma follows from theorem 1.7 in \([8]\).

**Lemma 2.2.** The function
\[ f_3(x) = x^2(\psi(x) - \ln(x)) + \frac{x}{2} \quad (2.12) \]
is strictly decreasing and convex from \((0, \infty)\) onto \((-1/12, 0)\).

### 3 Proof of the main theorem

**Proof of Theorem 1.3**

(1). According to \((2.6)\), the inequality \((1.13)\) can be written as
\[ -\beta < \frac{1}{\ln(n + a) - \psi(n + a)} - 2(n + a) < -\alpha. \]

By the Lemma 2.1(1), we know that the sequence
\[ f_1(n + a) = \frac{1}{\ln(n + a) - \psi(n + a)} - 2(n + a), \quad (n \in \mathbb{N}) \]
is strictly decreasing. This leads to
\[ -\frac{1}{3} = \lim_{n \to \infty} f_1(n) < f_1(n) \leq f_1(1) = \frac{1}{\ln(1 + a) - \psi(1 + a)} - 2(1 + a) \]

Hence the best possible constants are
\[ \alpha_1 = 2(1 + a) - \frac{1}{\psi(1 + a) - \ln(1 + a)}, \quad \beta_1 = \frac{1}{3}. \]

(2). According to \((2.7)\), the inequality \((1.15)\) can be written as
\[ 1 - \frac{\beta}{2} < \frac{1}{2(\psi(n + a) - \ln(n + a - 1))} - (n + a - 1) \leq 1 - \frac{\alpha}{2}. \]

5
By the Lemma 2.2, the sequence
\[ \tilde{f}_2(n + a - 1) = \frac{1}{2\psi(n + a) - \ln(n + a - 1)} - (n + a - 1) \quad (n \geq 2) \]
is strictly decreasing. Hence
\[ \frac{1}{6} = \lim_{n \to \infty} f_2(n) < f_2(n) \leq \max\{f_2(a), f_2(1 + a), f_2(2 + a)\}, \]
let \( d := \max\{f_2(a), f_2(1 + a), f_2(2 + a)\} \), hence the best possible constants are
\[ \alpha_2 = 2(1 - d), \quad \beta_2 = \frac{5}{3}. \quad (3.1) \]

**Proof of Theorem 1.4** According to (2.6) - (2.7), the inequality (1.17) - (1.18) can be written as
\[ \alpha_3 \leq (n + a)^2 (\ln(n + a) - \psi(n + a)) - \frac{(n + a)}{2} < \beta_3, \]
\[ \alpha_4 < (n + a - 1)^2 (\psi(n + a - 1) - \ln(n + a - 1)) + \frac{(n + a - 1)}{2} \leq \beta_4. \]

By the Lemma 2.2 we know that the sequence
\[ \tilde{f}_3(n + a - 1) = (n + a - 1)^2 (\psi(n + a - 1) - \ln(n + a - 1)) + \frac{(n + a - 1)}{2}, \quad (n \in \mathbb{N}) \]
is strictly decreasing, and \( \lim_{n \to \infty} f_3(n) = -1/12 \). Hence, the best possible constants are
\[ \alpha_3 = (1 + a)^2 [\ln(1 + a) - \psi(1 + a)] - \frac{1 + a}{2}, \quad \beta_3 = \frac{1}{12}, \]
\[ \alpha_4 = -\frac{1}{12}, \quad \beta_4 = (a)^2 [\psi(a) - \ln(a)] + \frac{a}{2} \]

**Remark 3.1.** (1). Taking \( a = 1 \) in Theorem 1.4 (2), then we get the inequality (1.14) with the constants \( \alpha = (2\gamma - 1)/(1 - \gamma) \) and \( \beta = 1/3 \) are the best possible.

(2). Taking \( a = 1 \) in the inequality (1.18) of Theorem 1.4 then we get the inequality (1.5) with the constants \( a = 1/12 \) and \( \beta = \gamma - 1/2 \) are the best possible.

(3). From Theorem 1.4 we know that the constants 1/3 and 5/3 are best possible in Theorem 1.7 for any \( a > 0 \).

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