Time-like detonation in presence of magnetic field

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We study the effect of magnetic field in an implosion process achieved by radiation. A time-varying magnetic field is seen to affect the continuous transition of space-like detonation to time-like detonation at the core of implosion region. A constant magnetic field does not affect the detonation, and the effect of exponentially decaying field is not significant. The oscillating varying magnetic field has a significant effect in increasing the volume of the time-like detonation volume of the core of implosion and also modify the time of the implosion process. This transition can have significant outcome both theoretically and experimentally in the areas of high energy hadronization of quark-gluon plasma (QGP) matter and Inertial Confinement Fusion efforts of fuels.

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I. INTRODUCTION

The hydrodynamics of shock waves has found numerous applications in the field of fluid dynamics, astrophysics, high energy physics, and cosmology. A shock wave is a wave where the disturbance in the medium propagates faster than the local speed of sound. At either side of the interference, the thermodynamic variables vary discontinuously. It occurs when there is a rapid compression or expansion of the system. The mass, momentum, and energy conservation laws across the surface lead to the Rankine-Hugoniot (RH) and Taub equation connecting the properties of the fluid on the two sides of the discontinuity. The normal vector of the surface of discontinuity is space-like (SL), and the wave propagation velocity is less than the speed of light.

Almost four decades later it was realized that under some condition the discontinuity of the surface could also be time-like (TL) and there can be a rapid phase transition. It was argued that if a system undergoes a rapid rarefaction, bubbles can form at different spatial points which are causally disconnected. If the thickness of the surface forming bubbles is very thin, the boundary between the two phases of matter becomes TL. The inflationary model of the universe was thought to be one such example.

Such treatment was successful in describing the sudden and rapid hadronization of quark-gluon plasma in high energy physics which would have been missed by general SL fronts. In such dense matter, an implosion induced by fast burning can smoothly take a SL detonation to TL detonation.

TL detonation found limited use in astrophysics and in most practical purposes the shocks are slow. However, recently there has been experimental efforts to observe Inertial confinement fusion. The experiments failed due to the appearance of Rayleigh-Taylor surface instabilities. Such instabilities can be avoided if the detonation front moves with the speed of light, which in turn can be achieved by radiation. Therefore, such a theoretical model can have an experimental application as well.

In this work, we carry forward the theoretical work by Csernai and show the effect of magnetic field on the continuous transition from SL to TL detonation in implosion carried via radiation. In Section II we calculate the TL detonation front due to radiation. In section III we introduce magnetic field in our calculation. In section IV we present our result and finally in section V we draw our conclusion from our results.

II. TIME-LIKE DETONATION DUE TO RADIATION

Here we have assumed that there is a spherical core filled with matter having vanishing opacity. The core is surrounded by rapidly igniting shell whose radiation is responsible for the heating of the core. When the temperature reaches a specific value, it follows an exothermic transition. The compression of the fuel is not taken into account, and the heating of the inner core is assumed to be due to isotropic radiation. The radius of the shell (R) remain unchanged, R = Constant. The shell is ignited at time t₀ = 0 at all points simultaneously. Q is the heat that the shell radiates in unit time through a unit surface and C be the fraction of heat absorbed by the matter. Then at a distance r from the center of the core (ignoring the opacity of core and measuring the distance in units of R and time in the units of R/c) the change in heat per unit time is

\[
\frac{dQ}{dt} = CQ \int_0^t d\tau \int_0^{2\pi} d\phi \int_0^0 d(cos \theta) \frac{\delta(\tau - |\vec{r}'|)}{|\vec{r}'|^2} \delta \quad (1)
\]

where \( r' = R^2 + r^2 - 2rr \cos \theta \) (only radiation that have reached inside the radius \( r \) will contribute to heating of
the matter). Taking \( R = 1 \);
\[
\frac{dQ}{dt} = CQ \left( \frac{2\pi}{r} \int_{0}^{r} \int_{0}^{2\pi} d\phi \int_{0}^{\pi} d(cos \theta) \right) \frac{\delta(r - (1 + r^2 - 2r \cos \theta))}{1 + r^2 - 2r \cos \theta}.
\]

From the property of the \( \delta \) function we can write
\[
\delta(g(cos \theta)) = \frac{\delta(cos \theta - \cos \theta_0)}{g'(cos \theta_0)}
\]
and \( \theta_0 \) is the angle for which \( g(cos \theta) \) is zero.
\[
g(cos \theta) = \tau - (1 + r^2 - 2r \cos \theta) \frac{\delta}{r}
\Rightarrow g'(cos \theta) = \frac{1 + r^2 - \tau^2}{2r}.
\]

For \( \cos \theta_0 \)
\[
\tau - (1 + r^2 - 2r \cos \theta_0) \frac{\delta}{r} = 0
\Rightarrow \cos \theta_0 = \frac{1 + r^2 - \tau^2}{2r}.
\]

Therefore, eqn 2 becomes dependent on \( ct \) distance covered by the radiation in the range \((1 - r)\) to \((1 + r)\)
\[
\frac{dQ}{dt} = \frac{2\pi CQ}{r} \int_{1-r}^{a} \frac{2\pi CQ}{r} \left[ \ln \frac{\tau}{a-r} \right]
\]
where
\[
a = \begin{cases} 
1 - r & t < (1 - r) \\
1 + r & t > (1 + r) \\
t & (1 - r) < t < (1 + r)
\end{cases}
\]
The equation can be defined as
\[
\frac{dQ}{dt} = \frac{2\pi CQ}{r} \ln \frac{\tau}{1-r} r < (1 - r) \ln \frac{\tau}{1+r} r > (1 + r).
\]

If we ignore compression and assume \( C_v = \) constant, we can write
\[
\frac{dT}{C_v} = \frac{dQ}{dt} \Rightarrow T(r, t) = \frac{1}{C_v} \int \left( \frac{dQ}{dt} \right) dt
\]

\[
T(r, t) = \frac{2\pi CQ}{C_v r} \int_{0}^{t} dt \left\{ \begin{array}{cc} 
0 & t < (1 - r) \\
\ln \frac{r}{1-r} & (1 - r) < t < (1 + r) \\
\ln \frac{1+r}{1-r} & t > (1 + r)
\end{array} \right.
\]

On solving the above integral, we finally have
\[
T(r, t) = \frac{2\pi CQ}{C_v r} \left\{ \begin{array}{cc}
0 & t \ln \frac{1+r}{1-r} - t + 1 + r \\
\ln \frac{1+r}{1-r} - 2r & t > (1 + r).
\end{array} \right. \
\]

Thus if \( t > 1 + r \) and \( r \to 0 \) then
\[
T(0, t) = \lim_{r \to 0} T(r, t)
\]

which can be simplified as
\[
T(0, t) = \frac{2\pi CQ}{C_v} \lim_{r \to 0} \left[ \ln \frac{1+r}{r} - \ln \frac{1-r}{r} - 2 \right]
\Rightarrow T(0, t) = \frac{4\pi CQ}{C_v} (t - 1)
\]

The discontinuity surface is determined by contour \( T(r, t) = T_c \). The critical point \((r_c, t_c)\), at which the SL and TL discontinuities converge, is determined by the condition
\[
\left( \frac{\partial r}{\partial t} \right)_{T_c} = \left( \frac{\partial T}{\partial t} \right)_{T_c} / \left( \frac{\partial T}{\partial r} \right)_{T_c} = 1.
\]

Substituting the value for \( t > (1 + r) \) from eqn 4 we have
\[
t_c = \frac{\ln \frac{1+r}{r}}{\ln \frac{1+r}{1-r} - 1 - \frac{1}{r}}
\]

III. TL DETONATION IN THE PRESENCE OF MAGNETIC FIELD

Magnetic Work

To add magnetic field in the first law of Thermodynamics, we use Maxwell’s fields. The energy generated within a volume \( V \) in time \( \delta t \), by \( \text{am} \) electric field \( \varepsilon \) acting on current density \( j \) is given by [10]
\[
\delta W = -\delta t \int_V j \cdot \varepsilon \, dV
\]
For the quasi static and reversible system, work done by the system
\[
\delta W = \delta t \int_V j \cdot \varepsilon \, dV.
\]
Using Maxwell’s Equation (Ampere’s law in differential form) and after simplification, we get
\[
\delta W = \delta t \left\{ \frac{c}{4\pi} \left[ \int \Delta \cdot (H \times \varepsilon) \, dV \right. \right.
\left. + \int H \cdot (\Delta \times \varepsilon) \, dV \right] - \frac{1}{4\pi} \int \frac{\delta D}{\delta t} \varepsilon \, dV \right\}
\]

\[
(8)
\]
\[ \int \Delta \cdot (H \times \varepsilon) \, dV \rightarrow \text{using Gauss Theorem} \rightarrow \text{surface integral, for a large distance it can be neglected.} \]

Using Maxwell’s Equation (Faraday’s law) and taking only magnetic field part, eqn 5 reduces to

\[ \delta W = -\frac{1}{4\pi} \int_V H \cdot \delta B \, dV \]  \hspace{1cm} (9)

Using the definition of magnetization density \( M \), as \( H \) the volume of magnetized matter. Talking only about the negative part of the integral, we have

\[ \delta W = -\frac{1}{4\pi} \left[ \int_V B \delta B \, dV + 4\pi \int_{\partial V} M \delta B \, dV \right]. \]

Here, the term \(-\frac{1}{4\pi} \int_V B \delta B \, dV\) is total field energy integrated over all space and \( \int_{\partial V} M \delta B \, dV \) is the integral over the volume of magnetized matter. Talking only about the work done by the magnetized matter, we can write

\[ \delta W = -M \, dB \]  \hspace{1cm} (10)

The first law of Thermodynamics in terms of enthalpy \( \mathcal{H} \) can be written as

\[ T \, dS = d\mathcal{H} - V \, dP + M \, dB. \]  \hspace{1cm} (11)

The heat capacity at constant magnetic field \( B \) is defined as

\[ C_B = \left( \frac{\partial H}{\partial T} \right)_B = T \left( \frac{\partial S}{\partial T} \right)_B \]  \hspace{1cm} (12)

Considering an adiabatic(isentropic) process, the change in temperature is given by

\[ dT = \left( \frac{\partial T}{\partial B} \right)_S \, dB. \]  \hspace{1cm} (13)

Also from the definition of Gibb’s potential

\[ dG = -S \, dT + V \, dP - M \, dB \Rightarrow \left( \frac{\partial S}{\partial B} \right)_T = \left( \frac{\partial M}{\partial T} \right)_B \]

eqn 13 is redefined as

\[ dT = -\frac{T}{C_B} \left( \frac{\partial M}{\partial T} \right)_B \, dB. \]  \hspace{1cm} (14)

A. Temperature Due to Magnetic Field

Assuming that the matter interact with magnetic field, the average magnetization can be given by

\[ \langle M \rangle = N \mu^2 B \frac{\partial M}{\partial T} = -N \mu^2 B \frac{k_B T^2}{k_B T^2} \]

the heat capacity becomes

\[ C_B = \frac{N \mu^2 B^2}{k_B T^2}. \]

Using the definition of \( C_B \) and \( \frac{\partial M}{\partial T} \) in eqn 13 and integrating, we have

\[ \ln \frac{T(r,t)}{T_0} = \int_0^t \left( \frac{dB}{dt'} \right) \frac{dt'}{B}. \]  \hspace{1cm} (15)

Next we will discuss the effect of some particular magnetic field profiles in the temperature contour \( T(r,t) = T_c \) defined above.

1. Exponential Decaying Field

The magnetic field profile is defined as

\[ B = \frac{\mu_0 NI}{2\pi r} \exp(-\alpha t) \]

\[ \Rightarrow \frac{dB}{dt} = -\frac{\alpha \mu_0 NI}{2\pi r} \exp(-\alpha t) \]

From eqn 15 the temperature at some \( r, t \) is given by

\[ T(r,t) = T_0 \exp(-\alpha t); \forall \, r. \]  \hspace{1cm} (16)

For \( t > 1 + r \) the overall contour expression becomes

\[ T(r,t) = \frac{2\pi CQ}{C_v r} \left[ t \ln \frac{1 + r}{1 - r} - 2r \right] + T_0 \exp(-\alpha t) \]  \hspace{1cm} (17)

In the limit \( r \to 0 \), eqn 17 reduces to

\[ T(0,t) = \frac{4\pi CQ}{C_v} (t - 1) + T_0 \exp(-\alpha t). \]  \hspace{1cm} (18)

Inserting \( t = 0 \) in eqn 17 we have

\[ T(r,0) = -\frac{4\pi CQ}{C_v} + T_0 \]

. Being far away from the ignited shells, we assume that the temperature \( T(r,0) \) is zero at \( t = 0 \). Hence, we have

\[ sT_0 = \frac{4\pi CQ}{C_v} = K \]  \hspace{1cm} (19)

a constant. The point where TL and SL discontinuity surfaces converge is determined by the condition

\[ \ln \frac{1 + r_c}{1 - r_c} - 2 \alpha r_c \exp(-\alpha t_c) = 0 \]

and the critical temperature \( T_c \) is given by

\[ T_c = \frac{2\pi CQ}{C_v} \left[ \frac{1}{r_c} \left( t_c \ln \frac{1 + r_c}{1 - r_c} - 2r_c \right) + 2 \exp(-\alpha t_c) \right]. \]  \hspace{1cm} (21)

The time taken by the core to be heated up to temperature \( T_c \) is given by the equation

\[ \frac{4\pi CQ}{C_v} [(t - 1) + \exp(-\alpha t)] = T_c \]  \hspace{1cm} (22)
2. Oscillating Field

For the oscillating field with cosine profile is given by
\[
B = \frac{\mu_0 NI}{2\pi r} \cos \alpha t.
\]

The temperature at some \( r, t \) is given by
\[
T(r, t) = T_0 \cos \alpha t; \quad \forall \ r
\] (23)

and for \( t > (1 + r) \) the temperature contour takes the form
\[
T(r, t) = \frac{2\pi CQ}{C_v} \left[ t \ln \frac{1 + r}{1 - r} - 2r \right] + T_0 \cos \alpha t.
\] (24)

In the limit \( r \to 0 \) eqn (24) becomes
\[
T(0, t) = \frac{4\pi CQ}{C_v} (t - 1) + T_0 \cos \alpha t.
\] (25)

\( T_0 \) is again the same constant defined in eqn (19).

The expressions for critical point \( (r_c, t_c) \) is calculated in similar manner as done for the exponential case.

\[
\ln \frac{1 + r_c}{1 - r_c} - 2\alpha r_c \sin \alpha t_c = 1
\] (26)

The critical temperature has the form
\[
T_c = \frac{2\pi CQ}{C_v} \left[ \frac{2}{r_c} \left( t_c \ln \frac{1 + r_c}{1 - r_c} - 2r_c \right) + 2 \cos \alpha t_c \right]
\] (27)

And the time taken by the core to be heated up to temperature \( T_c \) is given by the equation
\[
\frac{4\pi CQ}{C_v} (t - 1) + \cos \alpha t = T_c.
\] (28)

Similarly we calculate the critical point \( (r_c, t_c) \), critical temperature \( T_c \) and time \( t \) for the sinusoidal magnetic field profile. We only write the final expression for them below.

\[
T(r, t) = \frac{2\pi CQ}{C_v r} \left[ t \ln \frac{1 + r}{1 - r} - 2r \right] + T_0 \sin \alpha t.
\] (29)

\[
\ln \frac{1 + r_c}{1 - r_c} + 2\alpha r_c \cos \alpha t_c = 1
\] (30)

\[
T_c = \frac{2\pi CQ}{C_v} \left[ \frac{1}{r_c} \left( t_c \ln \frac{1 + r_c}{1 - r_c} - 2r_c \right) + 2 \sin \alpha t_c \right]
\] (31)

\[
\frac{4\pi CQ}{C_v} (t - 1) + \sin \alpha t = T_c
\] (32)

![FIG. 1. Contour \( T(r,t) = T_c \) for non magnetic and magnetic curves are drawn. The values of the normalized parameters are \( K = 1 = \frac{4\pi CQ}{C_v} \), \( \alpha = 1.0 \) and \( T = 3 \) (in unit of \( \frac{2\pi CQ}{C_v} \)).](image)

IV. RESULTS

The line \( t = t_c \) separates the SL and TL path of the discontinuity surface \( T(r,t) = T_c \). This discontinuity is formed at \( r = R \) at time \( t = 0 \) and then propagates inward. This process initially proceeds slowly, but then accelerates up by the radiative heat transfer and at \( r_c \) it goes over smoothly into a TL discontinuity.

For the non-magnetic implosion, for \( r_c = 0.5 \) the critical time comes out to be \( t_c = 2.34 \) (from eqn (7)) and the critical temperature \( T(r_c, t_c) \) is

\[
T(r_c, t_c) = \frac{2\pi CQ}{C_v r_c} \left[ t_c \ln \frac{1 + r_c}{1 - r_c} - 2r_c \right] = 3.20 \left( \frac{2\pi CQ}{C_v} \right)
\]

The time required to heat the center of core up to a temperature \( T_c \) (from eqn (19), is

\[
T_c = \frac{4\pi CQ}{C_v} (t - 1) \Rightarrow 2(t - 1) = 3.20 \Rightarrow t = 2.6
\]

For the magnetic field induced implosion we will treat the three cases separately. The first being the exponentially decaying magnetic field. Starting with a given value for \( r_c \) we can calculate \( t_c \). With these values we can then calculate the critical temperature \( T_c \) and time \( t \) taken by the core to gain a temperature \( T_c \) from eqn (21) (21) and (22)

For \( r_c = 0.5 \) the \( t_c \) is 2.072, and using this two values, we have \( T_c = 4.8 \left( \frac{2\pi CQ}{C_v} \right) \) and \( t = 3.36 \).

The oscillating magnetic field can either have cos variation or a sin variation. From equation (26) (27) and (28)
for the cos varying field, with \( r_c = 0.5 \) the critical time is \( t_c = 0.805 \), \( T_c = 1.115 \left( \frac{2\pi CQ}{c} \right) \) and \( t = 1.91 \). For the sin varying field with \( r_c = 0.5 \) gives \( t_c = 1.818 \) (using eqns. [30] [31] and [32], and the critical temperature \( T_c = 3.934 \left( \frac{2\pi CQ}{c} \right) \) and time \( t = 2.107 \).

We have plotted our result in fig 1. In our calculation the normalization of parameter \( K \) and \( T \) is set as \( K = 1 = \left( \frac{2\pi CQ}{c} \right) \) and \( T = 3 \) (in unit of \( \frac{2\pi CQ}{c} \)). The curve without any magnetic effect show the SL to TL detonation transition at \( r < 0.5 \) and \( t > 3 \).

The curve representing the exponentially decaying magnetic field follows the non-magnetic curve with the same transition point. The Cosine curve lies much above these two curves. The time like detonation region for such magnetic field and radiation-induced implosion lies inside the region bounded by \( r < 0.55 \) and \( t > 4 \). A large fraction of the core has TL detonation front for such magnetic field profile. However, the most interesting case is observed for the sinusoidally varying magnetic field. There is a smooth but continuous transition in the curve, at low \( t \) and \( r \). Initially at large \( r \) the value of \( t \) is much smaller (with respect to other curves), however, it steadily goes to much higher value of \( t \) at \( r < 0.65 \). The transition from SL to TL detonation happens at much smaller \( r \) (\( r \approx 0.4 \)), and the TL detonation volume shrinks.

Apart from \( K \) and \( T \), other input parameters that control this transition from SL to TL detonation is \( \alpha \). \( \alpha \) is the amplitude of the temporal variation of the magnetic fields. In fig 2 we have plotted how the transition from SL to TL detonation changes with a change in \( \alpha \) for exponentially decaying magnetic field. With the increase in \( \alpha \), the curves move to higher \( t \) values for same \( r \) without altering the shape of the curve. It signifies that the time required to heat the core up to the critical temperature increases with an increase in the magnetic field. The volume of the implosion region with TL detonation region remains the same as the shape of the curves does not change.

In fig 3 we demonstrate the effect of change of \( \alpha \) on the cosine varying magnetic field. At low \( \alpha \) the slope of the curve is very stiff (TL region is small), and as we increase \( \alpha \) the slope decreases and flattens around \( \alpha = 1 \). Increasing the value of \( \alpha \) beyond 1 shifts the curve to a small value of \( t \) without significant change in its slope. This trend continues until \( \alpha = 1.82 \), beyond which, there occurs a discontinuity in the curve. At \( \alpha = 1.83 \) the discontinuity appears at lower \( r \) values and as \( \alpha \) increases the discontinuity shifts outwards to higher values of \( r \). Beyond \( \alpha = 1.8 \), the spatial nature of TL detonation remains the same but the temporal part changes. The discontinuity in the curve signifies that the time required
to heat the particular $r = r_d$ (where the discontinuity lies) is large. There appears some potential barrier at this $r_d$ due to the magnetic field. The radiation experiences some blocking effect from the magnetic field and takes a lot of time to heat that particular $r_d$.

This feature is also present in the sinusoidally varying magnetic field as can be seen from fig.4. As $\alpha$ increases from lower to a higher value the smooth transition region moves towards the core of the implosion region and disappears at $\alpha = 2$. At such $\alpha$ the shape of the curve is similar to the shape of non-magnetic and exponentially decaying magnetic field curves. However, if $\alpha$ is increased to much higher values, a discontinuity in the curve appears (with a peak like feature), which shifts towards higher $r$ with an increase in $\alpha$. The magnetic field again shows some blocking effect on the radiation which takes a longer time in heating that particular $r_d$ where the discontinuity lies.

V. SUMMARY & CONCLUSION

To summarize, we have studied the effect of time-varying magnetic field on the smooth and continuous transition from SL to TL detonation in an implosion induced by radiation. The heating of the core arises due to the rapidly igniting explosive shells which surrounds it. The central region is swiftly heated to a very high temperature by incoming radiation from all directions. On top of this, we employ a time-varying magnetic field at the core. A constant magnetic field doesn’t effect the heating dynamics of the core.

The magnetic field has a significant effect on the dynamics of continuous transition from SL to TL detonation depending on the nature of field variation. An exponential decaying magnetic field does not affect the dynamics of this transition effectively. However, an oscillating field (sin or cos) can effect the dynamics of the change significantly. By the application of the oscillating field the volume of the TL detonation can be tuned (can be made larger or smaller). Increasing the amplitude of the temporal variation of the magnetic field induces a discontinuous jump in the curve, where for a particular radius inside the core the heat transfer due to radiation takes a significantly long time. The effect of the magnetic field can have enormous significance particularly in the experimental [6–8, 11] and theoretical [12, 13] study of Inertial Confinement Fusion of fuel. The change in the core volume of TL detonation gets rid of unwanted RT instabilities. However, the time of achieving the critical temperature may also play an important role here.

The magnetic field effect can also affect the hadronization of QGP volume involving detonation in high energy experiments. The emergence of a discontinuity due to the application of oscillating magnetic field is a phenomenon which has not been observed or expected, and further study is needed to understand it, which is our present ongoing endeavor.

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