Towards a classification of CMC-1 Trinoids in hyperbolic space via conjugate surfaces

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Abstract

We derive necessary conditions on the parameters of the ends of a CMC-1 trinoid in hyperbolic 3-space \( \mathbb{H}^3 \) with symmetry plane by passing to its conjugate minimal surface. Together with [Dan03], this yields a classification of generic symmetric trinoids. We also discuss the relation to other classification results of trinoids in [BPS03] and [UY00].

To obtain the result above, we show that the conjugate minimal surface of a catenoidal CMC-1 end in \( \mathbb{H}^3 \) with symmetry plane is asymptotic to a suitable helicoid.

1 Introduction

A minimal surface in \( \mathbb{R}^3 \) can be presented by its Weierstrass data, i.e. as a map \( \Phi_W : \Sigma \to \mathbb{R}^3 \), where \( \Sigma \) is a Riemann surface, and \( \Phi_W \) depends on \( (g, \omega) \), a meromorphic function and a holomorphic 1-form on \( \Sigma \).

Given a minimal surface, one can consider its associate (minimal) surface, which is determined by the Weierstrass data \( (g, i\omega) \).

Bryant found a representation of constant mean curvature 1 (CMC-1) surfaces in \( \mathbb{H}^3 \) depending on the same data (see [Bry87]). Therefore, we call CMC-1 surfaces in \( \mathbb{H}^3 \) Bryant surfaces. A Bryant surface has a minimal cousin, the minimal surface determined by the same data \( (g, \omega) \).

Given a Bryant surface, we define its conjugate (minimal) surface to be the associate minimal surface of its minimal cousin.

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Under this construction, a principal geodesic (i.e. a geodesic which is also a curvature line) on the Bryant surface corresponds to a straight line on its conjugate surface.

We define \( I := (-\frac{1}{4}, \infty) \setminus \{0\} \), and introduce helicoids \( H_\lambda \), catenoids \( C^{W}_\lambda \), and catenoid cousins \( C_\lambda \) parametrized by \( \lambda \in I \), such that:

The helicoid \( H_\lambda \) is the associate minimal surface of \( C^{W}_\lambda \), and \( C^{W}_\lambda \) is the minimal cousin of the Bryant surface \( C_\lambda \).

It is known that an end of a Bryant surface is asymptotic to some catenoid cousin or to a horosphere ([CHR01, Thm. 10]).

Clearly, one would expect that the conjugate surface of a catenoidal Bryant end is asymptotic to a suitable helicoid. However, this is not immediate, since the Bryant cousin relation is given by a second-order description only. For the similar situation of relating CMC-1 surfaces in \( \mathbb{R}^3 \) to minimal surfaces in \( S^3 \), there exists a first-order description. Using this, it is possible to conclude that asymptotics is preserved in this case (see [GKS01]).

For our situation, we show in section 3 that if a catenoidal end has a symmetry plane, then the asymptotics is indeed preserved:

**Theorem 1.1.** Let \( E' \) be a symmetric Bryant end asymptotic to \( C_\lambda \) for some \( \lambda \in I \). Then the conjugate minimal surface \( E'^c \) is asymptotic to \( H_\lambda \).

In section 4 we turn our attention to CMC-1 trinoids in \( \mathbb{H}^3 \). I.e., we examine Bryant surfaces of genus zero with three ends, all of which are catenoidal.

We study symmetric trinoids, i.e. trinoids which have a symmetry plane (determined by the asymptotic boundary points of their ends).

It follows from the classification by [UY00] that every (generic) trinoid is symmetric; since we present a different approach to this moduli problem, we do not use this result. One should look for a direct geometric proof that every properly immersed CMC-1 surface in \( \mathbb{H}^3 \) of genus zero and three ends has a symmetry plane.

A symmetric trinoid can be cut open along its symmetry plane to obtain two simply connected pieces. The conjugate surface of such a piece is a minimal surface bounded by three lines. Surfaces of this kind were already examined by Riemann (see [Ric61, sec. 17] or [Dar87]).

Using Theorem 1.1 this yields a necessary condition on the parameters of a generic trinoid:

Let \( J := (0, \infty) \setminus \{\pi\} \); for a real number \( \varphi \), we call

\[
 r(\varphi) := \min_{n \in \mathbb{Z}} |\varphi + 2n\pi|
\]
the reduced angle of \( \varphi \). Furthermore, let \( \mathcal{T} \) be the set of interior points of the tetrahedron with vertices \((\pi, 0, 0), (0, \pi, 0), (0, 0, \pi), (\pi, \pi, \pi)\). Then we have:

**Theorem 1.2.** If there exists a symmetric trinoid corresponding to the parameter triple \((\varphi_1, \varphi_2, \varphi_3) \in (J \setminus \pi \mathbb{Z})^3\), then for the triple of reduced angles holds (in the generic case):

\[
(r(\varphi_1), r(\varphi_2), r(\varphi_3)) \in \mathcal{T}.
\]

On the other hand, minimal surfaces bounded by three lines are constructed in [Dan03]. His main result is:

**Theorem 1.3 ([Dan03, Thm. 49]).** Let \((\varphi_1, \varphi_2, \varphi_3) \in (J \setminus \pi \mathbb{Z})^3\), and assume that \((r(\varphi_1), r(\varphi_2), r(\varphi_3))\) lies in \(\mathcal{T}\).

Under a certain polynomial condition (in the \(\varphi_i\)), there is a corresponding symmetric trinoid which arises from a minimal disk bounded by three lines.

In section 6, we compare the conditions given by the theorems above to the conditions found in [BPS03] and [UY00], and find that they are essentially the same:

**Corollary 1.4.** The conditions of Theorems 1.2 and 1.3 are equivalent to those given by [BPS03].

For symmetric parameter triples \((\varphi, \varphi, \varphi)\) with \(\varphi \in (\pi/3, \pi)\), one can construct the minimal surface using a sequence of Plateau solutions, and show that it corresponds to a trinoid; for details, see [Bal03].

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2 Preliminaries

In this section, we present the material from the beginning of section 1 in more detail.

First, we recall the Weierstrass representation for minimal surfaces in \(\mathbb{R}^3\) and the Bryant representation for CMC-1 surfaces in \(\mathbb{H}^3\):
The Weierstrass representation Theorem says that every minimal surface can be conformally parametrized as

\[
\Phi_W(z) = \text{Re} \oint_{z_0}^z ((1 - g^2)\omega, i(1 + g^2)\omega, 2g\omega),
\]

(1)

where \( z \) is in \( \Sigma \), the parametrizing Riemann surface (possibly with boundary), and \( g \) (resp. \( \omega \)) is a meromorphic function (resp. a holomorphic 1-form) on \( \Sigma \). Furthermore, \( g \) has a pole of order \( k \) in \( z \) if and only if \( \omega \) has a zero of order \( 2k \) in \( z \).

The function \( g \) has a geometric meaning: it is the stereographic projection of the Gauss (or normal) map of the minimal surface \( \Phi_W \).

The pair \( (g, \omega) \) is called the Weierstrass data of \( \Phi_W \).

Conversely, Weierstrass data on a Riemann surface \( \Sigma \) defines a minimal immersion from the universal cover \( \tilde{\Sigma} \) into \( \mathbb{R}^3 \) via (1).

To a minimal surface \( \Phi_W \) with Weierstrass data \( (g, \omega) \), one has its associate surface \( \Phi_{\bar{W}} \), which is given by the Weierstrass data \( (g, i\omega) \); it turns out that \( \Phi_W \) and \( \Phi_{\bar{W}} \) are (locally) isometric. Note that if \( \Phi_W \) is defined on \( \Sigma \), it may happen that \( \Phi_{\bar{W}} \) is defined on \( \tilde{\Sigma} \) only.

For more details on the Weierstrass representation, we refer the reader to [Oss86, §8].

In his seminal paper [Bry87], Bryant showed that there is a representation of CMC-1 surfaces in \( \mathbb{R}^3 \) using exactly the same data as the Weierstrass representation. Thus, to a minimal surface \( \Phi_W \) with Weierstrass data \( (g, \omega) \), one obtains a Bryant cousin \( \Phi_B \); vice versa, every CMC-1 surface in \( \mathbb{R}^3 \) has a minimal cousin. The surfaces \( \Phi_W \) and \( \Phi_B \) are (locally) isometric, and their Gauss maps agree.

**Definition 2.1.** Following Rosenberg, we define a Bryant surface to be an immersed CMC-1 surface in \( \mathbb{R}^3 \).

**Definition 2.2.** Given a simply connected Bryant surface \( M \), we define its conjugate surface \( M^c \) as the associate surface of \( M \)'s minimal cousin (in \( \mathbb{R}^3 \)).

Since the Bryant relation is a special case of Lawson's correspondence ([UY92]), a principal geodesic (i.e. a geodesic which is also a curvature line) corresponds to a principal geodesic under the Bryant cousin relation. Under the associate construction, principal geodesics go to straight lines. Thus principal geodesics on \( M \) are mapped to straight lines by \( M^c \).

**Example 2.3.** We introduce our notation for the helicoids, the catenoids, and the catenoid cousins:
Parametrize the surfaces by \( \Sigma = \mathbb{C} \): For \( 0 \neq \lambda \in \mathbb{R} \), the catenoid \( C^W_\lambda \) is the minimal surface with Weierstrass data \( g = \exp(z) \), \( \omega = \lambda \exp(-z)dz \), and the helicoid \( H_\lambda \) is its associate surface, with Weierstrass data \( g = \exp(z) \), \( \omega = \lambda i \exp(-z)dz \).

The formula for \( H_\lambda \) is
\[
H_\lambda(x + iy) = 2\lambda \left( \begin{array}{c}
\sinh x \sin y \\
-\sinh x \cos y \\
y
\end{array} \right).
\] (2)

If \( \lambda \in I \), where \( I := (-\frac{1}{4}, \infty) \setminus \{0\} \), we call the Bryant cousin of \( C^W_\lambda \) a Catenoid Cousin \( C_\lambda \). Formulas for catenoid cousins \( C_\lambda \) in the upper halfspace model \( (\mathbb{H}^3 = \{(u + iv, w) \mid u, v \in \mathbb{R}, w > 0\}) \) are given in [Ros02, sec. 11]:

The surfaces are again parametrized by \( \mathbb{C} \); every line with constant imaginary part parametrizes a principal geodesic from the end of \( C_\lambda \) at 0 to the end at \( \infty \) in \( \mathbb{H}^3 \). Set \( a := \sqrt{1 + 4\lambda} \); then the formula for \( C_\lambda(x + iy) \) is given by
\[
u + iv = \frac{-\lambda(e^x + e^{-x})e^{ax}}{(\frac{1}{2} + \lambda - \frac{1}{2}a) e^{-x} + (\frac{1}{2} + \lambda + \frac{3}{2}a) e^x e^{iay}},
\]
\[w = \frac{a e^{ax}}{(\frac{1}{2} + \lambda - \frac{1}{2}a) e^{-x} + (\frac{1}{2} + \lambda + \frac{3}{2}a) e^x}.
\]

Note that the parametrization of \( C_\lambda \) is periodic with period \( \frac{2\pi i}{\sqrt{1 + 4\lambda}} \).

Let \( J := (0, \infty) \setminus \{\pi\} \), and define the bijective function \( \tilde{\varphi} : I \to J \) by \( \tilde{\varphi}(\lambda) := \frac{\pi}{\sqrt{1 + 4\lambda}} \).

We remark that the Catenoids (and Catenoid Cousins) \( C^W_\lambda \) can alternatively be described by the Weierstrass data \( g = z^\alpha, \omega = \frac{1-\alpha}{4\alpha} z^{-1-\alpha} \) on \( \mathbb{C}^* \), where \( \pi \alpha = \tilde{\varphi}(\lambda) \); see [ST01, Ex. 1.5].

3 Symmetric catenoidal Bryant ends and their conjugate surfaces

In this section, we show that the conjugate minimal surface of a catenoidal Bryant end with a symmetry plane is asymptotic to the corresponding helicoid.

**Definition 3.1.** An **annular Bryant end** is a Bryant surface with domain \( \{0 < |z| \leq 1\} \) (or equivalently any other punctured disk with boundary).

Recall that a properly embedded Bryant annular end in \( \mathbb{H}^3 \) is asymptotic to some catenoid cousin or to a horosphere [CHR01, Thm. 10].
Definition 3.2. An annular Bryant end is called *catenoidal* if it is properly embedded and asymptotic to a catenoid cousin. An annular Bryant end is called *symmetric* if it is properly embedded and has a symmetry plane.

Definition 3.3. A *minimal end bounded by rays* is a properly immersed minimal surface in $\mathbb{R}^3$ with domain $\{0 < |z| \leq 1, \text{Im} z \geq 0\}$, such that $[-1, 0)$ and $(0, 1]$ are mapped to (monotonically parametrized) rays.

For a real number $\varphi$, we call $r(\varphi) := \min_{n \in \mathbb{Z}} |\varphi + 2n\pi|$ the reduced angle of $\varphi$.

The following Lemma is a slight generalization of [Dan03, L. 7]:

Lemma 3.4. Let $X$ be a minimal end bounded by horizontal rays with vertical limit normal for $z \to 0$. Assume that $X$ is contained in a vertical slab (i.e. the vertical component of $X$ is bounded), that the stereographic projection $g$ of the Gauss map of $X$ satisfies $g \sim z^\alpha$ for $z \to 0$ with $0 < \alpha \neq 1$, and that the vertical components of the boundary rays are $|\tilde{\varphi}^{-1}(\pi \alpha)| \pi \alpha$ apart. Then $X$ is asymptotic to (part of) $IH_\lambda$ for $\lambda = \tilde{\varphi}^{-1}(\pi \alpha)$ and some orientation preserving isometry $I$ of $\mathbb{R}^3$.

Proof. First we note that we can conclude from the proof of [Dan03, L. 7] that the angle between the boundary rays is the reduced angle $r(\pi \alpha)$. Additionally, observe that the (vertical) distance of the boundary rays is by assumption the distance of two lines in $H_\lambda$, where one has to be rotated by angle $\pi \alpha$ in $H_\lambda$ to be mapped to the other one (cf. formula (2)).

If the boundary rays are not parallel, the claim is just [Dan03, L. 7]. The case of parallel boundary rays is not covered there; however, its proof still works in this case by our assumptions on the limit normal, $X$ being contained in a slab, and the vertical distance of the rays. □

Proof of Theorem 1.1. It suffices to consider one symmetric piece $E$ of $E'$ bounded by principal geodesics (the curves of intersection with the symmetry plane). Let $E^c$ denote the conjugate surface of this half. We assume $E^c$ to be parametrized by $D := \{0 < |z| \leq 1, \text{Im} z \geq 0\}$. By [CHR01], $E'$ has a well-defined limit normal, which we may assume to be vertical.

Then $E^c$ also has a vertical limit normal, so it is a minimal end bounded by horizontal rays. We show that $E^c$ is contained in a vertical slab:

By [ST01], we may assume the Weierstrass data of $E'$ to be of the form $g = z^\alpha (g_0 + zg_1(z)), \quad \omega = z^{-1-\alpha} (w_0 + zw_1(z))$ with $g_0, w_0 \in \mathbb{C}$ such that $g_0 w_0 = \frac{1-\alpha^2}{4\alpha}$, and holomorphic functions $g_1, w_1$ on $\{|z| \leq 1\}$ (where $\pi \alpha = \tilde{\varphi}(\lambda)$, in particular $0 < \alpha \neq 1$).
Choose \( z_0 \in (0, 1) \subset D \); the third component of \( E^c \) is the negative of the imaginary part of the following integral:

\[
\oint_{z_0}^{z} 2g\omega = 2 \oint_{z_0}^{z} \xi^{-1} \left( g_0w_0 + \xi \left( g_0w_1(\xi) + w_0g_1(\xi) + \xi w_1(\xi)g_1(\xi) \right) \right) d\xi
\]

\[
= \frac{1 - \alpha^2}{2\alpha} \oint_{z_0}^{z} \xi^{-1} d\xi + 2 \oint_{z_0}^{z} g_0w_1(\xi) + w_0g_1(\xi) + \xi w_1(\xi)g_1(\xi) d\xi
\]

\[
= C(z)
\]

Hence, \( E^c \) is contained in a vertical slab, since \( C \) is bounded on \( D \) and the first summand corresponds to the third component of \( H_\lambda \). Observe that the imaginary part of the first summand above is 0 for \( z \in (0, 1] \) and constant for \( z \in [-1, 0) \). We show that \( \text{Im} \ C(z) = 0 \) for \( z \in (0, 1] \), since \( z_0 \in (0, 1] \), and a horizontal ray is parametrized. Similarly, \( \text{Im} \ C(z) \equiv C_2 \) for \( z \in [-1, 0) \) since this parametrizes another horizontal ray. Thus \( \text{Im} \oint_{1/n}^{-1/n} g_0w_1(\xi) + w_0g_1(\xi) + \xi w_1(\xi)g_1(\xi) d\xi \) is constant (i.e. independent from \( n \) and the path in \( D \) from \( \frac{1}{n} \) to \( -\frac{1}{n} \)) and we have

\[
C_2 = \text{Im} \lim_{n \to \infty} \oint_{1/n}^{^{-1/n}} g_0w_1(\xi) + w_0g_1(\xi) + \xi w_1(\xi)g_1(\xi) d\xi = 0.
\]

This shows that the two boundary rays of \( E^c \) have positive vertical distance, which is equal to the distance of corresponding lines on \( H_\lambda \). Now the conclusion follows via Lemma 3.4. \( \square \)

**Corollary 3.5.** Let \( E' \) be a symmetric Bryant end which is asymptotic to \( C_\lambda \), and let \( E \) be a symmetric piece of \( E' \) as above. If \( \varphi := \tilde{\varphi}(\lambda) \not\in \pi \mathbb{Z} \), we have:

The boundary rays \( l_1, l_2 \) of \( E^c \) are contained in \( IH_\lambda \) for some orientation-preserving isometry \( I \) of \( \mathbb{R}^3 \). In particular, the angle between the ends of \( l_1 \) and \( l_2 \) is \( r(\varphi) \). The distance of these two lines is \( h(\varphi) := |\lambda|\varphi \).

**Proof.** First we note that \( h : J \to \mathbb{R} \) is well-defined, because \( \tilde{\varphi} \) is a bijective function (in fact \( h(\varphi) = |\frac{\pi^2}{4\varphi} - \frac{\pi}{4}| \)).

The claim follows immediately from the proof of Theorem 1.1 and the formulas for helicoids. \( \square \)

In case \( \tilde{\varphi}(\lambda) \in \pi \mathbb{Z} \), the boundary rays are parallel, and we have a lower bound on their distance (by the distance of parallel lines in the corresponding helicoid).
4 Trinoids

Definition 4.1. We define a trinoid to be a properly immersed Bryant surface of genus zero with three ends, all of which are catenoidal. A symmetric trinoid is a trinoid \( T \) which has a symmetry plane \( P \) such that the asymptotic endpoints of \( T \) are contained in the asymptotic boundary of \( P \).

Denote by \( \mathcal{M} \) the space of symmetric trinoids with ends marked by 1, 2, 3, up to isometry (respecting the marks of the ends).

Observe that the symmetry plane \( P \) is uniquely determined if the asymptotic endpoints are distinct.

Pictures of trinoids can be found at http://www-sfb288.math.tu-berlin.de/~bobenko/Trinoid/webimages.html; see also [BPS03].

Definition 4.2. We can define the map \( \Psi : \mathcal{M} \to J^3 \) sending a trinoid to the triple \((\varphi_1, \varphi_2, \varphi_3) \in J^3\), where \( \varphi_i = \tilde{\varphi}(\lambda_i) \), and \( \lambda_i \) is the parameter of end \( i \).

Lemma 4.3. Any properly embedded Bryant surface \( M \) of genus zero with three ends is a symmetric trinoid.

Proof. By Theorem [CHR01 Thm. 12], every end is catenoidal, and by [CHR01 Thm. 11], the three asymptotic boundary points are distinct and \( M \) is a bigraph over the plane containing them.

We expect that the Lemma above generalizes to Alexandrov-embedded Bryant surfaces.

Note that a trinoid is a map \( S^2 \setminus \{x_1, x_2, x_3\} \to \mathbb{H}^3 \), where \( x_1, x_2, x_3 \) are distinct and correspond to the ends 1, 2, 3 respectively.

Lemma 4.4. For a symmetric trinoid \( M \in \mathcal{M} \), there is a unique principal geodesic of \( M \) joining \( x_1 \) to \( x_2 \), which we denote by \( l_{12} \). Similarly, there is a unique principal geodesic \( l_{23} \) joining \( (x_2, x_3) \) and a unique principal geodesic \( l_{31} \) joining \( (x_1, x_2) \).

Considering \( l_{12}, l_{23}, l_{31} \) as subsets of \( S^2 \), we have that \( S^2 \setminus (l_{12} \cup l_{23} \cup l_{31} \cup \{x_1, x_2, x_3\}) \) consists of exactly two components.

Proof. It is known that a principal geodesic is contained in a plane of symmetry of \( M \) (cf. [ST01 Prop. 3.2]). We conclude that the three lines we are looking for need to be contained in \( P \), the symmetry plane of \( M \) from the definition.
Consider the graph $G$ in $S^2$ with vertices $V := \{x_1, x_2, x_3\}$, and edges the principal geodesics of $M$ contained in $P$ which start or end in $V$ (observe that both asymptotic ends of such a principal geodesic are in $V$).

Since every end of $M$ is embedded, every vertex has degree two. Edges cannot intersect: Tangential contact is excluded by uniqueness of geodesics, and transversal intersection is impossible since $M$ intersects $P$ orthogonally near every point of $G$.

Thus, $G$ consists of one, two, or three loops in $S^2$. Reflection in $P$ maps every component of $S^2 \setminus G$ to an other component. Since all elements of $V$ are fixed points of this reflection, $G$ consists of one loop only. □

**Corollary 4.5.** Consider a symmetric trinoid $M$ and its symmetry plane $P$. Then there is a neighborhood $N$ of $l_{12} \cup l_{23} \cup l_{31} \cup \{x_1, x_2, x_3\}$ in $S^2$ such that $M(N) \cap P = M(l_{12} \cup l_{23} \cup l_{31})$. In particular: Near its boundary, each component of $S^2 \setminus (l_{12} \cup l_{23} \cup l_{31} \cup \{x_1, x_2, x_3\})$ is mapped to a component of $H^3 \setminus P$.

If $M$ is embedded, each component of $S^2 \setminus (l_{12} \cup l_{23} \cup l_{31} \cup \{x_1, x_2, x_3\})$ is mapped into a halfspace of $H^3 \setminus P$. □

Given a symmetric trinoid $M$, we can (by an orientation-preserving isometry) assume that its symmetry plane is the equatorial plane $E = \{x_3 = 0\}$ of the Poincaré disk model (lying inside $\mathbb{R}^3$). Further, we can assume that the ends are marked increasingly if one looks from above (i.e. the direction of positive $x_3$).

**Definition 4.6.** Given a symmetric trinoid $M$, we divide its domain $S^2 \setminus \{x_1, x_2, x_3\}$ into two components along $l_{12}, l_{23}, l_{31}$, and we define $M^+$ to be the restriction of $M$ to the closure of the component which is mapped to the upper half space near $l_{12}, l_{23}, l_{31}$, if $M$ is put in the Poincaré model in the way explained above.

So $M^+$ is a map $\overline{D} \setminus \{x_1, x_2, x_3\} \to H^3$, where $x_1, x_2, x_3$ are distinct points in $\partial D$ (and $D$ is the closed unit disk).

We choose the orientation on $D$ and its boundary as depicted in Figure 1.

Note that $M^+$ is well-defined up to orientation-preserving hyperbolic isometries leaving the upper half space near $l_{12}, l_{23}, l_{31}$, if $M$ is put in the Poincaré model invariant.

Define $M^c := (M^+)^c$ to be the conjugate minimal surface of $M^+$.

Since principal geodesics on a Bryant surface correspond to straight lines on its conjugate minimal surface, we have:

**Lemma 4.7.** Let $M$ be a symmetric trinoid. Then $M^c$ is a minimal surface bounded by three straight lines. □
Figure 1: The domain of $M^+$ and $M^c$.

We mention another interesting fact about $M^+$:

**Proposition 4.8.** If $M^+$ is embedded, then so is $M$.

**Proof.** Assume the symmetry plane of $M$ to be the equatorial plane $E$ as before. We apply the Alexandrov-reflection technique (see, for example, [LR85]): Using a (continuous) family of planes which foliate the upper half-space, we conclude that the normal of $M^+$ at any point $p \in M^+ \cap E$ has non-positive vertical coordinate.

Similarly, we use a family of planes foliating the lower half-space to find that for a point $p \in M^+ \cap E$, the normal has to have non-negative vertical coordinate.

Thus, every component of $M \cap E$ is a principal geodesic, i.e. a curve of planar reflection (by [ST01, Prop. 3.2]). So $M^+$ is cut off wherever it reaches $E$ (observe that there are no closed principal geodesics since $M$ has genus zero), and it does not intersect the lower half-space; so $M$ is embedded.

5 Necessary conditions on the constellation of boundary lines

In this section, we use the information about the constellation of lines which bound $M^c$ to obtain a necessary condition on the parameter triple $\Psi(M)$ in the generic case.
Definition 5.1. A triple \((\lambda_1, \lambda_2, \lambda_3) \in I^3\) is called a parameter triple.

A triple of oriented lines \((l_{12}, l_{23}, l_{31})\) in \(\mathbb{R}^3\) is called admissible constellation if

(i) There exists a parameter triple \((\lambda_1, \lambda_2, \lambda_3) \in I^3\) and orientation-preserving isometries \(I_1, I_2, I_3\) of \(\mathbb{R}^3\) such that

\[
\{l_{12}, l_{31}\} \subset I_1(H_{\lambda_1}), \quad \{l_{23}, l_{12}\} \subset I_2(H_{\lambda_2}), \quad \{l_{31}, l_{23}\} \subset I_3 H_{\lambda_3}.
\]

(ii) Rotating \(l_{i(i+1)}\) inside \(I_i(H_{\lambda_i})\) maps \(l_{i(i+1)}\) to \(l_{i(i+2)}\) with the opposite orientation (for \(i \in \mathbb{Z}_3\)).

(iii) The distance of \(l_{i(i+1)}\) and \(l_{i(i+2)}\) is \(h \circ \tilde{\varphi}(\lambda_i)\) (for \(i \in \mathbb{Z}_3\)).

A triple \((\varphi_1, \varphi_2, \varphi_3) \in J^3\) of angles is called admissible, if there exists an admissible constellation with parameter triple \((\tilde{\varphi}^{-1}(\varphi_1), \tilde{\varphi}^{-1}(\varphi_2), \tilde{\varphi}^{-1}(\varphi_3))\).

An admissible triple is called generic if there is a corresponding admissible constellation such that the lines are not contained in parallel planes. The triple is called parallel otherwise.

Remark 5.2. Note that a general triple of three oriented lines is determined (up to the action of \(SO(3)\)) by the oriented distances and the angles. We have the restriction that the distance and angle match, i.e. every pair of lines can be put into a suitable helicoid.

We define \(T\) to be the set of interior points of the tetrahedron with vertices \((\pi, 0, 0), (0, \pi, 0), (0, 0, \pi), (\pi, \pi, \pi)\).

Sketches of admissible constellations can be found in \[Bal03\].

From Corollary 3.5, we have:

Lemma 5.3. For any symmetric trinoid \(M \in \mathcal{M}\) with \(\tilde{\varphi}(\lambda_i) \notin \pi\mathbb{Z}\) for \(i \in \{1, 2, 3\}\), the triple \(\Psi(M)\) is admissible.

Proof of Theorem 5.4. A triple \((\varphi_1, \varphi_2, \varphi_3) \in J^3\) is a generic admissible triple if and only if \((r(\varphi_1), r(\varphi_2), r(\varphi_3)) \in T\). For every triple of that kind, there are exactly two generic admissible constellations of lines in \(\mathbb{R}^3\) (modulo \(SO(3)\)).

Proofs can be found in \[Dan03\ Prop. 9\]; and \[Bal03\]. Essentially, the conditions on the angles correspond to the condition that the directions of the lines form a spherical triangle (after identifying the unit tangent spheres of \(\mathbb{R}^3\) via parallel translation).

Proof of Theorem 1.2. The theorem follows from Lemma 5.3 and Theorem 5.4.
Remark 5.5. One can show that an admissible triple corresponds either to generic or to parallel constellations, and that the triple of reduced angle lies in the boundary of \( \mathcal{T} \) in the parallel case, see [Bal03]. Hence the name generic is justified.

6 Comparing to related results

In this section, we compare the conditions obtained by [Dan03] and our results with the results in in [BPS03] and [UY00].

Consider the presentation of catenoid cousins in [Bry87, Ex. 2]. Bryant parametrizes catenoid cousins with a parameter \(-\frac{1}{2} < \mu_B \neq 0\).

**Lemma 6.1.** The catenoid cousin given by Bryant’s parameter \(\mu_B\) is \(C_\lambda\), where \(\lambda = \tilde{\phi}^{-1}(\pi(2\mu_B + 1))\).

**Proof.** Bryant computes the total curvature of a catenoid cousin to be \(-4\pi(2\mu_B + 1)\). A standard catenoid has total curvature \(-4\pi\). Since a Bryant surface is locally isometric to its minimal cousin, a catenoid cousin \(C_\lambda\) has total curvature \(-4\pi \cdot \frac{1}{\sqrt{1+4\lambda}}\) (see Example 2.3). The claim follows. \(\square\)

Next, we trace back the relationship between our parameters and the parameters in [BPS03].

In [BPS03, sec. 4], catenoid cousins are parametrized by a parameter \(0 < \lambda_{BPS} \neq \frac{1}{2}\). Comparing the formulas for catenoid cousins given by Bryant and [BPS03], we obtain \(\lambda_{BPS} = \mu_B + \frac{1}{2}\); hence, the catenoid cousin described by the parameter \(\lambda_{BPS}\) is \(C_\lambda\), where

\[
\lambda = \lambda(\lambda_{BPS}) = \tilde{\phi}^{-1}(2\pi\lambda_{BPS}).
\]

They consider \(\{|(\lambda_{BPS,i})|\}\), where \(|\cdot|\) stands for the fractional part of a number in \([-\frac{1}{2}, \frac{1}{2})\).

The main result in [BPS03] is:

**Theorem 6.2 ([BPS03, Prop. 2]).** For given parameters \(p_i, q_i\), where \(i \in \{0, 1, \infty\}\), in the generic case, it is necessary for the existence of a trinoid that the numbers \(\Delta_i := |(\lambda_{BPS,i})|\) satisfy the conditions

\[
\begin{align*}
\Delta_0 + \Delta_1 + \Delta_\infty &> \frac{1}{2} \\
\Delta_0 + \Delta_1 - \Delta_\infty &< \frac{1}{2} \\
\Delta_0 - \Delta_1 + \Delta_\infty &< \frac{1}{2} \\
\Delta_0 + \Delta_1 + \Delta_\infty &< \frac{1}{2};
\end{align*}
\]
This condition is sufficient if furthermore, certain holomorphic spinors $P$ and $Q$ have no common zeroes.

They also show that their classification is equivalent to [UY00, Thm. 2.6].

Observe that the “generic case” in [BPS03] means that the case of half-integer $\lambda_{BPS,i}$ is excluded (cf. formula (6.3), and the remark at the bottom of page 18), so their class of generic trinoids is slightly larger than ours.

In [Dan03, Thm. 49], the trinoids from the classification of Umehara-Yamada (or equiv. Bobenko et al.) are constructed via minimal surfaces bounded by a generic constellation of three lines.

In view of [BPS03], we find that $|\{\lambda_{BPS}\}|$ corresponds to our notion of reduced angle, i.e. we have $r(\tilde{\varphi} \circ \lambda(\lambda_{BPS})) = 2\pi|\{\lambda_{BPS}\}|$. So the necessary conditions of Theorem 6.2 are the same as those in Theorem 1.2.

Comparing our Theorem 1.2 to the main theorem of [BPS03], the condition about the common zeroes of $P, Q$ is preventing our condition from being sufficient. In [Dan03], this additional condition is that his polynomial “$\varphi$” of degree two has no double root (for the equivalence, see [Dan03, proof of L. 16, and page 31]). This condition avoids singular points on the minimal surface and the trinoid.

Hence, Corollary 1.4 follows.

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