A Uniqueness theorem for black holes with Kaluza-Klein asymptotic in 5D Einstein-Maxwell gravity

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Abstract

In the present paper we prove a uniqueness theorem for stationary multi black hole configurations with Kaluza-Klein asymptotic in a certain sector of 5D Einstein-Maxwell gravity. We show that such multi black hole configurations are uniquely specified by the interval structure, angular momenta of the horizons, magnetic charges and the magnetic flux. A straightforward generalization of the uniqueness theorem for 5D Einstein-Maxwell-dilaton gravity is also given.

1 Introduction

In the last decade we have witnessed a remarkable advent of the higher dimensional gravity and especially of higher dimensional black holes. Many interesting black hole solutions with amazing properties were discovered in various gravity theories in higher dimensional spacetimes. The accumulation of black hole solutions naturally raises the question of their classification. The general classification for arbitrary spacetime dimensions and for all known gravity theories is formidable task which will probably need the efforts of generations. However, in some cases the full classification is possible. In $n = 4$ spacetime dimensions, asymptotically flat, stationary vacuum or electrovac black hole solutions in Einstein gravity are completely characterized by their asymptotic charges—mass, angular momentum, and electric (or magnetic) charge $[1, 2, 3, 4, 5, 6, 7]$ (see also $[8]$). The asymptotically flat, static, vacuum and electro-vacuum black holes in arbitrary dimensions were classified in $[9, 10]$. The complete classification of stationary black holes in more than $n = 4$ spacetime dimensions is at present an open problem. However, in a recent paper $[11]$, a partial classification was achieved for asymptotically flat, vacuum (non-extremal) black hole solutions under the assumption that the number of commuting axial Killing fields is sufficiently large. The particular case considered there was $n = 5$, and the number of axial Killing fields required was

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two. Under this assumption, it was shown how to construct from the given solution a certain set of invariants consisting of a set of real numbers and a collection of integer-valued vectors. These data were called the "interval structure" of the solution. It determines in particular the horizon topology, which could be either $S^3$, $S^1 \times S^2$ or a Lens-space $L(p, q)$. It was then demonstrated that the interval structure together with the asymptotic charges gives a complete set of invariants of the solutions, i.e., if these data coincide for two given solutions, then the solutions are isometric. The generalization of [11] for a certain sector of 5D Einstein-Maxwell gravity was done in [12]. In the sector under consideration, the 5D asymptotically flat Einstein-Maxwell black holes are classified in terms of their interval structure, angular momentum and the magnetic charges associated with the generators of $H_2(M)$. Uniqueness theorems were also proven for certain cases of the five-dimensional minimal supergravity [13]-[15]. The uniqueness of the 5D extremal vacuum black holes was considered in [16].

Fortunately, a full classification can also be achieved for Kaluza-Klein black holes in the higher dimensional Einstein gravity [17]. The vacuum Kaluza-Klein black holes are again fully classified in terms of their interval structure and angular momenta.

In this paper, we generalize the analysis of [17] to include Maxwell field. More precisely we generalize [17] to a certain, completely integrable sector of 5D Einstein-Maxwell gravity. We restrict ourselves to 5 dimensions where we are free from technical complications and where we can demonstrate the main idea in pure form. The naive expectation is that the generalization of uniqueness theorem for Einstein-Maxwell black holes with Kaluza-Klein asymptotic, can be done along the lines of the similar generalization in the asymptotically flat case, in other words in terms of the interval structure, angular momenta and the magnetic charges. However, this is not the case. It was shown in [18] (see also [19]) that, in the general case, the interval structure and the local and asymptotic charges are insufficient to fully classify the Einstein-Maxwell black holes with Kaluza-Klein asymptotic. The very uniqueness theorem was formulated in [18] which states that the Kaluza-Klein black holes in Einstein-Maxwell gravity are fully specified by their interval structure, global and local charges, angular momenta and magnetic fluxes. The novelty in comparison with the asymptotically flat case is the appearance of the magnetic fluxes in the conditions of the theorem.

Here we give mathematically more precise version of this theorem and its proof in 5 dimensions. The mathematical technique of the proof is the same with that in the asymptotically flat case [12] which requires σ-model presentation (on a symmetric space) of the dimensionally reduced field equations. In order to have symmetric space σ-model form of the dimensionally reduced equations, as in the asymptotically flat case, certain additional conditions are imposed upon the Maxwell field and the axial Killing fields. The extra assumptions placed upon the Killing fields imply that the electric charges, and some of the angular momenta of the horizons vanish. They also imply that the possible interval structures are limited. In particular, the horizons topologies can only be either $S^3$ or $S^2 \times S^1$.

Non-trivial Einstein-Maxwell black holes with Kaluza-Klein asymptotic satisfying our assumption have been found by [18, 19].

The paper is organized as follows. In the section 1 following [17], for completeness we give in concise form the necessary mathematical base. In section 2 we consider the extra...
assumptions imposed on the Killing fields and the Maxwell field, the consequences of them and the dimensionally reduced Einstein-Maxwell equations. The main result is presented in section 3. In the Discussion we comment on possible generalizations and some tricky cases.

2 Stationary Einstein-Maxwell black holes in 5 dimensions

Let $(M, g, F)$ be a 5-dimensional, analytic, stationary black hole spacetime satisfying the Einstein-Maxwell equations

\begin{align}
R_{ab} &= \frac{1}{2} \left( F_{ac} F_{b}^{c} - \frac{g_{ab}}{6} F_{cd} F^{cd} \right), \\
\nabla_a F^{ab} &= 0 = \nabla_{[a} F_{bc]}.
\end{align}

Let $\xi$ be the asymptotically timelike Killing field, $\xi g = 0$, which we assume is normalized so that $\lim g(\xi, \xi) = -1$ near infinity. We assume also that the Maxwell tensor is invariant under $\xi$, in the sense that $\xi g = 0$. We consider 5-dimensional spacetime $M$ that has four asymptotically flat large dimensions and one asymptotically small extra dimension. More precisely we consider 5-dimensional spacetime with asymptotic region $M_{\infty} = \mathbb{R}^{3,1} \times S^1$ and asymptotic metric

\begin{equation}
g = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + d\phi^2 + O(r^{-1})
\end{equation}

where $x_i$ are the standard Cartesian coordinates on $\mathbb{R}^3$, $\phi$ is the standard periodic coordinate on $S^1$ with a period $2\pi$. Here $O(r^{-1})$ stands for all metric components that drop off at least as $r^{-1}$ in the radial coordinate $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

The domain of outer communications is defined by

\begin{equation}
<< M >> = i^+(M_{\infty}) \cap i^-(M_{\infty})
\end{equation}

where $i^\pm(M_{\infty})$ denote the chronological future/past of $M_{\infty}$.

Here we will assume the existence of 2 further axial Killing fields $\zeta$ and $\eta$ which are mutually commuting and commute with the asymptotically timelike Killing field $\xi$, have periodic orbits with period $2\pi$ and leave the Maxwell tensor $F$ invariant, i.e. $\xi F = \xi g F = 0$. We also assume that the Killing filed $\eta$ is associated with the compact dimension and that in the asymptotic region $M_{\infty}$ the Killing fields $\zeta$ and $\eta$ take the standard form

\begin{align}
\zeta &= x_1 \partial / \partial x_2 - x_2 \partial / \partial x_1, \\
\eta &= \partial / \partial \phi.
\end{align}

The group of isometries is hence $G = \mathbb{R} \times U(1)^2$, where $\mathbb{R}$ stands for the flow of $\xi$ while $U(1)^2$ corresponds to the commuting flows of axial Killing fields.

As a part of our technical assumptions we further assume that [17]:

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a) \(<\!< M >\!\! >\) contains an acausal, spacelike, connected hypersurface \(\Sigma\) asymptotic to a \(t = \text{const}\) slice in the asymptotic region \(M_\infty\), whose closure has as its boundary \(\partial \Sigma = \bigcup_i H_i\) cross sections of the horizons.

b) The horizon cross sections are compact and the horizons are non-degenerate.

c) The orbits of the Killing field \(\xi\) are complete.

d) \(<\!< M >\!\! >\) is globally hyperbolic.

Due to the symmetries of the spacetime the natural space to work on is the orbit (factor) space \(\hat{M} = <\!< M >\!\! > / \mathcal{G}\), where \(\mathcal{G}\) is the isometry group. The structure of the factor space is described by the following theorem which is a straightforward generalization of the corresponding theorem in [17]:

**Theorem:** Let \((M, g)\) be a stationary, asymptotically Kaluza-Klein, 5-dimensional black hole spacetime with isometry group \(\mathcal{G} = \mathbb{R} \times U(1)^2\) satisfying the technical assumptions stated above. Then the orbit space \(\hat{M} = <\!< M >\!\! > / \mathcal{G}\) is a 2-dimensional manifold with boundaries and corners homeomorphic to a half-plane. Some boundary segments \(I_i \subset \partial \hat{M}\) correspond to the quotients of the horizons \(H_i = H_i / \mathcal{G}\), while the remaining segments \(I_j\) correspond to the various axes, where a linear combination \(a_\zeta(I_j) \zeta + a_\eta(I_j) \eta = 0\) and \(\mathbf{a}(I_j) = (a_\zeta(I_j), a_\eta(I_j)) \in \mathbb{Z}^2\). For adjacent intervals \(I_j\) and \(I_{j+1}\) (not including the horizons) the vectors \(\mathbf{a}(I) = (a_\zeta(I), a_\eta(I))\) are subject to the following constraint

\[
|\det \begin{pmatrix} a_\zeta(I_j) & a_\zeta(I_{j+1}) \\ a_\eta(I_j) & a_\eta(I_{j+1}) \end{pmatrix} | = 1. \tag{7}
\]

In the interior of \(\hat{M}\) there is a naturally induced metric \(\hat{g}\) which has signature ++. We denote derivative operator associated with \(\hat{g}\) by \(\hat{D}\). Let us now consider the Gramm matrix of the Killing fields \(G_{IJ} = g(K_I, K_J)\), where \(K_0 = \xi, K_1 = \zeta\) and \(K_3 = \eta\). Then the determinant \(\rho^2 = |\det G|\) defines a scalar function \(\rho\) on \(\hat{M}\) which, as well known, is harmonic, \(\hat{D}^a \hat{D}_a \rho = 0\) as a consequence of the Einstein-Maxwell field equations. It can be shown that \(\rho > 0\), \(\hat{D}_a \rho \neq 0\) in the interior of \(\hat{M}\) and that \(\rho = 0\) on \(\partial \hat{M}\). We may define a conjugate harmonic function \(z\) on \(\hat{M}\) by \(dz = \hat{\star} d\rho\), where \(\hat{\star}\) is the Hodge dual on \(\hat{M}\). The functions \(\rho\) and \(z\) define global coordinates on \(\hat{M}\) identifying the orbit space with the upper complex half-plane

\[
\hat{M} = \{z + i \rho \in \mathbb{C}, \rho \geq 0\} \tag{8}
\]

with the boundary corresponding to the real axis. The induced metric \(\hat{g}\) is given in these coordinates by

\[
\hat{g} = \Omega^2(\rho, z)(d\rho^2 + dz^2), \tag{9}
\]

\(\Omega^2\) being a conformal factor.

The above theorem allows us to introduce the notion of *interval structure*. The orbit space of the domain of outer communication by the isometry group is a half plane \(\hat{M} = \{z + i \rho, \rho > 0\}\) and its boundary \(\partial \hat{M}\) is divided into a finite number of intervals \(I_j\):
To each interval we associate its length \( l(I_j) \) and a vector \( \mathbf{a}(I_j) = (a_{\zeta}(I_j), a_{\eta}(I_j)) \in \mathbb{Z}^2 \) (subject to (7)) when the interval does not correspond to a horizon. To intervals corresponding to the orbit spaces \( \mathcal{H}_i \) of the horizons we associate zero vector \((0, 0)\). The data \( l(I_j) \) together with \( \mathbf{a}(I_j) = (a_{\zeta}(I_j), a_{\eta}(I_j)) \) are called interval structure. The vectors \( \mathbf{a}(I_j) = (a_{\zeta}(I_j), a_{\eta}(I_j)) \) corresponding to the outermost intervals \((-\infty, z_1)\) and \((z_{N+1}, +\infty)\) must be \((1, 0)\) and \((1, 0)\) since the spacetime is asymptotically Kaluza-Klein.

Furthermore, we have the following theorem about the topology of the horizons \([11, 12, 17]\):

**Theorem:** Under the assumptions made above the horizon cross sections \( \mathcal{H}_i \) must be topologically either \( S^2 \times S^1 \), \( S^3 \) or a Lens space \( L(p, q) \) \((p, q \in \mathbb{Z})\). Here \( p \) is given by \( p = \det(a_{h_i-1}, a_{h_i+1}) \) where \( a_{h_i-1} \) and \( a_{h_i+1} \) are vectors adjacent on the \( i \)-th horizon \( \mathcal{H}_i \). The topology of \( \mathcal{H}_i \) is \( S^2 \times S^1 \) for \( p = 0 \), \( S^3 \) for \( p = \pm 1 \) and \( L(p, q) \) in the other cases.

3 Dimensionally reduced Einstein-Maxwell equations, magnetic charges and magnetic flux

In the present paper we will not consider the most general 5D Einstein-Maxwell black holes. We will focus ourselves to black holes in a certain sector of 5D Einstein-Maxwell gravity which is known to be completely integrable \([20]\). The simplifying assumptions we make in addition to the general hypothesis stated above are the following:

1) We assume that the axial Killing field \( \eta \) is orthogonal to the other Killing fields, \( g(\zeta, \eta) = g(\xi, \eta) = 0 \), and that it is also hypersurface orthogonal, \( \eta \wedge d\eta = 0 \).

2) About the Maxwell 2-form \( F \) we assume that the following conditions are satisfied

\[
i_\zeta F = i_\zeta F = i_\eta \ast F = 0.
\]

Let us consider the consequences of the assumptions 1) and 2). The first consequence of 1) is that the angular momentum associated with \( \eta \) of every horizon \( \mathcal{H}_i \), defined\footnote{In the present paper the angular momenta are defined up to irrelevant numerical factor.} by

\[
J^I_\eta = \int_{\mathcal{H}_i} \ast d\eta
\]

is zero, \( J^I_\eta = 0 \). Secondly, since the Killing vector \( \eta \) is orthogonal to \( \zeta \), if at some spacetime point we have \( a_\zeta \zeta + a_\eta \eta = 0 \), then either \( (a_\zeta, a_\eta) = (0, 0) \) or \( (a_\zeta, a_\eta) = (1, 0) \), \( (a_\zeta, a_\eta) = (0, 1) \), or both axial Killing fields \( \zeta \) and \( \eta \) vanish. Thus the assumption 1) restricts the possible interval structures. However, the known exact solutions fall in these restricted
interval structures. In turn, the possible topologies of the horizons are also restricted and they are either $S^2 \times S^1$ or $S^3$. This immediately follows from the theorem about the topologies of the horizons.

Now let us consider the consequences of the assumption 2). From $i_\xi F = 0$ it follows that the electric charge of every horizon $\mathcal{H}_i$, defined by

$$q^i = \int_{\mathcal{H}_i} \star F$$  \hspace{1cm} (13)

is zero, $q^i = 0$. Furthermore, all the equations in assumption 2) show that the Maxwell field is completely characterized by the 1-form $i_\eta F$. It is easy to see that this form is closed.

Indeed we have

$$d(i_\eta F) = i_\eta dF = 0.$$  \hspace{1cm} (14)

Proceeding further we define the twist 1-form by

$$\omega = \star (\zeta \wedge \eta \wedge d\zeta) = i_\eta i_\xi \star d\zeta.$$

Using the equations of assumption 2) and the fact that the Killing fields commute, one can show that the twist 1-form is closed,

$$d\omega = 0.$$  \hspace{1cm} (15)

Both 1-forms $\omega$ and $f = i_\eta F$ are invariant under the spacetime symmetries and therefore they induce corresponding 1-forms $\hat{\omega}$ and $\hat{f}$ on the orbit space $\hat{M}$, which are still closed. Since the orbit space $\hat{M}$ is simply connected, there exist globally defined potentials $\chi$ and $\lambda$ such that $\hat{\omega} = d\chi$ and $\hat{f} = d\lambda$ on $\hat{M}$. The potential $\lambda$ and $\chi$ play important role in writing down the dimensionally reduced Einstein-Maxwell equations on the orbit space. Let $u$, $w$ and $\Gamma$ be functions on $\hat{M}$ defined by

$$e^{2u} = g(\eta, \eta), \quad e^{-u+2w} = g(\zeta, \zeta), \quad e^{-u+2w+2\Gamma} = g(\nabla \rho, \nabla \rho).$$  \hspace{1cm} (16)

Then the Einstein-Maxwell equations are equivalent to the following set of equations on the orbit space $\hat{M}$ [20] :

$$\hat{D}^a \left( \rho \Phi_1^{-1} \hat{D}_a \Phi_1 \right) = 0,$$
$$\hat{D}^a \left( \rho \Phi_2^{-1} \hat{D}_a \Phi_2 \right) = 0,$$  \hspace{1cm} (17)

together with

$$-\rho^{-1}(\hat{D}^a \rho) \hat{D}_a \Gamma = \left[ \frac{3}{8} \text{Tr} \left( \hat{D}^a \Phi_1 \hat{D}^b \Phi_1^{-1} \right) + \frac{1}{4} \text{Tr} \left( \hat{D}^a \Phi_2 \hat{D}^b \Phi_2^{-1} \right) \right] [\hat{g}_{ab} - 2(\hat{D}_a z) \hat{D}_b z]$$
$$-\rho^{-1}(\hat{D}^a \rho) \hat{D}_a \Gamma = \left[ \frac{3}{4} \text{Tr} \left( \hat{D}^a \Phi_1 \hat{D}^b \Phi_1^{-1} \right) + \frac{1}{4} \text{Tr} \left( \hat{D}^a \Phi_2 \hat{D}^b \Phi_2^{-1} \right) \right] (\hat{D}_a \rho) \hat{D}_b z,$$

where the matrix fields are defined in terms of $u, w, \lambda, \chi$ by

$$\Phi_1 = \begin{pmatrix} e^u + \frac{1}{3} \lambda^2 e^{-u} & \frac{1}{\sqrt{3}} \lambda e^{-u} \\ \frac{1}{\sqrt{3}} \lambda e^{-u} & e^{-u} \end{pmatrix},$$  \hspace{1cm} (18)
and
\[
\Phi_2 = \begin{pmatrix}
e^{2w} + 4\chi^2e^{-2w} & 2\chi e^{-2w} \\
2\chi e^{-2w} & e^{-2w}
\end{pmatrix}.
\]
\[\tag{19}\]

The first two equations state that each of the matrix fields \(\Phi_1\) and \(\Phi_2\) satisfies the equations of a 2-dimensional sigma-model. The matrix fields are real, symmetric, with determinant equal to 1 on the interior of \(\hat{M}\). The equations (17) are decoupled from the sigma-model equations and determine the function \(\Gamma\).

Before closing this section we shall introduce the magnetic charges and the magnetic flux associated with the interval structure. The magnetic charges are defined by
\[
Q[C_k] = \int_{C_k} F
\]
\[\tag{20}\]
where \(C_k, k = 1, 2, \ldots\) are all the topologically inequivalent, non-contractible, closed 2-surfaces in the domain of outer communications. The explicit construction of \(C_k\) is as follows [12]. We consider all possible curves \(\hat{\gamma}_k, k = 1, 2, \ldots\) in \(\hat{M}\) with the property that \(\hat{\gamma}_k\) starts on an interval labeled \((0, 1)\), and ends on another interval labeled \((0, 1)\), with no interval with label \((0, 1)\) in between. If we now lift \(\hat{\gamma}_k\) to a curve \(\gamma_k\) in \(<< M >>\), and act with all isometries generated by \(\eta\) on the image of this curve, then we generate a closed 2-surface \(C_k\) in \(<< M >>\), which is topologically a 2-sphere for all \(k\). We may repeat this by replacing \(\hat{\gamma}_k, k = 1, 2, \ldots\) with a set of curves each starting on an interval labeled \((1, 0)\), and ending on another interval labeled \((1, 0)\), with no interval with label \((1, 0)\) in between. If we again lift these curves to curves in \(<< M >>\), and act with all isometries generated by \(\zeta\), then we generate a set of topologically inequivalent closed 2-surfaces \(\bar{C}_l, l = 1, 2, \ldots\) in \(<< M >>\), each of which is topologically a 2-sphere. It may be seen that the set of 2-surfaces \(\{C_k, \bar{C}_l\}\) forms a basis of \(H_2(<< M >>)\).

The magnetic charges \(Q[\bar{C}_l]\) are not needed and in fact vanish, due to assumptions 1) and 2) of this section.

The magnetic flux \(\Psi^+\) is defined by
\[
\Psi^+ = \int_{C^+} F
\]
\[\tag{21}\]
where \(C^+\) is a 2-surface with the topology of disk which is constructed as follows [18]. Let us consider the rightmost interval \([z_{N-1}, z_N)\) with vector \((0, 1)\) and the semi-infinite interval \(\hat{\gamma}^+ = [z_N, +\infty)\). We lift \(\hat{\gamma}^+\) to a curve \(\gamma^+\) in \(<< M >>\), and act on it with the isometries generated by \(\eta\). Since \(\eta|_{z_N} = 0\) the generated 2-surface has disc topology. In the same way we can define the magnetic flux associated with the leftmost interval with vector \((0, 1)\). However, both fluxes are not independent and satisfy the relation \(\Psi^+ + \Psi^- = -2\pi \sum_k Q[C_k]\) (see for example [19]).

\[3\] In other words we consider the rightmost bubble.
4 Uniqueness theorem

The central result of the present paper is given in the following theorem

Uniqueness Theorem: Consider two stationary, asymptotically Kaluza-Klein, Einstein-Maxwell black hole spacetimes of dimension 5, having one time-translation Killing field and two axial Killing fields and satisfying all technical assumptions stated above. We also assume that the Killing and Maxwell fields satisfy the assumptions 1) and 2) above, implying that \(a(I_j) = (1, 0)\) or \((0, 1)\), and \(\mathcal{H}_i = S^3\) or \(S^1 \times S^2\), and \(q_i^j = 0 = J_i^j\) for the solutions. If the two solutions have the same interval structures, same horizon angular momenta \(J_i^j\), the same magnetic charges \(Q[C_1]\) for all 2-cycles \(C_i\), and same magnetic fluxes \(\Psi^+\), then they are isometric.

Remark: This uniqueness theorem obviously holds also in the case when the solutions do not possess any horizon. As an explicit example we may give the solutions describing magnetized bubbles [18].

Proof: Consider two solutions \((M, g, F)\) and \((\tilde{M}, \tilde{g}, \tilde{F})\) as in the statement of the theorem. We use the same "tilde" notation to distinguish any quantities associated with the two solutions. Since the interval structures of both solutions are the same, \(<M>\) and \(<\tilde{M}>\) can be identified as manifolds. Thus, we may assume that \(<\tilde{M}> = <M>\), and that \(\tilde{\xi} = \xi, \tilde{\zeta} = \zeta\) and \(\tilde{\eta} = \eta\). We may also assume that \(\tilde{\rho} = \rho\) and \(\tilde{z} = z\). As a consequence of these identifications, it is possible to combine the divergence identities (16) to the following Mazur identities

\[
\mathcal{D}^a (\rho \mathcal{D}_a \sigma_m) = \rho \tilde{g}^{ab} \text{Tr} (N_{ma} N_{mb})
\]

where \(m = 1, 2\) and

\[
\sigma_m = \text{Tr} (\Phi_m \Phi_m^{-1} - I), \quad N_{(m)a} = \tilde{S}_m^{-1} (\Phi_m^{-1} \mathcal{D}_a \Phi_m - \Phi_m^{-1} \mathcal{D}_a \Phi_m) S_m.
\]

Here the matrices \(S_m\) and \(\tilde{S}_m\) are defined by \(\Phi_m = S_m^T S_m\) and \(\Phi_m = \tilde{S}_m^T \tilde{S}_m\). The key and nice point about the Mazur identities (22) is that the right hand sides are nonnegative while the left hand sides are total divergences.

At this stage it is convenient to view \(\rho\) and \(z\) as cylindrical coordinates in an auxiliary space \(\mathbb{R}^3\) consisting of the points \(X = (\rho \cos \varphi, \rho \sin \varphi, z)\). It is also convenient to view \(\sigma_m\) as rotationally symmetric functions on the auxiliary space \(\mathbb{R}^3\). Then, according to the Mazur identities we have

\[
\Delta \sigma_m \geq 0, \quad \mathbb{R}^3 \setminus \{z - axis\}
\]

where \(\Delta\) is the ordinary Laplacian on \(\mathbb{R}^3\). Furthermore, the functions \(\sigma_m\) are nonnegative, \(\sigma_m \geq 0\). Indeed, we have

\[
\sigma_1 = \text{Tr} \left[ \Phi_1^{-1} \Phi_1 - I \right] = \left( e^\mu - e^{\tilde{\mu}} \right)^2 + \frac{1}{3} \left( \tilde{\lambda} - \lambda \right)^2 \geq 0
\]
and
\[ \sigma_2 = \text{Tr} \left[ \Phi_2^{-1} \Phi_2 - I \right] = \frac{(e^{2w} - e^{2\tilde{w}})^2}{e^{2w}e^{2\tilde{w}}} + 4 \frac{\dot{\chi} - \chi^2}{e^{2w}e^{2\tilde{w}}} \geq 0. \] (26)

According to the maximum principle [21, 22], if \( \sigma_m \) are globally bounded above on the entire \( \mathbb{R}^3 \) including the z-axis and infinity where they vanish, then they vanish identically. In order to show that \( \sigma_m \) are bounded we must consider the behavior of \( \sigma_m \) on (i) the horizons, (ii) on the axes of \( \xi \) and \( \eta \), (iii) near infinity and (iv) on the corners.

(i) Obviously, on the open intervals corresponding to the horizons \( \sigma_m \) are bounded. Indeed, neither \( e^u \) nor \( e^w \) vanish, since both Killing fields \( \zeta \) and \( \eta \) are non-vanishing on the open intervals corresponding to the horizons.

(ii) We first consider open intervals corresponding to \( \zeta = 0 \) and \( \eta \neq 0 \), in other words intervals with vector \( a = (1, 0) \). For such intervals \( e^{2u} = g(\eta, \eta) \neq 0 \) and \( g(\zeta, \zeta) = e^{2w-u} \rightarrow 0 \) which means that \( e^{2w} \rightarrow 0 \). Moreover, the smoothness of the solution requires \( e^{2w} = O(\rho^2) \) near the considered intervals. From the explicit forms of the functions \( \sigma_m \) it is clear that only \( \sigma_2 \) is potentially unbounded. The first term in \( \sigma_2 \) is obviously bounded. In order to show that the second term in \( \sigma_2 \) is also bounded we shall consider the behavior of the twist potential near the axis [12, 17]. From the fact that the twist 1-form \( \omega \) vanishes on any axis by definition, the twist potential \( \chi \) is constant on \( z \)-axis outside the intervals corresponding to the horizons. The difference between the constant value \( \chi_i \) of the twist potential on the \( z \)-axis left and right to a given horizon is [17]

\[ \chi_i(\rho = 0, z_{h+1}) - \chi_i(\rho = 0, z_h) = \frac{1}{(2\pi)^2} J_\zeta^h \] (27)

where \( J_\zeta^h \) is the angular momentum of the horizon. The same formula holds for the tilde solution

\[ \tilde{\chi}_i(\rho = 0, z_{h+1}) - \tilde{\chi}_i(\rho = 0, z_h) = \frac{1}{(2\pi)^2} \tilde{J}_\zeta^h. \] (28)

By assumption we have \( J_\zeta^h = \tilde{J}_\zeta^h \) which means that \( \chi_i - \chi_i = \text{const} \) on the \( z \)-axis outside the intervals corresponding to the horizons. Since \( \chi \) is defined up to a constant, we can chose this constant so that \( \tilde{\chi}_i = \chi_i \) on the \( z \)-axis (outside the intervals corresponding to the horizons). This together with the fact that \( \omega = d\chi \) also vanishes on the axes of \( \xi \) and \( \eta \), shows that \( \tilde{\chi}_i - \chi_i = O(\rho^2) \) near these axes, which in turn implies that the second term in \( \sigma_2 \) is bounded.

Let us now consider the second case when \( \eta = 0 \) and \( \zeta \neq 0 \), i.e. open intervals with vector \( a = (0, 1) \). In the case under consideration the smoothness of the solutions requires \( e^u = O(\rho) \) and \( e^{2w} = O(\rho) \) near the point where \( \eta = 0 \) and \( \zeta \neq 0 \). These behaviors guarantee that \( \sigma_2 \) is bounded. It is also clear that the first term in \( \sigma_1 \) is bounded. In order to show that the second term in \( \sigma_1 \) is bounded we shall consider the behavior of the potential \( \lambda \) near the points where \( \eta = 0 \) and \( \zeta \neq 0 \). Since \( i_\eta F \) vanishes on the axes of \( \eta \), it follows that \( \lambda \) is constant on these axes. The difference between the constant value \( \lambda_i \) on two neighbor \( \eta \)-axes connected by the curve \( \gamma_i \) is given by
\[ \lambda_{i+1} - \lambda_i = \int_{C_i} d\lambda = \frac{1}{2\pi} \int_{C_i} F = \frac{1}{2\pi} Q[C_i] \quad (29) \]

and a similar expression for the tilde solution

\[ \tilde{\lambda}_{i+1} - \tilde{\lambda}_i = \frac{1}{2\pi} \tilde{Q}[C_i]. \quad (30) \]

From these expressions and our assumption that \( Q[C_i] = \tilde{Q}[C_i] \), we conclude that \( \tilde{\lambda}_i - \lambda_i = \text{const} \) on the axes of \( \eta \). Since \( \lambda \) is defined up to a constant we can choose this constant so that \( \tilde{\lambda}_i = \lambda_i \) on the \( \eta \)-axes. This, together with the fact that \( d\lambda \) also vanishes on the \( \eta \)-axes implies that \( \tilde{\lambda} - \lambda = O(\rho^2) \) near the axes of \( \eta \) and therefore the second term in \( \sigma_1 \) is also bounded.

(iii) Let us consider the behavior of \( \sigma_m \) near infinity. In order to show that \( \sigma_2 \) is bounded near infinity, we use that both metrics are asymptotically Kaluza-Klein and have the same asymptotic angular momenta, \( \tilde{J}_\zeta = \sum_i \tilde{J}_i = J_\zeta = \sum_i J_i \). The fact that \( J_\zeta = \sum_i J_i \) (and the same expression for the tilde solution) can be proven by using condition 2) and by applying Gauss theorem to the definition formula of the asymptotic angular momentum

\[ J_\zeta = \int_{S^2 \times S^1} \star d\zeta \quad (31) \]

where the integration is performed over a surface at infinity. As a consequence of \( \tilde{J}_\zeta = J_\zeta \), one can show that [17]

\[ e^{2\tilde{w}} - e^{2w} = O(r^{-1}), \quad \tilde{\chi} - \chi = O(r^{-1}). \quad (32) \]

Hence we find that \( \sigma_2 |_{\infty} = 0 \).

One can show that \( \lambda \) has the following asymptotic behavior

\[ \lambda = \lambda_\infty + O(r^{-1}) \quad (33) \]

where \( \lambda_\infty \) is a constant. Now taking into account this asymptotic and the asymptotic of \( e^u \), namely \( e^u \to 1 \), for \( \sigma_1 \) we find

\[ \sigma_1 |_{\infty} = \frac{1}{3} (\tilde{\lambda}_\infty - \lambda_\infty)^2. \quad (34) \]

From the fact that \( \tilde{\lambda} = \lambda \) on the axes of \( \eta \) we can not conclude that \( \tilde{\lambda}_\infty = \lambda_\infty \) since the spacetime is asymptotically Kaluza-Klein and no axis of \( \eta \) reaches infinity. At this stage namely we must use our assumption that both solutions have the same magnetic flux. Calculating the magnetic flux we find
\[ \Psi^+ = \int_{C^+} F = 2\pi \int_{z_N}^{\infty} d\lambda = 2\pi (\lambda_\infty - \lambda(\rho = 0, z_N)) \]  
(35)

and a similar expression for the tilde solution:

\[ \tilde{\Psi}^+ = 2\pi \left( \tilde{\lambda}_\infty - \lambda(\rho = 0, z_N) \right). \]  
(36)

Here \( z_N \) is the right boundary of the rightmost axis of \( \eta \). By assumption \( \tilde{\Psi}^+ = \Psi^+ \) which means that \( \tilde{\lambda}_\infty = \lambda_\infty \). Therefore we find that \( \sigma_1 |_{\infty} = 0 \).

(iv) We must also consider the behavior of \( \sigma_m \) at the corners. The continuity argument shows that \( \sigma_m \) are bounded on the corners.

Summarizing, we have shown that the functions \( \sigma_m \), \( m = 1, 2 \) are bounded above on the entire \( \mathbb{R}^3 \) including the \( z \)-axes and infinity, where \( \sigma_m \) vanish. Therefore, by the maximum principle \([21, 22]\), \( \sigma_m \) vanish identically. Consequently, it immediately follows that \( \tilde{u} = u \), \( \tilde{w} = w \), \( \tilde{\chi} = \chi \), \( \tilde{\lambda} = \lambda \) and \( \tilde{\Gamma} = \Gamma \). From these equalities, as in the asymptotically flat case \([12]\), one can show that \( \tilde{g} = g \) and \( \tilde{F} = F \). This completes the proof.

**Remark:** We can consider the other case when the Killing field \( \zeta \) is hypersurface orthogonal, \( \zeta \wedge d\zeta = 0 \), and when the electromagnetic field is along the noncompact direction \( \zeta \), i.e. when the Maxwell 2-form \( F \) satisfies the conditions \( i_\zeta F = i_\eta F = i_\zeta \ast F = 0 \). In this case one can show that the black hole configurations are fully determined only in terms of the interval structure, the angular momenta \( J^i_\eta \) of the horizons and the magnetic charges \( Q[\tilde{C}_k] \).

## 5 Discussion

Let us discuss some generalizations of our result. The proven uniqueness theorem can be extended to the 5D Einstein-Maxwell-dilaton gravity which can be derived from the Lagrangian

\[ L = \ast R - 2d\phi \wedge \ast d\phi - \frac{1}{2} e^{-2\alpha \phi} F \wedge \ast F \]  
(37)

where \( \phi \) is the dilaton field and \( \alpha \) is the dilaton coupling parameter.

The \( \sigma \)-model presentation of the dimensionally reduced 5D Einstein-Maxwell-dilaton equations with the restrictions 1) and 2), was given in \([23]\). On this base and applying the mathematical technique of the present work, one can prove the following uniqueness theorem

**Uniqueness Theorem:** Consider two stationary, asymptotically Kaluza-Klein, Einstein-Maxwell-dilaton black hole spacetimes of dimension 5, having one time-translation Killing field and two axial Killing fields and satisfying all technical assumptions stated above. We

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\(^4\)In the expression of \( \Psi^+ \) we have taken into account that \( \tilde{\lambda}(\rho = 0, z_N) = \lambda(\rho = 0, z_N) \).

\(^5\)The angular momenta \( J^i_\zeta \) are zero.
also assume that the Killing and Maxwell fields satisfy the assumptions 1) and 2) above, implying that $\mathbf{a}(I) = (1,0)$ or $(0,1)$, and $\mathcal{H}_i = S^3$ or $S^1 \times S^2$, and $q^i = 0 = J^i_\eta$ for the solutions. If the two solutions have the same interval structures, same horizon angular momenta $J^i_\xi$, the same magnetic charges $Q[C_j]$ for all 2-cycles $C_i$, same value of the dilaton field at infinity $\phi_\infty$ and same magnetic fluxes $\Psi^+$, then they are isometric.

As a part of the technical assumptions in this theorem we obviously assume that the dilaton field is invariant under the spacetime symmetries, $\mathcal{L}_\xi \phi = \mathcal{L}_\zeta \phi = \mathcal{L}_\eta \phi = 0$.

The restrictions 1) and 2) play very important role in the present uniqueness theorem as well as in the uniqueness theorem for the asymptotically flat case [12]. Due to these restrictions the target space of the potentials is a symmetric space which ensures the complete integrability of the considered sector and the existence of Mazur identities which are a key moment in the proof. The natural step in generalizing the uniqueness theorems for Kaluza-Klein and asymptotically flat black holes is to remove the restrictions 1) and 2), in other words to consider the general case when the electromagnetic field is completely excited. In the general case, however, it seems that the dimensionally reduced 5D Einstein-Maxwell(-dilaton) gravity equations do not possess large enough group of symmetries which could ensure complete integrability and the existence of Mazur identities. So, in the general case the mathematical technique used for proving the uniqueness theorem in the present paper and in [12], is not applicable. New technique must be used in order to prove the uniqueness theorems in the general case [24].

Contrary to the 5D Einstein-Maxwell(-dilaton) gravity, the 5D minimal supergravity is completely integrable in the general case when the electromagnetic field is fully excited [25] (see also [26] and [27]). This fact shows that the uniqueness theorem of the present paper can be easily generalized within the framework of the 5D minimal supergravity, of course with the corresponding technical complications and extensions. We hope to give the formal mathematical results in a future work.

Finally, we would like to comment on following. Do the collection of the interval structure, local and asymptotic charges (and the magnetic fluxes) and angular momenta always fully determine the black hole solutions? Fortunately or unfortunately the answer seems to be "NO". As our preliminary numerical calculations show there could exist many (even infinitely many) black hole solutions with the same interval structure, angular momenta and local and asymptotic charges. Such a behavior is observed in some dilaton gravity models coupled to the electromagnetic field with an appropriate dilaton coupling function [28].

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