In this paper, we address the inverse scattering problem for a class of signals that have a compactly supported reflection coefficient. The problem boils down to the solution of the Gelfand-Levitan-Marchenko (GLM) integral equations with a kernel that is bandlimited. By adopting a sampling theory approach to the associated Hankel operators in the Bernstein spaces, a constructive proof of existence of a solution of the GLM equations is obtained under various restrictions on the nonlinear impulse response (NIR). The formalism developed in this article also lends itself well to numerical computations yielding algorithms that are shown to have algebraic rates of convergence. In particular, the use Whittaker-Kotelnikov-Shannon sampling series yields an algorithm that converges as $O(N^{-1/2})$ whereas the use of Helms and Thomas (HT) version of the sampling expansion yields an algorithm that converges as $O(N^{-m/2})$ for any $m > 0$ provided the regularity conditions are fulfilled. The complexity of the algorithms depend on the linear solver used. The use of conjugate-gradient (CG) method yields an algorithm of complexity $O(N_{\text{iter}}N^2)$ per sample of the signal where $N$ is the number of sampling basis functions used and $N_{\text{iter}}$ is the number of CG iterations involved. The HT version of the sampling expansions facilitates the development of algorithms of complexity $O(N_{\text{iter}}N\log N)$ (per sample of the signal) by exploiting the special structure as well as the (approximate) sparsity of the matrices involved. The algorithms are numerically validated using Schwartz class functions as NIRs that are either bandlimited or effectively bandlimited. The results suggest that the HT variant of our algorithm is spectrally convergent for an input of the aforementioned class.

I. INTRODUCTION

In this paper, we address the inverse scattering problem for a class of signals such that the continuous part of their nonlinear Fourier spectrum has a compact support and the discrete part is empty. Such signals are called nonlinearly bandlimited in analogy with bandlimited signals in conventional Fourier analysis and they are entirely radiative in nature by definition. Such signals lend themselves well to the design of a fast inverse nonlinear Fourier transform algorithm [1, 2] in the differential approach of inverse scattering. In practical applications, such signals are often used as a convenient approximation of signals that have an effectively localized continuous spectrum. This problem, for instance in the Hermitian class, arises in the design of nonuniform fiber Bragg gratings to compensate for second and third order dispersion in optical fibers [3, 4]. The target reflection coefficient in these problems is a compactly supported chirped profile. In the non-Hermitian class, the design of grating-assisted co-directional couplers, a device used to couple light between two different guided modes of an optical fiber (see [5, 6] and references therein) requires the solution of a similar problem. Such signals have also attracted interest in optical communication where it is proposed to encode information in the continuous part of nonlinear Fourier spectrum in an attempt to mitigate nonlinear signal distortions at higher power levels [7].

In all of the applications mentioned above, accuracy of the numerical algorithms form a bottleneck either at higher powers in the non-Hermitian class or at reflectivities approaching unity in the Hermitian class. There is a vast amount literature on numerical methods for the solution of the Gelfand-Levitan-Marchenko (GLM) integral equations notable among them are the integral layer-peeling [8], Töplitz inner-bordering [9–11] and the Nyström method [12]. From a practical viewpoint, these algorithms work for a large class of problems; however, these methods cannot provide accuracies up to the machine precision with the exception of the method due to Trogdon and Olver [13]. This method relies on the formulation of the inverse scattering problem as a Riemann-Hilbert problem and it has been demonstrated to be spectrally convergent. Its domain of application is not limited to the class of signals considered in this article; however, the complexity of this algorithm remains high at the same time it is somewhat complicated to implement.

The inverse scattering problem is generally formulated on an unbounded domain which poses a serious problem for the underlying quadrature schemes in the Nyström method or for the overlap integrals in the degenerate Kernel method (see Atkinson [14] for an introduction to these methods). In this paper, following Khare and George [15] (see also Vishal Vaibhav [16]), we propose a sampling theory based approach to the discretization of the GLM equations which has the advantage that the basis functions are naturally adapted to unbounded domains. The bandlimited nature of the functions facilitate accurate quadrature on unbounded domains [17]. It is important to emphasize that the method thus obtained requires sampling of the impulse response on an equispaced grid which has some clear advantages in preserving the inherent symmetries of the system.

In the sampling approach presented in this paper, we use the classical Whittaker-Kotelnikov-Shannon sampling series and the Helms and Thomas (HT) version [18, 19] of the sampling expansion. The associated basis functions provide a natural framework for the representation of the Hankel operators involved which makes the theoretical analysis related to existence of solution or issues of convergence somewhat easier. Further, the Bernstein spaces [17] provide a convenient setting for rigorous analysis of the GLM equations.

The algorithms presented in this article are shown to have algebraic orders of convergence. In particular, the use the...
HT version of the sampling expansion affords an accuracy of $O(N^{-m-1/2})$ (provided certain regularity conditions are fulfilled) where $N$ is the number of basis function used and $m > 0$ is a parameter that can be chosen arbitrarily. The complexity of these algorithms depends on the linear solver used. In order to compute one sample of the signal with a direct solver, the algorithm would require $O(N^3)$ operations whereas an iterative solver based on the conjugate-gradient method yields the same result in $O(N_{iter}N^2)$ operations. It must be emphasized that at any step, good seed solutions are readily available (from the previous step) when the signal is being computed on a sufficiently fine grid so that quantity $N_{iter}$ does not become prohibitively large. Further, the HT version of the sampling series leads to a dramatic decrease in complexity within the iterative approach if one takes into account the special structure as well as the (approximate) sparsity of the matrices involved. The sparsity structure can be controlled by introducing a tolerance $\epsilon > 0$ which introduces an error of $O(N\epsilon)$ while reducing the complexity to $O(N_{iter}\epsilon N \log N)$ per sample of the signal.

The rest of the paper is organized as follows: Sec. II discusses the GLM equations in the functional spaces introduced in Sec. II A. The exposition is organized such that the properties of the Hankel operators are studied in Sec. II B which is then used to discuss the GLM equation in Sec. II C. Sec. II D discusses the application of the HT version of the sampling expansion. Sec. III deals with the numerical and algorithmic aspects of the ideas developed in the preceding section. Sec

II. GELFAND-LEVITAN-MARCHENKO EQUATIONS WITH BANDLIMITED KERNELS

The coupled Gelfand-Levitan-Marchenko (GLM) integral equations arise in connection with the inverse scattering problem for the Hermitian as well as the non-Hermitian $2 \times 2$ Zakharov-Shabat scattering problem [20–22]. As stated earlier, we consider a class of signals such that its reflection coefficient $\rho(\xi)$ ($\xi \in \mathbb{R}$) has a compact support, say, in $[-\sigma, \sigma]$ where $\sigma$ is referred to as the bandlimiting parameter. The nonlinear impulse response (NIR), defined by

$$p(\tau) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} \rho(\xi)e^{i\xi \tau}d\xi,$$  

is, evidently, a bandlimited function.

Let $q(t)$ denote the inverse nonlinear Fourier transform (NFT) of $\rho(\xi)$. The GLM equations corresponding to $p(\tau)$ can be stated as

$$\kappa A_2^\prime(\tau, t) = \int_{\tau}^{\infty} A_1(\tau, s)p(s + t)ds,$$  

$$A_1^\prime(\tau, t) = p(\tau + t) + \int_{\tau}^{\infty} A_2(\tau, s)p(s + t)ds,$$  

where $\kappa = +1$ (Hermitian scattering problem) and $\kappa = -1$ (non-Hermitian scattering problem). The scattering potential is recovered from

$$q(\tau) = -2A_1(\tau, \tau),$$  

together with the estimate

$$\int_{\tau}^{\infty} |q(s)|^2 ds = 2\kappa A_2(\tau, \tau).$$  

Let us enumerate two interesting properties that will be useful later:

- **Shift in time domain:** If $q(t)$ is the inverse NFT of $\rho(\xi)$, then the inverse NFT of $\rho(\xi)e^{2i\xi t_0}$ is $q(t + t_0)$.
- **Scaling in frequency domain:** If the inverse NFT of $\rho(\xi)$ is $q(t)$, then the inverse NFT of $\rho(A\xi)$ is $A^{-1}q(t/\lambda)$.

The shifting property allows us to fix $\tau = t_0$ in (2) and simply keep varying the variable $t_0$ to obtain the scattering potential over the entire real line. Therefore, we may set $\tau = 0$ in (2) without the loss of generality and focus on the following form of the GLM equations:

$$\kappa A_2^\prime(\tau, t) = \int_{\tau}^{\infty} p(s + t)A_1(s)ds,$$

$$A_1^\prime(\tau, t) = p(t) + \int_{\tau}^{\infty} p(s + t)A_2(s)ds,$$  

A. Preliminaries

The set of real numbers (integers) is denoted by $\mathbb{R}$ ($\mathbb{Z}$) and the set of non-zero positive real numbers (integers) by $\mathbb{R}^+$ ($\mathbb{Z}^+$). The set of complex numbers are denoted by $\mathbb{C}$, and, for $\xi \in \mathbb{C}$, $\Re(\xi)$ and $\Im(\xi)$ refer to the real and the imaginary parts of $\xi$, respectively. The complex conjugate of $\xi \in \mathbb{C}$ is denoted by $\xi^*$ and $\mathbb{R}$ denotes its square root with a positive real part. The upper-half (lower-half) of $\mathbb{C}$ is denoted by $\mathbb{C}_+$ ($\mathbb{C}_-$) and it closure by $\overline{\mathbb{C}_+}$ ($\overline{\mathbb{C}_-}$).

The Fourier transform of a function $f(t)$ is defined as

$$F(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}} f(t)e^{-i\xi t}dt.$$  

The characteristic function of a set $\Omega \subset \mathbb{R}$ is denoted by

$$\chi_{\Omega} = \begin{cases} 1, & x \in \Omega, \\ 0, & \text{otherwise}. \end{cases}$$  

The Lebesgue spaces over the domain $\Omega \subset \mathbb{R}$ are denoted by $L^p(\Omega)$ ($1 \leq p \leq \infty$) and corresponding norm by $\| \cdot \|_{L^p(\Omega)}$. If the domain is not mentioned, it assumed to be $\mathbb{R}$ unless otherwise stated.

For a rigorous analysis of the GLM equations with bandlimited kernels, it is convenient to work with the Bernstein spaces [17, Chap. 2] (also see [23, Chap. 3]). $B_{0}^\nu$ with $1 \leq \nu \leq \infty$, defined as the class of entire functions of exponential type-$\sigma$ whose restriction to the $\mathbb{R}$ belong to $L^\nu$. Further, these spaces satisfy the following embedding property: $B_{0}^\nu \subset \subset B_{0}^{\nu'} \subset B_{0}^\nu$ where $1 \leq \nu \leq \nu' \leq \infty$. For $f \in B_{0}^\nu$ and any $h > 0$, the following inequality holds

$$\|f\|_{L^\nu} \leq \sup_{n \in \mathbb{Z}} \left( \sum_{n \in \mathbb{Z}} |f(t - nh)|^\nu \right)^{1/\nu} \leq (1 + \sigma h)\|f\|_{L^\nu}. \quad (7)$$
This inequality proves extremely useful in establishing certain bounds and it appears mostly with the parameter \( h = \pi/\sigma \), the grid spacing for Nyquist sampling of \( \sigma \)-bandlimited functions. Further, we recall from Boas [24, Thm. 6.7.1], if \( f \in \mathcal{B}_{\sigma}^\nu \) \((1 \leq \nu < \infty)\), then
\[
\int_{-\infty}^{\infty} |p(x + iy)|^2 \, dx \leq e^{\|p\|^2_2} \int_{-\infty}^{\infty} |p(x)|^2 \, dx, \tag{8}
\]
and \( \lim_{|t| \to \infty} f(t) = 0 \).

Next let us define the Hardy classes \( \mathcal{H}^2 \) which are a class of functions analytic in upper (lower) half of the complex plane such that the expressions (which qualify as norms)
\[
\|f\|_{\mathcal{H}^2} = \sup_{p \in \mathbb{R}} \left( \int_{p} \int_{p} f(x + iy)^2 \, dx \right)^{1/2},
\]
\[
\|f\|_{\mathcal{H}^2} = \sup_{p \in \mathbb{R}} \left( \int_{p} \int_{p} f(x + iy)^2 \, dx \right)^{1/2},
\]
are bounded, respectively. The Paley-Wiener theorem allows one to characterize these spaces solely in terms of their boundary functions as follows:
\[
\mathcal{H}^2 = \{ f \in L^2 \mid \mathcal{F}^{-1}[f]\} = 0, \tag{10}
\]
\[
\mathcal{H}^2 = \{ f \in L^2 \mid \mathcal{F}^{-1}[f]\} = 0, \tag{11}
\]
so that \( L^2 = \mathcal{H}^2 \oplus \mathcal{H}^2 \). For \( f \in L^2 \), the decomposition into \( \mathcal{H}^2 \) reads as
\[
f^{(1)} = \left( \mathcal{F} \circ \chi_{\Omega_+} \circ \mathcal{F}^{-1} \right) f = \frac{1}{2} \left( f + i \mathcal{H}[f] \right), \tag{12}
\]
\[
f^{(2)} = \left( \mathcal{F} \circ \chi_{\Omega_-} \circ \mathcal{F}^{-1} \right) f = \frac{1}{2} \left( f - i \mathcal{H}[f] \right), \tag{13}
\]
respectively.

**Lemma II.1.** If \( p \in \mathcal{B}_{\sigma}^\nu \), then \( \rho \in L^1 \cap L^2 \) with support in \([-\sigma, \sigma]\).

**Proof.** If \( p \in \mathcal{B}_{\sigma}^\nu \), then \( \rho \in L^1 \) with support in \([-\sigma, \sigma]\). Then, using Cauchy-Schwartz inequality, we have
\[
\left( \int_{-\sigma}^{\sigma} |p(x)|^2 \, dx \right)^{1/2} \leq \sqrt{2\sigma} \int_{-\sigma}^{\sigma} |p(x)|^2 \, dx \leq \sqrt{2\sigma} \|p\|_{L^2}. \tag{14}
\]
\[
\left( \int_{-\sigma}^{\sigma} |p(x)|^2 \, dx \right)^{1/2} \leq \sqrt{2\sigma} \int_{-\sigma}^{\sigma} |p(x)|^2 \, dx \leq \sqrt{2\sigma} \|p\|_{L^2}. \tag{15}
\]

The functions in \( \mathcal{B}_{\sigma}^\nu \) can be regarded as Fourier-Laplace transforms of certain class of distributions supported in \([-\sigma, \sigma]\). If \( \rho(x) \) is a function of bounded variation on \([-\sigma, \sigma]\), denoted by \( \mathcal{BV}(-\sigma, \sigma) \), such that \( \rho(-\sigma + 0) = \rho(\sigma - 0) \), then \( \rho(z) \) satisfies the following estimate
\[
|\rho(z)| \leq \frac{C}{1 + |z|^2} e^{\sigma |\text{Im}(z)|}, \quad z \in \mathbb{C}, \tag{16}
\]
for some \( C > 0 \). Such function belong \( B^2 \) but not \( B^2 \). Further, if \( \rho \in C^0_{\nu}(\mathbb{R}) \) with support in \([-\sigma, \sigma]\), then, there exists a \( C_{\nu} > 0 \) such that \( [25] \)
\[
|\rho(z)| \leq \frac{C}{1 + |z|^2} e^{\sigma |\text{Im}(z)|}, \quad z \in \mathbb{C}. \tag{17}
\]

**B. Hankel Operators with Bandlimited Kernels**

Let \( \Omega_+ = [0, \infty) \) and define the Hankel operator
\[
\mathcal{P}[g](t) = \int_{\Omega_+} p(t + s)g(s) ds, \quad t \in \Omega_+. \tag{18}
\]

The field underlying the image of \( \mathcal{P} \) can be extended to the entire complex plane. Let \( \Omega_- = (-\infty, 0) \). For convenience, we may also work with the form \( \mathcal{P}[g](t) = \mathcal{P}[g](-t) \) so that, for \( g \) supported in \( \Omega_+ \),
\[
\mathcal{P}[g](t) = \int_{\Omega_+} \tilde{p}(t - s)g(s) ds = (\tilde{p} \ast g)(t), \quad t \in \Omega_+, \tag{19}
\]
where \( \tilde{p}(t) = p(-t) \) and "\ast" denotes convolution. In the Fourier domain, the Hankel operator \( \mathcal{P} \) can be expressed as
\[
\mathcal{H}_{\rho} = \mathcal{F} \circ \chi_{\Omega_+} \circ \mathcal{F}^{-1} \tag{20}
\]
so that
\[
\mathcal{H}_{\rho}[G](\xi) = \left( \mathcal{F} \circ \chi_{\Omega_+} \circ \mathcal{F}^{-1} \right) [\rho G](\xi), \quad \xi \in \mathbb{R}, \tag{21}
\]
where \( G(\xi) = \mathcal{F}[g](\xi) \) with \( g \) supported in \( \Omega_+ \).

**Proposition II.2 (Boundness of Hankel operators).** Define \( I_\nu = \|p\|_{\mathcal{H}^2} \).

(a) If \( p \in L^1 \), then (15) defines a bounded linear operator \( \mathcal{P} : L^1(\Omega_+) \to L^1(\Omega_+) \) for \( \nu = 1, 2 \).

(b) If \( p \in \mathcal{B}_{\sigma}^\nu \), then (15) defines a bounded linear operator \( \mathcal{P} : L^1(\Omega_+) \to \mathcal{B}_{\sigma}^\nu \) for \( \nu = 1, 2 \).

(c) If \( p \in \mathcal{B}_{\sigma}^\nu \) with \( \rho \in L^1 \), then (15) and (18) define bounded linear operators \( \mathcal{P} : L^2(\Omega_+) \to \mathcal{B}_{\sigma}^\nu \) and \( \mathcal{H}_{\rho} : \mathcal{H}^2 \to \mathcal{H}^2 \), respectively.

**Proof.** In order to prove (a), consider \( g \in L^1(\Omega_+) \), then
\[
\|\mathcal{P}[g]\|_{L^1(\Omega_+)} \leq \int_0^\infty \left( \int_0^\infty |p(y + s)g(s)| \, ds \right) \, dy,
\]
\[
\leq \int_0^\infty \left( \int_0^\infty |p(y + s)| g(s) \, ds \right) \, dy,
\]
\[
\leq I_1 \int_0^\infty |g(s)| \, ds = I_1 \|g\|_{L^1(\Omega_+)}.
\]

Next, let \( g \in L^2(\Omega_+) \), then
\[
\|\mathcal{P}[g]\|_{L^2(\Omega_+)} \leq \int_0^\infty \left( \int_0^\infty |p(y + s)g(s)| \, ds \right)^2 \, dy,
\]
\[
\leq \int_0^\infty \int_0^\infty |p(y + s)| |g(s)| \, ds \, \int_0^\infty |p(y + s)| |g(s)| \, ds \, dy,
\]
\[
\leq I_1 \int_0^\infty \left( \int_0^\infty |p(y + s)| \, ds \right)^2 \, dy \leq I_1^2 \|g\|_{L^2(\Omega_+)}.
\]
Therefore, we have
\[
\|\mathcal{P}[g]\|_{L^2(\Omega_+)} \leq I_1 \|g\|_{L^2(\Omega_+)}, \quad \nu = 1, 2. \tag{22}
\]
This completes the proof of statement (a). Note that it is also possible to show that
\[
\|P[g]\|_{L^r} \leq \|p\|_{L^r} \|g\|_{L^r, \Omega}, \quad \nu = 1, 2. \tag{20}
\]

To prove (b), let \( z = x + iy \in \mathbb{C} \); then, the analyticity property of \( P[g](z) \) follows from the analyticity of \( p(z) \) and the Bernstein’s inequality [23, Chap. 3] which ensures that its derivative \( P' \in B_{1}^{1} \). The boundedness of \( \mathcal{P} : L^r(\Omega_{+}) \to L^r \) follows from (20). What remains to show is that \( P[g](z) \) is of exponential type-\( \sigma \). Observing
\[
|P[g](z)| \leq C e^{\nu|\nu|} \int_{0}^{\infty} |g(s)|ds, \tag{21}
\]
the result follows for \( \nu = 1 \). For \( \nu = 2 \), we have
\[
|P[g](z)|^2 \leq \int_{0}^{\infty} \int_{0}^{\infty} p(z + s)g(s)ds \tag{22}
\]
\[
\leq \int_{0}^{\infty} |p(z + s)|^2 ds \int_{0}^{\infty} |g(s)|^2 ds.
\]
From Boas [24, Thm. 6.7.1] and using the fact that \( B_{1}^{1} \subset B_{2}^{2} \), we have
\[
\int_{0}^{\infty} |p(x + s)|^2 ds \leq e^{2\nu|\nu|} \int_{0}^{\infty} |p(x + s)|^2 ds,
\]
which yields the estimate
\[
|P[g](z)| \leq C e^{\nu|\nu|} \|g\|_{L^r, \Omega}, \tag{23}
\]
for some \( C > 0 \). Finally, for any \( g \in L^1 \) and \( h > 0 \),
\[
\sup_{n \in \mathbb{N}} \left( \sum_{n \in \mathbb{N}} |P[g](t - nh)| \right) \leq \sup_{n \in \mathbb{N}} \left( \sum_{n \in \mathbb{N}} |p(t - nh)| \right) \int_{0}^{\infty} |g(s)|ds \leq (1 + \pi h) \|p\|_{L^r} \|g\|_{L^r, \Omega}. \tag{24}
\]

To prove (c), first consider \( H_\nu \). It is an elementary exercise to verify that, for any \( G \in H_2^r \),
\[
\|H_\nu[G]\|_{L^2} \leq \|p\|_{L^r} \|G\|_{L^2}. \tag{25}
\]
Turning to \( \mathcal{P} \), if \( p \in B_{1}^{1} \), then it is straightforward to see that \( \mathcal{P}[g] \in B_{2}^{2} \) for any \( g \in L^2 \) with support in \( \Omega_{+} \). Using Plancherel’s theorem, we have
\[
\|\mathcal{P}[g]\|_{L^2} \leq (2\pi)^{-1/2} \|pG\|_{L^2} \leq \|p\|_{L^r} \|G\|_{L^r, \Omega}. \tag{26}
\]

In the following, we would like to develop a representation of the Hankel operators with bandlimited kernels by exploiting the sampling expansion of such functions. The translates of the sinc function form an orthonormal basis in \( B_{2}^{2} \). Let us introduced the normalized form of these basis function as
\[
\psi_n(t) = \frac{\sqrt{\pi}}{\pi} \sin[\sigma(t - t_n)] = \frac{\sqrt{\pi}}{\pi} \sin(\sigma t - n\pi), \tag{27}
\]
so that the orthonormality condition can be stated as
\[
\int_{-\infty}^{\infty} \psi_m(t)\psi_n(t)dt = \delta_{mn}. \tag{28}
\]
For \( p \in B_{1}^{1} \), we can write
\[
p(t + s) = \sum_{n \in \mathbb{Z}} \sqrt{\frac{\pi}{\sigma}} \left( 1 + \frac{n\pi}{\sigma} \right) \psi_n(s). \tag{29}
\]
This series converges absolutely and uniformly with respect to \( s \in \mathbb{R} \) where we have fixed \( t \in \mathbb{R} \). Using this representation in (15), we have
\[
\mathcal{P}[g](t) = \sum_{n \in \mathbb{Z}} \sqrt{\frac{\pi}{\sigma}} p \left( 1 + \frac{n\pi}{\sigma} \right) \int_{0}^{\infty} g(s)\psi_n(s)ds \tag{30}
\]
\[
eq \sum_{n \in \mathbb{Z}} \sqrt{\frac{\pi}{\sigma}} p \left( 1 + \frac{n\pi}{\sigma} \right) \hat{g}_n. \tag{31}
\]
In view of the expression above, we introduce the Hankel operator
\[
\mathcal{F}[g](y) = \int_{0}^{\infty} \psi_0(y + s)g(s)ds = \hat{g}(y), \tag{32}
\]
then \( \hat{g}_n = \hat{g}(-n\pi/\sigma) \). Note that \( \psi_0 \in B_{1}^{1} \) with its Fourier transform \( \sqrt{\pi/\sigma} \chi_{[-\pi, \pi]} \in L^\infty \), therefore, it is a bounded linear operator from \( L^2(\Omega_+) \) to \( B_{1}^{1} \) (see Lemma II.2). Also, we have
\[
\|\mathcal{F}\|_{L^2} \leq \sqrt{\pi/\sigma} \|\chi_{[-\pi, \pi]}\|_{L^2} = \sqrt{\pi/\sigma}. \tag{33}
\]
In the following, we assume that \( g \in B_{2}^{2} \). Let us show that the series on the right hand side of (28) converges absolutely and uniformly for \( t \in \mathbb{R} \). Observing that \( \|\hat{g}_n\| \leq \|g\|_{L^2} \), we have
\[
\sum_{n \in \mathbb{Z}} \sqrt{\frac{\pi}{\sigma}} |p \left( 1 + \frac{n\pi}{\sigma} \right) \hat{g}_n| \leq \|g\|_{L^2} \sum_{n \in \mathbb{Z}} \sqrt{\frac{\pi}{\sigma}} |p \left( 1 + \frac{n\pi}{\sigma} \right)| \tag{34}
\]
\[
\leq (1 + \pi) \|g\|_{L^2} \|p\|_{L^r}, \tag{35}
\]
which follows from (7) and the fact that \( p \in B_{1}^{1} \). Further, the truncated series
\[
\mathcal{P}_N[g](t) = \sum_{|n| \leq N} \sqrt{\frac{\pi}{\sigma}} p \left( 1 + \frac{n\pi}{\sigma} \right) \hat{g}_n, \quad N \in \mathbb{R}^{+}, \tag{36}
\]
converges in \( L^r \)-norm (\( \nu = 1, 2 \)). To see this, consider
\[
\|\mathcal{P}_N[g] - \mathcal{P}_{N-1}[g]\|_{L^r} \leq \sqrt{\pi/\sigma} \|p\|_{L^r} (|\hat{g}_{-N}| + |\hat{g}_{-N}|). \tag{37}
\]
On account of \( \hat{g} \in B_{1}^{1} \),
\[
\lim_{N \to \infty} \|\hat{g}_{-N}\| = 0. \tag{38}
\]
In particular, we have \( |\hat{g}_{-N}| \leq (1/\pi)|g|_{L^r}N^{-1/2} \). The expression on right hand side in (33) goes to 0 as \( N \to \infty \); therefore, \( \mathcal{P}_N[g] \) is a Cauchy sequence in \( L^r \) and the limit belongs to \( B_{1}^{1} \).
The sampling series in (27) also holds for \( p \in \mathbb{B}_2^\alpha \). In order to prove the absolute and uniform convergence of the series in (28) with respect to \( t \in \mathbb{R} \), we first prove that \( (g_n)_{n \geq 0} \) belongs to \( \ell^2 \). On account of \( \hat{g} = g_n \in \mathbb{B}_2^\alpha \), we have
\[
\|\hat{g}\|_{\ell^2} = \left( \sum_{n \geq 1} |g_n|^2 \right)^{1/2} \leq (1 + \pi)|\|\hat{g}\|_{L^2} \leq (1 + \sqrt{\pi/\alpha})\|g\|_{L^2},
\]
which follows from (7) and (30) so that
\[
\sqrt{\frac{\pi}{\alpha}} \left( \sum_{n \geq 1} \left| p \left( t + \frac{n\pi}{\alpha} \right) \hat{g}_n \right|^2 \right)^{1/2} \leq \sqrt{\frac{\pi}{\alpha}} \left( \sum_{n \geq 1} \left| p \left( t + \frac{n\pi}{\alpha} \right) \right|^2 \right)^{1/2} \|\hat{g}\|_{\ell^2}
\leq (1 + \pi)^2 (\pi/\sigma)\|g\|_{L^2} \|p\|_{L^2},
\]
The convergence of \( \mathcal{P}_N[g] \) in \( L^2(\Omega, \sigma) \) follows the same line of reasoning as that of the previous case. The discussion so far can be summarized in the following proposition:

**Proposition II.3.** For \( p \in \mathbb{B}_2^\alpha \), the partial sums defined in (32) converge absolutely and uniformly (with respect to \( t \in \mathbb{R} \)) for every \( g \in \mathbb{B}_2^\alpha \) as \( N \to \infty \). Moreover, the partial sums also converge in the \( L^2 \)-norm. If \( p \in \mathbb{B}_1^\alpha \), then the partial sums converge in the \( L^1 \)-norm.

Next, we would like to address the problem of estimating the truncation error which is given by
\[
T_N(t) = \sum_{|n| \leq N} \sqrt{\frac{\pi}{\alpha}} p \left( t + \frac{n\pi}{\alpha} \right) \hat{g}_n, \quad N \in \mathbb{R}^+, \quad (34)
\]
under a stronger decay condition on \( p \).

**Proposition II.4.** For \( p \) satisfying an estimate of the form
\[
|p(z)| \leq \frac{C}{(1 + |z|^{k+1}) e^{\sigma |d_t(z)|}}, \quad z \in \mathbb{C},
\]
where \( k \geq 1 \) and \( g \) satisfying a similar estimate with the index \( k' \geq 1 \), the truncation error \( T_N(t) \) of the partial sums in (32) satisfies the estimate
\[
|T_N(t)| \leq \frac{2(\sigma/\pi)^k E_k(t)}{(N + 1)^k \sqrt{1 - 4^{-k}}} \frac{C'}{\sqrt{N + 1}},
\]
for some constant \( C' > 0 \), where
\[
E_k(t) = \sqrt{\frac{\pi}{\alpha}} \left( \int_\mathbb{R} s^k |p(s + t)|^2 ds \right)^{1/2}.
\]

**Proof.** Using Cauchy-Schwartz inequality in (34), we have
\[
|T_N(t)|^2 \leq \frac{\pi}{\alpha} \left( \sum_{|n| \leq N} \left| p \left( t + \frac{n\pi}{\alpha} \right) \right|^2 \right)^{1/2} \left( \sum_{|n| \leq N} |\hat{g}_n|^2 \right). \quad (38)
\]
Observing that \( p(t) \in \mathbb{B}_2^\alpha \) and following Jagerman [19], we can obtain the estimate
\[
|T_N(t)| \leq \frac{2(\sigma/\pi)^k E_k(t)}{(N + 1)^k \sqrt{1 - 4^{-k}}} \left( \sum_{|n| \geq 0} |\hat{g}_n|^2 \right)^{1/2}, \quad (39)
\]
for any \( g \in \mathbb{B}_2^\alpha \). Noting that \( g \in \mathbb{B}_1^\alpha \) and
\[
\hat{g}_N = \int_0^\infty \psi_N(s) g(s) ds = g(N\pi/\sigma) - \int_0^\infty \psi_N(s) g(s) ds,
\]
\[|\hat{g}_N| \leq |g(N\pi/\sigma)| + \sqrt{\sigma/\pi} N^{-1} \int_0^\infty |g(s)| ds = O(N^{-1}),\]
therefore, \( |\hat{g}_{N+}e| = O(N^{-1}) \). Now,
\[
\sum_{|n| > N} \frac{1}{N^2} \leq 2 \int_{N+1}^\infty \frac{ds}{s^2} = \frac{2}{N + 1},
\]
so that
\[
\left( \sum_{|n| > N} |\hat{g}_n|^2 \right)^{1/2} \leq \frac{C'}{\sqrt{N + 1}}.
\]
for some constants \( C' > 0 \). Plugging-in this estimate in (38), the result follows. \( \square \)

It is interesting to note that the truncation error does not improve by strengthening the regularity condition of \( g \) because \( k' \) does not feature in the estimate.

The representation in (28) can also be used to define a linear operator on \( \ell^2 \)
\[
\mathcal{P}[\alpha](t) = \sum_{n \in \mathbb{N}} \sqrt{\frac{\pi}{\alpha}} p \left( t + \frac{n\pi}{\alpha} \right) \alpha_n, \quad \alpha \in \ell^2. \quad (40)
\]
The infinite series converges absolutely and uniformly for \( t \in \mathbb{R} \) if \( p \in \mathbb{B}_2^\alpha \). Let us show that it defines a bounded linear operator from \( \ell^2 \to \mathbb{B}_2^\alpha \) if \( p \in L^\infty \). Rewriting the infinite series in the Fourier domain, we have
\[
\|\sum_{n \in \mathbb{N}} \sqrt{\frac{\pi}{\alpha}} p \left( t + \frac{n\pi}{\alpha} \right) \alpha_n \|_{L^2} = \frac{1}{2\pi} \sqrt{\frac{\pi}{\alpha}} \int_{-\sigma}^{\sigma} \pi \left( \sum_{n \in \mathbb{N}} \alpha_n e^{i(n\pi/\alpha)\xi} \right) \rho(\xi) e^{i\xi} d\xi.
\]
Using Plancherel’s theorem and the fact that \( \text{supp } \rho \subset [-\sigma, \sigma] \), we have
\[
\left\| \sum_{n \in \mathbb{N}} \sqrt{\frac{\pi}{\alpha}} p \left( t + \frac{n\pi}{\alpha} \right) \alpha_n \right\|_{L^2} = \frac{1}{\sqrt{2\pi}} \left\| \sum_{n \in \mathbb{N}} \alpha_n e^{i(n\pi/\alpha)\xi} \right\|_{L^2[-\sigma, \sigma]} \leq \sqrt{\pi/\alpha} \|p\|_{L^\infty} \|\alpha\|_{\ell^2}.
\]
These observations are summarized in the following proposition:

**Proposition II.5.** The operator \( \mathcal{P} \) defined by (40) with \( p \in \mathbb{B}_2^\alpha \) defines a bounded linear operator from \( \ell^2 \to \mathbb{B}_2^\alpha \) if \( p \in L^\infty \).

For the solution of the inverse scattering problem, the spectral properties of the Hankel operators are relevant. Let us state the following result which appears in somewhat general form in [26, Thm. 8.10].
The Hankel operator $\mathcal{P}$ defined by (15) with $p \in B^2_\rho$ is compact on $L^2(\Omega_\rho)$, if there exists a function $\varphi \in C(\mathbb{R})$ with support in $[-\sigma, \sigma]$ such that $p(t)$ agrees with $F_\varphi(t)$ on $\Omega_\rho$.

**Corollary II.7.** The Hankel operator $\mathcal{P}$ defined by (15) is compact on $L^2(\Omega_\rho)$ if $p \in B^2_\rho$ or, alternatively, if $p(z)$ is an entire function such that

$$|p(z)| \leq \frac{C}{(1 + |z|)^{k_+}}, \quad z \in \mathbb{C},$$

holds for some $k > 0$.

**Proof.** In first cases, $p \in B^2_\rho$ ensures that $\varphi \in C(\mathbb{R})$. In the second case, the estimate simply ensures that $p \in B^2_\rho$. \hfill $\square$

**C. Sampling Approach to Inverse Scattering**

The Hermitian conjugate of $\mathcal{P}$, denoted by $\mathcal{P}^\dag$, with respect to the inner product in $L^2(\Omega_\rho)$ works out to be

$$\mathcal{P}^\dag[g](y) = \int_0^\infty p^r(y + s)g(s)ds.$$

(41)

Define $\mathcal{H} = \mathcal{P}^\dag \circ \mathcal{P}$, so that

$$\mathcal{H}[g](y) = \int_0^\infty ds \int_0^\infty dx \ p^r(y + s)p(s + x)g(x)$$

$$= \int_0^\infty \mathcal{K}(y, x)g(x)dx,$$

(42)

where the kernel function $\mathcal{K}(y, x; t)$ is given by

$$\mathcal{K}(y, x; t) = \int_0^\infty ds \ p^r(y + s)p(s + x).$$

(43)

The properties of the operator $\mathcal{H}$ can be deduced easily from that of $\mathcal{P}$. If $p \in B^2_\rho$, the operator $\mathcal{H}$ defines a bounded linear operator on $L^2(\Omega_\rho)$, $(\nu = 1, 2)$ with the estimate

$$\|\mathcal{H}[g]\|_{L^2(\Omega_\rho)} \leq \|f\|_{L^2(\Omega_\rho)}.$$
Define

\[ \hat{p}_l = \int_0^\infty p(s)\psi_l(s)ds, \]

\[ M_{lm} = \sqrt{\frac{\pi}{\sigma}} \int_0^\infty p \left( s + \frac{m\pi}{\sigma} \right) \psi_l(s)dt, \tag{50} \]

so that

\[ \begin{cases} \kappa \alpha_l \pi = \sum_{m \in \mathbb{Z}} M_{lm} \alpha_m, \\ \alpha_l^\pi = \hat{p}_l + \sum_{m \in \mathbb{Z}} M_{lm} \alpha_m^\pi, \end{cases} \quad l \in \mathbb{Z}, \tag{51} \]

which can be stated in a compact form by introducing the infinite column vectors \( \alpha_j = (\alpha_j^m)_{m \in \mathbb{Z}} \) and \( \hat{p} = (\hat{p}_n)_{n \in \mathbb{Z}} \):

\[ \kappa \alpha_l \pi = M \alpha_1, \quad \alpha_l^\pi = \hat{p} + M \alpha_2. \tag{52} \]

Before attempting to solve the GLM equations, we analyze the properties of the “mass” matrix \( M \). Using expansion equations, we can write

\[ M_{lm} = \frac{\pi}{\sigma} \sum_{n \in \mathbb{Z}} Q_{ln} p \left( \pi n + \frac{\pi}{\sigma} \right) = \sum_{n \in \mathbb{Z}} Q_{ln} \mathcal{P}_{nm}, \tag{53} \]

where \( \mathcal{P} \) is the Hankel matrix defined by

\[ \mathcal{P}_{nm} = \frac{\pi}{\sigma} p \left( \pi n + \frac{\pi}{\sigma} \right), \tag{54} \]

and, \( Q \) is a real symmetric matrix defined by

\[ Q_{nl} = \int_0^\infty \psi_n(s)\psi_l(s)ds, \tag{55} \]

which we would refer to as the quadrature matrix. An estimate for the values of each its entries can be easily obtained using the Cauchy-Schwarz inequality:

\[ |Q_{nl}| \leq ||\psi_n||_{L^2(\alpha_1)} ||\psi_l||_{L^2(\alpha_1)} < 1. \tag{56} \]

It turns out that the entries of \( Q \) can be computed exactly in terms of the Sine and Cosine integrals (see Appendix A):

\[ Q_{ln} = \begin{cases} \frac{1}{\pi} - \frac{1}{2\pi} \text{Si}(2n\pi), \quad l = n \\ \frac{1}{2\pi} \frac{\text{Si}(2n\pi)}{n} - \text{Cin}(2l\pi) - \text{Cin}(2n\pi), \quad l \neq n \end{cases}. \tag{57} \]

Using this quadrature matrix, we can also write

\[ \hat{p}_l = \sqrt{\frac{\pi}{\sigma}} \sum_{n \in \mathbb{Z}} Q_{ln} p \left( \frac{n\pi}{\sigma} \right) = \sum_{n \in \mathbb{Z}} Q_{ln} p_n, \tag{58} \]

where \( p_n = \sqrt{\pi/\sigma} p(n\pi/\sigma) \).

**Proposition II.9.** Assume \( p \in \mathcal{B}_2^2 \) with \( \rho \in \mathbf{L}_\infty^\circ \) and let the infinite matrices \( M, \mathcal{P} \) and \( Q \) be defined by (50), (54) and (55), respectively.

(a) The matrix \( M \) defines a bounded linear operator on \( \ell^2 \). (b) The real symmetric matrix \( Q \) is positive definite and defines a bounded positive linear operator on \( \ell^2 \).

(c) The complex symmetric matrix \( \mathcal{P} \) defines a bounded Hankel matrix on \( \ell^2 \).

**Proof.** Let \( p \in \mathcal{B}_2^2 \). Then, for \( l, m \in \mathbb{Z} \), we have

\[ M_{lm} = \frac{\pi}{\sigma} \int_0^\infty p \left( \frac{s + m\pi}{\sigma} \right) \left( \ln \frac{s}{\sigma} \right) ds. \]

For any \( \alpha \in \ell^2 \), let us note that

\[ C(s) \equiv \frac{\pi}{\sigma} \sum_{l \in \mathbb{Z}} \alpha_l p \left( \frac{s + m\pi}{\sigma} \right) \in \mathcal{B}_2^2, \]

so that

\[ \left( \sum_{l \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} M_{lm} \alpha_m \right)^2 \leq \left( \sum_{l \in \mathbb{Z}} \left| \int_0^\infty \psi_l \left( \frac{s - \ln \sigma}{\sigma} \right) C(s)ds \right| \right)^2 \]

\[ \leq (1 + \pi) ||\mathcal{P}||_2 ||\alpha||_2 \]

which follows from (7), Prop. II.2 and (30). Using similar arguments for \( Q_{lm} \), we have

\[ \left( \sum_{l \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} Q_{lm} \alpha_m \right)^2 \leq (1 + \pi) ||\mathcal{P}||_2 ||\alpha||_2. \]

This shows that the symmetric real matrix \( Q \) is bounded. It also turns out to be a positive definite matrix:

\[ \alpha^\top Q \alpha = \int_0^\infty \left| \sum_{n \in \mathbb{Z}} \alpha_n \psi_n(s) \right|^2 ds \leq ||\alpha||_2^2, \]

yielding \( ||Q||_2 \leq 1 \).

In order to prove the last statement, let us note that the symbol of the Hankel matrix \( \mathcal{P} \) can be worked out to be \( \rho(\sigma^2/\pi), \pi - 1 \in [-\pi, \pi] \) therefore, on account of \( \rho \in \mathbf{L}_\infty^\circ \), the matrix \( \mathcal{P} \) turns out to be a bounded Hankel matrix. \( \square \)

The solution of the coupled system in (51) can be obtained by defining the \( 2 \times 2 \) block matrix equation

\[ \begin{pmatrix} I & -M \alpha \pi \\ -\kappa M^\ast & I \end{pmatrix} \begin{pmatrix} \alpha_l \pi \\ \alpha_2 \pi \end{pmatrix} = \begin{pmatrix} \hat{p} \pi \\ 0 \end{pmatrix}. \tag{59} \]

Using \( M = Q \mathcal{P} \), we may symmetrize the linear system by introducing \( \alpha_j = Q^{1/2} \beta_j, \quad j = 1, 2 \) so that

\[ \begin{pmatrix} I & -\kappa Q^{1/2} \mathcal{P} Q^{1/2} \\ -\kappa Q^{1/2} \mathcal{P} Q^{1/2} & I \end{pmatrix} \begin{pmatrix} \beta_1 \pi \\ \beta_2 \pi \end{pmatrix} = \begin{pmatrix} Q^{1/2} \pi \beta_1 \pi \\ 0 \end{pmatrix}, \]

where the column vector \( \beta = \sqrt{\pi/\sigma} \rho(n\pi/\sigma)_{n \in \mathbb{Z}} \). Putting \( \mathcal{G} = Q^{1/2} \mathcal{P} Q^{1/2} \), we have

\[ \begin{pmatrix} I & -\kappa \mathcal{G} \pi \\ -\kappa \mathcal{G} \pi & I \end{pmatrix} \begin{pmatrix} \beta_1 \pi \\ \beta_2 \pi \end{pmatrix} = \begin{pmatrix} Q^{1/2} \pi \beta_1 \pi \\ 0 \end{pmatrix}. \tag{60} \]
The block matrix on the right hand side is Hermitian for \( \kappa = +1 \) and it can be reduced to a Hermitian form for the case \( \kappa = -1 \) by rearranging:

\[
\begin{pmatrix}
-I & \mathcal{G}^* \\
\mathcal{G} & I
\end{pmatrix}
\begin{pmatrix}
\beta_1 \\
\beta_2
\end{pmatrix}
= 
\begin{pmatrix}
-Q^{1/2} p^* \\
0
\end{pmatrix}.
\]

On eliminating \( \beta_2 \), we have

\[
(I - \kappa \mathcal{G}^*) \beta_1 = Q^{1/2} p^*.
\]

Setting \( \kappa = +1 \), in order for \( (I - \mathcal{G}^*) \) to be invertible, it suffices to have \( \|p\|_2 < 1 \).

**Remark II.2.** The symbol of the Hankel matrix \( \mathcal{P} \) can be worked out to be \( \rho(\sigma \theta / \pi), \theta \in [-\pi, \pi] \); therefore, the requirement \( \|\mathcal{P}\|_\infty < 1 \) can be fulfilled if \( |\rho| < 1 \) for all \( \theta \in [-\pi, \pi] \).

Setting \( \kappa = -1 \), the infinite matrix \( \mathcal{G}^* \mathcal{G} \) defines a positive linear operator; therefore, \( (I - \mathcal{G}^*) \) invertible. Finally, we have the potential given by

\[
q(0) = -2A_1(0) = -2p^*(0) - 2 \sum_{n \in \mathbb{Z}} \frac{\pi}{\sigma} p^* \left( \frac{n \pi}{\sigma} \right) \alpha_n^{(2),*} = -2p^*(0) - 2p^* Q^{1/2} \mathcal{G}(I - \kappa \mathcal{G}^*)^{-1} Q^{1/2} p^*,
\]

together with its \( L^2 \)-norm on \( \Omega_+ \) as

\[
\|q\|_{L^2(\Omega_+)}^2 = 2 \kappa A_2(0) = 2 \sum_{n \in \mathbb{Z}} \frac{\pi}{\sigma} p^* \left( \frac{n \pi}{\sigma} \right) \alpha_n^{(1),*} = 2p^* \alpha_1^* = 2p^* Q^{1/2} (I - \kappa \mathcal{G}^*)^{-1} Q^{1/2} p.
\]

Turning to the numerical aspects, let us introduce the truncated version of the GLM equations that can be implemented as a numerical scheme. To this end, define

\[
k_2^{(N)*}(t) = \sum_{|n| \leq N} \sqrt{\frac{\pi}{\sigma}} p \left( t + \frac{n \pi}{\sigma} \right) \alpha_n^{(1),*},
\]

\[
A_1^{(N)}(t) = p(t) + \sum_{|n| \leq N} \sqrt{\frac{\pi}{\sigma}} p \left( t + \frac{n \pi}{\sigma} \right) \alpha_n^{(2),*}.
\]

so that

\[
\begin{aligned}
\kappa_2^{(N)}(t) &= \sum_{|n| \leq N} \mathcal{M}_n \alpha_n^{(1),*}, \\
\alpha_1^{(N)}(t) &= p_1 + \sum_{|n| \leq N} \mathcal{M}_n \alpha_n^{(2),*},
\end{aligned}
\]

or, equivalently,

\[
\kappa_2^{(N)}(t) = \mathcal{M}_N \alpha_1^{(N)}, \quad \alpha_1^{(N)} = \tilde{p}_N + \mathcal{M}_N \alpha_2^{(N)},
\]

which simplifies to

\[
(I_N - \kappa \mathcal{M}_N^* \mathcal{M}_N) \alpha_1^{(N)} = \tilde{p}_N.
\]

The potential is then obtained from

\[
q(0) \approx -2p^*(0) - 2p_1^\dagger \alpha_2^{(N)*},
\]

with its \( L^2 \)-norm on \( \Omega_+ \) as

\[
\|q\|_{L^2(\Omega_+)}^2 \approx 2p_N^\dagger \alpha_1^{(N)*}.
\]

Let us note that by introducing a truncated quadrature matrix \( \mathcal{Q}_N \), the linear system can be symmetrized by putting \( \mathcal{G}_N = Q_N^1 \mathcal{P}_N Q_N^1 \) in the same manner as described earlier. For the moment, we hold off the symmetrization process as it involves quadrature errors on account of the fact that the quadrature formula with finite \( Q_N \) is not exact. Now we turn to the convergence analysis of the numerical procedure described above. To this end, we consider the total numerical error \( \kappa_1^{(N)}(t)(j = 1, 2) \) given by

\[
\kappa_2^{(N)}(t) = \sum_{|n| \leq N} \sqrt{\frac{\pi}{\sigma}} p \left( t + \frac{n \pi}{\sigma} \right) \alpha_n^{(1),*} \approx \sum_{|n| \leq N} \sqrt{\frac{\pi}{\sigma}} p \left( t + \frac{n \pi}{\sigma} \right) \alpha_n^{(1),*},
\]

\[
R_1^{(N)}(t) = \sum_{|n| \leq N} \sqrt{\frac{\pi}{\sigma}} p \left( t + \frac{n \pi}{\sigma} \right) \alpha_n^{(2),*}. \quad \text{(71)}
\]

Define

\[
E_N(t) = \sqrt{\frac{\pi}{\sigma}} \left( \sum_{|n| \leq N} \left| p \left( t + \frac{n \pi}{\sigma} \right) \right| \right)^{1/2}, \quad \text{(72)}
\]

then, it is easy work out the estimates

\[
|R_2^{(N)}(t)| \leq (1 + \pi) \sqrt{\pi / \sigma} \|p\|_2 \|\alpha_1 - \alpha_1^{(N)}\|_2^2 + E_N(t) \left( \sum_{|n| > N} |\alpha_n^{(1),*}|^2 \right)^{1/2},
\]

\[
|R_1^{(N)}(t)| \leq (1 + \pi) \sqrt{\pi / \sigma} \|p\|_2 \|\alpha_2 - \alpha_2^{(N)}\|_2^2 + E_N(t) \left( \sum_{|n| > N} |\alpha_n^{(2),*}|^2 \right)^{1/2}. \quad \text{(73)}
\]

By a slight abuse of notations, let the vectors \( \alpha_2^{(N)} \) (\( j = 1, 2 \)) and \( \tilde{p}_N \) represent infinite dimensional vectors with entries corresponding to \( |n| > N \) taken to be identically zero. Then, it is straightforward to work out

\[
\begin{pmatrix}
\alpha_1^{(N)*} - \alpha_1^{(N)*} \\
\alpha_2^{(N)*} - \alpha_2^{(N)*}
\end{pmatrix}
= 
\begin{pmatrix}
-I & -\mathcal{M}^* \\
\mathcal{M} & I
\end{pmatrix}^{-1}
\begin{pmatrix}
\tilde{p} - \tilde{p}_N \\
0
\end{pmatrix}
= 
\begin{pmatrix}
(I - \kappa \mathcal{M}^* \mathcal{M})^{-1}[\tilde{p} - \tilde{p}_N] \\
\kappa \mathcal{M} \mathcal{M}^* (I - \kappa \mathcal{M}^* \mathcal{M})^{-1}[\tilde{p} - \tilde{p}_N]
\end{pmatrix}.
\]

(74)
Therefore, under the conditions that ensure \((I - \kappa MM^*)\) is invertible, it follows that, for some constant \(C > 0\), the estimates

\[
\|\alpha_j - \alpha_j^{(N)}\|_c \leq C\|\hat{p} - \hat{p}_N\|_c, \quad j = 1, 2, \tag{75}
\]

hold. The estimates (73) and (75), allow us to conclude that the truncated system converges to the true solution under the aforementioned conditions. It is possible to make precise statements about the rate of convergence with respect to the number of basis functions \(2N + 1\) of the truncated system if we strengthen the regularity condition on \(p\):

**Proposition II.10.** For \(p\) satisfying an estimate of the form

\[
|p(z)| \leq \frac{C}{(1+|z|)^2} e^{2|m|\pi z}, \quad z \in \mathbb{C}, \tag{76}
\]

we have for fixed \(t \in \mathbb{R},

\[
|R_j^{(N)}(t)| = \mathcal{O}(N^{-1/2}), \quad j = 1, 2,
\]

where \(2N + 1\) is the number of basis functions used in the sampling expansion.

**Proof.** It is easy to see that the solution of the GLM equations exists under the conditions prescribed in the proposition. Under the same conditions, following the methods discussed in the last section, it is also easy to show that

\[
\|\hat{p} - \hat{p}_N\|_c = \mathcal{O}(N^{-1/2}),
\]

and \(E_N(t) = \mathcal{O}(N^{-1})\) (see Prop. II.4) so that from (73) and (75) the result follows.

\[
\square
\]

1. Quadrature Errors

Let us observe that the entries of the mass matrix \(M\) require a quadrature method which works well on infinite domains and is capable of providing higher orders of convergence depending on the regularity of \(p\). The quadrature matrix \(Q\) exploits the sampling expansion to achieve this goal. Fortunately, the need for a numerical quadrature is avoided by computing the integrals exactly. However, truncation of this quadrature matrix introduces numerical errors in computing \(M\). Note that the same difficulties also arise in the computation of the vector \(\hat{p}\). In order to quantify these errors, we consider \(2M + 1\) number of basis functions and define

\[
P_i^{(M)}(x) = \sqrt{\frac{\pi}{\sigma}} \sum_{|n| \leq M} Q_{in} p\left(\frac{m\sigma}{\pi}\right),
\]

\[
M_{in}^{(M)} = \frac{\pi}{\sigma} \sum_{|n| \leq M} Q_{in} p\left(m\sigma + n\frac{\pi}{\sigma}\right). \tag{77}
\]

The total numerical error as a result of truncation of the linear system as well as the quadrature matrix can be written as

\[
\begin{align*}
(\alpha_1^{(N)} - \alpha_1^{(N,M)}) & = (\alpha_1^{(N)} - \alpha_1^{(N)}) + (\alpha_1^{(N)} - \alpha_1^{(N,M)}) + (\alpha_1^{(N,M)} - \alpha_1^{(N,M)}), \\
(\alpha_2^{(N)} - \alpha_2^{(N,M)}) & = (\alpha_2^{(N)} - \alpha_2^{(N)}) + (\alpha_2^{(N)} - \alpha_2^{(N,M)}) + (\alpha_2^{(N,M)} - \alpha_2^{(N,M)}),
\end{align*}
\tag{78}
\]

where \(\alpha^{(N,M)}\) refers to the solution of the linear system constructed using \((2N + 1) \times (2M + 1)\) quadrature matrix. The relevant quantities of this linear system are labeled as \(M_{in}^{(M)}\) and \(\hat{p}_N^{(M)}\) whose meanings are self-evident. We have already dealt with the difference \(\alpha_j - \alpha_j^{(N)}\); let us then turn to the difference \(\alpha_j^{(N)} - \alpha_j^{(N,M)}\), \(j = 1, 2\), which is given by

\[
(\alpha_1^{(N)} - \alpha_1^{(N,M)}) = \left(\frac{I - \kappa MM^*}{\kappa(M^{(M)} - M^{(N)})}\right)^{-1} (M - M^{(M)}) \alpha_2^{(N)} + (\hat{p}_N - \hat{p}_N^{(M)}).
\tag{79}
\]

In view of the expression above, it suffices to estimate \((M - M^{(M)})\) and \((\hat{p}_N - \hat{p}_N^{(M)})\):

**Lemma II.11.** Let \(p\) satisfy an estimate of the form

\[
|p(z)| \leq \frac{C}{(1+|z|)^{k+1}} e^{2|m|\pi z}, \quad z \in \mathbb{C}, \tag{80}
\]

where \(k \geq 1\). Let the quadrature matrix \(Q\) be truncated to the size \((2N + 1) \times (2M + 1)\).

\(\alpha\) Let \(\hat{p}_N^{(M)}\) denote the approximation to \(\hat{p}_N\) using the quadrature matrix \(Q\), then the estimate

\[
\|\hat{p}_N - \hat{p}_N^{(M)}\|_c \leq \frac{2(\sigma/\pi)^k E_k}{(M + 1)^4 \sqrt{1 - 4^{-k}}}, \tag{81}
\]

holds where

\[
E_k = \sqrt{\frac{\pi}{\sigma}} \left(\int_{-\alpha/\sigma}^{\alpha/\sigma} s^{2k} |p(s)|^2 ds\right)^{1/2}.
\]

\(\beta\) Let \(M_{in}^{(M)}\) denote the approximation to \(M_N\) using the quadrature matrix \(Q\), then the estimate

\[
\|M_N - M_N^{(M)}\|_c \leq \frac{2(\sigma/\pi)^k |\hat{p}_N|_{\mathcal{C}^k(-\alpha,\alpha)}}{(M + 1)^4 \sqrt{1 - 4^{-k}}}, \tag{82}
\]

holds.

**Proof.** To prove (a), we observe that

\[
\|\hat{p}_N - \hat{p}_N^{(M)}\|_c \leq \|Q\|_c \sqrt{\frac{\pi}{\sigma}} \left(\sum_{|m| \leq M} \left| p\left(\frac{m\sigma}{\pi}\right)\right|^2\right)^{1/2}. \tag{83}
\]

The result then follows by noting that \(\|Q\|_c < 1\) and using Jagelman’s estimate [19] for the remaining expression above. To prove (b), let \(\alpha \in \hat{c}^\circ\), then

\[
\|M_N - M_N^{(M)}\|_c \leq \|Q\|_c \sqrt{\frac{\pi}{\sigma}} \left(\sum_{|m| \leq M} \left| p\left(\frac{m\sigma}{\pi}\right)\right|^2\right)^{1/2}.
\]
Define
\[ C(s) = \sum_{n \in \mathbb{Z}} p\left( s + \frac{m\pi}{\sigma} \right) a_n, \] (84)
then \( s^k C(s) \in L^2 \) and
\[ E_k = \left( \int_{\mathbb{R}} s^{2k} |C(s)|^2 ds \right)^{\frac{1}{2}} \leq \| \hat{\phi}_k \|_{L^2(-\sigma, \sigma)} \| \alpha \|_{L^2}. \] (85)

Now, observing that \( \| \mathcal{Q} \|_{L^2} < 1 \) and
\[ \left| \sum_{|m| > M} \left| \sum_{|n| \leq N} p\left( m + n \frac{\pi}{\sigma} \right) a_n \right|^2 \right|^{\frac{1}{2}} \leq \left| \sum_{|m| > M} \left| C\left( \frac{k\pi}{\sigma} \right) \right|^2 \right|^{\frac{1}{2}}, \] (86)
the result follows by using Jagerman’s estimate [19]. \( \square \)

Let us conclude this section with the following remark. Based on the estimates obtained above, it is clear that choice \( M = N \) does not alter the rate of convergence. Besides, this choice makes it possible to symmetrize the truncated linear system which ensures that it is well conditioned. In particular, the uniform boundedness of the inverse of
\[ \begin{pmatrix} I & -M_N^{(N)} M_N^{(N)} I \\ -\kappa M_N^{(N)} & I \end{pmatrix} \]
can be established exactly in the manner we treated the infinite case.

D. Modified Sampling Approach

The slow convergence of the sampling series motivates us to consider modified versions of the sampling theorem which facilitate faster convergence at the cost of sampling beyond the Nyquist rate. A modified version of the sampling series was proposed by Helms and Thomas [18, 19] which can be described as follows: Introducing the bandlimiting parameter \( \sigma' \) and \( \delta \in (0, 1) \) such that
\[ \sigma' = \frac{\sigma}{1 - \delta}, \] (87)
where \( \sigma \) defines the support for the reflection coefficient \( \rho \), i.e., \( \text{supp} \rho \subseteq [-\sigma, \sigma] \). Let us define
\[ \begin{cases} \sigma'_- = (1 - \delta) \sigma' = \sigma, \\ \sigma'_+ = (1 + \delta) \sigma' = \left( \frac{1 + \delta}{1 - \delta} \right) \sigma. \end{cases} \] (88)
The sinc basis functions are then modified by a multiplier of the form
\[ \theta(t) = \left| \frac{\sin \left( \frac{\pi}{\delta} \sigma' t \right) }{ \frac{\pi}{\delta} \sigma' t } \right|^m, \ m \in \mathbb{Z}_+, \] (89)
in order to accelerate the convergence of the sampling series. Let us define the basis functions
\[ \phi_n(t) = \frac{\sqrt{\pi}}{\sigma'} \sin[\sigma'(t - t_n)]\theta(t - t_n) \] (90)
\[ = \sqrt{\pi} \left( \frac{\sin \left( \frac{\pi}{\delta} \sigma' (t - t_n) \right)}{ \frac{\pi}{\delta} \sigma' (t - t_n) } \right)^m. \] Clearly, \( \phi_n(t) \in B_{\sigma'}^1 \). For fixed \( t \), the Helms and Thomas [18, 19] expansion of \( p(t + s) \) reads as
\[ p(t + s) = \sum_{n \in \mathbb{Z}} \sqrt{\frac{\pi}{\sigma'}} \left( \frac{\sin \left( \frac{\pi}{\delta} \sigma' (t + s - t_n) \right)}{ \frac{\pi}{\delta} \sigma' (t + s - t_n) } \right)^m \phi_n(s). \] (91)

This series converges under much weaker conditions on \( p \), however, we still restrict ourselves to the case of \( p \in B_{\sigma'}^1 = B_{\sigma'}^\nu \) with \( \nu = 1, 2 \). For our purpose, it suffices to note that \( \{ \phi_n \}_{n \in \mathbb{Z}} \) spans \( B_{\sigma'}^2 \).

With this modified basis function, the operator defined by (29),
\[ \mathcal{F}[g](\cdot) \equiv \int_0^\infty \phi_0(y + s) g(s) ds, \] (92)
now defines a bounded linear operator from \( L^1(\Omega_+) \) to \( L^1 \) with
\[ \| \mathcal{F} \|_{L^1} \leq \int_{\Omega_+} |\phi_0(s)| ds < \| \theta \|_{L^2}. \] (93)
Moreover, we note that \( \mathcal{F} \) defines a bounded linear operator from \( L^1(\Omega_+) \) to \( B_{\sigma'}^1 \). Following Jagerman [19], an improved version of the result presented in Prop. II.4 is as follows.

Proposition II.12. For \( p \) satisfying an estimate of the form
\[ |p(z)| \leq \frac{C}{(1 + |z|)^{k + 1}}, \quad z \in \mathbb{C}, \] (94)
where \( k \geq 1 \) and \( g \) satisfying a similar estimate with the index \( k' \geq m \), the truncation error \( T_N(t) \) of the partial sums in (32) with the basis functions defined by (90) satisfies the estimate
\[ |T_N(t)| \leq \frac{2(\sigma'/\pi)^2 E_k(t)}{C'} \frac{C'}{(N + 1)^k \sqrt{1 - 4^{-k} (N + 1)^{m+1/2}}}, \] (95)
for some constant \( C' > 0 \).

Proof. Noting that \( g \in B_{\sigma'}^1 \) so that \( \theta g \in B_{\sigma'}^1 \). Consequently,
\[ |\hat{g}| \leq |g| (N\pi/\sigma') + \sqrt{\frac{\sigma' \pi}{\delta}} \left( \frac{\pi}{\delta} \right)^m N^{-m-1} \int_{-\infty}^0 |g(s)| ds \] (96)
\[ = \sigma(N^{-1-m}), \] (97)
therefore, \( \| \hat{g} \|_{L^2} = \sigma(N^{-1-m}) \). Now,
\[ \sum_{|n| > N} \frac{1}{N^{2m+2}} \leq 2 \int_{N+1}^\infty \frac{ds}{s^{2m+2}} = \frac{2}{(2m+1)(N+1)^{2m+1}}, \] (98)
so that
\[
\left( \sum_{|n|>N} |\hat{q}_n|^2 \right)^{1/2} \leq \frac{C'}{(N+1)^{m+1/2}},
\]
for some constants \(C' > 0\). Other details of the proof are same as that of Prop. II.4; therefore, we omit it. \(\square\)

Working from equations (74) and (75), if we assume \(M_N\) to be exact, then the proposition II.10 can be modified as follows:

**Proposition II.13.** Let \(p\) satisfy an estimate of the form
\[
|p(z)| \leq \frac{C}{(1 + |z|)^{k+1}} e^{\sigma'|\ln|z|}, \quad z \in \mathbb{C},
\]
where \(k \geq m + 1/2, m \in \mathbb{Z}_+\). Then, for fixed \(t \in \mathbb{R}\) and with the basis functions defined by (90), we have
\[
|P_j^{(N)}(t)| = \mathcal{O}(N^{-1/2-m}), \quad j = 1, 2,
\]
where \(2N + 1\) is the number of basis functions used in the sampling expansion.

We conclude this section with a discussion of the quadrature method for computing the entries of the quadrature matrix
\[
Q_{nl} = \int_0^{\pi} \phi_n(s)\phi_l(s) ds, \quad n, l \in \mathbb{Z},
\]
where \(\{\phi_n(s)\}\) are the new basis functions introduced in this section. By a straightforward application of the sampling theorem, a simple quadrature rule can be worked out as follows:
\[
Q_{nl}^{(M)} = \frac{1}{2\sigma_+} \sum_{k=-M}^M \phi_n \left( \frac{k\pi}{2\sigma_+} \right) \phi_l \left( \frac{k\pi}{2\sigma_+} \right) \left[ \frac{\pi}{2} - \text{Si}(k\pi) \right], \quad n, l \in \mathbb{Z}.
\]
(97)

Defining the matrices \(\mathcal{D}\) and \(\Phi\),
\[
\mathcal{D}_{nk} = \frac{1}{2\sigma_+} \left[ \frac{\pi}{2} - \text{Si}(k\pi) \right], \quad -M \leq m, n \leq M,
\]
\[
\Phi_{kl} = \phi_l \left( \frac{k\pi}{2\sigma_+} \right), \quad -M \leq k \leq M, -N \leq l \leq N.
\]
(99)
The quadrature matrix truncated to size \(N \times N\) can be written as
\[
Q_N^{(M)} = \Phi^T \mathcal{D} \Phi.
\]
(100)
The quadrature errors in the present case can be treated in exactly the same manner as before and given that the new basis functions have better decay properties, the results contained in the lemma II.11 remain valid.

### III. NUMERICAL AND ALGORITHMIC ASPECTS

Based on the analysis presented in the earlier sections, the discrete system to be solved has the form
\[
\begin{pmatrix}
1 & -Q p^* \\
-\kappa Q p^* & I
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix}
= \begin{pmatrix}
Q p \\
0
\end{pmatrix},
\]
where \(Q \in \mathbb{R}^{N \times N}\) is a symmetric positive-definite matrix, \(P \in \mathbb{C}^{N \times N}\) is a Hankel matrix and \(\kappa \in [+1, -1]\). We symmetrize the linear system by introducing \(v_j = Q^{1/2} u_j, j = 1, 2\) so that
\[
\begin{pmatrix}
1 & -Q^{1/2} \mathcal{P} Q^{1/2} \\
-\kappa Q^{1/2} \mathcal{P}^* Q^{1/2} & I
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2
\end{pmatrix}
= \begin{pmatrix}
Q^{1/2} p \\
0
\end{pmatrix}.
\]
Putting \(G = Q^{1/2} \mathcal{P} Q^{1/2}\), we have
\[
\begin{pmatrix}
1 & -G \\
-\kappa G^* & I
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2
\end{pmatrix}
= \begin{pmatrix}
Q^{1/2} p \\
0
\end{pmatrix},
\]
(102)
which reduces to
\[
(1 - \kappa G^*) u_1 = Q^{1/2} p.
\]
(103)
which is numerically well conditioned on account of the positive definite nature of \((1 - \kappa G^*)\) (for \(\kappa = +1\), we assume \(||G|| < 1\)). This linear system must be solved in order to compute \(q(0)\). In order to obtain \(q(nh)\), where \(h (> 0)\) is the step size and \(n \in \mathbb{Z}\), we must translate \(\mathcal{P}(\tau)\) by \(2nh\), and, compute \(\mathcal{P}^*\) and \(\mathcal{P}\). If a direct method of solving a system of linear equation is used, then the complexity of computation per sample of the potential works out to be \(\mathcal{O}(N^3)\) (excluding the cost of computing \(\mathcal{P}\) and \(\mathcal{P}^*\)). However, it is also possible to use an iterative method such as the conjugate-gradient (CG) method to solve the linear system leading to a complexity of \(\mathcal{O}(N_{iter} N^2)\). Note that the initial seed for such a procedure can be obtained by using a direct solver at any fixed \(n\), say, \(n = 0\). The CG-iterations in the subsequent step can be seeded by using the solution obtained in the last step. We choose the threshold for convergence of the CG iterations to be \(10^{-12}\) unless otherwise stated.

The discussion above is valid for both of the methods proposed in the earlier sections. For the sake of convenience, let us label the methods by the basis functions used in their respective sampling expansions. The first method uses the Whittaker-Kotelinkov-Shannon (WKS) sampling series which consists of translates of the sinc function. The basis is completely determined by the bandlimiting parameter \(\sigma\); therefore, we label this method by “WKS_\sigma”. The second method uses the Helms and Thomas [18, 19] version of the sampling series. The basis in this case is completely determined by three parameters: the guard-band parameter \(\delta\), the index \(m\) of the convergence accelerating function and the bandlimiting parameter \(\sigma'\) given by
\[
\sigma' = \sigma / (1 - \delta), \quad \delta < 1.
\]
(104)
Therefore, we label this method by “HT_{\sigma'}^{m,\delta}”. In the formalism adopted in Sec. II.D, \(\sigma' = \sigma\). Here, we restrict ourselves to the choice \(m = 4\) and \(\delta = 0.4\). We may often drop the subscripts and superscripts in these labels for the sake of brevity if additional information about the underlying basis is not relevant. Finally, let us also note that the basis functions involved in each of the methods are symmetrically translated copies of the zero index basis function, i.e., setting the number of basis functions to be \(N = 2N_{\text{shift}} + 1\), the maximum translation about the origin is given by \([\delta_{\text{shift}} = (\pi/\sigma) N_{\text{shift}}\) and \([\delta_{\text{shift}} = (\pi/\sigma') N_{\text{shift}}\) for WKS and HT, respectively.
Now, the estimate of computational complexity provided above excludes the cost of computing the samples of $p(\tau)$ needed to compute $\mathcal{P}$ and $p$ at each of the steps\(^1\). In view of the fact that function evaluations are, in general, expensive computationally, our algorithm should ensure that they are used optimally. Given that $\mathcal{P}$ is a Hankel matrix, it suffices to compute the first column and the last row which amounts to $2N - 1$ evaluations of $p(\tau)$. The vector $p$ is related to the column vector $(\mathcal{P}_k, 0)$; therefore, it does not require additional evaluations of the impulse response. Let the translation of $p(\tau)$, for the method WKS, be in the steps of $2h$ determined by

$$n_{os}h = \frac{\pi}{\sigma}, \quad h > 0, \quad n_{os} \geq 1,$$

where $n_{os}$ is referred to as the over-sampling factor. Consequently, the nodes over which one needs to sample $p(\tau)$ are of the form $\tau_j = jh$, $j \in \mathbb{Z}$. If the potential is supposed to be determined over the grid $t_k = kh$ where $k = -K, -K + 1, \ldots, K - 1, K$, and the number of basis functions is $N$, then the impulse response must be sampled at the grid points

$$\begin{cases} \tau_j = jh, & j = -N, -N + 1, \ldots, N - 1, N, \\ \mathcal{N} = n_{os}N + 2K. \end{cases}$$

For the method HT, $\sigma$ is replaced by $\sigma'$ in (105) while all the other aspects remain the same. For the examples considered in this section, we set $\sigma = 1$ and $n_{os} = 10$ unless otherwise stated.

With regard to the input required for the aforementioned methods, let us note that the nonlinear impulse response may not be available in a closed form. In fact, the inverse NFT is defined to take the reflection coefficient $\rho(\xi)$ as input. The samples of the impulse response can then be computed using the FFT algorithm with an appropriately large over-sampling factor. Alternatively, if extremely high accuracy is demanded, we may use methods that are specially de-

\(^1\) By steps we mean progressive translation of $p(\tau)$.
Bessel functions. In our tests, we have employed the latter
matter to the nonlinear impulse response in terms of the spherical
Legendre-Gauss-Lobatto (LGL) quadrature is used to ob-
gral [28, Chap. 3]. One such method, attributed to Bakhvalov
and Vasil’eva [29], is described in the Appendix B where
transferred symmetrically about the origin and the number of basis functions is
with
\[ N_{\text{shift}} \]
\[ \sigma \]
\[ \sigma' \]
\( \mu = 10, t_{\text{ref}} = 0 \)
\( \mu = 20, t_{\text{ref}} = 0 \)
\( \mu = 30, t_{\text{ref}} = 0 \)
\[ N_{\text{quad}} = 2000 \]
\[ N_{\text{quad}} = 1000 \]
\[ N_{\text{shift}} = 500 \]
\[ N_{\text{shift}} = 600 \]
\[ N_{\text{shift}} = 600 \]
\[ N_{\text{quad}} = 1000 \]
\[ N_{\text{quad}} = 2000 \]
\( \mu = 10, 20, 30 \) at the positions
\( \mu = 10, 20, 30 \) and the positions
\( t_{\text{ref}} = 0 \) (bottom row).
The basis functions used in either of these methods
are translated symmetrically about the origin and the number of basis functions is
In order to make the potential effectively
bandlimited, the scale parameter is chosen to be \( a = 80/\pi, 100/\pi, 150/\pi \), for \( \mu = 10, 20, 30 \), respectively. For the case \( \mu = 30 \), the number of
LGL quadrature nodes is \( N_{\text{quad}} = 2000 \) while for the rest \( N_{\text{quad}} = 1000 \). Before the error plateaus, the slope of the error curve is consistent with
an exponential rate of convergence.

FIG. 3. The figure shows the convergence analysis of the algorithms labelled WKS\( _\sigma \) and HT\( _{\sigma'}^{(m, \delta)} \) for the chirped secant-hyperbolic potential with \( \mu = 10, 20, 30 \) at the positions \( t_{\text{ref}} = 50 \) (top row) and \( t_{\text{ref}} = 0 \) (bottom row). The basis functions used in either of these methods
are translated symmetrically about the origin and the number of basis functions is \( N = 2N_{\text{shift}} + 1 \). In order to make the potential effectively
bandlimited, the scale parameter is chosen to be \( a = 80/\pi, 100/\pi, 150/\pi \), for \( \mu = 10, 20, 30 \), respectively. For the case \( \mu = 30 \), the number of
LGL quadrature nodes is \( N_{\text{quad}} = 2000 \) while for the rest \( N_{\text{quad}} = 1000 \). Before the error plateaus, the slope of the error curve is consistent with
an exponential rate of convergence.

FIG. 4. The figure shows the absolute error in the computed potential for the case of chirped secant-hyperbolic potential with \( \mu = 10, 20, 30 \). The computations are carried out with \( N_{\text{shift}} = 500 \) for the first two cases while the same for the last one is \( N_{\text{shift}} = 600 \). The number of LGL quadrature nodes for the first two cases is \( N_{\text{quad}} = 1000 \) while for the last one \( N_{\text{quad}} = 2000 \).

signed for highly oscillatory integrals such as the Fourier inte-
gral [28, Chap. 3]. One such method, attributed to Bakhvalov
and Vasil’eva [29], is described in the Appendix B where
Legendre-Gauss-Lobatto (LGL) quadrature is used to ob-
tain the nonlinear impulse response in terms of the spherical
Bessel functions. In our tests, we have employed the latter
method with the number of LGL nodes set to \( N_{\text{quad}} = 1000 \)
unless otherwise stated.

Now we turn to to the error analysis of the proposed meth-
ods. In these tests, we restrict ourselves to the case \( \kappa = -1 \). For the purpose of convergence analysis, we choose the
chirped secant-hyperbolic potential [30]:

\[ q(t) = A_0 \frac{\exp[-2i\mu A_0 \log(\cosh t)]}{\cosh(t)} \]

which is not a nonlinearly bandlimited signal, however, it can

FIG. 5. The figure shows a chirped bump function defined by (110) where \( A_0 = 10, \sigma = 1 \) and the chirp parameter \( \mu \in \{10, 20, 30\} \).

FIG. 6. The figure shows the numerically computed scattering potential corresponding to the chirped bump function as reflection coefficient depicted in Fig. 5.

FIG. 7. The figure shows the convergence analysis of the algorithms \( \text{WKS}_{\sigma} \) and \( \text{HT}_{(m, \delta)} \) for the chirped bump function as reflection coefficient defined by (110) with \( A_0 = 10 \) and \( \mu \in \{10, 20, 30\} \). The error is quantified by (111) (with the reference solution computed using HT with \( N_{\text{shift}} = 2000 \) and \( N_{\text{quad}} = 6000 \)) and the number of basis functions, symmetrically translated about the origin, is \( N = 2N_{\text{shift}} + 1 \). The slope of the error curve for the method WKS is unambiguously consistent with a second order of convergence. For the method HT, an algebraic rate of convergence better that \( O(N^{-10}) \) can be obtained using a linear fit before the error plateaus.
be considered effectively bandlimited. We assume that $\mu \geq 1$ so that the discrete spectrum is empty. The reflection coefficient is given by

$$
\rho(\xi) = \frac{A_0 e^{-2\mu A_0 (\log 2)}}{\Gamma \left( \frac{1}{2} - \frac{\omega}{2\mu} \right) \Gamma \left( \frac{1}{2} - \frac{2\omega}{A_0 \mu} \right) \Gamma \left( \frac{1}{2} - i\xi - iA_0 \mu \right)} \times \\
\frac{\Gamma \left( \frac{1}{2} - i\xi + iA_0 \mu \right) \Gamma \left( \frac{1}{2} - i\xi - iA_0 \mu \right)}{\Gamma \left( \frac{1}{2} - i\xi + i\lambda \right) \Gamma \left( \frac{1}{2} - i\xi - i\lambda \right)},
$$

where $\lambda = A_0 \mu \sqrt{1 - \mu^{-2}}, \quad \omega = \frac{2}{1 + \sqrt{1 - \mu^{-2}}}.$

As $|\xi| \to \infty$, the reflection coefficient decays as $\sim e^{-\pi|\xi|}$. We set $A_0 = 1$ and let $\mu \in \{10, 20, 30\}$. The potential corresponding to these choices of the parameters is shown Fig. 1 with the corresponding reflection coefficient shown in Fig. 2. In the tests, we take the input as $\rho(\alpha\xi)$ where $\alpha$ is large enough so that $\sigma = 1$, effectively. Given that the scattering potential at any point on the chosen grid can be computed independently of other points, it suffices to test the convergence of the methods at any arbitrary point, say, $t_{\text{ref}}$. We then quantify the error.

---

2 For a reflection coefficient which is not compactly supported, if $|\rho(\xi)| \leq C(1 + |\xi|)^{-\nu}$ for $\xi \in \mathbb{R}$ and some $\nu > 0$, then

$$
|\rho(\tau) - \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(\xi) e^{i\xi\tau} d\xi| \leq \frac{C}{\nu\pi(1 + \sigma)^{\nu}}, \quad \tau \in \mathbb{R}.
$$

Therefore, by choosing $\sigma$ large enough one can consider $\rho(\xi)$ as effectively bandlimited.
FIG. 10. The figure shows the contour plots of \( \log_{10}|Q_{mn}| \) for the method WKS (left) and the method HT\(_{(m, \delta)}\).

by

\[
\epsilon_{\text{ref}} = \frac{|q(t_{\text{ref}}) - q^{(\text{num.})}(t_{\text{ref}})|}{|q(t_{\text{ref}})|}.
\] (109)

Note that, for the determination of the rate of convergence, we resort to a direct solver for the linear system involved in order to avoid all possible sources of error.

The results of the convergence analysis are shown in Fig. 3. It turns out that the rate of convergence in these examples is superior than what is theoretically predicted. Both the methods exhibit exponential rate of convergence before plateauing of the error curves takes place. Note that the best accuracy achievable is remarkably close to the machine precision. Next, we may also want to examine the pointwise error in the computed potential over a set of grid points in order to ascertain if the CG iteration converges to the right solution. This is tested in Fig. 2 which is consistent with the error levels reported in the convergence analysis.

The next example is of a compactly supported reflection coefficient, the chirped “bump function”:

\[
\rho(\xi) = A_0 \exp \left[ -\frac{1}{1 - (\frac{\xi}{\sigma})^2n} + i\mu \left( \frac{\xi}{\sigma} \right)^2 \right] \chi_{[-\sigma,\sigma]}.
\] (110)

We set \( \sigma = 1, \quad A_0 = 10, \quad n = 1 \) and let \( \mu \in \{10, 20, 30\} \).

The potential corresponding to these choices of the parameters is shown in Fig. 6 with the corresponding reflection coefficient shown in Fig. 5. In the absence of a closed form solution of the inverse scattering problem, we choose to quantify the error by

\[
\epsilon_{\text{ref}} = \frac{|q^{(\text{ref.})}(t_{\text{ref}}) - q^{(\text{num.})}(t_{\text{ref}})|}{|q^{(\text{ref.})}(t_{\text{ref}})|},
\] (111)

where \( q^{(\text{ref.})} \) is the solution obtained using the method HT\(_{(m, \delta)}\) with \( N_{\text{shift}} = 2000 \) and \( N_{\text{quad.}} = 6000 \). The results of the error analysis in this example must be interpreted with caution because \( \epsilon_{\text{ref.}} \) is not the true numerical error. The results of this numerical experiment is shown in Fig. 7 where the method WKS shows an algebraic rate of convergence (which also turns out to be superior than what was predicted). However, the convergence behavior of HT is not immediately confirmed an algebraic rate because it seems to change to an exponential rate. To clarify this, let us compare the methods HT\(_{(\sigma, m)}\), \( m \in \{2, 4, 8\} \), for the chirped bump function as reflection coefficient defined by (110) with \( A_0 = 10, \quad n = 5 \) and \( \mu = 30 \). The results are shown in Fig. 9 where the plot on the right seems to confirm the earlier observation that convergence behavior might be exponential. Based on these observations it reasonable to expect that the HT method exhibits exponential convergence for Schwartz class impulse response. A theoretical justification for these observations is not available yet and we hope to address this in the future.

The pointwise error over a set of grid points is shown in Fig. 8. The reference solution in this case is computed using the fast inverse NFT reported in [1] with \( 2^{21} \) number of samples and the step-size is \( 2^{-10} \)-th of that used in the WKS or the HT method. The degree of agreement with the reference solution is consistent with the convergence behavior determined earlier.

A. Fast Solver using a Sparse Approximation

In this section, we would like to discuss how a fast variant of the method HT\(_{(m, \delta)}\) can be obtained by introducing a tolerance \( \epsilon \) to approximate its dense quadrature matrix with a sparse banded matrix. This idea is motivated by the contour plot of the quadrature matrix in Fig. 10. Clearly, the quadrature matrix for the method HT exhibits an effectively banded structure compared to that of WKS. The nature of the contour map of \( Q \) for HT can be easily understood as follows: Recall-
Then, without loss of generality, we can assume \( \pi \leq m \) and let \( m, n < 0 \), we have

\[
|Q_{nl}| \leq \left( \frac{m}{\pi \delta} \right)^m |n|^{-m} |\varphi_s| L^2. \tag{113}
\]

If only \( n < 0 \), then

\[
|Q_{nl}| \leq \left( \frac{m}{\pi \delta} \right)^m |n|^{-m} |\varphi_s| L^2. \tag{114}
\]

Appealing to the symmetric nature of \( Q \), similar conclusion holds for \( l < 0 \). Therefore, the dense part of the matrix \( Q \) falls in the quadrant where \( n, l > 0 \). Consider \( n, l > 0 \) and \( n \neq l \). Then, without loss of generality, we can assume \( n < l \) so that

\[
|Q_{nl}| \leq \int_0^{\infty} |\varphi_n(s)\varphi_l(s)| ds + \int_0^{\infty} \int_{m+l+1}^{\infty} |\varphi_n(s)\varphi_l(s)| ds \leq 2\left( \frac{m}{\pi \delta} \right)^m |n|^{-m} |\varphi_s| L^2 \tag{115}
\]

For \( \epsilon > 0 \),

\[
|n - l| \geq \left( \frac{m}{\pi \delta} \right)^{1/m} \left( \frac{\epsilon}{\pi} \right)^{1/m},
\]

ensures that \( |Q_{nl}| \leq \epsilon \). Based on the preceding inequalities, one can define the number of dominant diagonals, say, \( 2N_{\text{band}} \) by

\[
N_{\text{band}}(\epsilon) = \left[ \left( \frac{2m}{\pi \delta} \right) \left( \frac{\epsilon}{\pi} \right)^{1/m} \right] + 1, \tag{116}
\]

where \( \lceil x \rceil \) denotes the integral part of \( x \in \mathbb{R}_+ \). While this estimate is important as it sets the upper bound\(^3\), it is not so useful in practice because it greatly overestimates the number of dominant diagonals. Given that the quadrature matrix needs to be computed only once, it is rather easy to check the entries directly and determine the sparsity of this matrix. We choose to set this tolerance to be \( \epsilon = 10^{-12} \). Let \( Q_f \) and \( Q_f^{1/2} \) denote the banded matrices derived from the dense matrices \( Q \) and \( Q_f^{1/2} \), then the linear system in (103) can be approximated by

\[
(I - \kappa Q_f^{1/2} P Q_f^{1/2} P) u_{1,\epsilon} = Q_f^{1/2} p. \tag{117}
\]

where \( u_{1,\epsilon} \) approximates \( u_1 \). Let us now estimate the cost of one CG iteration if the matrix-vector multiplications involved are carried out in a cascaded fashion. The cost of multiplying \( Q_f \) or \( Q_f^{1/2} \) with a vector is \( \mathcal{O}(N_{\text{band}}N) \), the cost of multiplying \( P \) or \( P_f \) with a vector is \( \mathcal{O}(N \log N) \) (where we exploit the fact that they are Hankel matrices). Therefore the total cost of one CG iteration is \( \mathcal{O}(N \log N) + \mathcal{O}(N_{\text{band}}N) \). In the asymptotic limit \( \log N \gg N_{\text{band}} \) so that the cost works out to be \( \mathcal{O}(N \log N) \). Therefore, the total cost per sample of the scattering potential works out to be \( \mathcal{O}(N_{\text{iter}}N \log N) \). Finally, let us observe that the approximation introduced above adds an error of \( \mathcal{O}(N \epsilon) \) to the original error estimates.

The fast method obtained above can be tested against the original method to determine its convergence and run-time behavior. The results of the numerical experiment with the chirped secant-hyperbolic profile (\( \mu = 10 \)) is shown in Fig. 11. Here the average run-time is the run-time per sample averaged over the number of basis functions \( N \in \{2^5, \ldots, 2^{11}\} \). Note that the improvement in the complexity comes at a price of accuracy as evidenced by somewhat early plateauing of error in Fig. 11.

### IV. Conclusion

To conclude, we have presented a sampling theory approach to inverse scattering transform which is shown to

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3 This bound can facilitate a search based algorithm to look for more precise value of the number of dominant diagonals. We leave these issues for future research.
achieve algebraic orders of convergence provided the regularity conditions on the input data is fulfilled. The convergence behavior observed in the numerical experiments with Schwartz class (bandlimited or effectively bandlimited) impulse response tends to exhibit exponential orders of convergence. We hope to improve our theoretical estimates to explain these observations in the future. The complexity of the proposed algorithms depend on the linear solvers used. A conjugate gradient based iterative solver exhibits a complexity of $O(N^2 \log N)$ per sample of the signal computed where $N$ is the number of sampling basis functions used. Using a variant of the classical sampling series due to Helms and Thomas, we were able to achieve a complexity of $O(N \log N)$ by exploiting the Hankel symmetry and approximately banded structure of the matrices involved. The bandedness of the so-called quadrature matrix can be controlled by a tolerance $\epsilon$ which introduces an error of $O(N \epsilon)$ in the computed solution.

Finally, let us remark that, apart from the avenues of improvement mentioned above, one can identify several other ways the performance of the proposed algorithms can be improved. The first one has to do with the nature of the basis functions itself. We know from the work of Kaiblinger and Madych [31] that orthonormal sampling functions with rapid decay can be designed which can potentially reduce the errors committed in arriving at an effectively sparse quadrature matrix. Secondly, the seed for iterative solvers is obtained by using a direct solver at least once in order to start the algorithm when computing the signal over a grid. In a parallel implementation this would no longer be a good choice; therefore, our algorithm can benefit greatly from a cheaper method of “guessing” the seed.

[1] V. Vaibhav, Phys. Rev. E 98, 013304 (2018).
[2] V. Vaibhav, Phys. Rev. E 96, 063302 (2017).
[3] J. Skaar, B. Sahlgren, P.-Y. Fonjallaz, H. Storøy, and R. Stubbie, Opt. Lett. 23, 933 (1998).
[4] R. Feced, M. N. Zervas, and M. A. Muriel, IEEE J. Quantum Electron. 35, 1105 (1999).
[5] R. Feced and M. N. Zervas, J. Opt. Soc. Am. A 17, 1573 (2000).
[6] J. K. Brenne and J. Skaar, J. Lightwave Technol. 21, 254 (2003).
[7] S. K. Turitsyn, E. Prilepsky, T. L. Se, W. Wahl, L. L. Frumin, M. Kamalian, and S. A. Derevyanko, Optica 4, 307 (2017).
[8] A. Rosenthal and M. Horowitz, IEEE J. Quantum Electron. 39, 1018 (2003).
[9] O. V. Belai, L. L. Frumin, E. V. Podivilov, and D. A. Shapiro, J. Opt. Soc. Am. B 24, 1451 (2007).
[10] L. L. Frumin, O. V. Belai, E. V. Podivilov, and D. A. Shapiro, J. Opt. Soc. Am. B 32, 290 (2015).
[11] V. Vaibhav, J. Phys. A: Math. Theor. 51, 425201 (2018).
[12] C. van der Mee, S. Seatzu, and D. Theis, Calcolo 49, 59 (2007).
[13] T. Trogdon and S. Olver, Proc. Royal Soc. Lond. A.
[14] [15] E. Atkinson, The numerical solution of integral equations of the second kind, Cambridge monographs on applied and computational mathematics 4 (Cambridge University Press, Cambridge, 2009).
[16] K. Khare and N. George, J. Phys. A: Math. Gen. 36, 10011 (2003).
[17] V. Vaibhav, (2018), arXiv:1804.04713[math.NA].
[18] F. A. Marvasti, ed., Nonuniform Sampling: Theory and Practice, 1st ed., Information Technology: Transmission, Processing and Storage (Springer US, New York, 2001).
[19] H. D. Helms and J. B. Thomas, Proceedings of the IRE 50, 179 (1962).
[20] D. Jagerman, SIAM Journal on Applied Mathematics 14, 714 (1966).
[21] V. E. Zakharov and A. B. Shabat, Sov. Phys. JETP 34, 62 (1972).
[22] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Stud. Appl. Math. 53, 249 (1974).
[23] M. Ablowitz and H. Segur, Solitons and the Inverse Scattering Transform (Society for Industrial and Applied Mathematics, Philadelphia, 1981).
[24] S. M. Nikol’ski˘ı, Approximation of Functions of Several Variables and Imbedding Theorems, 1st ed., Grundlehren der mathematischen Wissenschaften (Springer-Verlag, Heidelberg, 1975).
[25] R. P. Boas, Entire Functions, Pure and Applied Mathematics (Academic Press Inc., New York, 1954).
[26] K. Yosida, Functional Analysis, 2nd ed., Die Grundlehren der mathematischen Wissenschaften (Springer, Berlin, 1995).
[27] V. Peller, Hankel Operators and Their Applications, 1st ed., Springer Monographs in Mathematics (Springer-Verlag, New York, 2003).
[28] C. S. Kubrusly, Spectral Theory of Operators on Hilbert Spaces, 1st ed. (Birkhäuser, Basel, 2012).
[29] P. J. Davis and P. Rabinowitz, Methods of numerical integration, Computer science and applied mathematics (Academic Press, San Diego, 1984).
[30] N. S. Bakhvalov and L. G. Vasil’eva, USSR Comput. Math. Math. Phys. 8, 241 (1968).
[31] A. Tovbis, S. Venakides, and X. Zhou, Commun. Pure Appl. Math. 57, 877 (2004).
[32] N. Kaiblinger and W. R. Madych, Appl. Comput. Harmon. Anal. 21, 404 (2006).
[33] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, eds., NIST Handbook of Mathematical Functions (Cambridge University Press, New York, 2010).
[34] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, Spectral Methods: Fundamentals in Single Domains (Springer-Verlag, Heidelberg, 2007).
[35] C. W. Clenshaw, Math. Comput. 9, 118 (1955).
[36] P. Deuflhard, Computing 17, 37 (1976).

Appendix A: The quadrature matrix

The entries of the quadrature matrix, denoted by $Q$, are defined as

$$Q_{ln} = \int_{0}^{\infty} \psi_l(s) \psi_n(s) ds, \quad l, n \in \mathbb{Z}. \quad (A1)$$
It is possible to compute these integrals in terms of the Sine and the Cosine integrals which are defined as [32, Chap. 6]

\[
\text{Si}(t) = \int_0^t \frac{\sin s}{s} \, ds, \quad \text{Ci}(t) = -\int_t^\infty \frac{\cos s}{s} \, ds,
\]

\[
(A2)
\]

where \( \gamma \) is the Euler’s constant. The diagonal entries of \( Q \) works out to be

\[
Q_{nn} = \frac{1}{\pi} \int_0^\infty \frac{\sin^2 t}{(t - m\pi)(t - n\pi)} \, dt = \frac{1}{\pi} \sin(2m\pi),
\]

\[
(A4)
\]

and

\[
\text{Si}(t) = \text{Si}(t) - \frac{\pi}{2}, \quad \text{Ci}(t) = -\text{Ci}(t) + \log t + \gamma,
\]

\[
(A3)
\]

Turning to the off-diagonal elements, we have

\[
Q_{mn} = \frac{(-1)^{m+n}}{\pi} \int_0^\infty \frac{\sin^2 t}{(m-n\pi)(t - n\pi)} \, dt, \quad m \neq n.
\]

\[
(A5)
\]

Note that the integrand of \( Q_{mn} \) is an entire function of \( t \). For the moment, assuming that the origin does not coincide with \( m\pi \) or \( n\pi \), one can deform the contour of integration to write

\[
2\pi^2(m-n)(-1)^{m+n}Q_{mn}
= \lim_{R \to \infty} \int_0^R \left[ \frac{1}{(t - m\pi)} - \frac{1}{(t - n\pi)} \right] dt
- \int_0^\infty \left[ \frac{1}{(t - m\pi)} - \frac{1}{(t - n\pi)} \right] \cos(2t) dt
= -\log\left(\frac{m}{n}\right) + \text{Ci}(2m\pi) - \text{Ci}(2n\pi),
\]

which yields

\[
Q_{mn} = -\frac{(-1)^{m+n}}{2\pi^2(m-n)} \left[ \text{Ci}(2m\pi) - \text{Ci}(2n\pi) \right], \quad m \neq n.
\]

\[
(A6)
\]

Note that the final result does not have any singularities; therefore, we conclude that it is valid for all \( m, n \in \mathbb{Z}, m \neq n \). Using the symmetry properties of the Sine and Cosine integrals, we have

\[
Q_{mn} = \begin{cases} \frac{1}{2} - \frac{1}{2}\gamma, & \text{if } m = n, \\ \frac{1}{2\pi^2(m-n)} \left[ \text{Ci}(2m\pi) - \text{Ci}(2n\pi) \right], & \text{if } m \neq n. \end{cases}
\]

\[
(A7)
\]

Appendix B: Computing the nonlinear impulse response

The input to the inverse NFT is the reflection coefficient \( \rho(\xi) \); however, the GLM equation based approach requires us to compute the nonlinear impulse response which is defined by

\[
p(\tau) = \frac{\sigma}{2\pi} \int_0^1 \rho(\sigma\xi)e^{i\sigma t} \, d\xi.
\]

\[
(B1)
\]

Ordinarily this integral can be computed quite efficiently using the FFT algorithm which is based on the trapezoidal rule. For large values of the quantity \( \sigma\tau \), the accuracy of the trapezoidal rule may degrade; therefore, if extremely high degree of accuracy is demanded we must turn to other alternatives. It is well known that Gauss-type quadrature schemes tend to perform poorly in computing these integrals on account of the oscillatory nature of the integrand which deviates considerably from polynomials, specially for larger values of \( \sigma\tau \). There is a vast amount of literature devoted to treating such problems, for instance, see [28, Section 2.10] and the references therein. Here, we would like to choose the method due to Bakhvalov and Vasil’eva [29] which begins with the series expansion

\[
\rho(\sigma\xi) = \sum_{n=0}^\infty \hat{\rho}_n L_n(\xi), \quad \xi \in (-1, 1),
\]

\[
(B2)
\]

where \( L_n(t) \) denotes the Legendre polynomials. Using the Legendre-Gauss-Lobatto (LGL) nodes, a finite dimensional approximations of \( \hat{\rho}_n \) can be obtained via the Legendre transform [33]. Let \( j_n(t) \) denote the spherical Bessel function of the first kind [32, Chap. 10]. Now, in order to obtain the exact result, we recall the identity

\[
\int_{-1}^1 L_n(\xi)e^{i\xi t} \, d\xi = 2\pi i j_n(\sigma t).
\]

\[
(B3)
\]

Plugging (B2) into (B1), we have

\[
p(\tau) = \frac{\sigma}{\pi} \sum_{n=0}^\infty \hat{\rho}_n e^{i\sigma t} j_n(\sigma t).
\]

\[
(B4)
\]

With precomputed LGL nodes and associated weights, the complexity of obtaining \( \hat{\rho}_n \), \( n = 0, 1, N_{\text{quad}} - 1 \), is \( O(N_{\text{quad}}^2) \) excluding the cost of evaluating \( \rho(\xi) \). For an efficient method of evaluation of the resulting series for \( p(\tau) \), one may use the Clenshaw’s algorithm [34, 35] which makes efficient use of the recurrence relation for the spherical Bessel functions.