Orthogonal Laurent polynomials in unit circle, 
extended CMV ordering and 2D Toda type integrable hierarchies

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Abstract

Orthogonal Laurent polynomials in the unit circle and the theory of Toda-like integrable systems are connected using the Gauss–Borel factorization of a Cantero–Moral–Velázquez moment matrix, which is constructed in terms of a complex quasi-definite measure supported in the unit circle. The factorization of the moment matrix leads to orthogonal Laurent polynomials in the unit circle and the corresponding second kind functions. Jacobi operators, 5-term recursion relations and Christoffel–Darboux kernels, projecting to particular spaces of truncated Laurent polynomials, and corresponding Christoffel–Darboux formulae are obtained within this point of view in a completely algebraic way. Cantero–Moral–Velázquez sequence of Laurent monomials is generalized and recursion relations, Christoffel–Darboux kernels, projecting to general spaces of truncated Laurent polynomials and corresponding Christoffel–Darboux formulae are found in this extended context. Continuous deformations of the moment matrix are introduced and is shown how they induce a time dependent orthogonality problem related to a Toda-type integrable system, which is connected with the well known Toeplitz lattice. Using the classical integrability theory tools the Lax and Zakharov–Shabat equations are obtained. The dynamical system associated with the coefficients of the orthogonal Laurent polynomials is explicitly derived and compared with the classical Toeplitz lattice dynamical system for the Verblunsky coefficients of Szegő polynomials for a positive measure. Discrete flows are introduced and related to Darboux transformations. Finally, the representation of the orthogonal Laurent polynomials (and its second kind functions), using the formalism of Miwa shifts, in terms of $\tau$-functions is presented and bilinear equations are derived.

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1 Introduction

This paper puts focus on orthogonal Laurent polynomials in the unit circle (OLPUC), a subject with strong links to that of orthogonal polynomials in the unit circle (OPUC), see [61], [56] and [57]. Despite that is well established that this matter is as source of interesting problems and applications in approximation theory, we are mainly interested in its connections with the general theory of integrable systems.

Let us introduce here some notation that will be used along this article. We will denote the unit circle by $\mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \}$, $\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}$ stands for the unit disk and $\Lambda_{[p,q]} := \text{span}\{ z^{-p}, z^{-p+1}, \ldots, z^q \}$ denotes the linear space of complex Laurent polynomials with their corresponding restrictions on their degrees while $\Lambda_{[\infty]}$ for the infinite set of Laurent polynomials. When $z \in \mathbb{T}$ we will use the parametrization $z = e^{i\theta}$ with $\theta \in [0, 2\pi)$.

A complex Borel measure $\mu$ supported in $\mathbb{T}$ is said to be positive definite if it maps measurable sets into non-negative numbers. When the measure $\mu$ is absolutely continuous with respect to the Lebesgue measure of the circle $d\theta$, according to the Radon–Nikodym theorem, it can be always expressed using a complex weight function $w$, so that $d\mu(\theta) = w(\theta)d\theta$. If in addition the measure is positive definite then the weight function should be a non-negative function on $\mathbb{T}$. For notational simplicity we will use, whenever it is convenient, the complex notation $d\mu(z) = ie^{i\theta}d\mu(\theta)$. If $\mu$ is a positive Borel measure supported in $\mathbb{T}$, then the OPUC or Szegö polynomials are monic polynomials $P_n$ of degree equal or less than $n$ that satisfy the following system of equations, the orthogonality relations,

$$\int_{\mathbb{T}} P_n(z)z^{-k}d\mu(z) = 0, \quad k = 0, 1, \ldots, n - 1. \quad (1)$$

It is well known the deep connections between orthogonal polynomials in the real line (OPRL) in $[-1, 1]$ and OPUC (e.g. [31, 17]). Let us observe that for this analysis the use of spectral theory techniques requires the study of the operator of multiplication by $z$. The study of the matrix associated to this operator leads to recurrence laws. Both, OPRL and OPUC have recurrence laws, but the main difference is that in the real case three term recurrence laws provide a tridiagonal matrix, the so called Jacobi operator, while in the circle case one is lead to a Hessenberg matrix [37], leading to a more involved scenario to deal with than the Jacobi case (as it is not a sparse matrix with a finite number of non vanishing diagonals). More precisely, the study of the recurrence relations for the OPUC requires the definition of the reciprocal or reverse Szegö polynomials $P_l^*(z) := z^lP_l(z^{-1})$ and the reflection or Verblunsky coefficients $\alpha_l := P_l(0)$.

\[\text{Schur parameters is another usual name. The definition is not unique and } \alpha_l := -\overline{P_{l+1}(0)} \text{ is another common definition.}\]
With these elements the recursion relations for the Szegő polynomials can be written as

\[
\begin{pmatrix}
    P_l \\
    P_l^*
\end{pmatrix} = \begin{pmatrix}
    z & \alpha_l \\
    z\bar{\alpha}_l & 1
\end{pmatrix} \begin{pmatrix}
    P_{l-1} \\
    P_{l-1}^*
\end{pmatrix}.
\]

(2)

There has been a relevant number of studies on the zeroes of the OPUC, see for example [10, 14, 15], or [32, 35, 47, 51], which have interesting applications to signal analysis theory, see [39, 41, 52, 53]. Despite of that the situation is still far from the corresponding state of the art in the OPRL context. A second important issue is the fact that the set of Szegő polynomials is in general not dense in the Hilbert space \( L^2(\mathbb{T}, \mu) \). As it follows from Szegő’s theorem it holds that for a nontrivial probability measure on \( \mathbb{R} \) with Verblunsky coefficients \( \{\alpha_n\}_{n=0}^\infty \) the Szegő’s polynomials are dense in \( L^2(\mathbb{T}, \mu) \) if and only if \( \prod_{n=0}^\infty (1 - |\alpha_n|^2) = 0 \). If \( \mu \) is an absolutely continuous probability measure then the Kolmogorov’s density theorem follows: polynomials are dense in \( L^2(\mathbb{T}, \mu) \) if and only if the so called Szegő’s condition \( \int_{\mathbb{T}} \log(\|\theta\|)d\theta = -\infty \) is fulfilled [60].

Orthogonal Laurent polynomials in the real line (OLPRL), where introduced in [42, 43] in the context of the strong Stieltjes moment problem, i.e finding a positive measure \( \mu \) such that its moments

\[
m_j = \int_{\mathbb{R}} x^j d\mu(x) \quad j = 0, \pm 1, \pm 2, \ldots
\]

are known in advance. When the moment problem has a solution, there exist polynomials \( \{Q_n\} \) such that

\[
\int_{\mathbb{R}} x^{-n+j}Q_n(x)d\mu(x) = 0, \quad j = 0, \ldots, n-1,
\]

(4)

which are called Laurent polynomials or L-polynomials. The theory of Laurent polynomials on the real line was developed in parallel with the theory of orthogonal polynomials, see [22, 29, 10] and [50]. Orthogonal Laurent polynomials theory was carried from the real line to the circle [62] and subsequent works broadened the matter (e.g. [25, 21, 23, 24]), treating subjects like recursion relations, Favard’s theorem, quadrature problems, and Christoffel–Darboux formulae.

The analysis of OLPUC, and specially the use of the Cantero–Moral–Velázquez (CMV) [21] representation is very helpful in the study of a number properties of Szegő polynomials. Different reasons support this statement, for example as we mentioned while the OLPUC are always dense in \( L^2(\mathbb{T}, \mu) \) this is not true in general for the OPUC, see [18] and [25], and also the bijection between OLPUC in the CMV representation and the ordinary Szegő polynomials allows to replace the complicated recurrence relations with a five-term recurrence relation more alike to the structure of the OPRL. The representation of the operator of multiplication by \( z \) is much more natural using CMV matrices than using Hessenberg matrices. This is the main motivation for us in order to take CMV matrices as a essential element in our scheme. Other papers have reviewed and broadened the study of CMV matrices, see for example [58, 41]. Alternative or generic orders in the base used to span \( \Lambda_l[\infty] \) can be found in [24].

The approach to the integrable hierarchies that we use here is based on the Gauss–Borel factorization. The seminal paper of M. Sato [55] and further developments performed by the Kyoto school [26, 28] settled the Lie-group theoretical description of the integrable hierarchies. It was M. Mulase, in the key paper [49], the one who made the connection between factorization problems, dressing procedures and integrability. In this context, K. Ueno and T. Takasaki [63] performed an analysis of the Toda-type hierarchies and their soliton-like solutions. In a series of papers, M. Adler and P. van Moerbeke [3, 9], made clear the connection between the Lie-group factorization, applied to Toda-type hierarchies (what they call discrete K-P) and the Gauss–Borel factorization applied to a moment matrix that comes from an orthogonality problems; thus, the corresponding orthogonal polynomials are closely related to specific solutions of the integrable hierarchy. See [45] and [11] for further developments in relation with the factorization problem, multicomponent Toda lattices and generalized orthogonality. In the paper [8] a profound study of the OPUC and the Toda-type associated lattice, called the Toeplitz lattice (TL), was performed. A relevant reduction of the equations of the TL has been found by L. Golinskii [36] in the context of Schur flows when the measure is invariant.
under conjugation, another interesting paper on this subject is [35]. The Toeplitz Lattice has been proved equivalent to the Ablowitz–Ladik lattice (ALL), [11 2], and that work has been generalized to the link between matrix orthogonal polynomials and the non-Abelian ALL in [20]. Both of them have to deal with the Hessenberg operator for the multiplication by $z$.

Our aim is to explore the connection between Toda-type integrable systems and orthogonality in the circle from a different point of view. As we proof in this paper the CMV representation is a bridge between the factorization techniques used in [12] and the circular case. We will see that many results obtained in [12] about Christoffel–Darboux (CD) formulae, continuous and discrete deformations, and $\tau$-function theory can be extended to the circular case under the suitable choice of moment matrices and shift operators.

Let us recall the reader that measures and linear functionals are closely connected; given a linear functional $L$ on $\Lambda_\infty$ we define the corresponding moments of $L$ as $c_n := L[z^n]$ for all the integer values of $n \in \mathbb{Z}$. The functional $L$ is said to be Hermitian whenever $c_{-n} = \overline{c_n}$, $\forall n \in \mathbb{Z}$. Moreover, the functional $L$ is defined as quasi-definite (positive definite) when the principal submatrices of the Toeplitz moment matrix $(\Delta_{i,j})$, $\Delta_{i,j} := c_{i-j}$, associated to the sequence $c_n$ is non-singular (positive definite), i.e. $\forall n \in \mathbb{Z}, \Delta_n := \det(c_{i-j})_{i,j=0}^n \neq 0(> 0)$. Some aspects on quasi-definite functionals and their perturbations are studied in [13, 19].

It is known [34] that when the linear functional $L$ is Hermitian and positive definite there exist a finite positive Borel measure with a support lying on $\mathbb{T}$ such that $L[f] = \int f d\mu$, $\forall f \in \Lambda_\infty$. In addition, an Hermitian positive definite linear functional $L$ defines a sesquilinear form $\langle \cdot, \cdot \rangle_L : \Lambda_\infty \times \Lambda_\infty \rightarrow \mathbb{C}$ as $\langle f, g \rangle_L = L[\overline{g}f]$, $\forall f, g \in \Lambda_\infty$. Two Laurent polynomials $\{f, g\} \subset \Lambda_\infty$ are said to be orthogonal with respect to $L$ if $\langle f, g \rangle_L = 0$. From the properties of $L$ it is easy to see that $\langle \cdot, \cdot \rangle_L$ is a scalar product and if $\mu$ is the positive finite Borel measure associated to $L$ we are lead to the corresponding Hilbert space $L^2(\mathbb{T}, \mu)$, the closure of $\Lambda_\infty$. As we mentioned before $\{P_l\}_{l=0}^\infty$ denotes the set of monic orthogonal polynomials, $\deg P_l \leq l$, for a positive measure $\mu$ satisfying [1] and therefore $\{P_l\}_{l=0}^\infty$ is an orthogonal basis of the space of truncated polynomials $\Lambda_{[0,q]}$.

In this work we allow for a more general setting assuming that $L$ is just quasi-definite, which is associated to a corresponding quasi-definite complex measure $\mu$, see [33]. As before a sesquilinear form $\langle \cdot, \cdot \rangle_L$ is defined for any such linear functional $L$; thus, we just have the linearity (in the first entry) and skew-linearity (in the second entry) properties. However, we have no symmetry allowing the interchange of the two arguments. We formally broaden the notion of orthogonality and say that $f$ is orthogonal to $g$ if $\langle f, g \rangle_L = 0$, but we must be careful as in this more general situation it could happen that $\langle f, g \rangle_L = 0$ but $\langle g, f \rangle_L \neq 0$.

The layout of this paper goes as follows. In [2] we present the application of the Gauss–Borel factorization of a CMV moment matrix to the construction of OLPUC, associated second kind functions, 5-term recursion relations and CD formulae. In [33] we perform a similar work using a more general sequence that the one used in [21]. This allows us to study snake-shaped (as are denoted in [24]) recursion formulae. Moreover, the CD formulae we derive for these extended cases is the kernel for the orthogonal projection to the general space of truncated Laurent polynomials, $\Lambda_{[p,q]}$ with $p, q \in \mathbb{N}$. This is a large generalization of the CMV situation as the possible spaces of truncated Laurent polynomials is very particular, namely either $\Lambda_{[l,l]}$ or $\Lambda_{[l+1,l]}$ with $l \in \mathbb{N}$. To conclude, in [4] we study the deformations of the moment matrices to obtain the integrable equations associated, the representation of the OLPUC and its associated second kind functions using $\tau$-functions and corresponding bilinear identities.

2 Orthogonal Laurent polynomials in the circle, $LU$ factorization and the CMV ordering

In this section we use the Gauss–Borel, also known as $LU$ or Gaussian, factorization of an infinite dimensional matrix, that we refer to as moment matrix, to derive bi-orthogonal Laurent polynomials in the unit circle (BOLPUC) to the given measure $\mu$—that, as we will see, are closely related to the Szegő polynomials—, associated second kind functions in terms of the Fourier series of the measure, their recursion relations and
corresponding CD formulae very much in the spirit of [12]. The key idea is to use the results of [21] to construct a very specific moment matrix that will lead to the mentioned results.

2.1 Biorthogonal Laurent polynomials

We are ready to, using a CMV moment type matrix, find a set of bi-orthogonal Laurent polynomials, its connection with Szegő polynomials and corresponding determinantal expressions. For this aim, let us consider the basic object fixing the CMV order of the Fourier family \( \{z^j\}_{j \in \mathbb{Z}} \). This order allows us to work in realm of the semi-infinite matrices avoiding in this way the less convenient scenario of bi-infinite matrices, very much as in the multiple OPRL situation [12].

**Definition 1.** We denote

\[
\chi_1(z) := (1, 0, z, 0, z^2, 0, \ldots)^T, \quad \chi_2(z) := (0, 1, 0, z, 0, z^2, \ldots)^T, \quad \chi_a(z) := z^{-1}\chi_a(z^{-1}), \quad a = 1, 2.
\]

\[
\chi(z) := \chi_1(z) + \chi_2(z) = (1, z^{-1}, z, z^{-2}, \ldots)^T, \quad \chi^*(z) := \chi_1^*(z) + \chi_2(z) = (z^{-1}, 1, z^{-2}, z, \ldots)^T.
\]

With these sequences at hand and with a given quasi-definite complex Borel measure \( \mu \) supported on \( \mathbb{T} \) we define the CMV moment matrix

**Definition 2.** The CMV moment matrix is the following semi-infinite complex-valued matrix\(^2\)

\[
g := \oint_{\mathbb{T}} \chi(z)\chi(z)^\dagger d\mu(z). \tag{5}
\]

The LU factorization, which will play an important role in what follows, is

\[
g = S_1^{-1}S_2, \tag{6}
\]

where \( S_1 \) is a normalized\(^3\) lower triangular matrix and \( S_2 \) is an upper triangular matrix. Hence we write

\[
S_1 = \begin{pmatrix}
1 & 0 & 0 & \cdots \\
(S_1)_{10} & 1 & 0 & \cdots \\
(S_1)_{20} & (S_1)_{21} & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad S_2 = \begin{pmatrix}
(S_2)_{00} & (S_2)_{01} & (S_2)_{02} & \cdots \\
0 & (S_2)_{11} & (S_2)_{12} & \cdots \\
0 & 0 & (S_2)_{22} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

This Gaussian factorization for the moment matrix makes sense if all the principal minors are non-singular, which is precisely the quasi-definiteness condition for the measure \( \mu \). More on the algebraic Gauss–Borel (or LU) factorization and its connection with integrable systems can be read in [30].

With the aid of these matrices we consider the sequences

\[
\Phi_{1,1} := S_1\chi_1, \quad \Phi_{1,2} := S_1\chi_2^*, \quad \Phi_{2,1} := (S_2^{-1})^\dagger\chi_1, \quad \Phi_{2,2} := (S_2^{-1})^\dagger\chi_2^*.
\]

which can be written as semi-infinite vectors

\[
\Phi_{1,a}(z) = \begin{pmatrix}
\varphi_{1,a}^{(0)}(z) \\
\varphi_{1,a}^{(1)}(z) \\
\vdots
\end{pmatrix}, \quad \Phi_{2,a}(z) = \begin{pmatrix}
\varphi_{2,a}^{(0)}(z) \\
\varphi_{2,a}^{(1)}(z) \\
\vdots
\end{pmatrix},
\]

\(^2\)The reader should notice that if \( \mu \) is a positive measure then \( g \) is a definite positive Hermitian matrix; i.e., \( g = g^\dagger \).

\(^3\)The coefficients in the main diagonal are equal to the unity.
for \( a = 1, 2 \). The corresponding components \( \varphi_{b,1}^{(l)} \) are polynomials of degree \( l \) in variable \( z \), while \( \varphi_{b,2}^{(l)} \) are polynomials of degree \( l + 1 \) in the variable \( z^{-1} \) which vanish at \( z = \infty \). Inspired by the multiple orthogonal case \[12\] we define the following sequences of Laurent polynomials

\[
\Phi_1(z) = S_1 \chi(z) = \Phi_{1,1} + \Phi_{1,2}, \quad \Phi_2(z) = (S_2^{-1})^T \chi(z) = \Phi_{2,1} + \Phi_{2,2},
\]

which are semi-infinite vectors that we write in the form

\[
\Phi_1(z) = \begin{pmatrix} \varphi_{1}^{(0)}(z) \\ \varphi_{1}^{(1)}(z) \\ \vdots \end{pmatrix}, \quad \Phi_2(z) = \begin{pmatrix} \varphi_{2}^{(0)}(z) \\ \varphi_{2}^{(1)}(z) \\ \vdots \end{pmatrix}
\]

with its coefficients being Laurent polynomials.

As we said in \[11\] the measure \( \mu \) has an associated sesquilinear form \( \langle \cdot, \cdot \rangle_\varphi \) acting on any pair of Laurent polynomials in \( T \), \( \varphi_1(z) \) and \( \varphi_2(z) \), as

\[
\langle \varphi_1, \varphi_2 \rangle_\varphi := \oint_T \varphi_1(z) \bar{\varphi}_2(z^{-1}) d\mu(z).
\]

From the definition is clear that \( g \) (whose principal minors should not vanish) is the matrix associated to \( \langle \cdot, \cdot \rangle_\varphi \). The reader can check that the principal minors of \( g \) are exactly the Toeplitz minors \( \Delta_n \) (one matrix is obtained from the other using permutations). As we are going to use quasi-definite measures the factorization condition will hold in the subsequent work.

We recall the reader that given two linear spaces \( V, V' \) and a sesquilinear form

\[
\langle \cdot, \cdot \rangle : \quad V \times V' \rightarrow \mathbb{C}, \quad \langle x, y \rangle \mapsto \langle x, y \rangle_\varphi
\]

we say that sets \( X \subset V \) and \( Y \subset V' \) are bi-orthogonal if \( \langle x, y \rangle_\varphi = 0 \) for all \( x \in X \) and \( y \in Y \).

**Theorem 1.** The sets of Laurent polynomials \( \{\varphi_1^{(l)}\}_{l=0}^{\infty} \) and \( \{\varphi_2^{(l)}\}_{l=0}^{\infty} \) are bi-orthogonal in the unit circle with respect to the sesquilinear form defined in \[11\], that is

\[
\langle \varphi_1^{(l)}, \varphi_2^{(k)} \rangle_\varphi = \oint_T \varphi_1^{(l)}(z) \bar{\varphi}_2^{(k)}(z^{-1}) d\mu(z) = \delta_{l,k} \quad l, k = 0, 1, \ldots
\]

**Proof.** We compute

\[
\oint_T \Phi_1(z) \Phi_2(z^{-1})^\top d\mu(z) = \oint_T \Phi_1(z) \Phi_2(z)^\dagger d\mu(z) \\
= S_1 \left[ \oint_T \chi(z) \chi(z)^\dagger d\mu(z) \right] S_2^{-1} \\
= \mathbb{I}.
\]

Orthogonality relations \[5\] can alternatively be expressed as follows

\[
\langle \varphi_1^{(2l)}, z^k \rangle_\varphi = \oint_T \varphi_1^{(2l)}(z) z^{-k} d\mu(z) = 0, \quad k = -l, \ldots, l-1, \\
\langle \varphi_1^{(2l+1)}, z^k \rangle_\varphi = \oint_T \varphi_1^{(2l+1)}(z) z^{-k} d\mu(z) = 0, \quad k = -l, \ldots, l, \\
\langle z^k, \varphi_1^{(2l)} \rangle_\varphi = \oint_T \varphi_1^{(2l)}(z^{-1}) z^k d\mu(z) = 0, \quad k = -l, \ldots, l-1, \\
\langle z^k, \varphi_1^{(2l+1)} \rangle_\varphi = \oint_T \varphi_1^{(2l+1)}(z^{-1}) z^k d\mu(z) = 0, \quad k = -l, \ldots, l.
\]

\[\square\]
Proposition 1. Given a positive definite measure \( \mu \) there exists \( h_l \in \mathbb{R}, \ l = 0, 1, \ldots \), such that
\[
\varphi_2^{(l)} = h_l^{-1} \varphi_1^{(l)}
\]

Proof. See Appendix A. \hfill \Box

Therefore, are proportional Laurent polynomials and bi-orthogonality \( \square \) implies that \( \{\varphi_1^{(l)}\}_{l=0}^{\infty} \) and \( \{\varphi_2^{(l)}\}_{l=0}^{\infty} \) are sets of orthogonal Laurent polynomials for the positive measure \( \mu \), that is
\[
\langle \varphi_1^{(l)}, \varphi_1^{(k)} \rangle_\mu = \int_\mathbb{T} \varphi_1^{(l)}(z)\varphi_1^{(k)}(z^{-1})d\mu(z) = \delta_{l,k} h_l \quad l, k = 0, 1, \ldots
\]
\[
\langle \varphi_2^{(l)}, \varphi_2^{(k)} \rangle_\mu = \int_\mathbb{T} \varphi_2^{(l)}(z)\varphi_2^{(k)}(z^{-1})d\mu(z) = \delta_{l,k} h_l^{-1} \quad l, k = 0, 1, \ldots
\]

We are now ready to write the relationship between these CMV Laurent polynomials and the Szegő polynomials \( P_l \) introduced previously.

Proposition 2. If the measure \( \mu \) is positive definite we have the following identifications between the CMV Laurent polynomials, the Szegő polynomials and their reciprocals
\[
\varphi_1^{(2l)}(z) = z^{-l} P_{2l}(z), \quad \varphi_1^{(2l+1)}(z) = z^{-l-1} P_{2l+1}^*(z).
\]

Proof. To check it just observe that
\[
\int_\mathbb{T} z^l \varphi_1^{(2l)}(z) z^{-k}d\mu(z) = 0, \quad k = 0, \ldots, 2l - 1.
\]
Hence, \( z^l \varphi_1^{(2l)}(z) \) has the same orthogonality relations that \( P_{2l} \) and both are monic polynomials of degree \( 2l \); uniqueness leads to their identification. In a similar way we proceed for the odd polynomials. Indeed,
\[
\int_\mathbb{T} z^{l+1} \varphi_1^{(2l+1)}(z) z^{-k}d\mu(z) = 0, \quad k = 1, \ldots, 2l + 1,
\]
that is, \( z^{l+1} \varphi_1^{(2l+1)}(z) \) has the same orthogonality relations that the polynomial \( P_{2l+1}^* \) (that makes them proportional) and both are equal to 1 at \( z = 0 \), consequently they are the same. \hfill \Box

Using the Verblunsky coefficients\( ^4 \) we can write
\[
\varphi_1^{(2l)}(z) = \alpha_{2l} z^{-l} + \cdots + z^l,
\]
\[
\varphi_1^{(2l+1)}(z) = z^{-l-1} + \cdots + \bar{\alpha}_{2l+1} z^l.
\]

For later use, and in addition to the reflection coefficients, it is also useful to define the sequence \( \rho_l := \sqrt{1 - |\alpha_l|^2} \), related to \( \{h_l\}_{l=0}^{\infty} \) by
\[
\rho_l^2 = \frac{h_l}{h_{l-1}},
\]
valid for \( l > 0 \), with \( \rho_0 = 0 \).

In the general quasi-definite case, we can perform a very similar construction. We write
\[
\varphi_1^{(2l)}(z) = \alpha_{2l}^{(1)} z^{-l} + \cdots + z^l,
\]
\[
\varphi_1^{(2l+1)}(z) = z^{-l-1} + \cdots + \alpha_{2l+1}^{(1)} z^l,
\]
\[
\varphi_2^{(2l)}(z) = \bar{\alpha}_{2l}^{(2)} z^{-l} + \cdots + \bar{\alpha}_{2l+1}^{(2)} z^l,
\]
\[
\varphi_2^{(2l+1)}(z) = \bar{h}_{2l+1}^{(1)} z^{-l-1} + \cdots + \bar{\alpha}_{2l+1}^{(1)} z^l.
\]

\( ^4 \)See the Introduction.
and also \( \rho_0^2 := 0, \rho_l^2 := \frac{b_l}{n-1}, l = 1, 2, \ldots \) The Hermitian case can be considered a particular reduction where \( \alpha_i(2) = \alpha_i(1) \). The reason for the notation stands in the following fact. Given a quasi-definite measure \( \mu \) we can find two families of monic orthogonal polynomials such that

\[
\int_T P^{(1)}_l(z) z^{-k} \, d\mu(z) = 0 \quad k = 0, 1, \ldots, l - 1, \tag{14}
\]
\[
\int_T z^k P^{(2)}_l(z^{-1}) \, d\mu(z) = 0 \quad k = 0, 1, \ldots, l - 1.
\]

If we call \( \alpha_i(1) = P^{(1)}_l(0) \), and \( \alpha_i(2) = P^{(2)}_l(0) \) then these coefficients are related with the Laurent Polynomial coefficients and we can obtain the quasi-definite version of Proposition 2 that is

\[
\varphi^{(2)}_1(z) = z^{-l} P^{(1)}_{2l}(z), \quad \varphi^{(2l+1)}_1(z) = z^{-l-1} P^{(2)}_{2l+1}(z), \tag{15}
\]
\[
\varphi^{(2)}_2(z) = \overline{h}_{2l}^{-1} z^{-l} P^{(2)}_{2l}(z), \quad \varphi^{(2l+1)}_2(z) = \overline{h}_{2l+1}^{-1} z^{-l-1} P^{(1)}_{2l+1}(z).
\]

In what follows \( g^{[l]} := \sum_{i,j=0}^{l-1} g_{i,j} E_{i,j} \) denotes the \( l \times l \) truncated moment matrix and \( \chi^{[l]} \) is the truncated vector consisting of the \( l \) first components of \( \chi \). With this notation we can express the bi-orthogonal Laurent polynomials in different ways:

**Proposition 3.** The following expressions hold true

\[
\varphi^{(l)}_1(z) = \chi^{(l)}(z) - (g_{l,0} \quad g_{l,1} \quad \ldots \quad g_{l,l-1}) (g^{[l]})^{-1} \chi^{[l]}
\]
\[
= (S_2)_{ll} \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix} (g^{[l+1]})^{-1} \chi^{[l+1]}, \quad l \geq 1, \tag{16}
\]

and

\[
\varphi^{(l)}_2(z) = (S_2)_{ll}^{-1} \left( (\chi^{(l)})^* - (\chi^{[l]})^* (g^{[l]})^{-1} \chi^{[l]} \right)
\]
\[
= (\chi^{[l+1]})^* (g^{[l+1]})^{-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad l \geq 1. \tag{19}
\]

\[
= \frac{1}{\det g^{[l+1]}} \det \begin{pmatrix} g_{0,0} & g_{0,1} & \cdots & g_{0,l} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l} \\ \vdots & \vdots & \ddots & \vdots \\ g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l} \\ \chi^{(0)}(z) & \chi^{(1)}(z) & \cdots & \chi^{(l)}(z) \end{pmatrix}, \quad l \geq 1. \tag{20}
\]

**Proof.** See Appendix [A] \[\square\]

Similar expressions hold for \( \varphi^{(l)}_{\alpha,1} \) replacing \( \chi \) by \( \chi_1 \), \( a = 1, 2 \), and for \( \varphi^{(l)}_{\alpha,2} \) replacing \( \chi \) by \( \chi^*_2 \), \( a = 1, 2 \).
2.2 Second kind functions

In this subsection we introduce the second kind functions associated with the orthogonal Laurent polynomials discussed before. First we present determinantal expressions, then we connect them with the Fourier series of the measure and also with corresponding Cauchy/Caratheodor transforms.

**Definition 3.** The partial second kind sequences are given by

\[
C_{1,1}(z) := (S_1^{-1})^\dagger \chi_1^*(z), \quad C_{1,2}(z) := (S_1^{-1})^\dagger \chi_2^*(z), \quad C_{2,1}(z) := S_2 \chi_1^*(z), \quad C_{2,2}(z) := S_2 \chi_2^*(z).
\]

and the second kind sequences

\[
C_1(z) := (S_1^{-1})^\dagger \chi_1^*(z), \quad C_2(z) := S_2 \chi_1^*(z).
\]

Observe that

\[
C_1 = C_{1,1} + C_{1,2}, \quad C_2 = C_{2,1} + C_{2,2}.
\]

We have the expressions as semi-infinite vectors, being its coefficients what we call second kind functions

\[
C_{1,a}(z) := \begin{pmatrix} C_{1,a}^{(0)}(z) \\ \vdots \\ C_{1,a}^{(1)}(z) \end{pmatrix}, \quad C_{2,a}(z) := \begin{pmatrix} C_{2,a}^{(0)}(z) \\ \vdots \\ C_{2,a}^{(1)}(z) \end{pmatrix}, \quad C_1(z) := \begin{pmatrix} C_1^{(0)}(z) \\ \vdots \end{pmatrix}, \quad C_2(z) := \begin{pmatrix} C_2^{(0)}(z) \\ \vdots \end{pmatrix}.
\]

The expressions just given for the second kind functions are, in principle, formal, as the matrix products lead to series, not necessarily convergent, instead of finite sums. In fact (as we will show in Proposition 5) they are well defined in terms of the bi-orthogonal Laurent polynomials and truncated Fourier series of the measure, with convergence ensured in some annulus centered at the origin of the complex plane. The coefficients of these sequences are called second kind functions.

One can find analogous determinantal type expressions as those for the OLPUC given in Proposition 3 if we define

\[
\Gamma_{1,j}^{(l)} := \sum_{k \geq l} g_{jk} \chi^*(k), \quad \Gamma_{2,j}^{(l)} := \sum_{k \geq l} g_{jk}^\dagger \chi^*(k).
\]

**Proposition 4.** The second kind functions have the following determinantal expressions for \(l \geq 1\)

\[
C_1^{(l)}(z) = \frac{1}{\det g^{[l+1]}} \det \begin{pmatrix} g_{0,0} & g_{0,1} & \cdots & g_{0,l} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l} \\ \vdots & \vdots & \ddots & \vdots \\ g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l} \\ \Gamma_1^{(l)}(z) & \Gamma_1^{(l)}(z) & \cdots & \Gamma_1^{(l)}(z) \end{pmatrix},
\]

and

\[
C_2^{(l)}(z) = \frac{1}{\det g^{[l]}} \det \begin{pmatrix} g_{0,0} & g_{0,1} & \cdots & g_{0,l-1} & \Gamma_2^{(l)}(z) \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l-1} & \Gamma_2^{(l)}(z) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-1} & \Gamma_2^{(l)}(z) \end{pmatrix}.
\]

**Proof.** See Appendix A. \(\square\)
Now, we introduce the following definition

**Definition 4.** 1. For the bi-orthogonal Laurent polynomials \( \varphi_1^{(l)} \) and \( \varphi_2^{(l)} \) we use the notation \( \varphi_2^{(l)}(e^{i\theta}) = \sum_{|k| \leq \infty} \varphi_{2,k}^{(l)} e^{ik\theta} \) and \( \varphi_1^{(l)}(e^{i\theta}) = \sum_{|k| \leq \infty} \varphi_{1,k}^{(l)} e^{ik\theta} \)

2. Let

\[
c_n = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-in\theta} d\mu(\theta), \quad F_\mu(u) = \sum_{n=-\infty}^{\infty} c_n u^n,
\]

be the Fourier coefficients and the Fourier series of the measure \( \mu \).

3. For each integer \( k \) we introduce the following truncated Fourier series

\[
F_{\mu,k}^{(+)}(z) := \sum_{n \geq -k} c_n z^n, \quad F_{\mu,k}^{(-)}(z) := \sum_{n < -k} c_n z^n.
\]

**Observations**

1. It holds that \( c_n(\bar{\mu}) = c_{-n}(\mu) \), hence \( c_{-n} = c_n \) for real measures. Consequently,

\[
F_{\mu,k}^{(+)}(z) = F_{\mu,-k-1}^{(-)}(z^{-1}), \quad F_{\mu,k}^{(-)}(z) = F_{\mu,-k-1}^{(+)}(z^{-1}), \quad F_\mu(z) = F_\mu(z^{-1}).
\]

2. The Fourier series always converges in \( D'(\mathbb{T}) \), the space of distributions in the circle, so that \( \int_{0}^{2\pi} F_\mu(\theta) f(\theta) d\theta = \int_{0}^{2\pi} f(\theta) d\mu(\theta) \), \( \forall f \in D(\mathbb{T}) \), here \( D(\mathbb{T}) \) denotes the linear space of test functions on the circle. For an absolutely continuous measure \( d\mu(\theta) = w(\theta) d\theta \) we can write \( d\mu(\theta) = F_\mu(\theta) d\theta \).

3. We will also consider the Laurent series \( F_\mu(z) = \sum_{n=-\infty}^{\infty} c_n z^n \), for \( z \in \mathbb{C} \). Notice that \( F_{\mu,k}^{(+)} + F_{\mu,k}^{(-)} = F_\mu \).

4. Let \( D(0; r, R) = \{ z \in \mathbb{C} : r < |z| < R \} \) denote the annulus around \( z = 0 \) with interior and exterior radii \( r \) and \( R \) and \( R_{\pm} := (\limsup_{n\to\infty} \sqrt{|c_{\pm n}|})^{1/2} \). Then, according to the Cauchy–Hadamard theorem, we have

- The series \( F_{\mu,k}^{(+)}(z) \) converges uniformly in any compact set \( K, K \subset D(0; 0, R_{+}) \).
- The series \( F_{\mu,k}^{(-)}(z) \) converges uniformly in any compact set \( K, K \subset D(0; 0, R_{-}, \infty) \).
- The series \( F_\mu(z) \) converges uniformly in any compact set \( K, K \subset D(0; R_{-}, R_{+}) \).

5. According to F. and M. Riesz theorem if \( c_n = 0, n < 0 \), then \( \mu \) is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{T} \). In fact if \( f(z) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{d\mu(\theta)}{1-ze^{-i\theta}} \), then \( w(\theta) = \lim_{r \to 1} f(r e^{i\theta}) \in L^1(\mathbb{T}) \) and \( d\mu = w(\theta) d\theta \); therefore, in this case we have \( f(z) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{w(\theta)}{1-ze^{-i\theta}} du \), a holomorphic function in \( \mathbb{D} \). Moreover, the set of \( w(\theta) = f(e^{i\theta}) \in L^2(\mathbb{T}) \) with \( c_n = 0, n < 0 \), is isometric to the set \( H^2 \) of holomorphic functions in \( \mathbb{D} \) with limits when \( r \to 1 \) in \( L^2(\mathbb{T}) \), observe that \( w = \sum_{n=0}^{\infty} c_n e^{n1\theta} = F_\mu \).

**Proposition 5.** The partial second kind functions can be expressed as

\[
C_{1,1}^{(l)} = 2\pi \sum_{|k| \leq \infty} \varphi_{2,k}^{(l)} z^{-k-1} F_{\mu,-k-1}^{(-)}(z), \quad R_- < |z| < \infty, \quad C_{1,2}^{(l)} = 2\pi \sum_{|k| \leq \infty} \varphi_{2,k}^{(l)} z^{-k-1} F_{\mu,-k-1}^{(+)}(z), \quad 0 < |z| < R_+,
\]

\[
C_{2,1}^{(l)} = 2\pi \sum_{|k| \leq \infty} \varphi_{1,k}^{(l)} z^{-k-1} F_{\mu,k}^{(+)}(z), \quad R_+^{-1} < |z| < \infty, \quad C_{2,2}^{(l)} = 2\pi \sum_{|k| \leq \infty} \varphi_{1,k}^{(l)} z^{-k-1} F_{\mu,k}^{(-)}(z), \quad 0 < |z| < R_-^{-1},
\]

and the second kind functions as

\[
C_1^{(l)} = 2\pi \varphi_2^{(l)}(z^{-1}) z^{-1} F_\mu(z), \quad R_- < |z| < R_+, \quad C_2^{(l)} = 2\pi \varphi_1^{(l)}(z^{-1}) z^{-1} F_\mu(z^{-1}), \quad R_-^{-1} < |z| < R_-^{-1}. \tag{25}
\]

\(^5\) Here \( \sum_{|k| \leq \infty} \) is used just to indicate that the sum is finite.
Proof. From the formal definition of $C_{a,b}$, $a, b = 1, 2$, and the aid of the Gaussian factorization of the moment matrix $g$, we have

$$C_{1,1} = (S_2^{-1})^\dagger \int_T \chi(u)\chi(u)\text{d}\mu(u)\chi_1^*(z), \quad C_{1,2} = (S_2^{-1})^\dagger \int_T \chi(u)\chi(u)\text{d}\mu(u)\chi_2(z)$$

$$C_{2,1} = S_1 \int_T \chi(u)\chi(u)\text{d}\mu(u)\chi_1^*(z), \quad C_{2,2} = S_1 \int_T \chi(u)\chi(u)\text{d}\mu(u)\chi_2(z),$$

We recall that $((S_2^{-1})^\dagger \chi(u))^{(l)} = \varphi_2^{(l)}(u)$ and $(S_1 \chi(u))^{(l)} = \varphi_1^{(l)}(u)$ and expand the matrix products involved, without any interchange of integrals and summation symbols, to get

$$C_{1,1}^{(l)} = \sum_{|k| \leq \infty} \varphi_{2,k}^{(l)} \left( \sum_{n=0}^{\infty} \left[ \int_0^{2\pi} e^{i(k-n)\theta} \text{d}\bar{\mu}(\theta) \right] z^{-n-1} \right), \quad C_{1,2}^{(l)} = \sum_{|k| \leq \infty} \varphi_{2,k}^{(l)} \left( \sum_{n=0}^{\infty} \left[ \int_0^{2\pi} e^{i(k+n+1)\theta} \text{d}\bar{\mu}(\theta) \right] z^n \right)$$

$$C_{2,1}^{(l)} = \sum_{|k| \leq \infty} \varphi_{1,k}^{(l)} \left( \sum_{n=0}^{\infty} \left[ \int_0^{2\pi} e^{i(k-n)\theta} \text{d}\mu(\theta) \right] z^{-n-1} \right), \quad C_{2,2}^{(l)} = \sum_{|k| \leq \infty} \varphi_{1,k}^{(l)} \left( \sum_{n=0}^{\infty} \left[ \int_0^{2\pi} e^{i(k+n+1)\theta} \text{d}\mu(\theta) \right] z^n \right).$$

Using the Fourier coefficients $c_n$ we write

$$C_{1,1}^{(l)} = 2\pi \sum_{|k| \leq \infty} \varphi_{2,k}^{(l)} \left( \sum_{n=0}^{\infty} c_{k-n} z^{-n-1} \right), \quad C_{1,2}^{(l)} = 2\pi \sum_{|k| \leq \infty} \varphi_{2,k}^{(l)} \left( \sum_{n=0}^{\infty} c_{k+n+1} z^n \right), \quad C_{2,1}^{(l)} = 2\pi \sum_{|k| \leq \infty} \varphi_{1,k}^{(l)} \left( \sum_{n=0}^{\infty} c_{k-n} z^{-n-1} \right), \quad C_{2,2}^{(l)} = 2\pi \sum_{|k| \leq \infty} \varphi_{1,k}^{(l)} \left( \sum_{n=0}^{\infty} c_{k-n-1} z^n \right),$$

from where the desired result follows. \hfill \Box

Example. Assume that

$$c_n = \frac{1}{2|n|!} + \delta_{n,0} \left( e + \frac{1}{2} \right)$$

so that

$$d\mu(\theta) = w(\theta) d\theta, \quad w = e + \sum_{n=0}^{\infty} \frac{\cos n\theta}{n!} = e + e^{\cos \theta} \cos(\sin \theta),$$

where the weight $w$ is a smooth positive $2\pi$-periodic function. Then

$$F_{\mu,k}^{(+)} = \begin{cases} e + \frac{1}{2} \left( \sum_{n=0}^{k} \frac{z^n}{n!} + e^z \right), & k \geq 0, \\ \frac{1}{2} \left( -\sum_{n=0}^{k-1} \frac{z^n}{n!} + e^z \right), & k < 0, \end{cases}$$

$$F_{\mu,k}^{(-)} = \begin{cases} \frac{1}{2} \left( -\sum_{n=0}^{k} \frac{z^{n-1}}{n!} + e^{z-1} \right), & k \geq 0, \\ e + \frac{1}{2} \left( \sum_{n=0}^{k-1} \frac{z^n}{n!} + e^{z-1} \right), & k < 0, \end{cases}$$

with annulus of convergence $D = \{ z \in \mathbb{C} : 0 < |z| < 1 \}$. Notice that in this example

$$F_{\mu}(z) = e + \frac{e^z + e^{z-1}}{2}$$

and therefore

$$C_{1}^{(l)}(z) = 2\pi z^{-1} \varphi_2^{(l)}(z^{-1}) F_{\mu}(z), \quad C_{2}^{(l)}(z) = 2\pi z^{-1} \varphi_1^{(l)}(z^{-1}) F_{\mu}(z).$$
Proposition 6. The formal series for $\Gamma_{a,j}^l(z)$ (where $a = 1, 2$) defined in Proposition 4 can be expressed in terms of the Fourier series of $\mu$ and consequently are convergent in corresponding annulus in the complex plane. More precisely

$$\Gamma_{1,j}^l(z) = 2\pi z^{-j-1} \left( F_{j+1}^+(l, \mu)(z^{-1}) + F_{j-1}^-(l, \mu)(z^{-1}) \right), \quad R_+^{-1} < |z| < R_-^{-1},$$

$$\Gamma_{2,j}^l(z) = 2\pi z^{-j-1} \left( \tilde{F}_{j+1}^-(l, \mu)(z) + \tilde{F}_{j-1}^+(l, \mu)(z) \right), \quad R_- < |z| < R_+.$$

where $J(j) = [(-1)^{a(j)-1} \frac{j}{2}]$, being $[p]$ the integer part of $p$.

**Proof.** See Appendix A. \qed

Notice that for $l = 0$ we have

$$\begin{align*}
\Gamma_{1,j}^0(z) &= 2\pi z^{-j-1} F_{\mu}(z^{-1}), \quad R_+^{-1} < |z| < R_-^{-1}, \\
\Gamma_{2,j}^0(z) &= 2\pi z^{-j-1} \tilde{F}_{\mu}(z), \quad R_- < |z| < R_+.
\end{align*}$$

Now we will justify the name we have given to these functions and show a Cauchy integral representation of them. We will prove it in two different scenarios depending on the measure. First, with a more measure theory taste, assuming that $\mu$ is positive and using the Lebesgue dominated convergence theorem. Second, with a more complex analysis taste, considering absolutely continuous complex measures $d\mu = w(\theta)d\theta$ when $w$ is a continuous complex function.

**Theorem 2.** Assume a positive measure $d\mu(\theta)$ or a complex measure $d\mu(\theta) = w(\theta)d\theta$ with $w$ a continuous function. Then, the second kind functions can be written as the following Cauchy integrals

$$\begin{align*}
C_{1,1}^{(l)} &= z^{-1} \int_T \frac{\varphi_2^{(l)}(u)}{u - z^{-1}} d\mu(u), \quad |z| > 1, \\
C_{1,2}^{(l)} &= -z^{-1} \int_T \frac{\varphi_2^{(l)}(u)}{u - z^{-1}} d\mu(u), \quad |z| < 1, \\
C_{2,1}^{(l)} &= z^{-1} \int_T \frac{\varphi_1^{(l)}(u)}{u - z^{-1}} d\mu(u), \\
C_{2,2}^{(l)} &= -z^{-1} \int_T \frac{\varphi_1^{(l)}(u)}{u - z^{-1}} d\mu(u).
\end{align*}$$

**Proof.** From the definition of $C_{a,b}$ and the aid of the Gaussian factorization of the moment matrix $g$, $(S_{1}^{-1})^\dagger = (S_{2}^{-1})^\dagger g^\dagger$ or $S_2 = S_1 g$ we get

$$\begin{align*}
C_{1,1} &= (S_{2}^{-1})^\dagger \int_T \chi(u) \varphi(\theta)^\dagger d\mu(u) \chi_{\mu}(z) = \sum_{n=0}^{\infty} \left( \int_T \Phi_2(u) u^{-n} z^{-n-1} d\mu(u) \right), \\
C_{1,2} &= (S_{2}^{-1})^\dagger \int_T \chi(u) \varphi(\theta)^\dagger d\mu(u) \chi_{\mu}(z) = \sum_{n=0}^{\infty} \left( \int_T \Phi_2(u) u^{n+1} z^n d\mu(u) \right), \\
C_{2,1} &= S_1 \int_T \chi(u) \varphi(\theta)^\dagger d\mu(u) \chi_{\mu}(z) = \sum_{n=0}^{\infty} \left( \int_T \Phi_1(u) u^{-n} z^{-n-1} d\mu(u) \right), \\
C_{2,2} &= S_1 \int_T \chi(u) \varphi(\theta)^\dagger d\mu(u) \chi_{\mu}(z) = \sum_{n=0}^{\infty} \left( \int_T \Phi_1(u) u^{n+1} z^n d\mu(u) \right).
\end{align*}$$

For the series in these expressions we have:

1. The series $\sum_{n=0}^{\infty} u^{-n} z^{-n-1}$ converges uniformly in the $u$ variable in any compact set $K \subset \{u \in \mathbb{C} : |u| > |z|^{-1} \}$ to $(z - u^{-1})^{-1}$ and if $|z| > 1$ then we can take $K$ such that $\mathbb{T} \subset K$.

2. The series $\sum_{n=0}^{\infty} u^{n+1} z^n$ converges uniformly in the $u$ variable in any compact set $K \subset \{u \in \mathbb{C} : |u| < |z|^{-1} \}$ to $-(z - u^{-1})^{-1}$ and if $|z| < 1$ then we can take $K$ such that $\mathbb{T} \subset K$.

\*The reader can check that $z^{(j)} = \chi^{(j)}(z)$.
Let us assume a positive measure. The corresponding $m$-th partial sums are
\[\sum_{n=0}^{m} u^{-n} z^{-n-1} = \frac{z^{-1} - (u z)^{-m-1}}{1 - (u z)^{-1}}, \quad \sum_{n=0}^{m} u^{n+1} z^n = u \frac{1 - (u z)^{m+1}}{1 - (u z)}.
\]

If we write $u = e^{i\theta}$ and $z = |z| e^{i \arg z}$ we have
\[
\left|1 - \frac{(u z)^{-m-1}}{1 - (u z)^{-1}}\right|^2 = \frac{1 - 2|z|^{-(m+1)} \cos((m + 1)(\theta + \arg z)) + |z|^{-(2m+1)}}{1 - 2|z|^{-1} \cos(\theta + \arg z) + |z|^{-2}}.
\]

For $|z|^{-1} < 1$ we have the following inequalities
\[0 < 1 - 2|z|^{-(m+1)} \cos((m + 1)(\theta + \arg z)) + |z|^{-(2m+1)} \leq (1 + |z|^{-(m+1)})^2 < 4,
\]
\[0 < 1 - 2|z|^{-1} \cos(\theta + \arg z) + |z|^{-2} \geq (1 - |z|^{-1})^2,
\]
so that, for $u \in \mathbb{T}$, we infer
\[
\left|z^{-1} \frac{1 - (u z)^{-m-1}}{1 - (u z)^{-1}}\right| < \frac{2|z|^{-1}}{1 - |z|^{-1}}, \quad |z| > 1.
\]

Similarly, we conclude that
\[
\left|\frac{u - (u z)^{m+1}}{1 - u z}\right| < \frac{2}{1 - |z|}, \quad |z| < 1.
\]

Thus, for $u \in \mathbb{T}$ and $a = 1, 2$, we have the control bounds
\[
\left|\sum_{n=0}^{m} \varphi_a^{(l)}(u) u^{-n} z^{-n-1}\right| < \frac{2}{1 - |z|} \varphi_a^{(l)}(u), \quad |z| > 1,
\]
\[
\left|\sum_{n=0}^{m} \varphi_a^{(l)}(u) u^{n+1} z^n\right| < \frac{2}{1 - |z|} \varphi_a^{(l)}(u), \quad |z| < 1.
\]

Consequently, as the Laurent polynomials $\varphi_a$ are measurable functions in $\mathbb{T}$, the Lebesgue dominated convergence theorem leads to the stated result.

Finally, if we assume that $d\mu = w(\theta) d\theta$, with $w$ a continuous complex function; we can always write $w(\theta) d\theta = F(u) \frac{du}{iu}$, $u = e^{i \theta}$, with $F$ a continuous function on $\mathbb{T}$. Then, recalling the uniform convergence of the geometrical series involved and the fact that the Laurent polynomials $\varphi_a$ are continuous functions on $\mathbb{T}$, we can interchange integral and series symbols arriving to the expressions

1. $\int_{\mathbb{T}} \left( \sum_{n=0}^{\infty} \Phi_a(u) u^{-n} z^{-n-1} \right) d\mu(u) = \sum_{n=0}^{\infty} \left( \int_{\mathbb{T}} \Phi_a(u) u^{-n} z^{-n-1} d\mu(u) \right)$ for $|z| > 1$.

2. $\int_{\mathbb{T}} \left( \sum_{n=0}^{\infty} \Phi_a(u) u^{n+1} z^n \right) d\mu(u) = \sum_{n=0}^{\infty} \left( \int_{\mathbb{T}} \Phi_a(u) u^{n+1} z^n d\mu(u) \right)$ for $|z| < 1$.

\[\square\]

The result motivates the name given to these functions [16]. These expressions can be also written as Geronimus transforms, for that aim we just need to recall that for $u \in \mathbb{T}$
\[
\frac{1}{z - u^{-1}} = \frac{1}{2z} \left(1 + \frac{u + z^{-1}}{u - z^{-1}}\right),
\]

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and therefore for \( l \geq 1 \)

\[
C_{1,1}^{(l)} = \frac{1}{2z} \oint_T \frac{u + z^{-1}\varphi_2^{(l)}(u)}{u - z^{-1}} \varphi_2^{(l)}(u) d\mu(u), \quad C_{2,1}^{(l)} = \frac{1}{2z} \oint_T \frac{u + z^{-1}\varphi_1^{(l)}(u)}{u - z^{-1}} \varphi_1^{(l)}(u) d\mu(u), \quad |z| > 1,
\]

\[
C_{1,2}^{(l)} = -\frac{1}{2z} \oint_T \frac{u + z^{-1}\varphi_2^{(l)}(u)}{u - z^{-1}} \varphi_1^{(l)}(u) d\mu(u), \quad C_{2,2}^{(l)} = -\frac{1}{2z} \oint_T \frac{u + z^{-1}\varphi_1^{(l)}(u)}{u - z^{-1}} \varphi_1^{(l)}(u) d\mu(u), \quad |z| < 1.
\]

For \( l = 0 \), we just obtain (up to constants)

\[
\frac{1}{2z}(|\mu| + C(z^{-1})),
\]

where \( C(z) \) is the Carathéodory transform of the measure:

\[
C(z) := \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta).
\]

**Example.** Let us assume that for \(|z| > 1\) we have

\[
C_{2,1}^{(l)} = z^{-1} \oint_T \frac{u \varphi_1^{(l)}(u)}{u - z^{-1}} F(u) \frac{du}{1u}, \tag{27}
\]

and that the only singularity of \( F \) inside \( \mathbb{D} \) can lay exclusively at \( z = 0 \). On the one hand, as \( F \) has a Laurent expansion of the form \( F(z) = \sum_{k \in \mathbb{Z}} F_k z^k \) and there no singularities different from \( z = 0 \) we conclude that the coefficients of the Laurent series are just the Fourier coefficients of the measure:

\[
F_k = \frac{1}{2\pi i} \int_T \frac{F(u)}{u^{k+1}} du = \frac{1}{2\pi} \int_0^{2\pi} F(e^{i\theta}) e^{-ik\theta} d\theta = c_k.
\]

On the other hand, as the Laurent polynomial \( \varphi_1 \) has its singularities at \( z = 0 \), and \( z^{-1} \in \mathbb{D} \), then the set of singularities of the integrand that lay in \( \mathbb{D} \) in (27) is \( \{0, z^{-1}\} \). Hence, the residue theorem gives

\[
C_{2,1}^{(l)} = 2\pi z^{-1} \left( \text{Res}_{u=z^{-1}} \left( \frac{\varphi_1^{(l)}(u) F(u)}{u - z^{-1}} \right) + \text{Res}_{u=0} \left( \frac{\varphi_1^{(l)}(u) F(u)}{u - z^{-1}} \right) \right).
\]

We now evaluate the residues, the first one may be computed noticing that \( \varphi_1^{(l)}(u) F(u) \) is analytic at \( u = z^{-1} \)

\[
2\pi z^{-1} \text{Res}_{u=z^{-1}} \left( \frac{\varphi_1^{(l)}(u) F(u)}{u - z^{-1}} \right) = \varphi_1^{(l)}(z^{-1}) F(z^{-1}) = C_1^{(l)},
\]

where we have used Proposition 5. For the residue at \( u = 0 \) we use the Laurent series around this point

\[
\varphi_1^{(l)}(u) = \sum_k \varphi_{1,k}^{(l)} u^k, \quad F(u) = \sum_j c_j u^j, \quad \frac{1}{u - z^{-1}} = -\sum_{n=0}^{\infty} z^{n+1} u^n,
\]

so that

\[
2\pi z^{-1} \text{Res}_{u=0} \left( \frac{\varphi_1^{(l)}(u) F(u)}{u - z^{-1}} \right) = -\sum_{n \geq 0} \varphi_{1,k}^{(l)} c_j z^{n+1} = -\sum_{n=0}^{\infty} \varphi_{1,k}^{(l)} c_{-k-n-1} z^{n+1} = -C_{2,2}^{(l)}
\]

and we finally get

\[
C_{2,1} = C_2 - C_{2,2},
\]

in agreement with (22).

The application of the residue theorem to the formulae in Theorem 2 leads to expressions for the second kind functions in terms of residues:

\footnote{This property does not hold if \( F \) has different singularities from \( z = 0 \) inside \( \mathbb{D} \), as we can not choose \( T \) as the integration circuit to apply the Cauchy formula for the coefficients of the Laurent series.}
Proposition 7. Let us assume \( d\mu(\theta) = F(u)\frac{du}{u} \) then:

1. When \( F \) is an analytic function in \( \mathbb{C} \setminus \mathbb{D} \) but for a set of isolated singularities, and if denote by \( \{ z_{j,-} \}_{j=0}^{p_-} \) the set of distinct points obtained from the union of \( z_0,- = 0 \) and the set of singularities of \( F \) at \( \mathbb{C} \setminus \mathbb{D} \), then, for \( z \not\in \mathbb{T} \),

\[
C_{1,1}^{(l)}(z) = 2\pi z^{-1} \left[ \varphi_2^{(l)}(z^{-1})F(z)\theta(|z| - 1) - \text{Res}_{u=0} \left( \varphi_2^{(l)}(u)\bar{F}(u^{-1}) \right) \right] z + \sum_{j=1}^{p_-} \left( \frac{\varphi_2^{(l)}(z_{j,-}^{-1})}{z_{j,-}^{-1} - z^{-1}} \text{Res}_{u=z_{j,-}} \bar{F}(u^{-1}) \right),
\]

\[
C_{1,2}^{(l)}(z) = 2\pi z^{-1} \left[ \varphi_2^{(l)}(z^{-1})F(z)\theta(1 - |z|) + \text{Res}_{u=0} \left( \varphi_2^{(l)}(u)\bar{F}(u^{-1}) \right) \right] z - \sum_{j=1}^{p_-} \left( \frac{\varphi_2^{(l)}(z_{j,-}^{-1})}{z_{j,-}^{-1} - z^{-1}} \text{Res}_{u=z_{j,-}} \bar{F}(u^{-1}) \right)\]

2. If \( F \) is an analytic function in \( \mathbb{D} \) but for a set of isolated singularities, and denote by \( \{ z_{j,+} \}_{j=0}^{p_+} \) the set of different points obtained from the union of \( z_0 = 0 \) and the set of singularities of \( F \) at \( \mathbb{D} \). Then, for \( z \not\in \mathbb{T} \),

\[
C_{2,1}^{(l)}(z) = 2\pi z^{-1} \left[ \varphi_1^{(l)}(z^{-1})F(z^{-1})\theta(|z| - 1) - \text{Res}_{u=0} \left( \varphi_1^{(l)}(u)F(u) \right) \right] z + \sum_{j=1}^{p_+} \left( \frac{\varphi_1^{(l)}(z_{j,+})}{z_{j,+} - z^{-1}} \text{Res}_{u=z_{j,+}} F(u) \right),
\]

\[
C_{2,2}^{(l)}(z) = 2\pi z^{-1} \left[ \varphi_1^{(l)}(z^{-1})F(z^{-1})\theta(1 - |z|) + \text{Res}_{u=0} \left( \varphi_1^{(l)}(u)F(u) \right) \right] z - \sum_{j=1}^{p_+} \left( \frac{\varphi_1^{(l)}(z_{j,+})}{z_{j,+} - z^{-1}} \text{Res}_{u=z_{j,+}} F(u) \right).\]

To conclude this section we give some summation rules that are derived using the geometrical series

Proposition 8. The OLPUC and its corresponding partial second kind functions satisfy

\[
\sum_{l=0}^{\infty} \overline{\varphi_{a,1}^{(l)}(z)} \varphi_{a,1}^{(l)}(z') = \frac{1}{z - z'}, \quad |z'| > |z|,
\]

\[
\sum_{l=0}^{\infty} \overline{\varphi_{a,2}^{(l)}(z)} \varphi_{a,2}^{(l)}(z') = -\frac{1}{z - z'}, \quad |z'| < |z|,
\]

\[
\sum_{l=0}^{\infty} \overline{\varphi_{a,1}^{(l)}(z)} \varphi_{a,2}^{(l)}(z') = \sum_{l=0}^{\infty} \overline{\varphi_{a,2}^{(l)}(z)} \varphi_{a,1}^{(l)}(z') = 0.
\]

for \( a = 1, 2 \).

Proof. See Appendix A.

\]

2.3 Recursion relations

We are about to derive, using the Gaussian factorization, the CMV recursion relations and obtain in this way the well known CMV five diagonal Jacobi type matrix for the recursion of the Szegő polynomials. Let us begin with the following

Definition 5. Given the canonical basis for semi-infinite matrices \( E_{i,j}, i, j \in \mathbb{Z}_+ \), we define the projections

\[
\Pi_1 := \sum_{j=0}^{\infty} E_{2j,2j}, \quad \Pi_2 := \sum_{j=0}^{\infty} E_{2j+1,2j+1},
\]
and the matrices

\[
\Lambda_1 := \sum_{j=0}^{\infty} E_{2j,2j}, \quad \Lambda_2 := \sum_{j=0}^{\infty} E_{1+2j,3+2j}, \quad \Lambda := \sum_{j=0}^{\infty} E_{j,j+1}, \quad \Upsilon := \Lambda_1 + \Lambda_2^\top + E_{1,1}\Lambda^\top.
\]

The matrix \(\Upsilon\),

\[
\Upsilon = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

is a central object in this paper, as its dressing–its orbit by conjugations–gives the pentadiagonal CMV Jacobi type matrix.

It is immediate to check that

**Proposition 9.** The following relations hold

\[
\begin{align*}
\Lambda_1 \chi(z) &= z\Pi_1 \chi(z), & \Lambda_2 \chi(z) &= z^{-1}\Pi_2 \chi(z), \\
\Lambda_1^\top \chi(z) &= (z^{-1}\Pi_1 - E_{0,0}\Lambda)\chi(z), & \Lambda_2^\top \chi(z) &= (z\Pi_2 - E_{1,1}\Lambda^\top)\chi(z), \\
\Upsilon \chi(z) &= z\chi(z), & \Upsilon^\top \chi(z) &= z^{-1}\chi(z).
\end{align*}
\]

(28)

(29)

With the aid of these conditions we characterize the moment matrix \(g\) as verifying a symmetry constraint, which we call string equation, from where the recursion as well as the CD formulae will be derived. The symmetry is detailed in the following

**Proposition 10.** The CMV moment matrix fulfils the following condition

\[
\Upsilon g = g \Upsilon.
\]

(30)

**Proof.** To prove the previous result we proceed as follows

\[
\begin{align*}
\Upsilon g &= \oint_T (\Lambda_1 + \Lambda_2^\top + E_{1,1}\Lambda^\top)\chi(z)\chi(z)^\dagger d\mu(z) = \oint_T z\chi(z)\chi(z)^\dagger d\mu(z) \\
&= \oint_T \chi(z)(z^{-1}\chi(z))^\dagger d\mu(z) \\
&= \oint_T \chi(z)((\Lambda_1^\top + \Lambda_2 + E_{0,0}\Lambda)\chi(z))^\dagger d\mu(z) \\
&= g(\Lambda_1 + \Lambda_2^\top + \Lambda^\top E_{0,0}) \\
&= g(\Lambda_1 + \Lambda_2^\top + E_{1,1}\Lambda^\top) \\
&= g \Upsilon.
\end{align*}
\]

\[\square\]
We now proceed to dress the $\Upsilon$ matrix in two ways.

**Definition 6.** We use the notation

$$J_1 := S_1 \Upsilon S_1^{-1}, \quad J_2 := S_2 \Upsilon S_2^{-1}.$$  

The following proposition is trivially derived from the above definition and

**Proposition 11.**

$$J_1 = J_2.$$  

Consequently, we introduce the CMV Jacobi type matrix

**Definition 7.** We define $J := J_1 = J_2$.

The matrix $J$ has a five diagonal structure; as easily follows when one observes that $J_1$ has zero coefficients over the third upper-diagonal and that $J_2$ has all its coefficients equal to zero under the third lower-diagonal. More specifically, the structure is

$$J = \begin{pmatrix} \ast & \ast & 0 & 0 & 0 & 0 & 0 & \cdots \\ + & \ast & \ast & 0 & 0 & 0 & 0 & \cdots \\ 0 & + & \ast & \ast & 0 & 0 & 0 & \cdots \\ 0 & 0 & + & \ast & \ast & 0 & 0 & \cdots \\ 0 & 0 & 0 & + & \ast & \ast & 0 & \cdots \\ 0 & 0 & 0 & 0 & + & \ast & \ast & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where, $\ast$ is a possibly non-vanishing term and $+$ is a positive term. In fact, using the $LU$ factorization problem we are able to completely characterize $J$ in terms of the Verblunsky coefficients.

**Proposition 12.** 1. The non-vanishing coefficients of $J$ are

$$J_{2k,2k-1} = -\alpha_{2k}^{(1)} \alpha_{2k+1}^{(1)}, \quad J_{2k,2k} = -\alpha_{2k}^{(2)} \alpha_{2k+1}^{(1)}, \quad J_{2k,2k+1} = -\alpha_{2k+2}^{(1)}, \quad J_{2k,2k+2} = 1,$$

$$J_{2k+1,2k-1} = \rho_{2k+1} \rho_{2k}, \quad J_{2k+1,2k} = \rho_{2k+1} \alpha_{2k}^{(2)}, \quad J_{2k+1,2k+1} = -\alpha_{2k+1}^{(2)} \rho_{2k+2}, \quad J_{2k+1,2k+2} = \alpha_{2k+1}^{(2)}.$$  

2. We have the recursion relations

$$z \varphi_1^{(2k)} = \varphi_1^{(2k+2)} - \alpha_{2k+2} \varphi_1^{(2k+1)} - \alpha_{2k+1} \varphi_1^{(2k)} - \alpha_{2k+1} \varphi_1^{(2k)}, \quad z \varphi_1^{(2k+2)} = \alpha_{2k+1} \varphi_1^{(2k+1)} + \rho_{2k+1} \varphi_1^{(2k+2)} + \rho_{2k+2} \varphi_1^{(2k+1)},$$  

(31)

$$z \varphi_1^{(2k+1)} = \alpha_{2k+1} \varphi_1^{(2k+1)} - \alpha_{2k+1} \alpha_{2k+2} \varphi_1^{(2k+1)} + \rho_{2k+1} \alpha_{2k} \varphi_1^{(2k+1)} + \rho_{2k+2} \rho_{2k+1} \varphi_1^{(2k+1)},$$  

(32)

**Proof.** 1. See Appendix A.

2. We have

$$z \varphi_1^{(2k)} = J_{2k,2k+2} \varphi_1^{(2k+2)} + J_{2k,2k+1} \varphi_1^{(2k+1)} + J_{2k,2k} \varphi_1^{(2k)} + J_{2k,2k-1} \varphi_1^{(2k-1)},$$

$$z \varphi_1^{(2k+1)} = J_{2k+1,2k+2} \varphi_1^{(2k+2)} + J_{2k+1,2k+1} \varphi_1^{(2k+1)} + J_{2k+1,2k} \varphi_1^{(2k)} + J_{2k+1,2k-1} \varphi_1^{(2k-1)}.$$  

$\square$
As $J_1 \Phi_1 = S_1 \mathcal{Y} S_1^{-1} \chi(z) = z \Phi_1$, the sequences $\Phi_1, \Phi_2$ have a five term recurrence formula. However, although there are five non-vanishing diagonals the recurrence relations do not have more than four non-zero terms, explicitly

$$
\begin{align*}
\varphi_1^{(2k)} &= J_{2k,2k+2} \varphi_1^{(2k+2)} + J_{2k,2k+1} \varphi_1^{(2k+1)} + J_{2k,2k} \varphi_1^{(2k)} + J_{2k,2k-1} \varphi_1^{(2k-1)} \\
&= \alpha_{2k} \varphi_1^{(2k+1)} - \alpha_{2k+1} \varphi_1^{(2k-1)} - \rho_{2k+1} \varphi_1^{(2k)} - \rho_{2k+1} \varphi_1^{(2k-1)}, \\
\varphi_1^{(2k+1)} &= J_{2k+1,2k+2} \varphi_1^{(2k+3)} + J_{2k+1,2k+1} \varphi_1^{(2k+2)} + J_{2k+1,2k} \varphi_1^{(2k+1)} + J_{2k+1,2k-1} \varphi_1^{(2k)} + J_{2k+1,2k-2} \varphi_1^{(2k-1)} \\
&= \alpha_{2k+1} \varphi_1^{(2k+2)} - \alpha_{2k+2} \varphi_1^{(2k)} - \rho_{2k+2} \varphi_1^{(2k+1)} + \rho_{2k+1} \varphi_1^{(2k)},
\end{align*}
$$

(33)

to them we can add the truncated relations for $k = 0, 1$

$$
\begin{align*}
\varphi_1^{(0)} &= \alpha_{1} \varphi_1^{(1)} - \alpha_{1} \varphi_1^{(0)} \\
\varphi_1^{(1)} &= \alpha_{2} \varphi_1^{(2)} - \alpha_{1} \varphi_1^{(1)} + \rho_{1} \varphi_1^{(0)}.
\end{align*}
$$

It is also possible to build recursion relations multiplying by $z^{-1}$, thus we have

**Proposition 13.** The OLPUC have the following recursion relations

$$
\begin{align*}
z^{-1} \varphi_1^{(2k)} &= \alpha_{2k} \varphi_1^{(2k+1)} - \alpha_{2k+1} \varphi_1^{(2k)} + \rho_{2k} \varphi_1^{(2k)} - \rho_{2k+1} \varphi_1^{(2k-1)} + \rho_{2k-1} \varphi_1^{(2k-2)}, \\
z^{-1} \varphi_1^{(2k+1)} &= \alpha_{2k+1} \varphi_1^{(2k+2)} - \alpha_{2k+2} \varphi_1^{(2k)} - \rho_{2k+1} \varphi_1^{(2k+1)} + \rho_{2k} \varphi_1^{(2k)} - \rho_{2k+1} \varphi_1^{(2k-1)},
\end{align*}
$$

(34)

(35)

where we have to add the truncated relation

$$
z^{-1} \varphi_1^{(0)} = \alpha_{1} \varphi_1^{(1)} - \alpha_{1} \varphi_1^{(0)}.
$$

**Proof.** Using that $S_1 \mathcal{Y} S_1^{-1} \Phi_1 = S_1 \mathcal{Y} S_1^{-1} \chi(z) = z^{-1} \Phi_1$ we need to calculate the coefficients of $S_1 \mathcal{Y} S_1^{-1} = S_2 \mathcal{T} S_2^{-1}$ as we did with $J_1$, to obtain the desired result. $lacksquare$

With the previous result we can get

**Proposition 14.** The coefficients $\rho_k^2$ verify the following relations

$$
\rho_k^2 = 1 - \alpha_k \alpha_{k+1}, \quad k \geq 0.
$$

**Proof.** See Appendix A. $lacksquare$

The following results were found previously in [21] using an alternative derivation. Recalling the explicit form of the CMV Jacobi type matrix $J$ provided in Proposition [22] we get

**Proposition 15.** When the measure $\mu$ is positive the recursion relations for the OLPUC can be expressed in terms of the Verblunsky coefficients as follows

$$
\begin{align*}
z \varphi_1^{(2k)} &= \varphi_1^{(2k+2)} - \alpha_{2k} \varphi_1^{(2k+1)} - \alpha_{2k+1} \varphi_1^{(2k)} - \rho_{2k+1} \varphi_1^{(2k-1)} - \rho_{2k+1} \varphi_1^{(2k-2)}, \\
z \varphi_1^{(2k+1)} &= \alpha_{2k+1} \varphi_1^{(2k+2)} - \alpha_{2k+2} \varphi_1^{(2k+1)} - \rho_{2k+1} \varphi_1^{(2k+1)} - \rho_{2k+1} \varphi_1^{(2k)},
\end{align*}
$$

(36)

where $k \geq 0$.

Using the Szegő polynomials and their reciprocals recursion relations [51] and [52] can be expressed like

$$
\begin{align*}
z P_{2k} &= z^{-1} (P_{2k+2} - \alpha_{2k+2} (k) P_{2k+1}^{*}) + \alpha_{2k} \alpha_{2k+1} P_{2k} - (1 - |\alpha_{2k}|^2) \alpha_{2k+1} P_{2k-1}^{*}, \\
z P_{2k+1}^{*} &= \alpha_{2k+1} P_{2k+2} - \alpha_{2k+1} \alpha_{2k+2} P_{2k+1}^{*} + \alpha_{2k+1} (1 - |\alpha_{2k+1}|^2) \alpha_{2k} P_{2k} + (1 - |\alpha_{2k+1}|^2) (1 - |\alpha_{2k}|^2) P_{2k-1}^{*},
\end{align*}
$$

(38)

(39)

relations that can be obtained also with the classical Szegő recurrence formulae and their reciprocals in $\mathbb{T}$. 

18
2.4 Projection operators and the Christoffel–Darboux kernel

Here we discuss the CD kernel; i.e., the integral kernel of the quasi-orthogonal projection, according to the sesquilinear form $(\cdot, \cdot)_\mathcal{L}$ defined by the measure $\mu$, to the space of OLPUC.

**Definition 8.** We use the following notation

$$\Lambda^{[l]} := \mathbb{C}\{\chi^{(0)}, \ldots, \chi^{(l-1)}\} = \begin{cases} \Lambda_{[k,k-1]}, & l = 2k, \\ \Lambda_{[k,k]}, & l = 2k + 1. \end{cases}$$

(40)

Notice that

$$\Lambda^{[l]} = \mathbb{C}\{\varphi_1^{(0)}, \ldots, \varphi_1^{(l-1)}\} = \mathbb{C}\{\varphi_2^{(0)}, \ldots, \varphi_2^{(l-1)}\}.$$

Associated with these spaces of truncated Laurent polynomials we consider the following related spaces, quasi-orthogonal complements,

$$(\Lambda^{[l]})^\perp := \left\{ \sum_{l \leq k < \infty} c_k \varphi_1^{(k)}, c_k \in \mathbb{C} \right\}, \quad (\Lambda^{[l]})^\perp_1 := \left\{ \sum_{l \leq k < \infty} c_k \varphi_2^{(k)}, c_k \in \mathbb{C} \right\}.$$

Formally, we can express the following bi-quasi-orthogonality relations

$$\langle \Lambda^{[l]}, (\Lambda^{[l]})^\perp \rangle_\mathcal{L} = 0, \quad \langle (\Lambda^{[l]})^\perp, \Lambda^{[l]} \rangle_\mathcal{L} = 0,$$

and the corresponding splittings

$$\Lambda_{[\infty]} = \Lambda^{[l]} \oplus (\Lambda^{[l]})^\perp_1 = \Lambda^{[l]} \oplus (\Lambda^{[l]})^\perp_2,$$

induce the associated quasi-orthogonal projections

$$\pi_1^{(l)} : \Lambda_{[\infty]} \rightarrow \Lambda^{[l]}, \quad \pi_2^{(l)} : \Lambda_{[\infty]} \rightarrow \Lambda^{[l]}.$$

The reader should notice that we cannot properly talk of an orthogonal complement and an orthogonal projection if the measure is not positive and consequently we do not have an scalar product. If $\mu$ is a positive measure then $(\Lambda^{[l]})^\perp_1 = (\Lambda^{[l]})^\perp_2 = (\Lambda^{[l]})^\perp$ and both projections are truly orthogonal and coincide.

**Definition 9.** The CD kernel is defined by

$$K^{[l]}(z, z') := \sum_{k=0}^{l-1} \varphi_1^{(k)}(z') \overline{\varphi_2^{(k)}(z)}. \quad (41)$$

As the CD kernel is expressed in terms of Laurent polynomials the definition makes sense as long as $z, z' \neq 0$. This is the kernel of the integral representation of the projections $\pi_1^{(l)}, \pi_2^{(l)}$:

**Proposition 16.** The integral representation

$$\begin{align*}
(\pi_1^{(l)} f)(z') &= \oint_T K^{[l]}(z, z') f(z) d\mu(z), & \forall f \in \Lambda_{[\infty]}, \\
(\pi_2^{(l)} f)(z) &= \oint_T K^{[l]}(z, z') \bar{f}(z') d\mu(z'), & \forall f \in \Lambda_{[\infty]},
\end{align*}$$

holds.

---

*In case that we have a positive measure $\mu$ then we can define the orthonormal Laurent polynomials $\tilde{\varphi}^{(l)}(z) = (h_1) \cdot \varphi_1^{(l)}(z) \cdot h_2^{-1} \varphi_2^{(l)}(\overline{z})$ so that $K^{[l]}(z, z') = \sum_{k=0}^{l-1} h_k \tilde{\varphi}_1^{(k)}(z') \tilde{\varphi}_2^{(k)}(\overline{z}) = \sum_{k=0}^{l-1} h_k \varphi_1^{(k)}(z') \overline{\varphi}_2^{(k)}(z) = \sum_{k=0}^{l-1} \varphi_2^{(k)}(z') \overline{\varphi}_2^{(k)}(\overline{z})$.*
Proof. It follows from the bi-orthogonality condition \([3]\).

This CD kernel has the reproducing property

**Proposition 17.** The kernel \(K^{[l]}(z, z')\) fulfills

\[
K^{[l]}(z, z') = \int_T K^{[l]}(z, u) K^{[l]}(u, z')\, d\mu(u).
\]

Proof. See Appendix \([A]\). \(\square\)

### 2.5 Associated Laurent polynomials

In order to find CD formulae for the CD kernel just discussed we need the following definitions introducing what we call associated Laurent polynomials

**Definition 10.** In the \(l\) even case the associated Laurent polynomials are

\[
\varphi^{(l)}_{1, +1} := \chi^{(l)} - (g_{l, 0} g_{l, 1} \cdots g_{l, l-1}) (g^{[l]})^{-1} \chi^{[l]},
\]

\[
\varphi^{(l)}_{1, -2} := e_{l-1} (g^{[l]})^{-1} \chi^{[l]},
\]

\[
\varphi^{(l)}_{2, +2} := \chi^{(l+1)} - (g_{0, l+1} g_{1, l+1} \cdots g_{l-1, l+1}) ((g^{[l]})^{-1})^{\dagger} \chi^{[l]},
\]

\[
\varphi^{(l)}_{2, -1} := e_{l-2} ((g^{[l]})^{-1})^{\dagger} \chi^{[l]},
\]

while for the \(l\) odd case they are

\[
\varphi^{(l)}_{1, +1} := \chi^{(l+1)} - (g_{l+1, 0} g_{l+1, 1} \cdots g_{l+1, l-1}) (g^{[l]})^{-1} \chi^{[l]},
\]

\[
\varphi^{(l)}_{1, -2} := e_{l-2} (g^{[l]})^{-1} \chi^{[l]},
\]

\[
\varphi^{(l)}_{2, +2} := \chi^{(l)} - (g_{0, l} g_{1, l} \cdots g_{l-1, l}) ((g^{[l]})^{-1})^{\dagger} \chi^{[l]},
\]

\[
\varphi^{(l)}_{2, -1} := e_{l-1} ((g^{[l]})^{-1})^{\dagger} \chi^{[l]}.
\]

The associated polynomials can be expressed in terms of the Laurent polynomials in different alternative manners

**Theorem 3.** If \(\mu\) is a positive measure the associated Laurent polynomials in the \(l\) even case are

\[
\varphi^{(l)}_{1, +1}(z) = \varphi^{(l)}_{1}(z),
\]

\[
\varphi^{(l)}_{2, +2}(z) = z^{-1} h_l \varphi^{(l)}_{2}(z^{-1}),
\]

\[
= z^{-1} (\alpha_l h_l \varphi^{(l)}_{2}(z) + h_{l-1} \rho_l^2 \varphi^{(l-1)}_{2}(z)),
\]

\[
= h_{l+1} \varphi^{(l+1)}_{2}(z) - h_l \alpha_l \varphi^{(l)}_{2}(z),
\]

and in the \(l\) odd case

\[
\varphi^{(l)}_{2, +2}(z) = h_{l-1} \varphi^{(l)}_{2}(z),
\]

\[
\varphi^{(l)}_{1, +1}(z) = \varphi^{(l)}_{1}(z),
\]

\[
\varphi^{(l-1)}_{2, -1}(z) = h_{l-1} \varphi^{(l-1)}_{2}(z^{-1}),
\]

\[
= z (\alpha_l \varphi^{(l)}_{1}(z) + \rho_l^2 \varphi^{(l-1)}_{1}(z)),
\]

\[
= \varphi^{(l+1)}_{1}(z) - \alpha_{l+1} \varphi^{(l)}_{1}(z),
\]

\[
= h_{l-1} \alpha_l \varphi^{(l-1)}_{1}(z) + h_{l-2} \varphi^{(l-2)}_{1}(z).\]

Proof. 1. To prove (42) and (45) we proceed as follows. On the one hand, when \(l\) is even \([48]\) implies

\[
\int_T z^{j} \varphi^{(l)}_{2, +2}(z) z^{-j} d\mu(z) = 0,
\]

\[
j = -\frac{l}{2} + 1, \ldots, \frac{l}{2},
\]

and on the other hand, due to the Hermitian property of the scalar product, it follows for \(\varphi^{(l)}_{2}\) that

\[
\int_T \varphi^{(l)}_{2}(z^{-1}) z^{-j} d\mu(z) = 0,
\]

\[
j = -\frac{l}{2} + 1, \ldots, \frac{l}{2}.
\]
Hence, $z\phi_{2,+2}^{(l)} \in \Lambda_{[l/2, l/2]}$ and solves the same linear system of equations that $\tilde{\phi}_2^{(l)}(z^{-1}) \in \Lambda_{[l/2, l/2]}$ does. Consequently both Laurent polynomials are proportional. The equality is obtained from the coefficients in the power $z^{-i}$. In a similar way, from (49) we see that
\[
\oint T z^j z^{-1} \phi_{2,-1}^{(l-1)}(z^{-1}) d\mu(z) = 0, \quad j = -\frac{l}{2}, \ldots, -\frac{l}{2} - 1, \quad \oint T z^j z^{-1} \phi_{2,-1}^{(l-1)}(z^{-1}) d\mu(z) = 1,
\]

this means that $z\phi_{2,-1}^{(l-1)}(z) \in \Lambda_{[l/2-1, l/2]}$ has the same orthogonality relations and normalization condition that $\tilde{\phi}_2^{(l-1)}(z^{-1}) \in \Lambda_{[l/2-1, l/2]}$, so they coincide. Analogously, in the odd case we obtain (45).

2. For (43) and (46) we argue in the following manner. Using orthogonality relations for $\phi_0^{(l)}(z^{-1})$ and $\phi_2^{(l)}(z^{-1})$, we conclude that
\[
\phi_1^{(l)}(z^{-1}) \in \text{span}\{\phi_1^{(l)}, \phi_2^{(l-1)}\},
\]
\[
\phi_2^{(l-1)}(z^{-1}) \in \text{span}\{\phi_2^{(l)}, \phi_2^{(l-1)}\},
\]
and identifying coefficients
\[
z\phi_{2,+2}^{(l)}(z) = h_l \phi_{2}^{(l)}(z^{-1}) = \bar{\alpha}_l \phi_{1}^{(l)}(z) + \rho_l^2 \phi_{1}^{(l-1)}(z) = \bar{\alpha}_l h_l \phi_{2}^{(l)}(z) + \rho_l^2 h_{l-1} \phi_{2}^{(l-1)}(z),
\]
\[
z\phi_{2,-1}^{(l)}(z) = \phi_{2}^{(l-1)}(z^{-1}) = \rho_l^2 \phi_{2}^{(l-1)}(z) - \alpha_l \phi_{2}^{(l-1)}(z),
\]
that concludes the proof of (43), (46) follows similarly.

3. Finally, we proceed now to prove (44) and (47). For the even case we compute the following integral
\[
\oint T z^j \phi_{2,+2}^{(l)}(z) d\mu(z) = \oint T \chi^{[l]}(z) \chi^{(l+1)}(z) d\mu(z) -
\]
\[
- \oint T \chi^{[l]}(z) \chi^{[l]}(z)^T d\mu(z) (g^{[l]})^{-1} g_{l-1,l+1}^T
\]
\[
= (g_{0,l+1} \quad g_{1,l+1} \quad \ldots \quad g_{l-1,l+1})^T - (g_{0,l+1} \quad g_{1,l-1} \quad \ldots \quad g_{l-1,l+1})^T
\]
\[
= (0 \quad 0 \quad \ldots \quad 0),
\]
which written componentwise reads
\[
\oint T z^j \phi_{2,+2}^{(l)}(z) d\mu(z) = 0, \quad j = -\frac{l}{2}, \ldots, -\frac{l}{2} - 1.
\]

It also follows from the definition that $\phi_{2,+2}^{(l)}(z^{-1}) \in \Lambda_{[l/2, l/2]}$, and $(\phi_{2,+2}^{(l)} - z^{\frac{l}{2}+1}) \in \Lambda_{[l/2, l/2]}$. For the other associated Laurent polynomials, the orthogonality relations are
\[
\oint T z^j \phi_{2,-1}^{(l-1)}(z) d\mu(z) = 0, \quad j = -\frac{l}{2}, \ldots, -\frac{l}{2} - 2, \quad \oint T z^j \phi_{2,-1}^{(l-1)}(z) d\mu(z) = 1.
\]
To get this result we proceed as before
\[
\oint T \chi^{[l]}(z) \chi^{[l]}(z)^T d\mu(z) = (g^{[l]})^{-1} e_{l-2} = e_{l-2},
\]
that is the matrix version of the orthogonality relations.
These orthogonality relations lead to
\[ \varphi_{2,+2}^{(l)} = a_l \varphi_{2}^{(l+1)} + b_l \varphi_{2}^{(l)}, \]
\[ \varphi_{2,-1}^{(l)} = c_{l-1} \varphi_{2}^{(l-1)} + d_{l-1} \varphi_{2}^{(l-2)}. \]

Let us prove this statement. As \( \varphi_{2,+2}^{(l)} \in \Lambda_{\left[ \frac{1}{2} + \frac{1}{2} \right]} \) then \( \varphi_{2,+2}^{(l)} \in \text{span}\{ \varphi_{2}^{(0)}, \varphi_{2}^{(1)}, \ldots, \varphi_{2}^{(l+1)} \} \), but due to the orthogonality relations all the coefficients vanish except for the ones corresponding to \( \varphi_{2}^{(l+1)} \) and \( \varphi_{2}^{(l)} \). Comparing the coefficients of \( z^{-\frac{1}{2} - 1} \) and \( z^{\frac{1}{2}} \) we get the system of equations
\[ 1 = (S_2)^{-1}_{l+1,l+1} a_l + 0, \]
\[ 0 = a_l (S_2)^{-1}_{l+1,l+1} \bar{a}_{l+1} + (S_2)^{-1}_{l,l} b_l, \]
from where we conclude \( a_l = (S_2)^{-1}_{l+1,l+1} \) and \( b_l = -\bar{a}_{l+1} (S_2)^{-1}_{l,l} \). Now, we notice that \( \varphi_{2,-1}^{(l-1)} \in \text{span}\{ \varphi_{2}^{(0)}, \varphi_{2}^{(1)}, \ldots, \varphi_{2}^{(l-1)} \} \) and also that the orthogonality relations imply that
\[ \varphi_{2,-1}^{(l-1)} \perp \text{span}\{ \varphi_{2}^{(0)}, \varphi_{2}^{(1)}, \ldots, \varphi_{2}^{(l-3)} \}. \]

Therefore, \( \varphi_{2,-1}^{(l-1)} \in \text{span}\{ \varphi_{2}^{(l-2)}, \varphi_{2}^{(l-1)} \} \) and we only need to find the expression of the associated Laurent polynomials as a linear combination of these two Laurent polynomials. For that aim, we take the complex conjugate, multiply by \( \varphi_{2,-1}^{(l-1)} \) and \( \varphi_{2,-1}^{(l-2)} \), and integrate to obtain
\[ c_{l-1} = \oint_T \varphi_{2,-1}^{(l-1)}(\bar{z}) \varphi_{2,-1}^{(l-1)}(\bar{z}) d\mu(z) = \oint_T z^{-\frac{1}{2} - 1} \varphi_{2,-1}^{(l-1)}(\bar{z}) d\mu(z) + \bar{a}_{l-1} \oint_T z^{\frac{1}{2} - 1} \varphi_{2,-1}^{(l-1)}(\bar{z}) d\mu(z) = 0 + \bar{a}_{l-1}, \]
\[ d_{l-1} = \oint_T \varphi_{2,-1}^{(l-2)}(\bar{z}) \varphi_{2,-1}^{(l-1)}(\bar{z}) d\mu(z) = \oint_T z^{\frac{1}{2} - 1} \varphi_{2,-1}^{(l-1)}(\bar{z}) d\mu(z) = 1, \]
so we conclude \( c_{l-1} = \alpha_{l-1} \) and \( d_{l-1} = 1 \). For the odd case one proceeds in an analogous form.

A different proof for the same formula can be found in Appendix A.

Finally we give determinantal expressions for these polynomials

**Proposition 18.** The associated Laurent polynomials have the following determinantal expressions

\[
\varphi_{1,+a}^{(l)}(z) = \frac{1}{\det g^{[l]}} \det \begin{pmatrix} g_{0,0} & g_{0,1} & \cdots & g_{0,l-1} & \chi^{(0)}(z) \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l-1} & \chi^{(1)}(z) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-1} & \chi^{(l-1)}(z) \\ g_{l+a,0} & g_{l+a,1} & \cdots & g_{l+a,l-1} & \chi^{(l+a)}(z) \end{pmatrix}, \quad l \geq 1. \quad (50)
\]

\[
\varphi_{1,-a}^{(l)}(z) = \frac{(-1)^{l+l-a}}{\det g^{[l+1]}} \det \begin{pmatrix} g_{0,0} & g_{0,1} & \cdots & g_{0,l-a-1} & \chi^{(0)}(z) \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l-a-1} & \chi^{(1)}(z) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{l-a,0} & g_{l-a,1} & \cdots & g_{l-a,l-1} & \chi^{(l-a)}(z) \end{pmatrix}, \quad l \geq 1. \quad (51)
\]

and

\[
\tilde{\varphi}_{2,+a}^{(l)}(z) = \frac{1}{\det g^{[l]}} \det \begin{pmatrix} g_{0,0} & g_{0,1} & \cdots & g_{0,l-a-1} & \chi^{(0)}(z) \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l-a-1} & \chi^{(1)}(z) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-a-1} & \chi^{(l-1)}(z) \\ (\chi^{(0)}(z))^\dagger & (\chi^{(1)}(z))^\dagger & \cdots & (\chi^{(l-1)}(z))^\dagger & (\chi^{(l+a)}(z))^\dagger \end{pmatrix}, \quad l \geq 1. \quad (52)
\]
The following CD formula holds

\[
\varphi_{2,-a}(\bar{z}) = \frac{(-1)^{l+a}}{\det g^{l+1}} \left( \begin{array}{cccc}
g_{0,0} & g_{0,1} & \cdots & g_{0,l} \\
g_{1,0} & g_{1,1} & \cdots & g_{1,l} \\
\vdots & \vdots & \ddots & \vdots \\
g_{l-a-1,0} & g_{l-a-1,1} & \cdots & g_{l-a-1,l} \\
g_{l-a+1,0} & g_{l-a+1,1} & \cdots & g_{l-a+1,l} \\
\vdots & \vdots & \ddots & \vdots \\
g_{l,0} & g_{l,1} & \cdots & g_{l,l} \\
\end{array} \right),
\]

\[l \geq 1.\] (53)

\(\text{Proof.}\) See Appendix A. \qed

### 2.6 The Christoffel–Darboux formula

To obtain CD formula in this context we need a number of preliminary lemmas. First we consider a version of the Aitken–Berg–Collar theorem [59].

**Lemma 1.** The following ABC type formula

\[K^{[l]}(z, z') = \chi^{[l]}(z) (g^{[l]})^{-1} \chi^{[l]}(z')\]

is fulfilled.

\(\text{Proof.}\) See the Appendix A. \qed

The CD formula can be obtained using the previous expressions for the CD kernel.

**Lemma 2.** For the CD kernel one has

\[(z' - \bar{z}^{-1})K^{[l]}(z, z') = \chi^{[l]}(z) (g^{[l]})^{-1} \chi^{[l]}(z') - \bar{z}^{-1} \chi^{[l]}(z) (g^{[l]})^{-1} \chi^{[l]}(z') - (\chi^{[l]}(z)^\dagger (g^{[l]})^{-1} g^{[l]} \chi^{[l]}(z') - \chi^{[l]}(z)^\dagger (g^{[l]})^{-1} g^{[l]} \chi^{[l]}(z') - \chi^{[l]}(z)^\dagger (g^{[l]})^{-1} \chi^{[l]}(z') - \chi^{[l]}(z') \chi^{[l]}(z) - \chi^{[l]}(z') \chi^{[l]}(z))\]

\(\text{Proof.}\) See Appendix A. \qed

The reader can easily check

**Lemma 3.** If \(l\) is an even number

\[\Upsilon^{[l, \geq l]} = E_{l-2,l-1} = e_{l-2} e_0^\top, \quad \Upsilon^{[\geq l, l]} = E_{l+1-l, l-1} = e_1 e_{l-1}^\top,\]

while for the \(l\) odd case he have

\[\Upsilon^{[l, \geq l]} = E_{l-1, l-1} = e_{l-1} e_1^\top, \quad \Upsilon^{[\geq l, l]} = E_{l-1, l-2} = e_0 e_{l-2}^\top.\]

**Theorem 4.** The following CD formula holds

\[K^{[l]}(z, z') = \frac{\varphi_{2,-2}(\bar{z}) \varphi_{1,-2}(z') - \varphi_{1,-2}(z') \varphi_{2,-2}(\bar{z})}{(1 - z' \bar{z})}, \quad z' \bar{z} \neq 1.\] (54)
Proof. The proof of (54) relies in Lemmas 2 and 3. Let us first study the $l$ even case; with Lemma 2 and 3 we obtain a more explicit expression for the CD kernel given by

$$(z^{-1} - z')K^{[l]}(z, z') = \chi^{(l+1)}(z)\bar{\chi}^{[l]}(z') - \chi^{[l]}(z)\bar{\chi}^{(l+1)}(z') - \chi^{[l]}(z)\bar{\chi}^{[l]}(z') - e_{l-1}^\top g_{\geq l}(g_{\geq l})^{-1}\chi^{[l]}(z'),$$

then using Definition 10 for the associated Laurent polynomials and Proposition 3 we conclude our claim that leads to (54).

For the $l$ odd case, reasoning again with Lemmas 2 and 3 we obtain the expression

$$(z^{-1} - z')K^{[l]}(z, z') = \chi^{(l)}(z)\bar{\chi}^{[l]}(z') - \chi^{[l]}(z)\bar{\chi}^{(l)}(z') - \chi^{[l]}(z)\bar{\chi}^{[l]}(z') - e_{l-1}^\top g_{\geq l}(g_{\geq l})^{-1}\chi^{[l]}(z').$$

and recalling with Definition 10 we immediately get the claimed result. \hfill \Box

Recalling the different expressions for the associated Laurent polynomials in Theorem 3, one easily notice that

**Corollary 1.** For a positive measure $\mu$ the CD kernel can be written in the following alternative forms. In the $l$ even case it can be written as

$$K^{[l]}(z, z') = \frac{\varphi_1^{(l)}(z^{-1})\varphi_2^{(l-1)}(z') - \varphi_1^{(l)}(z')\varphi_2^{(l-1)}(z^{-1})}{1 - z\bar{z}}$$

(55)

$$= \frac{(\bar{\alpha}_l\varphi_1^{(l)}(z) + \rho_l^2\varphi_1^{(l-1)}(z))\varphi_2^{(l-1)}(z') - \varphi_1^{(l)}(z')\rho_l^2\varphi_2^{(l)}(z)}{1 - z\bar{z}},$$

(56)

$$= \frac{z(\chi^{(l+1)}(z) - \alpha_{l+1}\varphi_1^{(l)}(z))\varphi_2^{(l-1)}(z') - \varphi_1^{(l)}(z')z(\alpha_{l-1}\varphi_2^{(l-1)}(z) + \varphi_2^{(l-2)}(z))}{1 - z\bar{z}},$$

(57)

while in the $l$ odd case it can be written as

$$K^{[l]}(z, z') = \frac{\bar{z}\varphi_1^{(l)}(z)\varphi_2^{(l-1)}(z^{-1}) - \varphi_1^{(l)}(z^{-1})\varphi_2^{(l-1)}(z)}{1 - z\bar{z}}$$

(58)

$$= \frac{z\bar{z}\varphi_1^{(l)}(z)(\rho_l^2\varphi_2^{(l)}(z') - \alpha_l\varphi_2^{(l-1)}(z')) - (\alpha_l\varphi_1^{(l)}(z') + \rho_l^2\varphi_1^{(l-1)}(z'))\varphi_2^{(l-1)}(z)}{1 - z\bar{z}}$$

(59)

$$= \frac{z\varphi_1^{(l)}(z)(\alpha_{l-1}\varphi_2^{(l-1)}(z') + \varphi_2^{(l-2)}(z')) - (\varphi_1^{(l+1)}(z') - \alpha_{l+1}\varphi_1^{(l)}(z'))z\varphi_2^{(l-1)}(z)}{1 - z\bar{z}}.$$

(60)

Formulae (56) and (59) were found in [25], however the other expressions, to the best of our knowledge, are new.

### 3 Extended CMV ordering and orthogonal Laurent polynomials

This section is devoted to an extension of the CMV ordering that allows to extend CMV results to more general situations. The main result is an extension of the CD formula allowing for projecting kernels to general spaces of Laurent polynomials, avoiding the CMV restriction on degrees.

\footnote{However, the authors use the orthonormal sequence instead of the dual monic orthogonal sequences}
3.1 Extending the CMV ordering

Let us consider a vector \( \vec{n} \in \mathbb{Z}_+^2 \), \( \vec{n} = (n_+, n_-) \), and the associated alternated sequence with \( n_+ \) positive increasing powers of \( z \) followed by \( n_- \) negative decreasing powers of \( z \), that is

\[
\chi_{\vec{n}}(z) := (1, z, \ldots, z^{n_+-1}, z^{-1}, z^{-2}, \ldots, z^{-n_-}, z^{n_+}, z^{n_++1}, \ldots)\top.
\]

The CMV case presented above corresponds to the particular choice \( n_+ = n_- = 1 \). Given \( l \geq 0 \) if \( \chi_{\vec{n}}^{(l)} \) is a non-negative power of \( z \) we say that \( a(l) = 1 \) and if \( \chi_{\vec{n}}^{(l)} \) is a negative power of \( z \) we define \( a(l) = 2 \). In addition, for any \( l \geq 0 \) we will denote \( \nu_+(l) \) (\( \nu_-(l) \)) as the number of elements in the set \( \{\chi_{\vec{n}}^{(l')}, 0 \leq l' \leq l\} \) with \( a(l') = 1 \) (\( a(l') = 2 \)). That is

\[
\nu_+(l) := \#\{\chi_{\vec{n}}^{(l')}, a(l') = 1, 0 \leq l' \leq l\}, \quad \nu_-(l) := \#\{\chi_{\vec{n}}^{(l')}, a(l') = 2, 0 \leq l' \leq l\}. \tag{61}
\]

We will use the notation

\[
\vec{\nu} = (\nu_+, \nu_-), \quad |\vec{\nu}| := \nu_+ + \nu_-, \quad |\vec{n}| := n_+ + n_-.
\]

Additional expressions for \( (61) \) can be found using the Euclidean division, \( ([12], [16]) \), as we now explain. For any given \( l \geq 0 \) there exists unique non-negative integers \( q(l) \) and \( r(l) \) such that

\[
\begin{align*}
\nu_+(l) & = \begin{cases} 
q(l)n_+ + r(l) + 1, & a(l) = 1, \\
(q(l) + 1)n_+, & a(l) = 2,
\end{cases} \\
\nu_-(l) & = \begin{cases} 
q(l)n_-, & a(l) = 1, \\
q(l)n_- + r(l) + 1, & a(l) = 2,
\end{cases}
\end{align*}
\]

so that

\[
\begin{align*}
|\vec{\nu}(l)| & = l + 1. \quad \text{If} \quad \{e_k\}_{k=0}^\infty \quad \text{is the formal canonical basis of} \quad \mathbb{R}^\infty \quad \text{we consider} \quad \{e_a(k)\}_{k=0}^\infty, \quad \text{with} \quad a = 1, 2, \quad \text{defined as}
\end{align*}
\]

\[
\begin{align*}
e_1(\nu_+(l) - 1) & := e_l, \\
e_2(\nu_-(l) - 1) & := e_l,
\end{align*}
\]

these are new labels for \( \{e_k\}_{k=0}^\infty \) adapted to \( \vec{n} \). Given a non-negative integer \( l \) there exist a unique non-negative integer \( k \) and a number \( a \in \{1, 2\} \) such that \( e_a(k) = e_l \). This basis allows for a natural decomposition of \( \chi_{\vec{n}} \) using positive and negative powers. In particular

\[
\chi_{\vec{n},a}(z) := \sum_{k=0}^\infty e_a(k)z^k, \quad a = 1, 2, \quad \chi_{\vec{n}} = \chi_{\vec{n},1} + \chi_{\vec{n},2}^*.
\]

With those sequences we define the extended CMV moment matrix

**Definition 11.** The extended moment matrix is the following semi-infinite matrix

\[
g_{\vec{n}} := \int_T \chi_{\vec{n}}(z)\chi_{\vec{n}}(z)\top d\mu(z). \tag{62}
\]

Notice that this moment matrix is a definite positive Hermitian matrix if \( \mu \) is positive. The Gaussian factorization for the semi-infinite matrix \( g_{\vec{n}} \) is

\[
g_{\vec{n}} = (S_{\vec{n},1})^{-1}S_{\vec{n},2},
\]

25
where $S_{n,1}$ is a normalized lower triangular matrix and $S_{n,2}$ is an upper triangular matrix. The associated sequences of Laurent polynomials are

$$
\Phi_{n,1}(z) := S_{n,1} \chi_{n}(z), \quad \Phi_{n,2}(z) := (S_{n,2}^{-1})^{\dagger} \chi_{n}(z),
$$

where

$$
\Phi_{n,1}(z) = \begin{pmatrix}
\varphi_{n,1}^{(0)}(z) \\
\varphi_{n,1}^{(1)}(z) \\
\vdots
\end{pmatrix}, \quad \Phi_{n,2}(z) = \begin{pmatrix}
\varphi_{n,2}^{(0)}(z) \\
\varphi_{n,2}^{(1)}(z) \\
\vdots
\end{pmatrix}.
$$

As in the CMV case, the sets of Laurent polynomials $\{\varphi_{n,1}^{(l)}\}_{l=0}^{\infty}$ and $\{\varphi_{n,2}^{(l)}\}_{l=0}^{\infty}$ are bi-orthogonal with respect to the sesquilinear form $\langle \cdot, \cdot \rangle_\mathcal{F}$, that is

$$
\langle \varphi_{n,1}^{(l)}, \varphi_{n,2}^{(k)} \rangle_\mathcal{F} = \int_T \varphi_{n,1}^{(l)}(z) \overline{\varphi_{n,2}^{(k)}(z^{-1})} d\mu(z) = \delta_{l,k}, \quad l, k = 0, 1, \ldots
$$

In addition if the measure $\mu$ is positive we have that $\varphi_{n,2}^{(l)} = h_{l}^{-1} \varphi_{n,1}^{(l)}$ and both families are proportional Laurent polynomials. In addition

$$
\langle \varphi_{n,1}^{(l)}, \varphi_{n,2}^{(k)} \rangle_\mathcal{F} = \int_T \varphi_{n,1}^{(l)}(z) \overline{\varphi_{n,2}^{(k)}(z^{-1})} d\mu(z) = \delta_{l,k} h_l, \quad l, k = 0, 1, \ldots
$$

so in this case $\{\varphi_{n,1}^{(l)}\}_{l=0}^{\infty}$ is a set of orthogonal Laurent polynomials with $\varphi_{n,1}^{(l)}(z) \in \Lambda_{[\nu_{-}(l), \nu_{+}(l) - 1]}$. The orthogonality relations (63) read as follows

$$
\langle \varphi_{n,1}^{(l)}, z^k \rangle_\mathcal{F} = \int_T \varphi_{n,1}^{(l)}(z) z^{-k} d\mu(z) = 0, \quad k = -\nu_{-}(l - 1), \ldots, \nu_{+}(l - 1) - 1.
$$

In terms of the Szegö polynomials we have

**Proposition 19.** For a positive measure $\mu$ we have the following identifications between the extended CMV Laurent polynomials, the Szegö polynomials and their reciprocals

$$
z^{\nu_{-}(l)} \varphi_{n,1}^{(l)}(z) = P_l(z), \quad a(l) = 1,
$$

$$
z^{\nu_{-}(l)} \varphi_{n,1}^{(l)}(z) = P_l^*(z), \quad a(l) = 2.
$$

**Proof.** See Appendix A.

The reader should notice that in the CMV case, $n_+ = n_- = 1$, for $l = 2k$ we have $a(l) = 1$, $\nu_{-}(l) = k$ and $\nu_{+}(l) = k + 1$, and when $l = 2k + 1$ then $a(l) = 2$, $\nu_{-}(l) = \nu_{+}(l) = k + 1$.

### 3.2 Functions of the second kind

Here we just give a brief description account of this extended case

**Definition 12.** The partial second kind sequences with the extended ordering are given by

$$
C_{n,1,1}(z) := (S_{n,1}^{-1})^\dagger \chi_{n,1}^*(z), \quad C_{n,1,2}(z) := (S_{n,1}^{-1})^\dagger \chi_{n,2}(z), \quad C_{n,2,1}(z) := S_{n,2} \chi_{n,1}^*(z), \quad C_{n,2,2}(z) := S_{n,2} \chi_{n,2}(z).
$$

and the second kind sequences

$$
C_{n,1}(z) := (S_{n,1}^{-1})^\dagger \chi_{n}^*(z), \quad C_{n,2}(z) := S_{n,2} \chi_{n}(z).
$$
Generalized determinantal formulae can be obtained

Proposition 20. The extended second kind functions have the following determinantal expressions for $l \geq 1$.

$$
C_{n,1}^{(l)}(z) = \frac{1}{\det g_{n}^{[l+1]}} \det \begin{pmatrix} g_{n,0,0} & g_{n,0,1} & \cdots & g_{n,0,l} \\ g_{n,1,0} & g_{n,1,1} & \cdots & g_{n,1,l} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n,l-1,0} & g_{n,l-1,1} & \cdots & g_{n,l-1,l} \\ \Gamma_{n,2,0}^{(l)} & \Gamma_{n,2,1}^{(l)} & \cdots & \Gamma_{n,2,l}^{(l)} \end{pmatrix},
$$

(67)

and

$$
C_{n,2}^{(l)}(z) = \frac{1}{\det g_{n}^{[l]}} \det \begin{pmatrix} g_{n,0,0} & g_{n,0,1} & \cdots & g_{n,0,l-1} \\ g_{n,1,0} & g_{n,1,1} & \cdots & g_{n,1,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n,l,0} & g_{n,l,1} & \cdots & g_{n,l,l-1} \\ \Gamma_{n,2,0}^{(l)} & \Gamma_{n,2,1}^{(l)} & \cdots & \Gamma_{n,2,l}^{(l)} \end{pmatrix},
$$

(68)

where $\Gamma_{n,1,j}^{(l)} := \sum_{k \geq l} g_{n,j,k} X_{n}^{(k)}$ and $\Gamma_{n,2,j}^{(l)} := \sum_{k \geq l} g_{n,j,k} X_{n}^{(k)}$.

The same can be said about the relationship between the second kind functions, the Fourier series of the measures, and the integral representation, that can be found in

Proposition 21. The partial second kind functions can be expressed as

$$
C_{n,1,1}^{(l)} = 2\pi \sum_{|k| \leq \infty} \varphi_{n,2,k}^{(l)} z^{-k-1} F_{\mu,-k-1}^{(-)}(z), \quad R_{-} < |z| < R_{+} \quad C_{n,1,2}^{(l)} = 2\pi \sum_{|k| \leq \infty} \varphi_{n,2,k}^{(l)} z^{-k-1} F_{\mu,-k-1}^{(+)}(z), \quad 0 < |z| < R_{+}
$$

$$
C_{n,2,1}^{(l)} = 2\pi \sum_{|k| \leq \infty} \varphi_{n,1,k}^{(l)} z^{-k-1} F_{\mu,k}^{(+)}(z^{-1}), \quad R_{-}^{-1} < |z| < R_{+}^{-1} \quad C_{n,2,2}^{(l)} = 2\pi \sum_{|k| \leq \infty} \varphi_{n,1,k}^{(l)} z^{-k-1} F_{\mu,k}^{(-)}(z^{-1}), \quad 0 < |z| < R_{+}^{-1}
$$

and the second kind functions as

$$
C_{n,1}^{(l)} = 2\pi \varphi_{n,2}^{(l)}(z^{-1}) z^{-1} F_{\mu}(z), \quad R_{-} < |z| < R_{+} \quad C_{n,2}^{(l)} = 2\pi \varphi_{n,1}^{(l)}(z^{-1}) z^{-1} F_{\mu}(z^{-1}) \quad R_{-}^{-1} < |z| < R_{+}^{-1}.
$$

(69)

and in

Proposition 22. Assume a positive measure $\mu$ or a complex measure $w(\theta)d\theta$ with $w$ a continuous function. Then, the second kind sequences can be written as the following Cauchy integrals

$$
C_{n,1,1,1}^{(l)} = z^{-1} \int_{T} \frac{u \varphi_{n,2}^{(l)}(u)}{u - z^{-1}} d\mu(u), \quad C_{n,1,2}^{(l)} = z^{-1} \int_{T} \frac{u \varphi_{n,1}^{(l)}(u)}{u - z^{-1}} d\mu(u), \quad |z| > 1,
$$

$$
C_{n,1,2,1}^{(l)} = -z^{-1} \int_{T} \frac{u \varphi_{n,2}^{(l)}(u)}{u - z^{-1}} d\mu(u), \quad C_{n,2,2}^{(l)} = -z^{-1} \int_{T} \frac{u \varphi_{n,1}^{(l)}(u)}{u - z^{-1}} d\mu(u), \quad |z| < 1.
$$

3.3 Recursion relations

As we already commented above the recursion relations among extended Laurent polynomials are more involved than in the CMV case.
Definition 13. Given \( \vec{n} \in \mathbb{Z}_+^2 \) we define the projections

\[
P_{\vec{n},a} := \sum_{k=0}^{\infty} e_a(k)e_a(k)^\top, \quad a = 1, 2,
\]
and the shift matrices

\[
\Lambda_{\vec{n},a} := \sum_{k=0}^{\infty} e_a(k)e_a(k+1)^\top,
\]
\[
\Upsilon_{\vec{n}} := \Lambda_{\vec{n},1} + \Lambda_{\vec{n},2}^\top + E_{\vec{n}+,\vec{n}+}(\Lambda^\top)^{n+}.
\]

In the context of section 3 recursion relations can be obtained using the same technique. Our objective is to have an expression for the multiplication by \( z \) and by \( z^{-1} \) using the shift operators.

Proposition 23. 1. The moment matrix \( g_{\vec{n}} \) has the following symmetry

\[
\Upsilon_{\vec{n}}g_{\vec{n}} = g_{\vec{n}}\Upsilon_{\vec{n}}. \tag{70}
\]

2. The operator of multiplication by \( z \) has a \( |\vec{n}| + 3 \) diagonal band structure in the basis given by \( \Phi_{\vec{n},1} \) or \( \Phi_{\vec{n},2} \)

Proof. See Appendix A. \( \square \)

Now we introduce the following associate integers. For \( a = 1, 2 \) we shall call \( l_{+a} \) and \( l_{-a} \) to the smallest (largest) integer \( l' \) that verifies \( l' \geq l (l' \leq l) \) and \( a_1(l') = a \). For instance, in the previous case with \( n_+ = n_- = 1 \), if \( l \) is an even number then \( l_{+2} = 0, l_{+1} = l + 1, \) and \( l_{-2} = l - 1, \) in the case that \( l \) is an odd number then \( l_{+2} = l - 1, l_{+1} = l + 1, \) and \( l_{-2} = l - 2, l_{-1} = l - l - 1 \). This structure leads to the following recursion laws for \( k \geq 1 \) and \( 0 \leq l \leq |\vec{n}| - 1 \) (the \( k = 0 \) case corresponds to the truncated recurrence relations).

\[
z\phi_{\vec{n},1}^{(|\vec{n}|k+l)} = J_{\vec{n},0}^l(k)\phi_{\vec{n},1}^{(|\vec{n}|k+l+1),+1} + J_{\vec{n},1}^l(k)\phi_{\vec{n},1}^{(|\vec{n}|k+l+1),+1-1} + \ldots + J_{\vec{n},m_{\vec{n}}(l)-1}(k)\phi_{\vec{n},1}^{(|\vec{n}|k+l-1),-2},
\]
where \( m_{\vec{n}}(l) \) is the number of non vanishing terms in the recurrence formulae. The coefficients \( J_{\vec{n},j}^l(k) \) are again labeled with the index \( j \) that accounts for the \( m_{\vec{n}}(l) \) non vanishing coefficients for each \( l \); as there are only \( |\vec{n}| \) “different” recursion relations (the equivalent to the recurrences for the odd and even polynomials) then \( l = 0, 1, \ldots, |\vec{n}| - 1 \). The connection with the elements of the Jacobi operator is the following \( J_{\vec{n},j}^l(k) = J_{\vec{n},|\vec{n}|k+l,|\vec{n}|k+l+1} - J_{\vec{n},|\vec{n}|k+l,|\vec{n}|k+l+1-j} \). The reader can check that \( m_{\vec{n}}(l) = (|\vec{n}|k+l+1)_{+1} - (|\vec{n}|k+l-1)_{-2} + 1 \). Due to the fact that \( (|\vec{n}|k+l+1)_{+1} |\vec{n}|k+l+n_- + 1 \) and that \( (|\vec{n}|k+l-1)_{-2} |\vec{n}|k+l-1-n_+ \) then \( m_{\vec{n}}(l) \leq |\vec{n}| + 3 \) that agrees with the structure of \( |\vec{n}| + 3 \) diagonals. Furthermore it is possible to calculate \( m_{\vec{n}}(l) \) more explicitly and show that actually it does not depend on \( k \). We can see that

\[
(|\vec{n}|k+l+1)_{+1} = \begin{cases} |\vec{n}|k+l+1, & 0 \leq l \leq n_+ - 2, \\ |\vec{n}|(k+1), & n_+ - 1 \leq l \leq |\vec{n}| - 1, \end{cases}
\]
\[
(|\vec{n}|k+l-1)_{-2} = \begin{cases} |\vec{n}|k-1, & 0 \leq l \leq n_+, \\ |\vec{n}|k+l-1 & n_+ + 1 \leq l \leq |\vec{n}| - 1, \end{cases}
\]
and consequently

\[
m_{\vec{n}}(l) = \begin{cases} l + 3 & l = 0, \ldots, n_+ - 2, \\ |\vec{n}| + 2 & l = n_+ - 1, n_+, \\ |\vec{n}| - l + 2 & l = n_+ + \ldots, |\vec{n}| - 1. \end{cases}
\]

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so in fact \( m_\nu(l) \leq |\bar{n}| + 2 \).

The expressions for the coefficients \( J^{l}_{\bar{n},p}(k) \) with \( l = 0, \ldots, n_+ \) are

\[
J^{l}_{\bar{n},p}(k) = \begin{cases} 
  (S^{-1}_{\bar{n},1})(|\bar{n}| k + l + 1)_1 + (|\bar{n}| k + l + 1)_1 + p & p = 0, \ldots, (|\bar{n}| k + l + 1)_1 - (|\bar{n}| k + l) - 1 \\
  (S^{-1}_{\bar{n},2})(|\bar{n}| k + l)_2 + (S^{-1}_{\bar{n},2})(|\bar{n}| k + l)_2 - 2 - (|\bar{n}| k + l + 1)_1 - p & p = (|\bar{n}| k + l + 1)_1 - (|\bar{n}| k + l), \ldots, m_\nu(l) - 1,
\end{cases}
\]

and for \( l = n_+, \ldots, |\bar{n}| - 1 \), the expressions are

\[
J^{l}_{\bar{n},p}(k) = \begin{cases} 
  (S^{-1}_{\bar{n},1})(|\bar{n}| k + l)_1 + (S^{-1}_{\bar{n},1})(|\bar{n}| k + l)_1 + p & p = 0, \ldots, (|\bar{n}| k + l)_1 - (|\bar{n}| k + l) \\
  (h_\nu)(|\bar{n}| k + l)(S^{-1}_{\bar{n},2})(|\bar{n}| k + l)_2 - 2 - (|\bar{n}| k + l + 1)_1 - p & p = (|\bar{n}| k + l)_1 - (|\bar{n}| k + l + 1), \ldots, m_\nu(l) - 1.
\end{cases}
\]

The particular case of \( n_- = n_+ = 1 \) gives the 5 diagonal CMV matrix with only four non-vanishing coefficients. As a consequence, the standard CMV case has the shortest possible recurrence formula.

### 3.4 The Christoffel–Darboux kernel

We discuss the CD kernel for this extended case

**Definition 14.** For each non-negative integer \( l \) we define the set of truncated Laurent polynomial subspace as the following span

\[
\Lambda^{[l]}_{\bar{n}} := \mathbb{C}\{\chi^{(0)}_{\bar{n}}, \ldots, \chi^{(l-1)}_{\bar{n}}\} = \Lambda_{[\nu_- (l-1), \nu_+ (l-1) - 1]}.
\]

As before, we define *quasi-orthogonal* subspaces

\[
(\Lambda^{[l]}_{\bar{n}})^{\perp 1} := \left\{ \sum_{l \leq k < \infty} c_k \varphi^{(k)}_{\bar{n},2}, c_k \in \mathbb{C} \right\}, \quad (\Lambda^{[l]}_{\bar{n}})^{\perp 2} := \left\{ \sum_{l \leq k < \infty} c_k \varphi^{(k)}_{\bar{n},1}, c_k \in \mathbb{C} \right\},
\]

so that the following *bi-quasi-orthogonality* relations are satisfied

\[
\langle (\Lambda^{[l]}_{\bar{n}}), (\Lambda^{[l]}_{\bar{n}})^{\perp 1} \rangle_{\nu} = 0, \quad \langle (\Lambda^{[l]}_{\bar{n}})^{\perp 2}, (\Lambda^{[l]}_{\bar{n}}) \rangle_{\nu} = 0,
\]

and the corresponding splittings

\[
\Lambda_{[\infty]} = \Lambda^{[l]}_{\bar{n}} \oplus (\Lambda^{[l]}_{\bar{n}})^{\perp 1} = \Lambda^{[l]}_{\bar{n}} \oplus (\Lambda^{[l]}_{\bar{n}})^{\perp 2},
\]

that induce the associated projections

\[
\pi^{(l)}_{\bar{n},1} : \Lambda_{[\infty]} \to \Lambda^{[l]}_{\bar{n}}, \quad \pi^{(l)}_{\bar{n},2} : \Lambda_{[\infty]} \to \Lambda^{[l]}_{\bar{n}},
\]

hold. In the positive definite case this extended version allows for the projection in more general spaces of truncated Laurent polynomials. Recall that for the CMV case the space of truncated polynomials given in \( \Box \) includes only a very particular class of these truncations, excluding the majority of cases. The introduction of the extended ordering allows to include in the discussion all the possible situations of truncation. In fact the space \( \Lambda_{[p,q]} \) can be achieved in a number of ways, and always by the choice \( n_+ = q + 1, n_- = p \).

**Definition 15.** The CD kernel is

\[
K^{[l]}_{\bar{n}}(z, z') := \sum_{k=0}^{l-1} \varphi^{(k)}_{\bar{n},1}(z') \varphi^{(k)}_{\bar{n},2}(\bar{z}),
\]

and, whenever the measure \( \mu \) is positive definite, we have the equivalent expressions

\[
K^{[l]}_{\bar{n}}(z, z') = \sum_{k=0}^{l-1} h^{-1}_k \varphi^{(k)}_{\bar{n},1}(z') \bar{\varphi}^{(k)}_{\bar{n},1}(z) = \sum_{k=0}^{l-1} h_k \varphi^{(k)}_{\bar{n},2}(z') \bar{\varphi}^{(k)}_{\bar{n},2}(\bar{z}) = \sum_{k=0}^{l-1} \varphi^{(k)}_{\bar{n}}(z') \bar{\varphi}^{(k)}_{\bar{n}}(z).
\]
Proceeding as in the CMV ordering we conclude the following results

**Proposition 24.**

1. This is the integral kernel of the integral representation of the projections $\pi_{n,1}^{(l)}, \pi_{n,2}^{(l)}$

   $$(\pi_{n,1}^{(l)} f)(z') = \oint_{T} K_{n}^{[l]}(z, z') f(z) d\mu(z), \quad \forall f \in \Lambda_{[\infty]}.$$  

   $$(\pi_{n,2}^{(l)} f)(z) = \oint_{T} K_{n}^{[l]}(z, z') \tilde{f}(z') d\mu(z'), \quad \forall f \in \Lambda_{[\infty]}.$$  

2. This CD kernel $K_{n}^{[l]}(z, z')$ has the reproducing property

   $$K_{n}^{[l]}(z, z') = \oint T K_{n}^{[l]}(z, u) K_{n}^{[l]}(u, z') d\mu(u).$$

3. The following version of the ABC theorem holds

   $$K_{n}^{[l]}(z, z') = \chi_{n}^{[l]}(z) \tilde{g}_{n}^{[l]}(z')^{-1} \chi_{n}^{[l]}(z').$$

4. The CD kernel has the following expression

   $$(z' - \bar{z}^{-1})K_{n}^{[l]}(z, z') = \chi_{n}^{[l]}(z) \tilde{g}_{n}^{[l]}(z')^{-1} \chi_{n}^{[l]}(z') - \bar{z}^{-1} \chi_{n}^{[l]}(z) \tilde{g}_{n}^{[l]}(z')^{-1} \chi_{n}^{[l]}(z')$$

   $$= (\chi_{n}^{[l]}(z) \tilde{g}_{n}^{[l]}(z')^{-1} g_{n}^{[l]}(z')^{-1} - \chi_{n}^{[l]}(z) \tilde{g}_{n}^{[l]}(z')^{-1} g_{n}^{[l]}(z')^{-1} \chi_{n}^{[l]}(z') -$$

   $$- \chi_{n}^{[l]}(z) \tilde{g}_{n}^{[l]}(z')^{-1} g_{n}^{[l]}(z')^{-1} \chi_{n}^{[l]}(z') - \chi_{n}^{[l]}(z) \tilde{g}_{n}^{[l]}(z')^{-1} g_{n}^{[l]}(z')^{-1} \chi_{n}^{[l]}(z')).$$

The integers $l_{1, \pm a}$ can be used to calculate the $\Upsilon$ blocks in this case. The reader can check

**Proposition 25.** The formula for $\Upsilon_{n}^{[l]}$ is the following

$$\Upsilon_{n}^{[l]} = e_{(l-1)_{-1}} c_{l+1-l}, \quad \Upsilon_{n}^{[l]} = e_{l+2-l} c_{(l-1)_{-2}}.$$  

Thus, the expression of the CD kernel is

$$(\bar{z}^{-1} - z')K_{n}^{[l]}(z, z') = (\chi_{n}^{[l]}(z) - \chi_{n}^{[l]}(z) \tilde{g}_{n}^{[l]}(z')^{-1} g_{n}^{[l]}(z')^{-1} e_{l+2-l} e_{l+1-n}^{-1} (\chi_{n}^{[l]}(z') -$$

$$- \chi_{n}^{[l]}(z) \tilde{g}_{n}^{[l]}(z')^{-1} g_{n}^{[l]}(z')^{-1} \chi_{n}^{[l]}(z)'),$$

(72)

that suggests the definition of the following associated polynomials

**Definition 16.** The associated Laurent polynomials are defined by

$$\varphi_{n,1,+a}^{(l)} := \chi_{n}^{(l+a)} - (\tilde{g}_{n,1+a,0} \cdots \tilde{g}_{n,1+a,1+l}) (g_{n}^{[l+1]})^{-1} \chi_{n}^{[l+1]}, \quad \varphi_{n,1,-a}^{(l)} := e_{l-a}^{\top} (g_{n}^{[l+1]})^{-1} \chi_{n}^{[l+1]},$$

$$\varphi_{n,2,+a}^{(l)} := \chi_{n}^{(l+a)} - (\tilde{g}_{n,0,l+a} \cdots \tilde{g}_{n,1,l+a}) ((g_{n}^{[l+1]})^{-1} \chi_{n}^{[l+1]}, \quad \varphi_{n,2,-a}^{(l)} := e_{l-a}^{\top} (g_{n}^{[l+1]})^{-1} \chi_{n}^{[l+1]},$$

where $a = 1, 2$.

It is easy to see that $\varphi_{n,1,+a(l)}^{(l)} = \varphi_{n,1+a(l)}^{(l)}$ and $\varphi_{n,2,+a(l)}^{(l)} = \varphi_{n,1,a(l)}^{(l)}$ and $\varphi_{n,2,a(l)}^{(l)} = \varphi_{n,2,-a(l)}^{(l)}$.

**Theorem 5.** For the associated Laurent polynomials $\varphi_{n,+,a}, \varphi_{n,-a}$ we have two alternative expressions.

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1. The reciprocal type form (valid for positive definite cases)

\[ \varphi_{\bar{n},1,+2}(z) = \varphi_{\bar{n},2,+2}(z) = z^{\nu_+ (l) - \nu_- (l) - 2} \varphi_{\bar{n},1}(z^{-1}), \]
\[ \varphi_{\bar{n},1,-2}(z) = \varphi_{\bar{n},2,-2}(z) = z^{\nu_+ (l) - \nu_- (l) - 1} \varphi_{\bar{n},2}(z^{-1}), \]

when \( a(l) = 1 \) and

\[ \varphi_{\bar{n},1,1+}(z) = \varphi_{\bar{n},2,1+}(z) = z^{\nu_+ (l) - \nu_- (l)} \varphi_{\bar{n},1}(z^{-1}), \]
\[ \varphi_{\bar{n},1,-1+}(z) = \varphi_{\bar{n},2,-1+}(z) = z^{\nu_+ (l) - \nu_- (l) - 1} \varphi_{\bar{n},2}(z^{-1}), \]

for \( a(l) = 2 \).

2. The linear combination form (valid for quasi-definite cases)

\[ \varphi_{\bar{n},1,+a}(z) = (S_1^{-1})_{l+a,l+a} \varphi_{\bar{n},1}^{(l+a)} + (S_1^{-1})_{l+a,l+a-1} \varphi_{\bar{n},1}^{(l+a-1)} + \cdots + (S_1^{-1})_{l,a,l_a} \varphi_{\bar{n},1}^{(l)}, \]
\[ \varphi_{\bar{n},2,+a}(z) = (\tilde{S}_2)_{l+a,l+a} \varphi_{\bar{n},2}^{(l+a)} + (\tilde{S}_2)_{l+a,l+a-1} \varphi_{\bar{n},2}^{(l+a-1)} + \cdots + (\tilde{S}_2)_{l,a,l_a} \varphi_{\bar{n},2}^{(l)}, \]
\[ \varphi_{\bar{n},1,-a}(z) = (S_2^{-1})_{l-a,l-a} \varphi_{\bar{n},1}^{(l-a)} + (S_2^{-1})_{l-a,l-a+1} \varphi_{\bar{n},1}^{(l-a+1)} + \cdots + (S_2^{-1})_{l-a-l_a} \varphi_{\bar{n},1}^{(l)}, \]
\[ \varphi_{\bar{n},2,-a}(z) = (\tilde{S}_1)_{l-a,l-a} \varphi_{\bar{n},2}^{(l-a)} + (\tilde{S}_1)_{l-a,l-a+1} \varphi_{\bar{n},2}^{(l-a+1)} + \cdots + (\tilde{S}_1)_{l-a-l_a} \varphi_{\bar{n},2}^{(l)}, \]

Proof. 1. Let us suppose that \( a(l) = 1 \). In that case we have \( \varphi_{\bar{n},1,1+} = \varphi_{\bar{n},1} \) and \( \varphi_{\bar{n},1,1+2} \in \Lambda_{[\nu_+ (l) - 1, \nu_+ (l) - 2]} \). Consequently, \( z^{\nu_+ (l) - \nu_- (l) + 2} \varphi_{\bar{n},1,1+2} \in \Lambda_{[\nu_1 (l) - 1, \nu_1 (l)]} \) and \( a(l) = 1, \nu_+ (l) = \nu_+ (l - 1), \nu_- (l) = \nu_- (l - 1) \). For the dual polynomials \( \Phi_{\bar{n},2,1} = \Phi_{\bar{n},2} \) and \( \Phi_{\bar{n},2,2} \in \Lambda_{[\nu_1 (l), \nu_1 (l)]} \), hence \( z^{\nu_+ (l) - \nu_+ (l) + 1} \Phi_{\bar{n},2,2} \in \Lambda_{[\nu_1 (l) - 1, \nu_- (l) - 1]} \). Using (79) we conclude that the following orthogonality relations hold true
\[
\int_{T} z^{\nu_+ (l) - \nu_+ (l) + 2} \varphi_{\bar{n},1,1+2}(z) z^{-k} \, d\mu(z) = 0, \quad k = -\nu_+ (l - 1) + 1, \ldots, \nu_- (l - 1),
\]
\[
\int_{T} z^{\nu_+ (l) - \nu_+ (l) + 1} \varphi_{\bar{n},2,2}(z) z^{-k} \, d\mu(z) = 0, \quad k = -\nu_+ (l - 1) + 1, \ldots, \nu_- (l - 1),
\]
and we get the result.

Now let us suppose that \( a(l) = 2 \). In this case we have \( \varphi_{\bar{n},1,2+} = \varphi_{\bar{n},1} \) and \( \varphi_{\bar{n},1,1+1} \in \Lambda_{[\nu_+ (l) - 1, \nu_+ (l)]} \). Consequently, \( z^{\nu_+ (l) - \nu_+ (l) + 1} \varphi_{\bar{n},1,2+} \in \Lambda_{[\nu_1 (l) - 1, \nu_- (l)]} \). Now, as \( a(l) = 2 \), we have \( \nu_+ (l) = \nu_+ (l - 1) \) and \( \nu_- (l) = \nu_- (l - 1) + 1 \). For the dual polynomials \( \Phi_{\bar{n},1,2} = \Phi_{\bar{n},2} \) and \( \Phi_{\bar{n},2,1} \in \Lambda_{[\nu_1 (l), \nu_1 (l)]} \), so \( z^{\nu_+ (l) - \nu_+ (l) + 1} \Phi_{\bar{n},1,2+} \in \Lambda_{[\nu_1 (l) - 1, \nu_- (l)]} \). Now using again (79) we get
\[
\int_{T} z^{\nu_+ (l) - \nu_+ (l) + 1} \varphi_{\bar{n},1,2+}(z) z^{-k} \, d\mu(z) = 0, \quad k = -\nu_+ (l - 1) + 1, \ldots, \nu_- (l - 1),
\]
\[
\int_{T} z^{\nu_+ (l) - \nu_+ (l) + 1} \Phi_{\bar{n},1,2+}(z) z^{-k} \, d\mu(z) = 0, \quad k = -\nu_+ (l - 1) + 1, \ldots, \nu_- (l - 1),
\]
and
\[
\int_{T} z^{\nu_+ (l) - \nu_+ (l) + 1} \Phi_{\bar{n},1,2+}(z) z^{-\nu_- (l) - 1} \, d\mu(z) = 1.
\]

2. For \( \varphi_{\bar{n},1,+a} \) direct computation gives
\[
\int_{T} \varphi_{\bar{n},1,+a}(z) x_{\bar{n}}^{\bar{a}}(z) \, d\mu(z) = \int_{T} (x_{\bar{n}}^{\bar{a}}(z) - (g_{\bar{n},l+a} 0 \ g_{\bar{n},l+a,1} \cdots g_{\bar{n},l+a,l-1}) g_{\bar{n}}^{\bar{a}}(z) x_{\bar{n}}^{\bar{a}}(z) \, d\mu(z)
\]

= \begin{pmatrix} g_{\bar{n},l+a,0} & g_{\bar{n},l+a,1} & \cdots & g_{\bar{n},l+a,l-1} \end{pmatrix} \begin{pmatrix} 0 & g_{\bar{n},l+a,0} & g_{\bar{n},l+a,1} & \cdots & g_{\bar{n},l+a,l-1} \end{pmatrix} = 0,
and for $\varphi_{n,2-a}^{(l)}$ we have
\[ \int_T \chi_n^{[l+1]}(z) \bar{\varphi}_{n,2-a}(\bar{z}) d\mu(z) = \int_T \chi_n^{[l+1]}(z) \bar{\varphi}_{n,2-a}(\bar{z}) (g_{n}^{[l+1]})^{-1} e_{l-a} d\mu(z) = e_{l-a}, \]
so that we get orthogonality relations for the associated polynomials
\[ \int_T \varphi_{n,1+a}^{(l)}(z) z^{-k} d\mu(z) = 0, \quad k = -\nu_-(l - 1), \ldots, \nu_+(l - 1) - 1, \]
\[ \int_T \chi_n^{(k)}(z) \bar{\varphi}_{n,2-a}(\bar{z}) d\mu(z) = \delta_{k,l-a}, \quad k = 0, 1, \ldots, l. \] (79)
Therefore,
\[ \varphi_{n,1+a}^{(l)} \in \text{span}\{ \varphi_{n,1}^{(l)}, \varphi_{n,1}^{(l+1)}, \ldots, \varphi_{n,1}^{(l+a)} \}, \]
i.e., $\varphi_{n,1+a}^{(l)} = \sum_{j=l}^{l+a} A_j^{(l)} \varphi_{n,1}^{(j)}$ for a set of coefficients $\{A_j^{(l)}\}$. Comparing the powers of $z$ that appear in the subsequence $\{\chi_j^{(j)}\}_{l \leq j \leq l+a}$ on both sides of the equation, the following linear system of equations is obtained
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
(S_1)_{l+a,l+a-1} & 1 & 0 & 0 & 0 & 0 \\
(S_1)_{l+a,l+a-2} & (S_1)_{l+a-1,l+a-2} & 1 & 0 & 0 & 0 \\
(S_1)_{l+a,l+a-3} & (S_1)_{l+a-1,l+a-3} & \cdots & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(S_1)_{l,a,l} & (S_1)_{l+a-1,l} & \cdots & (S_1)_{l+2,l} & (S_1)_{l+1,l} & 1
\end{pmatrix}
\begin{pmatrix}
A_{l+a}^{(l)} \\
A_{l+a-1}^{(l)} \\
\vdots \\
A_{l}^{(l)}
\end{pmatrix}
= \begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix}
\]
calling $M$ the coefficient matrix, the solution can be written as
\[
\begin{pmatrix}
A_{l+a}^{(l)} \\
A_{l+a-1}^{(l)} \\
\vdots \\
A_{l}^{(l)}
\end{pmatrix}
= \begin{pmatrix}
(M^{-1})_{0,0} \\
(M^{-1})_{1,0} \\
\vdots \\
(M^{-1})_{l-a,l,0}
\end{pmatrix}
\]
From the structure of $M$ we conclude that
\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 1 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix}
M^T
= \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
(S_1)_{l+1,l} & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
(S_1)_{l+a-1,l} & (S_1)_{l+a-1,l+1} & \cdots & 1 & 0 \\
(S_1)_{l+a,d} & (S_1)_{l+a,l+1} & \cdots & (S_1)_{l+a,l+a-1} & 1
\end{pmatrix}
\]
From the triangular structure of $S_1$ we deduce that $(M^{-1})_{i,j} = (S_1^{-1})_{i+l,j+l}$ for $i, j = 0, 1, \ldots, l+a - l$ and consequently $(M^{-1})_{j,0} = (S_1^{-1})_{l+a-1,l+a-j} = (S_1^{-1})_{l+a,1-a-j}$ for $i, j = 0, 1, \ldots, l+a - l$, which proves (73). The expression for (78) is obtained using a similar technique. Using again (73) we conclude that $\varphi_{n,2-a}^{(l)} \in \text{span}\{ \varphi_{n,2}^{(l-a)}, \varphi_{n,2}^{(l+1)}, \ldots, \varphi_{n,2}^{(l+a)} \}$; i.e., $\varphi_{n,2,a}^{(l)} = \sum_{j=l-a}^{l} B_j^{(l)} \varphi_{n,2}^{(j)}$. Bi-orthogonality and normalization properties imply
\[
\tilde{B}_j^{(l)} = \int_T \varphi_{n,1}^{(j)}(z) \varphi_{n,2-a}^{(l)}(z) d\mu(z) = (S_1)_{j,l-a}, \quad j = l-a, \ldots, l,
\]
that proves (78). The other two equations are obtained using the same idea.
Theorem 6. The CD formula for the extended ordering is the following

\[
K_n^{[l]}(z, z') = \frac{\tilde{z}^{\nu_+}(l)-\nu_-(l)-1}{(1-z\bar{z})} \frac{\varphi_n^{(l)}(z)+\varphi_n^{(l-1)}(z')}{\varphi_n^{(l-1)}(z')},
\]

and consequently the final result is the following

Theorem 6. The CD formula for a positive Borel measure \( \mu \) is

\[
K_n^{[l]}(z, z') = \frac{\tilde{z}^{\nu_+}(l)-\nu_-(l)-1}{(1-z\bar{z})} \frac{\varphi_n^{(l)}(z)+\varphi_n^{(l-1)}(z')}{\varphi_n^{(l-1)}(z')},
\]

We get the followings corollaries when we have a positive Borel measure \( \mu \)

Corollary 2. Given a positive measure \( \mu \), the CD kernel can be expressed using

\[
K_n^{[l]}(z, z') = \frac{\tilde{z}^{\nu_+}(l)-\nu_-(l)-1}{(1-z\bar{z})} \frac{\varphi_n^{(l)}(z)+\varphi_n^{(l-1)}(z')}{\varphi_n^{(l-1)}(z')},
\]

in the case \( a(l) = a(l-1) = 1 \),

\[
K_n^{[l]}(z, z') = \frac{\tilde{z}^{\nu_+}(l)-\nu_-(l)-1}{(1-z\bar{z})} \frac{\varphi_n^{(l)}(z)+\varphi_n^{(l-1)}(z')}{\varphi_n^{(l-1)}(z')},
\]

in the case \( a(l) = 1, a(l-1) = 2 \),

\[
K_n^{[l]}(z, z') = \frac{\tilde{z}^{\nu_+}(l)-\nu_-(l)-1}{(1-z\bar{z})} \frac{\varphi_n^{(l)}(z)+\varphi_n^{(l-1)}(z')}{\varphi_n^{(l-1)}(z')},
\]

in the case \( a(l) = 2, a(l-1) = 1 \),

\[
K_n^{[l]}(z, z') = \frac{\tilde{z}^{\nu_+}(l)-\nu_-(l)-1}{(1-z\bar{z})} \frac{\varphi_n^{(l)}(z)+\varphi_n^{(l-1)}(z')}{\varphi_n^{(l-1)}(z')},
\]

in the case \( a(l) = a(l-1) = 2 \).

and

Corollary 3. The CD formula for a positive Borel measure \( \mu \) can be expressed in terms on the Szegő polynomials as

\[
K_n^{[l]}(z, z') = \frac{\tilde{z}^{\nu_+}(l)-\nu_-(l)-1}{(1-z\bar{z})} \frac{\varphi_n^{(l)}(z)+\varphi_n^{(l-1)}(z')}{\varphi_n^{(l-1)}(z')},
\]

in the case \( a(l) = a(l-1) = 2 \).

and

4 Associated 2D Toda type hierarchies

Here we analyze the link between the previous constructions on OLPUC and integrable systems of Toda type. Our driving idea is the presence of the Borel-Gauss factorization problem in the theoretical construction of both, OLPUC and Toda.
4.1 2D Toda flows

We will consider a set of complex deformation parameters $t = \{t_{1j}, t_{2j}\}_{j \in \mathbb{N}}$ and with it two semi-infinite matrices that we will define now.

**Definition 17.**

1. The deformation matrices are defined as follows
   
   $$ W_{1,0}(t) := \exp \left( \sum_{j=1}^{\infty} t_{1j} \Upsilon^j \right), \quad W_{2,0}(t) := \exp \left( \sum_{j=1}^{\infty} t_{2j} (\Upsilon^\top)^j \right), \quad (84) $$

2. For each $t$ we will consider the matrix $g(t)$
   
   $$ g(t) := W_{1,0}(t)g(W_{2,0}(t))^{-1}, $$

3. and the corresponding time dependant Gauss–Borel factorization
   
   $$ g(t) := W_{1,0}(t)g(W_{2,0}(t))^{-1}, \quad g(t) = (S_1(t))^{-1}S_2(t). $$

As we show now the deformed moment matrix is a moment matrix of a deformed measure.

**Proposition 26.** The “deformed” moment matrix can be understood as a moment matrix for a “deformed” (that is, a “time” dependant) measure given by

$$ d\mu(t, z) := \exp \left( \sum_{j=1}^{\infty} t_{1j}z^j - t_{2j}z^{-j} \right) d\mu(z). \quad (85) $$

**Proof.** First expanding the exponentials in (84) we obtain

$$ W_{1,0}(t) = \sum_{k=0}^{\infty} \sigma_1^k(t)\Upsilon^k, \quad (W_{2,0}(t))^{-1} = \sum_{l=0}^{\infty} \sigma_2^l(t)(\Upsilon^\top)^l, $$

then using the definition of $g$ and $g(t)$ we get the desired result

$$ W_{1,0}(t)gW_{2,0}(t)^{-1} = \sum_{k,l=0}^{\infty} \sigma_1^k(t)\Upsilon^k g(\Upsilon^\top)^l \sigma_2^l(t) = \oint T \sum_{k=0}^{\infty} \sigma_1^k(t)z^j \chi(z)\chi(z)^\dagger \sum_{l=0}^{\infty} \sigma_2^l(t)z^{-l} d\mu(z) $$

$$ = \oint T \chi(z)\chi(z)^\dagger \exp \left( \sum_{j=0}^{\infty} (t_{1j}z^j - t_{2j}z^{-j}) \right) d\mu(z). $$

From this result we conclude at least for absolutely continuous measures

$$ F_{\mu(t)} = \exp \left( \sum_{j=1}^{\infty} t_{1j}z^j - t_{2j}z^{-j} \right) F_{\mu(z)}, $$

---

10 In the framework of the theory on integrable systems these parameters are understood as an infinite set of times, being the independent variables in an associated nonlinear hierarchy of partial differential-difference equations.

11 We shall drop the subindex $\vec{n}$ from $g$ and $\Upsilon$ as the definitions are valid for any value of $\vec{n}$. It will be supposed that a particular $\vec{n}$ is chosen and the whole $\mathcal{S}$ will be built using that $\vec{n}$.

12 For the sake of notation simplicity in some situations we drop the time dependence of $S_1, S_2$ and they will not denote the factors within the Gauss–Borel factorization of the initial condition but for the “deformed” one. Consequently, in this section $S_1, S_2$ will always depend on “time” parameters.
from where we deduce that the radii defining the annulus of convergence is time independent; i.e., \( R_\pm(t) = R_\pm \).

Given a positive definite initial measure \( \mu \) in order to ensure that the evolved measure \( \mu(t) \) is also definite positive for all times is enough to request to the exponential to be real; i.e., setting \( t_{2j} = -t_{1j} \), so that

\[
\exp \left( \sum_{j=0}^{\infty} (t_{1j}z^j + \bar{t}_{1j}z^{-j}) \right) = \exp \left( \sum_{j=0}^{\infty} 2\Re(t_{1j}z^j) \right).
\]

4.2 Integrable Toda equations

**Definition 18.** Associated with the deformed Gauss–Borel factorization we consider

1. Wave semi-infinite matrices

\[
W_1(t) := S_1(t)W_{1,0}(t), \quad W_2(t) := S_2(t)W_{2,0}(t).
\]  

2. Partial wave and partial adjoint (denoted the adjoint by \(^\ast\)) wave semi-infinite vector functions

\[
\Psi_{1,1}(z,t) := W_1(t)\chi_1(z), \quad \Psi_{2,1}(z,t) := (W_2(t)^{-1})^\dagger \chi_1(z), \\
\Psi_{1,2}(z,t) := W_1(t)\chi_2(z), \quad \Psi_{2,2}(z,t) := (W_2(t)^{-1})^\dagger \chi_2(z), \\
\Psi_{1,1}^\ast(z,t) := (W_1(t)^{-1})^\dagger \chi_1^\ast(z), \quad \Psi_{2,1}(z,t) := W_2(t)\chi_1^\ast(z), \\
\Psi_{1,2}^\ast(z,t) := (W_1(t)^{-1})^\dagger \chi_2(z), \quad \Psi_{2,2}(z,t) := W_2(t)\chi_2(z),
\]

and wave and adjoint wave functions

\[
\Psi_1(z,t) := W_1(t)\chi(z) = (\Psi_{1,1} + \Psi_{1,2})(z,t), \quad \Psi_2^\ast(z,t) := (W_2(t)^{-1})^\dagger \chi(z) = (\Psi_{2,1} + \Psi_{2,2})(z,t), \\
\Psi_1^\ast(z,t) := (W_1(t)^{-1})^\dagger \chi^\ast(z) = (\Psi_{1,1}^\ast + \Psi_{1,2}^\ast)(z,t), \quad \Psi_2(z,t) := W_2(t)\chi^\ast(z) = (\Psi_{2,1} + \Psi_{2,2})(z,t).
\]  

3. Lax semi-infinite matrices

\[
L_1(t) := S_1(t)S_1(t)^{-1}, \quad L_2(t) := S_2(t)S_2(t)^{-1}.
\]

4. Zakharov–Shabat semi-infinite matrices

\[
B_{1,j} := (L_j^1)_+, \quad B_{2,j} := (L_j^2)_-,
\]

where the subindex \(+\) indicates the projection in the upper triangular matrices while the subindex \(-\) the projection in the strictly lower triangular matrices.

**Theorem 7.** For \( j, j' = 1, 2, \ldots \), the following differential relations hold

1. Auxiliary linear systems for the wave matrices

\[
\frac{\partial W_1}{\partial t_{1j}} = B_{1,j}W_1, \quad \frac{\partial W_1}{\partial t_{2j}} = B_{2,j}W_1, \quad \frac{\partial W_2}{\partial t_{1j}} = B_{1,j}W_2, \quad \frac{\partial W_2}{\partial t_{2j}} = B_{2,j}W_2.
\]

2. Linear systems for the wave and adjoint wave semi-infinite functions

\[
\frac{\partial \Psi_1}{\partial t_{1j}} = B_{1,j}^\dagger \Psi_1, \quad \frac{\partial \Psi_1}{\partial t_{2j}} = B_{2,j}^\dagger \Psi_1, \quad \frac{\partial \Psi_2^\ast}{\partial t_{1j}} = -B_{1,j}^\dagger \Psi_2^\ast, \quad \frac{\partial \Psi_2^\ast}{\partial t_{2j}} = -B_{2,j}^\dagger \Psi_2^\ast, \\
\frac{\partial \Psi_1^\ast}{\partial t_{1j}} = -B_{1,j}^\dagger \Psi_1, \quad \frac{\partial \Psi_1^\ast}{\partial t_{2j}} = -B_{2,j}^\dagger \Psi_1, \quad \frac{\partial \Psi_2}{\partial t_{1j}} = B_{1,j} \Psi_2, \quad \frac{\partial \Psi_2}{\partial t_{2j}} = B_{2,j} \Psi_2.
\]

\(^{13}\)Also called Baker, or Baker–Akhiezer, functions
3. **Lax equations**

\[
\begin{align*}
\frac{\partial L_1}{\partial t_{1j}} &= [B_{1,j}, L_1], & \frac{\partial L_1}{\partial t_{2j}} &= [B_{2,j}, L_1], & \frac{\partial L_2}{\partial t_{1j}} &= [B_{1,j}, L_2], & \frac{\partial L_2}{\partial t_{2j}} &= [B_{2,j}, L_2].
\end{align*}
\]

(93)

4. **Zakharov–Shabat equations**

\[
\begin{align*}
\frac{\partial B_{1,j}}{\partial t_{1j}'} - \frac{\partial B_{1,j}}{\partial t_{1j}} + [B_{1,j}, B_{1,j}'] &= 0, \\
\frac{\partial B_{2,j}}{\partial t_{2j}'} - \frac{\partial B_{2,j}}{\partial t_{2j}} + [B_{2,j}, B_{2,j}'] &= 0, \\
\frac{\partial B_{1,j}}{\partial t_{2j}'} - \frac{\partial B_{1,j}}{\partial t_{2j}} + [B_{1,j}, B_{2,j}'] &= 0.
\end{align*}
\]

(94)  

(95)  

(96)

**Proof.** The proof can be made using the same idea used in [12], so we do not repeat them here again. \(\Box\)

From the definition it is clear that the wave functions are associated to the OLPUC for the evolved measure

**Proposition 27.** The wave functions are linked to the OLPUC and the Fourier series of the measure trough

\[
\begin{align*}
\Psi_1^{(n)}(z, t) &= \varphi_1^{(n)}(z, t) e^{\sum_{j=1}^{\infty} t_{1j} z^j}, \\
(\Psi_2^{(n)})^{(n)}(z, t) &= \varphi_2^{(n)}(z, t) e^{-\sum_{j=1}^{\infty} t_{2j} z^{-j}}, \\
(\Psi_1^{(n)}(z, t))^{(n)} &= 2\pi \varphi_2^{(n)}(z^{-1}, t) z^{-1} F_\mu(t) (z) e^{-\sum_{j=1}^{\infty} t_{1j} z^{-j}}, \\
(\Psi_2^{(n)}(z, t))^{(n)} &= 2\pi \varphi_2^{(n)}(z^{-1}, t) z^{-1} F_\mu(t) (z^{-1}) e^{\sum_{j=1}^{\infty} t_{2j} z^j}.
\end{align*}
\]

Moreover, the wave functions are eigen-functions of the Lax and adjoint Lax matrices

\[
\begin{align*}
L_1 \Psi_1 &= z \Psi_1, & L_1^* \Psi_1 &= z \Psi_1^*, \\
L_2 \Psi_2 &= z \Psi_2, & L_2^* \Psi_2 &= z \Psi_2.
\end{align*}
\]

4.3 **CMV matrices and the Toeplitz lattice**

For the CMV ordering of the Laurent basis, Lax equations (93) can be written as a nonlinear dynamical system that is a version in the CMV context of the Toeplitz lattice discussed by Mark Adler and Pierre van Moerbeke [8]

**Proposition 28.** For the case \(\vec{n} = (1, 1)\) Lax equations (93) have as a consequence the following nonlinear dynamical system for the Verblunsky coefficients

\[
\begin{align*}
\frac{\partial \alpha^{(1)}_k}{\partial t_{11}} &= \alpha^{(1)}_{k+1} (1 - \alpha^{(1)}_k \tilde{a}^{(2)}_k), & \frac{\partial \alpha^{(1)}_k}{\partial t_{21}} &= -\tilde{a}^{(2)}_{k-1} (1 - \alpha^{(1)}_k \tilde{a}^{(2)}_k), \\
\frac{\partial \alpha^{(2)}_k}{\partial t_{11}} &= -\alpha^{(2)}_{k+1} (1 - \alpha^{(1)}_k \tilde{a}^{(2)}_k), & \frac{\partial \alpha^{(2)}_k}{\partial t_{21}} &= -\tilde{a}^{(2)}_{k+1} (1 - \alpha^{(1)}_k \tilde{a}^{(2)}_k),
\end{align*}
\]

(99)

with \(k = 1, 2, \ldots\).

**Proof.** See Appendix A. \(\Box\)
If the initial measure $\mu$ is positive definite then the sequences $\{\alpha_k^{(1)}\}$ and $\{\alpha_k^{(2)}\}$ are identical in $t = 0$. Furthermore, if we set $t_{21} = -\bar{t}_{11}$, the evolved measure is always real and there is only one family of time-dependent functions. That is the reduction studied by L. B. Golinskii \cite{30} in the context of Schur flows.

We have obtained a CMV version of the Toeplitz lattice, but for any $\bar{n}$ the integrable hierarchy obtained is always equivalent to the one discussed in \S. The reason for this fact relies in the observation that for any positive Borel measure $\mu$ and for any $\bar{n}$, there is a bijection between the set of OLPUC $\{\phi_{n,1}^{(l)}\}$ and the set of OPUC $\{P_l\}$. All the coefficients of any $P_l$ are determined in terms of the set of reflection coefficients $\{\alpha_l\}$, so the time evolution for the Szegő polynomials under Toda-type flows is determined by the evolution of the reflection coefficients. As the measure evolution does not depend on $\bar{n}$, it is natural to always obtain the very same evolution for the family $\{\alpha_l\}$ under the Toda flows. We conjecture that a similar result holds for the quasi-definite case.

4.4 Discrete flows

We now consider discrete flows associated to the moment matrix. Given two integers $s_1, s_2$ and $s := (s_1, s_2)$ is then possible to make a new deformation of the moment matrix that depends on $s$.

Definition 19. We introduce for each $s$ and the deformed moment matrix $g(s)$ and its deformed Gauss–Borel factorization

$$g(s) := D_{1,0}(s)g(D_{2,0}(s))^{-1},$$

$$g(s) = S_1^{-1}(s)S_2(s),$$

where $D_{1,0}(s), D_{2,0}(s)$ are discrete deformation operators to be determined later on.

We consider the operator $T_1$ responsible of the shift $s_1 \mapsto s_1 + 1$ and $T_2$ corresponding to the shift $s_2 \mapsto s_2 + 1$. Let us suppose that matrices $q_1, q_2$ exist and satisfy

$$T_1(D_{1,0}) = q_1D_{1,0},$$

$$T_2(D_{1,0}) = D_{1,0},$$

$$T_1(D_{2,0}) = D_{2,0},$$

$$T_2(D_{2,0}) = q_2D_{2,0},$$

then we define

$$\delta_1 := S_1(s)q_1S_1(s)^{-1},$$

$$\delta_2 := S_2(s)q_2S_2(s)^{-1}.$$  

If $\delta_1, \delta_2$ can be $LU$ factorized, then there exist semi-infinite matrices $\delta_{1,+}, \delta_{1,-}, \delta_{2,+}, \delta_{2,-}$ such that

$$\delta_1 = \delta_{1,-}^{-1}\delta_{1,+},$$

$$\delta_2 = \delta_{2,-}^{-1}\delta_{2,+}.$$  

In this case we introduce

$$\omega_1 := \delta_{1,+},$$

$$\omega_2 := \delta_{2,-}.$$  

Proposition 29. The operators $T_1, T_2$ and the matrices $S_1(s), S_2(s)$ satisfy the following equations

$$T_1(S_1(s))(S_1(s))^{-1} = \delta_{1,-},$$

$$T_2(S_1(s))(S_1(s))^{-1} = \delta_{2,-},$$

$$T_1(S_2(s))(S_2(s))^{-1} = \delta_{1,+},$$

$$T_2(S_2(s))(S_2(s))^{-1} = \delta_{2,+}.$$  

Proof. First using $T_1$ and $T_2$ on the factorization $D_{1,0}gD_{2,0}^{-1} = S_1^{-1}S_2$ we obtain

$$T_1(D_{1,0})gT_1(D_{2,0}^{-1}) = T_1(S_1^{-1})T_1(S_2) \Rightarrow (T_1(S_1)^{-1})^{-1}T_1(S_2)S_2^{-1} = \delta_1,$$

$$T_2(D_{1,0})gT_2(D_{2,0}^{-1}) = T_2(S_1^{-1})T_2(S_2) \Rightarrow (T_2(S_1)^{-1})^{-1}T_2(S_2)S_2^{-1} = \delta_2^{-1},$$  

then using the factorization for $\delta_1$ and $\delta_2$ and its uniqueness we can identify the upper and lower triangular parts and prove the claimed result.
It is also possible to define wave matrices $W_1$ and $W_2$ in this discrete context,

$$W_1 := S_1 D_{1,0}$$

$$W_2 := S_2 D_{2,0}.$$  

To ensure the consistency between both continuous and discrete flows we only need to replace sequences with $\lambda$, not, for the proof one only needs a slight modification of the one in [12].

**Theorem 8.**

1. The next linear system for $W_1$ and $W_2$ is satisfied

   $$T_a(W_{a'}) = \omega_a W_{a'}$$

   $$a, a' = 1, 2.$$  

(100)

2. The discrete versions of the Lax equations are the following

   $$T_a(L_{a'}) = \omega_a L_{a'} \omega_a^{-1}$$

   $$a, a' = 1, 2.$$  

(101)

3. The compatibility equations for the discrete flows in the linear system (100) are

   $$T_1(\omega_2) \omega_1 = T_2(\omega_1) \omega_2$$

(102)

if there are also continuous deformation parameters the mixed compatibility equations are

$$T_a(B_{1,j}) = \frac{\partial \omega_a}{\partial t_{1j}} \omega_a^{-1} + \omega_a B_{1,j} \omega_a^{-1}$$

$$a = 1, 2, \ j = 1, 2, \ldots$$

$$T_a(B_{2,j}) = \frac{\partial \omega_a}{\partial t_{2j}} \omega_a^{-1} + \omega_a B_{2,j} \omega_a^{-1}$$

$$a = 1, 2, \ j = 1, 2, \ldots$$

(103)

Now we give examples of some discrete flows operators. Let be $\{\lambda_1(j)\}_{j \in \mathbb{Z}}$ and $\{\lambda_2(j)\}_{j \in \mathbb{Z}}$ two complex sequences with $\lambda_1(j), \lambda_2(j) \in \mathbb{D}$, then

$$D^{(1)}_{1,0} := \begin{cases} 
  \Pi^{n_1}_{j=0}(\Upsilon - \lambda_1(j)\mathbb{I}) & n_1 > 0 \\
  \mathbb{I} & n_1 = 0 \\
  \Pi^{[n_1]}_{j=0}(\Upsilon - \lambda_1(-j)\mathbb{I})^{-1} & n_1 < 0
\end{cases}$$

$$(D^{(1)}_{2,0})^{-1} := \begin{cases} 
  \Pi^{n_2}_{j=0}(\Upsilon^\top - \lambda_2(j)\mathbb{I}) & n_2 > 0 \\
  \mathbb{I} & n_2 = 0 \\
  \Pi^{[n_2]}_{j=0}(\Upsilon^\top - \lambda_2(-j)\mathbb{I})^{-1} & n_2 < 0
\end{cases}$$

the evolution of the measure is then

$$d\mu(z, s) = D^{(1)}_{1}(z, s_1)(D^{(1)}_{2})^{-1}(z, s_2)d\mu(z)$$

where

$$D^{(1)}_{1}(z, s) = \begin{cases} 
  \Pi^{n_1}_{j=0}(z - \lambda_1(j)) & s_1 > 0 \\
  1 & s_1 = 0 \\
  \Pi^{[s_1]}_{j=0}(z - \lambda_1(-j))^{-1} & s_1 < 0
\end{cases}$$

$$(D^{(1)}_{2})^{-1}(z, s) = \begin{cases} 
  \Pi^{n_2}_{j=0}(z^{-1} - \lambda_2(j)) & s_2 > 0 \\
  1 & s_2 = 0 \\
  \Pi^{[s_2]}_{j=0}(z^{-1} - \lambda_2(-j))^{-1} & s_2 < 0
\end{cases}$$

in that case

$$q^{(1)}_1 = \Upsilon - \lambda_1(s_1 + 1)\mathbb{I}$$

$$q^{(1)}_2 = \Upsilon^\top - \lambda_2(s_2 + 1)\mathbb{I}$$

$$\delta^{(1)}_1 = L_1 - \lambda_1(s_1 + 1)\mathbb{I}$$

$$\delta^{(1)}_2 = L_2 - \lambda_2(s_2 + 1)\mathbb{I}$$

The evolution of the wave functions is associated to the evolved Laurent polynomials

$$\Psi_1(z, s) = W_1(s)\chi(z) = S_1(s)D_{1,0}(s)\chi(z) = \Phi_1(z, s)D^{(1)}_{1}(z, s),$$

$$\Psi_2^*(z, s) = (W_2(s)^{-1})^\dagger\chi(z) = (S_2(s)^{-1})^\dagger(D_{2,0}(s)^{-1})^\dagger\chi(z) = \Phi_2(z, s)(D^{(1)}_{2})^{-1}(z, s),$$

where $\Phi_1(z, t)$ and $\Phi_2(z, t)$ are the Laurent polynomials associated to the evolved measure.
Lemma 4. We have the following structure for the matrices $\omega_1, \omega_2$

\[
\begin{align*}
\omega_1 &= \omega_{1,0} + \omega_{1,1} \Lambda + \cdots + \omega_{1,n_+ + 1} \Lambda^{n_+ + 1} \\
\omega_2 &= \omega_{2,0} + \omega_{2,1} \Lambda^T + \cdots + \omega_{2,n_+ + 1} (\Lambda^T)^{n_+ + 1} \\
\omega_1^\dagger &= \rho_{1,0} + \rho_{1,1} \Lambda^T + \cdots + \rho_{1,n_- + 1} (\Lambda^T)^{n_- + 1} \\
\omega_2^\dagger &= \rho_{2,0} + \rho_{2,1} \Lambda + \cdots + \rho_{2,n_- + 1} \Lambda^{n_- + 1}
\end{align*}
\]

for some semi-infinite matrices

\[
\begin{align*}
\omega_{1,j} &= \mbox{diag}(\omega_{1,j}(0), \omega_{1,j}(1), \ldots) & j &= 0, \ldots, n_- + 1 \\
\omega_{2,j} &= \mbox{diag}(\omega_{2,j}(0), \omega_{2,j}(1), \ldots) & j &= 0, \ldots, n_+ + 1 \\
\rho_{1,j} &= \mbox{diag}(\rho_{1,j}(0), \rho_{1,j}(1), \ldots) & j &= 0, \ldots, n_- + 1 \\
\rho_{2,j} &= \mbox{diag}(\rho_{2,j}(0), \rho_{2,j}(1), \ldots) & j &= 0, \ldots, n_+ + 1
\end{align*}
\]

Proof. Immediate from (104).

Defining

\[\gamma_1(z, s) := z - \lambda_1(s_1 + 1) \quad \quad \gamma_2(z, s) := z^{-1} - \lambda_2(s_2 + 1)\]

the previous Lemma allows us to compute the action of the operators $T_1$ and $T_2$ on the OLPUC $\varphi_1^l(z, s), \varphi_2^l(z, s)$

Proposition 30. The following equations hold

\[
\begin{align*}
(T_1 \varphi_1^l) \gamma_1 &= \omega_{1,0}(l) \varphi_1^l + \omega_{1,1}(l) \varphi_1^{l+1} + \cdots + \omega_{1,n_- + 1}(l) \varphi_1^{l+n_- + 1} \\
(T_2 \varphi_1^l) \gamma_1 &= \omega_{2,0}(l) \varphi_1^l + \omega_{2,1}(l) \varphi_1^{l-1} + \cdots + \omega_{2,n_+ + 1}(l) \varphi_1^{l-n_- + 1} \\
\varphi_2^l &= \rho_{1,0}(l)(T_1 \varphi_2^l) + \rho_{1,1}(l)(T_1 \varphi_2^{l-1}) + \cdots + \rho_{1,n_- + 1}(l)(T_1 \varphi_2^{l-n_- - 1}) \\
\varphi_2^l &= \left( \rho_{2,0}(l)(T_2 \varphi_2^l) + \rho_{2,1}(l)(T_2 \varphi_2^{l+1}) + \cdots + \rho_{2,n_+ + 1}(l)(T_2 \varphi_2^{l+n_+ + 1}) \right) \gamma_2
\end{align*}
\]

Proof. The first two equations come from (100) and Lemma 4. For the last two equations we use that

\[\omega_{a}^\dagger T_a((W_{a'}^{-1})^\dagger) = (W_{a'}^{-1})^\dagger \quad \quad a, a' = 1, 2.\]

Another possible option that preserves the reality of the measure is using pairs of conjugate transforms as follows

\[D_{1,0}^{(2)} := \begin{cases}
\Pi_{s_1 = 0}^s (\Omega - \lambda_1(j) I)(\Omega^T - \tilde{\lambda}_1(j) I) & s_1 > 0 \\
I & s_1 = 0 \\
\Pi_{s_1 = 0}^{[s_1]} (\Omega - \lambda_1(-j) I)^{-1}(\Omega^T - \tilde{\lambda}_1(-j) I)^{-1} & s_1 < 0
\end{cases}\]

\[(D_{2,0}^{(2)})^{-1} := \begin{cases}
\Pi_{s_2 = 0}^s (\Omega^T - \lambda_2(j) I)(\Omega - \tilde{\lambda}_2(j) I) & s_2 > 0 \\
I & s_2 = 0 \\
\Pi_{s_2 = 0}^{[s_2]} (\Omega^T - \lambda_2(-j) I)^{-1}(\Omega - \tilde{\lambda}_2(-j) I)^{-1} & s_2 < 0
\end{cases}\]

the evolution of the measure is then

\[d\mu(z, s) = D_{1,0}^{(2)}(z, s_1)(D_{2,0}^{(2)})^{-1}(z, s_2)d\mu(z)\]
where
\[
\mathcal{D}_1^{(2)}(z, s) = \begin{cases} 
\Pi_{j=0}^{s_1} |z - \lambda_1(j)|^2 & s_1 > 0 \\
1 & s_1 = 0 \\
\Pi_{j=0}^{s_1} |z - \lambda_1(-j)|^{-2} & s_1 < 0
\end{cases}
\]
\[
(D_2^{(2)})^{-1}(z, s) = \begin{cases} 
\Pi_{j=0}^{s_2} |z^{-1} - \lambda_2(j)|^2 & s_2 > 0 \\
1 & s_2 = 0 \\
\Pi_{j=0}^{s_2} |z^{-1} - \lambda_2(-j)|^{-2} & s_2 < 0
\end{cases}
\] (104)
in that case
\[
q_1^{(2)} = (\Upsilon - \lambda_1(s_1 + 1)\mathbb{I})(\Upsilon^\top - \lambda_1(s_1 + 1)\mathbb{I}) \\
q_2^{(2)} = (\Upsilon - \lambda_2(s_2 + 1)\mathbb{I})(\Upsilon - \lambda_2(s_2 + 1)\mathbb{I}) \\
\delta_1^{(2)} = (L_1 - \lambda_1(s_1 + 1)\mathbb{I})(L_1^{-1} - \lambda_1(s_1 + 1)\mathbb{I}) \\
\delta_2^{(2)} = (L_2 - \lambda_2(s_2 + 1)\mathbb{I})(L_2^{-1} - \lambda_2(s_2 + 1)\mathbb{I})
\] (105)
Observe that these discrete flows lead to extended Geronimus transformations [46]. When the sequences \{\lambda_1(j)\}, \{\lambda_2(j)\} are constant and thus \(q_1, q_2\) are invariant under the action of \(T_1\) and \(T_2\) we can make an interpretation in terms of Darboux transformations in the context of [4]. In that case what we obtain is
\[
\delta_1 = \delta_{1,-}^{-1} \delta_{1,+} \\
T_1 \delta_1 = T_1(W_1)q_1 T_1(W_1^{-1}) = \omega_1 \delta_1 \omega_1^{-1} = \delta_{1,+} \delta_{1,-} = \delta_{1,+} \delta_{1,-}
\]
\[
\delta_2 = \delta_{2,-}^{-1} \delta_{2,+} \\
T_2 \delta_2 = T_2(W_2)q_2 T_2(W_2^{-1}) = \omega_2 \delta_2 \omega_2^{-1} = \delta_{2,-} \delta_{2,+} = \delta_{2,+} \delta_{2,-}
\]
that is a change in the \(LU\) factorization into \(UL\).

### 4.5 \(\tau\)-functions

As is well known \(\tau\)-functions is an essential ingredient of the theory of integrable systems. Not only for the use of Hirota of these functions in the construction of soliton solutions [38] but also for its relevant geometrical insight [26-28], also the bilinear equations discussed in the mentioned papers are fundamental in the construction of solutions. The determinantal expressions for the OLPUC, and the associated Laurent polynomials and the corresponding second kind functions lead to a \(\tau\)-function representation of these objects. For that aim one consider the action of coherent shifts in the time variables, the so called Miwa shifts

**Definition 20.**

1. The Miwa shifts are the following time shifts
\[
t \mapsto t \pm [w]_1 := t_{1j} \mapsto t_{1j} \pm \frac{w^j}{j}, \quad t_{2j} \mapsto t_{2j},
\]
2. And the Miwa dual shifts
\[
t \mapsto t \pm [w]_2 := t_{1j} \mapsto t_{1j}, \quad t_{2j} \mapsto t_{2j} \pm \frac{w^j}{j}.
\]

A very important property of this Miwa shifts is the form in what they act on the deformed measure

**Proposition 31.** The evolved measure \(\mu(z, t)\) has the following behaviour
\[
\mu(z, t \pm [w]_1) = \left(1 - \frac{z}{w}\right)^{\pm 1} \mu(z, t), \quad |z| < |w|,
\]
\[
\mu(z, t \pm [w]_2) = \left(1 - \frac{w}{z}\right)^{\pm 1} \mu(z, t), \quad |z| > |w|.
\]

**Proof.** Using the series expansion of the logarithm, the evolution factors change under Miwa time shifts like
\[
\exp \left( \sum_{j=1}^{\infty} (t_{1j} z^j - t_{2j} z^{-j}) \right) \mapsto \exp \left( \sum_{j=1}^{\infty} \left( (t_{1j} \pm \frac{1}{j w^j}) z^j - t_{2j} z^{-j} \right) \right) = \left(1 - \frac{z}{w}\right)^{\pm 1} \exp \left( \sum_{j=1}^{\infty} (t_{1j} z^j - t_{2j} z^{-j}) \right), \quad |z| < |w|
\]
\[
\exp \left( \sum_{j=1}^{\infty} (t_{1j} z^j - t_{2j} z^{-j}) \right) \mapsto \exp \left( \sum_{j=1}^{\infty} \left( (t_{1j} \pm \frac{w^j}{j}) z^{-j} \right) \right) = \left(1 - \frac{w}{z}\right)^{\pm 1} \exp \left( \sum_{j=1}^{\infty} (t_{1j} z^j - t_{2j} z^{-j}) \right), \quad |z| > |w|.
\]
\[\square\]
We introduce the main and associated $\tau$-functions as determinants

**Definition 21.** The $\tau$-function is

$$
\tau^{(0)}(t) := 1,
\quad \tau^{(l)}(t) := \det g^{[l]}(t), \quad l = 1, 2, \ldots,
$$

while the associated $\tau$-functions are

$$
\begin{align*}
\tau^{(l)}_{1, -a}(t) &:= (-1)^{t+L-a} \det \begin{pmatrix}
0_0 & 0_{1,1} & \cdots & 0_{l-a-1,1} & 0_{l-a+1,1} & \cdots & 0_{l,1} \\
0_{1,0} & 0_{1,1} & \cdots & 0_{l-a-1,1} & 0_{l-a+1,1} & \cdots & 0_{l,1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0_{l-1,0} & 0_{l-1,1} & \cdots & 0_{l-a-1,1} & 0_{l-a+1,1} & \cdots & 0_{l,1} \\
\end{pmatrix}, \quad l = |\vec{n}|, |\vec{n}| + 1, \ldots,
\\
\tau^{(l)}_{2, -a}(t) &:= (-1)^{t+L-a} \det \begin{pmatrix}
0_{0,1} & 0_{1,1} & \cdots & 0_{l-a,1} & 0_{l-a+1,1} & \cdots & 0_{l,1} \\
0_{1,0} & 0_{1,1} & \cdots & 0_{l-a,1} & 0_{l-a+1,1} & \cdots & 0_{l,1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0_{l-1,0} & 0_{l-1,1} & \cdots & 0_{l-a,1} & 0_{l-a+1,1} & \cdots & 0_{l,1} \\
\end{pmatrix}, \quad l = |\vec{n}|, |\vec{n}| + 1, \ldots,
\\
\tau^{(l)}_{2, a}(t) &:= \det \begin{pmatrix}
0_{0,1} & 0_{1,1} & \cdots & 0_{l-2,1} & 0_{l-2+a,1} \\
0_{1,0} & 0_{1,1} & \cdots & 0_{l-2,1} & 0_{l-2+a,1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0_{l-1,0} & 0_{l-1,1} & \cdots & 0_{l-2,1} & 0_{l-2+a,1} \\
\end{pmatrix}, \quad l = |\vec{n}|, |\vec{n}| + 1, \ldots,
\\
\tau^{(l)}_{1, a}(t) &:= \det \begin{pmatrix}
0_{0,1} & 0_{1,1} & \cdots & 0_{l-1,1} \\
0_{1,0} & 0_{1,1} & \cdots & 0_{l-1,1} \\
\vdots & \vdots & \ddots & \vdots \\
0_{l-2,0} & 0_{l-2,1} & \cdots & 0_{l-1,1} \\
\end{pmatrix}, \quad l = |\vec{n}|, |\vec{n}| + 1, \ldots.
\end{align*}
$$

To find expressions for the OLPUC in terms of these $\tau$-functions we need the following

**Lemma 5.** Let be $r_j(t)$ the $j$-th row of the matrix $g(t)$, then for $j \in \mathbb{Z}_+ \setminus \{0, n_+\}$

$$
\begin{align*}
r_j(t - [z^{-1}]_1) &= \begin{cases} r_j(t) - z^{-1}r_{(j+1)+1}(t) & a(j) = 1, \\
r_j(t) - z^{-1}r_{(j-1)-2}(t) & a(j) = 2, \\
\end{cases}
\\
r_j(t + [z]_2) &= \begin{cases} r_j(t) - zr_{(j-1)-1}(t) & a(j) = 1, \\
r_j(t) - zr_{(j+1)+2}(t) & a(j) = 2, \\
\end{cases}
\end{align*}
$$

is satisfied. For $j = 0$ or $j = n_+$ one has

$$
\begin{align*}
r_0(t - [z^{-1}]_1) &= r_0(t) - z^{-1}r_{1+1}(t) \\
r_0(t + [z]_2) &= r_0(t) - zr_{0+2}(t) \\
r_{n_+}(t - [z^{-1}]_1) &= r_{n_+}(t) - z^{-1}r_0(t) \\
r_{n_+}(t + [z]_2) &= r_{n_+}(t) - zr_{(n_++1)+2}(t)
\end{align*}
$$

*Proof.* Immediate from Proposition 31. \qed
Lemma 6. Given a set of covectors \( \{r_1, \ldots, r_n\} \) it can be shown that

\[
\bigwedge_{j=1}^{n} (z r_j - r_{j+1}) = \sum_{j=1}^{n+1} (-1)^{n+1-j} z^{j-1} \bigwedge_{j=1}^{n} r_j \wedge \cdots \wedge \hat{r}_j \wedge \cdots \wedge r_{n+1},
\]

where the notation \( \hat{r}_j \) means that we have erased the covector \( r_j \) in the wedge product \( r_1 \wedge \cdots \wedge r_{n+1} \).

Proof. It can be done directly by induction. \( \square \)

These two lemmas are the key property to characterize deformed OLPUC using \( \tau \)-functions.

Theorem 9. Given \( l \geq |\vec{n}| \), one has the following \( \tau \)-function representation of the OLPUC

\[
\varphi_1^{(l)}(z, t) = \varphi_1^{(l)}(z, t) = (S_2)_w \varphi_1^{(l)}(z, t) = z^{\nu_+(l)-1} \frac{\tau^{(l)}(t - [z]_1)}{\tau^{(l)}(t)}, \quad a(l) = 1,
\]

\[
\varphi_1^{(l)}(z, t) = \varphi_1^{(l)}(z, t) = (S_2)_w \varphi_1^{(l)}(z, t) = z^{\nu_-(l)} \frac{\tau^{(l)}(t + [z]_2)}{\tau^{(l)}(t)}, \quad a(l) = 2,
\]

for the other “+” associated polynomials we have

\[
\varphi_1^{(l)}(z, t) = z^{-\nu_+(l+1)} \frac{\tau^{(l)}(t - [z]_1)}{\tau^{(l+1)}(t)}, \quad a(l) = 1,
\]

\[
\varphi_1^{(l)}(z, t) = z^{-\nu_-(l+1)} \frac{\tau^{(l)}(t + [z]_2)}{\tau^{(l+1)}(t)}, \quad a(l) = 2.
\]

Proof. Let us prove (107). If \( a(l) = 1 \) we can use Lemma 6 with \( r_1 = r_{(l-1)\ldots 2} \) and \( r_n = r_{(l-1)\ldots 1} \) to expand

\[
z^{\nu_+(l)-1} \tau^{(l)}(t - [z]_1) = z^{\nu_+(l)-1} (M_{1l}^{(l+1)} + (-1)^{l+(l-1)-1} z^{-1} M_{(l-1)\ldots 1l}^{(l+1)} + \cdots + (-1)^{l+(l-1)-2} z^{-l} M_{(l-1)\ldots 2l}^{(l+1)})
\]

\[
= \sum_{j=0}^{l} (-1)^{l+j} M_{jl}^{(l+1)} \chi^{(j)}(z)
\]

\[
= \det(g^{[l]}(t))\varphi_1^{(l)}(z, t)
\]

\[
= \tau^{(l)}(t)\varphi_1^{(l)}(z, t).
\]

If \( a(l) = 2 \) the same procedure works with \( r_1 = r_{(l-1)\ldots 1} \) and \( r_n = r_{(l-1)\ldots 2} \). Now the expansion is

\[
z^{-\nu_-(l)} \tau^{(l)}(t + [z]_2) = z^{-\nu_-(l)} (M_{1l}^{(l+1)} + (-1)^{l+(l-1)-2} z M_{(l-1)\ldots 2l}^{(l+1)} + \cdots + (-1)^{l+(l-1)-1} z^l M_{(l-1)\ldots l}^{(l+1)})
\]

\[
= \sum_{j=0}^{l} (-1)^{l+j} M_{jl}^{(l+1)} \chi^{(j)}(z)
\]

\[
= \det(g^{[l]}(t))\varphi_1^{(l)}(z, t)
\]

\[
= \tau^{(l)}(t)\varphi_1^{(l)}(z, t).
\]
The proof of expressions in (108) can be performed using the same technique, expanding the right hand side. If \( a(l) = 1 \) we have to consider Lemma \( l \) with the rows \( r_1 = r_{(l-1)_2} \) and \( r_n = r_{(l-1)_3} \). In the case \( a(l) = 2 \) the rows \( r_1 = r_{(l-1)_2} \) and \( r_n = r_{(l-1)_3} \) should be considered.

To conclude we prove (109), repeating previous arguments with adequate \( \tau \)-functions, that is, in the case \( a(l) = 1 \)

\[
\varphi^{(l)}_{1,-2}(t - [z^{-1}]_1) = \varphi^{(l)}_{1,-2}(t - [z^{-1}]_1) = z^{\nu_+(l)-1}(t) - (\sum_{j=0}^{l} (-1)^{l-2+j} M_{j(l-2)}^{(l+1)} z^{(l+1)}(z) = \tau^{(l+1)}(t) \varphi^{(l)}_{1,-2}(z,t).
\]

and in the case \( a(l) = 2 \)

\[
\varphi^{(l)}_{1,-2}(t + [z]_2) = z^{\nu_-(l)}(t) - (\sum_{j=0}^{l} (-1)^{l-1+j} M_{j(l-1)}^{(l+1)} z^{(l+1)}(z) = \tau^{(l+1)}(t) \varphi^{(l)}_{1,-2}(z,t).
\]

To obtain the \( \tau \)-function representation of the dual Laurent polynomials \( \varphi^{(l)}_2 \) and their associated ones we would proceed using again Lemma \( l \) and Lemma \( l \) interchanging the role of rows and columns, that would lead to the final expression

**Theorem 10.** For any \( l \geq |\bar{n}| \) the dual Laurent polynomials \( \varphi^{(l)}_2 \) have the following expressions in terms of \( \tau \)-functions

\[
\varphi^{(l)}_2(z,t) = (S_2)_{ll}^{-1} \varphi^{(l)}_{2,-2}(z,t) = \varphi^{(l)}_{2,-2}(z,t) = z^{\nu_+(l)-1} \tau^{(l)}(t + [\bar{z}^{-1}]_2) = \tau^{(l+1)}(t) \varphi^{(l)}_{2,-2}(z,t), \quad a(l) = 1,
\]

\[
\varphi^{(l)}_2(z,t) = (S_2)_{ll}^{-1} \varphi^{(l)}_{2,+2}(z,t) = \varphi^{(l)}_{2,+2}(z,t) = z^{\nu_-(l)} \tau^{(l)}(t - [\bar{z}]_1) = \tau^{(l+1)}(t) \varphi^{(l)}_{2,+2}(z,t), \quad a(l) = 2,
\]

the “+” labeled associated polynomials can be written as

\[
\varphi^{(l)}_{2,+2}(z,t) = z^{\nu_+(l+1)-1} \tau^{(l+1)}(t - [\bar{z}]_1) = \tau^{(l+1)}(t) \varphi^{(l)}_{2,+2}(z,t), \quad a(l) = 1,
\]

\[
\varphi^{(l)}_{2,+1}(z,t) = z^{\nu_+(l+1)-1} \tau^{(l+1)}(t + [\bar{z}]_1) = \tau^{(l+1)}(t) \varphi^{(l)}_{2,+1}(z,t), \quad a(l) = 2,
\]

to conclude, the “−” labeled polynomials have the following representation

\[
\varphi^{(l)}_{2,-2}(z,t) = z^{\nu_+(l+1)-1} \tau^{(l+1)}(t + [\bar{z}]_2) = \tau^{(l+1)}(t) \varphi^{(l)}_{2,-2}(z,t), \quad a(l) = 1,
\]

\[
\varphi^{(l)}_{2,-1}(z,t) = z^{\nu_+(l+1)-1} \tau^{(l+1)}(t - [\bar{z}]_1) = \tau^{(l+1)}(t) \varphi^{(l)}_{2,-1}(z,t), \quad a(l) = 2.
\]
We will end this section with results regarding the \( \tau \)-function representation of the second kind functions in the way we did in \([12]\).

**Lemma 7.** The following identity

\[
\bigwedge_{j=1}^{n} \left( \sum_{i=0}^{\infty} r_{j+i} z^{-i} \right) = r_{1} \land \cdots \land r_{n-1} \land \left( \sum_{i=0}^{\infty} r_{n+i} z^{-i} \right)
\]

holds.

**Proof.** Use induction in \( n \). \( \square \)

**Theorem 11.** Let \( \mu \) be a positive measure supported in \( \mathbb{T} \), then the following statements hold true

1. The second kind functions have the following representation involving \( \tau \)-functions

\[
C_{1,1}^{(l)}(z, t) = \bar{z}^{-\nu_{+}(l+1)} \frac{\tau_{l+1}^{*(l+1)}(t + [\bar{z}]_1)}{\tau^{(l+1)}(t)}, \quad R_{-} < |z|,
\]

\[
C_{1,2}^{(l)}(z, t) = z^{\nu_{-}(l+2)-1} \frac{\tau_{l+2}^{*(l+1)}(t - [\bar{z}]_2)}{\tau^{(l+1)}(t)}, \quad |z| < R_{+},
\]

\[
C_{1}^{(l)}(z, t) = \frac{\bar{z}^{-\nu_{+}(l+1)} \tau_{l+1}^{*(l+1)}(t + [\bar{z}]_1) + z^{\nu_{-}(l+2)-1} \tau_{l+2}^{*(l+1)}(t - [\bar{z}]_2)}{\tau^{(l+1)}(t)}, \quad R_{-} < |z| < R_{+}.
\]

\[
C_{2,1}^{(l)}(z, t) = z^{-\nu_{+}(l+1)} \frac{\tau_{2,l+1}^{*(l+1)}(t - [z]_2)}{\tau^{(l)}(t)}, \quad R_{+}^{-1} < |z|,
\]

\[
C_{2,2}^{(l)}(z, t) = z^{\nu_{-}(l+2)-1} \frac{\tau_{2,l+2}^{*(l+1)}(t + [z]_1)}{\tau^{(l)}(t)}, \quad |z| < R_{+}^{-1}
\]

\[
C_{2}^{(l)}(z, t) = \frac{z^{-\nu_{+}(l+1)} \tau_{2,l+1}^{*(l+1)}(t - [z]_2) + z^{\nu_{-}(l+2)-1} \tau_{2,l+2}^{*(l+1)}(t + [z]_1)}{\tau^{(l)}(t)}, \quad R_{+}^{-1} < |z| < R_{-}^{-1}.
\]

2. For \( R_{-} < |z| < R_{+} \) the Fourier series of the measure can be expressed in terms of \( \tau \)-functions in the following way

\[
F_{\mu}(t)(z) = \frac{\tau_{2,l+1}^{*(l+1)}(t - [z]_2) + z^{-\nu_{+}(l+1)} \tau_{2,l+2}^{*(l+1)}(t + [z]_1)}{2\pi \tau^{(l)}(t - [\bar{z}]_1)} + \frac{\tau_{l+1}^{*(l+1)}(t + [\bar{z}]_1) + z^{\nu_{-}(l+2)-1} \tau_{l+2}^{*(l+1)}(t - [\bar{z}]_2)}{2\pi \tau^{(l)}(t - [z]_2)}, \quad a(l) = 1,
\]

\[
F_{\mu}(t)(z) = \frac{z^{\nu_{-}(l+2)-1} \tau_{2,l+2}^{*(l+1)}(t - [z]_2) + \tau_{2,l+1}^{*(l+1)}(t + [z]_1)}{2\pi \tau^{(l)}(t + [\bar{z}]_2)} + \frac{z^{\nu_{+}(l+1)} \tau_{2,l+1}^{*(l+1)}(t - [z]_2) + \tau_{l+2}^{*(l+1)}(t + [\bar{z}]_1)}{2\pi \tau^{(l)}(t + [z]_1)}, \quad a(l) = 2,
\]

**Proof.** We will prove \((112)\) only, and the proof of \((111)\) that goes analogously is left to the reader. The expression from Proposition \([4]\) can be arranged using the truncated columns of the moment matrix, that is
Using this notation
\[
g^{[l]}(t)C_{2,1}^{[l]}(z, t) = \tau^{[l]}(t)C_{2,1}^{[l]}(z, t) = c_0^{[l]} \land c_1^{[l]} \land \cdots \land c_{l-1}^{[l]} \land \sum_{j=l}^{\infty} c_j^{[l]} (\chi_1^{[l]})^j
\]
\[
= c_0^{[l]} \land c_1^{[l]} \land \cdots \land c_{l-1}^{[l]} \land \sum_{j=l+1}^{\infty} z^{-\nu+(j)} c_j^{[l]} \delta_{a(j),1}.
\]

We define \( Z_{+i} := \{ j \in Z_+, a(j) = i \} \), \( i = 1, 2 \) and notice that \( Z_+ = Z_{+1} \cup Z_{+2} \). The restrictions \( \nu_+|_{Z_{+1}}, \nu_-|_{Z_{+2}} \) of the mappings \( \nu_+, \nu_- : Z_+ \mapsto \mathbb{N} \) are bijections; hence, they have a well defined inverse, \( (\nu_+)^{-1} \) and \( (\nu_-)^{-1} \). Therefore,
\[
\tau^{[l]}(t)C_{2,1}^{[l]}(z, t) = c_0^{[l]} \land c_1^{[l]} \land \cdots \land c_{l-1}^{[l]} \land z^{-\nu+(l+1)} \sum_{j=0}^{\infty} z^{-j} c_j^{[l]} (\nu_+(l+1+j))
\]
\[
= z^{-\nu+(l+1)} c_0^{[l]} \land c_1^{[l]} \land \cdots \land c_{l-1}^{[l]} \land \sum_{j=0}^{\infty} z^{-j} c_j^{[l]} (\nu_+(l+1+j))
\]
\[
= z^{-\nu+(l+1)} \tau_{2,1}^{(l+1)} (t - [z^{-1}]_2),
\]
where the last step requires the use of Lemma 7 with \( c_0^{[l]}, c_1^{[l]}, \ldots, c_{l-1}^{[l]}, c_l^{[l]} \). Proceeding in a very similar way, we have
\[
\text{det}(g^{[l]}(t))C_{2,2}^{[l]}(z, t) = \tau^{[l]}(t)C_{2,2}^{[l]}(z, t) = c_0^{[l]} \land c_1^{[l]} \land \cdots \land c_{l-1}^{[l]} \land \sum_{j=l}^{\infty} c_j^{[l]} \chi_2^{[l]}
\]
\[
= c_0^{[l]} \land c_1^{[l]} \land \cdots \land c_{l-1}^{[l]} \land \sum_{j=l+2}^{\infty} z^{\nu(j)-1} c_j^{[l]} \delta_{a(j),2}
\]
\[
= c_0^{[l]} \land c_1^{[l]} \land \cdots \land c_{l-1}^{[l]} \land z^{\nu(l+2)-1} \sum_{j=0}^{\infty} z^j c_j^{[l]} (\nu_-(l+2+j))
\]
\[
= z^{\nu-(l+2)-1} c_0^{[l]} \land c_1^{[l]} \land \cdots \land c_{l-1}^{[l]} \land \sum_{j=0}^{\infty} z^j c_j^{[l]} (\nu_-(l+2+j))
\]
\[
= z^{\nu-(l+2)} \tau_{2,2}^{(l+1)} (t + [z]_1).
\]

with \( c_0^{[l]}, c_1^{[l]}, \ldots, c_{l-1}^{[l]} \) as adequate entries for Lemma 7. Finally, (113) and (114) are obtained combining equations (111) and (112) with Proposition 5 Theorem 9 and Theorem 10.

A comment on the convergence of the expressions in Theorem 11 and its proof is needed at this point. The main tool used in the proof is the series expansion of \((1 - z)^{-1}\), that is convergent only for \(|z| < 1\) but can be analytically extended outside \(\mathbb{T}\). For instance, the \(\tau\)-function expression for \(C_{2,1}^{[l]}\) is only strictly valid outside \(\mathbb{T}\). Nevertheless, it can be analytically extended inside the circle up to \(R_-\) that is where the series for \(C_{2,1}\) is convergent. As this extension is unique, we can talk about the analytically extended “Miwa-shifted” \(\tau\)-function. The same can be said about the other equalities involving \(\tau\)-functions; they are formally correct and convergent outside or inside \(\mathbb{T}\), but there is an analytical continuation for the shifted \(\tau\)-functions that converges where the Cauchy transforms do. The differences between expressions for the Cauchy transforms (111) and (112) and their equivalent in the real line (e.g. [9,12]) is due to the existence of a positive and a negative part in the Laurent expansion around \(z = 0\). The series expansion of \((1 - z)^{-1}\) generates the positive part and the expansion of \((1 - z^{-1})^{-1}\) gives a negative power series that generates the singular part of the Laurent expansion.
4.6 Bilinear equations

For the derivation of a bilinear identity we proceed similarly as we did in [12] proving several lemmas. For the first one, let $W_1, W_2$ be the wave matrices associated with the moment matrix $g$; so that, $W_1 g = W_2$. Then, we have

Lemma 8. The wave matrices associated with different times satisfy

$$W_1(t)W_1(t')^{-1} = W_2(t)W_2(t')^{-1}, \quad (114)$$

Proof. We consider simultaneously the following equations

$$W_1(t)g = W_2(t),$$
$$W_1(t')g = W_2(t'),$$

and we get

$$W_1(t)^{-1}W_2(t) = W_1(t')^{-1}W_2(t') = g,$$

and the result becomes evident.

Lemma 9. 1. For the vectors $\chi, \chi^*$ the following formulae hold

$$\text{Res}_{z=0} (\chi(\chi^*)^\top) = \text{Res}_{z=0} (\chi^*\chi^\top) = \mathbb{I},$$

2. [12] For any couple of semi-infinite matrices $U$ and $V$ we have

$$UV = \text{Res}_{z=0} \left( (U\chi)(V^\top\chi)^\top \right),$$
$$= \text{Res}_{z=0} \left( (U\chi^*)(V^\top\chi^\top)^\top \right), \quad (115)$$

We have the following

Theorem 12. For any $t, t'$

1. Wave functions satisfy

$$\text{Res}_{z=0} \left( \Psi_1(z, t)(\Psi_1^*(\bar{z}, t'))^\top \right) = \text{Res}_{z=0} \left( \Psi_2(z, t)(\Psi_2^*(\bar{z}, t'))^\top \right)$$

2. OLPUC fulfill

$$\text{Res}_{z=0} \left( \varphi_1^{(k)}(z, t)\varphi_2^{(l)}(z^{-1}, t')z^{-1}F_{\mu}(z)e^{\sum_{j=1}^{\infty}(t_{1j}z^{-j}-t_{2j}z^{-j})} \right)$$
$$= \text{Res}_{z=\infty} \left( \varphi_1^{(k)}(z, t)\varphi_2^{(l)}(z^{-1}, t')z^{-1}F_{\mu}(z)e^{\sum_{j=1}^{\infty}(t_{1j}z^{-j}-t_{2j}z^{-j})} \right), \quad (117)$$

Proof. 1. First we notice that (115) and (116) can be written as

$$UV = \text{Res}_{z=0} \left( (U\chi(z))(V^\top\chi^*(\bar{z}))^\top \right),$$
$$= \text{Res}_{z=0} \left( (U\chi^*(z))(V^\top\chi(\bar{z}))^\top \right)$$

If we set in (115) $U = W_1(t)$ and $V = W_1(t')^{-1}$ and in (116) we put $U = W_2(t)$ and $V = W_2(t')^{-1}$ attending to (114), recalling that $\Psi_1 = W_1\chi$, $\Psi_2 = W_2\chi^*$ and observing that $\Psi_1^* = (W_1^{-1})^\top\chi^*$ and $\Psi_2^* = (W_2^{-1})^\top\chi$ we get the stated bilinear equation for the wave functions.
2. We can substitute the expressions \([97]\) and \([98]\) to prove the second part of the result.

We can reformulate this result using the residue theorem

\[
\oint_{\gamma_0} \Psi_1^{(n)}(z, t)(\bar{\Psi}_1^*)^{(m)}(z, t')dz = \oint_{\gamma_0} \Psi_2^{(n)}(z, t)(\bar{\Psi}_2^*)^{(m)}(z, t')dz,
\]

\[
= \oint_{\gamma_{\infty}} \varphi_1^{(k)}(z, t)\varphi_2^{(l)}(z^{-1}, t')e^{\sum_{j=1}^{\infty}(t_{1j}z^{j}-t_{2j}z^{-j})}z^{-1}F_{\mu}(z)dz,
\]

Alternatively the bilinear equation can be expressed using \(\tau\)-functions

\[
\oint_{\gamma_0} \tau_1^{(l)}(t-[z^{-1}]_1)\tau_2^{(l+1)}(t+[z^{-1}]_1)+z^{\nu_{+}(l+1)+\nu_{-}(l+2)-1}\tau_1^{(l+1)}(t-[z]_2)\right) e^{\sum_{j=1}^{\infty}(t_{1j}-t_{2j})z^{j}} \frac{dz}{z}
\]

if \(a(l)=1\) and

\[
\oint_{\gamma_0} \tau_1^{(l)}(t+[z]_2)(z^{-\nu_{+}(l+1)-\nu_{-}(l+2)+1}\tau_1^{(l)}(t-[z]_1)+\tau_1^{(l+1)}(t-[z]_2)) e^{\sum_{j=1}^{\infty}(t_{1j}-t_{2j})z^{j}} \frac{dz}{z}
\]

if \(a(l)=2\).

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**Appendices**

**A Proofs**

**Proof of Proposition 1** In this case \(g\) is positive definite and Hermitian, if we write \(S_2 = h\hat{S}_2\), where \(h = \text{diag}(h_0, h_1, \ldots)\) is a diagonal matrix and \(\hat{S}_2 = \begin{pmatrix} 1 & (\hat{S}_2)_{01} & (\hat{S}_2)_{02} & \ldots \\ 0 & 1 & (\hat{S}_2)_{12} & \ldots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}\), the uniqueness of the factorization implies that \(\hat{S}_2 = (S_1^{-1})^\dagger\) and \(h_l \in \mathbb{R}\) and we get the stated result.
Proof of Proposition 3  Expressions (16), (17), (19) and (20) are obtained expressing the factorization problem as a system of equations. From (17) we deduce

\[(S_1)_{lk} = (S_2)_{ll}((g^{l+1})^{-1})_{l,k} = \frac{(S_2)_{ll}}{\det g^{l+1}}(-1)^{l+k}M_{k,l}^{(l+1)},\]

so that

\[\varphi_1^{(l)}(z) = \sum_{k=0}^{l}(S_1)_{lk}\chi^{(k)} = \frac{1}{\det g^{l+1}}\sum_{k=0}^{l}(-1)^{l+k}M_{k,l}^{(l+1)}\chi^{(k)},\]

as stated in (15). To prove (21) we consider

\[(S_2^{-1})_{kl} = ((g^{l+1})^{-1})_{k,l} = \frac{1}{\det g^{l+1}}(-1)^{l+k}M_{l,k}^{(l+1)},\]

so that

\[\bar{\varphi}_2^{(l)}(\bar{z}) = \sum_{k=0}^{l}(S_2^{-1})_{kl}\chi^{(k)} \dagger = \frac{1}{\det g^{l+1}}\sum_{k=0}^{l}(-1)^{l+k}M_{k,l}^{(l+1)}(\chi^{(k)}) \dagger,\]

that leads to (21).

Proof of Proposition 4  Using the definition for $C_1$, we have that

\[C_1^{(l)}(z) = \sum_{k \geq l}(S_1^{-1})_{lk}^\dagger \chi^{* (k)}(z) = \sum_{j=0}^{l}(S_2^{-1})_{lj}^\dagger \sum_{k \geq l}g_{jk}^\dagger \chi^{* (k)}(z) = \sum_{j=0}^{l}(S_2^{-1})_{lj}^\dagger \Gamma_2^{(l)}(\bar{z}),\]

so

\[\overline{C_1^{(l)}(z)} = \sum_{j=0}^{l}(S_2^{-1})_{lj} \Gamma_2^{(j)}(\bar{z}).\]

For the other set of functions we have

\[C_2^{(l)}(z) = \sum_{k \geq l}(S_2)_{l,k} \chi^{* (k)}(z) = \sum_{j=0}^{l}(S_1)_{lj} \sum_{k \geq l}g_{jk} \chi^{* (k)}(z) = \sum_{j=0}^{l}(S_1)_{lj} \Gamma_1^{(l)}(z).\]

Comparing the expressions with those in Proposition 3 we see that they are formally identical, so we conclude the stated result.

Proof of Proposition 6  Using the Fourier coefficients of the measure and the definition for $\Gamma_{a,j}^{(l)}$, we obtain

\[\Gamma_{1,j}^{(l)}(z) = \sum_{k \geq 0}g_{j,k} \chi^{* (k)}(z) = \sum_{k \geq 0}g_{j,k} \chi_1^{(k)}(z) + \sum_{k \geq 0}g_{j,k} \chi_2^{(k)}(z) = \sum_{k \geq 0} \int_0^{2\pi} e^{i(J(j)-k)\theta} d\mu(\theta) z^{-k-1} + \sum_{k \geq 0} \int_0^{2\pi} e^{i(J(j)+k+1)\theta} d\mu(\theta) z^k = 2\pi \sum_{k \geq 0} (c_{k-J(j)} z^{-k-1} + c_{-k-J(j)} z^k) = 2\pi z^{-J(j)-1} \left( \sum_{k \geq -J(j)} c_k (z^{-1})^k + \sum_{k \geq J(j)} c_k (z^{-1})^k \right) = 2\pi z^{-J(j)-1} F_\mu(z^{-1}).\]
It can be deduced as follows

\[ \Gamma_{2,j}^{(0)}(z) = \sum_{k \geq 0} g_{j,k}^* \chi_1^*(k)(z) + \sum_{k \geq 0} g_{j,k}^* \chi_2^*(k)(z) \]
\[ = \sum_{k \geq 0} \int_0^{2\pi} e^{i(j-j')k} d\theta \theta^{-k-1} + \sum_{k \geq 0} \int_0^{2\pi} e^{i(j+j')k} d\theta \theta^{-k} \]
\[ = 2\pi \sum_{k \geq 0} \left( c_{(j-j)k}^* z^{-k-1} + c_{(j+j)k} z^k \right) \]
\[ = 2\pi z^{-(j-j)-1} \left( \sum_{k < J(j)+1} c_k z^k + \sum_{k \geq J(j)+1} c_k z^k \right) \]
\[ = 2\pi z^{-(j-j)-1} \tilde{F}_\mu(z). \]

From here we obtain the rest of the expressions

\[ \Gamma_{1,j}^{(l)}(z) = 2\pi z^{-(j-j)-1} \left( \sum_{k \geq -J(j)+l} c_k (z^{-1})^k + \sum_{k < -J(j)-l} c_k (z^{-1})^k \right) = 2\pi z^{-(j-j)-1} \left( \tilde{F}_{j-l-1}^{(+)}(z^{-1}) + \tilde{F}_{j+l+1}^{(-)}(z^{-1}) \right), \]
\[ \Gamma_{2,j}^{(l)}(z) = 2\pi z^{-(j-j)-1} \left( \sum_{k < J(j)+1-l} c_k z^k + \sum_{k \geq J(j)+1+l} c_k z^k \right) = 2\pi z^{-(j-j)-1} \left( \tilde{F}_{j-l-1}^{(-)}(z) + \tilde{F}_{j-l-1}^{(+)}(z) \right). \]

**Proof of Proposition 8** From the definitions we have

\[ (C_{1,1})^{(j)}(z) \varphi_{1,1}(z') = \chi_1^*(z) S_1^{-1} S_1 \chi_1(z') = (\chi_1^*)^T(z) \chi_1(z') = \sum_{n=0}^{\infty} z^{-n-1}(z')^n, \]
\[ (C_{2,1})^{(j)}(z) \varphi_{2,1}(z') = \chi_1^*(z) S_2^{-1} S_2 \chi_1(z') = (\chi_1^*)^T(z) \chi_1(z') = \sum_{n=0}^{\infty} z^{-n-1}(z')^n, \]
\[ (C_{1,2})^{(j)}(z) \varphi_{1,2}(z') = \chi_2^*(z) S_1^{-1} S_1 \chi_2(z') = (\chi_2^*)^T(z) \chi_2(z') = \sum_{n=0}^{\infty} z^n(z')^{-n-1}, \]
\[ (C_{2,2})^{(j)}(z) \varphi_{2,2}(z') = \chi_2^*(z) S_2^{-1} S_2 \chi_2(z') = (\chi_2^*)^T(z) \chi_2(z') = \sum_{n=0}^{\infty} z^n(z')^{-n-1}, \]

which, after the study the region of convergence of the series involved, leads to the sated result. Then other identities derived from \((\chi_1^*)^T(z) \chi_2(z') = (\chi_2^*)^T(z) \chi_1(z') = 0\).

**Proof of Proposition 12** It can be deduced as follows

\[ J_{2k,2k+2} = (S_1 E_{2k,2k+2} S_1^{-1})_{2k,2k+2} = (S_1)_{2k,2k} (S_1^{-1})_{2k+2,2k+2} = 1, \]
\[ J_{2k,2k+1} = (S_1 E_{2k,2k+2} S_1^{-1})_{2k,2k+1} = (S_1)_{2k,2k} (S_1^{-1})_{2k+2,2k+1} = -(S_1)_{2k,2k} = -\alpha_{2k+2}^{(1)}, \]
\[ J_{2k,2k} = (S_2 E_{2k+1,2k-1} S_2^{-1})_{2k,2k} = (S_2)_{2k,2k+1} (S_2^{-1})_{2k-1,2k} = -\alpha_{2k+1}^{(2)} \alpha_{2k+1}^{(1)}, \]
\[ J_{2k,2k-1} = (S_2 E_{2k+1,2k-1} S_2^{-1})_{2k,2k-1} = (S_2)_{2k,2k+1} (S_2^{-1})_{2k-1,2k-1} = -\rho_{2k}^{(2)} \alpha_{2k+1}^{(1)}. \]

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\[ J_{2k+1,2k-1} = (S_2 E_{2k+1,2k-1} S_2^{-1})_{2k+1,2k-1} = (S_2)_{2k+1,2k+1}(S_2^{-1})_{2k,2k}(S_2)_{2k,2k}(S_2^{-1})_{2k-1,2k-1} = \rho_{2k+1}^2 \rho_{2k}^2, \]
\[ J_{2k+1,2k} = (S_2 E_{2k+1,2k-1} S_2^{-1})_{2k+1,2k} = (S_2)_{2k+1,2k+1}(S_2^{-1})_{2k-1,2k} = \rho_{2k+1}^2 \rho_{2k}^2, \]
\[ J_{2k+1,2k+1} = (S_1 E_{2k,2k+2} S_1^{-1})_{2k+1,2k+1} = -(S_1)_{2k+1,2k}(S_1)_{2k+2,2k+1} = -\alpha_2(2) \alpha_2(1), \]
\[ J_{2k+1,2k+2} = (S_1 E_{2k,2k+2} S_1^{-1})_{2k+1,2k+2} = \alpha_2(2). \]

**Proof of Proposition 14** For \( k = 0 \) the result comes from the definition of \( \rho_0^2 \). For \( k = 1,2 \) we have to use the truncated recursion relations,
\[
\begin{align*}
 z^{-1} \phi_1^{(0)} &= \phi_1^{(1)} - \alpha_1(2) \phi_1^{(0)}, \\
 z^{-1} \phi_1^{(1)} &= \phi_1^{(2)} - \alpha_2(2) \phi_1^{(1)} - \alpha_1(1) \phi_1^{(1)} - \rho_1^2 \phi_1^{(2)}, \\
 z \phi_1^{(0)} &= \phi_2^{(2)} - \alpha_1(1) \phi_1^{(1)} - \alpha_1(1) \phi_1^{(0)}, \\
 z \phi_1^{(1)} &= \phi_2^{(2)} - \alpha_2(2) \phi_1^{(2)} - \alpha_2(2) \phi_1^{(1)} + \rho_1^2 \phi_1^{(0)},
\end{align*}
\]
multiplying by \( z \) and integrating we obtain \( h_0 = h_1 - \alpha_1(2) \int \phi_1^{(0)} z d\mu \), and \( \int \phi_1^{(0)} z d\mu = -\alpha_1(1) h_0 \), from where we have \( h_0 = h_1 + \alpha_2(2) \alpha_1(1) h_0 \). Now multiplying by \( z^{-1} \) and integrating we obtain \( 0 = \alpha_1(2) h_2 - \alpha_2(2) \int \phi_1^{(1)} z^{-1} d\mu + \rho_1^2 \int \phi_1^{(0)} z^{-1} d\mu \), \( \int \phi_1^{(0)} z^{-1} d\mu = -\rho_1^2 \alpha_2 h_0 \) and \( \int \phi_1^{(0)} z^{-1} d\mu = -\alpha_2 h_0 \) leading to \( 0 = h_2 + \alpha_2(2) \alpha_2 h_1 - h_1 \).

The other \( k \geq 2 \) can be proved by induction. For odd case we multiply (31) by \( z^{k+1} \) to obtain
\[
0 = \alpha_2(2) h_{2k+1} + \alpha_2(2) \alpha_2(1) \int T \phi_1^{(2k)} z^{k+1} d\mu + \rho_2^2 \alpha_2(1) \int T \phi_1^{(2k-1)} z^{k+1} d\mu + \rho_2^2 \alpha_2(1) \int T \phi_1^{(2k-2)} z^{k+1} d\mu
\]
then multiplying by \( z^k \) the recurrence expressions (31) and (32) for \( z \phi_1^{(2k)} \), \( z \phi_1^{(2k-1)} \), \( z \phi_1^{(2k-2)} \) and integrating we substitute, to get
\[
0 = h_{2k+1} + \alpha_2(2) h_{2k} - \alpha_2(2) \alpha_2 h_{2k} - \rho_2^2 h_{2k},
\]
from where using the induction principle the result is proven. Should we want to obtain the rest of the equations (those with even \( k \)), then it is necessary to multiply by \( z^{-k-1} \) the odd recurrence relations for \( z \phi_1^{(2k+1)} \) (32) and use the same procedure.

**Proof from Proposition 17** From the definition we have
\[
\int T K^{[l]}(u, z', K^{[l]}(z, u) d\mu(u) := \int T \sum_{j=0}^{l-1} \phi_1^{(j)}(z') \bar{\phi}_2^{(j)}(\bar{u}) \sum_{k=0}^{l-1} \phi_1^{(k)}(u) \bar{\phi}_2^{(k)}(\bar{z}) d\mu(u)
\]
\[
= \sum_{j=0}^{l-1} \phi_1^{(j)} (z') \sum_{k=0}^{l-1} \bar{\phi}_2^{(k)} (\bar{z}) \int T \phi_1^{(k)}(u) \bar{\phi}_2^{(k)}(\bar{u}) d\mu(u)
\]
\[
= \sum_{j=0}^{l-1} \phi_1^{(j)} (z') \sum_{k=0}^{l-1} \bar{\phi}_2^{(k)} (\bar{z}) \delta_{j,k} = K^{[l]}(z, z').
\]
Alternate proof for Theorem 3 Due to Gerardo Ariznabarreta.

From the block factorization problem $g^{[l]} = (S_1^{[l]})^{-1} S_2^{[l]}$. As $S_2$ is upper-triangular, we can write $S_1^{[l], S_2^{[l]}} g^{[l]} + S_1^{[l], S_2^{[l]}} g^{[l]} = S_2^{[l]} = 0$ from where

$$S_1^{[l], S_2^{[l]}} g^{[l]} = -S_1^{[l], S_2^{[l]}} g^{[l]} (g^{[l]})^{-1},$$

then using the definition for $\varphi_1^{(l)}$ and using the previous formula, we get

$$\varphi_1^{(l)} = \chi^{(l)} + \sum_{j=0}^{l-1} (S_1^{[l], S_2^{[l]}})_{0,j} \chi^{(j)} = \chi^{(l)} - \sum_{r,j=0}^{l-1} \sum_{k=0}^{1} (S_1^{[l], S_2^{[l]}})_{0,k} (g^{[l]})_{k,r} (g^{[l]})_{r,j}^{-1} \chi^{(j)}$$

$$= \chi^{(l)} - \sum_{r,j=0}^{l-1} (g^{[l]})_{0,r} (g^{[l]})_{r,j}^{-1} \chi^{(j)} = \chi^{(l)} - \left( g_{l,0} \quad g_{l,1} \quad \cdots \quad g_{l,l-1} \right) (g^{[l]})^{-1} \chi^{(l)}.$$

In addition, we can express the formula for $\varphi^{(l+1)}$ in the following way as

$$\varphi^{(l+1)} = \chi^{(l+1)} + (S_1^{[l+1], S_2^{[l+1]}})_{0,l} \chi^{(l)} + \sum_{j=0}^{l-1} (S_1^{[l+1], S_2^{[l+1]}})_{0,j} \chi^{(j)}$$

$$= \chi^{(l+1)} + (S_1)_{l+1,l} \chi^{(l)} + \sum_{j=0}^{l-1} (S_1^{[l+1], S_2^{[l+1]}})_{l,j} \chi^{(j)}$$

$$= \chi^{(l+1)} + (S_1)_{l+1,l} \chi^{(l)} - \sum_{r,j=0}^{l-1} \sum_{k=0}^{1} (S_1^{[l+1], S_2^{[l+1]}})_{l,k} (g^{[l]})_{k,r} (g^{[l]})_{r,j}^{-1} \chi^{(j)}$$

$$= (S_1)_{l+1,l} (\chi^{(l)} - \left( g_{l,0} \quad g_{l,1} \quad \cdots \quad g_{l,l-1} \right) (g^{[l]})^{-1} \chi^{(l)}) +$$

$$+ (\chi^{(l+1)} - \left( g_{l+1,0} \quad g_{l+1,1} \quad \cdots \quad g_{l+1,l-1} \right) (g^{[l]})^{-1} \chi^{(l)}),$$

from where we obtain (if $l$ is odd) $\varphi_1^{(l+1)}(z) = \varphi_1^{(l+1)}(z) - \alpha_1 + \beta_1 \varphi_1^{(l)}(z)$.

Proof of Lemma 1 If we denote by $\Pi^{[l]} = \sum_{i=0}^{l-1} E_{i,i}$ (the projection over the first $l$ components) we find

$$K^{[l]}(z, z') = (\Pi^{[l]} \Phi_2(z)) (\Pi^{[l]} \Phi_1(z')) = \Phi_2(z) \Pi^{[l]} \Phi_1(z') = \chi(z) S_2^{-1} \Pi^{[l]} S_1 \chi(z').$$

From the block factorization $g^{[l]} = (S_1^{[l]})^{-1} S_2^{[l]}$ and its inverse $(g^{[l]})^{-1} = (S_2^{[l]})^{-1} S_1^{[l]}$ we can get an expression for $K^{[l]}$ using only finite size matrices, that is

$$K^{[l]}(z, z') = \chi(z) S_2^{-1} \Pi^{[l]} S_1 \chi(z')$$

$$= \chi(z) (S_2^{[l]})^{-1} S_1^{[l]} \chi(z')$$

$$= \chi(z) (g^{[l]})^{-1} \chi(z').$$

Proof of Lemma 2 The symmetry of $g$ in (30) can be expressed using the block structure

$$\gamma = \begin{pmatrix} \gamma^{[l]} & \gamma^{[l, l]} \\ \gamma^{[l, l]} & \gamma^{[l]} \end{pmatrix}, \quad g = \begin{pmatrix} g^{[l]} & g^{[l, l]} \\ g^{[l, l]} & g^{[l]} \end{pmatrix},$$

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With this block structure we get
\[ \Upsilon^{[\ell]} g^{[\ell]} + \Upsilon^{[\ell,\geq \ell]} g^{[\geq \ell]} = g^{[\ell]} \Upsilon^{[\ell]} + g^{[\ell,\geq \ell]} \Upsilon^{[\geq \ell]}, \]

or equivalently, recalling the Gaussian factorization, we arrive to
\[ (g^{[\ell]})^{-1} \Upsilon^{[\ell]} - \Upsilon^{[\ell]} (g^{[\ell]})^{-1} = (g^{[\ell]})^{-1} (g^{[\ell,\geq \ell]} \Upsilon^{[\geq \ell]} - \Upsilon^{[\ell,\geq \ell]} g^{[\geq \ell]}) (g^{[\ell]})^{-1}. \]

We have also the equations
\[ \Upsilon^{[\ell]} \chi^{[\ell]}(z) + \Upsilon^{[\ell,\geq \ell]} \chi^{[\geq \ell]}(z) = z \chi^{[\ell]}(z), \]
\[ \chi^{[\ell]}(z)^\dagger \Upsilon^{[\ell]} + \chi^{[\ell,\geq \ell]}(z)^\dagger \Upsilon^{[\geq \ell]} = z^{-1} \chi^{[\ell]}(z)^\dagger, \]
that leads to
\[ (z' - z^{-1}) K^{[\ell]}(z, z') = \chi^{[\ell]}(z)^\dagger (g^{[\ell]})^{-1} z' \chi^{[\ell]}(z') - z^{-1} \chi^{[\ell]}(z)^\dagger (g^{[\ell]})^{-1} \chi^{[\ell]}(z'). \]

**Proof of Proposition 19** If \( a(l) = 1 \) then \( z^{-1} \varphi_{1}^{(l)}(z) \) is a monic polynomial of degree \( \nu_{-}(l) + \nu_{+}(l) - 1 \), while when \( a(l) = 2 \) then \( z^{-1} \varphi_{1}^{(l)}(z) \) is a monic polynomial of degree \( \nu_{-}(l) + \nu_{+}(l) - 1 \). The orthogonality relations for \( z^{-1} \varphi_{1}^{(l)}(z) \) and \( z^{-1} \varphi_{1}^{(l)}(z) \) can be obtained from (65)

\[
\int_T z^{-1} \varphi_{1}^{(l)}(z) z^{-k} d\mu(z) = 0, \quad k = 0, \ldots, |\vec{\nu}(l)| - 1, \quad a(l) = 1, \quad (119)
\]

\[
\int_T z^{-1} \varphi_{1}^{(l)}(z) z^{-k} d\mu(z) = 0, \quad k = 0, \ldots, |\vec{\nu}(l)| - 1, \quad a(l) = 2,
\]

that means that
\[ z^{-1} \varphi_{1}^{(l)}(z) = P_{|\vec{\nu}(l)|-1}(z), \quad a(l) = 1, \]
\[ z^{-1} \varphi_{1}^{(l)}(z) = P_{|\vec{\nu}(l)|-1}(z), \quad a(l) = 2,
\]
recalling that \(|\vec{\nu}(l)| - 1 = l\) we get the desired result.

**Proof of Proposition 23**

1. The shift operators defined fulfill
\[ \Lambda_{\bar{\nu},1} \chi_{\bar{\nu}}(z) = z \Pi_{\bar{\nu},1} \chi_{\bar{\nu}}(z), \]
\[ \Lambda_{\bar{\nu},2} \chi_{\bar{\nu}}(z) = z^{-1} \Pi_{\bar{\nu},2} \chi_{\bar{\nu}}(z), \]
\[ \Lambda_{\bar{\nu},1} \chi_{\bar{\nu}}(z) = (z^{-1} \Pi_{\bar{\nu},1} - E_{0,0} \Lambda_{n}^{+}) \chi_{\bar{\nu}}(z), \]
\[ \Lambda_{\bar{\nu},2} \chi_{\bar{\nu}}(z) = (z \Pi_{\bar{\nu},2} - E_{n,n} \Lambda_{n}^{+} (\Lambda_{n}^{+})^{\dagger}) \chi_{\bar{\nu}}(z), \]
that means that
\[ (\Lambda_{\bar{\nu},1} + \Lambda_{\bar{\nu},2}^{\dagger} + E_{n,n} \Lambda_{n}^{+} (\Lambda_{n}^{+})^{\dagger}) \chi_{\bar{\nu}}(z) = z \chi_{\bar{\nu}}(z), \quad (\Lambda_{\bar{\nu},1}^{\dagger} + \Lambda_{\bar{\nu},2} + E_{0,0} \Lambda_{n}^{+}) \chi_{\bar{\nu}}(z) = z^{-1} \chi_{\bar{\nu}}(z), \]

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from there it follows that
\[
\begin{align*}
\mathcal{Y}_{\tilde{n}} g_{\tilde{n}} &= \oint_{\mathbb{T}} z \chi_{\tilde{n}}(z) \chi_{\tilde{n}}(z)^\dagger \, d\mu(z) \\
&= \oint_{\mathbb{T}} \chi_{\tilde{n}}(z)(z^{-1} \chi_{\tilde{n}}(z))^\dagger \, d\mu(z) \\
&= g_{\tilde{n}} (\Lambda_{\tilde{n},1}^\top + \Lambda_{\tilde{n},2} + E_{0,0} \Lambda^n)^\dagger \\
&= g_{\tilde{n}} (\Lambda_{\tilde{n},1}^\top + \Lambda_{\tilde{n},2} + E_{n_+,n_+}(\Lambda^\top)^n) \\
&= g_{\tilde{n}} \mathcal{Y}_{\tilde{n}}.
\end{align*}
\]

2. With the definitions
\[
\begin{align*}
J_{\tilde{n},1} := S_{\tilde{n},1} \mathcal{Y}_{\tilde{n}} S_{\tilde{n},1}^{-1}, & \quad J_{\tilde{n},2} := S_{\tilde{n},2} \mathcal{Y}_{\tilde{n}} S_{\tilde{n},2}^{-1}, \\
\end{align*}
\]
the use of Proposition 23 leads easily to
\[
J_{\tilde{n}} := J_{\tilde{n},1} = J_{\tilde{n},2}.
\]

(120)

The matrix $J_{\tilde{n},1}$ has $n_- + 1$ diagonals over the main diagonal and the matrix $J_{\tilde{n},2}$ has $n_+ + 1$ diagonals under the main diagonal (in both computations we have excluded the main diagonal itself), so both $J_{\tilde{n},1}, J_{\tilde{n},2}$ have $n_+ + n_- + 3$ diagonal bands.

**Proof of Proposition 28**  We have calculated previously $J = L_1$ and using the same method $L_2$ can be calculated, both are five-diagonal matrices given by

\[
L_1 = J = 
\begin{pmatrix}
\begin{array}{cccccccc}
-\alpha_1^{(1)} & -\alpha_2^{(1)} & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\rho_1^2 & -\alpha_2^{(1)} & -\alpha_2^{(2)} & \alpha_1^{(2)} & 0 & 0 & 0 & 0 & \cdots \\
0 & -\rho_2^2 & -\rho_3^2 & \alpha_2^{(2)} & \alpha_2^{(3)} & 0 & 0 & 0 & \cdots \\
0 & 0 & \rho_2^2 & \rho_3^2 & \rho_4^2 & \alpha_2^{(4)} & \alpha_2^{(5)} & \alpha_2^{(6)} & \cdots \\
0 & 0 & 0 & \rho_2^2 & \rho_3^2 & \rho_4^2 & \rho_5^2 & \rho_6^2 & \alpha_2^{(7)} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\end{pmatrix}
\]

\[
L_2 = 
\begin{pmatrix}
\begin{array}{cccccccc}
-\alpha_1^{(2)} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
-\rho_1^2 \alpha_2^{(2)} & -\alpha_2^{(1)} & -\alpha_2^{(2)} & \alpha_3^{(2)} & 0 & 0 & 0 & 0 & \cdots \\
\rho_1^2 \alpha_2^{(2)} & 1 & -\alpha_2^{(1)} & -\alpha_2^{(3)} & \alpha_3^{(3)} & 0 & 0 & 0 & \cdots \\
0 & 0 & \rho_2^2 \alpha_2^{(4)} & -\alpha_3^{(4)} & \alpha_4^{(4)} & \alpha_3^{(5)} & 0 & 0 & \cdots \\
0 & 0 & \rho_2^2 \alpha_2^{(4)} & \rho_3^2 \alpha_3^{(5)} & -\alpha_4^{(5)} & \alpha_5^{(5)} & \alpha_4^{(6)} & 0 & \cdots \\
0 & 0 & 0 & \rho_2^2 \alpha_2^{(4)} & \rho_3^2 \alpha_3^{(5)} & \rho_4^2 \alpha_4^{(6)} & \rho_5^2 \alpha_5^{(7)} & \rho_6^2 \alpha_6^{(8)} & \alpha_5^{(8)} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\end{pmatrix}
\]
as we are looking only for time flows associated to $t_{11}$ and $t_{21}$ then $B_{1,1} = (L_1)_+$ and $B_{2,1} = (L_2)_-$. Using (53) in the matrix elements $(L_1)_{k,k+1}$ for $k \geq 0$ we obtain

\[
\frac{\partial \alpha^{(1)}_{2k+2}}{\partial t_{11}} = -\frac{\partial (L_1)_{2k,2k+1}}{\partial t_{11}} = -[B_{1,1}, L_1]_{2k,2k+1} = \alpha^{(1)}_{2k+3}(1 - \alpha^{(1)}_{2k+2} \overline{\alpha}^{(2)}_{2k+2}) \\
\frac{\partial \alpha^{(2)}_{2k+2}}{\partial t_{21}} = -\frac{\partial (L_1)_{2k,2k+1}}{\partial t_{21}} = -[B_{2,1}, L_1]_{2k,2k+1} = \alpha^{(1)}_{2k+1}(1 - \alpha^{(1)}_{2k+2} \overline{\alpha}^{(2)}_{2k+2}) \\
\frac{\partial \alpha^{(1)}_{2k+1}}{\partial t_{11}} = \frac{\partial (L_1)_{2k+1,2k+2}}{\partial t_{11}} = [B_{1,1}, L_1]_{2k+1,2k+2} = -\overline{\alpha}^{(2)}_{2k}(1 - \alpha^{(1)}_{2k+1} \alpha^{(2)}_{2k+1}) \\
\frac{\partial \alpha^{(2)}_{2k+1}}{\partial t_{21}} = \frac{\partial (L_1)_{2k+1,2k+2}}{\partial t_{21}} = [B_{2,1}, L_1]_{2k+1,2k+2} = -\overline{\alpha}^{(2)}_{2k+2}(1 - \alpha^{(1)}_{2k+1} \overline{\alpha}^{(2)}_{2k+2})
\]

and now looking at $(L_1)_{2k,2k}$ and $(L_2)_{2k+1,2k+1}$ for $k \geq 0$ we obtain the rest of the equations

\[
\frac{\partial (\alpha^{(2)}_{2k} \overline{\alpha}^{(1)}_{2k+1})}{\partial t_{11}} = \frac{\partial (L_1)_{2k,2k}}{\partial t_{11}} = -[B_{1,1}, L_1]_{2k,2k} \Rightarrow \frac{\partial \alpha^{(1)}_{2k+1}}{\partial t_{11}} = \alpha^{(1)}_{2k+2}(1 - \alpha^{(1)}_{2k+1} \overline{\alpha}^{(2)}_{2k+1}) \\
\frac{\partial (\alpha^{(2)}_{2k} \overline{\alpha}^{(1)}_{2k+1})}{\partial t_{21}} = \frac{\partial (L_1)_{2k,2k}}{\partial t_{21}} = -[B_{2,1}, L_1]_{2k,2k} \Rightarrow \frac{\partial \alpha^{(1)}_{2k+1}}{\partial t_{21}} = \alpha^{(1)}_{2k}(1 - \alpha^{(1)}_{2k+1} \overline{\alpha}^{(2)}_{2k+1}) \\
\frac{\partial (\alpha^{(1)}_{2k+1} \overline{\alpha}^{(2)}_{2k+2})}{\partial t_{11}} = \frac{\partial (L_1)_{2k+1,2k+2}}{\partial t_{11}} = -[B_{1,1}, L_2]_{2k+1,2k+2} \Rightarrow \frac{\partial \alpha^{(2)}_{2k+2}}{\partial t_{11}} = -\overline{\alpha}^{(2)}_{2k+1}(1 - \alpha^{(1)}_{2k+2} \overline{\alpha}^{(2)}_{2k+2}) \\
\frac{\partial (\alpha^{(1)}_{2k+1} \overline{\alpha}^{(2)}_{2k+2})}{\partial t_{21}} = \frac{\partial (L_1)_{2k+1,2k+2}}{\partial t_{21}} = -[B_{2,1}, L_2]_{2k+1,2k+2} \Rightarrow \frac{\partial \alpha^{(2)}_{2k+2}}{\partial t_{21}} = -\overline{\alpha}^{(2)}_{2k+3}(1 - \alpha^{(1)}_{2k+2} \overline{\alpha}^{(2)}_{2k+2})
\]

considering all the equations we obtain (99).

**Proof of Proposition** (18) First let us look to (51). Using Definition (16) it can be expressed as

\[
\varphi^{(l)}_{1,a}(z) = \chi^{(l+a)}(z) - \sum_{i,j=0}^{l} g_{l,a,i}(g^{[l]}_{i,j})^{-1} \chi^{(j)}(z)
\]

\[
= \frac{1}{\det g^{[l]}} \left( \chi^{(l+a)}(z) \det g^{[l]} - \sum_{i,j=0}^{l} g_{l,a,i}(-1)^{i+j} M^{(l)}_{ij} \chi^{(j)}(z) \right)
\]

\[
= \frac{1}{\det g^{[l]}} \left( \chi^{(l+a)}(z) \det g^{[l]} + \sum_{l=0}^{l-1} \sum_{i,j=0}^{l} (-1)^{i+j} M^{(l)}_{ij} \chi^{(j)}(z) \right),
\]

that is the expansion of (50). Using the same idea with (53)

\[
\varphi^{(l)}_{2,-a}(z) = \sum_{j=0}^{l} \left( g^{[l+1]}_{l,-a} \right)^{-1} \chi^{(j)}(z) = \frac{1}{\det g^{[l+1]}} \sum_{j=0}^{l} (-1)^{l+1+j} M^{(l+1)}_{l,-a} \chi^{(j)}(z),
\]

we arrive at the expansion of (53) taking the complex conjugate. Both (51) and (52) can be proved using the same ideas.

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