Erasure/List Exponents for Slepian–Wolf Decoding

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Related Work on Error Exponents for S–W Decoding

- Gallager (1976, unpublished): same technique as in channel decoding.
- Csiszár, Körner & Marton (1977,80): universal achievability.
- Csiszár and Körner (1981): linear codes; expurgated exponents.
- Csiszár (1982) + Oohama & Han (1994): coded side information.
- Kelly & Wagner (2011): improvements at high rates.
In This Work

- Generalized decoding for the S-W problem: erasure and list decoding.
- Draper & Martinian (2007): List decoding – fixed & deterministic list.
- We analyze tradeoffs between exponents similarly as in Forney (1968).
- Erasure option: no decoding when the confidence level is low.
- List option: tentative candidates – final decision after further processing.
In This Work (Cont’d)

We analyze error exponents using two methods:

- The Gallager/Forney method.
- Distance enumeration – inspired by the random energy model (REM).

Second method:

- Always at least as tight.
- May be better by an arbitrarily large factor.
- Sometimes simpler and easier to calculate.

Variable–rate coding: improves the exponents.
Some Definitions

Let $(X, Y) \sim \prod_{i=1}^{n} P(x_i, y_i)$.

$x$ – source to be encoded.

$y$ – side info @ decoder.

**Encoder**: $f : \mathcal{X}^n \rightarrow \{0, 1, \ldots, M - 1\}$, $M = e^{nR}$.

$$z = f(x).$$

Random binning: For every $x \in \mathcal{X}^n$, $z$ is selected independently at random from $\{0, 1, \ldots, M - 1\}$. 
Some Definitions (Cont’d)

Erasure/list decoder: Given \( y \in \mathcal{Y}^n \) and \( z \), calculate for all \( \hat{x} \in f^{-1}(z) \):

\[
\frac{P(\hat{x}, y)}{\sum_{x' \in f^{-1}(z) \setminus \{\hat{x}\}} P(x', y)}.
\]

If \( \geq e^{nT} \), \( \hat{x} \) is a candidate.

- If there are no candidates – an erasure is declared.
- If there is exactly one candidate – ordinary decoding: \( \hat{x} = \text{candidate} \).
- If there is more than one candidate – a list is of all candidates is created.

Define \( \mathcal{E}_1 \) as the event where the real \( x \) is not a candidate.
Let \( E_1(R, T) = \text{exponent of Pr}\{\mathcal{E}_1\} \). The other exponent

\[
E_2(R, T) = \left\{ \begin{array}{cc}
\text{decoding error exp} & \text{erasure mode} \\
\text{expected list size exp} & \text{list mode}
\end{array} \right. = E_1(R, T) + T.
\]
Common Starting Point

\[
\Pr\{\mathcal{E}_1\} = \sum_{x,y} P(x, y) \mathcal{I}\left\{ \frac{e^{nT} \sum_{x' \neq x} P(x', y) \mathcal{I}[f(x') = f(x)]}{P(x, y)} > 1 \right\}
\]

\[
\leq \sum_{x,y} P(x, y) \left[ \frac{e^{nT} \sum_{x' \neq x} P(x', y) \mathcal{I}[f(x') = f(x)]}{P(x, y)} \right]^s
\]

\[
= e^{nsT} \sum_{x,y} P^{1-s}(x, y) \left[ \sum_{x' \neq x} P(x', y) \mathcal{I}[f(x') = f(x)] \right]^s.
\]
The Gallager/Forney Approach

Use

\[
\Pr\{\mathcal{E}_1\} \leq e^{nsT} \sum_{x,y} P^{1-s}(x,y) \left( \sum_{x' \neq x} P(x', y) \mathbb{I}[f(x') = f(x)] \right)^{s/\rho} \rho \\
\leq e^{nsT} \sum_{x,y} P^{1-s}(x,y) \left( \sum_{x' \neq x} P^{s/\rho}(x', y) \mathbb{I}[f(x') = f(x)] \right)^{\rho} \quad \rho \geq s
\]

and then, for the ensemble average, use Jensen’s inequality with the limitation \( \rho \leq 1 \).

Two potential points of losing exponential tightness:

- The inequality \((\sum_i a_i)^r \leq \sum_i a_i^r, 0 \leq r \leq 1\).
- Jensen’s inequality.
The Resulting Error Exponent

$$\Pr\{\mathcal{E}_1\} \leq e^{-nE_1(R,T)},$$

where

$$E_1(R, T) = \sup_{0 \leq s \leq \rho \leq 1} [E_0(\rho, s) + \rho R - sT],$$

with

$$E_0(\rho, s) = -\ln \left[ \sum_{y \in \mathcal{Y}} P(y) \sum_{x \in \mathcal{X}} P^{1-s}(x|y) \left( \sum_{x' \in \mathcal{X}} P^{s/\rho}(x'|y) \right)^{\rho} \right].$$
Extension to Variable–Rate Coding

Instead of fixed rate $R$, let the rate be $R(x)$, a function that depends on $x$ only via the type class (header + random binning in each type), e.g.,

$$R(x) = \frac{1}{n} \sum_{i=1}^{n} r(x_i).$$

Then, the above extends to:

$$\tilde{E}_1(R, T) = \sup_{0 \leq s \leq \rho \leq 1} \sup \{r: \mathbb{E}\{r(X)\} \leq R, r(x) > 0 \ \forall \ x \in \mathcal{X}\} \left[\tilde{E}_0(\rho, s) - sT\right],$$

where

$$\tilde{E}_0(\rho, s) = -\ln \left[\sum_{y \in \mathcal{Y}} P(y) \sum_{x \in \mathcal{X}} P^{1-s}(x|y) \left(\sum_{x' \in \mathcal{X}} P^{s/\rho}(x'|y)e^{-r(x')}\right)^{\rho}\right].$$

Closed–form optimization of $\{r(x)\}$ s.t. $\mathbb{E}\{r(X)\} \leq R$ is easy at least for $\rho = 1$ and the improvement in the exponent can be assessed.
Type Class Enumeration Method

Back to fixed–rate, instead of the above, use:

\[
\mathbb{E}\left\{ \sum_{x' \neq x} P(x'|y)I[f(x') = f(x)] \right\}^s = \mathbb{E}\left[ \sum_{T(x'|y)} P(x'|y)N(x'|x, y) \right]^s
\]

\[
\mathbb{E}\left\{ \sum_{T(x'|y)} P(x'|y)\mathbb{E}\{N^s(x'|x, y)\} \right\}
\]

where \( N(x'|x, y) \) is the type class enumerator:

\[
N(x'|x, y) = \left| T(x'|y) \cap f^{-1}[f(x)] \right|.
\]

\( \mathbb{E}\{N^s(x'|x, y)\} \) can be assessed using simple large–deviations considerations.
The Binary Case

Let $X$ and $Y$ be BSS’s connected via a BSC with crossover probability $p$. Then,

$$\Pr\{E_1\} \leq e^{-nE'_1(R,T)},$$

where

$$E'_1(R, T) = \sup_{s \geq 0} E'_1(R, T, s)$$

and

$$E'_1(R, T, s) = \begin{cases} 
    s(R - T) + \gamma(s) & (s, R) \in C \cup F \cup G \\
    s[R - T + D(h^{-1}(R)\|p)] + \gamma(s) & (s, R) \in B \\
    R - sT + \gamma(s) + \gamma(1 - s) & (s, R) \in A \cup D \cup E
\end{cases}$$

$$\gamma(s) = -\ln[p^{1-s} + (1 - p)^{1-s}]$$

and where the sets $A$–$G$ are defined in the following figure.

The analysis can be extended to general DMS’s.
Phase Diagram for $E'_1(R, T, s)$

$$R = h(p_s)$$

$$R = h(p)$$

$$p_s = \frac{p^s}{p^s + (1-p)^s}$$

$$R(s) = \frac{\gamma(s)}{s-1}$$

$$s = 1$$
Comparison Between $E_1(R, T)$ and $E'_1(R, T)$

$E'_1(R, T) \geq E_1(R, T)$ always.

For some regions in the plane $R$—$T$, $E'_1(R, T)$ may be larger than $E_1(R, T)$ by an arbitrarily large factor!

1. For $R > h(p)$ and $T < \ln \frac{p}{1-p}$:

\[
E_1(R, T) \leq R + |T| < \infty; \quad E'_1(R, T) = \infty.
\]

2. Consider the case of very weakly correlated sources, i.e., $p = \frac{1}{2} - \epsilon, |\epsilon| \ll 1$.

For $R \in [h(p), \ln 2]$ and $T = -\tau \epsilon^2$ with $\tau > 4$:

\[
E_1(R, T) \leq (\tau + 2)\epsilon^2, \quad E'_1(R, T) \geq \left[\frac{\tau(\tau + 8)}{16} - 1\right] \epsilon^2.
\]

Both examples work thanks to the fact that $s$ take arbitrarily large values, not just in $[0, 1]$. 

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Summary

- Trade-offs between random coding exponents for erasure/list decoding.
- The type–class enum. method is never worse and sometimes a lot better.
- Optimization range of $s$ is unlimited.
- Only one parameter to optimize, rather than two.
- Variable–rate encoding can be handled also.
- Extendable to the case where both $X$ and $Y$ are encoded (separately).