PARETO GENEALOgies ARISING FROM A POISSON BRANCHING EVOLUTION MODEL WITH SELECTION

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ABSTRACT. We study a class of coalescents derived from a sampling procedure out of \( N \) i.i.d. Pareto(\( \alpha \)) random variables, normalized by their sum, including \( \beta \)-size-biasing on total length effects (\( \beta < \alpha \)). Depending on the range of \( \alpha \), we derive the large \( N \) limit coalescents structure, leading either to a discrete-time Poisson-Dirichlet(\( \alpha, -\beta \)) \( \Xi \)-coalescent \( (\alpha \in [0,1]) \), or to a family of continuous-time Beta(\( 2 - \alpha, \alpha - \beta \)) \( \Lambda \)-coalescents \( (\alpha \in [1,2]) \), or to the Kingman coalescent \( (\alpha \geq 2) \). We indicate that this class of coalescent processes (and their scaling limits) may be viewed as the genealogical processes of some forward in time evolving branching population models including selection effects. In such constant-size population models, the reproduction step, which is based on a fitness-dependent Poisson Point Process with scaling power-law(\( \alpha \)) intensity, is coupled to a selection step consisting of sorting out the \( N \) fittest individuals issued from the reproduction step.

Running title: Pareto genealogies in a Poisson evolution model with selection.

Keywords: Pareto coalescents, scaling limits, Poisson-Dirichlet, Kingman and Beta coalescents, Poisson Point Process, evolution model including selection.

1. Introduction

We first investigate discrete-time finite coalescents derived from sampling from \( N \) i.i.d. Pareto(\( \alpha \)) random variables, normalized by their sum, with \( \alpha > 0 \). We include size-biasing on total length effects involving a parameter \( \beta < \alpha \). Depending on the range of \( \alpha \), we derive the large \( N \) limit coalescents structure: The case \( \alpha \in [0,1) \) leads to a discrete-time Poisson-Dirichlet(\( \alpha, -\beta \)) \( \Xi \)-coalescent (with no time-scaling). The case \( \alpha = 1 \) gives rise to a continuous-time beta(\( 1, 1 - \beta \)) \( \Lambda \)-coalescent, involving a logarithmic time scaling \( \log N \). The case \( \alpha \in (1, 2) \) leads to a continuous-time Beta(\( 2 - \alpha, \alpha - \beta \)) \( \Lambda \)-coalescent, involving a power-law time scaling according to \( N^{\alpha-1} \). The range \( \alpha \geq 2 \) gives rise to the standard Kingman coalescent (with time scaling \( N \) if \( \alpha > 2 \) and \( N/\log N \) in the critical case \( \alpha = 2 \)). We give for each case the exact speeds of convergence. We establish a loose link with a Generalized Central Limit Theorem for stable random variables and we briefly recall the main statistical features akin to general \( \Lambda \)-coalescents.

We indicate that the above special classes of coalescent processes (and their scaling limits) may be viewed as the genealogical processes of some forward in time evolving
branching population models including selection effects. These models are closely related in spirit to the additive exponential model discussed in Brunet et al [6], [7]. In the models we first consider, the size $N$ of the population is kept constant over the generations. Each alive individual is assigned some positive fitness $x > 0$. In each generation and for each of the $N$ offspring alive independently, the reproduction step is based on a fitness-dependent Poisson Point Process (PPP) with scaling power-law($\alpha$) intensity; this procedure assigns new fitnesses to the (potentially infinitely many) individuals of the next generation, in a multiplicative way. We call $f(x) = x^\alpha$ the output fitness of an individual with fitness $x$. The selection step then consists of sorting out the $N$ fittest individuals issued from the reproduction step\(^1\). The process is iterated independently over the subsequent generations. To the first large $N$ approximation, the logarithm of the mean output fitness within each generation $k$, scaled by the generation number $k$, is shown to shift to the right at speed $v_N = \log \log N$ as $k \to \infty$.

While adopting a sampling point of view based on the intensity of the PPP to compute the coalescence probabilities that some offspring is the one of a parental individual with given fitness, it is shown that the genealogy of the branching model with selection is in the domain of attraction of the beta(1, $1 - \beta$) coalescent in the large $N$ limit (Bolthausen-Sznitman coalescent if $\beta = 0$). It is also shown that the full class of the Pareto-coalescents discussed earlier can be obtained while considering the PPP which is the output image of the original one, given by the output map $f(x) = x^\alpha$ in fitness space. In this setup, the large $N$ limit computations of the coalescence and merging probabilities are based not on the fitnesses but on the output deformed fitnesses.

2. Coalescents derived from Pareto-Sampling

2.1. Pareto sampling and coalescents. Let $X_1, ..., X_N$ be $N$ i.i.d. Pareto($\alpha$) random variables (rvs) with $P(X_1 > x) = x^{-\alpha}$, $\alpha > 0$ and $x \geq 1$. Let $F_{X_1}(x) = 1 - P(X_1 > x)$ denote its probability distribution function (pdf). The density of $X_1$ is $f_{X_1}(x) = \alpha x^{-(\alpha+1)}$. Let $S_n := X_n/ \sum_1^N X_n$, $n = 1, ..., N$ define a random partition of the unit interval, upon normalizing the $X$s by their sum. The $S_n$s are identically distributed but not independent of course as they sum up to 1; by doing so, the unit interval $[0, 1]$ is thus broken into $N$ random pieces (subintervals or segments) of sizes $S_n$, $n = 1, ..., N$.

By sampling the $S$s, we mean that we draw independently $i$ uniform random variables on the unit interval with $i \leq N$, looking at the subintervals which are being hit in the process. From this procedure, for instance, the probability that the $i$–sample hits any one of the $S_n$s only once is

$$P_{i,1}^{(N)} = E \left( \sum_{n=1}^N S_n^i \right) = N E \left( S_1^i \right).$$

\(^1\)This particular way of introducing selection in a randomly evolving branching population with constant population size seems to appear first in [5]. It has nothing to do with the way selection is classically introduced in population genetics; see [28], [14] and [8].
Let $\Sigma_N := \sum_{n=1}^{N} X_n$ denote the partial sum of the $X$s. For the values of $\beta$ for which $E \left( \Sigma_{N}^{\beta} \right)$ exists, we can size-bias the latter probability by the total length $\Sigma_N$ and consider instead

$$P_{i,1}^{(N)} = \frac{E \left( \Sigma_{N}^{\beta} \sum_{n=1}^{N} S_{n}^{i} \right)}{E \left( \Sigma_{N}^{\beta} \right)} = \frac{NE \left( \Sigma_{N}^{\beta} \sum_{n=1}^{N} S_{n}^{i} \right)}{E \left( \Sigma_{N}^{\beta} \right)}.$$  

The latter event under consideration corresponds to an $i$ to 1 merger of some Markov coalescent process where $i$ particles are identified to a single one (share the same ancestor) whenever the $i$-sample hits the same subinterval of the unit partition. In this setup, $P_{i,1}^{(N)}$ is therefore the entry $(i,1)$ of its one-step transition matrix.

The quantity $c_N := P_{2,1}^{(N)}$, which is the probability that two individuals chosen at random out of $N$ share the same common ancestor, is called the coalescence probability.

Similarly we can define a $i$ to $j$ merger ($j \leq i$) by considering the event that the $i$ particles hit any size $-j$ subset of the segments $S$ constituting the partition of unity.

We get

$$P_{i,j}^{(N)} = \binom{N}{j} \sum_{i_1 + \ldots + i_j = i} \binom{i}{i_1 \ldots i_j} E \left( \prod_{l=1}^{j} S_{l}^{i_l} \right) = \binom{N}{j} \sum_{l=1}^{j} (-1)^{j-l} \binom{j}{l} E \left( (S_1 + \ldots + S_l)^i \right),$$

where the star-sum in (2) runs over the $i_l \geq 1$. The quantity $E \left( \prod_{l=1}^{j} S_{l}^{i_l} \right)$ is the probability of a $(i_1, \ldots, i_j)$ -- merger from $i$ to $j$. Using the same abuse of notation, we shall also write the size-biased version of the latter probability as

$$P_{i,j}^{(N)} = \binom{N}{j} \sum_{i_1 + \ldots + i_j = i} \binom{i}{i_1 \ldots i_j} \frac{E \left( \Sigma_{N}^{\beta} \prod_{l=1}^{j} S_{l}^{i_l} \right)}{E \left( \Sigma_{N}^{\beta} \right)}.$$  

Unless stated otherwise, whenever we speak in the sequel of $P_{i,j}^{(N)}$, we mean (3) and not (2).

Clearly, $\sum_{j=1}^{i} P_{i,j}^{(N)} = 1$ and so the matrix $P^{(N)}$ with entries $P_{i,j}^{(N)}$, $i = 1, \ldots, N$ and $j = 1, \ldots, i$, is a $N \times N$ lower-triangular stochastic matrix corresponding to some Markov discrete-time-$k$ coalescent (pure death) process, say $x_{k}^{(N)}$, with finite state-space and state $\{1\}$ absorbing. Let us first investigate the expression of the size-biased probability $P_{i,1}^{(N)}$, showing that its large $N$ estimate depends on the understanding of the $\beta$--moments of $\Sigma_N$.

2We abusively use the same notation $P_{i,1}^{(N)}$ in the size-biased setup as in (1) (corresponding to $\beta = 0$), to avoid overburden notations.

3A ‘true’ coalescent process takes values in the set of equivalence relations or partitions on $\{1, \ldots, N\}$ and we rather deal here and throughout with its block-counting counterpart.
Proposition 1. When $-\infty < \beta < \alpha < 2$, the size-biased probability of an $i$ to 1 merger reads

$$P_{i,1}^{(N)} = N\alpha \frac{E \left( \frac{\beta - \alpha}{\Sigma_{N-1}} \right) \Gamma (i - \alpha) \Gamma (\alpha - \beta)}{E \left( \frac{\beta - \alpha}{\Sigma_N} \right) \Gamma (i - \beta)}, \ i \geq 2. \tag{4}$$

Proof: We have $1/S_1 = 1 + \Sigma_{N-1}'/X_1$ where $\Sigma_{N-1}' := \sum_{n=2}^N X_n = \sum_{n=1}^{N-1} X_n =: \Sigma_{N-1}$. By conditioning on $X_1$, with $f_{\Sigma_{N-1}}$ the density of $\Sigma_{N-1}$, we get

$$P_{i,1}^{(N)} = \frac{N}{E \left( \frac{\beta - \alpha}{\Sigma_N} \right)} \int_1^\infty dx \cdot f_{X_1}(x) x^{\beta+1} \int_{1 + x/(x-1)}^\infty z^{\beta-1} f_{\Sigma_{N-1}} (x (z - 1)) \, dz.$$  

Reversing the integration and after two changes of variables

$$P_{i,1}^{(N)} = \frac{N}{E \left( \frac{\beta - \alpha}{\Sigma_N} \right)} \int_1^\infty dz \cdot z^{\beta-1} \int_{1/(z-1)}^{\infty} x^{\beta+1} f_{X_1}(x) f_{\Sigma_{N-1}} (x (z - 1)) \, dx$$

$$= \frac{N\alpha}{E \left( \frac{\beta - \alpha}{\Sigma_N} \right)} \int_1^\infty dz \cdot z^{\beta-1} (z-1)^{\alpha-\beta-1} \int_{N-1}^\infty s^{\alpha-\beta-1} f_{\Sigma_{N-1}} (s) \, ds$$

$$= N\alpha \frac{E \left( \frac{\beta - \alpha}{\Sigma_{N-1}} \right)}{E \left( \frac{\beta - \alpha}{\Sigma_N} \right)} \int_0^1 u^{-2} u^{1-\alpha} (1-u)^{\alpha-\beta-1} \, du.$$

When $\beta < \alpha < 2$, the latter integral is (upon adequate normalization by a beta function term $B (2-\alpha, \alpha-\beta)$) identified with the order $i-2$ moment of a beta$(2-\alpha, \alpha-\beta)$ rv. In this parameter range, for $i \geq 2$

$$\int_0^1 u^{-2} u^{1-\alpha} (1-u)^{\alpha-\beta-1} \, du = \frac{\Gamma (i - \alpha) \Gamma (\alpha - \beta)}{\Gamma (i - \beta)} =: B (i - \alpha, \alpha - \beta)$$

are well-defined. ◦

The Pareto rv $X_1$ has power-law tails with index $\alpha$. Therefore $E \left( X_1^\beta \right)$ only exists when $\beta < \alpha$, with $E \left( X_1^\beta \right) = \alpha / (\alpha - \beta)$. Clearly also, the tails of the sum $\Sigma_N$ obey $\Pr (\Sigma_N > s) \sim \frac{N}{s - \alpha} \Pr (X_1 > s) = \frac{N}{s - \alpha}$. Because $\Pr (X_1 > s) \sim \Pr (M_N > s) = 1 - (1 - s^{-\alpha})^N$ where $M_N = \max (X_1, \ldots, X_N)$, this means that for large $s$, the event $\Sigma_N > s$ is essentially determined by the event $M_N > s$. Note that, as a result, $\Sigma_N$ has the same tail index as $X_1$, indicating that the $\beta-$moment of $\Sigma_N$ only exists for $\beta < \alpha$.

We will show below from this, that large $N$ estimates of $E \left( \frac{\beta - \alpha}{\Sigma_N} \right)$ can be obtained.

As a result, for example, based on (4), whenever $1 < \alpha < 2$ and $\beta < \alpha$, it will be checked that

$$c_N := P_{2,1}^{(N)} = N\alpha \frac{E \left( \frac{\beta - \alpha}{\Sigma_{N-1}} \right) B (2-\alpha, \alpha-\beta) \propto N^{-(\alpha-1)} \rightarrow 0. \tag{4}$$
We will show that, in that case, for each \( i \geq 2 \), the limits \( \lim_{N \to \infty} c_N^{-1} P_{i,1}^{(N)} \) exist, with
\[
c_N^{-1} P_{i,1}^{(N)} \to_{N \to \infty} \frac{1}{B(2-\alpha,\alpha-\beta)} \frac{\Gamma(\alpha-\beta) \Gamma(i-\alpha)}{\Gamma(i-\beta)} = \int_0^1 u^{i-2} \Lambda(du) > 0,
\]
where \( \Lambda(du) \) has the density on \([0,1] \): \( u^{1-\alpha} (1-u)^{\alpha-\beta-1} / B(2-\alpha,\alpha-\beta) \), which is a beta\((2-\alpha,\alpha-\beta)\) density.

Because in the large \( N \) limit, simultaneous multiple collisions will be seen to be negligible, we conclude, using similar arguments to the ones in [36] and [30], that, in the range \( 1 < \alpha < 2 \) and \( \beta < \alpha \), a time-scaled version of the finite discrete-time-\( k \) coalescent \( x_k^{(N)} \) arising from size-biased sampling out of Pareto\((\alpha)\) partitions converges weakly (as \( N \to \infty \)) to a continuous-time-\( t \) \( \Lambda \)--coalescent \( x_t \) where \( \Lambda \) is a beta\((2-\alpha,\alpha-\beta)\) probability measure on \([0,1] \); namely \( x_t \overset{d}{=} \lim_{t \to \infty} x_{t/c_N}^{(N)} \).

In other words, if \( t \) is continuous-time, the appropriate scaling is \( k \to \lceil t/c_N \rceil \) (the integral part of \( t/c_N \)), showing that time \( t \) should be measured in units of \( N_c \overset{\text{def}}{=} c_N^{-1} \) (the effective population size). The limiting process \( x_t \) is a continuous-time pure death Markov process on \( \mathbb{N} = \{1,2,\ldots\} \), absorbed at state \( \{1\} \) and with transition rates from \( i \) to \( j \) given by
\[
c_N^{-1} P_{i,j}^{(N)} \to_{N \to \infty} \lambda_{i,j} := \binom{i}{j-1} \int_0^1 u^{i-j-1} (1-u)^{j-1} \Lambda(du), \, 1 \leq j < i.
\]
The rate terms \( \lambda_{i,j} \) may also be written as
\[
\lambda_{i,j} := \sum_{l=0}^{j-1} \frac{(-1)^l}{l!} \binom{i}{j-l-1} q_l u^{i-j+1},
\]
where \( q_l := \int_0^1 u^{l-2} \Lambda(du) \) are the \( l \)--moments of \( u^{-2} \Lambda(du) \) and also the rates of \( l \) to \( 1 \) mergers \( \lambda_{i,1} \).

2.2. Generalities on \( \Lambda \)--coalescents. Whenever one gets a family of rates \( \lambda_{i,j} \) as above for some finite probability measure \( \Lambda \) on \([0,1] \), one speaks of continuous-time-\( \Lambda \)--coalescents (see [34] for a precise definition). These \( \Lambda \)--coalescents are non-increasing pure death Markov processes, say \( x_t \), on the state-space \( \mathbb{N} \), and the \( \lambda_{i,j} \)s are the rates at which \( i \) to \( j < i \) mergers for \( x_t \) occur. The state \( \{1\} \) is absorbing. In such processes, multiple collisions of any order (when \( 1 \leq j < i \)) can occur, but never simultaneously. The total death rate at which some merger occurs, starting from state \( i \), is \( \lambda_i := \sum_{j=1}^{i-1} \lambda_{i,j} \). One can check that when \( \Lambda = \delta_0 \) (corresponding to the Kingman coalescent), \( \lambda_{i,j} \neq 0 \) only when \( j = i - 1 \) with \( \lambda_i = \lambda_{i,i-1} = (i \choose 2) \). The general expression of \( \lambda_i \) is
\[
\lambda_i = \int_{[0,1]} u^{-2} \left( 1 - (1-u)^i - iu (1-u)^{i-1} \right) \Lambda(du).
\]

\[\text{Here, because two processes are involved, the symbol } \overset{d}{=} \text{ means convergence of all the finite-dimensional distributions of } x_{t/c_N}^{(N)}, \, t \geq 0 \text{ to the ones of } x_t, \, t \geq 0.\]
When $\Lambda\{\{0\}\} = 0$ (excluding thereby the Kingman coalescent), the precise dynamics of $x_t$ when started at $i$ is given by $x_0 = i$ and

$$
(5) \quad x_t - x_0 = -\int_{(0,t] \times (0,1]} \left( B(x_{s-}, u) - 1 \right) B(x_{s-}, u) \geq 0 \right) \mathcal{N}(ds \times du)
$$

$$
= -\int_{(0,t] \times (0,1]} (B(x_{s-}, u) - 1)_{+} \mathcal{N}(ds \times du).
$$

Here, $x_+ = \max(x, 0)$, $\mathcal{N}$ is a random Poisson measure on $[0, \infty) \times (0, 1]$ with intensity $ds \times \frac{1}{u} \Lambda(du)$ and $B(x_{s-}, u) \sim \text{bin}(x_{s-}, u)$ is a binomial rv with parameters $(x_{s-}, u)$. As a result, with

$$
(6) \quad r(i) := \int_{(0,1]} (ui - 1 + (1 - u)i) u^{-2} \Lambda(du), \quad i > 0,
$$

upon taking the expectation in (5), it holds that

$$
\mathbf{E}(dx_t \mid x_{t-}) = -r(x_{t-}) dt.
$$

From this, the quantity $r(i)$ is the rate at which size $i$ blocks are being lost as time passes by. Clearly $r$ is also

$$
r(i) = i\lambda_i - \sum_{j=1}^{i-1} j\lambda_{i,j} = \sum_{j=1}^{i-1} (i - j) \lambda_{i,j}.
$$

Consequently, the reciprocal function $1/r(i)$ of the rate $r(i)$ interprets as the expected time spent by $x_t$ in a state with $i$ lineages and therefore $\sum_{j=2}^{x_0+1} 1/r(j)$ will give a large $i$ estimate of the expected time to the most recent common ancestor (the height of the coalescent tree):

$$
\tau_{i,1} := \inf\{t \in \mathbb{R}_+ : x_t = 1 \mid x_0 = i\}.
$$

There are lots of detailed studies in the literature (see a precise partial list below) on other functionals of $x_t$ such as the total branch length $L_i$ of the $\Lambda$-coalescent, its total external branch length $L_i^e$, the length $l_i$ of its external branch (the time till first collision of a branch chosen at random out of $i$), the number of collisions $C_i$ till time to most recent common ancestor,...

All these functionals obey some distributional identities which prove useful to obtain some insight on their limit laws as $i \to \infty$. Given $x_0 = i$, all involve the number $U_i$ of singletons taking part in the first collision occurring at time $T_i \sim \exp(\lambda_i)$, giving $x_{T_i} - 1 = i - U_i$ singletons not participating to the first collision (with $T_i$, $x_{T_i}$ independent). The quantity $U_i$ is important in itself because, due to $\mathbf{P}(U_i = j) = \lambda_{i,i-j+1}/\lambda_i$, $j = 2, .., i$

$$
r(i) = \lambda_i (\mathbf{E}(U_i) - 1)
$$

where

$$
\mathbf{E}(U_i) = \frac{i}{\lambda_i} \int_{(0,1]} \frac{1 - u - (1 - u)^i}{u(1 - u)} \Lambda(du).
$$

Provided $\Lambda$ has no atom at point $\{1\}$, the condition $\sum_{i=2}^{\infty} 1/r(i) < \infty$ is the necessary and sufficient condition for $x_t$ to come down from infinity, [38].
Clearly indeed, τ_{i,1} \overset{d}{=} T_i + \tau_{x_{T_i}}, L_i \overset{d}{=} iT_i + L_{x_{T_i}}, L_i^c \overset{d}{=} iT_i + L_{x_{T_i}}^c, C_i \overset{d}{=} 1 + C_{x_{T_i}}
and l_i \overset{d}{=} T_i + B_i l_{x_{T_i}}^c - 1 where \( B_i \) is a Bernoulli rv, given by: \( P(B_i = 1 \mid x_{T_i}) = (x_{T_i} - 1)/i \) with \( B_i l_{x_{T_i}}^c - 1 \) independent of \( T_i \).

Famous examples that we shall deal with in the sequel, include \( \Lambda \)-coalescents for which:
- \( \Lambda (du) = B(a,b) u^{a-1} (1 - u)^{b-1} 1_{[0,1]} (u) \) \( du \) with \( a, b > 0 \) and with \( B(a,b) \) the beta function: we get the beta(\( a,b \)) coalescent.
- \( \Lambda (du) = B(2 - \alpha, \alpha) u^{1-\alpha} (1 - u)^{\alpha-1} 1_{[0,1]} (u) \) \( du \), \( \alpha \in (0,1) \cup (1,2) \); this is the beta(\( 2 - \alpha, \alpha \)) coalescent.
- (Lebesgue) \( \Lambda (du) = 1_{[0,1]} (u) \) \( du \): this is the Bolthausen-Sznitman coalescent or beta(1,1) coalescent.
- \( \Lambda (du) = \delta_0 \) : we get the Kingman coalescent where only binary mergers can occur \( (j = i - 1) \) one at a time, \( [26] \).
- \( \Lambda (du) = \delta_1 \) : we get the star-shaped coalescent involving a single big collision. \( \diamond \)

Using the above distributional identities, it can be shown for instance that, for the Kingman coalescent and for large \( i \), to leading order of magnitude, rough estimates are: \( \tau_{i,1} \sim 2 (1 - 1/i) \) \( [10] \), \( L_i \sim 2 \log i \) \( [12] \), \( L_i^c \sim 2 \) \( [22] \), \( C_i = i - 1 \) and \( l_i \sim 1/i \) \( [9] \).

For the Bolthausen-Sznitman coalescent: \( \tau_{i,1} \sim \log \log i \) \( [17] \), \( L_i \sim i/\log i \) \( [12] \), \( C_i \sim i/\log i \) \( [21] \) and \( l_i \sim 1/\log i \) \( [15] \).

For the beta(\( 2 - \alpha, \alpha \)) coalescent with \( 0 < \alpha < 1 \), \( L_i \sim i \) \( [29] \) and \( L_i/L_i^c \) converges in probability to \( 1 \) \( [31] \) and \( l_i \sim O(1) \) \( [16] \).

For the beta(\( 2 - \alpha, \alpha \)) coalescent with \( 1 < \alpha < 2 \), \( L_i \sim 1/i^{\alpha - 2} \) \( [24] \), \( L_i^c \sim 1/i^{\alpha - 2} \) \( [11] \) and \( l_i \sim 1/i^{\alpha - 1} \) \( [10] \).

This information is important to grasp the general shape of the coalescent trees in each case.

### 3. GCLT for Pareto sums

In this Section, we first sketch a loose connection of the large \( N \) estimate of \( c_N \) with the Generalized Central Limit Theorem for random variables in the domain of attraction of stable laws, \( [31] \).

Let \( \Sigma N = \sum_{n=1}^{N} X_n \) be the partial sums of the i.i.d. \( X \)'s with Pareto(\( \alpha \)) distributions, \( \alpha > 0 \), on \( (1, \infty) \). Let \( S_{\alpha} \) be skewed \( \alpha \)-stable rvs with

\[
E(e^{-\lambda S_{\alpha}}) = e^{-\lambda^\alpha} \text{ if } \alpha \in (0,1), \lambda \geq 0
\]

\( \overset{d}{=} \) means equality in distribution between random variables.
the Laplace-Stieltjes transform (LST) of a one-sided $\alpha$-stable rv,

$$
E(e^{i\lambda S_1}) = e^{-|\lambda|(1-\text{sign}(\lambda)\frac{2}{\pi}\log|\lambda|)}
$$

the characteristic function (c.f.) of a skewed 1-stable Cauchy rv on $\mathbb{R}$

$$
E(e^{i\lambda S_\alpha}) = e^{-|\lambda|^n(1-\text{sign}(\lambda)\tan(\frac{\pi}{2})))}
$$

if $\alpha \in (1, 2), \lambda \in \mathbb{R}$

the characteristic function (c.f.) of a skewed $\alpha$-stable Cauchy rv on $\mathbb{R}$

$$
E(e^{i\lambda S_\alpha}) = e^{-\lambda^2/2} \text{ if } \alpha \geq 2, \lambda \in \mathbb{R}
$$

the c.f. of a standard normal rv on $\mathbb{R}$.

The following Generalized Central Limit Theorem (GCLT) then holds

**Theorem 2.** ([41], [42]): Let $\Sigma_N$ denote the partial sum sequence of $N$ i.i.d. Pareto($\alpha$) random variables. Then,

$$
\frac{\Sigma_N - a_N}{b_N} \xrightarrow{d_{N \to \infty}} S_\alpha
$$

where, with

$$
C_\alpha = \left( \Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right) \right)^{1/\alpha} \text{ if } \alpha \in (0, 2) \setminus \{1\},
$$

$$
C_1 = \frac{\pi}{2},
$$

$$
C_\alpha = \left( \frac{\alpha}{\alpha - 2} - \left( \frac{\alpha}{\alpha - 1} \right)^2 \right)^{1/2} \text{ if } \alpha > 2,
$$

$b_N$ is given by

$$
(i) \quad b_N = C_\alpha N^{\max(1/\alpha, 1/2)} \text{ if } \alpha \neq 2
$$

$$
(ii) \quad b_N = (N \log N)^{1/2} \text{ if } \alpha = 2
$$

and, with $\gamma$ the Euler constant and $E(X_1) = \mu := \alpha/(\alpha - 1)$, $a_N$ is given by

$$
(i) \quad a_N = 0 \text{ if } \alpha \in (0, 1)
$$

$$
(ii) \quad a_N = \frac{N\pi^2}{2} \int_1^{\infty} \sin\left(\frac{2x}{\pi N}\right) dF_{X_1}(x) \sim N \log N + N \left(1 - \gamma - \log \frac{2}{\pi}\right) \text{ if } \alpha = 1
$$

$$
(iii) \quad a_N = N\mu \text{ if } \alpha \in (1, \infty).
$$

When $\alpha \leq 1$, the characteristic values of $\Sigma_N$ can be guessed to be what they are claimed to be $(N^{1/\alpha}$ if $\alpha < 1$ and $N \log N$ if $\alpha = 1)$ while estimating $N \int_{1}^{m_N} xF_{X_1}(x) \, dx$ where $m_N$ is the mode of $M_N$ which is seen to grow like $N^{1/\alpha}$ (see [41]) and similarly for the fluctuation scaling term $b_N$ for $\alpha \leq 2$ (of order $N^{1/\alpha}$ if $\alpha < 2$ and $(N \log N)^{1/2}$ if $\alpha = 2$).

From these rough estimates, we would conclude that with $c_N \propto N \frac{E(S_\alpha)}{E(S_\alpha)}$, up to the leading order in $N$

$$
c_N \propto N \left( C_\alpha N^{1/\alpha} \right)^{\beta - \alpha} = O(1) \text{ if } \alpha \in (0, 1)
$$
\[ c_N \propto N \left( \frac{N \log N)^{\beta-1}}{(N \log N)^\beta} \right) \sim \frac{1}{\log N} \text{ if } \alpha = 1 \]
\[ c_N \propto N \left( \frac{(\mu N)^{\beta-\alpha}}{(\mu N)^\beta} \right) \sim \mu^{-\alpha} N^{-(\alpha-1)} \text{ if } \alpha \in (1, 2). \]

Depending on the values of \( \alpha \), we therefore anticipate

- \( \alpha \in (0, 1) \): Because in that case \( c_N \) is asymptotic to a constant, this suggests a limiting discrete-time coalescent. We will show below that it is not a discrete-time \( \Lambda \)-coalescent, rather it is a \( \Xi \)-coalescent of the Poisson-Dirichlet type with two parameters \((\alpha, -\beta)\).

\( \Xi \)-coalescents were first introduced in [30] and further studied in [37]. In sharp contrast with \( \Lambda \)-coalescents, multiple collisions can occur simultaneously at the same transition time. In their block-counting version, they are characterized by the set of numbers \( \phi_j(i_1, \ldots, i_j) \) defining the probabilities of an \((i_1, \ldots, i_j)\)-merger \((i_1 + \ldots + i_j = i)\), resulting when the \( \Xi \)-coalescent is discrete, in an \( i \) to \( j \leq i \) transition with probability \( P_{i,j} = \frac{1}{j!} \sum_{i_1 + \ldots + i_j = i} \phi_j(i_1, \ldots, i_j) \). The \( \phi_j(i_1, \ldots, i_j) \) can be written as

\[ \phi_j(i_1, \ldots, i_j) = \int_{\Delta_j} \prod_{l=1}^j u_l^{i_l-2} \Lambda_j(du_1, \ldots, du_j), \]

for some finite measures \( \Lambda_j \) with density on the \((j+1)\)-simplex

\[ \Delta_j = \left\{ (u_1, \ldots, u_j) \in [0,1]^j : u_1 + \ldots + u_j \leq 1 \right\}. \]

The set of measures \( \Lambda_j, j \geq 1 \), (characterized by their moments \( \phi_j \)), with values over the simplices \( \Delta_j \), completely characterize the \( \Xi \)-coalescent, [30].

In the simplest cases, with \( \langle u, u \rangle := \sum_{l \geq 1} u_l^2 \), it is also convenient (see [37]) to rewrite the \( \phi_j \)'s as

\[ \phi_j(i_1, \ldots, i_j) = \int_\Delta \sum_{k_1, \ldots, k_j \text{ all distinct}} \prod_{l=1}^j u_l^{i_l} \Xi(du)/\langle u, u \rangle, \]

where \( \Delta = \left\{ u := (u_1, \ldots, u_j) : u_1 \geq \ldots \geq u_l \geq \ldots \geq 0 : \sum_{l \geq 1} u_l \leq 1 \right\} \) and \( \Xi \) a finite measure concentrated on the subset \( \Delta^* \) of \( \Delta \) consisting of those \( u \) exactly summing to 1. Letting \( \nu(du) := \Xi(du)/\langle u, u \rangle \), the measure \( \nu \) on the infinite simplex \( \Delta \) is such that \( \nu(\Delta) < \infty \).

- \( \alpha = 1 \): A \( \Lambda \)-coalescent with logarithmic effective population size where \( \Lambda \) is a beta\((1, 1-\beta)\) probability measure with \( \beta < 1 \) as in [27] (reducing to the Bolthausen-Sznitman coalescent when avoiding size-biasing corresponding to \( \beta = 0 \)).

- \( \alpha \in (1, 2) \): A \( \Lambda \)-coalescent with \( N_c \propto N^{\alpha-1} \) where \( \Lambda \) is a beta\((2-\alpha, \alpha-\beta)\) measure with \( \beta < \alpha \). In the latter case, \( \beta = 0 \) leads to the standard beta\((2-\alpha, \alpha)\) coalescent.
In this Section, we compute the asymptotic behavior of the \( \beta \) in the cases

\begin{equation}
\text{Theorem 3.}
\end{equation}

conclusions based on the GCLT consistent\(^6\) discrete-time Poisson-Dirichlet

\begin{equation}
\rho(8)
\end{equation}

• We start with the case \( \alpha = 1, 2 \). For the values of \( \alpha \)

\begin{equation}
\text{Proof:}
\end{equation}

\begin{equation}
\text{Remark: The Kingman coalescent also occurs when dealing with sampling from some alternative random partition. For instance, would sampling be defined from a random partition of unity given by } S_n := X_n / \Sigma_1^n X_n, n = 1, ..., N \text{ where } X_1 \text{ now obeys the following gamma(\( \theta \)) density: } f_{X_1}(x) = \Gamma(\theta)^{-1} x^{\theta-1} e^{-\theta}, \theta, x > 0, \text{ then the law of } \Sigma_N = X_1 + ... + X_N \text{ is } f_{\Sigma_N}(x) = \Gamma(N\theta)^{-1} x^{N\theta-1} e^{-x}, \text{ independent of } S_1 \text{ and}
\end{equation}

\begin{equation}
P_{i,j}^{(N)} = \frac{\mathbb{E} \left( \Sigma_N^i S_n^j \right)}{\mathbb{E} (\Sigma_N^i)} = \frac{N \mathbb{E} \left( \Sigma_N^i S_n^j \right)}{\mathbb{E} (\Sigma_N^i)} = N \mathbb{E} (S_n^j)
\end{equation}

\begin{equation}
= N \frac{\Gamma(N\theta)}{\Gamma(\theta)} \frac{\Gamma(i+\theta)}{\Gamma(N\theta+\theta)} \sim N^{-(i-1)} \frac{\Gamma(i+\theta)}{\Gamma(\theta)} \theta^{-i}.
\end{equation}

Thus \( c_N = P_{2,1}^{(N)} = \frac{1 + \theta}{N^\theta} \to 0 \) together with \( d_N = P_{3,1}^{(N)} = \frac{1}{N^\theta} \frac{(1+d\theta)(2+d\theta)}{\theta} \). Because triple mergers are asymptotically negligible compared to binary ones (\( d_N/c_N \to 0 \)), the time-scaled limiting coalescent using \( c_N = \frac{1}{N^\theta} \propto N^{-1} \) is a Kingman coalescent. The prefactor \( \frac{1}{N^\theta} \) appearing in front of \( c_N \) is the ratio \( \rho/\mu^2 \) where \( \rho := \mathbb{E} (X_1^2) = \theta (\theta + 1) \) and \( \mu := \mathbb{E} (X_1) = \theta \) and \( c_N \) is independent of \( \beta \).

4. LARGE \( N \) ASYMPTOTIC ESTIMATION OF \( \mathbb{E} (\Sigma_N^\alpha) \) AND CONSEQUENCES

In this Section, we compute the asymptotic behavior of the \( \beta \)-moments of \( \Sigma_N \) in the cases \( \alpha \in (0, 1), \alpha \in (1, 2) \) and \( \alpha = 1, \alpha \geq 2 \), making the previous conclusions based on the GCLT consistent\(^6\).

• We start with the case \( \alpha \in (0, 1) \).

\begin{equation}
\text{Theorem 3. When } \alpha \in (0, 1), \text{ as } N \to \infty, \text{ } x_k \overset{d}{=} \lim_{N \to \infty} x_k^{(N)} \text{ exists and is a discrete-time Poisson-Dirichlet(\( \alpha, -\beta \)) } \Xi-\text{coalescent.}
\end{equation}

\begin{equation}
\text{Proof: For the values of } \beta \text{ for which it makes sense, we have}
\end{equation}

\begin{equation}
\mathbb{E} \left( \Sigma_N^\alpha \right) = \frac{1}{\Gamma(-\beta)} \int_0^\infty d\lambda \lambda^{-\beta-1} \mathbb{E} \left( e^{-\lambda \Sigma_N} \right) = \frac{1}{\Gamma(-\beta)} \int_0^\infty d\lambda \lambda^{-\beta-1} \mathbb{E} \left( e^{-\lambda X_1} \right)^N.
\end{equation}

When \( N \) is large, only the small \( \lambda \) approximation of \( \mathbb{E} \left( e^{-\lambda X_1} \right) \) to the latter integral contributes. For small \( \lambda \), we have

\begin{equation}
\mathbb{E} \left( e^{-\lambda X_1} \right) = \alpha \int_1^\infty x^{-(\alpha+1)} e^{-\lambda x} dx = 1 - \alpha \int_1^\infty x^{-(\alpha+1)} \left( 1 - e^{-\lambda x} \right) dx \sim 1 - \alpha \int_0^\infty x^{-(\alpha+1)} (1 - e^{-\lambda x}) dx \sim 1 - \Gamma(1 - \alpha) \lambda^\alpha \sim e^{\Gamma(1 - \alpha) \lambda^\alpha}.
\end{equation}

\(^6\)The technique we use is inspired from the one used in \cite{7} in a particular case. The author is indebted to B. Derrida for pointing this out to him.
Note that, when $\lambda$ is small

$$
E(X_i^e^{-\lambda X_i}) = (-1)^i \frac{d^i}{d\lambda^i} E(e^{-\lambda X_i}) \sim \alpha \Gamma(i - \alpha) \lambda^{\alpha - i}.
$$

Thus,

$$
E(\Sigma_N^i) \sim \frac{1}{\Gamma(-\beta)} \int_0^\infty d\lambda \cdot \lambda^{\beta - 1} e^{-N(1-\alpha)\lambda}.
$$

With the change of variables $u = N \Gamma(1-\alpha) \lambda^\alpha$, with $\beta < \alpha$, we get

$$
E(\Sigma_N^i) \sim \frac{N^{\beta/\alpha} \Gamma(1-\alpha)^{\beta/\alpha}}{\alpha \Gamma(-\beta)} \int_0^\infty du \cdot u^{\beta/\alpha - 1} e^{-u} = N^{\beta/\alpha} \frac{\Gamma(1-\alpha)^{\beta/\alpha} \Gamma(1-\beta/\beta)}{\Gamma(1-\beta)}.
$$

Finally, using the above large $N$ estimate of $E(\Sigma_N^i)$, the identity $\Gamma(x+1) = x\Gamma(x)$ and (4) with $i = 2$, when $\beta < \alpha < 1$, we obtain

$$
c_N = N\alpha \frac{E(\Sigma_{N-1}^\beta)}{E(\Sigma_N^\beta)} \Gamma(2-\beta) \frac{\Gamma(1-\beta) \Gamma(1-\beta/\beta) \Gamma(2-\alpha)}{\Gamma(2-\beta) \Gamma(1-\beta) \Gamma(1-\beta/\beta)} \Gamma(2-\alpha)
$$

Thus the coalescence probability $c_N$ converges to $c_N \in (0, 1)$. When $\alpha \in (0, 1)$, using again (4), we obtain more generally

$$
P_{i,j}^{(N)} \rightarrow_{N \rightarrow \infty} P_{i,j} = \frac{\Gamma(1-\beta) \Gamma(i-\alpha)}{\Gamma(1-\beta) \Gamma(i-\beta)}
$$

which are the probabilities to merge all $i$ particles in one step in the limiting discrete-time coalescent.

To derive the full transition probabilities of the limiting discrete-time-$k$ coalescent $x_k \overset{d}{=} \lim_{N \rightarrow \infty} x_k^{(N)}$, recalling that

$$
P_{i,j}^{(N)} = \binom{\alpha}{j} \sum_{i_1 + \ldots + i_j = i} \binom{i}{i_1 \ldots i_j} \frac{E(\Sigma_N^i \prod_{l=1}^j S_l^i)}{E(\Sigma_N^i)}
$$

we use

$$
E\left(\sum_{l=1}^j X_i^l e^{-\lambda X_i^l}\right) = \frac{1}{\Gamma(i-\beta)} \int_0^\infty d\lambda \cdot \lambda^{i-\beta-1} \prod_{l=1}^j E(X_i^l e^{-\lambda X_i^l}) E(e^{-\lambda X_i})^{N-j}
$$

$$
\sim \frac{\alpha^j}{\Gamma(i-\beta)} \prod_{l=1}^j \Gamma(i_l - \alpha) \int_0^\infty d\lambda \cdot \lambda^{i-\beta-1} \lambda^{\alpha_j - i} e^{-N\Gamma(1-\alpha)\lambda^\alpha}.
$$

Performing again the change of variables $u = N \Gamma(1-\alpha) \lambda^\alpha$, with $\beta < \alpha$, we get

$$
E\left(\sum_{l=1}^j S_l^i\right) \sim \alpha^{j-1} N^{\beta/\alpha - j} \Gamma(1-\beta) \Gamma(1-\alpha)^{\beta/\alpha - j} \Gamma(j-\beta/\beta) \prod_{l=1}^j \Gamma(i_l - \alpha).
$$

Using $\binom{\alpha}{j} \sim N^j/j!$ for large $N$, we finally get $P_{i,j}^{(N)} \rightarrow P_{i,j}$ where $(1 \leq j \leq i)$

$$
P_{i,j} = \frac{\alpha^j}{j!} (\Gamma(1-\beta) \Gamma(j-\beta/\beta) \prod_{l=1}^j \Gamma(i_l - \alpha)) \sum_{i_1 + \ldots + i_j = i} \frac{\Gamma(i_l - \alpha)}{\Gamma(1-\beta) i_l!}.
$$
These are the full transition probabilities of the limiting discrete-time-\(k\) coalescent \(x_k\) in the regime \(\alpha \in (0, 1)\) and \(\beta < \alpha\) (satisfying \(\sum_{j=1}^i P_{i,j} = 1\)). Note that \(P_{1,1}\) are the probabilities obtained previously and that the diagonal terms (the eigenvalues of \(P\)) read
\[
P_{i,i} = \alpha^{i-1} \frac{\Gamma (1 - \beta)}{\Gamma (1 - \beta / \alpha)} \frac{\Gamma (i - \beta / \alpha)}{\Gamma (i - \beta)}.
\]
Clearly this discrete-time coalescent is not a discrete \(\Lambda\)-coalescent as simultaneous multiple collisions can occur (it is a \(\Xi\)-coalescent). Clearly \(P_{i,j}\) is also
\[
P_{i,j} = \frac{1}{j!} \sum_{i_1 + \ldots + i_j = i}^* \binom{i}{i_1, \ldots, i_j} \phi_j (i_1, \ldots, i_j),
\]
where the \(\phi_j (i_1, \ldots, i_j)\)s define the probabilities of a \((i_1, \ldots, i_j)\)-merger \((i_1 + \ldots + i_j = i)\). These \(\phi_j (i_1, \ldots, i_j)\)s, which can be read from (10), may be written under the alternative form
\[
\phi_j (i_1, \ldots, i_j) = \alpha^{j-1} \frac{\Gamma (1 - \beta)}{\Gamma (1 - \beta / \alpha)} \frac{\Gamma (j - \beta / \alpha)}{\Gamma (j - \beta)} \prod_{l=1}^j \frac{\Gamma (i_l - \alpha)}{\Gamma (1 - \alpha)}
\]
where
\[
c_{j,\alpha,\beta} := \prod_{l=1}^j \frac{\Gamma ((l-1) \alpha + 1 - \beta)}{\Gamma (1 - \alpha) \Gamma (l \alpha - \beta)}.
\]
Thus (10) is also
\[
(11) \quad P_{i,j} = c_{j,\alpha,\beta} \frac{i!}{j!} \frac{\Gamma (\alpha j - \beta)}{\Gamma (i - \beta)} \sum_{i_1 + \ldots + i_j = i}^* \prod_{l=1}^j \frac{\Gamma (i_l - \alpha)}{i_l !}.
\]
Defining the finite Dirichlet measures \(\Lambda_j\) with density on the \((j + 1)\)-simplex \(\Delta_j\) given by:
\[
\Lambda_j (du_1, \ldots, du_j) = c_{j,\alpha,\beta} \prod_{l=1}^j \left( u_1^{1-\alpha} du_l \right) \left( 1 - \sum_{l=1}^j u_l \right)^{\alpha j - \beta - 1},
\]
we get
\[
\phi_j (i_1, \ldots, i_j) = c_{\infty} \int_{\Delta_j} \prod_{l=1}^j u_l^{i_l - 2} \Lambda_j (du_1, \ldots, du_j).
\]
The set of finite Dirichlet measures \(\Lambda_j\) with parameters
\[
(\theta_1 = 2 - \alpha, \ldots, \theta_j = 2 - \alpha, \theta_{j+1} = \alpha j - \beta)
\]
on the simplices \(\Delta_j\) completely characterize this limiting discrete-time coalescent. Note that \(\Lambda_1\) is a \(\beta(2 - \alpha, \alpha - \beta)\) probability measure.

One may rewrite the \(\phi_j\)s as (see [37] and [31])
\[
\phi_j (i_1, \ldots, i_j) = \int_{\Delta} \sum_{k_1, \ldots, k_j}^* \prod_{l=1}^j u_k^{i_l} \Xi (du) \prod_{u \neq u'} \frac{u u'}{i_l},
\]
where $\Delta = \left\{ u := (u_1, u_2, \ldots) : u_1 \geq \ldots \geq u_l \geq \ldots \geq 0 : \sum_{l \geq 1} u_l \leq 1 \right\}$ and $\Xi$ a measure on $\Delta$. Letting $\nu (du) := \Xi (du) / \langle u, u \rangle$, the measure $\nu$ on the infinite simplex $\Delta$ can be identified (see [31]) to the two-parameter Poisson-Dirichlet($\alpha, -\beta$) measure, with $\alpha \in [0, 1)$ and $\beta < \alpha$. It holds that $\nu (\Delta) = 1$. Poisson-Dirichlet measures enjoy many remarkable properties including a stick-breaking property, [35].

**Remarks:**

(i) From [11], the limiting situation $\alpha = 0$ also makes sense, leading to the one-parameter Poisson-Dirichlet($0, -\beta$) measure with $\beta < \alpha = 0$. In this case, from [11]

$$P_{i,j} = \frac{(-\beta)^j}{\Gamma (i - \beta)} s_{i,j},$$

where $s_{i,j} := \frac{n_i^j}{\beta!} \sum_{i_1 + \ldots + i_j = i} \prod_{l=1}^j \frac{1}{u_l}$ are the absolute first kind Stirling numbers.

(ii) Finally, avoiding size-biasing ($\beta = 0$) gives rise to the discrete Poisson-Dirichlet($\alpha, 0$) coalescent with one-parameter $\alpha \in (0, 1)$, appearing in [39].

- The case $\alpha \in (1, 2)$.

**Lemma 4.** When $\alpha \in (1, 2)$, with $\mu := \frac{\alpha}{\alpha - 1}$, for all $\beta < \alpha$

$$c_N := P_{2,1}^{(N)} \sim \alpha \mu^{-\alpha} B (2 - \alpha, \alpha - \beta) N^{-(\alpha - 1)} \rightarrow 0.$$

**Proof:** In this case, defining $a := \alpha - 1 \in (0, 1)$, after an integration by parts and using the previous estimate [8] substituting $a$ to $\alpha$

$$E \left( e^{-\lambda X_1} \right) = e^{-\lambda} - \frac{\lambda}{a} \int_1^{\infty} x^{-(a+1)} e^{-\lambda x} dx \sim e^{-\lambda} - \frac{\lambda}{a} (1 - \Gamma (1 - a) \lambda^a).$$

Thus, for small $\lambda$, with $\mu := \frac{\alpha}{\alpha - 1}$

$$E \left( e^{-\lambda X_1} \right) \sim 1 - \lambda \mu - \Gamma (1 - a) \lambda^a \sim e^{-\lambda \mu}.$$

Thus, consistently with the GCLT approach, to the dominant order

$$E \left( \sum_{N_{(\beta)}} \right) = \frac{1}{\Gamma (-\beta)} \int_0^\infty d\lambda \cdot \lambda^{-\beta-1} E \left( e^{-\lambda X_1} \right)^N \sim (N \mu)^\beta,$$

so that

$$c_N := N \alpha \frac{E \left( \sum_{N_{(\beta)-\alpha}} \right)}{E \left( \sum_{N_{(\beta)}} \right)} B (2 - \alpha, \alpha - \beta) \sim \alpha \mu^{-\alpha} B (2 - \alpha, \alpha - \beta) N^{-(\alpha - 1)}.$$

This suggests that

**Proposition 5.** When $\alpha \in (1, 2)$, upon scaling time using an effective population size $N_e = c_N^\frac{1}{\alpha}$ (with $c_N$ as in [12]), we obtain a limiting continuous-time $\Lambda -$coalescent: $x_t \overset{d}{=} \lim_{N \to \infty} x_{\lfloor t/c_N \rfloor}^{(N)}$, with $\Lambda$ a beta($2 - \alpha, \alpha - \beta$) probability measure, $\beta < \alpha$. If $\beta = 0$, we get the standard beta($2 - \alpha, \alpha$) coalescent.
Proof: To confirm this point, we will first evaluate a large $N$ estimate of $P_{i,j}^{(N)}$ defined in (2), in the range $\alpha \in (1, 2)$. Using the above small $\lambda$ estimate of $E(e^{-\lambda X_1})$ in the parameter range under concern, for $i \geq 2$, we get

$$E \left( X_i^j e^{-\lambda X_1} \right) = (-1)^i \frac{d^i}{d\lambda^i} E \left( e^{-\lambda X_1} \right) \sim \alpha \Gamma (i - \alpha) \lambda^{\alpha - i}.$$  

Thus, using (9),

$$E \left( \sum_{N} S_i^j \right) = \frac{1}{\Gamma (i - \beta)} \int_{0}^{\infty} d\lambda \cdot \lambda^{i - \beta - 1} E \left( X_i^j e^{-\lambda X_1} \right) E \left( e^{-\lambda X_1} \right)^{N - 1} \sim \alpha \frac{\Gamma (i - \alpha) \Gamma (\alpha - \beta)}{\Gamma (i - \beta)} \int_{0}^{\infty} d\lambda \cdot \lambda^{i - \beta - 1} \lambda^{\alpha - i} e^{-\mu \lambda N} = \alpha \frac{\Gamma (i - \alpha) \Gamma (\alpha - \beta)}{\Gamma (i - \beta)} \left( \mu N \right)^{\alpha - \beta}.$$  

Finally, we obtain

$$P_{i,j}^{(N)} \sim N^{-(\alpha - 1) - \frac{\alpha - \beta}{\mu} \frac{\Gamma (i - \alpha) \Gamma (\alpha - \beta)}{\Gamma (i - \beta)}},$$

showing that, with $c_N = P_{i,j}^{(N)} \sim N^{-(\alpha - 1)} \rightarrow 0$, for each $i$, $\lim_{N \rightarrow \infty} c_N^{-1} P_{i,j}^{(N)}$ exist and are strictly positive constants. More precisely,

$$c_N^{-1} P_{i,j}^{(N)} \sim \phi_1 (i) = \frac{1}{B(2, \alpha - \alpha - \beta)} \frac{\Gamma (\alpha - \beta) \Gamma (i - \alpha)}{\Gamma (i - \beta)} = \int_{0}^{\infty} \frac{\lambda X_1 e^{-\lambda X_1}}{B(2, \alpha - \alpha - \beta)} \left( \mu \lambda N \right)^{\alpha - \beta} d\lambda,$$

where $\Lambda_1 = \Lambda$ is a beta($2 - \alpha, \alpha - \beta$) probability measure.

To deal with the higher order terms, with $m := \{l \in \{1, \ldots, j\} : i_l \geq 2\}$, assuming $1 \leq j < i$, let us write (3) as

$$P_{i,j}^{(N)} = \sum_{m=1}^{j} \sum_{i_1 + \ldots + i_m = i - j + m} \binom{j}{m} \binom{i}{i_1 \ldots i_m} \frac{E \left( \sum_{N} S_i^j \prod_{l=1}^{m} S_{i_l}^j \prod_{l=m+1}^{i_j} S_l \right)}{E \left( \sum_{N} S_i^j \right)} = 0,$$

so that simultaneous multiple collisions cannot occur in the limit (actually the contribution of these simultaneous multiple collisions terms in $P_{i,j}^{(N)}$ is of order $O(c_N^2)$).

In the latter expression of $P_{i,j}^{(N)}$ therefore, only the term corresponding to $m = 1$ will contribute to the $O(c_N)$-order. Because, $E \left( X_1 e^{-\lambda X_1} \right) = -\frac{d}{\lambda} E \left( e^{-\lambda X_1} \right) \sim \mu$ and $E \left( X_1 e^{-\lambda X_1} \right) \sim \alpha \Gamma (i - \alpha) \lambda^{\alpha - i}$, we get

$$\frac{\sum_{N} S_i^j \prod_{l=2}^{i} S_l}{E \left( \sum_{N} S_i^j \right)} = \int_{0}^{\infty} d\lambda \lambda^{i - \beta - 1} E \left( X_i^j e^{-\lambda X_1} \right) \prod_{l=2}^{i} E \left( X_l e^{-\lambda X_1} \right) \frac{E \left( e^{-\lambda X_1} \right)^{N - j}}{E \left( \sum_{N} S_i^j \right)} \sim \frac{\alpha \mu^{j - 1} \Gamma (i - \alpha)}{E \left( \sum_{N} S_i^j \right) \Gamma (i - \beta)} \int_{0}^{\infty} d\lambda \lambda^{i - \beta - 1} \lambda^{\alpha - i} e^{-\mu \lambda N} \sim \frac{\alpha \mu^{j - 1} \Gamma (i - \alpha) \Gamma (\alpha - \beta + j - 1)}{E \left( \sum_{N} S_i^j \right) \Gamma (i - \beta) (\mu N)^{\alpha - \beta + j - 1}}.$$
This shows, using $(N)^j_j \sim N^j/j!$, (12) and after some elementary algebra, that for $j = 1, \ldots, i - 1$,

$$\lambda_{i,j} = \left( \frac{i}{j-1} \right) B(i-j+1+\alpha, \alpha-\beta+j-1) \frac{B(2, \alpha-\beta)}{B(2, \alpha-\beta)}$$

Thus, with $\beta < 1$ and because $\Gamma(1-\alpha) = \Gamma(-\epsilon) \sim_{0+} -1/\epsilon$,

$$E\left(e^{-\lambda X_1}\right) \sim 1 - \lambda - \frac{1}{\epsilon} \left( \lambda - \lambda^{1+\epsilon} \right) \sim 1 - \lambda + \lambda \log \lambda =: I(\lambda).$$

Thus,

$$E\left(\sum_{k=1}^{N} \right) \sim 1 + \frac{u}{\log N} \left( \log u - \log \log N - 1 \right).$$

This shows, using (N) $\lambda \sim N^j/j!$, (12) and after some elementary algebra, that for $j = 1, \ldots, i - 1$,

$$\lambda_{i,j} = \left( \frac{i}{j-1} \right) B(i-j+1+\alpha, \alpha-\beta+j-1) \frac{B(2, \alpha-\beta)}{B(2, \alpha-\beta)}$$

where $\Lambda = \Lambda_1 \sim \beta(2-\alpha, \alpha-\beta)$. As a result, $x_t \equiv \lim_{N \to \infty} x_t^{(N)}$ is a continuous-time-t pure death coalescent process on $\mathbb{N}$ with infinitesimal transition rates $\lambda_{i,j}$. \hfill \Box

- The case $\alpha = 1$. We view it as a limiting case of the previous analysis deriving from (13) when $\alpha \to 1^+$.

**Proposition 6.** When $\alpha = 1$, upon scaling time using a logarithmic effective population size $N_e \equiv c_N^{-1} \sim \log N$, the limiting process $x_t \equiv \lim_{N \to \infty} x_t^{(N)}$ exists and is a continuous-time-t pure death coalescent with $\Lambda$ a beta$(1, 1-\beta)$ probability measure, $\beta < 1$.

If $\beta = 0$, we get the standard Bolthausen-Sznitman coalescent with $\Lambda$ uniform.

**Proof:** Put indeed $\alpha = 1 + \epsilon$ in the previous small-$\lambda$ estimate (13) of $E\left(e^{-\lambda X_1}\right)$ in the parameter range $\alpha \in (1, 2)$, with $\epsilon > 0$ small. Then $\mu \sim 1 + 1/\epsilon$ and because $\Gamma(1-\alpha) = \Gamma(1-\epsilon) \sim_{1+} 1/\epsilon$,

$$E\left(e^{-\lambda X_1}\right) \sim 1 - \lambda - \frac{1}{\epsilon} \left( \lambda - \lambda^{1+\epsilon} \right) \sim 1 - \lambda + \lambda \log \lambda =: I(\lambda).$$

Thus, $I(\lambda) \sim \log \lambda \sim e^{-\lambda}$

$$E\left(\sum_{k=1}^{N} \right) \sim 1 + \frac{u}{\log N} \left( \log u - \log \log N - 1 \right).$$

This shows, using (N) $\lambda \sim N^j/j!$, (12) and after some elementary algebra, that for $j = 1, \ldots, i - 1$,

$$\lambda_{i,j} = \left( \frac{i}{j-1} \right) B(i-j+1+\alpha, \alpha-\beta+j-1) \frac{B(2, \alpha-\beta)}{B(2, \alpha-\beta)}$$

with $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function. In the latter estimate, we used that differentiating $\int_0^\infty du \cdot u^{-\beta} e^{-u} = \Gamma (1- \beta + \theta)$ with respect to the extra parameter $\theta$ and then putting $\theta = 0$ gives $\int_0^\infty du \cdot u^{-\beta} \log (u) e^{-u} = \Gamma' (1- \beta)$.

Finally, consistently with the GCLT approach

$$c_N = \frac{\mathbb{E} \left( \sum_{k=1}^{N} \right) \sim 1}{\log N}.$$
Clearly, with Λ a beta(1, 1 − β) probability measure and β < α = 1
\[ c_N^{-1} P^{(N)}_{i,1} \xrightarrow{N \to \infty} \int_0^1 u^{i-2} \Lambda(du) = \frac{\Gamma(2-\beta)\Gamma(i-1)}{\Gamma(i-\beta)} = \lambda_{i,1}. \]

Using similar arguments on higher order terms, showing that simultaneous multiple collisions do not contribute in the limit, one can easily show
\[ c_N^{-1} P^{(N)}_{i,j} \xrightarrow{N \to \infty} \int_0^1 u^{i-j-1} (1-u)^{j-1} \Lambda(du) \]
\[ = \frac{B(i-j,j-\beta)}{B(1,1-\beta)} =: \lambda_{i,j}, \quad j = 1, \ldots, i-1, \]
where \( \Lambda = \Lambda_1 \overset{d}{\sim} \text{beta}(1,1-\beta) \). This confirms that, when α = 1, upon scaling time using an effective logarithmic population size \( N_e = c_N^{-1} \log N \), we get a limiting continuous-time \( \Lambda \)-coalescent \( x_t \overset{d}{=} \lim_{N \to \infty} x^{(N)}_{t/[c_N]} \), where \( \Lambda \) is a beta(1, 1 − β) probability measure, \( \beta < 1 \).

- The case \( \alpha > 2 \).

**Proposition 7.** When \( \alpha > 2 \), for any value of \( \beta \), with \( \mu := \frac{\alpha}{\alpha - 1} > 0 \) and \( \rho := \frac{\alpha}{\alpha - 2} > 0 \), upon scaling time using a linear effective population size \( N_e = c_N^{-1} = N/\mu^2/\rho \), the limiting process \( x_t \overset{d}{=} \lim_{N \to \infty} x^{(N)}_{t/[c_N]} \) exists and is the continuous-time Kingman coalescent.

**Proof:** If \( \alpha > 2 \), \( \Sigma_N \) is in the domain of attraction of the normal law. As a result, for small \( \lambda \), with \( \mu := E(X_1) = \frac{\alpha}{\alpha - 1} \) and \( \rho := E(X_1^2) = \frac{\alpha}{\alpha - 2} \)

\[ E(e^{-\lambda X_1}) \sim 1 - \lambda \mu + \frac{1}{2} \rho \lambda^2 \sim e^{-\lambda \mu}. \]

From (17), for small \( \lambda \), \( E(X_1 e^{-\lambda X_1}) = -\frac{d}{d\lambda} E(e^{-\lambda X_1}) \sim \mu \) and \( E(X_1^2 e^{-\lambda X_1}) = \frac{d^2}{d\lambda^2} E(e^{-\lambda X_1}) \sim \rho. \)

We have \( c_N := P^{(N)}_{2,1} = N \frac{E(S_{\Sigma_N}^2)}{E(S_{\Sigma_N}^2)} \) with, from (9)

\[ E\left(S_{\Sigma_N}^{\beta_{\Sigma_N}}\right) \sim \frac{\rho}{\Gamma(2-\beta)} \int_0^\infty d\lambda \cdot \lambda^{1-\beta} e^{-\lambda \mu N} = \rho (\mu N)^{-\beta-2}, \]
\[ E\left(S_{\Sigma_N}^{\beta_{\Sigma_N}}\right) \sim \frac{1}{\Gamma(-\beta)} \int_0^\infty d\lambda \cdot \lambda^{-\beta-1} e^{-\lambda \mu N} = (\mu N)^{\beta}. \]

Thus,

\[ c_N = N \frac{E\left(S_{\Sigma_N}^{\beta_{\Sigma_N}}\right)}{E\left(S_{\Sigma_N}^{\beta_{\Sigma_N}}\right)} \sim \rho \mu^2 N^{-1} \]

goes to 0 as claimed. As in Proposition 5, simultaneous multiple collisions cannot contribute in the limit. Only the term in the expression of \( P_{i,j}^{(N)} \) corresponding to \( m = 1 \) will contribute to the \( O(c_N) \) —order and so we need to focus on a collision with a single \( i_1 \geq 2 \).

Now, from (17), \( E(X_1 e^{-\lambda X_1}) = -\frac{d}{d\lambda} E(e^{-\lambda X_1}) \sim \mu \) and \( E(X_1^2 e^{-\lambda X_1}) \sim \rho \delta_{i_1,2} \) with \( i_1 = i - j + 1 \geq 2 \). Proceeding again as in Proposition 5 therefore, only
effective transitions from $i$ to $j < i$ with $j = i - i_1 + 1 = i - 1$ are seen in the limit, corresponding to the binary mergers of a Kingman coalescent.

- $\alpha = 2$. To complete the picture, it remains to study the limiting critical case $\alpha = 2$.

**Lemma 8.** When $\alpha = 2$, for all values of $\beta$, the coalescence probability goes to 0 as $N \to \infty$ like

$$c_N \sim \frac{1}{2} \log \frac{N}{N}.$$  

**Proof:** We view the case $\alpha = 2$ as a limiting case of (17) as $\alpha \to 2^+$. When $\lambda$ is small and for $\alpha > 2$, we have

$$E \left( e^{-\lambda X_1} \right) \sim 1 - \lambda \mu + \frac{1}{2} \rho \lambda^2 - \Gamma (1 - \alpha) \lambda^\alpha.$$ 

Putting $\alpha = 2 + \varepsilon$, for $\varepsilon > 0$ small, using $\Gamma (-1 - \varepsilon) \sim \frac{1}{\varepsilon}$, with $\rho \sim \frac{2}{\varepsilon}$, when $\lambda$ is small, we have

(18) $E \left( e^{-\lambda X_1} \right) \sim 1 - 2 \lambda + \frac{1}{\varepsilon} \lambda^2 - \frac{1}{\varepsilon} \lambda^{2+\varepsilon} \sim 1 - 2 \lambda - \lambda^2 \log \lambda \sim -e^{-2\lambda}.$

Because $E \left( X_1 e^{-\lambda X_1} \right) = -\frac{d^2}{d\lambda^2} E \left( e^{-\lambda X_1} \right) \sim 2$ and $E \left( X_1^2 e^{-\lambda X_1} \right) = \frac{d^2}{d\lambda^2} E \left( e^{-\lambda X_1} \right) \sim -2 \log \lambda - 3$, we have

$$E \left( \Sigma^2_{N,S^2_1} \right) \sim \frac{1}{1 - \beta} \int_0^\infty d\lambda \cdot \lambda^{1-\beta} E \left( X_1^2 e^{-\lambda X_1} \right) E \left( e^{-\lambda X_1} \right)^N \sim \frac{1}{\Gamma \left( 2 - \beta \right)} \int_0^\infty d\lambda \cdot \lambda^{1-\beta} (2 \log \lambda + 3) e^{-2N\lambda} = (2N)^\beta.$$ 

The Euler integral with a logarithmic term inside appearing in the expression of $E \left( \Sigma^2_{N,S^2_1} \right)$ can be obtained while taking the derivative of $\int_0^\infty d\lambda \cdot \lambda^{1-\beta} \lambda^\varepsilon e^{-2N\lambda}$ with respect to the extra parameter $\theta$ and then putting $\theta = 0$ in the obtained expression. Observing therefore

$$\int_0^\infty d\lambda \cdot \lambda^{1-\beta} \log (\lambda) e^{-2N\lambda} = \Gamma' (2 - \beta) (2N)^{-(2-\beta)} - \Gamma (2 - \beta) (2N)^{-(2-\beta)} \log (2N),$$

to the dominant order in $N$,

$$E \left( \Sigma^2_{N,S^2_1} \right) \sim 2 (2N)^{-(2-\beta)} \log N,$$

leading to

$$c_N := P_{2,1}^{(N)} = \frac{E \left( \Sigma^2_{N,S^2_1} \right)}{E \left( \Sigma^2_{N} \right)} \sim \frac{1}{2} \log \frac{N}{N}.$$ 

**Remark:** A similar scaling behavior for $c_N$ was recently obtained in Theorem 2.4 of [20], dealing with coalescents arising from compound Poisson discrete reproduction models, in the critical case.
Proposition 9. When $\alpha = 2$, upon scaling time using an effective population size $N_e = c_N^{−1} = (2N)/\log N$, the limiting process $x_t \xrightarrow{d} \lim_{N \to \infty} x_t^{(N)}$ is the continuous-time $t$ Kingman coalescent.

Proof: Using (13), for small $\lambda$, we have

$$E(X_i)e^{-\lambda X_i} = (-1)^i \int_0^\infty d\lambda \cdot \lambda^i \lambda^{-i-2} \sim 2\Gamma(i-2) \lambda^{-i-2}, \quad i \geq 3,$$

to which one should add $E(X_1)e^{-\lambda X_1} \sim 2$ and $E(X_i^2)e^{-\lambda X_i} \sim -(2 \log \lambda + 3)$. For $i \geq 3$, we get

$$E\left(\sum_{i=1}^N \lambda S_i^2\right) = \frac{1}{\Gamma(i-\beta)} \int_0^\infty d\lambda \cdot \lambda^i \lambda^{-1-\beta} E(X_1)e^{-\lambda X_1} E(e^{-\lambda X_1})^{N-1} \sim \frac{2\Gamma(i-2)}{\Gamma(i-\beta)} \int_0^\infty d\lambda \cdot \lambda^{1-\beta} e^{-2\lambda N} = \frac{2}{(2N)^{2-\beta}}.$$

Thus, for all $i \geq 3$, as $N \to \infty$

$$c_N^{-1} P_{i,1}^{(N)} = N c_N^{-1} E\left(\sum_{i=1}^N \lambda S_i^2 \prod_{i=2}^j S_i\right) \sim \frac{1}{\log N} B(i-2,2-\beta) \to 0.$$

Therefore, due to the extra factor $\log N$ appearing in $c_N$, the transitions from $i \geq 3$ to 1 cannot be seen in the limit, nor (for the same reason) the transitions involving a single multiple collision with $i_1 \geq 3$, nor transitions involving simultaneous multiple collisions of any order. In fact, only the events involving a single multiple collision with $i_1 = i - j + 1 = 2$ (corresponding to transitions from $i$ to $j = i - 1$) will contribute in the limit. Indeed,

$$E\left(\sum_{i=1}^N \lambda S_i^2 \prod_{i=2}^j S_i\right) = \int_0^\infty d\lambda \cdot \lambda^i \lambda^{-1-\beta} E(X_1^2)e^{-\lambda X_1} \prod_{i=2}^j \frac{E(X_i)e^{-\lambda X_i} E(e^{-\lambda X_i})^{N-j}}{E(\sum_{i=1}^N \lambda S_i^2) \Gamma(i-\beta)}$$

$$\sim \frac{(-2)^j-1}{E(\sum_{i=1}^N \lambda S_i^2) \Gamma(i-\beta)} \int_0^\infty d\lambda \cdot \lambda^{i-\beta-1} (2 \log \lambda + 3) e^{-2\lambda N} \sim \frac{1}{2} N^{-i} \log N.$$

This shows, using

$$p_{i,j}^{(N)} \sim \binom{N}{j} \frac{i}{1} \frac{E(\sum_{i=1}^N \lambda S_i^2 \prod_{i=2}^j S_i)}{E(\sum_{i=1}^N \lambda S_i^2)}$$

with $j = i - 1, \binom{N}{j} \sim N^j/j!$, and after some elementary algebra, that

$$c_N^{-1} P_{i,i-1}^{(N)} \xrightarrow{N \to \infty} \lambda_{i,i-1} = \binom{i}{2}.$$

Remarks:

(i) Results of a similar flavor can be found in Schweinsberg’s work [39]. However, his model and techniques are different from ours because he considers large $N$ limiting coalescents obtained while sampling without replacement from a discrete super-critical Galton-Watson branching process, assuming the reproduction law of each offspring to exhibit power-law Zipf tails of index $\alpha$ (as in his notations). Note
that there is no parameter $\beta$ in the construction [39]. In the same spirit, results can also be found in Huillet-Möhle [19] where, following [13], $\Lambda-$coalescents are obtained as scaling limits of discrete extended Moran models, the skewed reproduction law of which displaying occasional extreme events with one individual allowed to produce a large amount of offspring. Whenever the reproduction law displays systematic extreme events, discrete coalescents were even shown to emerge in the large $N-$limit, but in this Moran context, they are only $\Lambda-$coalescents with multiple but no simultaneous collisions, [19]. This contrasts with the occurrence in our present work of discrete Poisson-Dirichlet $\Xi-$coalescents. In contrast also with our current work where coalescents are derived from a sampling procedure in the continuum, the coalescents considered in [39] and [19] were built from discrete reproduction laws at fixed population size $N$ and looking at their scaling limits $N \to \infty$.

We refer to these two works and to [7] for additional background on $\Lambda-$coalescent processes adapted to our purposes.

(ii) When $\alpha \geq 2$, neither the (large $N$ estimate of the) scaling constant $c_N$ nor the limiting (Kingman) coalescent depend on the bias parameter $\beta$.

(iii) We observe that the probability $c_N := P_{2,1}^{(N)}$ that two individuals chosen at random share the same common ancestor, defining the time-scale to derive the large-$N$ limits of the Pareto-coalescents all have the same large-$N$ order of magnitude as the length $l_N$ of an external branch chosen at random in the limiting $\Lambda-$coalescents $x_t (\alpha \in [1,2))$ or $\Xi-$coalescent $x_k (\alpha \in (0,1))$, or Kingman coalescent $x_t (\alpha > 2)$, started at $x_0 = N$. This curious fact is unexplained so far.

5. Forward in time selection model and genealogies

In this Section, in the spirit of [7], we indicate that the coalescent processes just discussed may be viewed as the genealogical processes of some forward in time evolving branching population models with selection. As it is often the case in population genetics, the process we are interested in is in the class of branching processes conditioned on having a fixed population size over each generation, in the spirit of [23].

5.1. A Poisson-point process model with selection. Start with $N$ individuals at generation $t = 0$ and assume that each individual has an initial fitness $x_n(0) > 0$, $n = 1, \ldots, N$.

To describe the state of the population at the next generation, assume first that, independently of one another, each individual potentially generates an infinite number of offspring along a Poisson point process (PPP) with intensity (or occupation) density

$$
\pi_{x_n(0)}(x) = -\pi_{x_n(0)}'(x) = \alpha x_n(0)^{\alpha} x^{-(\alpha+1)} \quad \text{(19)}
$$

\footnote{The $'$ symbol indicates derivative with respect to $x$.}
where \( \pi_{x_n(0)}(x) := (x/x_n(0))^{-\alpha} \), \( x > 0 \), \( \alpha > 0 \), \( n = 1, \ldots, N \). We observe that, with \( \pi(x) := x^{-\alpha} \)
\[
\pi_{x_n(0)}(x) = \frac{\pi(x)}{\pi(x_n(0))}
\]
and if we let \( \pi(x) = -\pi'(x) = \alpha x^{-(\alpha+1)} \), then \( \pi_{x_n(0)}(x) = x_n(0)^{\alpha} \pi(x) \).

So the fitnesses of the offspring of each individual is generated according to a PPP depending on the fitness of its parent. From these simple assumptions, and as conventional wisdom suggests, we get:

**Proposition 10.** In a PPP model for fitness-dependent offspring reproduction with occupation density \([12]\), the fitnesses of the offspring of some parental individual with fitness \( x_n(0) \) are \( x_n(0) \) times the fitnesses of the offspring of some canonical individual with unit fitness: In this sense, the larger the fitness \( x_n(0) \) of some individual is, the more he will, proportionally, produce offspring with large fitness.

**Proof:** Let \((\tau_n: n \geq 1)\) be the points of a standard homogeneous Poisson point process (PPP) on the half-line with rate 1, and let \( \pi^{-1}(s) = s^{-1/\alpha} \) be the decreasing inverse of \( \pi \). Then, with \( \pi_{x_n(0)}^{-1}(s) = x_n(0) \cdot s^{-1/\alpha} \), \( (\pi_{x_n(0)}^{-1}(\tau_n): n \geq 1) \) are the (ordered) points of the offspring PPP on the positive half-line (or here the fitness space) with occupation density \( \pi_{x_n(0)} \). So, the fitter the individuals, the fitter their offspring, proportionally to the parental fitness. \( \diamond \)

Let us also briefly emphasize that, avoiding the fitness dependence on the parent of the PPP, \((\pi^{-1}(\tau_n): n \geq 1)\) are just the (ordered) points of a PPP on the half-line with occupation density \( \pi \). Whenever, as in our case study here, the rate function \( \pi \) is not integrable up to \( x = 0 \), there are infinitely many such points, with 0 as an accumulation point whereas there is of course a finite Poisson (with mean \( \pi(\varepsilon) \)) number of them above some threshold \( \varepsilon > 0 \).

It is well-known that when \( \alpha \in (0, 1) \), with \( \pi(x) := x^{-\alpha} \) and \( \pi^{-1}(s) = s^{-1/\alpha} \), the positive cumulative rv
\[
\chi \overset{d}{=} \sum_{n \geq 1} \pi_{x_n(0)}^{-1}(\tau_n)
\]
is a one-sided \( \alpha \)-stable rv on \( (0, \infty) \) with LST \( E(e^{-\lambda\chi}) = e^{-\alpha\lambda^\alpha} \), \( \kappa = \Gamma(1 - \alpha) > 0 \), \( \lambda \geq 0 \).

When \( \alpha \in (1, 2) \), the law of \( \chi \) is the one of a positive Lamperti rv with LST \( E(e^{-\lambda\chi}) = e^{-c\lambda + \kappa\lambda^\alpha} \), \( \kappa = -\Gamma(1 - \alpha) > 0 \), \( c := E(\chi) > 0 \), \([27]\). When \( \alpha = 1 \), the law of \( \chi \) is the one of a positive Neveu rv with LST \( E(e^{-\lambda\chi}) = e^{\lambda \log \chi} \) (see \([32]\)); the latter may be viewed as a Lamperti rv in the limit \( \alpha \to 1^+ \), \([18]\).

Typically indeed, by the Lévy-Khintchine formula, \( \pi(x) \, dx \) stands for the Lévy measure for the non-negative jumps of \( \chi \) which is the value at time \( t = 1 \) of an infinitely divisible subordinator \( (\chi_t: t \geq 0) \) with LST \( E(e^{-\lambda\chi}) \), \([2]\). In \([20]\), the terms \( \pi_{x_n(0)}^{-1}(\tau_n) \) are thus the ranked jumps of \( \chi \) in its Lévy decomposition (see \([33]\) for example).
Reproduction step.
Because we consider the offspring of all the \( N \) initial individuals with fitnesses \( x_n (0), \ n = 1, ..., N \), the step-1 state of the whole population is thus obtained from a PPP with global equivalent occupation density

\[
\pi_{x_{N,\alpha} (0)} (x) = \alpha x^{-(\alpha+1)} \sum_{n=1}^{N} x_n (0)^{\alpha} = \alpha x_{N,\alpha} (0)^{\alpha} x^{-(\alpha+1)},
\]

where

\[
x_{N,\alpha} (0) := \left( \sum_{n=1}^{N} x_n^\alpha (0) \right)^{1/\alpha}
\]

is the global equivalent initial fitness of the whole population at generation 0 to consider.

Proposition 11. In a PPP with occupation density \( (19) \) for the descent of each individual with fitness \( x_n (0), \ n = 1, ..., N \), the occupation density of the population as a whole is given by \( (21) \), where \( x_{N,\alpha} (0) := \left( \sum_{n=1}^{N} x_n^\alpha (0) \right)^{1/\alpha} \) is the global equivalent fitness.

Proof: This follows from the superposition principle of Poisson point processes (see [25] p. 16). The fact that the intensity of the superposed PPP is in the same class as the one of a single PPP descending from \( x_n (0) \) is a remarkable scaling property of \( \pi_{x_n (0)} (x) \).

In this setup therefore, \( \pi_{x_{N,\alpha} (0)} (x) := -\pi'_{x_{N,\alpha} (0)} (x) \) stands for the occupation density that there is a point at \( x \) descending from any of the \( N \) initial individuals with fitnesses \( x_n (0), \ n = 1, ..., N \).

Note that the cumulated fitness of all first-generation offspring is

\[
\sum_{n \geq 1} \tau_{x_{N,\alpha}}^{-1} (\tau_n) d \tau_n = x_{N,\alpha} (0) \cdot \chi,
\]

where \( \chi \) is given by \( (20) \).

Selection step.
In order to model a population with fixed size over the generations, the final state of the population at time 1 is obtained while selecting the \( N \) individuals of the whole population whose fitnesses are the largest (the selection step), truncating therefore the latter sum to its \( N \) first terms.

The whole process (including reproduction and selection steps) is then iterated independently over the next generations.

From this definition of the process, if the \( x_n (k) \)'s are the fitnesses of the \( N \) fittest individuals at generation \( k \), the ordered ones \( x_{(n)} (k+1) \) at generation \( k+1 \) (\( x_{(1)} > ... > x_{(N)} \)) descending from the whole population at step \( k \) are given by \( x_{(n)} = \pi_{x_{N,\alpha} (0)}^{-1} (\tau_n) = x_{N,\alpha} (k) \tau_n^{-1/\alpha} \). Here the \( \tau_n \)'s are the ordered points of a standard Poisson process on the half-line with \( \tau_1 < ... < \tau_n < ... < \tau_N \). The law of \( \tau_n \) is thus
the one of an Erlang gamma($n$) rv with density $f_{\tau_n}(s) = s^{n-1}e^{-s}/\Gamma(n)$. Further, given $\tau_{N+1} = s$, the probability density of $\tau_1, \ldots, \tau_N$ is
\[
f_{\tau_1, \ldots, \tau_N}(s_1, \ldots, s_N \mid \tau_{N+1} = s) = \frac{N!}{s^N}1_{0<s_1<\ldots<s_N<s}
\]
and so
\[
f_{\tau_1, \ldots, \tau_N}(s_1, \ldots, s_N) = 1_{0<s_1<\ldots<s_N} \int_{s_N}^{\infty} dse^{-s} = e^{-s}N_{0<s_1<\ldots<s_N}.
\]
The joint law of the ordered $x_n(k+1)s$ is thus the one of the images $\tau_{x_{N,n}(k)}(\tau_n)s$, namely
\[
f_{x_{(1)}(k+1), \ldots, x_{(N)}(k+1)}(x_1, \ldots, x_N) = e^{-\tau_{x_{N,n}(k)}(x_N)} \prod_{n=1}^{N} \tau_{x_{N,n}(k)}(x_n) 1_{x_1>\ldots>x_N}.
\]
Clearly also (with the two terms in the right-hand side term mutually independent),
\[
x_{(N+1)}(k+1) \overset{d}{=} x_{N,n}(k) x_{N+1}^*(k+1),
\]
where $x_{N+1}^*(k+1)$ is the $(N+1)$–st largest point of a PPP with occupation density $\pi(x) = \alpha x^{-(\alpha+1)}$, $x, \alpha > 0$. By the image measure theorem, the density of $x_{N+1}^*(k+1) \overset{d}{=} \tau_{N+1}$ is obtained as a power-gamma density
\[
f_{x_{N+1}^*(k+1)}(x) = \frac{\alpha}{N!} x^{-(N+1)\alpha+1}e^{-x^{-\alpha}}, x > 0.
\]

Next, the conditional density of each $x_n(k+1)$ given $x_{(N+1)}(k+1) = x$ is
\[
f_{x_n(k+1)}(x_n \mid x_{(N+1)}(k+1) = x) = \frac{\pi(x_n)}{\pi(x)} 1_{x_n>x},
\]
showing that (with the two terms in the right-hand side term mutually independent), $x_n(k+1) \overset{d}{=} x_{N,n}(k) x_{N+1}^*(k+1) X_n(k+1)$ where $X_n(k+1)$ is a Pareto($\alpha$) distributed rv with density $f(x) = \alpha x^{-(\alpha+1)}$ on $(1, \infty]$. Putting all this together, we obtained

**Proposition 12.** Independently for each $k \geq 0$, with $x_{N+1}^*(k+1)$ having the power-gamma distribution [22] and $X_n(k+1)$ being Pareto($\alpha$) distributed and with the three right-hand side terms being mutually independent,
\[
x_n(k+1) \overset{d}{=} x_{N,n}(k) x_{N+1}^*(k+1) X_n(k+1), n = 1, \ldots, N,
\]
indicates how to update multiplicatively the fitness of the $n$–th individual at generation $k+1$, when the fitnesses of the previous generation are summarized in $x_{N,n}(k)$.

We also clearly have an update of the global equivalent fitness $x_{N,n}$ from step $k$ to step $k+1$ as:

**Corollary 13.** With $x_{N,n}(k) := \left(\sum_{n=1}^{N} x_n(k)\right)^{1/\alpha}$ the global equivalent fitness of the whole population at generation $k$, the following recursion holds
\[
x_{N,n}(k+1) \overset{d}{=} x_{N,n}(k) x_{N+1}^*(k+1) \left(\sum_{n=1}^{N} X_n(k+1)^{\alpha}\right)^{1/\alpha}.
\]

---

8We used the scaling property of Pareto($\alpha$) rvs $X$ on $(1, \infty)$ stating that $X \mid X > a \overset{d}{=} aX$. 
In the latter sum term, each $X_n^\alpha$ is thus a Pareto(1) distributed rv with density $f(x) = x^{-2}$ on $(1, \infty)$ and we need to sum $N$ of them independently which is reminiscent of (15).

**Large $N$ asymptotics of the $\alpha$–mean fitness.**

Defining

$$\langle x \rangle_{N,\alpha} (k) := \left( \frac{1}{N} \sum_{n=1}^{N} x_n^\alpha (k) \right)^{1/\alpha},$$

to be the generalized (Hölder) $\alpha$–mean of the fitnesses $x_n$, $n = 1, \ldots, N$, at generation $k$, it follows from (23) that

$$\langle x \rangle_{N,\alpha} (k+1) \overset{d}{=} \langle x \rangle_{N,\alpha} (k) x_{N+1}^* (k+1) \left( \sum_{n=1}^{N} X_n (k+1)^\alpha \right)^{1/\alpha}.
$$

By Jensen inequality, the Hölder $\alpha$–means $\langle x \rangle_{N,\alpha} (k)$ are non-decreasing functions of $\alpha$.

**Corollary 14.** The $\beta$–moments of $\langle x \rangle_{N,\alpha}$ at generation $k$ are given by

$$E \left( \langle x \rangle_{N,\alpha} (k)^\beta \right) = \langle x \rangle_{N,\alpha} (0)^\beta \left[ E \left( x_{N+1}^* \right) E \left( \sum_{n=1}^{N} X_n^\alpha \right)^{\beta/\alpha} \right]^k.
$$

**Proof:** This follows from the independence of the $x_{N+1}^*$ and the $X_n^\alpha$ and their i.i.d. character within each generation $k$ and from (25).

**Proposition 15.** For large $N$, with $v_N := \log \log N$

$$\frac{1}{k} \log \langle x \rangle_{N,\alpha} (k) \overset{a.s.}{\to} \frac{1}{\alpha} v_N.
$$

With

$$F_N (\beta) := -\frac{\beta}{\alpha} \log \log N - \frac{\beta}{\alpha \log N} (\psi (1 - \beta/\alpha) - \log \log N - 1)
$$

and $f_N (a), a < 0$, its Legendre transform, the large deviation regime is given by

$$\frac{1}{k} \log P \left( \frac{1}{k} \log \langle x \rangle_{N,\alpha} (k) \to a \right) \to f_N (a) \leq 0.
$$

**Proof:** Observing that $E \left( x_{N+1}^* \right) = \Gamma (N+1 - \beta/\alpha) / \Gamma (N+1) \overset{N \to \infty}{\sim} N^{-\beta/\alpha}$ and applying (15) giving the moments of a partial sum of $N$ i.i.d. Pareto(1) distributed rvs, it follows that, with $\beta < \alpha$

$$E \left( \sum_{n=1}^{N} X_n^\alpha \right)^{\beta/\alpha} \overset{N \to \infty}{\sim} \left( N \log N \right)^{\beta/\alpha} \left( 1 + \frac{\beta}{\alpha \log N} (\psi (1 - \beta/\alpha) - \log \log N - 1) \right).
$$

As a result, we get

$$\frac{E \left( \langle x \rangle_{N,\alpha} (k)^\beta \right)}{\langle x \rangle_{N,\alpha} (0)^\beta} \overset{N \to \infty}{\sim} \left( \log N \right)^{(\beta k)/\alpha} \left( 1 + \frac{\beta}{\alpha \log N} (\psi (1 - \beta/\alpha) - \log \log N - 1) \right)^k."
Thus, for large $N$, with
\[ F_N(\beta) = -\frac{\beta}{\alpha} \log \log N - \frac{\beta}{\alpha \log N} (\psi (1 - \beta/\alpha) - \log \log N - 1) \]
defining the concave thermodynamical ‘pressure’,
\[ -\frac{1}{k} \log E \left( \langle x \rangle_{N,\alpha}(k)^\beta \right) \xrightarrow{k \to \infty} F_N(\beta). \]
Thus, with $a = F'_N(\beta) < 0$, by the large deviation principle
\[ \frac{1}{k} \log P \left( \frac{1}{k} \log \langle x \rangle_{N,\alpha}(k) \to a \right) \xrightarrow{k \to \infty} f_N(a) \leq 0, \]
where $f_N(a) = \inf_{\beta < \alpha} (a \beta - F_N(\beta))$ is the concave Legendre transform of $F_N$, giving the large deviation rate function of $-\log \langle x \rangle_{N,\alpha}(k)/k$. In particular, for large $N$, with $v_N := \log \log N$
\[ \frac{1}{k} \log \langle x \rangle_{N,\alpha}(k) \xrightarrow{a.s.} k \to \infty \sim \frac{1}{\alpha} v_N, \]
gives the limiting right shift of the Hölder $\alpha$–mean fitness induced by selection effects.

Remarks:
(i) The limiting right-hand-side term in (27), although increasing very slowly with $N$, does not stabilize to a limit, in contrast to other similar models [1] of branching with selection where, in each generation, each individual produces only two offspring with randomly shifted fitnesses.

(ii) With $\alpha > 0$, let $f(x) := x^\alpha > 0$ define some (increasing) output map of the individuals fitnesses $x$, with $x > 0$. Defining
\[ \langle f(x) \rangle_N(k) := \frac{1}{N} \sum_{n=1}^{N} f(x_n)(k) = \frac{1}{N} \sum_{n=1}^{N} x_n(k)^\alpha \]
to be the mean output fitness in generation $k$ of the whole population, then, whatever $\alpha$, (27) is also
\[ \frac{1}{k} \log \langle f(x) \rangle_N(k) \xrightarrow{a.s.} k \to \infty \sim \frac{1}{\alpha} v_N, \]
interpreting the speed $v_N$ itself. This suggests that it is of interest to work not only on the fitnesses $x_n$ themselves (and their $\alpha$–mean $\langle x \rangle_{N,\alpha}$) but rather on some deformed version of the fitnesses $x_n^\alpha$ (and their standard mean $\langle f(x) \rangle_N$). Clearly $\langle f(x) \rangle_N$ itself obeys the recursion
\[ \langle f(x) \rangle_N(k + 1) \equiv \langle f(x) \rangle_N(k) \cdot x^\alpha_{N+1}(k + 1) \cdot \sum_{n=1}^{N} X_n(k + 1)^\alpha, \]
deriving again from (28) and the definition of the equivalent global fitness $x_{N,\alpha}(k)$. 
(iii) Note finally that, given $x_{N,\alpha}(k)$, the cumulative distribution function (cdf) of the fitness $x_{(1)}(k + 1)$ of the fittest individual among the $N$ individuals at generation $k + 1$ is given by

$$P_{x_{N,\alpha}(k)}(x_{(1)}(k + 1) \leq x) = P\left(\pi_{x_{N,\alpha}(k)}^{-1}(\tau_1) \leq x\right) = e^{-(x / x_{N,\alpha}(k))^{-\alpha}},$$

where $\tau_1$ is exp(1) distributed. The fitness of the fittest individual obeys

$$x_{(1)}(k + 1) \overset{d}{=} x_{N,\alpha}(k) Y(k + 1),$$

where $Y(k), k \geq 0$ is a sequence of i.i.d. Fréchet rvs with cdf $P(Y \leq x) = e^{-x^{-\alpha}}$. The conditional mean given $x_{N,\alpha}(k)$ of $x_{(1)}(k + 1)$ is $x_{N,\alpha}(k) \Gamma(1 - 1/\alpha)$ and its median value $x_{N,\alpha}(k)(\log 2)^{-1/\alpha}$. More generally, when $N$ is large and unconditionally, due to the recursion (24) on the $x_{N,\alpha}(k)$s:

$$E\left(x_{(1)}(k + 1)^{\beta}\right) \sim x_{N,\alpha}(0)^{\beta} \Gamma(1 - \beta/\alpha) e^{-k F_N(\beta)}.$$

5.2. Genealogies. Now we turn to the genealogies of this branching process with selection.

The beta$(1,1-\beta)$ and Bolthausen-Sznitman coalescents.

Recall that $\pi_{x_{N,\alpha}(k)}(x) := -\pi_{x_{N,\alpha}(k)}^{-1}(x)$ is the occupation density that there would be a point (an offspring) of the PPP at position (with fitness) $x$ at generation $k + 1$, given a global population state $x_{N,\alpha}(k)$.

Proposition 16. Looking backward in time, upon scaling time using $c_N \sim 1/\log N$, the genealogy of the branching model with selection is a beta$(1,1-\beta)$ coalescent, reducing to the Bolthausen-Sznitman coalescent if $\beta = 0$.

Proof: Suppose first $\beta = 0$. Given there is an offspring at $x$ at generation $k + 1$, the sampling probability that it would be an offspring of the individual with fitness $x_n(k)$ is thus

$$\frac{\pi_{x_n(k)}(x)}{\pi_{x_{N,\alpha}(k)}(x)} = \frac{\alpha x_n(k)^{\alpha} x^{-(\alpha+1)}}{\alpha x_{N,\alpha}(k)^{\alpha} x^{-(\alpha+1)}} = \frac{x_n(k)^{\alpha}}{x_{N,\alpha}(k)^{\alpha}},$$

which is independent of $x$. Observing from (23) and (24) that

$$x_n(k) \overset{d}{=} x_{N,\alpha}(k - 1) x_{N+1}(k) X_n(k), \quad n = 1, ..., N,$$

$$x_{N,\alpha}(k) \overset{d}{=} x_{N,\alpha}(k - 1) x_{N+1}(k) \left(\sum_{n=1}^{N} X_n(k)^{\alpha}\right)^{1/\alpha},$$

this random probability is also

$$\frac{X_n(k)^{\alpha}}{\sum_{n=1}^{N} X_n(k)^{\alpha}},$$

where, for each $k$ independently, the $X_n$s are i.i.d. Pareto$(\alpha)$ distributed on $(1, \infty)$. Because in each generation, parents generate offspring independently and independently of one another, upon averaging, the probability that, at generation $k + 1$, i
individuals share the same common ancestor is independent of $k$, with
\[ P_{i,1}^{(N)} := \sum_{n=1}^{N} E \left( \left( \frac{X_n(k)^\alpha}{\sum_{n=1}^{N} X_n(k)^\alpha} \right)^i \right) = NE \left( \left( \frac{X_1(k)^\alpha}{\sum_{n=1}^{N} X_n(k)^\alpha} \right)^i \right). \]

The $X_n(k)$s being i.i.d. Pareto($\alpha$) distributed rvs, the $X_n(k)$s are i.i.d. Pareto(1) distributed rvs and we are thus back to the results of Proposition 6 with $\beta = 0$, stating that with $c_N \sim 1/\log N$,
\[ c_N^{-1} P_{i,1}^{(N)} \to \int_0^1 u^{i-2} \Lambda(du) = \frac{\Gamma(2) \Gamma(i-1)}{\Gamma(i)} = \frac{1}{i-1}, \]
where $\Lambda \sim \text{beta}(1,1)$, uniform. We can proceed similarly to derive the probabilities $P_{i,j}^{(N)}$ that $i$ individuals have $j < i$ parents, behaving consistently with (16) with $\beta = 0$. We conclude that, whatever $\alpha$, in the large $N$ limit, the time-scaled genealogy of the branching model with selection is a Bolthausen-Sznitman coalescent process, obtained while sampling from $N$ Pareto(1) i.i.d. rvs. Would the sampling probabilities $P_{i,j}^{(N)}$ include a $\beta$-size biasing effect on total length $\sum_{n=1}^{N} X_n(k)^\alpha$, the genealogy of this branching model with selection would be a full beta($1-\beta$) coalescent, provided $\beta < 1$. While adopting this sampling point of view to compute the coalescence and merging probabilities, we therefore obtain a limiting genealogical coalescent process which is independent of $\alpha$.

**Genealogies from the output PPP.**

What now if we set that the occupation density that, at generation $k+1$, there is an offspring at $x$ descending from some individual with fitness $x_n(k)$ at generation $k$, is instead given by
\[ \pi_{x_n(k)}(x) := x_n(k) x^{-2}, \]
while distorting the original occupation intensity $\pi_{x_n(k)}(x) = x_n(k)^\alpha x^{-(\alpha+1)}$?

Then the occupation density that, at generation $k+1$, there is an offspring at $x$ descending from any individual of the whole population would take the form
\[ \pi_{x_{N,1}(k)}(x) := x_{N,1}(k) x^{-2}, \]
where $x_{N,1}(k) := \sum_{n=1}^{N} x_n(k)$ is the cumulative fitness in generation $k$.

If this were to be the case, given there is an offspring at $x$ at time $k+1$, the sampling probability that it is an offspring of the individual with fitness $x_n(k)$ would be, thanks to (23) and (24) with $\alpha = 1$:
\[ \frac{\pi_{x_n(k)}(x)}{\pi_{x_{N,1}(k)}(x)} = \frac{x_n(k) x^{-2}}{x_{N,1}(k) x^{-2}} = \frac{x_n(k)}{x_{N,1}(k)} \frac{d X_n(k)}{\sum_{n=1}^{N} X_n(k)}, \]
again independently of $x$. This probability now involves a normalized sum of the $X_n$s, which are i.i.d. Pareto($\alpha$) distributed and the strategy to compute the merging probabilities of the ancestral process will be modified.
Under this hypothesis indeed, the probability that, at generation \( k+1 \), \( i \) individuals share the same common ancestor reads

\[
P_{i,1}^{(N)} := \sum_{n=1}^{N} E \left( \left( \frac{X_n(k)}{\sum_{n=1}^{N} X_n(k)} \right)^i \right) = N E \left( \left( \frac{X_1(k)}{\sum_{n=1}^{N} X_n(k)} \right)^i \right) =: N E \left( S_1(k)^i \right),
\]

where \( S_1 \) is the normalized segment size now obtained from \( N \) i.i.d. Pareto(\( \alpha \)) distributed rvs, normalized by their sum \( \Sigma_N(k) := \sum_{n=1}^{N} X_n(k) \). We can proceed similarly to derive the probabilities \( P_{i,j}^{(N)} \) that \( i \) individuals have \( j < i \) parents and we are back to the studies of Sections 2 − 4. And we can as well \( \beta \)-size-bias these sampling probabilities on the total lengths \( \Sigma_N \). We call this sampling procedure the distorted sampling procedure.

**Proposition 17.** Looking backward in time, using a distorted size-biased sampling procedure, the genealogy of the branching model with selection is

- a continuous-time Kingman coalescent if \( \alpha \geq 2 \) (upon scaling time with \( c_N \propto 1/N \) if \( \alpha > 2 \) or \( c_N \propto \log N/N \) if \( \alpha = 2 \)).
- a continuous-time beta(\( 2−\alpha,\alpha−\beta \)) coalescent if \( \alpha \in (1, 2) \) and \( \beta < \alpha \) (upon scaling time with \( c_N \propto N^{−(\alpha−1)} \)).
- a continuous-time beta(1,1 − \( \beta \)) coalescent if \( \alpha = 1 \) and \( \beta < 1 \) (upon scaling time with \( c_N \propto 1/\log N \)).
- a discrete-time Poisson-Dirichlet(\( \alpha,−\beta \)) coalescent if \( \alpha \in (0, 1) \) and \( \beta < \alpha \).

**Proof:** It remains to interpret the distorted size-biased sampling procedure which is proposed to compute the merging probabilities of the ancestral process: Assume that the fitness dependent PPP describing the descent of an individual with fitness \( x_n(k) \) is now the output image of the original one, given by the canonical application \( f_{x_n(k)}(x) = x_n(k)(x/x_n(k))^{\alpha} \), \( f_{x_n(k)} : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \). Its gives rise to a new PPP with distorted intensity \( \pi_{f_{x_n(k)}}(dx) = x_n(k)x^{-2}dx \), the image measure of \( \pi_{x_n(k)}(dx) = \pi_{x_n(k)}(x)dx = tx_n(k)x^{-(\alpha+1)}dx \) by \( f_{x_n(k)} \) (as a result of the classical Campbell formula for PPPs, see [25] p. 28). This was the starting point in (28). The skewed computation in (29) of the probability that some offspring is descending from one individual with fitness \( x_n(k) \) is thus based not on the original PPP attached to \( x_n(k) \) but rather on a deformed version of it through \( f_{x_n(k)} \). We note that \( f_{x_n(k)}(x) \), as a function of the two arguments \( (x_n(k), x) \) is homogeneous with \( \lambda \alpha x_n(k) \left( \lambda^b x \right) = \lambda^{(1−\alpha)+bx}f_{x_n(k)}(x), \lambda > 0 \), leading obviously, if \( \lambda = x_n(k)^{−1/\alpha} \), \( b/\alpha = 1 \) and \( f(x) := f_1(x) = x^{\alpha} \) to: \( f_{x_n(k)}(x) = x_n(k)f(x/x_n(k)) \). The function \( f(x) = x^{\alpha} \) is the output fitness function introduced in Subsection 5.1, Remark(ii).

Using this distorted size-biased sampling procedure therefore, following the introductory arguments, the full class of the Pareto-coalescents (described in Sections 2 − 4) are obtained.

Suppose for instance \( \alpha \in (1, 2), \beta = 0 \). Based on the previous computations of Sections 2 − 4, we conclude that the large \( N \) distorted genealogy of the branching model with selection coincides (upon scaling time correspondingly: \( k \rightarrow [t/c_N] \)) with a beta(\( 2−\alpha,\alpha \)) coalescent process, obtained while sampling from \( N \) Pareto(\( \alpha \))
i.i.d. rvs, normalized by their sum. Would this probability involve a $\beta$—size biasing effect, the genealogy of this branching model with selection is identified to a beta$(2 - \alpha, \alpha - \beta)$ coalescent, $\beta < \alpha$.

If $\alpha \in [0, 1)$, $\beta < \alpha$, the obtained large $N$ genealogical coalescent will coincide with the discrete-time-$k$ Poisson-Dirichlet coalescent with parameters $\alpha$ and $-\beta$. Only when $\alpha = 1$ do we get as in [7] (upon scaling time logarithmically with $N$) the Bolthausen-Sznitman coalescent ($\beta = 0$) or more generally the beta$(1, 1 - \beta)$ coalescent, provided $\beta < 1$.

**Acknowledgments:** The author acknowledges partial support from the ANR Modélisation Aléatoire en Écologie, Génétique et Évolution (ANR-Manègne-09-BLAN-0215 project) and from the labex MME-DII (Modèles Mathématiques et Économiques de la Dynamique, de l’ Incertitude et des Interactions). The author is also indebted to his referees for pointing out some errors in an earlier version of the draft and for encouraging him to write down a more concise and complete version.

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