Cosmological Gravitomagnetism and Mach’s Principle

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The spin axes of gyroscopes experimentally define local non-rotating frames, i.e. the time-evolution of axes of inertial frames. But what physical cause governs the time-evolution of gyroscope axes? We consider linear perturbations of Friedmann-Robertson-Walker (FRW) cosmologies with \( k = 0 \), i.e. spatially flat. We ask: Will cosmological vector perturbations (i.e. vorticity or rotational perturbations) exactly drag the spin axes of gyroscopes relative to the directions of geodesics to quasars in the asymptotic unperturbed FRW space? Using Cartan’s formalism with local orthonormal bases we cast the laws of linear cosmological gravitomagnetism into a form showing the close correspondence with the laws of ordinary magnetism. Our results, valid for any equation of state and any form of the energy-momentum tensor for cosmological matter, are: 1) The dragging of a gyroscope axis by rotational perturbations of matter beyond the \( H \)-dot radius from the gyroscope is exponentially suppressed, where \( H \) is the Hubble rate, and dot is the derivative with respect to cosmic time. 2) If the perturbation of matter is a homogeneous rotation inside some radius around a gyroscope, then exact dragging of the gyroscope axis by the rotational perturbation is reached exponentially fast as the rotation radius gets larger than the \( H \)-dot radius. 3) For the most general linear cosmological perturbations the time-evolution of all gyroscope spin axes and the axis directions of all local inertial frames exactly follow a weighted average of the rotational motion of cosmological matter, i.e. there is exact frame-dragging everywhere. The weight function is the density of measured angular momentum of matter times (1/r) times the Yukawa force \((-d/dr)/(1/r) \exp(-\mu r))\), where \( r \) is the geodesic distance from the source to the gyroscope. The exponential cutoff is given by \( \mu^2 = -4(dH/dt) \). Except for the Yukawa cutoff the weight function is the same as in the integrated form of Ampère’s law. — Our results demonstrate (in first-order perturbations of any type for FRW cosmologies with \( k = 0 \)) the validity of Mach’s hypothesis that axes of local non-rotating frames precisely follow an average of the motion of cosmic matter.

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I. INTRODUCTION

A. The observational fact

In tests of general relativity in the solar system two types of things are compared: 1) On the one hand measurements of the precession of perihelia or of gyroscopes’ spin axes in Gravity Probe B relative to distant stars and quasars. 2) On the other hand the predictions from the solutions of Einstein’s equations for the solar system in asymptotic Minkowski space, which do not contain distant stars or quasars. In this comparison the assumption is made (often not spelled out) that in the solution of Einstein’s equations for the solar system the asymptotic Minkowski space is nonrotating relative to distant stars and quasars. This implicit assumption is tested to high accuracy by the comparison of the observed perihelion precession with the prediction of general relativity. This implicit assumption is a basic observational fact, which has been called “Mach 0” e.g. in \( \text{[2]} \). — Distant stars and quasars have proper motions relative to the uniform Hubble flow, but the angular parts are unmeasurable today. — We conclude that the measurements of perihelion shifts and the measurement undertaken by Gravity Probe B are tests of two things combined, on the one hand tests of Einstein’s equations, on the other hand tests of the principle “Mach 0”.

No explanation for the observational fact “Mach 0” is given in classical mechanics, special relativity, and general relativity for isolated systems in asymptotic Minkowski space, except by invoking an accident of initial conditions. In these theories one could have different initial conditions, where distant stars and quasars could be orbiting around us relative to our gyroscopes. — Within these three theories (and in the generic case of general relativity) the local non-rotating frames (axis directions of inertial frames) can be experimentally determined by axes of gyroscopes. Conversely the time evolution of gyroscope axes is dictated by the laws of inertia, i.e. the gyroscope axes cannot rotate with respect to local inertial axes. This is self-consistent: One uses two gyroscopes (with spin axes not aligned) to experimentally construct, i.e. operationally define, the local non-rotating frame, and one uses all other local gyroscopes and other experiments (e.g. Foucault’s pendulum) to test the laws of inertia with respect to rotations. — But the question remains: What physical cause governs the time-evolution of the axes of inertial frames and of gyroscopes? These three theories do not contain such a physical cause, quite to the contrary they contain an absolute element, namely the non-rotating frame in classical mechanics and special relativity, and the asymptotic non-rotating frame.
in general relativity for isolated systems in asymptotic Minkowski space. This absolute element is experimentally established by Newton’s bucket experiment and today by gyroscopes or the Sagnac effect in inertial guidance systems. This absolute element is equivalent (with respect to rotations) to Newton’s absolute space.

B. Mach’s Principle

In the 1870’s and 1880’s Ernst Mach stated forcefully the hypothesis (as an alternative to Newton’s absolute space) that the axes of non-rotating frames (i.e., axes of gyroscopes) in their time-evolution are determined by, are exactly dragged by, precisely follow “some average” of the motion of matter in the universe. Mach’s formulation is given in [2]. See also [3]. Today one generalizes “matter” to “matter-energy”, i.e. the principle states that there is exact frame-dragging for the axis directions of inertial frames by “some average” of the energy currents in the universe. This is what we take as the formulation of Mach’s principle at the level of linear cosmological perturbations. At the level of linear perturbations gravitational waves do not carry energy in the sense of a pseudo-energy-momentum tensor. — Mach’s Principle in this formulation follows from general relativity for linear cosmological perturbations of FRW universes with $k = 0$, as will be demonstrated by our result Eq. (2).

Many alternative versions have been proposed under the name of Mach’s principle by other authors after Mach [4]. We shall discuss Einstein’s proposal of 1918 at the end of sect. XII. An incomplete list of ten inequivalent versions has been given and briefly discussed in [2]. Six of these versions are definitely not satisfied for Einstein gravity in a cosmological context, and these six versions have absolutely no counterpart in the writings of Mach. Another version is satisfied in Einstein gravity, even in a non-cosmological context, but it is too weak: partial dragging as opposed to exact dragging, i.e. inertial frames influenced as opposed to fully determined by the motion of cosmic matter. Because of the many conflicting versions proposed under the name of Mach’s principle by other authors after Mach, some physicists have been under the impression that the issue is “confusing and ill-defined”. — The version, “the theory contains no absolute elements,” explained by J. Ehlers in [10], is intimately connected to Mach’s starting point (no absolute space) and formulated in a modern way. This version is a general requirement, but it is not a specific implementation in contrast to our result Eq. (2), which gives an explicit demonstration of how Mach’s principle is implemented within general relativity.

C. Gravitomagnetism

At the time of Mach there was no known mechanism by which matter in the universe could influence the motion of gyroscopes axes. With General Relativity came the needed mechanism, gravitomagnetism. Thirring in 1918 considered a rotating infinitely thin spherical shell with uniform surface mass density and total mass $M$, and he analyzed the partial dragging of the axes of inertial frames inside the rotating shell for points near the origin. In the weak field approximation, $G_N(M/R)_{shell} \ll 1$, and to first order in the angular velocity $\Omega_{shell}$ he found that the axes of local inertial systems (and gyroscopes) at all points near the origin rotate relative to asymptotic Minkowski space with the same angular velocity $\Omega_{gyro} = f_{drag}(\Omega_{shell})$. For the dragging fraction $f_{drag}$ he obtained $f_{drag} = 4/3 G_N(M/R)_{shell} \ll 1$.— Gravitomagnetic effects in the solar system are exceedingly small. The Lense-Thirring effect by the rotating earth on the spin axes of gyroscopes on Gravity Probe B is predicted to be only 43 milli-arc-sec per year. — Brill and Cohen and Lindblom and Brill analyzed dragging by spherical shells to first order in the angular velocity but to all orders in $G_N(M/R)_{shell}$. They found that inside the shell one has a Minkowski space which rotates relative to asymptotic Minkowski space. In the limit of $(M/R)_{shell}$ approaching the value for a black hole they found perfect dragging of inertial axes inside the rotating shell.

Exact dragging by the masses in the universe is required by Mach’s principle, not a small influence. Therefore one must go to cosmological models. A simple, cosmologically relevant model is a uniform rotational motion of cold matter with $\Omega_{matter}$ and mass density $\rho$ both constant out to a radius $R_{rot}$ around a gyroscope and $\Omega_{matter} = 0$ outside $R_{rot}$. Starting with a weak field perturbation on a Minkowski background this simple model is a superposition of a sequence of Thirring shells. This gives a dragging fraction which grows quadratically in $R_{rot}$, $f_{drag} = 2G_N(M/R)_{rot} = (8\pi/3)G_NM_{rot}R_{H}$. The Friedmann equation for $k = 0$ reads $H^2 = (8\pi/3)G\rho$, hence $2G_NM_{H}/R_{H} = 1$, where $M_{H}$ is the mass inside the Hubble radius $R_{H}$. This gives $f_{drag} = R_{rot}^2/R_{H}^2$. Although the weak field perturbation on a Minkowski background implies $f_{drag} = 2G_N(M/R)_{rot} \ll 1$, applying the formula beyond its region of validity would give $f_{drag} \rightarrow 1$ for $R_{rot} \rightarrow R_{H}$, a “satisfying degree of self-consistency” in the words of Misner, Thorne, and Wheeler, and $f_{drag} \gg 1$ for $R_{rot} \gg R_{H}$, the problem of ‘overdragging’. What is needed is an analysis using cosmological perturbation theory including superhorizon scales. This is given in this paper.

II. SUMMARY AND CONCLUSIONS

In this paper we analyze realistic cosmological models. This is in contrast to toy models involving one or many shells (embedded in FRW models), infinitely thin and rotating around axes through one point, which are contained in [11, 12, 13]. In models [11, 12] “almost all the matter of the universe” must be “redistributed on
In order to make frame dragging almost perfect, i.e. these toy models are extremely far from physical cosmology. Also we consider realistic cosmological matter of any type as opposed to the contrived energy-momentum tensors discussed in the literature. We start from a Friedmann-Robertson-Walker (FRW) cosmology which is spatially flat (i.e., $k = 0$). We first add the most general linear cosmological perturbations in the vorticity sector (vector perturbations), i.e., perturbations derived from divergenceless vector fields. We ask: Will rotational motions of cosmological matter exactly drag the axes of any gyroscope relative to the directions of geodesics from the gyroscope to quasars in the asymptotic unperturbed FRW space? In Sec. XII we also include scalar and tensor perturbations.

A conference paper about our work appeared in [18] and a preliminary version of a full article in [19]. A reaction [20] to our conference paper will be discussed in sect. VIII.

Our first result is that the dragging of axes of gyroscopes (i.e., axes of local inertial systems) by matter beyond the H-dot radius $R_{H/dot}$ is exponentially suppressed, where $R_{H/dot} \equiv (-dH/dt)^{-1/2}$, $H$ is the Hubble rate, and dot is the derivative with respect to cosmic time. In this sense physics here is decoupled from the asymptotic FRW universe.

Our second result refers to the simple model discussed above (uniform rotation of matter inside a radius $R_{rot}$ around a gyroscope). It states that the dragging fraction $f_{drag}$ approaches the value 1, i.e. exact dragging, exponentially fast as $R_{rot}$ gets larger than the H-dot radius. The dragging fraction as a function of $R_{rot}$ is shown in Fig. 1. The problem of ‘overdragging’ discussed in [14] is removed by the exponential suppression at super-$H$-dot scales, which is missing in [16] and other papers.— For linear cosmological perturbations both $\Omega_{\text{matter}}$ and $\Omega_{\text{gyro}}$ are infinitesimally small, but their ratio approaches the value 1 for $R_{rot}$ much larger than the H-dot radius.

Our third result concerns the most general vorticity perturbation in linear approximation. To understand the result one needs two crucial inputs from cosmological perturbation theory in the vector sector (see Sec. III):

1. The slicing of space-time in slices $\Sigma_t$ in the vorticity sector is unique, as emphasized by Bardeen [21].

2. The intrinsic geometry of 3-space remains unperturbed, i.e. for vorticity perturbations of FRW with $k = 0$ each slice $\Sigma_t$ remains a Euclidean 3-space with its global parallelism.

Our result states what specific average of the energy flow $\vec{J}$ in the universe determines the precession rate $\bar{\Omega}_{\text{gyro}}$ of gyroscopes,

$$\bar{\Omega}_{\text{gyro}} = \frac{1}{2} B_{\vec{r}}(r = 0) = 2G_{N} \int d^3r' \left\{ \frac{1}{r} [\vec{r}' \times \vec{J}_{\vec{r}}(\vec{r}')] \right\} \left( \frac{\partial}{\partial r'} \frac{e^{\mu r'}}{r'} \right), \quad (1)$$

where $\vec{r}'$ is the position of the source relative to the gyroscope. $\vec{J}_{\vec{r}}$ is the energy current, with $\vec{J}_{\vec{r}} = (\rho + p) \vec{v}$ for perfect fluids, and $B_{\vec{r}}$ is the gravitomagnetic field, of which the general operational definition is given in section [13]. All quantities in Eq. (1) are directly measurable, i.e. they have a gauge invariant meaning, as is explained after Eq. (12).— The weight function is the density of measured angular momentum of matter times $(1/r)\text{Y}_{\nu}(r)$. The exponential suppression is determined by $\mu = -4(dH/dt) = 16\pi G_{N}(\rho + p)$. Except for the exponential factor the weight function is the same as in the integrated form of Ampère’s law for ordinary magnetism.

Note the fundamental difference between cosmological gravitomagnetism, Eq. (11), and Ampère’s law. The latter does not hold in a rotating reference frame, unless one introduces fictitious forces. In contrast Eq. (11) remains valid, as it stands, in a frame which is rotating with an angular velocity $\vec{\Omega}$ relative to asymptotic quasars. This holds because both sides of Eq. (11) change by the same amount: On the right-hand side, $(\vec{r}' \times \vec{J}_{\vec{r}})$ changes by $(\rho + p)[\vec{r}' \times (\vec{\Omega} \times \vec{r}')]$. Carrying out the integration gives $\vec{\Omega}$ with a prefactor 1 for the change on the right-hand side, which is equal to the change on the left-hand side. — The fact that Eq. (11) holds in any frame which is rotating relative to asymptotic quasars establishes that the asymptotic inertial frame has no influence in cosmological gravitomagnetism; the time-evolution of local inertial axes is determined exclusively by the weighted average of cosmological matter flows.

Eq. (11) can be rewritten to show explicitly that the precession of any gyroscope, $\bar{\Omega}_{\text{gyro}}$, exactly follows, i.e.
is exactly dragged by a weighted average of \( \Omega_{\text{matter}} \) with a weight function \( W(r_{PQ}) \), where \( P \) and \( Q \) are the positions of the gyroscope and the source. For reasons of symmetry under rotations and space reflection, the velocity field on a shell of a given radius \( r_{PQ} \) can only contribute to the gyroscope’s precession through its term with \( \ell = 1 \), odd parity sequence, which is equivalent to a rigid rotation with the angular velocity \( \Omega_{\text{matter}}(r) \).

We obtain

\[
\Omega_{\text{gyro}} = \int_0^\infty dr \, \Omega_{\text{matter}}(r) \, W(r),
\]

\[
W(r) = \frac{1}{3} \mu^2 r^3 Y_\mu(r).
\]

This is our most important result. It shows that \( \Omega_{\text{gyro}} \) is the weighted average of \( \Omega_{\text{matter}} \), i.e. the evolution of inertial axes exactly follows the weighted average of cosmic matter motion. The weight function \( W(r) \) has its normalization (integral over \( r \)) equal to unity, as it must be for an averaging weight function in any problem. That the integral of the weight function is equal to unity crucially depends on the exponential cutoff in the Yukawa force \( Y_\mu(r) \). Details about Eqs. \( 2 \) are given in the paragraph with Eqs. \( 17 \) and \( 18 \).— Our results (valid for linear perturbations of a FRW universe with \( k = 0 \) and for any equation of state), particularly Eq. \( 2 \), are a clear demonstration of how Mach’s principle is implemented within cosmological general relativity.

Our aim is to obtain the laws of linearized cosmological gravitomagnetism in a form which shows the correspondence with electromagnetism in a \( (3+1) \)-dimensional split. In sect. \( IV \) we give the general operational definition for the gravitoelectric field \( \hat{E}_g \) and the gravitomagnetic field \( \hat{B}_g \) via measurements by any choice of fiducial observers (FIDOs). These operational definitions are generally valid, i.e. beyond perturbation theory. The operational definitions need Cartan’s formalism with local orthonormal bases (LONBs) and the corresponding fiducial observers. For our specific problem we choose FIDOs with spatial basis vectors fixed to directions of geodesics on \( \Sigma_t \) from the FIDO to quasars in the asymptotically unperturbed FRW universe. This construction is clearly independent of the chosen coordinate system, i.e. it is a gauge-invariant construction.

In sect. \( V \) we first give the general method to compute connection coefficients for local orthonormal bases in Cartan’s formalism. We apply this general method to obtain the connection coefficients for linear cosmological gravitomagnetism and the equations of motion for free-falling test particles.

In sect. \( VI \) we compute curvature in Cartan’s formalism. The crucial equation for cosmological gravitomagnetism is Einstein’s \( G_\mu\nu \)-equation for vorticity perturbations, the momentum constraint. It has the form of Ampère’s law, except that for vorticity perturbations of FRW space there is an additional term \(-4(dH/dt)\hat{A}_g\), which is responsible for the exponential suppression of super-\( H \)-dot scales,

\[
curl \hat{B}_g - 4(dH/dt)\hat{A}_g = -16\pi G_N \hat{J}_e.
\]

In contrast to the Ampère-Maxwell equation of ordinary electrodynamics, the Maxwell term \( \partial_t \hat{E} \) is absent in the time-dependent context of gravitomagnetodynamics; see the comments after Eq. \( 36 \). The solution of Eq. \( 1 \) is given in Eq. \( 11 \).

In sect. \( VII \) we give a detailed discussion of our three main results for Mach’s principle, which were summarized above.— We then discuss Einstein’s objection: “Mach conjectured that inertia would have to depend upon the interaction of masses ... and their interactions as the original concepts. The attempt at such a solution does not fit into a consistent field theory.” We point out the remarkable fact that the solution of Einstein’s local field equation for \( G_{0i} \), i.e. of the momentum constraint equation \( 6 \), has the form of an instantaneous action-at-a-distance, Eq. \( 11 \). See also refs. \[13\], \[22\].

In sect. \( VIII \) we explain that the density of measured angular momentum, formed from the LONB components \( T_{0i}^0 \), is the relevant input for Mach’s principle on the right-hand side of Einstein’s equation, and not the density of conserved angular momentum, e.g. the coordinate-basis component \( T_{0i}^0 \) which was considered by other authors, e.g. \[16\]. This is the reason why these and other authors did not obtain the Yukawa suppression beyond the \( H \)-dot radius. This Yukawa suppression is crucial for obtaining the normalization to unity in the weight function \( W(r) \) in Eq. \( 2 \), and it is crucial to remove the problem of ‘overdragging’.

In sect. \( IX \) we discuss Einstein’s objection “If you have a tensor \( T_{\mu\nu} \) and not a metric, then this does not meaningfully describe matter. ... The statement that matter by itself determines the metric ... is meaningless.” From this objection Einstein drew the conclusion that “one should no longer speak of Mach’s principle.”— His objection applies to coordinate-basis components \( T_{\mu\nu} \), but we explain why it does not apply to LONB components \( T_{ab} \). LONB components are determined directly by matter measurements using only \( n_{ab} = \text{diag}\{-1,+1,+1,+1\} \), which is available before having solved Einstein’s equations. In contrast \( T_{\mu\nu} \) needs matter measurements plus \( g_{\mu\nu} \), which is available only after having solved Einstein’s equations.

In sect. \( X \) we explain that the vanishing of local vorticity measured by non-rotating observers is not relevant as a test of Mach’s principle, because Mach’s principle requires an average over all matter in the universe with a given weight function. Therefore we conclude that the claim by Ozsváth and Schücking that their Bianchi IX model violates Mach’s principle is not conclusive.

In sect. \( XI \) we derive the boundary conditions needed when solving Einstein’s equations for a system in asymptotic Minkowski geometry, if one does not restrict the allowed coordinate systems to those which are asymptotically non-rotating. The boundary conditions are needed
to encode the effects of sources outside the system considered, the effects of cosmological sources. — For linear vorticity perturbations of FRW cosmologies no boundary conditions of the encoding type are needed because of the exponential suppression of super-\(H\)-dot scales. This result is in contrast to Einstein’s conclusion that the problem of inertia and of boundary conditions could only be solved if the universe is spatially closed.

In sect. XII we add scalar and tensor perturbations, and give the mathematical expression for Mach’s principle in this more general case. — We discuss Einstein’s formulation of Mach’s principle in 1918 \[3\]. For linear cosmological perturbations Einstein’s formulation is (1) too strong to be valid, as has been stated many times (gravitational waves can exist without matter), and (2) unnecessarily strong, as we shall explain, for Mach’s original purpose (the dragging of axis directions of inertial frames).

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Bardeen’s gauge-invariant combination $\Psi$ is defined by $\Psi \equiv B^{(1)} - \frac{i}{\kappa} H^{(1)}_T$ (in Bardeen’s notation). $B^{(1)}$ is the shift amplitude, and $H^{(1)}_T$ is the amplitude of the perturbation of the 3-metric chosen for the unperturbed 3-geometry. Evidently the gauge-invariant amplitude $\Psi$ is equal to the shift amplitude $B^{(1)}$ in the gauge which is fixed by $H^{(1)} = 0$, i.e. in the gauge, where (3) $g_{ij}$ is unperturbed. Bardeen emphasized in [26]: “Much has been made by some authors of the advantages of using gauge-invariant variables. However ...” We now transcribe Bardeen’s formulation [26] from the scalar sector to the vector sector: “The advantages of the gauge-invariant variable $\Psi$ are the advantages of working with the shift amplitude $B^{(1)}$ in the gauge where (3) $g_{ij}$ is unperturbed, no more and no less.”

The fact stated in the preceding sentence expressed in our notation: The shift vector $\beta^i$ in the Cartesian gauge (for vector perturbations of FRW with $k = 0$) is equal to Bardeen’s gauge-invariant variable $\Psi$ times his vector harmonic $Q^{(1)}(x)$ (for a given wave number and polarization) apart from the sign discussed above,

$$\beta^i(x, t) = A^i_e = -\Psi(t)Q^{(1)}(x). \quad (8)$$

Our gravito-magnetic vector potential $\vec{A}_g$ is directly proportional to Bardeen’s gauge-invariant combination $\Psi$. Working with our gauge, i.e. with comoving Cartesian (or spherical) coordinates for Euclidean 3-space, is totally equivalent to working with Bardeen’s gauge-invariant potential $\Psi$.

**IV. FIDUCIAL OBSERVERS AND GRAVITOMAGNETIC FIELD**

Our aim is to obtain the laws of linearized gravito-magnetism in a formulation analogous to electromagnetism with a split to $(3 + 1)$-dimensions. What is the operational definition of the gravitoelectric field $\vec{E}_g$ and gravitomagnetic field $\vec{B}_g$? According to the equivalence principle for a free-falling, non-rotating observer there are no gravitational forces at his position, $\vec{E}_g = 0$, $\vec{B}_g = 0$. It all depends on the choice of fiducial observers (FIDOs) with their local orthonormal bases (LONBs), see Thorne et al [27]. For FIDOs which are not free-falling and/or which are rotating (relative to local gyroscope axes) there are gravitational forces (inertial forces, fictitious forces, e.g. the Coriolis force).

The operational definitions of $\vec{E}_g$ and $\vec{B}_g$ are independent of perturbation theory. They involve a family of world lines of fiducial observers, FIDOs, of any given choice, such that through each space-time point $P$ there passes precisely one world line, i.e. the family of world lines forms a congruence. The measurements of a FIDO are directly encoded by the components (of momentum $\vec{p}$ and of gyroscope spin $\vec{S}$) with respect to the FIDO’s local orthonormal basis, where $\vec{e}_0(P) = \vec{u}_{\text{FIDO}}(P)$, and $\vec{e}_i(P)$ are given by any definite choice. Hats always refer to LONBs, and bars designate space-time vectors. — To convert local measurements by a FIDO to LONB-components $p^a$ and $S^a$ one only needs $\eta_{ab} = \text{diag}\{-1, +1, +1, +1\}$, but one does not need $g_{\mu\nu}$, which is not available before having solved Einstein’s equations. Therefore it is necessary to abandon vector components in a coordinate basis, to work with LONBs, and to use the formalism of Élie Cartan [24, 25].— Measuring $\{\vec{E}_g, \vec{B}_g\}$ involves FIDOs measuring first time-derivatives along their world lines, on the one hand of the momentum components $p_i$ of free-falling quasistatic test particles, and on the other hand of the spin components $S_i$ of gyroscopes carried along by the FIDOs,

$$\frac{d}{dt}p_i \equiv m\vec{E}_g^i \quad \text{free-falling quasistatic test particle,} \quad (9)$$

$$\frac{d}{dt}S_i \equiv -\frac{1}{2}[\vec{B}_g \times \vec{S}]_i \quad \text{gyro comoving with FIDO,} \quad (10)$$

where $t$ is the local time measured by the FIDO. Arrows denote 3-vectors in the tangent spaces spanned by the spatial legs of LONBs. $\vec{E}_g \equiv \vec{g}$ is the gravitational acceleration of free-falling quasistatic test particles relative to the FIDO. — The operational definitions of Eqs. (9) and (10) are the same as for a classical charged spinning test particle in an electromagnetic field except that the charge $q$ is replaced by $m$, and the gyromagnetic ratio $q/(2m)$ is replaced by $1/2$. Eq. (11) gives the angular velocity of precession of the gyroscope’s spin axis relative to the axes of the FIDO,

$$\vec{\Omega}_\text{gyro} \equiv -\frac{1}{2}\vec{B}_g, \quad (11)$$

which is an equivalent operational definition of $\vec{B}_g$. — For gyroscopes carried along by FIDOs the magnitude of $\vec{S}$ is constant, and for non-rotating FIDOs the components of the vector $\vec{S}$ are constant, because a gyroscope experiences no mechanical torques by construction and no gravitational torques by the equivalence principle.

From the operational definitions of $\{\vec{E}_g, \vec{B}_g\}$ it follows immediately that these fields are identical (apart from an overall sign) to the connection coefficients $(\omega^a_b)_{\parallel}$ for a displacement along the FIDO’s world line. — The connection 1-forms $\omega^a_b$ resp. their components in LONBs (= Ricci rotation coefficients), $(\omega^a_b)_\parallel$, are defined by

$$\nabla_a \vec{e}_b = \vec{e}_c(\omega^c_b)_a. \quad (12)$$

In words: the Ricci rotation coefficients $(\omega^a_b)_\parallel$ give the rotation resp. the Lorentz boost $(\omega^a_b)$ of the LONBs under parallel transport along $\vec{e}_a$. Parallel transport of $\vec{u} = \vec{e}_0$ along $\vec{u}$ is given by free fall, parallel transport for $\vec{e}_i$ is given by gyroscope axes. Relative to FIDOS the equations of motion for free-falling test particles (geodesic equation) and for spin axes of gyroscopes
With Eqs. (13) the operational definitions Eqs. (9, 11) are translated into the equivalent definitions involving connection coefficients with a displacement index \( \theta \), namely a Lorentz boost \( \omega_{\theta} \) per unit time (acceleration \( \ddot{\vec{g}} = \dddot{E}_g \)) resp a rotation angle \( \omega_{ij} \) per unit time (angular velocity \( \Omega_{ij} = -\frac{1}{2} B^i_{j} \)),

\[
(\omega_{\theta i})_0 = -E^i_0, \quad (\omega_{ij})_0 = -\frac{1}{2} B^i_{j},
\]

where \( B_{ij} \equiv \varepsilon_{ijk} B_k \), \( \Omega_{ij} \equiv \varepsilon_{ijk} \Omega_k \), and \( \Omega_{ij} = (\omega_{ij})_0 \). The last equation (angular velocity) gives the motivation for using the letter \( \omega \) for Ricci rotation coefficients (connection coefficients). Only displacements along \( \vec{a} \) appear in Eq. (13), because for quasistatic test particles (typically initially at rest at the position of the FIDO) the spatial separation from the FIDO grows only quadratically in time.

Our choice of FIDOs: The world lines of our FIDOs are at fixed \( x^i \) in our coordinates, which are fixed to quasars in the asymptotic FRW universe. The 3-velocity of our FIDOs relative to the normals on \( \Sigma_t \) is equal to the shift 3-vector \( \beta^i \) in Eq. (14).— The construction of the world-lines of FIDOS is equivalent to the gauge-invariant statement that our FIDOS are at rest in the Hubble frame given by asymptotic quasars on \( \Sigma_t \).— We choose the spatial basis vectors of our FIDOS, \( \vec{e}_i(\vec{P}) \), fixed to directions of geodesics on \( \Sigma_t \) from \( \vec{P} \) to quasars in the asymptotic FRW universe, again a gauge-invariant statement. Hence we fix \( \vec{e}_i(\vec{P}) \) to \( \vec{P} / \partial x^i \), i.e. in 4-space the directions of \( \vec{e}_i \) and \( \vec{e}_i \) differ by a pure Lorentz boost, see Eq. (15) below.

V. CONNECTION COEFFICIENTS AND EQUATIONS OF MOTION FOR MATTER IN GRAVITOMAGNETISM

Cartan’s formalism with local orthonormal tetrads is crucial for gravitomagnetism, because (1) the operational definitions of the gravito-electric and gravito-magnetic fields (independent of perturbation theory) are given by measurements of the corresponding FIDOS, Eqs. (9) - (11), and (2) the measured LONB-components of the gravito-electric and gravito-magnetic fields are identical to connection coefficients in the local orthonormal basis of the FIDO, Eqs. (13). A third reason, why it is necessary to work in Cartan’s formalism with local orthonormal tetrads, will be presented in section IX.

The computation of connection coefficients in Cartan’s formalism: We give a short, self-contained derivation of the formulae needed from Cartan’s formalism with LONB-tetrads. This should be helpful for readers only familiar with the literature on cosmological perturbation theory and with standard one-year graduate courses on General Relativity. Readers not interested in calculational methods can go directly to the results starting with Eq. (22).

A. Cartan’s first equation in LONB components

The first step is to express our choice of LONBs \( \vec{a}(\vec{P}) \), given at the end of the previous section, in terms of the coordinate bases \( \vec{e}_a(\vec{P}) = \partial / \partial x^a \), i.e. \( \vec{e}_\alpha = (e_\alpha)^a \vec{e}_a \). To first order in \( (\beta^i / c) \)

\[
\vec{e}_0 = \vec{e}_0, \quad \vec{e}_k = \frac{1}{ahk}(\vec{e}_k + \beta k \vec{e}_a).
\]

Since the spatial LONBs point in the same spatial directions as the spatial coordinate bases, we use latin letters from the middle of the alphabet both for spatial LONBs (with hat) and for spatial coordinate bases (without hat). The dual bases (basis 1-forms) \( \hat{\vec{a}} \) for LONBs are defined by \( \langle \hat{\vec{a}}, \vec{e}_b \rangle = \delta^a_b \), where \( \delta \) designates space-time 1-forms. The LONB 1-forms \( \hat{\vec{a}} \) are expanded in the coordinate basis 1-forms, \( \theta^a = dx^a \), i.e. \( \theta^a = (\theta^a)_\alpha \theta^\alpha \), by

\[
\hat{\vec{a}}^0 = \gamma^0 - \beta_k \hat{\vec{a}}^k, \quad \hat{\vec{a}}^k = a\gamma_k \hat{\vec{a}}^k.
\]

The coefficients of the inverse expansion, i.e. coordinate bases in terms of LONBs, are \( (e_\alpha)_a = (\theta^a)_\alpha \) resp. \( (\theta^a)_\alpha = (e_\alpha)^a \).

The second step is computing the exterior derivative \( d \) of the basis 1-forms, \( (d\theta^a)_{\alpha\beta} = \partial_\alpha (\theta^\beta)_\beta - \partial_\beta (\theta^\alpha)_\alpha \), where \( [\alpha\beta] \) must be in the coordinate basis. Then one converts to components \( [\vec{a} \vec{b}] \) in the LONB,

\[
(d\theta^a)_{\vec{a}\vec{b}} = -C_{\vec{a}\vec{b}}^\vec{c} \langle e_a(\theta^\alpha)_\beta \rangle = -(\omega^a_j \beta)[(e^a_j + (\theta^a)c][\theta(\alpha)_\beta]
\]

The coefficients \( C_{\vec{a}\vec{b}}^\vec{c} \) are identical to the commutation coefficients of the basis vectors,

\[
[e_a, e_b] \equiv C_{\vec{a}\vec{b}}^\vec{c} e_c.
\]

This is easily shown by using \( \partial_\alpha (\hat{\vec{a}}^0, \hat{\vec{a}}^k) = 0 = [\partial_\alpha (\theta^a)_\beta (e^a_j + (\theta^a)c][\theta(\alpha)_\beta]
\]

The third step is obtaining the connection coefficients from the commutation coefficients. The definition of the connection via basis 1-forms is \( (\nabla_\alpha \theta^a)_\beta = -(\omega^a_j \beta)(\theta(\alpha)_\beta)
\).

We take both the displacement index \( \alpha \) and the equation’s component index \( \beta \) in the coordinate basis, and we antisymmetrize in \( [\alpha\beta] \). This makes the Christoffel symbols \( \Gamma_{\alpha\beta\gamma} \) on the left-hand side disappear, since they are symmetric in \( \alpha, \beta \). Hence the left-hand side reduces to \( (d\theta^a)_{\alpha\beta} \). Dropping the equation’s component indices \( [\alpha, \beta] \) gives

\[
\ddot{d} \theta^a = -\omega^a_j \theta^j \theta^d.
\]
This is Cartan’s first equation. The wedge product (exterior product) of two 1-forms \( \tilde{\sigma} \) and \( \tilde{\rho} \) is \((\sigma \wedge \rho)_{ab} = \sigma_a \rho_b - \sigma_b \rho_a\). Taking Cartan’s first equation in LONB components gives the commutation coefficients, Eq. (17), on the left-hand side, and the right-hand side simplifies in LONB because \( \theta^i_h = \delta^i_h \). Hence Cartan’s first equation in LONB components is

\[
C^i_{ab} = (\omega^i_h)_a - (\omega^i_h)_b.
\]

This equation is easily solved for the rotation coefficients,

\[
(\omega^i_h)_a = \frac{1}{2} [C^i_{hba} + C^i_{hab} - C^i_{bae}],
\]

B. Connection coefficients for vorticity perturbations on a Minkowski background

To first order in \( \beta^i \) and with Cartesian spatial coordinates on \( \Sigma_t \), the commutation coefficients are very simple to compute, because only \( \theta^i_h = -\beta^i \) is space-time dependent, and the prefactors in Eq. (17), \( (e^a)_i \) and \( (e^b)_j \), can be set to 1. Hence \( C^i_{ab} = (d\beta)_{ab} \), and

\[
(\omega^i_h)_0 = - E^i_j = \partial_t \beta^i,
(\omega^i_j)_0 = - \frac{1}{2} B^i_j = - \frac{1}{2} (d\beta)_{ij} = (\omega^i_0)_j,
(\omega^i_j)_k = 0.
\]

All connection coefficients with respect to LONB’s are directly measurable (in contrast to Christoffel symbols, which refer to coordinate bases).

C. Connection coefficients for vorticity perturbations on FRW with \( \{k = 0, \pm 1\} \)

It is again straightforward to compute the commutation coefficients and the connection coefficients using Eqs. (17), (19), (21),

\[
(\omega^i_0)_i = - E^i_j = \frac{1}{a} \partial_t (a\beta^i),
(\omega^i_j)_0 = - \frac{1}{2} B^i_j = - \frac{1}{2} (d\beta)_{ij},
(\omega^i_0)_j = - \frac{1}{2} B^i_j + \delta^i_j H,
(\omega^i_j)_k = \delta^i_k (H \beta^j + \partial_t L_i) - \delta^i_k (H \beta^j + \partial_j L_i),
\]

where \( L_i = \log h_i \), \( \partial_t (a h_i)^{-1} \partial_i \), and \( H \) is the Hubble rate. In Eq. (20) there is no summation over repeated indices on the right-hand side.— The Hubble term in Eq. (25) follows directly by comparing the operational definitions of the Hubble rate and of the connection coefficients.

Since we work to first order in the vorticity perturbations (i.e. in \( \beta^i \)), we can identify \( \tilde{E}_g \) and \( \tilde{B}_g \) with vectors in the tangent spaces to \( \Sigma_t \). From Eqs. (23), (24), we see that the shift vector \( \tilde{\beta} \) must be identified with the gravitomagnetic vector potential \( \tilde{A}_g \). From Eqs. (20), (24) follow

\[
g_{0i} = \beta_i = (A^i_g)_i \quad (27)
\]

\[
\tilde{B}_g = \text{curl} \tilde{A}_g, \quad \tilde{E}_g = - \frac{1}{a} \partial_t (a \tilde{A}_g), \quad \text{curl} \tilde{E}_g + \frac{1}{a^2} \partial_t (a^2 \tilde{B}_g) = 0. \quad (29)
\]

Equations (28) and (29) are identical with the homogeneous equations for electromagnetism in FRW space-times with \( k = 0, \pm 1 \).

D. Equation of motion for free-falling test particles

The equation of motion (geodesic equation) for free-falling test particles of arbitrary velocities \( v \leq c \) (e.g. photons) in linear vorticity perturbations on a Minkowski background reads

\[
\frac{d}{dt}(p_i) = \varepsilon [\tilde{E}_g + (v \times \tilde{B}_g)]_i. \quad (30)
\]

This is identical with the Lorentz law for electromagnetism, except that the charge \( q \) is replaced by the energy \( \varepsilon \) of the test particle, \( \varepsilon = m_0 (1 - \beta^2)^{-1/2} \) for massive, and \( \varepsilon = h \nu \) for photons. With Eq. (11) and in a stationary gravitomagnetic field (\( \tilde{E}_g = 0 \)), Eq. (30) becomes

\[
\frac{d}{dt}(p_i) = - 2\varepsilon [\tilde{v} \wedge \tilde{f}_\text{gyro}]_i, \quad \text{the Coriolis force law.}
\]

Note that \( \tilde{f}_\text{gyro} \) is minus the rotation velocity of the FIDO relative to the gyroscopes’ axes. A homogeneous gravitomagnetic field can be transformed away completely by going to FIDOs which are nonrotating relative to the spin axes of local gyroscopes, i.e. in homogeneous rotation relative to the previous FIDOs. Physics in a homogeneous gravitomagnetic field is equivalent to physics on a merry-go-round in Minkowski space.— The centrifugal force is quadratic in \( \Omega \), therefore it is missing in linear perturbation theory.

For vorticity perturbations on a FRW background with \( k = 0 \) in Cartesian comoving LONBs we obtain

\[
\frac{1}{a} \frac{d}{dt}(ap_i) = \varepsilon [\tilde{E}_g + (v \times \tilde{B}_g) + H \tilde{v} \times (\tilde{\beta} \times \tilde{v})]_i. \quad (31)
\]

For FRW with \( k = \{0, \pm 1\} \) in spherical comoving LONBs there are additional terms from \( \partial_t L_i \) in Eq. (20). These terms are present even in the absence of vorticity perturbations and of Hubble expansion, because in a spherical basis for \( k = 0 \) spatial LONBs are not parallelized, and for \( k = \pm 1 \) global parallelization is impossible on \( \Sigma_t \).

VI. EINSTEIN’S EQUATIONS FOR GRAVITOMAGNETISM: AMPÈRE’S LAW

The computation of curvature in Cartan’s formalism: We give a short, self-contained derivation of the formul-
lac needed from Cartan’s formalism. Readers not interested in calculational methods can skip the first part of this section and go directly to the results starting with Eq. (36).

A. Cartan’s second equation in LONB components

The curvature 2-form $\tilde{\mathcal{R}}^{\tilde{a}}_\tilde{b}$ has LONB components $(\tilde{R}^{\tilde{a}}_\tilde{b})_{\tilde{c}\tilde{d}}$, which are the LONB components of the Riemann tensor $R^{\tilde{a}}_{\tilde{b}\tilde{c}\tilde{d}}$. The Riemann tensor can be operationally defined by the action of $(\nabla_\gamma \nabla_\delta - \nabla_\delta \nabla_\gamma)$ on the LONB 1-form $\tilde{\theta}^\tilde{a}$,

$$ (\nabla_\gamma \nabla_\delta - \nabla_\delta \nabla_\gamma) \tilde{\theta}^\tilde{a} = -\tilde{\theta}^\tilde{b} (\mathcal{R}^{\tilde{a}}_{\tilde{b} \gamma \delta}), $$

where the covariant derivatives $\nabla_\gamma$ and $\nabla_\delta$ must be in the coordinate basis. To compute the curvature 2-form we first use $\nabla_\delta \tilde{\theta}^\tilde{a} = (\omega^\tilde{a}_{\tilde{b} \delta}) \tilde{\theta}^\tilde{b}$. Acting on the right-hand side by $\nabla_\gamma$ gives two terms. One term comes from $\nabla_\gamma$ acting on $\tilde{\theta}^\tilde{b}$, and after antisymmetrization in $[\gamma \delta]$ it produces $-(\omega_{\delta \gamma}^\tilde{a} + \omega_{\gamma \delta}^\tilde{a}) \tilde{\theta}^\tilde{b}$. The other term comes from $\nabla_\gamma$ acting on the expansion coefficient (number field) $(\omega^\tilde{a}_\gamma)_{\delta}$, where it can be replaced by $\partial_\gamma$, and then antisymmetrization in $[\gamma \delta]$ it produces $-(d \omega^\tilde{a}_\gamma)_{\gamma \delta} \tilde{\theta}^\tilde{b}$. Hence we obtain

$$ \tilde{R}^{\tilde{a}}_\tilde{b} = \tilde{d} \tilde{\omega}^\tilde{a}_\tilde{b} + \tilde{\omega}^\tilde{a}_\gamma \wedge \tilde{\omega}^\tilde{b}_\delta, $$

which is Cartan’s second equation.

To obtain the LONB components of the first term of the right-hand side, $(d \omega^\tilde{a}_\gamma)_{\delta \gamma}$, we must first convert the connection components of Eqs. (38, 39) from the LONB to the coordinate basis, $(\omega^\gamma_\tilde{a})_{\delta} = (\omega^\gamma_\gamma)_{\delta} (\theta^\delta)_\gamma$, then take the exterior derivative, $\partial_\gamma (\omega^\gamma_\tilde{a})_{\delta} (\theta^\delta)_\gamma = [\gamma \leftrightarrow \delta]$, and then convert the result back from coordinate components to LONB components. The partial derivative of the product gives two terms, one with a partial derivative of $(\omega^\gamma_\tilde{a})_{\delta}$, the other with $\partial_\gamma (\theta^\delta)_\gamma$, which produces another connection-1-form component. In the second term of Cartan’s second equation these conversions from LONB to coordinate basis and back again cancel, since there is no derivative in between. The result is Cartan’s 2nd equation in LONB components,

$$ (R^{\tilde{a}}_{\tilde{b} \tilde{c} \tilde{d}})_{\tilde{c} \tilde{d}} = (\varepsilon_\gamma) \partial_\gamma (\omega^\gamma_\tilde{a})_{\delta \gamma} - (\omega^\gamma_\tilde{a})_{\gamma \delta} (\varepsilon_\delta) + (\omega^\gamma_\tilde{a})_{\gamma \delta} (\varepsilon_\gamma) - [\gamma \leftrightarrow \delta]. $$

B. Einstein equations for vorticity perturbations of Minkowski space

For linear vorticity perturbations of Minkowski space (with Cartesian coordinates for 3-space) all non-zero connection coefficients in Eqs. (38-39) are of first order in the perturbations. Therefore the second term of Cartan’s second equation can be neglected, and in the first term one need not distinguish components in LONB from components in the coordinate basis. For vorticity perturbations the important Einstein equation is the equation for $G^{\tilde{a} \tilde{b}} = R^{\tilde{a} \tilde{b}}$,

$$ R^{\tilde{a} \tilde{b}} = (\mathcal{R}^\tilde{a}_{\tilde{b} \gamma \delta}) = (d \omega^\tilde{a}_\gamma)_{\delta \gamma} = \frac{1}{2} (\text{curl} \tilde{B})_{\tilde{a}}. $$

Hence Einstein’s $G^{\tilde{a} \tilde{b}}$-equation for general, i.e. time-dependent, vorticity perturbations on a Minkowski background reads

$$ \text{curl} \tilde{B}_\gamma = -16 \pi G_N \tilde{J}_\gamma. $$

In contrast to the Ampère-Maxwell equation of ordinary electrodynamics, the Maxwell term $(\partial_\gamma \tilde{E}_\gamma)$ is absent in the time-dependent, i.e. non-stationary context of gravitomagnetodynamics. Eq. (40) is consistent without $(\partial_\gamma \tilde{E}_\gamma)$, because in the vorticity sector all vector fields are divergence-free, particularly div$\tilde{J}_\gamma = 0$. A term $\partial_\gamma \tilde{E}_\gamma / \partial t$ cannot be present in gravitomagnetism, otherwise Eq. (40) together with Faraday’s law for gravitomagnetism, Eq. (29), would erroneously predict gravitational vector waves.— Eq. (40) for gravito-magnetodynamics (i.e. for a non-stationary context) is identical to the original law of Ampère for stationary magnetism, except that the charge current $\tilde{J}_\gamma$ is replaced by the energy current $\tilde{J}_\gamma$, and the prefactor $4\pi$ is replaced by the prefactor $(-16\pi G_N)$.— The source $\tilde{J}_\gamma \equiv T^{\tilde{a} \tilde{b}}$ is the energy current density, which is equal to the momentum density (for $c = 1$). In linear perturbation theory for a perfect fluid $\tilde{J}_\gamma = (\rho + p) \tilde{v}_\gamma$.

The $G^{\tilde{a} \tilde{b}}$ equation is an equation at fixed time, a constraint equation, called momentum constraint, since the momentum density appears on the right-hand side of Eq. (40). To see the analogous structures of gravitomagnetism and electromagnetism, it is more instructive to formulate this constraint equation, as we have done in Eq. (36), via connection 1-forms, which involves $\delta_\gamma (\partial_\gamma \beta_\gamma - \partial_\gamma \beta_\delta)$, i.e. the gravitomagnetic field, than via the extrinsic curvature tensor $K_{ij}$, which involves $\partial_\gamma \beta_\delta + \partial_\gamma \beta_\delta$. Of course the resulting constraint, if written in terms of $\tilde{A}_\gamma = \tilde{\beta}$, is the same, $\Delta \tilde{A}_\gamma = 16 \pi G_N \tilde{J}_\gamma$.

The $\delta G^{\tilde{a} \tilde{b}}$ equation with the source $\delta T^{\tilde{a} \tilde{b}}$ is trivially fulfilled, since these two objects are 3-scalars and therefore vanish in the vector sector.— The source $\delta \tilde{J}_{ij}$ vanishes (for linear vorticity perturbations and perfect fluids), since it is of second order in the perturbation. The $\delta G_{ij}$ equations give $\partial_0 (\partial_0 \beta_0 + \partial_0 \beta_1) = 0$, i.e. the shear of the field $\tilde{\beta}$ has vanishing time-derivative.

C. The momentum constraint for vorticity perturbations of spatially flat FRW spaces

With Cartesian comoving coordinates for flat 3-space there are two new terms in the connection coefficients,
where we have used $\beta = \vec{A}_g$. The scale factor $a$ of the spatially flat FRW universe does not appear in this fixed-time equation. Note that ‘fixed-time equation’ means that there are no partial time-derivatives of the fields $\vec{B}_g(\vec{x}, t)$, etc. The coefficient $(dH/\dot{t})$ is merely a given input number at each time.--- Every symbol in this equation refers to the physical scale. This is a general fact: If we consider any fixed-time equation at any time $t_1$, we are free to set $a(t_1) = 1$, hence the scale factor disappears.

Friedmann’s equations give $H$-dot,

$$dH/\dot{t} = -4\pi G_N (\rho + p).$$

Since $(dH/\dot{t}) \leq 0$ for $(p/\rho) \geq -1$, we define the $H$-dot radius $R_{dH/\dot{t}}$ by $R_{dH/\dot{t}}^2 = (dH/\dot{t})^{-1}$, and we define $\mu$ by

$$\mu^2 \equiv -4(dH/\dot{t}) \equiv 4(R_{dH/\dot{t}})^{-2}. \quad (39)$$

We use $\vec{B}_g = \text{curl} \vec{A}_g$, and with div $\vec{A}_g = 0$ we have $(\text{curl} \text{curl} \vec{A}_g) = -\Delta \vec{A}_g$. Therefore Eq. 37 becomes

$$(-\Delta + \mu^2) \vec{A}_g = -16\pi G_N \vec{J}_e,$$

an elliptic equation. Concerning the absence of partial time-derivatives of $\vec{A}_g$ in Eq. 40 for the time-dependent context of gravitodynamics, see the comments after Eq. 59.--- The new term on the left-hand side, $-4H \vec{A}_g = (\mu^2 \vec{A}_g)$, is responsible for the exponential cutoff on scales larger than the $H$-dot radius, Eqs. 11, 12, which is crucial to remove the problem of ‘overdragging’, as shown in Fig. 1, and for obtaining a weight function with normalization to unity in our Eq. 2, which gives $\vec{A}_\text{gyro}$ as the weighted average of $\vec{A}_{\text{matter}}$. Bardeen gives two alternative choices for gauge-invariant amplitudes for the matter velocity, $v_s$ and $v_e$ in his Eqs. (3.21, 3.23).--- Bardeen uses Einstein’s momentum constraint in coordinate-basis components, $T^{00}$ on the source side, and this is directly proportional to $v_e$.--- In contrast we use the momentum constraint in LONB components, $T^{01}$, and this is directly proportional to Bardeen’s other gauge-invariant variable $v_s$.

The physical meaning of $v_s$ and $v_e$: The field with the amplitude $v_s$, used by Bardeen, is directly related (via its curl) to the angular velocity field of matter relative to axes of local gyroscopes all over $\Sigma_t$. But this has the severe drawback that $v_e$ is not measurable without prior knowledge of the solution of Einstein’s equations, $\gamma_{\mu
u}$, and therefore $\vec{A}_g$ and $\vec{B}_g$. Hence $v_e$, employed by Bardeen, cannot be used as a matter input for solving Einstein’s equations in the context of Mach’s principle and of dragging, where a measured input is needed before having solved Einstein’s equations. See also sects. VIII and IX.--- In contrast we use Bardeen’s other gauge-invariant amplitude $v_s$, which is directly related to the angular velocity of matter relative to the asymptotic unperturbed quasars. This is an input which is measurable without prior knowledge of $\gamma_{0i} = A_i^g$, i.e. before knowing the output of solving Einstein’s constraint equation.

The difference between the two gauge-invariant amplitudes for the matter velocity is given by $(v_e - v_s) = -\Psi$, Eq. (3.23) of Bardeen. The $\Psi$-term, a geometric term, which is on the right-hand side of Bardeen’s momentum constraint, must be moved to the left-hand side, the geometric side. With the prefactors in the momentum constraint and in our notation the $\Psi$-term appears as $\mu^2 \vec{A}_g$.

Both sides of our Eq. 40 are gauge-invariant, the left-hand side because our $\vec{A}_g$ is directly proportional to Bardeen’s gauge-invariant potential $\Psi$, Eq. 8, the right-hand side because our matter velocity $\vec{v}$ is directly proportional to Bardeen’s gauge invariant amplitude $v_s$.

VII. MACH’S PRINCIPLE

A. Evolution of inertial axes determined exclusively by cosmic matter flows

The solution of Eq. 40 is the Yukawa potential for $\vec{A}_g = \vec{B}$ in terms of the energy currents $\vec{J}_e$ at the same time,

$$\vec{A}_g(r^t, t) = -4G_N \int d^3\vec{r} \vec{J}_e(r^t, t) \exp(-\mu|\vec{r} - \vec{r}^t|)/|\vec{r} - \vec{r}^t|. \quad (41)$$

This is analogous to the formula for ordinary magnetostatics except for the exponential cutoff, i.e. the $1/\sqrt{r}$ potential is replaced by the Yukawa potential $(1/r)\exp(-\mu r)$. The Green function which is exponentially growing for $r^t \to \infty$ has been rejected on the standard grounds of field theory. This gives our first conclusion: The contributions of vorticity perturbations beyond the $H$-dot radius are exponentially suppressed.

The fundamental law of gravitodynamics in integral form for linear vorticity perturbations and for spatially flat FRW universes ($k = 0$), the equation for $\vec{B}_g$ and $\vec{A}_\text{gyro}$ in terms of sources at the same time, is

$$\vec{B}_g(P) = -4G_N \int d^3r [\vec{n}_{PQ} \times \vec{J}_e(Q)] Y_\mu(r) \vec{J}_e(Q), \quad (42)$$

$$\frac{dY_\mu}{dr} = \mu \vec{Y}_\mu = \text{Yukawa force}, \quad (43)$$

$$\vec{A}_\text{gyro} = -\frac{1}{2} \vec{B}_g(r^t). \quad (44)$$

where $\vec{n}_{PQ}$ is the unit vector pointing along the geodesic on $\Sigma_t$ from $P$ to $Q$, and $Q$ stands for “source point”. We have applied $\vec{B}_g = \text{curl} \vec{A}_g$ to Eq. 44.
Both \( \vec{B}_g = -2\Omega_{\text{gyro}} \) and the transverse velocity in \([\vec{n}_{PQ} \times \vec{J}_e(Q)]\) are measured relative to geodesics on \(\Sigma_t\) from the gyroscopic origin to quasars in the asymptotic FRW universe. Since \(\Sigma_t\) is a Euclidean 3-space, global parallelism on \(\Sigma_t\) can be used when comparing vectors at \(P\) and \(Q\). Although we use Cartesian coordinates for Euclidean 3-space, all quantities in Eq. (42) have a gauge invariant meaning, they are directly measurable, e.g. \(d^3r_{PQ}\) is the measured volume element, and \(r_{PQ}\) is the geodesic distance from \(P\) to \(Q\).—The scale factor \(a\) of the spatially flat FRW universe does not appear in the fixed-time equation (42), as discussed after Eq. (37); every symbol refers to the physical scale.

The only differences from Ampère’s law in integral form are:

1. the replacement of the current of electric charge \(\vec{J}_q\) by the measured energy current of matter \(\vec{J}_e\),
2. the factor \((-G_N)\), as in the transition from Coulomb’s law to Newton’s law,
3. the additional factor 4, which occurs in the transition from Ampère’s law of ordinary magnetism to gravitomagnetism,
4. the replacement of the \(1/r^2\) force in Ampère’s law by the Yukawa force with its exponential cutoff, Eq. (43), which occurs in the transition from a Minkowski background to gravitomagnetism on a FRW background with \(k = 0\).

An analogous Yukawa cutoff at super-\(H\)-dot scales in the scalar sector of cosmological perturbation theory occurs in the ‘uniform expansion gauge’, which is also called the ‘uniform Hubble gauge’; see Ref. [29].

Note the fundamental difference between cosmological gravitomagnetism, Eq. (12), and Ampère’s law. The latter does not hold in a rotating reference frame, unless one introduces fictitious forces. The same is true for the equations for electromagnetism in special relativity, and for general relativity of the solar system in asymptotic Minkowski space, where fictitious forces must be encoded by boundary conditions at spatial infinity as explained in Sec. XI. In contrast, for cosmological gravitomagnetism, Eq. (12) remains valid, as it stands, in a frame which is rotating with angular velocity \(\Omega\) relative to asymptotic quasars, as explained in sec. II after Eq. (1). This establishes the fact that the asymptotic inertial frame has no influence in cosmological gravitomagnetism. The local nonrotating frame (at any point \(P\)) is not needed as an input in applying Eq. (12), i.e. no absolute element is needed as an input. The time evolution of local inertial axes is an output, determined exclusively by the weighted average of cosmological energy flows.

Eqs. (12) - (14) state what specific average of the measured energy flow of matter out there in the universe determines the motion of gyroscopic axes here. This answers Mach’s question [30]:

“What share has every mass in the determination of direction ... in the law of inertia? No definite answer can be given by our experiences.”

The gravitomagnetic moment density \(\vec{\mu}_g\) per volume element is the analog of the magnetic moment density (with \(J_q\) replaced by the measured energy current \(J_e\)). The gravitomagnetic moment density appears as the source in Eq. (12), and it is equal to one half of the measured angular momentum density of matter \(\vec{L} = (p + p) \times (\vec{r} \times \vec{v})\) at the source point \(Q\) relative to the gyroscopic point \(P\),

\[
\vec{\mu}_g = \frac{1}{2}[\vec{r}_{PQ} \times \vec{J}_e(Q)] = \frac{1}{2}\vec{L}_{PQ}, \tag{45}
\]

\[
\vec{\Omega}_{\text{gyro}}(P) = -\frac{1}{2}\vec{\mu}_g(P) = 2G_N \int d^3r_{PQ} \frac{\vec{L}_{PQ}}{r_{PQ}} Y_μ(r_{PQ}). \tag{46}
\]

It is a matter of taste and choice of emphasis, whether the term ‘source’ for the gravitomagnetic field is used for (1) the measured energy current (= momentum density) \(\vec{J}_e\) as in the analogue of Ampère’s law (momentum constraint), Eqs. (36) and (37), or (2) the measured angular momentum density \(\vec{L} = (p + p) \times (\vec{r} \times \vec{v})\), as in the integrated form of the analogue of Ampère’s law, Eq. (10).

### B. Evolution of inertial axes exactly follows the average of cosmic matter flows

The gravitomagnetic moment density per \(r\)-interval, \(d\vec{\mu}_g/dr\), is the lowest multipole term, the \(ℓ = 1\) term, of the ‘odd parity sequence’, \(P = (-1)^{ℓ+1}\), in the multipole expansion of the vector source \(\vec{J}_e\) for \(r_{\text{obs}} = r_{\text{gyro}} = 0\) and \(r_{\text{source}} > 0\). Other multipoles cannot contribute to the gyroscopic’s precession, i.e. to the gravitomagnetic field \(\vec{B}_g\) at \(r = 0\), for reasons of symmetry under rotations and space reflection.—In first-order perturbation theory \((p + p)\) = constant all over space, hence also the velocity field at a given radius \(r\) can only contribute via the term with \(ℓ = 1\), odd parity sequence), which is equivalent to a rigid rotation with the angular velocity \(\Omega_{\text{matter}}(r)\). Using this fact and Eqs. (45) - (50) we obtain

\[
\vec{\Omega}_{\text{gyro}} = \frac{μ^2}{3} \int_0^\infty dr r \vec{Ω}_{\text{matter}}(r) C_μ(r), \tag{47}
\]

\[
C_μ(r) = r^2 Y_μ(r) = (1 + μr) \exp(-μr). \tag{48}
\]

This is our most important result. Eq. (17) shows that \(\vec{Ω}_{\text{gyro}}\) is the weighted average of \(Ω_{\text{matter}}\), i.e. the precession of a gyroscopic, \(Ω_{\text{gyro}}\), exactly follows, i.e. it is exactly dragged by the weighted average of \(Ω_{\text{matter}}\).—The weight function is normalized to unity, as it must be for an averaging weight function in any problem. This crucially
depends on the exponential cutoff in the Yukawa force $Y_\mu(r)$. The cutoff function $C_\mu(r)$ is 1 for $r \ll \mu^{-1}$ and goes to zero exponentially fast for $r \gg \mu^{-1}$. The weight function per logarithmic r-interval in Eq. (47) is $r^2 C_\mu(r)$, i.e. it grows quadratically until one reaches, roughly, the $R$-dot radius, and then it goes to zero exponentially.—

The conclusion is that Eq. (47) is a clear demonstration of how Mach’s principle is contained within cosmological general relativity.

When talking about the “angular velocity of cosmic matter” one must keep the following in mind: For a given arbitrary velocity field the densities of gravitomagnetic moment and of measured angular momentum at one source point $Q$ relative to the gyroscope point $P$, $\vec{\Omega}(Q; P)$, because one can add terms $r^2 Q f(Q)$ without changing $\vec{r}(Q)$.

C. The dragging fraction

The dragging fraction refers to the simple cosmological model discussed in the Introduction, a uniform (rigid) rotational motion with $\Omega_{\text{matter}}$ constant inside a radius $R_{\text{rot}}$ around a gyroscope and zero outside and with energy density and pressure spatially constant everywhere. The definition of the dragging fraction $f_{\text{drag}}$ is given by

$$\vec{\Omega}_{\text{gyro}} = f_{\text{drag}} \vec{\Omega}_{\text{matter}},$$

where $\vec{\Omega}_{\text{gyro}}$ and $\vec{\Omega}_{\text{matter}}$ are measured relative to geodesics on $\Sigma_t$ from the gyroscope to quasars in the asymptotic FRW space.— A better word than dragging fraction is voting power of the rotating matter (in contrast to the voting power of the non-rotating matter outside), because there is exact dragging of $\vec{\Omega}_{\text{gyro}}$ by a weighted average of $\vec{\Omega}_{\text{matter}}$ according to Eq. (47).—

Putting $\vec{\Omega}_{\text{matter}} = \text{constant}$ inside a radius $R_{\text{rot}}$ and $\vec{\Omega}_{\text{matter}} = 0$ outside in Eq. (47) we obtain the dragging fraction,

$$f_{\text{drag}}(R_{\text{rot}}) = \frac{\mu^2}{3} \int_0^{R_{\text{rot}}} dr r (1 + \mu r) \exp(-\mu r)$$

$$= 1 - \{\exp(-\mu r) [1 + \mu r + \mu^2 r^2/3]\}_{r=R_{\text{rot}}}.$$ (50)

The dragging fraction (voting power of the matter inside $R_{\text{rot}}$) is shown by the solid curve in Fig. (1). It grows quadratically with $R_{\text{rot}}$ until it reaches order of one for $R_{\text{rot}} = R_{\text{H}}/\dot{H}$. As $R_{\text{rot}}$ grows beyond $R_{\text{H}}/\dot{H}$, the dragging fraction approaches the exact value 1 exponentially fast, i.e. exact dragging of the gyroscope axes by the rotating matter. This holds for any equation of state.

The problem of it ‘overdragging’, shown by the dashed curve in Fig. (1), arises when extrapolating the results of gravitomagnetic perturbation theory on a Minkowski background beyond the region of validity, as noted in [14].— The solid line shows, how the problem of ‘over-dragging’ is removed by the exponential cutoff from cosmological perturbation theory for super-Hubble-dot scales, i.e. by the exponential cutoff in Eqs. (40) [12].—

Lynden-Bell et al [14] do not have an exponential cutoff, because they use the conserved angular momentum instead of the measured angular momentum as the source. See our discussion below in sect. VII.

Analogous anti-dragging effects in magnetostatics: A rotating charged spherical shell acting on magnetic dipole moments inside has the opposite sign in Ampère’s law compared to gravitomagnetism, Eq. (40), and this causes antidragging in the magnetostatic case.

Physics could have evolved differently after Mach (1883): Physicists at that time could have extended the correspondence between Coulomb’s law and Newton’s law and postulated gravitomagnetism in correspondence to Ampere’s magnetostatics. Their prediction of the dragging of inertial frames would have been too small by a factor 4 compared to the correct result for linear gravitomagnetism on a Minkowski background. For the simple case of a homogeneous rotation out to some radius $R$ of cold matter with $\rho = \text{constant}$ they would have obtained perfect dragging for $GM/(Rc^2) = 2$ and noted the problem of ‘overdragging’ for $GM/(Rc^2) > 2$. In the late 1920’s, for an approximately homogeneous and expanding universe with cold matter, this critical radius would have been identified with the Hubble radius (apart from a missing factor 2).

Measuring everything relative to axes of gyroscopes at one given location (instead of relative to quasars in the asymptotic FRW space) makes the the left-hand side of Eq. (40) vanish. Hence Eq. (40) reduces to the statement that the measured angular momentum of matter relative to gyroscopes at one given location will vanish after averaging with the weight $r^{-3}$ and with the exponential cutoff $C_\mu(r)$.— From the point of view of measurements it is preferable to measure relative to gyroscopes at one given location. But to see the structure of gravitomagnetism and its correspondence with electromagnetism most clearly, it is best to measure relative to quasars in the asymptotic FRW space.

D. The de Sitter limit

Dark energy with $p/\rho = -1$, i.e. a cosmological constant, does not contribute in Mach’s principle, Eq. (12), since there is no flow of energy associated with it, its energy current $J_c = (p + \rho) \vec{u}$ vanishes.— When considering a FRW universe with $k = 0$ and letting $(p/\rho)$ get closer and closer to $-1$ at fixed Hubble rate (hence fixed $\rho$) until we are arbitrarily close to a de Sitter universe, Mach’s principle continues to work (in linear perturbation theory). When taking this limit the prefactor $\mu^2 = (16\pi G_N)(p + \rho)$ in Eq. (50) gets smaller and smaller (and tends to zero), but the $H$-dot radius $\mu^{-1}$, which cuts off the otherwise quadratically divergent integral in...
Eq. (50), gets larger and larger such that perfect dragging is maintained for arbitrarily small $\mu^2$, i.e. arbitrarily close to a de Sitter universe.

E. Einstein’s objection: Interaction of masses

Einstein’s objection to Mach’s principle in his autobiographical notes of 1949 [31]:

"Mach conjectures that inertia would have to depend upon the interaction of masses, precisely as was true for Newton’s other forces, a conception which for a long time I considered as in principle the correct one. It presupposes implicitly, however, that the basic theory should be of the general type of Newton’s mechanics: masses and their interactions as the original concepts. The attempt at such a solution does not fit into a consistent field theory, as will be immediately recognized."

We have shown how this apparent difficulty is resolved in General Relativity, specifically in linear Gravitomagnetism: The relevant Einstein equation (the momentum constraint) has the form of Ampère’s law with a Yukawa cutoff, Eq. (47). It is a remarkable fact that Einstein’s local field equation for $G_{ij}$ has the solution Eq. (42), which has the form of an instantaneous action-at-a-distance. It says that the measured mass-energy flow out there in the universe does indeed determine the precession of gyroscope axes here.— See also refs. [12] [22].

VIII. MEASURED MATTER INPUT FOR EINSTEIN’S EQUATIONS

The input on the right-hand side of Eqs. (12) (13), which express Mach’s principle, is the measured angular velocity of matter (stars, galaxies, etc) relative to the geodesics on $\Sigma_t$ to quasars in a flat FRW space. No knowledge of the metric perturbation $\vec{\beta}$ is needed when determining the input for Eqs. (42) (47). The measured angular velocity of matter is a purely kinematic or kinetic input. Similarly the measured angular momentum density $\vec{L} = (\rho + p) (\vec{r} \times \vec{v}) = (\rho + p) (\vec{r}^2 \sin^2 \theta \vec{\Omega})$ is a purely kinetic input. We call it the kinetic angular momentum.— The geometric-dynamical output of solving Einstein’s $G_{ij}$ equation is $\vec{\beta} = \vec{A}_g$, the gravitomagnetic vector potential in Eqs. (40) (41), hence the gravitomagnetic force $\vec{B}_g$ in Eq. (42).— The measured, kinetic angular momentum must be distinguished from the canonical angular momentum, which we introduce in the following paragraphs.

Kinetic versus canonical momentum and angular momentum

The Einstein-Hilbert action for linear vorticity perturbations (i.e., gravitomagnetism) on a Minkowski background and for point particles gives

$$S = (16\pi G_N)^{-1} \int d^4 x \sqrt{g} (\text{curl} \vec{A}_g)^2 + \int dt \sum_n \left[ \frac{1}{2} m \dot{x}_n^2 + m \dot{x}_n \vec{A}_g(\vec{x}_n, t) \right].$$

(51)

It is natural to take the nonrelativistic expression ($v \ll c$) for matter when discussing linear perturbations. Except for the prefactor $-(16\pi G_N)^{-1}$ in the first term and the prefactor $m$ in the last term, the action is the same as for electromagnetism without the $(\partial_t \vec{A})^2$ term.

The Lagrangian gives the equations of motion via the standard Euler-Lagrange equations (as in classical mechanics and classical electrodynamics), if and only if the Lagrangian is defined by

$$S = \int dt L$$

(52)

without any metric factors in the integrand. Hence the Lagrangian for a point particle in a gravitomagnetic field is given by the square bracket of the matter term in Eq. (51).

The canonical momentum of Lagrangian mechanics is defined by $(p_{\text{can}})_k = \partial L / \partial \dot{x}^k$, where $k = 1, 2, 3$. From Eq. (51) we obtain in Cartesian coordinates

$${\vec{p}}_{\text{can}} = m (\dot{\vec{x}} + \vec{A}_g).$$

(53)

This is the same equation as in classical Lagrangian mechanics for point particles in an electromagnetic field, except that the electric charge $q$ is replaced by the mass $m$.— The kinetic momentum is $m\dot{\vec{x}}$. It can be used as the input for solving Einstein’s equations, because it is directly determined by measurements without prior knowledge of the gravitomagnetic vector potential $\vec{A}_g$, which is an output of solving Einstein’s equations. On the other hand the canonical momentum (for a given measured state of motion) depends on the gravitomagnetic vector potential $\vec{A}_g$. Therefore the canonical momentum cannot be used as an input for solving Einstein’s equations. The canonical momentum cannot be determined by a FIDO from matter measurements without prior knowledge of the 4-geometry.— In curvilinear coordinates of Euclidean 3-space the general definition of the canonical momentum, $(p_{\text{can}})_k = \partial L / \partial \dot{x}^k$, gives

$$(p_{\text{can}})_k = m g^{(3)}_{kn} (\dot{x}^n + A_{g}^n) \quad k, n = 1, 2, 3.$$  

(54)

From its definition via the Lagrangian, the canonical momentum is a 1-form in 3-space, i.e. it has a lower 3-index.— On the other hand we can also start from the
4-velocity $u^\nu$, which is the archetype of a 4-vector (tangent vector), multiply it with the mass to obtain the 4-momentum $p^\nu$, and pull down the 4-index with $g^{(4)}_{\mu \nu}$. For the spatial components $p_k$, this gives $p_k = m g^{(4)}_{k \nu} u^\nu = m g^{(4)}_{k \nu} (\delta^n + A^n)$ for $v \ll c$, which is the same as the canonical momentum in Eq. 7. — We note that in spherical coordinates $p_\phi$ has the physical meaning of canonical angular momentum around the z-axis.

Mathematically one can freely raise and lower indices of the vectors $\vec{p}$ resp. $\vec{v}$ using the metric tensor, or convert e.g. from lower indices to LONB components using $p_\mu = (e_\mu)^\mu p_\nu$. But physical quantities are prototypes for either LONB components, or 1-form components, or contravariant vector components. For example in spherical coordinates the angular velocity around the z-axis is given by an upper-index component, $v^\theta = \frac{dx^\theta}{dt}$, the canonical angular momentum around the z-axis is given by a lower-index component, $p_\theta = mg_{\theta \phi} \frac{dx^\phi}{dt} = m r^2 \sin^2 \theta [(dx^\phi/dt) + A^\phi]$, while the measured linear momentum in the direction $\vec{e}_\psi$ is given by a LONB component, $p_\phi = m \sqrt{g_{\phi \phi}} \frac{dx^\phi}{dt} = m r \sin \theta (dx^\phi/dt)$. The physics question dictates, which type of index is the relevant one.

In the special case of azimuthal symmetry, i.e. when $\vec{e}_\phi = \hat{\xi}$ is a rotational Killing vector, the canonical angular momentum $p_\phi = (\hat{\phi}, \hat{\xi})$ is conserved, while the measured, kinetic angular momentum of matter, $(r \sin \theta p_\phi)$, is not conserved because of the gravitoelectric induction field $F_{\vec{E}}$, Eq. 20. The canonical angular momentum is relevant for the time-evolution, i.e. for the dynamics and conservation laws, not for kinematics (measurements at a given time).

In the continuum description of matter (energy current density) the LONB components $T_{ab}$ can be used as input for solving Einstein’s equations, since they can be measured without knowing the output $\beta_k$ of Einstein’s equations. On the other hand the coordinate-basis components $T^ab$ cannot be used as an input, because they cannot be determined by matter measurements without prior knowledge of $\beta_k$.

In Mach’s principle the input is the observed angular velocities of cosmic matter around us as in Eq. 4 resp. the measured kinetic angular momenta $(\rho + p)(r \times \vec{v})$ as in Eq. 12. This has also been recognized by Lynden-Bell et al [16], who wrote (in section 2.4) “In Mach’s principle, we wish to relate the dragging not merely to the angular momentum distribution but also to the distribution of angular velocity of the 'stars' about us.” Unfortunately Lynden-Bell et al [16] did not follow up on this important observation, i.e. they did not solve the corresponding equation. Instead, in the rest of their paper, they used the canonical angular momentum $T^ab$ as the input on the right-hand side of Einstein’s equation. For the reasons explained in this section, measured kinetic matter input versus dynamical-geometric output of solving Einstein’s equations, we think that their approach would be a very large detour to obtain our Eq. 4, which is the most direct expression of the dragging of inertial frames.— If one follows their proposal, the $(\dot{H}A_\lambda)$—term on the left-hand side of the Einstein equation 27 is absent, and there is no exponential suppression factor multiplying the canonical angular momentum distribution. This version of the momentum constraint equation may be more relevant in other contexts.— In response to our conference paper [18] Bičák, Lynden-Bell, and Katz [21] gave arguments why they thought that the canonical angular momentum density should be considered more fundamental, although they agreed that “the problem with given angular velocities may be closer to Mach’s original principle.” They solved the equations for both input possibilities (canonical versus kinetic angular momentum) for $k = 0, \pm 1$.

IX. EINSTEIN’S OBJECTION: “IF YOU HAVE A TENSOR $T_{\mu \nu}$ AND NOT A METRIC ...”

Einstein’s objection to Mach’s Principle in his letter to Felix Pirani of 2 February 1954 [32], as quoted by Ehlers in [33]:

> “If you have a tensor $T_{\mu \nu}$ and not a metric, then this does not meaningfully describe matter. There is no theory of physics so far, which can describe matter without already the metric as an ingredient of the description of matter. Therefore within existing theories the statement that the matter by itself determines the metric is neither wrong nor false, but it is meaningless.”

From this argument Einstein drew the conclusion that “one should no longer speak of Mach’s principle at all”, quoted by Renn in [33].

We agree with Einstein’s statement as quoted by Ehlers in [33], as long as one refers to coordinate-basis components $T_{\mu \nu}$. Einstein’s valid criticism applies to $T^ab$ in the proposal of Bičák et al [21]. On the other hand, for LONB components $T_{ab}$ our conclusions are entirely different. We have already explained this issue in the context of linear gravitomagnetism in the previous section. We shall now discuss Einstein’s objection in a general context.

The metric in LONB components, the Lorentz metric $\eta_{ab} = \text{diag}(-1, +1, +1, +1)$, by itself does not encode specific information about the particular Riemann space at hand (e.g. perturbed FRW metric versus Schwarzschild metric), while the metric in coordinate-basis components $g_{\alpha \beta}(x)$ encodes all specific information about the particular Riemannian space at hand (curvature etc.), i.e. it is metric data specific to the particular geometry. Both objects, the universal Lorentz metric $\eta_{ab}$ and the case-specific data-set $g_{\alpha \beta}(x)$, are metrics, but they play totally different roles in the context of input versus output when solving Einstein’s equations.
In Cartan’s LONB formalism the case-specific data set which encodes the specific metric information (about the particular Riemann space at hand) is obtained by measuring the LONB components of the coordinate basis vectors $e_\mu$ all over the particular Riemann space, i.e., measuring $(e_\mu)_a^b$, which gives $g_{\mu\nu}(x)$ directly via $g_{\mu\nu}(x) = (e_\mu)_a^b (e_\nu)_b^a \eta_{ab}$.

In the LONB formalism a choice must be made for a specific field of LONBs, $\hat{e}_a(P)$. For a coordinate system $P \rightarrow x^P$, given on a manifold one class of choices for LONBs is obtained by an orthonormalization procedure starting from the coordinate basis, as we have done in Eq. (19).

Having the universal metric $\eta_{\hat{a}\hat{b}}$ available is analogous to a surveyor having meter sticks available as a universal tool before starting the work of surveying a specific coordinate basis, as we have done in Eq. (19).

The basic issue is not the fact that the universal metric tool $\eta_{\hat{a}\hat{b}}$ makes it possible to perform the work of measuring the case-specific data-set $g_{\mu\nu}(P)$ all over space-time. The basic issue is that such measurements are not needed as a case-specific input data-set, if one wants to solve Einstein’s equations and thereby predict $g_{\mu\nu}(P)$ for a grid of points.

The crucial question is: What are the needed case-specific “ingredients”, what is the needed case-specific input data-set for solving Einstein’s equations and thereby obtaining the case-specific metric output data-set?

The input data set for solving Einstein’s equations is obtained by performing the work of measuring the LONB components of the 4-momenta $\hat{p}$ of all particles in tangent spaces all over space-time, i.e., measuring $\hat{p}^\hat{a}$. Summing over all particles within one measured cm$^3$ around a point gives the LONB components $T_{\hat{a}\hat{b}}(P)$, the observed matter input data for solving Einstein’s equations. To obtain this matter input one only needs the universal Lorentz metric $\eta_{\hat{a}\hat{b}} = \text{diag}\{-1, +1, +1, +1\}$, but one does not need the case-specific geometric data $g_{\mu\nu}(x)$, which are the case-specific geometric output data from solving Einstein’s equations and not available at the input level.

We conclude that *Einstein’s objection* quoted above does not apply to LONB components $T_{\hat{a}\hat{b}}$, because no case-specific metric data-set is needed as an ingredient to obtain $T_{\hat{a}\hat{b}}$. If one wants to implement the phrase “measured matter properties (by themselves) act on space, telling it how to curve” one is forced to use LONB components. And for LONBs the most convenient formalism is the one of E. Cartan.

X. LOCAL VORTICITY MEASURED BY NON-ROTATING OBSERVERS

Local vorticity is defined as $\text{curl} \vec{\varepsilon}_{\text{fluid}}$ measured in the local inertial coordinate system which is comoving with the fluid, i.e. measured relative to the axes of local gyrosopes (“local compass of inertia”). In general coordinates the local vorticity measured by non-rotating observers is

$$\omega^\alpha = -\varepsilon^{\alpha\beta\gamma\delta} u_\beta \nabla_\gamma u_\delta,$$

where $\varepsilon^{0123} = -1$.

Mach’s principle, formulated as a general hypothesis by Mach [5] and made precise in Eq. (42), states that the gyroscope axes here follow the rotational energy currents $\hat{J}_a$ of matter in the universe averaged over a $r^{-2}$ weight and an exponential cutoff at the $H$-dot radius. In general the gyroscope axes here most definitely do not follow the motion of the *local fluid here*, i.e. relative to gyroscopes the local vorticity is nonzero according to Mach’s principle in general.

There is a special model, a rigid rotation of a fluid out to a rotation radius $R_{\text{rot}}$. In the limit $R_{\text{rot}}/R_{\text{diff}}/dt \rightarrow \infty$ this special model produces an unperturbed FRW universe, where the local vorticity measured by non-rotating observers vanishes.

Unfortunately many authors have considered the vanishing of the vorticity relative to the local compass of inertia to be a test for Mach’s principle. See e.g. Ozsváth and Schücking’s [34] solution and discussion of a Bianchi IX model for perturbations of the closed and static Einstein universe. In contrast we conclude that the vanishing of the vorticity relative to the local compass of inertia is not relevant as a test of Mach’s principle.

The claim of Ozsváth and Schücking [34] that their Bianchi IX model violates Mach’s principle is incorrect, because their claim is based on the non-vanishing of the *local vorticity*. Now the question is: Does the Bianchi IX model of Ozsváth and Schücking satisfy Mach’s principle (which involves an integral over energy flows in the universe)?

XI. BOUNDARY CONDITIONS FOR EINSTEIN’S EQUATIONS IN ASYMMETRIC MINKOWSKI SPACE

For problems in asymmetric Minkowski space Einstein’s equations by themselves are not sufficient to determine a solution, they must be supplemented by boundary conditions at spatial infinity.

An example is the Schwarzschild solution. Consider Earth in asymmetric Minkowski space. In General Relativity we are free to choose any coordinate system. We choose spherical coordinates with Earth-fixed axes, but all the same we write down the standard form of the Schwarzschild solution. This certainly satisfies Einstein’s differential equations for the Earth as a source. But it is not the correct solution. Just consider a geostationary satellite. With this solution the equation of motion (geodesic equation) predicts that the satellite will fall down. What is missing are the centrifugal forces, which arise from our choice of Earth-fixed axes.
We now can do one of two things: 1) Either we do not admit coordinate systems which (in the asymptotic region for the solar system) are rotating relative to quasars. Then the usual asymptotic condition $g_{\mu\nu} \to \eta_{\mu\nu}$ is adequate. But excluding a class of coordinate systems is against the spirit of general relativity. 2) Or we do admit coordinate systems which are rotating asymptotically relative to quasars, e.g. we admit coordinate systems with Earth-fixed axes. But then we need encoding boundary conditions in the asymptotic part of the solar system. The boundary conditions encode fictitious forces (e.g. the Coriolis force, which is equivalent to a homogeneous gravitomagnetic field) in the asymptotic part of the solar system. In other words the boundary conditions encode the effects of sources outside the system considered (outside the solar system), they encode the influence of the cosmological sources.

The necessity of boundary conditions at spatial infinity is familiar from electromagnetism. Boundary conditions are needed for fields, whose sources lie outside the system under discussion. In ordinary magnetism the sources external to the system could be coils with electric currents generating applied external magnetic fields for the experiment under consideration.

What is the general form of the boundary conditions in the region asymptotic to the solar system? The forms of the asymptotic force fields are familiar from the fictitious forces in classical mechanics arising in coordinate systems which are rotating relative to inertial systems. In the vector sector of linear perturbations (gravitomagnetism) two fictitious forces arise, since the centrifugal force $m[\vec{r} \times \vec{\Omega}]$ is absent at the level of linear perturbation theory:

1) The Coriolis force $2m[\vec{r} \times \vec{\Omega}]$, which corresponds to a homogeneous gravitomagnetic field,

$$\vec{B}_g(x, t) \to (\vec{B}_g)_{\text{asy}} = -2 \vec{\Omega}_{\text{gyro}}^{\text{asy}},$$

where $\vec{\Omega}^{\text{asy}}_{\text{gyro}}$ is the precession vector of the spin axes of asymptotic gyroscopes (i.e. axes of inertial frames) relative to the axes of our FIDO’s (e.g. Earth-fixed axes).

2) The fictitious force due to a non-constant angular velocity, $m[\vec{r} \times d\vec{\Omega}/dt]$, which corresponds to a gravitoelectric induction field,

$$\vec{E}_g(x, t) \to (\vec{E}_g)_{\text{asy}} = [\vec{r} \times \frac{d}{dt} \vec{\Omega}^{\text{asy}}_{\text{gyro}}].$$

The asymptotic $\vec{B}_g$ and $\vec{E}_g$ fields are solutions of the source-free Einstein equations, i.e. they do not have any sources within our system (solar system). Mach’s principle implies that the sources for these fields are given by cosmological matter (which is at infinity from the point of view of the solar system), as we have shown in Eqs. (23).

One must distinguish two types of boundary conditions: 1) Boundary conditions which allow to include and encode the contribution of sources outside the system considered. 2) Boundary conditions which exclude unphysical solutions, e.g. the exponentially growing solution of Eq. (10). Boundary conditions discussed in the context of Mach’s principle refer to the encoding type.

In this paper we have shown that in cosmological gravitomagnetism there is no need for boundary conditions of the encoding type, because of the exponential cutoff for the contributions by the measured energy-momentum tensor of matter.

Einstein and others had hoped for some time that one could implement the ideas of Mach by specifying some boundary conditions on the equations of General Relativity. But in 1934 Einstein concluded that the problem of boundary conditions (for Mach’s principle) could only be solved by going to a spatially closed universe:

“In my opinion the general theory of relativity can only solve this problem of inertia satisfactorily if it regards the world as spatially self-enclosed.”

In contrast we have shown in this paper that General Relativity for linear cosmological perturbations of a spatially flat FRW universe satisfies Mach’s principle without the need for boundary conditions of the encoding type. In a companion paper [23] we shall show that this is also true for perturbations of FRW universes with $k = -1$, i.e. for open, spatially hyperbolic universes.

XII. MACH’S PRINCIPLE IN THE PRESENCE OF SCALAR AND TENSOR PERTURBATIONS

Linear scalar perturbations cause weak gravitational lensing. The geodesics from the gyroscope to quasars (on $\Sigma_t$) will suffer geodesic deviation. This forces us to adapt the method of fixing the spatial axes of our FIDO’s, i.e. our LONB’s, as follows: We demand that the spatial axes of the LONB’s do not rotate relative to the average of the directions of geodesics (on $\Sigma_t$) to distant quasars, where the weight function in the average is the toroidal vector field $\tilde{X}^{-}_{\ell=1,m} = -i\tilde{L}Y_{\ell=1,m}$ [23], which is tangential to the 2-sphere,

$$0 = \int dr W(r) \int d\Omega(\tilde{X}^{-}_{\ell=1,m})^* \cdot \vec{v}_{\text{asy}}^{\text{tang}}.$$  

The FIDO at the origin determines $\vec{v}_{\text{asy}}^{\text{tang}}$ from the measured angular velocities of tangents to geodesics on $\Sigma_t$ from the origin to quasars in the asymptotic FRW universe. An observational window function in the asymptotic FRW universe is denoted by $W(r)$.

The vector field of distortions by weak gravitational lensing belongs to the scalar sector. This vector field is orthogonal (under integration over the 2-sphere) to the toroidal vector spherical harmonics $\tilde{X}^{-}_{\ell=1,m}$, because the two fields belong to the opposite parity sequence. The scalar sector has $P = (-1)\ell^2$; the toroidal vector spherical harmonics have $P = (-1)^{\ell+1}$. Therefore scalar perturbations cannot affect Mach’s principle, if the gyroscope
axes are measured relative to the average of the directions of geodesics to asymptotic quasars with the weight function $X_{\ell=1,m}$. Gravitational waves at the linear level in the perturbed region between asymptotic FRW space and the gyroscope also cannot affect the average of geodesics in Eq. (58), again because the tensor sector is orthogonal to the vector sector.

Gravitational waves going through the gyroscope cannot produce a gyroscope precession (relative to quasars in the asymptotic unperturbed universe). This is because the precession of the gyroscope only depends on the displacement of unit measured time along the worldline of the gyroscope. These three rotation coefficients form an axial 3-vector, which cannot be affected by a 3-tensor (gravitational wave sector) for reasons of symmetry under 3-rotations.

If angular velocities are measured relative to the gyroscope axes (not relative to distant quasars), one has the value $\Omega_{\text{gyro}} = 0$ on the left-hand side of Eq. (12), and there is no need to modify the formula due to scalar or tensor perturbations, because Eq. (12) automatically performs a projection on $\{\ell = 1, \text{odd parity sequence}\}$. Einstein in his paper “Prinzipien zur allgemeinen Relativitätstheorie” of 1918 [3] coined the term “Mach’s Principle” and gave the following formulation for it:

“The $G$-field is entirely determined ... by the energy tensor of matter.”

The $G$-field refers to the metric tensor field $g_{\mu\nu}(x)$.

For linear cosmological perturbations Einstein’s formulation is too strong to be valid, and unnecessarily strong for Mach’s original purpose of explaining (giving the physical cause for) the evolution of gyroscope axes (i.e. non-rotating frames): 1) Too strong, as has been stated many times, because $g_{\mu\nu}$ can contain gravitational waves, which can exist without matter being present. 2) Unnecessarily strong, as we shall now explain: Gravitational waves cannot affect gyroscope axes directly, because a tensor perturbation cannot affect the spin evolution, $d\hat{S}/dt$, which is an axial vector. And gravitational waves cannot affect the $\langle X_{\ell=1,m} \rangle$-average of the geodesics from a gyroscope to the asymptotic unperturbed quasars.

In our approximation of linear cosmological perturbations a third question remains unaddressed: Since the gyroscope’s precession is determined by the $\vec{B}_g$ field at the gyroscope’s position, is $\vec{B}_g$ determined by an equation which has a gravitational pseudo-angular-momentum tensor (in addition to the angular momentum of matter) as a source term? Such an equation could not be Einstein’s equation in the standard way of separating output curvature terms from the matter input.

XIII. OUTLOOK

1. The analysis of this paper has been extended to FRW universes with $k = +1$ (spatially closed, spherical) and with $k = -1$ (spatially open, hyperbolic) with analogous results. This will be presented in a companion paper [22].

2. Causality and astronomical observations make a formulation desirable, where $\vec{B}_g$ at the space-time point $P$ is given by an integral over sources on the backward light-cone of $P$.

3. Second order perturbation theory: Does perfect dragging of gyroscope axes still hold? What is the contribution of gravitational waves to dragging of gyroscope axes? — Can one derive exact results about perfect dragging of gyroscope axes?

4. Does the Bianchi IX model of Ozsváth and Schücking satisfy Mach’s principle (which involves an integral over energy flows in the universe)?

5. Accelerated frames of reference include not only rotating frames (extensively discussed by Mach), but also frames with linear acceleration relative to inertial frames. Does cosmological matter (with an exponential cutoff) cause perfect frame dragging with respect to linear accelerations?

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[1] http://einstein.stanford.edu
[2] H. Bondi and J. Samuel, gr-qc/9607009
[3] E. Mach, Die Mechanik in Ihrer Entwicklung: Historisch-Kritisch Dargestellt (Brockhaus, Leipzig, 1. Auflage 1883; 7., verbesserte und vermehrte Auflage 1912). English translation by T.J. McCormack, The Science of Mechanics: A Critical and Historical Account of Its Development (Open Court Publishing, La Salle, Ill., 6th American ed., 1960, based on the 9th German edition of 1933).
[4] E. Mach, Die Geschichte und die Wurzel des Satzes von der Erhaltung der Arbeit (Calve, Prague, 1872; 2nd ed., Johann Ambrosius Barth, Leipzig, 1879); History and Roots of the Principle of the Conservation of Energy (Open Court, Chicago, 1911).
[5] Reference [3], chap. 2, sec. 6, subsec. 5.
[6] I. Ciufolini and J. A. Wheeler, Gravitation and Inertia
(Princeton University Press, Princeton, NJ, 1995).

[7] J.B. Barbour and H. Pfister, editors, Mach’s Principle: From Newton’s Bucket to Quantum Gravity (Birkhäuser, Boston, 1995). Index of Different Formulations of Mach’s Principle on p. 530.

[8] For reviews of the history of Mach’s principle see also [6], particularly Refs. 12-19 in chapter 5 of [6].

[9] A. Einstein, Ann. Phys. (Leipzig) 55, 241 (1918).

[10] J. Ehlers in [7], p. 458.

[11] H. Thirring, Phys. Zeitschr. 19, 33 (1918); 22, 29(E) (1921).

[12] D.R. Brill and J.M. Cohen, Phys. Rev. 143, 1011 (1966).

[13] L. Lindblom and D.R. Brill, Phys. Rev. D 10, 3151 (1974).

[14] Ref. [24], sect. 21.12, Mach’s Principle and the Origin of Inertia, pp. 548-9.

[15] C. Klein, Class. Quantum Grav. 10, 1619 (1993), and references therein.

[16] D. Lynden-Bell, J. Katz, and J. Bičák, Mon. Not. R. Astron. Soc. 272, 150 (1995); 277, 1600(E) (1995), and references therein.

[17] T. Dolčel, J. Bičák, and N. Deruelle, Class. Quantum Grav. 17, 2719 (2000).

[18] C. Schmid, gr-qc/0201095.

[19] C. Schmid, gr-qc/0409026.

[20] J. Bičák, D. Lynden-Bell, and J. Katz, Phys. Rev. D 69, 064011 (2004).

[21] J.M. Bardeen, Phys. Rev. D 22, 1882 (1980).

[22] J. Katz, D. Lynden-Bell, and J. Bičák, Class. Quantum Grav. 15, 3177 (1998).

[23] C. Schmid, Mach’s Principle for Perturbations of Friedmann-Robertson-Walker Universes with $k = (±1,0)$ (in preparation).

[24] C.W. Misner, K.S. Thorne, and J.A. Wheeler, Gravitation (Freeman, San Francisco, 1973).

[25] H. Kodama and M. Sazaki, Progr. Theor. Phys. Suppl. 78, 1 (1984).

[26] J.M. Bardeen, in Cosmology and Particle Physics, Proc. CCAST Symposium held at Nanjing, People’s Republic of China, July 1988, ed. L.-Z. Fang and A. Zee (Gordon and Breach, New York, 1988). The quote is from p. 17.

[27] K.S. Thorne, R.H. Price, and D.A. MacDonald, editors, Black Holes, The Membrane Paradigm (Yale University Press, New Haven, CT, 1986).

[28] E. Cartan, Riemannian Geometry in an Orthogonal Frame, from lectures delivered by Elie Cartan at the Sorbonne in 1926-27 (World Scientific, Singapore, 2001).

[29] C. Schmid, D.J. Schwarz, and P. Widerin, Phys. Rev. D 59, 043517 (1999).

[30] Ref. [6], p. 77-80. Quoted in Ref. [7], p. 107.

[31] A. Einstein, “Autobiographical Notes” in Albert Einstein, Philosopher-Scientist, ed. P.A. Schilpp (The Library of Living Philosophers, Inc., Evanston, Illinois, 1949), p. 29.

[32] Albert Einstein (Princeton) to Felix Pirani (Cambridge, Eng.), 2 February 1954, Albert Einstein Archives, The Hebrew University of Jerusalem, Call no. 17-447. [3 typed sheets]

[33] Reference [6], p.93.

[34] I. Ozsváth, and E. Schücking, Nature (London) 193, 1168 (1962), and Ann. Phys. (N.Y.) 55, 166 (1969).

[35] A. Einstein, Essays in Science (Philosophical Library, New York, 1934), p.52; translated from Mein Weltbild (Querido Verlag, Amsterdam, 1933).