On Quantum Radiation in Curved Spacetime

She-Sheng Xue

ICRA, INFN and Physics Department, University of Rome “La Sapienza”, 00185 Rome, Italy

In the context of quantum field theories in curved spacetime, we compute the effective action of the transition amplitude from vacuum to vacuum in the presence of an external gravitational field. The imaginary part of resulted effective action determines the probability of vacuum decay via quantum tunneling process, giving the rate and spectrum of particle creations. We show that gravitational field polarizes vacuum and discretizes its spectrum for such a polarization gains gravitational energy. On the basis of gravitational vacuum polarization, we discuss the quantum origin of vacuum decay in curved spacetime as pair-creations of particles and anti-particles. The thermal spectrum of particle creations is attributed to (i) the CPT invariance of pair-creations (annihilations) from (into) vacuum and (ii) vacuum acts as a reserve with the temperature determined by gravitational energy-gain.

PACS numbers: 04.62.+v, 04.70.Dy

I. INTRODUCTION.

The issue of quantum field theories of elementary particles in curved spacetime has shown a tremendously important role of understanding quantum phenomenon of particle creations in the presence of gravitational field. The Hawking radiation is one of such phenomena occurring around black hole’s horizon. It has advocated a numerous studies in the last three decades. Briefly and broadly speaking, in these studies there are two approaches to the Hawking radiation. In the first, one considers an incoming and outgoing wave destoried in a collapse geometry, and the Hawking radiation can explicitly and implicitly be approached by an appropriate boundary condition. In the second, one treats a black hole immersed in a thermal bath, implying the Hawking radiation from black holes by the detail balance. These approaches do not very directly reveal the genuine quantum process taking place for the source of the Hawking radiation, although, the Hawking radiation is heuristically considered as pair-creations of particles and antiparticles by a quantum effect of tunneling, analogous to the Euler-Heisenberg-Schwinger process in QED. In recent studies, the semi-classical WKB-method for quantum tunneling process is adopted

*Electronic address: xue@icra.it*
to reveal the quantum tunneling nature of the Hawking radiation and its back reaction. For explicitly revealing the genuine origin of quantum radiation in curved spacetime, it is essential to analyze the effect in the context of quantum field theories. This is also very important for understanding microscopic origin of entropy in black hole thermodynamics and the problem of unitality.

In a path-integral framework of quantum field theories, we compute the effective action of transition amplitude from vacuum to vacuum in curved spacetime. With the Schwarzschild geometry, we obtain the imaginary part of effective action, which gives rise to the probability of vacuum decay via quantum tunneling process for pair-creations of particles and anti-particles. We discuss that this quantum emission is dynamically attributed to (i) quantum-field fluctuations of positive- and negative-energy virtual particles in vacuum are polarized by external gravitational field, (ii) such a vacuum polarization gains gravitational energy and (iii) effective mass-gap separating positive-energy particles from negative-energy particles is very small. The energy spectrum of vacuum is discretized by external gravitational field. The spectrum of particle creations is determined by quantum emissions(absorptions) from(into) vacuum, in unit of quanta of the discrete energy spectrum of vacuum.

II. GENERAL FORMULATION.

We assume that the structure of spacetime is described by the pseudo-Riemannian metric $g_{\mu\nu}$ associated with the line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \quad \mu, \nu = 0, 1, 2, 3,$$

and the spacetime point is coordinated by $x = (x^0, x^i) = (t, \vec{x})$. The special geometrical symmetries of the spacetime $\mathcal{S}$ are described by using Killing vectors $\xi^\mu$, which are solutions of Killing’s equation

$$\mathcal{L}_\xi g_{\mu\nu}(x) = 0, \quad \xi_{\mu;\nu} + \xi_{\nu;\mu} = 0,$$

where $\mathcal{L}_\xi$ is the Lie derivative along the vector field $\xi^\mu$, orthogonal to the spacelike hypersurface $\Sigma_t$ ($t=$constant) of the spacetime $\mathcal{S}$. A static observer $O$ is at rest in this hypersurface $\Sigma_t$. We consider quantum-field fluctuations interacting with curved spacetime.

In order to clearly illustrate physics content, we first consider a complex scalar field $\phi$ in curved spacetime. The simplest coordinate-invariant action is given by ($\hbar = c = G = k = 1$)

$$S = \frac{1}{2} \int d^4 x \sqrt{-g} \left[ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi^* + (m^2 + \xi \mathcal{R}) \phi \phi^* \right],$$
where $m$ is particle mass and $\mathcal{R}$ the Riemann scalar. The quantum scalar field $\phi$ can be in principle expressed in terms of a complete and orthogonal basis of quantum-field states $u_k(x)$:

$$\phi(x) = \sum_k \left( a_k u_k(x) + a_k^\dagger u_k^*(x) \right), \quad \left[ a_k, a_{k'}^\dagger \right] = \delta_{k,k'}$$

(4)

where $a_k^\dagger$ and $a_k$ are creation and annihilation operators of the $k$-th quantum-field state $u_k(x)$. This quantum field state obeys the following equation of motion,

$$(\Delta x + m^2 + \xi \mathcal{R}) u_k(x) = 0,$$

(5)

and appropriate boundary conditions for selected values of $k$. In Eq.(5), $\Delta x$ is the Laplacian operator in curved spacetime:

$$\Delta x = \frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu\nu} \partial_{\nu} \right).$$

(6)

In Eq.(4), we assume that $u_k(x)$ are positive energy ($\omega$) states, satisfying

$$\mathcal{L}_\xi u_k(x) = -i\omega u_k(x), \quad \omega > 0,$$

(7)

with respect to the timelike Killing vector field $\xi^\mu$ (2) associated to the static observer $O$.

We assume that in the remote past ($t \to -\infty$), massive dust is uniformly distributed that the spacetime is approximately flat. Quantum-field states $u_k(x)$ (5) in the remote past ($t \to -\infty$) are asymptotically free states $\bar{u}_k(x)$, obeying Eq.(5) for $g_{\mu\nu} \simeq (1, -1, -1, -1)$. Then, the quantum scalar field $\phi(x)$ is an asymptotically free field in the hypersurface $\Sigma_{-\infty}$ of the spacetime $\mathcal{S}$:

$$\phi_{\text{in}}(x) = \sum_k \left( \bar{a}_k \bar{u}_k(x) + \bar{a}_k^\dagger \bar{u}_k^*(x) \right), \quad \left[ \bar{a}_k, \bar{a}_{k'}^\dagger \right] = \delta_{k,k'}$$

(8)

where $\bar{x} \in \Sigma_{-\infty}$, $\bar{a}_k^\dagger$ and $\bar{a}_k$ are creation and annihilation operators of the $k$-th asymptotically free quantum-field state $\bar{u}_k(x)$. Corresponding Lie derivative along the Killing vector (2) is $\partial_t$, positive energy states are $\bar{u}_k(x)$, satisfying Eq.(7). Then we may construct the standard Minkowski space quantum vacuum state $|\bar{0}, \text{in}\rangle$:

$$\bar{a}_k |\bar{0}, \text{in}\rangle = 0, \quad \langle \bar{0}, \text{in}| \bar{a}_k^\dagger = 0.$$
in the remote future \((t \to +\infty)\) are asymptotical states \(\tilde{u}_k(x)\), obeying Eq.\((5)\) for a stationary geometry \(g_{\mu\nu}\). In the hypersurface \(\Sigma_{+\infty}\) of the spacetime \(\mathcal{S}\), the asymptotical quantum scalar field \(\phi\) in the remote future \((t \to +\infty)\) is expressed in terms of \(\tilde{u}_k(x)\):

\[
\phi_{\text{out}}(x) = \sum_k \left( \tilde{a}_k \tilde{u}_k(x) + \tilde{a}^\dagger_k \tilde{u}^*_k(x) \right), \quad \left[ \tilde{a}_k, \tilde{a}^\dagger_{k'} \right] = \delta_{k,k'}
\]  

(10)

where \(\vec{x} \in \Sigma_{+\infty}\), \(\tilde{a}_k^\dagger\) and \(\tilde{a}_k\) are creation and annihilation operators of the \(k\)-th quantum-field state \(\tilde{u}_k(x)\). Corresponding Lie derivative along the Killing vector is \(\xi^\mu\), positive energy states are \(\tilde{u}_k(x)\) satisfying Eq.\((7)\). Then we may construct the quantum vacuum state \(|\tilde{0},\text{out}\rangle\):

\[
\tilde{a}_k|\tilde{0},\text{out}\rangle = 0, \quad \langle \tilde{0},\text{out}|\tilde{a}^\dagger_k = 0,
\]  

(11)

in curved spacetime. \(|\tilde{0},\text{out}\rangle\) is an final quantum vacuum state in the remote future \((t \to +\infty)\) with respect to the same static observer \(O\).

It is worthwhile to note that \(\phi_{\text{out}}(x)(\{\tilde{u}_k(x)\})\) are not asymptotically free field(states), instead they are asymptotical field(states) in the presence of external stationary gravitational field, so that the final vacuum state \(|\tilde{0},\text{out}\rangle\) \((\text{IV})\) is different from the initial vacuum state \(|\bar{0},\text{in}\rangle\) \((\text{IX})\). As will be seen, such a difference is not only a unitary phase. The final vacuum state \(|\tilde{0},\text{out}\rangle\) \((\text{IV})\), may not necessarily be measured as devoid of particles, in contrast to the initial vacuum state \(|\bar{0},\text{in}\rangle\) defined by Eq.\((9)\) relating to the asymptotically free field \(\phi_{\text{in}}(x)\). In fact, as will be shown, the final vacuum state \(|\tilde{0},\text{out}\rangle\) is a quantum-field state of particle and antiparticle creations upon the initial vacuum state \(|\bar{0},\text{in}\rangle\). This indicates gravitational field interacting with quantum-field fluctuations of positive and negative energy states of the initial vacuum \(|\bar{0},\text{in}\rangle\), and the quantum scalar field evolves throughout intermediate states \(\phi(\vec{x},t)\) \((\text{II})\) for \(-\infty < t < +\infty\). We speculate that this evolution is adiabatic for the reasons that gravitational-field energy involved is very small and the evolution takes infinite time.

To deal with all possible intermediate states, represented by \(\phi(x)\) or \(u_k(x)\) in Eq.\((\text{II})\), for \(-\infty < t < +\infty\), we use path-integral representation to study the transition amplitude between the initial vacuum state and final vacuum state:

\[
\langle \tilde{0},\text{out}|\bar{0},\text{in}\rangle = \int [D\phi D\phi^*] \exp(iS),
\]  

(12)

where

\[
\int [D\phi D\phi^*] = \Pi_{-\infty<t<+\infty}\Pi_{\vec{x}\in\Sigma_t}\int [d\phi(\vec{x},t)\phi^*(\vec{x},t)].
\]  

(13)
The intermediate states contributions to the transition amplitude (12) can be formally path-integrated,

\[ \langle \bar{0}, \text{out}|0, \text{in} \rangle = \det^{-1}(M), \quad M = \Delta_x + m^2 + \xi R. \]  

(14)

This result is in terms of \( \phi_{\text{in}} \) (3) and \( \phi_{\text{out}} \) (10), which are not explicitly written. The effective action \( S_{\text{eff}} \) is defined as

\[ S_{\text{eff}} = -i \ln \langle \bar{0}, \text{out}|0, \text{in} \rangle, \]  

(15)

which relates to the phase of the \( S \)-matrix transition from the initial vacuum state \( |\bar{0}, \text{in}\rangle \) to the final vacuum state \( |0, \text{out}\rangle \). We focus on the calculations of the effective action (15) in this article.

In order to evaluate the path-integral (12) over all intermediate states \( \phi(x) \) (4), it is convenient to introduce operators \( \hat{X}_\mu \) and \( \hat{K}_\mu \) defined on the states \( |x\rangle \) and \( |k\rangle \):

\[ \hat{X}_\mu |x\rangle = x_\mu |x\rangle; \quad \hat{K}_\mu |k\rangle = k_\mu |k\rangle. \]  

(16)

They enjoy the canonical communication,

\[ [\hat{X}_\mu, \hat{K}_\nu] = -ig_{\mu\nu}. \]  

(17)

The states \( |x\rangle \) and \( |k\rangle \) satisfy:

\[ \langle x|x'\rangle = \delta(x - x'), \quad \int dx|x\rangle\langle x| = 1 \]
\[ \langle k|k'\rangle = 2\pi\delta(k - k'), \quad \int dk|k\rangle\langle k| = 1, \]  

(18)

and intermediate quantum-field state \( u_k(x) \) can be represented as

\[ u_k(x) = \langle x|k\rangle. \]  

(19)

Using these matrix notations, we write the operator \( \mathcal{M}(\hat{X}, \hat{K}) \) (14) as a hermitian matrix

\[ \mathcal{M}_{k,k'} = \int dx dx' \langle k|x\rangle \langle x'|\mathcal{M}(\hat{X}, \hat{K})|x\rangle \langle x'|k'\rangle \]
\[ = \int dx u_k^*(x) \mathcal{M}(x, \hat{K}) u_{k'}(x). \]  

(20)

In this representation, diagonalizing this hermitian matrix \( \mathcal{M}_{k,k'} \), we formally compute the effective action \( S_{\text{eff}} \) given in Eqs. (15,14):

\[ iS_{\text{eff}} = -\text{tr} \ln(\mathcal{M}) = -\int \frac{d^4 k}{(2\pi)^4} \ln \lambda_k^2, \]  

(21)
where the \( \{ \lambda_k^2 \} \) is the diagonal element of the matrix \( \lambda^2 \)

\[
\lambda_k^2 = \int \sqrt{-g} d^4 x \tilde{u}_k^*(x) M(x, \hat{K}) \tilde{u}_k(x),
\]

in the phase space \( (k) \) of the quantum field states \( \{ \tilde{u}_k(x) \} \). The operator \( \lambda_k^2 \) and the number of quantum-field states \( \int \sqrt{-g} d^4 x d^4 k / (2\pi)^4 \) are invariant in arbitrary coordinate systems, later is the Liouville theorem for the phase-space invariance. In Eqs. (21) and (22), the term that is independent of non-trivial geometry \( g_{\mu\nu} \) has been dropped.

### III. SPECTRUM OF VACUUM.

At the remote future \( (t \rightarrow +\infty) \), we assume the geometry of the spacetime '"S' outside of a massive object \( M \) \((r > 2M)\) is stationary and spherical, e.g., the Schwarzschild geometry,

\[
ds^2 = g_{tt} dt^2 + g_{rr} dr^2 + r^2 d\Omega,
\]

\[
-g_{tt} = (g_{rr})^{-1} = g(r) \equiv (1 - 2M/r)
\]

where \( \Omega \) is the spherical solid angle and \( r, \theta, \phi, t \) are the Schwarzschild coordinates. The static observer located at \( O \) at \( x = (t, r, \Omega) \), whose four velocity \( u_\mu \) and Killing vector \( \xi_\mu \) are given by,

\[
u_\mu = (g_{tt}(r), 0, 0, 0), \quad \xi_\mu = (g_{tt}(r), 0, 0, 0),
\]

which are orthogonal to the spacelike hypersurface \( \Sigma_t \) \((t = \text{constant}, \ x = (r, \Omega) \in \Sigma_t)\). We will respectively discuss the static observers locates at finite radius \( (r = r^* < \infty) \) and at infinity \( (r = r^* \rightarrow \infty) \) of the hypersurface \( \Sigma_t \). The Riemann scalar \( R \) = 0. The Laplacian operator \( \Box \) is given by:

\[
\Delta_x = g^{tt} \frac{\partial^2}{\partial t^2} + \frac{1}{r^2} \frac{\partial}{\partial r} r^2 g^{rr} \frac{\partial}{\partial r} + \frac{\hat{L}^2}{r^2}
\]

\[
= -g^{-1}(r) \frac{\partial^2}{\partial t^2} + g(r) \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right)
\]

\[
+ \frac{2M}{r^2} \frac{\partial}{\partial r} + \frac{\hat{L}^2}{r^2},
\]

where \( \hat{L}^2 \) is the angular momentum operator.

The final vacuum state \( |0, \text{out} \rangle \) of this geometry is usually denoted as the Schwarzschild vacuum \( |0_S \rangle \). The appropriate basis of asymptotical quantum field \( \phi_{\text{out}}(x) \) \( (10) \) is chosen as

\[
\tilde{u}_k(x) = \langle t, r, \theta, \phi | \omega, k_r, l, m \rangle = R_{l\omega}(r) Y_{lm}(\theta, \phi) e^{i \omega t},
\]
where \( k \) indicates a set of quantum numbers \((\omega, k_r, l, m)\). \( \omega \) is the energy-spectrum and \( Y_{lm}(\theta, \phi) \) is the standard spherical harmonic function: \( \hat{L}^2 Y_{lm}(\theta, \phi) = l(l+1)Y_{lm}(\theta, \phi) \). The radial function \( R_{l\omega}(r) \) obeys the following differential equation for \( r > 2M \),

\[
\left[ \omega^2 - g^2(r) \hat{k}_r^2 + i \frac{2M}{r^2} g(r) \hat{k}_r - g(r)V_l(r) \right] R_{l\omega}(r) = 0,
\]

(27)

where the hermitian radial momentum operator,

\[
\hat{k}_r = \frac{1}{ir} \frac{\partial}{\partial r} r, \quad (\hat{k}_r)^2 = - \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right),
\]

(28)

and potential

\[
V_l(r) = \frac{l(l+1)}{r^2} + \frac{2M}{r^3} + m^2.
\]

(29)

Eq. (27) is exactly equivalent to the Regge and Wheeler equation. The radial function \( R_{l\omega}(r) \) is orthogonal and asymptotically behaves as the Hankel function \( h_{l\omega}(r) \) for \( r \gg 2M \).

The matrix operator \( \mathcal{M} \) in Eq. (22) is given by,

\[
\mathcal{M}(r, \hat{k}_r) = \omega^2 - g^2(r) \hat{k}_r^2 + i \frac{2M}{r^2} g(r) \hat{k}_r - g(r)V_l(r).
\]

(30)

Eqs. (26-30) define a complex eigen-value problem to find the energy spectrum of the vacuum state (11) in an external gravitation field. The imaginary part of the operator \( \mathcal{M}(r, \hat{k}_r) \) (30), giving rise to an imaginary part of the effective action (15), implies that the gravitating quantum-field system (3) be energetically unstable and quantum-field tunneling could occur, leading to the productions of particles and antiparticles.

Up to an irrelevant constant in Eq. (21), the diagonal matrix \( \lambda^2_k \) (22) is given by:

\[
\lambda^2_k \equiv N_k^{-1} \int \sqrt{-g} d^4x \bar{u}_k(x)(\Delta x + m^2)u_k^*(x),
\]

\[
= N_k^{-1} \int dr r^2 R^*_{l\omega}(r) \mathcal{M}(r, \hat{K}_r) R_{l\omega}(r),
\]

(31)

where the normalization factor is

\[
N_k = \int dr r^2 R^*_{l\omega}(r)R_{l\omega}(r).
\]

(32)

The elements of the diagonal matrix \( \lambda^2_k \) (31) is related to the inverse propagator (spectrum) of particles in the external gravitation field. We write Eq. (31) as,

\[
\lambda^2_k = \omega^2 - \kappa^2_k - m^2_k + i2M\epsilon_k,
\]

(33)
where we define the average values $\kappa_k^2$, $\epsilon_k$ and effective mass-gap $m_k$ of the quantum field state $R_{l\omega}(r)$:

$$
\kappa_k^2 \equiv N_k^{-1} \int dr r^2 R_{l\omega}^*(r) g^2(r) \kappa_r^2 R_{l\omega}(r); \quad (34)
$$

$$
m_k^2 \equiv N_k^{-1} \int dr r^2 R_{l\omega}^*(r) g(r) V_l(r) R_{l\omega}(r); \quad (35)
$$

$$
\epsilon_k \equiv N_k^{-1} \int dr r^2 R_{l\omega}^*(r) \frac{g(r)}{r^2} \hat{k}_r R_{l\omega}(r). \quad (36)
$$

By the definition (36), the sign of $\epsilon$ is determined by the sign of the radial momentum $k_r$ of quantum field state $R_{l\omega}(r)$,

$$
k_r \equiv N_k^{-1} \int dr r^2 R_{l\omega}^*(r) \hat{k}_r R_{l\omega}(r). \quad (37)
$$

$\epsilon_k > 0$ indicates that the radial momentum of the state $R_{l\omega}(r)$ is positive ($k_r > 0$), standing for outgoing states of positive-energy particles $\omega > 0$. Whereas, $\epsilon_k < 0$ indicates that the radial momentum of the state $R_{l\omega}^*(r)$ is negative ($k_r < 0$), standing for incoming states of negative energy particles $\omega < 0$. This is in accordance with Feynman’s $\omega \rightarrow \omega \pm i2M\epsilon_k$ prescription for particles(+) and antiparticles(-), latter are outgoing states with positive energy ($\omega > 0$) and positive radial momentum, traveling backward in time. In this prescription, we have positive energy $\omega > 0$ and positive radial momentum $|k_r|$ (37) for both particles and antiparticles.

**IV. EFFECTIVE ACTION.**

Given $\lambda_k^2$ (31), we can write the effective action (21) as,

$$
iS_{\text{eff}} = -\int \frac{d^4k}{(2\pi)^4} \ln(\lambda_k^2) = -\int \frac{d\omega dk_r}{(2\pi)^2} \sum_{l,m} \ln(\lambda_k^2). \quad (38)
$$

Using the identity:

$$
\ln \frac{a}{b} = \int_0^\infty \frac{ds}{s} \left( e^{is(b+ie)} - e^{is(a+ie)} \right), \quad (39)
$$

we are able to write the effective action (38)

$$
S_{\text{eff}} = -i \int \frac{d\omega dk_r}{(2\pi)^2} \sum_{l,m} \int_0^\infty \frac{ds}{s} e^{is(\lambda_k^2+ie)} - (M = 0), \quad (40)
$$

where the expression of the second part indicated by $(M = 0)$ is the same as the expression of the first part with $M = 0$. The logarithmatic function in Eq. (38) is represented by an $s$-integration in Eq. (40) and infrared convergence at $s \rightarrow 0$ is insured by $i\epsilon$ prescription $(\epsilon \rightarrow 0)$. 
We begin with the computation of summing over quantum numbers “l, m” of angular momenta in Eq. (40). Eqs. (29, 30, 31) show that \( \lambda_k^2 \) depends on “l” in terms of \( l(l + 1)/r^2 \) and \( R_{l\omega}(r) \). We find that the l-dependence is dominated by the quadratic term \( l(l + 1)/r^2 \), in contrast with this, \( R_{l\omega}(r) \) is a smooth function in varying “l”. Thus, to achieve the leading contribution to the effective action (40), we approximate the quantum-field state \( R_{l\omega}(r) \) in \( \lambda_k^2 \) as:

\[
R_{l\omega}(r) \simeq R_{l^*\omega}(r)
\]  

where \( l^* \) is a particular value of angular quantum number, for an example, \( l^* = 0 \) for the spherically symmetric state. Then, in Eq. (40) we approximately have,

\[
\sum_{l,m} e^{is(\lambda_k^2 + i\epsilon)} \simeq e^{is(\Lambda_k^2 + i\epsilon)} \sum_{l,m} e^{-is\beta_l(l+1)},
\]

where

\[
\Lambda_k^2 = \lambda_k^2 + \beta_k l(l + 1) \\
\beta_k = \frac{N}{k^2} \int_0^\infty dr r^2 |R_{l\omega}(r)|^2 \left( \frac{g(r)}{r^2} \right).
\]

In addition, we introduce continuous and dimensionless transverse momenta \( \vec{k}_\perp \) and \( k_\perp^2 = l(l + 1) \) so that,

\[
\sum_{l,m} e^{-is\beta_l(l+1)} \simeq 4\pi \int \frac{d^2k_\perp}{(2\pi)^2} e^{-is\beta_k k_\perp^2} = \frac{1}{is\beta_k},
\]

where \( \beta_k \) is analytically continued to complex value and \( \text{Im}(\beta_k) < 0 \). As a result, we obtain the effective action (40),

\[
S_{\text{eff}} \simeq -\int d\omega dk_r \frac{1}{(2\pi)^2} \frac{1}{\beta_k} \int_0^\infty \frac{ds}{s^2} e^{is(\Lambda_k^2 + i\epsilon)} - (M = 0).
\]

In order to compute the integration over “s” in Eq. (46), we introduce a complex variable \( z = -1 + \delta \ (|\delta| \to 0) \), and use the following integral representation of the \( \Gamma(z) \)-function by an analytical continuation for \( \text{Im}(\alpha) > 0 \):

\[
\int_0^\infty e^{i\alpha s} s^{z-1} ds = (-i\alpha)^{-z} \Gamma(z).
\]

This analytical continuation is equivalent to the analytical continuation of dimensionality of the transverse momentum-space,

\[
\int \frac{d^2k_\perp}{(2\pi)^2} \to \int \frac{d^{2+\delta}k_\perp}{(2\pi)^{2+\delta}},
\]
in Eq. (45). In the neighborhood of singularity, where $|\delta| \to 0$ and $z \to -1$ in Eq. (47), we have

$$\Gamma(z) = -\frac{1}{\delta} - 1 + \gamma + O(\delta),$$

$$\alpha^{-z} = e^{-\delta \ln \frac{\alpha}{\mu^2}},$$

$$(-i)^{-z} = -ie^{i\theta(-\frac{\pi}{2} + 2\pi n')},$$

$$n' = 0, 1, 2, 3 \cdots \quad (49)$$

where the Euler constant $\gamma \simeq 0.577$ and $\mu$ is an arbitrary scale of dimensional transmutation in the dimensional regularization. The singular term $1/\delta$ corresponds to the ultra-violet divergence of summing over $(l, m)$ in Eq. (50).

Analogously to the treatment of dimensional and $\zeta$-functional regularization schemes, using Eqs. (47, 49) to calculate integration over “$s$” in Eq. (46) and keeping up to the order $O(\delta^0)$, we cast the effective action Eq. (46) to be:

$$S_{\text{eff}} = -\int \frac{d\omega dk}{(2\pi)^2} \Lambda_k^2 \beta_k \left[ \frac{\pi}{2} - 2\pi n' \right] + i \left( \frac{1}{\delta} + 1 - \gamma - \ln \frac{\Lambda_k^2}{\mu^2} \right) - (M = 0). \quad (50)$$

In the normal prescription of renormalization of quantum field theories, we consistently add an appropriate counterterm to cancel the ultra-violet divergent term $1/\delta$ at the energy-scale $\mu$ in the minimum subtraction scheme.

**V. IMAGINARY PART OF THE EFFECTIVE ACTION.**

We are interested in the imaginary part of the effective action for studying the vacuum decay via quantum tunneling through the effective energy gap $m_k$, leading to pair-creations of particles and anti-particles. Given $\Lambda_k^2$ and $\lambda_k^2$, we have

$$\text{Im} \left( \ln \frac{\Lambda_k^2}{\mu^2} \right) = 2\pi n'' + \theta, \quad n'' = 0, \pm 1, \pm 2, \pm 3 \cdots, \quad (51)$$

$$\theta = \tan^{-1} \left[ \frac{\text{Im}(\Lambda_k^2)}{\text{Re}(\Lambda_k^2)} \right], \quad (52)$$

$$\text{Im}(\Lambda_k^2) = 2M\epsilon_k, \quad (53)$$

$$\text{Re}(\Lambda_k^2) = \omega^2 - \kappa_k^2 - m_k^2, \quad (54)$$

where and henceforth the effective mass-gap $m_k^2$ is Eq. (35) without the angular term $l(l + 1)/r^2$. Using Eqs. (36, 41), we define,

$$\Pi_k \equiv \frac{\epsilon_k}{\beta_k} = \frac{\int dr g(r) R^*_\omega(r) \hat{k}_r R_{l\omega}(r)}{\int dr g(r) |R_{l\omega}(r)|^2}. \quad (55)$$
As a result, the imaginary part of the effective action \((50)\) is given by,

\[
\text{Im}(S_{\text{eff}}) = - \int \frac{d\omega dk_r}{(2\pi)^2} \left[ 2M|\Pi_k| \left( \frac{\pi}{2} + \theta - 2\pi n \right) + \frac{\text{Re}(\Lambda^2_k)}{\beta_k} \left( 1 - \frac{|\Lambda^2_k|}{\mu^2} \right) \right] - (M = 0), \tag{56}
\]

\(n = 1, 2, 3, 4, 5, \ldots\)

We take the absolute value \(|\Pi_k|\) of Eq.\((55)\) for both particles and antiparticles, as \(|k_r| > 0\) discussed in the end of section 3. The integer “\(n\)” in Eq.\((56)\) describes all possible quantum bound states of the vacuum in the presence of negative gravitational potential well.

These quantum bound states “\(n\)” of vacuum in the presence of gravitational potential well can be understood by an analogue of semi-classical quantization of a particle confined in a negative spherical potential \(U(r) < 0\),

\[
\int_{r_c}^{r_c} dr |p_r| = \int_{r_c}^{r_c} dr \sqrt{E - U(r)} \simeq (n + \frac{1}{2}), \tag{57}
\]

where \(E < 0\) is the negative energy of particles. \(p_r\) is the radial momentum conjugated to radius \(r\) and integration in Eq.\((57)\) is limited by the classical point: \(U(r_c) = E\).

Quantum-field states tunneling to infinity are free particles and antiparticles in mass-shell \(\text{Re}(\Lambda^2_k) = 0\). Whereas the quantum-field states \(R_{l\omega}(r)\) of vacuum in the presence of gravitational field are bound states due to negative gravitational potential well, this indicates for these states

\[
\text{Re}(\Lambda^2_k) = \omega^2 - \kappa_k^2 - m^2_k \simeq 0^- , \tag{58}
\]

which we will have more discussions in the last section. The approximate “mass-shell condition” \((58)\) leads to \(\theta \simeq -\frac{\pi}{2}\) in Eq.\((52)\), and gives an approximate dispersion relationship \(\omega \simeq \omega(|k_r|)\) between energy \(\omega\) and radial momentum \(|k_r|\) of states \(R_{l\omega}(r)\). This allows us to approximately make integration \(\int \frac{d\omega dk_r}{(2\pi)^2}\) in Eq.\((56)\) with a proper measure \(2\pi\delta[\text{Re}(\Lambda^2_k)]\) in the energy-momentum phase space \((\omega, |k_r|)\). As a result, the effective action \(\text{Im}(S_{\text{eff}}) \) \((50)\) is approximately given by,

\[
\text{Im}(S_{\text{eff}}) \simeq - (-4\pi M|\Pi_k|n) . \tag{59}
\]

per unit of the number of quantum-field states in the range of \(\omega \to \omega + d\omega\).
VI. VACUUM DECAY.

The imaginary part of $S_{\text{eff}}$ determines the probability of vacuum decay caused by quantum tunneling of virtual particles:

$$|\langle \tilde{0}, \text{out} | \tilde{0}, \text{in} \rangle|^2 = e^{-2\text{Im}S_{\text{eff}}},$$

per unit of the number of quantum-field states in $\omega \to \omega + d\omega$. We sum over all possible quantum states “$n$” in Eq.(56) with respect to Bose-distribution in the occupation of these quantum states of vacuum. The total probability of vacuum decay, leading to pair-creations of real particles and antiparticles, is then given as,

$$|\langle \tilde{0}, \text{out} | \tilde{0}, \text{in} \rangle|^2 \simeq \frac{1}{\exp (8M\pi|\Pi_k|) - 1}.$$  (61)

Regarding these outgoing states of particles as an outward radiation flux, Eq.(61) gives the spectrum of such a radiation, with respect to a static observer located at infinity. The characteristic energy scale (temperature) of the spectrum of radiation is the Hawking temperature $1/(8\pi M)$.

As a particular case, we adopt the spherically symmetric solution $l^* = 0$ in Eq.(41) and approximate $\frac{2M}{r} \simeq 0$ in the differential equation Eq.(27) so that the spherically symmetric solution $R_{0\omega}(r) \sim \frac{e^{ikr}}{r}$ is the eigenstate of the radial momentum operators $\hat{k}_r$ and $\hat{k}_r^2$:

$$\hat{k}_r R_{0\omega}(r) = k_r R_{0\omega}(r),$$
$$\hat{k}_r^2 R_{0\omega}(r) = k_r^2 R_{0\omega}(r).$$

(62)

We approximately have $\kappa^2 \simeq k_r^2$ and $m^2 \simeq m^2$ in Eqs.(34,35). As a result of this approximation and Eq.(55), we obtain Eq.(55):

$$|\Pi_k| \simeq |k_r| \simeq \sqrt{\omega^2 - m^2},$$

(63)

and the spectrum of outward radiation Eq.(61), for $m = 0$ massless radiative field, is the black-body spectrum of the Hawking radiation.

We consider a local observer $O$ (24) rest at $r^* > 2M$. With respect to this local observer, we introduce a spherical shell whose radial size $\Delta r \ll r^*$, indicating the variation of the external gravitational field is very small within the shell. We can approximately treat the external gravitational field as a constant field in the shell. On the other hand, $\Delta r$ is much larger than the infrared cutoff of quantum field theories. Given approximate constancy of gravitational field in the shell, we can
approximately make substitution $r \rightarrow r^*$ and $\int dr r^2 \rightarrow \Delta(r^*)^2$, in the definitions $N_k$ (32), $m_k^2$ (35) and $\beta_k$ (37), $\kappa_k^2$ (34), and $\epsilon_k$ (36) become:

$$k_r(r^*) = \left[ \frac{R^*_{l\omega}(r)\hat{k}_r R_{l\omega}(r)}{|R_{l\omega}(r)|^2} \right]_{r = r^*}, \tag{64}$$

$$\kappa_k^2(r^*) = g^2(r^*) \left[ \frac{R^*_{l\omega}(r)\hat{k}_r^2 R_{l\omega}(r)}{|R_{l\omega}(r)|^2} \right]_{r = r^*}; \tag{65}$$

$$\epsilon_k(r^*) = \frac{g(r^*)}{(r^*)^2} \left[ \frac{R^*_{l\omega}(r)\hat{k}_r R_{l\omega}(r)}{|R_{l\omega}(r)|^2} \right]_{r = r^*}. \tag{66}$$

With respect to this local observer, the effective mass-gap $m_k$ (35) is,

$$m_k^2(r^*) = g(r^*) V_I(r^*), \tag{67}$$

and the approximate “mass-shell condition” (58) becomes

$$\text{Re}(\Lambda_k^2)(r^*) = \omega^2 - \kappa_k^2(r^*) - m_k^2(r^*) \simeq 0^-,$$ \tag{68}

where $\kappa_k^2(r^*)$ is given by Eq.(65) and $m_k^2(r^*)$ by Eq.(67). And $\Pi_k$ (55) becomes,

$$\Pi_k(r^*) = \left[ \frac{R^*_{l\omega}(r)\hat{k}_r R_{l\omega}(r)}{|R_{l\omega}(r)|^2} \right]_{r = r^*} = k_r(r^*). \tag{69}$$

Analogously to Eq.(61), the probability of vacuum decay via pair-production is given by,

$$|\langle 0, \text{out} | \bar{0}, \text{in} \rangle|^2 \simeq \frac{1}{\exp(8M\pi|\Pi_k(r^*)|) - 1}, \tag{70}$$

where $\Pi_k(r^*) = k_r(r^*)$ (69) and $k_r(r^*)$ is determined by Eq.(68). For spherically symmetric case $l^* = 0$ (62), $r^* \gg 2M$, $m = 0$ and $|k_r(r^*)| \simeq \omega$, the probability of vacuum decay with respect to the local observer is

$$|\langle 0, \text{out} | \bar{0}, \text{in} \rangle|^2 \simeq \frac{1}{\exp(8M\pi|k_r|) - 1} \simeq \frac{1}{\exp \left( \frac{\omega_{\text{loc}}}{T_{\text{loc}}} \right) - 1}, \tag{71}$$

where $\omega_{\text{loc}}$ and $T_{\text{loc}}$ are the energy-spectrum and temperature:

$$\omega_{\text{loc}} = g^{-\frac{1}{2}}(r^*) \omega, \quad T_{\text{loc}} = \frac{1}{8\pi M} g^{-\frac{1}{2}}(r^*), \tag{72}$$

with respect to the local observer rest at $r^*$. $\omega_{\text{loc}}$ and $T_{\text{loc}}$ with respect to the local observer $\mathcal{O}$ are gravitationally red-shifted from their counterparts $\omega$ and $T$ with respect to the static observer at infinity.

It is worthwhile to compare the some results of this article with the WKB-method adopted in the brick wall model (3, 4), where the solution to Eq.(27) in the vicinity of black horizon was found.
The radial wave-vector \( k_r \) or \( k_r(r^*) \) we define is consistent with that given in refs.\(^5\). Actually, the value of the radial wave-vector \( k_r(r^*) \) determined by Eq.(68) is the exactly same as the wave-vector (8.9) obtained by directly solving Eq.(27) in refs.\(^3\). The oscillatory behaviour of the quantum field \( \phi \) (8.7) in refs.\(^3\) for describing quantum tunneling by the WKB-method coincides with the imaginary part of effective action (59) obtained in this article. The total number \( \nu \) of radial wave-solutions up to a given energy, discussed in refs.\(^5\), is equivalent to the radial quantum number “\( n \)” (energy-state) discussed in Eqs.(56,59). In both schemes, these states “\( n \)” are occupied by non-negative number of quanta \( |k_r| \) to give rise to the energy-spectrum of quantum radiation.

VII. DISCUSSIONS.

We turn to discussions of the dynamical reason for such a vacuum decay and particle creations. In the framework of quantum field theories, quantum-field fluctuations in vacuum indicate pair creations and annihilations of positive- and negative-energy virtual particles, represented by closed fermion-loops in Feynman diagrams. We point out that in the presence of external gravitational field, such a pair-creation process is energetically favorable for it gains a gravitational energy \( \delta E < 0 \). Setting \( \delta r \ll r^* \) is the quantum separation of positive- and negative-energy virtual particles in the radial direction, \( \delta t \) and \( \delta \omega \) are time and energy variation of the pair-creation process, we can approximately compute the gravitational energy gain \( \delta E \) per quantum state \( 2\pi \):

\[
\delta E = g^{\frac{1}{2}}(r^* - |\delta r|) \frac{|\delta \omega_{\text{loc}}|}{2\pi} - g^{\frac{1}{2}}(r^* + |\delta r|) \frac{|\delta \omega_{\text{loc}}|}{2\pi} \\
\simeq -\frac{M}{2\pi(r^*)^2} |\delta \omega||\delta r| g^{\frac{1}{2}}(r^*) \simeq -\frac{M}{2\pi(r^*)^2} g^{\frac{1}{2}}(r^*),
\]

(73)

where the first and second terms in Eq.(73) are gravitational energies respectively for positive \( (\delta \omega_{\text{loc}} > 0) \) and negative energy \( (\delta \omega_{\text{loc}} < 0) \) virtual particles. \( \delta \omega_{\text{loc}} = g^{\frac{1}{2}}(r^*) \delta \omega \) is the energy variation in a local rest frame of the static observer \( O \) at \( r^* \). In Eq.(74), we adopt \( |\delta \omega| \simeq |\delta k_r| \) and the Heisenberg uncertainty relationships:

\[
\delta t \delta \omega \simeq 1; \quad \delta r \delta k_r \simeq 1,
\]

(75)

which are invariant in any arbitrary coordinate frames. Analogously, given the gravitational potential \( -\frac{M}{r^*} \) in the Newtonian limit and \( \delta \omega \simeq \delta \omega_{\text{loc}} \), we have

\[
\delta E \simeq -\frac{M |\delta \omega_{\text{loc}}|}{2\pi r^* - |\delta r|} + \frac{M |\delta \omega_{\text{loc}}|}{2\pi r^* + |\delta r|} \\
\simeq -\frac{M}{2\pi(r^*)^2} |\delta \omega| |\delta r| \simeq -\frac{M}{2\pi(r^*)^2}.
\]

(76)
This shows that the gravitational field polarizes the vacuum by displacing $-\frac{\delta r}{2}$ for positive energy virtual particles and $\frac{\delta r}{2}$ for negative energy virtual particles, such a displacement gains gravitational energy. As a consequence of gravitational vacuum polarization, vacuum possibly decays via pair-creation process to create real particles and anti-particles, leading to quantum radiation in curved spacetime. This is analogous to the phenomenon of an external electric field polarizing vacuum, leading to possible pair-creations of particles and anti-particles, as described by the Schwinger mechanism.

This gravitational vacuum polarization is characterized by the gravitational energy-gain $|\delta E|$ with respect to the local observer at $r^*$. In the proper frame of a free-falling observer, whose acceleration $a = g \frac{1}{2} (r^*) \frac{M}{(r^*)^2}$ at $r^*$, the energy-gain is,

$$|\delta E| = \frac{M}{2\pi (r^*)^2} g \frac{1}{2} (r^*) = a \frac{1}{2\pi}. \quad (77)$$

This is reminiscent of the Unruh effect and indicates that its origin has the same dynamical nature of gravitational vacuum polarization. An accelerating observer finds gravitational vacuum polarization leading to pair-creations in his/her proper frame, as required by the equivalent principle.

With respect to an infinity observer, the gravitational energy-gain for pair-creations is $\frac{M}{2\pi (r^*)^2}$, which can be found in Eq.(74) consistently with Eq.(72). In the neighborhood of an eternal black hole horizon ($r^* \to 2M$), the gravitational energy-gain is maximum and in fact determines the Hawking temperature $1/(8\pi M)$, which is very small, compared with neutrino masses $m_\nu$. However, the effective mass-gap $m_k(r^*)$ vanishes in the vicinity of black hole’s horizon. As the result of this vanishing effective mass-gap, virtual particles that are in quasi zero-energy states $\omega \sim 0^−$ just bellow the zero-energy level of vacuum, by quantum-field fluctuations, turn to be real particles that are in quasi zero-energy states $\omega \sim 0^+$ just above the zero-energy level of vacuum, since it almost costs no energy for such a quantum tunneling process crossing the zero-energy level of vacuum and leading to pair-creations. These particles in quasi zero-energy states have a typical energy-scale $\frac{M}{2\pi (r^*)^2} (r^* \to 2M)$, $\frac{1}{8\pi M}$ the Hawking temperature. We emphasize three properties of particle pair-creation in the vicinity of black hole horizon: (i) gravitational energy-gain is maximum; (ii) the effective mass-gap of virtual particles in the vacuum is vanishing; (iii) these lead to the largest probability of particle pair-creations, the rate and spectrum of the Hawking radiation.

Due to the smallness of the effective mass-gap $m_k \sim 0$, real particles and antiparticles created, are in the mass-shell condition $\omega = |k_r|$, which are quasi zero-energy states $\omega \sim 0^+$. By the continuation of energy-momentum dispersion relation, virtual particles and antiparticles, which are
in quasi zero-energy states $\omega \sim 0^-$, must be in an approximate mass-shell condition $\omega \simeq -|k_r| \sim 0^-$. This justifies the “mass-shell condition” $\text{Re}(\Lambda_k^2) \simeq 0^-$ and $\text{Re}(\Lambda_k^2) < 0$ is due to virtual particles bound by negative gravitational potential.

In the region away from black hole’s horizon ($r^* > 2M$) and massless particles $m = 0$, the effective mass-gap $m_k(r^*) \sim \left(\frac{2M}{(r^*)^2}\right)\frac{1}{2}$ is extremely small and comparable with the gravitational energy-gain $\frac{M}{2\pi(r^*)^2}$. Pair-creation rate is given by Eq. (70) in very low-energy regime ($\sim 1/r^*$). This could be case for extremely low-energy emission of massless particles. With respect to an infinity observer, the characteristic energy of such low-energy emissions is given by

$$T = |\delta E| = \frac{M}{2\pi(r^*)^2} = \frac{1}{8\pi M} \left(\frac{2M}{r^*}\right)^2,$$

which is smaller than the Hawking temperature $\frac{1}{8\pi M}$. The power of such low-energy emissions (black body radiation) is

$$P = 4\pi(r^*)^2 \sigma_B T^4,$$

where $\sigma_B$ is the Stefan-Boltzmann constant. The life-time of a gravitational body $M$, describing its instability against this radiation, is approximately given by

$$\tau \simeq \frac{M}{P}.$$

We examine two examples: the Earth $\frac{2M_{\oplus}}{r_{\oplus}} = 7 \cdot 10^{-10}$ where $M_{\oplus}$ is Earth’s mass and $r_{\oplus}$ Earth’s radius, a proton $\frac{M_p}{r_p} = 8 \cdot 10^{-37}$ where $M_p$ is proton’s mass and $r_p$ proton’s classical radius. We find the temperature, power and life-time of the Earth ($\oplus$) and a proton ($p$):

$$T_{\oplus} \simeq 4 \cdot 10^{-20} K^\circ, \quad P_{\oplus} \simeq 7 \cdot 10^{-65} \text{ergs/sec}, \quad \tau_{\oplus} \simeq 2 \cdot 10^{85} \text{years};$$

$$T_p \simeq 2 \cdot 10^{-22} K^\circ, \quad P_p \simeq 3 \cdot 10^{-122} \text{ergs/sec}, \quad \tau_p \simeq 2 \cdot 10^{111} \text{years}.$$

Compared with the life-time of the Universe $\sim 10^{10}$ years, these show that gravitational bodies are stable against such quantum radiation attributed to gravitational vacuum-polarization. On the other hand, all known fermion masses ($m$) are not zero and much larger than $\sim 1/r^*$. As a consequence, the rate (70) of quantum tunneling effect (quantum radiation) is exponentially suppressed ($\sim e^{-8\pi Mm} \simeq 0$).

Attributed to the nature of quantum-field fluctuations of virtual particles tunneling through a very small effective mass-gap, vacuum decays, leading to pair-creations of real particles and antiparticles. Accordingly, the nature of quantum-field fluctuations of real particles and antiparticles, clearly implies the inverse process: pairs of real particles and antiparticles annihilate into
virtual particles and antiparticles in vacuum. The rate of pair-annihilation process must be the same as the rate of pair-creation process, as the CPT invariance is preserved in such processes. In these emission(creation) and absorption(annihilation) processes, the vacuum acts as a reserve of temperature given by the gravitational energy-gain Eq. (74). In the presence of gravitational field, the spectrum of this reserve(vacuum) is quantized and described by the integer \( n \) and quanta \( |k_r| \) (see Eqs. (56, 69)). Particle creations(emissions) from vacuum and annihilation(absorptions) into vacuum are permitted, if these processes take place in unit of quanta \( |k_r| \). The detail balance of these emission and absorption processes in unit of quanta \( |k_r| \), leads to the black-body spectrum (70) of particle creations. In general, such a spectrum is different from black-body one up to a gray factor, since \( \text{Re}(\Lambda^2_k) \) is not exactly zero for virtual particles in quasi zero-energy states (\( \omega \sim 0^- \)) and the approximation (11) is made. This is actually because of particle creation and annihilation processes scattered by the potential terms \( \frac{2M}{r^3} \), \( l(l + 1)/r^2 \) and \( g(r) \) in the effective mass-gap.

In this article, we discuss that with respect to a static local observer at \( r^* < \infty \) and a static observer at infinity, the transition amplitude between the vacuum state \( |\bar{0}, \text{in}\rangle \) (9) of curved spacetime and the vacuum state \( |\tilde{0}, \text{out}\rangle \) (11) of flat spacetime, showing the vacuum state \( |\bar{0}, \text{out}\rangle \) is unstable against the vacuum state \( |\bar{0}, \text{in}\rangle \). This is attributed to the nature of gravitational field polarizing the vacuum state \( |\tilde{0}, \text{in}\rangle \) and such polarizations gaining gravitational energy, leading to particle and antiparticle productions. As a consequence, the static observer \( \mathcal{O} \) (24) in curved spacetime measures the vacuum state \( |\tilde{0}, \text{out}\rangle \) (11) as thermal radiation of particle and antiparticle productions upon the vacuum state \( |\bar{0}, \text{in}\rangle \) (9). Although such an effect is in general extremely small (81, 82), it converts the gravitational energy (matter) into the energy of radiation fields.

[1] S.W. Hawking, Commun. Math. Phys. 43, 199 (1975);
G.W. Gibbons and S.W. Hawking, Phys. Rev. D 15 2752 (1977).
[2] N.D. Birrell and P.C.W. Davies, “Quantum field theory in curved space” Cambridge University Press (1982);
R.M. Wald “Quantum field theory in curved space and black hole thermodynamics” University of Chicago Press (1994);
V.P. Frolov and I.D. Novikov “Black hole physics” Kluwer Academic Publisher (1998) references there in.
[3] T. Damour and R. Ruffini, Phys. Rev. D. 15, 332 (1976);
H.C. Ohanian and R. Ruffini, “Gravitation and Spacetime” W.W. Norton & Company (1994);
R. Brout, S. Massar, R. Parentani and Ph. Spindel, Phys. Rep. 260, 329 (1995) references there in.
[4] J. Schwinger, Phys. Rev. 98 (1951) 714.

[5] G. t'Hooft, Nucl. Phys. B256 (1985) 727; Int. J. Mod. Phys. A11 (1996) 4623-4688.

[6] P. Kraus and F. Wilczek Nucl. Phys. B433 403 (1995), B437 231 (1995);
M. K. Parikh and F. Wilczek, Phys. Rev. Lett. 85, 5042 (2001) and references therein.

[7] for the brick wall model, see also,
R. B. Mann, L. Tarasov and A.I. Zelnikov, Class. Quantum Grav. 9 (1992) 1487;
A. Ghosh and P. Mitra, Phys. Rev. Lett. 73 (1994) 2521;
D. Kabat and M. J. Strassler, Phys. Lett. B329, (1994) 46;
L. Susskind and J. Uglum, Phys. Rev. D50 (1994) 2700;
J.L.F. Barbon and R. Emparan, Phys. Rev. D52 (1995) 4527;
J.-G. Demers, R. Lafrance and R.C. Myers, Phys. Rev. D52 (1995) 2245;
F. Belgiorno and S. Liberati, Phys. Rev. D53 (1996) 3172;
V.P. Frolov, D.V. Fursaev and A.I. Zelnikov, Phys. Rev. D54 (1996) 2711.

[8] W. G. Unruh, Phys. Rev. D 14 870 (1976).