LIE NILPOTENCY INDICES OF SYMMETRIC ELEMENTS UNDER ORIENTED INVOLUTIONS IN GROUP ALGEBRAS

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Abstract. Let $G$ be a group and let $F$ be a field of characteristic different from 2. Denote by $(FG)^+$ the set of symmetric elements and by $U^+(FG)$ the set of symmetric units, under an oriented classical involution of the group algebra $FG$. We give some lower and upper bounds on the Lie nilpotency index of $(FG)^+$ and the nilpotency class of $U^+(FG)$.

1. Introduction

Let $FG$ denote the group algebra of a group $G$ over a field $F$ with $\text{char}(F) = p \neq 2$. A homomorphism $\sigma : G \to \{\pm 1\}$ is called an orientation of the group $G$. Working in the context of $K$-theory, Novikov [10], introduced an oriented classical involution of the group algebra $FG$ by $(FG)^+$ and $(FG)^-$.

We denote $(FG)^+ = \{\alpha \in FG : \alpha^* = \alpha\}$ and $(FG)^- = \{\alpha \in FG : \alpha^* = -\alpha\}$ the set of symmetric and skew-symmetric elements of $FG$. Under $\ast$, respectively. We denote by $N$ the kernel of $\sigma$. It is obvious that the involution $\ast$ coincides on the group algebra $FN$ with the classical involution. It is easy to see that, as an $F$-module, $(FG)^+$ is generated by the set

$$S = \{g + g^{-1} : g \in N\} \cup \{g - g^{-1} : g \in G \setminus N, g^2 \neq 1\}$$

and $(FG)^-$ is generated by

$$L = \{g + g^{-1} : g \in G \setminus N\} \cup \{g - g^{-1} : g \in N, g^2 \neq 1\}.$$

Given $g_1, g_2 \in G$, we define the commutator $(g_1, g_2) = g_1^{-1}g_2^{-1}g_1g_2$ and recursively, $(g_1, \ldots, g_n) = ((g_1, \ldots, g_{n-1}), g_n)$ for $n$ elements $g_1, \ldots, g_n$ of $G$. By the commutator $(X, Y)$ of the subsets $X$ and $Y$ of $G$ we mean the subgroup of $G$ generated by all commutators $(x, y)$ with $x \in X$, $y \in Y$. In this way, we can define the lower central series of a nonempty subset $H$ of $G$ by: $\gamma_1(H) = H$ and $\gamma_{n+1}(H) = (\gamma_n(H), H)$, for $n \geq 1$. We say that $H$ is nilpotent if $\gamma_n(H) = 1$, for some $n$. For a nilpotent subset $H \subseteq G$ the number $\text{cl}(H) = \min\{n \in \mathbb{N}_0 : \gamma_{n+1}(H) = 1\}$ is called the nilpotency class of $H$. It can be proved that $H$ is a nilpotent set if and only if $H$ satisfies the group identity $(g_1, \ldots, g_n) = 1$ for some $n \geq 2$.

In an associative ring $R$, the Lie bracket on two elements $x, y \in R$ is defined by $[x, y] = xy - yx$. This definition is extended recursively via $[x_1, \ldots, x_{n+1}] = [[x_1, \ldots, x_n], x_{n+1}]$. For $X, Y \subseteq R$ by $[X, Y]$ we denote the additive subgroup generated by all Lie commutators $[x, y]$. 

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with $x \in X, y \in Y$. The lower Lie central series of a nonempty subset $S$ of $R$ is defined
inductively by setting $\gamma^1(S) = S$ and $\gamma^{i+1}(S) = [\gamma^i(S), S]$. We say that the subset $S$ is
Lie nilpotent if there exists a natural number $n$, such that $\gamma^n(S) = 0$. The smallest natural
number with the last property, denoted by $t(S)$, is called the Lie nilpotency index of $S$. It
is possible to show that $S$ is Lie nilpotent if and only if $S$ satisfies the polynomial identity
$[x_1, \ldots, x_n] = 0$ for some $n \geq 2$.

Given a nonempty subset $S$ of $R$, we let $S^{(1)} = R$, and then for each $i \geq 2$, let $S^{(i)}$ be the
(associative) ideal of $R$ generated by all elements of the form $[a, b]$, with $a \in S^{(i-1)}$, $b \in S$. We
say that $S$ is strongly Lie nilpotent if $S^{(i)} = 0$ for some $i$. The minimal $n$ for which $S^{(n)} = 0$
is called the upper Lie nilpotency index and denoted by $t^U(S)$. Clearly, strong Lie nilpotence
implies Lie nilpotence and $t(S) \leq t^U(S)$. Denote by $U(S)$ the set of units in the subset $S$ of
$R$ and suppose that it is nonempty. By the equality $(x, y) = 1 + x^{-1}y^{-1}[x, y]$, it is easy to
see that $\gamma_n(U(S)) \subseteq 1 + S^{(n)}$ for all $n \geq 2$. In consequence, the set of units of a strongly Lie
nilpotent subset $S$ is nilpotent, and

$$\text{cl}(U(S)) < t^U(S).$$

In 1973, Passi, Passman and Sehgal [12] showed that the group algebra $FG$ is Lie nilpotent
if and only if $G$ is nilpotent and $G^r$ is a finite $p$-group, where $p$ is the characteristic of $F$.
Actually, see [14], a group algebra is Lie nilpotent if and only if it is strongly Lie nilpotent.
Next, S.K. Sehgal characterized group algebras which are Lie $n$-Engel, for some $n$.

In 1993, Giambruno and Sehgal [6] began the study of Lie nilpotence of symmetric and
skew-symmetric elements under the classical involution. They proved that given a group $G$
without elements of order 2 and a field $F$ with $\text{char}(F) \neq 2$, if either $(FG)^+$ or $(FG)^-$ is Lie
nilpotent, then $FG$ is Lie nilpotent. This work was completed by G.T. Lee [8], for groups in
general. More specifically, he proved that the Lie nilpotence of the symmetric elements under
the classical involution is equivalent to the Lie nilpotence of $FG$ when the group $G$ does not
contain a copy of $Q_8$, the quaternion group of order 8 and he also characterized the group
algebras such that the set of symmetric elements is Lie nilpotent when $G$ contains a copy of
$Q_8$.

Recently, Castillo and Polcino Milies, see [5], studied Lie properties of the symmetric elements
under an oriented classical involution. They extended some previous results from [6], [8] and [9]. In particular, they gave some groups algebras such that the Lie nilpotence of the symmetric set implies the same property in the whole group algebra. Also, they obtained a complete characterization of the group algebras $FG$, such that $Q_8 \subseteq G$ and $(FG)^+$ is Lie nilpotent.

Lately, Z. Balogh and T. Juhász in [2] and [3] studied the Lie nilpotency index of $(FG)^+$
and the nilpotency class of the $U^+(FG)$ under the classical involution in group algebras. They
gave a necessary condition to the numbers $t((FG)^+)$ and $\text{cl}(U^+(FG))$ be maximal, as possible,
in a nilpotent group algebra. Also, they studied this two numbers to group algebras such that
$(FG)^+$ is Lie nilpotent but $FG$ is not.

In this article we study the Lie nilpotency index of $(FG)^+$ and the nilpotency class of
$U^+(FG)$ under an oriented classical involution. In the next section we give some preliminary
results. In the third section we study the numbers $t((FG)^+)$ and $\text{cl}(U^+(FG))$ in Lie nilpotent
group algebras. In the fourth section we study the case when $Q_8 \subseteq G$ and $(FG)^+$ is Lie
nilpotent.

Throughout this paper $F$ will always denote a field of characteristic not 2, $G$ a group and
$s$ a nontrivial orientation of $G$. In a number of places, all over this paper, we use arguments
from [2], [3] and [10]. Some of them are reproduced here for the sake of completeness.
2. Preliminaries

We recall the following result from [10].

**Lemma 2.1.** Let $R$ be a ring and $S$ a subset of $R$. Suppose, for some $i \geq 1$, that $S^{(i)} \subseteq zR$, where $z$ is central in $R$. Then for all $j > 0$, we have $S^{(i+j)} \subseteq zS^{(j)}$. In particular, for any positive integer $m$, $S^{(mi)} \subseteq z^m R$.

**Proof.** The proof is by induction on $j$. If $j = 1$, then $S^{(i+1)} \subseteq S^{(i)}$, there is nothing to do. Assume that $S^{(i+j)} \subseteq zS^{(j)}$. Take $a \in S^{(i+j)}$, $b \in S$. So $a = za_1$, for some $a_1 \in S^{(j)}$. Thus, $[a, b] = [za_1, b] = z[a_1, b] \in zS^{(i+j)}$, as we want to prove.

To get the second part, notice that $S^{(2i)} = S^{(i+i)} \subseteq zS^{(i)} \subseteq z^2 R$.

Suppose that $S^{((m-1)i)} \subseteq z^{m-1} R$. So $S^{(mi)} = S^{((m-1)i+i)} \subseteq zS^{((m-1)i)} \subseteq z^m R$. \hfill $\square$

Throughout this article we denote by $Q_8 = \langle x, y : x^4 = 1, x^2 = y^2, xy = x^{-1} \rangle$ the quaternion group of order 8. Castillo and Polcino Milies [5] characterized the group algebras of groups containing $Q_8$ and with a nontrivial orientation, such that $(FG)^+$ is Lie nilpotent. Here we prove that the conditions obtained by them are also satisfied when $(FG)^+$ is strongly Lie nilpotent.

**Theorem 2.1.** Let $F$ be a field of characteristic $p \neq 2$, $G$ a group with a nontrivial orientation $\sigma$ and $x, y$ elements of $G$ such that $\langle x, y \rangle \simeq Q_8$. Then $(FG)^+$ is strongly Lie nilpotent if and only if either

(i) $\text{char}(F) = 0$, $N \simeq Q_8 \times E$ and $G \simeq \langle Q_8, g \rangle \times E$, where $E^2 = 1$ and $g \in G \setminus N$ is such that $(g, x) = (g, y) = 1$ and $g^2 = x^2$; or,

(ii) $\text{char}(F) = p > 2$, $N \simeq Q_8 \times E \times P$, where $E^2 = 1$, $P$ is a finite $p$-group and there exists $g \in G \setminus N$ such that $G \simeq \langle Q_8, g \rangle \times E \times P$, $(g, x) = (g, y) = 1$ and $g^2 = x^2$.

**Proof.** If $(FG)^+$ is strongly Lie nilpotent, then $(FG)^+$ is Lie nilpotent and from [5] Theorem 4.2] we get (i) and (ii).

Conversely, assume that $|P| = p^n$. We claim that, $((FG)^+)^{(2p^n)} = 0$. The proof will be by induction on $n$. If $n = 0$, then $G \simeq \langle Q_8, g \rangle \times E$ and thus, from [5] Lemma 4.3], $(FG)^+$ is commutative. Assume that $|P| = p^n > 1$. Take $z \in \zeta(P)$ with $o(z) = p$, applying our inductive hypothesis on $\overline{G} = G/\langle z \rangle$. Then, $((FG)^+)^{(2p^{n-1})} = 0$. Thus

$((FG)^+)^{(2p^{n-1})} \subseteq \Delta(G, \langle z \rangle) = (z - 1)FG$.

By Lem[2.1]

$((FG)^+)^{(2p^n)} \subseteq (z - 1)^p FG = 0$,

as we claimed. \hfill $\square$

From the equality, $(x, y) = 1 + x^{-1}y^{-1}[x, y]$ we know that $\gamma_n(U^+(FG)) \subseteq 1 + ((FG)^+)^{(n)}$ and thus we get the following.

**Corollary 2.1.** Let $F$ be a field of characteristic different from 2. Assume that $Q_8 \subseteq G$ and $(FG)^+$ is Lie nilpotent. Then, $U^+(FG)$ is nilpotent.

We need the following easy observation.
Lemma 2.2. Let $G$ be a group, $H$ any subgroup and $A$ a normal subgroup such that $A \subseteq N$. If $(FG)^+$ is Lie nilpotent, then so are $(FH)^+$ and $(F(G/A))^+$. Furthermore, $t((FH)^+) \leq t((FG)^+)$ and $t((F(G/A))^+) \leq t((FG)^+)$. 

Proof. Note that $(FH)^+$ is a subset of $(FG)^+$, and thus it has the required properties.

Since $A$ is a normal subgroup contained in the kernel of the orientation $\sigma$, we can define in $F(G/A)$ an induced oriented classical involution from $\ast$ in $FG$ as follows:

$$
\left( \sum_{\bar{g} \in G/A} \alpha_g \bar{g} \right)^\ast = \sum_{\bar{g} \in G/A} \alpha_g \sigma(g) \bar{g}^{-1}.
$$

Now, simply observe that the symmetric elements in $F(G/A)$, under $\ast$, are linear combinations of terms of the form $gA + \sigma(g)\bar{g}^{-1}A$, with $g \in G$. That is, every element of $(F(G/A))^+$ is the homomorphic image of an element of $(FG)^+$ under the natural map $\varepsilon_A : FG \to F(G/A)$, defined by $\varepsilon_A(\sum_{g \in G} \alpha_g g) = \sum_{g \in G} \alpha_g g \bar{g}$.

So assume that $(FG)^+$ is Lie nilpotent, therefore there exists $n = t((FG)^+)$ such that $[\alpha_1, \ldots, \alpha_n] = 0$ for all $\alpha_i \in (FG)^+$. Let $\beta_1, \ldots, \beta_n \in (F(G/A))^+$. Thus

$$
[\beta_1, \ldots, \beta_n] = [\varepsilon_A(\alpha_1), \ldots, \varepsilon_A(\alpha_n)] \\
= \varepsilon_A([\alpha_1, \ldots, \alpha_n]) = \varepsilon_A(0) = 0.
$$

Consequently, $t((F(G/A))^+) \leq t((FG)^+)$. \hfill \Box

3. Lie nilpotent group algebras

In this section we assume that $FG$ is Lie nilpotent. By \cite{15}, $t^L(FG) \leq |G'| + 1$ and by \cite{4} the equality holds if and only if $G'$ is cyclic, or $G'$ is a noncentral elementary abelian group of order 4.

Note that a group $G$ of odd finite order has trivial orientation. Indeed, let $a$ be an element of $G$. So $1 = \sigma(a^{[G]}) = \sigma(a)^{[G]}$ and as $|G|$ is odd we get that $\sigma(a) = 1$. For the last reason when $G$ is a group of odd finite order, the involution $\ast$ is the classical involution. In this way, we can use the following result, that is a combination from \cite{2} Lemma 2 and \cite{3} Lemma 2.

Lemma 3.1. Let $G$ be a finite $p$-group with a cyclic derived subgroup. Then $t((FG)^+) \geq |G'| + 1$ and $\text{cl}(U^+(FG)) \geq |G'|$.

We recall that a group $G$ is called $p$-abelian if $G'$, the commutator subgroup of $G$, is a finite $p$-group and 0-abelian means abelian.

Theorem 3.1. Let $FG$ be a Lie nilpotent group algebra of odd characteristic and nontrivial orientation. Then, $t((FG)^+) = |G'| + 1$ if and only if $G'$ is cyclic. Moreover, assuming that $G$ is a torsion group, $\text{cl}(U^+(FG)) = |G'|$ if and only if $G'$ is cyclic.

Proof. Assume that $t((FG)^+) = |G'| + 1$. As $G'$ is a finite $p$-group, if $G'$ is not cyclic, from \cite{4}, we know that $t((FG)^+) \leq t^L(FG) < |G'| + 1$ and we get a contradiction. Thus, $G'$ is cyclic.

Conversely, suppose that $G'$ is cyclic. By the hypotheses, $G$ is a nilpotent $p$-abelian group and from \cite{1} Lemma 1 there exists a finite $p$-group $P$ which is isomorphic to a subgroup of factor group of $G$ and $P' \simeq G'$. Actually, from the proof of \cite{1} Lemma 1], we know that $P \simeq H/A$, where $A$ is a maximal torsion-free central subgroup of $G$. 

Assume that there exists \( g \in A \) such that \( \sigma(g) = -1 \). In this way, as \( G = N \cup gN \), we get \( G' = N' \). Using in \( FP \) the classical involution, by lemmas 3.1 and 2.2, we obtain that
\[
|G'| + 1 = |N'| + 1 = |P'| + 1 \leq t((FP)^+) \leq t((FN)^+) \leq t((FG)^+).
\]
In the other hand, suppose that \( A \subseteq N \). Then we can define an induced oriented classical involution in \( P \approx H/A \), from that one in \( FG \). Consequently,
\[
|G'| + 1 = |P'| + 1 \leq t((FP)^+) \leq t((FG)^+).
\]

The proof of the second part is similar. \( \square \)

4. Groups that contain a copy of \( Q_8 \)

We assume that \( Q_8 \subseteq G \) and \( (FG)^+ \) is Lie nilpotent. This means that the group algebra \( FG \) is not Lie nilpotent. Recently, this kind of group algebras was characterized by Castillo and Polcino Milies \cite{5}. This characterization is the same as in Theorem 2.1 so during this section we assume that \( G \) is as in that result. In this section, we will study the Lie nilpotency index of the symmetric elements under oriented classical involutions.

It is easy to show that
\[
g^m - 1 \equiv m(g-1) \pmod{\Delta(G)^2}.
\]
for every \( g \in G \) and any integer \( m \).

We begin with the following result.

**Lemma 4.1.** Consider \( FG \) with an oriented classical involution. Then
\[
((FG)^+)^{(n)} \subseteq FG\Delta(P)^n
\]
for all \( n \geq 2 \)

**Proof.** Recall that the symmetric elements are spanned as an \( F \)-module by the set
\[
S = \{z + z^{-1} : z \in N\} \cup \{z - z^{-1} : z \in G \setminus N\}.
\]
If \( z \in N \), then \( z = ah \) with \( a \in Q_8 \times E \) and \( h \in P \). Note that if \( a^2h = 1 \), then \( h = 1 \) and \( a^2 = 1 \). Thus, \( a \in \zeta(Q_8 \times E) \). Assuming \( a^2h \neq 1 \), follows that \( z + z^{-1} = ah + a^{-1}h^{-1} = ah + a^3h^{-1} = a(h + a^2h^{-1}) \).

Also, if \( z \in G \setminus N \); we can write \( z = gah \) with \( a \in Q_8 \times E \) and \( h \in P \). If \( a^2h = 1 \), then \( a^2 = h = 1 \). Again, \( a \in \zeta(Q_8 \times E) \) and thus \( z - z^{-1} = gah - g^{-1}a^{-1}h^{-1} = ga - g^{-1}a = ga(1 - g^2) \in \zeta(Q_8 \times E) \). Now we suppose that \( a^2h \neq 1 \) and we get the following cases:

1. If \( a^2 = 1 \) and \( h \neq 1 \), then \( z - z^{-1} = gah - g^{-1}a^{-1}h^{-1} = ag(h - g^2h^{-1}) \).
2. If \( a^2 \neq 1 \) and \( h = 1 \), then \( z - z^{-1} = gah - g^{-1}a^{-1}h^{-1} = ga - g^2a^3 = ga - ga = 0 \).
3. If \( a^2 \neq 1 \) and \( h \neq 1 \), then \( z - z^{-1} = gah - g^{-1}a^{-1}h^{-1} = agh - a^3g^3h^{-1} = ag(h - h^{-1}) \), because \( a^3g^3 = ag \).

From the above considerations, we obtain that
\[
S = A \cup B \cup C \cup \zeta(Q_8 \times E),
\]
where
\[
A = \{a(h + a^2h^{-1}) : a \in Q_8 \times E, h \in P \text{ and } a^2h \neq 1\},
B = \{ag(h - g^2h^{-1}) : a \in Q_8 \times E, h \in P \text{ and } (a^2 = 1 \text{ and } h \neq 1)\},
C = \{ag(h - h^{-1}) : a \in Q_8 \times E, h \in P \text{ and } (a^2 \neq 1 \text{ and } h \neq 1)\}.
\]
Lemma 4.3. \( a \in Q_8 \times E, \) such that \( a^2 \neq 1 \) we know that \( 1 + a^2 \) is symmetric and \( a^2 \in \zeta(Q_8 \times E). \) In this way,

\[
a(h + a^2 h^{-1}) + 1 + a^2 = a(h - 1) + a^3(h^{-1} - 1) + 1 + a + a^2 + a^3,
\]

where \( 1 + a + a^2 + a^3 \) is a central element in \( FG \) and \( a(h - 1) + a^3(h^{-1} - 1) \in FG\Delta(P). \) It is clear that, \( ag(h - h^{-1}) \in FG\Delta(P). \) Furthermore, if \( a^2 = 1 \) and \( h \neq 1, \) then \( ag(h - g^2 h^{-1}) = ag(h - 1) - ag^3(h^{-1} - 1) + a(g - g^{-1}) \in FG\Delta(P) + \zeta(FG). \)

So

\[
\bar{S} = A' \cup B \cup C \cup \zeta(Q_8 \times E),
\]

also spans \((FG)^+\) as an \( F\)-module, where

\[
A' = \{ a(h + a^2 h^{-1}) + 1 + a^2 : a \in Q_8 \times E, h \in P \text{ and } a^2 h \neq 1 \}
\]

and \( B, C \) are as above.

In consequence,

\[(FG)^+ \subseteq FG\Delta(P) + \zeta(FG). \tag{3}\]

The proof follows by induction on \( n. \) Indeed, if \( n = 2 \)

\[
[(FG)^+, (FG)^+] \subseteq [FG\Delta(P), FG\Delta(P)] \subseteq FG\Delta(P)^2.
\]

Suppose that the lemma is true for some \( n \geq 2. \) Take \( \alpha \in ((FG)^+)^{(n)} \) and \( \beta \in (FG)^+. \) So

\[
[\alpha, \beta] \in [FG\Delta(P)^n, FG\Delta(P)] \subseteq FG\Delta(P)^{n+1}.
\]

and we get that \(((FG)^+)^{(n+1)} \subseteq FG\Delta(P)^{n+1}\) as required. \( \square \)

Denote by \( c \) the central element of \( Q_8 \times E, \) such that \( (Q_8 \times E)^2 = \langle c \rangle. \) Given \( n \geq 2, \) we denote with \( M_n \) the \( F\)-subspace of the vector space \( FG \) generates by the set

\[
\{(h_1 - h_1^{-1}) \cdots (h_n - h_n^{-1})(1-c) a : h_1, \ldots, h_n \in P, a \in (Q_8 \times E) \setminus \zeta(Q_8 \times E)\}.
\]

To simplify, we write \( f_{1,\ldots,n} \) instead of \( (h_1 - h_1^{-1}) \cdots (h_n - h_n^{-1}). \)

Let \( S_n \) be the symmetric group of degree \( n \) and \( FS_n \) its group algebra over the field \( F. \) It is possible to define a group action of \( S_n \) on \( M_n \) via: for a \( \sigma \in S_n \) and a generator element \( f_{1,\ldots,n} (1-c) a \) of \( M_n \) let

\[
\sigma \cdot f_{1,\ldots,n} (1-c) a = f_{\sigma(1),\ldots,\sigma(n)} (1-c) a.
\]

Naturally, this group action on a generator set of \( M_n \) can be extended linearly to the whole \( M_n. \) We extend this group action to a group algebra action: for \( x = \sum_{\sigma \in S_n} \alpha_{\sigma} \sigma \in FS_n \) and \( z \in M_n, \) let

\[
x \cdot z = \sum_{\sigma \in S_n} \alpha_{\sigma} (\sigma \cdot z).
\]

For \( n \geq 2 \) we define the elements \( x_{2,n}, x_{3,n}, \ldots, x_{n,n} \) of \( FS_n \) recursively as:

\[
x_{2,n} = 1 + (2, 1), \tag{4}
\]

\[
x_{i,n} = x_{i-1,n} + x_{i-1,n}(i, i-1, \ldots, 1); \text{ for } 3 \leq i \leq n. \tag{5}
\]

Since \((FN)^+ \subseteq (FG)^+, \) from Lemma 4 and Lemma 5 in [3], we get the following results.

**Lemma 4.2.** \( x_{n,n} M_n \in \gamma^n((FG)^+)(1-c) \) for all \( n \geq 2. \)

**Lemma 4.3.** If \( |P| = p^k, \) then \( \tilde{P}(1-c) a \in \gamma^{k(p-1)}((FG)^+) \) for some \( a \in Q_8 \times E. \)
We recall that the augmentation ideal \( \Delta(P) \) of a finite \( p \)-group \( P \) is a nilpotent ideal, see [13 Theorem 6.3.1], we will denote by \( t_{\text{nil}}(P) \) its nilpotency index. Also, we remind that a finite \( p \)-group \( P \), is called powerful if \( P' \subseteq P^p \). Let \( P \) be a powerful group. We denote with \( D_i = D_i(FP) \) the \( i \)-th dimensional subgroup. By Theorem 5.5 in [7], \( D_1 = P \) and for \( n > 1 \),
\[
D_n = \left\langle (D_{n-1}, P), (D_{i(n)})^p \right\rangle.
\]
It can be showed that, \( (P^i)^p = P^{p^{i+j}} \) and \( (P^i, P) \subseteq P^{p^{i+1}} \) for every pair \( i, j \). So, if \( p^{i-1} < n \leq p^i \) then \( D_n = P^{p^i} \).

**Lemma 4.4.** Let \( P \) be a powerful group and \( h_i - 1 \in \Delta(P)^{k_i} \) and \( h_j - 1 \in \Delta(P)^{k_j} \), where \( k_i \) and \( k_j \) are positive integers. Then
\[
(h_i - 1)(h_j - 1) \equiv (h_j - 1)(h_i - 1) \pmod{\Delta(P)^{k_i+k_j+1}}. \tag{6}
\]

**Proof.** First, we prove that \( (D_i, D_j) \subseteq D_{i+j+1} \), for every \( i, j \). Take \( h_i \in D_i \) and \( h_j \in D_j \). We get the following equation
\[
(h_i, h_j) - 1 = h_i^{-1}h_j^{-1}((h_i - 1)(h_j - 1) - (h_j - 1)(h_i - 1)). \tag{7}
\]
If either \( i \) or \( j \), say \( i \), is not a power of \( p \), then \( h_i \in D_i = D_{i+1} \), so by (7), \( (h_i, h_j) - 1 \in \Delta(P)^{i+j+1} \); thus \( (h_i, h_j) \in D_{i+j+1} \). If both \( i \) and \( j \) are powers of \( p \), then \( i + j \) cannot be a power of \( p \) and consequently \( D_{i+j} = D_{i+j+1} \). By (7) follows \( (h_i, h_j) \in D_{i+j+1} \); therefore our claim is proved.

Let \( h_i - 1 \in \Delta(P)^{k_i} \) and \( h_j - 1 \in \Delta(P)^{k_j} \) for some positive integers \( k_i, k_j \). Then
\[
(h_i - 1)(h_j - 1) = (h_j - 1)(h_i - 1) + h_j h_i((h_i, h_j) - 1),
\]
and as \((h_i, h_j) \in D_{k_i+k_j+1} \), the result follows. \( \square \)

Now we can prove our main result in this section.

**Theorem 4.1.** Let \( F \) be a field of characteristic \( p > 2 \). Consider the group algebra \( FG \) with an oriented classical involution. Assume that \( Q_8 \subseteq G \), \( (FG)^+ \) is Lie nilpotent and the Sylow \( p \)-group \( P \) of \( G \) is of order \( p^m \), with \( m \geq 1 \). Then
\begin{itemize}
  \item[(i)] \( 1 + m(p - 1) \leq t((FG)^+) \leq t^{L_1((FG)^+)} \leq t_{\text{nil}}(P) \) and \( \text{cl}(U^+\langle FG \rangle) \leq t_{\text{nil}}(P) - 1 \).
  \item[(ii)] If \( t((FG)^+) = t_{\text{nil}}(P) \), then \( \text{cl}(U^+(FG)) + 1 = t((FG)^+) \).
  \item[(iii)] If \( P \) is powerful, then \( t((FG)^+) = t_{\text{nil}}(P) \).
  \item[(iv)] If \( P \) is abelian, then, for all \( k \geq 2 \), the \( F \)-space \( \gamma^k((FG)^+) \) is generated by the set
    \[
    \mathcal{M}_k = \{(h_i - h_i^{-1}) \cdots (h_k - h_k^{-1})(1 - a^2)a : h_i \in P, a \in (Q_8 \times E) \setminus \zeta(Q_8 \times E)\} \cup \\
    \{g(h_1 - h_1^{-1}) \cdots (h_k - h_k^{-1})(1 - a^2)a : h_i \in P, a \in (Q_8 \times E) \setminus \zeta(Q_8 \times E)\}.
    \]
\end{itemize}

**Proof.** From Theorem 2.1 we know that \( N \cong Q_8 \times E \times P \), where \( E^2 = 1 \), \( P \) is a finite \( p \)-group and there exists \( g \in G \setminus N \) such that \( G \cong \langle Q_8, g \rangle \times E \times P \), \( (g, x) = (g, y) = 1 \) and \( g^2 = x^2 \). By Lemma 4.3 there exists \( 0 \neq \tilde{P}(1 - c)a \in \gamma^{m(p - 1)}((FG)^+) \) for some \( a \in Q_8 \times E \). In this way, \( 1 + m(p - 1) \leq t((FG)^+) \). Furthermore, Lemma 4.4 implies that \( t^{L_1((FG)^+)} \leq t_{\text{nil}}(P) \).

To show (ii), consider the symmetric elements
\[
u_i = 1 - a_i(1 + a_i^2)^2 + x_i, \text{ where } x_i = a_i(h_i + a_i^2h_i^{-1}) \in S, a_i \in Q_8 \times E \text{ and } h_i \in P.
\]
Thus, \( u_i = 1 + a_i(h_i - 1) + a_i^2(h_i^{-1} - 1) \in 1 + FG \Delta(P) \). Since \( FG \Delta(P) \) is a nilpotent ideal, we get that \( 1 + FG \Delta(P) \) is a normal subgroup of \( U(FG) \) and in consequence \( u_i \) is a unit in \( FG \). We will prove, by induction, that
\[
(u_1, u_2, \ldots, u_n) \equiv 1 + [x_1, x_2, \ldots, x_n] \pmod{FG \Delta(P)^{n+1}}. \tag{8}
\]
Since $u_1^{-1}u_2^{-1} \equiv 1 \pmod{FG\Delta(P)}$, Lemma 4.1 implies that
\[
(u_1, u_2) = 1 + u_1^{-1}u_2^{-1}[u_1, u_2] = 1 + (u_1^{-1}u_2^{-1} - 1)[u_1, u_2] + [u_1, u_2] \equiv 1 + [u_1, u_2] \pmod{FG\Delta(P)^3}.
\]
We recall that $\hat{a}_i = 1 + a_i + a_i^2 + a_i^3$ and $1 + a_i^2$, for each $a_i \in Q_8 \times E$, are central elements of $FG$. So
\[
[u_1, u_2] = [1 - a_1(1 + a_1^2) + x_1, 1 - a_2(1 + a_2^2) + x_2] = [x_1, x_2] + [a_1(1 + a_1^2), a_2(1 + a_2^2)] - [x_1, a_2(1 + a_2^2)] - [a_1(1 + a_1^2), x_2] = [x_1, x_2],
\]
which proves the congruence (8) when $n = 2$.

Suppose that (8), is true to $n - 1$; that is
\[
(u_1, u_2, \ldots, u_{n-1}) \equiv 1 + [x_1, x_2, \ldots, x_{n-1}] \pmod{FG\Delta(P)^n}.
\]
Then, Lemma 4.1 and as $(u_1, u_2, \ldots, u_{n-1})^{-1}u_{n-1}^{-1} - 1 \in FG\Delta(P)$ imply
\[
(u_1, u_2, \ldots, u_n)
\begin{align*}
&= 1 + [(u_1, u_2, \ldots, u_{n-1})^{-1}u_{n-1}^{-1} - 1][(u_1, u_2, \ldots, u_{n-1}), u_n] + [(u_1, u_2, \ldots, u_{n-1}), u_n] \\
&\equiv 1 + [(u_1, u_2, \ldots, u_{n-1}), u_n] \pmod{FG\Delta(P)^{n+1}} \\
&\equiv 1 + [x_1, x_2, \ldots, x_{n-1}], 1 - a_n(1 + a_n^2) + x_n] \pmod{FG\Delta(P)^{n+1}} \\
&\equiv 1 + [x_1, x_2, \ldots, x_n] - [x_1, x_2, \ldots, x_{n-1}], a_n] \pmod{FG\Delta(P)^{n+1}} \\
&\equiv 1 + [x_1, x_2, \ldots, x_n] \pmod{FG\Delta(P)^{n+1}},
\end{align*}
\]
and the statement (8) is true for all $n \geq 2$.

Let $n = t_{\text{nil}}(P) - 1$. If $t((FG)^+) = t_{\text{nil}}(P)$, then there are $x_1, \ldots, x_n \in S$ such that $[x_1, \ldots, x_n] \neq 0$. Thus, by the congruence (8), $\gamma_n(U^+(FG)) \neq 1$. So $n \leq \text{cl}(U^+(FG))$. Moreover, we know that $\text{cl}(U^+(FG)) < t^1((FG)^+) \leq t_{\text{nil}}(P) = n + 1$ and we get (ii).

Assume that $P$ is powerful. Then, by Lemma 4.3 we obtain
\[
x_{n,n,f_1,\ldots,n}(1 - c)a \equiv 2^n f_{1,\ldots,n}(1 - c)a \pmod{FG\Delta(P)^{n+1}}.
\]
Furthermore, if $h_i - 1 \in \Delta(P)^{k_i}$, then by (2)
\[
h_i - h_i^{-1} = (h_i - 1) - (h_i^{-1} - 1) \equiv 2(h_i - 1) \pmod{\Delta(P)^{k_i+1}},
\]
thus
\[
x_{n,n,f_1,\ldots,n}(1 - c)a \equiv 2^n (h_i - h_i^{-1}) \cdots (h_n - h_n^{-1})(1 - c)a \\
\equiv 2^{2^n} (h_i - 1) \cdots (h_n - 1)(1 - c)a \pmod{FG\Delta(P)^{n+1}}.
\]
It is clear that, if $n < t_{\text{nil}}(P)$, there exist $h_1, \ldots, h_n \in P$ such that $\prod_{i=1}^{n}(h_i - 1) \neq 0$, and then $x_{n,n,Mn} \neq 0$. Thus, $t_{\text{nil}}(P) \leq t((FG)^+)$ and (iii) follows.

Finally, assume that $P$ is abelian. Let $a, b \in (Q_8 \times E) \setminus \zeta(Q_8 \times E)$, and $h_1, h_2 \in P$, such that $(a, b) \neq 1$. Then
\[
[a(h_1 + a^2h_1^{-1}), b(h_2 + b^2h_2^{-1})] = (h_2 + b^2h_2^{-1})(h_1 + a^2h_1^{-1})[a, b] \\
= (h_2 - h_2^{-1})(h_1 - h_1^{-1})(1 - c)ab.
\]
If $\alpha \in FP$, $h \in P$, then
\[
[\alpha(1-c)a, b(h+b^2h^{-1})] = \alpha(1-c)(h+b^2h^{-1})[a, b]
= (h-h^{-1})(1-c)^2ab = 2\alpha(h-h^{-1})(1-c)ab. \tag{11}
\]
\[
[a(h_1 + a^2h_1^{-1}), bg(h_2 - h_2^{-1})] = (h_2 - h_2^{-1})(h_1 + a^2h_1^{-1})[a, bg]
= g(h_2 - h_2^{-1})(h_1 + ch_1^{-1})(1-c)ab \tag{12}
\]
and
\[
[ag(h_1 - h_1^{-1}), bg(h_2 - h_2^{-1})] = (h_2 - h_2^{-1})(h_1 - h_1^{-1})[ag, bg]
= (h_2 - h_2^{-1})(h_1 - h_1^{-1})g^2(1-c)ab \tag{13}
\]
The equations (11), (12) and (13) imply that $\gamma^2((FG)^+) = \mathcal{M}_2$. Suppose that $
 \gamma^{-1}((FG)^+) = \mathcal{M}_{n-1}$ for some $n \geq 3$. Take $\alpha \in FP$, $h \in P$ and $a, b \in (Q_8 \times E) \setminus \zeta(Q_8 \times E)$, such that $(a, b) \neq 1$. We get the following equalities:
\[
[\alpha(1-c)a, gb(h-h^{-1})] = \alpha(h-h^{-1})(1-c)[a, gb]
= g\alpha(h-h^{-1})(1-c)[a, b]
= g\alpha(h-h^{-1})(1-c)^2ab \tag{14}
\]
and
\[
[g\alpha(1-c)a, gb(h-h^{-1})] = g^2\alpha(h-h^{-1})(1-c)[a, b]
= c\alpha(h-h^{-1})(1-c)[a, b]
= c\alpha(h-h^{-1})(1-c)^2ab \tag{15}
\]
By substituting $f_{1,\ldots,n-1}$ for $\alpha$ in (11), (14) and (15), we get that
\[
[\mathcal{M}_{n-1}, (FG)^+] = \mathcal{M}_n \text{ and therefore}
\gamma^n((FG)^+) = [\gamma^{-1}((FG)^+), (FG)^+] = [\mathcal{M}_{n-1}, (FG)^+] = \mathcal{M}_n,
\]
as we wanted to prove.

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