Weights for Monoids and Actions of Monoids

Lukasz Sienkiewicz,
Marek Zawadowski
Instytut Matematyki, Uniwersytet Warszawski

December 5, 2022

Abstract

The main objective of the paper is to define the category of monoids as a weighted limit. We also define the category of actions of monoids along the action of a monoidal category as a weighted limit.

1 Introduction

Weighted limits and colimits provide a uniform way to define many interesting operations on 2-categories. It has been known for more than 40 years [Law] that the Eilenberg-Moore object for a monad $T$ in a 2-category $K$ is a weighted limit.

This is an example of a ‘2-algebraic set’ (EM-object) over a 2-dimensional algebraic structure (monad) that can be defined in any sufficiently complete 2-category. One can think that there should be a similar definition of a ‘2-algebraic set of monoids’ over any monoidal category object, another 2-dimensional algebraic structure that can be defined in any sufficiently complete 2-category with finite product. That was a question Bob Pare asked in a personal communication with the second author many years ago, with a further comment ‘After all, a monoid is a bunch of objects and morphisms satisfying some identities’.

The main purpose of this paper is to provide a positive answer to this question by constructing the weight 2-functor $W$ for the category of monoids. We also construct the weight 2-functor $W_a$ for actions of monoids along an action of monoidal category, to show how one can present many-sorted algebraic structures as weighted limits.

In Sections 2 and 3 we present the weight 2-functor for categories of monoids ($W$) and actions of monoids ($W_a$), respectively. Then in section 4 we shall comment on these constructions. The paper ends with Appendix containing definitions of monoidal category object, action of a monoidal category object, the category of monoids and the category of actions of monoids.

Notation

In this paper weighted limits in 2-categories are always meant to be pseudo-limits (i.e., unique up to an iso) and we call them simply (weighted) limits. Cat is a 2-category of small 2-categories, functors, and natural transformations. 2CAT is the 3-category of 2-categories, i.e., with 2-categories as 0-cells, 2-functors as 1-cells, 2-natural transformations as 2-cells, and 2-modifications as 3-cells. Thus Cat is a 0-cell of 2CAT. By a 2-category with finite products we will always mean a 2-category with finite products of 0-cells. Let 2CAT$_x$ be the sub-3-category of 2CAT full on 2-transformations and 2-modifications, whose 0-cells are 2-categories with finite products, and 1-cells are 2-functors preserving finite products. Mon$_{st}$(Cat) is a 2-category of monoidal categories, strict monoidal functors, and monoidal natural transformation. Act$_{st}$(Cat) is a 2-category of actions of monoidal categories, strict morphisms of actions, and transformations of actions. See Appendix for details. Cat, Mon$_{st}$(Cat), and Act$_{st}$(Cat) are 0-cells in 2CAT$_x$.

$\omega$ denotes the set of natural numbers, $n = \{0, \ldots, n-1\}$, for $n \in \omega$.

1Note that how ‘small’ are our categories is up to us. The (weighted) limits that we are considering in the paper are always countable.
2 The weight 2-functor for the category of monoids

Let $M_n$ denote the universe of free monoid on $n$-generators, for $n \in \omega + 1$. The set $BW_n$ of binary words on letters $n$ is the free magma on two operations of arity 0 and 2, i.e., it is the least set such that

1. $\iota \in BW_n$, where $\iota$ is a distinguished element,
2. $k \in BW_n$, for $k \in n$,
3. $(u \circ v) \in BW_n$, for $u, v \in BW_n$.

We have a type function $ty : BW_\omega \to Lin$ associating to any tree $t$ the linear order of occurrences of the numbers in $t$, i.e., it is the composition of functions

$$ty : BW_\omega \to M_\omega \to Lin$$

where the first map is the obvious homomorphism (of magmas) and the second one associates the linear order of occurrences of numbers in the words. For $o \in ty(t)$, we denote by $|o| \in \omega$ the number corresponding to the occurrence $o$.

The 2-Lawvere theory $M$ for monoidal categories is a 2-category defined as follows. The objects of $M$ are natural numbers. The morphism in $M$

$$\vec{u} = (u_0, \ldots, u_{n-1}) : m \to n$$

is an $n$-tuple of binary word in $BW_m$, i.e., $u_i \in BW_m$, for $i \in n$. The identities are given by

$$\vec{u} : n \to n$$

where $u_i = i$, for $i \in n$ and $n \in \omega$. The $k$-projections

$$\pi_k : n \to 1$$

are given by the binary word $\pi_k = k$, for $k \in n$, , for $n \in \omega$. The composition of morphisms is defined by simultaneous substitution. Given two morphisms

$$(u_0, \ldots, u_{m-1}) : k \to m, \quad (v_0, \ldots, v_{n-1}) : m \to n$$

their composition is given by

$$(v_0 \{0\backslash u_0, \ldots, m-1 \backslash u_{m-1}\}, \ldots, v_{n-1} \{0\backslash u_0, \ldots, m-1 \backslash u_{m-1}\}) : k \to n.$$  

and will be shortened to

$$(v_i \{j \backslash u_j\}_{j \in m})_{i \in n} : k \to n.$$  

A unique 2-cell

$$\vec{u} \Rightarrow \vec{v} : m \to n$$

exists iff $ty(u_i) = ty(v_i)$, for $i \in n$. Thus hom-sets in $M$ are equivalence relations.

**Examples.** There are morphisms in $M(3, 1)$, i.e., 2-cells in $M$, between $(((1 \odot 2) \odot (\iota \odot 0)) \odot 1)$ and $(((1 \odot 2) \odot (\iota \odot (0 \odot 1))) \odot \iota)$ but not between $(0 \odot (2 \odot 0))$ and $(0 \odot (0 \odot 2))$.

The following pictures present binary words as binary trees with labelled leaves:

$$t_0 = ((\iota \odot 2) \odot (0 \odot 1)) \quad t_1 \quad t_2 \quad t_3$$

$$t_4 \quad t_5 = ((\iota \odot (1 \odot \iota)) \odot 1) \quad t_6 \quad t_7$$
They are 1-cells in $\mathbb{M}(3, 1)$. There are 2-cells between 1-cells $t_0$ and $t_1$, $t_2$ and $t_3$, $t_4$ and $t_5$, and there are no other 2-cells between these 1-cells. Moreover, the orders of occurrences can be represented as

$$ty(t_0) = \langle\langle 0, 2\rangle, \langle 1, 0\rangle, \langle 2, 1\rangle\rangle, \quad ty(t_5) = \langle\langle 0, 1\rangle, \langle 1, 1\rangle\rangle,$$

with the first coordinate in the occurrence determining the order and the second the number that is occurring.

The category of words $CW$ has binary words in $BW_1$ as objects, and a morphism $f : a \rightarrow b$ between two binary words $a, b \in BW_1$ is a monotone function $f : ty(a) \rightarrow ty(b)$, i.e., between occurrences of 0's in $a$ and $b$.

**Examples.** Thus $s_0 = (0 \cdot (\cdot 0))$, $s_1 = (((0 \cdot (\cdot 0)) \cdot (\cdot 0) \cdot (\cdot 0) \cdot (\cdot 0))$ are objects of $CW$. The morphism from $s_0$ to $s_1$ in $CW$ is monotone function

$$f : \langle\langle 0, 0\rangle, \langle 1, 0\rangle\rangle \rightarrow \langle\langle 0, 0\rangle, \langle 1, 0\rangle, \langle 2, 0\rangle, \langle 3, 0\rangle\rangle$$

between linear orders.

The weight 2-functor

$$W : \mathbb{M} \rightarrow \text{Cat}$$

is defined as follows. For $m \in \omega$

$$W(m) = CW^m.$$  

For $\bar{u} : m \rightarrow n \in \mathbb{M}$, the functor

$$W(\bar{u}) : CW^m \rightarrow CW^n,$$

is given on object $\bar{a} = (a_0, \ldots, a_{m-1}) \in CW^m$ by

$$W(\bar{u})(\bar{a}) = (u_i(j\backslash a_j)_{j \in \alpha})_{i \in \alpha},$$

i.e., we substitute $m$ objects from $BW_1$ into $n$ objects from $BW_m$ and we obtain $n$ objects in $BW_1$.

On morphism $\bar{f} : \bar{a} \rightarrow \bar{b} \in CW^m$ the functor $W$ is given by

$$W(\bar{a})(\bar{f}) = (\prod_{o \in ty(\bar{u})} f_{\bar{o}})_{\bar{i} \in \bar{n}},$$

where $\prod_{x \in X} Y_x$ is the ordered sum of posets $Y_x$ indexed by a poset $X$. In particular, if $X$ and $Y_x$ are linear orders, the sum is a linear order as well. $X \uplus Y$ is an ordered sum of two posets.

To a 2-cell

$$\alpha : \bar{u} \Rightarrow \bar{v} : m \rightarrow n$$

the 2-functor $W$ associates a natural transformation

$$W(\alpha) : W(\bar{u}) \rightarrow W(\bar{v}),$$

so that its $i$-th component at $\bar{a} \in CW^m$ is

$$W(\alpha)_{\bar{a}, i} = \prod_{o \in ty(\bar{u})} id_{ty(a_{\bar{o}})} : (u_i(j\backslash a_j)_{j \in \alpha}) \rightarrow (v_i(j\backslash a_j)_{j \in \alpha}),$$

for $i \in \alpha$.

The fact that $W(\alpha)$ is indeed a natural transformation follows from the commutativity of the following diagram:

$$\begin{array}{ccc}
W(\bar{a})(\bar{f}) = (\prod_{o \in ty(\bar{v})} f_{\bar{o}})_{\bar{i} \in \bar{n}} & \xrightarrow{W(\alpha)_{\bar{a}, i}} & (\prod_{o \in ty(\bar{u})} f_{\bar{o}})_{\bar{i} \in \bar{n}} = W(\bar{v})(\bar{f}) \\
(u_i(j\backslash a_j)_{j \in \alpha})_{\bar{i} \in \bar{n}} & \xrightarrow{W(\alpha)_{\bar{a}, i}} & (v_i(j\backslash b_j)_{j \in \alpha})_{\bar{i} \in \bar{n}}
\end{array}$$
in $W(n)$, which is an ordered sum of the squares

$$
\begin{array}{c}
\text{id}_{ty(a_0)} \\
\text{id}_{ty(b_0)}
\end{array}
\begin{array}{c}
a_{[0]} \\
b_{[0]}
\end{array}
\begin{array}{c}
a_{[0]} \\
b_{[0]}
\end{array}
\begin{array}{c}
f_{[0]} \\
f_{[0]}
\end{array}
$$

for $o \in ty(u_i) = ty(v_i)$ and $i \in n$.

**Theorem 2.1.** We have a equivalence of 2-categories

$$
\zeta : \text{Mon}_{st}(\text{Cat}) \rightarrow 2\text{CAT}_x(\mathbb{M}, \text{Cat})
$$

such that the triangle

$$
\begin{array}{c}
\text{Mon}_{st}(\text{Cat}) \\
\text{Cat.}
\end{array}
\begin{array}{c}
\zeta \\
\text{Lim}_W
\end{array}
\begin{array}{c}
2\text{CAT}_x(\mathbb{M}, \text{Cat})
\end{array}
$$

commutes up to a 2-equivalence. In other words, $\mathbb{M}$ is the 2-Lawvere theory for monoidal categories and $W : \mathbb{M} \rightarrow \text{Cat}$ is the weight 2-functor for the category of monoids.

**Proof.** $\mathbb{M}$ has finite products given by the sum: $m \times n = m + n$.

To see the first claim, we shall define the 2-functor

$$
\begin{array}{c}
\text{Mon}_{st}(\text{Cat}) \\
\text{Mon}_{st}(\text{Cat})
\end{array}
\begin{array}{c}
\zeta \\
\zeta
\end{array}
\begin{array}{c}
2\text{CAT}_x(\mathbb{M}, \text{Cat})
\end{array}
$$

from the 2-category of monoidal categories to the 2-category for finite product preserving 2-functors $F : \mathcal{M} \rightarrow \text{Cat}$, natural transformations and modifications between them.

Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category. We define a 2-functor

$$
\overline{\mathcal{M}} = \zeta(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho) : \mathbb{M} \rightarrow \text{Cat}
$$

as follows. For $n \in \omega$

$$
\overline{\mathcal{M}}(n) = \mathcal{M}^n.
$$

for $s : n \rightarrow 1$ in $\mathbb{M}$

$$
\overline{\mathcal{M}}(s) = \left\{ \begin{array}{ll}
I & \text{if } s = \iota, \\
\pi_k & \text{if } s = \iota \in n, \\
\otimes \circ (\overline{\mathcal{M}}(s_1), \overline{\mathcal{M}}(s_2)) & \text{if } s = (s_1 \circ s_2).
\end{array} \right.
$$

where $I : \mathcal{M}^n \rightarrow \mathcal{M}$ is the constant functor with value $I$, and $\pi_k : \mathcal{M}^n \rightarrow \mathcal{M}$ is the $k$-th projection. As $\overline{\mathcal{M}}$ is supposed to preserve finite products, it is enough to define it on morphisms with codomain 1.

If $\sigma : s \Rightarrow t : n \rightarrow 1$ is a 2-cell in $\mathbb{M}$, then $\overline{\mathcal{M}}(\sigma) : \overline{\mathcal{M}}(s) \Rightarrow \overline{\mathcal{M}}(t) : \overline{\mathcal{M}}(n) \Rightarrow \overline{\mathcal{M}}(1)$ is the unique formal (i.e., built from $\alpha, \lambda, \rho$) natural transformation between functors $\pi$ and $\overline{\mathcal{M}}$. Such a natural transformation exists and is unique by MacLane’s coherence theorem for monoidal categories.

In order to show that $W : \mathbb{M} \rightarrow \text{Cat}$ is indeed the weight 2-functor for the category of monoids, we shall construct the universal $W$-weighted cone $\tau$ over $\overline{\mathcal{M}}$ with the vertex being the category of monoids $\text{mon}(\mathcal{M}, \ldots) = \text{mon}(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$. As $W$ preserves finite products, it is enough to define the projections $\tau_{(-)} : \text{mon}(\mathcal{M}) \rightarrow \mathcal{M}$ indexed by the objects and morphisms of category $W(1)$.

For object $a \in W(1)$ and monoid $(\mathcal{M}, m, i)$, we put

$$
\tau_a(M, m, i) = \left\{ \begin{array}{ll}
I & \text{if } a = \iota, \\
\mathcal{M} & \text{if } a = 0, \\
\tau_{a_1}(M, m, i) \otimes \tau_{a_2}(M, m, i) & \text{if } a = (a_1 \circ a_2),
\end{array} \right.
$$
and for homomorphism $h : (M, m, i) \to (M', m', i')$

$$
\tau_a(h) = \begin{cases} 
1_I & \text{if } a = \iota, \\
\tilde{h} & \text{if } a = 0, \\
\tau_{a_1}(h)\otimes\tau_{a_2}(h) & \text{if } a = (a_1\circ a_2).
\end{cases}
$$

We still need to define $\tau$ on morphisms of $\mathcal{W}(1)$. We shall do it for the (unique) morphisms of the form $f_a : a \to 0$. The unique extension to all the morphisms in $\mathcal{W}(1)$ is again due to the MacLane's coherence theorem. We define

$$
\tau_{f_a}(M, m, i) = \begin{cases} 
i & \text{if } a = \iota, \\
1_M & \text{if } s = 0, \\
m \circ (\tau_{f_{a_1}}(M, m, i) \otimes \tau_{f_{a_2}}(M, m, i)) & \text{if } a = (a_1 \circ a_2).
\end{cases}
$$

The remaining details of the construction and verifications that $\tau$ is indeed a universal $\mathcal{W}$-weighted cone are left for the reader. □

### 3 The weight 2-functor for actions of monoids

In this section we construct the weight 2-functor for actions of monoids along an action of a monoidal category. As the Lawvere theory of actions of monoidal categories has two types, we shall be using two (disjoint) sets of natural numbers: the usual one $\omega$, and the set of ‘underlined’ natural numbers $\underline{\omega} = \{\underline{n} : n \in \omega\}$.

The set $BW_{*, n, n'}$ of **pointed binary words** on letters $n \in \omega + 1$ and $n' \in \omega + 1$ is the least set such that:

1. $\underline{k} \in BW_{*, n, n'}$, for $k \in n'$,
2. $(u \circ s) \in BW_{*, n, n'}$, for $u \in BW_{n}$ and $s \in BW_{*, n, n'}$.

We have a type function $ty_* : BW_{*, \omega, \omega} \to Lin_*$ associating to any pointed binary word $t$ the linear order with the right end-point of occurrences of the numbers in $t$. Note that in pointed binary words there is always one occurrence of an underlined number and it is the last occurrence in the word; so it is the top end-point in the linear order of occurrences. If $o \in ty_*(t)$, then $|o|$ denotes the number or underlined number corresponding to the occurrence $o$.

The 2-Lawvere (finite product) theory $\mathbb{A}$, for actions of monoidal category, is a 2-category defined as follows. The objects of $\mathbb{A}$ are pairs of natural numbers. The 1-cells in $\mathbb{A}$

$$(\underline{u}, \underline{s}) = (u_0, \ldots, u_{n-1}; s_0, \ldots, s_{n'-1}) : (m, m') \to (n, n')$$

is an $n$-tuple of binary words in $BW_{m}$, i.e., $u_i \in BW_{m}$, for $i \in n$ and $n'$-tuple of pointed binary words in $BW_{*, m, m'}$, i.e., $s_j \in BW_{*, m, m'}$, for $i' \in n'$. When convenient, we shorten the notation for morphisms to $(u; s)_{i,j} \in n, i' \in n'$ or even to $(u; s)$.

The identity is given by

$$(u_0, \ldots, u_{n-1}; s_0, \ldots, s_{n'-1}) : (n, n') \to (n, n'),$$

where $u_i = i$, for $i \in n$, and $s_{i'} = \underline{j'}$, for $i' \in n'$. The $k$-th projection is

$$\pi_k = (u_i) : (n, n') \to (1, 0)$$

where $u = k$, for $k \in n$, and the $k$-th projection is

$$\pi_k = (s) : (n, n') \to (0, 1)$$

where $s = \underline{k}$, for $k \in n'$.

The composition of morphisms is defined by simultaneous substitution. Given two morphisms

$$(u_0, \ldots, u_{m-1}; s_0, \ldots, s_{m'-1}) : (k, k') \to (m, m'), \quad (v_0, \ldots, v_{n-1}; t_0, \ldots, t_{n'-1}) : (m, m') \to (n, n')$$

The 2-Lawvere theory of actions of monoidal categories has two types, we shall be using...
their composition is given by
\[(u_i; j; v_i; j')_{i \in m} : (k, k') \to (n, n').\]

A unique 2-cell
\[\langle u_i; s_j \rangle \in W \Rightarrow (v_i; t_j) : (m, m') \to (n, m')\]
exists iff \(ty(u_i) = ty(v_i)\), for \(i \in n\), and \(ty(u_j) = ty(t_j)\), for \(j \in n'\).

The category of pointed words \(\text{CW}_*\) has \(\text{BW}_*, 1, 1\) as the set of objects, and a morphism \(f : a \to b\) between two pointed binary words \(a, b \in \text{BW}_*, 1, 1\) is monotone top end-point preserving function \(f : ty(a) \to ty(b)\), i.e., between occurrences of 0's and \(\overline{0}\) (occurrence of 0 can be sent to the occurrence \(\overline{0}\) but not vice versa).

The weight 2-functor
\[W_a : \mathcal{A} \to \text{Cat}\]
is defined as follows. For \(m, m' \in \omega\)
\[W_a(m, m') = CW^m \times CW^{m'}\]
For \((\bar{a}; \bar{s}) : (m, m') \to (n, n') \in \mathcal{A}\), the functor
\[W_a(\bar{a}; \bar{s}) : W_a(m, m') \to W_a(n, n'),\]
is given, for \((\bar{a}; \bar{b}) \in W_a(m, m')\), by
\[W_a(\bar{a}; \bar{s})(\bar{a}; \bar{b}) = (u_i; j; v_i; j')_{i \in m}
\]
and, on morphism \((\bar{f}; \bar{g}) : (\bar{a}; \bar{b}) \to (\bar{a}; \bar{b}) \in W_a(m, m')\), is given by
\[W_a(\bar{a}; \bar{s})(\bar{f}; \bar{g}) = \bigoplus_{o \in \text{ty}(u_i)} \bigoplus_{i \in \omega} f_{i|o} \bigoplus_{o \in \text{ty}(v_i)} \bigoplus_{i \in \omega} g_{i|o}\]
Note that \(W_a(\bar{a}; \bar{s})(\bar{f}; \bar{g})\) is well defined as the last summand of the right sum contains exactly one element. To a 2-cell
\[\alpha : (\bar{a}; \bar{s}) \Rightarrow (\bar{v}; \bar{t}) : (m, m') \to (n, n')\]
the 2-functor \(W_a\) associates a natural transformation
\[W_a(\alpha) : W_a(\bar{a}; \bar{s}) \to W_a(\bar{v}; \bar{t}) : W_a(m, m') \to W_a(n, n'),\]
so that its \(i\)-th component at \((\bar{a}; \bar{b}) \in W_a(m, m')\) is
\[W_a(\alpha)_{\bar{a}, \bar{b}, i} = \bigoplus_{o \in \text{ty}(u_i)} id_{ty(u_i)} : (u_i; j; a_j)_{j \in m} \to (v_i; j; a_j)_{j \in m},\]
for \(i \in n\), and its \(i'\)-th component is
\[W_a(\alpha)_{\bar{a}, \bar{b}, i'} = \bigoplus_{o \in \text{ty}(u_i), |o| = 0} id_{ty(u_i)} \bigoplus_{o \in \text{ty}(v_i), |o| = 2} id_{ty(u_i)} : (s_i; j; a_j; j')_{j \in m} \to (t_i; j; a_j; j')_{j \in m, j' \in m'},\]
for \(i' \in n'\).

The fact that \(W_a(\alpha)\) is indeed a natural transformation follows from the commutativity of the following diagram in \(\text{Cat}\):
for any morphism \( (f'; g) : (\vec{a}; \vec{b}) \to (\vec{a}'; \vec{b}') \) in \( W_a(n, n') \). And this can be checked in a similar way as in the case of the weight 2-functor for monoids \( W \).

**Theorem 3.1.** We have an equivalence of 2-categories

\[
\zeta_\alpha : \text{Act}_{st}(\text{Cat}) \to 2\text{CAT}_\times(A, \text{Cat})
\]

such that the triangle

\[
\begin{array}{ccc}
\text{Act}_{st}(\text{Cat}) & \xrightarrow{\zeta_\alpha} & 2\text{CAT}_\times(A, \text{Cat}) \\
\downarrow\text{act} & & \downarrow\text{Lim}_{W_a} \\
\text{Cat.} & & \text{Cat.}
\end{array}
\]

commutes up to a 2-isomorphism. In other words, \( A \) is the (2-sorted) 2-Lawvere theory for actions of monoidal categories and \( W_A : A \to \text{Cat} \) is the weight 2-functor for the category of actions of monoids along an action of a monoidal category.

**Proof.** \( A \) has finite products given by the sum: \( (n, m) \times (n', m') = (n + n', m + m') \).

Let \( A = (M \otimes, I, \alpha, \lambda, \rho, \star, \psi, \bar{\psi}) \) be an action a monoidal category \((M \otimes, I, \alpha, \lambda, \rho)\) on a category \( X \). As before, we shall describe the universal \( N\)-cone with the vertex being the category of actions \( \text{act}(A) = \text{act}(M \otimes, I, \alpha, \lambda, \rho, X, \star, \psi, \bar{\psi}) \).

We define a 2-functor

\[
\tilde{A} = \zeta_\alpha(A) : A \to \text{Cat}
\]

as follows. For \( n, n' \in \omega \)

\[
\tilde{A}(n, n') = M^n \times X^{n'}.
\]

For a morphism \( (u, \emptyset) : (n, n') \to (1, 0) \) in \( A \), we put

\[
\tilde{A}(u, \emptyset) = M(u) \circ \pi_M : M^n \times X^{n'} \to M
\]

where \( \pi_M : M^n \times X^{n'} \to M^n \) is the obvious projection, and for a morphism \( (\emptyset, s) : (n, n') \to (0, 1) \) in \( A \), we put

\[
\tilde{A}(\emptyset, s) = : M^n \times X^{n'} \to X
\]

to be

\[
\tilde{A}(\emptyset, s) = \begin{cases} 
\pi_M & \text{if } s = \emptyset, \\
(M(u') \circ \pi_M) \star \tilde{A}(\emptyset, s') & \text{if } s = (u' \circ s'),
\end{cases}
\]

As \( \tilde{A} \) is supposed to preserve finite products, it is enough to define it on morphisms with codomain \((1, 0)\) and \((0, 1)\).

If \( \sigma : (u, s) \Rightarrow (v, t) : (n, n') \to (1, 1) \) is a 2-cell in \( A \), then \( \tilde{A}(\sigma) : \tilde{A}(u, s) \Rightarrow \tilde{A}(v, t) \) is the unique formal (i.e., built from \( \alpha, \lambda, \rho, \psi, \bar{\psi} \)) natural transformation between these functors. Such a natural transformation exists and is unique, again, by MacLane’s coherence theorem for monoidal categories.

In order to show that \( W_a : A \to \text{Cat} \) is indeed the weight 2-functor for the category of monoids, we shall construct the universal \( W_a \)-weighted cone \( \kappa \) over \( \tilde{A} \) with the vertex being the category of actions of monoids along action of monoidal category \( \text{act}(A) = \text{act}(M \otimes, I, \alpha, \lambda, \rho, X, \star, \psi, \bar{\psi}) \). As \( W_a \) preserves finite products, it is enough to define the projections

\[
\kappa_{(-)} : \text{act}(A) \to M \quad \text{and} \quad \kappa_{(-)} : \text{act}(A) \to X
\]

indexed by the objects and morphisms of categories \( W_a(1, 0) \) and \( W_a(0, 1) \), respectively. Note that the projections \( \kappa_{a, \emptyset} \) and \( \kappa_{f_a, \emptyset} \) composed with \( \pi_M \) are the projections \( \tau_\alpha \tau_{f_a} \), respectively, from the universal \( W \)-cone for monoids \( \tau_\alpha \), defined in the previous proof.

For object \((a, \emptyset) \in W_a(1, 0)\) and action \((M, m, i, X, \alpha)\), we put

\[
\kappa_{a, \emptyset}(M, m, i, X, \alpha) = \tau_\alpha((M, m, i)),
\]
and for and for morphism of actions \((h, k) : (M, X, m, i, \alpha) \rightarrow (M', X', m', i', \alpha')\)

\[
\kappa_{a, \emptyset}(h, k) = \tau_a(h).
\]

To define \(\kappa\) on morphisms of \(W_a(1, 0)\) we again use projection \(\tau\). We shall define the natural transformation \(\kappa_{(h, 0)}\) for the (unique) morphisms of the form \((f_a, \emptyset) : (a, \emptyset) \rightarrow (0, \emptyset)\) in \(W_a(1, 0)\). The unique extension to all the morphisms in \(W_a(1, 0)\) is again due to the MacLane’s coherence theorem. We put

\[
\kappa_{f_a, \emptyset}(M, X, m, i, \alpha) = \tau_{f_a}(M, m, i).
\]

For object \((\emptyset, b) \in W_a(0, 1)\) and action \((M, m, i, X, \alpha)\), we put

\[
\kappa_{\emptyset, b}(M, m, i, X, \alpha) = \begin{cases} X & \text{if } b = \emptyset, \\ \tau_a(M, m, i) \ast \kappa_{\emptyset, b'}(M, m, i, X, \alpha) & \text{if } b = (a \circ b'), \end{cases}
\]

and for morphism of actions \((h, k) : (M, X, m, i, \alpha) \rightarrow (M', X', m', i', \alpha')\)

\[
\kappa_{\emptyset, b}(h, k) = \begin{cases} k & \text{if } b = \emptyset, \\ \tau_a(h) \ast \kappa_{\emptyset, b'}(h, k) & \text{if } b = (a \circ b'). \end{cases}
\]

We still need to define \(\kappa\) on morphisms of \(W_{\emptyset, b}(0, 1)\). We shall do it for the (unique) morphisms of the form \((\emptyset, g_b) : (\emptyset, b) \rightarrow (\emptyset, \emptyset)\). The unique extension to all the morphisms in \(W_{\emptyset, b}(1, 0)\) is again due to the MacLane’s coherence theorem. We define

\[
\kappa_{\emptyset, g_b}(M, m, i, i, \alpha) = \begin{cases} 1 & \text{if } b = \emptyset, \\ \alpha \circ (\tau_{f_a}(M, m, i) \ast \kappa_{\emptyset, g_b'}(M, m, i, \alpha)) & \text{if } b = (a \circ b'). \end{cases}
\]

The remaining details of the construction and verifications that \(\kappa\) is indeed a universal \(W_a\)-weighted cone are left for the reader. \(\square\)

4 Concluding remarks

Extension of the context.

To make is simple the paper is presented in the essentially simplest context in which considerations of these weights is meaningful. Now we shall comment on possible extension of the context.

We know that in \(\textbf{Cat}\) not only strict but also lax monoidal functors between monoidal categories induce functors between categories of monoids. This is due to the fact even if the composition \(H(\tau)\) of a (strict) universal cone \(\tau : \text{mon}(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho) \rightarrow \mathcal{M}\) composed with a lax monoidal functor

\[
(F, \overline{\varphi}, \varphi) : (\mathcal{M}, \otimes, I, \alpha, \lambda, \rho) \rightarrow (\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')
\]

is not a strict cone (i.e., the commutations of 1-cells holds up to a non-invertible 2-cells) we can ‘strictify’ such a cone to a cone \(\sigma : \text{mon}(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho) \rightarrow \mathcal{M}'\) that is strict and hence it induces a functor between categories of monoids by the universal property of the category of monoids \(\text{mon}(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')\). The strictified cone \(\sigma\) can be defined as follows. For object \(a \in \mathcal{W}(1)\) and monoid \((M, m, i)\), we put

\[
\sigma_a(M, m, i) = \begin{cases} I' & \text{if } a = \iota, \\ F(M) & \text{if } a = 0, \\ \sigma_{a_1}(M, m, i) \otimes \sigma_{a_2}(M, m, i) & \text{if } a = (a_1 \circ a_2), \end{cases}
\]

and for homomorphism \(h : (M, m, i) \rightarrow (M', m', i')\)

\[
\sigma_a(h) = \begin{cases} 1_{I'} & \text{if } a = \iota, \\ F(h) & \text{if } a = 0, \\ \sigma_{a_1}(h) \otimes \sigma_{a_2}(h) & \text{if } a = (a_1 \circ a_2). \end{cases}
\]

For the (unique) morphisms of the form \(f_a : a \rightarrow 0\), we define
\[ \sigma_{f_a}(M, m, i) = \begin{cases} 
F(i) \circ \varphi : I' \to H(M) & \text{if } a = \iota, \\
1_M & \text{if } s = 0, \\
H(m) \circ (\sigma_{f_{a_1}}(M, m, i) \otimes \sigma_{f_{a_2}}(M, m, i)) \circ \varphi_{\sigma_{a_1}(M, m, i), \sigma_{a_2}(M, m, i)} & \text{if } a = (a_1 \circ a_2). 
\end{cases} \]

The remaining details of the construction and verifications that \( \sigma \) is indeed a (strict) universal \( W \)-weighted cone over \( (M, \otimes, I, \alpha, \lambda, \rho) \) is a routine check.

Once we have the weight 2-functor for categories of monoids, we can use it to define the category of monoids objects over any monoidal object in any 2-category with finite products. As the strictification described above can be defined using the universal properties of finite products, it is still true that a lax monoidal morphism of monoidal category objects in any 2-category with finite products induces a 1-cell between objects of monoids.

### Algebra needs coalgebra

Affine algebraic sets over (set-based) algebraic structures (like rings, fields, groups, module etc.) can be defined as limits on the diagrams that involve finite power of the universe and some definable (polynomial) functions between them. Typically, these limits come from finite sets of equations

\[ f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n) \]

but we do not make any restrictions on the variables that occur on both sides of the equations so that we can consider equations like

\[ f(x, x, y) = g(x, y, y, y) \]

that use the same variable more than once, not necessarily the same number of times on each side (thus using diagonals), and we can also have equations

\[ m(x, x) = e \]

that have different variables occurring on different sides of the equation (thus using projections).

The limits giving rise to such algebraic sets can be chosen canonically, if we allow weights in their definitions. We shall ‘prove it’ by an example. Let \( A \) be a commutative ring in a complete category \( E \). Then the equation \( x^2 = y^3 \) defines a subobject \( Z \) of the square of the universe of \( A \) (also denoted by \( A \)). In the internal language it can be expressed as

\[ Z = \{ (a, b) \in A^2 | a^2 = b^3 \} \]

Let \( B = F[x, y]/x^2 - y^3 \) be the free commutative ring (in \( Set \)) on two generators \( x \) and \( y \) divided by the equation \( x^2 = y^3 \), and \( L_{cring} \) be the Lawvere theory for commutative rings. We have finite products preserving functors

\[ \bar{A} : L_{cring} \to \mathcal{E}, \quad \bar{B} : L_{cring} \to Set \]

corresponding to the rings \( A \) and \( B \). Then, the set \( Z \) is the weighted limit \( \text{Lim}_{\bar{B}}\bar{A} \). We show that it is the case if \( \mathcal{E} \) is the category of sets \( Set \). We have a sequence of isomorphisms

\[ Z = \{ (a, b) \in A^2 | a^2 = b^3 \} \cong \]

\[ \text{Hom}(B, A) \cong \]

\[ \text{Nat}(\bar{B}, \bar{A}) \cong \]

\[ \text{Nat}(\bar{B}, \text{Set}(1, \bar{A}(-))) \cong \]

\[ \text{Set}(1, \text{Lim}_{\bar{B}}\bar{A}) \cong \]

\[ \text{Lim}_{\bar{B}}\bar{A} \]

where \( \text{Hom} \) is the hom-set in the category of commutative rings.
Note that we also have a PROP\(^\text{2}\) for commutative rings \(\mathcal{C}_{\text{cring}}\) and hence symmetric monoidal functors
\[
\tilde{A} : \mathcal{C}_{\text{cring}} \to \mathcal{E}, \quad \tilde{B} : \mathcal{C}_{\text{cring}} \to \text{Set}
\]
corresponding to rings \(A\) and \(B\). The monoidal structure considered on both \(\mathcal{E}\) and \(\text{Set}\) is the finite product structure. However, it is not the case that \(\text{Lim}_{\tilde{B}} \tilde{A}\) is isomorphic to the object \(\mathbb{Z}\) (even if \(\mathcal{E}\) is \(\text{Set}\)), as natural transformations from \(\tilde{B}\) to \(\tilde{A}\) do not correspond to homomorphisms from \(B\) to \(A\) in this case. The reason for this is that \(\mathcal{C}_{\text{cring}}\) does not have projections and diagonals, a piece of coalgebra which was vital in the former argument.

In other words, to define the usual algebraic sets we use the coalgebra structure on this set with respect to the tensor being the usual cartesian product. This comonoid structure is usually not mentioned for good reasons: it is unique, if our tensor is the binary product. However, if we replace the product by some other tensor, we need to specify the comonoid structure separately, if we want to use it. In this sense to do algebra we need to use a bit of coalgebra. This must be taken into account when we define 2-algebraic structures, as well.

2-algebra needs 2-coalgebra

As we already learned from the previous discussion, it is not necessarily true that if we can identify internally an algebraic concept (ring), then we will be able to derive all the ‘algebraic sets’ related to it (set of solutions of equations). In fact, now we need to talk about ‘2-algebraic sets’ as the derived concepts will be categories or even 0-cells in a 2-category. The category \(s\Delta\) is not even a 2–PROP, (i.e., a strict symmetric monoidal 2-category whose 0-cells are natural numbers such that \(I = 0\) and \(n \otimes m = n + m\)) but the Eilenberg-Moore object has sufficiently simple structure that we are able to get it as a weighted limit from the functor with domain \(s\Delta\). Thus in this case no coalgebra is needed.

If we want to internalize the notion of a monoidal category, we have to have, in our ambient 2-category \(\mathcal{K}\), finite products of 0-cells or at least a 2-monoidal structure. Then we can easily define a 2-category \(\mathcal{P}\mathcal{M}\) which is a 2–PROP for monoidal categories, i.e., with the property that if \(\mathcal{K}\) is a 2-category with finite products, then the 2-monoidal functors from \(\mathcal{P}\mathcal{M}\) to \(\mathcal{K}\) correspond to monoidal categories in \(\mathcal{K}\). However, if we want to derive a ‘2-algebraic set’ of monoids from a monoidal category, the weighted limit of a monoidal 2-functor from \(\mathcal{P}\mathcal{M}\) is not enough. This is because when we look at the structure maps of monoids
\[m : M \otimes M \to M \quad i : I \to M\]
as ‘some kind of equalities’, they are not linear-regular (cf. \(\text{[SZ1]}\), \(\text{[SZ2]}\)) as in the left ‘equation’ a variable is repeated and on the right a variable is dropped. Thus this uses full force of the equational logic, not just the linear-regular part. Therefore to define internally the object of monoids either we need the internal version of the notion of a bi-monoidal category (by this we mean the categorification of the notion of a bi-monoid) or we need to define the internal notion of a monoidal category on the basis of finite product, i.e., not using 2–PROP’s but Lawvere 2-theories. In this paper, we had follow the latter approach but a further extension is still possible using the former.

5 Appendix.

5.1 Monoidal category and category of monoids

A monoidal category \((\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)\) consists of
1. a category \(\mathcal{M}\),
2. a functor \(\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}\), and object \(I \in \mathcal{M}\),
3. three natural isomorphisms
\[
\alpha : \otimes \circ (1_{\mathcal{M}} \times \otimes) \to \otimes \circ (\otimes \times 1_{\mathcal{M}}) : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M},
\]
\[
\lambda : \otimes \circ (I, 1_{\mathcal{M}}) \to 1_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}, \quad \rho : \otimes \circ (1_{\mathcal{M}}, I) \to 1_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}
\]
\(^2\text{PROP is a strict symmetric monoidal category whose objects are natural numbers such that } I = 0 \text{ and } n \otimes m = n + m.\)
such that, for objects $M_1$, $M_2$, $M_3$, $M_4$ in $\mathcal{M}$, the following two diagrams in $\mathcal{M}$ commute.

**A strict monoidal functor**

$F : (\mathcal{M}, \otimes, I, \alpha, \lambda, \rho) \rightarrow (\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$

is a 1-cell $F : \mathcal{M} \rightarrow \mathcal{M}'$ such that

$$F(I) = I', \quad F(M_1 \otimes M_2) = F(M_1) \otimes' F(M_2),$$

for $M_1$, $M_2$ in $\mathcal{M}$, and

$$F(\alpha) = \alpha'_F, F(\lambda) = \lambda'_F, \quad F(\rho) = \rho'_F.$$

**A monoidal transformation between two strict monoidal 1-cells**

$\tau : F \rightarrow F' : (\mathcal{M}, \otimes, I, \alpha, \lambda, \rho) \rightarrow (\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$

is a 2-cell $\tau : F \rightarrow F'$ in $\mathcal{K}$ such that

$$\tau_I = 1_{I'}, \quad \tau_{M_1} \otimes \tau_{M_2} = \tau_{M_1 \otimes M_2}.$$

In this way we have defined the 2-category $\text{Mon}_{st}(\text{Cat})$ of monoidal categories with strict monoidal functors and monoidal natural transformations. It is an object map of a 3-functor

$$\text{Mon}_{st} : 2\text{CAT}_{\times} \rightarrow 2\text{CAT}_{\times},$$

whose remaining parts of definition are left for the reader.

Given a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ in $\text{Cat}$, we define the category of monoids. The objects are monoids, i.e., triples $(M, m : M \otimes M \rightarrow M, i : I \rightarrow M)$ so that the following diagram
commutes. A homomorphism of monoids \( h : (M, m, i) \to (M', m', i') \) is a morphism \( h : M \to M' \) in \( \mathcal{M} \) such that the diagram:

\[
\begin{array}{ccc}
M \otimes M & \xrightarrow{m} & M \\
\downarrow{h \otimes h} & & \downarrow{h} \\
M' \otimes M' & \xrightarrow{m'} & M'
\end{array}
\]

commutes. In this way we have defined the category of monoids \( \text{mon}(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho) \) over monoidal category \((\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)\) in \( \text{Mon}_{\mathcal{M}}(\text{Cat}) \). It is an object map of a 2-functor

\[\text{mon} : \text{Mon}_{\mathcal{M}}(\text{Cat}) \to \text{Cat}\]

whose remaining parts of definition are again left for the reader.

### 5.2 Action of monoidal category and actions along action

A **monoidal action** \( (\mathcal{M}, \otimes, I, \alpha, \lambda, \rho, \mathcal{X}, \ast, \psi, \overline{\psi}) \) of a monoidal category \((\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)\) on a category \( \mathcal{X} \) consists of

1. a monoidal category \((\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)\),
2. a category \( \mathcal{X} \),
3. a functor \( \ast : \mathcal{M} \times \mathcal{X} \to \mathcal{X} \),
4. and two natural transformations

\[
\psi : \ast \circ (1_{\mathcal{M}} \times \ast) \to \ast \circ (\otimes \times 1_{\mathcal{M}}) : \mathcal{M} \times \mathcal{M} \times \mathcal{X} \to \mathcal{X},
\]

\[
\overline{\psi} : 1_{\mathcal{X}} \to \ast \circ (I \otimes \ast) : \mathcal{X} \to \mathcal{X},
\]

(where \( I \) denotes here the constant functor \( \mathcal{M} \to \mathcal{M} \) equal \( I \))

such that, for objects \( M_1, M_2, M_3 \) in \( \mathcal{M} \) and object \( X \) in \( \mathcal{X} \), the diagrams MA1, MA2, MA3 in \( \mathcal{M} \) below

\[
\begin{array}{c}
\xymatrix{M_1 \ast (M_2 \ast (M_3 \ast X)) \ar[dr]_{\psi_{M_1, M_2, M_3, X}} \ar[rr]^{\psi_{1_{M_1}, \psi_{M_2, M_3, X}}} & & M_1 \ast ((M_2 \otimes M_3) \ast X) \ar[dl]_{\psi_{M_1, M_2 \otimes M_3, X}} \ar[rr]^{\psi_{M_1, M_2, M_3, X}} & & (M_1 \otimes (M_2 \otimes M_3)) \ast X \ar[ll]_{\alpha_{M_1, M_2, M_3, X}} \ast X}
\end{array}
\]

MA1

\[
\begin{array}{c}
\xymatrix{M_1 \ast M_X \ar[dr]_{\psi_{M_1, \ast X}} \ar[rr]^{1_{M_1} \ast X} & & M_1 \ast M_X \ar[dl]_{\lambda_{M_1} \ast 1_X} \ar[rr]^{\lambda_{M_1} \ast 1_X} & & (I \otimes M_1) \ast X \ar[ll]_{\psi_{I, M_1, X}} \ast X}
\end{array}
\]

MA2

and

\[
\begin{array}{c}
\xymatrix{M_1 \ast I_X \ar[dr]_{1_{M_1} \ast \overline{\psi}_X} \ar[rr]^{1_{M_1} \ast X} & & M_1 \ast I_X \ar[dl]_{\rho_{M_1} \ast 1_X} \ar[rr]^{\rho_{M_1} \ast 1_X} & & (M_1 \otimes I) \ast X \ar[ll]_{\psi_{M_1, I, X}} \ast X}
\end{array}
\]

MA3

12
A (strict) morphism of monoidal actions

\[(F,G) : (\mathcal{M}, \otimes, I, \alpha, \lambda, \rho, \mathcal{X}, *, \psi) \rightarrow (\mathcal{M}', \otimes', I', \alpha', \lambda', \rho', \mathcal{X}', *, \psi')\]

consists of

1. a strict monoidal functor \(F : (\mathcal{M}, \otimes, I, \alpha, \lambda, \rho) \rightarrow (\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')\),
2. a functor \(G : \mathcal{X} \rightarrow \mathcal{X}'\)
3. such that
   \[F(M) \star' G(X) = G(M * X), \quad G(\psi) = \psi'_F \times F \times G, \quad G(\bar{\psi}) = \bar{\psi}'_G.\]

for objects \(M\) in \(\mathcal{M}\) and \(X\) in \(\mathcal{X}\).

A transformation of strict morphisms of actions

\[\tau : F \rightarrow F' : (\mathcal{M}, \otimes, I, \alpha, \lambda, \rho, \mathcal{X}, *, \psi) \rightarrow (\mathcal{M}', \otimes', I', \alpha', \lambda', \rho', \mathcal{X}', *, \psi')\]

is a pair of natural transformations \(\tau : F \rightarrow F'\) and \(\sigma : G \rightarrow G'\) such that

\[\tau_{M \star} \sigma_X = \tau_{M * X},\]

i.e.,

\[
\begin{array}{ccc}
F(M) \star' G(X) & \xrightarrow{=} & G(M * X) \\
\tau_{M \star} \sigma_X & & \tau_{M * X} \\
F'(M) \star' G'(X) & \xrightarrow{=} & G'(M * X)
\end{array}
\]

for objects \(M\) in \(\mathcal{M}\) and \(X\) in \(\mathcal{X}\).

In this way we have defined the 2-category \(\text{Act}_{st}(\text{Cat})\) of actions of monoidal categories with strict morphisms and 2-cells. It is an object map of a 3-functor

\[\text{Act}_{st} : 2\text{CAT}_X \rightarrow 2\text{CAT}_X,\]

whose remaining parts of definition are left for the reader.

Given an action of monoidal category \((\mathcal{M}, \otimes, I, \alpha, \lambda, \rho, \mathcal{X}, *, \psi)\) in \(\text{Cat}\), we define category \(\text{act}(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho, \mathcal{X}, *, \psi)\) the category of actions of monoids along the action of monoidal category \((\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)\) on objects of the category \(\mathcal{X}\). The objects are actions of monoids, i.e., 5-tuples \((M, X, m : M \otimes M \rightarrow M, i : I \rightarrow M, a : M * X \rightarrow X)\) so that \((M, m, i)\) is a monoid and moreover the following diagram

\[
\begin{array}{cccccc}
M \otimes (M * X) & \xrightarrow{\psi} & (M \otimes M) * X & \xrightarrow{m \otimes 1_X} & M * X & \xrightarrow{i * 1_X} & I * X \\
1 \otimes a & & & & a & & \phi \\
M * X & \xrightarrow{a} & X & & & & \\
\end{array}
\]

commutes. A homomorphism of actions \((h, k) : (M, X, m, i, a) \rightarrow (M', X', m', i', a')\) is a homomorphism of monoids \(h : (M, m, a) \rightarrow (M', m', a')\) and a morphism \(k : X \rightarrow X'\) in \(\mathcal{X}\) making the diagram

\[
\begin{array}{ccc}
M * X & \xrightarrow{a} & X \\
h * k & & k \\
M' * X' & \xrightarrow{a'} & X'
\end{array}
\]
commute. In this way we have defined the category of actions of monoids $\text{act}(\mathcal{M}, \otimes, I, \alpha, \rho, X, \star, \psi)$ over an action of a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \rho, X, \star, \psi)$ in $\text{Mon}_\text{st}(\text{Cat})$. It is an object map of a 2-functor

$$\text{act} : \text{Act}_\text{st}(\text{Cat}) \rightarrow \text{Cat}$$

whose remaining parts of definition are again left for the reader.

References

[Law] F. W. Lawvere, *Ordinal sums and equational doctrines*, in ‘Seminar on Triples and Categorical Homology Theory’, Lecture Notes in Mathematics, Vol. 86, pp. 141-155, Springer-Verlag, New York-Berlin, (1969).

[CWM] S. MacLane, *Categories for the working Mathematician*, Graduate Text in Math., Springer, 2nd ed, (1998).

[May1] J.P. May, *The Geometry of Iterated Loop Spaces*, Lectures Notes in Mathematics 271, Springer, (1972).

[May2] J.P. May, *Definitions: operads, algebras and modules*, in J-L. Loday, J. Stasheff, A. Voronov (eds.), Operads: Proceedings of Renaissance Conferences, Contemporary Mathematics 202, AMS, (1997).

[SZ1] S. Szawiel, M. Zawadowski, *Theories of analytic monads*, arXiv:1204.2703 [math.CT].

[SZ2] S. Szawiel, M. Zawadowski, *Monads of regular theories*, arXiv:1207.0121 [math.CT].