GENUS FIELDS OF FINITE ABELIAN EXTENSIONS

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ABSTRACT. In this paper we find the genus field of finite abelian extensions of the global rational function field. We introduce the term conductor of constants for these extensions and determine it in terms of other invariants. We study the particular case of finite abelian $p$–extensions and give an explicit description of their genus field.

1. INTRODUCTION

It was C. F. Gauss [10] the first one to consider what now is known as the genus field. The work of Gauss was in the context of binary quadratic forms. Later on this concept was translated into the context of quadratic number fields. In this way, originally, the definition of genus field was given for a quadratic extension of $\mathbb{Q}$. We have that for a quadratic number field $K$, the Galois group of $K_{\text{ge}}/K$, $K_{\text{ge}}$ denoting the genus field of $K$, is isomorphic to the maximal subgroup of exponent 2 of the ideal class group of $K$. It was proved by Gauss that if $s$ is the number of different positive finite rational primes dividing the discriminant $\delta_K$ of a quadratic number field $K$, then the 2–rank of the class group of $K$ is $2^{s-2}$ if $\delta_K > 0$ and there exists a prime $p \equiv 3 \mod 4$ dividing $\delta_K$ and $2^{s-1}$ otherwise.

Genus theory using class field theory was introduced by H. Hasse [12] for the special case of quadratic number fields. Hasse translated Gauss’ genus theory using characters. H. W. Leopoldt [18] generalized the results of Hasse determining the genus field $K_{\text{ge}}$ of an absolute abelian number field $K$. Leopoldt used Dirichlet characters to develop genus theory of absolute abelian extensions and related the theory of Dirichlet characters to the arithmetic of $K$.

The concept of genus fields for an arbitrary finite extension of the field of rational numbers was introduced by A. Fröhlich [7, 8, 9]. Fröhlich defined the genus field $K_{\text{ge}}$ of an arbitrary finite number field $K/\mathbb{Q}$ as $K_{\text{ge}} := K k^\ast$ where $k^\ast$ is the maximal abelian number field such that $K k^\ast/K$ is unramified. We have that $k^\ast$ is the maximal abelian number field contained in $K_{\text{ge}}$. The degree $[K_{\text{ge}} : K]$ is called the genus number of $K$ and the Galois group $\text{Gal}(K_{\text{ge}}/K)$ is called the genus group of $K$.

We have that if $K_H$ denotes the Hilbert class field of $K$, then $K \subseteq K_{\text{ge}} \subseteq K_H$ and $\text{Gal}(K_H/K)$ is isomorphic to the class group $\text{Cl}_K$ of $K$. The genus field $K_{\text{ge}}$ corresponds to a subgroup $G_K$ of $\text{Cl}_K$, that is, $\text{Gal}(K_{\text{ge}}/K) \cong \text{Cl}_K / G_K$. The subgroup

\textbf{Date:} October 21st., 2017.
2010 Mathematics Subject Classification. Primary 11R58; Secondary 11R60, 11R29.
Key words and phrases. Global function fields, ramification, genus fields, abelian $p$–extensions.
$G_K$ is called the principal genus of $K$ and $|Cl_K/G_K|$ is equal to the genus number of $K$.

X. Zhang [28] gave a simple expression of $K_{ge}$ for any abelian extension $K$ of $\mathbb{Q}$ using Hilbert ramification theory. M. Ishida [15] described the narrow genus field $K_{ge}$ of any finite extension of $\mathbb{Q}$. That is, Ishida allowed ramification at the infinite primes. Given a number field $K$, Ishida found two abelian number fields $k_1^*$ and $k_2^*$ such that $k_1^* = k_1^*k_2^*$ and $k_1^* \cap k_2^* = \mathbb{Q}$. The field $k_1^*$ is related to the finite primes $p$ such that at least one prime in $K$ above $p$ is tamely ramified.

We are interested in genus theory for global function fields. There is no direct proper notion of Hilbert class field because, since all the constant field extensions are abelian and unramified, the maximal constant extension is infinite abelian and unramified. On the other extreme, if the class number of a congruence function field $K$ is $h_K$ then there are exactly $h := h_K$ abelian extensions $K_1, \ldots, K_h$ of $K$ such that $K_i/K$ are maximal unramified with exact field of constants of each $K_i$ the same as the one of $K$, $\mathbb{F}_q$, the finite field of $q$ elements and $Gal(K_i/K) \cong Cl_{K,0}$ the group of classes of divisors of degree zero ([2, Chapter 8, page 79]).

There have been different notions of genus fields according to different Hilbert class field definitions. M. Rosen [24] gave a definition of Hilbert class fields of $K$, fixing a nonempty finite set $S_\infty$ of prime divisors of $K$. Using Rosen’s definition of Hilbert class field, it is possible to give a proper concept of genus fields along the lines of number fields.

R. Clement [6] found a narrow genus field of a cyclic extension of $k = \mathbb{F}_q(T)$ of prime degree $l$ dividing $q - 1$. She used the concept of Hilbert class field similar to that of a quadratic number field $K$: it is the finite abelian extension of $K$ such that the prime ideals of the ring of integers $O_K$ of $K$ splitting there are precisely the principal ideals generated by an element whose norm is an $l$-power. S. Bae and J. K. Koo [3] were able to generalize the results of Clement with the methods developed by Fröhlich [9]. They defined the narrow genus field for general global function fields and developed the analogue of the classical genus theory. B. Anglès and J.-F. Jaulent [1] used narrow $S$–class groups to establish the fundamental results, using class field theory, for the genus theory of finite extensions of global fields, where $S$ is a finite set of places.

G. Peng [23] explicitly described the genus theory for Kummer extensions $K$ of $k := \mathbb{F}_q(T)$ of prime degree $l$, based on the global function field analogue of the P. E. Conner and J. Hurrelbrink exact hexagon. C. Wittman [27] extended Peng’s results to the case $l \mid q(q - 1)$ and used his results to study the $l$–part of the ideal class groups of cyclic extensions of prime degree $l$ of $k$. S. Hu and Y. Li [14] described explicitly the genus field of an Artin–Schreier extension of $k$.

In [19, 20] it was developed a theory of genus fields of congruence function fields using Rosen’s definition of Hilbert class field. The methods used there were based on the ideas of Leopoldt using Dirichlet characters and it was given a general description of $K_{ge}$ in terms of Dirichlet characters. The genus field $K_{ge}$ was obtained for an abelian extension $K$ of $k$. The method was used to give $K_{ge}$ explicitly when $K/k$ is a cyclic extension of prime degree $l \mid q - 1$ (Kummer) or $l = p$ where $p$ is the characteristic (Artin–Schreier) and also when $K/k$ is a $p$–cyclic extension (Witt). Later on, the method was used in [5] to describe $K_{ge}$ explicitly when $K/k$ is a cyclic extension of degree $l^n$, where $l$ is a prime number and $l^n \mid q - 1$. 
In this paper we consider a finite abelian extension $K/k$. We find the genus field of $K$ with respect to $k$. Special consideration is given to the genus field of a finite abelian $p$–extension of $k$, where $p$ is the characteristic.

The study of elementary abelian $p$–extensions, and more generally abelian $p$–extensions, has been considered by numerous authors. These extensions appear in several contexts. In [22] O. Ore considered additive polynomials using composition as multiplication. With this operation these polynomials are known as twisted polynomials and this is one of the bases for Drinfeld modules. G. Lachaud [17] obtained an analogue of the Carlitz–Uchiyama bound for geometric BCH codes and some consequences for cyclic codes. His results are part of the analysis of the $L$–function of Artin–Schreier extensions. Garcia and Stichtenoth [11] studied field extensions $L/K$ given by an equation of the type $y^q - y = f(x) \in K(x)$ where $q$ is a power of $p$ and $\mathbb{F}_q \subseteq K$. Using a result of E. Kani [16] they obtained a formula relating the genus of the extension and the genus of the several subextensions of degree $p$. There are many fields of this kind having the maximum number of rational places allowed by Weil’s bound, but they proved that fixed $K$, this number of rational places is asymptotically bad. They also used these extensions to find a family of fields whose Weierstrass gap sequences are nonclassical.

In [4] we considered an additive polynomial $f(X)$ whose roots belong to the base field and we proved results analogous to the ones obtained by Garcia and Stichtenoth. More generally, we studied abelian extensions of type $C_{p^m}$, where $C_j$ denotes a cyclic group of order $j$, and such that the base field contains the finite field $\mathbb{F}_q$, with $q = p^n$. For instance, given an additive polynomial $f(X)$, we have that if the roots of $f$ are in the base field, any elementary abelian $p$–extension can be obtained by means of an equation of the type $f(X) = u$. Furthermore, all the subextensions of degree $p$ over the base field can be deduced from the equation $f(X) = u$.

We have studied genus fields in [19, 20, 21]. The general result we present here goes along the lines of the proof we presented in [19], but it is much simpler since now we consider in just one step the tame and the wild ramification of the infinite prime. In [19] we first studied the case of tame ramification of the infinite primes and next the general case. It turns out that it is possible to consider the general case in just one step and in fact this approach gives the genus field much faster and, in a way, more transparent. Furthermore, in [19] we restricted ourselves to geometric extensions. Here we consider general finite abelian extensions, not necessarily geometric.

We use this approach to study finite abelian $p$–extensions of $k$. Obtaining the genus field of this family of extensions is much more transparent than the way it was obtained in [19]. Our first main result is Theorem 2.2. As a corollary we obtain the general description of the genus field of abelian $p$–extensions in Theorem 2.3.

Our second main result is the description of what we call the conductor of constants of an abelian extension $K/k$. The classical Kronecker–Weber Theorem establishes that every finite abelian extension of $\mathbb{Q}$, the field of rational numbers, is contained in a cyclotomic field. Equivalently, the maximal abelian extension of $\mathbb{Q}$ is the union of all cyclotomic fields. In 1974, D. Hayes [13], proved the analogous result for rational congruence function fields. Hayes proved that the maximal abelian extension of $k$ is the composite of three linearly disjoint fields: the first one is the union of all cyclotomic function fields; the second one is the union of all
constant extensions and the third one is the union of all the subfields of the corresponding cyclotomic function fields, where the infinite prime is totally wildly ramified.

Given a finite abelian extension $K/k$, by the Kronecker–Weber Theorem, using the notations of Section 2, we have $K \subseteq n k(\Lambda_N)_m$ for some $n, m \in \mathbb{N}$ and $N \in R_T$. The minimum $N$ and $n$ can be found by class field theory by means of the conductor related to the finite primes and the infinite prime respectively. However $m$ does not belong to this category. In this paper we define the conductor of constants as the minimum $m$ satisfying this condition and describe $m$ in terms of some other invariants of the extension. This is given in Theorems 3.1 and 3.5.

The third main result is the explicit description of genus fields of finite abelian $p$–extensions of rational function fields in case we have enough constants. This is Theorem 5.1.

To describe the genus fields of finite abelian $p$–extensions of rational function fields without enough constants, we first prove a result on the genus field of a composite of finite abelian extensions of degree relatively prime to the order of the multiplicative group of the field of constants, which shows that the genus field of the composite is the composite of the respective genus fields. The description of the genus field of an arbitrary finite abelian extension of a global rational function field of degree relatively prime to the order of the multiplicative group of the field of constants is the final main result, Theorem 6.8.

2. THE GENUS FIELD

We will use the following notation. Let $k = k_0(T)$ be a global rational function field of characteristic $p$, where $k_0 = \mathbb{F}_q$. Let $R_T = \mathbb{F}_q[T]$ be the polynomial ring. Let $R_T^+$ denote the set of all monic irreducible polynomials in $R_T$. For $N \in R_T$, $k(\Lambda_N)$ denotes the $N$–th Carlitz cyclotomic function field. Let $P_{\infty}$ be the pole of the principal divisor $(T)$ in $k$, which we call the infinite prime. The maximal real subfield $k(\Lambda_N)^+\cap k(\Lambda_N)$ is the decomposition field of the infinite prime. For any field $L$ such that $k \subseteq L \subseteq k(\Lambda_N)$, the real subfield $L^+$ of $L$ is $L^+ := k(\Lambda_N)^+ \cap L$.

The general results on cyclotomic function fields can be consulted in [26, Chapter 12]. Let $K/k$ be a finite abelian extension. From the Kronecker–Weber Theorem, we have that there exist $n, m \in \mathbb{N}$ and $N \in R_T$ such that

$$K \subseteq n k(\Lambda_N)_m := L_n^m k(\Lambda_N)^m \mathbb{F}_q,$$

where $L_n^m$ denotes the subfield of $k(\Lambda_{d^n})$ of degree $q^n$ and $k_m := \mathbb{F}_q(T)$ is the extension of constants of $k$ of degree $m$. We have that $P_{\infty}$ is totally and wildly ramified in $L_n^m/k$. We also have that $P_{\infty}$ is totally inert in $k_m/k$.

For any finite abelian extension $F$ of $k$, $S_{\infty}(F)$ denotes the set of prime divisors of $F$ above $P_{\infty}$. For any finite abelian field extension $E/F$, let $e_{\infty}(E/F)$, $f_{\infty}(E/F)$ and $h_{\infty}(E/F)$ denote the ramification index, the inertia degree and the decomposition number of $S_{\infty}(F)$ in $E$ respectively. For $P \in R_T^+$, $e_P(E/F)$ denotes the ramification index of any prime in $F$ above $P$ in $E/F$. For any extension $F/k$, let $F_{\varphi\varepsilon}$ denote the genus field of $F$ over $k$ as presented in the introduction with $S = S_{\infty}(F)$. When $F/k$ is a finite abelian extension, $F_{\varphi\varepsilon}$ is the maximal abelian extension contained in the Hilbert class field of $F$. The symbol $C_d$ will denote the cyclic group of $d$ elements.
For any field $F$, $W_v(F)$ denotes the ring of Witt vectors of length $v$. The Witt operations will be denoted by $\oplus$ and $\otimes$.

Let $M := L_n k_m$. Then

\begin{equation}
\tag{2.1}
e_\infty(M/k) = q^n, \quad f_\infty(M/k) = m \quad \text{and} \quad h_\infty(M/k) = 1.
\end{equation}

We have $M \cap k(\Lambda_N) = k$. The general results on genus fields needed along this paper, can be found in [19, 20].

First, we present a new proof of the fact that if $K \subseteq k(\Lambda_N)$, then $K_{ge} \subseteq k(\Lambda_N)$.

**Theorem 2.1.** Let $k \subseteq K \subseteq k(\Lambda_N)$ for some $N \in R^+ \cap T$. Then $K_{ge} \subseteq k(\Lambda_N)$. Furthermore, if the group of Dirichlet characters of $K$ is $X$ and if $L$ is the field associated to $Y = \prod_{P \in R^+ \cap T} X_P$, then

$$K_{ge} = KL^+.$$

**Proof.** Let $F/K$ be an unramified abelian extension so that the elements of $S_\infty(K)$ are fully decomposed in $F/K$. In particular $P_\infty$ is tamely ramified.

By the Kronecker–Weber theorem, we have $F \subseteq K(\Lambda_M)_m$ for some $M \in R^+ \cap T$, $m \in \mathbb{N}$.

Let $I$ be the inertia group of $S_\infty(K)$ in $k(\Lambda_M)/k$ and let $B = k(\Lambda_M)^I$.

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (k) at (0,0) {$k$};
  \node (kM) at (2,0) {$k(\Lambda_M)_m$};
  \node (F) at (1,1) {$F$};
  \node (B) at (1,-1) {$B$};
  \node (I) at (0,1) {$I$};
  \node (Bm) at (1,-2) {$B_m$};
  \node (K) at (1,-2) {$K_m$};
  \node (km) at (2,-2) {$k_m$};
  \draw (k) -- (kM);
  \draw (k) -- (B);
  \draw (k) -- (I);
  \draw (kM) -- (F);
  \draw (B) -- (F);
  \draw (I) -- (F);
  \draw (B) -- (I);
  \draw (B) -- (Bm);
  \draw (K) -- (F);
  \draw (K) -- (I);
  \draw (K) -- (km);
\end{tikzpicture}
\end{figure}

Since the elements of $S_\infty(B)$ are of degree 1, they are fully inert in $B_m/B$. Furthermore, the elements of $S_\infty(B)$ are fully ramified in $k(\Lambda_M)/B$. Now, the elements of $S_\infty(K)$ are fully decomposed in $B/K$ so we obtain that $B$ is the decomposition field of $S_\infty(K)$ in $k(\Lambda_M)_m/K$. It follows that $F \subseteq B \subseteq k(\Lambda_M)$.

Let $Z$ be the group of Dirichlet characters associated to $F$. Since $F/K$ is unramified, it follows that $X \subseteq Z \subseteq Y$, that is, $F \subseteq L$ since $L$ is the maximal abelian extension contained in some cyclotomic function field such that $L/K$ is unramified in the finite primes. In particular, we may take $M = N$. Therefore $K_{ge} = L^D$ where $D$ is the decomposition group of $S_\infty(K)$ in $L/K$.

Now, $S_\infty(K)$ decompose fully in $KL^+/K$ since $P_\infty$ decomposes fully in $L^+/k$. Since $L/K$ is unramified, we have $KL^+ \subseteq L$ so that $KL^+/K$ is unramified. Hence $KL^+ \subseteq K_{ge}$ and we obtain that $KL^+ \subseteq K_{ge} \subseteq L$.

Finally, let us see that $S_\infty(KL^+)$ is fully ramified in the extension $L/KL^+$. In fact this follows from the fact that $L^+ \subseteq KL^+ \subseteq L$ and from that $S_\infty(L^+)$ is totally
ramified in $L/L^+$. Since $KL^+ \subseteq K_{ge} \subseteq L$ and $K_{ge}/KL^+$ is unramified, it follows that $K_{ge} = KL^+ \subseteq k(\Lambda_N)$. □

Our first main result is

**Theorem 2.2.** With the above notations, let $K/k$ be a finite abelian extension. Let

$$E := KM \cap k(\Lambda_N).$$

Then

$$K_{ge} = E^{H_1}_{ge} K = (E_{ge} K)^H,$$

where $H$ is the decomposition group of any prime in $S_{\infty}(K)$ in $E_{ge} K/K$, $H_1 := H|_{E_{ge}}$ and $H_2 := H_1|_{E}$. 

Let $d := f_{\infty}(EK/K)$. We have $H \cong H_1 \cong H_2 \cong C_d$ and $d|q - 1$. We also have $E_{ge} K/K_{ge}$ and $EK/E_{ge} K$ are extensions of constants of degree $d$. Finally, the field of constants of $K_{ge}$ is $F_{q^t}$, where $t$ is the degree of $S_{\infty}(K)$ in $K$.

**Proof.** The proof that the field of constants of $K_{ge}$ is $F_{q^t}$ is the same as the one in [19, Lemma 4.1]. We repeat the argument for the sake of completeness. Let $k_r$ be the extension of constants of $K$ of degree $r$. Since the degree of any element of $S_{\infty}(K)$ is $t$, the elements of $S_{\infty}(K)$ decompose into $\gcd(t, r)$ elements of $K_r$. Therefore the elements of $S_{\infty}(K)$ decompose fully if and only if $\gcd(t, r) = r$ if and only if $r|t$. The assertion follows.

Since $k(\Lambda_N) \cap M = k$ and $E = KM \cap k(\Lambda_N)$, from the Galois correspondence, between $k(\Lambda_N)/k$ and $k(\Lambda_N)M/M$, $E$ corresponds to $KM$. Hence $KM = EM$ corresponds to $E$. Thus

$$KM = EM.$$

Now $E \cap K \subseteq E_{ge} \cap K \subseteq k(\Lambda_N) \cap K = (KM \cap k(\Lambda_N)) \cap k(\Lambda_N) \cap K = E \cap k(\Lambda_N) \cap K = E \cap K$. Therefore

$$E \cap K = E_{ge} \cap K = k(\Lambda_N) \cap K.$$

We have $[E : k] = [EM : M] = [KM : M] = [K : K \cap M]$. Thus

$$[K : k] = [E : k][K \cap M : k].$$

Next, we will prove that $EK/K$ is unramified. First note that $E \subseteq EK \subseteq EKM = E \cdot EM = EM$. In the extension $M/k$, $P_{\infty}$ is the only ramified prime.
Hence in $KM/E$ the only possible ramified primes are those in $S_\infty(E)$. We also have that in the extension $KM/K$ the only possible ramified primes are the elements of $S_\infty(K)$ and since $K \subseteq EK \subseteq EM = KM$, the only possible ramified primes in $EK/K$ are those in $S_\infty(K)$.

From (2.1) we have

$$e_\infty(EK/K) | e_\infty(M/K \cap M)$$

and

$$e_\infty(M/K \cap M) | e_\infty(M/k) = q^n.$$

On the other hand, we have

$$e_\infty(EK/K) | e_\infty(E/E \cap K)$$

and

$$e_\infty(E/E \cap K) | e_\infty(k(\Lambda_N)/k) = q - 1.$$

Thus

$$e_\infty(EK/K) | \gcd(q^n, q - 1) = 1$$

and $EK/K$ is unramified.

Now, we have that

$$e_\infty(EK/K) f_\infty(EK/K) | e_\infty(E/E \cap K) f_\infty(E/E \cap K),$$

and

$$e_\infty(EK/K) = 1, f_\infty(E/E \cap K) = 1.$$ Therefore

$$f_\infty(EK/K) | e_\infty(E/E \cap K) \quad \text{and} \quad e_\infty(E/E \cap K) | q - 1.$$ Thus

$$f_\infty(EK/K) | q - 1.$$

Therefore we have that $EK/K$ is unramified, the inertia degree of $S_\infty(K)$ in $EK/K$ is $d = f_\infty(EK/K)$ and $d | q - 1$. Since $EG/E$ is unramified and $S_\infty(E)$ decomposes fully in $EG/E$, the same holds in $EG/K/EK$. In this way we obtain that $EG/K$ is an unramified extension and the inertia degree of $S_\infty(K)$ is $d$.

Recall that $H$ is the decomposition group of any prime in $S_\infty(K)$ in $EG/K$ and let $H_1 := H|_{EG}$. Observe that $|H| = d$. Since $EG \cap K = E \cap K$, from the Galois correspondence we obtain that $H \cong H_1, |H| = |H_1|$, and $EG^{H_1}K = (EG/K)^H$. Analogously, $H_2 \cong H_1$. Furthermore, $H_1 \subseteq I_\infty(k(\Lambda_N)/k) \cong C_{q-1}$, where $I_\infty$ denotes the inertia group of $P_\infty$. Therefore $H$ is a cyclic group, $H \cong H_1 \cong H_2 \cong C_d$.

Since $S_\infty(K)$ decomposes fully in $EG^{H_1}K/K$, it follows that

$$EG^{H_1}K \subseteq K_{gr}.$$

Let $E_1 := EE^{H_1}_{gr} \subseteq EG$. Now $H_1 \subseteq I_\infty(E/E \cap K)$, so $S_\infty(E^{H_1}_{gr})$ is fully ramified in $EG/E^{H_1}_{gr}$. Therefore $S_\infty(E_1)$ is fully ramified in $EG/E_1$. On the other hand $S_\infty(E)$ decomposes fully in $EG/E$. Hence $S_\infty(E_1)$ decomposes fully in $EG/E_1$. That is, $S_\infty(E_1)$ ramifies and decomposes fully in $EG/E_1$. Therefore

$$EG = E_1 = EE^{H_1}_{gr}.$$
It follows that

$$(E_{gt}K)^H = E_{gt}^{H_1}K \subseteq K_{gt} \quad \text{and} \quad EE_{gt}^{H_1} = E_{gt}.$$  

To prove the other containment, we define $C := K_{gt}M \cap k(\Lambda_N)$. We have

$$E \subseteq EM = KM \subseteq K_{gt}M, \quad E \subseteq k(\Lambda_N).$$

Therefore

$$E \subseteq K_{gt}M \cap k(\Lambda_N) = C, \quad \text{that is} \quad E \subseteq C.$$  

Furthermore, $E_{gt}^{H_1} \subseteq E_{gt}^{H_1}K \subseteq K_{gt} \subseteq K_{gt}M$ and $E_{gt}^{H_1} \subseteq E_{gt} \subseteq k(\Lambda_N)$. Thus $E_{gt}^{H_1} \subseteq K_{gt}M \cap k(\Lambda_N) = C$. Hence $E_{gt}^{H_1} \subseteq C$. Therefore

$$E_{gt} = EE_{gt}^{H_1} \subseteq C.$$  

(2.3)

\[\begin{array}{c}
k(\Lambda_N) \\
\downarrow \\
C \\
\downarrow \\
E_{gt}K \\
\downarrow \\
E_{gt}M \\
\downarrow \\
EM, KM \\
\downarrow \\
M \\
\downarrow \\
k = k(\Lambda_N) \cap M
\end{array}\]

Since $C = K_{gt}M \cap k(\Lambda_N)$, from the Galois correspondence we have $CM = K_{gt}M$. Now, since $K_{gt}/K$ is unramified and $S_\infty(K)$ decomposes fully, it follows that

$$CM/KM \quad \text{is unramified and} \quad S_\infty(KM) \quad \text{decomposes fully.}$$  

(2.4)

We now prove that $C/E$ is unramified. From (2.4) follows that $CM/KM$ is unramified. Now, in $KM = EM$ over $E$, the only ramified primes are those in $S_\infty(E)$ and they have ramification index equal to $q^n$. It follows that the only ramified primes in $CM/E$ are those in $S_\infty(E)$. Hence the only possible ramified primes in $C/E$ are those in $S_\infty(E)$. Now

$$e_\infty(C/E) \mid e_\infty(CM/E) = q^n \quad \text{and} \quad e_\infty(C/E) \mid e_\infty(k(\Lambda_N)/k) = q - 1.$$
so that

\[ e_∞(C/E) \mid \gcd(q^n, q - 1) = 1. \]

Therefore \( C/E \) is an unramified extension.

On the other hand, being \( S_∞(E) \) unramified in \( C/E \), \( S_∞(E) \) decomposes fully in \( C/E \) since \( C \subseteq k(\Lambda_N) \). It follows that \( C \subseteq E_{gc} \). From this and equation (2.3), we obtain

\[ C = E_{gc} \quad \text{and} \quad E_{gc}M = CM = K_{gc}M. \]

We have \( E_{gc}K \subseteq E_{gc}K_{gc} \). Since \( K_{gc}/K \) is unramified and \( S_∞(K) \) decomposes fully in \( K_{gc} \), the same holds in the extension \( E_{gc}K_{gc}/E_{gc}K \). In particular \( h_∞(E_{gc}K_{gc}/E_{gc}K) = [E_{gc}K_{gc} : E_{gc}K] \).

Now, in the extension \( E_{gc}M/E_{gc} \), the only ramified primes are those in \( S_∞(E_{gc}) \) and we have \( e_∞(E_{gc}M/E_{gc}) = q^n \) and \( f_∞(E_{gc}M/E_{gc}) = m \) because \( e_∞(E_{gc}/k) \mid q-1 \) which is relatively prime to \( q \), \( f_∞(E_{gc}/k) = 1 \), \( e_∞(M/k) = q^n \) and \( f_∞(M/k) = m \).

Let \( F_1 \) and \( F_2 \) two fields such that \( k \subseteq F_1 \subseteq F_2 \subseteq M \). Let \( R_i = E_{gc}F_i, \) \( i = 1, 2 \). Since \( f_∞(E_{gc}/k) = 1 \) and \( e_∞(E_{gc}/k) \mid q - 1 \), it follows from the Galois correspondence between \( M/k \) and \( E_{gc}M/E_{gc} \) that \( e_∞(R_i/E_{gc}) = e_∞(F_i/k) \) and that \( f_∞(R_i/E_{gc}) = f_∞(F_i/k) \), \( i = 1, 2 \). Therefore \( e_∞(F_2/F_1) = e_∞(R_2/R_1) \) and \( f_∞(F_2/F_1) = f_∞(R_2/R_1) \).

Since \( h_∞(M/k) = 1 \), we have \( h_∞(R_2/R_1) = 1 \). In particular

\[ R_1 \neq R_2 \iff F_1 \neq F_2 \iff e_∞(F_2/F_1) > 1 \text{ or } f_∞(F_2/F_1) > 1 \]

(2.5)

Since

\[ E_{gc} \subseteq E_{gc}K \subseteq E_{gc}K_{gc} \subseteq K_{gc}M = E_{gc}M, \]

\( S_∞(E_{gc}K) \) is unramified in \( E_{gc}K_{gc}/E_{gc}K \) and \( S_∞(E_{gc}K) \) decomposes fully, we obtain that \( e_∞(E_{gc}K_{gc}/E_{gc}K) = 1 \) and \( f_∞(E_{gc}K_{gc}/E_{gc}K) = 1 \). From (2.5), it follows that

\[ E_{gc}K_{gc} = E_{gc}K. \]

Therefore \( K_{gc} \subseteq E_{gc}K_{gc} = E_{gc}K \). Since \( E_{gc}K/K \) is unramified, if \( D \) is the decomposition group of \( S_∞(K) \) in \( E_{gc}K/K \), we obtain that \( K_{gc} = (E_{gc}K)^D \). Finally, we have

\[ f_∞(E_{gc}K/K) = f_∞(E_{gc}K/EK)f_∞(EK/K) = 1 \cdot d = d. \]

Hence \( D = H \) and \( K_{gc} = (E_{gc}K)^D = (E_{gc}K)^H = E_{H^2}K \).

Finally, it remains to show that \( E_{gc}K/K_{gc} \) and \( EK/EH^2K \) are extensions of constants.

Since \( K_{gc}M = E_{gc}M \) and \( E_{gc}K_{gc} = E_{gc}K \), we have

\[ K_{gc} = (E_{gc}K)^H \subseteq E_{gc}K \subseteq E_{gc}K_{gc} \subseteq E_{gc}K_{gc}M = E_{gc}M. \]
Set $F_1 = K_{gt} \cap M$ and $F_2 = E_{gt}K \cap M$. We have $d = [E_{gt}K : K_{gt}] = f_{\infty}(E_{gt}K/K_{gt}) = |F_2 : F_1| = e_{\infty}(F_2/F_1)f_{\infty}(F_2/F_1)h_{\infty}(F_2/F_1)$. Since $e_{\infty}(F_2/F_1) | q^n$ and $h_{\infty}(F_2/F_1) = 1$, it follows that

$$e_{\infty}(F_2/F_1) = e_{\infty}(E_{gt}K/K_{gt}) = 1 \quad \text{and} \quad f_{\infty}(F_2/F_1) = f_{\infty}(E_{gt}K/K_{gt}) = d.$$ 

Therefore $k \subseteq F_1 \subseteq F_2 \subseteq M$ and $e_{\infty}(F_2/F_1) = 1$.

Let $a$ and $b$ be such that $F_2 \subseteq F_1\kappa b L_a$. Let $A_1 = F_1 k_b \cap L_a$, $i = 1, 2$. Note that because $e_{\infty}(F_2/F_1) = 1$ and $F_1 k_b = A_i k_b$, $i = 1, 2$, are extensions of constants, we have $e_{\infty}(A_2/A_1) = 1$.

Since $L_a/k$ is totally ramified at $P_{\infty}$, it follows that $A_1 = A_2$. Therefore $F_2 k_b = F_1 k_b$ and $F_2/F_1$ is an extension of constants.

Recall $F_1 = K_{gt} \cap M$. We consider $K_{gt} \subseteq E_{gt}K \subseteq K_{gt}M = E_{gt}M$:

$$
\begin{array}{cccc}
K_{gt} & E_{gt}K & K&M = E_{gt}M \\
\downarrow & \downarrow & \downarrow & \downarrow \\
F_1 & F_2 & M & \\
\end{array}
$$

Therefore $K_{gt} \subseteq F_2 K_{gt} = E_{gt}K$. It follows that $E_{gt}K/K_{gt}$ is an extension of constants of degree $[E_{gt}K : K_{gt}] = |H| = d$.

The proof that $E/K$ is an extension of constants is completely similar. This finishes the proof of the theorem.

For the particular case of a finite abelian $p$–extension, we have that, on the one hand, $d \mid q - 1$ and, on the other hand, $d \mid [E : K]$. Since $K/k$ is a $p$–extension, we obtain from (2.2), that $E/k$ is also a $p$–extension. Finally, since $\text{Gal}(E/k) \rightarrow \text{Gal}(E/k) \times \text{Gal}(K/k)$, $\sigma \mapsto (\sigma|_E, \sigma|_K)$ is injective, it follows that $E/K$ is also a $p$–extension. Therefore $d \mid p^\alpha$ for some $\alpha$. Thus $d = 1$. We have proved

**Theorem 2.3.** With the above notations, let $K/k$ be a finite abelian $p$–extension. Let

$$E := KM \cap k(\Lambda_N).$$
Then $K_{ge} = E_{ge}K$ and $K_{ge}/k$ is an abelian $p$-extension.

Proof. The last assertion follows from the fact that $E_{ge}/k$ is also an abelian $p$-extension. \hfill $\square$

### 3. Conductor of Constants

Let $K$ be a finite abelian extension of $k$. By the Kronecker–Weber we have that there exist $n, m \in \mathbb{N}$ and $N \in R_T$ such that $K \subseteq n_k(\Lambda_N)_m$. The minima $n$ and $N$ satisfying this condition are given by class field theory by means of the local conductors of the extension $K/k$: $n$ for $\mathcal{P}_\infty$ and $N$ for the finite primes.

In this section we will determine the minimum $m$ satisfying the above condition and we will see that this $m$ is related to the number $d$ given in Theorem 2.2. The number $m$ will be called the conductor of constants of the abelian extension $K/k$.

First, let $n, m \in \mathbb{N}$ and $N \in R_T$ be such that $K \subseteq n_k(\Lambda_N)_m$ and where $m$ is the minimum with respect to this condition. Note that $m$ might depend on $n$ and $N$. Consider the following diagram of Galois extensions:

$$
\begin{array}{c}
\xymatrix{n_k(\Lambda_N) \ar@{-}[rr]^n \ar@{-}[rrd] \ar@{-}[rrdd] & & n_k(\Lambda_N)_m \\
& U = n_k(\Lambda_N)K & \\
k \ar@{-}[rr]^m & & k \ar@{-}[rr] & & k \ar@{-}[rr]_{k} & & k_m' \\
& K & \\
\end{array}
$$

That is, let $U := n_k(\Lambda_N)K$ and $k_{m'} := U \cap k_m$. From the Galois correspondence, we have that $U = n_k(\Lambda_N)K = n_k(\Lambda_N)k_{m'} = n_k(\Lambda_N)_m \supseteq K$.

Since $m$ is minimal, we obtain that $m' = m$. That is, $m$ is determined by the equality

$$
(3.1) \quad n_k(\Lambda_N)K = n_k(\Lambda_N)_m.
$$

Now, we will see that $m$ is independent of $n$ and of $N$. Let $n_i \in \mathbb{N}$, $N_i \in R_T$ and $m_i \in \mathbb{N}$ be the minimum such that $K \subseteq n_i k(\Lambda_{N_i})_{m_i}, i = 1, 2$.

Let $n_0 := \max\{n_1, n_2\}$, $N_0 = \text{lcm}[N_1, N_2]$ and $m_0 \in \mathbb{N}$ be minimum such that $K \subseteq n_0 k(\Lambda_{N_0})_{m_0}$. From (3.1), it follows that

$$
\begin{align*}
n_0 k(\Lambda_{N_0}) & = L_{m_0} \left( n, k(\Lambda_{N_i})k(\Lambda_{N_0}) \right) K = L_{m_0} \left( n, k(\Lambda_{N_i})K \right) k(\Lambda_{N_0}) \\
& = L_{m_0} \left( n, k(\Lambda_{N_i})_{m_i}k(\Lambda_{N_0}) \right) = n_0 k(\Lambda_{N_0})_{m_i}, \quad \text{and}
\end{align*}
$$

Therefore $m_1 = m_2 = m_0$.

So, we consider $K \subseteq n_k(\Lambda_N)_m$ with $m$ the minimum. Let $F := K \cap n_k(\Lambda_N)$ and consider the following Galois square (see (3.1))

$$
\begin{array}{c}
\xymatrix{n_k(\Lambda_N) \ar@{-}[rr]^m \ar@{-}[rrd] \ar@{-}[rrdd] & & n_k(\Lambda_N)_m = n_k(\Lambda_N)K \\
& F & \\
k \ar@{-}[rr]^m & & K \ar@{-}[rr]_{k} & & k \ar@{-}[rr]_{k} & & k_m' \\
& K & \\
\end{array}
$$
Let $t$ be the degree of $S_\infty(K)$ in $K$. That is, $t = f_\infty(K/k)$. We have
$$e_\infty(nk(\Lambda_N)) = 1,$$
$$f_\infty(nk(\Lambda_N)) = m.$$  
In particular
$$\{1\} = I_\infty(nk(\Lambda_N)) \subseteq I_\infty(K/F),$$
$$C_m \equiv D_\infty(nk(\Lambda_N)) \subseteq D_\infty(K/F).$$

Since $|K : F| = m$ and $m \leq |D_\infty(K/F)| \leq [K : F] = m$, it follows that
$$m = \min K/F \subseteq \max(K/F) \equiv C_m.$$ In particular we have $h_\infty(K/F) = 1$
and $h_\infty(nk(\Lambda_N)) = 1$.

On the other hand, we have
$$t = f_\infty(K/k) = f_\infty(K/F)f_\infty(F/k) = f_\infty(K/F) \cdot 1 = f_\infty(K/F),$$
that is, $f_\infty(K/F) = t$. Furthermore
$$e_\infty(K/F)f_\infty(K/F)h_\infty(K/F) = e_\infty(K/F) \cdot t \cdot 1 = m,$$
so that $e_\infty(K/F) = \frac{m}{t}$. Hence
$$(3.2) \quad m = [K : F] = f_\infty(K/F)e_\infty(K/F) = t e_\infty(K/F) = \frac{e_\infty(K/k)}{e_\infty(F/k)}.$$  

Now we shall investigate the relation between $m$ and $d = f_\infty(E_{ge}K/K_{ge})$ given
in Theorem 2.2. Recall that $M = L_nk_m$, $E = KM \cap k(\Lambda_N)$ and that $EM = KM$. We have
$$E_{ge} \subseteq E_{ge}K \subseteq E_{ge}KL_n \subseteq E_{ge}KM = E_{ge}EM = E_{ge}M.$$  

Let $A := E_{ge}K \cap M$ and $B := E_{ge}KL_n \cap M$. From the Galois correspondence
we have $E_{ge}K = E_{ge}A$ and $E_{ge}KL_n = E_{ge}B$.

We have $L_n \subseteq E_{ge}KL_n \cap M = B \subseteq M = L_nk_m$. Therefore $B/L_n$ is an extension
of constants. Say $B = L_nk_{m'}$ with $m'/m$. From the Galois correspondence, we obtain
$$K \subseteq E_{ge}KL_n = E_{ge}B = E_{ge}L_nk_{m'} \subseteq k(\Lambda_N)L_nk_{m'} = nk(\Lambda_N)_{m'}.$$  

Since $m$ is the minimum, $m' = m$, $B = M$ and $E_{ge}KL_n = E_{ge}M$. 

Now, $E_{ge}(AL_n) = (E_{ge}A)L_n = (E_{ge}K)L_n = E_{ge}M$. From the Galois correspondence
it follows that $AL_n = M$. We consider the following Galois square:

We have $f_\infty(AL_n/L_n) = f_\infty(M/L_n) = m$ and $e_\infty(AL_n/L_n) = e_\infty(M/L_n) = 1.$
Thus
$$\{1\} = I_\infty(AL_n/L_n) \subseteq I_\infty(A/A \cap L_n) \quad \text{and}$$
$$C_m \equiv D_\infty(AL_n/L_n) \subseteq D_\infty(A/A \cap L_n).$$
Because \([A : A \cap L_n] = [M : L_n] = m\), it follows that \(D_\infty(A/A \cap L_n) \cong C_m\). \(e_\infty(A/A \cap L_n) = 1\) and \(f_\infty(A/A \cap L_n) = m\). Therefore \(f_\infty(E_{gt} K/k) = f_\infty(E_{gt} K/K_{gt}) f_\infty(K_{gt}/K) f_\infty(K/k) = d \cdot t = dt = td\). Thus
\[
f_\infty(E_{gt} M/E_{gt} K) = \frac{f_\infty(E_{gt} M/k)}{f_\infty(E_{gt} K/k)} = \frac{m}{td}.
\]
Finally
\[
m = \frac{m}{td} = f_\infty(E_{gt} M/E_{gt} K) [E_{gt} M : E_{gt} K] = [M : A] = [L_n : A \cap L_n][L_n : k] = q^n.
\]
It follows that
\[
m = td p^s
\]
for some \(s \in \mathbb{N} \cup \{0\}\).
Furthermore, \(f_\infty(K_m/K) = \frac{m}{t} = e_\infty(K/F)\). Note that
\[
td = f_\infty(K/k) f_\infty(EK/K) = f_\infty(EK/k).
\]
We have obtained

**Theorem 3.1 (Conductor of constants 1).** Let \(K\) be a finite abelian extension of \(k\). Let \(n, m \in \mathbb{N}\) and \(N \in \mathcal{R}_F\) be such that \(K \subseteq n k(\Lambda_N)\) and such that \(m\) is minimum with this property. Then \(m\) is independent of \(n\) and \(N\). Let \(t = f_\infty(K/k)\) be the degree of the infinite primes of \(K\). Let \(M = L_n k_m, E = K M \cap k(\Lambda_N), F = K \cap n k(\Lambda_N)\) and \(d = f_\infty(EK/K) = f_\infty(E_{gt} K/K_{gt})\). Then
\[
n k(\Lambda_N) K = n k(\Lambda_N) m
\]
and
\[
m = [K : F] = te_\infty(K/F) = td p^s = f_\infty(EK/k)p^s
\]
for some \(s \geq 0\). In particular
\[
e_\infty(K/F) = dp^s = f_\infty(K_m/K).
\]

**Remark 3.2.** When \(p \nmid \frac{m}{t}\), in particular when \(K/k\) is tamely ramified at \(P_\infty\), we have \(s = 0\) and \(m = td\). In the general case, we may have \(s \geq 1\).

**Example 3.3.** Let \(p\) be any prime and let \(q = p\). Let \(X := 1/T\). We have \(L_1 := k(\Lambda_{X, 2})\) and \([L_1 : k] = p\). We have that \(L_1/k\) is an Artin–Schreier extension. It is not necessary to give the explicit description of \(L_1\), however for the convenience of the reader we give a generator of \(L_1\). Let \(\lambda\) be a generator of \(\Lambda_{X, 2}\) such that \(\lambda^{p-1}\) is a generator of \(k(\Lambda_{X, 2})^+ = L_1\). Now \(\lambda\) is a root of the cyclotomic polynomial \(\Psi_{X, 2}(u)\). We have that \(\Psi_{X, 2}(u) = \Psi_{X}(u^X)\) where \(u^X\) denotes the Carlitz action. Since \(\Psi_{X}(u) = u^p/u = u^{p^2 - 1} + X\), it follows that \(\Psi_{X, 2}(\lambda) = (\lambda^p + X\lambda)^{p-1} + X\). Set \(\mu := \lambda^{p-1}\) and \(\xi := \mu + X\). Then we obtain
\[
\xi^p - X\xi^{p-1} + X = 0.
\]
Finally, if \(\delta := 1/\xi\), then \(L_1 = k(\delta)\) with
\[
\delta^p - \delta = -1/X = -T, \quad \delta = \frac{T}{T\lambda^{p-1} + 1}.
\]
Let \( \alpha \) be a solution of \( y^p - y = 1 \). Then \( \mathbb{F}_p(\alpha) = \mathbb{F}_{p^p}, k_p = \mathbb{F}_p(\alpha)(T) = \mathbb{F}_{p^p}(T) \) and \( L_1k_p = k(\alpha, \delta) \). The \( p+1 \) extensions \( K/k \) of degree \( p \) over \( k \) such that \( k \subseteq K \subseteq L_1k_p \) are \( \{ k(\alpha + i\delta) \}_{i=0}^{p-1} \) and \( L_1 \). Set \( K := k(\alpha + \delta) \). Then \( K \neq k_p \) and \( K \neq L_1 \). Then \( K = k(z) \) with \( z^p - z = 1 - T \).

Let \( N \in R \) be arbitrary. Then \( K \subseteq L_1k_p \subseteq k(\Lambda_N) \) and \( K \subseteq k(\Lambda_N)_1 \). Therefore \( m = p \) and \( M = L_1k_p \). We have \( f_{\infty}(K/k) = 1, e_{\infty}(K/k) = p \). We also have \( E := KM \cap k(\Lambda_N) = M \cap k(\Lambda_N) = k \). Therefore \( E_{ge} = k \) and \( K_{ge} = E_{ge}K = K \). It follows that \( KE = K \) and \( f_{\infty}(KE/K) = d = 1 \). Hence \( td = 1 \neq m = p \). In this example \( s = 1 \).

We will compute \( m \) in another way. First, with the same proof as the one for Theorem 2.2 we obtain

**Theorem 3.4.** Let \( K/k \) be a finite abelian extension. Let

\[ R := K_m \cap n k(\Lambda_N) \].

Then

\[ K_{ge} = R_{ge}^{H_1} K = (R_{ge} K)^{H_2} \]

where \( H_1 \) is the decomposition group of any prime in \( S_{\infty}(K) \) in \( R_{ge} K/K \), \( H_2 := H_{R_{ge}} \), and \( K_{ge} = E_{ge} K = K \).

Let \( d^* := f_{\infty}(RK/K) \). We have \( H_1 \cong H_1' \cong H_2 \cong C_{d^*} \) and \( d^* | q - 1 \). We also have \( R_{ge} K/K_{ge} \) and \( RK/R^{H_2} \) are extensions of constants of degree \( d^* \). Finally, the field of constants of \( K_{ge} \) is \( \mathbb{F}_q^t \), where \( t \) is the degree of \( S_{\infty}(K) \) in \( K \).

---

Let now \( F = K \cap n k(\Lambda_N) \) and consider the following Galois squares

```
  n k(\Lambda_N) -------- n k(\Lambda_N)_m
  |                  |
  R                  K_m = R_m
  |  \               |
  k                  K
  |                  |
  k_m

  n k(\Lambda_N) -------- n k(\Lambda_N) K = n k(\Lambda_N)_m
  |                  |
  C                  R_m = K_m
  |                  |
  R = K_m \cap n k(\Lambda_N) -------- RK
  |                  |
  F = K \cap n k(\Lambda_N) -------- K
```
In particular, where $E$ is a power of $F$. Let $R = K_m \cap_{n} k(\Lambda_N)$, then $C = R$ and, from the Galois correspondence, we have $RK = R_m = K_m$.

It follows that the field of constants of $RK$ is $\mathbb{F}_q^m$. The field of constants of $RK_{ge}$ is also $\mathbb{F}_q^m$.

Now, the field of constants of $K_{ge}$ is $\mathbb{F}_q$. On the other hand we have that $RK_{ge}/R^H_{ge}K = K_{ge}$ is an extension of constants of degree $d^* = |H_1|$. Thus, the field of constants of $RK_{ge}$ is $\mathbb{F}_q^{d^*}$. It follows that $td^* = m$.

We have obtained

**Theorem 3.5** (Conductor of constants 2). Let $K$ be a finite abelian extension of $k$. Let $n, m \in \mathbb{N}$ and $N \in R_T$ be such that $K \subseteq n k(\Lambda_N)_m$ and such that $m$ is minimum with this property. Let $t = f_{\infty}(K/k) = f_{\infty}(K/F)$ be the degree of the infinite primes of $K$.

Let $R = K_m \cap_{n} k(\Lambda_N)$ and $d^* = f_{\infty}(RK/K)$. Then

$$m = te_{\infty}(K/F) = td^* = f_{\infty}(RK/k).$$

In particular

$$d^* = f_{\infty}(RK/K) = e_{\infty}(K/F).$$

□

**Remark 3.6.** From Theorems 2.2 and 3.4 follows that if $K \subseteq n k(\Lambda_N)_m$, then $K_{ge} \subseteq n k(\Lambda_N)_m$. In particular the conductors of constants of $K$ and of $K_{ge}$ are the same.

4. Genus fields of subfields of cyclotomic function fields

For an abelian extension $K/k$, the description of $K_{ge}$ depends on the description of $E_{ge}$ (Theorem 2.2). In this section we present some details in order to find $E_{ge}$. For the results and notation on Dirichlet characters we use, we refer to [26, Chapter 12]. Here $K$ denotes a field $k \subseteq K \subseteq k(\Lambda_N)$ for some $N \in R_T$ and $k = \mathbb{F}_q(T)$.

**Remark 4.1.** Let $k \subseteq K \subseteq k(\Lambda_N)$ and let $X$ be the group of Dirichlet characters associated to $K$. If $L$ is the field associated to $\prod_{p \in R_T^+} X_P$, then

$$K_{ge} = L^D,$$

where $D$ is the decomposition group of any prime $p \in S_\infty(K)$ in $L/K$.

**Proposition 4.2.** With the notation as above, let $X$ be the group of Dirichlet characters corresponding to $K$. Fix $P \in R_T^+$. Let $Y$ be a group of Dirichlet characters such that $Y = Y_P$, that is, for any $\chi \in Y$, the conductor of $\chi$ is a power of $P$: $\mathcal{F}_\chi = P^{\alpha_\chi}$ for some $\alpha_\chi \in \mathbb{N} \cup \{0\}$. Let $L$ be the field associated to $\langle X, Y \rangle$, that is, if $F$ is the field associated to $Y$, then $L = KF$. If $KF/K$ is unramified at $P$, then $Y \subseteq X_P$.

**Proof.** We have $|\langle X, Y \rangle_P| = e_P(KF/k) = e_P(KF/K)e_P(K/k) = e_P(K/k) = |X_P|$. Since $X_P \subseteq \langle X, Y \rangle_P$, it follows that $X_P = \langle X, Y \rangle_P$. Since $Y_P \subseteq \langle X, Y \rangle_P$, the result follows. □

**Corollary 4.3.** If $|Y| = |X_P|$, then $Y = X_P$. □

We apply Proposition 4.2 to Kummer extensions of $k$ and to finite abelian $p$–extensions of $k$. 
4.1. Kummer extensions. Let $K = k(\sqrt[γ]{D})$ be a Kummer extension with $K \subseteq k(\Lambda_D)$, that is, $t|q - 1$, $D \in R_T$ is a monic polynomial, $D$ is $t$–power free and $γ = (-1)^{deg D}$. Say $D = P_1^{α_1} \cdots P_r^{α_r}$, $r ≥ 1$, $1 ≤ α_i ≤ t-1$, $1 ≤ i ≤ r$, as a product of powers of monic irreducible polynomials. Set $d_i := gcd(α_i, t)$. Then $gcd(\frac{α_i}{d_i}, \frac{1}{d_i}) = 1$. Let $p_i$ be a prime in $K$ above $P_i \in R_T$. Set $β := \sqrt[γ]{D}$ so that $β^t = γD = γ^p_1 \cdots P_r$. We have that $e_i := e_{P_i}(K/k) = t/d_i$ (see [21, Subsection 5.2]).

Let $F_i = k(\sqrt[γ]{D}, \sqrt[γ]{P_i^{α_i/d_i}})$. Set $γ_i = (-1)^{deg P_i^{α_i/d_i}}$. Let $X$ be the group of Dirichlet characters associated to $K$. In fact $X$ is a cyclic group of order $t$ and let $X = (χ)$. Let $Y$ be the group of Dirichlet characters associated to $F_i$. Then $Y = Y_{p_i}$ and $|Y_{p_i}| = e_{p_i}(F_i/k) = t/d_i$ since $gcd(t/d_i, α_i/d_i) = 1$, and $|X_{p_i}| = e_{p_i}(K/k) = t/d_i = |Y_{p_i}|$.

We will see that $KF_i/K$ is unramified at $p_i$. We have

$$KF_i = k(\sqrt[γ]{D}, \sqrt[γ]{P_i^{α_i/d_i}}) = k(\sqrt[γ]{D}, \sqrt[γ]{\sqrt[γ]{P_i^{α_i}}})$$

$$= K(\sqrt[γ]{(-1)^{deg P_i^{α_i}} P_i^{α_i}}) = K(\sqrt[γ]{\sqrt[deg P_i^{α_i/d_i}]{P_i^{α_i/d_i}}})$$

and $P_i \nmid \frac{P_i^{α_i/d_i}}{D}$. Hence $P_i$ is unramified in $KF_i/K$. Therefore $Y_{P_i} = X_{P_i} = Y$.

It follows that the field associated to the group $\prod P_i X_P$ is $k(ξ_1, \ldots, ξ_r)$ where $ξ_i = \sqrt[γ]{γ_i P_i^{α_i/d_i}}$.

We have proved

**Theorem 4.4.** Let $X$ be the group of Dirichlet characters associated to $K = k(\sqrt[γ]{D})$ with $t|q - 1$, $D \in R_T$ and is $t$–power free, $D = P_1^{α_1} \cdots P_r^{α_r}$, $r ≥ 1$, $1 ≤ α_i ≤ t-1$, $1 ≤ i ≤ r$, $γ = (-1)^{deg D}$. Let $d_i = gcd(t, α_i)$, $1 ≤ i ≤ r$. Then the field associated to $\prod P_i X_P = \prod_{i=1}^r X_{P_i}$ is $L = k(ξ_1, \ldots, ξ_r)$ where $ξ_i = \sqrt[γ_i]{\sqrt[γ_i]{P_i^{α_i/d_i}}}$ and $γ_i = (-1)^{deg P_i^{α_i/d_i}}$. That is,

$$L = k(\sqrt[γ_i]{(-1)^{deg P_i^{α_i}} P_i^{α_i}}, \ldots, \sqrt[γ_i]{(-1)^{deg P_i^{α_i}} P_i^{α_i}})$$

and the genus field of $K$ is $K_{ge} = L^D$, where $D$ is the decomposition group of any prime $p \in S_{∞}(K)$ in $L/K$.

4.2. Abelian $p$–extensions. We consider now $K = k(\bar{y})$ where

$$\bar{y}^{α_i} - \bar{y} = δ_i + \cdots + δ_r$$

with $δ_i = (δ_{i,1}, \ldots, δ_{i,v})$ for some $v \in \mathbb{N}$, $δ_{i,j} \equiv \frac{Q_{i,j}}{P_i^{v+1}}$, $e_{i,j} ≥ 0$, $Q_{i,j} \in R_T$. Here we assume that $\mathbb{F}_{p^v} \subseteq k_0 = \mathbb{F}_q$ and that $K \subseteq k(\Lambda_N)$ for some $N \in R_T$.

Let $X$ be the group of characters associated to $K$. According to Schmid [25], the ramification index of $P_i$ in $K/k$ is determined by the first index $j$ such that we may write $δ_{i,j} = \frac{Q_{i,j}}{P_i^{v+1}}$ with $gcd(Q_{i,j}, P_i) = 1$, $e_{i,j} ≥ 0$ and $gcd(e_{i,j}, p) = 1$.

In other words, the ramification index of $P_i$ at $K/k$ depends only on $δ_i$ and not on $δ_1, \ldots, δ_{i-1}, δ_{i+1}, \ldots, δ_r$. Therefore, if $Y$ is the group of characters associated to $F_i = k(\bar{y}_i)$ with $\bar{y}_i^{α_i} - \bar{y}_i = δ_i$, $1 ≤ i ≤ r$,
we have \(|X_P| = |Y| = |Y_P|\). Furthermore, the extension \(KF_i = k(\bar{y}, \bar{y}_i) = k(\bar{y}, \bar{y} - \bar{y}_i) = K(\bar{y} - \bar{y}_i)\) is unramified at \(P_i\) over \(K\). It follows that the field associated to \(\prod_P X_P = \prod_{i=1}^r X_{P_i}\) is \(k(\bar{y}_1, \ldots, \bar{y}_r)\). Here the decomposition group \(D\) is trivial.

Then, we have

**Theorem 4.5.** \(\text{With the conditions as above, if } K = k(\bar{y}), \text{ then the field associated to } \prod_P X_P = \prod_{i=1}^r X_{P_i} \text{ is} \)

\[
L = k(\bar{y}_1, \ldots, \bar{y}_r)
\]

\(\text{and the genus field of } K \text{ is also} \)

\[
K_{\text{ge}} = k(\bar{y}_1, \ldots, \bar{y}_r).
\]

\(\square\)

5. **Explicit Description of Genus Fields of Abelian \(p\)-Extensions**

Let \(K/k\) be a finite abelian \(p\)-extension. Recall that \(k = k_0(T)\) with \(k_0 = \mathbb{F}_q\), say \(q = p^l\). We will assume that \(\mathbb{F}_p \subseteq k_0\), that is, \(u \mid l\).

Then we have

\[
\text{Gal}(K/k) \cong (\mathbb{Z}/p^\alpha \mathbb{Z}) \times \cdots \times (\mathbb{Z}/p^\alpha \mathbb{Z}) \quad \text{with} \quad 1 \leq \alpha_1 \leq \cdots \leq \alpha_u = v.
\]

There exist \(\bar{w}_1, \ldots, \bar{w}_u \in W_v(\bar{k})\) such that \(\bar{w}_p^0 - \bar{w}_i = \xi_i \in W_v(k)\), with \(K = k(\bar{w}_1, \ldots, \bar{w}_v)\). We also have that there exist \(\bar{y}_0 \in W_v(\bar{k})\) such that \(K = k(\bar{y}_0)\) with

\[
\bar{y}_0^{|u|} - \bar{y}_0 = \xi_0 \quad \text{for some} \quad \xi_0 \in W_v(k)
\]

(see [4, Theorem 8.5]). Here \(\bar{k}\) denotes an algebraic closure of \(k\).

Let \(P_1, \ldots, P_r \in R_T^+\) be the finite primes in \(k\) ramified in \(K\). From [4, Theorem 8.10] it follows that we may decompose \(\xi_0\) as

\[
\xi_0 = \delta_1 \ldots \delta_r \gamma,
\]

where \(\delta_{i,j} = \frac{Q_i}{P_j}, \quad e_{i,j} \geq 0, \quad Q_{i,j} \in R_T\) and if \(e_{i,j} > 0\), then \(e_{i,j} = \lambda_{i,j} p^{n_{i,j}}\), \(\gcd(\lambda_{i,j}, p) = 1\), \(0 \leq m_{i,j} < n\), \(\gcd(Q_{i,j}, P_i) = 1\) and \(\deg(Q_{i,j}) < \deg(P_i^{e_{i,j}})\), and \(\gamma_j = f_j(T) \in R_T\) with \(\deg f_j = \nu_j p^{m_j}\) and \(\gcd(q, \nu_j) = 1\), \(0 \leq m_j < n\) when \(f_j \not\in k_0\).

If the ramification index of \(P_i\) is \(p^{n_i} < p^n\), we may write \(\delta_i = (\delta_{i,1}, \ldots, \delta_{i,v}) = (0, \ldots, 0, \delta_{i,v-a_i+1}, \ldots, \delta_{i,v})\). In particular, \(\mathcal{P}_\infty\) decomposes fully in \(k(\bar{y}_i)/k\), where

\[
\bar{y}_i^{|u|} - \bar{y}_i = \delta_i \quad \text{(see [4, Theorem 8.13])}.
\]

Let \(\bar{z}^{n_u} - \bar{z} = \gamma\). In \(k(\bar{z})/k\) the only possible ramified prime is \(\mathcal{P}_\infty\). Note that if

\[
\bar{y} = \bar{y}_1 \ldots \bar{y}_r, \quad \text{then} \quad \bar{y}^{n_u} - \bar{y} = \xi_0 - \gamma = \delta_1 \ldots \delta_r
\]

and \(\mathcal{P}_\infty\) decomposes fully in \(k(\bar{y})/k\).

The first main result of this section is

**Theorem 5.1.** \(\text{With the above notation, let } E = KM \cap k(\Lambda_N). \text{ Then } E = k(\bar{y}), E_{\text{ge}} = k(\bar{y}_1, \ldots, \bar{y}_r)\) and \(K_{\text{ge}} = k(\bar{y}_1, \ldots, \bar{y}_r, \bar{z})\).

**Proof.** From the Galois correspondence \(EM = KM\). To prove that \(E = k(\bar{y})\) is equivalent to show that \(k(\bar{y})M = KM\) since \(k(\bar{y}) \subseteq k(\Lambda_N)\).
Now, $k(\bar{z}) \subseteq M$ since $M = L_n F_{q^m}(T)$ codifies all the inertia and all the ramification, which is totally wild, of $P_\infty$. We have

$$k(\bar{y})M = k(\bar{y})k(\bar{z})M \supseteq k(\bar{y}^\bullet \bar{z})M = KM.$$ 

Also,

$$KM = Kk(\bar{z})M = k(\bar{y}_0)k(\bar{z})M \supseteq k(\bar{y}_0^\bullet \bar{z})M = k(\bar{y})M.$$ 

Thus

$$KM = k(\bar{y})M \quad \text{and} \quad E = k(\bar{y}).$$

From [19] (see also Theorem 4.5) we obtain $E_{\mathfrak{g}^e} = k(\bar{y}_1, \ldots, \bar{y}_r)$. Finally

$$K_{\mathfrak{g}^e} = E_k K = k(\bar{y}_1, \ldots, \bar{y}_r)k(\bar{y}_0) = k(\bar{y}_1, \ldots, \bar{y}_r)k(\bar{y}_0 - \bar{y}_1^\bullet - \cdots - \bar{y}_r)$$

$$= k(\bar{y}_1, \ldots, \bar{y}_r)k(\bar{z}) = k(\bar{y}_1, \ldots, \bar{y}_r, \bar{z}).$$

This finishes the proof. 

**Remarks 5.2.** (a).- Observe that with the above conditions $[k(\bar{y}_i) : k] = e_{P_i}(K/k)$ and $[k(\bar{z}) : k] = e_{P_\infty}(K/k)$.

(b).- Note that the proof of Theorem 5.1 works even in the case that $\delta_i$ and $\bar{\gamma}$ are not in the reduced form described above. We only need that in each extension $\bar{y}_i - y_i = \bar{\delta}_i, 1 \leq i \leq r$ and $\bar{\delta}^\bullet - \bar{z} = \bar{\gamma}$ there is at most one prime ramifying.

From Theorem 2.3, the cases of Artin–Schreier and Witt extensions, and elementary abelian $p$–extensions are an immediate consequence of Theorem 5.1.

**Corollary 5.3** (Theorems 5.4 and 5.7 of [19]). Let $k = k_0(T)$.

(a).- Let $K = k(y)$ with

$$y^p - y = \alpha = \sum_{i=1}^r \frac{Q_i}{P_i^e} + f(T),$$

where $P_i \in R^\times_T, Q_i \in R_T, \gcd(P_i, Q_i) = 1, e_i > 0, p \nmid e_i, \deg Q_i < \deg P_i^e$, $1 \leq i \leq r, f(T) \in R_T, \text{with } p \nmid \deg f \text{ when } f(T) \not\in k_0$.

Then

$$K_{\mathfrak{g}^e} = k(y_1, \ldots, y_r, \beta),$$

where $y_i^p - y_i = \frac{Q_i}{P_i^e}, 1 \leq i \leq r$ and $\beta^p - \beta = f(T)$.

(b).- Let $K = k(\bar{y})$ where

$$\bar{y}_i^p - \bar{y}_i = \bar{\beta} = \bar{\delta}_1 + \cdots + \bar{\delta}_r + \bar{\mu},$$

with $\delta_{i,j} = \frac{Q_{i,j}}{P_i^e}, e_{i,j} \geq 0, Q_{i,j} \in R_T, \gcd(Q_{i,j}, P_i) = 1$ and if $e_{i,j} > 0$, then $p \nmid e_{i,j}$, and $\deg(Q_{i,j}) < \deg(P_i^{e_{i,j}})$, and $\mu_j = f_j(T) \in R_T$ with $p \nmid \deg f_j$ when $f_j \not\in k_0$.

Then

$$K_{\mathfrak{g}^e} = k(\bar{y}_1, \ldots, \bar{y}_r, \bar{z}),$$

where $\bar{y}_i^p - \bar{y}_i = \bar{\delta}_i, 1 \leq i \leq r$ and $\bar{\delta}^p - \bar{z} = \bar{\mu}.$
(c)- Assume that \( \mathbb{F}_{p^n} \subseteq k_0 \). Let \( K = k(y) \) with
\[
y^{p^n} - y = \sum_{i=1}^{r} \frac{Q_i}{P_i^{t_i}} + f(T),
\]
where \( P_i \in R_T^+ \), \( Q_i \in R_T \) and \( f(T) \in k_0[T] \).

Then
\[
K_{gt} = k(y_1, \ldots, y_r, z),
\]
where \( y_i^{p^n} - y_i = \frac{Q_i}{P_i^{t_i}}, 1 \leq i \leq r \) and \( \beta y^n - z = f(T) \). \( \square \)

6. General finite abelian extensions of \( k \)

Up to now we have given the explicit description of the genus fields of abelian \( p \)-extensions \( K \) of \( k = k_0(T) \) where \( k_0 = \mathbb{F}_q \) is such that \( \mathbb{F}_{p^n} \subseteq k_0 \) and \( K = k(\bar{y}) \) and \( \bar{y} \) is given by an equation of the form \( y^{p^n} - y = \beta \in W_m(k) \). When \( \mathbb{F}_{p^n} \nsubseteq k_0 \) the field \( K \) cannot be given by this type of equations.

In this section we give explicitly the description of \( K_{gt} \) where \( K/k \) is a finite abelian extension of degree \( t \) with \( \gcd(t, q - 1) = 1 \). The case \( t \mid q - 1 \) is treated in Subsection 4.1.

Remark 6.1. For any abelian extension \( K/k \) of degree \( t \) with \( \gcd(t, q - 1) = 1 \), we have that if \( E = KM \cap k(\Lambda_N) \), then \( [E : k] \mid t \) (see (2.2)). If \( X \) is the set of Dirichlet characters of \( E \), we have \( \gcd([X], q - 1) = \gcd([E : k], q - 1) = 1 \).

Since for any \( \chi \in X \) and any \( P \in R_T^+ \), we have that \( \chi_P^{|X|} = 1 \), we obtain that \( \gcd([E_{gt} : k], q - 1) = 1 \). In particular \( H = \{1\} \). Therefore \( K_{gt} = E_{gt}K \).

In general if \( K_1 \) and \( K_2 \) are two finite extensions of \( k \) we have
\[
(K_1)_{gt}(K_2)_{gt} \subseteq (K_1K_2)_{gt},
\]
but we may have \( (K_1)_{gt}(K_2)_{gt} \nsubseteq (K_1K_2)_{gt} \). In fact, let \( q > 2 \) and \( P, Q, R, S \in R_T \) be four different monic polynomials in \( R_T \). Set \( L_1 := k(\Lambda_{PQ})^+ \) and \( L_2 := k(\Lambda_{RS})^+ \). Then \( (L_1)_{gt} = L_i, i = 1, 2 \). Therefore \( (L_1)_{gt}(L_2)_{gt} = L_1L_2 \). On the other hand, \( (L_1L_2)_{gt} = k(\Lambda_{PQRS})^+ \nsubseteq (L_1)_{gt}(L_2)_{gt} \), see [21, Remark 3.7].

We will show that for finite abelian extensions of \( k \) of degree relatively prime to \( q - 1 \) we have equality. In particular if \( K_1 \) and \( K_2 \) are finite abelian \( p \)-extensions of \( k \), we have equality.

For a subfield \( K \subseteq k(\Lambda_N) \) for some \( N \in R_T \), denote by \( K'_{gt} \) the maximal abelian extension of \( K \) contained in \( k(\Lambda_N) \), unramified at the finite primes. We have (see Remark 4.1)
\[
(K_1)_{gt} = (K'_{gt})^D,
\]
where \( D \) is the decomposition group of any element of \( S_\infty(K) \) in \( K'_{gt}/K \).

Consider \( K_i \subseteq k(\Lambda_N) \), \( i = 1, 2 \) and let \( X_i \) be the group of Dirichlet characters associated to \( K_i \). Therefore \( Y = X_1X_2 = (X_1, X_2) \) is the group of Dirichlet characters associated to \( L = K_1K_2 \). Let \( P \in R_T^+ \). It is easy to see that
\[
\langle X_1, X_2 \rangle_P = \langle (X_1)_P, (X_2)_P \rangle.
\]
so that we obtain

\[ \prod_{p \in R^+} Y_p = \prod_{p \in R^+} \langle X_1, X_2 \rangle_p = \left( \prod_{p \in R^+} \langle X_1 \rangle_p \right) \cdot \left( \prod_{p \in R^+} \langle X_2 \rangle_p \right). \]

It follows that

\[ (K_1)^{\prime}_{ge} (K_2)^{\prime}_{ge} = (K_1 K_2)^{\prime}_{ge}. \]

We have proved

**Proposition 6.2.** For \( K_i \subseteq k(\Lambda_N), i = 1, 2, \) we have

\[ (K_i)^{\prime}_{ge}(K_i)^{\prime}_{ge} = (K_i K_i)^{\prime}_{ge}. \]

\( \square \)

**Corollary 6.3.** Let \( K_i \subseteq k(\Lambda_N), i = 1, 2 \) be such that \( K_1/k \) and \( K_2/k \) are finite abelian extensions of degrees relatively prime to \( q - 1 \). Then \( (K_1)^{\prime}_{ge}(K_2)^{\prime}_{ge} = (K_1 K_2)^{\prime}_{ge} \).

**Proof.** Since the decomposition groups of \( P_\infty \) in \( K_1/k \) in \( K_2/k \) and in \( K_1 K_2/k \) are the unit group, it follows from (6.1) that \( (K_1)^{\prime}_{ge} = (K_1)^{\prime}_{ge}, i = 1, 2 \) and \( (K_1 K_2)^{\prime}_{ge} = (K_1 K_2)^{\prime}_{ge} \). The result follows from Proposition 6.2.

\( \square \)

**Corollary 6.4.** Let \( K_i/k, i = 1, 2 \) be two finite abelian extensions of degrees relatively prime to \( q - 1 \). Then

\[ (K_1)^{\prime}_{ge}(K_2)^{\prime}_{ge} = (K_1 K_2)^{\prime}_{ge}. \]

**Proof.** Let \( k_0 = \mathbb{F}_{p^m}, K_i \subseteq L \subseteq \mathbb{F}_p \cdot k(\Lambda_N), i = 1, 2, \) and let \( M := L_{\cap \mathbb{F}_p^m}(T) \).

Set \( E_i := K_i \cap k(\Lambda_N), i = 1, 2 \) and \( E := K_1 K_2 M \cap k(\Lambda_N) \). Using the Galois correspondence, it can be proved that \( E = E_1 E_2 \).

From Corollary 6.3 we have \( E_{ge} = (E_1)_{ge} (E_2)_{ge} \). Therefore

\[ (K_1)_{ge}(K_2)_{ge} = (E_1)_{ge} K_1 \cdot (E_2)_{ge} K_2 = (E_1)_{ge}(E_2)_{ge} \cdot K_1 K_2 \]

\[ = E_{ge} \cdot K_1 K_2 = (K_1 K_2)_{ge}. \]

Thus \( (K_1)_{ge}(K_2)_{ge} = (K_1 K_2)_{ge} \).

\( \square \)

**Corollary 6.5.** Let \( K_i/k, i = 1, 2 \) be two finite abelian \( p \)-extensions. Then

\[ (K_1)^{\prime}_{ge}(K_2)^{\prime}_{ge} = (K_1 K_2)^{\prime}_{ge}. \]

As a consequence we obtain the description of the genus field of a finite abelian \( p \)-extension of \( k \).

**Corollary 6.6.** Let \( K/k \) be a finite abelian \( p \)-extension with Galois group \( \text{Gal}(K/k) = G \cong G_1 \times \cdots \times G_s \) with \( G_i \cong \mathbb{Z}/p^n \mathbb{Z}, 1 \leq i \leq s \). Let \( K \) be the composite \( K_1 \cdots K_s \) such that \( \text{Gal}(K_i/k) \cong G_i \). Let \( P_1, \ldots, P_t \) be the finite primes ramified in \( K/k \). Let \( K_i = k(\tilde{u}_i) \) be given by the equation

\[ u_i^{p_i} - a_i = \xi_i, \quad 1 \leq i \leq s. \]

Write each \( \tilde{\xi}_i \) as in (5.1) that is

\[ \tilde{\xi}_i = \delta_{i,1} + \cdots + \delta_{i,r} + \gamma_i, \]
such that all the components of $\vec{\delta}_{i,j}$ are written so that the degree of the numerator is less than the degree of the denominator, the support of the denominator is at most $\{P_j\}$ and the components of $\vec{\gamma}_i$ are polynomials. Let

\[
\vec{w}_{i,j}^p \cdot \vec{w}_{i,j} = \vec{\delta}_{i,j}, \quad 1 \leq i \leq s, \quad 1 \leq j \leq r
\]

and

\[
\vec{z}_{i}^p \cdot \vec{z}_{i} = \vec{\gamma}_i, \quad 1 \leq i \leq s.
\]

Then

\[K_{ge} = k(\vec{w}_{i,j}, \vec{z}_{i} \mid 1 \leq i \leq s, 1 \leq j \leq r).
\]

Proof. It is a consequence of Remarks 5.2 (b), Corollary 5.3 (b) and Corollary 6.5.

Next, we consider a cyclic extension $K/k$ of degree $t$ such that $\gcd(t, p(q-1)) = 1$. We have that $E = KM \cap k(\Lambda_N)$ satisfies that $[E : k]$ is relatively prime to $q - 1$. Hence $E_{ge}' = E_{ge}$ and $K_{ge} = E_{ge}K$. Thus, we have to describe $E_{ge}$.

**Proposition 6.7.** Let $E \subseteq k(\Lambda_N)$ be a cyclic extension of $k$ of degree $t$ relatively prime to $p(q-1)$. Let $P_1, \ldots, P_r \in R_T^+$ be the primes in $k$ ramifying in $E$. Then

\[E_{ge} = \prod_{j=1}^{r} F_j,
\]

where $k \subseteq F_j \subseteq k(\Lambda_{P_j})$ is the subfield of degree $a_j$ over $k$, $a_j$ is the order of $\chi_{P_j}$, and $\chi$ is the character associated to $E$.

Proof. It follows from the fact that $X = \langle \chi \rangle$ is the group of Dirichlet characters associated to $E$, $E_{ge}$ is the field corresponding to $\prod_{j=1}^{r} X_{P_j}$, $X_{P_j} = \langle \chi_{P_j} \rangle$ and $F_j$ is the field associated to $\chi_{P_j}$.

We have our final main result.

**Theorem 6.8.** Let $K/k$ be an abelian extension of degree $t$ with $\gcd(t, q - 1) = 1$. Let $P_1, \ldots, P_r \in R_T^+$ be the primes in $k$ ramifying in $K$. Let $E = KM \cap k(\Lambda_N) = E_0E_1 \cdots E_s$ where $E_i/k$ is a cyclic extension of degree $t_i$, $\gcd(t_i, p(q-1)) = 1$, $1 \leq i \leq s$ and $E_0/k$ is an abelian $p$-extension. Then

\[K_{ge} = E_{ge}K, \quad \text{where} \quad E_{ge} = (E_0)_{ge}(E_1)_{ge} \cdots (E_s)_{ge},
\]

$(E_0)_{ge}$ is given by Corollary 6.6 and $(E_i)_{ge} = \prod_{j=1}^{r} F_{i,j}$ is given by Proposition 6.7, $1 \leq i \leq s$.

Furthermore, let $b_{i,j} := [F_{i,j} : k]$. Then $F_j := \prod_{i=1}^{s} F_{i,j}$ is the subfield of $k(\Lambda_{P_j})$ of degree $b_j := \text{lcm}[b_{i,j}, 1 \leq i \leq s]$ over $k$. We have

\[K_{ge} = (E_0)_{ge}\left(\prod_{j=1}^{r} F_j\right)K.
\]
REFERENCES

[1] Anglès, Bruno; Jaulent, Jean-François, *Théorie des genres des corps globaux*, Manuscripta Math. **101**, no. 4, 513–532, (2000).

[2] Artin, Emil; Tate John, *Class field theory*, Benjamin, New York, 1967.

[3] Bae, Sung-han; Koo, Ja Kyung, *Genus theory for function fields*, J. Austral. Math. Soc. Ser. A **60**, no. 3, 301–310, (1996).

[4] Barreto-Castañeda, Jonny Fernando; Jarquín-Zárate, Fausto; Rzedowski-Calderón, Martha & Villa–Salvador, Gabriel, *Abelian $p$-extensions and additive polynomials*, https://arxiv.org/pdf/1606.02354.pdf.

[5] Bautista-Ancona, Víctor; Rzedowski-Calderón, Martha; Villa–Salvador, Gabriel, *Genus fields of cyclic $l$-extensions of rational function fields*, International Journal of Number Theory **9**, no. 5, 1249–1262, (2013).

[6] Clement, Rosario, *The genus field of an algebraic function field*, J. Number Theory **40**, no. 3, 359–375, (1992).

[7] Fröhlich, Albrecht, *The genus field and genus group in finite number fields*, Mathematika **6**, 40–46, (1959).

[8] Fröhlich, Albrecht, *The genus field and genus group in finite number fields, II*, Mathematika **6**, 142–146, (1959).

[9] Fröhlich, Albrecht, *Central extensions, Galois groups and ideal class groups of number fields*, Contemporary Mathematics, **24**, American Mathematical Society, Providence, RI, 1983.

[10] Gauss, Carl Friedrich, *Disquisitiones arithmeticae*, 1801.

[11] Garcia, Arnaldo & Stichtenoth, Henning, *Elementary Abelian $p$–Extensions of Algebraic Function Fields*, manuscripta math. **72**, 67–79 (1991).

[12] Hasse, Helmut, *Zur Geschlechtertheorie in quadratischen Zahlkörpern*, J. Math. Soc. Japan **3**, 45–51, (1951).

[13] Hayes, David, *Explicit Class Field Theory for Rational Function Fields*, Trans. Amer. Math. Soc. **189** (1974) 77–91.

[14] Hu, Su; Li, Yan, *The genus fields of Artin–Schreier extensions*, Finite Fields Appl. **16**, no. 4, 255–264, (2010).

[15] Ishida, Makoto, *The genus fields of algebraic number fields*, Lecture Notes in Mathematics, Vol. **555**, Springer-Verlag, Berlin-New York, 1976.

[16] Kani, Ernst, *Relations between the genera and between the Hasse-Witt invariants of Galois coverings of curves*, Canad. Math. Bull. **28**, 321–327, (1985), no. 3.

[17] Lachaud, Gilles, *Artin–Schreier curves, exponential sums, and the Carlitz-Uchiyama bound for geometric codes*, J. Number Theory **39**, 18–40, (1991), no. 1.

[18] Leopoldt, Heinrich W., *Zur Geschlechtertheorie in abelschen Zahlkörpern*, Math Nachr. **9**, 351–362, (1953).

[19] Maldonado–Ramírez, M.; Rzedowski–Calderón, M.; Villa–Salvador, G., *Genus Fields of Abelian Extensions of Congruence Rational Function Fields*, Finite Fields Appl. **20**, 40–54 (2013).

[20] Maldonado–Ramírez, M.; Rzedowski–Calderón, M.; Villa–Salvador, G., *Corrigendum to Genus fields of abelian extensions of rational congruence function fields* [Finite Fields Appl. **20** (2013) 40–54], Finite Fields Appl. **20**, 283–285 (2015).

[21] Maldonado–Ramírez, Myriam.; Rzedowski–Calderón, Martha.; Villa–Salvador, G., *Genus fields of congruence function fields*, Finite Fields Appl. **44**, 56–75 (2017).

[22] Ore, Oystein, *On a special class of polynomials*, Trans. Amer. Math. Soc. **35**, 559–584 (1933), no. 3.

[23] Peng, Guohua, *The genus fields of Kummer function fields*, J. Number Theory **98**, no. 2, 221–227, (2003).

[24] Rosen, Michael, *The Hilbert class field in function fields*, Exposition. Math. **5**, no. 4, 365–378, (1987).

[25] Schmid, Hermann Ludwig, *Zur Arithmetik der zyklischen $p$-Körper*, J. Reine Angew. Math. **176** (1936) 161–167.

[26] Villa Salvador, Gabriel Daniel, *Topics in the theory of algebraic function fields*, Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 2006.

[27] Wittmann, Christian, *l–class groups of cyclic function fields of degree $l$*, Finite Fields Appl. **13**, no. 2, 327–347, (2007).

[28] Zhang, Xianke, *A simple construction of genus fields of abelian number fields*, Proc. Amer. Math. Soc. **94**, no. 3 (1985) 393–395.
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