Using results on topological band theory of phases of matter and discrete symmetries, we study topological properties of band structure of physical systems involving spin $\frac{1}{2}$ and $\frac{3}{2}$ fermions. We apply this approach to study partial breaking in 4D $\mathcal{N} = 2$ gauged supergravity in rigid limit and we describe the fermionic gapless mode in terms of chiral anomaly. We study as well the homologue of the usual spin-orbit coupling $\vec{L}.\vec{S}$, that opens the vanishing band gap for free $s = \frac{1}{2}$ fermions; and show that is precisely given by the central extension of the $\mathcal{N} = 2$ supercurrent algebra in 4D spacetime. We also give comments on the rigid limit of Andrianopoli et al. obtained in [28] and propose an interpretation of energy bands in terms of a chiral gapless isospin $\frac{1}{2}$-particle (iso-particle). Other features, such as discrete T-symmetry in FI coupling space, effect of quantum fluctuations and the link with Nielson-Ninomiya theorem, are also studied.

Key words: Topological band theory, $\mathcal{N} = 2$ gauged supergravity, gapped gravitinos, partial breaking, chiral anomaly.

1. INTRODUCTION

In the few past years, there has been an intensive interest into topological band theory in the Brillouin Zone and in 3D effective Chern- Simons field theories in connection with the phases of matter [1–9]. This interest has been mainly concerning spin $s = \frac{1}{2}$ topological matter and spin 1 topological gauge theories in lower spacetime dimensions; this is because of their particular properties in dealing with condensed matter systems like topological
insulators and superconductors; and also for the role they play in quantum Hall effect as well as in the study of boundary states and anomalies \[10\]–\[17\]. By looking to extend some special features of these studies to systems with spacetime spins beyond \(\frac{1}{2}\), \(1\), we fall into supergravity like models where fermionic modes of higher spins such as relativistic spin \(\frac{3}{2}\) are also known to play a basic role. In this paper, we would like to explore some topological aspects of band theory of systems having spins less or equal to \(2\); and look for a physical model where topological properties obtained for spins \(s = \frac{1}{2}, 1\) can be extended to higher spins. A priori, physical systems with spins \(s \leq 2\) may exist in spacetime dimensions \(D = d + 1 \geq 3\) where spin \(\frac{3}{2}\) and 2 particle fields have non trivial gauge degrees of freedom; and to get started it is then natural to begin by fixing the full spins content of the physical system we are interested in here; and also define the hamiltonian model or the field equations describing the full dynamics. To find a physical system with higher spins where such kinds of studies may be relevant to; and also identify the appropriate approach that we can use as the starting point, we give here below two motivations: The first one concerns the choice of a particular system having fermions with different spins, say two types of fermionic spins \(s = \frac{1}{2}, \frac{3}{2}\); and the other regards the tools to use for approaching their band structure. First, by studying the constructions of \[17\]–\[19\], one comes out with the conclusion that several topological condensed matter statements based on spin \(\frac{1}{2}\) fermions may be approached by starting with Dirac equation of \((1 + d)\) relativistic theory. From this theory, one may engineer effective hamiltonians breaking explicitly the SO\((1, d)\) Lorentz symmetry by allowing non linear dispersion relations due to underlying lattice geometries and interactions. It follows from this description that quantities like fermionic gapless/gapped modes, chiral ones and edge states have interpretations in terms massless/massive states, quasi particles with exotic statistics and anomalies whose explanation requires the use of topological notions such as manifold boundaries, left/right windings, Berry connection and Nielson-Ninomiya theorem. To look for extending results on spin \(\frac{1}{2}\) topological matter to spin \(\frac{3}{2}\) gravitinos, we then have to go beyond Dirac equation; for instance by considering the Rarita-Schwinger equation of gravitinos and try to mimic the analysis done for spin \(\frac{1}{2}\). Even though this is an interesting direction to take \[20, 21\], we will not follow this path here because of the complicated \(\mathcal{N} = 2\) supergravity interactions making the field equations difficult to manage. Instead, we will rather use related equations given by extended supergravity Ward identities \[22, 23\]. The use of these Ward identities has been motivated by the question to what kind of physical systems the specific properties of the gravitino band structure may serve to. Recalling the role played by gravitinos in the spontaneous breaking of local supersymmetry, we immediately come to
the point that gapless and gapped gravitinos can be applied to study the problem of partial supersymmetry breaking of $\mathcal{N}$-extended vector-like theories. Indeed, in the example of the effective $\mathcal{N} = 2$ gauged supergravity in 4D spacetime, one has, in addition to bosons (with $s = 0, 1, 2$) and spin $\frac{1}{2}$ fermions, two gravitinos $(\psi_1^{\alpha \mu}, \psi_1^{\alpha \mu})$ forming an isospin $\frac{1}{2}$ particle; that is to say a doublet under the SU(2) R-symmetry involving pairs of gapless gravitino modes. The breaking $\mathcal{N} = 2 \to \mathcal{N} = 1$ requires then a partial lifting of the degeneracies of mode doublets, which, like in the case of condensed matter with spin $\frac{1}{2}$ fermions, may be achieved by turning on a kind of spin-orbit like coupling $\vec{L} \cdot \vec{S}$ [24]. The spin $\frac{3}{2}$ matter study offers therefore a good opportunity to identify what is the iso-particle hamiltonian including the homologue of $\vec{L} \cdot \vec{S}$ that induces partial breaking of supersymmetry. This coupling will be denoted like $\vec{\xi} \cdot \vec{I}$ where $\vec{\xi}$ plays the role of angular momentum $\vec{L}$ and the isospin $\vec{I}$ the role of the spin $\vec{S}$. In this regards, it interesting to recall that spontaneous partial breaking in $\mathcal{N} = 2$ supergravity may be done by superHiggs mechanism; which, in $\mathcal{N} = 1$ supermultiplet language, a massive $\mathcal{N} = 1$ gravitino multiplet can be created by merging three multiplets: a massless $\mathcal{N} = 1$ gravitino eating a massless $\mathcal{N} = 1$ U(1) multiplet and a $\mathcal{N} = 1$ chiral multiplet [25]. But here, the partial breaking will be done by the isospin-orbit coupling that opens the gap energy between the two gravitinos. In this study, we will show that the $\vec{\xi} \cdot \vec{I}$ coupling is precisely given by the central anomaly of the $\mathcal{N} = 2$ supercurrent algebra in 4D spacetime [26, 27].

The main purpose of this work is then to use results on topological band theory of fermionic matter and chiral anomalies as well discrete symmetries to study partial breaking in $\mathcal{N} = 2$ gauged supergravity in 4D. The spacetime fields of our system are given by the fields content of the standard $\mathcal{N} = 2$ supermultiplets; in particular the fields content of the gravity multiplet, $n_V$ vector multiplets and $n_H$ matter multiplets. To perform this study, we will use $\mathcal{N} = 2$ supergravity Ward identities in rigid limit as considered in [28]; and also study the partial breakings by using the topological approach along the Nielson-Ninomiy theorem and chiral anomaly. We study as well the effect of quantum harmonic fluctuations in the FI coupling space; and show that the result of [28] is not affected by quantum corrections provided a saturated condition holds.

The presentation is as follows: In section 2, we describe some tools on partial breaking in the rigid limit of $\mathcal{N} = 2$ supergravity theory and present the basic equations to start with. We also give some useful comments. In section 3, we derive the free hamiltonian of the iso-particles in $\mathcal{N} = 2$ gauged supergravity; work out the isospin-orbit coupling that opens the zero gap between the two gravitino zero modes and show how time reversing
symmetry $T$; and $PT$ (combined $T$ and parity $P$) can be implemented. In section 4, we study gapless and gapped gravitinos in $\mathcal{N} = 2$ gauged supergravity, describe the properties of partial supersymmetry breakings and their interpretation from the view of Nielson-Ninomiya theorem and chiral anomaly. We discuss also the effect of quantum fluctuations on partial breaking of $\mathcal{N} = 2$ supersymmetry. Section 5 is devoted to conclusion and comments.

2. RIGID LIMIT OF $\mathcal{N} = 2$ WARD IDENTITY: CASE $U(1)$ MODEL

Following [28], partial breaking of rigid and local extended supersymmetries is highly constrained; it can occur in a certain class of supersymmetric field theories provided one evades some no-go theorems [29–32]; see also [33–38]. In global 4D $\mathcal{N} = 2$ theories, this was first noticed in [26, 39]; and was explicitly realized in [40, 41] for a model of a self-interacting $\mathcal{N} = 2$ vector multiplet in the presence of $\mathcal{N} = 2$ electric and magnetic Fayet-Iliopoulos (FI) terms. There, it has been shown explicitly that the presence of electric $\vec{v}$ and magnetic $\vec{m}$ FI couplings is crucial to achieve partial breaking. The general conditions for $\mathcal{N} = 2$ partial supersymmetry breaking have been recently elucidated by L. Andrianopoli et al in [28] where it has been also shown that $\vec{v}$ and $\vec{m}$ should be non aligned ($\vec{v} \wedge \vec{m} \neq \vec{0}$).

Their starting point for deriving the general conditions for partial supersymmetry breaking in rigid limit\(^1\) was the reduced $\mathcal{N} = 2$ gauged supergravity Ward identity

$$V\delta^A_B + C^A_B = \sum_{i=1}^{n_V} \delta_B \lambda^i C \delta^A \lambda^i_C$$

where the spin $\frac{1}{2}$ fermions $\lambda^i_A$ and $\lambda_{iB} := \varepsilon^i_{BA} g_{ij} \hat{\lambda}^j A$ refer to the chiral and antichiral projections of the gauginos respectively. Here, the SO(1,3) spacetime spin index of the $\lambda^i_A$ fermions has been omitted for simplicity; while we have exhibited the two other indices $A$ and $i$. The $A = 1, 2$ refers to the isospin $\frac{1}{2}$ representation of the SU(2)\(_R\) symmetry of the $\mathcal{N} = 2$ supersymmetric algebra seen that $\mathcal{N} = 2$ gauginos are isodoublets under SU(2)\(_R\); this index is lowered and rised by the antisymmetric tensor $\varepsilon^{AB}$ and its inverse $\varepsilon_{BA}$. The index $i = 1, ..., n_V$ designates the number of $\mathcal{N} = 2$ vector multiplets in the Coulomb branch of the $\mathcal{N} = 2$ gauged supergravity theory. Notice also that the quantity $(\delta_B \lambda^i A)$ is a convention notation for the $\mathcal{N} = 2$ supersymmetric transformation of gauginos which is given by $\delta_{\text{susy}} \lambda^i A = (\delta_B \lambda^i A) \epsilon^B$ with the two fermions $\epsilon^A = (\epsilon^1, \epsilon^2)$ standing for the supersymmetric

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\(^1\) Rigid limit is implemented through a rescaling of the fields contents of the theory and the space time supercoordinates by using a dimensionless parameter as $\mu = \frac{A}{\Lambda_{\text{Planck}}}$. For explicit details see [28, 41].
transformation parameters. In the relation (2.1), the right hand side is restricted to the pure Coulomb branch and so corresponds to the rigid limit of the following local identities
\[ \sum_i \alpha_i \delta_B \lambda^C \delta^A \lambda_i C = \tilde{V} \delta_B^A - \sum_u \alpha_u \delta_B \zeta^u \delta^A \zeta_u - \sum_{\mu, \nu} \alpha_0 \delta^A \psi_\nu C \Gamma^{\mu \nu} \delta_B \psi_\mu^C \] (2.2)

The left hand side of (2.1) contains two basic terms namely the rigid limit of the scalar potential \( \mathcal{V} \delta_B^A \) and an extra traceless constant matrix,
\[ C_B^A = \tilde{\xi}. (\tilde{\tau})_B^A \ , \quad Tr C = 0 \] (2.3)

This hermitian traceless matrix can be interpreted as an anomalous central extension in the \( \mathcal{N} = 2 \) supersymmetric current algebra \[27, 29, 39\]; it only affects the commutator of two supersymmetry transformations of the gauge field \[29, 41\] and contains data on hidden gravity and matter sectors. Recall that the basic anticommutator of the \( \mathcal{N} = 2 \) supercurrent algebra is,
\[ \{ \mathcal{J}_0^A (x), \int d^3 y \mathcal{J}_0^B (y) \} = \delta_B^A \sigma_\mu \mathcal{T}_\mu^0 + C_B^A \] (2.4)

where \( \mathcal{J}_{\alpha A}^0 (x), \mathcal{J}_{\alpha A}^0 (x) \) and \( \mathcal{T}_\mu^0 (x) \) are the time components of the supersymmetric current densities \( \mathcal{J}_{\alpha A}^0, \mathcal{J}_{\alpha A}^0 \) and \( \mathcal{T}_\mu^0 \). The time component densities in the current superalgebra (2.4) are related to \( Q_{\alpha A}, \bar{Q}_{\alpha B} \) and \( P_\mu \) charges of the \( \mathcal{N} = 2 \) supersymmetric QFT in the usual manner; for example
\[ Q_{\alpha A} = \int d^3 x \mathcal{J}_{\alpha A}^0 \ , \quad P_\mu = \int d^3 x \mathcal{T}_\mu^0 \] (2.5)

obeying \( Q_B \bar{Q}^A + Q^A Q_B \sim \delta_B^A \sigma_\mu P_\mu \); the usual globally defined \( \mathcal{N} = 2 \) supersymmetric algebra with \( C_B^A \) constrained to vanish.

By comparing eq(2.1) with the general form of the Ward identities eq(2.2), we deduce that the \( C_B^A \) term captures the contribution of the fermion shifts to the Ward identity coming from the rigid limit of the hidden gravity \( (\delta^A \psi_\nu C \Gamma^{\mu \nu} \delta_B \psi_\mu^C) \) and the matter \( (\delta_B \zeta^u \delta^A \zeta_u) \) branches. For the simple example of an abelian \( U(1) \) gauge multiplet \( (n_V = 1) \), the anomaly isovector \( \tilde{\xi} = Tr (\tilde{\tau} C) \) has been realised in terms of the electric \( \tilde{v} \) and the magnetic \( \tilde{m} \) FI coupling constant isovectors of the Coulomb branch of the effective \( \mathcal{N} = 2 \) \( U(1) \) gauge theory as follows²
\[ \tilde{\xi} = \tilde{v} \wedge \tilde{m} \ , \quad \tilde{m} \neq \mathbb{R}^+ \tilde{v} \] (2.6)

² The exact expression found in [28] is \( \tilde{\xi} = 2 \tilde{v} \wedge \tilde{m} \). Here the factor 2 = \( (\sqrt{2})^2 \) has been absorbed by scaling the FI couplings.
obeying the remarkable property $\vec{\xi}.\vec{\nu} = 0$ and $\vec{\xi}.\vec{m} = 0$; see (2.13). Moreover, partial breaking of supersymmetry takes place at \([28, 29]\),

$$\mathcal{V} = |\vec{\xi}| \geq 0$$

(2.7)

This relation will be used later on when considering topological aspects of gapless fermions (subsection 4.1) as well as harmonic fluctuations (subsection 4.2); but before that let us give other comments regarding (2.6-2.7).

First, notice that in order to have a non zero $\vec{\xi}$, it is sufficient to take the following particular and simple choice

$$\vec{\nu} = \begin{pmatrix} \nu_x \\ 0 \\ 0 \end{pmatrix}, \quad \vec{m} = \begin{pmatrix} 0 \\ m_y \\ 0 \end{pmatrix}, \quad \vec{\xi} = \begin{pmatrix} 0 \\ 0 \\ \xi_z \end{pmatrix}$$

(2.8)

satisfying $\vec{\nu}.\vec{m} = 0$ and $\vec{\nu} \wedge \vec{m} \neq \vec{0}$. This particular choice shows that a quadratic term of type

$$\frac{\gamma_\perp}{2} |\vec{m}| \times |\vec{\nu}|$$

like the one appearing in eq(2.17), comes necessary for the contribution of the $\vec{\xi}$-direction normal to the $(\vec{\nu}, \vec{m})$ plane; that is to say:

$$\vec{\xi} = \vec{0} \quad \Rightarrow \quad \gamma_\perp = 0$$

(2.9)

This implication is obviously not usually true since for non orthogonal $\vec{\nu}$ and $\vec{m}$, we have $\vec{\nu}.\vec{m} = |\vec{m}| \times |\vec{\nu}| \cos \theta \neq 0$ as far as $\theta \neq \pm \frac{\pi}{2} \mod 2\pi$. The trick $\theta = \pm \frac{\pi}{2}$ will help us to detect the effect of $\vec{\xi}$ especially when studying quantum fluctuations around the $\mathcal{N} = 2$ supersymmetric ground states $\mathcal{V} = 0$ and $\mathcal{V} = |\vec{\xi}|$.

Second, observe that by setting $\tau_{[kl]} = \frac{1}{2} \varepsilon_{kln} \tau^n$, $\xi^{[kl]} = \varepsilon^{klm} \xi_n$ and $\varepsilon_{kln} \varepsilon^{klm} = 2 \delta^n_m$, it follows that $\xi^{[kl]} \tau_{[kl]} = \xi_i \tau^i$, and then the central extension matrix (2.3) can be also expressed like

$$C_B^A = \xi^{[kl]} (\tau_{[kl]})_B^A, \quad \tau_{[kl]} = \frac{1}{4i} [\tau_k, \tau_l]$$

(2.10)

This way of expressing $C_B^A$ is interesting since, supported by the dimensional argument, it gives an idea on how to realize the factor $\xi^{[kl]}$ in terms of the electrical $\nu^k$ and magnetic $m^l$ couplings of FI. Antisymmetry implies the natural factorisation

$$\xi^{[kl]} = \nu^k m^l - \nu^l m^k, \quad \xi^{[xy]} = \nu^x m^y - \nu^y m^x$$

(2.11)
which is nothing but the Andrianopoli et al factorisation (2.6). We expect that this trick can also help to find extension the \( \mathcal{N} = 2 \) realisation (2.6) to higher supergravities; in particular to \( \mathcal{N} = 4 \) theory in rigid limit where there is no matter branch; this generalisation will not be considered here. Notice that for the simple choice (2.8), we have the diagonal matrix,

\[
C_B^A = \begin{pmatrix}
\nu_x & m_y \\
0 & -\nu_x m_y
\end{pmatrix}
\] (2.12)

showing that in the rest frame, we have \( \delta^A_B \sigma_\mu J^{\mu 0} = \delta^A_B \mathcal{V} \) and then the \( \mathcal{N} = 2 \) current algebra (2.4) splits into two \( \mathcal{N} = 1 \) copies with right hand energy densities given by \( \mathcal{V} \pm \nu_x m_y \).

Third, a non vanishing \( \vec{\xi} \) requires in general non collinear \( \vec{\nu} \) and \( \vec{m} \) vectors; and so the unit vectors \( \vec{e}_\nu = \frac{\vec{\nu}}{|\vec{\nu}|}, \vec{e}_m = \frac{\vec{m}}{|\vec{m}|} \) generate a 2-dimensional plane with a normal vector given by \( \vec{e}_\xi = \frac{\vec{\xi}}{|\vec{\xi}|} \). These three vectors form altogether a 3D vector basis of \( \mathbb{R}^3 \) that we term as the 3D iso-space,

\[
\vec{e}_\nu \; ; \; \vec{e}_m \; ; \; \vec{e}_\xi \; , \; \vec{e}_\xi = \vec{e}_\nu \wedge \vec{e}_m
\] (2.13)

By the terminology 3D iso-space, we intend to use its similarity with the usual Euclidean space \( \mathbb{R}^3 \) of classical mechanics of point like particles to propose a physical interpretation of (2.6) by using the notion of isospin \( I = \frac{1}{2} \) particle; this will be done in section 3.

Notice moreover the following features:

- the relation (2.6) concerns an effective \( \mathcal{N} = 2 \) U(1) gauge theory with one gauge supermultiplet \( n_V = 1 \). The general expression of \( \vec{\xi} \), extending (2.6), as well as the general form of the scalar potential energy \( \mathcal{V} \) associated with generic U(1)\(^{n_V} \) effective gauge theories reduced to FI couplings, have been shown to be functions of characteristic data of the special geometry of the scalar manifold. They are given by the following factorisations

\[
\mathcal{V} = \frac{1}{2} \delta_{ab} \mathcal{P}^a \mathcal{M} \mathcal{P}^b \mathcal{N} \mathcal{P}^b \mathcal{N} \] (2.14)

\[
\vec{\xi}_a = \frac{1}{2} \varepsilon_{abc} \mathcal{P}^b \mathcal{C} \mathcal{M} \mathcal{P}^c \mathcal{N} \] (2.15)

where \( \mathcal{P}^a = (m^aI, \nu_I) \) are moment maps carrying quantum numbers of SU(2)\(_R \) \( \times \) SP(2\( n_V \), \( R \)); \( \mathcal{C} \mathcal{M} \mathcal{N} \) the metric of SP(2\( n_V \), \( R \)); and \( \mathcal{S}_{MN} \) is a symmetric matrix of the form

\[
\mathcal{S}_{MN} = \begin{pmatrix}
I + \mathcal{R} \mathcal{I}^{-1} \mathcal{R} & -\mathcal{R} \mathcal{I}^{-1} \\
-\mathcal{I}^{-1} \mathcal{R} & \mathcal{I}^{-1}
\end{pmatrix}
\] (2.16)
encoding data on the scalar manifold of the $\mathcal{N} = 2$ theory, see \cite{28,29} for more details.

For the example of an abelian $U(1)$ gauge model, the $\xi_a$ is as in eq (2.6) while the scalar potential (2.14) has also the following remarkable quadratic shape

$$V = \alpha |\vec{m}|^2 + \beta |\vec{v}|^2 + \frac{\gamma}{2} |\vec{m}| \times |\vec{v}|$$

(2.17)

with $4\alpha \beta > \gamma^2 > 0$ and $\alpha, \beta$ assumed positive for later use. Notice that here $\gamma$ should be viewed as the sum $\gamma_\parallel + \gamma_\perp$ with $\gamma_\parallel$ describing the coupling in the $(\vec{m}, \vec{v})$ plane and $\gamma_\perp$ in the normal $\vec{m} \wedge \vec{v}$ directions; see also eqs (3.11-3.12).

- By substituting (2.16) and $P^a = (m^a, \nu^a)$ into (2.14), we learn that the real parameters $\alpha, \beta$ and $\gamma$ in above scalar potential have indeed a geometric interpretation in terms of the effective prepotential $\mathcal{F}$ of the $\mathcal{N} = 2$ special geometry. For example the parameters $\frac{\gamma}{2}$ in (2.17) depends both on the real $\mathcal{R}$ and imaginary $\mathcal{I}$ parts of the second derivative of $\mathcal{F}$.

- The scalar potential (2.17) has a particular dependence on $|\vec{m}|$ and $|\vec{v}|$; it can be presented as quadratic form $V = P^i G_{ij} P^j$ with $P^i$ and metric $G_{ij}$ as follows

$$V = \begin{pmatrix} |\vec{m}| & |\vec{v}| \end{pmatrix} \begin{pmatrix} \alpha & \frac{\gamma}{2} \\ \frac{\gamma}{2} & \beta \end{pmatrix} \begin{pmatrix} |\vec{m}| \\ |\vec{v}| \end{pmatrix}$$

(2.18)

with $\det G = \alpha \beta - \frac{\gamma^2}{4}$. This form will diagonalised later on for explicit calculations.

- The above $V$ is might be viewed as a special potential; a more general expression would involve more free parameters as shown here below,

$$V = V_0 + g_a \nu^a + w_a m^a + B_{ab} \nu^a m^b + A_{ab} \nu^a \nu^b + C_{ab} m^a m^b$$

(2.19)

where $V_0$ is a number that depends on the VEVs of the scalar fields and the parameters of the effective $\mathcal{N} = 2$ theory like masses and gauge coupling constants. The $g_a$, $w_a$ are two isovectors scaling in same manner as the FI constants; and $A_{ab}, B_{ab}, C_{ab}$ are dimensionless real $3 \times 3$ matrices; $A_{ab}$ and $C_{ab}$ are symmetric; but $B_{ab}$ is a general matrix. These moduli may characterise as well the scalar manifold of the effective $\mathcal{N} = 2$ supergravity and likely external fields as suspected from table (3.2); see also eq (3.27) where $\vec{w}$ of the $w_a m^a$ is interpreted in terms of an external iso-magnetic field.
Finally, notice that by giving these somehow explicit details on the \( n_V = 1 \) theory, we intend to use its simple properties to derive the iso-particle proposal and build the isospin-orbit coupling in \( \mathcal{N} = 2 \) supergravity mentioned in the introduction. We will also use these tools to study the isospin \( \frac{1}{2} \) particle as well as hidden discrete symmetries that capture data on the topological phases of the right hand of the \( \mathcal{N} = 2 \) supersymmetry current algebra \((2.4)\).

3. ISOSPIN \( \frac{1}{2} \) PARTICLE PROPOSAL

The Andrianopoli et al realisation \((2.6)\) of the rigid anomaly isovector \( \tilde{\xi} = \tilde{\nu} \wedge \tilde{m} \) in effective \( \mathcal{N} = 2 \) supersymmetric gauge theory is interesting and is very suggestive; see figure \((1)\) for illustration. This is because of the wedge product \( \tilde{\nu} \wedge \tilde{m} \) that allows us to establish a correspondence between properties of partial supersymmetry breaking and the electronic band theory with \( \Delta_{soc} \tilde{L} \cdot \tilde{S} \) spin-orbit coupling turned on \((\Delta_{soc} \neq 0)\).

Indeed, the axial vector \( \tilde{\xi} = \tilde{\nu} \wedge \tilde{m} \), to which we refer below to as Andrianopoli et al orbital vector, has the same form of the usual angular momentum vector, 

\[
\tilde{L} = \tilde{r} \wedge \tilde{p}
\]  

(3.1)
of a 3D classical particle with coordinate position \( \vec{r} \) and momentum \( \vec{p} \). By comparing the \( \vec{\nu} \wedge \vec{m} \) formula of (2.6) with the above \( \vec{r} \wedge \vec{p} \), it follows that the FI electric coupling \( \vec{\nu} \) may be put in correspondence with the vector \( \vec{r} \); and the magnetic \( \vec{m} \) with the vector \( \vec{p} \). Hence, we have the following schematic picture linking the physics of classical particles (electron) to the physics of iso-particle of \( \mathcal{N} = 2 \) gauged supergravity (gravitinos and gauginos),

| vectors in \( \mathbb{R}^3 \): electron | \( \leftrightarrow \) | iso-vectors in \( \tilde{\mathbb{R}}^3 \): gravitinos |
|---|---|---|
| particle \( (\vec{r}; \vec{p}) \) | \( \leftrightarrow \) | \( (\vec{\nu}; \vec{m}) \) |
| \( (\vec{r} + \delta \vec{r}; \vec{p} + \delta \vec{p}) \) | \( \leftrightarrow \) | \( (\vec{\nu} + \delta \vec{\nu}; \vec{m} + \delta \vec{m}) \) |
| isotropy \( \text{SO}(3) \) | \( \leftrightarrow \) | R-symmetry \( \text{SU}(2) \) |
| orbital moment \( \vec{L} = \vec{r} \wedge \vec{p} \) | \( \leftrightarrow \) | orbital moment \( \vec{\xi} = \vec{\nu} \wedge \vec{m} \) |
| hamiltonian \( h(\vec{r}; \vec{p}) \) | \( \leftrightarrow \) | hamiltonian \( h(\vec{\nu}; \vec{m}) \) |
| spin \( \vec{S} \) | \( \leftrightarrow \) | isospin \( \vec{I} \) |
| gauge symmetry \( U(1)_{\text{em}} \) | \( \leftrightarrow \) | gauge symmetry \( U(1)_{\text{elec}} \times U(1)_{\text{mag}} \) |

On the left hand side of this table, the Euclidian \( \mathbb{R}^3 \) space is the usual 3d-space with \( \text{SO}(3) \) isotropy symmetry. In this real space lives bosons and fermions; in particular fermions with intrinsic properties like spin \( \frac{1}{2} \) particles with symmetry

\[
\text{SU}(2)_{\text{spin}} \sim \text{SO}(3)
\]

(3.3)

On right hand side, the \( \tilde{\mathbb{R}}^3 \) is an iso-space with isotropy symmetry \( \text{SO}(3)_R \) given by the R-symmetry \( \text{SU}(2)_R \) of the \( \mathcal{N} = 2 \) supersymmetric algebra. This is a global symmetry group that will be imagined here as a global isospin group \( \text{SU}(2)_{\text{isospin}} \) characterising the quasi-particle of figure (1). Thus, the homologue of the real space symmetry (3.3) is given by,

\[
\text{SU}(2)_{\text{isospin}} \sim \text{SU}(2)_R \sim \text{SO}(3)_R
\]

(3.4)

Matter in the iso-space \( \tilde{\mathbb{R}}^3 \) is then given by quasi-particles carrying isospin charges under \( \text{SU}(2)_R \); in particular the isospin \( I = \frac{1}{2} \) describing the two gravitinos and the \( n_V \) pairs of gauginos of the Coulomb branch of the \( \mathcal{N} = 2 \) gauged supergravity. Recall that in this theory, the particle content belongs to three \( \mathcal{N} = 2 \) supermultiplets namely the gravity \( \mathcal{G}_{\mathcal{N}=2} \), the vector \( \mathcal{V}_{\mathcal{N}=2} \) and the matter \( \mathcal{H}_{\mathcal{N}=2} \). The two first ones are recalled here after
\begin{table}
\centering
\begin{tabular}{|c|c|c|c|}
\hline
\textbf{\(\mathcal{N} = 2\) multiplets} & \textbf{field content in spin} & \textbf{spin }s & \textbf{isospin }I \\
\hline
\textbf{gravity }\textbf{G}_{\mathcal{N}=2} & graviton : 2 & 2 & 0 \\
& gravitinos \(\psi^A\) : \(2 \times \frac{3}{2}\) & \(\frac{3}{2}\) & \(\frac{1}{2}\) \\
& graviphoton \(A^1_\mu\) : 1 & 1 & 0 \\
\hline
\textbf{vector }\textbf{V}_{\mathcal{N}=2} & vector \(A^2_\mu\) : 1 & 1 & 0 \\
& gauginos \(\lambda^A\) : \(2 \times \frac{1}{2}\) & \(\frac{1}{2}\) & \(\frac{1}{2}\) \\
& scalars : \(2 \times 0\) & 0 & 0 \\
\hline
\end{tabular}
\end{table}

The field content includes the fermions (gravitinos and gauginos) having a non trivial isospin charge. It contains also two spin \(s = 1\) gauge fields \(A^M_\mu\) (graviphoton \(A^1_\mu\) and Coulomb \(A^2_\mu\)) with

\[ U (1)_{elec} \times U (1)_{mag} \] (3.6)

gauge transformations given by abelian isometries of the scalar manifold of the supergravity theory. The fermionic fields \(F^A = \psi^A, \lambda^A\) carry a unit \(U (1)_{elec} \times U (1)_{mag}\) charge; and interact with the gauge vector fields \(A^M_\mu\) through the minimal coupling \(D_\mu F^A\) where the covariant derivative \(D_\mu = \partial_\mu + \vartheta_\mu A^M_\mu\) with electric/magnetic coupling \(\vartheta_\mu\); see [31, 32, 40, 41, 44–46] for other features.

In (3.2), we have moreover an exotic variable \(\tau\), playing the role of the real time \(t\) of the left hand side of the table. This \(\tau\) may be imagined in terms of energy scale variable; and hence one is left with running couplings \(\vec{\nu} = \vec{\nu} (\tau)\) and \(\vec{m} = \vec{m} (\tau)\) with

\[ \vec{m} (\tau) \sim \frac{d\vec{\nu} (\tau)}{d\tau} \leftrightarrow \vec{p} (t) \sim \frac{d\vec{\tau} (t)}{dt} \] (3.7)

In what follows, we assume that the classical correspondence (3.2) is valid as well at quantum level and study the energy band properties of the isospin \(\frac{1}{2}\) particles (gravitinos and gauginos) of the \(\mathcal{N} = 2\) gauged supergravity.

### 3.1. Deriving the free Hamiltonian of iso-particle

Here, we use (3.2) to build the free Hamiltonian \(h = h(\nu, m)\) of the iso-particle and study its classical and quantum behaviours. We also comment on some interacting terms appearing in the scalar potential (2.19).
3.1.1. Classical description

Using the proposal (3.2), the free hamiltonian $h$ of the classical iso-particle is given by the scalar potential of the supergravity theory. It is just the energy density of the supergravity theory,

$$h = V(\vec{\nu}, \vec{m})$$  \hspace{1cm} (3.8)

Because this energy is quadratic in $\vec{m}$ and $\vec{\nu}$ as shown by the rigid limit of [28], the $h$ describes then the free dynamics of a classical iso-particle in the 6D phase space $\tilde{\mathbb{R}}^3 \times \hat{\mathbb{R}}^3$ parameterised by the FI coupling parameters. By using eqs (2.17,2.18), we have

$$h = \alpha |\vec{m}|^2 + \beta |\vec{\nu}|^2 + \frac{\gamma}{2} |\vec{m}| \times |\vec{\nu}|$$  \hspace{1cm} (3.9)

where $\alpha$, $\beta$ and the planar $\gamma_{\parallel}$ are three real parameters that have an interpretation in the special geometry of the scalar manifold of the $\mathcal{N} = 2$ effective theory. Here, they will be given an interpretation in terms of an effective mass $\mu$ and a frequency $\omega$ with relationships an in (3.17) and (3.22). Notice the following useful features:

- The above $h$ has the form of a classical harmonic oscillator energy $\frac{p^2}{2M} + \frac{M \omega^2}{2} x^2$; so one can take advantage from this feature to learn more about the properties of the iso-particle of the $\mathcal{N} = 2$ gauged supergravity.

- The notation $\gamma_{\parallel}$ in (3.9) is to distinguish it from another contribution $\gamma_{\perp}$ to be turned on later when switching on $\vec{\xi}$. By using the two types of vector products, a general quadratic term like $|\vec{m}| \times |\vec{\nu}|$ has in general the typical form

$$\frac{\gamma}{2} |\vec{m}| \times |\vec{\nu}| = \frac{\Lambda}{2} \vec{m}.\vec{\nu} + \frac{\Lambda'}{2} ||\vec{m} \wedge \vec{\nu}||$$  \hspace{1cm} (3.10)

showing that $\frac{\gamma}{2}$ may come from two sources: (i) from a scalar product like $\frac{\Lambda}{2} \vec{m}.\vec{\nu}$; and/or (ii) from the norm of the wedge product of the two vectors as follows

$$\frac{\Lambda}{2} \vec{m}.\vec{\nu} = \frac{\gamma_{\parallel}}{2} |\vec{m}| \times |\vec{\nu}|$$  \hspace{1cm} (3.11)

$$\frac{\Lambda'}{2} ||\vec{m} \wedge \vec{\nu}|| = \frac{\gamma_{\perp}}{2} |\vec{m}| \times |\vec{\nu}|$$  \hspace{1cm} (3.12)

- In order to fix a freedom in the signs of $\alpha$, $\beta$, $\gamma_{\parallel}$; we assume that the discriminant of the $G_{ij}$ metric of (3.9) is positive definite,

$$\det G_{ij} = \alpha \beta - \frac{\gamma_{\parallel}^2}{4} > 0$$  \hspace{1cm} (3.13)
As this discriminant is non sensitive to \((\alpha, \beta, \gamma) \rightarrow (-\alpha, -\beta, -\gamma)\); we restrict \(\alpha, \beta\) to be both positive; this constraint is also needed for \(h\) in order to be bounded from below; this an important condition for quantisation of fluctuation of couplings.

- The omission of the zero value in \(\det G\) is because for \(\alpha \beta - \frac{\gamma^2}{4} = 0\), the above hamiltonian \((3.9)\) reduces to

\[
h_{\eta} = Z_{\eta}^2 \quad \text{with} \quad Z_{\pm}^2 = (|\vec{m}| \sqrt{\alpha \pm |\vec{v}|} \sqrt{\beta})^2
\]

ruling out harmonic oscillations needed for quantum fluctuations; see eq\((3.22)\). Nevertheless, the saturated limit captures as well an interesting data; it will be discussed in subsection 4.2.

With these features in mind, we are now in position to deal with the hamiltonian \((3.9)\). To that purpose, we perform a linear change of variables \((|\vec{m}|, |\vec{v}|) \rightarrow (|\vec{m}'|, |\vec{v}'|)\) in order to put \(h\) into a normal form as follows

\[
h_0 = \frac{1}{2\mu} \sum_{a=1}^{3} (m'_a)^2 + \frac{\kappa}{2} \sum_{a=1}^{3} (\nu'_a)^2
\]

where now \(\frac{1}{2\mu} \vec{m}'^2\) stands for ”kinetic energy” and \(\frac{\kappa}{2} \vec{v}'^2\) for the ”potential energy”. The new \(|\vec{m}'|, |\vec{v}'|\) are related to the old \(|\vec{m}|, |\vec{v}|\) ones by

\[
|\vec{m}'| = a |\vec{v}| + b |\vec{m}| \quad , \quad |\vec{v}'| = c |\vec{v}| + d |\vec{m}|
\]

with \(ad - bc = 1\), that diagonalise the metric in \((2.18)\). The resulting positive mass \(\mu\) and \(\kappa = \mu \omega^2\) (oscillation frequency) are functions of the \(\alpha, \beta, \gamma\) parameters; their explicit expressions read as follows

\[
\frac{1}{\mu} = \alpha + \beta + \sqrt{(\alpha - \beta)^2 + \gamma^2} \quad , \quad \kappa = \alpha + \beta - \sqrt{(\alpha - \beta)^2 + \gamma^2}
\]

Notice that using the condition \(\gamma^2 < 4\alpha\beta\) and positivity of \(\alpha\) and \(\beta\), we have \((\alpha - \beta)^2 + \gamma^2 < (\alpha + \beta)^2\) and then \(\kappa > 0\). Notice also the following properties:

- The saturated value \((\gamma^2)_{\text{max}} = 4\alpha\beta\); then \((\det G_{ij})_{\text{max}} = 0\) and \((\kappa)_{\text{min}} \rightarrow 2 (\alpha + \beta) \omega_{\text{min}}^2 = 0\).

- Classically, the hamiltonian \((3.15)\) is positive and bounded from below,

\[
h \geq h_0 \quad , \quad h_0 = 0
\]
This vanishing lower value \( h_0 = 0 \) is important in the study of \( \mathcal{N} = 2 \) gauged supergravity in the rigid limit; since \( \langle V_{\text{class}} \rangle = h_0 = 0 \) corresponds to an exact \( \mathcal{N} = 2 \) rigid supersymmetric phase. This property requires \( \vec{m} = \vec{\nu} = \vec{0} \).

- By restricting \( \vec{m} \) and \( \vec{\nu} \) to the particular choice eq\((2.8)\), the free eq\((3.15)\) reduces to the hamiltonian of a 1-dimensional harmonic oscillator,

\[
h^{(1D)} = \frac{1}{2\mu} \left( m'_y \right)^2 + \frac{\kappa}{2} \left( \nu'_x \right)^2
\]

\((3.19)\)

In what follows, we use this simple expression to study harmonic fluctuations of the FI couplings around the supersymmetric vacuum \( m'_y = \nu'_x = 0 \).

3.1.2. Quantum effect

The free iso-particle studied above is classical. However, like spin \( s = \frac{1}{2} \) fermions in the real 3d space, the iso-particle has also intrinsic degrees of freedom namely an isospin \( I = \frac{1}{2} \), as shown on table (3.5), and a unit electric/magnetic charge \( \vartheta \) given by (3.6).

Assuming the classical correspondence (3.2) to also hold at the quantum level in the iso-space \( \hat{\mathbb{R}}^3 \), it follows that the fluctuations of the FI couplings may also be governed by \( |\Delta \vec{m}'| \times |\Delta \vec{\nu}'| \gtrsim \hbar \) in same manner as for the usual Heisenberg uncertainty \( |\Delta x| \times |\Delta p_x| \gtrsim \hbar \) which is expressed in terms of the usual phase space coordinates \((\vec{r}, \vec{p})\). If one accepts this assumption; then we cannot have exact \( m'_y = \nu'_x = 0 \) since \( |\Delta \nu'_x| \times |\Delta m'_y| \nless \hbar \); and so one expects \( \mathcal{N} = 2 \) supersymmetry in rigid limit to be broken by quantum effect since ground state energy is now positive definite

\[
\left\langle h^{(1D)}_{\text{quant}} \right\rangle > 0
\]

\((3.20)\)

In what follows, we restrict our study to exhibiting this quantum behaviour and to checking the breaking \( \mathcal{N} = 2 \rightarrow \mathcal{N} = 0 \). We will return to study this feature in subsection 4.2 when isospin-orbit coupling is switched on. There, we will also give details on the condition to have partial breaking \( \mathcal{N} = 2 \rightarrow \mathcal{N} = 1 \).

The quantum effect, due to fluctuations of \( \vec{m} \) and \( \vec{\nu} \) around the supersymmetric ground state \( h_0 = \langle V \rangle \), is induced by quantum isotropic oscillations with discrete energy \( \varepsilon^{\parallel}_{(n_x, n_y, n_z)} = \epsilon^{\parallel}_{n_x} + \epsilon^{\parallel}_{n_y} + \epsilon^{\parallel}_{n_z} \) and fundamental oscillation frequency

\[
\omega^{\parallel} = \sqrt{\frac{\kappa}{\mu}}
\]

\((3.21)\)
By using (3.17), we have,
\[ \omega_{\parallel}^2 = 4\alpha\beta - \gamma_{\parallel}^2 \]  
(3.22)

Observe that because of the minus sign, this \( \omega_{\parallel} \) vanishes for those parameters \( \alpha, \beta, \gamma_{\parallel} \) satisfying the degenerate condition \( \gamma_{\parallel}^2 = 4\alpha\beta \) which has been ruled out by the constraint eq(3.13). To illustrate the quantum effect for \( \omega_{\parallel} > 0 \), we consider the particular choice (2.8) bringing (3.15) to a 1-dimensional quantum oscillator with hamiltonian operator as,
\[ H_{(1D)} = \frac{\hbar}{2} \left( \left( \hat{m}_y \sqrt{\mu \omega_{\parallel}} \right)^2 + (\hat{p}_x \sqrt{\mu \omega_{\parallel}})^2 \right) \]  
(3.23)

It has a diagonal form \( \frac{\hbar \omega_{\parallel}}{2} (Y^2 + X^2) \), which by setting \( A = \frac{X + Y}{\sqrt{2}} \), reads as usual like
\[ H_{(1D)} = \hbar \omega_{\parallel} \left( A^\dagger A + \frac{1}{2} \right) \]  
(3.24)

with \( AA^\dagger - A^\dagger A = I \). The energy spectrum \( \epsilon_{\parallel}^{(n_x,n_y,n_z)} \) reduces to
\[ \epsilon_{\parallel} = \hbar \omega_{\parallel} \left( n + \frac{1}{2} \right) \geq \epsilon_0 \]  
(3.25)

with frequency \( \omega_{\parallel} \) given by (3.22). The lowest energy value is given by \( \epsilon_0 = \frac{\hbar \omega_{\parallel}}{2} \); it is non zero for non vanishing frequency \( \omega_{\parallel} \). Hence, the exact \( \mathcal{N} = 2 \) supersymmetry living at classical vacuum \( \langle \mathcal{V}_{\text{class}} \rangle = 0 \) gets completely broken by quantum effect
\[ \langle \mathcal{V}_{\text{quant}} \rangle = \frac{\hbar \omega_{\parallel}}{2} > 0 \]  
(3.26)

We end this subsection by giving two brief comments on interactions. The first interacting potential energy has the linear expression in \( \vec{m} \),
\[ h_{\text{int}}^{(R^3)} = -q\vec{m} \cdot \vec{A} \]  
(3.27)

and concerns the electric \( U(1)_{\text{elec}} \) gauge charge. This is a subgroup of the electric/magnetic \( U(1)_{\text{elec}} \times U(1)_{\text{mag}} \) local symmetry of the \( \mathcal{N} = 2 \) gauged supergravity induced by gauging two abelian isometries in the scalar manifold of the supergravity theory. The second interacting potential energy is given by the isospin-orbit coupling \( h_{\text{loc}}^{(R^3)} = \xi \vec{I} \cdot \vec{A} \) we are too particularly interested in here; it will be considered with details in the next subsection.

Regarding (3.27), it is derived by making two steps as follows: First, start from the interaction energy \( h_{\text{int}}^{(R^3)} = -e\vec{p} \cdot \vec{A} \) of an electrically charged particle with momentum \( \vec{p} \) moving in the presence of an external magnetic field \( \vec{B}_{\text{ext}} = \vec{\nabla} \wedge \vec{A} \). Then, use the correspondence (3.2) allowing to imagine \( -e\vec{p} \) in terms of the FI magnetic vector \( -q\vec{m} \) and the \( \vec{A} \) by an iso-vector \( \vec{\nabla} \). The obtained (3.27) describes just the term \( w_a m^a \) in eq(2.19) from which we learn that \( \vec{w} = -q\vec{A} \).
3.2. Isospin- orbit coupling

The proposal (3.2) has been useful for the physical interpretation of the rigid Ward identity in terms of an iso-particle hamiltonian with phase space coordinates (⃗ν, ⃗m). Thanks to the Andrianopoli et al formula ⃗ξ = ⃗ν ∧ ⃗m giving the orbital momentum of this iso-particle. Thanks also to the structure of the scalar potential 𝑉 which turns out to be nothing but its the free hamiltonian ℏ (3.15). In this subsection, we derive the isospin-orbit coupling

\[ ℏ_{ioc} = \vec{ξ} \cdot \vec{I} \] (3.28)

where ⃗I stands for the isospin vector and \( \vec{ξ} = \vec{ν} \wedge \vec{m} \). To that purpose, recall that in eq(2.1), the rigid \( C - \) anomaly matrix appears in the form of a hermitian traceless 2×2 matrix

\[ C = \begin{pmatrix} \xi_z & \xi_x - i\xi_y \\ \xi_x + i\xi_y & -\xi_z \end{pmatrix} = \vec{ξ} \cdot \vec{τ} \] (3.29)

that reads in terms of the \( \vec{τ} - \) Pauli matrices and the Andrianopoli et al orbital vector as follows

\[ C = (\vec{ν} \wedge \vec{m}) \cdot \vec{τ} \] (3.30)

This factorised form of \( C \) teaches us that it can be imagined as describing the coupling of two things namely the orbital isovector \( \vec{ξ} = \vec{ν} \wedge \vec{m} \) and the isospin vector

\[ \vec{I} = \frac{\vec{τ}}{2} \] (3.31)

In what follows, we give two other different, but equivalent, manners to introduce \( \vec{ξ} \cdot \vec{I} \). The first way relies on comparing \( ℏ_{ioc} = \vec{ξ} \cdot \vec{I} \) with the usual spin-orbit coupling \( ℏ_{soc} = \vec{L} \cdot \vec{S} \) of a particle with spin \( \vec{S} = \frac{\vec{τ}}{2} \) moving in real space \( \mathbb{R}^3 \) with coordinate vector \( \vec{r} \). The second manner extends the approach done in previous section for deriving the free hamiltonian (3.15) by including the isospin effect.

By comparing the effect of the spin orbit coupling \( \vec{L} \cdot \vec{S} \) in electronic systems and the effect of \( \vec{ξ} \cdot \vec{I} \) in partial breaking of \( \mathcal{N} = 2 \) supersymmetry; and by following [28, 29], we learn that when the central extension matrix is turned off, i.e: \( C = 0 \), then \( \mathcal{N} = 2 \) supersymmetry is preserved (two gapless gravitinos). However, it can be partially broken when it is turned on i.e: \( C \neq 0 \). This property can be viewed in terms of a non zero gap energy \( E_g \) between the two fermionic iso-doublets; including the two charges \( Q_L, Q_R \) of \( \mathcal{N} = 2 \) supersymmetry with expression as

\[ E_g \propto |\vec{ξ}| \] (3.32)
This is exactly what happens for the case of two states of spin $\frac{1}{2}$ fermions in electronic condensed matter systems when the spin-orbit coupling $\vec{L}.\vec{S}$ is taken into account. This $\vec{L}.\vec{S}$ coupling is known to open the zero gap between the two states of free electrons. From this link with electronic properties, we deduce a correspondence between the central matrix $C$ of the $\mathcal{N} = 2$ supercurrent algebra and the hamiltonian $h_{soc} = \vec{L}.\vec{S}$. This link reads explicitly like

$$\vec{\xi}.\vec{I} \leftrightarrow \vec{L}.\vec{S}$$

(3.33)

where the isospin $\vec{I}$ plays the role of the spin $\vec{S}$; and the Andrianopoli et al vector $\vec{\xi}$ the role of the angular momentum $\vec{L}$. Adding the isospin-orbit coupling term to the free (3.15), we get $H = \mathcal{V} + \vec{\xi}.\vec{I}$ that reads explicitly as follows

$$H = \frac{1}{2\mu}m^2 + \frac{\kappa}{2}r^2 + \vec{\xi}.\vec{I}$$

(3.34)

In matrix form, we have

$$H = \begin{pmatrix}
\mathcal{V} + \xi_z & \xi_x - i\xi_y \\
\xi_x + i\xi_y & \mathcal{V} - \xi_z
\end{pmatrix}$$

(3.35)

with eigenvalues: $E_{\pm} = \mathcal{V} \pm \sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2}$ and eigenstates as

$$|\eta_{\pm}\rangle \sim \begin{pmatrix}
\xi_z \pm \sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2} \\
\xi_x + i\xi_y
\end{pmatrix}$$

(3.36)

The second manner to introduce (3.28) is a purely algebraic approach. The key idea relies on thinking of the free energy density term of the two $I_z = \pm \frac{1}{2}$ isospin states as

$$h_B^A = \mathcal{V}\delta_B^A$$

(3.37)

To each of the $I_z = \pm \frac{1}{2}$ states, we have used (3.15) to derive its free hamiltonian; but this result is just the diagonal term of a general hamiltonian matrix $H$. The extension of $\mathcal{V}\delta_B^A$ to have more interactions is then naturally given by the Ward identity (2.1)

$$H_B^A = \mathcal{V}\delta_B^A + C_B^A$$

(3.38)

which is nothing but the right hand side of the $\mathcal{N} = 2$ supersymmetric current algebra (2.4) including the central matrix.

### 3.3. Discrete symmetries

From the rigid Ward identity of Andrianopoli et al (2.1), we also learn that exact $\mathcal{N} = 2$ supersymmetry requires $\vec{\xi} = \vec{0}$; no isospin-orbit coupling in our modeling. But this vanishing
value is just the fix point of the $\mathbb{Z}_2$ discrete symmetry acting on the anomaly isovector as follows

$$\mathbb{Z}_2 : \quad \vec{\xi} \rightarrow -\vec{\xi} \quad (3.39)$$

To figure out the meaning of this discrete transformation, we use eq(2.6) from which we learn that the minus sign can be generated in two manners; either by the change $(\vec{\nu}, \vec{m}) \rightarrow (-\vec{\nu}, \vec{m})$; or by $(\vec{\nu}, \vec{m}) \rightarrow (\vec{\nu}, -\vec{m})$. To derive the physical interpretation to these two kinds of $\mathbb{Z}_2$ discrete symmetries, we use the analogy between the FI couplings $(\vec{\nu}, \vec{m})$ and the classical phase coordinates $(\vec{r}, \vec{p})$. Promoting this correspondence to dynamical (running) couplings; say

$$\vec{r}(t) \leftrightarrow \vec{\nu}(\tau) \quad \vec{p}(t) \leftrightarrow \vec{m}(\tau) \quad (3.40)$$

it follows that the transformation (3.39) corresponds for example to the usual time reversing symmetry $T$ which maps the position $\vec{r}(t)$ and momentum $\vec{p}(t)$ respectively to $\vec{r}(-t)$ and $-\vec{p}(-t)$. On the side of the FI couplings, we then have the following action of the $T$-analogue on isotime $\tau$,

$$T : \quad \vec{\nu}(\tau) \rightarrow -\vec{\nu}(-\tau) \quad \vec{m}(\tau) \rightarrow -\vec{m}(-\tau) \quad (3.41)$$

Notice that the usual space parity $P$ which maps the $(\vec{r}, \vec{p})$ phase coordinates to $(-\vec{r}, -\vec{p})$ allows us, by using the $(\vec{r}, \vec{p}) \leftrightarrow (\vec{\nu}, \vec{m})$ correspondence, to write

$$P : \quad \vec{\nu}(\tau) \rightarrow -\vec{\nu}(\tau) \quad \vec{m}(\tau) \rightarrow -\vec{m}(\tau) \quad (3.42)$$

But this discrete $P$- transformation leaves $\vec{\xi} = \vec{\nu} \wedge \vec{m}$ invariant and so it is not relevant for partial breaking. However, the combined $PT$ transformation which acts like

$$PT : \quad \vec{\nu}(\tau) \rightarrow -\vec{\nu}(-\tau) \quad \vec{m}(\tau) \rightarrow +\vec{m}(-\tau) \quad (3.43)$$

does affect the sign of $\vec{\xi}$. This combination can be also used to think about the $\mathbb{Z}_2$ transformation (3.39). Actually, it corresponds to the second possibility to realize $\vec{\xi} \rightarrow -\vec{\xi}$ from (2.6). Therefore, exact $\mathcal{N} = 2$ supersymmetry, which corresponds to $\vec{\xi} = \vec{0}$, lives at the fix point of the $T$- reversing time transformation (3.39); or at the combined $PT$ given by (3.43); or both.
4. TOPOLOGICAL ASPECTS AND QUANTUM EFFECT

In this section, we first study the topological behaviour of gapless iso-particles of exact \( \mathcal{N} = 2 \) supersymmetry as well as the gapless chiral ones that remain after partial breaking. Then, we study the effect of quantum fluctuations on partial supersymmetry breaking.

4.1. Chiral anomaly

Setting \( H_{AB} = \sum_i \delta_B \lambda^C i \delta^A \lambda^C \), we can turn the rigid Ward identities (2.1) into the matrix equation

\[
H_{AB} = V \delta^A_B + C^A_B
\]

which is nothing but the Hamiltonian matrix (3.34). Multiplying both sides of this 2×2 matrix relation by \( \eta_A = (\eta_1, \eta_2)^T \) describing the two states of the isoparticle, we end with the eigenvalue equation \( H \eta = E \eta \) whose two eigenvalues are given by \( E_\pm = V \pm |\xi| \); the eigenstates \( \hat{\eta}_\pm \) associated with these \( E_\pm \) are linear combinations of \( \eta_1 \) and \( \eta_2 \), they read like

\[
\hat{\eta}_\pm = A_\pm \eta_1 + B_\pm \eta_2
\]

with amplitudes \( A_\pm \) and \( B_\pm \) as follows

\[
A_\pm = \frac{\xi_\pm \pm |\xi|}{\sqrt{2 (\xi_\pm - |\xi|)}} \quad , \quad B_\pm = \frac{\xi_\pm + i \xi_y}{\sqrt{2 (\xi_\pm - |\xi|)}}
\]

The determinant \( \text{det} H = \Delta \) that captures data on the singular points in the \( (V, |\xi|) \) plane is given by the product of the eigenvalues \( E_\pm \), it reads as

\[
\Delta = \left( V + |\xi| \right) \left( V - |\xi| \right)
\]

It is a function of two real quantities namely \( V \) and \( |\xi| \); but here below we will treated it as a parametric function of one variable like \( \Delta_\zeta (x) \). The choice of the variable \( x \) depends on the property we are interested to exhibit; see figure [2]. From the view of the scalar potential energy, the variable is given by \( x = V \); while \( \zeta = |\xi| \) is seen as a free parameter. From the view of the \( \vec{\xi} \) vector, we have the reverse picture; \( x = |\xi| \) is the variable while \( \zeta = V \) stands for a free parameter. In the first image, \( \text{det} H \) has two zeros at \( V_\pm = \pm |\xi| \); one positive \( V_+ \), that is visible in global supersymmetry sector; and a hidden negative \( V_- \).

In the second picture, the discriminant \( \text{det} H \) has zeros at \( |\xi_\pm| = +V \) for positive \( V \); and \( |\xi_-| = -V \) for negative \( V \). Let us express these two zeros in \( \mathbb{R}^3 \) like \( \vec{\xi}_\pm = \pm V \vec{n} \) with unit
FIG. 2: On left, the discriminant $\Delta_\xi (V)$ as a function of $V$ and parameter $|\xi|$. For $\xi \neq 0$; there are two zeros at $V = \pm |\xi|$; one visible in rigid limit. At each zero, say $V = |\xi|$; it lives a chiral gapless corresponding to a partially broken $\mathcal{N} = 2$ supersymmetric state. In the limit $\xi \to 0$, the two chiral gapless modes at $V = \pm |\xi|$ collide at the origin and form together a gapless isodoublet. On right, the $\Delta_\xi (\xi)$ as a function of $\xi$. For non zero $V$; chiral gapless mode live at each $\xi = \pm V \vec{n}$ merging for $V = 0$.

vector $\vec{n} = |\xi|$. The effective gap energy $E_g = E_+ - E_-$ between the two $E_\pm$ energy density bands is given by

$$E_g = 2 |\xi|$$  \hspace{1cm} (4.4)

It vanishes for $|\xi| = 0$ and then for $\xi = 0$. Because of the property $|\xi| \geq 0$, the zeros of det $H$ are of two kinds: simple for $|\xi| > 0$ and double for $|\xi| = 0$. At each simple zero lives a gapless fermionic mode (gravitino and gaugino) and a gapped one. For $|\xi_+| = + V$ with positive energy density $V$, we have the conducting band; and for $|\xi_-| = - V$ with negative $V$, we have the valence band. Notice that det $H = V^2 - \xi^2$ is conserved under\(^3\) the discrete change,

$$Z_2 : V \to - V \quad \Rightarrow \quad \xi_+ \to - \xi_-$$  \hspace{1cm} (4.5)

Its two zeros $V_\pm = \pm |\xi|$ are not fix points of $Z_2$ except the origin; they are interchanged as shown by (4.5); for instance properties at $\xi_-$ may be deduced from those at $\xi_+$.

Now, let us approach det $H$ from the view of the iso-space vector $\xi$; and consider the 2-spheres $S^2_+ V$ and $S^2_- V$, with a surface normal to $\vec{n}$, surrounding respectively the zeros

$$\xi_\pm = \pm V \vec{n}$$  \hspace{1cm} (4.6)

\(^3\) from charged particles view, this mapping from valence to conducting like bands and vice versa may be imagined as a CT transformation combining time reversing $T$ and charge conjugation $C$. 
The 2-sphere \( S_{+v}^2 \) is described by the vector \( \vec{p} = \vec{\xi} - \vec{\xi}_+ \) and the \( S_{-v}^2 \) by \( \vec{q} = \vec{\xi} - \vec{\xi}_- \). These 2-spheres should not be confused with the unit 2-sphere \( S_n^2 \): 

\[
S_n^2 : n_x^2 + n_y^2 + n_z^2 = 1
\]

(4.7)

associated with the unit vectors (2.13); but the three \( S_{+v}^2, S_{-v}^2, S_n^2 \) live all of them in the iso-space \( \tilde{\mathbb{R}}^3 \); and are related to each other by continuous mappings like,

\[
\pi_+ : S_{+v}^2 \to S_n^2, \quad \pi_- : S_{-v}^2 \to S_n^2
\]

(4.8)

Focussing for instance on \( S_{+v}^2 \), the continuity of \( \pi_+ \) shows that it has a winding \( w(S_{+v}^2) \) describing the net number of times \( S_{+v}^2 \) wraps the unit sphere \( S_n^2 \); the integer number \( w(S_{+v}^2) \) reflects just the mathematical property \( \pi_2(S^2) \cong \mathbb{Z} \). A similar thing can be said about \( S_{-v}^2 \); thanks to \( \mathbb{Z}_2 \) parity (4.5) under which gauge curvature \( F \) of underlying Berry connection \( A \) is odd; see (4.11) given below.

Moreover, each one of gapless state at the two zeros \( \vec{\xi}_\pm = \pm \mathcal{V} \vec{n} \) is anomalous in the sense that it has one gapless chiral mode and then violates the Nielson-Ninomiya theorem [17, 42, 43]. Recall that in theories that are free from chiral anomalies, the usual Nielson-Ninomiya theorem [17, 42] states that the sum of winding numbers \( w(S_i^2) \) around 2-spheres \( S_i^2 \), surrounding the \( \vec{\xi}_s \) zeros where live gapless modes, vanishes identically. Here, this statement reads explicitly like

\[
\sum_i w(S_i^2) = \sum_i \int_{S_i^2} \frac{\text{Tr}(F)}{2\pi} = 0
\]

(4.9)

where \( F \) is a gauge curvature whose explicit expression will be given below. For positive \( \mathcal{V} \), eq(4.3) has one zero given by an outgoing \( \vec{\xi}_+ = + \mathcal{V} \vec{n} \) with positive sense in normal \( \vec{n} \) direction; and then a 2-sphere \( S_{+v}^2 \) surrounding the point \( \vec{\xi}_+ = (\xi_+, \xi_{++}, \xi_+) \) has a positive winding number

\[
w(S_{+v}^2) = \int_{S_{+v}^2} \frac{\text{Tr}(F)}{2\pi} = 1
\]

(4.10)

Here, the curvature \( F \) is given by the following rank 2 antisymmetric tensor,

\[
F_{ab} = \frac{1}{2} \vec{n} \cdot \left( \frac{\partial \vec{n}}{\partial \xi^a} \wedge \frac{\partial \vec{n}}{\partial \xi^b} \right)
\]

(4.11)

The Nielsenn-Ninomiya theorem is then violated due to the existence of one gapless chiral moving mode; and so the partially broken theory has a chiral anomaly; only one of the two supersymmetric charges \( (\bar{Q}_L, \bar{Q}_R) \); say the right \( \bar{Q}_R \) is preserved; the left \( \bar{Q}_L \) is broken. For the incoming \( \vec{\xi}_- = - \mathcal{V} \vec{n} \), we have a negative winding number

\[
w(S_{-v}^2) = \int_{S_{-v}^2} \frac{\text{Tr}(F)}{2\pi} = -1
\]

(4.12)
This negative value follows from the mapping \( \vec{n} \to -\vec{n} \), due to eq(4.5), and using (4.11). For the special case where the VEV of the scalar potential vanish, \( \mathcal{V} = 0 \), the discriminant of the matrix (4.3) reduces to \( \det H = -\left| \vec{\xi} \right|^2 \) and its zero \( \left| \vec{\xi} \right| = 0 \) has a multiplicity 2. In this case, the Nielson-Ninomiya theorem reads as

\[
w \left( S_+^2 \right) + w \left( S_-^2 \right) = 1 - 1 = 0 \tag{4.13}
\]

At the fix point of the transformation (4.5), the two zeros collide at \( \left| \vec{\xi}_\pm \right| = 0 \). Then, the two effective gravitino zero modes with opposite chiralities form a massless doublet (a massless iso-particle) and \( \mathcal{N} = 2 \) supersymmetry gets restored.

| zeros of \( \det H \) | multiplicity of zeros | winding number | conserved SUSY charges |
|----------------------|----------------------|---------------|-----------------------|
| \( \left| \vec{\xi}_+ \right| = +\mathcal{V} \) | 1 | +1 | \( \hat{Q}_+ \) |
| \( \left| \vec{\xi}_- \right| = -\mathcal{V} \) | 1 | -1 | \( \hat{Q}_- \) |
| \( \left| \vec{\xi}_\pm \right| = 0 \) | 2 | 0 | \( \left( \hat{Q}_+ \right) \) |

(4.14)

4.2. Quantum fluctuation

Here, we study quantum fluctuations in the FI couplings around the partial breaking vacuum \( \langle \mathcal{V} \rangle = \left| \vec{\xi} \right| \) and comment on their effect by using the special choice (2.5). To that purpose, we use \( |\Delta \vec{m}'| \times |\Delta \vec{v}'| \sim h \) to promote the matrix equation (4.1) into an effective quantum eigenvalue matrix equation \( H |\eta\rangle = E |\eta\rangle \) that we split into two eigenvalues equations as follows

\[
H_+ |\eta_+\rangle = E_+ |\eta_+\rangle \\
H_- |\eta_-\rangle = E_- |\eta_-\rangle
\tag{4.15}
\]

In these relations, we have \( H_\pm = \hat{\mathcal{V}} \pm \hat{\vec{\xi}} \) where the hatted \( \hat{\mathcal{V}} \) and \( \hat{\vec{\xi}} \) refer to the quantised operators associated with \( \mathcal{V} \) and \( \left| \vec{\xi} \right| \) expressed in terms of the phase space vectors \( \vec{m} \) and \( \vec{v} \). For the particular FI coupling choice (2.8), we have \( |\Delta m'_y| \times |\Delta v'_x| \sim h \) and find, after repeating the steps between eqs(3.9) and (3.24), the two following quantum 1D- hamiltonians

\[
H^{(1D)}_\pm = \hbar \omega_\pm \left( A^\dagger A + \frac{1}{2} \right) \tag{4.16}
\]

describing two oscillators with different frequencies \( \omega_\pm \). Their energies are given by

\[
\epsilon_n^\pm = \hbar \omega_\pm \left( n + \frac{1}{2} \right) \quad \text{with}
\]

\[
\omega_\pm^2 = 4\alpha\beta - \left( \gamma_\parallel \pm \gamma_\perp \right)^2 \tag{4.17}
\]
with the remarkable minus sign. Notice that imposing the constraint (3.13) to both $|\eta_\pm\rangle$ eigenstates, we have

$$\alpha\beta - \frac{(\gamma_\parallel - \gamma_\perp)^2}{4} \geq 0, \quad \alpha\beta - \frac{(\gamma_\parallel + \gamma_\perp)^2}{4} \geq 0$$

(4.18)

leading to

$$0 \leq (\gamma_\parallel - \gamma_\perp)^2 \leq 4\alpha\beta, \quad 0 \leq (\gamma_\parallel + \gamma_\perp)^2 \leq 4\alpha\beta$$

(4.19)

and then to

$$-\alpha\beta \leq \gamma_\parallel \gamma_\perp \leq \alpha\beta$$

(4.20)

For the case where one of the bounds of the constraint eqs(4.19) is saturated; for example the upper bound of the squared deviation $(\gamma_\parallel - \gamma_\perp)^2 \leq 4\alpha\beta$ is saturated, we can fix one of the four parameters in terms of the three others like

$$(\gamma_\parallel - \gamma_\perp)^2 = 4\alpha\beta \quad \Rightarrow \quad \gamma_\perp = \gamma_\parallel \pm 2\sqrt{\alpha\beta}$$

(4.21)

By substituting back into (4.17), we end with the two energy spectrums: First, $\epsilon^-_n = \hbar\omega_-^\text{sat} (n + \frac{1}{2})$ with

$$\omega_-^\text{sat} = 4\alpha\beta - (\gamma_\parallel - \gamma_\perp)^2 = 0$$

(4.22)

describing gapless iso-particles (gravitinos/gauginos) with $E_- = 0$. This corresponds to the ground state $\langle V \rangle = |\xi\rangle$ where partial breaking takes place. Second, $\epsilon^+_n = \hbar\omega_+^\text{sat} (n + \frac{1}{2})$ with

$$(\omega_+^\text{sat})^2 = 4\alpha\beta - (\gamma_\parallel + \gamma_\perp)^2 = -4\gamma_\parallel \gamma_\perp > 0$$

(4.23)

They describe a gapped iso-particle. Thus, along with the gapless modes ($\omega_-^\text{sat} = 0$), we have gapped states with harmonics $n\omega_+^\text{sat}$. The $\epsilon^+_n$ energies are bounded as

$$\epsilon^+_n \geq \epsilon^+_0 = \frac{1}{2}\hbar\omega_+^\text{sat} > 0$$

(4.24)

with ground state energy $\epsilon^+_0$ corresponding to the classical $E_+ = 2 |\xi\rangle$; which is also the gap energy between the two polarisations of the iso-particle. As a conclusion of this subsection, quantum fluctuations in the FI coupling space with $\gamma_\perp = \gamma_\parallel \pm 2\sqrt{\alpha\beta}$ do not destroy the partial breaking supersymmetry of Andrianopoli et al rigid limit; this property holds for the saturated condition (4.21); otherwise quantum corrections break as well the residual $\mathcal{N} = 1$ supersymmetry.
5. CONCLUSION

In this paper, we have used results on topological band theory of usual $s = \frac{1}{2}$ matter to study partial breaking of $\mathcal{N} = 2$ gauged supergravity in rigid limit. By using supergravity Ward identities and results from $[28]$ and $[17, 19, 31]$, we have derived a set of interesting informations on band structure of gravitinos and gauginos in $\mathcal{N} = 2$ theory. Part of these information have been obtained from the proposal (3.2); and its quantum extension that we rephrase here below:

(i) the interpretation of the Andrianopoli realisation $\vec{\xi} = \vec{\nu} \wedge \vec{m}$ as an angular momentum vector of a quasi-particle with phase space coordinates $(\vec{\nu}, \vec{m})$ allowed us to think of the two gravitinos and the two gauginos in terms of classical isospin $\frac{1}{2}$ particles (iso-particles) charged under $U(1)_{\text{elec}} \times U(1)_{\text{mag}}$ gauge symmetry. As a consequence of this observation, the scalar potential $\mathcal{V}$ has been interpreted as the Hamiltonian (3.8-3.9) of free iso-particle and the central extension of the $\mathcal{N} = 2$ supercurrent algebra (2.4) as describing isospin-orbit coupling $\vec{\xi} \cdot \vec{I}$. This isospin-orbit interaction is the homologue of the usual spin-orbit coupling $\vec{L} \cdot \vec{S}$ in electronic systems of condensed matter. The proposal (3.2) allowed us also to derive two discrete symmetries $T$ and $TP$ capturing data on partial breaking of $\mathcal{N} = 2$ supersymmetry; see subsection 3.3 for details. Exact $\mathcal{N} = 2$ lives at the fix point of these symmetries. In summary, we can say that the classical properties of the isoparticle is given by the $\mathcal{N} = 2$ supersymmetric current algebra (2.3).

(ii) By using Nielson-Ninomiya theorem, we have developed the study of the topological property of fermionic gapless states given by zeros of the discriminant (4.3). The two bands of the rigid Ward operator $H$ are gapped except at isolated points in the phase space of the electric and magnetic coupling constants where supersymmetry is partially broken and where live a gapless chiral state with a chiral anomaly violating the Nielson-Ninomiya theorem. From the study of the properties of $H$; it follows that the gap energy is given by $E_g = 2 |\vec{\xi}|$ and vanishes for $|\vec{\xi}| = 0$; that is for vanishing central extension in the $\mathcal{N} = 2$ supercurrent algebra. Zero modes of $H$ and their properties like windings and conserved supersymmetric charges are as reported in table (4.14). At the particular point $\mathcal{V} = 0$, the discriminant $\det H$ reduces to $-|\vec{\xi}|^2$ and has an SU(2) singularity at the origin $\vec{\xi} = \vec{0}$. There, the Nielson-Ninomiya theorem $\sum_i w(S_i^2) = 0$ is trivially satisfied as shown on the table (4.14) and $\mathcal{N} = 2$ supersymmetry is exact with compensating chiral anomalies.

(iii) We have used the proposal (3.2) to study effect of quantum corrections induced by fluctuations of FI coupling constants (running couplings). We have found that quantum
effect in iso-space of FI couplings may break supersymmetry completely except for the saturated bounds (4.22) where half of oscillating modes disappear.

Finally, we would like to add that this approach might be helpful to explore the picture in higher supergravities; in particular for $\mathcal{N} = 4$; progress in this direction will be reported in a future occasion.

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