ON THE ZEROS OF GENERALIZED HURWITZ ZETA FUNCTIONS

T. CHATTERJEE AND S. GUN

Abstract. In this note, we prove the existence of infinitely many zeros of certain generalized Hurwitz zeta functions in the domain of absolute convergence. This is a generalization of a classical problem of Davenport, Heilbronn and Cassels about the zeros of the Hurwitz zeta function.

1. Introduction

For a real number \( \alpha > 0 \), the Hurwitz zeta function \( \zeta(s, \alpha) \) is defined by

\[
\zeta(s, \alpha) := \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s},
\]

where \( s \in \mathbb{C} \) with real part \( \Re(s) > 1 \). It has a meromorphic continuation to the complex plane \( \mathbb{C} \), its only pole being a simple pole at \( s = 1 \) with residue \( 1 \). From now on, we set \( s := \sigma + it \) with \( \sigma, t \in \mathbb{R} \).

In a classical paper, Davenport and Heilbronn [5] proved that if \( \alpha \neq 1/2, 1 \) is either rational or transcendental, then \( \zeta(s, \alpha) \) has infinitely many zeros with \( \sigma > 1 \). Since \( \zeta(s, 1) = \zeta(s) \) and \( \zeta(s, 1/2) = (2^s - 1)\zeta(s) \), they do not have zeros with \( \sigma > 1 \). On the other hand when \( \alpha \) is an algebraic irrational, Cassels [3] showed the existence of infinitely many zeros of \( \zeta(s, \alpha) \) for \( \sigma > 1 \).

For an illuminating account of this theme, see the recent article of Bombieri and Ghosh [1].

For a periodic arithmetic function \( f \) with period \( q \geq 1 \) and \( \alpha > 0 \), consider the \( L \)-function

\[
L(s, f, \alpha) := \sum_{n=0}^{\infty} \frac{f(n)}{(n+\alpha)^s},
\]

where \( s \in \mathbb{C} \) with \( \sigma > 1 \). This can be thought of as a generalization of the Hurwitz zeta function. In a recent work, Saias and Weingartner [6] showed that \( L(s, f, \alpha) \) has infinitely many zeros for \( \sigma > 1 \) when \( \alpha = 1 \) and \( L(s, f, 1) \) is not a product of \( L(s, \chi) \) and a Dirichlet polynomial, where \( \chi \) is a Dirichlet character. In this paper, we study the zeros of \( L(s, f, \alpha) \) for an arbitrary positive irrational number \( \alpha \) in the region \( \sigma > 1 \). See also a recent work...
of Booker and Thorne [2] where another generalization of the work of Saias and Weingartner to higher degree $L$-functions is considered.

Since $f$ is periodic with period $q \geq 1$, the generalized Hurwitz zeta function can be written as

$$L(s, f, \alpha) = q^{-s} \sum_{b=1}^{q} f(b) \zeta(s, (\alpha + b)/q),$$

for $s \in \mathbb{C}$ with $\sigma > 1$. This shows that $L(s, f, \alpha)$ extends meromorphically to the whole complex plane with a possible simple pole at $s = 1$ with residue $q^{-1} \sum_{b=1}^{q} f(b)$. Moreover, if we assume that $f$ is real valued and $L(s, f, \alpha)$ does have a pole at $s = 1$, we then have the following theorems about the existence of zeros of $L(s, f, \alpha)$.

**Theorem 1.1.** Let $\alpha$ be a positive transcendental number and $f$ be a real valued periodic arithmetic function with period $q \geq 1$. If $L(s, f, \alpha)$ has a pole at $s = 1$, then $L(s, f, \alpha)$ has infinitely many zeros for $\sigma > 1$.

**Theorem 1.2.** Let $\alpha$ be a positive algebraic irrational number and $f$ be a positive valued periodic arithmetic function with period $q \geq 1$. Also let $c$ be defined by

$$c := \frac{\max_{n} f(n)}{\min_{n} f(n)}.$$

Assume that $c < 1.15$ and $L(s, f, \alpha)$ has a pole at $s = 1$. Then $L(s, f, \alpha)$ has infinitely many zeros for $\sigma > 1$.

## 2. Notations and Preliminaries

From now on, we denote the field of algebraic numbers, the multiplicative group of non-zero real numbers, the set of positive real numbers and the set of non-negative integers by $\mathbb{Q}$, $\mathbb{R}^*$, $\mathbb{R}_+$ and $\mathbb{N}$ respectively.

The following theorem of Kronecker (see [4]) will play an important role in proving Theorem 1.1.

**Theorem 2.1.** (Kronecker). Let $\alpha_1, \ldots, \alpha_N$ be real numbers which are linearly independent over the integers and $\beta_1, \ldots, \beta_N$ be arbitrary real numbers. Then for any real number $T$ and $\delta > 0$, there exists a real number $t > T$ and integers $x_1, \ldots, x_N$ such that

$$|t\alpha_n - \beta_n - x_n| < \delta$$

for all $n = 1, \ldots, N$. 
Lemma 2.2. (Cassels [3]). Let $\alpha$ be a real algebraic irrational number and $K = \mathbb{Q}(\alpha)$. Also let $\mathfrak{a}$ be an integral ideal such that $\mathfrak{a}(\alpha \mathcal{O}_K)$ is an integral ideal. Then there exists an $N_0 > 10^6$ satisfying the following property:

for any $N > N_0$ and $M = \lfloor 10^{-6} N \rfloor$, there are at least $51M/100$ integers $n$ in $N < n \leq N + M$ such that $(n + \alpha)\mathfrak{a}$ is divisible by a prime ideal $\mathfrak{p}_n$ for which

$$\mathfrak{p}_n \nmid \prod_{\substack{m \leq N+M \\text{m} \neq n}} (m + \alpha)\mathfrak{a}.$$ 

Remark 2.1. It is not difficult to see that the proof of Cassels yields at least $27M/50$ integers $n$ in $N < n \leq N + M$ with the above property. We will make use of this fact in the proof of Theorem 1.2.

3. Proofs of Theorem 1.1 and Theorem 1.2

In order to prove Theorem 1.1 and Theorem 1.2, we shall need the following propositions.

Proposition 3.1. Let $\alpha > 0$ be any transcendental number and $N \geq 1$ be an integer. Also, let $g_1, \ldots, g_N$ be a sequence of complex numbers having absolute value 1. Then for any real number $T$ and $\epsilon > 0$, there exists a real number $t > T$ such that

$$|(n + \alpha)^{-it} - g_n| < \epsilon$$

for all $1 \leq n \leq N$.

Proof. Since $\alpha$ is transcendental, the numbers $\log(n + \alpha)$ are linearly independent over $\mathbb{Q}$. Write $g_n = e^{-i\alpha_n}$, where $\alpha_n$'s are real numbers. Let $\delta > 0$ be arbitrary. Then by Kronecker’s theorem, there exists a real number $t > T$ and integers $x_1, \ldots, x_N$ such that

$$\left| \frac{t}{2\pi} \log(n + \alpha) - \frac{\alpha_n}{2\pi} - x_n \right| < \frac{\delta}{2\pi}.$$ 

Multiplying both sides by $2\pi$, we get

$$|t \log(n + \alpha) - \alpha_n - 2\pi x_n| < \delta.$$ 

Hence we have

$$|e^{-it \log(n+\alpha)} - e^{-i\alpha_n}| < \epsilon$$

since $e^{-ix}$ is continuous. This completes the proof. □

Our next proposition shows that with a little modification in the properties of $g_n$, one can get a similar result as above when $\alpha$ is a positive algebraic number. In this case, consider the number field $K = \mathbb{Q}(\alpha)$. For each prime ideal $\mathfrak{p}$ in the ring of integers $\mathcal{O}_K$, let $\chi(\mathfrak{p})$ be a complex number
with $|\chi(p)| = 1$. We extend $\chi$ to any element $\gamma$ of the number field $K$ by setting
\[ \chi(\gamma) = \prod_p \chi(p)^{\nu_p} \quad \text{if} \quad \gamma \mathcal{O}_K = \prod_p p^{\nu_p}. \tag{1} \]

Here we have the following proposition.

**Proposition 3.2.** Let $N \in \mathbb{N}$, $\alpha \in \overline{\mathbb{Q}} \cap \mathbb{R}_+$. Also let $K = \mathbb{Q}(\alpha)$ and $\chi$ be as in (1). Then for any real number $T$ and $\epsilon > 0$, there exists a real number $t > T$ such that
\[ |(n + \alpha)^{-it} - \chi(n + \alpha)| < \epsilon \]
for all $0 \leq n \leq N$.

**Proof.** Consider the multiplicative subgroup $A$ of $\mathbb{R}^*$ generated by
\[ S := \{n + \alpha \mid 0 \leq n \leq N\}. \]
One can choose a $\mathbb{Z}$-basis $B := \{s_j \mid 1 \leq j \leq l\}$ of $A$. Then there exists an integer $M > 0$ such that for any $0 \leq n \leq N$, we have
\[ n + \alpha = \prod_{j=1}^l s_j^{u_j}, \]
where $u_j \in \mathbb{Z}$ with $|u_j| \leq M$. This implies that
\[ \chi(n + \alpha) = \prod_{j=1}^l \chi(s_j)^{u_j}. \tag{2} \]

By Kronecker’s theorem, for any $\epsilon > 0$, there exists a real number $t > T$ such that
\[ |s_j^{-it} - \chi(s_j)| < \epsilon/ML, \tag{3} \]
for all $s_j \in B$. Now for any $n + \alpha \in S$ and $\epsilon > 0$, we have by (2) and (3) that
\[ |(n + \alpha)^{-it} - \chi(n + \alpha)| = \prod_{j=1}^l |s_j^{-it u_j} - \prod_{j=1}^l \chi(s_j)^{u_j}|, \]
where $u_j \in \mathbb{Z}$ and $|u_j| \leq M$
\[ \leq M \sum_{j=1}^l |s_j^{-it} - \chi(s_j)| < \epsilon. \]
Hence the proposition. \hfill \Box

We now prove the following proposition which will play an important role in the proof of Theorem 1.2.
Proposition 3.3. Let \(0 < r_1 \leq r_2 \leq \cdots \leq r_n\) be real numbers. Then the set 
\[
\Delta_n := \{c_1r_1 + \cdots + c_nr_n \mid |c_i| = 1, \ c_i \in \mathbb{C}\}
\]
for \(n \geq 1\) is a closed annulus with outer radius \(R_n\) and inner radius 
\[
T_n := \begin{cases} 
    r_n - R_{n-1} & \text{if } R_{n-1} \leq r_n, \\
    0 & \text{otherwise.}
\end{cases}
\]
Here \(R_0 := 0\) and \(R_i := r_1 + \cdots + r_i\) for \(1 \leq i \leq n\).

Proof. Note that the set \(\Delta_n\) is compact, connected and invariant under rotation around the origin and hence the result follows by induction on \(n\). \(\square\)

Before we state our next proposition, we shall formulate a hypothesis which is integral to our proofs. Let \(f\) be a periodic arithmetic function with period \(q \geq 1\).

**Hypothesis:** For any \(\delta > 0\), there exists a function \(F(s)\) analytic in the region \(\Re(s) > 1\) satisfying the following properties:

1. There exists a \(\sigma_0\) with \(F(\sigma_0) = 0\) and \(1 < \sigma_0 < 1 + \delta\).
2. For any real number \(T\) and real numbers \(\epsilon, \theta > 0\), there exists a real number \(t > T\) such that 
   \[ |L(s + it, f, \alpha) - F(s)| < \epsilon \] for all \(\sigma > 1 + \theta\).

In this context, we have the following proposition.

**Proposition 3.4.** Let \(\alpha \in \mathbb{R}_+\) and \(f\) be as above. Assume the previous hypothesis. Then \(L(s, f, \alpha)\) has infinitely many zeros for \(\sigma > 1\).

Proof. Let \(T, \delta > 0\) be real numbers. We will show that there exists a zero \(s_1\) of \(L(s, f, \alpha)\) with \(1 < \Re(s_1) < 1 + \delta\) and \(\Im(s_1) > T\).

Let \(F(s)\) be a function corresponding to \(\delta\) in the hypothesis. By property (1) of \(F(s)\), it has a zero \(\sigma_0\) with \(1 < \sigma_0 < 1 + \delta\). Since \(F(s)\) is an analytic function, one can choose \(\delta_1 > 0\) such that \(1 + \delta_1 < 1 + \delta\) and \(1 + \delta_1 < \sigma_0\) with \(F(s) \neq 0\) for \(|s - \sigma_0| = \delta_1\). Set 
\[
\epsilon := \min_{|s-\sigma_0| = \delta_1} |F(s)| \quad \text{and} \quad \theta < \sigma_0 - \delta_1 - 1.
\]

Then \(\sigma_0 - \delta_1 > 1 + \theta\) and hence by property (2) of \(F(s)\), there exists a real number \(t > T\) such that 
\[
|L(s + it, f, \alpha) - F(s)| < |F(s)|
\]
on \(|s - \sigma_0| = \delta_1\). Thus by Rouche’s theorem, the function \(L(s + it, f, \alpha)\) has a zero \(s_1\) which gives a zero \(s_1 + it\) of \(L(s, f, \alpha)\). \(\square\)
In view of the above proposition, our task is to construct functions \( F(s) \) as described in the hypothesis.

3.1. **Proof of Theorem 1.1.** Let \( \alpha \) be a positive transcendental number. In this case, replacing \( f \) by \( -f \) if needed, we can assume that the residue \( \frac{1}{q} \sum_{b=1}^{q} f(b) \) of \( L(s, f, \alpha) \) is a positive real number.

Since \( L(s, f, \alpha) \) converges absolutely for \( \sigma > 1 \), for any \( \delta > 0 \), one can choose an integer \( m \) such that

\[
\sum_{n=0}^{m} \frac{f(n)}{(n+\alpha)^{1+\delta}} > \sum_{n=m+1}^{\infty} \frac{f(n)}{(n+\alpha)^{1+\delta}}
\]

Define

\[
F(s) := \sum_{n=0}^{\infty} \frac{f(n)\beta(n)}{(n+\alpha)^s}
\]

for \( \Re(s) > 1 \), where \( \beta \) is the arithmetic function defined by

\[
\beta(n) := \begin{cases} 
-1 & \text{if } n > m, \\
1 & \text{otherwise}.
\end{cases}
\]

By (4), it is clear that \( F(1+\delta) > 0 \). On the other hand, since \( L(s, f, \alpha) \) has a pole at \( s = 1 \), \( F(s) \rightarrow -\infty \) as \( s \rightarrow 1^+ \). Since \( F(s) \) is a real valued continuous function when \( s \) is real and \( s > 1 \), it follows that \( F(s) \) has a zero in the interval \((1, 1+\delta)\). Thus \( F(s) \) is an analytic function satisfying (1). It follows from Proposition 3.2 that given any real numbers \( T, \epsilon > 0 \), there exists a real number \( t > T \) such that

\[
|L(s + it, f, \alpha) - F(s)| < \epsilon \text{ for } \Re(s) > 1.
\]

Since \( F(s) \) is a function satisfying properties (1) and (2) in the hypothesis, Theorem 1.1 follows from Proposition 3.4.

3.2. **Proof of Theorem 1.2.** Let \( \alpha \) be a positive algebraic irrational number and \( f \) be a positive real valued periodic function satisfying the conditions of Theorem 1.2. Let \( \delta > 0 \) be fixed. We shall define for \( \Re(s) > 1 \), a function

\[
F(s) = \sum_{n=0}^{\infty} \frac{f(n)\chi(n+\alpha)}{(n+\alpha)^s},
\]

where \( \chi \) is a suitably chosen character on the group of fractional ideals of \( K = \mathbb{Q}(\alpha) \). Here \( \chi(n+\alpha) \) is the value of this character on the principal ideal \((n+\alpha)\mathcal{O}_K\).

Clearly, such a function is holomorphic in \( \Re(s) > 1 \). Furthermore, it follows from Proposition 3.2 that \( F(s) \) satisfies property (2).
We shall show that it is possible to define \( \chi \) suitably to ensure the existence of a \( \sigma \) with \( 1 < \sigma < \min(1 + \delta, 2) \) satisfying
\[
F(\sigma) = \sum_{n=0}^{\infty} \frac{f(n)\chi(n + \alpha)}{(n + \alpha)^\sigma} = 0. \tag{5}
\]
Then this function will also satisfy property (1) and thereby establish the theorem. The idea of constructing \( \chi \) with the above properties is similar to that of Cassels [3], however our proof admits the function \( f \) which did not appear in Cassels’s paper.

We first begin by setting \( N_1 = \lceil \max(N_0, 10^7, 10^7\alpha) \rceil \), where \( \lfloor x \rfloor \) is the integral part of a real number \( x \). Then since \( L(s, f, \alpha) \) has a pole at \( s = 1 \), there exists a \( \sigma \) such that
\[
N_1 \sum_{n=0}^{\infty} f(n)(n + \alpha) < 10^{-2} \sum_{n=N_1+1}^{\infty} f(n)(n + \alpha) \sigma
\]
and \( 1 < \sigma < \min(1 + \delta, 2) \).

We now define an infinite sequence of integer pairs \( N_j, M_j \) for \( j \geq 1 \) by
\[
M_j := \left\lfloor \frac{N_j}{10^6} \right\rfloor \text{ and } N_{j+1} := N_j + M_j.
\]
To prove (5), it is sufficient to show that we can construct a character \( \chi \) such that
\[
\left| \sum_{n=0}^{N_j} f(n)\chi(n + \alpha)(n + \alpha) \sigma \right| < 10^{-2} \sum_{n=N_j+1}^{\infty} f(n)(n + \alpha) \sigma \tag{6}
\]
for all \( j \).

Let \( \mathfrak{a} \) be the ideal denominator of \( \alpha \) so that \( \mathfrak{a}(n + \alpha) \) is an integral ideal for every integer \( n \). If \( p|\mathfrak{a} \) or \( p|(n + \alpha)\mathfrak{a} \) for \( n \leq N_1 \), we choose \( \chi(p) := 1 \). Then by (6), we see that (7) is true for \( j = 1 \).

Suppose that (7) is true for all integers \( \leq j \). Define two subsets of integers as follows:
\[
\mathfrak{A} := \left\{ n \mid N_j < n \leq N_{j+1}, \nexists \mathfrak{p}_n|(n + \alpha)\mathfrak{a} \text{ but } \mathfrak{p}_n \nmid \prod_{m \leq N_j+1, m \neq n} (m + \alpha)\mathfrak{a} \right\}
\]
and
\[
\mathfrak{B} := \{ n \mid N_j < n \leq N_{j+1}, n \notin \mathfrak{A} \}.
\]
By Cassels’s lemma and the choice of \( N_j \), we have \( |\mathfrak{A}| \geq 27M_j/50 \).

Note that if \( p|\prod_{m \leq N_j+1} (m + \alpha)\mathfrak{a} \), then there are three possibilities, namely \( p|\prod_{m \leq N_j} (m + \alpha)\mathfrak{a} \text{ or } p = \mathfrak{p}_n \text{ for some } n \in \mathfrak{A} \) or \( p \) is different from both of these types. By induction hypothesis, we already know the values of \( \chi(p) \)
when \( p \) is of the first type and we define \( \chi(p) := 1 \) when \( p \) is of the third type. Now we will define \( \chi(p) \) when \( p \) is of the second type in such a way that (7) is true for \( j + 1 \).

By the hypothesis of the theorem, we have

\[
\frac{f(n)}{f(m)} \leq 1.15 \quad \text{for all } n, m \in \mathbb{N},
\]

and hence

\[
\frac{f(n)}{(n + \alpha)^\sigma} < \frac{3f(m)}{(m + \alpha)^\sigma}
\]

for any \( n, m \in \mathfrak{A} \) as we have

\[
\frac{(n + \alpha)^\sigma}{(m + \alpha)^\sigma} < 2
\]

for any \( n, m \in \mathfrak{A} \). Since \( |\mathfrak{A}| > 5 \), it follows from Proposition 3.3 that

\[
\sum_{n \in \mathfrak{A}} \frac{f(n)\chi(n + \alpha)}{(n + \alpha)^\sigma}
\]

can take any value \( z \) with

\[
|z| \leq \sum_{n \in \mathfrak{A}} \frac{f(n)}{(n + \alpha)^\sigma} = S_3, \text{ say.}
\]

Write

\[
\Lambda := \sum_{n \leq N_j} \frac{f(n)\chi(n + \alpha)}{(n + \alpha)^\sigma} + \sum_{n \in \mathfrak{B}} \frac{f(n)\chi(n + \alpha)}{(n + \alpha)^\sigma}
\]

and set

\[
S_1 := \left| \sum_{n \leq N_j} \frac{f(n)\chi(n + \alpha)}{(n + \alpha)^\sigma} \right|
\]

\[
S_2 := \sum_{n \in \mathfrak{B}} \frac{f(n)}{(n + \alpha)^\sigma}.
\]

Also set

\[
z := \begin{cases} 
-\Lambda & \text{if } 0 < |\Lambda| \leq S_3, \\
-S_3\Lambda/|\Lambda| & \text{if } |\Lambda| > S_3, \\
0 & \text{if } \Lambda = 0.
\end{cases}
\]

Then by appropriate choice of \( \chi(n + \alpha) \) for \( n \in \mathfrak{A} \), we have

\[
\left| \sum_{n \leq N_{j+1}} \frac{f(n)\chi(n + \alpha)}{(n + \alpha)^\sigma} \right| \leq \max \{0, S_1 + S_2 - S_3\}.
\]
Since $|B| \leq 23M_j/50$ and $|A| \geq 27M_j/50$, we have
\begin{align*}
\frac{S_3}{S_2} & \geq \frac{27}{23c} \frac{(N_j + \alpha)^\sigma}{(N_j + 1 + \alpha)^\sigma} \\
& > \frac{27 \times 10^7 \times 10^7}{23(10^7 + 11)^2 \times 1.15} > \frac{101}{99}.
\end{align*}
This implies that
\begin{equation}
100(S_3 - S_2) > S_3 + S_2. \quad (8)
\end{equation}
Set
\[
S_4 := \sum_{n>N_j+1} f(n) \frac{\sigma(n)}{(n+\alpha)^\sigma}.
\]
By induction,
\[
S_1 < 10^{-2}(S_3 + S_2 + S_4).
\]
Thus by (8), we have
\[
S_1 + S_2 - S_3 < 10^{-2}S_4.
\]
This proves (7) by induction and hence we have (5). This proves the theorem.

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