States and Curves of Five-Dimensional Gauged Supergravity

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Abstract
We consider the sector of \( \mathcal{N} = 8 \) five-dimensional gauged supergravity with non-trivial scalar fields in the coset space \( SL(6,\mathbb{R})/SO(6) \), plus the metric. We find that the most general supersymmetric solution is parametrized by six real moduli and analyze its properties using the theory of algebraic curves. In the generic case, where no continuous subgroup of the original \( SO(6) \) symmetry remains unbroken, the algebraic curve of the corresponding solution is a Riemann surface of genus seven. When some cycles shrink to zero size the symmetry group is enhanced, whereas the genus of the Riemann surface is lowered accordingly. The uniformization of the curves is carried out explicitly and yields various supersymmetric configurations in terms of elliptic functions. We also analyze the ten-dimensional type-IIB supergravity origin of our solutions and show that they represent the gravitational field of a large number of D3-branes continuously distributed on hyper-surfaces embedded in the six-dimensional space transverse to the branes. The spectra of massless scalar and graviton excitations are also studied on these backgrounds by casting the associated differential equations into Schrödinger equations with non-trivial potentials. The potentials are found to be of Calogero type, rational or elliptic, depending on the background configuration that is used.
1 Introduction

Ungauged and gauged $\mathcal{N} = 8$ supergravities in five dimensions were constructed several years ago in [1] and [2, 3], following the analogous construction made in four dimensions in [4] and [5]. More recently it has become clear that solutions of five-dimensional gauged supergravity play an important rôle in the context of the AdS/CFT correspondence [6, 7, 8]. In particular the maximum supersymmetric vacuum state in five-dimensional gauged supergravity with $AdS_5$ geometry, originates from the $AdS_5 \times S^5$ solution in ten-dimensional type-IIB supergravity. The latter solution arises as the near horizon geometry of the solution representing the gravitational field of a large number of coincident D3-branes and has been conjectured to provide the correct framework for analyzing $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang–Mills for large $N$ and ’t Hooft coupling constant at the conformal point of the Coulomb branch.

The supergravity approach to gauge theories at strong coupling is applicable not only at conformality, but also away from it. In particular, when the six scalar fields of the $\mathcal{N} = 4$ supersymmetric Yang–Mills theory acquire Higgs expectation values we move away from the origin of the Coulomb branch and the appropriate supergravity solution corresponds to a multicenter distribution of D3-branes with the centers, where the branes are located, associated with the scalar Higgs expectation values in the gauge theory side. A prototype example of such D3-brane distributions is the two-center solution that has been studied in [3, 4, 10], whereas examples of continuous D3-brane distributions arise naturally in the supersymmetric limit of rotating D3-brane solutions [11, 12]. Concentrating on the case of continuous distributions, note that from a ten-dimensional type-IIB supergravity viewpoint the $SO(6)$ symmetry, associated with the round $S^5$-sphere, is broken because this sphere is deformed. On the other hand, from the point of view of five-dimensional gauged supergravity the deformation of the sphere is associated with the fact that some of the scalar fields in the theory are turned on. Hence, finding solutions of five-dimensional gauged supergravity might shed more light into the AdS/CFT correspondence as far as the Coulomb branch is concerned. Using such solutions, investigations of the spectrum of massless scalars excitations and of the quark-antiquark potential have already been carried out with sometimes surprising results [13, 14, 15]. Solutions of the five-dimensional theory are also important in a non-perturbative treatment of the renormalization group flow in gauge theories at strong coupling [16, 17, 18, 19].

An additional motivation for studying solutions of five-dimensional gauged supergravity is the fact that for a class of such configurations, four-dimensional Poincaré invariance is preserved. It turns out that our four-dimensional space-time can be viewed as being embedded non-trivially in the five-dimensional solution with a warp factor. This particular idea of our space-time as a membrane in higher dimensions is quite old [20] and has been recently revived with interesting phenomenological consequences on the mass hierarchy problem [21]. In that work, in particular, our four-dimensional world was embedded into the $AdS_5$ space from which a slice was cut out; it results into a normalizable graviton zero mode, but also to a continuum spectrum of massive ones above it with no
mass gap separating them. The use of more general solutions of five-dimensional gauged supergravity certainly creates more possibilities and in fact there are solutions with a mass gap that separates the massless mode from the massive ones [22].

This paper is organized as follows: In section 2 we present a brief summary of some basic facts about $\mathcal{N} = 8$ five-dimensional gauged supergravity with gauge group $SO(6)$. In particular, we restrict our attention to the sector of the theory where only the metric and the scalar fields associated with the coset space $SL(6, \mathbb{R})/SO(6)$ are turned on. In section 3 we find the most general supersymmetric configuration in this sector, which as it turns out, depends on six real moduli. Our solutions have a ten-dimensional origin within type-IIB supergravity and represent the gravitational field of continuous distributions of D3-branes in hyper-surfaces embedded in the transverse space to the branes. In section 4 we further analyze our solution using some concepts from the theory of algebraic curves and in particular Riemann surfaces. We find that our states correspond to Riemann surfaces with genus up to seven, depending on their symmetry groups, which are all subgroups of $SO(6)$. In section 5 we provide details concerning the geometrical origin of the supersymmetric states in five dimensions from a ten-dimensional point of view using various distributions of D3-branes in type-IIB supergravity. This approach yields explicit expressions for the metric and the scalar fields, and it can be viewed as complementary to the algebro-geometric classification of section 4 in terms of Riemann surfaces. In section 6 we consider massless scalar and graviton fluctuations propagating on our backgrounds. We formulate the problem equivalently as a Schrödinger equation in one dimension and compute the potential in some cases of particular interest. We also note intriguing connections of these potentials to Calogero models and various elliptic generalizations thereof. Finally, we end the paper with section 7 where we present our conclusions and some directions for future work.

2 Elements of five-dimensional gauged supergravity

$\mathcal{N} = 8$ supergravity in five dimensions involves 42 scalar fields parametrizing the non-compact coset space $E_{6(6)}/USp(8)$ that describes their couplings in the form of a non-linear $\sigma$-model [1]. In five-dimensional gauged supergravity the global symmetry group $E_{6(6)}$ breaks into an $SO(6)$ subgroup which corresponds to the gauge symmetry group of the resulting theory, and a non-trivial potential develops [2, 3]. In the framework of the AdS/CFT correspondence [4, 5] the supergravity scalars represent the couplings of the marginal and relevant chiral primary operators of the $\mathcal{N} = 4$ supersymmetric Yang–Mills theory in four dimensions. The invariance of the theory with respect to the gauge group, as well as the $SL(2, \mathbb{R})$ symmetry inherited from type-IIB supergravity in ten dimensions, restricts the scalar potential to depend on $42 - 15 - 3 = 24$ invariants of the above groups. However, it seems still practically impossible to deal with such a general potential. In this paper we restrict attention to the scalar subsector corresponding to the symmetric traceless representation of $SO(6)$, which parametrizes the coset $SL(6, \mathbb{R})/SO(6)$, and
set all other fields (except the metric) equal to zero. In this sector we will be able to find explicitly the general solution of the classical equations of motion that preserves supersymmetry.

The Lagrangian for this particular coupled gravity-scalar sector includes the usual Einstein–Hilbert term, the usual kinetic term for the scalars as well as their potential

\[ \mathcal{L} = \frac{1}{4} \mathcal{R} - \frac{1}{2} \sum_{i=1}^{5} (\partial \alpha_i)^2 - P. \]  

(2.1)

A few explanations concerning the scalar-field part of this action are in order. It has been shown that in this subsector the scalar potential \( P \) depends on the symmetric matrix \( SS^T \) only, where \( S \) is an element of \( SL(6, \mathbb{R}) \) [3] (for a recent discussion see also [17, 13]). Diagonalization of this matrix yields a form that depends only on five scalar fields. It is convenient, nevertheless, to represent this sector in terms of six scalar fields \( \beta_i, i = 1, 2, \ldots, 6 \) as [13]

\[ P = -\frac{1}{8R^2} \left( \sum_{i=1}^{6} e^{2\beta_i} - 2 \sum_{i=1}^{6} e^{4\beta_i} \right). \]  

(2.2)

where

\[
\begin{pmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4 \\
\beta_5 \\
\beta_6
\end{pmatrix} = \begin{pmatrix}
1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & 0 & 1/\sqrt{6} \\
1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} & 0 & 1/\sqrt{6} \\
-1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} & 0 & 1/\sqrt{6} \\
-1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} & 0 & 1/\sqrt{6} \\
0 & 0 & 0 & 1 & -\sqrt{2/3} \\
0 & 0 & 0 & -1 & -\sqrt{2/3}
\end{pmatrix} \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5
\end{pmatrix}.
\]  

(2.3)

Note that the \( 6 \times 5 \) matrix that relates the auxiliary scalars \( \beta_i \) with the \( \alpha_i \)'s is not unique; it only has to satisfy the condition \( \sum_i \beta_i = 0 \). The choice in (2.3) is particularly useful for certain computational purposes. It also has the property that if the fields \( \beta_i \) are canonically normalized, the five independent scalar fields \( \alpha_i \) will be canonically normalized as well, i.e. \( \sum_{i=1}^{5}(\partial \beta_i)^2 = 2 \sum_{i=1}^{5}(\partial \alpha_i)^2 \).

The form of the kinetic term for the scalars in (2.1) suggests that the metric in the corresponding coset space is taken to be \( \delta^{ij} \). This was explicitly shown for the case of only one scalar field in [17] and the general result was quoted without detailed explanation in [13]. One can generally prove this statement by first realizing that the kinetic term of these scalars can depend on two type of terms, namely \( \text{Tr}(\partial_{\mu}SS^{-1})\text{Tr}(\partial_{\nu}SS^{-1}) \) and \( \text{Tr}(\partial_{\mu}SS^{-1}\partial_{\nu}SS^{-1}) \). Since \( \partial_{\mu}SS^{-1} \) belongs to the algebra of \( SL(6, \mathbb{R}) \) the first term is zero because of the traceless condition. The second term gives a result proportional to \( \sum_{i=1}^{6}(\partial \beta_i)^2 = 2 \sum_{i=1}^{5}(\partial \alpha_i)^2 \), thus showing that the scalar kinetic term in (2.1) has indeed the above form. The equations of motion follow by varying the action (2.1) with respect to the five-dimensional metric and the scalar fields. Using the metric \( G_{MN} \), we have

\[ \frac{1}{4} R_{MN} = \frac{1}{2} \sum_{i=1}^{5} \partial_M \alpha_i \partial_N \alpha_i + \frac{1}{3} G_{MN} P, \]  

3
\[ D^2 \alpha_i = \frac{\partial P}{\partial \alpha_i} \quad (2.4) \]

There is a maximally supersymmetric solution of the above equations that preserves all 32 supercharges, in which all scalar fields are set zero and the metric is that of \( AdS_5 \) space. Then, the potential in (2.2) becomes \( P = -3/R^2 \) and equals by definition to the negative cosmological constant of the theory. This defines the length scale \( R \) that will be used in the following.

The coupled system of non-linear differential equations (2.4) is in general difficult to solve. In this paper we will be interested in solutions preserving four-dimensional Poincaré invariance \( ISO(1,3) \). Hence, we make the following ansatz for the five-dimensional metric

\[ ds^2 = e^{2A}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2) \quad (2.5) \]

where \( \eta_{\mu\nu} = \text{diag}(-1,1,1,1) \) is the four-dimensional Minkowski metric and the conformal factor \( e^{2A} \), as well as the scalar fields \( \alpha_i \), depend only on the variable \( z \). In addition, we demand that our solutions preserve supersymmetry. The corresponding Killing spinor equations, arising from the supersymmetry transformation rules for the 8 gravitinos and the 42 spin-1/2 fields, give rise to the first order equations [13]

\[ A' = \frac{2}{3R}e^{4A}W; \quad \alpha_i' = -\frac{1}{R}e^{4A}\frac{\partial W}{\partial \alpha_i}, \quad i = 1, 2, \ldots, 5 \quad (2.6) \]

where

\[ W = -\frac{1}{4} \sum_{i=1}^{6} e^{2\beta_i} \quad (2.7) \]

and the derivative is taken with respect to the coordinate \( z \). It is straightforward to check that all supersymmetric solutions satisfying the first order equations (2.6) also satisfy the second order equations (2.4). In doing so, it is convenient to use the alternative expression for the potential, instead of (2.2),

\[ P = \frac{1}{2R^2} \sum_{i=1}^{5} \left( \frac{\partial W}{\partial \alpha_i} \right)^2 - \frac{4}{3R^2} W^2 \quad (2.8) \]

3 The general supersymmetric solution

We begin this section with the construction of the most general solution of the non-linear system of equations (2.6) and discuss some of the general properties of the corresponding supersymmetric configurations. We also show how our solution can be lifted to ten dimensions in the context of type-IIB supergravity.

3.1 Five-dimensional solutions

It might still seem difficult to find solutions of the coupled system of equations (2.6) at first sight, due to non-linearity. It turns out, however, that this is not the case, but
instead it is possible to find the most general solution. In order to proceed further, we first compute the evolution of the auxiliary scalar fields $\beta_i$. Using (2.3) and (2.6) we find

$$
\beta_i' = \frac{e^A}{R} \left( \frac{2}{3} W + e^{2\beta_i} \right) = A' + \frac{1}{R} e^{2\beta_i + A}, \quad i = 1, 2, \ldots, 6, \tag{3.1}
$$

where for the last equality we have used the first equation in (2.6). This substitution results into six decoupled first order equations for the $\beta_i$’s which can be easily integrated, as we will soon demonstrate. Of course, after deriving the explicit solution for the $\beta_i$, we also have to check the self-consistency of this substitution.

Let us reparametrize the function $A(z)$ in terms of an auxiliary function $F(z/R^2)$ as follows

$$
e^A = \frac{1}{R} (-F'/2)^{1/3}. \tag{3.2}
$$

We have included a minus sign in this definition since, according to the boundary conditions that we will later choose, $F$ will be a decreasing function of $z$. Then, according to this ansatz, the general solution of (3.1) is given by

$$
e^{2\beta_i} = \frac{(-F'/2)^{2/3}}{F - b_i}, \quad i = 1, 2, \ldots, 6, \tag{3.3}
$$

where the prime denotes here the derivative with respect to the argument $z/R^2$. The $b_i$’s are six constants of integration, which, sometimes is convenient to order as

$$
b_1 \geq b_2 \geq \ldots \geq b_6, \tag{3.4}
$$

without loss of generality. Note that we may fix one combination of them to an arbitrary constant value because (3.2) determines the function $F$ up to an additive constant. Also, since the sum of the $\beta_i$’s is zero, we find that the function $F$ has to satisfy the differential equation

$$
(F')^2 = 4 \prod_{i=1}^{6} (F - b_i)^{1/2}, \tag{3.5}
$$

which thus contains all the information about the supersymmetric configurations and provides a non-trivial algebraic constraint. Using (3.2), (3.3) and (3.5) one may easily check that the first equation in (2.6) is also satisfied. If we insist on presenting the solution in the conformally flat form (2.5) the differential equation (3.5) needs to be solved to obtain $F(z/R^2)$. This will be studied in detail in section 4, as it is a necessary step for investigating the massless scalar and graviton fluctuations in section 6.

At the moment we present our general solution in an alternative coordinate system, where $F$ is viewed as the independent variable. Indeed, using (3.3), we obtain for the metric

$$
ds^2 = \frac{f^{1/6}}{R^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{R^2}{4 f^{1/3}} dF^2; \quad f = \prod_{i=1}^{6} (F - b_i), \tag{3.6}
$$

whereas the expression for the scalar fields in (3.3) becomes

$$
e^{2\beta_i} = \frac{f^{1/6}}{F - b_i}, \quad i = 1, 2, \ldots, 6. \tag{3.7}
$$
When the constants $b_i$ are all equal, our solution becomes nothing but $AdS_5$ with all scalar fields turned off to zero. In the opposite case, when all constants $b_i$ are unequal from one another, there is no continuous subgroup of $SO(6)$ preserved by our solution. If we let some of the $b_i$'s to coincide we restore various continuous subgroups of $SO(6)$ accordingly. As for the five scalar fields $\alpha_i$, they can be found using (2.3)

\[
\begin{align*}
\alpha_1 &= \frac{1}{2\sqrt{2}}(\beta_1 + \beta_2 - \beta_3 - \beta_4) \\
\alpha_2 &= \frac{1}{2\sqrt{2}}(\beta_1 + \beta_4 - \beta_2 - \beta_3) \\
\alpha_3 &= \frac{1}{2\sqrt{2}}(\beta_1 + \beta_3 - \beta_2 - \beta_4) \\
\alpha_4 &= \frac{1}{2}(\beta_5 - \beta_6) \\
\alpha_5 &= -\sqrt{\frac{3}{8}}(\beta_5 + \beta_6).
\end{align*}
\]

(3.8)

Note that imposing the reality condition on the scalars in (3.7) restricts the values of $F$ to be larger that the maximum of the constants $b_i$, which according to the ordering in (3.4) means that $F \geq b_1$. For $F \gg b_1$ the scalars tend to zero and $f \simeq F^6$, in which case the metric in (3.6) approaches $AdS_5$ as expected; put differently, in this limit $F \simeq 1/z$ close to $z = 0$ that is taken as the origin of the $z$-coordinate. For intermediate values of $F$ we have a flow in the five-dimensional space spanned by all scalar fields $\beta_i$. In general we may have $b_1 = b_2 = \ldots = b_n$, with $n \leq 6$, when $b_1$ is $n$-fold degenerate. In this case, the solution preserves an $SO(n)$ subgroup of $SO(6)$ and the flow is actually taking place in $6 - n$ dimensions. On the other hand, let us consider the case when $F$ approaches its lower value $b_1$. Then, the scalars in (3.7) are approaching

\[
\begin{align*}
e^{2\beta_i} &\simeq \begin{cases} 
f_0^{1/6}(F - b_1)^{(n-6)/6}, & \text{for } i = 1, 2, \ldots, n \\
f_0^{1/6} (b_1 - b_i)^{n/6}, & \text{for } i = n + 1, \ldots, 6
\end{cases},
\end{align*}
\]

(3.9)

where $f_0 = \prod_{i=n+1}^{6}(b_1 - b_i)$. Consequently, we have a one-dimensional flow in this limit since the scalar fields $\beta_i$ can be expressed in terms of a single (canonically normalized) scalar $\alpha$, as

\[
\begin{align*}
\beta &\simeq \frac{1}{\sqrt{3n(6 - n)}}(n - 6, \ldots, n - 6, n, \ldots, n) \alpha , \\
\alpha &\simeq \frac{\sqrt{n(6 - n)}}{4\sqrt{3}} \ln(F - b_1).
\end{align*}
\]

(3.10)

It is also useful to find the limiting form of the metric (3.6) when $F \to b_1$. Changing the variable to $\rho$ as

\[
F = b_1 + \left(\frac{(6 - n)f_0^{1/6}}{3R}\right)^{6/(6 - n)} \rho^{6/(6 - n)},
\]

(3.11)
the metric (3.6) becomes for $\rho \to 0^+$

$$
ds^2 \simeq d\rho^2 + \left( \frac{6-n}{3} \right)^n \frac{f_0}{R^{12-n}} \rho^{\frac{n}{6-n}} \eta_{\mu\nu} dx^\mu dx^\nu. \tag{3.12}
$$

Hence, at $\rho = 0$ (or equivalently at $F = b_1$) there is a naked singularity which has an interpretation, as we will see later in the ten-dimensional context, as the location of a distribution of D3-branes. It is instructive to compare this with the singular behaviour of non-conformal non-supersymmetric solutions found in [23]. A similar naked singularity was found there, but the corresponding metric near the singularity had a power law behaviour in $\rho$ with exponent equal to $1/2$, which coincides with the result in (3.12) only for $n = 2$.

### 3.2 Type-IIB supergravity origin

It is possible to lift our solution with metric and scalars given by (3.6) and (3.7) to a supersymmetric solution of type-IIB supergravity, where only the metric and the self-dual five-form are turned on. This proves that our five-dimensional solution is a true compactification of type-IIB supergravity on $S^5$. This is not a priori obvious because unlike the case of the $S^7$ compactification of eleven-dimensional supergravity to four dimensions [24], there is no general proof that the full non-linear five-dimensional gauged supergravity action can be fully encoded into the action or equations of motion of the type-IIB supergravity for the $S^5$ compactification. However, there is a lot of evidence that this is indeed the case and our result gives further support in its favour.

We will show that the ten-dimensional metric corresponds to the gravitational field of a large number of D3-branes in the field theory limit with a special continuous distribution of branes in the transverse six-dimensional space. Namely, the metric has the form

$$
ds^2 = H_0^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + H_0^{1/2} \left( dy_1^2 + dy_2^2 + \ldots + dy_6^2 \right), \tag{3.13}
$$

where $H_0$ is a harmonic function (yet to be determined) in the six-dimensional space transverse to the brane parametrized by the $y_i$ coordinates. However, instead of being asymptotically flat, the metric (3.13) will become asymptotically $AdS_5 \times S^5$ for large radial distances (or equivalently in the UV region using the terminology of the AdS/CFT correspondence). The ten-dimensional dilaton field is constant, i.e. $e^\Phi = g_s = \text{const.}$ and, as usual, the self-dual five-form is turned on. Under these conditions, the ten-dimensional solution breaks half of the maximum number of supersymmetries (see, for instance, [25]).

We proceed further by first performing the coordinate change in (3.13)

$$
y_i = Re^{\alpha_i} \hat{x_i} = (F - b_i)^{1/2} \hat{x_i}, \quad i = 1, 2, \ldots, 6, \tag{3.14}
$$

where the $\hat{x_i}$’s define a unit five-sphere, i.e. they obey $\sum_{i=1}^6 \hat{x_i}^2 = 1$. Various convenient bases for these unit vectors can be chosen, depending on the particular applications that
will be presented later. It can be shown that the flat six-dimensional metric in the transverse part of the brane metric (3.13) can be written as
\[
\sum_{i=1}^{6} dy_i^2 = R^2 e^{2A} d\hat{\sigma}^2 + \frac{e^{-2A}}{4R^2} \sum_{i=1}^{6} e^{2\beta_i} \hat{x}_i^2 dF^2 ,
\]
where the line element \(d\hat{\sigma}^2\) defines the metric of a deformed five-sphere given by
\[
d\hat{\sigma}^2 = \sum_{i=1}^{6} e^{-2\beta_i} (d\hat{x}_i)^2 , \quad \text{det} \hat{g} = \text{Vol}(S^5) \sum_{i=1}^{6} e^{2\beta_i} \hat{x}_i^2 .
\]

For later use, we have also written the expression for the determinant of the deformed five-sphere in (3.16). In computing this determinant we have used the fact that the sum of the \(\beta_i\)'s is zero. Note that a similar expression also holds for a general \(n\)-sphere.

The harmonic function \(H_0\) is determined by comparing the massless scalar equation \(\Box_{10} \Phi = 0\) for the ten-dimensional metric (3.13) with the equation arising using the five-dimensional metric (2.5), i.e. \(\Box_5 \Phi = 0\). In both cases one makes the ansatz that the solution does not depend on the sphere coordinates, i.e. \(\Phi = e^{ikz} \phi(z)\). Since the solutions for the scalar \(\Phi\) should be the same in any consistent truncation of theory, the resulting second order ordinary differential equations should be identical. A comparison of terms proportional to \(\phi(z)\) determines the function \(H_0(z)\) as follows,
\[
H_0^{-1} = \frac{1}{R^2 f^{1/2}} \sum_{i=1}^{6} \frac{\hat{x}_i^2}{F - b_i} = \frac{1}{R^4 f^{1/2}} \sum_{i=1}^{6} \frac{y_i^2}{(F - b_i)^2} ,
\]
where in the second equality the harmonic function \(H_0\) has been expressed in terms of the transverse coordinates \(y_i\). Comparison of the terms proportional to the first and second derivative of \(\phi(z)\) yields, using the expression for \(\text{det} \hat{g}\) in (3.16), an identity and provides no further information. The coordinate \(F\) is determined in terms of the transverse coordinates \(y_i\) as a solution of the algebraic equation
\[
\sum_{i=1}^{6} \frac{y_i^2}{F - b_i} = 1 .
\]

This is a sixth order algebraic equation for general choices of the constants \(b_i\), and its solution cannot be written in closed form. However, this becomes possible when some of the \(b_i\)'s coincide in such a way that the degree of (3.18) is reduced to four or less. Even then, the resulting expressions are not very illuminating and we will refrain from presenting them except in the simplest case in section 5 below.

The corresponding D3-brane solution that is asymptotically flat is obtained by replacing \(H_0\) in (3.13) by \(H = 1 + H_0\). Then, in this context, the length parameter \(R\) has a microscopic interpretation using the string scale \(\alpha'\), the string coupling \(g_s\), and the (large) number of D3-branes \(N\), as \(R^4 = 4\pi g_s N\alpha'^2\).

In the rest of this section we demonstrate for completeness the proof that the function \(H_0\), as defined in (3.17), is indeed harmonic in the six-dimensional transverse space.
spanned by \( y_i, i = 1, 2, \ldots, 6 \). This is not a trivial check since \( F \) that appears in (3.17) is itself a function of the transverse space coordinates \( y_i \) due to the condition (3.18). For notational convenience we define the functions

\[
A_m = \sum_{i=1}^{6} \frac{y_i^2}{(F - b_i)^m}, \quad B_m = \sum_{i=1}^{6} \frac{1}{(F - b_i)^m}.
\]  

(3.19)

Then, using (3.18) we determine the derivative of the function \( F(y) \)

\[
\partial_i F = 2 \frac{y_i}{A_2(F - b_i)}. \tag{3.20}
\]

Also, the first derivative of \( H_0 \) with respect to \( y_i \) turns out to be

\[
\partial_i H_0 = -f^{-1/2} \frac{B_1 y_i}{A_2^2(F - b_i)} - 2f^{-1/2} \frac{y_i}{A_2^2(F - b_i)^2} + 4f^{-1/2} \frac{A_3 y_i}{A_3^2(F - b_i)}. \tag{3.21}
\]

Taking the derivative with respect to \( y_i \) once more, summing over the free indices and after some algebraic manipulations, we obtain the desired result

\[
\sum_{i=1}^{6} \partial_i^2 H_0 = 2f^{-1/2} \left( \frac{B_2}{A_2^2} - \frac{B_1 A_3}{A_2^3} \right) - 2f^{-1/2} \left( \frac{B_2}{A_2^2} - \frac{B_1 A_3}{A_2^3} \right) + 16f^{-1/2} \left( \frac{A_4}{A_2^3} - \frac{A_3^2}{A_4^2} \right) - 16f^{-1/2} \left( \frac{A_4}{A_2^3} - \frac{A_3^2}{A_2^3} \right) = 0, \tag{3.22}
\]

where the terms appearing in the three different lines above arise from the three distinct terms of (3.21) respectively.

## 4 Riemann surfaces in gauged supergravity

In this section we will present the basic mathematical aspects of our general ansatz for the supersymmetric conditions of five-dimensional gauged supergravity and find the means to obtain explicit solutions in several cases by appealing to methods of algebraic geometry. In fact, we will classify all possible solutions according to symmetry groups (subgroups of \( SO(6) \)) and use the uniformization of algebraic curves that result in this approach for deriving the corresponding expressions. To simplify matters the parameter \( R \) will be set equal to 1, but it can be easily reinstated by appropriate scaling in \( z \).

### 4.1 Schwarz–Christoffel transform

A useful way to think about the differential equation for the unknown function \( F(z) \) is in the context of complex analysis. Suppose that \( z \) and \( F \) are extended in the complex
domain and let us consider a closed polygon in the $z$-plane, including its interior, and map it via a Schwarz–Christoffel transformation onto the upper half $F$-plane. This provides a one-to-one conformal transformation and it is assumed that $F(z)$ is analytic in the polygon and is continuous in the closed region consisting of the polygon together with its interior. Considering the behaviour of $dz$ and $dF$ as the polygon is transversed in the counter-clockwise direction, we know that the transformation is described as

$$\frac{dz}{dF} = A(F - b_1)^{-\varphi_1/\pi}(F - b_2)^{-\varphi_2/\pi} \cdots (F - b_n)^{-\varphi_n/\pi}, \quad (4.1)$$

where $A$ is some constant that changes by rescaling $F$. The vertices of the polygon are mapped to the points $b_1, b_2, \ldots, b_n$ on the real axis of the upper complex $F$-plane and the exponents $\varphi_i$ that appear in the transformation are the exterior (deflection) angles of the polygon at the corresponding vertices. When the polygon is closed their sum is $\varphi_1 + \varphi_2 + \ldots + \varphi_n = 2\pi$. Of course, without loss of generality, we may take one point (say $b_n$) to infinity. Letting $A = B/(-b_n)^{-\varphi_n/\pi}$ we see that as $b_n \to \infty$ the Schwarz–Christoffel transformation becomes

$$\frac{dz}{dF} = B(F - b_1)^{-\varphi_1/\pi}(F - b_2)^{-\varphi_2/\pi} \cdots (F - b_{n-1})^{-\varphi_{n-1}/\pi}, \quad (4.2)$$

where $B$ is another constant factor. To make contact with our problem we choose $n = 7$ and let the angles $\varphi_1 = \varphi_2 = \cdots = \varphi_6 = \pi/4$. Then, we arrive at the differential equation

$$\left(\frac{dz}{dF}\right)^4 = B^4(F - b_1)^{-1}(F - b_2)^{-1} \cdots (F - b_6)^{-1}, \quad (4.3)$$

which is the same as the one implied by our ansatz for the general solution of gauged supergravity (with $B = 1/2$).

The solutions of this equation are difficult to obtain in practice for generic values of the moduli $b_i$. We will investigate this problem in connection with the theory of algebraic curves in $C^2$ and we will see that in many cases explicit solutions can be given using the theory of elliptic functions. Before proceeding further we note that in our formulation we are looking for the map from the interior of the polygon onto the upper half-plane, $F(z)$, and not for the inverse transformation.

### 4.2 Symmetries and algebraic curves

If we extend the variable $z$ to the complex domain, as before, and set

$$x = 4F(z), \quad y = 4F'(z), \quad \lambda_i = 4b_i, \quad (4.4)$$

the Schwarz–Christoffel differential equation will become an algebraic curve in $C^2$,

$$y^4 = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_6). \quad (4.5)$$

This is a convenient formulation for finding solutions of the supersymmetry equations, but at the end we have to restrict to real values of $z$ and demand that the resulting
supergravity fields $\beta_i$ are also real. For generic values of the parameters $b_i$, so that they are all unequal and hence there is no symmetry in the solution of five-dimensional gauged supergravity, the genus of the curve can be easily determined (like in any other case) via the Riemann–Hurwitz relation. Recall that for any curve of the form

$$y^m = (x - \lambda_1)^{\alpha_1}(x - \lambda_2)^{\alpha_2} \cdots (x - \lambda_n)^{\alpha_n}, \quad (4.6)$$

which is reduced, i.e. the integers $m$ and $\alpha_i$ have no common factors, and all $\lambda_i$'s are unequal, the genus $g$ can be found by first writing the ratios

$$\frac{\alpha_1}{m} = \frac{d_1}{c_1}, \quad \ldots, \quad \frac{\alpha_n}{m} = \frac{d_n}{c_n}; \quad \frac{\alpha_1 + \cdots + \alpha_n}{m} = \frac{d_0}{c_0} \quad (4.7)$$
in terms of relatively prime numbers and then using the relation

$$g = 1 - m + \frac{m}{2} \sum_{i=0}^{n} \left(1 - \frac{1}{c_i}\right). \quad (4.8)$$

According to this the genus of our surface turns out to be $g = 7$ when all $b_i$ are unequal, and so it is difficult to determine explicitly the solution in the general case. However, by imposing some isometries in the solution of gauged supergravity the genus becomes smaller and hence the problem becomes more tractable. The presence of isometries manifests by allowing for multiple branch points in the general form of the algebraic curve, which in turn degenerates along certain cycles that effectively reduce its genus.

Note for completeness that if we had not taken $b_7$ to infinity in our discussion of the Schwarz–Christoffel transformation, we would have had an additional factor $(x - \lambda_7)^2$ in the equation of the algebraic curve because $\varphi_7 = \pi/2$ instead of $\pi/4$ that was chosen for the remaining $\varphi_i$'s. It can be easily verified that this does not affect the genus of the curve, as expected on general grounds.

Next, we enumerate all possible cases with a certain degree of symmetry that correspond to various subgroups of $SO(6)$; this amounts to various deformations of the round five-sphere, $S^5$, which is used for the compactification of the theory from 10 to 5 dimensions. Consequently, this will in principle determine the solution for the scalar fields in the remaining 5 dimensions as we will see later in detail. If all the branch points are different, the $SO(6)$ isometry of $S^5$ will be completely broken, whereas if all of them coalesce to the same point the maximal isometry $SO(6)$ will be manifestly present. The classification is presented below in an order of increasing symmetry or else in decreasing values of $g$.

(1) $SO(2)$: It corresponds to setting two of the $\lambda_i$ equal to each other and the remaining are all unequal. The curve becomes

$$y^4 = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)(x - \lambda_4)(x - \lambda_5)^2 \quad (4.9)$$

and its genus turns out to be $g = 5$.

(2) $SO(3)$: It corresponds to setting three of the $\lambda_i$ equal and all other remain unequal. The curve becomes

$$y^4 = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)(x - \lambda_4)^3 \quad (4.10)$$
and the genus turns out to be $g = 4$.

3) $SO(2) \times SO(2)$: In this case two pairs of $\lambda_i$ are mutually equal and the remaining two parameters are unequal. The curve becomes

$$y^4 = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)^2(x - \lambda_4)^2$$

(4.11)

and its genus is $g = 3$.

4) $SO(3) \times SO(2)$: In this case three $\lambda_i$ are equal and another two are also equal to each other. The curve becomes

$$y^4 = (x - \lambda_1)(x - \lambda_2)^2(x - \lambda_3)^3$$

(4.12)

and its genus is $g = 2$. Therefore we know that it can be cast into a manifest hyper-elliptic form by introducing appropriate bi-rational transformations of the complex variables.

5) $SO(4)$: It corresponds to setting four $\lambda_i$ equal to each other and the other two remain unequal. The curve becomes

$$y^4 = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)^4$$

(4.13)

and its genus is $g = 1$. It can also be cast into a manifest (hyper)-elliptic form as we will see shortly.

6) $SO(2) \times SO(2) \times SO(2)$: It corresponds to three different pairs of mutually equal $\lambda_i$, but in this case the curve is not irreducible, since $y^4 = (x - \lambda_1)^2(x - \lambda_2)^2(x - \lambda_3)^2$. The reduced form is

$$y^2 = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)$$

(4.14)

and clearly has genus $g = 1$ as it is written directly in (hyper)-elliptic form.

7) $SO(3) \times SO(3)$: In this case we have two groups of triplets with equal values of $\lambda_i$. The curve becomes

$$y^4 = (x - \lambda_1)^3(x - \lambda_2)^3$$

(4.15)

and its genus is $g = 1$. It can also be cast into a manifest (hyper)-elliptic form.

8) $SO(5)$: In this case five $\lambda_i$ are equal to each other and the last remains different. The curve becomes

$$y^4 = (x - \lambda_1)(x - \lambda_2)^5$$

(4.16)

and its genus is also $g = 1$ as before.

9) $SO(4) \times SO(2)$: It corresponds to separating the $\lambda_i$ into four equal and another two equal parameters. The curve becomes $y^4 = (x - \lambda_1)^2(x - \lambda_2)^4$, but it is not irreducible. The reduced form is

$$y^2 = (x - \lambda_1)(x - \lambda_2)^2$$

(4.17)

and has genus $g = 0$, as it can also be obtained by degenerating a genus 1 surface along its cycles. Therefore, we expect the solution to be given in terms of elementary functions.
This is the case of maximal symmetry in which all $\lambda_i$ are set equal to each other. The curve becomes $y^4 = (x - \lambda_1)^6$, whose reduced form is

$$y^2 = (x - \lambda_1)^3$$

and has genus $g = 0$ as before.

Of course, when certain cycles contract by letting various branch points to coalesce, the higher genus surfaces reduce to lower genus and a bigger symmetry group emerges in the solutions corresponding to gauged supergravity. For genus $g \leq 2$ one can always transform to a manifest hyper-elliptic form so that two sheets (instead of four) are needed for picturing the Riemann surface by gluing sheets together along their branch cuts. We will investigate in detail the cases corresponding to genus 0 and 1 surfaces since the solutions can be given explicitly in terms of elementary and elliptic functions respectively. Some results about the genus 2 case will also be presented. The other cases are more difficult to handle in detail even though the general form of the solution is known implicitly for all $g$ according to our ansatz.

### 4.3 Genus 0 surfaces

There are two genus 0 surfaces according to the previous discussion, namely the curve $y^2 = (x - \lambda_1)^3$ for the isometry group $SO(6)$ and the curve $y^2 = (x - \lambda_1)(x - \lambda_2)^2$ for the isometry group $SO(4) \times SO(2)$. According to algebraic geometry every irreducible curve $f(x, y) = 0$ with genus 0 is representable as a unicursal curve (straight line)

$$w = v$$

by means of a bi-rational transformation $x(v, w), y(v, w)$ and conversely $v(x, y), w(x, y)$. In our two examples the underlying transformations are summarized as follows:

(a) $SO(6)$: We have

$$x = vw + \lambda_1, \quad y = vw^2$$

and conversely

$$v = (x - \lambda_1)^2, \quad w = \frac{y}{x - \lambda_1}.$$  \hspace{1cm} (4.21)

(b) $SO(4) \times SO(2)$: We have

$$x = vw + \lambda_1, \quad y = w(vw + \lambda_1 - \lambda_2)$$

and conversely

$$v = \frac{(x - \lambda_1)(x - \lambda_2)}{y}, \quad w = \frac{y}{x - \lambda_2}.$$  \hspace{1cm} (4.23)

Of course, the first curve arises as special case of the second for $\lambda_2 \to \lambda_1$. 

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For general \( \lambda_1 \) and \( \lambda_2 \) we may use \( u \) as a (trivial) uniformizing complex parameter for the unicursal curve, i.e. \( v = u = w \). Then, the expressions for \( x \) and \( y \) yield

\[
4F(z) = u^2 + 4b_1, \quad 4 \frac{dF(z)}{dz} = u \left( u^2 + 4(b_1 - b_2) \right),
\]

where we have taken into account the rescaling \( x = 4F(z), y = 4F'(z), \lambda_i = 4b_i \) that was introduced earlier. So we can determine \( u \) as a function of \( z \) by simple integration since

\[
\frac{du}{dz} = \frac{1}{2} \left( u^2 + 4(b_1 - b_2) \right).
\]

In fact there are three different cases for generic values of \( b_1 \) and \( b_2 \). Choosing appropriately the integration constant, so that the resulting conformal factor \( e^{2A(z)} \) will behave like \( 1/z^2 \) as \( z \to 0 \), we have:

(i) \[
u = -2\sqrt{b_1 - b_2} \cot \left( \sqrt{b_1 - b_2} z \right), \text{ for } b_1 > b_2,
\]

(ii) \[
u = -2\sqrt{b_2 - b_1} \coth \left( \sqrt{b_2 - b_1} z \right), \text{ for } b_2 > b_1,
\]

(iii) \[
u = -\frac{2}{z}, \text{ for } b_1 = b_2.
\]

The first two cases correspond to the \( SO(4) \times SO(2) \) isometry and they are obtained by analytic continuation from one other, depending on the size of the \( b_i \)'s, whereas the last case has \( SO(6) \) isometry. Here, we do not assume any given ordering among the \( b_i \)'s. As for the functions \( F(z) \) we have respectively

\[
F(z) = (b_1 - b_2) \cot^2 \left( \sqrt{b_1 - b_2} z \right) + b_1, \quad (b_2 - b_1) \coth^2 \left( \sqrt{b_2 - b_1} z \right) + b_1, \quad \frac{1}{z^2} + b_1.
\]

Then, the expression for the conformal factor of the metric is

(i) \[
e^{2A(z)} = (b_1 - b_2) \frac{\cos^{2/3} \left( \sqrt{b_1 - b_2} z \right)}{\sin^2 \left( \sqrt{b_1 - b_2} z \right)}, \text{ for } b_1 > b_2,
\]

(ii) \[
e^{2A(z)} = (b_2 - b_1) \frac{\cosh^{2/3} \left( \sqrt{b_2 - b_1} z \right)}{\sinh^2 \left( \sqrt{b_2 - b_1} z \right)}, \text{ for } b_2 > b_1,
\]

(iii) \[
e^{2A(z)} = \frac{1}{z^2}, \text{ for } b_1 = b_2,
\]

which indeed behaves as \( 1/z^2 \) in all three cases for \( z \to 0 \).

The solution for the scalar fields \( \beta_i(z) \) of five-dimensional gauged supergravity follows by simple substitution into our ansatz. We have explicitly in each case

(i) \[
e^{2\beta_1(z)} = e^{2\beta_2(z)} = \frac{1}{\cos^{4/3} \left( \sqrt{b_1 - b_2} z \right)}.
\]
\[ e^{2\beta_3(z)} = e^{2\beta_4(z)} = e^{2\beta_5(z)} = e^{2\beta_6(z)} = \cos^{2/3} \left( \sqrt{b_1 - b_2 z} \right), \quad (4.33) \]

(ii) \[ e^{2\beta_1(z)} = e^{2\beta_2(z)} = \frac{1}{\cosh^{4/3} \left( \sqrt{b_2 - b_1 z} \right)}, \]
\[ e^{2\beta_3(z)} = e^{2\beta_4(z)} = e^{2\beta_5(z)} = \cosh^{2/3} \left( \sqrt{b_2 - b_1 z} \right), \quad (4.34) \]

(iii) \[ e^{2\beta_i(z)} = 1, \quad i = 1, \ldots, 6. \quad (4.35) \]

Equally well we could have transformed the genus 0 curves into the quadratic form \( Y^2 = 1 - X^2 \) using the following transformation for the curve \( y^2 = (x - \lambda_1)(x - \lambda_2)^2 \)
\[ x = \frac{1 + X}{1 - X} + \lambda_1, \quad y = \frac{Y}{1 - X} \left( \frac{1 + X}{1 - X} + \lambda_1 - \lambda_2 \right) \quad (4.36) \]
and conversely
\[ X = \frac{x - \lambda_1 - 1}{x - \lambda_1 + 1}, \quad Y = \frac{2y}{(x - \lambda_1 + 1)(x - \lambda_2)}. \quad (4.37) \]

In this case we can use another uniformizing complex parameter \( u \), so that \( X = \sin u \) and \( Y = \cos u \), and proceed as above. Either way, the uniformization problem is solved in terms of elementary functions, which in turn determine the function \( F(z) \) every time and hence the particular supersymmetric solutions of five-dimensional gauged supergravity.

### 4.4 Genus 1 surfaces

Recall first that given a genus 1 algebraic curve in its Weierstrass form
\[ w^2 = 4v^3 - g_2 v - g_3 \quad (4.38) \]
the uniformization problem is solved by introducing the Weierstrass function \( \mathcal{P}(u) \) and its derivative \( \mathcal{P}'(u) \) with respect to a complex parameter \( u \). Then, \( v = \mathcal{P}(u) \) and \( w = \mathcal{P}'(u) \) in which case the Weierstrass function satisfies the time independent KdV equation \( \mathcal{P}'''(u) - 12\mathcal{P}(u)\mathcal{P}'(u) = 0 \). The two periods of the elliptic curve are denoted by \( 2\omega_1 \) and \( 2\omega_2 \) and the Weierstrass function is double periodic with respect to them. Also the values of the Weierstrass function at the half-periods coincide with the roots of the algebraic equation \( 4v^3 - g_2 v - g_3 = 0 \), namely \( e_1 = \mathcal{P}(\omega_1), \quad e_2 = \mathcal{P}(\omega_1 + \omega_2) \) and \( e_3 = \mathcal{P}(\omega_2) \). Conversely, given the differential equation
\[ \left( \frac{dG(z)}{dz} \right)^2 = 4G^3(z) - g_2 G(z) - g_3 \quad (4.39) \]
the general solution is described in terms of the Weierstrass function \( G(z) = \mathcal{P}(\pm z + a) \), where \( a \) is the constant of integration and \( g_2, \ g_3 \) are related as usual to the periods of the elliptic curve. This may be seen by taking a new dependent variable \( u \) defined by the equation \( G = \mathcal{P}(u) \), when the differential equation reduces to \( (du/dz)^2 = 1 \); for this
recall that the number of roots of the equation $P(u) = c$ that lie in any cell depend only on $P(u)$ and not on $c$, which can assume arbitrary values, like for any other elliptic function.

We have four different curves with genus 1 that follow from the classification that we described above. It is known from algebraic geometry that any genus 1 surface is hyper-elliptic (in particular elliptic since $g = 1$), but we see that only the one that corresponds to the case of $SO(2) \times SO(2) \times SO(2)$ isometry is essentially written in such form with roots $e_1 = \lambda_1$, $e_2 = \lambda_2$ and $e_3 = \lambda_3$ (when $\lambda_1 + \lambda_2 + \lambda_3 = 0$). The other three curves can be transformed to $w^2 = 4v^3 - g_2v - g_3$ for appropriately chosen coefficients $g_2$ and $g_3$ provided that one performs the necessary bi-rational transformations of the complex variables $v(x, y)$, $w(x, y)$ and conversely $x(v, w)$, $y(v, w)$. Only then the solution can be easily deduced from the resulting genus 1 curve in its Weierstrass form using elliptic functions. This is precisely what we are about to describe in the sequel.

Note first that all three curves that correspond to the symmetry groups $SO(4)$, $SO(3) \times SO(3)$ and $SO(5)$ can be transformed into the same curve

$$Y^4 = (X - \lambda_1)(X - \lambda_2)$$

(4.40)

according to the following bi-rational transformations

$$X = x, \quad Y = \frac{y}{x - \lambda_3} \quad \text{for} \quad y^4 = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)^4,$$

(4.41)

$$X = x, \quad Y = \frac{(x - \lambda_1)(x - \lambda_2)}{y} \quad \text{for} \quad y^4 = (x - \lambda_1)^3(x - \lambda_2)^3,$$

(4.42)

$$X = x, \quad Y = \frac{y}{x - \lambda_2} \quad \text{for} \quad y^4 = (x - \lambda_1)(x - \lambda_2)^5,$$

(4.43)

respectively. Then, defining

$$X - \lambda_1 = \frac{\eta^2}{\zeta}, \quad Y = \frac{\eta}{\zeta}$$

(4.44)

we arrive at the curve $\eta^2(1 - \zeta^2) = (\lambda_1 - \lambda_2)\zeta^3$ in all three cases. This simplifies further by defining new variables $V$, $W$ so that

$$\eta = \frac{W}{V}, \quad \zeta + 1 = \frac{1}{V},$$

(4.45)

in which case the curve becomes $W^2(2V - 1) = (\lambda_1 - \lambda_2)V(1 - V)^3$. Finally, letting

$$V = \frac{2v}{\lambda_2 - \lambda_1} + 1, \quad W = \frac{1}{\lambda_1 - \lambda_2} \left(v + \frac{1}{4}(\lambda_1 - \lambda_2)\right),$$

(4.46)

we obtain the genus 1 curve in its standard Weierstrass form

$$w^2 = 4v^3 - g_2v - g_3 \quad \text{with} \quad g_2 = \frac{1}{4}(\lambda_1 - \lambda_2)^2, \quad g_3 = 0$$

(4.47)

for all three cases of interest. This is a non-degenerate Riemann surface with $g = 1$, but it is more special than the $SO(2) \times SO(2) \times SO(2)$ surface since the latter depends on three
parameters \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) instead of the two \( \lambda_1 \) and \( \lambda_2 \) that appear in the Weierstrass form for higher (non-abelian) symmetry. Actually, in the present case we have \( \omega_2/\omega_1 = i \), and so by introducing the modulus of elliptic integrals, \( k \), and its complementary value \( k' \), one finds \( k = k' = 1/\sqrt{2} \).

We summarize the bi-rational transformations that are needed to transform each one of the genus 1 surfaces into their Weierstrass forms according to the symmetry groups of the solutions that they represent:

(a) \( SO(2) \times SO(2) \times SO(2) \): The curve \( y^2 = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3) \) can be brought into the standard Weierstrass form \( w^2 = 4v^3 - g_2v - g_3 \) with

\[
g_2 = \frac{1}{36} \left( (\lambda_1 + \lambda_2 - 2\lambda_3)^2 - (\lambda_2 + \lambda_3 - 2\lambda_1)(\lambda_1 + \lambda_3 - 2\lambda_2) \right),
\]

\[
g_3 = -\frac{1}{432} (\lambda_1 + \lambda_2 - 2\lambda_3)(\lambda_2 + \lambda_3 - 2\lambda_1)(\lambda_1 + \lambda_3 - 2\lambda_2),
\]

using the simple transformation

\[
y = 4w, \quad x = 4v + \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3).
\]

(b) \( SO(3) \times SO(3) \): The curve \( y^4 = (x - \lambda_1)^3(x - \lambda_2)^3 \) also transforms into the Weierstrass form \( w^2 = 4v^3 - g_2v \) with \( g_2 = (\lambda_1 - \lambda_2)^2/4 \) using

\[
x = \lambda_1 - \frac{1}{v} \left( v + \frac{1}{4}(\lambda_1 - \lambda_2) \right)^2, \quad y = \frac{w^3}{8v^3},
\]

and conversely

\[
v = \frac{4(\lambda_1 - \lambda_2)}{y^2 + (x - \lambda_1)(x - \lambda_2)},
\]

\[
w = \frac{1}{2}(\lambda_1 - \lambda_2) \frac{(x - \lambda_1)(x - \lambda_2) y^2 - (x - \lambda_1)(x - \lambda_2)^2}{y^2 + (x - \lambda_1)(x - \lambda_2)^2}.
\]

(c) \( SO(4) \): The curve \( y^4 = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)^4 \) transforms into the Weierstrass form \( w^2 = 4v^3 - g_2v \) with \( g_2 = (\lambda_1 - \lambda_2)^2/4 \) using

\[
x = \lambda_1 - \frac{1}{v} \left( v + \frac{1}{4}(\lambda_1 - \lambda_2) \right)^2,
\]

\[
y = \frac{w^3}{2v} \left( \lambda_1 - \lambda_3 - \frac{1}{v} \left( v + \frac{1}{4}(\lambda_1 - \lambda_2) \right)^2 \right),
\]

and conversely

\[
v = \frac{1}{4}(\lambda_2 - \lambda_1) \frac{y^2 - (x - \lambda_1)(x - \lambda_3)^2}{y^2 + (x - \lambda_1)(x - \lambda_3)^2},
\]

\[
w = \frac{1}{2}(\lambda_2 - \lambda_1) \frac{y^2 - (x - \lambda_1)(x - \lambda_3)^2}{x - \lambda_3 y^2 + (x - \lambda_1)(x - \lambda_3)^2}.
\]
(d) SO(5): This case arises from SO(4) when \( \lambda_3 \to \lambda_2 \), and so

\[
x = \lambda_1 - \frac{1}{v} \left( v + \frac{1}{4} (\lambda_1 - \lambda_2) \right)^2, \quad y = -\frac{w}{2v^2} \left( v - \frac{1}{4} (\lambda_1 - \lambda_2) \right)^2.
\] (4.58)

Note that in the three last cases (b)–(d) one may choose \( v = P(u) \) and \( w = P'(u) \), where \( u \) is the uniformizing parameter of the same Riemann surface. Thus, the \( x \)'s \((= 4F(z))\) are the same functions of \( u \) in these three cases, given in terms of Weierstrass functions and their derivatives, but the \( y \)'s \((= 4F'(z))\) are all different as can be readily seen. This simply means that the variable \( z \) is not equal to the uniformizing parameter \( u \) of the genus 1 curve in its Weierstrass form, but rather a more complicated function \( u(z) \) that has to be found in each case separately by integration (in analogy with what we did in the \( g = 0 \) cases). This complication does not arise for the case (a), since there we can take \( z = u \) (more generally \((du/dz)^2 = 1\), as we have already seen). For (a) the solution has already been very simply expressed in terms of the Weierstrass functions \( x = 4P(u) + (\lambda_1 + \lambda_2 + \lambda_3)/3 \) and \( y = 4P'(u) \), though of another Riemann surface with different coefficients \( g_2 \) and \( g_3 \).

Next, we take into account the field redefinitions \( x = 4F(z), y = 4F'(z), \lambda_i = 4b_i \) and solve for \( z(u) \) and its inverse \( u(z) \), when this is possible in closed form, thus determining \( F(z) \) in each case of interest. Of course, the elliptic functions that appear, refer to the corresponding curves with \( g_2 \) and \( g_3 \) determined as above. Summarizing the results, including some technical details, we have:

- \( \text{SO}(2) \times \text{SO}(2) \times \text{SO}(2) \): We have already seen that the uniformizing parameter \( u \) equals to \( z \) and hence

\[
F(z) = P(z) + \frac{1}{3}(b_1 + b_2 + b_3).
\] (4.59)

According to this we find

\[
e^{2A(z)} = \left( \frac{1}{2} P'(z) \right)^{2/3},
\] (4.60)

and so the conformal factor of the metric behaves as \( 1/z^2 \) when \( z \) approaches 0. The solution for the scalar fields of gauged supergravity follows by substitution into our general ansatz. We have in fact

\[
e^{2\beta_1(z)} = e^{2\beta_2(z)} = \frac{(P'(z)/2)^{2/3}}{P(z) - e_1},
\]

\[
e^{2\beta_3(z)} = e^{2\beta_4(z)} = \frac{(P'(z)/2)^{2/3}}{P(z) - e_2},
\] (4.61)

\[
e^{2\beta_5(z)} = e^{2\beta_6(z)} = \frac{(P'(z)/2)^{2/3}}{P(z) - e_3},
\]

where

\[
e_i = b_i - \frac{1}{3}(b_1 + b_2 + b_3), \quad i = 1, 2, 3.
\] (4.62)
• $SO(3) \times SO(3)$: This is the next simple case to consider. The relation between the differentials $dz$ and $du$ can be found by first computing $4dF/du$ as a function of $u$; it turns out to be $-P^3(u)/4P^3(u)$. Then, using the expression for $y = 4dF(z)/dz$ we arrive at the simple relation

$$\frac{du}{dz} = -\frac{1}{2}$$

(4.63)

and so $u = -z/2$, up to an integration constant that is taken zero. Consequently,

$$F(z) = b_1 - \frac{1}{4P(z/2)}(P(z/2) + b_1 - b_2)^2,$$

(4.64)

which in turn implies the following result for the conformal factor of the metric

$$e^{2A(z)} = \left(\frac{P(z/2)}{4P(z/2)}\right)^2 = \frac{1}{4} \left(\frac{b_1 - b_2}{P(z/2)}\right)^2.$$  

(4.65)

The derivative of the Weierstrass function is taken with respect to its argument $z/2$. The conformal factor clearly approaches $1/z^2$ as $z \to 0$, which justifies our choice of the integration constant above.

As for the solution corresponding to the scalar fields of gauged supergravity, we obtain by substitution into our general ansatz the result

$$e^{2\beta_1(z)} = e^{2\beta_2(z)} = e^{2\beta_3(z)} = -\frac{P(z/2) - b_1 + b_2}{P(z/2) + b_1 - b_2},$$

$$e^{2\beta_4(z)} = e^{2\beta_5(z)} = e^{2\beta_6(z)} = -\frac{P(z/2) + b_1 - b_2}{P(z/2) - b_1 + b_2},$$

(4.66)

which completes the task. At this point we add a clarifying remark, which takes into account the discrete symmetry $x \to -x$ and $b_i \to -b_i$ of the underlying algebraic curves. The uniformization that gave rise to eq. (4.64) implies that as $z$ ranges from 0 to $2\omega_1$, $F(z)$ ranges from $-\infty$ to $b_2$ (provided that $b_1 > b_2$ so that $P(\omega_1) \equiv e_1 = b_1 - b_2$). If one applies the discrete symmetry mentioned above, eq. (4.64) will change accordingly so that $F(z)$ will range from $+\infty$ to $b_1$ (taken as the maximum of the two moduli parameters). This particular symmetry implies in turn that the expressions for the scalar fields $\exp(2\beta_i)$ get modified by simply changing the overall sign according to the defining relation (3.3). Hence, despite appearances, the fields $\beta_i(z)$ are real provided that $z$ is real with values in the range where $F(z)$ is bigger than the maximum of $b_1$ and $b_2$, as it is usually taken.

• $SO(5)$: This case is computationally more difficult to handle. Since $4dF/du = -P^3/4P^3(u)$ again as a function of $u$, we find the following relation between the differentials $dz$ and $du$,

$$\frac{du}{dz} = \frac{1}{2} \frac{P(u) - b_1 + b_2}{P(u) + b_1 - b_2}.$$  

(4.67)

Then, integrating over $u$ we arrive at the formula

$$\frac{b_2 - b_1}{2}z = \zeta(u) + \frac{1}{2} \frac{P'(u)}{P(u) - b_1 + b_2},$$  

(4.68)
up to an integration constant that should be determined by the asymptotic behaviour $e^{2A(z)} \to 1/z^2$ as $z$ approaches 0. Here $\zeta(u)$ is the Weierstrass zeta-function. Note that the above expression will somewhat simplify if one uses the identity

$$\zeta(u + \omega_1) - \zeta(\omega_1) = \zeta(u) + \frac{P'(u)}{2P(u) - b_1 + b_2}, \quad \text{for } b_1 > b_2,$$

(4.69)

$$\zeta(u + \omega_2) - \zeta(\omega_2) = \zeta(u) + \frac{P'(u)}{2P(u) - b_1 + b_2}, \quad \text{for } b_2 > b_1,$$

(4.70)

where $\omega_1$ and $\omega_2$ are the half-periods of the curve. In either case, it is not possible to invert the relation and explicitly find $u(z)$ in closed form.

We give the result for the conformal factor of the metric as a function of $u$,

$$e^{2A} = \frac{1}{4P(u)}(P(u) + b_1 - b_2)^{1/3}(P(u) - b_1 + b_2)^{5/3}.$$  

(4.71)

Similar expressions are obtained for the scalar fields of gauged supergravity,

$$e^{2\beta_1} = - \left( \frac{P(u) - b_1 + b_2}{P(u) + b_1 - b_2} \right)^{5/3},$$

$$e^{2\beta_2} = \ldots = e^{2\beta_n} = - \left( \frac{P(u) + b_1 - b_2}{P(u) - b_1 + b_2} \right)^{1/3}.$$  

(4.72)

Similar remarks apply here for the overall sign appearing in eq. (4.72), as for the scalar fields of the model $SO(3) \times SO(3)$, using the discrete symmetry $x \to -x, b_i \to -b_i$ of the underlying algebraic curve.

We mention for completeness that as $b_1 \to b_2$ the Riemann surface degenerates and one recovers the $SO(6)$ model that was already discussed. It might seem that this contradicts the relation between $u$ and $z$ at first sight, since the left hand side becomes zero irrespective of $z$. However, for elliptic functions in the degeneration limit $g_2 = g_3 = 0$ we have $P(u) = 1/u^2$ and $\zeta(u) = 1/u$ for all $u$, and so the right hand side also becomes zero irrespective of $u$; hence there is no problem in taking this limit.

**$SO(4)$:** In this situation the calculation becomes even more involved. We find that

$$\frac{du}{dz} = \frac{1}{2} \left( \frac{(P(u) - b_1 + b_2)^2 - 4(b_2 - b_3)P(u)}{(P(u) + b_1 - b_2)(P(u) - b_1 + b_2)} \right),$$

(4.73)

but as in the $SO(5)$ case it is still not possible to find explicitly $u(z)$ in closed form. Besides, the integrals are more difficult to perform when $b_2 \neq b_3$ and so the resulting expressions are not very illuminating in terms of algebraic geometry. We postpone the presentation of the corresponding configuration for the next section, where a more geometrical approach is employed for it.
4.5 Genus 2 surface

Here we have only one such curve corresponding to the isometry group $SO(3) \times SO(2)$, which is described by the algebraic equation $y^4 = (x - \lambda_1)(x - \lambda_2)^2(x - \lambda_3)^3$. According to algebraic geometry it can be brought into a manifest hyper-elliptic form by performing appropriate bi-rational transformations. To achieve this explicitly we consider the following sequence of transformations: First, let

$$x = X, \quad y = \frac{(X - \lambda_3)(X - \lambda_2)}{Y},$$

that brings the curve into the form

$$(X - \lambda_1)Y^4 = (X - \lambda_2)^2(X - \lambda_3).$$

The second step consists in performing the transformation

$$X - \lambda_2 = \frac{\eta^2}{\zeta}, \quad Y = \frac{\eta}{\zeta},$$

that transforms it further into the form

$$\eta^2(1 - \zeta^2) = (\lambda_1 - \lambda_2)\zeta + (\lambda_2 - \lambda_3)\zeta^3.$$

Next, we introduce $V$ and $W$ so that

$$\eta = \frac{W}{V}, \quad \zeta + 1 = \frac{1}{V},$$

and the algebraic curve simplifies to

$$W^2(2V - 1) = (\lambda_1 - \lambda_2)V^3(1 - V) + (\lambda_2 - \lambda_3)V(1 - V)^3.$$

Finally, as last step let us consider

$$V = v, \quad W = \frac{w}{2v - 1},$$

which turns the curve into the desired hyper-elliptic form of genus two

$$w^2 = v(v - 1)(2v - 1)\left((\lambda_3 - \lambda_1)v^2 - 2(\lambda_3 - \lambda_2)v + \lambda_3 - \lambda_2\right),$$

with five distinct roots when all $\lambda_i$ are different from each other.

Summarizing the sequence of the above operations, which are similar to the genus 1 examples, we have for the transformation $x(v, w), y(v, w)$ the final result

$$x = \lambda_2 - \frac{(\lambda_3 - \lambda_1)v^2 - 2(\lambda_3 - \lambda_2)v + \lambda_3 - \lambda_2}{2v - 1}, \quad y = (\lambda_1 - \lambda_3)\frac{vw}{(2v - 1)^2},$$

whereas for its inverse $v(x, y), w(x, y)$ we have

$$v = \frac{(x - \lambda_2)(x - \lambda_3)^2}{(x - \lambda_2)(x - \lambda_3)^2 + y}, \quad w = \frac{y}{x - \lambda_3} \frac{(x - \lambda_2)(x - \lambda_3)^2 - y^2}{x - \lambda_3(x - \lambda_2)(x - \lambda_3)^2 + y^2}.$$
and so it is bi-rational, as required. These formulae are useful for addressing the uniformization problem of the original form of the curve in terms of theta functions. However, the resulting solution for gauged supergravity is rather complicated in this algebro-geometric context and we postpone its presentation for the next section using a different approach.

Before concluding this section note that the $SO(3) \times SO(3)$ model, which arises as $\lambda_1 \rightarrow \lambda_2$, corresponds in this context to the curve $w^2 = (\lambda_3 - \lambda_2)v(v-1)(2v-1)$, which according to the Riemann–Hurwitz relation has genus 1 as required; letting $w \rightarrow w(v-1)$, we see that the cubic form $w^2 = (\lambda_3 - \lambda_2)v(v-1)(2v-1)$ results in this case. Also, the $SO(5)$ model arises as $\lambda_2 \rightarrow \lambda_3$ and it corresponds in this context to the curve $w^2 = (\lambda_2 - \lambda_1)v^3(v-1)(2v-1)$, which again has genus 1, as required; it transforms, in turn, into the cubic form $w^2 = (\lambda_2 - \lambda_1)v(v-1)(2v-1)$ under the transformation $w \rightarrow wv$. Last, the $SO(4) \times SO(2)$ model is described by $w^2 = (\lambda_2 - \lambda_1)v(v-1)(2v-1)^2$ as $\lambda_1 \rightarrow \lambda_3$. This has genus 0 and it can be brought into a manifest quadratic form $w^2 = (\lambda_2 - \lambda_1)v(v-1)$ using the transformation $w \rightarrow w(2v-1)$. However, the bi-rational transformation for $y$ is appearing singular now, and the same is also true for the fully symmetric $SO(6)$ model; notice that for both of them the original form of the curve is not irreducible. Hence, we assume that the $SO(3) \times SO(2)$ model has $\lambda_1, \lambda_2, \lambda_3$ all different from each other (in particular $\lambda_1 \neq \lambda_3$). In any event, all previous models with genus 0 and 1 arise as special cases of $SO(3) \times SO(2)$ apart from the $SO(2) \times SO(2) \times SO(2)$ and the $SO(4)$ models that have already been described.

## 5 Examples

The five- as well as the ten-dimensional forms of our solutions preserve four-dimensional Poincaré invariance $ISO(1,3)$ along the three-brane, but for general values of the constants $b_i$, they have no other continuous isometries. In order to obtain some continuous group of isometries we have to choose some of the $b_i$’s equal. By means of (3.7) the corresponding scalars $\beta_i$ are also equal to one another. In this section we work out explicitly the expression for the metric and the scalar fields for some cases of particular interest using the ten-dimensional geometric frame where $F$ is more conveniently regarded as a coordinate instead of using $z$. We will present the models with isometry groups $SO(3) \times SO(2)$ and its limiting cases $SO(3) \times SO(3)$ and $SO(5)$, as well as the cases with isometry groups $SO(2) \times SO(2) \times SO(2)$, $SO(4)$ and their limiting model $SO(4) \times SO(2)$. They describe all solutions with genus $\leq 2$ from the point of view of the previous section. The examples are ordered by starting from the more general configurations and then specializing to models with higher symmetry.

The variable $z$ is more natural to use for addressing the uniformization problem of the algebraic curves underlying in our solutions. In here, we adapt our presentation to the ten-dimensional type-IIB supergravity description for two reasons: first as an alternative method for constructing explicit forms of our supersymmetric configurations, and second
for providing a higher dimensional view point for the compactification to five space-time dimensions, and naturally for questions regarding the AdS/CFT correspondence. To avoid confusion note that certain choices of the moduli $b_i$ made in the sequel differ slightly from those made in the previous section, but this should cause no problem.

5.1 Solutions with $SO(3) \times SO(2)$ symmetry

In this case it is convenient to use a basis for the unit vectors $\hat{x}_i$ that define the five-sphere in such a way that it is in one to one correspondence with the decomposition of the vector representation $6$ of $SO(6)$ with respect to the subgroup $SO(3) \times SO(2)$, as $6 \rightarrow (3, 1) \oplus (1, 2) \oplus (1, 1)$. Hence, we choose

$$\begin{align*}
\hat{x}_1 &= \cos \theta \cos \psi , \\
(\hat{x}_2) &= \cos \theta \sin \psi \left(\frac{\cos \varphi_1}{\sin \varphi_1}\right) , \\
(\hat{x}_3) &= \sin \theta \sin \omega \left(\frac{\cos \varphi_2}{\sin \varphi_2}\right) , \\
(\hat{x}_4) &= \sin \theta \sin \omega \left(\frac{\cos \varphi_1}{\sin \varphi_1}\right) , \\
(\hat{x}_5) &= \sin \theta \sin \omega \left(\frac{\cos \varphi_2}{\sin \varphi_2}\right) , \\
\hat{x}_6 &= \sin \theta \cos \omega .
\end{align*}$$

(5.1)

It is also convenient to choose the constants $b_i$ as follows

$$b_1 = b_2 = b_3 = 0 , \quad b_4 = b_5 = -l_1^2 , \quad b_6 = -l_2^2 ,$$

(5.2)

where $l_1$ and $l_2$ are real constants, thus ordering now the moduli $b_i$ in an increasing order according to (3.4). We moreover adopt the change of variable $F = r^2$ with $r \geq 0$, which is legitimate as $b_{\text{max}} = 0$. Then, the corresponding ten-dimensional metric takes the form

$$
\begin{align*}
ds^2 &= H^{-1/2} \eta_{\mu \nu} dx^\mu dx^\nu + H^{1/2} \frac{\Delta}{\Delta_1 \Delta_2} \, dr^2 \\
&+ r^2 H^{1/2} \left[ (\sin^2 \theta + \cos^2 \theta (\Delta_1 \sin^2 \omega + \Delta_2 \cos^2 \omega)) d\theta^2 \\
&+ \cos^2 \theta d\Omega_2^2 + \sin^2 \theta (\Delta_1 \cos^2 \omega + \Delta_2 \sin^2 \omega) d\omega^2 + \sin^2 \theta \sin^2 \omega d\varphi_2^2 \\
&+ 2 \cos \theta \sin \theta \cos \omega \sin \omega (\Delta_1 - \Delta_2) d\theta d\omega \right],
\end{align*}
$$

(5.3)

where the various functions appearing in it are

$$\begin{align*}
\Delta_1 &= 1 + \frac{l_1^2}{r^2} , \quad \Delta_2 = 1 + \frac{l_2^2}{r^2} , \\
\Delta &= \Delta_1 \Delta_2 \cos^2 \theta + \sin^2 \theta (\Delta_1 \cos^2 \omega + \Delta_2 \sin^2 \omega) , \\
H &= 1 + \frac{R^4 \Delta_1^{1/2}}{r^4} ,
\end{align*}$$

(5.4)

and $d\Omega_2^2$ is the two-sphere metric

$$d\Omega_2^2 = d\psi^2 + \sin^2 \psi d\varphi_1^2 .$$

(5.5)
In terms of five-dimensional gauged supergravity, the five-dimensional metric (3.6) is described by the form

\[ ds^2 = \frac{\Delta_1^{1/3}}{R^2} \frac{\Delta_2^{1/6}}{r^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{R^2}{r^2 \Delta_1^{2/3} \Delta_2^{1/5}} dr^2 \]  

(5.6)

and the expressions for the scalars (3.7) become

\[ e^{2\beta_1} = e^{2\beta_2} = e^{2\beta_3} = \frac{\Delta_1}{\Delta_2^{1/6}} , \]

\[ e^{2\beta_4} = e^{2\beta_5} = \frac{\Delta_1^{1/3} \Delta_2^{-2/3}}{\Delta_1^{1/6}} , \]

\[ e^{2\beta_6} = \frac{\Delta_1^{1/3} \Delta_2^{-5/6}}{\Delta_1^{1/6}} . \]

The metric (5.3) has a singularity at \( r = 0 \) where the harmonic function \( H \) diverges. However, this is not a point-like singularity as it occurs for all possible values of the angular variables \( \theta, \omega \) and \( \varphi_2 \). Hence, (5.3) may be interpreted as representing the distribution of a large number of D3-branes inside a solid three-dimensional ellipsoid defined by the equation

\[ \frac{y_1^2 + y_2^2}{l_1^2} + \frac{y_3^2}{l_2^2} = 1 , \]

(5.8)

and the three-dimensional hyper-plane \( y_1 = y_2 = y_3 = 0 \). We note that, by analytic continuation on the \( l_i \)'s we may obtain brane distributions other than (5.8), but we will not elaborate more on this point.

### 5.2 Solutions with \( SO(3) \times SO(3) \) symmetry

In this case we may obtain the metric by just setting \( l_1 = l_2 \equiv l \) in (5.3), since then the symmetry is enhanced, from \( SO(3) \times SO(2) \) to \( SO(3) \times SO(3) \). The metric becomes

\[ ds^2 = H^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + H^{1/2} \frac{r^2 + \lambda^2 \cos^2 \theta}{r^2 + l^2} \, dr^2 + H^{1/2} \left[ (r^2 + l^2 \cos^2 \theta) d\theta^2 + r^2 \cos^2 \theta d\Omega_2^2 + (r^2 + l^2) \sin^2 \theta d\tilde{\Omega}_2^2 \right] , \]

(5.9)

where the harmonic function \( H \) that follows from the corresponding expression in (5.4) is

\[ H = 1 + \frac{R^4}{r(r^2 + l^2 \cos^2 \theta)(r^2 + l^2)^{1/2}} \]

(5.10)

and the two different line elements for the two-dimensional sphere appearing in (5.9) are

\[ d\Omega_2^2 = d\psi^2 + \sin^2 \psi d\varphi_2^2 , \quad d\tilde{\Omega}_2^2 = d\omega^2 + \sin^2 \omega d\varphi_2^2 . \]

(5.11)

The field theory limit form of the metric (5.9) (with the 1 omitted in (5.10)) has appeared before in [13].
In the description in terms of gauged supergravity, the five-dimensional metric (3.6) takes the form
\[ ds^2 = \frac{r(r^2 + l^2)^{1/2}}{R^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{R^2}{r^2 + l^2} dr^2 \] (5.12)
and the expressions for the scalars (5.7) simplify to
\[ e^{2\beta_1} = e^{2\beta_2} = e^{2\beta_3} = \left(1 + \frac{l^2}{r^2}\right)^{1/2}, \]
\[ e^{2\beta_4} = e^{2\beta_5} = e^{2\beta_6} = \left(1 + \frac{l^2}{r^2}\right)^{-1/2}. \] (5.13)
Note that the five-dimensional metric, as well as the corresponding scalar fields, take the equivalent form (4.65) and (4.66), respectively, when written in terms of the variable \( z \).

Specializing (5.8) to the case at hand with \( l_1 = l_2 = l \), we deduce that the metric (5.9) represents the distribution of a large number of D3-branes inside the solid three-dimensional ball
\[ y_4^2 + y_5^2 + y_6^2 = l^2, \] (5.14)
in the three-dimensional hyper-plane defined by \( y_1 = y_2 = y_3 = 0 \).

### 5.3 Solutions with \( SO(5) \) symmetry

In this case we may obtain the metric by just setting \( l_1 = 0 \) (and also redefining \( l_2 \equiv l \)) in (5.3), since then the symmetry is enhanced from \( SO(3) \times SO(2) \) to \( SO(5) \). However, in order to present a metric with manifest \( SO(5) \) symmetry, the basis (5.1) is not appropriate. A convenient basis for the unit vectors \( \hat{x}_i \) should be such that it is in one to one correspondence with the decomposition of the vector representation \( 6 \) of \( SO(6) \), with respect to \( SO(5) \), as \( 6 \rightarrow 5 \oplus 1 \). Hence we choose
\[
\begin{align*}
\left( \hat{x}_1 \right) & = \cos \theta \sin \psi \left( \cos \varphi_1 \right) \\
\left( \hat{x}_2 \right) & = \cos \theta \sin \psi \left( \sin \varphi_1 \right) \\
\left( \hat{x}_3 \right) & = \cos \theta \cos \psi \sin \omega \left( \cos \varphi_2 \right) \\
\left( \hat{x}_4 \right) & = \cos \theta \cos \psi \cos \omega \\
\hat{x}_5 & = \sin \theta \cos \omega \\
\hat{x}_6 & = \sin \theta.
\end{align*}
\] (5.15)
The metric becomes
\[ ds^2 = H^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + H^{1/2} \frac{r^2 + l^2 \cos^2 \theta}{r^2 + l^2} dr^2 \]
\[ + \frac{H^{1/2}}{r^2 + l^2} \left[ (r^2 + l^2 \cos^2 \theta) d\theta^2 + r^2 \cos^2 \theta d\Omega_4^2 \right], \] (5.16)
where the harmonic function is
\[ H = 1 + \frac{R^4 (r^2 + l^2)^{1/2}}{r^3 (r^2 + l^2 \cos^2 \theta)}, \] (5.17)
and the line element for the four-sphere is defined as
\[ d\Omega^2_4 = d\psi^2 + \sin^2 \psi d\varphi_1^2 + \cos^2 \psi (d\omega^2 + \sin^2 \omega d\varphi_2^2). \] (5.18)

The field theory limit form of the metric (5.16) (with the 1 omitted in (5.17)) has also appeared before in [13].

The five-dimensional metric (3.6) takes the form
\[ ds^2 = \frac{r^5/3 (r^2 + l^2)^{1/6}}{R^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{R^2}{r^{4/3}(r^2 + l^2)^{1/3}} dr^2, \] (5.19)
whereas the expressions for the scalars (5.7) become
\[ e^{2\beta_1} = e^{2\beta_2} = e^{2\beta_3} = e^{2\beta_4} = e^{2\beta_5} = e^{2\beta_6} = \left(1 + \frac{l^2}{r^2}\right)^{1/6}, \] (5.20)

The singularity of the metric (5.16) for \( r = 0 \) may be interpreted as due to the presence of D3-branes distributed along the \( y_6 \) axis. This can be also obtained from (5.8) in the limit \( l_1 \to 0 \) (and \( l_2 \equiv l \)). In this limit, \( y_4 \) and \( y_5 \) are forced to be zero and therefore imposing (5.8) leads to \( y_6 = l \). Hence, the distribution of D3-branes is taken over a segment of length \( l \).

### 5.4 Solutions with \( SO(2) \times SO(2) \times SO(2) \) symmetry

In this case it is convenient to use a basis for the unit vectors \( \hat{x}_i \) that define the five-sphere in such a way that it corresponds to the decomposition of the vector representation \( 6 \) of \( SO(6) \) with respect to the full Cartan subgroup \( SO(2) \times SO(2) \times SO(2) \), as \( 6 \to (2, 1, 1) \oplus (1, 2, 1) \oplus (1, 1, 2) \). Hence, we choose
\[
\begin{align*}
( \hat{x}_1 ) &= \sin \theta \begin{pmatrix} \cos \varphi_1 \\ \sin \varphi_1 \end{pmatrix}, \\
( \hat{x}_2 ) &= \cos \theta \sin \psi \begin{pmatrix} \cos \varphi_2 \\ \sin \varphi_2 \end{pmatrix}, \\
( \hat{x}_3 ) &= \cos \theta \cos \psi \begin{pmatrix} \cos \varphi_3 \\ \sin \varphi_3 \end{pmatrix}.
\end{align*}
\] (5.21)

We also make the choice
\[ b_1 = b_2 \equiv a_1^2, \quad b_3 = b_4 \equiv a_2^2, \quad b_5 = b_6 \equiv a_3^2, \] (5.22)
where \( a_i, i = 1, 2, 3 \) are some real constants.
Using the change of variable $F = r^2$ (with $r \geq a_1$), the metric is written as

$$ds^2 = H^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + H^{1/2} \frac{\Delta r^4}{f} \, dr^2$$

$$+ r^2 H^{1/2} \left( \Delta_1 d\theta^2 + \Delta_2 \cos^2 \theta d\psi^2 + 2 \frac{a_2^2 - a_3^2}{r^2} \cos \theta \sin \psi \sin \psi \, d\theta d\psi \right)$$

$$+(1 - \frac{a_1^2}{r^2}) \sin^2 \theta \, d\varphi_1^2 + (1 - \frac{a_2^2}{r^2}) \cos^2 \theta \, d\varphi_1^2 + (1 - \frac{a_3^2}{r^2}) \cos^2 \theta \, d\varphi_3^2$$

(5.23)

where the various functions are defined as

$$H = 1 + \frac{R_4^4}{r^4 \Delta} ,$$

$$f = (r^2 - a_1^2)(r^2 - a_2^2)(r^2 - a_3^2) ,$$

$$\Delta = 1 - \frac{a_1^2}{r^2} \cos^2 \theta - \frac{a_2^2}{r^2} (\sin^2 \theta \sin^2 \psi + \cos^2 \psi) - \frac{a_3^2}{r^2} (\sin^2 \theta \cos^2 \psi + \sin^2 \psi)$$

$$+ \frac{a_2^2 a_3^2}{r^4} \sin^2 \theta + \frac{a_1^2 a_3^2}{r^4} \cos^2 \theta \sin^2 \psi + \frac{a_1^2 a_2^2}{r^4} \cos^2 \theta \cos^2 \psi ,$$

(5.24)

$$\Delta_1 = 1 - \frac{a_1^2}{r^2} \cos^2 \theta - \frac{a_2^2}{r^2} \sin^2 \theta \sin^2 \psi - \frac{a_3^2}{r^2} \sin^2 \theta \cos^2 \psi ,$$

$$\Delta_2 = 1 - \frac{a_3^2}{r^2} \cos^2 \psi - \frac{a_3^2}{r^2} \sin^2 \psi .$$

The metric (5.23), together with the defining relations (5.24), corresponds to the supersymmetric limit of the most general non-extremal rotating D3-brane solution [27]. Using this interpretation, it turns out that $a_1, a_2$ and $a_3$ correspond to the three rotational parameters of the solution, after a suitable Euclidean continuation. We also note that the metric (5.23) corresponds to the extremal limit of the three-charge black hole solution found in [27], in ansaetze for solutions to $N = 8, D = 5$ gauged supergravity preserving an $U(1)^3$ subgroup of SO(6) [28].

The five-dimensional metric (2.5) takes the form

$$ds^2 = \frac{\prod_{i=1}^3 (r^2 - a_i^2)^{1/3}}{R^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{R^2 r^2}{\prod_{i=1}^3 (r^2 - a_i^2)^{2/3}} \, dr^2 ,$$

(5.25)

whereas the expressions for the scalar fields are

$$e^{2\beta_1} = e^{2\beta_2} = (r^2 - a_1^2)^{-2/3} (r^2 - a_2^1)^{1/3} (r^2 - a_3^2)^{1/3} ,$$

$$e^{2\beta_3} = e^{2\beta_3} = (r^2 - a_1^2)^{1/3} (r^2 - a_2^2)^{-2/3} (r^2 - a_3^2)^{1/3} ,$$

$$e^{2\beta_5} = e^{2\beta_6} = (r^2 - a_1^2)^{1/3} (r^2 - a_2^1)^{1/3} (r^2 - a_3^2)^{-2/3} ,$$

(5.26)

The relationship to elliptic functions is made explicit by first using the definition (4.62), which is rewritten here in terms of three parameters $a_i$ as

$$e_i = a_i^2 - \frac{1}{3} (a_1^2 + a_2^2 + a_3^2) , \quad i = 1, 2, 3 .$$

(5.27)
Then, the differential equation (5.3) has as solution the Weierstrass elliptic function $P$

$$F(z) = P(z/R^2) ,$$

which is the same as (1.59) after ignoring the irrelevant additive constant. The invariants of the curve that define the Weierstrass elliptic function $P$ are

$$g_2 = -4(e_1e_2 + e_2e_3 + e_3e_1) , \quad g_3 = 4e_1e_2e_3 . \tag{5.29}$$

Since the Weierstrass function $P$ is double periodic with half-periods $\omega_1$ and $\omega_2$ given by

$$\omega_1 = \frac{K(k)}{\sqrt{e_1 - e_3}} , \quad \omega_2 = \frac{iK(k')}{\sqrt{e_1 - e_3}} , \tag{5.30}$$

where $K$ is the complete elliptic integral of the first kind with modulus $k$ and complementary modulus $k'$, we arrive at the following identification in terms of the rotational parameters

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3} = \frac{a_2^2 - a_3^2}{a_1^2 - a_3^2} ,$$

$$k'^2 = 1 - k^2 = \frac{e_1 - e_2}{e_1 - e_3} = \frac{a_1^2 - a_2^2}{a_1^2 - a_3^2} . \tag{5.31}$$

Finally, after changing variable

$$r = \sqrt{a_1^2 - a_3^2} \text{sn} u , \quad u \equiv \sqrt{a_1^2 - a_3^2} z , \tag{5.32}$$

where $\text{sn}u$ is the Jacobi function, the metric (5.23) assumes the conformally flat form (2.3) with

$$e^A = \frac{\sqrt{a_1^2 - a_3^2}}{R} \frac{\text{cn}^{1/3}u \, \text{dn}^{1/3}u}{\text{sn}u} = \frac{1}{R} \left( \frac{P'(z/R^2)}{2} \right)^{1/3} . \tag{5.33}$$

The last equality describes precisely the result found in (1.61) using the algebro-geometric method of uniformization. Also, in terms of the variable $z$, the scalar fields (5.26) coincide with (1.61).

### 5.5 Solutions with $SO(4) \times SO(2)$ symmetry

These solutions can be obtained by letting $e_2 = e_3$ (equivalently $a_2 = a_3$) into the various expressions of the previous subsection. In this limit, by taking into account the change of radial variable as $r^2 \to r^2 + a_2^2$, the metric (5.23) becomes

$$ds^2 = H^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + H^{1/2} \frac{r^2 - r_0^2 \cos^2 \theta}{r^2 - r_0^2} \, dr^2$$

$$+ H^{1/2} \left( (r^2 - r_0^2 \cos^2 \theta) \left( \frac{dr^2}{r^2 - r_0^2} + d\theta^2 \right) + (r^2 - r_0^2) \sin^2 \theta d\varphi_1^2 + r^2 \cos^2 \theta d\varphi_2^2 \right) . \tag{5.34}$$
where $r_0^2 \equiv a_1^2 - a_2^2$ and the harmonic function is

$$H = 1 + \frac{R^4}{r_0^2(r^2 - r_0^2 \cos^2 \theta)}$$

$$= 1 + \frac{2R^4}{\sqrt{(r_0^2 - r_0^2)^2 + 4r_0^2r_2^2}(r_0^2 + r_0^2 + \sqrt{(r_0^2 - r_0^2)^2 + 4r_0^2r_2^2})},$$

(5.35)

where $r_0^2 = y_1^2 + \ldots + y_6^2$ and $r^2 = y_1^2 + y_2^2$. In the second line of (5.35) we have written for completeness the harmonic function $H$ in terms of the Cartesian coordinates by explicitly substituting the function $F$ as a solution of the condition (3.18). The result agrees with what was obtained previously in [11, 12]. The three-sphere line element that appears in (5.34) is given by

$$d\Omega_3^2 = d\psi^2 + \sin^2 \psi d\varphi_2^2 + \cos^2 \psi d\varphi_3^2.$$  

(5.36)

The five-dimensional metric (2.5) takes the form [22]

$$ds^2 = \frac{r^{4/3}(r^2 - r_0^2)^{1/3}}{R^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{R^2}{r^{2/3}(r^2 - r_0^2)^{2/3}} dr^2,$$

(5.37)

whereas the expressions for the scalar fields are given by

$$e^{2\beta_1} = e^{2\beta_2} = \left(1 - \frac{r_0^2}{r^2}\right)^{-2/3},$$

$$e^{2\beta_3} = e^{2\beta_4} = e^{2\beta_5} = e^{2\beta_6} = \left(1 - \frac{r_0^2}{r^2}\right)^{1/3}.$$

(5.38)

Assuming that $r_0^2 > 0$, we find that the metric (5.34) has a singularity at $r = r_0$ and $\theta = 0$. This is not a point-like singularity as it occurs for general values of $\psi$, $\varphi_2$ and $\varphi_3$. It describes the situation where the horizon of the non-extremal metric coincides with the singularity as one approaches the extremal limit. The singularity of the metric (5.34) can be interpreted as arising from the presence of D3-branes distributed on a spherical shell [11, 12] defined in the $y_1 = y_2 = 0$ hyper-plane by the equation

$$y_3^2 + y_4^2 + y_5^2 + y_6^2 = r_0^2.$$

(5.39)

In the case that $e_1 = e_2$ (equivalently $a_2 = a_1$) it turns out that the previous results apply with $r_0^2 = a_3^2 - a_1^2 < 0$. It is then appropriate to define a new positive parameter by just letting $r_0^2 \rightarrow -r_0^2$. Then, the singularity of the metric (5.34) occurs at $r = 0$ and may be interpreted as coming from the presence of D3-branes distributed over a disc [11, 12], whose boundary is defined in the $y_3 = y_4 = y_5 = y_6 = 0$ hyper-plane by the circle

$$y_1^2 + y_2^2 = r_0^2.$$

(5.40)

It is instructive to recover the metric of five-dimensional gauged supergravity corresponding to our solution as a limiting case of (5.33), in analogy with the limiting
description of the ten-dimensional metric (5.34). To comment on this, let us first consider the limiting case $e_3 = e_2$ (or equivalently $a_3 = a_2$), where the modulus $k \simeq 0$ and the elliptic curve degenerates along the $a$-cycle. Then, using the well known properties of the Jacobi functions $\cnu \simeq \cos u$, $\snu \simeq \sin u$ and $\dnu \simeq 1$, that are valid for $k \simeq 0$, we obtain from (5.33) that the conformal factor in the corresponding five-dimensional metric (2.5) is given by

$$e^{2A} = \frac{r_0^2 \cos^{2/3}(r_0 z/R^2)}{R^2 \sin^2(r_0 z/R^2)},$$

(5.41)

where the variable $u$ in (5.32) becomes $u = r_0 / R^2 z$, with $r_0^2 = a_1^2 - a_2^2$, when $a_3 = a_2$. Another limiting case arises when $e_2 = e_1$ (or equivalently $a_2 = a_1$), in which the complementary modulus $k' \simeq 0$ and the elliptic curve degenerates along the $b$-cycle. Then, using the properties of the Jacobi functions for $k' \simeq 0$ we have $\cnu \simeq 1 / \cosh u$, $\snu \simeq \tanh u$ and $\dnu \simeq 1 / \cosh u$. From (5.33) we obtain that the conformal factor of the corresponding five-dimensional metric (2.5) becomes

$$e^{2A} = \frac{r_0^2 \cosh^{2/3}(r_0 z/R^2)}{R^2 \sinh^2(r_0 z/R^2)},$$

(5.42)

where we have used the fact that the variable $u$ in (5.32) becomes $u = r_0 / R^2 z$, with $r_0$ now defined as $r_0^2 = a_2^2 - a_3^2$, when $a_2 = a_1$. We note that the conformal factors appearing in (5.41) and (5.42) are the same as those found before in [22]. They are also precisely the same factors as those appearing in (4.30) and (4.31) by appropriate identification of the parameters and after reinstating the scale factor $R$ into the equations.

### 5.6 Solutions with $SO(4)$ symmetry

In this case we choose four of the constants $b_i$ equal to each other as follows

$$b_1 = -l_1^2, \quad b_2 = -l_2^2, \quad b_3 = b_4 = b_5 = b_6 = 0.$$  

(5.43)

Using the basis (5.21) for the $\hat{x}_i$’s we find that the metric takes the form

$$ds^2 = H^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + H^{1/2} \frac{\Delta}{\Delta_1 \Delta_2} dr^2$$

$$+ r^2 H^{1/2} \left[ \sin^2 \theta + \cos^2 \theta (\Delta_1 \cos^2 \varphi_1 + \Delta_2 \sin^2 \varphi_1) \right] d\theta^2$$

$$+ \cos^2 \theta d\Omega_3^2 + \sin^2 \theta (\Delta_1 \sin^2 \varphi_1 + \Delta_2 \cos^2 \varphi_1) d\varphi_1$$

$$+ 2 \cos \theta \sin \theta \cos \varphi_1 \sin \varphi_1 (\Delta_2 - \Delta_1) d\theta d\varphi_1,$$

(5.44)

where

$$\Delta_1 = 1 + \frac{l_1^2}{r^2}, \quad \Delta_2 = 1 + \frac{l_2^2}{r^2},$$

$$\Delta = \Delta_1 \Delta_2 \cos^2 \theta + \sin^2 \theta (\Delta_1 \sin^2 \varphi_1 + \Delta_2 \cos^2 \varphi_1),$$

$$H = 1 + \frac{R^4 \Delta_1^{1/2} \Delta_2^{1/2}}{r^4 \Delta}.$$  

(5.45)
The five-dimensional gauged supergravity metric (2.5) becomes

\[ ds^2 = \frac{r^2 \Delta_1^{1/6} \Delta_2^{1/6}}{R^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{R^2}{r^2 \Delta_1^{1/3} \Delta_2^{1/3}} \, dr^2 , \]  

and the scalar fields are given by

\[ e^{2\beta_1} = \Delta_1^{-5/6} \Delta_2^{1/6} , \]
\[ e^{2\beta_2} = \Delta_1^{1/6} \Delta_2^{-5/6} , \]
\[ e^{2\beta_3} = e^{2\beta_4} = e^{2\beta_5} = e^{2\beta_6} = \Delta_1^{1/6} \Delta_2^{1/6} . \]  

The metric (5.44) has a singularity at \( r = 0 \), where the harmonic function \( H \) in (5.45) blows up. It can be interpreted as being due to a continuous distribution of D3-branes in the ellipsoidal disc defined by

\[ \frac{y_1^2}{l_1^2} + \frac{y_2^2}{l_2^2} = 1 , \]  

lying in the \( y_3 = y_4 = y_5 = y_6 = 0 \) hyper-plane. Note also that in the case when \( l_1 = l_2 \), the symmetry is enhanced from \( SO(4) \) to \( SO(4) \times SO(2) \). Then, the expressions for the metric (5.44) and the scalars fields (5.47) coincide with those found in (5.34) and (5.38) respectively using the identification \( r_0^2 = -l_1^2 = -l_2^2 \). Also, when one of the \( l_i \)’s becomes zero, the symmetry is enhanced from \( SO(4) \) to \( SO(5) \) and by a suitable change of coordinates one recovers the results of subsection 4.3.

6 Spectrum for scalar and spin-two fields

In this section we investigate the problem of solving the differential equations for the massless scalar field as well as for the graviton fluctuations in our general five-dimensional background metric (2.5). We formulate the problem in terms of an equivalent Schrödinger equation in a potential that depends on the particular background. Later in this section we will discuss explicitly some cases of particular interest and determine the exact form of the corresponding potentials.

6.1 Generalities

We begin with the massless scalar field equation \( \Box_5 \Phi = 0 \) in the background geometry (2.5). In the context of the AdS/CFT correspondence, the solutions and eigenvalues of this equation have been associated with the spectrum of the operator \( \text{Tr} F^2 \) [29, 7, 30]. On the other hand, the fluctuations of the graviton polarized in the directions parallel to the brane are associated with the energy momentum tensor \( T_{\mu\nu} \) on the gauge theory side [29, 7, 30]. A priori, one expects that the spectra of the two operators \( \text{Tr} F^2 \) and \( T_{\mu\nu} \) are different. However, as was shown in [22], when graviton fluctuations on a three-brane
embedded in a five-dimensional metric as in (2.5) are considered, the two spectra and the corresponding eigenfunctions coincide. In particular, in order to study the graviton fluctuations, the Minkowski metric $\eta_{\mu\nu}$ along the three-brane is replaced in (2.5) by $\eta_{\mu\nu} + h_{\mu\nu}$ and then the equations of motion (2.4) are linearized in $h_{\mu\nu}$. Reparametrization invariance allows to gauge-fix five functions. In the gauge $\partial_\mu h_{\mu\nu} = h_{\mu\mu} = 0$, where indices are raised and lowered using $\eta_{\mu\nu}$ and its inverse, the graviton fluctuations obey the equation $\Box_5 h_{\mu\nu} = 0$, which is the same equation as that for a scalar field [22] (the same observation has been made in a slightly different context in [31]). Hence, the spectra for the operators $\text{Tr} F^2$ and $T_{\mu\nu}$ indeed coincide. In what follows, $\Phi$ will denote either a massless scalar field or any component of the graviton tensor field.

We proceed further by making the following ansatz for the solution

$$\Phi(x, z) = \exp(ik \cdot x)\phi(z),$$

which represents plane waves propagating along the three-brane with an amplitude function that is $z$-dependent. The mass-square is defined as $M^2 = -k \cdot k$. Using the expression for the metric in (2.5), we find that the equation for $\phi(z)$ is

$$\phi'' + 3A'\phi' + M^2 \phi = 0,$$

This equation can be cast into a Schrödinger equation for a wavefunction $\Psi(z)$ defined as $\Psi = e^{3A/2}\phi$. We find

$$-\Psi'' + V\Psi = M^2\Psi,$$

with potential given by

$$V = \frac{9}{4}A'^2 + \frac{3}{2}A''.$$

So far our discussion is quite general and applies to all solutions of the system of equations (2.4). When the solutions are supersymmetric we may use (3.2), (3.3) and (3.5) to find alternative forms for the potential, namely

$$V = \frac{e^{2A}}{16R^2} \left[ 3\left( \sum_{i=1}^{6} e^{2\beta_i} \right)^2 - 8 \sum_{i=1}^{6} e^{4\beta_i} \right],$$

$$= \frac{f^{1/2}}{16R^4} \left[ 3\left( \sum_{i=1}^{6} \frac{1}{F - b_i} \right)^2 - 8 \sum_{i=1}^{6} \frac{1}{(F - b_i)^2} \right].$$

(6.5)

This expression for the potential depends, of course, on the variable $z$ through the function $F(z)$. Even without having knowledge of the explicit $z$-dependence of the potential, we may deduce some general properties about the spectrum in the various cases of interest. Further details will be worked out in the following subsection using the results of section 4.

In general, $F$ takes values between the maximum of the constants $b_i$ (which according to the ordering made in (3.4) is taken to be $b_1$) and $+\infty$. When $F \to \infty$, the five-dimensional space approaches $AdS_5$ and the potential becomes

$$V \simeq \frac{15F}{4R^4}, \quad \text{as} \quad F \to \infty,$$

(6.6)
and hence it is unbounded from above. Let us now consider the behaviour of the potential close to the other end, namely when \( F \to b_1 \). Consider the general case where \( b_1 \) appears \( n \) times, as in the corresponding discussion made at the end of subsection 3.1. Using (6.3) we find that the potential behaves (including the subscript \( n \) to distinguish the various cases) as
\[
V_n \simeq \frac{f_0^{1/2}}{16 R^4} n(3n - 8)(F - b_1)^{2 - n}, \quad \text{as } F \to b_1,
\]
with \( f_0 \) being a constant given, as before, by \( f_0 = \prod_i (b_1 - b_i) \). Hence, for the value \( n = 6 \), corresponding to \( AdS_5 \) the potential goes to zero and the spectrum is continuous. The same is true for the value \( n = 5 \) corresponding to the \( SO(5) \) symmetric model. For the case \( n = 4 \) the potential approaches a constant value with the metric given by (5.44). Using the definitions (5.43), the general expression for the the minimum value of the potential is in this case
\[
V_{4,\text{min}} = \frac{l_1 l_2}{R^4}.
\]
Therefore, although the spectrum is continuous, it does not start from zero, but there is a mass gap whose value squared is given by the minimum of the potential in (6.8). For the \( SO(4) \times SO(2) \) model, where the metric is given by (5.34) with \( r_0^2 < 0 \), the existence of a mass gap has already been proven in [13, 14]. For \( n = 3 \) the potential goes to \( +\infty \) as \( F \to b_1 \) and therefore the spectrum is not continuous but discrete. Quite generally we may show, using simple scaling arguments, that the typical unit of mass square is \( f_0^{1-\nu}/R^4 \). Hence, for \( n = 3 \), we expect that \( M^2 \) will be quantized in units of \( f_0^{1/3}/R^4 \). For \( n = 2 \) the potential goes to \( -\infty \) and there is the danger that it is unbounded from below. Nevertheless, at least for the \( SO(4) \times SO(2) \) symmetric model with metric given by (5.34) with \( r_0^2 > 0 \), it was shown before that \( M^2 \) is discrete and positive [13, 14].

### 6.2 Examples of potentials

#### 6.2.1 \( SO(6) \)

Let us consider first the massless scalar equation for the most symmetric case, namely when the background is given by the \( AdS_5 \) metric itself. In this case we have \( e^A = R/z \) and the potential becomes
\[
V(z) = \frac{15}{4 z^2}, \quad 0 \leq z < \infty,
\]
which obviously has a continuous spectrum for \( M^2 \). The corresponding Schrödinger equation can be transformed into a Bessel equation and the result for the amplitude of the fluctuations (6.1) is given by
\[
\phi_M = (MR)^{-3/2}(Mz)^2 J_2(Mz),
\]
where \( J_2 \) is the Bessel function of index 2, which is regular at the origin \( z = 0 \). The arbitrary overall constant is chosen so that the Dirac-type normalization condition is
satisfied
\[ \int_0^\infty dz e^{3A} \phi_M \phi_{M'} = \delta(M - M') \, . \] (6.11)

The measure factor \( e^{3A} \) in the integrand of the equation above is such that the Schrödinger wave function \( \Psi = e^{3A/2} \phi \) obeys a normalization condition similar to (6.11), but with measure 1.

### 6.2.2 \( SO(4) \times SO(2) \)

Consider now the first non-trivial case with metric given by (5.34) with \( r_0^2 > 0 \). The spectrum for massless scalar and graviton fluctuations has already been analyzed in [13, 14] and [22], respectively. We include this case here not only for completeness, but also because we will make connections with Calogero-type models later. The explicit form for the potential turns out to be

\[ V(z) = \frac{r_0^2}{R^4} \left( -1 - \frac{1}{4 \cos^2(r_0 z/R^2)} + \frac{15}{4 \sin^2(r_0 z/R^2)} \right) , \quad 0 \leq z \leq \frac{\pi R^2}{2r_0} , \] (6.12)

and clearly possesses the features we have discussed at the beginning of this section. In fact (6.12) belongs to a family of potentials called Pöschl–Teller potential of type I in the literature of elementary quantum mechanics. The solution for the massless scalar or graviton fluctuations is given by (6.1) with the quantized amplitude modes being given by [14]

\[ \phi_n = \sqrt{\frac{(2n + 3)r_0}{8R^2}} (1 - x)^2 P_n^{(2,0)}(x) , \quad x = 1 - 2\frac{r_0^2}{r^2} = \cos(2r_0 z/R^2) , \quad n = 0, 1, \ldots , \] (6.13)

where in general \( P_n^{(\alpha,\beta)} \) denote the classical Jacobi polynomials. Note that the arbitrary overall constant in (6.13) has been chosen so that the \( \phi_n \)'s are normalized to 1 with measure \( e^{3A} \), similar to (6.11), where \( A \) is now given by (5.41). The associated mass spectrum is

\[ M_n^2 = \frac{4r_0^2}{R^4}(n + 1)(n + 2) , \quad n = 0, 1, \ldots . \] (6.14)

Let us now turn to the case of the metric (5.34) with \( r_0^2 < 0 \) and replace \( r_0^2 \) by \(-r_0^2\). Then, the potential takes the form (also given in [22])

\[ V(z) = \frac{r_0^2}{R^4} \left( 1 + \frac{1}{4 \cosh^2(r_0 z/R^2)} + \frac{15}{4 \sinh^2(r_0 z/R^2)} \right) , \quad 0 \leq z < \infty , \] (6.15)

which is the so called Pöschl–Teller potential of type II. Note that this potential is related to the one appearing in (6.12) by analytic continuation \( r_0 \to ir_0 \), as expected. This potential approaches the value \( r_0^2/R^4 \) as \( z \to \infty \) and therefore its spectrum is continuous, but with a mass gap given by [13, 14]

\[ M_{\text{gap}}^2 = \frac{r_0^2}{R^4} . \] (6.16)
In this case the solution for the massless scalar and the graviton fluctuations is given by (6.1) with amplitude

$$\phi_q \sim x^{(q-1)/2}F_q(x) - x^{-(q+1)/2}F_{-q}(x), \quad 0 \leq x = \frac{r^2}{r_0^2 + r^2} = \frac{1}{\cosh^2(r_0 zu/R^2)} \leq 1.$$  (6.17)

The constant $q$ and the function $F_q(x)$ are related to the mass $M$ via hypergeometric functions as

$$F_q(x) = F\left(\frac{q - 1}{2}, \frac{q - 1}{2}, 1 + q; x\right), \quad q = \sqrt{1 - R^2 M^2}.$$  (6.18)

Note that the constant $q$ is purely imaginary due to the mass gap of the model.

### 6.2.3 $SO(2) \times SO(2) \times SO(2)$

Since this case has not been discussed in the literature, we will explain the derivation of the potential $V(z)$ in some detail. Using the variable $u = z/R^2$, for simplicity, we find according to the definition that the potential becomes

$$V(u) = \frac{1}{16R^4}P'^2(u)\left(3\left(\frac{1}{P(u) - e_1} + \frac{1}{P(u) - e_2} + \frac{1}{P(u) - e_3}\right)^2 - \frac{4}{(P(u) - e_1)^2} - \frac{4}{(P(u) - e_2)^2} - \frac{4}{(P(u) - e_3)^2}\right),$$  (6.19)

where $e_1 + e_2 + e_3 = 0$ for the roots of the corresponding elliptic curve. Then, using the addition theorem for the Weierstrass function we have

$$\frac{1}{4}\left(\frac{P'^2(u)}{(P(u) - e_1)^2} + \frac{P'^2(u)}{(P(u) - e_2)^2} + \frac{P'^2(u)}{(P(u) - e_3)^2}\right) = 3P(u) + P(u + \omega_1) + P(u + \omega_2) + P(u + \omega_1 + \omega_2)$$  (6.20)

and so a straightforward calculation yields the final result

$$V(z) = \frac{1}{4R^4}\left(15P\left(\frac{z}{R^2}\right) - P\left(\frac{z}{R^2} + \omega_1\right) - P\left(\frac{z}{R^2} + \omega_2\right) - P\left(\frac{z}{R^2} + \omega_1 + \omega_2\right)\right)$$  (6.21)

with the dependence on $R$ appearing now explicitly.

It is easy to see how the degeneration of the curve leads to the rational potential of the $SO(4) \times SO(2)$ model. Recall that for $e_1 \neq e_2 = e_3$, i.e. for elliptic modulus $k = 0$, the $\alpha$-cycle of the Riemann surface shrinks to zero size and the Weierstrass function simplifies to

$$P(u) = -\frac{3g_3}{2g_2} + \frac{9g_3}{2g_2 \sin^2\left(u\sqrt{\frac{g_{45}}{2g_2}}\right)}.$$  (6.22)

In this limiting case we have $9g_3/2g_2 = a_1^2 - a_2^2$, whereas $a_2 = a_3$, using the rotational parameters of our ten-dimensional solution. Since this combination equals to $r_0^2$, we
obtain the trigonometric function \( \sin(\omega_1 z/\xi) \). In this limit we also have \( \omega_1 = \pi/2\xi \) and \( \omega_2 = i\infty \), and so \( \mathcal{P}(u + \omega) \) will involve the function \( \cos(\omega_1 z/\xi) \), while the terms originating from \( \mathcal{P}(u + \omega_2) \) and \( \mathcal{P}(u + \omega_1 + \omega_2) \) contribute only to the constant. In this fashion we recover the potential of the \( \text{SO}(4) \times \text{SO}(2) \) model. Its hyperbolic counterpart appears when \( r_0^2 < 0 \) and so the new potential can be obtained by suitable analytic continuation.

Note finally that in the general case the potential becomes infinite at the Weierstrass points \( 0, \omega_1, \omega_2, \omega_1 + \omega_2 \), because the Weierstrass function blows up at \( 0 \) modulo the periods; put differently, some term of the potential becomes infinite at each one of these points. Unfortunately, we do not have complete grasp of the spectrum for the Schrödinger equation in this potential. We hope that its computation will be discussed elsewhere.

### 6.2.4 \( \text{SO}(3) \times \text{SO}(3) \)

This case also leads to a new form for the potential that has not been investigated before. Again, using for simplicity the parameter \( u = z/(2\xi) \), since for \( \xi = 1 \) the uniformizing parameter is \( u = -z/2 \), and the minus sign plays no rôle in \( V \), we have for our solution (with general \( \xi \))

\[
V(u) = \frac{3}{256\xi^4} \frac{\mathcal{P}''(u)}{\mathcal{P}^2(u)} \left( \left( \mathcal{P}(u) - b_1 + b_2 \right)^2 + \left( \mathcal{P}(u) + b_1 - b_2 \right)^2 + 18 \right). \tag{6.23}
\]

This follows by substitution of our algebro-geometric solution into the defining relation of the potential, after reinstating the \( \xi \)-dependence. The elliptic curve has presently \( g_3 = 0 \) and so \( e_2 = 0, e_1 = -e_3 \). By employing the identities, special to this surface,

\[
\mathcal{P}'(u + \omega_1) = -2e_1^2 \frac{\mathcal{P}'(u)}{(\mathcal{P}(u) - e_1)^2}, \quad \mathcal{P}'(u + \omega_2) = -2e_3^2 \frac{\mathcal{P}'(u)}{(\mathcal{P}(u) - e_3)^2}, \tag{6.24}
\]

where \( e_1^2 = e_3^2 = (b_1 - b_2)^2 \), we arrive after some calculation at the final result for the potential

\[
V(z) = \frac{3(b_1 - b_2)^4}{4\xi^4} \left( \frac{1}{\mathcal{P}^2 \left( \frac{z}{2\xi^2} + \omega_1 \right)} + \frac{1}{\mathcal{P}^2 \left( \frac{z}{2\xi^2} + \omega_2 \right)} + \frac{18}{\mathcal{P}' \left( \frac{z}{2\xi^2} + \omega_1 \right) \mathcal{P}' \left( \frac{z}{2\xi^2} + \omega_2 \right)} \right). \tag{6.25}
\]

Note that this potential also becomes infinite at the four Weierstrass points, but its structural dependence on elliptic functions seems to be different from the previous example. However, making use of some further identities (special to the curves with \( g_3 = 0 \)), it can be cast into a form proportional to

\[
\frac{15}{4} \mathcal{P} \left( \frac{z}{2\xi^2} \right) + \frac{3}{4} \mathcal{P} \left( \frac{z}{2\xi^2} + \omega_1 \right) + \frac{15}{4} \mathcal{P} \left( \frac{z}{2\xi^2} + \omega_1 + \omega_2 \right) + \frac{3}{4} \mathcal{P} \left( \frac{z}{2\xi^2} + \omega_2 \right), \tag{6.26}
\]

which is invariant under shifts with respect to \( \omega_1 + \omega_2 \). We leave the computation of the spectrum for the corresponding Schrödinger equation to future investigation, as before.
For the other examples we have been unable to derive the form of the potential in closed form, because there are no closed formulae for the solutions in terms of the variable \(z\) that appears naturally in the corresponding Schrödinger equation, or else in the algebro-geometric description of the various models as Riemann surfaces. We close this section with some general remarks concerning the rational and elliptic variations of Calogero-like models.

### 6.3 Comments

It is rather amusing that the Schrödinger problem one has to solve in \(z\) is of Calogero type. This is a characteristic feature of \(\text{AdS}_5\) and possibly of more general \(\text{AdS}\) spaces. For the maximally symmetric \(SO(6)\) model, in particular, the potential is precisely \(V(z) = 1/z^2\) (up to an overall scale) \([31]\). It can be seen that the solutions of the less symmetric \(SO(4) \times SO(2)\) model are also related to a Calogero problem, namely the three-body model in one dimension. Recall that the quantum states of the general three-body problem can be found by solving the time independent Schrödinger equation

\[
\left( -\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} + 2g \sum_{i,j=1}^{3} \frac{1}{(x_i - x_j)^2} + 6f \sum_{i,j,k=1}^{3} \frac{1}{(x_i + x_j - 2x_k)^2} - E \right) \Psi(x_1, x_2, x_3) = 0 ,
\]

for \(i \neq j \neq k\), where the \(x_i\)'s describe the coordinates of the three particles and \(g, f\) denote the strength of the two-body and three-body Calogero interactions respectively. In fact, from a group theory point of view, this potential describes couplings between the particles according to the root system of the simple Lie algebra \(G_2\). Introducing the center of mass coordinates

\[
\sqrt{2} r \sin \varphi = x_1 - x_2 , \quad \sqrt{6} r \cos \varphi = x_1 + x_2 - 2x_3 , \quad 3R_{\text{cm}} = x_1 + x_2 + x_3 , \quad (6.28)
\]

one obtains, after moding out the \(R_{\text{cm}}\)-dependence, a differential equation for the wavefunction \(\Psi(r, \varphi)\). It can be separated, as usual, into two independent equations for the radial and angular dependence of the wavefunctions

\[
\left( -\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{\lambda^2}{r^2} - E \right) X(r) = 0 , \quad (6.29)
\]

\[
\left( -\frac{\partial^2}{\partial \varphi^2} + \frac{9g}{\sin^2(3\varphi)} + \frac{9f}{\cos^2(3\varphi)} - \lambda^2 \right) \Phi(\varphi) = 0 , \quad (6.30)
\]

where \(\lambda^2\) is the separation constant, \(\Psi(r, \varphi) = X(r)\Phi(\varphi)\), and \(E\) is now the energy in the center of mass frame \([32]\). Of course, the Hamiltonian is Hermitian provided that the coupling constants \(g \geq -1/4\) and \(f \geq -1/4\). For \(f = 0\) only two-body interactions are present in the problem.

The integrability of the classical Calogero model persists quantum mechanically and helps us to determine its spectrum and wave eigenfunctions. In particular, the differential
equation for the angular dependence is solved in general as follows,

\[ \Phi_n(\varphi) = \sin^{\mu_1}(3\varphi)\cos^{\mu_2}(3\varphi)P_n^{\mu_1-1/2,\mu_2-1/2}(\cos(6\varphi)), \]

\[ \lambda_n^2 = 9(2n + \mu_1 + \mu_2)^2, \quad n = 0, 1, 2, \ldots, \tag{6.31} \]

where the \( \mu_i \)'s are introduced as

\[ g = \mu_1(\mu_1 - 1), \quad f = \mu_2(\mu_2 - 1). \tag{6.32} \]

The angle \( \varphi \) assumes the values between 0 and \( \pi/6 \); because of its dependence on the Cartesian coordinates \( x_i \), a particular value of \( \varphi \) gives a specific ordering of the three particles and hence the problem can be divided into sectors depending on the range of \( \varphi \). For a general overview of these issues, see for instance [33] and references therein. Interestingly enough, the Schrödinger problem that arose in studying the spectrum of quantum fluctuations for scalar and spin-two fields in the background of the \( SO(4) \times SO(2) \) model of five-dimensional gauged supergravity fits precisely into the integrable class of such Calogero potentials with \( \mu_1 = 5/2 \) and \( \mu_2 = 1/2 \), which thus attains its minimum value required by hermiticity. Note, however, that presently \( \mu_1 \neq \mu_2 \). To make exact contact with our problem for the \( SO(4) \times SO(2) \) model, first introduce the necessary rescaling with respect to \( R \), setting \( 3\varphi = r_0z/R^2 \), and then conclude that the mass spectrum is given in general by

\[ M_n^2 = \frac{r_0^2}{R^4} \left( \frac{\lambda_n^2}{9} - 1 \right), \quad n = 0, 1, 2, \ldots, \tag{6.33} \]

as there is also a constant term which is present now that shifts the energy levels. For the values \( \mu_1 = 5/2 \) and \( \mu_2 = 1/2 \) the spectrum (6.33) coincides with that in (6.14).

On the other hand, the elliptic generalization of the \( 1/z^2 \) potential arose historically more than a century ago in connection with the problem of finding ellipsoidal harmonics for the 3-dim Laplace equation. When one deals with physical problems connected to ellipsoids, like having sources with a general ellipsoidal distribution, the mathematical structure of spheres, cones and planes usually associated to polar coordinates gets replaced by the structure of confocal quadrics. Since the transformation from Cartesian coordinates is not singled valued, elliptic functions are employed for its proper description. Introducing uniformizing parameters associated with confocal coordinates, it turns out that the solutions of Laplace’s equation are obtained by separation of variables, in which case one arrives at the Lamé equation

\[ \left( -\frac{d^2}{dz^2} + n(n+1)P(z) - E \right) \Psi(z) = 0 \tag{6.34} \]

for harmonics of degree \( n \); \( E \) is a separation constant that appears in the mathematical analysis of the problem (for details see [34]). It is interesting to note that this particular Schrödinger problem was fully investigated much later in connection with finite zone potentials, Riemann surfaces and the KdV hierarchies (see for instance [35] and references.
therein), since the Weierstrass function satisfies the time independent KdV equation. It comes as no surprize that potentials consisting of Weierstrass functions also arise in our study, because the relevant configurations can be obtained from distributions of D3-branes on ellipsoids in ten dimensions, as it has already been noted in the geometrical setting of our solutions. In this sense, all potentials that occur in the supergravity models with genus \( g > 0 \) should be considered as appropriate generalizations of the original derivation of Lamé’s equation in a ten-dimensional IIB framework.

Multi-particle systems with two-body interactions described by \( \mathcal{P}(z_i - z_j) \) have also been studied extensively as integrable systems \(^{33}\). However, the trigonometric identities used earlier for expressing \( \sin 3\varphi \) and \( \cos 3\varphi \) in terms of \( \sin \varphi \) and \( \cos \varphi \) for the rational three-body Calogero model, thus arriving at a separation of the angular \( \varphi \) dependence, are not generalizable to elliptic functions. Hence, there is no analogous understanding of the Schrödinger equation that determines the spectrum of scalar and spin-two fields in the five-dimensional background of our elliptic configurations using many-body elliptic Calogero systems. To the best of our knowledge, the specific quantum problems that arise here have not been investigated and pose a set of interesting questions for future work.

We mention for completeness that the only problem which has been studied in detail among the class of potentials given by a sum of Weierstrass functions concerns the Schrödinger equation with

\[
V(z) = 2 \sum_{i=1}^{n} \mathcal{P}(z - z_i(t))
\]

(6.35)

when \( z_i(t) \) are moduli that evolve in time as elliptic Calogero particles with two-body interactions only, namely

\[
\frac{d^2 z_i(t)}{dt^2} = 4 \sum_{i \neq j} \mathcal{P}'(z_i - z_j).
\]

(6.36)

Such systems are naturally encountered in the description of elliptic solutions of the KP equation, in analogy with the rational solutions of the KP equation where the ordinary \( 1/z^2 \) Calogero models make their appearance (see for instance \(^{36}\)). A static solution is easily obtained by considering four such particles located at the corners of a parallelogram inside the fundamental domain of elliptic functions described by the points \( z_1 = 0, z_2 = \omega_1, z_3 = \omega_2 \) and \( z_4 = \omega_1 + \omega_2 \). In this case, their differences \( z_i - z_j \) equal to half-periods (modulo the periods) for all \( i \neq j \), so the derivative of the Weierstrass function vanishes there and the elliptic Calogero equations are trivially satisfied. Then, the potential for the Schrödinger equation becomes

\[
V(z) = 2 (\mathcal{P}(z) + \mathcal{P}(z + \omega_1) + \mathcal{P}(z + \omega_2) + \mathcal{P}(z + \omega_1 + \omega_2)),
\]

(6.37)

which by the way equals to \( 8\mathcal{P}(2z) \) and reduces to the usual Lamé equation with \( n = 1 \) after rescaling \( z \). The generalization to potentials consisting of similar Weierstrass terms but with more arbitrary relative coefficients, as in the \( SO(2) \times SO(2) \times SO(2) \) model, or as in the \( SO(3) \times SO(3) \) model, remain open for study and we hope to return elsewhere in view of their relevance in five-dimensional gauged supergravity.
7 Conclusions

In this paper we have analyzed the conditions for having supersymmetric configurations in five-dimensional gauged supergravity in the sector where only five scalar fields, associated with the coset space $SL(6, \mathbb{R})/SO(6)$, are turned on apart from the metric. These conditions were integrated using an ansatz for the conformal factor of the five-dimensional metric in terms of a function $F(z)$, and the scalar fields were subsequently determined provided that a certain non-linear differential equation for the function $F(z)$ could be solved. This approach provides a natural algebro-geometric framework in which Riemann surfaces and their uniformization play a prominent role. A key ingredient was the interpretation of the non-linear differential equation for $F(z)$ as a Schwartz–Christoffel transformation by extending the range of parameters to the complex domain. In fact, the general solution depends on six real moduli, which when they start coalescing lead to configurations with various symmetry groups. The case with maximal symmetry $SO(6)$ corresponds to the maximally supersymmetric solution of $AdS_5$ with all scalar fields set equal to zero. More generally, we have classified all such algebraic curves according to their genus, and associated symmetry groups (all being subgroups of $SO(6)$). We also made use of their uniformization for finding explicit forms of the supersymmetric states in terms of elliptic functions. The calculations have been carried out in detail for the models of low genus, but they can be extended to all other cases with higher genus (or else smaller symmetry groups).

There is an alternative description of our solutions in terms of type-IIB supergravity in ten dimensions, which is a natural place for discussing solutions of five-dimensional gauged supergravity via consistent truncations. This higher dimensional point of view is also interesting for addressing various questions related to the AdS/CFT correspondence and supersymmetric Yang-Mills theory in four space-time dimensions. We found that the algebraic classes of our five-dimensional configurations could be understood as representing the gravitational field of a large number of D3-branes continuously distributed on hypersurfaces embedded in the six-dimensional space that is transverse to the branes. The geometry of these hypersurfaces is closely related to the Riemann surfaces underly-ing in the algebro-geometric approach, as the distribution of D3-branes is taken to be in the interior of certain ellipsoids for the corresponding elliptic solutions. Also, as more and more scalar fields are turned on, the geometry of the five-dimensional sphere that appears in the ten-dimensional description of our states (together with the remaining five dimensions which are asymptotic to $AdS_5$ space) becomes deformed and respects less and less symmetry from the original $SO(6)$ symmetry group of the round $S^5$. In this geometrical approach, there is no need to perform the uniformization of Riemann surfaces, as the metric is formulated in another frame with $F(z)$ being the coordinate variable instead of $z$. The Schwarz–Christoffel transform describes precisely this particular change of coordinates, when it is restricted to real values. Then, the calculation reduces to finding appropriate harmonic functions that correspond to the continuous distribution of D3-branes. In any event, both approaches are equivalent to each other and complement
nicely the classification of the supersymmetric states that has been considered.

Finally, we have examined the spectra of the massless scalar and graviton fields on these backgrounds and found that they can be determined by a Schrödinger equation in one dimension, which is \( z \), with a potential that depends on the conformal factor of the five-dimensional metric. It is rather curious that all these potentials are essentially of Calogero type. In the fully symmetric \( SO(6) \) model, whose solution represents \( AdS_5 \), the potential is \( 1/z^2 \), which is a characteristic feature of Calogero systems. For other models with less symmetry, the potential turns out to be either a rational form of Calogero interactions or elliptic generalizations thereof depending on each case. Such generalized potentials were not investigated in the literature before and there are many questions that are left open concerning their integrability properties and the exact determination of the spectrum. We think that supersymmetric quantum mechanics could help to make progress in this direction.

It will be also interesting to consider in future study the precise characterization of all these states in connection with the representation theory of the complete supersymmetry algebra. Shrinking cycles that lower the genus of our algebraic curves and lead to enhancement of the symmetry group of the various models should have an interesting interpretation in more traditional terms, using the representations of supersymmetry and the associated multiplets. Moreover, the extension of our techniques to other theories of gauged supergravity, in particular in higher dimensions, seems possible and we hope to return to all these elsewhere.

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