Partially classical limit of the Nelson model

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Abstract

We consider the Nelson model which describes a quantum system of non-relativistic identical particles coupled to a possibly massless scalar Bose field through a Yukawa type interaction. We study the limiting behaviour of that model in a situation where the number of Bose excitations becomes infinite while the coupling constant tends to zero in a suitable sense. In that limit the appropriately rescaled Bose field converges to a classical solution of the free wave or Klein-Gordon equation depending on whether the mass of the field is zero or not, the quantum fluctuations around that solution satisfy the wave or Klein-Gordon equation and the evolution of the nonrelativistic particles is governed by a quantum dynamics with an external potential given by the previous classical solution.

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1 Introduction

Quantum theories are generally expected to reduce to the corresponding classical ones when suitable parameters converge to a limit which is usually taken to be zero. In ordinary quantum mechanics this parameter is identified with Planck’s constant $\hbar$. The comparison between those two types of theories was first considered by Schrödinger [9] and by Ehrenfest [1] for simple systems with a finite number of degrees of freedom, and later put on a firm mathematical basis by Hepp [6] for more general systems including some with an infinite number of degrees of freedom. The study of the transition from quantum descriptions to classical descriptions is a quite active field of research. However, while the literature concerning systems with a finite number of degrees of freedom is rather extensive, in the case of an infinite number of degrees of freedom very few examples have been analyzed in reasonable depth [6] [4] [3].

In this paper we consider that problem for the so called Nelson model, which describes a quantum system of nonrelativistic identical particles interacting with a real scalar field in space-time $\mathbb{R}^{3+1}$. In the formalism of second quantization for the particles the Hamiltonian of the system is taken to be

$$H(\psi, a) = (2M)^{-1} \int dx (\nabla \psi)^* (\nabla \psi) + \int dk \omega a^* a + \lambda \int dx \varphi \psi^* \psi$$

where $\omega(k) = (k^2 + \mu^2)^{1/2}$ with $\mu \geq 0$ ($\mu$ is the mass of the bosons), $\psi, \psi^*, a, a^*$ are Heisenberg field operators satisfying

$$\left\{ \begin{array}{l}
[\psi(t, x), \psi(t, x') ]_\mp = 0 \\
[\psi(t, x), \psi^*(t, x') ]_\mp = \delta(x - x')
\end{array} \right.$$ \hspace{1cm} (1.2)

and

$$\left\{ \begin{array}{l}
[a(t, k), a(t, k') ]_\mp = 0 \\
[a(t, k), a^*(t, k') ]_\mp = \delta(k - k')
\end{array} \right.$$ \hspace{1cm} (1.3)

and

$$\varphi(t, x) = (2\pi)^{-3/2} \int dk (2\omega(k))^{-1/2} \left( a(t, k)e^{ik\cdot x} + a^*(t, k)e^{-ik\cdot x} \right).$$ \hspace{1cm} (1.4)

The $-$ sign in (1.2) and (1.3) denotes commutators and the $+$ sign in (1.2) denotes anticommutators. The field $\psi$ can be either a boson or a fermion field. The time
The evolution of the fields $\psi$ and $a$ in the Heisenberg picture is given by the equations of motion

$$
\begin{align*}
\dot{i}\partial_t \psi &= [\psi, H]_-
onumber \\
\dot{i}\partial_t a &= [a, H]_-
onumber \\
\end{align*}
$$

which, using (1.1), can be explicitly written as

$$
\begin{align*}
\dot{i}\partial_t \psi &= -(2M)^{-1}(2\omega)^{-1/2}F(\psi^*\psi) \\
\dot{i}\partial_t a &= \omega a + \lambda(2\omega)^{-1/2}F(\psi^*\psi) \\
\end{align*}
$$

where $F$ denotes the Fourier transform. The initial conditions are denoted by $\psi(t = 0) = \psi_0$, $a(t = 0) = a_0$ and $\varphi(t = 0) = \varphi_0$. In the same vein as in [4] we want to study the classical limit of the scalar field, keeping however intact the quantum nature of the nonrelativistic particles. In the following we present a heuristic discussion of the problem which underlies the rigorous developments of the next sections.

The classical limit is obtained by considering the average of the field operators on a sequence of states which contain a number $n$ of scalar particles increasing to infinity. The traditional way to construct such a sequence is through the Weyl operators

$$
C(\alpha) = \exp \left( \int dk \left( a_0^* \alpha - a_0^* \pi \right) \right)
$$

which applied to the Fock vacuum of the scalar particles generate the coherent states for the operators $(a_0, a_0^*)$. The sequence of operators $C(n^{1/2}\alpha)$, where $n$ is a positive integer, applied to any fixed state meets the requirements of the previously mentioned sequence of states. The average of $\varphi$ on such states scales as $n^{1/2}$ so that, in order to obtain a finite non trivial limiting equation for (1.6) when $n$ converges to infinity, we need to relate $\lambda$ to $n$ according to $n = \lambda^{-2}$. Therefore this classical limit is at the same time a weak coupling limit. From now on we shall use $\lambda$ as a parameter which will eventually tend to zero. We introduce a real function of space-time $A$ conveniently written in Fourier transform as

$$
A(t, x) = (2\pi)^{-3/2} \int dk \left( 2\omega(k) \right)^{-1/2} \left( \alpha(t, k) e^{ikx} + \overline{\alpha(t, k)} e^{-ikx} \right),
$$

(1.9)

to be thought as the limit of the rescaled field $\lambda \varphi$ when $\lambda$ tends to zero. The equation of motion (1.7) for $a$ can be trivially rewritten as

$$
\dot{i}\partial_t \alpha + \dot{i}\partial_t (\lambda a - \alpha) = \omega \alpha + \omega (\lambda a - \alpha) + \lambda^2 (2\omega)^{-1/2}F(\psi^*\psi).
$$

(1.10)
In order to obtain a non-trivial limit for (1.10) when $\lambda$ converges to zero, we impose on $\alpha$ to be solution of the equation

$$i\partial_t \alpha = \omega \alpha .$$

(1.11)

By rewriting the equations (1.5) in terms of the variables $\psi$ and $a - \alpha\lambda$, with $\alpha\lambda(t,k) = \lambda^{-1}\alpha(t,k)$, we obtain

$$i\partial_t \psi = [\psi, K]_-

(1.12)$$

$$i\partial_t (a - \alpha\lambda) = [a - \alpha\lambda, K]_-

(1.13)$$

where

$$K = (2M)^{-1} \int dx (\nabla \psi)^*(\nabla \psi) + \int dk \omega (a - \alpha\lambda)^*(a - \alpha\lambda) + \int dx A \psi^*\psi + \int dx (\lambda \varphi - A) \psi^*\psi .

(1.14)$$

In order to study the limit of (1.12) and (1.13) when $\lambda$ tends to zero we have to change the initial conditions for $a - \alpha\lambda$. We define the new field variables $\theta(t)$ and $b(t)$ by

$$\theta(t) = C(\alpha\lambda(0))^* \psi(t) \ C(\alpha\lambda(0))

(1.15)$$

$$b(t) = C(\alpha\lambda(0))^* (a(t) - \alpha\lambda(t)) \ C(\alpha\lambda(0))

(1.16)$$

so that

$$\theta(0) = \psi_0$$

$$b(0) = a_0 .

(1.17)$$

The $b$'s are the quantum fluctuations around the classical solution $\alpha$. The equations (1.12) and (1.13) take the form

$$i\partial_t \theta = [\theta, L]_-

(1.18)$$

where

$$L = (2M)^{-1} \int dx (\nabla \theta)^*(\nabla \theta) + \int dk \omega b^*b + \int dx A \theta^*\theta \ + \lambda \int dx \theta^*\theta \left(F^{-1}(2\omega)^{-1/2}b + F(2\omega)^{-1/2}b^* \right) .

(1.19)$$
In (1.19) the only term containing explicitly $\lambda$ is expected to converge to zero with $\lambda$ so that the putative limiting equations of (1.18) become

\[
\begin{align*}
    i\partial_t \theta' &= -(2M)^{-1} \Delta \theta' + A \theta' \\
    i\partial_t b' &= \omega b'.
\end{align*}
\]  

(1.20)

The system (1.18) subject to the initial condition (1.17) is conveniently solved by using the transformation which connects the Schrödinger picture to the Heisenberg picture. The solution can be written as

\[
\begin{align*}
    \theta(t) &= W(t)^* \psi_0 \ W(t) \\
    b(t) &= W(t)^* \ a_0 \ W(t)
\end{align*}
\]  

(1.21)

where $W(t)$ is the unitarity propagator satisfying

\[
    i\partial_t W(t) = \left\{ (2M)^{-1} \int dx (\nabla \psi_0)^* (\nabla \psi_0) + \int dk \ \omega \ a_0^* \ a_0 \\
    + \int dx \ A \ \psi_0^* \psi_0 + \lambda \int dx \ \phi_0^* \phi_0 \right\} W(t)
\]  

(1.22)

and $W(0) = 1$. The Schrödinger propagator $W(t)$ is expressed in terms of the Schrödinger field operators, which coincide with the Heisenberg field operators at time $t = 0$. Similarly, the solution of the limiting system (1.20) subject to the conditions $\theta'(0) = \psi_0$ and $b'(0) = a_0$ is given by

\[
\begin{align*}
    \theta'(t) &= V(t)^* \ \psi_0 \ V(t) \\
    b'(t) &= V(t)^* \ \ a_0 \ V(t)
\end{align*}
\]  

(1.23)

where $V(t)$ is the unitary propagator satisfying

\[
    i\partial_t V(t) = \left\{ (2M)^{-1} \int dx (\nabla \psi_0)^* (\nabla \psi_0) + \int dk \ \omega \ a_0^* \ a_0 + \int dx \ A \ \psi_0^* \psi_0 \right\} V(t)
\]  

(1.24)

and $V(0) = 1$. It can be checked directly that

\[
W(t) = C (\alpha_\lambda(t))^* \ U(t) \ C (\alpha_\lambda(0))
\]  

(1.25)

where

\[
U(t) = \exp \left( -itH(\psi_0, a_0) \right) .
\]  

(1.26)

Now the previous perturbation problem in the coupling constant $\lambda$ reduces to comparing the two families of operators $W(t)$ and $V(t)$ and to showing that

\[
\lim_{\lambda \to 0} W(t) = V(t) .
\]  

(1.27)
That convergence will turn out to hold in the strong operator topology. It implies in particular that for any bounded suitably regular functions $R_j(a_0) \ j = 1, 2, \cdots, m$ and $R_j(\psi_0) \ j = m + 1, \cdots, \ell$ and for any family of times $\{t_j\} \ j = 1, 2, \cdots, \ell$

$$\lim_{\lambda \to 0} C(\alpha_\lambda(0))^* \prod_{j=1}^m R_j(a(t_j) - \alpha_\lambda(t_j)) \prod_{j=m+1}^\ell R_j(\psi(t_j)) C(\alpha_\lambda(0))$$

$$= \prod_{j=1}^m R_j(b'(t_j)) \prod_{j=m+1}^\ell R_j(\theta'(t_j))$$

in the strong sense. This convergence can be interpreted in terms of correlation functions in coherent states of the Bose field. In particular

$$s \lim_{\lambda \to 0} C(\alpha_\lambda(0))^* \prod_{j=1}^m R_j(\lambda a(t_j)) C(\alpha_\lambda(0)) = \prod_{j=1}^m R_j(\alpha(t_j)) \ .$$

In conclusion we expect that the weak coupling limit of the quantum theory defined by the Hamiltonian $H$ (see (1.1) averaged over coherent states scaling as $\lambda^{-1}$ is the quantum theory of nonrelativistic particles in an external potential $A$ solution of the equation

$$(\Box + \mu^2)A = 0 \ .$$

The function $A$ is the limit of the rescaled field $\lambda \varphi$ and can be interpreted as the wave function of the condensate of the excitations of the $\varphi$ field. The quantum fluctuations around $A$ represent a free Bose field of mass $\mu$.

In the previous presentation we have totally ignored the fact that the theory described by the Hamiltonian $H$ is ill defined. In order to make $H$ a bonafide selfadjoint operator a cut off has to be introduced. The removal of this cutoff, after subtraction of the (infinite) self-energy of the nonrelativistic particles, has been achieved by Nelson in [8] in the case $\mu > 0$ by using a dressing transformation introduced by E.P. Gross [5]. The case $\mu = 0$ has been subsequently treated by Fröhlich [2]. Our aim is to implement in a rigorous way the previously described limit when $\lambda$ tends to zero and for that purpose we rely heavily on [8]. For reasons of clarity the ideas behind the classical limit have been so far explained by using the second quantization scheme for the nonrelativistic particles. This has allowed us to treat the particles and the field on the same footing with a similar formalism. However, since the number of particles is conserved, we could have worked as well in the first quantized formalism. In the next sections we will follow this last option, namely we shall project the equations (1.1), (1.22), (1.24), (1.26) and (1.27) on
spaces with a fixed number of particles, keeping the same notation for \( H, W, V \) and \( U \).

This paper is organized as follows. In Section 2, we recall without proofs the results of Nelson that we need [8]. In Section 3 we construct the limiting dynamics expressed by the propagator \( V \). Finally in Section 4 we prove the announced convergence when \( \lambda \to 0 \). The main result is stated in Proposition 4.3.

We conclude this section by introducing some notation. We denote by \( \| \cdot \|_r \) the norm in \( L^r \equiv L^r(\mathbb{R}^3) \), \( 1 \leq r \leq \infty \), and by \( (\cdot, \cdot) \) the scalar product in \( L^2 \). We shall need the spaces \( L^2_s \) defined for any \( s \in \mathbb{R} \) by

\[
L^2_s(\mathbb{R}^3) = \{ u : < \cdot >^s u \in L^2 \}
\]

where \( < \cdot > = (1 + | \cdot |^2)^{1/2} \).

### 2 The quantum theory

In this section we describe the basic results concerning the model we are interested in. We follow closely Nelson’s presentation and we refer essentially to [8] for the proofs. We consider a system of \( p \) nonrelativistic identical particles of mass \( M \) interacting with a real possibly massless Bose field. From now on \( p \) is fixed and generic constants \( C \) in some of the subsequent estimates may depend on \( p \). The Hilbert space \( \mathcal{H} \) of the theory, which is the tensor product of \( L^2(\mathbb{R}^{3p}) \) and of the Fock space of the Bose field, can be equivalently taken as the direct sum of the Hilbert spaces

\[
\mathcal{H}_n = \{ \Psi_n : \Psi_n(x_1, \ldots, x_p; k_1, \ldots, k_n) \in L^2(\mathbb{R}^{3p+3n}) \} \quad n \geq 0
\]

with each \( \Psi_n \) symmetric in the variables \( k_1, \ldots, k_n \). The set of variables \( (x_1, x_2, \ldots, x_p) \) is denoted by \( X \). The possible symmetry properties, if any, of the \( \Psi_n \) in the \( X \) variables do not play any role in the problem. The scalar product of \( \Phi, \Psi \) in \( \mathcal{H} \) is denoted by \( < \Phi, \Psi > \) and the norm of \( \Phi \) by \( \| \Phi \| \). On \( \mathcal{H} \) we define formally the annihilation and creation operators for the Bose field by

\[
(a(k)\Psi)_n(X; k_1, \ldots, k_n) = (n + 1)^{1/2} \Psi_{n+1} (X; k, k_1, \ldots, k_n) \quad ,
\]

\[
(a^*(k)\Psi)_n(X; k_1, \ldots, k_n) = n^{-1/2} \sum_{j=1}^{n} \delta(k - k_j) \Psi_{n-1} (X; k_1, \ldots, \hat{k}_j, \ldots, k_n)
\]

where \( \hat{k}_j \) indicates that the variable \( k_j \) has been omitted. The field operator \( \varphi \) is defined by

\[
\varphi(x) = (2\pi)^{-3/2} \int dk \ (2\omega(k))^{-1/2} \left( a(k)e^{ik \cdot x} + a^*(k)e^{-ik \cdot x} \right) .
\]
For any $f \in L^\infty(I^3, L^2(I^3))$ we define (by formal integration) the operators

$$(a(f)\Psi)_n (X; k_1, \ldots, k_n) = (n+1)^{1/2} \int dk f(X, k)\Psi_{n+1} (X; k, k_1, \ldots, k_n) ,$$

$$(a^*(f)\Psi)_n (X; k_1, \ldots, k_n) = n^{-1/2} \sum_{j=1}^n f(X, k_j)\Psi_{n-1} (X; k_1, \ldots, \hat{k}_j, \ldots, k_n) .$$

The number operator $N$, defined by

$$(N\Psi)_n (X; k_1, \ldots, k_n) = n\Psi_n (X; k_1, \ldots, k_n) ,$$

counts the number of excitations of the Bose field. We denote by $C_0(N)$ the space of vectors in $H$ with a finite number of components different from zero. Standard estimates show that

$$\| a(f)\Psi \| \leq \| f; L^\infty(I^3, L^2(I^3)) \| \| N^{1/2}\Psi \| ,$$

$$\| a^*(f)\Psi \| \leq \| f; L^\infty(I^3, L^2(I^3)) \| \| (N+1)^{1/2}\Psi \| .$$

For brevity from now on we shall write the estimates concerning $a(f)$ and $a^*(f)$ as if $f$ did not depend on $X$, i.e. $f \in L^2(I^3)$. The general case will be recovered by replacing the norms of $f$ in $L^2(I^3)$ by the corresponding norms in $L^\infty(I^3, L^2(I^3))$ in the estimates. We now define the dynamics of the theory. The kinetic energy $H_{01}$ of the nonrelativistic particles is defined by

$$(H_{01}\Psi)_n (X; k_1, \ldots, k_n) = -(2M)^{-1} \sum_{j=1}^p \Delta_j \Psi_n (X; k_1, \ldots, k_n)$$

(2.1)

where $\Delta_j$ is the Laplace operator acting on the $x_j$ variable, while the kinetic energy $H_{02}$ of the Bose field is defined by

$$(H_{02}\Psi)_n (X; k_1, \ldots, k_n) = \sum_{\ell=1}^n \omega(k_\ell)\Psi_n (X; k_1, \ldots, k_n)$$

(2.2)

where $\omega(k) = (k^2 + \mu^2)^{1/2}$ and $\mu \geq 0$. We denote by $H_0$ their sum

$$H_0 = H_{01} + H_{02} .$$

(2.3)

It is well known that $H_0$ is self-adjoint on any $H_n$ and therefore on the whole space $H$. For any $\sigma$, $0 \leq \sigma < \infty$, we define the cutoff function $\chi_\sigma$ by $\chi_\sigma(k) = 1$ if $|k| \leq \sigma$, $\chi_\sigma(k) = 0$ if $|k| > \sigma$. The interaction energy $H_{I\sigma}$ with cutoff $\sigma$ is defined by

$$H_{I\sigma} = \lambda \sum_{j=1}^p \varphi_\sigma(x_j)$$
with
\[ \varphi_\sigma(x) = (2\pi)^{-3/2} \int dk \ (2\omega(k))^{-1/2} \chi_\sigma(k) \left( a(k)e^{ikx} + a^*(k)e^{-ikx} \right), \]
so that
\[ H_{I\sigma} = a(\mathcal{T}\chi_\sigma) + a^* (f\chi_\sigma) \quad (2.4) \]
where
\[ f = \sum_j f_j, \ f_j = f_0 \ e^{-ikx_j} \ (1 \leq j \leq p), \]
\[ f_0 = \lambda(2\pi)^{-3/2}(2\omega(k))^{-1/2}. \]
The sum of \( H_0 \) and \( H_{I\sigma} \) defines the total Hamiltonian with cutoff
\[ H_\sigma = H_0 + H_{I\sigma}. \quad (2.5) \]
If we take formally \( \sigma = \infty \), namely \( \chi_\sigma(k) = 1 \), the second quantized version of the expression given by (2.5) coincides with the expression (1.1).

The analysis of Nelson and Fröhlich is based on some estimates of \( a(f) \) and \( a^*(f) \) in terms of the operator \( H_0 \). The following set of estimates holds for all \( \mu \geq 0 \).

**Lemma 2.1.** Let \( f \in L^2(\mathbb{R}^3) \) with \( \omega^{-1/2}f \in L^2(\mathbb{R}^3) \). Then, for all \( \Psi, \Phi \in \mathcal{H} \), the following estimates hold:

\[ \| a(f)\Psi \| \leq \omega^{-1/2}f \| H_{02}^{1/2} \Psi \|, \quad (2.6) \]
\[ \| a^*(f)\Psi \| \leq \omega^{-1/2}f \| H_{02}^{1/2} \Psi \| + \| f \| \| \Psi \|, \quad (2.7) \]
\[ \| a(f)^2\Psi \| \leq \omega^{-1/2}f \| H_{02}^{1/2} \Psi \| + \| f \| \| \Psi \|^2, \quad (2.8) \]
\[ \| a^*(f)a(f)\Psi \| \leq \omega^{-1/2}f \| H_{02}^{1/2} \Psi \| + \| f \| \| \omega^{-1/2}f \| H_{02}^{1/2} \Psi \|, \quad (2.9) \]
\[ \| (a^*(f))^2\Psi \| \leq \omega^{-1/2}f \| H_{02}^{1/2} \Psi \| + 4 \| f \| \| \omega^{-1/2}f \| H_{02}^{1/2} \Psi \| \]
\[ + 2 \| f \| \| \Psi \|^2, \quad (2.10) \]
\[ \left| \langle \Phi, (a(f))^2 \Psi \rangle \right| \leq (3/2)^{1/2} \left( \| \omega^{-1/2}f \| H_{02}^{1/2} \Psi \| H_{02}^{1/2} \Phi \| \right. \]
\[ + \| \omega^{-1/4}f \| H_{02}^{1/2} \Psi \| \| \Phi \| \right). \quad (2.11) \]

Using (2.6) and (2.7) we obtain the following result.
Proposition 2.1. For any $\sigma < \infty$, the operator $H_\sigma$ is self-adjoint on $\mathcal{D}(H_0)$.

In order to remove the cut off $\sigma$, we use a dressing transformation which allows to change the domain of definition of the limiting Hamiltonian with respect to the domain of $H_0$. In addition to the upper cut off $\sigma$ we introduce a lower cut off $\sigma_0 < \sigma$ which we keep fixed and which eventually will be chosen sufficiently large. In analogy with (2.4), we define the operators

$$T_\sigma = a(\mathcal{g}X_\sigma) - a^*(gX_\sigma) ,$$
$$T = a(\mathcal{g}) - a^*(g) ,$$

where

$$g = \sum_j g_j , \quad g_j = g_0 e^{-ikx_j} \quad (1 \leq j \leq p) ,$$

$$g_0(k) = - (1 - \chi_{\sigma_0}(k)) \left( \omega(k) + (2M)^{-1}k^2 \right)^{-1} f_0(k) \quad = - (1 - \chi_{\sigma_0}(k)) \lambda \left( \omega(k) + (2M)^{-1}k^2 \right)^{-1} \left( 2\pi \right)^{-3/2} (2\omega(k))^{-1/2} .$$

Note in particular that $g_0$ and therefore also $g_j$ and $g$ belong to $L^2$. We identify the operators $T_\sigma$ and $T$ with their closures. We also define the operators

$$Q_\sigma = \exp (-T_\sigma) , \quad Q = \exp (-T) .$$

The operators $Q_\sigma$ and $Q$ are Weyl operators associated with the Bose field $(a, a^*)$. In addition, they depend on the coordinates $X$, and therefore they also act as operators in the tensor factor $L^2(\mathbb{R}^{3p})$ of $\mathcal{H}$.

The operators $T_\sigma$, $T$ and $Q_\sigma$, $Q$ enjoy the following properties.

Proposition 2.2.

1) The operators $iT_\sigma$ and $iT$ are essentially self-adjoint on $C_0(N)$. The operators $Q_\sigma$ and $Q$ are unitary.

2) $Q_\sigma \mathcal{D}(H_0^{1/2}) = \mathcal{D}(H_0^{1/2})$ and $Q_\sigma \mathcal{D}(H_0) = \mathcal{D}(H_0)$.

3) The following limit holds in the strong sense

$$s - \lim_{\sigma \to \infty} Q_\sigma = Q .$$
Upon formal transformation of $H_\sigma$ by the unitary operator $Q_\sigma$, we obtain for $\sigma_0 < \sigma$

$$(H_\sigma - pE_\sigma) Q_\sigma = Q_\sigma (H'_\sigma - pE_{\sigma_0})$$

(2.15)

where

$$E_\sigma = -\lambda^2 (2\pi)^{-3} \int dk \ (2\omega(k))^{-1} \left( \omega(k) + (2M)^{-1} k^2 \right)^{-1} \chi_\sigma(k),$$

(2.16)

$$H'_\sigma = H_0 + H_{1\sigma_0} + H'_{1\sigma} + H'_{2\sigma} + H'_{3\sigma},$$

(2.17)

$$H'_{1\sigma} = iM^{-1} \sum_{j=1}^p \left\{ \nabla_j \cdot \left( k\overline{\gamma}_j \chi_\sigma \right) + a^* (kg_j \chi_\sigma) \cdot \nabla_j \right\},$$

(2.18)

$$H'_{2\sigma} = (2M)^{-1} \sum_{j=1}^p \left\{ \left( a \left( k\overline{\gamma}_j \chi_\sigma \right) \right)^2 + \left( a^* (kg_j \chi_\sigma) \right)^2 + 2a^* (kg_j \chi_\sigma) a \left( k\overline{\gamma}_j \chi_\sigma \right) \right\},$$

(2.19)

$$H'_{3\sigma} = \sum_{1 \leq j < \ell \leq p} q_\sigma (x_j - x_\ell),$$

(2.20)

$$q_\sigma (x) = -2\lambda^2 (2\pi)^{-3} \int dk \ (2\omega(k))^{-1} \left( \omega(k) + (2M)^{-1} k^2 \right)^{-2} \left( \omega(k) + M^{-1} k^2 \right) \times \left( \chi_\sigma(k) - \chi_{\sigma_0}(k) \right) \cos kx.$$  

(2.21)

The following proposition summarizes the meaning of the equality (2.15).

**Proposition 2.3.** Let $0 < \sigma_0 < \sigma$. Then

1) The operator $H'_\sigma$ is self-adjoint on $\mathcal{D}(H_0)$.

2) The equality (2.15) holds on $\mathcal{D}(H_0)$.

In order to remove the cutoff $\sigma$ and to take later the limit $\lambda \to 0$, the estimates contained in the following lemma will be useful. They are a consequence of Lemma 2.1.

**Lemma 2.2.** For all $\sigma$ with $0 < \sigma_0 < \sigma$, for all $\Psi \in \mathcal{D}(H_0^{1/2})$, the following estimates hold:

$$|\langle \Psi, H_{1\sigma_0} \Psi \rangle| \leq \lambda \left( \varepsilon \left\| H_0^{1/2} \Psi \right\|^2 + C \varepsilon^{-1} \sigma_0 \left\| \Psi \right\|^2 \right),$$

$$|\langle \Psi, H'_{1\sigma} \Psi \rangle| \leq C \lambda \sigma_{0}^{-1/2} \left\| H_0^{1/2} \Psi \right\|^2,$$

$$|\langle \Psi, H'_{2\sigma} \Psi \rangle| \leq C \lambda^2 \left( \sigma_{0}^{-1} \left\| H_0^{1/2} \Psi \right\|^2 + \left\| \Psi \right\|^2 \right),$$

$$|\langle \Psi, H'_{3\sigma} \Psi \rangle| \leq C \lambda^2 \left( \sigma_{0}^{-1} \left\| H_0^{1/2} \Psi \right\|^2 + \left\| \Psi \right\|^2 \right).$$

11
\[ |<\Psi, H'_{3\sigma}\Psi>| \leq \lambda^2 \left( \varepsilon \| H_0^{1/2}\Psi \|^2 + C \varepsilon^{-1} \sigma_0^{-2} \| \Psi \|^2 \right). \]

Finally the above estimates and similar ones lead to the existence of a limiting operator \( H'_{\infty} = \lim_{\sigma \to \infty} H'_{\sigma} \) in the sense of the form defined by the operator \( 1 + H_0 \) and therefore to the existence of the renormalized Hamiltonian of the theory \( \widehat{H} \) defined by

\[ \widehat{H} = Q (H'_{\infty} - pE_{\sigma_0}) Q^* \quad (2.22) \]
(see (2.15)).

**Proposition 2.4.** Let \( \lambda_0 > 0 \). Then there exists \( \sigma_0 > 0 \) such that for all \( \lambda, |\lambda| \leq \lambda_0, \)

1) For all \( \sigma \) with \( \sigma_0 < \sigma \), for all \( \Psi \in \mathcal{D}(H_0^{1/2}) \), the following estimate holds

\[ |<\Psi, (H'_{\sigma} - H_0)\Psi>| \leq (1/2) \| H_0^{1/2}\Psi \|^2 + C \| \Psi \|^2. \]

2) For all \( \sigma_1, \sigma_2 \) with \( \sigma_0 < \sigma_1 < \sigma_2 \), for all \( \Psi \in \mathcal{D}(H_0^{1/2}) \), the following estimate holds

\[ |<\Psi, (H'_{\sigma_1} - H'_{\sigma_2})\Psi>| \leq \varepsilon(\sigma_1) \left( \| H_0^{1/2}\Psi \|^2 + \| \Psi \|^2 \right), \]
where \( \varepsilon(\sigma_1) \) tends to zero when \( \sigma_1 \) tends to infinity.

3) There exists a self-adjoint operator \( H'_{\infty} \) such that for all \( \sigma \) with \( \sigma_0 < \sigma \) and for all \( \Psi \in \mathcal{D}(H_0^{1/2}) \) the following estimate holds

\[ |<\Psi, (H'_{\sigma} - H'_{\infty})\Psi>| \leq \varepsilon(\sigma) \left( \| H_0^{1/2}\Psi \|^2 + \| \Psi \|^2 \right). \]

The operator \( H'_{\infty} \) is bounded from below and there exist two constants \( \rho \) and \( C \) such that

\[ C^{-1} \| (1 + H_0)^{1/2}\Psi \|^2 \leq |<\Psi, (\rho + H'_{\infty})\Psi>| \leq C \| (1 + H_0)^{1/2}\Psi \|^2 \]
for all \( \Psi \in \mathcal{D}(H_0^{1/2}) \). Furthermore

\[ s - \lim_{\sigma \to \infty} \exp (-itH'_{\sigma}) = \exp (-itH'_{\infty}). \quad (2.23) \]

The operator \( \widehat{H} \) is selfadjoint and

\[ s - \lim_{\sigma \to \infty} U_{\sigma}(t) = U(t) \equiv \exp \left( -it\widehat{H} \right) \quad (2.24) \]
where

\[ U_{\sigma}(t) \equiv \exp \left\{ -it (H_{\sigma} - pE_{\sigma}) \right\}. \quad (2.25) \]
Both limits (2.23) and (2.24) hold for any \( t \in \mathbb{R} \) uniformly on compact intervals.
3 The limiting theory

In this section we give a precise definition and we study the properties of the unitary propagator implicitly and formally defined by (1.24) in the second order formalism. Rewritten in the first order formalism the problem consists in solving the equation

\[
\begin{align*}
  i\partial_t V(t, s) &= (H_0 + A(t)) V(t, s) \\
  V(s, s) &= 1
\end{align*}
\]  

(3.1)

where

\[
(A(t)\Psi)_n (X; k_1, \cdots, k_n) = \sum_{j=1}^{p} A(t, x_j) \Psi_n (X; k_1, \cdots, k_n),
\]

(3.2)

\[
A(t, x) = (2\pi)^{-3/2} \int dk \, (2\omega(k))^{-1/2} \left( \alpha(k) e^{i(k \cdot x - \omega(k)t)} + \overline{\alpha(k)} e^{-i(k \cdot x - \omega(k)t)} \right).
\]

(3.3)

The function \( A \) defined by (3.3) is the function defined by (1.9) with \( \alpha(t, k) = \alpha(k) \exp(-i\omega(k)t) \), namely with \( \alpha(t) \) solution of (1.11). We define in addition

\[
(\dot{A}(t)\Psi)_n (X; k_1, \cdots, k_n) = \sum_{j=1}^{p} (\partial_t A)(t, x_j) \Psi_n (X; k_1, \cdots, k_n)
\]

(3.4)

which represents the time derivative of the family of operators \( A(t) \). We collect some properties of \( A \) in the next lemma.

**Lemma 3.1.** Let \( \alpha \in L^2_1 \). Then the operators \( A(t) \) and \( \dot{A}(t) \) satisfy the following estimates

\[
\| A(t)\Psi \| \leq C \| \omega^{1/2} \alpha \|_2 \| H^{1/4}_0 \Psi \|,
\]

(3.5)

\[
| < \Psi, A(t)\Psi > | \leq C \| \omega^{1/2} \alpha \|_2 < \Psi, H^{1/4}_0 \Psi >,
\]

(3.6)

\[
\| \dot{A}(t)\Psi \| \leq C \| \omega^{1/2} \alpha \|_2 \| \Psi \|^{1/4} \| H^3_0 \Psi \|^{3/4},
\]

(3.7)

\[
| < \Psi, \dot{A}(t)\Psi > | \leq C \| \omega^{1/2} \alpha \|_2 < \Psi, H^3_0 \Psi >,
\]

(3.8)

\[
\| (t - s)^{-1} (A(t) - A(s)) - \dot{A}(s) \Psi \| \leq C \| \omega^{1/2} \alpha \|_2 \| \Psi \|^{1/4} \| H^3_0 \Psi \|^{3/4}
\]

(3.9)

with

\[
m(s, t) = 1 - \int_0^1 e^{i\omega(s-t)\theta} d\theta.
\]

**Proof.** From the definition of \( A(t) \) and \( (\partial_t A)(t) \) we obtain

\[
\| A(t) \|_6 \leq C \| \omega^{1/2} \alpha \|_2
\]

(3.10)
by a Sobolev inequality and
\[ \| \partial_t A(t) \|_2 \leq C \| \omega^{1/2} \alpha \|_2 \] (3.11)
by the unitarity of the Fourier transform.

We first prove (3.5). Let \( X = (x_1, X') \). Then
\[
\int dx_1 |A(t, x_1)\Psi_n(x_1, X'; k_1, \ldots, k_n)|^2 \leq \| A(t) \|_6^2 \| \Psi_n(\cdot, X'; k_1, \ldots, k_n) \|_3^2
\]
\[ \leq C \| A(t) \|_6^2 \| \Delta_{1/4} \Psi_n(\cdot, X'; k_1, \ldots, k_n) \|_2^2 \]
by Hölder and Sobolev inequalities. Integrating over the variables \( X' \) and \( k_1, \ldots, k_n \)
and summing over \( n \) we obtain
\[
\| A(t) \Psi \| \leq C \| A(t) \|_6 \sum_{j=1}^p \| \Delta_{1/4} \Psi \|
\]
which implies (3.5) by (3.10). The proof of (3.6) is similar.

We next prove (3.7). We estimate
\[
\int dx_1 |(\partial_t A)(t, x_1)\Psi_n(x_1, X'; k_1, \ldots, k_n)|^2 \leq \| \partial_t A(t) \|_2^2 \| \Psi_n(\cdot, X'; k_1, \ldots, k_n) \|_\infty^2
\]
\[ \leq C \| (\partial_t A)(t) \|_2^2 \| \Psi_n(\cdot, X'; k_1, \ldots, k_n) \|_{1/2}^2 \| \Delta_1 \Psi_n(\cdot, X'; k_1, \ldots, k_n) \|_{3/2}^2 \]
by Hölder and Sobolev inequalities. Integrating over the variables \( X' \) and \( k_1, \ldots, k_n \)
and summing over \( n \) we obtain
\[
\| \dot{A}(t) \Psi \| \leq C \| (\partial_t A)(t) \|_2 \| \Psi \|^{1/4} \sum_{j=1}^p \| \Delta_j \Psi \|^{3/4}
\]
which implies (3.7) by (3.11). The proof of (3.8) is similar. We finally prove (3.9).

By application of (3.2), (3.3) and (3.4) we can write
\[
\left( (t-s)^{-1}(A(t) - A(s)) - \dot{A}(s) \right) \Psi_n(X; k_1, \ldots, k_n) =
\]
\[ = \sum_{j=1}^p B(s, t, x_j)\Psi_n(X; k_1, \ldots, k_n) \]
where
\[
B(s, t, x) = i(2\pi)^{-3/2} \int dk \left( 2\omega(k) \right)^{-1/2} \omega(k) \left\{ \alpha(k) e^{i(kx-\omega(k)s)} m(s, t) - \overline{\alpha(k)} e^{-i(kx-\omega(k)s)} \overline{m(s, t)} \right\}
\]
Now the remaining part of the proof is identical with that of (3.7).

We are now in condition to prove the existence and uniqueness of solutions $V(t, s)$ of (3.1). For that purpose we rely on a result of Kato [7].

**Proposition 3.1.** Let $\alpha \in L^2_{1/2}$ and let $A(t)$ be defined by (3.2) and (3.3). Then
1) For any $t \in \mathbb{R}$, $A(t)$ is a Kato perturbation of $H_0$, so that $H_0 + A(t)$ is self-adjoint on $\mathcal{D}(H_0)$.
2) There exists a family of unitary operators $V(t, s)$, $t, s \in \mathbb{R}$ with the following properties
   (a) $V(t, t) = 1$.
   (b) $V(t, s)$ $V(s, r) = V(t, r)$.
   (c) $V(t, s)$ is strongly continuous on $\mathbb{R} \times \mathbb{R}$.
   (d) $V(t, s)$ $\mathcal{D}(H_0) \subset \mathcal{D}(H_0)$
and for any compact interval $I$ there exists a constant $C_I$ such that
\[
\| (1 + H_0) V(t, s) \Psi \| \leq C_I \| (1 + H_0) \Psi \| \quad (3.12)
\]
for any $\Psi \in \mathcal{D}(H_0)$ and for all $t, s \in I$.
(e) For any $\Psi \in \mathcal{D}(H_0)$
\[
i \frac{d}{dt} V(t, s) \Psi = (H_0 + A(t)) V(t, s) \Psi.
\]
3) Uniqueness holds under the assumptions (a), (d) and (e).

**Proof.**

1) From (3.5) it follows that for any $t \in \mathbb{R}$ and for any $\Psi \in \mathcal{D}(H_0)$ the following inequality holds:
\[
\| A(t) \Psi \| \leq \varepsilon \| H_0 \Psi \| + C\varepsilon^{-1/3} \| \omega^{1/2} \alpha \|^{1/3} \| \Psi \|
\]
so that $A(t)$ is infinitesimally small with respect to $H_0$. Therefore, for any $t \in \mathbb{R}$, $H_0 + A(t)$ is self-adjoint on $\mathcal{D}(H_0)$.

2) The existence of $V(t, s)$ and its properties follows from Theorem 1 of [7] once we have verified the assumptions of the theorem. The only non trivial point consists in proving that for some $\rho$ and for any $t \in \mathbb{R}$ the operator
\[
S(t) \equiv \rho + H_0 + A(t)
\]
is an isomorphism of $D(H_0)$ onto $H$ and that, for any $\Psi \in D(H_0)$, $S(t)\Psi$ is continuously differentiable. From (3.5) it follows that there exist $\rho$ and $C$ such that

$$C^{-1} \| (1 + H_0)\Psi \| \leq \| (\rho + H_0 + A(t))\Psi \| \leq C \| (1 + H_0)\Psi \|$$

for any $\Psi \in D(H_0)$ and for any $t \in IR$. This leads to the isomorphism property. Since

$$S(t) - S(s) = A(t) - A(s)$$

the differentiability properties of $S$ are the differentiability property of $A$. By (3.9) we see that $S(t)\Psi$ is differentiable and that

$$\frac{d}{dt} S(t)\Psi = \dot{A}(t)\Psi.$$ 

The continuity of $\dot{A}(t)\Psi$ follows from a minor variation of (3.7).

3) To prove uniqueness let us suppose the existence of $V'(t, s)$ satisfying (a), (d) and (e). Then, for any $\Phi, \Psi \in D(H_0)$ the conditions (d) and (e) imply

$$\frac{d}{dt} < V'(t, s)\Phi, V(t, s)\Psi > = 0$$

where $V(t, s)$ is the family constructed in Part 2. On the other hand

$$\frac{d}{dt} < V(t, s)\Phi, V(t, s)\Psi > = 0$$

so that by the condition (a)

$$< V'(t, s)\Phi - V(t, s)\Phi, V(t, s)\Psi > = 0$$

which implies

$$V'(t, s)\Phi = V(t, s)\Phi.$$ 

\[ \square \]

4 The limit $\lambda \to 0$

In this section we prove the main result of this paper, namely the operator convergence when $\lambda \to 0$ announced in (1.27). As in Section 2, we use the first order formalism for the particles. We recall the definition of the Weyl operator (see (1.8) where it is written with $a_0$ instead of $a$)
\[ C(\alpha) = \exp (a^*(\alpha) - a(\alpha)) \]  
for any \( \alpha \in L^2 \). (Strictly speaking the operator in the exponential should be replaced by its closure). We now define (see (1.25))

\[ W(t, s) = C(\alpha_\lambda(t))^* U(t - s) C(\alpha_\lambda(s)) \]  
where

\[ \alpha(t, k) = \alpha(k) \exp(-i\omega t), \]  
\[ \alpha_\lambda = \lambda^{-1} \alpha \] and \( U \) is defined in (2.24). Although \( U \) and \( W \) depend on \( \lambda \), for brevity we shall omit that dependence. We intend to prove that \( W(t, s) \) converges strongly when \( \lambda \to 0 \) to the propagator \( V(t, s) \) defined in Proposition (3.1), uniformly for \( t, s \) in compact intervals. The following lemma collects some properties of the Weyl operators.

**Lemma 4.1.**

1) Let \( \alpha \in L^2 \). Then \( C(\alpha) \) is unitary and strongly continuous as a function of \( \alpha \in L^2 \). In addition, for any \( \Psi \in \mathcal{D}(a(\gamma)) \) with \( \gamma \in L^2 \), \( C(\alpha)\Psi \in \mathcal{D}(a(\gamma)) \) and the following identity holds :

\[ C(\alpha)^* a(\gamma) C(\alpha) \Psi = a(\gamma) \Psi + (\gamma, \alpha) \Psi. \]  
Similarly, for any \( \Psi \in \mathcal{D}(a^*(\gamma)) \), \( C(\alpha)\Psi \in \mathcal{D}(a^*(\gamma)) \) and the following identity holds :

\[ C(\alpha)^* a^*(\gamma) C(\alpha) \Psi = a^*(\gamma) \Psi + (\alpha, \gamma) \Psi. \]  

2) Let \( \alpha \in L^2_{1/2} \). Then \( C(\alpha)\mathcal{D}(H^1_{0/2}) = \mathcal{D}(H^1_{0/2}) \) and, for any \( \Psi \in \mathcal{D}(H^1_{0/2}) \), the following inequality holds :

\[ \| H^1_{0/2} C(\alpha) \Psi \| \leq \| H^1_{0/2} \Psi \| + \| \omega^{1/2} \alpha \| \| \Psi \|. \]  
Let \( \alpha \in L^2_1 \). Then \( C(\alpha)\mathcal{D}(H_0) = \mathcal{D}(H_0) \) and, for any \( \Psi \in \mathcal{D}(H_0) \), the following inequality holds :

\[ \| H_0 C(\alpha) \Psi \| \leq 2 \| H_0 \Psi \| + \left( \| \omega \alpha \| + 2 \| \omega^{1/2} \alpha \|^2 \right) \| \Psi \|. \]  

3) Let \( \alpha : t \to \alpha(t) \in C^1(\mathbb{R}, L^2) \) with \( \omega^{-1/2}d\alpha/dt \equiv \omega^{-1/2}\dot{\alpha} \in C(\mathbb{R}, L^2) \). Then, for any \( \Psi \in \mathcal{D}(H^1_{0/2}) \), \( C(\alpha(t))\Psi \) is differentiable in \( t \). Its derivative is given by

\[ \frac{d}{dt} C(\alpha(t))\Psi = C(\alpha(t)) \left( a^*(\dot{\alpha}(t)) - a(\overline{\dot{\alpha}(t)}) + i \text{Im}(\alpha(t), \dot{\alpha}(t)) \right) \Psi. \]
Proof.

1) The set of vectors \( C_0(N) \) is a domain of essential self-adjointness for \( i(a^*(\alpha) - a(\overline{\alpha})) \) so that \( C(\alpha) \) is unitary. In addition \( C_0(N) \) is a set of entire analytic vectors [10] for \( a^*(\alpha) - a(\overline{\alpha}) \), which leads to the continuity of \( C(\alpha)\Psi \) in \( \alpha \) for any \( \Psi \in C_0(N) \) by direct inspection. Strong continuity for any \( \Psi \) follows immediately. Using again the power series expansion of \( C(\alpha)\Psi \) for \( \Psi \in C_0(N) \) we can check immediately that (4.4) holds for such a \( \Psi \). An elementary argument of closure leads to (4.4) in general. The proof of the part concerning \( a^*(\gamma) \) is similar.

2) Let \( \alpha \in L^2 \). By power series expansion we check directly that, for any \( \Psi \in C_0(N) \cap \mathcal{D}(H_{02}) \), \( C(\alpha)\Psi \in \mathcal{D}(H_{02}) \) and that the following identity holds:

\[
H_{02} C(\alpha) \Psi = C(\alpha) \left( H_{02} + a^*(\omega \alpha) + a(\omega \overline{\alpha}) + \| \omega^{1/2} \alpha \|^2 \right) \Psi . \tag{4.9}
\]

Using (2.6) and (2.7) with \( f = \omega \alpha \) and the Schwarz inequality, we obtain (4.7). A standard approximation argument leads to the conclusion that \( C(\alpha)\mathcal{D}(H_{02}) \subset \mathcal{D}(H_{02}) \) and that (4.7) holds for any \( \Psi \in \mathcal{D}(H_{02}) \).

Similarly from (4.9), using (2.6), we obtain (4.6) for \( \Psi \in C_0(N) \cap \mathcal{D}(H_{02}) \). To conclude we apply an approximation argument first on \( \Psi \) and then on \( \alpha \).

3) The Weyl operators satisfy the following well known identity

\[
C(\alpha + \beta) = C(\alpha)C(\beta) \exp (i \text{Im}(\alpha, \beta)) \tag{4.10}
\]

which can be proved by power series expansion on \( C_0(N) \) and then extended to the whole Hilbert space \( \mathcal{H} \) by the unitarity of \( C(\alpha) \). Using (4.10) applied to \( \Psi \in C_0(N) \), we can write the identity

\[
(t - t_0)^{-1} (C(\alpha(t)) - C(\alpha(t_0))) \Psi = C(\alpha(t_0))(t - t_0)^{-1} \times \{ C(\alpha(t) - \alpha(t_0)) \exp (i \text{Im}(\alpha(t_0), \alpha(t) - \alpha(t_0))) - 1 \} \Psi \tag{4.11}
\]

which in the limit \( t \to t_0 \) yields (4.8). Let now \( \Psi \in \mathcal{D}(H_{02}^{1/2}) \). We write the integrated form of (4.8), namely

\[
C(\alpha(t))\Psi_j = C(\alpha(t_0))\Psi_j + \int_{t_0}^t ds \, C(\alpha(s)) \times \left( a^*(\dot{\alpha}(s)) - a(\overline{\dot{\alpha}(s)}) + i \text{Im}(\alpha(s), \dot{\alpha}(s)) \right) \Psi_j \tag{4.12}
\]

for a sequence \( \Psi_j \in C_0(N) \cap \mathcal{D}(H_{02}^{1/2}) \) such that \( \Psi_j \to \Psi \) and \( H_{02}^{1/2} \Psi_j \to H_{02}^{1/2} \Psi \). Using (2.6) and (2.7) we can take the limit \( j \to \infty \) in both sides of (4.12) and we
obtain (4.12) with \( \Psi_j \) replaced by \( \Psi \). By differentiation in \( t \) we obtain (4.8) in full
generality.

We continue the argument temporarily with the approximate theory defined by
the Hamiltonian \( H_\sigma - pE_\sigma \) (see Prop. 2.4, part 3) and we shall remove the cutoff \( \sigma \)
at the end. For that purpose, for any \( \sigma \) with \( \sigma_0 < \sigma \) we define

\[
Z_\sigma(t, s) = Q_\sigma^* C (\chi_\sigma \alpha_\lambda(t))^* U_\sigma(t - s) C (\chi_\sigma \alpha_\lambda(s)) Q_\sigma
\]  

(4.13)

where \( Q_\sigma \) and \( U_\sigma \) are defined by (2.14) and (2.25) respectively, and \( \alpha_\lambda(t) = \lambda^{-1} \alpha(t) \)
with \( \alpha \) given by (4.3). In addition we define \( A_\sigma(t) \) by (3.2) with \( A \) replaced by \( A_\sigma \)
where

\[
A_\sigma(t) = (2\pi)^{-3/2} \int dk \, (2\omega(k))^{-1/2} \chi_\sigma(k) \left( \alpha(k)e^{i(kx-\omega(k)t)} + \alpha(k)e^{-i(kx-\omega(k)t)} \right) .
\]  

(4.14)

In the following proposition we perform the basic computation which exhibits
the compensations among the terms containing the coupling constant \( \lambda \) in the
operator \( Z_\sigma(t, s) \).

**Proposition 4.1.** Let \( \alpha \in L^2_{1/2} \) and let \( \alpha(t) \) be given by (4.3). Let \( \Psi \in \mathcal{D}(H_0) \).
Then, for any \( \sigma \) with \( \sigma_0 < \sigma \), \( Z_\sigma(t, s)\Psi \) is differentiable in \( t \) with derivative given by

\[
i \frac{d}{dt} Z_\sigma(t, s)\Psi = (H'_\sigma - pE_\sigma_0 + A_\sigma(t)) Z_\sigma(t, s)\Psi
\]  

(4.15)

where \( H'_\sigma \) is given by (2.17).

**Proof.** We first remark that all the operators in the product defining \( Z_\sigma(t, s) \) leave
\( \mathcal{D}(H_0) \) invariant by Proposition 2.2, part 2, by Lemma 4.1, part 2 and by Proposition
2.1. From the fact that \( C(\chi_\sigma \alpha_\lambda(t))^* \) is strongly differentiable in \( \mathcal{D}(H_0) \) by Lemma
4.1, part 3 and that \( U_\sigma(t) \) is strongly differentiable in \( \mathcal{D}(H_0) \) by Proposition 2.1, it
follows that \( Z_\sigma(t, s)\Psi \) is differentiable and that its time derivative is given by

\[
in \frac{d}{dt} Z_\sigma(t, s)\Psi = Q_\sigma^* \left\{ -ia^* (\chi_\sigma \dot{\alpha}_\lambda(t)) + ia \left( \chi_\sigma \alpha_\lambda(t) \right) + \text{Im} (\chi_\sigma \alpha_\lambda(t), \dot{\alpha}_\lambda(t)) 
+ C (\chi_\sigma \alpha_\lambda(t))^* (H_\sigma - pE_\sigma) C (\chi_\sigma \alpha_\lambda(t)) \right\} Q_\sigma \; Z_\sigma(t, s)\Psi .
\]  

(4.16)

Using (4.4), (4.5) and (4.9), we continue (4.16) as

\[
\cdots = Q_\sigma^* \left\{ -ia^* (\chi_\sigma \dot{\alpha}_\lambda(t)) + ia \left( \chi_\sigma \alpha_\lambda(t) \right) + \text{Im} (\chi_\sigma \alpha_\lambda(t), \dot{\alpha}_\lambda(t)) 
+ C (\chi_\sigma \alpha_\lambda(t))^* (H_\sigma - pE_\sigma) C (\chi_\sigma \alpha_\lambda(t)) \right\} Q_\sigma \; Z_\sigma(t, s)\Psi .
\]  

(4.16)
\[ H_{\sigma} - pE_{\sigma} + a^*(\chi_{\sigma}\omega\alpha_\lambda(t)) + a\left(\chi_{\sigma}\omega\alpha_\lambda(t)\right) + \| \chi_{\sigma}\omega^{1/2}\alpha_\lambda(t) \|^2 + \mathcal{A}_{\sigma}(t) \bigg\} Q_{\sigma} Z_{\sigma}(t, s) \Psi \]
\[ = Q_{\sigma}^* \left( H_{\sigma} - pE_{\sigma} + \mathcal{A}_{\sigma}(t) \right) Q_{\sigma} Z_{\sigma}(t, s) \Psi \]

which yields (4.15) by (2.15) and the fact that \( Q_{\sigma} \) commutes with \( \mathcal{A}_{\sigma}(t) \).

\[ \square \]

The operator \( H'_{\sigma} - pE_{\sigma_0} + \mathcal{A}_{\sigma}(t) \) contains only positive powers of \( \lambda \) and, as a form, is equivalent to \( (1 + H_0) \) uniformly in \( \lambda \) for \( \lambda \) sufficiently small. More precisely we have the following lemma.

\textbf{Lemma 4.2.} Let \( \alpha \in L^2_{1/2} \) and let \( \alpha(t) \) be given by (4.3). Let \( \lambda_0 \) and \( \sigma_0 \) be as in Proposition 2.4. Then there exist two constants \( \rho \) and \( C \) such that

\[ C^{-1} \| (1 + H_0)^{1/2} \Psi \|^2 \leq < \Psi, (\rho + H'_{\sigma} - pE_{\sigma_0} + \mathcal{A}_{\sigma}(t)) \Psi > \leq C \| (1 + H_0)^{1/2} \Psi \|^2 \]

(4.17)

for all \( \sigma \) with \( \sigma_0 < \sigma \), \( \lambda \) with \( |\lambda| < \lambda_0 \), \( t \in \mathbb{R} \) and \( \Psi \in \mathcal{D}(H_0^{1/2}) \). The constants \( \rho \) and \( C \) depend on \( \alpha \) through the norm \( \| \omega^{1/2}\alpha \|_2 \).

\textbf{Proof.} The estimate (3.6) implies

\[ |< \Psi, \mathcal{A}_{\sigma}(t) \Psi >| \leq \varepsilon < \Psi, H_0 \Psi > + C\varepsilon^{-1/3} \| \omega^{1/2}\alpha \|_2^{4/3} \| \Psi \|^2 \]

which together with Proposition 2.4, part 1 and the definition (2.16) of \( E_{\sigma_0} \) yields (4.17).

\[ \square \]

Using Lemma 4.2 we now prove that \( Z_{\sigma}(t, s) \) satisfies a uniform boundedness property and has a strong limit when \( \sigma \) tends to infinity.

\textbf{Proposition 4.2} Let \( \alpha \in L^2_{1/2} \) and let \( \alpha(t) \) be given by (4.3). Let \( \lambda_0 \) and \( \sigma_0 \) be as in Proposition 2.4. Then

1) For any compact interval \( I \) there exists a constant \( C_I \) such that

\[ \| (1 + H_0)^{1/2} Z_{\sigma}(t, s) \Psi \| \leq C_I \| (1 + H_0)^{1/2} \Psi \| \]

(4.18)

for all \( \sigma \) with \( \sigma_0 < \sigma \), \( \lambda \) with \( |\lambda| \leq \lambda_0 \), \( t, s \in I \) and \( \Psi \in \mathcal{D}(H_0^{1/2}) \). The constant \( C_I \) depends on \( \alpha \) through the norm \( \| \omega^{1/2}\alpha \|_2 \).
2) For any \( t, s \) the following strong limit exists

\[
\lim_{\sigma \to \infty} Z_\sigma(t, s) = Q^*W(t, s)Q = Z(t, s)
\]

and \( Z(t, s) \) satisfies the same estimate (4.18) as \( Z_\sigma(t, s) \).

**Proof.** Part 1 We know already by Proposition 2.2, part 2, by Lemma 4.1, part 2 and by Proposition 2.1 that \( Z_\sigma(t, s)D(H_0^{1/2}) = D(H_0^{1/2}) \). Let

\[
M_\sigma(t) \equiv \rho + H'_\sigma - pE_\sigma + A_\sigma(t)
\]

where \( \rho \) is the constant that appears in Lemma 4.2 and let \( \Psi \in D(H_0) \). The function \( < Z_\sigma(t, s)\Psi, M_\sigma(t)Z_\sigma(t, s)\Psi > \) is differentiable in the variable \( t \). In fact the differentiability of \( Z_\sigma(t, s) \) is known by Proposition 4.1 and the differentiability of \( M_\sigma(t) \) is a consequence of the fact that \( \partial_t M_\sigma(t) = \dot{A}_\sigma(t) \). In addition since \( A_\sigma \) and \( \dot{A}_\sigma \) belong to \( L^\infty(I\mathbb{R}^3) \), the operators \( A_\sigma(t) \) and \( \dot{A}_\sigma(t) \) are bounded in \( \mathcal{H} \) and strongly continuous in \( t \). Therefore

\[
\frac{d}{dt} < Z_\sigma(t, s)\Psi, M_\sigma(t)Z_\sigma(t, s)\Psi > = < Z_\sigma(t, s)\Psi, \dot{A}_\sigma(t)Z_\sigma(t, s)\Psi > \quad (4.20)
\]

and by integration

\[
< Z_\sigma(t, s)\Psi, M_\sigma(t)Z_\sigma(t, s)\Psi > = < \Psi, M_\sigma(s)\Psi > \\
+ \int_s^t dt' < Z_\sigma(t', s)\Psi, \dot{A}_\sigma(t')Z_\sigma(t', s)\Psi > . \quad (4.21)
\]

Using the estimate (4.17) for the terms with \( M_\sigma \) and the estimate (3.8) for the term with \( A_\sigma \) we obtain

\[
\| (1 + H_0)^{1/2}Z_\sigma(t, s)\Psi \|^2 \leq C \left( \| (1 + H_0)^{1/2}\Psi \|^2 + \int_s^t dt' \| (1 + H_0)^{3/4}Z_\sigma(t', s)\Psi \|^2 \right)
\]

which yields trivially the linear inequality

\[
\| (1 + H_0)^{1/2}Z_\sigma(t, s)\Psi \|^2 \leq C \left( \| (1 + H_0)^{1/2}\Psi \|^2 + \int_s^t dt' \| (1 + H_0)^{1/2}Z_\sigma(t', s)\Psi \|^2 \right) . \quad (4.22)
\]

By integrating (4.22) we obtain (4.18) for \( \Psi \in D(H_0) \). We then extend (4.18) to all \( \Psi \in D(H_0^{1/2}) \) by continuity.
Part 2. By Proposition 2.2, part 3, by Lemma 4.1, part 1 and by Proposition 2.4, part 3, all operators in the product (4.13) of \( Z_\sigma(t, s) \) converge strongly when \( \sigma \) tends to infinity. The estimate (4.18) for \( Z(t, s) \) follows from that convergence and from the uniformity of the estimate for \( Z_\sigma(t, s) \) in \( \sigma \).

\[ \square \]

We are now in condition to take the limit \( \lambda \to 0 \).

**Proposition 4.3.** Let \( \alpha \in L^2_{1/2} \), let \( \alpha(t) \) be given by (4.3) and let \( A \) be defined by (3.3). Let \( W(t, s) \) be defined by (4.2) and let \( V(t, s) \) be defined in Proposition 3.1. Then the following strong limit exists

\[ s \lim_{\lambda \to 0} W(t, s) = V(t, s) \quad (4.23) \]

uniformly for \( t, s \) in compact intervals.

**Proof.** Let \( I \) be a compact interval and let \( t, s \in I \). Let \( \lambda_0 \) and \( \sigma_0 \) be as in Proposition 2.4. Let \( \sigma > \sigma_0 \) and \( \Psi \in D(H_0) \). We estimate the difference

\[ \| (Z_\sigma(t, s) - V(t, s)) \Psi \|^2 = 2 \text{Re} \left\{ \langle \Psi, \Psi \rangle - \langle Z_\sigma(t, s) \Psi, V(t, s) \Psi \rangle \right\} \]

\[ = -2 \text{Re} \int_s^t dt' \frac{d}{dt'} \langle Z_\sigma(t', s) \Psi, V(t', s) \Psi \rangle = 2 \text{Im} \int_s^t dt' \left\{ \langle (H'_\sigma - p\mathcal{E}_{\sigma_0} + \mathcal{A}_\sigma(t')) Z_\sigma(t', s) \Psi, V(t', s) \Psi \rangle - \langle Z_\sigma(t', s) \Psi, (H_0 + \mathcal{A}(t')) V(t', s) \Psi \rangle \right\} \]

\[ = 2 \text{Im} \int_s^t dt' < Z_\sigma(t', s) \Psi, ((H'_\sigma - H_0) - p\mathcal{E}_{\sigma_0} + \mathcal{A}_\sigma(t') - \mathcal{A}(t')) V(t', s) \Psi > \quad (4.24) \]

where we have used Proposition 3.1 and Proposition 4.1. We now apply the estimate (3.12) to \( V(t, s) \) and the estimate (4.18) to \( Z_\sigma(t, s) \), thereby obtaining

\[ \| (Z_\sigma(t, s) - V(t, s)) \Psi \|^2 \leq C_I \int_s^t dt' \left\{ \| (1 + H_0)^{-1/2}(H'_\sigma - H_0)(1 + H_0)^{-1/2} \| + p|E_{\sigma_0}| + \| (1 + H_0)^{-1/2} (\mathcal{A}(t') - \mathcal{A}_\sigma(t')) (1 + H_0)^{-1/2} \| \right\} \| (1 + H_0)^{1/2} \Psi \|^2 \quad (4.25) \]

with \( C_I \) uniform in \( \sigma > \sigma_0 \) and in \( \lambda, |\lambda| \leq \lambda_0 \). Now

\[ \| (1 + H_0)^{-1/2}(H'_\sigma - H_0)(1 + H_0)^{-1/2} \| \leq C\lambda, \]

\[ p|E_{\sigma_0}| \leq C\lambda^2 \]

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by Lemma 2.2 and by (2.16) respectively, and
\[ \| (1+H_0)^{-1/2}(A_\sigma(t') - A(t'))(1+H_0)^{-1/2} \| \leq C \| (1+H_0)^{-1}H_0^{1/4} \| \| (1-\chi_\sigma)\omega^{1/2}\alpha \|_2 \]
by the estimate (3.6), so that (4.25) implies
\[ \| (Z_\sigma(t,s) - V(t,s)) \Psi \|^2 \leq C_T \left( \lambda + \| (1-\chi_\sigma)\omega^{1/2}\alpha \|_2 \right) \| (1 + H_0)^{1/2} \Psi \|^2 \] (4.26)
with \( C_T \) uniform in \( \lambda \), \( |\lambda| \leq \lambda_0 \) and in \( \sigma > \sigma_0 \). Taking the limit \( \sigma \to \infty \) in (4.26) and using Proposition 4.2, part 2, we obtain
\[ \| (Z(t,s) - V(t,s)) \Psi \| \leq C_I \lambda^{1/2} \| (1 + H_0)^{1/2} \Psi \| \] (4.27)
From the identity
\[ W(t,s) - V(t,s) = Q Z(t,s)Q^* - V(t,s) \]
\[ = (Q-1)Z(t,s) + Q Z(t,s)(Q^* - 1) + Z(t,s) - V(t,s) \]
we obtain the estimate
\[ \| (W(t,s) - V(t,s)) \Psi \| \leq \| (Q-1)Z(t,s)\Psi \| + \| (Q^* - 1)\Psi \| \]
\[ + \| (Z(t,s) - V(t,s)) \Psi \| \] (4.28)
Now
\[ \| (Q^* - 1)\Psi \| \leq \| T\Psi \| \leq \| T(1 + H_0)^{-1/2} \| \| (1 + H_0)^{1/2} \Psi \| \]
so that from the definition (2.13) of \( T \) and from (2.6) (2.7),
\[ \| (Q^* - 1)\Psi \| \leq C \lambda \| (1 + H_0)^{1/2} \Psi \| \] (4.29)
where the linear dependence in \( \lambda \) comes from the linear dependence of \( T \) on \( g \) and therefore on \( \lambda \).

Similarly from (4.29) using (4.18) for \( Z(t,s) \) we estimate
\[ \| (Q-1)Z(t,s)\Psi \| \leq C_I \lambda \| (1 + H_0)^{1/2} \Psi \| \] (4.30)
By substituting (4.27), (4.29) and (4.30) into (4.28) we obtain
\[ \| (W(t,s) - V(t,s)) \Psi \| \leq C_I \lambda^{1/2} \| (1 + H_0)^{1/2} \Psi \| \] (4.31)
This proves the convergence of \( W(t,s)\Psi \) to \( V(t,s)\Psi \) for any \( \Psi \in \mathcal{D}(H_0) \) when \( \lambda \) converges to zero, uniformly for \( t,s \) in compact intervals. Convergence for any \( \Psi \in \mathcal{H} \) follows from the unitarity of \( W(t,s) \) and \( V(t,s) \).
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