Duality among competing orders in antiferromagnets and topological insulators: nonlinear sigma model approach

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Abstract. We revisit our framework for studying competing orders in quantum antiferromagnets (A. Tanaka and X. Hu, Phy. Rev. Lett. 95 (2005) 036402; Phys. Rev. B 74 (2006) 140407(R)), in which we showed that when two or more ordering tendencies are organized into a single composite order parameter, the corresponding effective sigma model action can generally host a new topological term. It has since been argued by several authors that such terms are indicative of novel critical behaviors. Here we reinforce this assertion by searching for nonvanishing fermionic bilinears (Golstone-Wilczek-type currents) associated with the same models. We arrive at a duality relation among the competing orders (e.g. a direct relation between the topological charge of the antiferromagnetic order sector and a Noether current associated with the rotational symmetry in the valence bond solid order sector), which becomes relevant upon approaching criticality. We also discuss the physical implications of this duality in the context of topological insulators.

1. Introduction

Several years ago[1, 2] the author showed how a new nonlinear sigma model description of competing orders within a two-dimensional antiferromagnet can emerge, when one chooses to start from massive Dirac fermions which accommodates a manifold of chiral-symmetry breaking states (each realization of chiral symmetry breaking corresponds to a different ordering pattern in the antiferromagnet). The possible connection of this theory to the problem of deconfined quantum criticality[3] has since been taken up by various authors[4]. Our sigma model framework, which builds on a general field theory prescription laid out by Abanov and Wiegmann[5], features a new Wess-Zumino-type topological term, and (1) can be generalized to arbitrary dimensions (2) and applies as well to other combinations of competing orders, e.g. to the competition between a quantum spin Hall state and s-wave superconductivity[6]. More recently, a general mathematical correspondence between this term and momentum space topological invariants (Chern numbers) employed in the study of topological insulators was laid out, which adds a further motivation for investigating the properties of this theory[7].

Unfortunately, the effective action itself is difficult to analyze, and has so far lead to little information on the detailed nature of the criticality which might arise out of the competition among multiple ordering tendencies. The Abanov-Wiegmann machinery, however, can also be used to obtain various Goldstone-Wilczek-type currents/fermion numbers[8] which are induced by order parameter configurations with nontrivial topology. Here we utilize this aspect of the
theory to derive nontrivial duality relations which hold between currents/charges associated with the symmetry of one order, and the topological charge of an opposing order. In the original context[1] of the 2d quantum antiferromagnet, for instance, we can show that when the valence-bond-solid (VBS) sector of the model possesses a rotational symmetry (which implies the existence of a conserved Noether current) –such as had been observed in several numerical studies on frustrated spin systems, the Skyrmion number of the Néel vector field configuration is conserved. Since the conservation of Skyrmion number is a central feature of the theory of deconfined quantum criticality[3], its appearance here suggests that further investigations along this direction can contribute some fresh insights into this intriguing phenomenon. The duality relation, when applied to the case of topological insulators can be given a more direct physical interpretation: it suggests that the Skyrmion number of the spin Hall effect order parameter is to be identified with a preformed Cooper pair, as anticipated in Ref. [6].

2. Competing orders as patterns of chiral symmetry breaking

We begin by summarizing the salient points of the forementioned framework[1] employed in our following discussions. In our original settings[1], our point of departure was the Affleck-Marston $\pi$-flux state of an antiferromagnet residing on a 2d square lattice. This state accommodates two inequivalent “valleys” of massless Dirac cones within the Brioullin zone, which can be lumped together into a 4-component Dirac spinor with spin indices. The latter would also describe (with suitable readjustment of the gamma matrix representation) the continuum limit of tight-binding electrons hopping on a 2d honeycomb lattice, i.e. graphene.

Let us now open up a mass gap at the Dirac points. Due to the enlarged 4-by-4 gamma matrix representation, there are two generators of chiral transformations, denoted below as $\gamma_3$ and $\gamma_5$, which together span a manifold of chiral symmetry breaking states. This chiral structure offers an interesting setting (which readily generalizes to other spatial dimensions) for analyzing the competition among various orders, each of which corresponds to a different manifestation of broken chiral symmetry. An example of particular interest is the case of competing antiferromagnetic (AF) and VBS orders, which can be incorprated into the fermionic model by setting the mass term to be [1]:

$$L_{mass} = \bar{\psi} m_{AF-VBS} \psi = m\bar{\psi}(\mathbf{N} \cdot \sigma + iN_{xVBS}\gamma_3 + iN_{yVBS}\gamma_5)\psi.$$  \hspace{1cm} (1)

Here the vector $\mathbf{N}$ is the AF order parameter, while $N_{xVBS}$ and $N_{yVBS}$ are the horizontal and vertical components of the VBS order parameter. Together they comprise a five-component composite order parameter $\tilde{\mathbf{N}} \equiv (\mathbf{N}, N_{xVBS}, N_{yVBS})$ whose length we set to be unity. The effective action for $\tilde{\mathbf{N}}$, valid below the energy scale $m$, can be obtained[1] by the Abanov-Wiegmann gradient expansion scheme[5], which systematically salvages contributions which may arise from anomalies. This leads us to the effective action $S_{eff}[\tilde{\mathbf{N}}] = S_{NL\sigma} + S_{WZ}$, where $S_{NL\sigma}$ is an $O(5)$ nonlinear sigma model, and $S_{WZ}$ the Wess-Zumino term (written in Euclidean space-time),

$$S_{WZ} = ik \int_{S_3 \times [0,1]} \tilde{\mathbf{N}} \wedge \omega.$$  \hspace{1cm} (2)

In the above, the map $\tilde{\mathbf{N}} : S_3 \rightarrow S_4$ has been extended to $\tilde{\mathbf{N}} : S_3 \times [0,1] \rightarrow S_4$ according to standard prescriptions for constructing a Wess-Zumino (WZ) term[9], $\omega$ is the normalized volume form $\sim \tilde{\mathbf{N}} \wedge d\tilde{\mathbf{N}} \wedge d\tilde{\mathbf{N}} \wedge d\tilde{\mathbf{N}} \wedge d\tilde{\mathbf{N}}$ on $S_3$, and finally, $\tilde{\mathbf{N}}^*$ is the pullback from $S_4$ to the base manifold $S_3 \times [0,1]$. The level $k$ for our case is unity.

It is clear that this method can be generalized to different choices of competing orders, provided they can all be expressed as chiral symmetry breaking terms of an underlying Dirac fermion system. A particularly interesting situation arises when one of the orders involve pairing
Furthermore, as mentioned above, the form of the action number, which is a hallmark of the original DQC scenario, is not manifest in the O(5) model et al[3]. However, an emergent U(1) symmetry corresponding to the conservation of Skyrmion a nonvanishing current by introducing a gauge field \( \tilde{a} \) contributions coming from each Dirac “valley” is responsible for this null result. We thus seek suggesting the exotic nature of the system, is not suited for submitting to a detailed analysis.

To gain further insight into the symmetry properties of the model \( \mathcal{L} = \bar{\psi}(i\partial + \tilde{m}_{AF-VBS})\psi \) (hereafter the Feynman slash will represent, as usual, the contraction with the space-time components of the gamma matrices, e.g. \( \not\partial \equiv \gamma_{\mu} \partial_{\mu} \)), we recall that in addition to deriving the effective action, the Abanov-Wiegman gradient expansion also enables us to evaluate fermionic quantum Hall state (i.e. a 2d topological insulator) and s-wave superconductivity is present. Here again the two orders comprise a 5 component vectorial order parameter with a unit norm. The authors of these papers find that the corresponding effective action reduces to \( S_{\text{eff}}[\vec{N}] \), which we had found for the AF-VBS competition. It is also straightforward to work out lower/higher dimensional analogues. Hence the present work, which primarily deals with 2d quantum antiferromagnets, also should offer some insights into similar problems in different dimensionalities, as well as in the field of topological insulators.

3. Fermion currents

It has been suggested [1, 4] that the presence of the term \( S_{WZ} \) accounts for the highly unconventional phenomenology of deconfined quantum criticality (DQC) proposed by Senthil et al[3]. However, an emergent U(1) symmetry corresponding to the conservation of Skyrmion number, which is a hallmark of the original DQC scenario, is not manifest in the O(5) model as it stands. This observation has motivated the search for a different fixed point theory[10]. Furthermore, as mentioned above, the form of the action \( S_{\text{eff}}[\vec{N}] = S_{NLO} + S_{WZ} \), while strongly suggesting the exotic nature of the system, is not suited for submitting to a detailed analysis.

To gain further insight into the symmetry properties of the model \( \mathcal{L} = \bar{\psi}(i\partial + \tilde{m}_{AF-VBS})\psi \) (hereafter the Feynman slash will represent, as usual, the contraction with the space-time components of the gamma matrices, e.g. \( \not\partial \equiv \gamma_{\mu} \partial_{\mu} \)), we recall that in addition to deriving the effective action, the Abanov-Wiegman gradient expansion also enables us to evaluate fermionic quantum Hall state (i.e. a 2d topological insulator) and s-wave superconductivity is present. Here again the two orders comprise a 5 component vectorial order parameter with a unit norm. The authors of these papers find that the corresponding effective action reduces to \( S_{\text{eff}}[\vec{N}] \), which we had found for the AF-VBS competition. It is also straightforward to work out lower/higher dimensional analogues. Hence the present work, which primarily deals with 2d quantum antiferromagnets, also should offer some insights into similar problems in different dimensionalities, as well as in the field of topological insulators.

Let us first illustrate the general strategy using the simpler case of a two-component Dirac fermion interacting with a three-component unit vector field \( n \), \( \mathcal{L}_{F} = \bar{\psi}(i\not\partial + m n \cdot \sigma)\psi \equiv \bar{\psi}D\psi \). (For this case there are only three gamma matrices, which can conveniently be identified with the Pauli matrices.) Couple the fermions to a source vector field \( a_{\mu} \) via the prescription \( i\partial_{\mu} \rightarrow i\not\partial + a_{\mu} \). The current \( j_{\mu} \equiv \langle \bar{\psi}\gamma_{\mu}\psi \rangle \) is then easily calculated by noting that \( \frac{\delta S_{WZ}}{\delta a_{\mu}} = \text{tr}[(D^{\dagger}D)^{-1}D^{\dagger}\delta D|_{a_{\mu}=0}] \) and by treating the operator \( (D^{\dagger}D)^{-1} \) in a gradient expansion. We find that

\[
\langle j_{\mu} \rangle = \epsilon_{\mu\nu\lambda} \frac{1}{4\pi} n \cdot \partial_{\nu} n \times \partial_{\lambda} n. \tag{3}
\]

Repeating this procedure for our fermionic action \( \mathcal{L} = \bar{\psi}(i\not\partial + \tilde{m}_{AF-VBS})\psi \), we immediately find that \( \langle j_{\mu} \rangle = 0 \). A quick inspection reveals that an exact sign cancellation between the contributions coming from each Dirac “valley” is responsible for this null result. We thus seek a nonvanishing current by introducing a gauge field \( \tilde{a}_{\mu} \) whose gauge charge has opposite signs at the two valleys. [We note some interesting parallels to stategies taken in graphene physics. The Haldane model[13], which exhibits a zero-field integer quantum Hall effect, is in essence a 2d quantum antiferromagnets, also should offer some insights into similar problems in different dimensionalities, as well as in the field of topological insulators.

4. Channels

Refs. [6, 11, 12], for example, consider 2d systems where the competition between a quantum spin Hall state (i.e. a 2d topological insulator) and s-wave superconductivity is present. Here again the two orders comprise a 5 component vectorial order parameter with a unit norm. The authors of these papers find that the corresponding effective action reduces to \( S_{\text{eff}}[\vec{N}] \), which we had found for the AF-VBS competition. It is also straightforward to work out lower/higher dimensional analogues. Hence the present work, which primarily deals with 2d quantum antiferromagnets, also should offer some insights into similar problems in different dimensionalities, as well as in the field of topological insulators.
\[
\epsilon_{\mu\nu\lambda} \frac{1}{4\pi} \mathbf{N} \cdot \partial_\mu \mathbf{N} \times \partial_\lambda \mathbf{N},
\]

which bears an apparent similarity to eq. (3).

Noting that the set of matrices \{\gamma_3, \gamma_5, i\gamma_{35}\} forms an SU(2) algebra, and recalling[1] that the pseudoscalars \(\bar{\psi}\gamma_3\psi\) and \(\bar{\psi}\gamma_5\psi\) are the fermionic expressions for the VBS order parameters, we can see that \(\gamma_{35}\) generates rotation within the VBS order parameter space, and that \(J_{\mu}^{35}\) is the Noether current associated with rotational symmetry (when such symmetry is present) within this space. In other words, when the VBS sector exhibits a U(1) rotational symmetry, the right hand side of the above equation, which is a quantity associated with the AF sector, is a conserved current. In the case where there is no amplitude fluctuation between the two sectors, the latter is, modulo a constant prefactor, the topological current of AF Skyrmion events. It is interesting to interpret this duality relation in light of numerical results related to the DQC[14, 15], where the U(1) rotational symmetry in the VBS sector was indeed observed when the system was (presumably) in proximity to the critical point.

Let us now apply the same argument to the case where the competition is between quantum spin Hall and superconducting orders[6]. Here the quantum spin Hall state is assumed to be induced as a result of strong electron correlations (i.e. a topological Mott insulator), and the spin quantization axis is treated as a dynamically fluctuating entity. The duality relation now states that the conserved charge associated with a U(1) symmetry within the superconducting sector is given by the Skyrmion topological charge of the opposing (quantum spin Hall) sector. When combined with the conjugation relation between the phase and the amplitude of the superconducting order parameter, we find that this is in accord with the assertion[6] that the Skyrmions of this system are actually preformed Cooper pairs, which when condensed lead to superconductivity. Duality relations similar to eq. (4) are readily obtained for other spatial dimensions, and will be reported elsewhere.

Finally we note that while preparing this report, we came across a preprint by Fu, Sachdev, and Xu[16] which also addresses in part some similar aspects of competing orders in 2d. Instead of assuming a Dirac-like fermionic system at the outset, these authors begin with a Hubbard model, and incorporate the band structure of the electron system. They also find a nonvanishing Berry phase-induced electron bilinear in the background of a finite Skyrmion density, but point out that the results depend on the band structure. It is interesting, in view of these results, to attempt to incorporate nonrelativistic electron dispersions into the approach that we have described in this article.

In summary, we have studied the Goldstone-Wilczek currents associated with Dirac fermion systems which describe the competition between different ordering tendencies in antiferromagnets and topological insulators. We found that the low effective actions of such systems, nonlinear sigma models with a topological term, have a nontrivial built-in duality between the competing orders. We expect this approach to be useful in gaining new insights into competing orders in quantum magnets and topological insulators.

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