Semantic Incompleteness of Hilbert System for a Combination of Classical and Intuitionistic Propositional Logic

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Abstract

This paper shows Hilbert system \((C + J)^-\), given by del Cerro and Herzig (1996) is semantically incomplete. This system is proposed as a proof theory for Kripke semantics for a combination of intuitionistic and classical propositional logic, which is obtained by adding the natural semantic clause of classical implication into intuitionistic Kripke semantics. Although Hilbert system \((C + J)^-\) contains intuitionistic modus ponens as a rule, it does not contain classical modus ponens. This paper gives an argument ensuring that the system \((C + J)^-\) is semantically incomplete because of the absence of classical modus ponens. Our method is based on the logic of paradox, which is a paraconsistent logic proposed by Priest (1979).

1 Introduction

This paper shows semantic incompleteness of Hilbert system \((C + J)^-\), given by del Cerro and Herzig [8]. This system was provided for a combination of intuitionistic and classical propositional logic. This combined logic has two implications: intuitionistic one (denoted by “\(\rightarrow_i\)”) and classical one (denoted by “\(\rightarrow_c\)’’). This logic also has falsum, conjunction, and disjunction as connectives which are common to intuitionistic and classical logic. Since this logic is constructed to be a combination of intuitionistic and classical logic, Hilbert system \((C + J)^-\) has to enable us to be a conservative extension of both logics.

The way of constructing the combination is easier to understand from a semantic respect. The semantics for this combination is given in [9][8], and the basic idea is adding classical implication into intuitionistic Kripke semantics, which means the satisfaction relation of classical implication, denoted by “\(\rightarrow_c\)” is given in a Kripke model as follows:

\[ w \models_M A \rightarrow_c B \ \iff \ w \models_M A \ \text{implies} \ w \models_M B, \]

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where $M$ is an intuitionistic Kripke model, $w$ is a possible world in $M$, and $R$ is a preorder equipped in $M$. The other parts of the Kripke semantics is the same as that of intuitionistic propositional logic.

The syntax $L$ consists of a countably infinite set $\text{Prop}$ of propositional variables and the following logical connectives: falsum $\bot$, disjunction $\lor$, conjunction $\land$, intuitionistic implication $\rightarrow_i$, and classical implication $\rightarrow_c$. We denote by $L_C$ (the syntax for the classical logic) and $L_J$ (the syntax for the intuitionistic logic) the resulting syntax dropping $\rightarrow_i$ and $\rightarrow_c$ from $L$, respectively.

The set $\text{Form}$ of all formulas in the syntax is defined inductively as follows:

$$A ::= p | \bot | A \lor A | A \land A | A \rightarrow_i A | A \rightarrow_c A,$$

where $p \in \text{Prop}$. We denote by $\text{Form}_C$ and $\text{Form}_J$ the set of all classical formulas and the set of all intuitionistic formulas, respectively. We define $\top ::= \bot \rightarrow_i \bot$, $\neg_c A ::= A \rightarrow_c \bot$, and $\neg_i A ::= A \rightarrow_i \bot$.

Let us move to the semantics for the syntax $L$.

**Definition 1.** A model is a tuple $M = (W, R, V)$ where

- $W$ is a non-empty set of possible worlds,
- $R$ is a preorder on $W$, i.e., $R$ satisfies reflexivity and transitivity,
- $V : \text{Prop} \rightarrow \mathcal{P}(W)$ is a valuation function satisfying the following heredity condition: $w \in V(p)$ and $wRv$ jointly imply $v \in V(p)$ for all possible worlds $w, v \in W$.

**Definition 2.** Given a model $M = (W, R, V)$, a possible world $w \in W$ and a formula $A$, the satisfaction relation $w \models_M A$ is inductively defined as follows:

$$w \models_M p \iff w \in V(p),$$

$$w \not\models_M \bot,$$

$$w \models_M A \land B \iff w \models_M A \text{ and } w \models_M B,$$

$$w \models_M A \lor B \iff w \models_M A \text{ or } w \models_M B,$$

$$w \models_M A \rightarrow_i B \iff \text{ for all } v \in W, (wRv \text{ and } v \models_M A \text{ jointly imply } v \models_M B),$$

$$w \models_M A \rightarrow_c B \iff w \models_M A \text{ implies } w \models_M B.$$  

Let $\Gamma$ be a set of formulas and $A$ be a formula. A formula $A$ is called a semantic consequence of $\Gamma$, written as $\Gamma \models A$, if, for all models $M = (W, R, V)$ and all worlds $w \in W$, $w \models_M C$ for all formulas $C \in \Gamma$ implies $w \models_M A$. A formula $A$ is valid if $\emptyset \models A$ holds.

From Definition 2, the following satisfaction relation for a formula whose main connective is $\neg_c$ or $\neg_i$ is obtained:

$$w \models_M \neg_c A \iff w \not\models_M A,$$

$$w \models_M \neg_i A \iff \text{ for all } v \in W, (wRv \text{ implies } v \not\models_M A).$$

A notion of heredity, which is an important notion in intuitionistic logic, can be defined in this Kripke semantics.

**Definition 1 (Heredity).** A formula $A$ satisfies heredity iff for any model $M$ and $w, v \in W$, $wRv$ and $w \models_M A$ jointly imply $v \models_M A$.

In pure intuitionistic logic, any formula satisfies heredity. However, if we add classical implication, there will exist a formula which does not satisfy heredity.

**Proposition 2.** A formula $\neg c p$ does not satisfy heredity.
Corresponding to Proposition 2, the following proposition also holds.

**Proposition 3.** Both $\neg c p \rightarrow c (q \rightarrow s c p)$ and $\neg c p \rightarrow s c p (q \rightarrow s c p)$ is invalid.

Proposition 3 implies an intuitionistic theorem $A \rightarrow s c (B \rightarrow s c A)$ is no longer a theorem in this combination. An argument about Propositions 2 and 3 was given in [15, 16].

Let us move to a proof theory given in [8]. Before giving the detail of their axiomatization, we introduce the notion of persistent formulas as follows:

$$E ::= \bot \mid p \mid A \rightarrow s c A \mid E \wedge E \mid E \vee E,$$

where $p \in \text{Prop}$ and $A \in \text{Form}$.  \[1\]

**Definition 4.** Hilbert system $(C + J)^{-}$ consists of axioms (CL), (CK), (ID), (CMP), and (PER) of Table 1 and rules (MPI) and (RCN) of Table 1. Hilbert system $C + J$ is the extended system of $(C + J)^{-}$ with the rule (MPC) of Table 1.

**Table 1: Hilbert Systems $(C + J)^{-}$ and $C + J$**

| Hilbert System $(C + J)^{-}$ |
|-----------------------------|
| (CL) All instances of classical tautologies |
| (CK) $(A \rightarrow_{s c} (B \rightarrow_{s c} C)) \rightarrow_{s c} ((A \rightarrow_{s c} B) \rightarrow_{s c} (A \rightarrow_{s c} C))$ |
| (ID) $A \rightarrow_{s c} A$ |
| (CMP) $(A \rightarrow_{s c} B) \rightarrow_{s c} (A \rightarrow_{s c} B)$ |
| (PER) $A \rightarrow_{s c} (B \rightarrow_{s c} A)$ \[1\] ‡: $A$ is persistent. |
| (MPI) From $A$ and $A \rightarrow_{s c} B$ we may infer $B$ |
| (RCN) From $A$ we may infer $B \rightarrow_{s c} A$ |

| Hilbert System $C + J$ |
|------------------------|
| All the axioms and rules of $(C + J)^{-}$ |
| (MPC) From $A$ and $A \rightarrow_{s c} B$ we may infer $B$ |

An important axiom is (PER). Recall that a formula $\neg c p \rightarrow_{s c} (q \rightarrow_{s c} \neg c p)$ is not valid in the Kripke semantics, as is described in Proposition 3. In order for this formula to be underivable, an antecedent formula $A$ of (PER) should be restricted to a persistent formula.

In the next section, we show Hilbert system $(C + J)^{-}$ is semantically incomplete. This semantic incompleteness is the result of the absence of (MPC). The system $C + J$ may be what del Cerro and Herzig [8] intended to provide, but (MPC) does not exist, which may be an unfortunate typo. It is reasonable that (MPC) is necessary, because $C + J$ is based on an idea of an axiomatization of conditional logic, which adds axioms and rules on the conditional on the top of classical tautologies and the rule of classical modus ponens (see, e.g., [6, 7, 10]).

The following proposition ensures the rule (MPI) can be deleted from Hilbert system $C + J$.

**Proposition 5.** If we drop (MPI) from $C + J$, (MPI) is derivable in the resulting system.

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\[1\] Del Cerro and Herzig [8] did not defined $\bot$ as a persistent formula. However, this is not an essential point, since $\bot$ is equivalent to $p \wedge \neg c p$, which is a persistent formula in the sense of [8]. This slight change allows us to state that all formulas of $L_J$ is persistent.
Proof. Suppose $A$ and $A \rightarrow_{1} B$ are theorems in the resulting system. By (CMP) and (MPC), $A \rightarrow_{c} B$ is obtained from $A \rightarrow_{1} B$. By applying (MPC) to $A$ and $A \rightarrow_{c} B$, $B$ is obtained, as is desired. 

Thus, in order to obtain $C + J$ from $(C + J)^{-}$, to replace (MPC) with (MPC) is sufficient.

2 Semantic Incompleteness of Hilbert System $(C + J)^{-}$

In this section, we show Hilbert system $(C + J)^{-}$ is semantically incomplete. We provide a formula $C$ such that $C$ is valid in the semantics described in Definitions 1 and 2 but $C$ is not a theorem of $(C + J)^{-}$. Our candidate for $C$ is $(p \land (p \rightarrow_{c} q)) \rightarrow_{1} q$. The following is easy to establish.

**Proposition 6.** The formula $(p \land (p \rightarrow_{c} q)) \rightarrow_{1} q$ is valid in Kripke semantics in Definitions 1 and 2.

What we need to show is to establish that $(p \land (p \rightarrow_{c} q)) \rightarrow_{1} q$ is not a theorem of $(C + J)^{-}$. For this purpose, we need to consider a non-standard semantics such that the soundness holds to the original system but $(p \land (p \rightarrow_{c} q)) \rightarrow_{1} q$ is not valid.

In order to make such a semantics, we utilize three-valued semantics for a paraconsistent logic by Priest (cf. [14]), i.e., the logic of paradox [13], which allows the third truth value of “both true and false” {0, 1} in addition to the values {0} ("false only") and {1} ("true only").

**Definition 3.** A valuation $v$ is a mapping from Prop to \{0, 1, 1\}. A valuation $v$ is uniquely extended to a function $\mathfrak{v}$ from the set Form of all formulas to \{0, 1, 1\} as follows:

| $\mathfrak{v}(\bot)$ | $\mathfrak{v}(\bot)$ |
|-----------------------|-----------------------|
| 1                     | 0                     |
| 1 $\in \mathfrak{v}(A \land B)$ iff 1 $\in \mathfrak{v}(A)$ and 1 $\in \mathfrak{v}(B)$, |
| 0 $\in \mathfrak{v}(A \land B)$ iff 0 $\in \mathfrak{v}(A)$ or 0 $\in \mathfrak{v}(B)$, |
| 1 $\in \mathfrak{v}(A \lor B)$ iff 1 $\in \mathfrak{v}(A)$ or 1 $\in \mathfrak{v}(B)$, |
| 0 $\in \mathfrak{v}(A \lor B)$ iff 0 $\in \mathfrak{v}(A)$ and 0 $\in \mathfrak{v}(B)$, |
| 1 $\in \mathfrak{v}(A \rightarrow_{c} B)$ iff 0 $\in \mathfrak{v}(A)$ or 1 $\in \mathfrak{v}(B)$, |
| 0 $\in \mathfrak{v}(A \rightarrow_{c} B)$ iff 1 $\in \mathfrak{v}(A)$ and 0 $\in \mathfrak{v}(B)$, |
| 1 $\in \mathfrak{v}(A \rightarrow_{1} B)$ iff 1 $\notin \mathfrak{v}(A)$ or 1 $\in \mathfrak{v}(B)$, |
| 0 $\in \mathfrak{v}(A \rightarrow_{1} B)$ iff 1 $\in \mathfrak{v}(A)$ and 0 $\in \mathfrak{v}(B)$, |

A consequence relation $\Sigma \models_{3} A$ is defined as: if 1 $\in \mathfrak{v}(B)$ holds for all $B \in \Sigma$ then 1 $\in \mathfrak{v}(A)$. We say that a formula $A$ is 3-valid if $\models_{3} A$ holds.

**Proposition 7.** For every valuation $v : \text{Prop} \rightarrow \{\{1\}, \{0, 1\}, \{0\}\}$ and every $A \in \text{Form}$, 1 $\in \mathfrak{v}(A)$ or 0 $\in \mathfrak{v}(A)$.

Proof. Fix any valuation $v : \text{Prop} \rightarrow \{\{1\}, \{0, 1\}, \{0\}\}$. By induction on $A$, we can obtain the desired statement.
T able 2: Three V alued Truth T able

| ∧   | t | b | f |
|-----|---|---|---|
| t   | t | b | f |
| b   | b | b | f |
| f   | f | f | f |

| ∨   | t | b | f |
|-----|---|---|---|
| t   | t | t | t |
| b   | t | b | b |
| f   | f | t | b |

| ¬_c | t | b | f |
|-----|---|---|---|
| t   | t | b | f |
| b   | b | b | f |
| f   | f | b | t |

| →_c | t | b | f |
|-----|---|---|---|
| t   | t | b | f |
| b   | b | b | f |
| f   | f | t | t |

| ⊥   |
|-----|
| t   |
| b   |
| f   |

Remark 1. Let us denote \{1\}, \{0, 1\}, \{0\} by t, b, f, respectively. Then the semantics above provides the three valued truth table, described in Table 2 where the values t and b are defined as “designated values”.

By recalling \(\neg_c A := A \rightarrow \bot\) and \(\neg_i A := A \rightarrow \bot\) respectively, we can also obtain the following satisfaction relation for negations:

\[
1 \in \overline{v} (\neg_c A) \text{ iff } 0 \in \overline{v} (A), \\
0 \in \overline{v} (\neg_c A) \text{ iff } 1 \in \overline{v} (A), \\
1 \in \overline{v} (\neg_i A) \text{ iff } 1 \notin \overline{v} (A), \\
0 \in \overline{v} (\neg_i A) \text{ iff } 1 \notin \overline{v} (A).
\]

Therefore, \(\neg_c\) is De Morgan negation (cf. [14]). We can also get the truth table for \(\neg_c\) and \(\neg_i\), described in Table 3.

Table 3: Truth Table for Negations

| A   | \neg_c A |
|-----|---------|
| t   | t       |
| b   | b       |
| f   | t       |

| A   | \neg_i A |
|-----|---------|
| t   | t       |
| b   | b       |
| f   | t       |

The set \{∧, ∨, ¬_c\} of logical connectives is exactly the same set of primitive logical connectives as the propositional part of the logic of paradox [13, 14]. It is remarked, however, that \(A \rightarrow_c B\) is defined as \(\neg_c A \lor B\) but \(\bot\) cannot be defined in terms of \{∧, ∨, ¬_c\} (if a formula B defined \(\bot\) would return the value b for a valuation sending all propositional variables to b, a contradiction). Therefore, in terms of three-valued semantics above, \{∧, ∨, →_c, \bot\}, i.e., the syntax of \(L_C\) is stronger than \{∧, ∨, ¬_c\}, i.e, the syntax for the logic of paradox.

The truth table for →_i is the same as that of an implication introduced as “internal implication” in [5]. This connective are studied also in [12].

Lemma 1. \(p, p \rightarrow_c q \not\models_3 q\)

Proof. Take a valuation \(v\) such that \(v(p) = \{0, 1\}\) and \(v(q) = \{0\}\). Then, \(\overline{v}(p \rightarrow_c q) = \{0, 1\}\). But, \(1 \notin v(q)\). Therefore, \(p, p \rightarrow_c q \not\models_3 q\). ■

While the logic of paradox [13, 14] has a different consequence relation from those of classical logic (as shown in Lemma 1), it is well-known that the logic of
paradox has the same theorems as those of classical logic (see, e.g., [14] p.310). We may also extend this fact to \( \{ \wedge, \vee, \rightarrow, \bot \} \), i.e., the syntax of \( \mathcal{L}_C \) as follows.

**Proposition 8.** Let \( A \in \text{Form}_C \). Then, \( A \) is a tautology in classical logic iff \( A \) is 3-valid.

**Proof.** The proof from right to left is trivial, since \( v : \text{Prop} \rightarrow \{ 0, 1 \} \) is regarded as a three valued valuation by regarding 0 and 1 with \( \{ 0 \} \) and \( \{ 1 \} \), respectively. Conversely, assume that \( A \) is a tautology and fix any valuation \( v : \text{Prop} \rightarrow \{ \{ 0 \}, \{ 0, 1 \}, \{ 1 \} \} \). Our goal is to show that \( 1 \in \overline{v}(A) \). Define a valuation \( v_1 \) from \( v \) by changing all outputs \( \{ 0, 1 \} \) of \( v \) to \( \{ 1 \} \). We regard \( v_1 \) as a two-valued valuation function by regarding \( \{ 0 \} \) and \( \{ 1 \} \) with 0 and 1, respectively. It is easy to see that \( v_1(p) \subseteq v(p) \) for all \( p \in \text{Prop} \). We also have \( \overline{v_1}(B) \subseteq \overline{v}(B) \) for all \( B \in \text{Form}_C \) by Definition 3 (recall that \( C \rightarrow_D \) \( D \) is equivalent with \( \neg C \lor D \)). Since \( A \) is a classical tautology, then \( 1 \in \overline{v_1}(A) \). By \( \overline{v_1}(A) \subseteq \overline{v}(A) \), we conclude that \( 1 \in \overline{v}(A) \). \( \blacksquare \)

**Lemma 2.** Let \( A \in \text{Form}_C \). If \( \models_A A \) then \( \models_A \sigma(A) \) for all uniform substitutions \( \sigma : \text{Prop} \rightarrow \text{Form} \).

**Proof.** Assume that \( \models_A A \). Fix any valuation \( v : \text{Prop} \rightarrow \{ \{ 1 \}, \{ 0, 1 \}, \{ 0 \} \} \) and any uniform substitution \( \sigma : \text{Prop} \rightarrow \text{Form} \). The goal is to show \( 1 \in \overline{v}(\sigma(A)) \).

Define \( v' : \text{Prop} \rightarrow \{ \{ 1 \}, \{ 0, 1 \}, \{ 0 \} \} \) as follows:

\[
\begin{align*}
1 &\in v'(p) \quad \text{ iff } \quad 1 \in \overline{v}(\sigma(p)), \\
0 &\in v'(p) \quad \text{ iff } \quad 0 \in \overline{v}(\sigma(p)),
\end{align*}
\]

for all \( p \in \text{Prop} \). By assumption, we have \( 1 \in \overline{v}(A) \). By induction on a formula \( B \), we can establish:

\[
\begin{align*}
1 &\in \overline{v}(B) \quad \text{iff} \quad 1 \in \overline{v}(\sigma(B)), \\
0 &\in \overline{v}(B) \quad \text{iff} \quad 0 \in \overline{v}(\sigma(B)).
\end{align*}
\]

Here, we only deal with the case where \( B \) is of the form \( C \rightarrow_D D \):

\[
\begin{align*}
1 &\in \overline{v}(C \rightarrow_D D) \quad \text{iff} \quad 1 \notin \overline{v}(C) \quad \text{or} \quad 1 \in \overline{v}(D), \\
&\quad \text{iff} \quad 1 \notin \overline{v}(\sigma(C)) \quad \text{or} \quad 1 \in \overline{v}(\sigma(D)), \quad \text{by induction hypothesis}, \\
&\quad \text{iff} \quad 1 \in \overline{v}(\sigma(C) \rightarrow_D \sigma(D)), \\
&\quad \text{iff} \quad 1 \in \overline{v}(\sigma(C \rightarrow_D D)).
\end{align*}
\]

\[
\begin{align*}
0 &\in \overline{v}(C \rightarrow_D D) \quad \text{iff} \quad 1 \in \overline{v}(C) \quad \text{and} \quad 0 \in \overline{v}(D), \\
&\quad \text{iff} \quad 1 \in \overline{v}(\sigma(C)) \quad \text{and} \quad 0 \in \overline{v}(\sigma(D)), \quad \text{by induction hypothesis}, \\
&\quad \text{iff} \quad 0 \in \overline{v}(\sigma(C) \rightarrow_D \sigma(D)), \\
&\quad \text{iff} \quad 0 \in \overline{v}(\sigma(C \rightarrow_D D)).
\end{align*}
\]

Since we have \( 1 \in \overline{v}(A) \), we conclude that \( 1 \in \overline{v}(\sigma(A)) \), as required. \( \blacksquare \)

**Lemma 3.** Let \( A \in \text{Form} \). If \( A \) is a theorem of \( (C + J)^- \), then \( A \) is 3-valid.

**Proof.** It suffices to show each axiom of \( (C + J)^- \) is 3-valid and each rule of the system preserves 3-validity.

6
Lemma 4. \( \models_3 (p \land (p \to q)) \to q \) iff \( p, p \to q \models_3 q \).

Proof. This follows from the equivalence: \( 1 \in \mathfrak{F}(A \to B) \) iff \( 1 \in \mathfrak{F}(A) \) implies \( 1 \in \mathfrak{F}(B) \).
Theorem 1. The formula \((p \land (p \rightarrow q)) \rightarrow_1 q\) is not a theorem in \((C + J)^-\).

Proof. Suppose \((p \land (p \rightarrow q)) \rightarrow_1 q\) is a theorem in \((C + J)^-\). By Lemma 3, \((p \land (p \rightarrow q)) \rightarrow_1 q\) is 3-valid. By Lemma 4, \(p, p \rightarrow q \models_3 q\) should hold. This is a contradiction with Lemma 1.

Corollary 1. Hilbert system \((C + J)^-\) is not semantically complete, i.e., there exists a formula \(C\) such that \(C\) is not a theorem of \((C + J)^-\) but \(C\) is valid in Kripke semantics in Definition 2.

Proof. By Proposition 6 and Theorem 7.

The argument described above implies, in order to obtain the completeness theorem, the rule \((\text{MPC})\) is necessary. If \((\text{MPC})\) is added, Theorem 1 will no longer hold. This is because Lemma 4 does not hold for \(C + J\), since \((\text{MPC})\) does not preserve 3-validity, which is a well-known feature of the logic of paradox.

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