Newton flows for elliptic functions III

Classification of $3^{rd}$ order Newton graphs

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Abstract

A Newton graph of order $r (\geq 2)$ is a cellularly embedded toroidal graph on $r$ vertices, $2r$ edges and $r$ faces that fulfils certain combinatorial properties (Euler, Hall). The significance of these graphs relies on their role in the study of structurally stable elliptic Newton flows - say $N(f)$ - of order $r$, i.e. desingularized continuous versions of Newton’s iteration method for finding zeros for an elliptic function $f$ (of order $r$). In previous work we established a representation of these flows in terms of Newton graphs. The present paper results into the classification of all $3^{rd}$ order Newton graphs, implying a list of all nine possible $3^{rd}$ order flows $N(f)$ (up to conjugacy and duality).

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1 Motivation and preliminaries

1.1 Newton flows vs. Newton graphs

Throughout this paper the connected graph $\mathcal{G}$ is a cellularly embedded toroidal graph in the torus $T$ of an abstract connected multigraph $\mathcal{G}$ (i.e., no loops) with $r$ vertices, $2r$ edges ($r \geq 2$) and thus $r$ faces; $r =$ order $\mathcal{G}$. We say that $\mathcal{G}$ has the $A(\text{angle})$-property if all angles at a vertex in the boundary of a face spanning a sector of this face, are well defined, strictly positive and

\footnote{i.e., each face is homeomorphic to an open $\mathbb{R}^2$-disk.}
The A-property has a combinatorial interpretation (Hall), cf. [2]. We say that \( \mathcal{G} \) has the E(Euler)-property if the boundary of each face, as subgraph of \( \mathcal{G} \), is Eulerian, i.e., admits a closed facial walk that traverses each edge only once and goes through all vertices. The graph \( \mathcal{G} \) is called a Newton graph if both the A-property and the E-property hold.

It is proved ([2]) that the geometrical dual (denoted \( \mathcal{G}^* \)) of a Newton graph \( \mathcal{G} \) is also Newtonian. The anti-clockwise permutation on the embedded edges at vertices of \( \mathcal{G} \) induces a clockwise orientation of the faces on the boundaries of the \( \mathcal{G} \)-faces, cf. Fig.1-(a). On its turn, the clockwise orientation of \( \mathcal{G} \)-faces gives rise to a clockwise permutation on the embedded edges at the vertices of \( \mathcal{G}^* \), and thus to an anti-clockwise orientation of \( \mathcal{G}^* \).

In the sequel \( \mathcal{G} \) and \( \mathcal{G}^* \) are always oriented in this way: \( \mathcal{G} \) clockwise (−), \( \mathcal{G}^* \) anti-clockwise (+). Altogether, we find: \( \mathcal{G}^* = \mathcal{G} \).

The significance of Newton graphs relies on the study of so called elliptic Newton flows: With \( f \) a non-constant elliptic (i.e., meromorphic, doubly periodic) function of order \( r \) (\( \geq 2 \)), we considered ([1],[2]) \( C^1 \)-vector fields (flows), denoted \( \overrightarrow{\mathcal{N}}(f) \), on \( T \) that are defined on each chart of \( T \) as a toroidal, desingularized version of the planar dynamical system given by

\[
\frac{dz}{dt} = -\frac{f(z)}{f'(z)}, z \in \mathbb{C},
\]

thereby focussing on qualitative features of phase portraits (families of trajectories). Here, zeros, poles and critical points [i.e., \( f' \) vanishes but \( f \) not] of \( f \) serve as resp. attractors, repellors and saddles. We emphasize that \( \overrightarrow{\mathcal{N}}(f) \) is not complex analytic.

The flow \( \overrightarrow{\mathcal{N}}(f) \) is called structural stable if its phase portrait is topologically invariant under small perturbations of the zeros and poles for \( f \). We obtained: (cf. [1],[3])

**Characterization:** \( \overrightarrow{\mathcal{N}}(f) \) is structurally stable iff there holds:

(i) all zeros, poles and critical points for \( f \) are simple,
(ii) the phase portrait does not exhibit “saddle connections”.

**Genericity:** \( \overrightarrow{\mathcal{N}}(f) \) is structurally stable for “almost all” functions \( f \).

**Duality:** If \( \overrightarrow{\mathcal{N}}(f) \) is structurally stable, then also \( \overrightarrow{\mathcal{N}}(\frac{1}{f}) = -\overrightarrow{\mathcal{N}}(f) \).

Let \( \overrightarrow{\mathcal{N}}(f) \) be structurally stable, then \( \mathcal{G}(f) \) is a toroidal graph with as vertices the attractors, as edges the unstable manifolds at saddles and as faces the basins of repulsion of the repellors for \( \overrightarrow{\mathcal{N}}(f) \). It turns out that \( \mathcal{G}(f) \) is a Newton graph of order \( r \) endowed with the clockwise orientation and moreover, \( \mathcal{G}(\frac{1}{f}) = -\mathcal{G}(f)^* \).

The main result obtained in [2] is:

**Representation and classification:** (all graphs and flows of order \( r \))

Given a Newton graph \( \mathcal{G} \), a structurally stable flow \( \overrightarrow{\mathcal{N}}(f_\mathcal{G}) \) exists such that:

\[
\mathcal{G}(f_\mathcal{G}) \sim \mathcal{G}, \ (\text{thus } \mathcal{G}^* = -\mathcal{G}(\frac{1}{f_\mathcal{G}}))
\]

and, if \( \mathcal{G}, \mathcal{H} \) are Newton graphs, then:

\[
\overrightarrow{\mathcal{N}}(f_\mathcal{G}) \sim \overrightarrow{\mathcal{N}}(f_\mathcal{H}) \iff \mathcal{G} \sim \mathcal{H},
\]

where, \( \sim \) in the l.h.s. stands for conjugacy [4] between Newton flows, and \( \sim \) in the r.h.s. for equivalency (i.e., an orientation preserving isomorphism [4]) between Newton graphs.

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2 In fact, we considered the system \( \frac{dz}{dt} = -(1+|f(z)|^4)^{-1}|f'(z)|^2 \frac{f(z)}{f'(z)} \) : a continuous version of Newton’s damped iteration method for finding zeros for \( f \), see [1].

3 \( f \) in an open and dense set of the set of all functions \( f \) of order \( r \) (w.r.t. an appropriate topology)

4 Two elliptic Newton flows are conjugate if an homeomorphism from \( T \) onto itself exists mapping the phase portrait of one flow onto that of the other, thereby respecting the orientations of the trajectories.

5 i.e., between the underlying abstract graphs, respecting the oriented faces of the embedded graphs.
\(G(f)\) is, so to say, the principal part of the phase portrait of the structurally stable flow \(N(f)\) and determines, in a qualitative sense, the whole phase portrait; see Fig. 1-(a), (b) for an illustration. In accordance with our philosophy (“focus on qualitative aspects”), conjugate flows are considered as equal. Note however, that by the above classification we have: \(N(f) \sim N(1f)\) iff \(G(f) \sim -G(1f)^*\), which is in general not true. Nevertheless, from our point of view it is reasonable to consider the dual flows \(N(f)\) and \(N(1f)\) as equal (since the phase portraits are equal, up to the orientation of the trajectories). So, the problem of classifying structurally stable elliptic Newton flows is reduced to the classification (under equivalency and duality) of Newton graphs.

If \(r = 2\), the \(A\)-property always holds\(^6\) and if \(r = 3\) the \(E\)-property implies the \(A\)-property, whereas, in case \(r = 4\), possibly the \(A\)-property holds, but not the \(E\)-property (cf. [2], Lemma 3.17, Remark 3.18).

So, to avoid a further analysis of the \(A\)-property, we only deal with the cases \(r = 2, 3\).

\[\Pi = \{\pi_v | \text{all vertices } v \text{ in } G\},\]

where the local rotation system \(\pi_v\) at \(v\) is the cyclic permutation of the edges incident with \(v\) such that \(\pi_v(e)\) is the successor of \(e\) in the anti-clockwise ordering around \(v\). The boundary of a face of \(G\) is formally described by the Face traversal procedure.

**Face traversal procedure:**
If \(e = (v'v'')\) stands for an edge, with end vertices \(v'\) and \(v''\), a \(\Pi\)-(facial) walk \(w\), on \(G\) is defined by: Let \(e_1 = (v_1v_2)\) be an edge. Then the closed walk \(w = v_1e_1v_2e_2v_3\cdots v_ke_kv_1\) determined by the requirement that, for \(i = 1, \cdots, \ell\), we have \(\pi_{v_{i+1}}(e_i) = e_{i+1}\), where \(e_{\ell+1} = e_1\) and \(\ell\) is minimal\(^8\), is the desired \(\Pi\)-(facial) walk.

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\(^6\)If \(N(f) \sim N(1f)\), and thus \(G(f) \sim G(1f)\), we call the flow \(N(f)\) and also the graph \(G(f)\) self-dual. More general: \(G\) is called self-dual if \(G \sim G^*\).

\(^7\)In \(r = 2\) we proved [2] that all structurally stable \(N(f)\) are mutually conjugate. So, it is to be expected that, in this case, all Newton graphs are equal; see also the forthcoming Remark 2.3.

\(^8\)Apparenting, such “minimal” \(l\) exists since \(G\) is finite. In fact, the first edge which is repeated in the same direction when traversing \(w\), is \(e_1\).
Note that each edge occurs either once in two different II-walks, or twice (with opposite orientations) in only one II-walk. $G$ has the $E$-property iff the first possibility holds for all II-walks. The dual $G^*$ admits a loop iff the second possibility occurs at least in one of the II-walks. Thus, we have:

Under the $E$-property for $G$, the dual $G^*$ has no loops and each $G$-edge is adjacent to different faces; in fact, any $G$-edge, say $e$, determines precisely one $G^*$-edge $e^*$ (and vice versa) so that there are $2r$ intersections $s = (e, e^*)$ of $G$- and $G^*$-vertices.

A crucial principle in our considerations, is

The Heffter-Edmonds-Ringel rotation principle:
By this principle, the graph $G$ is uniquely determined up to an orientation preserving isomorphism by its rotation system. In fact, consider for each II-walk $w$ of length $l$, a so-called II-polygon in the plane with $l$ sides labelled by the edges of $w$, so that each polygon is disjoint from the other polygons. These polygons can be used to construct (patching them along identically labelled sides) an orientable surface $S$ and in $S$ a 2-cell embedded graph $\mathcal{H}$ with faces determined by the polygons. Then $S$ is homeomorphic to $T$ and $\mathcal{H}$ isomorphic with $G$.

The clockwise oriented II-walks of $G$ determine a clockwise rotation system $\Pi^*$ for $G^*$ that - by the face traversal procedure - leads to anti-clockwise oriented $\Pi^*$-walks for $G^*$. Occasionally, $G$ and $G^*$ will be referred to as to the pair $(G, \Pi)$ resp. $(G^*, \Pi^*)$.

2 Classification of Newton graphs of order 3

Let $G(= (G, \Pi))$ be an arbitrary Newton graph of order $r$, and $G^*(= (G^*, \Pi^*))$ a geometrical dual of $G$. The graph $G^*$ is also a Newton graph of order $r$, see Subsection 1.1. The vertices and faces of $G$ are denoted by $v_i$, respectively by $F_{r+i}$. The $G^*$-vertex “located” in $F_{r+i}$ is denoted by $v^r_{r+i}$, and the $G^*$-face that “contains” $v_i$ by $F^r_i$, $i = 1, \cdots, r$. In forthcoming figures, the vertices $G$ and $G^*$ will be indexed by their indices in combination with the symbols $\circ$ and $\bullet$, respectively. Let $v_i \leftrightarrow \circ_i, v^r_{r+i} \leftrightarrow \bullet_{r+i}$. This induces an indexation of the faces of $G$ and $G^*$ as follows: $F_{r+i} \leftrightarrow \bullet_{r+i}$ and $F^r_i \leftrightarrow \circ_i$.

The edges of $G$ and the corresponding edges of $G^*$ are $e_k$, resp. $e^*_k, k = 1, \cdots, 2r$ (compare Subsection 1.2). The degrees of the $G$- and $G^*$-edges are denoted by $\delta_i = \text{deg}(v_i)$, resp. $\delta^*_i = \text{deg}(v^r_{r+i})$. Put $\delta = (\delta(G)) = (\delta_1, \cdots, \delta_r), \delta^* = (\delta(G^*)) = (\delta^*_1, \cdots, \delta^*_r)$ and note that there holds $\delta(G^*) = \delta^*(G)$ and $\delta^*(G^*) = \delta(G)$.

We consider the “common refinement” $G \wedge G^*$ of $G$ and $G^*$. This graph is defined by:

It has vertices on three levels:
Level 1: The vertices $v_i$ of $G$.
Level 2: The “intersections” $s_k$ of the pairs $(e_k, e^*_k)$, compare Subsection 1.2,
Level 3: The vertices $v^r_{r+i}$ of $G^*$, whereas each $G$-edge $e_k$ (each $G^*$-edge $e^*_k$) is partitioned into two $G \wedge G^*$-edges connecting $s_k = (e_k, e^*_k)$ with the end vertices of $e_k$ (of $e^*_k$). Moreover, there are no $G \wedge G^*$-connections between vertices on Level 1 and 3.

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Note that by assumption $G$ has no loops.

The abstract, directed graph underlying $G \wedge G^*$ is denoted by $P(G)$, see also the forthcoming Fig.1(i), Fig.10. For the significance of $P(G)$ within the theory of structurally stable dynamical systems, see the papers [2], [4].
Lemma 2.1. The following relations hold:

\[ 1 < \delta_i \leq 2r, 1 < \delta_i^* \leq 2r, \sum_{i=1}^{r} \delta_i = \sum_{i=1}^{r} \delta_i^* = 4r. \]

By construction, \( G \land G^* \) has precisely \( 4r \) faces, and moreover,

no \( s_k \) is connected by two \( G \land G^* \)-edges to the same \( v_i \) or the same \( v_i^* \).

Proof. Since both \( G \) and \( G^* \) are Newtonian (Subsection 1.1), it follows by the \( E \)-property that these graphs do not admit loops (cf. Subsection 1.2), whereas the \( A \)-property ensures the non-existence of vertices for \( G \) and \( G^* \) of degree 1.

From now on, let \( G \) be a 3rd order Newton graph. We adapt the notations: the \( G \)-edges will be denoted by \( a, b, c, d, e, f \), and the corresponding \( G^* \)-edges by \( a^*, b^*, c^*, d^*, e^*, f^* \). The \( G \land G^* \)-vertices determined by \( (a, a^*) \), \( (b, b^*) \), \( \cdots \) are denoted by respectively \( a, b, \cdots \).

Our aim is a complete classification (up to equivalency) of all graphs \( G \), where we use that, since \( r = 3 \), the \( E \)-property already implies that \( G \) is a Newton graph. (cf. Subsection 1.1)

We distinguish between the following three possibilities with respect to the boundaries (or \( \Pi \)-walks) of \( G \)-faces:

Case 1: The boundary of one of the \( G \)-faces, say \( \partial F_4 \), has six edges, i.e. \( \delta_4^* = 6 \).

Case 2: The boundary of one of the \( G \)-faces, say \( \partial F_4 \), has five edges, i.e. \( \delta_4^* = 5 \).

Case 3: Each boundary of the faces in \( G \) and \( G^* \) has four edges, i.e. \( \delta = \delta^* = (4, 4, 4) \).

By Lemma 2.1, the Cases 1, 2 and 3 are mutually exclusive and cover all possibilities. First we should check whether there exist graphs \( G \) that fulfill the conditions in the above cases, and, even so, to what extent \( G \) is determined by these conditions.

Ad Case 1: Because of the \( E \)-property, and since \( G \) has no loops, it is necessary for the existence of \( G \) that the \( \Pi \)-walk \( w_{F_4} \) of a possible face \( F_4 \) fulfils one of the following conditions:

Subcase 1.1: Traversing \( w_{F_4} \) once, each vertex appears precisely twice.

Subcase 1.2: Traversing \( w_{F_4} \) once, there is one vertex (say \( v_1 \)) appearing three times, one (say \( v_2 \)) appearing twice, and one (say \( v_3 \)) showing up only once.

The \( \text{(clockwise oriented)} \) “\( \Pi \)-polygon” for \( \partial F_4 \) has six sides, labelled \( a, b, \cdots, f \) and six “corner points”, labelled by the vertices \( v_1, v_2, v_3 \) (repetitions necessary). Identifying points related to the same \( G \)-vertex, brings us back to \( w_{F_4} \). Assume that the cyclic permutations of the edges in \( w_{F_4} \) that are incident with the same vertex are oriented \( \text{anti-clockwise} \) (compare the conventions in Subsection 1.1)

In Subcase 1.1 there are precisely two different - up to relabeling- possibilities for \( w_{F_4} \) according to the schemes: (see Fig. 2)

\[ w_{F_4} : v_1 a v_2 b v_3 c v_1 d v_2 e v_3 f v_1 a v_2 \]  \hspace{1cm} (3)

or

\[ w_{F_4} : v_1 a v_2 b v_3 c v_2 d v_1 e v_3 f v_1 a v_2. \]  \hspace{1cm} (4)

First, we focus on \( w_{F_4} \) given by Scheme (3), see Fig. 2(i). In the \( \text{(anti-clockwise)} \) cyclic permutation of the \( w_{F_4} \)-edges, incident with the same vertex, these edges occur in pairs,
determining a (positively oriented) sector of \( F_4 \). As an edge is always adjacent to two different faces (cf. Subsection 1.2), two \( F_4 \)-sectors at the same \( v_i \) are separated by facial sectors (at \( v_i \) not belonging to \( F_4 \) (cf. Fig. 2(i)). Since, moreover, the graph we are looking for, admits altogether twelve facial sectors, the cyclic permutation of the edges at \( v_i \) are as indicated in Fig. 2(i) and constitute a rotation system that -upto equivalency and relabeling- determines the graph, say \( G \), uniquely.

With the aid of the rotation system in Fig. 2(i) and applying the face traversal procedure, as sketched in Subsection 1.2, we find the closed walks \( w_{av_1cv_2}v_2av_1 \) and \( w_{dv_1fv_3}bv_2dv_1 \) defining the two other \( G \)-faces, say \( F_5 \), resp. \( F_6 \). (Note that each edge occurs twice in different walks, but with opposite orientation). Glueing together the facial polygons corresponding to \( F_4, F_5 \) and \( F_6 \), according to equally labeled sides and corner points, gives rise to the plane representations of \( G \) in Fig. 3(i).

![Figure 2: The two possibilities for \( w_{F_4} \) in Subcase 1.1.](image2)

![Figure 3: The two possible plane representations for \( G, G^* \) in Subcase 1.1.](image3)
From Fig. 3-(i) it follows that the rotation system for $G^*$ is as depicted in Fig. 4. With the aid of this figure we find, again by the face traversal procedure, the following closed subwalks in $G^*$: \[v_4^*a^*v_5^*c^*v_3^*d^*v_6^*f^*v_1^*a^*, \quad v_6^*b^*v_5^*d^*v_1^*c^*v_5^*b^* \text{ and } v_2^*f^*v_6^*b^*v_1^*e^*v_5^*e^*v_3^*f^*,\] defining the $G^*$-faces $F_1^*$, $F_2^*$, $F_3^*$ respectively. (Note that each edge occurs twice in different walks, but with opposite orientation). Glueing together the facial polygons corresponding to these faces according to equally labeled sides and corner points, yields the plane representations of $G^*$ in Fig. 3-(i).

Figure 4: The rotation systems for $G^*$, according to Scheme 3.

If we start from a $\Pi$-walk for $F_4$, according to the Scheme 4, we find (by the same argumentation as above) plane representations for $G$ and $G^*$; see Fig. 3-(ii).

Note that in all graphs in Fig. 3 the anti-clockwise (clockwise) orientation of the cyclic permutations of edges incident with the same vertex induces a clockwise (anti-clockwise) orientation of the faces.

In Subcase 1.2 there is precisely one - up to relabeling - possibility for $w_{F_4}$ according to the scheme:

$$w_{F_4} : v_1a v_2b v_3 c v_2 d v_1 e v_3 f v_1 a.$$  \hspace{1cm} (5)

In this case however, there are three pairs of $G$-edges at $v_1$ determining (positively measured) sectors of $F_4$. So, reasoning as in Subcase 1.1, there are two possibilities for the (anti clockwise) cyclic permutations of the $G$-edges at $v_1$ (and thus also two different rotation systems; see Fig. 5).

Figure 5: The two possible rotation systems in Subcase 1.2.

Starting from Fig. 5-(i) and applying the face traversal procedure, we find the facial walks $v_1f v_3 e v_1b v_2 c v_1 f$ and $v_1 a v_2 d v_1 a$, which together with Scheme (5) define the faces $F_5$, $F_6$ and $F_3$ respectively. Reasoning as in Subcase 1.1, we arrive at the plane realizations of $G$. 

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and $G^*$ as depicted in Fig.6(i). In the case of Fig.5(ii) the facial walks $v_1dv_2av_1bv_2cv_1d$ and $v_1ev_3f v_1c$, together with Scheme (5), define the faces $F_5$, $F_6$ and $F_4$ respectively. Reasoning as in Subcase 1.1, we obtain the plane representations for $G$ and $G^*$ as depicted in Fig.6(ii).

Note that both graphs $G$ in Fig.6 are self dual (cf. footnote 6, or note that $\delta(G) = \delta(G^*)$ and use Lemma 3.5 in [2]), but—by inspection of their rotation systems—are not equivalent (cf. Subsection 1.2).

Figure 6: The two possible plane representations for $G$, $G^*$ in Subcase 1.2.

Ad Case 2: Because the II-walk of $F_4$ has no loops and consists of an Euler trail on the five edges of $G$, there is only one—up to relabeling—possibility for $w_{F_4}$ (see Fig.7(i)):

$$w_{F_4} : v_1av_3bv_2cv_1dv_2ev_1a.$$ 

In contradistinction with the previous Case 1, now there is one edge, namely $f$, that is not contained in $w_{F_4}$. By Lemma 2.1, this edge must connect either $v_1$ to $v_2$ ($f : v_1 \leftrightarrow v_2$), or $v_1$ to $v_3$ ($f : v_1 \leftrightarrow v_3$), or $v_2$ to $v_3$ ($f : v_2 \leftrightarrow v_3$); compare Fig.7(ii) where we show the part of the abstract graph $P(G)$ underlying $G \wedge G^*$ that is determined by $\partial F_4$. To begin with, we focus on the first two sub cases.

Taking into account the various positions of $f$ with respect to local sectors of $F_4$ at $v_1$ and $v_2$ (when $f : v_1 \leftrightarrow v_2$), respectively $v_1$ and $v_3$ (when $f : v_1 \leftrightarrow v_3$), we find four respectively two possibilities for the rotation systems; see Fig.8. The Subcases $f : v_1 \leftrightarrow v_3$ and $f : v_2 \leftrightarrow v_3$ are not basically different. So, we may neglect the case $f : v_2 \leftrightarrow v_3$. Reasoning as in Case 1, the rotation systems in Fig.8 yield the possible planar representations of $G$ and $G^*$; see Fig.9. Note that—by inspection of their rotation systems—all graphs $G$ in this figure are different under orientation preserving isomorphisms, whereas only in the cases of Fig.9(iii) and (iv) these graphs are equal w.r.t. an orientation reversing isomorphism (apply

11 Relabeling $v_1 \leftrightarrow v_2$, $a \leftrightarrow b$ and $c \leftrightarrow e$, transforms the two configurations in Fig.8(v) and (vi) into configurations that generate (anti-clockwise oriented) rotation systems describing the case $f : v_2 \leftrightarrow v_3$. **

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the relabeling introduced in Footnote 11. Apparently, the graphs $\mathcal{G}$ and $\mathcal{G}^*$ (and thus also $\mathcal{G}^*$ and $\mathcal{G}$) in Fig.9(i), resp. Fig.9(v) are equal (under an orientation preserving isomorphism). The graphs $\mathcal{G}$ in Fig. 9(ii),(iii),(iv),(vi) are self-dual.

Figure 7: The $\Pi$-walk for $F_4$ in Case 2.

Figure 8: The possible rotation systems for $\mathcal{G}$ in Case 2.

Ad Case 3: Without loss of generality, there are a priori two possibilities for the $\Pi$-walks of an arbitrary face, say $F_4$; see Fig.10(i) and (ii). By Lemma 2.1 and by inspection of the corresponding partial graph $\mathcal{P}(\mathcal{G})$, the first possibility is ruled out. So, we focus on Fig.

Ad Case 3: Without loss of generality, there are a priori two possibilities for the $\Pi$-walks
Figure 9: The graphs $\mathcal{G}$ and $\mathcal{G}^*$ in Case 2.

Recall that two facial sectors at the same vertex $v_i$ are separated by facial sectors (at $v_i$) not belonging to $F_4$ and that in the actual case we have $δ = δ^* = (4, 4, 4)$. So, we find the rotation systems and the distribution of “local facial sectors” as depicted in Fig. 10(ii), where the roles of both $e$, $f$ and $F_5$, $F_6$ may be interchanged. Now, by the face traversal procedure we find:

Apart from relabeling and equivalency, there is only one (self dual) graph possible, Fig. 11.
Figure 10: Apriori possibilities for $w_{F_4}$ in Case 3.

$\delta(\mathcal{G}) = (4, 4, 4) = \delta(\mathcal{G}^*)$

Figure 11: The only possible graphs $\mathcal{G}$ and $\mathcal{G}^*$ in Case 3.
Now the representation result from Subsection 1.1 and the remark there about dual flows, together with the above analysis of the 3rd order Newton graphs yields:

**Theorem 2.2.** (Classification of third order Newton graphs)

- Apart from conjugacy and duality, there are precisely nine possibilities for the 3rd order structurally stable elliptic Newton flows. These possibilities are characterized by the Newton graphs in Fig. 13.

- If we add to Fig. 13 the duals of the graphs in Fig. 13 (i), (ii), (v), we obtain a classification under merely conjugacy, containing twelve different possibilities.

**Remark 2.3.** The Case \( r = 2 \).

By similar (even easier) arguments as used in the above Case \( r = 3 \), it can be proved that -up to equivalency-there is only one (self-dual) possibility for the 2nd order Newton graphs; see Fig. 12 (Note that in view of the E-property both facial walks of such graphs have length 4, whereas the role of the A-property is not relevant, see Subsection 1.1). For a different approach, see Corollary 2.13 in [2].

**Remark 2.4.** In case of degenerate elliptic functions, it is possible to describe the corresponding Newton flows by so-called pseudo Newton graphs, see our paper [3].

\[
\delta(G) = (4,4) = \delta(G')
\]

Figure 12: The 2nd order Newton graphs.
Figure 13: The graphs characterizing structurally stable elliptic Newton flows of order 3.

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