ABOUT LEVI-MALCEV THEOREM FOR HOMOGENEOUS BOL ALGEBRAS

Thomas B. Bouetou
École Nationale Supérieure Polytechnique,
B.P. 8390 Yaoundé, Cameroun
e-mail:tbouetou@polytech.uninet.cm

March 29, 2022

Abstract The fundamental ideas of the applicability of Levi-Malcev Theorem for Bol algebras, which plays a basic role in structural theory are outlined

1. Introduction

It is well known that Levi–Malcev theorem is valid for Lie algebras and Malcev algebras. But it is not valid for binary-Lie algebras (in classical sense) as well as for Bol algebras. Since the Levi–Malcev theorem plays the basic role in structural theories, it is useful to get some generalized version of such a theorem for Bol algebras. In this report the problem for so-called homogeneous Bol algebras with some examples will be discussed.

Let us note that the problem is important in applications to Differential geometry and Smooth Quasigroups and Loops theory (because that Bol algebra is the proper infinitesimal object for a smooth Bol loop, and any symmetric space, for example, is a smooth Bol loop).

It is also noted that, Bol loops and Bol algebras are significant in Mathematical Physics (chiral anomalies etc.).

Remark: The smooth Bol loops-Bol algebras theory is due to L.V. Sabinin, P.O. Mikheev. See, for example [2].

2. General definitions

Let consider \((B, \nabla)\) be an affinely connected space with \(R = 0\), and let \(X(x)\) be an infinitesimal affine transformation in \((B, \nabla)\) that means for every vector field \(Y \in B\) we have

\[ L_X \circ \nabla - \nabla \circ L_X = \nabla_{[X,Y]} \]

where \(L\) is a Lie differential in the direction of the vector field \(X\). Let us introduce in \(B\) a basis of vectori field such that for every \(X_1, \ldots, X_n\) then

\[ X(x) = \alpha^j(x)(X_j) |_x \]
and we can have

\[ [X_i, X_k] = -T_{ikj}^j X_j. \]

**Definition 1.** Any vector space \( V \) of \( \mathbb{B} \) with the operations

\[ \xi, \eta \to \xi \cdot \eta \in V, \]

\[ \xi, \eta, \zeta \to (\xi, \eta, \zeta) \in V \quad (\xi, \eta, \zeta \in V) \]

and identities

\[ \xi \cdot \xi = 0, \quad (\xi, \eta, \zeta) + (\eta, \zeta, \xi) + (\zeta, \xi, \eta) = 0, \]

\[ (\xi, \eta, \zeta) \chi - (\xi, \eta, \chi) \zeta + (\zeta, \chi, \xi) \cdot \eta - (\xi, \eta, \zeta) \cdot \chi = 0, \]

\[ (\xi, \eta, (\zeta, \chi, \omega)) = ((\xi, \eta, \chi, \omega) + (\zeta, (\xi, \eta, \chi), \omega) + (\zeta, \chi, (\xi, \eta, \omega)) \]

is called a Bol algebra.

Let \( I \) be an ideal of a Bol algebra \( V \) and denote \( I^{(1)} = (V, I, I) \) and \( I^{(k)} = (V, I^{(k-1)}, I^{(k-1)}) \).

**Definition 2.** The ideal \( I \) of a Bol algebra \( V \) is called weakly solvable if there exist \( k \), such that \( I^{(k)} = 0 \).

**Definition 3.** The weak radical \( RV \) of a Bol algebra is called a maximal weakly solvable ideal. The Bol algebra \( V \) is called semisimple if \( RV = 0 \).

**Definition 4.** Let \( G \) be a vector subspace of \( V \), we say that \( G \) is a Bol subalgebra if, \( a, b, c \in G \Rightarrow a \cdot b, (a, b, c) \in G \).

**Definition 5.** [Mikheev 1.] A local analytic loops \((B, \cdot, e)\) verify the G-property if there exist such a neighbourhood \( U \) of the point \( e \in B \) such that for any \( x \in U \) the loop \((B, \cdot, e) \) and \((B, 1^x e) \) are isomorphic.

**Theorem 1.** [Mikheev 1.] A local analytic Bol loop \((B, \cdot, e)\) posses a G-property if and only, if the corresponding affine connected space \((B, \Delta)\) is local homogeneous.

**Definition 6.** An arbitrary Bol algebra is said to be homogeneous if its an infinitesimal object for a local analytic Bol loop that posses a G-property.

Indeed we will have the following bilinears and trilinears operations:

\[ \xi, \eta \to \xi \cdot \eta \in T_e(B) \]

\[ (\xi \cdot \eta)^i = T^i_{jk}(e)x^j \eta^k \]

\[ (\xi, \eta, \tau)^i = (\Delta xT^i_{jk} + T^i_{jk}T^j_{sl})x^j \eta^k \tau^l \]

\[ [X_i, X_k] = -T^i_{ikj} X_j \]

which will verify the identities of the definition of Bol algebras.

Let’s \( V \) be a Bol algebra over the field of characteristic zero and let us assume that in \( V \) there is a radical \( RV \) and a semisimple subalgebra \( SMV \) such that

\[ V = SMV + RV. \]
This implies that \( RV \cap SMV = \{0\} \).

Hence \( V = RV \oplus SMV \) and \( SMV \cong V/RV \) and the Bol subalgebra \( SMV \) is isomorphic to the factor algebra of the algebra \( V \) by its radical.

Conversely, if \( V \) contains the Bol subalgebra \( SMV \) isomorphic to \( V/RV \) then the algebra \( SMV \) is semisimple. Indeed, \( SMV \cap RV = 0 \) and

\[
\dim V = \dim RV + \dim(V/RV) = \dim RV + \dim SMV.
\]

Hence, \( V = SMV + RV \). This lead us to state the following theorem:

**Theorem 2.** For every homogenous Bol algebra \( V \) over the field of characteristic zero there exist semi simple sub algebra \( SMV \) and a weak radical \( RV \) such that \( V = SMV + RV \).

Proof:

Let assume that \([RV]^2 = 0\) which means that if the radical is abelian, then \( RV \) is a submodule of the module \( SMV \) relatively to \( V \) by the application \( \exp(\text{ad}x) \), where \( \text{ad}x \) is an inner differential of Bol enveloping Lie algebra.

Since \([RV]^2 = 0\) then, \( RV \) belongs to the kernel of the representation of the Bol algebra \( V \) of the defined module \( RV \). Hence we obtain the induced representation of the algebra \( V = V/RV \). We have for the module the following operation: \( a \cdot b = \prod([a, b]_V), \ a \in V, \ b \in RV, \) and the commutator is taken from the minimal enveloping Lie algebra. If \([RV]^2 \neq 0\) then \( V = (V)/[RV]^2 \), and \( \dim V \leq \dim V \). By using the induction according to the dimension of the algebra, we can say that the statement is true for the algebra \( V \). Further, \( \overline{RV} = (RV/\overline{RV}^2) \) is a radical for the algebra \( \overline{V} \) and \( \overline{V}/\overline{RV} \cong V/RV \). Therefore, the algebra \( \overline{V} \) contain the subalgebra \( SMV \cong V/RV \) as a subalgebra. The algebra \( \overline{V} \) and \( SMV \) has the form \( K/RV \) where \( K \) is the subalgebra of Bol algebra \( V \), containing \([RV]^2 \). But, \([RV]^2 \) is the radical of the Bol algebra \( K \) and \((K)/[RV]^2 \cong (V/RV) \) such that \( \dim K \leq \dim V \), using the induction we obtain that \( K \) contain the subalgebra \( SMV \cong V/RV \), hence the result.

Further an example is given with a counter example[7].

For the type I of the classification given in [7], the Bol algebra have trilinear operation equal to zero and:

\[
e_1 \cdot e_3 = e_1 + e_2
\]

\[
e_2 \cdot e_3 = e_2
\]

\[
V \cdot V = < e_1 + e_2, e_2 >
\]

\[
SMV = \frac{V}{< e_1 + e_2, e_2 >} = < e_3 >.
\]

We see in the case

\[
V = < e_1 + e_2 > \oplus < e_3 >
\]

Counter example, let consider the Bol algebras of Type IV in [7],
\[ e_1 \cdot e_3 = x e_1 + p e_2 + e_3, \forall p, x \geq 0 \]

\[ (e_1, e_2, e_3) = e_1 \]

\[ (e_1, e_3, e_3) = e_1. \]

This Bol algebra is not homogeneous.

Hence

\[ V \cdot V = \langle e_1, e_2, e_3 \rangle = RV \]

\[ SMV = \frac{V}{\langle e_1, e_2, e_3 \rangle} = \langle e_2, e_3 \rangle. \]

We see that:

\[ RV \cap SMV \neq 0. \]

Hence the Levi-Malcev theorem cannot be applied.

References

1. P.O. Mikheev On G-properties for analytic Bol loops, Webs and Quasigroups, Kalinin, 1986. p. 54-59.

2. L.V. Sabinin, P.O. Mikheev, Quasigroups and differential geometry, in: Quasigroups and loops theory and applications. Berlin: Helderman Verlas, 1990.-pp. 357-430.

3. K. Yamaguti On the Lie triple system and its generalization J.sci. Hiroshima Univ. ser. A. 21. N.3 1958. p. 155-160.

4. W.G. Lister A structure theory of Lie triple system Trans. AMS. -1952.-T. 72.-p. 217-245.

5. B. Harris Cohomology of Lie triple systems and Lie algebras with involution Trans. AMS.-1951.-T. 71.-p. 148-162.

6. M. Kikkawa Remarks on solvability of Lie triple algebras Mem. Fac. Sci. Shim. Univ. N.13 -1979. p.17-22.

7. T.B. Bouetou On the geometry of Bol algebras Ph.D Thesis, Moscow University 1995.

8. V.T. Filipov Homogeneous Bol Algebras Sibirian Math. Journal Vol.35, N.4 1994, pp. 818-825.

9. A.S. Kleshchev Branching Rules for Modular Representation of Symmetric Groups I Journal of algebra 178, pp 493-511 (1995).