EXISTENCE AND LARGE TIME BEHAVIOUR FOR A STOCHASTIC MODEL OF A MODIFIED MAGNETOHYDRODYNAMIC EQUATIONS

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Abstract. In this paper we study a system of nonlinear Stochastic Partial Differential equations describing the motion of turbulent Non-Newtonian media in the presence of fluctuating magnetic field. The system is basically obtained by a coupling of the dynamical equations of a Non-Newtonian fluids having $p$-structure and the Maxwell equations. We mainly show the existence of weak martingale solutions and their exponential decay when time goes to infinity.

1. Introduction

Stochastic Partial Differential Equations (SPDEs for short) have now become very important tools in Hydrodynamics. They are used in the mathematical investigation towards the understanding of turbulent motions of fluids. The SPDEs governing turbulent fluids are obtained by adding noise terms to deterministic models. This approach is basically motivated by Reynolds’ work which stipulates that the velocity of a fluid particle in turbulent regime is composed of slow (deterministic) and fast (stochastic) components. While this belief was based on empirical and experimental data, Rozovskii and Mikulevicius were able to derive the models rigorously in their recent work [39], thereby confirming the importance of this approach in hydrodynamic turbulence. Indeed, it is pointed out in [39] (see also [28]) that some rigorous information on questions in Turbulence might be obtained from stochastic versions of the equations of fluid dynamics.

The mathematical study of SPDEs for hydrodynamics was initiated in the early 1970’s in the papers of Bensoussan and Temam [4], since then stochastic Navier-Stokes equations and SPDEs in general have been the object of intensive research which has generated several important results. We refer to, among others, [1], [7], [9], [15], [18], [19], [39], [12], [14], [15], [16], [51], [53].

Magnetohydrodynamics (MHD) is a branch of continuum mechanics which studies the motion of conducting fluids in the presence of magnetic fields. The system of Partial Differential Equations (PDEs) in MHD are basically obtained through the coupling of the dynamical equations of the fluids with the Maxwell’s equations which are used to take into account the effect of the Lorentz force due to the magnetic field (see for example [13]). Magnetohydrodynamics plays essential role in Astrophysics, Geophysics, Plasma Physics, the magnetic confinement device Tokamak in Thermonuclear Physics, and in many other branches of applied sciences. In these areas turbulent magnetohydrodynamic flows which are usually due to magnetic-field fluctuations are typical. Deterministic models of MHD have been the focus of investigation by many mathematicians. Several important results have been obtained, for instance in [30], [59], [51], [22], [17], [37], [38]; just to cite a few relevant papers. The reader can consult [23] for a recent and detailed account in the mathematical investigation of hydrodynamic turbulence. Many scientists have also considered stochastic model for MHD by adding noise terms to the dynamical equations of the fluids and Maxwell equations representing the magnetic-field fluctuations. The stochastic MHD equations were investigated in [58], [2], [52]. The authors in [58], [2] consider additive noises. Using Galerkin’s approximation and compactness method, the author
in [32] proved the existence of martingale weak solutions for the stochastic MHD equations in the presence of nonlinear multiplicative noise which do not satisfy the Lipschitz condition.

Due to the conventional belief that the Navier-Stokes equations are an accurate model for the motion of incompressible fluids in many practical situations, the majority of the above work have assumed that the fluids are Newtonian. However, there are a lot of conducting materials appearing in many practical and theoretical situations that cannot be characterized by Newtonian fluids. To describe these media one generally has to use (conducting) fluids models that allow the stress to be a nonlinear function of the strain rate. Fluids in the latter class are called Non-Newtonian fluids. We refer for example to the introduction of Biskamp’s book [6] for some examples of these Non-Newtonian conducting fluids. These facts motivated us to consider in the present manuscript a class of stochastic modified MHD equations which allows the constitutive law of the conducting fluids to exhibit a nonlinear relationship between the stress tensor and the strain rate. More precisely, for a final time $T > 0$ and a sufficiently smooth bounded domain $\mathcal{Q}$ in $\mathbb{R}^n$ ($n = 2, 3$) we describe the motion of randomly forced Non-Newtonian conducting fluids in a fluctuating magnetic field by the following system of stochastic partial differential equations:

\[
\begin{aligned}
d\mathbf{u} + (\nabla \mathbf{u} + \mu \mathbf{B} \times \mathbf{B} - \mathbf{u} \cdot \nabla \mathbf{B}) dt &= f_1(\mathbf{u}, \mathbf{B}, t) dt + g_1(\mathbf{u}, \mathbf{B}, t) dW_1, \\
d\mathbf{B} + (S \mathbf{u} \times \mathbf{B} - \mathbf{u} \cdot \nabla \mathbf{B}) dt &= f_2(\mathbf{u}, \mathbf{B}, t) dt + g_2(\mathbf{u}, \mathbf{B}, t) dW_2, \\
\nabla \cdot \mathbf{u} &= 0, \\
\mathbf{B} \cdot \mathbf{n} &= \text{curl} \mathbf{B} \\
\mathbf{u}(0) &= \mathbf{u}_0, \quad \mathbf{B}(0) = \mathbf{B}_0,
\end{aligned}
\]

where $\mathbf{u} = (u_i; i = 1, \ldots, n)$, $\mathbf{B} = (B_i; \ldots, n)$ and $P$ are unknown random fields defined on $\mathcal{Q} \times [0, T]$, representing, respectively, the fluid velocity, the magnetic field, the pressure, at each point of $\mathcal{Q} \times [0, T]$. $S$ and $\mu$ are positive constants depending on the Reynolds numbers of the fluid and magnetic fields, and the Hartman number. The terms $f_i(\mathbf{u}, \mathbf{B}, t)$ and $g_i(\mathbf{u}, \mathbf{B}, t) dW_i$ ($i = 1, 2$) are external forces depending on $\mathbf{u}$ and $\mathbf{B}$, where $W_i$ are cylindrical Wiener processes evolving on two Hilbert spaces $\mathcal{H}_i$. We assume they are mutually independent and identically distributed. The quantities $\mathbf{u}_0$ and $\mathbf{B}_0$ are given non random initial velocity and magnetic field, respectively. Finally, $\mathbf{T}$ designates the extra stress tensor of the Non-Newtonian fluid and we suppose that there exists a potential $\Sigma : \mathbb{R}^{n \times n}_{\text{sym}} \to \mathbb{R}^+_{0}$ and constants $\nu_1, \nu_2$ such that for some $p > 1$ and for all $l, k, i, j, t = 1, 2, \ldots, n$, $\mathbf{D}, \mathbf{E} \in \mathbb{R}^{n \times n}_{\text{sym}}$:

\[
\begin{aligned}
\Sigma(0) &= 0, & \frac{\partial \Sigma(0)}{\partial D_{kl}} &= \partial_{kl} \Sigma(0) = 0, & T_{kl}(\mathbf{D}) &= \partial_{kl} \Sigma(\mathbf{D}), \\
\partial_{ij} \partial_{kl} \Sigma(\mathbf{D}) E_{ij} E_{kl} &\geq \nu_1 (1 + \mathbf{D})^{p-2} |\mathbf{E}|^2, \\
\partial_{ij} \partial_{kl} \Sigma(\mathbf{D}) &\leq \nu_2 (1 + |\mathbf{D}|^2)^{p-2}.
\end{aligned}
\]

Here

\[
\mathbb{R}^{n \times n}_{\text{sym}} = \{ \mathbf{D} \in \mathbb{R}^{n \times n} : D_{ij} = D_{ji}, i, j = 1, 2, \ldots, n \}.
\]

The structure of the nonlinearity of problem (1) introduces a number of interesting features which are not present in their Newtonian counterparts such as the basic MHD or the Navier-Stokes equations; we refer for instance to the papers [32], [34], [35] which deal with interesting mathematical questions arising in similar fluids in the deterministic case. Besides the usual nonlinear terms of the MHD equations, (1) contains another nonlinear term of $p$-structure which exhibits the non-linear relationships between the reduced stress and the rate of strain of the conducting fluids. Because of this, the analysis of the behavior of the MHD model (1) tends to be much more complicated and subtle than that of the Newtonian MHD equations.
In the deterministic case, that is when \( g_1, g_2 \equiv 0 \), a variant of (1) (the tensor \( T \) is replaced by the \( p \)-Laplacian) and its stationary version were initially investigated by Samokhin in [50], [49], [47], and [48]. Later on Gunzburger et al in [24] and [25] considered a more general model by taking a tensor \( T \) which satisfies the assumption we have made above. The paper [25] dealt with the control of (1) and [50], [49], [47], [48], and [24] addressed the existence and uniqueness results of weak solution of (1). To the best of our knowledge, there is no known result for the stochastic equations (1). The purpose of the present paper is to prove some results related to problem (1) which are the stochastic analog of some of those obtained in [50, 24] for the deterministic case.

The following two points are our main goals:

1. We prove the existence of martingale weak solution for the stochastic system (1). We consider a sufficiently general forcing consisting of a regular part and a stochastic part both depending nonlinearly on the velocity of the fluids and the magnetic field, and we do not require the functions involved in the forcing term to satisfy the Lipschitz condition. The method for the proof uses a blending of Galerkin, compactness, and monotonicity methods. However, to our great surprise the well-known compactness method from previous related works to stochastic hydrodynamic equations (see, among others, [21]) seems not to apply to our situation. Indeed unlike in many previous scenarios (see, for instance, [21], [16]) the drift term of (1) belong to the Sobolev space \( W^{1,p^*}(0,T;W^*_s) \), \( p^* \in (1,2) \) which cannot be embedded into the fractional Sobolev space \( W^{s,2}(0,T;W^*_s) \) with \( s \in (0,\frac{1}{2}] \) to which the stochastic integral or diffusion term belongs. Also, it seems that there is no other space which can contain the two spaces we have mentioned above. Overcoming this problem requires an ingenious trick which consists of a careful use of results about compact sets in \( L^2(0,T;\mathbb{H}) \) without invoking any compactness result involving fractional Sobolev spaces. We also should notice that we do not use the usual martingale representation argument.

2. After obtaining the existence of a martingale weak solution of our model, we turn our attention to the study of its asymptotic behavior as the time \( t \) is large. For this purpose, we study the decay of the martingale weak solutions as time goes to infinity. We mainly prove that under some condition on the forcing term \( f_i \) and \( g_i, i = 1, 2 \) the couple \((u;B)\) converges to zero almost surely exponentially. To prove this result we mainly follow the idea in [11] and [12].

As far as we know the present article is the first to deal with (1). In this sense, many topics and problems still stand opened. Some examples of challenges we may address in future research are the existence of weak solution for all values of \( p \), the uniqueness of such weak solutions. We may also want to study the existence and uniqueness of the invariant measure whenever it is possible. These few examples of research topics are taken as an analogy of the problems still unsolved in the mathematical theory of Non-Newtonian as reported in [3], [32] and [33]. But due to the nature of the nonlinear terms involved in (1) all of these questions are very difficult and beyond the scope of this paper, thus we will just limit ourselves with giving a suitable mathematical setting for (1) and partial results related to the dynamics of the weak solutions. However, we hope that our work will find its applications elsewhere or at least motivate further research in the study of stochastic model for Non-Newtonian MHD.

The paper is structured as follows. In Section 2, we gather all the necessary tools and the hypotheses. In section 3 we state the result for the existence of weak probabilistic solution and we prove it by means of Galerkin methods. The exponential asymptotic behavior of these weak solutions are studied in the last section.
2. Preliminary: Notations and hypotheses

In this section we introduce the necessary notations and most of the hypotheses relevant for our analysis.

2.1. The deterministic framework. We introduce some notations and background following the mathematical theory of hydrodynamic equations such as Navier-Stokes equations (NSE) or MHD equations. For any $p \in [1, \infty)$, we denote by $L^p(Q)$ and $W^{m,p}(Q)$ the space of functions taking values in $\mathbb{R}^n$ such that each component belongs to the Lebesgue spaces $L^p(Q)$ and the Sobolev spaces $W^{m,p}(Q)$, respectively. For $p = 2$ we denote $W^{m,p}(Q)$ by $H^m(Q)$. We denote by $|\cdot|$ the $L^2$-norm, and by $(\cdot, \cdot)$ the $L^2$-inner product. The norm of $W^{p,m}(Q)$ is denoted by $|| \cdot ||_{m,p}$.

Now we introduce the following spaces

$$V_1 = \{ u \in C_0^\infty(Q) : \text{div } u = 0 \},$$
$$H_1 = \{ u \in L^2(Q) : \text{div } u = 0, u \cdot n = 0 \text{ on } \partial Q \},$$
$$V_{1,p} = \{ u \in W^{1,p} : \text{div } u = 0, u = 0 \text{ on } \partial Q \},$$
$$V_1 = \{ u \in H^1(Q) : \text{div } u = 0, u = 0 \text{ on } \partial Q \}.$$

We also set

$$V_2 = \{ B \in C_0^\infty(Q) : \text{div } B = 0; B \cdot n = 0 \text{ on } \partial Q \},$$
$$H_2 = \text{the closure of } V_2 \text{ in } L^2(Q),$$
$$V_2 = \{ B \in H^1(Q) : \text{div } B = 0; B \cdot n = 0 \text{ on } \partial Q \}.$$

Note that $H_1 = H_2$.

The spaces $H_i, i = 1, 2$ are equipped with the scalar product and norm induced by $L^2(Q)$.

Thanks to Poincaré’s inequality we can endow the space $V_{1,p}$ with the norm $||u||_{1,p}$ defined by

$$||u||_{1,p}^p = \int_Q |\nabla u|^p dx.$$

This norm is equivalent to the usual $W^{1,p}$-norm on $V_{1,p}$.

We equip the space $V_1$ with the norm $|| \cdot ||_1$ generated by the scalar product

$$(u, v)_1 = \int_Q \nabla u \cdot \nabla v dx.$$

Owing to Poincaré’s inequality, $|| \cdot ||_1$ and the usual $H^1(Q)$-norm are equivalent on $V_1$.

On $V_2$ we define the scalar product

$$(u, v)_2 = (\text{curl } u, \text{curl } v),$$

which coincides with the usual scalar product of $H^1(Q)$.

Let

$$V = V_{1,p} \times V_2,$$
$$H = H_1 \times H_2.$$ 

The space $H$ has the structure of a Hilbert space when equipped with the scalar product

$$(\Phi, \Psi) = (u, v) + (B, C),$$

for $\Phi = (u; B), \Psi = (v; C) \in H$.

The space $V$ is a Banach space with norm

$$||\Phi||_V = ||u||_{1,p} + ||B||_2,$$
for \( \Phi = (u; B) \in V \).

**Remark 2.1.** Note that this norm is equivalent to any norm of the form
\[
\|\Phi\|_V = C_1||u||_{1,p} + C_2||B||_2,
\]
where the constants \( C_1, C_2 \) depend only on \( S, \text{mes}(Q), p \). Here \( \text{mes}(Q) \) denotes the Lebesgue measure of \( Q \).

Throughout this work we set
\[
\|\Phi\|_{p,2}^V = \|u\|_{1,p} + \|B\|_2^2.
\]

For any Banach space \( X \) we denote by \( X^* \) its dual space and \( \langle \phi, u \rangle \) the value of \( \phi \in X^* \) on \( u \in X \).

Let \( A_2 \) be the linear operator from \( V_2 \) taking values into \( V_2^* \) (i.e, \( A_2 \in \mathcal{L}(V_2, V_2^*) \) ) defined by
\[
\langle A_2 B, C \rangle = S((B, C)),
\]
for any \( B, C \in V_2 \).

For any any \( u \in W^{1,p} \) we set
\[
\mathcal{E}(u) = \frac{1}{2} (|\nabla u| + (\nabla u)^T).
\]

Let us recall the following results whose proofs can be found in [32].

**Lemma 2.2 (Korn’s inequalities).** Let \( 1 < p < \infty \) and let \( Q \subset \mathbb{R}^n \) be of class \( C^1 \). Then there exist two positive constants \( K_p^i = K_p^i(Q), i = 1, 2 \) such that
\[
K_p^i ||u||_{1,p} \leq \left( \int_Q |\mathcal{E}(u)|^p dx \right)^{\frac{1}{p}} \leq K_p^i ||u||_{1,p},
\]
for any \( u \in V_{1,p} \).

We introduce a nonlinear mapping \( A_p \) from \( V_{1,p} \) into \( V_{1,p}^* \) by setting
\[
\langle A_p u, v \rangle = \int_Q T(\mathcal{E}(u)) \cdot \mathcal{E}(v) dx,
\]
for any \( u, v \in V_{1,p} \). Now we can define a nonlinear operator \( A \) from \( V \) into \( V^* \) by
\[
\langle A \Phi, \Psi \rangle = \langle A_p u, v \rangle + \langle A_2 B, C \rangle,
\]
for any \( \Phi = (u; B), \Psi = (v; C) \in V \). We state very important properties of \( A \) in the following

**Lemma 2.3.** Let \( T \) and \( \Sigma \) satisfy [23]-[33]. Then,

1. the operator \( A \) is monotone; that is,
   \[
   \langle A \Phi_1 - A \Phi_2, \Phi_1 - \Phi_2 \rangle \geq 0,
   \]
   for any \( \Phi_1, \Phi_2 \in V \).

2. There exists a constant \( \tilde{\nu} \) such that
   \[
   \langle A \Phi, \Phi \rangle \geq \tilde{\nu}||\Phi||_{V_*}^{p,2},
   \]
   for any \( \Phi \in V \).

3. Also, there exists a positive constant \( C \) such that
   \[
   ||A \Phi||_{V_*}^{p,2} \leq C(1 + ||\Phi||_{V_*}^{p,2}),
   \]
   for any \( \Phi \in V \). Here
   \[
   \frac{1}{p} + \frac{1}{p^*} = 1,
   \]
and
\[ ||\Psi||_{V^*}^{p^*,2} = ||\psi||_{V^*}^{p^*,2} + ||\phi||_{V^*}^{p^*,2}, \]
for \( \Psi = (\phi; \psi) \in V^* \).

Proof. It is known from [13] that for any \( p \geq 2 \) there exist positive constants \( \nu_i, i = 3, 4, 5 \), such that for all \( D, E \in \mathbb{R}^{n \times n} \):
\[
\begin{align*}
&\mathbf{T}(D) \cdot D \geq \nu_3(1 + ||D||^{p-2})||D||^2, \quad (15) \\
&||\mathbf{T}(D)|| \leq \nu_4(1 + ||D||)^{p-1}, \quad (16) \\
&(\mathbf{T}(D) - \mathbf{T}(E)) \cdot (D - E) \geq \nu_5||D - E||^2. \quad (17)
\end{align*}
\]

Therefore, it follows from (17) that
\[
\langle A_p u - A_p v, u - v \rangle = \int_Q [\mathbf{T}(\mathcal{E}(u)) - \mathbf{T}(\mathcal{E}(v))] \cdot [u - v] \, dx,
\]
\[
\geq \nu_5 \int_Q |\mathcal{E}(u) - \mathcal{E}(v)|^2 \, dx, \quad (18)
\]
for any \( u, v \in V_{1,p} \). It is easily seen that
\[
\langle A_2 B - A_2 C, B - C \rangle \geq S||B - C||_2^2, \quad (19)
\]
for any \( B, C \in V_2 \). Therefore, it follows from (18)-(19) that \( A \) is monotone. Now it follows from (15) that
\[
\langle A \Phi, \Phi \rangle = \int_Q \mathbf{T}(\mathcal{E}(u)) \cdot \mathcal{E}(u) \, dx + S||\mathbf{B}||_2^2,
\]
\[
\geq \nu_3 \int_Q (|\mathcal{E}(u)|^2 + |\mathcal{E}(u)|^p) \, dx + S||\mathbf{B}||_2^2,
\]
for any \( \Phi \in \mathcal{V} \). Owing to Korn’s inequalities we infer from the last estimate that
\[
\langle A \Phi, \Phi \rangle \geq \tilde{\nu}_3(||\mathbf{u}||_1^2 + ||\mathbf{u}||_{1,p}^p) + S||\mathbf{B}||_2^2,
\]
\[
\geq \tilde{\nu}_3||\mathbf{u}||_{1,p}^p + S||\mathbf{B}||_2^2,
\]
which implies that there exists a constant \( \tilde{\nu} \) such that (13) holds.

We have that
\[
||A_2 \mathbf{B}||_{V^*}^{p^*,2} \leq S^2||\mathbf{B}||_2^2. \quad (20)
\]

Also,
\[
||A_p \mathbf{u}||_{V^*}^{p^*,2} = \sup_{||\mathbf{v}||_{1,p} = 1} ||\langle A_p \mathbf{u}, \mathbf{v} \rangle||,
\]
\[
\leq \sup_{||\mathbf{v}||_{1,p} = 1} \left( \int_Q |\mathbf{T}(\mathcal{E}(u))|^p \, dx \right)^{1/p} \left( \int_Q |\mathcal{E}(\mathbf{v})|^p \, dx \right)^{1/p}.
\]

Thanks to Korn’s inequalities and (16) we have
\[
||A_p \mathbf{u}||_{V^*}^{p^*,2} \leq \sup_{||\mathbf{v}||_{1,p} = 1} \left( \int_Q |\mathbf{T}(\mathcal{E}(u))|^p \, dx \right)^{1/p} ||\mathbf{v}||_{1,p},
\]
\[
||A_p \mathbf{u}||_{V^*}^{p^*,2} \leq C \int_Q (1 + |\mathcal{E}(\mathbf{u})|^p) \, dx. \quad (21)
\]
By using Korn’s inequalities into (21), we can deduce from the resulting estimate and (20) that (14) holds. \( \square \)
For any $p \geq 2$ and $u, v, w \in W^{1,p}$, we set

$$b(u, v, w) = \int_Q u_i \frac{\partial v_j}{\partial x_i} w_j dx,$$

where summations over repeated indices are enforced. The trilinear form $b(u, v, w)$ is continuous on $H^1(Q) \times H^1(Q) \times H^1(Q)$. Moreover,

$$b(u, v, v) = 0,$$

$$b(u, v, w) = -b(u, w, v),$$

for any $u \in V_2, v, w \in H^1(Q)$. Since $V_1 \subset V_2$ and $V_{1,p} \subset V_2$, then (22) and (23) are also valid for any element $u$ in $V_1$ and $V_{1,p}$. For the proofs of the above properties and more information on the trilinear form $b(\cdot, \cdot, \cdot)$, we refer, for instance, to [60].

As proved in [31], the trilinear form $b(u, v, w)$ is also continuous in $V_{1,p} \times V_{1,p} \times V_{1,p}$ as long as

$$p \geq 1 + \frac{2n}{n+2}.$$  

Hereafter we always assume that (24) holds. Set

$$B_0(\Phi_1, \Phi_2, \Phi_3) = b(u_1, u_2, u_3) - \mu b(B_1, B_2, u_3) + \mu b(u_1, B_2, B_3) - \mu b(B_1, u_2, B_3),$$

for any $\Phi_i = (u_i, B_i) \in V, i = 1, 2, 3$. It follows from (22) and (23) that

$$B_0(\Phi_1, \Phi_2, \Phi_2) = 0,$$

$$B_0(\Phi_1, \Phi_2, \Phi_3) = -B_0(\Phi_1, \Phi_3, \Phi_2),$$

for any $\Phi_i \in V, i = 1, 2, 3$. Following the idea in [31], we choose $s > 1 + \frac{n}{2}$ and set

$$W_{1,s} = \text{closure of } V_1 \text{ in } H^s(Q),$$

$$W_{2,s} = \text{closure of } V_2 \text{ in } H^s(Q),$$

and

$$W_s = W_{1,s} \times W_{2,s}.$$  

The space $W_{1,s}, W_{2,s}$ will be equipped with the usual scalar product and norm of $H^s(Q)$ respectively denoted by $(\cdot, \cdot)_s$ and $\| \cdot \|_s$. We also use these symbols to denote the norm and scalar product of $W_s$. Identifying $H$ with its dual, we have the following Gelfand chain

$$W_s \subset V \subset H \subset V^* \subset W^*_s,$$

where each space is densely and compactly embedded into the next one.

Since $s - 1 > \frac{n}{2}$, $\frac{\partial \Phi}{\partial x_i}$ is an element of $L^\infty(Q)$ for any $\Phi_3 \in W_s$. Therefore

$$|B_0(\Phi_1, \Phi_2, \Phi_3)| = | - B_0(\Phi_1, \Phi_3, \Phi_2)|,$$

$$\leq C|\Phi_1||\Phi_2||\Phi_3|_{W_s},$$

for any $\Phi_1, \Phi_2 \in V$ and $\Phi_3 \in W_s$. From this we infer the existence of a continuous bilinear form $B(\cdot, \cdot)$ defined on $V \times V$ taking its values in $W_s$. This bilinear mapping satisfies the properties stated in the following

**Lemma 2.4.**  
(i) For any $\Phi_1, \Phi_2 \in V$ and $\Phi_3 \in W_s$,

$$\langle B(\Phi_1, \Phi_2), \Phi_3 \rangle = B_0(\Phi_1, \Phi_2, \Phi_3).$$

(ii) We have that

$$\langle B(\Phi_1, \Phi_2), \Phi_2 \rangle = 0,$$

for any $\Phi_1 \in V$ and $\Phi_2 \in W_s$. 


(iii) There exists a positive constant $C$ such that

$$||B(\Phi_1, \Phi_2)||_{W^s} \leq C||\Phi_1||\Phi_2||,$$

for any $\Phi_i \in V, i = 1, 2.$

Proof. All of the statements in the lemma were proved above (see the lines between (26)-(31)). □

2.2. Stochastic setting and some hypotheses. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $K_i, i = 1, 2$ two fixed Hilbert spaces with orthonormal basis $\{e_{k,i}: k \geq 1\}, i = 1, 2.$ We may define two cylindrical Wiener processes $W_i$ mutually independent by setting $W_i = \sum_{k=1}^{\infty} W_{k,i} e_{k,i}$ (see [14]), where $(W_{k,i}; k \geq 1)$ are infinite sequences of mutually independent and identically distributed standard real-valued Wiener processes. By $L^2(K_i, H_i)$ we denote the space of Hilbert-Schmidt operators from $K_i$ to $H_i$:

$$L^2(K_i, H_i) = \{ R \in L(K_i, H_i) : \sum_{k=1}^{\infty} |Re_{k,i}|^2 < \infty \}.$$

We endow the space $L^2(K_i, H_i), i = 1, 2$ with the norm defined by

$$||R||_{L^2,i} = \left( \sum_{k=1}^{\infty} |Re_{k,i}|^2 \right)^{\frac{1}{2}}.$$

Now let $K_{0,i}$ be another Hilbert space such that the embedding $K_{0,i} \subset K_i$ is Hilbert-Schmidt, and there exists $\Omega' \in \mathcal{F}$ with $P(\Omega') = 1$ and $W_i(\omega) \in C(0, T; K_{0,i})$ for any $\omega \in \Omega'.$ Given the Hilbert spaces $K_i$ and a cylindrical Wiener processes $W_i,$ it is always possible to construct $K_{0,i}$ (see, among others, [36, 14]). We denote by $K$ (resp., $K_0$) the cartesian product $K_1 \times K_2$ (resp., $K_{0,1} \times K_{0,2}$). The orthonormal basis of $K$ is $\{e_k = (e_{k,1}; e_{k,2}): k \geq 1\}$. Let us set

$$W = \begin{pmatrix} W_1 \\
W_2 \end{pmatrix},$$

and

$$L^2(K, H) = L^2(K_1, H_1) \times L^2(K_2, H_2).$$

The space $L^2(K, H)$ is a Hilbert space endowed with the natural scalar product of the cartesian product whose corresponding norm is

$$||R||_{L^2} = \sum_{i=1}^{2} ||R_i||_{L^2,i}.$$

The stochastic processes $W$ defines a cylindrical processes such that $W \in C(0, T; K_0)$ $P$-almost surely. Furthermore

$$W = \sum_{k=1}^{\infty} W^k \cdot e_k,$$

where $W^k = (W_{k,1}; W_{k,2}).$

Let $X^*_1 = V^*_1 \times [0, T]$ and $X^*_2 = V^*_2.$ Now we introduce the hypotheses on $f_i(u, B, t), g_i(u, B, t)$ that are relevant for the major part of the paper.

(F) We assume that $f_i : H \times [0, T] \rightarrow X^*_i (i = 1, 2)$ are nonlinear mappings such that

(a) they are measurable with respect to $t,$
(b) \((u; B) \mapsto f_i(u, B, t)\) are continuous for almost all \(t \in [0, T]\) and there exists a positive constant \(C\) such that
\[
||f_i(u, B, t)||_{X^1} \leq C(1 + |u| + |B|),
\]
for any \((u; B) \in \mathbb{H}\).

(G) We suppose that \(g_i : \mathbb{H} \times [0, T] \to L_2(\mathbb{K}, \mathbb{H})\) are nonlinear mappings such that
(a) the \(g_i\)’s are measurable with respect to \(t\),
(b) \((u; B) \mapsto g_i(u, B, t)\) are continuous for almost all \(t \in [0, T]\) and there exists a positive constant \(C\) such that
\[
||g_i(u, B, t)||_{L_{2,i}} \leq C(1 + |u| + |B|),
\]
for any \((u; B) \in \mathbb{H}\).

Now we introduce the operators
\[
f = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} : \mathbb{H}_1 \times \mathbb{H}_2 \times [0, T] \to \mathbb{V}^*.
\]
and
\[
g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} : \mathbb{H}_1 \times \mathbb{H}_2 \times [0, T] \to L_2(\mathbb{K}, \mathbb{H}).
\]

Then, modulo divergence freeness, the problem (1) can be rewritten as
\[
dy + [Ay + B(y, y)]dt = f(y, t)dt + g(y, t)dW
\]
\[
y(0) = y_0 \in \mathbb{H}
\]
where \(y = (u; B)\) is a solution of (1) and \(y_0 = (u_0; B_0)\). From now on, we will work with (37) - (38).

To close this preliminary let us introduce additional notations frequently used throughout the work. Let \((\Omega, \mathbb{F}, \mathbb{P})\) be any complete probability space equipped with a right-continuous and increasing filtration \((\mathbb{F}_t)_{t \in [0, T]}\) such that \(\mathbb{F}_0\) contains all null sets of \(\mathbb{F}\); such kind of filtration will be referred to as a “filtration satisfying the usual condition”. The mathematical expectation with respect to the probability measure \(\mathbb{P}\) is denoted by \(\mathbb{E}\). Let \(r \in [2, \infty)\) and \(L^{r/2}(\Omega, L^{2,p}(0, T; \mathbb{V}))\) the space of functions \(y = y(\omega, t, x)\) defined on \(\Omega \times [0, T]\) with values in \(\mathbb{V}\) such that:
(a) \(u(\omega, t, x)\) is measurable with respect to \((\omega, t)\) and for each \(t\) is \(\mathbb{F}_t\)-measurable in \(\omega\).
(b) \(u(\omega, t, x) \in X\), for almost all \((\omega, t)\) and
\[
\mathbb{E} \left(\int_0^T ||y(s)||_{\mathbb{V}^*}^2 ds\right)^{\frac{r}{2}} < C.
\]
Similarly we define \(L^r(\Omega, L^{\infty}(0, T; \mathbb{H}))\) as the set of all functions \(y = y(\omega, t, x)\) defined on \(\Omega \times [0, T]\) with values in \(\mathbb{H}\) such that:
(i) \(u(\omega, t, x)\) is measurable with respect to \((\omega, t)\) and for each \(t\) it is \(\mathbb{F}_t\)-measurable in \(\omega\).
(ii) \(u(\omega, t, x) \in X\), for almost all \((\omega, t)\) and
\[
\mathbb{E} \sup_{t \in [0,T]} |y(t)|^r < C.
\]
We also set
\[
L^2(\Omega, L^{p,2}(0, T; \mathbb{V})) = L^2(\Omega, L^p(0, T; \mathbb{V}_{1,p})) \times L^2(\Omega, L^2(0, T; \mathbb{V}_2)),
\]
and
\[
L^2(\Omega, L^{p,*,2}(0, T; \mathbb{V}^*)) = L^2(\Omega, L^{p,*(0, T; \mathbb{V}_{1,p}^*)}) \times L^2(\Omega, L^2(0, T; \mathbb{V}_2)),
\]
where \(p^* = \frac{p}{p-1}\).
3. Existence of martingale weak solution

In this section we state our first main result and give its proof. Before we proceed further we explicitly define what we mean by a (martingale) weak solution of (37)-(38).

**Definition 3.1.** A weak solution of (37)-(38) is a probabilistic system \((\Omega, \mathbb{F}, \{\mathbb{F}^t : t \geq 0\}, \mathbb{P}, W, y)\) where

1. \((\Omega, \mathbb{F}, \mathbb{P})\) is a probability space, \(\{\mathbb{F}^t : t \geq 0\}\) is an increasing filtration satisfying the usual condition
2. \(W\) is a cylindrical Wiener process adapted to \(\{\mathbb{F}^t : t \geq 0\}\),
3. \(y\) is an element of \(L^{r/2}(\Omega, L^{2-p}(0, T; \mathbb{V})) \cap L^{r}(\Omega, L^{\infty}(0, T; \mathbb{H}))\) for any \(r \in [2, \infty)\),
4. for any \(w \in \mathbb{W}_s\),
   \[
   (y(t), w) + \int_0^t ((Ay(s) + B(y(s), y(s)), w)) ds = (y_0, w) + \int_0^t \langle f(y(s), s), w \rangle ds + \int_0^t (g(y(s), s), w) dW
   \]
   almost surely and for all \(t \in [0, T]\).
5. The function \(y\) takes values in \(\mathbb{H}\) and is continuous with respect to \(t\ \mathbb{P}\)-almost surely.

The point (5) of our definition can be justified as follows. Owing to Lemmata [23 and 24] point (3) of Definition 3.1 and (39) \(y(t)\) can be written in the following form

\[
y(t) = y_0 + \int_0^t G(s)ds + \int_0^t S(s)W(s), \ t \in [0, T],
\]
where \(G \in L^2(\Omega, L^{r}, \mathbb{W}^*_s))\) and \(S \in L^2(\Omega \times [0, T]; \mathbb{H})\). Now it easily follows from [26 Chapter I, Theorem 3.2] that there exists \(\Omega^* \in \mathbb{F}\) such that \(\mathbb{P}(\Omega^*) = 1\) and for each \(w \in \Omega^*\) the function \(y(\cdot)\) takes values in \(\mathbb{H}\), and it is continuous in \(\mathbb{H}\) with respect ot \(t\). We can also use the argument in [40 Chapter 2, Page 42] to justify that \(y \in C(0, T; \mathbb{H})\) almost surely.

**Remark 3.2.** We should note that we use the notation

\[
\int_0^t (g(y(s), s), w) dW = \sum_{k=1}^\infty \int_0^t (g(y(s), s)e_k, w) dW^k.
\]

Our first main result is

**Theorem 3.3.** Let the conditions (2)-(4), (F), and (G) be satisfied and \(y_0 \in \mathbb{H}\). Then there exists at least a weak solution of (37)-(38) in the sense of Definition 3.1.

The proof of this statement is given in the next section and it relies very much on Galerkin, compactness and monotonicity methods.

3.1. Galerkin approximation and a priori estimates. In this subsection we introduce the Galerkin approximation scheme of our problem and derive a priori estimates for the solution.

We introduce the family of eigenfunctions

\[
((w_j, v))_s = \lambda_j^1(w_j, v), \forall v \in \mathbb{W}_{1,s},
\]
and

\[
((C_j, B))_s = \lambda_j^2(C_j, B), \forall B \in \mathbb{W}_{2,s}.
\]
Then, we can define a spectral problem on \(\mathbb{W}_s\) by setting

\[
\Psi_j = (\Psi_j^1, \Psi_j^2), \Psi_j^1 = w_j, \Psi_j^2 = C_j,
\]
and

\[(\Psi_j, w)_s = \sum_{k=1}^{2} \lambda^k_j(\Psi_j, w), \forall w \in \mathbb{W}_s.\]

We assume that \(\Psi_j, j = 1, 2, 3, \ldots\) form an orthonormal basis of \(\mathbb{W}_s\) which are complete in \(V\) and form an orthogonal basis of \(H\). Let \((\Omega, \mathbb{F}, \mathbb{P}), (\mathbb{W} = \sum_{i=1}^{\infty} W^i \cdot e_i \text{ is a cylindrical Wiener process evolving on } \mathbb{K})\) and \(m\) a positive integer. We equip the probability space \((\Omega, \mathbb{F}, \mathbb{P})\) with the natural filtration of \(\mathbb{W}\) which is denoted by \(\mathbb{F}\). We set \(\mathbb{W}^m_s = \text{Span}\{\Psi_j : j = 1, 2, \ldots, m\}\). We look for a sequence of stochastic processes \((y^m : m = 1, 2, \ldots) \subset \mathbb{W}^m_s\) such that

\[
d(y^m(s), \Psi_j) + (A y^m(s) + B(y^m(s), y^m(s)), \Psi_j)ds = (f(y^m(s), s), \Psi_j)ds + \sum_{i=1}^{m} (g(y^m(s), s) e_i, \Psi_j)d\bar{W}^i,
\]

\[
y^m(0) = y^m_0 \in \mathbb{W}^m_s,
\]

\[
y^m_0 \rightarrow y_0 \text{ in } H \text{ as } m \rightarrow \infty.
\]

Note that,

\[
y^m(t) = \sum_{j=1}^{m} ((y^m(t), \Psi_j))_s \Psi_j
\]

\[
= \sum_{k=1}^{2} \sum_{j=1}^{m} \lambda^k_j(y^m(t), \psi^k_j)\Psi^k_j.
\]

The system (41)-(43) is a system of stochastic differential equations in a finite dimensional Banach space. Under the constraints on \(A, B, f,\) and \(g\) this system satisfies the conditions of existence in [56] which do not require the Lipschitz condition on the coefficients. Hence a sequence of continuous functions \(y^m\) exists on a short interval \([0, T_m]\). It will follow from a priori estimates that \(y^m\) exists on \([0, T]\).

First we prove the following

**Lemma 3.4.** The sequence \((y^m : m = 1, 2, \ldots)\) satisfies

\[
\mathbb{E} \sup_{s \in [0,T]} |y^m(s)|^r < C,
\]

for any \(r \in [2, \infty).\)

**Proof.** Let \(M\) be a positive integer. We define a sequence of stopping times \(\tau_M\) by setting

\[
\tau_M = \inf\{s : |y^m(s)| + \left(\int_{0}^{s} ||y^m(t)||_V^2 dt\right)^{\frac{1}{2}} \geq M\} \wedge T.
\]
We shall use a modification of the argument used in [1]. Let $t \in [0, T \wedge \tau_M]$. By Itô’s formula, we have

$$|\mathbf{y}^m(t)|^2 + 2\int_0^t \langle A\mathbf{y}^m(s), \mathbf{y}^m(s) \rangle ds = |\mathbf{y}^m_0|^2 + 2\int_0^t \langle f(\mathbf{y}^m(s), s), \mathbf{y}^m(s) \rangle ds$$

$$+ \sum_{j=1}^m \sum_{k,i=1}^{m} \int_0^t \lambda_k^j(g(\mathbf{y}^m(s), s)e_i, \psi_k)^2 ds$$

$$+ 2\sum_{i=1}^m \int_0^t (g(\mathbf{y}^m(s), s)e_i, \mathbf{y}^m(s)) d\mathbf{W}^i. \tag{48}$$

Using (13) into this first equation implies

$$|\mathbf{y}^m(t)|^2 + 2\tilde{\nu} \int_0^t ||\mathbf{y}^m(s)||_{\mathbf{V}}^2 ds \leq |\mathbf{y}^m_0|^2 + 2\int_0^t ||f(\mathbf{y}^m(s), s)||_{\mathbf{V}} ||\mathbf{y}^m(s)||_{\mathbf{V}} ds$$

$$+ \sum_{k,i=1}^{m} \int_0^t ||g(\mathbf{y}^m(s), s)||_{L_2}^2 ds$$

$$+ 2\sum_{i=1}^m \int_0^t (g(\mathbf{y}^m(s), s)e_i, \mathbf{y}^m(s)) d\mathbf{W}^i. \tag{49}$$

For any $\varepsilon > 0$, we easily check that

$$||f(\mathbf{y}^m(s), s)||_{\mathbf{V}} ||\mathbf{y}^m(s)||_{\mathbf{V}} \leq C_{\varepsilon} ||f(\mathbf{y}^m(s), s)||_{\mathbf{V}}^2 + \varepsilon ||\mathbf{y}^m(s)||_{\mathbf{V}}^2 + C(\varepsilon, p). \tag{50}$$

Hence, owing to the assumptions on $f$ and $g$, we can derive from the last estimate and (19) that

$$|\mathbf{y}^m(t)|^2 + \tilde{\nu} \int_0^t ||\mathbf{y}^m(s)||_{\mathbf{V}}^2 ds \leq |\mathbf{y}^m_0|^2 + C \int_0^t (1 + |\mathbf{y}^m(s)|^2) ds$$

$$+ 2\sum_{i=1}^{m} \int_0^t (g(\mathbf{y}^m(s), s)e_i, \mathbf{y}^m(s)) d\mathbf{W}^i. \tag{51}$$

In view of (51), we have

$$\mathbb{E} \sup_{s \in [0, T \wedge \tau_M]} ||\mathbf{y}^m(s)||_{\mathbf{V}}^2 \leq |\mathbf{y}^m_0|^2 + C \mathbb{E} \int_0^{T \wedge \tau_M} (1 + |\mathbf{y}^m(s)|^2) ds$$

$$+ 2\sum_{i=1}^m \mathbb{E} \sup_{s \in [0, T \wedge \tau_M]} \left| \int_0^s (g(\mathbf{y}^m(\tau), \tau)e_i, \mathbf{y}^m(\tau)) d\mathbf{W}^i \right|. \tag{52}$$

Burkhölder-Davis-Gundy’s inequality enables us to derive the following estimates

$$2\sum_{i=1}^m \mathbb{E} \sup_{s \in [0, T \wedge \tau_M]} \left| \int_0^s (g(\mathbf{y}^m(\tau), \tau)e_i, \mathbf{y}^m(\tau)) d\mathbf{W}^i \right| \leq \frac{1}{2} \mathbb{E} \sup_{s \in [0, T \wedge \tau_M]} ||\mathbf{y}^m(s)||_{\mathbf{V}}^2 + C \mathbb{E} \int_0^{T \wedge \tau_M} (1 + |\mathbf{y}^m(s)|^2) ds,$$  \tag{53}$$

where we have used the assumption on $g$ to get (53). Now it follows from (52) and (53) that

$$\mathbb{E} \sup_{s \in [0, T \wedge \tau_M]} ||\mathbf{y}^m(s)||_{\mathbf{V}}^2 \leq |\mathbf{y}^m_0|^2 + C \mathbb{E} \int_0^{T \wedge \tau_M} (1 + |\mathbf{y}^m(s)|^2) ds \tag{54}$$
Since the second term of the left-hand side of (54) is positive, then it follows from Gronwall’s lemma that
\[ \mathbb{E} \sup_{s \in [0, t]} |y^m(s)|^2 < C, \] (55)
for any \( t \in [0, T_m] \). Taking (55) into (54) yields
\[ \mathbb{E} \int_0^{T_m} ||y^m(s)||^2_p ds \leq C. \] (56)
Since the constants \( C \) in (55)-(56) do not depend on \( m \) and \( M \), then we can show that \( \tau_m \not\to T \) \( \mathbb{P} \)-almost surely as \( M \to \infty \). Therefore, \( T_m = T \). We can conclude by passing to the limit in (55) and (56) that
\[ \mathbb{E} \sup_{s \in [0, T]} |y^m(s)|^2 < C, \] (57)
and
\[ \mathbb{E} \int_0^T ||y^m(s)||^2_p ds \leq C. \] (58)
Now let \( r > 2 \). Thanks to Itô’s formula, we derive from (48) that
\[ |y^m(t)|^r + r \int_0^t |y^m(s)|^{r-2} \langle Ay^m(s), y^m(s) \rangle ds \leq |y^m_0|^r + C \int_0^t |y^m(s)|^{r-2} \left[ \langle f(y^m(s), s), y^m(s) \rangle + ||g(y^m(s), s)||^2_{L^2} \right] ds + r \sum_{i=1}^m \int_0^t |y^m(s)|^{r-2}(g(y^m(s), s)e_i, y^m(s)) d\tilde{W}^i. \] (59)
Owing to (13), (50) and the assumptions on \( g \) and \( f \) we derive from the last estimate that
\[ |y^m(t)|^r + r\tilde{\nu} \int_0^t |y^m(s)|^{r-2}||y^m(s)||^2_p ds \leq |y^m_0|^r + C \int_0^t |y^m(s)|^{r-2} \left[ C\varepsilon(1 + |y^m(s)|^2 + \varepsilon||y^m(s)||^2_p + C(\varepsilon, p)) + C(1 + |y^m(s)|^2) \right] ds + r \sum_{i=1}^m \int_0^t |y^m(s)|^{r-2}(g(y^m(s), s)e_i, y^m(s)) d\tilde{W}^i. \] (60)
By an appropriate choice of \( \varepsilon \) we deduce from (60) that
\[ |y^m(t)|^r + \tilde{\nu} \int_0^t |y^m(s)|^{r-2}||y^m(s)||^2_p ds \leq |y^m_0|^r + C \int_0^t |y^m(s)|^{r-2}(1 + |y^m(s)|^2) ds + r \sum_{i=1}^m \int_0^t |y^m(s)|^{r-2}(g(y^m(s), s)e_i, y^m(s)) d\tilde{W}^i. \] (61)
It is not difficult to prove that
\[ |y^m(s)|^{r-2}(1 + |y^m(s)|^2) \leq C|y^m(s)|^r. \]
Thus, from (61) we see that
\[
\mathbb{E} \sup_{s \in [0,t]} |\mathbf{y}^m(s)|^r + \nu \mathbb{E} \int_0^t |\mathbf{y}^m(s)|^{r-2}||\mathbf{y}^m(s)||^2 d\sigma \leq C(|\mathbf{y}_0|^r, T, p) + C\mathbb{E} \int_0^t |\mathbf{y}^m(s)|^r d\sigma
\]
\[
\quad + r \sum_{i=1}^m \mathbb{E} \sup_{s \in [0,t]} \left| \int_s^t |\mathbf{y}^m(\tau)|^{r-2}(g(\mathbf{y}^m(\tau), \tau) e_i, \mathbf{y}^m(\tau)) d\mathbb{W}^i \right|.
\]
(62)

Thanks to Burkholder-Davis-Gundy’s inequality and the assumptions on \(g\) we derive from the last estimate that
\[
\mathbb{E} \sup_{s \in [0,t]} |\mathbf{y}^m(s)|^r + \nu \mathbb{E} \int_0^t |\mathbf{y}^m(s)|^{r-2}||\mathbf{y}^m(s)||^2 d\sigma \leq C(|\mathbf{y}_0|^r, T, p) + C\mathbb{E} \int_0^t |\mathbf{y}^m(s)|^r d\sigma
\]
\[
\quad + C\mathbb{E} \left( \int_0^t |\mathbf{y}^m(s)|^{r-2}(1 + |\mathbf{y}^m(s)|^2) d\sigma \right)^{\frac{r}{2}}.
\]
(63)

As before we can check that the third term of the right hand side of (63) can be bounded from above by
\[
\frac{1}{2} \mathbb{E} \sup_{s \in [0,t]} |\mathbf{y}^m(s)|^r + C\mathbb{E} \int_0^t |\mathbf{y}^m(s)|^r d\sigma.
\]
(64)

We infer from (63) and (64) that
\[
\mathbb{E} \sup_{s \in [0,t]} |\mathbf{y}^m(s)|^r + \nu \mathbb{E} \int_0^t |\mathbf{y}^m(s)|^{r-2}||\mathbf{y}^m(s)||^2 d\sigma \leq C(|\mathbf{y}_0|^r, T, p) + C\mathbb{E} \int_0^t |\mathbf{y}^m(s)|^r d\sigma.
\]
(65)

Making use of Gronwall’s inequality in (65) implies that
\[
\mathbb{E} \sup_{s \in [0,t]} |\mathbf{y}^m(s)|^r < C, \forall t \in [0, T] \text{ and } r > 2.
\]
(66)

Next recall that
\[
|\mathbf{y}^m(T)|^2 + C \int_0^T ||\mathbf{y}^m(s)||^2 d\sigma \leq |\mathbf{y}_0|^2 + C \int_0^T |\mathbf{y}^m(s)|^2 d\sigma + \left| \int_0^T \sum_{i=1}^m (g(\mathbf{y}^m(s), s), \mathbf{y}^m(s)) d\mathbb{W}^i \right|.
\]
(67)

Therefore, it is straightforward to check that
\[
\mathbb{E} \left( \int_0^T ||\mathbf{y}^m(s)||^2 d\sigma \right)^{\frac{r}{2}} \leq C(|\mathbf{y}_0|^r, r) + C\mathbb{E} \left( \int_0^T |\mathbf{y}^m(s)|^2 d\sigma \right)^{\frac{r}{2}}
\]
\[
\quad + \mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t \sum_{i=1}^m (g(\mathbf{y}^m(s), s), \mathbf{y}^m(s)) d\mathbb{W}^i \right|.
\]
(68)

And we derive that
\[
\mathbb{E} \left( \int_0^T ||\mathbf{y}^m(s)||^2 d\sigma \right)^{\frac{r}{2}} \leq C, \forall r \in (2, \infty).
\]
(69)

We easily conclude the lemma from (67), (58), (60) and (69).

Now, we derive a crucial estimate on the difference \(\mathbf{y}^m(t + \theta) - \mathbf{y}^m(t)\) in \(\mathbb{W}_{s}^p\).
Lemma 3.5. We assume that $t \mapsto y^m(t)$ is extended to zero outside the interval $[0, T]$. Then, there exists a positive constant $C$ such that

$$
\mathbb{P} \int_0^T \sup_{|\theta| \leq \epsilon} ||y^m(t + \theta) - y^m(t)||_{\mathcal{W}^s_x}^p \leq C \delta^{\frac{p-1}{p}},
$$

for any positive integer $m$ and $\delta \in (0, 1)$.

Proof. Noting that $\{\lambda^k\Psi^k_j : k = 1, 2, j = 1, 2, \ldots\}$ form a basis of $\mathcal{W}^s_x$, we introduce the projector

$$
P_m : \mathcal{W}^s_x \rightarrow \text{Span}\{\lambda^k\Psi^k_j : k = 1, 2, j = 1, 2, \ldots, m\}.
$$

It is well-known (see [31]) that

$$
||P_m||_{L(\mathcal{W}^s_x, \mathcal{W}^s_x)} \leq 1. \tag{70}
$$

Arguing as in [31] we can rewrite the system (41) in the following form

$$
y^m(t) = y^m_0 - \int_0^t P_m[Ay^m(s) + B(y^m(s), y^m(s))]ds + \int_0^t P_m f(y^m(s), s)ds + \int_0^t P_m g(y^m(s), s)d\bar{W}, \tag{71}
$$

where, for the sake of simplicity, we have set

$$
P_m \left(\int_0^t g(y^m(s), s)d\bar{W}\right) = \int_0^t P_m g(y^m(s), s)d\bar{W}.
$$

It follows from (71) that

$$
y^m(t + \theta) - y^m(t) = -\int_t^{t + \theta} P_m[Ay^m(s) + B(y^m(s), y^m(s))]ds + \int_t^{t + \theta} P_m f(y^m(s), s)ds + \int_t^{t + \theta} P_m g(y^m(s), s)d\bar{W}, \tag{72}
$$

for any $\theta \in (0, \delta)$ and $\delta \in (0, 1)$. From (72) and (71) we infer that

$$
||y^m(t + \theta) - y^m(t)||_{\mathcal{W}^s_x}^p \leq C \left(\int_t^{t + \theta} ||Ay^m(s)||_{\mathcal{W}^s_x}ds\right)^p + C \left(\int_t^{t + \theta} ||B(y^m(s), y^m(s))||_{\mathcal{W}^s_x}ds\right)^p + C \left(\int_t^{t + \theta} ||f(y^m(s), s)||_{\mathcal{W}^s_x}ds\right)^p + C \left(\int_t^{t + \theta} g(y^m(s), s)d\bar{W}\right)^p,
$$

$$
\leq CI_1(t, \theta) + CI_2(t, \theta) + CI_3(t, \theta) + CI_4(t, \theta), \tag{73}
$$

where

$$
I_1(t, \theta) = \left(\int_t^{t + \theta} ||Ay^m(s)||_{\mathcal{W}^s_x}ds\right)^p, \quad I_2(t, \theta) = \left(\int_t^{t + \theta} ||B(y^m(s), y^m(s))||_{\mathcal{W}^s_x}ds\right)^p,
$$

$$
I_3(t, \theta) = \left(\int_t^{t + \theta} ||f(y^m(s), s)||_{\mathcal{W}^s_x}ds\right)^p, \quad I_4(t, \theta) = \left(\int_t^{t + \theta} g(y^m(s), s)d\bar{W}\right)^p.
$$

For $I_1(t, \theta)$, we have

$$
I_1(t, \theta) \leq C \left(\int_t^{t + \theta} ||Ay^m(s)||_{\mathcal{W}^s_x}ds\right)^p.
$$

But,

$$
\int_t^{t + \theta} ||Ay^m(s)||_{\mathcal{W}^s_x}ds \leq \theta^{\frac{1}{p}} \left(\int_t^{t + \theta} ||Ay^m(s)||_{\mathcal{W}^s_x}^pds\right)^{\frac{1}{p}}.
$$
Therefore
\[ I_1(t, \theta) \leq C \theta^{p^*} \int_t^{t+\theta} \|Ay^m(s)\|_{L^p(V)}^2 ds, \]  
(74)
and thanks to (14) we have
\[ I_1(t, \theta) \leq C \theta^{p^*} \int_t^{t+\theta} (1 + \|y^m(s)\|_{L^p(V)}^2) ds 
\leq C \theta^{p^*} (\theta + \int_t^{t+\theta} \|y^m(s)\|_{L^p(V)}^2 ds). \]

Thus,
\[ \mathbb{E} \int_0^T \sup_{\theta \in (0, \delta)} I_1(t, \theta) dt \leq C \delta^{p^*} (\delta + \mathbb{E} \int_0^T \int_t^{t+\delta} \|y^m(s)\|_{L^p(V)}^2 ds dt), \]
\[ \leq C \delta^{p^*} (\delta + \int_0^T \mathbb{E} \int_t^{t+\delta} \|y^m(s)\|_{L^p(V)}^2 ds dt). \]

Owing to Lemma 3.4 we derive from this last inequality that
\[ \mathbb{E} \int_0^T \sup_{\theta \in (0, \delta)} I_1(t, \theta) dt \leq C \delta^{p^*}. \]  
(75)

As above we can check that
\[ I_2(t, \theta) \leq C \theta^{p^*} \int_t^{t+\theta} \|B(y^m(s), y^m(s))\|_{L^p(V)}^2 ds. \]

In view of (34) and Lemma 3.4 we have
\[ \mathbb{E} \int_0^T \sup_{\theta \in (0, \delta)} I_2(t, \theta) dt \leq C \delta^{p^*}. \]  
(76)

It follows from the assumptions on \( f \) and Lemma 3.4 that
\[ \mathbb{E} \int_0^T \sup_{\theta \in (0, \delta)} I_3(t, \theta) dt \leq C \delta^{p^*}. \]  
(77)

For \( I_4(t, \theta) \), we have that
\[ \mathbb{E} \int_0^T \sup_{\theta \in (0, \delta)} I_4(t, \theta) dt \leq C \int_0^T \mathbb{E} \left( \int_t^{t+\delta} \|g(y^m(s), s)\|_{L^2}^2 ds \right)^{p^*} dt, \]
where we have used Fubini’s theorem and Burkholder-Davis-Gundy’s inequality. By using the assumptions on \( g \), we see from the last estimate that
\[ \mathbb{E} \int_0^T \sup_{\theta \in (0, \delta)} I_4(t, \theta) dt \leq C \int_0^T \mathbb{E} \left( \int_t^{t+\theta} (1 + \|y^m(s)\|^2) ds \right)^{p^*} dt, \]
\[ \leq C \int_0^T \left( \delta + \delta \mathbb{E} \sup_{s \in [t, t+\delta]} |y^m(s)|^2 \right)^{p^*} dt. \]

Thanks to Lemma 3.4 we have that
\[ \mathbb{E} \int_0^T \sup_{\theta \in (0, \delta)} I_4(t, \theta) dt \leq C \delta^{p^*}, \]
\[ \leq C \delta^{p^*}. \]  
(78)
Now it follows from (73)-(78) that
\[ \mathbb{P} \left( \int_0^T \sup_{\theta \in (0, \delta)} \| y^m(t + \theta) - y^m(t) \|_{W^*_p}^{p^*} \leq C \delta^{\frac{p}{p^*}} \right), \]
for any positive integer \( m \) and \( \delta \in (0, 1) \). One can deal with the negative values of \( \theta \) by similar argument which completes the proof of the lemma.

\[ \square \]

**Remark 3.6.** We did not use the result in [21] because of the following reason. The drift term of (71) belongs almost surely to the Sobolev space \( W^{1,p^*}(0,T;W^*_p) \), \( p^* \in (1,2) \) which cannot be embedded into to the fractional Sobolev space \( W^{\alpha,2}(0,T;W^*_p) \), \( \alpha \in (0,\frac{1}{2}) \) to which the stochastic integral or diffusion term belongs. Also, it seems that there is no other space which can contain them.

### 3.2. Tightness property and passage to the limit

Throughout this subsection we set \( \mathcal{G}_1 = L^2(0,T;H) \) (resp., \( \mathcal{G}_2 = C(0,T;\mathbb{R}_0) \)) and we denote by \( \mathcal{B}(\mathcal{G}_1) \) (resp., \( \mathcal{B}(\mathcal{G}_2) \)) its Borel \( \sigma \)-algebra. The laws of \( \{y^m : m \in \mathbb{N}\setminus\{0\} \} \) is denoted by \( \Pi_{1,m} \). We have the following claim.

**Lemma 3.7.** The family \( \Pi_{1,m} \) is tight on \( \mathcal{G}_1 \).

The proof of this claim relies very much on the following result.

**Lemma 3.8.** Let \( \mu_n, \nu_n \) be sequences of positive numbers such that \( \mu_n, \nu_n \to 0 \) as \( n \to \infty \). The set
\[ Z = \left\{ z : \int_0^T \| z \|_{W^*_p}^{p^*} dt \leq L, \ |z(t)|^2 \leq K \ a.e. \ t, \ \sup_{|\theta| \leq \mu_n} \int_0^T |z(t + \theta) - z(t)|_{W^*_p}^{p^*} \leq \nu_n M \right\} \]
is a compact subset of \( L^2(0,T;H) \).

**Proof.** The argument of the proof is very similar to [54 Theorem 5], so we omit it. \( \square \)

**Proof of Lemma 3.7.** Let \( \delta > 0 \) and let \( L_\delta, K_\delta, M_\delta \) positive constants depending only on \( \delta \) to be fixed later. It follows from Lemma 3.8 that
\[ Z_\delta = \left\{ z : \int_0^T \| z \|_{W^*_p}^{p^*} dt \leq L_\delta, \ |z(t)|^2 \leq K_\delta \ a.e. \ t, \ \sup_{|\theta| \leq \mu_n} \int_0^T |z(t + \theta) - z(t)|_{W^*_p}^{p^*} \leq \nu_n M_\delta \right\} \]
is a compact subset of \( L^2(0,T;H) \) for any \( \delta > 0 \). Here we choose the sequence \( \mu_n, \nu_n \) so that
\[ \sum_{m} \frac{1}{\nu_n(\mu_m)}^{p^*} < \infty. \]
We have that
\[ \mathbb{P}(y^m \notin Z_\delta) \leq \mathbb{P} \left( \int_0^T \| y^m(s) \|_{W^*_p}^{p^*} ds \geq L_\delta \right) + \mathbb{P} \left( \sup_{s \in [0,T]} \| y^m(s) \|_{W^*_p}^{p^*} \geq K_\delta \right) \]
\[ + \mathbb{P} \left( \sup_{|\theta| \leq \mu_n} \int_0^T |y^m(t + \theta) - y^m(t)|_{W^*_p}^{p^*} dt \geq \nu_n M_\delta \right). \]

Thanks to Tchebychev’s inequality we have
\[ \mathbb{P}(y^m \notin Z_\delta) \leq \frac{1}{L_\delta} \mathbb{E} \int_0^T \| y^m(s) \|_{W^*_p}^{p^*} ds + \frac{1}{K_\delta} \mathbb{E} \sup_{s \in [0,T]} \| y^m(s) \|_{W^*_p}^{p^*} ds \]
\[ + \sum_{m} \frac{1}{\nu_n M_\delta} \mathbb{E} \sup_{|\theta| \leq \mu_n} \int_0^T |y^m(t + \theta) - y^m(t)|_{W^*_p}^{p^*} dt. \]
From Lemmata 3.3 and 3.5 it follows that
\[ \mathbb{P}(y^m \in Z_\delta) \leq \frac{C}{L_\delta} + \frac{C}{K_\delta} + \frac{C}{M_\delta} \sum_{m} \frac{1}{\nu_n(\mu_m)}^{\frac{p^*}{p^*}}. \]
By Choosing
\[ K_\delta = L_\delta = \frac{3C}{\delta} \quad \text{and} \quad M_\delta = \frac{3C}{\delta} \left( \sum \frac{\mu_m}{\nu_m} f_\nu \right) \]
we have that
\[ \mathbb{P}(y^m \notin Z_\delta) \leq \delta, \quad (79) \]
from which the proof of the lemma follows. \qed

For any \( m \in \mathbb{N} \setminus \{0\} \), we construct a family of Wiener processes evolving on \( K \) by setting
\[ \bar{W}_m = W, \quad \forall m \geq 1. \]
Note that it is possible to find a set \( \Omega' \in \mathcal{F} \) of measure zero such that \( W_m(\omega) \in C(0,T;\mathbb{K}_0) \) for any \( \omega \in \Omega \setminus \Omega' \) and \( m \in \mathbb{N} \setminus \{0\} \). The laws of \( \{W_m : m \in \mathbb{N} \setminus \{0\}\} \) is denoted by \( \Pi_{2,m} \) and \( \mathcal{M}(C(0,T;\mathbb{K}_0)) \) denotes the space of measures on \( (\mathcal{G}_2,\mathcal{B}(\mathcal{G}_2)) \).

**Theorem 3.9.** Let \( \mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2 \) and \( \mathcal{B}(\mathcal{G}) \) its Borel \( \sigma \)-algebra. The laws of \( (y^m; \bar{W}_m) \) are tight on \( \mathcal{G} \).

**Proof.** Endowed with the uniform convergence \( C(0,T;\mathbb{K}_0) \) is a Polish space, then it follows from [5 Theorem 6.8] that \( \mathcal{M}(C(0,T;\mathbb{K}_0)) \) endowed with the Prohorov’s metric is a separable and complete metric space. By construction the laws of \( \{\bar{W}_m : m \geq 1\} \) are reduced to an an element which belongs to \( \mathcal{M}(C(0,T;\mathbb{K}_0)) \). Therefore, invoking [13 Theorem 3.2] we easily deduce that the laws of the family \( \{\bar{W}_m : m \geq 1\} \) are tight on \( \mathcal{M}(C(0,T;\mathbb{K}_0)) \). Owing to this fact along with Lemma [3.7 and [29 Corollary 1.3], the laws of the joint processes \( (y^m; \bar{W}_m) \) are tight on \( \mathcal{G} \). \qed

It follows from Theorem 3.9 and Prohorov’s theorem [13] that there exists a subsequence \( \Pi^{m_j} \) of \( \Pi^m \) converging weakly (in the sense of measure) to a probability measure \( \Pi \). It emerges from Skorokhod’s embedding theorem [56] that we can find a new probability space \( (\Omega,\mathcal{F},\mathbb{P}) \) and random variables \( (y^{m_j}; W_{m_j}), (y; W) \) defined on this new probability space and taking values in \( \mathcal{G} \) such that:

The probability law of \( (y^{m_j}; W_{m_j}) \) is \( \Pi^{m_j} \),

(80)

The probability law of \( (y; W) \) is \( \Pi \),

(81)

\[ W_{m_j} \rightarrow W \text{ in } C(0,T;\mathbb{K}_0) \mathbb{P}\text{-a.s.}, \]

(82)

\[ y^{m_j} \rightarrow y \text{ in } L^2(0,T;\mathbb{H}) \mathbb{P}\text{-a.s.}. \]

(83)

Arguing as in [8] we can check that \( \{W_{m_j}\} \) is a sequence of cylindrical Brownian Motions evolving on \( \mathbb{K} \). We let \( \mathcal{F} \) be the \( \sigma \)-algebra generated by \( (W(s); y(s)), 0 \leq s \leq t \) and the null sets of \( \mathcal{F} \). We can show by arguing as in [8] (see also [14]) that \( W \) is an \( \mathcal{F}^t \)-adapted cylindrical Wiener process evolving in \( \mathbb{K} \). By the same argument as in [14] we can show that the following holds true

\[ (y^{m_j}(t), w) + \int_0^t (Ay^{m_j}(s) + B(y^{m_j}(s), y^{m_j}(s)), w)ds = (y^m_0, w) + \int_0^t (f(y^{m_j}(s), s), w)ds \]

\[ + \sum_{i=1}^{m_j} \int_0^t (g(y^{m_j}(s), s)e_i, w)dW_{m_j}^i, \]

(84)

for all \( w \in \mathbb{W}_s^{m_j} \) and for almost all \( (\omega, t) \in \Omega \times [0,T] \).
From (84) and Lemma 3.3 it follows that $\mathbf{y}^{m_j}$ satisfies the estimates

$$
E \sup_{t \in [0,T]} |\mathbf{y}^{m_j}(t)|^r < C, \tag{85}
$$

and

$$
\mathbb{E} \left( \int_0^T |\mathbf{y}^{m_j}(t)|^{p,2} dt \right)^{\frac{1}{p}} < C, \tag{86}
$$

for any $r \in [2, \infty)$. Here $E$ denotes the mathematical expectation with respect to $\mathbb{P}$. Thus modulo the extraction of a subsequence denoted again $\mathbf{y}^{m_j}$, we have

$$
\mathbf{y}^{m_j} \to \mathbf{y} \text{ weakly-}* \text{ in } L^r(\Omega, L^\infty(0, T; \mathbb{H})), \tag{87}
$$

$$
\mathbf{y}^{m_j} \to \mathbf{y} \text{ weakly in } L^2(\Omega, L^{p,2}(0, T, \mathbb{V})). \tag{88}
$$

Let us consider the positive nondecreasing function $\varphi(x) = x^4$, defined on $\mathbb{R}_+$. The function $\varphi$ obviously satisfies

$$
\lim_{x \to \infty} \frac{\phi(x)}{x} = \infty.
$$

Thanks to the estimates (85) we have

$$
\sup_{m_j \geq 1} \mathbb{E}(\phi(||\mathbf{y}^{m_j}||^2_{L^2(0,T;\mathbb{H})})) < \infty.
$$

Thanks to uniform integrability criteria in [27], Chapter 3, Exercise 6] we see that $||\mathbf{y}^{m_j}||^2_{L^2(0,T;\mathbb{H})}$ is uniform integrable with respect to the probability measure. Thanks to Vitali’s convergence theorem (see, for instance, [27], Chapter 3, Proposition 3.2) and (85), we obtain that

$$
\mathbf{y}^{m_j} \to \mathbf{y} \text{ strongly in } L^2(\Omega, L^2(0, T, \mathbb{H})). \tag{89}
$$

Thus modulo the extraction of a subsequence denoted again with the same symbols we have

$$
\mathbf{y}^{m_j} \to \mathbf{y} \text{ } d\mathbb{P} \otimes dt \text{ in } \mathbb{H}. \tag{90}
$$

In view of (90), the continuity of $f$ and the applicability of Vitali’s convergence theorem, we derive that

$$
\int_0^T P_{m_j} f(\mathbf{y}^{m_j}(s), s) ds \to \int_0^T f(\mathbf{y}(s), s) ds \text{ strongly in } L^2(\Omega, L^2(0, T, \mathbb{V})). \tag{91}
$$

Arguing as in [16] we can show that

$$
\sum_{i=1}^{m_j} \int_0^T (g(\mathbf{y}^{m_j}(s), s)e_i, \mathbf{w}) dW_{m_j}^i \to \sum_{i=1}^{\infty} \int_0^T (g(\mathbf{y}(s), s)e_i, \mathbf{w}) dW^i \tag{92}
$$

in $L^2(\Omega, L^2(0,T))$ for any $\mathbf{w} \in \mathbb{W}_s$. Thanks to (34), (35), and (36) $P_m B(\mathbf{y}^{m_j}, \mathbf{y}^{m_j})$ belongs to a bounded set of $L^2(\Omega, \mathbb{P}; L^2(0,T; \mathbb{W}_s^*))$. Taking advantage of (35) and (36), we will show that

$$
P_m B(\mathbf{y}^{m_j}, \mathbf{y}^{m_j}) \to B(\mathbf{y}, \mathbf{y}) \text{ weakly in } L^2(\Omega, \mathbb{P}; L^2(0,T; \mathbb{W}_s^*)). \tag{93}
$$

To this end let

$$
\mathbb{D} = \{ \Phi = \phi(\omega)\chi(t)\Psi_j : \phi(\omega) \in L^\infty(\Omega, \mathbb{P}), \chi(t) \in \mathbb{C}^\infty_c(0,T) \text{ and } j = 1, 2, \ldots \},
$$

where $\{\Psi_j : j = 1, 2, \ldots \}$ is defined in (10). This set is dense in $L^2(\Omega, \mathbb{P}; L^2(0,T; \mathbb{W}_s^*))$. Owing to [61], Proposition 21.23, the claim (93) is achieved if we prove that

$$
E \left( \phi(\omega) \int_0^T \langle B(\mathbf{y}^{m_j}(s), \mathbf{y}^{m_j}(s)) - B(\mathbf{y}(s), \mathbf{y}(s)), \Psi_j \rangle \chi(s) ds \right) \to 0,
$$

for any $\Phi = \phi(\omega)\chi(t)\Psi_j \in \mathbb{D}$. For this purpose, we rewrite the last identity in the following form

$$
E \left( \phi(\omega) \int_0^T \langle B(\mathbf{y}^{m_j}(s), \mathbf{y}^{m_j}(s)) - B(\mathbf{y}(s), \mathbf{y}(s)), \Psi_j \rangle \chi(s) ds \right) = I_1 + I_2.
$$
Throughout this section we suppose that there exist positive constants $\alpha(\cdot)$, $\beta(\cdot)$ satisfying
\[ 0 < \alpha(t) \leq M_\alpha e^{-\theta t}, \quad 0 < \beta(t) \leq M_\beta e^{-\theta t} \] (96)
and
\[ \langle f(y, t), y \rangle \leq \alpha(t) + (C_f + \beta(t))|y|^2, \] (97)
for any $t \in [0, \infty)$ and $y \in \mathbb{H}$.

We also assume that there exist positive constants $\zeta, M_\delta, M_\gamma$ and two positive functions $\gamma(\cdot), \delta(\cdot)$ such that
\[ \delta(t) \leq M_\delta e^{-\theta t}, \quad \gamma(t) \leq M_\gamma e^{-\theta t}, \] (98)
and
\[ ||g(y, t)||_{L_2}^2 \leq \gamma(t) + (\zeta + \delta(t))|y|^2, \]  
(99)
for any \( t \in [0, \infty) \) and \( y \in \mathbb{H} \).

We also suppose that
\[ 2\bar{\mu} < 2C_f + \zeta. \]  
(100)

**Theorem 4.1.** Under the conditions (99)-(100) any weak solution to (37)-(38) converges to zero almost surely exponentially.

**Proof.** Recall that
\[ y(t) = y_0 - \int_0^t \left[ A(y(s)) + B(y(s), y(s)) \right] ds + \int_0^t f(y(s), s) ds + \int_0^t g(y(s), s) dW, \]
from which and Itô’s formula we derive that
\[ |y(t)|^2 = |y_0|^2 - 2 \int_0^t \langle A(y(s)) - f(y(s), s), y(s) \rangle ds + \int_0^t ||g(y(s), s)||_{L_2}^2 ds \]
\[ + 2 \int_0^t (g(y(s), s), y(s)) dW. \]

Since \( 2\bar{\mu} < 2C_f + \zeta \) we can choose a constant \( a \in (0, \theta) \) such that \( 2\bar{\mu} < 2C_f + \zeta + a \). Hence, Itô’s formula implies
\[ e^{at}|y(t)|^2 = |y_0|^2 - 2 \int_0^t e^{as} \langle A(y(s)) - f(y(s), s), y(s) \rangle ds + \int_0^t e^{as} ||g(y(s), s)||_{L_2}^2 ds \]
\[ + a \int_0^t e^{as} |y(s)|^2 ds + 2 \int_0^t (g(y(s), s), y(s)) dW. \]

Since the mathematical expectation of the last term of the right hand side of this equation vanishes, then
\[ e^{at}E|y(t)|^2 = |y_0|^2 - 2 \int_0^t E(e^{as} \langle A(y(s)) - f(y(s), s), y(s) \rangle ds + \int_0^t e^{as}E||g(y(s), s)||_{L_2}^2 ds \]
\[ + a \int_0^t e^{as}E|y(s)|^2 ds \]  
(101)

Therefore, we can derive from (101) and (95) that
\[ e^{at}E|y(t)|^2 \leq \int_0^t (2M_\beta + 2M_\delta)e^{(a-\theta)s}|y(s)|^2 ds + (2C_f + \zeta + a - 2\bar{\mu}) \int_0^t e^{as}||y(s)||_{L_2}^2 ds \]
\[ + |y_0|^2 + \int_0^t (2M\alpha + 2M_\gamma)e^{(a-\theta)s} ds. \]  
(102)

By invoking Gronwall’s lemma we can infer the existence of \( M_0 = M_0(||y_0||^2) \) such that
\[ E|y(t)|^2 \leq M_0e^{-at}, \]  
(103)
for any \( t \geq 0 \).

Now let \( N \) be a positive integer. Itô’s formula yields
\[ |y(t)|^2 = |y(N)|^2 - 2 \int_N^t \langle A(y(s)) - f(y(s), s), y(s) \rangle ds + \int_N^t ||g(y(s), s)||_{L_2}^2 ds \]
\[ + 2 \int_N^t (g(y(s), s), y(s)) dW. \]
Owing to Burkholder-Davis-Gundy’s inequality we have

\[
E \sup_{N \leq t \leq N+1} \left| \int_N^t \left( g(y(s), s), y(s) \right) dW \right| \leq \eta_1 E \left( \int_N^{N+1} \| y(s) \|^2 \| g(y(s), s) \|^2_{L_2} ds \right)^{\frac{1}{2}},
\]

\[
\leq \eta_1 E \left( \sup_{N \leq t \leq N+1} |y(t)|^2 \int_N^{N+1} \| g(y(s), s) \|^2_{L_2} ds \right)^{\frac{1}{2}},
\]

\[
\leq \eta_2 \int_N^{N+1} E \| g(y(s), s) \|^2_{L_2} ds + \frac{1}{2} E \sup_{N \leq t \leq N+1} |y(t)|^2,
\]

where \( \eta_1, \eta_2 > 0 \). From this we deduce that

\[
E \sup_{N \leq t \leq N+1} |y(t)|^2 \leq 2 \int_N^{N+1} \left[ \alpha(s) + (C_f + \beta(s))E|y(s)|^2 \right] ds
\]

\[+ E|y(N)|^2 - 2\bar{\mu} \int_N^{N+1} E|y(s)|^2 ds \]

\[+ (1 + \eta_2) \int_N^{N+1} E \| g(y(s), s) \|^2_{L_2} ds. \tag{104} \]

The assumptions on \( f \) and \( g \) imply

\[
E \sup_{N \leq t \leq N+1} |y(t)|^2 \leq E|y(N)|^2 + (-2\bar{\mu} + 2C_f + \zeta) \int_N^{N+1} E|y(s)|^2 ds
\]

\[+ \int_N^{N+1} [2\alpha(s) + (1 + \eta_2)\gamma(s)] ds \]

\[+ \int_N^{N+1} [2\beta(s) + \eta_2(\zeta + \delta(s))]E|y(s)|^2 ds. \tag{105} \]

Thanks to (103) there exists a positive constant \( M_1 = M_1(|y_0|^2) \) such that

\[
E \sup_{N \leq t \leq N+1} |y(s)|^2 \leq M_1 e^{-aN}.
\]

Finally, the results follows from Borel-Cantelli’s lemma. \( \Box \)

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