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HARDY-SOBOLEV EQUATIONS ON COMPACT RIEMANNIAN MANIFOLDS

HASSAN JABER

Abstract. Let $(M, g)$ be a compact Riemannian Manifold of dimension $n \geq 3$, $x_0 \in M$, and $s \in (0, 2)$. We let $2^*(s) := \frac{2(n-s)}{n-2}$ be the critical Hardy-Sobolev exponent. We investigate the existence of positive distributional solutions $u \in C^0(M)$ to the critical equation

$$\Delta_g u + a(x)u = \frac{u^{2^*(s)-1}}{d_g(x, x_0)^s} \text{ in } M$$

where $\Delta_g := -\text{div}_g(\nabla)$ is the Laplace-Beltrami operator, and $d_g$ is the Riemannian distance on $(M, g)$. Via a minimization method in the spirit of Aubin, we prove existence in dimension $n \geq 4$ when the potential $a$ is sufficiently below the scalar curvature at $x_0$. In dimension $n = 3$, we use a global argument and we prove existence when the mass of the linear operator $\Delta_g + a$ is positive at $x_0$. As a byproduct of our analysis, we compute the best first constant for the related Riemannian Hardy-Sobolev inequality.

Let $(M, g)$ be a compact Riemannian Manifold of dimension $n \geq 3$ without boundary. Given $s \in (0, 2)$, $x_0 \in M$, and $a \in C^0(M)$, we consider distributional solutions $u \in C^0(M)$ to the equation

$$(1) \quad \Delta_g u + a(x)u = \frac{u^{2^*(s)-1}}{d_g(x, x_0)^s} \text{ in } M$$

where $2^*(s) := \frac{2(n-s)}{n-2}$ is the Hardy-Sobolev exponent. More precisely, let $H^2(M)$ be the completion of $C^\infty(M)$ for the norm $u \mapsto \|u\|_2 + \|\nabla u\|_2$. The exponent $2^*(s)$ is critical in the following sense: the Sobolev space $H^2(M)$ is continuously embedded in the weighted Lebesgue space $L^p(M, d_g(\cdot, x_0)^{−s})$ if and only if $1 \leq p \leq 2^*(s)$, and this embedding is compact if and only if $1 \leq p < 2^*(s)$.

There is an important literature on Hardy-Sobolev equations in the Euclidean setting of a domain of $\mathbb{R}^n$, in particular to show existence or non-existence of solutions, see for instance Ghoussoub-Kang [5], Ghoussoub-Yuan [7], Li-Ruf-Guo-Niu [15], Musina [16], Pucci-Servadei [17], Kang-Peng [12], and the references therein. In particular, in the spirit of Brezis-Nirenberg, Ghoussoub-Yuan [7] proved the existence of solutions for equations like $(1)$ when $n \geq 4$ and the potential $a$ achieves negative values at the interior singular point $x_0$. In the present manuscript, our objective is both to study the influence of the curvature when dealing with a Riemannian Manifold, and to tackle dimension $n = 3$.

We consider the functional

$$J(u) := \frac{\int_M (|\nabla u|^2_g + au^2) \, dv_g}{\left( \int_M \frac{|u|^{2^*(s)}}{d_g(x, x_0)^s} \, dv_g \right)^{1/2}} \quad ; \quad u \in H^2(M) \setminus \{0\},$$
which is well-defined due to the above-mentioned embeddings. Here $dv_g$ denotes the Riemannian element of volume. When the operator $\Delta_g + a$ is coercive, then, up to multiplication by a positive constant, critical points of the functional $J$ (if they exist) are solutions to equation (1). In the sequel, we assume that $\Delta_g + a$ is coercive. In the spirit of Aubin [1], we investigate the existence of solutions to (1) by minimizing the functional $J$: it is classical for this type of problem that the difficulty is the lack of compactness for the critical embedding. Since the resolution of the Yamabe problem (see [1], [19] and [24]), it is also well known that there exists a dichotomy between high dimension (see Aubin [1]) where the arguments are local, and small dimension (see Schoen [19]) where the arguments are global.

In the sequel, we let $\text{Scal}_g(x)$ be the scalar curvature at $x \in M$. We let $G_{x_0} : M \setminus \{x_0\} \to \mathbb{R}$ be the Green’s function at $x_0$ for the operator $\Delta_g + a$ (this is defined since the operator is coercive). In dimension $n = 3$, there exists $m(x_0) \in \mathbb{R}$ such that for all $\alpha \in (0, 1)$

$$G_{x_0}(x) = \frac{1}{\omega_{2d}_g(x, x_0)} + m(x_0) + O(d_g(x, x_0)^\alpha)$$

when $x \to x_0$.

Here and in the sequel, $\omega_k$ denote the volume of the canonical $k$-dimensional unit sphere $\mathbb{S}^k$, $k \geq 1$. The quantity $m(x_0)$ is refered to as the mass of the point $x_0 \in M$. Our main result states as follows:

**Theorem 1.** Let $x_0 \in M$, $s \in (0, 2)$, and $a \in C^0(M)$ be such that the operator $\Delta_g + a$ is coercive. We assume that

$$\left\{ \begin{array}{ll}
  a(x_0) < c_{n,s}\text{Scal}_g(x_0) & \text{if } n \geq 4 \\
  m(x_0) > 0 & \text{if } n = 3.
\end{array} \right.$$  

(2)

with $c_{n,s} := \frac{(n-2)(6-s)}{2(2n-2-s)}$. Then there exists a positive solution $u \in C^0(M) \cap H^2(M)$ to the Hardy-Sobolev equation (1). Moreover, $u \in C^{\theta, \theta}(M)$ for all $\theta \in (0, \min\{1, 2-s\})$ and we can choose $u$ as a minimizer of $J$.

As a consequence of the Positive Mass Theorem (see [20], [21]), we get (see Druet [3] and Proposition 2 in Section 4 below) that $m(x_0) > 0$ for $n = 3$ when $a \leq \text{Scal}_g/8$, with the additional assumption that $(M, g)$ is not conformally equivalent to the canonical $3$-sphere if $a \equiv \text{Scal}_g/8$.

Theorem 1 suggests some remarks. For equations of scalar curvature type, that is when $s = 0$, a similar result was obtained by Aubin [1] (for $n \geq 4$) and by Schoen [19] (see also Druet [3]) (for $n = 3$): however, when $s \in (0, 2)$, the problem is subcritical outside the singular point $x_0$, and therefore it is natural to get a condition at this point. Another remark is that, when $s = 0$, Aubin (see [1]) obtained the constant $c_{n,0}$ when $n \geq 4$, the potential $c_{n,0}\text{Scal}_g$ being such that the Yamabe equation is conformally invariant. When $s \in (0, 2)$, the critical equation enjoys no suitable conformal invariance due to the singular term $d_g(\cdot, x_0)^{n-s}$, and, despite our existence result involves the scalar curvature, one gets another constant $c_{n,s}$.

It is also to notice that, unlike the case $s = 0$, the solutions to equations like (1) are not $C^2$. This lead us to handle with care issues related to the maximum principle, for which we develop a suitable approach. As in Aubin, the minimization approach leads to computing some test-function estimates. However, unlike the case $s = 0$,
the terms involved in the expansion of the functional are not explicit and we need to collect them suitably to obtain the explicit value of $c_{n,s}$ above. The proof of Theorem 1 uses the best constant in the Hardy-Sobolev inequality. It follows from the Hardy-Sobolev embedding that there exist $A, B > 0$ such that

\begin{equation}
(3) \quad \left( \int_M \frac{|u|^2^*(s)}{d_g(x,x_0)^s} \, dv_g \right)^{\frac{2}{2+s}} \leq A \int_M |\nabla u|^2 \, dv_g + B \int_M u^2 \, dv_g
\end{equation}

for all $u \in H^2(M)$. We let $A_0(M, g, s, x_0)$ be the best first constant of the Riemannian Hardy-Sobolev inequality, that is

\begin{equation}
(4) \quad A_0(M, g, s, x_0) := \inf\{ A > 0 ; (3) \text{ holds for all } u \in H^2(M) \}.
\end{equation}

We prove the following:

**Theorem 2.** Let $(M, g)$ be a compact Riemannian Manifold of dimension $n \geq 3$, $x_0 \in M$, $s \in (0, 2)$ and $2^*(s) = \frac{2(n-s)}{n-2}$. Then

\[ A_0(M, g, s, x_0) = K(n, s), \]

where $K(n, s)$ is the optimal constant of the Euclidean Hardy-Sobolev inequality, that is

\begin{equation}
(5) \quad K(n, s)^{-1} := \inf_{\varphi \in C_c^\infty(\mathbb{R}^n) \setminus \{ 0 \}} \frac{\int_{\mathbb{R}^n} |\nabla \varphi|^2 \, dX}{\left( \int_{\mathbb{R}^n} \frac{|\varphi^{2^*(s)}|}{|X|} \, dX \right)^{\frac{2}{2+s}}}.
\end{equation}

Theorem 2 was proved by Aubin [2] for the case $s = 0$. The value of $K(n, s)$ is

\[ K(n, s) = \left( \frac{n-2}{n-s} \right)^{\frac{2-n}{2-s}} \left( \frac{1}{\Gamma(2(n-s)/2-s)} \right)^{\frac{2-n}{2-s}}. \]

It was computed independently by Aubin [2], Rodemich [18] and Talenti [22] for the case $s = 0$, and the value for $s \in (0, 2)$ has been computed by Lieb (see [14], Theorem 4.3).

A natural question is to know whether the infimum $A_0(M, g, s, x_0)$ is achieved or not, that is if there exists $B > 0$ such that equality (3) holds for all $u \in H^2(M)$ with $A = K(n, s)$. The answer is positive: this is the object of the work [11].

A very last remark is that Theorem 1 holds when $M$ is a compact manifold with boundary provided $x_0$ lies in the interior. In particular, we extend Ghoussoub-Yuan’s [7] result to dimension $n = 3$:

**Theorem 3.** Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^3$ and let $x_0 \in \Omega$ be an interior point. For $a \in C^0(\overline{\Omega})$ such that $\Delta + a$ is coercive, we define the Robin function as $R(x,y) := \omega^{-1}_x |x-y|^{-1} - G_x(y)$ where $G$ is the Green’s function for $\Delta + a$ with Dirichlet boundary condition. We assume that $R(x_0, x_0) < 0$. Then there exists a function $u \in C^{0,\theta}(\overline{\Omega})$ for all $\theta \in (0, \min\{1, 2-s\})$ to the Hardy-Sobolev equation

\[ \Delta u + a(x)u = \frac{u^{2^*(s)-1}}{|x-x_0|^s}, \quad u > 0 \quad \text{in } \Omega \text{ and } u = 0 \text{ on } \partial \Omega. \]

This paper is organized as follows. In Section 1, we prove Theorem 2. In Section 2, we prove a general existence theorem for solutions to equation (1). In Section 3, we compute the full expansion of the functional $J$ taken at the relevant test-functions.
for dimension $n \geq 4$. In Section 4, we perform the test-functions estimate for the specific dimension $n = 3$ and prove Theorems 1 and 3.

After this work was completed, we learned that Thiam [23] has independently studied similar issues.

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1. **The best constant in the Hardy-Sobolev inequality**

In this section, we will prove Theorem 2. For that, we begin by the following proposition:

**Proposition 1.** Let $(M, g)$ be a compact Riemannian Manifold of dimension $n \geq 3$, $x_0 \in M$, $s \in (0, 2)$. For any $\epsilon > 0$, there exists $B_\epsilon > 0$ such that

$$\left(\int_M \frac{|u|^{2^*(s)}(s)}{d_g(x, x_0)^s} dv_g\right)^{\frac{1}{2^*}} \leq (K(n, s) + \epsilon) \int_M |\nabla u|^2_g dv_g + B_\epsilon \int_M u^2 dv_g,$$  

for all $u \in H^{2^*_s}(M)$.

Thiam [23] proved a result in the same spirit with addition of an extra remainder term. The case $s = 0$ has been proved by Aubin [2] (see also [9], [10] for an exposition in book form). We adapt this proof to our case.

**Proof.** **Step 1:** Covering of $M$ by geodesic balls. For any $x \in M$, we denote as $\exp_x$ the exponential map at $x$ with respect to the metric $g$. In the sequel, for any $r > 0$ and $z \in M$, $B_r(z) \subset M$ denotes the ball of center 0 and of radius $r$ for the Riemannian distance $d_g$. For any $x \in M$ and any $\rho > 0$, there exist $r = r(x, \rho) \in (0, i_g(M)/2)$, $\lim_{\rho \to 0} r(x, \rho) = 0$ (here, $i_g(M)$ denotes the injectivity radius of $(M, g)$) such that the exponential chart $(B_{2r}(x), \exp_x^{-1})$ satisfies the following properties: on $B_{2r}(x)$, we have that

$$(1 - \rho)\delta \leq g \leq (1 + \rho)\delta,$$

$$(1 - \rho)\tilde{\mathbb{E}} dx \leq dv_g \leq (1 + \rho)\tilde{\mathbb{E}} dx,$$

$$D^\rho_p[T]\delta \leq |T|_g \leq D^\rho_p|T|_g, \text{ for all } T \in \chi(T^*M)$$

where $\lim_{\rho \to +\infty} D^\rho_p = 1$. $\chi(T^*M)$ denotes the space of 1-covariant tensor fields on $M$, $\delta$ is the Euclidean metric on $\mathbb{R}^n$, that is the standard scalar product on $\mathbb{R}^n$, and we have assimilated $g$ to the local metric $(\exp_x)^*g$ on $\mathbb{R}^n$ via the exponential map.

It follows from the compactness of $M$ that there exists $N \in \mathbb{N}$ (depending on $\rho$) and $x_1, \ldots, x_{N-1} \in M \setminus \mathbb{B}_{2r}(x_0)$ (depending on $\rho$) such that

$$M \setminus \mathbb{B}_{r_0}(x_0) \subset \bigcup_{m=1}^{N-1} \mathbb{B}_{r_m}(x_m),$$

where $r_0 = r(x_0, \rho)$ and $r_m = r(x_m, \rho)$, $r_m < r_0^2$, for $m = 0, \ldots, N - 1$.

**Step 2:** We claim that for all $\epsilon > 0$ there exists $\rho_0 = \rho_0(\epsilon) > 0$ such that $\lim_{\rho \to 0} \rho_0(\epsilon) = 0$ and for all $\rho \in (0, \rho_0)$, all $m \in \{0, \ldots, N - 1\}$ and all $u \in C^\infty_0(\mathbb{B}_{r_m}(x_m))$, we have that:

$$\left(\int_M \frac{|u|^{2^*(s)}(s)}{d_g(x, x_0)^s} dv_g\right)^{\frac{1}{2^*}} \leq (K(n, s) + \frac{\epsilon}{2}) \int_M |\nabla u|^2_g dv_g.$$
Indeed, it follows from (5) that for all \( \varphi \in C^\infty_c(\mathbb{R}^n) \):

\[
(8) \quad \left( \int_{\mathbb{R}^n} \left| \varphi \right|^{2^*(s)} \left| X \right|^{-\frac{2^*(s)}{2}} dX \right)^{\frac{1}{2^*(s)}} \leq K(n, s) \int_{\mathbb{R}^n} |\nabla \varphi|^2 dX.
\]

We consider \( \rho > 0, m \in \{0, \ldots, N\} \) and \( u \in C^\infty_c(B_{r_m}(x_m)) \) such that \( (B_{r_m}(x_m), \exp^{-1}) \) is an exponential card as in Step 1. We distinguish two cases:

**Case 2.1**: If \( m = 0 \) then using the properties of the exponential card \((B_{r_0}(x_0), \exp^{-1})\), developed in Step 1, and the Euclidean Hardy-Sobolev inequality (8), we write

\[
\left( \int_M \frac{|u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{1}{2^*(s)}} \leq (1 + \rho)^{\frac{1}{2^*(s)}} K(n, s) \int_{\mathbb{R}^n} |\nabla (u \circ \exp_{x_m})|^2 dX
\]

\[
\leq D^2_2(1 + \rho)^{\frac{1}{2^*(s)}} (1 - \rho)^{-\frac{s}{2}} K(n, s) \int_M |\nabla u|_g^2 dv_g.
\]

Letting \( \rho \to 0 \), we get (7), for all \( u \in C^\infty_c(B_{r_0}(x_0)) \), when \( m = 0 \). This proves (7) in the Case 2.1.

**Case 2.2**: If \( m \in \{1, \ldots, N - 1\} \) then for all \( x \in B_{r_m}(x_m) \), we have:

\[
d_g(x, x_0) \geq \lambda_0 > 0,
\]

with \( \lambda_0 = \frac{2^*}{2} - r_m \). Thanks to Hölder’s inequalities and inequalities of Gagliardo-Nirenberg-Sobolev, we can write that:

\[
\left( \int_M \frac{|u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{1}{2^*(s)}} \leq \frac{\text{vol}(B_{r_m}(x_m))^{2^*(s)}}{\lambda_0^{\frac{2^*(s)}{2}}} \left( \int_{B_{r_m}(x_m)} |u|^{2^*(s)} dv_g \right)^{\frac{1}{2^*(s)}}
\]

\[
\leq Q'_\rho \int_M |\nabla u|_g^2 dv_g,
\]

where \( \lim_{\rho \to 0} Q'_\rho = 0 \) and \( 2^* := \frac{2n}{n - 2} \) is the Sobolev exponent. Letting \( \rho \to 0 \), we get (7), for all \( u \in C^\infty_c(B_{r_0}(x_m)) \), when \( m \geq 1 \). This ends Step 2.

**Step 3**: We fix \( \epsilon > 0, \rho \in (0, \rho_0(\epsilon)) \) and \( x_1, \ldots, x_N \) as in Step 1 and Step 2. We consider now \((\alpha_m)_{m=0, \ldots, N-1}\) a \( C^\infty\)-partition of unity subordinate to the covering \((B_{r_m}(x_m))_{m=0, \ldots, N-1}\) of \( M \) and define, for all \( m = 0, \ldots, N - 1 \), a function \( \eta_m \) on \( M \) by

\[
\eta_m = \frac{\alpha_m^3}{\sum_{i=0}^{N-1} \alpha_i^3}.
\]

We can see easily that \((\eta_m)_{m=0, \ldots, N-1}\) is a \( C^\infty\)-partition of unity subordinate to the covering \((B_{r_m}(x_m))_{m=0, \ldots, N-1}\) of \( M \) s.t. \( \eta_m^\infty \in C^1(M) \), for every \( m = 0, \ldots, N - 1 \). We let \( H > 0 \) satisfying for each \( m = 0, \ldots, N - 1 \):

\[
|\nabla \eta_m^\infty|_g \leq H.
\]

**Step 4**: In this step, we will prove the Hardy-Sobolev inequality on \( C^\infty(M) \). Indeed, we let \( \epsilon > 0 \) and \((\eta_m)_{m=0, \ldots, N-1}\) be a \( C^\infty\)-partition of unity as in Step 3.
and consider \( u \in C^\infty(M) \). Since \( \frac{2^{(\alpha)}(s)}{2} > 1 \), we get that:

\[
\left( \int_M \frac{|u|^{2^{(s)}}}{d_g(x, x_0)^s} dv_g \right)^{\frac{s}{2^{(s)}}} \leq \left( \int_M \left| \sum_{m=0}^{N-1} \eta_m u^{2^{(s)}} \right| d_g(x, x_0)^s \right)^{\frac{s}{2^{(s)}}},
\]

\[
\leq \left\| \sum_{m=0}^{N-1} \eta_m u^2 \right\|_{L^{\frac{2^{(s)}}{s}}(M, d_g(x, x_0)^{-s})} \leq \sum_{m=0}^{N-1} \left\| \eta_m u^2 \right\|_{L^{\frac{2^{(s)}}{s}}(M, d_g(x, x_0)^{-s})}
\]

\[
\leq \sum_{m=0}^{N-1} \left( \int_M \frac{|\eta_m u^{2^{(s)}}}{d_g(x, x_0)^s} dv_g \right)^{\frac{s}{2^{(s)}}}.
\]

Using inequality (7) in Step 2 and by density \( (\eta_m u \in C^1(M)) \), we get that

\[
\left( \int_M \frac{|\eta_m u^{2^{(s)}}}{d_g(x, x_0)^s} dv_g \right)^{\frac{s}{2^{(s)}}} \leq (K(n, s) + \epsilon) \int_M |\nabla (\eta_m u)|^2 dv_g.
\]

Hence

\[
\left( \int_M \frac{|u|^{2^{(s)}}}{d_g(x, x_0)^s} dv_g \right)^{\frac{s}{2^{(s)}}} \leq (K(n, s) + \epsilon) \int_M \left( |\nabla u|^2 + 2\eta_m^2 |\nabla u| |\nabla \eta_m| + |u|^2 |\nabla \eta_m|^2 \right) dv_g.
\]

Using the Cauchy-Schwarz inequality and (9) from Step 3, we get that:

\[
\left( \int_M \frac{|u|^{2^{(s)}}}{d_g(x, x_0)^s} dv_g \right)^{\frac{s}{2^{(s)}}} \leq (K(n, s) + \epsilon) \int_M \left( |\nabla u|^2 + 2NH |\nabla u|_2 |u|_2 + NH^2 |u|_2^2 \right) dv_g.
\]

We choose now \( \epsilon_0 > 0 \) s.t.

\[
(K(n, s) + \epsilon) (1 + \epsilon_0) \leq K(n, s) + \epsilon.
\]

Since

\[
2NH |\nabla u|_2 |u|_2 \leq \epsilon_0 |\nabla u|_2^2 + \frac{(NH)^2}{\epsilon_0} |u|_2^2,
\]

then by combining (10) with (11) and (12), we get that:

\[
\left( \int_M \frac{|u|^{2^{(s)}}}{d_g(x, x_0)^s} dv_g \right)^{\frac{s}{2^{(s)}}} \leq (K(n, s) + \epsilon) \int_M |\nabla u|^2 dv_g + B \epsilon \int_M |u|^2 dv_g,
\]

where \( B \epsilon = \frac{(NH)^2}{\epsilon_0} + NH^2 (K(n, s) + \frac{\epsilon}{2}) \). This proves inequality (6) for functions \( u \in C^\infty(M) \). The inequality for \( H^2_\gamma(M) \) follows by density. This ends the proof of Proposition 1.

\[\Box\]

**Proof of Theorem 2:** We let \( A \in \mathbb{R} \) be such that there exists \( B > 0 \) such that inequality (3) holds for all \( u \in H^2_\gamma(M) \). Therefore, we have that

\[
\left( \int_M \frac{|u|^{2^{(s)}}}{d_g(x, x_0)^s} dv_g \right)^{\frac{s}{2^{(s)}}} \leq A \int_M |\nabla u|^2 dv_g + B \int_M u^2 dv_g.
\]
We consider $\phi \in C^\infty_c(\mathbb{R}^n)$ such that $\text{Supp } \phi \subset B_R(0)$, $R > 0$ and $(B_{p_0}(x_0), \exp_{x_0}^{-1})$ an exponential chart centered at $x_0$ with $p_0 \in (0, \delta_g(M))$. For all $\mu > 0$ sufficiently small ($\mu \leq \frac{\rho}{2R}$), we let $\phi_\mu \in C^\infty(B_{p_0}(x_0))$ be such that
\[
\phi_\mu(x) = \phi(\mu^{-1} \exp_{x_0}^{-1}(x))
\]
for all $x \in B_{p_0}(x_0)$. Applying, by density, (13) to $\phi_\mu$, we write:
\[
(14) \quad \left( \int_{B_{R}(x_0)} \frac{|\phi_\mu|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^\frac{1}{2^*} \leq A \int_{B_{R}(0)} |\nabla \phi_\mu|^2 dv_g + B \int_{B_{R}(0)} \phi_\mu^2 dv_g.
\]
For all $\epsilon > 0$, there exists $R_\epsilon > 0$ such that
\[
(1 - \epsilon)\delta \leq g \leq (1 + \epsilon)\delta
\]
in $B_{R_\epsilon}(x_0)$, where $g$ is assimilated to the local metric $(\exp_{x_0})^*g$ on $\mathbb{R}^n$. Then, for all $\mu > 0$ sufficiently small such that $R_\mu < R_\epsilon$, we get successively that :
\[
(15) \quad \int_{B_{R}(x_0)} \frac{|\phi_\mu|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \geq (1 - \epsilon)^{2}\mu^{-s} \int_{B_R(0)} \frac{\phi^{2^*(s)}(X)}{|X|^s} dX,
\]
\[
(16) \quad \int_{B_{R}(x_0)} |\nabla \phi_\mu|^2 dv_g \leq (1 + \epsilon)^{2}\mu^{s-2} \int_{B_R(0)} |\nabla \phi|^2 dX
\]
and
\[
(17) \quad \int_{B_{R}(x_0)} \phi_\mu^2 dv_g \leq (1 + \epsilon)^{2}\mu^{n} \int_{B_R(0)} \phi^2 dX.
\]
Plugging the estimates (15), (16) and (17) into (14), letting $\mu \to 0$ and then $\epsilon \to 0$, we get that
\[
(18) \quad \left( \int_{\mathbb{R}^n} \frac{\phi^{2^*(s)}(X)}{|X|^s} dX \right)^\frac{1}{2^*} \leq A \int_{\mathbb{R}^n} |\nabla \phi|^2 dX, \quad \text{for all } \phi \in C^\infty_c(\mathbb{R}^n).
\]
It then follows from the definition of $K(n, s)$ that $A \geq K(n, s)$. Therefore, it follows from the definition of $A_0(M, g, s, x_0)$ that $A_0(M, g, s, x_0) \geq K(n, s)$. By Proposition 1, we have that $A_0(M, g, s, x_0) \leq K(n, s)$. Therefore, $A_0(M, g, s, x_0) = K(n, s)$. This proves Theorem 2.

**Remark:** Proposition 1 does not allow to conclude whether $A_0(M, g, s, x_0)$ is achieved or not, that is of one can take $\epsilon = 0$ in (6). Indeed, in our construction, when $\epsilon \to 0$, $r_m \to 0$ and then $H \geq |\nabla \eta|^2 \to +\infty$ (see the proof of Proposition 1). This implies that $\lim_{\epsilon \to 0} B_\epsilon = +\infty$. Proving that $A_0(M, g, s, x_0)$ is achieved required different techniques and blow-up analysis: this is the object of the article [11].

2. A General Existence Theorem

This section is devoted to the proof of the following Theorem:

**Theorem 4.** Let $(M, g)$ be a compact Riemannian Manifold of dimension $n \geq 3$ without boundary. We fix $s \in (0, 2)$, $x_0 \in M$, and $a \in C^0(M)$ such that $\Delta_g + a$ is coercive. We assume that
\[
(18) \quad \inf_{u \in H^1_0(M) \setminus \{0\}} J(u) < \frac{1}{K(n, s)}
\]
Then the infimum of $J$ on $H^2_q(M) \setminus \{0\}$ is achieved by a positive function $u \in H^2_q(M) \cap C^0(M)$. Moreover, up to homothety, $u$ is a solution to (1) and $u \in C^{0,\theta}(M) \cap C^{1,\alpha}_{\text{loc}}(M \setminus \{x_0\})$ for all $\theta \in (0, \min\{1, 2 - s\})$ and $\alpha \in (0,1)$.

The existence of a minimizer of $J$ in $H^2_q(M) \setminus \{0\}$ has been proved independently by Thiam [23].

We prove Theorem 4 via the classical subcritical approach. For any $q \in (2, 2^*(s)]$, we define

$$J_q(u) := \frac{\int_M (|\nabla u|^2 + au^2) \, dv_g}{\left( \int_M \frac{|u|^q}{d_g(x,x_0)} \, dv_g \right)^{\frac{2}{q}}} ; \ u \in H^2_q(M),$$

and

$$H_q = \left\{ u \in H^2_q(M) ; \int_M d_g(x,x_0)^s \, dv_g = 1 \right\}.$$ 

Finally, we define:

$$\lambda_q = \inf_{u \in H^2_q(M) \setminus \{0\}} J_q(u).$$

We fix $q \in (2, 2^*(s))$. Since the embedding $H^2_q(M) \hookrightarrow L^q(M, d_g(\cdot, x_0)^{-s})$ is compact, there exists a minimizer for $\lambda_q^{\ominus}$. More precisely, there exists $u_q \in H^2_q(M) \setminus \{0\} \cap H_q$, $u_q \geq 0$ a.e. such that $u_q$ verifies weakly the subcritical Hardy-Sobolev equation:

$$\Delta_g u_q + au_q = \lambda_q^{\ominus} \frac{u_q^{q-1}}{d_g(x,x_0)^s} \ \text{in} \ M.$$ 

In particular, we have that $\lambda_q = J_q(u_q)$.

Now we proceed in several steps.

**Step 1:** We claim that the sequence $(\lambda_q)_q$ converge to $\lambda_{2^*(s)}$ when $q \to 2^*(s)$.

The proof follows the standard method described in [25] and [1] for instance. We omit the proof.

**Step 2:** As one checks, the sequence $(u_q)_q$ is bounded in $H^2_q(M)$ independently of $q$.

Therefore, there exists $u \in H^2_q(M)$, $u \geq 0$ a.e. such that, up to a subsequence, $(u_q)_q$ converge to $u$ weakly in $H^2_q(M)$ and strongly in $L^2(M)$, moreover, the convergence holds a.e. in $M$. It is classical (see [25] and [1]) that $u \in H^2_q(M)$ is a weak solution to

$$\Delta_g u + au = \lambda_{2^*(s)} \frac{u_{2^*(s)-1}}{d_g(x,x_0)^s} \ \text{in} \ M ; \ u \geq 0 \ \text{a.e. in} \ M.$$ 

**Step 3:** We claim that $u \neq 0$ is a minimizer of $J(s)$ and that $(u_q)_q \to u$ strongly in $H^2_q(M)$.

Indeed, it follows from the hypothesis (18) that there exists $\epsilon_0 > 0$ such that

$$\lambda_{2^*(s)}(K(n,s) + \epsilon_0) < 1.$$ 

Now from Proposition 1, we know that there exists $B_{\epsilon_0} > 0$ such that for all $q \in (2, 2^*(s))$:

$$\left( \int_M \frac{|u_q|^{2^*(s)}}{d_g(x,x_0)^s} \, dv_g \right)^{\frac{2}{2^*(s)}} \leq (K(n,s) + \epsilon_0) \int_M |\nabla u_q|^2 \, dv_g + B_{\epsilon_0} \int_M u_q^2 \, dv_g.$$ 

\begin{align}
(20) \quad \int_M \frac{|u_q|^{2^*(s)}}{d_g(x,x_0)^s} \, dv_g & \leq (K(n,s) + \epsilon_0) \int_M |\nabla u_q|^2 \, dv_g + B_{\epsilon_0} \int_M u_q^2 \, dv_g. 
\end{align}
Hölder inequality and $u_q \in H_q$ yield:

$$
\left( \int_M \frac{|u|^2(s)}{d_g(x, x_0)^s} dv_g \right)^{\frac{1}{2}} \geq 1 + o(1)
$$

where $o(1) \to 0$ when $q \to 2^*(s)$. Combining (20) and (21), we get:

$$
\left[ (K(n, s) + \epsilon_0) \lambda_q + B_\epsilon \int_M u^2 dv_g \right] \geq 1 + o(1),
$$

where $o(1) \to 0$ when $q \to 2^*(s)$. Letting $q \to 2^*(s)$ in the last relation, we write:

$$
(K(n, s) + \epsilon_0) \lambda_{2^*(s)} + B(\epsilon_0) \int_M u^2 dv_g \geq 1.
$$

It then follows from (19) that $B_\epsilon \int_M u^2 dv_g > 0$, and then $u \not\equiv 0$. It is then classical that $u \in H^2_1(M)$ is a minimizer and that $u_q \to u$ strongly in $H^2_1(M)$ when $q \to 2^*(s)$.

**Step 4:** We claim that $u \in C^{0, \theta}(M)$, for all $\theta \in (0, \min\{1, 2 - s\})$. Following the method used in [6] (see Proposition 8.1) inspired by the strategy developed by Trudinger [24] for the Yamabe problem, we get that $u \in L^p(M)$, for all $p \geq 1$. Defining $f_\alpha(x) := \frac{u(x)^{2^*(s) - 1}}{d(x, x_0)}$, we then get from Hölder inequality that $f_\alpha \in L^p(M)$, for all $p \in [1, \frac{n}{s})$. Since $\Delta_g u + au = f_\alpha$ and $u \in H^2_1(M)$ and $s \in (0, 2)$, it follows from standard elliptic theory (see [8]) that $u \in C^{0, \theta}(M)$, for all $\theta \in (0, \min\{1, 2 - s\})$.

**Step 5:** We claim that $u \in C^{1, \alpha}_{loc}(M \setminus \{x_0\})$, for all $\alpha \in (0, 1)$. Indeed, since $u \in L^p(M)$ for all $p > 1$ (see Step 4), we get that $f_\alpha \in L^p_{loc}(M \setminus \{x_0\})$ for all $p > 1$. Since $\Delta_g u + au = f_\alpha$ and $u \in H^2_1(M)$, then, up to taking $p > n$ sufficiently large, it follows from standard elliptic theory (see [8]) that $u \in C^{1, \alpha}_{loc}(M \setminus \{x_0\})$ for all $\alpha \in (0, 1)$.

**Remark:** If $a \in C^{0, \gamma}(M)$ for some $\gamma \in (0, 1)$ then, using the same argument as above, we get that $u \in C^{2, \gamma}_{loc}(M \setminus \{x_0\})$.

**Step 6:** We claim that $u > 0$ on $M$. Indeed, we consider $x_1 \neq x_0$ such that $B_{2r}(x_1) \subset M \setminus \{x_0\}$, with $r > 0$ sufficiently small and a function $h$ defined on $\mathbb{B}_{2r}(x_1)$ by $h(x) := a(x) - \lambda_{2^*(s)} \frac{|u(x)|^{2^*(s) - 2}}{d(x, x_0)}$. Clearly, we have that $h \in C^0(\overline{\mathbb{B}_{2r}(x_1)})$. Since $u \in H^2_1(\mathbb{B}_{2r}(x_1))$, $u \geq 0$ and $(\Delta_g + h) u = 0$ on $\mathbb{B}_{2r}(x_1)$. It then follows from standard elliptic theory (see [8], Theorem 8.20) that there exists $C = C(M, g, x_1, r) > 0$ such that $\sup_{\mathbb{B}_{r}(x_1)} u \leq C \inf_{\mathbb{B}_{r}(x_1)} u$. This implies that $u_{\mathbb{B}_{r}(x_1)} > 0$. Therefore, $u(x) > 0$ for all $x \in M \setminus \{x_0\}$.

We are left with proving that $u(x_0) > 0$. We argue by contradiction and we assume that $u(x_0) = 0$.

**Step 6.1:** We claim that $u$ is differentiable at $x_0$. Here again, we follow the method used in [6] (see Proposition 8.1). Since $u \in C^{0, \alpha}(M)$, for all $\alpha \in (0, \min\{1, 2 - s\})$ (from Step 4) and $u(x_0) = 0$ then for any $\alpha \in (0, \min\{1, 2 - s\})$, there exists a constant $C_1(\alpha) = C(M, g, \alpha) > 0$ such that

$$
|u(x)| \leq C_1(\alpha) d_g(x, x_0)^{\alpha}
$$

for all $x \in M$. Therefore, we have that

$$
\Delta_g u + au = f_u,
$$

where $f_u$ is the right-hand side of the equation.

We then have

$$
\int_M \frac{|u(x)|^{2^*(s)} - \frac{2^*(s) - 1}{2} |u(x)|^{2^*(s) - 2} |\nabla_g u(x)|^2}{d_g(x, x_0)^s} dv_g \geq 0.
$$

Since $u(x_0) = 0$, we have

$$
\int_M \frac{|u(x)|^{2^*(s)} - \frac{2^*(s) - 1}{2} |u(x)|^{2^*(s) - 2} |\nabla_g u(x)|^2}{d_g(x, x_0)^s} dv_g \geq 0.
$$

This implies that $\Delta_g u + au = f_u$, where $f_u$ is the right-hand side of the equation.

We then have

$$
\int_M \frac{|u(x)|^{2^*(s)} - \frac{2^*(s) - 1}{2} |u(x)|^{2^*(s) - 2} |\nabla_g u(x)|^2}{d_g(x, x_0)^s} dv_g \geq 0.
$$

Since $u(x_0) = 0$, we have

$$
\int_M \frac{|u(x)|^{2^*(s)} - \frac{2^*(s) - 1}{2} |u(x)|^{2^*(s) - 2} |\nabla_g u(x)|^2}{d_g(x, x_0)^s} dv_g \geq 0.
$$

This implies that $\Delta_g u + au = f_u$, where $f_u$ is the right-hand side of the equation.
where with (22), we have that

\begin{equation}
|f_u(x)| \leq \frac{C_2(\alpha)}{d_p(x, x_0)^{s-\alpha(2^*(s)-1)}}
\end{equation}

for all \( x \in M \setminus \{x_0\} \).

We claim that \( u \in C^{0,\alpha}(M) \), for all \( \alpha \in (0,1) \).

Indeed, we define \( \alpha_1 := \sup\{\alpha \in (0,1) : u \in C^{0,\alpha}(M)\} \) and \( N'_s = s - \alpha_1(2^*(s) - 1) \) and distinguish the following cases:

- **Case 6.1.1** \( N'_s \leq 0 \). In this case, up to taking \( \alpha \) close enough to \( \alpha_1 \), we get that \( f_u \in L^p(M) \), for all \( p \geq 1 \). It follows from (23) and standard elliptic theory that there exists \( \theta \in (0,1) \) such that \( u \in C^{1,\theta}(M) \). This proves that \( \alpha_1 = 1 \) in Case 6.1.1.

- **Case 6.1.2** \( 0 < N'_s < 1 \). In this case, up to taking \( \alpha \) close enough to \( \alpha_1 \), we get that \( f_u \in L^p(M) \), for all \( p < \frac{N'_s}{\alpha_1} \). Since \( 1 > N'_s \) then there exists \( p \in (n, \frac{n}{N'_s}) \), such that \( f_u \in L^p(M) \). Therefore, (23) and standard elliptic theory yield the existence of \( \theta \in (0,1) \) such that \( u \in C^{1,\theta}(M) \). This proves that \( \alpha_1 = 1 \) in Case 6.1.2.

- **Case 6.1.3** \( N'_s = 1 \). In this case, up to taking \( \alpha \) close enough to \( \alpha_1 \), we get that \( f_u \in L^p(M) \), for all \( p < n \). This implies that for any \( p \in (\frac{n}{2}, n) \), we have that \( f_u \in L^p(M) \). Equation (23) and standard elliptic theory then yields \( u \in C^{0,\theta}(M) \) for all \( \theta \in (0,1) \). This proves that \( \alpha_1 = 1 \) in Case 6.1.3.

- **Case 6.1.4** \( N'_s > 1 \). In this case, up to taking \( \alpha \) close enough to \( \alpha_1 \), we get that \( f_u \in L^p(M) \), for all \( p < \frac{n}{N'_s} \). Therefore, (23), \( N'_s \in (1,2) \) (because \( N'_s > 0 \) and \( s < 2 \)), and standard elliptic theory yield \( u \in C^{0,\theta}(M) \) for all \( \theta < 2 - N'_s \). It then follows from the definition of \( \alpha_1 \) that \( \alpha_1 \geq 2 - N'_s \). This leads to a contradiction with the definition of \( N'_s \). Then Case 6.1.4 does not occur.

These four cases imply that \( u \in C^{0,\alpha}(M) \), for all \( \alpha \in (0,1) \). This proves the claim.

In order to end Step 6.1, we proceed as the above, let \( N''_s = s - 2^*(s) + 1 \) and distinguish two cases:

- **Case 6.1.5** \( N''_s \leq 0 \). In this case, up to taking \( \alpha \) close enough to 1, we have that \( f_u \in L^p(M) \), for all \( p \geq 1 \). Therefore, (23) and elliptic theory yield \( u \in C^1(M) \). This proves Step 6.1 in Case 6.1.5.

- **Case 6.1.6** \( N''_s > 0 \). In this case, up to taking \( \alpha \) close enough to 1, we have that \( f_u \in L^p(M) \), for all \( p < \frac{n}{N''_s} \). Since \( 1 > N''_s \), there exists \( p \in (n, \frac{n}{N''_s}) \) such that \( f_u \in L^p(M) \). Therefore, it follows from (23) and elliptic theory that \( u \in C^1(M) \). This proves the claim of Step 6.1 in Case 6.1.6.

This ends Step 6.1.

**Step 6.2:** We prove the contradiction here. Since \( u \in C^1(M) \), we are able to follow the strategy of [8] (see Lemma 3.4) to adapt Hopf’s strong maximum principle. We let \( \Omega \subset M \setminus \{x_0\} \) be an open set such that \( x_0 \in \partial\Omega \) and \( \partial\Omega \) satisfies an interior sphere condition at \( x_0 \), then there exists an exponential chart \( (\mathbb{B}_{2r_y}(y), \exp_y^{-1}) \), \( y \in \Omega, r_y > 0 \) small enough such that \( \mathbb{B}_{r_y}(y) \cap \partial\Omega = \{x_0\} \). We consider \( C > 0 \) such that

\[ L_{g,C}(-u) := -((\Delta_g + C)(-u) \geq (\Delta_g + a)(u) \geq 0 \]

on \( \Omega \). We fix \( \rho \in (0, r_y) \) and introduce the function \( v_\rho \) defined on the annulus \( \mathbb{B}_{r_y}(y) \setminus \mathbb{B}_\rho(y) \) by \( v_\rho(x) = e^{-kr^2} - e^{-kr_y^2} \) where \( r := d_g(x, y) \) and \( k > 0 \) to be
Now we define \( \tilde{\lambda} \) determined. Now, if \( u \) is the smaller eigenvalue of \( g^{-1} \) then that for any \( x \in B_{\rho}(y) \setminus B_{\rho}(y) \) we have that:

\[
L_{g,C}v_\rho(x) \geq e^{-kr^2} \left[ 4k^2 \lambda(x)r^2 - 2k \left( \sum_{i=1}^{n} g^{ii} + \Gamma_0 r \right) - C \right]
\]

where \( \Gamma_0 = \Gamma_0(y) \). Hence we choose \( k \) large enough so that \( L_{g,C}v_\rho \geq 0 \) on \( B_{\rho}(y) \setminus B_{\rho}(y) \). Since \( -u < 0 \) on \( \partial B_{\rho}(y) \) then there exists a constant \( \epsilon > 0 \) such that \( -u + \epsilon v_\rho \leq 0 \) on \( \partial B_{\rho}(y) \). Thus we have \( -u + \epsilon v_\rho \in H^2_{\Gamma}(B_{\rho}(y) \setminus B_{\rho}(y)), -u + \epsilon v_\rho \leq 0 \) on \( \partial B_{\rho}(y) \) and \( L_{g,C}(-u + \epsilon v_\rho) \geq 0 \) on \( B_{\rho}(y) \setminus B_{\rho}(y) \). It follows from the weak maximum principle (see Theorem 8.1 in [8]) that

\[
-u + \epsilon v_\rho \leq 0, \quad \text{on } B_{\rho}(y) \setminus B_{\rho}(y)
\]

In the sequel, \( B_r(0) \) denotes a ball in \( (\mathbb{R}^n, \delta) \) centered at the origin and of radius \( r \). Now we define \( \tilde{u} = u \circ \exp_y \) and \( \tilde{v}_\rho = v \circ \exp_y \) on \( B_{\rho}(0) \). By (25), we get:

\[
\epsilon \tilde{v}_\rho \leq \tilde{u}, \quad \text{on } B_{\rho}(0) \setminus B_{\rho}(0)
\]

We define \( X_0 := \exp_y^{-1}(x_0) \). Since \( \tilde{u}(X_0) = \tilde{v}_\rho(X_0) = 0 \), then, by (26), we can write that

\[
\frac{\partial \tilde{u}}{\partial \nu}(X_0) := \liminf_{t \to 0^+} \frac{\tilde{u}(X_0 + tv) - \tilde{u}(X_0)}{t} \leq \epsilon \liminf_{t \to 0^+} \frac{\tilde{v}_\rho(X_0 + tv) - \tilde{v}_\rho(X_0)}{t} := \epsilon \frac{\partial \tilde{v}_\rho}{\partial \nu}(X_0),
\]

where \( \nu \) is the outer normal vector field on \( B_{\rho}(y) \). Therefore \( \frac{\partial u}{\partial \nu}(X_0) \leq \epsilon \frac{\partial v_\rho}{\partial \nu}(X_0) \), but \( \frac{\partial v_\rho}{\partial \nu}(x_0) = v_\rho'(R) \), it follows that

\[
\frac{\partial \tilde{u}}{\partial \nu}(X_0) \leq \epsilon v_\rho'(r_\rho) \leq 0.
\]

This is a contradiction since \( \min_M u = u(x_0) \) and therefore \( \nabla \tilde{u}(X_0) = \nabla u(x_0) = 0 \). This ends the proof of Step 6.

3. Test-functions for \( n \geq 4 \)

We consider the test-function sequence \( (u_\epsilon)_{\epsilon > 0} \) defined, for any \( \epsilon > 0 \), \( x \in M \), by

\[
u(x) = \left( \frac{\epsilon^{1-\frac{\alpha}{2}}}{\epsilon^{2-s} + d_\rho(x,x_0)^{2-s}} \right) \xrightarrow{\epsilon \to 0} \Phi
\]

the function \( \Phi \) defined on \( \mathbb{R}^n \) by

\[
\Phi(X) = (1 + |X|^{-2-s})^{-\frac{s}{2-s}}
\]

Since \( u_\epsilon \) is a Lipschitz function, we have that \( u_\epsilon \in H^2(M) \), for any \( \epsilon > 0 \). Given \( \rho \in (0,i_g(M)) \), where \( i_g(M) \) is the injectivity radius on \( M \), we recall that \( B_{\rho}(x_0) \) be the geodesic ball of center \( x_0 \) and radius \( \rho \). Cartan’s expansion of the metric \( g \) (see [13]) in the exponential chart \( (B_{\rho}(x_0), \exp_{x_0}^{-1}) \) yields

\[
\det(g)(x) = 1 - \frac{R_{\alpha\beta}(x_0)x_\alpha x_\beta}{3} + O(r^3),
\]

where the \( x_\alpha \)'s are the coordinates of \( x \), \( r^2 = \sum_{\alpha} (x_\alpha)^2 \) and \( (R_{\alpha\beta}) \) is the Ricci curvature. Integrating on the unit sphere \( S^{n-1} \) yields

\[
\int_{S^{n-1}} \sqrt{\det(g)}(r) \, d\theta = \omega_{n-1} \left[ 1 - \frac{\text{Scal}(x_0)}{6n} r^2 + O(r^3) \right].
\]
3.1. **Estimate of the gradient term.** At first, we estimate \( \int_M |\nabla u_\epsilon|^2 d\nu_g \). For that, we write for all \( x \in M \):

\[
|\nabla u_\epsilon|^2(x) = (n - 2)^2 \epsilon^{n-2} \frac{r^{2(1-s)}}{(\epsilon^2 - s + r^{2-s})^{\frac{2n-s}{2}}}
\]

where \( r = d_g(x, x_0) \). Therefore, using (27) and the change of variable \( t = r\epsilon^{-1} \), we get that

\[
\int_{\mathcal{B}_r(x_0)} |\nabla u_\epsilon|^2 d\nu_g = (n - 2)^2 \epsilon^{n-2} \omega_{n-1} \int_0^{r_0} \frac{r^{n+1} \left(1 - \frac{\text{Scal}_g(x_0)}{6n} \right) r^2 + O(r^3)}{t^{2s} (\epsilon^2 - s + r^{2-s})^{\frac{2(n-s)}{2}}} dt
\]

Straightforward computations yield

\[
\int_0^{+\infty} \frac{t^{n+1} dt}{t^{2s} (1 + t^{2-s})^{\frac{2(n-s)}{2}}} = (n - 2)^{-2} \omega_{n-1} \int_{\mathbb{R}^n} |\nabla \Phi|^2 dX,
\]

and

\[
\epsilon^2 \int_0^{+\infty} \frac{t^{n+3} dt}{t^{2s} (1 + t^{2-s})^{\frac{2(n-s)}{2}}} = \begin{cases} \\
\epsilon^2 (n - 2)^{-2} \omega_{n-1} \int_{\mathbb{R}^n} |X|^2 |\nabla \Phi|^2 dX & \text{if } n \geq 5, \\
\epsilon^2 \ln \frac{1}{\epsilon} & \text{if } n = 4, \\
O(\epsilon) & \text{if } n = 3.
\end{cases}
\]

Since

\[
\int_{M \setminus \mathcal{B}_r(x_0)} |\nabla u_\epsilon|^2 d\nu_g = O(\epsilon^{n-2}),
\]

when \( \epsilon \to 0 \), putting together (30) with (31) and (32) yield

\[
\int_M |\nabla u_\epsilon|^2 d\nu_g = \begin{cases} \\
\int_{\mathbb{R}^n} |\nabla \Phi|^2 dX - \frac{\text{Scal}_g(x_0)}{6n} \epsilon^2 + O(\epsilon^2) & \text{if } n \geq 5, \\
\int_{\mathbb{R}^n} |\nabla \Phi|^2 dX - \frac{\epsilon^2}{6n} \text{Scal}_g(x_0) \epsilon^2 \ln(\frac{1}{\epsilon}) + O(\epsilon^2) & \text{if } n = 4, \\
\int_{\mathbb{R}^n} |\nabla \Phi|^2 dX + O(\epsilon) & \text{if } n = 3.
\end{cases}
\]

Arguing as the above and using that \( a \in C^0(M) \), we get that:

\[
\int_M au_\epsilon^2 d\nu_g = \begin{cases} \\
e^2 a(x_0) \int_{\mathbb{R}^n} \Phi^2 dX + O(\epsilon^2) & \text{if } n \geq 5, \\
a(x_0) \omega_3 \epsilon^2 \ln \frac{1}{\epsilon} + O(\epsilon^2) & \text{if } n = 4, \\
O(\epsilon) & \text{if } n = 3.
\end{cases}
\]

and

\[
\int_M |u_\epsilon|^2 d\nu_g = \begin{cases} \\
\int_{\mathbb{R}^n} |\Phi|^2 dX - \epsilon^2 \frac{\text{Scal}_g(x_0)}{6n} \int_{\mathbb{R}^n} |X|^{2-s}|\Phi|^2 dX + O(\epsilon^2) & \text{if } n \geq 4, \\
\int_{\mathbb{R}^n} |\Phi|^2 dX + O(\epsilon) & \text{if } n = 3.
\end{cases}
\]
From Lieb [14], we know that \( \Phi \) is an extremal for (5), that is
\[
\frac{\int_{\mathbb{R}^n} |\nabla \Phi|^2 dX}{\left( \int_{\mathbb{R}^n} |\Phi|^{2^*} dX \right)^{\frac{n}{2^*}}} = K(n, s)^{-1}
\] (36)
Combining (33), (34) and (35) and this last equation, we obtain, for any \( \epsilon > 0 \), the following results :
\[
J(u_\epsilon) = K(n, s)^{-1} \left( 1 + \left\{ \begin{array}{ll}
(C_1(n, s)a(x_0) - C_2(n, s)\text{Scal}_g(x_0))e^2 + o(e^2) & \text{if } n \geq 5 \\
\omega_3(\int_{\mathbb{R}^n} |\nabla \Phi|^2 dX)^{-1} (a(x_0) - \frac{1}{6}\text{Scal}_g(x_0)) e^2 \ln \left( \frac{1}{\epsilon} \right) + O(\epsilon^2) & \text{if } n = 4 \\
O(\epsilon) & \text{if } n = 3
\end{array} \right. \right)
\] (37)
where
\[
C_1(n, s) := \frac{\int_{\mathbb{R}^n} |\Phi|^2 dX}{\left( \int_{\mathbb{R}^n} |\nabla \Phi|^2 dX \right)^{\frac{n}{2}}}
\]
\[
C_2(n, s) := \frac{1}{6n} \frac{\int_{\mathbb{R}^n} |X|^2 |\nabla \Phi|^2 dX}{\int_{\mathbb{R}^n} |\Phi|^2 dX} - \frac{2}{2^*(s)6n} \frac{\int_{\mathbb{R}^n} |X|^{2-s} |\Phi|^{2^*(s)} dX}{\int_{\mathbb{R}^n} |\Phi|^{2^*(s)} dX}
\]
Unlike the case \( s = 0 \), it is not possible to compute explicitly the constants \( C_1(n, s) \) and \( C_2(n, s) \). However, we are able to explicit their quotient, which is enough to prove our theorem. We need the following lemma taken from Aubin [1] :

**Lemma 1.** Let \( p, q \in \mathbb{R}_+^* \) such that \( p - q > 1 \) and assume that \( I^q_p = \int_0^{+\infty} \frac{e^t dt}{(1+t)^p} \), then
\[
I^q_{p+1} = \frac{p - q - 1}{p} I^q_p \quad \text{and} \quad I^q_{p+1} = \frac{q + 1}{p - q - 1} I^q_p
\]
Indeed, an integration by parts shows that \( I^q_p = \frac{p}{q+1} I^q_{p+1} \). On the other hand, we can easily see that \( I^q_p = I^q_{p+1} + I^q_{p+1} \). Together, the above relations yield the lemma.

We apply Lemma 1 to the computation of \( C_2(n, s)/C_1(n, s) \) when \( n \geq 5 \). We have that
\[
\frac{C_2(n, s)}{C_1(n, s)} = \frac{1}{6n} \frac{\int_{\mathbb{R}^n} |X|^2 |\nabla \Phi|^2 dX}{\int_{\mathbb{R}^n} \Phi^2 dX} - \frac{2}{2^*(s)6n} \frac{\int_{\mathbb{R}^n} |X|^{2-s} |\Phi|^{2^*(s)} dX}{\int_{\mathbb{R}^n} |\Phi|^{2^*(s)} dX}
\]
Independently
\[
\frac{\int_{\mathbb{R}^n} |X|^2 |\nabla \Phi|^2 dX}{\int_{\mathbb{R}^n} \Phi^2 dX} = \int_0^{+\infty} \frac{e^{n-3-2s} r^{n-1} dr}{(1+r^{2-s})^{\frac{2(n-2)}{2-s}}},
\]
up to taking \( t = r^{2-s} \) and using the Lemma 1, we get that :
\[
\frac{\int_{\mathbb{R}^n} |X|^2 |\nabla \Phi|^2 dX}{\int_{\mathbb{R}^n} \Phi^2 dX} = \frac{(n-2)^2}{2-s} \int_0^{+\infty} \frac{r^{n-3-2s} dr}{(1+t)(1+t)^2} = \frac{n(n-2)(n+2-s)}{2(2n-2-s)}
\] (39)
We follow the technique developed by Druet [3] for test-function in dimension \( n \geq 2 \). Let \( W \) be a compact Riemannian Manifold of dimension \( n \geq 2 \). The case of a manifold with boundary is discussed at the end of this section. We fix \( \rho \in (0, i_g(M)/2) \) and we consider a cut-off function \( \eta \in C^\infty_c(\mathbb{B}_2(x_0)) \) such that \( \eta \equiv 1 \) on \( \mathbb{B}_\rho(x_0) \). Then there exists

\[
\phi \in \mathcal{D}(M) \cap L^1_v S^p(M) \quad \text{such that} \quad \phi \equiv 1 \quad \text{on} \quad \mathbb{B}_\rho(x_0) \quad \text{and} \quad \phi \in C^\infty(M) \quad \text{outside} \quad \mathbb{B}_\rho(x_0).
\]

Moreover, if \( \rho \leq 1 \), then

\[
\inf_{v \in H^1(M) \setminus \{0\}} J(v) \leq K(n,s)^{-1}.
\]

Moreover, if \( n \geq 4 \) and \( a(x_0) < c_{n,s} \text{Scal}_g(x_0) \), where \( c_{n,s} \) is as (43), then inequality (44) is strict.

4. Test-functions: the case \( n = 3 \)

The argument used for \( n \geq 4 \) is local in the sense that the expansion (42) only involves the values of \( a \) and \( \text{Scal}_g \) at the singular point \( x_0 \). When \( n = 3 \), the first-order in (42) of Section 3 has an undetermined sign. It is well-known since Schoen [19] that the relevant quantity to use in small dimension is the mass, which is a global quantity.

We follow the technique developed by Druet [3] for test-function in dimension 3. The case of a manifold with boundary is discussed at the end of this section. We define the Green-function \( G_{x_0} \) of the elliptic operator \( \Delta_g + a \) on \( x_0 \) as the unique function strictly positive and symmetric verifying, in the sense of distribution,

\[
\Delta_g G_{x_0} + a G_{x_0} = D_{x_0},
\]

where \( D_{x_0} \) is the Dirac mass at \( x_0 \). We fix \( \rho \in (0, i_g(M)/2) \) and we consider a cut-off function \( \eta \in C^\infty_c(B_{2\rho}(x_0)) \) such that \( \eta \equiv 1 \) on \( B_\rho(x_0) \). Then there exists
\( \beta_{x_0} \in H^2_1(M) \) such that we can write \( G_{x_0} \) as follow:

\[
\omega_2 G_{x_0}(x) = \frac{\eta(x)}{d_g(x, x_0)} + \beta_{x_0}(x)
\]

for all \( x \in M \). According to (45) and (46), we have that

\[
\Delta_g \beta_{x_0} + a \beta_{x_0} = f_{x_0}
\]

where

\[
f_{x_0}(x) := -\Delta_g \left( \frac{\eta(x)}{d_g(x, x_0)} \right) - \frac{(a \eta)(x)}{d_g(x, x_0)} \quad \text{for all } x \in M \setminus \{x_0\}.
\]

In particular, for all \( p \in (1, 3) \), we have \( f_{x_0} \in L^p(M) \). Therefore, it follows from standard elliptic theory that \( \beta_{x_0} \in C^0(M) \cap C^1_{\text{loc}}(M \setminus \{x_0\}) \cap H^2_p(M) \) for all \( p \in (1, 3) \).

In particular, the mass satisfies \( m(x_0) = \beta_{x_0}(x_0) \). For any \( \epsilon > 0 \), we define, on \( M \), the function

\[
u_\epsilon = \eta u_\epsilon + \sqrt{\epsilon} \beta_{x_0},
\]

where \( u_\epsilon \) is the general test-function defined as (27). This section is devoted to computing the expansion of \( J(\nu_\epsilon) \). We compute the different terms separately.

4.1. **The leading term** \( f_M(\|\nabla v_\epsilon\|_g^2 + a v_\epsilon^2) dv_g \). Integration by parts and using the definition of \( v_\epsilon \), we write, for any \( \epsilon > 0 \), that:

\[
\int_M (\|\nabla v_\epsilon\|_g^2 + a v_\epsilon^2) dv_g = \int_M \eta^2 u_\epsilon \Delta_g u_\epsilon dv_g + \int_M u_\epsilon^2 \eta \Delta_g \eta dv_g - \int_M \eta (\nabla \eta, \nabla u_\epsilon^2) g dv_g + \int_M a \eta^2 u_\epsilon^2 dv_g + \int_M (\Delta_g \beta_{x_0} + a \beta_{x_0})(\epsilon \beta + 2 \sqrt{\epsilon} \eta u_\epsilon) dv_g.
\]

Writing \( u_\epsilon^2 \) in the form:

\[
u_\epsilon = \frac{\epsilon}{d_g(x, x_0)^2} + O(\epsilon^{5-2s}),
\]

with \( O(1) \in C^2(M \setminus B_\rho(x_0)) \) uniformly bounded with respect to \( \epsilon \), we obtain that

\[
\int_M u_\epsilon^2 \eta \Delta_g \eta dv_g = \epsilon \int_{M \setminus B_\rho(x_0)} \frac{\eta \Delta_g \eta}{d_g(x, x_0)^2} dv_g + o(\epsilon),
\]

and

\[
\int_M \eta (\nabla \eta, \nabla u_\epsilon^2) g dv_g = \epsilon \int_{M \setminus B_\rho(x_0)} \eta (\nabla \eta, \nabla \frac{1}{d_g(x, x_0)^2}) g dv_g + o(\epsilon).
\]

By integrating by parts, using (50) and since \( \partial_\nu \eta = 0 \) then we write

\[
\int_M u_\epsilon^2 \eta \Delta_g \eta dv_g - \int_M \eta (\nabla \eta, \nabla u_\epsilon^2) g dv_g = \epsilon \int_{M \setminus B_\rho(x_0)} \frac{\|\nabla \eta\|_g^2}{d_g(x, x_0)^2} dv_g + o(\epsilon)
\]

We have also that

\[
\int_M a \eta^2 u_\epsilon^2 dv_g = \epsilon \int_M \frac{a \eta^2}{d_g(x, x_0)^2} dv_g + R_1(\epsilon) + o(\epsilon),
\]

where, as in (30),

\[
R_1(\epsilon) = O \left( \epsilon^{3-s} \int_{B_\rho(x_0)} \frac{a \eta^2 dv_g}{d_g(x, x_0)^2(\epsilon^{2-s} + a \eta^2 + d_g(x, x_0)^2)^{2-s}} \right) = O \left( \epsilon^2 \int_0^{t_0} \frac{dt}{1 + t^{2-s}} \right) = o(\epsilon).
\]
This latest relation and (54) give that

\[ \int_M \alpha n^2 u^2 dv_g = \epsilon \int_M \frac{\alpha n^2}{d_g(x, x_0)^2} dv_g + o(\epsilon). \]

Writing now \( u_\epsilon \) in the form

\[ u_\epsilon(x) = \frac{\sqrt{\epsilon}}{d_g(x, x_0)} + O(\epsilon^{-s}), \]

with \( O(1) \in C^2(M \setminus B_\rho(x_0)) \) we get that

\[ \int_{M \setminus B_\rho(x_0)} \eta^2 u_\epsilon \Delta_g u_\epsilon dv_g = \epsilon \int_{M \setminus B_\rho(x_0)} \frac{\eta^2}{d_g(x, x_0)} \Delta_g \left( \frac{1}{d_g(x, x_0)} \right) dv_g + o(\epsilon). \]

Since \( u_\epsilon \) is radially symmetrical, denoting \( \Delta_\delta \) as the Laplacian in the Euclidean metric \( \delta \), we get with a change of variable and Cartan’s expansion of the metric (29) that

\[ \int_{B_\rho(x_0)} u_\epsilon \Delta_\delta u_\epsilon dv_g = \int_{\mathbb{R}^n} \Phi \Delta_\delta \Phi dX + o(\epsilon), \]

where \( \Phi \) is defined in (28). Since

\[ \Delta_g u_\epsilon = \Delta_\delta u_\epsilon - \partial_r (\ln det(g)) \partial_r u_\epsilon \]

in \( g \)-normal coordinates, we have that

\[ \int_{B_\rho(x_0)} u_\epsilon \Delta_g u_\epsilon dv_g = \int_{\mathbb{R}^n} \Phi \Delta_\delta \Phi dX + o(\epsilon) \]

when \( \epsilon \to 0 \). Similar computations to the ones we just developed give that

\[ \epsilon \int_{B_\rho(x_0)} \frac{d_g(x, x_0)^{1-s} \partial_r (\ln det(g))}{2(\epsilon^2 + d_g(x, x_0)^2 s)} dv_g = \epsilon \int_{B_\rho(x_0)} \frac{\partial_r (\ln det(g))}{2d_g(x, x_0)^2} dv_g + o(\epsilon) \]

\[ + O \left( \epsilon^{s-3} \int_{B_\rho(x_0)} d_g(x, x_0)^3 (\epsilon^2 + d_g(x, x_0)^2) dv_g \right). \]

Cartan’s expansion of the metric \( g \), (29) and to this latest relation yield

\[ \epsilon \int_{B_\rho(x_0)} \frac{d_g(x, x_0)^{1-s} \partial_r (\ln det(g))}{2(\epsilon^2 + d_g(x, x_0)^2 s)} dv_g = \epsilon \int_{B_\rho(x_0)} \frac{\partial_r (\ln det(g))}{2d_g(x, x_0)^2} dv_g + o(\epsilon) \]

(59)

Relations (57), (58) and (59) yield

\[ \int_M \eta^2 u_\epsilon \Delta_g u_\epsilon dv_g = \int_{\mathbb{R}^n} \Phi \Delta_\delta \Phi dX + \epsilon \int_{B_\rho(x_0)} \frac{\partial_r (\ln det(g))}{2d_g(x, x_0)^2} dv_g \]

\[ + \epsilon \int_{M \setminus B_\rho(x_0)} \frac{\eta^2}{d_g(x, x_0)} \Delta_g \left( \frac{1}{d_g(x, x_0)} \right) dv_g + o(\epsilon) \]

(60)
when $\epsilon \to 0$. At last, using again the expansion (60) of $u_\epsilon$, we obtain that:
\[
\int_M (\Delta g \beta_{x_0} + a \beta_{x_0})(\epsilon \beta_{x_0} + 2 \sqrt{\epsilon} u_\epsilon) dv_g = \epsilon \int_M (\Delta g \beta_{x_0} + a \beta_{x_0})(\beta_{x_0} + 2 \eta d_g(x, x_0)) dv_g + O\left(\epsilon^{4-s} \int_M (\Delta g \beta_{x_0} + a \beta_{x_0}) \frac{\eta}{d_g(x, x_0)} dv_g\right).
\]
The latest relation and (47) allow to write:
\[
\int_M (\Delta g \beta_{x_0} + a \beta_{x_0})(\epsilon \beta_{x_0} + 2 \sqrt{\epsilon} u_\epsilon) dv_g = \epsilon \int_M (\Delta g \beta_{x_0} + a \beta_{x_0})(\beta_{x_0} + 2 \eta d_g(x, x_0)) dv_g + o(\epsilon).
\]
Since $\beta_{x_0} \in C^0(M) \cap H^p(M)$ for all $p \in (\frac{3}{4}, 3)$, it follows from (45) and (46) that
\[
\int_M (\Delta g \beta_{x_0} + a \beta_{x_0})(\beta_{x_0} + 2 \eta d_g(x, x_0)) dv_g = \omega_2 \beta_{x_0}(x_0).
\]
Then the last couple of relations give that
\[
\int_M (\Delta g \beta_{x_0} + a \beta_{x_0})(\epsilon \beta_{x_0} + 2 \sqrt{\epsilon} u_\epsilon) dv_g = \epsilon \omega_2 \beta_{x_0}(x_0) + o(\epsilon)
\]
when $\epsilon \to 0$. Knowing, from (47) and (48), that
\[
\int_M (\Delta g \beta_{x_0} + a \beta_{x_0})(\beta_{x_0} + 2 \eta d_g(x, x_0)) dv_g = M_1 = \int_B \phi \Delta_g \phi dX + \epsilon \omega_2 \beta_{x_0}(x_0) + o(\epsilon)
\]
and combining (53), (55),(60) and (61) with (49), we get that
\[
\int_M (|\nabla u_\epsilon|^2 + u_\epsilon^2) dv_g = \int_{\mathbb{R}^n} \Phi \Delta_g \Phi dX + \epsilon \omega_2 \beta_{x_0}(x_0) + o(\epsilon)
\]
when $\epsilon \to 0$.

4.2. Estimate of $\int_M \frac{e^{\epsilon \phi(x)}}{M_3(x, x_0)} dv_g$. Since $s \in (0, 2)$ then $6 - 2s > 2$. Therefore there exists $C(s) > 0$ such that for all $X, Y \in \mathbb{R}$, we have:
\[
||X + Y|^{6-2s} - |X|^{6-2s} - (6 - 2s)XY|X|^{4-2s} \leq C(s) (|X|^{4-2s}Y^2 + |Y|^{6-2s})
\]
This allows to write
\[
\int_M \frac{e^{\epsilon \phi(x)}}{M_3(x, x_0)} dv_g = \int_M \frac{(\eta u_\epsilon + \sqrt{\epsilon} \beta_{x_0})^{6-2s}}{d_g(x, x_0)^s} dv_g
\]
\[
= \int_{B_\rho(x_0)} \frac{(\eta u_\epsilon + \sqrt{\epsilon} \beta_{x_0})^{6-2s}}{d_g(x, x_0)^s} dv_g + O(\epsilon^{3-s})
\]
\[
= \int_{B_\rho(x_0)} \frac{u_\epsilon^{6-2s} + (6 - 2s)u_\epsilon^{6-2s} - 2u_\epsilon \sqrt{\epsilon} \beta_{x_0}}{d_g(x, x_0)^s} dv_g + R_2(\epsilon) + o(\epsilon)
\]
(63)

where
\[
R_2(\epsilon) = O\left(\int_{B_\rho(x_0)} \frac{u_\epsilon^{4-2s} \epsilon \beta_{x_0} + \epsilon^{3-s} \beta_{x_0}^{6-2s}}{d_g(x, x_0)^s} dv_g\right) = o(\epsilon)
\]
(64)
Hence, the latest relation and (66) give that
\[\int_{B_\rho(x_0)} \frac{u_0^6 - 2s}{d_g(x, x_0)^s} dv_g = \int_{\mathbb{R}^n} \frac{\Phi^{2^*(s)}(X)}{|X|^s} dX + o(\epsilon),\]
when \(\epsilon \to 0\). Using that \(\beta_{x_0} \in C^0,\theta(M)\) for all \(\theta \in (0, 1)\), we get that
\[\int_{B_{\rho}(x_0)} \frac{(6 - 2s)u \rho^{-2s} \sqrt{c_{x_0}}}{d_g(x, x_0)^s} dv_g = \epsilon^{3-s}(6 - 2s)\beta_{x_0}(x_0)\omega_2 \int_0^\rho \frac{r^{2-s} dr}{(\epsilon^{2-s} + r^{2-s})^{\frac{2-s}{s}}} + o(\epsilon)\]
\[= \epsilon(6 - 2s)\beta_{x_0}(x_0)\omega_2 \int_0^\pi \frac{t^{2-s} dt}{(1 + t^{2-s})^{\frac{2-s}{s}}} + o(\epsilon)\]
\[= \epsilon(6 - 2s)\beta_{x_0}(x_0)\omega_2 \int_0^{+\infty} \frac{t^{2-s} dt}{(1 + t^{2-s})^{\frac{2-s}{s}}} + o(\epsilon),\]
when \(\epsilon \to 0\). Since \(\Delta_\theta \Phi = (3 - s)\frac{\Phi^{2^*(s) - 1}}{|X|^s}\) in \(\mathbb{R}^n\), a changes of variable and an integration by parts yields
\[\omega_2 \int_0^{+\infty} \frac{t^{2-s} dt}{(1 + t^{2-s})^{\frac{2-s}{s}}} = \int_{\mathbb{R}^n} \frac{\Phi^{2^*(s)-1}(X)}{|X|^s} dX = (3 - s)^{-1} \lim_{R \to +\infty} \int_{\partial B_R(0)} -\partial_\nu \Phi dX,\]
where \(\nu\) is the normal vector field on the Euclidean ball \(B_R(0)\). Since \(\partial_\nu \Phi = -|X|^{1-s}(1 + |X|^{2-s}) \frac{\Phi^{2^*(s)-1}}{|X|^s}\) for all \(X \in \mathbb{R}^n\), passing to the limit in (67) yields
\[\omega_2 \int_0^{+\infty} \frac{t^{2-s} dt}{(1 + t^{2-s})^{\frac{2-s}{s}}} = (3 - s)^{-1} \omega_2.\]
Hence, the latest relation and (66) give that
\[\int_{B_{\rho}(x_0)} \frac{(6 - 2s)u \rho^{-2s} \sqrt{c_{x_0}}}{d_g(x, x_0)^s} dv_g = \epsilon(2\beta_{x_0}(x_0)\omega_2) + o(\epsilon),\]
when \(\epsilon \to 0\). Combining (64), (65) and (68) with (63), we get that
\[\int_M \frac{\nu_0^{2^*(s)}}{d_g(x, x_0)^s} dv_g = \int_{\mathbb{R}^n} \frac{\Phi^{2^*(s)}(X)}{|X|^s} dX + \epsilon(2\beta_{x_0}(x_0)\omega_2) + o(\epsilon),\]
when \(\epsilon \to 0\).

4.3. Expansion of \(J(\nu_x)\) and proof of Theorem 1. Equality (69), (62) and (67) yield
\[J(\nu_x) = \frac{\int_M (|\nabla \nu_0|^2 + a\nu_0^2) dv_g}{\left(\int_M \frac{\nu_0^{2^*(s)}}{d_g(x, x_0)^s} dv_g\right)^{\frac{1}{2^*(s)}}}\]
\[= K(3, s)^{-1} \left(1 - \epsilon \frac{2\beta_{x_0}(x_0)\omega_2}{\int_{\mathbb{R}^n} |x|^{-s}\Phi^{2^*(s)}(x) dx} + o(\epsilon)\right)\]
when \(\epsilon \to 0\). Noting that \(m(x_0) = \beta_{x_0}(x_0)\), we then get the following as a consequence of (70):
Theorem 6. Let $(M, g)$ be a compact Riemannian Manifold of dimension $n = 3$. Let $a \in C^0(M)$ such that $\Delta_g + a$ is coercive, $x_0 \in M$ and $s \in (0, 2)$. Assume that that the mass at $x_0$ is positive, that is $\beta_{x_0}(x_0) > 0$. Then we have that
\[
\inf_{v \in H^2(\Delta_g + a) \setminus \{0\}} J(v) < K(n, s)^{-1}.
\]

Proof of Theorem 1. Theorem 1 follows from the existence result (Theorem 4) and the upper-bounds (Theorem 5 and Theorem 6).

Proof of Theorem 3. As one checks, the estimates (42) and (70) hold when $M$ is a smooth compact manifold with boundary provided $x_0$ lies in the interior. Then Theorem 1 extends to such a case, and Theorem 3 is a corollary.

4.4. Examples with positive mass.

Proposition 2. Let $(M, g)$ be a compact Riemannian Manifold of dimension $n = 3$. Let $a \in C^0(M)$ such that $\Delta_g + a$ is coercive, $x_0 \in M$ and $s \in (0, 2)$. If $\{a \leq c_{3,0}\text{Scal}_g\}$ or $\{a \equiv c_{3,0}\text{Scal}_g\}$ and $(M, g)$ is not conformally equivalent to the canonical $n$-sphere then we have that:
\[
\inf_{v \in H^2(M) \setminus \{0\}} J(v) < K(3, s)^{-1}.
\]

Indeed, the positivity of the mass in this case was proved by Druet [4]. We incorporate the proof for the sake of self-completeness.

Lemma 2. Let $(M, g)$ be a compact Riemannian Manifold of dimension $n = 3$. We consider $a, a' \in C^0(M)$ such that operators $\Delta_g + a$ and $\Delta_g + a'$ are coercive. We denote as $G_x, G'_x$ their respective Green’s function at any point $x \in M$. We assume that $a \leq a'$. Then $\beta_x > \beta'_x$ for all $x \in M$, where $\beta_x, \beta'_x \in C^{0,\theta}(M), \theta \in (0,1)$ are such that
\[
\omega_2G_x = \frac{\eta_x}{d_g(x, \cdot)} + \beta_x \quad \text{and} \quad \omega_2G'_x = \frac{\eta_x}{d_g(x, \cdot)} + \beta'_x.
\]

Proof. We fix $x \in M$ and we define $h_x = \beta'_x - \beta_x$, where $\beta'_x$ and $\beta_x$ are as in (71). Noting $L := \Delta_g + a$ and $L' := \Delta_g + a'$, we have that $L'(h_x) = -(a' - a)G_x \leq 0$. Since $h_x \in H^2(M)$ for all $p \in (1,3)$, then for all $y \in M$, Green’s formula yields
\[
h_x(y) = - \int_M G'_y(z)(a' - a)(z)G_x(z) \, dv_g(z).
\]

Therefore $h_x \leq 0$ since $a \leq a'$. Moreover, since $a \not\equiv a'$, we have that $h_x < 0$. This ends the proof. 

Proof of Proposition 2: We consider the operator $L^0 := \Delta_g + c_{3,0}\text{Scal}_g$, $\beta^0$ the mass of $(M, g)$ corresponding to $L^0$. The Positive Mass Theorem (see [20], [21]) gives that $\beta^0(x) > 0$, the equality being achieved only in the conformal class of the canonical sphere. It then follows from Lemma 2 that $\beta_{x_0}(x_0) > 0$ when $\{a \leq c_{3,0}\text{Scal}_g\}$ or $\{a \equiv c_{3,0}\text{Scal}_g\}$ and $(M, g)$ is not conformally equivalent to the unit $n$-sphere. It then follows from Theorem 6 that
\[
\inf_{v \in H^2(M) \setminus \{0\}} J(v) < K(3, s)^{-1}.
\]
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