Electromagnetic energy and negative asymmetry parameter in coated magneto-optical cylinders: Applications to tunable light transport in disordered systems

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We investigate electromagnetic scattering of normally irradiated gyrotropic, magneto-optical core-shell cylinders using Lorenz-Mie theory. A general expression for time-averaged electromagnetic energy inside a coated gyroelectric and gyromagnetic scatterer is derived. Using realistic material parameters for a silica core and InSb shell, we calculate the stored electromagnetic energy and the scattering anisotropy. We show that the application of an external magnetic field along the cylinder axis induces a drastic decrease in electromagnetic absorption in a frequency range in the terahertz, where absorption is maximal in the absence of the magnetic field. We demonstrate not only that the scattering anisotropy can be externally tuned by applying a magnetic field, but also that it reaches negative values in the terahertz range even in the dipolar regime. We also show that this preferential backscattering response results in an anomalous regime of multiple light scattering from a collection of magneto-optical core-shell cylinders, in which the extinction mean free path is longer than the transport mean free path. By additionally calculating the energy-transport velocity and diffusion coefficient, we demonstrate an unprecedented degree of external control of multiple light scattering, which can be achieved by either applying an external magnetic field or varying the temperature.

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I. INTRODUCTION

Electromagnetic (EM) scattering by small particles with sizes of the order of the wavelength has many applications not only in different areas of physics, such as meteorology, optical communications, sensing, and astrophysics, but also in interdisciplinary fields, such as biophysics. The advent of plasmonics and metamaterials has allowed for the discovery of novel scattering phenomena, which do not exist in naturally occurring materials such as plasmonic cloaking, unconventional Fano resonances, artificial magnetism, supersonic scattering, and the unprecedented control of the scattering directionality.

In particular, controlling the scattering direction crucially depends on the design optimization of the electric and magnetic responses of small particles. Most approaches to controlling the scattering direction rely on tailoring the electric structures of nanoparticles, as typically the electric response is dominant in natural media at optical frequencies. However, efforts have been made to propose and design metamaterials that support both electric and magnetic dipolar resonances, such as spheres and high permittivity cylinders. These efforts have allowed the achievement of zero-backward scattering and near-zero-forward scattering conditions (known as Kerker conditions), first theoretically predicted for hypothetical particles exhibiting both electric and magnetic dipolar resonances, and highly directional EM scattering. The observation of Kerker conditions relies on the interference of electric and magnetic dipoles in nanostructures. Alternative theoretical approaches for Kerker conditions, involving electric dipoles and quadrupoles, also exist. Recently, both broadband zero backward and near-zero forward scattering have been obtained even beyond the dipole limit, relaxing design constraints in practical experiments. Other mechanisms for achieving directional light scattering have also been proposed using magneto-optical materials.

In addition, achieving preferential backward scattering remains a challenge and has many applications in multiple light scattering. Preferential backscattering, which hardly occurs in natural media, is characterized by negative values of the asymmetry parameter, the average of the cosine of the scattering angle. Indeed, for small particles in the Rayleigh regime, scattering is dipolar so that the asymmetry parameter is close to 0. In contrast, Mie particles scatter strongly in the forward direction, the asymmetry parameter is close to 1. Negative asymmetry parameters have been reported in ferromagnetic particles and lossless dielectric nanospheres made of moderate permittivity materials, such as silicon or germanium nanospheres in the infrared region. In these cases, negative asymmetry parameters have been shown to lead to an unusual regime in multiple light scattering, in which the scattering mean free path is larger than the transport mean free path. This peculiar transport regime has also been demonstrated for multiple scatter-
material parameters of the coated cylinder are \((\varepsilon_q^1, \mu_q^1)\) for the core \((0 < r \leq a)\) and \((\varepsilon_q^2, \mu_q^2)\) for the shell \((a < r \leq b)\), where the gyroelectric and gyromagnetic tensors are, respectively,

\[
\varepsilon_q^1 = \begin{pmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & 0 \\
\varepsilon_{yx} & \varepsilon_{yy} & 0 \\
0 & 0 & \varepsilon_{zz}
\end{pmatrix} = \begin{pmatrix}
\varepsilon_{q||} & 0 & 0 \\
0 & \varepsilon_{q\perp} & 0 \\
0 & 0 & \varepsilon_{q\perp}
\end{pmatrix}, \tag{1}
\]

\[
\mu_q^1 = \begin{pmatrix}
\mu_{xx} & \mu_{xy} & 0 \\
\mu_{yx} & \mu_{yy} & 0 \\
0 & 0 & \mu_{zz}
\end{pmatrix} = \begin{pmatrix}
\mu_{q||} & 0 & 0 \\
0 & -\eta_q & 0 \\
0 & 0 & \mu_{q\perp}
\end{pmatrix}. \tag{2}
\]

Considering the time harmonic dependence \(e^{-i\omega t}\), where \(\omega\) is the angular frequency, one has the curl Maxwell equations \(\nabla \times (\mathbf{E}, \mathbf{H}) = i\omega(\mu_0 \mathbf{H} - \varepsilon_0 \mathbf{E})\). In cylindrical coordinates \((r, \phi, z)\), there are two irradiation schemes with analytical solutions, as depicted in Fig. 1: the TM polarization or \(p\) wave \((\mathbf{H}\parallel \hat{z})\), which provides the field components \(E_r\) and \(E_\phi\) in terms of partial derivatives of \(H_z\); and the TE polarization or \(s\) wave \((\mathbf{E}\parallel \hat{z})\), which leads to \(H_r\) and \(H_\phi\) in terms of partial derivatives of \(E_z\).

![Figure 1](image_url)

Figure 1. A center-symmetric core-shell, infinitely long circular cylinder normally irradiated with plane waves in the Voigt configuration \((\mathbf{B} \perp \mathbf{k})\). The core has optical properties \((\varepsilon_1, \mu_1)\) and radius \(a\), whereas the shell has \((\varepsilon_2, \mu_2)\) and radius \(b\). The surrounding medium is the vacuum \((\varepsilon_0, \mu_0)\). The incident EM fields \((\mathbf{E}_i^p, \mathbf{H}_i^p)\) and \((\mathbf{E}_i^s, \mathbf{H}_i^s)\) are on \(p\) or \(s\) polarizations, respectively. The applied external magnetic field satisfies \(|\mathbf{B}| \gg |\mu_0 \mathbf{H}_i|\).

For both polarizations, we define the following quanti-
ties, used throughout this text:

\[
\begin{align*}
\text{TM (p)} & \Rightarrow \begin{cases}
\varepsilon_p^q \equiv \varepsilon_q^p [1 - (\beta_p^q)^2], \\
\mu_p^q \equiv \mu_q^p, \\
\beta_p^q \equiv \gamma_q^p/\varepsilon_q^p,
\end{cases} \\
\text{TE (s)} & \Rightarrow \begin{cases}
\varepsilon_p^s \equiv \varepsilon_s^p, \\
\mu_p^s \equiv \mu_s^p [1 - (\beta_p^s)^2], \\
\beta_p^s \equiv \eta_s/\mu_s^p.
\end{cases}
\end{align*}
\]

With this simplified notation, for both \( p \) and \( s \) waves, \( \beta_q = 0 \) (also known as the Voigt parameter) describes an isotropic 2D medium with \( (\varepsilon_q, \mu_q) \). In the following, we focus on \( p \)-polarized waves and cylinders composed of gyroelectric materials, Eq. (11). The discussion for \( s \) polarization is analogous.

A. Electric and magnetic fields for \( p \) waves

The EM wave impinging on the cylinder is set as a monochromatic wave propagating with wave vector \( \mathbf{k} = -k \hat{x} \) and time harmonic dependence \( e^{-i\omega t} \). The scatterer geometry is depicted in Fig. 1. For \( p \)-polarized waves in cylindrical coordinate system \((r, \phi, z)\), we have the ansatz \([E_p^n(r, \phi), H_p^n(r, \phi)] = (\mathbf{E}_0, \mathbf{H}_0) e^{-ikr \cos \gamma}, \) where the electric and magnetic amplitudes are related by \( E_0 = H_0/\sqrt{\varepsilon_0/\mu_0}. \)

Expanding the incident EM field in vector cylindrical harmonics, we obtain for \( r > b \) the nonvanishing components

\[
\begin{align*}
E_p^r & = - \sum_{n=-\infty}^{\infty} E_n \frac{J_n(kr)}{kr} e^{in\phi}, \\
E_p^\phi & = -i \sum_{n=-\infty}^{\infty} E_n J_n'(kr) e^{in\phi}, \\
H_p^z & = \frac{k}{\omega \mu_0} \sum_{n=-\infty}^{\infty} E_n J_n(kr) e^{in\phi},
\end{align*}
\]

where \( E_n = E_0(-i)^n, k^2 = \omega^2\varepsilon_0\mu_0, \) and \( J_n \) is the cylindrical Bessel function. As a consequence, the nonvanishing components of the EM field scattered by the cylinder are, for \( r > b \) [1],

\[
\begin{align*}
E_s^r & = - \sum_{n=-\infty}^{\infty} A_n^p H_n^{(1)}(kr)/kr e^{in\phi}, \\
E_s^\phi & = -i \sum_{n=-\infty}^{\infty} A_n^p H_n^{(1)}(kr) e^{in\phi}, \\
H_s^z & = \frac{k}{\omega \mu_0} \sum_{n=-\infty}^{\infty} A_n^p H_n^{(1)}(kr) e^{in\phi},
\end{align*}
\]

where \( A_n^p \) is the scattering coefficient and \( H_n^{(1)} \) is the cylindrical Hankel function of the first kind. The form of the scattering coefficient depends on the material properties of the scatterer.

From Maxwell’s equations, one can show that the magnetic field \( \mathbf{H}_q = H_{qz} \hat{z} \) within the scatterer must satisfy the following Helmholtz equation [28]: \( \nabla^2 H_{qz} + \kappa_q^2 H_{qz} = 0 \), where \( (\kappa_q^2)^2 = \omega^2 \varepsilon_q^p \mu_q^p. \) The remaining EM field components are calculated from \( (\kappa_q^2)^2 E_{qr} = \omega \mu_q^p |\beta_q^p|^2 \partial / \partial r + (1/\gamma_q^p) \partial / \partial \phi) H_{qz} \) and \( (\kappa_q^2)^2 E_{q\phi} = -\omega \mu_q^p |\beta_q^p|^2 \partial / \partial r - (1/\gamma_q^p) \partial / \partial \phi) H_{qz} \). Explicitly, we have for the core region, \( q = 1 \) (\( 0 < r < a \)),

\[
\begin{align*}
E_p^r & = - \sum_{n=-\infty}^{\infty} E_n b_n^p J_n(k_r^p r, \beta_p^q) e^{in\phi}, \\
E_p^\phi & = -i \sum_{n=-\infty}^{\infty} E_n b_n^p J_n'(k_r^p r, \beta_p^q) e^{in\phi}, \\
H_p^z & = \frac{k_p^p}{\omega \mu_0} \sum_{n=-\infty}^{\infty} E_n b_n^p J_n(k_r^p r) e^{in\phi},
\end{align*}
\]

where, for the sake of simplicity, we define \( J_n(\rho, \beta) \equiv J_n'(\rho)/\rho \) and \( \tilde{J}_n(\rho, \beta) \equiv \beta J_n'(\rho) + n J_n(\rho)/\rho; \) and, for the shell region, \( q = 2 \) (\( a < r < b \)),

\[
\begin{align*}
E_p^r & = - \sum_{n=-\infty}^{\infty} E_n \left[ c_n^p J_n(k_r^p r, \beta_p^q) + d_n^p \tilde{J}_n(k_r^p r, \beta_p^q) \right] e^{in\phi}, \\
E_p^\phi & = -i \sum_{n=-\infty}^{\infty} E_n \left[ c_n^p J_n'(k_r^p r, \beta_p^q) + d_n^p \tilde{J}_n'(k_r^p r, \beta_p^q) \right] e^{in\phi}, \\
H_p^z & = \frac{k_p^p}{\omega \mu_0} \sum_{n=-\infty}^{\infty} E_n \left[ c_n^p J_n(k_r^p r) + d_n^p \tilde{J}_n(k_r^p r) \right] e^{in\phi},
\end{align*}
\]

where \( Y_n(\rho, \beta) \equiv Y_n'(\rho) + n Y_n(\rho)/\rho \) and \( \tilde{Y}_n(\rho, \beta) \equiv \beta Y_n'(\rho) + n Y_n(\rho)/\rho, \) with \( Y_n(\rho) \) being the cylindrical Neumann function.

The Lorenz-Mie coefficients \( a_n^p, b_n^p, c_n^p \) and \( d_n^p \) are obtained by imposing the boundary conditions at \( r = a \) and \( r = b \), reading:

\[
\begin{align*}
a_n^p & = \frac{\tilde{m}_p^2 J_n(k_r^p a)^2 - A_n^p Y_n(m_r^p a)^2 - J_n(y) a_n^p}{\tilde{m}_p^2 H_n^{(1)}(y) [J_n(m_r^p a)^2 - A_n^p Y_n(m_r^p a)^2] - H_n^{(1)}(y) a_n^p}, \\
b_n^p & = \frac{\tilde{m}_p^2 J_n(k_r^p b)^2 - A_n^p Y_n(m_r^p b)^2}{\tilde{m}_p^2 H_n^{(1)}(y) [J_n(m_r^p b)^2 - A_n^p Y_n(m_r^p b)^2] - H_n^{(1)}(y) a_n^p}, \\
c_n^p & = \frac{2i/(\pi y)}{\tilde{m}_p^2 H_n^{(1)}(y) [J_n(m_r^p b)^2 - A_n^p Y_n(m_r^p b)^2] - H_n^{(1)}(y) a_n^p}, \\
d_n^p & = -A_n^p e_n^p,
\end{align*}
\]

where the auxiliary functions are

\[
\begin{align*}
a_n^p & = J_n(m_r^p a, \beta_p^q) - A_n^p Y_n(m_r^p a, \beta_p^q), \\
a_n^p & = \frac{\tilde{m}_p^2 J_n(m_r^p a) J_n(m_r^p b, \beta_p^q) - \tilde{m}_p^2 J_n(m_r^p b) J_n(m_r^p a)}{\tilde{m}_p^2 J_n(m_r^p b) Y_n(m_r^p a, \beta_p^q) - \tilde{m}_p^2 J_n(m_r^p a) Y_n(m_r^p b, \beta_p^q)}.
\end{align*}
\]
with size parameters $x = ka$ and $y = kb$. The relative refractive and impedance indices are $m_q^p = k^2/k = \sqrt{\varepsilon_q^p/\mu_q^p}/(\varepsilon_0 \mu_0)$ and $\tilde{m}_q = \sqrt{\varepsilon_q^p/\mu_q^p}$, respectively ($m_q = \tilde{m}_q$ if $\mu_q = \mu_0$ [30]). Notice that parity symmetry $a_n = a_n$, $b_n = b_n$, $c_n = c_n$ and $d_n = d_n$ only holds if $\beta_1 = \beta_2 = 0$, which retrieves the isotropic result for the TM mode [37–39].

The corresponding expressions for $s$-polarized waves are analogous to the ones above. For the sake of completeness, the multipole expansions and the Lorenz-Mie coefficients $(a_n^q, b_n^q, c_n^q, d_n^q)$ are presented in Appendix A. These expressions are necessary to study cylinders composed of gyromagnetic materials, Eq. [2].

### B. Lorenz-Mie efficiencies and multiple scattering

The extinction and scattering efficiencies for cylindrical scatterers at normal incidence are directly calculated via $Q_{\text{sca}} = (2/y) \sum_{n=-\infty}^{\infty} |a_n|^2$ and $Q_{\text{ext}} = (2/y) \sum_{n=-\infty}^{\infty} \text{Re}(a_n)$, respectively, where $y = kb$ is the size parameter of the outer cylinder. They are defined as the respective cross section of a segment of the outer cylinder. They are defined as the respective cross section of a segment $L \gg b$ of the infinite cylinder in units of the geometrical cross section $2bL$. Rewriting these efficiencies to consider sums for $n \geq 1$, one has

$$Q_{\text{sca}} = \frac{2}{y} \left| a_0 \right|^2 + \sum_{n=1}^{\infty} \left( |a_{-n}|^2 + |a_n|^2 \right),$$  \hfill (21)

$$Q_{\text{ext}} = \frac{2}{y} \text{Re} \left[ a_0 + \sum_{n=1}^{\infty} (a_{-n} + a_n) \right],$$  \hfill (22)

with $Q_{\text{abs}} = Q_{\text{ext}} - Q_{\text{sca}}$ being the absorption efficiency. The differential scattering efficiency reads

$$\frac{\partial Q(\phi)}{\partial \phi} = \frac{2}{\pi y} \left| a_0 + \sum_{n=1}^{\infty} (a_{-n} e^{in\theta} + a_n e^{-in\theta}) \right|^2,$$  \hfill (23)

where $\theta = \pi - \phi$ is the scattering angle, so that $\theta = 0^\circ$ corresponds to forward scattering and $\theta = 180^\circ$ corresponds to backscattering. Here we consider the same convention as in Refs. [10] and [28], so that one obtains $Q_{\text{sca}}$ by integrating Eq. (23) in the range $[0, \pi]$ instead of $[0, 2\pi]$ [44]. The efficiencies for $p$ and $s$ polarizations are obtained by considering $a_0^p$ and $a_0^s$, respectively, where one must define the quantities $(\varepsilon_q, \mu_q, \beta_q)$ according to relations [30] or [44].

Some quantities calculated in the single scattering approach can be used to study multiple-scattering properties in the diffusive regime and for low concentrations of scatterers [12, 13]. In this regime, the scattering mean free path $\ell_{\text{sca}}$ is comparable to the size of the system and suffices $k\ell_{\text{sca}} \gg 1$. This situation is depicted in Fig. 2.

The asymmetry parameter $\langle \cos \theta \rangle$, which is related to the transferred linear momentum in the forward direction [1], is calculated from the relationship

$$Q_{\text{sca}}(\cos \theta) = \int_0^\pi \cos \theta \frac{\partial Q(\phi)}{\partial \phi} \, d\phi.$$

where $Q_{\text{sca}}$ and $\partial Q(\phi)/\partial \phi$ are defined in Eqs. (21) and (23), respectively, for a single-scattering process.

The transport mean free path $\ell^* = 1/(\rho \sigma^*)$, where $\rho$ is the density of particles in the host medium, and $\sigma^* = \sigma_{\text{ext}} - \sigma_{\text{sca}}(\cos \theta)$ is the transport cross section [44] with $\sigma_{\text{ext}}$ and $\sigma_{\text{sca}}$ being the extinction and scattering cross sections, respectively. Notice that here we take into account unavoidable losses to calculate $\ell^*$, as in Refs. [44–46]. For lossless scatterers $\sigma_{\text{ext}} = \sigma_{\text{sca}}$, so that $\ell^* = \ell_{\text{sc}}(1 - \cos \theta)$, where $\ell_{\text{sc}} = 1/(\rho \sigma_{\text{sca}})$ is the scattering mean free path. For a disordered 2D medium consisting of parallel cylindrical particles, as depicted in Fig. 2, we obtain

$$\ell^* = \frac{\pi/2}{f_{\text{pack}} (Q_{\text{ext}} - Q_{\text{sca}}(\cos \theta))},$$

where $f_{\text{pack}}$ is the filling fraction. It is convenient to define the extinction mean free path: $\ell_{\text{ext}} = \pi b/(2f_{\text{pack}}Q_{\text{ext}})$. In this 2D case in the Voigt configuration, the effective diffusion coefficient is $D = v_{\text{tp}} \ell^*/2$, where $v_{\text{tp}}$ is the energy-transport velocity. Note that $D$ does not depend explicitly on $B$, which does not apply to the Faraday configuration $(B \parallel k)$ [17]. From the weak disorder approximation of the Bethe-Salpeter equa-

![Figure 2](image-url)


\[ \frac{v_E}{c} \approx \frac{1}{\text{f}_\text{pack}(W/W_0 - 1)} \cdot \] (26)

where \( c = 1/\sqrt{\varepsilon_0 \mu_0} \) is the velocity of light in the host medium and \( W/W_0 \) is the energy-enhancement factor in a single scatterer, with \( W \) being the time-averaged internal EM energy \([17, 33, 38]\). Equation (26), originally calculated for spheres, is not restricted to low densities of scatterers \([12]\) and can successfully be applied to cylinders \([16]\). In the following, we analytically calculate \( W/W_0 \) for a gyroscopic coated cylinder for both \( p \) and \( s \) waves.

### III. THE EXACT ANALYTIC TIME-AVERAGED ENERGY WITHIN GYROTROPIC COATED CYLINDERS

The time-averaged EM energy density within a gyroelectric and gyromagnetic medium \((\mathcal{E}_q, \mu_q)\), given by Eqs. (11) and (2), is

\[
\langle u_q \rangle_t = \frac{1}{4} \varepsilon_{q,\perp} \left( |E_{q_r}|^2 + |E_{q_\theta}|^2 \right) + \varepsilon_{q,||} |E_{q_z}|^2
\]

\[
+ \varepsilon_{q,||} \left( |H_{q_r}|^2 + |H_{q_\theta}|^2 \right) + \varepsilon_{q,\perp} |H_{q_z}|^2
\]

\[
+ 2\text{Im} \left( \varepsilon_{q,\perp} E_{q_r} E_{q_\theta}^* + \varepsilon_{q,||} H_{q_r} H_{q_\theta}^* \right) \right),
\] (27)

where the effective energy coefficients, if the medium is weakly absorbing \([50]\), are \( \varepsilon_{q,\perp} = \partial[\text{Im}(\varepsilon_{q,\perp})]/\partial \omega \), \( \varepsilon_{q,||} = \partial[\text{Im}(\varepsilon_{q,||})]/\partial \omega \), \( \gamma_{q,\perp} = \partial[\text{Re}(\gamma_{q,\perp})]/\partial \omega \), and so forth. Equation (27) is simplified whether we consider \( p \) waves \((E_{q_z} = H_{q_r} = H_{q_\theta} = 0)\) or \( s \) waves \((H_{q_z} = E_{q_r} = E_{q_\theta} = 0)\).

From Eq. (27), the corresponding average EM energy in a segment \( L \) of a cylindrical shell \( l_1 \leq l \leq l_2 \) is, therefore \([32]\),

\[
W_l = \int_{-L/2}^{L/2} dz \int_0^{2\pi} d\phi \int_{l_1}^{l_2} dl \langle u_q \rangle_t. \] (28)

If the cylindrical shell \( l_1 \leq l \leq l_2 \) has the same optical properties as the surrounding medium \((\varepsilon_0, \mu_0)\), it follows that

\[
W_{0q} = \frac{\varepsilon_0}{2} |E_0|^2 \pi (l_2^2 - l_1^2) L, \] (29)

where \( E_0 \) is the electric amplitude of the incident wave.

The technical details involved in the analytical derivation of \( W_l \), with \( q = 1 \) for \((l_1, l_2) = (0, a)\) and \( q = 2 \) for \((l_1, l_2) = (a, b)\), are given in Appendix B.

Using the results in Appendix B let us consider the partial contributions to the internal energy: \( W_{q,\perp} = \int d^3r \varepsilon_{q,\perp} (|E_r|^2 + |E_\theta|^2)/4, \) \( W_{q,||} = \int d^3r \varepsilon_{q,||} |E_z|^2/4, \) \( W_{q,\perp} = \int d^3r \varepsilon_{q,\perp} (|E_r|^2 + |E_\theta|^2)/4, \) and so on. For both \( p \) or \( s \) polarizations, the partial contributions to the EM energy in the cylinder have the same analytical expression, but with the corresponding Lorenz-Mie coefficients and material parameters in the equations.

From Eq. (25), we obtain for the core region \((q = 1, l_1 = 0, l_2 = a)\)

\[
\frac{W_{1\perp}}{W_{01}} = \zeta_{1\perp} \sum_{n = -\infty}^{\infty} |c_n|^2 \mathcal{F}_{1,n}(J_J), \tag{30}
\]

\[
\frac{W_{1||}}{W_{01}} = \zeta_{1||} \sum_{n = -\infty}^{\infty} |c_n|^2 \mathcal{F}_{1,n}^2(J_J), \tag{31}
\]

where we have considered \((A, B) = (b_n, 0)\) into Eqs. (31) and (32) in Appendix B to obtain \( W_{1\perp}^2 \) and \( W_{1||}^2 \), respectively. The auxiliary functions \( \mathcal{F}_{1,n}^{(J_J)} \) and \( \mathcal{F}_{1,n}^{(J_J)} \) are obtained from Eqs. (33) or (34) and (35), respectively, and depend on the product of the Bessel functions. The EM energy within the core \((\mathcal{E}_0, \mu_0)\) is, therefore,

\[
W_1 = W_{1\perp} + W_{1||}. \tag{32}
\]

For the cylindrical shell \((q = 2, l_1 = a, l_2 = b)\), we obtain

\[
\frac{W_{2\perp}}{W_{02}} = \zeta_{2\perp} \sum_{n = -\infty}^{\infty} \left\{ |c_n|^2 \mathcal{F}_{2,n}^{(J_J)} \right\}, \tag{33}
\]

\[
+ 2\text{Re} \left[ c_n d_n^* \mathcal{F}_{2,n}^{(J_J)} \right] + |d_n|^2 \mathcal{F}_{2,n}^{(J_J)} \right\}, \tag{34}
\]

where we have considered \((A, B) = (c_n, d_n)\) into Eqs. (31) and (32) in Appendix B to achieve \( W_{2\perp}^2 \) and \( W_{2||}^2 \), respectively. The auxiliary functions \( \mathcal{F}_{2,n}^{(J_J)} \) and \( \mathcal{F}_{2,n}^{(J_J)} \) are defined in Appendix B where \( Z \) and \( \breve{Z} \) are any Bessel \((J_n)\) or Neumann \((Y_n)\) function. The EM energy within the shell \((\mathcal{E}_0, \mu_0)\) is

\[
W_2 = W_{2\perp}^2 + W_{2||}^2. \tag{35}
\]

To obtain the internal energy associated with \( p \) or \( s \) polarization schemes one must consider Eqs. (31) and \((b_n, c_n, d_n)\) or Eqs. (32) and \((b_n, c_n, d_n)\), respectively, and apply the relations:

\[
\text{TM (p)} \Rightarrow \begin{cases} \frac{\varepsilon_{q,\perp}}{\varepsilon_0}, \\ \frac{\varepsilon_{q,||}}{\varepsilon_0} \\ \frac{\varepsilon_{q,\perp}}{\varepsilon_0} \end{cases} \tag{36}
\]

\[
\text{TE (s)} \Rightarrow \begin{cases} \frac{\varepsilon_{q,\perp}}{\varepsilon_0}, \\ \frac{\varepsilon_{q,||}}{\varepsilon_0} \end{cases} \tag{37}
\]
The energy-enhancement factor $W_{1,2}/W_0$ within the scatterer, where $W_{1,2} = W_1 + W_2$ is the total internal energy and $W_0 = W_{01} + W_{02}$, is

$$W_{1,2} \over W_0 = S^2 W_1 \over W_{01} + (1 - S^2) W_2 \over W_{02}, \quad (38)$$

with $S = a/b$ being the aspect ratio.

In addition, since the internal field intensities are proportional to the power loss, we can write the absorption efficiency $Q_{\text{abs}}^p$ in terms of the partial energy contributions:

$$Q_{\text{abs}}^p = \frac{\text{Im} \left( S^2 \left[ \frac{\varepsilon_{\|} W_{p1}^{\text{||}}}{\varepsilon_{\|} \text{eff} W_{01}} + \frac{\varepsilon_{\perp} W_{p1}^{\text{\perp}}}{\varepsilon_{\perp} \text{eff} W_{01}} + \frac{\mu_{\|} W_{p1}^{\text{||}}}{\mu_{\|} \text{eff} W_{01}} \right] + (1 - S^2) \left[ \frac{\varepsilon_{\perp} W_{p2}^{\text{\perp}}}{\varepsilon_{\perp} \text{eff} W_{02}} + \frac{\mu_{\perp} W_{p2}^{\text{\perp}}}{\mu_{\perp} \text{eff} W_{02}} \right] \right)}{\pi \gamma}, \quad (39)$$

For $s$ waves, $Q_{\text{abs}}^s$ is obtained from Eq. (39) by replacing the symbols ($\varepsilon, \gamma, \mu$) with ($\mu, \eta, \varepsilon$) and the label $p$ with $s$.

It is worth mentioning that Eq. (39) provides an explicit connection between the internal energy and a measurable quantity, $Q_{\text{abs}}^s$.

### IV. DIELECTRIC MICROCYLINDERS WITH MAGNETO-OPTICAL COATINGS

So far our results are general and can be applied, e.g., to the study of coated gyro-magnetic materials and nanowires. Here we focus on a particular case: infinite coated gyroelectric cylinders irradiated with THz $p$ waves. Finite-size effects are known to weakly affect the scattering properties of cylinders provided their length is much larger than both their diameter and the incident wavelength [1-11, 51]. Provided these conditions are met, light is mostly scattered in the plane perpendicular to the cylinder axis [4]. Some technical details regarding the calculations are provided in Appendix C.

The cylinder is embedded in vacuum ($\varepsilon_0, \mu_0$) and consists of a dielectric core made of silica (SiO$_2$) ($\varepsilon_1 = 2.25\varepsilon_0$ and $\mu_1 = \mu_0$ in the far-infrared) coated with a cylindrical shell of indium antimonide (InSb), whose dielectric tensor [14] for $q = 2$ reads

$$\varepsilon_{\perp}^{\text{eff}}(\omega, B, T) = \varepsilon_0 - \varepsilon_{\|} \frac{\omega_p^2 (\omega + \Gamma) - \omega^2}{\omega (\omega + \Gamma)^2 - \omega_p^2}, \quad (40)$$

$$\varepsilon_{\parallel}^{\text{eff}}(\omega, B, T) = \varepsilon_0 - \frac{\omega_p^2}{\omega (\omega + \Gamma)^2 - \omega_p^2}, \quad (41)$$

$$\gamma_2(\omega, B, T) = \frac{\omega_p^2 \omega_c}{\omega (\omega + \Gamma)^2 - \omega_p^2}, \quad (42)$$

where $\varepsilon_{\infty} = 15.7$ is the high-frequency permittivity. The cyclotron frequency is $\omega_c = eB/m^*$, where $e$ is the electron charge, $B$ is the external dc magnetic field, and $m^* = 0.015m_e$ is the effective mass of free carriers, with $m_e$ being the bare mass of the electron. The plasma frequency and the collision frequency of carriers are, respectively, $\omega_p = \sqrt{N e^2/(\varepsilon_0 m^*)}$ and $\Gamma = e/(\mu_e m^*)$, where $N$ is the intrinsic carrier density and $\mu_e$ is the electron mobility. The intrinsic carrier density (in cm$^{-3}$) in undoped InSb is strongly dependent on the temperature and reads [54, 55]:

$$N(T) \approx 5.76 \times 10^{14} T^{3/2} \exp[-0.129/(k_b T)], \quad (43)$$

where $k_B$ is the Boltzmann constant (in eV K$^{-1}$). This expression, derived from the temperature variation of the Hall coefficient, agrees well with experimental data for 150 K $\leq$ T $\leq$ 300 K [33, 37]; for this reason we restrict our analysis to this temperature range. In addition, we employ a realistic empirical expression for the electron Hall mobility (in cm$^2$ V$^{-1}$ s$^{-1}$) [33]

$$\mu_e(T) \approx 7.7 \times 10^4 (T/300)^{-5/3}, \quad (44)$$

which has been experimentally validated in the temperature range 150 K $\leq$ T $\leq$ 300 K [35], which we consider here.

For the corresponding energy coefficients in Eq. (27), we consider the London approach [55] to deal with lossy Drude-Lorentz models [56]:

$$\varepsilon_{21}^{\text{eff}}(\omega) = \text{Re} \left[ \varepsilon_{21}^{\perp}(\omega) + \frac{2\omega}{\pi} \text{Im} \left[ \varepsilon_{21}^{\perp}(\omega) \right] \right], \quad (45)$$

$$\gamma_2^{\text{eff}}(\omega) = \text{Re} \left[ \gamma_2(\omega) + \frac{2\omega}{\pi} \text{Im} \left[ \gamma_2(\omega) \right] \right]. \quad (46)$$

We recall that $\varepsilon_2^{\perp}(\omega)$ does not contribute to the scattering by $p$ waves. The remaining energy coefficients are calculated by the usual Landau’s formula for lossless or weakly absorbing media [57]. For non-dispersive media, it is simply the real part.

From Eqs. (40) and (41), note that $\varepsilon_2^{\perp}(\omega, T, B) = \varepsilon_1^{\perp}(\omega, T, 0)$. Using relations [14], i.e., $\varepsilon_1^{\perp} = \varepsilon_2^{\perp}$ and $\mu_2^{\perp} = \mu_0$ (with $\beta_2^\| = 0$), one can readily verify that scattering for $s$ waves is insensitive to $B$. Indeed in the Rayleigh limit ($kB < 1$) for $s$ waves one has $|a_{s1}^{\|} | > |a_{s2}^{\|} |$ for nonmagnetic scatterers [1], and hence the overall scattering response depends on the bulk resonances of the InSb associated with $\varepsilon_2^{\perp}$. For this reason, we do not consider $s$ waves in our discussion. In addition, it is worth mentioning that oblique incidence would lead to cross-polarization coupling for higher-order modes, i.e. $s$ or $p$ waves would be scattered in a combination of both $s$ and $p$ polarization states [51]. Since the magneto-optical response is maximal for $p$ waves and vanishes for $s$ waves, oblique incidence would weaken the net magneto-optical effect due to radiation polarization conversion. For this reason, together with the fact that for normal incidence an analytical solution exists, we prefer to focus on the normal incidence case [53, 54].

In Figs. 3(a)–3(c), we show the asymmetry parameter $(\cos \theta)$, the energy-enhancement factor $W_{1,2}/W_0$, and $\mu_e(T)$.
the transport mean free path $\ell^*$, respectively, in a (SiO$_2$) core-shell (InSb) cylinder for $p$ waves as a function of the frequency and external magnetic field. We set $b = 2.5 \mu m$ (with aspect ratio $S = a/b = 0.35$), and room temperature ($T = 295$ K). The range of size parameters in Figs. (a)–(c) is $0.089 < kb < 0.17$, so that dipole contributions to the scattering ($n = 0$ and $n = \pm 1$) are dominant; in particular, the magnetic dipole contribution ($n = 0$) is negligible since $\mu_1 = \mu_2 = \mu_0$. Figure (a) shows that the application of an external magnetic field $B$ strongly affects the scattering directionality. Indeed, the presence of $B$ breaks the scattering isotropy of dipolar scattering, in contrast to what occurs for non-Faraday-active materials in the Rayleigh regime ($kb \ll 1$). In these materials $\langle \cos \theta \rangle \approx 0$ \cite{xia1997quantum} as a consequence of the typical isotropic dipolar scattering pattern, for which $Q_{e\omega}^\text{p} \propto |a_1^\text{p}|^2$. For magneto-optical materials $a_1^\text{p} \neq a_1^\text{p}$ for $B \neq 0$ [see the inset of Fig. (a)], leading to a strongly asymmetric, magnetic-field-dependent scattering pattern, as shown in Fig. (a). In particular, the two
peaks related to the dipole resonance for \( B = 0.0 \) T are essentially due to the presence of the dielectric core SiO\(_2\). We have verified that as \( a \to 0 \), only one peak remains for \( a_1^p = a_{-1}^p \) around \( f = 2.4 \) THz. Here, the dielectric core broadens the dipole resonance for \( B = 0.0 \) T.

\[
\theta_{\text{tot}} \approx \frac{1}{2} \arctan \left[ \frac{\text{Im} \left( a_1^p a_{-1}^* \right)}{\text{Re} \left( a_1^p a_{-1}^* \right)} \right].
\]

Conversely, for the stored EM energy, we have nonvanishing interference between the electric-field components \((E_{2x}, E_{2y})\) in the \( xy \) plane, as can be seen from the EM energy density expression [see Eq. (27)]. It is worth emphasizing that, in contrast to previous studies on directional scattering [13, 14, 31], our approach does not rely on magnetic resonances since \( a_1^p = 0 \). Rather, it is based on the magnetic-field dependence of electric dipolar resonances \( a_1^p \) and \( a_{-1}^p \).

The breaking of the degeneracy in the scattering coefficients \( a_1^p \neq a_{-1}^p \) in a magnetic field also shows up in the internal EM energy stored in the cylinder, \( W_{1.2} \), as shown in Fig. 3(b). In fact, by increasing \( B \) the internal resonances at \( a_1^p \) and \( a_{-1}^p \) become farther apart in frequency, leading to an increasing spectral gap in \( W_{1.2} \). As the internal energy is proportional to the absorption cross section, \( Q_{\text{abs}}^\text{int} \) for \( kb \ll 1 \) and weak absorption \([32, 63]\), Fig. 3(b) demonstrates a novel way to externally tune EM absorption by applying an external magnetic field. It is worth mentioning that this effect can be achieved for moderate magnetic fields (\( B \approx 0.5 \) T) and that \( B > 0 \) shifts \( a_1^p \) and \( a_{-1}^p \) to low and high frequencies, respectively; \( B < 0 \) does the opposite. In Fig. 3(c), the ratio \( \ell^*/\ell_{\text{ext}} \) is shown to demonstrate that a frequency band exists below approximately 2.4 THz in which the anomalous transport regime \( \ell^* < \ell_{\text{ext}} \) occurs. This band can be shifted to lower frequencies by varying \( B \) and results from the negative asymmetry parameters in the same frequency range, as shown in Fig. 3(a).

Figures 3(d)–3(f) demonstrate that it is possible to achieve directional scattering, which can be tuned by applying an external magnetic field, beyond the Rayleigh limit. Indeed, in Figs. 3(d)–3(f) (\( \text{cos} \theta \)), \( W_{1.2} \), and \( \ell^* \) are calculated, respectively, for the same system but now with \( b = 25 \mu m \) (\( S = a/b = 0.5 \)), and \( T = 250 \) K. For the frequency range 0.6 THz to 2.6 THz, size parameters are \( 0.31 < kb < 1.4 \), i.e., beyond the Rayleigh limit. In addition, by decreasing the temperature from 295 K to 250 K, absorption of the InSb coating also decreases significantly [see Eqs. (33) and (43)]. The overall result is that for this new set of parameters absorption is small beyond the Rayleigh limit, so that \( Q_{\text{abs}}^\text{int} \) is comparable to \( Q_{\text{ext}}^\text{int} \) for \( B = 0.0 \) T. In Fig. 3(d), we demonstrate that \( \text{cos} \theta \) becomes negative by applying \( B \) even for \( kb \approx 1 \). Figure 3(e) shows that the presence of \( B \) increases the magnetic dipole contribution \( a_1^p \) for low frequencies (1.2 THz) at the same time that it increases the electric dipole contribution \( a_{-1}^p \) for high frequencies (2.4 THz). The interference between electric and magnetic dipole contributions leads to a minimum in the internal energy around \( f \approx 1.6 \) THz as \( B \) increases. As shown in Fig. 3(f), this interference induces a band (1.3 THz to 2.0 THz).
of anomalous scattering in which \( \ell^* < \ell_{\text{ext}} \). Moreover, for \( B = 1.3 \, \text{T} \), \( \ell^* \approx \ell_{\text{sca}}/(1 - \langle \cos \theta \rangle) \) since absorption becomes very small in this frequency range, as can be verified by the inset in Fig. 3(f). This implies that there exists a transport regime in which \( \ell^* < \ell_{\text{sca}} \), with \( \ell^* \approx 0.8 \ell_{\text{sca}} \). It is worth mentioning that the application of the external magnetic field can suppress absorption in this frequency range, resulting in \( Q_{\text{sca}}/Q_{\text{ext}} \approx 1 \), as shown in the inset in Fig. 3(f). It is worth emphasizing that this anomalous scattering regime, induced by the external magnetic field, occurs with the inclusion of unavoidable losses and without consideration of any positional correlation among scatterers, in contrast to Refs. 31 and 32, respectively. In addition, for fixed frequency and material parameters, Fig. 4 shows that we can effectively tune the directional scattering pattern by applying \( \mathbf{B} \).

In Fig. 5 we investigate the impact of tunable scattering anisotropy in light transport in planes composed of identical, infinitely long magneto-optical core-shell cylinders, as depicted in Fig. 2. The parameters are the same as in Figs. 3(a) and 3(b): \( b = 2.5 \, \mu\text{m}, S = a/b = 0.35 \), and \( T = 295 \, \text{K} \). For a fixed packing fraction \( f_{\text{pack}} = 35\% \) we calculate the energy-transport velocity \( v_\parallel \) and the diffusion coefficient \( D = v_\parallel \ell^*/2 \). For fixed (room) temperature, Figs. 5(a) and 5(b) show that one can effectively tune light transport with an external magnetic field. Indeed, Figs. 5(a) and 5(b) reveal that the application of an external magnetic field up to \( B \approx 1.0 \, \text{T} \) leads to an increase in \( v_\parallel \) and \( D \), increasing diffusion in the plane. In particular, as the magnetic field is increased the diffusion coefficient \( D \) becomes maximal at a frequency band where a minimum at \( \ell^* \) and \( v_\parallel \approx 0.15c \), and hence \( D \), exists for \( B = 0.0 \, \text{T} \). Indeed, at \( B = 1.1 \, \text{T} \), the diffusion coefficient is two orders of magnitude greater than at \( B = 0.0 \, \text{T} \).

In Figs. 5(c) and 5(d), we calculate \( v_\parallel \) and \( D \) as a function of the frequency fixing all the aforementioned parameters, for \( B = 0.0 \, \text{T} \) and for different temperatures. The analysis of these figures reveals that tuning the light scattering and light propagation in-plane with the temperature is also possible. In fact, note that by increasing the temperature from \( T = 270 \, \text{K} \) to \( 300 \, \text{K} \) one broadens and shifts the band of minimum \( v_\parallel \) to high frequencies, and hence the diffusion coefficient \( D \). Also, as the temperature decreases (typically for \( T > 220 \, \text{K} \)), smaller magnetic fields are required to achieve a strong magneto-optical response in InSb at high frequencies, in
the THz range \( \omega \). This implies that, for \( T < 295 \text{ K} \), smaller magnetic fields (e.g., \( B \approx 0.5 \text{ T} \) instead of \( 1.0 \text{ T} \)) could be applied to obtain the same energy-transport enhancement exhibited in Figs. 5a and 5b. This strong dependence on the temperature facilitates the modulation of the EM energy transport, which can be enhanced or attenuated by \( B \) and shifted in frequency by varying temperature.

Although we have focused on InSb magneto-optical coatings, there are other materials that could possibly be used to achieve similar results. As alternatives to InSb, one could use, e.g., materials that are known to exhibit a low electron effective mass \( m^* \), and hence a high cyclotron frequency \( \omega_c \), such as InAs, HgTe, Hg\(_{1-x}\)Cd\(_x\)Te, PbTe, PbSe, PbS, and GaAs \( [57, 58] \). In cylindrical geometry, all these materials are expected to exhibit a strong magneto-optical effect under a normal incidence of \( p \) waves at high frequencies.

V. CONCLUSIONS

Using the Lorenz-Mie theory, we have calculated a set of analytically expressible results to completely describe the EM scattering by gyrotropic core-shell magneto-optical cylinders. A closed analytic expression has been derived for the EM energy stored inside the cylinder. For concreteness, using realistic material parameters for the silica core and InSb shell, we have calculated the stored EM energy and the scattering anisotropy. We have shown that the application of an external magnetic field induces a drastic decrease in EM absorption in a frequency window in the THz, where absorption is maximal in the absence of the magnetic field. We have demonstrated not only that the scattering anisotropy can be externally tuned by applying a magnetic field, but also that it can reach negative values in the THz even in the dipolar regime. This is due to the fact that the external magnetic field breaks the degeneracy between the first two electric Mie scattering coefficients, which, without the magnetic field, lead to isotropic scattering. We have shown that this also leads to an anomalous regime of multiple light scattering in a collection of magneto-optical core-shell cylinders, in which the scattering mean free path is longer than the transport mean free path in specific ranges in the THz. In our approach, we have demonstrated an unprecedented degree of external control of multiple light scattering, which can be tuned by either applying an external magnetic field or varying the temperature.

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Appendix A: Electric and magnetic fields for \( s \) waves

Let us briefly discuss the multipole expansions for TE mode or \( s \) polarization. According to Fig. 11, we have: \( \mathbf{E}_s = (\mathbf{E}_s, H_s) = (E_s \mathbf{z}, H_s) e^{-ikr \cos \phi} \), with \( k = -ik_0 \). By duality relations between electric and magnetic quantities, the EM fields for \( \mathbf{E}_s \) polarization in cylindrical coordinates \( (r, \phi, z) \) are readily obtained from Eqs. (9)–(10). First, we must redefine the material parameters according to Eq. (11), substituting \( (\varepsilon^p_{m}, \mu^p_{m}, \beta^p_{m}) \) with \( (\varepsilon^s_{m}, \mu^s_{m}, \beta^s_{m}) \). The field components are then obtained by replacing \( (\mathbf{E}^p, \mathbf{E}^p, \mathbf{H}^p) \) with \( (\mathbf{H}^s, -\mathbf{H}^s, p^{-1} \mathbf{E}^s) \) and \( (\alpha^p_{m}, \beta^p_{m}, \gamma^p_{m}, \delta^p_{m}) \) with \( (\alpha^s_{m}, \beta^s_{m}, \gamma^s_{m}, \delta^s_{m}) \), where \( p = \omega \mu_0 / k \) for the incident and scattered EM fields \( \{E, H\} = \{E_1, \ldots, E_{10}\} \) and \( p = \omega \mu_0^p / k^p \) for the internal fields \( \{q = 1 \text{ for Eqs. (11)} - (13) \text{ and } q = 2 \text{ for Eqs. (14)} - (16) \} \).

The TE coefficients are

\[
\alpha^s_{n} = \frac{J_n^\prime(y) [J_n(m_s^2 y) - A_n^s Y_n(m_s^2 y)] - \tilde{m}_s^2 J_n(y) \alpha^s_{n}}{H_n^s(y) [J_n(m_s^2 y) - A_n^s Y_n(m_s^2 y)] - \tilde{m}_s^2 H_n^s(y) \alpha^s_{n}}, \quad (A1)
\]

\[
b^s_{n} = \frac{c^s_{n} [J_n(m_s^2 x) - A_n^s Y_n(m_s^2 x)]}{J_n(m_s^2 x)}, \quad (A2)
\]

\[
c^s_{n} = \frac{2s/\pi y}{H_n^s(y) [J_n(m_s^2 y) - A_n^s Y_n(m_s^2 y)] - \tilde{m}_s^2 H_n^s(y) \alpha^s_{n}}, \quad (A3)
\]

\[
d^s_{n} = -A_n^s c^s_{n}, \quad (A4)
\]

where the new auxiliary functions are

\[
\alpha^s_{n} = J_n(m_s^2 y, \beta^s_{n}) - A_n^s Y_n(m_s^2 y, \beta^s_{n}) \end{equation}
\]

\[
\alpha^s_{n} = \frac{\tilde{m}_s J_n(m_s^2 x) J_n(m_s^2 y, \beta^s_{n}) - \tilde{m}_s \tilde{m}_s J_n(m_s^2 x, \beta^s_{n}) J_n(m_s^2 y)}{m_s^2 J_n(m_s^2 x) Y_n(m_s^2 x, \beta^s_{n}) - \tilde{m}_s \tilde{m}_s J_n(m_s^2 x, \beta^s_{n}) Y_n(m_s^2 x)}
\]

and \( m^s_q = \sqrt{\varepsilon^s_q \mu^s_q / (\varepsilon^s_q \mu^s_q)} \) and \( \tilde{m}_s^2 = \sqrt{\varepsilon^2 \mu_0 / (\varepsilon_0 \mu_0^2)} \).

Appendix B: Integrals of Bessel and Neumann functions

To calculate the stored EM energy \( W_q \) defined in Eq. 28, we perform volume integrations involving the product of Bessel and/or Neumann functions. By the recurrence relations \( n Z_{n-1}(p) = \rho Z_{n-1}(p) - \rho Z_n(p) \) and \( \rho Z_n(p) = n Z_{n-1}(p) - \rho Z_{n+1}(p) \), for any cylindrical Bessel
or Neumann functions \(Z_n\), we obtain
\[
2 \left[ |A_J(n, \beta) + B_Y(n, \beta)|^2 + |A_\tilde{J}(n, \beta) + B_\tilde{Y}(n, \beta)|^2 \right] \\
= |1 + \beta|^2 \left| A_J(n-1, \beta) + B_Y(n-1, \beta) \right|^2 \\
+ |1 - \beta|^2 \left| A_{\tilde{J}}(n+1, \beta) + B_{\tilde{Y}}(n+1, \beta) \right|^2 ,
\]
and
\[
4 Re \left\{ [A_J(n, \beta) + B_Y(n, \beta)]^* \left[ A_\tilde{J}(n, \beta) + B_\tilde{Y}(n, \beta) \right] \right\} \\
= |1 + \beta|^2 \left| A_J(n-1, \beta) + B_Y(n-1, \beta) \right|^2 \\
- |1 - \beta|^2 \left| A_{\tilde{J}}(n+1, \beta) + B_{\tilde{Y}}(n+1, \beta) \right|^2 .
\]

Equations (B1) and (B2) are suitable for simplifying the radial integrals of the field components. Indeed, according to Refs. [39], we define, for \(m_q \neq m_q^*\) (\(q = \{1, 2\}\)), the auxiliary function
\[
\mathcal{T}_{q,n}^{(ZZ)} = \frac{1}{(l_2^2 - l_1^2)} \int_{l_1}^{l_2} dr \, r Z_n(\rho_q) \bar{Z}_n(\rho_q^*) \\
= r^2 \left[ \rho_q^2 Z_n(\rho_q) \bar{Z}_n(\rho_q^*) - \rho_q Z_n(\rho_q) \bar{Z}_n(\rho_q^*) \right] \bigg|_{r=l_2}^{r=l_1},
\]
where \(Z_n\) and \(\bar{Z}_n\) are any cylindrical Bessel or Neumann functions, and \(l_1, l_2 \in \mathbb{R}\) are the integration limits. Using the L’Hospital rule, if \(m_q = m_q^*\) [i.e., \(\text{Im}(m_q) = 0\)], Eq. (B3) can be rewritten as
\[
\mathcal{T}_{q,n}^{(ZZ)} = \frac{1}{4(l_2^2 - l_1^2)} \left[ 2Z_n(\rho_q) \bar{Z}_n(\rho_q^*) \right. \\
- Z_{n-1}(\rho_q) \bar{Z}_{n-1}(\rho_q^*) - Z_{n+1}(\rho_q) \bar{Z}_{n+1}(\rho_q^*) \bigg|_{r=l_2}^{r=l_1} .
\]

Appendix C: Numerical calculation of the internal energy

Our numerical results are based on a computer code written for Scilab 5.5.2. For calculations, the infinite sums are truncated in \(n_{\text{max}} = \max(N_{\text{Mie}}, |m_1| |y|m_2|y| + (101 + y)^{1/2})\), where \(N_{\text{Mie}} = y + 4.05y^{1/3} + 2\) [67]. This value guarantees the convergence of the scattering quantities [68]. In particular, it is convenient to define the internal energy for \(n \geq 1\) to perform numerical calculations. To this end, we define the functions

\[
\mathcal{S}_{11}^+ = \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \left| (b_n - n^2 \rho_q^2) \left[ \frac{1}{2} - \beta_1^2 \right] \right|^2 \mathcal{T}_{1,n+1}^{(JJ)} + \left| (b_n - n^2 \rho_q^2) \left[ \frac{1}{2} + \beta_1^2 \right] \right|^2 \mathcal{T}_{1,n-1}^{(JJ)} \right\} , \\
\mathcal{S}_{11}^- = \sum_{n=1}^{\infty} \left( b_n - n^2 \rho_q^2 \right) \mathcal{T}_{1,n}^{(JJ)} , \\
\mathcal{S}_{21}^+ = \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \left[ (c_n - n^2 \rho_q^2) \left[ \frac{1}{2} + \beta_2^2 \right] \right|^2 \mathcal{T}_{2,n+1}^{(JJ)} + \left[ (c_n - n^2 \rho_q^2) \left[ \frac{1}{2} - \beta_2^2 \right] \right|^2 \mathcal{T}_{2,n-1}^{(JJ)} \right\} + \left( d_n - n^2 \rho_q^2 \right)^2 \mathcal{T}_{2,n}^{(YY)} , \\
+ 2Re \left\{ \left( c_n - n^2 \rho_q^2 \right)^2 + \left| c_n \rho_q^2 \right|^2 \mathcal{T}_{2,n}^{(YJ)} + \left( c_n \rho_q^2 \right)^2 \mathcal{T}_{2,n}^{(JJ)} \right\} , \\
\mathcal{S}_{21}^- = \sum_{n=1}^{\infty} \left\{ \left[ (c_n - n^2 \rho_q^2) \mathcal{T}_{2,n}^{(JJ)} + \left( d_n - n^2 \rho_q^2 \right)^2 \mathcal{T}_{2,n}^{(YY)} \right\} + 2Re \left\{ \left| c_n \rho_q^2 \right|^2 + \left| c_n \rho_q^2 \right|^2 \mathcal{T}_{2,n}^{(YJ)} \right\} .
\]

With this set of expressions, Eqs. (31) – (35) can be rewritten for both \(p\) and \(s\) waves, reading
\[ \frac{W_{1+}}{W_{-1+}} = \zeta_{1+} \left[ |b_0|^2 \left( 1 + |\beta_1|^2 \right) \mathcal{I}_{1+}^{(JJ)} + S_{1+}^+ \right], \]

\[ \frac{W_{1-}}{W_{-1-}} = -\zeta_{1-} \left[ 2|b_0|^2 \text{Re}(\beta_1) \mathcal{I}_{1-}^{(JJ)} + S_{1-}^- \right], \]

\[ \frac{W_{1\parallel}}{W_{-1\parallel}} = \zeta_{1\parallel} \left[ |b_0|^2 \mathcal{I}_{1\parallel}^{(JJ)} + S_{1\parallel} \right], \]

\[ \frac{W_{2+}}{W_{-2+}} = \zeta_{2+} \left[ |c_0|^2 \left( 1 + |\beta_2|^2 \right) \mathcal{I}_{2+}^{(JJ)} + 2\text{Re} \left[ c_0 d_0^* \left( 1 + |\beta_2|^2 \right) \mathcal{I}_{2+}^{(YY)} + |d_0|^2 \left( 1 + |\beta_2|^2 \right) \mathcal{I}_{2+}^{(YY)} + S_{2+}^+ \right] \right], \]

\[ \frac{W_{2-}}{W_{-2-}} = -\zeta_{2-} \left[ 2|c_0|^2 \text{Re}(\beta_2) \mathcal{I}_{2-}^{(JJ)} + 4\text{Re} \left[ c_0 d_0^* \text{Re}(\beta_2) \mathcal{I}_{2-}^{(YY)} + 2|d_0|^2 \text{Re}(\beta_2) \mathcal{I}_{2-}^{(YY)} + S_{2-}^- \right] \right], \]

\[ \frac{W_{2\parallel}}{W_{-2\parallel}} = \zeta_{2\parallel} \left[ |c_0|^2 \mathcal{I}_{2\parallel}^{(YY)} + 2\text{Re} \left[ c_0 d_0^* \mathcal{I}_{2\parallel}^{(YY)} + |d_0|^2 \mathcal{I}_{2\parallel}^{(YY)} + S_{2\parallel} \right] \right]. \]
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