REDUCTIONS OF GALOIS REPRESENTATIONS OF SLOPE $\frac{3}{2}$

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Abstract. We prove a zig-zag conjecture describing the reductions of irreducible crystalline two-dimensional representations of $G_{\mathbb{Q}_p}$ of slope $\frac{3}{2}$ and exceptional weights. This along with previous works completes the description of the reduction for all slopes less than 2. The proof involves computing the reductions of the Banach spaces attached by the $p$-adic LLC to these representations, followed by an application of the mod $p$ LLC to recover the reductions of these representations.

1. Introduction

Let $p$ be an odd prime. This paper is concerned with computing the reductions of certain crystalline two-dimensional representations of the local Galois group $G_{\mathbb{Q}_p}$. This problem is classical and important in view of its applications to Galois representations attached to modular forms.

1.1. Main Result. Let $k \geq 2$ be an integer and let $a_p$ lie in a finite extension $E$ of $\mathbb{Q}_p$. Assume that $v(a_p) > 0$, where $v$ is the $p$-adic valuation of $\mathbb{Q}_p$, normalized so that $v(p) = 1$. Let $V_{k,a_p}$ be the irreducible two-dimensional crystalline representation of $G_{\mathbb{Q}_p}$ defined over $E$ of weight $k \geq 2$ and positive slope $v(a_p)$. Let $\bar{V}_{k,a_p}$ be the semisimplification of the reduction of $V_{k,a_p}$ modulo the maximal ideal of the ring of integers of $E$. It is a two-dimensional semisimple representation of $G_{\mathbb{Q}_p}$ defined over $\overline{\mathbb{F}}_p$, and is independent of the choice of the lattice used to define the reduction.

For simplicity, we often write $v$ for the slope $v(a_p)$. The reduction $\bar{V}_{k,a_p}$ is known by classical work of Fontaine and Edixhoven [Edi92] and its subsequent extension by Breuil [Bre03b] for small weights $k \leq 2p + 1$, and for all large slopes $v > \left\lfloor \frac{k-1}{p-1} \right\rfloor$ by Berger-Li-Zhu [BLZ04]. There has been a spate of recent work computing the reduction $\bar{V}_{k,a_p}$ for small slopes. Buzzard-Gee [BG09], [BG13] treated the case of slopes $v$ in $(0, 1)$. The case of slopes $v$ in $(1, 2)$ was treated in [GG15], [BG15], but only under an assumption when $v = \frac{3}{2}$. The missing case of slope $v = 1$ was treated subsequently in [BGR18]. The general goal of this paper is to give a complete treatment of the case of slope $v = \frac{3}{2}$, thereby filling a gap in the literature computing the reduction for all small slopes less than 2.

More specifically, let us say that a weight $k$ is exceptional for a particular half-integral (and possibly integral) slope $v \in \frac{1}{2}\mathbb{Z}$ if

$$k \equiv 2v + 2 \mod (p - 1).$$

These weights turn out to be the hardest to treat in the sense that the reduction seems to take on more possibilities rather than just the generic answer for that slope. In [Gha19 Conjecture 1.1], a general zig-zag conjecture was made describing the reduction $\bar{V}_{k,a_p}$ for all exceptional weights for
all half-integral slopes $0 < v \leq \frac{p-1}{2}$. The conjecture specializes to known theorems when $v = \frac{1}{2}$ [BG13] and $v = 1$ [BGR18]. In this paper, we shall prove the zig-zag conjecture for slope $v = \frac{3}{2}$. In particular, this removes an assumption made for exceptional weights in [BG15] when $v = \frac{3}{2}$, giving a complete description of the reduction for slope $v = \frac{3}{2}$.

In order to state the main theorem, let us recall some recent history on the reduction problem for exceptional weights. To do this we introduce some standard notation. Let $\omega$ and $\omega_2$ be the mod $p$ fundamental characters of levels 1 and 2. Let $\text{ind}(\omega_2^b)$ be the (irreducible) mod $p$ representation of $G_{Q_p}$ obtained by inducing the $c$-th power of $\omega_2$ from the index 2 subgroup $G_{Q_p}^2$ of $G_{Q_p}$ (for $p + 1 \nmid c$). Let $\mu_\lambda$ be the unramified character of $G_{Q_p}$ mapping a (geometric) Frobenius at $p$ to $\lambda \in \overline{F}_p$. Let $r := k - 2$ and let $b \in \{1, 2, \cdots, p - 1\}$ represent the congruence class of $r \mod (p - 1)$. Then $b = 2v$ is a representative for the exceptional congruence class of weights $r \mod (p - 1)$. In particular, $b = 1, 2, 3, \cdots$, represent the exceptional congruence classes of weights $r \mod (p - 1)$ for the half-integral slopes $v = \frac{1}{2}, 1, \frac{3}{2}, \cdots, \frac{p-1}{2}$.

In [BG09], Buzzard-Gee showed that the reduction $\bar{V}_{k,a_p}$ is always irreducible for slopes $v$ in $(0, 1)$ (and isomorphic to $\text{ind}(\omega_2^{b+1})$), except possibly in the exceptional case $v = \frac{1}{2}$ and $b = 1$. This case was only treated later in [BG13], where the authors show that when a certain parameter, which we call $\tau$, is larger than another parameter, which we call $t$, a reducible possibility (namely $\omega \oplus \omega$ on inertia) occurs instead. More precisely, setting

$$\tau = v \left( \frac{a_p^2 - rp}{pa_p} \right),$$
$$t = v(1 - r),$$

it is shown in [BG13, Theorem A] that in the exceptional case $v = \frac{1}{2}$ and $b = 1$, there is a dichotomy:

$$\bar{V}_{k,a_p} \sim \begin{cases} 
\text{ind}(\omega_2^{b+1}), & \text{if } \tau < t \\
\mu_\lambda \cdot \omega^b \oplus \mu_{\lambda^{-1}} \cdot \omega, & \text{if } \tau \geq t,
\end{cases}$$

for $r > 1$, where $\lambda$ is a root of the quadratic equation

$$\lambda + \frac{1}{\lambda} = \frac{1}{1-r} \cdot \frac{a_p^2 - rp}{pa_p}.$$

In [BGR18], the reduction $\bar{V}_{k,a_p}$ was completely determined on the boundary $v = 1$ of the annulus $(0, 1)$, and was shown to be generically reducible instead. In the difficult exceptional case $v = 1$ and $b = 2$, the authors established, for $r > 2$, the following trichotomy:

$$\bar{V}_{k,a_p} \sim \begin{cases} 
\text{ind}(\omega_2^{b+1}), & \text{if } \tau < t \\
\mu_\lambda \cdot \omega^b \oplus \mu_{\lambda^{-1}} \cdot \omega, & \text{if } \tau = t \\
\text{ind}(\omega_2^{b+p}), & \text{if } \tau > t,
\end{cases}$$
where
\[ \tau = v \left( \frac{a_p^2 - \binom{r}{2}p^2}{p a_p} \right), \]
\[ t = v(2 - r), \]
and \( \lambda \) is a constant given by
\[ \lambda = \frac{2}{2 - r} \cdot \frac{a_p^2 - \binom{r}{2}p^2}{p a_p}. \]

Based on these results for slopes \( \frac{1}{2} \) and 1, and some computations in [Roz16] for a few larger half-integral slopes, one might guess that in the general exceptional case \( \tau = 2v \) for half-integral slope \( v \leq \frac{p - 1}{2} \), there are \( b + 1 \) possibilities for \( \bar{V}_{k,a_p} \), with various irreducible and reducible cases occurring alternately. A more concrete version of this guess (and several subtleties related to it) were outlined by one of the authors in a conjecture called the zig-zag conjecture, see Conjecture 1.1 of [Gha19]. That this conjecture is indeed true for slope \( v = \frac{3}{2} \) is the main theorem of this paper:

**Theorem 1.1.** If \( v = \frac{3}{2} \), then the zig-zag conjecture [Gha19] is true. More precisely, say \( p \geq 5 \), the slope \( v = \frac{3}{2} \) and \( b = 3 \), so that the congruence class of \( r \mod (p - 1) \) is exceptional. If \( r > b \),
\[ c = \frac{a_p^2 - (r - 2)\binom{r}{2}p^3}{p a_p}, \]
and
\[ \tau = v(c), \]
\[ t = v(b - r), \]
then the reduction \( \bar{V}_{k,a_p} \) satisfies the following tetrachotomy:

\[ \bar{V}_{k,a_p} \sim \begin{cases} \text{ind}(\omega_{b+1}^b), & \text{if } \tau < t \\ \mu_{\lambda_1} \cdot \omega^b \oplus \mu_{\lambda_1^{-1}} \cdot \omega, & \text{if } \tau = t \\ \text{ind}(\omega_{b+p}^b), & \text{if } t < \tau < t + 1 \\ \mu_{\lambda_2} \cdot \omega^{b-1} \oplus \mu_{\lambda_2^{-1}} \cdot \omega^2, & \text{if } \tau \geq t + 1, \end{cases} \]

where the \( \lambda_i \), for \( i = 1, 2 \), are constants given by
\[ \lambda_1 = \frac{b}{b - r} \cdot c, \]
\[ \lambda_2 = \frac{1}{\lambda_2} = \frac{b - 1}{(b - 1 - r)(b - r)} \cdot \frac{c}{p}. \]

Theorem 1.1 was proved in [Bre03b] for \( r = p + 2 \). Moreover, it was proved in [BG15, Theorem 1] under a relatively strong assumption (in the present context), namely \( \tau = v(c) = \frac{1}{2} \) is as small as possible. Indeed, condition (⋆) imposed in [BG15, Theorem 1.1] exactly says that the numerator of \( c \) above has minimal possible valuation 3. While the authors of [BG15] suspected that removing
condition (*) would require new ideas, they did not perhaps appreciate how difficult removing this condition would turn out to be. An important psychological step was taken in [BGR18]. Indeed, many of the techniques used in this paper to prove the tetrachotomy above for \( v = \frac{1}{2} \) substantially develop techniques used in [BGR18] to prove the trichotomy for \( v = 1 \), although there are many additional complications. In any case, since \( \tilde{V}_{k,a,p} \) was determined in [BG15] for all other slopes \( v \) in \((1,2)\) (even for \( p \geq 3 \)), we can finally state the following corollary.

**Corollary 1.2.** If \( p \geq 5 \), then the reduction \( \tilde{V}_{k,a,p} \) is known for all slopes \( v = v(a_p) \) less than 2.

1.2. **Proof of Theorem 1.1.** The proof of Theorem 1.1 uses the compatibility of the \( p \)-adic and mod \( p \) Local Langlands Correspondences [Bre03b], with respect to the process of reduction [Ber10]. This compatibility allows one to reduce the problem of computing \( \tilde{V}_{k,a,p} \) to a representation theoretic one, namely, to computing the reduction of a \( \text{GL}_2(\mathbb{Q}_p) \)-stable lattice in a certain unitary \( \text{GL}_2(\mathbb{Q}_p) \)-Banach space. The key ingredients in the argument have been recalled several times in earlier works, so let us only recall the main steps here.

Let \( G = \text{GL}_2(\mathbb{Q}_p) \) and let \( B(V_{k,a,p}) \) be the unitary \( G \)-Banach space associated to \( V_{k,a,p} \) by the \( p \)-adic Local Langlands Correspondence. The reduction \( \overline{B(V_{k,a,p})}^{ss} \) of a lattice in this Banach space coincides with the image of \( \tilde{V}_{k,a,p} \) under the (semisimple) mod \( p \) Local Langlands Correspondence defined in [Bre03b]. Since the mod \( p \) correspondence is by definition injective, to compute \( \tilde{V}_{k,a,p} \) it suffices to compute the reduction \( \overline{B(V_{k,a,p})}^{ss} \).

Let \( K = \text{GL}_2(\mathbb{Z}_p) \), a maximal compact open subgroup of \( G = \text{GL}_2(\mathbb{Q}_p) \), and let \( Z = \mathbb{Q}_p^\times \) be the center of \( G \). Let \( X = KZ \setminus G \) be the (vertices of the) Bruhat-Tits tree associated to \( G \). The module \( \text{Sym}^r \mathbb{Q}_p^2 \), for \( r = k-2 \), carries a natural action of \( KZ \), and the projection

\[
KZ \setminus (G \times \text{Sym}^{k-2} \mathbb{Q}_p^2) \to KZ \setminus G = X
\]
defines a local system on \( X \). The space

\[
\text{ind}^G_{KZ} \text{Sym}^{k-2} \mathbb{Q}_p^2
\]
consisting of all sections \( f : G \to \text{Sym}^{k-2} \mathbb{Q}_p^2 \) of this local system which are compactly supported mod \( KZ \) form a representation space for \( G \), which is equipped with a \( G \)-equivariant Hecke operator \( T \). Let \( \Pi_{k,a,p} \) be the locally algebraic representation of \( G \) defined by taking the cokernel of \( T - a_p \) acting on the above space of sections. Let \( \Theta_{k,a,p} \) be the image of the integral sections \( \text{ind}^G_{KZ} \text{Sym}^{k-2} \mathbb{Q}_p^2 \) in \( \Pi_{k,a,p} \). Then \( B(V_{k,a,p}) \) is the completion \( \hat{\Pi}_{k,a,p} \) of \( \Pi_{k,a,p} \) with respect to the lattice \( \Theta_{k,a,p} \). The completion \( \hat{\Theta}_{k,a,p} \), and sometimes by abuse of notation \( \Theta_{k,a,p} \) itself, is called the standard lattice in \( B(V_{k,a,p}) \). We have \( \overline{B(V_{k,a,p})}^{ss} \cong \hat{\Theta}_{k,a,p}^{ss} \cong \hat{\Theta}_{k,a,p}^{ss} \).

Thus, to compute \( \tilde{V}_{k,a,p} \), it suffices to compute the reduction \( \hat{\Theta}_{k,a,p}^{ss} \) of \( \Theta_{k,a,p} \). In order to do this, we need a bit more notation. Let \( V_r \) denote the \((r+1)\)-dimensional \( \overline{\mathbb{F}}_p \)-vector space of homogeneous polynomials \( P(X,Y) \) in two variables \( X \) and \( Y \) of degree \( r \) over \( \overline{\mathbb{F}}_p \). The group \( \Gamma = \text{GL}_2(\overline{\mathbb{F}}_p) \) acts on \( V_r \) by the formula \((a \ b \ c \ d) \cdot P(X,Y) = P(aX + cY, bX + dY) \), and \( KZ \) acts on \( V_r \) via projection
to $\Gamma$, with $(\frac{P^0}{G^0\Gamma}) \in \mathbb{Z}$ acting trivially. By definition of the lattice $\Theta_{k,a_p}$, there is a surjection
\[ \text{ind}_{KZ}^G \text{Sym}^{k-2p} G^0 \to \Theta_{k,a_p}, \]
which induces a surjective map
\[ \text{ind}_{KZ}^G V_r \to \Theta_{k,a_p}, \]
for $r = k - 2$. Thus to compute $\Theta_{k,a_p}$ one needs to understand the kernel of the map \((1.1)\). Let $\theta(X,Y) = X^p Y - X Y^p$. The action of $\Gamma$ on $\theta$ is via the determinant $D : \Gamma \to \mathbb{F}_p^\times$. Define the following $\Gamma$- (hence $KZ$-) submodules of $V_r$. First, let $\mathcal{V}_r^{**} = \{ P(X,Y) \in V_r : \theta^2[P] \}$ be the $\Gamma$-submodule of $V_r$ consisting of all polynomials divisible by $\theta^2$. Second, let $X_{r-1}$ be the $\Gamma$-submodule of $V_r$ generated by $X^{r-1} Y$. These submodules are important in computing the kernel of \((1.1)\) because of the following two useful facts [BG09, Remark 4.4]: if the slope $v < 2$, then $\text{ind}_{KZ}^G V_r^{**}$ lies in the kernel of \((1.1)\), and if the slope $v > 1$ and $r \geq 2p + 1$, then $\text{ind}_{KZ}^G X_{r-1}$ lies in the kernel of \((1.1)\). Thus, if $1 < v < 2$ and $r \geq 2p + 1$ (this is not a restriction, since as mentioned above, $r = p + 2$ was treated in [Bre03b]), the surjection \((1.1)\) factors through the map
\[ \text{ind}_{KZ}^G Q \to \Theta_{k,a_p}, \]
where
\[ Q = \frac{V_r}{X_{r-1} + V_r^{**}}, \]
for $r = k - 2$. The module $Q$ was studied in some detail in [BG15], and it is known that $Q$ has at most three Jordan-Hölder (JH) factors, which we call $J_1$, $J_2$ and $J_3$ (though these factors were called $J_2$, $J_0$ and $J_1$, respectively, in [BG15]).

Now, the mod $p$ Local Langlands Correspondence (mod $p$ LLC) says (roughly) that irreducible Galois representations $\hat{V}_{k,a_p}$ correspond to supersingular representations of the form $\text{ind}_{KZ}^G J$, for some irreducible $\Gamma$-module $J$, whereas reducible $\hat{V}_{k,a_p}$ correspond (generically) to a sum of two principal series representations of the form $\text{ind}_{KZ}^G J$ and $\text{ind}_{KZ}^G J'$, for some (possibly equal) ‘dual’ irreducible $\Gamma$-modules $J$, $J'$, the sum of whose dimensions is $p - 1 \mod (p - 1)$, and some $\lambda \in \mathbb{F}_p^\times$.

Thus, exactly one, or possibly exactly two, of the above JH factors $J_i$, $i = 1$, 2, 3, contribute to $\Theta_{k,a_p}$. Now, the sum of the dimensions of $J_2$ and $J_3$ is $p + 1$, and by the mod $p$ LLC, each of them gives the same irreducible Galois representation $\hat{V}_{k,a_p}$ when it occurs as the sole contributing factor to a supersingular $\Theta_{k,a_p}$. Also, the sum of the dimensions of the pair $(J_1, J_2)$ is $p - 1$ so a potential ‘duality’ occurs and these two JH factors may contribute together to $\Theta_{k,a_p}$ giving a reducible Galois representation $\hat{V}_{k,a_p}$. Finally, it turns out that the last JH factor $J_3 = V_{p-2} \otimes D^2$ is potentially ‘self-dual’, noting that twice its dimension is still $p - 1 \mod (p - 1)$, so that again it may give rise to a reducible Galois representation $\hat{V}_{k,a_p}$.

We are now ready to make some key observations. We claim that as $\tau$ varies through the rational line, the JH factors that contribute to $\Theta_{k,a_p}$ actually occur in the following order: first $J_1$ contributes, then both $J_1$ and $J_2$ contribute together, then only $J_2$ contributes, then only $J_3$ contributes, then finally $J_3$ contributes together with itself in a self-dual way. Moreover, we claim that the jumps when two JH factors contribute as in the two cases described above occur when $\tau$ takes specific
integral values, namely, \( \tau = t \) and \( \tau = t + 1 \), respectively. These claims are remarkable considering that \textit{a priori} there is no reason to expect that there should be any patterns in the way \( \tilde{\Theta}_{k,a_p} \) ‘selects’ JH factors. Indeed this selection of these JH factors seems to be the beginning of a more general conjectural zig-zag pattern among the JH factors (see \cite{Gha19} Conjecture 1.1). In the present case, it explains why the reduction \( \tilde{V}_{k,a_p} \) alternates between irreducible and reducible possibilities, with the reducible possibilities occurring exactly at the integer \( \tau = t \) and for \( \tau \geq t + 1 \).

All of this is best summarized with a picture. Let \( F_i \) be the subquotient of \( \tilde{\Theta}_{k,a_p} \) occurring as the image of \( \text{ind}_{G}^K J_i \), for \( i = 1, 2, 3 \). Then we prove that all the subquotients \( F_i \) vanish in \( \tilde{\Theta}_{k,a_p} \), except for the \( F_i \) occurring for \( \tau \) in the following regions:

\[
\begin{array}{cccccc}
F_1 & (F_1, F_2) & F_2 & (F_2, F_3) & (F_3, F_3)
\end{array}
\begin{array}{cccccccc}
t & t + \frac{1}{2} & t + 1
\end{array}
\]

More formally, we prove the following key symmetric nine-part proposition:

**Proposition 1.3.** Assume \( v = \frac{3}{2} \) and \( b = 3 \). Then the subquotients \( F_i \) of \( \tilde{\Theta}_{k,a_p} \) satisfy the following

1. **Around \( t \):**
   - \( \tau > t \implies F_1 = 0 \)
   - \( \tau = t \implies F_1 \leftarrow \frac{\text{ind } J_1}{T - \lambda_1} \) and \( F_2 \leftarrow \frac{\text{ind } J_2}{T - \lambda_1} \), with \( \lambda_1 = \frac{b}{b - r} \cdot c \)
   - \( \tau < t \implies F_2 = 0 \),

2. **Around \( t + \frac{1}{2} \):**
   - \( \tau > t + \frac{1}{2} \implies F_2 = 0 \)
   - \( \tau = t + \frac{1}{2} \implies F_2 \leftarrow \frac{\text{ind } J_2}{T} \) and \( F_3 \leftarrow \frac{\text{ind } J_3}{T} \)
   - \( \tau < t + \frac{1}{2} \implies F_3 = 0 \),

3. **Around \( t + 1 \):**
   - \( \tau > t + 1 \implies F_3 \leftarrow \frac{\text{ind } J_3}{T^2 + 1} \)
   - \( \tau = t + 1 \implies F_3 \leftarrow \frac{\text{ind } J_3}{T^2 - dt + 1} \), with \( d = \frac{b - 1}{(b - 1 - r)(b - r)} \cdot \frac{c}{p} \)
   - \( \tau < t + 1 \implies F_3 \leftarrow \frac{\text{ind } J_3}{T} \),

where \( \text{ind} = \text{ind}_{KZ}^G \).

The main theorem of this paper, Theorem 1.1, follows immediately from Proposition 1.3 and the fact that \( \tilde{\Theta}_{k,a_p} \) corresponds to \( \tilde{V}_{k,a_p} \) under the mod \( p \) LLC. Checking each of the nine statements in the proposition requires a substantial amount of work involving explicit computations with the Hecke operator on various polynomial valued functions on \( G \). The hardest part of the argument is coming up with the function in the first place. Once the functions are found, it is an elementary

\[\text{In fact, we can only prove this for } \tau \leq t, \text{ but this suffices.}\]

\[\text{The fifth statement actually follows easily from the mod } p \text{ LLC and the ninth statement.}\]
though lengthy process to cross-check that the computations involving the Hecke operator are correct. In this (arXiv) version of the paper (where space is less of a constraint), we have decided to provide complete details for the benefit of the interested reader.

We end with an outline of the paper. In Section 2 we recall some basics facts. In Section 3 we prove some combinatorial identities and in Section 4 we prove several useful telescoping lemmas in order to deal with the action of the Hecke operator at ‘infinity’. With these tools in hand, the statements about $F_1$, $F_2$ and $F_3$ in the proposition above are proved in Sections 5, 6 and 7 respectively. This then completes the proof of Proposition 1.3 and hence the proof of Theorem 1.1.

2. Basics

In this section, we recall some notation and well-known facts, see [Bre03b], [BG15], [BGR18].

2.1. Hecke operator $T$. Let $G = \text{GL}_2(\mathbb{Q}_p)$, $K = \text{GL}_2(\mathbb{Z}_p)$ be the standard maximal compact subgroup of $G$ and $Z = \mathbb{Q}_p^\times$ be the center of $G$. Let $R$ be a $\mathbb{Z}_p$-algebra and let $V = \text{Sym}^r R^2 \otimes D^*$ be the usual symmetric power representation of $KZ$ twisted by a power of the determinant character $D$, modeled on homogeneous polynomials of degree $r$ in the variables $X, Y$ over $R$. We require that $p \in Z$ acts trivially. We will denote $\text{ind}_{KZ}^G$ to mean compact induction. Thus $\text{ind}_{KZ}^G V$ consists of functions $f : G \to V$ such that $f(hg) = h \cdot f(g)$, for all $h \in KZ$ and $g \in G$, and $f$ is compactly supported mod $KZ$. Recall that $G$ acts on such functions by right translation: $(g' \cdot f)(g) = f(gg')$, for $g, g' \in G$. For $g \in G, v \in V$, let $[g, v] \in \text{ind}_{KZ}^G V$ be the function with support in $KZg^{-1}$ given by

$$g' \mapsto \begin{cases} g'g \cdot v, & \text{if } g' \in KZg^{-1} \\ 0, & \text{otherwise.} \end{cases}$$

One checks that $g' \cdot [g, v] = [g'g, v]$ and $[gh, v] = [g, h \cdot v]$, for all $g, g' \in G$, $h \in KZ$, $v \in V$. Any function in $\text{ind}_{KZ}^G V$ is a finite linear combination of functions of the form $[g, v]$, for $g \in G$ and $v \in V$. The Hecke operator $T$ is defined by its action on these elementary functions via

$$T([g, v(X, Y)]) = \sum_{\lambda \in \mathbb{F}_p} \left[ g \left( \begin{smallmatrix} p & [\lambda] \\ 0 & 1 \end{smallmatrix} \right), v \left( X, -[\lambda]X + pY \right) \right] + \left[ g \left( \begin{smallmatrix} 0 & 1 \\ 0 & p \end{smallmatrix} \right), v(pX, Y) \right],$$

where $[\lambda]$ denotes the Teichmüller representative of $\lambda \in \mathbb{F}_p$. We may write $T = T^+ + T^−$, where

$$T^+([g, v(X, Y)]) = \sum_{\lambda \in \mathbb{F}_p} \left[ g \left( \begin{smallmatrix} p & [\lambda] \\ 0 & 1 \end{smallmatrix} \right), v \left( X, -[\lambda]X + pY \right) \right],$$

$$T^−([g, v(X, Y)]) = \left[ g \left( \begin{smallmatrix} 1 & 0 \\ p & 1 \end{smallmatrix} \right), v(pX, Y) \right].$$

For $m = 0$, set $I_0 = \{0\}$, and for $m > 0$, let $I_m = \{[\lambda_0]+[\lambda_1]+\cdots+[\lambda_{m−1}]p^{m−1} : \lambda_i \in \mathbb{F}_p\} \subset \mathbb{Z}_p$, where the square brackets denote Teichmüller representatives. For $m ≥ 1$, there is a truncation map $[\cdot]_{m−1} : I_m \to I_{m−1}$ given by taking the first $m−1$ terms in the $p$-adic expansion above: for $m = 1$,
Thus a general element in $\text{ind}^G_{KZ}V$ is a finite sum of functions of the form $[g,v]$, with $g = g^0_{m,\lambda}$ or $g^1_{m,\lambda}$, for some $\lambda \in I_m$ and $v \in V$.

2.2. The mod $p$ Local Langlands Correspondence. For $0 \leq r \leq p-1$, $\lambda \in \mathbb{F}_p$ and $\eta : \mathbb{Q}_p^\times \to \mathbb{F}_p^\times$ a smooth character, let

$$\pi(r, \lambda, \eta) := \frac{\text{ind}^G_{KZ} \text{Sym}^r \mathbb{F}_p^2}{T-\lambda} \otimes (\eta \circ \det)$$

be the smooth admissible representation of $G$, known to be irreducible unless $(r, \lambda) = (0, \pm 1)$ or $(p-1, \pm 1)$, by the classification of irreducible representations of $G$ in characteristic $p$ in [BL94, BL95, Bre03a]. Breuil’s semisimple mod $p$ Local Langlands Correspondence [Bre03b, Def. 1.1] is given by:

- $\lambda = 0$: $\text{ind}(\omega_2^{r+1}) \otimes \eta \xrightarrow{LL} \pi(r, 0, \eta)$,
- $\lambda \neq 0$: $(\mu_\lambda \cdot \omega_2^{r+1} \oplus \mu_{\lambda-1}) \otimes \eta \xrightarrow{LL} \pi(r, \lambda, \eta)_{ss} \oplus \pi((p-3-r, \lambda^{-1}, \eta \omega^{r+1})_{ss}$,

where $\{0, 1, \ldots, p-2\} \ni [p-3-r] \equiv p-3-r \mod (p-1)$. For a more functorial description, see [Co10].

Recall that there is a locally algebraic representation of $G$ given by

$$\Pi_{k,a_p} = \frac{\text{ind}^G_{KZ} \text{Sym}^r \mathbb{Q}_p^2}{T-a_p},$$

where $r = k-2 \geq 0$ and $T$ is the Hecke operator. Consider the standard lattice in $\Pi_{k,a_p}$ given by

$$\Theta = \Theta_{k,a_p} := \text{image} \left( \text{ind}^G_{KZ} \text{Sym}^r \mathbb{Z}_p^2 \to \Pi_{k,a_p} \right) \simeq \frac{\text{ind}^G_{KZ} \text{Sym}^r \mathbb{Z}_p^2}{(T-a_p)(\text{ind}^G_{KZ} \text{Sym}^r \mathbb{Q}_p^2) \cap \text{ind}^G_{KZ} \text{Sym}^r \mathbb{F}_p^2}.$$

It is known [Ber10] that the semisimplification of the reduction of this lattice satisfies $\bar{\Theta}^{ss}_{k,a_p} \simeq LL(\bar{\Pi}^{ss}_{k,a_p})$, where $LL$ is the (semisimple) mod $p$ Local Langlands Correspondence above. Since the map $LL$ is clearly injective, it is enough to know $LL(\bar{\Pi}^{ss}_{k,a_p})$ to determine $\bar{\Pi}^{ss}_{k,a_p}$.

2.3. The structure of the quotient $Q$. Let $V_r = \text{Sym}^r \mathbb{F}_p^2$ be the usual symmetric power representation of $G := \text{GL}_2(F_p)$ (hence $KZ$, with $p \in Z$ acting trivially).

It follows directly from the definition of $\Theta_{k,a_p}$ in (2.4), that there is a surjection $\text{ind}^G_{KZ} \text{Sym}^r \mathbb{Z}_p^2 \to \Theta_{k,a_p}$, for $r = k-2$, whence a surjective map

$$\text{ind}^G_{KZ} V_r \to \bar{\Theta}_{k,a_p},$$

where $\bar{\Theta}_{k,a_p} = \Theta_{k,a_p} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$. 

Write $X_{k,a_p}$ for the kernel. A model for $V_r$ is the space of all homogeneous polynomials of degree $r$ in two variables $X$ and $Y$ over $\overline{\mathbb{F}}_p$ with the standard action of $\Gamma$, recalled in Section 1.2. Let $X_{r-1} \subset V_r$ be the $\Gamma$-(hence $KZ$-) submodule generated by $X^{r-1}$. Let $V_r^\ast$ and $V_r^{**}$ be the submodules of $V_r$ consisting of polynomials divisible by $\theta$ and $\theta^2$ respectively, for $\theta = X^pY - XY^p$. If $r \geq 2p + 1$, then Buzzard-Gee have shown [BG09, Remark 4.4]:

- $v(a_p) > 1 \implies \text{ind}_{KZ}^G X_{r-1} \subset X_{k,a_p}$,
- $v(a_p) < 2 \implies \text{ind}_{KZ}^G V_r^{**} \subset X_{k,a_p}$.

It follows that when $1 < v(a_p) < 2$, the map (2.5) induces a surjective map $\text{ind}_{KZ}^G Q \to \Theta_{k,a_p}$, where

$$Q := \frac{V_r}{X_{r-1} + V_r^{**}}.$$  

Note that unlike $V_r$ the length of $Q$ as a $\Gamma$-module is bounded independently of $r$. The $\Gamma$-module structure of $Q$ has been derived in [BG15, Proposition 6.4]. We have:

**Proposition 2.1.** Let $p \geq 5$, $r \geq 2p + 1$ and $r \equiv 3 \pmod{p-1}$.

1. If $r \not\equiv 3 \pmod{p}$, then there is an exact sequence

$$(2.6) \quad 0 \to V_{p-2} \otimes D^2 \to Q \to V_{p-4} \otimes D^3 \to 0,$$

and moreover the exact sequence above is $\Gamma$-split.

2. If $r \equiv 3 \pmod{p}$, then

$$(2.7) \quad 0 \to V_r^*/V_r^{**} \to Q \to V_{p-4} \otimes D^3 \to 0,$$

where $V_r^*/V_r^{**}$ is the non-trivial extension of $V_{p-2} \otimes D^2$ by $V_1 \otimes D$.

Let us now set some important notation. Let us denote the JH factors of $Q$ above as follows:

- $J_1 = V_{p-4} \otimes D^3$,
- $J_2 = V_1 \otimes D$,
- $J_3 = V_{p-2} \otimes D^2$.

(These JH factors of $Q$ were called $J_2$, $J_0$, $J_1$, respectively, in [BG15] and [BGR18].)

We now define some important subquotients of $\tilde{\Theta}_{k,a_p}$. Let

- $F_2$ denote the image of $\text{ind}_{KZ}^G J_2$ inside $\tilde{\Theta}_{k,a_p}$,
- $F_{2,3}$ denote the image of $\text{ind}_{KZ}^G (V_r^*/V_r^{**})$ inside $\tilde{\Theta}_{k,a_p}$,
- $F_3 := F_{2,3}/F_2$, and
- $F_1 := \tilde{\Theta}_{k,a_p}/F_2$.

So we have a short exact sequence

$$0 \to (F_2 \to F_{2,3} \to F_3) \to \tilde{\Theta}_{k,a_p} \to F_1 \to 0.$$  

Thus, to study the structure of $\tilde{\Theta}_{k,a_p}$ up to semisimplification it suffices to study the surjections $\text{ind}_{KZ}^G J_i \to F_i$ for $i = 1, 2, 3$. This will be done in detail in Sections 3, 6, 7 respectively. Note that even though $\tilde{\Theta}_{k,a_p}$ is not necessarily semisimple, $F_1$, $F_2$ and $F_3$ are semisimple.
We end with the following useful lemma.

**Lemma 2.2.** Let \( p \geq 3, r \geq 2p + 1, r \equiv b \mod (p - 1) \), with \( b = 3 \).

(i) The image of \( X^{r-i}Y^i \) in \( Q \) maps to \( 0 \in J_1 \), for \( 0 \leq i \leq b - 1 \), whereas the image of \( X^{r-b}Y^b \) in \( Q \) maps to \( X^{b-b-1} \) in \( J_1 \).

(ii) The image of \( \theta X^{r-p-1}, \theta Y^{r-p-1} \) in \( Q \) correspond to \( X^{b-2}, Y^{b-2} \), respectively, in \( J_2 \), so both map to \( 0 \in J_3 \).

(iii) The image of \( \theta X^{r-p-b+1}Y^{b-2} \) in \( Q \) maps to \( X^{p-b+1} \in J_3 \).

**Proof.** This is a special case of \([BG15\, Lemma\, 8.5]\) and \([BGR18\, Lemma\, 3.4]\), though the JH factors \( J_1, J_2, J_3 \) were called \( J_2, J_0, J_1 \), respectively in these papers. The proof is elementary and consists of explicit calculations with the maps given in \([Glo78\, (4.2)]\) and \([Bre03b\, Lem.\, 5.3]\). \(\square\)

### 3. Combinatorial Identities

In this section, we state several combinatorial identities.

We first state three lemmas on the congruence properties of binomial coefficients, which will later be useful in working with the operator \( T^- \), and to some extent with the operator \( T^+ \).

**Lemma 3.1.** Let \( p > 3 \). If \( r \equiv 3 \mod (p - 1) \) and \( t = v(r - 3) \), then \( p^i \binom{r}{i} \equiv 0 \mod p^{i+4} \), for \( i \geq 4 \).

**Proof.** We need to show that \( v(p^i \binom{r}{i}) \geq t + 4 \) for all \( i \geq 4 \). Now

\[
v(p^i \binom{r}{i}) = i + v(r(r-1) \cdots (r-i+1)) - v(i!).
\]

Since \( i \geq 4 \), we get that \( v(p^i \binom{r}{i}) \geq i + t - v(i!) \).

For \( i = 4 \), we have \( v(p^4 \binom{r}{4}) \geq 4 + t - v(4!) \). Since \( p \geq 5 \), we get that \( v(p^4 \binom{r}{4}) \geq t + 4 \). Similarly for \( i = 5 \), \( v(p^5 \binom{r}{5}) \geq 5 + t - v(5!) \). Since \( v(5!) \leq 1 \) for \( p \geq 5 \), we get that \( v(p^5 \binom{r}{5}) \geq t + 4 \). Now assume \( i \geq 6 \). The difference \( i - v(i!) \geq i - \left( \frac{t}{p} + \frac{t}{p^2} + \cdots \right) = i \left( 1 - \frac{1}{p^{i-1}} \right) \geq 6 \left( 1 - \frac{1}{4} \right) \geq 4 \), because \( i \geq 6 \) and \( p \geq 5 \). Therefore again \( i + t - v(i!) \geq t + 4 \). \(\square\)

**Lemma 3.2.** Let \( p > 3 \). If \( r \equiv 2 \mod (p - 1) \) and \( t = v(r - 2) \), then \( p^i \binom{r}{i} \equiv 0 \mod p^{i+3} \), for \( i \geq 3 \).

**Proof.** This is Lemma 2.6 in \([BGR18]\). \(\square\)

**Lemma 3.3.** Let \( p \geq 3 \). If \( r \equiv 1 \mod (p - 1) \) and \( t = v(s - 1) \), then \( p^i \binom{r}{i} \equiv 0 \mod p^{i+2} \), for \( i \geq 2 \).

**Proof.** This is Lemma 2.1 in \([BG13]\). \(\square\)

Second, we state several propositions on the congruence properties of sums of products of binomial coefficients. These will be useful in computing with the operator \( T^+ \). The proofs are similar to the proofs of similar results in \([BGR18]\), though we provide complete details of all new results here.

The first proposition will be used in the proof of Proposition \(5.1\).

**Proposition 3.4.** Let \( p > 3 \). If \( r = 3 + n(p - 1)p^i \), with \( t = v(r - 3) \) and \( n > 0 \), then we have:
\begin{align*}
(1) & \quad \sum_{0 < j < r, j \equiv 3 \mod (p-1)} \binom{r}{j} = \frac{3-r}{6(1-p)} \left( 6p^2 + 5p - 3\binom{2p+1}{p-1} \right) \\
& \quad \quad + \frac{1}{6} \left( \frac{3-r}{1-p} \right)^2 \left( -3p^2 - 3\binom{2p+1}{p-1} \right) \mod p^{t+3}. \\
(2) & \quad \sum_{j \equiv 3 \mod (p-1)} \binom{r}{j} \equiv \frac{pr(3-r)}{2} \mod p^{t+2}. \\
(3) & \quad \sum_{j \equiv 3 \mod (p-1)} \binom{j}{2} \binom{r}{j} \equiv 0 \mod p^{t+1}. \\
(4) & \quad \sum_{j \equiv 3 \mod (p-1)} \binom{j}{3} \binom{r}{j} \equiv \binom{j}{1} \mod p^t. \\
(5) & \quad p^i \sum_{j \equiv 3 \mod (p-1)} \binom{j}{i} \binom{r}{j} \equiv 0 \mod p^{t+4}, \forall i \geq 4.
\end{align*}

Proof. Let $0 \leq i < r$. Let $S_{i,r} = \sum_{0 < j < r, j \equiv 3 \mod (p-1)} \binom{j}{i} \binom{r}{j}$. Let $\sum_{i,r} = (p-1) \sum_{j \equiv 3 \mod (p-1)} \binom{j}{i} \binom{r}{j}$, then
\[ \sum_{i,r} = (p-1) \left( S_{i,r} + \binom{r}{i} \right). \]
Let $f_r(x) = (1 + x)^r = \sum_{j \geq 0} \binom{r}{j} x^j$, which we consider as a function from $\mathbb{Z}_p$ to $\mathbb{Z}_p$. Let
\[ g_{i,r}(x) = \frac{x^{i-3}}{i!} f_r^{(i)}(x) = \sum_{j \geq 0} \binom{j}{i} \binom{r}{j} x^{j-3}. \]
Let $\mu_{p-1}$ be the set of $(p-1)$-st roots of unity. Summing $g_{i,r}(x)$ as $x$ varies over all $(p-1)$-st roots of unity we get
\[ \sum_{\xi \in \mu_{p-1}} g_{i,r}(\xi) = \sum_{\xi \in \mu_{p-1}} \sum_{j \geq 0} \binom{j}{i} \binom{r}{j} \xi^{j-3} \]
\[ = \sum_{\xi \in \mu_{p-1}} \left( \sum_{j \equiv 3 \mod (p-1)} \binom{j}{i} \binom{r}{j} \xi^{j-3} + \sum_{j \equiv 3 \mod (p-1)} \binom{j}{i} \binom{r}{j} \xi^{j-3} \right). \]

We will be using the following easy fact quite frequently:
\[ (3.1) \quad \sum_{\lambda \in \mathbb{F}_p^*} [\lambda]^i = \begin{cases} p-1 & \text{if } p-1 \mid i, \\ 0 & \text{if } (p-1) \nmid i, \end{cases} \]
where $[\lambda]$ is the Teichmüller representative of $\lambda \in \mathbb{F}_p$. 

Using the above fact we get that
\[
\sum_{\xi \in \mu_{p-1}} g_{i,r}(\xi) = (p - 1) \sum_{j \equiv 3 \mod (p-1)} \left( \binom{r}{j} \binom{j}{i} \right),
\]
which is equal to \( \sum_{i,r} \). Let \( \mu' \) be the set of \((p - 1)\)-st roots of unity except \(-1\). Since \( g_{i,r}(x) = x^{i-3} \binom{i}{1}(1 + x)^{r-1} \),
\[
\sum_{i,r} = \sum_{\xi \in \mu_{p-1}} \left( \binom{r}{i} \right)^{\xi^{i-3}}(1 + \xi)^{r-i},
\]
which is equal to \( \sum_{i,r} \). Let \( \mu' \) be the set of \((p - 1)\)-st roots of unity except \(-1\). Since \( g_{i,r}(x) = x^{i-3} \binom{i}{1}(1 + x)^{r-1} \),
\[
\sum_{i,r} = \sum_{\xi \in \mu_{p-1}} \left( \binom{r}{i} \right)^{\xi^{i-3}}(1 + \xi)^{r-i}, \text{ as } r - i > 0.
\]
Let \( \xi \in \mu' \), clearly \( 1 + \xi \in \mathbb{Z}_p \). Since \( \xi \) is the unique Teichmüller lift of some element of \( \mathbb{F}_p^\times \) and the lift of \(-1 \mod p \) is \(-1 \mod p \), we see that \( \xi \not\equiv -1 \mod p \). Since \( 1 + \xi \not\in p\mathbb{Z}_p \), we have \( 1 + \xi \in \mathbb{Z}_p^\times \). As \( \mathbb{Z}_p^\times \cong \mu_{p-1} \times (1 + p\mathbb{Z}_p) \), we can write \((1 + \xi)^{p-1} = 1 + pz_\xi \), for some \( z_\xi \in \mathbb{Z}_p \).

Computing \( S_0, r \) mod \( p^{t+3} \):
\[
S_0, r = \sum_{\xi \in \mu'} \xi^{-3}(1 + \xi)^r
= \sum_{\xi \in \mu'} \xi^{-3}(1 + \xi)^3(1 + \xi)^{n(p-1)p'}
= \sum_{\xi \in \mu'} (1 + \xi^{-1})^3(1 + pz_\xi)^{np'},
\]
Computing \( S_0, r \) mod \( p^{t+3} \) we get
\[
(3.2) \quad S_0, r \equiv \sum_{\xi \in \mu'} (1 + \xi^{-1})^3 \left( 1 + np^{t+1}z_\xi + np^{t+2}z_\xi^2 \left( \frac{np^t - 1}{2} \right) \right) \mod p^{t+3}.
\]
Now \( \sum_{\xi \in \mu'} (1 + \xi^{-1})^3 = \sum_{\xi \in \mu'} (1 + 3\xi^{-1} + 3\xi^{-2} + \xi^{-3}) = (p - 2) + 3 - 3 + 1 = p - 1. \)

Specializing equation (3.2) at \( n = 1, t = 0 \), so \( r = p + 2 \), we get
\[
S_{0,p+2} \equiv \sum_{\xi \in \mu'} (1 + \xi^{-1})^3(1 + pz_\xi) \mod p^3
= (p - 1) \left( \binom{p + 2}{3} + 1 \right) \equiv p - 1 + p \sum_{\xi \in \mu'} (1 + \xi^{-1})^3 z_\xi \mod p^3
= \frac{p(p^2 - 1)(p + 2)}{6} \equiv p \sum_{\xi \in \mu'} (1 + \xi^{-1})^3 z_\xi \mod p^3.
\]
Simplifying, we get that
\[
(3.3) \quad p \sum_{\xi \in \mu'} (1 + \xi^{-1})^3 z_\xi \equiv -\frac{p(p + 2)}{6} \mod p^3.
\]
Again specializing equation (3.2) at \( n = 2, t = 0 \), so \( r = 2p + 1 \), we get

\[
\sum_{0, 2p+1} = \sum_{\xi \in \mu'} (1 + \xi^{-1})^3 (1 + 2pz \xi + p^2 z^2 \xi) \mod p^3
\]

\[
(p - 1) \left( \binom{2p + 1}{3} + \binom{2p + 1}{p + 2} + 1 \right) \equiv p - 1 + 2p \sum_{\xi \in \mu'} (1 + \xi^{-1})^3 z \xi + p^2 \sum_{\xi \in \mu'} (1 + \xi^{-1})^3 z^2 \xi \mod p^3.
\]

Using equation (3.3) in the above expression, we get

\[
\sum_{0, 2p+1} \equiv p \left( p - 1 \right)^2 \sum_{\xi \in \mu'} \xi^{-3} \xi^2 \xi^{-3} \xi^2 \xi^{-3} \xi^2 \mod p^3.
\]

Simplifying above we get that

(3.4) \[ p^2 \sum_{\xi \in \mu'} (1 + \xi^{-1})^3 z \xi \equiv p + (p - 1) \binom{2p + 1}{p + 2} \mod p^3. \]

Now using equations (3.3) and (3.4) in equation (3.2), we get

\[
(p - 1)(1 + S_{0, r}) \equiv p - 1 + np^t \left( \frac{-p(p + 2)}{6} \right) + \frac{np^t\left(np^t - 1\right)}{2} \binom{2p + 1}{p + 2} \mod p^{t+3}.
\]

So

\[
(p - 1)S_{0, r} \equiv \frac{-np^{t+1}(p + 2)}{6} + \binom{n^2p^{2t} - np^t}{2} \binom{2p + 1}{p + 2} \mod p^{t+3}.
\]

Replacing \( np^t \) by \( \frac{r - 3}{p - 1} \), we get

\[
(p - 1)S_{0, r} \equiv -\frac{1}{6} \binom{r - 3}{p - 1} \binom{2p + 1}{p + 2} \mod p^{t+3}.
\]

Finally, dividing by \( p - 1 \), we get

\[
S_{0, r} \equiv \frac{1}{6} \binom{3 - r}{1 - p} \left( 5p + 6p^2 - 3 \binom{2p + 1}{p + 2} \right) \mod p^{t+3},
\]

proving part (1).

Computing \( S_{1, r} \mod p^{t+2} \):

\[
\sum_{1, r} = \binom{r}{1} \sum_{\xi \in \mu'} \xi^{-2} (1 + \xi)^{r-1} = r \sum_{\xi \in \mu'} \xi^{-2} (1 + \xi)^{r} (1 + \xi)^{n(p-1)p^t}.
\]
\[
= r \sum_{\xi \in \mu',}^{\xi \in \mu',} (1 + \xi^{-1})^2 (1 + p z \xi)^{np'}.
\]

Computing \(\sum_{1,r} \text{mod } p^{t+2}\), we get

\[
(3.5) \quad \sum_{1,r} \equiv r \sum_{\xi \in \mu',}^{\xi \in \mu',} (1 + \xi^{-1})^2 (1 + np^{t+1} z \xi) \text{ mod } p^{t+2}.
\]

Now \(\sum_{\xi \in \mu'} (1 + \xi^{-1})^2 = \sum_{\xi \in \mu'} (1 + 2 \xi^{-1} + \xi^{-2}) = (p - 2) + 2 - 1 = p - 1\).

Specializing equation (3.5) at \(n = 1, t = 0, \) so \(r = p + 2\), we get

\[
\sum_{1,r} \equiv r \sum_{\xi \in \mu'}^{\xi \in \mu'} (1 + (1 + \xi^{-1})^2) \text{ mod } p^{t+2}.
\]

Simplifying, we get

\[
(3.6) \quad p \sum_{\xi \in \mu'}^{\xi \in \mu'} (1 + \xi^{-1})^2 z \xi \equiv -\frac{p}{2} \text{ mod } p^{t+2}.
\]

Now using (3.6) in equation (3.5), we get

\[
(p - 1)(S_{1,r} + r) \equiv r(p - 1) - \frac{nr p^{t+1}}{2} \text{ mod } p^{t+2},
\]

which is the same as \(S_{1,r} \equiv \frac{pr(3-r)}{2} \text{ mod } p^{t+2}\), proving part (2).

Computing \(S_{2,r} \text{ mod } p^{t+1}\):

\[
\sum_{2,r} = \left(\begin{array}{c} r \\ 2 \end{array}\right) \sum_{\xi \in \mu'}^{\xi \in \mu'} \xi^{-1} (1 + \xi)^{r-2}
\]

\[
= \left(\begin{array}{c} r \\ 2 \end{array}\right) \sum_{\xi \in \mu'}^{\xi \in \mu'} (1 + \xi^{-1})(1 + \xi)^{n(p-1)p'}
\]

\[
= \left(\begin{array}{c} r \\ 2 \end{array}\right) \sum_{\xi \in \mu'}^{\xi \in \mu'} (1 + \xi^{-1})(1 + p z \xi)^{np'}.
\]

Computing \(\sum_{2,r} \text{ mod } p^{t+1}\), we get

\[
(3.7) \quad \sum_{2,r} \equiv \left(\begin{array}{c} r \\ 2 \end{array}\right) \sum_{\xi \in \mu'}^{\xi \in \mu'} (1 + \xi^{-1}) \text{ mod } p^{t+1}.
\]

Now \(\sum_{\xi \in \mu'} (1 + \xi^{-1}) = p - 1\), so using this in equation (3.7), we get

\[
(p - 1) \left( S_{2,r} + \left(\begin{array}{c} r \\ 2 \end{array}\right) \right) \equiv \left(\begin{array}{c} r \\ 2 \end{array}\right) (p - 1) \text{ mod } p^{t+1},
\]

which is the same as \(S_{2,r} \equiv 0 \text{ mod } p^{t+1}\), proving part (3).

Computing \(S_{3,r} \text{ mod } p^t\):
Since \( n > 0, r > 3 \), hence \( r - i > 0 \), so

\[
\sum_{3,r} = \binom{r}{3} \sum_{\xi \in \mu'} (1 + \xi)^{r-3} = \binom{r}{3} \sum_{\xi \in \mu'} (1 + pz\xi)^{np'}. \]

Computing \( \sum_{3,r} \mod p' \), we get that \( \sum_{3,r} \equiv \binom{r}{3} (p - 2) \mod p' \). Therefore we get

\[
(p - 1) \left( S_{3,r} + \binom{r}{3} \right) \equiv \binom{r}{3} (p - 2) \mod p',
\]

which is the same as \( S_{3,r} \equiv \frac{(r)}{3} \mod p' \), proving part (4).

Computing \( p^i S_{i,r} \mod p'^{t+4} \), for \( i \geq 4 \):

Note that \( \binom{r}{i} \) divides \( \binom{r}{i} \binom{r}{j} \), since \( \binom{r}{i} \binom{r}{j} = \binom{r}{i} \binom{r-i}{j-i} \). By Lemma 3.3 we see that if \( i \geq 4 \), then

\[
p^i \binom{r}{i} \equiv 0 \mod p'^{t+4}, \]

\[
\therefore \sum_{0<j<r \mod (p-1)} p^i \binom{j}{i} \binom{s}{j} \equiv 0 \mod p'^{t+4}.
\]

Hence part (5) holds. \( \square \)

The next two propositions are used in the proof of Proposition 6.1.

**Proposition 3.5.** Let \( p > 3 \) and write \( s = 1 + n(p - 1)p^t \), with \( t \geq 0 \) and \( n > 0 \). Then we have:

(1) \( \sum_{j=1 \mod (p-1)} \binom{s}{j} \equiv 1 + np^{t+1} \mod p^{t+2} \).

(2) \( \sum_{j=1 \mod (p-1)} j \binom{s}{j} \equiv \frac{s(p - 2)}{p - 1} - snp^{t+1} \mod p^{t+2} \).

(3) For all \( i \geq 2, p^i \sum_{j=1 \mod (p-1)} \binom{j}{i} \binom{s}{j} \equiv 0 \mod p^{t+2} \).

**Proof.** This is Proposition 2.9 in [BGR18]. \( \square \)

**Proposition 3.6.** Let \( p > 3 \). If \( r = 3 + n(p - 1)p^t \), with \( t = v(r - 3) \) and \( n > 0 \), then we have:

(1) \( \sum_{1<j<r-2 \mod (p-1)} \binom{r}{j} \equiv 3 - r \mod p^{t+1} \).

(2) \( \sum_{1<j<r-2 \mod (p-1)} j \binom{r}{j} \equiv 0 \mod p^{t+1} \).

(3) \( \sum_{1<j<r-2 \mod (p-1)} \binom{j}{2} \binom{r}{j} \equiv 0 \mod p^t \).

**Proof.** This is Proposition 2.9 in [BGR18]. \( \square \)
\[
\sum_{1<j<r-2} \binom{j}{3} \binom{r}{j} \equiv \frac{(r)}{1-p} \mod p^t.
\]

\[
p^i \sum_{1<j<r-2} \binom{j}{i} \binom{r}{j} \equiv 0 \mod p^{t+4}, \forall i \geq 4.
\]

**Proof.** Let \(0 \leq i < r\). Let \(S_{i,r} = \sum_{1<j<r-2} \binom{j}{i} \binom{r}{j} \equiv 1 \mod (p-1)\) \((j)\cdot(r)\binom{r}{j}\). Let \(\sum_{i,r} = (p-1) \sum_{j \equiv 1 \mod (p-1)} \binom{j}{i} \binom{r}{j}\), so we get

\[
\sum_{i,r} = (p-1) (r^i) + S_{i,r}.
\]

Let \(f_r(x) = (1 + x)^r = \sum_{j \geq 0} \binom{r}{j} x^j\), which we consider as a function from \(\mathbb{Z}_p\) to \(\mathbb{Z}_p\). Let

\[
g_{i,r}(x) = \frac{x^{i-1}}{i!} f_r^{(i)}(x)
= \sum_{j \geq 0} \binom{j}{i} \binom{r}{j} x^{j-1}.
\]

Let \(\mu_{p-1}\) be the set of \((p-1)\)-st roots of unity. Summing \(g_{i,r}(x)\) as \(x\) varies over the \((p-1)\)-st roots of unity we get

\[
\sum_{\xi \in \mu_{p-1}} g_{i,r}(\xi) = \sum_{\xi \in \mu_{p-1}} \sum_{j \geq 0} \binom{j}{i} \binom{r}{j} \xi^{j-1}
= \sum_{\xi \in \mu_{p-1}} \left( \sum_{j \equiv 1 \mod (p-1)} \binom{j}{i} \binom{r}{j} \xi^{j-1} + \sum_{j \not\equiv 1 \mod (p-1)} \binom{j}{i} \binom{r}{j} \xi^{j-1} \right).
\]

Using (3.1), we get

\[
\sum_{\xi \in \mu_{p-1}} g_{i,r}(\xi) = (p-1) \sum_{j \equiv 1 \mod (p-1)} \binom{j}{i} \binom{r}{j},
\]

which is equal to \(\sum_{i,r}\). Let \(\mu'\) be the set of \((p-1)\)-st roots of unity except \(-1\). Since \(g_{i,r}(x) = \binom{r}{i} x^{i-1}(1 + x)^{r-i}\), we have

\[
\sum_{i,r} = \sum_{\xi \in \mu_{p-1}} \binom{r}{i} \xi^{i-1}(1 + \xi)^{r-i}
= \binom{r}{i} \sum_{\xi \in \mu'} \xi^{i-1}(1 + \xi)^{r-i}, \text{ as } r - i > 0.
\]

(3.8)

Also for \(\xi \in \mu'\), we can write \((1 + \xi)^{p-1} = 1 + pz\xi\), for some \(z_\xi \in \mathbb{Z}_p\).

Computing \(S_{0,r} \mod p^t\):

\[
\sum_{0,r} = \sum_{\xi \in \mu'} \xi^{-1}(1 + \xi)^r
\]
Now \( \sum_{\xi \in \mu'} \xi^{-1}(1 + \xi)^3 = \sum_{\xi \in \mu'} (\xi^{-1} + 3 + 3\xi + \xi^2) = 1 + 3(p - 2) + 3 - 1 = 3(p - 1) \), so using this in equation (3.9), we get

\[
(p - 1)(r + S_{0,r}) \equiv 3(p - 1) \mod p^{t+1},
\]

which is the same as \( S_{0,r} \equiv 3 - r \mod p^{t+1} \), proving part (1).

Computing \( S_{1,r} \mod p^{t+1} \):

\[
\sum_{1,r} = \binom{r}{1} \sum_{\xi \in \mu'} (1 + \xi)^r - 1 = r \sum_{\xi \in \mu'} (1 + \xi)^2 (1 + pz\xi)^{np'}
\]

\[
\equiv r \sum_{\xi \in \mu'} (1 + \xi)^2 \mod p^{t+1}.
\]

Now \( \sum_{\xi \in \mu'} (1 + \xi)^2 = \sum_{\xi \in \mu'} (1 + 2\xi + \xi^2) = p - 2 + 2 - 1 = p - 1 \), so using this in equation (3.10), we get

\[
(p - 1)(r + S_{1,r}) \equiv r(p - 1) \mod p^{t+1},
\]

\[
S_{1,r} \equiv 0 \mod p^{t+1},
\]

proving part (2).

Computing \( S_{2,r} \mod p^t \):

\[
\sum_{2,r} = \binom{r}{2} \sum_{\xi \in \mu'} \xi(1 + \xi)^{r-2}
\]

\[
= \binom{r}{2} \sum_{\xi \in \mu'} \xi(1 + \xi)(1 + \xi)^{r-3}
\]

\[
= \binom{r}{2} \sum_{\xi \in \mu'} (\xi + \xi^2)(1 + p\xi)^{np'}.
\]

Now \( \sum_{\xi \in \mu'} (\xi + \xi^2) = 1 - 1 = 0 \), so using this in equation (3.11), we get

\[
(p - 1)S_{2,r} \equiv 0 \mod p^t
\]

\[
S_{2,r} \equiv 0 \mod p^t,
\]

proving part (3).

Computing \( S_{3,r} \mod p^t \):
Since \( n > 0, r > 3, \) hence \( r - i > 0, \) so

\[
\sum_{3, r} = \binom{r}{3} \sum_{\xi \in \mu'} \xi^2 (1 + \xi)^{r-3}
\]

(3.12)

\[
= \binom{r}{3} \sum_{\xi \in \mu'} \xi^2 (1 + p\xi)^{np'}. \]

Now \( \sum_{\xi \in \mu'} \xi^2 = -1. \) Using this in equation (3.12), we get

\[
\sum_{3, r} \equiv \binom{r}{3} \sum_{\xi \in \mu'} \xi^2 \mod p^t
\]

\[(p - 1)S_{3, r} \equiv - \binom{r}{3} \mod p^t
\]

\[S_{3, r} \equiv \frac{r}{1 - p} \mod p^t,
\]

proving part \((4)\).

Computing \( p^i S_{i, r} \mod p^{t+4}, \) for \( i \geq 4:\)

Note that \( \binom{i}{j} \) divides \( \binom{i}{j} \binom{j}{i} \), since \( \binom{i}{j} \binom{j}{i} = \binom{i}{j} \binom{i}{j-i} \). By Lemma 3.1 we get that for \( i \geq 4, \)

\[p^i \binom{r}{i} \equiv 0 \mod p^{t+4}, \]

\[\therefore \sum_{1 < j < r - 2 \atop j \equiv 1 \mod (p-1)} p^i \binom{r}{j} \binom{r}{j} \equiv 0 \mod p^{t+4}, \]

proving part \((5)\).

The next proposition is used in the proofs of Proposition 6.3, Proposition 7.3 and Proposition 7.4.

**Proposition 3.7.** Let \( p > 3 \) and write \( r = 2 + n(p-1)p^t, \) with \( t = v(r-2) \) and \( n > 0. \) Then:

1. \[
\sum_{0 < j < r \atop j \equiv 2 \mod (p-1)} \binom{r}{j} \equiv \frac{p(2 - r)}{2} + 3p^2(2 - r) \frac{r^2(2 - r)^2}{2} \mod p^{t+3}. \]

2. \[
\sum_{0 < j < r \atop j \equiv 2 \mod (p-1)} j \binom{r}{j} \equiv \frac{pr(2 - r)}{1 - p} \mod p^{t+2}. \]

3. \[
\sum_{0 < j < r \atop j \equiv 2 \mod (p-1)} \binom{j}{2} \binom{r}{j} \equiv \frac{\binom{r}{2}}{1 - p} \mod p^{t+1}. \]

4. \[
p^i \sum_{0 < j < r \atop j \equiv 2 \mod (p-1)} j \binom{r}{i} \binom{r}{j} \equiv 0 \mod p^{t+3}, \text{ for all } i \geq 3. \]

**Proof.** We prove part \((1)\) of the proposition. Parts \((2), (3)\) and \((4)\) are in [BGR18], Proposition 2.8.
Let \( S_r = \sum_{2 \leq j < r, j \equiv 2 \mod (p-1)} \binom{r}{j} \). Let \( \sum_r = (p-1) \sum_{2 \leq j \leq r, j \equiv 2 \mod (p-1)} \binom{r}{j} \), then \( \sum_r = (p-1) (S_r + 1) \). Let 

\[
g_r(x) = x^{-2}(1+x)^r = \sum_{j \geq 0} \binom{r}{j} x^j.
\]

(3.13)

Let \( \mu_{p-1} \) be the set of \((p-1)\)-st roots of unity. Summing \( g_r(x) \) as \( x \) varies over all \((p-1)\)-st roots of unity we get 

\[
\sum_{\xi \in \mu_{p-1}} g_r(\xi) = \sum_{\xi \in \mu_{p-1}} \sum_{j \geq 0} \binom{r}{j} \xi^{-2} = \sum_{\xi \in \mu_{p-1}} \left( \sum_{j \equiv 2 \mod (p-1)} \binom{r}{j} \xi^{-2} + \sum_{j \not\equiv 2 \mod (p-1)} \binom{r}{j} \xi^{-2} \right) = (p-1) \sum_{2 \leq j \leq r, j \equiv 2 \mod (p-1)} \binom{r}{j}.
\]

by (3.1). So \( \sum_{\xi \in \mu_{p-1}} g_r(\xi) = \sum_r \). Let \( \mu' \) be the set of \((p-1)\)-st roots of unity except \(-1\). By (3.13), we have 

\[
\sum_r = \sum_{\xi \in \mu'} \xi^{-2}(1+\xi)^r.
\]

(3.14)

For \( \xi \in \mu' \), we can write \((1+\xi)^{p-1} = 1 + pz_\xi \), for some \( z_\xi \in \mathbb{Z}_p \). From (3.14), we get 

\[
\sum_r = \sum_{\xi \in \mu'} \xi^{-2}(1+\xi)^{2+n(p-1)p'} = \sum_{\xi \in \mu'} (1+\xi^{-1})^2(1+pz_\xi)^{np'}.
\]

Computing \( \sum_r \mod p^{t+3} \), we get 

\[
\sum_r = \sum_{\xi \in \mu'} (1+\xi^{-1})^2 \left( 1 + np^{t+1}z_\xi + np^{t+2} \left( \frac{np' - 1}{2} \right) z^2_\xi \right) \mod p^{t+3}.
\]

(3.15)

Now \( \sum_{\xi \in \mu'} (1+\xi^{-1})^2 = \sum_{\xi \in \mu'} (1+2\xi^{-1} + \xi^{-2}) = p - 2 + 2 - 1 = p - 1 \).

Specializing equation (3.15) at \( n = 1, t = 0 \), so \( r = p + 1 \), we get 

\[
\sum_{p+1} = \sum_{\xi \in \mu'} (1+\xi^{-1})^2(1+pz_\xi) \mod p^3
\]

\[
(p-1) \left( \binom{p+1}{2} + \binom{p+1}{p+1} \right) \equiv p - 1 + p \sum_{\xi \in \mu'} (1+\xi^{-1})^2z_\xi \mod p^3.
\]

Simplifying we get 

\[
p \sum_{\xi \in \mu'} (1+\xi^{-1})^2z_\xi \equiv \frac{-p}{2} \mod p^3.
\]

(3.16)
Again specializing (3.15) at \( n = 2, t = 0 \), so \( r = 2p \), we get

\[
\sum_{\xi \in \mu'} (1 + \xi^{-1})^2 (1 + 2p\xi + p^2 \xi^2) \mod p^3
\]

\[
(p - 1) \left( \binom{2p}{2} + \binom{2p}{p+1} + \binom{2p}{2p} \right) \equiv p - 1 + 2p \sum_{\xi \in \mu'} (1 + \xi^{-1})^2 \xi + p^2 \sum_{\xi \in \mu'} (1 + \xi^{-1})^2 \xi^2 \mod p^3.
\]

By using (3.16) in the above expression and simplifying further we get

(3.17) \[
p^2 \sum_{\xi \in \mu'} (1 + \xi^{-1})^2 \xi^2 \equiv 2p - 3p^2 + (p - 1) \left( \binom{2p}{p+1} \right) \mod p^3.
\]

Now using equations (3.16) and (3.17) in (3.15), we get

\[
(p - 1) (S_r + 1) \equiv p - 1 + np^t \left( \frac{-p}{2} \right) + np^t \left( \frac{np^t - 1}{2} \right) \left( 2p - 3p^2 + (p - 1) \left( \binom{2p}{p+1} \right) \right) \mod p^{t+3}.
\]

So

\[
S_r \equiv \frac{-np^{t+1}}{2(p-1)} + \frac{np^t(np^t - 1)}{2(p-1)} \left( 2p - 3p^2 + (p - 1) \left( \binom{2p}{p+1} \right) \right) \mod p^{t+3}.
\]

As \( \frac{1}{p^2}(2p - 3p^2 + (p - 1) \left( \binom{2p}{p+1} \right)) \equiv 1 \mod p \), we get

\[
S_r \equiv \frac{np^{t+1}}{2(1-p)} + \frac{np^{t+2}(np^t - 1)}{2(1-p)} \mod p^{t+3}
\]

\[
\equiv \frac{np^{t+1}}{2} + \frac{np^{t+2}}{2} + \frac{np^{t+2} - n^2 p^{2t+2}}{2} \mod p^{t+3}
\]

\[
= \frac{np^{t+1}}{2} + np^{t+2} - \frac{n^2 p^{2t+2}}{2} \mod p^{t+3}.
\]

Replacing \( np^t \) by \( \frac{2}{1-p} \), we get

\[
S_r \equiv \frac{(2 - r)p}{2(1-p)} + \frac{(2 - r)p^2}{1-p} - \frac{p^2}{2} \left( \frac{2 - r}{1-p} \right)^2 \mod p^{t+3}
\]

\[
\equiv \frac{(2 - r)p}{2} + \frac{(2 - r)p^2}{2} + (2 - r)p^2 - \frac{p^2(2 - r)^2}{2} \mod p^{t+3}
\]

\[
= \frac{(2 - r)p}{2} + \frac{3(2 - r)}{2} - \frac{p^2(2 - r)^2}{2} \mod p^{t+3},
\]

proving part (1) of the proposition.

Note in particular that \( S_r \equiv \frac{(2 - r)p}{2} \mod p^{t+2} \), which matches with part (1) of Proposition 2.8 in [BGR18].

The next proposition is used in the proofs of Proposition 2.11, Proposition 3.3, and Proposition 3.4.

**Proposition 3.8.** Let \( p > 3 \). If \( r = 3 + n(p - 1)p^t \), with \( t = v(r - 3) \) and \( n > 0 \), then we have:

(1) \[
\sum_{\substack{2 \leq j \leq r - 1 \atop j \equiv 2 \mod (p-1)}} \binom{r}{j} \equiv 3 \binom{r}{2} + \frac{5np^{t+1}}{2} \mod p^{t+2}.
\]
Using (3.1), we get
\[ \sum_{j \equiv 2 \pmod{p-1}} j \binom{r}{j} = r(3-r) \mod p^{t+1}. \]

(3) \[ \sum_{j \equiv 2 \pmod{p-1}} \binom{j}{2} \binom{r}{j} = 0 \mod p^{t+1}. \]

(4) \[ \sum_{j \equiv 2 \pmod{p-1}} \binom{j}{3} \binom{r}{j} = \frac{r(r-1)}{p-1} \mod p^{t+1}. \]

(5) \[ p^i \sum_{j \equiv 2 \pmod{p-1}} \binom{j}{i} \binom{r}{j} = 0 \mod p^{i+1}, \forall i \geq 4. \]

**Proof.** Let \( 0 \leq i < r \). Let \( S_{i,r} = \sum_{j \equiv 2 \pmod{p-1}} \binom{j}{2} \binom{r}{j} \). Let \( \sum_{i,r} = (p-1) \sum_{j \equiv 0 \mod{p-1}} \binom{j}{2} \binom{r}{j} \), then \( \sum_{i,r} = (p-1) \left( \binom{2}{2} + S_{i,r} \right) \). Let \( f_r(x) = (1 + x)^r = \sum_{j \geq 0} \binom{r}{j} x^j \), which we consider as a function from \( \mathbb{Z}_p \) to \( \mathbb{Z}_p \). Let
\[ g_{i,r}(x) = \frac{x^{i-2}}{i!} f_r(i) = \sum_{j \geq 0} \binom{j}{i} \binom{r}{j} x^{j-2}. \]

Let \( \mu_{p-1} \) be the \( (p-1) \)-st roots of unity. Summing \( g_{i,r}(x) \) as \( x \) varies over all \( (p-1) \)-st roots of unity we get
\[
\sum_{\xi \in \mu_{p-1}} g_{i,r}(\xi) = \sum_{\xi \in \mu_{p-1}} \sum_{j \geq 0} \binom{j}{i} \binom{r}{j} \xi^{j-2}
= \sum_{\xi \in \mu_{p-1}} \left( \sum_{j \equiv 2 \pmod{p-1}} \binom{j}{i} \binom{r}{j} \xi^{j-2} + \sum_{j \equiv 2 \pmod{p-1}} \binom{j}{i} \binom{r}{j} \xi^{j-2} \right).
\]

Using (3.1), we get
\[ \sum_{\xi \in \mu_{p-1}} g_{i,r}(\xi) = (p-1) \sum_{j \equiv 2 \pmod{p-1}} \binom{j}{i} \binom{r}{j}, \]
which is equal to \( \sum_{i,r} \). Let \( \mu' \) be the set of \( (p-1) \)-st roots of unity except \(-1\). Since \( g_{i,r}(x) = x^{i-2} \binom{r}{i} (1 + x)^{r-i} \), we have
\[ \sum_{i,r} = \sum_{\xi \in \mu_{p-1}} \binom{r}{i} \xi^{i-2}(1 + \xi)^{r-i}
= \binom{r}{i} \sum_{\xi \in \mu'} \xi^{i-2}(1 + \xi)^{r-i}, \text{ as } r - i > 0. \]

Also for \( \xi \in \mu' \), write \( (1 + \xi)^{p-1} = 1 + px_\xi \), for some \( x_\xi \in \mathbb{Z}_p \).
Computing $S_{0,r} \mod p^{t+2}$:

\[
\sum_{\xi \in \mu'} \xi^{-2}(1 + \xi)^r = \sum_{\xi \in \mu'} \xi^{-2}(1 + \xi)^3(1 + \xi)^{n(p-1)p'} = \sum_{\xi \in \mu'} \xi^{-2}(1 + \xi)^3(1 + p\xi)^{np'}.
\]

Computing $\sum_{0,r} \mod p^{t+2}$ we get

\[
(3.18) \quad \sum_{0,r} \equiv \sum_{\xi \in \mu'} \xi^{-2}(1 + \xi)^3(1 + np^{t+1}z\xi) \mod p^{t+2}.
\]

Now $\sum_{\xi \in \mu'} \xi^{-2}(1 + \xi)^3 = \sum_{\xi \in \mu'} (\xi^{-2} + 3\xi^{-1} + 3 + \xi) = -1 + 3 + 3(p - 2) + 1 = 3(p - 1)$.

Specializing equation (3.18) at $t = 0$, $n = 1$, so $r = p + 2$, we get

\[
(p - 1) \left( \binom{p+2}{2} + \binom{p+2}{p+1} \right) \equiv 3(p - 1) + p \sum_{\xi \in \mu'} \xi^{-2}(1 + \xi)^3z\xi \mod p^2.
\]

Simplifying we get

\[
(3.19) \quad p \sum_{\xi \in \mu'} \xi^{-2}(1 + \xi)^3z\xi \equiv -\frac{5p}{2} \mod p^2.
\]

Using equation (3.19) in equation (3.18), we get

\[
(p - 1) \left( \binom{r}{2} + S_{0,r} \right) \equiv 3(p - 1) + np^t \left( -\frac{5p}{2} \right) \mod p^{t+2}
\]

\[
\binom{r}{2} + S_{0,r} \equiv 3 - \frac{5np^{t+1}}{2(p - 1)} \mod p^{t+2}
\]

\[
S_{0,r} \equiv 3 - \binom{r}{2} + \frac{5np^{t+1}}{2} \mod p^{t+2},
\]

proving part (1).

Computing $S_{1,r} \mod p^{t+1}$:

\[
\sum_{1,r} = \binom{r}{1} \sum_{\xi \in \mu'} \xi^{-1}(1 + \xi)^{r-1} = r \sum_{\xi \in \mu'} \xi^{-1}(1 + \xi)^2(1 + p\xi)^{np'}. \]

Computing $\sum_{1,r} \mod p^{t+1}$ we get

\[
(3.20) \quad \sum_{1,r} \equiv r \sum_{\xi \in \mu'} \xi^{-1}(1 + \xi)^2 \mod p^{t+1}.
\]
Now \( \sum_{\xi \in \mu'} \xi^{-1}(1 + \xi)^2 = \sum_{\xi \in \mu'} (\xi^{-1} + 2 + \xi) = 1 + 2(p - 2) + 1 = 2(p - 1) \). Using this in equation \( (3.20) \), we get

\[
(p - 1) \left( 2 \binom{r}{2} + S_{1,r} \right) \equiv 2r(p - 1) \mod p^{t+1}.
\]

So

\[
S_{1,r} \equiv r(3 - r) \mod p^{t+1},
\]

proving part (2).

Computing \( S_{2,r} \mod p^{t+1} \):

\[
\sum_{2,r} = \binom{r}{2} \sum_{\xi \in \mu'} (1 + \xi)^{r-2}
\]

\[
= \binom{r}{2} \sum_{\xi \in \mu'} (1 + \xi)(1 + \xi)^{n(p-1)p'}
\]

\[
= \binom{r}{2} \sum_{\xi \in \mu'} (1 + \xi)(1 + p\xi)^{np'}.
\]

Computing \( \sum_{2,r} \mod p^{t+1} \) we get

\[
\sum_{2,r} \equiv \binom{r}{2} \sum_{\xi \in \mu'} (1 + \xi) \mod p^{t+1}.
\]

Now \( \sum_{\xi \in \mu'} (1 + \xi) = (p - 2) + 1 = p - 1 \). Using this in equation \( (3.21) \), we get

\[
\sum_{2,r} \equiv \binom{r}{2} (p - 1) \mod p^{t+1}
\]

\[
(p - 1) \left( \binom{r}{2} + S_{2,r} \right) \equiv \binom{r}{2} (p - 1) \mod p^{t+1},
\]

which is the same as \( S_{2,r} \equiv 0 \mod p^{t+1} \), proving part (3).

Computing \( S_{3,r} \mod p^{t+1} \):

Since \( n > 0 \), \( r > 3 \), hence \( r - i > 0 \), so

\[
\sum_{3,r} = \binom{r}{3} \sum_{\xi \in \mu'} \xi(1 + \xi)^{r-3}
\]

\[
= \binom{r}{3} \sum_{\xi \in \mu'} \xi(1 + p\xi)^{np'}.
\]

Computing \( \sum_{3,r} \mod p^{t+1} \) we get

\[
\sum_{3,r} \equiv \binom{r}{3} \sum_{\xi \in \mu'} \xi \mod p^{t+1}.
\]
Now $\sum_{\xi \in \mu'} \xi = 1$. Using this in equation (3.22), we get

$$\sum_{b_r} \equiv \binom{r}{3} \mod p^{t+1}$$

$$(p - 1)S_{b_r} \equiv \binom{r}{3} \mod p^{t+1},$$

which is the same as $S_{b_r} \equiv \binom{r}{p-1}$, proving part (4).

Computing $p^i S_{i,r} \mod p^{t+4}$ for $i \geq 4$:

As before $\binom{r}{i}$ divides $\binom{r}{j} \binom{r}{j}$, From Lemma 3.1 we get that for $i \geq 4$

$$p^i \binom{r}{i} \equiv 0 \mod p^{t+4},$$

and

$$\therefore p^i \sum_{2 \leq j < r - 1 \atop j \equiv 2 \mod (p-1)} \binom{j}{i} \binom{r}{j} \equiv 0 \mod p^{t+4},$$

proving part (5). □

The final proposition is a slight variant of the previous proposition. It is also used in the proof of Proposition 7.1.

**Proposition 3.9.** Let $p > 3$. If $r = 3 + n(p - 1)p^t$, with $t = v(r - 3)$ and $n > 0$, then one can choose integers $\beta_j$ for all $j \equiv 2 \mod (p - 1)$, with $2 \leq j < r - 1$, satisfying:

1. $\beta_j \equiv \binom{r}{j} \mod p^i$ for all $j$’s above.

2. $\sum_{2 \leq j < r - 1 \atop j \equiv 2 \mod (p-1)} \binom{j}{i} \beta_j \equiv 0 \mod p^{t+2-i}$ for $i = 0, 1$ and 2.

3. $p^3 \sum_{2 \leq j < r - 1 \atop j \equiv 2 \mod (p-1)} \binom{j}{3} \beta_j \equiv \frac{p^3 \binom{r}{3}}{p - 1} \mod p^{t+3}$.

4. For $i \geq 4$, we have $p^i \sum_{j \equiv 2 \mod (p-1)} \binom{j}{i} \beta_j \equiv 0 \mod p^{t+3}$.

**Proof.** The proof is similar to that of Lemma 7.2 in [BG15]. Define

(3.23) $\beta_j = \binom{r}{j}$ for $j \equiv 2 \mod (p - 1), 2 < j < r - 1, j \neq 2p$,

(3.24) $\beta_2 = - \sum_{2 \leq j < r - 1 \atop j \equiv 2 \mod (p-1)} b' \binom{r}{j} \mod p^{t+1}$, where $2b' \equiv 1 \mod p^{t+1},$

(3.25) $\beta_{2p} = - \sum_{2 \leq j < r - 1 \atop j \equiv 2 \mod (p-1) \atop j \neq 2p} \binom{r}{j} - \beta_2.$
Clearly $\beta_j \equiv \left( \begin{array}{c} r \\ j \end{array} \right) \mod p^t$, when $2 < j < r - 1$, $j \equiv 2 \mod (p - 1)$ and $j \neq 2p$.

Now for $j = 2$ we have

$$2\beta_2 = - \sum_{2 < j < r - 1 \atop j \equiv 2 \mod (p - 1)} 2b'j \left( \begin{array}{c} r \\ j \end{array} \right)$$

$$\equiv - \sum_{2 < j < r - 1 \atop j \equiv 2 \mod (p - 1)} j \left( \begin{array}{c} r \\ j \end{array} \right) \mod p^t,$$

(3.26)

because $2b' \equiv 1 \mod p^{t+1}$. Using Proposition 3.8 part (2) in equation (3.26), we get $2\beta_2 \equiv r(r - 1) \mod p^t$, which is the same as $\beta_2 \equiv \left( \begin{array}{c} r \\ 2 \end{array} \right) \mod p^t$.

For $j = 2p$ we have to show that

$$\beta_{2p} = - \sum_{2 < j < r - 1 \atop j \equiv 2 \mod (p - 1)} \left( \begin{array}{c} r \\ j \end{array} \right) - \beta_2 \equiv \left( \begin{array}{c} r \\ 2p \end{array} \right) \mod p^t,$$

which is the same as showing

$$\left( \begin{array}{c} r \\ 2 \end{array} \right) = \left( \begin{array}{c} r \\ 2p \end{array} \right) \mod p^t,$$

(3.27)

because $\beta_2 \equiv \left( \begin{array}{c} r \\ 2 \end{array} \right) \mod p^t$. By Proposition 3.8 part (1) mod $p^t$, the left hand side of equation (3.27) is

$$-3 + \left( \begin{array}{c} r \\ 2 \end{array} \right) + \left( \begin{array}{c} r \\ r - 1 \end{array} \right) = r - 3 + \left( \begin{array}{c} r \\ 2 \end{array} \right) \equiv \left( \begin{array}{c} r \\ 2 \end{array} \right) \mod p^t,$$

which is the right hand side of (3.27). Therefore $\beta_{2p} \equiv \left( \begin{array}{c} r \\ 2p \end{array} \right) \mod p^t$. This proves part (1) of the proposition.

For $i = 0$,

$$\sum_{2 < j < r - 1 \atop j \equiv 2 \mod (p - 1)} \beta_j = \beta_2 + \beta_{2p} + \sum_{2 < j < r - 1 \atop j \equiv 2 \mod (p - 1) \atop j \neq 2p} \left( \begin{array}{c} r \\ j \end{array} \right) = 0.$$

Therefore $\sum_{2 \leq j < r - 1 \atop j \equiv 2 \mod (p - 1)} \beta_j \equiv 0 \mod p^{t+2}$.

For $i = 1$,

$$\sum_{2 < j < r - 1 \atop j \equiv 2 \mod (p - 1)} j\beta_j = 2\beta_2 + 2p\beta_{2p} + \sum_{2 < j < r - 1 \atop j \equiv 2 \mod (p - 1) \atop j \neq 2p} j \left( \begin{array}{c} r \\ j \end{array} \right)$$

$$\equiv - \sum_{2 < j < r - 1 \atop j \equiv 2 \mod (p - 1) \atop j \neq 2p} j \left( \begin{array}{c} r \\ j \end{array} \right) + 2p\beta_{2p} + \sum_{2 < j < r - 1 \atop j \equiv 2 \mod (p - 1) \atop j \neq 2p} j \left( \begin{array}{c} r \\ j \end{array} \right) \mod p^{t+1}$$
\[ = -2p \binom{r}{2p} + 2p^2 \equiv 0 \mod p^{t+1}. \]

For \( i = 2 \),
\[
\sum_{2 \leq j < r-1 \atop j \equiv 2 \mod (p-1)} \binom{j}{2} \beta_j \equiv \sum_{2 \leq j < r-1 \atop j \equiv 2 \mod (p-1)} \binom{j}{r} \mod p^t.
\]

Using Proposition \( 3.8 \) part (3), we get
\[
\sum_{2 \leq j < r-1 \atop j \equiv 2 \mod (p-1)} \binom{j}{2} \beta_j \equiv \binom{r}{2} - r \binom{r-1}{2} \mod p^t
\]
\[
\equiv \binom{r}{2}(3 - r) \equiv 0 \mod p^t,
\]
proving part (2) of the proposition.

Since \( \beta_j \equiv \binom{j}{r} \mod p^t \), we have
\[
(3.28) \quad p^3 \sum_{2 \leq j < r-1 \atop j \equiv 2 \mod (p-1)} \binom{j}{3} \beta_j \equiv \sum_{2 \leq j < r-1 \atop j \equiv 2 \mod (p-1)} \binom{j}{r} \mod p^{t+3}.
\]

Using Proposition \( 3.8 \) part (4) in equation (3.28), we get
\[
p^3 \sum_{2 \leq j < r-1 \atop j \equiv 2 \mod (p-1)} \binom{j}{3} \binom{r}{j} \equiv p^3 \left( \frac{(r)}{3} \binom{r-1}{3} - r \binom{r-2}{3} \right) \mod p^{t+4}.
\]

Using Lemma \( 3.2 \) we get that
\[
p^3 \sum_{2 \leq j < r-1 \atop j \equiv 2 \mod (p-1)} \binom{j}{3} \beta_j \equiv \frac{p^3 (r)}{3} \binom{r-1}{3} \mod p^{t+3},
\]
proving part (3).

By part (1) of the proposition we know that \( \beta_j \equiv \binom{j}{r} \mod p^t \), hence we have the following
\[
p^i \sum_{2 \leq j < r-1 \atop j \equiv 2 \mod (p-1)} \binom{j}{i} \beta_j \equiv p^i \sum_{2 \leq j < r-1 \atop j \equiv 2 \mod (p-1)} \binom{j}{i} \binom{r}{j} \mod p^{t+i}.
\]

For \( i \geq 4 \), by Proposition \( 3.8 \) part (5), we get that
\[
(3.29) \quad p^i \sum_{2 \leq j < r-1 \atop j \equiv 2 \mod (p-1)} \binom{j}{i} \binom{r}{j} \equiv -p^i r \binom{r-1}{i} \mod p^{t+4}.
\]

Using Lemma \( 3.2 \) in equation (3.29), we finally get
\[
p^i \sum_{2 \leq j < r-1 \atop j \equiv 2 \mod (p-1)} \binom{j}{i} \beta_j \equiv 0 \mod p^{t+3} \text{ for all } i \geq 4,
\]
proving part (5). \( \square \)
4. Telescoping Lemmas

In this section we study the action of the Hecke operator on some simple functions supported mod $KZ$ on a line segment in the Bruhat-Tits tree whose length is linear in $t = v(3-r)$. Similar functions were used once in [BG13] and twice in [BGR18]. In this paper the importance of these functions will become clear as they will be used extensively to study the structure of the subquotients $F_1$, $F_2$ and $F_3$ of $\tilde{\Theta}_{k,a_p}$ at ‘infinity’.

Define

$$c = \frac{a_p^2 - (r - 2)(r^{-1})p^3}{pa_p},$$

$$\tilde{c} = \frac{a_p^2 - (\tilde{r})p^3}{pa_p},$$

and set $\tau = v(c)$ and $\tilde{\tau} = v(\tilde{c})$. These quantities will play a crucial role in everything that follows.

The following lemma explicates the relationship between $\tau$ and $\tilde{\tau}$.

**Lemma 4.1.** Let $p > 3$, $r = 3 + n(p-1)p^t$ with $t = v(r-3)$, and let the slope of $a_p$ be $v(a_p) = \frac{t}{2}$. We have:

(i) If $\tilde{\tau} < t + \frac{1}{2}$, then $\tilde{\tau} = \tau$.

(ii) If $\tilde{\tau} \geq t + \frac{1}{2}$, then $\tau \geq t + \frac{1}{2}$.

(iii) If $\tau < t + \frac{1}{2}$, then $\tau = \tilde{\tau}$.

(iv) If $\tau \geq t + \frac{1}{2}$, then $\tilde{\tau} \geq t + \frac{1}{2}$.

**Proof.** Write

$$c = \frac{a_p^2 - (r - 2)(r^{-1})p^3}{pa_p} = \frac{a_p^2 - (\tilde{r})p^3}{pa_p} + \frac{(\tilde{r})p^3 - (r - 2)(r^{-1})p^3}{pa_p} = \tilde{c} + \frac{2(3-r)(r^{-1})p^2}{3a_p}.$$

We use the non-archimedean properties of the valuation $v$. We have $\tau \geq \min(\tilde{\tau}, v(\frac{(3-r)(r^{-1})p^2}{a_p}))$. Moreover, since $v(\frac{(3-r)(r^{-1})p^2}{a_p}) \geq t + \frac{1}{2}$, we get that if $\tilde{\tau} < t + \frac{1}{2}$, then $\tau = \tilde{\tau}$ and if $\tilde{\tau} \geq t + \frac{1}{2}$, then $\tau \geq t + \frac{1}{2}$, proving part (i) and (ii). The proof of parts (iii) and (iv) are similar. □

We now state and prove several lemmas involving the action of the Hecke operators on simple functions supported mod $KZ$ on a line segment on the tree. All the lemmas are characterized by a simple common property: due to a telescoping cancellation property, the functions obtained after applying the Hecke operator have a very simple form with support mod $KZ$ on vertices very close to the origin of the tree.

We make some remarks about notation. Let $f$, $g \in \text{ind}_{KZ}^G \text{Sym}^r\mathbb{Q}_p^2$. By $f = g + O(p^s)$, for some $s \geq 0$, we shall mean that the difference $f - g \in \text{ind}_{KZ}^G \text{Sym}^r\mathbb{Z}_p^2$ is integral and is divisible by $p^s$. Also, in the proofs we will often compute in ‘radius $n$’. This means that we are computing the part of a function which is supported mod $KZ$ at distance $n$ from the origin of the tree. This ‘$n$’ is not to be confused with the ‘$n$’ occurring in the statements of the lemmas below.

The first lemma will be used in the proof of Proposition 5.1 and 6.1.
Lemma 4.2. Let \( p > 3, \ r \geq 2p + 1, \ r = 3 + n(p - 1)p^t, \) with \( t = v(r - 3), \ v(a_p) = \frac{3}{2}, \ c = \frac{a_p^2 - (5)p^3}{p \sigma_p}, \) \( v(\tilde{c}) = \tilde{v} \) and let \( \ell_0 = \min\{t, \tilde{v}\}. \) If

\[
\chi = \sum_{n=0}^{t} a_p^{\alpha}[g_{n,0}, Y^r - X^{r-3}Y^3],
\]

then

\[
(T - a_p)\chi = [\alpha, Y^r] + a_p[1, X^{r-3}Y^3] + p^{t+1}h_\chi + O(p^{\ell_0+2}),
\]

where \( h_\chi \) is an integral linear combination of the terms of the form \([g, X^r]\) and \([g, X^{r-1}Y]\), for \( g \in G \).

Proof. Write \( \chi = \sum_{n=0}^{t} \chi_n, \) where \( \chi_n := a_p^{\alpha}[g_{n,0}, Y^r - X^{r-3}Y^3], \) for \( n \geq 0. \) By the formula for the Hecke operator, in radius \(-1\) we have

\[
T^{-}\chi_0 = [\alpha, Y^r] - (pX)^{r-3}Y^3 = [\alpha, Y^r] + O(p^{\ell_0+3}),
\]

because \( p > 3, \) so \( r - 3 \geq (p - 1)p^t \geq 4p^t > t + 3 \geq \ell_0 + 3. \)

In radius \( 0 \) we have \(-a_p\chi_0 = -a_p[1, Y^r - X^{r-3}Y^3],

\[
T^{-}\chi_1 = a_p[0, Y^r - (pX)^{r-3}Y^3] = a_p[1, Y^r] + O(p^{\ell_0+3}).
\]

So finally in radius \( 0 \) we have

\[
-a_p\chi_0 + T^{-}\chi_1 = a_p[1, X^{r-3}Y^3] + O(p^{\ell_0+3}).
\]

Now for radius \( n \) with \( 1 \leq n \leq t - 1, \) we compute \( T^{-}\chi_{n+1} - a_p\chi_n + T^{+}\chi_{n-1}: \)

\[
T^{-}\chi_{n+1} = a_p^{n+1}[g_{n+1,0}, Y^r - (pX)^{r-3}Y^3] = a_p^{n+1}g_{n,0}, Y^r] + O(p^{\ell_0+3}).
\]

Also we have \(-a_p\chi_n = -a_p^{n+1}[g_{n,0}, Y^r - X^{r-3}Y^3] \) and

\[
T^{+}\chi_{n-1} = a_p^{n-1}
\left[ \sum_{\lambda \in \mathbb{F}_p} \left[ g_{n-1,0}^{\alpha,0,1,1}, (-[\lambda], X + pY)^r - X^{r-3}(-[\lambda], X + pY)^3 \right] \right]
\]

\[
= a_p^{n-1}
\left[ \sum_{\lambda \in \mathbb{F}_p} \left[ g_{n-1,0}^{\alpha,0,1,1}, (-[\lambda], X)^r + pr(-[\lambda], X)^{r-1}Y + p^2 \binom{r}{2} (-[\lambda], X)^{r-2}Y^2 \right. \right.
\]

\[
+ p^3 \binom{r}{3} (-[\lambda], X)^{r-3}Y^3 + \sum_{j \geq 4} p^j \binom{r}{j} (-[\lambda], X)^{r-j}Y^j - X^{r-3}(-[\lambda]^3 X^3 + 3p[\lambda]^2 X^2 Y
\]

\[
- 3p^2[\lambda] XY^2 + p^3 Y^3) \bigg] + g_{n-1,0}^{\alpha,0,1,0}, p^rY^r - p^3 X^{r-3}Y^3 \bigg). \]

By Lemma 3.1 we get

\[
T^{+}\chi_{n-1} = a_p^{n-1}
\left[ \sum_{\lambda \in \mathbb{F}_p} \left[ g_{n-1,0}^{\alpha,0,1,1}, (r - 3)p[\lambda]^2 X^{r-1}Y + \left( 3 - \binom{r}{2} \right)p^2[\lambda] X^{r-2}Y^2 \right. \right.
\]

\[
+ \left( \binom{r}{3} - 1 \right) p^3 X^{r-3}Y^3 \bigg] - g_{n,0}^{\alpha,0,1,0}, p^3 X^{r-3}Y^3 \bigg) + O(p^{\ell_0+4}).
\]
Note that \( v(3 - \binom{t}{j}) \geq t \) and \( v(\binom{t}{j} - 1) \geq t \). So we get

\[
T^+ \chi_{n-1} = a_p^{n-1}(r-3)p \sum_{\lambda \in \mathbb{F}_p^2} [\lambda]^2 [g_{n,p^{n-1} \lambda}, X^{r-1}Y] - a_p^{n-1}p^3 [g_{n,0}, X^{r-3}Y^3] + O(p^{t_0+2}).
\]

Therefore in radius \( n \), for \( 1 \leq n \leq t-1 \), we have

\[
T^- \chi_{n+1} - a_p \chi_n + T^+ \chi_{n-1} = a_p^{n+1} [g_{n,0}, X^{r-3}Y^3] - a_p^{n-1}p^3 [g_{n,0}, X^{r-3}Y^3] + p^{t+1}h_n + O(p^{t_0+2}),
\]

where \( h_n \) is an integral linear combination of terms of the form \([g, X^{r-1}Y]\) for \( g \in G \). As \( \tilde{c} = \frac{a_p^2 - \binom{t}{j} p^3}{pa_p^3} \),
so \( a_p^2 - p^3 = \tilde{c}pa_p + p^3 \left( \binom{t}{j} - 1 \right) \). Hence \( v(a_p^2 - p^3) = \min\{v(\tilde{c}) + \frac{2}{2}, t + 3\} > t_0 + 2 \). We finally get

(4.3)

\[
T^- \chi_{n+1} - a_p \chi_n + T^+ \chi_{n-1} = p^{t+1}h_n + O(p^{t_0+2}).
\]

Similarly for \( n = t \geq 1 \), we compute \(-a_p \chi_t + T^+ \chi_{t-1}:

\[
-a_p \chi_t = -a_p^{t+1} [g_{t,0}, Y^r - X^{r-3}Y^3].\]

Also

\[
T^+ \chi_{t-1} = a_p^{t-1} \left( \sum_{\lambda \in \mathbb{F}_p^2} [g_{t-1,0} g_{1,0}^0 \lambda, (-\lambda X + pY)^r - X^{r-3}(-\lambda X + pY)^3] \right)
\]

\[
= a_p^{t-1} \left( \sum_{\lambda \in \mathbb{F}_p^2} [g_{t-1,0} g_{1,0}^0 \lambda, (-\lambda X)^r + pr(-\lambda X)^{r-1}Y + p^2 \binom{r}{2} (-\lambda X)^{r-2}Y^2 \right.
\]

\[
+ p^3 \binom{r}{3} (-\lambda X)^{r-3}Y^3 + \sum_{j \geq 4} p^j \binom{r}{j} (-\lambda X)^{r-j}Y^j - X^{r-3}(-\lambda X^3 + 3p[\lambda]^2 X^2 Y
\]

\[
- 3p^2[\lambda]XY^2 + p^3 Y^3) \right) + [g_{t,0}^0, -p^3 X^{r-3}Y^3] + O(p^{t_0+4}),
\]

because from Lemma 3.3 \( p^j \binom{t}{j} \equiv 0 \ mod \ p^{j+4}, \forall j \geq 4 \). As \( v \left( 3 - \binom{t}{j} \right) \geq t \) and \( v \left( \binom{t}{j} - 1 \right) \geq t \), we get

\[
T^+ \chi_{t-1} = a_p^{t-1}(r-3)p \sum_{\lambda \in \mathbb{F}_p^2} [\lambda]^2 [g_{t,p^{t-1} \lambda}, X^{r-1}Y] - a_p^{t-1}p^3 [g_{t,0}, X^{r-3}Y^3] + O(p^{t_0+2}).
\]

As before, note that \( v(a_p^2 - p^3) > t_0 + 2 \), so we get

(4.4)

\[
-a_p \chi_t + T^+ \chi_{t-1} = p^{t+1}h_t + O(p^{t_0+2}),
\]

where \( h_t \) is an integral linear combination of the terms of the form \([g, X^r] \) and \([g, X^{r-1}Y]\), for \( g \in G \).

Finally in radius \( t + 1 \), we compute

(4.5)

\[
T^+ \chi_t = a_p^t \left( \sum_{\lambda \in \mathbb{F}_p^2} [g_{t+1,0} g_{1,0}^0 \lambda, (-\lambda X + pY)^r - X^{r-3}(-\lambda X + pY)^3] + [g_{t+1,0}, (pY)^r - p^3 X^{r-3}Y^3] \right)
\]

\[
= p^{t+1}h_{t+1} + O(p^{t_0+2}),
\]
where \( h_{t+1} \) is a linear combination of the terms of the form \([g, X^{r-1}Y]\) for \( g \in G \).

Using equations (4.1), (4.2), (4.3), (4.4), (4.5), we get

\[
(T - a_p)\chi = T^- \chi_0 - a_p \chi_0 + T^- \chi_1 + \sum_{n=1}^{t-1} (T^- \chi_{n+1} - a_p \chi_n + T^+ \chi_{n-1}) - a_p \chi_t + T^+ \chi_{t-1} + T^+ \chi_t
\]

\[
= [\alpha, Y^r] + ap[1, X^{r-3}Y^3] + p^{t+1} h_\chi + O(p^{t+2}),
\]

where \( h_\chi \) is an integral linear combination of the terms of the form \([g, X^r]\) and \([g, X^{r-1}Y]\), for \( g \in G \).

The following lemma will be used in the proof of Proposition 6.3.

**Lemma 4.3.** Let \( p > 3, \ r \geq 2p + 1, \ r = 3 + n(p - 1)p^t \), with \( t = v(r - 3), \ v(a_p) = \frac{3}{2} \) and \( c = \frac{a_p^2 - (r-2)(r-1)p^3}{p^{t+1}} \). Let \( \tau = v(c) \) and assume \( \tau > t + \frac{1}{2} \). If

\[
\xi = \frac{-1}{p^2(3-r)} \sum_{n=0}^{2^{t+1}} \left( \frac{a_p}{p} \right)^{n+1} [g_{n,0}, X^{r-2}Y^2 + (r-3)X^pY^{r-p} - (r-2)XY^{r-1}],
\]

then

\[
(T - a_p)\xi = \frac{a_p(r-2)}{p^2(3-r)} [\alpha, XY^{r-1}] + \frac{a_p^2}{p^3(3-r)} [1, X^{r-2}Y^2 + (r-3)X^pY^{r-p}] + O(\sqrt{p}).
\]

**Proof.** Let \( \xi_n = \frac{-1}{p^2(3-r)} \left( \frac{a_p}{p} \right)^{n+1} [g_{n,0}, X^{r-2}Y^2 + (r-3)X^pY^{r-p} - (r-2)XY^{r-1}] \). So we can write

\[
\xi = \sum_{n=0}^{2^{t+1}} \xi_n. \text{ We will compute } (T - a_p)\xi \text{ in several steps.}
\]

Computation in radius \(-1\):

\[
T^- \xi_0 = \frac{-1}{p^2(3-r)} \left( \frac{a_p}{p} \right) [\alpha, (pX)^{r-2}Y^2 + (r-3)(pX)^pY^{r-p} - (r-2)pX^{r-1}] + O(p^2),
\]

because \( p > 3, \) so \( r-2 \geq (p-1)p^t + 1 > t + 4 \).

Computation in radius 0:

\[
-a_p \xi_0 = \frac{a_p^2}{p^2(3-r)} [1, X^{r-2}Y^2 + (r-3)X^pY^{r-p} - (r-2)XY^{r-1}],
\]

\[
T^- \xi_1 = \frac{-1}{p^2(3-r)} \left( \frac{a_p}{p} \right)^2 [1, (pX)^{r-2}Y^2 + (r-3)(pX)^pY^{r-p} - (r-2)pX^{r-1}] + O(p^2).
\]

Therefore in radius 0 we get

\[
-a_p \xi_0 + T^- \xi_1 = \frac{a_p^2}{p^3(3-r)} [1, X^{r-2}Y^2 + (r-3)X^pY^{r-p}] + O(p^2).
\]
Computation in radius $n$, when $1 \leq n \leq 2t$:

$$-a_p\xi_n = \frac{a_p}{p^2(3-r)} \left( \frac{a_p}{p} \right)^{n+1} [g_{n,0}^0, X^{r-2}Y^2 + (r-3)X^pY^{r-p} - (r-2)XY^{r-1}],$$

$$T^{-\xi_{n+1}} = \frac{-1}{p^2(3-r)} \left( \frac{a_p}{p} \right)^{n+2} [g_{n+1,0}^0, (pX)^{r-2}Y^2 + (r-3)(pX)^pY^{r-p} - (r-2)pXY^{r-1}]$$

$$= \frac{r-2}{p(3-r)} \left( \frac{a_p}{p} \right)^{n+2} [g_{n,0}^0, XY^{r-1}] + O(p^2).$$

So $-a_p\xi_n + T^{-\xi_{n+1}} = \frac{a_p}{p^2(3-r)} \left( \frac{a_p}{p} \right)^{n+1} [g_{n,0}^0, X^{r-2}Y^2] + O(\sqrt{p}).$

We will calculate $T^+\xi_{n-1}$, for $1 \leq n \leq 2t + 2$:

$$T^+\xi_{n-1} = \frac{-1}{p^2(3-r)} \left( \frac{a_p}{p} \right)^n \sum_{\lambda \in \mathbb{F}_p} [g_{n-1,0}^0, -\lambda X + pY]^{r-2}(-\lambda X + pY)^2 + (r-3)X^p(-\lambda X + pY)^{r-p}$$

$$- (r-2)X(-\lambda X + pY)^{r-1}$$

$$= \frac{-1}{p^2(3-r)} \left( \frac{a_p}{p} \right)^n \sum_{\lambda \in \mathbb{F}_p^n} [g_{n-1,0}^0, -\lambda X + pY]^{r-2}(-\lambda X + pY)^2 + (r-3)X^p(-\lambda X + pY)^{r-p}$$

$$- (r-2)X(-\lambda X + pY)^{r-1} + [g_{n,0}^0, \partial X^{r-2}Y^2] + O(p^3).$$

The coefficient of $[g_{n,p^n-1,\lambda}^0, X^r]$ in $T^+\xi_{n-1}$ for $\lambda \neq 0$ is $0$. The coefficient of $[g_{n,p^n-1,\lambda}^0, X^{r-1}Y]$ in $T^+\xi_{n-1}$ for $\lambda \neq 0$ is

$$\frac{[\lambda]}{p^2(3-r)} \left( \frac{a_p}{p} \right)^n (2p + p(r-3)(r-p) - p(r-2)(r-1)) = [\lambda] \left( \frac{a_p}{p} \right)^n = O(\sqrt{p}).$$

The coefficient of $[g_{n,p^n-1,\lambda}^0, X^{r-2}Y^2]$ in $T^+\xi_{n-1}$ for $\lambda \neq 0$ is

$$\frac{-1}{p^2(3-r)} \left( \frac{a_p}{p} \right)^n \left( p^2 + p^2(r-3) \left( \frac{r-p}{2} \right) - p^2(r-2) \left( \frac{r-1}{2} \right) \right) = O(\sqrt{p}).$$

Finally using Lemma 3.2 for the remaining terms in $T^+\xi_{n-1}$, we get

$$(4.8) \quad T^+\xi_{n-1} = \frac{-1}{(3-r)} \left( \frac{a_p}{p} \right)^n [g_{n,0}^0, X^{r-2}Y^2] + O(\sqrt{p}).$$

Therefore in radius $n$ for $1 \leq n \leq 2t$, we have

$$T^{-\xi_{n+1}} - a_p\xi_n + T^+\xi_{n-1} = \frac{1}{(3-r)} \left( \frac{a_p}{p} \right)^n \left( \frac{a_p^2 - p^3}{p^3} \right) [g_{n,0}^0, X^{r-2}Y^2] + O(\sqrt{p}).$$

Now $a_p^2 - p^3 = pa_pc + (r-2)(r-1)p^3 - p^3$, so we get

$$(4.9) \quad v(a_p^2 - p^3) \geq \min \left( \tau + \frac{5}{2}, t + 3 \right).$$

As $\tau > t + \frac{1}{2}$, we get $v(a_p^2 - p^3) = t + 3$. Therefore

$$(4.10) \quad T^{-\xi_{n+1}} - a_p\xi_n + T^+\xi_{n-1} = O(\sqrt{p}).$$
Therefore in radius 2:

\[-a_p \xi_{2t+1} = \frac{a_p}{p^2(3-r)} \left( \frac{a_p}{p} \right)^{2t+2} [g_{2t+1,0}^0, X^{r-2}Y^2 + (r-3)X^pY^{r-p} - (r-2)XY^{r-1}] = O(\sqrt{p}).\]

Using equation (4.8), we get

\[T^+ \xi_{2t} = -\frac{1}{(3-r)} \left( \frac{a_p}{p} \right)^{2t+1} [g_{2t+1,0}^0, X^{r-2}Y^2] + O(\sqrt{p}) = O(\sqrt{p}).\]

Therefore in radius 2 + 1, we have

\[(4.11) \quad -a_p \xi_{2t+1} + T^+ \xi_{2t} = O(p).\]

Computation in radius 2 + 2:

By equation (4.8), we get

\[(4.12) \quad T^+ \xi_{2t+1} = O(p).\]

Finally, by equations (4.6), (4.7), (4.10), (4.11), (4.12), we get

\[(T - a_p) \xi = \frac{a_p(r-2)}{p^2(3-r)} [\alpha, XY^{r-1}] + \frac{a_p^2}{p^3(3-r)} [1, X^{r-2}Y^2 + (r-3)X^pY^{r-p}] + O(\sqrt{p}).\]

\[\square\]

The next two lemmas will be used in the proof of Proposition 7.1.

**Lemma 4.4.** Let \( p > 3, r \geq 2p + 1, r = 3 + n(p-1)p^t \), with \( t = v(r-3), v(a_p) = \frac{3}{2}, c = \frac{a_p^2 - (r-2)(r-1)p^t}{pa_p} \). Let \( \tau = v(c) \) and assume \( \tau \leq t \). If

\[\chi' = \sum_{n=0}^{t} a_p^n [g_{n,0}, [\lambda]^{-1}(Y^r - X^{r-3}Y^3)], \text{ for some } \lambda \neq 0,\]

then

\[(T - a_p)\chi' = [\alpha, [\lambda]^{-1}Y^r] + a_p [1, [\lambda]^{-1}X^{r-3}Y^3] + p^{t+1}h_{\chi'} + O(p^{\tau+2}),\]

where \( h_{\chi'} \) is an integral linear combination of terms of the form \([g, X^r]\) and \([g, X^{r-1}Y]\), for \( g \in G \).

**Proof.** Since \( \tau \leq t \), by Lemma 4.1 we have \( \tau = \hat{\tau} \). Let \( \lambda \in \mathbb{F}_p^x \), by Lemma 4.2 we have the following:

\[(T - a_p)\chi' = [\alpha, [\lambda]^{-1}Y^r] + a_p [1, [\lambda]^{-1}X^{r-3}Y^3] + p^{t+1}h_{\chi'} + O(p^{\tau+2}),\]

where \( h_{\chi'} \) is an integral linear combination of terms of the form \([g, X^r]\) and \([g, X^{r-1}Y]\), for some \( g \in G \).

\[\square\]

**Lemma 4.5.** Let \( p > 3, r \geq 2p + 1, r = 3 + n(p-1)p^t \), with \( t = v(r-3), v(a_p) = \frac{3}{2} \) and \( c = \frac{a_p^2 - (r-2)(r-1)p^t}{pa_p} \). Let \( \tau = v(c) \) and assume \( \tau \leq t \). If

\[\phi = \sum_{n=0}^{2t+1} \left( \frac{p^2}{a_p} \right)^n [g_{n,0}, X^{r-2}Y^2 - XY^{r-1}],\]

then...
where \( h_{\phi} \) is an integral linear combination of the terms of the form \([g, X^{r-1}Y]\) and \([g, XY^{r-1}]\), for some \( g \in G \).

**Proof.** Let \( \phi_n = \left( \frac{a^2}{a_p} \right)^n [g_{n,0}^0, X^{r-2}Y^2 - XY^{r-1}] \), so we can write \( \phi = \sum_{n=0}^{2t+1} \phi_n \). We will compute \((T - a_p)\phi\) in several steps.

Computation in radius \(-1\):

\[(4.13)\quad T^{-\phi_0} = [\alpha, (pX)^{r-2}Y^2 - pXY^{r-1}] = [\alpha, -pXY^{r-1}] + O(p^{t+4}),\]

because \( p > 3 \), so \( r - 2 \geq (p - 1)p^t + 1 > t + 4 \).

Computation in radius 0:

\[-a_p \phi_0 = -a_p[1, X^{r-2}Y^2 - XY^{r-1}], \]

\[T^{-\phi_1} = \frac{p^2}{a_p}[1, (pX)^{r-2}Y^2 - pXY^{r-1}] = \frac{-p^3}{a_p}[1, XY^{r-1}] + O(p^{t+4}).\]

So in radius 0 we have

\[-a_p \phi_0 + T^{-\phi_1} = -a_p[1, X^{r-2}Y^2] + \left( \frac{a^2_p - p^3}{a_p} \right)[1, XY^{r-1}] + O(p^{t+4}).\]

As \( \tau \leq t \), by equation (3), we have \( v(a^2_p - p^3) = \tau + \frac{5}{2} \).

Therefore

\[(4.14)\quad -a_p \phi_0 + T^{-\phi_1} = -a_p[1, X^{r-2}Y^2] + p^{t+1}[1, XY^{r-1}] + O(p^{t+4}).\]

Computation in radius \( n \) with \( 1 \leq n \leq 2t \):

\[-a_p \phi_n = -a_p \left( \frac{p^2}{a_p} \right)^n [g_{n,0}^0, X^{r-2}Y^2 - XY^{r-1}], \]

\[T^{-\phi_{n+1}} = \left( \frac{p^2}{a_p} \right)^{n+1} [g_{n,0}^0, (pX)^{r-2}Y^2 - pXY^{r-1}] = \left( \frac{p^2}{a_p} \right)^{n+1} [g_{n,0}^0, -pXY^{r-1}] + O(p^{t+4}).\]

So we get

\[-a_p \phi_n + T^{-\phi_{n+1}} = -a_p \left( \frac{p^2}{a_p} \right)^n [g_{n,0}^0, X^{r-2}Y^2] + \left( \frac{p^2}{a_p} \right)^n \left( \frac{a^2_p - p^3}{a_p} \right) [g_{n,0}^0, XY^{r-1}] + O(p^{t+4})\]

\[= -a_p \left( \frac{p^2}{a_p} \right)^n [g_{n,0}^0, X^{r-2}Y^2] + p^{t+2} [g_{n,0}^0, XY^{r-1}] + O(p^{t+4}).\]

We will compute \( T^+\phi_{n-1} \) for \( 1 \leq n \leq 2t + 2 \):

\[T^+\phi_{n-1} = \left( \frac{p^2}{a_p} \right)^{n-1} \sum_{\lambda \in \mathbb{F}_p} \left( g_{n-1,0}^0 \lambda^r, X^{r-2}(\lambda X + pY)^2 - X(\lambda X + pY)^{r-1} \right)\]
From equation (4.15) we get that
\begin{align*}
T^+ \phi_{n-1} &= p^2 \left( \frac{p^2}{a_p} \right)^{n-1} \left( \sum_{\lambda \neq 0} [g_{n,p^n-1}[\lambda], X^{r-2}(-[\lambda]X + pY)^2 - X(-[\lambda]X + pY)^{-1}] + [g_{n,0}, p^2 X^{r-2}Y^2] \right) \\
&+ O(p^{t+5}).
\end{align*}

Coefficient of \([g_{n,p^n-1}[\lambda], X^r]\) in \(T^+ \phi_{n-1}\) for \(\lambda \neq 0\) is 0.

Coefficient of \([g_{n,p^n-1}[\lambda], X^{r-1}Y]\) in \(T^+ \phi_{n-1}\) for \(\lambda \neq 0\) is \([\lambda]p \left( \frac{p^2}{a_p} \right)^{n-1} (r - 3) = O(p^{t+1}).

Coefficient of \([g_{n,p^n-1}[\lambda], X^{r-2}Y^2]\) in \(T^+ \phi_{n-1}\) for \(\lambda \neq 0\) is \(p^2 \left( \frac{p^2}{a_p} \right)^{n-1} (1 - (r^{-1})) = O(p^{t+2}).

Using Lemma 3.2, we have
\begin{align*}
T^+ \phi_{n-1} &= p^2 \left( \frac{p^2}{a_p} \right)^{n-1} [g_{n,0}, X^{r-2}Y^2] + p^{t+1} h'_n + O(p^{t+2}),
\end{align*}

where \(h'_n\) is an integral linear combination of the terms of the form \([g, X^{r-1}Y]\), for some \(g \in G\).

Therefore in radius \(n\), for \(1 \leq n \leq 2t\), we have
\begin{align*}
T^- \phi_{n+1} - a_p \phi_n + T^+ \phi_{n-1} &= p^{-1} \left( \frac{p^2}{a_p} \right)^{2t+1} [g_{n,0}, X^{r-1}Y^{-1}] + p^{t+1} h'_n + O(p^{t+2}).
\end{align*}

Computation in radius \(2t + 1\):
\begin{align*}
-a_p \phi_{2t+1} &= -a_p \left( \frac{p^2}{a_p} \right)^{2t+1} [g_{2t+1,0}, X^{r-2}Y^2 - XY^{-1}] = O(p^{t+2}).
\end{align*}

From equation (4.15) we get that \(T^+ \phi_{2t} = p^{t+1} h'_{2t+1} + O(p^{t+2})\), where \(h'_{2t+1}\) is an integral linear combination of the terms of the form \([g, X^{r-1}Y]\), for some \(g \in G\). Therefore in radius \(2t + 1\) we get
\begin{align*}
-a_p \phi_{2t+1} + T^+ \phi_{2t} &= p^{t+1} h'_{2t+1} + O(p^{t+2}).
\end{align*}

Computation in radius \(2t + 2\):

By equation (4.15), we get
\begin{align*}
T^+ \phi_{2t+1} &= p^{t+1} h'_{2t+2} + O(p^{t+2}),
\end{align*}

where \(h'_{2t+2}\) is an integral linear combination of the terms of the form \([g, X^{r-1}Y]\), for some \(g \in G\).

Finally by equations (4.13), (4.14), (4.10), (4.17), (4.15), we get
\begin{align*}
(T - a_p) \phi &= -[\alpha, pXY^{-1}] - a_p[1, X^{r-2}Y^2] + p^{t+1} h_\phi + O(p^{t+2}),
\end{align*}

where \(h_\phi\) is an integral linear combination of the terms of the form \([g, X^{r-1}Y]\) and \([g, XY^{-1}]\), for some \(g \in G\).

The next two lemmas will be used in the proof of Proposition 7.3.

**Lemma 4.6.** Let \(p > 3\), \(r \geq 2p + 1\), \(r = 3 + n(p - 1)p^t\), with \(t = v(r - 3)\), \(v(a_p) = \frac{5}{2}\) and \(c = \frac{a^{2-(r-2)(p^{-1})}}{p^t}\). Let \(v = v(c)\) and assume \(\tau > t + \frac{1}{2}\). If
\begin{align*}
\xi' &= \frac{1}{p^t} \sum_{n=1}^{2r+2} \left( \frac{a_p}{p} \right)^n [g_{n,0}, X^{r-2}Y^2 + (r-3)X^{pY^{-1}} - (r-2)XY^{-1}],
\end{align*}
Lemma 4.7. Let

\[(T - a_p)\xi' = \frac{a_p}{p^3} \{ 1, (2 - r)XY^{r - 1} \} - \frac{a_p^2}{p^3} [g_{1,0}^0, X^{r - 2}Y^2 + (r - 3)X^pY^{r - p}] + O(p^{t + \frac{1}{2}}).\]

Proof. We can write

\[\xi' = \frac{a_p}{p^3} \sum_{n=1}^{2t+2} \left( \frac{a_p}{p} \right)^{n-1} [g_{n,0}^0, X^{r - 2}Y^2 + (r - 3)X^pY^{r - p} - (r - 2)XY^{r - 1}]\]

\[= \frac{a_p}{p^3} \sum_{n=0}^{2t+1} \left( \frac{a_p}{p} \right)^n [g_{1,0}^0, X^{r - 2}Y^2 + (r - 3)X^pY^{r - p} - (r - 2)XY^{r - 1}].\]

From the definition of \(\xi\) in Lemma 4.3, we can write

\[\xi = \frac{-a_p}{p^3 (3 - r)} \sum_{n=0}^{2t+1} \left( \frac{a_p}{p} \right)^n [g_{n,0}^0, X^{r - 2}Y^2 + (r - 3)X^pY^{r - p} - (r - 2)XY^{r - 1}].\]

So we can rewrite \(\xi'\) in terms of \(\xi\) as

\[\xi' = \frac{a_p}{p^3} \left( \frac{p^3 (r - 3) [g_{1,0}^0]}{a_p} \right) = (r - 3)\xi_{g_{1,0}^0},\]

where \(\xi_{g_{1,0}^0} = g_{1,0}^0 \xi\).

\[\text{So } (T - a_p)\xi' = (r - 3) \{ T - a_p \} \xi_{g_{1,0}^0} = (r - 3)g_{1,0}^0 (T - a_p) \xi. \] By Lemma 4.3 we get

\[(T - a_p)\xi' = \left( r - 3 \right) \left( \frac{a_p (r - 2)}{p^2 (3 - r)} \right) [g_{1,0}^0, XY^{r - 1}] + \frac{a_p^2 (r - 3)}{p^3 (3 - r)} [1, X^{r - 2}Y^2 + (r - 3)X^pY^{r - p}] + O(p^{t + \frac{1}{2}})\]

\[= \frac{a_p}{p^2} [1, (2 - r)XY^{r - 1}] - \frac{a_p^2}{p^3} [1, X^{r - 2}Y^2 + (r - 3)X^pY^{r - p}] + O(p^{t + \frac{1}{2}}). \]

Lemma 4.7. Let \(p > 3, r \geq 2p + 1, r = 3 + n(p - 1)p^t, \) with \( t = v(p - 3), v(a_p) = \frac{3}{2} \) and \( c = \frac{a_p^2 (r - (2 - t))}{pa_p} \). Let \( \tau = v(c) \) and assume \( \tau \geq t + 1. \) If

\[\psi = \frac{1}{p^2} \sum_{n=1}^{t+1} a_p^n [g_{n,0}^0, [\mu]^{-1}(Y^r - X^{r - 3}Y^3)], \text{ for some } \mu \in \mathbb{F}_p^x,\]

then

\[(T - a_p)\psi = \frac{a_p}{p^2} [1, [\mu]^{-1} ([\mu]X + Y)^r] + \frac{a_p^2}{p^2} [g_{1,0}^0, [\mu]^{-1}X^{r - 3}Y^3] + O(p^{t + \frac{1}{2}}).\]

Proof. Let \( \mu \in \mathbb{F}_p^x \) and let \( \psi_n = \frac{a_p}{p^2} [g_{n,0}^0, [\mu]^{-1} (Y^r - X^{r - 3}Y^3)], \text{ for } n \geq 1. \) So we can write

\[\psi = \sum_{n=1}^{t+1} \psi_n.\]

Computation in radius 0:

\[(4.19) \quad T^{-1} \psi_1 = \frac{a_p}{p^2} [g_{1,0}^0, [\mu]^{-1} (Y^r - (pX)^{r - 3}Y^3) = \frac{a_p}{p^2} [1, [\mu]^{-1} ([\mu]X + Y)^r] + O(p^{t + \frac{1}{2}}),\]

because \( p > 3, \) so \( r - 3 \geq (p - 1)p^t > t + 3. \)
Computation in radius 1:
\[-a_p \psi_1 = \frac{a_p^2}{p^2} \left[ g_{1, [\mu]}^0, [\mu]^{-1}(Y^r - X^{r-3}Y^3) \right], \]
\[T^{-} \psi_2 = \frac{a_p^2}{p^2} \left[ g_{2, [\mu]}^0 \alpha, [\mu]^{-1}(Y^r - (pX)^{r-3}Y^3) \right] \]
\[= \frac{a_p^2}{p^2} \left[ g_{1, [\mu]}^0, [\mu]^{-1}Y^r \right] + O(p^{t+4}). \]

Therefore in radius 1 we get
(4.20) \[-a_p \psi_1 + T^{-} \psi_2 = \frac{a_p^2}{p^2} \left[ g_{1, [\mu]}^0, [\mu]^{-1}X^{r-3}Y^3 \right] + O(p^{t+4}). \]

Computation in radius \(n\), with \(2 \leq n \leq t\):
\[-a_p \psi_n = -\frac{a_p^{n+1}}{p^2} \left[ g_{n, [\mu]}^0, [\mu]^{-1}(Y^r - X^{r-3}Y^3) \right], \]
\[T^{-} \psi_{n+1} = \frac{a_p^{n+1}}{p^2} \left[ g_{n+1, [\mu]}^0 \alpha, [\mu]^{-1}(Y^r - (pX)^{r-3}Y^3) \right] \]
\[= \frac{a_p^{n+1}}{p^2} \left[ g_{n, [\mu]}^0, [\mu]^{-1}Y^r \right] + O(p^{t+\frac{3}{2}}). \]

So \(-a_p \psi_n + T^{-} \psi_{n+1} = \frac{a_p^{n+1}}{p^2} \left[ g_{n, [\mu]}^0, [\mu]^{-1}X^{r-3}Y^3 \right] + O(p^{t+\frac{3}{2}}). \)

We will compute \(T^+ \psi_{n-1}\), with \(2 \leq n \leq t + 2\).
\[T^+ \psi_{n-1} = \frac{a_p^{n-1}}{p^2} \left( \sum_{\lambda \in \varpi_p} \left[ g_{n-1, [\mu]}^0 g_{1, [\lambda]}^0, [\mu]^{-1}((-[\lambda]X + pY)^r - X^{r-3}(-[\lambda]X + pY)^3) \right] \right) \]
\[= \frac{a_p^{n-1}}{p^2} \left( \sum_{\lambda \neq 0} \left[ g_{n, [\mu]}^0 + p^{n-1}, [\mu]^{-1}((-[\lambda]X + pY)^r - X^{r-3}(-[\lambda]X + pY)^3) \right] \right) \]
\[+ \left[ g_{n, [\mu]}^0, -[\mu]^{-1}X^{r-3}Y^3 \right] \right) + O(p^{t+\frac{3}{2}}). \]

Coefficient of \([g_{n, [\mu]}^0 + p^{n-1}, [\lambda], X^r]\) in \(T^+ \psi_{n-1}\) for \(\lambda \neq 0\) is 0.
Coefficient of \([g_{n, [\mu]}^0 + p^{n-1}, [\lambda], X^{r-1}Y]\) in \(T^+ \psi_{n-1}\) for \(\lambda \neq 0\) is \([\mu]^{-1}[\lambda]^2 \left( \frac{a_p^{n-1}}{p} \right) (r - 3) = O(p^{t+\frac{3}{2}}). \)
Coefficient of \([g_{n, [\mu]}^0 + p^{n-1}, [\lambda], X^{r-2}Y^2]\) in \(T^+ \psi_{n-1}\) for \(\lambda \neq 0\) is \(-[\mu]^{-1}[\lambda]a_p^{n-1} \left( \binom{r}{2} - 3 \right) = O(p^{t+\frac{5}{2}}). \)
Coefficient of \([g_{n, [\mu]}^0 + p^{n-1}, [\lambda], X^{r-3}Y^3]\) in \(T^+ \psi_{n-1}\) for \(\lambda \neq 0\) is \([\mu]^{-1}p a_p^{n-1} \left( \binom{r}{3} - 1 \right) = O(p^{t+\frac{7}{2}}). \)

Using Lemma 3.1 for the remaining terms in \(T^+ \psi_{n-1}\), we finally get
(4.21) \[T^+ \psi_{n-1} = -pa_p^{n-1} \left[ g_{n, [\mu]}^0, [\mu]^{-1}X^{r-3}Y^3 \right] + O(p^{t+\frac{7}{2}}). \]

So we have
\[-a_p \psi_n + T^+ \psi_{n-1} = a_p^{n-1} \left( \frac{a_p^2 - p^3}{p^2} \right) \left[ g_{n, [\mu]}^0, [\mu]^{-1}X^{r-3}Y^3 \right] + O(p^{t+\frac{7}{2}}). \]
As $\tau \geq t + 1$, by equation (4.14), we have $v(a_p^2 - p^3) = t + 3$. Therefore in radius $n$, with $2 \leq n \leq t$, we get

\begin{equation}
T^{-\psi_{n+1}} - a_p\psi_n + T^{+\psi_{n-1}} = O(p^{t+\frac{1}{2}}). \tag{4.22}
\end{equation}

Computation in radius $t + 1$:

\[-a_p\psi_{t+1} = -\frac{a_p^{t+2}}{p^2}[g_{t+1,[\mu]}, [\mu]^{-1}(Y^r - X^{r-3}Y^3)] = O(p^{t+1}).\]

From equation (4.21) we get that $T^{+\psi_t} = O(p^{t+\frac{1}{2}})$. Therefore in radius $t + 1$ we have

\begin{equation}
-a_p\psi_{t+1} + T^{+\psi_t} = O(p^{t+\frac{3}{4}}). \tag{4.23}
\end{equation}

Computation in radius $t + 2$:

From equation (4.21) we get

\begin{equation}
T^{+\psi_{t+1}} = -pa^{t+1}_p[g_{t+2,[\mu]}, [\mu]^{-1}X^{r-3}Y^3] + O(p^{t+\frac{3}{2}}) = O(p^{t+\frac{1}{2}}). \tag{4.24}
\end{equation}

Finally, by equations (4.19), (4.20), (4.22), (4.23), (4.24), we get

\[(T - a_p)\psi(a_p^2/p^2[1, [\mu]^{-1} \{ [\mu]X + Y \}] + a_p^2/p^2[g_{1,[\mu]}, [\mu]^{-1}X^{r-3}Y^3] + O(p^{t+\frac{1}{2}})).\]

The last two lemmas of this section will be used in the proof of Proposition 7.3.

**Lemma 4.8.** Let $p > 3$, $r \geq 2p + 1$, $r = 3 + n(p - 1)p^\epsilon$, with $t = v(r - 3)$, $v(a_p) = \frac{3}{2}$ and $c = \frac{a_p^2 - (r-3)(r+1)p^\epsilon}{pa_p}$. Let $\tau = v(c)$ and assume $\tau < t + 1$. If

\[\xi'' = \frac{1}{pc} \sum_{n=1}^{2t+2} \left( \frac{a_p}{p} \right)^n [g_{n,0}, X^{r-2}Y^2 + (r - 3)XY^{r-p} - (r - 2)XY^{r-1}],\]

then

\[(T - a_p)\xi'' = \frac{a_p}{pc}[1, (2 - r)XY^{r-1}] - \frac{a_p^2}{p^2c}[g_{1,0}, X^{r-2}Y^2] + O(p^\epsilon),\]

where $\epsilon = \min\{t + 1 - \tau, \frac{1}{2}\}$.

**Proof.** We can write $\xi'' = \xi' \xi''$, where $\xi'$ is as in Lemma 4.6. Therefore by Lemma 4.6 we get

\[(T - a_p)\xi'' = \frac{p}{c}(T - a_p)\xi'' = \frac{a_p}{pc}[1, (2 - r)XY^{r-1}] - \frac{a_p^2}{p^2c}[1, X^{r-2}Y^2] + O(p^\epsilon),\]

because $\tau < t + 1$, so $\xi''O(p^{t+\frac{1}{2}}) = O(p^\epsilon)$, where $\epsilon = \min\{t + 1 - \tau, \frac{1}{2}\}$. \qed
Proposition 5.1. Let \( p > 3, \ r \geq 2p + 1, \ r = 3 + n(p - 1)p^t, \) with \( t = v(r - 3), \ v(a_p) = \frac{3}{2} \) and \( c = \frac{a^2 - (r-2)(r-1)p^3}{p^2a_p}. \) Let \( \tau = v(c) \) and assume \( \tau < t + 1. \) If

\[
\psi' = \frac{1}{pc} \sum_{n=1}^{t+1} a_p^n [g_{n,[\mu]}, [\mu]^{-1}(Yr - Xr-3Y^3)], \text{ for some } \mu \in \mathbb{F}_p^x,
\]

then

\[
(T - a_p)\psi' = \frac{a_p}{pc} [1, [\mu]^{-1}(X + Y)^r] + \frac{a_p^2}{pc} [g_{1,[\mu]}, [\mu]^{-1}Xr-3Y^3] + O(\sqrt{p}).
\]

Proof. We can write \( \psi' = \xi_\psi, \) where \( \psi \) is as in Lemma 4.7. Therefore, by Lemma 4.7, we get

\[
(T - a_p)\psi' = \frac{p}{c} (T - a_p)\psi
\]

\[
= \frac{a_p}{pc} [1, [\mu]^{-1}(X + Y)^r] + \frac{a_p^2}{pc} [g_{1,[\mu]}, [\mu]^{-1}Xr-3Y^3] + O(\sqrt{p}),
\]

because \( \tau < t + 1, \) so \( \frac{p}{c} O(p^{t+\frac{1}{2}}) = O(\sqrt{p}). \) \( \square \)

5. Behaviour of \( F_1 \)

We now begin to compute the structure of the subquotients of \( \Theta_{k,a_p}. \) In this section we compute the structure of the quotient \( F_1. \) Recall that \( \text{ind}_{KZ}^J J_1 \rightarrow F_1, \) where \( J_1 = V_{p-4} \otimes D^3. \)

In the next three sections we use the following notation:

- For a function \( f \in \text{ind}_{KZ}^G \text{Sym}^r \mathbb{Q}_p^2, \) we say that \( \text{\"f dies mod \ p\" if } f \in O(p^\epsilon), \) for some \( \epsilon > 0, \) so that the reduction \( \bar{f} \in \text{ind}_{KZ}^G \text{Sym}^r \mathbb{F}_p^2 \) is identically zero.

- For \( g \in G \) and \( \chi, \xi, \phi, \) etc. functions in \( \text{ind}_{KZ}^G \text{Sym}^r \mathbb{Q}_p^2, \) as in Section 4 we let \( \chi_g = g \cdot \chi, \)

\( \xi_g = g \cdot \xi, \phi_g = g \cdot \phi, \) etc. in \( \text{ind}_{KZ}^G \text{Sym}^r \mathbb{Q}_p^2. \)

Proposition 5.1. Let \( p > 3, \ r \geq 2p + 1, \ r = 3 \mod (p - 1) \) and assume \( v(a_p) = \frac{3}{2}. \) Let \( t = v(r - 3), \ c = \frac{a^2 - (r-2)(r-1)p^3}{p^2a_p} \) and \( \tau = v(c). \)

(i) If \( \tau > t, \) then \( F_1 = 0. \)

(ii) If \( \tau = t, \) then \( F_1 \) is a quotient of \( \pi(p-4, \lambda^{-1}, \omega^3), \) where \( \lambda = \frac{3}{3 - \hat{r} \left( \frac{a^2 - (r-2)(r-1)p^3}{p^2a_p} \right)} \in \mathbb{F}_p^x. \)

Proof. The proof is a variant of the proof of Proposition 6.4 in [BGRIS].

Recall \( \hat{c} = \frac{a^2 - (r-1)p^3}{p^2a_p} \) and \( \hat{\tau} = v(\hat{c}). \) We claim that it is sufficient to prove the proposition with \( \tau \) replaced by \( \hat{\tau} \) in (i) and (ii). Indeed, by Lemma 4.1 we see that if \( \hat{\tau} = t, \) then \( \hat{\tau} < t + \frac{1}{2}, \) so \( \tau = \hat{\tau}. \)

Similarly, if \( \hat{\tau} > t, \) then \( \tau = \hat{\tau} \) (when \( t < \hat{\tau} < t + \frac{1}{2} \)) or \( \tau \geq t + \frac{1}{2} \) (when \( \hat{\tau} \geq t + 0.5 \)), so in either case we have \( \tau > t. \)

Let \( S_i = \sum_{0 < j < r} \binom{r}{j} \binom{r}{j}, \) for \( i = 0, 1, 2. \) Note that \( S_i = S_{i,r} \) in Proposition 3.4.
Define the functions
\[
f_0 = \frac{p - 1}{pa_p(3 - r)} \left[ 1, \sum_{0 < j < r} \binom{r}{j} X^{r-j} Y^j - (A X^{r-3} Y^3 + B X^{r-p-2} Y^{p+2}) \right],
\]
\[
f_\infty = \frac{1}{p(3 - r)} \left( \sum_{x \in \mathbb{F}_p} \chi_{g^1_{t, \{\lambda\}}} + (1 - p) \chi_{g^0_{1, \{0\}}} \right),
\]
where \( A = \frac{(p^2 + 1) S_0 - S_1}{p - 1}, \ B = \frac{S_1 - 3 S_0}{p - 1}, \) and \( \chi \) is as in Lemma 4.2. Let \( f = f_0 + f_\infty. \)

We will compute \((T - a_p)f\) in several steps.

Computation in radius \(-1:\)
\[
T^{-} f_0 = \frac{p - 1}{pa_p(3 - r)} \left[ \alpha, \sum_{0 < j < r} \binom{r}{j} (p X)^{r-j} Y^j - (A (p X)^{r-3} Y^3 + B (p X)^{r-p-2} Y^{p+2}) \right]
\]
\[
= \frac{p - 1}{pa_p(3 - r)} \left[ \alpha, \sum_{0 < j < r} \binom{r}{j} X^{r-j} Y^j + O(p), \right]
\]
because \( v(A), v(B) \geq t + 1 \) and \( r - 3 > r - p - 2 \geq p - 1 > v(p a_p) = \frac{p}{2} \) as \( p > 3 \) and \( r \geq 2p + 1. \)

Using Lemma 3.1 we get
\[
(5.1) \quad T^{-} f_0 = O(p),
\]
because \( r - j \geq p - 1 \geq 4, \) for all \( j \) as above. Now
\[
(5.2) \quad -a_p f_0 = \frac{1 - p}{p(3 - r)} \left[ 1, \sum_{0 < j < r} \binom{r}{j} X^{r-j} Y^j - (A X^{r-3} Y^3 + B X^{r-p-2} Y^{p+2}) \right].
\]

Also
\[
T^{+} f_0 = \frac{p - 1}{pa_p(3 - r)} \left( \sum_{x \in \mathbb{F}_p} \chi_{g^1_{t, \{\lambda\}}} \sum_{0 < j < r} \binom{r}{j} X^{r-j} (-[\lambda] X + p Y)^j - (A X^{r-3} (-[\lambda] X + p Y)^3
\]
\[+ B X^{r-p-2}([\lambda] X + p Y)^{p+2}) \right)
\]
\[
= \frac{p - 1}{pa_p(3 - r)} \left( \sum_{x \in \mathbb{F}_p} \chi_{g^1_{t, \{\lambda\}}} \sum_{0 < j < r} \binom{r}{j} X^{r-j} \sum_{i=0}^{j} \binom{j}{i} (-[\lambda] X)^{j-i}(p Y)^i
\]
\[− (A X^{r-3} \sum_{i=0}^{3} \binom{3}{i} (-[\lambda] X)^{3-i}(p Y)^i + B X^{r-p-2} \sum_{i=0}^{p+2} \binom{p+2}{i} (-[\lambda] X)^{p+2-i}(p Y)^i) \right]
\]
+ \left[ g_{1,0}^0, \sum_{0<j<r} \frac{p^l}{pa_p(3-r)} \left( \sum_{\lambda \in F_p^*} p^{l\left( \frac{r}{j} \right)} X^{r-j} Y^j - \left( Ap^3 X^{r-3} Y^3 + Bp^{p+2} X^{r-p-2} Y^{p+2} \right) \right) \right]

= \frac{p-1}{pa_p(3-r)} \left( \sum_{\lambda \in F_p^*} g_{1,\lambda}^0 \sum_{i=0}^r p^l(-\lfloor \lambda \rfloor)^{3-i} \sum_{0<j<r} \binom{r}{j} \binom{r}{i} - A\binom{3}{i} - B\binom{p+2}{i} \right) X^{-i} Y^i

+ \left[ g_{1,0}^0, \sum_{0<j<r} \frac{p^l}{pa_p(3-r)} \left( \sum_{\lambda \in F_p^*} p^{l\left( \frac{r}{j} \right)} X^{r-j} Y^j - \left( Ap^3 X^{r-3} Y^3 + Bp^{p+2} X^{r-p-2} Y^{p+2} \right) \right) \right].

For \( \lambda \neq 0 \), the coefficients of \([g_{1,\lambda}^0, X^r], [g_{1,\lambda}^0, X^{r-1} Y], [g_{1,\lambda}^0, X^{r-2} Y^2] \) in \( T^f_0 \) are given by

\[-[\lambda]^3 \frac{p-1}{pa_p(3-r)} \left( \sum_{0<j<r} \binom{r}{j} - (A+B) \right),\]

\[-[\lambda]^2 \frac{p-1}{pa_p(3-r)} \left( \sum_{0<j<r} j \binom{r}{j} - (3A+(p+2)B) \right),\]

\[-[\lambda] \frac{p(p-1)}{pa_p(3-r)} \left( \sum_{0<j<r} \binom{j}{2} \binom{r}{j} - (3A+(p+2)B) \right),\]

respectively. From the way we defined \( A \) and \( B \), we get \( A+B = S_0 \), \( 3A+(p+2)B = S_1 \) and \( 3A+(p+2)B = 0 \) mod \( p^{t+1} \) as \( v(A) \geq t+1 \), \( v(B) \geq t+1 \), by parts (1) and (2) of Proposition 3.4.

Hence using Proposition 3.4 part (3) we get that the coefficients of \([g_{1,\lambda}^0, X^r], [g_{1,\lambda}^0, X^{r-1} Y] \) and \([g_{1,\lambda}^0, X^{r-2} Y^2] \) in \( T^f_0 \) die mod \( p \).

For \( \lambda \neq 0 \), \( i \geq 4 \), the coefficient of \([g_{1,\lambda}^0, X^{r-i} Y^i] \) in \( T^f_0 \) is given by

\[-[\lambda]^{3-i} \frac{p^l(p-1)}{pa_p(3-r)} \left( \sum_{0<j<r} \binom{j}{i} \binom{r}{j} - B\binom{p+2}{i} \right),\]

which dies mod \( p \) by Proposition 3.4 part (5) and the fact that \( v(pa_p(3-r)) = t+\frac{r}{2} \) and \( v(B) \geq t+1 \).

For \( j > 3 \), the coefficient of \([g_{1,0}^0, X^{r-j} Y^j] \) dies mod \( p \) again by using Lemma 3.1.

So finally using part (4) of Proposition 3.4 we get

\[(5.3) \quad T^f_0 = \frac{p-1}{pa_p(3-r)} \left( \sum_{\lambda \in F_p^*} g_{1,\lambda}^0 \frac{p^3}{1-p} X^{r-3} Y^3 \right) + \left[ g_{1,0}^0, p^3 \binom{r}{3} X^{r-3} Y^3 \right] + O(p)\]
Now we do a small computation which will be used in computing \((T - a_p)f_\infty\) just below:

\[
\frac{1}{p(3-r)} \left( \sum_{\lambda \in \mathbb{F}_p^\times} [g_{1,\lambda}^0, Y^r] + (1-p)[g_{1,0}^0, Y^r] \right) = \frac{1}{p(3-r)} \left( \sum_{\lambda \in \mathbb{F}_p^\times} \left[ \begin{array}{c} 1 \\ \lambda \end{array} \right], Y^r \right) + (1-p)[1, Y^r]
\]

\[
= \frac{1}{p(3-r)} \left( \sum_{\lambda \in \mathbb{F}_p^\times} [1, (\lambda X + Y)^r] + (1-p)[1, Y^r] \right)
\]

\[
= \frac{p-1}{p(3-r)} \left[ 1, \sum_{0 \leq j < r \atop j \equiv 3 \mod (p-1)} \binom{r}{j} X^{r-j} Y^j \right],
\]

using the fact \([3.1]\).

Now from the \(f_\infty\) part of \(f\), we get

\[
(T - a_p)f_\infty = \frac{1}{p(3-r)} \left( \sum_{\lambda \in \mathbb{F}_p^\times} [g_{1,\lambda}^0, Y^r] + a_p[g_{1,0}^0, X^{r-3}Y^3],
\]

\[
+ (1-p)(g_{1,0}^0, Y^r] + a_p[g_{1,0}^0, X^{r-3}Y^3]) \right) + h + O(p),
\]

by Lemma \([4.2]\) where \(h\) is an integral linear combination of functions of the form \([g, X^r]\) and
\([g, X^{r-1}Y]\), for some \(g \in G\). By equation \((5.4)\), we get

\[
(T - a_p)f_\infty = \frac{p-1}{p(3-r)} \left[ 1, \sum_{0 \leq j < r \atop j \equiv 3 \mod (p-1)} \binom{r}{j} X^{r-j} Y^j \right] + \frac{a_p}{p(3-r)} \sum_{\lambda \in \mathbb{F}_p^\times} [g_{1,\lambda}^0, X^{r-3}Y^3]
\]

\[
+ \frac{a_p(1-p)}{p(3-r)} [g_{1,0}^0, X^{r-3}Y^3] + h + O(p).
\]

Using equations \((5.4)\), \((5.2)\), \((5.3)\) and \((5.5)\), we get

\[
(T - a_p)(f_0 + f_\infty) = \left( \frac{a_p}{p(3-r)} + \frac{-p^2(r)}{(3-r)a_p} \right) \sum_{\lambda \in \mathbb{F}_p^\times} [g_{1,\lambda}^0, X^{r-3}Y^3]
\]

\[
+ \left( \frac{a_p(1-p)}{p(3-r)} + \frac{(p-1)p^2(r)}{a_p(3-r)} \right) [g_{1,0}^0, X^{r-3}Y^3]
\]

\[
+ \frac{p-1}{p(3-r)} [1, (AX^{r-3}Y^3 + BX^{r-p-2}Y^{p+2})] + h + O(p).
\]

By part (i) of Lemma \([2.2]\) we get that \(X^r, X^{r-1}Y\) map to 0 inside \(J_1 = V_{p-4} \otimes D^3\) and \(X^{r-3}Y^3\) maps to \(X^{p-4} \in J_1\). Since \(\theta = X^pY - XY^p\) divides \(X^{r-3}Y^3 - X^{r-p-2}Y^{p+2}\), we see \(X^{r-p-2}Y^{p+2}\)
also maps to $X^{p^{-4}}$ in $J_1$. Thus the image of $(T - a_p)f$ in $\ind_{K^Z}^G J_1$ is

$$\frac{(T - a_p)f}{(T - a_p)^2} = \frac{a_p^2 - (\frac{3}{p})p^3}{p a_p (3 - p)} \sum_{\lambda \in \mathbb{F}_p} [g_{1, \lambda}, X^{p^{-4}}] + \frac{(A + B)(p - 1)}{p(3 - p)} [1, X^{p^{-4}}].$$

By Proposition 3.41 part (1),

$$\frac{(A + B)(p - 1)}{p(3 - p)} = \frac{S_0(p - 1)}{p(3 - r)}
\equiv -\frac{1}{6} \left[ 6p + 5 - \frac{3}{p} \left( 2p + 1 \right) \right] + \frac{3 - r}{6(p - 1)} \left[ -3p - 3 + \frac{3}{p} \left( 2p + 1 \right) \right] \mod p$$

$$\equiv -\frac{1}{2} \left( \frac{5}{6} - 1 \right) \equiv -\frac{1}{2} \mod p,$$

since $\frac{1}{p} \left( 2p + 1 \right) \equiv 1 \mod p$. Therefore the image of $(T - a_p)f$ in $\ind_{K^Z}^G J_1$ is

$$\frac{1}{3}(\tilde{\lambda} T - 1)[1, X^{p^{-4}}],$$

where $\tilde{\lambda} = \frac{3}{3 - r} \left( \frac{a_p^2 - (\frac{3}{p})p^3}{p a_p} \right)$. Recall $\lambda = \frac{3}{3 - r} \left( \frac{a_p^2 - (r - 2)(\frac{1}{2})p^3}{p a_p} \right)$. Since

$$\frac{3}{3 - r} \left( \frac{a_p^2 - (r - 2)(\frac{1}{2})p^3}{p a_p} \right) = \frac{3}{3 - r} \left( \frac{a_p^2 - (\frac{3}{p})p^3}{p a_p} \right) + \frac{3}{3 - r} \left( \frac{a_p^2 - (r - 2)(\frac{1}{2})p^3}{p a_p} \right),$$

and as $v \left( \frac{3}{3 - r} \left( \frac{a_p^2 - (\frac{3}{p})p^3}{p a_p} \right) \right) \geq \frac{1}{2}$, we get that

$$\lambda = \tilde{\lambda}.$$

If $\tilde{\tau} > t$, then $\lambda = 0$, so we have that $F_1 = 0$ because $(T - a_p)f = \frac{1}{3}[1, X^{p^{-4}}]$ is zero in $\hat{\Theta}_{k, a_p}$ and $[1, X^{p^{-4}}]$ generates $\ind_{K^Z}^G J_1$.

If $\tilde{\tau} = t$, then by (5.6), $F_1$ is a quotient of $\pi(p - 4, \lambda^{-1}, \omega^3)$ which is the same as saying that $F_1$ is a quotient of $\pi(p - 4, \lambda^{-1}, \omega^3)$, by (5.7).

$$\Box$$

6. Behaviour of $F_2$

Next we study the structure of the sub $F_2$ of $\hat{\Theta}_{k, a_p}$. Recall $\ind_{K^Z}^G J_2 \to F_2$, where $J_2 = V_1 \otimes D$.

**Proposition 6.1.** Let $p > 3, r \geq 2p + 1, r = 3 + n(p - 1)p^t$ with $t = v(r - 3)$ and $v(a_p) = \frac{3}{2}$. Let $c = \frac{a_p^2 - (r - 2)(\frac{1}{2})p^3}{p a_p}$ and $\tau = v(c)$.

(i) If $\tau < t$, then $F_2 = 0$.

(ii) If $\tau = t$, then $F_2$ is a quotient of $\pi(1, \lambda, \omega)$, where $\lambda = \frac{1}{3 - r} \left( \frac{a_p^2 - (r - 2)(\frac{1}{2})p^3}{p a_p} \right) \in \mathbb{F}_p^\times$.

Proof. The proof is a variant of the proof of Proposition 6.7 in [BGR18].

As in the proof of Proposition 6.1, we claim that it is sufficient to prove the above proposition with $\tau$ replaced by $\tilde{\tau}$ in (i) and (ii), where $\tilde{\tau} = v(\tilde{c})$ and $\tilde{c} = \frac{a_p^2 - (\frac{3}{p})p^3}{p a_p}$. Indeed, by Lemma 4.1 we see that if $\tilde{\tau} < t$, then $\tau = \tilde{\tau}$, so $\tau < t$. Similarly if $\tilde{\tau} = t$, then $\tau = \tilde{\tau}$, so $\tau = t$. 

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We first define ‘building block’ functions as follows.

Let \( A = [1, X^{r-1}Y] \), \( B = \left[ 1, \sum_{1<j<r-2 \atop j \equiv 1 \mod (p-1)} \binom{r-2}{j} X^{r-j}Y^j \right] \) and \( C = [1, X^{r-p}Y^p] \).

Also for \( g \in G \), let
\[
\phi_g = \left[ g, \sum_{1<j<r-2 \atop j \equiv 1 \mod (p-1)} \binom{r}{j} X^{r-j}Y^j - S_0X^{r-p}Y^p \right] \quad \text{and} \quad \psi_g = \sum_{\mu \in \mathbb{F}_p^\times} [\mu]^{-2} \chi_{g_1, [\alpha]}
\]
where \( S_0 = \sum_{1<j<r-2 \atop j \equiv 1 \mod (p-1)} \binom{r}{j} \) and \( \chi \) is as in Lemma 4.2.

Now consider the function \( f = f_0 + f_1 + f_\infty \), with
\[
f_0 = \frac{1 - p}{p^3} \left( A + \frac{\binom{r}{2}p^3}{3a^2}B \right) + \frac{p - 1}{a_p} C,
\]
\[
f_1 = \frac{-1}{3a_p} \left( \sum_{\lambda \in \mathbb{F}_p^\times} \Phi_{g_1, [\lambda]} + (1 - p)\Phi_{g_1, 0} \right),
\]
\[
f_\infty = \frac{1}{3} \left( \frac{1}{1 - p} \sum_{\lambda \in \mathbb{F}_p^\times} \Psi_{g_1, [\lambda]} + \Psi_{g_1, 0} \right).
\]

We now compute \( (T - a_p)f \) in several steps.

In radius \(-1\) we have
\[
T^{-1}A = [\alpha, (pX)^{r-1}Y] = O(p^{t+5}),
\]
\[
T^{-1}B = \left[ \alpha, \sum_{1<j<r-2 \atop j \equiv 1 \mod (p-1)} \binom{r-2}{j} (pX)^{r-j}Y^j \right] = O(p^{t+4}),
\]

since \( p^{r-j} \binom{r-2}{j} = p^2 \binom{r-j-2}{r-2-j} \), and \( r - j - 2 \geq p - 1 > 3 \) for all \( j \) as above, so by Lemma 3.3 we get \( p^{r-j-2} \binom{r-2}{r-2-j} \equiv 0 \mod p^{t+2} \). Also
\[
T^{-1}C = [\alpha, (pX)^{r-p}Y^p] = O(p^{r-p}).
\]

As \( v(\tilde{e}) \leq t \), we get
\[
(6.1) \quad T^{-1}f_0 = O(p^3),
\]
hence it dies \( \mod p \).

In radius \(-1\) we compute \(-a_p f_0 + T^{-1}f_1 \). Now
\[
T^{-1}\Phi_g = \left[ g\alpha, \sum_{1<j<r-2 \atop j \equiv 1 \mod (p-1)} \binom{r}{j} (pX)^{r-j}Y^j - S_0(pX)^{r-p}Y^p \right].
\]
For all $j$ with $1 < j < r - 2$ and $j \equiv 1 \pmod{(p-1)}$, we have $r - j > p > 4$, so by Lemma 3.1 we get that $p^{r-j} \equiv 0 \pmod{p^4}$. Also by Proposition 3.6 part (1), we have $v(S_0) \geq t$. So we get

\begin{equation}
T^{-}\Phi_g = \left[ ga, \left( \frac{r}{r-2} \right) p^2 X^2 Y^{r-2} \right] + O(p^{t+4}).
\end{equation}

For $\lambda \neq 0$, by substituting $g_{\lambda,1}^0$ for $g$ in (6.2), we get

\begin{equation}
T^{-}\Phi_{g_{\lambda}^0} = \left[ \frac{r}{r-2} \right] p^2 X^2 (|\lambda| X + Y)^{r-2} + O(p^{t+4})
\end{equation}

Substituting $g_{1,0}^0$ for $g$ in (6.2), we get

\begin{equation}
T^{-}\Phi_{g_{1,0}^0} = \left[ 1, \left( \frac{r}{2} \right) p^2 \sum_{i=0}^{r-2} \binom{r-2}{i} |\lambda|^{r-2-i} X^{r-i} Y^i \right] + O(p^{t+4}).
\end{equation}

By equation (6.3) and (6.4), we get

\begin{equation}
T^{-}f_1 = \frac{p^2 \binom{r}{3}}{3a_p \hat{c}} \left( \sum_{\lambda \in \mathbb{F}_p} \binom{r-2}{i} |\lambda|^{r-2-i} X^{r-i} Y^i \right) + (1 - p)[1, X^2 Y^{r-2}] + O(p^2)
\end{equation}

\begin{equation}
= \frac{p^2 \binom{r}{3}}{3a_p \hat{c}} \left[ 1, \sum_{1 \leq j < r-2 \atop j \equiv 1 \pmod{(p-1)}} \binom{r-2}{j} X^{r-j} Y^j \right] + O(p^2).
\end{equation}

We also have

\begin{equation}
-a_p f_0 = \frac{-a_p (1 - p)}{p \hat{c}} [1, X^{r-1} Y] - \frac{p^2 (1 - p) \binom{r}{3}}{3a_p \hat{c}} \left[ 1, \sum_{1 \leq j < r-2 \atop j \equiv 1 \pmod{(p-1)}} \binom{r-2}{j} X^{r-j} Y^j \right]
\end{equation}

\begin{equation}
= -(p - 1)[1, X^{r-p} Y^p].
\end{equation}

So finally by equations (6.5) and (6.6), in radius 0 we get

\begin{equation}
-a_p f_0 + T^{-}f_1 = -[1, X^{r-1} Y - X^{r-p} Y^p] + O(p),
\end{equation}

because the coefficient of $[1, X^{r-1} Y]$ term from $-a_p f_0$ and $T^{-}f_1$ is

\begin{equation}
\frac{a_p (1 - p)}{p \hat{c}} + \frac{p^2 (1 - p) \binom{r}{3} (r-2)}{3a_p \hat{c}} = \frac{p-1}{\hat{c}} \left( \frac{a_p^2 - \binom{3}{2} p^3}{p a_p} \right) \equiv -1 \pmod{p}.
\end{equation}

In radius 1, we compute $T^+ f_0 - a_p f_1 + h_{\infty,1}$, where $h_{\infty,1}$ denotes the part of $(T - a_p)f_\infty$ that lives in radius 1. We have

\begin{equation}
T^+ A = \sum_{\lambda \in \mathbb{F}_p} [g_{\lambda,1}^0, X^{r-1}(-|\lambda| X + pY)].
\end{equation}
Also
\[
T^+ B = \sum_{\lambda \in \mathbb{F}_p} \left[ g_{1,[\lambda]}^0, \sum_{1 < j < r-2 \atop j \equiv 1 \mod (p-1)} \binom{r-2}{j} X^{r-j} (-[\lambda] X + pY)^j \right] \\
= \sum_{\lambda \in \mathbb{F}_p} \left[ g_{1,[\lambda]}^0, \sum_{1 < j < r-2 \atop j \equiv 1 \mod (p-1)} \binom{r-2}{j} X^{r-j} \sum_{i=0}^j \binom{j}{i} (-[\lambda] X)^{j-i} (pY)^i \right] \\
+ \left[ g_{1,0}^0, \sum_{1 < j < r-2 \atop j \equiv 1 \mod (p-1)} p^j \binom{r-2}{j} X^{r-j} Y^j \right] \\
= \sum_{\lambda \in \mathbb{F}_p} \left[ g_{1,[\lambda]}^0, \sum_{i=0}^r p^i (-[\lambda])^{1-i} \sum_{1 < j < r-2 \atop j \equiv 1 \mod (p-1)} \binom{j}{i} \binom{r-2}{j} X^{r-j} Y^j \right] + O(p^{t+2}),
\]
because by Lemma 3.3 the coefficient of \([g_{1,0}^0, X^{r-j} Y^j]\) is \(p^j \binom{r-2}{j} \equiv 0 \mod p^{t+2}\), for all \(1 < j < r-2\) and \(j \equiv 1 \mod (p-1)\).

By Proposition 3.5 parts (1), (2), (3) and Lemma 3.3 the coefficients of \([g_{1,[\lambda]}^0, X^r], [g_{1,[\lambda]}^0, X^{r-1} Y], [g_{1,[\lambda]}^0, X^{r-1} Y^i]\) for \(\lambda \neq 0\) and \(i \geq 2\) in \(T^+ B\) are
\[
-[\lambda] \sum_{1 < j < r-2 \atop j \equiv 1 \mod (p-1)} \binom{r-2}{j} \equiv -[\lambda](-(r-2) - 1 + 1 + np^{t+1}) \equiv -[\lambda](2 - r + np^{t+1}) \mod p^{t+2},
\]
\[
p \sum_{1 < j < r-2 \atop j \equiv 1 \mod (p-1)} j \binom{r-2}{j} \equiv p \left( -2(r-2) + \frac{(r-2)(p-2)}{p-1} \right) \equiv \frac{-p^2(r-2)}{p-1} \mod p^{t+2},
\]
\[
(-[\lambda])^{1-i} p^i \sum_{1 < j < r-2 \atop j \equiv 1 \mod (p-1)} \binom{j}{i} \binom{r-2}{j} \equiv 0 \mod p^{t+2},
\]
respectively. Therefore we can write
\[
(6.9) \quad T^+ B = \sum_{\lambda \in \mathbb{F}_p} \left[ g_{1,[\lambda]}^0, -[\lambda](2 - r + np^{t+1}) X^r + \frac{p^2(r-2)}{1-p} X^{r-1} Y \right] + O(p^{t+2}).
\]
Finally
\[
(6.10) \quad T^+ C = \sum_{\lambda \in \mathbb{F}_p} \left[ g_{1,[\lambda]}^0, X^{r-p}(-[\lambda] X + pY)^p \right] = \sum_{\lambda \in \mathbb{F}_p} \left[ g_{1,[\lambda]}^0, -[\lambda] X^r \right] + O(p^2).
\]
So from equations (6.3), (6.9) and (6.10), we get

\[
T^+ f_0 = \frac{1-p}{pc} \sum_{\lambda \in \mathbb{F}_p} [g^0_{1,\lambda}, X^{r-1}(-[\lambda]X + pY)] \\
+ \left( \frac{1-p}{pc} \right) \left( \frac{(p')^3}{3a_p^2} \right) \sum_{\lambda \in \mathbb{F}_p^\times} [g^0_{1,\lambda}, Y - [\lambda](2 - r) + np^{t+1})X^r + \frac{p^2(r-2)}{1-p}X^{r-1}Y] \\
+ \frac{p-1}{ap} \sum_{\lambda \in \mathbb{F}_p^\times} [g^0_{1,\lambda}, -[\lambda]X^r] + O(\sqrt{p}).
\]

Now, the coefficient of \([g^0_{1,0}, X^r]\) in \(T^+ f_0\) is 0.

Coefficient of \([g^0_{1,\lambda}, X^r]\) for \(\lambda \neq 0\) in \(T^+ f_0\) is

\[
- [\lambda] \left( \frac{1-p}{pc} \right) - [\lambda] \left( \frac{1-p}{pc} \right) \left( \frac{(p')^3}{3a_p^2} \right) \left( 2 - r + np^{t+1} - [\lambda] \frac{p-1}{ap} \right) \\
= (1-p)[\lambda] \left( \frac{-a_p^2 + \frac{(p')^3}{3a_p^2}}{pa_p^2} \right) + \frac{1}{ap} \left( n[\lambda](1-p)(p')^t+3 \right) \\
= -\frac{n[\lambda](1-p)(p')^t+3}{3a_p^2},
\]

which is integral because \(v(\tilde{c}) \leq t\).

Coefficient of \([g^0_{1,0}, X^{r-1}Y]\) in \(T^+ f_0\) is \(\left( \frac{1-p}{pc} \right) p = \frac{1-p}{c}\).

Coefficient of \([g^0_{1,\lambda}, X^{r-1}Y]\) for \(\lambda \neq 0\) in \(T^+ f_0\) is

\[
p \left( \frac{1-p}{pc} \right) + \frac{1-p}{pc} \left( \frac{(p')^3}{3a_p^2} \right) \left( \frac{p^2(r-2)}{1-p} \right) \\
= \frac{1-p}{c} + \frac{p}{c} \left( \frac{(p')^3}{a_p^2} \right) \frac{1}{c} + \frac{p}{c} \left( \frac{pa_p(-\tilde{c})}{a_p^2} \right) = \frac{1}{c} \left( \frac{p^2}{a_p} \right)
\]

So we get

\[
T^+ f_0 = \left( \frac{1-p}{c} \right) [g^0_{1,0}, X^{r-1}Y] + \frac{1}{c} \sum_{\lambda \in \mathbb{F}_p^\times} [g^0_{1,\lambda}, X^{r-1}Y] + h + O(\sqrt{p}),
\]

where \(h\) is an integral linear combination of terms of the form \([g, X^r]\), for some \(g \in G\). We can rewrite this as

\[
T^+ f_0 = \frac{1-p}{3c} \left[ g^0_{1,0}, \left( \frac{r}{1} \right) X^{r-1}Y \right] + \frac{1}{3c} \sum_{\lambda \in \mathbb{F}_p^\times} \left[ g^0_{1,\lambda}, \left( \frac{r}{1} \right) X^{r-1}Y \right] + \frac{3-r}{3c} \sum_{\lambda \in \mathbb{F}_p^\times} \left[ g^0_{1,\lambda}, X^{r-1}Y \right] \\
+ h + O(\sqrt{p}).
\]
We also have

\[-a_p f_1 = \frac{1 - p}{3c} \left[ a_1, 0 \right] \left[ \sum_{1 \leq j \leq r - 2} \binom{r}{j} X^{r-j} Y^j - S_0 X^{r-p-2} \right] + \frac{1}{3c} \sum_{\lambda \in \mathbb{F}_p^*} \left[ g_{1, \lambda}^0, \sum_{1 \leq j \leq r - 2} \binom{r}{j} X^{r-j} Y^j - S_0 X^{r-p-2} \right].\]

By Proposition 3.6 part (1), we have \(S_0 \equiv 3 - r \mod p^{t+1}\). As \(v(c) \leq t\), we can write

\[-a_p f_1 = \frac{1 - p}{3c} \left[ a_1, 0 \right] \left[ \sum_{1 \leq j \leq r - 2} \binom{r}{j} X^{r-j} Y^j \right] + \frac{1}{3c} \sum_{\lambda \in \mathbb{F}_p^*} \left[ g_{1, \lambda}^0, \sum_{1 \leq j \leq r - 2} \binom{r}{j} X^{r-j} Y^j \right] - \frac{3 - r}{3c} \sum_{\lambda \in \mathbb{F}_p^*} [g_{1, \lambda}^0, X^{r-p-2}] + O(p).\]

(6.12)

Now combining the radius 1 terms from equations (6.11) and (6.12), we get

\[T^+ f_0 - a_p f_1 = \frac{1 - p}{3c} \left[ a_1, 0 \right] \left[ \sum_{1 \leq j \leq r - 2} \binom{r}{j} X^{r-j} Y^j \right] + \frac{1}{3c} \sum_{\lambda \in \mathbb{F}_p^*} \left[ g_{1, \lambda}^0, \sum_{1 \leq j \leq r - 2} \binom{r}{j} X^{r-j} Y^j \right] + \frac{3 - r}{3c} \sum_{\lambda \in \mathbb{F}_p^*} [g_{1, \lambda}^0, X^{r-p-2}] + h + O(\sqrt{p}).\]

(6.13)

We now compute \((T - a_p) \Psi_g\). Using Lemma 4.2 as \(\bar{r} \leq t\), we have

\[(T - a_p) \Psi_g = \sum_{\mu \in \mathbb{F}_p^*} [\mu]^{-2} \left[ \sum_{\bar{r}-2} \left[ g_{g1, [\mu]}(X, Y) \right] + a_p [g_{g1, [\mu]} X^{r-3} Y^3] + p^{t+1} h_x + O(p^{\bar{r}+2}) \right] \]

\[= \sum_{\mu \in \mathbb{F}_p^*} [\mu]^{-2} g, \left[ [\mu] X + Y \right]^{r} + a_p \sum_{\mu \in \mathbb{F}_p^*} [\mu]^{-2} [g_{g1, [\mu]}, X^{r-3} Y^3] + O(p^{\bar{r}+1}) \]

\[= (p - 1) \left[ g, \sum_{1 \leq j \leq r - 2} \binom{r}{j} X^{r-j} Y^j \right] + a_p \sum_{\mu \in \mathbb{F}_p^*} [\mu]^{-2} [g_{g1, [\mu]}, X^{r-3} Y^3] + O(p^{\bar{r}+1}),\]

by (3.1). Using the above computation, we have

\[(T - a_p) f_{\infty} = \frac{1}{3c(1 - p)} \sum_{\lambda \in \mathbb{F}_p^*} \left[ (p - 1) \left[ g_{1, \lambda}^0, \sum_{1 \leq j \leq r - 2} \binom{r}{j} X^{r-j} Y^j \right] + \frac{1}{3c} \left[ (p - 1) \left[ g_{1, 0}, \sum_{1 \leq j \leq r - 2} \binom{r}{j} X^{r-j} Y^j \right] \right] \right] \]

\[\quad + a_p \sum_{\lambda \in \mathbb{F}_p^*} [\mu]^{-2} [g_{g1, [\mu]} g_{1, [\mu]}, X^{r-3} Y^3] + \frac{1}{3c} \left[ (p - 1) \left[ g_{1, 0}, \sum_{1 \leq j \leq r - 2} \binom{r}{j} X^{r-j} Y^j \right] \right].\]
\[ + a_p \sum_{\mu \in \mathbb{F}_p} [\mu]^{-2} [g_{1,0}^0 g_{1,[\mu]}^0] X^{r-3}Y^3 \bigg) + O(p). \]

Rearranging the terms, we finally have

\[ (T-a_p)f_{\infty} \]

\[ = - \frac{1}{3c} \left( 1 - p \right) \left[ g_{1,0}^0 \sum_{j \equiv 1 mod (p-1)} 1 \leq j \leq r - 2 \choose j \right] X^{r-j}Y^j \] \[ + \sum_{\lambda \in \mathbb{F}_p^*} \sum_{j \equiv 1 mod (p-1)} 1 \leq j \leq r - 2 \choose j \right] X^{r-j}Y^j \bigg) \]

\[ + \frac{a_p}{3c} \left( \sum_{\mu \in \mathbb{F}_p} [\mu]^{-2} [g_{1,0}^0 g_{1,[\mu]}^0] X^{r-3}Y^3 \right) + \frac{1}{1 - p} \sum_{\lambda \in \mathbb{F}_p^*} \sum_{\mu \in \mathbb{F}_p} [\mu]^{-2} [g_{1,0}^0 g_{1,[\mu]}^0] X^{r-3}Y^3 \bigg) + O(p). \]

Combining everything together, from equations (6.13) and (6.14), in radius 1 we have

\[ T^+ f_0 - a_p f_1 + h_{\infty,1} = \frac{3 - r}{3c} \sum_{\lambda \in \mathbb{F}_p} [g_{1,[\lambda]}^0] X^{r-1}Y - X^{r-p}Y^p \] \[ + h + O(\sqrt{p}). \]

Finally, in radius 2 we compute \( T^+ f_1 + h_{\infty,2} \), where \( h_{\infty,2} \) is the radius 2 part of \( (T-a_p)f_{\infty} \). We have

\[ T^+ \Phi_g = \sum_{\mu \in \mathbb{F}_p} \left[ gg_{1,[\mu]}^0, \sum_{j \equiv 1 mod (p-1)} 1 \leq j \leq r - 2 \choose j \right] X^{r-j}(-[\mu]X + pY)^j - S_0X^{r-p}(-[\mu]X + pY)^p \] \[ = g_{1,0}^0 \sum_{\lambda \in \mathbb{F}_p^*} \sum_{j \equiv 1 mod (p-1)} 1 \leq j \leq r - 2 \choose j \right] X^{r-j}Y^j + [\mu]S_0X^r \] \[ + \left[ gg_{1,0}^0, \sum_{j \equiv 1 mod (p-1)} 1 \leq j \leq r - 2 \choose j \right] X^{r-j}Y^j \bigg) + O(p^{t+2}), \]

since \( v(S_0) \geq t \), by Proposition 3.6 part (I).

By using Lemma 3.1 for the \( \mu = 0 \) terms in (6.16), we get

\[ T^+ \Phi_g = \sum_{\mu \in \mathbb{F}_p^*} \left[ gg_{1,[\mu]}^0, \sum_{i=0}^r p^i(-[\mu])^{1-i} \sum_{j \equiv 1 mod (p-1)} 1 \leq j \leq r - 2 \choose j \right] X^{r-j}Y^j + [\mu]S_0X^r \] \[ + O(p^{t+2}). \]

By the definition of \( S_0 \), the coefficient of \([gg_{1,[\mu]}^0], X^r\) for \( \mu \neq 0 \) in \( T^+ \Phi_g \) vanishes since

\[ -[\mu] \sum_{j \equiv 1 mod (p-1) 1 \leq j \leq r - 2 \choose j \right] + [\mu]S_0 = 0. \]
By Proposition 3.6 parts (2), (3), (4) and (5), the coefficients of \([gg^0_{g,[\mu]}, X^{r-1}Y], [gg^0_{g,[\mu]}, X^{r-2}Y^2], [g^0_{1,[\mu]}, X^{r-3}Y^3], [g^0_{1,[\mu]}, X^{r-i}Y^i]\) (for \(i \geq 4\)) in \(T^+\Phi_g\), for \(\mu \neq 0\), are
\[
p \sum_{1 < j \leq r-2 \atop j \equiv 1 \bmod (p-1)} j \binom{r}{j} \equiv 0 \bmod p^{t+2},
\]
\[-\mu^{-1} p^2 \sum_{1 < j \leq r-2 \atop j \equiv 1 \bmod (p-1)} \binom{j}{2} \binom{r}{j} \equiv 0 \bmod p^{t+2},
\]
\[-\mu^{-2} p^3 \sum_{1 < j \leq r-2 \atop j \equiv 1 \bmod (p-1)} \binom{j}{3} \binom{r}{j} \equiv \frac{[\mu]^{-2} \binom{r}{3} p^3}{1-p} \bmod p^{t+2},
\]
\[-\mu^{-1} p^i \sum_{1 < j \leq r-2 \atop j \equiv 1 \bmod (p-1)} \binom{j}{i} \binom{r}{j} \equiv 0 \bmod p^{t+4}, \forall i \geq 4,
\]
respectively. As \(v(a_p \mu) \leq t + \frac{3}{2}\), we get that
\[
\frac{-1}{3a_p c} T^+\Phi_g = \frac{-p^3 \binom{r}{3}}{3a_p c(1-p)} \sum_{\mu \in \mathbb{F}_p^\times} [\mu]^{-2}[g^0_{g,[\mu]}, X^{r-3}Y^3] + O(\sqrt{p})
\]
(6.17)
\[
= \frac{-a_p}{3c(1-p)} \sum_{\mu \in \mathbb{F}_p^\times} [\mu]^{-2}[g^0_{g,[\mu]}, X^{r-3}Y^3] + O(\sqrt{p}),
\]
since
\[
\frac{-p^3 \binom{r}{3}}{3a_p c(1-p)} = \frac{pa_p \mu - a_p^2}{3a_p c(1-p)} \equiv \frac{-a_p}{3c(1-p)} \bmod p.
\]
So by equation (6.17), we get
(6.18)
\[
T^+ f_1 = \frac{-a_p}{3c} \left( \frac{1}{1-p} \sum_{\lambda \in \mathbb{F}_p^\times} \sum_{\mu \in \mathbb{F}_p^\times} [\mu]^{-2}[g^0_{g,[\lambda]g^0_{g,[\mu]}, X^{r-3}Y^3}] + \sum_{\mu \in \mathbb{F}_p^\times} [\mu]^{-2}[g^0_{g,[\mu]}, X^{r-3}Y^3] \right) + O(\sqrt{p}).
\]
Therefore from equations (6.14) and (6.18), in radius 2 we have
(6.19)
\[
T^+ f_1 + h_{\infty,2} = O(\sqrt{p}).
\]
So putting everything together by equations (6.1), (6.7), (6.15) and (6.19), we have
\[
(T - a_p) f = \frac{3 - r}{3c} \sum_{\lambda \in \mathbb{F}_p^\times} [g^0_{1,[\lambda]}, X^{r-1}Y - X^{r-p}YP] - [1, X^{r-1}Y - X^{r-p}YP] + h + O(\sqrt{p})
\]
\[
= \frac{3 - r}{3c} \sum_{\lambda \in \mathbb{F}_p^\times} [g^0_{1,[\lambda]}, \theta X^{r-p-1}] - [1, \theta X^{r-p-1}] + h + O(\sqrt{p}),
\]
where \( h \) is an integral combination of terms of the form \([g, X^r]\), for some \( g \in G \). Now the term \( h \) dies in \( \text{ind}_{KZ}^G Q \). Moreover, by part (ii) of Lemma 2.2 we see that \( \theta X^{r-p-1} \) is identified with \( X \in J_2 = V_1 \otimes D \). So the image of \((T - a_p)f\) in \( \text{ind}_{KZ}^G Q \) actually lies in \( \text{ind}_{KZ}^G J_2 \) and is given by

\[
\frac{3 - r}{3\xi} \sum_{\lambda \in \mathbb{F}_p} [g_{1,\lambda}^0, X] - [1, X] = \left( \frac{1}{\lambda} T - 1 \right) [1, X],
\]

where \( \lambda = \frac{3 - r}{3\xi} \left( \frac{a^2 - (\xi)p^3}{p^{a_p}} \right) \).

If \( \tau < t \), then \( \lambda^{-1} = 0 \), so we have that \( F_2 = 0 \) because \( (T - a_p)f = [1, X] \) which is zero in \( \Theta_{k, a_p} \) and \( [1, X] \) generates \( \text{ind}_{KZ}^G J_2 \).

If \( \tau = t \), then \( F_2 \) is a quotient of \( \pi(1, \lambda, \omega) \) which is the same as saying that \( F_2 \) is a quotient of \( \pi(1, \lambda, \omega) \), by (6.7).

We now prove a lemma which will be used to show that \( F_2 = 0 \), when \( \tau > t + \frac{1}{2} \) (see Proposition 6.3).

**Lemma 6.2.** Let \( T \) be the Hecke operator and \( J_2 = V_1 \otimes D \). If \( h = \sum_{\lambda \in \mathbb{F}_p} [g_{1,\lambda}^0, Y] \in \text{ind}_{KZ}^G J_2 \), then \( h \notin T(\text{ind}_{KZ}^G J_2) \).

**Proof.** Recall that the vertices of the Bruhat-Tits tree for \( G = \text{GL}_2(\mathbb{Q}_p) \) are represented by the matrices \( g_{n,\lambda}^0 \) and \( g_{m,\mu}^1 \) for \( n, m \geq 0 \), \( \lambda \in I_n \) and \( \mu \in I_m \) (cf. Section 2). Let \( f \in \text{ind}_{KZ}^G J_2 \). If \( f \) is supported mod \( KZ \) on matrices of the first kind, then we say that \( f \) is supported on the positive side of the tree and if \( f \) is supported mod \( KZ \) on matrices of the second kind, then we say that \( f \) is supported on the negative side of the tree.

Suppose there exists an \( f \in \text{ind}_{KZ}^G J_2 \) such that \( Tf = h \). Let \( f = f_+ + f_- \), where \( f_+ \) and \( f_- \) are the parts of \( f \) supported mod \( KZ \) on the positive and the negative side of the tree respectively.

We first show that \( \text{supp}(T f_-) \) mod \( KZ \) does not contain matrices of the form \( g_{n,\lambda}^0 \), for \( n \geq 1 \) and \( \lambda \in I_n \). As \( T = T^+ + T^- \), we first claim that \( \text{supp}(T^+ f_-) \) mod \( KZ \) does not contain matrices of the form \( g_{n,\gamma}^0 \), for \( n \geq 1 \) and \( \gamma \in I_n \). Indeed, if \( \text{supp}(T^+ f_-) \) mod \( KZ \) contains some matrix \( g_{n,\gamma}^0 \), then we have, by (2.2),

\[
\begin{pmatrix}
1 & 0 \\
p \mu & p^{m+1}
\end{pmatrix}
\begin{pmatrix}
\lambda \\
p \mu
\end{pmatrix} =
\begin{pmatrix}
p^n \gamma \\
n \mu
\end{pmatrix}
\]

for some \( m \geq 0 \), \( n \geq 1 \), \( \mu \in I_m \), \( \gamma \in I_n \), \( k \in K = \text{GL}_2(\mathbb{Z}_p) \) and \( z = (\frac{z}{z})^0 \in Z \). Comparing the valuations of the determinants in equation (6.20), we get \( v(\det(z)) = 2v(z) = m - n + 2 \), so

\[
v(z) = \frac{m - n + 2}{2}.
\]

Multiplying by \( (g_{n,\gamma}^0)^{-1} \) on both sides of equation (6.20) and further multiplying out the matrices, we get

\[
p^{-n} \begin{pmatrix}
p - p^2 \gamma \mu & [\lambda] - p[\lambda] \gamma \mu - \gamma p^{m+1} \\
p^{n+2} \mu & p^{n+1} \mu [\lambda] + p^{m+n+1}
\end{pmatrix} = k z.
\]
Multiplying equation (6.22) by \(z^{-1}\) and comparing the valuation of the \((1,1)\)-th entry, we get

\[
v(k_{(1,1)}) = -n + 1 - \frac{m - n + 2}{2} = \frac{-m - n}{2},
\]

by (5.21). As \(m \geq 0\) and \(n \geq 1\), we get that \(v(k_{(1,1)}) < 0\), which is a contradiction since \(k_{(1,1)} \in \mathbb{Z}_p\), proving the claim.

Next we claim that \(\text{supp}(T^-f_-) \mod KZ\) does not contain matrices of the form \(g^0_{n,\gamma}\), for \(n \geq 0\) and \(\gamma \in I_n\). Indeed if \(\text{supp}(T^-f_-) \mod KZ\) does contain such a matrix, then we have, by (2.3),

\[
\begin{pmatrix}
1 & 0 \\
p^\mu & p^{m+1}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & p
\end{pmatrix}
= \begin{pmatrix}
p^n & \gamma \\
0 & 1
\end{pmatrix}kz,
\]

(6.23)

where \(m, n \geq 0, \mu \in I_m, \gamma \in I_n, k \in \text{GL}_2(\mathbb{Z}_p)\) and \(z \in \mathbb{Z}\). Comparing the valuations of the determinants in equation (6.23), we get

\[
v(z) = \frac{m - n + 2}{2}.
\]

(6.24)

Multiplying both sides of equation (6.23) by \((g^0_{n,\gamma})^{-1}\) and further multiplying out the matrices, we get

\[
p^{-n}\begin{pmatrix}1 - p\gamma\mu & -p\gamma p^{m+2} \\
p^n+1\mu & p^{m+n+2}
\end{pmatrix} = kz.
\]

(6.25)

Multiplying equation (6.25) by \(z^{-1}\) and comparing the valuation of the \((1,1)\)-th entry, we get

\[
v(k_{(1,1)}) = -n - \frac{m - n + 2}{2} = \frac{-m - n}{2} - 1,
\]

by equation (6.24). As \(m, n \geq 0\), we get that \(v(k_{(1,1)}) < 0\), which is a contradiction since \(k_{(1,1)} \in \mathbb{Z}_p\), proving the claim. Therefore \(\text{supp}(Tf_-) \mod KZ\) does not contain matrices of the form \(g^0_{n,\mu}\), for \(n \geq 1\).

Let \(f_{n,+}\) denote the radius \(n\) part of \(f_+\). We now claim that \(f_+ = f_{0,+}\). Suppose not, then there exists an \(m \geq 1\) such that \(f_{m,+} \neq 0\) and \(f_{l,+} = 0\), for all \(l > m\). From our initial assumption we know that

\[
Tf_- + Tf_+ = h.
\]

(6.26)

As \(m\) is the non-zero maximum radius that appears in \(f_+\), the radius \(m + 1\) part of the left hand side of (6.26) is \(T^+f_{m,+}\), since \(\text{supp}(Tf_-) \mod KZ\) does not contain \(g^0_{n,\lambda}\) for \(n \geq 1\). Comparing the radius \(m + 1 \geq 2\) parts of (6.26), we get

\[
T^+f_{m,+} = 0,
\]

(6.27)

since \(h\) lies in radius \(1 < 2\). Write \(f_{m,+} = \sum_{\mu} \sum_{\lambda \in \mathbb{F}_p} [g^0_{m,\mu,\gamma}] \cdot A_{\mu}X + B_{\mu}Y\), for some \(A_{\mu}, B_{\mu} \in \mathbb{F}_p\). Now

\[
T^+f_{m,+} = \sum_{\lambda \in \mathbb{F}_p} \sum_{\mu} [g^0_{m,\mu,\gamma}] \cdot A_{\mu}X + B_{\mu}(-[\lambda]X + pY)
\]

\[
= \sum_{\lambda \in \mathbb{F}_p} \sum_{\mu} [g^0_{m+1,\mu+p\gamma}] \cdot (A_{\mu} - [\lambda]B_{\mu})X.
\]
By (6.24), we have

$$A_\mu - [\lambda]B_\mu = 0,$$

for all $\mu \in I_m$ and $\lambda \in \mathbb{F}_p$. Substituting $\lambda = 0, 1$ in the above equation we get $A_\mu = B_\mu = 0$, for all $\mu \in I_m$, implying $f_{m,+} = 0$, which is a contradiction, proving out claim that $f_+ = f_{0,+}$.

Write $f_{0,+} = [1, AX + BY]$, for some $A, B \in \bar{\mathbb{F}}_p$. From (6.25), we get in radius 1 that

$$T^+ f_{0,+} = h,$$

since $\text{supp}(T f_-)$ mod $KZ$ does not contain vertices of the form $g_{1,\lambda}^0$, for $\lambda \in I_1$. So we have

$$T^+ f_{0,+} = \sum_{\lambda \in \mathbb{F}_p} [g_{1,[\lambda]}^0, AX + B(-[\lambda]X + pY)]$$

$$= \sum_{\lambda \in \mathbb{F}_p} [g_{1,[\lambda]}^0, (A - [\lambda]B)X].$$

As $T^+ f_{0,+} = h$, we get a contradiction because $h$ is supported on functions of the form $[g, v]$ where $v$ has only $Y$ terms. Therefore $h \notin T(\text{ind}_{KZ} J_2)$. \qed

**Proposition 6.3.** Let $p > 3, r \geq 2p + 1, r = 3 + n(p - 1)p^4$, with $t = v(r - 3)$ and $v(a_p) = \frac{3}{2}$. Let $c = \frac{r^2 - (r-2)^2(r-1)p^3}{p^3}$ and $\tau = v(c)$. If $\tau > t + \frac{1}{2}$, then $F_2 = 0$.

**Proof.** Define the function $f = f_0 + f_\infty$, with

$$f_0 = \frac{(p-1)(r-2)}{p^2(3-r)} \left[ 1, \sum_{2 \leq j < r-1 \atop j \equiv 2 \text{ mod } (p-1)} \binom{r-1}{j} X^{r-j} Y^j \right] + \frac{r-2}{2p} [1, X^{r-2} Y^2],$$

$$f_\infty = \sum_{\lambda \in \mathbb{F}_p} \xi g_{1,\lambda}^0 + (1-p)\xi g_{1,\alpha}^0,$$

where $\xi$ is as in Lemma 4.3.

We compute $(T - a_p)f$ in several steps.

Computation in radius $-1$:

$$T^- f_0 = \frac{(p-1)(r-2)}{p^2(3-r)} \left[ \alpha, \sum_{2 \leq j < r-1 \atop j \equiv 2 \text{ mod } (p-1)} \binom{r-1}{j} (pX)^{r-j} Y^j \right] + \frac{r-2}{2p} [\alpha, (pX)^{r-2} Y^2]$$

(6.28) $$= O(p^2),$$

since $p^{r-j} \binom{r-1}{j} = p \left( p^{r-j-1} \binom{r-1}{j-1} \right)$, and $r - j - 1 \geq p - 1 > 3$ for all $j$ as above, so by Lemma 3.2 we get $p^{r-j-1} \binom{r-1}{j-1} \equiv 0 \text{ mod } p^{j+3}.$
Computation in radius 0:

\[ -a_p f_0 = \frac{-a_p(p-1)(r-2)}{p^2(3-r)} \left[ 1, \sum_{2 \leq j < r-1 \atop j \equiv 2 \, \text{mod} \, (p-1)} \binom{r-1}{j} X^{r-j} Y^j \right] + O(\sqrt{p}). \]

Let \( h_{0,\infty} \) be the radius 0 part of \((T - a_p)f_\infty\). By Lemma 4.3 we get

\[
h_{0,\infty} = \frac{a_p(r-2)}{p^2(3-r)} \left( \sum_{\lambda \in \mathbb{F}_p} [g^0_{1,[\lambda]} \alpha, XY^{r-1}] + (1-p)[g^0_{1,0} \alpha, XY^{r-1}] \right)
\]

\[
= \frac{a_p(r-2)}{p^2(3-r)} \left( (p-1) \left[ 1, \sum_{2 \leq j < r-1 \atop j \equiv 2 \, \text{mod} \, (p-1)} \binom{r-1}{j} (X)^{r-j} Y^j \right] \right) + (1-p)[1, XY^{r-1}]
\]

\[ = \frac{a_p(r-2)(p-1)}{p^2(3-r)} \left[ 1, \sum_{2 \leq j < r-1 \atop j \equiv 2 \, \text{mod} \, (p-1)} \binom{r-1}{j} X^{r-j} Y^j \right].
\]

So in radius 0, from equations (6.29) and (6.30), we get

\[ (6.31) \quad -a_p f_0 + h_{0,\infty} = O(\sqrt{p}). \]

Computation in radius 1:

\[ T^+ f_0 = \frac{(p-1)(r-2)}{p^2(3-r)} \sum_{\lambda \in \mathbb{F}_p^\times} g^0_{1,[\lambda]} \psi \sum_{\ell \equiv 2 \, \text{mod} \, (p-1)} \binom{r-1}{j} X^{r-j} (-[\lambda] X + pY)^j \]

\[ + \frac{r-2}{2p} \sum_{\lambda \in \mathbb{F}_p^\times} [g^0_{1,[\lambda]} X^{r-2} (-[\lambda] X + pY)^2] \]

\[ = \frac{(p-1)(r-2)}{p^2(3-r)} \sum_{\lambda \in \mathbb{F}_p^\times} g^0_{1,[\lambda]} \psi \sum_{\ell \equiv 2 \, \text{mod} \, (p-1)} \binom{r-1}{j} X^{r-j} \sum_{i=0}^j \binom{j}{i} (-[\lambda] X)^{j-i} (pY)^i \]

\[ + \frac{(p-1)(r-2)}{p^2(3-r)} \left[ g^0_{1,0} \psi \sum_{\ell \equiv 2 \, \text{mod} \, (p-1)} p^i \binom{r-1}{j} X^{r-j} Y^j \right] \]

\[ + \frac{r-2}{2p} \sum_{\lambda \in \mathbb{F}_p^\times} [g^0_{1,[\lambda]} \lambda^2 X^{r-2} - 2[\lambda] pX^{r-1} Y] + O(p) \]

\[ = \frac{(p-1)(r-2)}{p^2(3-r)} \sum_{\lambda \in \mathbb{F}_p^\times} g^0_{1,[\lambda]} \sum_{i=0}^r (-[\lambda])^{2-i} p^i \sum_{2 \leq j < r-1 \atop j \equiv 2 \, \text{mod} \, (p-1)} \binom{j}{i} \binom{r-1}{j} X^{r-j} Y^i \]
\[
\begin{align*}
+ \frac{(p-1)(r-2)}{(3-r)} \left[ g_{1,0}^0 \binom{r-1}{2} X^{r-2}Y^2 \right] + \frac{r-2}{2p} \sum_{\lambda \in \mathbb{F}_p^*} [g_{1,0}^0, \lambda] X^r - 2[\lambda]pX^{r-1}Y] + O(p),
\end{align*}
\]

since by Lemma 3.2, \( p^i \binom{r-1}{j} \equiv 0 \mod p^{i+3} \) for all \( 2 < j < r - 1 \).

By Proposition 3.4, part (1), we can write
\[
(6.32) \quad \sum_{2 \leq j < r-1 \atop j \equiv 2 \mod (p-1)} \binom{r-1}{j} = \frac{p(3-r)}{2} + \frac{p^2r(3-r)}{2} \mod p^{i+3}.
\]

Now by (6.32) and Proposition 3.7, parts (2), (3) and (4), the coefficients of \([g_{1,0}^0, X^r], [g_{1,0}^0, X^{r-1}Y], [g_{1,0}^0, X^{r-2}Y^2]\) and \([g_{1,0}^0, X^{r-i}Y^i] \) for \( i \geq 3 \) and \( \lambda \neq 0 \) in \( T^+f_0 \) are
\[
\frac{\lambda^2(p-1)(r-2)}{p^2(3-r)} \sum_{2 \leq j < r-1 \atop j \equiv 2 \mod (p-1)} \binom{r-1}{j} \equiv \frac{\lambda^2(p-1)(r-2)}{p^2(3-r)} \left( \frac{p(3-r)}{2} + \frac{p^2r(3-r)}{2} \right)
\]
\[
+ \frac{\lambda^2(2)}{2p} \mod p
\]
\[
\equiv \frac{\lambda^2(2)(1-r)}{2} \mod p,
\]
\[
\frac{[-\lambda](p-1)(r-2)}{p(3-r)} \sum_{2 \leq j < r-1 \atop j \equiv 2 \mod (p-1)} j^{r-1} - [\lambda](r-2) = \frac{-\lambda(p-1)(r-2)}{p(3-r)} \left( \frac{p(3-r)}{1-p} \right)
\]
\[
- [\lambda](r-2) \mod p
\]
\[
= \lambda^2(r-2)^2 \mod p,
\]
\[
\frac{(p-1)(r-2)}{3-r} \sum_{2 \leq j < r-1 \atop j \equiv 2 \mod (p-1)} \binom{j}{2} \binom{r-1}{j} \equiv \frac{(p-1)(r-2)}{3-r} \left( \frac{r-1}{1-p} \right) \mod p
\]
\[
= \frac{-(r-2)(r-1)}{3-r} \mod p,
\]
\[
\frac{(-\lambda)^{2-i}(p-1)(r-2)p^i}{p^2(3-r)} \sum_{2 \leq j < r-1 \atop j \equiv 2 \mod (p-1)} \binom{j}{i} \binom{r-1}{j} \equiv 0 \mod p,
\]
respectively. Note that the coefficients of \([g_{1,0}^0, X^r] \) and \([g_{1,0}^0, X^{r-1}Y] \) are integral. So we get
\[
(6.33) \quad T^+f_0 = \frac{-(r-2)(r-1)}{3-r} \left( \sum_{\lambda \in \mathbb{F}_p^*} [g_{1,0}^0, X^{r-2}Y^2] + (1-p)[g_{1,0}^0, X^{r-2}Y^2] \right) + h + O(p),
\]
where \( h \) is an integral linear combination of the terms of the form \([g, X^r] \) and \([g, X^{r-1}Y] \), for some \( g \in G \).
Let $h_{1,\infty}$ be the radius 1 part of $(T - a_p)f_\infty$. By Lemma 6.2 we get

\[(6.34)\]

\[h_{1,\infty} = \frac{a_p^2}{p^3(3-r)} \left( \sum_{\lambda \in \mathbb{F}_p} [g_{1,[\lambda]}^0, X^{r^2}Y^2 + (r - 3)X^p Y^{r-p}] + (1 - p)[g_{1,0}^0, X^{r^2}Y^2 + (r - 3)X^p Y^{r-p}] \right).\]

From equations (6.33) and (6.34), in radius 1 we get

\[(6.35)\]

\[T^+ f_0 + h_{1,\infty} = \left( \frac{a_p^2 - (r - 2)(r-1)p^3}{p^3(3-r)} \right) \left( \sum_{\lambda \in \mathbb{F}_p} [g_{1,[\lambda]}^0, X^{r^2}Y^2] + (1 - p)[g_{1,0}^0, X^{r^2}Y^2] \right) - \frac{a_p^2}{p^3} \sum_{\lambda \in \mathbb{F}_p} [g_{1,[\lambda]}^0, X^{r^2}Y^{p}] + h + O(p).\]

So from equations (6.28), (6.31) and (6.35), we get

\[(T - a_p)f = \frac{a_p^2}{p^3(3-r)} \left( \sum_{\lambda \in \mathbb{F}_p} [g_{1,[\lambda]}^0, X^{r^2}Y^2] + (1 - p)[g_{1,0}^0, X^{r^2}Y^2] \right) - \frac{a_p^2}{p^3} \sum_{\lambda \in \mathbb{F}_p} [g_{1,[\lambda]}^0, X^{r^2}Y^p]\]

\[+ h + O(\sqrt{p}).\]

Now, the image of $h$ in $\text{ind}^{G}_{KZ} Q$ is 0. Also as $\Gamma > t + \frac{1}{2}$, the image of $(T - a_p)f$ in $\text{ind}^{G}_{KZ} Q$ is

\[(T - a_p)f = \frac{-a_p^2}{p^3} \sum_{\lambda \in \mathbb{F}_p} [g_{1,[\lambda]}^0, X^{r^2}Y^p] \mod X_{r-1}\]

\[(6.36)\]

\[= \frac{-a_p^2}{p^3} \sum_{\lambda \in \mathbb{F}_p} [g_{1,[\lambda]}^0, X^{r^2}Y^p - XY^{r-1}] \mod X_{r-1},\]

since $XY^{r-1} \in X_{r-1}$. Now it follows from part (ii) of Lemma 2.2 that $\theta Y^{r-p-1} = X^{p} Y^{r-p} - XY^{r-1}$ is identified with $Y \in J_2 = V_1 \otimes D$. So the image of $(T - a_p)f$ in $\text{ind}^{G}_{KZ} Q$ actually lies in $\text{ind}^{G}_{KZ} J_2$ and is given by

\[(T - a_p)f = \frac{-a_p^2}{p^3} \sum_{\lambda \in \mathbb{F}_p} [g_{1,[\lambda]}^0, Y].\]

By Lemma 6.2 we get that $(T - a_p)f \notin T(\text{ind}^{G}_{KZ} J_2)$, so it is non-zero in $\text{ind}^{G}_{KZ} J_2$. But $(T - a_p)f$ maps to zero in $\hat{\Theta}_{k,a_p}$. So the kernel of the map $\text{ind}^{G}_{KZ} J_2 \rightarrow F_2$ is not zero. Since $\text{ind}^{G}_{KZ} J_2$ is irreducible, the kernel is all of $\text{ind}^{G}_{KZ} J_2$. Therefore $F_2 = 0$. \qed

We remark that when $t = 0$ we know $F_2 = 0$ always, by Proposition 2.1 part (1), whereas Proposition 6.3 only shows that $F_2 = 0$ when $\Gamma > \frac{1}{2}$. \hfill \red{\Box}
7. Behaviour of $F_3$

Finally, we study the structure of the subquotient $F_3$ of $\bar{\Theta}_{k,a_p}$ which is the hardest to treat. Recall that $\text{ind}_{KZ}^G J_3 \to F_3$, where $J_3 = V_{p-2} \otimes D^2$.

One might expect that $F_3 = 0$ for $\tau < t + \frac{1}{2}$, but have not been able to prove this. However, for our purposes it will suffice to show that $F_3 = 0$ for $\tau \leq t$.

**Proposition 7.1.** Let $p > 3$, $r \geq 2p + 1$, $r = 3 + n(p-1)p^t$, with $t = v(r-3)$ and $v(a_p) = \frac{3}{2}$. Let $c = \frac{a_{p}^2 - (r-2)\left(\frac{r-1}{p}\right)p^3}{pa_p}$ and $\tau = v(c)$. If $\tau \leq t$, then $F_3 = 0$.

**Proof.** The reader should compare the proof with the proof of Theorem 8.6 in [BG15].

Note that $\tau \leq t$ forces $t$ to be at least 1, since $\tau \geq \frac{1}{2}$.

Define

$$f_0 = \frac{p-1}{a_p c} \left[ 1, \sum_{2 \leq j < r-1 \atop j \equiv 2 \text{ mod } (p-1)} \beta_j X^{r-j}Y^j \right],$$

$$f_\infty = \frac{1}{c} \sum_{\lambda \in \mathbb{F}_p^\times} \chi'_\lambda \left( \frac{r(p-1)}{pc} \phi g_{\lambda,0}^t, \right),$$

where the $\beta_j$ are defined in the proof of Proposition 3.9 and $\chi'$, $\phi$ are as in Lemma 4.4 and Lemma 4.5, respectively. Let $f = f_0 + f_\infty$.

Computation in radius $-1$:

$$T^{-1}f_0 = \frac{p-1}{a_p c} \left[ \alpha, \sum_{2 \leq j < r-1 \atop j \equiv 2 \text{ mod } (p-1)} \beta_j (pX)^{r-j}Y^j \right] = O(p^2),$$

(7.1)

since $r-j \geq p > 4$, for all $j$ as above, so by Proposition 3.9 part (1), we have $p^{r-j}\beta_j \equiv p^{r-j}(\ell_0^j) \mod p^{t+4}$ and by Lemma 3.1 we have $p^{r-j}(\ell_0^j) \equiv 0 \mod p^{t+4}$.

Computation in radius 0:

$$-a_pf_0 = \frac{1-p}{c} \left[ 1, \sum_{2 \leq j < r-1 \atop j \equiv 2 \text{ mod } (p-1)} \beta_j X^{r-j}Y^j \right].$$

(7.2)

Let $h_{0,\infty}$ be the radius 0 part of $(T - a_p)f_\infty$. By Lemma 4.4 and Lemma 4.5, we get

$$h_{0,\infty} = \frac{1}{c} \sum_{\lambda \in \mathbb{F}_p^\times} \left[ g_{\lambda,0}^0, [\lambda]^{-1}Y^r \right] - \frac{r(p-1)}{pc} \left[ g_{\lambda,0}^0, pX Y^{r-1} \right] + h_{\phi} + O(p)$$

$$= \frac{1}{c} \sum_{\lambda \in \mathbb{F}_p^\times} [1, [\lambda]^{-1}([\lambda]X + Y)^r] - \frac{r(p-1)}{c} [1, X Y^{r-1}] + h_{\phi} + O(p)$$
By the definition of radius 0, we have since 2

by \((3.1)\), where \(h_\phi\) is an integral linear combination of the terms of the form \([g, X^{r-1}Y]\) and \([g, XY^{r-1}]\). So in radius 0, by equations \((7.2)\), \((7.3)\) and the definition of \(\beta_j\) in \((3.23)\) for \(j \neq 2, 2p\), we get

\[
(7.4) \quad -a_pf_0 + h_{0, \infty} = \frac{1-p}{c} \left[ 1, \left( \frac{r}{2} \right) X^{r-2}Y^2 + \left( \frac{r}{2p} \right) X^{r-2p}Y^{2p} \right] + h_\phi + O(p).
\]

By the definition of \(\beta_2\) in \((3.24)\), we have

\[
\beta_2 - \left( \frac{r}{2} \right) = -\frac{1}{2} \left( \sum_{2<j<r-1 \atop j \equiv 2 \mod (p-1)} 2b'j \left( \frac{r}{j} \right) + r(r-1) \right)
\]

\[
= -\frac{1}{2} \left( \sum_{2<j<r-1 \atop j \equiv 2 \mod (p-1)} j \left( \frac{r}{j} \right) \right) \mod p^{t+1},
\]

since \(2b' \equiv 1 \mod p^{t+1}\). By Proposition \((3.8)\) part \((2)\), we get

\[
(7.5) \quad \beta_2 - \left( \frac{r}{2} \right) = \frac{r(r-3)}{2} \mod p^{t+1}.
\]

By the definition of \(\beta_{2p}\) in \((3.25)\), we get

\[
\beta_{2p} - \left( \frac{r}{2p} \right) = -\sum_{2<j<r-1 \atop j \equiv 2 \mod (p-1)} \left( \frac{r}{j} \right) - \beta_2 - \left( \frac{r}{2p} \right)
\]

\[
= -\sum_{2<j<r-1 \atop j \equiv 2 \mod (p-1)} \left( \frac{r}{j} \right) + r - \left( \frac{r}{2} \right) - \frac{r(r-3)}{2} \mod p^{t+1},
\]

by \((7.5)\). Now, by Proposition \((3.8)\) part \((1)\), we get

\[
(7.6) \quad \beta_{2p} - \left( \frac{r}{2p} \right) = r - 3 - \frac{r(r-3)}{2} = \frac{(r-3)(2-r)}{2} = \frac{3-r}{2} \mod p^{t+1},
\]

since \(2-r \equiv -1 \mod p\) as \(t \geq 1\). Since \(v(c) \leq t\), by using equations \((7.5)\) and \((7.6)\) in \((7.4)\), in radius 0, we have

\[
-a_pf_0 + h_{0, \infty} = \frac{r-3}{2c} \left[ 1, rX^{r-2}Y^2 - X^{r-2p}Y^{2p} \right] + h_\phi + O(p)
\]

\[
= \frac{r-3}{2c} \left[ 1, 3X^{r-2}Y^2 - X^{r-2p}Y^{2p} \right] + h_\phi + O(p)
\]

\[
(7.7) \quad = \frac{r-3}{2c} \left[ 1, 2X^{r-2}Y^2 + \theta X^{r-2p-2}Y + \theta X^{r-2p-1}Y^p \right] + h_\phi + O(p),
\]

since \(r \equiv 3 \mod p\).
Computation in radius 1:

\[ T^+ f_0 = \frac{p - 1}{a_pf_c} \sum_{\lambda \in \mathbb{F}_p} \left[ g_{1,\lambda}^0, \sum_{2 \leq j < r-1 \mod (p-1)}^{r} \beta_j X^{r-j} \right] \]

\[ = \frac{p - 1}{a_pf_c} \sum_{\lambda \in \mathbb{F}_p} \left[ g_{1,\lambda}^0, \sum_{i=0}^{r} (-[\lambda])^{2-i} p^i \sum_{2 \leq j < r-1 \mod (p-1)}^{p} \beta_j \left( \frac{j}{i} \right) X^{r-j} Y^i \right] \]

\[ + \frac{p - 1}{a_pf_c} \left[ g_{1,0}^0, \sum_{2 \leq j < r-1 \mod (p-1)}^{2} p^j \beta_j X^{r-j} Y^j \right]. \]

By Proposition 3.9 parts (2), (3) and (4), the coefficients of \([g_{1,\lambda}^0, X^r], [g_{1,\lambda}^0, X^{r-1} Y], [g_{1,\lambda}^0, X^{r-2} Y^2], [g_{1,\lambda}^0, X^{r-3} Y^3]\) and \([g_{1,\lambda}^0, X^{r-i} Y^i]\) for \(i \geq 4\) and \(\lambda \neq 0\) in \(T^+ f_0\) are

\[ \frac{[\lambda]^2 (p-1)}{a_pf_c} \sum_{2 \leq j < r-1 \mod (p-1)}^{r} \beta_j \equiv 0 \mod \sqrt{p}, \]

\[ -\frac{p[\lambda](p-1)}{a_pf_c} \sum_{2 \leq j < r-1 \mod (p-1)}^{r} j \beta_j \equiv 0 \mod \sqrt{p}, \]

\[ \frac{p^2 (p-1)}{a_pf_c} \sum_{2 \leq j < r-1 \mod (p-1)}^{r} \left( \frac{j}{2} \right) \beta_j \equiv 0 \mod \sqrt{p}, \]

\[ \frac{p^3 (-[\lambda])^{-1} (p-1)}{a_pf_c} \sum_{2 \leq j < r-1 \mod (p-1)}^{r} \left( \frac{j}{3} \right) \beta_j \equiv -[\lambda]^{-1} p^3 \beta_3 \mod p, \]

\[ \frac{p^i (-[\lambda])^{2-i} (p-1)}{a_pf_c} \sum_{2 \leq j < r-1 \mod (p-1)}^{r} \left( \frac{j}{i} \right) \beta_j \equiv 0 \mod p, \]

respectively. The coefficient of \([g_{1,0}^0, X^{r-j} Y^j]\) for \(2 < j < r-1\) and \(j \equiv 2 \mod (p-1)\) in \(T^+ f_0\) is

\[ \left( \frac{p - 1}{a_pf_c} \right) p^j \beta_j \equiv 0 \mod p^2, \]

since by Proposition 3.9 part (1) and Lemma 3.1 \(p^j \beta_j \equiv p^j \left( \frac{j}{i} \right) \equiv 0 \mod p^{j+4} \).

The coefficient of \([g_{1,0}^0, X^{r-2} Y^2]\) in \(T^+ f_0\) is

\[ \left( \frac{p - 1}{a_pf_c} \right) p^2 \beta_2 \equiv \frac{p^2 (p-1) \beta_3}{a_pf_c} \mod \sqrt{p}, \]
by Proposition 3.9 part (I). So we can write

\[ T^+ f_0 = -\frac{p^3(p)}{ap^c} \sum_{\lambda \epsilon \mathbb{F}_p^*} \left[ g_{1,[\lambda]}^0, [\lambda]^{-1}X^{-3}Y^3 \right] + \frac{p^2(p-1)(\tau)}{ap^c} \left[ g_{1,0}, X^{-r-2}Y^2 \right] + O(\sqrt{p}). \]

Let \( h_{1,\infty} \) be the radius 1 part of \((T - a_p)f_\infty\). By Lemma 4.3 and Lemma 4.5 we get

\[ h_{1,\infty} = \frac{a_p}{c} \sum_{\lambda \epsilon \mathbb{F}_p^*} \left[ g_{1,[\lambda]}^0, [\lambda]^{-1}X^{-3}Y^3 \right] - \frac{apr(p-1)}{pc} \left[ g_{1,0}, X^{-r-2}Y^2 \right] + h'_\phi + O(p), \]

where \( h'_\phi \) is an integral linear combination of the terms of the form \([g, X^{-r}Y] \) and \([g, XY^{r-1}]\), for some \( g \in G \).

The coefficient of \([g_{1,0}, X^{-r-2}Y^2]\) for \( \lambda \neq 0 \) in \( T^+ f_0 + h_{1,\infty} \) is

\[ [\lambda]^{-1} \left( \frac{a_p}{c} - \left( \frac{\tau}{p} \right) p^3 \right) = \frac{[\lambda]^{-1}}{c} \left( \frac{a_p^2 - (\tau)p^3}{ap} \right) \equiv 0 \text{ mod } p, \]

since \( \tau = \bar{\tau} \) by Lemma 4.1 part (iii).

The coefficient of \([g_{1,0}, X^{-r-2}Y^2]\) in \( T^+ f_0 + h_{1,\infty} \) is

\[ \frac{p^2(p-1)(\tau)}{ap^c} - \frac{apr(p-1)}{pc} = \frac{p - 1}{c} \left( \frac{p^3(\tau) - a_p^2}{pa_p} \right) \equiv 3(p-1) \left( \frac{a_p^2 - p^3}{pa_p} \right) \equiv 3 \left( \frac{a_p^2 - p^3}{pa_p} \right) \text{ mod } p, \]

since \( r \equiv 1 \) mod \( p^\ell \). Note that, by (4.9) and our hypothesis \( \tau \leq t \) we have \( v(a_p^2 - p^3) = \tau + \bar{\tau} \), so the constant \( 3 \left( \frac{a_p^2 - p^3}{pa_p} \right) \) is a unit.

So finally in radius 1 we get

(7.8) \[ T^+ f_0 + h_{1,\infty} = 3 \left( \frac{a_p^2 - p^3}{pa_p} \right) \left[ g_{1,0}, X^{-r}Y^2 \right] + h'_\phi + O(\sqrt{p}). \]

From equations (7.7), (7.8) and (7.8), we get

(7.9) \[ (T - a_p)f = \frac{r - 3}{2c} \left[ 1, 2X^{-2}Y^2 + \theta X^{-r-2}Y + \theta X^{-r-2}Y^2 \right] + 3 \left( \frac{a_p^2 - p^3}{pa_p} \right) \left[ g_{1,0}, X^{-r}Y^2 \right] + 3 \left( \frac{a_p^2 - p^3}{pa_p} \right) \left[ g_{1,0}, X^{-r}Y^2 \right] + h + O(\sqrt{p}), \]

where \( h = h'_\phi + h'_\phi \).

Now, \( h \) dies in \( \text{ind}_{KZ}^G Q \). The image of \( X^{-r-2}Y^2 \) in \( Q \) is the same as that of \( X^{-r}Y + Y^{-r-1} \), which is

(7.10) \[ \theta(X^{-r-2}Y + \ldots + Y^{-r-1}) \equiv \theta \left( \frac{r - p - 2}{p - 1} X^{-r-2}Y + Y^{-r-1} \right) \text{ mod } V_r^{**}. \]

By parts (iii) and (ii) of Lemma 2.2 \( \theta X^{-r-2}Y \) and \( \theta Y^{-r-1} \) map to \( X^{-2} \) and 0 in \( J_3 = V_{p-2} \otimes D^2 \), respectively. Therefore

(7.11) \[ X^{-2} \mapsto (2-r)X^{-2} \]
in $J_3$. As $t \geq 1$, we get that $X^{r-2}Y^2$ maps to $-X^p-2$ in $J_3$. Also $\theta X^{r-2p-1}Y^p$ maps to $X^{p-2}$ in $J_3$ because $\theta X^{r-2p-1}Y^p \equiv \theta X^{r-p-2}Y \mod \theta^2$, so $2X^{r-2}Y^2 + \theta X^{r-p-2}Y + \theta X^{r-2p-1}Y^p$ maps to 0 in $J_3$. So the image of $(T-a_p)f$ in $\text{ind}_{KZ}^G J_3$ is given by

$$3 \begin{pmatrix} p^3 - a_p^2 \\ -pa_p \end{pmatrix} [g_{1,0}^0, X^{p-2}].$$

But $(T-a_p)f$ maps to zero in $\Theta_{k,a_p}$. Therefore $F_3 = 0$ since $[g_{1,0}^0, X^{p-2}]$ generates $\text{ind}_{KZ}^G J_3$. □

Remark 7.2. We make some remarks to show that our computations in the proof of Proposition 7.1 are consistent with previous results. By Proposition 2.1, part (2), as $t \geq 1$, we have $V_{1,0}^0 \subset Q$, where $V_{1,0}^0$ is the non-trivial extension of $J_3 = V_{p-2} \otimes D^2$ by $J_2 = V_1 \otimes D$.

Suppose $\tau < t$. Then the first term on the right hand side of (7.9) dies mod $p$, so by equation (7.10), we have $(T-a_p)f = [g_{1,0}^0, v]$, where $v = 3 \begin{pmatrix} a_p^2 - p^3 \\ -pa_p \end{pmatrix} \theta(-X^{r-p-2}Y + Y^{r-p-1}) \in V_{1,0}^0$. As $V_{1,0}^0$ is the non-trivial extension of $J_3$ by $J_2$ and $v$ projects to a non-zero vector in $J_3$, we see that $v$ generates $V_{1,0}^0$, so $[g_{1,0}^0, v]$ generates $\text{ind}_{KZ}^G V_{1,0}^0$, since $[g_{1,0}^0, v]$ is an elementary function (i.e., supported mod $KZ$ on one matrix). This shows that $\text{ind}_{KZ}^G V_{1,0}^0 \mapsto F_{2,3}$ is the zero map. In particular $F_2 = 0$ because $F_2 \subset F_{2,3}$, and we recover Proposition 6.1 part (i), showing $F_2$ vanishes for $\tau < t$.

This argument does not work for $\tau = t$. Indeed when $\tau = t$, by (7.9), we see that $(T-a_p)f$ is supported mod $KZ$ in radii 0 and 1, so is not an elementary function and so does not necessarily generate $\text{ind}_{KZ}^G V_{1,0}^0$. So $F_2$ does not necessarily vanish when $\tau = t$. This is consistent with Proposition 6.1 part (ii).

Proposition 7.3. Let $p > 3$, $r \geq 2p+1$, $r = 3 + n(p-1)p^t$, with $t = v(r-3)$ and $v(a_p) = \frac{3}{2}$. Let $c = a_p^{2r} - r^2(r-1)^3$ and $\tau = v(c)$. If $\tau < t + 1$, then the map from $\text{ind}_{KZ}^G J_3$ to $F_3$ factors through the image of $T$, i.e., $\pi(p-2,0,\omega^3) \mapsto F_3$.

Proof. If $v(2r) > 0$, then $t = v(r-3) = 0$ and $\tau = \frac{3}{2}$, i.e., $v(a_p^{2r} - (r-1)^3(r-2)p^3) = 3$. So condition (*) holds in the main result (Theorem 1.1) of [BG15] and the proposition follows from the first part of the proof of Theorem 9.1 in [BG15]. Note that $J_3$ for us is $J_1$ in [BG15].

So throughout this proof we will assume that $v(2r) = 0$.

Let

$$S_{t-1} = \sum_{\substack{2 \leq j < r-1 \\ j \equiv 2 \mod (p-1)}} \binom{r-1}{j},$$

(7.12)

$$T_{t-1} = \sum_{\substack{2 \leq j < r-1 \\ j \equiv 2 \mod (p-1)}} j \binom{r-1}{j}.$$

Define

$$f_0 = \frac{1-p}{ca_p} \left[ 1 - \sum_{\substack{2 \leq j < r-1 \\ j \equiv 2 \mod (p-1)}} \binom{r-1}{j} X^{r-2}Y^j \right] + \left( \frac{p-1}{ca_p} \right) S_{t-1}[1, X^{r-2}Y^2]$$
\[ f_1 = \frac{-1}{cp(2 - r)} \left( \sum_{\lambda \in F_p} [g_{1,\lambda}^0, X^{r-2}Y^2 - (r - 2)XY^{r-1}] + (1 - p)[g_{1,0}^0, X^{r-2}Y^2 - (r - 2)X^{r-1}] \right) \]

\[ + \frac{1 - p}{cp(2 - r)} \left( \sum_{\lambda \in F_p^*} [g_{1,\lambda}^0, \sum_{2 \leq j < r - p} \binom{r}{j} X^{r-j}Y^j] + (1 - p) \left[ g_{1,0}^0, \sum_{2 \leq j < r - p} \binom{r}{j} X^{r-j}Y^j \right] \right) \]

\[ - \frac{3(3 - r)}{2c(2 - r)} \left( \sum_{\lambda \in F_p} [g_{1,\lambda}^0, X^{r-2}Y^2] + (1 - p)[g_{1,0}^0, X^{r-2}Y^2] \right), \]

\[ f_\infty = \frac{pr - 2}{(2 - r)^2} \left( \sum_{\lambda \in F_p^*} \xi'^p g_{1,\lambda}^0 + (1 - p)\xi'' \right) + \frac{1}{r - 2} \sum_{\mu \in F_p} \left( \sum_{\lambda \in F_p^*} \psi' g_{1,\lambda}^0 + (1 - p)\psi' \right), \]

where \( \xi'' \) and \( \psi' \) are as in Lemma 4.8 and Lemma 4.9. Let \( f = f_0 + f_1 + f_\infty \).

Computation in radius \(-1\):

\[ T^{-f_0} = \frac{(1 - p)}{ca_p} \left[ \alpha, \sum_{2 \leq j < r - 1} \binom{r - 1}{j} (pX)^{r-j}Y^j \right] + \left( \frac{p - 1}{ca_p} \right) S_{r-1}[\alpha, (pX)^{r-2}Y^2] \]

\[ + \left( \frac{2S_{r-1} - T_{r-1}}{ca_p} \right) [\alpha, (pX)^{r-2}Y^2 - (pX)^{r-p-1}Y^{p+1}] \]

(7.13) = \( O(p) \),

since \( r - j - 1 \geq p - 1 > 3 \) for all \( j \) as above so by Lemma 3.2, \( p^{r-j-1}(r-j-1)^{r-j-1} \equiv 0 \mod p^{l+4} \)
and by Proposition 3.7 part (1) and (2), we have \( v(S_{r-1}) = v(T_{r-1}) = t + 1 \), so \( p^{r-2}S_{r-1} \equiv p^{r-p-1}(2S_{r-1} - T_{r-1}) \equiv 0 \mod p^{l+4} \).

Computation in radius \(0\):

\[ -a_pf_0 = \frac{p - 1}{c} \left[ 1, \sum_{2 \leq j < r - 1} \binom{r - 1}{j} X^{r-j}Y^j \right] + \left( \frac{1 - p}{c} \right) S_{r-1}[1, X^{r-2}Y^2] \]

\[ - \left( \frac{2S_{r-1} - T_{r-1}}{c} \right) [1, X^{r-2}Y^2 - X^{r-p-1}Y^{p+1}] \]

(7.14) = \( \frac{p - 1}{c} \left[ 1, \sum_{2 \leq j < r - 1} \binom{r - 1}{j} X^{r-j}Y^j \right] + O(p^\delta) \),
for \( \delta := t + 1 - \tau > 0 \), because \( v(S_{r-1}) = v(T_{r-1}) = t + 1 \) by Proposition 3.1, part (1) and (2) and the fact that \( v(c) = \tau < t + 1 \). Now

\[
T^{-1} f_1 = \frac{-1}{c(p(2-r))} \left( \sum_{\lambda \in \mathbb{F}_p} [g_{1,\lambda}] \alpha, (pX)^{t-2Y^2} - (r - 2)pXY^{r-1} \right) \\
+ (1 - p)[g_{1,0}] \alpha, (pX)^{t-2Y^2} - (r - 2)pXY^{r-1} \right) \\
+ \frac{1 - p}{c(p(2-r))} \left( \sum_{\lambda \in \mathbb{F}_p} \left[ g_{1,\lambda} \alpha, \sum_{2 \leq j \leq r-p, j \equiv 2 \mod (p-1)} \binom{r}{j} (pX)^{r-jY^j} \right] \\
+ (1 - p) \left[ g_{1,0} \alpha, \sum_{2 \leq j \leq r-p, j \equiv 2 \mod (p-1)} \binom{r}{j} (pX)^{r-jY^j} \right] \\
- \frac{3(3-r)}{2c(2-r)} \left( \sum_{\lambda \in \mathbb{F}_p} [g_{1,\lambda}] \alpha, (pX)^{t-2Y^2} \right) + (1 - p)[g_{1,0}] \alpha, (pX)^{t-2Y^2} \right) \\
= \frac{-1}{c} \left( \sum_{\lambda \in \mathbb{F}_p} [1, X(\lambda)X + Y)^{r-1}] + (1 - p)[1, XY^{r-1}] \right) + O(p^2),
\]

since \( r - 2 > t + 4 \), by Lemma 3.1, \( p^{r-j} \left( \binom{r}{j} \right) \equiv 0 \mod p^{t+4} \) for all \( j \) as above, and \( v(c) < t + 1 \). By 3.1, we have

\[
(7.15) \quad T^{-1} f_1 = \frac{1 - p}{c} \left[ 1, \sum_{2 \leq j \leq r-1, j \equiv 2 \mod (p-1)} \binom{r-1}{j} X^{r-jY^j} \right] + O(p^2).
\]

So in radius 0 by equations (7.14) and (7.15), we get

\[
(7.16) \quad -a_p f_0 + T^{-1} f_1 = O(p^{\delta'}),
\]

with \( \delta' := \min(\delta, 2) \).

Computation in radius 1:

\[
T^+ f_0 = \left( \frac{1 - p}{ca_p} \right) \left[ \sum_{\lambda \in \mathbb{F}_p} [g_{1,\lambda}] \alpha, \sum_{2 \leq j \leq r-1, j \equiv 2 \mod (p-1)} \binom{r}{j} X^{r-j(-\lambda)X + pY^j} \right] \\
+ \left( \frac{p - 1}{ca_p} \right) S_{r-1} \sum_{\lambda \in \mathbb{F}_p} [g_{1,\lambda}] X^{r-2(-\lambda)X + pY^2} \\
+ \left( \frac{2S_{r-1} - T_{r-1}}{ca_p} \right) \sum_{\lambda \in \mathbb{F}_p} [g_{1,\lambda}] X^{r-2(-\lambda)X + pY^2} - X^{r-p-1(-\lambda)X + pY^{p+1}}
\]

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\[
\begin{align*}
&= \left(1 - \frac{p}{ca_p}\right) \sum_{\lambda \in \mathbb{F}_p^x} \left[ g_{1,[\lambda]}^0 \sum_{i=0}^{r} (-[\lambda])^{2-i} \frac{p^i}{i!} \sum_{j \equiv 2 \mod (p-1)} \binom{r-1}{j} \binom{r}{j} X^{r-i} Y^i \right] \\
&+ \left(1 - \frac{p}{ca_p}\right) \left[ g_{1,0}^0 p^2 \binom{r-1}{2} X^{r-2} Y^2 \right] + \left(\frac{p-1}{ca_p}\right) S_{r-1} \sum_{\lambda \in \mathbb{F}_p^x} \left[ g_{1,[\lambda]}^0 [\lambda]^2 X^r - 2p[\lambda] X^{r-1} Y \right] \\
&+ \left(\frac{2S_{r-1} - T_{r-1}}{ca_p}\right) \sum_{\lambda \in \mathbb{F}_p^x} \left[ g_{1,[\lambda]}^0 [\lambda] p(p-1) X^{r-1} Y \right] + O(\sqrt{p}),
\end{align*}
\]

by Lemma 3.2, \( p^j (r-1) \equiv 0 \mod p^{j+3} \) for \( 2 < j < r-1 \) and using the fact that \( v(S_{r-1}) = v(T_{r-1}) = t + 1 \).

By the definition of \( S_{r-1}, T_{r-1} \) and Proposition 3.7 part (3), part (4) and the fact that \( v(S_{r-1}) = v(T_{r-1}) = t + 1 \), the coefficients of \([g_{1,[\lambda]}^0, X^r], [g_{1,0}^0, X^{r-1} Y], [g_{1,[\lambda]}^0, X^{r-2} Y^2], [g_{1,0}^0, X^{r-1} Y]^i \) for \( i \geq 3 \) and \( \lambda \neq 0 \) in \( T^+ f_0 \) are

\[
\begin{align*}
&= \left(1 - \frac{p}{ca_p}\right) \binom{r-1}{2} \binom{r}{2} X^{r-2} Y^2 \equiv 0, \\
&= \left(1 - \frac{p}{ca_p}\right) p^2 \sum_{2 \leq j < r-1 \mod (p-1)} \binom{r-1}{j} \binom{r-1}{j} \equiv \frac{p^2(r-1)}{ca_p} \mod \sqrt{p}, \\
&= \left(1 - \frac{p}{ca_p}\right) (-[\lambda])^{2-i} \frac{p^i}{i!} \sum_{j \equiv 2 \mod (p-1)} \binom{r-1}{j} \binom{r}{j} \equiv 0 \mod \sqrt{p},
\end{align*}
\]

respectively. Therefore we get

\[
(7.17) \quad T^+ f_0 = \frac{p^2(r-1)}{ca_p} \left( \sum_{\lambda \in \mathbb{F}_p^x} \left[ g_{1,[\lambda]}^0, X^{r-2} Y^2 + (1-p)[g_{1,0}^0, X^{r-2} Y^2] \right] \right) + O(\sqrt{p}).
\]

Now

\[
-a_p f_1 = \frac{a_p}{cp(2-r)} \left( \sum_{\lambda \in \mathbb{F}_p^x} \left[ g_{1,[\lambda]}^0, X^{r-2} Y^2 - (r-2)XY^{r-1} \right] + (1-p)[g_{1,0}^0, X^{r-2} Y^2 - (r-2)XY^{r-1}] \right) \\
+ \frac{(p-1)a_p}{cp(2-r)} \left\{ \sum_{\lambda \in \mathbb{F}_p^x} \left[ \sum_{2 \leq j \leq r-p \mod (p-1)} \binom{r}{j} X^{r-j} Y^j \right] \right\} \\
(7.18) \quad + (1-p) \left[ g_{1,0}^0, \sum_{2 \leq j \leq r-p \mod (p-1)} \binom{r}{j} X^{r-j} Y^j \right] + O(\sqrt{p}),
\]
since \( v \left( \frac{3(3-r)a_p}{2(2-r)} \right) > \frac{1}{2} \) as \((2-r)\) is a unit.

Let \( h_{1,\infty} \) be the radius 1 part of \((T - a_p)f_{\infty} \). By Lemma 4.8 and Lemma 4.9 we get

\[
h_{1,\infty} = \frac{pr - 2}{(2 - r)^2} \left( \frac{a_p}{pc} \right) \left( \sum_{\lambda \in \mathbb{F}_p^*} [g_{1,[\lambda]}^0, (2 - r)XY^{r - 1}] + (1 - p)[g_{1,0,0}^0, (2 - r)XY^{r - 1}] \right)
+ \frac{1}{r - 2} \left( \frac{a_p}{pc} \right) \left( \sum_{\lambda \in \mathbb{F}_p^*} \left( g_{1,[\lambda]}^0, [\mu]^{-1}([\mu]X + Y)^r \right) + (1 - p)[g_{1,0,0}, [\mu]^{-1}([\mu]X + Y)^r] \right) + O(p^r),
\]

for \( \epsilon = \min(\delta, \frac{1}{2}) > 0 \). By (8.1), we can rewrite the above as

\[
h_{1,\infty} = \frac{pr - 2}{(2 - r)^2} \left( \frac{a_p}{pc} \right) \left( \sum_{\lambda \in \mathbb{F}_p^*} [g_{1,[\lambda]}^0, (2 - r)XY^{r - 1}] + (1 - p)[g_{1,0,0}^0, (2 - r)XY^{r - 1}] \right)
+ \frac{p - 1}{r - 2} \left( \frac{a_p}{pc} \right) \left( \sum_{\lambda \in \mathbb{F}_p^*} \left( g_{1,[\lambda]}^0, \sum_{2 \leq j \leq r - 1 \atop j \equiv 2 \mod (p-1)} \binom{r}{j} X^{r-j}Y^j \right) \right)
+ (1 - p) \left( g_{1,0,0}^0, \sum_{2 \leq j \leq r - 1 \atop j \equiv 2 \mod (p-1)} \binom{r}{j} X^{r-j}Y^j \right) + O(p^r)
= -\frac{a_p}{pc} \left( \sum_{\lambda \in \mathbb{F}_p^*} [g_{1,[\lambda]}^0, XY^{r - 1}] + (1 - p)[g_{1,0,0}^0, XY^{r - 1}] \right)
+ \frac{a_p(p - 1)}{(r - 2)pc} \left( \sum_{\lambda \in \mathbb{F}_p^*} \left( g_{1,[\lambda]}^0, \sum_{2 \leq j \leq r - p \atop j \equiv 2 \mod (p-1)} \binom{r}{j} X^{r-j}Y^j \right) \right) + (1 - p) \left[ \sum_{\lambda \in \mathbb{F}_p^*} \left( g_{1,0,0}^0, \sum_{2 \leq j \leq r - p \atop j \equiv 2 \mod (p-1)} \binom{r}{j} X^{r-j}Y^j \right) \right]
+ O(p^r).
\]

(7.19)

So finally in radius 1 by (7.17), (7.18) and (7.19), we get

\[
T^+f_0 - a_pf_1 + h_{1,\infty} = \left( \frac{p^2(r - 1)}{c^2p} \right) \left( \sum_{\lambda \in \mathbb{F}_p^*} [g_{1,[\lambda]}^0, X^{r-2}Y^2] + (1 - p)[g_{1,0,0}^0, X^{r-2}Y^2] \right) + O(p^r)
= \frac{1}{2 - r} \left( \sum_{\lambda \in \mathbb{F}_p^*} [g_{1,[\lambda]}^0, X^{r-2}Y^2] + (1 - p)[g_{1,0,0}^0, X^{r-2}Y^2] \right) + O(p^r)
\]

(7.20)

\[
= \frac{1}{2 - r} \sum_{\lambda \in \mathbb{F}_p^*} [g_{1,[\lambda]}^0, X^{r-2}Y^2] + O(p^r).
\]
Computation in radius 2:

\[
T^+ f_1 = -\frac{1}{cp(2-r)} \sum_{\lambda \in \mathbb{F}_p^*} \sum_{\mu \in \mathbb{F}_p} [g_{2,[\lambda]+p[\mu]}, X]^{r-2}(-[\mu]X + pY)^2 - (r-2)X(-[\mu]X + pY)^{r-1}]
- \frac{(1-p)}{cp(2-r)} \sum_{\mu \in \mathbb{F}_p} [g_{2,p[\mu]}, X]^{r-2}(-[\mu]X + pY)^2 - (r-2)X(-[\mu]X + pY)^{r-1}
\]

\[
+ \frac{(1-p)^2}{cp(2-r)} \sum_{\lambda \in \mathbb{F}_p^*} \sum_{\mu \in \mathbb{F}_p} \left[ g_{0,\lambda}^0 \sum_{r \in \mathbb{F}_p} \binom{r}{j} X^{r-j}(-[\mu]X + pY)^j \right]
\]

\[
+ \frac{(1-p)^2}{cp(2-r)} \sum_{\lambda \in \mathbb{F}_p^*} \sum_{\mu \in \mathbb{F}_p} \left[ g_{0,\lambda}^0 \sum_{r \in \mathbb{F}_p} \binom{r}{j} X^{r-j}(-[\mu]X + pY)^j \right]
\]

\[
- \frac{3(3-r)}{2c(2-r)} \sum_{\lambda \in \mathbb{F}_p^*} \sum_{\mu \in \mathbb{F}_p} [g_{2,[\lambda]+p[\mu]}, X]^{r-2}(-[\mu]X + pY)^2]
\]

\[
- \frac{3(3-r)(1-p)}{2c(2-r)} \sum_{\mu \in \mathbb{F}_p} [g_{2,p[\mu]}, X]^{r-2}(-[\mu]X + pY)^2.
\]

By using Lemma 3.2 in the first sum, Lemma 3.1 for \(\mu = 0\) in the third and fourth sums and by our assumption \(v(c) < t + 1\), we get

\[
T^+ f_1 = -\frac{1}{cp(2-r)} \sum_{\lambda \in \mathbb{F}_p^*} \sum_{\mu \in \mathbb{F}_p} \left[ g_{0,\lambda}^0 [\mu]^{2(3-r)}X - [\mu]pr(3-r)X^{r-1}\right]
\]

\[
+ p^2 \left( 1 - (r-2) \binom{r-1}{2} \right) X^{r-2}Y^2
\]

\[
+ \frac{(1-p)(p-1)}{cp(2-r)} \sum_{\mu \in \mathbb{F}_p} \left[ g_{0,\lambda}^0 \binom{r}{j} X^{r-j}(-[\mu]X + pY)^j \right]
\]

\[
+ p^2 \left( 1 - (r-2) \binom{r-1}{2} \right) X^{r-2}Y^2
\]

\[
- \frac{1}{cp(2-r)} \left( \sum_{\lambda \in \mathbb{F}_p^*} [g_{2,[\lambda]}^0, p^2 X^{r-2}Y^2] + (1-p)[g_{2,0}, p^2 X^{r-2}Y^2] \right)
\]

\[
+ \frac{(1-p)^2}{cp(2-r)} \sum_{\lambda \in \mathbb{F}_p^*} \sum_{\mu \in \mathbb{F}_p} \left[ g_{0,\lambda}^0 \binom{r}{j} \sum_{i=0}^r (-[\mu])^{2-i} p^i \sum_{2 \leq j \leq r-p \mod (p-1)} \binom{j}{i} X^{r-i}Y^i \right]
\]

\[
+ \frac{(1-p)^2}{cp(2-r)} \sum_{\mu \in \mathbb{F}_p} \left[ g_{0,\lambda}^0 \binom{r}{j} \sum_{2 \leq j \leq r-p \mod (p-1)} \binom{j}{i} X^{r-i}Y^i \right].
\]
Recall $\delta = t + 1 - \tau > 0$. By Proposition 3.3, part (1), (2), (3), (4), (5) and Lemma 3.2, the coefficients of $[g_{2, [\lambda] + p \mu}], X^r$, $[g_{2, [\lambda] + p \mu}], X^{r-1} Y$, $[g_{2, [\lambda] + p \mu}], X^{r-2} Y^2$, $[g_{2, [\lambda] + p \mu}], X^{r-3} Y^3$ and $[g_{2, [\lambda] + p \mu}], X^{r-i} Y^i$ for $i \geq 4, \lambda \neq 0, \mu \neq 0$ in $T^+ f_1$ are

\[
\frac{[\mu]^2 (r-3)}{c(2-r)} + \frac{[\mu]^2 (1-p)}{c(2-r)} \sum_{2 \leq j \leq r-p} \binom{r}{j} - \frac{3[\mu]^2 (3-r)}{2c(2-r)} \\
= \frac{[\mu]^2 (r-3)}{c(2-r)} + \frac{[\mu]^2 (1-p)}{c(2-r)} (3-r) + \frac{5(3-r)p}{2(1-p)} \mod p^\delta \\
= \frac{[\mu]^2 (r-3)}{c(2-r)} (1-1) + \frac{[\mu]^2 (3-r)}{c(2-r)} \left(-1 + \frac{5}{2} - \frac{3}{2}\right) \mod p^\delta \\
= 0 \mod p^\delta,
\]

\[
\frac{[\mu] r (3-r)}{c(2-r)} - \frac{[\mu] (1-p)}{c(2-r)} \sum_{2 \leq j \leq r-p} j \binom{r}{j} \\
= \frac{[\mu] r (3-r)}{c(2-r)} - \frac{[\mu] (1-p)}{c(2-r)} \left(2 \binom{r}{2} + r(3-r) - (r-1) \binom{r}{r-1}\right) \mod p^\delta \\
= 0 \mod p^\delta,
\]

\[
-\frac{p}{c(2-r)} \left(1 - (r-2) \binom{r-1}{2}\right) + \frac{p(1-p)}{c(2-r)} \sum_{2 \leq j \leq r-p} \binom{j}{2} \binom{r}{j} \\
= -\frac{p}{c(2-r)} \left(1 - (r-2) \binom{r-1}{2}\right) + \frac{p(1-p)}{c(2-r)} \left(\binom{r}{2} - \binom{r-1}{2} \binom{r}{r-1}\right) \mod p^{\delta+1} \\
= -\frac{p}{c(2-r)} \left((3-r)(r^2 - 2r + 2)\right) + \frac{p(1-p)}{c(2-r)} \left(\frac{r(r-1)(3-r)}{2}\right) \mod p^{\delta+1} \\
= 0 \mod p^\delta,
\]

\[
\frac{[\mu]^{-1} p^2 (p-1)}{c(2-r)} \sum_{2 \leq j \leq r-p} \binom{j}{3} \binom{r}{j} \\
= \frac{[\mu]^{-1} p^2 (p-1)}{c(2-r)} \left(\frac{r}{p-1} - \binom{r-1}{r-1}\right) \mod p \\
= \frac{[\mu]^{-1} p^2 (p-1)}{c(2-r)} \mod p,
\]
\[
\frac{(1 - p)\mu^2}{c^2 - r} \left( \begin{pmatrix} r \choose i 
\sum_{\lambda \in \mathbb{F}^p_\lambda} \left( \begin{pmatrix} r \choose i \right) \right) \equiv 0 \mod p,
\]
respectively.

The coefficients of \([g_{2, p}[\mu], X^r], [g_{2, p}[\mu], X^{r-1}Y], [g_2^0, X^{r-2}Y^2], [g_2^0, X^{r-3}Y^3] \) and \([g_{2, p}[\mu], X^{r-i}Y^i] \) for \( i \geq 4, \mu \neq 0 \) in \( T^+ f_1 \) are \( (1 - p) \) times the coefficients above. So we have

\[
T^+ f_1 = \left( \frac{(1 - p)p(r-\mu)}{c^2 - r} - \frac{p^2}{c^2 - r} \right) \left( \begin{pmatrix} r \choose i \right) \sum_{\lambda \in \mathbb{F}^p_\lambda} \left( \begin{pmatrix} r \choose i \right) \right) + (1 - p)[g_{2, p}[\mu], X^{r-3}Y^3] \right) + O(p^\delta),
\]
where \( \delta = \min(\delta, 1) > 0 \).

Let \( h_{2, \infty} \) be the radius 2 part of \((T - a_p)f_\infty \). By Lemma 4.8 and Lemma 4.9 we get

\[
h_{2, \infty} = \frac{2 - pr}{(2 - r)^2} \left( \frac{a_p^2}{c^2 + r} \right) \left( \begin{pmatrix} r \choose i \right) \sum_{\lambda \in \mathbb{F}^p_\lambda} \left( \begin{pmatrix} r \choose i \right) \right) + (1 - p)[g_{2, p}[\mu], X^{r-2}Y^2] \right) + O(p^\epsilon),
\]
where \( \epsilon = \min(\delta, \frac{\delta}{2}) > 0 \).

The coefficient of \([g_{2, [\lambda]}, X^{r-2}Y^2] \) for \( \lambda \neq 0 \) in \( T^+ f_1 + h_{2, \infty} \) is

\[
\frac{(1 - p)p(r-\mu)}{c^2 - r} - \frac{p}{c^2 - r} \right) \left( \begin{pmatrix} r \choose i \right) \sum_{\lambda \in \mathbb{F}^p_\lambda} \left( \begin{pmatrix} r \choose i \right) \right) \equiv 0 \mod p^\epsilon,
\]
for \( \epsilon = \min(\delta, \frac{\delta}{2}) > 0 \), since \( v(a_p^2 - p^3) \geq \min(t + \frac{5}{2}, t + 3) \) by (4.4) and \( v(a_p^2 - (r-2)p^3) \geq \min(t + \frac{5}{2}, t + 3) \) by a similar argument, and \( \tau < t + 1 \).

Similarly, the coefficient of \([g_{2, 0}, X^{r-2}Y^2] \) in \( T^+ f_1 + h_{1, \infty} \) is \( (1 - p) \) times the computation above, so is \( 0 \mod p^\epsilon \).

The coefficient of \([g_{2, [\lambda]+p[\mu]}, X^{r-3}Y^3] \) for \( \lambda \neq 0, \mu \neq 0 \) in \( T^+ f_1 + h_{2, \infty} \) is

\[
[\mu]^{-1} \left( \frac{p^2}{c^2 - r} + \frac{a_p^2}{c^2 + r} \right) \equiv \left( \begin{pmatrix} r \choose i \right) \right) \equiv 0 \mod p^\epsilon,
\]
since as before \( v(a_p^2 - (r-3)^2p^3) \geq \min(t + \frac{5}{2}, t + 3) \) and \( \tau < t + 1 \).
Similarly, the coefficient of \([g^0_{2, p[\lambda]}] X^{r-3} Y^3] \) for \( \mu \neq 0 \) in \( T^+ f_1 + h_{2, \infty} \) is \((1 - p)\) times the computation above, so is 0 mod \( p \). Therefore in radius 2 we get

\[
T^+ f_1 + h_{2, \infty} = O(p^\epsilon),
\]

with \( \epsilon > 0 \).

By (7.13), (7.16), (7.20) and (7.21), we get

\[
(T - a_p) f = \frac{1}{2 - r} \sum_{\lambda \in \mathbb{F}_p} [g^0_{1, [\lambda]}] X^{r-2} Y^2] + O(p^\epsilon),
\]

with \( \epsilon > 0 \).

By (7.11), \( X^{r-2} Y^2 \) maps to \((2 - r) X^{p-2} \) in \( J_3 = V_{p-2} \otimes D^2 \). So the image \((T - a_p) f \) inside \( \text{ind}^G_{KZ} J_3 \) is

\[
\sum_{\lambda \in \mathbb{F}_p} [g^0_{1, [\lambda]}] X^{p-2} = T[1, X^{p-2}].
\]

As \((T - a_p) f \) maps to zero in \( \tilde{\Theta}_{k, a_p} \) and \([1, X^{p-2}] \) generates \( \text{ind}^G_{KZ} J_3 \), we get that \( T(\text{ind}^G_{KZ} J_3) \subset \text{ker}(\text{ind}^G_{KZ} J_3 \rightarrow F_3) \). Therefore, if \( \tau < t + 1 \), then \( F_3 \) is a quotient of \( \text{ind}^G_{KZ} J_3 \).

Finally, we give the structure of \( F_3 \) for \( \tau \geq t + 1 \).

**Proposition 7.4.** Let \( p > 3 \), \( r \geq 2p + 1 \), \( r = 3 + n(p - 1)p^t \), with \( t = v(r - 3) \) and \( v(a_p) = \frac{3}{2} \). Let

\[
c = \frac{a^2_2 - (r - 2)(r - 1)p^3}{p a_p^t}
\]

and \( \tau = v(c) \).

(i) If \( \tau = t + 1 \), then \( F_3 \) is a quotient of

\[
\frac{\text{ind}^G_{KZ} J_3}{T^2 - dT + 1},
\]

where

\[
d = \frac{2c}{(2 - r)(3 - r)p}.
\]

(ii) If \( \tau > t + 1 \), then \( F_3 \) is a quotient of

\[
\frac{\text{ind}^G_{KZ} J_3}{T^2 + 1}.
\]

Thus in both cases \( F_3 \) is a quotient of \( \pi(p - 2, \lambda, \omega^2) \oplus \pi(p - 2, \lambda^{-1}, \omega^2) \), where

\[
\lambda + \frac{1}{\lambda} = d.
\]

**Proof.** The idea of the proof is similar to the proof of Proposition [7.3]

Note that \( \tau \geq t + 1 \) forces \( v(2 - r) = 0 \). Indeed, \( v(2 - r) > 0 \) implies \( \tau = \frac{1}{2} \).

Let \( S_{r-1} = \sum_{2 \leq j \leq r-1} \binom{r-1}{j} \) and \( T_{r-1} = \sum_{2 \leq j \leq r-1} j \binom{r-1}{j} \) be as in (7.12).
Define
\[
f_0 = \frac{2(p-1)}{(2-r)(3-r)p} \left( 1, \sum_{2 \leq j < r-1 \mod (p-1)} \binom{r-1}{j} X^{r-j} Y^j \right) - S_{r-1}[1, X^{r-2} Y^2] \]
\[
- \frac{2(2S_{r-1} - T_{r-1})}{(2-r)(3-r)p} [1, X^{r-2} Y^2 - X^{r-p+1} Y^{p+1}],
\]
\[
f_1 = \frac{2}{(2-r)^2(3-r)p^2} \left( \sum_{\lambda \in \mathbb{F}_p^*} [g_{1,\lambda}^0, X^{r-2} Y^2 - (r-2)XY^{r-1}] + (1-p)[g_{1,0}^0, X^{r-2} Y^2 - (r-2)XY^{r-1}] \right) \]
\[
- \frac{2(1-p)}{(2-r)^2(3-r)p^2} \left( \sum_{\lambda \in \mathbb{F}_p^*} \left[ g_{1,\lambda}^0, \sum_{2 \leq j \leq r-p \mod (p-1)} \binom{r}{j} X^{r-j} Y^j \right] + (1-p) \left[ g_{1,0}^0, \sum_{2 \leq j \leq r-p \mod (p-1)} \binom{r}{j} X^{r-j} Y^j \right] \right) \]
\[
+ \frac{3}{(2-r)^2 p} \left( \sum_{\lambda \in \mathbb{F}_p^*} [g_{1,\lambda}^0, X^{r-2} Y^2] + (1-p)[g_{1,0}^0, X^{r-2} Y^2] \right),
\]
\[
f_\infty = \frac{2(2-p r)}{(2-r)^3(3-r)} \left( \sum_{\lambda \in \mathbb{F}_p^*} \xi_{g_{1,\lambda}^0}^0 + (1-p)\xi_{g_{1,0}^0}^0 \right) + \frac{2}{(2-r)^2(3-r)} \sum_{\mu \in \mathbb{F}_p^*} \left( \sum_{\lambda \in \mathbb{F}_p^*} \psi_{g_{1,\lambda}^0} + (1-p)\psi_{g_{1,0}^0} \right),
\]
where \(\xi'\) and \(\psi\) are as in Lemma \(\ref{lem:xi psi}\) and Lemma \(\ref{lem:psi psi}\) respectively. Let \(f = f_0 + f_1 + f_\infty\).

The careful reader will notice that the functions \(f_0, f_1, f_\infty\) and therefore \(f\) above are \(\frac{2c}{(2-r)(3-r)p}\) times the corresponding functions defined in the proof of Proposition \(\ref{prop:S_r-1 T_r-1}\) noting \(\xi'' = \frac{2c}{c} \xi'\) and \(\psi' = \frac{2c}{c} \psi\). Essentially, we have replaced the constant \(c\) by \(3-r)p\), which allows us to investigate the case \(\tau \geq t+1\) using the functions for \(\tau < t+1\). Thus, many of the computations in this proof are similar to those in the proof of Proposition \(\ref{prop:S_r-1 T_r-1}\). However, the computations must necessarily be more complicated, since now \((T-a_p) f\) contributes in radii 0 and 2 and not just in radius 1.

Computation in radius \(-1\):

\[
T^{-1} f_0 = \frac{2(p-1)}{(2-r)(3-r)p} \left( \alpha, \sum_{2 \leq j < r-1 \mod (p-1)} \binom{r-1}{j} (pX)^{r-j} Y^j \right) - S_{r-1}[\alpha, (pX)^{r-2} Y^2] \]
\[
- \frac{2}{(2-r)(3-r)p} (2S_{r-1} - T_{r-1})[\alpha, (pX)^{r-2} Y^2 - (pX)^{r-p+1} Y^{p+1}] \tag{7.22}
\]
\[
\quad = O(p),
\]
since \(r-j-1 \geq p-1 > 3\) for all \(j\) as above, so by Lemma \(\ref{lem:p^r-1}\) \(p^{r-j-1} \equiv 0 \mod p^{t+4}\) and by Proposition \(\ref{prop:S_r-1 T_r-1}\) parts (1) and (2), we have \(v(S_{r-1}) = v(T_{r-1}) = t+1\), so \(p^{r-2} S_{r-1} \equiv p^{r-p-1}(2S_{r-1} - T_{r-1}) \equiv 0 \mod p^{t+4}\).
Computation in radius 0:

\[ -a_p f_0 = \frac{-2(p-1)}{(2-r)(3-r)p} \left( \sum_{j=2}^{r-1} \binom{r-1}{j} X^{r-j}Y^j - S_{r-1}[1, X^{r-2}Y^2] \right) \]

(7.23) \[ + \frac{2(2S_{r-1} - T_{r-1})}{(2-r)(3-r)p} [1, X^{r-2}Y^2 - X^{r-p-1}Y^{p+1}] \]

Now

\[ T^{-} f_1 = \frac{2}{(2-r)^2(3-r)p^2} \left( \sum_{\lambda \in \mathbb{F}_p^*} [g_{1,\lambda}^0, -(r-2)pXY^{r-1}] + (1-p)[g_{1,0}^0, -(r-2)XY^{r-1}] \right) \]

\[ - \frac{2(1-p)}{(2-r)^2(3-r)p^2} \left( \sum_{\lambda \in \mathbb{F}_p^*} \left[ g_{1,\lambda}^0, \sum_{j=2}^{r-p} \binom{r-1}{j} (pX)^{r-j}Y^j \right] \right) + \left( \sum_{\lambda \in \mathbb{F}_p^*} [1, X^{r-2}Y^2 - X^{r-p-1}Y^{p+1}] \right) + O(p) \]

since \( r - 2 \geq t + 4 \) and by Lemma 3.1, \( p^{-j} \left( \frac{r}{r-j} \right) \equiv 0 \mod p^{t+4} \) since \( r-j \geq 4 \) for \( 2 \leq j \leq r-p \).

By (3.1), we have

(7.24) \[ T^{-} f_1 = \frac{2(p-1)}{(2-r)(3-r)p} \left[ 1, \sum_{j=2}^{r-1} \binom{r-1}{j} X^{r-j}Y^j \right] + O(p). \]

By equations (7.23), (7.24), we get

\[ -a_p f_0 + T^{-} f_1 = \frac{2(p-1)S_{r-1}}{(2-r)(3-r)p^2} [1, X^{r-2}Y^2] + \frac{2(2S_{r-1} - T_{r-1})}{(2-r)(3-r)p} [1, X^{r-2}Y^2 - X^{r-p-1}Y^{p+1}] + O(p). \]

By Proposition 3.7 parts (1) and (2) \mod p^{t+2}, in radius 0, we get

\[ -a_p f_0 + T^{-} f_1 = \frac{-1}{2-r} [1, X^{r-2}Y^2] + \frac{2}{2-r} \left( 1 - \frac{p-1}{1-p} \right) [1, X^{r-2}Y^2 - X^{r-p-1}Y^{p+1}] + O(p) \]

\[ = \frac{-1}{2-r} [1, X^{r-2}Y^2] + 2[1, X^{r-2}Y^2 - X^{r-p-1}Y^{p+1}] + O(p) \]

(7.25) \[ = \frac{-1}{2-r} [1, X^{r-2}Y^2] + 2[1, X^{r-p-2}Y] + O(p). \]
Computations in radius 1:

\[
T^+ f_0 = \frac{2(p-1)}{(2-r)(3-r)} \sum_{\lambda \in \mathbb{F}_p^*} \left[ g_{1,|\lambda|}^0, \sum_{2 \leq j < r-1 \atop j \equiv 2 \mod (p-1)} \binom{r-1}{j} X^{r-j}(-|\lambda|X + pY)^j \right]
\]

\[
= \frac{2(p-1)}{(2-r)(3-r)} \sum_{\lambda \in \mathbb{F}_p^*} [g_{1,|\lambda|}^0, X^{r-2}(-|\lambda|X + pY)^2]
\]

\[
= \frac{2(2S_{r-1} - T_{r-1})}{(2-r)(3-r)} \sum_{\lambda \in \mathbb{F}_p^*} [g_{1,|\lambda|}^0, X^{r-2}(-|\lambda|X + pY)^2 - X^{r-p-1}(-|\lambda|X + pY)^{p+1}]
\]

\[
= \frac{2(p-1)}{(2-r)(3-r)} \sum_{\lambda \in \mathbb{F}_p^*} \left[ g_{1,|\lambda|}^0, \sum_{i=0}^{r} (-|\lambda|)^{2-i} p^i \sum_{2 \leq j < r-1 \atop j \equiv 2 \mod (p-1)} \binom{r-1}{j} \binom{j}{i} X^{r-iY^j} \right]
\]

\[
= \frac{2(p-1)}{(2-r)(3-r)} \sum_{\lambda \in \mathbb{F}_p^*} \left[ g_{1,|\lambda|}^0, p^2 \binom{r-1}{2} X^{r-2Y^2} \right]
\]

since, by Lemma 3.2, for \(2 < j < r - 1\), we have \(p^i \binom{r-1}{j} \equiv 0 \mod p^{i+3}\), and by Proposition 3.7 parts (1) and (2), we have \(v(S_{r-1}) = v(T_{r-1}) = t + 1\).

By Proposition 3.7 parts (3) and (4), the coefficients of \([g_{1,|\lambda|}^0, X^r], [g_{1,|\lambda|}^0, X^{r-1}Y], [g_{1,|\lambda|}^0, X^{r-2}Y^2]\) and \([g_{1,|\lambda|}^0, X^{r-i}Y^j]\) for \(i \geq 3\) and \(\lambda \neq 0\) in \(T^+ f_0\) are

\[
\frac{2|\lambda|^2(p-1)}{(2-r)(3-r)} (S_{r-1} - S_{r-1}) = 0,
\]

\[
\frac{-2|\lambda|(p-1)}{(2-r)(3-r)} (T_{r-1} - 2S_{r-1} + 2S_{r-1} - T_{r-1}) = 0,
\]

\[
\frac{2(p-1)}{(2-r)(3-r)} \left( p^2 \sum_{2 \leq j < r-1 \atop j \equiv 2 \mod (p-1)} \binom{j}{2} \binom{r-1}{j} \right) \equiv \frac{-2p\binom{r-1}{2}}{(2-r)(3-r)} \mod \sqrt{p},
\]

\[
\frac{2(-|\lambda|)^2(p-1)}{(2-r)(3-r)} \left( p^i \sum_{2 \leq j < r-1 \atop j \equiv 2 \mod (p-1)} \binom{j}{i} \binom{r-1}{j} \right) \equiv 0 \mod \sqrt{p},
\]

respectively. Therefore we get

\[
T^+ f_0 = \frac{-2p\binom{r-1}{2}}{(2-r)(3-r)} \left( \sum_{\lambda \in \mathbb{F}_p^*} [g_{1,|\lambda|}^0, X^{r-2}Y^2] + (1-p)[g_{1,0}^1, X^{r-2}Y^2] \right) + O(\sqrt{p}).
\]
Now

\[-a_p f_1 = \frac{-2a_p}{(2-r)2^3(3-r)p^2} \left( \sum_{\lambda \in \mathbb{F}_p^*} [g_{1,[\lambda]}^0, X^{r-2}Y^2 - (r-2)XY^{r-1}] \right.

\[+ (1-p)[g_{1,0}^0, X^{r-2}Y^2 - (r-2)XY^{r-1}] \]

\[+ \frac{2a_p(1-p)}{(2-r)^2(3-r)p^2} \left( \sum_{\lambda \in \mathbb{F}_p^*} [g_{1,[\lambda]}^0, \sum_{j=2 \mod (p-1)}^r X^{r-j}Y^j] \right) \]

\[(7.27) \quad + (1-p) \left[ g_{1,0}^0, \sum_{2 \leq j \leq r-p \mod (p-1)} \binom{r}{j} X^{r-j}Y^j \right] + O(\sqrt{p}), \]

since \( v \left( \frac{2a_p}{p^2} \right) = \frac{1}{2} > 0. \)

Let \( h_{1,\infty} \) be the radius 1 part of \((T - a_p)f_\infty\). By Lemma 4.6 and Lemma 4.7 we get

\[h_{1,\infty} = \frac{2(2-pr)}{(2-r)^3(3-r)} \left( \frac{a_p}{p^2} \right) \left( \sum_{\lambda \in \mathbb{F}_p^*} [g_{1,[\lambda]}^0, (2-r)XY^{r-1}] \right. \]

\[+ (1-p)[g_{1,0}^0, (2-r)XY^{r-1}] \]

\[+ \frac{2}{(2-r)^2(3-r)} \left( \frac{a_p}{p^2} \right) \sum_{\mu \in \mathbb{F}_p^*} \left( \sum_{\lambda \in \mathbb{F}_p^*} [g_{1,[\lambda]}^0, [\mu]^{-1}(\mu X + Y)^r] \right) \]

\[\left. + (1-p)[g_{1,0}^0, [\mu]^{-1}(\mu X + Y)^r] \right) + O(\sqrt{p}). \]

By (3.1), we get

\[h_{1,\infty} = \frac{2a_p(2-pr)}{(2-r)^2(3-r)p^2} \left( \sum_{\lambda \in \mathbb{F}_p^*} [g_{1,[\lambda]}^0, XY^{r-1}] + (1-p)[g_{1,0}^0, XY^{r-1}] \right. \]

\[+ \frac{2a_p(p-1)}{(2-r)^2(3-r)p^2} \left( \sum_{\lambda \in \mathbb{F}_p^*} g_{1,[\lambda]}^0, \sum_{2 \leq j \leq r-1} \binom{r}{j} X^{r-j}Y^j \right) + (1-p) \left[ g_{1,0}^0, \sum_{2 \leq j \leq r-1} \binom{r}{j} X^{r-j}Y^j \right] \]

\[+ O(\sqrt{p}). \]

Now combining the coefficient of \([g_{1,[\lambda]}^0, XY^{r-1}]\) from the first and second sums, we get

\[h_{1,\infty} = \frac{2a_p}{(2-r)(3-r)p^2} \left( \sum_{\lambda \in \mathbb{F}_p^*} [g_{1,[\lambda]}^0, XY^{r-1}] + (1-p)[g_{1,0}^0, XY^{r-1}] \right) \]
\[
+ \frac{2a_p(p-1)}{(2-r)^2(3-r)p^2} \left( \sum_{\lambda \in \mathbb{F}_p^*} \sum_{2 \leq j \leq r-p, \ j \equiv 2 \mod (p-1)} g_{1,[\lambda]}^0 \binom{r}{j} X^{r-j}Y^j \right) + (1-p) \left( \sum_{\lambda \in \mathbb{F}_p^*} \sum_{2 \leq j \leq r-p, \ j \equiv 2 \mod (p-1)} \binom{r}{j} X^{r-j}Y^j \right)
\]

(7.28)

\[+ O(\sqrt{p}).\]

So in radius 1, by (7.28), (7.27), (7.25), we get

\[
T^+ f_0 - a_p f_1 + h_{1,\infty}
= \left( -\frac{2p^r}{(2-r)^2} - \frac{2a_p}{(2-r)^2(3-r)p^2} \right) \left( \sum_{\lambda \in \mathbb{F}_p^*} [g_{1,[\lambda]}^0, X^{r-2}Y^2] + (1-p)[g_{1,0}^0, X^{r-2}Y^2] \right) + O(\sqrt{p})
\]

(7.29)

\[
= -\frac{2c}{(2-r)^2(3-r)p} \left( \sum_{\lambda \in \mathbb{F}_p^*} [g_{1,[\lambda]}^0, X^{r-2}Y^2] \right) + O(\sqrt{p}).
\]

Computation in radius 2:

\[
T^+ f_1 = \frac{2}{(2-r)^2(3-r)p^2} \sum_{\lambda \in \mathbb{F}_p^*} \sum_{\mu \in \mathbb{F}_p^*} [g_{2,[\lambda]+p[\mu]}^0, X^{r-2}(-[\mu]X + pY)^2 - (r - 2)X(-[\mu]X + pY)^{r-1}]
\]

\[
+ \frac{2(1-p)}{(2-r)^2(3-r)p^2} \sum_{\mu \in \mathbb{F}_p^*} \left[ g_{2,p[\mu]}^0, X^{r-2}(-[\mu]X + pY)^2 - (r - 2)X(-[\mu]X + pY)^{r-1} \right]
\]

\[
- \frac{2(1-p)}{(2-r)^2(3-r)p^2} \sum_{\lambda \in \mathbb{F}_p^*} \sum_{\mu \in \mathbb{F}_p^*} \left[ g_{2,[\lambda]+p[\mu]}^0, \sum_{2 \leq j \leq r-p, \ j \equiv 2 \mod (p-1)} \binom{r}{j} X^{r-j}(-[\mu]X + pY)^j \right]
\]

\[
- \frac{2(1-p)^2}{(2-r)^2(3-r)p^2} \sum_{\mu \in \mathbb{F}_p^*} \left[ g_{2,p[\mu]}^0, \sum_{2 \leq j \leq r-p, \ j \equiv 2 \mod (p-1)} \binom{r}{j} X^{r-j}(-[\mu]X + pY)^j \right]
\]

\[
+ \frac{3}{(2-r)^2p} \sum_{\lambda \in \mathbb{F}_p^*} \sum_{\mu \in \mathbb{F}_p^*} [g_{2,[\lambda]+p[\mu]}^0, X^{r-2}(-[\mu]X + pY)^2]
\]

\[
+ \frac{3(1-p)}{(2-r)^2p} \sum_{\mu \in \mathbb{F}_p^*} [g_{2,p[\mu]}^0, X^{r-2}(-[\mu]X + pY)^2].
\]

By using Lemma 3.22 and the fact that \( r - 1 \geq t + 5 \) in the first and second sums, Lemma 3.31 for \( \mu = 0 \) in the third and fourth sums, we get

\[
T^+ f_1 = \frac{2}{(2-r)^2(3-r)p^2} \sum_{\lambda \in \mathbb{F}_p^*} \sum_{\mu \in \mathbb{F}_p^*} \left[ g_{2,[\lambda]+p[\mu]}^0, [\mu]^2(3-r)X^r - [\mu]pr(3-r)X^{r-1}Y 
\]

\[
+ p^2 \left( 1 - (r-2) \left( \frac{r-1}{2} \right) \right) X^{r-2}Y^2 \right]
\]
By Proposition 3.8, parts (1), (2), (3), (4), (5) and Lemma 3.2 the coefficients of \([g_{2,\lambda}^0, \mu X^r], [g_{2,\lambda}^0 + p \mu, X^{r-1}Y], [g_{2,\lambda}^0, \mu X^r - 2p \mu X^{r-1}Y], [g_{2,\lambda}^0, \mu X^r - 2p \mu X^{r-2}Y], [g_{2,\lambda}^0, \mu X^r - 3Y^3], [g_{2,\lambda}^0 + p \mu, X^{r-1}Y^i] \) for \(i \geq 4, \lambda \neq 0 \) and \( \mu \neq 0 \) in \( T^+ f_1 \) are

\[
\frac{2[\mu]^2}{(2 - r)^2p^2} - \frac{2[\mu]^2(1 - p)}{(2 - r)^2(3 - r)p^2} \left( \sum_{2 \leq j \leq r-p \mod (p-1)} \binom{r}{j} \right) + \frac{3[\mu]^2}{(2 - r)^2p^2} = \frac{2[\mu]^2}{(2 - r)^2p^2} \left( \frac{2[\mu]^2(1 - p)}{(2 - r)^2(3 - r)p^2} \left( 3 - r + \frac{5(3 - r)p}{2(1 - p)} \right) + \frac{3[\mu]^2}{(2 - r)^2p^2} \right) \mod Z_p
\]

\[
= \frac{2[\mu]^2}{(2 - r)^2p^2} \left( 1 - 1 \right) - \frac{2[\mu]^2}{(2 - r)^2p^2} \left( -1 + 5 \frac{3}{2} - \frac{3}{2} \right) \mod Z_p
\]

\[
= 0 \mod Z_p,
\]

\[
-\frac{2[\mu]^r}{(2 - r)^2p} + \frac{2[\mu]^r(1 - p)}{(2 - r)^2(3 - r)p} \left( \sum_{2 \leq j \leq r-p \mod (p-1)} \binom{r}{j} \right) - \frac{6[\mu]}{(2 - r)^2p^2}
\]
respectively.

The coefficients of \([g_{2,p,[\mu]}, X^r], [g_{2,p,[\mu]}, X^{r-1}Y], [g_{2,p,[\mu]}, X^{r-2}Y^2], [g_{2,p,[\mu]}, X^{r-3}Y^3]\) and \([g_{2,p,[\mu]}, X^{r-i}Y^i]\) for \(i \geq 4, \mu \neq 0\) in \(T^+ f_1\) is \((1 - p)\) times the coefficients above. So, finally we get

\[
T^+ f_1 = \frac{1}{2 - r} \sum_{\lambda \in \mathbb{F}_p} \sum_{\mu \in \mathbb{F}_p^*} [g_{2,[\lambda]+[p], X^{r-2}Y^2}] + \
\frac{2}{2 - r} \left(1 - (1 - p) \binom{\mu}{r}\right) \sum \left[ g_{2,[\lambda], X^{r-2}Y^2} \right] + (1 - p) \left[ g_{2,0, X^{r-2}Y^2} \right]
\]
where $h$ is an integral linear combination of the terms of the form $[g, X^r]$ and $[g, X^{r-1}Y]$, for some $g \in G$.

Let $h_{2, \infty}$ be the radius 2 part of $(T - a_p)f_\infty$. By Lemma 4.6 and Lemma 4.7 we have

$$h_{2, \infty} = \frac{2(2 - pr)}{(2 - r)^2(3 - r)} \left( \frac{-a_p^2}{p^3} \right) \left( \sum_{\lambda \in \mathbb{F}_p^*} [g_{2,[\lambda]}^0, X^{r-2}Y^2] + (r - 3)X^pY^{r-p} \right)$$

$$+ (1 - p)[g_{2,0}^0, X^{r-2}Y^2 + (r - 3)X^pY^{r-p}]$$

$$+ \frac{2}{(2 - r)^2(3 - r)} \left( \frac{a_p^2}{p^3} \right) \left( \sum_{\lambda \in \mathbb{F}_p^*} [g_{2,[\lambda]+p[\mu]}^0, [\mu]^{-1}X^{r-3}Y^3] + (1 - p)[g_{2,p[\mu]}^0, [\mu]^{-1}X^{r-3}Y^3] \right)$$

(7.31)

$$+ O(\sqrt{p}).$$

By (7.30) and (7.31), the coefficient of $[g_{2,[\lambda]}^0, X^{r-2}Y^2]$ for $\lambda \neq 0$ in $T^+f_1 + h_{2, \infty}$ is

$$2 \left( 1 - (1 - p)\left( \frac{c}{\lambda} \right) \right) \frac{2(2 - pr)a_p^2}{(2 - r)^2(3 - r)}$$

$$= \frac{2(p^3 - (pa_p + (r - 2)(\frac{r}{2})p^3))}{(2 - r)^2(3 - r)p^3} - \frac{2(1 - p)r(pa_p + (r - 2)(\frac{r}{2})p^3)}{(2 - r)^3(3 - r)p^3}$$

$$= \frac{2(1 - (r - 2)(\frac{r}{2}))(r - 2 - 1)}{(2 - r)^2(3 - r)} - \frac{2(1 - p)r(\frac{r}{2})(r - 2 - 1)}{(2 - r)^3(3 - r)} \mod \sqrt{p}$$

$$= \frac{1}{2 - r} \mod \sqrt{p},$$

since $v(c) \geq t + 1$.

Similarly, the coefficient of $[g_{2,0}^0, X^{r-2}Y^2]$ in $T^+f_1 + h_{2, \infty}$ is $(1 - p)$ times the coefficient above, i.e., is still $\frac{1}{2 - r} \mod \sqrt{p}$.

By (7.30) and (7.31), the coefficient of $[g_{2,[\lambda]+p[\mu]}^0, [\mu]^{-1}X^{r-3}Y^3]$ for $\lambda \neq 0$ and $\mu \neq 0$ is

$$\frac{-2p(\frac{r}{2})}{(2 - r)^2(3 - r)} + \frac{2a_p^2}{(2 - r)^2(3 - r)p^2} = \frac{2(pa_p + (r - 2)(\frac{r}{2})p^3 - (\frac{r}{2})p^3)}{(2 - r)^2(3 - r)p^2} \equiv 0 \mod p.$$

Similarly the coefficient of $[g_{2,p[\mu]}^0, [\mu]^{-1}X^{r-3}Y^3]$ for $\mu \neq 0$ in $T^+f_1 + h_{2, \infty}$ is $(1 - p)$ times the coefficient above, i.e., is still 0 mod $p$. 
Therefore by (7.30) and (7.31), in radius 2, we get
\[ T^+ f_1 + h_{2,\infty} = \frac{1}{2 - r} \sum_{\lambda \in \mathbb{F}_p} \sum_{\mu \in \mathbb{F}_\mu} [g_{2,\lambda}^0, p^r]\ X^{r-2}Y^2 \]
\[ + \frac{2(2-pr)a^2}{(2-r)^3p^3} \sum_{\lambda \in \mathbb{F}_p} [g_{2,\lambda}^0, X^{p^r-p}] + h + O(\sqrt{p}). \]
\[ (7.32) \]

Putting everything together, from (7.22), (7.25), (7.29) and (7.32), we finally get

\[ (T - a_p)f = \frac{-1}{2-p}[1, X^{r-2}Y^2] + 2[1, \theta X^{r-p-2}Y] - \frac{2c}{(2-r)^2(3-r)p} \left( \sum_{\lambda \in \mathbb{F}_p} [g_{1,\lambda}^0, X^{r-2}Y^2] \right) \]
\[ + \frac{1}{2 - r} \sum_{\lambda \in \mathbb{F}_p} \sum_{\mu \in \mathbb{F}_\mu} [g_{2,\lambda}^0, p^r], X^{r-2}Y^2 \]
\[ + \frac{2(2-pr)a^2}{(2-r)^3p^3} \sum_{\lambda \in \mathbb{F}_p} [g_{2,\lambda}^0, X^{p^r-p}] + h + O(\sqrt{p}). \]
\[ (7.33) \]

We project to \( Q = \frac{V}{\lambda \in \mathbb{F}_p} \). All the polynomials occurring in (7.33) are in fact in the image of \( V^* \) in \( Q \). Indeed in \( Q \), we have \( X^{r-2}Y^2 = X^{r-2}Y^2 - XY^{r-1} = \theta((2-r)X^{r-p-2} + Y^{r-p-1}) \), by (7.10), and \( X^{p^r-p} = X^{p^r-p} - XY^{r-1} = \theta Y^{r-p-1} \).

By Proposition 2.1 part (1), if \( t = 0 \), then the image of \( V^* \) in \( J_3 = V_{p-2} \otimes D^2 \), and by part (2), if \( t > 0 \), then the image of \( V^* \) in \( Q \) is which has \( J_3 \) as a further quotient. So in either case we may project to \( J_3 = V_{p-2} \otimes D^2 \). Now \( h \) maps to zero in \( \text{ind}^G_{KZ}Q \) and therefore to zero in \( \text{ind}^G_{KZ}J_3 \). Similarly \( X^{p^r-p} \) maps to zero in \( J_3 \), since \( \theta Y^{r-p-1} \) maps to 0 in \( J_3 \), by part (ii) of Lemma 2.2.

Also, by part (iii) of the same lemma, \( \theta X^{r-p-2}Y \) maps to \( X^{p^r-2} \) in \( J_3 \). Finally, by (7.11), \( X^{r-2}Y^2 \) maps to \( (2-r)X^{p^r-2} \) maps to \( J_3 \).

Therefore, the image of \( (T - a_p)f \) in \( \text{ind}^G_{KZ}J_3 \) is

\[ (T - a_p)f = (-1 + 2)[1, X^{p^r-2}] - \frac{2c}{(2-r)(3-r)p} \sum_{\lambda \in \mathbb{F}_p} [g_{1,\lambda}^0, X^{p^r-2}] + \sum_{\lambda \in \mathbb{F}_p} \sum_{\mu \in \mathbb{F}_\mu} [g_{2,\lambda}^0, p^r], X^{p^r-2} \]
\[ = (T^2 - dT + 1)[1, X^{p^r-2}], \]
where \( d = \frac{2c}{(2-r)(3-r)p} \). As \( (T - a_p)f \) maps to zero in \( \tilde{\Theta}_{KZ} \), and \( [1, X^{p^r-2}] \) generates \( \text{ind}^G_{KZ}J_3 \), we get that \( (T^2 - dT + 1)\text{ind}^G_{KZ}J_3 \subset \ker(\text{ind}^G_{KZ}J_3) \to F_3 \).

Therefore if \( \tau = t + 1 \), then \( F_3 \) is a quotient of \( \frac{\text{ind}^G_{KZ}J_3}{T^2-dT+1} \) and if \( \tau > t + 1 \), then \( F_3 \) is a quotient of \( \frac{\text{ind}^G_{KZ}J_3}{T^2+1} \), since \( d = 0 \). \( \square \)

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