Radial power-law position-dependent mass: cylindrical coordinates, separability and spectral signatures

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Received 17 May 2011
Published 10 August 2011
Online at stacks.iop.org/JPhysA/44/355303

Abstract

We discuss the separability of the position-dependent-mass (PDM) Hamiltonian in cylindrical coordinates in the framework of a radial power-law PDM. We consider two particular radial mass settings: a harmonic oscillator type and a Coulombic type. We subject the radial harmonic oscillator-type mass to a radial harmonic oscillator potential and the radial Coulombic mass to a radial Coulombic potential. Azimuthal symmetry is assumed and spectral signatures of various z-dependent interaction potentials are reported.

PACS numbers: 03.65.Ge, 03.65.Ca

1. Introduction

The von Roos Hamiltonian [1] is known to describe quantum particles with position-dependent mass (PDM), \( M(\vec{r}) = m_0 m(\vec{r}) \). Over the last few years, the PDM Hamiltonians have inspired research attention [2–32] because of their applicability in the study of the many-body problem, semiconductors, quantum dots, quantum liquids, etc. The kinetic energy operator in the von Roos Hamiltonian (with \( m_0 = \hbar = 1 \) units),

\[
H = -\frac{1}{2} [m(\vec{r})^\alpha \vec{\nabla} m(\vec{r})^\beta \cdot \vec{\nabla} m(\vec{r})^\gamma + m(\vec{r})^\alpha \vec{\nabla} m(\vec{r})^\beta \cdot \vec{\nabla} m(\vec{r})^\gamma] + V(\vec{r}),
\]

admits an ordering ambiguity manifested by the non-uniqueness representation of the kinetic energy operator, which would, in effect, introduce a profile change in the effective potential as the values of the parameters \( \alpha, \beta \) and \( \gamma \) change (cf, e.g., [25–29]). Here, \( \alpha, \beta \) and \( \gamma \) are called the von Roos ordering ambiguity parameters satisfying the von Roos constraint \( \alpha + \beta + \gamma = -1 \). Nevertheless, an interesting and comprehensive background on the issue of the consistency and usefulness of the PDM Schrödinger equation was given by Lévy-Leblond [32]. Therein, his work is devoted to sustaining and strengthening the conclusions that not only the use of PDM gives correct approximation, but it is also a conceptually consistent approach.
It is, however, constructive to mention that the continuity conditions at the abrupt heterojunction between two crystals implied that $\alpha = \gamma$; otherwise for $\alpha \neq \gamma$ the wavefunctions vanish at the boundaries and the heterojunction plays the role of impenetrable barrier (cf. e.g., Mustafa and Mazharimousavi [10] and Koc et al [28]). Eliminating in the process Gora’s and Williams’ ($\beta = \gamma = 0$, $\alpha = -1$), and Li’s and Kuhn’s ($\beta = \gamma = -1/2$, $\alpha = 0$) known parametric sets. Moreover, Dutra’s and Almeida’s [9] reliability test classifies the parametric set of Ben Daniel and Duke ($\alpha = \gamma = 0$, $\beta = -1$) as a set to-be-discarded for it yields imaginary eigenvalues. This would leave us with Zhu’s and Kroemer’s ($\alpha = \gamma = -1/2$, $\beta = 0$) and Mustafa’s and Mazharimousavi’s ($\alpha = \gamma = -1/4$, $\beta = -1/2$) ordering ambiguity parameters that are classified as ‘good’ parametric sets, so to speak. Nevertheless, we have observed (cf. e.g., [29]) that the physical and/or mathematical admissibility of a given ambiguity parametric set depends also on the form of the PDM and/or the form of the interaction potential. In the forthcoming methodical proposal, we shall work with the ambiguity parameters as they are without any classification as to which set is ‘good’ or ‘to-be-discarded’.

Very recently, Mustafa [31] has considered the von Roos Hamiltonian (1) using cylindrical coordinates. Therein, we sought some manifestly feasible separability through the suggestion that the PDM is only radial-dependent (i.e. $m(\rho) = m$. $M(\rho, \varphi, z) = M(\rho) = 1/\rho^2$), where azimuthal symmetrization is granted through a proper assumption of the interaction potential. The spectral signatures of different $z$-dependent interaction potential settings on the radial Coulombic and radial harmonic oscillator interaction potentials’ spectra are reported for impenetrable walls at $z = 0$ and $z = L$, for a Morse, non-Hermitian $PT$-symmetrized Scarf II and non-Hermitian $PT$-symmetrized Samsonov interaction models.

In this work, we offer a parallel azimuthal symmetrization along with a more general (though still only radial-dependent) power-law-type PDM (i.e. $M(\rho, \varphi, z) = M(\rho) = b\rho^{2\gamma+1}/2$). Obviously, $\nu = -3/2$ and $b = 2$ yield $M(\rho) \sim 1/\rho^2$ which is, under the current forthcoming settings, a special case of $M(\rho) = b\rho^{2/4} \sim 1/\rho^2$ that shall not be repeated here. Instead, we shall use $\nu = -1$ and $\nu = 1/2$ that yield quantum particles endowed with PDMs of a Coulombic type, $M(\rho) = b\rho^{-1}/2$, and a harmonic oscillator type, $M(\rho) = b\rho^2/2$, respectively. To the best of our knowledge, such PDM settings have not been considered elsewhere.

To make this work self-contained, we recollect (in section 2) the most relevant and vital relations (namely, equations (2)–(5) below) that have been readily reported in [31] for cylindrical coordinates separability and exact solvability of the PDM-Hamiltonian (1). In the same section, we discuss the separability in the framework of a manifestly radial power-law PDM and contemplate on the feasible separabilities. In section 3, we consider two particular radial mass settings; a harmonic oscillator type, $M(\rho, \varphi, z) = M(\rho) = g(\rho) = b\rho^2/2$, and a Coulombic type, $M(\rho, \varphi, z) = M(\rho) = g(\rho) = b\rho^{-1}/2$. We subject the radial harmonic oscillator-type mass to move under the influence of a radial harmonic oscillator potential field $\tilde{V}(\rho) = a^2\rho^2/4$ and the radial Coulombic mass to a radial Coulombic potential $\tilde{V}(\rho) = -2A/\rho$. The spectral signatures of (i) two impenetrable walls at $z = 0$ and $z = L$ provided by the potential well $\tilde{V}(z) = 0$ for $0 < z < L$ and $\tilde{V}(z) = \infty$ elsewhere, (ii) a Morse-type [31] interaction $\tilde{V}(z) = D(e^{-2z} - 2e^{-\epsilon z})$, $D > 0$, and (iii) a trigonometric Rosen–Morse [32] potential $\tilde{V}(z) = U. \cot^2(\pi z/d)$, $z \in [0, d]$, are reported in the same section. Our concluding remarks are given in section 4.

2. Cylindrical coordinates and the radial power-law PDM framework

Following our recent work [31] on cylindrical coordinates separability and exact solvability of the PDM Hamiltonian (1), we again consider the PDM and the interaction potential to
take the forms $m(\vec{r}) = M(\rho, \varphi, z) = g(\rho) f(\varphi) k(z)$ and $V(\vec{r}) = V(\rho, \varphi, z)$, respectively. We have shown (see Mustafa [31] for more details on this issue) that the corresponding PDM Schrödinger equation $[H - E]\Psi(\rho, \varphi, z) = 0$

$$\Psi(\rho, \varphi, z) = R(\rho) \Phi(\varphi) Z(z); \quad \rho \in (0, \infty), \quad \varphi \in (0, 2\pi), \quad z \in (-\infty, \infty),$$

(2)

would imply

$$0 = 2g(\rho)f(\varphi)k(z)[E - V(\rho, \varphi, z)] + \left[ \frac{R'(\rho)}{R(\rho)} - \left( \frac{g'(\rho)}{g(\rho)} - \frac{1}{\rho} \right) \frac{R'(\rho)}{R(\rho)} \right] + \frac{\zeta}{2} \left( \frac{g'(\rho)}{g(\rho)} \right)^2 - \frac{1}{2g(\rho)} \left( \frac{g'(\rho)}{g(\rho)} + \frac{g''(\rho)}{g(\rho)} \right)
+ \left[ \frac{Z''(z)}{Z(z)} - \frac{k'(z)}{k(z)} \right] \frac{\zeta}{2} \left( \frac{k'(z)}{k(z)} \right)^2
+ \frac{1}{\rho^2} \left[ \frac{\Phi''(\varphi)}{\Phi(\varphi)} - \frac{f'(\varphi)}{f(\varphi)} \frac{\Phi'(\varphi)}{\Phi(\varphi)} \right] + \frac{\zeta}{2} \left( \frac{f'(\varphi)}{f(\varphi)} \right)^2 - \frac{1}{2} \frac{f''(\varphi)}{f(\varphi)},$$

(3)

where

$$\zeta = \alpha(\alpha - 1) + \gamma(\gamma - 1) - \beta(\beta + 1).$$

(4)

To facilitate and ease separability, we have suggested that the interaction potential satisfies an obviously ‘manifested-by-equation (3)’ general identity of the form

$$2MV(\rho, \varphi, z) = 2g(\rho)f(\varphi)k(z)V(\rho, \varphi, z) = \tilde{V}(\rho) + \tilde{V}(\varphi) + \frac{1}{\rho^2} \tilde{V}(\varphi).$$

(5)

Hereby, we may remind the reader that in [31] we have used $g(\rho) = 1/\rho^2$ along with $f(\varphi) = 1 = k(z)$ as one of the options that secured separability of the problem at hand.

In the search for a more general recipe, however, we choose to eliminate the first-order derivatives $Z(z), \Phi'(\varphi)$ and $R'(\rho)$. At this point, the elimination of the first-order derivatives of $Z(z)$ and $\Phi(\varphi)$ is achieved through the substitutions

$$Z(z) = \sqrt{k(z)Z(z)} \quad \text{and} \quad \Phi(\varphi) = \sqrt{f(\varphi)\Phi(\varphi)},$$

(6)

and imply that

$$\frac{Z''(z)}{Z(z)} - \frac{k'(z)}{k(z)} \frac{Z'(z)}{Z(z)} = \frac{3}{4} \left( \frac{k'(z)}{k(z)} \right)^2 + \frac{1}{2} \frac{k''(z)}{k(z)} + \tilde{Z}(z),$$

(7)

and

$$\frac{\Phi''(\varphi)}{\Phi(\varphi)} - \frac{f'(\varphi)}{f(\varphi)} \frac{\Phi'(\varphi)}{\Phi(\varphi)} = \frac{3}{4} \left( \frac{f'(\varphi)}{f(\varphi)} \right)^2 + \frac{1}{2} \frac{f''(\varphi)}{f(\varphi)} + \tilde{\Phi}(\varphi).$$

(8)

On the other hand, the elimination of the first-order derivative of $R(\rho)$ may be sought through the substitutions

$$\rho U(\rho) \quad \text{and} \quad g(\rho) = \frac{b}{2} \rho^{2\nu + 1}, \quad \nu, b \in \mathbb{R}$$

(9)

(with the restriction that $b$ is a non-zero constant to avoid triviality) to imply that

$$\frac{R'(\rho)}{R(\rho)} - \left( \frac{g'(\rho)}{g(\rho)} - \frac{1}{\rho} \right) \frac{R'(\rho)}{R(\rho)} = \frac{U''(\rho)}{U(\rho)} - \frac{\nu(\nu + 1)}{\rho^2}.$$
Two particular radial settings: $g(\rho) = b\rho^{2+1}$ and $g(\rho) = b\rho^{-1}/2$

In this section, we consider the PDM $M(\rho, \varphi, z) = g(\rho)$ to indulge a radial harmonic oscillator $g(\rho) = b\rho^{2}/2$ (i.e. $\nu = 1/2$) and the radial Coulombic $g(\rho) = b\rho^{-1}/2$ (i.e. $\nu = -1$) forms. For keeping this work simple and instructive, we shall consider the radial harmonic oscillator $g(\rho) = b\rho^{2}/2$ accompanied by a radial harmonic oscillator-type interaction $\tilde{V}(\rho) = a^2\rho^2/4$ and the radial Coulombic $g(\rho) = b\rho^{-1}/2$ accompanied by a radial Coulombic $\tilde{V}(\rho) = -2A/\rho$. We shall moreover report the spectral signatures of different $\tilde{V}(z)$ potentials on the overall spectrum.
3.1. The radial harmonic oscillator $g(\rho) = b\rho^2/2$

The choice of $\nu = 1/2$ along with $\tilde{V}(\rho) = a^2\rho^2/4$ would imply that equation (14) reads

$$E = \frac{a^2}{4b} - \frac{1}{4b} \left[ \frac{k_z^2}{2n_\rho + \sqrt{(m^2 + 3) - 2(\zeta - \beta)}} \right]^2,$$

and (15), in turn, yields

$$|\tilde{\ell}_{1/2}| = \sqrt{(m^2 + 3) - 2(\zeta - \beta)}.$$  

(18)

Obviously, equation (17) has exact eigenvalues in the form

$$k_z^2 = -\sqrt{(a^2 - 4bE)}[2n_\rho + |\tilde{\ell}_{1/2}| + 1]^2,$$

and implies that

$$E = \frac{a^2}{4b} - \frac{1}{4b} \left[ \frac{k_z^2}{2n_\rho + \sqrt{(m^2 + 3) - 2(\zeta - \beta)}} \right]^2.$$  

(20)

We observe that an auxiliary constraint

$$(\zeta - \beta) = a(\alpha - 1) + \gamma(\gamma - 1) - \beta(\beta + 2) \leq (m^2 + 3)/2$$

(21)

on the ambiguity parameters is manifested here by the requirement that $E \in \mathbb{R}$.

3.1.1. Spectral signatures of some $\tilde{V}(z)$ potentials on the radial harmonic oscillator spectrum.

Recollect [31] that if our PDM particle is trapped to move between two impenetrable walls at $z = 0$ and $z = L$ under the influence of a

$$\tilde{V}(z) = \begin{cases} 0; & 0 < z < L \\ \infty; & \text{elsewhere,} \end{cases}$$

(22)

one would find that $K_z = n_z\pi/L$, $n_z = 1, 2, 3, \ldots$ (see [31] for more details on this issue).

This would, in effect, give the spectral signature of $\tilde{V}(z)$ of (22) on the overall spectrum

$$E = \frac{a^2}{4b} - \frac{1}{4b} \left[ \frac{(n_z\pi/L)^2}{2n_\rho + \sqrt{(m^2 + 3) - 2(\zeta - \beta)}} \right]^2,$$

(23)

for a PDM particle of $M(\rho, \varphi, z) = M(\rho) = b\rho^2/2$ moving in a potential of the form

$$V(\rho, \varphi, z) = \frac{a^2}{4b} + \begin{cases} 0; & 0 < z < L \\ \infty; & \text{elsewhere.} \end{cases}$$

(24)

Next, let us subject this PDM particle to move in a Morse-type [31] interaction $\tilde{V}(z) = D(e^{-2\epsilon z} - 2 e^{-\epsilon z})$, $D > 0$. In this case

$$k_z^2 = \left( \frac{\sqrt{D}}{\epsilon} - \tilde{n}_z - \frac{1}{2} \right)^2,$$

(25)

Therefore, a PDM quantum particle endowed with $M(\rho, \varphi, z) = M(\rho) = b\rho^2/2$ and subjected to an interaction potential of the form

$$V(\rho, \varphi, z) = \frac{a^2}{4b} + \frac{D}{b\rho^2}(e^{-2\epsilon z} - 2 e^{-\epsilon z}), \quad D > 0,$$

(26)

would admit exact energy eigenvalues given by

$$E = \frac{a^2}{4b} - \frac{1}{4b} \left[ \frac{(\sqrt{D}/\epsilon - \tilde{n}_z - \frac{1}{2})}{2n_\rho + \sqrt{(m^2 + 3) - 2(\zeta - \beta)}} \right]^2.$$  

(27)
Now, let $M(\rho, \varphi, z) = M(\rho) = b \rho^2 / 2$ move under the influence of a trigonometric Rosen–Morse potential $\tilde{V}(z) = U_0 \cot^2(\pi z / d)$, $z \in [0, d]$, where $U_0$ and $d$ are two positive parameters. In this case,

$$ V(\rho, \varphi, z) = \frac{a^2}{4b} + \frac{U_0}{b \rho^2} \cot^2(\pi z / d), \quad z \in [0, d], \quad (28) $$

$$ k_z^2 = \frac{1}{d^2} [Cd + \tilde{n} \pi]^2 - U_0, \quad C = \frac{\pi}{2d} \left( 1 + \sqrt{1 + \frac{4U_0 d^2}{\pi^2}} \right) \quad (29) $$

(see Ma et al [32] for more details; note that one should consider $2\mu = \hbar = 1$ of Ma as proper parametric mapping into our settings) and

$$ E = \frac{a^2}{4b} - \frac{1}{4b} \left[ \frac{[Cd + \tilde{n} \pi]^2 / d^2 - U_0}{2n_\rho + \sqrt{(m^2 + 3) - 2(\xi - \beta) + 1}} \right]^2. \quad (30) $$

3.2. The radial Coulombic $g(\rho) = b \rho^{-1/2}$

Now consider the PDM particle to have a radial Coulombic-type mass of the form $M(\rho, \varphi, z) = M(\rho) = b \rho^{-1/2}$, (i.e. $\nu = -1$) and subjected to move in a radial Coulombic potential $\tilde{V}(\rho) = -2 \tilde{A} / \rho$. In this case,

$$ V(\rho, \varphi, z) = -\frac{\tilde{A}}{\rho} + \rho \tilde{V}(z), \quad (31) $$

and equation (14) yields

$$ \left[ -\tilde{a}^2 + \frac{\tilde{\ell}_-^2 - 1/4}{\tilde{\rho}^2} - \frac{2(\tilde{b} E + \tilde{A})^2}{\tilde{\rho}} \right] U(\rho) = -k_z^2 U(\rho), \quad \tilde{b} = b/2, \quad (32) $$

with

$$ |\tilde{\ell}_-| = \sqrt{(m^2 + 3/4) - (\xi - \beta)^2} \quad (33) $$

and

$$ k_z = \pm \frac{\tilde{b} E + \tilde{A}}{(n_\rho + |\tilde{\ell}_-| + 1)}, \quad (34) $$

which, in turn, results in

$$ E = \pm \frac{k_z}{b} (n_\rho + \sqrt{(m^2 + 3/4) - (\xi - \beta)/2 + 1}) - \frac{\tilde{A}}{b}, \quad (35) $$

with the auxiliary constraint

$$ (\xi - \beta) = \alpha(\alpha - 1) + \gamma(\gamma - 1) - \beta(\beta + 2) \leq (2m^2 + 3/2) \quad (36) $$

on the ambiguity parameters that secure the reality of $E$. Nevertheless, two branches of energies are obviously obtained. Moreover, the spectral signature of $k_z$ on the overall spectrum is obtained through the solution of equation (13).
3.2.1. Spectral signatures of some $\tilde{V}(\rho)$ potentials on the radial Coulombic spectrum. If we subject our radial Coulombic PDM particle $M(\rho, \varphi, z) = M(\rho) = b\rho^{-1}/2$ to move in $V(\rho, \varphi, z) = -\tilde{A}/b + \rho\tilde{V}(z)/b$, where $\tilde{V}(z)$ is given by (22), it will admit exact energies of the form

$$E = \pm \frac{n_\pi}{bL}(n_\rho + \sqrt{(m^2 + 3/4) - (\xi - \beta)^2 + 1}) - \frac{\tilde{A}}{b}, \quad n_\rho = 1, 2, 3, \ldots$$  \quad (37)

Moreover, if this PDM particle is subjected to move in a Morse-type [31] interaction $\tilde{V}(z) = D(e^{-2\epsilon z} - 2e^{-\epsilon z})$, $D > 0$, then in this case,

$$V(\rho, \varphi, z) = -\frac{\tilde{A}}{b} + \frac{\rho}{b}D(e^{-2\epsilon z} - 2e^{-\epsilon z}), \quad D > 0,$$

and the exact energies are of the form

$$E = \pm \frac{1}{b} \left( \frac{\sqrt{D}}{\epsilon} - \tilde{n}_\rho - \frac{1}{2} \right)(n_\rho + \sqrt{(m^2 + 3/4) - (\xi - \beta)^2 + 1}) - \frac{\tilde{A}}{b}, \quad \tilde{n}_\rho = 0, 1, 2, 3, \ldots$$

where $\tilde{n}_\rho = 0, 1, 2, 3, \ldots$. Obviously, the condition $\left( \frac{\sqrt{D}}{\epsilon} - \tilde{n}_\rho - \frac{1}{2} \right) > 0$ is manifested here and ought to be enforced; otherwise complex pairs of energy eigenvalues are obtained in the process.

Next, let $M(\rho, \varphi, z) = M(\rho) = b\rho^{-1}/2$ move under the influence of a trigonometric Rosen–Morse potential $\tilde{V}(z) = U_z\cos^2(\pi z/d)$, $z \in [0, d]$. Then,

$$V(\rho, \varphi, z) = -\frac{\tilde{A}}{b} + \frac{\rho}{b}U_z\cos^2(\pi z/d), \quad z \in [0, d],$$

and

$$E = \pm \sqrt{\left[ C\tilde{d} + \tilde{n}_\pi \right]^2 - U_z d^2} \left( n_\rho + \sqrt{(m^2 + 3/4) - (\xi - \beta)^2 + 1} \right) - \frac{\tilde{A}}{b}.$$ \quad (41)

4. Concluding remarks

We have recollected the most relevant and vital relations (equations (2)–(5) above) that have been readily reported by Mustafa [31] for cylindrical coordinates separability of the PDM Hamiltonian in (1), where the PDM setting was considered in the form $M(\rho, \varphi, z) = g(\rho)f(\varphi)k(z) = g(\rho) = 1/\rho^2$ under azimuthally symmetric settings.

In this work, however, we offered a more general power-law radial PDM recipe $M(\rho, \varphi, z) = g(\rho) = b\rho^{2u+1}/2; u, b \in \mathbb{R}$, within which $M(\rho, \varphi, z) = g(\rho) = 1/\rho^2$ of [31] represents a special case (the results and examples reported therein hold true and yet document additional examples on the applicability of the current methodical proposal, therefore). Moreover, the structure of the position-dependent energy term $b\rho^{2u+1}f(\varphi)k(z)E$ in (11) suggests that there are four feasible cases towards separability: (i) $f(\varphi) = 1 = k(z)$, (ii) $k(z) = 1 = g(\rho)$ and $f(\varphi) \neq 1$, (iii) $f(\varphi) = 1 = g(\rho)$ and $k(z) \neq 1$ and (iv) $u = -3/2$, $k(z) = 1$, $f(\varphi) \neq 1$ (which would break azimuthal symmetry as in case (ii), of course). Therefore, the separability of (3) may be facilitated by the forms of the position-dependent mass and the interaction potential $V(\rho, \varphi, z)$. These are not the only cases to secure separability of (3), so to speak.

We have considered two particular mass settings: a radial harmonic oscillator-type, $M(\rho, \varphi, z) = M(\rho) = g(\rho) = b\rho^{2}/2$, and a radial Coulombic type, $M(\rho, \varphi, z) = M(\rho) = g(\rho) = b\rho^{-1}/2$. We have observed that for the Coulombic case two branches of energies are obtained, each of which is a ‘mirror-reflection’ of the other about the zero-energy axis.
Moreover, when we subjected the radial harmonic oscillator mass to a radial harmonic oscillator potential \( V(\rho) = a^2 \rho^2 / 4 \) and the radial Coulombic mass to a radial Coulombic potential \( V(\rho) = -2A/\rho \), only constant shifts in the energies were observed (i.e. a shift \( (a^2/4b) \) for the radial harmonic oscillator mass and \( (-A/b) \) for the radial Coulombic mass, documented in (20) and (35), respectively). That is, the radial interaction potentials \( \tilde{V}(\rho) \) considered for the two over-simplified examples here provided no quantization recipe at all (i.e. they have only introduced constant shifts to the energies but not discrete quantum energy shifts). This is because of the form of the general interaction potential \( V(\rho, \varphi, z) \) that we have adopted in (5).

Yet, auxiliary constraints on the ambiguity parameters (see (21) for the harmonic oscillator and (36) for the Coulombic) are observed mandatory to secure the reality of \( E \). Hereby, if \( m = 0 \) is considered in (21) and (36) as a reference test, then one would observe that only Gora’s and Williams’ ambiguity parametric set \( (\beta = \gamma = 0, \alpha = -1) \) fails to provide real energies (i.e. \( \sqrt{3}/2 - \langle \xi - \beta \rangle \in \mathbb{C} \)). We contemplate that more auxiliary constraints on the ambiguity parameters should be anticipated for different, though exactly solvable, power-law-type radial masses (within our methodical proposal, of course). Furthermore, the spectral signatures of different \( \tilde{V}(z) \) interactions on the overall spectrum are also reported. Namely, the spectral signatures of (i) two impenetrable walls at \( z = 0 \) and \( z = L \) provided by the potential well \( \tilde{V}(z) = 0 \) for \( 0 < z < L \) and \( \tilde{V}(z) = \infty \) elsewhere, (ii) a Morse-type \([31]\) interaction \( \tilde{V}(z) = D(e^{-2\alpha z} - 2 e^{-\alpha z}) \), \( D > 0 \), and (iii) a trigonometric Rosen–Morse \([32]\) potential \( \tilde{V}(z) = U, \cot^2(\pi z/d), z \in [0, d] \).

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