Symmetries of Multipartite Entanglement Evolution in Many-Sided Local Channels

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Symmetries of the initial state of a quantum system and the quantum channels, which simultaneously affect parts of the system, can significantly simplify the description of the entanglement evolution. Using concurrence as the entanglement measure, we study the entanglement evolution of few qubit systems, when each of the qubits is affected by a local channel independently of the others. We show that, for low-rank density matrices of the final quantum state, such complex entanglement dynamics can be completely described by a combination of independent factors representing the evolution of entanglement of the initial state, when just one of the qubits is affected by a local channel.

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I. INTRODUCTION

Current development of quantum technologies suggests quantum entanglement as the exclusive resource for many potential applications, such as quantum teleportation, superdense coding, quantum cryptography and quantum computing [1]. However, apart from entanglement generation [2] and detection [3], a successful practical utilization of entanglement-based quantum technologies demands efficient protocols for entanglement protection from detrimental environmental influence [4] and its recovery after a possible partial loss [5,6]. The construction of the protocols, in turn, requires exact methods for entanglement quantitative description as well as clear understanding of the fundamental laws of the entanglement evolution. The lack of accurate entanglement measure for multipartite entangled systems imposes serious limitations on our ability to describe complex entanglement dynamics. Therefore, any realistic situation, when the description of the entanglement evolution of a multipartite quantum system can be simplified, is of great practical importance.

An important example of such a simplified description of complex entanglement dynamics was given by Konrad et al. [10]. It was shown, in particular, that the entanglement evolution of an arbitrary pure two-qubit state can be completely described by two factors, which are given by the initial entanglement of the pure state and the entanglement dynamics of the maximally entangled state. Subsequently, this result has been extended to the cases of high-dimensional bipartite systems [11,12], mixed initial states [13] and multiqubit systems [15]. All these results, however, were obtained under assumption that just one subsystem of the entangled system undergoes the action of an environmental channel (i.e. the system is affected by a single-sided channel). In practice, however, it is often required to distribute parts of an entangled system between several remote recipients [16]. In this case, each subsystem is coupled locally with some environmental channel, i.e. the quantum system is the subject of many-sided channels. Recently, we analyzed the entanglement dynamics of initially pure three-qubit Greenberger-Horne-Zeilinger (GHZ) state, when each qubit is simultaneously affected by a noisy channel [17]. We showed that, in some cases, the entanglement dynamics of the three-qubit system in many-sided channels can be completely described by factors, which represent the evolution of the entangled system in single-sided channels. Similar result has been independently obtained by Man et al. [18] for generalized multiqubit GHZ states. Moreover, using so-called G-concurrence [19] as the entanglement measure, Gheorghiu and Gour [20] have recently shown that the average loss of entanglement induced by many-sided local channels is independent on the initial state and is completely defined by the local channels.

In this paper, we analyze the entanglement evolution of two-, three- and four-qubit systems affected by local many-sided channels. Using Wootters’s concurrence [21] for two qubits and its extension to higher dimensions [22] for multiqubit systems, we show that, for low-rank density matrices of the final state, such a complex entanglement dynamics can be completely described by factors representing single-sided entanglement dynamics of the initial state. For two qubits, in particular, we show that the above factorization can be achieved, if the rank of the final state density matrix is two. For multiqubit systems, in contrast, the factorization is possible for density matrices with rank no higher then four. On analytical examples and by numerical simulations we show that the factorization is independent on the initial (pure) quantum state of the qubit system and the local channels, as far as the above rank conditions are fulfilled. Since it is generally difficult to generate a low rank final state density matrix out of an arbitrary initial multiqubit state, for three and four qubit systems, we shall assume first that the initial state is a maximally entangled state, either GHZ or W state. The symmetry of the initial states allows us generating final state density matrices with all possible ranks for arbitrary local channels. Later we shall

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relieve this latter assumption extending our results to the case of arbitrary initially pure state of the qubits.

This work is organized as follows. In the next section we shall briefly describe the entanglement measures of use and introduce the quantum operation formalism \cite{23,22} that allows us to access the state dynamics of quantum systems under the action of local noisy channels. In Sec. \ref{III} we step-by-step analyze the entanglement dynamics of two-, three- and four-qubit systems affected by local many-sided channels and show examples when such a complex entanglement dynamics can be factorized on terms representing single-sided entanglement evolution. We conclude in Sec. \ref{IV} with a summary of our results and a discussion of their possible implications to theoretical and experimental description of the entanglement dynamics.

\section{Concurrence and Quantum State Dynamics}

\subsection{The entanglement measure}

It has been found difficult to quantify the entanglement of mixed many-partite states, and no general solution is known \cite{1} apart from Wootters’ concurrence \cite{21}. The Wootters’ concurrence allows us to compute the entanglement of an arbitrary state of a two-qubit system, which is given by the density matrix \( \rho \), as \( C_W = \max\{0, \lambda^1 - \lambda^2 - \lambda^3 - \lambda^4\} \). Here \( \lambda^i \) are the square roots of the four eigenvalues of the non-Hermitian matrix \( \rho (\sigma_y \otimes \sigma_y)^\dagger (\sigma_y \otimes \sigma_y) \), if taken in decreasing order. It is important to note that this matrix is obtained from the density matrix \( \rho \) by simultaneous inversion of the single-qubit subsystems with the help of the only generator \( \sigma_y \) of the SO(2) group.

Various extensions of Wootters’ concurrence have been worked out over the years \cite{1}. Ot et al. \cite{22}, in particular, suggested a generalization of Wootters’ concurrence for bipartite states, if the dimensions of the associated Hilbert subspaces are larger than two. For a \( d_1 \otimes d_2 \)-dimensional quantum system, this concurrence can be written as

\[
C = \sqrt{\frac{d_1(d_1-1)/2 d_2(d_2-1)/2}{\sum_{m=1}^{d_1} \sum_{n=1}^{d_2} (C_{mn})^2},
\]

where each term \( C_{mn} \) is given by

\[
C_{mn} = \max\{0, \lambda^1_{mn} - \lambda^2_{mn} - \lambda^3_{mn} - \lambda^4_{mn}\}.
\]

Here, the \( \lambda^k_{mn}, k = 1,4 \) are the square roots of the four nonvanishing eigenvalues of the matrix \( \rho \bar{\rho}_{mn} \), if taken in decreasing order. These matrices \( \rho \bar{\rho}_{mn} \) are formed by means of the density matrix \( \rho \) and its complex conjugate \( \rho^* \), and are further transformed by the operators \( S_{mn} = L_m \otimes L_n \) as: \( \bar{\rho}_{mn} = S_{mn} \rho^* S_{mn} \). In this notation, moreover, \( L_m \) are the \( d_2(d_2-1)/2 \) generators of the group \( \text{SO}(d_1) \), while the \( L_n \) are the \( d_2(d_2-1)/2 \) generators of the group \( \text{SO}(d_2) \).

Although the bipartite concurrence \cite{11} reduces to Wootters’ concurrence for the special case of two qubits, in general it is an approximate entanglement measure, which provides limited information about entanglement of the bipartite system \cite{23}. While the dimensionality of the Hilbert space of a two-qubit system is four, the inversion of an arbitrary state \( \rho \) is unambiguously defined by the single generator of the SO(2) group. In higher dimensions, however, there is no unique way to invert a given quantum state \cite{23}. Ambiguous choice of the state inversion leads to the summation over all possible \( d_1(d_1-1) d_2(d_2-1) / 4 \) state inversions in Eq. \( \text{(1)} \) in all \( 2 \otimes 2 \)-dimensional subspaces of the original \( d_1 \otimes d_2 \)-dimensional Hilbert state space of the bipartite system. The main consequence of such approximation for state inversion is that there may be only four nonzero eigenvalues of the matrix \( \rho \bar{\rho}_{mn} \), while the other \( d_1 d_2 - 4 \) eigenvalues of this matrix always vanish.

In spite of the above limitations, concurrence \text{(1)} has been shown to be quite powerful measure of entanglement \cite{8,12,14}. Using the bipartite concurrence Li et al. \cite{20} formulated an analytical lower bound for multiqubit concurrence, which is given by a squared sum of the bipartite concurrences computed for all possible bi-partitioning of the multiqubit system. For three qubits, in particular, the lower bound can be written in terms of the three bipartite concurrences that correspond to possible cuts of two qubits from the remaining one, i.e.

\[
\tau_3(\rho) = \frac{1}{3} ((C^{12|3})^2 + (C^{13|2})^2 + (C^{23|1})^2).
\]

This lower bound, moreover, has been used to describe the entanglement dynamics of three-qubit states under the action of certain multi-sided noisy channels \cite{24,25}. On particular analytical examples and by numerical simulations, it has been shown that for three-qubit density matrices with rank no higher then four, the lower bound \text{(3)} provides adequate description of the entanglement evolution irrespective from system-channel coupling rate and for all times of interaction. For density matrices with higher ranks, however, the lower bound vanishes after a finite time, while the quantum states it is applied to are not separable, i.e. possess certain amount of entanglement. This behavior of the lower bound \text{(3)} is not the consequence of the entanglement sudden death \cite{29}, but is induced by the approximate character of the bipartite concurrence \text{(1)} as an entanglement measure.

\subsection{Quantum Operation Formalism}

Quantum operation formalism is a very general and prominent tool to describe how a quantum system has been influenced by its environment. According to this formalism the final state of the quantum system, that is
coupled to some environmental channel, can be obtained from its initial state with the help of (Kraus) operators
\[ \rho_{\text{fin}} = \sum_i K_i \rho_{\text{ini}} K_i^\dagger, \]
and the condition \[ \sum_i K_i^\dagger K_i = I \] is fulfilled. Note that we consider only such system-environment interactions that can be associated with completely positive trace-preserving maps [24].

If the quantum system of interest consists of just a single qubit, which is subjected to some environmental channel \( A \), then an arbitrary quantum operation associated with the channel’s action can be expressed with the help of at most four operators [24]. Let us define the four operators through the Pauli matrices as
\[ K_1(a_1) = \frac{a_1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K_2(a_2) = \frac{a_2}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]
\[ K_3(a_3) = \frac{a_3}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad K_4(a_4) = \frac{a_4}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]
where \( a_i \) are real parameters and the condition \[ \sum_{i=1}^4 a_i^2 = 1 \] holds. Following standard notations [24], the channel with parameters \( a_1 \neq 0, a_2 \neq 0 \) and \( a_3 = a_4 = 0 \) is called bit flip (BF); if \( a_1 \neq 0, a_4 \neq 0 \) and \( a_2 = a_3 = 0 \) the channel is called phase flip (PF), while the channel with \( a_1 \neq 0, a_3 \neq 0 \) and \( a_2 = a_4 = 0 \) is bit-phase flip (BPF). These notations and definitions shall be used along the paper.

III. FACTORIZATION OF CONCURRENCE

As we mentioned in the Introduction, our ability to simplify the description of the entanglement evolution strongly depends on the symmetries of the qubits-environment system. Therefore, at the beginning we shall assume that the quantum system is initially prepared in a maximally entangled state, i.e. in a Bell state for two qubits and either in GHZ or W state for three and four qubits. At the same time we do not impose any limitations on the single-qubit channels, apart from their representation by operators [13]. Moreover, if in the text, for example, two single-qubit channels \( A \) and \( B \) are both BF channels, it is assumed that in general these channels are not equivalent, i.e. \( a_1 \neq b_1 \) and \( a_2 \neq b_2 \).

A. Two qubits

Suppose, each qubit of the two-qubit system, initially prepared into a Bell state \( |\phi\rangle = 1/\sqrt{2} (|00\rangle + |11\rangle) \), is affected by the BF channels \( A \) and \( B \). Using the definitions for the BF channel [5] and Wootters’s concurrence we obtain that
\[ C_W(|A \otimes B| |\phi\rangle \langle \phi|) = C_W([1 \otimes A] |\phi\rangle \langle \phi|) C_W([1 \otimes B] |\phi\rangle \langle \phi|), \]
i.e. the entanglement dynamics of the two-qubit system affected by the two-sided channel \([A \otimes B]\) is completely defined by the action of single-sided channels \([1 \otimes A]\) and \([1 \otimes B]\) on the initial state \(|\phi\rangle\). It is easy to verify that the factorization (8) holds true also if both channels \( A \) and \( B \) are PF or BPF.

In all the cases above the final state density matrix \([A \otimes B]|\phi\rangle \langle \phi|\) has rank two. We have numerically generated channels \( A \) and \( B \) and confirmed that for any final density matrix with rank two, the factorization (8) remains valid irrespective of the channels \( A \) and \( B \). Moreover, the factorization (8) is possible even when the initial state is not the maximally entangled state, but an arbitrary pure state. For example, one may verify that if \(|\varphi\rangle = 1/\sqrt{2} (\alpha |00\rangle + \sqrt{1 - \alpha^2} |11\rangle)\) and the channels \( A \) and \( B \) are both BF, the final state density matrix is of rank two and Eq. (8) is still a valid decomposition. For nonmaximally entangled initial states \(|\varphi\rangle\), however, an additional factor \( C_W(\varphi) \) appears in the right hand side of Eq. (9). This factor describes only the initial entanglement of these states and does not cause any impact on the description of the entanglement dynamics.

B. Three qubits

In contrast to two qubits, there are two maximally entangled states for three-qubit systems [30], which are typically written in the computational basis as
\[ |\text{GHZ}_3\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle), \]
\[ |W_3\rangle = \frac{1}{\sqrt{3}} (|010\rangle + |001\rangle + |100\rangle). \]

The entanglement dynamics of these states in local channels can be described, for example, by the bipartite concurrence \( C_{12|3} \) that correspond to the separation of two qubits of the three-qubit system from the remaining one. Let us consider the three-qubit system initially prepared in the GHZ state (7). If each of the qubits is simultaneously affected by the PF channels, the final state \([A \otimes B \otimes C]|\text{GHZ}_3\rangle\) \( |\text{GHZ}_3\rangle\) is given by rank-2 density matrix. In this case, the bipartite concurrence \( C_{12|3} \) can be factorized in a way similar to Eq. (9), i.e.
\[ C_{12|3}([A \otimes B \otimes C]|\psi\rangle \langle \psi|) = C_{12|3}(A) C_{12|3}(B) C_{12|3}(C), \]
where
\[ A \equiv [1 \otimes 1 \otimes A]|\psi\rangle \langle \psi|, \]
\[ B \equiv [1 \otimes 1 \otimes B]|\psi\rangle \langle \psi|, \]
\[ C \equiv [1 \otimes 1 \otimes C]|\psi\rangle \langle \psi|, \]
where \(|\psi\rangle\) is the initial state. If, in contrast, the three qubits are initially prepared into the GHZ state (7) and are all simultaneously affected
by BF channels, or two of the qubits undergo the action of
PF channels, while the remaining one is the subject of
BF or BPF, the final state density matrix has rank four
and the bipartite concurrence $C^{12\mid 3}$ can be factorized as

$$C^{12\mid 3}(A \otimes B \otimes C \mid \psi \rangle \langle \psi \mid) =$$

$$C^{12\mid 3}(A) \ C^{12\mid 3}(C) + C^{12\mid 3}(B) \ C^{12\mid 3}(C) . \quad (11)$$

This factorization (11) remains valid also for the rank-
3 density matrix $[A \otimes B \otimes C \mid W_4 \rangle \langle W_4 \mid$, where three
PF channels affect the initial W state $\tilde{W}$. By numerical
generation of the channels $A, B$ and $C$, we checked that
the factorization (11) is true for arbitrary channels
$A, B$ and $C$ affecting one of the initial states $\tilde{W}$ or $\tilde{W}$,
if the rank of the final state density matrix is three or
four. We also obtained a few numerical evidences that the
factorizations (9) and (11) are true even for nonmaxi-
mally entangled pure states of three-qubit systems. As
in the case of two qubits, which we discussed in the pre-
vious section, an additional factor appears in the right
hand side of Eqs. (9) and (11), if the initial state is not a
maximally entangled state. This factor gives the initial
entanglement of the three-qubit state.

Eqs. (9) and (11) have very different structure of the
right hand side. This can be understood from the de-
inition of the bipartite concurrence $C^{12\mid 3}$. As we men-
tioned in Sec. 4A because of lack of the unique state
inversion in higher dimensions, the bipartite concurrence
takes into account all possible state inversions in all $2 \otimes 2$
-dimensional subspaces of the original Hilbert state space.
The bipartite concurrence $C^{12\mid 3}$ separates the three-qubit
system on two subsystems, so that two of three qubits
are unified in one subsystem. If the GHZ state (7) is
affected by three PF channels $A, B$ and $C$, both the
two-qubit subsystem affected by channels $A$ and $B$, and
the single-qubit subsystem affected by channel $C$ lie in
two-dimensional subspaces of the Hilbert space of three-
qubits. In this special case the state inversion is unique,
and there is only single term in the right hand side of
Eq. (9). If, in contrast, the GHZ state (7) is affected by
three channels $A, B$ and $C$, so that the final state
density matrix has rank four, the two-qubit subsystem
subjected to the action of the channels $A$ and $B$ be-
longs to a four-dimensional subspace, while the single-
qubit subsystem affected by channel $C$ is the subject of
a two-dimensional subspaces. This leads to two terms in
the factorization (11) caused by unambiguous choice of
the two-dimensional subspace from the four-dimensional
subspace of the two-qubit subsystem.

C. Four qubits

Entanglement dynamics of three-qubit systems can be
quantified only with the bipartite concurrence $C^{12\mid 3}$$
where two qubits are separated from the remaining one.
For four-qubit system, in contrast, we may consider two
bipartite concurrences $C^{12\mid 3\mid 4}$ and $C^{12\mid 3\mid 4}$ and have a bet-
ter insight of the multiqubit entanglement dynamics in
many-sided channels and possible ways to simplify its
description through the factorization. As in the previous
section we shall focus on the two maximally entangled
states, i.e.

$$|GHZ_4\rangle = \frac{1}{\sqrt{2}} (|0000\rangle + |1111\rangle) , \quad (12)$$

$$|W_4\rangle = \frac{1}{2} (|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle) . \quad (13)$$

Let us assume that the four-qubit GHZ state (12) is
affected by four PF channels. The final state density
matrix $[A \otimes B \otimes C \otimes D \mid |GHZ_4\rangle \langle GHZ_4\mid \rangle$ has rank two. In
this special case both bipartite concurrences $C^{12\mid 3\mid 4}$ and
$C^{12\mid 3\mid 4}$ factorize as

$$C^{12\mid 3\mid 4}(A \otimes B \otimes C \otimes D \mid |\psi \rangle \langle \psi \mid) =$$

$$C^{12\mid 3\mid 4}(A) \ C^{12\mid 3\mid 4}(B) \ C^{12\mid 3\mid 4}(C) \ C^{12\mid 3\mid 4}(D) , \quad (14)$$

where $D$ is defined by analogy with Eqs. (10) as $D =
[1 \otimes 1 \otimes 1 \otimes D \mid |\psi \rangle \langle \psi \rangle$. This factorization mimics the
structure of Eqs. (10) and (11), which were obtained for
two- and three-qubit systems, i.e. there is only single
term in the right hand side of this equation.

If, however, the final state density matrix has rank four,
the concurrences $C^{12\mid 3\mid 4}$ and $C^{12\mid 3\mid 4}$ factorize differen-
tly. The rank four density matrix can be obtained, for
example, by the action of three PF channels and single
BF channel on the initial GHZ state (12) or by action of
four PF channels on W state (13). In these cases we have

$$C^{12\mid 3\mid 4}(A \otimes B \otimes C \otimes D \mid |\psi \rangle \langle \psi \mid) =$$

$$C^{12\mid 3\mid 4}(A) \ C^{12\mid 3\mid 4}(D) + C^{12\mid 3\mid 4}(B) \ C^{12\mid 3\mid 4}(D) +$$

$$C^{12\mid 3\mid 4}(C) \ C^{12\mid 3\mid 4}(D) , \quad (15)$$

$$C^{12\mid 3\mid 4}(A \otimes B \otimes C \otimes D \mid |\psi \rangle \langle \psi \mid) =$$

$$C^{12\mid 3\mid 4}(A) \ C^{12\mid 3\mid 4}(C) + C^{12\mid 3\mid 4}(A) \ C^{12\mid 3\mid 4}(D) +$$

$$C^{12\mid 3\mid 4}(B) \ C^{12\mid 3\mid 4}(C) + C^{12\mid 3\mid 4}(B) \ C^{12\mid 3\mid 4}(D) . \quad (16)$$

The structure of these equations is similar to Eq. (11)
for three qubits and can be explained using the interpre-
tation given in the previous section: each factor in the
right hand side of these equations represent a particular
selection of a $2 \otimes 2$-dimensional subspace of the original
four-qubit state space.

Because a general four-qubit mixed state density
matrix has rank 16, it is difficult to generate low-rank den-
sity matrices even from the maximally entangled states.
Therefore, very few numerical results have been obtained
to confirm the factorizations (15)-(16) for an arbitrary
local channels and only for initial maximally entangled
states (12) and (13).
IV. RESULTS AND DISCUSSION

We have shown that the complex entanglement evolution of multiqubit systems in many-sided noisy channels can be factorized on terms representing the entanglement evolution in single-sided channels. It has been argued and confirmed on analytical examples and by numerical simulations that this factorization is independent on the local noisy channels and initial pure states of the qubit systems, and is solely defined by the rank of the final state density matrices. If the rank of the final state density matrix is two [cf. Eqs. (9), (10) and (11)], the complex multi-sided entanglement dynamics is given by a product of factors describing the single-sided entanglement evolution. For rank-3 and rank-4 density matrices [cf. Eqs. (11), (12) and (13)], in contrast, the entanglement dynamics is represented by a sum of terms giving different combinations of products representing the single-sided entanglement evolution.

In all the factorization equations mentioned above, the ranks of the density matrices in the right hand side are no higher then the ranks of the density matrices in the left hand side. This means that these equations provide low-rank decompositions of the mixed state density matrix with respect to (Wootter’s or bipartite) concurrence. These decompositions can be, in principle, used to approximate the complex entanglement dynamics, when the rank of the final state density matrix is higher then four. If, for example, the initial three-qubit GHZ state \(|\psi\rangle\) is affected by three BPF channels \(A, B\) and \(C\) (for simplicity we assume that these channels are identical, i.e. \(a_1 = b_1 = c_1 = \sqrt{1-p}\) and \(a_3 = b_3 = c_3 = \sqrt{p}\)), the final state density matrix \([A \otimes B \otimes C] |\psi\rangle \langle \psi| (|\psi\rangle\langle \psi|)\) has rank 8. The entanglement dynamics of this three-qubit state can be directly described with the lower bound [3]. If, in contrast, we assume that the right hand side of Eq. (9) is a valid decomposition for each bipartite concurrence \(C^{abc}\) (where \(a, b, c = 1, 3 \text{ and } a \neq b \neq c \neq a\)) the multi-sided entanglement dynamics is simply given by \(\tau_3 \left( [A \otimes B \otimes C] |\psi\rangle \langle \psi| (|\psi\rangle\langle \psi|) \right) = (1 - 2p)^3\). If, in contrast, we assume that the right hand side of Eq. (11) is a valid decomposition for the bipartite concurrence, then \(\tau_3 \left( [A \otimes B \otimes C] |\psi\rangle \langle \psi| (|\psi\rangle\langle \psi|) \right) = (1 - 2p)^2\).

Fig. 1 shows the comparison between the direct application of the lower bound to the final state density matrix \([A \otimes B \otimes C] |\psi\rangle \langle \psi| (|\psi\rangle\langle \psi|)\) and the behavior of the lower bound taking into account decompositions (9) and (11). The approximations of the lower bound with help of the factorizations (9) and (11) provide adequate description of the entanglement dynamics of the three-qubit state. Moreover, the lower bound vanish for \(p \approx 0.31\) due to the drawback of its construction, which we discussed in Sec. III and because of its application to the rank-8 density matrix \([A \otimes B \otimes C] |\psi\rangle \langle \psi| (|\psi\rangle\langle \psi|)\). At the same time the approximations exploiting Eqs. (9) and (11) nullify only for \(p = 0.5\). Using a separability criteria \(\mathcal{S}_3\) one may check that the state \([A \otimes B \otimes C] |\psi\rangle \langle \psi| (|\psi\rangle\langle \psi|)\), where the channels are the BPF, become separable only for \(p = 0.5\). Thus, the factorization approximations provide even better description of the long-time entanglement dynamics then the original lower bound [3] in applications to high-rank mixed state density matrices.

The main difficulty to construct the approximations using Eqs. (9) and (11) is to choose which of these equations should be used for the approximation, since they provide different description of the entanglement dynamics. In principle, it should be possible to construct an entanglement measure \(E\) that approximates a complex multi-sided entanglement dynamics \(E ([A \otimes \ldots \otimes Z] |\psi\rangle \langle \psi|)\) by a function of terms \(E(A), \ldots, E(Z)\) describing the single-sided entanglement dynamics, where \(A, \ldots, Z\) are defined by analogy with Eq. (10). However, the construction of such an entanglement measure requires further study and, therefore, shall be discussed elsewhere.

Finally, the derived factorization equations as well as the approximations to the entanglement dynamics based on these equations can substantially simplify experimental detection of the entanglement of a system undergoing complex multi-sided entanglement dynamics as well as experimental testing of realistic communication channels. It is known that the complexity of an entanglement witness (i.e. the number of local measurements) strongly depends on the rank of the density matrix representing the quantum state of interest. The factorization of the complex multi-sided entanglement dynamics on terms providing single-sided entanglement evolution allows reducing the complexity of the corresponding entanglement witnesses and thus simplify the procedure of entanglement detection and experimental quantitative description.
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