Graph Automorphism Shuffles from Pile-Scramble Shuffles

Kengo Miyamoto1 · Kazumasa Shinagawa1,2

Received: 1 September 2021 / Accepted: 20 February 2022 / Published online: 2 April 2022 © Ohmsha, Ltd. and Springer Japan KK, part of Springer Nature 2022

Abstract
A pile-scramble shuffle is one of the most effective shuffles in card-based cryptography. Indeed, many card-based protocols are constructed from pile-scramble shuffles. This paper aims to study the power of pile-scramble shuffles. In particular, for any directed graph \( G \), we introduce a new protocol called “a graph shuffle protocol for \( G \)”, and show that it can be implemented using pile-scramble shuffles only. Our proposed protocol requires \( 2(n + m) \) cards, where \( n \) and \( m \) are the numbers of vertices and edges of \( G \), respectively. The number of pile-scramble shuffles is \( k + 1 \), where \( 1 \leq k \leq n \) is the number of distinct degrees of vertices of \( G \). As an application, a random cut for \( n \) cards, which is also an important shuffle, can be implemented by \( 3n \) cards and two pile-scramble shuffles.

Keywords Secure computation · Card-based cryptography · Pile-scramble shuffles · Graph automorphisms

Introduction

Background

Let \( X, Y \) be finite sets, \( n \in \mathbb{N} \) a natural number, and \( f : X^n \rightarrow Y \) a function. Suppose that \( n \) players each having \( x_i \in X \) as input wish to know an output value \( f(x_1, x_2, \ldots, x_n) \in Y \) without revealing anything about their own inputs beyond the output value to other players. Secure computation protocols can solve this kind of situation. Secure computation, which was formalized by Yao [62, 63], plays an
important role in cryptography (cf. see the survey on secure computation by Lindell [21]).

Card-based cryptography [7, 8] is a kind of secure computation, which uses a deck of physical cards. Given a sequence of face-down cards (which is typically an encoding of input \((x_1, x_2, \ldots, x_n) \in X^n\)), a card-based protocol transforms it to an output sequence (which is typically an encoding of output \(f(x_1, x_2, \ldots, x_n) \in Y\)) by a bunch of physical operations on cards. One of the features of card-based cryptography is that it allows us to understand intuitively the correctness and security of a protocol, since we can actually perform the protocol by hands. For this reason, it is expected to be used as an educational material. Indeed, some universities [6, 22, 27] have actually used card-based cryptography as an educational material.

In card-based protocols, a shuffle, which is a probabilistic rearrangement, is allowed to apply to a sequence of cards. It is considered as the most crucial operation in card-based protocols since randomness from shuffles is the primary tool to obtain the security of protocols. Among shuffles, a \((pile)\) random cut (RC), a random bisection cut (RBC), and a pile-scramble shuffle (PSS) are the most effective shuffles\(^1\) in card-based cryptography. Indeed, most card-based protocols are constructed with these shuffles only (cf. protocols with RCs only [2, 3, 7, 8, 13, 17, 19, 24, 25, 34, 36, 37, 42, 43, 49, 52, 56, 61], protocols with RBCs only [29–31, 33, 38–40, 48, 53], protocols with PSSs only [4, 11, 16, 35, 41, 44, 45, 50, 51, 54], protocols with RCs and RBCs only [1, 15, 18, 26, 59, 60], protocols with RCs and PSSs only [5, 9, 20, 46, 47, 57, 58], and protocols with RBCs and PSSs only [12, 14, 28, 55]). With this background, it is essential to study further what can be done by these shuffles.

**Contribution**

In this paper, we show that graph shuffles can be implemented with PSSs. Let \(G\) be a directed graph.\(^2\) A graph shuffle for \(G\) is a shuffle that arranges a sequence of cards according to an automorphism of \(G\) chosen uniformly at random. Our main contribution is to construct a card-based protocol that achieves a graph shuffle for any graph \(G\). We call this a graph shuffle protocol for \(G\). The number of cards in our protocol is \(2(n + m)\), where \(n\) and \(m\) are the numbers of vertices and edges of \(G\), respectively. The number of shuffles (i.e., PSSs) in our protocol is \(|\text{Deg}_G| + 1\), where \(\text{Deg}_G\) is the set of vertex degree of \(G\) (see “Graph shuffle”). We remark that our protocol has one drawback: it requires to compute a graph isomorphism between \(G\) and its isomorphic graph \(G'\). In general, computing a graph isomorphism is a complex computational task (see also Remark 3.3). We conjecture that computing a graph isomorphism is inherent in implementing a graph shuffle. We left it as an open problem whether computing a graph isomorphism can be removed or not.

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\(^1\) Our classification focuses on the group structure of permutations: the cyclic groups (RCs) and the symmetric groups (PSSs). Since RBCs are historically important shuffles and the intersection of RCs and PSSs, we classify them as RC, RBC, and PSS.

\(^2\) We regard undirected graphs as directed graphs by identifying each undirected edge with two directed edges with opposite directions.
A class of graph shuffles includes many interesting shuffles (see “Implication of our protocol”). Indeed, RCs, RBCs, and PSSs are special cases of graph shuffles. In particular, a RC is a graph shuffle for a directed cycle graph. A straightforward corollary of our main result is that a RC can be implemented with PSSs. Since a PSS can be implemented with RCs (cf. see Crépeau and Kilian’s [7] idea for generating a random fixed-free permutation), PSSs and RCs are essentially equivalent from the viewpoint of feasibility. It is worthwhile to mention the importance of the fact that RCs are implementable by PSSs. From the theoretical viewpoint, this shows that every protocol with RCs is transformed into a protocol with PSSs and vice versa. From the practical viewpoint, you can choose whether to use RCs or PSSs as shuffles in a protocol execution. To execute a RC by hand, we need to ensure that everyone must be able to verify that the rearrangement is indeed a cyclic shift while hiding the rearrangement itself. On the other hand, a PSS can be done by a rearrangement of piles in a completely randomly fashion although it requires physical envelopes as an additional tool. Which shuffle can be easily executable depends on a situation, and thus, there should be some cases that PSSs are more desirable than RCs.

Due to the importance of the result of RC, we improve a graph shuffle protocol for a directed cycle graph. In particular, for the directed cycle graph with $n$ vertices, we design a graph shuffle protocol with $3n$ cards while the general protocol requires $4n$ cards.

We also improve a graph shuffle protocol for an undirected cycle graph. A graph shuffle for the undirected cycle graph is equivalent to the dihedral shuffle, which is introduced by Niemi and Renvall [36]. For the undirected cycle graph with $n$ vertices, we design a graph shuffle protocol with $3n$ cards while the general protocol requires $6n$ cards.

**Related Works**

Koch and Walzer [17] showed that uniform closed shuffles (see Definition 2.1) can be implemented with RCs only. It is an essential milestone for implementing uniform closed shuffles. Since graph shuffles are uniformly closed, Koch and Walzer’s method allows that every graph shuffle can be done by RCs. However, we point out that their protocol requires each party somehow to generate a uniformly random element of a given group in the party’s head. This action is not allowed in the Mizuki–Shizuya model [32] which is known as the standard computational model of card-based cryptography. From this viewpoint, our protocol for graph shuffles and their protocol are based on different models of card-based cryptography. Our motivation is to implement a subclass of uniform closed shuffles in the Mizuki–Shizuya model. Besides the theoretical aspect, it is worthwhile to note that removing a randomness generation in the head brings a practical benefit for security because it is not clear how close the
distribution of random elements generated in the head will be to the distribution of truly random elements.

**Preliminaries**

In this section, we collect some fundamentals in card-based cryptography; see [32] for example.

**Cards**

Throughout this paper, we deal with physical cards with the symbol “?” on the backs. We use two collections of cards: black-cards $\begin{array}{c} 1 \ 2 \ 3 \end{array}$ and red-cards $\begin{array}{c} 1 \ 2 \ 3 \end{array}$ as follows:

| black-cards | red-cards |
|-------------|-----------|
| $\begin{array}{c} 1 \ 2 \ 3 \ 4 \ 5 \ 6 \end{array}$ | $\begin{array}{c} 1 \ 2 \ 3 \ 4 \ 5 \ 6 \end{array}$ |
| $\begin{array}{c} ? \ ? \ ? \ ? \ ? \ \end{array}$ | $\begin{array}{c} ? \ ? \ ? \ ? \ ? \ \end{array}$ |

We distinguish between the natural number $i$ (written in red) and the natural number $i$ (written in black). We denote by $\mathbb{N}_{\text{red}}$ the set of all natural numbers written in red, i.e., $\mathbb{N}_{\text{red}} = \{1, 2, 3, \ldots\}$. The set $\mathbb{N}_{\text{red}}$ is a totally ordered set using the natural order on $\mathbb{N}$. We define a totally order $\preceq$ on $\mathbb{N} \cup \mathbb{N}_{\text{red}}$ by

- $x, y \in \mathbb{N}$ and $x \preceq y$, where $\alpha = x$ and $\beta = y$,
- $\bar{x}, \bar{y} \in \mathbb{N}_{\text{red}}$ and $x \preceq y$, where $\alpha = \bar{x}$ and $\beta = \bar{y}$, or
- $\bar{x} \in \mathbb{N}_{\text{red}}$ and $y \in \mathbb{N}$, where $\alpha = \bar{x}$ and $\beta = y$.

A deck $D$ is a non-empty multiset such that $\{?\} \cap D = \emptyset$. Let $D$ be a deck. An expression $\begin{array}{c} x \ ? \end{array}$ (resp. $\begin{array}{c} ? \ x \end{array}$) with $x \in D$ is said to be a face-up card (resp. a face-down card) of $D$. A lying card $y$ of $D$ is the face-up card $y = \frac{x}{?}$ of $D$ or the face-down card $y = \frac{?}{x}$ of $D$, and in this case, we set atom$(y) = x$. A card-sequence from $D$ is a list of lying cards of $D$, say $(x_1, \ldots, x_n)$, such that $\{\text{atom}(x_i) \mid i = 1, 2, \ldots, n\} = D$ as multisets. For a card-sequence $x$, we write $x_i$ for the $i$-th term. A face-up card $\frac{x}{?}$ is represented by $\begin{array}{c} x \end{array}$ and a face-down card $\frac{?}{x}$ is represented by $\begin{array}{c} ? \end{array}$. Given a card $x$ with the expression $\frac{z}{y}$, we write front$(x) = y$, back$(x) = z$, and $\text{swap}(x) = \frac{y}{z}$. For a card-sequence $x = (x_1, \ldots, x_n)$ and a subset $T \subseteq \{1, 2, \ldots, n\}$, we define an operator $\text{turn}_T(\_)$ by

$$\text{turn}_T(x) = (y_1, \ldots, y_n), \quad y_i = \begin{cases} \text{swap}(x_i) & \text{if } i \in T, \\ x_i & \text{if } i \notin T. \end{cases}$$
The card-sequence \( \text{front}(x) = (\text{front}(x_1), \ldots, \text{front}(x_n)) \) is called the visible sequence of \( x \). Let \((T, \mathcal{G})\) be a pair of a collection of subsets of \( \{1, 2, \ldots, n\} \) (i.e., \( T \subseteq 2^{\{1,2,\ldots,n\}} \)) and a probability distribution on \( T \). Now, we also define an operation \( \text{rflip}_{(T,\mathcal{G})}(-) \) associated with the pair \((T, \mathcal{G})\) by

\[
\text{rflip}_{(T,\mathcal{G})}(x) = \text{turn}_T(x),
\]

where \( T \) is chosen from \( T \) depending on the probability distribution \( \mathcal{G} \). Note that if \( T = \{T\} \) with a subset \( T \subseteq \{1, 2, \ldots, n\} \), then \( \text{rflip}_{(T,\mathcal{G})}(-) = \text{turn}_T(-) \).

### Shuffles

For a natural number \( n \in \mathbb{N} \), we denote by \( \mathfrak{S}_n \) the symmetric group of degree \( n \), that is, the group whose elements are all bijective maps from \( \{1, 2, \ldots, n\} \) to itself, and whose group multiplication is the composition of functions. An element of the symmetric group is called a permutation.

Given a card-sequence \( x = (x_1, \ldots, x_n) \) and \( \sigma \in \mathfrak{S}_n \), we have a card-sequence \( \sigma(x) \) in the natural way:

\[
\sigma(x) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}).
\]

Now, we recall an operation on a card-sequence which is called a “shuffle”. Roughly speaking, a shuffle is a probabilistic reordering operation on a card-sequence. Let \((\Pi, \mathcal{F})\) be a pair of a subset of \( \mathfrak{S}_n \) and a probability distribution on \( \Pi \). For a card-sequence \( x = (x_1, \ldots, x_n) \), an operation \( \text{shuffle}_{(\Pi,\mathcal{F})}(-) \) associated with the pair \((\Pi, \mathcal{F})\) is defined by

\[
\text{shuffle}_{(\Pi,\mathcal{F})}(x) = \sigma(x).
\]

Here, \( \sigma \) is chosen according to the probability distribution \( \mathcal{F} \) on \( \Pi \). Note that when we apply a shuffle to a card-sequence, no one knows which permutation was actually chosen. We also note that if \( \Pi = \{\sigma\} \) for some \( \sigma \in \mathfrak{S}_n \), then \( \text{shuffle}_{(\Pi,\mathcal{F})}(-) = \sigma(-) \).

**Definition 2.1** A shuffle \( \text{shuffle}_{(\Pi,\mathcal{F})} \) is said to be uniform closed if \( \Pi \) is closed under the multiplication of the symmetric group, and \( \mathcal{F} \) is the uniform distribution on \( \Pi \).

All shuffles dealt with this paper are uniform closed shuffles.

**Example 2.2**

1. For a sequence of \( \ell \) cards, suppose that a subsequence of the sequence is divided into \( n \) piles of \( m \) cards. (It holds \( \ell \geq nm \).) A pile-scramble shuffle (PSS for short) is a uniform closed shuffle that completely randomly permutes \( n \) piles. The following shuffle is an example of a PSS:
We use $PSS_{(n,m)}$ to denote a PSS for $n$ piles each having $m$ cards. We remark that $PSS_{(n,m)}$ can be easily implemented by putting each pile into each physical envelope and then permuting them.

(2) Let $\pi_k \in \mathfrak{S}_n$ be the permutation

$$\pi_k = \begin{pmatrix} 1 & 2 & \ldots & k & k+1 & \ldots & n \\ n-k+1 & n-k+2 & \ldots & n & 1 & \ldots & n-k \end{pmatrix},$$

and set $\Pi = \{\pi_k \mid k = 1, 2, \ldots, n\}$. This uniform closed shuffle $\text{shuffle}_{(\Pi,F)}$ is called a random cut (RC for short).

**Protocols**

Mizuki and Shizuya [32] define the formal definition of a card-based protocol via an abstract machine. In this section, we recall the definition of a card-based protocol and introduce a shuffle protocol, which is a particular card-based protocol realizing a shuffle.

**Card-Based Protocols**

To put it briefly, a “protocol” is a Turing machine that chooses one of the following operations to be applied to a card-sequence $x$: turning ($x \mapsto \text{flip}_{(T,G)}(x)$) or shuffling ($x \mapsto \text{shuffle}_{(\Pi,F)}(x)$).

For a deck $D$, the set of all card-sequences from $D$ will be denoted by $\text{Seq}^D$. Then, the visible sequence set $\text{Vis}^D$ is defined as the set of all sequences $\text{front}(x)$ for $x \in \text{Seq}^D$. We also define the sets of the actions:

$turn^n = \{\text{turn}_T(-) \mid T \subseteq \{1, 2, \ldots, n\}\}$,

$perm^n = \{\sigma(-) \mid \sigma \in \mathfrak{S}_n\}$,

$SP^n = \{\text{shuffle}_{(\Pi,F)}(-) \mid F$ is a probability distribution on $\Pi \in 2^{\mathfrak{S}_n}\}$, and

$TP^n = \{\text{flip}_{(T,G)}(-) \mid G$ is a probability distribution on $T \subseteq 2^{\{1, 2, \ldots, n\}}\}$.

A protocol is a Markov chain, that is, a stochastic model describing a sequence of possible actions in which the probability of each action depends only on the state attained in the previous event. Let $Q$ be a finite set with two distinguished states, which are called an initial state $q_0$ and a final state $q_f$. 
Definition 2.3 A card-based protocol is a quadruple \( \mathcal{P} = (D, U, Q, A) \), where \( U \subseteq \text{Seq}^D \) is an input set and \( A \) is a partial action function

\[
A : (Q \setminus \{q_t\}) \times \text{Vis}^D \rightarrow Q \times (\text{turn}^n \cup \text{perm}^n \cup \text{SP}^n \cup \text{TP}^n),
\]

which depends only on the current state and visible sequence, specifying the next state and an operation on the card-sequence from \( (\text{turn}^n \cup \text{perm}^n \cup \text{SP}^n \cup \text{TP}^n) \), such that \( A(q_0, \text{front}(x)) \) is defined if \( x \in U \). For a state \( q \in Q \setminus \{q_t\} \) and a visible sequence \( y = \text{front}(x) \in \text{Vis}^D \) such that \( A(q, y) = (q', \text{act}_{q,y}) \), we obtain the next state \( (q', \text{front}(\text{act}_{q,y}(x))) \). By the above process, if we have \( (q_t, X) \in Q \times \text{Vis}^D \) for some \( X \in \text{Vis}^D \), the protocol \( \mathcal{P} \) terminates.

Let \( \mathcal{P} = (D, U, Q, A) \) be a card-based protocol. For an execution of \( \mathcal{P} \) with an input card-sequence \( x^{(0)} \in U \), we obtain a sequence of results of actions as follows:

\[
(q_0, x^{(0)}) \rightarrow (q_1, x^{(1)}) \rightarrow (q_2, x^{(2)}) \rightarrow (q_3, x^{(3)}) \rightarrow \ldots,
\]

where \( x^{(i)} = \text{act}_{q_{i-1}, \text{front}(x^{(i-1)})}(x^{(i-1)}) \) for \( i \geq 1 \). Here, \( q_i \) \((i = 0, 1, 2, \ldots)\) are not necessarily distinct. If the action function value \( A(q_i, x^{(i)}) \) is undefined for some \( i \in \mathbb{N} \), we say that “\( \mathcal{P} \) aborts at Step \( i \) in the execution”. Note that even for the same input card-sequence \( x^{(0)} \), the obtained chains may be different for each execution. If the protocol \( \mathcal{P} \) terminates for an input card-sequence \( x^{(0)} \), then we have a chain of results as follows:

\[
(q_0, x^{(0)}) \rightarrow (q_1, x^{(1)}) \rightarrow (q_2, x^{(2)}) \rightarrow (q_2, x^{(3)}) \rightarrow \ldots \rightarrow (q_f, x^{(f)}).
\]

In this case, \( x^{(0)} \) is called an initial sequence, \( x^{(f)} \) is called a final sequence, and the sequence

\[
(y^{(0)}, y^{(1)}, \ldots, y^{(f)}),
\]

where \( y^{(i)} = \text{front}(x^{(i)}) \), is called a visible sequence-trace of \( \mathcal{P} \). We denote by \( \text{Fin}(\mathcal{P}) \) the set of all final sequences, which is obtained by \( \mathcal{P} \).

Example 2.4 Let us consider the following. Take the deck \( D = \{1, 2, 3, 4\} \), and hence use as follows:

front:  
\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
? & ? & ? & ? \\
\end{array}
\]

back:  
\[
\begin{array}{cccc}
? & ? & ? & ? \\
? & ? & ? & ? \\
\end{array}
\]

Now, we give a card-based protocol \( \mathcal{P} = (D, \{\left(\begin{array}{c}
1 \\
? \\
\end{array}\right), \left(\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
? \\
? \\
\end{array}\right)\}, \{q_0, q_1, q_2, q_f\}, A) \) such that
In this case, the card-sequence \([1, 2, 3, 4]\) is changed by the protocol \(P\) as follows:

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\rightarrow & ? & ? & ? & ? \\
\rightarrow & ? & ? & ? & ? \\
\rightarrow & ? & ? & ? & ? \\
\end{array}
\]

Thus, the final sequence is \(？？？1？\).

**Shuffle Protocols**

A shuffle protocol\(^3\) is a card-based protocol realizing a shuffle operation. It takes a card-sequence \(x = (x_1, x_2, \ldots, x_n)\) such that \(\text{back}(x) = (？？？...)\) as input and outputs \(y = (y_1, y_2, \ldots, y_n)\) such that \(y = \sigma(x)\) for a permutation \(\sigma\) is chosen from \(\mathbb{S}_n\) depending on some probability distribution:

\[
\begin{array}{cccc}
? & ? & \cdots & ? \\
\rightarrow & h_1 & h_2 & \cdots & h_k \\
\rightarrow & ? & ? & \cdots & ? \\
\end{array}
\]

where \(h\) is a card-sequence of helping cards. Informally speaking, the correctness requires \(y = \text{shuffle}_{(1,5)}(x)\) and the security requires that no one learns nothing about the chosen permutation \(\sigma \in \Pi\).

**Definition 2.5** Let \(D_{\text{inp}}, D_{\text{help}}\) be decks, \(U_{\text{inp}}\) an input set from \(D_{\text{inp}}\), and \(h \in \text{Seq}^{D_{\text{help}}}\) a card-sequence from \(D_{\text{help}}\). We define an input set \(U\) from \(D = D_{\text{inp}} \cup D_{\text{help}}\) by \(U = \{(x, h) \mid x \in U_{\text{inp}}\}\). A card-based protocol \(P = (D, U, Q, A)\) is said to be a shuffle protocol if the following conditions are satisfied:

(a) \(P\) always terminates within a fixed number of steps, i.e., it is a finite-runtime protocol;
(b) for any input sequence \((x, h) \in U\) and for any final sequence \(y \in \text{Fin}(P)\) of the form \(y = (x', h)\), there exists a permutation \(\sigma \in \mathbb{S}_{|P_{\text{seq}}|}\) such that \(x' = \sigma(x)\);
(c) for any input sequence \((x, h) \in U\), any card contained in \(x\) has not been turned at any step of a protocol execution.

\(^3\) Koch and Walzer [17] considered a similar notion and proposed a protocol for any uniform closed shuffles. The main difference of their model and our model is that their model allows a randomness generation in the head (see “Related works” in Introduction).
We say that $\mathcal{P}$ implements a shuffle shuffle $\sigma_{[\Pi,\mathcal{F}]}$ if every permutation $\sigma$ in (b) belongs to $\Pi$ and it is chosen according to the distribution $\mathcal{F}$. We say that $\mathcal{P}$ is secure if for any $x \in U_{\text{inp}}$, a random variable of $\sigma$ is stochastically independent of the random variable of the visible sequence-trace of $\mathcal{P}$.

### Graph Shuffle Protocols

In this section, we construct a card-based protocol called the graph shuffle protocol for a directed graph. First, we introduce a graph shuffle in the next subsection. Second, we construct the graph shuffle protocol which is a shuffle protocol for any graph shuffle in the next following subsection. We note that our protocol requires PSSs only.

#### Graph Shuffle

First, we recall some fundamentals from graph theory; for example, see [10].

A directed graph is a quadruple $G = (V_G, E_G, s_G, t_G)$ consisting of two sets $V_G, E_G$ and two maps $s_G, t_G : E_G \to V_G$. Each element of $V_G$ (resp. $E_G$) is called a vertex (resp. an edge). Note that there might be two or more edges from $a$ to $b$ for some $a, b \in V_G$, that is, $G$ admits multiple edges. For an edge $e \in E_G$, we call $s_G(e)$ (resp. $t_G(e)$) the source (resp. the target) of $e$. We will commonly write $a \xrightarrow{e} b$ or $e : a \to b$ to indicate that an edge $e$ has the source $a$ and the target $b$, and identify $e$ with a pair $(s_G(e), t_G(e))$. A directed graph $G$ is finite if two sets $V_G$ and $E_G$ are finite sets. In this paper, a graph means a finite directed graph with $n$ vertices and $m$ edges.

Let $G$ be a graph. For a vertex $v \in V_G$, we define the following three functions:

$$\text{in}(v) = |\{ e \in E_G \mid v = t_G(e) \}|, \quad \text{out}(v) = |\{ e \in E_G \mid v = s_G(e) \}|, \quad \text{and} \quad \text{deg}(v) = \text{in}(v) + \text{out}(v).$$

The number $\text{deg}(v)$ is called the degree of $v$. We set $\text{Deg}_G = \{ \text{deg}(v) \mid v \in V_G \}$.

For graphs $G$ and $G'$, a pair $f = (f_0, f_1) : G \to G'$ consisting of maps $f_0 : V_G \to V_{G'}$ and $f_1 : E_G \to E_{G'}$ is a morphism of graphs if $(f_0 \times f_0) \circ (s_G \times t_G) = (s_{G'} \times t_{G'}) \circ f_1$ holds. In addition, if $f_0$ and $f_1$ are bijective, $f$ is called an isomorphism of graphs. In this case, we say that $G$ and $G'$ are isomorphic as graphs. In other words, two graphs $G$ and $G'$ are isomorphic as graphs when $x \xrightarrow{f} y$ in $G$ exists if and only if $f_0(x) \xrightarrow{f_1} f_0(y)$ exists in $G'$. We denote by $\text{Iso}(G, G')$ the set of all isomorphisms from $G$ to $G'$, and $\text{Iso}_0(G, G')$ the set of all $f_0$ such that $(f_0, \ast) \in \text{Iso}(G, G')$. For a graph $G$, an isomorphism from $G$ to itself is called an automorphism. We denote by $\text{Aut}(G)$ the set of all automorphisms of $G$, and $\text{Aut}_0(G)$ the set of all $f_0$ such that there exists $(f_0, f_1) \in \text{Aut}(G)$. Then it is obvious that $\text{Aut}(G)$ is a group by the composition of maps. Furthermore, the group structure of $\text{Aut}(G)$ induces the group structure on $\text{Aut}_0(G)$. Note that, if $G$ has no multiple edges, then an automorphism $f = (f_0, f_1) \in \text{Aut}(G)$ is determined by $f_0$. In the case that $G$ is an undirected graph, one can transform $G$ into the following directed graph $G$:

$$\begin{align*}
G' = (V_{G'}, E_{G'}) = (V_G, E_G, s_G, t_G) \quad &\text{with} \\
E_{G'} = \{ (u, v) \mid (u, v) \in E_G \text{ or } (v, u) \in E_G \}. 
\end{align*}$$
\[ V_G = V_G, \quad E_G = \{ i \to j, \; j \to i \mid (i, j) \in E_G \}. \]

**Definition 3.1** Let \( G \) be a graph. The uniform closed shuffle shuffle\(_{(Aut(G), \mathcal{E})}\) is called the **graph shuffle** for \( G \) over \( n \) cards. (Recall that \( G \) has \( n \) vertices.)

**Graph Shuffle Protocols**

In this subsection, we construct a graph shuffle protocol, which is a shuffle protocol of the graph shuffle for a graph \( G = (V_G, E_G, \mathcal{S}_G, t_G) \). We set \( V_G = \{1, 2, \ldots, n\} \). Let \( D_{\text{inp}} = \{x_1, x_2, \ldots, x_n\} \) be any deck, and \( U_{\text{inp}} \) any input set from \( D_{\text{inp}} \). We set a card-sequence \( h \) of helping cards as follows:

\[
0 \quad 2 \quad 3 \quad \cdots \quad \pi \quad 1 \quad \cdots \quad 1 \quad 2 \quad 2 \quad 3 \quad \cdots \quad 3 \quad \cdots \quad n \quad \cdots \quad n .
\]

Thus, the deck of helping cards is \( D_{\text{help}} = \{0, 2, \ldots, \pi, 1^\text{deg(1)}, 2^\text{deg(2)}, \ldots, n^\text{deg(n)}\} \), where the superscript denotes the number of symbols in the deck \( D_{\text{help}} \). The deck \( D \) is the union of \( D_{\text{inp}} \) and \( D_{\text{help}} \) as multisets and it consists of \( 2(n + m) \) symbols.

For an input card-sequence \( x \in U_{\text{inp}} \), our protocol proceeds as follows:

1. **Place the cards as follows.**

\[
\begin{array}{cccccccc}
? & ? & \cdots & ? & ? & ? & ? & ? \\
\times & & & & & & & \\
\end{array}
\]

2. **For each \( i \), we define \( \text{pile}[i] \) by**

\[
\text{pile}[i] = \left( \frac{?}{\pi}, \frac{?}{1}, \ldots, \frac{?}{i} \right) = \left( \frac{?}{i}, \frac{?}{i}, \ldots, \frac{?}{i} \right).
\]

Arrange the card-sequence as \((x, \text{pile}[1], \text{pile}[2], \text{pile}[3], \ldots, \text{pile}[n])\), that is:

\[
\begin{array}{cccccccc}
? & ? & \cdots & ? & ? & ? & ? & ? \\
\times & \text{pile}[1] & \text{pile}[2] & \text{pile}[3] & \text{pile}[n] & & & \\
\end{array}
\]

3. **For each \( d \in \text{Deg}_G \), we set \( V_G^{(d)} = \{v_1^{(d)}, v_2^{(d)}, \ldots, v_{\ell_d}^{(d)}\} \) for all vertices with degree \( d \), and apply \( \text{PSS}_{(\ell_d, d+1)} \) to the card-sequence \((\text{pile}[v_1^{(d)}], \text{pile}[v_2^{(d)}], \ldots, \text{pile}[v_{\ell_d}^{(d)}])\). Then we obtain a card-sequence**

\[
\begin{array}{cccccccc}
? & ? & \cdots & ? & ? & ? & ? & ? \\
\times & \text{pile}[\alpha_1] & \text{pile}[\alpha_2] & \text{pile}[\alpha_3] & \text{pile}[\alpha_n] & & & \\
\end{array}
\]

Let \( \sigma \in \mathfrak{S}_n \) be the chosen permutation such that \( \alpha_i = \sigma^{-1}(i) \).
(4) For each $i \in V_G$ and $j \rightarrow k \in E_G$, we set $\text{vertex}[i] = \left( \frac{?}{\alpha_i}, x_j \right)$ and $\text{edge}[j \rightarrow k] = \left( \frac{?}{\alpha_j}, \frac{?}{\alpha_k} \right)$, respectively. Arrange the card-sequence\(^4\) as follows:

\[
\begin{array}{cccccccc}
\text{vertex}[1] & \text{vertex}[2] & \cdots & \text{vertex}[n] & \text{edge}[e_1] & \text{edge}[e_2] & \cdots & \text{edge}[e_m]
\end{array}
\]

where $E_G = \{e_1, e_2, \ldots, e_m\}$.

(5) Apply $\text{PSS}_{(m+n, 2)}$ to the card-sequence as follows:

\[
\begin{array}{cccccccc}
\text{vertex}[1] & \text{vertex}[2] & \cdots & \text{edge}[e_m]
\end{array} \rightarrow \begin{array}{cccc}
? & ? & ? & \cdots & ? & ? & ? & ? & ?
\end{array}
\]

(6) For each pile, turn over the left card, and if it is a black-card, turn over the right card. Then sort $n + m$ piles\(^5\) so that the left card is in ascending order via $\leq$ as follows:

\[
\begin{array}{cccccccc}
i_1 \leq i_2 \leq i_3 \leq \cdots \leq i_m.
\end{array}
\]

(7) We define a graph $G'$ by $V_{G'} = V_G$ and $E_{G'} = \{i_1 \rightarrow j_1, i_2 \rightarrow j_2, i_3 \rightarrow j_3, \ldots, i_m \rightarrow j_m\}$.

(8) Take an isomorphism $\psi : G' \rightarrow G''$, and set $\beta_i := \psi_0^{-1}(i)$. Let $y_i$ be the right next card of $\beta_i$, and $y = (y_1, y_2, \ldots, y_n)$. Arrange the card-sequence as follows:

\[
\begin{array}{cccccccc}
y & ? & ? & \cdots & ? & 1 & 2 & 3 & \cdots & n & \cdots & n & \cdots & n & \cdots & n & \cdots
\end{array}
\]

The output card-sequence for the input $x$ is $y$.

**Remark 3.2** Regarding the number of cards, the number of cards in the proposed protocol is $2n + 2m$, of which $n + 2m$ are helping cards. As for the number of shuffles, it is $|\text{Deg}_G| + 1$, and all of them are PSSs. We remark that the PSSs in Step (3) can be executed in parallel.

**Remark 3.3** In Step (8), given two isomorphic graphs $G$ and $G'$, we need to solve the problem of finding one specific isomorphism between them. However, no polynomial-time algorithm for this problem has been found so far in general. On the other hand, there exist polynomial-time algorithms to find isomorphisms for some specific

---

\(^4\) Note that this rearrangement is possible without looking under the cards since the subscripts of $\alpha_i$ are public information.

\(^5\) The order of the pairs of helping cards is not essential.
Let $x = (x_1, x_2, \ldots, x_n)$ be an input sequence and $y = (y_1, y_2, \ldots, y_n)$ a random variable of an output sequence of the protocol when $x$ is given as input. Fix a graph $G'$, which is defined in Step (7) following the opened result in Step (6). Let $\sigma \in S_n$ be a random variable of the permutation chosen by the PSSs in Step (3) such that $\alpha_i = \sigma^{-1}(i)$ for all $i \in V_G$. Since an edge $i \rightarrow j \in E_G$ of $G$ corresponds to an edge $\alpha_i \rightarrow \alpha_j = \sigma^{-1}(i) \rightarrow \sigma^{-1}(j) \in E_{G'}$ of $G'$, there is an isomorphism $\phi = (\phi_0, \phi_1) \in \text{Iso}(G, G')$ such that $\phi_0 = \sigma^{-1}$. From the property of the PSSs, the permutation $\phi_0$ is a uniform random variable on $\text{Iso}_0(G, G')$. Let $\psi = (\psi_0, \psi_1) \in \text{Iso}(G', G)$ be a random variable of the isomorphism chosen in Step (8).

We first claim that $y = \psi_0 \circ \phi_0(x)$. This is shown by observing a sequence of red cards $\overline{i}_1 \overline{i}_2 \overline{i}_3 \cdots \overline{i}_n$. Hereafter, for the sake of clarity, we do not distinguish face-up and face-down and use “$\overline{i}$” to denote the red card $i$. In Step (1), the sequence of red cards omitting other cards is $(\overline{1}, \overline{2}, \ldots, \overline{n})$. In Steps (3), (6), and (8), it is arranged as follows:

$$(\overline{1}, \overline{2}, \ldots, \overline{n}) \xrightarrow{\phi_0^{-1}} (\overline{\alpha_1}, \overline{\alpha_2}, \ldots, \overline{\alpha_n}) \xrightarrow{\phi_0} (\overline{1}, \overline{2}, \ldots, \overline{n}) \xrightarrow{\psi_0} (\overline{\beta_1}, \overline{\beta_2}, \ldots, \overline{\beta_n}).$$

Since the input sequence $x$ is arranged as $((\overline{\alpha_1}, x_1), (\overline{\alpha_2}, x_2), \ldots, (\overline{\alpha_n}, x_n))$ in Step (3), the permutation $\psi_0 \circ \phi_0$ is applied to $x$. Thus, it holds $y = \psi_0 \circ \phi_0(x)$. We note that $\psi \circ \phi_0$ is an automorphism of $G$.

It remains to prove that the distribution of $\psi_0 \circ \phi_0 \in \text{Aut}_0(G)$ is uniformly random. We note that given the graph $G'$, the distributions of $\phi_0$ and $\psi_0$ are independent. This is because the choice of $\psi_0$ depends on the opened symbols in Step (6) only, and they are independent of $\phi_0$ due to the PSS in Step (5). Thus, we can change the order of choice without harming the distributions of $\phi_0, \psi_0$: first, $\psi_0$ is chosen, and then $\phi_0$ is chosen. Since the distribution of $\phi_0 \in \text{Iso}_0(G, G')$ is uniformly random, it is sufficient to show that the function

$$\Phi : \text{Iso}_0(G, G') \rightarrow \text{Aut}_0(G)$$

$$\phi_0 \mapsto \psi_0 \circ \phi_0$$

is bijective.

We first prove that $\Phi$ is injective. Suppose that $\Phi(\phi'_0) = \Phi(\phi''_0)$ for some $\phi'_0, \phi''_0 \in \text{Iso}_0(G, G')$, that is, $\psi_0 \circ \phi'_0 = \psi_0 \circ \phi''_0$. Since $\psi_0$ is a bijection, $\phi'_0 = \phi''_0$ holds. Thus, $\Phi$ is injective. We next prove that $\Phi$ is surjective. For any $\tau \in \text{Aut}_0(G)$, we have

$$\tau = \psi_0 \circ \phi_0^{-1} \circ \tau = \Phi(\psi_0^{-1} \circ \tau).$$

It yields that $\Phi$ is surjective. Therefore, $\Phi$ is bijective.
This shows that the distribution of \( \psi \circ \sigma^{-1} \) is uniformly random, and hence our protocol is correct.

**Proof of Security**

In the proof of the correctness, we have already claimed that the distribution of the opened symbols in Step (6) is independent of \( \sigma \) due to the PSS in Step (5).

Since cards are opened in Step (6) only, this shows a distribution of the permutation \( \psi \circ \sigma^{-1} \in \text{Aut}(G) \) is independent of the distribution of the visible sequence-trace of our protocol. Therefore, our protocol is secure.

**Example of Our Protocol for a Graph**

Let \( G \) be a directed graph with 5 vertices as follows:

\[
G = \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5
\end{array} \\
\begin{array}{c}
e_3 \\
e_1 \\
e_2 \\
e_4 \\
e_5 \\
e_6
\end{array}
\end{array}
\]

We perform our graph shuffle protocol for \( G \). Let \( D_{\text{inp}} \) be an arbitrary deck with \( D_{\text{inp}} = \{x_1, x_2, x_3, x_4, x_5\} \). The card-sequence \( h \) of helping cards is defined as follows:

\[
h = [1234511222333345].
\]

Set \( D_{\text{help}} = \{1, 2, 3, 4, 5, 1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 5\} \) and \( D = D_{\text{inp}} \cup D_{\text{help}} \). For an input card-sequence \( x = (x_1, x_2, x_3, x_4, x_5) \in U \), the graph shuffle protocol proceeds as follows:

1. Place the cards such as:

\[
\begin{array}{c}
? ? ? ? ? \\
x_1 \ x_2 \ x_3 \ x_4 \ x_5 \\
\end{array}
\begin{array}{c}
1234511222333345 \\
h
\end{array}
\]

2. Arrange the card-sequence as follows:

\[
\begin{array}{c}
? ? ? ? ? \\
x_1 \ x_2 \ x_3 \ x_4 \ x_5 \\
\end{array}
\begin{array}{c}
12 \ 1 \\
opile[1] \pile[2] \pile[3] \pile[4] \pile[5] \\
3 \ 3 \ 3 \ 3 \ 3
\end{array}
\]

3. Perform \( \text{PSS}_{(2,4)} \) and \( \text{PSS}_{(2,2)} \) as follows:
Let \( \alpha_3 = 3 \), we have the following card-sequence:

\[
\begin{array}{cccccccc}
? & ? & ? & ? & ? & ? & ? & ? \\
\alpha_1 & \alpha_1 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 \\
\end{array}
\rightarrow
\begin{array}{cccccccc}
? & ? & ? & ? & ? & ? & ? & ? \\
\alpha_1 & \alpha_1 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 \\
\end{array}
\]

By setting \( \alpha_3 = 3 \), we have the following card-sequence:

\[
\begin{array}{cccccccc}
? & ? & ? & ? & ? & ? & ? & ? \\
\alpha_1 & \alpha_1 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 \\
\end{array}
\rightarrow
\begin{array}{cccccccc}
? & ? & ? & ? & ? & ? & ? & ? \\
\alpha_1 & \alpha_1 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 \\
\end{array}
\]

(4) Arrange the card-sequence as follows:

\[
\begin{array}{cccccccc}
? & ? & ? & ? & ? & ? & ? & ? \\
\alpha_1 & \alpha_1 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 \\
\end{array}
\rightarrow
\begin{array}{cccccccc}
? & ? & ? & ? & ? & ? & ? & ? \\
\alpha_1 & \alpha_1 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 \\
\end{array}
\]

(5) Apply PSS\(_{(11,2)}\) to the card-sequence as follows:

\[
\begin{array}{cccccccc}
? & ? & ? & ? & ? & ? & ? & ? \\
\alpha_1 & \alpha_1 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 \\
\end{array}
\rightarrow
\begin{array}{cccccccc}
? & ? & ? & ? & ? & ? & ? & ? \\
\alpha_1 & \alpha_1 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 \\
\end{array}
\]

(6) For each pile, turn over the left card, and if it is a black-card, turn over the right card. The following card-sequence is an example outcome:

\[
\begin{array}{cccccccc}
5 & ? & 1 & 3 & 2 & 3 & 4 & ? \\
\alpha_1 & \alpha_1 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 \\
\end{array}
\rightarrow
\begin{array}{cccccccc}
5 & ? & 1 & 3 & 2 & 3 & 4 & ? \\
\alpha_1 & \alpha_1 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 \\
\end{array}
\]

Sort 11 piles so that the left card is in ascending order via \( \preceq \) as follows:

\[
\begin{array}{cccccccc}
? & ? & ? & ? & ? & ? & ? & ? \\
\alpha_1 & \alpha_1 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 \\
\end{array}
\rightarrow
\begin{array}{cccccccc}
? & ? & ? & ? & ? & ? & ? & ? \\
\alpha_1 & \alpha_1 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 \\
\end{array}
\]

(7) Define a graph \( G' \) by \( V_{G'} = \{1, 2, 3, 4, 5\} \) and \( E_{G'} = \{1 \rightarrow 3, 1 \rightarrow 2, 2 \rightarrow 3, 2 \rightarrow 1, 3 \rightarrow 4, 3 \rightarrow 5\} \);

\[
G' = \begin{array}{ccc}
1 & \rightarrow & 3 \\
2 & \rightarrow & 4 \\
3 & \rightarrow & 5
\end{array}
\]

(8) Take an isomorphism \( \psi : G \rightarrow G' \) defined by

\[
1 \mapsto 2, \quad 2 \mapsto 1, \quad 3 \mapsto 3, \quad 4 \mapsto 4, \quad 5 \mapsto 5.
\]

Arrange the above card-sequence as follows:
The output card-sequence for the input \( x \) is \((y'_2, y'_1, y'_3, y'_4, y'_5)\).

### Implication of Our Protocol

In this subsection, we consider several interesting graph shuffles.

We first observe that a RC for \( n \) cards are graph shuffles for the directed \( n \)-cycle graph \( C_n \) (see “The \( n \)-cycle graph”). Since it holds \( 2n + 2m = 4n \) and \( |\text{Deg}_G| + 1 = 2 \), a RC can be done by \( 4n \) cards and two PSSs. In “The \( n \)-cycle graph”, the number of cards is improved to \( 3n \). We remark that our graph shuffle protocol works even for a sequence of piles each having equivalent number of face-down cards. Thus, a pile-shifting shuffle (i.e., a pile-version of RC) can be done by the same number of helping cards. In particular, for a pile-shifting shuffle for \( n \) piles of \( m \) cards, it can be done by \( nm + 3n \) cards and two PSSs. We note that PSSs and RBCs are graph shuffles for graphs with no edges in this sense.

A graph shuffle for the undirected \( n \)-cycle graph \( C_n \) is equivalent to the dihedral shuffle, which is introduced by Niemi and Renvall [36]. Since it holds \( 2n + 2m = 5n \) and \( |\text{Deg}_G| + 1 = 2 \), our result implies that a RC can be done by \( 5n \) cards and two PSSs. In “The undirected \( n \)-cycle”, the number of cards is improved to \( 3n \), although the number of PSSs is increased to three.

For a cyclic group \( \Pi = \langle (1 2)(3 4 5 6) \rangle \), a uniform closed shuffle \((\text{shuffle}, \Pi, \mathcal{F})\) is a graph shuffle for \( G \) where \( V_G = \{1, 2, 3, 4, 5, 6\} \) and \( E_G = E_1 \cup E_2 \cup E_3 \) with \( E_1 = \{1 \rightarrow 2, 2 \rightarrow 1\} \), \( E_2 = \{3 \rightarrow 4, 4 \rightarrow 5, 5 \rightarrow 6, 6 \rightarrow 4\} \), and \( E_3 = \{1 \rightarrow 3, 1 \rightarrow 5, 2 \rightarrow 4, 2 \rightarrow 6\} \). Since it holds \( \text{Aut}_G(G) = \langle (1 2)(3 4 5 6) \rangle \), we can conclude that a graph shuffle for \( G \) is equivalent to a uniform closed shuffle \((\text{shuffle}, \Pi, \mathcal{F})\). Since it holds \( 2n + 2m = 32 \) and \( |\text{Deg}_G| + 1 = 3 \), our result implies that it can be done by \( 32 \) cards and three PSSs. By generalizing this idea, for any cyclic group \( \Pi = \langle \pi \rangle \), a uniform closed shuffle \((\text{shuffle}, \Pi, \mathcal{F})\) is a graph shuffle for some graph.

### Efficiency Improvements for Graph Shuffles for Cycles

In this section, we implement efficient graph shuffle protocols for some specific graph classes. In particular, we improve the number of cards in our protocol.

#### The \( n \)-Cycle Graph

First, we consider the \( n \)-cycle graph \( \vec{C}_n \):

\[
\vec{C}_n = \begin{array}{cccccccc}
1 & \rightarrow & 2 & \rightarrow & \cdots & \rightarrow & n-1 & \rightarrow & n
\end{array}
\]
The graph shuffle for $\vec{C}_n$ is equivalent to a RC of $n$ cards since the automorphism group $\text{Aut}(\vec{C}_n)$ is isomorphic to the cyclic group of degree $n$. For example, the graph shuffle for $C_n$ when $n = 4$ is given as follows:

$$
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1 \\
3 & 4 & 1 & 2 \\
4 & 1 & 2 & 3
\end{array}
\rightarrow
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1 \\
3 & 4 & 1 & 2 \\
4 & 1 & 2 & 3
\end{array},
$$

where each sequence is obtained with probability $1/4$. If we apply our graph shuffle protocol for $C_n$ proposed in “Graph shuffle protocol”, we need $4n$ cards. In this subsection, we propose a graph shuffle protocol for $C_n$ with $3n$ cards only.

Before describing the improved protocol, we shortly mention how to improve the number of cards. The idea\(^6\) is to remove the red cards by making a pile of $(x_i, a_i, a_{i+1})$ instead of a pile of $(x_i, \vec{a})$ and a pile of $(a_i, a_{i+1})$ in the previous protocol. Since all vertices of $C_n$ have the same degree, all piles of $(x_i, a_i, a_{i+1})$ have the same number of cards and thus the final randomization (corresponding to Step (5) in the previous protocol) can be done by a single PSS.

Let $D_{\text{inp}} = \{x_1, x_2, \ldots, x_n\}$ be an arbitrary deck and $D_{\text{help}} = \{1, 1, 2, 2, 3, 3, \ldots, n, n\}$ a deck of the symbols of $2n$ helping cards. The sequence of helping cards $h$ is defined as follows:

$$h = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 & \cdots & n & n \end{pmatrix}.$$

For $i = 1, 2, \ldots, n$, we set $\text{pile}[i] = \left( \frac{?}{i}, \frac{?}{i} \right)$.

(1) Place the $3n$ cards as follows:

$$
\begin{array}{cccccccc}
? & ? & ? & \cdots & ? & 1 & 1 & 2 & 2 & 3 & 3 & \cdots & n & n \\
\times
\end{array}
$$

(2) Arrange the card-sequence as follows:

$$
\begin{array}{cccccccc}
? & ? & ? & \cdots & ? & ? & ? & ? & ? & ? & \cdots & ? & ? \\
\times
\end{array}
\begin{array}{cccc}
\text{pile}[1] & \text{pile}[2] & \text{pile}[3] & \cdots & \text{pile}[n] \\
\end{array}
$$

Apply PSS\(_{(n,2)}\) to $(\text{pile}[1], \text{pile}[2], \ldots, \text{pile}[n])$ and then we obtain the card-sequence as follows:

---

\(^6\) We remark that this idea works for every graphs such that all vertices have the same degree.
where \( \{a_1, a_2, \ldots, a_n\} = \{1, 2, \ldots, n\} \).

(3) Arrange the card-sequence as follows:

\[
\begin{array}{cccccccc}
? & ? & ? & ? & ? & ? & ? & ? \\
\alpha_1 & \alpha_1 & \alpha_2 & \alpha_2 & \alpha_3 & \alpha_3 & \alpha_3 \end{array}
\]

(4) Apply PSS\(_{(\alpha, 3)}\) to the card-sequence as follows:

\[
\begin{array}{cccccccc}
? & ? & ? & ? & ? & ? & ? & ? \\
x_1 & \alpha_1 & \alpha_2 & x_2 & \alpha_2 & \alpha_3 & x_3 & \alpha_3 & \alpha_4 & \alpha_4 & \alpha_5 \\
\end{array}
\]

(5) For all piles, turn over the second and third cards. Let \( a_i, b_i \in \{1, 2, \ldots, n\} \) be the opened symbols of the second and third cards, respectively, in the \( i \)-th pile as follows:

\[
\begin{array}{cccccccc}
? & a_1 & b_1 & ? & a_2 & b_2 & ? & a_3 & b_3 \\
\end{array}
\]

(6) Arrange \( n \) piles so that \( (c_1, d_1) = (a_1, b_1) \), \( d_i = c_{i+1} \), \( 1 \leq i \leq n - 1 \), and \( d_n = c_1 \) as follows:

\[
\begin{array}{cccccccc}
? & c_1 & d_1 & ? & c_2 & d_2 & ? & c_3 & d_3 \\
y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} & y_{12} \\
\end{array}
\]

After that, we arrange the card-sequence as follows:

\[
\begin{array}{cccccccc}
? & ? & ? & ? & ? & ? & ? & ? \\
y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} & y_{12} \\
\end{array}
\]

Then the output card-sequence is \( y = (y_1, y_2, \ldots, y_n) \).

We show the correctness of the protocol. Let \( x = (x_1, \ldots, x_n) \) be an input sequence. Assume that the protocol outputs the sequence \( y = (y_1, \ldots, y_n) \) when \( x \) is given as input. First, we see that \( y = \sigma(x) \) for some \( \sigma \) in the cyclic group of degree \( n \). For \( i = 1, \ldots, n \), we set \( P_i = (x_i, a_i, a_{i+1}) \), where \( a_{n+1} = a_1 \), in Step (3) and put \( P = (P_1, P_2, \ldots, P_n) \). Let \( Q = (Q_1, \ldots, Q_n) = (P_{\sigma^{-1}(1)}, \ldots, P_{\sigma^{-1}(n)}) \) for some \( \sigma \in \mathbb{S}_n \). Then, \( Q \) is obtained by \( \sigma \) in the cyclic group of degree \( n \) if and only if the third entry of \( Q_i \) and the second entry of \( Q_{i+1} \) are same for any \( 1 \leq i \leq n - 1 \). It follows that the components of obtained sequence in Step (6) are sorted in a cyclic fashion of \( P \). Therefore, \( y \) is equal to \( \sigma(x) \) for some \( \sigma \) in the cyclic group of degree \( n \). Note that each element \( \sigma \) of the cyclic group is determined by \( \sigma^{-1}(1) \), and it is determined by \( d_1 \). For each \( k \in \{1, 2, \ldots, n\} \), the probability that \( k = d_1 \) is \( \frac{1}{n} \) since \( d_1 \) is dependent on the PSS in Step (4) only. Thus, the distribution of \( \sigma \) is uniformly random, and hence the protocol is correct.

We show the security of the protocol. Assume that \( \sigma \in \mathbb{S}_n \) and \( \tau \in \mathbb{S}_n \) are chosen in Steps (2) and (4), respectively. Then, the first card in the \( i \)-th pile in Step (5)
is $x_{i-1(0)}$. On the other hand, the second and third cards in the $i$-th pile in Step (5) are $a_i = \tau^{-1}\sigma^{-1}(i)$ and $b_i = \tau^{-1}\sigma^{-1}(i + 1)$. Here, we consider $n + 1$ as 1. This implies that these opened symbols $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ do not allow us to guess the first card of any pile since $\sigma$ is chosen uniformly at random in Step (4). Therefore, the protocol is secure.

**The Undirected $n$-Cycle**

Next, we consider the undirected $n$-cycle graph $C_n$:

$$C_n = \begin{array}{cccc}
1 & 2 & \cdots & n-1 & n
\end{array}$$

Recall that we regard undirected edge as two directed edges with opposite directions (see the paragraph just before Definition 3.1). The automorphism group $\text{Aut}(C_n)$ is isomorphic to the dihedral group of degree $n$. For example, the graph shuffle for $C_n$ when $n = 4$ is given as follows:

$$
\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1 \\
3 & 4 & 1 & 2 \\
4 & 1 & 2 & 3 \\
4 & 3 & 2 & 1 \\
3 & 2 & 1 & 4 \\
2 & 1 & 4 & 3 \\
1 & 4 & 3 & 2 
\end{pmatrix}
\xrightarrow{\text{graph shuffle}}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1 \\
3 & 4 & 1 & 2 \\
4 & 1 & 2 & 3 \\
4 & 3 & 2 & 1 \\
3 & 2 & 1 & 4 \\
2 & 1 & 4 & 3 \\
1 & 4 & 3 & 2 
\end{pmatrix}
$$

where each sequence is obtained with probability 1/8. If we apply the graph shuffle for $C_n$, we need $6n$ cards. In this subsection, we propose a graph shuffle protocol for $C_n$ with $3n$ cards only.

For an undirected $n$-cyclic graph, even though it has $2n$ edges, the number of cards corresponding to edges is reduced to $n$ pairs of cards using the symmetry of the graph. This improvement is done by the pile-scramble shuffle in Step (3) in the below protocol.

Let $D_{\text{inp}} = \{x_1, x_2, \ldots, x_n\}$ be an arbitrary deck and $D_{\text{help}} = \{1, 1, 2, 2, 3, 3, \ldots, n, n\}$ a deck of the symbols of $2n$ helping cards. The sequence of helping cards $h$ is defined as follows:

$$h = \begin{pmatrix}
1 & 1 & 2 & 2 & 3 & 3 & \cdots & n & n
\end{pmatrix}.$$

For $i = 1, 2, \ldots, n$, we set $\text{pile}[i] = \left(\frac{2}{i}, \frac{2}{i}\right).$
(1) Place the $3n$ cards as follows:

\[
\begin{array}{cccccc}
?? & ?? & \cdots & ? & 1 & 1 \\
\end{array}
\]

(2) Arrange the card-sequence as follows:

\[
\begin{array}{cccccc}
?? & ?? & \cdots & ? & 1 & 1 \\
\end{array}
\]

Apply $PSS_{(n,2)}$ to (pile[1], pile[2], ..., pile[n]) and then we obtain the card-sequence as follows:

\[
\begin{array}{cccccc}
?? & ?? & \cdots & ? & 1 & 1 \\
\end{array}
\]

where $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} = \{1, 2, \ldots, n\}$.

(3) Arrange the card-sequence as follows:

\[
\begin{array}{cccccc}
?? & ?? & \cdots & ? & 1 & 1 \\
\end{array}
\]

Apply $PSS_{(2,n)}$ to the rightmost card-sequence of $2n$ cards as follows:

\[
\begin{array}{cccccc}
?? & ?? & \cdots & ? & 1 & 1 \\
\end{array}
\]

Then we obtain the following card-sequence:

\[
\begin{array}{cccccc}
?? & ?? & \cdots & ? & 1 & 1 \\
\end{array}
\]

where $\{((\alpha_1, \alpha_2, \ldots, \alpha_n), (\alpha_2, \ldots, \alpha_n, \alpha_1))\} = \{\beta_1, \beta_2, \ldots, \beta_n, \gamma_1, \gamma_2, \ldots, \gamma_n\}$.

(4) Arrange the card-sequence as follows:

\[
\begin{array}{cccccc}
?? & ?? & \cdots & ? & 1 & 1 \\
\end{array}
\]

(5) Apply $PSS_{(n,3)}$ to the card-sequence as follows:

\[
\begin{array}{cccccc}
?? & ?? & \cdots & ? & 1 & 1 \\
\end{array}
\]

(6) For all piles, turn over the second and third cards. Let $a_i, b_i \in \{1, 2, \ldots, n\}$ be the opened symbols of the second and third cards, respectively, in the $i$-th pile as follows:
Arrange $n$ piles so that $(c_1, d_1) = (a_1, b_1)$, $d_i = c_{i+1}$, $(1 \leq i \leq n-1)$, and $d_n = c_1$ as follows:

$$\begin{array}{c}
\begin{array}{c}
\? a_1 b_1 \? a_2 b_2 \? a_3 b_3 \cdots \? a_n b_n
\end{array}
\end{array}$$

Then arrange the card-sequence as follows:

$$\begin{array}{c}
\begin{array}{c}
\? c_1 d_1 \? c_2 d_2 \? c_3 d_3 \cdots \? c_n d_n
\end{array}
\end{array}$$

Then the output card-sequence is $(y_1, y_2, \ldots, y_n)$. We first show the correctness of the protocol. Let $x = (x_1, \ldots, x_n)$ be an input sequence. Assume that the protocol outputs the sequence $y = (y_1, \ldots, y_n)$ when $x$ is given as input. Observe that if we apply a graph shuffle for $C_n$ to $x$, the output sequence is one of the following sequences:

$$(x_k, x_{k+1}, \ldots, x_n, x_1, x_2, \ldots, x_{k-1}), \quad (x_k, x_{k-1}, \ldots, x_1, x_n, x_{n-1}, \ldots, x_{k+1}),$$

for some $k \in \{1, 2, \ldots, n\}$. We denote by Cyc($k$) and Rev($k$) the former sequence and the latter sequence, respectively. To show the correctness of the protocol, we see that $y$ is one of Cyc($k$) and Rev($k$) for some $k = 1, \ldots, n$. For $i = 1, \ldots, n$, we set $P_i = (x_i, \beta_i, \gamma_i)$ and put $P = (P_1, \ldots, P_n)$. Suppose that $(a_1, \ldots, a_n)$ is equal to $(\beta_1, \ldots, \beta_n)$ in Step (3). In this case, it holds $\gamma_i = \beta_{i+1}$ for any $i \in \{1, 2, \ldots, n\}$, where $\beta_{n+1} = \beta_1$. It follows from the above equations and the argument in the proof of the correctness of the protocol in “The n-cycle graph” that $y = \text{Cyc}(k)$ for some $k$. Similarly, if $(a_1, \ldots, a_n) = (\gamma_1, \ldots, \gamma_n)$ in Step (3), the equations $\gamma_i = \beta_{n-i+1}$ hold for any $i \in \{1, 2, \ldots, n\}$. This implies that $y = \text{Rev}(k)$ for some $k$.

Next, we show that the distribution of $y$ is uniform. Assume that $n = 2$. We note that Cyc(1) = Rev(1) and Cyc(2) = Rev(2). Then, the candidates appearing as a result of Step (4) are:

$$\begin{array}{c}
\begin{array}{c}
\? \? \? \? \? \? \?, \quad \? \? \? \? \? \? \?
\end{array}
\end{array}$$

and these each have a probability of $\frac{1}{2}$. Thus, the probabilities that $y = (x_1, x_2)$ and $y = (x_2, x_1)$ are same. Now, we assume that $n \geq 3$. In this case, for any $k = 1, \ldots, n$, all sequences Cyc($k$) and Rev($k$) are distinct. To get $y = \text{Cyc}(k)$, it requires that $(\alpha_1, \ldots, \alpha_n) = (\beta_1, \ldots, \beta_n)$ in Step (3) and $\sigma^{-1}(1) = k$, where $\sigma$ is the chosen permutation in Step (5). Hence, the probability that $y = \text{Cyc}(k)$ is $\frac{1}{2k}$. Similarly, the probability that $y = \text{Rev}(k)$ is also $\frac{1}{2k}$. This shows that the protocol is correct.

We show the correctness of the protocol. Assume that $\sigma \in \mathcal{S}_n$ and $\tau \in \mathcal{S}_n$ are chosen in Step (2) and Step (5), respectively. Then, the first card in the $i$-th pile in Step
(6) is $x_{\tau^{-1}(i)}$. On the other hand, the second and third cards in the $i$-th pile in Step (5) are depending on the result of Step (3), and they are determined as follows. If $(\alpha_1, \ldots, \alpha_n) = (\beta_1, \ldots, \beta_n)$, then $a_i = \tau^{-1}\sigma^{-1}(i)$ and $b_i = \tau^{-1}\sigma^{-1}(i+1)$, otherwise, $a_i = \tau^{-1}\sigma^{-1}(i+1)$ and $b_i = \tau^{-1}\sigma^{-1}(i)$. Here, we consider $n + 1$ as 1. In either case, these open symbols $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ do not allow us to guess the first card of any pile since $\sigma$ is chosen uniformly at random in Step (5). Therefore, the protocol is secure.

Conclusions and Future Works

In this paper, we show that any graph shuffle can be done by PSSs. In particular, we need $2(n + m)$ cards and $|\text{Deg}_G| + 1$ PSSs, where $n$ and $m$ are the numbers of vertices and edges of $G$, respectively. We left as open problems (1) to remove the computation of an isomorphism between two isomorphic graphs in a graph shuffle protocol keeping everything efficient and (2) to find another interesting applications for our graph shuffle protocol. We hope that this research direction (i.e., constructing a non-trivial shuffle from the standard shuffles such as RCs, RBCs, and PSSs) will attract the interest of researchers on card-based cryptography and new shuffle protocols will be proposed in future work.

**Funding** The first author K. Miyamoto was partly supported by JSPS KAKENHI 20K14302. The corresponding author K. Shinagawa was partly supported by JSPS KAKENHI 21K17702.

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