Fluctuation Theorem for Partially-masked Nonequilibrium Dynamics

Naoto Shiraishi and Takahiro Sagawa
Department of Basic Science, The University of Tokyo,
3-8-1 Komaba, Meguro-ku, Tokyo 153-8902, Japan
(Dated: March 21, 2014)

We establish a novel generalization of the fluctuation theorem for partially-masked nonequilibrium dynamics. We divide total entropy production into two contributions from observable transitions and from masked ones, and show that each part of entropy production individually satisfies the fluctuation theorem. Our result reveals the fundamental properties of a broad class of autonomous nanomachines as well as non-autonomous ones. In particular, our result gives a unified fluctuation theorem for both autonomous and non-autonomous Maxwell’s demons, where mutual information plays a crucial role.

PACS numbers: 05.70.Ln, 05.40.-a, 89.70.-a, 87.10.Mn.

Introduction.—In modern nonequilibrium statistical physics, the fluctuation theorem (FT) is significant to characterize the foundation of thermodynamic irreversibility [1-4]. FT has revealed that entropy production is directly related to the probability of the observed trajectory and that of its time-reversal. The entropy production is measured by observing the microscopic trajectories, which has been experimentally demonstrated in a variety of systems [3-11].

In many of nonequilibrium systems, however, it is not necessarily possible to observe all of the microscopic trajectories. For example, it is experimentally difficult to observe all degrees of freedom of complicated artificial [12-14] and biological [15-17] nanomachines. If a part of the observed system is masked, one cannot determine the total amount of entropy production. In such situations, is it still possible to obtain a universal nonequilibrium relation like FT?

In this Letter, we reveal the universal property of partially-masked nonequilibrium dynamics. We divide all possible transitions between microscopic states into two groups; one consists of observable transitions and the other consists of masked, non-observable transitions. We define the observable entropy production and the masked one such that their sum is equal to the total entropy production. Surprisingly, we can show that FT holds for the two parts of the entropy production individually, which is regarded as a novel generalization of FT. The obtained FT can reproduce the previous results on Maxwell’s demon [18-19].

Our result is straightforwardly applicable to quite a broad class of nanomachines in thermal environment, such as autonomous Maxwell’s demons and sensing systems [20-27], ion exchangers [28], molecular motors [29], bacterial chemotaxis [30,31], circadian clocks [32], and enzymes [33]. In the following, we focus on a version of autonomous demons [20-23] and ion exchangers [28] in order to show the power of our general result. In particular, our result reveals the crucial role of mutual information in an autonomous demon. Our result leads to a novel direction to understand autonomous stochastic machines in terms of nonequilibrium thermodynamics and information theory.

Setting.—A thermodynamic system obeys continuous-time Markov jump process for time interval $0 \leq t \leq T$. We assume that the number of states of the system is finite. The transition (i.e., jump) from state $w$ to state $w'$ is written as $w \rightarrow w'$, to which we assign transition probability $P(w \rightarrow w'; t)$ that depends on time $t$ in general. The dynamics of the system is described by the master equation

$$\frac{\partial P(w,t)}{\partial t} = J(w,t) := \sum_{w'} J(w' \rightarrow w; t), \quad (1)$$

where $P(w,t)$ is the probability of $w$ at time $t$, and $J(w' \rightarrow w; t) := P(w', t) P(w' \rightarrow w; t) - P(w; t) P(w \rightarrow w'; t)$ is the probability flux from $w'$ to $w$. We assume that the system is attached with a single heat bath at inverse temperature $\beta$. Due to the local detailed balance condition, the heat absorbed by the system from the bath during transition $w' \rightarrow w$ at time $t$ is given by $Q(w' \rightarrow w; t) = -(1/\beta) \ln (P(w' \rightarrow w; t)/P(w \rightarrow w'; t))$.

Let $\Gamma$ be an observed trajectory of the dynamics, in which transitions occur $N$ times at $t = t_1, t_2, \cdots, t_N$. The state during time interval $t_i \leq t < t_{i+1}$ is denoted by $w_i$ with $t_0 := 0$ and $t_{N+1} := T$. In particular, the initial and final states are denoted by $w_0$ and $w_N$, respectively. The total entropy production along trajectory $\Gamma$ is then

![FIG. 1. (Color online) Schematic of an example of a Markov jump process, where the circles indicate the microscopic states and the arrows indicate the paths of possible transitions. The red arrows indicate the observable transitions in $\Omega$, while the black ones indicate the masked transitions in $\Omega'$.](image-url)
given by \( \sigma_{\text{tot}} := -\beta \sum_{i=1}^{N} Q(w_{i-1} \rightarrow w_i; t_i) + \Delta s \), where the stochastic entropy at time \( t \) is given by \( s(w, t) := -\ln P(w, t) \), and its change is given by \( \Delta s := s(w_N, t) - s(w_0, 0) \).

We now divide the dynamics into the observable part and the masked one. Let \( \Omega \) be the set of observable transitions, which is a subset of all possible transitions. Let \( \Omega^c \) be the complement set of \( \Omega \), which corresponds to the masked transitions. For example, in the case of Fig. 1, \( \Omega \) consists of four transitions colored by red, while \( \Omega^c \) consists of eight transitions colored by black. The choice of \( \Omega \) is assumed to be time-dependent in general. Corresponding to the division of transitions, \( J(w, t) \) can be divided as \( J(w, t) = J_\Omega(w, t) + J_{\Omega^c}(w, t) \), where \( J_\Omega(w, t) \) is defined as

\[
J_\Omega(w, t) := \sum_{\{w' \mid (w' \rightarrow w) \in \Omega\}} J(w' \rightarrow w; t),
\]

and \( J_{\Omega^c}(w, t) \) is defined in the same manner.

Main Result.—To discuss the main result, we introduce the concept of the partial entropy production \( \sigma_\Omega \) associated with the observable transitions in \( \Omega \):

\[
\sigma_\Omega := -\beta Q_\Omega + \Delta s_\Omega.
\]

The right-hand side (rhs) consists of the following two terms. First, \( Q_\Omega \) is the heat absorbed by the system during the transitions in \( \Omega \):

\[
Q_\Omega := \sum_{i=1}^{N} Q(w_{i-1} \rightarrow w_i; t_i)\delta_\Omega(w_{i-1} \rightarrow w_i),
\]

where \( \delta_\Omega(w_{i-1} \rightarrow w_i) \) takes 1 if \( w_{i-1} \rightarrow w_i \) is in \( \Omega \), and 0 otherwise. Second, \( \Delta s_\Omega \) is the change in the stochastic entropy induced by the transitions in \( \Omega \):

\[
\Delta s_\Omega := s_{\Omega, \text{jump}} - \int_0^T \frac{J_\Omega(w(t), t)}{P(w(t), t)} dt,
\]

where \( w(t) \) represents the state at time \( t \). The first term on the rhs in Eq. 5 represents the change in the stochastic entropy due to the realized jumps in \( \Omega \):

\[
s_{\Omega, \text{jump}} := \sum_{i=1}^{N} (s(w_i, t_i) - s(w_{i-1}, t_i)) \delta_\Omega(w_{i-1} \rightarrow w_i).
\]

The second term on the rhs in Eq. 5 represents the change in the stochastic entropy due to the time evolution of the probability distribution induced by the transitions in \( \Omega \), which is a part of \( \partial s(w, t)/\partial t = -\langle J_\Omega(w, t) + J_{\Omega^c}(w, t) \rangle/P(w, t) \). If \( \Omega \) includes all of the transitions (i.e., no transition is masked), \( \sigma_\Omega \) is equal to the total entropy production \( \sigma_{\text{tot}} \), because in this case \( \Delta s_\Omega = \Delta s \).

We also define \( \sigma_{\Omega^c} \) for the masked transitions in the same manner to Eq. 3. We can then show that

\[
\sigma_{\text{tot}} = \sigma_\Omega + \sigma_{\Omega^c},
\]

which is the crucial property of the definition 4: the total entropy production is additively decomposed into the observable and masked parts. We stress that it is highly nontrivial whether each of \( \sigma_\Omega \) and \( \sigma_{\Omega^c} \) satisfies FT individually. Here, we indeed have that

\[
\langle e^{-\sigma_\Omega} \rangle = 1, \quad \langle e^{-\sigma_{\Omega^c}} \rangle = 1,
\]

which is the main result in this Letter. Before going to the proof of Eq. 8, we will show its significance by applying it to autonomous nanomachines in the following.

Example 1: autonomous Maxwell’s demon.—First, we consider a model of autonomous Maxwell’s demon, which is a simplification of models discussed in Refs. [20, 21, 37]. Suppose that a particle is transported between two particle baths: H with high density and L with low density (see Fig. 2). Between the baths, there is a single cite where at most a single particle can come in. Let \( x \in \{0, 1\} \) be the number of the particle in the cite. In addition, we consider a wall that plays the role of the demon. The wall is inserted between the cite and one of the baths. Let \( y \in \{l, r\} \) be the position of the wall corresponding to left or right. If \( y = l \) (\( y = r \)), the wall prohibits the jump of the particle between the cite and the bath H (L). The whole state is written as \( w := (x, y) \). Correspondingly, we denote \( w_i := (x_i, y_i) \). We assume that the probability of \( y = l \) is higher (lower) than \( r \) if \( x = 0 \) (\( x = 1 \)). Intuitively, the wall measures \( x \) and then changes its own state depending on the measurement result, which enables the particles to move from L to H against the chemical potential difference.

To discuss the meaning of Eq. 8 in this system, we introduce the entropy production associated with \( x \), \( \sigma_x := -\beta Q_x + s(x_N, t) - s(x_0, 0) \), and the mutual information that quantifies the correlation between \( x \) and \( y \). The stochastic mutual information between the particle and the wall is given by \( I_1(x; y) := \)
in $P(x, y, t)/P(x, t)P(y, t)$ \cite{18, 19}, whose ensemble average gives the mutual information \cite{34}. The change in the mutual information associated with the dynamics of the particle is given by

$$\Delta I_x := I_{x, \text{jump}} + \int_0^T F_x(x(t), y(t), t)dt. \quad (9)$$

Here, $I_{x, \text{jump}}$ represents the change in the mutual information induced by jumps between $x$:

$$I_{x, \text{jump}} := \sum_{i=1}^N \left( I_t(x_i; y_i) - I_t(x_{i-1}; y_{i-1}) \right) \delta_{y_i, y_{i-1}}, \quad (10)$$

where $\delta$ is Kronecker delta. With notation $J_{x', x}^y(t) := J((x', y) \rightarrow (x, y); t)$, $F_x(x, y, t)$ is defined as

$$F_x(x, y, t) := \frac{1}{P(x, y, t)} \sum_{x'} J_{x', x}^y(t) - \frac{1}{P(x, t)} \sum_{y, x'} J_{x', x}^y(t), \quad (11)$$

which represents the change in the mutual information induced by the time evolution of the probability distribution induced by transitions between $x$, where $F_x(x, y, t) + F_y(x, y, t) = \partial I_t(x; y)/\partial t$ with $F_y(x, y, t)$ defined in the same manner to $F_x(x, y, t)$.

We now apply Eq. (3) to this model. We set $\Omega$ to transitions between $x$ (i.e., $\Omega := \{(0, r) \equiv (1, r), (0, l) \equiv (1, l)\}$). Then, $Q_\Omega$ describes the heat absorbed by the particles (i.e., $Q_x = Q_\Omega$). We also obtain

$$-s\Omega_{\text{jump}}$$

$$= I_{x, \text{jump}} - \sum_{i=1}^N \left( s(x_i, t_i) - s(x_{i-1}, t_i) \right)$$

$$= I_{x, \text{jump}} - s(x_N, T) + s(x_0, 0) + \int_0^T \frac{\sum_{y, x'} J_{x', x}^y(t)}{P(x(t), t)} dt,$$

and hence

$$\sigma_\Omega = \sigma_x - \Delta I_x. \quad (13)$$

Then Eq. (8) reduces to

$$\langle e^{-\sigma_x + \Delta I_x} \rangle = 1, \quad (14)$$

in which mutual information contributes to FT on an equal footing with the entropy production associated with the particles. Notably, for any bipartite system described as $w = (x, y)$ with time-dependent transition rates, Eq. (14) holds with completely the same derivation. We call the system autonomous when the transition rates are time-independent. Therefore, Eq. (14) provides a unified view on autonomous and non-autonomous demons, where mutual information is a resource of the entropy decrease of a subsystem. In fact, Eq. (14) can reproduce a previously-obtained FT for non-autonomous demons \cite{18, 19} (see Supplemental Material).

FIG. 3. (Color online) Numerical test of Eq. (14). (a) A histogram of $-\sigma_x + I_{x, \text{jump}}$ (blue) and $-\sigma_x + I_{x, \text{jump}} + \int F_x(x, y, t)dt$ (red) on $R = 3.5$ with 10000 numerical trials. (b) $\langle e^{-\sigma_x + I_{x, \text{jump}}} \rangle$ (blue) and $\langle e^{-\sigma_x + I_{x, \text{jump}} + \int F_x(x, y, t)dt} \rangle$ (red) with the change in $R$. The system is in equilibrium on $R = 2$; the larger $R$ is, the larger the stationary flux becomes.

Using Jensen’s inequality, Eq. (14) leads to a second law-like inequality

$$\langle \sigma_x \rangle - \sum_{x', y} J_{x', x} (I_t(x'; y) - I_t(x; y)) \geq 0, \quad (15)$$

which implies that the entropy production rate of the particles is bounded by the mutual information flow. This inequality has recently obtained in Refs. \cite{22, 23}. Note that this inequality does not include any contribution from $F_x(x, y, t)$, because the ensemble average of $F_x(x, y, t)$ is shown to be zero.

While the ensemble average of $F_x(x, y, t)$ vanishes, it is needed in the ensemble average in Eq. (14). We explicitly show this point with numerical simulation. Set the parameters $P(1 \rightarrow 0|r) = P(0 \rightarrow 1|l) = 1$, $P(0 \rightarrow 1|r) = P(1 \rightarrow 0|l) = 2$, $P(r \rightarrow 1|l) = P(l \rightarrow r|0) = 1$, $P(1 \rightarrow r|1) = P(r \rightarrow 0|0) = R$, $T = 10$, and set the initial state at its stationary state. We obtain the probability distribution of $-\sigma_x + I_{x, \text{jump}}$ (blue) and that of $-\sigma_x + I_{x, \text{jump}} + \int F_x(x, y, t)dt$ (red) on $R = 3.5$. As shown in Fig. 3, the variance of the distribution of $-\sigma_x + I_{x, \text{jump}} + \int F_x(x, y, t)dt$ is larger than that of $-\sigma_x + I_{x, \text{jump}}$. Since the tails of the distributions make significant contribution in Eq. (14), $\langle e^{-\sigma_x + I_{x, \text{jump}}} \rangle$ deviates from one as $R$ increases, while $\langle e^{-\sigma_x + I_{x, \text{jump}} + \int F_x(x, y, t)dt} \rangle$ stays at one in agreement with Eq. (14).

Example 2: $H^+/Ca^{2+}$ exchanger—Next, we consider the $H^+/Ca^{2+}$ exchanger \cite{28}, which binds and releases ions to transport $Ca^{2+}$ between the inside and the outside of a membrane by using a density gradient of $H^+$. Figure 4 illustrates the states of the exchanger. The state $w$ of the exchanger is given by $w = (a, b)$, where $a = 0.1$ respectively represents that the exchanger is in the inward or outward facing state, and $b \in \{H^+, Ca^{2+} \}$ represents the ion attached to the exchanger. Correspondingly, we denote $w_i = (a_i, b_i)$.

Our general result \cite{5} enables us to evaluate the amount of the entropy exchange between the inward and the outward facing states. We define the entropy exchange from the outside to the inside of the membrane...
FIG. 4. (Color online) State space of the H$^+$/Ca$^{2+}$ exchanger. By one counterclockwise rotation, one H$^+$ is exported to the outside of the membrane and one Ca$^{2+}$ is imported to the inside of the membrane. We set the red arrows as the observable transitions in $\Omega$.

during the $i$-th transition as
$$s_{0,i}^{ch} := s(w_{i-1}, t_i)g(a_{i-1}, a_i),$$
where $g(a_{i-1}, a_i)$ takes 1 for $(a_{i-1}, a_i) = (0, 1), -1$ for
$(a_{i-1}, a_i) = (1, 0)$, and 0 for $a_{i-1} = a_i$. Note that $s_{0,i}^{ch}$
takes non-zero values only if the value of $a$ changes. We also define
$s_0(w, t) := s(w, t)\delta_{0,a}$. We set $\Omega = \{w \rightarrow w'|a' = 0\}$ (i.e., red arrows in Fig. 4). As shown below, $\Delta s_\Omega$ satisfies
$$\Delta s_\Omega = \sum_{i=1}^{N} s_{0,i}^{ch} + s_0(w_N, T) - s_0(w_0, 0),$$
and hence Eq. (8) reduces to
$$(e^{\beta Q_0 - \sum s_{0,i}^{ch} - s_0(w_0, 0)}) = 1,$$
where $Q_0 := \sum_{i=1}^{N} Q(w_{i-1} \rightarrow w_i, t_i)\delta_{0,a}$ is the
absorbed heat associated with $a = 0$. Here, Eq. (17) shows the validity of the definition of the entropy exchange as
Eq. (16). It is remarkable that Eq. (17) includes quantities for the whole time interval but only for times when jumps between $a = 0$ and $a = 1$ occur.

We prove Eq. (17) as follows. It is easy to show that
$$-\frac{J_\Omega(w, t)}{P(w, t)} = \frac{\partial s(w, t) / \partial t}{s(w, t)} \quad \text{for } a = 0$$
$$0 \quad \text{for } a = 1.$$ (19)

We take $k, l (1 \leq k < l \leq N)$ such that $a_k = a_l = 1$ and
$a_i = 0 (k + 1 \leq i \leq l - 1)$. Then, it follows from Eq. (19)
that $\Delta s_\Omega$ for $t_k \leq t \leq t_l$ is calculated as
$$\sum_{i=k+1}^{l-1} \left( s(w_i, t_i) - s(w_{i-1}, t_i) + \int_{t_i}^{t_{i+1}} \frac{d}{dt} s(w_i, t) dt \right)$$
$$= \sum_{i=k+1}^{l} s_{0,i}^{ch},$$ (20)

because both hand sides are equal to $s(w_k, t_{k+1}) - s(w_{l-1}, t_l)$. If $a_k = 1$ and $a_i = 0 (k + 1 \leq i \leq N)$, $\Delta s_\Omega$
for $t_k \leq t \leq T$ is calculated as $\sum_{i=k+1}^{N} s_{0,i}^{ch} + s(w_N, T)$.

We also note that there have been other approaches to partially-accessible systems in terms of course-graining
of the states in the state space. The relation between these approaches and ours merits further study. Authors thank Sosuke Ito and Kyogo Kawaguchi for fruitful discussion and useful comments. This work is supported by JSPS KAKENHI Grant Nos. 25800217 and 22340114.

[1] D. J. Evans, E. G. D. Cohen, and G. P. Morriss. Phys. Rev. Lett. 71, 2401 (1993).
[2] C. Jarzynski, Phys. Rev. Lett. 78, 2690 (1997).
[3] G. E. Crooks, Phys. Rev. E 60, 2721 (1999).
[4] C. Jarzynski, J. Stat. Phys. 98, 77 (2000).
[5] U. Seifert, Phys. Rev. Lett. 95, 040602 (2005).
[6] U. Seifert, Rep. Prog. Phys. 75, 126001 (2012).
[7] G. M. Wang, E. M. Sevick, E. Mittag, D. J. Searles and D. J. Evans, Phys. Rev. Lett. 89, 050601 (2002).
[8] J. Liphardt, S. Dumont, S. B. Smith, I. Tinoco Jr., and C. Bustamante, Science 296, 1832 (2002).
[9] C. Jarzynski, J. Stat. Phys. 98, 77 (2000).
[10] K. Hayashi, H. Ueno, R. Iino, and H. Noji, Phys. Rev. Lett. 104, 218103 (2010).
[11] J. V. Koski, T. Sagawa, O-P. Saira, Y. Yoon, A. Kutvonen, P. Solinas, M. Mottonen, T. Ala-Nissila, and J. P. Pekola, Nature Physics 9, 644 (2013).
[12] V. Serreli, C-F. Lee, E. R. Kay, and D. A. Leigh, Nature 445, 523 (2007).
[13] V. Balzani, A. Credi, S. Silvi, and M. Venturi, Chem. Soc. Rev. 35, 1135 (2006).
[14] J-Q. Liu and T. Nakano, IEEE TRANSACTIONS ON SYSTEMS, MAN, AND CYBERNETICS. 42, 357 (2012).
[15] S. Toyabe, H. Ueno, and E. Muneyuki, Europhys. Lett. 97, 40004 (2012).
[16] E. A. Korobkova, T. Emonet, H. Park, and P. Chuzel, Phys. Rev. Lett. 96, 058105 (2006).
[17] Y. Ding and T. Dale, TREND in Biochemical Sciences 27, 7 (2002).
[18] T. Sagawa and M. Ueda, Phys. Rev. Lett. 109, 180602 (2012).
[19] T. Sagawa and M. Ueda, New Jour. Phys. 15, 125012 (2013).
[20] K. Sekimoto, Physica D 205, 242 (2005).
[21] P. Strasberg, G. Schaller, T. Brandes, and M. Esposito, Phys. Rev. Lett. 110, 040601 (2013).
[22] D. Hartich, A. C. Barato, and U. Seifert, J. Stat. Mech. P02016 (2014).
[23] J. Horowitz and M. Esposito, arXiv: 1402.3276 (2014).
[24] D. Mandal and C. Jarzynski, Proc. Nat. Ac. Sci. 109, 11641 (2012).
[25] J. M. Horowitz, T. Sagawa, and J. M. R. Parrondo, Phys. Rev. Lett. 111, 010602 (2013).
[26] A. C. Barato, D. Hartich, and U. Seifert, Phys. Rev. E 87, 042104 (2013).
[27] S. Ito and T. Sagawa, Phys. Rev. Lett. 111, 180603 (2013).
[28] T. Nishizawa, S. Kita, A. D. Maturana, N. Furuya, K. Hirata, G. Kasuya, S. Ogasawara, N. Dohmoe, T. Iwamoto, R. Ishitani, and O. Nureki, Science 341, 168 (2013).
[29] D. Lacoste, A. W. C. Lau, and K. Mallick, Phys. Rev. E 78, 011915 (2008).
[30] Y. Tu, Proc. Nat. Ac. Sci. 105, 11737 (2008).
[31] G. Lan, P. Sartori, S. Neumann, V. Sourjik, and Y. Tu, Nature Phys. 8, 422 (2012).
[32] J. S. van Zon, D. K. Lubensky, P. R. H. Altena, and P. R. ten Wolde, Proc. Nat. Ac. Sci. 104, 7420 (2007).
[33] U. Seifert, Eur. Phys. J. E 34, 26 (2011).
[34] T. M. Cover and J. A. Thomas, “Elements of Information Theory” (John Wiley and Sons, New York, 1991).
[35] R. Kawai, J. M. R. Parrondo, and C. Van den Broeck, Phys. Rev. Lett. 98, 080602 (2007).
[36] A. Gomez-Marin, J. M. R. Parrondo, and C. Van den Broeck, Phys. Rev. E 78, 011107 (2008).
[37] M. Esposito, Phys. Rev. E 85, 041125 (2012).
[38] K. Kawaguchi and Y. Nakayama, Phys. Rev. E 88, 022147 (2013).
[39] M. Esposito and J. M. R. Parrondo, arXiv: 1310.2987 (2013).
We reproduce the FT for non-autonomous Maxwell’s demons from Eq. (14) in the main text: 
\[ \langle e^{-\sigma_x + \Delta I_x} \rangle = 1. \]
We consider a bipartite system with state \( w = (x, y) \). Intuitively, \( x \) is the state of the engine and \( y \) is the state of the memory of the demon. We assume that the transition rates satisfy
\begin{align*}
P(x \to x'; t|y) &= 0 \quad (T_{2i} \leq t < T_{2i+1}), \quad (S.1) \\
P(y \to y'; t|x) &= 0 \quad (T_{2i+1} \leq t < T_{2i+2}), \quad (S.2)
\end{align*}
with \( 0 = T_0 < T_1 < T_2 < \cdots < T_{2M} = T \) (See also Fig. S.1). In other words, only \( y \) can change in time interval \( T_{2i} \leq t < T_{2i+1} \), where a measurement is performed by the demon; the state of the outcome is registered on the memory. Whereas, only \( x \) can change in time interval \( T_{2i+1} \leq t < T_{2i+2} \), where feedback control is performed; the engine evolves depending on the outcome registered on the memory.

We apply Eq. (14) to this situation and calculate \( \Delta I_x \). While \( \Delta I_x \) is equal to zero for time interval \( T_{2i} \leq t < T_{2i+1} \),
\[ F_x(x, y, t) = \frac{\partial}{\partial t} I_x(x; y) \quad (S.3) \]
holds for time interval \( T_{2i+1} \leq t < T_{2i+2} \), because the probability distribution \( P(x, y, t) \) changes only by transitions between \( x \) during this time interval. Therefore, \( \Delta I_x \) for \( T_{2i+1} \leq t < T_{2i+2} \) becomes
\[ \Delta I_x = I_{x, \text{jump}} + \int_{T_{2i+1}}^{T_{2i+2}} \frac{\partial}{\partial t} I_x(x; y)dt = I_{T_{2i+2}}(x; y) - I_{T_{2i+1}}(x; y). \quad (S.4) \]
We then transform Eq. (14) into
\[ \langle e^{-\sigma_x + \sum I_{T_{2i+2}}(x; y) - I_{T_{2i+1}}(x; y)} \rangle = 1, \quad (S.5) \]
which is equivalent to the FT obtained in Refs. [18, 19].

\[ \]