Non-monotone risk functions for learning

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Abstract

In this paper we consider generalized classes of potentially non-monotone risk functions for use as evaluation metrics in learning tasks. The resulting risks are in general non-convex and non-smooth, which makes both the computational and inferential sides of the learning problem difficult. For random losses belonging to any Banach space, we obtain sufficient conditions for the risk functions to be weakly convex, and to admit unbiased stochastic directional derivatives. We then use recent work on stochastic optimization of weakly convex functionals to obtain non-asymptotic guarantees of near-stationarity for Hilbert hypothesis classes, under assumptions that are weak enough to capture a wide variety of feedback distributions, including potentially heavy-tailed losses and gradients.

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1 Introduction

Arguably the most ubiquitous learning model is that which has been called the “general setting of the learning problem” by Vapnik [26]. In this setting, we have a hypothesis set \( \mathcal{H} \) of candidates to choose from, and for any fixed candidate \( h \in \mathcal{H} \), the learner receives feedback in the form of \( n \) random losses \( L_1(h), \ldots, L_n(h) \), typically assumed to be independent copies of a random variable \( L(h) \). Working on a probability space \( (\Omega, \mathcal{F}, \mu) \), the central goal in this traditional setting is to minimize the expected loss \( \mathbb{E}_\mu L(h) \) giving the abstract notion of “good

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off-sample generalization” a concrete mathematical meaning. In particular, it emphasizes good off-sample performance on average. While this is clearly a natural formulation, the emphasis on average performance is a non-trivial value judgement, which may or may not be appropriate for any given real-world learning problem.

Let us broaden our context slightly. The learning problem by its nature is inherently uncertain, and to reflect this uncertainty related to performance, the expected loss \( E_{\mu} L(h) \) is the classical approach to quantify the risk incurred by the candidate \( h \in \mathcal{H} \). In considering different approaches to quantify risk, let us break the problem down into its core parts. Leaving the notions of loss and hypothesis class abstract for the moment, we have a probability measure \( \mu \), a random variable \( Z \), and a map \( Z \mapsto E_{\mu} Z \). We call this map a risk function, since it is tasked with providing a concise numerical “summary” of the risk captured by the probability distribution of \( Z \).\(^1\) In mapping a random variable to a single real-valued summary, inevitably most information about the distribution is discarded. The decision of what information to keep is a direct reflection of how “performance” should be understood and evaluated. Naturally, many alternative risk functions have been proposed and studied in the literature. For example, the risk function \( Z \mapsto E_{\mu} Z + E_{\mu}(Z - E_{\mu} Z)^2 \) dates back to the mean-variance model of Markowitz [12] for financial portfolio optimization, and provides a lucid reflection of a preference for the least uncertainty (measured by the variance) at a given (average) performance level. More generally, a large class of risk functions admits an interpretation as being expressible in the form

\[
\text{risk} = \text{location} + \text{deviation}
\]

where the “location” refers to a point of central tendency of \( Z \), and the “deviation” refers to the tendency of \( Z \) to deviate from a given point, and either of the summands may be fixed to zero in special cases.\(^2\) We proceed by considering a generalized “deviation” function

\[
\text{dev}_{\mu}(\theta; Z) := \eta E_{\mu} \rho \left( \frac{Z - \theta}{\sigma} \right)
\]

where \( \rho : \mathbb{R} \to \mathbb{R} \) and \( \eta, \sigma > 0 \) are all controllable parameters. Leaving integrability conditions aside for the time being, note that in the special case of \( \rho(u) = u^2 \), the minimizer of \( \text{dev}_{\mu}(\theta; Z) \) in \( \theta \) is precisely the mean \( E_{\mu} Z \), and the minimum achievable value of \( \text{dev}_{\mu} \) is thus equal to the variance \( E_{\mu}(Z - E_{\mu} Z)^2 \). In the case of \( \rho(u) = |u| \), any median of \( Z \) will minimize \( \text{dev}_{\mu}(\cdot; Z) \). With these special cases in mind, minimizing over an arbitrary permissible set \( \Theta \subset \mathbb{R} \), let us use the map

\[
Z \mapsto \inf_{\theta \in \Theta} [E_{\mu} Z + \text{dev}_{\mu}(\theta; Z)]
\]

(2)

to construct one natural class of risk functions.\(^3\) When we say a “class” of risk functions, we mean all risk functions that can be constructed based on the difference choices of \( \rho, \eta, \) and \( \sigma \) which determine \( \text{dev}_{\mu}(\cdot; Z) \), as well as the permissible set \( \Theta \). This quantification of risk keeps the focus on average performance, but gives us significant freedom in how the uncertainty

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\(^1\)While the alternative term “risk measure” is frequently used in the literature, we follow the nomenclature of Ruszczyński and Shapiro [21], reserving the term “measure” for its traditional meaning of a countably additive set function [2, Ch. 1].

\(^2\)Numerous examples are given in representative works by Ruszczyński and Shapiro [21, 22].

\(^3\)This class does not appear in the well-known work of Ruszczyński and Shapiro [21], but it can be found in Ruszczyński and Shapiro [22].
associated with deviations are defined. For ease of reference, we refer to the class of risk functions specified by (2) as \textit{class 1}.

Another approach to quantifying risk may be considered natural. Instead of naively adding the customized deviation to the classical location, we can use this deviation to simultaneously derive a customized location metric, by using risk functions of the form

\[ Z \mapsto \inf_{\theta \in \Theta} [\theta + \text{dev}_\rho(\theta; Z)] . \]

That is, assuming the infimum is achieved, the location term is precisely the point about which we are measuring deviations. In the special case of \( \rho(u) = u^2 \) with \( \Theta = \mathbb{R} \), we obtain the classical mean-variance risk function. We refer to the class of risk functions defined by (3) as \textit{class 2}.

In this work, we consider learning problems that are formulated based on the families of risk functions specified by (2) and (3). Written explicitly, the ultimate goal of the learning problem in this setting amounts to minimization of one of the following two risks:

\[
\begin{align*}
R_1(h) &= \inf_{\theta \in \Theta} [E_\mu L(h) + \text{dev}_\rho(\theta; L(h))] \quad (4) \\
R_2(h) &= \inf_{\theta \in \Theta} [\theta + \text{dev}_\rho(\theta; L(h))] . \quad (5)
\end{align*}
\]

Our goal in this paper is to provide formal guarantees for practical learning algorithms, stated in terms of these risk classes.

\section{Background and setup}

\subsection{Preliminaries}

\textbf{General notation (probability)} Underlying all our analysis is a probability space \((\Omega, \mathcal{F}, \mu)\).\footnote{For basic measure-theoretical facts supporting our main arguments, we use Ash and Döllens-Dade \cite{ash} as a well-established and accessible reference. We will cite the exact results that pertain to our arguments in the main text as they become necessary.}

All random variables, unless otherwise specified, will be assumed to be \(\mathcal{F}\)-measurable functions with domain \(\Omega\). Integration using \(\mu\) will be denoted by \(E_\mu \cdot Z = \int_\Omega Z(\omega) \mu(d\omega)\), and \(\mathbb{P}\) will be used as a generic probability function, typically representing \(\mu\) itself, or the product measure induced by a sample of random variables on \((\Omega, \mathcal{F}, \mu)\). We use \(\mathcal{L}_2 := \mathcal{L}_2(\Omega, \mathcal{F}, \mu)\) to denote the set of all square-\(\mu\)-integrable functions.\footnote{Strictly speaking, this is the set of all equivalence classes of square-\(\mu\)-integrable functions, where \(f \in \mathcal{L}_2(\Omega, \mathcal{F}, \mu)\) represents all functions that are equal \(\mu\)-almost everywhere.}

\textbf{General notation (normed spaces)} Let \(\mathcal{V}\) denote an arbitrary vector space. When we call \(\mathcal{V}\) a normed (linear) space, we are referring to \((\mathcal{V}, \| \cdot \|)\), where \(\| \cdot \| : \mathcal{V} \to \mathbb{R}\) denotes the relevant norm. For any normed space \(\mathcal{V}\), we shall denote by \(\mathcal{V}^*\) the usual dual space of \(\mathcal{V}\), namely all continuous linear functionals defined on \(\mathcal{V}\). The space \(\mathcal{V}^*\) is equipped with the norm \(\| v^* \| := \sup\{ v^*(u) : \forall u \in \mathcal{V}, \| u \| \leq 1 \}\). We shall use the notation \(\langle \cdot, \cdot \rangle\) to represent the “coupling” function between \(\mathcal{V}\) and \(\mathcal{V}^*\), that is for any \(u \in \mathcal{V}\) and \(v^* \in \mathcal{V}^*\), we will write \(\langle u, v^* \rangle := v^*(u)\). For any sequence \((x_n)\) of elements \(x_1, x_2, \ldots \in \mathcal{V}\), we denote convergence of \((x_n)\) to some element \(x'\) by \(x_n \to x'\). When we take limits and do not specify a particular sequence, for example writing \(x \to x'\), then this refers to any sequence (of elements from \(\mathcal{V}\)) that converges to \(x'\). In the special case of real-valued sequences (where \(\mathcal{V} \subset \mathbb{R}\)), if we write \(x_n \to x'_+\) (respectively \(x_n \to x'_-\)), this refers to all sequences from above (resp. below), i.e.,
any convergent sequence such that \( x_n \geq x' \) (resp. \( x_n \leq x' \)) for all \( n \). We denote the open ball of radius \( r > 0 \) centered at \( x_0 \in \mathcal{V} \) by \( B(x_0; r) := \{ x \in \mathcal{V} : \|x - x_0\| < r \} \). We denote the extended real line by \( \mathbb{R} \). On normed space \( \mathcal{V} \), we denote the interior of a set \( U \subset \mathcal{V} \) by \( \text{int} \ U \) (all \( x \in U \) such that \( B(x; \delta) \subset U \) for some \( \delta \)).

**General terminology**  On any normed linear space \( \mathcal{V} \), a set \( A \subset \mathcal{V} \) is said to be **compact** if for any sequence of elements in \( A \), there exists a sub-sequence which converges on \( A \).

We denote the **effective domain** of an extended real-valued function \( f \) by \( \text{dom} \ f := \{ x : f(x) < \infty \} \). We call a convex function \( f : \mathcal{V} \to \mathbb{R} \) **proper** if \( f > -\infty \) and \( \text{dom} \ f \neq \emptyset \). We say that \( f \) is **coercive** if \( \|x\| \to \infty \) implies \( f(x) \to \infty \).

For a function \( f : \mathcal{X} \to \mathcal{Y} \), with \( \mathcal{X} \) and \( \mathcal{Y} \) being normed spaces, we say \( f \) is (locally) **Lipschitz** at \( x_0 \in \mathcal{X} \) if there exists \( \delta > 0 \) and \( \lambda > 0 \) such that \( x_1, x_2 \in B(x_0; \delta) \) implies \( \|f(x_1) - f(x_2)\| \leq \lambda \|x_1 - x_2\| \). We say \( f \) is \( \lambda \)-Lipschitz on \( \mathcal{X} \) if this property holds with a common coefficient \( \lambda \) for all \( x_0 \in \mathcal{X} \).

**Semi-continuous functions**  We say that a function \( f \) is **lower semi-continuous**\(^7\) (LSC) at a point \( x \) if for each \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( \|x - x'\| < \delta \) implies \( f(x') > f(x) - \varepsilon \). If \( -g \) is LSC, then we say \( g \) is **upper** semi-continuous (USC). The property that \( f \) is LSC at a point \( x \) is equivalent\(^8\) to the property that for any sequence \( x_n \to x \), we have

\[
\liminf_{n \to \infty} f(x_n) \leq f(x). \tag{6}
\]

Ordinary continuity is equivalent to being both USC and LSC, but many of the added generality of these weaker notions of continuity is often useful.

**Differentiability**  We start by introducing some common notions of directional differentiability at a high level of generality.\(^9\) Let \( \mathcal{X} \) and \( \mathcal{Y} \) be normed linear spaces, \( U \subset \mathcal{X} \) an open set, and \( f : \mathcal{X} \to \mathcal{Y} \) a function of interest. The **radial derivative** of \( f \) at \( x \in U \) in direction \( u \) is defined

\[
f'_r(x; u) := \lim_{\alpha \to 0^+} \frac{f(x + \alpha u) - f(x)}{\alpha}. \tag{7}
\]

A slight modification to this gives us the (Hadamard) **directional derivative** of \( f \) at \( x \in U \) in direction \( u \):

\[
f'(x; u) := \lim_{(\alpha, \alpha') \to (0^+, u)} \frac{f(x + \alpha u) - f(x)}{\alpha}. \tag{8}
\]

When \( f'_r(x; u) \) exists for all directions \( u \), we say that \( f \) is **radially differentiable** at \( x \). Identically, when \( f'(x; u) \) exists for all directions \( u \), we say that \( f \) is **directionally differentiable** at \( x \). When the map \( u \mapsto f'_r(x; u) \) is continuous and linear, we say that \( f \) is **Gateaux differentiable** at \( x \). When the map \( u \mapsto f'(x; u) \) is continuous and linear, we say \( f \) is **Hadamard differentiable** at \( x \). If \( f \) is Hadamard differentiable, then it is Gateaux differentiable. The converse does not hold in general, but if \( f \) is Lipschitz on a neighbourhood of \( x \in U \), then radial differentiability and directional differentiability (at \( x \)) are equivalent.\(^{10}\)

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\(^{6}\)For example, the function \( f(x) = x^2 \) is coercive, but \( f(x) = \exp(x) \) is not.

\(^{7}\)Nice references on semi-continuity: Ash and Doléans-Dade [2, Appendix 2], Luenberger [11, Ch. 2], Barbu and Precupanu [3, Sec. 2.1], Penot [14, Ch. 1].

\(^{8}\)Ash and Doléans-Dade [2, Thm. A2.2], Penot [14, Lem. 1.18].

\(^{9}\)We follow basic notation and terminology of the authoritative text by Penot [14].

\(^{10}\)Penot [14, Prop. 2.25].
When we simply say that a function $f : X \to Y$ is differentiable at $x \in U$, we mean that there exists a function $f'(x)(\cdot) : X \to Y$ that is linear, continuous, and which satisfies
\[
\lim_{\|u\| \to 0} \frac{\|f(x + u) - f(x) - f'(x)(u)\|}{\|u\|} = 0.
\] (9)

This property is often referred to as Fréchet differentiability. When $f$ is differentiable at $x$, the map $f'(x)$ is uniquely determined.\(^{11}\) In the special case where $Y \subset \mathbb{R}$, the linear functional represented by $f'(x) \in \mathbb{R}^n$ is called the gradient of $f$ at $x$. Differentiability is also closely related to directional differentiability: if $f$ is Gateaux differentiable on $U$ and the map $x \mapsto f'(x; \cdot)$ is continuous at $x$, then $f$ is differentiable at $x$.\(^{12}\)

**Sub-differentials** Let $\mathcal{V}$ be any normed linear space. If $f : \mathcal{V} \to \mathbb{R}$ is a (proper) convex function, the sub-differential of $f$ at $x \in \text{dom } f$ is defined
\[
\partial f(x) := \{ v^* \in \mathcal{V}^* : f(u) - f(x) \geq \langle u - x, v^* \rangle, u \in \mathcal{V} \} \quad (10)
\] and
\[
\partial f(x) := \{ v^* \in \mathcal{V}^* : f'_c(x; u) \geq \langle u, v^* \rangle, u \in \mathcal{V} \}. \quad (11)
\]

This local requirement is much weaker than the condition characterizing the MR-sub-differential, and clearly we have $\partial f(x) \subset \partial f(x)$. When $f$ is assumed to be locally Lipschitz, another class of sub-differentials is often useful. Define the Clarke directional derivative of $f$ at $x$ in the direction $u$ by
\[
f'_c(x; u) := \limsup_{(\alpha, x') \to (0^+, x)} \frac{f(x' + \alpha u) - f(x')}{\alpha}.
\] (12)

The corresponding Clarke sub-differential is defined as
\[
\partial_C f(x) := \{ v^* \in \mathcal{V}^* : f'_c(x; u) \geq \langle u, v^* \rangle, u \in \mathcal{V} \}.
\] (13)

In the special case where $f$ is convex, all the sub-differentials coincide, i.e., $\partial f(x) = \partial f(x) = \partial_C f(x).$\(^{15}\) We say that a function $f$ is sub-differentiable at $x$ if its sub-differential (in any sense) at $x$ is non-empty. Finally, a remark on notation when using set-valued functions like $x \mapsto \partial_C f(x)$. When we write something like “we have $\{ u, \partial_C f(x) \} \geq g(u)$,” it is the same as writing “we have $\{ u, v^* \} \geq g(u)$ for all $v^* \in \partial_C f(x)$.” This kind of notation will be used frequently.

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\(^{11}\)See for example Luenberger \[11, \text{Ch. 7}\] or Penot \[14, \text{Ch. 2}\].

\(^{12}\)Penot \[14, \text{Prop. 2.51}\].

\(^{13}\)See Penot \[14, \text{Thm. 3.22}\] for this fact.

\(^{14}\)We follow Penot \[14, \text{Sec. 4.1}\] for terms and notation here.

\(^{15}\)This follows from Penot \[14, \text{Prop. 5.34}\]. See also Penot \[14, \text{Sec. 4.1.1, Exercise 1}\].
2.2 Generalized convexity

Let $\mathcal{X}$ be a normed linear space. Take an open set $U \subset \mathcal{X}$ and fix some point $x_0 \in U$. For a function $f : \mathcal{X} \to \mathbb{R}$ and parameter $\gamma \in \mathbb{R}$, say that there exists $\delta > 0$ such that for all $x, x' \in B(x_0; \delta)$ and $\alpha \in (0, 1)$, we have

$$f(\alpha x + (1 - \alpha)x') \leq \alpha f(x) + (1 - \alpha)f(x') + \frac{\gamma \alpha(1 - \alpha)}{2} \|x - x'\|^2. \quad (14)$$

When $\gamma \geq 0$, we say $f$ is $\gamma$-weakly convex at $x_0$. When $\gamma \leq 0$, we say $f$ is $(-\gamma)$-strongly convex at $x_0$. When (14) holds for all $x_0 \in U$, we say that $f$ is $\gamma$-weakly/strongly convex on $U$. The special case of $\gamma = 0$ is the traditional definition of convexity on $U$.

The ability to construct a quadratic lower-bounding function for $f$ is closely related to notions of weak/strong convexity. Consider the following condition: given $\gamma \in \mathbb{R}$, there exists $\delta > 0$ such that for all $x, x' \in B(x_0; \delta)$ we have

$$f(x') \geq f(x) + \langle x' - x, \partial_C f(x) \rangle - \frac{\gamma}{2} \|x - x'\|^2. \quad (15)$$

Here $\partial_C f$ denotes the Clarke sub-differential of $f$, defined by (13). Let us assume henceforth that $\mathcal{X}$ is Banach, $f$ is locally Lipschitz, and $\partial_C f(x)$ is non-empty for all $x \in U$. For any $\gamma \in \mathbb{R}$, it is straightforward to show that (14) $\implies$ (15) holds. Since (15) gives us a lower bound on both $f(x)$ and $f(x')$ for any $x$ and $x'$ close enough to $x_0$, adding up the inequalities immediately implies

$$\langle x - x', \partial_C f(x) - \partial_C f(x') \rangle \geq -\gamma \|x - x'\|^2. \quad (16)$$

When $\mathcal{X}$ is Banach and $f$ is locally Lipschitz, it is straightforward to show that (16) $\implies$ (14) is valid. As such, for Banach spaces and locally Lipschitz functions, we have that the conditions (14), (15), and (16) are all equivalent for the general case of $\gamma \in \mathbb{R}$.

Let us consider one more closely related property on the same open set $U \subset \mathcal{X}$:

$$x \mapsto f(x) + \frac{\gamma}{2} \| x \|^2 \text{ is convex on } U. \quad (17)$$

In the special case where $\mathcal{X}$ is a real Hilbert space and the norm $\| \cdot \|$ is induced by the inner product as $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$, then for any $x, x' \in U$ and $\alpha \in \mathbb{R}$, the equality

$$\alpha \| x \|^2 + (1 - \alpha) \| x' \|^2 = \| \alpha x + (1 - \alpha)x' \|^2 + \alpha(1 - \alpha) \| x - x' \|^2 \quad (18)$$

is easily checked to be valid. In this case, the equivalence (14) $\iff$ (17) follows from direct verification using (18).

The facts above are summarized in the following result:

**Proposition 1** (Characterization of generalized convexity). Consider a function $f : \mathcal{X} \to \mathbb{R}$ on normed linear space $\mathcal{X}$. When $\mathcal{X}$ is Banach and $f$ is locally Lipschitz, then with respect to open set $U \subset \mathcal{X}$ we have the following equivalence:

$$(14) \iff (15) \iff (16).$$

When $\mathcal{X}$ is Hilbert, this equivalence extends to (17).

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16 See for example Nesterov [13, Ch. 3].

17 Penot [14, Prop. 5.3].

18 See for example Daniilidis and Malick [6, Thm. 3.1]; in particular their proof of (i) $\implies$ (iii). Their result is stated for $\mathcal{X} = \mathbb{R}^d$ and locally Lipschitz $f$, but the proof easily generalizes to Banach spaces. See also the remarks following their proof about how the local Lipschitz condition can be removed.

19 Just apply the argument for (iii) $\implies$ (i) employed by Daniilidis and Malick [6, Thm. 3.1], and strengthen their argument by using a more general form of Lebourg’s mean value theorem [14, Thm. 5.12].

20 Bauschke and Combettes [5, Cor. 2.15].

21 See also Davis and Drusvyatskiy [7, Lem. 2.1] for a similar result when $\mathcal{X} = \mathbb{R}^d$ and $f$ is LSC.
2.3 Function composition on normed spaces

Next we consider the properties of compositions involving functions which are smooth and/or convex. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed linear spaces. Let $g : \mathcal{X} \to \mathcal{Y}$ and $h : \mathcal{Y} \to \mathbb{R}$ be the maps used in our composition, and denote by $f := h \circ g$ the composition, i.e., $f(x) = h(g(x))$ for each $x \in \mathcal{X}$. Our goal will be to present sufficient conditions for the composition $f$ to be weakly convex on an open set $U \subset \mathcal{X}$, in the sense of (14). If we assume simply that $h$ is convex, fixing any point $x_0 \in U$ such that $h$ is sub-differentiable at $g(x_0)$, it follows that for any choice of $x \in \mathcal{X}$ we have

$$f(x) = h(g(x)) \geq h(g(x_0)) + \langle g(x) - g(x_0), \partial h(g(x_0)) \rangle. \quad (19)$$

Let us further assume that $h$ is $\lambda_0$-Lipschitz, and $g$ is smooth in the sense that it is differentiable on $U$ and the map $x \mapsto g'(x)$ is $\lambda_1$-Lipschitz. For readability, denote the derivative $g'(x_0) : \mathcal{X} \to \mathcal{Y}$ by $g_0'(\cdot) := g'(x_0)(\cdot)$. Taking any choice of $v_0 \in \partial h(g(x_0))$, we can write

$$\langle g(x) - g(x_0), v_0 \rangle = \langle g_0'(x - x_0), v_0 \rangle + \langle g(x) - g(x_0) - g_0'(x - x_0), v_0 \rangle$$
$$\geq \langle g_0'(x - x_0), v_0 \rangle - \|g(x) - g(x_0) - g_0'(x - x_0)\|v_0\|$$
$$\geq \langle g_0'(x - x_0), v_0 \rangle - \lambda_1 \|x - x_0\|^2 v_0\|$$
$$\geq \langle g_0'(x - x_0), v_0 \rangle - \frac{\lambda_0 \lambda_1}{2} \|x - x_0\|^2 \quad (20)$$

The first inequality follows from the definition of the norm for linear functionals and the fact that $\partial h(g(x_0)) \subset \mathcal{Y}^*$. The second inequality follows from a Taylor approximation for Banach spaces (Proposition 20), using the smoothness of $g$. The final equality follows from the fact that for convex functions, the Lipschitz coefficient implies a bound on all sub-gradients, see (44). To deal with the remaining term, note that we can write

$$\langle g_0'(x - x_0), \partial h(g(x_0)) \rangle = \langle x - x_0, (g_0')^*(\partial h(g(x_0))) \rangle = \langle x - x_0, \partial h(g(x_0)) \circ g_0' \rangle. \quad (21)$$

To explain the notation here, we use $(\cdot)^*$ to denote the adjoint, namely $A^*(y^*) := y^* \circ A$, induced by any continuous linear map $A : \mathcal{X} \to \mathcal{Y}$, defined for each $y^* \in \mathcal{Y}^*$. The special case we have considered here is where $Au = g_0'(u)$, noting that differentiability means that the map $u \mapsto g_0'(u)$ is continuous and linear. Recalling the desired form of (15), we need to establish a connection with $\partial C f(x_0)$. If we further assume that $g$ is locally Lipschitz, then we have

$$\partial C f(x_0) \subset \partial C h(g(x_0)) \circ g_0' = \partial h(g(x_0)) \circ g_0', \quad (22)$$

where the equality follows from the coincidence of sub-differentials in the convex case, and the key inclusion follows from direct application of a generalized chain rule. Taking these facts together yields the following result.

**Proposition 2** (Weak convexity for composite functions). Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. Let $g : \mathcal{X} \to \mathcal{Y}$ be locally Lipschitz and $\lambda_1$-smooth on an open set $U \subset \mathcal{X}$. Let $h : \mathcal{Y} \to \mathbb{R}$ be convex and $\lambda_0$-Lipschitz on $g(U) \subset \mathcal{Y}$. Furthermore, let $g(U) \subset \text{dom} \ h$. Then, the composite function $f := h \circ g$ is $\gamma$-weakly convex on $U$, with $\gamma \leq \lambda_0 \lambda_1$.

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22 Written explicitly, for any $u \in \mathcal{X}$, we have $\langle u, A^*(y^*) \rangle = A^*(y^*)u = y^*(Au) = \langle Au, y^* \rangle$. If $A : \mathcal{X} \to \mathcal{Y}$ is linear and continuous, this implies that the adjoint $A^*$ is a continuous linear map from $\mathcal{Y}^*$ to $\mathcal{X}^*$. For more general background: Luenberger [11, Ch. 6], Penot [14, Ch. 1].

23 See for example Penot [14, Thm. 5.13(b)]. This inclusion holds as long as $g$ is strictly differentiable [14, Defn. 2.54], a property implied by the smoothness we have assumed.
under the Hilbert space assumption, we have that for any \( x_0 \) extends a result of Drusvyatskiy and Paquette \cite{Drusvyatskiy2018}

These insights extend readily to the setting of weak convexity. Under the assumption that \( \mathcal{X} \) is a Hilbert space, let \( f : \mathcal{X} \to \mathbb{R} \) be \( \gamma \)-weakly convex on \( \mathcal{X} \). Trivially we can write

\[
f(x') + \frac{1}{2\beta} ||x - x'||^2 = \left( f(x') + \frac{\gamma}{2} ||x - x'||^2 \right) + \frac{1}{2} \left( \frac{1}{\beta} - \gamma \right) ||x - x'||^2.
\]

If we write \( f_{\gamma,x}(u) := f(u) + (\gamma/2)||x-u||^2 \) and \( \beta_\gamma := (\beta^{-1} - \gamma)^{-1} \) for readability, then as long as \( \beta_\gamma > 0 \) we have for all \( x \in \mathcal{X} \) that \( \text{env}_\beta(f)(x) = \text{env}_{\beta_\gamma}(f_{\gamma,x}(x)) \) and \( \text{prox}_\beta f(x) = \text{prox}_{\beta_\gamma}(f_{\gamma,x}(x)) \). By leveraging Proposition 1 under the Hilbert space assumption, we have that for any \( x \in \mathcal{X} \), the function \( f_{\gamma,x}(\cdot) \) is convex. This means that as long as \( \beta_\gamma > 0 \), namely whenever \( \gamma < \beta^{-1} \), all the standard results available for the case of convex functions can be brought to bear on the problem.\footnote{One can just apply standard arguments such as given by Bauschke and Combettes \cite[Prop. 12.30]{Bauschke2011}, while utilizing the weak convexity property described. See also Drusvyatskiy and Paquette \cite[Lem. 4.3]{Drusvyatskiy2018}, Davis and Drusvyatskiy \cite[Lem. 2.2]{Davis2017}, and Poliquin and Rockafellar \cite[Thm. 4.4]{Poliquin1998}.}

Of particular importance to us is the fact that when \( f \) is LSC and \( \gamma \)-weakly convex, the Moreau envelope is differentiable, with gradient

\[
(\text{env}_\beta f)'(x) = \frac{1}{\beta} \left( x - \text{prox}_\beta f(x) \right),
\]

well-defined for all \( \beta < \gamma^{-1} \) and \( x \in \mathcal{X} \). We will be interested in finding stationary points of \( f \), namely those \( x \in \mathcal{X} \) such that \( 0 \in \partial_C f(x) \). From the basic properties of the envelope and

\[\footnote{See for example Bauschke and Combettes \cite[Ch. 12 and 24]{Bauschke2011}. For Banach spaces, modified notions of “proximity” measured using Bregman divergences have also been developed \cite[24]{Bauschke2011}. See also Jourani et al. \cite{Jourani2013} for more analysis of the Moreau envelope in more general spaces.} \]

Remark 3. The preceding Proposition 2 extends a result of Drusvyatskiy and Paquette \cite[Lem. 4.2]{Drusvyatskiy2018} from the case where \( \mathcal{X} \) and \( \mathcal{Y} \) are finite-dimensional Euclidean spaces, to the general Banach space setting considered here. For the classical case of Euclidean spaces, exact chain rules are well-known \cite[Ch. 10.B]{Bauschke2011}.
proximal mapping, for $\gamma$-weakly convex $f$ we have
\[ \text{dist}(0; \partial C f(\prox_{\beta} f(x))) \leq \|(env_{\beta} f)'(x)\|. \] (27)
That is, for any point $x \in X$, the point $\prox_{\beta} f(x) \in X$ is approximately stationary. The degree of precision is controlled by the gradient of $env_{\beta} f$ evaluated at $x$. In addition, it follows immediately from (26) that
\[ \|x - \prox_{\beta} f(x)\| = \beta \|(env_{\beta} f)'(x)\|. \] (28)
Since one trivially also has $f(\prox_{\beta} f(x)) \leq f(x)$, the norm of the gradient of $env_{\beta} f$ evaluated at $x$ also tells us how far we are from a point (namely $\prox_{\beta} f(x) \in X$) which is no worse than $x$ in terms of function value. These basic facts directly motivate the use of the Moreau envelope norm to quantify algorithm performance.\textsuperscript{27}

3 Theoretical analysis

In doing analysis of learning algorithm performance with respect to the risks (4) and (5), the properties of both the loss and the risk function (to which the loss is passed) play an important role. We separate our analysis into two sub-sections: in section 3.1, we consider just the risk functions, and in section 3.2 we consider risks induced by a composition of losses and risk functions. To keep things readable, we introduce the following notation:
\[
\tilde{r}_1(Z, \theta) := E_{\mu} Z + \text{dev}_{\mu}(\theta; Z) \quad (29)
\]
\[
\tilde{r}_2(Z, \theta) := \theta + \text{dev}_{\mu}(\theta; Z). \quad (30)
\]
The functions $\tilde{r}_1 : Z \times \mathbb{R} \to \mathbb{R}$ and $\tilde{r}_2 : Z \times \mathbb{R} \to \mathbb{R}$ just represent the intermediate step taken in the risk function computations, so that (2) and (3) are respectively $\inf\{\tilde{r}_1(Z, \theta) : \theta \in \Theta\}$ and $\inf\{\tilde{r}_2(Z, \theta) : \theta \in \Theta\}$.

3.1 Properties of risk function class

The most important part of the intermediate functions $\tilde{r}_1$ and $\tilde{r}_2$ is the deviation map $(Z, \theta) \mapsto \text{dev}_{\mu}(\theta; Z)$. We shall assume that $Z$ comes from a set $\mathcal{Z}$, and the basic running requirement we make on $\mathcal{Z}$ is as follows:

\textbf{A0.} $\mathcal{Z}$ is a linear space of $\mathcal{F}$-measurable random variables.\textsuperscript{28}

We proceed by looking at how basic properties of $\rho$ impact the function $\text{dev}_{\mu} : \mathbb{R} \times \mathcal{Z} \to \mathbb{R}$ and the risk functions that are based upon it.

\textbf{Proposition 4 (Measurability).} If $\rho$ is Borel-measurable, then for any choice of $Z \in \mathcal{Z}$ and $\theta \in \mathbb{R}$, we have that $\text{dev}_{\mu}(\theta; Z)$, $\tilde{r}_1(Z, \theta)$ and $\tilde{r}_2(Z, \theta)$ are $\mathcal{F}$-measurable.

\textbf{Proof.} Since $Z \in \mathcal{Z}$ is by assumption measurable, the measurability of $\rho$ as a function on the real line implies the measurability of the map $\omega \mapsto \rho((Z(\omega) - \theta)/\sigma)$, and directly implies the desired result.\textsuperscript{29}

\textsuperscript{27}This is highlighted in works such as Drusvyatskiy and Paquette [8] and Davis and Drusvyatskiy [7].
\textsuperscript{28}That is, for each $Z \in \mathcal{Z}$ we have that $Z : \Omega \to \mathbb{R}$ satisfies $Z^{-1}((-\infty, a]) := \{\omega : Z(\omega) \leq a\} \in \mathcal{F}$, for all $a \in \mathbb{R}$.
\textsuperscript{29}This uses basic results on measurability of compositions, e.g., Ash and Doléans-Dade [2, Lem. 1.5.7].
For the remainder of this section, we shall assume A0, and that ρ is Borel-measurable.

**Proposition 5** (Convexity). If ρ is non-negative and proper convex, then devμ, ˜r1, and ˜r2 are all proper convex.

**Proof.** If ρ is convex, then for any α ∈ (0, 1), Z1, Z2 ∈ Z and θ1, θ2 ∈ R, we have

\[
ρ\left(\frac{(αZ_1 + (1-α)Z_2) - (αθ_1 + (1-α)θ_2)}{σ}\right) = ρ\left(\frac{Z_1 - θ_1}{σ} + (1-α)\frac{Z_2 - θ_2}{σ}\right)
\leq αρ\left(\frac{Z_1 - θ_1}{σ}\right) + (1-α)ρ\left(\frac{Z_2 - θ_2}{σ}\right).
\]

Convexity of devμ follows immediately from monotonicity and additivity of the integral.\(^{30}\) As for propriety, we have devμ > −∞ trivially since ρ ≥ 0 and μ is a measure. Furthermore, since ρ is proper, there exists θ0 ∈ R such that −∞ < ρ(θ0/σ) < ∞. Since Z is a linear space, we can take 0 ∈ Z to obtain devμ(θ0; 0) < ∞, implying that dom devμ ≠ ∅. Thus devμ is also proper. The convexity of ˜r1 and ˜r2 follow immediately from the convexity of devμ and the fact that the sum of a convex and affine function is convex. The propriety of ˜r2 follows from the propriety of devμ. Finally, since ˜r1(0, θ0) = 0 + devμ(θ0; 0) < ∞, we have that ˜r1 is also proper. \(\square\)

**Proposition 6** (Lower semi-continuity). If ρ is non-negative and LSC, then devμ is LSC.

**Proof.** Let (Z_n) and (θ_n) respectively be sequences on Z and R. As we take n → ∞, say Z_n → Z pointwise, for some Z ∈ X, and θ_n → θ ∈ R. Since by assumption ρ is LSC on R, using (6) we have (again, pointwise) that

\[
ρ\left(\frac{Z - θ}{σ}\right) \leq \liminf_{n→∞} ρ\left(\frac{Z_n - θ_n}{σ}\right).
\]

Writing ρ_n := ρ((Z_n - θ_n)/σ) for each n ≥ 1 and ρ_σ = ρ((Z - θ)/σ), it follows that

\[
E_ρ ρ_σ ≤ E_ρ \left(\liminf_{n→∞} ρ_n\right) \leq \liminf_{n→∞} (E_ρ ρ_n).
\]

(31)

The former inequality follows from monotonicity of the integral, and the latter inequality follows from an application of Fatou’s inequality, which is valid since ρ_n ≥ 0.\(^{31}\) Taking both ends of (31) together, since the choice of sequences (Z_n) and (θ_n) were arbitrary, it follows again from the equivalence (6) that the functional (Z, θ) → E_µ ρ((Z - θ)/σ) is LSC on Z × R. Clearly devμ is also LSC. \(\square\)

**Remark 7** (Implications for the existence of minima). Note that the above proposition only relies upon the LSC property of ρ, it does not require convexity. With lower semi-continuity in hand, it becomes quite routine to prove that the infimum is achieved in concrete cases of interest. For example, the generalized Weierstrass theorem tells us that on any normed linear space, the infimum of any LSC function is achieved on any compact subset.\(^{32}\) At the very least, this tells us that if Θ is bounded and closed, then both risk functions (2) and (3) are well-defined.\(^{33}\)

---

\(^{30}\)Ash and Doléans-Dade [2, Thm. 1.5.9, Sec. 1.6].

\(^{31}\)Ash and Doléans-Dade [2, Lem. 1.6.8].

\(^{32}\)Luenberger [11, Sec. 2.13], Barbu and Precupanu [3, Sec. 2.1.2].

\(^{33}\)This is immediate for (2). For (3), since ˜r2 is a linear combination of a continuous function and a LSC function, it is also LSC and thus the result holds.
Remark 8 (Links to LSC property for risk functions). It is well-known that the supremum of any family of LSC functions is itself LSC.\(^{34}\) On the other hand, the nature of the infimum of families of LSC functions is not quite so simple.\(^{35}\) For our purposes, if \(\Theta\) is bounded and closed, then under the assumptions of Proposition 6, we have that both risk functions (2) and (3) are in fact LSC on \(Z\).\(^{36}\) Considering the previous Remark, this LSC property opens the door to obtaining sufficient conditions for the existence of minimizers of \(Z \mapsto \inf \{\tilde{r}_j(Z, \theta) : \theta \in \Theta\}\) on subsets of \(Z\).

We turn our attention to notions of differentiability and sub-differentials. For points \(Z, Z' \in Z, \theta, \theta' \in \mathbb{R}\) and scalar \(\alpha \neq 0\), consider the following generalized difference quotients:

\[
q_j(\alpha) := \frac{\tilde{r}_j(Z + \alpha Z', \theta + \alpha \theta') - \tilde{r}_j(Z, \theta)}{\alpha}.
\]  

(32)

If the limit \(\lim_{\alpha \to 0_+} q_j(\alpha)\) exists, this is the radial derivative of \(\tilde{r}_j\) at \((Z, \theta)\) in the direction \((Z', \theta')\). The quantity \(q_j(\alpha)\) is ideal, in the sense that since \(\mu\) will be unknown in practice, it can never be observed directly. It will only be accessible via random samples of the radial derivative. For both \(\tilde{r}_1\) and \(\tilde{r}_2\), the core component in this random sample will be

\[
Q(\alpha) := \frac{1}{\alpha} \left( \rho \left( \frac{Z - \theta + \alpha (Z' - \theta')}{\sigma} \right) - \rho \left( \frac{Z - \theta}{\sigma} \right) \right).
\]  

(33)

More explicitly, we have \(\mathbb{E}_\mu(Z' + \eta Q(\alpha)) = q_1(\alpha)\) and \(\mathbb{E}_\mu(\theta + \eta Q(\alpha)) = q_2(\alpha)\) for each \(\alpha \neq 0\), up to integrability. As long as \(\rho\) is convex, these random samples afford us an unbiased estimate of the true radial derivative, as the following result makes precise.

Proposition 9. Under the assumptions of Proposition 5, for any \(j \in \{1, 2\}\), we have that \(\tilde{r}_j\) is radially differentiable on \(\text{int}(\text{dom} \tilde{r}_j)\), and that we have access to unbiased random samples of the radial derivatives in that

\[
\mathbb{E}_\mu \lim_{\alpha \to 0_+} (Z' + \eta Q(\alpha)) = \lim_{\alpha \to 0_+} q_1(\alpha)
\]

\[
\mathbb{E}_\mu \lim_{\alpha \to 0_+} (\theta' + \eta Q(\alpha)) = \lim_{\alpha \to 0_+} q_2(\alpha)
\]

for any choice of directions \(Z' \in Z\) and \(\theta' \in \mathbb{R}\).

Proof. It follows from Proposition 5 that if \(\rho\) is proper convex, so is \(\tilde{r}_j\) for all \(j \in \{1, 2\}\). This implies that for any choice of \((Z, \theta) \in \text{dom} \tilde{r}_j\), the map \(\alpha \mapsto q_j(\alpha)\) is non-decreasing on the positive reals, in the sense that \(\alpha_1 > \alpha_2 > 0\) implies \(q_j(\alpha_1) \geq q_j(\alpha_2)\), regardless of the choice of directions \(Z' \in Z\) and \(\theta' \in \mathbb{R}\).\(^{37}\) Propriety guarantees \(q_j(\alpha) > -\infty\) for all \(\alpha \neq 0\), and under the stronger condition that \((Z, \theta) \in \text{int}(\text{dom} \tilde{r}_j)\), then for small enough \(\alpha > 0\), we also have \(q_j(\alpha) < \infty\), and using monotonicity we can show \(\lim_{\alpha \to 0_+} q_j(\alpha) < \infty\).\(^{38}\) That is, \(\tilde{r}_j\) is radially differentiable on \(\text{int}(\text{dom} \tilde{r}_j)\), for \(j \in \{1, 2\}\).

If we take \((Z, \theta) \in \text{int}(\text{dom} \tilde{r}_j)\) (for any \(j \in \{1, 2\}\)), then from the preceding paragraph, there exists a \(\alpha > 0\) small enough that \(\mathbb{E}_\mu Q(\pi) < \infty\). Furthermore, \(Q(\pi)\) is \(\mu\)-almost surely finite.\(^{39}\) Constraining ourselves to this event of probability 1, keeping \(\tilde{r}\) fixed, let \((\alpha_n)\) be a sequence such that \(\tilde{r} > \alpha_1 > \alpha_2 > \cdots > 0\). Analogous to the deterministic case above,

\(^{34}\)See for example Barbu and Precupanu [3, Cor. 2.6].

\(^{35}\)See Penot [14, Prop. 1.22] for general-purpose sufficient conditions.

\(^{36}\)Penot [14, Cor. 1.23].

\(^{37}\)For a detailed proof of this fact, see the exposition leading up to Proposition 21 in the appendix.

\(^{38}\)Recall (47) in the appendix.

\(^{39}\)Ash and Doléans-Dade [2, Thm. 1.6.6].
the convexity of $\rho$ implies that $\infty > Q(\mathbf{1}) \geq Q(\alpha_1)$ for all integer $n$ on this “good” event, and also that $Q(\alpha_n) \downarrow \lim_{n \to \infty} Q(\alpha) > -\infty$. Using this monotonicity and the fact that $E_\mu Q(\mathbf{1}) < \infty$, we can apply the monotone convergence theorem. This subsequently implies that $E_\mu Q(\alpha_n) \downarrow E_\mu \lim_{\alpha \to \infty} Q(\alpha)$, and thus we conclude that

$$
\lim_{\alpha \to \infty} q_j(\alpha) = \begin{cases} 
\lim_{\alpha \to \infty} \left( Z' + \eta Q(\alpha) \right), & \text{when } j = 1 \\
\lim_{\alpha \to \infty} \left( \theta + \eta Q(\alpha) \right), & \text{when } j = 2.
\end{cases}
$$

Since the choice of $Z' \in \mathbb{Z}$ and $\theta' \in \mathbb{R}$ was arbitrary, this concludes the proof.

When we require $Z$ to be a Banach space, and that $\rho$ be both convex and LSC, then the situation is much simpler, and we can easily obtain additional insights, as follows.

**Proposition 10** (Sub-differentiability). Let $Z$ be a Banach space. If $\rho$ is proper convex and LSC, then we have that $\tilde{r}_j$ is continuous and sub-differentiable on $\text{int}(\text{dom} \tilde{r}_j)$, for all $j \in \{1, 2\}$.

**Proof.** For any $j$, by Propositions 5 and 6, we have that $\tilde{r}_j$ is proper convex and LSC. Any convex LSC function on a Banach space is continuous at every point in the interior of its effective domain. It thus follows that $\text{int}(\text{dom} \tilde{r}_j) \subset \{(Z, \theta) \in \mathbb{Z} \times \mathbb{R} : \partial \tilde{r}_j(Z, \theta) \neq \emptyset\}$. □

**Remark 11** (Non-monotone risk functions). Due to work such as Ruszczyński and Shapiro [21, Sec. 3], the continuity and sub-differentiability of any risk function which is convex and monotone is well-known for a large class of Banach spaces. We have made no assumptions on $\rho$ that would imply monotonicity; indeed, it is easy to construct an example where $Z_1, Z_2 \in \mathbb{Z}$ satisfy $Z_1 \geq Z_2$ $\mu$-almost surely but $\text{inf}\{\tilde{r}_j(Z_1, \theta) : \theta \in \Theta\} < \text{inf}\{\tilde{r}_j(Z_2, \theta) : \theta \in \Theta\}$. As such, the existing results for convex monotone risk functions cannot be applied here.

**Remark 12** (Risk functions under unbounded $\Theta$). Under the setup of Proposition 10, clearly we can construct families of proper convex LSC risk functions of the form $Z \mapsto \inf\{\tilde{r}_j(Z, \theta) : \theta \in \Theta\}$, whenever $\Theta$ is bounded and closed. What can we say about such functions when $\Theta$ is unbounded, say the case of $\Theta = \mathbb{R}$? If we also assume that $\rho$ is coercive, one can use Propositions 5 and 6 to establish conditions under which the function $\theta \mapsto \text{dev}_\mu(\theta; Z)$ is proper convex, LSC, and coercive on $\mathbb{R}$, for all $Z \in \mathbb{Z}$. This alone is sufficient to imply that $\text{inf}\{\tilde{r}_1(Z, \theta) : \theta \in \mathbb{R}\} > -\infty$, i.e., risk functions of the form (2) are well-defined even when $\Theta = \mathbb{R}$. This situation is less straightforward for $\tilde{r}_2$, since for any fixed $Z \in \mathbb{Z}$, even if $\rho$ is coercive, if it grows too slowly, it is always possible that the function $\theta \mapsto \tilde{r}_2(Z, \theta)$ decrease without bound as $\theta \to -\infty$.

**Remark 13** (Classical risk and deviation axioms). In the financial risk literature, the seminal work of Artzner et al. [1] gives an axiomatic characterization of risk functions that can be considered “coherent,” while Rockafellar et al. [17] characterize functions which capture the intuitive notion of “deviation,” and establish a lucid relationship between coherent risks and...
their deviation class. What can we say about our risk functions classes in terms of these traditional axioms? For some $Z \in \mathcal{Z}$, let $\theta_1(Z)$ and $\theta_2(Z)$ respectively achieve the infimum in (2) and (3), i.e., we have

$$E_\mu Z + \text{dev}_\mu(\theta_1(Z); Z) = \inf_{\theta \in \Theta} \tilde{r}_1(Z, \theta)$$
$$\theta_2(Z) + \text{dev}_\mu(\theta_2(Z); Z) = \inf_{\theta \in \Theta} \tilde{r}_2(Z, \theta).$$

If we are to view the entire left-hand side of these equations as risk functions (of $Z$), then in general, clearly they need not fall into the classical class of coherent risk functions. More precisely, while convexity of $\rho$ is enough to imply convexity, as we discussed in Remark 11, except in very special cases (e.g., where $\rho$ is such that $x \mapsto x + \rho(x)$ is non-decreasing on $\mathbb{R}$), monotonicity cannot be guaranteed for either class. In addition to convexity, we also assume that $\rho$ is differentiable, and that in differentiating $\tilde{r}_j$ we can take the derivative under the integral $E_\mu$.\footnote{On when we can reverse the order of operations, see Ash and Doléans-Dade\cite[Sec. 1.6]{ss} for a simple example.} First order optimality conditions then take the form

$$E_\mu \rho' \left( \frac{Z - \theta_j(Z)}{\sigma} \right) = \begin{cases} 0, & \text{when } j = 1 \\ \tilde{Z}_\eta, & \text{when } j = 2. \end{cases}$$

Since convexity of $\rho$ on $\mathbb{R}$ is equivalent to $\rho'$ being monotone non-decreasing on $\mathbb{R}$, clearly $Z \geq Z'$ implies $\theta_j(Z) \geq \theta_j(Z')$ for $j \in \{1, 2\}$. That is, while the entire output of the risk function is not monotone, the “pivot” $\theta_1(Z)$ and “location” part $\theta_2(Z)$ are clearly monotone.

Furthermore, these points are both translation equivariant in that for any fixed $a \in \mathbb{R}$, we have that $\theta_j(Z + a) = \theta_j(Z) + a \in \arg\min \{\tilde{r}_j(Z + a, \theta) : \theta \in \Theta\}$. Note that we also have $\rho_\sigma(\theta_j(Z + a); Z + a) = \rho_\sigma(\theta_j(Z); Z)$ for any $a \in \mathbb{R}$, so the translation invariance of the deviation part holds. Non-negativity of $\rho$ implies that $\text{dev}_\mu \geq 0$, and if we also and assume $\rho(0) = 0$ and restrict $\Theta = [0, b)$ for some $b \in \mathbb{R}$, then we also have $\text{dev}_\mu(\theta_j(0); 0) = 0$. Positive homogeneity of the risk functions does not hold by default, but it can be easily enforced (in principle) if the parameters $\sigma$ and $\eta$ are properly modified.\footnote{That is, when passed $\alpha Z$ instead of $Z$, by scaling $\sigma \propto \alpha$ and $\eta \propto \alpha$, we can achieve $\inf_{\theta \in \Theta} \tilde{r}_j(\alpha Z, \theta) = \alpha \inf_{\theta \in \Theta} \tilde{r}_j(Z, \theta)$.}

As such, we see that the location and deviation terms of the risk functions being considered here satisfy some key axioms of coherent risks and deviations, respectively.

### 3.2 Learning guarantees

To bring our analysis into the context of learning problems, we need some mechanism for introducing random, data-driven feedback to the learning algorithm. Let $(\Omega, \mathcal{F}, \mu)$ be the underlying probability space, and let $\mathcal{H}$ be our hypothesis class of interest, assumed to be a normed linear space. For any candidate $h \in \mathcal{H}$, we say that any non-negative random-variable depending on $h$ is a loss incurred by $h$. More precisely, we denote a loss incurred by $h$ by any $\mathcal{F}$-measurable function $L(h; \cdot) : \Omega \to \mathbb{R}_+$. When a particular realization $\omega \in \Omega$ is important, we will write $L(h; \omega)$, but otherwise, for readability we will typically write $L(h) := L(h; \cdot)$. For concreteness, as a running assumption, we will assume that

$$E_\mu |L(h)|^2 = \int_\Omega |L(h; \omega)|^2 \mu(d\omega) < \infty \tag{34}$$

for all $h \in \mathcal{H}$. It follows that $\{L(h) : h \in \mathcal{H}\} \subset L_2(\Omega, \mathcal{F}, \mu)$. Thus the map $h \mapsto L(h)$ takes us from $\mathcal{H}$ to $L_2(\Omega, \mathcal{F}, \mu)$. 
Loss-specific terminology To ensure our use of formal terms is clear, we apply the definitions of section 2.1 to losses here. We shall typically suppress the dependence on $\omega \in \Omega$ in directional derivatives and gradients, writing $L'_h(h; g) \equiv L'_h(h; g; \cdot)$, $L'(h; g) \equiv L'(h; g; \cdot)$, and $L'(h) \equiv L'(h; \cdot)$, for all $h \in \mathcal{H}$ and $g \in \mathcal{G}$. Let $H \subset \mathcal{H}$ be an open set. We say that $L$ is Gateaux differentiable at $h \in H$ if the radial derivative $L'_h(h; g)$ exists for all directions $g \in \mathcal{G}$, $\mu$-almost surely. We say that $L$ is directionally differentiable at $h \in H$ if the directional derivative $L'(h; g)$ exists for all directions $g \in \mathcal{G}$, $\mu$-almost surely. On this “good” event of probability 1, if the map $h \mapsto L'(h; g)$ is linear and continuous, we say $L$ is Hadamard differentiable at $h$. We say that $L$ is (Fréchet) differentiable at $h \in H$, if there exists a function $L'(h) : \mathcal{H} \to \mathbb{R}$ that is linear, continuous, and which satisfies (9) $\mu$-almost surely. We say that $L$ is $\lambda$-Lipschitz at $h \in \mathcal{H}$ if there exists a $\delta > 0$ such that $\|h - h'\| < \delta \implies \|L(h) - L(h')\| \leq \lambda \|h - h'\|$. With the running assumption about second moments, this amounts to requiring

$$
\|h - h'\| < \delta \implies \mathbb{E}_{\mu} |L(h) - L(h')|^2 \leq \lambda^2 \|h - h'\|^2.
$$

We say that $L$ is weakly $\lambda$-smooth at $h \in H$ if $L$ is Gateaux differentiable and the map $h \mapsto L'_h(h; \cdot)$ is $\lambda$-Lipschitz $\mu$-almost surely at $h$. That is, if for small enough $\delta > 0$ we have

$$
\|h - h'\| < \delta \implies \|L'_h(h; \cdot) - L'_h(h'; \cdot)\| \leq \lambda \|h - h'\|.
$$

Note that the norm used here is the operator norm applied to the linear map $L'_h(h; \cdot)$. In order to bridge the risks $R_1$ and $R_2$ defined in (4) and (5) with the results in section 3.1 for generic risk functions, note that we have

$$
R_j(h) = \inf_{\theta \in \Theta} \tilde{r}_j(L(h), \theta), \quad j \in \{1, 2\}.
$$

That is, we have a simple composition $(h, \theta) \mapsto (\tilde{r}_j(L(h), \theta))$ before minimizing over $\theta \in \Theta$. Using the preliminary results of section 2.3, it is straightforward to obtain sufficient conditions for this composite function to be weakly convex.

**Proposition 14.** Let the hypothesis class $\mathcal{H}$ be Banach. Let the loss $L$ be locally Lipschitz and weakly $\lambda_1$-smooth on $\mathcal{H}$. Let $\rho$ be proper convex and $\lambda_0$-Lipschitz on $\mathbb{R}$. Then, the function $(h, \theta) \mapsto \tilde{r}_j(L(h), \theta)$ is $\gamma$-weakly convex with $\gamma \leq (1 + \eta \lambda_0/\sigma) \max\{1, \lambda_1\}$, for any choice of $j \in \{1, 2\}$.

**Proof.** Recall the generic result given in Proposition 2 for the weak convexity of generic composite functions. Our proof here amounts to checking that the assumptions of Proposition 2 are satisfied for the composition $f := f_2 \circ f_1$, with $f_1 : \mathcal{H} \times \mathbb{R} \to \mathcal{L}_2(\Omega, \mathcal{F}, \mu) \times \mathbb{R}$ and $f_2 : \mathcal{L}_2(\Omega, \mathcal{F}, \mu) \times \mathbb{R} \to \mathbb{R}$ defined as

$$
F_1(h, \theta) := (L(h), \theta)
$$

$$
F_2(Z, \theta) := \tilde{r}_j(Z, \theta)
$$

for $j \in \{1, 2\}$.

To start, let us consider the properties of $F_1$. Since $L$ is locally Lipschitz and Gateaux differentiable, it follows that $L$ is also Hadamard differentiable. Since the map $h \mapsto L'_h(h; \cdot) = \lambda$-Lipschitz at $h \in \mathcal{H}$ if there exists a $\delta > 0$ such that $\|h - h'\| < \delta \implies \|L(h) - L(h')\| \leq \lambda \|h - h'\|$. With the running assumption about second moments, this amounts to requiring

$$
\|h - h'\| < \delta \implies \mathbb{E}_{\mu} |L(h) - L(h')|^2 \leq \lambda^2 \|h - h'\|^2.
$$

48In our particular setting with losses here, the norm used in the numerator of (9) will be the $\mathcal{L}_2$ norm.

49Penot [14, Prop. 2.25].
L′(h; ·) is continuous (by weak smoothness), it follows that L is (Fréchet) differentiable.\textsuperscript{50} Since (h, θ) → F_1(h, θ) = (L(h), θ) just passes θ through the identity, trivially the second component is also differentiable, and the differentiability of both components thus implies F_1 is differentiable.\textsuperscript{51} Furthermore, the local Lipschitz property of L is clearly retained by the map (h, θ) → F_1(h, θ) = (L(h), θ). Evaluating the gradients we have F_1′(h, θ)(g, r) = (L′(h)(g), r), and thus using a typical product space norm we have
\[ \|F_1′(h_1, θ_1) - F_1′(h_2, θ_2)\| = \|L′(h_1) - L′(h_2)\| + |θ_1 - θ_2|. \]
By weak smoothness of L, it follows that F_1 is max{1, λ_1}-smooth (in the sense defined in section 2.3) µ-almost surely.

Next, let us look at properties of f_2. Note that using the Lipschitz property of ρ, we obtain
\[ |f_2(Z_1, θ_1) - f_2(Z_2, θ_2)| \leq \left\{ \begin{array}{ll} |θ_1 - θ_2| + (1 + \frac{2λ_0}{σ}) E_μ|Z_1 - Z_2|, & \text{when } j = 1 \\ (1 + \frac{νλ_0}{σ}) |θ_1 - θ_2| + \frac{2λ_0}{σ} E_μ|Z_1 - Z_2|, & \text{when } j = 2. \end{array} \right. \]
To obtain bounds in terms of the correct norm, note that
\[ E_μ|Z_1 - Z_2| \leq μ(Ω)\sqrt{E_μ|Z_1 - Z_2|^2} = \sqrt{E_μ|Z_1 - Z_2|^2}, \]
which follows from the fact that μ is a probability, and a simple application of Hölder’s inequality.\textsuperscript{52} Plugging this into the previous inequality, and noting that it holds for any choice of Z_1, Z_2 ∈ L_2(Ω, F, μ) and θ_1, θ_2 ∈ ℝ, it follows that f_2 is (1 + ηλ_0/σ)-Lipschitz on L_2(Ω, F, μ) × ℝ, and F_1(H × ℝ) ⊂ dom f_2, for all cases j ∈ {1, 2}. Furthermore, from Proposition 5, we have that convexity of ρ implies that f_2 is convex.

Taking the above points together, if we consider the good event of probability 1 where F_1 satisfies the desired properties, direct application of Proposition 2 to the map (h, θ) → (f_2 ∘ F_1)(h, θ) yields the desired result.

Remark 15. The result in the preceding Proposition 14 is rather useful since it does not require the loss to be convex. When the loss is convex, the analysis becomes somewhat simpler and stronger arguments are naturally possible; composite risks under convex losses and convex, monotone risk functions is the setting considered by Ruszczyński and Shapiro [21, Sec. 3.2], for example.

We have established conditions under which the intermediate joint objective r_j(L(h), θ) is weakly convex, and characterized this weak convexity with respect to properties of the underlying risk function and data distribution. Since the data distribution μ is unknown, we can never actually compute r_j(L(h), θ). Any learning algorithm will only have access to feedback of a stochastic nature which provides incomplete, noisy information. Our next task is to establish conditions under which the feedback available to the learner is “good enough” to ensure reasonable performance guarantees.

**Unbiased stochastic feedback** Let us make our notion of feedback more precise. First, to keep the subsequent analysis as readable as possible, we introduce the following notation:\textsuperscript{53}
\[ F_1(h, θ) := (L(h), θ) \]
\[ f_j(x, θ) := \left\{ \begin{array}{ll} x + ηρ\left(\frac{x - θ}{σ}\right), & \text{when } j = 1 \\ θ + ηρ\left(\frac{x - θ}{σ}\right), & \text{when } j = 2. \end{array} \right. \]
\textsuperscript{50}Penot [14, Prop. 2.51].
\textsuperscript{51}Penot [14, Prop. 2.52].
\textsuperscript{52}Ash and Doléans-Dade [2, Sec. 2.4].
\textsuperscript{53}Note that f_j in the case of j = 2 here should not be confused with f_2 from the proof of Proposition 14.
For each $h \in \mathcal{H}$ and $\theta \in \mathbb{R}$, $F_1(h, \theta)$ is a random vector. We shall assume that for any $h \in \mathcal{H}$, the learner can obtain independent random samples of the loss $L(h)$ and the associated gradient $L'(h)$. Since $F_1'(h, \theta) = (L'(h), 1)$ by which we mean $F_1'(h, \theta)(g, r) = (L'(h)(g), r)$ for all $g \in \mathcal{H}$ and $r \in \mathbb{R}$, clearly the learner can also independently sample from $F_1(h, \theta)$ and $F_1'(h, \theta)$. If $\rho$ is convex, then $f_j$ is convex (for $j \in \{1, 2\}$). Furthermore, if $\rho$ is LSC, then recalling the proof of Proposition 10, it follows that $f_j$ is continuous and sub-differentiable on $\text{int}(\text{dom } f_j)$. Since $\rho$, $\eta$, and $\sigma$ are all known by the learner, for any $(x, \theta) \in \text{int}(\text{dom } f_j)$ the learner will reasonably be able to acquire an element from $\partial f_j(x, \theta)$. Thus if $(L(h), \theta) \in \text{int}(\text{dom } f_j) \mu$-almost surely, it follows that the learner can sample from $\partial f_j(L(h), \theta) \circ F_1'(h, \theta)$. This is the stochastic feedback available to the learner, and when we ask that it be “good enough,” this means we require it to be an unbiased estimator of the (Clarke) sub-differential of $\tilde{r}_j$. The following result gives mild conditions under which this is achieved.

**Proposition 16.** Under the conditions of Proposition 14, for any $h \in \mathcal{H}$ and $\theta \in \mathbb{R}$, as long as $E_\mu |L(h; \cdot)| < \infty$, the stochastic sub-differential is an unbiased estimator in that

$$E_\mu [\partial f_j(L(h), \theta) \circ F_1'(h, \theta)] \subset \partial_C \tilde{r}_j(L(h), \theta)$$

for all $j \in \{1, 2\}$.

**Proof.** Using the weak smoothness of $L$, with probability 1, the map $(h, \theta) \mapsto F_1'(h, \theta)$ is continuous, plus $F_1$ is locally Lipschitz and $\partial_C F_1(h, \theta) = \{F_1'(h, \theta)\}$. Furthermore, since $F_1'(h, \theta)(g, r) = (L'(h)(g), r)$, the linearity of $L'(h)(\cdot)$ implies that $F_1'(h, \theta)(\mathcal{H} \times \mathbb{R}) = \mathbb{R}^2$ $\mu$-almost surely. Since $F_1$ and $f_j$ are (locally) Lipschitz, the facts we have just laid out imply a strong chain rule. That is, it holds $\mu$-almost surely that

$$\partial_C (f_j \circ F_1)(h, \theta) = \partial_C f_j(L(h), \theta) \circ F_1'(h, \theta)$$

$$= \partial f_j(L(h), \theta) \circ F_1'(h, \theta),$$

where the second equality follows from the convexity of $f_j$.

Next, the Lipschitz property of $\rho$ implies that $f_j$ is $\lambda$-Lipschitz for some $\lambda > 0$ (the actual value is not important). Using this Lipschitz nature of $f_j$ and the fact that $\partial_C F_1(h, \theta) = \{F_1'(h, \theta)\}$ implies $\partial_C L(h) = \{L'(h)\}$, we have

$$|f_j \circ F_1)(h, \theta)(g, r)| \leq \lim_{(\alpha, h, h') \to (0^+, h)} \lambda \left( \lim_{(\alpha, h', \theta') \to (0^+, h)} \frac{|(f_j \circ F_1)(h' + \alpha g, \theta' + \alpha r) - (f_j \circ F_1)(h', \theta')|}{\alpha} \right)$$

$$\leq \lambda \left( \lim_{(\alpha, h', \theta') \to (0^+, h)} \frac{|L(h' + \alpha g) - L(h')|}{\alpha} + |r| \right)$$

Thus, using the fact that $\tilde{r}_j(h, \theta) = E_\mu (f_j \circ F_1)(h, \theta)$ and $E_\mu |L'(h)(g)| < \infty$ for all $h, g \in \mathcal{H}$, we have

$$E_\mu (f_j \circ F_1)'(h, \theta; \cdot) = (\tilde{r}_j)'(h, \theta; \cdot)$$

by a direct application of dominated convergence.\(^{56}\)

\(^{54}\)Penot [14, Prop. 5.6].\(^{55}\)Penot [14, Prop. 5.13].\(^{56}\)Ash and Doléans-Dade [2, Thm. 1.6.9].
To conclude, taking $G(h, \theta) \in \partial f_j(L(h), \theta) \circ F'_i(h, \theta)$, by (39) it follows that we have $G(h, \theta) \in \partial C_\psi(f_j \circ F'_i)(h, \theta)$, and thus by definition of Clarke sub-differential, monotonicity of the integral, and finally (40), we obtain
\[ E_\mu G(h, \theta)(\cdot) \leq E_\mu (f_j \circ F'_i \circ C_\psi)(h, \theta; \cdot) = (\tilde{r}_j)_C(h, \theta; \cdot). \] (41)

Linearity of $E_\mu G(h, \theta)(\cdot)$ follows from the linearity of both $G(h, \theta)(\cdot)$ and the integral. Finally, applying (41) we have
\[ \sup_{\|u, r\|=1} E_\mu G(h, \theta)(g, r) \leq \sup_{\|g, r\|=1} (\tilde{r}_j)_C(h, \theta; u, r) < \infty, \]
where finiteness holds because $\tilde{r}_j$ is locally Lipschitz.\(^{57}\) Thus $E_\mu G(h, \theta) \in (\mathcal{H} \times \mathbb{R})^*$, and with (41) we have $E_\mu G(h, \theta) \in \partial C_\psi \tilde{r}_j(h, \theta)$ as desired. \( \square \)

**Remark 17.** The validity of interchanging the operations of (sub-)differentiation and expectation is a topic of fundamental importance in stochastic optimization and statistical learning theory. A useful, modern reference on this topic is included in Ruszczyński and Shapiro [20, Ch. 2]. A classical reference is Rockafellar and Wets [18]; see also Rockafellar [16] for a look at measurability of convex integrands. The interchangeability problem appears in various places in the literature over the years, see for example Shapiro [23] as well as Kall and Mayer [10, Ch. 3, Rmk. 2.2]. See also Ruszczyński and Shapiro [21, Eqn. (3.9)], who refer to generalized versions of a classic result due to Strassen [25].

**Algorithm analysis** Since we have established reasonable conditions under which the learner has access to unbiased estimates of the joint risk sub-differential, it is now possible to leverage stationarity results for iterative procedures that run on information of this sort. Given an arbitrary initial value $(\hat{h}_0, \theta_0) \in \mathcal{H} \times \Theta$, we consider an iterative update that generates $\hat{h}_{t+1}$ and $\hat{\theta}_{t+1}$ for each step $t \geq 0$ based on the stochastic sub-differential feedback
\[ G_t \in \partial f_j(L(\hat{h}_t), \hat{\theta}_t) \circ F'_i(\hat{h}_t, \hat{\theta}_t). \] (42)

As a nascent example, consider the special case in which $\mathcal{H}$ is a Hilbert space. Let $C \subset \mathcal{H} \times \Theta$ be a closed, convex set ($C = \mathcal{H} \times \Theta$ is allowed). All Hilbert spaces are reflexive Banach spaces, and $G_t \in (\mathcal{H} \times \mathbb{R})^*$ can be uniquely identified with an element of $\mathcal{H} \times \mathbb{R}$, for which we use the same notation $G_t$. Consider a traditional first-order update with step-size $\alpha_t > 0:\]
\[ (\hat{h}_{t+1}, \hat{\theta}_{t+1}) = \Pi_C \left( (\hat{h}_t, \hat{\theta}_t) - \alpha_t G_t \right). \] (43)

Here feasibility is enforced by projecting the post-update iterate to $C$, denoted by $\Pi_C$. To evaluate performance, we utilize $\text{env}_\beta \tilde{r}_j : (\mathcal{H} \times \mathbb{R}) \rightarrow (\mathcal{H} \times \mathbb{R})$, the Moreau envelope of the joint risk, recalling the general definition given in (24). In particular, since Proposition 14 gives us sufficient conditions for $\tilde{r}_j$ to be $\gamma$-weakly convex, it follows that whenever $\beta^{-1} > \gamma$, the gradient $(\text{env}_\beta \tilde{r}_j)'(h, \theta)$ is well-defined. Denoting the partial sequence $G_t := (G_0, \ldots, G_t)$, we formalize our assumptions as follows:

A1. All the assumptions of Proposition 16 hold. In particular, $\rho$ is $\lambda_0$-Lipschitz, the loss is weakly $\lambda_1$-smooth, and $E_\mu |L(h; \cdot)| < \infty$ for all $h \in \mathcal{H}$.

A2. $\mathcal{H}$ is a Hilbert space, and $C \subset \mathcal{H} \times \Theta$ is a closed and convex set.

A3. The feedback (42) satisfies $E[G_t \mid G_{[t-1]}] = E_\mu G_t$ for all $t > 0$.\(^{58}\)
Algorithm 1 Projected sub-gradient method with randomized output.

**inputs:** initial point \((\hat{h}_0, \hat{\theta}_0) \in C \subset H \times \Theta\), step sizes \((\alpha_t)\), and max iterations \(n\).

**for** \(t \in \{0, \ldots, n-1\} \) do 

Sample \(G_t\) via \((42)\).

Update \((\hat{h}_t, \hat{\theta}_t) \mapsto (\hat{h}_{t+1}, \hat{\theta}_{t+1})\) via \((43)\).

**end for**

Sample \(T \in \{0, \ldots, n-1\}\) with probabilities \(P\{T = t\} = \alpha_t/(\sum_{k=0}^{n-1} \alpha_k), t \in [n-1]\).

**return:** \((\tilde{h}_{[n]}, \tilde{\theta}_{[n]}) := (\hat{h}_T, \hat{\theta}_T)\).

A4. Second moments are bounded as \(\mathbf{E}_\mu \|G_t\|^2 \leq \kappa^2\) for all \(t\).

The precise procedure of interest is summarized in Algorithm 1. The following result gives a performance guarantee for this algorithm in terms of the Moreau envelope of the joint risk.

**Proposition 18** (Closeness to a nearly-stationary point). Under assumptions A1–A4, let \(\tilde{h}_{[n]}\) and \(\tilde{\theta}_{[n]}\) denote the output of Algorithm 1. Fixing \(\gamma := (1 + \eta\lambda_0/\sigma) \max\{1, \lambda_1\}\) and writing \(\tilde{r}_j := \inf_{(h, \theta) \in C} \{\tilde{r}_j(h, \theta) : (h, \theta) \in C\}\), for any choice of \(\tilde{r}_j < 1/\gamma\), taking expectation over all the feedback \(G_{[n-1]}\), we have

\[
\mathbf{E} \left\| (\text{env}_\beta \tilde{r}_j)\langle \tilde{h}_{[n]}, \tilde{\theta}_{[n]} \rangle\right\|^2 \leq \left(\frac{1}{1 - \beta \gamma}\right) \frac{\text{env}_\beta \tilde{r}_j(\hat{h}_0, \hat{\theta}_0) - \tilde{r}_j^* + \gamma \kappa^2 \sum_{t=0}^{n-1} \alpha_t^2/2}{\sum_{t=0}^{n-1} \alpha_t}
\]

for each \(j \in \{1, 2\}\).

**Proof.** With all the results established thus far, this proof has just two simple parts. First, we need to show that the objective function of interest is weakly convex, and that we have access to unbiased estimates of the sub-differential; this is done here. Once this has been established, the remaining part just has us applying recent results from the literature for non-asymptotic control of the envelope gradient norm.

To begin, the assumptions of Proposition 14 are satisfied by A1, which implies that \(\tilde{r}_j\) is \(\gamma\)-weakly convex, and that we can take \(\gamma = (1 + \eta\lambda_0/\sigma) \max\{1, \lambda_1\}\). Furthermore, the \(\mu\)-integrability assumption on \(L'(h)\) lets us use Proposition 16 to ensure that feedback drawn from \((42)\) is such that \(\mathbf{E}_\mu G_t = \partial C \tilde{r}_j(\hat{h}_t, \hat{\theta}_t)\) for all \(t\). Furthermore, using A3 implies that \(\mathbf{E}[G_t | G_{[t-1]}] \in \partial C \tilde{r}_j(\hat{h}_t, \hat{\theta}_t)\) for all \(t\), since Algorithm 1 uses \(G_t\) sampled via \((42)\).

The desired result follows from an application of Davis and Drusvyatskiy [7, Thm. 3.1], where their objective function \(f\) corresponds to our \(\tilde{r}_j\) in either case of \(j \in \{1, 2\}\). While their proof is given for the case of \(H = \mathbb{R}^d\), if we assume A2, leverage the characterization of weak convexity (our Proposition 1), and replace their Lemma 2.2 with our (26), one can readily show that their result extends to arbitrary Hilbert spaces using the usual induced norm. Thus with the moment bound A4 in hand, their proof generalizes, and their result can be applied to Algorithm 1, for objective function \(\tilde{r}_j\), which has just proved to be \(\gamma\)-weakly convex. The desired result follows immediately. \(\square\)

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\(^{57}\)Penot [14, Prop. 5.2(b)].

\(^{58}\)The expectation on the left-hand side is with respect to the joint distribution of \(G_t\) conditioned on \(G_{[t-1]}\).

\(^{59}\)It also relies on the observation that a *proximal* stochastic sub-gradient update using the indicator function of \(C\) as a regularizer is equivalent to the *projected* sub-gradient update we do here.

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Let us briefly summarize some direct take-aways from Proposition 18. Writing $\Delta_0 := \text{env}_3 \tilde{r}_j(\tilde{h}_0, \tilde{\theta}_0) - \tilde{r}_j^*$ for readability, if $\Delta_0, \gamma$, and $\kappa$ are known (upper bounds will of course suffice), then constructing step sizes as $\alpha_t^2 \geq \Delta_0/(n\gamma\kappa^2)$, if we set $\beta = 1/(2\gamma)$, it follows that

$$E \| (\text{env}_{1/(2\gamma)} \tilde{r}_j)'(\tilde{h}_{[n]}, \tilde{\theta}_{[n]}) \|^2 \leq \sqrt{\frac{2\gamma\kappa^2\Delta_0}{n}}.$$ 

Fixing some desired precision level of

$$\sqrt{E \| (\text{env}_{1/(2\gamma)} \tilde{r}_j)'(\tilde{h}_{[n]}, \tilde{\theta}_{[n]}) \|^2} \leq \varepsilon,$$

the sample complexity is $O(\varepsilon^{-4})$. With enough samples to achieve this $\varepsilon$-precision, recalling the general properties of the Moreau envelope from section 2.4, we have that the algorithm output $(\tilde{h}_{[n]}, \tilde{\theta}_{[n]})$ is $(\varepsilon/2\gamma)$-close to a $\varepsilon$-nearly stationary point. More precisely, we have that writing

$$(h^*_n, \theta^*_n) := \text{prox}_{1/(2\gamma)} \tilde{r}_j \left( \tilde{h}_{[n]}, \tilde{\theta}_{[n]} \right),$$

it follows that

$$E \left[ \text{dist} \left( 0; \partial_C \tilde{r}_j(h^*_n, \theta^*_n) \right) \right] \leq \varepsilon, \text{ and } E \left\| (\tilde{h}_{[n]}, \tilde{\theta}_{[n]}) - (h^*_n, \theta^*_n) \right\| = \frac{\varepsilon}{2\gamma}.$$

These non-asymptotic guarantees of being close to a “good” point extend to the function values of the risks (4) and (5), since we are close to a candidate $h^*_n$ whose risk value can be no worse than

$$E \left[ R_j(h^*_n) \right] \leq E \left[ \tilde{r}_j(h^*_n, \theta^*_n) \right] \leq E \left[ \tilde{r}_j(\tilde{h}_{[n]}, \tilde{\theta}_{[n]}) \right].$$

We remark that these learning guarantees hold for a class of risks that are in general non-convex and need not even be differential, let alone satisfy smoothness requirements.

### 3.3 Examples

**Example 19.** While stated with a somewhat high degree of abstraction, let us give a concrete example to emphasize that the assumptions of Proposition 14 are readily satisfied under natural and important learning settings. Consider the regression problem, where we observe random pairs $(X,Y) \sim \mu$, assuming that $X$ is a finite-dimensional real-valued random vector, and $Y$ is a real-valued random variable, related to the inputs by the relation $Y = h^*(X) + \epsilon$, where $\epsilon$ is a zero-mean random noise term. For simplicity, let $h^*$ be a continuous linear map, and let $\mathcal{H}$ be the set of all continuous linear maps on the space that $X$ is distributed over. Finally, let the loss by the squared error, such that

$$L(h) = (h(X) - Y)^2 = (\langle X, h - h^* \rangle - \epsilon)^2$$

$$L'(h)(u) = 2\langle X, h - h^* \rangle - 2\epsilon \langle u, X \rangle.$$

Since we make almost no assumptions on the nature of the underlying noise distribution, clearly both the losses and the “gradients” can be unbounded and heavy-tailed. Fix any $h_0 \in \mathcal{H}$, and note that for any $h \in \mathcal{H}$, we have

$$L(h) - L(h_0) = \langle X, h - h^* \rangle^2 - \langle X, h_0 - h^* \rangle^2 - 2\epsilon \langle X, h - h_0 \rangle$$

$$= \langle X, h - h_0 \rangle \langle X, (h - h^*) + (h_0 - h^*) \rangle - 2\epsilon \langle X, h - h_0 \rangle.$$
Absolute values can be bounded above as
\[
|L(h) - L(h_0)| \leq \|X\|^2 \|h - h_0\| (\|h - h^*\| + \|h_0 - h^*\|) + 2|\epsilon|\|X\|\|h - h_0\|.
\]

It follows immediately that as long as \(E_\mu \|X\|^4 < \infty\), we have that the local Lipschitz property (35) of the loss is satisfied, for arbitrary choice of \(h_0\).

As for the weak smoothness requirement on the loss, note that
\[
\|L'(h) - L'(h_0)\| = \sup_{\|u\|=1} \langle u, L'(h) - L'(h_0) \rangle = \sup_{\|u\|=1} 2\langle (X, h - h_0) \rangle \langle u, X \rangle \leq 2\|X\|^2 \|h - h_0\|.
\]

Thus, if the random inputs \(X\) are \(\mu\)-almost surely bounded, the desired smoothness condition (36) holds. Note that this does not preclude heavy-tailed losses and gradients since no additional assumptions have been made on the noise term.

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In this section, we included some standard results that are leveraged in the main paper.

A.1 Useful results based on Lipschitz properties

Let $\mathcal{X}$ be a normed linear space, and let $f : \mathcal{X} \to \mathbb{R}$ be convex and $\lambda$-Lipschitz. If $f$ is sub-differentiable at a point $x$, then using the definition of the sub-differential, we have that

$$|\langle x' - x, \partial f(x) \rangle| \leq |f(x') - f(x)| \leq \lambda \|x' - x\|.$$
It immediately follows that
\[ \| \partial f(x) \| \leq \lambda. \] (44)

That is, all sub-gradients of \( f \) at \( x \) have norm no greater than the Lipschitz coefficient \( \lambda \).

Let \( \mathcal{X} \) and \( \mathcal{Y} \) be Banach spaces, and let \( f : \mathcal{X} \to \mathcal{Y} \) be differentiable on \( U \subset \mathcal{X} \), an open set. Further, assume that the derivative is \( \lambda \)-Lipschitz on \( U \), that is, for each \( x, x' \in U \), we have \( \| f'(x) - f'(x') \| \leq \lambda \| x - x' \| \). First-order Taylor approximations have direct analogues in this general setting, as the following result shows.\(^{60}\)

**Proposition 20.** Let \( f : \mathcal{X} \to \mathcal{Y} \) be differentiable on an open set \( U \subset \mathcal{X} \), with \( \mathcal{X} \) and \( \mathcal{Y} \) assumed to be Banach. If \( f'\) is \( \lambda \)-Lipschitz on \( U \), then for any \( x, u, v \in U \) such that \( x + u \in U \), we have
\[ \| f(x + u) - f(x) - f'(x)(u) \| \leq \frac{\lambda}{2} \| u \|^2. \]

### A.2 Radial derivatives of convex functions

Say a function \( f : \mathcal{V} \to \mathbb{R} \) is convex. Take any \( u, v \in \text{dom} \, f \), and any scalar \( c \geq 0 \) such that \( v + c(v - u) \in \text{dom} \, f \). Then writing \( u' := v + c(v - u) \), note that
\[ v = \frac{1}{1 + c} (u' + cu) = (1 - \beta)u' + \beta u, \]
where \( \beta := c/(1 + c) \in [0, 1) \). By convexity we have \( f((1 - \beta)u' + \beta u) \leq (1 - \beta)f(u') + \beta f(u) \).

Filling in definitions and rearranging we have
\[ f(v + c(v - u)) - f(v) \geq c(f(v) - f(u)). \] (45)

Note that this can be done for any pair of \( u, v \) and scalar \( c \) that keeps the relevant points on the domain. Clearly this property is necessary for convexity, but it is in fact also sufficient.\(^{61}\)

For any function \( f : \mathcal{V} \to \mathbb{R} \) and open set \( U \subset \mathcal{X} \), fix a point \( x \in U \). We denote the difference quotient of \( f \) at \( x \), incremented in the direction \( u \), modulated by scalar \( \alpha \neq 0 \) as
\[ q(\alpha) := q(\alpha; f, x, u) := \frac{f(x + au) - f(x)}{\alpha}. \] (46)

Consider the map \( g(t) := f(x + tu) - f(x) \), with all elements but \( t \geq 0 \) fixed. When \( f \) is convex, direct inspection immediately shows that \( t \to g(t) \) is convex. For any \( 0 \leq t_1 < t_2 \), take some \( t' \in (t_1, t_2) \). Clearly, there exists a \( \beta \in (0, 1) \) such that \( t' = \beta t_1 + (1 - \beta)t_2 \). Then, we have
\[ \frac{g(t') - g(t_1)}{t' - t_1} = \frac{g(\beta t_1 + (1 - \beta)t_2) - g(t_1)}{(1 - \beta)(t_2 - t_1)} \leq \frac{\beta g(t_1) + (1 - \beta)g(t_2) - g(t_1)}{(1 - \beta)(t_2 - t_1)} = \frac{g(t_2) - g(t_1)}{t_2 - t_1}, \]
where the inequality follows from convexity of \( g \). If we use this inequality in the special case of \( t_1 = 0 \), alongside the basic relation \( q(\alpha) = (g(\alpha) - g(0))/\alpha \), it immediately follows that \( \alpha \to q(\alpha) \) is monotone (non-decreasing) on the positive reals. Furthermore, the set \( \{ q(\alpha) : \alpha > 0 \} \) is bounded below. To see this, take some \( \gamma > 0 \) small enough that \( x - \gamma u \in \text{dom} \, f \), and note that by direct application of convexity and the basic property (45), it follows that
\[ \left( \frac{\alpha}{\gamma} \right) (f(x) - f(x - \gamma u)) \leq f \left( x + \frac{\alpha}{\gamma} (x - (x - \gamma u)) \right) - f(x) = f(x + \alpha u) - f(x). \]

\(^{60}\)See for example Luenberger [11, Sec. 7.3, Prop. 2–3] and Nesterov [13, Ch. 1].

\(^{61}\)For example, see Nesterov [13, Thm. 3.1.1].
That is, dividing both sides by $\alpha$, we have
\[
\left(\frac{1}{\gamma}\right) (f(x) - f(x - \gamma u)) \leq \frac{f(x + \alpha u) - f(x)}{\alpha} = q(\alpha; f, u).
\] (47)

Since the choice of $\gamma > 0$ depends only on $x$ and $u$, and is free of $\alpha$, it follows that the set $\{q(\alpha) : \alpha > 0\}$ is bounded below, as desired. Using this boundedness alongside the monotonicity of $\alpha \mapsto q(\alpha)$, we have that the infimum is finite. Thus, recalling the definition (7) of the radial derivative of $f$ at $x$ in the direction $u$, since we have
\[
f'_r(x; u) = \lim_{\alpha \to 0_+} \frac{f(x + \alpha u) - f(x)}{\alpha} = \inf \{q(\alpha; f, x, u) : \alpha > 0\},
\] (48)

it follows immediately that the radial derivative always exists (i.e., $f'(x; u) \in \mathbb{R}$). Note also that using convexity, direct inspection shows that for all $u$ we have
\[
f(u) - f(x) \geq f'(x; u - x).
\] (49)

Furthermore, it is easily verified that whenever $x \in \text{dom } f$, the map $u \mapsto f'(x; u)$ is sub-additive and positively homogenous, i.e., a sub-linear functional.\footnote{This means that the Hahn-Banach theorem can be applied to construct a linear functional $g$ bounded above as $g(u) \leq f'(x; u)$, for all $u$. See for example [11, Sec. 5.4] or Ash and Doléans-Dade [2, Thm. 3.4.2]. This $g$ is not necessarily a sub-gradient of $f$ at $x$, since it need not be continuous in general; such functions are sometimes called \textit{algebraic sub-gradients} [21, Sec. 3].}

The basic facts of interest here are summarized in the following proposition.

**Proposition 21** (Difference quotients for convex functions). Let $\mathcal{V}$ be a vector space. If function $f : \mathcal{V} \to \mathbb{R}$ is proper and convex, then it is radially differentiable on $\text{int}(\text{dom } f)$.

**Proof.** The desired result follows immediately from previous discussion leading up to (48), and the fact that if $x$ is an interior point of the effective domain of $f$, it follows that for any $u \in \mathcal{V}$, we can find a $\gamma > 0$ small enough that $x - \gamma u \in \text{dom } f$, which means we can apply the lower bound of (47) to the difference quotients $q(\alpha; f, x, u)$ indexed by $\alpha > 0$. \qed