On Petrie cycle and Petrie tour partitions of 3- and 4-regular plane graphs

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Abstract
Given a plane graph \( G = (V, E) \), a Petrie tour of \( G \) is a tour \( P \) of \( G \) that alternately turns left and right at each step. A Petrie tour partition of \( G \) is a collection \( \mathcal{P} = \{P_1, \ldots, P_p\} \) of Petrie tours so that each edge of \( G \) is in exactly one tour \( P_i \in \mathcal{P} \). A Petrie tour \( P \) is called a Petrie cycle if all its vertices are distinct. A Petrie cycle partition of \( G \) is a collection \( \mathcal{C} = \{C_1, \ldots, C_p\} \) of Petrie cycles so that each vertex of \( G \) is in exactly one cycle \( C_i \in \mathcal{C} \). In this paper, we study the properties of 3-regular plane graphs that have Petrie cycle partitions and 4-regular plane multi-graphs that have Petrie tour partitions. Given a 4-regular plane multi-graph \( G = (V, E) \), a 3-regularization of \( G \) is a 3-regular plane graph \( G_3 \) obtained from \( G \) by splitting every vertex \( v \in V \) into two degree-3 vertices. \( G \) is called Petrie partitionable if it has a 3-regularization that has a Petrie cycle partition. The general version of this problem is motivated by a data compression method, tristrip, used in computer graphics. In this paper, we present a simple characterization of Petrie partitionable graphs and show that the problem of determining if \( G \) is Petrie partitionable is NP-complete.

Keywords: Plane graph; Petrie cycles; Petrie tours; left-right paths

1. Introduction
Throughout this paper, \( G = (V, E) \) denotes an undirected connected plane graph, which may have multiple edges, but no self-loops. Given a vertex \( v \in V \) and an edge \( e = (u, v) \in E \), the left-edge of \( e \) (at \( v \)) is the edge \( e_1 = (u_1, v) \) that follows \( e \) (at \( v \)) in clockwise (cw) direction, the right-edge of \( e \) (at \( v \)) is the edge \( e_2 = (u_2, v) \) that follows \( e \) (at \( v \)) in counter-clockwise (ccw) direction.

A walk of \( G \) is a sequence \( P = v_0e_1v_1e_2 \ldots e_kv_k \) where \( v_i \) are vertices of \( G \) (may be repeated) and \( e_i = (v_{i-1}, v_i) \) are distinct edges of \( G \). The length of \( P \) is \( k \). If \( v_0 = v_k \), \( P \) is called a tour. A walk (tour, respectively) consisting of distinct vertices is called a path (cycle, respectively). A walk is called a Petrie walk if the edge \( e_{i+1} \) is alternately the left- and the right-edge of \( e_i \) for \( 1 \leq i < k \). A tour \( P \) is called a Petrie tour if it is a Petrie walk, and the alternating left- and right-edge condition also holds for \( e_{k-1}, e_k, \) and \( e_1 \). Petrie path and Petrie cycle are defined similarly. Petrie walks are also called left-right paths and studied in Shank (1975).

A Petrie cycle partition of \( G \) is a set \( \mathcal{C} = \{C_1, \ldots, C_p\} \) of Petrie cycles such that each vertex of \( G \) is in exactly one \( C_i \in \mathcal{C} \). If \( \mathcal{C} \) consists of a single cycle \( C_1 \), \( C_1 \) is call a Petrie Hamiltonian cycle of \( G \). The properties of graphs with Petrie Hamiltonian cycle have been studied in Fouquet et al. (1982), Ivančo and Jendroň (1999), Ivančo et al. (1994).

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A Petrie tour partition of a graph $G$ is a set $\mathcal{P} = \{P_1, \ldots, P_q\}$ of Petrie tours such that each edge of $G$ is in exactly one $P_i \in \mathcal{P}$. If $\mathcal{P}$ consists of a single tour $P_1$, $P_1$ is called a Petrie Eulerian tour of $G$. The properties of graphs with Petrie Eulerian tours have been studied in Kidwell and Bruce Richter (1987), Žitnik (2002).

In this paper, we study the properties of 3-regular plane graphs that have Petrie cycle partitions and the properties of 4-regular plane graphs that have Petrie tour partitions. For reasons that will become clear later, the 3-regular plane graph contains no multiple edges, but 4-regular plane graphs may contain multiple edges. We describe a simple characterization for 3-regular plane graphs with Petrie cycle partitions. This extends the results on plane graphs with Petrie Hamiltonian cycles in Fouquet et al. (1982), Ivančo and Jendroľ (1999), Ivančo et al. (1994). We also study the properties of Petrie partitionable graphs (as defined in the abstract). The general version of this problem is motivated by a data compression method, tristrip, used in computer graphics. We present a nice characterization of such graphs and show that the problem of determining if $G$ is Petrie partitionable is NP-complete.

The present paper is organized as follows. Section 2 introduces the definitions and the motivation of these problems in computer graphics. Section 3 discusses the Petrie cycle partition of 3-regular plane graphs. Section 4 considers the Petrie tour partition of 4-regular plane graphs. The results in Sections 3 and 4 are relatively easy generalizations of known results in Fouquet et al. (1982), Ivančo and Jendroľ (1999), Ivančo et al. (1994), Kidwell and Bruce Richter (1987), Žitnik (2002). To the best of our knowledge, they have not been published in literature. Since they are of independent interests and also needed by the discussion in Section 5, we include these results here. Section 5 describes a nice characterization of Petrie partitionable 4-regular plane graphs. In Section 6, we show the problem of determining if a input 4-regular plane graph is Petrie partitionable is NP-complete. Section 7 describes some open problems and concludes the paper.

2. Definitions and Motivations

In this section, we give definitions and preliminary results. We use standard terminology in Bondy and Murty (1979). Let $G = (V, E)$ be an undirected graph with $n$ vertices and $m$ edges. The degree of a vertex $v \in V$, denoted by $\deg(v)$, is the number of edges incident to $v$. $G$ is called 3-regular (4-regular, respectively), if $\deg(v) = 3$ ($\deg(v) = 4$, respectively) for all $v \in V$. For a subset $E_1 \subseteq E$, the subgraph of $G$ induced by $E_1$ consists of $E_1$ as its edge set and the set of the vertices incident to the edges in $E_1$ as its vertex set. $G$ is bipartite if $V$ can be partitioned into two subsets $V_1$ and $V_2$ such that no two vertices in $V_1$ are adjacent and no two vertices in $V_2$ are adjacent. A $k$-vertex-coloring of $G$ is a coloring of $V$ by $k$ colors so that any two adjacent vertices have different colors. A $k$-edge-coloring of $G$ is a coloring of $E$ by $k$ colors so that any two edges incident to the same vertex have different colors.

A plane graph $G$ is a graph embedded in the plane without edge crossings (i.e. an embedded planar graph). The embedding of a plane graph $G$ divides the plane into a number of regions called faces. The unbounded region is the exterior face. Other regions are interior faces. $\mathcal{F}$ denotes the set of the faces of $G$. For each face $F \in \mathcal{F}$, $\deg(F)$ is the number of edges on the boundary of $F$. It is well known that a plane graph $G$ is bipartite if and only if $\deg(F)$ is even for all $F \in \mathcal{F}$ (Bondy and Murty 1979). A $k$-face-coloring of a plane graph $G$ is a coloring of its faces by $k$ colors so that any two faces sharing an edge as their common boundary have different colors.

A plane graph $G$ is called a triangulation (quadrangulation, respectively), if $\deg(F) = 3$ ($\deg(F) = 4$, respectively) for all faces $F \in \mathcal{F}$. The dual graph $G^* = (V^*, E^*)$ of a plane graph
G = (V, E) is defined as follows: For each face F of G, V^* has a vertex v_F. For each edge e in G, G^* has an edge e^* = (v_{F_1}, v_{F_2}) where F_1 and F_2 are the two faces of G with e on their common boundary. e^* is called the dual edge of e. The mapping e ↔ e^* is a one-to-one correspondence between E and E^*. G is a triangulation (quadrangulation, respectively) if and only if G^* is 3-regular (4-regular, respectively).

2.1 Motivation
The problem studied in this paper is motivated by a data compression technique used in computer graphics. 3D objects are often represented by triangular meshes. For large 3D objects, this takes too much space. The Stripification problem is closely related to the Petrie cycle partition problem as follows. Let G = (V, E) be the dual graph of G̃. Clearly, G is 3-regular. For each face F of G̃, let v_F denote the...
vertex in \( G \) corresponding to \( F \). Consider a sequence of faces \( \mathcal{T} = F_1 \ldots F_t \) of \( \tilde{G} \). It is easy to see that \( \mathcal{T} \) is a tristrip (or tristrip cycle) of \( \tilde{G} \) if and only if the corresponding sequence \( v_{F_1} \ldots v_{F_t} \) is a Petrie path (or Petrie cycle) in \( G \). (See Figure 1A and B.) Hence, the problem of finding a minimum tristrip cycle partition for the faces of \( \tilde{G} \) is the same as the problem of finding a minimum Petrie cycle partition for \( G \).

In computer graphics, 3D objects are also represented by quadrangular meshes (see Bommes et al. 2012, 2013; Dong et al. 2006). For our purpose, this is just a plane quadrangulation \( \tilde{G} = (\tilde{V}, \tilde{E}) \). If we add a chord into each face of \( \tilde{G} \), it becomes a plane triangulation \( \tilde{G}_3 \) which is called a triangular extension of \( \tilde{G} \). Since each face \( F \) of \( \tilde{G} \) has degree 4, there are two ways to add a chord into \( F \). If \( \tilde{G} \) has \( f \) faces, it has \( 2^f \) triangular extensions. One way to represent \( \tilde{G} \) is first convert it to a plane triangular extension \( \tilde{G}_3 \) by adding chords into its faces and then represent \( \tilde{G}_3 \) by using tristrips or tristrip cycles (Estkowski et al. 2002). The question is: which of those \( 2^f \) triangular extensions can be partitioned into a minimum number of tristrips (or tristrip cycles)?

A special version of this problem is closely related to the Petrie tour partition problem. Consider the dual graph \( G = (V, E) \) of \( \tilde{G} \). Clearly, \( G \) is 4-regular. Consider a vertex \( v \in V \) corresponding to a face \( F \) in \( \tilde{G} \). Let \( e_1, e_2, e_3, e_4 \) be the four edges incident to \( v \) in cw order. The split operation at \( v \) splits \( v \) into two degree-3 vertices \( v' \) and \( v'' \) as shown in Figure 2. There are two ways to split \( v \). They correspond to the two ways of adding a chord into the face \( F \). Let \( G_3 \) be the 3-regular plane graph obtained by performing split operation at every vertex of \( G \). We call \( G_3 \) a 3-regularization of \( G \). The edge \( (v', v'') \) of \( G_3 \) introduced by splitting a vertex \( v \) is denoted by \( e(v) \) and called a split edge of \( G_3 \).

Suppose \( G \) has a Petrie tour partition \( \mathcal{P} = \{P_1, \ldots, P_q\} \). Consider any vertex \( v \in V \) with four incident edges \( e_1, e_2, e_3, e_4 \in E \). Two tours \( P_i \) and \( P_j \) in \( \mathcal{P} \) visit \( v \) (possibly \( P_i = P_j \)). We split \( v \) so that \( P_i \) and \( P_j \) are still tours after splitting. (See Figure 2B and C.) Do this at every vertex \( v \in V \). Let \( G_3 \) be the resulting 3 regularization of \( G \). It is easy to see that \( \mathcal{P} = \{P_1, \ldots, P_q\} \) is a Petrie cycle partition of \( G_3 \). Thus, if \( G \) has a Petrie tour partition, then \( G \) has a 3-regularization \( G_3 \) with a Petrie cycle partition \( \mathcal{P} \), where every edge \( e \) of \( G \) belongs to a Petrie cycle in \( \mathcal{P} \). For its application in computer graphics, this restriction is not necessary. Figure 10A shows a 4-regular plane graph \( G \) which has no Petrie tour partition (as we will see later). However, it has a 3-regularization \( G_3 \) (shown in Figure 10B) which has a Petrie cycle partition with a single Petrie Hamiltonian cycle \( C \). (Some edges of \( G \) are in \( C \). Some are not.) This motivates:

**Definition 1.** A 4-regular plane graphs \( G \) is called Petrie partitionable if it has a 3-regularization with a Petrie cycle partition.
A main interest of this paper is to characterize the Petrie partitionable graphs. In computer graphics applications, the problem is to find a 3-regularization $G'$ of $G$ so that the faces of $G'$ can be partitioned into tristrips and/or tristrip cycles. The NP-hardness result in Estkowski et al. (2002) suggests that this problem might be NP-hard also. The problem considered in this paper is a restricted version of the general problem: partition into Petrie cycles only. In contrast to the more general problem, this restriction leads to a simple characterization. The focus of our study is on the graph-theoretical properties of these problems. The insights obtained here may help to find more efficient algorithms for solving the general problem.

3. Characterization of 3-Regular Plane Graphs with Petrie Cycle Partition

In this section, $G = (V, E)$ always denotes a 3-regular connected plane simple graph (i.e. $G$ has no self-loops nor parallel edges). Suppose $G$ has a Petrie cycle partition $\mathcal{C} = \{C_1, \ldots, C_p\}$. For any vertex $v \in V$, two edges incident to $v$ belong to a cycle $C_i \in \mathcal{C}$ and its third incident edge is not in any $C_j \in \mathcal{C}$. We call the third edge a non-cycle edge with respect to $\mathcal{C}$.

A 3-regular plane graph $G$ is called a multi-3-gon if all of its faces have degrees divisible by 3. The following lemma was proved in Ivančo and Jendroľ (1999), Ivančo et al. (1994).

**Lemma 1.** If a 3-regular plane graph has a Petrie Hamiltonian cycle, then it is a multi-3-gon.

The following lemma generalizes Lemma 1. The proof is essentially the same as in Ivančo and Jendroľ (1999). We include it here for completeness.

**Lemma 2.** Any 3-regular plane graph $G$ with a Petrie cycle partition must be a multi-3-gon.

**Proof.** Let $\mathcal{C} = \{C_1, \ldots, C_p\}$ be a Petrie cycle partition of $G$. Consider any face $F$ of $G$ and a cycle $C_i \in \mathcal{C}$ that travels at least one edge of $F$. Suppose $C_i$ travels two edges $e_1 = (u, v)$ and $e_2 = (v, w)$ from $u$ to $v$ to $w$ where $e_1$ is not an edge of $F$, but $e_2$ is an edge of $F$. Let $e_3 = (w, x)$ and $e_4 = (x, y)$ be the two edges of $F$ following $e_2$ in the direction from $v$ to $w$. When $C_i$ travels from $e_1$ to $e_2$, suppose it turns left at $v$ (see Figure 3A). Since $C_i$ is a Petrie cycle, it turns right at $w$ and travels $e_3$. Then, $C_i$ must make a left-turn at $x$. So $e_4$ is a non-cycle edge with respect to $\mathcal{C}$. If $C_i$ turns right at $v$, it must turn left at $w$ and travel $e_3$ (Figure 3B). Then, $C_i$ must turn right at $x$. So $e_4$ must be a non-cycle edge with respect to $\mathcal{C}$.

Thus, the edges on the boundary of $F$ can be partitioned into subpaths of length 3: The first two edges of a subpath belong to a Petrie cycle $C_i \in \mathcal{C}$, while the third edge of the subpath is a non-cycle edge. Hence, the length of $F$ must be a multiple of 3. \qed
If $G = (V, E)$ has a proper 3-edge-coloring $\lambda : E \to \{1, 2, 3\}$, the Heawood valuation (or simply valuation) associated with $\lambda$ is a mapping $\lambda^* : V \to \{-1, 1\}$ defined as follows. For any vertex $v \in V$, if the three edges incident to $v$ are colored 1, 2, 3 in $cw$ order, then $\lambda^*(v) = 1$. Otherwise, define $\lambda^*(v) = -1$. The following lemma is well known (Ringel 1959, p. 18, Theorem 5):

**Lemma 3.** A 3-regular plane graph $G = (V, E)$ has a 3-edge-coloring if and only if there exists a mapping $\kappa : V \to \{-1, 1\}$ such that the sum of the values $\kappa(v)$ for all vertices on the boundary of any face $F$ of $G$ is divisible by 3. If $\kappa$ is such a mapping, then there exists a 3-edge-coloring $\lambda$ of $G$ such that its associated valuation $\lambda^* = \kappa$.

**Theorem 1.** Every 3-regular multi-3-gon $G = (V, E)$ has exactly three Petrie cycle partitions.

*Proof.* Define a mapping $\kappa : V \to \{-1, 1\}$ by setting $\kappa(v) = 1$ for all $v \in V$. Since $G$ is a multi-3-gon, the condition for $\kappa$ in Lemma 3 is satisfied. Thus, $G$ has a 3-edge-coloring $\lambda$ such that its associated valuation $\lambda^*(v) = \kappa(v) = 1$ for all $v \in V$.

Let $E_i (i = 1, 2, 3)$ be the subset of the edges of color $i$ in the coloring $\lambda$. Then, each $E_i$ is a perfect matching of $G$. Let $G_{12}$ be the subgraph of $G$ induced by the edge set $E_1 \cup E_2$. Clearly each vertex in $G_{12}$ has degree 2. Hence, $G_{12}$ is a collection $C_{12} = \{C_1, \ldots, C_p\}$ of disjoint cycles. For each $C_i \in C$, the edges of $C_i$ alternate between $E_1$ and $E_2$. Imagine we travel along $C_i$ passing three consecutive edges $e_1 = (u, v), e_2 = (v, w), e_3 = (w, x)$, where $e_1, e_2 \in E_1$ and $e_3 \in E_2$. If $\lambda^*(v) = 1$ implies $C_i$ turns left at $v$. $\lambda^*(w) = 1$ implies $C_i$ turns right at $w$. Repeating this argument, we see $C_i$ alternately turns left and right. Thus, $C_i$ is a Petrie cycle and $C_{12}$ is a Petrie cycle partition of $G$.

Similarly, we can define the subgraph $G_{13}$ induced by $E_1 \cup E_3$ and the subgraph $G_{23}$ induced by $E_2 \cup E_3$. By the same argument, each of them defines a distinct Petrie cycle partition of $G$.

Next we show $G$ has at most three distinct Petrie cycle partitions. Pick any vertex $v$ of $G$ with three incident edges $e_1, e_2, e_3$. Consider any Petrie cycle partition $\mathcal{C} = \{C_1, \ldots, C_p\}$ of $G$. Assume $e_1$ and $e_2$ belong to $C_i \in \mathcal{C}$ and $e_3$ is a non-cycle edge with respect to $\mathcal{C}$. Since $C_i$ is a Petrie cycle, and two edges $e_1$ and $e_2$ are in $C_i$, we can uniquely trace all edges of $C_i$ by alternately turning left and right. Any vertex $w$ of $C_i$ has a non-cycle edge $(w, x)$ incident to it. The two other edges incident to $x$ belong to a Petrie cycle $C_j \in \mathcal{C}$ (possibly $C_j = C_i$). Thus, we can uniquely trace all edges of $C_j$. Since $G$ is connected, we can trace all Petrie cycles in $\mathcal{C}$ by repeating this process. (Note that there is no guarantee this process can successfully lead to a Petrie cycle partition of $G$). In summary, the fact that an edge $e_3$ incident to $v$ is a non-cycle edge uniquely determines entire $\mathcal{C}$. If we pick $e_1$ or $e_2$ as a non-cycle edge, we can determine at most two other Petrie cycle partitions. So $G$ has at most three distinct Petrie cycle partitions.

On the other hand, we have shown $G$ has at least three distinct Petrie cycle partitions. Thus, $G$ has exactly three Petrie cycle partitions. (This also shows the process described above always produces a valid Petrie cycle partition of $G$).

Figure 4 shows a 3-regular multi-3-gon plane graph $G$ and two Petrie cycle partitions $\mathcal{C}_{12}$ and $\mathcal{C}_{23}$. $\mathcal{C}_{12}$ contains two Petrie cycles. $\mathcal{C}_{23}$ contains a single Petrie Hamiltonian cycle. We end this section by proving the following::

**Theorem 2.** A 3-regular plane graph $G$ has a Petrie cycle partition if and only if it is a multi-3-gon. Such $G$ has exactly three Petrie cycle partitions, which can be found in linear time.

*Proof.* The proof immediately follows from Lemma 2 and Theorem 1. The implementation of the algorithm follows the proof of Theorem 1. First, we find a 3-edge-coloring of $G$ by 3 colors $\{1, 2, 3\}$ so that, for every vertex $v$ of $G$, the three edges incident to $v$ are colored by 1, 2, 3 in $cw$ order. To do this, pick any vertex $v$ and color its three incident edges this way. Then, we can propagate the colors to other edges in $G$. Because $G$ is a multi-3-gon, this process never causes
Figure 4. (A) Petrie cycle partition $C_{12}$; (B) Petrie cycle partition $C_{23}$. (The thick lines are the edges in Petrie cycles. The thin lines are non-cycle edges.)

color conflicts. Once the 3-edge-coloring is obtained, the Petrie cycle partitions $C_{12}, C_{13}, C_{23}$ can be easily obtained. Entire algorithm clearly takes linear time.

4. Characterization of 4-Regular Plane Graphs with Petrie Tour Partition

In this section, we study Petrie tour partitions of a 4-regular plane graph $G$. The special case of this problem where the Petrie tour partition contains only one tour (i.e. a Petrie Eulerian tour) was studied in Žitnik (2002). We generalize the results in Žitnik (2002). Throughout this section, $G = (V, E)$ denotes a 4-regular connected plane graph with no self-loops, but may have parallel edges.

4.1 Characterization

Let $P = e_1 \ldots e_k$ be a Petrie walk of $G$. Let $P^* = e_1^* \ldots e_k^*$ be the sequence of the dual edges $e_i^*$ in the dual graph $G^*$. The following simple observation is crucial to our results.

Observation 1. (Žitnik 2002) If $P$ is a Petrie tour of $G$, then the sequence $P^*$ of the dual edges is a Petrie tour of the dual graph $G^*$.

This observation is illustrated in Figure 5A. The sequence 12345 is a Petrie walk in $G$. The sequence $F_1 F_2 F_3 F_4 F_5$ is the corresponding Petrie walk in the dual graph $G^*$. In this figure, the thick solid lines are the edges of the Petrie walk $P$ of $G$. The thin solid lines are the edges of $G$ but not in $P$. The dashed thick lines are the dual edges in the dual Petrie walk $P^*$ of $G^*$. The dotted thin lines are the edges of $G^*$ but not in $P^*$.

Based on Observation 1, Žitnik (2002) showed that a 4-regular plane graph with a Petrie Eulerian tour must be bipartite. The following lemma generalizes this result.

Lemma 4. If $G = (V, E)$ has a Petrie tour partition, then $G$ must be bipartite.

Proof. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_q\}$ be a Petrie tour partition of $G$. By Observation 1, each $P_i^*$ is a Petrie tour in the dual graph $G^* = (V^*, E^*)$. Because the edge set $E$ of $G$ one-to-one corresponds to the dual edge set $E^*$ of $G^*$, the Petrie tours in $\mathcal{P}^* = \{P_1^*, \ldots, P_q^*\}$ partition the edge set $E^*$. Consider any node $v_F$ in $G^*$. When a Petrie tour $P_i^* \in \mathcal{P}$ visits $v_F$, two edges in $P_i^*$ are consumed: 

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one enters \( v_F \), one leaves \( v_F \). Hence, every node in \( V^* \) has even degree in \( G^* \). Namely every face \( F \) in \( G \) has even degree, as to be shown.

Let \( n, m, f \) denote the number of vertices, edges, and faces of \( G \), respectively. Since \( G \) is 4-regular, we have 
\[
2m = \sum_{v \in V} \text{deg}(v) = 4n. 
\]
Let \( f_{2i} \) \((i \geq 1)\) be the number of faces of \( G \) with degree \( 2i \). By counting the sum of the degrees of the faces of \( G \), we also have 
\[
2m = \sum_{F \text{ is a face of } G} \text{deg}(F) = \sum_{i \geq 1} 2i f_{2i}. 
\]
By Euler formula: 
\[
m = n + f - 2. 
\]
Putting these equations together, we have: (i) \( m = 2n \); (ii) \( f = n + 2 \); and (iii) \( f_{2i} = 4 + \sum_{i \geq 3} (i - 2) f_{2i} \).

Note that \( f_2 \geq 4 \) is the number of degree-2 faces of \( G \), and a degree-2 face is a pair of parallel edges. This explains why we have not restricted to simple graphs in this section.

Consider a tour \( P \) of \( G \). Since \( G \) is 4-regular, at every vertex of \( P \), there are three ways to continue the tour: turn left, go straight, or turn right. A tour \( S \) of \( G \) consisting of only going-straight steps is called a straight tour (or an \( S \)-tour). It is easy to see the edge set \( E \) of \( G \) can be uniquely partitioned into \( S \)-tours. Denote this partition by \( S(G) = \{S_1, \ldots, S_k\} \). An \( S \)-tour may visit a vertex of \( G \) twice.

An \( S \)-tour is called simple if it is a cycle in \( G \). Figure 5B shows a 4-regular plane graph \( G \). \( S(G) \) contains three \( S \)-tours: \( S_1 \) and \( S_2 \) are simple. \( S_3 \) is not. Two \( S \)-tours are said independent if they do not intersect. The following theorem was proved in Jaeger and Shank (1981):

**Theorem 3.** Let \( G = (V, E) \) be a 4-regular plane graph, and let \( \mathcal{S}(G) = \{S_1, \ldots, S_k\} \) be the set of \( S \)-tours of \( G \). Then, \( G \) is bipartite if and only if (i) all \( S \)-tours \( S_i \in \mathcal{S}(G) \) are simple; and (ii) \( \mathcal{S} \) can be partitioned into two subsets \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) such that each \( \mathcal{S}_i \) \((i = 1, 2)\) consists of mutually independent \( S \)-tours.

By Lemma 4 and Theorem 3, all graphs with a Petrie tour partition have a special structure: the set \( \mathcal{S}(G) \) is partitioned into two subsets \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \); \( \mathcal{S}_1 \) is a collection of simple cycles; \( \mathcal{S}_2 \) is also a collection of simple cycles; and the two sets of cycles are overlaid with each other. Such graphs can be complex: Even if \( \mathcal{S}_1 \) has only one cycle \( S_1 \) and \( \mathcal{S}_2 \) has only one cycle \( S_2 \), \( S_1 \) and \( S_2 \) can cross each other many times in complex ways.

In the following, we show that every 4-regular bipartite plane graph \( G = (V, E) \) has exactly two distinct Petrie tour partitions.

Since \( G \) is bipartite, we can color the vertices of \( G \) by two colors red and green. Note that (1) the boundary of every face of \( G \) has the same number of red and green vertices, and (2) every vertex \( v \) is incident to exactly four faces. These two facts imply the number of red vertices equals the number of green vertices.

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**Figure 5.** (A) An example of Observation 1. (B) A graph \( G \) and its \( S \)-tours \( \mathcal{S}(G) = \{S_1, S_2, S_3\} \).
Since \( G \) is 4-regular, we can color the faces of \( G \) using two colors white and black. (The number of the white faces and the number of the black faces of \( G \) may be different).

**Definition 2.** Let \( v \) be a vertex of \( G \) with four incident edges \( e_i \) \( (1 \leq i \leq 4) \) in cw order and four incident faces \( F_i \) \( (1 \leq i \leq 4) \) where \( F_1, F_3 \) are white and \( F_2, F_4 \) are black. Assume \( e_i, e_{i+1} \) \( (1 \leq i \leq 4) \) are the edges of \( F_i \) (see Figure 6A.)

1. The **white merge** operation at \( v \) is (Figure 6B):
   - Replace \( v \) by two new vertices \( v' \) and \( v'' \);
   - Make the edges \( e_1 \) and \( e_4 \) incident to \( v' \); and make \( e_2 \) and \( e_3 \) incident to \( v'' \).
2. The **black merge** operation at \( v \) is (Figure 6C):
   - Replace \( v \) by two new vertices \( v' \) and \( v'' \);
   - Make the edges \( e_1 \) and \( e_2 \) incident to \( v'' \); and make \( e_3 \) and \( e_4 \) incident to \( v' \).

Note that after either merge operation at \( v \), the two new vertices \( v' \) and \( v'' \) have degree 2. After the white merge operation at \( v \), the two white faces \( F_1 \) and \( F_3 \) become one face. After the black merge operation at \( v \), the two black faces \( F_2 \) and \( F_4 \) become one face.

**Definition 3.**

1. The **red-white-merge graph**, denoted by \( G_{rwm} \), is the graph obtained from \( G \) by applying the white merge operation at every red vertex of \( G \) and the black merge operation at every green vertex of \( G \). (See Figure 7B.)
2. The **red-black-merge graph**, denoted by \( G_{rbm} \), is the graph obtained from \( G \) by applying the black merge operation at every red vertex of \( G \) and the white merge operation at every green vertex in \( G \). (See Figure 7C.)

By construction, every vertex \( v \) in \( G_{rwm} \) has degree 2 and the edge set of \( G_{rwm} \) one-to-one corresponds to the edge set of \( G \). The same properties also hold for \( G_{rbm} \). We can similarly define the **green-black-merge graph** \( G_{gbm} \) and the **green-white-merge graph** \( G_{gwm} \). Obviously, \( G_{rbm} = G_{gwm} \) and \( G_{rwm} = G_{gbm} \).

**Theorem 4.** Every 4-regular plane bipartite graph \( G \) has exactly two Petrie tour partitions.

**Proof.** Consider the graph \( G_{rbm} \). Since every vertex in \( G_{rbm} \) has degree 2, \( G_{rbm} \) is a disjoint union of simple cycles. Let \( \mathcal{C} = \{C_1, \ldots, C_q\} \) be these cycles. Note that the edge set of \( G_{rbm} \) one-to-one corresponds to the edge set of \( G \). For each cycle \( C_i \in \mathcal{C} \), let \( P_i \) be the sequence of the edges of \( G \) corresponding to the edges of \( C_i \). Then, \( P_i \) is a tour of \( G \) alternately traveling red and green vertices. Imagine we travel \( P_i \) so that the black faces are on right side. By the construction of \( G_{rbm} \),

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**Figure 6.** (A) A vertex \( v \) and its incident edges and faces; (B) After the white merge operation at \( v \); (C) After the black merge operation at \( v \).
Figure 7. (A) $G$; (B) The red-white-merge graph $G_{rwm}$; (C) The red-black-merge graph $G_{rbm}$.

$P_i$ always turns left at red vertices and right at green vertices (see Figure 7C). Hence, $P_i$ is a Petrie tour of $G$. Let $\mathcal{P}_{rbm} = \{P_1, \ldots, P_q\}$. Since the edge set of $G_{rbm}$ one-to-one corresponds to the edge set of $G$, every edge of $G$ belongs to exactly one $P_i \in \mathcal{P}_{rbm}$. Thus, $\mathcal{P}_{rbm}$ is a Petrie tour partition of $G$. Similarly, we can show the red-white-merge graph $G_{rwm}$ corresponds to another Petrie tour partition $\mathcal{P}_{rwm}$ of $G$.

Next we show $\mathcal{P}_{rbm}$ and $\mathcal{P}_{rwm}$ are the only Petrie tour partitions of $G$. Let $\mathcal{Q} = \{Q_1, \ldots, Q_t\}$ be any Petrie tour partition of $G$. Since $G$ is bipartite, each $Q_i \in \mathcal{Q}$ alternately travels red and green vertices. Consider any tour $Q_i \in \mathcal{Q}$ and three consecutive edges $e_1 = (u, v)$, $e_2 = (v, w)$ and $e_3 = (w, x)$ of $Q_i$, where $u, w$ are green; $v, x$ are red. Let $F_1$ be the face with $e_1$ and $e_2$ on its common boundary. Let $F_2$ be the face with $e_2$ and $e_3$ on its common boundary.

**Case 1:** $Q_i$ turns left at the red vertex $v$ between $e_1$ and $e_2$ and $F_1$ is a white face. Since $Q_i$ is a Petrie tour, it turns right at the green vertex $w$ between $e_2$ and $e_3$. Since $F_1$ and $F_2$ share $e_2$ as common boundary and $F_1$ is white, $F_2$ must be black. This corresponds to performing the black merge operation at the red vertex $v$, and the white merge operation at the green vertex $w$. Repeating this argument, we see that all $Q_i \in \mathcal{Q}$ are obtained by performing the black merge operation at all red vertices and performing the white merge operation at all green vertices of $G$. Thus, $\mathcal{Q}$ is the same as the Petrie tour partition $\mathcal{P}_{rbm}$.

**Case 2:** $Q_i$ turns left at the red vertex $v$ between $e_1$ and $e_2$ and $F_1$ is a black face. By using same argument as in Case 1, we can show $\mathcal{Q}$ is the same as $\mathcal{P}_{rwm}$.

**Case 3:** $Q_i$ turns right at the red vertex $v$ between $e_1$ and $e_2$ and $F_1$ is a black face. By using same argument as in Case 1, we can show $\mathcal{Q}$ is the same as $\mathcal{P}_{rwm}$.

**Case 4:** $Q_i$ turns right at the red vertex $v$ between $e_1$ and $e_2$ and $F_1$ is a white face. By using same argument as in Case 1, we can show $\mathcal{Q}$ is the same as $\mathcal{P}_{rbm}$.

This completes the proof.

Figure 7A shows a 4-regular plane bipartite graph $G$. Figure 7B shows the graph $G_{rwm}$ corresponding to a Petrie tour partition of $G$ with a single Petrie Eulerian tour. Figure 7C shows the graph $G_{rbm}$ corresponding to a Petrie tour partition of $G$ with three Petrie tours.

We end this subsection by proving the following:

**Theorem 5.** A 4-regular plane graph $G$ has a Petrie tour partition if and only if it is bipartite. Such $G$ has exactly two Petrie tour partitions, which can be found in linear time.

**Proof.** The proof immediately follows from Lemma 4 and Theorem 4. The linear time implementation of the algorithm can be done as follows.
First we construct the dual graph $G^*$ of $G$. Color the vertices of $G$ red and green. Color the faces of $G^*$ white and black. Consider a vertex $v$ of $G$. Let $e_1, e_2, e_3, e_4$ be the four edges incident to $v$ in cw order. Split $v$ into two vertices $v'$ and $v''$. If $v$ is red, make its two edges bounding a black face incident to $v'$ and its other two edges bounding a black face incident to $v''$. If $v$ is green, make its two edges bounding a white face incident to $v'$ and its other two edges bounding a white face incident to $v''$. The resulting graph is the red-white-merge graph $G_{rbm}$. By Theorem 4, this is a Petrie tour partition of $G$. The other Petrie tour partition $G_{rbm}$ of $G$ can be obtained similarly. The whole process clearly takes linear time.

4.2 Graph theoretic interpretation of the size of Petrie tour partition

In this subsection, we explain the meaning of the size of Petrie tour partitions of $G$.

**Definition 4.** The white graph $G_{white}^* = (V_{white}^*, E_{white}^*)$ of $G$ is defined as follows. (To avoid confusion, we call the members of $V_{white}^*$ nodes and the members of $E_{white}^*$ lines.)

- The node set $V_{white}^*$ is the set of the white faces of $G$.
- Let $vF_1$ and $vF_2$ be two nodes in $V_{white}^*$ corresponding to the two white faces $F_1$ and $F_2$ of $G$. $vF_1$ and $vF_2$ are connected by a line $e = (vF_1, vF_2) \in E_{white}^*$ if and only if $F_1$ and $F_2$ share a vertex $v$ on their boundary. We denote this edge by $e(v)$. Moreover, if $v$ is red, $e(v)$ is called a red line. If $v$ is green, $e(v)$ is called a green line.
- The white-red subgraph of $G_{white}^*$, denoted by $G_{white,red}^*$, is the subgraph of $G_{white}^*$ induced by its red lines. The white-green subgraph of $G_{white}^*$, denoted by $G_{white,green}^*$, is the subgraph of $G_{white}^*$ induced by its green lines.

We can embed $G_{white}^*$ as follows: Place the node $vF$ corresponding to a white face $F$ in the center of $F$. Draw the line $e(v) = (vF_1, vF_2)$ as a curve connecting two nodes $vF_1$ and $vF_2$, passing through the vertex $v$ that defines $e(v)$. Clearly $G_{white}^*$ is a plane graph. Figures 9A and B show the graph $G_{white,red}^*$ and $G_{white,green}^*$, respectively, overlaid with $G$. Since the numbers of red and green vertices in $G$ are the same, the number of red lines in $G_{white,red}^*$ and the number of green lines in $G_{white,green}^*$ are the same.

**Definition 5.** The black graph $G_{black}^* = (V_{black}^*, E_{black}^*)$ of $G$ is defined as follows:

- The node set $V_{black}^*$ is the set of the black faces of $G$.
- Let $vF_1$ and $vF_2$ be two nodes in $V_{black}^*$ corresponding to the two black faces $F_1$ and $F_2$ of $G$. $vF_1$ and $vF_2$ are connected by a line $e = (vF_1, vF_2) \in E_{black}^*$ if and only if $F_1$ and $F_2$ share a common vertex $v$ of $G$ on their boundary. We denote this edge by $e(v)$. Moreover, if $v$ is red, $e(v)$ is called a red line. If $v$ is green, $e(v)$ is called a green line.
- The black-red subgraph of $G_{black}^*$, denoted by $G_{black,red}^*$, is the subgraph of $G_{black}^*$ induced by its red lines. The black-green subgraph of $G_{black}^*$, denoted by $G_{black,green}^*$, is the subgraph of $G_{black}^*$ induced by its green lines.

It is known the graphs $G_{white}^*$ and $G_{black}^*$ are dual graphs to each other (Berman and Shank 1979; Kidwell and Bruce Richter 1987).

Consider the white-red subgraph $G_{white,red}^*$ and a node $vF \in V_{white,red}^*$ corresponding to a white face $F$ of $G$. Let $e_1, \ldots, e_t$ be the lines in $G_{white,red}^*$ incident to $vF$ in cw order. Each line $e_i$ passes a red vertex $v_i$ on the boundary of $F$. A pair of consecutive lines defines an angle $\theta_i = (e_i, e_{i+1})$ of $G_{white,red}^*$ (1 \leq i \leq t$ where $e_{t+1} = e_1$). The subpath of $F$ for $\theta_i$, denoted by $p(\theta_i)$, is the clockwise
subpath on the boundary of $F$ between the vertex $v_i$ and $v_{i+1}$. Note that $\bigcup_{i=1}^{t} p(\theta_i)$ is the boundary of $F$ (see Figure 8A.)

Embed $G_{\text{white,red}}^*$ in the plane (without the embedding of $G$). Let $D$ be a connected component of $G_{\text{white,red}}^*$. Each face (including the exterior face) of $D$ is called an elementary circuit of $D$. The elementary circuit number of $D$, denoted by $ecn(D)$, is the number of elementary circuits of $D$. If $D$ has $a$ nodes and $b$ lines, then $ecn(D) = b - a + 2$. Let $D_1, \ldots, D_t$ be the connected components of $G_{\text{white,red}}^*$. All elementary circuits of each $D_i$ are the elementary circuits of $G_{\text{white,red}}^*$. The elementary circuit number of $G_{\text{white,red}}^*$ is defined to be $ecn(G_{\text{white,red}}^*) = \sum_{i=1}^{t} ecn(D_i)$. For example, if $G_{\text{white,red}}^*$ is a tree, then $ecn(G_{\text{white,red}}^*) = 1$. As another example, if $G_{\text{white,red}}^*$ has two connected components $D_1$ and $D_2$ where $D_1$ is a tree and $D_2$ is the union of a spanning tree and two non-tree edges, then $ecn(D_1) = 1$, $ecn(D_2) = 3$ and $ecn(G_{\text{white,red}}^*) = 4$. In general, we have:

**Fact 1.** If $G_{\text{white,red}}^*$ has $a$ nodes, $b$ lines and $k$ connected components, then $ecn(G_{\text{white,red}}^*) = b - a + 2k$.

Let $c$ be an elementary circuit of $G_{\text{white,red}}^*$. Imagine we travel along $c$ in cw order so that the interior of $c$ is on the right side. (If $c$ is the exterior face, we travel $c$ in ccw order so that the exterior of $c$ is on the right side.) Let $\Theta(c) = \theta_1, \ldots, \theta_t$ be the sequence of angles of $c$ we encounter on the right side of $c$. For each $i (1 \leq i \leq t)$, let $p(\theta_i)$ be the subpath for $\theta_i$. Let $p(c)$ be the concatenation of $p(\theta_i) (1 \leq i \leq t)$. Note that $p(c)$ is a tour of $G$ and is called the Petrie tour associated with $c$.

Figure 8B illustrates these terms. The white-red subgraph $G_{\text{white,red}}^*$ is a single tree $D_1$. So $ecn(G_{\text{white,red}}^*) = 1$, and it has only one elementary circuit $c$ which is just walking around $D_1$ in ccw order. Three angles $\theta_1, \theta_2, \theta_3$ of $c$ are shown. $p(\theta_1) = \{(u, v), (v, w)\}$, $p(\theta_2) = \{(w, x), (x, y)\}$, $\theta_3$ is defined by a leaf node $v_{F_1}$ of $D_1$. So $p(\theta_1) = \{(y, z), (z, y)\}$ (the two parallel edges on the boundary of $F_1$).

**Theorem 6.** Let $G$ be a 4-regular bipartite plane graph. Let $G_{\text{white,red}}^*$ and $G_{\text{white,green}}^*$ be the subgraphs defined above.

1. Let $\Gamma_1 = \{c_1, \ldots, c_t\}$ be the set of elementary circuits of $G_{\text{white,red}}^*$. For each $c_i (1 \leq i \leq t)$, let $p(c_i)$ be the Petrie tour associated with $c_i$. The union $\bigcup_{i=1}^{t} p(c_i)$, which is called the graph associated with $G_{\text{white,red}}^*$, is the same as the red-white-merge graph $G_{\text{rwm}}$.

2. Let $\Gamma_2 = \{c'_1, \ldots, c'_s\}$ be the set of elementary circuits of $G_{\text{white,green}}^*$. For each $c'_i (1 \leq i \leq s)$, let $p(c'_i)$ be the Petrie tour associated with $c'_i$. The union $\bigcup_{i=1}^{s} p(c'_i)$, which is called the graph associated with $G_{\text{white,green}}^*$, is the same as the green-white-merge graph $G_{\text{gwm}}$.

![Figure 8](https://doi.org/10.1017/S0960129522000238) Published online by Cambridge University Press
Proof. We only prove the Statement 1. The proof of Statement 2 is similar.

The red-white-merge graph $G_{rwm}$ is a collection $\mathcal{C} = \{C_1, \ldots, C_q\}$ of cycles. It is uniquely characterized by the following properties:

1. Each edge of $G$ belongs to exactly one $C_i \in \mathcal{C}$.
2. For each red vertex $v$ in $C_i$, the two edges in $C_i$ incident to $v$ belong to a back face of $G$.
3. For each green vertex $w$ in $C_i$, the two edges in $C_i$ incident to $w$ belong to a white face of $G$.

We show the graph $\bigcup_{i=1}^q p(c_i)$ also satisfies these properties.

1. Every edge $e$ of $G$ is on the boundary of exactly one white face $F$ of $G$. So it belongs to exactly one subpath $p(\theta)$ for an angle $\theta$ at node $v_F$. Every angle $\theta$ of $G^*_{white,red}$ belongs to exactly one elementary circuit $c_i$ of $G^*_{white,red}$. So each edge $e$ of $G$ belongs to exactly one Petrie tour $p(c_i)$ associated with $c_i$.
2. For each red vertex $v$ in $p(c_i)$, by construction, the two edges in $p(c_i)$ incident to $v$ belong to a black face of $G$.
3. Similarly, for each green vertex $w$ in $p(c_i)$, the two edges in $p(c_i)$ incident to $w$ belong to a white face of $G$.

This proves Statement 1.

Figure 9C shows the graph $G_{rwm}$ which is also the graph associated with the white-red subgraph $G^*_{white,red}$. It is a Petrie tour partition of $G$ consisting of a single Petrie Eulerian tour. Figure 9D shows the graph $G_{gwm}$ which is also the graph associated with the white-green subgraph $G^*_{white,green}$. It is a Petrie tour partition of $G$ consisting of three Petrie tours.

The following theorem immediately follows from Theorem 6.

**Theorem 7.** Given a 4-regular plane bipartite graph $G$, the size of the minimum Petrie tour partition of $G$ is $\min\{ecn(G^*_{white,red}), ecn(G^*_{white,green})\}$.

Note: The following Theorem was proved in Žitnik (2002).

**Theorem 8.** Let $G = (V, E)$ be a 4-regular plane bipartite graph. Then, $G$ has a Petrie Eulerian tour if and only if the following conditions hold.
The number of black faces and the number of white faces in $G$ are the same.
Either the subgraph $G_{\text{white, red}}^*$ or the subgraph $G_{\text{white, green}}^*$ is a spanning tree of $G^*$.

Note that Theorem 7 generalizes Theorem 8: if the two conditions in Theorem 8 hold, then either $ecn(G_{\text{white, red}}^*) = 1$ or $ecn(G_{\text{white, green}}^*) = 1$.

The proof of the following theorem is similar to the proof of Theorem 6.

**Theorem 9.** Let $G$ be a 4-regular bipartite plane graph. Let $G_{\text{black, red}}^*$ and $G_{\text{black, green}}^*$ be the subgraphs as defined above.

1. Let $\Gamma_3 = \{c_1, \ldots, c_t\}$ be the set of elementary circuits of $G_{\text{black, red}}^*$. For each $c_i$ ($1 \leq i \leq t$), let $p(c_i)$ be the Petrie tour associated with $c_i$. Then, the union $\bigcup_{i=1}^{t} p(c_i)$ is the same as the red-black-merge graph $G_{\text{rbm}}$.
2. Let $\Gamma_4 = \{c'_1, \ldots, c'_s\}$ be the set of elementary circuits of $G_{\text{black, green}}^*$. For each $c'_i$ ($1 \leq i \leq s$), let $p(c'_i)$ be the Petrie tour associated with $c'_i$. Then, the union $\bigcup_{i=1}^{s} p(c'_i)$ is the same as the green-black-merge graph $G_{\text{gbm}}$.

5. **Characterization of Petrie Partitionable 4-Regular Plane Graphs**

In this section, $G$ always denotes a 4-regular plane graph, not necessarily bipartite. $\mathcal{S}(G) = \{S_1, \ldots, S_k\}$ denotes the set of S-tours of $G$.

**Definition 6.** Let $G$ be a 4-regular plane graph, and $G_3$ be a 3-regularization of $G$. A Petrie cycle partition $\mathcal{C}_3$ is called full if every edge of $G$ belongs to a cycle $C_i \in \mathcal{C}_3$. (This implies all split edges of $G_3$ are non-cycle edges with respect to $\mathcal{C}_3$).

A full Petrie cycle partition of $G_3$ corresponds to a Petrie tour partition of $G$. So the problem considered in Section 4, characterizing $G$ with Petrie tour partitions, is to determine when $G$ has a 3-regularization $G_3$ that has full Petrie cycle partitions. In computer graphics applications, the restriction to full Petrie cycle partitions of $G_3$ is not necessary. In this section, we study the general problem: characterize the Petrie partitionable graphs. In other words, determine when $G$ has a 3-regularization $G_3$ that has Petrie cycle partition (full or not).

Figure 10A shows a 4-regular plane graph $G$ with three S-tours $S_1, S_2, S_3$ where each pair of them intersect. By Lemma 4 and Theorem 3, $G$ has no Petrie tour partitions. Figure 10B shows a 3-regularization $G_3$ of $G$ with a Petrie cycle partition $\mathcal{C}_3$, consisting of a single Petrie Hamiltonian cycle $C$ (denoted by thick lines in the figure).

We have the following simple characterization of Petrie partitionable graphs.

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**Theorem 10.** A 4-regular plane graph $G$ with $S$-tour set $\mathcal{S}(G)$ is Petrie partitionable if and only if the following hold: (i) All $S$-tours in $\mathcal{S}(G)$ are simple, and (ii) $\mathcal{S}(G)$ can be partitioned into at most three subsets $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2$ of mutually independent $S$-tours.

**Proof.** By Theorem 2, $G$ is Petrie partitionable if and only if $G$ has a 3-regularization $G_3$ that is a multi-3-gon. By Theorems 5 and 3 and the remark after Definition 6, $G$ has a 3-regularization $G_3$ with a full Petrie cycle partition if and only if all $S$-tours in $\mathcal{S}(G)$ are simple and $\mathcal{S}(G)$ can be partitioned into two subsets of mutually independent $S$-tours.

Suppose $G$ has a 3-regularization $G_3$ that is a multi-3-gon. By Lemma 3, there exists a 3-edge-coloring $\lambda$ of $G_3$ (with three colors 0, 1, 2) such that the valuation $\kappa(u) = 1$ for all vertices $u$ in $G_3$. In other words, the three edges in $G_3$ incident to $u$ are colored by 0, 1, 2 in cw order. Consider any vertex $v$ in $G$. Let $e_1, e_2, e_3, e_4$ be the edges in $G$ incident to $v$ in cw order. Let $e(v) = (v', v'')$ be the split edge in $G_3$ corresponding to $v$. Because the edge color pattern around $v'$ and $v''$, it is clear that $\lambda(e_1) = \lambda(e_3), \lambda(e_2) = \lambda(e_4)$ and $\lambda(e_1) \neq \lambda(e_2)$. Treat $\lambda$ as an edge coloring of $G$, and let $\mathcal{S}_i (i \in [0, 1, 2])$ be the set of edges in $G$ with color $i$. Clearly, each $\mathcal{S}_i$ is a collection of mutually independent simple $S$-tours of $G$. Thus, $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2$ satisfy the condition of the theorem.

Now suppose the set of $S$-tours $\mathcal{S}(G) = \{\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2\}$ of $G$ satisfies the condition of the theorem. For each edge $e$ of $G$, if $e$ belongs to an $S$-tour in $\mathcal{S}_i (i \in [0, 1, 2])$ color $e$ by color $i$. Consider any vertex $v$ in $G$ and let $e_1, e_2, e_3, e_4$ be the edges in $G$ incident to $v$ in cw order. Let $F_1, F_2, F_3, F_4$ be the faces of $G$ incident to $v$ where $e_i, e_{i+1}$ are on the boundary of $F_i$. For each $i = 1, 2, 3, 4$, call the triple $\alpha = (e_i, v, e_{i+1})$ an angle of $F_i$ and denote it by $\alpha \in F_i$. Let $\mathcal{A}$ be the set of all angles of $G$. Define a mapping $\pi: \mathcal{A} \rightarrow \{-1, +1\}$ as follows. Consider any angle $\alpha = (e_i, v, e_{i+1}) \in \mathcal{A}$, let $c_i$ and $c_{i+1}$ be the color of $e_i$ and $e_{i+1}$, respectively. Define: $\pi(\alpha) = +1$ if $c_{i+1} - c_i \equiv +1 \pmod{3}$; and $\pi(\alpha) = -1$ if $c_{i+1} - c_i \equiv -1 \pmod{3}$.

For any face $F$ of $G$, clearly $\sum_{\alpha \in F} \pi(\alpha) \equiv 0 \pmod{3}$. Consider any vertex $v$ in $G$, with four angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ incident to $v$ in cw order. Clearly, $\pi(\alpha_1) = \pi(\alpha_3); \pi(\alpha_2) = \pi(\alpha_4)$; and $\pi(\alpha_1) \neq \pi(\alpha_2)$. We now define a 3-regularization $G_3$ of $G$ as follows:

- If $\pi(\alpha_1) = \pi(\alpha_3) = -1$, we split $v$ in a way so that the faces $F_1$ and $F_3$ share the split edge $e(v)$ on their common boundary.
- If $\pi(\alpha_2) = \pi(\alpha_4) = -1$, we split $v$ in a way so that the faces $F_2$ and $F_4$ share the split edge $e(v)$ on their common boundary.

For any face $F'$ of $G_3$, let $F$ be the face in $G$ corresponding to $F'$. For each vertex $v$ incident to $F$ in $G$, let $\alpha$ be the angle of $F'$ incident to $v$. By the construction of $G_3$, if $\pi(\alpha) = -1$, $v$ is split into two vertices in $G_3$ incident to $F'$; if $\pi(\alpha) = +1$, $v$ corresponds to one vertex in $G_3$ incident to $F'$. Thus, the degree of $F'$ in $G_3$ is: $\sum_{\alpha \in F} \pi(\alpha) = +1 \sum_{\alpha \in F} \pi(\alpha) = -1 \equiv \sum_{\alpha \in F} \pi(\alpha) = +1 \equiv \sum_{\alpha \in F} \pi(\alpha) \equiv 0 \pmod{3}$.

Since $F'$ is any face in $G$, $G_3$ is a multi-3-gon, as to be shown.

\[\square\]

6. Determining if a 4-Regular Plane Graph is Petrie Partitionable is NP-Complete

**Definition 7.** The Petrie Partitionability of 4-regular plane graphs (PP4R for short) problem is: given a 4-regular plane graph $G$, determine if $G$ is Petrie partitionable or not.

**Definition 8.** The Planar Graph 3-Colorability (PG3C for short) problem is: given a plane graph $H$, determine if $H$ has a vertex coloring using 3 colors.

In this section, we show the following:
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Theorem 11. The PP4R problem is NP-Complete.

Proof. It is known PG3C problem is NP-complete (Garey et al. 1976). We reduce the PG3C problem to the PP4R problem. This will establish the NP-completeness of the PP4R problem.

Let $H = (V_H, E_H)$ be a plane graph. We construct a 4-regular plane graph $G = (V_G, E_G)$ as follows. For each degree-$k$ vertex $u$ in $V_H$, we draw a flower shaped cycle $c_u$ with $k$ petals around it. (See Figure 11A). Each petals is associated with an edge incident to $u$ in $H$. If $(u, v)$ is an edge in $E_H$, then the petal of the cycle $c_u$ associated with the edge $(u, v)$ intersects with the petal of the cycle $c_v$ associated with $(u, v)$ at two points. (See Figure 11B). Do this for every vertex in $V_H$ in such a way that these are the only intersection points of the cycles $c_u$'s. So there are $2k$ points on the cycle $c_u$, and $c_u$ is divided by these points into $2k$ cycle segments.

The vertices of $G$ are these intersection points, and the edges of $G$ are these cycle segments. From the construction, the following facts hold:

- $G$ is a 4-regular plane graph, which can be constructed from $H$ in polynomial time.
- Let $\mathcal{S}(G) = \{ S_1, \ldots, S_k \}$ be the set of $S$-tours of $G$. Each $S$-tour in $\mathcal{S}(G)$ is just a cycle $c_u$ for some vertex $u$ in $H$. In other words, $\mathcal{S}(G) = \{ c_u \mid u \in V_H \}$. Hence all $S$-tours of $H$ are simple.

By Theorem 10, $G$ is Petrie partitionable if and only if $\mathcal{S}(G)$ can be partitioned into three independent subsets $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$, which is true if and only if the graph $H$ is 3 vertex colorable. (If the cycle $c_u \in \mathcal{S}_i$ ($i = 1, 2, 3$), then we color the vertex $u$ by the color $i$ in $H$. This defines a valid 3 vertex coloring of $H$.) Hence, $G$ is Petrie partitionable if and only if $H$ is 3 vertex colorable. This completes the polynomial time reduction from the PG3C problem to the PP4R problem and the proof of the theorem.

7. Conclusion and Open Problems

We studied the problems of partitioning 3-regular plane graphs by Petrie cycles and partitioning 4-regular plane graphs by Petrie tours. We found simple characterizations for graphs having such partitions. This leads to simple linear time algorithms for finding minimum partitioning.

For 4-regular plane graphs, we discovered a nice characterization of Petrie partitionable graphs. We also showed that the problem of determining if a input 4-regular plane graph is Petrie partitionable is NP-complete.

The general version of these problems is motivated by applications in computer graphics, which require finding minimum partitioning of 3-regular plane graphs by Petrie paths and/or Petrie cycles, and finding minimum partitioning of 4-regular plane graphs by Petrie walks and/or Petrie...
tours. The general version of the problem is NP-complete. It is interesting to see if the insights discovered in this paper can lead to better heuristic algorithms and/or more efficient exact algorithms for solving the general version of these problems.

For the Petrie cycle partition problem, we considered only the 3-regular plane graphs. This is motivated by its connection to computer graphics applications. But it is also an interesting graph-theoretic problem for $r$-regular plane graphs with $r=4$ or 5. Is there a simple characterization for such graphs with Petrie cycle partitions?

Conflicts of interest. The authors declare none.

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