Deterministic spatial search using alternating quantum walks

S. Marsh* and J. B. Wang†

Department of Physics, The University of Western Australia, Perth, Australia
(Dated: April 12, 2021)

This paper examines the performance of spatial search where the Grover diffusion operator is
replaced by continuous-time quantum walks on a class of interdependent networks. We prove that
for a set of optimal quantum walk times and marked vertex phase shifts, a deterministic algorithm
for structured spatial search is established that finds the marked vertex with 100% probability. This
improves on the Childs & Goldstone spatial search algorithm on the same class of graphs, which we
show can only amplify to 50% probability. Our method uses $\sqrt{\frac{\pi}{2^{n-1}}}$ marked vertex phase shifts
for an $N$-vertex graph, making it comparable with Grover’s algorithm for unstructured search. It is
expected that this new framework can be readily extended to deterministic spatial search on other
families of graph structures.

I. INTRODUCTION

Quantum search on a spatially structured database via
a continuous-time quantum walk (CTQW) is an impor-
tant and broad problem in quantum computation [1–5],
which aims to find an unknown marked vertex on an un-
derlying graph of specified topology. Many spatial struc-
tures studied so far admit a quadratic speedup
for continuous-time spatial search using the Childs &
Goldstone (CG) algorithm [2, 6–10], where the proba-
bility of measuring the marked vertex $|\omega\rangle$ can be amplified
in $O(1)$ in $O(\sqrt{N})$ time for an $N$-vertex graph. However,
with the exception of the complete graph, the vast major-
ity of graphs studied previously have a success probability
less than 1 [10].

More recently, with the significant research interests
in the Quantum Approximate Optimisation Algorithm
(QAOA) [11, 12], searching for a marked element has
been treated variationally as a combinatorial optimisa-
tion problem having the objective function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is the marked element } \omega, \\ 0 & \text{otherwise}. \end{cases} \quad (1)$$

In QAOA-based algorithms the aim is to optimise the
parameters of a quantum circuit to maximise the expecta-
tion value of the output state with respect to the objec-
tive function. Specifically, for database search, Morales
et al. [13] show that Grover’s algorithm can be learnt
variationally with a parameterised Grover diffusion oper-
ator, and Jiang et al. [14] demonstrate that replacing the
diffusion operator with a parameterised transverse field
operator can also result in an efficient search algorithm.

These results for two different so-called ‘mixing oper-
ators’ suggest a generalisation for a gate-model search
framework. Motivated by the fact that the above mix-
ers are equivalent to a CTQW on the complete graph

$|\vec{t}, \vec{\theta}\rangle = \left( \prod_{j=1}^{p} U_{w}(t_j) U_f(\theta_j) \right) |s\rangle. \quad (2)$

Here, starting in the equal superposition $|s\rangle$, applications
of $U_f(\theta) = e^{-i\theta|\omega\rangle\langle\omega|}$ perform a controlled phase shift $\theta$
to the marked vertex, and each $U_w(t) = e^{-itA}$ applies a
quantum walk for time $t$ over a graph having adjacency
matrix $A$. Our aim is to determine values for $\vec{t}$ and $\vec{\theta}$
that maximise the overlap with the marked element after $p$
iterations. This is a specific case of the Quantum Walk
Optimisation Algorithm applied to search [15, 16].

Expressing the spatial search problem in this frame-
work has a number of benefits. As in [14], it can result
in a mixing circuit that uses substantially fewer gates
than the standard Grover diffusion operator, having ad-
vantages for NISQ hardware. More generally, hardware
with limited qubit connectivity can benefit from a mixing
operator that suits the couplings. In addition, in terms of
quantum optimisation, studying the query complexity of
a particular mixer on the search problem gives an analyti-
cal baseline for the mixer’s performance on more complex
combinatorial optimisation problems. Finally, there are
interesting advantages over the $CG$ spatial search scheme
[2]. As well as being naturally suited to gate model cir-
cuit implementation, for the class of graphs we study
in this paper deterministic search can be achieved using
the alternating phase-walk formulation. This is, to
the authors’ knowledge, the first example of determinis-
tic search on a spatially structured database that goes
beyond the direct use of the generalised Grover diffusion
operator [17, 18]. It improves on the $CG$ algorithm on
the same class of graphs, which we will show only reaches
50% success probability.

In this paper, we focus on finding closed-form expres-
sions for parameters $(\vec{t}, \vec{\theta})$ for a particular class of interde-
pendent networks to achieve efficient deterministic quan-
tum spatial search. A $2n$-vertex interdependent network

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* samuel.marsh@research.uwa.edu.au
† jingbo.wang@uwa.edu.au
In the CIIN graph the dynamics can be reduced analytically to the consideration of a 4-dimensional ‘walk subspace’. In addition, since CIINs are vertex-transitive, for the purposes of analysis we can set the marked vertex as $|\omega\rangle = |0\rangle$ and the resulting algorithm will apply regardless to an arbitrary marked vertex. Consequently, relative to the marked vertex $|0\rangle$, there are four distinct categories of vertex that together define the walk basis $B = \{|b_1\rangle, |b_2\rangle, |b_3\rangle, |b_4\rangle\}$:

- the marked node $|b_1\rangle = |\omega\rangle = |0\rangle$,
- the opposite node $|b_2\rangle = |\bar{\omega}\rangle = |n\rangle$,
- the equal superposition over the unmarked vertices on the same side as $|\omega\rangle$,
- the equal superposition over the vertices on the same side as $|\bar{\omega}\rangle$, excluding $|\bar{\omega}\rangle$.

The above basis can be obtained simply by symmetry arguments, or alternatively by applying systematic dimensionality reduction via the Lanczos algorithm as per [20].

We initialise the system in the equal superposition, which in the reduced walk basis has the form

$$|s\rangle = \frac{1}{\sqrt{2n}} (1, 1, \sqrt{n-1}, \sqrt{n-1}) ,$$

where approximately half the probability is contained in each of the last two vertex groups. A proof for the adjacency matrix taking this form can be found in Appendix A.

To determine the reduced adjacency matrix $A_{\text{full}} \mapsto A$, $\langle b_i | A_{\text{full}} | b_j \rangle$ is calculated to obtain

$$A = \begin{pmatrix}
0 & 1 & \sqrt{n-1} & 0 \\
\frac{1}{\sqrt{n-1}} & 0 & 0 & \frac{\sqrt{n-1}}{}
\end{pmatrix} .$$

This reduced adjacency matrix can itself be treated as an undirected but weighted graph as illustrated in Fig. 2.

### III. APPLYING $CG$ SPATIAL SEARCH

We first briefly explore the performance of the $CG$ algorithm [2] on CIINs, in order to form a basis for comparison with our algorithm. The results in this section are consistent with the general analysis of optimality conditions for the $CG$ algorithm given in [10]. In $CG$ spatial...
where \( A \) is the graph adjacency matrix and \(|\omega\rangle\langle\omega|\) is the oracular marking Hamiltonian. The Laplacian is often used in place of the adjacency matrix, however, CIINs are regular graphs so the dynamics are equivalent up to a global phase. The aim for \( CG \) spatial search is to find a critical value of the variational parameter \( \gamma = \gamma^* \) that will induce ‘fast’ rotation between the initial superposition and a state having \( O(1) \) overlap with the marked state.

In the \( CG \) spatial search framework, we define our Hamiltonian as

\[
H = -\gamma A - |\omega\rangle\langle\omega|
\]

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(reaching $O(1)$ success probability after $O(\sqrt{N})$ time), it is somewhat unsatisfying that the search algorithm is ‘ignoring’ half of the interdependent network. One could heuristically argue that the well-known result of $\sim 100\%$ probability for $CG$ spatial search on the complete graph is being ‘exploited’ on half of the CIIN, rather than the CIIN itself fundamentally admitting efficient $CG$ spatial search.

IV. ALTERNATING WALK ALGORITHM

In this section, we present a new approach to evolve the equal superposition to the marked vertex state. We will introduce the core algorithm, which analogous to Grover’s algorithm has failure rate going to zero as $n \to \infty$. Then, in the same vein as the Grover-Long algorithm [17], we show that the search can be made fully deterministic by parametrising the phase shifts.

A. Quantum walk on CIIN

First, we examine some essential properties of quantum walks on CIINs. Quantum walk on a CIIN is $2\pi$-periodic, since the graph is regular with integer adjacency matrix eigenvalues $-2, 0, (n - 2)$ and $n$ [21]. We observe the state of the system after a quantum walk is applied starting in the single marked vertex, $U_w(t)|\omega\rangle$. The overlap of each basis state after a quantum walk for an 18-vertex example is shown in Fig. 4. It is worth to note that for any $k \in \mathbb{Z}$,

$$U_w\left(\frac{2\pi k}{n}\right)|\omega\rangle = \cos\left(\frac{2\pi k}{n}\right)|\omega\rangle - i\sin\left(\frac{2\pi k}{n}\right)|\tilde{\omega}\rangle. \quad (15)$$

That is, for specific walk times a vertex only interacts with its opposite vertex, with probability in the other vertex groups destructively interfering. We also observe from Eq. (15) that the probability in the opposite vertex is maximised when $k = \lfloor n/4 \rfloor$. Clearly, the graph exhibits perfect state transfer [22] from one vertex to its opposite when $n \mod 4 = 0$.

B. The dual basis

The core of the CIIN search algorithm is to rotate between eigenstates of the adjacency matrix, spanning a two-dimensional subspace of the walk space. Hence for the purposes of a detailed mathematical analysis, we introduce a new basis consisting of the eigenstates of $A$. These basis elements are simply the eigenvectors of the original (and significantly, unperturbed) adjacency matrix in the walk basis, which are straightforward to compute:

$$|b_1^*\rangle = \frac{1}{\sqrt{2n}}(1, 1, \sqrt{n - 1}, \sqrt{n - 1}) = |s\rangle,$$  \hspace{1cm} (16)

$$|b_2^*\rangle = \frac{1}{\sqrt{2n}}(-1, 1, -\sqrt{n - 1}, \sqrt{n - 1}), \quad (17)$$

$$|b_3^*\rangle = \frac{1}{\sqrt{2n}}(\sqrt{n - 1}, -\sqrt{n - 1}, -1, 1), \quad (18)$$

$$|b_4^*\rangle = \frac{1}{\sqrt{2n}}(-\sqrt{n - 1}, -\sqrt{n - 1}, 1, 1). \quad (19)$$

The equal superposition is an eigenvector of any regular graph, and hence is the first basis state in the dual basis. In this basis, the marked state takes the form

$$|\omega\rangle = \frac{1}{\sqrt{2n}}(1, -1, \sqrt{n - 1}, -\sqrt{n - 1}), \quad (20)$$

and the initial state is

$$|s\rangle = (1, 0, 0, 0). \quad (21)$$

We then have

$$U_w(t) = \exp\{-it\text{diag}(n, n - 2, -2, 0)\},$$ \hspace{1cm} (22)

$$U_f(\theta) = \exp\{-i\theta|\omega\rangle\langle\omega|\}. \quad (23)$$

Thus in the dual basis the two unitaries take opposite roles: in contrast to the original walk basis, here the quantum walk only applies a phase difference to the basis states, whilst the marking unitary now performs the ‘mixing’.

FIG. 4: Dynamics of a quantum walk on an 18-vertex CIIN, starting from the marked element. There is $2\pi$-periodicity, and for multiple values of $t$ the amplitude is suppressed in one or more vertex groups.
C. Approximate algorithm

The algorithm aims to rotate the system from the initial superposition \(|s\rangle\) to the entangled state

\[
|+\omega\rangle = \frac{1}{\sqrt{2}}(|\omega\rangle + |\bar{\omega}\rangle) = \frac{1}{\sqrt{2}}(|b_1\rangle + |b_2\rangle). \tag{24}
\]

To design an appropriate iteration, observe that in the dual basis,

\[
|+\omega\rangle = \frac{1}{\sqrt{n}}|b_1^*\rangle - \sqrt{\frac{n-1}{n}}|b_2^*\rangle, \tag{25}
\]

so the system should be rotated from the initial state towards the state \(|b_1^*\rangle\). In order to achieve this, consider the parametrised unitary

\[
U(t, k) = U_w(t)U_f(\pi)U_w(\frac{2\pi k}{n})U_f(\pi) \tag{26}
\]

where \(k\) is an integer. In the dual basis, we find that when \(j = 2, 3\),

\[
\langle b_j^* | U(t, k) | b_j^* \rangle = \langle b_j^* | U(t, k) | b_j^* \rangle = 0 \tag{27}
\]

and thus \(U(t, k)\) restricts to the desired 2-dimensional subspace \(|b_1^*\rangle, |b_2^*\rangle\). Consequently, we define the iterate

\[
U = U_w(t_2)U_f(\pi)U_w(t_1)U_f(\pi) \tag{29}
\]

where

\[
t_1 = \frac{2\pi}{n} \left\lfloor \frac{n}{4} \right\rfloor, \tag{30}
\]

\[
t_2 = \frac{2}{n} \arctan \left( \frac{n-2}{\sqrt{n}} \tan t_1 \right). \tag{31}
\]

These parameters to \(U(t, k)\) are chosen to maximise \(|\langle b_j^* | U(t, k) | b_j^* \rangle|^2\), i.e. to maximise the rotation towards the \(|b_1^*\rangle\) state. For large \(n\), the parameters converge to \(t_1 \approx \frac{\pi}{2}\) and \(t_2 \approx \frac{\pi}{2}\).

In the 2-dimensional subspace \(|b_1^*\rangle, |b_2^*\rangle\), the iterate takes the form

\[
U = \left( \begin{array}{cc}
\frac{n+e^{2\pi i/4} - e^{2\pi i/4}}{\sqrt{n-1}+e^{2\pi i/4}} & \frac{-\sqrt{n-1}+e^{2\pi i/4}}{\sqrt{n-1}+e^{2\pi i/4}} \\
\end{array} \right).
\]

If a global phase is introduced such that \(\arg \langle b_1^* | U | b_1^* \rangle = 0\), we find that this matrix has eigenphases

\[
\lambda_{\pm} = \pm \arcsin \left( 2\sqrt{\frac{n-1}{n}} \sin t_1 \right) \tag{33}
\]

with corresponding eigenstates

\[
|v_{\pm}\rangle = \frac{1}{\sqrt{2}} \left( |b_1^*\rangle \mp e^{-i\pi/2} |b_2^*\rangle \right). \tag{34}
\]

Using the eigensystem of \(U\) to diagonalise and compute the matrix power,

\[
U^p |b_1^*\rangle = \cos(p\lambda_+)|b_1^*\rangle - ie^{int_1/2} \sin(p\lambda_+)|b_2^*\rangle. \tag{35}
\]

Hence, when \(p = \frac{1}{\lambda_+} \arccos \frac{1}{\sqrt{n}}\) we reach the state

\[
|\psi\rangle = \frac{1}{\sqrt{n}} |b_1^*\rangle - ie^{int_2/2} \left( \frac{n-1}{n} \right) |b_2^*\rangle, \tag{36}
\]

which is nearly the desired entangled state \(|+\omega\rangle\) given in Eq. (25), but with a different phase on \(|b_2^*\rangle\). Observe that applying a final quantum walk for time \(t_3\) to this state gives

\[
U_w(t_3) |\psi\rangle = \frac{1}{\sqrt{n}} |b_1^*\rangle - ie^{int_2+2it_3} \left( \frac{n-1}{n} \right) |b_2^*\rangle, \tag{37}
\]

and thus with \(t_3 = \frac{\pi}{2} - \frac{\pi}{2}\) the state \(|+\omega\rangle\) is reached. We illustrate a numerical simulation of this evolution for a 2048-vertex system in Fig. 5.

Reaching the entangled state \(|+\omega\rangle\) is sufficient to solve the search problem with 100% success as \(N \to \infty\). There are two approaches:

1. One can measure the system to obtain either the marked vertex or the vertex ‘opposite’ it: the marking oracle can be queried one more time to confirm which. If the system is measured to be in state \(|x\rangle\), where \(x\) is then verified to not be the marked vertex, then \(|\omega\rangle = |x + n \mod N\rangle\).

2. For a fully coherent algorithm, observe from Eq. (15) that when \(k = \left\lfloor \frac{n}{2} \right\rfloor\),

\[
U_w \left( \frac{2\pi k}{n} \right) |\omega\rangle \approx \frac{1}{\sqrt{2}}(|\omega\rangle - i|\bar{\omega}\rangle). \tag{38}
\]

Thus, \(|\omega\rangle \approx U_w \left( \frac{2\pi}{n} \left\lfloor \frac{n}{2} \right\rfloor \right) U_f(x/2) + \omega\).
Consequently, this algorithm achieves essentially 100% probability (as $n$ increases) of measuring the marked element after

$$\frac{2}{\lambda_+} \arccos \frac{1}{\sqrt{n}} + 1 \approx \frac{\pi}{2} \sqrt{n} = \frac{\pi}{2\sqrt{2}} \sqrt{N}$$

It is straightforward to verify the eigenphases of $U(-\theta)U(\theta)$ as

$$\lambda_{\pm} = \pm 2 \arcsin \left( \frac{2\sqrt{n} - 1}{n} \sin \frac{\theta}{2} \right)$$

with eigenstates

$$\frac{1}{\sqrt{2}} (|b_4^+\rangle \pm e^{-i\gamma} |b_4^-\rangle)$$

where $\gamma = \arctan \left( \frac{n-2}{n} \tan \frac{\theta}{2} \right)$. This means that again, the system is rotated between the initial state and the fourth eigenstate of the adjacency matrix. However, in this case, the speed of rotation is controlled by the $\theta$ parameter. Using the eigensystem to find the state of the system after $p$ iterations of $U(-\theta)U(\theta)$ gives

$$(U(-\theta)U(\theta))^p |s\rangle = \cos p \lambda_+ |s\rangle + i e^{-i\gamma} \sin p \lambda_+ |b_4^+\rangle$$

and thus with $p = \frac{\arccos \frac{1}{\sqrt{n}}}{\lambda_+}$ the state

$$\frac{1}{\sqrt{n}} |s\rangle + i e^{-i\gamma} \sqrt{\frac{n-1}{n}} |b_4^+\rangle$$

is reached. Note that solving for $\theta$ gives Eq. (41). As before, to tune the phase difference and reach $|\psi\rangle$ exactly, a final quantum walk for time $t_3 = \frac{\pi}{2n} - \frac{2}{\lambda_+}$ is applied.

The final step, as before, is to map $|\psi\rangle \mapsto |\omega\rangle$. Here, we show that two phase queries are sufficient to perform the deterministic mapping from entangled to marked. It is easiest to work in the reverse direction, starting in the marked state (the conjugate transpose of a phase-walk evolution is simply the operators in reverse order with all parameters negated). Consider the following evolution

$$U_w(\frac{2\pi j}{n})U_f(\phi)U_w(\frac{2\pi k}{n})|\omega\rangle$$

with $j, k \in \mathbb{Z}$. Calculating the resultant state gives (dropping the global phase),

$$\cos \frac{\phi}{2} \left( \cos \frac{2(j+k+1) \pi}{n} |\omega\rangle - i \sin \frac{2(j+k+1) \pi}{n} |\tilde{\omega}\rangle \right)$$

$$- \sin \frac{\phi}{2} \left( i \cos \frac{2(j-k+1) \pi}{n} |\omega\rangle + \sin \frac{2(j-k+1) \pi}{n} |\tilde{\omega}\rangle \right).$$

In order to have equal probability in each state, the parameters $(\phi, j, k)$ must then satisfy the requirement

$$\cos^2 \frac{\phi}{2} \cos \frac{4\pi (j+k)}{n} + \sin^2 \frac{\phi}{2} \cos \frac{4\pi (j-k)}{n} = 0$$

Solving for $\phi$ in general,

$$\phi = 2 \arctan \sqrt{\frac{\cos \left( \frac{4\pi (j+k)}{n} \right)}{\cos \left( \frac{4\pi (j-k)}{n} \right)}}.$$
FIG. 6: Comparing the dynamics of the (a) approximate and (b) deterministic algorithms on a 24-vertex instance. The horizontal lines represent the evolution target of $\langle \psi | b^*_1 \rangle^2 = \frac{1}{n}$ (and $\langle \psi | b^*_4 \rangle^2 = \frac{n-1}{n}$). In the exact algorithm, the speed of evolution can be manipulated such that the target is reached exactly at $2p = 4$, i.e. after two iterations of $U(\theta)U(\theta)$.

Thus for a given $n$, to obtain a real-valued phase rotation we need to find an integer pair $j$ and $k$ such that the expression inside the square root is non-negative. This is always possible: one choice that satisfies this requirement for all $n \geq 8$ is $j = \lfloor n/8 \rfloor$ and $k = \lceil n/8 \rceil$. This maps $|\omega\rangle \mapsto \frac{1}{\sqrt{2}} (|\omega\rangle + e^{i\gamma} |\tilde{\omega}\rangle)$ with

$$
\gamma = \arccot \left( \sqrt{\frac{\sin \left( \frac{4\pi j}{n} \right)}{\cos \left( \frac{4\pi k}{n} \right)}} - 1 \right),
$$

and so a final phase shift $U_w(-\gamma)$ can be applied to eliminate the phase difference and obtain the Bell-like state $|+\rangle$ exactly. This gives a deterministic mapping between the marked and entangled states.

Hence, we obtain a deterministic algorithm to find the marked vertex on a CIIN for any value of $N$. Since this reduces to the prior algorithm when $\theta = \pi$, the algorithmic complexity is the same, requiring approximately $\frac{\pi}{2\sqrt{2}} \sqrt{N}$ oracle queries. A simulation of the approach is shown in Fig. 6 for a small 24-vertex CIIN.

E. A different path

We now demonstrate the flexibility of the framework by showing a different ‘path’ through the search subspace that leads to the marked element, while still retaining the quadratic quantum speedup and 100% success probability. Here, assume that $n$ is odd. Then define the simple iterate

$$
U_o = U_w(\frac{\pi}{2})U_f(\pi).
$$

In the dual basis, $U^2_o$ takes the form

$$
U^2_o = \begin{pmatrix}
\frac{n-2}{n} & 0 & \alpha & \beta \\
0 & \frac{n-2}{n} & -i^n \alpha & \frac{n-2}{n} \\
-i^n \beta & \frac{n-2}{n} & 0 & \frac{n-2}{n} \\
i^n \beta & \alpha & \frac{n-2}{n} & 0
\end{pmatrix}
$$

where $\alpha = (-1 + i^n) \frac{n-1}{n}$ and $\beta = (1 + i^n) \frac{\sqrt{n-1}}{n}$. It immediately follows by observation that a rotation is induced in the subspace spanned by the initial state $|b^*_1\rangle$ and

$$
|\xi\rangle = \frac{1 + i^n}{2} (|b^*_3\rangle + i^n |b^*_4\rangle).
$$
Expressing $U_o^2$ in the $\{|b_1^\dagger\rangle, |\xi\rangle\}$ basis gives
\[
U_o^2 = \left( \frac{n-2}{n} - \frac{2\sqrt{n-1}}{n} \right) \frac{\sin \theta}{
\sqrt{\frac{n-2}{n} - \frac{2\sqrt{n-1}}{n}}}
\]
which is recognisable as the Grover iteration, with eigenvalues $\pm 2 \arcsin \frac{1}{\sqrt{n} \gamma}$ and eigenstates
\[
|\xi\rangle = \frac{1}{\sqrt{2}} (|b_1^\dagger\rangle + |\xi\rangle).
\]
Hence, after $\frac{\pi}{2} \arcsin \frac{2}{\sqrt{n}} \approx \frac{\pi}{4} \sqrt{n}$ iterations of $U_o^2$, the state $|\xi\rangle$ is reached. Now we observe that (again dropping the global phase)
\[
U_w(-\frac{\pi n}{4})|\xi\rangle = \frac{1}{\sqrt{2}} (|b_1^\dagger\rangle - |b_1\rangle) = \sqrt{\frac{n-1}{n}} |b_1\rangle - \frac{1}{\sqrt{n}} |b_3\rangle
\]
and thus this state has $O(1)$ overlap with the marked state. The results of a numerical simulation on a 2050-vertex CIIN (where $n = 1025$) are shown in Fig. 7.

Note that this evolution, as before, can be made deterministic by slightly slowing the rotation rate such that the number of iterations required is integer-valued. We relegate the proof to Appendix B, but one approach for determination is
\[
U_o(\theta) = (U_w(\frac{\pi}{2}) U_f(-\theta))(U_w(\frac{\pi}{2}) U_f(\theta))^2
\]
where
\[
|\xi\rangle = U_w(\theta)^p |b_1^\dagger\rangle
\]
with $\theta = 2 \arcsin \left( \frac{n-1}{n} \sin \frac{\pi}{4p} \right)$. Here, to make the angle $\theta$ real-valued, again $p \geq \frac{\pi}{\sqrt{n}}$ is required.

To perform the final step and map completely to the marked state $|\omega\rangle \mapsto |\omega\rangle$, we follow a prescription very similar to the previous section, where $|+\omega\rangle \mapsto |\omega\rangle$ was carried out. As before, for specific walk times $t$ the evolution can be constrained to the desired 2-dimensional subspace. In this case, when $n$ mod 2 = 1 the choice $t = \pi$ induces rotation between $|b_1\rangle$ and $|b_3\rangle$:
\[
U_w(\pi) (\alpha |b_1\rangle + \beta |b_3\rangle) = \frac{2\sqrt{n-1}}{n} (\beta |b_1\rangle + \alpha |b_3\rangle) + \frac{n-2}{n} (\alpha |b_1\rangle + \beta |b_3\rangle).
\]}
Thus, a phase angle $\phi = 2 \arcsin \left( \frac{n^{3/2}}{4(n-2)^{1/2}} \right)$ is chosen such that
\[
U_w(\pi) U_f(\phi) U_w(\pi) |b_1\rangle = \sqrt{\frac{n-1}{n}} |b_1\rangle + \frac{e^{-i\gamma}}{\sqrt{n}} |b_3\rangle.
\]
with $\gamma = \arctan \left( \frac{n^{2}\cot \frac{\pi}{4}}{n^2 - 8n + 8} \right)$. After a final controlled phase shift $U_f(\gamma)$, we match up with the state reached in the previous section. Thus, overall we have shown that two efficient and unique rotations through the search space are possible, with both having 100% theoretical success probability of measuring the marked element after $\frac{\pi}{2\sqrt{2}} \sqrt{N}$ iterations.

V. QUANTUM CIRCUIT IMPLEMENTATION

In practice, an efficient spatial search algorithm also requires an efficient physical implementation of the associated quantum walks [23], for example, through an efficient quantum circuit [24–27]. Since the time-evolution operator of a CTQW on an unperturbed CIIN can be fast-forwarded, i.e. the quantum walk $U_w(t)$ can be simulated in constant-time with respect to $t$ [28], this makes this search approach amenable for implementation as a gate model quantum algorithm. The quantum circuit to implement a quantum walk on a $2^{m+1}$-vertex CIIN, which consists of two complete graphs of size $2^m$ with identity interconnections, is given in Fig. 8. The rotation gate used in the circuit is defined as the generic two-phase rotation gate [28],
\[
R(\theta, \phi) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\phi} \end{pmatrix}
\]

The first wire controls the identity interconnections, whilst the other $m$ wires represent the complete graph $\mathbb{K}_{2^m}$. Notably, in the quantum circuit diagram, the last $m$ wires of the circuit are exactly the generalised Grover diffusion operator, where a circuit for comparison can be found in [29]. As with ordinary Grover search, if one wishes to implement a quantum circuit for a 2-vertex CIIN where $n$ is not a power of two, the arbitrary-modulus Quantum Fourier Transform $F_n$ can be used to efficiently diagonalise $\mathbb{K}_n$ and replace the $\mathbb{K}_{2^m}$ component of the circuit [15, 30]. In reality, however, it is more convenient to round up the database size.

Hence, this work also provides motivation to develop fast-forwarded quantum circuits for continuous-time quantum walks on other graphs, in order to study the corresponding gate-model database search algorithm.

VI. DISCUSSION AND CONCLUSION

In this work, we have illustrated the benefits of a novel spatial search algorithm for finding a marked vertex on a particular class of interdependent networks.
The algorithm interleaves the two components of the spatial search Hamiltonian, performing alternating controlled phase shifts of the marked element followed by continuous-time quantum walks. We demonstrate that deterministic search can be achieved, and the number of oracle queries $\frac{\pi}{2\sqrt{N}}\sqrt{N}$ is comparable to Grover’s algorithm for unstructured database search. We also observe that the walk and phase parameters do not converge to zero even as the number of iterations approaches infinity, indicating that this is not simply a Trotterised discretisation of standard spatial search.

The approach can be implemented as both a gate-model algorithm (c.f. Grover’s algorithm) and an analog search algorithm (c.f. $CG$ spatial search). Consequently, an interesting property of the framework is that the efficiency can be studied both in the query model and in terms of the total walk time. To compare to Grover’s algorithm, the number of calls to $U_f(\theta)$ is summed. To obtain the total ‘evolution time’ $T$ spent quantum walking, as per spatial search, the walk times is totalled. Quadratic quantum speedup is attained for CIINs in both cases, and time-efficiency would automatically be attained for any other $2\pi$-periodic graph that has $O(\sqrt{N})$ query complexity. As an additional note, we also observe that there is the potential for some (necessarily dense) graphs to not have an efficient quantum circuit implementation for $U_w(t)$, and yet admit efficient gate-model search in terms of query complexity.

Although in this work the main focus is on a particular class of interdependent networks, we stress that this approach appears to work more generally on undirected graphs that admit efficient spatial search. Our work motivates the study of alternating phase-walk versions of spatial search on other graphs. The general pattern is to use the dual basis to compose phase-walk iterations that sequentially restrict to smaller subspaces. Although it is not immediately clear how to determine appropriate walk time and phase shift parameters given an arbitrary graph, there appears to be a strong connection to the spectrum of the adjacency matrix in terms of perfect state transfer and graph periodicity [21].

**ACKNOWLEDGMENTS**

This research was supported by a Hackett Postgraduate Research Scholarship and an Australian Government Research Training Program Scholarship at The University of Western Australia. We thank Leonardo Novo for valuable insight and suggestions, and Lyle Noakes for his continuous support and discussions.

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**Appendix A: Proof of reduced adjacency matrix**

Here, we prove the form of the adjacency matrix in the reduced walk basis. The reduction starts with

\[
A_{\text{full}} = \begin{pmatrix} \mathbb{K}_n & \mathbb{I} \\ \mathbb{I} & \mathbb{K}_n \end{pmatrix}
\]  

(A1)

where \( \mathbb{K}_n \) is the adjacency matrix of the complete graph, i.e., an all-ones matrix with zeroes on the diagonal. We primarily use the property that for \( 0 \leq x < n \)

\[
A_{\text{full}}|x\rangle = |n + x\rangle + \sum_{j=0}^{n-1} |j\rangle
\]

(A2)

and for \( n \leq x < N \)

\[
A_{\text{full}}|x\rangle = |x - n\rangle + \sum_{j=n}^{N-1} |j\rangle.
\]

(A3)

It is sufficient to determine the action on \( |b_1\rangle \) and \( |b_2\rangle \), with the action on the other two basis states following by symmetry. Hence,

\[
A_{\text{full}}|b_1\rangle = \sum_{j=0}^{n-1} |j\rangle + |n\rangle = \sqrt{n - 1}|b_3\rangle + |b_2\rangle
\]

(A4)

\[
A_{\text{full}}|b_3\rangle = \frac{1}{\sqrt{n - 1}} \sum_{j=1}^{n-1} A_{\text{full}}|j\rangle
\]

(A5)

\[
= \frac{1}{\sqrt{n - 1}} \sum_{j=1}^{n-1} \left( \sum_{j=0}^{n-1} |j\rangle + |n + x\rangle \right)
\]

\[
= \frac{1}{\sqrt{n - 1}} \sum_{j=0}^{n-1} |0\rangle - |j\rangle + \sum_{j=0}^{n-1} |j\rangle
\]

(A6)

and thus the adjacency matrix takes the form as shown in Eq. (7).

**Appendix B: Proof of second approach to deterministic search**

We use the iteration as per Eq. (58). First, define the following basis:

\[
|c_1\rangle = |b_1\rangle, \\
|c_2\rangle = \frac{1}{2} (-1 + i^n)|b_3\rangle + \frac{1}{2} (1 + i^n)|b_4\rangle.
\]

(B1) 

(B2)

In this basis the reduced iterate takes the following form:

\[
\begin{pmatrix}
\frac{4(n-1)\cos(\theta)+(n-2)^2}{2}\frac{2e^{-i\theta}(-1+e^{i\theta})\sqrt{n-1}(e^{i\theta}+n-1)}{n^2}
\end{pmatrix}
\]

(B3)

which again can be verified to have eigenphases \( \lambda_{\pm} = \pm 2 \arcsin \left( \frac{2\sqrt{n-1}}{n} \sin \frac{\theta}{2} \right) \) and corresponding eigenstates

\[
\frac{1}{\sqrt{2}} \left( \pm e^{-i \arctan \left( \frac{(n-2)}{n} \tan \frac{\theta}{2} \right)} |c_1\rangle + |c_2\rangle \right).
\]

(B4)

Thus, using this diagonalisation to compute the matrix power,

\[
U(\theta)^p |c_1\rangle = \cos(p\lambda_+)|c_1\rangle + ie^{i \arctan \left( \frac{(n-2)}{n} \tan \frac{\theta}{2} \right)} \sin(p\lambda_+)|c_2\rangle.
\]

(B5)

Hence, with \( p = \frac{n}{x} \) the system is mapped to \( |c_2\rangle \). Solving to find \( \theta \) in terms of \( p \) gives

\[
\theta = 2 \arcsin \left( \frac{n}{2\sqrt{n-1}} \sin \left( \frac{\pi}{2p} \right) \right)
\]

(B6)

as required.