GEOMETRIC PROPERTIES OF \( \varphi \)-UNIFORM DOMAINS

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Abstract. We consider proper subdomains \( G \) of \( \mathbb{R}^n \) and their images \( G' = f(G) \) under quasiconformal mappings \( f \) of \( \mathbb{R}^n \). We compare the distance ratio metrics of \( G \) and \( G' \); as an application we show that \( \varphi \)-uniform domains are preserved under quasiconformal mappings of \( \mathbb{R}^n \). A sufficient condition for \( \varphi \)-uniformity is obtained in terms of the quasi-symmetry condition. We give a geometric condition for uniformity: If \( G \subset \mathbb{R}^n \) is \( \varphi \)-uniform and satisfies the twisted cone condition, then it is uniform. We also construct a planar \( \varphi \)-uniform domain whose complement is not \( \psi \)-uniform for any \( \psi \).

1. Introduction and Main Results

Classes of subdomains of the Euclidean \( n \)-space \( \mathbb{R}^n \), \( n \geq 2 \), occur often in geometric function theory and modern mapping theory. For instance, the boundary regularity of a conformal mapping of the unit disk onto a domain \( D \) depends on the properties of \( D \) at its boundary. Similar results have been established for various classes of functions such as quasiconformal mappings and mappings with finite distortion. In such applications, uniform domains and their generalizations occur \([Ge99, GO79, GH12, Ko09, Va71, Va88, Va91, Va98, Vu88]\); \( \varphi \)-uniform domains have been recently studied in \([KLSV14]\).

Let \( \gamma : [0,1] \to G \subset \mathbb{R}^n \) be a path, i.e. a continuous function. All the paths \( \gamma \) are assumed to be rectifiable, that is, to have finite Euclidean length (notation-wise we write \( \ell(\gamma) < \infty \)).

Let \( G \subset \mathbb{R}^n \) be a domain and \( x, y \in G \). We denote by \( \delta_G(x) \), the Euclidean distance from \( x \) to the boundary \( \partial G \) of \( G \). When the domain is clear, we use the notation \( \delta(x) \). The \( j_G \) metric (also called the distance ratio metric) \([Vu85]\) is defined by

\[
j_G(x,y) := \log \left( 1 + \frac{|x - y|}{\delta(x) \land \delta(y)} \right),
\]

where \( a \land b = \min\{a,b\} \). A slightly different form of this metric was studied in \([GO79]\). The quasi hyperbolic metric of \( G \) is defined by the quasi hyperbolic length minimizing property

\[
k_G(x,y) = \inf_{\gamma \in \Gamma(x,y)} \ell_k(\gamma), \quad \ell_k(\gamma) = \int_{\gamma} \frac{|dz|}{\delta(z)},
\]

where \( \ell_k(\gamma) \) is the quasi hyperbolic length of \( \gamma \) (cf. \([GP76]\)) and \( \Gamma(x,y) \) is the set of all rectifiable curves joining \( x \) and \( y \) in \( G \). For a given pair of points \( x, y \in G \), the infimum is always attained \([GO79]\), i.e., there always exists a quasi hyperbolic geodesic \( J_G[x,y] \) which minimizes the above integral, \( k_G(x,y) = \ell_k(J_G[x,y]) \) and furthermore with the property that the distance is additive on the geodesic: \( k_G(x,y) = k_G(x,z) + k_G(z,y) \) for all \( z \in J_G[x,y] \). It
also satisfies the monotonicity property: \( k_{G_1}(x, y) \leq k_{G_2}(x, y) \) for all \( x, y \in G_2 \subset G_1 \). If the domain \( G \) is emphasized we call \( J_G[x, y] \) a \( k_G \)-geodesic. Note that for all domains \( G \),
\[
j_G(x, y) \leq k_G(x, y)
\]
for all \( x, y \in G \) [GP76].

In 1979, uniform domains were introduced by Martio and Sarvas [MS79]. A domain \( G \subset \mathbb{R}^n \) is said to be uniform if there exists \( C \geq 1 \) such that for each pair of points \( x, y \in G \) there is a path \( \gamma \subset G \) with
\[
(i) \quad \ell(\gamma) \leq C |x - y|; \quad \text{and}
(ii) \quad \delta(z) \geq \frac{1}{C} [\ell(\gamma[x, z]) \wedge \ell(\gamma[z, y])] \quad \text{for all } z \in \gamma.
\]
Subsequently, Gehring and Osgood [GO79] characterized uniform domains in terms of an upper bound for the quasihyperbolic metric as follows: a domain \( G \) is uniform if and only if there exists a constant \( C \geq 1 \) such that
\[
k_G(x, y) \leq C j_G(x, y)
\]
for all \( x, y \in G \). As a matter of fact, the above inequality appeared in [GO79] in a form with an additive constant on the right hand side; it was shown by Vuorinen [Vu85, 2.50] that the additive constant can be chosen to be 0. This observation leads to the definition of \( \varphi \)-uniform domains introduced in [Vu85].

**Definition 1.1.** Let \( \varphi : [0, \infty) \to [0, \infty) \) be a strictly increasing homeomorphism with \( \varphi(0) = 0 \). A domain \( G \subset \mathbb{R}^n \) is said to be \( \varphi \)-uniform if
\[
k_G(x, y) \leq \varphi \left( \frac{|x - y|}{\delta(x) \wedge \delta(y)} \right)
\]
for all \( x, y \in G \).

An example of a \( \varphi \)-uniform domain which is not uniform is given in Section 4. That domain has the property that its complement is not \( \psi \)-uniform for any \( \psi \).

Väisälä has also investigated the class of \( \varphi \)-domains [Va91] (see also [Va98] and references therein) and pointed out that \( \varphi \)-uniform domains are nothing but uniform under the condition that \( \varphi \) is a slow function, i.e. \( \varphi(t)/t \to 0 \) as \( t \to \infty \).

In the above definition, uniform domains are characterized by the quasi-convexity (i) and twisted-cone (ii) conditions. In Section 3 we show that the former can be replaced by \( \varphi \)-uniformity, which may in some situations be easier to establish.

**Theorem 1.2.** If a domain \( G \subset \mathbb{R}^n \) is \( \varphi \)-uniform and satisfies twisted cone condition, then it is uniform.

Let \( G \subset \mathbb{R}^n \) be a domain and \( f : G \to f(G) \subset \mathbb{R}^n \) be a homeomorphism. The linear dilatation of \( f \) at \( x \in G \) is defined by
\[
H(f, x) := \limsup_{r \to 0} \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|} : |x - y| = r \right\} \inf \left\{ \frac{|f(x) - f(z)|}{|x - z|} : |x - z| = r \right\}.
\]
We adopt the definition of \( K \)-quasiconformality from Väisälä [Va71]. If \( f \) is \( K \)-quasiconformal then \( \sup_{x \in G} H(f, x) \leq c(n, K) < \infty \).

In Section 2 we study \( \varphi \)-uniform domains in relation to quasiconformal and quasisymmetric mappings. Gehring and Osgood [GO79] Theorem 3 and Corollary 3], proved that uniform
domains are invariant under quasiconformal mappings of $\mathbb{R}^n$. Our next theorem extends this result to the case of $\varphi$-uniform domains.

**Theorem 1.3.** Suppose that $G \subsetneq \mathbb{R}^n$ is a $\varphi$-uniform domain and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a quasiconformal mapping which maps $G$ onto $G' \subsetneq \mathbb{R}^n$. Then $G'$ is $\varphi_1$-uniform for some $\varphi_1$.

2. **Quasiconformal and quasi-symmetric mappings**

In general, quasiconformal mappings of a uniform domain do not map onto a uniform domain. For example, by the Riemann Mapping Theorem, there exists a conformal mapping of the unit disk $D = \{ z \in \mathbb{R}^2 : |z| < 1 \}$ onto the simply connected domain $D \setminus [0, 1)$. Note that the unit disk $D$ is $(\varphi)$-uniform whereas the domain $D \setminus [0, 1)$ is not. However, this changes if we consider quasiconformal mappings of the whole space $\mathbb{R}^n$ [GO79]. In this section we provide the analogue for $\varphi$-uniform domains.

We notice that the quasihyperbolic metric and the distance ratio metric have similar natures in several senses. For instance, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Möbius mapping that takes a domain onto another, then $f$ is $2$-bilipschitz with respect to the quasihyperbolic metric [GP76]. Counterpart of this fact with respect to the distance ratio metric can be obtained from the proof of [GO79, Theorem 4] with the bilipschitz constant $2$.

**Lemma 2.1.** [GO79, Theorem 3] For $n \geq 2$ and $K \geq 1$ there exists a constant $c$ depending only on $n$ and $K$ such that, if $f : G \to G'$ is a $K$-quasiconformal mapping of $G \subsetneq \mathbb{R}^n$ onto $G' \subsetneq \mathbb{R}^n$, then

$$k_{G'}(f(x), f(y)) \leq c \max\{k_G(x, y), k_G(x, y)^\alpha\}$$

for all $x, y \in G$, where $\alpha = K^{1/(1-n)}$.

We obtain an analogue of Lemma 2.1 for $j_G$ with the help of the following result of Gehring’s and Osgood’s:

**Lemma 2.2.** [GO79, Theorem 4] For $n \geq 2$ and $K \geq 1$ there exist constants $c_1$ and $d_1$ depending only on $n$ and $K$ such that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a $K$-quasiconformal mapping which maps $G \subsetneq \mathbb{R}^n$ onto $G' \subsetneq \mathbb{R}^n$, then

$$j_{G'}(f(x), f(y)) \leq c_1 j_G(x, y) + d_1$$

for all $x, y \in G$.

In order to investigate the quasiconformal invariance property of $\varphi$-uniform domains we reformulate Lemma 2.2 in the form of the following lemma. We make use of both the above lemmas in the reformulation.

**Lemma 2.3.** For $n \geq 2$ and $K \geq 1$ there exists a constant $C$ depending only on $n$ and $K$ such that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a $K$-quasiconformal mapping which maps $G$ onto $G'$, then

$$j_{G'}(f(x), f(y)) \leq C \max\{j_G(x, y), j_G(x, y)^\alpha\}$$

for all $x, y \in G$, where $\alpha = K^{1/(1-n)}$. 
Proof. Without loss of generality we assume that $\delta(x) \leq \delta(y)$ for $x, y \in G$. Suppose first that $y \in G \setminus B^n(x, \delta(x)/2)$. Since $|x - y| \geq \delta(x)/2$, it follows that $j_G(x, y) \geq \log(3/2)$. By Lemma 2.2 we obtain

$$j_G'(f(x), f(y)) \leq \left(c_1 + \frac{d_1}{\log(3/2)}\right) j_G(x, y).$$

Suppose then that $y \in B^n(x, \delta(x)/2)$. By [Vu88] Lemma 3.7 (2), $k_G(x, y) \leq 2j_G(x, y)$. Hence we obtain that

$$j_G'(f(x), f(y)) \leq k_G'(f(x), f(y)) \leq c \max\{k_G(x, y), k_G(x, y)\}$$

$$\leq 2c \max\{j_G(x, y), j_G(x, y)\},$$

where the first inequality always holds by (1.1) and the second inequality is due to Lemma 2.1.

As a consequence of Lemmas 2.1 and 2.3, we prove our main result Theorem 1.3 about the invariance property of $\varphi$-uniform domains under quasiconformal mappings of $\mathbb{R}^n$.

Proof of Theorem 1.3. By Lemma 2.3 there exists a constant $C$ such that

$$j_G(x, y) \leq C \max\{j_G'(f(x), f(y)), j_G'(f(x), f(y))\}$$

for all $x, y \in G$. Define $\psi(t) := \varphi(e^t - 1)$. Then

$$k_G'(f(x), f(y)) \leq c \max\{k_G(x, y), k_G(x, y)\}$$

$$\leq c \max\{\psi(j_G(x, y)), \psi(j_G(x, y))\}$$

$$\leq c \max\{\psi(C \max\{j_G'(f(x), f(y)), j_G'(f(x), f(y))\}), \psi(C \max\{j_G'(f(x), f(y)), j_G'(f(x), f(y))\})\},$$

where the first inequality is due to Lemma 2.1, the second inequality holds by hypothesis, and the last inequality is due to (2.1). Thus, $G'$ is $\varphi_1$-uniform with

$$\varphi_1(t) = c \max\{\psi(C \max\{\log(1 + t), \log(1 + t)\}), \psi(C \max\{\log(1 + t), \log(1 + t)\})\},$$

where $\alpha = K^{1/(1-n)}$. 

A mapping $f : (X_1, d_1) \to (X_2, d_2)$ is said to be $\eta$-quasi-symmetric ($\eta$-QS) if there exists a strictly increasing homeomorphism $\eta : [0, \infty) \to [0, \infty)$ with $\eta(0) = 0$ such that

$$\frac{d_2(f(x), f(y))}{d_2(f(y), f(z))} \leq \eta\left(\frac{d_1(x, y)}{d_1(y, z)}\right)$$

for all $x, y, z \in X_1$ with $x \neq y \neq z$. Here ($X_1, d_1$) and ($X_2, d_2$) are metric spaces.

Note that $L$-bilipschitz mappings are $\eta$-QS with $\eta(t) = L^2t$ and $\eta$-QS mappings have the linear dilitation bounded by $\eta(1)$. It is pointed out in [TV80] that quasiconformal mappings are locally quasi-symmetric.

The following result gives a sufficient condition for $G$ to be a $\varphi$-uniform domain.

Proposition 2.4. If the identity mapping $id : (G, j_G) \to (G, k_G)$ is $\eta$-QS, then $G$ is $\varphi$-uniform for some $\varphi$ depending on $\eta$ only.
Proof. By hypothesis we have

$$\frac{k_G(x,y)}{k_G(x,z)} \leq \eta \left(\frac{j_G(x,y)}{j_G(x,z)}\right)$$

for all $x, y, z$ with $x \neq y \neq z$. Choose $z \neq y$ such that $\delta(z) = e^{-1}\delta(y)$. Then

$$j_G(y,z) = k_G(y,z) = \log \left(1 + \frac{|y-z|}{\delta(z)}\right) = \log \left(\frac{\delta(y)}{\delta(z)}\right) = 1.$$  

Hence, by the hypothesis we conclude that

$$k_G(x,y) \leq \eta(j_G(x,y)).$$

This shows that $G$ is $\varphi$-uniform. \hfill $\square$

**Question 2.5.** Is the converse of Proposition 2.4 true?

### 3. The $\varphi$-Uniform Domains Which Are Uniform

A domain $G \subset \mathbb{R}^n$ is said to satisfy the twisted cone condition, if for every $x, y \in G$ there exists a rectifiable path $\gamma \subset G$ joining $x$ and $y$ such that

$$\min \{\ell(\gamma[z, x]), \ell(\gamma[z, y])\} \leq c \delta(z) \text{ for all } z \in \gamma$$

and for some constant $c > 0$. Sometimes we call the path $\gamma$ a twisted path. Domains satisfying the twisted cone condition are also called John domains (see for instance [GHM89, He99, KL98, NV91]). If, in addition, $\ell(\gamma) \leq c|x - y|$ holds then the domain $G$ is uniform. Note that the path $\gamma$ in the definition of the twisted cone condition may be replaced by a quasihyperbolic geodesic (see [GHM89]).

We observe from Section 1 that a $\varphi$-uniform domain need not be uniform (or quasi-convex). Nevertheless, a $\varphi$-uniform domain satisfying the twisted-cone condition is uniform.

**Proof of Theorem 1.2.** Assume that $G$ is $\varphi$-uniform and satisfies the twisted cone condition (3.1). Let $x, y \in G$ be arbitrary and $\gamma$ be a twisted path in $G$ joining $x$ and $y$. Choose $x', y' \in \gamma$ such that $\ell(\gamma[x, x']) = \ell(\gamma[y, y']) = \frac{1}{10}|x - y|.$

Now, by the cone condition, we have

$$\delta(x') \geq \frac{1}{c} \min \{\ell(\gamma[x, x']), \ell(\gamma[x', y])\} \text{ and } \delta(y') \geq \frac{1}{c} \min \{\ell(\gamma[x, y']), \ell(\gamma[y, y'])\}.$$

By the choice of $x'$ and $y'$, on one hand, we see that

$$\ell(\gamma[x', y]) \geq |x' - y| \geq |x - y| - |x - x'| \geq \frac{8}{10}|x - y|.$$

On the other hand, $\ell(\gamma[x, x']) = \frac{1}{10}|x - y|$. The same holds for $x$ and $y$ interchanged. Thus,

$$\min \{\delta(x'), \delta(y')\} \geq \frac{1}{10c}|x - y|.$$

To complete the proof, our idea is to prove the following three inequalities:

$$\begin{cases}
 k_G(x, x') \leq a_1 j_G(x, x') \leq b_1 j_G(x, y); \\
 k_G(x', y') \leq b_2 j_G(x, y); \\
 k_G(y', y) \leq a_3 j_G(y', y) \leq b_3 j_G(x, y)
\end{cases}$$

for some constants $a_i, b_i, i = 1, 2, 3$. Finally, the inequality

$$k_G(x,y) \leq k_G(x,x') + k_G(x',y') + k_G(y',y) \leq c j_G(x, y)$$
with \((c = b_1 + b_2 + b_3)\) would conclude the proof of the theorem. It is sufficient to show the first two lines in (3.3), as the third is analogous to the first.

We start with a general observation: if \(j_G(z, w) < \log \frac{3}{2}\), then \(z\) lies in the ball \(B(w, \frac{1}{2}\delta(w))\), and we can connect the points by the segment \([z, w] \subset G\). Furthermore, due to [Vu88 Lemma 3.7 (2)] \(k_G(z, w) \leq 2 j_G(z, w)\). Thus in each inequality between the \(k\) and \(j\) metrics, we may assume that \(j_G(z, w) \geq \log \frac{3}{2}\) for all \(z, w \in G\).

First we prove the second line of (3.3). Since \(G\) is \(\varphi\)-uniform and \(\varphi\) is an increasing homeomorphism,

\[ k_G(x', y') \leq \varphi \left( \frac{|x' - y'|}{\min\{\delta(x'), \delta(y')\}} \right) \leq \varphi(12c), \]

where the triangle inequality \(|x' - y'| \leq |x' - x| + |x - y| + |y - y'|\) and the relation (3.2) are applied to obtain the second inequality. On the other hand, \(j_G(x, y) \geq \log \frac{3}{2}\) for all \(x, y \in G\).

Then we consider the first line of (3.3): \(k_G(x, x') \leq a_1 j_G(x, x') \leq b_1 j_G(x, y)\). The second inequality is easy to prove. Indeed, we have

\[ j_G(x, x') = \log \left( 1 + \frac{|x - x'|}{\min\{\delta(x), \delta(x')\}} \right) < \log \left( 1 + \frac{(1 + c)|x - y|}{\delta(x)} \right) \leq (1 + c) j_G(x, y), \]

where the first inequality holds since \(|x - x'| \leq \frac{1}{10}|x - y|\) and \(\min\{\delta(x), \delta(x')\} \geq \delta(x)/(1 + c)\).

Fix a point \(z \in \gamma\) with \(\ell(\gamma[z, x]) = \frac{1}{3}\delta(x)\) and denote \(\gamma_1 = \gamma[z, x]\). Assume for the time being that \(x' \not\in \gamma_1\). Clearly, \(k_G(x, x') \leq k_G(\gamma_1) + k_G(\gamma_2)\), where \(\gamma_2 = \gamma[z, x']\). For \(w \in \gamma_1\) we have \(\delta(w) \geq \frac{1}{2}\delta(x)\), and for \(w \in \gamma_2\), the twisted cone condition and the fact \(\ell(\gamma[w, y]) \geq 9\ell(\gamma[x, w])\) together give \(\delta(w) \geq (1/c) \ell(\gamma[x, w])\). Thus we find that

\[ k_G(\gamma_1) \leq \frac{\ell(\gamma_1)}{2 \delta(x)} = \frac{1}{\log \frac{3}{2}} j_G(x, x') \leq \frac{1}{\log \frac{3}{2}} (1 + c) j_G(x, y). \]

Furthermore,

\[ k_G(\gamma_2) \leq \int_{\ell(\gamma[x, x'])}^{\ell(\gamma[z, x])} \frac{dt}{t} = c \log \frac{\ell(\gamma[x, x'])}{\ell(\gamma[z, x])} = c \log \frac{10}{\frac{1}{2} \delta(x)} \leq c j_G(x, y). \]

So the inequality is proved in this case. If, on the other hand, \(x' \in \gamma_1\), then we set \(z = x'\) and repeat the argument of this paragraph for \(\gamma_1\), since \(\gamma_2\) is empty in this case.

This completes the proof of our theorem. \(\square\)

4. Complement of \(\varphi\)-uniform domains

In [KLSV14 Section 3] the following question was posed: Are there any bounded planar \(\varphi\)-uniform domains whose complementary domains are not \(\varphi\)-uniform? In this section we show that the answer is “yes”.

**Proposition 4.1.** There exists a bounded \(\varphi\)-uniform Jordan domain \(D \subset \mathbb{R}^2\) such that \(\mathbb{R}^2 \setminus D\) is not \(\psi\)-uniform for any \(\psi\).

**Proof.** Fix \(0 < u < t < v < 1\). Let \(R_k = (x_k, x_k + u^k) \times [0, v^k]\) be the rectangle, \(k \geq 1\). The parameter \(x_k\) is chosen such that \(x_1 = 0\) and \(x_{k+1} = x_k + u^k + t^k\). At the top of each rectangle
we place a semi-disc $C_k$ with radius $u_k^k/2$ and center on the midpoint of the top side of $R_k$. Set $s = u/(1-u) + t/(1-t)$. With these elements we define $D$, shown in Figure 1, by

$$D := ((0, s) \times (-2, 0)) \cup (\cup_k R_k) \cup (\cup_k C_k).$$

**Figure 1.** The $\phi$-uniform domain $D$ constructed in the proof of Proposition 4.1.

Let us show that $D$ is $\phi$-uniform. For $x \in R_k$ and $y \in R_l$, $k > l$, let $d := \min\{\delta(x), \delta(y)\}$. We choose a polygonal path $\gamma$ as follows: from $x$ the shortest line segment to the medial axis of $R_k$, then horizontally at $y = -d$ and finally from the medial axis of $R_l$ to $y$ along the shortest line segment (see again Figure 1). The lengths of the vertical and horizontal parts are at most $2\nu^k + d$, $2\nu^l + d$ and $|x - y| + \nu^{k+l}/2$. The line segments joining $x$ and $y$ to the medial axis have length at most $\nu^k/2$ and $\nu^l/2$. The whole curve is at distance at least $d$ from the boundary. Thus

$$k_D(x,y) \leq k_D(\gamma) \leq \frac{\ell(\gamma)}{d} \leq \frac{3\nu^k + 3\nu^l + 2d + |x - y|}{d} \leq \frac{8\nu^k + |x - y|}{d}.$$

On the other hand,

$$\frac{|x - y|}{\delta(x) \wedge \delta(y)} \geq \frac{t^k}{d}.$$

Let $\varphi(\tau) := \tau + 8\tau^\alpha$, where $\alpha$ is such that $t^\alpha = \nu$. Then $k_D(x,y) \leq \varphi(|x-y|/\delta(x) \wedge \delta(y))$. The case when $x, y \in R_k$ or in the base rectangle are handled similarly, although they are simpler. Thus we conclude that $D$ is $\varphi$-uniform.

We show then that $\mathbb{R}^2 \setminus \overline{D}$ is not $\psi$-uniform for any $\psi$. We choose $z_k = (x_{k+1} - t^{k}/2, t^k)$ in the gap between $R_k$ and $R_{k+1}$. Then

$$\frac{|z_k - z_{k+1}|}{\delta(z_k) \wedge \delta(z_{k+1})} \leq \frac{t^k/2 + u^{k+1} + t^{k+1}/2 + t^k}{t^{k+1}} = \frac{3}{2t} + \frac{1}{2} + \left(\frac{u}{t}\right)^{k+1} \leq \frac{3}{2t} + \frac{3}{2}.$$
On the other hand, a curve connecting these points has length at least $v^k - t^k/2$, so that

$$k_{\mathbb{R}^2\setminus \mathcal{D}}(z_k, z_{k+1}) \geq \int_{t^k/2}^{v^k - t^k/2} \frac{dx}{x} = \log \frac{v^k - t^k/2}{t^k/2} \to \infty$$

as $k \to \infty$. Therefore, it is not possible to find $\psi$ such that $k_{\mathbb{R}^2\setminus \mathcal{D}}(z_k, z_{k+1}) \leq \psi(\frac{|z_k - z_{k+1}|}{\delta(z_k) \wedge \delta(z_{k+1})})$, as claimed.

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