Discrete Spectrum of the Graviton in the $AdS^5$ Black Hole Background

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The discrete spectrum of fluctuations of the metric about an $AdS^5$ black hole background are found. These modes are the strong coupling limit of so called glueball states in a dual 3-d Yang-Mills theory with quantum numbers $J^{PC} = 2^{++}, 1^{--}, 0^{++}$. For the ground state modes, we find the mass relation: $m(0^{++}) < m(2^{++}) < m(1^{--})$. Contrary to expectation, the mass of our new $0^{++}$ state ($m^2 = 5.4573$) associated with the graviton is smaller than the mass of the $0^{++}$ state ($m^2 = 11.588$) from the dilaton. In fact the dilatonic excitations are exactly degenerate with our tensor $2^{++}$ states. We find that variational methods gives remarkably accurate mass estimates for all three low-lying levels while a WKB treatment describes the higher modes well.

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1. Introduction

The Maldacena conjecture [1] and its further extensions [2][3] allow us to compute quantities in a strongly coupled gauge theory from its dual gravity description. In particular, Witten [4] has pointed out if we compactify the 4-dimensional conformal super Yang Mills (SYM) to 3 dimensions using anti-periodic boundary conditions on the fermions, then we break supersymmetry and conformal invariance and obtain a theory that has interesting mass scales. In Refs. [5] and [6], this approach was used to calculate a discrete mass spectrum for $0^{++}$ states associated with $Tr[F^2]$ at strong coupling by solving the dilaton’s wave equation in the corresponding gravity description. Although the theory at strong coupling is really not pure Yang-Mills, since it has additional fields, some rough agreement was claimed between the pattern of glueball masses thus obtained and the masses computed for ordinary 3-d Yang-Mills by lattice calculations. In Ref. [5] the calculation was extended to solving the modes of the vector and two form fields with certain polarizations. Several other investigations have expanded our knowledge of the relation between string theory and gauge theories, in these and related directions [7].

Here we consider the discrete spectrum for the graviton field which is dual to the energy momentum tensor of the Yang-Mills field theory. This is a very interesting case, since a 5-d graviton field has five independent on-shell states that in a dual correspondence to a 4-d Yang-Mills theory would account for the five helicity components of a massive $2^{++}$ state lying on the leading diffractive (Pomeron) trajectory [8][9]. In the present analysis, these five states are mapped by dimensional reduction to a reducible multiplet with quantum numbers $2^{++}, 1^{-+}, 0^{++}$ in a 3-d Yang Mills theory [10]. Contrary to expectations, we find that the mass gap defined by the lowest $0^{++}$ state in the graviton multiplet is lower than mass gap for the dilaton. In fact our $2^{++}$ states are exactly degenerate with the lowest dilaton states, which shall be denoted as $\tilde{0}^{++}$, while the $1^{-+}$ equation gives the most massive states. Remarkably these mass relations, 

$$m(0^{++}) < m(\tilde{0}^{++}) = m(2^{++}) < m(1^{-+})$$

for the lowest modes are reminiscent of weak coupling glueball spectra from lattice gauge theory or the MIT bag model. Extensions to the graviton multiplet for states dual to the 4-d Yang Mills theory will be reported separately.

Quite apart from the issue of how closely the gauge theory obtained from the AdS description at strong coupling agrees “accidentally” with ordinary Yang-Mills, it is a very
interesting fact that one obtains discrete masses representing long lived states in the theory from a very different looking calculation done with the gravity variables. On the gravity side, if we compactify the $AdS_5$ metric on a large $S^1$ then the metric is a periodic version of the original $AdS^5 \times S^5$ metric. At high temperature (small radius of the $S^1$) this metric is replaced by an AdS black hole metric. This geometry has stationary modes for the fields of string theory (and in particular the fields of supergravity), and it is conjectured that the spectrum of these modes gives the mass spectrum for the compactified gauge theory on $R^3 \times S^1$. Understanding the properties of such a Yang-Mills theory even at strong coupling is interesting.

In this paper we carry out the calculation of the discrete modes for the perturbations of the gravitational metric. We restrict the perturbations to be independent of the $S^5$ factor in the metric and constant in the co-ordinate for the compact $S^1$. Our results are presented as follows.

In sec II, we derive three different wave equations that arise from fluctuations in the gravitational metric. These correspond to spin-2, spin-1 and spin-0 perturbations from the viewpoint of the rotation group in the 3-dimensional noncompact space:

(i) The spin-2 equation can be mapped by a change of variables to the equation for a massless scalar, and so the energy levels found are the same as those for the dilaton field $\phi$.

(ii) The spin-1 equation gives energy levels that are somewhat higher than the spin-2 case.

(iii) We first obtain the spin-0 equation in a gauge which has only diagonal components of the metric. We obtain a third order equation for the perturbation. We observe that there is a one parameter residual gauge freedom left in the ansatz, and we use the form of this gauge mode to reduce the third order equation to a second order equation for the perturbation. The situation here is similar to the analysis of polar perturbations of the 3+1 Schwarzschild hole in a diagonal gauge, where a third order system is also found. We comment on the gauge freedom present in the latter case; this discussion may illuminate some of the mysteries associated with the existence of the second order ‘Zerilli equation’ [11] for the above mentioned polar perturbations of the Schwarzschild metric. As a check on our result, we then study the perturbation in an alternative gauge, similar to one used by Regge and Wheeler [12]. In this latter gauge we directly obtain a second order equation, which is seen to be equivalent to the equation obtained by the first method.
In Sec III, we solve these equations numerically. We comment of the issue of the boundary condition to be used at the horizon of the black hole: The physics we impose is that there be no flux transfer across the horizon of the black hole, in agreement with the boundary condition used in Ref. [3]. (In Appendix B, we also show that the desired boundary conditions at $r = 1$ and $r = \infty$ together allow us to treat these equations in a Sturm-Liouville approach.) We present accurate numerical solutions to the first 10 levels. The spin-0 equation gives energy levels that are lower than the spin-1 and spin-2 equations.

In Sec IV, we consider approximation methods to gain insight into the mass relations. The ground state mass is also given in terms of a very simple variational ansatz and the entire spectrum approximated very well by a low-order WKB approximation. We also demonstrate how the boundary condition at the horizon can be implemented more effectively with the help of an “effective angular momentum”. (Technical details are postponed to Appendix B on the variational method, Appendix C on the Schrodinger form of the wave equations and Appendix D on the derivation of the WKB expansion.)

In Sec V, we discuss the results and make a few cautionary remarks on the difficulties of the comparison of this strong coupling spectrum with glueballs as computed in lattice gauge theory or classified in bag models. We recommend a more thorough analysis of the complete set of spin-parity states for the entire bosonic supergravity multiplet and its extension to 4-d Yang Mills models.

1.1. AdS/CFT correspondence at Finite Temperature

Let us review briefly the proposal for getting a 3-d Yang-Mills theory dual to supergravity. One begins by considering Type IIB supergravity in Euclidean 10-dimensional spacetime with the topology $M_5 \times S^5$. The Maldacena conjecture asserts that IIB superstring theory on $AdS^5 \times S^5$ is dual to the $\mathcal{N} = 4$ SYM conformal field theory on the boundary of the $AdS$ space. The metric of this spacetime is

$$ds^2/R_{ads}^2 = r^2(d\tau^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{dr^2}{r^2} + d\Omega_5^2,$$

where the radius of the $AdS$ spacetime is given through $R_{ads}^4 = g_s N\alpha'^2$ ($g_s$ is the string coupling and $l_s$ is the string length, $l_s^2 = \alpha'$). The Euclidean time is $\tau = ix_0$. To break conformal invariance, following [4], we place the system at a nonzero temperature described by a periodic Euclidean time $\tau = \tau + 2\pi R_0$. The metric correspondingly changes, for small enough $R_0$, to the non-extremal black hole metric in $AdS$ space. For large black hole
temperatures, the stable phase of the metric corresponds to a black hole with radius large compared to the $AdS$ curvature scale. To see the physics of discrete modes we may take the limit of going close to the horizon of this large hole, whereupon the metric reduces to that of the black 3-brane. This metric is (we scale out all dimensionful quantities)

$$ds^2 = f(r)dr^2 + f^{-1}(r)dt^2 + r^2(dx_1^2 + dx_2^2 + dx_3^2) + d\Omega_5^2,$$  \hspace{1cm} (1.2)

where

$$f(r) = r^2 - \frac{1}{r^2} \hspace{1cm} (1.3)$$

On the gauge theory side, we have a $N = 4$ susy theory corresponding to the $AdS$ spacetime, but with the $S^1$ compactification we have this theory on a circle with antiperiodic boundary conditions for the fermions. Thus supersymmetry is broken and massless scalars are expected to acquire quantum corrections.

From the view point of a 3-d theory, the compactification radius acts as an UV cut-off. Before the compactification the 4-d theory was conformal, and was characterized by a dimensionless effective coupling $(g_{YM}^{(4)})^2N \sim g_sN$. After the compactification the theory is not conformal, and the radius of the compact circle provides a length scale. Let this radius be $R$. Then a naive dimensional reduction from 4-d Yang-Mills to 3-d Yang-Mills, would give an effective coupling in the 3-d theory equal to $(g_{YM}^{(3)})^2N = (g_{YM}^{(4)})^2N/R$. This 3-d coupling has the units of mass. If the dimensionless coupling $(g_{YM}^{(4)})^2N$ is much less than unity, then the length scale associated to this mass is larger than the radius of compactification, and we may expect the 3-d theory to be a dimensionally reduced version of the 4-d theory. Unfortunately the dual supergravity description applies at $(g_{YM}^{(4)})^2N >> 1$, so that the higher Kaluza-Klein modes of the $S^1$ compactification have lower energy than the mass scale set by the 3-d coupling. Thus we do not really have a 3-d gauge theory with a finite number of additional fields.

One may nevertheless expect that some general properties of the dimensionally reduced theory might survive the change between small and large coupling. For this purpose we look at the pattern of masses and spins that are obtained for the fields that are singlets under the $S^1$ - i.e. we ignore the Kaluza-Klein modes of the $S^1$. In keeping with earlier work, we also restrict ourselves to modes that are singlets of the $SO(6)$, since non-singlets under the $S^1$ and the $SO(6)$ can have no counterparts in a dimensionally reduced $QCD_3$.\[4\]
2. Wave Equations

2.1. Field equations and the equilibrium configuration

We consider Type IIB supergravity in Euclidean 10-dimensional spacetime. The spacetime will have the topological form $M_5 \times S^5$. For the perturbations that we are interested in, we can set to zero the dilaton $\phi$, the axion $C$ and the 2-form fields $B^{NSNS}, B^{RR}$. The metric and the 4-form gauge field satisfy the Einstein equation

$$R_{\hat{\mu}\hat{\nu}} = \frac{1}{6} F_{\hat{\mu}\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\kappa}} F^*_{\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\kappa}}$$

(2.1)

and the self-duality condition for the field strength of the four form

$$F_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} = \frac{1}{5!} \epsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\lambda}} F^*_{\hat{\lambda}\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\kappa}}$$

(2.2)

**Notations:** Indices with a ‘hat’ are 10-d indices. The coordinates of the $S^5$ will be called $y^\alpha$, and the indices here will be $\alpha, \beta, \ldots$. The coordinates of $M$ will be $x^\mu$, and the indices will be $\mu, \nu, \ldots$. The $\epsilon$ tensor is real in spacetime with Lorentzian signature, and thus pure imaginary in Euclidean signature. Thus $F$ will be real on $S^5$, and imaginary on $M_5$.

The equilibrium configuration about which we perturb will be given by the metric

$$ds^2 = f(r)dr^2 + f^{-1}(r)dr^2 + r^2(dx_1^2 + dx_2^2 + dx_3^2) + d\Omega_5^2$$

(2.3)

where

$$f(r) = r^2 - \frac{1}{r^2}$$

(2.4)

The last term in (2.3) gives the metric of $S^5$ and we have chosen units to make this a sphere of unit radius. The 5-form field strength with indices in the $S^5$ is a constant times the volume form on $S^5$. By the self duality condition (2.2) we have

$$F_{\mu\rho\sigma\tau\kappa} F^*_{\nu} = -2\Lambda g_{\mu\nu}$$

(2.5)

where

$$2\Lambda = \frac{1}{5} F_{\alpha\beta\gamma\delta\epsilon} F^{\alpha\beta\gamma\delta\epsilon} = -\frac{1}{5} F_{\lambda\rho\sigma\tau\kappa} F^{\lambda\rho\sigma\tau\kappa}$$

(2.6)

is a positive constant.
2.2. Ansatz for the perturbations

We wish to consider fluctuations of the metric of the form

\[ g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}(x) \]  

(2.7)

Thus the perturbations will have no dependence on the coordinates of the \(S^5\). Further, we wish to keep unchanged all the other fields in the theory. It is easy to see that \(\phi, C, B^{NSNS}, B^{RR}\) can be held fixed, since they arise quadratically in the action, and vanish in the equilibrium configuration. (The metric above is the Einstein metric, and so there is no linear coupling of \(h\) to the dilaton.) It is also consistent to hold fixed the geometry of the \(S^5\) and the 5-form field-strength on the \(S^5\). Keeping these quantities fixed keeps unchanged the flux of \(F\) on \(S^5\), which is one constraint that the perturbations must respect. The self duality condition (2.2) can then regarded as determining the value of \(F\) on \(M\). But then we see that we will still get after such a perturbation:

\[ F_{\mu\rho\sigma\tau\kappa} F^\rho\sigma\tau\kappa_{\nu} = -2\Lambda g_{\mu\nu} \]  

(2.8)

with \(\Lambda\) the same constant as in (2.6). The Einstein equations then give

\[ R_{\mu\nu} = -\frac{1}{3}\Lambda g_{\mu\nu} \]  

(2.9)

Here the Ricci tensor \(R_{\mu\nu}\) can be computed solely from the 5-d metric of \(M\); all contributions involving the \(S^5\) variables are zero. Thus if we find a solution to (2.9), we have a consistent solution to the entire supergravity problem. With the expansion (2.7) we get

\[ -\frac{1}{2} h_{\mu\nu;\lambda}^\lambda - \frac{1}{2} h_{\lambda;\mu\nu}^\lambda + \frac{1}{2} h_{\mu\lambda;\nu}^\lambda + \frac{1}{2} h_{\nu\lambda;\mu}^\lambda + (D - 1) h_{\mu\nu} = 0 \]  

(2.10)

where \(D = 5\).

To see how many kinds of modes we expect, consider first the graviton in 5-d flat spacetime with coordinates \(x^i, i = 1, \ldots, 5\), and signature \((-++++)\). Then the graviton modes can be brought by gauge freedom to a form

\[ h_{\mu\nu} = \epsilon_{\mu\nu} e^{ik(x_2 - x_1)}, \quad \mu, \nu = 3, 4, 5, \quad \epsilon^\mu = 0 \]  

(2.11)

Thus there are 5 independent polarizations of the graviton.

In the spacetime (2.3) the coordinates \(x_1, x_2, x_3\) project to the 3-d spacetime of the gauge theory that we wish to study, \(\tau\) is direction of the gauge theory that is compactified in
obtaining the 3-d theory from the 4-d theory, and \( r \) is the radial coordinate of the spacetime that loosely speaking governs the scale at which the gauge theory is being studied. Our perturbations will have the form

\[
h_{\mu\nu} = \epsilon_{\mu\nu}(r)e^{-mx_3}
\]

where we have chosen to use \( x_3 \) as a Euclidean time direction to define the glueball masses of the 3-d gauge theory. We start with fixing for the gravitational perturbations the gauge

\[
h_{3\mu} = 0
\]

though later we will have occasion to consider changes to another gauge as well.

### 2.3. Wave equations for the perturbations

From the above ansatz and the metric \((2.3)\) we see that we have an \( SO(2) \) rotational symmetry in the \( x_1 - x_2 \) space, and we can classify our perturbations with respect to this symmetry. We then find the following categories of perturbations:

(a) Spin-2: There are two linearly independent perturbations which form the spin-2 representation of the above \( SO(2) \):

\[
h_{12} = h_{21} = q_T(r)e^{-mx_3}, \quad \text{all other components zero} \quad (2.14)
\]

\[
h_{11} = -h_{22} = q_T(r)e^{-mx_3}, \quad \text{all other components zero} \quad (2.15)
\]

The Einstein equations \((2.10)\) give,

\[
r^2(1 - \frac{1}{r^4})q''_T(r) + r(1 + \frac{3}{r^4})q'_T(r) + \left[ \frac{m^2}{r^2} - 4(1 + \frac{1}{r^4}) \right]q_T(r) = 0. \quad (2.16)
\]

Defining

\[
\phi_T = \frac{q_T}{r^2}
\]

Eq. \((2.16)\) becomes

\[
r^2(1 - \frac{1}{r^4})\phi''_T(r) + r(5 - \frac{1}{r^4})\phi'_T(r) + \frac{m^2}{r^2}\phi_T(r) = \phi_{T;\lambda} = 0, \quad (2.18)
\]

which is the free wave equation, and thus the same equation as that satisfied by the dilaton (with constant value on the \( S^5 \)). These dilaton modes have been studied in [5] for example, and we therefore will obtain the same energy levels for the
spin-2 perturbations as those obtained for the dilaton, e.g., for the ground state, $m_{T,0}^2 \simeq 11.588$.

(b) Spin-1:

$$h_{i\tau} = h_{\tau i} = q_V(r)e^{-mx^3}, \quad i = 1 \text{ or } i = 2, \quad \text{all other components zero} \quad (2.19)$$

The Einstein equations (2.10) give,

$$r^2(1 - \frac{1}{r^4})q''_V(r) + r(1 - \frac{1}{r^4})q'_V(r) + \left[\frac{m^2}{r^2} - 4(1 - \frac{1}{r^4})\right]q_V(r) = 0 \quad (2.20)$$

(c) Spin-0: Based on the symmetries we choose an ansatz where the nonzero components of the perturbation are

$$h_{11} = h_{22} = q_1(r)e^{-mx^3}$$

$$h_{\tau\tau} = -2q_1(r)f_r(r)e^{-mx^3} + q_2(r)e^{-mx^3} \quad \quad (2.21)$$

$$h_{rr} = q_3(r)e^{-mx^3}$$

where $f(r)$ is given in (2.4). We have chosen to express $h_{\tau\tau}$ in such a way so that the part involving $q_1$ is traceless, which we expect is to dominate at large $r$.

The analysis of the Einstein equations are given in Appendix A, where they are derived in the above gauge and also in a different gauge which is less directly related to the flat space form, (2.11), but more easy to solve:

$$h_{\tau\tau} = q_1(r)e^{-mx^3}$$

$$h_{rr} = q_2(r)e^{-mx^3}$$

$$h_{r3} = h_{3r} = q_3(r)e^{-mx^3} \quad \quad (2.22)$$

The field equation is, for $q_3 \equiv q_S(r)$,

$$\left\{p_2(r)\frac{d^2}{dr^2} + p_1(r)\frac{d}{dr} + p_0(r)\right\}q_S(r) = 0, \quad (2.23)$$

where

$$p_2(r) = r^2(r^4 - 1)^2[3(r^4 - 1) + m^2r^2]$$

$$p_1(r) = r(r^4 - 1)[3(r^4 - 1)(5r^4 + 3) + m^2r^2(7r^4 + 5)] \quad (2.24)$$

$$p_0(r) = 9(r^4 - 1)^3 + 2m^2r^2(3 + 2r^4 + 3r^8) + m^4r^4(r^4 - 1).$$
3. Numerical Solution

To calculate the discrete spectrum for our three equation, one must apply the correct boundary conditions at $r = 1$ and $r = \infty$. From the indicial equations, the asymptotic value of the two linearly independent solutions at infinity are

\[
q_T(r) \sim r^{-2}, r^2, \\
q_V(r) \sim r^{-2}, r^2, \\
q_S(r) \sim r^{-3}, r^{-1},
\]

and at $r = 1$ are

\[
q_T(r) \sim 1, \log(r - 1), \\
q_V(r) \sim r - 1, (r - 1)\log(r - 1), \\
q_S(r) \sim (r - 1)^{-1}, (r - 1)^{-1}\log(r - 1).
\]

In all cases the appropriate boundary condition at $r = 1$ is the one without the logarithmic singularity. At $r = \infty$ the least singular boundary is required to have a normalizable eigenstate. Matching these two boundary conditions results in a discrete set of eigenvalues $m^2_n$, where $n$ is the number of zeros in the wave function inside the interval $r \in (1, \infty)$. We solved the eigenvalue equations by the shooting method, integrating from $r_1 \approx 1$ to large $r_\infty \approx \infty$. The parameter $m^2$ is adjusted by applying Newton’s method to satisfy the condition $\lim_{r \to \infty} \frac{q(r)}{r} = 0$. To begin the integration one needs the function and its first derivative at $r = 1$. Thus expanding to the next significant order for the regular solution at $r = 1$, the initial conditions for the tensor were taken to be $q_T(r_1) = 1$, $q_T'(r_1) = 2 - m^2/4$ and for the vector to be $q_V(r_1) = 0$, $q_V'(r_1) = 1$. For the scalar equation, we found it convenient to define an new function, $f_S(r) = (r^2 - 1)q_S(r)/r$, so that the initial value is nonsingular and only the correct solution at infinity vanishes. This leads to initial conditions, $f_S(r_1) = 1$, $f_S'(r_1) = -m^2/4$.

Below we present in Table 1 the first 10 states using this shooting method. The spin-2 equation is equivalent to the dilaton equation solved in Refs. [5] and [6], so the excellent agreement with earlier values validates our method. In all of the equation we use a standard Mathematica routine with boundaries taken to be $x = r^2 - 1 = \epsilon$ and $1/x = \epsilon$ reducing $\epsilon$ gradually to $\epsilon = 10^{-6}$. Note that since all our eigenfunctions must be even in $r$ with nodes spacing in $x = r^2 - 1$ of $O(m^2)$, the variable $1/x$ is a natural way to measure the distance to the boundary at infinity. For both boundaries, the values of $\epsilon$ was varied.
Glueball $m_n^2$ by Shooting Technique

| level | $J^{PC} = 0^{++}$ | $J^{PC} = 1^{-+}$ | $J^{PC} = 2^{++}$ |
|-------|-------------------|-------------------|-------------------|
| n= 0  | 5.4573            | 18.676            | 11.588            |
| n= 1  | 30.442            | 47.495            | 34.527            |
| n= 2  | 65.123            | 87.722            | 68.975            |
| n= 3  | 111.14            | 139.42            | 114.91            |
| n= 4  | 168.60            | 203.99            | 172.33            |
| n= 5  | 237.53            | 277.24            | 241.24            |
| n= 6  | 317.93            | 363.38            | 321.63            |
| n= 7  | 409.82            | 461.00            | 413.50            |
| n= 8  | 513.18            | 570.11            | 516.86            |
| n= 9  | 628.01            | 690.70            | 631.71            |

Table 1: Radial Excitations for Glueballs associated with the Gravitation tensor field.

to demonstrate that they were near enough to $r = 1$, and $\infty$ so as not to substantially effect the answer.

4. Main Features of Glueball Spectrum

There are two intriguing aspects of our numerical results for the glueball spectrum. The first concerns the pattern of low-mass states, *e.g.*, for the respective ground states, one has: $m_{S,0}^2 < m_{T,0}^2 < m_{V,0}^2$. The second interesting feature is the asymptotic growth for the masses of their radial excitations. In order to provide further insights concerning these features, we turn next to approximation schemes appropriate for discussing: (1) ground state masses, and (2) high mass excitations. We end this section by providing a heuristic picture for understanding these main features.

4.1. Low-Mass Spectrum

Eq. (2.18) for tensor modes can be expressed in a standard “Sturm-Liouville” form

$$
\left( -\frac{d}{dx} x(x+1)(x+2) \frac{d}{dx} + w(x) \right) \phi(x) = \frac{m^2}{4} \phi(x),
$$

where we have found it more convenient to use $x \equiv r^2 - 1$ as the new variable. For tensor modes, the potential term is in fact absent, *i.e.*, $w_T(x) = 0$. Eq. (2.20) for vector
modes can also be brought into this form, with \( \phi_V(x) \equiv q_V(r)/\sqrt{r^4-1} \) and \( w_V(x) = 1/(x(x+1)(x+2)) \). As a Sturm-Liouville system finding various excitations can be carried out by applying the minimum principle.

Using the fact that the ground state energy in such a system corresponds to the absolute minimum of its energy functional, we can prove rigorously an inequality

\[
m_T^2,0 < m_V^2,0,
\]

between the masses for the tensor and the vector ground states. This inequality follows immediately from the fact that the potential function for vector modes, \( w_V(x) \), is strictly positive over the range \( 0 < x < \infty \). (See Appendix B.)

We next apply a variational approach to obtain upper bounds for the respective ground state masses. Simple trial wave functions satisfying the respective tensor and vector boundary conditions at \( x = 0 \) and at \( x \to \infty \) are \( \bar{\phi}_T(x) = (x+1)^{-2} \) and \( \bar{\phi}_V(x) = (x+1)^{-3} \sqrt{x(x+2)} \). One readily obtains the following simple upper bounds for the ground state masses:

\[
m_T^2,0 \leq 12, \quad \text{and} \quad m_V^2,0 \leq 20.
\]

These values compare very well with the precise numerical answers: \( m_T^2,0 = 11.588 \) and \( m_V^2,0 = 18.676 \). One can easily improve these already impressive bounds by using a better variational ansatz. For instance, using \( \phi_T(x) = (x+1/x+\lambda)\bar{\phi}_T(x) \) and \( \phi_V(x) = (x+1/x+\lambda)\bar{\phi}_V(x) \), with \( \lambda \) as variational parameter, we obtain

\[
m_T^2,0 \leq 11.588, \quad \text{for} \quad \lambda = 2.0857,
\]

\[
m_V^2,0 \leq 18.687, \quad \text{for} \quad \lambda = 1.1729.
\]

The situation for scalar modes is slightly more involved. Nevertheless, an analogous analysis can also be carried out, as explained in Appendix B. For scalar wave-functions, \( \phi_S(x) \equiv (r^4-1)q_S(r)/r \), the simplest choice for ground-state trial function consistent with boundary conditions at \( x = 0 \) and \( x \to \infty \) is \( \bar{\phi}_S(x) = \text{constant} \), which leads to an upper bound for the scalar ground-state mass:

\[
m_S^2,0 \leq 5.5213.
\]

Observe immediately that this upper bound is already very close to the exact numerical result: \( m_S^2,0 = 5.4573 \). An improved variational bound can obviously be obtained, as was done for both the tensor and the vector ground states. It is also important to note that this simple variational upper bound for the scalar ground state is already much lower than the mass for the tensor ground state.
4.2. WKB Estimates For High-Mass Excitations

We next focus on high-mass excitations by carrying out a WKB analysis. We begin by first converting our Sturm-Liouville type equations into “radial” type Schroedinger equations,

\[
\left(-\hbar^2 \frac{d^2}{dx^2} + V(x; m^2)\right)\psi(x) = E\psi(x),
\]

(4.7)

where \( E \to 0^- \). In keeping with the standard practice of WKB methods we have also introduced a formal expansion parameter \( \hbar \), which at the end of the calculation we set back to \( \hbar = 1 \). The relations between the new set of wave functions, \( \psi_V(x), \psi_T(x) \) and \( \psi_S(x) \), and the previous set, \( \phi_V(x), \phi_V(x) \) and \( \phi_S(x) \), are given in Appendix C. Note that, in this approach, eigenvalue parameter \( m^2 \) appears as a parameter in an effective potential, and it controls the “strength” of “central force” necessary in forming “zero-energy” bound states. For all three cases, when \( m^2 \) becomes large, it can be shown that each potential approaches a smooth large-\( m \) limit:

\[
V(x; m^2) \to m^2 V_0(x) + 0(1),
\]

\[
V_0(x) = -\frac{1}{4x(x+1)(x+2)}.
\]

(4.8)

By dividing Eq. (4.7) by \( m^2 \), we observe that, in the large \( m \)-limit, \( \hbar \) and \( m \) appear naturally in a simple ratio, \( \hbar/m \). Therefore, a large-\( m \) limit is formally the same as a “small”-\( \hbar \) limit. (Strictly speaking, this is not the case for our scalar equation.) Or less formally stated, as we observe in our numerical solutions, the mass eigenvalues \( m^2_n \) increase monotonically with the number of zeros \( n \) in the wave function so that the large eigenvalues expansion is also a short distance limit, as expected.

For bound-state problems, the WKB consistency condition for eigenvalues \( m = m_n \) for \( n = 0, 1, 2, \cdots \), is given as an expansion in \( \hbar \),

\[
(n + \frac{1}{2}) \pi \hbar = I_0(m) + \hbar^2 I_2(m) + \hbar^4 I_4(m) + \cdots \quad ,
\]

(4.9)

where the coefficients \( I_{2k}(m) \) are expressed as an integral between two classical turning points over integrand expressed in term of the potential. Since the leading behavior of the integrals are \( I_{2k}(m) \sim m^{1-2k} \) at large \( m \), we obtain an asymptotic expansion in \( 1/m \) at \( \hbar = 1 \). Systematically expanding in \( 1/m \), the WKB condition takes the form:

\[
(n + \frac{1}{2}) \pi = s_0 m + s_1 + \frac{s_2}{m} + \cdots ,
\]

(4.10)
where coefficients $s_i$ are easily calculable.

We have shown in Appendix D that, in order to obtain the first two coefficients, $s_0$ and $s_1$, it is sufficient to consider only the leading order WKB contribution,

$$I_0(m) = \int_{x_L}^{x_R} dx \sqrt{-V(x; m^2) - \frac{1 + 4\epsilon}{4x^2}}, \quad (4.11)$$

where $x_L(m)$ and $x_R(m)$ are two classical turning points with $\epsilon \to 0^+$. The shift from $V(x; m^2)$ to $V(x; m^2) + \frac{1}{4x^2}$, first discussed by K. E. Langer [13], is commonly known as the “Langer correction”. It incorporates the correction at the singular boundary when there is a centrifugal term, $l(l+1)/x^2$, in a radial wave equation. Furthermore we have added an $\epsilon$ term, $1/4x^2 \to (1 + 4\epsilon)/4x^2$, as a simple procedure for properly implementing the threshold behavior for $\psi(x)$ at $x = 0$. (See Appendix D.)

¿From Eq. (4.11), one notes that $I_0(m)$ can be a non-trivial function of $m$ through its dependence in $V(x; m^2)$ and on the two classical turning points, $x_L(m)$ and $x_R(m)$. Since $V \sim m^2$ for $m$ large, it follows that $I_0(m) = m(I_{0,0} + m^{-1}I_{0,1} + m^{-2}I_{0,2} + \cdots)$ at large $m$. Consequently $s_0 = I_{0,0}$, and $s_1 = I_{0,1}$. (To obtain $s_2$, one will also need to calculate $I_2$.)

Truncating Eq.(4.10) to this order and solving for $m^2$, one obtains the following expansion for the mass spectrum,[14]

$$m_n^2/\mu^2 \simeq n^2 + \delta n + \gamma, \quad (4.12)$$

where $\mu^2$, $\delta$, and $\gamma$ are related to $s_0$, $s_1$ and $s_2$ by $\mu = \pi/s_0$, $\delta = 1 - 2s_1/\pi$, and $\gamma = \delta^2/4 - 2s_0s_2/\pi^2$.

We have carried out this WKB analysis for all three cases, and have obtained the following interesting results. First, we have found that the leading coefficient, $\mu^2$, is universal, i.e.,

$$\mu^{-1} = \frac{2}{\pi} \int_0^\infty dx \sqrt{-V_0(x)} = \frac{1}{4} B\left(\frac{1}{2}, \frac{1}{2}\right), \quad (4.13)$$

is the same for tensor, vector, and scalar modes. Therefore, it is also the same as that for the dilaton modes, with $\mu^2 \simeq 5.742$. Second, for the term linear in $n$, simple integer coefficients are obtained for both the tensor and vector modes:

$$\delta_T = 3 \quad \delta_V = 4. \quad (4.14)$$

However, for scalar modes, a more intricate calculation is needed. We find that

$$\delta_S = \frac{1}{\pi} \int_0^{u_0} \frac{du}{u^{3/2}} \left(1 - \sqrt{1-3u^2-9u^3}\right) + \frac{2}{\pi \sqrt{u_r}} + 1 \simeq 3.018, \quad (4.15)$$

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where $u_r$ is the positive root of $9u_r^3 + 3u_r^2 - 1 = 0$, $u_r \simeq 0.392$. Putting these together, we have

$$
m_T^2 \simeq 5.742 \left(n^2 + 3n + \gamma_T\right),
$$
$$
m_V^2 \simeq 5.742 \left(n^2 + 4n + \gamma_V\right),
$$
$$
m_S^2 \simeq 5.742 \left(n^2 + 3.018n + \gamma_S\right).
$$

In Figure 1, we show a comparison on a log-log plot of our WKB prediction to our direct numerical results up to $n = 9$ for all three distinct modes. Interestingly, except for the lowest scalar mass, our WKB results agree with the actual data extremely well, with constants, $\gamma_T \simeq 2.011$, $\gamma_V \simeq 3.283$, and $\gamma_S \simeq 1.25$ fitted to the asymptotic form at large $n$. Higher order WKB results and other related work will be reported elsewhere.

![Figure 1: Comparison of WKB approximation with the levels computed in Table 1.](image)

In spite of this impressive agreement between the low order WKB calculations and the exact numerical results, a cautionary note should be added. As pointed out in Ref [5], the question of convergence of $1/m$-expansion for the logarithm of wave functions, $\phi_V(x)$, $\phi_V(x)$ and $\phi_S(x)$, remains to be addressed.
4.3. Heuristic Picture:

It is possible to provide a qualitative understanding for both the low-mass pattern, $m_{S,0}^2 < m_{T,0}^2 < m_{V,0}^2$, and the asymptotic high-mass excitations by making use of a quantum mechanical analogy. For $m$ large, Eq. (4.8) formally corresponds to a “screened” attractive Coulomb potential, with a “bare” charge $Q$ at the core,

$$Q = -\frac{m^2}{8} + 0(1). \quad (4.17)$$

More generally, we have shown in Appendix C that, for all three cases, each potential, $V(x; m^2)$, behaves near $x \sim 0$ as

$$V(x; m^2) \simeq \frac{L^2}{x^2} + \frac{Q}{x} + 0(1). \quad (4.18)$$

This naturally suggests that we identify $L^2 = l(l + 1)$ as angular momentum squared and $Q$ as the unscreened bare charge at the core. We find that $l_V = 0$, and $l_T = l_S = -\frac{1}{2}$.

Our boundary condition for $\psi$ at $x = 0$ corresponds to choosing $\psi(x) \sim x^{l+1}$ over $\sim x^{-l}$. For $\text{Re } l \geq -\frac{1}{2}$, this would correspond to always selecting the “regular” solutions. In order to avoid the degeneracy ambiguity for the case where $l_T = l_S = -\frac{1}{2}$, we shall in what follows always assume that $l_T = l_S = -\frac{1}{2} + \epsilon$ and consider the limit $\epsilon \to 0^+$.

When we include the “Langer correction”, $V \to V + (\frac{1}{4} + \epsilon)(\frac{1}{x^2})$, the angular momentum factor, $L^2 = l(l + 1)$, becomes $\tilde{L}^2 = (l + \frac{1}{2})^2 + \epsilon$. For $x \to 0$, we find that

$$\begin{align*}
\tilde{L}_V^2 &= \frac{1}{4}, & Q_V &= -\frac{m^2 - 6}{8}, \\
\tilde{L}_T^2 &= \epsilon, & Q_T &= -\frac{m^2 - 12}{8}, \\
\tilde{L}_S^2 &= \epsilon, & Q_S &= -\frac{m^2 - 2 + 16m^{-2}}{8}.
\end{align*} \quad (4.19)$$

At large $m^2$, (4.19) leads to (4.17), in agreement with Eq. (4.8). More generally, since $V(x; m^2) \to m^2 V_0(x)$ for all three cases, this explains our finding that the coefficient $\mu^2$ is universal. The next order correction, $\delta$, will depend on how $V(x; m^2)$ deviates away from $m^2 V_0(x)$, both at $x \sim 0$ and at $x \sim \infty$. Indeed, as we show in Appendix D that our results for $\delta_T$, $\delta_V$ and $\delta_S$ follow directly after we have identified the leading asymptotic behavior of $V$ at $x = 0$ and at $x = \infty$.

Turning next to the low-mass spectrum. In this analog picture, the vector ground state is more massive since it corresponds to a higher angular momentum state when compared
to the tensor ground state, $\tilde{L}_V^2 > \tilde{L}_S^2$, and therefore a stronger binding, (larger $m^2$), is required in order to form a zero-energy bound state. In contrast, the tensor modes and the scalar modes have the same angular momenta, $\tilde{L}_T^2 = \tilde{L}_S^2 = \epsilon > 0$. We find that the scalar “potential-well” is “deeper” than that for the tensor modes, especially when $m^2$ is small. Indeed, whereas the Coulomb core for tensor modes only become attractive when $m^2 \geq 12$, it is always attractive for the scalar modes, with minimum attraction at $m^2 = 4$. Therefore, less binding is required in forming a zero-energy scalar bound state, and it is consistent with the fact that the scalar ground state is lighter than the tensor mode.

5. Discussion

We have computed the discrete spectrum of metric perturbations in an $AdS^5$ black hole background. We would like to end by briefly discussing our results and making a few cautionary remarks on the difficulties of the comparison of this strong coupling spectrum with glueball masses as computed in lattice gauge theory or classified in bag models.

As stressed in Ref. [6], in order to obtain these discrete modes, a non-singular boundary condition must be applied at the horizon of the black hole and requiring normalizibility of wave-functions alone is insufficient for such a purpose. A deeper understanding on the physics of the boundary condition used would be highly desirable. Our discussion has also been restricted to modes that are singlets under the SO(6) symmetry and the “finite temperature” Kluza Klein U(1) gauge group. As a guide to finding states that might survive the dimensionally reduced limit, concentrating on states that are non-singlets in $SO(6) \times U(1)$ appears reasonable.

Earlier comparisons for $QCD_3$ have noted an apparent agreement between the ratio, $m(0^-)/m(0^{++}) = 1.45 \pm 0.03$, found in weak coupling lattice simulations and the ratio, $m(0^-)/m(0^{++}) \simeq 1.50$, found at strong coupling from supergravity, using the lowest state of the pseudoscalar mode ($B_{\mu\nu}$) and the lowest state of the dilaton ($\phi$). Our new lower mass $0^{++}$ state will somewhat upset this quantitative correspondence. With a much smaller value for the scalar mass, this supergravity mass-ratio becomes 3.19 which is too large by a factor of two.

However, the earlier agreement was inexplicably good in any case. From the point of view of a 3-d Yang-Mills theory at weak coupling the lowest $0^{++}$ state should couple to both the operator $S = Tr[F_{\mu\nu}^2(x)]$ and the helicity-0 component of the energy-momentum tensor $T_{\mu\nu} = Tr[F_{\mu\nu}^2(x)] - \frac{1}{4}\eta_{\mu\nu}S$, which are associated with the dilaton and the scalar...
component of the metric fluctuations respectively. Indeed each of these operators have
contributions from the $\tau$-component of $A_\mu$ along the $S^1$ compact direction, which from
the viewpoint of 3-d Yang-Mills theory is an adjoint Higgs field. The state associated with
the Higgs mode must decouple at weak coupling if there is a pure Yang-Mills fixed point.

A second feature that has previously been noted is the similarity in the level spacing
of the excitation spectrum of supergravity states and the spectrum of the excited glueball
spectrum in lattice simulations. In our view, this is even more problematic. For one thing,
we have so far ignored Kluza-Klein excitations in the compact $S^1$ direction that have a
level spacing given by $M_{KK} = 1/R \equiv 2b$, where $R$ is the size of the compactified circle
and the parameter $b$ so defined has units of mass. This spacing is of the same order as the
mass gap for the lowest $0^{++}$ state: $m(0^{++}) \simeq \sqrt{5.457 \, b} \simeq 2.3 \, b$. This suggests that the
proper identification of the excitation spectrum of the supergravity modes with states in
Yang-Mills theory is non-trivial. In quarkless QCD, all mass splittings are controlled by a
single mass scale, $\Lambda_{QCD}$ in 4-d or $g_3^2N$ in 3-d, which one may equally well take as the string
tension, $\sigma$. In the strong coupling limit on the lattice, this scale is replaced by the UV
cut-off (or inverse lattice spacing $a^{-1}$). In these units both the lattice glueball masses and
the string tension diverge as $\log(g^2N)$ or $\log(a g_3^2 N)$, whereas in the supergravity scenario
at strong coupling the putative 3-d “glueball” masses go to constants while AdS string
tensions diverges as

$$\sigma = \frac{\sqrt{4\pi g_s N} \, b^2}{2\pi}, \quad (5.1)$$

in leading order in strong coupling. It may be that lattice glueball excitations are transverse
stringy fluctuations that would not survive in the the supergravity approximation. In that
case, from a lattice view point, there should be no infinite tower of “glueball” states. The
supergravity spectrum describes radial (5-th dimensional) excitations that either do not
survive the weak coupling finite $N$ limit or are mixed with stringy states. It is difficult to
see what should be the precise comparison between these two sets of excitations since they
have quite different physical features. We consider the understanding of the correspondence
of this tower of states with Yang-Mills theory at the boundary to be an important unsolved
issue.

In spite of these reservations, we believe that a comparison between the supergravity
results and lattice as well as bag model results does offer opportunities to uncover phe-
nomenologically interesting features. For instance, in the mass relations for the lowest
states as a function of their quantum numbers, we do see a pattern reminiscent of lattice glueball spectra. Indeed the pattern,

\[ m(0^{++}) < m(0^{++}) = m(2^{++}) < m(1^{-+}) \]

is generic to both lattice calculations in 3-d and bag calculations, which are in excellent agreement with each other. We find that the classification of states with low-dimension color singlet fields in a bag picture for glueballs, \textit{i.e.}, in terms of “valence-glue”, follows the same pattern as the expected in the AdS/CFT correspondence. A fuller understanding of these relationships is certainly worth pursuing. First it is evident from our analysis above that all 128 lowest bosonic modes for the supergraviton multiplet that are \( SO(6) \times U(1) \) singlets need to be found. For 3-d Yang-Mills, it may be also more fruitful to make comparisons with high temperature lattice QCD results, where the constant KK adjoint field in the compact \( \tau \) direction still remains.

Our current exercise needs to be extended to schemes for 4-d QCD such as the finite temperature versions of \( AdS^7 \times S^4 \). As has been suggested elsewhere, the goal may be to find that background metric that has the phenomenologically best strong coupling limit. This can then provide an improved framework for efforts to find the appropriate formulation of a QCD string and for addressing the question of Pomeron intercept in QCD. In addition, a more thorough analysis of the complete set of spin-parity states for the entire bosonic supergravity multiplet and its extension to 4-d Yang-Mills models is also worthwhile. Results on these computations will be reported in a future publication.

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\textbf{Note added:} A brief version of our results was presented in talks by R. Brower at Lattice ’99 (Pisa, July 1999) and also at “QCD and Multiparticle Production–ISMD99”, (Brown, August 1999). While this paper was being finalized, a paper appeared, which has a considerable overlap with our results.
Appendix A. Field Equation for Spin-0 Perturbation:

Let us start with the gauge where the nonzero components of the spin-0 perturbation are

\[ h_{11} = h_{22} = q_1(r)e^{-mx_3} \]
\[ h_{\tau\tau} = -2q_1(r)\frac{f(r)}{r^2}e^{-mx_3} + q_2(r)e^{-mx_3} \]
\[ h_{rr} = q_3(r)e^{-mx_3} \] (A.1)

The Einstein equations (2.10) give

\[ q_3 = -\frac{1}{(r^4 - 1)^2(3r^4 - 1)}[2r^8 q_2(r) - r^5(r^4 - 1)q_{2,\tau}(r) + 4(r^4 - 1)q_1(r)]e^{-mx_3} \] (A.2)

and one first order and one second order equation in \( q_1 \) and \( q_2 \):

\[ 4r(3r^4 - 1)q_{1,\tau} + 3r^5(r^4 + 1)q_{2,\tau} - 8(3r^4 + 1)q_1 \]
\[ -[6r^4(r^4 - 1) + m^2 \frac{r^6(3r^4 - 1)}{r^4 - 1}]q_2 = 0 \] (A.3)

\[ 3r^2(r^8 - 1)q_{1,rr} + r[3(r^4 + 3)(r^4 + 1) - 4m^2 r^2]q_{1,\tau} \]
\[ +[-12(r^4 + 1)^2 + m^2 r^2(3r^4 - 1)]q_1 - m^2 r^6[\frac{m^2 r^2}{r^4 - 1} + 3]q_2 = 0 \] (A.4)

We can find \( q_2 \) from (A.4) and substitute it into (A.3) to obtain a third order equation for \( q_1 \):

\[ r^3(r^4 - 1)^2[m^2 r^2 + 3(r^4 - 1)]q_{1,rrr} \]
\[ +r^2(r^4 - 1)[-m^4 r^4 + 2m^2 r^2(r^4 + 5) + 3r^2(r^4 - 1)^2(3r^4 + 5)]q_{1,rr} \]
\[ +r[-4m^4 r^4 + m^2 r^2(23 - 6r^4 - r^8) - 3(r^4 - 1)^2(3r^4 + 13)]q_{1,\tau} \]
\[ +[-m^6 r^6 + m^4 r^4(r^4 + 3) - 2m^2 r^2(13 + 2r^4 + r^8) + 48(r^4 - 1)^2]q_1 = 0 \] (A.5)

It may appear strange that we have obtained a third order system (which needs three constants of integration) instead of a second order system as in the spin-2 and spin-1 cases above. The source of this extra freedom is that the gauge (2.13) there is a one parameter freedom left in the present ansatz:

\[ x_3 \rightarrow x_3 + \epsilon a(r)e^{-mx_3} \]
\[ r \rightarrow r + \epsilon \frac{ma(r)}{r f(r)} e^{-mx_3} \] (A.6)

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\[ a(r) = \exp\left[ \int \frac{dr}{r}(2 + \frac{k^2}{f}) \right] \]  
(A.7)

This freedom gives a change in \( q_1 \) proportional to

\[ q_1^{\text{gauge}} \equiv -\frac{2 - mr^2(1 - r^2)m^2/4}{(1 + r^2)m^2/4} \]  
(A.8)

Substituting

\[ q_1 = q_1^{\text{gauge}} \tilde{q}_1 \]  
(A.9)

we get a second order equation

\[
\begin{align*}
& \quad \left( r^4 - 1 \right)^2 \left[ -m^2 r^2 - 3(r^4 - 1) \right] \tilde{q}_{1,rr} \\
& + r(r^4 - 1) \left[ -2m^4 r^4 - m^2 r^2 (17r^4 - 5) - 3(r^4 - 1)(9r^4 - 1) \right] \tilde{q}_{1,r} \\
& - \left[ m^6 r^6 + 4m^4 r^4 (3r^4 - 1) + 4m^2 r^2 (5r^4 - 1) (2r^4 - 1) + 3(r^4 - 1)^2 (15r^4 + 1) \right] \tilde{q}_1 = 0
\end{align*}
\]  
(A.10)

A similar third order system of equations arises in the study of polar perturbations of the Schwarzschild metric, in the gauge where the metric is constrained to be diagonal. One discovers a combination of fields that ‘remarkably’ satisfy a second order equation, the ‘Zerilli equation’ \([11]\). But we can see that such a reduction is to be expected because here again there is a one parameter family of diffeomorphisms that preserves the gauge condition. Consider a metric of the form

\[ ds^2 = k(r) dt^2 + k^{-1}(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]  
(A.11)

Then if we are looking at perturbations with frequency \( \omega \) in time \( t \), then we have the following diffeomorphisms which leaves the metric within the chosen gauge (off diagonal components zero):

\[
\begin{align*}
\theta & \to \theta + h(r) \frac{\partial}{\partial \theta} P_l(\theta) e^{\omega t} \\
r & \to r - r^2 \left( \frac{h(r)}{r^2} \right)_r P_l(\theta) e^{\omega t} \\
t & \to t - \omega h(r) P_l(\theta) e^{\omega t}
\end{align*}
\]  
(A.12)

where for infinitesimal \( \epsilon \)

\[ h(r) = \epsilon r k(r)^{1/2}. \]  
(A.13)

(\( P_l(\theta) \) is the associated Legendre polynomial in \( \cos \theta \).) We can use this solution to reduce the third order system to a second order equation.
But for the Schwarzschild metric one can choose a different gauge, where such a residual gauge freedom does not exist. Such a gauge was used in [12]. We thus look for a gauge in our problem which would also have no residual gauge symmetry, and so directly yield a second order equation.

Consider the gauge where the nonzero components of the perturbation are

\[
\begin{align*}
    h_{\tau\tau} &= q_1(r)e^{-mx^3} \\
    h_{rr} &= q_2(r)e^{-mx^3} \\
    h_{r3} &= h_{3r} = q_3(r)e^{-mx^3}
\end{align*}
\] (A.14)

The Einstein equations (2.10) determine \( q_2 \) and \( q_1 \) in terms of \( q_3 \):

\[
q_2 = -\frac{r^5(r^4-1)q_{1,r} + 2r^8q_1}{(r^4-1)^2(3r^4-1)}
\] (A.15)

\[
q_1 = \frac{2(r^4-1)}{mr^5[m^2r^2 + 3(r^4-1)]}[3r(r^8-1)q_{3,r} + (3(r^4+1)(3r^4+1) + m^2r^2(3r^4-1))q_3]
\] (A.16)

and give a second order equation for:

\[
\begin{align*}
    r^2(r^4-1)^2(3(r^4-1) + m^2r^2)q_{3,rr} + r(r^4-1)[3(r^4-1)(5r^4+3) + m^2r^2(7r^4+5)]q_{3,r} \\
    + [9(r^4-1)^3 + 2m^2r^2(3 + 2r^4 + 3r^8) + m^4r^4(r^4-1)]q_3 = 0.
\end{align*}
\] (A.17)

This is Eq. (2.23), where we have re-labeled \( q_3 \) by \( q_S \). This second order equation is related to the second order equation (A.10) by the transformation

\[
\tilde{q}_1 = \frac{(1 + r^2)^{m^2/4}}{r^2(1 - r^2)^{m^2/4}}q_S
\] (A.18)

Appendix B. Sturm-Liouville Approach and Variational Analysis:

Eq. (4.1) is in the standard Sturm-Liouville form,

\[
\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + w(x)\right)\phi_n(x) = m_n^2\sigma(x)\phi_n(x),
\] (B.1)

where \( \tau(x) = x(x+1)(x+2) \) and \( \sigma(x) = \frac{1}{4} \). The set of eigenfunctions, \( \{\phi_n\} \), provides a complete orthonormal basis: \( <\phi_n|\phi_m> = \int_0^\infty dx \sigma(x)\phi_n(x)\phi_m(x) = \delta_{m,n} \). It is well-known that solving for this set of eigenfunctions and their corresponding eigenvalues is equivalent to finding stationery points of the following ”energy functional”

\[
\Gamma[\phi] \equiv \frac{\int_0^\infty dx[\tau(x)\phi'(x)^2 + w(x)\phi(x)^2]}{\int_0^\infty dx \sigma(x)\phi(x)^2}.
\] (B.2)
At each stationery point, $\phi_n$, one has $\Gamma[\phi_n] = m_n^2$. Since both $\tau(x)$ and $w(x)$ are positive, $\Gamma[\phi]$ is bounded from below and is in fact positive. In particular, the square of the mass for the lowest state, $m_0^2 = \Gamma[\phi_0]$, is the absolute minimum of $\Gamma[\phi]$, and $m_0^2 > 0$.

Let us denote $\Gamma_{(T/V)}[\phi]$ as the energy functional for the tensor and vector modes respectively. We shall next apply this formalism to obtain a simple and yet important mass inequality:

$$m_{V,0}^2 > m_{T,0}^2.$$ 

Denote the the lowest tensor and vector masses by $m_{V,0}$ and $m_{T,0}$ respectively. Since each corresponds to the absolute minimum of its energy functional, it follows that

$$0 < m_{T,0}^2 = \Gamma_T[\phi_0^T] \leq \Gamma_T[\phi]; \quad 0 < m_{V,0}^2 = \Gamma_V[\phi_0^V] \leq \Gamma_V[\phi]. \quad (B.3)$$

Alternatively, $m_{V,0}^2$ can also be expressed as $m_{V,0}^2 = <w_V>_0 + \Gamma_T[\phi_0^V]$, where

$$<w_V>_0 \equiv \frac{\int_0^\infty dx w_V(x)\phi_0^V(x)^2}{\int_0^\infty dx \sigma(x)\phi_0^V(x)^2}.$$ 

Note that $w_V(x) = 1/(x(x+1)(x+2))$ is a positive function for $0 \leq x < \infty$; it follows that $<w_V>_0$ is positive, and

$$m_{V,0}^2 > \Gamma_T[\phi_0^V]. \quad (B.4)$$

On the other hand, since $m_{T,0}^2$ is the absolute minimum of $\Gamma_T$, we also have the following inequality,

$$\Gamma_T[\phi_0^V] > \Gamma_T[\phi_0^T] = m_{T,0}^2. \quad (B.5)$$

From (B.4) and (B.5), the mass inequality between vector and tensor ground states, $m_{V,0}^2 > m_{T,0}^2$, Eq. (4.2), follows.

It is also worth pointing out that, in re-deriving these eigenvalue equations from the energy functional, it is necessary to impose boundary conditions

$$\tau(x)\phi(x)\phi'(x) \rightarrow 0, \quad (B.6)$$

both for $x \rightarrow 0$ and $x \rightarrow \infty$. This states more clearly what boundary condition is to be used at the horizon of the black hole. In particular, this rules out the solution where $\phi(x) \sim \log x$ at $x = 0$. To be more precise, by examining our differential equations, Eq. (4.1), both
at \( x = 0 \) and at \( x \to \infty \), Eq. (B.6) requires that tensor and vector wave-functions must behave as

\[
\phi_T(x) \to x^{-2}, \quad \phi_V(x) \to x^{-2}, \quad x \to \infty,
\]
\[
\phi_T(x) \to \text{constant}, \quad \phi_V(x) \to x^{1/2}, \quad x \to 0.
\]

We next address the question of scalar modes. Let us first convert Eq. (2.18) into

\[
\hat{L}_S \phi_S(x) \equiv \left( -\left( \frac{d}{dx} \tau_S(x; m^2) \frac{d}{dx} \right) + w_S(x; m^2) \right) \phi_S(x) = 0,
\]

(B.8)

where

\[
\tau_S(x; m^2) = \frac{x(x+2)}{[3x(x+2) + m^2(x+1)]}, \quad w_S(x; m^2) = \frac{m^2[(3x^2 + 6x + 4) - m^2(x+1)]}{4(x+1)[3x(x+2) + m^2(x+1)]^2}.
\]

(B.9)

Here, \( \phi_S(x) \) is related to \( q_S(r) \) in (2.18) by \( \phi_S(x) = \frac{x(x+2)}{\sqrt{x+1}} q_S(r) \). Since \( m^2 \) enters into the differential equation in a rather non-trivial fashion, we are unable to directly convert this into a Sturm-Liouville problem with \( m^2 \) as eigenvalues. Let us consider instead the following eigenvalue problem:

\[
\hat{L}_S \phi_n(x) = E_n \sigma_S(x) \phi_n(x),
\]

(B.10)

where \( E_0 < E_1 < E_2 < \cdots \), and \( \sigma_S(x) \) is at this point unspecified. This in turn can be obtained via a Sturm-Liouville type variational approach, i.e., finding stationery points of the following energy functional:

\[
\Gamma_S[\phi] \equiv \int_0^\infty dx (\tau_S(x; m^2)|\phi(x)'|^2 + w_S(x; m^2)|\phi(x)|^2) \int_0^\infty dx \sigma_S(x)|\phi(x)|^2.
\]

(B.11)

Note that, here, \( m^2 \) is to be treated as a parameter. To have a properly defined stationery conditions, we require \( \tau_S(x; m^2)\phi(x)\phi(x)' = 0 \), both at \( x = 0 \) and \( x \to \infty \). This, in turn, fixes our boundary conditions. A careful examination indicates that the desired boundary conditions for \( \phi_S(x) \) are

\[
\phi_S(x) \to \text{constant},
\]

(B.12)

for both \( x \to \infty \) and \( x \to 0 \).

Since \( w_S(x; m^2) \) is no longer positive, neither is \( \Gamma_S[\phi] \). Nevertheless, \( \Gamma_S[\phi] \) is still bounded from below since \( w_S(x; m^2) \) is. As such, we identify \( E_n^2(m^2) = \Gamma_S[\phi_n] \), in increasing order, at each local minimum, \( \phi_n \). In particular, \( E_0(m^2) \) is the absolute minimum of
\( \Gamma_S \), although it no longer is positive in general. This “generalized Sturm-Liouville problem” reduces to our scalar problem if a zero-energy solution exists. To be more precise, one needs to answer the following question: “For what values of \( m_j^2, j = 0, 1, 2, \ldots \), will there be a zero-energy solution where \( E_j^2(m_j^2) = 0 \) ?” As a minimum problem, we are searching for \( m_0^2 \) for which the absolute minimum of \( \Gamma_S(\phi) \) vanishes, i.e., \( \min \Gamma_S[\phi; m_0^2] = 0 \). Turning this into a variational problem, we require that, for \( m^2 > m_0^2 \),

\[
0 = E_0 \leq \frac{\int_0^\infty dx (\tau_S(x; m^2) |\phi(x)|^2 + w_S(x; m^2) |\phi(x)|^2)}{\int_0^\infty dx \sigma_S(x)|\phi(x)|^2}, \tag{B.13}
\]

for suitably chosen trial wave-functions. For each choice of \( \phi(x) \), the value of \( m^2 \) at which the integral on the right vanishes provides an upper bound for \( m_0^2 \). In arriving at our bound for the scalar ground-state mass, Eq. (4.6), the trial function \( \phi_S(x) = \text{constant} \), was used.

**Appendix C. Radial Schroedinger Representation and Effective Potentials:**

In order to bring our ODE, Eq. (4.1), into the radial Schroedinger form, Eq. (4.7), we perform the following transformation:

\[
\begin{align*}
\psi_T(x) &= (x(x+1)(x+2))^{\frac{1}{2}} \phi_T(x), \\
\psi_V(x) &= (x(x+1)(x+2))^{\frac{1}{2}} \phi_V(x), \\
\psi_S(x) &= \left( \frac{m^2 x(x+2)}{3x(x+2) + m^2(x+1)} \right)^{\frac{1}{2}} \phi_S(x).
\end{align*} \tag{C.1}
\]

The resulting effective potentials for all three cases are:

\[
\begin{align*}
V_T(x; m^2) &= m^2 V_0(x) + \frac{3(x^2 + 2x + 2)}{4x(x+1)^2(x+2)} - \frac{1}{x^2(x+1)^2(x+2)^2}, \\
V_V(x; m^2) &= m^2 V_0(x) + \frac{3(x^2 + 2x + 2)}{4x(x+1)^2(x+2)}, \\
V_S(x; m^2) &= m^2 V_0(x) + \frac{m^2(3x^2 + 6x + 2)}{2x(x+1)(x+2)[m^2(x+1) + 3x(x+2)]} + \Delta V_S,
\end{align*} \tag{C.2}
\]

where a rather involved expression has to be added for the scalar potential,

\[
\Delta V_S = -\left( \frac{(x+1)^2 + 1}{4x^2(x+2)^2} \right) \left( m^4 + \frac{4m^2x(x+1)(x+2)(3x+1)^2+7}{((x+1)^2+1)^2+4x(x+2)} \right), \tag{C.3}
\]

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Let us briefly comment on these effective potentials, \( \tilde{V}(x; m^2) = V(x; m^2) + (1/4+\epsilon)(1/x^2) \), in various limits,

(a) \( m^2 \to \infty \): For all three effective potentials,

\[
\tilde{V}(x; m^2) = m^2 V_0(x) + 0(1). \tag{C.4}
\]

(b) \( x \to 0 \): In going from \( V \) to \( \tilde{V} \), for \( x \sim 0 \), we have \( \tilde{V}(x; m^2) \simeq \tilde{L}^2/x^2 + Q/x + 0(1) \), where \( \tilde{L}^2 = (l + 1/2)^2 + \epsilon \). Those \( \tilde{L}^2 \) and \( Q \) values given in Eq. (4.19) follow from

\[
\tilde{V}_T(x; m^2) = \frac{\epsilon}{x^2} - \frac{m^2 - 12}{8x} + 0(x^0), \tag{C.5}
\]

\[
\tilde{V}_V(x; m^2) = \frac{1}{4x^2} - \frac{m^2 - 6}{8x} + 0(x^0), \tag{C.5}
\]

\[
\tilde{V}_S(x; m^2) = \frac{\epsilon}{x^2} - \frac{(m^2 - 2 + 16 m^2)}{8x} + 0(x^0).
\]

(c) \( x \to \infty \):

\[
\tilde{V}_T(x; m^2) = \frac{1}{x^2} - \frac{m^2 + 6}{x^3} + 0(x^{-4}),
\]

\[
\tilde{V}_V(x; m^2) = \frac{1}{x^2} - \frac{m^2 + 6}{x^3} + 0(x^{-4}), \tag{C.6}
\]

\[
\tilde{V}_S(x; m^2) = \frac{1}{4x^2} - \frac{m^2}{12x^3} + 0(x^{-4}).
\]

From these, one concludes that each “Coulomb” bare charge, \( Q \), is always screened at large distance; an effective potential vanishing as \( 1/x^2 \) at large distance is overall “neutral”.

**Appendix D. WKB Treatment:**

In a standard WKB treatment for a one-dimensional quantum mechanics problem, one relies on having a small parameter, “\( \hbar \)”, which allows a systematically expansion for the logarithm of the wave-function, \( \phi(x) \sim \exp[\pm i \hbar \int^x dx / \sqrt{\epsilon - V(x)} + \cdots] \). For a problem involving a spherically symmetric potential, \( V(x) \), one has a one-dimensional radial Schroedinger equation involving an effective potential, e.g., Eq. (4.4), where \( V(x) \to V_{eff}(x) = V(x) + l(l + 1) \hbar^2/x^2 \). In solving the radial equation, normalizability requires that the wave-function \( \psi \sim x^{l+1} \) as \( x \to 0^+ \), which also corresponds to our ”non-singular” boundary condition at the black hole horizon. Note that this exact threshold behavior, \( \psi(x) \sim e^{(l+1) \log x} \), would normally appear only in the next to leading order in a WKB expansion.
It is indeed possible to arrange so that the threshold behavior is properly implemented in the leading order by working with a “rapidity” variable, \( y \equiv \log x \), where \( y \) is now unrestricted, \(-\infty < y < \infty\), and the radial Schroedinger Equation becomes \((-x^2/dy^2 + U(y))\tilde{\psi}(y) = \tilde{E}\tilde{\psi}(y)\), where \( \tilde{\psi}(y) = e^{-\frac{y}{2}}\psi(x)\), \( \tilde{E} \equiv Ee^{2y} \), and

\[
U(y) \equiv e^{2y}\tilde{V}_{eff}(x) \equiv x^2(V_{eff}(x) + \frac{\hbar^2}{4x^2}) = x^2(V(x) + \frac{(l + \frac{1}{2})^2\hbar^2}{x^2}),
\]

Note that, for \( U(y) \) and \( \tilde{V}_{eff}(x) \), the combination \( l(l + 1)h^2 \) has become \( (l + \frac{1}{2})^2\hbar^2 \), which correlates precisely with the appropriate threshold behavior for \( \tilde{\psi} \sim x^{l+\frac{1}{2}} \). For an attractive potential \( V(x) \) which is less singular than \( \frac{1}{x} \) at the origin, this new effective potential \( U(y) \) will have an attractive well, with \( U(y) \rightarrow (l + \frac{1}{2})^2\hbar^2 \) when \( y \rightarrow -\infty \), whose bound state energies can be calculated via a standard WKB approximation. For instance, the leading order WKB contribution is \( (\frac{n + \frac{1}{2}}{2})\pi\hbar = \int_{y_L}^{y_R} dy\sqrt{\tilde{E} - U(y)} \), where \( n = 0, 1, 2, \ldots \), \( e^{y_L} = x_L \) and \( e^{y_R} = x_R \).

Returning to our problem; with \( E \rightarrow 0^- \), and \( 1/m \) playing the role of \( \hbar \), we arrive at

\[
I_0(m) = \frac{1}{m} \int_{y_L(m)}^{y_R(m)} dy\sqrt{-U(y;m)}.
\]  

(D.1)

By changing variable from \( y \) back to \( x \), Eq. (D.1) can also be written as Eq. (4.11) where where \( \tilde{V}_{eff}(x;m) \) enters. Let us turn next to the expansion of \( I_0(m) \) in \( 1/m \), \( I_0(m) \approx I_{0,0} + \frac{1}{m}I_{0,1} + \frac{1}{m^2}I_{0,2} + \cdots \). Before proceeding, we separate the integral in Eq. (D.1) , into two,

\[
I_0(m) = I_L(m) + I_R(m) = \frac{1}{m} \int_{y_L(m)}^{0} dy\sqrt{-U(y;m)} + \frac{1}{m} \int_{0}^{y_R(m)} dy\sqrt{-U(y;m)}.
\]

(D.2)

This separation at \( y = 0 \) is of course arbitrary; our final result will be independent of this choice. It is convenient to separate \( U(y;m) \) into two pieces, \( U(y;m) = m^2(U_0(y) + m^{-2}U_1(y;m^2)) \), which will be used in Eq. (D.2). To expand in \( 1/m \), one needs to know how \( U_0 \) and \( U_1 \) behave as \( y \rightarrow \pm \infty \).

Let us begin by treating the case of tensor modes with reasonable amount of details. Consider \( I_L(m) \) and let us examine the lower integration limit, \( y_L(m) \). With both \( \epsilon \sim 0^+ \) and \( 1/m \rightarrow 0 \), the left-turning point is determined by the asymptotic limit of \( U(y;m^2) \) as \( y \rightarrow -\infty \), \( U(y;m^2) \rightarrow \epsilon - (m^2/8 - 3/2)e^y \). That is, \( e^{y_L} \approx \frac{8\epsilon}{m^2-12} \). As \( \epsilon \rightarrow 0^+ \), for \( m \) large, it follows that the left-turning point moves to to minus infinity, \( y_L \rightarrow -\infty \). More
explicitly, this is a consequence of the fact that $\tilde{L}_T^2 = \epsilon \to 0$. We further note that, with both $U_0(y)$ and $U_1(y)$ vanish as $0(e^y)$ as $y \to -\infty$, the ratio $U_1(y)/U_0(y)$ remains finite; it is justified to expand the integrand in $m^{-2}$ and one obtains
\[ I_L(m) = \int_{-\infty}^{0} dy \sqrt{-U_0(y)} + 0(1/m^2) = \int_{0}^{1} dx \sqrt{-V_0(x)} + 0(1/m^2). \tag{D.3} \]

We turn next to $I_R(m)$, which can be more conveniently written as
\[ I_R(m) = \int_{1}^{x_R} \frac{dx}{x} \sqrt{-x^2V_0(x)} + \int_{1}^{x_R} \frac{dx}{x} \sqrt{-x^2V_0(x)} \left\{ \sqrt{1 + \frac{1}{m^2} \tilde{V}_1(x; m^2)} - 1 \right\}. \tag{D.4} \]

The upper limit is determined by $\tilde{V}(x_R) = m^2 (V_0(x_R) + \frac{1}{m^2} \tilde{V}_1(x_R)) = 0$. Since $V_0(x) \to -1/4x^3$ and $\tilde{V}_1(x) \to \frac{1}{x^2}$, it follows that $x_R(m) \simeq m^2/4$. The first integral can be expressed as a difference: $\int_{1}^{\infty} dx \sqrt{-V_0(x)} - \int_{x_R}^{\infty} dx \sqrt{-V_0(x)}$, where, for $1/m$ small, $\int_{x_R}^{\infty} dx \sqrt{-V_0(x)} \simeq x_R^{-1/2} \simeq 2/m$. For the second integral, the main contribution now comes from region where $x = 0(m^2)$. Let us change variable from $x$ to $u = xR \simeq 4x/m^2$, and, in the limit of large $m$, this expression becomes $\frac{1}{m} \int_{0}^{1} \frac{du}{u^{3/2}} (\sqrt{1-u} - 1) + 0(1/m^2) = (\frac{2}{m})(1 - \frac{\pi}{2}) + 0(\frac{1}{m^2})$. Putting things together, we thus have,
\[ I_R(m) = \int_{1}^{\infty} dx \sqrt{-V_0(x)} - \frac{\pi}{m} + 0(\frac{1}{m^2}). \tag{D.5} \]

Combining this with that for $I_L$, we have, for tensor modes,
\[ I(m) \simeq \int_{0}^{\infty} dx \sqrt{-V_0(x)} - \frac{\pi}{m} + 0(\frac{1}{m^2}) = \frac{1}{4} B\left(\frac{1}{4}, \frac{1}{2}\right) - \frac{\pi}{m} + 0(\frac{1}{m^2}). \tag{D.6} \]

This leads to Eq. (4.13) for $\mu$, and $\delta_T = 3$ for tensor modes.

Turning next to the vector modes. Again, separating $I(m)$ into two parts, Eq. (D.2), an identical analysis for $I_R(m)$ can be carried out as was done for the tensor modes, leading to the expansion, Eq. (D.3). However, since $U(y) \to 1/4$ as $y \to -\infty$, (corresponding to the situation where $\tilde{L}_V = 1/4$, Eq. (D.3) no longer holds for the vector modes. One finds, instead,
\[ I_L(m) = \int_{0}^{1} dx \sqrt{-V_0(x)} - \frac{\pi}{2m} + 0(\frac{1}{m^2}). \tag{D.7} \]

Combining this with that for $I_R$, we have, for vector modes,
\[ I(m) \simeq \int_{0}^{\infty} dx \sqrt{-V_0(x)} - \frac{3\pi}{2m} + 0(\frac{1}{m^2}). \tag{D.8} \]
This leads to Eq. (4.13) for $\mu$, and $\delta_V = 4$ for vector modes.

We complete this analysis by treating the scalar modes. Since $\tilde{L}_S^2 = \tilde{L}_I^2 = 0$, one finds that Eq. (D.3) holds for scalars modes also. As for $I_R(m)$, we need to work with Eq. (D.4) with $\tilde{V}(x; m^2) = 1/(4x^2) + V(x; m^2)$, where $V$ for scalar modes is given by the rather involved expression in (C.2) and (C.3). Let us consider the second integral first, where the main contribution now comes from region where $x \sim m^2$. Changing variable from $x$ to $u \equiv x/m^2$, and, in the limit of large $m$, this expression becomes

$$\frac{1}{2m} \int_0^{u_R} \frac{du}{u^{3/2}} \left( \frac{\sqrt{1-3u^2-9u^3}}{1+3u} - 1 \right) + O\left( \frac{1}{m^2} \right),$$

where $u_R$ is the positive root of the $1-3u^2-9u^3 = 0$, $u_R \simeq 0.3915$. The first integral can be expressed as a difference: $\int_1^\infty dx \sqrt{-V_0(x)} - \int_x^{\infty} dx \sqrt{-V_0(x)}$, where, for $1/m$ small, $\int_x^{\infty} dx \sqrt{-V_0(x)} \simeq 1/(m\sqrt{u_R})$. Combining these with that for $I_L$, we have

$$I(m) \simeq \int_0^\infty dx \sqrt{-V_0(x)} + \frac{s_1}{m} + O\left( \frac{1}{m^2} \right),$$

where $s_1 \simeq -(2.018)(\pi/2)$, thus leading to $\delta_S = 3.018$, Eq. (4.15).
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