Generalized Moving Horizon Estimation for Nonlinear Systems with Robustness to Measurement Outliers

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Abstract—The accuracy of moving horizon estimation (MHE) significantly degrades under measurement outliers. Existing methods usually formulate combinatorial optimization problems to address this issue and are restricted to linear systems to ensure computational tractability. To overcome these limitations, this paper proposes a generalized MHE (GMHE) approach that formulates MHE as a maximum a posteriori estimation problem and extends the standard MHE with a generalized loss function. The proposed approach avoids the high computational complexity of existing methods and has no restriction on the system models. We demonstrate that the standard MHE is a special case of GMHE, where the loss function uses the Kullback-Leibler (KL) divergence between the empirical distribution of the observations and the assumed likelihood. Because KL divergence is sensitive to outliers, we replace it with a robust $\beta$-divergence and name the corresponding GMHE as $\beta$-MHE. We prove that for the case of linear Gaussian systems, the gross error sensitivity of the $\beta$-MHE remains bounded, which demonstrates its robustness against outliers. The effectiveness of $\beta$-MHE is further demonstrated on systems subject to outliers with Gaussian distribution and the student distribution, respectively.

I. INTRODUCTION

Obtaining accurate state estimation from noisy measurements and the knowledge of system dynamics has been crucial for system control, and has received significant attention in different domains such as signal processing, robotics as well as econometrics.

Moving horizon estimation (MHE), which is referred to as the standard MHE in this paper, has proven to be a promising method to acquire a precise estimate for nonlinear systems [1]. It is formulated as a receding horizon optimization problem over recent measurements. MHE allows handling various noise distributions and explicitly considers the constraints on states and noise. In fact, it reduces to Kalman filter (KF) in the absence of constraints when considering only the last measurement.

The accuracy of the state estimate will degrade when the assumed measurement generation mechanism deviates from the true mechanism, and the misspecification of the measurement model could be caused by sensor malfunction or heavy-tailed noise. Measurements from such misspecification are usually termed outliers and should be carefully handled.

Measurement outliers have attracted enormous interest in the statistics community. In the field of optimal filtering, there are also a large number of works focusing on handling outliers, especially the outlier-robust KF methods. For example, by employing the M-estimate method, a batch-mode maximum likelihood-type KF was proposed in [2]. The maximum correntropy KF was derived by adopting the robust maximum correntropy criterion as the optimality criterion instead of using the minimum mean square error criterion [3]. More recently, an advanced outlier-robust Kalman filtering framework was developed by introducing a statistical similarity measure, which has the potential to reveal the relationships between different existing methods [4].

The derivation of those methods relies heavily on the linearity property of recursive prediction equations, which is inapplicable to nonlinear systems, thus making them hard to extend to optimization-based MHE approaches. An alternative strategy was proposed to develop a robust moving-horizon estimator, in which a mixed-integer optimization problem was formulated to handle the measurement outliers [5], [6]. However, this algorithm assumes bounded values of the measurement noise and expects intermittent outliers. In addition, it only works for linear systems to guarantee the stability and suffers from a high computational burden as the mixed-integer optimization problem is well-known to be NP-hard.

To address measurement outliers, one promising approach is Generalized Bayesian Inference (GBI) [7], which is gaining increased attention in recent years. This approach treats standard Bayesian inference as an optimization procedure, the object of which consists of a data dependent loss and the discrepancy between the prior and posterior belief distributions. Inspired by the principle of GBI, we propose a novel framework of generalized moving horizon estimation (GMHE) that preserves robustness under measurement outliers. Noticeably, GMHE extends the standard MHE by using a generalized, measurement-dependent loss function. We further design $\beta$-MHE algorithm based on the $\beta$-divergence to attenuate the effects of outliers under the framework of GMHE. Specifically, the contributions of this paper can be summarized as follows:

1) To avoid the complexity of NP-hardness and linearity assumption of mixed-integer optimization in existing robust MHE methods [5], [6], we formulate MHE as a maximum a posteriori problem and develop the framework of GMHE to tackle the measurement outliers.
2) We demonstrate that the standard MHE can be obtained by choosing the measurement dependent loss function of GMHE as the KL divergence from the empirical distribution of the observations to the assumed likelihood. We then propose the $\beta$-MHE by using the $\beta$-divergence as the loss function of GMHE, which is more robust than KL divergence against outliers.

3) To the best of the authors’ knowledge, we are the first to obtain the analytical form of the influence functions of the MHE methods, forming the basis for analyzing their robustness. More importantly, we prove that for linear Gaussian systems, the gross error sensitivity of $\beta$-MHE is bounded, which demonstrates its robustness.

The remainder of this paper is organized as follows: Section II presents the technical background of GBI and MHE. Section III proposes the GMHE framework and demonstrates the robust loss function design. Section IV performs the robust analysis for our proposed methods. Finally, the numerical results are provided in Section V and the conclusions are drawn in Section VI.

**Notation:** $D_{KL}(\cdot, \cdot)$ and $D_\beta(\cdot, \cdot)$ are the Kullback–Leibler divergence and the $\beta$-divergence of two probability distributions, respectively. $\delta(\cdot, \cdot)$ represents the Dirac function. A function $\alpha$ is a $K_+$ function if it is continuous, strictly monotone increasing, $\alpha(x) > 0$ for $x \neq 0$, and $\alpha(0) = 0$. $M \succeq 0$ indicates that the matrix $M$ is positive semi-definite, while $M \succ 0$ indicates that the matrix $M$ is positive-definite. $\|x\|$ represents the $l_2$-norm of the vector $x$, in other words, $\|x\| = \sqrt{x^T x}$. For $M \succeq 0$, $\|x\|_M$ is short for $\sqrt{x^T M x}$. $\|M\|_F$ is the Frobenius norm of matrix $M$. Besides, $I_{n \times n}$ is the identity matrix with dimension $n$.

II. TECHNICAL BACKGROUND

A. Generalized Bayesian Inference

Standard Bayesian inference calculates the posterior distribution $p_{pos}(x|y)$ utilizing a prior probability $\pi(x)$ and a likelihood function $g(y|x)$ derived from a statistical model of the observed data, as shown in (1). It implicitly assumes that such a statistical model is well-specified; in particular, the observations are generated from the assumed likelihood function $g(y|x)$.

$$p_{pos}(x|y) = \frac{\pi(x)g(y|x)}{p(y)},$$

$$p(y) = \int_x \pi(x)g(y|x)dx. \quad (1)$$

In real-world scenarios, this assumption is often violated. GBI is such an approach proposed to deal with the discrepancy between the true data-generating mechanism (DGM) and the assumed likelihood [7]. GBI can be described as a solution to the optimization problem (2), and it is proved that (1) can be viewed as a special case of GBI. Specifically, a belief distribution $q$ can be constructed by solving

$$q = \arg \min_{\nu} L(\nu; \pi, y), \quad (2)$$

where $L(\nu; \pi, y)$ is the loss function. However, not all loss functions allow deriving the Bayes-type updating rules. To obtain such a type of updating rules, the loss function needs to be specified as the sum of a “data term” and a “regularization term” [7]

$$L(\nu; \pi, y) = L_{data}(\nu; y) + L_{reg}(\nu; \pi). \quad (3)$$

It is shown that the form of (3) that satisfies the von Neumann–Morgenstern utility theorem and Bayesian additivity is given by

$$L(\nu; \pi, y) = \int \ell(x,y)\nu(x)dx + D_{KL}(\nu||\pi).$$

This formation leads to a Bayes-type updating rule given by

$$q(x) = \frac{\pi(x) G(y|x)}{Z},$$

$$Z = \int_x \pi(x)G(y|x)dx$$

with $G(y|x) := \exp(-\ell(x,y))$, where $\ell(x, y)$ is the loss function representing the discrepancy between the observed information and the assumed likelihood function.

B. Bayesian View on MHE Method

Consider the stochastic systems (4), where $\{x_t\}$ is the set of unobserved states of a Markov process and $\{y_t\}$ is the set of noisy observations. Optimal state estimation infers the true state $x_t$ of the stochastic systems given the current and past measurements $y_{1:t}$.

$$x_0 \sim \pi_0(x_0),$$

$$x_t|x_{t-1} \sim f_t(x_t|x_{t-1}),$$

$$y_t|x_t \sim g_t(y_t|x_t). \quad (4)$$

Bayesian filtering aims to calculate the posterior marginal distributions $q_t(x_t|y_{1:t})$ or posterior joint distributions $q_t(x_0:t|y_{1:t})$ that can be factorized as (5). When both the transition distributions $f_t$ and the likelihood functions $g_t$ are linear-Gaussian, $q_t(x_t|y_{1:t})$ and $q_t(x_0:t|y_{1:t})$ can be calculated analytically by the well-known KF. However, when $f_t$ or $g_t$ are nonlinear or non-Gaussian, the posterior distributions are generally intractable. To this effect, particle filter employs sequential Monte Carlo method to approximate the posterior joint distribution [8].

$$q_t(x_0:t|y_{1:t}) = \pi_0(x_0) \prod_{k=1}^{t} g_k(y_k|x_k) f_k(x_k|x_{k-1}). \quad (5)$$

In many applications, the point estimate of the state is of most interest rather than the entire density. One question is which kind of point estimate is desirable, the mean or the mode? Since the conditional density is generally asymmetric and potentially multi-modal for nonlinear systems, calculating the mode of the posterior distribution is a more reasonable choice [9]. Full information estimation (FIE) tries to acquire the mode of the posterior distribution

$$\hat{x}_{0:t} = \arg \max_{x_{0:t}} q_t(x_0:t|y_{1:t}). \quad (6)$$
The computational burden of solving the full information estimator (6) grows as more measurements become available. To ensure computational tractability, MHE employs only a finite number of past measurements and leverages the logarithm transformation to simplify the problem. Its problem formulation can be written as

\[
\hat{x}_{t-T:t} = \arg \min_{x_{t-T:t}} J(x_{t-T:t}),
\]

(7a)

\[
J(x_{t-T:t}) = \Gamma(x_T) + \sum_{k=t-T+1}^{t} \{k_k(x_k, x_{k-1}) + h_k(y_k, x_k)\},
\]

(7b)

where

\[
\Gamma(x_T) \approx -\log q_{t-T}(x_{0:t-T}|y_{1:t-T}),
\]

(8a)

\[
k_k(x_k, x_{k-1}) = -\log f_k(x_k|x_{k-1}),
\]

(8b)

\[
h_k(y_k, x_k) = -\log g_k(y_k|x_k).
\]

(8c)

Here \(\Gamma(x_T)\) is called arrival cost (also called initial penalty) and \(C_t(x_t, x_{t-1}, y_t) = k_t(x_t, x_{t-1}) + h_t(y_t, x_t)\) is the stage cost function. Arrival cost is a fundamental concept in MHE. It serves as an equivalent statistic for summarizing the past data \(y_{1:t-T}\) and is typically chosen as the quadratic form \(\|x_{t-T} - \hat{x}_t\|^2_{P_{t-T}^{-1}}\), where \(\hat{x}_t\) is the optimal estimate solved before and \(P_{t-T}\) can be calculated via Extended KF [1] or Unscented KF (UKF) [10].

Remark 1. The problem formulation of MHE is conventionally written in the form of the optimal control problem with optimization variables being \(x_{1:T}\) and \(\xi_{1:T-1}\) (\(\xi_t\) represents the process noise at time \(t\)) underlying the state space model. However, in this paper, we choose an equivalent set of optimization variables, \(x_{t-T:t}\), for narrative simplicity and clarity.

### III. Generalized Moving Horizon Estimation

In this section, we propose the GMHE algorithm that generalizes the standard MHE algorithm in the sense of a generalized likelihood function based on the GBI theory. Then a robust loss function is designed to tackle the misspecified system models.

#### A. GMHE Algorithm

In section II, we have illustrated that GBI replaces the standard likelihood function \(q(y|x)\) with a generalized likelihood function \(G(y|x) = \exp(-\ell(x, y))\). Similar to GBI, if we define the sequence of the generalized likelihoods as \(G_t(y_t|x_t) = \exp(-\ell_t(x_t, y_t))\), then the joint generalized posterior distribution \(q_t(x_{0:t}|y_{1:t})\) can be decomposed as the product of \(G_t(y_t|x_t)\) and \(f_t(x_t|x_{t-1})\), i.e.,

\[
q_t(x_{0:t}|y_{1:t}) = \pi_0(x_0) \prod_{k=1}^{t} G_k(y_k|x_k) f_k(x_k|x_{k-1}).
\]

Here we only consider the unconstrained MHE for simplicity.

Essentially, we fix the other parts of the standard MHE but generalizes (8c) in the MHE objective to

\[
h_k(y_k, x_k) = -\log G_k(y_k|x_k).
\]

(9)

We denote this new method as GMHE, and the steps to be carried out are sketched in Algorithm 1. The computational overhead of Algorithm 1 is comparable to the standard MHE algorithm considering the negligible difference between (9) and (8c) in the problem formulation.

#### B. Loss Function Design with Robustness to Outliers

We have shown how \(G_t(y_t|x_t) = \exp(-\ell_t(x_t, y_t))\) generalizes \(g_t(y_t|x_t)\) in section III-A, then we will discuss how to design the loss function \(\exp(-\ell_t(x_t, y_t))\) to resist the contamination of the measurement outliers.

The GMHE can naturally reduce to the standard MHE when the loss function \(\ell_t(x_t, y_t)\) is restricted to \(-\log g_t(y_t|x_t)\). The inherent mechanism of \(-\log g_t(y_t|x_t)\) is that it specifies the Kullback–Leibler (KL) divergence between the true DGM, which is denoted as \(p_{\text{true}}(y_t)\), and the assumed likelihood function \(g_t(y_t|x_t)\).

Because the logarithm operation is sensitive to outliers, substituting the KL divergence with a more robust divergence, such as the \(\beta\)-divergence, can make the inference more robust [11] [12]. An intuitive illustration can be seen in Fig. 1. From Fig. 1, we can see that the KL divergence grows faster when one distribution moves away from the other. Besides, the \(\beta\)-divergence becomes more sensitive as \(\beta\) approaches zero.

The \(\beta\)-divergence between the true data distribution \(p_{\text{true}}(\cdot)\) and the assumed likelihood \(g_t(\cdot|x_t)\) is

\[
\mathcal{D}_\beta(p_{\text{true}}(\cdot), g_t(\cdot|x_t)) = \frac{1}{\beta} \int p_{\text{true}}(y)\beta+1dy - \frac{\beta+1}{\beta} \int p_{\text{true}}(y)g_t(y|x_t)\beta+1dy + \int g_t(y|x_t)\beta+1dy.
\]

(10)
In general, $\beta \in (0, 1)$, and
\[
\lim_{\beta \to 0} D_\beta(\cdot, \cdot) = D_{KL}(\cdot, \cdot). \tag{11}
\]
The term $\frac{1}{\beta} \int p_{\text{true}}(y)^{\beta+1} dy$ in (10) can be neglected because it is a constant value independent of $x_t$ and $y_t$. Also, scaling (10) using $\frac{1}{\beta+1}$ will not change any result. Thus, the minimizer of $D_\beta p_{\text{true}}(\cdot, g_t(\cdot|x_t))$ equals to
\[
\arg \min_{x_t} \left\{ -\frac{1}{\beta} \int p_{\text{true}}(y) g_t(y|x_t)^2 dy + \frac{1}{\beta+1} \int g_t(y|x_t)^{\beta+1} dy \right\}. \tag{12}
\]

In practice, the true data distribution $p_{\text{true}}(y)$ in (12) is inaccessible and should be replaced by the empirical likelihood $p_{\text{emp}}(y) = \delta(y - y_t)$. As a result, a modified function $\ell_t^\beta(x_t, y_t)$ related to the $\beta$-divergence can be obtained by substituting $p_{\text{true}}(y)$ with $p_{\text{emp}}(y)$, i.e.,
\[
\ell_t^\beta(x_t, y_t) = -\frac{1}{\beta} g_t(y_t|x_t)^2 + \frac{1}{\beta+1} \int g_t(y|x_t)^{\beta+1} dy. \tag{13}
\]
When choosing $G_t(y_t|x_t)$ in (9) as the general “$\beta$ likelihood”
\[
G_t^\beta(y_t|x_t) := \exp \left( -\ell_t^\beta(x_t, y_t) \right), \tag{14}
\]
we finally obtain the MHE with robustness to measurement outliers and name it as $\beta$-MHE for simplicity.

Similar to (11), $\beta$-MHE converges to the standard MHE when $\beta$ approaches zero. There is a trade-off between robustness and efficiency in selecting the $\beta$ value. Generally speaking, small $\beta$ values are suitable for mild model misspecification, while it is better to choose a large value of $\beta$ when the measurement model markedly deviates from the actual model.

IV. ROBUSTNESS ANALYSIS OF THE MHE METHODS

In section III, we have demonstrated that $\beta$-divergence surpasses KL divergence under measurement outliers intuitively. In this section, we measure the robustness of $\beta$-MHE via the lens of influence function, which quantifies the effect of infinitesimal perturbations in the data on some estimated statistics. Specifically, we derive the influence function of $\beta$-MHE and investigate the maximum impact of the contaminated observation for different estimators in the linear Gaussian case.

A. Influence function of the MHEs

Influence function is a classic technique from robust statistics that measures the effects of data contamination on the estimated statistics [13]. It is defined as
\[
\text{IF}(z, \hat{x}, g) = \frac{\partial}{\partial \epsilon} \left[ \hat{x} \left( (1 - \epsilon)g + \epsilon \delta(z) \right) \right]_{\epsilon=0},
\]
where $\hat{x} \left( (1 - \epsilon)g + \epsilon \delta(z) \right)$ is a special notation representing the state estimate derived from the likelihood $g(y|x)$ with the contaminated empirical data distribution $p_{\text{em}}(y') = (1 - \epsilon)\delta(y' - y) + \epsilon \delta(y' - z)$. Although general influence function analysis has been extensively carried out in statistics [14], there is lack of study on the influence function of MHE. Before we present our main result, we make following two assumptions:

Assumption 1. The solution to (7a)(7b) exists for all $T \in \mathbb{N}$. The sufficient conditions for the existence of solutions are well studied in [1]. For example, it requires the stage function $C_t(x_t, x_{t-1}, y_t) = h_t(x_t, x_{t-1}) + h_t(y_t, x_t)$ to be bounded by two $K_+$-functions.

Assumption 2. $J(x_{t-1:T})$ is twice continuously differentiable with respect to $x_{t-1:T}$ and its second derivative at the local minimum is nonsingular.

Assumption 1 requires that the problem definition is well-posed and Assumption 2 guarantees to find an analytical form of influence function. Our main result is presented in Theorem 1.

Theorem 1. Under Assumption 1 and 2, the influence function of $\beta$-MHE for a general system (4) can be written as
\[
\text{IF}(z, \hat{x}_{t-1:T}, G_t^\beta(y_t|x_t)) = M_1^{-1}(\hat{x}_{t-1:T},) \cdot M_2(z, \hat{x}_{t-1:T}),
\]
\[
M_1(\hat{x}_{t-1:T}) := \frac{\partial^2 [J(x_{t-1:T})]}{\partial x_{t-1:T}^2} \bigg|_{x_{t-1:T} = \hat{x}_{t-1:T}}, \tag{15a}
\]
\[
M_2(z, \hat{x}_{t-1:T}) := \sum_{k=t-T+1}^{t} \left. \frac{\partial \mathcal{K}_t^\beta(x_k, y_k)}{\partial x_{t-1:T}} \right|_{x_{t-1:T} = \hat{x}_{t-1:T}}, \tag{15b}
\]
where $J(x_{t-1:T})$ satisfies (7b)(8a)(8b)(14) and
\[
\mathcal{K}_t^\beta(x_t, y_t) = \frac{1}{\beta} g_t(y_t|x_t)^{\beta} - \frac{1}{\beta} g_t(z|x_t)^{\beta}.
\]

Proof. Consider that the measurement model is contamination. Here we suppose this assumption holds for both the uncontaminated and contaminated empirical distributions.
nated by outliers, and
\[ p_{\text{em}}(y) = (1 - \epsilon)\delta(y - y_t) + \epsilon\delta(y - z). \] (16)
Substituting (16) for \( p_{\text{true}}(y) \) in (12), the loss function
\( \ell_t^\beta(xy|x_t, y_t) \) defined in (13) is converted to
\[ \ell_t^{\beta, z, \epsilon}(x_t, y_t) = -\frac{1 - \epsilon}{\beta} g_t(y_t|x_t)^\beta - \frac{\epsilon}{\beta} g_t(z|x_t)^\beta + \frac{1}{\beta + 1} \int g_t(y_t|x_t)^{\beta + 1} dy_t. \]
If we replace \( G_t^\beta(y|x_t) \) defined in (14) with \( G_t^{\beta, z, \epsilon}(y|x_t) := \exp(-\ell_t^{\beta, z, \epsilon}(x_t, y_t)) \), the state estimate with contaminated data is
\[ \hat{x}_{t-T|t} = \arg \min_{x_{t-T|t}} \{ J^{\beta, z, \epsilon}(x_{t-T|t}) \}, \]
where
\[ J^{\beta, z, \epsilon}(x_{t-T|t}) = \Gamma(x_{t-T}) + \sum_{k=t-T+1}^t \left\{ k_k(x_k, x_{k-1}) + \ell_k^{\beta, z, \epsilon}(x_k, y_k) \right\}. \]
Here \( \Gamma(x_{t-T}) \) satisfies (8a), and \( k_1(x_t, x_{t-1}) \) satisfies (8b).
By the assumption of the differentiability of \( J^{\beta, z, \epsilon}(x_{t-T|t}) \), we obtain
\[ \frac{\partial\{ J^{\beta, z, \epsilon}(x_{t-T|t}) \}}{\partial x_{t-T|t}^{t}} = 0. \] (17)
According to the definition,
\[ \mathbb{I}(z, \hat{x}_{t-T|t}, G_t^\beta(y|x_t)) = \frac{\partial \hat{x}_{t-T|t}}{\partial \epsilon} \bigg|_{\epsilon=0}. \]
Taking the derivative of both sides of (17) with respect to \( \epsilon \) at \( \epsilon = 0 \) and defining \( K_t^{\beta, z, \epsilon}(x_t, y_t) = \frac{\partial J^{\beta, z, \epsilon}(x_t, y_t)}{\partial \epsilon} \bigg|_{\epsilon=0}, \) we have
\[ \frac{\partial\{ J(x_{t-T}) \}}{\partial x_{t-T}^{t}} \bigg|_{x_{t-T}^{t} = \hat{x}_{t-T|t}} \cdot \mathbb{I}(z, \hat{x}_{t-T|t}, G_t^\beta(y|x_t)) + \sum_{k=t-T+1}^t \frac{\partial\{ K_t^{\beta, z, \epsilon}(x_k, y_k) \}}{\partial x_{t-T|t}^{t}} \bigg|_{x_{t-T}^{t} = \hat{x}_{t-T|t}} = 0. \]
Therefore, (15a) holds given the Assumption 2 that the Hessian matrix \( M_t(\hat{x}_{t-T|t}) \) is nonsingular.

**Corollary 1.** Similar results hold for the standard MHE when \( K_t^{\beta, z, \epsilon}(x_t, y_t) \) is restricted to \( K_t^\beta(x_t, y_t) = \log g_t(y|x_t) - \log g_t(z|x_t) \) and \( G_t^\beta(y|x_t) \) is reduced to \( g_t(y|x_t) \).

**Proof.** Let \( \beta \to 0 \) and we will obtain this result directly.

**B. Gross Error Sensitivity of MHEs for Linear Gaussian Systems**

Gross error sensitivity measures the maximum change a small perturbation to the likelihood \( g \) at a point \( z \) can induce to the estimate \( \hat{x} \). It is defined as
\[ \gamma(\hat{x}, g) = \sup_z \| \mathbb{I}(z, \hat{x}, g) \|. \]
We consider the typical linear Gaussian systems
\[ x_{t+1} = Ax_t + \xi_t, \]
\[ y_t = Cx_t + \zeta_t, \]
\[ \xi_t \sim \mathcal{N}(0, Q), \]
\[ \zeta_t \sim \mathcal{N}(0, R), \]
\[ x_0 \sim \mathcal{N}(0, P_0), \]
where \( x_t \in \mathbb{R}^n, y_t \in \mathbb{R}^m, Q \succeq 0 \) and \( R \succ 0 \). The arrival cost and the stage cost function of the standard MHE method are
\[ \Gamma(x_{t-T}) = \| x_{t-T} - \hat{x}_{t-T} \|_{P_{t-T}}^2, \]
\[ k_t(x_t, x_{t-1}) = \| x_t - Ax_{t-1} \|_{Q_{t-1}}^2, \]
\[ h_t(y_t, x_t) = \| y_t - Cx_t \|_{R_t}^2. \]
Here \( \hat{x}_{t-T} \) is the past optimal estimate and \( P_{t-T} \) can be calculated recursively by solving the matrix Riccati equation
\[ P_{t+1} = Q + AP_tA^T - AP_tC^T(R + CP_tC^T)^{-1}CP_tA^T. \]
The problem setting of the \( \beta \)-MHE method is similar to that of the standard MHE except that \( h_t(y_t, x_t) \) in (19) is adjusted to
\[ h_t(y_t, x_t) = -\frac{1}{\beta} \sqrt{2(\pi)^{\frac{3m}{2}}} e^{-\frac{\beta}{2} \| y_t - Cx_t \|_{R_t}^2} + \frac{1}{(\beta + 1)^{\frac{3}{2}} (2\pi)^{\frac{m}{2}}} | R_t |^{\frac{m}{2}}. \]
Our next result characterizes the robustness properties for the linear Gaussian MHE methods.

**Theorem 2.** Under Assumptions 1 and 2, the gross error sensitivity of the standard MHE for system (18) is infinite while for that of the \( \beta \)-MHE, it is bounded. More specifically, we have
\[ \| \gamma(\hat{x}_{t-T}, G_t^\beta(y|x_t)) \| \leq 2\sqrt{T} \cdot \| M_t^{-1}(\hat{x}_{t-T}) \|_F \cdot \rho_{\max}, \]
where \( M_t(\hat{x}_{t-T}) \) is defined as (15b), and
\[ \rho_{\max} := \max_{t-T \leq s \leq t} \sup_z \| \rho(\hat{x}_s, z) \|, \]
\[ \rho(\hat{x}_s, z) := -\frac{C_t^T R_t^{-1}}{\sqrt{2(\pi)^{\frac{3m}{2}}} | R_t |^{\frac{m}{2}}} (z - C_t \hat{x}_s) e^{-\frac{\beta}{2} \| z - C_t \hat{x}_s \|_{R_t}^2}. \] (21)

**Proof.** First, we will prove that the gross error sensitivity of the standard MHE is infinite. We observe that
\[ \| M_2(z, \hat{x}_{t-T}) \| \leq \| M_1(\hat{x}_{t-T}) \|_F \cdot \| \mathbb{I}(z, \hat{x}_{t-T}, y_t) \|. \]
For the standard MHE, we have
\[ \sum_{k=t-T+1}^t \frac{\partial K_k^\beta(x_k, y_k)}{\partial x_{t-T}} \bigg|_{x_{t-T} = \hat{x}_{t-T}} = 0. \] (22)
and
\[\sum_{k=i-T+1}^{T} \frac{\partial[K^x_k(x_k, y_k)]}{\partial x_{t-i}} \bigg|_{x_{t-i} = \hat{x}_{t-i}} = C^TR^{-1}(y_{t-i} - C\hat{x}_{t-i}) \]
\[-C^TR^{-1}(z - C\hat{x}_{t-i}), \quad i = 0, 1, ..., T - 1. \]

(23)

Thus
\[
\|\gamma(\hat{x}_{t-T:t}, g_t(y_t|x_t))\| = \sup_z \|M_2(z, \hat{x}_{t-T:t})\| \geq \sup_z \|M_1(\hat{x}_{t-T:t})\|_F
\]
\[= \infty. \]

In the next step, we will prove that the gross error sensitivity of the \( \beta \)-MHE is bounded. We find
\[
\|\Pi^\beta(z, \hat{x}_{t-T:t})\| \leq \|M_1(\hat{x}_{t-T:t})\|_F \cdot \|M_2(z, \hat{x}_{t-T:t})\|.
\]

Similar to (22) and (23), we have
\[
\sum_{k=i-T+1}^{T} \frac{\partial[K^z_k(x_k, y_k)]}{\partial x_{t-i}} \bigg|_{x_{t-i} = \hat{x}_{t-i}} = 0,
\]
and
\[
\sum_{k=i-T+1}^{T} \frac{\partial[K^z_k(x_k, y_k)]}{\partial x_{t-i}} \bigg|_{x_{t-i} = \hat{x}_{t-i}} = \frac{C^TR^{-1}}{\sqrt{(2\pi)^{bn}|R^|}}(y_{t-i} - C\hat{x}_{t-i})e^{-\frac{1}{2}\|y_{t-i} - C\hat{x}_{t-i}\|_2^2}
\]
\[-\frac{C^TR^{-1}}{\sqrt{(2\pi)^{bn}|R^|}}(z - C\hat{x}_{t-i})e^{-\frac{1}{2}\|z - C\hat{x}_{t-i}\|_2^2}, \quad i = 0, 1, ..., T - 1. \]

Define the function \( \rho(\hat{x}, z) \) as in (21) and we observe that \( \lim_{z \to \infty} \rho(\hat{x}, z) = 0 \). Because \( \|\rho(\hat{x}, z)\|^2 \) is continuous with respect to \( z \), it is bounded, i.e.,
\[
\|\rho(\hat{x}, z)\| \leq \rho_{\text{max}}, \quad \forall z.
\]

Because
\[
\|M_2(z, \hat{x}_{t-T:t})\|^2 \leq \sum_{i=0}^{T-1} \left( \sum_{k=i-T+1}^{T} \frac{\partial[K^z_k(x_k, y_k)]}{\partial x_{t-i}} \bigg|_{x_{t-i} = \hat{x}_{t-i}} \right)^2 \leq 4T \cdot \rho_{\text{max}}^2,
\]

we get
\[
\|\Pi^\beta(z, \hat{x}_{t-T:t}, G_t^\beta(y_t|x_t))\| \leq 2\sqrt{T} \cdot \|M_1^{-1}(\hat{x}_{t-T:t})\|_F \cdot \rho_{\text{max}}. \]

(24)

Taking the supremum on both sides of (24) arrives at (20), which finally ends the proof.

\[\square\]

Remark 2. Due to the selected arrival cost, choosing the MHE horizon length \( T = 1 \) recovers the standard KF.

Therefore, the conclusion also suits the robustness analysis of KF.

V. Simulations

In this section, two simulations will be conducted to illustrate the performance and verify the robustness of \( \beta \)-MHE. Throughout, we report the root mean squared error (RMSE) as the goodness-of-fit measure. RMSE is a frequently used measure of the differences between the state estimation \( \hat{x}_t \) and the actual state \( x_t \), and it is computed as
\[
RMSE = \sqrt{\frac{\sum_{t=1}^{N_{\text{step}}} \|x_t - \hat{x}_t\|^2}{n \cdot N_{\text{step}}}},
\]

where \( N_{\text{step}} \) is the time length of the trajectory and \( n \) is the dimension of the state. Simulations are run on a laptop equipped with AMD Ryzen 7 processor, 16.0 GB RAM, and 512 GB SSD. Meanwhile, the optimization problems are formulated in CasADi [15], which is an open-source nonlinear optimization tool.

A. Linear System Case: Wiener Velocity Model

Wiener velocity model is well-known in the field of target tracking, where the velocity is modelled as the Wiener process [16]. Discretizing it with \( \Delta t = 0.1 \) yields a linear Gaussian stochastic model, which can be represented by (18).

The state \( x = [p_x \quad p_y \quad \dot{p}_x \quad \dot{p}_y]^T \) consists of the position \( p_x, p_y \) and the velocity \( \dot{p}_x, \dot{p}_y \). The parameters defined in (18) are shown as
\[
A = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} \Delta t^2 & 0 & \Delta t \cdot \Delta t & 0 \\ 0 & \Delta t^2 & 0 & \Delta t \cdot \Delta t \\ \Delta t^2 & 0 & \Delta t & 0 \\ 0 & \Delta t^2 & 0 & \Delta t \end{bmatrix}, \quad R = \mathbb{I}_{2 \times 2}.
\]

The measurements are contaminated with \( \epsilon_t \sim \mathcal{N}(0, 100^2 \cdot \mathbb{I}_{2 \times 2}) \) with probability \( \rho_c \). We compare the proposed \( \beta \)-MHE, for a range of values of \( \beta \), to the standard MHE with a Gaussian likelihood and the KF. RMSE is evaluated over 100 runs of Monte Carlo experiments with 200 steps (\( N_{\text{step}} = 200 \)). Fig. 2 shows the box plot of RMSE for different methods when \( \rho_c = 0.2 \). We find that the average RMSE (ARMSE) decreases as the \( \beta \) increases until it reaches its minimum at \( \beta = 10^{-4} \), then rises. When \( \beta \leq 10^{-4} \), our proposed \( \beta \)-MHE significantly outperforms both the KF and the standard MHE predictively. Besides, we plot the estimation error of 4 states for different methods in Fig. 3, and \( \beta \) is chosen as \( 10^{-4} \). The average computation time of each step for KF, the standard MHE, and the \( \beta \)-MHE are 0.023 ms, 2.44 ms, and 2.75 ms respectively, which reveals the computational burden of the proposed \( \beta \)-MHE is on par with the standard MHE.
where $k_1 = 0.16$, $k_2 = 0.0064$, $\xi_t \sim \mathcal{N}(0, 10^{-4}I_{2 	imes 2})$, and $x_0 \sim \mathcal{N}(0, I_{2 	imes 2})$. A pressure gauge measures the total pressure of the system as the species react, i.e.,

$$y_t = P_{A,t} + P_{B,t} + \xi_t,$$

where $\xi_t \sim \mathcal{N}(0, 0.01)$. We simulate measurement outliers as the impulsive noise drawing from a Student’s $t$ distribution with $\nu = 1$ degrees of freedom. Strictly speaking, we define $\xi_t \sim p_c \cdot t_{\nu=1}(0, 0.01) + (1 - p_c) \cdot \mathcal{N}(0, 0.01)$. We evaluate the results for our proposed $\beta$-MHE, the standard MHE, and UKF for different contamination probabilities. As shown in Fig. 4, our method surpasses UKF and the standard MHE under different contamination probabilities and remains effective when there are no outliers. Besides, the estimation error of the two states is illustrated in Fig. 5, where $\beta$ is chosen as $10^{-4}$ and $p_c = 0.25$. It is noticeable that the estimation error of UKF and standard MHE varies considerably, whereas for $\beta$-MHE, it remains relatively stable. Besides, the average computation time of each step for UKF, the standard MHE, and the $\beta$-MHE are 15 ms, 451 ms, and 495 ms respectively, which reveals the computational burdens of the proposed $\beta$-MHE and the standard MHE are at the same level for nonlinear systems.

Fig. 4: Violin plot of RMSE for UKF, the standard MHE, and $\beta$-MHE with different contamination probabilities. Wider sections of the violin plot represent a higher probability that members will take on the given value; the skinnier sections represent a lower probability. Besides, the MHE horizon length $T = 3$, and “■” represents the ARMSE.

B. Nonlinear System Case: Isothermal Gas-phase Reactor Model

Consider an isothermal gas-phase reactor where the reversible reaction $2A \rightleftharpoons B$ takes place [17]. An initial amount of $A_r$ and $B_r$ are charged to the reactor, but the composition of the original mixture is not known accurately. The state $x$ includes the partial pressures, i.e., $x = [P_{A,t} \ P_{B,t}]^T$. The discrete-time version of the gas-phase reactor model with the one-step Euler method ($\Delta t = 0.1$) is

$$P_{A,t+1} = P_{A,t} + (-2k_1 P_{A,t}^2 + 2k_2 P_{B,t}) \cdot \Delta t + \xi_{1,t},$$

$$P_{B,t+1} = P_{B,t} + (k_1 P_{A,t}^2 - k_2 P_{B,t}) \cdot \Delta t + \xi_{2,t}.$$

Fig. 5: (a) Estimation error of $P_{A,t}$. (b) Estimation error of $P_{B,t}$. The solid lines correspond to the mean, and the shaded regions correspond to 95% confidence interval over 100 runs.

VI. CONCLUSION

We propose a GMHE approach that formulates MHE as a maximum a posteriori estimation problem and extends
the standard MHE with a generalized loss function. GMHE significantly reduces the computational overhead compared with existing methods that use combinatorial optimization strategy. We then propose the $\beta$-MHE by leveraging the $\beta$-divergence to resist measurement outliers. An analytical form of the influence function is derived for MHEs, and we prove that the gross error sensitivity is bounded for $\beta$-MHE, which demonstrates its robustness.

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