Quantum gravity vacuum and invariants of embedded spin networks

A. Miković *

Departamento de Matemática e Ciências de Computação, Universidade Lusófona, Av. do Campo Grande, 376, 1749-024, Lisboa, Portugal

Abstract

We show that the path integral for the three-dimensional SU(2) BF theory with a Wilson loop or a spin network function inserted can be understood as the Rovelli-Smolin loop transform of a wavefunction in the Ashtekar connection representation, where the wavefunction satisfies the constraints of quantum general relativity with zero cosmological constant. This wavefunction is given as a product of the delta functions of the SU(2) field strength and therefore it can be naturally associated to a flat connection spacetime. The loop transform can be defined rigorously via the quantum SU(2) group, as a spin foam state sum model, so that one obtains invariants of spin networks embedded in a three-manifold. These invariants define a flat connection vacuum state in the q-deformed spin network basis. We then propose a modification of this construction in order to obtain a vacuum state corresponding to the flat metric spacetime.

*E-mail address: amikovic@ulusofona.pt
1 Introduction

In a recent review of loop quantum gravity [1], Smolin has given a comprehensive description of a vacuum state for a quantum general relativity (GR) theory with a positive cosmological constant $\Lambda$. This state can be defined in the loop representation as a linear combination of the q-deformed spin network states, where the coefficients are given by the loop transforms of the Kodama wavefunction. These coefficients can be understood as the invariants of embedded spin networks in the spatial manifold $\Sigma$ determined by the path integral of the $SU(2)$ Chern-Simons (CS) theory. These invariants are generalizations of the Jones polynomial [2], and can be expressed as the Kaufman bracket invariants, i.e. as evaluations of the q-deformed spin networks [3, 4], or more generally as the Reshetikhin-Turaev (RT) invariants [5].

However, there exists another set of spin network invariants for the three-dimensional manifolds, which are also determined by a path integral over the connections, but instead of the CS action one uses the BF theory action [6, 7, 8]. The purpose of this paper is to demonstrate that these new invariants can be understood as the loop transforms of a flat connection vacuum wavefunction which corresponds to quantum GR with zero cosmological constant. This vacuum wavefunction is given by a product of the delta functions of $F$, where $F$ is an $SU(2)$ two-form field strength, and therefore that wavefunction is annihilated by the GR constraints in the Ashtekar connection representation. One can then naturally associate this wavefunction to a classical vacuum solution with $F = 0$. There are many such solutions, and we will be interested in constructing a quantum state describing the flat space-time metric solution. Our approach is an extension of a partially reduced phase space quantization approach of Baez [9], where he imposes classically the $F = 0$ constraint and then considers the wavefunctions on the moduli space of flat connections.

In section 2 we show how the BF theory path integral for a spin network can be represented as the Rovelli-Smolin loop transform of a flat connection vacuum wavefunction in the Ashtekar connection representation. We then show how the constraints of quantum GR for zero cosmological constant annihilate this vacuum wavefunction, and how to construct the corresponding vacuum state in the spin network basis. In section 3 we define the BF theory spin network path integrals as the spin foam model state sum invariants for the quantum $SU(2)$ group. In section 4 we present our conclusions and
discuss our results, including a discussion on the relationship between the $F = 0$ classical solutions and the flat spacetime metric solution. We then propose a modification of our construction in order to obtain a flat spacetime metric vacuum state.

2 Ashtekar formulation

The Ashtekar formulation of GR [10], can be understood as a special choice of variables in the canonical formulation of GR based on the triad one forms $e^a = e^a_i dx^i$, where $a = 1, 2, 3$, and $x^i$ are the coordinates on the spatial three-manifold $\Sigma$. The three-metric is given by $h_{ij} = e^a_i e^a_j$ and the spacetime manifold is given by $\Sigma \times \mathbb{R}$. Let $e^a_i$ be the inverse triad components, and let $p^a_i$ be the corresponding canonically conjugate momenta. Then the Ashtekar variables can be defined as

$$A^a_i = \omega^a_i(e) + i \frac{p^a_i}{\sqrt{h}}, \quad E^i_a = \sqrt{h} e^i_a,$$  \hspace{1cm} (1)

where the coefficient $i = \sqrt{-1}$, $h = \det(h_{ij})$ and the $\omega(e)$ is the torsion-free $SO(3)$ spin connection. Hence the new connection $A$ is complex, and the advantage of this unusual choice is that the Hamiltonian constraint becomes polynomial in the new canonical variables $(-iA, E)$, i.e.

$$C_H = \epsilon_{abc} (E^a_i F^{ij} F^{kc}_{ij} + \Lambda \epsilon_{ijk} E^a_i E^b_j E^c_k),$$  \hspace{1cm} (2)

where $F$ is the curvature two form associated to the $SU(2)$ connection $A$ and $\Lambda$ is the four-dimensional cosmological constant. The Gauss and the diffeomorphism constraints are also polynomial in the new variables

$$C_a = \partial_i E^a_i + \epsilon_{abc} A^b_i E^{ic}, \quad C_i = F^a_{ij} E^j_a.$$  \hspace{1cm} (3)

Because of this simplification, one can adopt a Dirac quantization strategy where the pair $(A, E)$ is represented by the operators

$$\hat{A}^a_i = A^a_i, \quad \hat{E}^i_a = \frac{\delta}{\delta A^a_i},$$  \hspace{1cm} (4)

acting on the wavefunction(al) $\Psi(A)$, such that

$$\hat{C}_a \Psi = 0, \quad \hat{C}_i \Psi = 0, \quad \hat{C}_H \Psi = 0.$$  \hspace{1cm} (5)
where the $\hat{C}$ operators are obtained by substituting (4) in the classical expressions in a chosen order. The $SU(2)$ connection $A$ is complex, and the reality properties of the expressions (1) require that the wavefunction $\Psi(A)$ should be a holomorphic functional of $A$, i.e. independent of $\bar{A}$ [11]. This holomorphy also implies that in order to construct a wavefunction of the triads $e$, it is sufficient to consider only the real $A$’s [11].

Hence by constructing a gauge invariant and a diffeomorphism invariant functional of the $SU(2)$ connection $A$ defined on the three-manifold $\Sigma$, one will solve the Gauss and the diffeomorphism constraints. Therefore the problem is reduced to analyzing the action of the Hamiltonian constraint operator $\hat{C}_H$. For example, the Kodama wavefunction

$$\Psi_K[A] = \exp \left( \frac{1}{2\Lambda} \int_{\Sigma} Tr (A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \right) = e^{ks_{CS}[A]} , \quad (6)$$

which is an exponent of the Chern-Simons action, is gauge and diffeomorphism invariant functional and satisfies the Hamiltonian constraint for a non-zero cosmological constant [12].

A more advanced strategy of quantization is to use the Rovelli-Smolin loop representation [13], i.e. if we think of $\Psi(A)$ as $\langle A|\Psi\rangle$, then one can introduce the loop states $|\gamma\rangle$ associated to the Wilson loop functionals $W[\gamma, A] = Tr P \exp \int_{\gamma} A$, and define the loop representation as

$$\Psi(\gamma) = \langle \gamma|\Psi\rangle = \int \mathcal{D}A \langle \gamma|A\rangle \langle A|\Psi\rangle = \int \mathcal{D}A W[\gamma, A] \Psi(A) \quad . \quad (7)$$

If $\Psi(A)$ is a solution of the gauge and the diffeomorphism constraints (3), then $\Psi(\gamma)$ will be a three-manifold invariant of the loop $\gamma$. More generally, one can replace the $\gamma$ with a labelled graph $\Gamma$, and hence obtain an invariant of the embedded spin network. In the case of the Kodama wavefunction, one can argue that $\Psi(\Gamma)$ should be the invariant given by the Kauffman bracket [14]†. This is a consequence of the fact that the CS action defines a topological theory on $\Sigma$, and the path integral (7) defines an observable of the theory [2].

†This argument is straightforward in the case of the Euclidian GR, when the Ashtekar connection is real and the Kodama wavefunction is $\exp(ikS_{CS})$. In the Lorentzian case, $A$ is complex, and the Kodama wavefunction is $\exp(kS_{CS})$, so that a factor of $i$ is missing. There is no rescaling $A \rightarrow cA$, where $c$ is a complex number, which will give that factor, so that the argument is that a Wick rotation $k \rightarrow ik$ of the Euclidian result has to be done.
However, there exists another topological theory in three dimensions, closely related to the CS theory, namely the BF theory (for a review and references see [15]). It is also a theory of flat connections, but its phase space is bigger, because it includes a one form $B$ field. The BF $SU(2)$ theory can be considered as the three-dimensional Euclidian GR with a zero cosmological constant [15]. The partition function of this theory

$$Z_{BF} = \int \mathcal{D}A \mathcal{D}B \ e^{i \int \text{Tr}(B \wedge F)} ,$$

(8)

can be defined via simplicial decomposition of the manifold as a spin foam state sum [16, 17, 15]. This gives the Ponzano-Regge state sum model[18]. A topologically invariant way to regularize the Ponzano-Regge state sum is via the $SU(2)$ quantum group, so that one obtains the Turaev-Viro state-sum model [19].

Given these properties, one is led to consider the following quantity

$$\Psi_0(\gamma) = \int \mathcal{D}A \mathcal{D}B W[\gamma, A] e^{i \int \text{Tr}(B \wedge F)} ,$$

(9)

i.e. the BF theory analog of the CS Wilson loop expectation value. This quantity should be gauge and diffeomorphism invariant, and it can be defined via simplical decomposition of $\Sigma$ as a spin foam state sum [6, 7, 8]. This state sum can be then regularized in a topologically invariant way via the quantum $SU(2)$ group.

The form of the path-integral $\Psi_0$ is such that after performing the $B$ integration we obtain

$$\Psi_0(\gamma) = \int \mathcal{D}A W[\gamma, A] \delta(F) ,$$

(10)

where

$$\delta(F) = \int \mathcal{D}B e^{i \int \text{Tr}(B \wedge F)} = \prod_{x \in \Sigma} \prod_{i,j} \delta(e^{F_{ij}(x)}) .$$

(11)

These are heuristic expressions, which have to be properly defined (see the next section), but for the moment if we think of $\delta(F)$ as a functional of the connection $A$, then $\delta(F) = \Psi_0(A)$ is gauge invariant due to group delta function identity $\delta(g e^F g^{-1}) = \delta(e^F)$. Furthermore we have

$$\hat{C}_i \Psi_0 = \frac{\delta}{\delta A^j_i(x)} F^a_{ij}(x) \prod_{y \in \Sigma} \prod_{k,l} \delta(e^{F_{kl}(y)}) = 0 ,$$

(12)
and
\[ \hat{C}_H \Psi_0 = \epsilon^{abc} \frac{\delta}{\delta A_i^a(x)} \frac{\delta}{\delta A_j^b(y)} F_{ijc}(x) \prod_{y \in \Sigma} \prod_{k,l} \delta(e^{F_{kl}(y)}) = 0, \]
(13)
due to \( F(x)\delta(e^{F(x)}) = 0 \). Hence \( \Psi_0(A) \) is gauge and diffeomorphism invariant functional which satisfies the Hamiltonian constraint for the zero cosmological constant.

This heuristics implies that the invariant (9) is a loop transform of a quantum gravity \( F = 0 \) vacuum state \( \Psi_0(A) = \delta(F) \). This vacuum state can be naturally associated to the space of \( F = 0 \) classical solutions of the Einstein equations. A particular \( F = 0 \) solution is a flat metric space when \( A = \omega(e) \). Hence understanding the properties of the \( \delta(F) \) state will be a first step toward constructing a flat metric vacuum state.

One can construct a flat connection vacuum state in the diffeomorphism invariant basis of spin network states \(|\Gamma\rangle\) [20], as
\[ |0\rangle = \sum_{\Gamma} |\Gamma\rangle \langle \Gamma|0\rangle, \]
(14)
where
\[ \langle \Gamma|0\rangle = \Psi_0(\Gamma) = \int \mathcal{D}A W[\Gamma, A] \delta(F) = \int \mathcal{D}A \mathcal{D}B W[\Gamma, A] e^{i\int Tr(B^\wedge F)} , \]
(15)
is a spin network invariant to be defined, and \( W[\Gamma, A] \) is the spin network generalization of the Wilson loop functional, i.e. a product of the edge holonomies contracted by the vertex intertwiners.

3 Spin-foam state sum representation

Let us now write the spin foam state sum representation of the spin network invariant (15) in order to better understand its properties. This path integral can be defined via the discretization procedure provided by a simplicial decomposition of \( \Sigma \). Let \( \mathcal{C} \) be a simplicial complex associated to \( \Sigma \), and let \( \mathcal{C}^* \) be the dual complex. We denote by \( \tau, \Delta \) and \( \epsilon \) a tetrahedron, a triangle and an edge of \( \mathcal{C} \) respectively, and by \( v, l \) and \( f \) the corresponding dual cells, i.e. a vertex, a link and a face. Let the \( B \) one-form be piece-wise constant in the 3-polytopes formed by the pairs \((\epsilon, f)\), where \( f = \epsilon^* \), such that the only
non-zero spatial components of $B$ are those parallel to the vectors $e^\dagger$. Then

$$\int_\Sigma \text{Tr} \left( B \wedge F \right) = \sum_{\epsilon \in \mathcal{C}} \text{Tr} \left( B_{\epsilon} F_{\epsilon} \right),$$

where

$$B_{\epsilon} = \int_\epsilon B, \quad F_{\epsilon} = \int_\epsilon F.$$

Given the variables (17), we can define the path integral (8) as

$$Z = \int \prod_l dA_l \prod_\epsilon dB_{\epsilon} e^{i \sum_\epsilon \text{Tr} (B_{\epsilon} F_{\epsilon})},$$

where $A_l = \int_l A$. We will fix the measures of integration $dA$ and $dB$ such that

$$Z = \int \prod_l dg_l \prod_f \delta \left( \prod_{l \in \partial f} g_l \right),$$

where $g_l = e^{A_l}$. Note that (19) is a discretized form of the path integral $\int D A \delta (F)$ and therefore $\prod_f \delta \left( \prod_{l \in \partial f} g_l \right) = \prod_f \delta \left( e^{F_f} \right)$ can be considered as a discretized form of the wavefunction $\delta (F)$.

By using the group theory formulas

$$\delta (g) = \sum_j \text{dim} j \chi_j (g) = \sum_{2j = 0}^{\infty} (2j + 1) \chi_j (g),$$

and

$$\int_{SU(2)} dg D^{(j_1)}_{\alpha_1 \beta_1} (g) D^{(j_2)}_{\alpha_2 \beta_2} (g) D^{(j_3)}_{\alpha_3 \beta_3} (g) = C^{j_1 j_2 j_3}_{j_1 j_2 j_3} \left( C^{j_1 j_2 j_3}_{\beta_1 \beta_2 \beta_3} \right)^*,$$

where $D^{(j)}$ is the representation matrix in the representation $j$, $\chi_j$ is the corresponding trace and $C$’s are the Clebsh-Gordan coefficients, or the $3j$ intertwiners, as well as the associated graphical calculus [17, 15], one arrives at

$$Z = \sum_{j_f} \prod_f \text{dim} j_f \prod_v A_{\text{tet}} (j_{f_1(v)}, \ldots, j_{f_6(v)}).$$

$A_{\text{tet}}$ is the tetrahedron spin network amplitude, given by a contraction of four $3j$ tensors\(^8\), which is also known as the $(6j)$ symbol. The terms in the sum (22)

\(^1\)This is an approach to the discretization introduced in [23]. It is a more convenient approach for treating non-linear actions then the usual approach where the $B$ field is taken to be a distribution concentrated along the edges $\epsilon$ [17, 15].

\(^2\)For a tensorial approach to spin networks see [21].
can be interpreted as the amplitudes for a colored two-complex associated to $\mathcal{C}^*$, which is called the spin-foam [15]. The Ponzano-Regge state sum formula can be obtained by relabelling the sum (22) as a sum over the irreducible representations (irreps) which are associated to the simplices dual to the spin foam, i.e. the edges and the tetrahedrons of the manifold triangulation.

$$Z = \sum_{j_\epsilon} \prod_{\epsilon} \dim j_\epsilon \prod_\tau A_{\text{tet}}(j_{e_1(\tau)}, \ldots, j_{e_6(\tau)}) = \sum_{j_\epsilon} \prod_{\epsilon} (2j_\epsilon + 1) \prod_\tau (6j_\epsilon)(\tau). \quad (23)$$

The expression (23) is not finite, because the infinite sum over the $SU(2)$ irreps $j$ diverges. Formally, the sum (23) is topologically invariant, because it is invariant under the three-dimensional Pachner moves. It is possible to regularize this sum while preserving the topological invariance, by replacing the $SU(2)$ irreps with the irreps of the quantum group $SU_q(2)$ for $q = e^{i\pi/n}$, where $n$ is an integer greater than 2 [19]. Then for each $n$ there are finitely many irreps, up to $j = \frac{n-2}{2}$. Each face (edge) amplitude $\dim j$ is replaced by $\dim_q j$, while each vertex (tetrahedron) amplitude $(6j)$ is replaced by $(6j)_q$.

In this way one obtains the Turaev-Viro state sum invariant

$$Z_{TV} = d_k^{-2V} \sum_{j_\epsilon} \prod_{\epsilon} \dim_q j_\epsilon \prod_\tau (6j)_q(\tau), \quad (24)$$

where $d_k^2 = \sum_{j=0}^{k/2} (\dim_q j)^2$, $V$ is the number of tetrahedra and the $(6j)_q$ symbol is appropriately normalized.

In the case of the path integral (15), we will use the formula

$$\Psi_0[\Gamma] = \int \prod_t dg_t \prod_f \delta \left( \prod_{l \in \partial f} g_l \right) W_\Gamma(g_{v'}, s_{v'}, i_{v'}) \quad , \quad (25)$$

where $W_\Gamma$ is the simplical version of the spin network function for the spin network $\Gamma$. This embedded spin network is specified by labelling a subset $\{l'\}$ of the dual edges by the irreps $\{s_{l'}\}$, as well as by labelling the corresponding set of vertices $\{v'\}$ with the appropriate intertwiners $i_{v'}$. One can associate more than one irrep to a given link, in order to obtain the spin nets of the valence higher than four. By using (20), (21) as well as

$$\int_{SU(2)} dg D_{\alpha_1\beta_1}^{(j_1)}(g) \cdots D_{\alpha_n\beta_n}^{(j_n)}(g) = \sum_\iota C_{\alpha_1 \cdots \alpha_n}^{j_1 \cdots j_n(\iota)} \left( C_{\beta_1 \cdots \beta_n}^{j_1 \cdots j_n(\iota)} \right)^*, \quad n > 3 \quad , \quad (26)$$
for the edges which carry the irreps of the spin network, where $C^{(i)}$ are the intertwiners for $n$ irreps, one obtains

$$\Psi_0(\Gamma) = \sum_{j_f, \iota_t} \prod_f \dim j_f \prod_{v' \in V \setminus V_{\Gamma}} (6j)(v') \prod_{v' \in V_{\Gamma}} \tilde{A}_{tet}(j_f(v'), s_{v'}(v'), \iota_{v'}, \iota_{v'}) \quad (27)$$

The amplitude $\tilde{A}_{tet}$ is the evaluation for the spin network consisting of the tetrahedral graph labelled by the $j_f$ irreps, with an additional vertex labelled by the intertwiner $\iota_{v'}$, which is connected by the edges labelled by the irreps $s$ to the tetrahedral graph vertices $[6, 7]$. This type of the spin network amplitudes also appears in the case of the spin foam models proposed for matter [21], and the spin foam amplitude (27) is a special case of a more general spin foam amplitude proposed for an embedded open spin network [21, 22].

As in the case of the partition function, the expression (27) is a divergent sum. One can then regularize it by using the $SU_q(2)$ irreps. The reason why this procedure works, is that the representations of $SU(2)$ and the representations of $SU_q(2)$ form the same algebraical structure, i.e. the tensor category (see [8] for a review and references). The difference is that the $SU_q(2)$ category is a more general type of a tensor category, i.e. a ribbon tensor category, whose algebraical properties are defined by the invariance properties of ribbon tangles under the isotopies in $R^3$ or $S^3$. The 3$j$ and the 6$j$ symbols, as well as the other spin network amplitudes can be interpreted as morphisms in this category, i.e. the maps among the objects of the category and the compositions of these maps. These maps can be represented by the open and the closed spin network diagrams. The relations among the intertwiner morphisms which imply the topological invariance are preserved and well-defined in the quantum group case. Hence the quantum group version of the embedded spin network state sum (27) should be a topological invariant.

Therefore a well-defined invariant for $\Gamma$ will be given by the quantum group version of the spin foam amplitude (27)

$$\Psi_0(\Gamma) = d_k^{-2V} \sum_{j_f, \iota_t} \prod_f \dim_q j_f \prod_{v' \in V \setminus V_{\Gamma}} (6j)_q(v') \prod_{v' \in V_{\Gamma}} \tilde{A}_{tet}^q(j_f(v'), s_{v'}(v'), \iota_{v'}, \iota_{v'}) \quad . (28)$$

The face and the vertex amplitudes are now given by the evaluations of the corresponding q-deformed spin networks. Nontrivial dual edge (triangle) amplitudes may appear, although these can be reabsorbed into normalization
factors of the vertex amplitudes. In order to determine these normalization factors it will be necessary to check the corresponding identities for the simplex amplitudes associated to the invariance under the motions of the graph in the manifold.

4 Discussion and Conclusions

In order for (28) to be well-defined, the spin network \( \Gamma \) should be framed, i.e. \( \Gamma \) should be a ribbon graph spin network. The invariant (28) could be interpreted as a coefficient of a state \( |0_q\rangle \) in the diffeomorphism invariant basis of q-deformed spin network states \( |\Gamma_q\rangle \) [1], i.e.

\[
|0_q\rangle = \sum_{\Gamma_q} |\Gamma_q\rangle \langle \Gamma_q|0_q\rangle = \sum_{\Gamma_q} |\Gamma_q\rangle \Psi_0(\Gamma) .
\]  

(29)

The q-deformed spin network states can be defined via the q-deformed loop algebra [14], and these states have the same labels as the non-deformed ones, but the q-deformed states have the restriction that the maximal label \( j \) is given by \( (n-2)/2 \) where \( q = \exp(i\pi/n) \). Since the q-deformed spin networks are naturally associated to the quantum GR with \( \Lambda \propto 1/n \) [14, 1], then we expect that the action of the \( \Lambda = 0 \) Hamiltonian constraint (HC) operator on the state (29) should give a state with the basis coefficients of order \( 1/n^2 \) or higher. Verifying this would require the knowledge of the matrix elements of the HC operator in the q-deformed spin network basis.

Given that the \( \Lambda = 0 \) quantum GR is naturally formulated in the non-deformed \( (q = 1) \) spin network basis [20, 1], one would like to construct a flat connection vacuum state in that basis. Since \( q = e^{i\pi/n} \), a natural way would be to consider a large \( n \) limit of (29), i.e.

\[
|0\rangle = \lim_{n \to \infty} |0_q\rangle .
\]

(30)

It would be interesting to study the action of the HC operator on the state (30), given that the matrix elements of a HC operator corresponding to the constraint (2) for \( \Lambda = 0 \) are known in the non-deformed spin network basis [24].

Note that the state \( |0\rangle \) would determine a wavefunction \( \Psi_0(A) \) via

\[
\Psi_0(A) = \langle A|0\rangle = \sum_{\Gamma} \langle A|\Gamma\rangle \langle \Gamma|0\rangle = \sum_{\Gamma} W[\Gamma, A] \Psi_0(\Gamma) \\
= \sum_{\Gamma} W_\Gamma(g_1, \cdots, g_L) \Psi_0(\Gamma) ,
\]

(31, 32)
where \( g \) are the group elements associated to the spin network edge holonomies \( P \exp \int_\gamma A \). This expression defines an inverse loop transform. It would be important to understand the mathematical properties of this type of expression, because it could serve as a definition of the vacuum wavefunction which was given by the heuristic expression \( \prod_{x \in \Sigma} \delta(e^{F_x}) \). Related to this, one would like to see what would be the result of the action of the constraint operators on the expression (32).

In the case of \( \Lambda \neq 0 \) quantum GR, one can construct in the \( \Gamma_q \) basis a vacuum state

\[
|0_K\rangle = \sum_{\Gamma_q} |\Gamma_q\rangle \langle \Gamma \rangle_K ,
\]

whose expansion coefficients are given by the Kauffman bracket evaluation of the spin network \( \Gamma \) [1]. The state \( |0_q\rangle \) is different from the state \( |0_K\rangle \) because the invariants \( \Psi_0(\Gamma) \) and \( \langle \Gamma \rangle_K \) are not the same. The invariant \( \Psi_0(\Gamma) \) is more complicated, since it is a sum of products of the Kauffman brackets, which are the \( q \)-deformed spin network evaluations. Also, \( \Psi_0(\Gamma) \) is by construction sensitive to the topology of the manifold, while \( \langle \Gamma \rangle_K \) is essentially an invariant for a spin network embedded in \( S^3 \) or \( R^3 \). Since \( \Psi_0(\Gamma) \) corresponds to \( \Lambda = 0 \) quantum GR, this raises a question how to generalize the formula (33) to the case of arbitrary manifold topology.

This can be done by replacing the basis coefficients of the state \( |0_K\rangle \) by the RT invariants for the \( \Gamma \)'s \( \langle \Gamma \rangle_{RT} [5] \). These invariants correspond to the CS theory path integrals, whose construction is based on the surgery representation of the manifold \( \Sigma \) and the representation theory of \( SU_q(2) \). The RT invariant for \( \Gamma \) does not have a state-sum representation, although it is related to the TV invariant for \( \Gamma \) [8, 25, 26]. The TV invariant is given by the TV partition function for a manifold triangulation where there is no summation over the edge (face) irreps belonging to the embedded spin network. The TV and the RT invariants are related because the TV partition function is the square of the modulus of the RT partition function. The fact that the invariant \( \Psi_0(\Gamma) \) has a very different form compared to the corresponding RT or TV invariants, which are naturally associated to the \( \Lambda \neq 0 \) quantum GR, is consistent with our conjecture that \( \Psi_0(\Gamma) \) describes a state in the \( \Lambda = 0 \) quantum GR.

In order to interpret physically the vacuum states \( |0_q\rangle \) and \( |0\rangle \) we need the complex connection representation and the requirement of the holomorphy of the corresponding wavefunction. Although a very elegant formalism, it is difficult to implement the reality conditions by using a scalar product.
We believe that this is not such a big problem, because the holomorphic representation seems to capture the classical reality conditions, at least in the case of finitely many degrees of freedom [11]. Also, the implementation of the reality conditions via the usual scalar product presupposes that the scalar products of the physical states are finite. We believe that this assumption is too strong, because it is based on the interpretation of the square of the modulus of the wavefunction of the universe as a probability, which is a meaningless concept.

Given the $|0_K\rangle$ or the $|0_q\rangle$ state, one is wondering is it possible to define an inverse loop transforms for them. On the technical side, since the algebra of functions on a Hopf algebra is sufficiently well understood [8], it should be straightforward to generalize the expression (32) to the case of functions on the quantum group. On the conceptual side, the resulting connection representation would correspond to a gauge theory based on the quantum group, whose properties are not well understood.

The $F = 0$ wavefunction for the $\Lambda = 0$ GR has a simple form in the connection representation, i.e. $\Psi_0(A) = \delta(F)$. One can then naturally associate a space of $F = 0$ classical solutions to this state. However, it is clear that no particular $F = 0$ classical solution can be associated to it, and hence we cannot associate this state to a flat metric vacuum because a metric flatness condition was never imposed, even classically. An $F = 0$ classical solution will be a flat metric solution when $A = \omega(e)$ constraint is imposed.

Therefore if we want to construct a state which corresponds to a flat metric spacetime, we would need to impose a flat-metric condition quantum mechanically. In order to do this, let us consider the equation

$$\left(\hat{A}_i^a - \hat{\omega}_i^a(e)\right) |\Phi_0\rangle = 0 .$$  \hspace{1cm} (34)

In the “$E$” representation this equation takes a simple form

$$\left(-\frac{\delta}{\delta E_i} - \omega_i^a(\epsilon)\right) \Phi_0(\epsilon) = 0 ,$$  \hspace{1cm} (35)

which has a solution

$$\Phi_0(\epsilon) = e^{-\Omega(\epsilon)} , \quad \Omega(\epsilon) = \int_{\Sigma} d^3x \, E^i_a E_b^j \partial_k \frac{e^{\epsilon c}}{\sqrt{h}} \Delta^{k\ell}_{ij} \epsilon^{abc} ,$$  \hspace{1cm} (36)

where $\Delta$ is a tensor made from the $\delta$’s such that

$$\frac{\delta \Omega}{\delta E_i^a(x)} = \omega_i^a(e) .$$  \hspace{1cm} (37)
In the “A” representation this solution will be given by

\[ \Phi_0(A) = \int \mathcal{D}E \ e^{i \int AEd^3x} \Phi_0(E) , \]  

(38)

and let us then consider the following loop transform

\[ \langle \Gamma|0_h \rangle = \int \mathcal{D}A W[\Gamma, A] \delta(F) \Phi_0(A) . \]  

(39)

This suggests a flat metric vacuum wavefunction \( \Psi_0(A) = \delta(F) \Phi_0(A) \). However, although it is annihilated by the constraint operators, \( \Psi_0 \) is not annihilated by the \( \hat{A} - \hat{\omega}(E) \) operator. In order to understand this, note that the functional integral (39) can be also interpreted as a functional integral over the flat connections \( A_0 \)

\[ \langle \Gamma|0_h \rangle = \int \mathcal{D}A_0 W[\Gamma, A_0] \Phi_0(A_0) , \]  

(40)

so that (39) can be interpreted as a loop transform in the partially reduced phase space quantization approach of Baez [9]. Since \( \Phi_0(A_0) \) is annihilated by \( \hat{A} - \hat{\omega}(E) \), it would make sense to define a flat metric vacuum state as

\[ |0_h \rangle = \sum_{\Gamma} |\Gamma\rangle \langle \Gamma|0_h \rangle . \]  

(41)

Given the representation (39), one could hope that the coefficients \( \langle \Gamma|0_h \rangle \) could be defined by using the same approach as in the \( \Phi_0 = 1 \) case, i.e. by using a simplicial path-integral representation to define them as three-dimensional topologically invariant state-sums for the \( SU(2) \) quantum group at a root of unity. However, the main problem in this approach will be the non-polynomiality of the \( \Omega(E) \) term, and hence one should resort to some sort of approximation. The simplest thing one can do is to approximate \( \Phi_0 \) by a coherent state. For example one can take

\[ \Omega(E) = \int \Sigma d^3x \sum_{i,a} (E_{ai}^i - \delta_{ai}^i)^2 , \]  

(42)

so that \( \Phi_0 = e^{-\Omega(E)} \) will be peaked around the flat metric triads \( e_{ai}^a = \delta_{ai}^a \). However, the expression (42) is not a diffeomorphism invariant. One can try to replace it by a diffeomorphism invariant analog, for example

\[ \Omega(E) = \int \Sigma d^3x (\det(E_{ai}^i) - 1)^2 . \]  

(43)
However, (43) is not quite a diffeomorphism invariant, since $\det(E)$ is a scalar density. Still, given a polynomial $\Omega(E)$, one can use the path-integral techniques developed for the simplical BF theories with interactions [17, 23] in order to obtain a state sum for a given triangulation of $\Sigma$. This state sum will not be topologically invariant if the initial $\Omega(E)$ was not a diffeomorphism invariant, and therefore some sort of averaging over the topologically equivalent triangulations would have to be performed.

Note that this approach suggests relaxing the condition that $\Psi_0(A)$ should be annihilated by $\hat{A}$ to requiring that $\Psi_0(A)$ should be sufficiently peaked around the flat metric triads, which means using a coherent state $\Phi_0$. Then the expressions (39) and (41) also make sense in the Dirac quantization formalism.

In the case when $\Lambda \neq 0$, one would need to replace the Kodama wavefunction $\Psi_K(A)$ by $\Psi_K(A)\Phi_0(A)$ in order to construct a flat metric vacuum because the de-Sitter classical vacuum solution found in [1] satisfies $F_{ij}^a - \Lambda \epsilon_{ijk} E^{ka} = 0$ and $R_3 = 0$, where $R_3$ is the spatial curvature, while the Kodama state corresponds only to $F_{ij}^a - \Lambda \epsilon_{ijk} E^{ka} = 0$ solutions without the flat metric condition.

Matter could be included by perturbing the vacuum state via the ansatz $\Psi_0(A) \rightarrow \Psi_0(A)\chi[A, \phi]$, where $\phi$ stands for the matter degrees of freedom, see [1]. In a similar manner one can also study the gravitational perturbations, i.e. the gravitons. Besides exploring the physical implications of the presence of matter, it would be also interesting to see what kind of invariants one would obtain by using the spin network formulation and by defining the corresponding path-integrals via generalized spin foam state sums. These can be then compared to the generalized spin foam state sum amplitudes proposed for matter in [21, 22, 23].

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