UNIVERSALITY THEOREMS FOR LINKAGES IN THE MINKOWSKI PLANE

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Abstract. A mechanical linkage is a mechanism made of rigid rods linked together by flexible joints, in which some vertices are fixed and others may move. The partial configuration space of a linkage is the set of all the possible positions of a subset of the vertices. We characterize the possible partial configuration spaces of linkages in the Minkowski plane. We also give a proof of a differential universality theorem in the Minkowski plane: for any manifold $M$ which is the interior of a compact manifold with boundary, there is a linkage which has a configuration space diffeomorphic to the disjoint union of a finite number of copies of $M$.

1. Introduction

A mechanical linkage is a mechanism made of rigid rods linked together by flexible joints. Mathematically, we consider a linkage as a marked graph: lengths are assigned to the edges, and some vertices are pinned down while others may move.

A realization of a linkage $\mathcal{L}$ in a manifold $M$ is a function which sends each vertex of the graph to a point of $M$, respecting the lengths of the edges. The configuration space $\text{Conf}_M(\mathcal{L})$ is the set of all realizations. Intuitively, the configuration space is the set of all the possible states of the mechanical linkage. This supposes, classically, the ambient manifold $M$ to have a Riemannian structure: thus the configuration space may be seen as the space of “isometric immersions” of the metric graph $\mathcal{L}$ in $M$.

Here we will always deal with (non-trivially) marked connected graphs, that is, a non-empty set of vertices have fixed realizations (in fact, when $M$ is homogeneous, considering a linkage without fixed vertices only adds a translation factor to the configuration space). Hence, our configurations spaces will be compact even if $M$ is not compact, but rather complete.

Most existing studies deal with the special case where $M$ is the Euclidean plane $\mathbb{R}^2$ (see for instance [Far08]), and some with the higher dimensional Euclidean case (see [Kin98]), or that of polygonal linkages in the standard 2-sphere (see [KM+99]), or in the hyperbolic plane (see [KM96]).

Motivation. Our idea here is to relax positiveness of both the length structure on the graph, and the Riemannian metric on the manifold, and see what happens: instead of a Riemannian metric, we will assume $M$ has a pseudo-Riemannian one. This framework extension is mathematically natural, and may be related to the problem of the embedding of causal sets in physics, but the most important (as well as exciting) fact for us is that configuration spaces are (a priori) no longer compact, and we want to see what new spaces we get in our new setting.

The Minkowski plane $\mathbb{M}$. In the present paper, we will in fact restrict ourselves to the simple flat case where $M$ is a linear space endowed with a non-degenerate quadratic form, and more specially to the 2-dimensional case. Therefore, $M$ is the (Lorentz-)Minkowski plane, i.e. $\mathbb{R}^2$ endowed with a non-degenerate indefinite quadratic form. We denote the “space coordinate” by $x$ and the “time coordinate” by $t$.

Universality theorems. When $M$ is the Euclidean plane or the Minkowski plane, a configuration space is an algebraic set. This set is smooth for a generic length structure on the underlying graph.

Universality theorems tend to state that, playing with mechanisms, we get any algebraic set of $\mathbb{R}^n$, and any manifold, as a configuration space! In contrast, it is a hard task to understand the topology or geometry of the configuration space of a given, even simple mechanism.

In order to be more precise, it will be useful to introduce partial configuration spaces: for $W$ a subset of the vertices of $\mathcal{L}$, one defines $\text{Conf}_M^W(\mathcal{L})$ as the set of realizations of the subgraph induced by $W$ that extend to realizations of $\mathcal{L}$. One has in particular a restriction map: $\text{Conf}_M(\mathcal{L}) \to \text{Conf}_M^W(\mathcal{L})$. 

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If $W = \{a\}$ is a vertex of $L$, its partial configuration space is its workspace, i.e., the set of all its positions in $M$ corresponding to all realizations of $L$.

**Euclidean planar linkages.** Now regarding the algebraic side of universality, the history starts (and almost ends) in 1876 with the well-known Kempe’s Theorem [Kem76]:

**Theorem 1.** Any algebraic curve of the Euclidean plane, intersected with a Euclidean ball, is the workspace of some vertex of some mechanical linkage.

As for the differentiable side, Kapovich and Millson proved [KM02]:

**Theorem 2.** Any compact connected smooth manifold is diffeomorphic to one connected component of the configuration space of some linkage in the Euclidean plane. More precisely, there is a configuration space whose components are all diffeomorphic to the given differentiable manifold.

Jordan and Steiner proved a weaker version of this theorem with more elementary techniques [JS99]. Thurston gave lectures on a similar theorem in the 1980’s but never wrote a proof.

**Minkowski planar linkages.** Our goal here is to prove the natural and expected generalizations of both the algebraic and differentiable previous universality theorems in the case of linkages in the Minkowski plane. More precisely, we prove the same statements without assuming compactness:

**Theorem 3.** Let $A$ be a semi-algebraic subset of $M^n$ (identified with $\mathbb{R}^{2n}$). Then, $A$ is a partial configuration space of some linkage $L$ in $M$. When $A$ is algebraic, one can choose $L$ such that the restriction map $\text{Conf}_M(L) \rightarrow A$ is a smooth finite trivial covering.

Conversely, any partial configuration space of any linkage is clearly a semi-algebraic subset of $M^n$ (defined by polynomials of degree 2), so this theorem characterizes the sets which are partial configuration spaces (see Definition 7 for the definition of “semi-algebraic”).

In particular, Kempe’s theorem extends (globally) to the Minkowski plane: any algebraic curve is the workspace of one vertex of some linkage.

When $A$ is an algebraic set which is not a smooth manifold, and the restriction map is a smooth finite trivial covering, this implies that the whole configuration space is not smooth either. In this case, a function is said to be “smooth” if it is the restriction of a smooth function defined on the ambient $\mathbb{R}^k$.

**Remark.** If the restriction map is injective, then it is a bijective algebraic morphism from $\text{Conf}_M(L)$ to $A$. However, it is not necessarily an algebraic isomorphism between $\text{Conf}_M(L)$ and $A$! In fact, it is true for non-singular complex algebraic sets that bijective morphisms are isomorphisms, but this is no longer true in the real algebraic case (see for instance [Mum95], Chapter 3).

**Theorem 4.** For any differentiable manifold $M$ with finite topology, i.e., diffeomorphic to the interior of a compact manifold with boundary, there is a linkage in the Minkowski plane with a configuration space whose components are all diffeomorphic to $M$. More precisely, there is a partial configuration space $\text{Conf}_M^W(L)$ which is diffeomorphic to $M$ and such that the restriction map $\text{Conf}_M(L) \rightarrow \text{Conf}_M^W(L)$ is a smooth finite trivial covering.

**Some questions.** Our results suggest naturally the following:

1. Besides the 2-dimensional case, are the previous results true for any (finite-dimensional) linear space endowed with a non-degenerate quadratic form? Any such space contains a Euclidean or a Minkowski plane, and it is likely that the adaptation of 2-dimensional proof hides no surprise.

2. In our definition of linkages, we allow some edges to have imaginary lengths (they are “timelike”). Is it possible to require the graphs of Theorems 3 and 4 to be spacelike, i.e. require all their edges to have real length?

3. Let $M$ be the interior of a compact manifold with boundary. We know there is a linkage whose configuration space is diffeomorphic to the sum of a finite number of copies of $M$. Is it possible to choose this sum trivial, that is, with exactly one copy of $M$? (This question is also open on the Euclidean plane.)
How to go from the algebraic universality to the differentiable one? The differentiable universality Theorem (Theorem 4) will follow from the algebraic universality Theorem (Theorem 3) once we know which smooth manifolds are diffeomorphic to algebraic sets. In 1952, Nash [Nas52] proved that for any smooth compact manifold $M$, one may find an algebraic set which has one component diffeomorphic to $M$. In 1973, Tognoli [Tog73] proved that there is in fact an algebraic set $M$ which is diffeomorphic to $M$ (a proof may be found in [AK92]). In the non-compact case, Akbulut and King [AK81] proved that every smooth manifold which is obtained as the interior of a compact manifold (with boundary) is diffeomorphic to an algebraic set. Note that conversely, any non-singular algebraic set is diffeomorphic to the interior of a compact manifold (with boundary).

**Ingredients of the proof.** There are essentially three technical as well as conceptual tools: functional linkages – combination of elementary linkages – regular inputs. Basically, we adapt the ideas of Kapovich and Millson [KM02] to the Minkowski plane.

**Functional linkages.** One major ingredient in the proof of Theorem 3 is the notion of functional linkages. Here we enrich the graph structure by marking two new vertex subsets $P$ and $Q$ playing the role of inputs and outputs, respectively. If the partial realization of $Q$ is determined by the partial realization of $P$, by means of a function $f : \text{Conf}^P_M(\mathcal{L}) \rightarrow \mathcal{M}^Q$ (called the input-output function), then we say that we have a functional linkage for $f$ (here $\mathcal{M}$ is the Minkowski plane $\mathbb{M}$). The Peaucellier linkage is a famous historical example: it is functional for an inversion with respect to a circle.

**Combination.** A major step in the proof consists in proving the existence of functional linkages associated to any given polynomial $f$. This will be done by “combining” elementary functional linkages. We define combination so that combining two functional linkages for the functions $f_1$ and $f_2$ provides a functional linkage for $f_1 \circ f_2$.

**Elementary linkages.** All the work then concentrates in proving the existence of linkages for suitable elementary functionals (observe that even for elementary linkages one uses combination of more elementary ones).

1. The linkages for geometric operations:
   a. The robotic arm linkage (Section 4.1): one of the most basic linkages, used everywhere in our proofs and in robotics in general.
   b. The rigidified square (Section 4.2): a way of getting rid of degenerate configurations of the square using a well-known construction.
   c. The Peaucellier inversor (Section 4.3): this famous linkage of the 1860’s has a slightly different behavior in the Minkowski plane but achieves basically the same goal.
   d. The partial $t_0$-line linkage (Section 4.4): it is obtained using a Peaucellier linkage, but does not trace out the whole line.
   e. The $t_0$-integer linkage (Section 4.5): it is a linkage with a discrete configuration space.
(f) The $t_0$-line linkage (Section 4.6): we draw the whole line by combining the two previous linkages.

(g) The horizontal parallelizer (Section 4.7): it forces two vertices to have the same ordinate, and it is obtained by combining several line linkages.

(h) The diagonal parallelizer (Section 4.8): its role is similar to the horizontal parallelizer but its construction is totally different.

(2) The linkages for algebraic operations, which realize computations on the $t = 0$ line:

(a) The average function linkage (Section 5.1): it computes the average of two numbers, and is obtained by combining several of the previous linkages.

(b) The adder (Section 5.2): it is functional for addition on the $t = 0$ line, and is obtained from several average function linkages.

(c) The square function linkage (Section 5.3): it is functional for the square function and is obtained by combining the Peaucellier linkage (which is functional for inversion) with adders. This linkage is a bit difficult to obtain because we want the inputs to be able to move everywhere on the line, while the inversion is of course not defined at $x = 0$.

(d) The multiplier (Section 5.4): it is functional for multiplication and is obtained from square function linkages.

(e) The polynomial linkage (Section 5.5): it is obtained by combining adders and multipliers. It is functional for a fixed polynomial function $f$. This linkage is used to prove the universality theorems: if the outputs are fixed to 0, the inputs are allowed to move exactly in $f^{-1}(0)$.

Regular inputs. In our theorems, we need the restriction map $\text{Conf}_M(\mathcal{L}) \to \text{Conf}_P^M(\mathcal{L})$ to be a smooth finite trivial covering. In the differential universality Theorem, it ensures in particular that the whole configuration space consists in several copies of the given manifold $M$. The set of regular inputs $\text{Reg}_P^M(\mathcal{L})$ is the set of all realizations of the inputs which admit a neighborhood onto which the restriction map is a smooth finite covering. We have to be very careful, because even for quite simple linkages such as the robotic arm, the restriction map is not a smooth covering everywhere! There are mainly two possible reasons for the restriction map not to be a smooth covering:

(1) One realization of the inputs may correspond to infinitely many realizations of the whole linkage (for example, when the robotic arm in Section 4.1 has two inputs fixed at the same location, the workspace of the third vertex is a whole circle).

(2) Even if it corresponds only to a finite number of realizations, these realizations may not depend smoothly on the inputs (for example, when the robotic arm in Section 4.1 is stretched).

New difficulties in the Minkowski case. While the results are similar, the linkages used in [KM02] require major changes to work correctly. Here are some of the difficulties in the Minkowski plane:

(1) The Minkowski plane $\mathbb{M}$ is not isotropic: its directions are not all equivalent. Indeed, these directions have a causal character in the sense that they may be spacelike, lightlike or timelike. For example, one needs different linkages in order to draw spacelike, timelike and lightlike lines.

(2) On the Euclidean plane, two circles $C(x, r)$ and $C(x', r')$ intersect if and only if $|r - r'| \leq \|x - x'\| \leq r + r'$, but in the Minkowski plane, the condition of intersection is much more complicated to state (see Section 3.2).

(3) In the Euclidean plane, one only has to consider compact algebraic sets. Applying a homothety, one may consider such a set to be inside a small neighborhood of zero, which makes the proof easier. Here, the algebraic sets are no longer compact, so we have to work with mechanisms which are able to deal with the whole plane.
2. Generalities on linkages

In the present section, we develop generalities on linkages which apply to any ambient pseudo-Riemannian manifold $\mathcal{M}$, in particular to the case where $\mathcal{M}$ is a linear space endowed with a non-degenerate quadratic form $q$. (In all the following sections, $\mathcal{M}$ will be the Minkowski plane $\mathbb{M}$). There is no standard distance structure associated to such a form, but we will here argue by a naive algebraic analogy and define a distance as:

$$\delta : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}^+ \cup i\mathbb{R}^+$$

$$(x, y) \mapsto \sqrt{q(y-x)}.$$

Accordingly, the length structure of the linkage will be generalized by taking values in $\mathbb{R}^+ \cup i\mathbb{R}^+$ (instead of $\mathbb{R}^+$) as follows:

**Definition 5.** A linkage $\mathcal{L}$ in $\mathcal{M}$ is a graph $(V, E)$ together with:

1. A function $l : E \rightarrow \mathbb{R}^+ \cup i\mathbb{R}^+$ (which gives the length of each edge);
2. A subset $F \subseteq V$ of fixed vertices (represented by $\blacksquare$ on the figures);
3. A function $\phi_0 : F \rightarrow \mathcal{M}$ which indicates where the edges of $F$ are fixed;

When the linkage is named $\mathcal{L}_1$, we usually write $\mathcal{L}_1 = (V_1, E_1, l_1, \ldots)$ and name its vertices $a_1, b_1, c_1, \ldots$. If the linkage $\mathcal{L}_1$ is a copy of the linkage $\mathcal{L}$, the vertex $a_1 \in V_1$ corresponds to the vertex $a \in V$, and so on.

**Definition 6.** Let $\mathcal{L}$ be a linkage in $\mathcal{M}$. A realization of a linkage $\mathcal{L}$ in $\mathcal{M}$ is a function $\phi : V \rightarrow \mathcal{M}$ such that:

1. For each edge $v_1v_2 \in E$, $\delta(\phi(v_1), \phi(v_2)) = l(v_1v_2)$;
2. $\phi|_F = \phi_0$.

**Remark.** On the figures of this paper, linkages are represented by abstract graphs. The edges are not necessarily represented by straight segments, and the positions of the vertices on the figures do not necessarily correspond to a realization (unless otherwise stated).

**Definition 7.** An algebraic subset of $\mathbb{R}^n$ is a set $A \subseteq \mathbb{R}^n$ such that there exist $m \in \mathbb{N}$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ a polynomial such that $A = f^{-1}(0)$.

A semi-algebraic subset of $\mathbb{R}^n$ (see [BCR98]) is a set $B$ such that there exists $N \geq n$ and an algebraic set $A$ of $\mathbb{R}^N$ such that $B = \pi(A)$, where $\pi$ is the natural projection

$$\pi : \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^{N-n} \rightarrow \mathbb{R}^n$$

$$(x, y) \mapsto x.$$

Finally, we define the (semi-)algebraic subsets of $\mathbb{M}^n$ by identifying $\mathbb{M}^n$ with $(\mathbb{R}^2)^n = \mathbb{R}^{2n}$.

**Definition 8.** Let $\mathcal{L}$ be a linkage in $\mathcal{M}$. Let $W \subseteq V$. The partial configuration space of $\mathcal{L}$ in $\mathcal{M}$ with respect to $W$, written $\text{Conf}^W_M(\mathcal{L})$, is the following set of functions from $W$ to $\mathcal{M}$:

$$\text{Conf}^W_M(\mathcal{L}) = \{ \phi|_W \mid \text{realization of } \mathcal{L} \}.$$

In particular, $\text{Conf}_M(\mathcal{L})$ is the set of all realizations of $\mathcal{L}$. It is called the configuration space of $\mathcal{L}$.

**Definition 9.** A marked linkage is a tuple $(\mathcal{L}, P, Q)$, where $P$ and $Q$ are subsets of the set $V$ (the set of the vertices of $\mathcal{L}$). $P$ is called the “input set” and its elements, called the “inputs”, are represented by $\blacksquare$ on the figures. $Q$ is called the “output set” and its elements, called the “outputs”, are represented by $\bigtriangleup$ on the figures.

The input map $p : \text{Conf}_M(\mathcal{L}) \rightarrow \mathcal{M}^P$ is the map induced by the natural projection $\mathcal{M}^V \rightarrow \mathcal{M}^P$ (the restriction map). In other words, for all $\phi \in \text{Conf}_M(\mathcal{L})$, we define $p(\phi) = \phi|_P$.

Likewise, we define the output map $q : \text{Conf}_M(\mathcal{L}) \rightarrow \mathcal{M}^Q$ by $q(\phi) = \phi|_Q$. 


Then we say that $\psi$ with a different set of inputs and outputs:

**Definition 10.** We say that $\mathcal{L}$ is a functional linkage for the input-output function $f : \text{Conf}_{\mathcal{M}}(\mathcal{L}) \rightarrow \mathcal{M}^Q$ if

$$\forall \phi \in \text{Conf}_{\mathcal{M}}(\mathcal{L}) \quad f(p(\phi)) = q(\phi).$$

For example, the Peaucellier linkage on the plane (Figure 1) is functional for the inversion with respect to a circle: the vertex $b$ (the output) is the image of the vertex $a$ (the input) by an inversion.

2.1. Regularity.

**Definition 11.** Let $\mathcal{L}$ be a linkage. Let $W \subseteq V$ and $\psi \in \text{Conf}_{\mathcal{M}}(\mathcal{L})$. Let $\pi_W$ be the restriction map

$$\pi_W : \text{Conf}_{\mathcal{M}}^W(\mathcal{L}) \rightarrow \text{Conf}_{\mathcal{M}}^L(\mathcal{L}).$$

We say that $\psi$ is a regular input for $W$ if there exists an open neighborhood $U \subseteq \text{Conf}_{\mathcal{M}}^L(\mathcal{L})$ of $\psi$ such that $\pi_W|_{\pi_W^{-1}(U)}$ is a finite smooth covering.

We write $\text{Reg}_{\mathcal{M}}^W(\mathcal{L}, W) \subseteq \text{Conf}_{\mathcal{M}}^L(\mathcal{L})$ the set of regular inputs for $W$. When $W$ is the set $V$ of all vertices of $\mathcal{L}$, we simply write $\text{Reg}_{\mathcal{M}}^L(\mathcal{L})$.

Roughly speaking, $\psi$ is a regular input for $W$ if it determines a finite number of realizations $\phi$ of $W$, and if these configurations are determined smoothly with respect to $\psi$ (in other words, $\pi_W^{-1}$ is a smooth multivalued function in a neighborhood of $\psi$).

Note that, in Definition 11, we do not require $U$ or $\pi_W^{-1}(U)$ to be a smooth manifold.

The following fact is simple but essential:

**Proposition 12.** For any $W_1, W_2 \subseteq V$, we have

$$\text{Reg}_{\mathcal{M}}^L(\mathcal{L}, W_1) \cap \text{Reg}_{\mathcal{M}}^L(\mathcal{L}, W_2) \subseteq \text{Reg}_{\mathcal{M}}^L(\mathcal{L}, W_1 \cup W_2).$$

Therefore, in practice, when we want to prove that $\text{Reg}_{\mathcal{M}}^L(\mathcal{L}) = \text{Conf}_{\mathcal{M}}^L(\mathcal{L})$, we only have to prove that $\text{Reg}_{\mathcal{M}}^L(\mathcal{L}, \{v\}) = \text{Conf}_{\mathcal{M}}^L(\mathcal{L})$ for all $v \in V$.

2.2. Changing the input set. In this proposition, we take a linkage, then consider the same linkage with a different set of inputs $P$ and analyse the impact on $\text{Reg}_{\mathcal{M}}^P(\mathcal{L})$.

**Proposition 13.** Let $\mathcal{L}_1 = (V_1, E_1, l_1, F_1, \phi_{01}, P_1, Q_1)$. Let $P_2 \subseteq V_1$ and $\mathcal{L}_2 = (V_1, E_1, l_1, F_1, \phi_{02}, P_2, Q_1)$. Then $\text{Reg}_{\mathcal{M}}^{P_2}(\mathcal{L}_2)$ contains

$$\left\{ \psi \in \text{Conf}_{\mathcal{M}}^{P_2}(\mathcal{L}_2) \mid \forall \phi \in p_2^{-1}(\psi) \quad p_1(\phi) \in \text{Reg}_{\mathcal{M}}^{P_1}(\mathcal{L}_1) \right\} \cap \text{Reg}_{\mathcal{M}}^{P_2}(\mathcal{L}_2, P_1).$$

**Proof.** It is a simple consequence of the fact that the composition of two smooth functions is a smooth function.

2.3. Combining linkages. This notion is essential to construct complex linkages from elementary ones. The proofs in this section are straightforward and left to the reader.

Let $\mathcal{L}_1 = (V_1, E_1, l_1, F_1, \phi_{01}, P_1, Q_1)$ and $\mathcal{L}_2 = (V_2, E_2, l_2, F_2, \phi_{02}, P_2, Q_2)$ be two linkages, $W_1 \subseteq V_1$, and $\beta : W_1 \rightarrow V_2$.

The idea is to construct a new linkage $\mathcal{L}_3 = \mathcal{L}_1 \cup_{\beta} \mathcal{L}_2$ as follows:

**Step 1.** Consider $\mathcal{L}_1 \cup \mathcal{L}_2$, the disjoint union of the two graphs $(V_1, E_1)$ and $(V_2, E_2)$.
Step 2. Identify some vertices of $V_1$ with some vertices of $V_2$ via $\beta$, without removing any edge.

Since linkages are graphs which come with an additional structure, we need to clarify what happens to the other elements ($l$, $F$, $\phi_0$, $P$, $Q$). In particular, note that the inputs of $L_2$ which are in $\beta(W_1)$ are not considered as inputs in the new linkage $L_3$.

**Definition 14** (Combining two linkages). We define $L_3 = L_1 \cup_{\beta} L_2 = (V_3, E_3, l_3, F_3, \phi_03, P_3, Q_3)$ in the following way:

1. $V_3 = (V_1 \setminus W_1) \cup V_2$;
2. $E_3 = (E_1 \cap (V_1 \setminus W_1)^2) \cup (E_2 \cap V_2^2) \cup \{v\beta(v') \mid v \in V_1 \setminus W_1, v' \in W_1, vv' \in E_1\}$
   \[ \cup \{\beta(v)\beta(v') \mid v, v' \in W_1, vv' \in E_1\}; \]
3. For all $v_1, v_1' \in V_1 \setminus W_1, w_1, w_1' \in W_1, v_2, v_2' \in V_2$, we define
   \[ l_3(v_1v_1') = l_1(v_1v_1'), \quad l_3(v_1\beta(w_1)) = l_1(v_1w_1), \quad l_3(v_2v_2') = l_2(v_2v_2'); \quad l_3(\beta(w_1)\beta(w_1')) = l_1(w_1w_1'); \]
4. $F_3 = (F_1 \setminus W_1) \cup \beta(F_1 \cap W_1) \cup F_2$;
5. $\phi_{03}|F_1|W_1 = \phi_{01}|F_1|W_1$, \quad $\phi_{03} \circ \beta = \phi_{01}|W_1$, \quad $\phi_{03}/F_2 \setminus \beta(W_1) = \phi_{02}/F_2$;
6. $P_3 = (P_1 \setminus W_1) \cup \beta(P_1 \cap W_1) \cup (P_2 \setminus \beta(W_1))$;
7. $Q_3 = (Q_1 \setminus W_1) \cup Q_2$.

The combination of two linkages is prohibited in the following cases:

1. There exist $a_1, b_1 \in F_1 \cap W_1$ such that $\beta(a_1) = \beta(b_1)$ and $\phi_{01}(a_1) \neq \phi_{01}(b_1)$ (two vertices are fixed at different places but should be attached to the same other vertex).
2. There exist $a_1, b_1 \in W_1$ such that $a_1b_1 \in E_1$, $\beta(a_1)\beta(b_1) \in E_2$, and $l_1(a_1b_1) \neq l_2(\beta(a_1)\beta(b_1))$ (two edges of different lengths should joint one couple of vertices).

**Example.** We consider the two identical linkages $L_1$ and $L_2$:

![Diagram](diagram.png)

The inputs of $L_1$ are $a_i, b_i$ and the output is $c_i$.

To combine the two linkages, we set $W_1 = \{c_1\}$ and $\beta(c_1) = a_2$. Then $L_3 := L_1 \cup_{\beta} L_2$ is the following linkage:

![Diagram](diagram2.png)

The inputs of $L_3$ are $a_1, b_1, b_2$ and the output is $c_2$.

The following proposition relates the sets $\text{Conf}_M(L_1), \text{Conf}_M(L_2), \text{Conf}_M(L_3)$.

**Proposition 15.** Let $L_1, L_2$ be two linkages, $W_1 \subseteq V_1, \beta : W_1 \rightarrow V_2$, and $L_3 = L_1 \cup_{\beta} L_2$ be defined as in Definition 14. Then

\[
\text{Conf}_M(L_3) = \{ \phi_3 \in \mathcal{M}^{V_3} \mid \exists (\phi_1, \phi_2) \in \text{Conf}_M(L_1) \times \text{Conf}_M(L_2) \quad \phi_1|V_1\setminus W_1 = \phi_3|V_1\setminus W_1, \quad \phi_1|W_1 = \phi_3|\beta(W_1) \circ \beta, \quad \phi_2 = \phi_3|V_2 \}.
\]
The following proposition shows that combining linkages often means composing functions.

**Proposition 16.** Let $\mathcal{L}_1$, $\mathcal{L}_2$ be two linkages with $\text{card}(Q_1) = \text{card}(P_2)$.

Assume that $\mathcal{L}_1$ is a functional linkage for $f_1 : \text{Conf}^{P_2}_{\mathcal{M}}(\mathcal{L}_1) \rightarrow \mathcal{M}^{Q_1}$ and that $\mathcal{L}_2$ is a functional linkage for $f_2 : \text{Conf}^{P_3}_{\mathcal{M}}(\mathcal{L}_2) \rightarrow \mathcal{M}^{Q_2}$. Let $W_1 = Q_1$, $\beta : W_1 \rightarrow P_2$ a bijection, and $L_3 = L_1 \cup \beta L_2$.

The bijection $\beta$ induces a bijection $\beta'$ between $\mathcal{M}^{Q_1}$ and $\mathcal{M}^{P_2}$.

Then $L_3$ is functional for $f_2 \circ \beta' \circ f_1|_{\text{Conf}^{P_3}_{\mathcal{M}}(L_3)}$.

The following proposition relates the sets $\text{Reg}^{P_1}_{\mathcal{M}}(L_1)$, $\text{Reg}^{P_2}_{\mathcal{M}}(L_2)$, $\text{Reg}^{P_3}_{\mathcal{M}}(L_3)$ defined in Definition 11.

**Proposition 17.** Let $\mathcal{L}_1$, $\mathcal{L}_2$ be two linkages, $W_1 \subseteq V_1$, $\beta : W_1 \rightarrow P_2$, and $\mathcal{L}_3 = L_1 \cup \beta L_2$. Suppose that $\psi \in \text{Conf}^{P_3}_{\mathcal{M}}(L_3)$ satisfies both of the following properties:

1. $\exists \psi_1 \in \text{Reg}^{P_1}_{\mathcal{M}}(L_1)$ $\psi_1|_{P_1 \setminus W_1} = \psi_3|_{P_1 \setminus W_1}$, $\psi_1|_{P_1 \cap W_1} = \psi_3|_{\beta(P_1 \cap W_1) \circ \beta}$;
2. $\forall \phi \in p_3^{-1}(\psi_3)$ $\phi|_{P_2} \in \text{Reg}^{P_2}_{\mathcal{M}}(L_2)$.

Then $\psi \in \text{Reg}^{P_3}_{\mathcal{M}}(L_3)$.

### 3. Generalities on the Minkowski Plane

#### 3.1. Notation.

The Minkowski plane $\mathbb{M}$ is $\mathbb{R}^2$ equipped with the bilinear form $\varphi \left( \begin{pmatrix} x \\ t \end{pmatrix}, \begin{pmatrix} x' \\ t' \end{pmatrix} \right) = xx' - tt'$. The pseudo-norm $\|\|$ is defined by $\|a\|^2 = \varphi(a, a)$ and $\|a\| \in \mathbb{R}^+ \cup i\mathbb{R}^+$ for all $a \in \mathbb{M}$.

Let $\alpha \in \mathbb{M}$. We write $x_\alpha$ and $t_\alpha$ the usual coordinates in $\mathbb{R}^2$, so that $\|\alpha\|^2 = x_\alpha^2 - t_\alpha^2$ for all $\alpha \in \mathbb{M}$.

Sometimes, it will be more convenient to use *lightlike coordinates*, defined by $y_\alpha = x_\alpha + t_\alpha$ and $z_\alpha = x_\alpha - t_\alpha$, so that $\|\alpha\|^2 = y_\alpha z_\alpha$.

We write $I = \left\{ \begin{pmatrix} x \\ t \end{pmatrix} \in \mathbb{M} \mid t = 0 \right\}$.

In the Minkowski plane, hyperbolae play a central role (instead of circles in the Euclidean plane): for any $\alpha \in \mathbb{M}$ and $r^2 \in \mathbb{R}$, the hyperbola $H(\alpha, r)$ is the set of all $\gamma \in \mathbb{M}$ such that $\delta(\alpha, \gamma)^2 = r^2$.

#### 3.2. Intersection of Two Hyperbolae.

Let $\alpha_0, \alpha_1 \in \mathbb{M}$ and $r_0^2, r_1^2 \in \mathbb{R}$. We write $d = \|\alpha_0 - \alpha_1\|$. The aim of this section is to determine the cardinality of $I = H(\alpha_0, r_0) \cap H(\alpha_1, r_1)$.

**Proposition 18.** If $\alpha_0 \neq \alpha_1$, $r_0^2 \neq 0$ and $r_1^2 \neq 0$, we have $\text{card}(I) = 2$.

**Proof.** We write $y_0 = y_{\alpha_0}$ and $z_0 = z_{\alpha_0}$. We may assume $\alpha_1 = 0$ and $y_0 \neq 0$. $I$ is the set of the solutions of the system in $(y, z)$:

\[
\begin{cases}
yz = r_1^2 \\
(y - y_0)(z - z_0) = r_0^2
\end{cases}
\]

which is equivalent to

\[
\begin{cases}
yz = r_1^2 \\
y_0z^2 - (y_0z_0 + r_1^2 - r_0^2)z + r_1^2z_0 = 0
\end{cases}
\]

Thus, $z$ is one of the roots of a polynomial of degree 2 and $y$ is fully determined by $z$, so there are at most two solutions to the system. \qed

**Proposition 19.**

1. If $r_0^2r_1^2 < 0$ and $d^2 \neq 0$, then $\text{card}(I) = 2$. Moreover, if $d'$ is the distance between the two points of $I$, then $d^2d'^2 < 0$.
2. If $r_0^2r_1^2 < 0$ and $d^2 = 0$, then $\text{card}(I) = 1$.
3. If $r_0^2r_1^2 > 0$ and $r_0^2d^2 < 0$, then $\text{card}(I) = 2$.

**Proof.** Examine the following figures. \qed
3.3. The case of equality in the triangle inequality. In the Minkowski plane, the triangle inequality is not always valid, but the equality case is the same as in the Euclidean plane.

**Proposition 20.** Let \( \alpha, \beta \in \mathbb{M} \). If \( \|\alpha\| + \|\beta\| = \|\alpha + \beta\| \), then \( \alpha \) and \( \beta \) are colinear.

**Proof.** We have

\[
(\|\alpha\| + \|\beta\|)^2 = \|\alpha + \beta\|^2
\]

\[
\|\alpha\|^2 + \|\beta\|^2 + 2\|\alpha\|\|\beta\| = \|\alpha\|^2 + \|\beta\|^2 + 2\varphi(\alpha, \beta)
\]

\[
\varphi(\alpha, \beta) = \|\alpha\|\|\beta\|.
\]

Therefore, the discriminant of the polynomial function

\[
\lambda \mapsto \|\beta\|^2 \lambda^2 + 2\varphi(\alpha, \beta) \lambda + \|\alpha\|^2
\]

is zero. Thus, \( \|\alpha + \lambda \beta\|^2 \) is either nonnegative for all \( \lambda \) or nonpositive for all \( \lambda \). This means that \( \alpha \) and \( \beta \) are not linearly independant.

\[\boxdot\]

3.4. The dual linkage. Let \( L_1 \) be a linkage in the Minkowski plane. We define the reflection

\[
s : \mathbb{C} \rightarrow \mathbb{C}
\]

\[
a + ib \mapsto b + ia.
\]

We construct \( L_2 \), the dual linkage of \( L_1 \), by

1. \( V_2 = V_1 \)
2. \( E_2 = E_1 \)
3. \( F_2 = F_1 \)
4. \( P_2 = P_1 \)
5. \( Q_2 = Q_1 \)
6. \( l_2 = s \circ l_1 \)
7. \( \phi_{02} = s \circ \phi_{01} \) (with \( \mathbb{R}^2 \) identified to \( \mathbb{C} \)).

For all \( W \subseteq V_1 \), this linkage satisfies

\[
\text{Conf}^W_M(L_2) = s(\text{Conf}^W_M(L_1)).
\]

\[
\text{Reg}^P_M(L_2) = s(\text{Reg}^P_M(L_1)).
\]
4. The linkages for geometric operations

4.1. The robotic arm linkage.

We let $P = \{a, b\}$ and $F = \emptyset$. We assume $l_1 \neq 0$ and $l_2 \neq 0$. We translate Proposition 19 in terms of linkages.

Proposition 21. (1) If $l_1^2$ and $l_2^2$ have different signs,

$$\text{Conf}_{\mathbb{M}}^P(\mathcal{L}) \supseteq \text{Reg}_{\mathbb{M}}^P(\mathcal{L}) \supseteq \{ \psi \in \mathbb{M}^P \mid \|\psi(a) - \psi(b)\| \neq 0 \}$$

(2) If $l_1^2$ and $l_2^2$ have the same sign,

$$\text{Conf}_{\mathbb{M}}^P(\mathcal{L}) \supseteq \text{Reg}_{\mathbb{M}}^P(\mathcal{L}) \supseteq \{ \psi \in \mathbb{M}^P \mid \|\psi(a) - \psi(b)\|^2 \cdot l_1^2 < 0 \}$$

(3) More generally, let $\psi \in \text{Conf}_{\mathbb{M}}^P(\mathcal{L})$. If the intersection $\mathcal{H}(\psi(a), l_2) \cap \mathcal{H}(\psi(b), l_1)$ contains exactly two elements, then $\psi \in \text{Reg}_{\mathbb{M}}^P(\mathcal{L})$.

Proof. When the intersection $\mathcal{H}(\psi(a), l_2) \cap \mathcal{H}(\psi(b), l_1)$ contains exactly two elements, it means that the intersections are obtained from simple roots of a polynomial of degree 2 (see the proof of Proposition 18). Therefore, locally, the roots depend smoothly on the coefficients. □

4.2. The rigidified square linkage. This linkage is well-known on the Euclidean plane. It is the common solution to the problem of degenerate configurations of the square. It is very useful to notice that it does behave in the same way in the Minkowski plane.

We now explain why we need to rigidify square linkages. If one considers the ordinary square linkage (see the following figure), there are many realizations $\phi$ in which $\phi(a)\phi(b)\phi(c)\phi(d)$ is not a parallelogram (we call these realizations degenerate realizations of the square).

In degenerate realizations, two vertices are sent to the same point of $\mathbb{M}$. To avoid degenerate realizations, we add two vertices and five edges to the square $abcd$. We call this operation “rigidifying the square".
This linkage is called the **rigidified square**. The input set is \( P = \{a, c\} \).

We assume \( l \neq 0 \).

**Proposition 22.**

1. For all \( \phi \in \text{Conf}_{\mathbb{M}}(\mathcal{L}) \) we have
   \[
   \phi(b) - \phi(a) = \phi(c) - \phi(d)
   \]
   \( (\phi(a)\phi(b)\phi(c)\phi(d) \) is a “standard parallelogram”).

2. For all \( \phi \in \text{Conf}_{\mathbb{M}}(\mathcal{L}) \) such that \( \phi(b) \neq \phi(d) \) and \( \phi(a) \neq \phi(c) \), we have \( \phi|_P \in \text{Reg}^P_{\mathbb{M}}(\mathcal{L}) \). In particular, \( \text{Reg}^P_{\mathbb{M}}(\mathcal{L}) \) contains
   \[
   \left\{ \psi \in \mathbb{M}^{[a,c]} \mid ||\psi(a) - \psi(c)|| \cdot l^2 < 0 \right\}.
   \]

**Proof.**

1. Let \( \phi \in \text{Conf}_{\mathbb{M}}(\mathcal{L}) \). From the equality case in the triangle inequality, we have \( \phi(f) = \frac{\phi(a) + \phi(d)}{2} \) and \( \phi(c) = \frac{\phi(b) + \phi(e)}{2} \).

   **Case 1:** \( \phi(a) = \phi(c) \). In this case, \( ||\phi(f) - \phi(c)|| = ||\phi(f) - \phi(a)|| \). Therefore,
   \[
   ||\phi(c) - \phi(e)|| + ||\phi(f) - \phi(c)|| = ||\phi(f) - \phi(e)||,
   \]
   so \( \phi(e), \phi(c) \) and \( \phi(f) \) are aligned, so \( \phi(b), \phi(c) \) and \( \phi(d) \) are aligned and therefore \( \phi(b) - \phi(a) = \phi(c) - \phi(d) \).

   **Case 2:** \( \phi(d) = \frac{\phi(a) + \phi(c)}{2} \). We have
   \[
   ||\phi(b) - \phi(a)|| + ||\phi(b) - \phi(c)|| = ||\phi(c) - \phi(a)|| \quad (= 2l),
   \]
   so \( \phi(a), \phi(c) \) and \( \phi(b) \) are aligned, thus \( \phi(d) = \phi(b) \). We are taken back to the first case.

   **Case 3:** \( \phi(d) \neq \frac{\phi(a) + \phi(c)}{2} \) and \( \phi(a) \neq \phi(c) \). Let \( I = \mathcal{H}(\phi(a), l) \cap \mathcal{H}(\phi(c), l) \). We have \( \phi(d) \in I \), \( \phi(b) \in I \), and \( \text{card}(I) \leq 2 \).

   We have \( \phi(a) + \phi(c) - \phi(d) \in I \) if \( \phi(a) + \phi(c) - \phi(d) = \phi(d) \) then we are in the second case.

   If not, we have \( I = \{ \phi(d), \phi(a) + \phi(c) - \phi(d) \} \) and therefore, either \( \phi(b) = \phi(d) \) (this is again the first case) or \( \phi(b) = \phi(a) + \phi(c) - \phi(d) \), i.e. \( \phi(b) - \phi(a) = \phi(c) - \phi(d) \).

2. This is a consequence of Proposition 21.

\[\square\]

4.3. The Peaucellier inversor.
We choose \( r, R, l > 0 \). We let \( F = \{ a \} \) and \( \phi_0(a) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \). The input set is \( P = \{ e \} \). The output set is \( Q = \{ d \} \). The square \( bdce \) is rigidified (see Section 4.2), but for convenience, we do not draw on the figure the vertices which are necessary for the rigidification.

![Figure 2. Two realizations of the same Peaucellier inversor in the Minkowski plane.](image)

In the Euclidean plane, it is well-known that the Peaucellier linkage is a functional linkage for the inversion with respect to the circle \( C \left( 0, \sqrt{R^2 - r^2} \right) \), that is, the function \( \alpha \mapsto -\frac{R^2 - r^2}{\|\alpha\|^2}\alpha \). In the Minkowski plane, we will prove that it is essentially functional for inversion with respect to the hyperbola \( H \left( 0, \sqrt{R^2 + r^2} \right) \). More precisely, it is functional for the function \( \alpha \mapsto -\frac{R^2 + r^2}{\|\alpha\|^2}\alpha \) (in the version of the Peaucellier inversor which we choose, a "-" sign appears).

**Proposition 23.** For all \( \phi \in \text{Conf}_M(\mathcal{L}) \), we have \( \|\phi(e)\| \neq 0 \) and \( \phi(d) = -\frac{R^2 + r^2}{\|\phi(e)\|^2}\phi(e) \).

**Proof.** Let \( \phi \in \text{Conf}_M(\mathcal{L}) \). We know that \( \phi(b) \neq \phi(c) \) (because of the lengths of the edges between \( b, c \) and \( g \)) so the intersection of the two hyperbolae \( H(\phi(e), R) \) and \( H(\phi(a), ir) \) is exactly \( \{ \phi(b), \phi(c) \} \). Moreover, \( \|\phi(e)\| \neq 0 \) because of the lengths of the edges between \( a, e \) and \( b \).

Then, \((y_{\phi(b)}, z_{\phi(b)})\) and \((y_{\phi(c)}, z_{\phi(c)})\) are the two solutions of the following system with unknown \((y, z)\):

\[
\begin{align*}
yz &= -r^2 \\
(y - y_{\phi(e)})(z - z_{\phi(e)}) &= R^2.
\end{align*}
\]

This system is equivalent to

\[
\begin{align*}
yz &= -r^2 \\
y_{\phi(e)}z^2 + (y_{\phi(e)}z_{\phi(e)} - r^2 - R^2)z + r^2z_{\phi(e)} &= 0.
\end{align*}
\]

We deduce that

\[
z_{\phi(b)} + z_{\phi(c)} = z_{\phi(e)} - \frac{r^2 + R^2}{y_{\phi(e)}}
\]

and similarly

\[
y_{\phi(b)} + y_{\phi(c)} = y_{\phi(e)} - \frac{r^2 + R^2}{z_{\phi(e)}}
\]

which gives the desired result, since \( \phi(d) = \phi(b) + \phi(c) - \phi(e) \). \( \square \)
Proposition 24. For this linkage, $\text{Conf}_{M}^{p}(\mathcal{L})$ (that is, the workspace of the vertex $e$) contains the spacelike cone
\[ \{ \alpha \in M \mid \| \alpha \|^2 > 0 \} . \]

Proof. This is a consequence of Proposition 19. \hfill \square

Proposition 25. For this linkage, $\text{Reg}_{M}^{p}(\mathcal{L}) = \text{Conf}_{M}^{p}(\mathcal{L})$.

Proof. We give a detailed proof in order to illustrate the use of Proposition 17. This method is the key to many proofs concerning $\text{Reg}_{M}^{p}(\mathcal{L})$, later in this paper.

The Peaucellier inversor may be seen as the combination of the following linkages:
\[ \mathcal{L}_{1}: \text{a robotic arm} \{ a_{1}, c_{1}, e_{1} \} \text{ with one input } e_{1} \text{ and one fixed vertex } a_{1}, \text{ one edge } \{ a_{1}, c_{1} \} \text{ of length } il; \]
\[ \mathcal{L}_{2}: \text{a robotic arm} \{ a_{2}, b_{2}, e_{2} \} \text{ with one input } e_{2} \text{ and one fixed vertex } a_{2}, \text{ one edge } \{ a_{2}, b_{2} \} \text{ of length } ir \text{ and one edge } \{ b_{2}, e_{2} \} \text{ of length } R; \]
\[ \mathcal{L}_{3}: \text{a rigidified square} \{ b_{3}, d_{3}, c_{3}, e_{3} \} \text{ with inputs } b_{3}, c_{3} \text{ and four edges of length } R; \]
\[ \mathcal{L}_{4}: \text{a robotic arm} \{ b_{4}, g_{4}, e_{4} \} \text{ with inputs } b_{4}, e_{4}, \text{ one edge } \{ b_{4}, g_{4} \} \text{ of length } l \text{ and one edge } \{ g_{4}, e_{4} \} \text{ of length il}. \]

We combine the linkages in the following way (observe that the name of the vertices are chosen so that each $\beta_{i}$ preserves the letters and only changes indices):

1. Let $W_{1} = \{ c_{1}, e_{1} \}$. Let $\beta_{1}(c_{1}) = c_{3}$, $\beta_{1}(e_{1}) = e_{3}$. Let $\mathcal{L}_{5} = \mathcal{L}_{1} \cup_{\beta_{1}} \mathcal{L}_{3}$. The input set of $\mathcal{L}_{5}$ is $P_{6} = \{ e_{3}, b_{3} \}$.
2. Let $W_{2} = \{ b_{2}, e_{2} \}$. Let $\beta_{2}(b_{2}) = b_{3}$, $\beta_{2}(e_{2}) = e_{3}$. Let $\mathcal{L}_{6} = \mathcal{L}_{2} \cup_{\beta_{2}} \mathcal{L}_{5}$. The input set of $\mathcal{L}_{6}$ is $P_{6} = \{ e_{3} \}$.
3. Let $W_{6} = \{ b_{6}, c_{6} \}$. Let $\beta_{6}(b_{6}) = b_{4}$, $\beta_{6}(c_{6}) = c_{4}$. Let $\mathcal{L}_{7} = \mathcal{L}_{6} \cup_{\beta_{6}} \mathcal{L}_{4}$. The input set of $\mathcal{L}_{7}$ is $P_{7} = \{ e_{3} \}$.

The linkage $\mathcal{L}_{7}$ is exactly the Peaucellier linkage.

Let $\psi \in \text{Conf}_{M}^{p}(\mathcal{L}_{1})$ such that the intersection $\mathcal{H}(0, ir) \cap \mathcal{H}(\psi(e_{1}), R)$ has cardinality 2. Proposition 21 and Proposition 13 show that $\psi \in \text{Reg}_{M}^{p}(\mathcal{L}_{1})$.

We may naturally identify $\text{Conf}_{M}^{\{e_{3}, b_{4}\}}(\mathcal{L}_{7})$ with a subset $C$ of $\text{Conf}_{M}^{p}(\mathcal{L}_{5})$ (identifying $b_{4}$ with $b_{3}$). Let us show that $C$ is in fact a subset of $\text{Reg}_{M}^{p}(\mathcal{L}_{5})$ using Proposition 17. Let $\psi \in C$, and let $\phi \in \text{Conf}_{M}^{\{e_{3}, b_{4}\}}(\mathcal{L}_{7})$ such that $\phi(e_{3}) = \psi(e_{3})$ and $\phi(b_{4}) = \psi(b_{3})$. Let $\psi_{1} \in M^{(e_{1})}$ defined by $\psi_{1}(e_{1}) = \psi(e_{3})$; since $\phi(b_{4}) \neq \phi(c_{3})$, the intersection $\mathcal{H}(0, ir) \cap \mathcal{H}(\psi_{1}(e_{1}), R)$ has cardinality at least 2, but it is in fact exactly 2 from Proposition 18. Therefore, $\psi_{1} \in \text{Reg}_{M}^{p}(\mathcal{L}_{1})$, so the first hypothesis of Proposition 17 is satisfied. For the second hypothesis, we need to show that $\phi|_{P_{3}} \in \text{Reg}_{M}^{p}(\mathcal{L}_{3})$. We know that $\phi(b_{4}) \neq \phi(c_{4})$, and from Proposition 23, we also know that $\phi(e_{3}) \neq \phi(d_{4})$. Therefore, Proposition 22 tells us that $\phi|_{P_{3}} \in \text{Reg}_{M}^{p}(\mathcal{L}_{3})$. The two hypotheses of Proposition 17 are satisfied, so $C \in \text{Reg}_{M}^{p}(\mathcal{L}_{5})$.

In the same way, one may show that $\text{Conf}_{M}^{\{e_{3}\}}(\mathcal{L}_{7}) \subseteq \text{Reg}_{M}^{p}(\mathcal{L}_{6})$, and finally, that $\text{Conf}_{M}^{\{e_{3}\}}(\mathcal{L}_{7}) \subseteq \text{Reg}_{M}^{p}(\mathcal{L}_{7})$, so $\text{Reg}_{M}^{p}(\mathcal{L}_{7}) = \text{Conf}_{M}^{p}(\mathcal{L}_{7})$. \hfill \square

Proposition 26. For all $\phi \in \text{Conf}_{M}(\mathcal{L})$ we have the equivalence
\[ \phi(d) \in \mathcal{H}
\begin{pmatrix}
0 \\
-1
\end{pmatrix}
\iff y_{\phi(e)} - z_{\phi(e)} = -(R^{2} + r^{2}). \]
Proof. Let $\phi \in \text{Conf}_M(L)$. The following lines are equivalent:

$$\phi(d) \in \mathcal{H}\left(\begin{pmatrix} 0 \\ -1 \end{pmatrix}, i\right)$$

$$(y_{\phi(d)} + 1)(z_{\phi(d)} - 1) = -1$$

$$\left(-\frac{(R^2 + r^2)}{\|\phi(e)\|^2}\right)y_{\phi(e)} + 1 \left(-\frac{(R^2 + r^2)}{\|\phi(e)\|^2}\right)z_{\phi(e)} - 1 = -1$$

$$y_{\phi(e)} - z_{\phi(e)} = -(R^2 + r^2).$$

$\square$

4.4. The partial $t_0$-line linkage.

[Diagram of the partial $t_0$-line linkage]

$R = r = \frac{1}{\sqrt{2}}; l > 0; F = \{a, f\}; \phi_0(a) = \left(\begin{pmatrix} 5 \\ t_0 + 1/2 \end{pmatrix}, \phi_0(f) = \left(\begin{pmatrix} 5 \\ t_0 - 1/2 \end{pmatrix}; P = \{e\}.$

Proposition 27. The workspace of $e$, $\text{Conf}_M(L)$, is contained in the line $t = t_0$, but does not necessarily contain the whole line. More precisely

$$\left\{ \left(\begin{pmatrix} x \\ t \end{pmatrix} \in \mathbb{M} \mid t = t_0, |x - 5| > 1/2 \right) \subseteq \text{Reg}_P(L) = \text{Conf}_P(L) \subseteq \left\{ \left(\begin{pmatrix} x \\ t \end{pmatrix} \in \mathbb{M} \mid t = t_0 \right\}.$$

Proof. We apply Propositions 24, 25 and 26. $\square$

The dual of this linkage is called the partial $x_0$-line linkage.

4.5. The $t_0$-integer linkage. This linkage contains four vertices $a, b, c, d$ which are restricted to move on $I$ (the $x$-axis) using a partial $t_0$-line linkages. More precisely, the linkage with four vertices on the figure below is combined with four partial $t_0$-line linkages $L_i, i = 1 \ldots 4$, to form the $t_0$-integer linkage. The combination mappings $\beta_i$ send $a, b, c$ and $d$ respectively to the inputs $e_i$.

[Diagram of the $t_0$-integer linkage]

Take $F = \{a\}; \phi_0(a) = \left(\begin{pmatrix} 0.5 \\ t_0 \end{pmatrix}; P = \emptyset.$

We have

$$\text{Conf}_M^{[d]}(L) = \left\{ \left(\begin{pmatrix} -3 \\ t_0 \end{pmatrix}, \left(\begin{pmatrix} -2 \\ t_0 \end{pmatrix}, \ldots, \left(\begin{pmatrix} 3 \\ t_0 \end{pmatrix}, \left(\begin{pmatrix} 4 \\ t_0 \end{pmatrix}\right) \right) \right\}.$$

Moreover, $\text{Conf}_M(L)$ is a finite set so $\text{Reg}_P(L) = \text{Conf}_P(L)$.

We will use this linkage twice to construct more complex linkages. In Section 4.6, we could have used a simpler linkage with a configuration space of cardinality 2 instead of 8, but we need it to have cardinality at least 7 in Section 5.3.
4.6. **The \(t_0\)-line linkage.** This linkage traces out the whole horizontal line \(t = t_0\); it contains a vertex \(e\), the input, such that

\[
\text{Conf}_{\mathcal{M}}^{(e)}(\mathcal{L}) = \{ \alpha \in \mathcal{M} | t_\alpha = t_0 \}.
\]

The idea is to combine a partial \(t_0\)-line linkage with a \(t_0\)-integer linkage. Here is how we construct the \(t_0\)-line linkage. Let:

- \(\mathcal{L}_1\) a \((t_0 = \frac{1}{2})\)-integer linkage.
- \(\mathcal{L}_2\) a \((t_0 = -\frac{1}{2})\)-integer linkage.
- \(\mathcal{L}_3\) the combination (disjoint union) of the two linkages \(\mathcal{L}_1\) and \(\mathcal{L}_2\).
- \(\mathcal{L}_4\) a linkage similar to a partial \(t_0\)-line linkage, with the only difference that \(F_4 = \emptyset\) instead of \(F_4 = \{ a_4, f_4 \}\).
- \(W_3 = \{ d_1, d_2 \}\) and \(\beta(d_1) = a_4, \beta(d_2) = f_4\).
- \(\mathcal{L}_5 = \mathcal{L}_3 \cup \beta \mathcal{L}_4\). We have as desired

\[
\text{Conf}_{\mathcal{M}}^{(e_5)}(\mathcal{L}_5) = \{ \alpha \in \mathcal{M} | t_\alpha = t_0 \}.
\]

Using Proposition 17, we also obtain \(\text{Reg}_{\mathcal{M}}^{P}(\mathcal{L}) = \text{Conf}_{\mathcal{M}}^{P}(\mathcal{L})\).

For future reference, we let \(a := a_1, f := f_1, e := e_4\).

The dual of this linkage is called the \(x_0\)-line linkage.

4.7. **The horizontal parallelizer.** This linkage has the input set \(P = \{e_3, e_4\}\). It satisfies

\[
\text{Reg}_{\mathcal{M}}^{P}(\mathcal{L}) = \text{Conf}_{\mathcal{M}}^{P}(\mathcal{L}) = \left\{ \psi \in \mathcal{M}^{(e_3, e_4)} | t_\psi(e_4) = t_\psi(e_4) \right\}.
\]

Let:

- \(\mathcal{L}_1\) and \(\mathcal{L}_2\) two \((x_0 = 0)\)-line linkages;
- \(\mathcal{L}_3, \mathcal{L}_4\) two linkages similar to \((t_0 = 0)\)-line linkages, but with \(F_3, F_4 = \emptyset\);
- \(\mathcal{L}_5\) the combination of \(\mathcal{L}_1\) and \(\mathcal{L}_2\);
- \(W_3 = \{ a_3, f_3 \}\), \(\beta(a_3) = e_1, \beta(f_3) = e_2\), and \(\mathcal{L}_6 = \mathcal{L}_3 \cup \beta \mathcal{L}_5\);
- \(W_4 = \{ a_4, f_4 \}\), \(\beta(a_4) = e_1, \beta(f_4) = e_2\), and \(\mathcal{L}_7 = \mathcal{L}_4 \cup \beta \mathcal{L}_6\).

\(\mathcal{L}_7\) is the desired linkage.

For future reference, we let \(a := e_3\) and \(b := e_4\).

The dual of this linkage is called *vertical parallelizer*.

4.8. **The diagonal parallelizer.**

\[
P = \{a, b\}, F = \{g, f\}, \phi_0(f) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \phi_0(g) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

In this section, we use lightlike coordinates (see Section 3.1).

**Proposition 28.** We have

\[
\text{Reg}_{\mathcal{M}}^{P}(\mathcal{L}) = \text{Conf}_{\mathcal{M}}^{P}(\mathcal{L}) = \left\{ \psi \in \mathcal{M}^{P} | y_\psi(a) = y_\psi(b) \right\}.
\]
Proof. The point is that for $\alpha_1, \alpha_2 \in \mathbb{M}$ such that $y_{\alpha_1} = y_{\alpha_2}$ and $\alpha_1 \neq \alpha_2$, the intersection $\mathcal{H}(\alpha_1, 0) \cap \mathcal{H}(\alpha_2, 0)$ is a straight line and more precisely

$$\mathcal{H}(\alpha_1, 0) \cap \mathcal{H}(\alpha_2, 0) = \{ \gamma \mid y_\gamma = y_{\alpha_1} \}.$$ 

First, we prove the inclusion $\text{Conf}_P^\mathbb{M}(L) \subseteq \{ \psi \in \mathbb{M}^P \mid y_{\psi(a)} = y_{\psi(b)} \}$. 

For all $\phi \in \text{Conf}_\mathbb{M}(L)$, $\phi(c) \in \mathcal{H}(\phi(g), 0) \cap \mathcal{H}(\phi(f), 0)$ and $z_{\phi(f)} = z_{\phi(g)}$, so $z_{\phi(c)} = 0$. Likewise, $z_{\phi(d)} = 0$. 

Since $\phi(e) \in \mathcal{H}(\phi(d), 0)$ and $\phi(e) \notin \mathcal{H}(\phi(c), 0)$, we have $y_{\phi(d)} = y_{\phi(e)}$ and $\phi(d) \neq \phi(e)$. 

Therefore, since $\phi(a) \in \mathcal{H}(\phi(d), 0) \cap \mathcal{H}(\phi(e), 0)$, we have $y_{\phi(a)} = y_{\phi(d)}$. Likewise, $y_{\phi(b)} = y_{\phi(d)}$ and finally, $y_{\phi(a)} = y_{\phi(b)}$. 

Now, let us prove the inclusion $\text{Conf}_P^\mathbb{M}(L) \supseteq \{ \psi \in \mathbb{M}^P \mid y_{\psi(a)} = y_{\psi(b)} \}$. Let $\psi \in \mathbb{M}^{(a,b)}$ such that $y_{\psi(a)} = y_{\psi(b)}$. We construct $\phi \in \text{Conf}_\mathbb{M}(L)$ such that $\phi_{\{a,b\}} = \psi$. Let $\phi(d) \in \mathbb{M}$ such that $z_{\phi(d)} = 0$ and $y_{\phi(d)} = y_{\psi(a)}$. Let $\phi(e) = \phi(d) + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\phi(c) = \phi(d) + \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ (in $(x,t)$ coordinates). Then it is easy to check that $\phi \in \text{Conf}_\mathbb{M}(L)$. 

Moreover, it is easy to see that $\text{Reg}_P^\mathbb{M}(L) = \text{Conf}_P^\mathbb{M}(L)$. 

□

5. Linkages for algebraic operations

5.1. The average function linkage. The average function linkage is a linkage with the input set $P = \{a,b\}$ and the output set $Q = \{c\}$ which is a functional linkage for the function

$$f : \mathbb{I}^2 \rightarrow \mathbb{I}$$

$$(x_1, x_2) \mapsto \frac{x_1 + x_2}{2},$$

with $\text{Reg}_P^\mathbb{M}(L) = \text{Conf}_P^\mathbb{M}(L) = \mathbb{I}^P$.

Recall that by “$L$ is a functional linkage for $f$”, we mean that for all $\psi \in \text{Conf}_P^\mathbb{M}(L)$

$$x_{\psi(c)} = \frac{x_{\psi(a)} + x_{\psi(b)}}{2}.$$
The vertices $a$, $b$, $c$ are restricted to move on the line $I$ using $(t_0 = 0)$-line linkages: this means that the linkage in the figure above is combined with three $(t_0 = 0)$-line linkages. Likewise, the points $e$, $d$ and $c$ are restricted to have the same $x$ coordinate using a vertical parallelizer. The square adbe is rigidified. Thus, the actual average function linkage has much more than 5 vertices, but many of them are not represented on the figure above.

It is left to the reader to check that this linkage is the desired functional linkage.

5.2. The adder. The adder is a linkage with the input set $P = \{a_1, b_1\}$ and the output set $Q = \{b_2\}$ which is a functional linkage for the function

\[
\begin{align*}
    f : \mathcal{I}^2 &\rightarrow \mathcal{I} \\
    (x_1, x_2) &\mapsto x_1 + x_2,
\end{align*}
\]

with $\text{Reg}_M^P(\mathcal{L}) = \text{Conf}_M^P(\mathcal{L}) = \mathcal{I}^P$.

It is constructed using two average function linkages $\mathcal{L}_1$ and $\mathcal{L}_2$. Let $W_1 = \{c_1\}$, $\beta(c_1) = c_2$, $F_2 = \{a_2\}$, $\phi_{a2}(a_2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\mathcal{L} = \mathcal{L}_1 \cup_{\beta} \mathcal{L}_2$.

Note that we may obtain a functional linkage for subtraction by setting $P = \{b_2, b_1\}$ and $Q = \{a_1\}$.

5.3. The square function linkage. The square linkage is a linkage with the input set $P = \{a\}$ and the output set $Q = \{b\}$. It is a functional linkage for the function

\[
\begin{align*}
    I &\rightarrow I \\
    x &\mapsto x^2,
\end{align*}
\]

with $\text{Reg}_M^P(\mathcal{L}) = \text{Conf}_M^P(\mathcal{L}) = \mathcal{I}^P$.

To construct it, recall the algebraic trick described by Kapovich and Millson in [KM02]:

\[
\forall x \in \mathbb{R} \setminus \{-0.5, 0.5\} \quad x^2 = 0.25 + \frac{1}{x-0.5} - \frac{1}{x+0.5}.
\]

We have to find another trick to obtain a formula which works for every $x \in \mathbb{R}$.

To do this, notice that for all $x$ and $x'$ in $\mathbb{R}$ we have the identity

\[
x^2 = 2(x + x')^2 + 2(x')^2 - (x + 2x')^2.
\]

Thus the expression $x^2$ can be rewritten

\[
(1) \quad 2\left(0.25 + \frac{1}{x+x'-0.5}\right) + 2\left(0.25 + \frac{1}{x' - 0.5} - \frac{1}{x+0.5}\right) - \left(0.25 + \frac{1}{x+2x'-0.5} - \frac{1}{x+2x'+0.5}\right).
\]

Moreover, for all $x \in \mathbb{R}$ there exists an $x' \in \{-3, -2, \ldots, 3, 4\}$ such that

\[
\{x + x', x + 2x', x'\} \cap \{-0.5, 0.5\} = \emptyset.
\]

Start with a $(t_0 = 0)$-integer linkage $\mathcal{L}_1$: think of the vertex $d_1$ as the number $x'$. Let $\mathcal{L}_2$ be the linkage $\mathcal{L}_1$ to which we add new fixed vertices at $\begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0.25 \\ 0 \end{pmatrix}$, and a new mobile vertex which will represent $x$ and will be the input of the linkage (we do not add any new edge for now). Since Expression 1 is the composition of additions, subtractions and inversions, one may combine $\mathcal{L}_2$ with linkages for addition, substraction and inversion (for the inversion, use the Peaucellier invesor), in the
spirit of Proposition 16, so that the output of the new linkage \( L \) corresponds to Expression 1. This is the desired linkage.

5.4. The multiplier. The multiplier is a linkage with the input set \( P = \{a, b\} \) and the output set \( Q = \{c\} \) which is a functional linkage for the function

\[
f : \mathcal{I}^2 \to \mathcal{I}
\]

\[
(x_1, x_2) \mapsto x_1 x_2,
\]

with \( \text{Reg}^P_{\text{SM}}(L) = \text{Conf}^P_{\text{M}}(L) = \mathcal{I}^P \).

We simply construct the multiplier by combining square function linkages and adders, using the identity

\[
\forall x_1, x_2 \in \mathbb{R} \quad x_1 x_2 = \frac{1}{4} \left( (x_1 + x_2)^2 - (x_1 - x_2)^2 \right).
\]

5.5. The polynomial linkage. Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be a polynomial. We identify \( \mathbb{R} \) with \( \mathcal{I} \).

The polynomial linkage is a functional linkage for the function \( f \) with \( \text{card}(P) = n \) and

\[
\text{Reg}^P_{\text{SM}}(L) = \text{Conf}^P_{\text{M}}(L) = \mathcal{I}^P.
\]

The polynomial linkage is obtained by combining adders and multipliers (we use Proposition 16). The coefficients are represented by fixed vertices.

Note that the restriction map \( p \) is a finite covering over the simply connected set \( \mathcal{I}^P \), so it is a finite trivial covering.

**Example.** To illustrate the general case, we give the following example. Let \( n = 2, m = 1, f(x, y) = 2x^3y + \pi \).

To construct the polynomial linkage for \( f \), start with a linkage \( L \) consisting of two fixed vertices \( a, b \) with \( \phi_0(a) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \) and \( \phi_0(b) = \begin{pmatrix} \pi \\ 0 \end{pmatrix} \), but also two vertices \( c, d \) which are the inputs and correspond respectively to the variables \( x \) and \( y \).

Combine this linkage with a multiplier: the combination mapping \( \beta \) sends \( c \) to one of the inputs of the multiplier and \( d \) to the other one. The linkage (still called \( L \)) is now functional for \( (x, y) \mapsto xy \).

Combine the new linkage with another multiplier: the combination mapping \( \beta \) sends \( c \) to one of the inputs and the output of \( L \) to the other one. The new linkage \( L \) is functional for \( (x, y) \mapsto x^2y \).

Repeating this process once, we obtain a functional linkage for \( x^3y \), and then for \( 2x^3y \) (using the vertex \( c \)).

Finally, combine the linkage \( L \) with an adder: the combination mapping \( \beta \) sends the output of \( L \) to one of the inputs, and \( b \) to the other input.

6. End of the proof of Theorem 3

Let \( n \in \mathbb{N} \). We are given \( A \) a semi-algebraic subset of \( (\mathbb{R}^2)^n \), but we first assume that \( A \) is in fact an algebraic subset of \( (\mathbb{R}^2)^n \), defined by a polynomial \( f : \mathbb{R}^{2n} \to \mathbb{R}^m \) (so that \( A = f^{-1}(0) \)).

Take a polynomial linkage \( L \) for \( f \). We name the elements of the input set: \( P = \{a_1, \ldots, a_{2n}\} \). Fix the outputs to the origin: precisely, replace \( F \) by \( F \cup Q \) and let

\[
\forall a \in Q \quad \phi_0(a) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

\( L \) is not yet the desired linkage: since \( L \) has \( 2n \) inputs, the partial configuration space \( \text{Conf}^P_{\text{M}}(L) \) is a subset of \( (\mathbb{R}^2)^{2n} \) (which is contained in \( \mathcal{I}^{2n} \)), while \( A \) is a subset of \( (\mathbb{R}^2)^n \) (in particular, we are looking for a linkage with \( n \) inputs). To obtain \( \text{Conf}^P_{\text{M}}(L) = A \subseteq (\mathbb{R}^2)^n \), we have to modify \( L \) in the following way.

(1) With several \( (x_0 = 0) \)-line linkages and diagonal parallelizers, extend the linkage \( L \) to a new one with new vertices \( c_2, c_4, c_6, \ldots, c_{2n} \) such that for all realization \( \phi \) and for all \( k \in \{1, \ldots, n\} \)

\[
x_{\phi(c_{2k})} = 0;
\]
\[ y_{\phi}(c_{2k}) = y_{\phi}(a_{2k}) \quad \text{(i.e.)} \quad x_{\phi}(c_{2k}) + t_{\phi}(c_{2k}) = x_{\phi}(a_{2k}) + t_{\phi}(a_{2k}) \].

(2) With several vertical and horizontal parallelizers, extend this linkage to a new one with vertices \( d_2, d_4, d_6, \ldots, d_{3n} \) such that for all realization \( \phi \) and for all \( k \in \{1, \ldots, n\} \)

\[ x_{\phi}(d_{2k}) = x_{\phi}(a_{2k-1}) \]

\[ t_{\phi}(d_{2k}) = t_{\phi}(c_{2k}) \].

Thus, for all realization \( \phi \) and all \( k \in \{1, \ldots, n\} \), we have \( x_{\phi}(d_{2k}) = x_{\phi}(a_{2k-1}) \) and \( t_{\phi}(d_{2k}) = t_{\phi}(a_{2k}) \).

**Figure 4.** A partial realization of the four vertices \( a_1, a_2, c_2, d_2 \). We have \( x_{\phi}(d_2) = x_{\phi}(a_1) \) and \( t_{\phi}(d_2) = x_{\phi}(a_2) \).

Let \( P = \{d_2, d_4, \ldots, d_{2n}\} \). We obtain as desired \( \operatorname{Reg}_{M}^{P}(L) = \operatorname{Conf}_{M}^{P}(L) = A \).

In particular, the restriction map \( p \) is a finite covering. Moreover, it is trivial as the restriction of a trivial covering (see Section 5.5).

Finally, if \( A \) is any semi-algebraic set of \( (\mathbb{R}^2)^n \), then \( A \) is the projection of an algebraic set \( B \) of \( (\mathbb{R}^2)^N \) for some \( N \geq n \). Construct the linkage \( L_1 \) such that \( \operatorname{Conf}_{M}^{P}(L_1) = B \) and remove the unnecessary inputs. Then

\[ \operatorname{Conf}_{M}^{P}(L) = A. \]

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