STRING-LIKE STRUCTURES IN COMPLEX
KERR GEOMETRY

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ABSTRACT. The Kerr geometry is represented as being created by a source moving along an analytical complex world-line. The equivalence of this complex world-line and an Euclidean version of complex strings (hyperbolic strings) is discussed. It is shown that the complex Kerr source satisfies the corresponding string equations. The boundary conditions of the complex Euclidean strings require an orbifold-like structure of the world-sheet. The related orbifold-like structure of the Kerr geometry is discussed.

1. Introduction

Much attention has been paid recently to the relations of the two-dimensional black holes and strings [1]. In this paper we shall consider the four-dimensional Kerr geometry and derive some string-like structures for its complex source.

The Kerr solution [2] is well known as the field of a rotating black hole. However, for the case of a large angular momentum \( |a| = L/m \geq m \) the its horizons are all absent and a naked ring-like singularity appears, along which space branches into two sheets.

This naked singularity has many unpleasant manifestations, and must be hidden inside a rotating disk-like source [3,4]. The solution with \( |a| \gg m \) displays some remarkable features suggesting certain relationships with the spinning elementary particles [3-6]. The real structure of the Kerr metric source has itself some quite exotic properties [4] and contains a string-like closed vortex line resembling the superconducting strings of Nielsen-Olesen and Witten [7].

The aim of this paper is to show some connections between the complex structure of the Kerr geometry and an Euclidean version of string theory. We find, that from a complex point of view the source of the Kerr geometry represents a complex Euclidean string propagating in complex Minkowski space.

We start from the complex structure of the Kerr geometry and describe the complex world-line representation [8]. The Kerr geometry can then be represented as a retarded time solution generated by a source moving along a complex world line. We use the Kerr-Schild formalism [2], which is close to the geometrical twistor description. The necessary conditions for the nonstationary generalization of the complex world-line representation are obtained.
The complex world line is really a two-dimensional surface, equivalent a two-dimensional field theory [9], and may be considered as an intermediate object between a particle and a Euclidean version of a string [10-12]. The peculiarities of these Euclidean strings are analyzed, and it is shown that the boundary conditions of the complex Euclidean string require an orbifold-like structure of the world-sheet. The corresponding orbifold structure of the complex Kerr geometry is discussed. It links the retarded and advanced folds of the complex light cone, which plays a very important role in the structure of the Kerr geometry. Such an orbifold-like structure was recently suggested for two-dimensional black holes by Witten [1].

In the last section we shall briefly discuss some generalizations.

2. The complex world-line representation of the Kerr geometry

We use the Kerr-Schild form of the Kerr-Newman metric [2]

\[ g_{ik} = \eta_{ik} + h_{ik}k_i^k; \] (2.1)

where \( \eta_{ik} = \text{diag}(-1,1,1,1) \) is the auxiliary Minkowski metric in Cartesian coordinates \((x,y,z,t)\), and \( h = (2mr - e^2)/(r^2 + a^2 \cos^2 \theta) \). The vector null field \( k_i(x) \) is a geodesic and shear-free principal null congruence and may be represented in the form

\[ K(x) = k_i dx^i = P^{-1}(du + Yd\xi + Yd\bar{\xi} - Y\bar{Y}dv), \] (2.2)

where the coordinates used are null \( u = (z + t)/\sqrt{2}; \) \( v = (z - t)/\sqrt{2}; \) \( \xi = (x + iy)/\sqrt{2}; \) \( \bar{\xi} = (x - iy)/\sqrt{2}; \) and \( Y \) is a complex projective spinor coordinate, \( Y = \psi^1/\psi^0 \). The complex world line representation was first introduced by Lind and Newman [8]. It shows that the Kerr-Newman solution may be considered as a retarded time solution generated by a "complex point" source [8,6]. The source propagates in complex Minkowski space \( \mathbb{C}M^4 \) along a complex "world line" \( X_o^i(\tau) \), \( i = 0,1,2,3 \), parametrized by a complex time parameter \( \tau = t + i\sigma = X^0(\tau) \). An important role in this retarded time construction is played by the complex light cones \( \mathcal{K} \), whose apexes lie on the complex "world line" \( X_o(\tau) \).

2.1 Complex light cone

The complex light cone \( \mathcal{K} \), written in spinor form

\[ \mathcal{K} = \{ x : x = X_o^i(\tau) + \psi_L^i \sigma_{AA}^i \bar{\psi}_R \} \] (2.3)

may be split into two families of null planes: "left" (\( \psi_L \) =const; \( \bar{\psi}_R \) -var.) and "right" (\( \psi_R \) =const; \( \psi_L \) -var.). These are the only two-dimensional planes which are wholly contained in the complex null cone. The rays of the principal null congruence \( K \) of the Kerr geometry are the tracks of these complex null planes (right or left) on the real slice of Minkowski space. The light cone equation in the Kerr-Schild metric coincides with the corresponding equation in Minkowski space because the null directions \( k^i \) are null in both metrics, \( g_{ij} \) and \( \eta_{ij} \).

This splitting of the light cone on the null planes has a close connection with the twistor. By introducing the projective parameter \( Y = \psi^1/\psi^0 \) the complex light cone equation

\[ (\xi - \xi_o)(\bar{\xi} - \bar{\xi}_o) = -(u - u_o)(v - v_o) \]
splits into two linear equations\(^1\)

\[
\begin{align*}
\xi - \xi_o &= Y(v - v_o), \\
-Y(\bar{\xi} - \bar{\xi}_o) &= (u - u_o),
\end{align*}
\]  
\(\text{(2.4)}\)

describing the "left" complex null planes (the null rays in the real space). Another splitting

\[
\begin{align*}
-\bar{Y}(\xi - \xi_o) &= (u - u_o), \\
(\bar{\xi} - \bar{\xi}_o) &= \bar{Y}(v - v_o),
\end{align*}
\]  
\(\text{(2.5)}\)

gives the "right" complex null planes. The three parameters

\[
Y, \quad \lambda_1 = u + Y\bar{\xi}, \quad \lambda_2 = \xi - Yv 
\]  
\(\text{(2.6)}\)

are indeed projective twistor coordinates.\(^2\)

The equations for the "left" null planes of the complex light cone with apex at the point \(X_o\) take the form

\[
\begin{align*}
\lambda_1 &= \lambda_1^o = u_o + Y\bar{\xi}_o, \\
\lambda_2 &= \lambda_2^o = \xi_o - Yv_o.
\end{align*}
\]  
\(\text{(2.7)}\)

The "left" null planes of the complex light cones form a complex congruence or regulus which generates all the rays of the principal null congruence \(K\) in real space. The ray with polar direction \(\theta, \phi\) is the real track of the "left" plane corresponding to \(Y = \exp i\phi \tan(\theta/2)\) and belonging to the cone which is placed at the point \(X_o\) corresponding to \(\sigma = a \cos(\theta)\). The parameter \(\sigma = Im\tau\) has a meaning only in the range \(-a \leq \sigma \leq a\) where the cones have real slices.

Thus, the complex world line \(X_o(t, \sigma)\) represents a restricted two-dimensional surface or strip, in complex Minkowski space, and is really a world-sheet. It may be considered as a complex open string with a Euclidean parametrization \(\tau = t + i\sigma, \bar{\tau} = t - i\sigma\), and with end points \(X_o(t, \pm a)\).

### 2.2 The complex world lines and the open Euclidean strings

The complex world line, or world-sheet, is an intermediate object between particle and string [10,11].\(^3\) The complex source of the Kerr-Newman solution corresponds to a straight analytical world-line in the complex Minkowski space \(CM^4\)

\[
X_o^i(\tau) = X_o^i(0) + U_o^i\tau. \quad \text{(2.8)}
\]

In the Euclidean version, the general string solution has to be a sum of the left and right modes, taking the form

\[
X^i(t, \sigma) = X_L^i(\tau) + X_R^i(\bar{\tau}). \quad \text{(2.9)}
\]

\(^1\)It is a generalization of the Veblen and Ruse construction [13] which has been used for the geometrical representation of spinors.

\(^2\)In the standard twistor form \((1, Y, \lambda_1, \lambda_2) = (Z^0, Z^1, Z^2, Z^3)/Z^0\), where \(Z = (\psi^B, \pi_A)\); \(\pi_A = X_{AB}^i\psi^B\).

\(^3\)The corresponding complex Euclidean strings have also been considered in recent papers of Oogury and Vafa [12], where they were called "hyperbolic".
It seems most natural to use the complex conjugate world-line as the right mode, leading to the real string solution \( X^i(t, \sigma) \) However, the string solutions have also to satisfy the constraints

\[
(\partial_\tau X)^2 = 0; \quad (\partial_{\bar{\tau}} X)^2 = 0,
\]

(2.10)

where

\[
\tau = t + i\sigma; \quad \bar{\tau} = t - i\sigma; \quad \partial_\tau = (\partial_t - i\partial_\sigma)/2; \quad \partial_{\bar{\tau}} = (\partial_t + i\partial_\sigma)/2,
\]

(2.11)

and the boundary conditions for \( \sigma = \pm a \) are

\[
\text{Re}\left[\frac{\partial L_e}{\partial (\partial_\tau X^i)}\right]|_{\sigma=\pm a} = 0.
\]

(2.12)

This equations puts certain strong restrictions on the solutions, with the result that the real solutions for the open Euclidean strings are absent. Nevertheless, one can find a consistent formulation of the complex closed Euclidean strings, in which the complex analytical straight worldlines form closed strings satisfying the corresponding full system of string equations.

3. Closed complex Euclidean strings with an orbifold-like world-sheet

One can consider Euclidean strings in complex Minkowski space \( CM^4 \) as a complex objects with a Hermitian Lagrangian,

\[
L = -\eta_{ij}(\partial_\tau X^i \partial_\tau \bar{X}^j + \partial_\tau X^i \partial_{\bar{\tau}} \bar{X}^j),
\]

(3.1)

The general solution is given as a sum of the right and left modes,

\[
X^i(t, \sigma) = X^i_L(\tau) + X^i_R(\bar{\tau}),
\]

(3.2)

which are not necessarily complex conjugates of each other. The constraints take the form

\[
\eta_{ij}\partial_\tau X^i_L \partial_\tau \bar{X}^j_R = 0; \quad \eta_{ij}\partial_\tau X^i_L \partial_{\bar{\tau}} \bar{X}^j_R = 0.
\]

(3.3)

An oscillator expansion of the Euclidean string is

\[
X_L = X_L(0) + P\tau + \omega^{-1} \sum (1/n)\alpha_n \exp\{-in\omega\tau\},
\]

\[
X_R = X_R(0) + P\bar{\tau} + \omega^{-1} \sum (1/n)\tilde{\alpha}_n \exp\{-in\omega\bar{\tau}\},
\]

(3.4)

where a periodic time dependence is proposed with a period \( T = 2\pi/\omega \). The expansion contains the hyperbolic basis function, which are not orthogonal over the string length. For the parameter \( \sigma \) it is convenient to use an interval \([-a, a]\). The complex Euclidean strings can not be open, since the boundary conditions for open strings, \((\partial_\sigma X)^2|_{\sigma=\pm a} = 0\) lead only to a trivial solution.

Attempts to introduce boundary conditions for closed complex Euclidean strings meet obstacles too, because it is impossible to introduce the same boundary conditions for the real and imaginary part of a string. However, the problem may be resolved when the world sheet admits an orbifold-like structure [11,14,15]. The general idea is
very simple: it is necessary to modify the world-sheet to include the twisting boundary conditions for the imaginary part of the complex string. A more precise formulation is given by the following.

To construct an orbifold it is necessary to double the interval \( \Sigma = [-a, a] \) for the parameter \( \sigma = \text{Im} \tau \). Two copies of the interval, \( \sigma^+ \in \Sigma^+ = [-a, a] \) and \( \sigma^- \in \Sigma^- = [-a, a] \), must be joined to form an oriented circle, \( S^1 = \Sigma^+ \cup \Sigma^- \) which may be parametrized by a periodic coordinate \( \theta \sim \theta + 2\pi \), and represents a covering space for the original interval \( \Sigma \). For an explicit correspondence one can use the projection \( S^1 \) on \( \Sigma \),

\[
\sigma = a \cos \theta. \tag{3.5}
\]

The original string is then parametrized by \( 0 \leq \theta \leq \pi \), and \( \pi \leq \theta \leq 2\pi \) covers the string a second time in the opposite direction.

A group \( G = \mathbb{Z}_2 \); \( g \in G \), acts on the circle as \( g\theta = \theta + \pi \); \( g^2 = 1 \), and induces the transformation

\[
g\sigma^+ = -\sigma^-; \quad g\sigma^- = -\sigma^+; \quad g\tau^+ = \bar{\tau}^-; \quad g\tau^- = \bar{\tau}^+. \tag{3.6}
\]

The string modes \( X_L(\tau_L) \) and \( X_R(\bar{\tau}_R) \) can be extended on the circle for \( \pi \leq \theta \leq 2\pi \) to form a closed string by means the well-known kind of extrapolation [15]

\[
X_L(g\tau_+) = X_R(\bar{\tau}_+); \quad X_R(g\bar{\tau}_+) = X_L(\tau_+) \tag{3.7}
\]

The group \( G \) is a symmetry of the theory and therefore acts on the string variables,

\[
gX_L(\tau_\pm) = X_L(g\tau_\pm) = X_R(\bar{\tau}_\pm), \quad gX_R(\bar{\tau}_\pm) = X_R(g\bar{\tau}_\pm) = X_L(\tau_\pm).
\]

One can see that the group acts by interchanging the left and right modes. The time coordinate \( t \) may be transformed by projection on the circle \( S^1 \) so that the world-sheet forms the torus \( T^2 = S^1 \times S^1 \). The action of the time reversing element \( h \), of \( \mathbb{Z}_2 \) is \( ht = -t, \quad h^2 = 1 \). The elements \( g \) and \( h \) commute with each other and act together as \( gh\tau_L = -\tau_R; gh\tau_R = -\tau_L; \quad h\tau_L = -g\tau_R; h\tau_R = -g\tau_L \). The quotient space \( T/\mathbb{Z}_2 \) is the orbifold. The action \( g \) on the orbifold is defined by the projection (3.5)

\[
g\sigma = -\sigma, \quad g\tau = \bar{\tau}, \quad gX(\tau) = X(g\tau) = X(\bar{\tau}). \tag{3.8}
\]

The actions of \( gh \) and \( h \) are

\[
gh\tau = -\tau, \quad gh\bar{\tau} = -\bar{\tau}, \quad h\tau = -g\tau, \quad h\bar{\tau} = -g\bar{\tau}. \tag{3.9}
\]

The Hilbert space of the string solutions \( H \) on the orbifold have to be decomposed on two subspaces: the \( H_+ \) space of the even functions,

\[
X_{ev}(t, \sigma) = (X + gX)/2, \quad gX_{ev} = X_{ev}, \tag{3.10}
\]

and the \( H_- \) space of the odd functions,

\[
X_{odd}(t, \sigma) = (X - gX)/2; \quad gX_{odd} = -X_{odd}. \tag{3.11}
\]
As a result the even solutions are closed on the orbifold and have the form
\[ X_{ev} = (X_L(\tau) + X_R(\bar{\tau}) + X_L(\bar{\tau}) + X_R(\tau))/2, \] (3.12)
whereas the odd solutions,
\[ X_{odd} = (X_L(\tau) + X_R(\bar{\tau}) - X_L(\bar{\tau}) - X_R(\tau))/2, \] (3.13)
form the closed strings on the orbifold with twisted boundary conditions. A general complex string solution on the orbifold may be represented as the sum of the even and odd parts
\[ X(t, \sigma) = X_{ev} + X_{odd} = (X + \tilde{g}X)/2 + (X - \tilde{g}X)/2. \] (3.14)
The \( gh \) transformations do not mix the analytic and antianalytic parts of the functions on the orbifold, but the functions may be decomposed into the even and odd parts with respect of the \( gh \) transformations \( X(\tau) = X_{gh^+}(\tau) + X_{gh^-}(\tau) \). In the terms of the \( gh \) decomposition the general even and odd solutions for a closed string are
\[ X_{ev} = 2ReX_{gh^+} + 2iImX_{gh^-}; \quad X_{od} = 2ReX_{gh^-} + 2iImX_{gh^+}. \] (3.15)

The complex constraints (3.3) then take the form
\[ g_{ij}\left[ \partial_\tau X^i \partial_\tau X^j + \partial_\nu JmX^i \partial_\nu JmX^j \right] = 0, \] (3.16)
which is a generalization of the real constraints, taking into account that the imaginary components are the independent degrees of freedom.

These constraints admit one interesting class of solutions with \( X_R = 0 \),
\[ X^i(t, \sigma) = X^i_L(\tau), \] (3.17)
which can be seen from the complex form of constraints (3.3). These solutions contain only left modes like the Schild null strings or the bosonic sector of the heterotic string, and correspond to analytical world-sheets. In spite of the absence of a right mode the solutions are nontrivial and contain both even and odd parts. It is interesting that the straight analytical world-line corresponding to the stationary Kerr solution satisfies the constraints as well as the arbitrary analytical complex world-lines.\(^4\) Thus, the world-line corresponding to the complex source of the Kerr solution satisfies the string equations on the orbifold.

4. **Orbifold-like structure of the complex Kerr geometry**

To obtain a real cut of the complex Kerr-Schild space for every \( \tau \) we shall use a null vector
\[ \mathcal{M}^i = \psi^i\bar{\psi} \] (4.1)
of the complex light cone, linking points of the complex world line \( X_o(\tau) \) with points of the real slice. If a cone has a real slice, than for every normalized spinor \( \psi \) (with \( \psi\bar{\psi} = 1 \)) there exists a spinor
\[ \tilde{\psi} = \sigma^a(X^a_o - \bar{X}^a_o)\bar{\psi}, \quad (a = 1, 2, 3), \] (4.2)
\(^4\)In the case of real strings the stationary solutions do not satisfy the constraints.
such that $X_o + \mathcal{M}$ is a point on the real slice, i.e.

$$X_o + \mathcal{M} = \bar{X}_o + \bar{\mathcal{M}}. \quad (4.3)$$

The explicit form of $\mathcal{M}$ is the following

$$\mathcal{M}^i = (X_o^i - \bar{X}_o^i)/2 + i\epsilon^{i o j k} k_k (X_{o j} - \bar{X}_{o j})/2, \quad (4.4)$$

where $k_i = \psi \sigma_i \bar{\psi}$ is the real vector of the principal null congruence. One can also add to $\mathcal{M}$ a real vector directed along the null ray $r k^i$. The null vectors $k^i$ and $\mathcal{M}^i$ belong to the "left" null plane.

One can find even and odd structures in the Kerr geometry. The real "center" of the Kerr geometry, $X_{re}(t, \sigma) = (X_{o L}(\tau) + \bar{X}_{o L}(\bar{\tau}))/2$, is an even object. The imaginary part of the world-line $X_{im}(t, \sigma) = (X_{o L}(\tau) - \bar{X}_{o L}(\bar{\tau}))/2$ has odd parity with respect to complex time, but it also contains an even constant $X_{o L}(0)$. The imaginary degrees of freedom in the Kerr geometry may be included in a compactification scheme based on the constrained $CP^3$ sigma-model [16]. The light cones, which are adjoined to every point of the complex world line, can be considered as an "internal" space of the Kerr string in analogy with the compactification in string models.

One can introduce relative coordinates for the real points of the Kerr-Schild space,

$$Z^i = X^i - X^i_o, \quad i = 0, 1, 2, 3, \quad (4.5)$$

which include the imaginary part of the complex world-line $Im Z^i = -Im X^i_o$ and satisfy the light cone constraints,

$$Z_i Z^i = 0, \quad (4.6)$$

In the constrained sigma-model the $Z^i$ are considered as homogeneous coordinates for the complex projective space $CP^3$,

$$\xi^\alpha = Z^\alpha / Z^0, \quad \alpha = 1, 2, 3. \quad (4.7)$$

The space may be endowed with a Kähler metric in the standard manner,

$$g_{\alpha \bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \mathcal{M}^{fs}, \quad (4.8)$$

It is a metric on the compact space of the complex null rays defined by the Fubini-Study potential,

$$\mathcal{M}^{fs} = \ln(1 + \xi \bar{\xi}). \quad (4.9)$$

For every real point of the Kerr-Schild space $X^i$ one can find both the "left" retarded time $\tau_{L}$ and a point of the complex world-line $X_{o L}(\tau_{L})$ defined by the intersection of the world-line with the "left" null plane $(2.4)$. This plane is spanned by the null vectors $\mathcal{M}^i$ and $k^i$. The "right" plane $(2.5)$ contains the vector $\mathcal{M}^i$, but it also defines the congruence $l^i = \bar{\psi} \sigma^i \bar{\psi}$ through the equation $\bar{Y} = \psi / \bar{\psi}$. It should be mentioned that the "right" and "left" planes change places under the action of the gh - transformation $(v - v_o) \leftrightarrow (u - u_o)$ and a redefinition $\bar{Y} = -1/Y$. On the other hand the gh-transformations change the retarded and advanced folds of the complex light cone, as can be seen from the alternative splitting of the complex light cone,

$$t - \tau = \pm \bar{\tau}; \quad gh \bar{\tau} = -\bar{\tau}, \quad (4.10)$$
where \((\bar{r})^2 = (x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2\) When \(t\) is real, the transformation \(g\) acts on \(\bar{r}\) as complex conjugation. These transformations induce the orbifold structure on the \(CP^3\) by defining an equivalence between positive and negative folds relating \(Z^0\) to \(-Z^0 = g h Z^0\), and by introducing two new folds of the orbifold corresponding to \(\pm Z^0 = g(\pm Z^0)\). This orbifold may be described in \(CP^3\) by the quartic equation

\[
[Z_1^2 + Z_2^2 + Z_3^2 - g Z_0^2](Z_1^2 + Z_2^2 + Z_3^2 - Z_0^2) = 0. \tag{4.11}
\]

This equation corresponds to a peculiar case of the \(K3\) surface, a known example of the Calabi-Yau manifolds \([14,18]\). Notice, that the complex light cone plays an important role in the structure of the Calabi-Yau spaces. It is a "conifold" \([17]\), or a singular limit of many Calabi-Yau spaces. The procedure of "small resolution" of the singularities is produced by introducing the extra \(CP^1\) space \(Y = \psi^1/\psi^0\) and splitting the constraints \([17]\). One can note, that this is an exact copy of the above (see eq.(2.4), (2.5)) splitting of the complex light cone on the left (or right) twistor null planes. It not only removes the singularity but also annihilates the multiple connectedness caused by the group of discrete symmetry.

5. Generalizations

We will discuss two generalizations of the previous sections:

5.1 A nonstationary generalization of the complex world-line representation

The complex Kerr source corresponds to the straight analytical word-line, but from the point of view of string excitations the oscillating world lines are very interesting. The problem of the existence of corresponding solutions is open. One of the problems involved in obtaining the nonstationary generalizations of the Kerr-Schild solutions is the question of the existence of the corresponding principal null congruencies (which have to be both geodesic and shear-free).

In this section we shall briefly discuss the nonstationary generalization of congruencies, following \([20]\). According to Kerr’s theorem \([2,19]\) the geodesic and shear-free congruencies are defined by eq.(2.2), where the function \(Y(X)\) is a solution of the equation

\[
F(Y, \lambda_1, \lambda_2) = 0, \tag{5.1}
\]

\(F\) being an analytical function. The singularities of the solutions occur as caustics in \(Y\), satisfying the equations \(F = 0; \quad \partial_Y F = 0\). In the case when the singularities are confined to a bounded region \([21,6]\) the \(F\) is, at most, quadratic in \(Y\) and the equation \(F(Y) = 0\) may be solved explicitly, giving an explicit form for the principal null congruence. The function \(F\) may be represented in a form

\[
F = (\partial_Y \lambda_1^0)(\lambda_2 - \lambda_2^0) - (\partial_Y \lambda_2^0)(\lambda_1 - \lambda_1^0), \tag{5.2}
\]

which is convenient for the nonstationary generalizations because the parameters \(\lambda_A^0\) and \(\partial_Y \lambda_A^0\) are the functions of the complex world line parameters \(X_o(\tau)\) and therefore depend on \(\tau\).
Following the general scheme described in [2], one can show that the shear-free and geodesic conditions for the congruence

\[(\partial_{\bar{\xi}} - Y \partial_u)Y = 0; \quad (\partial_\nu + Y \partial_{\bar{\xi}} + \bar{Y} \partial_{\xi} - Y \bar{Y} \partial_u)Y = 0.\]  

(5.3)

lead to a differential equation,

\[Z^{-1}(\bar{Y} - \phi)dY = \phi(d\xi - Y dv) + (du + Y d\bar{\xi}),\]  

(5.4)

where \(Z = (\partial_{\xi} - \bar{Y} \partial_{u})Y\) is the complex expansion of the congruence, and \(\phi\) is an arbitrary solution of the equation

\[(\partial_{\bar{\xi}} - Y \partial_u)\phi = 0.\]  

(5.5)

By using the form (5.2) one can obtain the following equations which are necessary for the congruence to be both geodesic and shear-free,

\%(\partial_{\bar{\xi}} - Y \partial_u)\tau = 0;\]  

(5.6)

\[\lambda_1 = \lambda_1^0(\tau); \quad \lambda_2 = \lambda_2^0(\tau).\]  

(5.7)

These equations may be considered as constraints on the retarded time parameter \(\tau\). Equations (5.7) are equivalent to (2.4) and imply that the points \(X\) and \(X_o(\tau)\) belong to the same "left" null plane of the complex light cone. Consequently, the retarded time \(\tau = \tau_L(X)\) have to be defined by the point of intersection of the "left" null plane with the world line \(X_o\), or by the "left" projection of the \(X\) on the world line along the corresponding "left" null planes. Thus, in the nonstationary case the complex world line representation acts via the "left" retarded time \(\tau_L\), which must be used in the parameters of equation (5.2).

Equation (5.6) gives an extra necessary condition on the world-line \(X_o(\tau)\). It has to be an analytical function of \(\tau\). Remarkably, this condition coincides with the necessary condition on the world-line to be a solution of the complex Euclidean string equations. In the real space these nonstationary solutions for the congruence \(K\) correspond to a moving singular ring, the displacement and orientation of which depend on the real and imaginary parts of the complex world line \(X_o(\tau_L)\).

5.2 Super-extension.

One can extend complex time parameter on the complex superspace by introducing the (2,0)- world-sheet supersymmetry. The complex (2,0)- supertime coordinates [22] \(\tau, \bar{\tau}, \xi^+, \bar{\xi}^+\) may be unite in the ciral superfield,

\[T = \tau + \eta_+(\tau, \bar{\tau})\xi^+ + i\xi^+\bar{\xi}^+,\]  

(5.8)

and then the analytical complex world-line turns out to be the chiral superfunction of supertime,

\[X^i(T) = x^i(\tau) + \eta^i_+(\tau, \bar{\tau})\xi^+ + i\xi^+\bar{\xi}^+\partial_{\tau}x^i(\tau).\]  

(5.9)
The $(2,0)$-supercovariant derivatives are
\[ D_+ = \partial_{\xi^+} + i\bar{\xi}^+ \partial_\tau; \quad \bar{D}_+ = \partial_{\bar{\xi}^+} + i\xi^+ \partial_\tau. \tag{5.10} \]

The superfield action corresponding to (3.1) takes the form
\[ S = \frac{-1}{4\pi\alpha'} \int d\tau d\bar{\tau} d\xi d\bar{\eta}_i d\chi^i (D_+ D_+ \chi^i) = \]
\[ = \frac{-1}{\pi\alpha'} \int d\tau d\bar{\tau} \left[ (\partial_\tau x^i_o) (\partial_\tau x^i_o) - i(\bar{\eta}_i + \partial_\tau \eta^i_+) - (\bar{\eta}_i + \partial_\tau \eta^i_+) \right] \tag{5.11} \]
and gives rise to the Klein-Gordon action for $x^i_o$ and a Dirac action for $\eta^i_+$.  

There exists a hint that the problem of the nonstationary generalization of the Kerr-Newman solution and the problem of its superextension are related problems.

It is known that the dynamical extension of the Kerr-Newman solution runs into many serious problems. One of the obstacles is connected with the radiative character of such solutions and in consequence with the corresponding violation of the flat asymptotic behavior of the space-time, since the stress tensor does not decrease rapidly enough at the large distances. One can hope that the corresponding supersolutions could be asymptotically flat, because in the vacuum region, at large distances from source, the supersymmetry could be restored. This ought to guarantee both the vanishing of the stress tensor and the asymptotically flat behavior at infinity, in spite of the radiative character of nonstationary solutions.

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