PHASE TRANSITION OF OSCILLATORS AND TRAVELLING WAVES IN A CLASS OF RELAXATION SYSTEMS

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Abstract. The main purpose of this article is to investigate the phase transition of oscillation solutions and travelling wave solutions in a class of relaxation systems as follows

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \pm u(u-a)(u-b) - v + D \frac{\partial^2 u}{\partial x^2}, \quad a \neq b, \\
\frac{\partial v}{\partial t} &= \varepsilon (mu + nv + p), \quad 0 < \varepsilon \ll 1,
\end{aligned}
\]

where \(a, b, m, n, p\) are parameters in this system. By using the orbit analysis method of planar dynamical system and the homoclinic bifurcation theory, the phase transitions of the solitary oscillators, kink oscillators, periodic oscillators and travelling waves in the relaxation system above are studied. Various critical parameters of the phase transition are obtained under different parametric conditions, while various sufficient conditions to guarantee the existence of the above oscillation solutions and travelling waves are given. As some applications, this paper studied the FitzHugh-Nagumo equation, the van der Pol-equation and the Winfree generic system.

1. Introduction. The relaxation oscillator is a type of limit cycle which repeatedly alternates between two states and also alternates the speeds between fast and slow. It is applied to dynamical systems in diverse areas of science such as geothermal geysers, networks of firing nerve cells, thermostat controlled heating systems, chemical reactions and the beating human heart, see [2], [17], [13]. The general form of the oscillation system is widely studied in [14] and [8], which is shown as follows

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= f(u, v, \lambda), \\
\frac{\partial v}{\partial t} &= \varepsilon g(u, v, \lambda), \quad 0 < \varepsilon \ll 1,
\end{aligned}
\]

where \(u\) is called fast variable, and \(v\) is slow variable, \(\lambda = \lambda(\lambda_1, \lambda_2, \lambda_3, \ldots)\) is the parameter in the system (1). More detailed introductions and results of (1) can be found in [11] and [6]. In this paper we shall focus on the relaxation oscillators of the following class

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \pm u(u-a)(u-b) - v + D \frac{\partial^2 u}{\partial x^2}, \quad a \neq b, \\
\frac{\partial v}{\partial t} &= \varepsilon (mu + nv + p), \quad 0 < \varepsilon \ll 1.
\end{aligned}
\]

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The system (2) generates from many famous relaxation oscillation systems including the Fitzhugh-Nagumo model (see [11], [5]), the van der-Pol equation (see [14]) and the Winfree generic system (see [17]), moreover some physical phenomena tell us there are relaxation oscillators in these systems under some parametric conditions.

As we know, the system with homoclinic orbit or heteroclinic orbit is structurally unstable, so we can deduced that the solitary oscillation solutions and the kink oscillation solutions may disappear for the perturbations of the system (2). Based on the orbit analysis theories including the homoclinic bifurcation theory we established a method to study the phase transition of the oscillation solutions and travelling waves in (2). Various critical parameters of the phase transition are obtained, and also various sufficient conditions to guarantee the existence of the different types of oscillation solutions are given. The main conclusions in this paper is shown as follows:

1. In system (1) let the parameter \( D = 0 \) and the parameters \( b, m, n, p \) be constant. If \( a \) is an adjusted parameter for the system (2), then \( a = 0 \) is a critical point of the system (1), and there is a fast-slow relaxation oscillator bifurcates from the point \( a = 0 \).

2. In system (1) let the parameter \( D = 0 \) and the parameters \( a, b \) be constant, and \( m, n, p \) are the adjusted parameters of the system (2), with the application to Andronov-Leontovich Theorem, there are two critical points \( p_0 = \frac{mAB + nA}{n} \) and \( p_1 = \frac{mAB + nA}{n} \) in this case, and there is a fast-slow relaxation oscillator bifurcates from the point \( p_0 \) and \( p_1 \).

3. Note \( D = 0 \), the parameters \( a, b \) be constant, if \( m, n \) are the adjusted parameters of the system (2), then some bifurcation theorems of the oscillators are also obtained in this case.

4. Let \( D \neq 0 \), let the parameters \( b \) be constant, and \( a \) is the an adjusted parameter of the system (2), then the bifurcation theorem of the travelling waves is given and we applied it to study the Fitzhugh-Nagumo equation.

As the applications of the method we obtained, we study the following equations, they are all famous relaxation oscillation systems in nonlinear science. The theorems in (1), (2), (3), (4) are applied to the model FHN model, Van der Pol equation, and the Winfree generic system.

2. The phase transition of oscillators and travelling waves.

2.1. Preliminary knowledge. Before presenting our results, we give some background ideas and notations, which can also be found in [14], [1], [15]. Relaxation oscillators are characterized by two alternating processes on different time scales: a long relaxation period during which the system approaches an equilibrium point, alternating with a short impulsive period in which the equilibrium point shifts, the typical relaxation oscillators approach \( u(x, t) \) is a type of limit cycle of (1) which repeatedly alternates between two states and also alternates the speeds between fast and slow. Suppose that \( u(x, t) \) is a continuous solution of (1) for \( t \in (-\infty, +\infty) \), and \( \lim_{t \to \infty} u(x, t) = c, \lim_{t \to -\infty} u(x, t) = d \), where \( c \) and \( d \) are both equilibrium states of (1). It is well known that

\( \text{(D1)} \) \( u(x, t) \) is called a solitary oscillator solution if \( c = d \), which is a homoclinic
orbit in the phase plane.

(D2) \( u(x, t) \) is called a kink oscillator solution if \( c \neq d \), which is a heteroclinic orbit in the phase plane.

(D3) \( u(x, t) \) is called a periodic oscillator if and only if it is a limit circle in the phase plane.

The homoclinic orbit in system (1) leads to the existence of the solitary oscillation solution of both \( u \) and \( v \) in (1), and the heteroclinic orbit of the system (1) leads to the existence of the kink oscillation solution of both \( u \) and \( v \) in (1), which can be shown in Figure 1.

2.1.1. The homoclinic bifurcation. Consider a 2-D autonomous system of ordinary differential equations

\[
\begin{align*}
\frac{\partial u}{\partial t} &= f(u, v, \lambda), \\
\frac{\partial v}{\partial t} &= g(u, v, \lambda),
\end{align*}
\]

where \( \lambda \) is a parameter, \( f \) and \( g \) are smooth functions, and \((u, v) \in \mathbb{R}^2\). Denote by \( \varphi^t_\lambda \) the flow of the (3), let \((u_0, v_0)\) be an equilibrium of the system (3) and it is a saddle point, let the eigenvalues be \( \mu_1(0), \mu_2(0) \) at \( \lambda = 0 \), and \( \mu_2(0) < 0 < \mu_2(0) \). The saddle quantity \( \sigma_0(\sigma_0 = \mu_1(0) + \mu_2(0)) \) of a hyperbolic equilibrium is the sum of the real parts of its eigenvalues, where \( \mu_2 \) is an unstable eigenvalue and \( \mu_1 \) is a stable eigenvalue.

An orbit \( \Gamma^0 \) starting at a point \((u, v)\) in \( \mathbb{R}^2 \) is called homoclinic orbit to the equilibrium point \((u_0, v_0)\) of (3) at \( \lambda = 0 \) if \( \lim_{t \to \pm \infty} \varphi^t_\lambda(u, v) = (u_0, v_0) \). Generically, we assume \( \sigma_0 \neq 0 \) in the system (3). A homoclinic orbit to a hyperbolic equilibrium of (3) is structurally unstable, which means that the phase portrait in the neighborhood of \( \Gamma^0 \cup (u_0, v_0) \) becomes topological nonequivalent to the original one, as we shall see, in [7], it is shown that the homoclinic orbit simply disappears for perturbations of the system.

The presence of a homoclinic orbit implies a global codimension-one bifurcation of (3) at \( \lambda = 0 \), since the homoclinic orbit disappears for all sufficiently small perturbation. Moreover, the disappearance of a homoclinic orbit leads to the creation or destruction of one (or more) limit cycle nearby. When such a cycle approaches the homoclinic orbit \( \Gamma^0 \) as \( \lambda \to 0 \), its period tends to infinity, which is shown in Figure 2, a homoclinic orbit forms at \( \lambda = 0 \), and a periodic orbit exists for \( \lambda > 0 \) but not for \( \lambda < 0 \). As \( \lambda \to 0^+ \), the period \( T \) of the periodic orbit diverges to infinity.

The Andronov–Leontovich Theorem (see [2], [13], [16]) is a basic tool to study the bifurcation of the oscillators, which is shown as follows.
Theorem 2.1. (Andronov-Leontovich, 1971) For any generic one-parameter system (3) having a saddle equilibrium point \((u_0, v_0)\) with a homoclinic orbit \(\Gamma^0\) at \(\lambda = 0\), the separated function is \(\beta(x)\) (which is defined in [14]), if the following assumptions hold true

(A1) The saddle quantity \(\sigma^0 = \mu_1(0) + \mu_2(0) \neq 0\);
(A2) \(\beta'(0) \neq 0\);

then there exists a neighbourhood of \(\Gamma^0 \cup (u_0, v_0)\) in which a unique limit cycle \(L_{\lambda}\) bifurcates from \(\Gamma^0\) as \(\lambda\) passes through zero.

2.1.2. The stability of the orbits. In the system (3), at the case \(\lambda = 0\), if the saddle quantity \(\sigma^0\) of the homoclinic orbit \(\Gamma^0\) is negative, then \(\Gamma^0\) is a stable orbit, in the other word, it can attract the orbits around it, and also the limit cycle \(L_{\lambda}\) bifurcates from \(\Gamma^0\) is stable. However, if the saddle quantity \(\sigma^0\) is positive, then \(\Gamma^0\) is an unstable orbit, and the limit cycle \(L_{\lambda}\) bifurcates from \(\Gamma^0\) is unstable. The stability of the orbits leads to the stability of the oscillators, which means, the stable oscillator attracts the oscillators around it as \(t \to +\infty\), the unstable oscillator repels the oscillators around it as \(t \to +\infty\).

2.2. The phase transition of oscillation solutions. In the case \(D = 0\), we study the phase transition of oscillation solutions in the following two dynamical systems:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -u(u-a)(u-b) - v, \quad a \neq b, \\
\frac{\partial v}{\partial t} &= \varepsilon (mu + nv + p), \quad 0 < \varepsilon \ll 1.
\end{align*}
\]

And

\[
\begin{align*}
\frac{\partial u}{\partial t} &= u(u-a)(u-b) - v, \quad a \neq b, \\
\frac{\partial v}{\partial t} &= \varepsilon (mu + nv + p), \quad 0 < \varepsilon \ll 1.
\end{align*}
\]

In the system (4), remark that the curve \(\Gamma = (u, v) : v = -u(u-a)(u-b)\), and the line \(\ell = (u, v) : mu + nv + p = 0\).

In the system (4), remark that \(f(u) = -u^3 + (a+b)u^2 - abu\), \(u_1\) and \(u_2\) are the roots of the equation \(f'(u) = 0\), where

\[
\begin{align*}
u_1 &= \frac{1}{3}(a + b - \sqrt{a^2 + b^2 - ab}), \\
u_2 &= \frac{1}{3}(a + b + \sqrt{a^2 + b^2 - ab}).
\end{align*}
\]

Note \(f(u_1) = A = -u_1^3 + (a+b)u_1^2 - abu_1\) and \(f(u_2) = B = -u_2^3 + (a+b)u_2^2 - abu_2\), \(u_A\) is a root of the equation \(f(u) = A\) such that \(u_A \neq u_1\), and also \(u_B\) is a root of the equation \(f(u) = B\) such that \(u_B \neq u_2\). Then we have the following theorems:
Theorem 2.2. In the system (4), let \( m, n, a, b \) be constant, and \( m > 0, n > 0, a^2 + b^2 \neq 0 \), if \( p \) is an adjusted parameter in the system (4), then we have the following conclusions:

(1) If the system (4) satisfies the following conditions:
\[
\begin{align*}
-\frac{m}{n} &> f'(u_A), \\
-\frac{m}{n} u_2 + \frac{mu_A + nA}{n} &< B, \\
-\frac{m}{n} u_B + \frac{mu_B + nB}{n} &> B,
\end{align*}
\]
then \( p = p_0 = \frac{mu_A + nA}{n} \) is the critical point of the system (4), namely if \( p = p_0 \), there is a solitary oscillator solution in (4), if \( p < p_0 \), there is a fast-slow relaxation oscillator bifurcates from the solitary oscillator solution in (4);

(2) If the system (4) satisfies the following conditions:
\[
\begin{align*}
-\frac{m}{n} &> f'(u_B), \\
-\frac{m}{n} u_1 + \frac{mu_B + nB}{n} &> A, \\
-\frac{m}{n} u_A + \frac{mu_B + nB}{n} &< A,
\end{align*}
\]
then \( p = p_1 = \frac{mu_B + nB}{n} \) is the critical point of the system (4). If \( p = p_1 \), there is a solitary oscillator solution in (4), if \( p > p_1 \), there is a fast-slow relaxation oscillator solution bifurcates from the solitary oscillator solution in (4).

Proof. We applied the Theorem 2.1 to proof (1), the proof of (2) in this theorem is similar to (1). Figure 3 is the phase plane of the system (2.2), we see the curve \( \Gamma \) divided the \( uv \) plane into two parts, it is easy to check that on the upper part of the curve \( \Gamma, \frac{du}{dt} < 0 \) holds true, and on the lower half part of the curve \( \Gamma, \frac{du}{dt} > 0 \) holds true. On the upper part of the line \( \ell, \frac{dv}{dt} > 0 \) holds true, and on the lower half part of the curve \( \ell, \frac{dv}{dt} < 0 \) holds true, then it is easy to find that there is a homoclinic orbit in (4) if \( p = p_0 = \frac{mu_A + nA}{n} \). Let \( H(h_1, h_2) \) is the crossing point of \( \Gamma \) and \( \ell \) if \( p = p_0 \), and also \( H'(h'_1, h'_2) \) is the crossing point of \( \Gamma \) and \( \ell \) if \( p < p_0 \).

![Figure 3](image-url)

It is easy to check that the point \( H \) is a saddle point in (4), the saddle quantity
\[
\sigma_0 = -3u_A^2 + 2(a + b)u_A - ab + \varepsilon n < 0.
\]
The separated function is \( \beta(p-p_0) = h'_0 - h'_1 \), it is obvious that \( \beta'(0) > 0 \). With the application of Andronov-Leontovich Theorem, the presence of a homoclinic orbit implies a global codimension-one bifurcation of (4) at \( p = p_0 \), since the homoclinic orbit disappears for all sufficiently small \( |p - p_0| \). Moreover, the disappearance of a homoclinic orbit leads to the creation or destruction of one limit cycle nearby such that the cycle approaches the homoclinic orbit as \( p \to p_0 \), its period tends to infinity. It implies that there is a fast-slow relaxation oscillator in (4) if \( p < p_0 \). 

**Theorem 2.3.** In the system (4), let \( b > 0, p = 0, m > 0, n < 0 \), and the parameters \( b, m, n, p \) be constant, let \(- \frac{m}{n} > \frac{B}{2} \), if \( a \) is an adjusted parameter for the system (4), then the following conclusions hold true:

1. If \( a > 0 \), there is no oscillator solution in (4);
2. If \( a = 0 \), there is a solitary oscillator solution in (4);
3. If \( a < 0 \), there is a fast-slow relaxation oscillator that bifurcates from the solitary oscillator solution in (4).

**Proof.** In the phase plane of the system (4), on the upper part of the curve \( \Gamma \), \( \frac{du}{dt} < 0 \) holds true, and on the lower half part of the curve \( \Gamma \), \( \frac{du}{dt} > 0 \) holds true. On the upper part of the line \( \ell \), \( \frac{dv}{dt} < 0 \) holds true, and on the lower half part of the curve \( \Gamma \), \( \frac{dv}{dt} > 0 \) holds true. If \( a = 0 \), \( O \) is an equilibrium point, the flow from the point \( O \) arrived at the point \( A \) fast on the line \( OA \), and then come to the point \( B \) slowly on the curve \( \Gamma \), later it leaves the point \( B \) to arrived at the point \( C \) fast on the line \( BC \), and then come back to the point \( O \) on the curve \( \Gamma \), that is, there is a homoclinic orbit \( OABC \) in (4) in figure 4.

![Figure 4](image)

Which leads to there is a solitary oscillator solution in system (4), and also in case \( a < 0 \), \( O \) is still an equilibrium point, the flow from the point \( A \) arrived at the point \( B \) fast on the line \( AB \), and then come to the point \( C \) slowly on the curve \( \Gamma \), later it leaves the point \( C \) to arrived at the point \( D \) fast on the line \( CD \), and then come back to the point \( A \) slowly on the curve \( \Gamma \), that is, there is a limit cycle \( ABCD \) in (4) in figure 5, which leads to there is a periodic oscillator solution in system (4).

**Corollary 1.** In the system (5), let \( b > 0, p = 0, m > 0, n > 0 \), and the parameters \( b, m, n, p \) be unchanged, if \( a \) is the an adjusted parameter for the system (5), then we have the following conclusions:

1. If \( a > 0 \), there is no oscillator solution in (5);
2. If \( a = 0 \), there is a solitary oscillator solution in (5);
3. If \( a < 0 \), there is a fast-slow relaxation oscillator that bifurcates from the solitary oscillator solution in (5).
Theorem 2.4. In the system (4), let $a, b, p$ be constant, and $b > a \geq 0$, $p = 0$, $m > 0$, $n < 0$, the points $M_1$ and $M_2$ are the crossover points of the curve $\Gamma$ and the line $\ell$, which is shown in the following figure. If $0 < -\frac{m}{n} < \frac{B_2}{u_2}$, there is a kink oscillator solution in (4), which satisfies $\lim_{t \to -\infty} (u, v) = M_1$ and $\lim_{t \to +\infty} (u, v) = M_2$

Proof. If $0 < -\frac{m}{n} < \frac{B_2}{u_2}$, then figure 6 is the phase plane of the system (2.2) in this case, it is easy to check that on the upper part of the curve $\Gamma$, $\frac{du}{dt} < 0$ holds true, and on the lower half part of the curve $\Gamma$, $\frac{du}{dt} > 0$ holds true. On the upper part of the line $\ell$, $\frac{dv}{dt} < 0$ holds true, and on the lower half part of the curve $\ell$, $\frac{dv}{dt} > 0$ holds true, $M_1$ and $M_2$ are two equilibrium points, the flow from the point $M_1$ arrived at the point on curve $\Gamma$ fast, which is shown in figure 6, then it arrived at the point $M_2$ fast on the curve $\Gamma$, that is, there is a heteroclinic orbit in (4), which leads to the existence of a kink oscillator solution in (4).

Theorem 2.5. In the system (4), if $n = 0, m > 0$, we have the following conclusions:

(1) If $-\frac{m}{n} < u_1$ or $-\frac{m}{n} > u_2$, there is no oscillator solution in (4);
(2) If $-\frac{m}{n} = u_1$ or $-\frac{m}{n} = u_2$, there is a solitary oscillator solution in (4);
(3) If $u_2 > -\frac{m}{n} > u_1$, there is a fast-slow relaxation oscillator solution that bifurcates from the solitary oscillator solution in (4).

Proof. Case 1. $-\frac{m}{n} < u_1$. Figure 7 is the phase plane of the system (2.2) in this case, it is easy to check that on the upper part of the curve $\Gamma$, $\frac{du}{dt} < 0$ holds true, and on the lower half part of the curve $\Gamma$, $\frac{du}{dt} > 0$ holds true. On the left part of the line $\ell$, $\frac{dv}{dt} < 0$ holds true, and on the right half part of the curve $\ell$, $\frac{dv}{dt} > 0$ holds
true. It is easy to find that there is no homoclinic orbit or periodic orbit in Figure 7.

**Case 2.** $-\frac{p}{m} = u_1$. Figure 8 is the phase plane of the system (2.2) in this case, it is easy to check that on the upper part of the curve $\Gamma$, $\frac{du}{dt} < 0$ holds true, and on the lower half part of the curve $\Gamma$, $\frac{du}{dt} > 0$ holds true. On the left part of the line $\ell$, $\frac{dv}{dt} < 0$ holds true, and on the right half part of the curve $\ell$, $\frac{dv}{dt} > 0$ holds true. It is easy to find that there is a homoclinic orbit in Figure 8.

**Case 3.** $u_2 > -\frac{p}{m} > u_1$. Figure 9 is the phase plane of the system (2.2) in this case, it is easy to check that on the upper part of the curve $\Gamma$, $\frac{du}{dt} < 0$ holds true, and on the lower half part of the curve $\Gamma$, $\frac{du}{dt} > 0$ holds true. On the left part of the line $\ell$, $\frac{dv}{dt} < 0$ holds true, and on the right half part of the curve $\ell$, $\frac{dv}{dt} > 0$ holds true. It is easy to find that there is a periodic orbit in Figure 9.

**Corollary 2.** In the system (5), if $n = 0, m < 0$, we also have the following conclusions:

1. If $-\frac{p}{m} < u_1$ or $-\frac{p}{m} > u_2$, then there is no oscillator solution in (5);
2. If $-\frac{p}{m} = u_1$ or $-\frac{p}{m} = u_2$, then there is a solitary oscillator solution in (5);
3. If $u_2 > -\frac{p}{m} > u_1$, then there is a fast-slow relaxation oscillator solution bifurcates from the solitary oscillator solution in (5).
3. The applications. Based on the theory we established in this paper, we studied the phase transition of the oscillators and travelling waves in the following systems, all of them are famous Relaxation Oscillation systems.

3.1. The application to Fitzhugh-Nagumo equation. The general Fitzhugh-Nagumo model (see [5]) is shown as follows,

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= -u(u - g)(u - 1) - v + D\frac{\partial^2 u}{\partial x^2}, \\
\frac{\partial v}{\partial t} &= \varepsilon (eu - rv + k), \quad 0 < \varepsilon \ll 1,
\end{aligned}
\]  

(15)

where \( u \) is a function of \( x \) and \( t \). The model is a family of two-variable reaction diffusion equations that capture the essential properties of spatially distributed excitatory media, which also described the propagation of electrical signals in nerve axons and other biological tissues (see [11],[6],[4]). Here \( u(x, t) \) is the voltage inside the axon at position \( x \in \mathbb{R} \) and time \( t \). The first equation in (15) is Kirchhoff’s law, expressing that the change \( \frac{\partial^2 u}{\partial x^2} \) of the current \( \frac{\partial u}{\partial x} \) along the axon is compensated by the currents passing through the cell membrane: a capacitance based current \( \frac{\partial v}{\partial t} \) and a resistance based current \( -u(u - g)(u - 1) - v \). \( v \) describes a part of the transmembrane current that passes through slowly adapting ion channels. The oscillators and travelling waves of Fitzhugh-Nagumo equation is widely studied in many papers, including [3],[15],[9],[4],[10].

In [17], the original Fitzhugh-Nagumo model was formulated as a reduction of the Hodgkin-Huxley model without the diffusion effect of the \( u \), so firstly we studied the following system

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= -u(u - g)(u - 1) - v, \\
\frac{\partial v}{\partial t} &= \varepsilon (eu - rv + k), \quad 0 < \varepsilon \ll 1.
\end{aligned}
\]  

(16)

Let

\[
\begin{aligned}
u_1 &= \frac{1}{3}(g + 1 - \sqrt{g^2 + 1 - g}), \\
u_2 &= \frac{1}{3}(g + 1 + \sqrt{g^2 + 1 - g}).
\end{aligned}
\]  

(17)

(18)

Note that \( u_A \) is a root of the equation \( f(u) = -u_1^3 + (g + 1)u_1^2 - gu_1 \) such that \( u_A \neq u_1 \), \( u_B \) is a root of the equation \( f(u) = -u_2^3 + (g + 1)u_2^2 - gu_2 \) such that \( u_B \neq u_2 \). Then the Corollary 3 follows from the Theorem 2.3.

Corollary 3. In the system (16), let \( e, r, k \) be constant and \( e > 0, k = 0, r > 0. \) If \( \frac{e}{r} > \frac{B}{u_2} \) then \( g \) is the an adjusted parameter in the system (16), we have the
following conclusions:
(1) If $g > 0$, there is no oscillator solution in (16);
(2) If $g = 0$, there is a solitary oscillator solution in (16);
(3) If $g < 0$, there is a fast-slow relaxation oscillator solution in (16);

Corollary 4. In the system (16), let $r, g, e$ be constant, and $e > 0, r < 0$. If $k$ is an adjusted parameter of the system (16), then we have the following conclusions:
(1) If the system (16)satisfies the following conditions:
\[ \frac{e}{r} > f'(u_A), \]
\[ \frac{e}{r} u_2 - \frac{e u_A - r A}{r} < B, \]
\[ \frac{e}{r} u_B - \frac{e u_A - r A}{r} > B. \]
Then $k = k_0 = -\frac{eu_A + r A}{r}$ is the critical point of the system (16). If $k = k_0$, there is a solitary oscillator solution in (16), if $k < k_0$, there is a fast-slow relaxation oscillator solution bifurcates from the solitary oscillator solution in (16);
(2) If the system (16)satisfies the following conditions:
\[ \frac{e}{r} > f'(u_B), \]
\[ \frac{e}{r} u_1 + \frac{-eu_B + r B}{r} > A, \]
\[ \frac{e}{r} u_A + \frac{-eu_B + r B}{r} < A. \]
then $k = k_1 = -\frac{eu_B + r B}{r}$ is the critical point of the system (16). If $k = k_1$, there is a solitary oscillator solution in (16), if $k > k_1$, there is a fast-slow relaxation oscillator that bifurcates from the solitary oscillator solution in (16).

The Corollary 4 follows from the Theorem 2.2.

Corollary 5. In the system (16), let $g, k$ be constant, and $1 > g > 0, k = 0, e > 0, r > 0$, the points $M_1$ and $M_2$ are the crossover points of the curve $\Gamma = (u, v) : v = u(g - u)(u - 1)$ and the line $\ell = (u, v) : eu - r v = 0$, if $0 \leq \frac{e}{r} \leq \frac{B}{u_2}$, then there is a kink oscillator solution in (16), which satisfies $\lim_{t \to -\infty} (u, v) = M_1$ and also $\lim_{t \to +\infty} (u, v) = M_2$.

The Corollary 5 follows from the Theorem 2.4.

Corollary 6. In the system (16), if $r = 0, g, e > 0$, we have the following conclusions:
(1) If $-\frac{k}{e} < u_1$ or $-\frac{k}{e} > u_2$, there is no oscillator solution in (16);
(2) If $-\frac{k}{e} = u_1$ or $-\frac{k}{e} = u_2$, there is a solitary oscillator solution in (16);
(3) If $u_2 > -\frac{k}{e} > u_1$, there is a fast-slow relaxation oscillator that bifurcates from the solitary oscillator solution in (16).

The Corollary 6 follows from the Theorem 2.5.

In the next section, we will study the bifurcation of travelling waves in system (15) with $D \neq 0$ (which is studied in [6], [15], [18]), the method can be found in [17], some other papers including [19] also studied the bifurcation of travelling waves.

Theorem 3.1. In the system (15), let $g$ be an adjusted parameter. Then we have the following conclusions:
(1) If \( g < 0 \), there are three wave solutions \( u_1(z), u_2(z), u_3(z) \) in system (16), where \( c = \sqrt{2D(g + 1)} \) such that

\[
\begin{align*}
  u_1(-\infty) &= 0, u_1(+\infty) = g, \\
  u_2(-\infty) &= 1, u_2(+\infty) = 0, \\
  u_3(-\infty) &= g, u_3(+\infty) = 1.
\end{align*}
\]

(2) If \( 0 < g < 1 \), there are six wave solutions \( u_1(z), u_2(z), u_3(z), u_4(z), u_5(z), u_6(z) \) in system (16), where \( c_1 = \sqrt{2D(g - \frac{1}{2})} \), \( c_2 = \sqrt{2D(1 - g)} \) such that the wave speed of \( u_1(z), u_2(z), u_3(z) \) is \( c_1 \), the wave speed of \( u_4(z), u_5(z), u_6(z) \) is \( c_2 \).

\[
\begin{align*}
  u_1(-\infty) &= 0, u_1(+\infty) = g, u_2(-\infty) = 1, u_2(+\infty) = g, \\
  u_3(-\infty) &= 1, u_3(+\infty) = 0, u_4(-\infty) = g, u_4(+\infty) = 0, \\
  u_5(-\infty) &= g, u_5(+\infty) = 1, u_6(-\infty) = 0, u_6(+\infty) = 1.
\end{align*}
\]

(3) If \( g > 1 \), there are six wave solutions \( u_1(z), u_2(z), u_3(z), u_4(z), u_5(z), u_6(z) \) in system (16), where \( c_1 = \sqrt{2D(g - 2)} \), \( c_2 = \sqrt{2D(2 - g)} \) such that the wave speed of \( u_1(z), u_2(z), u_3(z) \) is \( c_1 \), the wave speed of \( u_4(z), u_5(z), u_6(z) \) is \( c_2 \).

\[
\begin{align*}
  u_1(-\infty) &= 0, u_1(+\infty) = 1, u_2(-\infty) = g, u_2(+\infty) = 1, \\
  u_3(-\infty) &= g, u_3(+\infty) = 0, u_4(-\infty) = 1, u_4(+\infty) = 0, \\
  u_5(-\infty) &= 1, u_5(+\infty) = g, u_6(-\infty) = 0, u_6(+\infty) = g.
\end{align*}
\]

Proof. In reason of \( 0 < \varepsilon << 1 \) is very small, here we let \( \frac{dv}{dt} = 0 \), which means \( v = \text{constant} \), without loss of generality, let \( v = 0 \), then (2) is changed to the following style.

\[
\frac{\partial u}{\partial t} = -u(u-g)(u-1) + D \frac{\partial^2 u}{\partial z^2}.
\]

The typical travelling wave approach \( u(x, t) = U(z) \) with the wave variable \( z = x - ct \) leads to

\[
D\ddot{U} + c\dot{U} - U(U-g)(U-1) = 0,
\]

which can be transformed into

\[
\begin{cases}
  \dot{U} = V, \\
  \dot{V} = -\frac{D}{2}V + \frac{1}{D}U(U-g)(U-1).
\end{cases}
\]

The system obviously has three stationary points \((g,0),(1,0),(0,0)\), let \( c > 0, 1 > g > 0 \), it is easy to know that \((0,0)\) and \((1,0)\) are saddle-points, \((g,0)\) is stable point. For a travelling wave, we look for a connection of \((0,0)\) to \((1,0)\) (for this, the \( u \)-value has to exceed \( g \) somewhere). So we want to have the boundary conditions

\[
\begin{cases}
  \dot{U} = AU(U-1), \ A < 0, \\
  U(-\infty) = 0, U(+\infty) = 1.
\end{cases}
\]

Then we look for \( A \) and \( c \) to make (37) satisfying (35), by direct calculation, we obtained

\[
D\ddot{U} + c\dot{U} - U(U-g)(U-1) = U(U-1)[U(2DA^2 - 1) + cA + g - DA^2] = 0.
\]

Hence we must have

\[
2DA^2 - 1 = 0 \text{ and } cA + g - DA^2 = 0,
\]

where \( A \) and \( c \) (the wave speed) are uniquely determined:

\[
A = -\sqrt{\frac{1}{2D}} \text{ and } c = \sqrt{2D}(g - \frac{1}{2}).
\]
If \( A \) and \( c \) are chosen in this way, the solutions of (37) can satisfy the full equation. Notice that the sign of \( c \) may change: If \( g > \frac{1}{2} \), then \( c > 0 \). If \( g < \frac{1}{2} \) then \( c < 0 \). Based on the same method, we can also obtained the wave solution \( \bar{U}(z) \), where \( z = x - ct \), such that \( \bar{U}(\infty) = 1, \bar{U}(\infty) = 0 \), then the boundary conditions is shown as follows

\[
\begin{align*}
\dot{\bar{U}} &= \bar{A} \bar{U}(\bar{U} - 1) \bar{A} > 0, \\
\bar{U}(\infty) &= 0, \bar{U}(\infty) = 1.
\end{align*}
\]

(41)

Then we obtained \( \bar{A} = \frac{1}{\sqrt{2g}}, \bar{c} = \sqrt{2D} \left( \frac{1}{2} - g \right) \). If \( g > \frac{1}{2} \), then \( \bar{c} < 0 \). If \( g < \frac{1}{2} \) then \( \bar{c} > 0 \). If \( g = \frac{1}{2} \), the wave solutions \( U \) and \( \bar{U} \) are called Certain traveling wave solution.

The same method can be also used to study the cases \( g < 0 < 1 \) and also \( 1 < g \). In the case \( g < 0 \), it is easy to check that \( (0, 0) \) is stable point, \( (g, 0) \) is unstable point, \( (b, 0) \) is saddle point, we also have the wave solution \( U(x - ct) \), such that \( U(\infty) = g, U(\infty) = 1 \). In the case \( 0 < 1 < g \), it is easy to check that \( (0, 0) \) and \( (g, 0) \) is saddle point, \( (1, 0) \) is stable point. Based on the same method, it is easy to obtained the existence of the travelling waves in these cases. The proof is complete.

3.2. The application to the van der Pol equation. The Van der Pol equation was originally proposed by the Dutch electrical engineer and physicist Balthasar van der Pol while he was working at Philips, see [14] and [8], the Van der Pol equation is a famous relaxation oscillator. It is a non-conservative oscillator with non-linear damping. It evolves in time according to the second-order differential equation

\[
\ddot{x} + x = \mu(1 - x^2)\dot{x}.
\]

(42)

where \( x \) is the position coordinate, which is a function of the time \( t \), \( \mu \) is a huge scalar parameter indicating the nonlinearity and the strength of the damping. Now, we consider the Forced Van der Pol equation, that is, the Van der Pol equation takes the original function and adds a driving function \( A \sin(\omega t) \) to give a differential equation of the form (see [8])

\[
\ddot{x} + x - \mu(1 - x^2)\dot{x} - A \sin(\omega t) = 0.
\]

(43)

The two-dimensional form of the Forced Van der Pol equation is shown as follows

\[
\begin{align*}
\frac{\partial x}{\partial t} &= \mu(y + x - \frac{1}{3}x^3), \\
\frac{\partial y}{\partial t} &= \frac{1}{\mu}(x - A \sin(\omega t)).
\end{align*}
\]

(44)

Let \( p = -A \sin(\omega t) \), the parameter \( p \) is related to \( t \), which means that the bifurcation of the oscillators is in connection with time, in the other word, the phase transition may occurs as time goes by. According to the previous conclusions, we obtained the following theorem.

**Theorem 3.2.** In the system(44), we have the following conclusion

(1) If \( p = -1 \) or \( p = 1 \), there is a solitary oscillator of (44);

(2) If \(-1 < p < 1 \), there exists a fast-slow relaxation oscillator that bifurcates from the solitary oscillator solution in (44);

(3) If \( p < -1 \) or \( p > 1 \), there is no oscillator solution in (44).

The theorem 3.2 follows from the Theorem 2.5.
3.3. The application to the Winfree generic system. Winfree at the year 1996 related travelling waves to aspects of movement in heart muscle and nerves which are excitable, he used a generic excitable reaction - diffusion system to study it, see [17]. This paper studied the generic system without the diffusion effect, the equation is shown as follows

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{1}{\varepsilon} \left( u - \frac{1}{3} u^3 - v \right), \\
\frac{\partial v}{\partial t} &= \varepsilon \left( u - \frac{1}{2} v + \beta \right).
\end{align*}
\]

(45)

**Theorem 3.3.** In the system (45), we have the following conclusions

1. If \( \beta = \frac{1}{3} \) or \( \beta = -\frac{1}{3} \), there is a solitary oscillator of (45);
2. If \( -\frac{1}{3} < \beta < \frac{1}{3} \), there exists a fast-slow relaxation oscillator that bifurcates from the solitary oscillators solution in (45);
3. If \( \beta < -\frac{1}{3} \) or \( \beta > \frac{1}{3} \), there is no oscillator solution in (45).

The theorem 3.3 follows from the Theorem 2.5.

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