GLOBAL SOLUTIONS TO ROTATING MOTION OF ISENTROPIC FLOWS WITH CYLINDRICAL SYMMETRY

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Abstract. We are concerned with global weak solutions to the isentropic compressible Euler equations with cylindrically symmetric rotating structure, in which the origin is included. Due to the presence of the singularity at the origin, only the case excluding the origin $|\vec{x}| \geq 1$ has been considered by Chen-Glimm [4]. The convergence and consistency of the approximate solutions are proved by using $L^\infty$ compensated compactness framework and vanishing viscosity method. Moreover, if the blast wave initially moves outwards, and if initial density and velocity decay to zero at certain algebraic rate near the origin, then the density and velocity decay at the same rate for any positive time.

1. Introduction

In this paper, we prove the existence result for the global entropy solutions to two dimensional isentropic compressible Euler equations with cylindrically symmetric rotating flow including the origin. The compressible Euler equations are of the following conservative form:

\[
\begin{align*}
\rho_t + \nabla \cdot \vec{m} &= 0, \quad \vec{x} \in \mathbb{R}^2, \\
\vec{m}_t + \nabla \cdot \left( \frac{\vec{m} \otimes \vec{m}}{\rho} \right) + \nabla p &= 0, \quad \vec{x} \in \mathbb{R}^2.
\end{align*}
\]

(1.1)

We are interested in cylindrically symmetric flow to system (1.1) with the form:

\[
(\rho, \vec{m})(\vec{x}, t) = \left( \rho(x,t), m(x,t) \frac{-x_2, x_1}{x}, \tilde{m}(x,t) \frac{2 \rho}{\gamma} \right), \quad x = |\vec{x}|.
\]

(1.2)

Then $(\rho(x,t), m(x,t), \tilde{m}(x,t))$ in (1.2) is governed by the one-dimensional Euler equations with geometric source terms:

\[
\begin{align*}
\rho_t + \frac{1}{x} m_x &= 0, \quad x \in [0, +\infty), \quad t > 0, \\
\frac{m_t + \left( \frac{m^2}{\rho} + p(\rho) \right)}{x} &= -\frac{1}{x} m - \frac{\tilde{m}^2}{\rho}, \quad x \in [0, +\infty), \quad t > 0, \\
\tilde{m}_t + \frac{\tilde{m} \tilde{m}}{\rho x} &= -\frac{2}{x} \tilde{m}, \quad x \in [0, +\infty), \quad t > 0.
\end{align*}
\]

(1.3)

Here $\rho, m, \tilde{m}$ and $p(\rho)$ represent the density, the normal and tangential momentum and the pressure of the gas separately. For polytropic gas, $p(\rho) = p_0 \rho^\gamma$, with $p_0 = \frac{\rho_0^2}{\gamma}, \quad \theta = \frac{\rho_0^2}{\gamma}$ and adiabatic exponent $\gamma > 1$. 

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Various studies on spherical or cylindrical symmetric flow have been aroused attention in recent decades, motivated by many important physical phenomena, such as the stellar dynamics including supernova formation, inertial confinement fusion. See [11, 12, 16] and references therein.

Consider the initial value problem for (1.3) with geometric structure:
\[
\begin{align*}
\frac{v_t}{v} + F(v)\cdot x &= G(x,v), x \in [0, +\infty), t \in [0, +\infty), \\
v|_{t=0} &= v_0(x), x \in [0, +\infty),
\end{align*}
\] (1.4)
where the vector \( v = (\rho, m, \tilde{m})^T \), the flux \( F(v) = (m, m^2 \rho + p(\rho), \frac{m \tilde{m}}{\rho})^T \), the source term \( G(x,v) = (a(x)m, a(x)m^2 - \tilde{m}^2, a(x)2m\tilde{m})^T \), with \( a(x) = -\frac{1}{x^2} \).

Here, we consider the Cauchy problem of the compressible Euler equation (1.1) with cylindrical symmetric initial data
\[
(\rho, m, \tilde{m})|_{t=0} = (\rho_0(x), m_0(x), \tilde{m}_0(x)), x \geq 0.
\] (1.5)

There have been extensive studies on one dimensional isentropic gas dynamics. The first global existence of weak entropy solution with large initial data was established in Diperna [15] by introducing vanishing viscosity method for \( \gamma = 1 + \frac{2}{2n+1} \), where \( n \geq 2 \) is any positive integer. Ding-Chen-Luo [13] and Chen [1] extended the result for general values \( \gamma \in (1, \frac{5}{3}] \) by using Lax-Friedrichs scheme. Lions-Perthame-Tadmor [21] and Lions-Perthame-Souganidis [22] dealt with the case \( \gamma > \frac{5}{3} \). When \( \gamma = 1 \), Huang-Wang [19] obtained global existence of \( L^\infty \) entropy solutions of isothermal gases by adopting analytic extension method.

For the spherical symmetry solutions of compressible Euler equations, Makino et al. studied the global weak entropy solution outside a solid ball for \( \gamma = 1 \). The global existence for general \( \gamma \) was first studied by Chen-Glimm [5], and then by Tsuge [31]. Chen [3] proved a global existence theorem with large \( L^\infty \) data having only non-negative initial velocity. Recently, Huang-Li-Yuan [17] proved the global existence of weak solutions of the isentropic Euler equations with spherical symmetry including the origin by \( L^\infty \) compensated compactness framework and the vanishing viscosity method. Chen et al. [9,10] proved the global existence of finite-energy entropy solution by using \( L^p \) compensated compactness framework. For the cylindrical symmetric flow, Chen-Glimm [4] considered the case excluding the origin and extended the analysis in [5] to \( 3 \times 3 \) system. Chen-Wang [6] applied the shock capturing schemes to the compressible Euler-Possion equations with geometrical structure in semiconductor devices. More interesting and relevant results can be found in [3,7,8,20,23,30,33,34] and references therein.

Almost all the results in previous work developed numerical scheme to construct approximate solutions and used lengthy estimates. In this paper, we apply the vanishing viscosity method together with the Smoller’s invariant region theory [29] of quasilinear parabolic equations tackling with nonlinear singular terms to get uniform a priori estimates of approximate solutions. There are two technical difficulties to be overcome. One of the difficulties for the spherical or cylindrical symmetry flow is the singularity formed at the origin \( x = 0 \). There are complicated resonance between the characteristic mode.
and the geometric mode. Following the method proposed in [17], we introduce the spacial scaling transformation \( \rho = \hat{\rho} x^\gamma, m = \hat{m} x^d, \hat{m} = \hat{m} x^d \). Next, we introduce a new variable \( \xi = \ln x \) and transform the original system into (3.3). Then, viscous perturbations \((\varepsilon \hat{\rho}_{\xi}, \varepsilon \hat{m}_{\xi}, \varepsilon \hat{\rho}_{\xi})\) are added in the new system for \((\hat{\rho}, \hat{m}, \hat{m})\). Finally, we get uniform estimates of approximate solutions independent of viscosity \(\varepsilon\) using an enhanced maximum principle for parabolic system (See Lemma 2.1). We remark that Lemma 2.1 is quite powerful when we estimate the parabolic systems with source terms. Another difficulty is from the third equation in (1.3), due to the size of the system being \(3 \times 3\). There is an extra tangential direction we need to deal with. The extra field is linear degenerate generating from the angular momentum. Different from spherical symmetric flow, extra estimate is needed on tangential momentum \(\hat{m}\), which satisfies a parabolic equation. See (3.3). In fact, when \(\hat{m} = 0\), the equations could be reduced to the gas dynamics with spherical symmetric flow. Following the idea of quasi-decoupling method and framework developed by Chen [2], and the method furthered used in Chen-Glimm [4] and Chen-Wang [6], as the case excluding the origin, we shall get strong convergence of the tangential momentum \(\hat{m}\). This framework in [2] is useful for studying the limiting behavior of approximate solutions to general hyperbolic conservation laws. The initial oscillations could propagate along the corresponding linear degenerate field. The rest of the argument is standard. We observe that if the blast wave initially move outwards, that is, the normal velocity is non-negative, and the tangential velocity is allowed to be both positive and negative, then both the density and velocities are continuous at the origin. Besides, if densities and velocities initially tend to zero at certain algebraic rate near the origin, then the density and velocity tend to zero at the same rate for any positive time.

We define the weak entropy solution and state our main result as follows.

**Definition 1.1.** A measurable vector function \(v(x, t) = (\rho(x, t), m(x, t), \hat{m}(x, t))\) is called a global entropy solution of the Cauchy problem (1.4) provided that

\[
\int_0^{+\infty} \int_0^{+\infty} (v \varphi_t + F(v) \varphi_x + G(x, v) \varphi) \, dx \, dt + \int_0^{+\infty} v_0(x) \varphi(x, 0) \, dx = 0
\]

holds for any test function \(\varphi \in C_0^1((0, +\infty) \times [0, +\infty))\), and for the weak entropy pair \((\eta, q)\), the inequality

\[
\eta(v)_t + q(v)_x - \nabla \eta(v) \cdot G(x, v) \leq 0
\]

holds in the sense of distributions.

**Theorem 1.1.** (Existence) Let \(\gamma > 1\), and \(d = (\theta + 1)c > 0, c = \frac{1}{\theta}\). Assume that there exist two positive constants \(M_1, M_2\) such that

\[
0 \leq \rho_0(x), \quad \frac{m_0}{\rho_0} + \rho_0^\theta \leq M_1 x, \quad \frac{m_0}{\rho_0} - \rho_0^\theta \geq 0, \quad \frac{\hat{m}_0}{\rho_0} \leq M_2 x, \quad \text{a.e. } x \in [0, +\infty).
\]

Then there exists a global entropy solution of (1.3)-(1.5) satisfying

\[
0 \leq \rho(x, t) \leq Cx^c, \quad 0 \leq m(x, t) \leq C\rho(x, t)x, \quad |\hat{m}(x, t)| \leq C\rho(x, t)x,
\]

for a.e. \((x, t) \in [0, +\infty) \times \mathbb{R}^+\), where \(C\) depends on \(M_1, M_2, T\).
Remark 1.1. To the best of the author’s knowledge, the entropy solution obtained above is the first result on the entropy solution to the Cauchy problem of isentropic gas dynamics system with cylindrical symmetry including the origin.

Remark 1.2. The initial condition are allowed to be unbounded when $x$ is sufficiently large in Theorem 1.1.

The rest of the paper is organized as follows. In Section 2, we describe our approach of constructing approximate solutions by adding artificial viscosity. Some necessary formulas for the system are also given in this section. In Section 3, the uniform upper bound estimate for the approximate solutions $(\rho^\varepsilon, m^\varepsilon, \tilde{m}^\varepsilon)$ is proved and then the $H^{-1}_{loc}$ compactness of entropy pair, consistency and entropy inequality are obtained, and finally the proof of Theorem 1.1 is completed by applying $L^\infty$ compactness framework established in [13, 15, 22].

2. Formulations

First, we recall some basic notations for the system (1.1) with eigenvalues:

$$
\lambda_1 = \frac{m}{\rho} - \theta \rho^\theta, \quad \lambda_2 = \frac{m}{\rho} + \theta \rho^\theta, \quad \lambda_3 = \frac{m}{\rho},
$$

where $\theta = \frac{\gamma-1}{2}$, and the corresponding right eigenvectors are

$$
r_1 = \begin{bmatrix} \lambda_1 \\ -\tilde{m} \\ \tilde{m} \rho \end{bmatrix}, \quad r_2 = \begin{bmatrix} 1 \\ \lambda_2 \\ \tilde{m} \rho \end{bmatrix}, \quad r_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
$$

The Riemann invariants $(w, z, \omega)$ are given by

$$
w = \frac{m}{\rho} + \rho^\theta, \quad z = \frac{m}{\rho} - \rho^\theta, \quad \omega = \frac{\tilde{m}}{\rho},
$$

satisfying $\nabla w \cdot r_1 = 0, \nabla z \cdot r_2 = 0, \nabla \omega \nabla F(v) = \lambda_3 \nabla \omega$, where $\nabla = (\partial_{\rho}, \partial_{m}, \partial_{\tilde{m}})$ is the gradient with respect to the phase-space coordinates $v$. Eigenvalue $\lambda_3$ is linearly degenerate, i.e., $\nabla \lambda_3 \cdot r_3 \equiv 0$. A pair of functions $(\eta, q) : \mathbb{R}^3 \mapsto \mathbb{R}^2$ is called an entropy-entropy flux pair of system (1.3) or (1.4) if it satisfies

$$
\nabla q(v) = \nabla \eta(v) \nabla \left[ \begin{array}{c}
\frac{m^2}{\rho} + p(\rho) \\
\frac{m}{\rho} \end{array} \right].
$$

When

$$
\eta \bigg|_{\rho=0, \frac{m}{\rho}, \frac{\tilde{m}}{\rho} \text{ fixed}} = 0,
$$

$\eta(\rho, m, \tilde{m})$ is defined to be weak entropy of system (1.3).

The mechanical energy and mechanical energy flux, $(\eta^*, q^*)$,

$$
\eta^*(\rho, m, \tilde{m}) = \frac{m^2 + \tilde{m}^2}{2\rho} + \frac{p \rho \gamma}{\gamma - 1}, \quad q^*(\rho, m, \tilde{m}) = \frac{m^3 + m \tilde{m}^2}{2\rho^2} + \frac{\gamma p \rho \gamma - 1}{\gamma - 1} m,
$$

is a strictly convex weak entropy pair of system (1.3).

Next, we will introduce a variant of invariant region theory for a decoupled parabolic system which can be used to deal with general nonlinear source terms. In the following,
we extend the maximum principle to quasilinear parabolic systems and apply it to prove
the global existence of isentropic compressible Euler equations with cylindrical symmetry. Readers could refer to [17, 18] for details.

**Lemma 2.1.** (Maximum principle) Let \( p(x,t), q(x,t), (x,t) \in [a,b] \times [0,T] \) be any bounded classical solutions of the following quasilinear parabolic system

\[
\begin{aligned}
p_t + \mu_1 p_x &= \varepsilon p_{xx} + a_{11} p + a_{12} q + R_1, \\
q_t + \mu_2 q_x &= \varepsilon q_{xx} + a_{21} p + a_{22} q + R_2,
\end{aligned}
\]

with initial-boundary data

\[
\begin{align*}
p(x,0) &\leq 0, \quad q(x,0) \geq 0, \quad \text{for } x \in [a,b], \\
p(a,t) &\leq 0, \quad q(a,t) \geq 0, \quad \text{for } t \in [0,T], \\
p(b,t) &\leq 0, \quad q(b,t) \geq 0, \quad \text{for } t \in [0,T],
\end{align*}
\]

where

\[
\mu_i = \mu_i(x,t,p(x,t),q(x,t)), \quad a_{ij} = a_{ij}(x,t,p(x,t),q(x,t)),
\]

and the source terms

\[
R_i = R_i(x,t,p(x,t),q(x,t),p_x(x,t),q_x(x,t)), \quad i, j = 1, 2, \forall (x,t) \in [a,b] \times [0,T].
\]

\( \mu_i, a_{ij} \) are bounded with respect to \((x,t,p,q) \in [a,b] \times [0,T] \times K, \) where \( K \) is an arbitrary compact subset in \( \mathbb{R}^2. \) \( a_{12}, a_{21}, R_1, R_2 \) are continuously differentiable with respect to \( p,q. \)

Assume the following conditions hold:

- \((C1): When p = 0 and q \geq 0, there is a_{12} \leq 0; \) When \( q = 0 \) and \( p \leq 0, there is a_{21} \leq 0; \)
- \((C2): When p = 0 and q \geq 0, there is R_1 = R_1(x,t,0,q,\zeta,\eta) \leq 0; \) When \( q = 0 \) and \( p \leq 0, there is R_2 = R_2(x,t,p,0,\zeta,\eta) \geq 0. \)

Then for any \((x,t) \in [a,b] \times [0,T], \)

\[
p(x,t) \leq 0, \quad q(x,t) \geq 0.
\]

### 3. Proof of Theorem 1.1

#### 3.1. Uniform estimates.

Here, we consider \((1.3) \) on a cylinder \((a,b) \times \mathbb{R}^+, \) with \( \mathbb{R}^+ = [0, +\infty), a := a(\varepsilon) = -\frac{1}{m\varepsilon}, \lim_{\varepsilon \to 0} b(\varepsilon) = \infty. \) We set the transformation to the equation \((1.3): \)

\[
\rho = \tilde{\rho}x^\varepsilon, m = \tilde{m}x^d, \tilde{m} = \hat{m}x^d.
\]

Motivated by \[(3.1), \] we take \( d = (\theta + 1)c = c + 1, c = \frac{1}{7} > 0. \) Then we can transform \((1.3) \) into following form:

\[
\begin{aligned}
\tilde{p}_t + \tilde{m}_x x &= - (d + 1)\tilde{m}, \\
\tilde{m}_t + \left( \frac{\tilde{m}^2}{\rho} + p(\tilde{\rho}) \right) x &= - (2d - c + 1)\frac{\tilde{m}^2}{\rho} + \frac{\tilde{m}^2}{\rho} - (2d - c)p(\tilde{\rho}), \\
\hat{m}_t + \left( \frac{\tilde{m}\hat{m}}{\rho} \right) x &= - (2d - c + 2)\frac{\tilde{m}\hat{m}}{\rho}, \quad x \in [a(\varepsilon), b(\varepsilon)], t > 0.
\end{aligned}
\]
We introduce a new variable $\xi = \ln x$. Then (3.2) can be transformed into the following form:

\[
\begin{aligned}
\dot{\rho} + \dot{m} = -(d+1)m, \\
\dot{m} + \left( \frac{m^2}{\rho} + p(\rho) \right) \xi = -(2d - c + 1)\frac{m^2}{\rho} + \frac{\dot{m}^2}{\rho} - (2d - c)p(\rho), \\
\dot{m} + \left( \frac{\dot{m}}{\rho} \right) \xi = -(2d - c + 2)\frac{\dot{m}m}{\rho} + \varepsilon \dot{m}\xi, \\
\end{aligned}
\]

(3.3)

We approximate (3.3) by adding the artificial viscosity as follows:

\[
\begin{aligned}
\dot{\rho} + \dot{m} = -(d+1)m + \varepsilon \dot{\rho}\xi, \\
\dot{m} + \left( \frac{m^2}{\rho} + p(\rho) \right) \xi = -(2d - c + 1)\frac{m^2}{\rho} + \frac{\dot{m}^2}{\rho} - (2d - c)p(\rho) + \varepsilon \dot{m}\xi, \\
\dot{m} + \left( \frac{\dot{m}}{\rho} \right) \xi = -(2d - c + 2)\frac{\dot{m}m}{\rho} + \varepsilon \dot{m}\xi, \\
\end{aligned}
\]

(3.4)

The initial-boundary value conditions for (3.4) are given by

\[
\begin{aligned}
(\dot{\rho}, \dot{m}, \dot{\rho})|_{t=0} = (\dot{\rho}_0(x), \dot{m}_0(x), \dot{\rho}_0(x)) \\
= (\dot{\rho}_0(x) + \varepsilon \dot{\rho}(x) + \varepsilon)(\dot{\rho}_0(x) + \varepsilon) \chi, (\dot{m}_0(x) + \varepsilon)(\dot{\rho}_0(x) + \varepsilon) \chi) * \bar{\varepsilon}, \\
x \in [a(\varepsilon), b(\varepsilon)]; \\
(\dot{\rho}, \dot{m}, \dot{\rho})|_{x=a(\varepsilon)} = (\dot{\rho}_0(a(\varepsilon)), \dot{m}_0(a(\varepsilon)), \dot{\rho}_0(a(\varepsilon))) = (\dot{\rho}_0(a(\varepsilon)), 0, 0), \\
(\dot{\rho}, \dot{m}, \dot{\rho})|_{x=b(\varepsilon)} = (\dot{\rho}_0(b(\varepsilon)), \dot{m}_0(b(\varepsilon)), \dot{\rho}_0(b(\varepsilon))) ; t > 0,
\end{aligned}
\]

(3.5)

where $\bar{\varepsilon}$ is the standard mollifier, $\chi = \chi[2a(\varepsilon), b(\varepsilon)]$ is the characteristic function and the parameter $\varepsilon > 0$ is small.

Next, we will derive the uniform bound of the approximate solution $\dot{\rho}, \dot{m}, \dot{\rho}$ by the maximum principle, i.e., Lemma 2.1 and then derive the $L^\infty$ bound for $\rho, m, \dot{m}$. Using Riemann invariants’ definition, we obtain that

\[
\begin{aligned}
w = \frac{m}{\rho} + \rho^\theta = \left( \frac{m}{\rho} + \rho^\theta \right) x := \tilde{w}x, \\
z = \frac{m}{\rho} - \rho^\theta = \left( \frac{m}{\rho} - \rho^\theta \right) x := \tilde{z}x,
\end{aligned}
\]

\[
\omega = \frac{\dot{m}}{\rho} = \frac{\dot{m}}{\rho} x := \tilde{\omega}x.
\]

Similarly, we have

\[
\begin{aligned}
\lambda_1 = \frac{m}{\rho} - \theta \rho^\theta = \left( \frac{m}{\rho} - \theta \rho^\theta \right) x := \tilde{\lambda}_1x, \\
\lambda_2 = \frac{m}{\rho} + \theta \rho^\theta = \left( \frac{m}{\rho} + \theta \rho^\theta \right) x := \tilde{\lambda}_2x,
\end{aligned}
\]

\[
\lambda_3 = \frac{m}{\rho} x := \tilde{\lambda}_3x.
\]
Then we can approximate (3.3) by adding the same artificial viscosity:

\[
\begin{align*}
\rho_t + m_x &= -\frac{1}{x}m + \varepsilon \left[ \rho_{xx}x^2 + (d - 3c)\rho_x + c^2 \rho \right], \\
m_t + \left( \frac{m^2}{\rho} + p(\rho) \right)_x &= -\frac{1}{x}m^2 + \varepsilon \left[ m_{xx}x^2 - (c + d)m_x + d^2m \right] \\
\tilde{m}_t + \left( \frac{\tilde{m}}{\rho} \right)_x &= -\frac{2}{x}m\tilde{m} + \varepsilon \left[ \tilde{m}_{xx}x^2 - (c + d)\tilde{m}_x + d^2\tilde{m} \right],
\end{align*}
\]

(3.6)

It is noted that system (3.4) is equivalent to system (3.6). Therefore it is suffice to prove the existence of the approximate solutions of (3.6).

Now, we transform (3.4) into the following form:

\[
\begin{align*}
\tilde{w}_t + \tilde{\lambda}_2 \tilde{w}_x &= \varepsilon \tilde{w}_{x\xi} + 2\varepsilon \frac{\tilde{\rho}_x}{\tilde{\rho}} \tilde{w}_x - \varepsilon \theta(\theta + 1)\rho^{\theta - 2}\rho_x^2 \\
&\quad - \frac{\tilde{m}^2}{\rho^2} - \theta(d + 1)\rho^{\theta - 1}\tilde{m} - (2d - c)\frac{\theta^2}{\gamma} \rho^{2\theta} + \tilde{m}^2 \rho^2, \\
\tilde{z}_t + \tilde{\lambda}_1 \tilde{z}_x &= \varepsilon \tilde{z}_{\xi\xi} + 2\varepsilon \frac{\tilde{\rho}_x}{\tilde{\rho}} \tilde{z}_x + \varepsilon \theta(\theta + 1)\rho^{\theta - 2}\rho_x^2 \\
&\quad - \frac{\tilde{m}^2}{\rho^2} + \theta(d + 1)\rho^{\theta - 1}\tilde{m} - (2d - c)\frac{\theta^2}{\gamma} \rho^{2\theta} + \tilde{m}^2 \rho^2, \\
\tilde{\omega}_t + \tilde{\lambda}_3 \tilde{\omega}_x &= \varepsilon \tilde{\omega}_{\xi\xi} + 2\varepsilon \frac{\tilde{\rho}_x}{\tilde{\rho}} \tilde{\omega}_x - 2\frac{\tilde{m}\tilde{m}}{\rho^2}.
\end{align*}
\]

(3.7)

We first consider the \(L^\infty\) estimate of \(\dot{\omega}\) satisfying the third equation in (3.7),

\[
\dot{\omega}_t + \left( \tilde{\lambda}_3 - 2\varepsilon \frac{\tilde{\rho}_x}{\tilde{\rho}} \right) \dot{\omega}_x = \varepsilon \dot{\omega}_{\xi\xi} - 2\frac{\tilde{m}\tilde{m}}{\rho^2}.
\]

(3.8)

From the initial assumption (3.15) on \(\dot{\omega}\), there hold

\[
\left| \frac{\tilde{m}_0}{\rho_0} \right| \leq M_2, \left| \frac{\tilde{m}}{\rho}(a(\varepsilon), t) \right| \leq M_2 a(\varepsilon), \left| \frac{\tilde{m}}{\rho}(b(\varepsilon), t) \right| \leq M_2 b(\varepsilon).
\]

Then, we have

\[
|\dot{\omega}_0| \leq M_2, |\dot{\omega}(a(\varepsilon), t)| \leq M_2, |\dot{\omega}(b(\varepsilon), t)| \leq M_2.
\]

Then using the standard maximum principle for (3.8), we have

\[
||\dot{\omega}||_{L^\infty} \leq C,
\]

(3.9)

where \(C\) is dependent on initial data \(||\rho_0, \tilde{m}_0||_{L^\infty}\), which is independent of \(\varepsilon\). We will prove the \(L^\infty\) bound of \(\rho, \tilde{m}\) by Lemma 2.1 and by the positiveness of \(\frac{\tilde{m}}{\rho}\). We define the control functions \((\phi, \psi) = (M_3 + Ct + 2\varepsilon, 0)\), and the modified Riemann invariants \((\tilde{w}, \tilde{z})\) as

\[
\tilde{w} = \tilde{w} - \phi, \quad \tilde{z} = \tilde{z} + \psi.
\]

(3.10)
The system (3.17) becomes
\[
\begin{cases}
\dot{w}_t + \hat{\lambda}_2 \dot{w}_\xi = \varepsilon \dot{w}_\xi \varepsilon + 2 \varepsilon \frac{\bar{\rho}_\xi}{\rho} \dot{w}_\xi - \varepsilon \theta(\theta + 1) \rho^\theta - \rho^2 \hat{\rho}_\xi^2 \\
+ \left[ -\frac{\bar{m}^2}{\rho^2} - \theta(d + 1) \rho^\theta \hat{z} - \theta(d + 1) \rho^{2\theta} - \theta^2 c \rho^{2\theta} + \hat{\omega}^2 - C \right], \tag{3.11}
\end{cases}
\]
\[
\begin{cases}
\dot{z}_t + \bar{\lambda}_1 \dot{z}_\xi = \varepsilon \dot{z}_\xi \varepsilon + 2 \varepsilon \frac{\bar{\rho}_\xi}{\rho} \dot{z}_\xi + \varepsilon \theta(\theta + 1) \rho^{\theta - 2} \rho^2 \hat{\rho}_\xi^2 \\
+ \left[ -\left( \frac{\bar{w} + \hat{z}}{2} \right)^2 + \theta(d + 1) \frac{\bar{w}^2 - \hat{z}^2}{4} - \theta^2 c \left( \frac{\bar{w} - \hat{z}}{2} \right)^2 + \hat{\omega}^2 \right].
\end{cases}
\]

Then the above system (3.11) can be written as
\[
\begin{cases}
\dot{w}_t + \left( \hat{\lambda}_2 - 2 \varepsilon \frac{\bar{\rho}_\xi}{\rho} \right) \dot{w}_\xi = \varepsilon \dot{w}_\xi \varepsilon + a_{11} \dot{w} + a_{12} \dot{z} + R_1, \tag{3.12}
\end{cases}
\]
\[
\begin{cases}
\dot{z}_t + \left( \bar{\lambda}_1 - 2 \varepsilon \frac{\bar{\rho}_\xi}{\rho} \right) \dot{z}_\xi = \varepsilon \dot{z}_\xi \varepsilon + a_{21} \dot{w} + a_{22} \dot{z} + R_2,
\end{cases}
\]
where
\[
a_{11} = 0, \quad a_{12} = -\theta(d + 1) \rho^\theta \leq 0,
\]
\[
a_{21} = 0, \quad a_{22} = \frac{1}{4} \left[ -1 - \theta(d + 1) - \theta^2 c \right] \hat{z} + \frac{1}{2} \left( \theta^2 c - 1 \right) \bar{w},
\]
\[
R_1 = -\frac{\bar{m}^2}{\rho^2} - \theta(d + 1) \rho^{2\theta} - \theta^2 c \rho^{2\theta} - \varepsilon \theta(\theta + 1) \rho^{\theta - 2} \rho^2 \hat{\rho}_\xi^2 + (\hat{\omega}^2 - C) \leq 0,
\]
\[
R_2 = \frac{1}{4} \left[ -1 + \theta(d + 1) - \theta^2 c \right] \bar{w}^2 + \varepsilon \theta(\theta + 1) \rho^{\theta - 2} \rho^2 \hat{\rho}_\xi^2 + \hat{\omega}^2 \geq \frac{1}{4} \theta \bar{w}^2 \geq 0.
\]

By the initial and boundary data,
\[
\frac{m_0}{\rho_0} + \rho^\theta_0 \leq (M_3 + 2\varepsilon)x, \quad \frac{m_0}{\rho_0} - \rho^\theta_0 \geq 0, \quad x \in [a(\varepsilon), b(\varepsilon)],
\]
\[
\left( \frac{m}{\rho} + \rho^\theta \right) \big|_{x = a(\varepsilon)} \leq (M_3 + 2\varepsilon)a(\varepsilon), \quad \left( \frac{m}{\rho} - \rho^\theta \right) \big|_{x = a(\varepsilon)} \geq 0, \tag{3.13}
\]
\[
\left( \frac{m}{\rho} + \rho^\theta \right) \big|_{x = b(\varepsilon)} \leq (M_3 + 2\varepsilon)b(\varepsilon), \quad \left( \frac{m}{\rho} - \rho^\theta \right) \big|_{x = b(\varepsilon)} \geq 0,
\]
we obtain that
\[
\bar{w}(\xi, 0) = \bar{w}(\xi, 0) - M_3 - 2\varepsilon - C \bar{t} \leq 0, \quad \dot{\bar{z}}(\xi, 0) = \bar{z}(\xi, 0) \geq 0, \quad \ln(a(\varepsilon)) \leq \xi \leq \ln(b(\varepsilon)),
\]
\[
\bar{w}(\ln(a(\varepsilon)), t) \leq (M_3 + 2\varepsilon), \quad \bar{z}(\ln(a(\varepsilon)), t) = 0, \quad \text{for } t > 0,
\]
\[
\bar{w}(\ln(b(\varepsilon)), t) \leq (M_3 + 2\varepsilon), \quad \bar{z}(\ln(b(\varepsilon)), t) = 0, \quad \text{for } t > 0.
\]

By the Lemma 2.1,
\[
\bar{w}(\xi, t) \leq C(T), \quad \bar{z}(\xi, t) \geq 0.
\]

Hence, we obtain
\[
0 \leq \bar{p}(x, t) \leq C(T), \quad 0 \leq \bar{m}(x, t) \leq C\bar{p}(x, t), \quad |\bar{m}(x, t)| \leq C\rho(x, t),
\]
\[
i.e., \quad \text{for } a.e. (x, t) \in [a(\varepsilon), b(\varepsilon)] \times \mathbb{R}^+,
\]
\[
0 \leq \rho(x, t) \leq C x^c, \quad 0 \leq m(x, t) \leq C \rho(x, t)x, \quad |\bar{m}(x, t)| \leq C\rho(x, t)x, \tag{3.14}
\]
where $C$ depends on $M_3, T$. This completes the proof of the following theorem.

**Theorem 3.1.** *(L∞ estimate for Cylindrically Symmetry Problem)* Let $\gamma > 1$. For any positive constants $c$ and $d$ satisfying $d = (\theta + 1)c > 0, c = \frac{1}{\gamma}$. Assume that there exists a positive constant $M_3$ such that the initial and boundary data satisfy

$$
\frac{m_0}{\rho_0} + \rho_0^\theta \leq (M_3 + 2\varepsilon)x, \quad \frac{m_0}{\rho_0} - \rho_0^\theta \geq 0, \quad \left| \frac{\dot{m}_0}{\rho_0} \right| \leq M_2, \text{ a.e. } x \in [a(\varepsilon), b(\varepsilon)],
$$

$$
\left( \frac{m}{\rho} + \rho^\theta \right) |_{x=a(\varepsilon)} \leq (M_3 + 2\varepsilon)a(\varepsilon), \quad \left( \frac{m}{\rho} - \rho^\theta \right) |_{x=a(\varepsilon)} \geq 0,
$$

$$
\left( \frac{m}{\rho} + \rho^\theta \right) |_{x=b(\varepsilon)} \leq (M_3 + 2\varepsilon)b(\varepsilon), \quad \left( \frac{m}{\rho} - \rho^\theta \right) |_{x=b(\varepsilon)} \geq 0,
$$

$$
\left| \frac{\dot{m}}{\rho} \right| (a(\varepsilon), t) \leq M_2(a(\varepsilon)), \quad \left| \frac{\dot{m}}{\rho} \right| (b(\varepsilon), t) \leq M_2(b(\varepsilon)), \quad t > 0.
$$

(3.15)

Then for the solution of (3.6), (3.13), the following estimates hold

$$
0 \leq \rho(x, t) \leq C_x^c, \quad 0 \leq m(x, t) \leq C\rho(x, t)x, \quad |\dot{m}(x, t)| \leq C\rho(x, t)x,
$$

(3.16)

for a.e., $(x, t) \in [a(\varepsilon), b(\varepsilon)] \times \mathbb{R}^+$, where $C$ depends on $M_2, M_3, T$.

**3.2. Lower bound estimate.** Therefore, we obtain the following global existence of approximate solutions, whose proof is similar to that in [17, 18] and we obtain that

$$
\tilde{\rho} \geq e^{-C(\varepsilon, t)}.
$$

**Theorem 3.2.** For any time $T > 0$, there exist positive constants $C$ and $\varepsilon_0$ such that for $0 < \varepsilon < \varepsilon_0$, the initial-boundary value problem (3.6), (3.13) has following $L^\infty$ estimates

$$
e^{-C(\varepsilon, T)x^c} \rho(x, t) \leq \rho^\varepsilon(x, t) \leq Cx^c, \quad 0 \leq \dot{m}(x, t) \leq C\rho^\varepsilon(x, t)x,
$$

where $C$ is independent of $x$.

3.3. $H_{loc}^{-1}$ compactness of the entropy pair. Denote $\Pi_T = (a(\varepsilon), b(\varepsilon)) \times [0, T]$, for any $T \in (0, \infty)$. Let $K \subset \Pi_T$ be any compact set, and choose $\varphi \in C^\infty_c (\Pi_T)$ such that $\varphi|_K = 1$, and $0 \leq \varphi \leq 1$. Multiplying $(3.14)$ by $\nabla \eta \varphi$ with $\eta^*$ the mechanical entropy, we obtain

$$
\varepsilon \int \int_{\Pi_T} \varphi(p_x, m_x, \dot{m}_x) \nabla^2 \eta^*(p_x, m_x, \dot{m}_x)^\top x^2 dx dt
$$

$$
= \int \int_{\Pi_T} \left[ \eta^*_\rho \left( -\frac{1}{x} m + \varepsilon(d - 3c)\rho p_x x + \varepsilon c^2 \rho \right) + \eta^*_m \left( -\frac{1}{x} \frac{m^2}{\rho} - \varepsilon(c + d)m_x x + \varepsilon d(c + 1)m \right) \right. \left. + \eta^*_m \left( -\frac{2}{x} \frac{\dot{m}}{\rho} - \varepsilon(c + d)\dot{m}_x x + \varepsilon d(c + 1)\dot{m} \right) \right] \varphi
$$

$$
+ \eta^* \varphi_t + q^* \varphi_x + \varepsilon \eta^*(x^2 \varphi)_{xx} dx dt.
$$

(3.17)

Direct calculation shows that

$$
(p_x, m_x, \dot{m}_x) \nabla^2 \eta^*(p_x, m_x, \dot{m}_x)^\top = p_0 \gamma \rho^{\gamma-2} \rho_x^2 + \rho(u_x^2 + \tilde{u}_x^2).
$$
Note that
\[\eta^*_m\varepsilon_m x(c + d) \leq \frac{\varepsilon p_0 \gamma}{8} \rho^{\gamma - 2} \rho^2 x^2 + \frac{\varepsilon p u_2^2}{2} x^2 + C_\varepsilon \rho u^2,\]
\[\eta^*_\rho \varepsilon_{\rho x}(d - 3c)x \leq \frac{\varepsilon p_0 \gamma}{4} \rho^{\gamma - 2} \rho^2 x^2 + C_\varepsilon \rho^{2(\gamma - 1)} u^4 + C_\varepsilon \rho^2,\]
\[\eta^*_m \varepsilon \bar{m}_x x(c + d) \leq \frac{\varepsilon p_0 \gamma}{8} \rho^{\gamma - 2} \rho^2 x^2 + C_\varepsilon \rho^2 - \gamma u^4 + \frac{\varepsilon p u_2^2}{2} x^2 + C_\varepsilon \rho u^2,\]
we get
\[\frac{\varepsilon}{2} \int_\Pi_T \varphi(\rho, m, \bar{m}) \nabla^2 \eta^*(\rho, m, \bar{m})^\top x^2 dxdt\]
\[\leq \int \int_\Pi_T \left( C_\varepsilon \rho^{2 - \gamma} u^4 + C_\varepsilon \rho u^2 + C_\varepsilon \rho^{2(\gamma - 1)} \right) \varphi dxdt\]
\[+ \int \int_\Pi_T \eta^*_\varepsilon \varphi_t + q^* \varphi_x + \varepsilon \eta^*(x^2 \varphi)_{xx} + \left( \frac{m^2 + \tilde{m}^2}{2 \rho^2} - \frac{p_0 \gamma}{\gamma - 1} \right) \frac{1}{x} m \varphi - \frac{1}{x^2} \rho^2 \varphi dxdt\]
\[\leq C(\varphi),\]
where the constant \(C(\varphi)\) depends on \(\varphi\). Thus we have arrived that
\[\varepsilon(\rho, m, \bar{m}) \nabla^2 \eta^*(\rho, m, \bar{m})^\top \in L^1_{\text{loc}}(\Pi_T),\]
i.e.,
\[\varepsilon \rho^{\gamma - 2} \rho^2 x^2 + \varepsilon \rho(u_2^2 + \tilde{u}_x^2) \in L^1_{\text{loc}}(\Pi_T).\]
(3.18)
For any weak entropy-entropy flux pairs, we have
\[\eta_t + q_x = \varepsilon \eta_{xx} x^2 - \varepsilon(\rho, m, \bar{m}) \nabla^2 \eta(\rho, m, \bar{m})^\top x^2\]
\[+ \left( \eta^*_\rho \left( - \frac{1}{x} m + \varepsilon c^2 \rho \right) + \eta^*_m \left( - \frac{1}{x} m^2 \rho + \varepsilon d(c + 1) \tilde{m} \right) + \eta^*_\rho \left( - \frac{2}{x} \frac{m \tilde{m}}{\rho} + \varepsilon d(c + 1) \tilde{m} \right) \right)\]
\[+ \varepsilon \left( \eta^*_\rho \rho_x (d - 3c) - \eta^*_m m_x (c + d) - \eta^*_\rho (c + d) \bar{m}_x \right) =: \sum_{i=1}^4 J_i.\]
(3.19)
From (3.18), it is obvious that \(J_1\) is compact in \(H^2_{\text{loc}}(\Pi_T)\). The proof is similar with the argument in [21] and will not be reproduced here. For \(\gamma > 2, \rho^{\gamma - 2} \rho^2\) is degenerate near the vacuum. Here, we assume that \(1 < \gamma \leq 2\). For any weak entropy, the Hessian matrix \(\nabla^2 \eta\) can be controlled by \(\nabla^2 \eta^*\), refer to Lions et al. [22],
\[(\rho, m, \bar{m}) \nabla^2 \eta(\rho, m, \bar{m})^\top \leq (\rho, m, \bar{m}) \nabla^2 \eta^*(\rho, m, \bar{m})^\top,\]
therefore \(J_2\) is bounded in \(L^1_{\text{loc}}(\Pi_T)\) and by the Sobolev embedding theorem, this term is compact in \(W^{1, \alpha}_{\text{loc}}(\Pi_T)\) for some \(1 < \alpha < 2\). For \(J_3\),
\[|J_3| = \left| \eta^*_\rho \left( - \frac{1}{x} m + \varepsilon c^2 \rho \right) + \eta^*_m \left( - \frac{1}{x} m^2 \rho + \varepsilon d(c + 1) \tilde{m} \right) + \eta^*_\rho \left( - \frac{2}{x} \frac{m \tilde{m}}{\rho} + \varepsilon d(c + 1) \tilde{m} \right) \right| \leq C,\]
we obtain that \(J_3\) is bounded in \(L^1_{\text{loc}}(\Pi_T)\). For the last term \(J_4\), we get
\[|J_4| \leq C \varepsilon \left( \rho^{\gamma/2 - 1} |\rho_x| + \rho^2 \left( |u_x| + |\tilde{u}_x| \right) \right) x.\]
Using (3.18), we have $J_4$ is compact in $H^{-1}_{loc}(\Pi T)$. Therefore,

$$J_4$$ compact in $W^{-1,\alpha}_{loc}(\Pi T)$ for some $1 < \alpha < 2$.

It is easy to see that

$$\eta_t + q_x$$ is compact in $W^{-1,\alpha}_{loc}(\Pi T)$.

Next, we shall utilize Murat’s Lemma to conclude our result.

**Lemma 3.1.** (Murat [28]) Let $\Omega \subseteq \mathbb{R}^n$ be an open set, then

$$(\text{compact set of } W^{-1,q}_{loc}(\Omega)) \cap (\text{bounded set of } W^{-1,r}_{loc}(\Omega)) \subset (\text{compact set of } H^{-1}_{loc}(\Omega)),$$

where $1 < q \leq 2 < r$.

Combining Lemma [3.1] we obtain that

$$\eta_t + q_x$$ is compact in $H^{-1}_{loc}(\Pi T)$ (3.21)

holds for all weak entropy-entropy flux pairs, independent of $\tilde{m}$.

**Remark 3.1.** (3.21) still holds for the case that $\gamma > 2$ by a similar discussion as in [24,32]. We assume $1 < \gamma \leq 2$ for simplicity.

### 3.4. Strong convergence and consistency

Combining the compactness in (3.21) and the compactness framework established in [4,13,15,22], we obtain that strong convergency of the approximate solutions. There exists a subsequence of $(\rho^\varepsilon, m^\varepsilon, \tilde{m}^\varepsilon)$ (still adopt the same notation by $(\rho^\varepsilon, m^\varepsilon, \tilde{m}^\varepsilon)$) such that

$$(\rho^\varepsilon, m^\varepsilon, \tilde{m}^\varepsilon) \to (\rho, m, \tilde{m})$$

in $L^p_{loc}(\Pi T)$, $p \geq 1$. (3.22)

Here, applying the quasidecoupling method and framework for the limiting behavior of approximate solutions in [2], it is easy to show the strong convergence of $\tilde{m}^\varepsilon$.

Furthermore, it is shown that $\rho = O(1)x^c$ and $m = O(1)x^d$, $\tilde{m} = O(1)x^d$ so that the terms on right hand sides of the equation (1.3), that is, $\frac{m}{x}$ and $\frac{m^2 - \tilde{m}^2}{xp}$, and $\frac{2m\tilde{m}x}{xp}$ are integrable near the origin with respect to the spacial variable $x$. As in [5,8], we can show that $(\rho, m, \tilde{m})$ is an entropy solution to the problem (1.3). Therefore, this completes the proof of Theorem 1.1.

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