Interior derivative estimates for the Kähler-Ricci flow

Morgan Sherman* and Ben Weinkove†

Abstract

We give a maximum principle proof of interior derivative estimates for the Kähler-Ricci flow, assuming local uniform bounds on the metric.

1 Introduction

Let \((M,\hat{\omega})\) be a Kähler manifold of complex dimension \(n\). Let \(\omega = \omega(t)\) be a solution of the Kähler-Ricci flow on \(M \times [0,T]\), for some \(T > 0\):

\[
\frac{\partial}{\partial t} \omega = -\text{Ric}(\omega), \quad \omega|_{t=0} = \omega_0,
\]

with \(\omega_0\) a smooth initial Kähler metric.

Fix a point \(p \in M\) and denote by \(B_r \subset M\) the open ball centered at \(p\) of radius \(r\) for \(0 < r < 1\) with respect to \(\hat{\omega}\). We assume that \(r\) is sufficiently small so that \(B_r\) is contained in a single holomorphic coordinate chart. Our main result is as follows:

Theorem 1.1. Let \(N > 1\) satisfy

\[
\frac{1}{N} \hat{\omega} \leq \omega \leq N \hat{\omega}, \quad \text{on } B_r \times [0,T].
\]

Then for each \(m = 0,1,2,\ldots\) there exist constants \(C\) and \(C_m\) depending only on \(\hat{\omega}\) and \(T\) such that on \(B_{r/2} \times (0,T]\),

(i) \(\hat{\nabla}_\omega^2 \omega \leq C \frac{N^3}{r^2 t}\), for \(\hat{\nabla}\) the covariant derivative with respect to \(\hat{\omega}\).

(ii) \(|\text{Rm}|^2_\omega \leq C_0 \frac{N^8}{r^4 t^2}\).

(iii) \(\nabla^m_R \text{Rm} \leq C_m \left(\frac{N^4}{r^2 t}\right)^{m+2}\) for \(m = 1, 2, \ldots\), for \(\nabla_R\) the real covariant derivative with respect to the metric \(\omega\).

Moreover, if we allow the constants \(C\) and \(C_m\) to depend also on \(\omega_0\) then the estimates (i), (ii) and (iii) hold with each factor of \(t\) on the right hand side replaced by \(1\).

We prove this result using the maximum principle. Note that by work of Shi \[17\] \[18\] it was already known that a bound on curvature as in (ii) implies (iii) (nevertheless, we include a proof here, for the sake of completeness). Theorem 1.1 implies the following:

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Corollary 1.2. Let $N > 1$ satisfy
\[
\frac{1}{N} \hat{\omega} \leq \omega \leq N \hat{\omega}, \quad \text{on } B_r \times [0, T].
\] (1.3)

Then for each $m = 0, 1, 2, \ldots$ there exist constants $C_m, \alpha_m, \beta_m$ and $\gamma_m$ depending only on $m, \hat{\omega}$ and $T$ such that
\[
|\nabla^m_R \omega| \hat{\omega} \leq C_m \frac{N^{\alpha_m}}{r^{\beta_m} \gamma_m}, \quad \text{on } B_{r/2} \times (0, T),
\] (1.4)

Moreover, if we allow the constants $C_m, \alpha_m$ and $\beta_m$ to depend also on $\omega_0$ then (1.4) holds with $\gamma_m = 0$.

Namely, a local uniform estimate for the metric along the Kähler-Ricci flow implies local derivative estimates to all orders. This fact in itself is not new. Indeed the local PDE theory of Evans-Krylov [10, 14] can be applied to the Kähler-Ricci flow equation (see for example [7] or the generalization in [12]). The key point here is to establish this via Theorem 1.1 whose proof uses only elementary maximum principle arguments.

The form of the estimate (1.4) may be useful for applications and does not seem to be written down explicitly elsewhere in the literature. When considering the Kähler-Ricci flow on projective varieties, it is often the case that one obtains a uniform estimate for the metric $\omega$ away from a subvariety (see for example [19, 20, 21, 22, 23, 26, 29]). Theorem 1.1 can be used to replace global arguments. To illustrate, suppose that $\omega = \omega(t)$ solves the Kähler-Ricci flow on a compact Kähler manifold $M$ and there exists an analytic hypersurface $D \subset M$ whose associated line bundle $[D]$ admits a holomorphic section $s$ vanishing to order 1 along $D$. Assume that
\[
\frac{1}{C} \|s\|^p_H \hat{\omega} \leq \omega \leq \frac{C}{\|s\|^p_H} \hat{\omega}, \quad \text{on } (M \setminus D) \times [0, T]
\] (1.5)

for some positive constants $C$ and $\alpha$, where $H$ is a Hermitian metric on $[D]$. An elementary argument shows that Theorem 1.1 implies the existence of $C_m, \alpha_m$ and $\gamma_m$ such that
\[
|\nabla^m_R \omega| \hat{\omega} \leq \frac{C_m}{r^{\beta_m} \gamma_m \|s\|_H}, \quad \text{on } (M \setminus D) \times (0, T)
\] (1.6)

for each $m = 1, 2, \ldots$. Moreover we can take $\gamma_m = 0$ if we allow $C_m$ and $\alpha_m$ to depend on the initial metric $\omega_0$. Estimates of the form of (1.6) are used for example in [21, 22]. In particular, Corollary 1.2 gives an alternative proof of the results in Section 4 of [21].

Finally we remark that since our result is completely local, we may and do assume that $M = \mathbb{C}^n$, $p = 0$ and $\hat{\omega}$ is the Euclidean metric. We will write $g$ and $\hat{g}$ for the Kähler metrics associated to $\omega$ and $\hat{\omega}$. All magnitudes $| \cdot |$ are taken with respect to the metric $g$. We shall use the letter $C$ (as well as $C', C''$, etc.) for a uniform constant (depending only on $m, \hat{\omega}$, and $T$) which may differ from line to line.

In Sections 2, 3 and 4 we prove parts (i), (ii) and (iii) of Theorem 1.1 respectively. In Section 5 we give a proof of Corollary 1.2.
2 Bound on the first derivative of the metric

In this section we prove the estimate on the first derivative of the metric $g$, establishing part (i) of Theorem 1.1. This gives a local parabolic version of the well-known Calabi ’3rd order’ estimate [3] for the complex Monge-Ampère equation (used by Yau [27] in his solution of the Calabi conjecture). There exist now many generalizations of Calabi’s estimate (see for example [6, 24, 25, 28]). A global parabolic Calabi estimate was applied to the case of the Kähler-Ricci flow in [4]. Phong-Sesum-Sturm [16] later gave a neat and explicit computation in which we will make use of here for our local estimate.

We wish to bound the quantity

$$S = |\hat{\nabla} g|^2 = g^{i\ell} g^{k\ell} \hat{\nabla}_i g_{\ell p} \hat{\nabla}_j g_{\ell q}$$

where we write $\hat{\nabla}$ for the covariant derivative with respect to $\hat{g}$. Write $r_0 = r$ and let $\psi$ be a nonnegative $C^\infty$ cut-off function that is identically equal to 1 on $B_{r_1}$ and vanishes outside $B_r$, where $r_0 > r_1 > r/2$. We may assume that

$$|\nabla \psi|^2, |\Delta \psi| \leq C N r^{-2},$$

(2.2)

where $\Delta = \nabla^j \nabla_j = g^{p q} \nabla_p \nabla_q$. Thus

$$(\partial_t - \Delta) (\psi^2 S) \leq \psi^2 (\partial_t - \Delta) S + C N r^{-2} S + 2 |\nabla \psi|^2, \nabla S|,$$

(2.3)

where we are writing $\langle \nabla F, \nabla G \rangle = g^{i \ell} \partial_i F \partial_{\ell} G$ for functions $F, G$. Following the notation in [16], we introduce the endomorphism $h^i_k = \hat{g}^{i \ell} g_{\ell k}$ and let $X$ be the tensor with components $X_k^i = (\nabla_i h \cdot h^{-1})^k_i$, so that $S = |X|^2$. Note that $X$ is the difference of the Christoffel symbols of $g$ and $\hat{g}$.

An application of Young’s inequality gives

$$2 |\langle \nabla \psi, \nabla S \rangle| \leq \psi^2 (|\nabla X|^2 + |\nabla X|^2) + C N r^{-2} S.$$

(2.4)

We now use the evolution equation for $S$ derived by Phong-Sesum-Sturm (see equation (2.51) of [16]) which, in the case where $\omega$ is Euclidean, has the simple form:

$$(\partial_t - \Delta) S = - (|\nabla X|^2 + |\nabla X|^2).$$

(2.5)

Combining (2.3, 2.4, 2.5) we find

$$(\partial_t - \Delta) (\psi^2 S) \leq C N r^{-2} S.$$

(2.6)

We now need to use the evolution equation for $\text{tr} \ h$ from [4], which is a parabolic version of an estimate from [11, 27]. More precisely, we can apply equations (2.28) and (2.31) of [16] and use the fact that the fixed metric is Euclidean to obtain

$$(\partial_t - \Delta) (\text{tr} \ h) = - \hat{g}^{i \ell} g^{k \ell} \hat{\nabla}_i g_{\ell p} \hat{\nabla}_j g_{\ell q}.$$

(2.7)
Hence
\[(\partial_t - \Delta)(\text{tr} h) \leq -\frac{S}{N}.\] (2.8)

Let \(f(t)\) denote either the function \(t\) or the constant 1. Then \(0 \leq f(t) \leq \max(T, 1)\) and \(f'(t) = 1\) or \(0\) so that we get, for any positive constant \(B\),
\[(\partial_t - \Delta)(f(t)\psi^2 S + B \text{ tr} h) \leq C \frac{N}{r^2} S - \frac{B}{N} S.
\]

Let \(B = \frac{N^2}{r}(C + 1)\). Then by the maximum principle, the maximum of \(f(t)\psi^2 S + B \text{ tr} h\) on \(\overline{B_r} \times [0, T]\) can only occur at \(t = 0\) or on the boundary of \(\overline{B_r}\), where \(\psi = 0\). Since \(\text{ tr} h \leq nN\), we have
\[S \leq C \frac{N^3}{f(t)r^2} \text{ on } \overline{B_{r_1}} \times (0, T].\] (2.9)
giving part (i) of Theorem 1.1.

3 Bound on curvature

We now prove part (ii) of Theorem 1.1. For global estimates of this type, see for example [5, 15]. We fix a smaller radius \(r_2\) satisfying \(r_1 > r_2 > r/2\). In this section we let \(\psi\) be a cut-off function, identically 1 on \(\overline{B_{r_2}}\) and identically 0 outside \(B_{r_1}\). As before we may assume \(|\Delta \psi|, |\nabla \psi|^2 \leq CN/r^2\) for some uniform constant \(C\). Calculate
\[
(\partial_t - \Delta) R_{i\overline{jk}} = - R_{ji}^{\overline{pq}} R_{p\overline{lk}} + R_{ji}^{\overline{pq}} R_{j\overline{pq}l} - R_{i\overline{pq}} R_{j\overline{pq}l} = - R_{i\overline{pq}} R_{j\overline{pq}l},
\]
and therefore (cf. [13])
\[(\partial_t - \Delta) |\text{Rm}|^2 \leq -|\nabla \text{Rm}|^2 - |\nabla \text{Rm}|^2 + C|\text{Rm}|^3,
\] (3.2)
where we are writing \(|\text{Rm}|^2 = R_{i\overline{jk}} R_{i\overline{jk}}\) etc.

As before we set \(f(t) = t, 1\). We introduce the function
\[\tilde{S} = f S + C_1 N \text{ tr} h\] (3.3)
where \(C_1\) is a large uniform constant. Note that by (2.9) we have \(\tilde{S} \leq C \frac{N^3}{r^2}\) at every \((x, t) \in \overline{B_{r_1}} \times [0, T]\). Furthermore \(\tilde{S}\) satisfies
\[(\partial_t - \Delta) \tilde{S} \leq - f(|\nabla X|^2 + |\nabla X|^2) - C_2 \tilde{S}\] (3.4)
where \(C_2 = C_1 - f' \gg 1\) is uniform. Let \(K = C_3 N^4/r^2\) where \(C_3 \gg 1\) is a uniform constant. Note that we may assume \(K/2 \leq K - \tilde{S} \leq K\). We will establish our bound for \(|\text{Rm}|\) by using a maximum principle argument for the function \(F = f^2 \frac{|\text{Rm}|^2}{K - \tilde{S}} + \tilde{B}\tilde{S}\) where \(\tilde{B} = C_4/N^3\)
with $C_4 \gg 1$ uniform. We begin by computing

\[
(\partial_t - \Delta) \left( \psi^2 \frac{|\text{Rm}|^2}{K - \tilde{S}} \right) = -\Delta \psi^2 \frac{|\text{Rm}|^2}{K - \tilde{S}} + \psi^2 \frac{\partial_t - \Delta |\text{Rm}|^2}{K - \tilde{S}} + \psi^2 \frac{\partial_t - \Delta \tilde{S}}{(K - \tilde{S})^2} |\text{Rm}|^2
- 2\psi^2 \frac{|\nabla \tilde{S}|^2 |\text{Rm}|^2}{(K - \tilde{S})^3} - 4\text{Re} \frac{\psi \langle \nabla \psi, \nabla |\text{Rm}|^2 \rangle}{K - \tilde{S}}
- 4\text{Re} \frac{\psi \langle \nabla \psi, \nabla \tilde{S} \rangle |\text{Rm}|^2}{(K - \tilde{S})^2} - 2\text{Re} \frac{\psi^2 \langle \nabla |\text{Rm}|^2, \nabla \tilde{S} \rangle}{(K - \tilde{S})^2}
\]

and thus

\[
(\partial_t - \Delta) \left( \frac{\psi^2 |\text{Rm}|^2}{K - \tilde{S}} \right) \leq \frac{1}{(K - \tilde{S})^2} \left[ |\Delta \psi^2|(K - \tilde{S}) |\text{Rm}|^2 + \psi^2 (K - \tilde{S}) (C |\text{Rm}|^3 - |\nabla \text{Rm}|^2 - |\nabla |\text{Rm}|^2) \right.
+ \psi^2 (-f |\nabla X|^2 - f |\nabla X|^2 - C_2 S) |\text{Rm}|^2 - 2\psi^2 \frac{|\nabla \tilde{S}|^2 |\text{Rm}|^2}{K - \tilde{S}}
+ 16|\nabla \psi|^2 (K - \tilde{S}) |\text{Rm}|^2 + \frac{1}{2} \psi^2 (K - \tilde{S}) |\nabla |\text{Rm}|^2 + \frac{1}{2} \psi^2 (K - \tilde{S}) |\nabla \text{Rm}|^2
+ \frac{4}{K - \tilde{S}} \psi^2 |\nabla \tilde{S}|^2 |\text{Rm}|^2 + \frac{1}{2} \psi^2 (K - \tilde{S}) |\nabla |\text{Rm}|^2 + \frac{1}{2} \psi^2 (K - \tilde{S}) |\nabla |\text{Rm}|^2 \left].
\]

(3.5)

We wish to bound (3.6) in terms of $|\text{Rm}|^2$. Label the terms (1), (2), ..., (16). Then the bad terms are (1), (2) and (9) through (16) while the remaining terms are all good. One sees that (1) + (9) + (13) $\leq C \frac{N^2}{r^2} |\text{Rm}|^2$ while $[(10) + (11) + (15) + (16)] + [(3) + (4)] \leq 0$ and (12) + $\frac{1}{2}(8) \leq 0$. It remains only to bound the terms (2) and (14). For (2) we argue as follows: we may assume that at a maximum for the function $F$ we have a lower bound of the form

\[
f |\text{Rm}| \geq CK, \quad C \gg 1
\]

(3.7)

for if not we can apply a maximum principle argument immediately: At any $(x, t) \in \overline{B_{r_2}} \times (0, T]$ we would have $F$ is at most $CK + C/r^2$ which implies that

\[
f^2 |\text{Rm}|^2 \leq C \frac{N^8}{r^4} \text{ on } \overline{B_{r_2}} \times (0, T].
\]

Now since $\hat{\omega}$ is Euclidean we have

\[
|\nabla \omega|^2 = |\text{Rm} - \text{Rm}|^2 = |\text{Rm}|^2.
\]

(3.8)

Hence, using (3.7), we have (2) + $\frac{1}{2}(6) \leq 0$. Finally, to control (14) we use

\[
|\nabla \tilde{S}|^2 \leq 4 f^2 S(|\nabla X|^2 + |\nabla X|^2) + 2nC_1^2 N^4 S.
\]

(3.9)

Here we have made use of a well-known estimate (computed in [27]) which implies that $|\text{tr} h|^2 \leq nN^2 S$. Now we find (14) + $\frac{1}{2}(5) + (6) + (7) \leq 0$ if in $K = C_3 N^4/r^2$ we choose
\[ C_3 \gg C_1. \] In total then we have
\[ (\partial_t - \Delta) \left( \frac{\psi^2 |\text{Rm}|^2}{K - \tilde{S}} \right) \leq C \frac{N^3}{N^3} |\text{Rm}|^2. \] (3.10)

Therefore
\[ (\partial_t - \Delta) \left( \frac{\psi^2 f^2 |\text{Rm}|^2}{K - \tilde{S}} + \tilde{B} \tilde{S} \right) \leq -f \frac{N^3}{N^3} |\text{Rm}|^2, \] (3.11)

if in \( \tilde{B} = C_4 / N^3 \) we pick \( C_4 \) large enough. This implies that the maximum of \( F \) on \( B_{m+1} \times [0, T] \) can only occur at \( t = 0 \) or on the boundary of \( B_r \), where \( \psi = 0 \). Hence \( F \) is bounded above by \( C/r^2 \). Therefore at any \( (x, t) \) in \( B_{m+2} \times [0, T] \) we have \( f^2 |\text{Rm}|^2 \leq C \frac{N^3}{f(t)^2 r^4} \).

Comparing with our comments following (3.7) we arrive at the following estimate:
\[ |\text{Rm}|^2 \leq C \frac{N^8}{f(t)^2 r^4} \quad \text{on} \quad B_{m+2} \times (0, T). \] (3.12)

4 Higher order estimates

We finish the proof of Theorem 1.1 by establishing bounds on the derivatives of curvature, following the basic idea of Shi [17, 18] (cf. [2, 8, 9]). Our setting here is slightly different from that of Shi, where it is assumed that curvature is uniformly bounded (independent of \( t \)) but that (1.2) does not necessarily hold. Although the result we need can be recovered from what is known in the literature, we include the short proof for the sake of completeness.

Fix a sequence of radii \( r = r_0 > r_1 > r_2 > \ldots > r/2 \). For a fixed \( m \) we will denote by \( \psi \) a cutoff function which is zero outside \( B_{m+1} \) and identically 1 on \( B_{m+2} \).

We now work in real coordinates, writing, in this section, \( \nabla \) for the real covariant derivative \( \nabla \). Write \( \nabla^m \) for \( \nabla \nabla \ldots \nabla \) (\( m \) times). The key evolution equation we need is due to Hamilton [13]:
\[ (\partial_t - \Delta) |\nabla^m \text{Rm}|^2 = -|\nabla^{m+1} \text{Rm}|^2 + \sum_{i+j=m} \nabla^i \text{Rm} \ast \nabla^j \text{Rm} \ast \nabla^m \text{Rm}, \] (4.1)

where we are writing \( S \ast T \) to denote a linear combination of the tensors \( S \) and \( T \) contracted with respect to the metric \( g \). To clarify (4.1), we take \( \Delta \) here to be the complex Laplacian, which, acting on functions, is half the usual Riemannian Laplace operator. When comparing to the formula in [13] note that Hamilton’s Ricci flow equation includes a factor of 2 which is not present in our equation (1.1).

We will show inductively that
\[ |\nabla^m \text{Rm}|^2 \leq C \left( \frac{N^4}{f(t)^2 r^4} \right)^{m+2} \quad \text{on} \quad B_{m+2} \times (0, T) \] (4.2)

for every \( m \geq 0 \), the base case \( m = 0 \) having already been established in Section 3. Assume (4.2) holds for every value \( \leq m \). Let \( A = N^4 / r^4 \). We will apply the maximum principle argument to the function
\[ F = \psi^2 f^{m+2} |\nabla^m \text{Rm}|^2 + B f^m |\nabla^{m+1} \text{Rm}|^2 \] (4.3)
where $B = C_1 A$ with $C_1 \gg 1$ a large uniform constant. Let $(x_0, t_0) \in \overline{B_{r_{m+1}}} \times [0, T]$ be the point at which $F$ achieves a maximum. We may assume that $(x_0, t_0)$ lies in $B_{r_{m+1}} \times (0, T]$, otherwise, by the inductive hypothesis, we are finished. Suppose first that $f^{m+2} |\nabla^m Rm|^2 \leq A^{m+2}$ at the point $(x_0, t_0)$. Then at any $(x, t) \in \overline{B_{r_{m+2}}} \times [0, T]$ we have

$$f^{m+2} |\nabla^m Rm|^2 \leq A^{m+2} + f^{m+1} B |\nabla^{m-1} Rm|^2 \bigg|_{(x_0, t_0)}, \tag{4.4}$$

and our claim follows by the inductive hypothesis. Otherwise we have

$$f^{m+2} |\nabla^m Rm|^2 > A^{m+2} \text{ at } (x_0, t_0). \tag{4.5}$$

We note that by the inductive hypothesis we always have

$$|\nabla^i Rm| |\nabla^j Rm| \leq C(A/f)^{i+j+2} \text{ when } i, j < m. \tag{4.6}$$

At $(x_0, t_0)$,

$$0 \leq (\partial_t - \Delta) F \leq C \psi^2 f^{m+1} |\nabla^m Rm|^2 + |\Delta \psi^2| f^{m+2} |\nabla^m Rm|^2 - \psi^2 f^{m+2} |\nabla^{m+1} Rm|^2 + C \psi^2 f^{m+2} (A/F)^{m+2} |\nabla^m Rm| + C f^{m+2} (A/F)^{m+1} \nabla^m Rm| |\nabla^{m+1} Rm| + C B f^{m+1} (A/F)^{m+1} |\nabla^{m-1} Rm| \leq C f^{m+1} A |\nabla^m Rm|^2 + C f^{m+1} A^{m+2} |\nabla^m Rm| - C_1 A f^{m+1} |\nabla^m Rm|^2 + C A^{m+3} f^{-1} \leq - f^{m+1} A |\nabla^m Rm|^2 + C A^m f^{-1} \tag{4.7}$$

where the final inequality follows from $(1.5)$ and by taking the uniform constant $C_1$ in $B = C_1 A$ uniformly large enough. Hence $f^{m+2} |\nabla^m Rm|^2 \leq C' A^{m+2}$ at $(x_0, t_0)$ and then, arguing in a similar way to $(4.4)$ above, this completes the inductive step. Thus $(4.2)$ is established.

## 5 Proof of Corollary (1.2)

There are various ways to deduce Corollary (1.2) from Theorem (1.1). We could directly apply standard local parabolic theory (as discussed in [5] [15] for example), or the method in [8]. However, in our setting, we do not even need that $g(t)$ is a solution of a parabolic equation and instead we use an argument similar to one in [21] which uses only standard linear elliptic theory and some embedding theorems.

Fix a time $t \in (0, T]$. Regarding $g_{ij}$ as a set of $n^2$ functions, we consider the equations

$$\hat{\Delta} g_{ij} = - \sum_k R_{kkij} + \sum_{k,p,q} g^{kp} \partial_k g_{ij} \partial_p g_{pq} =: Q_{ij}. \tag{5.1}$$

where $\hat{\Delta} = \sum_k \partial_k \partial_k$. For each fixed $i, j$, we can regard $(5.1)$ as Poisson’s equation $\hat{\Delta} g_{ij} = Q_{ij}$.
For the purposes of this section we will say that a quantity $Z$ is \textit{uniformly bounded} if there exist constants $C, \alpha, \beta, \gamma$ depending only on $\hat{\omega}$ and $T$ such that $Z \leq CN^\alpha r^{-\beta} t^{-\gamma}$. In the case when the constants may depend on $\omega_0$, we insist that $\gamma = 0$.

Let $r = r_0 > r_1 > \cdots > r/2$ be as above. Fix $p > 2n$. From what we have proved, each $\|Q_{ij}\|_{L^p(B_{r_2})}$ is uniformly bounded. Applying the standard elliptic estimates for the Poisson equation (see for example Theorem 9.11 of [11]) to (5.1) we see that the Sobolev norm $\|g_{ij}\|_{L^p(B_{r_3})}$ is uniformly bounded. Morrey’s embedding theorem (Theorem 7.17 of [11]) gives that $\|g_{ij}\|_{C^{\kappa}(B_{r_4})}$ is uniformly bounded for some $0 < \kappa < 1$.

The key observation we now need is that the $m$th derivative of $Q_{ij}$ can be written as a finite sum $\sum_s A_s \ast B_s$ where each $A_s$ or $B_s$ is either a covariant derivative of $R_m$ or a quantity involving derivatives of $g$ up to order at most $m+1$. Hence if $g$ is uniformly bounded in $C^{m+1+\kappa}$ then each $Q_{ij}$ is uniformly bounded in $C^{m+\kappa}$, after possibly passing to a slightly smaller ball.

Applying this observation with $m = 0$ we see that each $\|Q_{ij}\|_{C^{\kappa}(B_{r_4})}$ is uniformly bounded. The standard Schauder estimates for the Poisson equation (Theorem 4.8 of [11]) give that $\|g_{ij}\|_{C^{2+\kappa}(B_{r_5})}$ is uniformly bounded.

We can now apply a bootstrapping argument. Applying the observation with $m = 1$ we see that $Q_{ij}$ is uniformly bounded in $C^{1+\kappa}$ on a slightly smaller ball and so on. This completes the proof of the corollary.

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* Department of Mathematics, California Polytechnic State University, San Luis Obispo, CA 93407

† Department of Mathematics, University of California San Diego, La Jolla, CA 92093