Closed-loop frequency analyses of reset systems

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Abstract

Today, linear controllers cannot satisfy requirements of high-tech industry due to fundamental limitations like the waterbed effect. This is one of the reasons why nonlinear controllers, such as reset elements, are receiving increased attention. To analyze reset elements in the frequency domain, researchers use the Describing Function (DF) method. However, it cannot accurately predict the closed-loop frequency responses of the system because it neglects high order harmonics. To overcome these barriers, this paper proposes a mathematical framework to model the closed-loop frequency responses of reset systems including the closed-loop high order harmonics. Furthermore, pseudo-sensitivities for reset systems are defined to make their analyses more straight-forward. In addition, a user-friendly toolbox is developed based on the proposed approach to facilitate frequency analyses of reset systems. To show the effectiveness of the method, multiple illustrative examples on a high-tech precision positioning stage are used to compare the results of the closed-loop frequency responses obtained using our proposed method with DF method. The results demonstrated that the proposed method is significantly more precise than the DF method. Indeed, this developed toolbox can enable reset controllers to be widely-used in industry and academia.

Key words: frequency domain analyses, reset controllers; pseudo-sensitivities, toolbox, describing function, High Order Sinusoidal Input Describing Function (HOSIDF)

1 Introduction

Proportional Integral Derivative (PID) controllers are used in more than 90% of the cases in industry (Samad et al. 2019, Dastjerdi et al. 2018, O’Dwyer 2009, Chen 2006). However, technology developments in cutting edge industry have control requirements that cannot be fulfilled with PID controllers. The main barrier which restricts the performance of these controllers is the water-bed effect (Middleton 1991, Schmidt et al. 2014). To overcome this problem, linear controllers should be substituted with non-linear ones. Reset controller is one such non-linear controller which has attracted much attention due to its simple configuration (Clegg 1958, Beker et al. 2004, Nešić et al. 2005, Aangenent et al. 2010, Forni et al. 2011, Villaverde et al. 2011, Baños and Barreiro 2011, Van Loon et al. 2017, Nair et al. 2018). Many researchers have used reset controllers to improve system performances (Wu et al. 2007, Pavlov et al. 2013, Panni et al. 2014, Hazeleger et al. 2016, Beerens et al. 2019, Saikumar et al. 2019b, Saikumar et al. 2019c).

A traditional reset controller consists of a linear element whose states are reset to zero when its input crosses zero (input zero-crossing resetting law). The simplest reset controller is the Clegg integrator which is a linear integrator with a reset mechanism (Clegg 1958). To provide more design freedom and applicability, reset controllers have been extended to First Order Reset Elements (FORE) (Zaccarian et al. 2005, Horowitz and Rosenbaum 1975) and Second Order Reset Elements (SORE) (Hazeleger et al. 2016). These reset elements are utilized to introduce new compensators to achieve huge performance enhancement (Hunnekens et al. 2014, Van den Eijnden et al. 2018, Palanikumar et al. 2018, Chen et al. 2019, Valério et al. 2019, Saikumar et al. 2019c). Nevertheless, reset controllers produce high order harmonics which have negative effects on the performance of the system. In order to reduce negative effects of high order harmonics of reset systems, several techniques such as non-zero reset values (Baños and Barreiro 2011, Horowitz and Rosenbaum 1975), reset band (Barreiro et al. 2014, Baños and Davó 2014), fixed reset instants (the time in which the rest action happens), and PI + CI configuration (Vidal and Baños 2008, Nair et al. 2018, HosseinNia et al. 2014) are introduced.
Frequency domain analyses are preferred in industry since all system performance characteristics can be easily ascertained. Describing Function (DF) is one of the methods for studying non-linear controllers in the frequency domain, and this has been widely used in literature for reset controllers as well. The DF method is a quasi-linear approximation of the steady-state output of a non-linear system considering only the first harmonic of the Fourier series expansion. The general formulation of the DF for reset controllers in the open-loop is presented in (Guo et al. 2009). Although the DF gives an insight into behavior of reset elements, it is not accurate enough to estimate the closed-loop frequency behavior of reset systems because of two main reasons. First, the high order harmonics are neglected. Second, unlike linear controllers, it is not possible to precisely predict the closed-loop behavior using DF of the open-loop. It is due to reset instants (time instants when reset happens) in the closed-loop being significantly different from those found in the open-loop. Hence, in this paper, the closed-loop frequency response of reset systems including high order harmonics will be found directly considering closed-loop reset instants. Moreover, pseudo-sensitivities are defined to combine all harmonics to facilitate analyzing reset systems in the closed-loop configuration.

In this article, first, preliminaries related to frequency analyses of reset controllers are presented in Section 2. In Section 3, a method to obtain closed-loop frequency responses of reset systems including high order harmonics for stable linear plants is developed, and pseudo-sensitivities are defined. Then, a user-friendly toolbox is developed to easily utilize the proposed approach on practical systems. In Section 4, the performance of our proposed methods is assessed through several illustrative examples. Finally, some concluding remarks and suggestions for future studies are presented in section 5.

2 Preliminaries

In this section, reset elements and their frequency responses are briefly presented. Note that in all the following methods it is assumed that the system is stable (i.e. bounded input-bounded output).

The state-space equation of a reset element is

\[
\begin{align*}
\dot{x}_R(t) &= A_R x_R(t) + B_R r(t) & r(t) \neq 0 \\
x_R(t^+) &= A_R x_R(t) & r(t) = 0 ,
\end{align*}
\]

in which \(A_R, B_R, C_R, \) and \(D_R\) are dynamic matrices of the controller, \(A_R\) is the reset matrix. In order to find DF of a general reset element, a sinusoidal reference \(r(t) = a_0 \sin(\omega t)\) is applied, and its output is approximated with the first harmonic of the Fourier series expansion of the steady-state output. Therefore, the state-space equation of the reset element (1) can be re-written as

\[
\begin{align*}
\dot{x}_R(t) &= A_R x_R(t) + a_0 B_R \sin(\omega t) & t \neq t_k \\
x_R(t^+) &= A_R x_R(t) & t = t_k ,
\end{align*}
\]

in which \(t_k = \frac{k \pi}{\omega}\) is the reset instant. According to (Guo et al. 2009), DF is found as follows:

\[
A_{DF} = \frac{a_1(\omega) e^{i \theta_1(\omega)}}{a_0} = C_R j\omega I - A_R)^{-1} (I + j \theta(\omega)) B_R + D_R,
\]

where \(\theta(\omega)\) is

\[
\theta(\omega) = \frac{2a_0}{\omega} (I + e^{i \omega t})(I + A_R e^{i \omega t})^{-1} A_R (I + e^{i \omega t})^{-1} - I) (\omega^2 I + A_R^2)^{-1}
\]

Recently, a new tool called Higher-Order Sinusoidal Input Describing Functions (HOSIDF) (Fig. 1) for studying non-linearities in the frequency domain is introduced by (Nuij et al. 2006). In that method, a non-linear system is considered as a virtual harmonic generator, and HOSIDF is defined in the following way (Nuij et al. 2006):

\[
H_n(j \omega) = \frac{a_n(\omega) e^{i \theta_n(a_0, \omega)}}{a_0}.
\]

in which \(a_n\) is the \(n^{th}\) component of the Fourier series expansion of the steady-state output of the system for a sinusoidal input. This framework has been extended for reset control.
In summary, the DF method just provides information about the first order harmonic of the system by which the first order harmonic of the closed-loop is approximated using linear sensitivity function relations. As a result, there is error between the actual first order harmonic of the closed-loop and the estimation using the DF method. Moreover, the HOSIDF method only presents high order harmonics of the system in the open-loop, and it cannot be used to find high order harmonics of the closed-loop. Consequently, the DF method is not reliable to assess the closed-loop performance of the reset systems, particularly in precision applications. Hence, it is necessary to develop a method to get the frequency behavior of the closed-loop configuration of reset controllers to predict their performances.

3 Closed-loop frequency responses

In this section, we obtain closed-loop frequency responses of reset elements which control Single Input Single Output (SISO) Linear Time Invariant (LTI) Bounded Input Bounded Output (BIBO) systems. Figure 2 shows the general block diagram of reset systems in the closed-loop configuration. In this study, as was shown in Fig. 2, the complete closed-loop system is considered as a virtual harmonic generator, and all harmonics are calculated applying Fourier series expansion to the steady-state output of the system.

Remark 1 Different sequences of control filters do not result in a similar response when there is a non-linear filter among them. Therefore, in this study, the linear parts of the controller before and after the reset element are considered as $\mathcal{C}_{\mathcal{L}}$ and $\mathcal{C}_{\mathcal{L}}$, respectively (Fig. 2). If $\mathcal{C}_{\mathcal{L}} = 1$, the system has zero-error crossing reseting law. Otherwise, based on $\mathcal{C}_{\mathcal{L}}$, different reset laws (e.g. $e(t) = 0$, $\int e(t) = 0$) can be emerged and taken into consideration by the proposed method.

3.1 Reference tracking analyses

In this part, the frequency response of the system (Fig. 2) from input $r(t)$ to output $y(t)$, error $e(t)$, and control input $u(t)$ will be found in absence of noise $w(t)$ and disturbance $d(t)$ (i.e. $r(t) = a_0 \sin(\omega t)$ and $d = w = 0$). To this respect, we construct the state-space representation of the closed-loop system shown in Fig. 2. The state-space representation of the linear controller $\mathcal{C}_{\mathcal{L}}$ is as

\[
\begin{align*}
\dot{x}_{\mathcal{L}}(t) &= A_{\mathcal{L}} x_{\mathcal{L}}(t) + B_{\mathcal{L}} e(t) \\
u_{\mathcal{L}}(t) &= C_{\mathcal{L}} x_{\mathcal{L}}(t) + D_{\mathcal{L}} e(t),
\end{align*}
\]

(7)

in which $A_{\mathcal{L}}, B_{\mathcal{L}}, C_{\mathcal{L}},$ and $D_{\mathcal{L}}$ are dynamic matrices of controller $\mathcal{C}_{\mathcal{L}}$, and $e(t)$ is the error of the system ($r(t) - y(t)$). The state-space equation of the reset element of the system $\mathcal{R}$ is

\[
\begin{align*}
\dot{x}_{\mathcal{R}}(t) &= A_{\mathcal{R}} x_{\mathcal{R}}(t) + B_{\mathcal{R}} u_{\mathcal{R}}(t) \\
u_{\mathcal{R}}(t) &= C_{\mathcal{R}} x_{\mathcal{R}}(t) + D_{\mathcal{R}} u_{\mathcal{R}}(t),
\end{align*}
\]

(8)

and state-space equation of the linear controller $\mathcal{C}_{\mathcal{L}}$ is

\[
\begin{align*}
\dot{x}_{\mathcal{L}}(t) &= A_{\mathcal{L}} x_{\mathcal{L}}(t) + B_{\mathcal{L}} u_{\mathcal{L}}(t) \\
u(t) &= C_{\mathcal{L}} x_{\mathcal{L}}(t) + D_{\mathcal{L}} u_{\mathcal{L}}(t),
\end{align*}
\]

(9)

where $A_{\mathcal{L}}, B_{\mathcal{L}}, C_{\mathcal{L}},$ and $D_{\mathcal{L}}$ are dynamic matrices of controller $\mathcal{C}_{\mathcal{L}}$. The plant (to avoid overly of complex equations, we only consider linear plants with strictly proper transfer...
function) is represented as

$$\begin{align*}
\dot{x}_G(t) &= A_Gx_G(t) + B_Gu(t) \\
y(t) &= C_Gx_G(t)
\end{align*}$$

(10)

in which $A_G$, $B_G$, and $C_G$ are dynamic matrices of the plant $G$. Now, substituting $e(t)$ with $r(t) - y(t)$ in (7), and using equations (8) to (10), the general equation of the closed-loop system (Fig. 2) in the state-space format is obtained as

$$\begin{align*}
\dot{x}(t) &= Ax(t) + Br(t) \quad u_{x_1}(t) \neq 0 \\
x(t^+) &= A_r x(t) \quad u_{x_2}(t) = 0, \\
y(t) &= Cx(t)
\end{align*}$$

(11)

where

$$A = \begin{bmatrix}
A_{x_1} & 0 & 0 & -B_{x_1}C_G \\
B_{x_1}C_{x_1} & A_{x_1} & 0 & -B_{x_1}D_{x_1}C_G \\
B_{x_2}D_{x_1}C_{x_1} & B_{x_2}C_{x_1} & A_{x_2} & -B_{x_2}D_{x_1}C_G \\
B_{x_2}D_{x_2}D_{x_1}C_{x_1} & B_{x_2}D_{x_2}C_{x_1} & B_{x_2}C_{x_1} & A_{x_2} - B_{x_2}D_{x_2}D_{x_1}C_G
\end{bmatrix},$$

$$B = \begin{bmatrix}
B_{x_1} \\
B_{x_1}D_{x_1} \\
B_{x_2}D_{x_1}A_{x_1} \\
B_{x_2}D_{x_2}D_{x_1}
\end{bmatrix}, \quad A_r = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & A_{x_1} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.$$
\[ \xi_{k+1} = A_r x'(t_{k+1}) = A_r \left[ e^{A(t_{k+1} - t_k)} \left( \xi_k + \psi(t_k) \right) - \psi(t_{k+1}) \right] \]  

(23)

Now, putting \( k = 0 \) in (22), (23), (16), and (17), and considering \( \xi_0 = 0 \), we find that \( t_1' = t_1 \) and \( \xi_1' = -\xi_1 \). Following the same procedure, we substitute \( k = 1 \) in (16) and (22), and we get

\[
D_{\xi_2} \sin(\omega t_2) - C_{\xi_2} \psi(t_2) = -C_{\xi_2} e^{A(t_2 - t_1)} \left( \xi_1 + \psi(t_1) \right) \\
D_{\xi_1} \cos(\omega t_2) - C_{\xi_1} \psi(t_2) = C_{\xi_1} e^{A(t_2 - t_1)} \left( -\xi_1 - \psi(t_1) \right).
\]

Therefore, it is found out that \( t_1' = t_2 \), and using (23) and (17), we achieve

\[
\xi_2' = A_r \left[ e^{A(t_2 - t_1)} \left( -\xi_1 + \psi(t_1) \right) - \psi(t_2) \right] = -\xi_2.
\]

Similarly, we prove that

\[
t_2 = t_3' \quad \xi_3 = -\xi_3' \\
t_3 = t_4' \quad \xi_4 = -\xi_4' \\
\vdots 
\]

Therefore, (21) becomes

\[
x'(t) = -e^{A(t - t_i)} \left( \xi_k + \psi(t_k) \right) + \psi(t) = -x(t), \quad t \in (t_k, t_{k+1}] \\
\Rightarrow y'(t) = -Cx(t) = -y(t)
\]

which means \( y(t) \) is an odd function.

Therefore, even terms of \( T_n \) are zero, and we just need to calculate its odd terms.

**Theorem 1** The frequency response of \( \tilde{y} \) (the complementary sensitivity) for any stable closed-loop reset system (11) is

\[
T_n(j\omega) = \begin{cases} 
\frac{j\omega C}{\pi} (A - j\omega l)^{-1} \left( \sum_{i=1}^{m} \mathcal{R}_{(1,i)} \right) - C(j\omega l + A) \mathcal{F} & n = 1 \\
\frac{j\omega C}{\pi} (A - jn\omega l)^{-1} \left( \sum_{i=1}^{m} \mathcal{R}_{(i,n)} \right) & \text{odd } n > 1 \\
0 & \text{even } n > 1
\end{cases}
\]

in which

\[
\mathcal{R}_{(i,n)} = \left( \frac{e^{A(t_{si} - t_{si-1})} - I}{e^{jn\omega t_{si-1}} - I} \right) \left( \xi_{si-1} + \psi(t_{si-1}) \right)
\]

\[
t_{si} = \{ s_{i,j} \in t_k, i \in \mathbb{Z}^+ | t_{si0} \leq t_{si} \leq t_{si1}, i \leq m \}.
\]

(27)

**Proof.** Using (14), (20) is rewritten as

\[
T_n(j\omega) = \frac{j\omega C}{\pi} \sum_{i=1}^{m} \left( \int_{t_{si0}}^{t_{si1}} \mathcal{R}_{(i,n)}(t)e^{-j\omega t} dt - \int_{t_{si0}}^{t_{si1}} \psi(t)e^{-j\omega t} dt \right).
\]

(28)

where

\[
\mathcal{F}_{n-1}(t) = e^{A(t-t_{n-1})} \left( \xi_{n-1} + \psi(t_{n-1}) \right).
\]

(29)

Then, the two integrals of (28) are calculated as

\[
\int_{t_{si0}}^{t_{si1}} \mathcal{R}_{(i,n)}(t)e^{-j\omega t} dt = (A - jn\omega I)^{-1} \mathcal{R}_{(i,n)}
\]

and

\[
\int_{t_{si0}}^{t_{si1}} \psi(t)e^{-j\omega t} dt = \frac{\pi (I - jA \omega I)}{2} \mathcal{S} \quad n = 1 \\
0 \quad n \geq 2
\]

(30)

(31)

Now, substituting (30) and (31) in (28), we obtain

\[
T_n(j\omega) = \begin{cases} 
\frac{j\omega C}{\pi} (A - j\omega l)^{-1} \left( \sum_{i=1}^{m} \mathcal{R}_{(1,i)} \right) - C(j\omega l + A) \mathcal{F} & n = 1 \\
\frac{j\omega C}{\pi} (A - jn\omega l)^{-1} \left( \sum_{i=1}^{m} \mathcal{R}_{(i,n)} \right) & \text{odd } n > 1 \\
0 & \text{even } n > 1
\end{cases}
\]

(32)

Finally, using (32) and considering Lemma 1, Theorem 1 is proved.

To find the frequency response of \( \tilde{e} \) (the sensitivity frequency response) for the reset systems, we apply Fourier series expansion to \( e(t) \) and obtain

\[
S_n(j\omega) = \frac{1}{\pi} \int_{t_{si0}}^{t_{si1}} e(t)e^{-j\omega t} dt = \frac{j\omega}{\pi} \int_{t_{si0}}^{t_{si1}} (r(t) - Cx(t))e^{-j\omega t} dt
\]

in which \( S_n \) represents the \( n^{th} \) harmonic of the sensitivity frequency response.

**Corollary 1.1** Using the same procedure, the frequency response of \( \tilde{u} \) for any stable reset system (11) is obtained as

\[
S_n(j\omega) = \begin{cases} 
1 - \frac{j\omega C}{\pi} (A - j\omega l)^{-1} \left( \sum_{i=1}^{m} \mathcal{R}_{(1,i)} \right) + C(j\omega l + A) \mathcal{S} & n = 1 \\
\frac{j\omega C}{\pi} (A - jn\omega l)^{-1} \left( \sum_{i=1}^{m} \mathcal{R}_{(i,n)} \right) & \text{odd } n > 1 \\
0 & \text{even } n > 1
\end{cases}
\]

(34)

Also, we apply Fourier series expansion to the control input of the system \( u(t) \) (13) to get the frequency response of \( \tilde{u} \) (the control sensitivity) as

\[
CS_n(j\omega) = \frac{1}{\pi} \int_{t_{si0}}^{t_{si1}} u(t)e^{-j\omega t} dt = \frac{j\omega}{\pi} \int_{t_{si0}}^{t_{si1}} (C_u x(t) + D_u x(t))e^{-j\omega t} dt
\]

(35)
in which \( CS_n \) represents the \( n \)th harmonic of the control sensitivity of the closed-loop configuration.

**Corollary 1.2** For LTI stable plants, the frequency response of \( \frac{\mu}{r} \) (the control sensitivity) for reset systems (11) is obtained using \( CS_n(j \omega) = \frac{T_n(j \omega)}{G(n j \omega)} \) as

\[
CS_n(j \omega) = \begin{cases} 
  j \omega C (j \omega A - j \omega I)^{-1} (\sum_{i=1}^{m} \Re_{((i,0))}) - \pi C (j \omega A + 1) \Re & n = 1 \\
  j \omega C (j \omega A - j \omega I)^{-1} (\sum_{i=1}^{m} \Re_{((i,0))}) & \text{odd } n > 1 \\
  0 & \text{even } n > 1
\end{cases}
\]

**3.2 Noise rejection analyses**

In this part, the frequency response of the system from noise \( w(t) \) to error \( e(t) \) and control input \( u(t) \) will be found in absence of reference \( r(t) \) and disturbance \( d(t) \) (i.e. \( w(t) = \sin(\omega t) \) and \( r = d = 0 \)). For this purpose, we can substitute \( r(t) \) with \( -w(t) = -\sin(\omega t) \) in (11). Thus,

\[
\frac{y}{w} = -T_n, \quad \frac{u}{w} = -CS_n, \quad \text{and} \quad \frac{e}{w} = T_n.
\]

**3.3 Disturbance rejection analyses**

In this part, the frequency response of the system from disturbance \( d(t) \) to error \( e(t) \) and control input \( u(t) \) will be found in absence of reference \( r(t) \) and noise \( w(t) \) (i.e. \( d(t) = \sin(\omega t) \) and \( r = w = 0 \)) with a procedure similar to that in Section 3.1. Most of previous equations remain the same, but some of them are changed as elaborated in following.

The state-space equations of the plant in the presence of the disturbance are

\[
\begin{align*}
    \dot{x}_d(t) &= A_d x_d(t) + B_d (u(t) + d(t)) \\
    y_d(t) &= C_d x_d(t)
\end{align*}
\]

Furthermore, substituting \( e(t) \) with \( -y_d(t) \) in (7) to (9) and (38), (11) is changed to

\[
\begin{cases}
    \dot{x}(t) = A x(t) + B_d d(t) & u_{\not{x}}(t) \neq 0 \\
    \dot{x}^*(t) = A_r x(t) & u_{\not{x}}(t) = 0, \\
    y_d(t) = C x(t)
\end{cases}
\]

Remark 2 When \( C_{\not{x}} = 1 \), the columns related to \( x_{\not{x}}(t) \) in \( A, A_r, \) and \( C \) are removed. Also, the rows related to \( x_{\not{x}}(t) \) are removed in \( x(t), B, \) and \( B_{\not{x}}. \) In addition, when \( C_{\not{x}} = 1 \) the columns related to \( x_{\not{x}}(t) \) in \( A, A_r, \) and \( C \) are removed, and the rows related to \( x_{\not{x}}(t) \) are removed in \( x(t), B, \) and \( B_{\not{x}}. \)

\[
u_{\not{x}}(t) \text{ and } u(t) \text{ in the case of disturbance rejection analyses are}
\]

\[
u_{\not{x}}(t) = C_{\not{x}} x(t) \]

and

\[
u(t) = C_{\not{x}} x(t).
\]

Furthermore, matrix \( B \) in (15) has to be replaced with \( B_{\not{x}} \) as

\[
\psi_{\not{x}}(t) = (\omega t \cos(\omega t) + 1) \sin(\omega t) \Re_{\not{x}}, \quad \Re_{\not{x}} = (\omega t^2 + 1)^{-1} B_{\not{x}}.
\]

Since \( u_{\not{x}}(t) \) is changed in the presence of the disturbance, recursive algorithm (16) is changed to

\[
e(t_{k+1}) = 0 \Rightarrow u_{\not{x}}(t_{k+1}) = 0 \Rightarrow 0
\]

\[
C_{\not{x}} x(t_{k+1}) = C_{\not{x}} e^{A_{\not{x}}(t_{k+1} - t_k)} (x_{\not{x}}(t_k) + \psi_{\not{x}}(t_k)).
\]

The frequency response of \( \frac{e}{d} \) (the process sensitivity) for reset systems is achieved applying Fourier series expansion to (39) as

\[
PS_n(j \omega) = \frac{\int_{-\infty}^{\infty} e(t) e^{-jn\omega t} dt}{\int_{-\infty}^{\infty} \sin(\omega t) e^{-jn\omega t} dt} = \frac{j \omega C \int_{-\infty}^{\infty} x(t) e^{-jn\omega t} dt}{\pi}
\]

in which \( PS_n \) represents the \( n \)th harmonic of the process sensitivity of the closed-loop configuration.

**Corollary 1.3** The frequency response of \( \frac{e}{d} \) (the process sensitivity) for any stable reset system (39) is calculated as

\[
PS_n(j \omega) = \begin{cases} 
  j \omega C \pi (j \omega A - j \omega I)^{-1} (\sum_{i=1}^{m} \Re_{((i,0))}) + C(j \omega A + 1) \Re & n = 1 \\
  j \omega C \pi (j \omega A - j \omega I)^{-1} (\sum_{i=1}^{m} \Re_{((i,0))}) & \text{odd } n > 1 \\
  0 & \text{even } n > 1
\end{cases}
\]

In order to see effects of the disturbance on the control input, we apply Fourier series expansion to the control input of the system \( u(t) \) (42) to get the frequency response of \( \frac{\mu}{d} \) in the presence of disturbance as

\[
CS_{\not{d}}(j \omega) = \frac{\int_{-\infty}^{\infty} \mu(t) e^{-jn\omega t} dt}{\int_{-\infty}^{\infty} \sin(\omega t) e^{-jn\omega t} dt} = \frac{j \omega C \int_{-\infty}^{\infty} x(t) e^{-jn\omega t} dt}{\pi}
\]
in which $CS_{n}(\omega)$ represents the $n^{th}$ harmonic of the control sensitivity due to the disturbance of the closed-loop configuration.

**Corollary 1.4** For LTI stable plants, the frequency response of $\frac{u}{d}$ (the control sensitivity due to disturbance) for reset systems (46) is obtained using

$$CS_{n}(\omega) = \begin{cases} \frac{-PS_{1}(j\omega)}{G(j\omega)} - 1 & n = 1 \\ \frac{-PS_{n}(j\omega)}{G(nj\omega)} & n > 1 \end{cases}$$

as

$$CS_{n}(\omega) = \begin{cases} j\omega C(A-j\omega)^{-1}(\sum_{i=1}^{n}\xi_{i}B_{i}) - \pi C(j\omega + A)\frac{C}{\pi} & \text{odd } n > 1 \\ 0 & \text{even } n > 1 \end{cases}$$

### 3.4 Pseudo-sensitivities for reset systems

Error $e(t)$ and control input $u(t)$ analyses are two main factors while designing a controller. In linear systems, these analyses are done using the closed-loop transfer functions (Schmidt et al. 2014). As was discussed in Section 1, although reset systems are analyzed using the DF of reset controllers in the closed-loop sensitivity equations, these approximations are not precise enough because these equations are not applicable for non-linear controllers due to the existence of high order harmonics. On the other hand, in some cases, it is not trivial to analyze reset controllers considering all harmonics separately. In this part, in order to make analyses of reset systems more straightforward, we combine all harmonics into one frequency function for each closed-loop frequency response.

Since the tracking error of the reset system is the summation of all harmonics of the system, it is a periodic function with the period of the first harmonic of the system ($\frac{2\pi}{\omega}$) as shown in figure 3. Thus, from the perspective of precision, it is possible to define a pseudo-sensitivity frequency response as the ratio of the maximum tracking error of the system ($r(t) = \sin(\omega t), w = d = 0$) to the magnitude of the reference at each frequency.

**Definition 1** Pseudo-sensitivity $S_{\infty}$

$$\forall \omega \in \mathbb{R}^+ \exists! t_{\max} \in (t_{\min}, t_{\max}) \forall t \in (t_{\min}, t_{\max}) : e(t_{\max}) = e_{\max} \geq e(t)$$

$$\Rightarrow S_{\infty}(j\omega) = \max_{t_{\min} \leq t \leq t_{\max}} \frac{r(t) - y(t)}{|r|} = \max_{t_{\min} \leq t \leq t_{\max}} \sin(|\omega t|) - Cx(t) = e_{\max} e^{j\phi_{\text{max}}},$$

**Definition 2** Pseudo-process sensitivity $PS_{\infty}$

$$\forall \omega \in \mathbb{R}^+ \exists! t_{\max} \in (t_{\min}, t_{\max}) \forall t \in (t_{\min}, t_{\max}) : e(t_{\max}) = e_{\max} \geq e(t)$$

$$\Rightarrow PS_{\infty}(j\omega) = \max_{t_{\min} \leq t \leq t_{\max}} \frac{(-y(t))}{|d|} = \max_{t_{\min} \leq t \leq t_{\max}} (-Cz(t)) = e_{\max} e^{j\phi_{\text{max}}},$$

where $\phi_{\text{max}} = \frac{\pi}{2} - \omega t_{\max}$, and

$$t_{\max} \in \{t_{\text{ext}} \mid e(t_{\text{ext}}) = 0, t_{\text{ext}} \leq t_{\text{ext}} \leq t_{\text{ext}} \} \cup \{t_{\max} \mid i \in \mathbb{Z}, 0 \leq i \leq m\}.$$
Since \( e(t) \) in the presence of the disturbance is equal to \(-y_d(t)\), and using (39), (50) is changed to

\[
\dot{e}(t_{ext}) = 0 \Rightarrow Cx(t_{ext}) = C(Ax(t_{ext}) + B \sin(\omega t_{ext})) = 0 \Rightarrow CB \sin(\omega t_{ext}) = CA \left( \psi(t_{ext}) - e^{A(t_{ext}-t_0)} (\xi_{ss,m} + \psi(t_{ss,m})) \right).
\]

\( t_{ext} \in (t_{ss,m}, t_{ss}) \), \( i = \{ i | i \leq m \} \).

(51)

Besides, the pseudo-complementary sensitivity is defined in a similar way as the ratio of the maximum error of the system due to the noise \((w(t) = \sin(\omega t), r = d = 0)\) to the magnitude of the noise at each frequency.

**Definition 3** Pseudo-complementary sensitivity \( T_{w} \)

\[
\forall \omega \in \mathbb{R}^+ \exists! t_{maxw} \in (t_{ss,m}, t_{ss}) \mid \forall t \in (t_{ss,m}, t_{ss}) : e(t_{maxw}) = e_{maxw} \geq e(t)
\]

\[
\Rightarrow T_{w}(j\omega) = \frac{\max_{t_{ss,m} \leq t \leq t_{ss}} y(t)}{|w|} = \max_{t_{ss,m} \leq t \leq t_{ss}} (Cx(t)) = e_{maxw}/\varphi_{maxw}
\]

where \( \varphi_{maxw} = \frac{\pi}{2} - \omega t_{maxw} \), and

\( t_{maxw} \in \{ t_{ext} | t_{ext} = 0, t_{ss,m} \leq t_{ext} \leq t_{ss} \} \cup \{ t_{ss} | i \in \mathbb{Z}, 0 \leq i \leq m \} \).

Moreover, in the presence of the noise, (50) is changed to

\[
\dot{e}(t_{ext}) = 0 \Rightarrow Cx(t_{ext}) = C(Ax(t_{ext}) + B \sin(\omega t_{ext})) = 0 \Rightarrow CB \sin(\omega t_{ext}) = CA \left( \psi(t_{ext}) - e^{A(t_{ext}-t_0)} (\xi_{ss,m} + \psi(t_{ss,m})) \right).
\]

\( t_{ext} \in (t_{ss,m}, t_{ss}) \), \( i = \{ i | i \leq m \} \).

(52)

Likewise, knowing the maximum magnitude of the controller output is necessary to avoid the saturation of amplifiers in practical systems. Thus, in the same way, the pseudo-control sensitivity is defined as the ratio of the maximum control input to the magnitude of the reference at each frequency.

**Definition 4** Pseudo-control sensitivity \( CS_{w} \)

\[
\forall \omega \in \mathbb{R}^+ \exists! t_{maxw} \in (t_{ss,m}, t_{ss}) \mid \forall t \in (t_{ss,m}, t_{ss}) : u(t_{maxw}) = u_{maxw} \geq u(t)
\]

\[
\Rightarrow CS_{w}(j\omega) = \frac{\max_{t_{ss,m} \leq t \leq t_{ss}} u(t)}{|d|} = \max_{t_{ss,m} \leq t \leq t_{ss}} (Cux(t) + Dux \sin(\omega t)) = u_{maxw}/\varphi_{maxw}
\]

where \( \varphi_{maxw} = \frac{\pi}{2} - \omega t_{maxw} \), and

\( t_{maxw} \in \{ t_{ext} | t_{ext} = 0, t_{ss,m} \leq t_{ext} \leq t_{ss} \} \cup \{ t_{ss} | i \in \mathbb{Z}, 0 \leq i \leq m \} \).

Extrema of \( u(t) \) are found using (13) and (14) as

\[
u(t_{ext}) = 0 \Rightarrow-Cux(t_{ext}) = Dux \omega \cos(\omega t_{ext}) \Rightarrow Dux \omega \cos(\omega t_{ext}) + Cux B \sin(\omega t_{ext}) = CA \left( \psi(t_{ext}) - e^{A(t_{ext}-t_0)} (\xi_{ss,m} + \psi(t_{ss,m})) \right).
\]

\( t_{ext} \in (t_{ss,m}, t_{ss}) \), \( i = \{ i | i \leq m \} \).

(53)

Similarly, the pseudo-control sensitivity due to the noise is defined as the ratio of maximum control input to the magnitude of the noise.

**Definition 5** Pseudo-control sensitivity of noise \( CS_{wn} \)

\[
\forall \omega \in \mathbb{R}^+ \exists! t_{maxw} \in (t_{ss,m}, t_{ss}) \mid \forall t \in (t_{ss,m}, t_{ss}) : u(t_{maxw}) = u_{maxw} \geq u(t)
\]

\[
\Rightarrow CS_{wn}(j\omega) = \frac{\max_{t_{ss,m} \leq t \leq t_{ss}} u(t)}{|w|} = \max_{t_{ss,m} \leq t \leq t_{ss}} (Cux(t) - Dux \sin(\omega t)) = u_{maxw}/\varphi_{maxw}
\]

where \( \varphi_{maxw} = \frac{\pi}{2} - \omega t_{maxw} \).

**Remark 3** As explained before, for noise rejection analyses, we can consider \((r(t) = -\sin(\omega t)\) and \(w = d = 0\)). Therefore, similar to linear controller transfer functions \((\frac{u}{r} = \frac{-u}{w})\),

\[
CS_{wn} = -CS_{w}.
\]

(54)

In linear control theory, the transfer function of \( \frac{u}{d} \) is equal to \(-\frac{e}{d} \). However, this relation does not hold for the defined pseudo-sensitivities due to the non-linear nature of the system. Hence, another pseudo-control sensitivity is defined as the ratio of maximum control input in the presence of the disturbance to the magnitude of the disturbance.

**Definition 6** Pseudo-control sensitivity of disturbance \( CS_{dn} \)

\[
\forall \omega \in \mathbb{R}^+ \exists! t_{maxd} \in (t_{ss,m}, t_{ss}) \mid \forall t \in (t_{ss,m}, t_{ss}) : u(t_{maxd}) = u_{maxd} \geq u(t)
\]

\[
\Rightarrow CS_{dn}(j\omega) = \frac{\max_{t_{ss,m} \leq t \leq t_{ss}} u(t)}{|d|} = \max_{t_{ss,m} \leq t \leq t_{ss}} (Cux(t) + Dux \sin(\omega t)) = u_{maxd}/\varphi_{maxd}
\]

where \( \varphi_{maxd} = \frac{\pi}{2} - \omega t_{maxd} \), and

\( t_{maxd} \in \{ t_{ext} | t_{ext} = 0, t_{ss,m} \leq t_{ext} \leq t_{ss} \} \cup \{ t_{ss} | i \in \mathbb{Z}, 0 \leq i \leq m \} \).

In addition, extrema of \( u(t) \) in the presence of the disturbance are obtained utilizing (39) as follows:

\[
u(t_{ext}) = 0 \Rightarrow-Cux(t_{ext}) = Dux \omega \cos(\omega t_{ext}) \Rightarrow Dux \omega \cos(\omega t_{ext}) + Cux B \sin(\omega t_{ext}) = CA \left( \psi(t_{ext}) - e^{A(t_{ext}-t_0)} (\xi_{ss,m} + \psi(t_{ss,m})) \right).
\]

\( t_{ext} \in (t_{ss,m}, t_{ss}) \), \( i = \{ i | i \leq m \} \).

(55)

To wrap up, the pseudo-sensitivities which relate \( r(t), d(t), \) and \( w(t) \) to the error \( e(t) \) and control input \( u(t) \) of the system are found. These relations are essential for analysing reset systems in the frequency domain.
3.5 High frequency analyses

The previous relations may be computationally expensive and may require small time steps at high frequencies. In order to simplify the previously defined relations, the reset instants at high frequency are approximated. In frequencies above the crossing over frequency, due to the attenuating behavior of the closed-loop system, the error \( e(t) \) is prevailed by the reference. As a result, the reset instant line, which triggers the reset integral, is dominated by the first order harmonic of \( u_{\omega 1}(t) \) in those frequencies. If we consider

\[
u_{\omega 1} = u_{\omega 1} \sin(\omega t + \phi_{\omega 1})
\]
as the first harmonic of \( u_{\omega 1}(t) \), then,

\[
∀ \varepsilon \in (0, 1) \exists \omega_0 \in \mathbb{R}^+ | \forall \omega \geq \omega_0 : \forall t \in (t_{ss}, t_{ss}, t_{ss}) \Rightarrow |u_{\omega 1}(t) - u_{\omega 1}(t)| \leq \varepsilon
\]  

(56)

Therefore, if \( \varepsilon \) is chosen small enough, the reset instants \( t_k \) for \( \omega \geq \omega_0 \) can be obtained through

\[
t_k \approx k \pi - \frac{\phi_{\omega 1}}{\omega}
\]

(57)
in which \( \phi_{\omega 1} \) can be precisely approximated as

\[
\phi_{\omega 1} \approx \angle \left( \frac{C_{\omega 1}(j\omega)}{1 + C_{\omega 1} C_{DF} C_{\omega 2} G(j\omega)} \right)
\]

(58)

where \( C_{DF} \) is the DF of \( C_{\omega} \) obtained using (3).

**Remark 4** The precision of the approximation depends on the magnitude of \( \varepsilon \). The smaller the value of \( \varepsilon \) is, the more precise the approximation is.

Utilizing (57), (11) for \( \omega \geq \omega_0 \) becomes

\[
\begin{align*}
\dot{x}(t) &= A x(t) + B \sin(\omega t) & t \neq \frac{k \pi - \phi_{\omega 1}}{\omega} \\
x(t^+) &= A x(t) & t = \frac{k \pi - \phi_{\omega 1}}{\omega} \\
y(t) &= C x(t)
\end{align*}
\]

(59)

To simplify the problem, we consider \( r = \sin(\omega t - \phi_{\omega 1}) \) as a reference, so (59) can be re-written as

\[
\begin{align*}
\dot{x}(t) &= A x(t) + B \sin(\omega t - \phi_{\omega 1}) & t \neq \frac{k \pi - \phi_{\omega 1}}{\omega} \\
x(t^+) &= A x(t) & t = \frac{k \pi - \phi_{\omega 1}}{\omega} \\
y(t) &= C x(t)
\end{align*}
\]

(60)

Thus, \( \psi(t) \) (15) is changed to

\[
\psi_{\omega}(t) = (\omega I \cos(\omega t - \phi_{\omega 1}) + A \sin(\omega t - \phi_{\omega 1})) G.
\]

(61)

Moreover, we can consider \( t_{ss} = \{0, \frac{\pi}{\omega}, \frac{2\pi}{\omega}, \ldots\} \), and using similar approach described in (Guo et al. 2009), we get

\[
\xi_{ss} = -\xi_{ss1} = \frac{-A_r(I + e^{\frac{j\pi}{\omega}})\psi_{\omega}(0)}{I + A_r e^{\frac{j\pi}{\omega}}}. 
\]

(62)

Having \( t_{ss} \) and \( \xi_{ss} \), (27) is calculated as

\[
\sum_{i=1}^{2} \mathcal{R}(i, n) = -2(e^{\frac{j\pi}{\omega}} + I)(\xi_{ss} + \psi_{\omega}(0))
\]

(63)

**Corollary 1.5** For \( \omega \geq \omega_0 \), using (26) and (63), the frequency response of \( \frac{u}{r} \) for any stable reset system (60) is obtained as

\[
T_n(j\omega) = \begin{cases} 
C(A - j\omega I)^{-1} \theta_{\omega}(\omega) - C(j\omega I + A) \mathcal{F} & n = 1 \\
C(A - j\omega I)^{-1} \theta_{\omega}(\omega) & odd n > 1 \\
0 & even n > 1
\end{cases}
\]

(64)
in which

\[
\theta_{\omega}(\omega) = \frac{-2 j \omega e^{j\phi_{\omega 1}}}{\pi} (I - (I + Ae^{\frac{j\pi}{\omega}})^{-1}(A_r(I + e^{\frac{j\pi}{\omega}}))) \psi_{\omega}(0)
\]

(65)

Moreover, using a similar approach in Section 3.1, the sensitivity and control sensitivity frequency responses are achieved.

**Corollary 1.6** For \( \omega \geq \omega_0 \), the frequency responses of \( e_r \) and \( \frac{u}{r} \) for any stable reset system (60) are obtained as

\[
S_n(j\omega) = \begin{cases} 
1 - C(A - j\omega I)^{-1} \theta_{\omega}(\omega) + C(j\omega I + A) \mathcal{F} & n = 1 \\
-C(A - j\omega I)^{-1} \theta_{\omega}(\omega) & odd n > 1 \\
0 & even n > 1
\end{cases}
\]

(66)

and

\[
CS_n(j\omega) = \begin{cases} 
j\omega C(A - j\omega I)^{-1} \theta_{\omega}(\omega) - \pi C(j\omega I + A) \mathcal{F} & n = 1 \\
\pi C(j\omega I + A)^{-1}B_c & odd n > 1 \\
0 & even n > 1
\end{cases}
\]

(67)

In addition, this methodology holds for disturbance rejection analyses. Similarly, if we consider \( d = \sin(\omega t - \phi_{\omega 1}) \), (60) is

\[
\begin{align*}
\dot{x}(t) &= A x(t) + B \sin(\omega t - \phi_{\omega 1}) & t \neq \frac{k \pi}{\omega} \\
x(t^+) &= A x(t) & t = \frac{k \pi}{\omega} \\
y_d(t) &= C x(t)
\end{align*}
\]

(68)
In the presence of disturbance, \( \varphi_{u1} \) is approximated by

\[
\varphi_{u1} \approx \angle \left( \frac{-C_{s} G(j \omega)}{1 + C_{s} C_{DF} G(j \omega)} \right),
\]

and \( \psi_{\varphi} \) (43) is changed to

\[
\psi_{\varphi}(t) = (\omega t \cos(\omega t - \varphi_{u1}) + A \sin(\omega t - \varphi_{u1})) e^{\psi \omega t}. \tag{70}
\]

As a result, (62) becomes

\[
\xi_{ss} = -\xi_{u1} = \frac{-A_{r}(1 + e^{\psi \omega t}) \psi_{\varphi}(0)}{1 + A_{r} e^{\psi \omega t}}. \tag{71}
\]

**Remark 5** These \( \xi_{ss} \) and \( \xi_{u1} \) can be used to obtain pseudo-sensitivities which are defined in Section 3.4.

**Corollary 1.7** For \( \omega \geq \omega_{n} \), the frequency responses of \( \frac{e}{d} \) and \( \frac{u}{d} \) for any stable reset system are

\[
PS_{\varphi}(j \omega) = \begin{cases} 
C(j \omega - A)^{-1} \theta_{\varphi}(\omega) + C(j \omega + A) \varphi \omega & n = 1 \\
C(j \omega - A)^{-1} \theta_{\varphi}(\omega) & \text{odd } n > 1 \\
0 & \text{even } n > 1
\end{cases}
\]

and

\[
CS_{\varphi}(j \omega) = \begin{cases} 
\frac{j \omega C(A - j \omega)}{\pi C_{C}(j \omega - A_{n})^{-1} B_{C}} - 1 & n = 1 \\
\frac{j \omega C(A - j \omega)}{\pi C_{C}(j \omega - A_{n})^{-1} B_{C}} & \text{odd } n > 1 \\
0 & \text{even } n > 1
\end{cases}
\]

in which

\[
\theta_{\varphi}(\omega) = \frac{-2 j \omega e^{j \varphi_{u1}}}{\pi} (1 + e^{\psi \omega t}) (I + (I + A_{r} e^{\psi \omega t})^{-1} A_{r} (I + e^{\psi \omega t}) \psi_{\varphi}(0)) \tag{74}
\]

### 3.6 Toolbox

All the presented theoretical results have been integrated into an open source toolbox, which is developed using Matlab software. This tool facilitates the implementation of reset control theory which are developed based on our proposed HOSIDF. The toolbox and its help documents are presented in the supplementary file.

### 4 Illustrative examples

In this section, two illustrative examples are provided to show the effectiveness of the developed theory. The Spyder

![Fig. 4. 3 DOF planar precision positioning Spyder stage. Voice coil actuators 1A, 1B and 1C control 3 masses (indicated as 3) which are constrained by leaf flexures. The 3 masses are connected to central mass (indicated by 2) through leaf flexures. Linear encoders (indicated by 4) placed under masses 3 provide position feedback](image)

![Fig. 5. The open-loop frequency responses of \( C_{P(CI)D} \) and \( C_{PID} \)](image)

setup (Fig. 4) (Dastjerdi et al. 2018, Saikumar et al. 2019a) is selected as the benchmark plant for this purpose. After identification (Hou 2019), the transfer function of this stage is found as

\[
G(s) = \frac{9602}{s^2 + 4.27s + 7627} \tag{75}
\]

4.1 Digging more into Clegg integrator

In this part, the frequency responses of two PID controllers with the same base linear system, such that one of them has a Clegg integrator instead of a linear integrator, are compared. The linear PID controller is

\[
C_{PID}(s) = k_{p} \left( 1 + \frac{\omega_{n}}{s} \right) \left( \frac{s}{\omega_{n}} + 1 \right), \tag{76}
\]
while its precision is lower than $C_{P_{ID}}$. Unlike DF analyses, the proposed method can explain why the $C_{P_{ID}}$ outperforms the $C_{P(CI)D}$ in the sense of precision and control output. The defined pseudo-sensitivities and harmonics of the sensitivities are shown in Fig. 7. As shown in Fig. 7a, the reference tracking of the $C_{P_{ID}}$ is better than $C_{P(CI)D}$ at low frequencies considering the defined pseudo-sensitivity $S_m$ which explains the time responses depicted in Fig. 6a. Moreover, $S_m$ predicts precisely the maximum tracking error of the system at 1Hz, which is the frequency that the experiments (Fig. 6) are performed. In addition, the defined pseudo-control sensitivity ($CS_m$) is much higher than the DF approximation and control sensitivity of $C_{P_{ID}}$ which explains why the control input of $C_{P(CI)D}$ is larger than $C_{P_{ID}}$ (Fig. 7c). Furthermore, ($CS_m$) gives the maximum value of control input of the system accurately (Fig. 7c). Therefore, for avoiding saturation problems, designers should consider the $CS_m$ instead of DF when they use reset controllers. Besides, the disturbance rejection of $C_{P_{ID}}$ is better than $C_{P(CI)D}$ at low frequencies. Furthermore, as shown in Fig. 7d, the non-linearity of the Clegg integrator does not have much influence on high frequencies, and the noise attenuating behavior of both controllers is the same.

To sum up, although it has been believed that Clegg integrator outperforms simple integrator based on DF, the proposed theory shows while the phase margin may have been improved by using Clegg integrator, the tracking performance is reduced and larger control input signal is produced. The proposed method can explain time behaviour of reset systems which cannot be justified by the DF analyses. In addition, the proposed method can obtain closed-loop performances of reset systems more precisely than the DF method.

4.2 Performance of $C_{gLp}$

In this part, one of the new compensator which is developed using reset elements is analyzed using the proposed and DF method. This compensator is constant gain Lead in phase ($C_{gLp}$) which uses a reset filter (FORE) and Proportional Derivative (PD) filter in series (Saikumar et al. 2019c, Palanikumar et al. 2018). The DF of the FORE is depicted in figure 8 using (3). As it is shown, if the FORE is approximated with its DF, the combination of PD and FORE produces a compensator whose gain is constant while providing positive phase.

Fig. 6. The time domain responses of $C_{P(CI)D}$ and $C_{P_{ID}}$ for 1 Hz sinusoidal reference

and the reset controller which has a Clegg integrator instead of linear integrator is

$$C_{P(CI)D}(s) = k_p \left( 1 + \frac{s}{\omega_c} \right) \left( \frac{s}{\omega_d} + 1 \right)$$

(77)

Considering 100 Hz as the bandwidth ($\omega_c$), the control knob parameters are tuned based on a rule of thumb in (Schmidt et al. 2014, Krijnen et al. 2017, Dastjerdi et al. 2018) as

$$k_p = \frac{1}{3|G(j\omega_c)|} = 13, \quad \omega_0 = \frac{\omega_c}{10} = 20\pi,$$
$$\omega_c = 3\omega_c = 600\pi, \quad \omega_d = \frac{\omega_c}{3} = \frac{200\pi}{3}. \quad \text{(78)}$$

The open-loop frequency response of $C_{P(CI)D}$ and $C_{P_{ID}}$ are shown in Fig. (5) using the DF. Based on DF of the open-loop, it is expected that the tracking performance and disturbance rejection of $C_{P(CI)D}$ would be better than $C_{P_{ID}}$ while their control input and noise attenuation performance would almost be the same. However, the time domain results (Fig. 6) disprove this expectation. In this figure, a sinusoidal wave $r(t) = 100\sin(2\pi t)$ is applied to the systems, and their error and control input are depicted using Simulink in Matlab. It is seen that the output of $C_{P(CI)D}$ is much larger than $C_{P_{ID}}$.  

\[ e(t) \quad u(t) \]
Fig. 7. The closed-loop frequency responses of $C_{P(I/D)}$ and $C_{PID}$ consist of DF approximation ($\infty$, DF), the first harmonic ($1$), and the pseudo-sensitivity ($\infty$).

Fig. 8. The DF based frequency behavior of a (CgLp) compensator.

Fig. 9. The open-loop frequency response of $C_{g_1}$ and $C_{g_2}$. 
In order to study effects of changing sequence on the performance of these compensators, two controllers having CgLp compensators are considered for study. The configuration of both controller is as:

\[ C_g(s) = k_p \left( \frac{1}{s} + \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \frac{1}{\omega_3^2} \right) \left( \frac{s}{\omega_0} + 1 \right) \left( \frac{s}{\gamma} + 1 \right) \]  

(79)

In these controllers, \( \gamma \) is used to reset the states to non-zero values as \( A_r = \gamma I \). The parameters of these two controllers are the same and tuned such that two controllers have the bandwidth of 100 Hz and phase margin of 30\(^\circ\) considering the DF method. Based on the method described in (Hou 2019), the control parameters are \( k_p = 46.8 \), \( \omega_0 = \frac{\omega_1}{10} = 20\pi \), \( \omega_1 = 500\pi \), \( \omega_2 = 1000\pi \), \( \omega_3 = 281.5 \), \( \omega_d = 93.5 \) and \( \gamma = -0.5 \). The only difference between these two controllers is the sequence of their filters. The sequence filters of \( C_{g1} \) is proportional-FORE-lead-integrator, and the sequence of \( C_{g2} \) is proportional-lead-FORE-integrator. The open-loop frequency responses of both configurations are depicted in Fig. 9. Both controllers have the same DF, but their high order harmonics are different which causes different closed-loop responses. In the first configuration, the jump signal will first be generated by reset action and then be differentiated consequently. As a result, the derivative of the jump signal leads to large magnitudes of high order harmonics. In Fig. 10, the closed-loop frequency responses of both controllers are depicted. As it is observed, there are big differences between the DF and proposed theory. The proposed theory can investigate the effects of the sequence on the performance of reset controllers while the DF of both controllers are the same. This difference in the magnitude of high order harmonics in the open-loop (Fig. 9) leads to discrepancies between closed-loop frequency responses of the two controllers. As shown in Fig. 10a, \( C_{g2} \) has better tracking performance than \( C_{g1} \) based on the defined pseudo-sensitivity, and its third harmonic is less than the third harmonic of \( C_{g1} \) around the bandwidth. Also, \( C_{g2} \) has better disturbance rejection than \( C_{g1} \) as shown in Fig. 10b. Since the derivative of a jump signal produces large magnitude, \( C_{g2} \) has smaller control input in comparison with \( C_{g1} \) in both disturbance rejection and reference tracking as shown in Fig. 10c and 10d. Note, as discussed before, unlike linear controllers and the DF analyses, the control sensitivity due to the disturbance \( \frac{d}{d} \) is not better than linear integrator in the sense of precision and control input. The time domain results were consistent with the proposed method results. Second, closed-loop frequency behavior of CgLp compensators considering different sequences of their filters were achieved by the experiments, our proposed method, and the DF. The results illustrated that the proposed method can predict closed-loop frequency behaviors of the reset systems more precisely than the DF method. Besides, this method can consider effects of different sequences of filters on the performances of reset systems while DF does not have this ability. All in all, this toolbox gives more insights into the closed-loop frequency behaviors of reset systems so that designers can tune reset controllers appropriately. Therefore, it facilitates the use of reset controllers in industry and academia.

| Table 1: Comparison between experiment and theory results of \( S_{Cg\infty} \) |
|------------------|------------------|------------------|
| Reference       | 5 Hz             | 10 Hz            | 20 Hz            |
| \( S_{ce} \) (experiment) | -25(dB)       | -25(dB)           | -18(dB)          |
| \( S_{ce} \) (theory)      | -30(dB)         | -30(dB)           | -20(dB)          |
| \( S_{DF} \)               | -43(dB)         | -43(dB)           | -32(dB)          |

To show the effectiveness of this approach, two illustrative examples were given. First, the performance of a Clegg controller on a high-tech positioning stage was obtained using the DF and our proposed method, and their results were compared with Simulink results. It is revealed that unlike DF analyses, the time domain performance of Clegg integrator is not better than linear integrator in the sense of precision and control input. The time domain results were consistent with the proposed method results. Second, closed-loop frequency behavior of CgLp compensators considering different sequences of their filters were achieved by the experiments, our proposed method, and the DF. The results illustrated that the proposed method can predict closed-loop frequency behaviors of the reset systems more precisely than the DF method. Besides, this method can consider effects of different sequences of filters on the performances of reset systems while DF does not have this ability. All in all, this toolbox gives more insights into the closed-loop frequency behaviors of reset systems so that designers can tune reset controllers appropriately. Therefore, it facilitates the use of reset controllers in industry and academia.

\[ \text{References} \]

1 The way of obtaining the closed-loop frequency response using the proposed toolbox is explained in the movie which is part of the supplementary file.
Fig. 10. The closed-loop frequency responses of $C_{g1}$ and $C_{g2}$ consist of DF approximation ($\cdot$, DF), the first harmonic ($\cdot$, 1), the third harmonic ($\cdot$, 3), and the pseudo-sensitivity ($\cdot$, $\infty$)

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