PSEUDOSCALAR MASS EFFECTS IN DECAYS OF TAUS WITH THREE PSEUDOSCALAR MESONS

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Abstract

We study the effects of non-vanishing pseudoscalar masses in \( \tau \) decays into three mesons. The hadronic matrix elements are obtained by using the generalized structure of the chiral currents with nonvanishing pseudoscalar masses and implementing the low-lying resonances in the different channels. We demonstrate that suitable angular distributions are sensitive to the mass effects in the chiral Langrangian. Numerical results for the relevant structure functions are given for the decay modes \( \tau \to \nu \pi^\pm \pi^\mp \), \( \nu K^- \pi^- \pi^+ \) and \( \nu K^- \pi^- K^+ \).

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1 Introduction

In this paper we investigate the effects of the non-vanishing pseudoscalar meson masses on predictions which use chiral symmetry for calculating tau decays into neutrino and three pseudoscalar mesons (pions and/or kaons). The usual philosophy employed to derive the hadronic matrix elements is to introduce resonances in the different channels with constant couplings. These couplings are then fixed by the low energy theorems which follow from the spontaneously broken chiral symmetry of QCD and can be calculated using effective chiral Lagrangians [1, 2, 3, 4, 5]. But note that all these authors use the chiral Lagrangian for exactly massless pseudoscalar mesons, which results in a completely transverse (i.e., conserved) hadronic amplitude with no spin-0 contribution. All what is then known about the size of the spin-0 amplitude is the PCAC (partial conserved axial current) argument that the spin-0 amplitude should be suppressed by a factor $m^2_\pi/Q^2$ relative to the spin-1 amplitude ($Q^2$ is the invariant mass squared of the hadronic system). The chiral Lagrangian for massive pseudoscalar mesons, however, is also well known [3] and so it would seem to be the best approach to start from the generalized chiral limit which it describes and then include the resonances. This is what we do in this present paper. Of course the hadronic current predicted by such a model is no longer transverse and results in a nonvanishing spin-0 amplitude.

Now the question arises whether and how the effects of the spin-0 amplitude can be measured. In many experimental analyses of $\tau \to 3\pi\nu_\tau$ decays, up to now the spin-0 part of the hadronic system has been neglected. This was justified by the above mentioned PCAC suppression of this scalar part relative to the spin-1 part. Assuming the mean $Q^2$ to be about $m^2_{\pi^0}$ in the decay $\tau \to 3\pi\nu_\tau$, the spin-0 amplitude is expected to be about 1% of that of the spin-1 amplitude and therefore the relative contribution to the total rate for $\tau \to 3\pi\nu$ to be of the order of $10^{-4}$. For decays modes with kaons the corresponding quantity $m^2_K/Q^2$ could be as large as 10%, but even then the contribution to the total width would not be more than a few percent. Arguments along these lines show that the scalar contributions are not measurable in the total decay width. But of course these are only very crude order of magnitude arguments which could be modified by higher-lying resonances. However, an inspection of the effects of the $\pi(1300)$ has shown no important enhancement [4].

Therefore we must consider other quantities for a measurement of the scalar form factor. It appears that angular distributions [3, 4] are sensitive to the interference of spin-0 and spin-1 form factors and could make the measurement of the spin-0 form factor in high statistic experiments possible. We will therefore study in this paper if this is actually feasible.

This paper is organized as follows: In Sec. 2 we review the kinematics, the general form of the hadronic current and the definitions of the angular moments. In Sec. 3 we present the generalized chiral predictions for all the ten possible decay channels with pions and kaons in the final state. In Sec. 4 we implement vector and axial-vector resonances, give general formulae for the form factors and the specialization to the three most interesting modes, viz. to $\tau \to$ either $\pi^-\pi^-\pi^+$ or $K^-\pi^-\pi^+$ or $K^-\pi^-K^+$. The numerical results for these three channels are presented in Sec. 5, and in Sec. 6 we draw our conclusions.
2 Hadronic Current and Angular Distributions

We consider the decay

\[ \tau \to M_1(k_1)M_2(k_2)M_3(k_3)\nu_\tau \]  

The hadronic matrix element involved can be parametrized in terms of four form factors:

\[ J^\mu = \langle M_1(k_1)M_2(k_2)M_3(k_3)| (V^\mu - A^\mu)|0 \rangle \]

\[ = \left[ F_1(Q^2, s_1, s_2)(k_1 - k_3)\nu + F_2(Q^2, s_1, s_2)(k_2 - k_3)\nu \right] \left[ g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2} \right] \]

\[ + F_3(Q^2, s_1, s_2)\epsilon^{\mu\alpha\beta\gamma}k_{1\alpha}k_{2\beta}k_{3\gamma} + F_S(Q^2, s_1, s_2)Q^\mu \]  

where \( V^\mu \) and \( A^\mu \) are the vector and axialvector quark currents, respectively,

\[ Q = k_1 + k_2 + k_3 \]  

and the invariant masses \( s_j \) are defined by

\[ s_1 = (k_2 + k_3)^2 \]

\[ s_2 = (k_1 + k_3)^2 \]  

The differential decay rate is obtained from

\[ d\Gamma(\tau \to \nu_\tau 3h) = \frac{G^2}{4m_\tau} \langle \cos^2 \theta_e \rangle \{ L^\mu \nu \} \{ H_\mu \nu \} d\text{PS}^{(4)} \]  

where \( L^\mu \nu = M^\mu (M_\nu)^\dagger \), \( M^\mu \) is the leptonic current and \( H^\mu \nu = J^\mu (J^\nu)^\dagger \).

The considered decays are most easily analyzed in the hadronic rest frame \( \vec{k}_1 + \vec{k}_2 + \vec{k}_3 = \vec{Q} = 0 \). The orientation of the hadronic system is characterized by three Euler angles (\( \alpha, \beta \) and \( \gamma \)) as introduced in [3, 4].

Note that in current \( e^+ + e^- \to \tau^+ \tau^- \to \nu_\tau \text{3 mesons} \) experiments only two out of the three Euler angles are measurable. The reason for this is that the rest frame of the \( \tau \) can not be reconstructed or equivalently the direction of flight of the \( \tau \) in the hadronic rest frame is not known. Instead the direction of the Labframe in hadronic rest frame is introduced. The measurable Euler angles are defined by

\[ \cos \beta = \hat{n}_L \cdot \hat{n}_\perp \]  

\[ \cos \gamma = -\hat{n}_L \cdot \hat{k}_3 \| \hat{n}_L \times \hat{n}_\perp \| \]  

where (\( \hat{a} \) denotes a unit three-vector)

- \( n_L \) is the direction of the labframe in the hadronic restframe. Note that \( \hat{n}_L = -\hat{n}_Q \), with \( \hat{n}_Q \) the direction of the hadrons in the labframe,

- \( \hat{n}_\perp = \hat{k}_1 \times \hat{k}_2 \), the normal to the plane defined by the momenta of particles 1 and 2.
Performing the integration over the momentum of the unobserved neutrino and the Euler angle $\alpha$ we obtain the differential decay width for a polarized $\tau$ [3, 4]:

$$d\Gamma(\tau \to 3h) = \frac{G^2}{2m_\tau} (\cos^2 \theta_c) \left\{ \sum_X \bar{L}_X W_X \right\} \times$$

$$\frac{1}{(2\pi)^3} \frac{1}{64} \frac{(m_\tau^2 - Q^2)^2}{m_\tau^2} \frac{dQ^2}{Q^2} \frac{ds_1}{2\pi} \frac{ds_2}{2\pi} \frac{d\gamma}{2} \frac{d\cos \beta}{2} \frac{d\cos \theta}{2}$$

(8)

The main advantage of working in the hadronic rest frame is that the product $L_{\mu \nu} H^{\mu \nu}$ reduces to a sum of 16 hadronic structure functions ($W_X$)

$$L_{\mu \nu} H^{\mu \nu} \to \sum_X \bar{L}_X W_X$$

(9)

The angle $\theta$, a kinematical angle, can be seen as the angle between the unmeasured direction of flight of the $\tau$ in the labframe and $\vec{Q}$ in the $\tau$ restframe, and is obtained from the hadronic energy in the labframe $E_h$ by [3, 4]

$$\cos \theta = \frac{(2x m_\tau^2 - m_\tau^2 - Q^2)}{(m_\tau^2 - Q^2) \sqrt{1 - 4m_\tau^2/s}}$$

(10)

with

$$x = 2 \frac{E_h}{\sqrt{s}} \quad s = 4E_{\text{beam}}^2$$

(11)

Finally, the angle between the unmeasured tau direction and that of the lab in the hadron restframe ($\vec{Q} = 0$) is needed (for $K_i$ in Eqn. (19) and it can also computed from the energy $E_h$:

$$\cos \psi = \frac{x(m_\tau^2 + Q^2) - 2Q^2}{(m_\tau^2 - Q^2) \sqrt{x^2 - 4Q^2/s}}$$

(12)

Different combinations of the hadronic form factors can be measured by considering moments

$$\langle f(\beta, \gamma) \rangle \propto \int f(\beta, \gamma) \sum_X L_X H^X d\cos \beta d\gamma$$

(13)

which have been defined in [4].

The simplest moment ($f(\beta, \gamma) = 1$) is proportional to the angular integrated rate:

$$\langle 1 \rangle \propto (2K_1 + 3K_2)(W_A + W_B) + 3K_2 W_{SA}$$

(14)

where $K_i(\theta, \psi, P)$ are known functions of kinematical variables $\theta$ and $\psi$ and the $\tau$ polarisation $P$ [4]. The hadronic structure functions $W_X$ depends on $Q^2, s_1$ and $s_2$. $W_{SA}$ and $W_A + W_B$ are closely related to the spin-0 and spin-1 part of the spectral functions:

$$\rho_0(Q^2) = \frac{1}{2} \frac{1}{(4\pi)^4} \frac{1}{Q^4} \int ds_1 ds_2 W_{SA}$$

(15)

$$\rho_1(Q^2) = \frac{1}{6} \frac{1}{(4\pi)^4} \frac{1}{Q^4} \int ds_1 ds_2 (W_A + W_B)$$

(16)
and
\[ \Gamma(\tau \to 3h) = \frac{G^2}{4m_\tau} (g_V^2 + g_A^2) \left( \cos^2 \theta_c - \sin^2 \theta_c \right) \frac{1}{(4\pi)} \int dQ^2 \left( m_\tau^2 - Q^2 \right)^2 \left\{ \rho_0 + \left( 1 + \frac{2Q^2}{m_\tau^2} \right) \rho_1 \right\} \] (17)

Note that the spin-0 contribution is very small as compared to the spin-1 part \([1]\).

In order to measure spin-0 contributions we have to consider interference of spin-0 and spin-1 terms. Therefore the following moments are of interest
\[ f(\beta, \gamma) = \cos \beta \]
\[ f(\beta, \gamma) = \sin \beta \cos \gamma \]
\[ f(\beta, \gamma) = \sin \beta \sin \gamma \] (18)

In the notation of \([4]\) these moments are proportional to
\[ \langle \cos \beta \rangle \propto K_3(\theta, \psi, P) W_E - K_2(\theta, \psi, P) W_{SF} \]
\[ \langle \sin \beta \sin \gamma \rangle \propto -K_3(\theta, \psi, P) W_G - K_2(\theta, \psi, P) W_{SD} \]
\[ \langle \sin \beta \cos \gamma \rangle \propto -K_3(\theta, \psi, P) W_I + K_2(\theta, \psi, P) W_{SB} \] (19)

where \(K_i(\theta, \psi, P)\) are known functions of kinematical variables \(\theta\) and \(\psi\) and the \(\tau\) polarisation \(P\) \([4]\).

The needed hadronic structure functions \(W_X\) are related to the hadronic form factors. Let us consider the hadronic restframe with \(z\) and \(x\) axis are aligned with \(\vec{n}_\perp\) and \(\vec{k}_3/|\vec{k}_3|\), respectively. In this frame the momenta of the hadrons are given as follows:
\[ k_3^\mu = (E_3, k_3^x, 0, 0) \]
\[ k_2^\mu = (E_2, k_2^x, k_2^y, 0) \]
\[ k_1^\mu = (E_1, k_1^x, -k_2^y, 0) \] (20)

Then the following variables are useful to express the hadronic structure functions \(W_X\)
\[ x_1 = k_1^x - k_3^x \]
\[ x_2 = k_2^x - k_3^x \]
\[ x_3 = k_1^y = -k_2^y \]
\[ x_4 = -x_3 k_3^y \] (21)

Using these variables the following results hold
\[ W_A = (x_1^2 + x_3^2) |F_1|^2 + (x_2^2 + x_3^2) |F_2|^2 + 2(x_1 x_2 - x_3^2) \text{Re} (F_1 F_2^*) \]
\[ W_B = x_4^2 |F_3|^2 \]
\[ W_{SA} = Q^2 |F_4|^2 \]
\[ W_E = -2x_3(x_1 + x_2) \text{Im} (F_1 F_2^*) \]
\[ W_G = -2x_4 \left[ x_1 \text{Re}(F_1 F_3^*) + x_2 \text{Re}(F_2 F_3^*) \right] \]
\[ W_I = -2x_3x_4 \left[ \text{Re}(F_1 F_3^*) - \text{Re}(F_2 F_3^*) \right] \]
\[ W_{SB} = 2\sqrt{Q^2} \left[ x_1 \text{Re}(F_1 F_4^*) + x_2 \text{Re}(F_2 F_4^*) \right] \]
\[ W_{SD} = 2x_3 \left[ \text{Re}(F_1 F_4^*) - \text{Re}(F_2 F_4^*) \right] \]
\[ W_{SF} = -2\sqrt{Q^2}x_4 \text{Im}(F_3 F_4^*) \]

(22)

Before discussing our model we would like to remind the reader some properties of the structure functions

- Let us consider the case of the decay into three pions. Due to \(G\)-parity conservation the form factor \(F_3\) vanishes. Using Eqn. (22) we observe that \(W_G\) and \(W_I\) are vanishing and therefore in this case (three pions) a nonvanishing \(\langle \sin \beta \cos \gamma \rangle\) and or \(\langle \sin \beta \sin \gamma \rangle\) yields a clean signature of the presence a scalar contribution.

- \(\langle \cos \beta \rangle\) yields a measurement of the parity violation in \(\tau\) decay and for pions \(W_{SF} = 0\).

- In general a measurement of the scalar parts is only possible if the spin-0 structure functions \(W_{SA,SB,SD,SB}\) in Eqn. (22) are comparable with the spin-1 structure functions at least in some kinematical areas.

3 The Chiral Limit

The generalized chiral limit, ie. with nonvanishing pseudoscalar masses, is most conveniently described by the effective Lagrangian

\[ \mathcal{L}^{(2)} = \frac{f_\pi^2}{4} \text{tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{f_\pi^2 \mu}{2} \text{tr}(MU^\dagger + U^\dagger M) \]

(23)

where \(U\) is the exponential of the pseudoscalar fields,

\[ U = \exp \left\{ \frac{\sqrt{2}i}{f_\pi} \begin{pmatrix} \pi^0/\sqrt{2} + \eta/\sqrt{6} & \pi^+ & K^+ \\ \pi^- & -\pi^0/\sqrt{2} + \eta/\sqrt{6} \end{pmatrix} \begin{pmatrix} K^- \\ K^0 \end{pmatrix} \right\} \]

(24)

\(M\) the quark mass matrix

\[ M = \text{diag}(m_u, m_d, m_s) \]

(25)

and

\[ f_\pi = 93.3\text{MeV}. \]

(26)
Table 1: Parameters for the respective three meson channels (The charge conjugated channels are obtained by reversing the sign of $A^{(123)}$.)

| $M_1 M_2 M_3$ | $A^{(123)}$ | $G^{(123)}$ | $m^2_{(123)}$ | $X^{(123)}$ |
|----------------|--------------|--------------|----------------|--------------|
| $\pi^- \pi^- \pi^+$ | $\cos \theta_c$ | 1 | $m_\pi$ | $2m_\pi$ |
| $\pi^- \pi^- \pi^-$ | $\cos \theta_c$ | 1 | $m_\pi$ | $m_\pi$ |
| $K^- \pi^- K^+$ | $-1/2 \cos \theta_c$ | 1 | $m_\pi$ | $m_\pi + m_K^2$ |
| $K^0 \pi^- \bar{K}^0$ | $-1/2 \cos \theta_c$ | 1 | $m_\pi$ | $m_\pi + m_K^2$ |
| $K^- \pi^0 K^0$ | $3/(2\sqrt{2}) \cos \theta_c$ | 0 | $m_\pi^2$ | 0 |
| $\pi^0 \pi^0 K^0$ | $1/4 \sin \theta_c$ | 1 | $m_K^2$ | $2(m_\pi^2 + m_K^2)$ |
| $K^- \pi^- \pi^+$ | $-1/2 \sin \theta_c$ | 1 | $m_K^2$ | $m_\pi^2 + m_K^2$ |
| $\pi^- \bar{K}^0 \pi^0$ | $3/(2\sqrt{2}) \sin \theta_c$ | 0 | $m_K^2$ | 0 |
| $K^- K^- K^+$ | $\sin \theta_c$ | 1 | $m_K^2$ | $2m_K^2$ |
| $K^- \bar{K}^0 K^0$ | $-1/2 \sin \theta_c$ | 1 | $m_K^2$ | $2m_K^2$ |

From this Lagrangian the matrix element $H_{\mu}^{\text{chiral}}$ of the axial weak hadronic current $A^\mu$ between the hadronic vacuum and a state with three pseudoscalar mesons $M_1, \ldots M_3$ is derived as

$$H_{\mu}^{\text{chiral}} = \frac{2\sqrt{2}}{3f_\pi} A^{(123)} \left\{ \left[ (k_1 - k_3)^\mu - \frac{1}{2} (q + k_2) \cdot (k_1 - k_3) q^\mu \right] \right.$$

$$+ G^{(123)} \left[ (k_2 - k_3)^\mu - \frac{1}{2} (q + k_1) \cdot (k_2 - k_3) q^\mu \right] \right.$$

$$+ \frac{1}{2} \frac{X^{(123)}}{Q^2 - m^2_{(123)}} q^\mu \right\}$$

The values for $A^{(123)}$, $G^{(123)}$, $m_{(123)}$ and $X^{(123)}$ for the respective channels are found in Tab. 1. Note that in the strict chiral limit, i.e., when $m_u = m_d = m_s = 0$, the matrix element, which in this limit we will denote by $H_{\mu}^{\text{chiral,0}}$, becomes transverse, because then $m_{(123)}$ and $X^{(123)} = 0$ and $1/2(q + k_2) \cdot (k_1 - k_3) = q \cdot (k_1 - k_2)$ (and similarly for $(1 \leftrightarrow 2)$).

### 4 Implementation of Resonances

The chiral current of the last section is the $O(P^2)$ theorem of low energy QCD and can only be expected to be correct for very small momentum transfers (very small compared with typical resonances masses such as $m_\rho^2$, say). For larger momenta, effects of the order of $O(P^4)$, $O(P^6)$, ... must successively be taken into account, and when the momentum transfers can actually become larger than the resonance masses, all orders in $O(P^2)$ must be summed up. The leading effect of this series may be described by Breit-Wigner
resonances $BW(s)$:

$$BW_X(s) = \frac{m_X^2}{m_X^2 - im_X \Gamma_X(s) - s} = \sum_{n=0}^{\infty} \left( \frac{s + im_X \Gamma_X(s)}{m_X^2} \right)^n$$  \hspace{1cm} (28)$$

And so the usual approach to extrapolate to higher momenta is to write down a current including Breit Wigners describing possible final state resonances in such a way that in the low energy limit, where all terms of higher order than $O(P^2)$ are neglected, the current reduces to the correct chiral limit. Note that in the chiral counting not only the pseudoscalar momenta $k_1$, $k_2$ and $k_3$, but also the pseudoscalar masses $m_\pi$, $m_K$ and the coupling to the external gauge field of the $W$ count as $O(P)$. And so taking the limit of the order $O(P^0)$ for the amplitude means taking the limit of the order $O(P^0)$ for the form factors. It turns out that this limit is always obtained from our formulae by putting the involved Breit-Wigner resonance factors equal to one. Let us consider the three pion case first for simplicity. The vector meson dominance model of Fig. 1 gives the following current:

$$H_\mu = C^{(3\pi)} \left\{ BW_{a_1}(Q^2) \left( g^{\mu\nu} - \frac{q_{\mu} q_{\nu}}{m_{a_1}^2} \right) \Gamma_{\nu\alpha} BW_\rho(s_2) \right\}$$

$$\times \left( g^{\alpha\beta} - \frac{(k_1 + k_3)_{\alpha}(k_1 + k_3)_{\beta}}{m_\rho^2} \right) (k_1 - k_3)_{\beta} + (1 \leftrightarrow 2) \right\}$$  \hspace{1cm} (29)$$

Here $C^{(3\pi)}$ is an overall factor (product of couplings and the like). $\Gamma_{\mu\nu}$ is proportional to the vertex describing the coupling of the $a_1$ to the $\rho\pi$. The most general form for $\Gamma_{\mu\nu}$ for off-shell particles contains five form factors. The ansatz in Ref. \[1\] corresponds to the transverse form

$$\Gamma_{\mu\nu}(a_1(q_\mu) \to \rho(k_\nu)\pi) = g_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{Q^2}$$  \hspace{1cm} (30)$$

With this ansatz for $\Gamma_{\mu\nu}$ and the normalization

$$C^{(3\pi)} = \frac{2\sqrt{2}}{3f_\pi} A^{(3\pi)} = \cos \theta_C \frac{2\sqrt{2}}{3f_\pi}$$  \hspace{1cm} (31)$$

the hadronic current $H^\mu$ reduces to the strict chiral limit of vanishing quark masses, i.e. the transverse current $H^\mu_{\text{chiral,0}}$, in the limit of neglecting all terms of higher order than $O(P^2)$.

If we want $H^\mu$ to reduce to the generalized chiral limit of Eqn. (27), we have to modify the $q_{\mu}q_{\nu}$ term in $\Gamma_{\mu\nu}$:

$$\Gamma_{\mu\nu} \to g_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{Q^2 - m_\pi^2} \approx g_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{Q^2} \left( 1 + \frac{m_\pi^2}{Q^2} \right)$$  \hspace{1cm} (32)$$

and add a non-resonant contribution proportional to $X^{(123)}$ by hand. The hadronic current for the three pion decay mode then becomes:

$$H^\mu = C^{(3\pi)} \left\{ BW_{a_1}(Q^2) \left( g^{\mu\nu} - \frac{q_{\mu} q_{\nu}}{m_{a_1}^2} \right) \left( g_{\nu\alpha} - \frac{q_{\nu} q_{\alpha}}{Q^2 - m_\pi^2} \right) BW_\rho(s_2) \right\}$$

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If we have a three-particle axial resonance $A$, the normalization is obtained from the chiral limit:

\[
C^{(3\pi)} = \frac{1}{1 + \beta} \{ BW_\rho(s) + \beta BW_\rho'(s) \}
\]  

(see Ref. [4], Eqn. (20)), and get the final result for the form factors:

\[
\begin{align*}
F_1^{(3\pi)} &= C^{(3\pi)} BW_{a_1}(Q^2) T_\rho(s_2) \\
F_2^{(3\pi)} &= C^{(3\pi)} BW_{a_1}(Q^2) T_\rho(s_1) \\
F_S^{(3\pi)} &= C^{(3\pi)} \frac{m_\pi^2}{Q^2 - m_\pi^2} \left\{ \frac{Q^2 - m_{a_1}^2}{m_{a_1}^2 Q^2} BW_{a_1}(Q^2) \right. \\
&\quad \left. \times \left[ \frac{s_3 - s_1}{2} T_\rho(s_2) + \frac{s_3 - s_2}{2} T_\rho(s_1) \right] + 1 \right\}
\end{align*}
\]  

(33)

Now we want to generalize our results to the case of different pseudoscalars with $m_j \neq m_k^2$. We have to modify the axialvector(A)-vector(V)-pseudoscalar(P) coupling in the following way:

\[
\Gamma_{\mu\nu} \left( A(q_\mu) \to V(k_\mu) P \right) \to g_{\mu\nu} - \frac{q_\mu q_\nu}{Q^2 - m_{123}^2} + \frac{1}{2} \frac{q_\mu k_\nu}{Q^2 - m_{123}^2}
\]  

(36)

If we have a three-particle axial resonance $A$ and two two-particle vector resonances $V_{13}$ and $V_{23}$ in the $s_2$ and the $s_1$ channels, respectively, the hadronic current is given by:

\[
H^\mu = C^{(123)} BW_A(Q^2) BW_{V_{13}}(s_2) \left\{ (k_1 - k_3)^\mu - q^\mu \frac{q \cdot (k_1 - k_3)(m_A^2 - m_{123}^2)}{m_A^2 (Q^2 - m_{123}^2)} \right. \\
&\quad - \frac{m_1^2 - m_3^2}{m_{V_{13}}^2} \left[ (k_1 + k_3)^\mu - \frac{q^\mu}{m_A^2 (Q^2 - m_{123}^2)} \right] \\
&\quad \times \left( q \cdot (k_1 + k_3)(m_A^2 - m_{123}^2) + \frac{1}{2} (s_2 - m_{V_{13}}^2)(Q^2 - m_A^2) \right) \left[ \right\} + (1 \leftrightarrow 2) + C^{(123)} \frac{X^{(123)}}{2} \frac{1}{Q^2 - m_{123}^2} q^\mu
\]  

(37)

The normalization is obtained from the chiral limit:

\[
C^{(123)} = \frac{2\sqrt{2}}{3f_\pi} A^{(123)}
\]  

(38)
If we allow for two resonances \( V_{13} \) and \( V'_{13} \) in \( s_2 \) with a relative strength defined by \( \beta_{13} \) (equivalent to Eqn. (34)) and similarly for \( V_{23} \) and \( V'_{23} \) with \( \beta_{23} \), we get the general formulae for the form factors:

\[
F_1^{(123)} = C^{(123)} BW_A(Q^2) \left\{ \frac{BW_{V_{13}}(s_2)}{1 + \beta_{13}} \left( 1 - \frac{1}{3} \frac{m_1^2 - m_3^2}{m_{V_{13}}^2} \right) \right. \\
+ \frac{\beta_{13}}{1 + \beta_{13}} BW_{V'_{13}}(s_2) \left( 1 - \frac{1}{3} \frac{m_1^2 - m_3^2}{m_{V'_{13}}^2} \right) \\
+ \frac{2}{3} \left( m_2^2 - m_3^2 \right) \left\{ \frac{1}{1 + \beta_{13}} BW_{V_{13}}(s_1) + \frac{\beta_{13}}{1 + \beta_{13}} BW_{V'_{13}}(s_1) \right\} \left. \right\} \\
F_2^{(123)} = C^{(123)} BW_A(Q^2) \left\{ \frac{BW_{V_{23}}(s_1)}{1 + \beta_{23}} \left( 1 - \frac{1}{3} \frac{m_2^2 - m_3^2}{m_{V_{23}}^2} \right) \right. \\
+ \frac{\beta_{23}}{1 + \beta_{23}} BW_{V'_{23}}(s_1) \left( 1 - \frac{1}{3} \frac{m_2^2 - m_3^2}{m_{V'_{23}}^2} \right) \\
+ \frac{2}{3} \left( m_1^2 - m_3^2 \right) \left\{ \frac{1}{1 + \beta_{23}} BW_{V_{23}}(s_2) + \frac{\beta_{23}}{1 + \beta_{23}} BW_{V'_{23}}(s_2) \right\} \left. \right\} \\
F_S^{(123)} = \frac{C^{(123)}}{2(Q^2 - m_{(123)}^2)} \left\{ X^{(123)} + \frac{BW_A(Q^2)(Q^2 - m_A^2)}{m_A^2 Q^2} \right. \\
\times \left( \frac{1}{1 + \beta_{13}} BW_{V_{13}}(s_2) \left( m_{(123)}^2(Q^2 - 2s_1 - s_2 + 2m_1^2 + m_2^2) \right) \right. \\
- \left. \frac{m_2^2 - m_3^2}{m_{V_{13}}^2} [m_{(123)}^2(Q^2 + s_2 - m_2^2) - Q^2(s_2 - m_{V_{13}}^2)] \right) \\
+ \frac{\beta_{13}}{1 + \beta_{13}} BW_{V'_{13}}(s_2) \times \left( V_{13} \rightarrow V'_{13} \right) + (1 \leftrightarrow 2) \left. \right\} \quad \text{(39)}
\]

For the channels where \( C^{(123)} = 0 \), the Breit-Wigner resonance factors \( BW_{V_{13}} \) and \( BW_{V'_{13}} \) must be put equal to zero.

Note that in the case of exact \( SU(3) \) flavour symmetry we always have \( m_1^2 = m_2^2 = m_3^2 \), in which case the form factors \( F_1 \) and \( F_2 \) retain the form they have in the case of exact \( SU(3)_L \otimes SU(3)_R \) chiral symmetry. This is of course also true for the three pion decay mode with three equal masses. The scalar form factor \( F_S \), on the other hand, is always non-zero once the full chiral symmetry is broken, whether or not the flavour symmetry still holds. \( F_S \) gets a non-resonant contribution which is proportional to pseudoscalar masses squared \( (X^{(123)}) \) and a resonant contribution which is proportional to the off-shellness \( (Q^2 - m_A^2) \) of the axial three particle resonance.

We will now apply these formulae to the channels \( K^-\pi^-\pi^+ \) and \( K^+\pi^+K^- \), taking into account \( \rho \) and \( \rho' \) resonances in \( \pi^+\pi^- \) and \( K^+K^- \) with relative strength \( \beta \), as in Eqn. (34) and a single \( K^* \) resonance in \( K\pi \). The relevant equations are:

\[
F_1^{(K2\pi)} = -\frac{\sqrt{2} \sin \theta_C}{3 f_{\pi}} BW_{K_1}(Q^2) BW_{K^*}(s_2) \left( 1 - \frac{1}{3} \frac{m_{K_1}^2 - m_{\pi}^2}{m_{K^*}^2} \right)
\]
\[
F_2^{(K^2\pi)} = -\frac{\sqrt{2}\sin\theta_C}{3f_\pi} BW_{K_1}(Q^2) \left\{ T_\rho(s_1) + \frac{2m_K^2 - m_\pi^2}{3m_K^2} BW_{K^*}(s_2) \right\}
\]

\[
F_S^{(K^2\pi)} = -\frac{\sqrt{2}\sin\theta_C}{3f_\pi} \frac{1}{2(Q^2 - m_\pi^2)} \left( m_\pi^2 + m_K^2 \right) - \frac{BW_{K^*}(s_2)}{m_K^2} \left( m_K^2 - m_\pi^2 \right) \left( m_K^2(Q^2 - 2s_1 - s_2 + 2m_K^2 + m_\pi^2) \right)
\]

\[
\times \left[ BW_{K^*}(s_2) \left( m_K^2 - m_\pi^2 \right) (Q^2 - 2s_1 - s_2 + 2m_K^2 + m_\pi^2) \right)
\]

Note that the anomalous form factor \( F_3 \) is not affected by the pseudoscalar masses, since the anomaly is a short distance effect. Therefore in the numerical discussion in the next section we use the same anomalous form factors as in Ref. \[1\].

Finally, we do not consider scalar resonances in this paper. For the inclusion of a \( J^P = 0^- \) resonance \( (\pi') \) and its possible effects we refer to Refs. \[1\] [4].

5 Numerical Results

We will start the numerical discussion by giving the integrated decay rates \( \Gamma^{(abc)} \), normalized in the usual way to the electronic width \( \Gamma_e \) of the tau. The total width gets three contributions (cf. Eqns. \[13\]—\[17\] and \[22\]):

\[
\Gamma^{(abc)} = \Gamma_n^{(abc)} + \Gamma_a^{(abc)} + \Gamma_S^{(abc)} \tag{42}
\]

\( \Gamma_n^{(abc)} \) is the “normal” contribution resulting from the form factors \( F_1 \) and \( F_2 \), \( \Gamma_a^{(abc)} \) is the anomalous contribution resulting from \( F_3 \), and \( \Gamma_S^{(abc)} \) is the scalar contribution from
For the $3\pi$ channel, $\tau^- \rightarrow \nu \pi^- \pi^- \pi^-$, we use the parametrization of [1] (see also [2]) for the Breit-Wigners $BW_\omega$ and $T_\rho$. In the case of the channel $\tau^- \rightarrow \nu K^- \pi^- \pi^+$, the parametrizations of the Breit-Wigner factors $BW_{K^*}$, $BW_{K}$, and $T_\rho(s_i)$ are taken from [3], where Eqn. (35) in [2] is used for the $T_{K^*}$. Breit-Wigner in the three body resonance, which occurs in the form factor $F_3$. The other parameters can also be found in [1]. The parametrization for the decay $\tau^- \rightarrow \nu K^- \pi^- K^+$ is also taken from [1] and [2].

With these parametrizations we get the following results:

| Channel $(abc)$ | $\Gamma^{(abc)}(abc)$ | $\Gamma^{(abc)}_{\text{Tot}}$ | $\Gamma^{(abc)}_{\nu}$ | $\Gamma^{(abc)}_{\pi}$ |
|----------------|-------------------------|-------------------------------|------------------------|------------------------|
| $\pi^- \pi^- \pi^+$ | 0.356 | 0.356 | 0 | 0.0000073 |
| $K^- \pi^- \pi^+$ | 0.0327 | 0.0313 | 0.00137 | 0.0000033 |
| $K^- \pi^- K^+$ | 0.0061 | 0.0037 | 0.0023 | 0.0000013 |

We find that the relative contribution of the scalar part is of the order of $10^{-5}$ in the $3\pi$ case, and $10^{-4}$ in the channels $K^- \pi^- \pi^+$ and $K^- \pi^- K^+$. Note that especially in the modes with kaons our results for the scalar part are actually much smaller than the naive application of the PCAC argument would indicate (cf. the estimates in the introduction). So we find that in no case the scalar contribution could be measured in the total decay width. In the modes with mesons of different masses (ie. $K^- \pi^- \pi^+$ and $K^- \pi^- K^+$), the form factors $F_1$ and $F_2$ are also modified by the inclusion of the pseudoscalar mass effects, but the numerical size of this effect is negligible (less than 1%). Considering the large uncertainties in the predictions for the rate which result from details of the Breit-Wigner parametrizations [1], it is clear that also these effects on the total rate can not be used to see the pseudoscalar mass effects experimentally.

So we have to consider angular distributions as suitable observables. We will therefore present now numerical results for the spin-0-spin-1 interference structure functions $W_{SB,SD,SF}$ and the pure spin-0 structure function $W_{SA}$ and compare them with the pure spin-1 structure functions $W_{E,G}$ and $W_I$. In particular we will concentrate on the $Q^2$ distribution of the structure functions, i.e. we integrate over the Dalitz-plot variables $s_1$ and $s_2$. Note that the most interesting moments for our analysis in Eqn. (19) projects only on a linear combination of one spin-1 and one spin-0 structure function.

There are two possible effects contributing to the scalar form factor: The pseudoscalar masses, considered in the present paper, and scalar ($J^P = 0^-$) three particle resonances [1] [4]. The $Q^2$ dependence of the structure functions for a possible $J^P = 0^-$ resonance is already discussed in [4]. As we will show below the $Q^2$ dependence of the impacts on the structure functions of these different effects are very different: The scalar resonance contribution is peaked around the resonance mass, whereas the pseudoscalar mass effects are large at low $Q^2$. So by measuring the $Q^2$ distributions these two effects can be distinguished.

Let us start with the $3\pi$ channel: $\tau^- \rightarrow \nu \pi^- \pi^- \pi^-$. As already mentioned, due to G-parity conservation $W_G$ and $W_I$ are vanishing in this case and a nonvanishing contribution to the moments $\langle \sin \beta \cos \gamma \rangle$ and $\langle \sin \beta \sin \gamma \rangle$ yields a clean signature of the presence of a scalar contribution and a measurement would allow to analyse the scalar...
form factor $F_s$ in detail. In Fig. 2a we show the $Q^2$ distribution of the $s_1, s_2$ integrated structure functions $W_{SA,SB,SD}$ normalized to $W_{tot} = W_A + W_{SA}$. Note that $W_B$ vanishes in the three pion channel. One observes a sizable contribution of the scalar form factor only at low $Q^2$ values ($Q^2 < 0.8 GeV^2$). This is in contrast to a possible scalar resonance contribution to the scalar form factor where the scalar resonance is peaked around the $\pi'$ resonance mass, see Fig. 4 in [4]. A measurement of the $Q^2$ dependence would therefore allow to disentangle these two possible contributions. It is clear that the scalar form factor effect is strongly enhanced by the interference with the larger spin-1 form factors $F_{1,2,3}$, whereas the pure spin-0 structure function remains small over the whole $Q^2$ range. For comparison, we show the normalized spin-1 structure function $W_E/W_{tot}$ in Fig. 2b as a function of $Q^2$. As mentioned before this ratio is closely related to the parity violating asymmetry [3, 4]. Note that in contrast to the figures in [3, 4], we have taken the spin-0 contribution in the normalization $W_{tot}$ into account.

Let us now discuss the numerical effect of the nonvanishing meson masses to the Cabibbo suppressed $\tau \rightarrow \nu K^-\pi^-\pi^+$ channel. In this case there are contributions to all structure functions in Eqn. (22). Fig. 3a shows the $Q^2$ dependence of the $s_1, s_2$ integrated structure functions $W_{SA,SB,SD, SF}$ normalized to $W_{tot} = W_A + W_{SF}$. Like in the three pion case a sizable contribution of the scalar form factor is only observable at low $Q^2$ values and the pure spin-0 contribution remains small over the whole $Q^2$ range. The results for the normalized spin-1 structure functions are shown in Fig. 3b. (A detailed discussion of the latter structure functions can also be found in [2].) We find that the spin-1 structure function $W_I$ is rather small in the region where the spin-0-spin-1 interference structure function $W_{SB}$ becomes large, so the scalar part can indeed be measured.

Finally we present results for the Cabibbo allowed decay $\tau \rightarrow \nu K^-\pi^-K^+$. In Fig. 4a we show the results for the normalized scalar structure functions $W_{SA,SB,SD, SF}$ normalized to $W_{tot} = W_A + W_B + W_{SA}$. For $Q^2$ values below $1.6 GeV^2$ the mass effects are fairly large. For comparison, predictions for the pure spin-1 structure functions are shown in Fig. 4b. (The latter have also been discussed in [2].) We find again that in the important low $Q^2$ region the spin-0-spin-1 interference structure functions are comparable with the corresponding pure spin-1 structure functions.

6 Conclusions

We have shown how the well-known approach of extrapolating from the chiral limit to higher energies by Breit-Wigner resonances can be generalized by extrapolating from massive rather than massless pseudoscalar mesons. The inclusion of the pseudoscalar masses does not change the predictions for the integrated decay rate significantly. Nevertheless the non-conservation of the axial current leads to a non-vanishing scalar form factor which can be measured in angular distributions by suitable spin-0-spin-1 interference effects and can be distinguished clearly from a pseudoscalar resonance by the $Q^2$ distribution. Therefore we now urge for a careful experimental analysis of the scalar

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2In the three pion case the moment $\cos \beta$ is combined with an energy ordering $\text{sign}(s_2 - s_2)$ to account for Bose Symmetry, see [3, 4].
form factor in these tau decays, which would enhance our understanding of the structure of the hadronic current and which would also be important for other analyses which make certain assumptions about the scalar form factor (eg. the measurement of the tau polarization by using the three pion decay).

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Figure captions

Fig. 1 diagrams for the vector meson dominance model of the decay $\tau^- \rightarrow \pi^- \pi^- \pi^+$.

Fig. 2 $Q^2$ dependence of $s_1, s_2$ integrated structure functions for the decay channel $\tau \rightarrow \nu \pi^- \pi^- \pi^+$:
   a) $W_{SA}, W_{SB}, W_{SD}$ (solid, dashed, dotted) normalized to $W_{tot}$.
   b) $W_E$ normalized to $W_{tot}$.

Fig. 3 $Q^2$ dependence of $s_1, s_2$ integrated structure functions for the decay channel $\tau \rightarrow \nu K^- \pi^- \pi^+$:
   a) $W_{SA}, W_{SB}, W_{SD}, W_{SF}$ (solid, dashed, dotted, dashed-dotted)
      normalized to $W_{tot}$.
   b) $W_E, W_G, W_I$ (solid, dashed, dotted) normalized to $W_{tot}$.

Fig. 4a,b same as Fig. 3 for the decay channel $\tau \rightarrow \nu K^- \pi^- K^+$. 