Paths, Demazure Crystals
and Symmetric Functions

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Abstract
We review the path realization of Demazure crystals and discuss Demazure characters in the light of symmetric functions.

1 Introduction
Let $U_q(g)$ be a quantum affine Lie algebra. Representation of $U_q(g)$ at $q = 0$ is well described by the crystal base theory [Kai], [KMN1], [KMN2]. For example, consider the irreducible highest weight $U_q(g)$-module $V(\lambda)$ for any dominant integral weight $\lambda$ of level $l$. At $q = 0$ its crystal $B(\lambda)$ admits a parametrization in terms of paths. The latter is the combinatorial object that arose in the studies of solvable lattice models [DJKMO1], [DJKMO2] by Baxter’s corner transfer matrix method [Bax]. Given a perfect crystal $B$ of level $l$, a path is an element of the semi-infinite tensor product $\cdots B \otimes B$. It must obey some boundary condition on the left tail, which is uniquely specified from $\lambda$ and $B$. Letting $\mathcal{P}(\lambda, B)$ denote the set of such paths, one has an isomorphism of crystals $\psi : B(\lambda) \rightarrow \mathcal{P}(\lambda, B)$. These features [KMN1], [KMN2] will be summarized in section 2.

In [Ka2], Kashiwara showed that for each Weyl group element $w$ there exists a finite subset $B_w(\lambda) \subset B(\lambda)$ that corresponds to the crystal of the Demazure module $V_w(\lambda) \subset V(\lambda)$. Then a natural question arises: What kind of paths are contained in the image $\psi(B_w(\lambda))$? This was answered in [KMOU] for a class of $w$ obeying certain conditions. The result is given for each value of the ‘mixing

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index \( \kappa \in \mathbb{Z}_{>1} \) that reflects some property of \( \lambda \) and \( B \). For simplicity we shall exclusively consider \( \kappa = 1 \) case in this paper and refer to \( \text{[KMOU]} \) for \( \kappa \) general case. Then roughly speaking there occur only those paths corresponding to a tensor product of some subset \( B^{(j)}_q \subset B \) and finitely many \( B' \)'s. The precise description will be given in section 3, which constitutes the first main contents in this report. The result may also be viewed as a combinatorial explanation of the tensor product structure in the Demazure modules observed in \( \text{[3]} \). In section 4 we shall give an example from \( \widehat{\mathfrak{sl}}_n \). Results on the other classical affine Lie algebras are available in \( \text{[KMOTU1]} \).

Section 5 is devoted to our another main topic in this paper, namely, the characters of \( V_w(\lambda) \) in the light of symmetric functions. By Theorem 4.2 the Demazure characters provide a \( q \)-analogue of the products of classical characters, that is, Schur functions. We then relate them with the Kostka-Foulkes polynomials and Milne polynomial. Results on some Demazure characters of the other classical affine Lie algebras are available in \( \text{[KMOTU2]} \).

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### 2 Perfect Crystals and Demazure Modules

First we fix the notations following \( \text{[KMN]} \). \( U_q(\mathfrak{g}) \) is the quantized universal enveloping algebra of an affine Lie algebra \( \mathfrak{g} \). Let \( \{\alpha_i\}_{i\in I}, \{\beta_i\}_{i\in I} \) and \( \{\Lambda_i\}_{i\in I} \) denote the set of simple roots, coroots and fundamental weights. \( P \) is the weight lattice and \( P_+ = \{ \lambda \in P \mid \langle \lambda, \beta_i \rangle \geq 0 \text{ for any } i \} \). \( V(\lambda) \) is the irreducible highest weight module of highest weight \( \lambda \in P_+ \) and \( (\mathcal{L}(\lambda), \mathcal{B}(\lambda)) \) is its crystal base (which was originally denoted by \( (L(\lambda), B(\lambda)) \) in \( \text{[KMN]} \)). For the notation of a finite-dimensional representation of \( U_q(\mathfrak{g}) \), we follow section 3 in \( \text{[KMN]} \). For instance, \( P_{cl} \) is the classical weight lattice, \( U_q(\mathfrak{g}) \) is the subalgebra of \( U_q(\mathfrak{g}) \) generated by \( e_i, f_i, q^h \) (\( h \in (P_{cl})^* \)) and \( \text{Mod}^f(\mathfrak{g}, P_{cl}) \) is the category of finite-dimensional \( U_q(\mathfrak{g}) \)-modules which have the weight decompositions. We set \( P_{cl}^+ = \{ \lambda \in P_{cl} \mid \langle \lambda, \beta_i \rangle \geq 0 \text{ for any } i \} \simeq \sum \mathbb{Z}_{\geq 0} \Lambda_i \) and \( (P_{cl}^+)_l = \{ \lambda \in P_{cl}^+ \mid \langle \lambda,c \rangle = l \} \), where \( c \) is the canonical central element. Assume \( V \) in \( \text{Mod}^f(\mathfrak{g}, P_{cl}) \) has a crystal base \( (L,B) \). For an element \( b \) of \( B \), we set \( \varepsilon_i(b) = \max\{n \geq 0 \mid e_i^n b \in B \} \), \( \varepsilon(b) = \sum_i \varepsilon_i(b) \Lambda_i \) and \( \varphi_i(b) = \max\{n \geq 0 \mid f_i^n b \in B \} \), \( \varphi(b) = \sum_i \varphi_i(b) \Lambda_i \).

Let \( B \) be a perfect crystal of level \( l \). We refer Definition 4.6.1 in \( \text{[KMN]} \) for its definition. For \( \lambda \in (P_{cl}^+)_l \), let \( b(\lambda) \in B \) be the element such that \( \varphi(b(\lambda)) = \lambda \).
From the definition of perfect crystal, such a $b(\lambda)$ exists and is unique. Let $\sigma$ be the automorphism of $(P^+_\mathfrak{g})_I$ given by $\sigma\lambda = \varepsilon(b(\lambda))$. We set $b_k = b(\sigma^{k-1}\lambda)$ and $\lambda_k = \sigma^k\lambda$. Then perfectness assures that we have the isomorphism of crystals

$$B(\lambda_{k-1}) \simeq B(\lambda_k) \otimes B.$$ 

Iterating this isomorphism, we have

$$\psi_k : B(\lambda) \simeq B(\lambda_k) \otimes B^\otimes k.$$ 

Defining the set of paths $P(\lambda, B)$ by

$$P(\lambda, B) = \{ p = \cdots \otimes p(2) \otimes p(1) \mid p(j) \in B, p(k) = \overline{b}_k \text{ for } k \gg 0 \},$$ 

we see that there is an isomorphism of crystals $\psi : B(\lambda) \rightarrow P(\lambda, B)$. In particular, the image $\psi(u_{\lambda})$ of the highest weight vector $u_{\lambda} \in B(\lambda)$ is given by $p = \cdots \otimes \overline{b}_k \otimes \cdots \otimes \overline{b}_2 \otimes \overline{b}_1$. We call $p$ the ground-state path. One can explicitly describe the weights and the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on $P(\lambda, B)$ by the energy function and the signature rule. See sections 1.3 and 1.4 of [KMOU].

Now we proceed to Demazure crystals. Let $\{r_i\}_{i \in I}$ be the set of simple reflections, and let $W$ be the Weyl group. For $w \in W$, $l(w)$ denotes the length of $w$, and $\prec$ denotes the Bruhat order on $W$. Let $U^+_q(\mathfrak{g})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $e_i$’s. For $\lambda \in (P^+_\mathfrak{g})_I$, we consider the irreducible highest weight $U^+_q(\mathfrak{g})$-module $V(\lambda)$ as before. Let $V_w(\lambda)$ denote the $U^+_q(\mathfrak{g})$-module generated by the extremal weight space $V(\lambda)_{w,\lambda}$. These modules $V_w(\lambda)$ ($w \in W$) are called the Demazure modules. They are finite dimensional subspaces of $V(\lambda)$. Let $(L(\lambda), B(\lambda))$ be the crystal base of $V(\lambda)$. In [K2] Kashiwara showed that for each $w \in W$, there exists a subset $B_w(\lambda)$ of $B(\lambda)$ such that

$$\frac{V_w(\lambda) \cap L(\lambda)}{V_w(\lambda) \cap qL(\lambda)} = \bigoplus_{b \in B_w(\lambda)} Q b.$$ 

Furthermore, $B_w(\lambda)$ has the following recursive property.

If $r_i w \succ w$, then

$$B_{r_i w}(\lambda) = \bigcup_{n \geq 0} \tilde{f}_i^n B_w(\lambda) \setminus \{0\}. \quad (2.1)$$ 

We call $B_w(\lambda)$ a Demazure crystal.

### 3 Main Theorem

Let us present the main theorem in [KMOU] in the case $\kappa = 1$. For the definition of the mixing index $\kappa$, see section 2.3 of [KMOU].

Let $\lambda$ be an element of $(P^+_\mathfrak{g})_I$, and let $B$ be a classical crystal. For the theorem, we need to assume four conditions (I-IV).
B is perfect of level l.

Thus, we can assume an isomorphism between $\mathcal{B}(\lambda)$ and the set of paths $\mathcal{P}(\lambda, B)$. Let $\overline{\mathcal{P}} = \cdots \otimes \overline{b}_2 \otimes \overline{b}_1$ denote the ground state path. Fix a positive integer $d$. For a set of elements $i^{(j)}_a$ ($j \geq 1, 1 \leq a \leq d$) in $\mathcal{I}$, we define $B^{(j)}_a$ ($j \geq 1, 0 \leq a \leq d$) by

$$B_0^{(j)} = \{ \overline{p}_j \}, \quad B_a^{(j)} = \bigcup_{n \geq 0} \tilde{f}^n_{i^{(j)}_a} B_{a-1}^{(j)} \setminus \{0\} \quad (a = 1, \ldots, d).$$ (3.1)

(II) For any $j \geq 1$, $B_d^{(j)} = B$.

(III) For any $j \geq 1$ and $1 \leq a \leq d$, $(\lambda_j, h_{i^{(j)}_a}) \leq \varepsilon_{i^{(j)}_a}(b)$ for all $b \in B_{a-1}^{(j)}$.

We now define an element $w^{(k)}$ of the Weyl group $W$ by

$$w^{(0)} = 1, \quad w^{(k)} = r_{i^{(j)}_a} w^{(k-1)} \quad \text{for} \quad k > 0,$$

where $j$ and $a$ are fixed from $k$ by $k = (j - 1)d + a, j \geq 1, 1 \leq a \leq d$.

(IV) $w^{(0)} \prec w^{(1)} \prec \cdots \prec w^{(k)} \prec \cdots$.

See [KMOU], [KMOTU1] on how to check the last condition.

Now the main statement in [KMOU] is

**Theorem 3.1** ([KMOU]) Under the assumptions (I-IV), we have

$$\mathcal{B}_{w^{(k)}}(\lambda) \simeq u_{\lambda_j} \otimes B_a^{(j)} \otimes B^{\otimes (j-1)}.$$

The proof is done by showing the recursion relation (2.1) in the path realization. The paths on the RHS enjoy ‘full fluctuations’ over $\mathcal{B}$ in the first $j - 1$ steps, while at the $j$-th step they are allowed only ‘partial fluctuations’ over $B_{a-1}^{(j)} \subset B$. After that, they are completely frozen to the ground state path $\overline{\mathcal{P}}$.

4 **Example from $\widehat{\mathfrak{sl}_n}$**

In this section, we describe an example in the $\widehat{\mathfrak{sl}_n}$ case. We begin with fixing notations. We use the cyclic notation for $\alpha_i, h_i, \Lambda_i, r_i, e_i, f_i$, etc, that is, we consider their subscripts $i$ belong to $\mathbb{Z}/n\mathbb{Z}$. Let $V^{k,l}$ be the irreducible highest weight $U_q(\mathfrak{sl}_n)$-module with highest weight $l \Lambda_k$. It turns out that $V^{k,l}$ admits $U'_q(\widehat{\mathfrak{sl}_n})$ actions, and has a crystal $B^{k,l}$.

We describe the explicit actions of $\hat{e}_i, \hat{f}_i$ when $k = 1$ (symmetric tensor case). As a set, $B^{1,l}$ is described as $B^{1,l} = \{(x_0, \ldots, x_{n-1}) \in \mathbb{Z}_{\geq 0}^n \mid \sum_{i=0}^{n-1} x_i = l\}$. The
actions of $\tilde{e}_i, \tilde{f}_i$ are defined as follows.

$$
\tilde{e}_i(x_0, \cdots, x_{n-1}) = \begin{cases} (x_0, \cdots, x_{i-1} + 1, x_i - 1, \cdots, x_{n-1}) & (i \neq 0) \\ (x_0 - 1, \cdots, x_{n-1} + 1) & (i = 0) \end{cases} \quad (4.1)
$$

$$
\tilde{f}_i(x_0, \cdots, x_{n-1}) = \begin{cases} (x_0, \cdots, x_{i-1} - 1, x_i + 1, \cdots, x_{n-1}) & (i \neq 0) \\ (x_0 + 1, \cdots, x_{n-1} - 1) & (i = 0) \end{cases} \quad (4.2)
$$

If the right hand side contains a negative component, we should understand it as 0.

**Example 4.1** Let $n = 2$. Under the identification of the elements $(2, 0) \leftrightarrow 00, (1, 1) \leftrightarrow 01, (0, 2) \leftrightarrow 11$, $B^{1,2}$ is described as follows.

$$
B^{1,2} \quad 00 \overset{1}{\leftrightarrow} 01 \overset{1}{\leftrightarrow} 11
$$

Here $b \overset{1}{\to} b'$ means $b' = \tilde{f}_i b$.

We are to show $\lambda = l \Lambda_0$ and $B = B^{1, l}$ satisfies the four conditions in section 3. Firstly, as listed in [KMN2], $B^{1, l}$ is perfect of level $l$. (Note that we deal with the $(A^{m}_{n-1}, B(l\Lambda_1))$ in their notation.) $\lambda_j$ and $\bar{\theta}_j$ are given by $\lambda_j = l \Lambda_{j-1}$ and $\bar{\theta}_j = (0, \cdots, 0, \bar{\theta}_j, 0, \cdots, 0)$ $(r = -j \bmod n, 0 \leq r \leq n - 1)$. Set $d = n - 1$, and define $j^{(j)}_a \in \mathbb{Z}/n\mathbb{Z}$ by $j^{(j)}_a = a - j$. To illustrate, we consider the case $j = 1$.

From the definition (4.3) and the rule (4.2), we easily obtain

$$
B^{(1)}_0 = \{(0, \cdots, 0, l)\}, \quad B^{(1)}_1 = \{(x_0, 0, \cdots, 0, x_{n-1}) \mid x_0 + x_{n-1} = 1\}, \cdots,
$$

$$
B^{(1)}_a = \{(x_0, \cdots, x_{a-1}, 0, \cdots, 0, x_{n-1}) \mid x_0 + \cdots + x_{a-1} + x_{n-1} = l\}, \cdots.
$$

Thus, we have $B^{(1)}_d = B$. For $j > 1$, the situation is the same, and $B^{1, l}$ satisfies (II).

**Example 4.2** Let $n = 3$ and $l = 2$. Under the identification of the elements $(2, 0, 0) \leftrightarrow 00, (1, 1, 0) \leftrightarrow 01, (1, 0, 1) \leftrightarrow 02, (0, 2, 0) \leftrightarrow 11, (0, 1, 1) \leftrightarrow 12, (0, 0, 2) \leftrightarrow 22$, $B^{(1)}_a$ $(a = 0, 1, 2)$ are given by

$$
B^{(1)}_0 = \{22\}, \quad B^{(1)}_1 = B^{(1)}_0 \sqcup \{00, 02\}, \quad B^{(1)}_2 = B^{(1)}_1 \sqcup \{01, 11, 12\}(= B).
$$

The third condition is obviously cleared, since $\langle \lambda_j, h_{j^{(j)}_a} \rangle = \langle l \Lambda_{j-1}, h_{a-j} \rangle = 0$ for $1 \leq a \leq d$. By definition, we have

$$
w^{(0)} = 1, \quad w^{(K)} = r_{K-1} \cdots r_1 r_0 \quad \text{for} \quad K > 0,
$$

and the last condition is also true. Therefore, from Theorem 3.3 we have

$$
B^{(K)}_{w^{(K)}}(l\Lambda_0) \simeq u_{\Lambda_{-j}} \otimes B^{(j)}_a \otimes B^{j-1}. \quad (4.3)
$$

Here $j$ and $a$ are determined from $K$ by $K = (j - 1)d + a$, $j \geq 1, 1 \leq a \leq d$, and $u_{\Lambda_{-j}}$ is identified with $\cdots \otimes \bar{b}_{j+2} \otimes \bar{b}_{j+1}$.
Example 4.3  Let $n = 2, l = 1$, then $d = 1$. Under the identification $(1, 0) \leftrightarrow 0, (0, 1) \leftrightarrow 1$, illustrated in Figure 1 is the Demazure crystal $B_{r_0 r_1 r_0} (\Lambda_0)$. The symbol $\otimes$ is omitted.

Let us assume $K = Ld$ for some $L \in \mathbb{Z}_{\geq 0}$. Then, (4.3) turns out to be

$$B_{w(l,d)} (l \Lambda_0) \simeq u_{l \Lambda_{-L}} \otimes (B^{1,l}) \otimes L.$$  

(4.4)

For this fixed $L$, consider a subalgebra $U_q(\mathfrak{sl}_n) = \langle e_i, f_i, t_i (i \neq -L) \rangle$ of $U_q(\mathfrak{sl}_n)$. Noting that $Q(q) u_{l \Lambda_{-L}}$ is a trivial $U_q(\mathfrak{sl}_n)$-module, we see the Demazure crystal $B_{w(l,d)} (l \Lambda_0)$ is isomorphic to $(B^{1,l}) \otimes L$ as $U_q(\mathfrak{sl}_n)$-crystals. This is also true for $q \neq 0$, namely, we have the following theorem.

Theorem 4.1

$$V_{w(l,d)} (l \Lambda_0) \simeq (V^{1,l}) \otimes L \text{ as } U_q(\mathfrak{sl}_n)-modules.$$  

Here $U_q(\mathfrak{sl}_n) = \langle e_i, f_i, t_i (i \neq -L) \rangle$.

As crystals, we already have (4.4). Thus it is enough to show the $U_q(\mathfrak{sl}_n)$-invariance of $V_{w(l,d)} (l \Lambda_0)$. For this purpose, we cite two propositions from [Ka2].

Proposition 4.1  ([Ka2] Lemma 3.2.1 (i)) If $r_i w \prec w$, then

$$f_i V_w (\lambda) \subset V_w (\lambda).$$

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Figure 1: The Demazure crystal $B_{r_0 r_1 r_0} (\Lambda_0)$. 

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Proposition 4.2 ([Ka2] Corollary 3.2.2) If \( w = r_{i_1} \cdots r_{i_l} \) is reduced, then

\[
V_w(\lambda) = \sum_{k_1, \ldots, k_l \geq 0} Q(q) f_{i_1}^{k_1} \cdots f_{i_l}^{k_l} u_\lambda.
\]

Let us prove Theorem 4.1. We assume \( n \geq 3 \). (The \( n = 2 \) case is simple.) By definition, \( e_i V_{w(L_0)}(\lambda_0) \subset V_{w(L_0)}(\lambda_0) \subset V_{w(L_0)}(\lambda_0) \) for \( i \neq -L \). If \( r_i w(L_0) \neq u(L_0) \), the statement is a direct consequence of Proposition 4.1. (This case includes \( \lambda_0 \).) Assume \( r_i w(L_0) \neq w(L_0) \). Since \( w(L_0) = r_{L-1} \cdots r_1 r_0 \) is reduced, \( r_i w(L_0) = r_{L-1} \cdots r_1 r_0 \) should be also reduced. Using the braid relation \( r_j r_{j+1} r_j = r_{j+1} r_j r_{j+1} \) and the condition \( i \neq -L, -L-1 \), we can check \( r_i w(L_0) = w(L_0) r_{i+1} \). Since \( r_i w(L_0) = r_{i-1} \cdots r_1 r_0 \) and \( w(L_0) r_{i+1} = r_{L-1} \cdots r_1 r_0 \) are both reduced, from Proposition 4.2 we have

\[
V_{r_i w(L_0)}(\lambda_0) = \sum_{k \geq 0} f_{i}^k V_{w(L_0)}(\lambda_0),
\]

\[
V_{w(L_0)}(r_{i+1} \lambda_0) = \sum_{k_0, k_1, \ldots, k_L \geq 0} Q(q) f_{L-1}^{k_0} \cdots f_{i+1}^{k_1} u_{\lambda_0}.
\]

From \( V_{r_i w(L_0)}(\lambda_0) = V_{w(L_0)}(r_{i+1} \lambda_0) \), we can conclude the invariance under \( f_i \). The theorem is proved.

These facts admit straightforward generalization to arbitrary \( k \) cases. To define the corresponding Weyl group sequence, for \( k (1 \leq k \leq n-1) \) and \( i \in \mathbb{Z}/n\mathbb{Z} \), we set

\[
R_i^{(k)} = (r_{i+(n-k-1)} \cdots r_{i+1} r_{i-(k-1)}) \cdots
\]

\[
\cdots (r_{i+1} r_{i-1})\cdots (r_{i+(n-k)}) \cdots r_{i+1} r_i).
\]

There are \( k \) blocks in \( R_i^{(k)} \), and in each block there are \( (n-k) \) simple reflections. From the relations among fundamental reflections, \( R_i^{(k)} \) admits another expression.

\[
R_i^{(k)} = (r_{i-(k-1)} \cdots r_{i-1} \cdots r_{i+(n-k-1)}) \cdots
\]

\[
\cdots (r_{i+1} r_{i+1}) \cdots (r_{i+1} r_{i+1}) \cdots r_{i-1} r_i).
\]

In this case, there are \( (n-k) \) blocks, and in each block there are \( k \) simple reflections. We take \( d \) to be \( k(n-k) \) blocks, and let \( w(L_0) \) be determined recursively by

\[
w^{(0)} = 1, \quad w^{((L+1)d)} = R_{-kL}^{(k)} w^{(Ld)}.
\]
Example 4.4 Explicit expression of $w^{(Ld)}$.

$$n = 2, \quad k = 1 \quad w^{(d)} = r_0, w^{(2d)} = r_1 r_0, w^{(3d)} = r_0 r_1 r_0, \cdots.$$  

$$n = 3, \quad k = 1 \quad w^{(d)} = r_1 r_0, w^{(2d)} = r_0 r_2 r_1 r_0, w^{(3d)} = r_2 r_1 r_0 r_2 r_1 r_0, \cdots.$$  

$$k = 2 \quad w^{(d)} = r_2 r_1 r_0, w^{(2d)} = r_0 r_1 r_2 r_0, w^{(3d)} = r_1 r_2 r_0 r_1 r_2 r_0, \cdots.$$  

$$n = 4, \quad k = 1 \quad w^{(d)} = r_2 r_1 r_0, w^{(2d)} = r_1 r_0 r_3 r_2 r_1 r_0, w^{(3d)} = r_0 r_3 r_2 r_1 r_0 r_3 r_2 r_1 r_0, \cdots.$$  

$$k = 2 \quad w^{(d)} = r_2 r_3 r_0, w^{(2d)} = r_3 r_0 r_1 r_2 r_3 r_0, w^{(3d)} = r_0 r_1 r_2 r_3 r_0 r_1 r_2 r_3 r_0, \cdots.$$  

$\mathfrak{g}$-modules

Theorem 4.2 We have

$$B_{w^{(Ld)}}(\Lambda_0) \simeq u_{\Lambda_0} \otimes (B^{k,l})^L,$$

$$V_{w^{(Ld)}}(\Lambda_0) \simeq (V^{k,l})^L$$ as $U_q(\mathfrak{g})$-modules,

where $U_q(\mathfrak{g}) = \langle e_i, f_i, t_i (i \neq -kL) \rangle$.

5 Demazure Characters and Symmetric Functions

In this section, we consider the characters of the Demazure modules we have seen in the previous section. Using the automorphism coming from the Dynkin diagram symmetry, Theorem 4.3 turns out to be the following.

$$B_{w^{(Ld)}}(\Lambda_{kL}) \simeq u_{\Lambda_{kL}} \otimes (B^{k,l})^L,$$

$$V_{w^{(Ld)}}(\Lambda_{kL}) \simeq (V^{k,l})^L$$ as $U_q(\mathfrak{g})$-modules.

Note that $w^{(Ld)}$ is also changed suitably. Here and in what follows, $U_q(\mathfrak{g})$ always means the subalgebra of $U_q(\mathfrak{g})$ generated by $e_i, f_i, t_i (i \neq 0)$.

By definition, the character of the Demazure module $V_{w^{(Ld)}}(\lambda)$ reads as

$$\text{ch} V_{w^{(Ld)}}(\lambda) = \sum_{\mu} \sharp (B_{w^{(Ld)}}(\lambda))_{\mu} e^\mu.$$  

Here $(B_{w^{(Ld)}}(\lambda))_{\mu}$ is the set of elements in $B_{w^{(Ld)}}(\lambda)$ of weight $\mu$, and $\mu$ runs over all weights. Consider the character of $V_{w^{(Ld)}}(\Lambda_{kL})$ given above. Since it has the $U_q(\mathfrak{g})$-invariance, its character has the following form.

$$e^{-\lambda_0} \text{ch} V_{w^{(Ld)}}(\Lambda_{kL}) = \sum_{\lambda \vdash kL} K_{\lambda}(q) s_{\lambda}.$$  

(5.3)
Here $\lambda$ runs over all partitions of $klL$ having at most $n$ parts, $K_\lambda(q)$ is some polynomial in $q$, and $s_\lambda$ is the Schur function considered as a character of $\mathfrak{sl}_n$. $q$ stands for $e^{-\delta}$, where $\delta$ is the null root of $\hat{\mathfrak{sl}}_n$. In view of (5.2), we have
\[
\left(e^{-l\Lambda_0}chV_{w(l;L)}(l\Lambda_{kL})\right)_{q=1} = s_{(k)}^L.
\]
Thus, (5.3) can be viewed as a $q$-analogue of $s_{(k)}^L$.

**Example 5.1** We consider the case given in Example 4.3. To adapt the rule in this section, we apply the automorphism of the Dynkin diagram.

\[
e^{-\Lambda_0}chV_{r_1r_0r_1}(\Lambda_1) = (1 + q)s_{(21)} + q^2s_{(3)}.
\]

Let us examine the polynomial $K_\lambda(q)$. First, we focus on the case of $k = 1$ (symmetric tensor case). The following theorem was suggested by A.N. Kirillov.

**Theorem 5.1** If $k = 1$, we have
\[
K_\lambda(q) = q^{-E_0}K_{\lambda(L)}(q),
\]
\[
E_0 = l\alpha(L - \frac{n}{2}(a + 1)) \quad (a = \left\lfloor \frac{L}{n} \right\rfloor).
\]
where $K_{\lambda\mu}(q)$ is the Kostka-Foulkes polynomial.

This is a direct consequence of the following expression shown in [NY]. (See also [FF].)

\[
K_{\lambda(L)}(q) = \sum q\sum_{j=1}^{L-1}j^H(b_{i+1} \otimes b_i),
\]
where the sum is over all elements $b_L \otimes \cdots \otimes b_1$ in $(B^{1;L})^{\otimes L}$ which are killed by $\check{e}_i$ ($i \neq 0$) and have weight $\sum_{i=1}^{n-1}(\lambda_i - \lambda_{i+1})\lambda_i$. $H$ stands for the so-called energy function. Recalling the Milne polynomial (see p73 of [Ki] and references therein)

\[
M_\mu(x; q) = \sum s_\lambda(x)K_{\lambda\mu}(q),
\]
we see the Demazure character (5.3) for $k = 1$ turns out to be the Milne polynomial $M_{(L)}(x; q)$ up to a power of $q$.

For general $k$, we have not obtained concrete results yet. We only mention some generalizations of the Kostka-Foulkes polynomial and Milne polynomial. For the former, there exists a $q$-analogue of the multiplicity of the irreducible component $V_\lambda$ in the tensor product $V_{\mu_1} \otimes \cdots \otimes V_{\mu_N}$ when all $\mu_i$ have a rectangular shape. (See (2.35) of [Ki].) For the latter, there exists a $q$-analogue of products of Schur functions by Lascoux, Leclerc and Thibon [LLT]. (They call it $H$ function.) It would be a fascinating problem to relate them with Demazure characters where $q$ has its own meaning as the degree along the null root.

We finish with presenting an inhomogeneous version of (5.3) and Theorem 5.1.
Theorem 5.2  For a partition \( \mu = (\mu_1, \cdots, \mu_m) \) of \(|\mu| = \mu_1 + \cdots + \mu_m\), we set
\[
\omega_\mu = R_{\mu_1}^ (\mu_1) R_{\mu_1+\mu_2} (\mu_2) \cdots R_{|\mu|} (\mu_m).
\]
Then we have
\[
e^{-\Lambda_0} \text{ch} V_{\omega_\mu}(\Lambda_{|\mu|}) = q^{c(T_\mu)} \sum_{\lambda \vdash |\mu|, \lambda_1 \leq n} K_{\lambda'}(q^{-1}) s_\lambda.
\]
Here \( c(T_\mu) \) is the Lascoux-Schützenberger charge of the tableau \( T_\mu \), whose shape is \((n^p r) \) \(|\mu| = pn + r, 0 \leq r < n \). The contents of \( T_\mu \) are filled in the natural way from left to right from the first row. \( \lambda' \) stands for the transposition of \( \lambda \).

For the definition of the LS charge, see chapter III of [M].

Example 5.2  Set \( n = 4, \mu = (321) \). Then \( |\mu| = 6, \omega_\mu = r_1 r_2 r_3 r_1 r_0 r_2 r_1 r_0 r_3 r_2 \), \( T_\mu = \begin{array}{cccc}
1 & 1 & 1 & 2 \\
2 & 3 & & 
\end{array} \). We have
\[
e^{-\Lambda_0} \text{ch} V_{\omega_\mu}(\Lambda_2) = (1 + q) s_{(212)} + q s_{(313)} + q s_{(23)} + q^2 s_{(321)}.
\]

We sketch the proof of Theorem 5.2. Firstly, we note
\[
B_{\omega_\mu}(\Lambda_{|\mu|}) \simeq u_{\Lambda_0} \otimes B^{\mu_1,1} \otimes B^{\mu_2,1} \otimes \cdots \otimes B^{\mu_m,1}.
\]
This is a consequence of an inhomogeneous version of Theorem 3.1. We also note that \( V_{\omega_\mu}(\Lambda_{|\mu|}) \) is \( U_q(\mathfrak{sl}_n) \)-invariant. Next, we refer to the following expression of the Kostka-Foulkes polynomial from [NY].
\[
K_{\lambda'}(q^{-1}) = \sum_p q^{E(p)},
\]
where \( p = u_{\Lambda_0} \otimes b_1 \otimes \cdots \otimes b_m \) runs over all elements in \( B_{\omega_\mu}(\Lambda_{|\mu|}) \) such that \( \hat{e}_i p = 0 \) \((i \neq 0) \), \( \text{wt} p = \sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1}) \) \( \text{mod} \ Z \delta \). The energy \( E(p) \) of a path \( p = u_{\Lambda_0} \otimes b_1 \otimes \cdots \otimes b_m \) is defined by
\[
E(p) = \sum_{j=1}^{m} \sum_{i=1}^{j-1} H_{\Lambda_{\mu_i}, \Lambda_{\mu_j}} (b_i \otimes b_j(i+1)).
\]
Now we have to explain two things: the energy function \( H \) and the definition of \( b_j(i+1) \). Both come from [NY]. Let us consider the following isomorphism of crystals.
\[
B_1 \otimes B_2 \simeq B_2 \otimes B_1, \quad b_1 \otimes b_2 \mapsto b_2' \otimes b_1'.
\]
Up to a constant shift, the energy function \( H \) on \( B_1 \otimes B_2 \) is determined by

\[
H(\tilde{e}_i(b_1 \otimes b_2)) = \begin{cases} 
H(b_1 \otimes b_2) + 1 & \text{if } i = 0, \tilde{e}_0(b_1 \otimes b_2) = \tilde{e}_0b_1 \otimes b_2, \\
H(b_1 \otimes b_2) - 1 & \text{if } i = 0, \tilde{e}_0(b_1 \otimes b_2) = b_1 \otimes \tilde{e}_0b_2, \\
= H(b_1 \otimes b_2) & \text{otherwise.}
\end{cases}
\]

If \( B_1 = B^{\mu_1,1} \) and \( B_2 = B^{\mu_2,1} \), we write \( H = H_{\Lambda_{\mu_1},\Lambda_{\mu_2}} \). On the other hand, \( b_j^{(i)} \) \((i \leq j)\) is defined recursively by \( b_j^{(i)} = b_j \) and

\[
B^{\mu_1,1} \otimes B^{\mu_2,1} \simeq B^{\mu_2,1} \otimes B^{\mu_1,1},
\]

\[
b_i \otimes b_j^{(i+1)} \mapsto b_j^{(i)} \otimes b_i.
\]

Since the crystal graph \( B_{w_\mu}(\Lambda_{|\mu|}) \) is connected, the proof reduces to

**Proposition 5.1** If \( \tilde{e}_ip \neq 0 \), then

\[
E(\tilde{e}_ip) = \begin{cases} 
E(p) - 1 & (i = 0), \\
E(p) & (i \neq 0).
\end{cases}
\]

The case of \( i \neq 0 \) is clear. For \( i = 0 \), we need

**Lemma 5.1** Let \( p = u_{\Lambda_0} \otimes b_1 \otimes \cdots \otimes b_m \) and \( \tilde{e}_0p = u_{\Lambda_0} \otimes b_1 \otimes \cdots \tilde{e}_0b_k \otimes \cdots \otimes b_m \neq 0 \). Then we have \( k \neq 1 \), and

\[
E^{(j)}(\tilde{e}_0p) = \begin{cases} 
E^{(j)}(p) - 1 & (j = k), \\
E^{(j)}(p) & (j \neq k).
\end{cases}
\]

Here \( E^{(j)}(p) = \sum_{i=1}^{j-1} H_{\Lambda_{\mu_1},\Lambda_{\mu_2}}(b_i \otimes b_j^{(i+1)}) \).

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