Computational Model of Rib-Reinforced Plate

Ilya Grishin\textsuperscript{1}[0000-0002-9014-2680], Rashit Kayumov\textsuperscript{1}[0000-0003-0711-9429], Gennady Ivanov\textsuperscript{1} and Olga Petropavlovckikh\textsuperscript{1}

\textsuperscript{1}Kazan State University of Architecture and Engineering, Kazan, Russia
E-mail: il6357grishin@yandex.ru

Abstract. The article provides a methodology for constructing a calculation model that allows determining displacements in rectangular plates, reinforced by transverse and longitudinal stiffeners. The calculation model is based on the solution of the Sophie Germain’s equation by the Bubnov-Galerkin methods and successive approximations with the representation of the reactive action of the ribs on the plate in the form of a concentrated forces system. Based on the described computational model, the results of the corresponding algorithm implementation and estimation of its convergence are presented. The need to develop a calculation model is explained by the fact that the main cause of cracks in the asphalt concrete coatings of metal bridges with an orthotropic plate is the wheel load. It reaches its maximum values for short periods of time and does not allow the stresses arising in asphalt concrete to relax. At this stage of the development of the calculation model a single-layer plate is considered supported by longitudinal stiffeners. However, since the bridge bed is a multilayer structure, further development of the design model of the multilayer slab reinforced by longitudinal and transverse stiffeners will be necessary. It will help in the joint work of metal flooring and pavement. In addition, it is necessary to take into account the temperature stress factor.

Keywords: asphalt pavement of bridges, temperature stresses, orthotropic plate, Sophie Germain’s equation, method of successive approximations.

1 Introduction
One of the main problems of asphalt concrete pavements is the appearance of cracks. This is typical for roads [1-6] and bridges [7-13]. One of the reasons for the appearance of cracks can be temperature deformations [14-16]. However, they have the feature that they grow along with seasonal temperature fluctuations over a long time. And taking into consideration that asphalt concrete is a material with pronounced rheological properties [17, 18] it will lead to relaxation of most of the stresses. Thus, a significant contribution to the formation of cracks will be made by loads increasing to their maximum values in short periods of time. When it comes to the asphalt concrete pavement of bridges wheel load is considered. This problem is especially relevant in the case of metal bridges with an orthotropic plate, where the orthotropic plate floor bends relatively to the longitudinal and transverse stiffeners. Subsequently it leads to the tensile stresses appearance in the asphalt concrete of the bridge sheet and in these places.

Thus, in addition to computational models that allow taking into account the temperature stresses [15]. It is necessary to develop a computational model that involves accounting the wheel effect on a
multilayer plate made up of orthotropic metal flooring and bridge layers, reinforced by longitudinal and transverse stiffeners.

At this stage, we are developing a calculation model that takes into account only one layer of the plate, namely the metal flooring, as well as the impact of longitudinal stiffeners. In future, this model will be improved and the transverse stiffeners and layers of the bridge canvas will be taken into consideration, if it is possible using this approach.

2 Materials and methods

2.1 Equation of rectangular plate bending

It is proposed to use the Sophie Germain equation (Fig. 1) as the main relation describing the bending of the plate $w(x, y)$:

$$D\left(\partial^4 w(x,y)/\partial x^4+2\partial^4 w(x,y)/\partial x^2 \partial y^2+(\partial^4 w(x,y)/\partial y^4)\right)=\Sigma P_i \delta(x_i,y_i),$$

where $D=Eh^3/(12(1-\nu^2))$ – plate cylindrical stiffness;

$E$ – the modulus of elasticity of the plate material, N/m$^2$;

$h$ – plate thickness, m;

$\nu$ – plate Poisson's ratio;

$P_i$ – i-th force applied to the plate, N;

$\delta(x_i,y_i)$ – Dirac delta-function, equal to zero everywhere except for the point with coordinates $(x_i,y_i)$, where it is equal to unity;

$w(x,y)$ – vertical movements of the median plane of the plate, m.

The boundary conditions for supporting the slab are taken into account by selecting the corresponding function $w(x,y)$. At this stage, a search was made for a solution for hinged support of the plate, as shown in figure 1, which is provided by a function of the following form:

$$w(x,y)=\sum_{i=1}^m \sum_{j=1}^n A_{ij} \cdot \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b}.$$ (2)

Coefficients $A_{ij}$ we found by substituting (2) in (1) and integrating the found equation with respect to $x$ and $y$ in the interval $0\leq x\leq a$ and $0\leq y\leq b$, having previously multiplied both its parts by $\sin(i\pi x/a)\sin(j\pi y/b)$. Thus, we get $m\cdot n$ linear algebraic equations, which is sufficient to obtain the coefficients $A_{ij}$. In the case of articulation, $A_{ij}$ are expressed explicitly:

$$A_{ij} = \frac{\sum_{i=1}^k P_i \sin \frac{i\pi x_l}{a} \sin \frac{j\pi y_l}{b}}{D\pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \frac{ab}{4}}.$$ (3)
2.2 Consideration of the longitudinal ribs influence

Consider a plate supported, for example, by one edge (figure 2a). At this stage, the influence of the rib is taken into account by the system of vertical forces \( F_i \) (figure 2b). In future, if necessary, it will be possible to take into account the horizontal component of the reactive action of the rib, since equation (1) allows this. When required, the amount of force expressed by the parameter \( mn \) can be increased, thereby improving the approximation of the vertical component of the reactive action of the rib.

The forces \( F_i \) are determined by the method of successive approximations, which includes the following steps:

- **Step 1.** When exposed to a plate without an edge of a certain system of forces \( P \), at the points of application of forces \( F_i \) deflections \( d_{i1} \) will occur. If we consider the edge as a beam (figure 3), then creating beam points with the corresponding coordinates \( y_i \) of the deflections \( d_{i1} \), we need the application of the force system \( F_{i1} \). At step 1 we get: \( d_{i1}, F_{i1} \).

![Figure 2. The scheme of replacing the effects of the ribs on the plate with reactive forces.](image)

![Figure 3. The scheme of step 1 to obtain values \( d_{i1} \) and \( F_{i1} \).](image)

- **Step k.** It makes sense for \( k>1 \). The plate is still affected by the system of external forces \( P \) (figure 4), but in addition, the system of reactive forces of the rib \( F_{ik-1} \) from step \( k-1 \). As a result, we obtain deflections and new values of reactive forces: \( d_{ik}, F_{ik} \).

Thus, we get two sequences: \( \{d_{ik}\} \) and \( \{F_{ik}\} \), where \( \{x_k\} \) means a sequence of values with variable index \( k \). In this case, \( d_{ik} \) and \( F_{ik} \) denote the system of deflections and forces. If the sequence \( \{F_{ik}\} \) tends to a certain value as \( k \to \infty \), for example \( F_{i} \), we have a situation where the deflections of the plate and the ribs are equal at the points of application of reactive forces. The forces of influence of the rib on
the plate are equal to the force of the plate on the rib as well. This justifies the possibility to use the successive approximations method and makes it possible to take longitudinal ribs into account, since the reactive force systems \( F_i \) for each rib can be applied to the plate.

![Figure 4](image1.png)

**Figure 4.** The scheme of step \( k \) for obtaining the values of \( d_{ik} \) and \( F_{ik} \).

### 2.3 Determination of the reactive forces of the rib on the plate according to the given deflections

We determine the reactive forces under the assumption that the ribs work as beams using the principle of superposition. When force of \( F_i = F \) is applied to the beam at the point \( y = x, \) the deflection functions are expressed as follows (figure 5):

\[
W_1(y, x) = \frac{-(b-x)y^3}{6b} + \left(\frac{(b-x)^2}{6b}\right)y/(EJ),
\]

\[
W_2(y, x) = \frac{-(b-x)y^3}{6b} + \left(\frac{(b-x)^2}{6b}\right)y/(EJ),
\]

where \( b \) – rib height, m;

\( J \) – moment of inertia of the cross section of the rib, m^4.

![Figure 5](image2.png)

**Figure 5.** Scheme of deflection functions.

Next we use the matrix notation of the following form: \( \{a_1, ..., a_m\} \) – matrix row of length \( m; \)

\[
\begin{bmatrix}
  a_{11} & \cdots & a_{1m} \\
  \vdots & \ddots & \vdots \\
  a_{n1} & \cdots & a_{nm}
\end{bmatrix}
\]

– rectangular matrix of size \( n \) by \( m \); \( \{a_1, ..., a_m\} \) – matrix column of height \( n \).

Then, if the system of forces \( (F_1, ..., F_m) = F \) acts on the beam at points with coordinates \( (x_1, ..., x_m) = x, \) the deflections \( (d_1, ..., d_m) = d \) are defined as follows:

\[
d = W \times F,
\]

where \( \times \) – a symbol denoting matrix multiplication;
\[ W = \begin{bmatrix} W_1(x_1, x_1) & \cdots & W_1(x_1, x_{nn}) \\ \vdots & \ddots & \vdots \\ W_n(x_{nn}, x_1) & \cdots & W_n(x_{nn}, x_{nn}) \end{bmatrix} \]  

- matrix of size \( nn \) by \( nn \), where element \( a_{ij} = W_1(x_i, x_j) \) if \( j \geq i \) and element \( a_{ij} = W_2(x_i, x_j) \) if \( j < i \).  

Then, knowing the matrix of deflections \( d \), we can unambiguously find the matrix of efforts creating them, as \( F = W^{-1} \times d \). This is the solution to the problem that arises at each step mentioned in the previous paragraph.

### 3 Results and discussion

Considering what is described in paragraphs 1-3, we can set an algorithm that implements the successive approximations method. Below are the results of such algorithm realization in the MATLAB system. The following notation is used:  

- \( a, b, h \) – dimensions in accordance with figure 1;  
- \( x_1, y_1 \) – coordinates of the external force \( P_1 \) application;  
- \( x_2, y_2 \) – coordinates of the external force \( P_2 \) application;  
- \( P_1, P_2 \) – the magnitude of external forces, while the forces are positive in the direction along the \( Z \) axis;  
- \( J, \nu, E, D \) – in accordance with the notation of paragraph 1;  
- \( nn \) – is the number of reactive forces of the rib;  
- \( n \) – in accordance with (2) under the assumption that \( n = m \);  
- \( rp \) – is the number of longitudinal ribs.

Table 1 and figure 6 shows the results with the following parameters:  

- \( a = 2 \) m, \( b = 1 \) m, \( h = 0.018 \) m, \( x_1 = 3a/8 \) m, \( y_1 = b/2 \) m, \( x_2 = 5a/8 \) m, \( y_2 = b/2 \) m, \( P_1 = -5000 \) N, \( P_2 = -5000 \) N, \( J = 8 \times 10^{-6} \) m\(^4\), \( \nu = 0.2 \), \( E = 2 \times 10^{11} \) N/m\(^2\), \( nn = 6 \), \( n = 10 \), \( rp = 1 \).

#### Table 1. Deflection convergence results in 51 steps.

| step | \( d_1 \times 10^4 \) | \( d_2 \times 10^4 \) | \( d_3 \times 10^4 \) | \( d_4 \times 10^4 \) | \( d_5 \times 10^4 \) | \( d_6 \times 10^4 \) |
|------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1    | -5.22194        | -9.62787        | -12.2655        | -12.2655        | -9.62787        | -5.22194        |
| 2    | -4.03966        | -7.35454        | -9.24479        | -9.24479        | -7.35454        | -4.03966        |
| 3    | -3.11606        | -5.64059        | -7.06026        | -7.06026        | -5.64059        | -3.11606        |
| 4    | -2.41676        | -4.36675        | -5.45793        | -5.45793        | -4.36675        | -2.41676        |
| 5    | -1.89517        | -3.42292        | -4.27703        | -4.27703        | -3.42292        | -1.89517        |
| 6    | -1.50824        | -2.72455        | -3.40504        | -3.40504        | -2.72455        | -1.50824        |
| 7    | -1.22179        | -2.20806        | -2.76067        | -2.76067        | -2.20806        | -1.22179        |
| 8    | -1.00991        | -1.82618        | -2.28437        | -2.28437        | -1.82618        | -1.00991        |
| 9    | -0.85324        | -1.54383        | -1.93227        | -1.93227        | -1.54383        | -0.85324        |
| 10   | -0.7374         | -1.33509        | -1.67196        | -1.67196        | -1.33509        | -0.7374         |
| 11   | -0.65175        | -1.18076        | -1.47951        | -1.47951        | -1.18076        | -0.65175        |
| 12   | -0.58843        | -1.06666        | -1.33724        | -1.33724        | -1.06666        | -0.58843        |
| 13   | -0.54162        | -0.98231        | -1.23205        | -1.23205        | -0.98231        | -0.54162        |
| 14   | -0.50701        | -0.91995        | -1.15429        | -1.15429        | -0.91995        | -0.50701        |
| 15   | -0.48143        | -0.87385        | -1.09679        | -1.09679        | -0.87385        | -0.48143        |
| 16   | -0.46251        | -0.83976        | -1.05429        | -1.05429        | -0.83976        | -0.46251        |
| 17   | -0.44853        | -0.81456        | -1.02287        | -1.02287        | -0.81456        | -0.44853        |
| 18   | -0.43819        | -0.79593        | -0.99964        | -0.99964        | -0.79593        | -0.43819        |
| 19   | -0.43054        | -0.78216        | -0.98246        | -0.98246        | -0.78216        | -0.43054        |
| 20   | -0.42489        | -0.77197        | -0.96976        | -0.96976        | -0.77197        | -0.42489        |
| 21   | -0.42072        | -0.76444        | -0.96037        | -0.96037        | -0.76444        | -0.42072        |
| 22   | -0.41763        | -0.75888        | -0.95343        | -0.95343        | -0.75888        | -0.41763        |
| 23   | -0.41534        | -0.75476        | -0.9483         | -0.9483         | -0.75476        | -0.41534        |
| 24   | -0.41365        | -0.75172        | -0.94451        | -0.94451        | -0.75172        | -0.41365        |
Figure 6. The deflection of a plate with one edge, $a = 2\text{m}$, $b = 1\text{m}$, maximum deflection of $2.8\times10^{-4}\text{m}$.
Figure 7 shows the calculation results for the following parameters: $a=2$ m, $b=1$ m, $h=0.018$ m, $x_1=3a/8$ m, $y_1=b/2$ m, $x_2=5a/8$ m, $y_2=b/2$ m, $P_1=-5000$ N, $P_2=-5000$ N, $J=8\cdot10^{-6}$ m$^4$, $v=0.2$, $E=2\cdot10^{11}$ N/m$^2$, $mm=6$, $n=10$. $rp=4$.

Figure 8 shows the calculation results for the following parameters: $a=2$ m, $b=2$ m, $h=0.018$ m, $x_1=3a/8$ m, $y_1=b/2$ m, $x_2=5a/8$ m, $y_2=b/2$ m, $P_1=-5000$ N, $P_2=-5000$ N, $J=8\cdot10^{-6}$ m$^4$, $v=0.2$, $E=2\cdot10^{11}$ N/m$^2$, $mm=6$, $n=10$. $rp=4$.

Figure 7. The deflection of a plate with four ribs, $a = 2$ m, $b = 1$ m, maximum deflection of $6.9\cdot10^{-5}$ m.

Figure 8. The deflection of a plate with four ribs, $a = 2$ m, $b = 2$ m, maximum deflection $3.9\cdot10^{-4}$ m.

4 Conclusions

Based on the above reasoning and calculation examples, we can conclude that the use of the successive approximations method in solving the Sophie Germain’s equation can be successfully used to determine displacements in plates supported by longitudinal and transverse edges. This method has the advantage of considering the features of the plate and eliminating the need for a detailed examination of local sections, when using engineering calculation methods [19]. At the same time, it is much simpler than, for example, the finite element method [20] and is available for understanding by an engineer who does not have a special mathematical background. This allows avoiding some errors related to the convergence of the method.

A further area of research in this sphere will be connected with transverse stiffeners, considering the bridge web in the work of a multilayer plat, and the possibility of taking into account the pinched edges of the plate.

References

[1] Boutonnet M, Savard Y, Hornych P 2004 Pavement damage by cracking under severe frost
conditions. In: Proceedings of 5th International RILEM Conference on Cracking in Pavements: Mitigation, Risk Assessment and Prevention pp. 341-348. RILEM Publications SARL.

[2] Chen X, Huang W, Yang J 2008 Cracking of wearing courses on steel orthotropic bridge decks. In: Proceedings of 6th International RILEM Conference on Cracking in Pavements: Mechanisms, Modeling, Detection, Testing and Case Histories pp. 907-912. Springer Netherlands. doi: 10.1201/9780203882191.ch89

[3] Medani T O, Huurman M, Houben L J M, Molenaar A A A 2004 A proposed fatigue based design methodology for asphaltic mixes applied on orthotropic steel bridges. In: Proceedings of 5th International RILEM Conference on Cracking in Pavements: Mitigation, Risk Assessment and Prevention pp. 529-536. RILEM Publications SARL.

[4] Nishizawa T, Himeno K, Uchido K, Nomura K 2004 Longitudinal surface cracking in asphalt pavements on steel bridge decks. In: Proceedings of 5th International RILEM Conference on Cracking in Pavements: Mitigation, Risk Assessment and Prevention pp. 537-544. RILEM Publications SARL.

[5] Vater E J, Recknagel S 2004 Bond strength and crack bridging behaviour of bridge deck surfacings for concrete bridges. In: Proceedings of 5th International RILEM Conference on Cracking in Pavements: Mitigation, Risk Assessment and Prevention pp. 521-528. RILEM Publications SARL.

[6] Xu W, Zhang N 2008 Experimental and numerical simulation study on the crack of steel orthotropic bridge deck pavement. In: Proceedings of 6th International RILEM Conference on Cracking in Pavements: Mechanisms, Modeling, Detection, Testing and Case Histories pp. 899-906. Springer Netherlands.

[7] Zinchenko E V, Ovchinnikov I G, Ilchenko E D 2014 Comparative analysis of the applied structures of the road dishwashers Construction of Olympic venues. Part 2. Basic damage to the pavement of the bridge. Naukovedenie. URL: https://naukovedenie.ru/PDF/39KO514.pdf.

[8] Kohler E, Kannekanti V 2008 Influence of the coefficient of thermal expansion on the cracking of jointed concrete pavements. In: Proceedings of 6th International RILEM Conference on Cracking in Pavements: Mechanisms, Modeling, Detection, Testing and Case Histories pp. 79-88. Springer Netherlands. doi: 10.1201/9780203882191.ch7

[9] Ovchinnikov I G, Ovchinnikov I I 2014 Bridge pavements: domestic and foreign experience. Naukovedenie. URL: https://naukovedenie.ru/PDF/37KO514.pdf.

[10] Telegin M A, Ovchinnikov I G 2013 Using of influence surfaces of stresses when analyzing spatial work of spans orthotropic slabs with closed longitudinal ribs. Roads and Bridges, pp 175-186 URL: http://rosdornii.ru/files/10-01-14/12.pdf.

[11] Telegin M A, Ovchinnikov I G 2014 Research of simultaneous working of a steel orthotropic plate with road pavement on it at their various parameters. Russian journal of transport engineering, URL: https://t-s.today/PDF/02TS215.pdf. doi: 10.15862/02TS215

[12] Pokorski P, Radziszewski P, Sarnowski M 2016 Fatigue life of asphalt pavements on bridge decks. Procedia Engineering, pp 556-562 URL: https://core.ac.uk/download/pdf/82301367.pdf. doi: 10.1016/j.proeng.2016.08.191

[13] Kokkalis A, Panetsos P 2015 Asphalt bridge deck pavement behavior, the Egnatia experience. Bituminous Mixtures and Pavements VI, pp 789-793. URL: https://www.researchgate.net/publication/300151056_Asphalt_bridge_deck_pavement_behavior_the_Egnatia_experience. doi: 10.1201/b18538-112

[14] Grishin I V, Kayumov R A, Ivanov G P 2013 Experimental research of asphaltic concrete rheological properties at different temperatures Izvestiya KGASU 2(24), pp 99-107.

[15] Grishin I V, Kayumov R A, Ivanov G P 2011 About stress state computation of steel bridge asphaltic pavement laid on orthotropic slab Izvestiya KGASU 3(17), 171-178.

[16] Pszczola M, Jacezewski M, Szydlowski C 2019 Assessment of Thermal Stresses in Asphalt
Mixtures at Low Temperatures Using the Tensile Creep Test and the Bending Beam Creep Test. *Applied Science* 9(5), URL: [https://www.researchgate.net/publication/331374349_Assessment_of_Thermal_Stresses_in_Asphalt_Mixtures_at_Low_Temperatures_Using_the_Tensile_Creep_Test_and_the_Bending_Beam_Creep_Test](https://www.researchgate.net/publication/331374349_Assessment_of_Thermal_Stresses_in_Asphalt_Mixtures_at_Low_Temperatures_Using_the_Tensile_Creep_Test_and_the_Bending_Beam_Creep_Test). doi: 10.20944/preprints201902.0009.v1

[17] Yin Y, Huang W, Lv J, Ma X, Yan J 2017 Unified Construction of Dynamic Rheological Master Curve of Asphalts and Asphalt Mixtures. *International Journal of Civil Engineering* 16(1). doi: 10.1007/s40999-017-0256-x

[18] Iskakbayev A, Teltayev B, Oliviero C, Yensebayeva G. 2018 Determination of Nonlinear Creep Parameters for Hereditary Materials. *Applied Sciences* 8(5), 10.3390/app8050760. doi: 10.3390/app8050760

[19] Hambly E C 2014 Bridge deck behaviour. 2nd edn. Taylor & Francis, New York.

[20] Vican J 2013 Numerical Analysis of the Bridge Orthotropic Deck Time Dependent Resistance. *Komunikacie* 14(3), pp 112-117.