AN INTRIGUING HYPERELLIPTIC SHIMURA CURVE QUOTIENT OF GENUS 16

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Abstract. Let $F$ be the maximal totally real subfield of $\mathbb{Q}(\zeta_{32})$, the cyclotomic field of 32nd roots of unity. Let $D$ be the quaternion algebra over $F$ ramified exactly at the unique prime above 2 and 7 of the real places of $F$. Let $\mathcal{O}$ be a maximal order in $D$, and $X^D_0(1)$ the Shimura curve attached to $\mathcal{O}$. Let $C = X^D_0(1)/\langle w_D \rangle$, where $w_D$ is the unique Atkin-Lehner involution on $X^D_0(1)$. We show that the curve $C$ has several striking features. First, it is a hyperelliptic curve of genus 16, whose hyperelliptic involution is exceptional. Second, there are 34 Weierstrass points on $C$, and exactly half of these points are CM points; they are defined over the Hilbert class field of the unique CM extension $E/F$ of class number 17 contained in $\mathbb{Q}(\zeta_{64})$, the cyclotomic field of 64th roots of unity. Third, the normal closure of the field of 2-torsion of the Jacobian of $C$ is the Harbater field $N$, the unique Galois number field $N/\mathbb{Q}$ unramified outside 2 and $\infty$, with Galois group $\text{Gal}(N/\mathbb{Q}) \cong F_{17} = \mathbb{Z}/17\mathbb{Z} \rtimes (\mathbb{Z}/17\mathbb{Z})^\times$. In fact, the Jacobian $\text{Jac}(X^D_0(1))$ has the remarkable property that each of its simple factors has a 2-torsion field whose normal closure is the field $N$. Finally, and perhaps the most striking fact about $C$, is that it is also hyperelliptic over $\mathbb{Q}$.

1. Introduction

Let $F$ be the maximal totally real subfield of $\mathbb{Q}(\zeta_{32})$, the cyclotomic field of 32nd roots of unity. Recall that 2 is totally ramified in $F$, and let $\mathfrak{p}$ be the unique prime above it. Let $D$ be the quaternion algebra defined over $F$ ramified exactly at $\mathfrak{p}$ and 7 of the real places of $F$. Let $\mathcal{O}$ be a maximal order in $D$, and $X^D_0(1)$ the Shimura curve attached to $\mathcal{O}$. Let $C = X^D_0(1)/\langle w_D \rangle$, where $w_D$ is the unique Atkin-Lehner involution on $X^D_0(1)$. Let $\text{Jac}(X^D_0(1))$ and $\text{Jac}(C)$ be the Jacobians of $X^D_0(1)$ and $C$, respectively. In this note, we show that $C$ has several striking properties. First, we prove the following theorem (Theorem 5.13).

**Theorem A.** The curve $C$ is a hyperelliptic curve of genus 16 defined over $\mathbb{Q}$.

We first show that $C$ is hyperelliptic over $F$ (Theorem 5.9), then we apply a descent argument from [SV16] to show that both the curve and the hyperelliptic involution are defined over $\mathbb{Q}$. For the first part, we simply count the number of Weierstrass points on $C$. This count yields that $C$ has 34 Weierstrass points, the maximum number for a hyperelliptic curve of genus 16 by the Weierstrass gap theorem. Half of those Weierstrass points are CM points defined over the Hilbert class field of the unique CM extension $E/F$ of class number 17 contained in $\mathbb{Q}(\zeta_{64})$, the cyclotomic field of 64th roots of unity.

The second part requires that we determine the automorphism group $\text{Aut}(C)$ of $C$ as a curve over $F$. We do this by exploiting the Čerednik-Drinfeld 2-adic uniformisation of $X^D_0(1)$ and the fact that the automorphism group of a stable curve

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injects into an admissible subgroup of the automorphism group of its dual graph (see [DM69] and §4.6 for the definition of admissibility). A careful study of the dual graph of the stable model of $C$ over the completion of $F$ at $p$ then yields that $\text{Aut}(C) = \mathbb{Z}/2\mathbb{Z}$. As a result, we get that the only non-trivial automorphism of $C$ is the hyperelliptic involution, which in this case must be exceptional since the curve $C$ is obtained as the quotient of $X_0^D(1)$ by the unique Atkin-Lehner involution $w_D$.

Our second result concerns the field of 2-torsion of $\text{Jac}(C)$. It is known that 17 is the smallest odd integer which can occur as the degree of a number field $K/\mathbb{Q}$ for which 2 is the only finite prime which ramifies. That there is no such integer less than 17 follows from work of Jones [Jon10]. On the other hand, Harbater [Har94] proves that there is a unique Galois number field $N/\mathbb{Q}$ unramified outside 2 and $\infty$, with Galois group $\text{Gal}(N/\mathbb{Q}) \simeq F_{17} = \mathbb{Z}/17\mathbb{Z} \rtimes (\mathbb{Z}/17\mathbb{Z})^\times$. So, the fixed field of the Sylow 2-subgroup of $F_{17}$ is a number field of degree 17 in which 2 is the only ramified finite prime. Noam Elkies provides a degree 17 polynomial whose splitting field is $N$. The computation which led to that polynomial stemmed from a discussion initiated by Jeremy Rouse. In the context of that discussion, it is natural to asked whether there is a curve defined over $\mathbb{Q}$, with good reduction away from 2, whose field of 2-torsion is the Harbater field $N$. The following theorem provides an affirmative answer to that question (Theorem 6.1).

**Theorem B.** The field of 2-torsion of $\text{Jac}(C)$ is the Harbater field $N$.

The fact that the Harbater field can be realised as the field of 2-torsion of a hyperelliptic curve of rather large genus, with good reduction outside 2, seems rather remarkable to us. For that reason, we think that it would be very interesting to find a defining equation for $C$ over $\mathbb{Q}$. This is a question of independent interest that we hope to consider in the future.

In fact, we prove a slightly stronger result than Theorem B. Namely, we give two proofs of the following (Theorem 6.4).

**Theorem C.** Let $A$ be a simple factor of $\text{Jac}(X_0^D(1))$. Then the normal closure of the field of 2-torsion of $A$ is the Harbater field $N$.

The second proof of Theorem C uses congruences. Namely, let $S_2^D(1)$ be the space of automorphic forms of level (1) and weight 2 over the quaternion algebra $D$, and $T$ be the Hecke algebra acting on $S_2^D(1)$. We show that there are two congruence classes modulo 2 among the newforms in $S_2^D(1)$, whose associated mod 2 residual Galois representations have the same image $D_{17}$. These two congruence classes are permuted by $\text{Gal}(F/\mathbb{Q})$. As a result, we get that the normal closure of the field of 2-torsion of every simple factor of $\text{Jac}(X_0^D(1))$ is the Harbater field $N$. Interestingly, the existence of these two distinct congruence classes modulo 2 turns out to have the following amusing consequence: The connectedness of $\text{Spec}(T)$, which is obtained by an argument à la Mazur [Maz77, Proposition 10.6], cannot arise from a single congruence modulo 2. In other words, the existence of the Harbater field as the normal closure of the field of 2-torsion of $\text{Jac}(X_0^D(1))$ is an obstruction to the connectedness of $\text{Spec}(T)$ being achieved via a unique congruence modulo 2. This is due to the tautological reason that the semi-direct product $F_{17} = D_{17} \rtimes \mathbb{Z}/8\mathbb{Z}$ is non-split. In fact, we show that the connectedness of $\text{Spec}(T)$ is given by two different congruences modulo 3 and 5.

Our initial interest in the curve $X_0^D(1)$ stems from a conjecture of Benedict H. Gross which states that, for any prime $p$, there is a non-solvable number field $K/\mathbb{Q}$ ramified at $p$ (and possibly at $\infty$) only. In [Dem09], we proved that conjecture for $p = 2$ by using Hilbert modular forms of level (1) and weight 2 over $F$. Theorem C implies that none of the simple factors of $\text{Jac}(X_0^D(1))$ has a 2-torsion field that can be used to provide an affirmative answer to the Gross conjecture for number
fields given that $N$ is solvable. Amusingly, it turns out that the simple factors of $\text{Jac}(X_0^D(1))$ are more interesting in relation to other conjectures of Gross [Gro16] which concern modularity of abelian varieties not of $\text{GL}_2$-type. Indeed, functorially, these simple factors are related to abelian varieties defined over $\mathbb{Q}$ with small or even trivial endomorphism rings, but which acquire extra endomorphisms over $F$, as we explain later (see also [CD17]).

The outline of the paper is as follows. In Section 2, we recall the necessary background on Weierstrass points and hyperellipticity. In Section 3, we recall the necessary background on arithmetic groups in quaternion algebras, and compliment this by discussing optimal embeddings into maximal arithmetic Fuchsian groups. In Section 4, we review the theory of Shimura curves, especially their $p$-adic uniformisation. Finally, in Sections 5 and 6, we discuss our example, its Jacobian and the connection of their 2-torsion fields with the Harbater field.

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2. Background on Weierstrass points

Throughout this section, $X$ is a smooth projective curve of genus $g \geq 2$ defined over a field $k$ of characteristic 0, with algebraic closure $\overline{k}$.

2.1. Definition and properties. Let $P$ be a point on $X$. We say that $P$ is a Weierstrass point if there exists a differential form $\omega \in H^0(X, \Omega^1_X)$ such that $\text{ord}_P(\omega) \geq g$. We let $W$ be the set of all Weierstrass points on $X$. Alternatively, one can describe $W$ as follows. Let $D$ be a divisor on $X$, and $L(D)$ the Riemann-Roch space associated to $D$, i.e.

$$L(D) := \{ f \in k(X)^\times : \text{div}(f) + D \geq 0 \} \cup \{0\}.$$  

By the Riemann-Roch Theorem, $L(D)$ is finite dimensional, and we let $\ell(D)$ be its dimension.

**Proposition 2.1.** Let $P$ be a point on $X$. Then, $P \in W$ if and only if $\ell(gP) \geq 2$.

**Proof.** This is a consequence of the Riemann-Roch Theorem [HS00, §A.4].

The gap sequence associated to a Weierstrass point $P$ is the set

$$G(P) := \{ n \in \mathbb{Z}_{\geq 0} : \ell(nP) = \ell((n-1)P) \}.$$  

The weight of the Weierstrass point $P$ is defined by

$$w(P) := \left( \sum_{n \in G(P)} n \right) - \frac{g(g+1)}{2}.$$
Theorem 2.2. Let $P$ be a point on $X$. Then $P$ is a Weierstrass point if and only if $w(P) \geq 1$, and $\sum w(P)P$ belongs to the complete linear system
\[
\left\lfloor \frac{g(g+1)}{2} K_X \right\rfloor,
\]
where $K_X$ is a canonical divisor on $X$. In particular, we have that
\[
\sum_{P \in \mathcal{W}} w(P) = g(g^2 - 1).
\]

Proof. See [FK92, §III.5] or [HS00, Exercise A.4.14].

2.2. Hyperellipticity. We recall that $X$ is a hyperelliptic curve if there is a degree 2 map $\phi : X \to \mathbf{P}^1$ defined over $\overline{k}$. In that case, $\phi$ is unique (up to automorphisms of $\mathbf{P}^1$). The map $\phi$ induces a degree 2 extension $k(X)/k(\mathbf{P}^1)$, which is Galois since $\text{char}(k) = 0$. So, this gives rise to a map $i : X \to X$ called the hyperelliptic involution. We say $X$ is hyperelliptic over $k$ if $\phi$ is defined over $k$, or equivalently if $i$ is defined over $k$. The following is a well-known classical result.

Proposition 2.3. Let $X$ be a curve of genus $g \geq 2$ defined over a field $k$ of characteristic 0, and $\mathcal{W}$ the set of Weierstrass points of $X(\overline{k})$. Then, we have
\[
2g + 2 \leq \#\mathcal{W} \leq g^3 - g.
\]
Furthermore, $X$ is hyperelliptic if and only if $\#\mathcal{W} = 2g + 2$. In that case, the branch points are the Weierstrass points.

Proof. See [FK92, §III.5] or [HS00, Exercise A.4.14].

2.3. Galois action. Let $\mathcal{W}$ be the set of all Weierstrass points over $X(\overline{k})$, then $\mathcal{W}$ is preserved by the action of $\text{Gal}(\overline{k}/k)$. In particular, when $X$ is a hyperelliptic curve, this action factors through the symmetric group $S_{2g+2}$.

3. Arithmetic Fuchsian groups

From now on, $F$ is a totally real number field of degree $g$. We denote the real embeddings of $F$ by $v_1, \ldots, v_g$. We let $\mathcal{O}_F$ be the ring of integers of $F$. We let $D$ be a quaternion algebra defined over $F$, and fix a maximal order $\mathcal{O}$ in $D$. Let $v$ be a place of $F$, and $F_v$ the completion of $F$ at $v$. We recall that $D$ is said to be ramified at $v$ if $D_v := D \otimes F_v$ is a division quaternion algebra. We let $S_\infty$ (resp. $S_f$) be the set of archimedean places (resp. finite places) where $D$ is ramified; and set $S = S_\infty \cup S_f$. We let $r = \#S_f$.

3.1. Fuchsian groups. From now on, we assume that $D$ is ramified at all but one archimedean places; namely, that $S_\infty = \{v_1, \ldots, v_g\}$. This means that, we have $D \otimes \mathbf{R} \simeq M_2(\mathbf{R}) \times \mathbf{H}^{g-1}$, where $\mathbf{H}$ is the Hamilton quaternion algebra over $\mathbf{R}$. We let $j_1 : D \otimes v_1 \mathbf{R} \to M_2(\mathbf{R})$ be the projection onto the factor corresponding to $v_1$. We will also denote the map induced on the unit groups by $j_1 : (D \otimes v_1 \mathbf{R})^\times \to \text{GL}_2(\mathbf{R})$. We let
\[
\mathcal{O}_1 := \{ x \in \mathcal{O} : \text{Nrd}(x) = 1 \};
\]
\[
\mathcal{O}^\times := \{ x \in \mathcal{O} : \text{Nrd}(x) \in \mathcal{O}_F^\times \};
\]
\[
\mathcal{O}_F^\times := \{ x \in \mathcal{O} : \text{Nrd}(x) \in \mathcal{O}_F^\times \};
\]
We let $\Gamma^1$ (resp. $\Gamma$) be the image of $\mathcal{O}_1$ (resp. $\mathcal{O}_F^\times$) in $\text{PGL}_2^+(\mathbf{R}) := \text{GL}_2^+(\mathbf{R})/\mathbf{R}^\times$, where
\[
\text{GL}_2^+(\mathbf{R}) := \{ \gamma \in \text{GL}_2(\mathbf{R}) : \det(\gamma) > 0 \}.
\]
We recall that $\Gamma^1$ is an arithmetic Fuchsian group, i.e. a discrete subgroup of $\text{PSL}_2(\mathbb{R})$. The commensurability class of $\Gamma^1$, consists of all the subgroups $\Gamma' \subset \text{PGL}^+_2(\mathbb{R})$ that are commensurable with $\Gamma^1$, i.e. such that $\Gamma' \cap \Gamma^1$ has finite index in both $\Gamma'$ and $\Gamma^1$. Any Fuchsian group that is commensurable to an arithmetic Fuchsian group is itself arithmetic. So, the commensurability class of $\Gamma^1$ is independent of the embedding $j_1$. We define it simply as the commensurability class of $\mathcal{O}^1$ in $D^\times/F^\times$, and denote it by $\mathcal{E}(D)$. In $\mathcal{E}(D)$, one is particularly interested in those groups $\Gamma'$ with minimal covolume. Borel [Bor81] shows that, up to conjugacy, there are finite many such groups, and gives their covolume purely in terms of the number theoretic data used in defining them. These groups are called maximal arithmetic Fuchsian groups, and are the main objects of interest to us in this section.

We recall that the normaliser of $\mathcal{O}$ inside $D$ is defined by

$$N_D(\mathcal{O}) := \{ x \in D^\times : x\mathcal{O} = \mathcal{O}x \}.$$ 

We set

$$N_D(\mathcal{O})_+ := \{ x \in N_D(\mathcal{O}) : \text{Nrd}(x) \in F^+_1 \},$$

and we let $\Gamma_\mathcal{O}$ be its image inside $D^\times/F^\times$. We will use the same notation for the image of $N_D(\mathcal{O})_+$ inside $\text{PGL}^+_2(\mathbb{R})$.

**Theorem 3.1** (Borel [Bor81]). Every maximal arithmetic Fuchsian group in $\mathcal{E}(D)$ is of the form $\Gamma_\mathcal{O}$, where $\mathcal{O}$ is a maximal order in $D$. In that case, the covolume of $\Gamma_\mathcal{O}$ is given by

$$\text{Vol}(\Gamma_\mathcal{O}\backslash \mathfrak{H}) = \frac{8\pi D^{3/2} \zeta_F(2)}{(4\pi^2)^d |H : F^\times|} \prod_{q \in S_f} (Nq - 1),$$

where $H = \{ \text{Nrd}(x) : x \in N_D(\mathcal{O})_+ \}$. In particular, it depends only on $F$ and $S_f$.

**Proof.** See Borel [Bor81, §8.4].

### 3.2. The Atkin-Lehner group

We define the Atkin-Lehner group

$$W := N_D(\mathcal{O})/F^\times \mathcal{O}^\times.$$ 

By the Skolem-Noether Theorem [Vig80, Chap. II, Théorème 2.1], $W$ can be identified with the group of automorphisms of $\mathcal{O}$. It is generated by the classes $[u] \in W$ such that $(u)$ is a principal two-sided ideal whose norm is supported at the prime ideals in $S_f$. By the Hasse-Schilling-Maass Theorem [Vig80, Chap. III, Théorème 5.7], $W$ is a finite elementary abelian 2-group. So, there is a positive integer $r$ such that

$$W \cong (\mathbb{Z}/2\mathbb{Z})^r.$$ 

We define the positive Atkin-Lehner groups

$$W_+ := N_D(\mathcal{O})_+/F^\times \mathcal{O}_+^\times,$$

$$W^1 := N_D(\mathcal{O})/F^\times \mathcal{O}_+^1.$$ 

There is a split exact sequence

$$1 \to \mathcal{O}_F^\times/(\mathcal{O}_F^\times)^2 \to W^1 \to W_+ \to 1,$$

which gives an isomorphism

$$W^1 \cong (\mathcal{O}_F^\times)^s \times W_+ \cong (\mathbb{Z}/2\mathbb{Z})^s,$$

where $s \leq (n-1) + r$. The rank $s$ of $W^1$ can be determined from the Dirichlet unit theorem and the fact that the image of $W_+$ inside $W$ is generated by those principal two-sided ideals whose norms are totally positive and supported at $S_f$. 


3.3. Optimal embeddings. Let $E/F$ be a CM extension, i.e. a totally imaginary quadratic extension. By [Vig80, Chap. III, Théorème 3.8], $E$ embeds into $D$ if and only if, every finite place $v \in S_f$ is ramified or inert in $E$. The following theorem will be very useful for us.

**Theorem 3.2.** Let $E/F$ be a CM extension, and $\sigma : E \hookrightarrow D$ an embedding. Let $\alpha \in E \setminus F$, and $\text{disc}(\alpha) = \text{Tr}_{E/F}(\alpha)^2 - 4N_{E/F}(\alpha)$. Then, up to conjugation, $\sigma(\alpha) \in N_D(O)_+$ if and only if $\text{disc}(\alpha)/N_{E/F}(\alpha) \in O_F$, and $N_{E/F}(\alpha) \in F_+^\times$ is supported at $S_f$ modulo squares.

**Proof.** This follows from Chinburg-Friedman [CF99, Lemma 4.3] (see also Maclachlan [Mac06, Theorem 3.1]).

Let $E/F$ be a CM extension, and $\mathcal{O}$ an $O_F$-order in $E$. An **optimal embedding** of $\mathcal{O}$ in $D$ is a homomorphism $\iota : E \hookrightarrow D$ such that $\iota(\mathcal{O}) = \iota(E) \cap \mathcal{O}$. We denote the set of optimal embeddings of $\mathcal{O}$ into $O$ by $\text{Emb}(\mathcal{O}, O)$. We fix an embedding $E \hookrightarrow D$. Then, by the Skolem-Noether Theorem, every embedding of $E$ into $D$ is of the form $(x \mapsto \alpha x \alpha^{-1})$ for some $\alpha \in D^\times$. So, we can identify $\text{Emb}(\mathcal{O}, O)$ with the coset space $E^\times/\mathcal{E}(\mathcal{O}, O)$ where

$$
\mathcal{E}(\mathcal{O}, O) := \{ \alpha \in D^\times : \alpha E \alpha^{-1} \cap O = \mathcal{O} \}
$$

$$= \{ \alpha \in D^\times : E \cap \alpha^{-1} O \alpha = \alpha^{-1} O \}.
$$

Conjugation induces a right action of $N_D(O)/F_+^\times$ on $\text{Emb}(\mathcal{O}, O)$. For any subgroup $\Gamma^\times \subset \Gamma \subset N_D(O)/F^\times$, we let $\text{Emb}(\mathcal{O}, \mathcal{O}; \Gamma)$ be the set of $\Gamma$-conjugacy classes of optimal embeddings. Similarly, if $O^\times \subset G \subset N_D(O)$, we let $\text{Emb}(\mathcal{O}, \mathcal{O}; G) := \text{Emb}(\mathcal{O}, \mathcal{O}; G)$, where $G$ is the image of $G$ in $N_D(O)/F^\times$. The set $\text{Emb}(\mathcal{O}, \mathcal{O}; \Gamma)$ is finite since $\Gamma$ has finite index in $N_D(O)/F^\times$. We denote its cardinality by $m(\mathcal{O}, \mathcal{O}; \Gamma)$. There are formula for $m(\mathcal{O}, \mathcal{O}; O^\times)$, see for example [Vig80, Chap. II, §3 and Chap. III, §55] or [Voi18, §30]. The following lemma can be used to get $m(\mathcal{O}, \mathcal{O}; G)$ for any subgroup $O^\times \subset G \subset O^\times$.

**Lemma 3.3.** Let $O^\times \subset G \subset O^\times$ be a subgroup. Then we have

$$m(O, O; G) = m(O, O; O^\times)[\text{Nrd}(O^\times) : \text{Nrd}(G) N_{E/F}(O^\times)].$$

**Proof.** See Voight [Voi18, Lemma 30.3.14]. (We note that the statement in Vignéras [Vig80, Chap. III, Corollaire 5.13] is only correct with the inclusion $G \subset N_D(O)$ replaced by $G \subset O^\times$.)

Here we are interested in the case when $O^\times_+ \subset G \subset N_D(O)_+$. In particular, we want $\text{Emb}(\mathcal{O}, \mathcal{O}; N_D(O)_+)$ when $O$ is a maximal order in $D$.

**Lemma 3.4.** Let $O^\times_+ \subset G \subset N_D(O)_+$ be a subgroup. Then we have

$$m(O, O; O^\times_+) = m(O, O; G)[\text{Nrd}(G) : \text{Nrd}(G) \cap N_{E/F}(E^\times) O^\times_+] .$$

**Proof.** There is a natural surjection

$$E^\times/\mathcal{E}(\mathcal{O}, O)/O^\times_+ \to E^\times/\mathcal{E}(\mathcal{O}, O)/G.$$

To prove the lemma, we need to understand the fibres of this map. For $\alpha \in \mathcal{E}(\mathcal{O}, O)$, the fibre of $E^\times \alpha G$ is

$$T := E^\times \alpha G/O^\times_+ \simeq (\alpha E^\times \alpha^{-1} \cap G) \backslash G/O^\times_+ .$$

It is enough to show that the cardinality of $T$ is independent of $\alpha$. To see this, we recall that the reduced norm $\text{Nrd} : D_+^\times \to F_+^\times$ induces a map

$$\phi : (\alpha E^\times \alpha^{-1} \cap G) \backslash G/O^\times_+ \to \text{Nrd}(G)/\text{Nrd}(G) \cap N_{E/F}(E^\times) O^\times_+ ,$$

which is a bijection since $\ker(\phi) = O^\times_+ \subset O^\times \subset G \subset N_D(O)_+$. 

Alternatively, we can observe that \( \mathcal{O}_\mathbb{Q}^\times \) is a normal subgroup of \( G \). So, we can identify \( (\alpha E^\alpha M \cap G) \backslash G / \mathcal{O}_\mathbb{Q}^\times \) with a subgroup of \( W_+ \). This means that \( \# T \) divides \( \# W_+ \), and is always a power of 2. \( \square \)

Let \( \hat{\mathcal{O}} := \mathcal{O} \otimes \hat{\mathbb{Z}} \) and \( \hat{D} := D \otimes \hat{\mathbb{Q}} \), where \( \hat{\mathbb{Z}} \) and \( \hat{\mathbb{Q}} \) are the finite adèles of \( \mathbb{Z} \) and \( \mathbb{Q} \), respectively. For any finite place \( v \), let \( \mathcal{O}_v^\times \subset G_v \subset N_{D_v}(\mathcal{O}_v) \) be a subgroup, and \( \hat{G} := \prod_{v < \infty} G_v \). We would like to understand the global embedding numbers of the group \( \hat{G} \), or \( G := \hat{G} \cap D_\mathbb{Q}^\times \). Since \( D \) satisfies the Eichler condition, we have \( D_\mathbb{Q}^\times / \hat{D}_\mathbb{Q}^\times \simeq \text{Cl}_F^+ \), where \( \text{Cl}_F^+ \) is the narrow class group of \( F \). Let \( h = \# \text{Cl}_F^+ \), and

\[
\hat{D}_\mathbb{Q}^\times = \prod_{i=1}^{h} D_\mathbb{Q}^\times g_i \hat{\mathcal{O}}^\times,
\]

where \( g_i \in \hat{D}_\mathbb{Q}^\times \), \( i = 1, \ldots, h \), and \( g_1 = 1 \). Then, for each \( i \), \( \mathcal{O}_i := g_i \hat{\mathcal{O}} g_i^{-1} \cap D \) is a maximal order, and \( N_{D}(\mathcal{O}_i) = g_i N_{\hat{D}}(\hat{\mathcal{O}}) g_i^{-1} \cap D \). Letting \( G_i := g_i \hat{G} g_i^{-1} \cap D_\mathbb{Q}^\times \), we have \( (\mathcal{O}_i)^+ \subset G_i \subset N_{D}(\mathcal{O}_i)^+ \).

For \( \hat{G} = \hat{D}_\mathbb{Q}^\times \), there are formulas for global optimal embeddings numbers (see [Vig80, Chap. III, §5] or [Voi18, §30]). For \( \mathcal{O}_\mathbb{Q}^\times \subset \hat{G} \subset N_{\hat{D}}(\hat{\mathcal{O}}) \), we have the following theorem.

**Theorem 3.5.** Keeping the notations above, let \( G := G_1 \) and \( h_\mathcal{O} \) be the class number of \( \mathcal{O} \). Then we have

\[
\sum_{i=1}^{h} m(\mathcal{O}, \mathcal{O}_i; G_i) = \frac{2h_\mathcal{O}}{[H : H \cap N_{\mathbb{E}/\mathbb{F}}(E^\times)\mathcal{O}_\mathbb{F}^{\times +}]} \prod_q m(\mathcal{O}_q, \mathcal{O}_q; \mathcal{O}_q^\times),
\]

where \( H := \text{Nrd}(G) \).

**Proof.** By applying Lemma 3.3 with \( G = \mathcal{O}_\mathbb{Q}^\times \), we have

\[
m(\mathcal{O}, \mathcal{O}; \mathcal{O}^\times) = m(\mathcal{O}, \mathcal{O}; \mathcal{O}^\times)[\text{Nrd}(\mathcal{O}^\times) : \text{Nrd}(\mathcal{O}_\mathbb{Q}^\times) N_{\mathbb{E}/\mathbb{F}}(\mathcal{O}^\times)]
\]

\[
= m(\mathcal{O}, \mathcal{O}; \mathcal{O}^\times)[\text{Nrd}(\mathcal{O}^\times) : \mathcal{O}_\mathbb{F}^{\times +}] = 2m(\mathcal{O}, \mathcal{O}, \mathcal{O}^\times).
\]

The latter equality follows from the fact that \( D \) ramified at all but one archimedian place, the Norm Theorem [Vig80, Chap. III, Théorème 4.1] and the Dirichlet unit theorem.

Now we return to the situation \( \mathcal{O}_\mathbb{Q}^\times \subset G \subset N_{\mathcal{O}_\mathbb{Q}}^\times, \mathcal{O}_\mathbb{Q} \). Combining the above identity with Lemma 3.4, we have

\[
2m(\mathcal{O}, \mathcal{O}; \mathcal{O}^\times) = m(\mathcal{O}, \mathcal{O}; G)[\text{Nrd}(G) : \text{Nrd}(G) \cap N_{\mathbb{E}/\mathbb{F}}(E^\times)\mathcal{O}_\mathbb{F}^{\times +}].
\]

A similar identity holds for the other maximal orders. In other words, for each maximal order \( \mathcal{O}_i \), we have

\[
2m(\mathcal{O}, \mathcal{O}_i; \mathcal{O}_i^\times) = m(\mathcal{O}, \mathcal{O}_i; G_i)[\text{Nrd}(G_i) : \text{Nrd}(G_i) \cap N_{\mathbb{E}/\mathbb{F}}(E^\times)\mathcal{O}_\mathbb{F}^{\times +}].
\]

However, the group \( \text{Nrd}(G_i) \) is independent of \( i \) again by the Norm Theorem. Hence setting \( H := \text{Nrd}(G) \), we get

\[
2m(\mathcal{O}, \mathcal{O}_i; \mathcal{O}_i^\times) = m(\mathcal{O}, \mathcal{O}_i; G_i)[H : H \cap N_{\mathbb{E}/\mathbb{F}}(E^\times)\mathcal{O}_\mathbb{F}^{\times +}].
\]

So, we have

\[
\sum_{i=1}^{h} m(\mathcal{O}, \mathcal{O}_i; G_i) = \frac{2}{[H : H \cap N_{\mathbb{E}/\mathbb{F}}(E^\times)\mathcal{O}_\mathbb{F}^{\times +}]} \sum_{i=1}^{h} m(\mathcal{O}, \mathcal{O}_i; \mathcal{O}_i^\times).
\]

We then apply [Vig80, Chap. III, Théorème 5.11] or [Voi18, Theorem 30.7.3] to conclude the proof. \( \square \)
3.4. Torsion in maximal arithmetic groups. From now on, we will assume that the field $F$ has narrow class number one. However, the results discussed here can be easily adapted to any field by following [Voi18, §§31 and 39] given that our maximal orders do not satisfy the selectivity condition in [CF99].

Since $F$ has narrow class number one, under the assumptions of Theorem 3.5, we have

$$m(\mathcal{O}_F, \mathcal{O}; G) = \frac{2h_D}{|H : H \cap N_{E/F}(E^*)\mathcal{O}_F^\times|} \prod q m(\mathcal{O}_q, \mathcal{O}_q; \mathcal{O}_q^\times).$$

**Theorem 3.6.** Let $q \geq 2$ be an integer. For $q \geq 3$, let $E = F[\zeta_q]$, where $\zeta_q$ is a primitive $q$-th root of unity, and let $\mathcal{S}_q$ be the set of $\mathcal{O}_F$-orders defined by

$$\mathcal{S}_q := \{ O_F[\zeta_q] \subset \mathcal{O} \subset O_E : \# \mathcal{O}_\text{tors}^\times = q \}.$$

For $q = 2$, let $\mathcal{S}_q$ be a set of representatives for the norms of elements in $G$ in $\text{Nrd}(W_+)$, and let $\mathcal{S}_q$ be the set of $\mathcal{O}_F$-orders defined by

$$\mathcal{S}_q := \bigcup_{\alpha \in \mathcal{O}_F[\sqrt{-n}]} \{ O_F[\sqrt{-n}] \subset \mathcal{O} \subset O_E \}.$$

Then the number of elliptic points of order $e_q$ in $G$ is given by

$$e_q := \frac{1}{2} \sum_{\mathcal{O} \in \mathcal{S}_q} m(\mathcal{O}, \mathcal{O}; G).$$

**Proof.** The proof is essentially an adaptation of the discussion of [Voi18, §39.4] (see also [Vig00, Chap. IV, Section 2]); the only difference arises from the elliptic points that are fixed by the Atkin-Lehner group $W_+$. However, the number of 2-torsion elliptic elements can be computed by combining Theorem 3.2 and §3.2. □

**Remark 3.7.** There seems to be very little discussion on the number of elliptic elements (or optimal embeddings) in maximal arithmetic Fuchsian groups. The only literature we could find on this topic is from Michon [Mic81] and Vignéras [Vig80, Chap. IV, §3] for $F = \mathbb{Q}$, and Maclachlan [Mac06, Mac09] for $[F : \mathbb{Q}] > 1$. In the latter case, however, the presentation is very different than ours. Our results are stated in a way as to draw the most parallel with optimal embeddings in Fuchsian groups, which correspond to Shimura curves, given that there is an abundance of literature in this case (see [Voi18, §30] and references therein).

3.5. Genus formula. Let $\Gamma$ be a Fuchsian group of signature $(g; e_1, \ldots, e_r)$, then the quotient $\Gamma \backslash \mathfrak{H}$ is a compact Riemann surface, whose volume is given by

$$\text{Vol}(\Gamma \backslash \mathfrak{H}) = 2\pi \left( 2g - 2 + \sum_{i=1}^r \left( 1 - \frac{1}{e_i} \right) \right).$$

When $\Gamma = \Gamma_\mathcal{O}$ is maximal in some commensurability class $\mathcal{C}(D)$, the volume depends only on $F$ and $S_f$ according to Theorem 3.1. So all maximal arithmetic Fuchsian groups in the commensurability class $\mathcal{C}(D)$ have the same signature, and we can compute their genus by combining the volume formula in Theorem 3.1 with the results of §3.4 (at least when $F$ has narrow class number one).

4. Shimura curves

We keep the notations of Section 3. Here, we summarise the necessary backgrounds on canonical models and $p$-adic uniformisation of Shimura curves. Our main references are [BC91, BZ, Car86, Nek12, Sij13]. We view $F$ as a subfield of $\mathbb{C}$ via the embedding $v_1 : F \hookrightarrow \mathbb{C}$. 

4.1. Complex uniformisation. Let $U = \prod q U_q \subset \mathfrak{O}^*$ be a compact open subgroup, such that $U_p$ is maximal. We consider the quotient

$$X_U(C) := D^\times \setminus X \times \hat{D}^\times / U,$$

where $X := P^1(C) - P^1(R) = \mathfrak{H} = \mathfrak{H}^+ \cup \mathfrak{H}^-$, and $\mathfrak{H}^-$ and $\mathfrak{H}^+$ are the lower and upper Poincaré half planes. Since $D$ is a division algebra, $X_U(C)$ is a Riemann surface.

There is a right action of $\hat{D}^\times$ on $X \times \hat{D}^\times$ by conjugation. For each $g \in \hat{D}^\times$, this induces an isomorphism of complex curves

$$X_U(C) \xrightarrow{\sim} X_{g^{-1}Ug}(C).$$

By the strong approximation theorem, we have the following bijections

$$D^\times \setminus \hat{D}^\times / U \simeq D^\times \setminus \{\pm 1\} \times \hat{D}^\times / U \simeq F_+^\times \setminus \hat{F}^\times / \text{Nrd}(U).$$

By class field theory, there is a unique abelian extension $F_U$ of $F$ such that the Artin map induces an isomorphism

$$\text{Art}_F : \text{Gal}(F_U/F) \simeq F_+^\times \setminus \hat{F}^\times / \text{Nrd}(U).$$

So the set $F_+^\times \setminus \hat{F}^\times / \text{Nrd}(U)$ is a Galois set. Therefore, there is a finite étale scheme $\tilde{T}_U$ defined over $F$ such that

$$\tilde{T}_U(F_U) = \tilde{T}_U(F) = \tilde{T}_U(C) = F_+^\times \setminus \hat{F}^\times / \text{Nrd}(U).$$

Shimura [Shi70] shows that $X_U(C)$ admits a canonical model defined over $F$ (see also [Del71]). Namely, we have the following result.

**Theorem 4.1.** There is a curve $X_U$ defined over $F$, called a canonical model, which satisfies the following properties:

(i) The set of complex points of $X_U$ is $X_U(C)$, i.e.

$$(X_U \otimes_{F,\nu} C)(C) = X_U(C).$$

(ii) For a compact open $U' \subset U$, the morphism $X_{U'}(C) \to X_U(C)$ is induced by an $F$-morphism $X_{U'} \to X_U$.

(iii) For each $g \in \hat{D}^\times$, the morphism $X_U(C) \to X_{g^{-1}Ug}(C)$ is induced from a $F$-morphism $X_U \to X_{g^{-1}Ug}$.

(iv) The morphism $X_U(C) \to \tilde{T}_U(C)$, has connected fibres, and is induced by a morphism of $F$-schemes $X_U \to \tilde{T}_U$. In particular, the group of connected component $\pi_0(X_U)$ is a finite étale group scheme over $F$ such that $\pi_0(X_U)(C) = \pi_0(X_U(C)) = \tilde{T}_U(C)$, where $\pi_0(X_U(C))$ is the group of connected components of $X_U(C)$.

**Proof.** This is essentially a summary of the properties of canonical models of Shimura curves listed in [Car86, §§1.1 and 1.2].

Theorem 4.1 (iv) is known as the Shimura reciprocity law, it implies that $X_U$ is an irreducible scheme, which is not geometrically irreducible in general. However, when $\text{Nrd}(U) = \mathfrak{O}^\times$, then $X_U$ is geometrically irreducible since we assume that $F$ has narrow class number one.

We define the **adelic Atkin-Lehner group** by $\hat{W} := N_{\hat{D}}(U)/\hat{F}^\times U$. By making use of the weak approximation theorem, one can show that

$$\hat{W} \simeq \prod_{q \in \mathfrak{S}_f \cup S_0} \mathbb{Z}/2\mathbb{Z},$$

where $S_0$ is the set of primes where $U_q$ is non-maximal.

**Corollary 4.2.** The group $\hat{W}$ acts on $X_U(C)$. This action is induced from an action of $\hat{W}$ on $X_U$ defined over $F$. In particular, if $W' \subseteq \hat{W}$ is a subgroup, then the quotient $X_U/W'$ is defined over $F$. 
Proof. Every element \( g \in \hat{W} \) defined an automorphism of \( X_U(\mathbb{C}) \). By Theorem 4.1 (iii), this automorphism descends to \( F \).

When there is an integral ideal \( \mathfrak{R} \) coprime with the discriminant \( \text{disc}(D) \) of \( \mathcal{O} \), and an Eichler order \( \mathcal{O}_0(\mathfrak{R}) \subset \mathcal{O} \) of level \( \mathfrak{R} \) such that \( U = \hat{\mathcal{O}_0(\mathfrak{R})}^\times \), we will denote the Shimura curve \( X_U \) by \( X_U^D(\mathfrak{R}) \), or simply write \( X_U^D(1) \) when \( \mathfrak{R} = (1) \).

4.2. Bruhat-Tits tree. Let \( T_p \) be the Bruhat-Tits tree attached to \( \text{GL}_2(F_p) \). Its set of vertices \( V(T_p) \) consists of maximal \( \mathcal{O}_{F_p} \)-orders in \( M_2(F_p) \); two vertices being adjacent if their intersection is an Eichler order of level \( p \). Let \( \mathcal{E}(T_p) \) denote the set of ordered edges of \( T_p \), i.e., the set of ordered pairs \((s, t)\) of adjacent vertices of \( T_p \). If \( e = (s, t) \), the vertex \( s \) is called the source of \( e \) and the vertex \( t \) is called its target; they are denoted by \( s(e) \) and \( t(e) \) respectively.

The Atkin-Lehner involution \( \iota : \mathcal{E}(T_p) \to \mathcal{E}(T_p) \) sends the edge \( e = (s, t) \) to the opposite edge \( \bar{e} = (t, s) \). We let \( \mathcal{E}(T_p) = \mathcal{E}(T_p)/\langle \iota \rangle \) be the set of non-oriented edges.

The tree \( T_p \) is endowed with a natural left action of \( \text{PGL}_2(F_p) \) by isometries corresponding to conjugation of maximal orders by elements of \( \text{GL}_2(F_p) \). This action is transitive on both \( V(T_p) \) and \( \mathcal{E}(T_p) \).

4.3. \( p \)-adic uniformisation. Let \( \overline{T_p} \) be an algebraic closure of \( F_p \), and \( C_p := \overline{T_p} \) be a fixed completion of \( T_p \). Let \( \mathcal{H}_p \) be \( p \)-adic upper half plane. This is the formal scheme over \( \text{Spf}(\mathcal{O}_{F_p}) \) defined in [BC91, §1.3].

\[
\mathcal{H}_p := \mathbb{P}^1(C_p) - \mathbb{P}^1(F_p).
\]

The scheme \( \mathcal{H}_p \) admits a natural action by the group \( \text{GL}_2(F_p) \), which factors through the adjoint group \( \text{PGL}_2(F_p) \). We let

\[
\mathcal{H}_p^{\text{ur}} = \mathcal{H}_p \times_{\text{Spf}(\mathcal{O}_{F_p})} \text{Spf}(\mathcal{O}_{F_p}^{\text{ur}}).
\]

Let \( B \) be the totally definite quaternion algebra defined over \( F \) whose set of ramified finite places is \( S \setminus \{p\} \) so that \( B_p \simeq M_2(F_p) \). (Note that this means that the set of ramified archimedean places of \( B \) is \( S_\infty \cup \{v_1\} \).) We write \( \hat{B} = B_p \times B_p \) and \( \hat{D} = D_p \times D_p \), and we fix an isomorphism \( \varphi : D_p \sim \hat{B} \). We let \( K = K_p \times K_p \) be a compact open subgroup of \( \hat{B}^\times \) such that \( K_p \simeq \text{GL}_2(\mathcal{O}_{F_p}) \) and \( \varphi(U^p) = K^p \). We also let

\[
K_0^p := \begin{cases} (a, b) \in K_p : c \equiv 0 \mod p \end{cases},
\]

and \( K_0(p) = K_0^p \times K^p \).

Since \( U_p \) is the maximal compact open subgroup of \( D_p^\times \), the norm map induces an isomorphism \( D_p^\times / U_p \sim F_p^\times / \mathcal{O}_{F_p}^\times \) (see [Vig80, Chap. II, Lemme 1.5]). The group \( B_p^\times \) acts on \( F_p^\times / \mathcal{O}_{F_p}^\times \) through its reduced norm map \( \text{Nrd} : B_p^\times \to F_p^\times \). We obtain a corresponding action of \( B_p^\times \) on \( D_p^\times / U_p \). This, together with the isomorphism \( \varphi \), gives an action of \( \hat{B}^\times \) on \( \hat{D}^\times / U \).

Theorem 4.3 (Čerednik-Drinfel’d). There exist a model \( \mathcal{M} \) of \( X_U \) over \( \mathcal{O}_{F_p} \), and an isomorphism of formal schemes

\[
\hat{\mathcal{M}}^{\text{ur}} = \mathcal{M} \times_{\text{Spf}(\mathcal{O}_{F_p})} \text{Spf}(\mathcal{O}_{F_p}^{\text{ur}}) \simeq B^\times \setminus \mathcal{H}_p^{\text{ur}} \times \hat{D}^\times / U,
\]

where \( \hat{\mathcal{M}} \) is the completion of \( \mathcal{M} \) along its special fibre.

Proof. See [BZ, Theorem 3.1].
4.4. The dual graph. The dual graph associated to $B^x \backslash \mathcal{M}_p^x \times \hat{D}^x / U$ is the weighted graph

$$\mathcal{G} := B^x \backslash T_p \times \hat{D}^x / U.$$ 

The vertices of $\mathcal{V}(\mathcal{G})$ and oriented edges $\mathcal{E}(\mathcal{G})$ of $\mathcal{G}$ are given respectively by

$$\mathcal{V}(\mathcal{G}) := B^x \backslash \mathcal{V}(T_p) \times \hat{D}^x / U, \quad \mathcal{E}(\mathcal{G}) := B^x \backslash \mathcal{E}(T_p) \times \hat{D}^x / U.$$ 

We define the weight of a vertex $v \in \mathcal{V}(\mathcal{G})$ to be $\# \text{Stab}_{B^x / F^x}(v)$, and the weight of an edge $e \in \mathcal{E}(\mathcal{G})$ to be $\# \text{Stab}_{B^x / F^x}(e)$. For a vertex $v$, we let Star($v$) denote the set of all edges containing $v$.

**Proposition 4.4.** The maps

$$\vartheta_1 : (B_p^x / F_p^x K_p) \times (D_p^x / U_p) \times (D_p^x / U^p) \to (B_p^x / K_p) \times (B_p^x / K^p)$$

$$(x_p, y_p, y^p) \mapsto (x_p, \text{ord}_p(\text{Nrd}(y_p)), \varphi(y^p)),$$

and

$$\vartheta_2 : (B_p^x / F_p^x K_p^0) \times (D_p^x / U_p) \times (D_p^x / U^p) \to (B_p^x / K^0) \times (B_p^x / K^p)$$

$$(x_p, y_p, y^p) \mapsto (x_p, \text{ord}_p(\text{Nrd}(y_p)), \varphi(y^p))$$

induce an isomorphism of bipartite graphs

$$\mathcal{V}(\mathcal{G}) = B^x \backslash \mathcal{V}(T_p) \times \hat{D}^x / U \cong (B^x \backslash \hat{B}^x / K) \times \mathbb{Z} / 2\mathbb{Z},$$

$$\mathcal{E}(\mathcal{G}) = B^x \backslash \mathcal{E}(T_p) \times \hat{D}^x / U \cong (B^x \backslash \hat{B}^x / K_0(p)) \times \mathbb{Z} / 2\mathbb{Z}$$

as follows: We write $\mathcal{V}(\mathcal{G}) = \mathcal{V} \cup \mathcal{V}' \cong B^x \backslash \hat{B}^x / K \sqcup B^x \backslash \hat{B}^x / K$, and we let the adjacency matrix in the basis $\mathcal{V} \cup \mathcal{V}'$ be given by the matrix

$$\begin{bmatrix}
0 & T_p \\
T_p & 0
\end{bmatrix},$$

where $T_p$ is the Hecke operator at $p$ acting on the Brandt module $M := \mathbb{Z}[B^x \backslash \hat{B}^x / K]$. In that identification, the action of the Atkin-Lehner involution $w_p$ on $\mathcal{V}(\mathcal{G})$ is given by the matrix

$$\begin{bmatrix}
0 & 1_M \\
1_M & 0
\end{bmatrix}.$$ 

**Proof.** See [Sij13, Propositions 3.1.8 and 3.1.9], [Nek12, §1.5] or [Kur79, §4]. □

**Remark 4.5.** In the isomorphism of Proposition 4.4, the set of non-oriented edges is given by

$$\mathcal{E}(\mathcal{G}) = B^x \backslash \mathcal{E}(T_p) \times \hat{D}^x / U \cong B^x \backslash \hat{B}^x / K_0(p).$$

The following result is an essential ingredient in the description of the special fibre of the Cerednik-Drinfeld’s model described in Theorem 4.3. As we will see later, it is also useful in understanding the automorphism group of the curve $X_U$.

**Theorem 4.6.** Let $\mathcal{M}$ be the scheme in Theorem 4.3. Then, we have the following:

(i) $\mathcal{M} \otimes_{\mathcal{O}_{F^p}} \mathcal{O}_{F^p \times}$ is a normal, proper, flat and semi-stable scheme over $\mathcal{O}_{F^p \times}$.
(ii) The special fibre of $\mathcal{M} \otimes_{\mathcal{O}_{F^p}} \mathcal{O}_{F^p \times}$ is reduced. Its components are rational curves, and all its singular points are ordinary double points.
(iii) The weighted dual graph associated to $\mathcal{M} \otimes_{\mathcal{O}_{F^p}} \mathcal{O}_{F^p \times}$ is the graph $\mathcal{G}$ described in Proposition 4.4.
(iv) Let $\mathcal{H}$ be a connected component of $\mathcal{G}$, and $\mathcal{M}_{\mathcal{H}}$ the corresponding irreducible component of $\mathcal{M}$. Then the arithmetic genus of $\mathcal{M}_{\mathcal{H}}$ is given by the Betti number $1 + \# \mathcal{E}(\mathcal{H}) - \# \mathcal{V}(\mathcal{H})$.

**Proof.** See [Nek12, Proposition 1.5.5] or [Kur79, Proposition 3.2]. □
4.5. **Special fibre of** $\mathcal{M} \otimes \mathcal{O}_{F_p} \mathcal{O}_{F_p^2}$. The curve $\mathcal{M}$ is an admissible curve over $\mathcal{O}_{F_p}$ in the following sense:

(i) $\mathcal{M} \otimes \mathcal{O}_{F_p} \mathcal{O}_{F_p^2}$ is a normal, proper, flat and semistable scheme over $\mathcal{O}_{F_p^2}$.

(ii) Each irreducible component has a smooth generic fibre.

(iii) The completion of the local ring of $\mathcal{M} \otimes \mathcal{O}_{F_p} \mathcal{O}_{F_p^2}$ at each of its singular points $x$ is isomorphic, as an $\mathcal{O}_{F_p^2}$-algebra, to $\mathcal{O}_{F_p}[[X,Y]]/(XY - \varpi^w)$, where $w = w(x) \in \{1, 2, 3, \ldots\}$.

(iv) Each vertex $v \in \mathcal{V}(\mathcal{G})$ corresponds to an irreducible component $C_v$ of the special fibre $\mathcal{M} \otimes \mathcal{O}_{F_p}$ $k(p)$.

(v) Each edge $e = \{v, v'\} \in \mathcal{E}(\mathcal{G})$ corresponds to a singular point in $x_e \in C_v \cap C_{v'}$.

The dual graph encodes the following combinatorial data of the special fibre.

4.6. **Automorphism groups.** An automorphism of weighted graph $\mathcal{G}$ is an automorphism of graphs which preserves the weights of the edges. We will denote the group of such automorphisms by $\text{Aut}(\mathcal{G})$. We note that there is a natural inclusion $\text{Aut}(\mathcal{G}) \subset \text{Aut}^s(\mathcal{G})$, where $\text{Aut}^s(\mathcal{G})$ is the automorphism group of the underlying simple graph to $\mathcal{G}$.

For the next statement, we recall the notion of admissibility from [KR08]. We say that an element $\omega \in \text{Aut}(\mathcal{G})$ is admissible if there is no vertex $v \in \mathcal{V}(\mathcal{G})$ fixed by $\omega$ such that $\text{Star}(v)$ has at least 3 edges also fixed by $\omega$. We say that a subgroup $H \subset \text{Aut}(\mathcal{G})$ is admissible if every non-trivial element $\omega \in H$ is admissible.

**Proposition 4.7.** Let $W' \in \hat{W}$ be a subgroup. Then, we have the following:

1. The dual graph of $(\mathcal{M}/W') \otimes \mathcal{O}_{F_p} \mathcal{O}_{F_p^2}$ is the graph $\mathcal{G}' = (\mathcal{G}/W')^\ast$, where $^\ast$ means we remove all loops from the quotient graph $\mathcal{G}/W'$.

2. Assume that the genus of $X_U/W'$ is at least 2, and let $\mathcal{G}_{st}$ be the dual graph of the stable model $(\mathcal{M}/W')_{st}$ of $(\mathcal{M}/W')^\ast$. Then there is a natural injection $\varrho : \text{Aut}(X_U/W') \hookrightarrow \text{Aut}(\mathcal{G}_{st})$ whose image $\text{im}(\varrho)$ lies in an admissible subgroup.

**Proof.** Part (1) follows from general properties of Mumford curves. From [DM69, Lemmas 1.12 and 1.16], and universal properties of stable models, there is an injection $\varrho : \text{Aut}(X_U/W') \hookrightarrow \text{Aut}(\mathcal{G}_{st})$. To prove Part (2), we only need to show that every non-trivial element in the image of $\varrho$ is admissible. To this end, let $\omega \in \text{Aut}(X_U/W')$ be such that $\varrho(\omega)$ fixes a vertex $v$, and at least 3 edges in $\text{Star}(v)$. Then, since every automorphism of the projective line, which fixes at least 3 points is the identity, the restriction $\omega|_{C_v}$ is the identity, where $C_v$ is the irreducible component associated to $v$. This would imply that, as an automorphism of the Riemann surface $(X_U/W')(\mathbb{C})$, $\omega$ fixes more than $2g(X_U/W') + 2$ points, where $g(X_U/W')$ is the genus of $X_U/W'$. Hence $\omega$ must be the identity. Therefore, if $\omega$ is non-trivial, then $\varrho(\omega)$ must be admissible. \qed

5. **The hyperelliptic Shimura quotient curve**

5.1. **The quaternion algebra.** Let $F = \mathbb{Q}(\alpha) = \mathbb{Q}(\zeta_{32} + \zeta_{32}^{-1})$ be the maximal totally real subfield of the cyclotomic field of the 32nd roots of unity. This field is defined by the polynomial $x^8 - 8x^6 + 20x^4 - 16x^2 + 2$. Let $\sigma$ be a generator of
Then we have:

The space $S$ in [DD08, DV13, GV11]. We let the Hilbert Modular Forms Package in Magma and $24$ respectively. (We note that all the computations have been performed using correspondence, we have isomorphisms of Hecke modules of automorphic forms of level $p$ be the space of automorphic forms of level $(1)$ and weight $2$ on $p$ of level $p$ implemented by Voight [Voi05], but we will not need such a map here.

Embedding $K \hookrightarrow F$ is the unique prime of class number $17$. For later, we observe that $\beta$ the real places $v$ signature $+(p \text{ prime } F)$ Gal$(L_f/K_f)$. We recall the following diagram

| Newform | Coefficient field | $L_f$ | Fixed field | $K_f = L_f^\beta$ | Gal$(L_f/K_f)$ |
|---------|------------------|------|-------------|------------------|--------------|
| $f, f'$ | $\mathbb{Q}(\zeta_{15})^+$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{Z}/4\mathbb{Z}$ |
| $g, g'$ | Quartic subfield of $\mathbb{Q}(\zeta_{64})^+$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{Z}/2\mathbb{Z}$ |
| $h$ | Ray class field of modulus $r = (\frac{1}{2}(c^2 - 16c + 25))$ | $\mathbb{Q}(x) := \mathbb{Q}[x]/(r(x))$, $\mathbb{Z}/8\mathbb{Z}$ |

Relations $\sigma f = f^r$ and $\sigma^2 f = f^r$, $\sigma g = g^r$ and $\sigma^2 g = g^r$, $\sigma h = h^r$

Gal$(F/\mathbb{Q})$. Let $\mathcal{O}_F$ be the ring of integers of $F$. Let $v_1, \ldots, v_8$ be the real places of $F$. We consider the quaternion algebra $D/F$ ramified at $v_2, \ldots, v_8$ and the unique prime $p$ above $2$. More concretely, we have $D = (\frac{u - 1}{x})$, where $u = -\alpha^2 + \alpha$ has signature $(+, -, \ldots, -)$. Let $\mathcal{O}$ be the maximal order in $D$ given by

$$\mathcal{O} := \mathcal{O}_F[1, i, (\alpha^7 + \alpha^6 + \alpha^4 + 1) + \alpha^7 i + j, (\alpha^7 + \alpha^6 + \alpha^4 + 1) i + k].$$

We also let $B/F$ be the totally definite quaternion algebra ramified exactly at all the real places $v_1, \ldots, v_8$, and fix a maximal order $\mathcal{O}_B$ in $B$.

5.2. The CM field and its embedding. We recall the following diagram

The subfield $K := \mathbb{Q}(\beta) = \mathbb{Q}(i(\zeta_{64} + \zeta_{64}^{-1}))$ is the unique CM extension of $F$ with class number $17$. For later, we observe that $\beta^2 = \alpha$, where $p = (2 + \alpha)$. Since $p$ is the unique prime of $F$ that ramifies in both $K$ and $D$, we see that $K$ is a splitting field of $D$ by [Vig80, Chap. III, Théorème 3.8]. It is possible to compute an explicit embedding $K \hookrightarrow D$ using the Quaternion Algebras Package in magma [BCP97] implemented by Voight [Voi05], but we will not need such a map here.

5.3. The spaces of forms. Let $S_2(p)^{\text{new}}$ be the new subspace of Hilbert cusp forms of level $p$ and weight $2$, this is a $40$-dimensional space. Let $S_2^{D}(1)$ (resp. $S_2^B(p)^{\text{new}}$) be the space of automorphic forms of level $(1)$ and weight $2$ on $D$ (resp. new subspace of automorphic forms of level $p$ and weight $2$ on $B$.) By the Jacquet-Langlands correspondence, we have isomorphisms of Hecke modules

$$S_2(p)^{\text{new}} \simeq S_2^{D}(1) \simeq S_2^B(p)^{\text{new}}.$$ 

The space $S_2(p)^{\text{new}}$ decomposes into $5$ Hecke constituents of dimensions $4, 4, 4, 4$ and $24$ respectively. (We note that all the computations have been performed using the Hilbert Modular Forms Package in magma [BCP97], the algorithms are described in [DD08, DV13, GV11].) We let $f, f', g, g'$ and $h$ be newforms in those constituents. Then we have:

(i) The forms $f$ and $f'$ have the same coefficient field $L_f = L_{f'}$, which is the real quartic field $\mathbb{Q}(\zeta_{15})^+$ given by $x^4 + x^3 - 4x^2 - 4x + 1$. They satisfy the relations $\tau f = f^r$ and $\tau^2 f = f^r$, where $\tau$ is a generator of Gal$(L_f/\mathbb{Q})$.  

Table 1. Newforms of level $p$ and weight $2$ on $F = \mathbb{Q}(\zeta_{64})^+$
(ii) The forms \( g \) and \( g' \) have the same coefficient field \( L_g = L_{g'} \), which is the real quartic subfield of \( \mathbb{Q}(\zeta_5)^+ \) given by \( x^4 + 19x^3 - 59x^2 + 19x + 1 \). They satisfy the relations \( \bar{g} = g' \) and \( \bar{\bar{g}} = \bar{g}^\tau \), where \( \tau \) is a generator of \( \text{Gal}(L_g/\mathbb{Q}) \).

(iii) The coefficient field of the form \( h \) is a field \( L_h \) of degree 24, which is cyclic over the field \( K_h = \mathbb{Q}(\sqrt{c}) \) defined by \( c^3 + c^2 - 229c + 167 = 0 \). More precisely, it is the ray class field of conductor \( c = (\frac{1}{2}(c^2 - 16c + 25)) \). The form \( h \) satisfies the relation \( \bar{h} = h^\tau \), where \( \tau \) is a generator of \( \text{Gal}(L_h/K_h) \).

(We summarise that data in Table 1, and the relations among the forms.) Let \( w \) and \( w_D \) be the Atkin-Lehner involutions acting on \( S_2(p)_{\text{new}} \) and \( S^D_2(1) \) respectively. The Atkin-Lehner involution \( w \) acts as follows:

\[
wf = -f, \quad w^2f = -f', \quad wg = -g, \quad wg' = -g', \quad wh = h.
\]

We recall that \( w_D = -w \).

5.4. The Shimura curve and its quotient. Let \( X^D_0(1) \) be the Shimura curve attached to \( \mathcal{O} \). Let \( w_D \) be the Atkin-Lehner involution at \( \mathfrak{p} \), and \( C := X^D_0(1)/\langle w_D \rangle \). We can canonically identify \( S^D_2(1) \) with the the space of 1-differential forms on \( X^D_0(1) \). From the discussion in \( \S 5.3 \), it follows that \( X^D_0(1) \) is a curve of genus 40; and that \( C \) is a curve of genus 16.

Theorem 5.1. The curves \( X^D_0(1) \) and \( C \) have the respective signatures \( (40; 3^{18}, 16^1) \) and \( (16; 2^{17}, 3^6, 32^1) \).

Proof. The complex points of the curve \( X^D_0(1) \) are determined by the quotient \( \Gamma_1(\delta) \), where \( \Gamma_1 \) is the image of \( \mathcal{O}^1 \) inside \( \text{PSL}_2(\mathbb{R}) \). So it is a Shimura curve. So, we can compute the signature of \( X^D_0(1) \) using the Shimura Curves Package in \texttt{magma} [BCP97], which was implemented by Voight [Voi09]. This gives that \( X^D_0(1) \) has signature \((40; 3^{18}, 16^1)\).

The curve \( C = X^D_0(1)/\langle w_D \rangle \) is given by the maximal arithmetic Fuchsian group \( \Gamma_0(\mathcal{O}) \). It is not a Shimura curve. Although Voight has implemented algorithms for computing with maximal arithmetic Fuchsian groups, they are not publicly available yet. So, we compute the signature of \( C \) by using the results of Section 3.

Let \( q \geq 2 \) be an integer. Then, by Theorem 3.2, \( \Gamma_0(\mathcal{O}) \) contains an elliptic element of order \( q \) if and only if the following three conditions are satisfied:

1. \( 2 \cos(2\pi/q) \in F \);
2. no prime \( q \in S_f \) splits in \( E = F[\zeta_q] \);
3. the ideal generated by \( 2 + 2\cos(2\pi/q) \) is supported at \( S_f \) modulo squares.

It is enough to test all integers \( q \) between 3 and 64. The only \( q \geq 3 \) which satisfy these conditions are: 3, 4, 6, 8, 16 and 32.

For \( q = 4, 8, 16 \) or 32, we have \( E = F[\zeta_q] = \mathbb{Q}(\zeta_{32}) \). In that case, the only \( \mathcal{O}_E \)-order which contains \( \mathcal{O}_F[\zeta_{32}] \) and optimally embeds into \( D \) is the maximal order \( \mathcal{O}_D \). By Theorem 3.6, we get that \( e_{32} = 1 \).

For \( q = 3, 5, 7, 13, 17, 19, 23, 29, 31 \). Then, the only \( \mathcal{O}_E \)-order which contains \( \mathcal{O}_F[\sqrt{-n}] \) and optimally embeds into \( D \) is also the maximal order \( \mathcal{O} := \mathcal{O}_E \). We have \( h_D = 9 \). Now since the prime \( \mathfrak{p} \) is inert in the relative extension \( E/F \), we have \( [H : H \cap N_E/F(E^\infty)\mathcal{O}_F^+] = 2 \). So, by Theorem 3.6, we get that \( e_3 = 9 \).

Finally, for \( q = 2 \), we have \( W^1 = W_+ = \mathbb{Z}/2\mathbb{Z} \) since \( F \) has narrow class number one and there is a unique prime in \( S_f \); namely, the prime \( \mathfrak{p} \) above 2. So the unique CM extension \( E/F \) which satisfies the condition of Theorem 3.2 is the extension \( K \) discussed in Subsection 5.2. Recall that the ideal \( \mathfrak{p} \) is generated by the totally positive element \( n = 2 + \alpha \). The only \( \mathcal{O}_E \)-order which contains \( \mathcal{O}_F[\sqrt{-n}] \) and optimally embeds into \( D \) is also the maximal order \( \mathcal{O} := \mathcal{O}_K \). We
have \( h_D = 17 \). Now since the prime \( p \) is ramified in the relative extension \( K/F \), we have \( [H : H \cap N_{E/F}(E^x)\mathcal{O}_F^\times] = 1 \). So, by Theorem 3.6, we get that \( e_2 = 17 \).

So we conclude that there are 3 classes of elliptic elements in \( \Gamma_C \) of orders 2, 3 and 32, with respective multiplicities 17, 9 and 1.

By the volume formula in Theorem 3.1 and the genus formula in Subsection 3.5, the genus \( g \) of the curve \( C \) must satisfy the equality
\[
\frac{\text{Vol}(\Gamma_C \backslash \mathcal{D})}{2\pi} = \frac{1455}{32} = 2(g - 1) + 17 \left( 1 - \frac{1}{2} \right) + 9 \left( 1 - \frac{1}{3} \right) + \left( 1 - \frac{1}{32} \right).
\]
Solving this, we get that \( g = 16 \). Hence the curve \( C \) has signature \((16; 2^{17}, 3^{9}, 32^1)\).

\[\Box\]

**Lemma 5.2.** The curve \( X^p_0(1) \) and the Atkin-Lehner involution \( w_D \) are both defined over \( \mathbb{Q} \). In particular, the curve \( C \) descends to \( \mathbb{Q} \).

**Proof.** Since \( \sigma(p) = p \) and the ray class group of modulus \( pv_2\cdots v_8 \) is trivial, the curve \( X^p_0(1) \) is defined over \( F \) by [DN67, Corollary], and the field of moduli is \( \mathbb{Q} \).

The field \( \mathbb{Q}(\zeta_{32}) \) is a splitting field for \( D \) whose class number is one. So, there is a unique CM point attached to the extension \( \mathbb{Q}(\zeta_{32})/F \), and it is defined over \( F \). Therefore, by [SV16, Corollary 1.9], the curve \( X^p_0(1) \) descends to \( \mathbb{Q} \).

Alternatively, by using the moduli interpretation in [Car86], or the more recent work [TX16], one can show that both \( X^p_0(1) \) and \( w_D \) are defined over \( \mathbb{Q} \).

\[\Box\]

5.5. **The dual graph of the quotient curve.** The dual graph \( G' \) of the curve \( \mathcal{M}/\langle w_p \rangle \) is displayed in Figure 1. It was computed by using Proposition 4.4 and Proposition 4.7. The computations combine both Magma [BCP97] and Sage [Sag19].

**Lemma 5.3.** The automorphism group of \( G' \) is \( \text{Aut}(G') \cong \mathbb{Z}/4\mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z})^4 \).

**Proof.** We compute the dual graph \( G' \) of the quotient \( \mathcal{M}/\langle w_p \rangle \) using Proposition 4.4 and Proposition 4.7. Let \( B \) be the definite quaternion algebra defined in §5.1. Then, by Proposition 4.4 and Proposition 4.7, the dual graphs \( G \) and \( G' \) are determined by the Brandt module \( M_B := \mathbb{Z}[B^\times \backslash \hat{B}^\times / \hat{\mathcal{O}}_B^\times] \). In this case, the class number of the maximal order \( \mathcal{O}_B \) is 58, and a basis of this module is given by equivalence classes of \( \mathcal{O}_B \)-right ideals. We let \( v_1, v_2, \ldots, v_{58} \) be such a basis, which we order so that the weights of the elements are in decreasing order.

We get the following sequence of weights: \( 32, 24, 16, 8^2, 4^3, 4^2, 6^2 \) and \( 1^40 \), where the exponent indicates the number of times each weight is repeated. Similarly, we compute the set of edges, and we obtain the following sequence for their weights: \( 32, 16^2, 8^6, 4^6, 2^{12} \) and \( 1^{28} \). By combining this with the Hecke operator \( T_p \), we obtain the graph \( G' \) in Figure 1.

We compute the automorphism group \( \text{Aut}^+(G') \) of the underlying simple graph using Magma, and check that every element in \( \text{Aut}^+(G') \) preserves the weights of the edges, i.e. that \( \text{Aut}(G') = \text{Aut}^+(G') \).

To determine the group structure of \( \text{Aut}(G') \), we first check that there is a unique normal subgroup of \( \text{Aut}(G') \) which is isomorphic to \( (\mathbb{Z}/2\mathbb{Z})^4 \). Finally, we show that there is a unique cyclic subgroup of order 4 whose intersection with \( (\mathbb{Z}/2\mathbb{Z})^4 \) is the neutral element.

\[\Box\]

**Remark 5.4.** As a byproduct of the computation of \( G' \), we check that
\[
1 + \#\mathcal{E}(G') - \#\mathcal{V}(G') = 1 + 73 - 58 = 16,
\]
which is the genus of \( \mathcal{M}/\langle w_p \rangle \), or equivalently \( C \).

Let \( (\mathcal{M}/\langle w_p \rangle)_{st} \) be the stable model of \( \mathcal{M}/\langle w_p \rangle \), and \( \mathcal{G}_{st} \) its dual graph. By definition, \( (\mathcal{M}/\langle w_p \rangle)_{st} \) is stable if for all \( v \in \mathcal{V}(\mathcal{G}_{st}) \), we have \( \# \text{Star}(v) \geq 3 \). So, we obtain \( (\mathcal{M}/\langle w_p \rangle)_{st} \) by blowing down all components \( C_v \) associated to a vertex \( v \) such that \( \# \text{Star}(v) < 3 \). On graphs, this corresponds to doing the following:
a) For all \( v \in V(G') \), with \# \( \text{Star}(v) = 1 \), remove \( v \) and all edges in \( \text{Star}(v) \);

b) For each \( v \in V(G') \), with \# \( \text{Star}(v) = 2 \), contract the chain \( v' - v - v'' \) to \( v' - v'' \) with \( w(e) = w(e') + w(e'') \).

By applying this process to the curve \( \mathcal{M}/\langle w_p \rangle \), and then relabelling the resulting graph, we obtain the stable model whose dual graph \( G_{st} \) is given by Figure 2. The graph \( G_{st} \) has 30 vertices and 45 edges so that

\[
1 + \#\mathcal{E}(G_{st}) - \#V(G_{st}) = 1 + 45 - 30 = 16.
\]

**Lemma 5.5.** Let \((\mathcal{M}/\langle w_p \rangle)_{st}\) be the stable model of \( \mathcal{M}/\langle w_p \rangle \), and \( G_{st} \) its dual graph. Then, \( G_{st} \) is a connected graph such that \( \text{Aut}(G_{st}) = \text{Aut}(G') \).

**Proof.** This follows from a direct calculation. \( \square \)

**Lemma 5.6.** Every admissible subgroup of \( \text{Aut}(G_{st}) \) of exponent 2 has order 2.

**Proof.** First, we note that, since the degree of the Hecke operator \( T_p \) is 3, and \((\mathcal{M}/\langle w_p \rangle)_{st}\) is stable, \# \( \text{Star}(v) = 3 \) for each \( v \in V(G_{st}) \).

In the notations of Figure 2, we label the vertices 1, 2, \ldots, 30. There are 19 permutations of order 2 in \( \text{Aut}(G_{st}) \subset S_{30} \). Of those 19 permutations, there are exactly 4 with the same support of length 28. Each of the remaining 17 has a support whose length belongs to \( \{2, 4, 6, 8\} \). The permutations of length 28 fix the vertices \( v = 1 \) and \( v' = 2 \). So, they must be admissible since a non-admissible element must fix at least 4 different vertices. For each of remaining 17 permutations, one easily sees that the complement of its support contains a vertex \( v \) and its \( \text{Star}(v) \), meaning that it cannot be admissible.

To conclude the proof of the lemma, we let \( \sigma_i, i = 1, 2, 3, 4 \) be the 4 admissible permutations obtained above, and we check that \( \sigma_i \sigma_j \) is not admissible for \( i \neq j \). \( \square \)

**Lemma 5.7.** There is an injection \( \text{Aut}(C) \hookrightarrow H \) into an admissible subgroup of \( \text{Aut}(G_{st}) \) of exponent 2. In particular \( \text{Aut}(C) \) has order at most 2.

**Proof.** In Subsection 5.7, we will show that the endomorphism ring of each of the simple factor of Jac(\( C \)) is a totally real field. (This follows from the decomposition (2).) Using this, we see that \( \text{Aut}(C) \subset (\mathbb{Z}/2\mathbb{Z})^4 \). So, by Proposition 4.7, \( \text{Aut}(C) \) injects into an admissible subgroup \( H \) of \( \text{Aut}(G_{st}) \) of exponent 2. By Lemma 5.6, \( H \) has order at most 2. \( \square \)

**Remark 5.8.** The graph \( G' \) of the integral model \( \mathcal{M}/\langle w_p \rangle \) (see Figure 1) is an example of a graph whose automorphism group does not have an element that is admissible. Indeed, it is easy to see that every element of \( \text{Aut}(G') \) must fix the vertex \( v_4 \) and the 3 edges of weight 8 contained in \( \text{Star}(v_4) \). However, the vertex \( v_4 \) and \( \text{Star}(v_4) \) are removed when we blow down \( \mathcal{M}/\langle w_p \rangle \) to obtain the stable model \((\mathcal{M}/\langle w_p \rangle)_{st}\) (see Figure 2). This example shows that [KR08, Proposition 3.4] is incorrect as stated and needs to be modified slightly.

5.6. **Hyperellipticity of the curve \( C \).** We are now ready to prove one of our main results.

**Theorem 5.9.** The curve \( C \) is hyperelliptic over \( F \).

**Proof.** Let \( \gamma \in \Gamma_C \) be an elliptic element of order 2, and \( P \) a fixed point by \( \gamma \). Then \( P \) is a CM point by construction, and \( \gamma \) acts on the local ring \( \mathcal{O}_{C,P} \) as an involution. More specifically, letting \( t \) be a uniformiser at \( P \), we see that \( \gamma \) acts on \( t \) as:

\[
t(t \mod t^2) \mapsto -t \mod t^2.
\]
This forces any global differential form in $H^0(C,\Omega^1_C)$, which vanishes at $P$, to vanish to even order. We claim that this implies that $P$ is a Weierstrass point. To prove this, we use Riemann Roch.

Let $K_C$ the canonical divisor. Then, we have $\ell(K_C - 2P) = \ell(K_C - P)$, i.e. every differential that vanishes at $P$ vanishes to order 2. By Riemann-Roch, we have

$$\ell(K_C - 2P) - \ell(P) = \deg(K_C - 2P) - g + 1 = (2g - 4) - g + 1 = g - 3;$$

and

$$\ell(K_C - P) - \ell(2P) = \deg(K_C - P) - g + 1 = (2g - 3) - g + 1 = g - 2.$$

So, if $\ell(K_C - 2P) = \ell(K - P)$, then $\ell(2P) - \ell(P) = 1$. Now, since $\mathcal{Z}(P)$ is the space of constant functions, we see that $\mathcal{Z}(2P)$ must be non-trivial, and thus $P$ is a (hyperelliptic) Weierstrass point.

From the above argument, it follows that $C$ has 17 hyperelliptic Weierstrass points that are all CM. By Shimura reciprocity law, these CM points are all defined over the Hilbert class field $H_K$ of $K$. Let $M$ be the normal closure of $H_K$ over $F$. Then $[M : F] = 34$ and $\text{Gal}(M/F) \simeq D_{17}$; and the action of $\text{Gal}(\overline{F}/F)$ on the set of Weierstrass points $\mathcal{W}$ must factor through it (see Subsection 2.3). Therefore, we must have $\#\mathcal{W} = 34$. In other words, $C$ has 34 hyperelliptic Weierstrass points. Since $C$ has genus 16, it must therefore be hyperelliptic by Proposition 2.3.

**Remark 5.10.** It follows from the proof of Theorem 5.9 that half of the Weierstrass points on $C$ are CM, while the remaining half are non CM. This means that the hyperelliptic involution must necessarily be exceptional. However, this should be expected since $C = X_0^D(1)/\langle w_D \rangle$, where $w_D$ is the unique Atkin-Lehner involution acting on $X_0^D(1)$.

**Theorem 5.11.** The automorphism group of the curve $C$ is $\text{Aut}(C) = \mathbb{Z}/2\mathbb{Z}$. 

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**Figure 1.** The dual graph $G'$ of the quotient curve $\mathcal{M}/\langle w_p \rangle$. 

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**Figure 2.** The dual graph $G''$ of the quotient curve $\mathcal{M}/\langle w_p \rangle$. 

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**Figure 3.** The dual graph $G'''$ of the quotient curve $\mathcal{M}/\langle w_p \rangle$. 

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**Figure 4.** The dual graph $G''''$ of the quotient curve $\mathcal{M}/\langle w_p \rangle$.
Proof. By Theorem 5.9, the group $\text{Aut}(C)$ is non-trivial since it contains the hyperelliptic involution. By Lemma 5.7, it injects into an admissible subgroup $H$ of $\text{Aut}(\mathcal{G}_{st})$ of order 2. □

Remark 5.12. By Theorem 5.11, $\text{Aut}(C) = \mathbb{Z}/2\mathbb{Z}$, so that $\text{Aut}(X^D_0(1)) = (\mathbb{Z}/2\mathbb{Z})^s$, with $1 \leq s \leq 2$. We note that $s = 2$ if and only if the hyperelliptic involution on $C$ comes from an exceptional automorphism on $X^D_0(1)$. We also note that it is conjectured that there are only finitely many Shimura curves $X$ defined over $\mathbb{Q}$ such that $\text{Aut}(X)$ contains an exceptional automorphism (see [KR08]). This conjecture would imply that there are very few Shimura curve quotients defined over $\mathbb{Q}$ which have automorphisms arising from exceptional automorphisms. However, analogues of this conjecture have barely been explored over totally real fields.

Theorem 5.13. The curve $C$ is hyperelliptic over $\mathbb{Q}$.

Proof. Since $C$ descends to $\mathbb{Q}$, it is enough to show that the hyperelliptic involution $\iota : C \to C$ also descends to $\mathbb{Q}$. By Theorem 5.11, $\text{Aut}(C)/\langle \iota \rangle$ is trivial. Furthermore, the field $\mathbb{Q}(\zeta_{32})$ is a splitting field for $D$ whose class number is one. So the CM point attached to the extension $\mathbb{Q}(\zeta_{32})/F$ is defined over $F$. So $C$ descends to $\mathbb{Q}$ as a hyperelliptic curve by [SV16, Proposition 4.8]. □

Remark 5.14. One should be able to compute an equation for $C$ by using [VW14]. However, currently, the strategy for doing so is not fully implemented. It should also be possible to use a generalisation of the $p$-adic approach discussed in [FM14], which was inspired by [Kur79, Kur94]. Given that the determination of equations for
Shimura curves defined over totally real fields is one question that is of independent interest in its own right, we hope to return to this in the future.

**Remark 5.15.** We note that Michon [Mic81] (and also unpublished work of Ogg) provides a complete list of all hyperelliptic Shimura curves with square-free level defined over $\mathbb{Q}$. Shimura curves defined over $\mathbb{Q}$ which admit hyperelliptic quotients have also been investigated quite a bit, see for example [Mol12, GM16, GY17] and references therein. In contrast, there has been very little work on these types of questions for Shimura curves defined over totally real fields $F$ larger than $\mathbb{Q}$. This makes Theorem 5.9 of the more striking. Indeed, not only does it give one of the few examples of Shimura curves with a hyperelliptic quotient over a totally real field, but also one whose genus is larger than most known examples over $\mathbb{Q}$.

### 5.7. The Jacobian varieties $\text{Jac}(X_0^D(1))$ and $\text{Jac}(C)$

In this section, we explain the connection between the simple factors of $\text{Jac}(X_0^D(1))$ and the conjectures in [Gro16]. There is more in [CD17], where this connection is established via lifts of Hilbert modular forms.

From the discussion in Subsections 5.3 and 5.4, we have the decomposition for $\text{Jac}(X_0^D(1))$ over $F$ (up to isogeny):

$$\text{Jac}(X_0^D(1)) = A_f \times A_f' \times A_g \times A_g',$$

From (1), and the fact that $w_D = -w$, we see that

$$\text{Jac}(C) = A_f \times A_f' \times A_g \times A_g'.$$

The fourfolds $A_f$ and $A_f'$ (resp. $A_g$ and $A_g'$) are Galois conjugate. We will see later that one of consequences of the compatibility between the base change action and Hecke orbits is that the decompositions (1) and (2) descend to subfields of $F$.

**Theorem 5.16.** The abelian variety $A_h$ descends to a 24-dimensional variety $B_h$ defined over $\mathbb{Q}$, with good reduction outside 2, such that $\text{End}_{\mathbb{Q}}(B_h) \otimes \mathbb{Q} = K_h$ and

$$L(B_h, s) = \prod_{\Pi' \in [\Pi_h]} L(\Pi', s),$$

where $\pi_h$ is the automorphic representation of $\text{GL}_2(\mathbf{A}_F)$ attached to $h$, $\Pi_h$ it lifts to $\text{GSpin}_{17}(\mathbf{A}_\mathbb{Q})$, and $[\Pi_h]$ the Hecke orbit of $\Pi_h$.

**Proof.** By Table 1, there exists a generator $\tau \in \text{Gal}(L_h/K_h)$ such that $^\tau h = h^\tau$. So, by [CD17, Theorem 5.4], $\pi_h$ lifts to an automorphic representation $\Pi_h$ on a split form of $\text{GSpin}_{17}(\mathbf{A}_\mathbb{Q})$, with field of rationality the cubic field $K_h$. The Hecke orbit $[\Pi_h]$ of $\Pi_h$ has 3 elements, and by functoriality

$$L(B_h, s) = \prod_{\Pi' \in [\Pi_h]} L(\Pi', s),$$

It follows that $\text{End}_{\mathbb{Q}}(B_h) \otimes \mathbb{Q} = K_h$. Since the level of the form $h$ is the unique prime $p$ above 2, $B_h$ has good reduction outside 2. \qed

Now, we turn to the quotient $C := X_0^D(1)/\langle w_D \rangle$.

**Theorem 5.17.** The abelian varieties $A_f$ and $A_f'$ (resp. $A_g$ and $A_g'$) descend to pairwise conjugate fourfolds $B_f$ and $B_f'$ (resp. $B_g$ and $B_g'$) over $\mathbb{Q}(\sqrt{2})$, with trivial endomorphism rings, such that

$$L(B_f, s) = L(\Pi_f, s)$$

and $L(B_f', s) = L(\Pi_f', s)$,

$$L(B_g, s) = L(\Pi_g, s)$$

and $L(B_g', s) = L(\Pi_g', s)$,
where $\pi_f, \pi_{f'}, \pi_g$ and $\pi_{g'}$ are the automorphic representations of $GL_2(A_F)$ attached to $f$, $f'$, $g$ and $g'$, respectively; and $\Pi_f, \Pi_{f'}, \Pi_g$ and $\Pi_{g'}$ their respective lifts to $GSpin_9/Q(\sqrt{2})$. They have good reduction outside $(\sqrt{2})$.

**Proof.** The identities in Table 1, combined with [CD17, Theorem 5.4], implies that $\pi_f, \pi_{f'}, \pi_g$ and $\pi_{g'}$ indeed lift to automorphic representations $\Pi_f, \Pi_{f'}, \Pi_g$ and $\Pi_{g'}$ on $GSpin_9/Q(\sqrt{2})$ with field of rationality $Q$. Consequently, the fourfolds $A_f, A_{f'}, A_g$ and $A_{g'}$ descend to pairwise conjugate fourfolds $B_f$ and $B_{f'}$ (resp. $B_g$ and $B_{g'}$) such that

$$\text{End}_{Q(\sqrt{2})}(B_f) = \text{End}_{Q(\sqrt{2})}(B_{f'}) = \text{End}_{Q(\sqrt{2})}(B_g) = \text{End}_{Q(\sqrt{2})}(B_{g'}) = \mathbb{Z}.$$ 

The equalities of $L$-series follow by functoriality. For the same reason as above, the fourfolds have good reduction outside $(\sqrt{2})$. 

**Remark 5.18.** The decomposition (1) is only true a priori over $F$. However, Theorem 5.16 and Theorem 5.17 imply that it descends to $Q(\sqrt{2})$. In fact, the products $A_f \times A_{f'}$ (resp. $A_g \times A_{g'}$) further descend to $Q$. And so, the decomposition (1) will descend to $Q$ if we group them accordingly.

### 5.8. The connectedness of Spec($T$).

Let $T$ be the $\mathbb{Z}$-subalgebra of $\text{End}_{C}(S^D_2(1))$ acting on $S^D_2(1)$. We recall that $S^D_2(1)$ is isomorphic to $S_2(p)^{\text{new}}$ as a Hecke module.

**Proposition 5.19.** Spec($T$) is connected.

**Proof.** The curve $X^D_0(1)$ is a Shimura curve of prime level, and each Hecke constituent appears with multiplicity one. So, the proof in [Maz77, Proposition 10.6] applies readily.

The following two propositions determine the congruences which realise the connectedness of Spec($T$).

**Proposition 5.20.** The forms $f$, $f'$, $g$ and $g'$ are congruent modulo 5.

**Proof.** The prime 5 is totally ramified in $L_f = L_{f'}$. Let $\mathfrak{P}_5$ be the unique prime above it, and $\rho_{f,5}, \rho_{f',5} : \text{Gal}(\overline{Q}/F) \to \text{GL}_2(O_{L_f}, \mathfrak{P}_5)$ the $\mathfrak{P}_5$-adic representations attached to $f$ and $f'$, respectively. By reduction modulo $\mathfrak{P}_5$, we get two representations $\overline{\rho}_{f,5}, \overline{\rho}_{f',5} : \text{Gal}(\overline{Q}/F) \to \text{GL}_2(F_5)$. From Table 1, we have and $\sigma^5 f' = f'$. Also, since $\mathfrak{P}_5$ is totally ramified in $L_f$, we have $\tau(\mathfrak{P}_5) = \mathfrak{P}_5$. It follows that $\overline{\rho}_{f,5} = \overline{\rho}_{f',5}$ is a base change from $Q(\sqrt{2})$. The prime 5 is also totally ramified in $L_g = L_{g'}$. With obvious notations, the same argument as above shows that $\overline{\rho}_{g,5} = \overline{\rho}_{g',5}$ is also a base change from $Q(\sqrt{2})$.

By using the multiplicity one argument in [BCD+18, §6], we show that $\overline{\rho}_{f,5} \simeq \overline{\rho}_{g,5}$. This implies that $f, f', g$ and $g'$ are congruent modulo 5.

**Proposition 5.21.** The forms $f$, $f'$ and $h$ are congruent modulo 3.

**Proof.** There is a unique prime $\mathfrak{P}_3$ above 3 in $L_f = L_{f'}$; it has inertia degree 2 and ramification degree 2. Let $\rho_{f,3}, \rho_{f',3} : \text{Gal}(\overline{Q}/F) \to \text{GL}_2(O_{L_f}, \mathfrak{P}_3)$ the $\mathfrak{P}_3$-adic representations attached to $f$ and $f'$, respectively. By reduction modulo $\mathfrak{P}_3$, we get two representations $\overline{\rho}_{f,3}, \overline{\rho}_{f',3} : \text{Gal}(\overline{Q}/F) \to \text{GL}_2(F_3)$. From Table 1, we have and $\sigma^3 f' = f'$. Also, since $\mathfrak{P}_3$ is the unique prime above 3 in $L_f$, we have $\tau(\mathfrak{P}_3) = \mathfrak{P}_3$. It follows that $\overline{\rho}_{f,3} = \overline{\rho}_{f',3}$ is a base change from $Q(\zeta_3)^7$.

In the cubic subfield $K_h$ of $L_h$, the prime 3 factors as $(3) = p_3^2 p_3'$, where $p_3$ has inertia degree 1, and $p_3'$ inertia degree 2. The prime $p_3'$ is totally ramified in $L_h$. We let $\mathfrak{P}_3'$ be the unique prime above it, and $\rho_{h,3} : \text{Gal}(\overline{Q}/F) \to \text{GL}_2(O_{L_h}, \mathfrak{P}_3')$ the $\mathfrak{P}_3'$-adic representation attached to $h$. By reduction modulo $\mathfrak{P}_3'$, we get a representation $\overline{\rho}_{h,3} : \text{Gal}(\overline{Q}/F) \to \text{GL}_2(F_3)$. From Table 1, we have and $\overline{\sigma}^h h = h$. Also, since $\mathfrak{P}_3'$
is the unique prime above $p_3$ in $L_h$, we have $\tau(\mathfrak{P}_3^\prime) = \mathfrak{P}_3'$. It follows that $\tilde{\rho}_{h,3}$ is also a base change from $\mathbb{Q}(\zeta_{16})^\times$.

By using the multiplicity one argument in [BCD+18, §6], we show that $\tilde{\rho}_{f,3} \cong \tilde{\rho}_{h,3}$. This implies that $f$, $f'$ and $h$ are congruent modulo 3. □

6. The 2-torsion field of $\text{Jac}(X_{16}^D(1))$ and the Harbater field

The main result of this section establishes that every simple factor of $\text{Jac}(X_{16}^D(1))$ has a 2-torsion field whose normal closure is the Harbater field. We start with the following theorem.

**Theorem 6.1.** Let $N$ the field of 2-torsion of $\text{Jac}(C)$ over $\mathbb{Q}$. Then $N$ is the Harbater field.

*Proof.* Keeping the notations in the proof of Theorem 5.9, $N$ is the normal closure of $M$. It follows from this, and direct calculations, that $\text{Gal}(N/\mathbb{Q}) \cong F_{17}$. By construction, $N$ is unramified outside 2 and $\infty$. However, by [Har94, Theorem 2.25], there is a unique Galois number field unramified outside 2 and $\infty$, with Galois group $F_{17}$. So $N$ must be the Harbater field. □

**Remark 6.2.** The field $N$ is the splitting field of the polynomial

$$H := x^{17} - 2x^{16} + 8x^{13} + 16x^{12} - 16x^{11} + 64x^9 - 32x^8 - 80x^7 + 32x^6 + 40x^5 + 80x^4 + 16x^3 - 128x^2 - 2x + 62.$$  

This polynomial was computed by Noam Elkies following a mathoverflow.net [ER15] discussion initiated by Jeremy Rouse. We thank David P. Roberts for bringing this discussion to our attention.

6.1. The mod 2 Hecke eigensystems. Let $T_f$, $T_f'$, $T_g$, $T_g'$ and $T_h$ be the $\mathbb{Z}$-subalgebras acting on the Hecke constituents of $f$, $f'$, $g$, $g'$ and $h$ respectively.

From the discussion in Subsection 5.3, we have

$$T \otimes \mathbb{Q} = (T_f \otimes \mathbb{Q}) \times (T_f' \otimes \mathbb{Q}) \times (T_g \otimes \mathbb{Q}) \times (T_g' \otimes \mathbb{Q}) \times (T_h \otimes \mathbb{Q}) = L_f \times L_{f'} \times L_g \times L_{g'} \times L_h.$$

By direct calculations, we get the following:

- $[O_{L_f} : T_f] = [O_{L_{f'}} : T_{f'}]$ divides 3,
- $[O_{L_g} : T_g] = [O_{L_{g'}} : T_{g'}] = 1$,
- $[O_{L_h} : T_h]$ divides $3 \cdot 5^6$.

Therefore $T \otimes \mathbb{Z}_2$ decomposes into $\mathbb{Z}_2$-algebras as

$$T \otimes \mathbb{Z}_2 = (T_f \otimes \mathbb{Z}_2) \times (T_f' \otimes \mathbb{Z}_2) \times (T_g \otimes \mathbb{Z}_2) \times (T_g' \otimes \mathbb{Z}_2) \times (T_h \otimes \mathbb{Z}_2).$$

The prime 2 is inert in $L_f = L_{f'}$, and $L_g = L_{g'}$, so the first four factors are local $\mathbb{Z}_2$-algebras. Let $m_f$, $m_{f'}$, $m_g$ and $m_{g'}$ be the corresponding maximal ideals. Then, by the identities in Table 1, we have $\sigma(m_f) = m_{f'}$ and $\sigma^2(m_f) = \tau_f(m_f)$ for some $\tau_f \in \text{Gal}(\mathbb{F}_{16}/\mathbb{F}_2)$; and $\sigma(m_g) = m_{g'}$ and $\sigma^2(m_g) = \tau_g(m_g)$ for some $\tau_g \in \text{Gal}(\mathbb{F}_{16}/\mathbb{F}_2)$. We let $\theta_f, \theta_{f'}, \theta_g, \theta_{g'} : T \otimes \mathbb{Z}_2 \to \mathbb{F}_{16}$ be the corresponding mod 2 Hecke eigensystems.

Next, we recall that $L_h$ is the ray class field of conductor $c = (\frac{1}{2} (c^2 - 16c + 25))$ over the field $K_h = \mathbb{Q}(\zeta_c)$, with $c^3 + c^2 - 229c + 167 = 0$. The prime 2 is totally ramified in $K_h$. Letting $p_2 = \mathcal{P} \mathcal{P}'$, where $\mathcal{P}$ and $\mathcal{P}'$ are inert primes, and $\tau(\mathcal{P}) = \mathcal{P}'$. Therefore, there are two maximal ideals $m_h$ and $m'_h$ in $T_h \otimes \mathbb{Z}_2$ such that $\sigma(m_h) = m'_h$ and $\sigma^2(m_h) = \tau_h(m_h)$. We let $\theta_h, \theta'_h : T \otimes \mathbb{Z}_2 \to \mathbb{F}_{16}$ be the resulting two mod 2 Hecke eigensystems.
Proposition 6.3. The forms $f, f', g, g'$ and $h$ give rise to two mod 2 Hecke eigensystems $\theta$ and $\theta'$ that $\theta' = \theta \circ \sigma$ and $\theta \circ \sigma^2 = \bar{\tau} \circ \theta$, where $\text{Gal}(F_{16}/F_2) = \langle \bar{\tau} \rangle$.

Up to relabelling, we have $\theta = \theta_f = \theta_g = \theta_h$, and $\theta' = \theta_f' = \theta_g' = \theta_h'$. 

Proof. We will apply the multiplicity one argument in [BCD18, §6] to deduce that, up to relabelling, $\theta_f = \theta_g = \theta_h$, and $\theta_f' = \theta_g' = \theta_h'$. Let $M$ be the underlying $F_2$-module to $T \otimes F_2$. Then, the pair $(\theta_f, \theta_f')$ comes from two simple Hecke constituents of dimension 4 over $F_2$ that are conjugate by the action of $\text{Gal}(F/Q)$. These Hecke constituents belong to the socle $S$ of $M$, i.e. the largest semi-simple $T \otimes F_2$-submodule of $M$. Likewise for the pairs $(\theta_g, \theta_g')$ and $(\theta_h, \theta_h')$. Let $T'$ be the $Z$-subalgebra of $T$ generated by the Hecke operators $T_p$, with $Np \leq 1000$. We view $M$ as a $T' \otimes F_2$-module, and let $S'$ be its socle. By direct calculations in magma, we show that $S'$ has two irreducible constituents, and each constituent has dimension 4 and multiplicity one. Furthermore, each of those constituents decomposes into 4 one-dimensional Hecke constituents over $T' \otimes F_{16}$. This means that $S'$ must necessarily be the socle of $M$ viewed as a $T' \otimes F_2$-module, and that its $T' \otimes F_{16}$-decomposition is also the $T \otimes F_{16}$-decomposition of $S$. By comparing these one-dimensional $F_{16}$-valued Hecke eigensystems with the reduction modulo 2 of the Hecke eigenvalues of the newforms in $S_{2}^{\mathbb{Q}}(1)$, we see that $\theta_f = \theta_g = \theta_h$, and $\theta_f' = \theta_g' = \theta_h'$, up to relabelling. The identities $\theta' = \theta \circ \sigma$ and $\theta \circ \sigma^2 = \bar{\tau} \circ \theta$ follow from the relations between the forms. □

6.2. The fields of 2-torsion of the simple factors of $\text{Jac}(X_{0}^{H}(1))$.

Theorem 6.4. Let $A$ be a simple factor of $\text{Jac}(X_{0}^{H}(1))$, and $M = F(A[2])$ the field of 2-torsion of $A$. Then, the normal closure of $M$ is the Harbater field $N$.

We will give two proofs of this result, starting with the simplest one.

First proof of Theorem 6.4. In light of Proposition 6.3, it is enough to prove this for the simple factors of $\text{Jac}(C)$. To this end, recall that

$$\text{Jac}(C) = A_f \times A_{f'} \times A_g \times A_{g'}.$$ 

So the field of 2-torsion of $\text{Jac}(C)$ is the compositum of the field of 2-torsion of its simple factors. But, again by Proposition 6.3, $A_f$ and $A_g$ have the same field of 2-torsion. It is the field $M_\theta$ cut out by the Galois representation attached to the Hecke eigensystem $\theta$. Similarly, $A_{f'}$ and $A_{g'}$ have the same field of 2-torsion, the field $M_{\theta'}$ cut out by the Galois representation attached to $\theta'$. Since $\theta$ and $\theta'$ are interchanged by $\text{Gal}(F/Q)$, we must have $M_{\theta'} = M_{\theta}$. Therefore $M_\theta$ and $M_{\theta'}$ have the same normal closure, and it must equal $N$ by Theorem 6.1. □

For the second proof, we need the following result.

Proposition 6.5. Let $\theta$ and $\theta'$ be the Hecke eigensystems in Proposition 6.3, and $\bar{\rho}, \bar{\rho} : \text{Gal}(\overline{Q}/F) \to \text{GL}_2(F_{16})$ the mod 2 Galois representations attached to them. Then, there are characters $\chi, \chi'$ : $\text{Gal}(\overline{Q}/K) \to F_{2}^{\times}$, with trivial conductor such that $\bar{\rho} = \text{Ind}_{K}^{F} \chi$ and $\bar{\rho}' = \text{Ind}_{K}^{F} \chi'$. 

Proof. We already computed the Hecke constituents of the space $S_{2}(1)$ in [Dem09]. The mod 2 Hecke eigensystems in that case have coefficient fields $F_{2^s}$ where $s = 1, 2, 8$. Therefore, since $\theta$ has coefficient field $F_{16}$, it cannot arise from an eigenform of level 1. By the Serre conjecture for totally real fields (the totally ramified case) [GS11], it must appear on the quaternion algebra $D'$ with level (1) and non-trivial weight. The same is true for $\theta'$. In fact, the analysis conducted in the proof of Proposition 6.3 also shows that they are the only eigensystems that can appear at that weight. (We note that there are only two Serre weights in this case.)
Let $\chi : \text{Gal}(\overline{Q}/K) \to \mathbb{F}_2^\times$ be a character with trivial conductor such that $\chi^4 \neq \chi$, where $\text{Gal}(K/F) = \langle s \rangle$. By class field theory, we can identify $\chi$ with its image under the Artin map. Since $\chi$ is unramified, it must factor as $\chi : K^\times \backslash \mathcal{A}_K^\times / \text{Cl}_K \to \mathbb{F}_2^\times$. Furthermore, since $\text{Cl}_K \cong \mathbb{Z}/17\mathbb{Z}$, we must have $\chi : K^\times \backslash \mathcal{A}_K^\times \to \mathbb{F}_2^\times$, and the representation $\bar{\rho}_\chi := \text{Ind}_{F}^{K} \chi : \text{Gal}(\overline{Q}/F) \to \text{GL}_2(\mathbb{F}_{16})$ has coefficients in $\mathbb{F}_{16}$. So, $\bar{\rho}_\chi$ has level (1) and non-trivial weight by the argument above. Therefore, it must be isomorphic to a Galois conjugate of $\bar{\rho}$. Up to relabelling, we can assume that $\bar{\rho} \simeq \bar{\rho}_\chi$. Since $\theta$ and $\theta'$ are $\text{Gal}(F/Q)$-conjugate, there is also a character $\chi' : K^\times \backslash \mathcal{A}_K^\times \to \mathbb{F}_2^\times$ such that $\bar{\rho}' \simeq \bar{\rho}_{\chi'}$.

Alternatively, we can show that $\theta$ appears on $D'$ with the non-trivial weight without using the fact that it has coefficients in $\mathbb{F}_{16}$. Indeed, we have

$$\bar{\rho}_\chi|_{D} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$  

Let $K_0$ be the completion of $K$ at $\mathfrak{p}$, the unique prime above $p$. Since $K = F[\beta]$, and $\beta^2 = -2 - \alpha$ is a generator of $\mathfrak{p}$, then we have $K_0 = F_p[\sqrt{\alpha}]$, where $\alpha$ is a unimodular of $F_p$. Therefore, $\bar{\rho}_\chi|_{D} \simeq \bar{\rho}_{\chi'}$ doesn’t arise from a finite flat group scheme. Hence, $\bar{\rho}_\chi$ must have non-trivial weight. \hfill $\square$

We are now ready for the second proof of Theorem 6.4.

Second proof of Theorem 6.4. Let $\bar{\rho}_\theta, \bar{\rho}_{\theta'} : \text{Gal}(\overline{Q}/F) \to \text{GL}_2(\mathbb{F}_{16})$ be the mod 2 Galois representations attached to the eigensystems $\theta$ and $\theta'$. By Proposition 6.5, $\bar{\rho}_\theta$ and $\bar{\rho}_{\theta'}$ are dihedral and we have that $\text{im}(\bar{\rho}_\theta) = \text{im}(\bar{\rho}_{\theta'}) = D_{17}$. Let $M_\theta, M_{\theta'}$ be the fields cut out by $\bar{\rho}_\theta$ and $\bar{\rho}_{\theta'}$; and $N_\theta$ and $N_{\theta'}$ the normal closure of $M_\theta$ and $M_{\theta'}$, respectively. By Proposition 6.3, we have $M_{\theta'} = M_{\theta'}^\sigma$, hence $N_{\theta} = N_{\theta'}$. Also, by construction $M_\theta$ and $M_{\theta'}^\sigma$ are unramified extension of $K$. So, by uniqueness of the Hilbert class field, we must have $M_\theta = M_{\theta'}^\sigma = M_\beta M_{\theta'}^\sigma = H_K$, where $M_\beta M_{\theta'}^\sigma$ is the compositum of $M_\theta$ and $M_{\theta'}^\sigma$; and $H_K$ is the Hilbert class field of $K$. Since $\theta \circ \sigma^2 = \overline{\theta} \circ \theta$, we have

$$\text{Gal}(N_\theta/Q) = D_{17} \rtimes \mathbb{Z}/8\mathbb{Z} = F_{17}.$$  

Again by [Har94, Theorem 2.25], we must have $N = N_\theta = N_{\theta'}$. \hfill $\square$

Remark 6.6. From Theorem 6.4, we see that none of the fourfolds $A_f, A_{f'}, A_g$ or $A_{g'}$ can be the Jacobian of a hyperelliptic curve since the action of $\text{Gal}(\overline{Q}/F)$ on the points of 2-torsion cannot factor through $S_{10}$ (see Subsection 2.3). However, as we explained earlier, $A_f, A_{f'}, A_g$ and $A_{g'}$ descend, separately, into pairwise conjugate abelian varieties over $\mathbb{Q}(\sqrt{2})$. And the products $A_f \times A_{f'}$ and $A_g \times A_{g'}$ are 8-dimensional abelian varieties which further descend to $Q$. So, we conclude with the following questions. Do there exist hyperelliptic curves $C_f$ and $C_g$ defined over $F$ such that, up to isogeny, we have

$$\text{Jac}(C_f) = A_f \times A_{f'}, \text{ and } \text{Jac}(C_g) = A_g \times A_{g'}.$$  

If so, do these two curves descend to $Q$ as well? We were asked these two questions by Noam Elkies in an email. An affirmative answer to them would mean that the Harbater field is given by a curve of genus 8, which is much smaller. In that case, the hyperelliptic polynomials will have degree 17, the same as that of the Elkies polynomial displayed earlier.
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