THE FAILURE OF THE FRACTAL UNCERTAINTY PRINCIPLE FOR
THE WALSH–FOURIER TRANSFORM

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Abstract. We construct \( \delta \)-regular sets with \( \delta \geq \frac{1}{2} \) for which the analog of the Bourgain–Dyatlov Fractal Uncertainty Principle fails for the Walsh–Fourier transform.

1. The Fractal Uncertainty principle for the Fourier transform

This note explores the so-called Fractal Uncertainty principle, a fundamental result in Fourier analysis with far-reaching consequences in the spectral theory of hyperbolic surfaces.

Definition 1.1. Let \( X \subset \mathbb{R} \) be a nonempty closed set. Consider the constants \( \delta \in [0, 1) \), \( C_R \geq 1 \) and \( 0 \leq \alpha_0 \leq \alpha_1 \leq \infty \). We say that \( X \) is \( \delta \)-regular with constant \( C_R \) on scales \( \alpha_0 \) to \( \alpha_1 \) if there is a Borel measure \( \mu_X \) supported on \( X \) such that

- for each interval \( I \) of size \( |I| \in [\alpha_0, \alpha_1] \), we have \( \mu_X(I) \leq C_R|I|^\delta \)
- if additionally \( I \) is centered at a point in \( X \), then \( \mu_X(I) \geq C_R^{-1}|I|^\delta \).

We will denote by \(|X|\) the Lebesgue measure of \( X \).

Examples of regular sets will be discussed in Section 3. At this point, we only mention that \( \delta \)-regular sets need to have small Lebesgue measure, more precisely (see Lemma 2.9 in [2])

\[
|X| \leq 24C_R^2\epsilon_1^{1-\delta}. \tag{1}
\]

The following Fractal Uncertainty principle for the Fourier transform

\[
\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x \xi} dx
\]

was proved in [2]. It refines earlier versions due to Dyatlov-Zahl [3] and Bourgain–Dyatlov [1].

Theorem 1.2. Let \( \delta \in [0, 1) \), \( C_R \geq 1 \) and \( N \geq 1 \). Assume that

- \( X \subset [0, 1] \) is \( \delta \)-regular with constant \( C_R \) on scales \( \frac{1}{N} \) to 1
- \( Y \subset [0, N] \) is \( \delta \)-regular with constant \( C_R \) on scales 1 to \( N \).

Then there exist constants \( \beta > 0 \) and \( C \), both depending only on \( \delta \) and \( C_R \), such that for each \( f \in L^2(\mathbb{R}) \) with Fourier transform supported on \( Y \) we have

\[
\|f\|_{L^2(X)} \leq CN^{-\beta}\|f\|_{L^2(\mathbb{R})}. \tag{2}
\]

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When $\delta < \frac{1}{2}$, this theorem has an easy proof that also provides an explicit value for $\beta$. For reader’s convenience, we recall this argument below. If $\hat{f}$ is supported on $Y$ we have

\[
\|f\|_{L^2(X)} \leq |X|^{1/2} \|f\|_{L^\infty(\mathbb{R})}
\]

\[
\leq |X|^{1/2} \|\hat{f}\|_{L^1(\mathbb{R})}
\]

\[
= |X|^{1/2} \|\hat{f}\|_{L^1(Y)}
\]

\[
\leq |X|^{1/2} |Y|^{1/2} \|\hat{f}\|_{L^2(\mathbb{R})}
\]

\[
= |X|^{1/2} |Y|^{1/2} \|f\|_{L^2(\mathbb{R})}.
\]

If $X$ and $Y$ are as in the theorem, then (1) implies that $|X|^{1/2} |Y|^{1/2} \leq C N^{-\beta}$, $\beta = \frac{1}{2} - \delta$.

On the other hand, the proof from [2] in the case $\delta \geq \frac{1}{2}$ is very involved. At its heart, it relies both on the multi-scale structure of regular sets, and on the following unique continuation result (Lemma 3.2 in [2]).

**Lemma 1.3.** Let $I$ be a non overlapping collection of intervals of size 1 and let $c_0 > 0$. For each $I \in \mathcal{I}$, let $I'' \subset I$ be an interval of size $c_0$. Then there exists a constant $C$ depending only on $c_0$ such that for all $r \in (0, 1)$, $0 < \kappa \leq e^{-C/r}$ and $f \in L^2(\mathbb{R})$ with $\hat{f}$ compactly supported, we have

\[
\sum_{I \in \mathcal{I}} \|f\|_{L^2(I')}^2 \leq C \left( \sum_{I \in \mathcal{I}} \|f\|_{L^2(I'')}^2 \right)^{\kappa} \|e^{2\pi r |\xi|} |\hat{f}(\xi)|\|_{L^2(\mathbb{R})}^{2(1-\kappa)}.
\]

In the next section we recall the details about the Walsh transform, a closely related, though technically simpler analog of the Fourier transform. We will construct sets $X$ and $Y$ as in Theorem 1.2 with regularity $\delta \geq \frac{1}{2}$, such that the Fractal Uncertainty Principle fails when the Walsh transform replaces the Fourier transform. This fundamental difference between the behavior of the two transforms explains why the proof in [2] is so complicated. The argument in [2] must necessarily rely not just on the fine structure of the regular sets, but also on the stronger form of the Uncertainty Principle that governs the Fourier world. This has to do with the fact that there is no (nontrivial) compactly supported function whose Fourier transform is also compactly supported. Lemma 1.3 is a manifestation of this principle.

In the next section we will see that there are compactly supported functions whose Walsh transforms are also compactly supported. This easily shows the failure of Lemma 1.3 and ultimately of Theorem 1.2 in the Walsh framework. Our main result, Theorem 3.1 is proved in the last section.

2. **The Walsh transform**

Let $\mathbb{Z}_2 = \{-1, 1\}$ with addition modulo 2 and Haar measure splitting the mass evenly between $-1$ and 1. We consider the infinite product group $G = \prod_1^\infty \mathbb{Z}_2$ equipped with the product Haar measure. This is sometimes referred to as the Cantor group.

Let $\mathcal{D} = \{j 2^{-i} : 0 \leq j \leq 2^i\}$ be the dyadic numbers in $[0, 1]$. They have zero Lebesgue measure. The map

\[
\Phi : G \to [0, 1], \quad \Phi(a_1, a_2, \ldots) = \sum_{k=-\infty}^{1} a_k 2^k
\]
is almost bijective – if \( x \in [0, 1] \setminus D \), \( \Phi^{-1}\{\{x\}\} \) consists of one point – measurable, and maps the Haar measure on \( G \) to the Lebesgue measure \(| \cdot |\) on \([0,1]\). This suggests a natural way to identify \( G \) with \(([0,1], \oplus, | \cdot |)\), where \( \oplus \) is defined as follows. Given \( x, y \in [0, 1] \setminus D \),

\[
x \oplus y = \sum_{k \leq -1} c_k 2^k, \quad c_k = x_k + y_k \pmod{2}.
\]

See Sec 2.2 in [5] for details.

The characters on \( G \) are the so-called Walsh functions. For \( n \geq 0 \) the \( n \)-th Walsh function \( W_n : [0, 1) \to \{-1, 1\} \) is defined recursively by the formula

\[
W_0 = 1_{[0,1)}
\]

\[
W_{2n}(x) = W_n(2x) + W_n(2x-1)
\]

\[
W_{2n+1}(x) = W_n(2x) - W_n(2x-1).
\]

In particular,

\[
W_1(x) = \begin{cases} 1, & 0 \leq x < \frac{1}{2} \\ -1, & \frac{1}{2} \leq x < 1 \end{cases}
\]

\[
W_2(x) = \begin{cases} 1, & x \in [0, \frac{1}{4}) \cup [\frac{3}{4}, \frac{1}{2}) \\ -1, & x \in [\frac{1}{4}, \frac{1}{2}) \cup [\frac{3}{4}, 1) \end{cases}
\]

\[
W_3(x) = \begin{cases} 1, & x \in [0, \frac{1}{4}) \cup [\frac{3}{4}, 1) \\ -1, & x \in [\frac{1}{4}, \frac{3}{4}) \end{cases}
\]

In many ways, the functions \( W_n \) resemble the (Fourier) system of exponentials \( e^{2\pi inx} \). For example, the functions \((W_n)_{n \geq 0}\) form an orthonormal basis for \( L^2([0,1]) \). See Sec 4.1 [5] for more details.

The Walsh–Fourier coefficients of a function \( f : [0, 1] \to \mathbb{C} \) are given by

\[
\mathcal{F}_{W} f(n) = \int f(x) W_n(x) dx, \quad n \geq 0.
\]

To get a greater perspective on the role of the Walsh system and its closeness to the Fourier system of exponentials, we introduce a new operation. For \( x, y \in [0, \infty) \) having unique representations (that is, for Lebesgue almost all pairs \((x, y)\))

\[
x = \sum_{k=-\infty}^{\infty} x_k 2^k, \quad y = \sum_{k=-\infty}^{\infty} y_k 2^k,
\]

we define

\[
x \otimes y := \sum_{k=-\infty}^{\infty} c_k 2^k
\]

where

\[
c_k = \sum_{j \in \mathbb{Z}} x_j y_{k-j} \pmod{2}.
\]

We note that this sum is always finite. From now on, we will implicitly ignore the zero measure dyadic points.
Define the function \( e_W : [0, \infty) \to \{-1, 1\} \) such that \( e_W(x) = 1 \) when \( x_{-1} = 0 \) and \( e_W(x) = -1 \) when \( x_{-1} = 1 \). This 1-periodic function is the Walsh analogue of \( e^{2\pi i x} \). It is easy to check that
\[
W_n(x) = e_W(x \otimes n)1_{[0,1]}(x). \tag{3}
\]

We may introduce the Walsh (also called Walsh-Fourier) transform of a compactly supported function \( f : [0, \infty) \to \mathbb{C} \) to be the function
\[
\mathcal{F}_W f : [0, \infty) \to \mathbb{C}, \quad \mathcal{F}_W f(y) := \int_{(0,\infty)} e_W(x \otimes y) f(x)dx.
\]
The Walsh–Fourier inversion formula takes the form \( \mathcal{F}_W \circ \mathcal{F}_W = id \).

It is worth noting that
\[
e_W(x \otimes y) = e_W(x \otimes z)
\]
whenever \( x \in [0, 1) \) and \( n \leq y, z < n + 1 \). Consequently, if \( f \) is supported on \([0, 1]\) then \( \mathcal{F}_W f \) is constant on intervals \([n, n + 1)\). This explains why for such functions the Walsh–Fourier coefficients completely characterize the function \( f \).

While the Walsh transform behaves very similar to the Fourier transform, it has one notable feature that makes it easier to work with. This has to do with the fact that there are (plenty of) compactly supported functions whose Walsh transforms are also compactly supported. A quick computation shows that for each dyadic interval \( I = [l2^k, (l + 1)2^k) \) we have
\[
\mathcal{F}_W 1_I(y) = |I|1_{[0,|I|^{-1}]}(y)e(x_I \otimes y), \tag{4}
\]
where \( x_I \) is an arbitrary element of \( I \). Because of this feature, typically the results that hold in the Fourier case are expected to also hold in the Walsh setting, with the argument in the latter case being cleaner, less technical. The approach of first proving results in the Walsh setting and then “transferring” them to the Fourier world was successfully employed in the time-frequency analysis of modulation invariant operators, starting with \([6]\). The interested reader may consult the survey paper \([3]\), which explores a few different arguments for the Walsh analog of Carleson’s Theorem and contains some relevant references.

In this paper we present an example that goes against the aforementioned philosophy. We show that a fundamental result that holds for the Fourier transform is in fact false for the Walsh transform.

### 3. The main result

The “textbook” example of regular sets can be constructed as follows. Fix integers \( 0 < M < L \). Let \( S \) be a collection of subsets \( S \) of \( \{0, 1, \ldots, L - 1\} \) with cardinality \( M \). We create a collection of nested sets \( X_1, X_2, \ldots \) as follows. Pick \( S_1 \in S \) and let
\[
A_1 = L^{-1}S_1, \quad X_1 = A_1 + [0, L^{-1}].
\]
Next, for each \( a \in A_1 \), choose some \( S_{2,a} \in S \) and define
\[
A_{2,a} = a + L^{-2}S_{2,a}, \quad A_2 = \bigcup_{a \in A_1} A_{2,a}, \quad X_2 = A_2 + [0, L^{-2}].
\]
The rest of the construction is recursive. Assume we have constructed \( A_j \) and \( X_j \) for \( 1 \leq j \leq n - 1 \). For each \( a \in A_{n-1} \), choose some \( S_{n,a} \in S \) and define
\[
A_{n,a} = a + L^{-n}S_{n,a}, \quad A_n = \bigcup_{a \in A_{n-1}} A_{n,a}, \quad X_n = A_n + [0, L^{-n}].
\]
Note that \(X_n \subset [0,1]\) consists of \(M^n\) intervals \(I \in \mathcal{I}_{X_n}\) of length \(L^{-n}\). Also, \(X_n\) is 
\[
\frac{\log M}{\log L} - \text{regular on scales } \frac{1}{L^n} \text{ to } 1, \text{ with constant } C_n \text{ satisfying the uniform bound } C_n \leq C(M, L), \text{ where } C(M, L) \text{ depends only on } M, L.
\]
The reader may check that Definition \ref{def:regular} is satisfied with the measure \(\mu_{X_n}\) given by 
\[
\mu_{X_n}(I) = \frac{1}{M^n}, \text{ for each } I \in \mathcal{I}_{X_n}.
\]
We specialize this construction as follows. Fix the positive integers \(m_1\) and \(m_2 \geq m_1\). We consider a set as above with \(M = 2^{m_2}\) and \(L = 2^{m_1+m_2}\). The collection \(\mathcal{S}\) will consist of only the set \(S = \{k2^{m_1}, 0 \leq k \leq 2^{m_2} - 1\}\).

More precisely, define 
\[
A_n = \left\{ \sum_{i=1}^{n} \frac{k_{n-i+1}2^{m_1}}{L^i} : 0 \leq k_1, \ldots, k_n \leq 2^{m_2} - 1 \right\}
\]
and 
\[
X_n = A_n + [0, L^{-n}]. \tag{5}
\]
Then \(X_n \subset [0,1]\) is \(\frac{m_2}{m_1+m_2}\)-regular on scales \(L^{-n}\) to 1, with constant \(C_n\) uniformly bounded in \(n\).

Define also the dilate 
\[
Y_n = L^n X_n = \{ L^n x : x \in X_n \}.
\]
Note that \(Y_n\) is the union of intervals of length 1 and \(Y_n \subset [0, L^n]\). It is \(\frac{m_2}{m_1+m_2}\)-regular on scales 1 to \(L^n\), with the same constant \(C_n\) as \(X_n\).

\textbf{Theorem 3.1.} The (real) vector space \(V_{X_n,Y_n}\) of all functions 
\[
f : [0,1] \to \mathbb{R}, \text{ supp } f \subset X_n, \text{ supp } \mathcal{F}_W f \subset Y_n
\]
has dimension at least \(2^n(m_2-m_1)\). In particular, for each \(n \geq 1\) there is a function \(f_n\) (other than the zero function) with \(\mathcal{F}_W f_n\) supported on \(Y_n\) such that 
\[
\|f_n\|_{L^2(X_n)} = \|f_n\|_{L^2([0,1])}.
\]

Fixing \(m_1, m_2\) and letting \(n \to \infty\) shows that the Walsh analog of (2) fails to hold for any \(\beta > 0\), when \(\delta \geq \frac{1}{2}\).

We remark that the restriction \(m_2 \geq m_1\) is needed in Theorem 3.1, as it is equivalent with the lower bound \(\delta \geq \frac{1}{2}\) for the regularity of \(X_n, Y_n\). When \(\delta < \frac{1}{2}\), Theorem \ref{thm:regular} remains true in the Walsh framework and the argument from the first section for the Fourier case translates to the Walsh case, too.

4. PROOFS

We start by proving a sequence of lemmas.

\textbf{Lemma 4.1.} For \(x, y \in [0, \infty)\) and \(l \in \mathbb{Z}\) we have
\[
(2^l x) \otimes y = x \otimes (2^l y).
\]

\textit{Proof.} If 
\[
x = \sum_{k \in \mathbb{Z}} x_k 2^k, \quad y = \sum_{k \in \mathbb{Z}} y_k 2^k
\]
then 
\[
2^l x = \sum_{k \in \mathbb{Z}} x_{k-l} 2^k, \quad 2^l y = \sum_{k \in \mathbb{Z}} y_{k-l} 2^k
\]
and
\[(2^j x \otimes y)_k = \sum_{j \in \mathbb{Z}} (2^j x)_j y_{k-j} = \sum_{j \in \mathbb{Z}} x_{j-l} y_{k-j} = \sum_{j \in \mathbb{Z}} x_j y_{k-j-l} = (x \otimes (2^l y))_k.\]

Combining this lemma with (3) and (4) reveals that if \(I = \left[\frac{k}{L^n}, \frac{k+1}{L^n}\right] \subset [0, 1]\) then
\[
\mathcal{F}_{W^1_I}(y) = L^{-n}W_k(y)_{L^n}.
\]

Lemma 4.2. The functions \(W_0, W_1, \ldots, W_{2^m-1}\) span the vector space
\[
C_m = \{ f : [0, 1] \to \mathbb{R} : f \text{ constant on dyadic intervals of length } 2^{-m}\}.
\]

Proof. An easy induction argument based on the recursive formula for \(W_n\) shows that \(W_0, W_1, \ldots, W_{2^m-1} \in C_m\). The vector space \(C_m\) has dimension \(2^m\), and since \(W_0, W_1, \ldots, W_{2^m-1}\) are linearly independent (being orthogonal), they form a basis for this space. \(\square\)

The recursive definition of \(W_n\) also immediately implies the following periodicity property.

Lemma 4.3. The function \(W_{k2^l}\) is \(2^{-l}\) periodic, if \(k, l\) are positive integers. Moreover, when \(x \in [0, 2^{-l}]\) we have
\[W_{k2^l}(x) = W_k(x2^l)\]

The combination of the last two lemmas yields the following result.

Proposition 4.4. Consider the (real) vector space of all \(F : [0, 1] \to \mathbb{R}\) having the following two properties for some positive integers \(l, m\)

\((P1): F\) is \(2^{-l}\) periodic

\((P2): F\) is constant on dyadic intervals of length \(2^{-l-m}\).

Then this vector space coincides with the span of the Walsh functions \(W_{k2^l}\), for \(0 \leq k \leq 2^m - 1\).

Let us recall that \(L = 2^{m_1+m_2}\). Rescaling the above result gives:

Corollary 4.5. For \(1 \leq i \leq n\), consider the (real) vector space \(V_{i,n}\) of all functions \(F_i : [0, L^n] \to \mathbb{R}\) such that

\((P1): F_i\) is \(\frac{L^i}{2^{m_1}}\) periodic

\((P2): F_i\) is constant on dyadic intervals of length \(L^{i-1}\).

Then \(V_{i,n}\) coincides with the span of the rescaled Walsh functions \(W_{kL^n-i2^{m_1}}(\frac{y}{L^n})\), for \(0 \leq k \leq 2^{m_2} - 1\).
Let $\mathcal{V}_{X_n}$ be the (real) vector space spanned by the Walsh transforms $\mathcal{F}_{W}1_{I}$ of all intervals $I$ of length $L^{-n}$ in $X_n$. According to (5) and (6) this is the same as the vector space spanned by the rescaled Walsh functions

$$W_{\sum_{i=1}^{n} k_{n-i+1}2^{m_1}L^{-i}} \left( \frac{y}{L^n} \right) : 0 \leq k_1, \ldots, k_n \leq 2^{m_2} - 1.$$  

Note that $\mathcal{V}_{X_n}$ is a proper subset of the family of Walsh transforms of functions supported on $X_n$. We are going to search for functions in $\mathcal{V}_{X_n}$ that are supported on $Y_n$.

**Lemma 4.6.** For each $k, k' \in \mathbb{Z}$

$$W_k W_{k'} = W_{k \oplus k'}.$$  

**Proof.**

$$W_k(x)W_{k'}(x) = (-1)^{(x \oplus k)_{-1}}(-1)^{(x \oplus k')_{-1}} = (-1) \sum_{j} x_j^{k_{-1-j}}(-1) \sum_{j} x_j^{k'_{-1-j}}$$

$$= (-1) \sum_{j} x_j^{(k \oplus k')_{-1-j}} = W_{k \oplus k'}(x).$$

\[\square\]

Combining the last lemma and corollary we get:

**Proposition 4.7.** The space $\mathcal{V}_{X_n}$ coincides with the collection of arbitrary finite sums of products of arbitrary functions $F_i \in \mathcal{V}_{i,n}$.

**Proof.** Note that since $k_{n-i+1}2^{m_1} < L$ we have

$$\sum_{i=1}^{n} k_{n-i+1}2^{m_1}L^{-i} = \bigoplus_{i=1}^{n} k_{n-i+1}2^{m_1}L^{-i},$$

where the factors on the right hand side are summed using $\oplus$ rather than $+$. Thus

$$W_{\sum_{i=1}^{n} k_{n-i+1}2^{m_1}L^{-i}} \left( \frac{y}{L^n} \right) = \prod_{i=1}^{n} W_{k_{n-i+1}2^{m_1}L^{-i}} \left( \frac{y}{L^n} \right).$$

\[\square\]

We now prove Theorem 3.1 by induction. It suffices to show that the vector space of those $F$ supported on $Y_n$, that are in the span of the rescaled Walsh functions in (7), has dimension at least $2^{n(m_2-m_1)}$.

Let us start with the base case $n = 1$. Using the characterization from Proposition 4.7 it suffices to prove that the vector space

$$\{ F \in \mathcal{V}_{1,1} : \text{supp} \ F \subset Y_1 \}$$

has dimension $2^{m_2-m_1}$. The functions $F$ in this space are $2^{m_2}$ periodic and constant on all intervals $[l, l+1)$. Since $Y_1$ contains exactly $2^{m_2-m_1}$ unit intervals in $[0, 2^{m_2}]$ (these are $I_k = [k2^{m_1}, k2^{m_1} + 1], 0 \leq k \leq 2^{m_2-m_1} - 1$), and since

$$Y_1 = \bigcup_{0 \leq k' \leq 2^{m_1} - 1} ((I_0 \cup I_1 \cup \ldots \cup I_{2^{m_2-m_1} - 1}) + k'2^{m_2}),$$

it is immediate that the values of $F$ on $I_0 \ldots I_{2^{m_2-m_1}-1}$ may be chosen arbitrarily. This verifies the base case of the induction.
Next, let us prove the theorem for \( n \geq 2 \), assuming its validity for \( n - 1 \). We write
\[
Y_n = L Y_{n-1} \cap Z_n, \quad Z_n = \bigcup_{k \leq L^n} [k 2^{m_1}, k 2^{m_1} + 1].
\]

Let \( V_1 \) be the vector space of those \( F_1 \in V_{1,n} \) that are supported on \( Z_n \). Note first that this has dimension \( 2^{m_2 - m_1} \), since there are \( 2^{m_2 - m_1} \) unit intervals in \( Z_n \) that lie in the periodicity interval \([0, 2^{m_2}]\) associated with \( V_{1,n} \). Pick \( 2^{m_2 - m_1} \) functions \( H \) in the span of \( W_{k, L^{n-1} 2^{m_1}} \), with \( 0 \leq k \leq 2^{m_2} - 1 \), such that the rescaled functions \( H(\frac{y}{L^n}) \) form a basis for \( V_1 \).

By the induction hypothesis, we may find a subset consisting of \( 2^{(n-1)(m_2 - m_1)} \) linearly independent functions \( G \) in the span of
\[
W_{\sum_{i=1}^{n-1} k_{n-i} 2^{m_1} L^{n-i-1}} : 0 \leq k_1, \ldots, k_{n-1} \leq 2^{m_2} - 1
\]
such that each \( G(\frac{y}{L^n}) \) is supported on \( Y_{n-1} \). So \( G(\frac{y}{L^n}) \) is supported on \( L Y_{n-1} \).

Because of \((8)\) and since
\[
W_{\sum_{i=1}^{n} k_{n-i+1} 2^{m_1} L^{n-i}}(\frac{y}{L^n}) = W_{k_n L^{n-1} 2^{m_1}}(\frac{y}{L^n}) W_{\sum_{i=1}^{n-1} k_{n-i} 2^{m_1} L^{n-i-1}}(\frac{y}{L^n})
\]
is supported on \( Y_n \), we conclude that there are at least \( 2^{n(m_2 - m_1)} \) linearly independent functions in \( V_{X_n} \) (recall these are functions spanned by the functions in \((7)\)) that are supported on \( Y_n \). We thus have
\[
\dim V_{X_n, Y_n} \geq 2^{n(m_2 - m_1)}.
\]

**Remark 4.8.** The argument shows \( F = F_{X_1} Y_n \in V_{X_n, Y_n} \).

**References**

[1] Bourgain, Jean; Dyatlov, Semyon *Fourier dimension and spectral gaps for hyperbolic surfaces* Geom. Funct. Anal. 27 (2017), no. 4, 744-771
[2] Bourgain, Jean; Dyatlov, Semyon *Spectral gaps without the pressure condition* Ann. of Math. (2) 187 (2018), no. 3, 825-867
[3] Demeter, Ciprian *A guide to Carleson’s theorem* Rocky Mountain J. Math. 45 (2015), no. 1, 169-212
[4] Dyatlov, Semyon; Zahl, Joshua *Spectral gaps, additive energy, and a fractal uncertainty principle* Geom. Funct. Anal. 26 (2016), no. 4, 1011-1094
[5] Folland, Gerald B. *A course in abstract harmonic analysis. Studies in Advanced Mathematics.* CRC Press, Boca Raton, FL, 1995. x+276 pp.
[6] Thiele, C., *The quartile operator and pointwise convergence of Walsh series*, Trans. Amer. Math. Soc. 352 (2000), no. 12, 5745-5766

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