$\mathcal{N} = 1^*$ model superpotential revisited
(IR behaviour of $\mathcal{N} = 4$ limit)

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Abstract

The one-loop contribution to the superpotential, in particular the Veneziano-Yankielowicz potential in $\mathcal{N} = 1$ supersymmetric Yang-Mills model is discussed from an elementary field theory method and the matrix model point of view. Both approaches are based on the Renormalization Group variation of the superconformal $\mathcal{N} = 4$ supersymmetric Yang-Mills model.
1 Introduction

Several attempts have been made to derive from dynamical principles the superpotential of $\mathcal{N} = 1$ SUSY gauge models including the celebrated Veneziano-Yankielowicz (VY) potential for the pure $\mathcal{N} = 1$ Super Yang-Mills (SYM) model [1].

Within the QFT framework one can distinguish two approaches i.e. either to apply the generalized Konishi anomaly and the corresponding anomalous Ward-Takahashi identity (AWTI) [2] or to try to compute the potential directly from the microscopic Lagrangian with an appropriate regularization scheme.

The former approach leads to the formal equivalence [2] with the highly successful Matrix Model method due to Dijkgraaf and Vafa (DV) [3], originally proposed as a spin-off from string theory, whereas in the latter, one generally makes appeal to the instanton calculus [4] so as to take into account “non perturbative” effects like the VY potential.

The instanton method is known to produce some ambiguities in certain cases [5] although, in the celebrated Seiberg-Witten model, agreement with the instanton method is considered as a proof of correctness of both methods.

On the other hand, there are also a few attempts to compute low energy quantities like the superpotential by making use of elementary diagrammatic methods which exploit the covariant supersymmetric Feynman rule [6] previously developed for the purpose of the “perturbative” derivation of the DV correspondence.

This latter gives an efficient way to extract information about the low energy holomorphic quantities (“F terms”).

The impression here is that one may obtain the superpotential of the system made up of gluons and some additional “matter” but that it is difficult to deal with pure gluonic systems [7].

In [8], a diagrammatic derivation of VY superpotential for the pure $\mathcal{N} = 1$ SYM model has been attempted; the authors have limited themselves to the case of the $SU(2)$ gauge group.

The central observation in [8] is that the superpotential of the superconformal $\mathcal{N} = 4$ SYM model in four dimensions is “trivial” in the sense that it receives no contributions from non-trivial holomorphic terms [9]. The $\mathcal{N} = 4$ SYM model is assumed to be UV finite and so is its mass deformed version ($\mathcal{N} = 1^\ast$ model) [10]. Superconformal symmetry of the undeformed $\mathcal{N} = 4$ SYM implies $\beta(g) = 0$; therefore the gauge coupling constant does not vary with the energy scale.

Applying a Renormalization Group (RG) -type argument [11] one may try to compute the (holomorphic) $\mathcal{N} = 1$ potential as the difference from the trivial $\mathcal{N} = 4$ superpotential.

In [8], one has obtained the VY superpotential for $SU(2)$ pure $\mathcal{N} = 1$ SYM. However, there are some ambiguities in the computations presented in [8].

First of all, the insufficient analysis of anticommuting external field ($W_\alpha$ or $\lambda_\alpha$) makes
the generalization to $N_c > 2$ difficult.

Moreover, the limiting procedure to evaluate the “difference” of potentials

$$\Delta V = V_{N=1} - V_{N=4}$$

and reduce it to the VY form as the regularizing mass parameter $\mu$ flows from large $M_0$ (corresponding to $N = 1$ SYM) to $M \sim 0$ (corresponding to $N = 4$ SYM) lacks of mathematical rigour.

Indeed at the end of computation, the one-loop potential takes the form

$$\text{const} \, W^2 \log \left\{ \frac{1 + \overline{\alpha} g_0^2(W^2/M^3)}{1 + \overline{\alpha} g_0^2[W^2/(M_0 M^2)]} \right\}, \quad (1)$$

with $\overline{\alpha}$ being a numerical constant.

The VY form of the potential can appear if one assumes $\overline{\alpha} g_0^2(W^2/M^3) \gg 1 \gg \overline{\alpha} g_0^2[W^2/(M_0 M^2)]$ and, hence, approximate it with

$$\text{const} \, W^2 \log \left( \overline{\alpha} g_0^2(W^2/M^3) \right), \quad (2)$$

On the other hand, a naive IR limit $M \rightarrow 0$ makes the whole potential logarithmically divergent.

Moreover, in order to arrive at the quoted result, eq. (1), the authors [8] have adopted the Gaussian approximation i.e. the effective coupling obtained in the intermediate stage of computation has been truncated beyond the quadratic term.

As we see later, the justification for such “Gaussian approximation” also depends on the smallness of $\overline{\alpha} g_0^2[W^2/(M_0 M^2)]$.

The same RG-type approach (with respect to $N = 4$ SYM model) has been used with complete success in Matrix Model computation [14]. In this paper, the authors have applied the Matrix Model method by Dijkgraaf and Vafa [3] to the same model parametrized by the “floating” mass $\mu (M \leq \mu \leq M_0)$.

The crucial point here is again to appeal to the assumed triviality of the holomorphic part of the $N = 4$ superpotential, corresponding to the $M \rightarrow 0$ limit. In this way, the authors of [14] were able to uniquely determine the overall coefficient $C_{\hat{N}}$ for the measure of the matrix integral.

In particular, their computation shows that the leading term as $\hat{N} \rightarrow \infty$ exhibits a smooth limit as $M \rightarrow 0$. Thus one can define $C_{\hat{N}}$ in such a way that the $M \rightarrow 0$ limit gives the required boundary condition without any ambiguity. It was shown that this definition of the matrix integral yields the complete $N = 1^*$ superpotential including the VY term as well as the perturbative corrections.
Adding the symmetry breaking potential at the beginning, one can also successfully deal with a spontaneously broken $\mathcal{N} = 2$ SYM model, obtaining the perturbative version of Seiberg-Witten solution [16].

In this approach, moreover, one does not appear to encounter any of the ambiguities plaguing the instantonic computation of VY potential.

It should be emphasized that in the Matrix Model computation, the use of the same idea of triviality of the $\mathcal{N} = 4$ superpotential does not lead to any IR divergence so long as one is interested in leading $\hat{\mathcal{N}}$ results.

The paper is organized as follows: section (2) is devoted to retracing the results of [8] and [14] and their reanalysis. In section (3) we will attempt to construct a more convincing diagrammatic computation of the $SU(N_c)$ SYM superpotential from the QFT point of view, while section (4) will contain our conclusions.

2 QFT and Matrix Model derivation of Superpotential

In this section, we will briefly review and compare the methods of [8] and [14].

2.1 ERG approach to $\mathcal{N} = 1$ Superpotential

In [8], one starts with the microscopic action for the $\mathcal{N} = 1^*$ model with gauge group $G = SU(N_c)$.

$$S_{\mathcal{N}=1^*}(V, \Phi_i, \Phi^*_i; g_0) = \frac{1}{16} \int d^4x d^2\theta \frac{1}{g_0^2} W^2 + h c$$

$$+ 2N_c \int d^4x d^2\theta d^2\bar{\theta} \sum_{i=1}^3 \bar{\Phi}_i e^{g_0 V} \Phi_i$$

$$+ \frac{ig_0}{\sqrt{2}} \int d^4x d^2\theta f_{abc} \epsilon^{ijk} \Phi_i^a \Phi_j^b \Phi_k^c + h c$$

$$+ \frac{1}{2} \int d^4x d^2\theta \sum_{i=1}^3 M_i^2 \Phi_i^2 + h c,$$  \hspace{1cm} (3)

where $\frac{1}{g_0^2} = \frac{1}{g_0^2} + \frac{ig_0}{8\pi^2}$ (canonical representation). Note that in the original presentation [8] the authors have used the so-called holomorphic representation while here we will be using the canonical representation.

For large $M_i^0 (\equiv M_0)$, this model can be regarded as a $\mathcal{N} = 1$ SYM model, regularized by a mass deformed $\mathcal{N} = 4$ SYM [11].
It is believed that the model is free of UV divergences for an arbitrary set of masses \((M_{i0})\) just as the original superconformal model without any mass deformation \([10]\).

The pure \(\mathcal{N}=1\) SYM can be realized in the limit \(M_{i0} = M_0 \to \infty\) and \(g_0 \to 0\) with

\[
\Lambda_{\mathcal{N}=1} = \frac{M_0}{g_0^{2/3}} \exp -\frac{8\pi^2}{3N_cg_0^2}\tag{4}
\]

held fixed.

On the other hand, the \(M_{i0} = M \to 0\) limit at fixed \(g_0\) should realize the \(\mathcal{N}=4\) SYM for any \(g_0\).

In \([8]\) the following three-stage procedure has been used to obtain the superpotential.

2.1.1 The holomorphic reduction

Following \([6]\), one can approximately integrate out the antichiral components \((\Phi_i)_{i=1,2}\) (we regard \(\vec{\Phi}_3\) and \(\bar{\Phi}_3\) as external at this stage), thus obtaining the effective \(\vec{\Phi}_1, \bar{\Phi}_2\) action written in momentum space\([4]\):

\[
\frac{1}{2} \int d^4p d^2\pi \Phi_{ia}^*(p,\pi)(-p^2 + \pi_\alpha \hat{W}_\alpha + \hat{A}(\Phi_3) + M_0)_{ia,jb} \Phi_{jb}^*(-p,\pi), \tag{5}
\]

where

\[
\hat{W}_\alpha \to \hat{W}_{ia,jb} = W_{ia,jb} = F_{ab}^c \delta_{ij},
\]

\[
\hat{A} \to \hat{A}_{ia,jb} = g_0 \Phi_3F_{ab}^c \epsilon_{ij} \equiv g_0 \Phi_3 \cdot \vec{F} \otimes \sigma_2.
\]

Note that in writing \((5)\), the chiral field \(\vec{\Phi}_3\) too is treated as if it were constant. However, it is easy to see that, for large \(M_0\), only the lowest frequency components of \(\vec{\Phi}_3\) contribute when integrated with respect to \(\vec{\Phi}_1\) and \(\bar{\Phi}_2\). Eq. \((5)\) gives the effective holomorphic propagator of \(\vec{\Phi}_1\) and \(\bar{\Phi}_2\) (valid only for the evaluation of low energy amplitudes).

2.1.2 Exact Renormalization Group reduction.

In general, one can transform a path integral with an action like \((5)\) into another of similar form, where the regularizing parameter \(M_0\) has been changed to \(M < M_0\). This is equivalent to K. Wilson’s decimation method in lattice models \([12]\) and to the variation of the cutoff in continuum QFT \([13]\). The simple formula for implementing this change is

\[\text{Following the lead in [6], we will Fourier transform all superspace coordinates}\]
Zinn-Justin’s transformation \[17\]:

$$\int \mathcal{D}\Phi \exp \left[ -\frac{1}{2} \int \Phi^*(-p) \left( \frac{1}{D_1(p) + D_2(p)} \Phi^*(p) - V(\Phi) \right) \right] =$$

$$\int \mathcal{D}\Phi \mathcal{D}\Phi' \exp \left[ -\frac{1}{2} \int \Phi^*(-p) D^{-1}_1(p) \Phi^*(p) +$$

$$-\frac{1}{2} \int \Phi^*(-p) D^{-1}_2(p) \Phi^*(p) - V(\Phi + \Phi') \right] \tag{6}$$

where $D_i(p)$’s are the regulated propagators.

In our case, the situation is somewhat simpler as the effective action, eq. \[5\], is quadratic in $\vec{\Phi}_{1,2}$. Then the required equivalent of eq. \[6\] is:

$$\int \prod_{i=1,2} \mathcal{D}\vec{\Phi}_i \exp \frac{i}{2} \int d^4 p d^2 \pi \Phi_i^a [-p^2 + \pi_\alpha \tilde{W}^\alpha + \tilde{A}(\Phi_3) + M_0]_{ia,jb} \Phi_j^* \tag{7}$$

In eq. \[7\] $\vec{\Phi}$ and $\vec{\Phi}'$ have been diagonalised in order to cancel the mixed product in \[6\].

As explained in \[8\], the first term in RHS, with reduced mass $M$, will reproduce, in the vanishing $M$ limit, the amplitude for the $\mathcal{N} = 4$ SYM, while the second term should contribute the non-trivial part of the “Wilsonian action” $S_{M_0} - S_{\mathcal{N}=4}$.

The gaussian integral over $\Phi'_{1,2}$ has been exactly computed in \[8\] in the case of $N_c = 2$. We only quote the result:

$$\frac{W_1 W_2}{8\pi^2} \left\{ \log \left( \frac{M}{M_0} \right)^2 + \log \left( \frac{1 + \phi^2/M^2}{1 + \phi^2/M_0^2} \right) +$$

$$+ 2 \left( \frac{\phi_1^2 + \phi_2^2}{M^2} \right) \left( \frac{M}{\phi} \right)^3 \left\{ \tan^{-1} \left( \frac{\phi}{M} \right) - \left( \frac{\phi}{M} \right) \right\} - (M \leftrightarrow M_0) \right\} +$$

$$- (W_1 \leftrightarrow W_2) \tag{8}$$

where $W_\alpha = W_\alpha^3$ (diagonal component), $(\vec{\Phi}_3)^2 \equiv \vec{\phi}^2 = \phi_1^2 + \phi_2^2 + \phi_3^2$. 

5
2.1.3 Integration over $\Phi_3$

In order to arrive at the superpotential as a function of the gluon supermultiplet only, one must integrate over $\Phi_3 \equiv \tilde{\phi}$, with the effective coupling given in eq. (8).

To this end, one may try to apply the same procedure used for $D\Phi_1 D\Phi_2$, i.e. first integrate out $\Phi_3$ and then apply the ERG transformation, $M_0 \to M$. Dealing with a non-quadratic action, eq. (8), one must in principle apply Zinn-Justin’s formula (6) in its unsimplified form. However, in [8], a Taylor expansion up to the second order was performed first and, then, the previous procedure to the resulting quadratic action for $\Phi_3$ was applied.

The quadratic approximation to (8) is

$$4 W_1 W_2 \left[ \log \left( \frac{M}{M_0} \right)^2 + \left( \frac{\phi}{M} \right)^2 - \frac{2}{3} \frac{\phi_1^2 + \phi_2^2}{M^2} \right] - (W_1 \leftrightarrow W_2)$$

(omitting $M_0^{-2}$ terms) and the result after $D\Phi_3$ integration is

$$\exp \frac{i}{4 \cdot 16\pi^2} \int W^2 \left\{ \log \left( \frac{M}{M_0} \right) + \log \left( \frac{1 + \frac{g_0^2 W^2}{32\pi^2 \cdot 3M^3}}{1 + \frac{g_0^2 W^2}{32\pi^2 \cdot 3M_0 M^3}} \right) \right\}. \quad (9)$$

If one adds the $\log(M/M_0)^2$ contribution from (8), and the gauge kinematical term in $S_{N=4}$, eq. (9) takes the form

$$\exp \frac{2i}{128\pi^2} \int W^2 \left\{ \log \left( \frac{M}{\Lambda} \right)^3 + \log \left( \frac{1 + \frac{g_0^2 W^2}{32\pi^2 \cdot 3M^3}}{1 + \frac{g_0^2 W^2}{32\pi^2 \cdot 3M_0 M^3}} \right) + \frac{i\vartheta_0}{2} \right\}. \quad (10)$$

To conclude that the potential is of VY type, in the vanishing $M/M_0$ limit, one must be able to ascertain

$$\log \left( \frac{1 + \alpha g_0^2 W^2/M^3}{1 + \alpha g_0^2 W^2/(M_0 M^3)} \right) \sim \log \left( \frac{\alpha g_0^2 W^2}{M^3} \right) \quad (11)$$

If one can justify this assumption, then the superpotential for $N = 1$ SYM (with $N_c = 2$) takes the VY form

$$W_{\text{eff}} = \frac{1}{128\pi^2} \int \left\{ 2 \log \left( \frac{S}{3\Lambda^3 \cdot 32\pi^2} \right) + i\vartheta_0 \right\} S \quad (S \equiv W^2) \quad (12)$$

If we look for the extrema of (12), we find

$$\langle W^2/(32\pi^2) \rangle \sim (\pm \exp(-i\vartheta_0/2)\Lambda^3). \quad (13)$$
where
\[
\Lambda' = \left( \frac{3}{e} \right)^{1/3} \Lambda
\]  
(14)

Note that no use of instantons has been made to obtain the results (12)-(13).

### 2.2 Matrix Model method to compute \( \mathcal{N} = 1 \) potential

It has been suggested in [3] that the large \( \hat{N} \) limit of a certain Matrix Model can reproduce the holomorphic superpotential of a wide class of gauge field theories.

At the beginning, it was believed that such a correspondence was limited to the perturbative corrections to the superpotential, thus excluding \( \mathcal{N} = 1 \) VY potential [2]. This “inability” was related to the fact that one could not determine unambiguously the overall coefficient for the matrix integral measure [15].

Kawai and his collaborators, after establishing the direct correspondence between DV methods and certain generalizations of gauge field theories (on non-commutative space time) [18], tried to use the triviality of \( \mathcal{N} = 4 \) superpotential as a boundary condition for determining the unknown overall coefficient of the matrix measure [14].

In [14], one starts with a \( \hat{N} \)-dimensional hermitian Matrix Model characterized by the tree-level potential
\[
S_m = \frac{\hat{N}}{g_m} \text{tr} \left( \Phi_1 [\Phi_2, \Phi_3] + W(\Phi_1) + \frac{M_2}{2} \Phi_2^2 + \frac{M_3}{2} \Phi_3^2 \right)
\]  
(15)

and the Dijkgraaf-Vafa-type free energy
\[
Z = \exp \left( - \frac{\hat{N}^2}{g_m^2} F_m \right) = C_{\hat{N}} \int d\Phi_1 d\Phi_2 d\Phi_3 \exp(-S_m).
\]  
(16)

To determine the overall coefficient \( C_{\hat{N}} \), one considers the specific form (\( \mathcal{N} = 1^* \) model),
\[
S_m = \frac{\hat{N}}{g_m} \text{tr} \left( \Phi_1 [\Phi_2, \Phi_3] + \frac{M_1}{2} \Phi_1^2 + \frac{M_2}{2} \Phi_2^2 + \frac{M_3}{2} \Phi_3^2 \right),
\]  
(17)

and tries to fix \( C_{\hat{N}} \) by demanding that the \( \mathcal{N} = 4 \) SYM limit (\( M_i \to 0 \)) reproduce the “trivial” model

\[
\mathcal{F}_{\mathcal{N}=4} = \lim_{M_i \to 0} \mathcal{F}_{\mathcal{N}=1^*} = \frac{\pi g_0 g_m^2}{\hat{N} c}.
\]

Evaluating the matrix integral for small \( M_i \), one obtains
\[
Z_{\mathcal{N}=1^*} \approx C_{\hat{N}} J_{\hat{N}} \left( \frac{2\pi}{\hat{N}} \right)^{\hat{N}^2} \left( \frac{2\pi g_m}{\hat{N} M_1 M_2 M_3} \right)^{\hat{N}/2}
\]  
(18)
where $J_N$ is the result of the integral over the “angular variables” of a hermitian matrix$^2$.

The remarkable fact about eq. (18) is that the leading term, of order $\hat{N}^2$, is smooth in the $M_i \to 0$ limit (constant), and the IR divergent term is subleading in $\hat{N}$.

This fact guarantees the successful outcome of Kawai’s scheme and the result is

$$C_{\hat{N}} = \left( \frac{\hat{N}^3 g^2_0}{(2\pi)^3 g^{2/3} m^2} \right)^{\hat{N}^2/2} e^{-\pi \gamma_0 \hat{N}^2/N_c}. \quad (19)$$

Now we can assume that this value, for given $(\hat{N}, g_m)$, be valid for any potential $W(\Phi_1)$ in eq. (15).

As has been shown explicitly in [14], choosing $W(\Phi_1) = (M_1/2)\Phi_2$ and $M_i \equiv M_0 \to \infty$, one can obtain the superpotential of $\mathcal{N} = 1$ SYM following the DV prescription [3],

$$W_{\text{SYM}}^{\text{eff}} = N_c \frac{\partial}{\partial S} \left[ \frac{S^2}{2} \log \left( \frac{e^{3/2} \Lambda^3}{S} \right) \right] = N_c S \left[ 1 - \log \left( \frac{S}{\Lambda^3} \right) \right], \quad (20)$$

which is none other than the VY potential [1].

### 2.2.1 $\mathcal{N} = 1^*$ models

One can also generalize [14] the above computation to the case of an arbitrary $W(\Phi_1)$ in eq. (15). In particular, in the simple case that $W(\Phi_1) = (M_1/2)\Phi_2$, one obtains the superpotential for $\mathcal{N} = 1^*$ model which include both the VY term and perturbative corrections.

The corresponding matrix integral is given by

$$Z = C_{\hat{N}} \int d\Phi_1 \, d\Phi_2 \, d\Phi_3 \exp \left[ -S_{\mathcal{N}=1^*}(\Phi_i; M_i) \right], \quad (21)$$

where

$$S_{\mathcal{N}=1^*} = \frac{\hat{N}}{g_m} \text{tr} \left( g_0 \Phi_1[\Phi_2, \Phi_3] + \frac{1}{2} \sum_{i=1}^3 M_i \Phi_i^2 \right), \quad (22)$$

and $C_{\hat{N}}$ is as in (19). One is now interested in the small gauge coupling, $g_0$, and large but finite $M_i$ region.

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$^2$As is usual,

$$\int d\tilde{\Phi} = J_{\hat{N}} \int d\tilde{\lambda} \Delta^2(\lambda),$$

where the Van Der Monde determinant for the Gaussian Unitary Ensemble takes the form $\Delta^2(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)^2$. 

8
After integrating over two of the three $\Phi$’s and rescaling the diagonal elements of the remainder, one can rewrite $\mathcal{Z}_{N=1^*}$ in the following form

$$\mathcal{Z}_{N=1^*} = e^{-\pi \tau_0 \frac{g_m^2}{N_c}} \cdot \int d \vec{\lambda} e^{-\frac{1}{2} \sum_{i=1}^{N_c} \lambda_i^2} \prod_{i<j} \left\{ \frac{(\lambda_i - \lambda_j)^2}{1 + \gamma (\lambda_i - \lambda_j)^2} \right\}, \quad (23)$$

where $\gamma \equiv g_0^2 g_m / (\hat{N} M_0^3)$ and $M_0 \equiv (M_1 M_2 M_3)^{1/3}$.

One can evaluate $(23)$, expanding it in terms of the “small” parameter $\gamma$.

Thus one can compute the DV free energy, $F_m$, as

$$F_m = \frac{g_m^2}{2} \left[ \log \left( \frac{g_0^2 S}{M_0^3 e^{3/2}} \right) - \frac{2 \pi \tau_0}{N_c} \right] + \frac{g_m^2}{N^2} \log \int d \vec{\lambda} e^{-\frac{1}{2} \sum_{i=1}^{N_c} \lambda_i^2} \prod_{i<j} \left\{ \frac{(\lambda_i - \lambda_j)^2}{1 + \gamma (\lambda_i - \lambda_j)^2} \right\}. \quad (24)$$

In the planar limit, exploiting the DV correspondence ($\hat{N} \to \infty$, $g_m \sim S$)

$$F_m^{(\text{planar})} = -\frac{S^2}{2} \left[ \log \left( \frac{g_0^2 S}{M_0^3 e^{3/2}} \right) - \frac{2 \pi \tau_0}{N_c} \right] - \frac{S^2}{N^2} \left\{ -\gamma \left\langle \sum_{i>j} (\lambda_i - \lambda_j)^2 \right\rangle \right. +$$

$$+ \frac{\gamma^2}{2} \left\{ \left[ \sum_{i>j} (\lambda_i - \lambda_j)^2 \right]^2 + \left\langle \left[ \sum_{i>j} (\lambda_i - \lambda_j)^2 \right]^2 \right\rangle \right\} +$$

$$+ \left\langle \sum_{i>j} (\lambda_i - \lambda_j)^4 \right\rangle \right\} \ldots \} =$$

$$= -\frac{S^2}{2} \left[ \log \left( \frac{g_0^2 S}{M_0^3 e^{3/2}} \right) - \frac{2 \pi \tau_0}{N_c} \right] +$$

$$+ S^2 \left\{ \frac{g_0^2 S}{M_0^3} + \frac{7}{2} \left( \frac{g_0^2 S}{M_0^3} \right)^2 - 23 \left( \frac{g_0^2 S}{M_0^3} \right)^3 + \ldots \right\}, \quad (25)$$

where

$$\left\langle \{ \ldots \} \right\rangle \equiv \frac{\int d \vec{\lambda} \Delta^2(\lambda) e^{-\frac{1}{2} \sum_{i=1}^{N_c} \lambda_i^2} \{ \ldots \}}{\int d \vec{\lambda} \Delta^2(\lambda) e^{-\frac{1}{2} \sum_{i=1}^{N_c} \lambda_i^2}}. \quad (26)$$

The effective potential is given by

$$W_{\text{eff}}(S) = N_c \frac{\partial F_m^{(\text{planar})}}{\partial S} = -N_c S \left[ \log \left( \frac{S}{\Lambda^3} \right) - 1 \right] +$$

$$-N_c S \left[ -3 \frac{g_0^2 S}{M_0^3} + 14 \left( \frac{g_0^2 S}{M_0^3} \right)^2 - 115 \left( \frac{g_0^2 S}{M_0^3} \right)^3 + \ldots \right], \quad (27)$$

which agrees with previous results [19].
2.2.2 $\mathcal{N} = 2$ models

As shown in [14], with exactly the same definition of the integration measure one can deal with spontaneously broken models.

In this case, one has to add to the tree-level action, eq. (22), a symmetry breaking term.

In order to analyse the breaking pattern

$$U(N) \to U(N_1) \times U(N_2) \quad N_1 + N_2 = N,$$

it suffices to introduce the cubic term

$$\Delta W(\Phi_1) = \epsilon \text{tr} \left( \frac{1}{3} \Phi_1^3 - v^2 \Phi_1 \right), \quad (28)$$

and take the $\epsilon \to 0$ limit at the end of computation. Then, one can study the $\mathcal{N} = 2^*$ model which can go over to $\mathcal{N} = 2$ SYM (Seiberg-Witten model) in the infinite mass limit.

One sets $M_2 = M_3 = \Lambda_0 \gg M_1 \sim 0$ and $\langle \Phi \rangle_{cl} = \pm v$.

The relevant matrix integral is again

$$Z_{\mathcal{N}=2^*} = C_N \int d\Phi_1 d\Phi_2 d\Phi_3 \exp \left[ S_{\mathcal{N}=1^*} + \Delta W(\Phi_1) \right]. \quad (29)$$

To study $\mathcal{N} = 2$ SYM, for instance, one can go over to the limit that $\Lambda_0 \to \infty$ and integrate out $\Phi_2, \Phi_3$. In this case, since the cubic term can be neglected with respect to the $\Phi_2, \Phi_3$ mass terms, the only trace of the $\mathcal{N} = 4$ regularization is the coefficient $C_N$.

Eq. (29) is then reduced to the matrix integral with a single matrix

$$Z = C_N \left[ \frac{2 \pi g_m}{N \Lambda_0} \right]^{\hat{N}^2} \int d\Phi \exp \left[ -\frac{\hat{N}}{g_m} \text{tr} \left( \epsilon \left( \frac{1}{3} \Phi^3 - v^2 \Phi \right) \right) \right]$$

$$= C_N J_{\hat{N}} \left[ \frac{2 \pi g_m}{N \Lambda_0} \right]^{\hat{N}^2} \int d\phi \Delta^2(\phi) \exp \left[ -\frac{\hat{N} \epsilon}{g_m} \sum_{i=1}^{\hat{N}} \left( \frac{1}{3} \phi_i^3 - v^2 \phi_i \right) \right] \quad (30)$$

Such integral has been already studied [16] except for the explicit reference to $C_N$.

On the other hand, in the $\mathcal{N} = 4$ approach by Kawai et al. [14], the unambiguous definition of $C_N$ leads to a very clear interpretation of dynamical cutoffs appearing in the computation. This is particularly important when dealing with the non-perturbative formulation of the SW model where UV divergences appear in the corresponding matrix integral.
To analyze further the matrix integral (30) one, as usual, picks up the classical vacua of the potential (28)
\[ \langle \Phi \rangle_{cl} = \pm v \]  
and constructs the series of vacua, characterized by \( \hat{N}_1 \) eigenvalues at \( v \) and \( \hat{N}_2 \) eigenvalues at \(-v\) \( (\hat{N}_1 + \hat{N}_2 = \hat{N}) \).

Expanding the action (30) around the respective classical vacua
\[ S' = \frac{\hat{N}}{g_m} \sum_{i=1}^{\hat{N}_1} \left[ \frac{2\epsilon v}{2} p_i^2 + \frac{\epsilon}{3} p_i^3 \right] + \frac{\hat{N}}{g_m} \sum_{j=1}^{\hat{N}_2} \left[ -\frac{2\epsilon v}{2} q_j^2 + \frac{\epsilon}{3} q_j^3 \right] \]  
and taking account of all possible ways to choose \( \hat{N}_1 \) out of \( \hat{N} \) eigenvalues, \( Z \) can be written as
\[ Z = \left( \frac{\hat{N}}{\hat{N}_1} \right) \Lambda_0^{\hat{N}_2} e^{-\frac{\pi \epsilon V}{N_c} \hat{N}^2} \int d\vec{p} d\vec{q} \prod_{i=1}^{\hat{N}_1} \prod_{j=1}^{\hat{N}_2} (2v + p_i + q_j)^2 \prod_{1 \leq i < l \leq \hat{N}_1} (p_i - p_l)^2 \prod_{1 \leq j < k \leq \hat{N}_2} (q_j - q_k)^2 e^{-S'}. \]

Wick-rotating the \( \vec{q} \) variables and with a suitable rescaling
\[ Z = \left( \frac{\hat{N}}{\hat{N}_1} \right) \Lambda_0^{\hat{N}_2} e^{-\frac{\pi \epsilon V}{N_c} \hat{N}^2} \left( \frac{\alpha V}{\hat{N}_1 N_2} \right) \int d\vec{p} \prod_{1 \leq i < l \leq \hat{N}_1} (p_i - p_l)^2 e^{-\sum_{i=1}^{\hat{N}_1} \frac{p_i^2}{\beta}} \times \]
\[ \times \prod_{1 \leq j < k \leq \hat{N}_2} (q_j - q_k)^2 e^{-\sum_{j=1}^{\hat{N}_2} \frac{q_j^2}{\beta}} \times \]
\[ \times \left\{ \prod_{i=1}^{\hat{N}_1} \prod_{j=1}^{\hat{N}_2} \left[ 1 + \frac{p_i - q_j}{\beta} \right]^2 e^{-\sum_{i=1}^{\hat{N}_1} \frac{p_i^2}{\beta} e^{-\sum_{j=1}^{\hat{N}_2} \frac{q_j^2}{\beta}}} \right\}, \]

where \( V = (v - (-v)) = 2v \), \( \alpha = (\hat{N}_1 V/g_m)^{1/2} \) and \( \beta = (V/\alpha) \).

Here, one can consider \((1/\beta)\) as the “small” parameter for the perturbative expansion of \( Z \) as for the corresponding free energy \( F_m \).

In the planar limit \((N \to \infty \text{ with } g_m \hat{N}_i/\hat{N} \to S_i\) \( (i = 1, 2) \) the first few terms of \( F_m \) are the following:

**0th order**
\[ F_m^{(0)} = \frac{\pi \tau_0}{\hat{N}_c} (S_1 + S_2)^2 + (S_1 + S_2)^2 \log \frac{\Lambda_0}{V} + \sum_{i=1}^{2} \frac{S_i^2}{2} \log \left( \frac{e^{3/2} \epsilon V^3}{S_i} \right). \]
This is essentially equal to the expression found in [14] except for the fact that there one assumes \( \Lambda_0 \approx V \), while here \( \Lambda_0 \) is clearly identified with the regularization mass from the mass-deformed \( \mathcal{N} = 4 \) SYM models (\( \mathcal{N} = 2^* \) models).

Thus \( \Lambda_0 \) is related to the dynamical cutoff for \( \mathcal{N} = 2 \) SYM, \textit{i.e.}

\[
\Lambda_{\mathcal{N}=2}/\Lambda_0 = \exp[-8\pi^2/(2N_c g_0^2)]
\]

and the kinematical term of the mass deformed \( \mathcal{N} = 4 \) SYM is equal to

\[
\pi \tau_0 = N \log[\Lambda_{\mathcal{N}=2}/\Lambda_0].
\]

Eq. (35) can be rewritten in term of the physical quantities for \( \mathcal{N} = 2 \) SYM,

\[
F_m^{(0)} = (S_1 + S_2)^2 \log \frac{\Lambda_{\mathcal{N}=2}}{V} - \sum_{i=1}^{2} \frac{S_i^2}{2} \log \left( \frac{S_i}{\epsilon^3/\epsilon V^3} \right).
\]

The corresponding superpotential is

\[
W_{\text{eff}}^{(0)} = \sum_{i=1}^{2} N_i \frac{\partial F}{\partial S_i} = N_c (S_1 + S_2) \log \left( \frac{\Lambda_{\mathcal{N}=2}}{V} \right)^2 - \sum_{i=1}^{2} N_i S_i \left[ \log \left( \frac{S_i}{\epsilon V^3} \right) - 1 \right] = \sum_i N_i S_i \left[ \log \left( \frac{\Lambda_3^i}{S_i} \right) + 1 \right]
\]

The last equality in the above is obtained by introducing the low energy cutoffs \( (\Lambda_i)_{i=1,2}^3 \) defined by

\[
\epsilon V^3 \left( \frac{\Lambda_{\mathcal{N}=2}}{V} \right)^{\frac{2N_i}{N_1}} = \Lambda_i^3
\]

More explicitly

\[
\Lambda_1^{3N_1} = \epsilon^{N_1} V^{N_1 - 2N_2} \Lambda_{N=2}^{2N_1}, \\
\Lambda_2^{3N_2} = \epsilon^{N_1} V^{N_2 - 2N_1} \Lambda_{N=2}^{2N_2},
\]

which agrees with the definition introduced in [23].

\textbf{Higher order terms in 1/\beta}

In order to calculate higher order corrections in \( 1/\beta \) we have to expand the partition function:

\[
\mathcal{Z} \sim \left( \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} \left[ 1 + \frac{(p_i - i q_j)}{\beta} \right] \right)^2 \exp \left( - \sum_{i=1}^{N_1} \frac{p_i^3}{3\beta} \right) \exp \left( - \sum_{j=1}^{N_2} \frac{(i q_j)^3}{3\beta} \right)
\]
where

\[ \left\langle (\ldots) \right\rangle \equiv \int \frac{d\vec{p}}{\mathcal{N}} \prod_{1 \leq i < j \leq \hat{N}_1} (p_i - p_j)^2 \exp \left[ -\sum_{m=1}^{\hat{N}_1} \frac{P_m^2}{2} \right] \]

\[ \int \frac{d\vec{q}}{\mathcal{N}} \prod_{1 \leq j < k \leq \hat{N}_2} (q_j - q_k)^2 \exp \left[ -\sum_{n=1}^{\hat{N}_2} \frac{Q_n^2}{2} \right] (\ldots). \] (43)

Computing \( F_m \) to order \( 1/\beta^2 \) and \( (1/\beta^2)^2 \) respectively, one obtains, in the planar limit

\[ \left( \frac{1}{\beta^2} \right) : \quad F^{(1)}_m(S_1, S_2) = -\frac{1}{V^3 \epsilon} \left[ \frac{2}{3} S_1^3 - \frac{2}{3} S_2^3 - 5 S_1 S_2 (S_1 - S_2) \right] \]

\[ \left( \frac{1}{\beta^2} \right)^2 : \quad F^{(1)}_m(S_1, S_2) = -\frac{1}{V^6 \epsilon} \left[ \frac{8}{3} S_1^4 - \frac{8}{3} S_2^4 - \frac{91}{3} S_1 S_2 (S_1^2 + S_2^2) + 59 S_1^2 S_2^2 \right] \]

These results are essentially equivalent to those in [16, 19].

### 2.2.3 Non perturbative approach to SW model

The perturbative treatments in the Matrix Model approach, described in sections 2.2.1 and 2.2.2, can be converted to the exact (“non perturbative”) results by means of the large \( \hat{N} \) matrix technology [21].

The coefficient \( C_{\hat{N}} \) of the matrix integral measure introduced in [14] is still relevant in this case.

As is well known, in the analytical treatment of the matrix integral in large-\( \hat{N} \) limit, a certain integral of the matrix resolvent \( R(x) \) is divergent [20], which necessitates the introduction of cutoffs.

The coefficient \( C_{\hat{N}} \), as defined in section 2.2, gives a simple and consistent interpretation of such cutoffs in terms of the regularizing mass in the original mass deformed \( \mathcal{N} = 4 \) SYM model.

We will illustrate this in the case of Seiberg-Witten type model [24] with \( \mathcal{N} = 2 \) SUSY and gauge symmetry broken as:

\[ SU(N_c) \to [U(1)]^{N_c-1}. \] (44)

Here, we mostly follow the presentation of [26].

The relevant matrix partition function is given by

\[ Z = e^{-\frac{\hat{S}_m}{\beta_m}} = C_{\hat{N}} \int d\Phi \exp \left[ -S_{\mathcal{N}=1} + W(\Phi) \right]. \] (45)
Here $S_{N=1^*}$, given by eq. (22), is the tree-level potential for the mass-deformed $\mathcal{N} = 4$ SYM with masses for chiral fields $\vec{M} = (0, \Lambda_0, \Lambda_0)$, and $W(\Phi_i)$ is the symmetry breaking inducing term. One has

$$W'(x) \propto \prod_{i=1}^{N_c} (x - a_i), \quad \sum a_i = 0. \quad (46)$$

After integrating out the heavy components $\Phi_2$ and $\Phi_3$ and diagonalizing $\Phi_1$, eq. (45) is reduced to

$$Z = \frac{1}{(\Lambda_0)^{N_c^2/2}} e^{-\pi \tau_0 \hat{N}^2 / N_c} \int d\phi \Delta^2(\phi) \exp -\frac{\hat{N}}{g_m} \sum_i W(\phi_i). \quad (47)$$

Rescaling $\phi_i$ with the regularizing mass $\Lambda_0$, i.e. $\phi_i \equiv \Lambda_0 \lambda_i$, one has

$$Z = e^{-\pi \tau_0 \hat{N}^2 / N_c} \int d\lambda \Delta^2(\lambda) \exp -\frac{\hat{N}}{g_m} \sum_i W(\lambda_i), \quad (48)$$

where $\widetilde{W}(\lambda_i)$ is again a polynomial with $\widetilde{W}'(x) \propto \prod_{i=1}^{N_c} (x - \tilde{a}_i)$. If we write the large-$\hat{N}$ matrix integral as

$$\int d\lambda \Delta^2(\phi) \exp -\frac{\hat{N}}{g_m} \sum_i \widetilde{W}(\lambda_i) \equiv e^{-\frac{\hat{N}^2}{g_m} \mathcal{F}_m}, \quad (49)$$

then the physical matrix free-energy is

$$\mathcal{F}_m = \frac{\pi \tau_0}{N_c} g_m^2 + \mathcal{F}_m, \quad (50)$$

where the first term comes from the factor $C_N$, while $\mathcal{F}_m$ can be written in terms of the density $\bar{\rho}(\lambda)$ of matrix eigenvalues in the large-$\hat{N}$ limit

$$-\frac{\hat{N}^2}{g_m^2} \mathcal{F}_m = g_m \sum_i \int_{A_i} d\lambda \bar{\rho}(\lambda) \widetilde{W}(\lambda) - g_m^2 \sum_{i,j} \int_{A_i} d\lambda \int_{A_j} d\lambda' \bar{\rho}(\lambda) \bar{\rho}(\lambda') \log |\lambda - \lambda'|. \quad (51)$$

$(A_i)_{i=1}^{N_c}$ are the intervals where the eigenvalues are concentrated and correspond to the cuts on the real axis for the matrix resolvent

$$R(x) = g_m \sum_i \int_{A_i} d\lambda \frac{\bar{\rho}(\lambda)}{(x - \lambda)}, \quad (52)$$
They are analytically defined by the auxiliary function

$$Y^2 = W'^2 + f_{N_c-1} = \prod_{i=1}^{N_c}(x - a_i^+)(x - a_i^-), \quad (53)$$

which gives the corresponding branch points and $A_i = [a_i^-, a_i^+]$. $f_{N_c-1}$ is the $(N_c-1)$-th order polynomial defined by

$$f_{N_c-1} = \frac{4g_m}{N} \left\langle \text{tr}(\tilde{W}'(\Phi) - \tilde{W}'(x)) \right\rangle_{\Phi}. \quad (54)$$

As usual, one considers the model in terms of the independent input parameters $a_1, \ldots, a_{N_c-1}, \ (\sum_{i=1}^{N_c} a_i = 0)$, and

$$S_i = \frac{1}{2\pi i} \oint_{A_i} R(x) \, dx, \quad i = 1, \ldots N_c \quad (\sum_{i=1}^{N_c} S_i = g_m). \quad (55)$$

To study the variation of $\tilde{F}_m$ with respect to the collective variables $S_i$'s, one needs to consider the structure of the Riemann surface defined by the behaviour of $R(x)$ with respect to the cuts $A_i$,

$$R(x + i\epsilon) - R(x - i\epsilon) = -2\pi i \rho(x), \quad x \in A_i' \quad \text{(gluing conditions)}. \quad (56)$$

The above defines the two-sheeted surface connected with $N_c$ tubes. $R(x)$ and $Y(x)$ are single valued functions on this surface. One can show, from eq. (51), that

$$\frac{\partial \tilde{F}_m}{\partial S_i} = \frac{1}{2} \int_{C_i} Y \, dx + \text{const.} = \int_{C_i} R \, dx + \text{const.}, \quad (57)$$

where $C_i$ is the line connecting the points at infinity $P(x_+ = \Lambda'_0 \rightarrow \infty)$ on the first sheet and $Q(x_+ = \Lambda'_0 \rightarrow \infty)$ on the second sheet passing through the cut $A_i$. One can get rid of the unknown const. by rewriting eq. (57) as

$$\frac{\partial \tilde{F}_m}{\partial S_i} - \frac{\partial \tilde{F}_m}{\partial S_j} = \frac{1}{2} \oint_{B_{i,j}} Y \, dx = \oint_{B_{i,j}} R \, dx. \quad (58)$$

The closed curve $B_{i,j}$ is given by $B_{i,j} = C_i - C_j$. $\Lambda'_0$ represents the U-V cutoff necessary to compute the matrix model.
Seiberg-Witten theory

The Seiberg-Witten theory is given by the extremum condition for the effective action (DV action)

\[ W_{\text{eff}} = \sum_{h=1}^{N_c} \frac{\partial}{\partial S_h} F_m \]

\[ = \sum_{h=1}^{N_c} \frac{\partial}{\partial S_h} \tilde{F}_m + N_c \frac{\partial}{\partial g_m} \left( \frac{\pi i \tau_0}{N_c} g_m^2 \right) \]

\[ = 2\pi i \tau_0 \sum_{h=1}^{N_c} S_h + \sum_{h=1}^{N_c} \frac{\partial}{\partial S_h} F_m \] \quad (59)

i.e.

\[ \frac{\partial}{\partial S_j} \sum_{h=1}^{N_c} \left( 2\pi i \tau_0 S_h + \frac{\partial \tilde{F}_m}{\partial S_h} \right) = 0 \quad j = 1, 2, \ldots, N_c. \] \quad (60)

This can be rewritten as

\[ \left( \frac{\partial}{\partial S_i} - \frac{\partial}{\partial S_{i+1}} \right) \sum_{h=1}^{N_c} \frac{\partial \tilde{F}}{\partial S_h} = 0 \quad i = 1, \ldots, N_c - 1, \] \quad (61)

\[ 2\pi i \tau_0 + \frac{\partial}{\partial S_i} \sum_{h=1}^{N_c} \frac{\partial \tilde{F}}{\partial S_h} = 0. \] \quad (62)

From the expression for \( \partial F / \partial S_i \) given above, eq. (58), these equations reduce to the following equations for the resolvent \( R \):

\[ \oint_{B_{i,i+1}} \sum_h \frac{\partial R}{\partial S_h} = 0 \] \quad (63)

\[ \oint_{C_{i}} \sum_h \frac{\partial R}{\partial S_h} + 2\pi i \tau_0 = 0. \] \quad (64)

The \( \tau_0 \) term in eq. (64) comes from the \( C_N \) coefficient of the matrix integral measure. The bare coupling constant \( \tau_0 \) codifies the regularization condition of our model in the original QFT form, i.e. the \( N = 2^* \) model with mass \( M = (0, \Lambda_0, \Lambda_0) \), which defines the regularized \( N = 2 \) SYM in the limit \( \Lambda_0 \to \infty, g_0 \to \infty \), with the \( N = 2 \) dynamical cutoff

\[ \Lambda_{N=2}^2 = \Lambda_0^2 e^{-8\pi^2/g_0^2 N_c}. \] \quad (65)
kept constant.

Thus, in such regularization scheme

$$2\pi i\tau_0 = 2N_c \log \left( \frac{\Lambda_{N=2}}{\Lambda_0} \right). \quad (66)$$

It has been shown \[25\] that in the particular breaking pattern, eq. (44), the function $f_{N=1}$, introduced in eq. (54), has to reduce to a constant. In more detail, conditions (63, 64) are solved by the particular choice of inputs (see, e.g., \[26\]):

$$\mathcal{Y}^2 = \tilde{W}'^2 - 4 \Lambda^{2N_c} \quad (\Lambda = \text{const.}). \quad (67)$$

that is the Seiberg-Witten curve \[24\]. Under this choice, for $x \to \infty$

$$R \sim \frac{1}{x^{N_c}} \quad \text{on the first sheet},$$
$$R \sim x^{N_c} \quad \text{on the second}.$$

From eq. (67), one can write

$$\sum \frac{\partial R}{\partial S_h} = - \frac{d}{dx} \log R(x) \quad (68)$$

from which one can easily appreciate that the condition (63) is satisfied. As for eq. (64), one has

$$\int_{C_1} \sum \frac{\partial R}{\partial S_h} = - \log R \big|_P$$

Now

$$\left( \log R \right)_P = \log(W' - \sqrt{W'^2 - 4\Lambda^{2N_c}}) \bigg|_{x \sim \Lambda_0}$$
$$\sim \log \left\{ \Lambda_0^{N_c} - \Lambda_0^{-N_c} [1 - 2 (\Lambda/\Lambda_0')^{2N_c}] \right\}$$
$$= \log \left[ \frac{2\Lambda^{2N_c}}{\Lambda_0^{2N_c}} \right],$$

$$\left( \log R \right)_Q = \log(W' + \sqrt{W'^2 - 4\Lambda^{2N_c}}) \bigg|_{x \sim \Lambda_0'}$$
$$\sim \log \left\{ \Lambda_0^{N_c} + \Lambda_0^{-N_c} [1 - \ldots] \right\}$$
$$= \log \left[ 2\Lambda_0'^{N_c} \right].$$
and

\[ \log R^p_Q = \log \left( \frac{\Lambda}{\Lambda'} \right)^{2N_c}. \]  

(70)

This will be exactly cancelled by the $2\pi i \tau_0$ term in eq. (64) if

\[
\left\{ \begin{array}{l}
\Lambda = \Lambda_{\mathcal{N}=2} \\
\Lambda' = \Lambda_0
\end{array} \right. 
\]

(71)

Thus one can conclude that the SW ansatz is the solution of extremum conditions for our matrix model if the cutoffs $\Lambda'_0$ and $\Lambda$ in the matrix model computation are identified in terms of the $\mathcal{N} = 2^*$ regularization of $\mathcal{N} = 2$ SYM.

For the rest of the calculation, one can go along the standard SW construction of the prepotential [20].

### 3 An improved QFT derivation of superpotential

In this section, we would like to describe an improved computation of the superpotential for the general $SU(N_c)$ SYM model.

We deal with the model given by eq. (3) and apply the techniques outlined in sections 2.1.1 and 2.1.2, i.e. covariant supersymmetric Feynman rules and “ERG” variation of regularizing mass parameters.

With respect to eq. (1), we have seen in section 2 that integrating out $(\bar{\Phi}_a^i)_{a=1}^{N_c^2-1}$ $\quad$ $(i = 1, 2)$ reproduces the holomorphic action (5) which is quadratic in $\Phi_i^a$.

In this section, however, we will make use of the fact that the actions to be dealt are always quadratic and take a shortcut substituting Z-J transformation (6) with a simple rescaling transformation of the field variables.

Thus we apply the following transformation written in momentum space

\[
\Phi^*_i(p, \pi) \rightarrow \left\{ (-p^2 + \pi \hat{W} + \hat{A} + M)^{1/2}(-p^2 + \pi \hat{W} + \hat{A} + M_0)^{-1/2} \right\}_{ia,jb} \Phi^*_j(p, \pi) \]  

(72)

The corresponding Jacobian is

\[
\left\{ \det \left[ (-p^2 + \pi \hat{W} + \hat{A} + M)(-p^2 + \pi \hat{W} + \hat{A} + M_0)^{-1} \right] \right\}^{-1/2} = \exp \left( -\frac{i}{2} \int d^4 p d^2 \pi \text{ tr} \left[ \log(p^2 + \pi \hat{W} + \hat{A} + M) - \log(p^2 + \pi \hat{W} + \hat{A} + M_0) \right] \right) 
\]

(the 4-momenta are Wick rotated) where again $\hat{A}(\bar{\Phi}_3)$ is treated as if it were a constant matrix (see section 2.1).
Note that in the present situation, the Konishi anomaly is the simple consequence of our holomorphic Feynman rule. \[8\]

The resulting intermediate effective action (as a functional of $W$ and $\hat{\Phi}_3$) is given by

\[-\frac{1}{2}\int d^4p \, d^2\pi \, \text{tr} \left[ \log(\hat{\Gamma} + M_0) - \log(\hat{\Gamma} + M) \right], \tag{73}\]

with $\hat{\Gamma} \equiv p^2 + \pi \hat{W} + \hat{A}$. Effecting the fermionic integration yields

\[-\frac{1}{8}\int d^4p \left\{ \text{tr} \left[ (p^2 + \mu + \hat{A})^{-1} \hat{W}_1 (p^2 + \mu + \hat{A})^{-1} \hat{W}_2 \right] - (\hat{W}_1 \leftrightarrow \hat{W}_2) \right\}_{\mu = M_0} =
\]

\[-\frac{1}{8}\int d^4p \frac{1}{(p^2 + \mu)^2} \text{tr} \left[ \left( 1 + \frac{\hat{A}}{p^2 + \mu} \right)^{-1} \hat{W}_1 \left( 1 + \frac{\hat{A}}{p^2 + \mu} \right)^{-1} \hat{W}_2 - (\hat{W}_1 \leftrightarrow \hat{W}_2) \right]_{\mu = M} \equiv S^{(1)}(M_0) - S^{(1)}(M). \tag{74}\]

The matrices $\hat{W}$ and $\hat{A}$ are defined in eq. \[\text{(5)}\].

As in the previous section, we consider the effective action \[\text{(74)}\] only up to the quadratic term in $\hat{\Phi}_3$. Then

\[S^{(1)}(\mu) \sim \int \frac{d^4p}{(p^2 + \mu)^2} \, \text{tr}_{\text{adj}} \left\{ (\hat{W}_1 \cdot \hat{W}_2) - \frac{g_0^2/8}{(p^2 + \mu)^2} \left[ (F \cdot \phi)^2 (F \cdot W_1)(F \cdot W_2) + (F \cdot W_1)(F \cdot \phi)^2 (F \cdot W_2) + (F \cdot W_1)(F \cdot W_2)(F \cdot \phi)(F \cdot W_2) \right] - (W_1 \leftrightarrow W_2) \right\} \]

(where it is understood that the momentum integration should be done only after taking the difference $S^{(1)}(M_0) - S^{(1)}(M)$.

Making use of the commutation relation between $F \cdot \phi$’s and $F \cdot W$’s, one can rewrite it as

\[S^{(1)}(\mu) \sim 2 \int \frac{d^4p}{(p^2 + \mu)^2} \, \text{tr}_{\text{adj}} \left\{ (\hat{W}_1 \cdot \hat{W}_2) - \frac{g_0^2/8}{(p^2 + \mu)^2} \left[ 3(F \cdot \phi)^2 (F \cdot W_1)(F \cdot W_2) + \frac{N_c}{2} f_{abc} W_1^a \phi_c f_{ade} W_2^d \phi_e \right] - (1 \leftrightarrow 2) \right\} \tag{75}\]

Introducing a Cartan-Weyl basis for $SU(N_c)$, $S^{(1)}(\mu)$ can be written as

\[S^{(1)}(\mu) \sim 2 \int \frac{d^4p}{(p^2 + \mu)^2} \left\{ \text{tr}(\hat{W}^2) + \frac{g_0^2/8}{(p^2 + \mu)^2} \left[ 3 \sum_\alpha (F \cdot \phi)^2_{\alpha\alpha} (\hat{\alpha} \cdot \hat{W})^2 - N_c \sum_\alpha (\hat{\alpha} \cdot \hat{W})^2 \phi_{-\alpha} \phi_\alpha \right] \right\} \tag{76}\]
where $\bar{\alpha}$ refers to the roots and $\bar{W}$ is taken to belong to the Cartan subalgebra.

In our low energy (potential) approximation only the charged components of the chiral scalar contributes to the effective action, so one may replace $(F \cdot \phi)^2$ by $(E_{\beta \phi \beta})_{\alpha \alpha}$, where $E_\alpha$ are the ladder operators in the SU($N_c$) algebra. Then

$$\sum_{\alpha} (F \cdot \phi)^2_{\alpha \alpha} (\bar{\alpha} \cdot \bar{W})^2 = \sum_{\alpha} (E_{\beta \phi \beta})_{\alpha \alpha} (\bar{\alpha} \cdot \bar{W})^2 = \sum_{i>j} [N_c (W_i^2 + W_j^2) + W^2] \phi_{\mu_i - \mu_j} \phi_{\mu_j - \mu_i}$$ (77)

where $\mu_i (i = 1, ..., N_c)$ are the weights of the fundamental representation of SU($N_c$).

As for the second term (the commutator term)

$$- N_c \sum_{\alpha} (\bar{\alpha} \cdot \bar{W})^2 \phi_{- \alpha} \phi_\alpha = - N_c \sum_{i>j} [W_i - W_j]^2 \phi_{\mu_i - \mu_j} \phi_{\mu_j - \mu_i}$$ (78)

Eq. (76) becomes

$$S^{(1)}(\mu) = 2 \int \frac{d^4 p}{(p^2 + \mu)^2} \left\{ \tau (\bar{W}^2) - \frac{g_0^2}{8} \frac{(p^2 + \mu)^2}{(p^2 + \mu)^2} \cdot \sum_{i<j} \left[ 2N_c (W_i^2 + W_j^2 + W_i W_j) + 3W^2 \right] \phi_{\mu_i - \mu_j} \phi_{\mu_j - \mu_i} \right\}.$$ (79)

Thus $S^{(1)}(M_0) - S^{(1)}(M)$ consists of two terms:

1. **Constant ($\bar{\Phi}_3$ independent) term**

$$2 \int \frac{d^4 p}{(p^2 + \mu)^2} \left. \tau (\bar{W}^2) \right|_M = \frac{2N_c W^2}{16 \pi^2} \int \frac{d \tau}{(\tau + \mu)^2} \left|_M^M_0 \right. = \frac{N_c W^2}{16 \pi^2} \log \left( \frac{M}{M_0} \right)^2$$ (80)

2. **$\bar{\Phi}_3$ mass term**

$$\int d^4 p \frac{2g_0^2/8}{(p^2 + \mu)^4} \sum_{i<j} \left[ 2N_c \omega_{ij} + 3W^2 \right] \phi_{\mu_i - \mu_j} \phi_{\mu_j - \mu_i} \bigg|^{\mu=M_0}_{\mu=M} = \frac{2g_0^2}{16 \pi^2 8 \cdot 6M^2} \sum_{i<j} (2N_c \omega_{ij} + 3W^2) \phi_{\mu_i - \mu_j} \phi_{\mu_j - \mu_i}$$ (81)

where $\omega_{ij} \equiv W_i^2 + W_j^2 + W_i W_j$ and we have omitted the term proportional to $1/M_0^2$.

Thus after integrating $\bar{\Phi}_1$ and $\bar{\Phi}_2$, the effective action for $\bar{\Phi}_3 = \phi$ is, up to the quadratic term,

$$\int d^4 p d^2 \pi \frac{1}{2} \sum_{i>j} \left\{ -p^2 + \pi \bar{W} + \left[ \mu + \frac{2}{32 \pi^2} \frac{g_0^2}{8} \frac{1}{3M^2} \left( 2N_c \omega_{ij} + 3W^2 \right) \right] \phi_{\mu_i - \mu_j} \phi_{\mu_j - \mu_i} \right\}$$ (82)
This corresponds to eq. 8 of section 2.1 in the case of $N_c = 2$. Note however that, due to the different treatment with respect to the anticommuting external field $\vec{W}^{(\alpha)}$, eq. (82) would not go over formally to eq. 8 simply with $N_c = 2$ and $W_1 \cdot W_2 = W^2/2$ (see below).

To obtain the final form of the effective potential one integrates over $\Phi_3$ with the approximate quadratic action (82).

Here one has again to apply the RG transformation to separate the contribution for $N = 4$ SYM $S^{(2)}(M)$ ($M \to 0$) and “Wilsonian part”, $S^{(2)}(M_0) - S^{(2)}(M)$. Because of the gaussian approximation, one can repeat the same arguments used for integrating out $\Phi_1$ and $\Phi_2$, making use of an appropriate rescaling transformation.

The resulting non zero contribution to the effective action, $S^{(2)}$ (function of $\vec{W}$ only), is

$$S^{(2)}(\mu) = \frac{1}{4} \int d^4 p \, d^2 \pi \sum_{i \neq j} \pi^\alpha (W_i - W_j) \pi^\beta (W_i - W_j) \times$$

$$\times \left[ p^2 + \mu + \frac{g_0^2}{64\pi^2} \cdot 3M^2 \right]^{-2}$$

$$= \frac{1}{8} \int d^4 p \sum_{i,j} (W_i^2 + W_j^2 - W_i W_j - W_j W_i) \times$$

$$\times \left[ p^2 + \mu + \frac{g_0^2 W^2}{64\pi^2 M^2} + \frac{g_0^2 N_c \omega_{ij}}{32\pi^2 \cdot 3M^2} \right]^{-2}$$

(83)

If one expands the last expression in powers of $\omega_{ij}$, then the generic term is of the form

$$S_n \equiv \text{const} \int d^4 p \sum_{i,j} (W_i^2 + W_j^2 - W_i W_j - W_j W_i) \left[ p^2 + \mu + \frac{g_0^2 W^2}{64\pi^2 M^2} \right]^{-2} \times$$

$$\times (-)^n (n+1) \left[ \frac{g_0^2 W^2}{32\pi^2 \cdot 3M^2} \right]^n \omega_{ij}^n \quad (n = 0, 1, 2, \ldots)$$

(84)

Here one must remember that $W_i^\alpha, W_j^\beta$ are anticommuting fields. One consequence of this is of course that

$$(W^2)^\nu = 0 \quad \text{for} \quad \nu \geq N_c$$

(85)

However, let us assume that $N_c$ is sufficiently large so that one can attach unambiguous meaning to the function defined by the series

$$\sum_{\nu=0}^{\infty} C_\nu (W^2)^\nu$$

(86)
to any desired order.

The important point is the fact that in (84), for any given pair \( i \neq j \), one cannot have the product of more than four \( W_i \) and \( W_j \), i.e. \( S_n = 0 \) except for \( n = 0 \) or \( n = 1 \).

Then eq. (84) reduces to

\[
S^{(2)}(\mu) = \frac{1}{8} \int d^4 p \sum_{i,j} (W_i^2 + W_j^2 - W_i W_j - W_j W_i) \times \\
\times \left[ p^2 + \mu + \frac{g_0^2 W^2}{64\pi^2 M^2} \right]^{-2} \left( 1 - \frac{2g_0^2 N_c \omega_{ij}}{32\pi^2 M^2} \right) \\
= \frac{N_c}{8} \int d^4 p \left[ p^2 + \mu + \frac{g_0^2 W^2}{64\pi^2 M^2} \right]^{-2} \left( W^2 - \frac{g_0^2 (W^2)^2}{64\pi^2 M^2} \right) \quad (87)
\]

Note that, to arrive at the above, one has used the completeness relation for the \( SU(N_c) \) weights

\[
\sum_{i=1}^{N_c} \mu^A_i \mu^B_i = \delta_{AB}/2 
\] (88)

Thus, the superpotential can be determined from

\[
S^{(2)}(M_0) - S^{(2)}(M) = \frac{N_c}{8 \cdot 16\pi^2} \int_0^\infty \frac{\tau d\tau}{\tau + \mu + \frac{g_0^2 W^2}{64\pi^2 M^2}} \left[ W^2 - \frac{g_0^2 (W^2)^2}{32\pi^2 M^2} \right] \left. \right|_{\mu=M_0}^{\mu=M} \\
= \frac{N_c W^2}{8 \cdot 16\pi^2} \left\{ \log \left( \frac{M + \frac{g_0^2 W^2}{64\pi^2 M^2}}{M_0 + \frac{g_0^2 W^2}{64\pi^2 M^2}} \right) + \frac{1}{2} \frac{g_0^2 W^2}{64\pi^2 M^2} \right\} \quad (89)
\]

Again one has omitted \( 1/M_0^2 \) term.

Naturally, eq. (89) will not go over to eq. (9) of section 2.1 when \( N_c = 2 \). This is, as mentioned above, due to the different treatment of the anticommuting fields. Roughly speaking, eq. (9) makes unambiguous sense only up to \( \mathcal{O}(W^2) \).

### 3.1 IR limit of superpotential and VY form

According to the assumptions discussed in the Introduction, eq. (89) should lead to the superpotential for \( \mathcal{N} = 1 \) SYM (or more generally, to the perturbative expansion of \( \mathcal{N} = 1^* \) model with small parameter \( g_0^2 W^2/M_0^3 \)) when \( M \to 0 \), that is when the residual terms should reproduce the superpotential for \( \mathcal{N} = 4 \) SYM, i.e. a triviality.
The second term in the last line of eq. (89) will go over to constant as \( M \to 0 \) (or, more appropriately, as \( \sim M^3/W^2 \to 0 \)), so we may substitute its contribution for small \( M \) by
\[
\frac{N_c}{8 \cdot 16\pi^2} W^2
\] (90)

To the one loop integral (89), one must add the constant term (80) and the kinematical term in \( S_{N=4} \) SYM which is equal to
\[
\frac{1}{16\pi^2} \int W^2 = \frac{1}{16\pi^2} \frac{N_c}{8} \left\{ \log \left[ \left( \frac{M_0}{\Lambda} \right)^3 \frac{1}{g_0^2} \right] + \frac{i\vartheta_0}{N_c} \right\} \int W^2
\] (91)

The final result is then
\[
\frac{N_c}{128\pi^2} \int W^2 \left\{ \log \left[ \left( \frac{M}{\Lambda} \right)^3 \frac{1}{g_0^2} \right] + \log \left( \frac{1 + \frac{g_0^2 W^2}{64\pi^2 M_0^2 M^3}}{1 + \frac{g_0^2 W^2}{64\pi^2 M_0^2 M^3}} \right) + 1 + \frac{i\vartheta_0}{N_c} \right\}
\] (92)

To go further and deduce the VY form of the potential, one has to follow the same argument as in section 2.1 [8] [cf. eq. (80)]: if one is allowed to conclude that the second, logarithmic term in eq. (92) can be replaced for “small \( M \)” by
\[
\sim \log \left( \frac{g_0^2 W^2}{64\pi^2 M^3} \right),
\] (93)
then the whole of eq. (92) can be reduced to VY form:
\[
\sim \frac{N_c}{128\pi^2} \int W^2 \left\{ \log \left[ \frac{W^2}{128\pi^2} \frac{2\Lambda^3}{e} \right] + \frac{i\vartheta_0}{N_c} \right\}.
\] (94)

Eq. (94) should be compared with the standard expression
\[
\frac{N_c}{128\pi^2} \int W^2 \left\{ \log \left[ \frac{W^2}{128\pi^2} \frac{\Lambda^3}{e} \right] + \frac{i\vartheta_0}{N_c} \right\}
\] (95)

Now, to assert
\[
\log \left[ \frac{1 + \alpha g_0^2 W^2/M^3}{1 + \alpha g_0^2 W^2/(M_0 M^2)} \right] \sim \log \left( \frac{\alpha g_0^2 W^2/M^3}{M_0 M^2} \right)
\] (96)
\((M_0 \gg M, M \sim 0, \alpha = \text{numerical constant})\) it is necessary and sufficient to have
\[
\alpha g_0^2 W^2/M^3 \gg 1 \quad \text{and} \quad \alpha g_0^2 W^2/(M_0 M^2) \equiv (M/M_0)\alpha g_0^2 W^2/M^3 \ll 1 \quad (97)
\]
Substituting to $W^2$ its “desired” mean value $W^2 \sim \Lambda^3$, (96) becomes

$$g_0^2(\Lambda/M)^3 \gg 1 \quad \text{and} \quad (M/M_0)g_0^2(\Lambda/M)^3 \ll 1$$

(98)

or

$$(M_0/M) \gg g_0^2(\Lambda/M)^3 \gg 1$$

(99)

By the definition of the dynamical cutoff of $\mathcal{N} = 1$ SYM

$$g_0^2(\Lambda/M)^3 = (M_0/M)^3 \exp \left[ -8\pi^2/(N_c g_0^2) \right]$$

In the end, one needs the simultaneous inequalities

$$(M_0/M)^2 \exp \left[ -8\pi^2/(N_c g_0^2) \right] \ll 1 \quad \text{and} \quad (M_0/M)^3 \exp \left[ -8\pi^2/(N_c g_0^2) \right] \gg 1$$

or

$$\exp \left[ \frac{1}{3} \frac{8\pi^2/(N_c g_0^2)}{8\pi^2/(N_c g_0^2)} \right] \ll (M_0/M) \ll \exp \left[ \frac{1}{2} \frac{8\pi^2/(N_c g_0^2)}{8\pi^2/(N_c g_0^2)} \right]$$

(100)

Does this last set of inequalities make sense for values of $M_0/M$ between 1 and $\infty$?

Naturally, this makes sense only for

$$\exp \left[ \frac{8\pi^2/(N_c g_0^2)}{8\pi^2/(N_c g_0^2)} \right] \gg 1.$$

(101)

Only then $(M/M_0)$ can flow into $M/M_0 \to 0$. (101) implies the ’t Hooft coupling should be small

$$\lambda^2 = N_c g_0^2 \to 0$$

On the other hand, in the strong coupling regime,

$$\exp \left[ \frac{8\pi^2/(N_c g_0^2)}{8\pi^2/(N_c g_0^2)} \right] \ll 1,$$

(102)

the inequalities (100) do not make any sense ($M/M_0 \sim 1$) and the present computational scheme collapses.

### 3.2 Non-Gaussian corrections

The above discussion is relevant also for another difficulty raised in section 2.1, i.e. the justification of the gaussian approximation.

The full effective action for $\Phi_3 = \phi$, after $\Phi_1$ and $\Phi_2$ have been integrated out, looks like

$$\int \left( \frac{1}{2} \phi \left( \hat{\Gamma} + M_0 \right) \phi + W^2 \zeta^4 \left( \frac{g_0}{M} \phi \right) \right)$$

(103)
where \( \hat{\Gamma} = -p^2 + \pi \hat{W} + (1/32\pi^2)(g_0^2/3M^2)W^2 \).

To simplify the discussion, here we have taken the special case of \( N_c = 2 \), the generalization to arbitrary \( N_c \) being straightforward.

The rescaling transformation, which can be written as

\[
\phi \rightarrow \sqrt{\frac{\hat{\Gamma} + M}{\hat{\Gamma} + M_0}} \phi
\]

transforms eq. (103) to

\[
\int \left\{ \frac{1}{2} \phi \left( \hat{\Gamma} + M_0 \right) \phi + W^2 \zeta' \left( \frac{g_0}{M} \sqrt{\frac{\hat{\Gamma} + M}{\hat{\Gamma} + M_0}} \phi \right) \right\}
\]

From eq. (104), one can read off the components of relevant Feynman graphs. An internal line connecting any two vertices in \( \zeta' \) looks like

\[
\sim \left( \frac{g_0}{M} \right)^2 \frac{1}{\hat{\Gamma} + M_0}.
\]

Thus, for a vacuum graph with \( I \) internal lines, \( V \) vertices and \( L \) loops, its value is proportional to

\[
F(I, V, L) \propto S^L S^V \left[ \left( \frac{g_0^2}{M} \right)^2 \frac{1}{\hat{\Gamma} + M_0} \right]^I = S \left( \frac{S(g_0/M)^2}{\hat{\Gamma} + M_0} \right)^I \quad \text{since } L + V = I - 1
\]

\[
\sim S \left( \frac{g_0^2 W^2}{M_0 + g_0^2 W^2} \right)^I = S \left( \frac{g_0^2 W^2}{M_0 + \frac{g_0^2 W^2}{32\pi^2 3M^2}} \right)^I
\]

Thus, again, the error due to the gaussian approximation is negligible only if (cfr. eq. (97))

\[
\frac{g_0^2 W^2}{M_0 M^2} \ll 1.
\]

4 Conclusions

In this note, we have attempted to explain the VY potential of \( \mathcal{N} = 1 \) SYM model with gauge group \( SU(N_c) \) starting with the microscopic Lagrangian and covariant supersymmetric ("holomorphic") Feynman rules, valid for low energy external states.

Instead of the usual instanton expansion, we have applied a Renormalization Group-inspired method of varying the regularizing mass \( \mu \) (\( M_0 \geq \mu \geq M \)). Taking the limit that
$M \to 0$, one hopes to deduce the potential in question as the difference with respect to the holomorphic superpotential of $\mathcal{N} = 4$ SYM model which is assumed to be trivial.

In the end, we have obtained, with a more or less reliable approximation, the superpotential of pure $\mathcal{N} = 1$ SYM. Note that this is obtained essentially as one loop effect, as has been stated or conjectured several times previously [1].

Indeed, the RG method allows one to extract the convergent expression for such a one loop integral, which can then lead to VY form for the case of pure SYM.

However, we have to conclude also that our method, while qualitatively correct, does not arrive at the precision and generality of the Matrix Model approach. As we have seen in [2], the Matrix Model can be applied to much wider class of problems.

Once one accepts the prescription of [3] with the fixed integration measure of [14] one can obtain the superpotential of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ models without any ambiguity.

On the other hand, in [18] the direct correspondence (without going over to the superstring theory or M-theory) has been shown between supersymmetric gauge field theory and matrix model.

More explicitly, one can map the $\mathcal{N} = 1$ $U(N)$ SYM + adjoint matter on non-commutative space time on to the large $\hat{N}$ limit of certain super matrix models.

Since the former at low energies goes over to the usual GFT model on commutative space-time except for the quantities with UV divergence, the authors of [18] apply this correspondence to derive directly the Dijkgraaf-Vafa method.

In the present note, the main object of the discussion is the so-called $\mathcal{N} = 1^*$ model - mass deformed $\mathcal{N} = 4$ SYM - which is free of UV divergences. Thus the demonstration of Kawai et al. must apply in a very simple way.

Then, there must be a simple QFT method which reproduces the Matrix Model results for quantities like superpotentials.

One possibility is that our present method does not take sufficient account of the IR structure of the $\mathcal{N} = 4$ model. Indeed, we did not find the “miracle” corresponding to the large $\hat{N}$ matrix result quoted in section [3].

At the same time, the problem of singular external fields like $S \equiv W^2$ with $S^{N_c} = 0$ ($S$ is however “bosonic”) appears not to have been completely cleared. In fact, one must recognize that the discussion on IR limit at the end of section [3] is still not entirely satisfactory. For instance, for the simplest case of $N_c = 2$, one can even invent a “proof” of the desired IR limit. Namely, by making use of the fact that $(W^2)^2 \equiv 0$ for $N_c = 2$, one can easily show that

$$\log \left\{ \frac{1 + A/M^3}{1 + A/(M_0 M^2)} \right\} + \log \left( \frac{M}{\Lambda} \right)^3 \approx \left( 1 - \frac{M}{M_0} \right) \left\{ \log \left( 1 + \frac{A}{M^3} \right) + \log \left( \frac{M}{\Lambda} \right)^3 \right\} + \frac{M}{M_0} \log \left( \frac{M}{\Lambda} \right)^3,$$
where $A \propto W^2$. The last expression has a finite $M \to 0$ limit which is precisely equal to $\log(A/\Lambda^3)$ as desired.

This “demonstration” only shows that we did not yet establish the exact rule for dealing with “classical” quantities like $S \equiv W^2$ and their functions, quite apart from the quantum effect discussed in [22]. These problems need further investigations.

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In these approaches, VY potential comes out as a one-loop effect. This idea has been made more precise by noting the link between $\mathcal{N} = 1$ SYM in 4D and $\mathcal{N} = 2$ Sigma-model in 2D, through the compactification of the former in $T^3$.

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