Large Banach spaces with no infinite equilateral sets

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Abstract
A subset of a Banach space is called equilateral if the distances between any two of its distinct elements are the same. It is proved that there exist nonseparable Banach spaces (in fact of density continuum) with no infinite equilateral subset. These examples are strictly convex renormings of $\ell_1([0,1])$. A wider class of renormings of $\ell_1([0,1])$ which admit no uncountable equilateral sets is also considered.

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1 | INTRODUCTION

Let $\mathcal{X}$ be a Banach space and $r > 0$ a real. A subset $\mathcal{Y} \subseteq \mathcal{X}$ is called $r$-equilateral if $\|x - y\| = r$ for any two distinct $x, y \in \mathcal{Y}$; it is called equilateral if it is $r$-equilateral for some real $r > 0$.

As shown by Brass and by Dekster [1, 3] for each $k \in \mathbb{N} \setminus \{0,1\}$ there is $d(k) \in \mathbb{N}$ such that every normed space of dimension $d(k)$ admits a $k$ element $r$-equilateral set. However, the smallest value of $d(k)$ is unknown and it is an open conjecture that $d(k)$ can take the value $k - 1$ for each $k \in \mathbb{N} \setminus \{0,1\}$ [20]

The above results of [1, 3] imply that any infinite dimensional Banach space contains arbitrarily large finite equilateral sets. In fact, by a result of Shkarin [16] every finite ultrametric space (a metric space where the distance $d$ satisfies $d(x, z) \leq \max(d(x, y), d(y, z))$ for any points $x, y, z$) isometrically embeds in any infinite dimensional Banach space. A surprising result was obtained by Terenzi who proved in [19] that there are infinite dimensional (separable) Banach spaces with no infinite equilateral sets (for other spaces with this property see [5, 18]). On the other hand Mercourakis and Vassiliadis proved that any Banach space containing an isomorphic copy of $c_0$ admits an infinite equilateral set [13] and Freeman, Odell, Sari and Schlumprecht proved that every uniformly smooth Banach space admits an infinite equilateral set [4]. The difference between the
example of Terenzi and the latter Banach spaces should be seen not only in the context of the geometry of Banach spaces but also in the context of infinite combinatorics, in particular the applicability of Ramsey methods in Banach spaces.

A natural problem if every nonseparable Banach space admits an uncountable equilateral set has been considered in [11, 13, 14]. The first author constructed in [11] a consistent example of a nonseparable Banach space which does not admit an uncountable equilateral set. It is of the form $C(K)$, where $K$ is Hausdorff and compact. However, it was also proved in [11] that it is consistent that no such Banach space of the form $C(K)$ exists. This showed that the problem whether a nonseparable Banach space of the form $C(K)$ admits an uncountable equilateral set is undecidable [11].

The main result of this paper is that there are absolute (under no extra set-theoretic assumption) examples of nonseparable Banach spaces with no uncountable equilateral set (necessarily not of the form $C(K)$). Moreover, they do not even admit an infinite equilateral set and have density continuum.†

Our approach is to transfer some parts of the Terenzi arguments from [19] to the nonseparable setting. He considered a renorming of $\ell^1$, where the norm is given for any $x \in \ell^1$ by

$$\|x\| = \|x\|_1 + \sqrt{\sum_{i \in \mathbb{N}} x(i)^2 / 2^i}.$$ 

We consider renormings of $\ell^1([0, 1])$ where the norm is defined for any $x \in \ell^1([0, 1])$ by

$$\|x\|_T = \|x\|_1 + \|T(x)\|_{\mathcal{X}},$$

where $T : \ell^1([0, 1]) \to \mathcal{X}$ is an injective operator into a Banach space $\mathcal{X}$. Renormings of $\ell^1([0, 1])$ similar but different to ours were already employed in, for example, [6] to obtain Banach spaces not admitting certain subsets. The foundation of our main result is the following:

**Theorem 1.** Suppose that $\mathcal{X}$ is a Banach space with a strictly convex norm and that $T : \ell^1([0, 1]) \to \mathcal{X}$ is a compact bounded injective operator. Then the equivalent renorming $(\ell^1([0, 1]), \| \cdot \|_T)$ of $(\ell^1([0, 1]), \| \cdot \|_1)$ admits no infinite equilateral set.

Since there exist operators as in the hypothesis of the above theorem (Lemma 5) we obtain:

**Corollary 2.** There is a Banach space of density continuum which does not admit an infinite equilateral set.

In particular, this solves the question of whether there is a nonseparable Banach space with no uncountable equilateral set ([11, 14], Problem 293 of [7]). Another absolute construction of a nonseparable Banach space with no uncountable equilateral set is being presented at the same time in a paper by the first author [12]. However, that is a renorming of a space $C_0(K)$ for $K$ locally compact and scattered, so it is $c_0$-saturated (by [15]). Since a result in [13] says that any Banach

† We do not know if the density continuum is the maximal possible. The only result bounding the densities of Banach spaces with no uncountable equilateral sets was obtained by Terenzi in [19] using essentially an Erdős–Rado type argument which is an uncountable version of the Ramsey theorem: If the density of a Banach space $\mathcal{X}$ is bigger than $2^{(2^\omega)}$, then $\mathcal{X}$ admits an uncountable equilateral set.
space which contains an isomorphic copy of \( c_0 \) admits an infinite equilateral set, we conclude that spaces of \([12]\) admit such infinite sets.

By an argument of Terenzi from \([19]\), given any equilateral set \( \mathcal{Y} \) in a Banach space \( \mathcal{X} \) we may assume that it is a 1-equilateral set by scaling it. Considering \( \{ y - y_0 : y \in \mathcal{Y} \setminus \{y_0\} \} \) for any \( y_0 \in \mathcal{Y} \) we may assume that it is a 1-equilateral set included in the unit sphere of \( \mathcal{X} \). Thus equilateral sets are related to the questions concerning separation of points in the spheres of Banach spaces (see, for example, \([8]\) for references). Recall that a subset \( \mathcal{Y} \) of a Banach space \( \mathcal{X} \) is called \( \delta \)-separated if \( \|y - y'\| \geq \delta \) for all distinct \( y, y' \in \mathcal{Y} \). It is called \((\delta+)\)-separated if \( \|y - y'\| > \delta \) for all distinct \( y, y' \in \mathcal{Y} \). By Remark 3.16 \([8]\) the unit sphere of every renorming of \( \ell_1([0,1]) \) contains a subset \( \mathcal{Y} \) of cardinality continuum such that \( \|y - y'\| \geq 1 + \varepsilon \) some \( \varepsilon > 0 \) and for every two distinct \( y, y' \in \mathcal{Y} \).

After proving Theorem 1 in Section 3 we consider renormings \( \|T\| \) of \( \ell_1([0,1]) \) as in (\( \bigcirc \)) for any injective \( T \) with separable range. Some of such renormings admit many infinite equilateral sets (see Remark 10). We obtain:

**Theorem 3.** Suppose that \( \mathcal{X} \) is a Banach space and that \( T : \ell_1([0,1]) \to \mathcal{X} \) is bounded linear operator which is injective and has separable range and that \( r > 0 \). Then \( \ell_1([0,1]), \|T\| \) has the following property: Any \( r \)-separated subset \( \mathcal{Y} \subseteq \ell_1([0,1]) \) of regular uncountable cardinality has a subset \( \mathcal{Z} \subseteq \mathcal{Y} \) of the same cardinality which is \((r+)\)-separated. In particular \( \ell_1([0,1]), \|T\| \) does not admit any uncountable equilateral set.

The above property for renormings induced by \( T \) injective compact and with strictly convex range is a consequence of Theorem 1 and some partition calculus results (see Corollary 7). Also this property is much stronger than not having uncountable equilateral sets (see Remark 11).

A close link between separated subsets in the sphere and Auerbach bases was demonstrated in \([8]\). In fact Godun’s renorming of \( \ell_1([0,1]) \) was designed to prove that the space has no fundamental Auerbach system \([6]\). Nevertheless, we do not know if our spaces admits an uncountable Auerbach system.

Let us also remark that considering Banach spaces without large equilateral sets which have renormings admitting large equilateral sets (obviously \( \ell_1([0,1]) \) admits equilateral sets of cardinality continuum in the standard norm) is sometimes necessary to obtain examples of the former kind. For example, by a result of Swanepoel \([17]\) any infinite dimensional Banach space has an equivalent renorming which admits an infinite equilateral set (see also \([13]\)). Moreover by the results of \([13]\) the existence of a biorthogonal system of cardinality \( \kappa \) in a Banach space \( \mathcal{X} \) implies the existence of an equivalent renorming of \( \mathcal{X} \) which admits equilateral set of cardinality \( \kappa \). This means by a result of Todorcevic \([21]\) that it is consistent that every nonseparable Banach space has an equivalent renorming which admits an uncountable equilateral set. We do not know however if it is consistent that there is a nonseparable Banach space without an equivalent renorming which admits uncountable equilateral sets. The densities of such an example could not exceed the continuum (by a result of W. Johnson that any Banach space of density bigger than continuum admits an uncountable biorthogonal system cf. \([10, Theorem 2.1]\)). If at all possible, the construction for density equal to any consistent value of the continuum would not be easy, as the examples in the literature of Banach spaces which do not admit uncountable biorthogonal sets have reached only the density \( \omega_2 \) so far \([2]\). Note also that it remains open if there are (even consistent) Banach spaces (or even renormings of \( \ell_1(\kappa) \)) of densities in the interval \( (2^{\omega}, 2^{2\omega}) \) which do not admit infinite or uncountable equilateral sets.
2 | PRELIMINARIES AND NOTATION

The notation and terminology are standard. The notation $A^B$ represents the set of all functions from a set $B$ into a set $A$. Given a set $A$ by $[A]^2$ we mean the collection of all two-element subsets of $A$. When $f : [A]^2 \to B$, we say that $A' \subseteq A$ is $b$-monochromatic for $b \in B$ if $f([A']^2) = \{b\}$; a set is called monochromatic if it is $b$-monochromatic for some $b \in B$. The symbols $\omega_1, \omega_2$ denote the first and the second uncountable cardinals, respectively. The set of all natural numbers and the set of all rational numbers are denoted by $\mathbb{N}$ and $\mathbb{Q}$, respectively.

All Banach spaces considered here are over the reals. Whenever $(\mathcal{X}, \| \cdot \|_\mathcal{X})$ is a Banach space we will refer only to $\mathcal{X}$ if $\| \cdot \|_\mathcal{X}$ is clear from the context. We will consider norms $\| \cdot \|_1$ and $\| \cdot \|_2$ defined as usual for a real sequence $(x(i))_{i \in \mathbb{N}}$ by $\|x\|_1 = \sum_{i \in \mathbb{N}} |x(i)|$ and $\|x\|_2 = \sqrt{\sum_{i \in \mathbb{N}} x(i)^2}$. By $\ell_1(A)$ for a set $A$ we mean all functions $x \in \mathbb{R}^A$ such that $\|x\|_1 < \infty$ with $\| \cdot \|_1$ norm and by $\ell_2$ as all functions $x \in \mathbb{R}^\mathbb{N}$ such that $\|x\|_2 < \infty$ with $\| \cdot \|_2$ norm. The dual space to $\ell_1([0,1])$ is $\ell_\infty([0,1])$ together with the action

$$\langle \phi, x \rangle = \sum_{t \in [0,1]} \phi(t)x(t)$$

for any $\phi \in \ell_\infty([0,1])$ and $x \in \ell_1([0,1])$. By a support of $x \in \ell_1(A)$ we mean $\{a \in A : x(a) \neq 0\}$; it is denoted $\text{supp}(x)$. If $x \in \ell_1(A)$ and $B \subseteq A$, then by $x|B$ we mean the coordinatewise product of $x$ and the characteristic function of $B$.

Recall that a Banach space $(\mathcal{X}, \| \cdot \|_\mathcal{X})$ is strictly convex if $\|x + y\| = \|x\| + \|y\|$ implies that there is $\lambda > 0$ such that $x = \lambda y$ for any $x \neq 0 \neq y$. It is well known that the norm on $\ell_2$ is strictly convex. We say that two norms $\| \cdot \|$ and $\| \cdot \|'$ on a Banach space $\mathcal{X}$ are equivalent if there are constants $c, C > 0$ such that $c\|x\| \leq \|x\'| \leq C\|x\|$ for every $x \in \mathcal{X}$. This is equivalent to the fact that the identity is an isomorphism between $(\mathcal{X}, \| \cdot \|)$ and $(\mathcal{X}, \| \cdot \|')$.

**Lemma 4.** Suppose that $\mathcal{X}, \mathcal{Y}$ are Banach spaces and $T : \mathcal{X} \to \mathcal{Y}$ is a bounded linear operator. Then the norm $\| \cdot \|_T$ on $\mathcal{X}$ given by $\|x\|_\mathcal{X} = \|T(x)\|_\mathcal{Y}$ for $x \in \mathcal{X}$ is equivalent to the norm $\| \cdot \|_\mathcal{X}$. If $T$ is injective and $\mathcal{Y}$ is strictly convex, then $\| \cdot \|_T$ is strictly convex.

**Proof.** We have

$$\|x\|_\mathcal{X} \leq \|T(x)\|_\mathcal{Y} \leq (1 + \|T\|)\|x\|_\mathcal{X}.$$ 

For strict convexity suppose that $\|x + y\|_T = \|x\|_T + \|y\|_T$ for some nonzero $x, y \in \mathcal{X}$. So $\|x + y\|_\mathcal{X} + \|T(x + y)\|_\mathcal{Y} = \|x\|_\mathcal{X} + \|y\|_\mathcal{X} + \|T(x)\|_\mathcal{Y} + \|T(y)\|_\mathcal{Y}$. By the triangle inequality this means that $\|x + y\|_\mathcal{X} = \|x\|_\mathcal{X} + \|y\|_\mathcal{X}$ and $\|T(x + y)\|_\mathcal{Y} = \|T(x)\|_\mathcal{Y} + \|T(y)\|_\mathcal{Y}$. The injectivity of $T$ yields $T(x) \neq 0 \neq T(y)$. The strict convexity of $\mathcal{Y}$ yields $\lambda > 0$ such that $T(x) = \lambda T(y)$. The injectivity of $T$ gives $x = \lambda y$.

Let $(I_i)_{i \in \mathbb{N}}$ be an enumeration of all subintervals of $[0,1]$ with rational end-points. We define $x_i^* = \chi_{I_i} \in \ell_\infty([0,1])$, where $\chi_{I_i}$ is the characteristic function of $I_i$. Given a nonzero $x \in \ell_1([0,1])$ we find $t \in [0,1]$ such that $x(t) \neq 0$ and an open interval $I_i \ni t$ such that $\sum \{|x(t')| : t' \in I_i \setminus \{t\} \} < |x(t)|$. Then $(x_i^*, x) \neq 0$. This shows that $\{x_i^* : i \in \mathbb{N}\}$ is total for $\ell_1([0,1])$, that is, $x_i^*(x) = 0$ for each $i \in I$ implies that $x = 0$. Observe that $\|x_i^*\| = 1$ for each $i \in \mathbb{N}$.
Lemma 5. There is a bounded compact injective operator \( T : \ell_1([0,1]) \to \ell_2 \).

Proof. Define \( T \) by

\[
T(x) = \left( \frac{x_i^*(x)}{2^i} \right)_{i \in \mathbb{N}}
\]

for any \( x \in \ell_1([0,1]) \). As \( x_i^* \)'s form a total set, the operator is injective. It is also clear that the values of \( T \) are in \( \ell_2 \) and the operator is bounded with its norm \( \sqrt{2} \), as \( \| x_i^* \| = 1 \) for each \( i \in \mathbb{N} \).

For the compactness, use again the fact that the norms of \( x_i^* \)'s are 1 and so \( T \) can be approximated in the operator norm by finite rank operators which are \( T \) up to the \( k \)-th coordinate and later 0 for \( k \in \mathbb{N} \). As compact operators form a closed ideal this proves the compactness of \( T \). \( \square \)

3 | PROOF OF THE MAIN RESULT

Theorem 1. Suppose that \( \mathcal{X} \) is a Banach space with a strictly convex norm and that \( T : \ell_1([0,1]) \to \mathcal{X} \) is a compact bounded injective operator. Then the equivalent renorming \((\ell_1([0,1]), \|\cdot\|_T)\) of \((\ell_1([0,1]), \|\cdot\|_1)\) admits no infinite equilateral set.

Proof. Suppose that \( \{x_n : n \in \mathbb{N}\} \) is equilateral in \( \ell_1([0,1]) \) with the norm \( \|\cdot\|_T \). We will derive a contradiction. By scaling it, we may assume that it is 1-equilateral. As the supports of \( x_n \)'s are countable, they are all included in some countable \( A \subseteq [0,1] \). So we need to prove that the corresponding renorming of the separable \( \ell_1(A) \) does not admit an infinite equilateral set.

By the compactness of \( T \), passing to a subsequence we may assume that \( \{T(x_n) : n \in \mathbb{N}\} \) converges in the norm \( \|\cdot\|_{\mathcal{X}} \) to \( z \in \mathcal{X} \). As the range of \( T \) is not closed, \( z \) does not need to belong to it.

Since we work now with separable \( \ell_1(A) \) and \( (x_n)_{n \in \mathbb{N}} \) is bounded, by passing to a subsequence we may assume that \( (x_n)_{n \in \mathbb{N}} \) converges pointwise to \( y \in \ell_1(A) \), that is for every \( t \in A \) the sequence \( (x_n(t))_{n \in \mathbb{N}} \) converges to 0. Moreover \( (T(x_n'))_{n \in \mathbb{N}} \) converges in the norm \( \|\cdot\|_{\mathcal{X}} \) to \( z' = z - T(y) \).

Let \( x_n' = x_n - y \) for every \( n \in \mathbb{N} \). Then \( \|x_n' - x_m'\|_T = \|x_n - x_m\|_T = 1 \) for all distinct \( n, m \in \mathbb{N} \) and for every \( t \in A \) the sequence \( (x_n'(t))_{n \in \mathbb{N}} \) converges to 0. Moreover \( (T(x_n'))_{n \in \mathbb{N}} \) converges in the norm \( \|\cdot\|_{\mathcal{X}} \) to \( z' = z - T(y) \).

Fix \( m \in \mathbb{N} \) and \( \varepsilon > 0 \). Choose a finite \( F \subseteq A \) such that \( \|x_n' - x_m'\|_1 < \varepsilon/4 \). As \( (x_n'(t))_{n \in \mathbb{N}} \) converges to 0, for each \( t \in F \), for sufficiently large \( n \in \mathbb{N} \) we have \( \|x_n'|_F\|_1 \leq \varepsilon/4 \) and so we have

\[
\begin{align*}
\|x_n'\|_1 &\leq \|x_n' - x_m'|_F\|_1 + \varepsilon/4, \\
\|x_m'|_1 &\leq \|x_m'|_F\|_1 + \varepsilon/4, \\
\|x_n' - x_m'\|_1 &\geq \|(x_n' - x_m'|_F) - x_m'|_F\|_1 - \varepsilon/2, \\
\|x_n' - x_m'|_F\|_1 &\geq \|(x_n' - x_m'|_F)\|_1 + \|x_m'|_F\|_1 \text{ as these vectors have disjoint supports,}
\end{align*}
\]

so we obtain

\[
\begin{align*}
\|x_n'\|_1 + \|x_m'|_1 - \|x_n' - x_m'\|_1 &\leq \|x_n' - x_m'|_F\|_1 + \|x_m'|_F\|_1 - \|(x_n' - x_m'|_F) - x_m'|_F\|_1 + \varepsilon = \varepsilon.
\end{align*}
\]
So for every $m \in \mathbb{N}$ we have

$$\lim_{n \to \infty} (\|x'_n\|_1 + \|x'_m\|_1 - \|x'_n - x'_m\|_1) = 0. \quad (*)$$

For $n \in \mathbb{N}$ define

$$c_n = 1/2 - \|x'_n\|_1.$$

**Claim 6.** $\lim_{n \to \infty} c_n = 0.$

**Proof of the Claim.** Since the sequence of $c_n$'s is bounded (as the $x'_n$'s form an equilateral set), by passing to a subsequence we may assume that it is converging to $c$. First suppose that $c > 0$. Let $k \in \mathbb{N}$ and $\varepsilon > 0$ be such that $\|x'_n\| < 1/2 - \varepsilon$ for all $n > k$. Then by passing to a subsequence we may assume that $\|T(x'_n) - T(x'_m)\|_{\mathcal{X}} \leq \varepsilon$ for all $n, m \in \mathbb{N}$ as $(T(x'_n))_{n \in \mathbb{N}}$ converges to $z'$ in $\mathcal{X}$. Fixing $m > k$ by the triangle inequality we obtain

$$\|x'_n - x'_m\|_T = \|x'_n - x'_m\|_1 + \|T(x'_n - x'_m)\|_{\mathcal{X}} \leq \|x'_n\|_1 + \|x'_m\|_1 + \|T(x'_n) - T(x'_m)\|_{\mathcal{X}} < 2(1/2 - \varepsilon) + \varepsilon \leq 1$$

contradicting the fact that $x'_n$'s form a 1-equilateral set.

Now suppose that $c < 0$. Let $k \in \mathbb{N}$ and $\varepsilon > 0$ be such that $\|x'_n\| > 1/2 + \varepsilon$ for all $n > k$. Fixing $m > k$ by $(*)$ we may find $n \in \mathbb{N}$ such that $\|x'_n\|_1 + \|x'_m\|_1 - \|x'_n - x'_m\|_1 \leq \varepsilon$. So

$$\|x'_n - x'_m\|_T > \|x'_n - x'_m\|_1 = (\|x'_n - x'_m\|_1 - \|x'_n\|_1 - \|x'_m\|_1) + (\|x'_n\|_1 + \|x'_m\|_1) > -\varepsilon + 2(1/2 + \varepsilon) \geq 1$$

contradicting the fact that $x'_n$'s form a 1-equilateral set. This completes the proof of the claim.

For distinct $m, n \in \mathbb{N}$ we have

$$\|x'_n - x'_m\|_T = \|x'_n - x'_m\|_1 + \|T(x'_n) - T(x'_m)\|_{\mathcal{X}} = 1,$$

so subtracting $1 = 1/2 + 1/2 = (\|x'_n\|_1 + c_n) + (\|x'_m\|_1 + c_m)$ from both sides of the second equality, we get

$$-c_n - c_m - \|x'_n\|_1 - \|x'_m\|_1 + \|x'_n - x'_m\|_1 + \|T(x'_n) - T(x'_m)\|_{\mathcal{X}} = 0. \quad (***)$$

Fixing any $m \in \mathbb{N}$, by going to infinity with $n \in \mathbb{N}$ by $(*)$ and Claim 6 we obtain

$$\|z' - T(x'_m)\|_{\mathcal{X}} = c_m. \quad (***)$$

Defining $u_m = T(x'_m) - z'$ and combining $(***)$ and $(***)$ we obtain for distinct $n, m \in \mathbb{N}$

$$\|x'_n - x'_m\|_1 - \|x'_n\|_1 - \|x'_m\|_1 = \|u_n\|_{\mathcal{X}} + \|u_m\|_{\mathcal{X}} - \|u_n - u_m\|_{\mathcal{X}}.$$
By the triangle inequality the right-hand side of the above is non-negative while the left-hand side is non-positive which implies that both of the expressions are equal to zero. In particular, the left-hand side is zero.

Note that \( u_n \neq u_m \) for distinct \( n, m \in \mathbb{N} \) because otherwise we would have \( T(x'_n) = T(x'_m) \), which implies \( x'_n = x'_m \) by the injectivity of \( T \) and this contradicts the fact that \( x'_n \)'s form a 1-equilateral set. So at most one \( u_n \) can be zero. By passing to an infinite subset we may assume that all are nonzero, so that we can apply the definition of the strict convexity.

By the strict convexity of the norm in \( \mathcal{X} \) we obtain \( \lambda_{m,n} > 0 \) such that \( u_m = \lambda_{m,n} u_n \). If \( \lambda_{m,n} = 1 \) for some distinct \( m, n \in \mathbb{N} \), then \( T(x'_n) = T(x'_m) \) which contradicts the injectivity of \( T \). Otherwise \( (z' - T(x'_m)) = \lambda_{m,n} (z' - T(x'_n)) \) which gives \( z'(1 - \lambda_{m,n}) = -\lambda_{m,n} T(x'_n) + T(x'_m) \) and so

\[
z' = T \left( \frac{1}{1 - \lambda_{m,n}} (x'_m - \lambda_{m,n} x'_n) \right).
\]

By the injectivity of \( T \) this means for any distinct \( k, l \in \mathbb{N} \)

\[
\frac{1}{1 - \lambda_{m,n}} (x'_m - \lambda_{m,n} x'_n) = \frac{1}{1 - \lambda_{k,l}} (x'_k - \lambda_{k,l} x'_l),
\]

in particular that

\[
x'_m = \frac{1 - \lambda_{m,n}}{1 - \lambda_{k,l}} (x'_k - \lambda_{m,n} x'_l) + \lambda_{m,n} x'_n.
\]

That means that \( \{x'_n : n \in \mathbb{N}\} \) spans a two-dimensional space (fix \( k = n \neq l \) and vary \( m \)). However such a space cannot admit an infinite equilateral set as its ball is compact and so any bounded sequence has a convergent subsequence. This is the required contradiction.

\[\square\]

**Corollary 7.** Suppose that \( \mathcal{X} \) is a Banach space with a strictly convex norm and that \( T : \ell_1([0,1]) \to \mathcal{X} \) is a compact bounded injective operator. Then the equivalent renorming \((\ell_1([0,1]), \|\cdot\|_T)\) of \((\ell_1([0,1]), \|\cdot\|_1)\) has the following property: Any infinite \( r \)-separated subset \( \mathcal{Y} \subseteq \ell_1([0,1]) \) has a subset \( \mathcal{Z} \subseteq \mathcal{Y} \) of the same cardinality which is \((r+)-separated.

**Proof.** Define a function \( c : [\mathcal{Y}]^2 \to \{0,1\} \) by putting \( c([y, y']) = 0 \) if and only if \( \|y - y'\|_T = r \). First consider the case when \( \mathcal{Y} \) is countable. By Ramsey’s [9, Problem 24.1] theorem there is an infinite monochromatic subset of \( \mathcal{Y} \). It cannot be 0-monochromatic as \((\ell_1([0,1]), \|\cdot\|_T)\) has no infinite equilateral sets by Theorem 1. A 1-monochromatic infinite subset is the required one. Now consider an uncountable regular cardinality of \( \mathcal{Y} \). A version of Dushnik Miller theorem says that for any uncountable cardinal \( \kappa \) for any \( f : [\kappa]^2 \to \{0,1\} \) there is either an infinite 0-monochromatic set or a 1-monochromatic subset of cardinality \( \kappa \) [9, Problem 24.13]. So apply this to \( c \) and use the fact that \((\ell_1([0,1]), \|\cdot\|_T)\) has no infinite equilateral sets by Theorem 1. A 1-monochromatic infinite subset is the required one. \[\square\]

## 4 REFORMINGS INDUCED BY INJECTIVE SEPARABLE RANGE OPERATORS

In this section we prove that a substantial part of the property of Corollary 7 holds for much bigger class of renormings of \( \ell_1([0,1]) \) than those considered in Theorem 1.
Theorem 3. Suppose that $\mathcal{X}$ is a Banach space and that $T : \ell_1([0,1]) \to \mathcal{X}$ is a bounded linear operator which is injective and has separable range and that $r > 0$. Then $(\ell_1([0,1]), \|\cdot\|_T)$ has the following property: Any $r$-separated subset $\mathcal{Y} \subseteq \ell_1([0,1])$ of regular uncountable cardinality has a subset $\mathcal{Z} \subseteq \mathcal{Y}$ of the same cardinality which is $(r+)$-separated. In particular $(\ell_1([0,1]), \|\cdot\|_T)$ does not admit any uncountable equilateral set.

Proof. Let $\{d_n : n \in \mathbb{N}\}$ be a dense subset of the range of $T$. Suppose that $x$ is a regular uncountable cardinal and $\mathcal{Y} = \{x_\alpha : \alpha < x\} \subseteq \ell_1([0,1])$ is $r$-separated in the norm $\|\cdot\|_T$. As the supports of $x_\alpha$s are countable, their union has cardinality at most $x$. So we may assume that $\mathcal{Y} \subseteq \ell_1(A)$ for some $A \subseteq [0,1]$ of cardinality $x$. Let $A = \{t_\xi : \xi < x\}$.

Now we need to define a certain function $M$. The domain of $M$ will consist of 5-tuples of the form $(\varepsilon, q, F, s, n)$, where $\varepsilon, q > 0$ are rationals, $F$ is a finite subset of $[0,1]$, $s \in \mathbb{Q}_F$ and $n \in \mathbb{N}$. Note that there are only countably many such $\varepsilon, q, n$ and given a finite $F \subseteq [0,1]$ there are only countably many choices for $s \in \mathbb{Q}_F$.

The function $M$ will assume values in $x$. It is defined as follows: If there is $\alpha < x$ such that

(a) $\|d_n - T(x_\alpha)\|_\mathcal{X} < \varepsilon/10$,
(b) $\|\|x_\alpha|[0,1] \setminus F\|_1 - q\| < 2\varepsilon/10$,
(c) $\|s - (x_\alpha|F)\|_1 < \varepsilon/10$,

then we choose minimal such $\alpha$. Define $M(\varepsilon, q, F, s, n)$ as the minimal ordinal less than $x$ such that $\supp(x_\alpha) \subseteq \{t_\xi : \xi < M(\varepsilon, q, F, s, n)\}$. If there is no such $\alpha$, then we define $M(\varepsilon, q, F, s, n)$ anyhow.

Claim 8.

$$C = \{\delta < x : \forall (\varepsilon, q, F, s, n) \in \text{dom}(M) \ [F \subseteq \{t_\xi : \xi < \delta\} \Rightarrow M(\varepsilon, q, F, s, n) < \delta]\}$$

is unbounded in $x$

Proof of the Claim. Fix $\delta_0 < x$. By recursion define a strictly increasing $(\delta_n)_{n \in \mathbb{N}}$ in $x$ such that $M(\varepsilon, q, F, s, n) < \delta_{n+1}$ whenever $(\varepsilon, q, F, s, n)$ is in the domain of $M$ and $F \subseteq \delta_n$. Given $\delta_n$ there are less than $x$ many elements in the domain of $M$ such that $F \subseteq \delta_n$ as there are only less than $x$ such $F$s, so by the regularity of $x$ the next $\delta_{n+1}$ can be taken as the supremum of all the values under $M$ of such elements. One sees that $\delta = \sup\{\delta_n : n \in \mathbb{N}\}$ is in $C$. As $\delta_0$ was arbitrary, this completes the proof of the claim.

For any $\delta \in C$ and $\alpha < x$ we define

$$x_{\alpha, \delta} = x_\alpha|\{t_\xi : \delta < \xi < x\}.$$

Claim 9. If $x_{\alpha, \delta} \neq 0$, then $\|x_{\alpha, \delta}\|_1 \geq r/2$ for each $\alpha < x$ and $\delta \in C$.

Proof of the Claim. Fix $\delta \in C$ and $\alpha < x$ such that $x_{\alpha, \delta} \neq 0$. Fix a rational $\varepsilon > 0$. We will show that there is $\beta < x$ such that $\|\|x_{\beta} - x_{\alpha}\|_T - 2\|x_{\alpha, \delta}\|_1\| < \varepsilon$. As $\{x_\alpha : \alpha < x\}$ is an $r$-separated set and $\varepsilon > 0$ is arbitrary, this is sufficient. Find

(1) $q \in \mathbb{Q}$ such that $|q - \|x_{\alpha, \delta}\|_1| < \varepsilon/10$, 

(2) $\varepsilon > 0$ such that $\|x_{\beta} - x_{\alpha}\|_T < \varepsilon/10$,

(3) $\delta < x$ such that $\|x_{\beta} - x_{\alpha, \delta}\|_1 < \varepsilon/10$,

(4) $\delta < x$ such that $\|x_{\alpha, \delta}\|_1 < \varepsilon/10$,

(5) $\delta < x$ such that $\|x_{\alpha, \delta}\|_1 < \varepsilon/10$.
(2) \( n \in \mathbb{N} \) such that 
\[
\|d_n - T(x_\alpha)\|_{\ell^\infty} < \varepsilon/10,
\]

(3) finite \( F \subseteq \{t_\xi : \xi < \delta\} \) such that \( \|x_\alpha|([0,1] \setminus F)\|_1 - q| < 2\varepsilon/10, \)

(4) \( s \in \mathcal{Q}^F \) such that \( \|s - (x_\alpha|F)\|_1 < \varepsilon/10. \)

So \( x_\alpha \) satisfies the following formulae when substituted in place of \( x; \)

(5) \( \|d_n - T(x)\|_{\ell^\infty} < \varepsilon/10, \)

(6) \( \|x|([0,1] \setminus F)\|_1 - q| < 2\varepsilon/10, \)

(7) \( \|s - (x|F)\|_1 < \varepsilon/10. \)

As \( F \subseteq \{t_\xi : \xi < \delta\} \) since \( \delta \in \mathcal{C} \), we have \( M(\varepsilon, q, F, s, n) < \delta. \) As there is \( x_\gamma \) (for \( \gamma = \alpha \)) which satisfies (5)–(7) when substituted for \( x, \) by the definition of \( M \) there is \( \beta \) such that \( x_\beta \) satisfies (5)–(7) when substituted for \( x \) and \( x_\beta \) has its support included in \( \{t_\xi : \xi < M(\varepsilon, q, F, s, n)\} \) and in particular in \( \{t_\xi : \xi < \delta\}. \)

Now we estimate \( \|x_\beta - x_\alpha\|_{T}: \)

(8) 
\[
\|T(x_\beta - x_\alpha)\|_{\ell^\infty} = \|T(x_\beta) - T(x_\alpha)\|_{\ell^\infty} \leq \|T(x_\beta) - d_n\|_{\ell^\infty} + \|d_n - T(x_\alpha)\|_{\ell^\infty} \leq 2\varepsilon/10
\]

by (2) and (5) for \( x_\beta \) in place of \( x. \)

(9) 
\[
\|x_\beta - x_\alpha\|_1 = \|x_\beta|F - x_\alpha|F\|_1 + \|x_\beta|\{t_\xi : \xi < \delta\} \setminus F\|_1 - \|x_\alpha|\{t_\xi : \xi < \delta\} \setminus F\|_1 + \|x_\alpha,\delta\|_1
\]

since the support of \( x_\beta \) is included in \( \{t_\xi : \xi < \delta\}. \) Conditions (4) and (7) for \( x_\beta \) in place of \( x \) imply that

(10) 
\[
\|x_\beta|F - x_\alpha|F\|_1 \leq 2\varepsilon/10.
\]

Conditions (1) and (3) imply that 
\[
\|x_\alpha|\{t_\xi : \xi < \delta\} \setminus F\|_1 < 3\varepsilon/10
\]

and so by (6) for \( x_\beta \) in place of \( x \) and the fact that the support of \( x_\beta \) is included in \( \{t_\xi : \xi < \delta\} \) we conclude that

(11) 
\[
q - 5\varepsilon/10 \leq \|x_\beta|\{t_\xi : \xi < \delta\} \setminus F - x_\alpha|\{t_\xi : \xi < \delta\} \setminus F\|_1 \leq q + 5\varepsilon/10,
\]

so by (1) and (8)–(11) we conclude that

\[
2\|x_{\alpha,\delta}\|_1 - 6\varepsilon/10 \leq \|x_\beta - x_\alpha\|_T \leq 2\|x_{\alpha,\delta}\|_1 + \varepsilon,
\]

which completes the proof of the Claim.

Now note that as \( \{x_\alpha : \alpha < \kappa\} \) is discrete, it cannot be contained in any subspace of the form \( \ell_1^\prime(\{t_\xi : \xi < \delta\}) \) for \( \delta < \kappa \) which has density less than \( \kappa. \) So for \( \delta \in \mathcal{C} \) we can find \( \alpha_\delta < \kappa \) such that
\( x_{\alpha, \delta} \neq 0 \) and moreover we may make sure that \( \alpha_\delta \neq \alpha_{\delta'} \) for any \( \delta < \delta' \) in \( C \). Next we find \( C' \subseteq C \) of cardinality \( \kappa \) such that

\[
\text{supp}(x_{\alpha, \delta'}) \subseteq \{ t_\xi : \delta' \leq \xi < \delta \}
\]

for any \( \delta' < \delta \) and \( \delta, \delta' \in C' \). This can be done by recursion taking at the inductive step the next \( \delta \in C \) such that the supports of the previous \( x_{\alpha, \delta'} \)'s are included in \( \{ t_\xi : \xi < \delta \} \).

Now we will consider two cases, the first when there is \( \theta \in C' \) such that for any \( \delta, \delta' \in C' \) such that \( \theta < \delta' < \delta \) we have

\[
\text{supp}(x_{\alpha, \delta'}) \cap \text{supp}(x_{\alpha, \delta}) = \emptyset.
\]

Then by Claim 9 we have

\[
\| x_{\alpha_\delta} - x_{\alpha_{\theta'}} \|_1 \geq \| x_{\alpha, \delta'} \|_1 + \| x_{\alpha_\delta} \|_1 \geq r
\]

for every \( \theta < \delta' < \delta \). Since \( T \) is injective \( \| x_{\alpha_\delta} - x_{\alpha_{\theta'}} \|_T > \| x_{\alpha_\delta} - x_{\alpha_{\theta'}} \|_1 \geq r \), so we obtain that \( Z = \{ x_{\alpha_\delta} : \delta \in C' \}, \delta > \theta \} \).

The second case is when there is no \( \theta \in C' \) as in the first case. Then by recursion we can construct \( C'' \subseteq C' \) and (\( \theta_\delta : \delta \in C'' \)) such that

\[
\text{supp}(x_{\alpha, \delta'}) \cap \text{supp}(x_{\alpha, \delta}) \neq \emptyset
\]

for every \( \delta' < \delta \) with \( \delta, \delta' \in C'' \). Indeed, having constructed less then \( \kappa \) elements \( \delta' \in C'' \) whose supremum is \( \theta < \kappa \) we consider all \( \delta \in C' \) which are above \( \theta \). Since there is no \( \theta \in C' \) as in the first case we find \( \theta < \delta' < \delta \) with \( \delta', \delta \in C' \) such that \( \text{supp}(x_{\alpha_{\delta'}, \delta'}) \cap \text{supp}(x_{\alpha_{\delta}}) \neq \emptyset \). To complete the recursion take \( \delta \) as the next element of \( C'' \) and take as \( t_{\theta_\delta} \) any element of the above intersection.

Since \( \kappa \) is a cardinal of uncountable cofinality, by passing to a subset of cardinality \( \kappa \) we may assume that there is \( \varepsilon > 0 \) such that \( | x_{\alpha_{\delta}}(t_{\theta_\delta}) | \geq \varepsilon \) for every \( \delta \in C'' \).

Now we will use the following version of the Dushnik–Miller theorem: If \( \kappa \) is a regular uncountable cardinal and \( c : [\kappa]^2 \to \{0, 1\} \), then either there is a 0-monochromatic set for \( c \) which has its order type equal to \( \omega + 1 \) or there a 1-monochromatic subset of cardinality \( \kappa \) for \( c \) [9, Problem 24.32]. We define \( c : [C'']^2 \to \{0, 1\} \) by \( c(\delta', \delta) = 0 \) for \( \delta' < \delta \) if and only if

\[
\| x_{\alpha_{\delta}} - x_{\alpha_{\delta'}} \|_T = r.
\]

So it is enough to prove that there is no 0-monochromatic set of order type \( \omega + 1 \). Suppose \( \{ \delta_n : n \in \mathbb{N} \} \cup \{ \delta_\omega \} \) forms such a set, where \( \delta_\xi < \delta_\eta \) if \( \xi < \eta \) for all \( \xi, \eta < \omega + 1 \). Since the supports of \( x_{\alpha_\delta} \)'s for \( \delta \in C'' \) are pairwise disjoint by (12) there is \( n \in \mathbb{N} \) such that

\[
\| x_{\alpha_{\delta_\omega}} | \text{supp}(x_{\alpha_{\delta_n}}, \delta_n) \|_1 \leq \varepsilon/2.
\]

Also by (13) we have \( t_{\theta_\delta_\omega} \notin \text{supp}(x_{\alpha_{\delta_n}} \delta_n) \cup \text{supp}(x_{\alpha_{\delta_\omega}} \delta_\omega), \) where the union is disjoint by (12). So by Claim 9 we have

\[
\| x_{\alpha_{\delta_\omega}} - x_{\alpha_{\delta_n}} \|_T > \| x_{\alpha_{\delta_\omega}} - x_{\alpha_{\delta_n}} \|_1 \geq \| x_{\alpha_{\delta_n}} \delta_\omega \|_1 + | x_{\alpha_{\delta_\omega}}(t_{\theta_\delta_\omega}) | + \| x_{\alpha_{\delta_n}} \delta_n \|_1 - \varepsilon/2 \geq r.
\]
which contradicts the choice of \( \{ \delta_n : n \in \mathbb{N} \} \cup \{ \delta_\omega \} \) as 0-monochromatic. So the set of vectors \( x_{\alpha, \delta} \), for \( \delta \) in the 1-monochromatic set of cardinality \( \kappa \), is the desired \( \mathcal{Z} \).

\[ \square \]

**Remark 10.** Some of the renormings considered in Theorem 3 admit many infinite equilateral sets. For example, we can identify \( \ell_1([0,1]) \) with \( \ell_1 \oplus \ell_1([0,1]) \) and define \( T' : \ell_1 \oplus \ell_1([0,1]) \to \ell_1 \oplus \ell_2 \) by \( T'(x,y) = (x,T(y)) \), where \( T \) is as in Lemma 5.

**Remark 11.** Let us remark that the property of the spaces from Theorem 3 is much stronger than not admitting uncountable equilateral sets. To see this consider \( \ell_\infty([0,1]^2) \) with the usual \( \| \cdot \|_\infty \) norm. Let \( c : [0,1]^2 \to \{0,1\} \) be Sierpiński’s coloring with no uncountable monochromatic subset [9, Problem 24.23]. So \( c \in \ell_\infty([0,1]^2) \) and consider \( f_t \in \ell_\infty([0,1]^2) \) defined by

\[
f_t(\{r,s\}) = \begin{cases} 
  c(\{r,s\}) & \text{if } t = \min(\{r,s\}) \\
  -c(\{r,s\}) & \text{if } t = \max(\{r,s\}) \\
  0 & \text{otherwise.}
\end{cases}
\]

For distinct \( t, t' \in [0,1] \) the intersection of supports of \( f_t \) and \( f_{t'} \) is included in \( \{\{t,t'\}\} \). For \( t < t' \) the value of \( f_t - f_{t'} \) at \( \{t,t'\} \) is \( 1 - (-1) = 2 \) if \( c(\{t,t'\}) = 1 \) and it is 0 otherwise, so

\[
\|f_t - f_{t'}\| = \begin{cases} 
  1 & \text{if } c(\{t,t'\}) = 0 \\
  2 & \text{if } c(\{t,t'\}) = 1.
\end{cases}
\]

This means that \( \{f_t : t \in [0,1]\} \) is a 1-separated subset of the unit sphere which does not admit any uncountable equilateral subset but also there is no uncountable subset which is (1+)-separated. Note that any countable 1-separated set \( \{x_n : n \in \mathbb{N}\} \) in any Banach space contains either an infinite 1-equilateral set or an infinite (1+)-separated set by Ramsey’s theorem [9, Problem 24.1].

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