Euler complexes and geometry of modular varieties

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To Iosif Bernstein for his 60th birthday

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1 Introduction

1.1 Summary

In [G2], [G3], [G5] we described a mysterious connection between the depth $m$ multiple polylogarithms at $N$-th roots of unity

\[ \text{Li}_{n_1,\ldots,n_m}(z_1,\ldots,z_m) := \sum_{0<k_1<\cdots<k_m} z_1^{k_1} \cdots z_m^{k_m}, \quad z_1^N = 1, \]  

(1)

and the modular variety

\[ Y_1(m, N) := \Gamma_1(m, N) \backslash SL_m(\mathbb{R}) / SO_m \]  

(2)

for the congruence subgroup $\Gamma_1(m; N)$ of $GL_m(\mathbb{Z})$ stabilizing the row $(0, \ldots, 0, 1)$ modulo $N$.

The multiple polylogarithms at $N$-th roots of unity provide us all periods of the mixed Hodge structure on the pronilpotent completion of the fundamental group $\pi_1(G_m - \mu_N, v_0)$, where $\mu_N$ is the group of all $N$-th roots of unity and $v_0 = \partial/\partial t$ is the standard tangent vector at zero.

The refined version of the above connection describes the structure of the motivic fundamental group $\pi_1^M(G_m - \mu_N, v_0)$. In particular, in the $l$-adic realization it relates the image of the Galois group acting on the pro-$l$ fundamental group $\pi_{1(l)}^M(G_m - \mu_N, v_0)$ with the geometry of the modular varieties (2), for all $m$.

In this paper we give an explanation of the story for $m = 2$. Recall that the modular complex for $GL_2(\mathbb{Z})$ is the chain complex of the modular triangulation of the hyperbolic plane $H_2$, see Fig. 1, placed in degrees $[1, 2]$:

![Figure 1: The modular triangulation of the hyperbolic plane.](image)

We assume, for simplicity, that $n_1 = n_2 = 1$, i.e. work with the double logarithm $\text{Li}_{1,1}(z_1, z_2)$. The structure of the (motivic) double logarithm at $N$-th roots of unity was described in [G2] by a length two complex, called the level $N$ cyclotomic complex. The above connection in the double logarithm case is described by a surjective homomorphism

The modular complex for $GL_2(\mathbb{Z}) \rightarrow$ The level $N$ cyclotomic complex.

(3)

It is factorized via the coinvariants of the action of the group $\Gamma_1(N)$ on the modular complex.

For every modular curve, we define an Euler complex datum. It is given by a map of complexes of vector spaces

The modular complex for $GL_2(\mathbb{Z}) \rightarrow$ The weight two motivic complex on the modular curve.

(4)

Its image is called the Euler complex on the modular curve. Passing to its second cohomology we recover the Beilinson-Kato Euler system in $K_2$ of the tower of modular curves ([B], [Ka]).
We show that the map (3) can be obtained as the composition of the map (4) on the modular curve \( Y_1(N) \) followed by the specialization at a cusp.

At the same time Euler complexes suggest a new twist of the story. Let us restrict the Euler complex to the point corresponding to a CM elliptic curve \( E_K \) with complex multiplication by the ring of integers \( \mathcal{O}_K \) in an imaginary quadratic field \( K \), and pass to a subcomplex corresponding to the \( \mathcal{N} \)-torsion points, where \( \mathcal{N} \) is an ideal of \( \mathcal{O}_K \). We relate the complex obtained to the geometry of the three dimensional modular hyperbolic orbifold

\[
\mathcal{Y}_1(\mathcal{N}) := \Gamma_1(\mathcal{N}) \setminus SL_2(\mathbb{C}) / SU(2)
\]

where \( \Gamma_1(\mathcal{N}) \subset GL_2(\mathcal{O}_K) \) is the subgroup of matrices stabilising the row \((0, 1)\) modulo \( \mathcal{N} \). The relationship is very precise when \( K \) is the field of Gaussian or Eisenstein numbers, and the ideal \( \mathcal{N} \) is prime. However it is still rather elusive for other imaginary quadratic fields.

In a sequel to this paper we will show that the obtained complex is closely related to the \( \mathbb{Q}_l \)-part of the image of Galois group acting on the pro-\( l \) completion of the fundamental group of \( E_K - \{ \mathcal{N} \text{-torsion points} \} \). Thus we get new examples of the mysterious connection between Galois groups and geometry of modular varieties.

The two examples discussed in the paper can be seen as higher analogs of the theory of cyclotomic/elliptic units – we discuss this analogy in the final part of the Introduction.

Finally, there is a similar story in the depth two and arbitrary weight situation. Since in this paper we want to present the picture in the simplest form, we will elaborate it elsewhere.

### 1.2 The double logarithm at roots of unity and modular curves ([G2])

The simplest way to state this connection goes via the dilogarithm story, which we recall now.

Let \( F \) be an arbitrary field and \( \mathbb{Z}[F^* - \{1\}] \) the free abelian group generated by elements \( \{x\} \), where \( x \in F^* - \{1\} \). Then there is a version of the Bloch complex

\[
\delta_2 : B_2(F) \longrightarrow \Lambda^2 F^*, \quad \delta_2 \{x\} = (1 - x) \wedge x
\]

where \( B_2(F) \) is a certain quotient of \( \mathbb{Z}[F^* - \{1\}] \), see Section 2.1. Recall the dilogarithm function

\[
\text{Li}_2(z) = -\int_0^z \log(1 - t) \frac{dt}{t}. \quad \text{Im} \text{Li}_2(z) + \arg(1 - z) \cdot \log|z|
\]

It has a single-valued version \( \mathcal{L}_2(z) := \text{Im} \text{Li}_2(z) + \arg(1 - z) \cdot \log|z| \), providing a homomorphism

\[
B_2(\mathbb{C}) \longrightarrow \mathbb{R}, \quad \{z\} \longmapsto \mathcal{L}_2(z).
\]

We will put the Bloch group in a wider conceptual framework in Section 1.3.

The **motivic double logarithms** at \( N \)-th roots of unity were defined ([G5]) as elements

\[
\text{Li}_{1,1}^M(a, b) \in B_2(\mathbb{Q}(\zeta_N)) \otimes \mathbb{Q}, \quad a^N = b^N = 1, \quad \zeta_N = e^{\frac{2\pi i}{N}}.
\]

They are motivic avatars of the numbers \( \text{Li}_{1,1}(a, b) \). Setting \( c := (ab)^{-1} \), one has

\[
\delta_2 : \text{Li}_{1,1}^M(a, b) \longmapsto (1 - a) \wedge (1 - b) + (1 - b) \wedge (1 - c) + (1 - c) \wedge (1 - a).
\]
Therefore there is a complex
\[ \delta : C_2(N) \rightarrow \Lambda^2 C_1(N). \] (9)
where \( C_2(N) \) is the subspace spanned by the elements (7), and \( C_1(N) \) is the \( \mathbb{Q} \)-subspace in \( \mathbb{Q}(\zeta_N)^* \otimes \mathbb{Q} \) generated by the cyclotomic \( N \)-units \( 1 - \zeta_N^a \). We call it the \textit{cyclotomic complex}.

To explain the connection with modular curves, recall the modular complex for \( GL_2(\mathbb{Z}) \):
\[ M_*^{(2)} := M_1^{(2)} \rightarrow M_2^{(2)}. \] (10)
Here \( M_1^{(2)} \) is the group generated by the oriented triangles, and \( M_2^{(2)} \) is generated by the oriented geodesics of the modular triangulation of the hyperbolic plane shown on Fig 1. It is a complex of \( GL_2(\mathbb{Z}) \)-modules. Let \( \Gamma \) be a subgroup of \( GL_2(\mathbb{Z}) \). Projecting the modular complex onto the modular curve \( Y_\Gamma := H_2/\Gamma \), we get a triangulation of the latter. Its chain complex is identified with \( M_*^{(2)} \otimes \mathbb{Q} \).

Let \( \hat{C}_1(N) = C_1(N) \oplus \mathbb{Q} \). Using the embedding \( \Lambda^2 C_1(N) \hookrightarrow \Lambda^2 \hat{C}_1(N) \), we extend the cyclotomic complex (9) to a complex
\[ C_2(N) \rightarrow \Lambda^2 \hat{C}_1(N). \] (11)

We defined in [G2] a canonical map from the modular complex to this complex:
\[ \begin{array}{ccc}
M_1^{(2)} & \rightarrow & M_2^{(2)} \\
\downarrow & & \downarrow \\
C_2(N) & \rightarrow & \Lambda^2 \hat{C}_1(N)
\end{array} \] (12)
It enjoys the following properties. The right vertical arrow is surjective. It is an isomorphism for a prime \( N \). The space \( C_2(N) \) contains a subspace isomorphic to \( K_3(\mathbb{Q}(\zeta_N))_{\mathbb{Q}} \), where \( A_\mathbb{Q} := A \otimes \mathbb{Q} \); it is generated by the values of the motivic dilogarithm \( \text{Li}_2^M \) at \( N \)-th roots of unity. It is the kernel of the bottom arrow. Modulo the subspace \( K_3(\mathbb{Q}(\zeta_N))_{\mathbb{Q}} \), the left vertical map is surjective for all \( N \), and is an isomorphism for a prime \( N \). Therefore for a prime level \( N = p \), the map of complexes (12) gives rise to an isomorphism of the top complex with the quotient of the bottom one by the subspace \( K_3(\mathbb{Q}(\zeta_N))_{\mathbb{Q}} \).

### 1.3 Mixed Tate motives and the Bloch group

Let \( S \) be a set of primes in a number field \( F \), and \( \mathcal{O}_{F,S} \) the ring of \( S \)-integers in \( F \). Then there is an abelian \( \mathbb{Q} \)-category \( M_T(\text{Spec}\mathcal{O}_{F,S}) \) of mixed Tate motives over \( \text{Spec}\mathcal{O}_{F,S} \), defined in [DG]. It is equipped with a canonical fiber functor
\[ \omega : M_T(\text{Spec}\mathcal{O}_{F,S}) \rightarrow \mathbb{Q} \text{-vector spaces}. \]

Let \( L_\bullet(\text{Spec}\mathcal{O}_{F,S}) \) be the Lie algebra of its derivations, called the fundamental Lie algebra of \( \mathcal{O}_{F,S} \).\(^1\) It is graded by negative integers. The functor \( \omega \) provides an equivalence between the category of mixed Tate motives over \( \text{Spec}\mathcal{O}_{F,S} \) and the category of graded finite dimensional modules over the fundamental Lie algebra. Let us denote by \( L_\bullet(\text{Spec}\mathcal{O}_{F,S}) \) the graded dual of the fundamental Lie algebra. It is a positively graded Lie coalgebra. We will usually omit \( \text{Spec} \) from notation.

\(^1\)We apologize for the abuse of notation: \( L_2(\mathbb{Z}) \) denotes the Bloch-Wigner, and both notation are rather standard.
Let $B_n$ in degree $n$ of the standard cochain complex of the fundamental Lie coalgebra. The cyclotomic complex (9) is a subcomplex in the weight two part of logarithms at $p$-th roots of unity is smaller then $L_2$ to $\Lambda^2\mathcal{O}_{F,S} \otimes \mathbb{Q}$. Furthermore, there is a canonical isomorphism (Lemma 2.2)

$$\mathcal{B}_2(F)_Q \sim \mathcal{L}_2(F).$$

Let $\mathcal{B}_2(\mathcal{O}_{F,S})_Q$ be the biggest subspace in $\mathcal{B}_2(F)_Q$ which is mapped by the differential $\delta_2$ to $\Lambda^2\mathcal{O}_{F,S} \otimes \mathbb{Q}$. Then the isomorphism (14) restricts (Lemma 2.3) to an isomorphism

$$\mathcal{B}_2(\mathcal{O}_{F,S})_Q \sim \mathcal{L}_2(\mathcal{O}_{F,S}).$$

Let us introduce the two cyclotomic schemes

$$S_N := \text{Spec} \mathbb{Z}[\zeta_N][\frac{1}{N}], \quad S_N^{un} := \text{Spec} \mathbb{Z}[\zeta_N].$$

The cyclotomic complex (9) is a subcomplex in the weight two part

$$\mathcal{L}_2(S_N) \rightarrow \Lambda^2\mathcal{L}_1(S_N)$$

of the standard cochain complex of the fundamental Lie coalgebra $\mathcal{L}_2(S_N)$. Moreover, if $N = p$ is a prime, we can replace here $S_p$ by $S_p^{un}$. Analysing the diagram (12) we arrive at the following

**Conclusion 1 ([G2]).** For a prime $p$, the subspace of $\mathcal{L}_2(S_p^{un})$ generated by the motivic double logarithms at $p$-th roots of unity is smaller then $\mathcal{L}_2(S_p^{un})$; the quotient is isomorphic to $H^1_{\text{cusp}}(\Gamma_1(p), \mathbb{Q})$.

Below we develop an analog of this picture related to a CM elliptic curve, see Section 1.6.

### 1.4 Relation with the motivic fundamental group of $\mathbb{G}_m - \mu_N$

The unipotent motivic fundamental group $\pi_1^M(\mathbb{G}_m - \mu_N, \nu_0)$ is not a group but rather a pro-unipotent Lie algebra object in the abelian category of mixed Tate motives over the scheme $S_N$ ([DG]). Thus applying the fiber functor $\omega$ to the (pro-) mixed Tate motive $\pi_1^M(\mathbb{G}_m - \mu_N, \nu_0)$ we get a graded Lie algebra $\omega(\pi_1^M(\mathbb{G}_m - \mu_N, \nu_0))$ over $\mathbb{Q}$. The fundamental Lie algebra $L_\bullet(S_N)$ acts by its derivations, providing a canonical Lie algebra homomorphism

$$L_\bullet(S_N) \rightarrow \text{Der}(\omega(\pi_1^M(\mathbb{G}_m - \mu_N, \nu_0))).$$

Let us denote by $C_\bullet(\mu_N)$ its image. We called it the cyclotomic Lie algebra. Let $C_\bullet(\mu_N)^\vee$ be the dual Lie coalgebra. One has

$$C_{-1}(\mu_N)^\vee = (\text{the group of cyclotomic units in } S_N) \otimes \mathbb{Q} = \mathcal{O}_{S_N} \otimes \mathbb{Q}.$$ 

We proved in [G5] that

$$C_{-2}(\mu_N)^\vee = C_2(N)$$

and the dual to the commutator map $[*, *] : \Lambda^2C_{-1}(\mu_N) \rightarrow C_{-2}(\mu_N)$ is identified with the cyclotomic complex (9). Thus the cyclotomic complex describes the weight 2 part of the image of the motivic Galois group acting on $\pi_1^M(\mathbb{G}_m - \mu_N, \nu_0)$. In general, the cyclotomic Lie algebra $C_\bullet(\mu_N)$ is described by the motivic multiple polylogarithms at $N$-th roots of unity.
1.5 Euler complexes on modular curves

In this paper we develop this story as follows. Using a construction from Section 6 of [G1], we define a modular deformation of the complex (9), called the Euler complex. It is a subcomplex of the Bloch complex (6) for the field of functions on the modular curve $Y(N)$. (In fact it is a subcomplex of the Bloch complex of the modular curve from Definition 2.1). We show in Proposition 2.16 that there is a canonical map

$$\text{the modular complex} \longrightarrow \text{the Euler complex on } Y(N).$$  \hspace{1cm} (16)

One gets a similar map for other modular curves, e.g. $Y_1(N)$. Taking its specialization at a cusp $\infty$ on $Y_1(N)$ obtained by projection of the point $\infty$ from the hyperbolic plane, we recover the cyclotomic complex (9): the specialization provides a surjective morphism of complexes:

$$\text{The Euler complex on } Y_1(N) \longrightarrow \text{The level } N \text{ cyclotomic complex } (9).$$ \hspace{1cm} (17)

This map intertwines the maps from the modular complex to the Euler and cyclotomic complexes. So we arrive at the commutative diagram on Fig. 2. This explains why the modular curve $Y_1(N)$

![Figure 2: Relating the modular, Euler and cyclotomic complexes.](image)

appears in the study of the motivic double logarithm at roots of unity.

**Generalizations.** Recall the standard cochain complex

$$C_* (\mu_N)^\vee \longrightarrow \Lambda^2 C_* (\mu_N)^\vee \longrightarrow \Lambda^3 C_* (\mu_N)^\vee \longrightarrow \ldots$$ \hspace{1cm} (18)

of the Lie algebra $C_* (\mu_N)$. The first map is dual to the commutator map, and the others are defined using the Leibniz rule. The grading of the Lie algebra $C_* (\mu_N)$ provides a weight grading of the complex. The cyclotomic complex (9) is isomorphic to the weight two part of the standard cochain complex (18) of the cyclotomic Lie algebra $C_* (\mu_N)$.

The Lie algebra $C_* (\mu_N)$ has a depth filtration. So the complex (18) inherits the depth filtration. The depth $m$ part of $C_* (\mu_N)$ is described by motivic multiple polylogarithms of the depth $\leq m$.

In [G3], see also [G4], [G5], we generalized the diagonal arrow in Fig. 2 to $GL_m(\mathbb{Z})$, for any positive integer $m$. Namely, we defined the rank $m$ modular complex and constructed a map from this complex, tensored by $S^{w-m} V_m$, where $V_m$ is the standard representation of $GL_m$, to the depth $m$, weight $w$ part of the standard cochain complex of the cyclotomic Lie algebra $C_* (\mu_N)$.

What might play the role of the Euler complex for $m > 2$? Observe that the modular varieties for $m > 2$ are not algebraic varieties.
1.6 Euler complexes, CM points, and modular hyperbolic 3-folds

Let $\mathcal{O}_K$ be the ring of integers in the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$, and $\mathcal{N}$ an ideal in $\mathcal{O}_K$. The group $GL_2(\mathcal{O}_K)$ acts discretely on the hyperbolic 3-space $\mathcal{H}_3$. Recall the subgroup $\Gamma_1(\mathcal{N})$ of $GL_2(\mathcal{O}_K)$ stabilizing the row vector $(0,1)$ modulo $\mathcal{N}$, and the corresponding modular orbifold $\mathcal{Y}_1(\mathcal{N}) := \Gamma_1(\mathcal{N})\backslash \mathcal{H}_3$. Let $E_K$ be an elliptic curve with the endomorphism ring $\mathcal{O}_K$.

Restricting the Euler complex on $Y(\mathcal{N})$, where $\mathcal{N}$ is the norm of $\mathcal{N}$, to a point corresponding to the curve $E_K$, and taking the subcomplex corresponding to the $\mathcal{N}$-torsion points, we get a complex which is

- Related to the geometry of the three-dimensional modular orbifold $\mathcal{Y}_1(\mathcal{N})$. The relationship is most precise when the ideal $\mathcal{N}$ is prime, and $K$ is the field of Gaussian or Eisenstein numbers.
- Related to the simplest (depth two, weight two) non-abelian quotient of the image of the Galois group acting on the pro-$l$ fundamental group of $E_K - E_K[\mathcal{N}]$.

Below we elaborate the first connection. The second will be discussed in a sequel to this paper.

The modular and elliptic units. Recall that the classical modular unit is an invertible function $\theta_q(z)$ on the modular curve $Y_1(\mathcal{N})$ constructed as follows. Let $q = e^{2\pi i \tau}, \text{Im}(\tau) > 0, \quad z = e^{2\pi i \xi}, \quad \xi = \alpha_1 \tau + \alpha_2, \quad \alpha_1, \alpha_2 \in \frac{1}{\mathcal{N}} \mathbb{Z}^2 - \mathbb{Z}^2$

Set

$$\theta_q(z) = -q^{\frac{1}{2}} B_2(\alpha_1) \cdot e^{2\pi i \alpha_2(\alpha_1 - 1)/2} (1 - z) \prod_{n=1}^{\infty} (1 - q^nz)(1 - q^nz^{-1})$$  \hspace{1cm} (19)

where $B_2(x) := x^2 - x + \frac{1}{6}$ is the second Bernoulli polynomial. Changing $(\alpha_1, \alpha_2)$ by an element of $\mathbb{Z}^2$, we alter $\theta_q(z)$ by multiplication by an $\mathcal{N}$-th root of unity. So $\theta_q(z)^\mathcal{N}$ is a well defined invertible function on $Y_1(\mathcal{N})$.

Let $E$ be an elliptic curve over an arbitrary field $k$. Then, given an $\mathcal{N}$-torsion point $z$ on $E$ defined over a field $k_z$, one can define an element $\theta_E(z)$ such that $\theta_E(z)^\mathcal{N} \in k_z^\times$ (see [GL], and Section 2.3 below). More generally, if $E$ is an elliptic curve over a base $S$, and $z$ is an $\mathcal{N}$-torsion section, there is an element $\theta_E(z)$ such that $\theta_E(z)^\mathcal{N} \in \mathcal{O}_S$. There are the following interesting special cases:

- If $E$ is the universal elliptic curve $\mathcal{E}_\mathcal{N}$ over $Y_1(\mathcal{N})$ we get the described above modular unit.
- If $E$ is the CM curve $E_K$, and $\mathcal{N}$ is an ideal of $\mathcal{O}_K$ with the norm $\mathcal{N}$, the elements $\theta_E(z)$ for all $\mathcal{N}$-torsion points $z$ of $E_K$ generate an abelian extension $K_{\mathcal{N}}$ of $K$. Moreover, $\theta_E(z)$ is an $\mathcal{N}$-unit in $K_{\mathcal{N}}$, called an elliptic unit.

The elements $\theta_E(a,b,c)$. For any triple of torsion points $a,b,c$ on an elliptic curve $E$ over an arbitrary field $k$ with $a + b + c = 0$ we construct an element

$$\theta_E(a,b,c) \in B_2(k') \otimes \mathbb{Q}$$

where $k'$ is the field generated over $k$ by the coordinates of the points $a,b$. An explicit construction of this element is given by a reciprocity law from Section 6 of [Gl], which strengthens Suslin’s reciprocity law for the Milnor $K_3$-group of the function field of $E$. One has the following key formula

$$\delta : \theta_E(a,b,c) \mapsto \theta_E(a) \wedge \theta_E(b) + \theta_E(b) \wedge \theta_E(c) + \theta_E(c) \wedge \theta_E(a).$$  \hspace{1cm} (20)
Double modular units and Euler complexes. In particular, let $\mathcal{E}_N$ be the universal elliptic curve over the modular curve $Y(N)$. Then a pair of torsion sections $a, b$ of $\mathcal{E}_N$ provides an element $\theta_{\mathcal{E}_N}(a, b, c)$. It can be thought of as a double modular unit: its coproduct is a sum of the wedge products of the classical modular units $\theta_{\mathcal{E}_N}(a)$. We define the Euler complex as the subcomplex of the complex (6) on $Y(N)$ spanned by $\theta_{\mathcal{E}_N}(a, b, c)$ in degree 1, and $\theta_{\mathcal{E}_N}(a) \wedge \theta_{\mathcal{E}_N}(b)$ in degree 2, for all torsion sections $a, b$ as above.

An analytic version of the double modular units. Let $X$ be a complex algebraic variety with the function field $\mathbb{C}(X)$. An element $\beta$ of the Bloch group $\mathcal{B}_2(\mathbb{C}(X))$ gives rise to a multivalued analytic function $\text{Li}_2(\beta)$ at the generic point of $X(\mathbb{C})$. Namely, if $\beta = \sum_i n_i \{ f_i(z) \}_2$, we set $\text{Li}_2(\beta) = \sum_i n_i \text{Li}_2(f_i(z))$. So the element $\theta_{\mathcal{E}_N}(a, b, c)$ gives rise to a multivalued analytic function $\tilde{\theta}_{\mathcal{E}_N}(a, b, c)$ at the generic point of the modular curve $Y(N) \otimes_{\mathbb{Q}(\mu_N)} \mathbb{C}$. It follows from (20) that its differential is

$$d\tilde{\theta}_{\mathcal{E}_N}(a, b, c) = \text{Cycle}_{a, b, c} \left( \log \theta_{\mathcal{E}_N}(a) d \log \theta_{\mathcal{E}_N}(b) - \log \theta_{\mathcal{E}_N}(b) d \log \theta_{\mathcal{E}_N}(a) \right).$$

So elements $\theta_{\mathcal{E}_N}(a, b, c)$ allow to express the integral of the 1-form on the right via the dilogarithm.

Double elliptic units and modular 3-folds. Let us specialize the Euler complex on $Y(N)$ to the point corresponding to a CM curve $E_K$ with complex multiplication by $\mathcal{O}_K$, and consider only $\mathcal{N}$-torsion points, where $\mathcal{N}$ is an ideal of $\mathcal{O}_K$. Let us spell the definition of the obtained complex.

Definition 1.1 The $\mathbb{Q}$-vector space $\mathcal{C}_2(\mathcal{N})$ is the subspace of $\mathcal{B}_2(K_\mathcal{N})_\mathbb{Q}$ generated by the elements $\theta_{\mathcal{E}}(a, b, c)$ when $a, b, c$ run through all $\mathcal{N}$-torsion points of $E$ such that $a + b + c = 0$.

Let us denote by $\mathcal{C}_1(\mathcal{N})$ the group of $\mathcal{N}$-units in the field $K_\mathcal{N}$ tensored by $\mathbb{Q}$. It is known to be generated by the elliptic units $\theta_{\mathcal{E}}(a)$, when $a$ runs through the $\mathcal{N}$-torsion points of the curve $E_K$.

It follows from (20) that we get a complex

$$\mathcal{C}_2(\mathcal{N}) \xrightarrow{\delta} \Lambda^2 \mathcal{C}_1(\mathcal{N}).$$

(21)

Let $\mathcal{H}_3 := \mathcal{H}_3 \cup K \cup \{ \infty \}$. We relate this complex with the geometry of the modular hyperbolic orbifold $\Gamma_1(\mathcal{N}) \backslash \mathcal{H}_3^*$, as follows. The hyperbolic space $\mathcal{H}_3$ has a classical Bianchi tessellation on geodesic polyhedrons invariant under the action of $GL_2(\mathcal{O}_K)$. For example, for the ring of Gaussian integers it is a tessellation on octahedron’s and for the ring of Eisenstein integers it is tessellation on tetrahedrons, see Fig. 3 and Fig. 4.

Projecting the Bianchi tessellation onto the modular orbifold $\Gamma_1(\mathcal{N}) \backslash \mathcal{H}_3^*$ we get a cell decomposition of the latter. Let $\mathcal{V}(\mathcal{N})$ be its integral chain complex, called the Bianchi complex of $\Gamma_1(\mathcal{N}) \backslash \mathcal{H}_3^*$. In Section 5 we relate the complex (21) with the complex $\mathcal{V}(\mathcal{N})$. Our results are complete when $\mathcal{N} = \mathcal{P}$ is a prime ideal, and $d = -1, -3$. In these cases we define a canonical morphism of complexes

$$\begin{array}{ccccccc}
V_3(\mathcal{P}) & \rightarrow & V_2(\mathcal{P}) & \rightarrow & V_1(\mathcal{P}) & \rightarrow & V_0(\mathcal{P}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_3(\mathcal{K}_\mathcal{P})_\mathbb{Q} & \rightarrow & C_2(\mathcal{P}) & \rightarrow & C_1(\mathcal{P}) & \rightarrow & C_0(\mathcal{P})
\end{array}$$

(22)

(where $v$ is the residue map related to the valuation defined by $\mathcal{P}$) and prove that it is almost a quasiisomorphism – see a precise statement in Theorem 5.7. It is an analog of homomorphism (12). Here is a corollary. Set $\mathcal{C}^\text{un}_1(\mathcal{P}) := \mathcal{O}_{K_\mathcal{P}}^* \otimes \mathbb{Q}$. Clearly $\text{Im} \delta \subset \Lambda^2 \mathcal{C}^\text{un}_1(\mathcal{P})$ in the bottom line of (22).
Theorem 1.2 Let \( d = -1 \) or \( d = -3 \), and \( \mathcal{P} \) be a prime ideal in \( \mathcal{O}_K \). Then there is an isomorphism
\[
H^2_{\text{cusp}}(\Gamma_1(\mathcal{P}), \mathbb{Q}) = \text{Coker}(C_2(\mathcal{P}) \rightarrow \Lambda^2C^\text{un}_1(\mathcal{P})).
\]

Double elliptic units and mixed Tate motives over \( \mathcal{O}_{K_N} \). Set
\[
S_N := \text{Spec} \mathcal{O}_{K_N}[\frac{1}{N}], \quad S^\text{un}_N := \text{Spec} \mathcal{O}_{K_N}.
\]
Recall the fundamental Lie coalgebra \( L_\bullet(S_N) \). It follows from (14), the very definition of \( C_2(\mathcal{N}) \) as a subspace of \( B_2(K_N) \), the key formula (20), and Lemma 2.2 below that there is an inclusion
\[
i : C_2(\mathcal{N}) \hookrightarrow L_2(S_N).
\]
It gives rise to a homomorphism of complexes
\[
\begin{align*}
    C_2(\mathcal{N}) &\rightarrow \Lambda^2C_1(\mathcal{N}) \\
i \downarrow &\quad \downarrow = \\
    L_2(S_N) &\rightarrow \Lambda^2L_1(S_N)
\end{align*}
\]
where the right arrow is provided by the isomorphism (13) for the scheme \( S_N \), and the bottom arrow dualises the commutator map. The kernel of the bottom arrow is identified with \( K_3(K_N) \). Moreover, if \( \mathcal{P} \) is a prime ideal, then we have \( i : C_2(\mathcal{P}) \hookrightarrow L_2(S^\text{un}_\mathcal{P}) \), and Theorem 1.2 implies

Corollary 1.3 Let \( d = -1 \) or \( d = -3 \), and \( \mathcal{P} \) be a prime ideal in \( \mathcal{O}_K \). Then there is an isomorphism
\[
H^2_{\text{cusp}}(\Gamma_1(\mathcal{P}), \mathbb{Q}) \sim \frac{L_2(S^\text{un}_\mathcal{P})}{i(C_2(\mathcal{P})) + K_3(K_\mathcal{P})}. \tag{24}
\]

Therefore we arrive at the CM analog of the Conclusion 1 in Section 1.3.

Conclusion 2. Let \( d = -1 \) or \( d = -3 \) and \( \mathcal{P} \) be a prime ideal in \( \mathcal{O}_K \). Then the subspace of \( L_2(S^\text{un}_\mathcal{P}) \) generated by the double elliptic units at \( \mathcal{P} \)-torsion points of \( E_K \) and \( K_3(K_\mathcal{P}) \) is smaller than \( L_2(S^\text{un}_\mathcal{P}) \); the quotient is isomorphic to \( H^2_{\text{cusp}}(\Gamma_1(\mathcal{P}), \mathbb{Q}) \).

Coda. The cyclotomic units provide a finite index subgroup in the unit group of a cyclotomic field. There are two generalizations, highlighted in Conclusions 1 and 2, where the unit group is replaced by the \( \mathbb{Q}(2) \)-part of a fundamental Lie coalgebra, the role of the cyclotomic unit subgroup is played by an explicitly defined “double units” subspace of the fundamental Lie coalgebra, and the gap between them is isomorphic to the cuspidal cohomology of certain modular varieties.

In both cases the “double units” subspaces can be obtained by specializations of the first groups of the Euler complexes. On the other hand the Beilinson-Kato Euler system is delivered by the second cohomology groups of the Euler complexes. Thus both parts of the Euler complex are important.

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2 Euler complexes on modular curves

2.1 The weight two motivic complex and the dilogarithm

Let $F$ be an arbitrary field and $\mathbb{Z}[[\mathbb{P}^1(F)]]$ the free abelian group generated by the elements $\{ x \}$ where $x \in \mathbb{P}^1(F)$. Consider a homomorphism $\delta_2 : \mathbb{Z}[[\mathbb{P}^1(F)]] \to \Lambda^2 F^*$, $\{ x \} \mapsto (1 - x) \wedge x$, $\{ 0 \}, \{ 1 \}, \{ \infty \} \mapsto 0$.

Let $R_2(F)$ be the subgroup of $\mathbb{Z}[[\mathbb{P}^1(F)]]$ generated by the following elements. Let $X$ be a connected curve over $F$ and $f_i$ rational functions on $X$ such that $\sum_i (1 - f_i) \wedge f_i = 0$ in $\Lambda^2 F(X)^*$. Then $R_2(F)$ is generated by the elements $\sum_i \{ f_i(0) \} - \{ f_i(1) \}$ for all possible curves and functions $f_i$ as above.

One can show that $\delta_2(R_2(F)) = 0$, so setting $B_2(F) := \mathbb{Z}[[\mathbb{P}^1(F)]] / R_2(F)$ we get a complex (a version of the Bloch complex, see [G6])

$$\delta_2 : B_2(F) \to \Lambda^2 F^*.$$

There is a more explicit version $B_2(F)$ of the group $B_2(F)$. Denote by $R_2(F)$ the subgroup of $\mathbb{Z}[[\mathbb{P}^1(F)]]$ generated by

$$\{ 0 \}, \{ \infty \} \text{ and } \sum_{i=1}^5 (-1)^i \{ r(x_1, ..., \hat{x}_i, ..., x_5) \},$$

where $(x_1, ..., x_5)$ runs through all 5-tuples of distinct points in $\mathbb{P}^1(F)$, where $r(\ldots)$ is the cross-ratio. The Bloch group $B_2(F)$ is the quotient of $\mathbb{Z}[[\mathbb{P}^1(F)]]$ by the subgroup $R_2(F)$. We get the Bloch complex

$$\delta_2 : B_2(F) \to \Lambda^2 F^*.$$

According to the theorems of Matsumoto and Suslin, one has for this complex:

$$\text{Coker}\delta_2 = K_2(F), \quad \text{Ker}\delta_2 \otimes \mathbb{Q} = K_3^{\text{ind}}(F)_{\mathbb{Q}}. \quad (25)$$

One can show that $R_2(F) \subset R_2(F)$. Thus there is a map $i : B_2(k) \to B_2(k)$ induced by the identity map on the generators. According Proposition 6.1 of [G1], this map is an isomorphism modulo torsion for a number field $F$. The function $L_2$ provides a homomorphism $B_2(\mathbb{C}) \to \mathbb{R}$, $\{ z \} \mapsto L_2(z)$.

**Definition 2.1** The Bloch group $B_2(X)$ of an irreducible scheme $X$ with the field of functions $F_X$ is the largest subgroup of $B_2(F_X)$ which has the property that $\delta_2(B_2(X)) \subset \Lambda^2 O_X^*$.

By the very definition, there is a commutative diagram, where the vertical arrows are embeddings:

$$B_2(X) \xrightarrow{\delta_2} \Lambda^2 O_X^* \xrightarrow{\delta_2} \Lambda^2 F_X^*$$

**Lemma 2.2** Let $F$ be a number field. Then there is a canonical isomorphism

$$B_2(F)_{\mathbb{Q}} \xrightarrow{\sim} L_2(F). \quad (26)$$
Proof. Recall the canonical isomorphism \( \mathcal{L}_1(F) = F_3^* \). Let us define a canonical map \( \mathbb{Z}[F] \to \mathcal{L}_2(F) \). We assign to a generator \( \{x\} \) a framed mixed Tate motive \( \mathbb{L}_2^M(x) \) defined as follows. Consider \( \mathbb{P}^2 \) with a projective coordinate system \((z_0, z_1, z_2)\). Let \( L_i \) be the line defined by the equation \( z_i = 0 \). We get a coordinate triangle \((L_0, L_1, L_2)\). Consider another triple of lines \((M_0, M_1, M_2)\), where \( M_0 := \{z_1 + z_2 = 1\}, M_1 := \{z_1 = 1\}, M_2 := \{z_2 = x\} \). Let us assume \( x \neq 0 \). Let us blow up all points where three of the \( L \) and \( M \)-lines intersect: the vertices of the \( L \)-triangle at infinity, the vertex of the \( M \)-triangle on the line \( L_2 \), and, if there is a vertex of the \( L \)-triangle on the line \( L_1 \), i.e. \( x = 1 \), this vertex as well. Denote by \( \mathbb{P}^2 \) the obtained surface. Then we get an \( M \)-pentagon \( \mathcal{M} \) there, formed by the strict preimage of the \( K \)-cohomology in the degree 2 are given by \( K \). Further, we get \( L \)-triangle at infinity. Further, we get \( M \)-polygon \( \mathcal{L}_0 \) (it is a pentagon if \( x = 1 \) and 4-gon otherwise). We define now a mixed Tate motive by using the formula

\[
\mathbb{L}_2^M(x) := H^2(\mathbb{P}^2 - \mathbb{L}, \mathcal{F}_1 - (\mathbb{L} \cap \mathcal{M})).
\]

Its framing, and interpretation of the obtained object as a mixed Tate motive is standard, see for example [G7]. Finally, according to the standard formalism, the framed mixed Tate motive \( \mathbb{L}_2^M(x) \) gives rise to an element of \( \mathcal{L}_2(F) \).

It is well known, and easy to prove, that the coproduct \( \delta \mathbb{L}_2^M(x) \) equals \((1 - x) \wedge x\). So we get a commutative diagram, where the right arrow is an isomorphism after \( \otimes \mathbb{Q} \):

\[
\begin{array}{ccc}
\mathbb{Z}[F] & \xrightarrow{\delta} & \Lambda^2 F^* \\
\downarrow & & \downarrow \\
\mathcal{L}_2(F) & \xrightarrow{\delta} & \Lambda^2 \mathcal{L}_1(F)
\end{array}
\]

Further, for a complex embedding \( \sigma : F \to \mathbb{C} \) the real period of \( \mathbb{L}_2^M(\sigma(x)) \) is given by \( \mathcal{L}_2(\sigma(x)) \). This, the injectivity of the regulator map on \( K_3(F)_\mathbb{Q} \), and Suslin’s theorem (25) imply that the subspace \( \mathcal{R}_2(F)_\mathbb{Q} \subset \mathbb{Q}[F] \) is killed by the left arrow. Thus we get a commutative diagram

\[
\begin{array}{ccc}
\mathcal{B}_2(F)_\mathbb{Q} & \xrightarrow{\delta} & \Lambda^2 F^* \\
\downarrow & & \downarrow \\
\mathcal{L}_2(F) & \xrightarrow{\delta} & \Lambda^2 \mathcal{L}_1(F)
\end{array}
\]

For a number field \( F \) we have \( \mathcal{K}_2(F)_\mathbb{Q} = 0 \) Thus the horizontal arrows are epimorphic \( \otimes \mathbb{Q} \). Further, for a number field \( K_3^{ind}(F)_\mathbb{Q} = K_3(F)_\mathbb{Q} \), and so the kernels of both horizontal arrows are isomorphic to \( K_3(F)_\mathbb{Q} \). Thus the left arrow is an isomorphism. The lemma is proved.

Lemma 2.3 There is a canonical isomorphism

\[
\mathcal{B}_2(\mathcal{O}_{F,S})_\mathbb{Q} \sim \mathcal{L}_2(\mathcal{O}_{F,S}). \tag{27}
\]

Proof. By the very definition the category of mixed Tate motives over \( \text{Spec} \mathcal{O}_{F,S} \) is a subcategory of the one over \( \text{Spec} F \). Thus there is an inclusion of the Lie coalgebras \( \mathcal{L}_\bullet(\mathcal{O}_{F,S}) \hookrightarrow \mathcal{L}_\bullet(F) \). It provides a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & K_3(\mathcal{O}_{F,S})_\mathbb{Q} & \longrightarrow & \mathcal{L}_2(\mathcal{O}_{F,S}) & \longrightarrow & \Lambda^2 \mathcal{L}_1(\mathcal{O}_{F,S}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & K_3(F)_\mathbb{Q} & \longrightarrow & \mathcal{L}_2(F) & \longrightarrow & \Lambda^2 \mathcal{L}_1(F) & \longrightarrow & 0
\end{array}
\]

where the horizontal lines are exact since the fundamental Lie algebras are free, and their first cohomology in the degree 2 are given by \( K_3 \) of the corresponding scheme. Since \( K_3(F)_\mathbb{Q} = K_3(\mathcal{O}_{F,S})_\mathbb{Q} \), \( \mathcal{L}_2(\mathcal{O}_{F,S}) \) is the biggest subspace of \( \mathcal{L}_2(F) \) which maps by the cobracket to \( \Lambda^2 \mathcal{L}_1(\mathcal{O}_{F,S}) \). The lemma follows.
2.2 A reciprocity law for $K^M_3$ of the function field of an elliptic curve ([G1])

The Chow dilogarithm. Let $f_1,f_2,f_3$ be arbitrary rational functions on a complex curve $X$. Set

$$r_2(f_1,f_2,f_3) := \text{Alt}_3\left( \frac{1}{6} \log |f_1| d \log |f_2| \wedge d \log |f_3| - \frac{1}{2} \log |f_1| d \arg f_2 \wedge d \arg f_3 \right)$$

where $\text{Alt}_3$ is the alternation of $f_1,f_2,f_3$. The crucial property of this form is the following:

$$dr_2(f_1,f_2,f_3) = \text{Re} \left( d \log f_1 \wedge d \log f_2 \wedge d \log f_3 \right).$$

Then the integral

$$\frac{1}{2\pi i} \int_{X(\mathbb{C})} r_2(f_1,f_2,f_3)$$

converges, and provides a homomorphism

$$\Lambda^3 \mathbb{C}(X)^* \to \mathbb{R}, \quad f_1 \wedge f_2 \wedge f_3 \mapsto \text{the integral (28)}.$$

Let $X$ be a regular curve over an algebraically closed field $k$ and $F := k(X)^*$. Let us define a morphism of complexes

$$\begin{array}{cccc}
B_2(F) \otimes F^* & \delta_2 \wedge Id & \Lambda^3 F^* \\
\downarrow \text{Res} & \downarrow \text{Res} & \\
B_2(k) & \delta_2 & \Lambda^2 k^* \\
\end{array}$$

Here $\text{Res} := \sum_x \text{res}_x$, where $\text{res}_x$ is the local residue homomorphism for the valuation $v_x$ on $F$ corresponding to a point $x$ of $X(k)$. The local residue maps are defined as follows. We have $\text{res}_x((y)_2 \otimes z) = 0$ unless $v_x(y) = 0$. In the latter case $\text{res}_x((y)_2 \otimes z) = v_x(z)((\overline{y})_2_2$, where $\overline{y}$ denotes projection of $y$ to the residue field of $F$ for the valuation $v_x$. The map $\text{res}_x : \Lambda^3 F^* \to \Lambda^2 k^*$ is determined uniquely by the following two conditions: $\text{res}_x(z_1 \wedge z_2 \wedge z_3) = 0$ if $v_x(z_1) = v_x(z_2) = v_x(z_3) = 0$ and $\text{res}_x(z_1 \wedge z_2 \wedge z_3) = \overline{z}_2 \wedge \overline{z}_3$ if $v_x(z_1) = 1$, $v_x(z_2) = v_x(z_3) = 0$.

Let us present the projective plane $\mathbb{P}^2$ as the projectivisation of the vector space $V_3$. A linear functional $l \in V_3^*$ is called a linear homogeneous function on $\mathbb{P}^2$. If $X$ is an elliptic curve in $\mathbb{P}^2$, any rational function $f$ on $X$ can be written as a ratio of products of linear homogeneous functions:

$$f = \frac{l_1 \cdots l_k}{l_{k+1} \cdots l_{2k}}.$$ 

This is checked by induction on the total number of zeros and poles using that $X$ is degree three plane curve.

Let us return to an arbitrary curve $X$. For three points $a,b,c$ and a divisor $D = \sum n_i(x_i)$ on a line set

$$\{r(a,b,c,D)\}_2 := \sum_i n_i \{r(a,b,c,x_i)\}_2.$$ 

Here $r(a,b,c,d)$ is the cross-ratio of four points on $P^1$, normalized so that $r(\infty,0,1,x) = x$.

Let $L_i$ be the line $l_i = 0$ in the plane, $D_i$ the divisor $L_i \cap X$, and $l_{ij} := L_i \cap L_j$. The following is Theorem 6.14 in [G1].
Theorem 2.4 Let \( k \) be an algebraically closed field. Let \( X \) be either an elliptic curve, or a rational curve over \( k \), and \( F := k(X)^* \). Then there is a group homomorphism \( h : \Lambda^3 F^* \to B_2(k) \) satisfying the following conditions:

a) \( h(k^* \wedge \Lambda^2 F^*) = 0 \), and the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{B}_2(F) \otimes F^* & \xrightarrow{\delta_3} & \Lambda^3 F^* \\
\text{Res} \downarrow & & \downarrow \text{Res} \\
\mathcal{B}_2(k) & \xrightarrow{\delta_2} & \Lambda^2 k^*
\end{array}
\] (30)

b) If \( X \) is an elliptic curve, then for any linear homogeneous functions \( l_0, \ldots, l_3 \) one has

\[
h(l_1/l_0 \wedge l_2/l_0 \wedge l_3/l_0) = -\sum_{i=0}^{3} (-1)^i \{ r(l_{i0}, \ldots, \hat{l}_i, \ldots, l_3, D_i) \}_2.
\] (31)

c) If \( X = \mathbb{P}^1 \), \( t \) is a natural parameter on it, and \( a_i \in \mathbb{P}^1(k) \), then

\[
h(\frac{t-a_1}{t-a_0} \wedge \frac{t-a_2}{t-a_0} \wedge \frac{t-a_3}{t-a_0}) = -\{ r(a_0, a_1, a_2, a_3) \}_2.
\] (32)

d) If \( X \) is a defined over \( \mathbb{C} \) then

\[
\frac{1}{2\pi i} \int_{X(\mathbb{C})} r_2(f_1 \wedge f_2 \wedge f_3) = \mathcal{L}_2 \left( h(f_1 \wedge f_2 \wedge f_3) \right).
\] (33)

e) The map \( h \) is \( \text{Gal}(F/k) \)-invariant.

Remark. By Suslin’s reciprocity law for \( K^M_3(F) \), the projection of \( \text{Res}(\Lambda^3 F^*) \subset \Lambda^2 k^* \) to \( K_2(k) \) is zero. Since by Matsumoto’s theorem \( K_2(k) = \text{Coker}(\delta_2) \), one has \( \text{Res}(\Lambda^3 F^*) \subset \text{Im}(\delta_2) \). However \( \text{Ker}(\delta_2) \) is nontrivial, so the problem is to lift naturally the map \( \text{Res} \) to a map \( h \).

Let \( P_E \) be the abelian group of principal divisors on \( E \). Theorem 2.4 provides us a map

\[
h : \Lambda^3 P_E \longrightarrow B_2(k).
\]

Lemma 2.5 Let \( E \) be an elliptic curve over an arbitrary field \( k \). Let \( D_1, D_2, D_3 \) be principal divisors rational over an extension \( k' \) of \( k \). Then \( h(D_1 \wedge D_2 \wedge D_3) \subset B_2(k') \).

Proof. Let \( D = \sum n_i(x_i) \) be a divisor on \( E \) as in the Lemma. We can decompose it into a fraction of products of linear homogeneous functions \( l_i \) defined over \( k' \) as follows. Let \( l_{x,y} \) (resp. \( l_x \)) be a linear homogeneous equation of the line in \( \mathbb{P}^2 \) through the points \( x \) and \( y \) on \( E \) (resp. \( x \) and \( -x \)). The divisor of the function \( l_{x,y}/l_{x+y} \) is \( (x) + (y) - (x+y) - (0) \). If \( D = (x) + (y) + D_1 \), we write \( f = l_{x,y}/l_{x+y} \cdot f' \), so \( f' = (0) + (x+y) + D_1 \). After a finite number of such steps we get the desired decomposition. It follows that in formula (31) the cross-ratios in the right hand side lie in the field \( k' \). The lemma is proved.
2.3 The modular and elliptic units revisited

Let $E$ be an elliptic curve over an arbitrary field $k$, and $J(k)$ the group of $k$-points of the Jacobian $J$ of $E$. We defined in Section 2 of [GL] an abelian group $B_2(E)(k)$, usually denoted simply by $B_2(E)$, which has the following properties:

a) There is an exact sequence of abelian groups

$$0 \longrightarrow k^* \longrightarrow B_2(E) \xrightarrow{p} S^2J(k) \longrightarrow 0. \quad (34)$$

b) There is a canonical (up to a choice of a sixth root of unity) surjective homomorphism

$$h : \mathbb{Z}[E(k)] \longrightarrow B_2(E)$$

whose projection to $S^2J(k)$ is given by $\{a\} \mapsto a \cdot a$.

c) The group $B_2(E)$ is functorial: an automorphism $A$ of $E$ induces an automorphism of the extension (34), which acts as the identity on the subgroup $k^*$, and whose action on $S^2J(k)$ is induced by the map $A : J(k) \longrightarrow J(k)$.

d) If $k$ is a local field then there is a homomorphism $H_k : B_2(E) \longrightarrow \mathbb{R}$ whose restriction to the subgroup $k^*$ is given by $a \mapsto -\log |a|$.

The group $B_2(E)$ is a motivic avatar of the theta-functions on $E$.

Corollary 2.6 Let $a$ be an $N$-torsion element of $E(k)$. Then there is a well defined element

$$\theta_E(a) \in k^* \otimes \mathbb{Z}[\frac{1}{N}].$$

For any automorphism $A$ of $E$ we have $\theta_E(A(a)) = \theta_E(a)$.

Proof. The element $p \circ h(a)$ is annihilated by $N$. Thus we have $Nh(a) \in k^*$. Taking its $N$-th root we get $\theta_E(a)$. The second claim follows immediately from the property c). The corollary is proved.

We call $\theta_E(a)$ the elliptic unit corresponding to the torsion point $a$ of $E$. If $E$ is defined over $\mathbb{C}$, it is given by $\theta_q(z)$ in (19), where $q$ is the modulus of $E$, and $z$ is an $N$-torsion point on $E$.

The following result for the first statement follows from Theorem 4.1 (or Corollary 4.3) in [GL]; for the second statement see Theorem 4.1 on page 43 of [KL].

Lemma 2.7 The elliptic units $\theta_E(a)$ satisfy the following distribution relations: Given an isogeny $\psi : E \rightarrow E'$, one has

$$\prod_{\psi(t) = t'} \theta_E(t) = \theta_{E'}(t'), \quad t' \neq 0; \quad (35)$$

$$\prod_{\psi(t) = 0} \theta_E(t) = \left( \frac{\Delta_{E'}}{\Delta_E} \right)^{1/12}. \quad (36)$$

The 12-th root of the $\Delta$-function $\Delta(\tau)$ is defined canonically as $\Delta(\tau)^{1/12} = 2\pi e^{2\pi i/12} \prod_{n=1}^{\infty} (1 - q^n)^2$. 

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2.4 The elements $\theta_E(a_1, a_2, a_3)$

Let $E$ be an elliptic curve over an algebraically closed field $k$. For any $N$-torsion points $a, b$ on $E$ the divisor $N(\{a\} - \{b\})$ is principal, so there exists a (non zero) function $f_{a,b}$ on $E$ such that $\text{div} f_{a,b} = N(\{a\} - \{b\})$. It is well defined up to a non zero constant factor.

**Definition 2.8** Let $a_1, a_2, a_3$ be $N$-torsion points on $E$. Then

$$
\theta_E(a_1 : a_2 : a_3) := -\frac{1}{N^3} \sum_{x \in E[N]} h(f_{a_1,x}, f_{a_2,x}, f_{a_3,x}) \in B_2(k)_Q
$$

(37)

where we sum over all $N$-torsion points of $E$.

Clearly this element is invariant under the shift $a_i \mapsto a_i + a$ and skew symmetric in $a_i$. We will show in Lemma 2.12 that this definition does not depend on the choice of $N$.

We extend the definition of the elements $\theta_E(a_1 : a_2 : a_3)$ by linearity to a map

$$
\theta_E : \Lambda^3 \mathbb{Z}[E[N]] \rightarrow B_2(k)_Q.
$$

The restriction of this map to degree zero $N$-torsion divisors on $E$ is given by a simpler formula:

**Lemma 2.9** One has

$$
\theta_E(\{a_1\} - \{b_1\} : \{a_2\} - \{b_2\} : \{a_3\} - \{b_3\}) = -\frac{1}{N^3} h(f_{a_1,b_1}, f_{a_2,b_2}, f_{a_3,b_3}).
$$

**Proof.** Clearly, $f_{a_1,b_1} = f_{a_1,x} - f_{b_1,x}$. The lemma follows immediately from this.

Observe that given a triple of elements $a_1, a_2, a_3$ of an abelian group $A$ such that $a_1 + a_2 + a_3 = 0$, there are elements $b_1, b_2, b_3 \in A$ such that $a_i = b_{i+1} - b_i$, where the index $i$ is modulo 3. The triple $(b_1, b_2, b_3)$ is defined uniquely up to a shift $b_i \mapsto b_i + b$. So if $(a_1, a_2, a_3)$ are torsion points of $E$ such that $a_1 + a_2 + a_3 = 0$, we can define an element $\theta_E(a_1, a_2, a_3)$ such that

$$
\theta_E(b_2 - b_1, b_3 - b_2, b_1 - b_3) = \theta_E(b_1 : b_2 : b_3).
$$

**Remark.** Below we add to the elements $\theta_E(a)$, defined for $a \neq 0$, a formal variable $\theta_E(0)$. Therefore, strictly speaking, in the formulas below we work in $\Lambda^2(k^* \otimes \mathbb{Q} \oplus \mathbb{Q})$ where the element 1 in the extra summand $\mathbb{Q}$ corresponds to $\theta_E(0)$. However one easily checks that in the final formulas there will be no $\theta_E(0)$.

The basic properties of the elements $\theta_E(a_1, a_2, a_3)$ are listed in the theorem below.

**Theorem 2.10** a) The differential of the element $\theta_E(a_1, a_2, a_3)$ in the Bloch complex is given by

$$
\delta_2 : \theta_E(a_1, a_2, a_3) \rightarrow \theta_E(a_1) \wedge \theta_E(a_2) + \theta_E(a_2) \wedge \theta_E(a_3) + \theta_E(a_3) \wedge \theta_E(a_1).
$$

(38)

b) The elements $\theta_E(a_1, a_2, a_3)$ satisfy the following relations:

- The dihedral symmetry relations:

$$
\theta_E(a_1, a_2, a_3) = (-1)^{\text{sign}(\sigma)} \theta_E(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}), \quad \sigma \in S_3,
$$

(39)

$$
\theta_E(a_1, a_2, a_3) = \theta_E(\varepsilon(a_1), \varepsilon(a_2), \varepsilon(a_3)), \quad \varepsilon \in \text{Aut}(E).
$$

(40)
The distribution relations: Given an isogeny $\psi : E \to E'$, one has
\[
\sum_{\psi(a_i) = a'_i} \theta_E(a_1, a_2, a_3) = \theta_{E'}(a'_1, a'_2, a'_3), \quad a'_i \neq 0; \tag{41}
\]

Proof. a) The formula is equivalent to the following one, which we are going to prove:
\[
\delta_2 : \theta_E(a_1 : a_2 : a_3) \to \tag{42}
\]
\[
\theta_E(a_1 - a_2) \land \theta_E(a_2 - a_3) + \theta_E(a_2 - a_3) \land \theta_E(a_3 - a_1) + \theta_E(a_3 - a_1) \land \theta_E(a_1 - a_2).
\]
Denote the right hand side of (42) by $C(a_1 : a_2 : a_3)$. Let us prove first that
\[
\delta_2 \circ \theta_E(\{a_1\} - \{x\} : \{a_2\} - \{x\} : \{a_3\} - \{x\}) = \tag{43}
\]
\[
C(a_1 : a_2 : a_3) - C(x : a_2 : a_3) + C(a_1 : x : a_3) - C(a_1 : a_2 : x).
\]
Using $f_{a_1,x} = \theta_E(t - a_1)^N/\theta_E(t - x)^N$, we get
\[
\frac{1}{N^3} \text{Res}(f_{a_1,x} \land f_{a_2,x} \land f_{a_3,x}) = C(a_1 : a_2 : a_3) - C(x : a_2 : a_3) + C(a_1 : x : a_3) - C(a_1 : a_2 : x).
\]
Thanks to (30), this implies (43). Now we deduce formula (42) from this. We claim that
\[
\sum_{x \in E[N]} C(x : a_2 : a_3) = 0. \tag{44}
\]
Having this formula, and taking the sum of formula (43) over all $x \in E[N]$, we get formula (42).

We will obtain formula (44) as a special case of the following simple general statement.

Let $A$ and $B$ be abelian groups. Let
\[
\Phi : \Lambda^3 \mathbb{Z}[A] \to B, \quad \{a\} \land \{b\} \land \{c\} \mapsto \Phi(a : b : c).
\]
be a group homomorphism.

Lemma 2.11 Let us assume that the group $A$ is finite and the map $\Phi$ satisfies the following properties:
\[
\Phi(-a : b : c) = \Phi(a : b : c), \quad \Phi(a : b : c) = \Phi(a + x : b + x : c + x) \quad \text{for any} \ x \in A.
\]
Then one has modulo 2-torsion
\[
\sum_{x \in A} \Phi(a : b : x) = 0.
\]

Proof. Indeed, one has
\[
\sum_{x \in A} \Phi(a : b : x) = \sum_{x \in A} \Phi(a - (a + b) : b - (a + b) : x - (a + b)) = \]
\[
\sum_{y \in A} \Phi(-b : -a : y) = - \sum_{y \in A} \Phi(a : b : y).
\]
The Lemma is proved.

Since $\theta_E(y) = \theta_E(-y)$, and $a_2, a_3$ are $N$-torsion points, the element $C(a_1 : a_2 : a_3)$ satisfies the dihedral symmetry relations, and by definition is invariant under the shift of the arguments. Thus formula (44) follows from Lemma 2.11. The part a) of the theorem is proved.

b) The dihedral symmetry relations follow from Theorem 2.4 and the definition.

Let us prove the distribution relations. Thanks to the distribution relations for elliptic units, the difference between the left and the right hand sides of (41) is killed by the differential $\delta_2$. Since the kernel of the differential $\delta_2$ on the group $B_2$ is rigid by the very definition, the difference is constant on the modular curve: Indeed, take the modular curve as the curve $X$ in the definition in Section 2.1. So it remains to check that it is zero at a single point. As such one can take a CM point corresponding to a certain number field $K$ extending $K$ (use Lemma 2.5). Thanks to the injectivity of the regulator map, it is enough to show that the dilogarithm kills this element and its Galois conjugated. Now the claim follows from (33) plus the fact that the Chow dilogarithm satisfies the distribution relations on the nose. The claim is proved. The theorem is proved.

**Remark.** Observe that $\theta_E(a, -a, 0) = 0$ since by the dihedral relations one has $\theta_E(a, -a, 0) = -\theta_E(-a, a, 0)$. Formula (38) makes sense if $a_1 = 0$. Indeed, thanks to $\theta_E(a) = -\theta_E(-a)$, it gives

$$\delta_2 \circ \theta_E(0, a, -a) = \theta_E(a, -a) = \theta_E(0) \land \theta_E(a) \lor \theta_E(-a) = \theta_E(a) \land \theta_E(-a) = 0.$$  

**Lemma 2.12** The element (37) does not depend on the choice of $N$. Namely, if $N | M$, then the elements defined by formula (37) using $N$ or $M$ coincide.

**Proof.** For $M$-torsion points $a, b$, denote by $f_{a,b}^{(M)}$ a function with the divisor $M(\{a\} - \{b\})$. Given $M$-torsion points $a_1, a_2, a_3, x$ one has, by Lemma 2.9,

$$-\frac{1}{M^3} h(f_{a_1,x}^{(M)}, f_{a_2,x}^{(M)}, f_{a_3,x}^{(M)}) = \theta_E(a_1 : a_2 : a_3) - \theta_E(x : a_2 : a_3) + \theta_E(a_1 : x : a_3) - \theta_E(a_1 : a_2 : x).$$

Assume that $a_i, x$ are $N$-torsion points. Averaging over $x \in E[N]$, and using Lemma 2.11, we get

$$\frac{1}{N^2} \sum_{x \in E[N]} -\frac{1}{M^3} h(f_{a_1,x}^{(M)}, f_{a_2,x}^{(M)}, f_{a_3,x}^{(M)}) = \theta_E(a_1 : a_2 : a_3).$$

Since $(f_{a_i,x}^{(M)})^{M/N} = f_{a_i,x}^{(N)}$, the lemma follows.

Below we will need the following special case of the definition of elements $\theta_E(a : b : c)$ for the modular curve $Y_1(N)$. Recall that points of $Y_1(N)$ parametrize pairs $(E, p)$, where $p$ is an $N$-torsion point of $E$, generating the subgroup $E[N]$. So for any residues $\alpha_i \in \mathbb{Z}/N\mathbb{Z}$ there are elements $\theta_E(\alpha_1 p : \alpha_2 p : \alpha_3 p)$.

**Lemma 2.13** One has

$$\theta_E(\alpha_1 p : \alpha_2 p : \alpha_3 p) = \frac{1}{N} \sum_{\beta \in \mathbb{Z}/N\mathbb{Z}} h(f_{\alpha_1 p, \beta p}^{(N)}, f_{\alpha_2 p, \beta p}^{(N)}, f_{\alpha_3 p, \beta p}^{(N)}).$$

**Proof.** Follows from Lemmas 2.11 and 2.9 the same way as Lemma 2.12.
2.5 Euler complexes on modular curves

Recall the full level \( N \) modular curve \( Y(N) \). It is defined over the cyclotomic field \( \mathbb{Q}(\mu_N) \). Denote by \( F_{Y(N)} \) its function field. Let \( \mathcal{E}_N \) be the universal elliptic curve over \( Y(N) \).

Recall that for a torsion section \( a \), the element \( \theta_E(a) \) is a modular unit, i.e. lies in \( \mathcal{O}_{Y(N)}^* \otimes \mathbb{Z}[\frac{1}{N}] \).

**Definition 2.14** The \( \mathbb{Q} \)-vector space \( E^1_N \) is generated by the elements

\[
\theta_E(a_0 : a_1 : a_2) \in B_2(F_{Y(N)})_\mathbb{Q}
\]

when \( a_i \)’s run through all \( N \)-torsion points of the universal elliptic curve \( \mathcal{E}_N \).

The \( \mathbb{Q} \)-vector space \( E^2_N \) is generated by the elements

\[
\theta_E(a) \wedge \theta_E(b) \in \Lambda^2 \mathcal{O}_{Y(N)}^* \otimes \mathbb{Q}
\]

when \( a, b \) run through all non-zero \( N \)-torsion points of the universal elliptic curve \( \mathcal{E}_N \).

It follows from the formula (43) that we get a complex \( E^* \), placed in degrees \([1, 2]\):

\[
E_N^* : \quad E^1_N \xrightarrow{\delta} E^2_N.
\]

We call it the Euler complex of the modular curve \( Y(N) \).

**Lemma 2.15** For any three \( N \)-torsion points \( a, b, c \) of \( E \) one has

\[
\theta_E(a : b : c) \in B_2(Y(N)) \otimes \mathbb{Q}.
\]

**Proof.** Follows from the definition of \( B_2(Y(N)) \) formula (43) and the fact that \( \theta_E(a) \in \mathcal{O}_{Y(N)}^* \).

Thus the Euler complex is a subcomplex of the Bloch complex of the modular curve \( Y(N) \):

\[
\begin{array}{ccc}
E^1_N & \rightarrow & E^2_N \\
\downarrow & & \downarrow \\
\big(B_2(Y(N)) & \rightarrow & \Lambda^2 \mathcal{O}_{Y(N)}^* \big)_\mathbb{Q}.
\end{array}
\]

Since \( \mathcal{O}_{Y(N)}^* \otimes \mathbb{Q} \) is generated by the elements \( \theta_E(a) \), when \( a \in E[N] \), the right arrow in the diagram above is an isomorphism, and the left one is inclusion by definition. When it is an isomorphism? In other words, when \( B_2(Y(N)) \otimes \mathbb{Q} \) is generated over \( \mathbb{Q} \) by the elements \( \theta_E(a, b, c) \), where \( a, b, c \in E[N] \)?

Recall the modular complex

\[
M^*_2 := M^1_2 \rightarrow M^2_2.
\]

(45)

Let \( \Gamma \) be a subgroup of \( GL_2(\mathbb{Z}) \). Projecting the modular complex onto the modular curve \( Y_{\Gamma} := \mathcal{H}_2/\Gamma \), we get the modular triangulation of the latter. Its chain complex is identified with \( M^*_2 \otimes \mathbb{Q} \).

**Proposition 2.16** For every positive integer \( N \) there is a canonical surjective homomorphism of complexes

\[
\begin{array}{ccc}
M^1_2 & \rightarrow & M^2_2 \\
\downarrow \lambda^1_N & & \downarrow \lambda^2_N \\
E^1_N & \rightarrow & E^2_N
\end{array}
\]

providing a map of the modular complex for \( Y(N) \) to the corresponding Euler complex:

\[
\lambda^*_N : M^*_2 \otimes \mathcal{G}(N) \mathbb{Q} \rightarrow E^*_N.
\]

(47)
Proof. The modular complex for $Y(N)$ has the following description:

$$Z[\Gamma(N)\backslash GL_2(\mathbb{Z})] \otimes_{D^1} \chi_1 \rightarrow Z[\Gamma(N)\backslash GL_2(\mathbb{Z})] \otimes_{D^2} \chi_2.$$  

Here $D^1$ is the subgroup of order 12 of $GL_2(\mathbb{Z})$ stabilizing the modular triangle with vertexes $(0, 1, \infty)$. Further, $D^2$ is the subgroup of order 8 of $GL_2(\mathbb{Z})$ stabilizing the geodesic $(0, \infty)$. The characters $\chi_1$ and $\chi_2$ are given by the determinant. Observe that $\Gamma(N)\backslash GL_2(\mathbb{Z}) = GL_2(\mathbb{Z}/N\mathbb{Z})$. A point of the universal elliptic curve $\mathcal{E}_N$ is given by a triple $(E; p_1, p_2)$, where $E$ is an elliptic curve, and $(p_1, p_2)$ is a basis in the abelian group $E[N]$. The maps $\lambda^1_N$ and $\lambda^2_N$ act on the generator parametrized by a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}/N\mathbb{Z})$$

as follows:

$$\lambda^1_N : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \theta_E(ap_1 + cp_2, bp_1 + dp_2, -(a + b)p_1 - (c + d)p_2); \quad (48)$$

$$\lambda^2_N : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \theta_E(ap_1 + cp_2) \land \theta_E(bp_1 + dp_2). \quad (49)$$

Now the claim that the map (48) gives rise to a group homomorphism $\lambda^1_N$ given by the left arrow in (46) is equivalent to the dihedral symmetry relations (39). The claim that the map (49) gives rise to a group homomorphism $\lambda^2_N$ given by the right arrow in (46) follows from $\theta_E(-a) = \theta_E(a)$ and the skew-symmetry of the wedge product. The proposition is proved.

Summarizing, we arrive at the diagram on Fig 2.

Corollary 2.17 The map $\lambda^2_N$ gives rise to a canonical homomorphism

$$\lambda_N : H^1(\Gamma(N), \mathbb{Z}) \rightarrow K_2(Y(N)) \otimes \mathbb{Z}[\frac{1}{N}],$$

The $H^2$ cohomology groups of the Euler complexes for various $N$ deliver the Beilinson-Kato Euler system on the tower of modular curves. Let us explain this in more detail.

The Euler complex datum on a modular curve $Y(N)$ is a homomorphism of complexes (47). The Euler complex is the image of this homomorphism. The source of the map (47) does not depend on the choice of the modular curve. Therefore each element $\gamma$ of the modular complex gives rise to a collection of elements $\lambda_N(\gamma)$ in Euler complexes. In particular, if $\gamma$ is the element corresponding to the geodesic $(0, \infty)$ on $H_2$, the elements $\lambda_N(\gamma)$, projected to $K_2(Y(N))$, give rise to an Euler system in $K_2$ – this is deduced from the results of Kato [Ka].

3 Specialization of Euler complexes at a cusp

3.1 Some identities in the Bloch group

Let $k$ be an arbitrary field. We define the motivic dilogarithm $Li^M_2(x) := \{x\}_2 \in B_2(k)$. Following Section 2.1 of [G2], the motivic double logarithm is the following element of the Bloch group:

$$Li^M_1(x, y) := \{\frac{xy - y}{1 - y}\}_2 - \{\frac{y}{y - 1}\}_2 - \{xy\}_2 \in B_2(k).$$
This definition was suggested by a similar identity between the corresponding multivalued analytic functions, easily proved by differentiation, see loc. cit..

It is easy to check the crucial formula (8) for the coproduct $\delta_2 \text{Li}_{1,1}(a, b)$, where $a^N = b^N = 1$.

Let us introduce a standard homogeneous notation for the double logarithm:

$$ I_{1,1}^M(a_1 : a_2 : a_3) := \text{Li}_{1,1}(a_3/a_1, a_2/a_2). \quad (50) $$

It is a motivic version of the iterated integral $I(a_0 : a_1 : a_2) = \int_0^{a_2} \frac{dt}{t-a_1} \circ \frac{dt}{t-a_2}$.

**Lemma 3.1** Let $a_i^N = 1$. Then one has the following identity in $B_2(k)$ modulo 6-torsion:

$$ \sum_{i=0}^3 (-1)^i I_{1,1}^M(a_0 : \ldots : \hat{a}_i : \ldots : a_3) = \{r(a_0, a_1, a_2, a_3)\}_2. $$

**Proof.** One has

$$ I_{1,1}^M(a_1 : a_2 : a_3) = \left\{ \frac{a_3(a_2 - a_1)}{a_1(a_2 - a_3)} \right\}_2 - \left\{ \frac{a_3}{a_3 - a_2} \right\}_2 - \left\{ \frac{a_3}{a_1} \right\}_2 $$

After the alternation the last two terms on the right in this formula will disappear. Since

$$ r(0, a_2, a_3, a_1) = \frac{a_3(a_2 - a_1)}{a_1(a_2 - a_3)} $$

and $\{r(a_{\sigma(1)}, ..., a_{\sigma(4)})\}_2 = (-1)^{|\sigma|} \{r(a_1, ..., a_4)\}_2$ modulo 6-torsion, what remains is the five-term relation. The lemma is proved.

**Lemma 3.2** Let $a_i^N = 1$. Then one has in $B_2(k) \otimes \mathbb{Q}$

$$ \frac{1}{N} \sum_{x^N=1} \{r(a_1, a_2, a_3, x)\}_2 = I_{1,1}^M(a_1 : a_2 : a_3). \quad (51) $$

**Proof.** Lemma 2.11 implies that

$$ \sum_{x^N=1} I_{1,1}^M(a_1 : a_2 : x) = 0. $$

This and Lemma 3.1 imply the identity. The Lemma is proved.

### 3.2 The specialization map

Let $\mathcal{O}$ be a discrete valuation ring with a uniformizer $q$, the residue field $k$, the valuation homomorphism $v$, and the fraction field $F$. There is a canonical homomorphism $\mathcal{O} \to k, f \mapsto f$. We define a specialization map

$$ \text{Sp}_q : F^* \to k^*, \quad f \mapsto \frac{f}{q^v(f)}. $$

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Lemma 3.3  The specialization maps

\[ \text{Sp}_q\{f\}_2 = \begin{cases} \overline{f} & \text{if } f \in \mathcal{O}^* \\ 0 & \text{otherwise} \end{cases} , \quad \text{Sp}_q(f_1 \wedge f_2) := \text{Sp}_q f_1 \wedge \text{Sp}_q f_2 \]

gives rise to a homomorphism of complexes, modulo 2-torsion,

\[
\begin{array}{ccc}
B_2(F) & \longrightarrow & \Lambda^2 F^* \\
\downarrow \text{Sp}_q & & \downarrow \text{Sp}_q \\
B_2(k) & \longrightarrow & \Lambda^2 k^*
\end{array}
\]  \hfill (52)

Proof. Let us show that the specialization map commutes with the differentials. Indeed, set \( f = q^n f_0 \) where \( f_0 \in \mathcal{O}^* \). Then if \( n \neq 0 \), one has \( \text{Sp}_q\{f\}_2 = 0 \). On the other hand, if \( n > 0 \), then \( \text{Sp}_q(1 - q^n f_0) \wedge q^n f_0 = 1 \wedge \overline{f}_0 = 0 \). Similarly, if \( n < 0 \), we get \( \text{Sp}_q((1 - f) \wedge f) = (1 - f)/q^{(1-f)} \wedge 1 = 0 \). We left to the reader to check that the specialization map is a well defined map \( B_2(F) \longrightarrow B_2(k) \). The lemma is proved.

Therefore there is a specialization homomorphism of complexes corresponding to a local parameter \( q \) at a \( k \)-point \( x \) of a curve \( X \) over \( k \).

3.3 Specialization of the Euler complex at a cusp on \( Y_1(N) \)

Let \( \infty \) be the cusp on a modular curve obtained by projection of the point \( \infty \in \mathcal{H}_2 \). Then, for some integer \( N \), \( q = \exp(2\pi i \tau / N) \) is a local parameter at this cusp. Using it, we define the specialization map \( \text{Sp}_q \) at the cusp \( \infty \).

Theorem 3.4  a) The specialization map \( \text{Sp}_q \) at the cusp \( \infty \) on \( Y_1(N) \) provides a surjective homomorphism from the Euler complex on \( Y_1(N) \) to the level \( N \) cyclotomic complex:

\[
\begin{array}{ccc}
\mathcal{E}^1(Y_1(N)) & \longrightarrow & \mathcal{E}^2(Y_1(N)) \\
\downarrow \text{Sp}_q & & \downarrow \text{Sp}_q \\
C_2(N) & \longrightarrow & \Lambda^2 C_1(N)
\end{array}
\]  \hfill (53)

It intertwines the maps of the modular complex to the Euler and cyclotomic complexes.

b) If \( N = p \) is a prime number, it gives rise to an isomorphism of complexes

\[
\begin{array}{ccc}
\mathcal{E}^1(Y_1(p)) & \longrightarrow & \mathcal{E}^2(Y_1(p)) \\
= \downarrow \text{Sp}_q & = \downarrow \text{Sp}_q \\
C_2(p) & \longrightarrow & \Lambda^2 \hat{C}_1(p)
\end{array}
\]  \hfill (54)

Proof. a) The specialization of modular units \( \theta_E(z) \) on \( Y(N) \), in the notation of formula (19), is:

\[ \text{Sp}_q \theta_E(z) = \begin{cases} (1 - e^{2\pi i \alpha_2}) & \text{if } \alpha_1 = 0, \alpha_2 \neq 0 \\ 1 & \text{otherwise} \end{cases} \text{ modulo torsion.} \]  \hfill (55)
For the modular curve $Y_1(N)$, there are modular units $\theta_E(\alpha p)$, where $(E,p)$ is a point of $Y_1(N)$, and $\alpha \in \mathbb{Z}/N\mathbb{Z} - 0$. We get

$$\text{Sp}_q \theta_E(\alpha p) = (1 - e^{2\pi i \alpha}) \mod \text{torsion}. \quad (56)$$

So the specialization of a modular unit at the cusp $\infty$ is a cyclotomic $N$-unit. This implies that the right arrow in (53) is surjective.

**Lemma 3.5** One has

$$\text{Sp}_q \theta_E(\alpha_1 p, \alpha_2 p, \alpha_3 p) = \text{Li}_{1,1}^M(\zeta_{N_1}^{\alpha_1}, \zeta_{N_2}^{\alpha_2}), \quad \zeta_N = \exp(2\pi i / N).$$

**Proof.** Both the left and the right hand sides are zero if one of $\alpha_i$ is zero modulo $N$. So we may assume $\alpha_i$ are non zero.

One can realize the family $(E,p)$ of pairs (an elliptic curve plus a point $p$ of order $N$ on $E$) parametrized by a neighborhood of a cusp $\infty$, as a family of plane elliptic curves $E_q$ with an $N$-torsion section $p$, degenerating at $q = 0$ to a nodal curve with the torsion point $\zeta_N$. Then $f_{\alpha_1 p, b} \wedge f_{\alpha_2 p, b} \wedge f_{\alpha_3 p, b}$, where $b$ is a non zero multiple of $p$, degenerates, modulo $N$-torsion, to an element $f_{\zeta_{N_1}^{\alpha_1}} \wedge f_{\zeta_{N_2}^{\alpha_2}} \wedge f_{\zeta_{N_3}^{\alpha_3}}$ on $\Lambda^3 \mathbb{Q}(\mathbb{G}_m)^\ast$. It remains to apply Lemma 2.13, formula (32), and identity (51). (The reader may compare this with the proof of the formula (41)). The Lemma is proved.

It follows from Lemma 3.5 that the specialization map intertwines the maps of the modular complex to the Euler and cyclotomic ones.

b) If $p$ is a prime, the only relations between the modular units, as well as the cyclotomic units, are the parity relations $\theta_E(-a) = \theta_E(a)$, and similar ones for the cyclotomic case. Therefore we get a map of complexes which is surjective by definition, and is an isomorphism on the right. We claim that the left arrow is injective. The specialization map, restricted to the kernel of the differential in the Euler complex, is an isomorphism. Indeed, the Bloch group $B_2(X)$ is rigid by its very definition. So any element of $\text{Ker}\delta_2$ is constant on $X$. The specialization of a constant on $X - x$ equals to its value at any nearby point. So we get an isomorphism of complexes. The theorem is proved.

### 4 Imaginary quadratic fields and tessellations of the hyperbolic space

Let $O_K$ be the ring of integers in the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$. Following the work of Cremona [C], we recall the classical description (due to Bianchi [B]) of the fundamental domains for the group $GL_2(O_K)$ acting on the hyperbolic space $H_3$ for $d = 1,2,3,7,11$. Using this, we describe explicitly the chain complex of the corresponding tessellation of the hyperbolic plane.

In the next section we relate this complex with the Euler complex restricted to a CM point of the modular curve corresponding to an elliptic curve $E_K$ with the endomorphism ring $O_K$.

#### 4.1 Bianchi tessellations of the hyperbolic space

Let $H_3$ be the three dimensional hyperbolic space. The group $GL_2(\mathbb{C})$ acts on $H_3$ through its quotient $PGL_2(\mathbb{C})$. We present the hyperbolic space as an upper half space

$$H_3 = \{(z,t) : z \in \mathbb{C}, t \in \mathbb{R}, t > 0\}.$$
Let $K$ be an imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$, where $d$ is a square-free positive integer. Let $\mathcal{O}_K$ be the ring of integers in $K$. Then $GL_2(\mathcal{O}_K)$ acts on $\mathcal{H}_3$ discretely through its quotient $PGL_2(\mathcal{O}_K)$. Extending $\mathcal{H}_3$ by the set of cusps $K \cup \{\infty\}$ on the boundary, we form $\mathcal{H}_3^*: = \mathcal{H}_3 \cup K \cup \{\infty\}$.

The kernel of the $GL_2(\mathcal{O}_K)$-action on $\mathcal{H}_3$ is the center of $GL_2(\mathcal{O}_K)$. It is isomorphic to the group of roots of unity in $K$, which coincides with the units group $\mathcal{O}_K^*$. It is the subgroup $\{\pm 1\}$, unless $d = 1, 3$, i.e. $\mathcal{O}_K$ is the ring of Gaussian or Eisenstein integers, and its order is 4 or 6 respectively. The group $SL_2(\mathcal{O}_K)$ is the kernel of the determinant map $GL_2(\mathcal{O}_K) \rightarrow \mathcal{O}_K^*$. Observe that $PSL_2(\mathcal{O}_K)$ is an index two subgroup in $PGL_2(\mathcal{O}_K)$.

The Bianchi tessellation and the modular complex. The Bianchi tessellation is a classical $GL_2(\mathcal{O}_K)$-invariant tessellation of $\mathcal{H}_3^*$ on geodesic polyhedrons. It can be defined by the following construction, due to Voronoi. Let $L$ be a free rank two $\mathcal{O}_K$-module. Set $V := L \otimes \mathcal{O}_K \mathbb{C}$. We realize $\mathcal{H}_3$ as the quotient of the cone of positive definite Hermitian forms on $V^*$ modulo $\mathbb{R}_{>0}$. A non-zero vector $l \in L$ gives rise to a degenerate Hermitian form $\varphi_l$, where $\varphi_l(f) := |\langle f, l \rangle|^2$ on $V^*$. Let us consider the set of forms $\varphi_l$ when $l$ run through the set of all non zero primitive vectors in the lattice $L$. The convex hull of this set is an infinite polyhedron. Its projection to $\mathcal{H}_3$ provides a tessellation of the latter. It is the Bianchi tessellation. The set of its vertexes coincides with the set of cusps $K \cup \{\infty\}$. The interiors of the cells of dimension $\geq 1$ are inside of $\mathcal{H}_3$. The stabilizers of the cells are finite.

**Definition 4.1** The modular complex $M_{\mathcal{O}_K}^*$ for $GL_2(\mathcal{O}_K)$ is the chain complex of the Voronoi tessellation of the hyperbolic space $\mathcal{H}_3$ for the group $GL_2(\mathcal{O}_K)$:

$$M_{\mathcal{O}_K}^0 \xrightarrow{\partial} M_{\mathcal{O}_K}^1 \xrightarrow{\partial} M_{\mathcal{O}_K}^2 \xrightarrow{\partial} M_{\mathcal{O}_K}^3.$$

The extended modular complex $M_{\mathcal{O}_K}^{*}$ is the chain complex of the Voronoi tessellation of $\mathcal{H}_3^*$ for the group $GL_2(\mathcal{O}_K)$:

$$M_{\mathcal{O}_K}^0 \xrightarrow{\partial} M_{\mathcal{O}_K}^1 \xrightarrow{\partial} M_{\mathcal{O}_K}^2 \xrightarrow{\partial} M_{\mathcal{O}_K}^3.$$

Here $M_{\mathcal{O}_K}^i$ is the group generated by the oriented $(3 - i)$-cells of the tessellation.

The extended modular complex is a complex of right $GL_2(\mathcal{O}_K)$-modules.

### 4.2 Bianchi tessellations for Euclidean fields

Now let us assume that

$$K$$ is one of the five Euclidean field $\mathbb{Q}(\sqrt{-d})$, where $d = 1, 2, 3, 7, 11\ldots$$ (57)

The Voronoi tessellation in this case was considered by Bianchi [Bi]. Its main features are the following:

- The edges of the Bianchi tessellation are obtained from a single geodesic $\mathcal{G} = (0, \infty)$ by the action of the group $GL_2(\mathcal{O}_K)$.
- The three dimensional cells of the Bianchi tessellation are obtained from a single basic geodesic polyhedron $\mathbb{B}_d$ by the action of the group $GL_2(\mathcal{O}_K)$. 

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There is also a basic triangle $T = (0, 1, \infty)$. The orbits of $T$ give all the two dimensional cells of the Voronoi tessellation if and only if $d = 1, 3$.

Reflecting the basic polyhedron by the faces, and repeating this procedure infinitely many times with the obtained polyhedrons, we recover the Bianchi tessellation of $\mathcal{H}_3$. Below we describe the basic polyhedrons for the Euclidean fields.

**The Bianchi tessellation for the Gaussian integers.** The units group $\mathbb{Z}[i]^*$ is generated by $i$. The basic polyhedron $B_1$ is the geodesic octahedron described by its vertexes:

$$B_1 = (0, 1, i, 1 + i, (1 + i)/2, \infty).$$

![Figure 3: An octahedron of the Bianchi tessellation; on the right shown the view from the infinity.](image)

**The Bianchi tessellation for the Eisenstein integers.** Let $\mathbb{Z}[\rho]$, where $\rho = \frac{1+\sqrt{-3}}{2} = \exp(\pi i/3)$, be the ring of Eisenstein integers. Its units group is cyclic of order 6, generated by $\rho$. The basic polyhedron is the geodesic tetrahedron

$$B_3 = (0, 1, \rho, \infty).$$

![Figure 4: A tetrahedron of the Bianchi tessellation for the Eisenstein integers; the right picture shows the base viewed from infinity.](image)
The Bianchi tessellations for the Euclidean fields with \( d = 2, 7, 11 \). We present on Figure 5 the plans of the basic polyhedrons \( \mathcal{B}_2, \mathcal{B}_7, \mathcal{B}_{11} \). We set there
\[
\theta := \sqrt{-2}, \quad \alpha := (1 + \sqrt{-d})/2, \quad d = 7, 11
\]
Each of the plans shows the finite vertexes of a basic polyhedron as the vertexes of the plan, and the projections of the non-vertical geodesic edges of the polyhedron from the infinity as the edges of the plan. Observe that in the addition to the triangular faces we have quadrilateral faces for \( d = 2 \) and \( d = 7 \), and hexagonal faces for \( d = 11 \). The action of the group \( \mathbb{D}_d^0 \) on the set of faces of \( \mathcal{B}_d \) has two orbits for \( d = 2, 7, 11 \). The vertical faces of the polyhedrons project to the exterior sides of the plan. So for \( d = 2 \) we have two triangular vertical faces, and two quadrilateral ones. For \( d = 7 \) we have two quadrilateral vertical faces, and one triangular. For \( d = 11 \) we have two hexagonal vertical faces, and one triangular. So the plans describe the fundamental polyhedrons completely.

4.3 Describing the modular complexes for \( d = 1, 3 \)
Let \( K \) be a Euclidean imaginary quadratic field. Then the two properties of the Bianchi tessellation in the subsection 4.2 just mean that the groups \( \mathcal{M}_0^{O_K} \) and \( \mathcal{M}_2^{O_K} \) are right \( GL_2(O_K) \)-modules with one generator. A generator can be picked as follows:
- For \( \mathcal{M}_0^{O_K} \): the oriented polyhedron \( \mathcal{B}_d \).
- For \( \mathcal{M}_2^{O_K} \): the oriented geodesic \( G = (0, \infty) \).
The \( GL_2(O_K) \)-module \( \mathcal{M}_1^{O_K} \) has one generator if and only if \( d = 1, 3 \). In the latter case we have \( \mathcal{M}_1^{O_K} \) is generated by the oriented basic triangle \( \mathcal{T} = (0, 1, \infty) \).

Let \( \mathbb{D}_K^k \) be the stabilizer of the corresponding generator of \( \mathcal{M}_k^{O_K} \) up to a sign. These groups are described as follows:
(i) For any imaginary quadratic number field $K$ we have

$$\mathbb{D}^2_{O_K} := \text{the stabilizer of the geodesic } \mathbb{G} \cong \text{the semidirect product of } S_2 \text{ and } O_K^* \times O_K^*,$$

where $S_2$ acts by permuting the factors. The subgroup $\mathbb{D}^2_{O_K} \subset GL_2(O_K)$ consists of transformations $$(\sigma; \varepsilon_1, \varepsilon_2) : (v_1, v_2) \mapsto (\varepsilon v_{\sigma(1)}, \varepsilon v_{\sigma(2)}); \quad \varepsilon_i \in O_K^*, \quad \sigma \in S_2.$$

(ii) For any imaginary quadratic number field $K$ there is an isomorphism

$$\mathbb{D}^1_{O_K} := \text{the stabilizer of the geodesic triangle } \mathbb{T} \cong S_3 \times O_K^*.$$

It can be described as follows. Let $L_2$ be a free rank two $O_K$-module.

Choose a basis $(v_1, v_2)$ of $L_2$. Then $GL_2(O_K)$ is the automorphism group of $L_2$ written in this basis. We define a vector $v_0$ of $L_2$ by the formula

$$v_0 + v_1 + v_2 = 0.$$ 

Then $GL_2(O_K)$ is the automorphism group of $L_2$ written in this basis. We define a vector $v_0$ of $L_2$ by the formula

$$v_0 + v_1 + v_2 = 0.$$ 

Let $S_3$ be the group of permutations of the set $\{0, 1, 2\}$. The subgroup $\mathbb{D}^1_{O_K}$ consists of transformations

$$(v_1, v_2) \mapsto (\varepsilon v_{\sigma(1)}, \varepsilon v_{\sigma(2)}), \quad \varepsilon \in O_K^*, \quad \sigma \in S_3.$$ 

(iii) We define the group $\mathbb{D}^0_{O_K}$ only for the Euclidean fields. It is the stabilizer in $GL_2(O_K)$ of the basic polyhedron $\mathbb{B}_d$. Here is its description for the Gaussian and Eisenstein integers:

$$\mathbb{D}^0_{\mathbb{Z}[i]} \cong (\text{the symmetry group of the octahedron}) \times O_{\mathbb{Z}[i]}^*,$$

$$\mathbb{D}^0_{\mathbb{Z}[\rho]} \cong (\text{the symmetry group of the tetrahedron}) \times O_{\mathbb{Z}[\rho]}^*.$$ 

The group $\mathbb{D}^k_{O_K}$ stabilizes the corresponding generator of $M^k_{O_K}$ up to a sign. The sign provides a homomorphism $\chi_k : \mathbb{D}^k_d \to \mathbb{Z}$. Summarizing, we get

**Proposition 4.2** Let $K = \mathbb{Q}(\sqrt{-d})$ and $d = 1, 3$. Then we have for $k = 1, 2, 3$:

$$M^k_{O_K} = \mathbb{Z}[GL_2(O_K)] \otimes \mathbb{Z} \chi_k.$$ 

The action of the group $GL_2(O_K)$ is induced by the multiplication from the left.

5 Euler complexes, CM elliptic curves, and modular hyperbolic 3-folds

Let us restrict the Euler complex $\mathbb{E}^*_N$ to a CM point of $X(N)$ corresponding to a CM curve $E_K$ with complex multiplication by $O_K$, and then specialize further, by considering only $N$-torsion points.

5.1 Euler complexes for CM elliptic curves and geometry of modular orbifolds

*Elliptic units for a CM elliptic curve.* Let $K$ be an imaginary quadratic field, and $E_K$ a CM elliptic curve with the endomorphism ring $O_K$. Let us assume that the class number of $K$ is one. Then the curve $E_K$ is defined over $K$. The set of complex points of $E_K$ is given by $\mathbb{C}/\sigma(O_K)$, where $\sigma : K \hookrightarrow \mathbb{C}$ is an embedding.

Recall that $\text{Aut}(E_K) = O_K^*$. We denote by $\varepsilon(a)$ the image of a point $a$ under the action of the automorphism of $E_K$ corresponding to $\varepsilon \in O_K^*$. Let $\mathcal{N}$ be an ideal of $O_K$. The group $O_K^*$ acts by automorphisms of the group of $\mathcal{N}$-torsion points of $E_K$.

The following result is well known, see [KL].
**Lemma 5.1** Let $K$ be an imaginary quadratic field, $E_K$ the corresponding CM elliptic curve, and $\mathcal{N}$ an ideal in $\mathcal{O}_K$. Then the only relations between the elliptic units corresponding to $\mathcal{N}$-torsion points of $E_K$ are the distribution relations (35) and the symmetry relations provided by Corollary 2.6:

$$\theta_E(\varepsilon(a)) = \theta_E(a), \quad \varepsilon \in \mathcal{O}_K^*.$$  

(58)

In particular if $\mathcal{N}$ is a prime ideal in $\mathcal{O}_K$, the only relations between the corresponding elliptic units are the symmetry relations (58).

Let $K_\mathcal{N}$ be the field obtained by adjoining to $K$ the $N$-th roots of unity and elliptic units $\theta_E(a)$, where $a$ runs through all nonzero $\mathcal{N}$-torsion points of the curve $E_K$. The field $K_\mathcal{N}$ coincides with the ray class field corresponding to the ideal $\mathcal{N}$, see [KL], Chapter 11.

**Definition 5.2** Set

$$C_1^\text{un}(\mathcal{N}) := \left( \text{The group of elliptic units in } \mathcal{O}_{K_\mathcal{N}} \right) \otimes \mathbb{Q} \cong \mathcal{O}_{K_\mathcal{N}}^* \otimes \mathbb{Q},$$

$$C_1(\mathcal{N}) := \left( \text{The group of elliptic } \mathcal{N}\text{-units in } \mathcal{O}_{K_\mathcal{N}} \right) \otimes \mathbb{Q} \cong \mathcal{O}_{K_\mathcal{N}}[\frac{1}{N}]^* \otimes \mathbb{Q},$$

$$\tilde{C}_1(\mathcal{N}) := C_1(\mathcal{N}) \oplus \mathbb{Q}.$$

In the last formula the factor $\mathbb{Q}$ formally corresponds to the non-existing element $\theta_E(0)$. We abuse notation and denote the element $1 \in \mathbb{Q}$ there by $\theta_E(0)$.

**Proposition 5.3** Let $a_1, a_2, a_3$ be three $\mathcal{N}$-torsion points of $E$ with $a_1 + a_2 + a_3 = 0$. Then $\theta_E(a_1, a_2, a_3) \in B_2(K_\mathcal{N})_\mathbb{Q}$.

**Proof.** By the very definition, $\theta_E(a_1, a_2, a_3) \in B_2(\mathbb{Q})_\mathbb{Q}$. Recall that $B_2(\mathbb{Q})_\mathbb{Q} = B_2(\mathbb{Q})_\mathbb{Q}$. The rest follows from Lemma 2.5. The proposition follows.

**Definition 5.4** The $\mathbb{Q}$-vector space $C_2(\mathcal{N})$ is the subspace of $B_2(K_\mathcal{N})_\mathbb{Q}$ generated by the elements $\theta_E(a_1, a_2, a_3)$ when $a_1, a_2, a_3$ run through all $\mathcal{N}$-torsion points of $E$ with $a_1 + a_2 + a_3 = 0$.

It follows from (38) that there are the following complexes, placed in degrees $[1, 2]$:

$$E^\bullet(\mathcal{N}) := C_2(\mathcal{N}) \xrightarrow{\delta_2} \Lambda^2C_1(\mathcal{N}); \quad \hat{E}^\bullet(\mathcal{N}) := C_2(\mathcal{N}) \xrightarrow{\delta_2} \Lambda^2\tilde{C}_1(\mathcal{N}).$$  

(59)

**Lemma 5.5** Let $\mathcal{N} = \mathcal{P}$ be a prime ideal. Then there is one more complex:

$$E^\bullet_{un}(\mathcal{P}) := C_2(\mathcal{P}) \xrightarrow{\delta_2} \Lambda^2C^\text{un}_1(\mathcal{P});$$  

(60)

Let $\Gamma_1(\mathcal{N})$ be the subgroup of $GL_2(\mathcal{O}_K)$ fixing the row vector $(0, 1)$ modulo $\mathcal{N}$. Set

$$Y_1(\mathcal{N}) := \Gamma_1(\mathcal{N}) \backslash \mathcal{H}_3, \quad X_1(\mathcal{N}) := \Gamma_1(\mathcal{N}) \backslash \mathcal{H}_3^*.$$

The Galois group $\text{Gal}(K_{\mathcal{P}}/K)$. Let $\mathbb{F}_\mathcal{P} = \mathcal{O}_K / \mathcal{P}$ be the residue field of a prime ideal $\mathcal{P}$. Let $\mu_{\mathcal{P}}$ be the projection of the group of units to $\mathbb{F}_\mathcal{P}$. The Galois group $\text{Gal}(K_{\mathcal{P}}/K)$ is canonically isomorphic to $\mathbb{F}^*_\mathcal{P} / \mu_{\mathcal{P}}$. Indeed, $K_{\mathcal{P}}$ is the ray class field corresponding to the ideal $\mathcal{P}$, and $h_K = 1$. 

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The diamond operators. Let \( N(G) \) the normalizer of a group \( G \). Then the group

\[
D_P := N(\Gamma_1(P))/\Gamma_1(P)
\]

acts on \( \mathcal{Y}_1(P) \) preserving the Bianchi tessellation. Hence it acts on the modular complex. The group \( D_P \) is naturally isomorphic to \( \mathbb{F}_p^*/\mu_P \). It consists of the diamond operators \( \langle a \rangle \), where \( a \in \mathbb{F}_p^*/\mu_P \). Therefore there are canonical isomorphisms

\[
D_P = \mathbb{F}_p^*/\mu_P = \text{Gal}(K_P/K).
\]

Given a prime ideal \( P \), we define a map

\[
\Lambda^2\hat{\mathcal{C}}_1(P) \xrightarrow{\delta'} C_1(P) \oplus C_1(P).
\]

by setting

\[
\delta' : \theta_E(a) \wedge \theta_E(b) \mapsto \begin{cases} 
0 \oplus \theta_E(a) - \theta_E(b) & \text{if } a, b \neq 0 \\
-\theta_E(b) \oplus \theta_E(b) & \text{if } a = 0, b \neq 0.
\end{cases}
\] (61)

Then clearly the composition \( \delta' \circ \delta_2 = 0 \). Further, there is a map \( C_1(P) \rightarrow \mathbb{Q}, \theta_E(a) \mapsto 1 \). Taking the sum of the two copies of this map, we get a map \( \Sigma : C_1(P) \oplus C_1(P) \rightarrow \mathbb{Q} \). So we get the following complex, placed in degrees [1, 4]:

\[
C_2(P) \xrightarrow{\delta_2} \Lambda^2\hat{\mathcal{C}}_1(P) \xrightarrow{\delta'} C_1(P) \oplus C_1(P) \xrightarrow{\Sigma} \mathbb{Q}.
\] (62)

**Lemma 5.6** The complex (62) is quasiisomorphic to the complex \( \mathbf{E}^\bullet_{\text{un}}(P) \).

**Proof.** There is an obvious map of complexes

\[
\begin{array}{ccc}
C_2(P) & \xrightarrow{\delta_2} & \Lambda^2\hat{\mathcal{C}}_1(P) \\
= & \downarrow & \downarrow \\
C_2(P) & \xrightarrow{\delta_2} & \Lambda^2\hat{\mathcal{C}}_1(P) \xrightarrow{\delta'} C_1(P) \oplus C_1(P) \xrightarrow{\Sigma} \mathbb{Q}.
\end{array}
\]

It provides the desired quasiisomorphism. Indeed, it follows from the very definition that \( \hat{\mathcal{C}}_1(P) = \mathcal{C}_1^\text{un}(P) \oplus \mathbb{Q} \oplus \mathbb{Q} \). It is easy to see from this and the definition of the map \( \delta' \) that bottom complex is exact in degree 3, and the kernel of the map \( \delta' \) is \( \Lambda^2\mathcal{C}_1^\text{un}(P) \). The lemma is proved.

**Theorem 5.7** Let us assume that \( K \) is either Gaussian or Eisenstein field, i.e. \( d = 1, 3 \). Let \( P \) be a prime ideal in \( \mathcal{O}_K \). Choose a \( P \)-torsion point \( z \) of the curve \( E_K \) generating \( E_K[\mathcal{P}] \).

Then there exists a surjective homomorphism of complexes

\[
\begin{array}{ccc}
\mathbb{Q} \otimes_{\Gamma_1(P)} \left( M_{\mathcal{O}_K}^1 \xrightarrow{\partial} M_{\mathcal{O}_K}^2 \xrightarrow{\partial} M_{\mathcal{O}_K}^3 \right) & \\
\downarrow \theta^{(1)} & \downarrow \theta^{(2)} & \downarrow \theta^{(3)} \\
C_2(P) & \xrightarrow{\delta_2} & \Lambda^2\hat{\mathcal{C}}_1(P) \xrightarrow{\delta'} C_1(P) \oplus C_1(P)
\end{array}
\] (63)

which intertwines the action of the group \( D_P \) with the action of \( \text{Gal}(K_P/K) \).

Moreover the maps \( \theta^{(2)} \) and \( \theta^{(3)} \) are isomorphisms.
Proof. Since $\mathcal{O}_K$ has the class number one, one has

$$\Gamma_1(\mathcal{P}) \backslash GL_2(\mathcal{O}_K) = \mathbb{F}_p^2 - \{0,0\} = \{(\alpha, \beta) \in \mathbb{F}_p^2 - \{0,0\}\}.$$  

We identify it with the set of nonzero rows $\{(\alpha, \beta, \gamma)\}$ with $\alpha + \beta + \gamma = 0$. Then

$$Q \otimes_{\Gamma_1(\mathcal{P})} M^1_{\mathcal{O}_K} = \mathbb{Z}[\Gamma_1(\mathcal{P}) \backslash GL_2(\mathcal{O}_K)] \otimes_{\mathbb{D}^1_{\mathcal{O}_K}} \chi_1 =$$

$$\mathbb{Z}[\{(\alpha, \beta, \gamma) \in \mathbb{F}_p^3 - 0 \mid \alpha + \beta + \gamma = 0\}]$$

where the dihedral symmetry relations are the following:

$$(\alpha_1, \alpha_2, \alpha_3) = (-1)^{\text{sgn}(\sigma)}(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \alpha_{\sigma(3)}), \quad \sigma \in S_3,$$

$$(\alpha_1, \alpha_2, \alpha_3) = (\varepsilon \alpha_1, \varepsilon \alpha_2, \varepsilon \alpha_3), \quad \varepsilon \in \mathcal{O}_K^*.$$

$$Q \otimes_{\Gamma_1(\mathcal{P})} M^3_{\mathcal{O}_K} = \mathbb{Z}[\Gamma_1(\mathcal{P}) \backslash GL_2(\mathcal{O}_K)] \otimes_{\mathbb{D}^2_{\mathcal{O}_K}} \chi_2 =$$

$$\mathbb{Z}[\{(\alpha, \beta) \in \mathbb{F}_p^2 - 0\}]$$

$$\quad \underbrace{(\alpha, \beta) = - (\beta, \alpha) = (\varepsilon \alpha, \varepsilon \beta)}_{\varepsilon \in \mathcal{O}_K^*}, \quad \varepsilon \in \mathcal{O}_K^*.$$  

$$Q \otimes_{\Gamma_1(2,\mathcal{P})} M^3_{\mathcal{O}_K} = \mathbb{Z}[\Gamma_1(\mathcal{P}) \backslash GL_2(\mathcal{O}_K)/B(\mathcal{O}_K)] =$$

the free abelian group with generators $[\beta, 0]$ and $[0, \beta]$, where $\beta \in \mathbb{F}_p^* / \mu_p$.

Here $B$ is group of upper triangular matrices.

Then the differential in the top complex in (84) is described as follows:

$$\partial : (\alpha, \beta, \gamma) \mapsto (\alpha, \beta) + (\beta, \gamma) + (\gamma, \alpha);$$

$$\partial : (\alpha, \beta) \mapsto \begin{cases} 
0 \oplus [0, \alpha] - [0, \beta] & \text{if } \alpha, \beta \neq 0 \\
[-\beta, 0] \oplus [0, \beta] & \text{if } \alpha = 0, \beta \neq 0.
\end{cases}$$

To define the map of complexes $\theta^{(1)}$ we pick a nonzero $\mathcal{P}$-torsion point $z$ of the curve $E_K$. Then there is an isomorphism of abelian groups

$$\mathbb{F}_p \longrightarrow E_K[\mathcal{P}], \quad \alpha \mapsto \alpha z.$$  

Now we proceed as follows.

(i) Let $\mathcal{F}_p := \mathbb{Q}[\mathbb{F}_p^*/\mu_p]$ be the $\mathbb{Q}$-vector space generated by the set $\mathbb{F}_p^*/\mu_p$. We denote by $[\alpha]$ the generator of $\mathcal{F}_p$ corresponding to an element $\alpha \in \mathbb{F}_p^*$, so $[\varepsilon \alpha] = [\alpha]$. According to Proposition 5.2 there is an isomorphism

$$\mathcal{F}_p \xrightarrow{\sim} \left(\text{The group of } \mathcal{P}\text{-elliptic units in } \mathcal{O}_{K_p}^*\right) \otimes \mathbb{Q}, \quad [\alpha] \mapsto \theta_E(\alpha z).$$

(ii) Using the identifications (64) - (66) we define the desired map of complexes as follows:

$$\theta^{(1)} : (\alpha, \beta, \gamma) \mapsto \theta_E(\alpha z, \beta z, \gamma z),$$
\[ \theta^{(2)} : (\alpha, \beta) \mapsto \theta_E(\alpha z) \wedge \theta_E(\beta z), \]
\[ \theta^{(3)} : [\beta_1, 0] + [0, \beta_2] \mapsto \theta_E(\beta_1 z) \oplus \theta_E(\beta_2 z). \]

The map \( \theta^{(1)} \) is well defined thanks to the dihedral symmetry relations for \( \theta_E(a, b, c) \); the maps \( \theta^{(2)} \) and \( \theta^{(3)} \) are well defined thanks to the symmetry relations for \( \theta_E(a) \). It follows easily from (67)-(68), the formula (38), and (61) that the map \( \theta^* \) is a homomorphism of complexes. It is surjective by Theorem 2.10 b) and Proposition 5.2. Since \( \mathcal{P} \) is a prime ideal, thanks to Lemma 5.1 there are no distribution relations between the elliptic units, and hence the maps \( \theta^{(2)} \) and \( \theta^{(3)} \) are isomorphisms.

It follows from the very definition that the map \( \theta^* \) intertwines the actions of the groups \( \mathcal{D}_\mathcal{P} \) and \( \text{Gal}(K_{\mathcal{P}}/K) \). The theorem is proved.

**Remark.** Let us remove dependence on the choice of a \( \mathcal{P} \)-torsion point \( z \) in the theorem. Let \( \mu(1) \) be the \( \text{Gal}(K_{\mathcal{P}}/K) \)-module of non-trivial \( \mathcal{P} \)-torsion points on the curve \( E_K \). Then there is a canonical morphism

the top complex in (63) \( \rightarrow \) the bottom complex in (63) \( \otimes_{\text{Gal}(K_{\mathcal{P}}/K)} \mu(1) \).

It is given by \( \theta^*_z \otimes z^{-1} \), where \( \theta^*_z \) is the above map of complexes defined using a torsion point \( z \).

### 5.2 The rational cohomology of \( \Gamma_1(\mathcal{P}) \)

If \( \Gamma \) is a torsion free finite index subgroup of \( GL_2(\mathcal{O}_K) \), we have an isomorphism

\[ H^*(\Gamma, \mathbb{Q}) = H^*(\Gamma \backslash \mathcal{H}_3, \mathbb{Q}). \]

If we understood the right hand side as the cohomology group of the orbifold \( \mathcal{Y}_\Gamma := \Gamma \backslash \mathcal{H}_3 \), this formula remains valid for any subgroup of \( GL_2(\mathcal{O}_K) \). Let us recall few basic facts about the cohomology groups \( H^*(\Gamma, \mathbb{Q}) \). See [Hal] and references there for further details.

The orbifold \( \mathcal{Y}_\Gamma \) is compactified by \( \mathcal{X}_\Gamma := \Gamma \backslash \mathcal{H}_3^\infty \). The complement \( \mathcal{X}_\Gamma - \mathcal{Y}_\Gamma \) consists of a finite number of cusps. The orbifold \( \mathcal{Y}_\Gamma \) has a boundary \( \partial \mathcal{Y}_\Gamma \) given by disjoint union of two dimensional orbifold tori. These tori are parametrized by the cusps. Restriction to the boundary provides a map

\[ \text{Res} : H^*(\mathcal{Y}_\Gamma, \mathbb{Q}) \rightarrow H^*(\partial \mathcal{Y}_\Gamma, \mathbb{Q}). \]

The kernel of the restriction map is the cuspidal part \( H^*_{\text{cusp}}(\Gamma, \mathbb{Q}) \) of the cohomology. The image is called the Eisenstein part of the cohomology, and denoted \( H^*_{\text{Eis}}(\Gamma, \mathbb{Q}) \). So there is an exact sequence

\[ 0 \rightarrow H^*_{\text{cusp}}(\Gamma, \mathbb{Q}) \rightarrow H^*(\Gamma, \mathbb{Q}) \rightarrow H^*_{\text{Eis}}(\Gamma, \mathbb{Q}) \rightarrow 0. \]

The cuspidal cohomology satisfies the Poincare duality, so we have

\[ \dim H^1_{\text{cusp}}(\Gamma, \mathbb{Q}) = \dim H^2_{\text{cusp}}(\Gamma, \mathbb{Q}). \]

Let \( M \) be a threefold with boundary \( \partial M \). Then it follows from the Poincare duality that the image of the restriction map \( \text{Res} : H^*(M) \rightarrow H^*(\partial M) \) is a Lagrangian subspace in \( H^*(\partial M) \).

Therefore if \( \Gamma \) is torsion free then the image of the restriction map is a Lagrangian subspace in the boundary cohomology \( H^*(\partial \mathcal{Y}_\Gamma, \mathbb{Q}) \).

It follows from this that if \( \Gamma \) is torsion free then the homological Euler characteristic is zero:

\[ \chi_h(\Gamma) = \sum (-1)^i \dim H^i(\Gamma, \mathbb{Q}) = 0. \]
Indeed, clearly \( H^0_{\text{Eis}}(\Gamma, \mathbb{Q}) = \mathbb{Q} \). Since the image of the restriction map (73) is Lagrangian, this implies that there is a natural isomorphism

\[
H^2_{\text{Eis}}(\Gamma, \mathbb{Q}) = \text{Ker}\left( \mathbb{Q}[\text{cusps of } \Gamma] \xrightarrow{\Sigma} \mathbb{Q} \right).
\]

Observe that identifying \( H^2_{\text{Eis}}(\Gamma, \mathbb{Q}) \) with the Borel-Moore homology \( H^2_{BM}(\mathcal{Y}_\Gamma, \mathbb{Q}) \), the boundary map is given by the “boundary at infinity” of the 1-cycle representing the class, which is a linear combination of cusps of total degree zero.

Finally, \( \dim H^1(\partial \mathcal{Y}_\Gamma) = 2 \times (\text{the number of cusps of } \Gamma) \), and therefore \( \dim H^1_{\text{Eis}}(\Gamma, \mathbb{Q}) \) is equal to the number of cusps of \( \Gamma \). However if \( \Gamma \) has torsion the situation is more complicated.

**Example.** The modular 3-fold \( \mathcal{Y}_1(\mathcal{P}) \) is a three dimensional orbifold. Indeed, the subgroup of diagonal matrices \( \text{diag}(\epsilon, 1) \), where \( \epsilon \in \mathcal{O}_K^* \), is a torsion subgroup in \( \Gamma_1(\mathcal{P}) \). The complement \( \mathcal{X}_1(\mathcal{P}) - \mathcal{Y}_1(\mathcal{P}) \) consists of \( 2|\mathbb{F}_{\mathcal{P}}^*/\mu_{\mathcal{P}}| \) cusps. The natural covering \( \mathcal{X}_1(\mathcal{P}) \to \mathcal{X}_0(\mathcal{P}) \) is unramified of degree \( |\mathbb{F}_{\mathcal{P}}^*/\mu_{\mathcal{P}}| \). There are two cusps on \( \mathcal{X}_0(\mathcal{P}) \): the 0 and \( \infty \) cusps. So there are \( |\mathbb{F}_{\mathcal{P}}^*/\mu_{\mathcal{P}}| \) cusps over 0 as well as over \( \infty \).

**Lemma 5.8** Let \( \mathcal{P} \) be a prime ideal in \( \mathcal{O}_K \). Then

\[
\begin{align*}
H^1_{\text{Eis}}(\Gamma_1(\mathcal{P}), \mathbb{Q}) &= 0, \\
H^2_{\text{Eis}}(\Gamma_1(\mathcal{P}), \mathbb{Q}) &= \text{Ker}\left( \mathbb{Q}[\text{cusps of } \Gamma_1(\mathcal{P})] \xrightarrow{\Sigma} \mathbb{Q} \right).
\end{align*}
\]

**Proof.** In this case the boundary cohomology group \( H^1 \) is zero. Indeed, the cusps, and hence the boundary components are parametrized by the \( \Gamma_1(\mathcal{P}) \)-orbits on the set of Borel subgroups in \( GL_2(\mathcal{O}_K) \). Consider the boundary component corresponding to the standard (upper triangular) Borel subgroup of \( GL_2(\mathcal{O}_K) \). We identify \( H^1 \) of the corresponding boundary component with the unipotent radical \( U(\mathcal{O}_K) = \mathcal{O}_K \). The torsion element \( \text{diag}(\epsilon, 1) \) acts on it by multiplication on \( \epsilon \). Taking \( \epsilon = -1 \), we see that it kills the boundary \( H^1 \). However on \( H^2 \) it acts as the identity. The lemma follows.

**Remark.** This lemma is consistent with Horozov’s computation of the homological Euler characteristic of \( \Gamma_1(\mathcal{P}) \), see Theorem 0.5 in [H]. Namely,

\[
\dim H^2_{\text{Eis}}(\Gamma_1(\mathcal{P}), \mathbb{Q}) = 2|\mathbb{F}_{\mathcal{P}}^*/\mu_{\mathcal{P}}| - 1,
\]

and thus we get

\[
\chi_h(\Gamma_1(\mathcal{P})) = 1 - \dim H^1_{\text{Eis}}(\Gamma_1(\mathcal{P}), \mathbb{Q}) + \dim H^2_{\text{Eis}}(\Gamma_1(\mathcal{P}), \mathbb{Q}) = 2|\mathbb{F}_{\mathcal{P}}^*/\mu_{\mathcal{P}}|
\]

which coincides with the result obtained in loc. cit.

### 5.3 Applications

**Corollary 5.9** Let \( K = \mathbb{Q}(\sqrt{-d}) \) where \( d = 1, 3 \), and let \( \mathcal{P} \) be a prime ideal in \( \mathcal{O}_K \). Then there are canonical isomorphisms

\[
\begin{align*}
\hat{\theta}^{(2)}_h : H^2(\Gamma_1(\mathcal{P}), \mathbb{Q}) &\xrightarrow{\sim} H^2(\hat{\mathcal{E}}^{\bullet}(\mathcal{P})), \\
\theta^{(2)}_{\text{cusp}} : H^2_{\text{cusp}}(\Gamma_1(\mathcal{P}), \mathbb{Q}) &\xrightarrow{\sim} H^2(\mathcal{E}^{\bullet}_{\text{un}}(\mathcal{P})).
\end{align*}
\]
**Proof.** The isomorphism (76) follows immediately from Theorem 5.7. Indeed, take the stupid truncation in degrees \( \leq 2 \) of the two complexes in (63). Then \( H^2 \) of the truncated top complex is \( H^2(\Gamma_1(\mathcal{P}), \mathbb{Q}) \), while \( H^2 \) of the truncated bottom complex is \( H^2(\mathcal{E}^*(\mathcal{P})) \).

To prove (77) observe that \( H^2 \) of the top complex delivers the left hand side in (77). By Lemma 5.6, the \( H^2 \) of the bottom complex gives the right hand side.

Here is another argument for this. There is a natural map

\[
H^2(\mathcal{E}^*(\mathcal{P})) \to \text{Ker}(\mathcal{C}_1\mathcal{P} \oplus \mathcal{C}_1\mathcal{P} \xrightarrow{\Sigma} \mathbb{Q}).
\]

The right hand side can be identified with \( H^2_{\text{Eis}}(\Gamma_1(\mathcal{P}), \mathbb{Q}) \), see Section 5.2, e.g. (75), and thus the kernel is identified with \( H^2_{\text{cusp}}(\Gamma_1(\mathcal{P}), \mathbb{Q}) \). The corollary is proved.

The commutativity of the diagram (63) implies that the composition

\[
M^0_{\mathcal{O}_K} \xrightarrow{\partial} M^1_{\mathcal{O}_K} \xrightarrow{\theta^{(1)}} \mathcal{C}_2(\mathcal{P})
\]

provides a natural map

\[
M^0_{\mathcal{O}_K} \to \text{Ker} \left( \mathcal{C}_2(\mathcal{P}) \xrightarrow{\delta_2} \Lambda^2 \mathcal{C}_1(\mathcal{P}) \right) = H^1(\mathcal{E}^*(\mathcal{P})).
\]  

**Definition 5.10** \( \alpha(K_P) \) is the image of the map (78).

Since \( K_P \) is a number field, \( B_2(K_P)_\mathbb{Q} = B_2(K_P)_\mathbb{Q} \). So Ker\( \delta_2 = K_3(K_P)_\mathbb{Q} \) by Suslin’s theorem. Thus there is an embedding

\[
H^1(\mathcal{E}^*(\mathcal{P})) \subset K_3(K_P)_\mathbb{Q}.
\]

So we immediately get

**Corollary 5.11** Under the same assumptions as in Corollary 5.9, there is a homomorphism

\[
\theta_1^{(1)} : H^1_{\text{cusp}}(\Gamma_1(\mathcal{P}), \mathbb{Q}) \to \frac{H^1(\mathcal{E}^*(\mathcal{P}))}{\alpha(K_P)} \subset K_3(K_P)_\mathbb{Q}. \tag{79}
\]

**Proof.** Indeed, there is a map

\[
\theta_1^{(1)} : H^1(\Gamma_1(\mathcal{P}), \mathbb{Q}) \to \frac{H^1(\mathcal{E}^*(\mathcal{P}))}{\alpha(K_P)}.
\]  

Since \( H^1_{\text{Eis}}(\Gamma_1(\mathcal{P}), \mathbb{Q}) = 0 \), one has \( H^1_{\text{cusp}} = H^1 \), so the map (80) coincides with the needed map (79).

**5.4 Relations between the elements** \( \theta_E(a_1, a_2, a_3) \)

To describe the vector space \( \mathcal{C}_2(\mathcal{N}) \) one needs to know all relations between the elements \( \theta_E(a_1, a_2, a_3) \).

There is a decomposition

\[
\mathcal{Y}_1(\mathcal{P}) = \bigcup_{\gamma \in \Gamma_1(\mathcal{P}) \backslash \text{PGL}_2(\mathcal{O}_K)} \gamma \mathcal{E}_g
\]  

of the orbifold \( \mathcal{Y}_1(\mathcal{P}) \) into a union of finite number of geodesic polyhedrons, obtained by projecting the Bianchi tesselation of \( H_3 \) corresponding to \( \mathcal{O}_K \) onto \( \mathcal{Y}_1(\mathcal{P}) \). We call it the Bianchi decomposition. Each polyhedron of the Bianchi decomposition of \( \mathcal{Y}_1(\mathcal{P}) \) gives rise to a relation between the elements.
\[ \theta_E(a_1, a_2, a_3), \text{ where } a_i \text{ are } \mathcal{P}\text{-torsion points of } E_K. \] Namely, the sum of the elements corresponding via the map \( \theta^{(1)} \) to the faces of the polyhedron \( \gamma \mathbb{B}_d \) is an element

\[ \theta_E^{(1)}(\partial(\gamma \mathbb{B}_d)) \in K_3(K_{\mathcal{P}})_\mathbb{Q}. \]

We call the quotient of the space of relations between the elements \( \theta_E(a_1, a_2, a_3) \), where \( a_1, a_2, a_3 \) run through the nonzero \( \mathcal{P}\text{-torsion points in } E_K \), modulo the subspace generated by the dihedral symmetry relations and the described above relations corresponding to the polyhedrons of the decomposition (81), the space of sporadic level \( \mathcal{P} \) relations between the elements \( \theta_E(a_1, a_2, a_3) \).

**Corollary 5.12** Let \( \mathcal{P} \) be a prime ideal in \( \mathcal{O}_K \). Then the space of level \( \mathcal{P} \) sporadic relations between the elements \( \theta_E(a_1, a_2, a_3) \) is identified with the kernel of the map \( \theta^{(1)}_h \).

To describe the relation corresponding to a polyhedron of the decomposition (81) precisely one needs to know explicitly the element of \( K_3(K_{\mathcal{P}}) \) corresponding to this polyhedron.

**Conjecture 5.13** The element \( \theta_E^{(1)}(\partial(\gamma \mathbb{B}_d)) \) is zero.

Summarizing, there are three types of the relations between the elements \( \theta_E(a_1, a_2, a_3) \):

(i) The dihedral symmetry relations (39) - (40) corresponding to symmetries of the geodesic triangles in the Bianchi decomposition of the modular orbifold \( \mathcal{Y}_1(\mathcal{P}) \).

(ii) The relations provided by the polyhedrons of the Bianchi decomposition (81).

(iii) Sporadic relations are described by the map (79), that is,

\[ \theta^{(1)}_h : H^1_{\text{cusp}}(\Gamma_1(\mathcal{P}), \mathbb{Q}) \rightarrow K_3(K_{\mathcal{P}})_\mathbb{Q}/\alpha(K_{\mathcal{P}}). \] (82)

The Hecke algebra acts on the left. It is hard to imaging an action of the Hecke algebra on the right.

**Problem 5.14** Determine the map (82).

**Appendix.** Here is another way to assign to translations of the basic polyhedron elements of \( K_3 \otimes \mathbb{Q} \); this time they lie in the one dimensional space \( K_3(K) \).

**Lemma 5.15** For any \( \gamma \in \text{PGL}_2(\mathcal{O}_K) \), the geodesic polyhedron \( \gamma \mathbb{B}_d \) provides an element

\[ [\gamma \mathbb{B}_d] \in K_3(K)_\mathbb{Q}. \] (83)

The value of the regulator map \( K_3(K) \rightarrow \mathbb{R} \) on this element is the volume of the polyhedron \( \mathbb{B}_d \). Thus the element (83) does not depend on \( \gamma \).

**Proof.** Cut the polyhedron \( \gamma \mathbb{B}_d \) into ideal geodesic simplices. We asssine to an ideal geodesic simplex \( I(\infty, 0, 1, z) \) with vertices at \( \infty, 0, 1, z \), where \( z \in \overline{\mathbb{Q}} \), the element \( \{z\}_2 \) of the Bloch group \( B_2(\overline{\mathbb{Q}}) \). Let \([\gamma \mathbb{B}_d]\) be the sum of the elements corresponding to the simplices of the decomposition. It lies in \( B_2(K) \) since the vertices of the simplices can be taken at the cusps, which are identified with \( K \cup \{\infty\} \). Let us show that the differential \( \delta_2 \) in the Bloch complex kills this element.

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The Dehn invariant of the ideal geodesic simplex \( I(\infty, 0, 1, z) \) equals
\[ \log |1 - z| \otimes \arg(z) - \log |z| \otimes \arg(1 - z) \in \mathbb{R} \otimes S^1 = (\Lambda^2 \mathbb{C}^*)^- \]
where \(-\) stands for the antiinvariants of the complex conjugation acting on \( \Lambda^2 \mathbb{C}^* \). A union of finite number of translations of the polyhedron \( \mathbb{B}_d \) gives (a multiple of) the fundamental cycle of the threefold \( Y_1(\mathcal{P}) \). The Dehn invariant of the latter is zero. Thus the Dehn invariant of \( \mathbb{B}_d \) is torsion.

According to the Lobachevsky formula one has
\[ \text{vol} I(\infty, 0, 1, z) = L_2(z) \]

Since the regulator map on \( K_3(K)_\mathbb{Q} \) is given by the dilogarithm function \( L_2(z) \), we get the second claim of the Lemma. Now the injectivity of the regulator map on \( K_3(K)_\mathbb{Q} \) implies that the element (83) does not depend on \( \gamma \). The Lemma is proved.

**Remark.** The element \([\gamma \mathbb{B}_d]\) can not be a non-zero multiple of \( \theta(1) \). Indeed, consider the sum of (translations of) basic polyhedrons providing the fundamental cycle of the threefold \( Y_1(\mathcal{P}) \). Then the sum of the corresponding elements \([\theta(\gamma \mathbb{B}_d)]\) is zero. Indeed, the contribution of each polyhedron is provided by the triangles forming its boundary, and the boundary of the fundamental cycle is zero. On the other hand the sum of the elements \([\gamma \mathbb{B}_d]\) corresponding to the polyhedrons entering to the fundamental cycle is, by Lemma 5.15, a non-zero multiple of \([\mathbb{B}_d]\).

### 5.5 Generalizations

**Theorem 5.16** Let \( \mathcal{O}_K \) be the ring of Gaussian or Eisenstein integers. Let \( \mathcal{N} \) be an ideal in \( \mathcal{O}_K \). Then there exists a canonical surjective homomorphism of complexes
\[ Q \otimes_{\Gamma_1(\mathcal{N})} \left( M^1_{\mathcal{O}_K} \xrightarrow{\partial} M^2_{\mathcal{O}_K} \right) \downarrow \theta^{(1)} \downarrow \theta^{(2)} \]
\[ C_2(\mathcal{N}) \xrightarrow{\delta_2} \Lambda^2 \hat{C}_1(\mathcal{N}) \]

The maps \( \theta^{(*)} \) intertwine the action of the group \( \mathcal{D}_N \) with the action of \( \text{Gal}(K\mathcal{N}/K) \).

**Proof.** We follow the proof of Theorem 5.7 with necessary modifications. One has
\[ \Gamma_1(\mathcal{N}) \backslash GL_2(\mathcal{O}_K) = \{ (\alpha, \beta) \in (\mathcal{O}_K/\mathcal{N})^2 | (\alpha, \beta, \mathcal{N}) = 1 \} \]

We identify it with the set of rows \( \{ (\alpha, \beta, \gamma) \} \) with \( \alpha + \beta + \gamma = 0 \), and \( \alpha, \beta, \gamma \) are modulo \( \mathcal{N} \) and have no common divisors with \( \mathcal{N} \). Then
\[ Q \otimes_{\Gamma_1(\mathcal{N})} M^1_{\mathcal{O}_K} = \frac{\mathbb{Z}[\alpha, \beta, \gamma] \in (\mathcal{O}_K/\mathcal{N})^3 | \alpha + \beta + \gamma = 0, (\alpha, \beta, \mathcal{N}) = 1}{\text{the dihedral symmetry relations}} \]
\[ Q \otimes_{\Gamma_1(\mathcal{N})} M^2_{\mathcal{O}_K} = \frac{\mathbb{Z}[\alpha, \beta] \in (\mathcal{O}_K/\mathcal{N})^2 | (\alpha, \beta, \mathcal{N}) = 1}{(\alpha, \beta) = -\beta, (\beta, \alpha) = (\varepsilon \alpha, \varepsilon \beta), \varepsilon \in \mathcal{O}_K} \]
It would be very interesting to find the “right” generalisation of these results for other imaginary quadratic fields. The above construction has an obvious generalization: the subcomplex of the modular complex, given by the \( GL_2(\mathcal{O}_K) \)-submodules generated by \( \mathbb{T} \) and \( \mathbb{G} \), is mapped to the complex \( C_2(N) \xrightarrow{\delta_2} \Lambda^2\mathcal{C}_1(N) \). However the \( GL_2(\mathcal{O}_K) \)-submodules generated by \( \mathbb{T} \) and \( \mathbb{G} \) do not give, in general, the whole modular complex. For example, let \( K = \mathbb{Q}(\sqrt{-2}) \). Then the \( GL_2(\mathcal{O}_K) \)-submodule generated by \( \mathbb{G} \) is \( M^2_{\mathcal{O}_K} \). However there are 2-cells of the Bianchi tessellation which are not in the \( GL_2(\mathcal{O}_K) \)-orbit of \( \mathbb{T} \), e.g. the geodesic 4-gon with vertices at \( (1, \frac{1+\theta}{2}, 1 + \theta, \infty) \). This suggests that there might exist a natural construction of an element of \( B_2(S_N) \), which goes under the differential \( \delta_2 \) to the sum of four wedge products of the elliptic units corresponding to the sides of this 4-gon, and similarly for other 2-cells in the \( GL_2(\mathcal{O}_K) \)-orbit of this 4-gon.

Another question is whether/when the 1-cycles provided by the \( GL_2(\mathcal{O}_K) \)-orbits of \( \mathbb{G} \) generated the Borel-Moore \( H_1 \) of the corresponding modular 3-fold for general \( K \). If so, this would imply that for a prime ideal \( \mathfrak{P} \) of \( \mathcal{O}_K \) there is a natural surjective map from \( \Lambda^2\mathcal{O}_K^{\mathfrak{P}} \) to \( H^2_{\text{cusp}} \) of the corresponding modular 3-fold, just like in Conclusion 2.

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