LOCAL TO GLOBAL PRINCIPLES FOR GENERATION TIME OVER NOETHER ALGEBRAS

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Abstract. In the derived category of modules of a Noether algebra a complex $G$ is said to generate a complex $X$ if the latter can be obtained from the former by taking finitely many summands and cones. The number of cones needed in this process is the generation time of $X$. In this paper we present some local to global type results for computing this invariant, and also discuss some applications of these results.

1. Introduction

The goal of this paper is to develop techniques for computing generation time of complexes over Noether algebras. In the derived category of a Noether algebra $R$, a complex $G$ generates a complex $X$ if the latter is obtained from the former by taking direct summands and mapping cones. The minimal number of cones required is the generation time, or level, of $X$ with respect to $G$ and denoted $\text{level}^G_R(X)$. Bondal and van den Bergh introduced the notion of generation in [BvdB03], see also [Rou08]. The notation and terminology of level are adopted from [ABIM 10].

Level has connections to other, more familiar, invariants. When $G = R$ and $X$ is a finitely generated module, the level of $X$ with respect to $R$ is the projective dimension of $X$, see [Chr98]. When $X$ is a complex with finitely generated homology, $\text{level}^R_R(X)$ is bounded above by the projective dimension of $X$—that is the minimal length of a projective resolution—but typically the level is smaller.

When $R$ is a semilocal ring with Jacobson radical $J(R)$, the level with respect to $G = R/J(R)$ of a module is the Loewy length. Level with respect to $G$ gives an extension of this notion to complexes.

Unlike projective dimension and Loewy length, levels behave better under functors of derived categories. This is because such a functor need not map projectives to projectives or semisimple modules to semisimple modules. This flexibility afforded by levels becomes useful.

Despite their utility, there are few results on the behavior of level even under functors induced by a change of rings. This paper tracks the behavior of level under standard commutative algebra operations, notably localizations and completions.

There are two main theorems.

Theorem 1. Let $\varphi: R \to S$ be a faithfully flat ring map with $R$ a commutative noetherian ring and $S$ a noetherian ring, so that $R$ acts centrally on $S$. For any objects $G$ and $X$ in $\text{D}_f(R)$, there is an equality

$$\text{level}^G_R(X) = \text{level}^S_{\varphi^*}(G)(\varphi^*(X))$$

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where \( \varphi^* = S \otimes_R^L \) is the functor between the derived categories.

This especially applies for a local ring \((R, \mathfrak{m})\) and \( S = \widehat{R} \) its \( \mathfrak{m} \)-adic completion. The result and its corollary are given in 3.11 and 3.12.

The second result considers the localizations of a commutative, noetherian ring. While a localization need not be faithful, a morphism is zero if and only if its localization at any prime ideal is zero. This leads to:

**Theorem 2.** Let \( R \) be a Noether algebra with center \( Z(R) \). For any objects \( G \) and \( X \) in \( D_f(R) \), there is an equality

\[
\text{level}^{G^*}_{R^*}(X) = \sup \left\{ \text{level}^{G^*}_{R^*}(X_p) \middle| p \in \text{Spec}(Z(R)) \right\}.
\]

Moreover if \( \text{level}^{G^*}_{R^*}(X_p) < \infty \) for all prime ideals \( p \) of \( Z(R) \), then \( \text{level}^{G^*}_{R^*}(X) < \infty \).

This statement, contained in 4.4 and 4.5, should be compared with, and extends the result of Bass and Murthy [BM67, Lemma 4.5] that a finitely generated module has finite projective dimension if it has finite projective dimension locally.

For a commutative noetherian ring Theorems 1 and 2 reduce computing level to the derived category of complete local rings and have the following applications:

From Hopkins’ [Hop87, Theorem 11] and Neeman’s [Nee92, Lemma 1.2] result about perfect complexes, we deduce in 5.4 that for complexes \( X, Y \) of finite injective dimension with finitely generated bounded homology, \( X \) generates \( Y \) if and only if the support of \( X \) contains the support of \( Y \).

One can also track the behavior of proxy small, introduced in [DGI06]. A complex \( X \) is proxy small if \( X \cong 0 \) or it generates a perfect complex \( Y \neq 0 \) with the same support as \( X \). We prove that \( X \) is proxy small if and only if it is proxy small locally, see 6.3, and \( X \) is proxy small precisely when it is proxy small under a faithfully flat base change, see 6.5.

By [Pol18], proxy small objects in \( D_f(R) \) characterize whether a local ring \( R \) is a complete intersection. We conclude that proxy small objects also characterize whether a ring is locally a complete intersection.

The main tool to prove Theorems 1 and 2 is a converse coghost lemma proved by Oppermann and Šťovíček [OŠ12]. A map is coghost if it cannot be detected by post-composition with any suspension of \( G \). The coghost index of \( X \) with respect to \( G \) is the minimal number \( n \) for which every \( n \)-fold composition of coghost maps that ends at \( X \) is zero. It is well-known that the coghost index is less than or equal to the level, see [Kel65]. By the converse coghost lemma one has an equality in the bounded derived category of a Noether algebra.

Section 2 recalls definitions of level and the coghost index. There we state the converse coghost lemma and discuss some aspects of the proof, as well as deduce a converse ghost lemma whenever the ring has a dualizing complex. In Section 8 the behavior of level and the coghost index are tracked under the functor given by tensoring with a complex of finite flat dimension. Here Theorem 1 is proved. Then localizations are discussed in Section 4. Applications are discussed in Section 5 and proxy smallness in Section 6.

## 2. Level and coghost

In this section, we recall the definition of levels and the (co)ghost index from [BvdB03] and [ABIM10]. Then the converse coghost lemma is stated, and a converse ghost lemma is proved.
Level. Let \( \mathcal{T} \) be a triangulated category and \( \mathcal{C} \) a subcategory of \( \mathcal{T} \). Then

1. \( \text{add}(\mathcal{C}) \) denotes the smallest strictly full subcategory of \( \mathcal{T} \) containing \( \mathcal{C} \) that is closed under finite direct sums and suspensions, and
2. \( \text{smd}(\mathcal{C}) \) denotes the smallest strictly full subcategory of \( \mathcal{T} \) containing \( \mathcal{C} \) that is closed under direct summands.

If \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are subcategories of \( \mathcal{T} \), then \( \mathcal{C}_1 \star \mathcal{C}_2 \) is the strictly full subcategory containing all objects \( X \), such that there exists an exact triangle

\[
Y \to X \to Z \to \Sigma Y
\]

in \( \mathcal{T} \) with \( Y \in \mathcal{C}_1 \) and \( Z \in \mathcal{C}_2 \). Set

\[
\mathcal{C}_1 \circ \mathcal{C}_2 := \text{smd}(\text{add}(\mathcal{C}_1) \star \text{add}(\mathcal{C}_2)) .
\]

Recall \( \mathcal{S} \) is a thick subcategory of \( \mathcal{T} \), if it is a triangulated subcategory and is closed under direct summands.

Definition 2.1. For a subcategory \( \mathcal{C} \) of \( \mathcal{T} \) the \( n \)-th thickening is

\[
\text{thick}^n_T(\mathcal{C}) := \begin{cases} 
\{0\} & n = 0 \\
\text{smd}(\text{add}(\mathcal{C})) & n = 1 \\
\text{thick}^{n-1}_T(\mathcal{C}) \circ \text{thick}^1_T(\mathcal{C}) & n \geq 2
\end{cases}.
\]

The level of an object \( X \) in \( \mathcal{T} \) with respect to \( \mathcal{C} \) is

\[
\text{level}^T_{\mathcal{C}}(X) := \inf\{n \geq 0 | X \in \text{thick}^n_T(\mathcal{C})\}.
\]

The union of all thickenings of a subcategory \( \mathcal{C} \) is the smallest thick subcategory of \( \mathcal{T} \) containing \( \mathcal{C} \), denoted \( \text{thick}_T(\mathcal{C}) \). Thus the thickenings give a filtration of \( \text{thick}_T(\mathcal{C}) \). Level behaves nicely with respect to direct sums and exact triangles, see [ABIM10, Lemma 2.4]. In the following, we are interested in the generation by one object \( G \), and in this case we write \( \text{level}^G \) for \( \text{level}^T_{\{G\}} \).

Remark 2.2. Let \( \mathcal{T} \) be a triangulated category and \( \mathcal{S} \) a thick subcategory. For \( G \) and \( X \) in \( \mathcal{S} \) one has

\[
\text{thick}^n_T(G) = \text{thick}^n_S(G) \quad \text{and} \quad \text{level}^G_T(X) = \text{level}^G_S(X).
\]

Given an exact functor \( f: \mathcal{S} \to \mathcal{T} \) of triangulated categories. For objects \( X \) and \( G \) of \( \mathcal{S} \) we have the inequality

\[
\text{level}^G_S(X) \geq \text{level}^{f(G)}_T(f(X)).
\]

This paper investigates cases in which this is an equality.

Coghost. To show that \( (2.3) \) is an equality, we utilize the connection between level and the coghost index given by the converse coghost lemma.

Definition 2.4. Let \( \mathcal{C} \) be a full subcategory of \( \mathcal{T} \). A morphism \( f: X \to Y \) is \( \mathcal{C} \)-coghost, if the natural transformation

\[
f^*: \text{Hom}^*_\mathcal{T}(Y, -) \to \text{Hom}^*_\mathcal{T}(X, -)
\]

vanishes on \( \mathcal{C} \).

A map \( f: X \to Y \) is \( n \)-fold \( \mathcal{C} \)-coghost, if it can be written as a composition of \( n \) \( \mathcal{C} \)-coghost maps. The coghost index with respect to \( \mathcal{C} \) is defined by

\[
\text{cogin}^n_T(\mathcal{C})(Y) := \inf\{n| \text{all } n \text{-fold } \mathcal{C} \text{-coghost maps } X \to Y \text{ are zero}\}.
\]
Just as for level we are interested in the case where $C$ consists of one object $G$. Then we write $\text{cogin}^C$ for $\text{cogin}^G$.

Unlike level, the coghost index may depend on the ambient category. More precisely given a thick subcategory $T \subseteq \mathcal{U}$, one has

$$\text{cogin}^C_T(X) \leq \text{cogin}^G_T(X)$$

for $G$, $X$ in $T$. We do not know whether this is an equality.

Similar to coghost maps, ghost maps are the maps that become zero by pre-composition with any suspension of $G$. Then the ghost index is

$$\text{gin}^C_T(X) := \inf \{n \mid \text{all } n\text{-fold } C\text{-ghost maps } X \to Y \text{ are zero} \}.$$  

The ghost and coghost maps are dual to each other in the sense that $f$ is ghost in $T$ if and only if $f$ is coghost in $T^{\text{op}}$. The same holds for the ghost and coghost index:

$$\text{gin}^C_T(X) = \text{cogin}^G_{T^{\text{op}}}(X).$$

It is well known that the level and the coghost index always satisfy

$$\text{cogin}^C_T(X) \leq \text{level}^C_T(X).$$

This is called the coghost lemma, see [Kel65]. The same inequality holds when replacing cogin by gin. We do not know whether the level and the coghost index (or the ghost index) are equal in every triangulated category. Some partial converses are known. If every object has a left/right approximation by a direct sum of suspension of $G$ then the converse holds, see [BeI08].

In the bounded derived category of a ring $R$, Christensen [Chr98] showed that the level and the ghost index with respect to $R$ are the same. The case that is relevant for this paper was proven by Oppermann and Šťovíček [OŠ12]. They show that for bounded derived categories of Noether algebras the level and the coghost index agree for any generator. This result is discussed below.

**Converse coghost lemma.** A ring $R$ is a Noether algebra if its center $Z(R)$ is noetherian and $R$ is a finitely generated module over $Z(R)$.

For example, given a finitely generated module $M$ over a commutative noetherian ring $A$, the endomorphism ring $\text{Hom}_A(M, M)$ is a Noether algebra, where $A$ lies in the center.

If $R$ is a Noether algebra, and $M$, $N$ are finitely generated left modules, then the groups $\text{Ext}^n_R(M, N)$ are finitely generated modules over $Z(R)$.

For a noetherian ring $R$ the derived category of left $R$-modules is denoted $\mathcal{D}(R)$. The subcategory of complexes with finitely generated homology, that is $H_i(X) = 0$ for $|i| \gg 0$ and $H_i(X)$ finitely generated for all $i$, is denoted $\mathcal{D}_f(R)$. This is a thick subcategory of $\mathcal{D}(R)$. By (2.2) it does not matter whether thick and level are calculated in $\mathcal{D}(R)$ or $\mathcal{D}_f(R)$ (or any bounded above/below derived category of (finitely generated) left $R$-modules) for objects $G$ and $X$ in $\mathcal{D}_f(R)$. They only depend on $R$, so we write

$$\text{thick}^n_R(C) = \text{thick}^n_{\mathcal{D}(R)}(C) \quad \text{and} \quad \text{level}^C_R(X) = \text{level}^C_{\mathcal{D}(R)}(X).$$

It is not known, whether a similar statement as (2.2) holds for the coghost index. There are more coghost maps in the ambient category, so there could be a non-zero $n$-fold coghost map, that does not lie in the thick subcategory.

The following theorem is due to [OŠ12 Theorem 24].
Theorem 2.7 (Converse coghost lemma). Let $R$ be a Noether algebra and $G$ an object in $\text{D}_f(R)$. Then for any $X$ in $\text{D}_f(R)$

$$\text{cogin}^G_{\text{D}_f(R)}(X) = \text{level}^G_R(X).$$

Given $n = \text{level}^G_R(X)$ this result guarantees the existence of a non-zero $(n-1)$-fold composition of $G$-coghost maps. This turns out to be useful since, under a faithful functor, this composition stays non-zero. So if the functor preserves coghost maps, the coghost index behaves the opposite way from the level, that is it does not decrease. The level does not increase under a change of rings as described in (2.3).

We will give an outline of the proof, because some of the intermediate results are used to prove Theorem 2.

In the proof of the converse coghost lemma, they first show

$$(2.8) \quad \text{level}^{\text{Prod}_+(G)}_R(X) = \text{cogin}_{\text{D}_f(R)}^G(X)$$

for $G$ in $\text{D}_f(R)$ and $X$ in $\text{D}_+(R)$. Here $\text{D}_+(R)$ is the category of bounded below complexes with finitely generated homology, that is $H_i(X) = 0$ for $i \ll 0$ and $H_i(X)$ finitely generated for all $i$, and $\text{Prod}_+(G)$ the smallest full subcategory of $\text{D}_+(R)$, that contains $G$ and is closed under all products that exist in $\text{D}_+(R)$.

The equality is proved by finding a left approximation of every object in $\text{D}_+(R)$ by an object of $\text{Prod}_+(G)$. This fact becomes important for Section 4, so here is the precise statement.

Lemma 2.9. Fix $G$ in $\text{D}_f(R)$. Then every object $X$ in $\text{D}_+(R)$ has a left $\text{Prod}_+(G)$-approximation: There exists a map $m(X): X \to H$ with $H$ in $\text{Prod}_+(G)$, such that any map $X \to H'$ with $H'$ in $\text{Prod}(G)$ factors through $m(X)$. □

As noted before, when every object has a left approximation the converse coghost lemma holds, see for example [BFK12, 2.14]. This shows (2.8).

An object $C$ in an additive category $\mathcal{C}$ is cocompact, if for any family of objects $X$ in $\mathcal{C}$, whose product exists in $\mathcal{C}$, the natural map

$$\bigoplus_{X \in \mathcal{X}} \text{Hom}_{\mathcal{C}}(X, C) \to \text{Hom}_{\mathcal{C}} \left( \prod_{X \in \mathcal{X}} X, C \right)$$

is an isomorphism.

By [OS12, Theorem 18] the cocompact objects in $\text{D}_+(R)$ are precisely the bounded complexes, that is all the objects in $\text{D}_f(R)$.

Lemma 2.10. If $X$ and $G$ are cocompact objects in the triangulated category $\mathcal{T}$, then

$$\text{level}^{\text{Prod}(G)}_{\mathcal{T}}(X) = \text{level}^G_{\mathcal{T}}(X),$$

where $\text{Prod}(G)$ is the smallest full subcategory of $\mathcal{T}$, that contains $G$ and is closed under all products that exist in $\mathcal{T}$. □

It remains to show the coghost index in $\text{D}_f(R)$ is the same as the coghost index in $\text{D}_+(R)$. Given a non-zero composition of $G$-coghost maps in $\text{D}_+(R)$, one constructs a non-zero composition of $G$-coghost maps in $\text{D}_f(R)$. For this, the objects in the composition are replaced by their projective resolutions and then truncated, so they become perfect. To show the induced maps on the truncations are $G$-coghost maps it is crucial that the truncations are perfect. This gives the following:
Lemma 2.11. For $G$-coghost maps $f^i: X^i \to X^{i-1}$ in $D_+(R)$ for $1 \leq i \leq n$ with $X = X^0$ in $D_f(R)$, there exists a commutative diagram

\[
\begin{array}{cccccc}
X^n & \xrightarrow{f^n} & X^{n-1} & \xrightarrow{f^{n-1}} & \cdots & \xrightarrow{f^1} & X \\
Y^n & \xrightarrow{g^n} & Y^{n-1} & \xrightarrow{g^{n-1}} & \cdots & \xrightarrow{g^1} & Y
\end{array}
\]

with $Y^i$ perfect and the horizontal maps $G$-coghost. Moreover the top row is zero if and only if the bottom row is zero.

This concludes the proof of the converse coghost lemma.

All the steps but the last of this proof can be adjusted by replacing coghost with ghost maps and $D^+(R)$ by $D^-(R)$. In the last step, the projective resolution is replaced by an injective resolution. But in general, there do not exist enough injective modules in the category of finitely generated modules. So it is not possible to truncate injective resolutions and stay in $D_f(R)$. It seems a converse ghost lemma cannot be proven similarly. It is still possible to establish a converse ghost lemma if the ring has a dualizing complex.

Dualizing complex. Let $d: S \to T^{op}$ and $d': S \to T^{op}$ be a duality of triangulated categories in the sense that $d$ and $d'$ are contravariant functors, and $dd' \cong \text{id}_S$ and $d'd \cong \text{id}_T$. The duality interchanges ghost and coghost maps, so that

\[
\begin{align*}
gin^S_G(X) &= \text{cogin}^{d(G)}(d(X)) \\
cogin^S_G(X) &= \text{gin}^{d(G)}(d(X)).
\end{align*}
\]

Thus the converse coghost lemma holds for $G$ in $S$ if and only if the converse ghost lemma holds for $d(G)$ in $T$.

The dualizing complex gives a class of dualities on the derived categories.

Definition 2.13. [CFH06, Definition 1.1] Let $S$ be a left noetherian ring and $R$ a right noetherian ring. A dualizing complex of the ordered pair $(S, R)$ is a complex $\omega$ of $S$-$R$-bimodules, such that

1. $\omega$ is a bounded complex of injective modules over $S$ and $R^{op}$,
2. $H(\omega)$ is finitely generated over $S$ and $R^{op}$,
3. there exists a quasi-isomorphism $P \to \omega$ where $P$ a bounded below complex of projective modules over $S$ and $R^{op}$, and
4. the canonical morphisms

\[
S \to \text{RHom}_{R^{op}}(\omega, \omega) \quad \text{and} \quad R \to \text{RHom}_S(\omega, \omega)
\]

are quasi-isomorphisms.

If $R$ is additionally left noetherian, there exists a contravariant auto-equivalence (see [IK06, 3.4])

\[
D_f(S) \xrightarrow{\text{RHom}_S(-, \omega)} D_f(R^{op}) \xrightarrow{\text{RHom}_{R^{op}}(-, \omega)} D_f(S).
\]

These functors send ghost maps to coghost maps and reverse.

Theorem 2.14 (Converse ghost lemma). Let $S$ be left noetherian and $R$ a Noether algebra with $\omega$ a dualizing complex of $(S, R)$. Fix $G$ in $D_f(S)$. Then for any $X$ in $D_f(S)$ one has

\[
\text{gin}^G_{D_f(S)}(X) = \text{level}^G_S(X).
\]
Proof. Set \((-\dagger) = \mathbb{R}\text{Hom}_S(\_\, , \omega)\) and \((-\dagger)^\prime = \mathbb{R}\text{Hom}_{R^{op}}(\_\, , \omega)\). These functors are a duality in the sense above. Thus by (2.12)

\[
\text{level}_R^G(X) = \text{level}_{R^{op}}^G(X^{\dagger}) = \text{cogin}_{D^b(S)}^G(X) = \text{gin}_{D_f^b(R)}^G(X)
\]

where the converse coghost lemma 2.7 gives the equality in the middle. \(\square\)

If \(R\) is a commutative noetherian ring, the definition of a dualizing complex of \(\langle R, R \rangle\) coincides with Grothendieck’s definition of a dualizing complex [Har66, V §2]. Then \(R\) has a dualizing complex if and only if it is the homomorphic image of a Gorenstein ring of finite Krull dimension (see [Kaw02, Corollary 1.4]). So for any such ring, the converse ghost lemma also holds.

3. Finite flat dimension

In this section we look at cases when level is unchanged by the functor \(W \otimes_R^L -\) for a complex of \(S\)-\(R\)-bimodules \(W\).

Let \(\mathcal{F}\) be the class of all flat (not necessarily finitely generated) left \(R\)-modules. Then \(\text{thick}_R(\mathcal{F})\) consists of all \(R\)-complexes of finite flat dimension.

**Lemma 3.1.** Let \(R\) and \(S\) be noetherian rings, \(X \in D_f(S)\) and \(Y\) a complex of \(S\)-\(R\)-bimodules and \(W \in D(R)\). Assume one of the following conditions is satisfied

1. \(X\) is perfect, or
2. \(Y\) a bounded above, that is \(Y_i = 0\) for \(i \gg 0\), and \(W \in \text{thick}_R(\mathcal{F})\).

Then the natural morphism of complexes of abelian groups

\[
\mathbb{R}\text{Hom}_S(X, Y) \otimes_R^L W \to \mathbb{R}\text{Hom}_S(X \otimes_R^L W)
\]

is a quasi-isomorphism.

**Proof.** The proof of this lemma is standard. For (1), the claim holds for \(X = R\) and thus by induction on the level with respect to \(R\) for any perfect complex. For (2), one first proves the claim for flat modules and then uses induction on the level of \(W\) with respect to \(\mathcal{F}\). \(\square\)

3.2. For the rest of the section suppose \(R\) is a commutative noetherian ring and \(S\) a noetherian ring. Let \(W\) be a complex of \(S\)-\(R\)-bimodules, such that \(W\) is a bounded complex of finitely generated projective \(S\)-modules and has finite flat dimension over \(R\). Additionally let the left and right action of \(R\) on \(\text{Hom}_S(W, W)\) be the same, that is the canonical map \(R \to \text{Hom}_S(W, W)\) is central.

This gives adjoint functors

\[
\text{D}(R) \xrightarrow{t = W \otimes_R^L -} \text{D}(S)
\]

and \(t\) restricts to a functor from \(D_f(R)\) to \(D_f(S)\). We track how coghost maps behave under the functor \(t\), when restricted to \(D_f(R) \to D_f(S)\).

**Lemma 3.4.** As an \(R\)-complex \(\text{Hom}_S(W, W)\) lies in \(\text{thick}_R(\mathcal{F})\).

**Proof.** Since \(W = \text{Hom}_S(S, W)\) lies in \(\text{thick}_R(\mathcal{F})\), the complex \(\text{Hom}_S(P, W)\) lies in \(\text{thick}_R(\mathcal{F})\) for any perfect complex \(P\) over \(S\). In particular \(\text{Hom}_S(W, W)\) lies in \(\text{thick}_R(\mathcal{F})\). \(\square\)
Lemma 3.5. For any $X, Y \in D_\mathcal{F}(R)$, there is a quasi-isomorphism

$$R\text{Hom}_R(X, Y) \otimes^L_R \text{Hom}_S(W, W) \simeq R\text{Hom}_S(t(X), t(Y)).$$

Proof. Since $R$ is commutative, the left $R$-action on $Y$ induces a right $R$-action on $Y$. Also the left and right $R$-action on $\text{Hom}_S(W, W)$ are the same, so that there is a natural isomorphism

$$Y \otimes^L_R \text{Hom}_S(W, W) \cong \text{Hom}_S(W, W) \otimes^L_R Y.$$

One has the following equivalences

$$R\text{Hom}_R(X, Y) \otimes^L_R \text{Hom}_S(W, W) \cong R\text{Hom}_R(X, \text{Hom}_S(W, W) \otimes^L_R Y) \cong R\text{Hom}_R(X, h(t(Y))) \cong R\text{Hom}_S(t(X), t(Y)).$$

The last step holds by the adjunction in (3.3). \qed

The next lemma shows how coghost maps act under the functor $t$.

Lemma 3.6. Let $V$ be in thick$_R(\mathcal{F})$ with level$_R^t(V) \leq l$ and $G$ in $D_\mathcal{F}(R)$. Then for any $l$-fold $G$-coghost map $f: X \to Y$ in $D_\mathcal{F}(R)$, the map $H(R\text{Hom}_R(f, G) \otimes^L_R V)$ is zero.

Proof. If $f: X \to Y$ is a $l$-fold $G$-coghost map, then $R\text{Hom}_R(f, G)$ is a $l$-fold $R$-ghost map. Extending the argument of [AIN18] Lemma 2.6] to complexes of finite flat dimension, one has $R\text{Hom}_R(f, G) \otimes^L_R V$ is a $R$-ghost map. In particular $H(R\text{Hom}_R(f, G) \otimes^L_R V)$ is zero. \qed

Corollary 3.7. If level$_R^t(\text{Hom}_S(W, W)) \leq l$ and $f$ is a $l$-fold $G$-coghost in $D_\mathcal{F}(R)$, then $t(f)$ is $t(G)$-coghost.

Proof. By Lemma 3.6 we have $H(R\text{Hom}_R(f, G) \otimes^L_R \text{Hom}_S(W, W)) = 0$, and by Lemma 3.5

$$H(R\text{Hom}_R(f, G) \otimes^L_R \text{Hom}_S(W, W)) = \text{Hom}^*_S(t(f), t(G)).$$

Thus $t(f)$ is $t(G)$-coghost. \qed

From the corollary it follows that if $\text{Hom}_S(W, W)$ is isomorphic in $D(R)$ to a direct sum of suspensions of flat modules, the functor $t$ preserves coghost maps. This does not imply that it also preserves the coghost index. For that the functor $t$ needs to be faithful.

Lemma 3.8. If $\text{Hom}_S(W, W) \in \text{add}(\mathcal{F})$ and $t$ is faithful, then for $X$ and $G$ in $D_\mathcal{F}(R)$

$$\text{cogin}^G_{D_\mathcal{F}(R)}(X) \leq \text{cogin}^f_{D_\mathcal{F}(S)}(f(X)).$$

Proof. Given a non-zero $n$-fold $G$-coghost map $f$. The map $t(f)$ is a $n$-fold $t(G)$-coghost map, and it is non-zero, because $t$ is faithful. \qed

Theorem 3.9. Suppose $R$ is a commutative noetherian ring and $S$ is a noetherian ring. Let $W$ be a complex of $S$-$R$-bimodules and set

$$t = W \otimes^L_R : D_\mathcal{F}(R) \to D_\mathcal{F}(S).$$

Assume
W is a bounded complex of finitely generated projective $S$-modules,
- $W$ has finite flat dimension over $R$,
- the natural map $R \to \text{Hom}_S(W, W)$ is central,
- $\text{Hom}_S(W, W) \in \text{add}(\mathcal{F})$, and
- $t$ is faithful.

Then for any $G, X$ in $D_f(R)$ one has
\[
\text{level}^G_R(X) = \text{level}^{t(G)}_S(t(X)).
\]

**Proof.** We have the (in)equalities
\[
\begin{align*}
\text{level}^G_R(X) &= \text{cogin}^G_{D_f(R)}(X) \\
&\leq \text{cogin}^{t(G)}_{D_f(S)}(f(X)) \\
&\leq \text{level}^{t(G)}_S(f(X))
\end{align*}
\]
where the equality holds by the converse coghost lemma 2.7. The first inequality holds by 3.8, and the second by (2.6). The opposite inequality holds by (2.3). \qed

Note in this proof that the converse coghost lemma does not need to hold in $D_f(S)$. We only require it to hold in $D_f(R)$.

**Ring maps.** An important class of examples for which Theorem 3.9 applies comes from faithfully flat ring maps $\varphi: R \to S$ with $R$ commutative noetherian and $S$ noetherian. This induces the functor $\varphi^*: S \otimes_R -: D_f(R) \to D_f(S)$.

**Lemma 3.10.** If $S$ is faithfully flat as an $R$-module, the functor $\varphi^*$ is faithful.

**Proof.** Since $\varphi$ is faithful the map of abelian groups
\[
\text{Hom}_{D_f(R)}(X, Y) \hookrightarrow S \otimes_R \text{Hom}_{D_f(R)}(X, Y)
\]
is injective. Because $S$ is flat, one has
\[
S \otimes_R \text{Hom}_{D_f(R)}(X, Y) \cong H(S \otimes_R \text{RHom}_R(X, Y)) \cong \text{Hom}_{D_f(R)}(X, \varphi^*(Y)) \cong \text{Hom}_{D_f(S)}(\varphi^*(X), \varphi^*(Y)).
\]
The last equivalence holds by adjunction. \qed

If $R$ acts centrally on $S$, then the functor $\varphi^*$ with $W = S$ satisfies all the conditions of Theorem 3.9. The following answers a question posed in [DGJ06, Remark 9.6].

**Corollary 3.11.** Let $\varphi: R \to S$ be a faithfully flat ring map with $R$ a commutative ring and $S$ a noetherian ring, so that $R$ acts centrally on $S$. For $X, G \in D_f(R)$, one has
\[
\text{level}^G_R(X) = \text{level}^{\varphi^*(G)}_{\varphi^*(X)}(X).
\]
In particular the level remains unchanged after completion.

**Corollary 3.12.** Let $(R, m, k)$ be a local ring and let $\widehat{(\cdot)}$ be the completion with respect to $m$. Then for any $X, G$ in $D_f(R)$ we have
\[
\text{level}^G_R(X) = \text{level}^{\varphi^*(G)}_{\varphi^*(X)}(X).
\]
4. A LOCAL TO GLOBAL PRINCIPLE

In this section we investigate the behavior of level and finite generation in the derived category of a Noether algebra under the localization at prime ideals of the center.

**Localization in the derived category.** Let \( R \) be a Noether algebra with center \( Z(R) \) and \( p \) a prime ideal of \( Z(R) \). For any left \( R \)-module \( M \) one has

\[
M_p \cong R_p \otimes_R M \cong (Z(R)_p \otimes_{Z(R)} R) \otimes_R M \cong Z(R)_p \otimes_{Z(R)} M
\]
as a left module over \( Z(R)_p \otimes_{Z(R)} R \cong R_p \). Since \( Z(R) \to Z(R)_p \) is flat so is the ring map \( R \to R_p \). These maps need not be faithful.

An \( R \)-module \( M \) is zero if and only if \( M_m \) is zero for all maximal ideals. Thus a map of \( R \)-modules \( f \) is zero if and only if \( f_m = 0 \) for all maximal ideals \( m \). The same holds for maps in the derived category:

**Lemma 4.1.** Let \( f : X \to Y \) be a morphism in \( D_f(R) \). Then the following conditions are equivalent

1. \( f = 0 \) in \( D(R) \),
2. \( f_p = 0 \) in \( D(R_p) \) for all \( p \in \text{Spec}(Z(R)) \), and
3. \( f_m = 0 \) in \( D(R_m) \) for all \( m \in \text{Max}(Z(R)) \).

**Proof.** (1) \( \implies \) (2) and (2) \( \implies \) (3) are obvious.

For (3) \( \implies \) (1): Since \( X \) and \( Y \) lie in \( D_f(R) \), we have

\[
\text{Hom}_{D_f(R)}(X, Y)_m = \text{Hom}_{D(R_m)}(X_m, Y_m).
\]
Now \( [f_m] = [f]_m = 0 \) for all maximal ideals \( m \) if and only if \( [f] = 0 \). \( \Box \)

**Lemma 4.2.** Given the map \( f : X \to Y \) in \( D_f(R) \). The subset

\[
\{ p \in \text{Spec}(Z(R)) | f_p = 0 \text{ in } D(R) \}
\]
of \( \text{Spec}(Z(R)) \) is open in the Zariski topology.

**Proof.** The map \( f \) fits in an exact triangle

\[
X \xrightarrow{f} Y \xrightarrow{g} Z \to \Sigma X.
\]
Applying \( \text{Hom}_{D(R)}(X, -) \) to this exact triangle gives the long exact sequence

\[
\cdots \to \text{Hom}_{D(R)}(X, X) \xrightarrow{f_*} \text{Hom}_{D(R)}(X, Y) \xrightarrow{g_*} \text{Hom}_{D(R)}(X, Z) \to \cdots .
\]
Then \( f = 0 \) if and only if \( \ker(g_*) = 0 \). Since \( X \) and \( Y \) are in \( D_f(R) \), one has \( \ker(g_*)_p = \ker((g_p)_*) \). A module being zero is a open property, so the set

\[
\{ p \in \text{Spec}(Z(R)) | f_p = 0 \} = \{ p \in \text{Spec}(Z(R)) | \ker(g_*)_p = 0 \}
\]
is open in the Zariski topology. \( \Box \)

**Lemma 4.3.** Fix \( G \) in \( D_f(R) \). Then for any \( X \in D_f(R) \), one has

\[
\text{cogin}^G_{D_f(R)}(X) \leq \sup \left\{ \text{cogin}^{G_m}_{D_f(R_m)}(X_m) \bigg| m \in \text{Max}(Z(R)) \right\}
\]

\[
\leq \sup \left\{ \text{cogin}^{G_p}_{D_f(R_p)}(X_p) \bigg| p \in \text{Spec}(Z(R)) \right\}.
\]

**Proof.** Given a \( n \)-fold \( G \)-coghost map \( f \). Then \( f_m \) is a \( n \)-fold \( G_m \)-coghost map by \( \text{Lemma 4.1} \). If \( f_m = 0 \) for all maximal ideals \( m \), then \( f = 0 \) by \( \text{Lemma 4.1} \). This proves the first inequality. The second is obvious. \( \Box \)
Local to global principle. The next theorem shows that the level can be calculated locally.

**Theorem 4.4.** Let $R$ be a Noether algebra. Fix $G$ and $X$ in $D_f(R)$. Then

$$\text{level}^G_R(X) = \sup \left\{ \text{level}^G_{R_p}(X_p) \big| p \in \text{Spec}(Z(R)) \right\}$$

$$= \sup \left\{ \text{level}^G_{R_m}(X_m) \big| m \in \text{Max}(Z(R)) \right\}.$$ 

**Proof.** Given a prime ideal $p$, there exists a maximal ideal $m \supseteq p$ and by (2.3)

$$\text{level}^G_R(X) \geq \text{level}^G_{R_p}(X_p) \geq \text{level}^G_{R_m}(X_m).$$

So it is enough to show the claim for maximal ideals. By the converse coghost lemma 2.7 and Lemma 4.3 one has

$$\text{level}^G_R(X) = \text{cogin}^G_{D_f(R)}(X) \leq \sup \left\{ \text{cogin}^G_{D_f(R_m)}(X_m) \big| m \in \text{Max}(Z(R)) \right\}$$

$$= \sup \left\{ \text{level}^G_{R_m}(X_m) \big| m \in \text{Max}(Z(R)) \right\}.$$ 

For the opposite inequality, by (2.3)

$$\text{level}^G_R(X) \geq \text{level}^G_{R_m}(X_m)$$

holds for all maximal ideals $m \in \text{Max}(Z(R))$. \qed

In [BM67, Lemma 4.5] it is proved that a module $M$ has finite projective dimension if and only if $M_p$ has finite projective dimension for all prime ideals $p$. This was extended to perfect complexes by [AIL10, Theorem 4.1]. The next result generalizes this to level with respect to any generator $G$. This complements Theorem 4.4 in that it is not only possible to compute level locally, but also to check finiteness of level locally.

**Theorem 4.5.** Let $R$ be a Noether algebra. Suppose $G$ and $X$ are objects in $D_f(R)$. Then for any integer $n$ the set

$$\left\{ p \in \text{Spec}(Z(R)) \big| \text{level}^G_{R_p}(X_p) \leq n \right\} \subseteq \text{Spec}(Z(R))$$

is Zariski open. Moreover the following conditions are equivalent

1. $\text{level}^G_R(X) < \infty$,
2. $\text{level}^G_{R_p}(X_p) < \infty$ for all $p \in \text{Spec}(Z(R))$, and
3. $\text{level}^G_{R_m}(X_m) < \infty$ for all $m \in \text{Max}(Z(R))$.

**Proof.** We use an idea from the proof of the converse coghost lemma [OS12], recalled in Section 2.

Set $X = X^0$. For $i \geq 0$ we define inductively objects $X^{i+1}$ and $G$-coghost maps

$$f^{i+1}: X^{i+1} \to X^i \ \text{in} \ D_+(R).$$

By Lemma 2.9 there exists a left Prod$_G(G)$-approximation $m(X^i): X^i \to H^i$, which we complete to an exact triangle

$$X^{i+1} \xrightarrow{f^{i+1}} X^i \xrightarrow{m(X^i)} H^i \to \Sigma X^{i+1}.$$
Since \( n(X^i) \) is a left \( \text{Prod}_+(G) \)-approximation, the map \( f^{i+1} \) is \( G \)-cohost. These exact triangles can be rewritten as

\[
\begin{array}{c}
X = X^0 \leftrightarrow X^1 \leftrightarrow X^2 \leftrightarrow \cdots \\
H^0 \quad +1 \quad +1 \quad +1
\end{array}
\]

This is the dual notion of an Adams resolution as in [Chr98, 4], we will call it Adams coresolution. By (2.8) and Lemma 2.10 one has

\[
\text{level}^G_R(X) \geq \text{cogin}^G_{D_+(R)}(X) \geq \inf \{ n | f^1 \circ \cdots \circ f^n = 0 \text{ in } D(R) \}.
\]

By the octahedral axiom \( \text{cone}(f^1 \circ \cdots \circ f^1) \in \text{thick}^n_R(\text{Prod}_+(G)) \). Thus by 2.10

\[
\text{level}^G_R(X) = \inf \{ n | f^1 \circ \cdots \circ f^n = 0 \text{ in } D(R) \}.
\]

Then by 2.11 there exist \( G \)-cohost maps \( g^i: Y^i \to Y^{i-1} \) with \( Y^0 = X \) and \( Y^i \) perfect, such that a composition \( g^1 \circ \cdots \circ g^n \) is zero if and only if \( f^1 \circ \cdots \circ f^n \) is zero. That gives

\[
\text{level}^G_R(X) = \inf \{ n | g^1 \circ \cdots \circ g^n = 0 \}.
\]

While products need not localize in general, the products in \( D_+(R) \) localize. The reason is that if a product exists, then by [OS12, Proposition 13] it is the componentwise product and one may assume in each component the product is finite. Thus \( \text{Prod}_+(G_p) = \text{Prod}_+(G) \).

Since the functor \( D_f(R) \to D_f(R_p) \) is full, the localization of a left \( \text{Prod}_+(G) \)-approximation in \( D_+(R) \) is a left \( \text{Prod}_+(G_p) \)-approximation in \( D_+(R_p) \). So the Adams coresolution of \( X \) is a Adams coresolution of \( X_p \) in \( D_+(R_p) \). The truncation used in 2.11 descends to the localization, so that \( (f^1 \circ \cdots \circ f^n)_p \) is zero if and only if \( (g^1 \circ \cdots \circ g^n)_p \) is zero. This gives

\[
\mathcal{V}_n = \{ p \in \text{Spec}(R) | \text{level}^{G_p}_R(X_p) \leq n \} = \{ p \in \text{Spec}(R) | (g^1 \circ \cdots \circ g^n)_p = 0 \}
\]

is open by Lemma 4.2.

For the second part \((2) \iff (3) \) and \((1) \implies (2) \) is clear. For \((2) \implies (1) \) assume \( \text{level}^{G_p}_R(X_p) \) is finite for all prime ideals \( p \in \text{Spec}(Z(R)) \). That is the union of all \( \mathcal{V}_n \) is \( \text{Spec}(Z(R)) \). Now the \( \mathcal{V}_n \) form an ascending chain of open sets. Since \( Z(R) \) is noetherian, the space \( \text{Spec}(Z(R)) \) is noetherian and the chain stabilizes. So there exists an \( N \), such that \( \mathcal{V}_n = \mathcal{V}_N \) for \( n \geq N \). Thus \( \text{Spec}(Z(R)) = \mathcal{V}_N \), and \( \text{level}^{G_p}_R(X_p) \leq N \) for all prime ideals \( p \). By Theorem 4.4 then \( \text{level}^G_R(X) < \infty \). \( \square \)

**Local to global principal for upper bounds.** One way to think about level of \( X \) with respect to \( G \) is as the generation time for \( X \) when using \( G \) as a building block. It is interesting to know whether \( G \) generates every object and if there is an upper limit of the generation times for all objects.

**Definition 4.6.** An object \( G \) in \( \mathcal{T} \) is a strong generator of \( \mathcal{T} \) if \( \text{thick}^n_\mathcal{T}(G) = \mathcal{T} \) for some \( n \). The generation time of \( G \) is defined by

\[
\Theta_\mathcal{T}(G) = \inf \{ n \geq 0 | \text{thick}^{n+1}_{\mathcal{T}}(G) = \mathcal{T} \}.
\]

The generation time is shifted by one from the level. That is

\[
\text{level}^G_R(X) \leq \Theta_\mathcal{T}(G) + 1.
\]
To detect whether an object is a strong generator locally, one has to be able to lift objects from the localizations.

**Lemma 4.7.** For any prime ideal \( p \in \text{Spec}(Z(R)) \), the functor \( \text{D}_f(R) \to \text{D}_f(R_p) \) is essentially surjective.

**Proof.** Every finitely generated module over \( R_p \) can be lifted to a finitely generated module over \( R \). Also any \( R_p \)-linear map can be lifted to a \( R \)-linear map. Given a sequence

\[
X \xrightarrow{f} Y \xrightarrow{g} Z
\]

of finitely generated modules over \( R_p \) with \( g \circ f = 0 \). It can be lifted to a sequence

\[
\tilde{X} \xrightarrow{\tilde{f}} \tilde{Y} \xrightarrow{\tilde{g}} \tilde{Z}
\]

over \( R \). It is not necessarily \( \tilde{g} \circ \tilde{f} = 0 \), but one has \( ((\tilde{g} \circ \tilde{f})(\tilde{X}))_p = 0 \). Since \( X \) is finitely generated there exists \( r \in R \setminus p \), such that \( r \cdot ((\tilde{g} \circ \tilde{f})(\tilde{X})) = 0 \). Replacing \( \tilde{g} \) by \( r \tilde{g} \) gives a sequence whose composition is zero. Since \( r \) is a unit in \( R_p \) this sequence localizes to the original sequence. Thus inductively any bounded complex of finitely generated \( R_p \)-modules lifts to a complex of finitely generated \( R \)-modules. \( \square \)

It is possible to detect a strong generator locally, through the generation time has to have an upper bound for all localizations.

**Theorem 4.8.** Let \( R \) be a Noether algebra. Fix \( G \) in \( \text{D}_f(R) \) and a positive integer \( N \). Then the following are equivalent

1. \( G \) is a strong generator of \( \text{D}_f(R) \) with \( \text{gldim}(G) \leq N \),
2. \( G_p \) is a strong generator of \( \text{D}_f(R_p) \) with \( \text{gldim}(G_p) \leq N \) for all prime ideals \( p \in \text{Spec}(Z(R)) \), and
3. \( G_m \) is a strong generator of \( \text{D}_f(R_m) \) with \( \text{gldim}(G_m) \leq N \) for all maximal ideals \( m \in \text{Max}(Z(R)) \).

**Proof.** (2) \( \implies \) (3) is obvious. For (1) \( \implies \) (2): Given any \( X \) in \( \text{D}_f(R_p) \). By 4.7 there exists \( Y \) in \( \text{D}_f(R) \) with \( Y_p = X \). One has

\[
\text{level}^G_{R_p}(X) \leq \text{level}^G_{R}(Y) \leq \text{gldim}(G) + 1 \leq N + 1.
\]

So \( G_p \) is a strong generator of \( \text{D}_f(R_p) \) with generation time \( \leq N \).

It remains to show (3) \( \implies \) (1). For any \( X \) in \( \text{D}_f(R) \) we have by 4.4

\[
\text{level}^G_X = \sup \left\{ \text{level}^G_{R_m}(X_m) \mid m \in \text{Max}(Z(R)) \right\}
\]

\[
\leq \sup \left\{ \text{gldim}(G_m) \mid m \in \text{Max}(Z(R)) \right\} + 1 \leq N + 1,
\]

and so \( G \) is a strong generator of \( \text{D}_f(R) \) with \( \text{gldim}(G) \leq N \). \( \square \)

This statement does not hold without a uniform bound on local generation time.

**Example 4.9.** In Appendix A1 Nagata constructed a commutative noetherian ring \( R \) of infinite Krull dimension, such that \( R_m \) is regular and of finite Krull dimension for all maximal ideals \( m \). So

\[
\text{gldim}(R_m) = \dim(R_m) = \dim(R_m) < \infty
\]

for any maximal ideal \( m \). But

\[
\text{gldim}(R) = \dim(R) = \infty.
\]
So just because $G_m$ is a strong generator of $D_f(R_m)$ for any $m$, does not mean $G$ is a strong generator of $D_f(R)$.

5. Applications

The results 3.12 1.4 and 1.5 show that one can reduce working with level to complete local rings.

**Corollary 5.1.** Given a commutative noetherian ring $R$, and $G$, $X$ in $D_f(R)$, one has
\[
\text{level}^G_{R}(X) = \sup \left\{ \text{level}^{G_m}_{R_m}(X_m) \mid m \in \text{Max}(R) \right\}
\]
and $\text{level}^G_{R}(X) < \infty$ if and only if $\text{level}^{G_m}_{R_m}(X_m) < \infty$ for all maximal ideals $m$. Here $\hat{R}$ denotes the completion in $R_m$ with respect to its maximal ideal $mR_m$. \hfill \Box

**Theorem of Hopkins and Neeman for complexes of finite injective dimension.** For a complex $X$, let
\[
\text{Supp}_R(X) := \text{Supp}_R(H(X)) = \{ p \in \text{Spec}(R) \mid H(X)_p \neq 0 \}
\]
be the support of $X$. For perfect complexes Hopkins [Hop87 Theorem 11] and Neeman [Nee92, Lemma 1.2] prove the statement:

**Theorem 5.2.** Let $R$ be a commutative noetherian ring, and let $X$, $Y$ be perfect complexes over $R$. Suppose $\text{Supp}_R(X) \subseteq \text{Supp}_R(Y)$. Then $\text{level}^Y_{R}(X) < \infty$. \hfill \Box

Using the dualizing complex introduced in Section 2 one gets a connection between the complexes of finite projective dimension and the complexes of finite injective dimension (see [Rob80, Chapter 3]).

**Lemma 5.3.** Assume $R$ has a dualizing complex $\omega$. Then there is an equivalence
\[
\begin{array}{c}
\text{Perf}(R) \\
\text{RHom}\left(R,-,\omega\right)
\end{array}
\begin{array}{c}
\text{K}_{h,f}(R-\text{Inj}) \\
\text{RHom}\left(R,-,\omega\right)
\end{array}
\]
where $\text{K}_{h,f}(R-\text{Inj})$ the homotopy category of all bounded complexes of injective $R$-modules with finitely generated homology. \hfill \Box

A commutative noetherian ring $R$ need not have a dualizing complex, but every complete local ring has a dualizing complex. Using this we get a similar statement to 5.4 for complexes in $\text{K}_{h,f}(R-\text{Inj})$:

**Theorem 5.4.** Let $R$ be a commutative noetherian ring, and let $X$, $Y \in \text{K}_{h,f}(R-\text{Inj})$ with $\text{Supp}_R(X) \subseteq \text{Supp}_R(Y)$. Then $\text{level}^Y_{R}(X) < \infty$.

**Proof.** Let $p \in \text{Spec}(R)$ be any prime ideal. Then $X_p$ and $Y_p$ are complexes of finite injective dimension with finitely generated homology. Also localization preserves the inclusion of their support and by 1.5 $\text{level}^Y_{R_p}(X_p) < \infty$ if and only if $\text{level}^Y_{R_p}(X_p) < \infty$ for all prime ideals $p$. So we may assume $R$ is local.

Let $\hat{R}$ denote the completion with respect to the maximal ideal and $k$ the residue field of $R$. By [AF01, 5.5(1)], $X \in \text{K}_{h,f}(R-\text{Inj})$ if and only if $\text{RHom}_R(k, X)$ is a bounded above complex. Since $X \in \text{K}_{h,f}(R-\text{Inj})$, it is in particular in $D_f(R)$, so that $\hat{X} = X \otimes^L \hat{R}$. Then
\[
\text{RHom}_R(k, \hat{X}) \cong \text{RHom}_R(k, \hat{X}) \cong \text{RHom}_R(k, X) \otimes^L \hat{R}
\]
and thus $X \in K_{b,f}(R\text{-Inj})$ if and only if $\hat{X} \in K_{b,f}(\hat{R}\text{-Inj})$.

It is well known, that 
$$(^a\varphi)^{-1}(\text{Supp} \hat{R}(X)) = (^a\varphi)^{-1}(\text{Supp} \hat{R}(H(X))) = \text{Supp} \hat{R}(\hat{R} \otimes R H(X)) = \text{Supp} \hat{R}(\hat{X})$$

where $\varphi: R \to \hat{R}$ is the canonical ring homomorphism and $^a\varphi: \text{Spec}(\hat{R}) \to \text{Spec}(R)$ the induced map. So completion preserves the inclusion of the support.

Last we have $\text{level}^R_Y(X) < \infty$ if and only if $\text{level}^\hat{R}_Y(\hat{X}) < \infty$ by (6.2) Thus without loss of general we assume $R$ is a complete local ring.

Now $R$ has a dualizing complex $\omega$. Set $(-)^\dagger = \text{RHom}_R(-, \omega)$. Then the complexes $X^\dagger$ and $Y^\dagger$ are perfect by (6.3). Since $X$ has finitely generated homology, one has $(X^\dagger)_p = (X_p)^\dagger$ and thus $\text{Supp} \hat{R}(X^\dagger) \subseteq \text{Supp} \hat{R}(X)$. Since $(-)^\dagger$ is an auto-equivalence, the supports are equal. The same holds for $Y$, so $\text{Supp} \hat{R}(Y^\dagger) \subseteq \text{Supp} \hat{R}(Y^\dagger)$. By (6.2) one has $\text{level}^R_Y(X^\dagger) < \infty$, and thus $\text{level}^R_Y(X) < \infty$. □

6. Virtual and proxy smallness

In the derived category $D(R)$ of a noetherian ring, the perfect complexes are precisely the compact—also called small—objects. That is the perfect complexes are precisely the complexes $P$ for which the functor 
$$\text{RHom}_R(-, P): D(R) \to D(R)$$

commutes with direct sums. There are two notions on how to describe complexes that are almost small, see [DGJ06].

**Definition 6.1.** A complex $X$ in $D(R)$ is virtually small, if $X \simeq 0$ or there exists $Y \not\simeq 0$ in $D(R)$, such that 
\begin{equation}
\text{level}^R_Y(X) < \infty \quad \text{and} \quad \text{level}^R_Y(Y) < \infty.
\end{equation}

If additionally $\text{Supp}_R(X) = \text{Supp}_R(Y)$ then $X$ is proxy small.

By (6.2) a complex is small if and only if it is small locally. Similarly we track the behavior of proxy small complexes under localization.

**Proposition 6.3.** Let $R$ be a commutative noetherian ring and $X$ in $D_f(R)$. Then $X$ is proxy small if and only if $X_p$ is proxy small for all $p \in \text{Spec}(R)$.

**Proof.** For the if direction: By [DGJ06] 4.4] a complex $X$ is proxy small if and only if for the Koszul complex $K(I)$ on the ideal $I$, where $V(I) = \text{Supp} \hat{R}(X)$, one has 
$$\text{level}^X_R(K(I)) < \infty.$$

For any prime ideal $p$, we have $K(I)_p \simeq K(I_p)$. So by (6.3) $X$ is proxy small if $X_p$ is proxy small for all prime ideals $p$.

For the only if direction, let $Y$ be a perfect complex, such that 
$$\text{level}^Y_R(Y) < \infty \quad \text{and} \quad \text{Supp} \hat{R}(X) = \text{Supp} \hat{R}(Y).$$

Then for any $p \in \text{Supp} \hat{R}(X)$, one has $Y_p \not\simeq 0$ and $\text{level}^X_R(Y_p) < \infty$. If $p \not\in \text{Supp} \hat{R}(X)$, then $X_p \simeq 0$. Thus $X_p$ is proxy small for any prime ideal $p$. □

Virtual smallness does not behave in the same way. If $Y$ is a perfect complex, that is build by $X$, and it does not have the same support, then for some $p \in \text{Supp} \hat{R}(X)$, $Y_p \simeq 0$. Thus if the complex $X$ is virtually small, the localizations $X_p$ need not be. But since $Y \not\simeq 0$, there exists some maximal ideal $m$, for which $Y_m \not\simeq 0$. On the
other hand it is enough that $X_m$ is virtually small for some maximal ideal $m$, for $X$ to be virtually small.

**Proposition 6.4.** Let $R$ be a commutative noetherian ring and $X \neq \emptyset$ a complex over $R$. Then $X_m \neq \emptyset$ is virtually small for some maximal ideal $m$ if and only if $X$ is virtually small.

**Proof.** For the only if direction: Let $Y \neq \emptyset$ be a perfect complex, such that $\text{level}_R^X(Y) < \infty$. Since $Y$ lies in $D_f(R)$, there exists a maximal ideal $m \in \text{Supp}_R(Y)$. In particular $X_m \neq \emptyset$. Then $Y_m \neq \emptyset$ is a perfect complex and $\text{level}_{R_m}^X(Y_m) < \infty$. Thus $X_m$ is virtually small.

For the if direction: By [DGI06, 4.5] the complex $X$ is virtually small if and only if there exists a maximal ideal $m \in \text{Supp}_R(X)$, such that for the Koszul complex $K(m)$ on $m$ one has

$$\text{level}_R^X(K(m)) < \infty.$$  

By hypothesis there exists $m \in \text{Supp}_R(X)$, such that $X_m$ is virtually small and thus

$$\text{level}_{R_m}^X(K(m)_m) < \infty.$$  

For $p \neq m$ one has $K(m)_p = 0$. So by Theorem 4.5

$$\text{level}_R^X(K(m)) < \infty.$$  

Since the Koszul complex $K(m)$ is perfect, $X$ is virtually small. \hfill \square

We can also track the behavior of virtually and proxy small under a faithfully flat ring map.

**Proposition 6.5.** Let $\varphi: R \rightarrow S$ be a faithfully flat ring map of commutative noetherian rings and $X \in D_f(R)$.

1. $X$ is proxy small if and only if $\varphi^*(X) := X \otimes_R^L S$ is proxy small in $D(R)$.

2. If $X$ is virtually small, then $\varphi^*(X)$ is virtually small in $D(R)$.

**Proof.** By 3.10 $\varphi^*$ is faithful. So $X \simeq 0$ if and only if $\varphi^*(X) \simeq 0$. Then we may assume $X \neq 0$. Let $I$ be an ideal in $R$, such that $V(I) = \text{Supp}_R(X)$. Given that $S$ is faithfully flat over $R$, it is well known that

$$K(I) \otimes_R^L S = K(I \otimes_R S) \quad \text{and} \quad \text{Supp}_S(\varphi^*(X)) = V(I \otimes_R S).$$

Then by 3.11 one has

$$\text{level}_R^X(K(I)) = \text{level}_S^{\varphi^*(X)}(K(I \otimes_R S)).$$

Now $X$ is proxy small if and only if $\text{level}_R^X(K(I)) < \infty$ and $\varphi^*(X)$ is proxy small if and only if $\text{level}_S^{\varphi^*(X)}(K(I \otimes_R S))$. This shows the claim of (1).

For (2), let $Y \neq \emptyset$ be a perfect complex, such that $\text{level}_R^X(Y) < \infty$. By 3.10 the functor $\varphi^*$ is faithful, so $\varphi^*(Y) \neq \emptyset$ and by 2.3

$$\text{level}_S^{\varphi^*(X)}(\varphi^*(Y)) \leq \text{level}_R^X(Y) < \infty.$$  

So $\varphi^*(X)$ is virtually small. \hfill \square

The properties virtually and proxy small can be used to give a categorical description of a complete intersection. A local ring $(R, m, k)$ is a complete intersection, if its $m$-adic completion $\hat{R}$ is of the form $\hat{R} = Q/(f_1, \ldots, f_c)$ where $Q$ is a regular local ring and $f_1, \ldots, f_c$ a regular sequence in $Q$. 

A commutative noetherian ring \( R \) is a locally complete intersection if for any prime ideal \( p \) the ring \( R_p \) is a complete intersection. Using [Pol18, Theorem 5.4] and 6.3 we get a characterization of locally complete intersections.

**Theorem 6.6.** For a commutative noetherian ring \( R \) the following are equivalent

1. \( R \) is a locally complete intersection, and
2. every object in \( D_f(R) \) is proxy small.

**Proof.** Assume \( R \) is a locally complete intersection. That is \( \hat{R}_p \) is a quotient of a regular local ring. By [DGI06, Theorem 9.4], every object in \( D_f(\hat{R}_p) \) is proxy small. Then by 6.3 and 6.5 every object in \( D_f(R) \) is proxy small.

For the opposite direction, by 4.7 the functor \( D_f(R) \rightarrow D_f(R_p) \) is essentially surjective and thus since every object in \( D_f(R) \) is proxy small, so is every object in \( D_f(R_p) \). Then by [Pol18, Theorem 5.2] \( R_p \) is a complete intersection. \( \square \)

In [Pol18, Theorem 5.4] Pollitz proved that (1) holds if and only if every object in \( D_f(R) \) is virtually small.

Over a local ring \( (R, m, k) \) a complex \( X \in D_f(R) \) has finite CI-dimension, if there exist local homomorphisms \( R \rightarrow R' \leftarrow Q \), such that

- \( R \rightarrow R' \) is faithfully flat,
- \( Q \rightarrow R' \) is surjective and the kernel is generated by a regular sequence, and
- \( \text{fd}_Q(R' \otimes_R^L X) < \infty \).

This was first introduced by [AGP97] and extended to complexes by [SW04].

Theorem 3.11 answers the question raised in [DGI06, 9.6 Remarks]. So we can complete the proof that a complex of finite CI-dimension is virtually small. This has been proven by a different method by [Ber09]. Using 6.3 we can strengthen the result to the following.

**Proposition 6.7.** Every complex in \( D_f(R) \) of finite CI-dimension is proxy small.

**Proof.** Let \( X \) be a complex in \( D_f(R) \) of finite CI-dimension and let \( R \rightarrow R' \leftarrow Q \) be a diagram of local homomorphisms satsifying the required conditions. Then \( R' \otimes_R^L X \) has finite homology over \( R' \) and in particular over \( Q \). So \( R' \otimes_R^L X \) is a perfect complex over \( Q \). Then by [DGI06, Theorem 9.1] the complex \( R' \otimes_R^L X \) is proxy small over \( R' \) and by 6.5 (1) \( X \) is proxy small in \( D(R) \). \( \square \)

The condition given in Theorem 6.6 to test whether a ring is a locally complete intersection, is difficult to use: It is hard to check whether every bounded complex with finite homology is proxy small. For some rings it is possible to reduce this to checking one object for proxy smallness.

Given a \( k \)-algebra \( R \), the enveloping algebra of \( R \) is \( R^e = R \otimes_k R \). Then \( R^e \) acts on \( R \) diagonally.

**Theorem 6.8.** Let \( k \) be a field and \( R \) a \( k \)-algebra essentially of finite type over \( k \). Then the following are equivalent

1. \( R \) is a locally complete intersection, and
2. \( R \) is proxy small in \( D(R^e) \).

**Proof.** Both conditions are local conditions. So it is enough to show for a local ring \( R \) of finite type over \( k \) that \( R \) is a complete intersection if and only if \( R \) is proxy small in \( D(R^e) \).
Since \( R \) is a complete intersection, so is \( R^e \) by [Avr99, 5.11]. Then by [6.6] every object in \( D_f(R^e) \) is proxy small and thus \( R \) is proxy small in \( D(R^e) \).

For the converse direction, assume \( R \) is proxy small in \( D(R^e) \). That is there exists a non-zero complex \( P \) in \( D(R^e) \), such that

\[
\text{level}^{R^e}_{R^e}(P) < \infty \quad \text{and} \quad \text{level}^{R}_{R}(P) < \infty \quad \text{and} \quad \text{Supp}_{R^e}(P) = \text{Supp}_{R^e}(R).
\]

Let \( X \in D_f(R) \). By [6.6] it is enough to show \( X \) is proxy small in \( D(R) \). Any complex \( Y \) in \( D(R^e) \) has a left and a right \( R \)-action. Thus \( Y \otimes_R^L X \) has a left \( R \)-action through the left \( R \)-action of \( Y \). This induces the exact functor

\[- \otimes_R^L X : D(R^e) \to D(R)\]

and by (2.3) one has

\[
\text{level}^{R} (P \otimes^L_R X) < \infty \quad \text{and} \quad \text{level}^X_R (P \otimes^L_R X) < \infty.
\]

The object \( R \otimes_k X \) is a direct sum of suspensions of \( R \). Let \( \text{Add}(R) \) be the subcategory of all such complexes. Then

\[ R \otimes_k X \in \text{Add}(R) \quad \text{and so} \quad \text{level}^{\text{Add}(R)}_R (P \otimes^L_R X) < \infty.\]

In particular \( P \otimes^L_R X \) has a finite resolution by projective modules. Since \( P \otimes^L_R X \) is built by \( X \), it has finite homology. Thus \( P \otimes^L_R X \) has a finite resolution of finitely generated projective modules, that is it is perfect.

It remains to show \( P \otimes^L_R X \) has the same support as \( X \). A localizing subcategory generated by an object \( X \) in \( D(R) \) is the smallest triangulated subcategory of \( D(R) \), that is closed under summands, sums and contains \( X \). By [Nee92, Theorem 2.8] two complexes with finitely generated homology have the same support if and only if they generate the same localizing subcategory. Now since \( P \) and \( R \) have the same support over \( R^e \), they have the same localizing subcategories in \( D(R^e) \). So the localizing subcategories of \( P \otimes^L_R X \) and \( R \otimes^L_R X = X \) have the same localizing subcategories in \( D(R) \) and in fact they have the same support over \( R \). \( \square \)

This characterization is similar to the characterization of smooth ring: If \( k \) is a field and \( R \) a \( k \)-algebra essentially of finite type over \( k \), then \( R \) is smooth if and only if \( R \) is small in \( D(R^e) \).

Remark 6.9. As in Theorem 6.8, let \( k \) be a field and \( R \) a \( k \)-algebra of essentially finite type over \( k \). If \( R \) is a locally complete intersection, and \( Q \to R \) is a surjective map of \( k \)-algebras with \( Q \) a regular ring and kernel \( I \), then \( R \) generates the small object \( R \otimes^L_Q R \) in \( D(R^e) \). Adapting the argument of [DGJ06, Theorem 9.1] the generation time is bound above by

\[
\sup \{ \text{codim}(R_m) | m \in \text{Max}(R) \} + 1 \leq \text{level}^{R^e}_{R^e}(R \otimes^L_Q R),
\]

and using [ABIM10, Theorem 11.3] bound below by

\[
\text{level}^{R^e}_{R^e}(R \otimes^L_Q R) \leq \sup \{ \text{height}(I_m) | m \in \text{Max}(R) \} + 1.
\]

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