We investigate the equivalence between two different parametrizations of fields in cosmology – the so-called Jordan frame and Einstein frame – in the framework of a general scalar-tensor theory. While it is clear that both parametrizations are mathematically equivalent at the level of the classical action, the question about their mathematical equivalence at the quantum level as well as their physical equivalence is still a matter of debate in cosmology. We analyze whether the mathematical equivalence still holds when the first quantum corrections are taken into account.

We therefore explicitly calculate the one-loop divergences in both parametrizations by using the generalized Schwinger-DeWitt algorithm and compare both results. We find that the quantum corrections do not coincide and hence induce an off-shell dependence on the parametrization. An explanation of the origin of this frame dependence is suggested to be found within a geometrical approach in the more general field theoretical framework. This approach also implies that the one-loop results in the Jordan frame and Einstein frame should coincide on-shell, which we have shown explicitly for a simple cosmological background. Finally, we discuss the physical implications of this analysis and its consequences for cosmology.

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I. INTRODUCTION

The cosmological models based on scalar fields non-minimally coupled to gravity [1]–[21] have recently become more popular again, in particular because of inflationary models, in which the inflation is driven by the non-minimally coupled Standard Model Higgs boson [10]–[20].

The renormalization group running that connects the present low-energy vacuum of the Standard Model with the high-energy phase during inflation is based on quantum corrections and is essential for the numerical predictions of the non-minimal Higgs inflation model [12]–[14] [15]. While the one-loop running seems to be not accurate enough to ensure the agreement of these models with the latest LHC results [22]–[23] (a difference of roughly 10 GeV between the predicted and measured values for the Higgs mass), it turned out that the two-loop running [18] brings the lower end of the predicted Higgs mass spectrum very close to the measured value of $M_H \sim 126$ GeV [22]–[23].

Although it is an appealing feature of these models that they are falsifiable and indeed produce numerical predictions that are in agreement with the latest results of the satellite PLANCK [24] as well as with the recently announced Higgs mass [22]–[23], one should first clarify the basic principles of the model before worrying about exact numerical values. Some of these principle questions have already been discussed recently. Among them the multiplet nature of the Standard Model complex Higgs $SU(2)$ doublet [25]–[27] and the question of unitarity for the high energy phase during inflation [28]–[36]. However, the probably most fundamental problem, connected to the question of the equivalence of different field parametrizations, still remains unsolved. The purpose of this paper is to address this problem.

In cosmological models with a scalar field non-minimally coupled to gravity usually two special parametrizations of fields are used. These are the so-called “Jordan frame” and “Einstein frame”. This is not restricted to the non-minimal Higgs inflation model, which is only one special representative in the class of scalar-tensor theories. There is an ongoing debate with quite a long history about the equivalence of these parametrizations, see e.g. [37]–[43]. This debate can be subdivided into the question of mathematical and physical equivalence. Let us first focus on the mathematical aspects.

While it is rather easy to check explicitly that the mathematical description in the two frames is equivalent at the tree-level, it is not so obvious whether this equivalence will still hold at the quantum level. There has been much activity in analyzing the frame equivalence of cosmological observables and perturbations [44]–
Although the cosmological perturbations are quantized, this only corresponds to a “mode-by-mode” analysis and does not involve the quantum divergences that arise due to loop effects. Therefore, the analysis of cosmological perturbations does not answer the question whether the renormalization and the back-reaction on the background, on which these perturbations propagate, depends on the chosen parametrization. In particular, in the model of non-minimal Higgs inflation [10]–[20], the renormalization group running was crucial for the derivation of testable predictions. The running, in turn, is governed by the logarithmic UV-divergences of the theory which determine the beta functions.

This paper provides a natural application of the result obtained in our preceding paper [51], where we have calculated the one-loop divergences in the Jordan frame for a general scalar-tensor theory. Here, we will use this result in order to address the question of equivalence at the quantum level by explicitly calculating the first quantum corrections in the Einstein frame and Jordan frame parametrizations.

The strategy of our calculation can be summarized as follows: We choose to start in the Jordan frame parametrization. In the first calculation, we compute the one-loop divergences directly in the Jordan frame. In the second calculation, we first transform the tree-level action from the Jordan frame parametrization to the Einstein frame parametrization. Then, we calculate the one-loop divergences in the Einstein frame parametrization and finally express the obtained quantum result again in the Jordan frame parametrization, in order to compare the two results obtained by quantizing in different frames.

The question of equivalence at the quantum level then simply boils down to the question whether the following diagram commutes or not.

![Diagram](image)

**FIG. 1:** Transition between the Jordan frame (JF) and the Einstein frame (EF) at the tree-level and Jordan frame (QJF) and Einstein frame (QEF) at the quantum level.

We find that the diagram does not commute, which implies that already the first quantum corrections induce a frame dependence.

The paper is structured as follows:

In Sec. III we will perform the transformation of the Jordan frame tree-level action to the Einstein frame and calculate the one-loop divergences in the Einstein frame by using the generalized Schwinger-DeWitt method [52, 53].

In Sec. IV we will express the Einstein frame one-loop result obtained in Sec. III in terms of the Jordan frame parametrization by applying the inverse transformation back to the Jordan frame. We then compare it with the Jordan frame one-loop divergences obtained in Sec. III in order to explicitly show the frame-dependence of quantum corrections.

In Sec. V we analyze the origin of this frame dependence in a more general field theoretical context by following a geometrical approach for the construction of the effective action. This will provide a new geometrical perspective on the quantization ambiguity.

In Sec. VI we discuss the on-shell equivalence that is suggested by the geometrical approach.

In Sec. VII we comment on the explicit on-shell comparison of the specific model under consideration.

In Sec. VIII we discuss the relation between the geometrical approach and other attempts that try to fix the quantization ambiguities in the simpler context of ordinary quantum mechanics.

Finally, in Sect. IX we conclude with a discussion of the obtained results and their implications for cosmology. We also comment on the status of the cosmological debate about the physical equivalence of the two frames and present a new viewpoint on this debate, which is based on the geometrical approach discussed in Sec. V.

The explicit conformal transformation formulas for all expressions that appear in the calculations are collected in Appendix A. The rather long expressions for all the coefficients that enter the divergent one-loop contributions to the effective action in the Jordan frame and Einstein frame are presented in Appendix B and Appendix C respectively. The technical details of the explicit one-loop calculation in the Einstein frame can be found in Appendix D.

## II. JORDAN FRAME CALCULATION

Consider the action \(^1\) for a general scalar-tensor theory

\[
S = \int_M d^4x \sqrt{g} \left( U R - \frac{1}{2} G g^{\mu\nu} \partial_{\mu} \Phi \partial_{\nu} \Phi - V \right)
\]

\(^1\) We choose to work with the conventions \(\text{sign}(g_{\mu\nu}) = +2\), \(g := \text{det}g_{\mu\nu}\), \(R_{\mu\nu} := R^a_{\mu\alpha\nu} \sigma^a\), \(R^a_{\mu\nu\beta\gamma} := \partial_\alpha \Gamma^a_{\mu\nu\beta\gamma} - ..., \Box := g^{\mu\nu} \nabla_\mu \nabla_\nu\)
of a non-minimally coupled $O(N)$-symmetric scalar multiplet $\phi^a$ with the modulus
\[ \phi := \sqrt{\delta_{ab} \Phi^a \Phi^b}, \quad a = 1, \ldots, N. \] (2)

The field dependent couplings $U(\phi)$, $G(\phi)$ and $V(\phi)$ are invariant with respect to the rotations in the isotopic $N$-dimensional space. In [51] the divergent part of the one-loop effective action was calculated in a closed form by making use of the generalized Schwinger-DeWitt technique [52] [53]. All the calculations were carried out consistently in the Jordan frame parametrization.

Here, we will make use of this general result by restricting ourselves to the case of a single scalar field $N = 1$ with the action
\[ S^I = \int_M d^4 x \sqrt{g} \left( U_R - \frac{1}{2} G g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V \right), \] (3)
and investigate whether the calculation of the one-loop divergences in different frames will lead to different results.

Using the MathTensor package [54] to reduce the general result obtained in [51] to the single field case $N = 1$, the one-loop divergences calculated in the Jordan frame are easily obtained to read

\[
\Gamma_{\text{1-loop}}^{\text{div}} = \frac{1}{32\pi^2(\omega - 2)} \int d^4 x \sqrt{g} \left\{ U_{1\text{-loop}}^I R + G_{1\text{-loop}}^I (\phi, \phi' \phi^\nu) + V_{1\text{-loop}}^I + \alpha_1^I R_{\mu\nu} R^{\mu\nu} + \alpha_2^I R^2 + \alpha_3^I R (\Box \phi) \\
+ \alpha_4^I R (\phi, \phi' \phi^\nu) + \alpha_5^I R^{\mu\nu} \phi_{\mu\nu} + \alpha_6^I (\Box \phi) (\phi, \phi' \phi^\nu) + \alpha_7^I (\phi, \phi' \phi^\nu)^2 \right\}. \] (4)

Here, $1/(\omega - 2)$ is a pole in dimension, with $\omega = d/2$ being half the dimension $d = 4$ of space-time. The explicit form of the individual coefficients $U_{1\text{-loop}}^I$, $G_{1\text{-loop}}^I$ and $\alpha_i^I$, $i = 1, \ldots, 8$ can be found in Appendix B. In the main text, we only focus on the one-loop corrections to the effective potential $V_{1\text{-loop}}^I$. This is not only the most important structure regarding cosmological applications, it is also the only structure that does not contain any space-time derivatives and thereby can never receive any contributions from other structures due to integration by parts. Since the basis in the space of independent invariants (scalar contractions of field operators, space-time derivatives and curvature terms) used to express the final result is up to some extent a matter of choice, the effective potential can serve as a unique indicator to test the quantum equivalence of frames in the following sense: It is already sufficient to show the non-equivalence of the effective potential calculated in different frames, in order to show that the quantum result is frame-dependent. The one-loop corrections to the potential in the Jordan frame are explicitly given by

\[
\sqrt{g} V_{1\text{-loop}}^I = \sqrt{g} \left\{ V^2 \left[ 2 s^2 \frac{(U')^4}{U^4} - 2 s \frac{(U')^2}{U^2} + \frac{5}{U^2} \right] + V V' \left[ -8 s^2 \frac{(U')^3}{U^3} + 4 s \frac{U'}{U^2} + 2 V V'' s^2 \frac{(U')^2}{U^2} \right] \right. \\
+ \left. (V')^2 \left[ 8 s^2 \frac{(U')^2}{U^2} - 2 s \frac{U'}{U} \right] - 4 V V'' s^2 \frac{U'}{U} + \frac{1}{2} \left( V'' s^2 \right)^2 \right\}. \] (5)

Here, a prime denotes a derivative with respect to the Jordan frame field $\phi$ and $s(\phi)$ is a particular combination of the field dependent generalized potentials $G$ and $U$
\[ s = \frac{U}{GU + 3 (U')^2}. \] (6)

A similar calculation in the Jordan frame for a single scalar field was already performed before in [55] 2. As discussed in [51], a few discrepancies in several coefficients $\alpha_i$ remain when comparing the result obtained in [54] with the limiting case $N = 1$ of the general result derived in [51]. This, however, does not affect the most

2 See also [56]-[59] for similar one-loop calculations in the context of dilaton gravity.
important structures and in particular not the effective potential, so that the conclusion drawn here remain valid independent of these discrepancies.

III. EINSTEIN FRAME CALCULATION

Let us now transform the tree-level action [3] to the Einstein frame. In the absence of any additional matter fields apart from the scalar field, the Einstein frame is defined as the particular parametrization of fields \( (g_{\mu\nu}, \phi) \) such that [3] formally resembles the ordinary Einstein-Hilbert action with a minimally coupled scalar field. The action can be divided into two steps. First, a conformal transformation of the metric field \( g_{\mu\nu} \to \hat{g}_{\mu\nu} = f(\phi) g_{\mu\nu} \) is performed in order to remove the non-minimal coupling term. Then, an additional reparametrization of the scalar field \( \phi \to \hat{\phi} \) is performed in such a way that the kinetic term acquires the standard canonically normalized form.

In order to find the explicit transformation law that connects the Jordan frame parametrization with the Einstein frame parametrization, we have to investigate how the non-minimal term in the tree-level action [3] changes under a conformal transformation \( g_{\mu\nu} \to \hat{g}_{\mu\nu} \) with a field dependent conformal factor \( f(\phi) \). Using the general conformal transformation laws provided in Appendix A, we find

\[ U \sqrt{g} R = U \sqrt{\hat{g}} \left( f \hat{R} + \frac{3}{2} f^{-1} f_{,\nu} f_{;\nu} - 3 f_{,\nu} f_{;\nu} \right) . \] (7)

All quantities expressed in terms of the Einstein frame parametrization are denoted by a hat. In order to remove the non-minimal coupling (keeping only the Einstein-Hilbert term associated with some constant \( U_0 \)) we have to choose

\[ f = \frac{U_0}{U} . \] (8)

Under this conformal transformation, the potential and the kinetic term in [3] will be simply rescaled by powers of the conformal factor [8].

\[ -\sqrt{g} V = \sqrt{\hat{g}} \left( \frac{U_0}{U} \right)^2 V = \sqrt{\hat{g}} \hat{V} , \] (9)

\[ -\frac{1}{2} \sqrt{g} G g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} = -\frac{1}{2} \sqrt{\hat{g}} \left( \frac{U_0}{U} \right)^2 G \hat{g}^{\mu\nu} \phi_{,\mu} \phi_{,\nu} . \] (10)

In the last equality of [10], we have absorbed the conformal factor by a redefinition of the potential

\[ \hat{V}(\phi) := U_0^2 \frac{V(\phi)}{U^2(\phi)} . \] (11)

Integration by parts of the last term in [7], with the choice [8], leads to

\[ U \sqrt{g} R = U_0 \sqrt{\hat{g}} \hat{R} - \frac{3}{2} \left( \frac{U'}{U} \right)^2 \phi_{,\mu} \phi_{,\mu} . \] (12)

Thus, the right hand side of [10] receives an extra contribution, leading to the transformed kinetic term

\[ -\frac{1}{2} \sqrt{g} G g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} = -\frac{1}{2} \sqrt{\hat{g}} \hat{g}^{\mu\nu} \phi_{,\mu} \phi_{,\nu} . \] (13)

with the field dependent quantity

\[ M(\phi) := \left( \frac{U_0}{U} \right) \left( \frac{G U + 3 (U')^2}{U} \right) . \] (14)

In order to normalize the kinetic term, we can again make use of the fact that \( \phi \) is just a configuration space variable and perform an additional field reparametrization

\[ \phi \to \hat{\phi} . \] (15)

The condition for the field transformation law is fixed by [13]. We must therefore find a solution \( \tilde{\phi} \) to the equation

\[ M(\phi) \tilde{g}^{\mu\nu} \phi_{,\mu} \phi_{,\nu} = \hat{g}^{\mu\nu} \phi_{,\mu} \phi_{,\nu} . \] (16)

This can equivalently written as the condition

\[ \left( \frac{\partial \tilde{\phi}}{\partial \phi} \right)^2 = \left( \frac{U}{U_0} \right) \left( \frac{U}{G U + 3 (U')^2} \right) . \] (17)

Later on, we will also need the inverse relation

\[ \left( \frac{\partial \phi}{\partial \tilde{\phi}} \right)^2 = \left( \frac{U_0}{U} \right) \left( \frac{U_0}{G U + 3 (U')^2} \right) . \] (18)

After the conformal transformation with the conformal factor [8], the field transformation [18] and the redefinition of the potential [11] can be used in order to express the action [3] in terms of the Einstein frame parametrization

\[ \hat{S} = \int_{\mathcal{M}} d^4 x \sqrt{\hat{g}} \left( U_0 \hat{R} - \frac{1}{2} \hat{g}^{\mu\nu} \partial_{\mu} \hat{\phi} \partial_{\nu} \hat{\phi} - \hat{V} \right) . \] (19)

We can now proceed and use the generalized Schwinger-DeWitt algorithm [53] in order to calculate the one-loop divergences for the action [19] in the Einstein frame parametrization. We will perform the calculation in two different ways. First, we repeat the explicit calculation that was done already in [59]. A brief summary of the details of that calculation can be found in Appendix L. Second, we again make use of the general Jordan frame result for a \( O(N) \)-symmetric multiplet of scalar fields, obtained in [51], reduce it to the single field case \( N = 1 \) and set in addition \( U = U_0 \) and \( G = 1 \). This should lead to the same result, since the general Jordan frame result contains the Einstein frame result as a special case. For both ways of calculating, we obtain the same result.
Here, the derivative $\partial_\phi$ has to be computed with respect to the Einstein frame field $\hat{\phi}$. This result coincides with the one already obtained in [59], except for the $V^2$ structure: instead of the correct $5\hat{\Gamma}^\text{loop}$, there was a wrong prefactor $5/2\hat{V}^2$, which was already noted by the authors of [59] in [9]. In particular, the divergent one-loop contribution to the effective potential calculated in the Einstein frame parametrization is given by

$$\sqrt{g} V_1^\text{loop} = \sqrt{\hat{g}} \left( \frac{1}{2} (\partial^2_{\phi^2} \hat{V})^2 - \frac{2}{U_0^2} (\partial_{\phi^2} \hat{V})^2 + \frac{5}{U_0^2} \hat{V}^2 \right).$$

(21)

IV. EINSTEIN FRAME QUANTUM RESULT EXPRESSED IN THE JORDAN FRAME

The next step consists of expressing the one-loop divergences (20), derived from the tree-level action in the Einstein frame parametrization and the superscript $E$ (Jordan frame) parametrization, in terms of the original Jordan frame parametrization. Thus, we need to perform the inverse transformation

$$\hat{\Gamma}_1^\text{div, E} = \frac{1}{32\pi^2(\omega - 2)} \int d^4x \sqrt{g} \left\{ \frac{43}{60} \hat{R}_{\mu\nu} \hat{R}^\mu\nu + \frac{1}{40} \hat{R}^2 - \frac{1}{6} \hat{R} (\partial_\phi^2 \hat{V}) + \frac{1}{2} (\partial_\phi^2 \hat{V})^2 - U_0^{-1} \left[ \frac{1}{3} \hat{R} (\hat{\varphi}, \nu), \nu \right] \right\}.$$

(20)

The divergent one-loop contribution to the effective potential calculated in the Einstein frame parametrization and expressed in terms of the Jordan frame parametrization then reads

$$\sqrt{g} V_1^E = \sqrt{\hat{g}} \left\{ V^2 \left[ s^4 \left( \frac{6G' (U')^3 U''}{U^3} - \frac{3G' (U')^5}{U^4} + \frac{(G')^2 (U')^2}{2U^2} - \frac{18 (U')^6 U''}{U^3} + \frac{18 (U')^4 (U'')^2}{U^4} + \frac{9 (U')^8}{2U^6} \right) \right] + s^3 \left( -\frac{2G' U' U''}{U^2} + \frac{5G' (U')^3}{U^3} + \frac{36 (U')^4 U''}{U^4} - \frac{12 (U')^2 (U'')^2}{U^3} - \frac{15 (U')^6}{U^5} \right) \right\}.$$

(26)
This quantity should now be compared with the divergent one-loop contribution to the effective potential directly calculated in the Jordan frame $V_{1\text{-loop}}$ that was given in \([3]\). Defining the difference $\Delta V_{1\text{-loop}} := V_{E\text{-loop}} - V_{1\text{-loop}}$, it is easy to see that $\Delta V_{1\text{-loop}} \neq 0$.

Thus, we have shown that the off-shell one-loop divergences calculated in the Jordan frame parametrization and in the Einstein frame parametrization do not lead to the same result. Therefore, already the first quantum corrections induce a frame dependence.

\[ \Delta V_{1\text{-loop}} \]

V. ORIGIN OF FRAME DEPENDENCE

In the preceding section we showed the parametrization dependence of quantum corrections for the case of a single scalar field non-minimally coupled to gravity by a direct comparison of the corresponding one-loop divergences calculated in the two cosmological Jordan frame and Einstein frame parametrizations.

However, it would be interesting to obtain a deeper understanding of the origin of this result in order to provide an explanation for this parametrization dependence. Moreover, it would be desirable to make a more general statement, not only restricted to this (albeit very general) cosmological model.

For the following discussion it is convenient to adapt the (hyper)condensed DeWitt notation \([52]\), in which all basic fields, together with their discrete space-time and internal indices, are collected in the generalized field

\[ \phi^i := \phi^A(x) := \begin{pmatrix} g_{\mu\nu}(x) \\ \varphi(x) \\ \vdots \end{pmatrix}. \]

In the case of our model, $\phi^i$ would simply consist of the metric field $g_{\mu\nu}(x)$ and the scalar field $\varphi(x)$, but the formalism is more general so that it easily extends to all kinds of additional fields. The discussion simplifies, if we consider only bosonic fields without internal gauge symmetries, but the same line of argumentation can be extended easily for the more general case of fermionic matter and gauge fields \([50]\). The capital indices $A, B, \ldots$ can be thought of as “super-indices” collecting the different fields contained in the action. In order to take care of the functional nature of field space, we will adopt an even more condensed notation by including the continuous space-time points in the “DeWitt-index” $i = (A, x)$. Thus, whenever there is a summation of the indices $i, j, k, \ldots$, it is implicitly assumed that this summation also includes integration over the space-time arguments $x$. Geometrically, the field $\phi^i$ can be thought of representing a point in the configuration space of fields $C$ (or “field space” for short), viewed as a differentiable manifold. Different parametrizations of $\phi^i$ can be thought of as describing the same point in $C$ in terms of different coordinate systems. Hence, two different parametrizations $\phi^i$ and $\tilde{\phi}^i$ still denote one and the same point in $C$, which is simply described by a different set of coordinates in field space. Since the choice of coordinates should have no physical meaning at all, we must check whether the formalism is covariant with respect to the diffeomorphisms of field space. The situation is analogous to the one in general relativity, but there one has to deal with space-time diffeomorphisms. For simplicity, we also restrict the field space to have vanishing curvature, i.e. we restrict ourselves to the case of “curvilinear parametrizations” in a flat field space. In general, however, the field space can have curvature.

Adopting this geometric viewpoint, we can locate the
origin of frame dependence at the one-loop level by the following consideration: The theory is described by the classical action $S$, which is a “true scalar” with respect to field space and therefore transforms in a manifestly covariant way $S[\phi] = \bar{S}[\bar{\phi}]$ with respect to coordinate changes in field space $\phi \to \bar{\phi}$. Similarly, the first functional derivative $S_i := \delta S/\delta \phi^i$ transforms in a covariant way as a vector in field space. However, already the second functional derivative does not transform like a tensor anymore. Since the one-loop contributions to the effective action are proportional to the second functional derivative

$$\Gamma_{1\text{-loop}} = \frac{1}{2} \text{Tr} \ln \frac{\delta^2 S}{\delta \phi^i \delta \phi^j},$$

(29)

in order to restore covariance, the “partial” functional derivative $\delta/\delta \phi^i$ must be replaced by a covariant functional derivative $\nabla_i$

$$\frac{\delta^2 S}{\delta \phi^i \delta \phi^j} \to \nabla_i \nabla_j S = \frac{\delta^2 S}{\delta \phi^i \delta \phi^j} - \Gamma_{ij} \frac{\delta S}{\delta \phi^k}.$$

(30)

Here, $\Gamma_{ij}$ is a configuration space connection, which is at the moment not specified explicitly. But since we have left aside the problem of separating gauge transformations from parametrizations, we can simply think of it as being constructed from a configuration space metric $G_{ij}$ in a similar way as the Christoffel symbol $\Gamma_{\mu \nu}^\rho$ is constructed from the space-time metric $g_{\mu \nu}$ in general relativity. However, due to the functional nature of field space, in general, not all concepts of general relativity can simply be lifted up to field space.

This geometrical approach to the effective action was developed by Vilkovisky in [60] and its covariant construction was denoted the “unique effective action”. Later, this concept was applied to several physical models in [61]. In [60], the source of the non-covariance was traced back to the defining equation for the effective action. The usual way to derive the effective action within the path integral approach is to start with the partition function

$$Z[J] := \int \mathcal{D}\phi \, e^{\frac{i}{\hbar} \Gamma[\langle \phi \rangle]} J + \phi \rangle, \quad \text{(31)}$$

which is a functional of the source $J$. $Z[J]$ is also denoted generating functional because all $n$-point Greens functions can be obtained from it by functional differentiation. Defining a new functional $W[J]$ via

$$Z[J] := e^{\frac{i}{\hbar} W[J]}, \quad \text{(32)}$$

one can show that functional differentiation of $W[J]$ only generates connected Greens functions. Using the definition of the mean field

$$\langle \phi \rangle^k := \frac{\delta W[J]}{\delta J^k} \bigg|_{J=0}, \quad \text{(33)}$$

one can perform a functional Legendre transformation

$$\Gamma[\langle \phi \rangle] := W[J] - J_k \langle \phi \rangle^k. \quad \text{(34)}$$

This new functional $\Gamma[\langle \phi \rangle]$ is the generating functional for the one-particle irreducible Greens functions and defines the effective action. $\Gamma[\langle \phi \rangle]$ is a functional of the mean field $\langle \phi \rangle^k$ and the reason why it is denoted effective action can be traced back to the relation

$$e^{\frac{i}{\hbar} \Gamma_1[\langle \phi \rangle]} = \int \mathcal{D}\phi \, e^{\frac{i}{\hbar} \left[ S[\phi] + \frac{\delta^2 \langle \phi \rangle}{\delta \langle \phi \rangle^k} \langle \phi \rangle^k \right]} \langle \phi \rangle^k \right]^2. \quad \text{(35)}$$

This is a complicated functional integro-differential equation which for almost all cases can only be solved by iteration in powers of $\hbar$. At first order, one obtains the one-loop contribution [29]. The non-covariance of the formalism is connected with the coordinate difference $\langle \phi \rangle^k - \langle \phi \rangle^k$ in [35] which has no geometrical meaning. In order to restore covariance, in [60], it was suggested to replace this difference by a geometrical meaningful object

$$\langle \phi \rangle^k - \langle \phi \rangle^k \to \nabla k \sigma[\phi, \langle \phi \rangle]. \quad \text{(36)}$$

Here, the concept of the “world function”, originally introduced in [61] in the context of space-time, was elevated to field space in [60]. In the context of field theory it is defined by

$$2 \sigma[\phi, \langle \phi \rangle] = \langle \text{geod. dist. between } \phi \text{ and } \langle \phi \rangle^2 \rangle. \quad \text{(37)}$$

In [60], it was argued that the configuration space metric could be uniquely fixed by certain physical assumptions, which ultimately would allow to construct the “unique effective action”. This covariant construction would then be independent of the field parametrization in which the quantization procedure was performed. Independently of whether it is possible or not to realize such a construction in an unambiguous way, we can still derive an important conclusion that follows from the geometrical analysis alone:

First, it seems obvious that the cosmological Jordan frame and Einstein frame parametrizations just correspond to two very special choices of coordinate systems in field space. Thus, the geometrical formalism is, in particular, applicable to the cosmological debate about the equivalence of Jordan frame and Einstein frame at the quantum level [11]. Moreover, from this geometrical point of view the often discussed physical (non-)equivalence between the Jordan frame and the Einstein frame seems to be a “phantom discussion” in the following sense: The covariant analysis suggests that there is simply no point in looking for physical arguments in favor or against one or another parametrization. Within a covariant formalism such an undertaking would be meaningless in the same sense as it would be meaningless to ask whether it is more physical to describe a mechanical system in terms of Cartesian or spherical coordinates. Thus, the geometrical framework suggests that the reason for the frame dependence is not to be found in the
the naive definition of the effective action can be traced back to the extra factor

$$-\Gamma^k_{ij} \frac{\delta S}{\delta g^{ik}},$$  \hfill (38)$$

which arises if one writes out the second covariant functional derivative in terms of the partial functional derivative in (30). Going on-shell means using the equations of motion

$$S_{,i} = \frac{\delta S}{\delta \varphi^i} = 0. \hfill (39)$$

Thus, the extra factor (38) vanishes on-shell, which in turn means that the difference between the covariant and the naive definition of the one-loop effective action should also vanish on-shell. In principle, we can view this as a self-consistency check for the geometric approach, or vice versa, if we believe in the correctness of the geometrical viewpoint, we can view this as a cross check for all explicit applications, in particular for the discussion of the equivalence of Jordan frame vs. Einstein frame.

Independently of the on-shell discussion, one should bear in mind that essential off-shell information is needed in the calculation of physical quantities such as the renormalization group flow of the couplings determined by the beta functions.

\section{VII. On-shell Comparison of Jordan vs. Einstein Frame Quantization}

The explicit on-shell comparison of $\Gamma^J_{1-loop}$ and $\Gamma^E_{1-loop}$ found in Sec. \[ and Sec. \[ is difficult. The equations of motion in the Jordan frame are easily obtained by the first variation of the Jordan frame action \[. Variation with respect to $g_{\mu\nu}$ gives the field equation for the metric field

$$g_{\alpha\beta} (\nabla \varphi)^2 + \frac{U'}{U} \varphi_{,\alpha\beta} - \frac{U'}{U} g_{\alpha\beta} \Box \varphi - \frac{1}{2} g_{\alpha\beta} \frac{V}{U}. \hfill (40)$$

number of independent structures in the effective actions
Variation with respect to $\varphi$ yields one additional scalar equation, the Klein-Gordon equation for the scalar field

$$\square \varphi = \left( \frac{U'}{G} - \frac{1}{2} \frac{G'}{G} \right) (\nabla \varphi)^2 + \frac{V'}{G}.$$  

These formulas show, that the equations of motion relate scalar invariants containing space-time derivatives with scalar invariants containing no space-time derivatives. Therefore, the coefficients of the invariants containing space-time derivatives will contribute in a non-trivial way to the one-loop divergences of the effective potential. Ultimately, these on-shell contributions will alter the comparison between the effective potential calculated in the Jordan and Einstein frame and should lead to an on-shell frame independence as suggested by the geometric arguments of Sec. [VI].

The process of explicitly eliminating the dependent structures by the equations of motion in the particular model under consideration is rather tedious. The system is non-linear in the invariants and requires to alternately use (41)--(45) and integration by parts in an iterative procedure several times, in order to reduce the different invariant structures to a minimal independent set.

We will, however, provide a detailed calculation and prove the on-shell frame independence explicitly for the special case of a canonically normalized $G(\varphi) = 1$, constant background scalar field $\nabla \varphi = 0$. This particular background implies that no structure that involves a derivative of the scalar field can appear in the divergent one-loop contribution to the effective action (4). Moreover, from the form of the transformation to the Einstein frame scalar field [18], it is clear that this also implies $\nabla \varphi = 0$. Thus, the one-loop divergences calculated in the Jordan frame [4] and in the Einstein frame [20] reduce to the following structures respectively

$$
\Gamma^{\text{div}, \text{J}}_{1\text{-loop}}[g, \varphi] = \frac{1}{32 \pi^2 (\omega - 2)} \int d^4x \sqrt{g} \left\{ U^2_{1\text{-loop}} R + V^2_{1\text{-loop}} + \alpha^2_{1} R_{\mu\nu} R^{\mu\nu} + \alpha^2_{1} R^2 \right\}, 
$$

$$
\Gamma^{\text{div}, \text{E}}_{1\text{-loop}}[g, \varphi] = \frac{1}{32 \pi^2 (\omega - 2)} \int d^4x \sqrt{g} \left\{ U^2_{1\text{-loop}} \hat{R} + V^2_{1\text{-loop}} + \alpha^2_{1} \hat{R}_{\mu\nu} \hat{R}^{\mu\nu} + \alpha^2_{1} \hat{R}^2 \right\}.
$$

From the Jordan frame equations of motion [41], [42] and [45], we obtain in the $\nabla \varphi = 0$ case, the following identities: The trace of the equations of motion for the metric field $g_{\mu\nu}$ reduces to

$$R \triangleq 2 \frac{V}{U},$$

while the Klein-Gordon equation for the scalar field yields

$$R \triangleq \frac{V'}{U'}. $$

Combining these two equations leads to a relation that allows to express $V$ in terms of $U$

$$V \triangleq U^2.$$ 

Inserting this again in the trace equation, we find

$$R \triangleq 2U.$$ 

Finally, using these results in [42], we find

$$R_{\mu\nu} R^{\mu\nu} \doteq U^2,$$ 

where the wedge over the equality sign indicates that we have made use of the equations of motion. The conformal transformations [AS] and [A10] become simple scaling relations since the additional derivative structures are absent in the case of the constant background scalar field,

$$\hat{R} = \left( \frac{U_0}{U} \right)^2 R + \text{terms}[\nabla \varphi],$$ 

$$\hat{R}_{\mu\nu} \hat{R}^{\mu\nu} = \left( \frac{U_0}{U} \right)^2 R_{\mu\nu} R^{\mu\nu} + \text{terms}[\nabla \varphi].$$ 

It remains to express the Einstein frame coefficients $U^2_{1\text{-loop}}, V^2_{1\text{-loop}}, \alpha^2_{1}, \alpha^2_{2}$ in terms of the Jordan frame field $\varphi$, in order to find $U^2_{1\text{-loop}}, V^2_{1\text{-loop}}, \alpha^2_{1}, \alpha^2_{2}$. For $G(\varphi) = 1$, the suppression function [6] becomes

$$s(\varphi) = \frac{U}{U + 3 (U')^2}.$$ 

The Einstein frame coefficients $\hat{U}_{1\text{-loop}}^E$, $\hat{V}_{1\text{-loop}}^E$, $\hat{\alpha}_1^E$, $\hat{\alpha}_2^E$ only involve the Einstein frame potential $\hat{V}$ and first and second derivatives thereof $\partial_\varphi \hat{V}$ and $\partial^2_\varphi \hat{V}$,

\[
\begin{align*}
\hat{U}_{1\text{-loop}}^E & = -\frac{1}{6} (\partial^2_\varphi \hat{V}) - \frac{13}{3} \hat{V} U_0, \\
\hat{V}_{1\text{-loop}}^E & = \frac{1}{2} \left( \partial^2_\varphi \hat{V} \right)^2 - \frac{2}{U_0} (\partial_\varphi \hat{V})^2 + \frac{5}{U_0^2} \hat{V}^2, \\
\hat{\alpha}_1^E & = \frac{43}{60}, \\
\hat{\alpha}_2^E & = \frac{1}{40}.
\end{align*}
\] (56) (57) (58) (59)

With the definition (11) and the on-shell relation (50), we find

\[
\hat{V} = \frac{U_0^2}{U^2} V \triangleq U_0^2.
\] (60)

Therefore, the relevant transformation formulas (25) and (26) for the derivatives $\partial_\varphi \hat{V}$ and $\partial^2_\varphi \hat{V}$ imply $\partial_\varphi \hat{V} \equiv 0$ and $\partial^2_\varphi \hat{V} \equiv 0$. Thus, the on-shell Einstein frame coefficients expressed in terms of the Jordan frame parametrization become

\[
\begin{align*}
U_{1\text{-loop}}^J & \triangleq -U \left\{ U^2 \left[ 6(U'')^2 + U'' + 13 \right] + 117 (U')^4 + 7U (U')^2 (10 - 3U'') \right\} \\
V_{1\text{-loop}}^J & \triangleq U^2 \left\{ U^2 \left[ 2(U'')^2 + 5 \right] + 47 (U')^4 + 4U (U')^2 (7 - 2U'') \right\} \\
\alpha_1^J & \triangleq \frac{2 (U')^2}{3(U')^2 + U} + \frac{43}{60}, \\
\alpha_2^J & \triangleq \frac{U^2 \left[ 60(U'')^2 + 20U'' + 3 \right] - 213 (U')^4 - 2U (U')^2 [90U'' + 71]}{120 \left( 3(U')^2 + U \right)^2}.
\end{align*}
\] (61) (62) (63) (64)

Combining (61)-(64) and (50)-(52), we obtain the on-shell version of the Einstein frame one-loop divergences (47) expressed in terms of the Jordan frame parametrization

\[
\Gamma_{1\text{-loop}}^{\text{div}, E}[g, \varphi] \triangleq \frac{1}{32 \pi^2 (\omega - 2)} \int d^4x \sqrt{g} \left\{ -\frac{57}{20} U^2 \right\}.
\] (65)

In order to compare this with the on-shell version of the one-loop divergences calculated in the Jordan frame (46), we have to find the on-shell values of the coefficients (45), (B1), (B3) and (B4). Using again, the relations (50)-(52) and $G(\varphi) = 1$, we find

\[
\begin{align*}
U_{1\text{-loop}}^J & \triangleq -U \left\{ U^2 \left[ 6(U'')^2 + U'' + 13 \right] + 117 (U')^4 + 7U (U')^2 (10 - 3U'') \right\} \\
V_{1\text{-loop}}^J & \triangleq U^2 \left\{ U^2 \left[ 2(U'')^2 + 5 \right] + 47 (U')^4 + 4U (U')^2 (7 - 2U'') \right\} \\
\alpha_1^J & \triangleq \frac{2 (U')^2}{3(U')^2 + U} + \frac{43}{60}, \\
\alpha_2^J & \triangleq \frac{U^2 \left[ 60(U'')^2 + 20U'' + 3 \right] - 213 (U')^4 - 2U (U')^2 [90U'' + 71]}{120 \left( 3(U')^2 + U \right)^2}.
\end{align*}
\] (66) (67) (68) (69)

Combining the on-shell coefficients (66)-(69) with the on-shell values for the corresponding structures (50)-(52), we obtain for the on-shell Jordan frame one-loop divergences (46)

\[
\Gamma_{1\text{-loop}}^{\text{div}, J}[g, \varphi] \triangleq \frac{1}{32 \pi^2 (\omega - 2)} \int d^4x \sqrt{g} \left\{ -\frac{57}{20} U^2 \right\}.
\] (70)

Thus, a very non-trivial cancellation between the complicated coefficients (66)-(69) ensures the on-shell coincidence of the one-loop divergences (65) and (70). This is in perfect agreement with the formal implications of the geometric approach to the effective action and supports the viewpoint that the quantum frame dependence can indeed be traced back to the non covariant definition of the effective action with respect to the configuration space of fields. Note that the classical background, representing the solution to the equations of motion with $V = U^2$ and $\nabla \varphi = 0$ is nothing but the maximally symmetric de Sitter space-time. The problem of the on-shell coincidence of the one-loop effective action, calculated in the Jordan frame and in the Einstein frame, was also studied in [57].
VIII. NON-COVARIANT FORMALISM IN THE PRESENCE OF TESTABLE PREDICTIONS

In order to discuss some aspects of the problem without the complicated functional nature of field space, we can try to focus on the conceptual points by considering a similar problem in the familiar context of ordinary quantum mechanics. The advantage is that we only have to deal with generalized coordinates instead of fields and therefore with a finite number of degrees of freedom. As usual, such a simple analogy has its limits and might even be misleading in some aspects. Nevertheless, we believe that it can help to understand a complicated problem in a simpler context. In particular, the problem of a non-covariant quantization formalism is already present in this much simpler setting.

Consider the system of a single hydrogen atom. Classically the radial symmetric Lagrangian for the relative motion is given by

\[ L = \frac{1}{2} m \dot{r}^2 - V(r) . \] (71)

Here, \( m := \frac{m_p + m_e}{2} \) is the reduced mass, \( V(r) \) is the radial symmetric potential, \( \vec{r}(t) \) is the relative position vector between proton and electron and \( r = \sqrt{x^2 + y^2 + z^2} \) is the modulus of \( \vec{r} \). In Cartesian coordinates \( \vec{r} = (x, y, z) \) and therefore

\[ L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(r) . \] (72)

Following the canonical quantization procedure, we first calculate the generalized momenta, perform the Legendre transformation, and finally obtain the Hamiltonian

\[ H = \frac{1}{2m} (p_x^2 + p_y^2 + p_z^2) + V(r) . \] (74)

We have indicated the spherical parametrization by a subscript \( s \). The calculation of the generalized momenta gives

\[ p_r = \frac{\partial L_s}{\partial \dot{r}} = m \dot{r}, \quad p_\theta = \frac{\partial L_s}{\partial \dot{\theta}} = m r^2 \dot{\theta}, \quad p_\varphi = \frac{\partial L_s}{\partial \dot{\varphi}} = m r^2 \sin^2 \theta \dot{\varphi} . \] (80)

Inverting these relations and performing again the Legendre transformation, we finally obtain the Hamiltonian operator

\[ \hat{H} = -\frac{\hbar^2}{2m} (\partial_x^2 + \partial_y^2 + \partial_z^2) + V(r) \]
\[ := -\frac{\hbar^2}{2m} \Delta(x, y, z) + V(r) . \] (77)

Due to the symmetry of the problem, it is convenient to use a spherical coordinate system. If we perform the coordinate transformation, we have to calculate the Laplace operator in spherical coordinates

\[ \Delta = r^{-2} \partial_r (r^2 \partial_r) + (r^2 \sin^2 \theta)^{-1} [\sin \theta \partial_\theta (\sin \theta \partial_\theta) + \partial_\varphi^2] . \] (78)

This gives of course the well known textbook result. More important, the prescribed procedure yields the correct spectrum for the hydrogen atom, which is in accordance with experimental results. However, from a theoretical point of view, there is nothing which could have prevented us from quantizing the theory in a different coordinate system and to only afterwards transform to spherical coordinates. Since the spherical coordinate system is distinguished in the sense that it is adapted best to the symmetries of the problem, we will investigate the consequences that arise if we quantize the system directly in spherical coordinates. Consider again (72) and transform to spherical coordinates \( (r, \theta, \varphi) \),

\[ L_s = \frac{m}{2} (r^2 + \varphi^2 + r^2 \sin^2 \theta \varphi^2) - V(r) . \] (79)

We have indicated the spherical parametrization by a subscript \( s \). The calculation of the generalized momenta gives

\[ p_r = \frac{\partial L_s}{\partial \dot{r}} = m \dot{r}, \quad p_\theta = \frac{\partial L_s}{\partial \dot{\theta}} = m r^2 \dot{\theta}, \quad p_\varphi = \frac{\partial L_s}{\partial \dot{\varphi}} = m r^2 \sin^2 \theta \dot{\varphi} . \] (80)

Performing the naive quantization scheme by representing the classical configuration space variable \( r, \theta \) and \( \varphi \) by multiplication operators and the momenta by derivatives with respect to the corresponding configuration space variables,\n
\[ \hat{p}_r = -i \hbar \partial_r, \quad \hat{p}_\theta = -i \hbar \partial_\theta, \quad \hat{p}_\varphi = -i \hbar \partial_\varphi , \] (82)

we obtain the spherical Hamilton operator

\[ \hat{H}_s = -\frac{\hbar^2}{2m} \Delta_s(r, \theta, \varphi) + V(r) \] (83)
with the corresponding Laplacian
\[
\Delta_s(r, \theta, \varphi) := \partial_r^2 + r^{-2} \partial_\theta^2 + (r^2 \sin^2 \theta)^{-1} \partial_\varphi^2 .
\] (84)
Comparison of \((78)\) with \((81)\) shows that the two Hamiltonians \(\hat{H}\) and \(\hat{H}_s\) are different, therefore leading to different spectra and thus to different physical predictions.

To summarize, we have used a non-covariant quantization procedure to quantize our theory once in Cartesian coordinates and once in spherical coordinates. In both cases, we have expressed the final Hamilton operator in terms of spherical coordinates – once before and once after quantization. Both procedures lead to a different quantum theory and thus to different experimental predictions.

One could argue that the non-covariant quantization procedure, used here, could be justified a posteriori by comparison with experimental data that would select a specific coordinate system to be used for quantization. However, we would like to emphasize that from a theoretical point of view no coordinate system should be distinguished a priori by the formalism. In principle, we could have started with an arbitrary curvilinear coordinate system in configuration space (for simplicity assuming again no curvature of configuration space) and quantize the theory with the prescribed naive method. The experiment would then only afterwards “select” the quantization in Cartesian coordinates. However, that this “choice” would lead to the correct experimental result was not clear a priori – it could have been different. In the absence of any experiment guidance, this ambiguity poses a serious problem.

Instead, we could avoid this problem from the very beginning by using a covariant formalism. A covariant formalism would automatically lead to a unique result, independent of the coordinates system used for quantization. Instead of comparing the predictions of different parametrizations within a non-covariant theory, we could follow a geometrical approach in a similar manner as discussed in the field theoretical context in Sec. V. This would imply the replacement of the partial derivatives by covariant ones \(\hat{\partial}_q = -i \hbar \hat{\partial}_q \rightarrow -i \hbar \nabla_q\) which would in turn lead to the covariant Laplace-Beltrami operator
\[
\Delta_{LB} := g^{ij} \nabla_i \nabla_j = (\sqrt{g})^{-1} \partial_q [\sqrt{g} g^{ij} \partial_q ] .
\] (85)
This would automatically yield the correct unique result. Here \(g_{ij}\) is the metric on configuration space and \(g^{ij}\) is its inverse.

Of course, the quantum mechanical problem discussed above is well known and usually one “excludes” the quantization in spherical coordinates, since the resulting momentum operators are not self-adjoint.

In contrast to the geometrical approach, the requirement of self-adjointness has the drawback of being a “case-by-case” criterion that can only exclude certain ways of quantizing rather than providing a general constructive instruction for quantization.

The geometrical approach provides a general constructive “quantization recipe” that is applicable without any reference to symmetries. In this case symmetries may only simplify the calculations but are not necessary to obtain the quantized theory.

The geometrical approach, however, does not come without problems. Similar to the field theoretical context discussed in Sec. V the covariant Laplace operator involves the geometry of the configuration space, i.e. its metric. In our analysis of Sec. V we only dealt with curvilinear coordinates in a flat configuration space. In general, the configuration space will be curved, inducing an additional ambiguity. As has already been discussed in [67], the resulting covariant Laplace operator \(\Delta_{LB} \rightarrow \Delta_{LB} + cR\) will involve the Ricci curvature \(R\) of the configuration space manifold. The constant of proportionality \(c\) for this term will not be fixed by the requirement of covariant quantization. Therefore, additional criteria seem to be required in order to fix this ambiguity. Moreover, one could think of endowing the configuration space manifold with all kinds of geometrical structure such as e.g. torsion.

The problem of covariant quantization is also closely related to the well-known factor ordering ambiguity\(^3\), in particular, for terms involving a derivative coupling, e.g. terms like \(px\). In the field theoretical context a term describing a non-minimal coupling to gravity corresponds to such a coupling of the scalar field to the derivatives of the metric field.

**IX. CONCLUSION**

We have calculated the divergent part of the one-loop effective action for a general scalar-tensor theory involving a single scalar field non-minimally coupled to gravity in the cosmological Jordan frame and Einstein frame parametrizations. The individual results were obtained by applying the Schwinger-DeWitt method [52, 53] separately to the same action for each parametrization. Then, we have expressed both results in the Jordan frame parametrization in order to compare them. We found that the two off-shell results do not coincide. This implies that the classical equivalence between the two frames is destroyed already by the first quantum corrections and shows that the one-loop divergences are indeed frame dependent.

We have proposed an explanation of this result in the more general field theoretical context by the geometrical definition of the effective action, developed in [69]. By identifying the two cosmological frames as two coordinate systems in the configuration space of fields, we have traced back the quantum non-equivalence of frames to

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\(^3\) Consider e.g. a term \(\propto px\) in the Hamilton function. At the classical level, \(x\) and \(p\) commute and thus \(xp = px\). This does no longer hold for the corresponding combination of operators \(\hat{x} \hat{p} \hat{\hat{x}} \neq \hat{\hat{p}} \hat{\hat{x}} \hat{\hat{x}}\) in the Hamilton operator.
the non-covariance of the mathematical formalism, used to calculate the one-loop divergences. As has been already emphasized in [41], from the covariant geometrical point of view, the cosmological debate about a physically preferred frame seems to become meaningless, as the origin of the discrepancy between the frames at the quantum level is to be found in the mathematical formalism rather than in the physical properties of a specific frame. Thus, the controversial debate about a physically preferred parametrization in cosmology seems to become obsolete, in the sense that the discussion about whether the Einstein frame or the Jordan frame should be the “correct physical frame” has found a natural resolution by a simple explanation: there is no preferred physical frame. Any frame is as good as any other, as long as we work in a covariant formalism.

See also the recently appeared paper [68], where this idea, proposed by us already in [41], was applied to the model of non-minimal Higgs inflation [10]–[20].

In order to discuss the implications of this result in a more familiar context, we have also analyzed a simple quantum mechanical analogy. There, a very similar picture arises which can be helpful in order to understand the underlying difficulties with the general field theoretical case. In particular, we have discussed how the presence of experimental data can be used in order to exclude certain parametrizations in a non-covariant quantization formalism at the level of a “case-by-case” study.

The question of frame dependence could also be interesting for the recently investigated “quantum tunneling of the universe” in scalar-tensor theories, see e.g. [69] [70].

By general considerations, the geometrical approach implies that – at least at the one-loop level – the quantum corrections in both frames should coincide on-shell. We have carried out an explicit analysis for the case of a constant background scalar field which confirms this by a non-trivial cancellation of contributions from different structures. This supports the geometrical viewpoint and can also serve as an additional cross check of the off-shell results for the one-loop divergences obtained in [51]–[59]. In this context, it should be emphasized again that the on-shell divergences will not be sufficient for many important physical applications that involve essential off-shell quantities, such as e.g. the beta functions which determine the renormalization group flow or the effective potential in the context of spontaneous symmetry breaking.

Finally, we would like to mention some critical points that go beyond the scope of the present work. It is a difficult question, whether the perturbative approach followed in the present paper is meaningful at all, given the perturbatively non-renormalizable character of the gravitational interaction. The final answer about the equivalence of frames might find its natural resolution in a complete theory of quantum gravity that could admit a non-perturbative solution to this problem. In the absence of a fully developed quantum gravitational theory, however, the perturbative treatment seems to be the only straightforward practical approach. In addition, the application of the loop-truncated effective action is a standard technique in many different areas of physics, which shows that the parametrization problem is interesting by itself and not only restricted to its cosmological application.

Another critical point is related to the question of multi-loop calculations in the non-geometric formalism. For higher loops it seems that one cannot simply use the same tree-level transformation rules between the Jordan frame and Einstein frame parametrizations in order to relate and compare the quantum corrected results in the ordinary formalism. Instead, the relevant transformations should be constructed order by order in perturbation theory. This means that one has to find a quantum corrected transformation between the off-shell effective actions. However, it does not seem to be clear how to construct such a transformation explicitly. Moreover, even if there might exist such a quantum corrected transformation between different parametrizations, which could be written down explicitly for the specific case under study, such a quantum corrected transformation is of no practical use as long as there is no universal principle of how to obtain this transformation in general. This becomes obvious in view of the fact that beside the cosmological Jordan frame and Einstein frame parametrizations, there are infinitely many other equally valid parametrizations. Similarly, in order to perform the on-shell comparison of the n-loop corrected effective actions, calculated in different frames within the non-geometric approach, one should use the (n − 1)-loop corrected effective equations of motion. This rather complicated iterative strategy also suggests that it would be desirable to use the perturbative geometric approach, which avoids the construction of complicated quantum transformations that relate different parametrizations by providing a unique result order by order in perturbation theory.

In this paper, we have focused on the pre-logarithmic coefficients of the one-loop divergences only. There is the additional problem of the conformal anomaly that also affects the logarithmic structure [11]. This could be a hint that the problem is not only connected with the frame dependent calculation of the counterterms but in addition with the procedure of renormalization itself, see e.g. [23] for a discussion of this point.

As we have mentioned already in the Introduction, the equivalence between Jordan and Einstein frame is a controversial topic in cosmology. For the particular model of non-minimal Higgs inflation, it was recently claimed in [71] that no indication for a frame dependence of the one-loop effective potential has been found. The result of [71] therefore seems to be in contradiction with the quantum parametrization dependence found in this paper. In [71], the one-loop Coleman-Weinberg potential was calculated, including neither the contributions of Higgs loops nor of graviton loops. Such an approach might seem to be justified in the effective field theoretical framework of non-minimal Higgs inflation with a strong non-minimal
coupling \( \xi \). In this case, the Higgs propagator is suppressed by powers of the function \( \xi \) that scales like \( s \sim 1/6\xi \) for high energies and graviton loops are suppressed by powers of the Planck mass. Such an analysis seems, however, to essentially miss the complicated and interesting part of the calculation: the graviton-scalar mixing that is a consequence of the non-minimal coupling to gravity. Instead, in our work we have presented a fully (space-time) covariant calculation of the one-loop divergences that contribute to the effective potential, including graviton loops. We performed the calculation for a general background \((g_{\mu \nu}, \bar{\phi})\), a general potential \(V(\phi)\) and a general non-minimal coupling \(U(\phi)\). We believe that the inclusion of these contributions is crucial in order to settle the issue of frame dependence.

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Appendix A: Conformal Transformation

Let us consider the following conformal transformation of the metric field \( g_{\mu \nu} \) in four spacetime dimensions

\[
\frac{\delta g_{\mu \nu}}{\delta g_{\mu \nu}} = f \dot{g}_{\mu \nu},
\]

and its inverse

\[
g^{\mu \nu} = f^{-1} \dot{g}^{\mu \nu}.
\]

The square root of the determinant transforms as

\[
\sqrt{g} = f^2 \sqrt{\dot{g}}.
\]

In order to calculate the transformation of the Riemann tensor, it is necessary to first calculate the behavior of the Christoffel symbol and its first derivative under the conformal transformation \((A1)\).

\[
\Gamma^\alpha_{\mu \nu} = \tilde{\Gamma}^\alpha_{\mu \nu} + \frac{1}{2} f^{-1} \left( \delta^\alpha_{\mu} f_{,\nu} + \delta^\alpha_{\nu} f_{,\mu} - \dot{g}^{\alpha \gamma} \dot{g}_{\mu \nu} f_{,\gamma} \right),
\]

\[
\Gamma^\alpha_{\mu \nu, \beta} = \tilde{\Gamma}^\alpha_{\mu \nu, \beta} + \frac{1}{2} f^{-1} \left( \delta^\alpha_{\mu} f_{,\nu, \beta} + \delta^\alpha_{\nu} f_{,\mu, \beta} - \dot{g}^{\alpha \gamma} \dot{g}_{\mu \nu, \beta} f_{,\gamma} - \dot{g}^{\alpha \gamma} \dot{g}_{\mu \nu, \beta} f_{,\gamma} - \dot{g}^{\alpha \gamma} \dot{g}_{\mu \nu} f_{,\gamma, \beta} \right.
\]

\[\left. - \frac{1}{2} f^{-2} \left( \delta^\alpha_{\mu} f_{,\nu} + \delta^\alpha_{\nu} f_{,\mu} - \dot{g}^{\alpha \gamma} \dot{g}_{\mu \nu} f_{,\gamma} \right) \right). \tag{A5}\]

In order to facilitate the calculation of the Riemann tensor, Ricci tensor and Ricci scalar, it is convenient to use a Riemannian normal coordinate system for the intermediate calculations in which \(\Gamma^\alpha_{\mu \nu, \beta} = 0\) but \(\Gamma^\alpha_{\mu \nu, \beta} \neq 0\).

The final result for the conformal transformation of the Riemann tensor is then found to be

\[
R^\alpha_{\beta \gamma \delta} = \tilde{R}^\alpha_{\beta \gamma \delta} + \frac{1}{4} f^{-2} \left( 3f_{,\gamma} f^{,\gamma} \dot{g}_{\beta \delta} - 3f_{,\delta} f^{,\delta} \dot{g}_{\gamma \beta} - 3f_{,\beta} f^{,\beta} \dot{g}_{\gamma \delta} - 3f_{,\gamma} f^{,\gamma} \dot{g}_{\beta \delta} \right.
\]

\[\left. + 3f_{,\beta} f^{,\beta} \dot{g}_{\gamma \delta} + 3f_{,\delta} f^{,\delta} \dot{g}_{\gamma \beta} - f_{,\nu} f_{,\nu} \dot{g}_{\beta \delta} \dot{g}_{\gamma} \right) \]

\[+ \frac{1}{2} f^{-2} \left( f_{,\gamma} \dot{g}_{\beta \delta} - f_{,\delta} \dot{g}_{\gamma \beta} \right). \tag{A6}\]

Contracting the first and the third indices the result for the Ricci tensor is

\[
R_{\alpha \beta} = \tilde{R}_{\alpha \beta} + \frac{3}{2} f^{-2} f_{,\alpha} f_{,\beta} - f^{-1} f_{,\alpha} f_{,\beta} - \frac{1}{2} f^{-1} f_{,\nu} f_{,\nu} \dot{g}_{\alpha \beta}. \tag{A7}\]

The transformation of the Ricci scalar follows from

\[
R = \dot{g}^{\alpha \beta} R_{\alpha \beta} = f^{-1} \tilde{R} + \frac{3}{2} f^{-3} f_{,\nu} f_{,\nu} f_{,\nu} - 3f^{-2} f_{,\nu} f_{,\nu}. \tag{A8}\]

In the divergent part of the one-loop contributions to the effective action also the structures \(R^{\alpha \beta \gamma \delta} R_{\alpha \beta \gamma \delta}, R^{\alpha \beta} R_{\alpha \beta}\) and \(R^2\) appear. By making use of the Gauss-Bonnet identity

\[
\frac{\delta}{\delta g_{\mu \nu}} \left[ \int d^4 x \sqrt{g} \left( R^{\alpha \beta \gamma \delta} R_{\alpha \beta \gamma \delta} + 4 R^{\alpha \beta} R_{\alpha \beta} - R^2 \right) \right] = 0\tag{A9},
\]

the structure \(R^{\alpha \beta \gamma \delta} R_{\alpha \beta \gamma \delta}\) can be expressed in terms of the structures \(R^{\alpha \beta} R_{\alpha \beta}\) and \(R^2\). The transformation laws for these two structures are given by

\[
R^{\alpha \beta} R_{\alpha \beta} = f^{-2} \tilde{R}^{\alpha \beta} R_{\alpha \beta} - f^{-3} \left( \tilde{R} f_{,\nu} f_{,\nu} + 2 \tilde{R}^{\alpha \beta} f_{,\alpha} f_{,\beta} \right)
\]

\[+ f^{-4} \left( f_{,\nu} f^{,\nu} f_{,\nu} + 3 \tilde{R}^{\alpha \beta} f_{,\alpha} f_{,\beta} + 2 f_{,\nu} f_{,\nu} f_{,\nu} f_{,\nu} \right)
\]

\[+ f^{-5} \left( \frac{1}{2} f_{,\nu} f_{,\nu} f_{,\nu} f_{,\nu} + f_{,\alpha} f_{,\alpha} f_{,\beta} f_{,\beta} \right)
\]

\[+ \frac{9}{4} f^{-6} f^{,\alpha} f^{,\alpha} f^{,\beta} f^{,\beta} \tilde{R} \] . \tag{A10}

\[ R^2 = f^{-2} \dot{R}^2 - 6 f^{-3} \dot{R} f_{;\nu} + f^{-4} \left( 3 \dot{R} f_{;\nu} f^{;\nu} + 9 f_{;\nu} f_{;\mu} - 9 f^{-5} f_{;\nu} f^{;\nu} f_{;\mu} + \frac{9}{4} f^{-6} f_{;\nu} f^{;\nu} f_{;\mu} f^{;\mu} \right) \]  

Appendix B: Jordan Frame Coefficients

In this Appendix we present the explicit form of the coefficients for the Jordan frame effective action \([1]\)

\[ U_{1\text{-loop}}^J = V \left[ s^3 \left( -\frac{18G'(U')^3}{U^3} U'' + \frac{9G'(U')^5}{U^4} - \frac{5(G')^2}{2U^2} \left( \frac{U''}{U^3} \right)^2 - \frac{90(U')^6}{U^5} U'' - \frac{18(U')^4}{U^4} + \frac{99(U')^8}{2U^6} \right) \right. 
+ s^2 \left( \frac{G''}{U^2} + \frac{5G'U'U''}{2U^2} + \frac{11G'(U')^3}{2U^3} \right) \left( \frac{U''}{U^3} \right)^2 + \frac{147(U')^4}{U^4} U'' + \frac{9(U')^2}{U^3} U'' - \frac{105(U')^6}{2U^6} \right) 
+ s \left( \frac{G'}{U^3} - \frac{17(U')^3}{4U^4} + \frac{37(U')^4}{2U^4} + \frac{G'}{U^2} + \frac{17(U')^2}{U^3} + \frac{3U''}{U^2} \right) + V' \left[ s^3 \left( \frac{36G'(U')^2}{U^2} U'' \right) 
+ \frac{18G'(U')^4}{U^3} + \frac{5(G')^2}{U^2} U' + \frac{180(U')^5}{U^4} U'' + \frac{36(U')^3}{U^3} (U'')^2 - \frac{199(U')^7}{U^5} \right] 
+ s \left( \frac{G'}{2U} + \frac{14U'U''}{U^2} \right) - \frac{47(U')^3}{2U^4} - \frac{3U''}{2U^2} \right) \Big] 
+ V'' \left[ s^3 \left( \frac{11(U')^2}{U^2} - \frac{U''}{U} \right) - \frac{2}{U} \right] + V'' \left[ s^2 \left( \frac{G'}{2} + \frac{9(U')^3}{2U^2} \right) - \frac{sU''}{U} \right] , \]  

\[ \alpha_1^J = \frac{43}{60} + 2 s \frac{(U')^2}{U} , \]  

\[ \alpha_2^J = \frac{1}{40} + s^2 \left( -\frac{2(U')^2 U''}{U} + \frac{2(U')^4}{U^2} + \frac{(U'')^2}{2} \right) + s \left( \frac{U''}{6} - \frac{4(U')^2}{3U} \right) , \]  

\[ \alpha_3^J = s^2 \left( \frac{G'U''}{2} - \frac{G'(U')^2}{U} + \frac{9(U')^3}{2U^2} - \frac{9(U')^5}{U^3} \right) + s \left( \frac{G'}{12} - \frac{U'U''}{U} + \frac{19(U')^3}{4U^2} \right) - \frac{3U''}{U} , \]
\[ \alpha_4 = s^3 \left( \frac{-15G'(U')^3 U''}{2U^2} + \frac{3G' U' (U'')^2}{U} + \frac{1}{4} (G')^2 U'' + \frac{3G' (U')^5}{U^3} - \frac{(G')^2 (U'')^2}{2U} + \frac{81 (U')^6 U''}{4U^4} \right) \\
- \frac{27 (U')^4 (U'')^2}{U^3} + \frac{9 (U')^2 (U'')^3}{2U^2} - \frac{9 (U')^8}{U^5} \right) + s^2 \left( \frac{G'U'U''}{U} + \frac{5G' (U')^3}{4U^2} + \frac{(G')^2}{24} - \frac{12 (U')^4 U''}{U^3} \right) \\
+ \frac{18 (U')^2 (U'')^2}{U^2} + \frac{15 (U')^6}{8U^4} - \frac{3 (U'')^3}{U} \right) + \left( -\frac{G'U'}{2U} - \frac{7 (U')^2 U''}{4U^2} + \frac{(U')^4}{2U^3} - \frac{(U'')^2}{U} \right) - \frac{G}{3U} \\
- \frac{19 (U')^2}{4U^2} - \frac{2U''}{U} , \right) \] 

\[ \alpha_5 = s^2 \left( -\frac{G'U'U''}{U} - \frac{3G' (U')^3}{U^2} - \frac{15 (U')^4 U''}{U^3} - \frac{6 (U')^2 (U'')^2}{U^2} + \frac{9 (U')^6}{U^4} \right) \\
+ \left( \frac{G'U'}{U} + \frac{10 (U')^2 U''}{U^2} - \frac{5 (U')^4}{U^3} + \frac{2 (U'')^3}{U^2} \right) - \frac{(U')^2}{2U''} - \frac{2U''}{U} , \right) \] 

\[ \alpha_6 = s^2 \left( \frac{9G' (U')^3}{4U^2} + \frac{(G')^2}{8} + \frac{81 (U')^6}{8U^4} \right) + \left( \frac{-G'U'}{2U} - \frac{15 (U')^4}{2U^3} \right) + \frac{27 (U')^2}{4U^2} - \frac{U''}{U} , \right) \] 

\[ \alpha_7 = s^3 \left( \frac{9G' (U')^4 U''}{U^3} + \frac{9G' (U')^2 (U'')^2}{2U^2} + \frac{3 (G')^2 U''}{2U} + \frac{45G' (U')^6}{8U^4} + \frac{3 (G')^2 (U'')^3}{8U^2} + \frac{(G')^3}{8} \right) \\
- \frac{81 (U')^7 U''}{2U^5} + \frac{81 (U')^5 (U'')^2}{2U^4} - \frac{81 (U')^9}{8U^6} \right) + s^2 \left( -\frac{3G' (U')^2 U''}{4U^2} - \frac{3G' (U'')^2}{2U} + \frac{3G' (U')^4}{2U^3} \right) \\
- \frac{(G')^2 U'}{4U} + \frac{153 (U')^3 U''}{2U^4} - \frac{63 (U')^3 (U'')^2}{2U^3} + \frac{45 (U')^7}{4U^5} \right) + \left( \frac{G'U''}{2U} + \frac{G' (U')^2}{2U} - \frac{9 (U')^3 U''}{U^3} \right) \\
+ \frac{15U' (U'')^2}{2U^2} + \frac{9 (U')^5}{2U^4} \right) + \frac{G'}{2U} - \frac{9GU'}{2U^2} + \frac{15U'U''}{U^2} - \frac{15 (U')^3}{2U^3} - \frac{U''}{U} , \right) \] 

\[ \alpha_8 = \left( \frac{81 (U')^{12}}{32U^8} - \frac{81U'' (U')^{10}}{4U^7} - \frac{27G' (U')^9}{8U^6} + \frac{243 (U'')^2 (U')^8}{4U^6} + \frac{81G'U'' (U')^7}{4U^5} - \frac{81 (U'')^3 (U')^6}{U^5} \right) \\
+ \frac{27 (G')^2 (U')^6}{16U^4} - \frac{81G'U'' (U')^5}{4U^4} + \frac{81 (U'')^4 (U')^4}{4U^3} - \frac{(G')^3 (U')^3}{8U^2} \right) \right) s^4 + \left( \frac{9 (U')^{10}}{4U^7} + \frac{135U'' (U')^8}{8U^6} \right) \\
- \frac{3G' (U')^7}{2U^5} - \frac{405 (U'')^2 (U')^6}{4U^5} + \frac{57G'U'' (U')^5}{4U^4} - \frac{261 (U'')^3 (U')^4}{2U^4} + \frac{(G')^2 (U')^4}{4U^3} + \frac{39G' (U')^2 (U')^3}{U^3} \right) s^3 + \left( \frac{45 (U')^8}{4U^6} \right) \\
- \frac{27 (U')^4 (U')^2}{U^3} + \frac{23 (G')^2 U'' (U')^2}{U^2} - \frac{9G' (U')^3 U'}{4U} - \frac{3 (G')^2 (U'')^2}{4U} \right) s^2 + \left( \frac{45 (U')^8}{4U^6} \right) \\
- \frac{141U'' (U')^6}{4U^4} + \frac{29G' (U')^5}{4U^4} - \frac{3U'' (U')^5}{4U^3} - \frac{135 (U'')^2 (U')^4}{2U^3} - \frac{G' (U')^4}{8U^4} - \frac{67G'U'' (U')^3}{4U^3} \right) \\
- \frac{6U'' U'' (U')^3}{U^3} - \frac{39 (U'')^3 (U')^2}{2U^2} - \frac{(G')^2 (U')^2}{U^2} - \frac{G'U'' (U')^2}{2U^2} - \frac{G' U'' (U')^2}{U^2} + \frac{4G' (U')^2 U''}{U^2} \right) \\
+ \frac{9 (U'')^4}{2U^2} + \frac{(G')^2 (U'')^2}{2U} \right) s^2 + \left( \frac{25 (U')^6}{2U^5} - \frac{33U'' (U')^4}{2U^4} - \frac{17G' (U')^3}{2U^3} + \frac{9 (U'')^3}{2U^3} + \frac{133 (U')^2 (U')^2}{4U^3} \right)
+ \frac{G'' (U')^2}{2U^2} + \frac{7 G' U'' U'}{U^2} - \frac{U'''^2 U''}{U^2} - \frac{(U'')^3}{2U^2} + \left( \frac{G''}{4U} - \frac{G'''}{2U} \right) s + \frac{145 (U')^4}{16 U^4} + \frac{27 G (U')^2}{4 U^3} \\
+ \frac{7 (U'')^2}{2U^2} - \frac{3 G' U'}{U^2} + \frac{3 (U'')^2 U''}{U^3} + \frac{2 G'' U''}{U^2} - \frac{3 U' U'''}{2U^2} + \frac{5 G^2}{4 U^2} . \quad (B10)

**Appendix C: Einstein Frame Coefficients**

In this Appendix we present the explicit form of the coefficients in (20), expressed in terms of the Jordan frame parametrization. We start with the two most important remaining structures

\[ U_{1-\text{loop}}^E = V \left[ s^2 \left( -\frac{G' U'}{6U} - \frac{(U')^2 U''}{2U^2} + \frac{(U')^4}{4U^3} \right) + s \left( \frac{U''}{3U} - \frac{5 (U')^2}{6U^2} \right) - \frac{13}{3U} \right] + V' \left[ s^2 \left( \frac{G' U'}{12} + \frac{U' U''}{2U} - \frac{(U')^3}{4U^2} \right) + \frac{7 s U'}{12U} \right] - s \frac{V''}{6} , \quad (C1) \]

\[ G_{1-\text{loop}}^E = V' \left[ s^2 \left( \frac{3 G' U' U'}{4U} + \frac{9 (U')^2 U''}{2U^2} - \frac{9 (U')^4}{4U^3} \right) + \frac{2 s (U')^2}{U^2} - \frac{2}{U} \right] + V' \left[ s^3 \left( -\frac{6 G' (U')^2 U'}{U^2} \right) + \frac{3 G' (U')^4}{U^3} - \frac{(G')^2 U' U''}{2U} + \frac{18 (U')^5 U''}{U^4} - \frac{18 (U')^3 (U'')^2}{U^3} - \frac{9 (U')^7}{2U^5} \right] + s^2 \left( \frac{G'' U''}{4U} - \frac{19 G' (U')^2}{8U^2} - \frac{18 (U')^3 U''}{U^3} + \frac{3 U' U''}{2U^2} + \frac{13 (U')^5}{8U^4} + \frac{3 (U')^3 (U'')^2}{2U^2} \right) + s \left( \frac{G'}{U} + \frac{35 U' U''}{4U^2} - \frac{65 (U')^3}{8U^3} - \frac{6 U'}{U^2} \right) + V \left[ s^2 \left( \frac{12 G' (U')^3 U''}{U^3} - \frac{6 G' (U')^5}{U^4} + \frac{(G')^2 (U')^2}{U^2} - \frac{36 (U')^6 U''}{U^5} + \frac{36 (U')^4 (U'')^2}{U^4} + \frac{9 (U')^8}{U^6} \right) \right] + s^2 \left( -\frac{2 G'' (U')^2}{2U^2} - \frac{3 G' U'' U''}{2U^2} + \frac{13 G' (U')^3}{4U^3} + \frac{63 (U')^4 U''}{2U^4} - \frac{12 (U')^2 (U'')^2}{U^3} - \frac{51 (U')^6}{4U^5} - \frac{3 U''' (U')^3}{U^3} \right) + s \left( -\frac{2 G' U'}{U^2} - \frac{37 (U')^2 U''}{2U^3} + \frac{49 (U')^4}{4U^4} + \frac{U'' U'}{U^2} \right) + G \frac{U'}{U^2} + \frac{25 (U')^2}{2U^3} + \frac{4 U''}{U^2} \right] - \frac{V''}{2U} , \quad (C2) \]

The results for these structures coincide with those derived in [59]. The coefficients of the remaining structures are given by

\[ \alpha_4^E = -\frac{G}{3U} - \frac{5}{24} \frac{(U')^2}{U^2} - \frac{19}{12} \frac{U''}{U} , \quad (C6) \]

\[ \alpha_5^E = 0 , \quad (C7) \]

\[ \alpha_6^E = \frac{19}{8} \frac{(U')^2}{U^2} , \quad (C8) \]

\[ \alpha_7^E = \frac{G U'}{U^2} + \frac{5}{8} \frac{(U')^3}{U^3} + \frac{19}{4} \frac{U' U''}{U^2} , \quad (C9) \]
\[ a_8^E = \frac{5}{4} G^2 + \frac{7}{8} \frac{G \,(U')^2}{U^3} + \frac{331}{32} \frac{(U')^4}{U^4} + \frac{5}{8} \frac{G \, U''}{U^2} + \frac{19}{8} \frac{(U'')^2}{U^2}. \] 

Appendix D: Calculation of the Einstein Frame effective action

In this Appendix, we have dropped the convention to denote a field in the Einstein frame parametrization by a hat in order not to overload the notation. The result of the divergent part for the one-loop contributions to the effective action can be expressed in terms of the heat Kernel coefficients \( a_2 \) of the gauge-fixed action and \( a_2^{(Q)} \) of the ghost action.

For a general differential operator of the form

\[ \hat{F} := \Box + \hat{P} - \frac{1}{6} \hat{R} \hat{I}, \]  

acting with a gauge transformation on the gauge fixing condition \( \text{(D5)} \) gives rise to the ghost operator

\[ Q_\sigma^\rho := \frac{\delta (\chi_\xi)^\rho}{\delta \xi_\sigma} = \Box \delta_\sigma^\rho + \hat{R}_\sigma^\rho. \]  

Here, \( \xi_\mu \) is the vector field pointing in the direction of the Lie-dragging. The differential operator defined by \( \text{(D4)} \) has the formal structure

\[ F_{AB} = C_{AB} \Box + 2 \Gamma_{AB}^\sigma \nabla_\sigma + W_{AB}. \]  

where the different parts are ordered according to the number of derivatives acting on the perturbations. The individual parts can be read off from the result of \( \text{(D4)} \).
The second step consists in removing the part of (D17) and as well as to a modified commutator (bundle) curvature that is now defined with respect to the new covariant derivative $\hat{D}_\mu$.

\[
[D_\mu, \hat{D}_\nu] \phi = \hat{R}_{\mu \nu} \phi = (\hat{R}_{\mu \nu})^B_{\mu \nu B} \phi.
\]  

(D19)

With these modifications and by absorbing a factor of $1/6 \hat{R}_{\delta A B}$ into the definition of the potential part, we can bring the operator $[D_\delta]$ into its minimal form (D2). By using (D18), we can express this operator again in terms of the original background derivative $\nabla_\mu$ and we find the following result for the potential part and the commutator curvature

\[
\hat{P} = P_A^B = W_A^B + \frac{1}{6} \hat{R}_{A B} - (\nabla_\mu \Gamma_A^B) \mu \\
- \hat{g}_{\mu \nu} \Gamma^A_{C B} \Gamma^C_{\nu \mu} \\
+ 2 \hat{g}_{\mu \nu} \Gamma^A_{C B} \Gamma^C_{\nu \mu},
\]

(D20)

Here the commutator curvature $\hat{R}_{\mu \nu}^B$ with respect to the original background derivative $\nabla_\mu$ is defined by

\[
[\hat{\nabla}_\mu, \hat{\nabla}_\nu] \phi = \hat{R}^0_{\mu \nu} \phi .
\]

(D22)

Using (D9)–(D11), the explicit form of the connection part in (D17) is given by

\[
\hat{\Gamma}^\epsilon_{\alpha \beta} := \Gamma^\epsilon_{\alpha \beta} := (C^{-1})^A C^B \Gamma^E_{\alpha \beta} \\
\begin{pmatrix}
0 \\
\delta^\epsilon_{\alpha \beta} \hat{\nabla}_\epsilon \\
-\hat{G}^{\delta \mu \nu} \hat{\nabla}_\delta \hat{\nabla}_\mu \hat{\nabla}_\nu \\
-\frac{1}{2} \hat{g}^{\gamma \delta} \hat{V}' \\
-\frac{1}{6} \hat{R} + (\hat{\nabla}_\nu \hat{\nabla}_\delta \hat{\nabla}_\mu \hat{\nabla}_\lambda \hat{\nabla}_\sigma \\
\end{pmatrix},
\]

(D23)

and the potential part by

\[
\hat{W} := \hat{W}_A^B := (C^{-1})^A C^B W_C^B \\
\begin{pmatrix}
\hat{P}_{\alpha \beta}^\delta \\
-2 \hat{G}^{\delta \mu \nu} (\hat{\nabla}_\mu \hat{\nabla}_\nu \hat{\nabla}_\delta \hat{\nabla}_\mu \hat{\nabla}_\nu) \\
-\frac{1}{2} \hat{g}^{\gamma \delta} \hat{V}' \\
-\frac{1}{2} \hat{g}^{\gamma \delta} \hat{g}_{\delta \mu} \hat{V}' \\
\end{pmatrix},
\]

(D24)

Substituting (D23) and (D24) into (D20) and (D21), we find the explicit expressions
\[
\hat{R}_{\mu\nu} = R^A_{\, \mu\nu} = R^0_{\mu\nu} + 2 \nabla_{[\mu} \tilde{g}_{\nu]} \epsilon G^B_{\, \mu\nu} + 2 \tilde{g}_{[\mu} \tilde{g}_{\nu]} \rho \Gamma^C_{\, \epsilon} \Gamma^B_{\, \rho} = \\
\left( -2 \delta_{(\alpha} \hat{R}^{\beta)} - 2 \tilde{g}_{[\mu} \tilde{g}_{\nu]} \rho_{\beta \epsilon} \hat{G}^{\gamma \delta \eta} \varphi_{, \tau \varphi_{, \eta}} + 2 \nabla_{[\mu} \tilde{g}_{\nu]} \rho_{\epsilon} \hat{G}^{\delta \lambda \epsilon} \varphi_{, \lambda} \right). \quad \text{(D26)}
\]

Since a trace appears in the final expression \[\text{(D1)}\], we have to calculate the trace of the square of these quantities:

\[
\text{tr}(\hat{P} \hat{P}) = 3 \hat{R}_{\alpha\beta\gamma\delta} \hat{R}^{\alpha\beta\gamma\delta} - 6 \hat{R}_{\mu\nu} \hat{R}^{\mu\nu} + \frac{119}{36} R^2 \\
- \frac{5}{6} \hat{R} \hat{R}^{\mu\nu} + \frac{11}{4} \left( \hat{R}^{\mu\nu} \right)^2 + 10 \hat{V}^2 \\
- \frac{26}{3} \hat{V} \hat{V} - \frac{1}{3} \hat{V} \hat{V}^{\mu\nu} - 4 \hat{V}^{\mu\nu} + \hat{V}^{\mu\nu} \\
+ 2 \hat{V}^\mu \nabla_\mu \varphi + 2 \hat{V} (\varphi, \varphi^{\mu\nu}) - 2 \hat{V}^{\mu\nu} (\varphi, \varphi^{\mu\nu}) \\
+ \varphi^{\mu\nu} \varphi^{\mu\nu} + \frac{1}{2} (\Box \varphi)^2. \quad \text{(D27)}
\]

\[
\text{tr}(\hat{\mu}_{\mu\nu} \hat{\mu}^{\mu\nu}) = - 6 \hat{R}_{\alpha\beta\gamma\delta} \hat{R}^{\alpha\beta\gamma\delta} + \hat{R} (\varphi, \varphi^{\mu\nu}) \\
+ 2 \hat{R}_{\mu\nu} (\varphi^{\mu\nu}) - \frac{3}{2} \left( \varphi^{\mu\nu} \right)^2 \\
- 4 \hat{\mu}_{\mu\nu} \varphi^{\mu\nu} + \frac{1}{2} (\Box \varphi)^2. \quad \text{(D28)}
\]

The trace over the identity \( \text{tr}(\hat{1}) = 11 \) is composed of the 10 degrees of freedom contained in the metric field \( g_{\mu\nu} \) plus one degree of freedom from the scalar field \( \varphi \).

It remains to calculate the ghost contribution. Following the same steps as for the operator \( \text{(D8)} \), we obtain

\[
\text{tr}^{(Q)} (1) = 4, \quad \text{(D29)}
\]

\[
\text{tr}^{(Q)} (\hat{F}^2) = \hat{R}_{\alpha\beta} \hat{R}^{\alpha\beta} + \frac{4}{9} R^2, \quad \text{(D30)}
\]

\[
\text{tr}^{(Q)} (\hat{\mu}_{\mu\nu} \hat{\mu}^{\mu\nu}) = - \hat{R}_{\alpha\beta\gamma\delta} \hat{R}^{\alpha\beta\gamma\delta}. \quad \text{(D31)}
\]

Using \( \text{(D1)} \), inserting the results \( \text{(D27)} - \text{(D31)} \) in the formulas for the \( a_2 \)-coefficient \( \text{(D3)} \), performing an integration by parts of the structures \( \nabla_{[\mu} \nabla_{\nu]} (\nabla^{\mu} \nabla^{\nu} \varphi) \), \( V (\Box \varphi) \) and making use of the topological Gauss-Bonnet identity \( \text{(A9)} \), we obtain the final result for the effective action in the Einstein frame parametrization \( \text{(20)} \).
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