Spectrum of partial automorphisms of regular rooted tree

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Abstract
We study properties of eigenvalues of a matrix associated with a randomly chosen partial automorphism of a regular rooted tree. We show that asymptotically, as the level number goes to infinity, the fraction of non-zero eigenvalues converges to zero in probability.

Keywords Partial automorphism · Inverse semigroup · Eigenvalues · Delta measure · Random element · Uniform distribution

1 Introduction

We study spectral properties of the semigroup of partial automorphisms of a regular $n$-level rooted tree. Here by a partial automorphism we mean a root-preserving injective tree homomorphism defined on a connected subtree. This semigroup was studied, in particular, in [4, 5].

For the group of automorphisms of a regular rooted tree, a similar question was studied in [1]. Namely, it was shown that the spectral measure $\Theta_1^n$ of a randomly chosen element $\sigma$ of the $n$-fold wreath product of symmetric group $S_d$ converges weakly in probability to the normal Lebesgue measure $\lambda$ on the unit circle, i.e., for any trigonometric polynomial $f$,

$$\lim_{n \to \infty} \mathbb{P} \left\{ \int_C f(x) \Theta_1^n(dx) \neq \int_C f(x) \lambda(dx) \right\} = 0.$$ 

In contrast, for partial automorphisms of a binary rooted tree, in [3] it was shown that the uniform distribution $\Xi_n$ on eigenvalues of the action matrix converges weakly in probability to $\delta_0$ as $n \to \infty$, where $\delta_0$ is the delta-measure concentrated at 0. In this...
article we generalize this result for a regular rooted tree of any degree. Specifically, denote by \( B_n = \{ v^n_i \mid i = 1, \ldots, d^n \} \) the set of vertices of the \( n \)-th level of the \( n \)-level \( d \)-regular rooted tree. To a randomly chosen partial automorphism \( y \), assign the action matrix \( A_y = \left( 1_{y(v^n_i) = v^n_j} \right)_{i,j=1}^{d^n} \) (the \((i,j)\)th entry equals 1 if \( y \) maps \( v^n_i \) to \( v^n_j \) and 0 if \( y(v^n_i) \) is not equal to \( v^n_j \) or undefined). Let

\[ \Xi_n = \frac{1}{d^n} \sum_{k=1}^{d^n} \delta_{\lambda_k} \]

be the uniform distribution on eigenvalues of \( A_y \). We show that \( \Xi_n \) converges weakly in probability to \( \delta_0 \) as \( n \to \infty \), where \( \delta_0 \) is the delta-measure concentrated at 0.

The rest of the paper is organized as follows. Sect. 2 contains basic facts on partial wreath product of semigroups and its connection with the semigroup of partial automorphisms of a regular rooted tree. The main result is stated and proved in Sect. 3.

### 2 Preliminaries

For a set \( X = \{1, 2, \ldots, d\} \) consider the set \( \mathcal{S}_d \) of all partial bijections. This set forms an inverse semigroup under the natural composition law, namely, \( f \circ g : \text{dom}(f) \cap f^{-1} \text{dom}(g) \ni x \mapsto g(f(x)) \) for \( f, g \in \mathcal{S}_d \). A detailed description of this semigroup can be found in [2, Chapter 2].

Recall the definition of a partial wreath product of semigroups. Let \( S \) be an arbitrary semigroup. For functions \( f : X \supset \text{dom}(f) \to S \) and \( g : X \supset \text{dom}(g) \to S \) define the product \( fg \) as:

\[ \text{dom}(fg) = \text{dom}(f) \cap \text{dom}(g), \quad (fg)(x) = f(x)g(x) \text{ for all } x \in \text{dom}(fg). \]

For \( a \in \mathcal{S}_d, f : \text{dom}(f) \to S \), define \( f^a \) as:

\[ (f^a)(x) = f(x^a), \quad \text{dom}(f^a) = \{ x \in \text{dom}(a) ; x^a \in \text{dom}(f) \}, \]

where \( x^a = a(x) \).

**Definition 1** The partial wreath \( n \)-th power of the semigroup \( \mathcal{S}_d \) is defined recursively as a semigroup

\[ P_1 = \mathcal{S}_d, \]

\[ P_n = (P_{n-1}) \rtimes_p \mathcal{S}_d = \{ (f, a) \mid a \in \mathcal{S}_d, \ f : \text{dom}(a) \to P_{n-1} \}, \ n \geq 2, \]

with multiplication defined by

\[ (f, a) \cdot (g, b) = (fg^a, ab), \]
where \( \mathcal{P}_{n-1} \) is the partial wreath \((n - 1)\)-th power of semigroup \( \mathcal{I}_d \).

It is well-known that the partial wreath product of inverse semigroups is an inverse semigroup \([6, \text{Lemmas 2.22 and 4.6}]\). Denote by \( \mathcal{P}_n \) the \( n \)-th partial wreath power of \( \mathcal{I}_d \).

**Proposition 1** \([4]\) The number of elements in the semigroup \( \mathcal{P}_n \) is equal to

\[
N_n = \sum_{a \in \mathcal{I}_d} (N_{n-1})^{\text{rank}(a)} = S(N_{n-1}) = S(S \ldots (S(1)) \ldots),
\]

where \( S(x) = \sum_{i=1}^{d} \binom{d}{i}^2 i! t^i \).

**Remark 1** Let \( T \) be an \( n \)-level \( d \)-regular rooted tree. We define a partial automorphism of a tree \( T \) as a root-preserving isomorphism \( y : T_1 \to T_2 \) between its subtrees \( T_1 \) and \( T_2 \) containing root. Denote by \( \text{dom}(y) := T_1 \), \( \text{ran}(y) := T_2 \) the domain and the image of \( y \), respectively. Let \( \text{PAut} T \) be the set of all partial automorphisms of \( T \). Obviously, \( \text{PAut} T \) forms a semigroup under natural composition law. It was proved in \([4, \text{Theorem 1}]\) that the partial wreath power \( \mathcal{P}_n \) is isomorphic to \( \text{PAut} T \). From now on, we will call elements of the semigroup \( \mathcal{P}_n \) partial automorphisms.

### 3 Asymptotic behavior of a spectral measure of a regular rooted tree

Let \( T \) be an \( n \)-level \( d \)-regular rooted tree and \( \text{PAut} T \) its semigroup of partial automorphisms. We identify \( y \in \mathcal{P}_n \) with a partial automorphism from \( \text{PAut} T \). Let \( B_n \) be the set of vertices of the \( n \)-th level of \( T \). It is clear that \( |B_n| = d^n \).

Let us enumerate the vertices of \( B_n \) by positive integers from 1 to \( d^n \):

\[
B_n = \{ v_i^n \mid i = 1, \ldots, d^n \}.
\]

To a randomly chosen partial automorphism \( y \in \mathcal{P}_n \), we assign the matrix

\[
A_y = \left( \mathbf{1}_{y(v_i^n) = v_j^n} \right)_{i,j=1}^{d^n}.
\]

In other words, the \((i, j)\)th entry of \( A_y \) is equal to 1 if the transformation \( y \) maps \( i \) to \( j \), and 0, otherwise.

**Remark 2** In the automorphism group of a tree such a matrix describes completely the action of an automorphism. Unfortunately, for semigroups this is not the case, as shown in the example below. That is why we cannot use the technique developed by Evans in \([1]\). Also, the generalization of result from \([3]\) is not straightforward, despite the result is similar.
Example 1 Consider a binary tree and the partial automorphism \( y \in \mathcal{P}_2 \), which acts in the following way

![Diagram of a binary tree](image)

(where solid lines indicate the domain of \( y \)). Then the corresponding matrix for \( x \) is

\[
A_y = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

We can see from \( A_y \) that \( y \) maps \( v_1^1 \) to \( v_1^2 \). However, from \( A_y \) we cannot infer the action of \( y \) on \( v_2^1 \), in particular, whether \( v_2^1 \) belongs to the domain of \( y \).

Let \( \chi_y(\lambda) \) be the characteristic polynomial of \( A_y \) and \( \lambda_1^y, \ldots, \lambda_{d^n}^y \) its roots, respecting multiplicity. Let

\[
\Xi_n^y = \frac{1}{d^n} \sum_{k=1}^{d^n} \delta_{\lambda_k^y}
\]

denote the uniform distribution on eigenvalues of \( A_y \).

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete probability space, which is rich enough to carry all the random objects defined henceforth.

Our main result is the following theorem.

**Theorem 1** Let \( Y_n: \Omega \rightarrow \mathcal{P}_n \) be a randomly chosen element of \( \mathcal{P}_n \) with uniform distribution. For any function \( f \in C(D) \), where \( D = \{ z \in \mathbb{C} \mid |z| \leq 1 \} \) is a unit disc,

\[
\int_D f(x) \Xi_n^y(dx) \xrightarrow{\mathbb{P}} f(0), \quad n \rightarrow \infty.
\]  

(2)

In other words, \( \Xi_n^y \) converges weakly in probability to \( \delta_0 \) as \( n \rightarrow \infty \), where \( \delta_0 \) is the delta-measure concentrated at 0.
Proof For a partial automorphism \( y \in \mathcal{P}_n \), let \( \eta_n(y) = \mathbb{E}_n^y(0) \) be the fraction of zero eigenvalues of \( A_y \), and let \( \xi_n(y) = 1 - \eta_n(y) \) denote a fraction of non-zero eigenvalues. We have to prove that

\[
\eta_n(Y_n) \xrightarrow{p} 1, \quad n \to \infty,
\]
or, equivalently,

\[
\xi_n(Y_n) \xrightarrow{p} 0, \quad n \to \infty.
\]

Thanks to the Markov inequality, it is enough to show that

\[
\mathbb{E}\xi_n(Y_n) \to 0, \quad n \to \infty.
\]

For \( y \in \mathcal{P}_n \), denote by

\[
S(y) = \left\{ j : v^y_j \in \text{dom } y^m \text{ for all } m \geq 1 \right\}
\]
the set of indices of vertices of the bottom level, which “survive” under the action of all powers of \( y \), and define the ultimate rank of \( y \) by \( \text{rk}(y) = \# S(y) \). It can be shown similarly to [3, Lemma 1] that

\[
\xi_n(y) = \frac{\text{rk}(y)}{d^n},
\]

whence

\[
\mathbb{E}\xi_n(Y_n) = \frac{R_n}{d^n N_n},
\]

where \( R_n = \sum_{y \in \mathcal{P}_n} \text{rk}(y) \).

Also define \( \text{rank } y = \# (\text{dom } y \cap B_n) \).

Recalling that

\[
\mathcal{P}_n = \mathcal{P}_{n-1} \wr_p \mathcal{I}_d,
\]
we can identify \( y \in \mathcal{P}_n \) with an element \( a_y \in \mathcal{I}_d \) and a collection \( \left\{ y_x \in \mathcal{P}_{n-1}, x \in \text{dom } a \right\} \).

We can write

\[
R_n = \sum_{a \in \mathcal{I}_d} R_n(a), \quad \text{where } R_n(a) = \sum_{y \in \mathcal{P}_n : a_y = a} \text{rk}(y).
\]

A partial transformation \( a \in \mathcal{I}_d \) is a product of disjoint cycles \( (x_1 \ldots x_k) \) and chains \( [x_1 \ldots x_k] \), that is \( a(x_i) = x_{i+1}, 1 \leq i \leq k - 1 \) and \( x_k \notin \text{dom } a \).
If \( x \in X \) belongs to a chain, then no elements in \( B_n \) under \( x \) will belong to \( S(y) \) and contribute to \( \text{rk}(y) \).

If \( x = x_1 \) belongs to a cycle \((x_1, \ldots, x_k)\), then the number of elements surviving under \( x \) is

\[
\text{rk}(y_{x_1} \cdots y_{x_k}).
\]

As a result, if \( a \) contains \( r \) cycles \((x_{i_1}, \ldots, x_{i_{c_j}}), i = 1, \ldots, r\), then

\[
\text{rk} y = \sum_{i=1}^{r} c_i \text{rk} (y_{x_{i_1}} \cdots y_{x_{i_{c_j}}}).
\]

Therefore,

\[
R_n(a) = \sum_{i=1}^{r} c_i \sum_{y_1, \ldots, y_{c_j} \in \mathcal{P}_{n-1}} \text{rk} (y_1 \cdots y_{c_j}).
\] (3)

For convenience the rest of the proof is split into lemmas.

\[\Box\]

**Lemma 1** For \( n \geq 2 \)

\[
R_n(a) \leq R_{n-1} \left( \frac{N_{n-1}}{d} \right)^{\text{rank}(a)-1} \text{rank}(a).
\]

**Proof** The element \( y_1 \) can be decomposed into a product of idempotent \( e_{y_1} \) on the domain of \( y_1 \) and an automorphism \( \sigma_{y_1} \). Then

\[
\sum_{y_1, \ldots, y_{c_j} \in \mathcal{P}_{n-1}} \text{rk} (y_1 \cdots y_{c_j}) = \sum_{y_1, \ldots, y_{c_j} \in \mathcal{P}_{n-1}} \text{rk} (e_{y_1} \sigma_{y_1} y_2 \cdots y_{c_j}).
\]

Since \( \sigma_{y_1} \) is an automorphism, then \( \sigma_{y_1} y_2 \) is a bijection on \( \mathcal{P}_{n-1} \), so

\[
\sum_{y_1, \ldots, y_{c_j} \in \mathcal{P}_{n-1}} \text{rk} (e_{y_1} \sigma_{y_1} y_2 \cdots y_{c_j}) = \sum_{y_1, \ldots, y_{c_j} \in \mathcal{P}_{n-1}} \text{rk} (e_{y_1} y_2 \cdots y_{c_j}).
\]

Further \( S(e_{y_1} y_2 \cdots y_{c_j}) \subset \text{dom} y_1 \cap S(y_2 \cdots y_{c_j}) \), therefore,

\[
\sum_{y_1, \ldots, y_{c_j} \in \mathcal{P}_{n-1}} \text{rk} (e_{y_1} y_2 \cdots y_{c_j}) \leq \sum_{y_1, \ldots, y_{c_j} \in \mathcal{P}_{n-1}} \sum_{k=1}^{d^n} 1_{v^n_k \in \text{dom} y_1} 1_{v^n_k \in S(y_2 \cdots y_{c_j})}.
\]

By symmetry, the sum \( \sum_{y_1 \in \mathcal{P}_{n-1}} 1_{v^n_k \in \text{dom} y_1} \) does not depend on \( k \). Hence
\[
\sum_{y_1 \in \mathcal{P}_{n-1}} 1_{v_1^p \in \text{dom } y_1} = \frac{1}{d^n} \sum_{j=1}^{d^n} \sum_{y_1 \in \mathcal{P}_{n-1}} 1_{v_j^p \in \text{dom } y_1} = \frac{1}{d^n} \sum_{y \in \mathcal{P}_{n-1}} \text{rank}(y) =: \frac{1}{d^n} R'_{n-1}.
\]

Consequently,
\[
\sum_{y_1, \ldots, y_{c_i} \in \mathcal{P}_{n-1}} \text{rk} \left( y_1 \cdots y_{c_i} \right) \leq \frac{R'_{n-1}}{d^n} \sum_{y_2, \ldots, y_{c_i} \in \mathcal{P}_{n-1}} \text{rk} \left( y_2 \cdots y_{c_i} \right) \\
\leq \cdots \leq \left( \frac{R'_{n-1}}{d^n} \right)^{c_i-1} \sum_{y_{c_i} \in \mathcal{P}_{n-1}} \text{rk} \left( y_{c_i} \right) = \left( \frac{R'_{n-1}}{d^n} \right)^{c_i-1} R_{n-1}.
\]

Note that \( R'_{n-1} \leq dN_{n-1} \), since the rank of every element from \( \mathcal{P}_{n-1} \) is not greater than \( d^{n-1} \). Combining this with the above inequality and using Eq. (3), we get for \( n \geq 2 \)
\[
R_n(a) \leq \sum_{i=1}^{r} c_i \left( \frac{R'_{n-1}}{d^n} \right)^{c_i-1} R_{n-1} \leq R_{n-1} \sum_{i=1}^{r} c_i \left( \frac{d^{n-1} N_{n-1}}{d^n} \right)^{c_i-1} \\
= R_{n-1} \sum_{i=1}^{r} c_i \left( \frac{N_{n-1}}{d} \right)^{c_i-1} \leq R_{n-1} \sum_{i=1}^{r} c_i \left( \frac{N_{n-1}}{d} \right)^{\text{rank}(a)-1} \\
= R_{n-1} \left( \frac{N_{n-1}}{d} \right)^{\text{rank}(a)-1} \sum_{i=1}^{r} c_i \leq R_{n-1} \left( \frac{N_{n-1}}{d} \right)^{\text{rank}(a)-1} \text{rank}(a).
\]

**Lemma 2** For \( n \to \infty \)
\[
\frac{R_n}{d^n N_n} \leq \frac{R_{n-1}}{d^{n-1} N_{n-1}}
\]
with \( \limsup_{n \to \infty} r_n \leq 1/d \).

**Proof** Using Lemma 1, we get
\[
\frac{R_n}{d^n N_n} = \frac{1}{d^n N_n} \sum_{a \in \mathcal{J}_d} R_n(a) \leq \frac{1}{d^n N_n} \sum_{a \in \mathcal{J}_d} R_{n-1} \left( \frac{N_{n-1}}{d} \right)^{\text{rank}(a)-1} \text{rank}(a) \\
= \frac{R_{n-1}}{d^n N_n} \sum_{a \in \mathcal{J}_d} \left( \frac{N_{n-1}}{d} \right)^{\text{rank}(a)-1} \text{rank}(a) = r_n \frac{R_{n-1}}{d^{n-1} N_{n-1}},
\]

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where
\[ r_n = \frac{1}{dN_n} \sum_{a \in \mathcal{J}_d} \frac{N_{n-1}^{\text{rank}(a)}}{d^{\text{rank}(a)-1}} - \text{rank}(a) \leq \frac{1}{dN_n} \sum_{a \in \mathcal{J}_d} (N_{n-1})^{\text{rank}(a)} = \frac{1}{d} . \]

Here we have used (1) and the fact that for any \( a \in \mathcal{J}_d \), \( \text{rank}(a) \leq d^{\text{rank}(a)-1} \). \( \square \)

As a result,
\[ \frac{R_n}{d^n N_n} \to 0, \quad n \to \infty, \]
exponentially fast. The proof of Theorem 1 is now immediate.

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