Geometric Responses on A Curved Surface Embedded in 4D Euclidean Space

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We consider a 2D curved surface embedded in 4D Euclidean space, and give the effective Hamiltonian describing noninteracting electrons with orbital spin by introducing a confining potential. We find that the geometry of embedding in 4D space, a nonzero torsion, constructs the gauge structure of effective dynamics, and the SO(2) symmetry of confining potential effectively converts an extrinsic orbital angular momentum into an intrinsic orbital spin. We further find that the geometric response of orbital spin contributes quantum Hall effect, the adiabatic response to torsion deformation provides Hall viscosity presented as a simultaneous occurrence of multiple Hall conductances.

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Introduction.—The quantum Hall effect (QHE) was observed in two-dimensional (2D) systems at low temperature and in strong magnetic field\(^[1, 2]\). For quantum Hall (QH) states, the geometrical and topological features are standing-long projects. A prominent feature is the quantized Hall conductance, which results from the topological characteristics of QH states determined by magnetic singularities\(^[3]\), that is seen as a transversal response to the electromagnetic field. An important progress of the QHE in recent years is the Hall viscosity, which was originally defined by the metric perturbation\(^[4]\), that is seen as an adiabatic response to gravitational anomaly\(^[5, 6]\) or framing anomaly\(^[7]\), and was measured in graphene\(^[8]\). Alternatively, the Hall viscosity is also created by an inhomogenous electromagnetic field\(^[9]\), in which the representation of magnetic singularities is local. To describe those QH phenomena, topological invariants are employed because Landau’s symmetry breaking theory is no longer valid. In the light of topological structure, a nonzero QH conductance was realized by parity anomaly in a 2D condensed-matter lattice model\(^[10, 11]\), or by the topological charge of defects\(^[12]\). In the absence of an external magnetic field, the quantum anomalous Hall (QAH) state\(^[13, 14]\) can show more geometrical structures, thus the associated geometric responses can be evidently represented by orbital spin\(^[15]\).

The second can be also accomplished in magnetic topological insulator film\(^[16, 17]\). In this letter, the Hall conductance is seen as a geometric response to the torsion deformation different from the usual magnetic flux deformation.

A 2D curved surface \(\mathcal{M}^2\) embedded in three-dimensional (3D) Euclidean space \(\mathbb{R}^3\) is characterized by curvature, while embedded in \(\mathbb{R}^4\) by curvature and torsion. In a particular case\(^[18, 19]\), the torsion plays the role of a gauge connection\(^[20]\). A pure mathematical connection is unphysical and immeasurable\(^[21, 22]\), but it becomes measurable and physical in the presence of orbital spin\(^[18]\). The response of orbital spin to torsion leads many universal features to be observables for noninteracting electrons confined to \(\mathcal{M}^2\). One of the most representative features is that the Hall conductance can be seen in an adiabatic response to torsion. The result provides a way to understand the experimentally researching of 4D QH physics\(^[23, 24]\).

In the present letter, we consider noninteracting electrons confined to a curved surface embedded in \(\mathbb{R}^4\). The system is tried to solve two problems. One is to investigate the contribution of the motion defined in high-dimensional space to the effective theory confined on \(\mathcal{M}^2\). The other is to generate an inhomogeneous geometric gauge field, in which the quantized Hall conductance and the Hall viscosity are induced. The first problem can shed light on having a glimpse of the relationship between the topological structure defined in high-dimensional space and the gauge structure of the effective Hamiltonian defined in the real low-dimensional space. The second can provide an useful tool to probe the subtle geometrical features of QH states, such as Hall viscosity, and display the associated geometric singularity.

Effective Hamiltonian. — For the curved surface \(\mathcal{M}^2\) embedded in 4D Euclidean space \(\mathbb{R}^4\) (sketched in Fig. 1), we use two tangent coordinate variables \(r\) and \(s\) to describe \(\mathcal{M}^2\), and employ two coordinate variables \(q_3\) and \(q_4\) to construct the subspace normal to \(\mathcal{M}^2\) with an adapted frame. The four coordinate variables are orthogonal each other. In the original subspace consisting of points close to and on \(\mathcal{M}^2\), an free electron can be described by the Hamiltonian, that is

\[
H = -\frac{\hbar^2}{2m^*} \frac{1}{\sqrt{G}} \partial_i \sqrt{G} G^{ij} \partial_j,
\]
which can well approximately describe noninteracting electrons. Here the metric tensor $G_{ij}$ does not only depend on the tangent variables $r$ and $s$, but also on the normal ones $q_3$ and $q_4$. In comparison with the reduced metric tensor $g_{ab}$ defined on $\mathbb{M}^2$, which just depends on $r$ and $s$, it is strikingly different that the contents of $G_{ij}$ are enriched by the embedding of $\mathbb{M}^2$ in $\mathbb{R}^4$. According to the Hamiltonian (1), the effective Hamiltonian describing electron confined on $\mathbb{M}^2$ can be given in the thin-layer quantization formalism [25, 26] with $q_3$ and $q_4$ being reduced. In order to discover more geometric effects, the reduction of two normal coordinates can be accomplished by confining one coordinate variable. In the light of the geometric formula [18, 19], the confining potential can be selected as a radial harmonic potential, $V_c = m^* \omega_0^2 \rho^2/2$, where polar coordinates ($\rho, \theta$) are used to replace $q_3$ and $q_4$. The effective Hamiltonian reads

$$H_{\text{eff}} = -\frac{\hbar^2}{2m^*}[(\partial_\rho + \frac{e_l}{\hbar} A_s)^2 + \partial_\theta^2] + V_g,$$  

where $e_l = \hbar l$ is a quantized orbital angular momentum, or a topological charge, describing the winding number of electron moving around the s-axis in the normal space $\mathbb{N}^2$ spanned by ($\rho, \theta$). The emergence of the topological charge $e_l$ in Eq. (2) results from the extrinsic orbital angular momentum defined in $\mathbb{N}^2$, is eventually given by the reduced not constrained $\theta$ dimension. Converting an extrinsic variable into intrinsic is entirely determined by the SO(2) symmetry of $V_c$. The orbital spin preserved in the effective dynamics leads to the wavefunction, describing the electron confined on $\mathbb{M}^2$, having particular topological structure. In other words, the intrinsic orbital spin can come from high-dimensional degrees of freedom, the topological property of wavefunction can be also from high-dimensional space.

In Eq. (2), $V_g$ is the well-known geometric potential [26] as below

$$V_g = -\frac{\hbar^2}{4m^* \tau^2},$$

wherein $\tau$ is locally the torsion of $\mathbb{M}^2$. By comparing to the result given in Refs. [27, 28], $V_g$ is a locally attractive scalar potential, without the repulsive component. The geometric potential $V_g$ and geometric gauge potential $A_s$ are both important ingredients to confirm the consistency of $H_{\text{eff}}$. The minimally coupling of $e_l$ and $A_s$ in $H_{\text{eff}}$ (2) is the key result of the present letter. The derivations of $H_{\text{eff}}$ can be found in Supplemental Materials.

Gauge Structure. — In the effective Hamiltonian (2), $A_s$ plays a role of gauge potential that determines the U(1) gauge structure. The strength and representation content of $A_s$ are defined by a spin connection, that is the normal fundamental form of $\mathbb{M}^2$ with the definition $n_3 \cdot \partial_s n_4 = \tau/\eta$, where $\tau$ denotes the twisted angle on the plane spanned by $n_3$ and $n_4$ around the s-axis with increasing a unit length in the z-direction. Due to $n_3$ and $n_4$ normal to $\mathbb{M}^2$, $A_s$ results from the embedding of $\mathbb{M}^2$ in $\mathbb{R}^4$, and it can be taken as a local rotation of the normal space $\mathbb{N}^2$ around a point of $\mathbb{M}^2$ around the s-axis. The axis point of rotation does not belong to $\mathbb{N}^2$, the rotation is therefore nontrivial singularity, the normal space $\mathbb{N}^2$ has a particular topological structure. Under an infinitesimal rotation around the s-axis, $R = e^{-i\theta}$, the wave function $|\psi\rangle$ and $A_s$ transform as

$$|\psi\rangle \rightarrow |\psi'\rangle = R|\psi\rangle,$$

$$A_s \rightarrow A'_s = A_s + \partial_\theta \theta.$$  

These transformations show that the geometry intrinsic to $\mathbb{M}^2$ embedded in $\mathbb{R}^4$ constructs the U(1) gauge structure of $H_{\text{eff}}$.

Analogous to the electromagnetic field, the orbital spin $\hbar l$ plays the role of electric charge, also known as a topological charge that describes the winding number of electron moving around a singular point of $\mathbb{N}^2$, the singular point belongs to $\mathbb{M}^2$, not $\mathbb{N}^2$. Specifically, the orbital motion is originally defined in $\mathbb{N}^2$, and $A_s$ describes the rotation of the normal space $\mathbb{N}^2$ around a point of $\mathbb{M}^2$ with increasing $s$. It is straightforward to couple $A_s$ and $\hbar l$ likely the minimally coupling of the electromagnetic field. In the case, the original extrinsic variable $\hbar l$ is converted into an intrinsic one by introducing the confining potential. The SO(2) symmetries of confining potential are preserved in the wavefunction that inherits the topological properties of $\mathbb{N}^2$.

As a consequence, the geometric gauge potential is provided by the embedding of $\mathbb{M}^2$ in $\mathbb{R}^4$, and the orbital spin is defined by the SO(2) symmetries of confining potential. Further, they are from the high-dimensional space, that is the dependence of $q_3$, $q_4$ and their associated derivatives. As a bold assumption, the geometry intrinsic to $\mathbb{M}^2$ can be employed to investigate the quantum physical phenomena in $\mathbb{R}^4$ [23, 24].
**Geometric Magnetic Field.**— With the nonvanishing geometric gauge potential $A_s$, the effective magnetic field can be given by

$$B_n = \frac{-2 \hbar}{e} \frac{w^3 r}{(1 + w^2 r^2)^2},$$  \hspace{1cm} (5)

where the subscript $n$ denotes the direction normal to $M^2$ in the usual three-dimensional Euclidean space $\mathbb{R}^3$. The geometric magnetic field $B_n$ is local and nonuniform, a function depends on $r$, the distance of the point on $M^2$ to the $z$-axis. As shown in Fig. 2, at $r = \frac{\sqrt{3}}{3}d$ the geometric magnetic field has minimum value, at which the electrons with nonvanishing orbital spin will feel the strongest effective Lorentz force provided by $B_n$, and that the electrons with different orbital spins can be quickly separated. For positive orbital spin, the electron tends to concentrate to the outer edge of $M^2$, for negative one the electron tends to the inner edge. These results can be seen as the response of the orbit spin of electron to torsion, the effective Lorentz force. The consequence is reminiscent of the quantum Hall effect. There is a striking feature that the quantum Hall effect is generated by the geometric properties of $M^2$ embedded in $\mathbb{R}^4$ not external magnetic field. It is worth mentioning that the geometric magnetic field can have considerable strength when $d$ takes a suitable value, such as $B_{n_{\text{max}}} \approx -10T$ for $d = 10nm$.

![Figure 2](image.png)

**FIG. 2.** The geometric magnetic field $B_n$ versus $r$. The scale unit for $r$-axis is the pitch $d$. The scale unit of the geometric magnetic field is $2\frac{\hbar}{e}$ with $d = 1$.

**Geometric Phase.**— In the presence of the geometric gauge potential $A_s$, an electron with nonzero orbital spin confined to $M^2$ moving along the $s$-axis will gain an additional geometrical phase as

$$\Delta \phi_g = l \tau \int_{z_0}^{z_e} dz,$$  \hspace{1cm} (6)

where we have considered $ds = \eta dz$, $z_0$ and $z_e$ denote the start and end values of integral, respectively. Obviously, the geometrical phase has $\Delta \phi_g = l \tau$ for an unit length of $z$-axis. Its sign depends on the sign of the orbital spin, and it will decrease when the distance from the $z$-axis increases. As a subsequence, the geometrical phase can be adjusted by designing the geometry and size of $M^2$.

**Quantum Hall Effect.**— In the geometric magnetic field $B_n$, an electron with orbital spin feels an effective Lorentz force, $\mathbf{F} = e_s v_s B_n$, where $v_s$ is the $s$ component of velocity. The force direction is $e_r$, for a positive orbital spin, $-e_r$, for negative, and the force strength is proportional to the absolute value of orbital spin. Subsequently, the electrons with positive orbital spin accumulate to the outer side edge of $M^2$, those with negative orbital spin close to the inner side edge. The redistribution of electrons generates an inhomogenous electric field $E_r$. Under the situation, the electrons will feel Coulomb forces. For final balance, $e \mathcal{E}_r = e_n v_s B_n$, where $\mathcal{E}_r$ is the electric field induced by the geometric magnetic field, and its fine structure is constructed by the orbital spin $|29|$. Due to that the $s$ component of current can be $j_s = e_n v_s$, and $j_s = \sigma_{sr} \mathcal{E}_r$, the Hall conductivity $|9|$ would be

$$\sigma_{sr} = \frac{e}{\Phi_0} \nu,$$  \hspace{1cm} (7)

where $\Phi_0 = h/e$ is the effective flux quantum, that is the effective magnetic flux contained within the area $2\pi \ell_B^2$, $\nu = n2\pi \ell_B^2$ denotes a filling factor, describing the electrons coupled to one flux quantum $\Phi_0$, wherein $n$ is the electron density, $\ell_B = \sqrt{1/|\Phi_0|}$ stands for an effective magnetic length.

In terms of Eq. (2), for fixed $z$ the periodicity of the helical curve determines that the $s$ component of kinetic energy is quantized as $k_s = (mw + l\tau)/\eta$. Equivalently, the quantum motion of $r$-dimension has an effective background potential, that is

$$U(r) = -\frac{\hbar^2}{2m^*} \frac{w^2}{\eta^2} \left( \frac{1}{4} - \frac{m + \frac{l}{\sqrt{2\eta}}}{\frac{1}{2}} \right)^2,$$  \hspace{1cm} (8)

where $m$ is a new quantum number for the periodicity of $z$. This potential is a sum of two contributions: an attractive part and a repulsive one. For $|m + \frac{l}{\sqrt{2\eta}}| < \frac{1}{2}$, the action of $U(r)$ qualifies it as an antizentrifugal potential to concentrate electrons toward the inner rim. As $|m + \frac{l}{\sqrt{2\eta}}| > \frac{1}{2}$, the behavior of $U(r)$ qualifies as a centrifugal potential to push electrons toward the outer rim. This finite size effect and the above geometric effects together determine the emergence of QH phenomenon. Specically, in the present letter the finite size effect and the quantization of $z$-component kinetic energy have to be considered. The above discussion is given in Supplemental Materials.

Compared with the usual QH effect, the present QH effect is induced by torsion, not external electromagnetic field. The topological properties of QH states are given by the orbital spin preserved in $B_{\text{eff}}$ by coupling with torsion. Those nontrivial properties are from the bound
states of ρ-dimension, they are nontrivial irreducible fundamental representations of SO(2) in ℍ². In the presence of torsion, a geometric gauge potential, the Landau-like energy level splitting are exhibited. The associated energy levels increase as the torsion increases. In the case without nonzero torsion, the states below the Fermi energy \( E_F \) are fully occupied at zero temperature. By adiabatically increasing the torsion, the energy level for the maximum orbital spin will first reach \( E \), a channel for Hall current appears. If the torsion continues to increase, the energy for the second orbital spin will reach \( E_F \), a new channel would appear.

Do a metaphor, the orbital spin provides channels for electron moving in \( s \) direction without resistance, the geometric property, torsion, plays the role of key to open the channels. The orbital spin transport, driven by an adiabatic change of the torsion, is a fundamental probe of QH states with topological characterization, complementary to the more familiar electromagnetic response. The orbital spin highlights the topological properties of the QH states and serves as an ideal setting to probe their geometric properties. And the orbital spin provides a new way to investigate the geometry of topological states, to encode the external geometric characterization of states in the intrinsic degree of freedom. In other words, the intrinsic orbital spin provides a useful tool to probe the geometric responses and to find more subtle features of QH states.

**Hall Viscosity.**—A Hall viscosity is determined by the nondissipative part of the stress response to metric perturbation [4], and it can be also created by an inhomogeneous electric field [9], or special boundary conditions [30]. The metric perturbation leads to the quantum geometry of guiding center [31] that is related to the Hall viscosity. In the present letter, the torsion of \( M^2 \) has the normal fundamental form that is antisymmetric in the metric tensor \( G_{ij} \), and that describes the nondissipative part of the stress response in the discussed system. And the torsion-induced gauge potential \( A_μ \), an Abelian SO(2) orbit connection, thus plays the role of the gravitational Abelian Chern-Simons action [15] to contribute a Hall viscosity. Specifically, the dependence of torsion on \( r \) provides a nonvanishing \( r \)-gradient of stress that deforms the shape of cyclotron orbit defined by both the geometric magnetic field \( B_μ \), and the quantum number of orbital spin \( l \) together. This deformation provides that the position of guiding-center satisfies the non-commutative relationship \( [X_0, X_a] = i\epsilon_{ab}\ell_B^2 \), here \( X_a \) denotes the position coordinate of the guiding-center, and \( \ell_B \) is the effective magnetic length [31]. In terms of the charge conservation law, the Hall viscosity can be obtained as

\[
\eta^A = \frac{l}{2\pi\ell_B^2}. \tag{9}
\]

This result is in nice agreement with that given by Read [32] as \( \bar{n} = \frac{1}{\pi\ell_B^2} \) denotes the electronic density. The associated derivations is shown in Supplemental Materials.

![FIG. 3. With a certain twisted coefficient, two Landau-like energies cross with \( E_F \) at different \( r \) positions.](image)

In the helical submanifold, the torsion response contributes the quantized Hall conductance like the electromagnetic response, and the dependence of \( r \) contributes the Hall viscosity. In the inhomogeneous geometric magnetic field \( B_0 \), the Landau-like level decreases as \( r \) increases as described in Fig. 3. Therefore, the two energy levels of \( l = 1, 2 \) can cross with \( E_F \) at different positions of \( r \) simultaneously. In other words, there are two channels for a same Hall conductance, that is sketched in Fig. 4, as long as the electric field induced by the inhomogeneous distribution of electrons is sufficient strong to fully balance the effective magnetic field \( B_0 \) and the effective background potential \( U(\mathbf{r}) \). After all, the simultaneous occurrence of multiple Hall conductances results from Hall viscosity, that is the \( r \)-dependence of torsion. In other words, the simultaneous occurrence phenomena is a manifestation of Hall viscosity.

![FIG. 4. The two channels of \( l = 1, 2 \) for Hall conductance.](image)

More specifically, the contribution of the two channels for Hall current has slightly difference, the difference shows more the details of Hall viscosity. The inner
channel can contributes a slightly larger current, and the outer channel contributes a slightly smaller current. The different current between two channels can be expressed as
\[
\delta I = I_0 \frac{1}{\sqrt{1 + w^2 r_1^2}} - \frac{1}{\sqrt{1 + w^2 r_2^2}},
\]
where \( I_0 \) is a constant describing a current contributed by the electrons with a fixed orbital spin in a flat surface, \( r_{1,2} \) denote the position of two channels of \( l = 1, 2 \), respectively.

**Conclusions and Discussions.**—The embedding of 2D curved surface in 4D Euclidean space can provide a non-vanishing torsion, a SO(2) spin connection, as a geometric gauge potential in the effective dynamics. The minimally coupling is accomplished via a topological charge, that is an orbital spin originally defined in \( R^4 \). The presence of the orbital spin in the effective Hamiltonian determines the topological properties of the states describing electron confined to \( M^2 \). The topological properties are presented by a torsion as QH conductance. The \( r \) dependence of torsion contributes a Hall viscosity, and displays the simultaneous occurrence of multiple channels for a same Hall conductance. For QH states of systems with SO(2) invariance, the Hall viscosity can be, in principle, determined from geometric response alone. Hence, by measuring the geometric response of the system to inhomogeneous geometric magnetic field, one can determine the Hall viscosity. In other words, the orbital spin and torsion can be used to construct the topological structure of QH states, and to study the QH viscosity. As a conclusion, the geometry of a 2D curved surface embedded in high-dimensional space can provide an effective gauge field, and an intrinsic dimension can come from a high-dimensional space. These results provide a way to study the quantum physics defined in \( R^4 \). In this letter, the spin of electron is not considered. The electron with spin and orbital spin confined to \( M^2 \) will be discussed in another paper.

Experimentally, the torsion can be provided by disclination [33], screw dislocation [34], or dispiration [35], and the electrons with orbital spin can be generated by spiral phase plate [36], or by a versatile holographic reconstruction technique [37]. The geometry of 2D curved surface can be used to manipulate the phase singularity of electron with orbital spin [38, 39].

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