Taxonomy of Clifford $\mathbf{Cl}_{3,0}$ subgroups: Choir and band groups

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Abstract. We list the subgroups of the basis set of $\mathbf{Cl}_{3,0}$ and classify them according to three criteria for construction of universal Clifford algebras: (1) each generator squares to $\pm 1$, (2) the generators within the group anti-commute, and (3) the order of the resulting group is $2^{n+1}$, where $n$ is the number of nontrivial generators. Obedient groups we call choirs; disobedient groups, bands. We classify choirs by modes and bands by rhythms, based on canonical equality. Each band generator has a transposition (number of other generators it commutes with). The band’s transposition signature is the band’s chord. The sum of transpositions divided by twice the number of generator pair combinations is the band’s beat. The band’s order deviation is the band’s disorder. For $n \leq 3$, we show that the $\mathbf{Cl}_{3,0}$ basis set has 21 non-isomorphic subgroups consisting of 9 choirs and 12 bands.

1 Introduction

a. Generators of Clifford Algebra. Clifford (geometric) algebra is usually presented by taking a set of $n$ unit vectors,

$$\mathbf{B}_{n,1} = \{e_1, e_2, \ldots, e_n\},$$

and defining the vector elements to be orthonormal via the following relations:\[1\]:

$$e_j^2 = \begin{cases} +1, & 1 \leq j \leq p, \\ -1, & p + 1 \leq j \leq (p + q = n), \end{cases}$$

$$e_j e_k = -e_k e_j,$$

where $j$ and $k$ are nonnegative integers. The set of all possible products of these $n$ vectors form a group, and from this group we form a group algebra called Clifford (Geometric) Algebra $\mathbf{Cl}_{p,q}$.

In the language of combinatorial group theory\[2\], we say that $\mathbf{B}_{n,1}$ is the set of generators and Eqs. (2) to (4) are the defining relations. The word generator suggests a mathematical machinery for producing a group, as an electric generator produces electricity. But the defining relations do not provide this machinery as electromagnetic theory do not make an electromagnet. What we need is a mathematical machinery for listing down all the elements of the geometric group, the group consisting of all basis elements in geometric algebra, with their + and − signs included.

To answer this need, we shall use a set algebra formalism with two operations: multiplication (juxtaposition) and set union ($\cup$). This algebra is already known, but remains unused for listing group elements.

In group theory, the product of two subgroups $\mathbf{F}$ and $\mathbf{H}$ of group $\mathbf{G}$ is defined as

$$\mathbf{FH} = \{ f^h | \text{for all } f \in \mathbf{F}, h \in \mathbf{H} \}. \quad (5)$$

Scherphuis\[3\] commented that this definition need not be restricted to subgroups: $\mathbf{F}$ and $\mathbf{H}$ need only be subsets of $\mathbf{G}$. If we adopt this view, then we see, for example, that the distributivity property holds:

$$H_1 (H_2 \cup H_3) = (H_1 H_2) \cup (H_1 H_3), \quad (6)$$

$$H_2 \cup (H_3 H_1) = (H_2 H_1) \cup (H_3 H_1), \quad (7)$$

where $H_1, H_2, H_3 \in \mathbf{G}$. Thus, it is indeed possible to create a set algebra over set union.

Johnson\[4\] may have some idea of this set multiplication when he wrote the Clifford algebraic set equation

$$\{1, e_2\} \otimes \{1, e_1\} = \{1, e_1, e_2, e_1 e_2\}. \quad (8)$$

But his $\otimes$ operator appears to mean differently: find all the linearly independent products of the elements of the sets $\{1, e_1\}$ and $\{1, e_2\}$. Clearly, this operator does not always satisfy Eq. (8) (it would if we choose $e_2 e_1$ instead of $e_1 e_2$, but this decision is arbitrary).

Using the set algebra we propose, we can show, for example, that the set $\mathbf{C}_{2,0}$ of all basis elements in $\mathbf{Cl}_{2,0}$
can be expressed in two ways:

\[ C_{2,0} = \{1, -1\}\{1, e_1\}\{1, e_2\}, \quad (9) \]
\[ = \{1, -1\}\{1, e_1\}\{1, i\}, \quad (10) \]

where \( i = e_1e_2 \). Notice the similarity of Eq. (9) with Eq. (8), save for the presence of \( \{1, -1\} \), which prevents us from worrying about the order of the factors. The set product form in Eq. (9) or (10) is the mathematical machinery we need for listing the elements of a geometric group like \( C_{2,0} \). From this group we construct a geometric group algebra.

b. Signature of Clifford Algebra. But what geometric algebra do we choose from all the possible dialects of \( Cl_{p,q} \)? Is \( e_j^2 = +1 \) or \(-1 \) or both?

The axiom \( e_j^2 = -1 \) is a good choice if we wish to illustrate the natural continuity of Clifford algebra \( Cl_{0,n} \) with Hamilton’s quaternions. In fact, this is what Hamilton chose in 1878. But in 1882, he changed his mind and wrote \( e_j^2 = 1 \). What made Hamilton change his mind, we can only guess. Perhaps, it is the notion that the square of a vector must be its magnitude squared; quaternions fail this test. This is what Cayley in 1871 pointed out to the quaternion defender Tait. And this is also what Gibbs in 1870’s thought before his word war with Tait. So in deference to the opinions of Clifford, Cayley, and Gibbs, we shall adopt the axiom \( e_j^2 = 1 \). Thus, the algebra that we shall use is not \( Cl_{0,n} \) but \( Cl_{n,0} \).

By restricting ourselves to positive signedatured Clifford algebras, we claim that we can still obtain other Clifford algebras. In \( Cl_{3,0} \), for example, we can construct basis sets isomorphic to that of \( Cl_{1,2}, Cl_{2,0}, Cl_{1,1}, Cl_{0,2}, Cl_{1,0}, Cl_{0,1}, Cl_{0,0} \). These basis sets we shall call choirs. Basis sets that do not construct Clifford algebras we shall call bands. We shall see that from the basis set of \( Cl_{3,0} \), we can form 21 non-isomorphic subgroups consistings of 9 choirs and 12 bands.

c. Outline. We shall divide the body of the paper into five sections. The first section is Introduction. In the second section, we shall discuss vector products via logic gate operations and Walsh functions, and use these to prove the group properties of the basis set of Clifford algebra \( Cl_{n,0} \). In the third section, we shall list the axioms of set algebra with two operations: juxtaposition multiplication and set union. We shall define and investigate geometric groups, which are similar to Clifford basis groups, except that their generators are allowed to commute and their squares are allowed to be negative. In the fourth section, we shall list the geometric subgroups of the basis set of \( Cl_{3,0} \) and classify them into choirs and bands. We shall arrange choirs by modes and bands by rhythms. We shall distinguish bands by their Clifford signature, (dis)order of elements, and number of commuting generators. The last section is Conclusions.

2 Clifford Basis Group

2.1 Axioms

Let us define

\[ G_{n,1} = \{\pm e_1, \pm e_2, \ldots, \pm e_j, \ldots, \pm e_n\}. \quad (11) \]

The quantity \( e_j \) where \( j \in \{1, 2, \ldots, n\} \) is called a vector or ray and \(-e_j \) is the vector opposite to it.

Let us assume that the six vectors in the set \( G_{n,1} \) satisfy the three axioms for the geometric product (denoted by juxtaposition multiplication):

a. Associativity. For all \( j, k, \ell \in \{1, 2, 3\} \),

\[ (e_j e_k)e_\ell = e_j(e_k e_\ell). \quad (12) \]

That is, regrouping the factors do not affect the product.

b. Orthogonality. For \( j, k \in \{1, 2, 3\} \),

\[ e_j e_k = -e_k e_j. \quad (13) \]

Geometrically, \( e_j e_k \) is the oriented area defined by the vectors (rays) \( e_j \) and \( e_k \) connected head-to-tail according to the order of the factors. Thus, \( e_j e_k \) is the same area but oriented opposite. This explains the negative sign.

c. Norm. For \( j \in \{1, 2, 3\} \),

\[ e_j^2 = 1. \quad (14) \]

Geometrically, Eq. (14) states that the length of \( e_j \) is one unit.

2.2 Theorems

From these three axioms we can prove three theorems:

a. Power. The raising of a vector \( e_j \) for \( j \in \{1, 2, 3\} \) to a nonnegative integer power \( b_j \) results to either the real number 1 or the vector itself:

\[ e_j^{b_j} = 1, \quad b_j \equiv 0 \mod 2, \]
\[ = e_j, \quad b_j \equiv 1 \mod 2. \]

That is, if the power is even, \( e_j^{b_j} = 1 \); if odd, \( e_j^{b_j} = e_j \).

One corollary to this theorem is that the inverse of \( e_j^{b_j} \) is itself:

\[ (e_j^{b_j})^{-1} = e_j^{-b_j} = e_j^{b_j}. \quad (17) \]

That is,

\[ (e_j^{b_j})^{-1} e_j^{b_j} = e_j^{b_j} e_j^{-b_j} e_j^{b_j} = e_j^{2b_j} = 1, \quad (18) \]

because \( 2b_j \equiv 0 \mod 2 \).

b. XOR. The product of \( e_j^{b_j} \) and \( e_j^{b_j} \) is

\[ e_j^{b_j} e_j^{b_j} = e_j^{b_j + b_j}. \quad (19) \]
There are four possibilities:

\[ e_j^{b_j} e_k^{b_k} = e_j^{b_j} e_k^{b_k}, \quad b_j = 0, b_k = 0; \quad (20) \]

\[ e_k^{b_k} e_j^{b_j}, \quad b_j = 0, b_k = 1; \quad (21) \]

\[ e_k^{b_k} e_j^{b_j}, \quad b_k = 1, b_j = 0; \quad (22) \]

\[ -e_k^{b_k} e_j^{b_j}, \quad b_k = 1, b_j = 1. \quad (23) \]

Thus, in terms of the XOR gate, we write

\[ e_j^{b_j} e_k^{b_k} = e_j^{b_j} \text{XOR} e_k^{b_k}. \quad (24) \]

**2.3 Group Properties**

Let \( G_n \) be a set whose elements are of the form

\[ \hat{g} = s_1 e_1^{b_1} e_2^{b_2} \ldots e_k^{b_k} \ldots e_n^{b_n}, \quad (30) \]

where \( s_\hat{g} = \pm 1 \) is the sign of \( \hat{g} \) and the bit power \( b_k \in \{0, 1\} \) for \( k \in \{1, 2, \ldots, n\} \). Our aim is to show that \( G_n \) satisfies the four group properties under geometric product: closure, associativity, identity, and inverse\[9\]. We shall call \( G_n \) as the Clifford basis group with \( n \) vector generators.

**a. Closure.** Let \( \hat{g} \) and \( \hat{g}' \) be two elements of \( G_n \). Their product is

\[ \hat{g} \hat{g}' = s_\hat{g} s_{\hat{g}'} e_1^{b_1} e_2^{b_2} \ldots e_{k_1}^{b_{k_1}} \ldots e_k^{b_k} \ldots e_n^{b_n} \ldots e_{k_1}^{b_{k_1}} \ldots e_k^{b_k} \ldots e_n^{b_n}. \quad (31) \]

Using the theorems in Eqs. (19) and (29), Eq. (31) becomes

\[ \hat{g} \hat{g}' = W_{\hat{g} \hat{g}'} s_\hat{g} s_{\hat{g}'} e_1^{b_1} e_2^{b_2} \ldots e_k^{b_k} e_{k+1}^{b_{k+1}} \ldots e_n^{b_n}. \quad (32) \]

where \( W_{\hat{g} \hat{g}'} \) is a Walsh function,

\[ W_{\hat{g} \hat{g}'} = (-1)^{M_{\hat{g} \hat{g}'}}, \quad (33) \]

\[ M_{\hat{g} \hat{g}'} = \sum_{k=1}^{n-1} b_k \sum_{j=k+1}^{n} b_j. \quad (34) \]

Because Eq. (24) is true and \( W_{\hat{g} \hat{g}'} = \pm 1 \), then \( \hat{g} \hat{g}' \) in Eq. (31) is an element of \( G_n \). Thus, \( G_n \) is closed under juxtaposition multiplication.

The \( Cl_{n,0} \) Walsh function in Eq. (33) is given in Biasbas\[10\], while that for \( Cl_{0,n} \) was derived by Vahlen (1897)\[11\] and rederived by Hagmark and Lounesto\[12\].

The general Walsh function for \( Cl_{p,q} \) is derived by Brauer and Weyl (1935)\[13\].

**b. Associativity.** For \( \hat{g}_1, \hat{g}_2, \hat{g}_3 \in G_n \),

\[ \hat{g}_1 (\hat{g}_2 \hat{g}_3) = (\hat{g}_1 \hat{g}_2) \hat{g}_3. \quad (35) \]

because products of the basis vectors in the set \( B_{n,1} \) in Eq. (11) are associative.

**c. Identity.** The unit real number 1 is the unit element of \( G_n \), because

\[ 1 = e_1^{0} e_2^{0} \ldots e_n^{0}, \quad (36) \]

which is of the form of an element of \( G_n \). Hence,

\[ 1 \hat{g} = \hat{g} 1 = \hat{g}. \quad (37) \]

**d. Inverse.** For every \( \hat{g} \in G_n \), there exists \( \hat{g}^{-1} \in G_n \) such that

\[ \hat{g} \hat{g}^{-1} = \hat{g}^{-1} \hat{g} = 1. \quad (38) \]

From the definition of \( \hat{g} \) in Eq. (30) and the identity \( e_k^{b_k} = 1 \) from Eq. (11), we can immediately see that

\[ \hat{g}^{-1} = s_\hat{g} e_n^{b_n} \ldots e_k^{b_k} \ldots e_2^{b_2} e_1^{b_1}. \quad (39) \]

Using the theorem in Eq. (24), Eq. (30) may be written as

\[ \hat{g}^{-1} = s_\hat{g} W_{\hat{g}^{-1}} e_1^{b_1} e_2^{b_2} \ldots e_k^{b_k} \ldots e_n^{b_n}, \quad (40) \]

where

\[ W_{\hat{g}^{-1}} = (-1)^{M_{\hat{g}^{-1}}}, \quad (41) \]

\[ M_{\hat{g}^{-1}} = \sum_{k=1}^{n-1} b_k \sum_{j=k+1}^{n} b_j. \quad (42) \]

Because \( W_{\hat{g}^{-1}} = \pm 1 \), then \( \hat{g}^{-1} \in G_n \).

### 3 Geometric Group

**3.1 Set Axioms**

Let \( F, H \subseteq G \). We define the product of the sets \( F \) and \( H \) as follows:

\[ FH = \{ \hat{f} \hat{h} \mid \hat{f} \in F, \hat{h} \in H \}. \quad (43) \]

From this definition we can construct an algebra for the set \( G \) with respect to juxtaposition multiplication and set union \( \cup \).
3.2 Set Theorems

From these five set properties, we can derive two theorems:

a. **Associativity.** If $H_1, H_2, H_3 \in G$, then

$$H_1(H_2H_3) = H_1H_2H_3,$$

$$H_1 \cup (H_2 \cup H_3) = (H_1 \cup H_2) \cup H_3.$$  

b. **Commutativity.** If $H_1, H_2 \in G$, then

$$H_1 \cup H_2 = H_2 \cup H_1.$$  

c. **Identity.** There exists an identity $\{1\} \in G$ and an empty set $\{} \in G$, such that for all $H \subseteq G$,

$$H \{1\} = \{1\}H = H,$$

$$H \cup \{} = \{} \cup H = H.$$  

d. **Distributivity.** If $H_1, H_2, H_3 \in G$, then

$$H_1(H_2 \cup H_3) = H_1H_2H_3,$$

$$H_2 \cup H_3H_1 = H_2H_1 \cup H_3H_1.$$  

e. **Inverse.** If $g \in G$, then there exists a $g^{-1} \in G$ such that

$$gg^{-1} = g^{-1}g = 1,$$

which is a group property. Sets containing more than one element has no multiplicative inverse. There is also no set that when united to a nonempty set gives the empty set $\{}$.

3.3 Generators of a Geometric Group

Let $G$ be a group and let $H \subseteq G$ defined in Eq. (54) satisfy two conditions:

- For all $h_j \in H$,  

  $$h_j^2 = \pm 1.$$  

- For all $h_j, h_k \in H$, $h_j$ and $h_k$ either commute or anticommute:

  $$h_jh_k = \pm h_kh_j.$$  

We claim that the set

$$G_H = \{\pm 1\}{1, h_1}{1, h_2} \cdots \{1, h_m\}$$

generated from $H$ is a group. We call $G_H$ as the geometric group generated by the set $H$.

**Proof.** We verify that $G_H$ satisfies the closure, associativity, identity, and inverse properties of a group:

a. **Closure.** To prove that $G_H$ is closed, we use the definition in Eq. (56):

$$G_H^2 = G_H.$$  

The left side of Eq. (56) may be expanded as

$$G_H^2 = \{\pm 1\}{1, h_1}{1, h_2} \cdots \{1, h_m\} \cdot$$

$$\cdot \{\pm 1\}{1, h_1}{1, h_2} \cdots \{1, h_m\}.$$
where the centered dot (·) means that the other set factors are below the line. Using the theorems in Eqs. (58) and (67), we may move the middle {±1} to the place beside the leftmost {±1} to arrive at

\[ G_H = \{±1\} \{1, h_1\} \{1, h_2\} \cdots \{1, h_m\} : \\
\{1, h_1\} \{1, h_2\} \cdots \{1, h_m\}, \tag{65} \]
because \{±1\} is idempotent.

Now, we move \{±1\} again near the middle part. Because \(h_1\) commutes or anticommutes with \(h_m\) by Eq. (61), then

\[ \{±1\} \{\hat{h}_m\} \{1, \hat{h}_1\} = \{±1, \pm \hat{h}_m, \pm \hat{h}_1, \pm \hat{h}_m \hat{h}_1\} = \{±1, \pm \hat{h}_m, \pm h_1, \pm h_1 \hat{h}_m\} = \{±1\} \{1, \hat{h}_1\} \{1, \hat{h}_m\}. \tag{66} \]

Then we move \{±1\} again back to the leftmost side, so that Eq. (65) becomes

\[ G_H^2 = \{±1\} \{1, h_1\} \{1, h_2\} \cdots \{1, h_1\} \cdots \{1, h_m\} \{1, h_2\} \cdots \{1, h_m\}. \tag{67} \]

In general, we have the following lemma: if a set is of the form given in Eq. (62) and the condition in Eq. (61) holds, the order of the set factors does not matter.

Applying this lemma for \(G_H^2\) in Eq. (65), we get

\[ G_H^2 = \{±1\} \{1, h_1\}^2 \{1, h_2\}^2 \cdots \{1, h_m\}^2. \tag{68} \]

Because \(\hat{h}_j^2 = ±1\) by Eq. (60), then

\[ \{±1\} \{1, \hat{h}_j\}^2 = \{±1, ± \hat{h}_j\} = \{±1\} \{1, \hat{h}_j\}, \tag{69} \]

so that Eq. (68) becomes

\[ G_H^2 = \{±1\} \{1, h_1\} \{1, h_2\} \cdots \{1, h_m\} = G_H, \tag{70} \]

which is what we wish to prove. \(\text{QED}\)

b. Associativity. We may rewrite the group \(G_H\) in Eq. (62) as

\[ G_H = \{±1\} \{\hat{h}_1^0, \hat{h}_1\} \{\hat{h}_2^0, \hat{h}_2\} \cdots \{\hat{h}_m^0, \hat{h}_m\}. \tag{71} \]

Thus, every \(\hat{A} \in G_H\) is of the form

\[ \hat{A} = s_A \hat{h}_1^{a_1} \hat{h}_2^{a_2} \cdots \hat{h}_j^{a_j} \cdots \hat{h}_m^{a_m}, \tag{72} \]

where \(s_\hat{A} = ±1\) is the sign of \(\hat{A}\) and \(a_j \in \{0, 1\}\) is the bit power of generator \(\hat{h}_j\).

If \(\hat{B}, \hat{C} \in G_H\), then

\[ \hat{A}(\hat{B}\hat{C}) = (\hat{A}\hat{B})\hat{C}, \tag{73} \]
because the factors of \(\hat{A}, \hat{B}, \) and \(\hat{C}\) are elements of the group \(G\), and the products of the elements in a group are associative. \(\text{QED}\)

c. Identity The identity element of \(G_H\) is the real number 1:

\[ \hat{h}_1^0 \hat{h}_2^0 \cdots \hat{h}_m^0 = 1, \tag{74} \]

so that for all \(\hat{A} \in G_H\),

\[ \hat{A}1 = 1\hat{A} = \hat{A}. \tag{75} \]

d. Inverse From the conditions in Eqs. (60) and (61), we see that the inverse of \(\hat{A} \in G_H\) is either \(\hat{A}\) itself or \(-\hat{A} \in G_H\), so that

\[ \hat{A}^{-1} = \hat{A}^{-1}. \tag{76} \]

### 4 Groups and Subgroups

The Clifford basis group \(C_{p,q}\) consists of \(n = p + q\) anti-commuting generators, with \(p\) generators that square to +1 and \(q\) generators that square to −1. This group \(C_{p,q}\) forms the basis for the Clifford group algebra \(Cl_{p,q}\).

Our aim in this section is to describe the Clifford (Pauli) basis group \(C_{3,0}\) and enumerate its geometric subgroups in a hierarchical order for \(n \leq 3\).

#### 4.1 Subscript Notation

We shall use the following notations for the following geometric groups with one nontrivial generator \(\neq -1\):

\[ E_a = \{±1\} \{1, e_a\}, \tag{77} \]

\[ E_{ab} = \{±1\} \{1, e_a e_b\} \equiv \{±1\} \{1, e_{ab}\}, \tag{78} \]

\[ E_{abc} = \{±1\} \{1, e_a e_b e_c\} \equiv \{±1\} \{1, e_{abc}\}, \tag{79} \]

where \(a, b, c \in \{1, 2, 3\}\) are distinct integers. Note that the product of these groups is also a geometric group. For example, the Clifford basis group \(C_{3,0}\) may be represented as

\[ E_a E_b E_c = \{±1\}^3 \{1, e_a\} \{1, e_b\} \{1, e_c\} = \{±1\} \{1, e_a e_b e_c\} \{1, e_{abc}\} = \{±1\} \{1, e_{abc}\} \{1, e_{abc}\} = C_{3,0}. \tag{80} \]

#### 4.2 Four Group Relations

There are four relations that characterize groups:

a. Isomorphic. Two geometric groups are isomorphic (\(\cong\)) if there exists a one-to-one mapping between their generators, which preserve the relationship between the generators (commutation and anticommutation) and the square of each generator \(±1\). For example,

\[ E_a \cong C_{1,0}, \tag{81} \]

\[ E_{ab} \cong C_{0,1}. \tag{82} \]
b. Similar. Two geometric groups are similar (≡) if they have the same generators, except for the labelling of the generator subscripts. For example,

\[ E_{abc} \approx E_{acb} \approx E_{bac}. \]  

(83)

but

\[ E_{ab}E_{bc} \not\approx E_{ab}E_{cd}, \]  

(84)

because the mapping \( b \mapsto \{b, d\} \) is not one-to-one. Note that all similar groups are isomorphic.

c. Equivalent. Two geometric groups are equivalent (\( \equiv \)) if they have the same canonical listing of elements, except for the labelling of the element subscripts. (The listing of a group is said to be canonical if all elements of the group are expressed as products of the vector generators of the Clifford group \( C_{n,0} = E_{a_1}E_{a_2}\cdots E_{a_n} \).) For example,

\[ E_aE_{abc} = \{\pm 1\}\{1, e_a, \hat{e}_{ac}, \hat{e}_{abc}\}, \]  

(85)

\[ E_{ab}E_{abc} = \{\pm 1\}\{1, e_{ac}, e_{ab}, \hat{e}_{abc}\}. \]  

(86)

If we map the ordered pair \((a, c) \mapsto (c, a)\), then Eq. (85) becomes

\[ E_aE_{abc} :\mapsto E_cE_{cba} = \{\pm 1\}\{1, e_c, \hat{e}_{ab}, \hat{e}_{abc}\} = E_{ab}E_{abc}. \]  

(87)

Thus,

\[ E_aE_{abc} \equiv E_{ab}E_{abc}. \]  

(88)

d. Equal. Two geometric groups are equal (=) if they have the same canonical listing of elements. For example,

\[ E_aE_b = \{\pm 1\}\{1, e_a\}\{1, e_b\} = \{\pm 1\}\{1, e_a, e_b, \hat{e}_{ab}\} = \{\pm 1\}\{1, e_a\}\{1, e_{ab}\} = E_{a}E_{ab}. \]  

(89)

Note that all equivalent groups can be made equal after relabelling the subscripts.

Note also that if two single-generator groups are similar, then they are equal. Thus, we may write Eq. (85) as a strict equality:

\[ E_{abc} = E_{acb} = E_{bac}. \]  

(90)

This is obvious from the definition of \( E_{abc} \) in Eq. (79).

### 4.3 Hierarchy of Subgroups of \( C_{3,0} \)

Let us enumerate the subgroups of \( C_{3,0} \) according to the number of their nontrivial generators. We shall limit ourselves to at most \( n = 3 \) generators—though the maximum is \( n = 7 \)—to simplify the tables. (Also, no Clifford basis subgroups can be formed for \( C_{n,0} \) if the number of generators of the subgroup exceed \( n \).)

a. One Generator. There are four \( C_{3,0} \) subgroups with one nontrivial generator:

\[ E_a = \{\pm 1\}\{1, e_a\} \cong C_{1,0}. \]  

(91)

\[ E_{ab} = \{\pm 1\}\{1, e_{ab}\} \cong C_{0,1}. \]  

(92)

\[ E_{abc} = \{\pm 1\}\{1, e_{abc}\} \cong C_{0,1}. \]  

(93)

Thus, the Clifford group \( C_{1,0} \) may be generated by a vector \( e_a \); the Clifford group \( C_{0,1} \), by a bivector \( \hat{e}_{ab} \) or a trivector \( \hat{e}_{abc} \). These groups are order four, with two linearly independent basis elements each.

Removing the trivial group

\[ C_{0,0} = \{\pm 1\} = \{1, -1\} \]  

(94)

from Eq. (91), we obtain

\[ \{1, e_a\} \cong C_{0,0}. \]  

(95)

Note that \( C_{0,0} = \{\pm 1\} \) is the group basis for the algebra of \( \mathbb{R} \) of real numbers.

b. Two Generators. For geometric groups with two nontrivial generators, we shall order them by writing the product of single generator groups as \( E_AE_B \) for \( |A| \leq |B| \), where \( A \) and \( B \) are the multivector ranks of the generators. For example, \( E_{ab} \) is rank 2 (bivector generated group), while \( E_{abc} \) is rank 3 (trivector generated group). Thus, we write \( E_{ab}E_{abc} \) and not \( E_{abc}E_{ab} \). Note that the subscripts could not be equal, \( A \neq B \), because

\[ E_AE_B = E_AE_A = E_A. \]  

(96)

which is a group with only one nontrivial generator.

There are six \( C_{3,0} \) subgroups with two nontrivial generators:

\[ E_aE_b = \{\pm 1\}\{1, e_a, e_b, \hat{e}_{ab}\}, \]  

(97)

\[ E_aE_{ab} = \{\pm 1\}\{1, e_a, e_b, \hat{e}_{ab}\} = E_aE_b, \]  

(98)

\[ E_aE_{bc} = \{\pm 1\}\{1, e_a, e_{bc}, \hat{e}_{abc}\}, \]  

(99)

\[ E_{ab}E_{abc} = E_{ab}E_{abc}, \]  

(100)

\[ E_{ab}E_{ac} = \{\pm 1\}\{1, \hat{e}_{ab}, \hat{e}_{ac}, \hat{e}_{bc}\}, \]  

(101)

\[ E_{ab}E_{abc} = \{\pm 1\}\{1, e_a, e_{ab}, e_{abc}\} \equiv E_{bc}E_{abc} = E_{bc}E_{abc}. \]  

(102)

Notice that \( E_aE_b \) and \( E_aE_{ab} \) are canonically equal groups. And so are the groups \( E_aE_{bc}, E_aE_{abc}, E_aE_{abc}, \) and \( E_{ab}E_{abc} \), except that they are not isomorphic to Clifford basis groups, because their respective generators commute.

Notice, too, that the set \( E_{ab}E_{ac} \) is the set of quaternion basis:

\[ i = \hat{e}_{ab}, \]  

(103)

\[ j = \hat{e}_{bc}, \]  

(104)

\[ k = \hat{e}_{ac}. \]  

(105)
c. **Three Generators.** There are ten subgroups with three nontrivial generators:

\[
\{ \pm 1 \} \{ e_a, e_b, e_c, e_{ab}, e_{ac}, e_{bc}, e_{abc} \}, \quad (106)
\]

\[
E_a E_b E_c = E_a E_b, \quad (107)
\]

\[
E_a E_b E_{ac} = E_a E_b E_c, \quad (108)
\]

\[
E_a E_b E_{ac} = E_a E_b E_c, \quad (109)
\]

\[
E_a E_b E_{ac} = E_a E_b E_c, \quad (110)
\]

\[
E_a E_b E_{ac} = E_a E_b E_c, \quad (111)
\]

\[
E_a E_b E_{ac} = E_a E_b E_c, \quad (112)
\]

\[
E_a E_b E_{ac} = E_a E_b E_c, \quad (113)
\]

\[
E_a E_b E_{ac} = E_a E_b E_c, \quad (114)
\]

\[
E_a E_b E_{ac} = E_a E_b E_c, \quad (115)
\]

Notice that no two of these groups are isomorphic, though some of them are equal. Notice, too, that though the groupings, however, do not follow Dionysius’s pattern of 3-3-3 but of 1-3-3-2. Notice, too, that there are four chant modes, with the mode number corresponding to the number of generators.

b. **Bands** There are two ways to arrange the bands. One way is to arrange them according to signature of the Clifford algebra they failed to obey (see Table 6). Another way is to arrange them according to *rhythms;* bands in the same rhythm are canonically equal (=). We shall designate in each rhythm a *leader.* (See Tables 7 to 10)

Three useful measures for describing bands are disorder, chord, and beat.

The *disorder* \( \Phi \) of a group of order \( 2^n \) with \( n \) nontrivial generators is

\[
\Phi = \log_2 \frac{2^{n+1}}{2^m} = n + 1 - m. \quad (116)
\]

Notice that disorder is measured with respect to \( 2^{n+1} \), which is the order prescribed for choirs. Choirs have a disorder \( \Phi = 0 \).

The number of generators in the same group that the generator commutes with (going into another’s place without making a – sign) other than itself is called the generator’s transposition number or simply *transposition.*

The series of transpositions of each generator describes the band’s chord \( X \).

The sum \( T \) of the transpositions of the band’s generators divided by the total possible permutations of \( n \) of generators taken 2 at a time is the band’s *beat:*

\[
B = T \frac{(n-2)!}{n!} = \frac{T}{n(n-1)}. \quad (117)
\]

Bands have a maximum beat of \( B = 1 \). Choirs have a beat of \( B = 0 \). Note that beat for single-generator groups is not defined.

Because we associated choir groups with the angelic hierarchy, let us, for mnemonic purposes, associate band groups with the demonic lowerarchy, as suggested by their dominant number 6-6-6. Beatless bands with a unity disorder are \( \{ 1, e_a \} \), \( E_a E_b E_{ab} \), and \( E_{ab} E_{ac} E_{bc} \). Most bands have no unity disorder but have nonzero beats. One band stands out with a disorder of one and a beat of one: \( E_a E_{bc} E_{abc} \).

There are still many demonic bands whose number of generators fall in the interval \( 4 \leq n \leq 7 \). They are legion.

5 Conclusion

In this paper, we showed that the basis set of the Clifford \( Cl_{n,0} \) algebra forms a group under juxtaposition multiplication. This group with \( 2^{n+1} \) elements is generated by \( n \) anticommuting vectors that square to \(+1\). Using set algebra, we showed that from this basis set we can construct geometric groups whose generators not only square to \(+1\), but also commute or anticommute with other generators in the group.


To illustrate this claim, we enumerated all the subgroups of the basis set of $\mathbf{Cl}_{3,0}$ according to the number $n$ of their nontrivial generators, for $n \leq 3$. We classified the subgroups according to three criteria: (1) the square of each generator is $\pm 1$, (2) the generators within the group anticommute, and (3) the order of the resulting group is $2^{n+1}$. All obey the first rule, but some groups fail in the second or third rule or both. Obedient groups we called choirs; disobedient groups, bands.

The choirs form the basis of Clifford algebras of arbitrary signature for $p + q = n$. Canonically equal choirs we grouped into modes. Bands, on the other hand, do not form the basis of Clifford algebras because of their disobedience. We distinguished bands according to disorder, chord, and beat. Canonically equal bands we classified under one rhythm.

A similar taxonomic system may be made for the basis set of the Clifford (Dirac) algebra $\mathbf{Cl}_{4,0}$. But this may be a Herculean task, requiring days and weeks of pen and paper computations. A computer may be necessary.

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Table 1: Hierarchy of choir groups and their angelic names

| n  | Choir     | Sign | Name          |
|----|-----------|------|---------------|
| 0  | $C_{0.0}$ | ±1   | Seraphim      |
| 1  | $C_{1.0}$ | $E_a$| Cherubim      |
| 1  | $C_{0.1}$ | $E_{ab}$| Thrones    |
| 1  | $C_{0.1}$ | $E_{abc}$| Virtues    |
| 2  | $C_{2.0}$ | $E_aE_b$| Dominations |
| 2  | $C_{1.1}$ | $E_aE_{ab}$| Powers |
| 2  | $C_{0.2}$ | $E_{ab}E_{abc}$| Principalities |
| 3  | $C_{1.0}$ | $E_aE_bE_c$| Archangels |
| 3  | $C_{1.2}$ | $E_aE_{ab}E_{abc}$| Angels |

Table 2: Mode 0 by $\{\pm 1\}$

| n  | Choir     | Sign |
|----|-----------|------|
| 0  | $C_{0.0}$ | ±1   |

Table 3: Mode 1 led by $E_a$

| n  | Choir     | Sign |
|----|-----------|------|
| 1  | $C_{1.0}$ | $E_a$   |
| 2  | $C_{0.1}$ | $E_{ab}$| − |
| 3  | $C_{0.1}$ | $E_{abc}$| − |

Table 4: Mode 2 led by $E_aE_b$

| n  | Choir     | Sign |
|----|-----------|------|
| 2  | $C_{2.0}$ | $E_aE_b$| ++ |
| 2  | $C_{1.1}$ | $E_aE_{ab}$| ++ |
| 2  | $C_{0.2}$ | $E_{ab}E_{abc}$| −− |

Table 5: Mode 3 led by $E_aE_bE_c$

| n  | Choir     | Sign |
|----|-----------|------|
| 3  | $C_{1.0}$ | $E_aE_bE_c$| ++ + |
| 3  | $C_{1.2}$ | $E_aE_{ab}E_{abc}$| −− − |

Table 6: Lowerarchy of band groups arranged according to signature $(p, q)$, disorder $\Phi$, chord $X$, and beat $B$

| n  | Band     | Sign | $\Phi$ | $X$ | $B$ |
|----|----------|------|--------|-----|-----|
| 1  | $C_{1.0}$| $\{1, e_a\}$| + | 1 | |

Table 7: The band $\{1, e_a\}$ with disorder $\Phi = 1$, undefined chord $X$, and undefined beat $B$

| n  | Band     | Sign | $\Phi$ | $X$ | $B$ |
|----|----------|------|--------|-----|-----|
| 1  | $C_{1.0}$| $\{1, e_a\}$| + | 1 | |

Table 8: Rhythm of $E_aE_{bc}$

| n  | Band     | Sign | $\Phi$ | $X$ | $B$ |
|----|----------|------|--------|-----|-----|
| 2  | $C_{2.1}$| $E_aE_{bc}$| +− | 0 | (1, 1) | 2/2 |

Table 9: Rhythm of $E_aE_{ab}$

| n  | Band     | Sign | $\Phi$ | $X$ | $B$ |
|----|----------|------|--------|-----|-----|
| 3  | $C_{2.1}$| $E_aE_{ab}$| +− | 0 | (0, 0, 0) | 0/6 |

Table 10: Rhythm of $E_aE_{abc}$

| n  | Band     | Sign | $\Phi$ | $X$ | $B$ |
|----|----------|------|--------|-----|-----|
| 3  | $C_{2.1}$| $E_aE_{abc}$| +− | 0 | (1, 0, 1) | 2/6 |
| 3  | $C_{2.1}$| $E_aE_{abc}$| +− | 0 | (1, 1, 2) | 4/6 |
| 3  | $C_{1.2}$| $E_aE_{ab}E_{bc}$| +− | 0 | (1, 1, 2) | 4/6 |
| 3  | $C_{0.3}$| $E_aE_{ab}E_{abc}$| −− | 0 | (1, 1, 2) | 4/6 |