Posterior Convergence of Nonparametric Binary and Poisson Regression Under Possible Misspecifications

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Abstract

In this article, we investigate posterior convergence of nonparametric binary and Poisson regression under possible model misspecification, assuming general stochastic process prior with appropriate properties. Our model setup and objective for binary regression is similar to that of Ghosal and Roy (2006) where the authors have used the approach of entropy bound and exponentially consistent tests with the sieve method to achieve consistency with respect to their Gaussian process prior. In contrast, for both binary and Poisson regression, using general stochastic process prior, our approach involves verification of asymptotic equipartition property along with the method of sieve, which is a manoeuvre of the general results of Shalizi (2009), useful even for misspecified models. Moreover, we will establish not only posterior consistency but also the rates at which the posterior probabilities converge, which turns out to be the Kullback-Leibler divergence rate. We also investigate the traditional posterior convergence rates. Interestingly, from subjective Bayesian viewpoint we will show that the posterior predictive distribution can accurately approximate the best possible predictive distribution in the sense that the Hellinger distance, as well as the total variation distance between the two distributions can tend to zero, in spite of misspecifications.

Keywords: Binary/Poisson regression; Cumulative distribution function; Infinite dimension; Kullback-Leibler divergence rate; Misspecification; Posterior convergence.

1 Introduction

The situation for applicability of nonparametric regression is frequently encountered in many practical scenarios where no parametric model fits the data. In particular, non-parametric regression for binary dependent variables is very common for various branches of statistics like medical and spatial statistics, whereas nonparametric version of Poisson regression is being used recently in many non-trivial scenerios such as for analyzing the likelihood and severity of vehicle crashes (Ye et al. (2018)). Interestingly, despite vast applicability of both the binary as well as Poisson regression, it seems that the available literature on nonparametric Poisson regression is scarce in comparison to the available literature on nonparametric binary regression. The Bayesian approach to nonparametric binary regression problem has been accounted for in Diaconis and Freedman (1993). An account of posterior consistency for Gaussian process prior in nonparametric binary regression modeling can be found in Ghosal and Roy (2006), where the authors suggested that similar consistency results should hold for nonparametric Poisson regression model setup. Literature on consistency results for nonparametric Poisson regression is very limited. Pillai et al. (2007) have obtained consistency results for Poisson regression using an approach similar to that of Ghosal and Roy (2006) under certain assumptions, but so far without explicit specifications and detail on prior. On the other hand, our approach will be based on results on Shalizi (2009), which is much different from Ghosal and Roy (2006) and capable of handling model misspecification. Unlike the previous works, the approach of Shalizi...
(2009) also enables us to investigate the rate at which the posterior converges, which turns out to be the Kullback-Leibler (KL) divergence rate, and also the traditional posterior convergence rate.

In this article, we investigate posterior convergence of nonparametric binary and Poisson regression where the nonparametric regression is modeled as some suitable stochastic process. In the binary situation, we consider a similar setup as that of Ghosal and Roy (2006), where the authors have considered binary observations with response probability as an unknown smooth function of a set of covariates, which was modeled using Gaussian process. Here we will consider a binary response variable $Y$ and a $d$-dimensional covariate $x$ belonging to a compact subset. The probability function is given by $p(x) = P(Y = 1 | X = x)$ along with a prior for $p$ induced by some appropriate stochastic process $\eta(x)$ with the relation $p(x) = H(\eta(x))$ for a known, non-decreasing and continuously differentiable cumulative distribution function $H(\cdot)$. We will establish a posterior convergence theory for nonparametric binary regression under possible misspecifications based on the general theory of posterior convergence of Shalizi (2009). Our theory also includes the case of misspecified models, that is, if the true regression function is not even supported by the prior. This approach to Bayesian asymptotics also permits us to show that the relevant posterior probabilities converge at the KL divergence rate, and that the posterior convergence rate with respect to KL-divergence is just slower than $\frac{1}{n}$, where $n$ denotes the number of observations. We further show that even in the case of misspecification, the posterior predictive distribution can approximate the best possible predictive distribution adequately, in the sense that the Hellinger distance, as well as the total variation distance between the two distributions can tend to zero.

For nonparametric Poisson regression, given $x$ in the compact space of covariates, we model the mean function $\lambda(x)$ as $\lambda(x) = H(\eta(x))$, where $H$ is a continuously differentiable function. Again, we investigate the general theory of posterior convergence, including misspecifications, rate of convergence of the posterior distribution and the usual posterior convergence rate, in Shalizi’s framework.

The rest of our paper is structured as follows. In Section 2 we provide a brief overview and intuitive explanation of the main assumptions and results of Shalizi (2009) suitable for our approach. The basic premises for nonparametric binary and Poisson regression are provided in Sections 3 and 4, respectively. The required assumptions and their discussions are provided in Section 5. In Section 6, our main results on posterior convergence of binary and Poisson regression are provided, while Section 8 details the consequences of misspecifications. Concluding remarks are provided in Section 9.

The technical details are presented in the Appendix. Specifically, details of the necessary assumptions and results of Shalizi (2009) are provided in Appendix A. The detailed proofs of verification of Shalizi’s assumptions are provided in Appendix B and Appendix C for binary and Poisson regression setups, respectively.

2 An outline of the main assumptions and results of Shalizi

Let the set of random variables for the response be denoted by $Y_n = (Y_1, Y_2, \ldots, Y_n)$. For a given parameter space $\Theta$, let $f_\theta(Y_n)$ be the observed likelihood and $f_{\theta_0}(Y_n)$ be the true likelihood. We assume $\theta \in \Theta$ but the truth $\theta_0$ need not be in $\Theta$, thus allowing possible misspecification.

The KL divergence $KL(f, g) = \int f \log(\frac{f}{g})$ is a measure of divergence between two probability densities $f$ and $g$. The KL divergence is related to likelihood ratios, since by the Strong Law of Large Numbers (SLLN) for independent and identical (i.i.d) situations,

$$\frac{1}{n} \sum_{i=1}^{n} \log \left[ \frac{f(Y_i)}{g(Y_i)} \right] \rightarrow KL(f, g).$$
For every $\theta \in \Theta$, the KL divergence rate is given by:

$$h(\theta) = \lim_{n \to \infty} \frac{1}{n} E \left[ \log \left\{ \frac{f_{\theta_0}(Y_n)}{f_\theta(Y_n)} \right\} \right].$$

(2.1)

The key ingredient associated with the approach of Shalizi (2009) for proving convergence of the posterior distribution of $\theta$ is to show that the asymptotic equipartition property holds. To illustrate, let us consider the following likelihood ratio:

$$R_n(\theta) = \frac{f_{\theta}(Y_n)}{f_{\theta_0}(Y_n)}.$$  

(2.2)

If we think of the iid setup, $h(\theta)$ reduces to the KL divergence between the true and the hypothesized model. For each $\theta \in \Theta$, “asymptotic equipartition” property is as follows:

$$\lim_{n \to \infty} \frac{1}{n} \log |R_n(\theta)| = -h(\theta),$$

(2.3)

Here “asymptotic equipartition” refers to dividing up $\log |R_n(\theta)|$ into $n$ factors for large $n$ such that all the factors are asymptotically equal. For illustration, in the iid scenario, each factor converges to the same KL divergence between the true and the postulated model. The purpose of asymptotic equipartition is to ensure that relative to the true distribution, the likelihood of each $\theta$ decreases to zero exponentially fast, with rate being the KL divergence rate.

for $A \subseteq \Theta$, let

$$h(A) = \text{ess inf}_{\theta \in A} h(\theta);$$  

(2.4)

$$J(\theta) = h(\theta) - h(\Theta);$$  

(2.5)

$$J(A) = \text{ess inf}_{\theta \in A} J(\theta),$$  

(2.6)

where $h(A)$ roughly represent the minimum KL-divergence between the postulated and the true model over the set $A$. If $h(\Theta) > 0$, it indicates model misspecification. However, as we shall show, model misspecification need not always imply that $h(\Theta) > 0$. One such counter example is also given in Chatterjee and Bhattacharya (2020).

Observe that, for $A \subset \Theta$, $J(A) > 0$. For the prior, it is required to construct an appropriate sequence of sieve sets $\mathcal{G}_n \rightarrow \Theta$ as $n \rightarrow \infty$ such that:

1. $h(\mathcal{G}_n) \rightarrow h(\Theta)$, as $n \rightarrow \infty$.
2. $\pi(\cdot | Y_n) \geq 1 - \alpha \exp (-\beta n)$, for some $\alpha > 0$, $\beta > 2h(\Theta)$;

The sets $\mathcal{G}_n$ can be interpreted as the sieves in the sense that, the behaviour of the likelihood ratio and the posterior on the sets $\mathcal{G}_n$ essentially carries over to $\Theta$.

Let $\pi(A \mid Y_n)$ denote the posterior distribution of $\theta$ given $Y_n$. Then with the above notions, verification of (2.3) along with several other technical conditions (details given in Appendix A) ensure that any $A \subseteq \Theta$ for which $\pi(A) > 0$,

$$\lim_{n \to \infty} \pi(A \mid Y_n) = 0,$$

(2.7)

almost surely, provided that $h(A) > h(\Theta)$. The latter $h(A) > h(\Theta)$ implies positive KL-divergence in $A$, even if $h(\Theta) = 0$. That is, $A$ is the set in which the postulated model fails to capture the true model in terms of the KL-divergence. Hence, expectedly, the posterior probability of that set converges to zero.
Under mild assumptions, it also holds that

\[
\lim_{n \to \infty} \frac{1}{n} \log \pi(A|Y_n) = -J(A),
\]

almost surely. This result shows that the rate at which the posterior probability of \( A \) converges to zero is about \( \exp(-nJ(A)) \). From the above results it is clear that the posterior concentrates on sets of the form \( N_{\epsilon} = \{ \theta : h(\theta) \leq h(\Theta) + \epsilon \} \), for any \( \epsilon > 0 \).

Shalizi addressed the rate of posterior convergence as follows. Letting \( N_{\epsilon_n} = \{ \theta : h(\theta) \leq h(\Theta) + \epsilon_n \} \), where \( \epsilon_n \to 0 \) such that \( n\epsilon_n \to \infty \), Shalizi showed, under an additional technical assumption, that almost surely,

\[
\lim_{n \to \infty} \pi(N_{\epsilon_n}|Y_n) = 1.
\]

Moreover, it was shown by Shalizi that the squares of the Hellinger and the total variation distances between the posterior predictive distribution and the best possible predictive distribution under the truth, are asymptotically almost surely bounded above by \( h(\Theta) \) and \( 4h(\Theta) \), respectively. That is, if \( h(\Theta) = 0 \), then this allows very accurate approximation of the true predictive distribution by the posterior predictive distribution.

## 3 Model setup and preliminaries of the binary regression

Let \( Y \in \{0, 1\} \) be a binary outcome variable and \( X \) a vector of covariates. Suppose \( Y_1, Y_2, \ldots, Y_n \in \{0, 1\}^n \) are some independent binary responses conditional on unobserved covariates \( X_1, X_2, \ldots, X_n \in \mathcal{X} \subset \mathbb{R}^d \). We assume that the covariate space \( \mathcal{X} \) is compact. Let \( Y_n = (Y_1, Y_2, \ldots, Y_n)^T \) be the binary response random variables against the covariate vector \( X_n = (X_1, X_2, \ldots, X_n)^T \). The corresponding observed values will be denoted by \( y_n = (y_1, y_2, \ldots, y_n) \) and \( x_n = (x_1, x_2, \ldots, x_n) \) respectively. Let the model be specified as follows: for \( i = 1, 2, \ldots, n \):

\[
Y_i|X_i \sim Binomial(1, p(X_i))
\]

\[
p(x) = H(\eta(x))
\]

\[
\eta(\cdot) \sim \pi_{\eta},
\]

where \( \pi_{\eta} \) is the prior for some suitable stochastic process. Note that the prior for \( p \) is induced by the prior for \( \eta \). Our concern is to infer about the success probability function \( p(x) = P(Y = 1|X = x) \) when the number of observations goes to infinity. We will assume that the functions \( \eta \) have continuous first partial derivatives. We denote this class of functions by \( C'(\mathcal{X}) \). We do not assume the truth \( \eta_0 \) in \( C'(\mathcal{X}) \), allowing misspecification. The link function \( H \) is a known, non-decreasing, continuously differentiable cumulative distribution function on the real line \( \mathbb{R} \). It is widely accepted to assume the function \( H(\cdot) \) to be known as part of model assumption. For example, in logistic regression we choose the standard logistic cumulative distribution function as the link function, whereas in probit regression \( H \) is chosen to be the standard normal cumulative distribution function \( \phi \). More discussion on link function along with several other examples can be found in Choudhuri et al. (2007), Newton et al. (1996), Gelfand and Kuo (1991). A Bayesian method for estimation of \( p \) has been provided in Choudhuri et al. (2007). In has been shown in Ghosal and Roy (2006) that the sample paths of the Gaussian processes can well approximate a large class of functions and hence it is not essential to consider additional uncertainty in the link function \( H \).

Let \( \mathcal{C} \) be the counting measure on \( \{0, 1\} \). Then according to the model assumption, the conditional density of \( y \) given \( x \) with respect to \( \mathcal{C} \) will be represented by the density function \( f \) as follows:

\[
f(y|x) = p(x)^y (1 - p(x))^{1-y}.
\]
The prior for \( f \) will be denoted by \( \pi \). Let \( f_0 \) and \( p_0 \) denote truth density and success probability, respectively. Then under the truth, the joint density is:

\[
f_0(y|x) = p_0(x)^y (1 - p_0(x))^{1-y}.
\]  

(3.5)

One of the main objectives of this article is to show consistency of the posterior distribution of \( p \) treated as parameter arising from the parameter space \( \Theta \) specified as follows:

\[
\Theta = \{ p(\cdot) : p(x) = H(\eta(x)), \eta \in C'(X) \},
\]  

(3.6)

or simply, \( \Theta = C'(X) \).

4 Model setup and preliminaries of Poisson regression

For Poisson regression model setup, let \( Y \in \mathbb{N} \) be a count outcome variable and \( X \) a vector of covariates. Here \( \mathbb{N} \) denote the set of non negative integers. Suppose \( Y_1, Y_2, \ldots, Y_n \in \mathbb{N}^n \) are some independent responses conditional on covariates \( X_1, X_2, \ldots, X_n \in X \subset \mathbb{R}^d \). We assume that the covariate space \( X \) is compact. Let \( Y_n = (Y_1, Y_2, \ldots, Y_n)^T \) be the response random variables against the covariate vector \( X_n = (X_1, X_2, \ldots, X_n)^T \). The corresponding observed values will be denoted by \( y_n = (y_1, y_2, \ldots, y_n) \) and \( x_n = (x_1, x_2, \ldots, x_n) \) respectively. Let the parameter space be specified as follows:

\[
\Lambda = \{ \lambda(\cdot) : \lambda(x) = H(\eta(x)), \eta \in C'(X) \}.
\]  

(4.1)

The link function \( H \) is a known, non-negative continuously differentiable function on \( \mathbb{R} \). We equivalently define the parameter space as \( \Theta = C'(X) \). Thus, in what follows, we shall use both \( \Lambda \) and \( \Theta \) to denote the parameter space, depending on convenience. Then the model is specified as follows: for \( i = 1, 2, \ldots, n \),

\[
Y_i|X_i \sim \exp (-\lambda(X_i)) \frac{(\lambda(X_i))^y}{y!}
\]  

(4.2)

\[
\lambda(x) = H(\eta(x));
\]  

(4.3)

\[
\eta(\cdot) \sim \pi_{\eta}.
\]  

(4.4)

Similar to binary regression, here our concern will be to infer about \( \lambda(x) \) when the number of observations goes to infinity. We do not assume the truth \( \eta_0 \) in \( C'(X) \) as before, allowing misspecification.

Now, suppose \( \mathcal{C} \) be the counting measure on \( \mathbb{N} \). According to the model assumption for Poisson regression, the conditional density of \( y \) given \( x \) with respect to \( \mathcal{C} \) will be represented by density function \( f \) as follows:

\[
f(y|x) = \exp (-\lambda(x)) \frac{(\lambda(x))^y}{y!}.
\]  

(4.5)

The prior for \( f \) will be denoted by \( \Pi \). Let \( f_0 \) and \( \lambda_0 \) denote truth density and true mean function, respectively. Again, one of our main aims is to establish consistency of the posterior distribution of \( \lambda \) treated as parameter arising from \( \Lambda \).

5 Assumptions and their discussions

We need to make some appropriate assumptions for establishing convergence of both the binary and Poisson regression models equipped with stochastic process prior. The latter also requires suitable assumptions. Many of the assumptions are similar to those taken in Chatterjee and
Bhattacharya (2020). Hence the purpose of such assumptions will be as discussed in Chatterjee and Bhattacharya (2020), which we shall briefly touch upon here.

**Assumption 1.** \( \mathcal{X} \) is a compact, \( d \)-dimensional space, for some finite \( d \geq 1 \) equipped with a suitable metric.

**Assumption 2.** Recall that in our notation, \( \mathcal{C}'(\mathcal{X}) \) denotes the class of continuously partially differentiable function on \( \mathcal{X} \). In other words, the functions \( \eta \in \mathcal{C}'(\mathcal{X}) \) are continuous on \( \mathcal{X} \) and for such functions the limit

\[
\eta'_j(x) = \frac{\partial \eta(x)}{\partial x_j} = \lim_{h \to 0} \frac{\eta(x + h\delta_j) - \eta(x)}{\partial h}
\]

exists for each \( x \in \mathcal{X} \) and is continuous \( \mathcal{X} \). Here \( \delta_j \) is the \( d \)-dimensional vector with the \( j \)-th element as 1 and all the other elements as zero.

**Assumption 3.** The priors for \( \eta \) is chosen such that for \( \beta > 2h(\Theta) \),

\[
\pi \left( \|\eta\| \leq \exp \left( (\beta n)^{1/4} \right) \right) \geq 1 - c_\eta \exp (-\beta n);
\]

\[
\pi \left( \|\eta'_j\| \leq \exp \left( (\beta n)^{1/4} \right) \right) \geq 1 - c_{\eta'_j} \exp (-\beta n), \text{ for } j = 1, \ldots, d;
\]

where \( c_\eta \) and \( c_{\eta'_j}; j = 1, \ldots, d \), are positive constants.

We treat the covariates as either random (observed or unobserved) or non-random (observed). Accordingly, in Assumption 4 below we provide conditions pertaining to these aspects.

**Assumption 4.** (i) \( \{x_i : i = 1, 2, \ldots\} \) is an observed or unobserved sample associated with an iid sequence associated with some probability measure \( Q \), supported on \( \mathcal{X} \), which is independent of \( \{y_i : i = 1, 2, \ldots\} \)

(ii) \( \{x_i : i = 1, 2, \ldots\} \) is an observed non-random sample. In this case, we consider a specific partition of the \( d \)-dimensional space \( \mathcal{X} \) into \( n \) subsets such that each subset of the partition contains at least one \( x \in \{x_i : i = 1, 2, \ldots\} \) and has Lebesgue measure \( \frac{L}{n} \), for some \( L > 0 \).

**Assumption 5.** The truth function \( \eta_0 \) is bounded in sup norm. In other words, the truth \( \eta_0 \) satisfies the following for some constant \( \kappa_0 \):

\[
\|\eta_0\|_\infty < \kappa_0 < \infty
\]

Observe that in general \( \eta_0 \not\in \mathcal{C}'(\mathcal{X}) \). For random covariate \( X \), we assume that \( \eta_0(X) \) is measurable.

**Assumption 6.** For binary regression model set up we assume a uniform positive lower bound \( \kappa_B \) for \( \min\{p(\cdot), 1 - p(\cdot)\} \). In other words, for all \( p \in \Theta \),

\[
\inf\{\min (p(x), 1 - p(x)) : x \in \mathcal{X} \} \geq \kappa_B > 0,
\]

where \( \Theta \) as defined in expression 3.6.

**Assumption 7.** For Poisson regression model set up we assume a uniform positive lower bound \( \kappa_P \) for \( \lambda(\cdot) \). In other words, for all \( \lambda \in \Lambda \),

\[
\inf\{\lambda(x) : x \in \mathcal{X} \} \geq \kappa_P > 0,
\]

where \( \Lambda \) is as defined in expression 4.1.
5.1 Discussion of the assumptions

Assumption 1 is on compactness of \( \mathcal{X} \), which guarantees that continuous functions on \( \mathcal{X} \) will have finite sup-norms.

Assumption 2 is as taken in Chatterjee and Bhattacharya (2020) for the purpose of constructing appropriate sieves in order to show posterior convergence results. More precisely, Assumption 2 is required for to ensure that \( \eta \) is Lipschitz continuous in the sieves. Since a differentiable function is Lipschitz if and only if its partial derivatives are bounded, this serves our purpose, as continuity of the partial derivatives of \( \eta \) guarantees the boundedness in the compact domain \( \mathcal{X} \). In particular, if \( \eta \) is a Gaussian process, conditions presented in Adler (1981), Adler and Taylor (2007), Cramer and Leadbetter (1967) guarantee the above continuity and smoothness properties required by Assumption 2. We refer to Chatterjee and Bhattacharya (2020) for more discussion about this.

Assumption 3 is required for ensuring that the complements of the sieves have exponentially small probabilities. In particular, this assumption is satisfied if \( \eta \) is a Gaussian process, even if \( \exp \left( \beta n^{1/4} \right) \) is replaced with \( \sqrt{\beta n} \).

Assumption 4 is for the covariates \( x_i \), accordingly as they are considered an observed random sample, unobserved random sample, or non-random. Note that thanks to the strong law of large numbers (SLLN), given any \( \eta \) in the complement of some null set with respect to the prior, and given any sequence \( \{ x_i : i = 1, 2, \ldots \} \) Assumption 4 (i) ensures that for any integrable function \( g \), as \( n \to \infty \),

\[
\frac{1}{n} \sum_{i=1}^{n} g(x_i) \to \int_{\mathcal{X}} g(x) dQ(X) = E_{X} [g(X)] \quad {\text{(say)}},
\]

where \( Q \) is some probability measure supported on \( \mathcal{X} \).

Assumption 4 (ii) ensures that \( \frac{1}{n} \sum_{i=1}^{n} g(x_i) \) is a particular Riemann sum and hence \( 5.5 \) holds with \( Q \) being the Lebesgue measure on \( \mathcal{X} \). We continue to denote the limit in this case by \( E_{X} [g(X)] \).

Assumption 5 is equivalent to the Assumption(T) of Ghosal and Roy (2006). Assumption 5 actually implies that \( p_0(x) = H(\eta_0(x)) \) is bounded away from 0 and 1 and hence the corresponding truth function \( \eta_0 \), given by \( \eta_0(x) = H^{-1}(p_0(x)) \) is uniformly bounded above and below.

As \( \eta_0 \) is uniformly bounded above and below, hence \( p_0(x) = H(\eta_0(x)) \) will also be bounded away from 0 and 1. For the Poisson regression model set up it follows that \( \|\lambda_0\|_{\infty} < \infty \).

It is to be noted that here we do not require to assume that \( p_0 \in \Theta \) or \( \lambda_0 \in \Lambda \), allowing model misspecifications.

Observe that, similar to Pillai et al. (2007) we need the parameter space for Poisson regression to be bounded away from zero (Assumption 7). As pointed out in Pillai et al. (2007), we cannot bypass this and as such these are not a mere pathway towards our proof. This is because, if almost all observations in a sample from a Poisson distribution are zero, then it impossible to extract the information about the (log) mean. Hence we must require at least some condition to make it bound away from zero. Similar argument also applicable for binary regression, which is reflected in Assumption 6.

It is important to remark that Assumptions 6 and 7 are necessary only to validate Assumption (S6) of Shalizi, and unnecessary elsewhere. The reasons are clarified in Remarks 1 and 2. Although many of our proofs would be simpler if Assumptions 6 and 7 were used, we reserved these assumptions only to validate Assumption (S6) of Shalizi.

To achieve Assumptions 6 and 7, we set, for all \( x \in \mathbb{R} \),

\[
H(x) = \kappa_B \mathbb{I}_{\{G(x) \leq \kappa_B\}}(x) + G(x) \mathbb{I}_{\{\kappa_B < G(x) \leq 1-\kappa_B\}}(x) + (1 - \kappa_B) \mathbb{I}_{\{G(x) \geq 1-\kappa_B\}}(x),
\]

(5.6)
for the binary case, where $0 < \kappa_B < 1/2$, and

$$H(x) = \kappa_B \mathbb{I}_{\{G(x) \leq \kappa_P\}}(x) + G(x)\mathbb{I}_{\{G(x) > \kappa_P\}}(x),$$

where $\kappa_P > 0$. In (5.6), $G$ is a continuously differentiable distribution function on $\mathbb{R}$ and in (5.7), $G$ is a non-negative continuously differentiable function on $\mathbb{R}$.

6 Main results on posterior convergence

Here we will state a summary of our main results regarding posterior convergence of nonparametric binary regression and Poisson regression. The key results associated with the asymptotic equipartition property are provided in Theorems 1–4, proofs of which are provided in Appendix B (for binary regression) and in Appendix C (for Poisson regression).

**Theorem 1.** Let $Q$ and the counting measure $\mathfrak{C}$ on $\{0,1\}$ be the measures associated with the random variable $X$ and the binary random variable $Y$ respectively. Denote $E_{X,Y}(\cdot) = \int \int d\mathfrak{C} dQ$ and $E_X(\cdot) = \int \int dQ$. Then under the nonparametric binary regression model, under Assumption 4, the KL divergence rate $h(p)$ exists for $p \in \Theta$, and is given by

$$h(p) = \left[ E_X \left( p_0(X) \log \frac{p_0(X)}{p(X)} \right) + E_X \left( (1-p_0(X)) \log \frac{1-p_0(X)}{1-p(X)} \right) \right].$$

Alternatively, $h(p)$ admits the following form:

$$h(p) = E_{X,Y} \left( f_0(X,Y) \log \frac{f_0(X,Y)}{f(X,Y)} \right),$$

where $f$ and $f_0$ are as defined in (3.4) and (3.5).

**Theorem 2.** Let $Q$ and the counting measure $\mathfrak{C}$ on $\mathbb{N}$ be associated with the random variable $X$ and the count random variable $Y$, respectively. Denote $E_{X,Y}(\cdot) = \int \int d\mathfrak{C} dQ$ and $E_X(\cdot) = \int \int dQ$. Then under the nonparametric Poisson regression model, under Assumption 4, the KL divergence rate $h(\lambda)$ exists for $\lambda \in \Lambda$, and is given by

$$h(\lambda) = \left[ E_X \left( \lambda(X) - \lambda_0(X) \right) + E_X \left( \lambda_0(X) \log \frac{\lambda_0(X)}{\lambda(X)} \right) \right].$$

**Theorem 3.** Under the nonparametric binary regression model and Assumption 4, the asymptotic equipartition property holds, and is given by

$$\lim_{n \to \infty} \frac{1}{n} \log [R_n(p)] = -h(p).$$

The convergence is uniform on any compact subset of $\Theta$.

**Theorem 4.** Under the nonparametric Poisson regression model and Assumption 4, the asymptotic equipartition property holds, and is given by

$$\lim_{n \to \infty} \frac{1}{n} \log [R_n(\lambda)] = -h(\lambda).$$

The convergence is uniform on any compact subset of $\Lambda$.

Theorems 1 and 3 for binary regression and Theorems 2 and 4 for Poisson regression ensure that conditions (S1) to (S3) of Shalizi (2009) hold, and (S4) holds for both binary and Poisson regression because of compactness of $\mathcal{X}$ and continuity of $H$ and $\eta$. The detailed proofs are presented in Appendix B.4 and Appendix C.4, respectively.
We construct the sieves $G_n$ for binary regression model set up as follows:

$$G_n = \{ \eta \in C'(X) : \|\eta\| \leq \exp((\beta n)^{1/4}), \|\eta_j\| \leq \exp((\beta n)^{1/4}); j = 1, 2, \ldots, d \}$$  \hspace{1cm} (6.6)

It follows that $G_n \to \Theta$ as $n \to \infty$, where the parameter space $\Theta$ is given by (3.6). In a similar manner, we construct the sieves $G_n$ for binary regression as follows:

$$G_n = \{ \lambda(\cdot) : \lambda(x) = H(\eta(x)), \eta \in C'(X), \|\eta\| \leq \exp((\beta n)^{1/4}), \|\eta_j\| \leq \exp((\beta n)^{1/4}); j = 1, 2, \ldots, d \}. \hspace{1cm} (6.7)$$

Then similarly it will also follow that $G_n \to \Lambda$ as $n \to \infty$, where the parameter space $\Lambda$ is given by (4.1).

Assumption 3 ensures that for binary regression, $\Pi(G_n^c) \leq \alpha \exp(-\beta n)$ for some $\alpha > 0$ and similarly $\Pi(G_n^c) \leq \alpha \exp(-\beta n)$ for Poisson regression. Now, these results, continuity of $h(\theta)$, $h(\lambda)$ (the proofs of continuity of $h(p)$ and $h(\lambda)$ follows using the same techniques as in Appendices B.1 and C.1), compactness of $G_n$, $G_n$ and the uniform convergence results of Theorems 3 and 4, together ensure (S5) for both the model setups.

Now, as pointed out in Chatterjee and Bhattacharya (2020), we observe that the aim of assumption (S6) is to ensure that (see the proof of Lemma 7 of Shalizi (2009)) for every $\epsilon > 0$ and for all sufficiently large $n$,

$$\frac{1}{n} \log \int_{G_n} R_n(p) \, d\pi(p) \leq h(G_n) + \epsilon, \quad \text{almost surely.} \hspace{1cm} (6.8)$$

As $h(G_n) \to h(\Theta)$ as $n \to \infty$, it is enough to verify that for every $\epsilon > 0$ and for all $n$ sufficiently large,

$$\frac{1}{n} \log \int_{G_n} R_n(p) \, d\pi(p) \leq h(\Theta) + \epsilon, \quad \text{almost surely.} \hspace{1cm} (6.9)$$

First we observe that

$$\frac{1}{n} \log \int_{G_n} R_n(p) \, d\pi(p) \leq \frac{1}{n} \sup_{p \in G_n} \log R_n(p). \hspace{1cm} (6.10)$$

For large enough $\kappa > h(\Theta)$, consider $S = \{ p : h(p) \leq \kappa \}$.

**Lemma 1.** $S = \{ p : h(p) \leq \kappa \}$ is a compact set.

**Proof.** First recall that the proof of continuity of $h(p)$ in $p$ follows easily using the same techniques as in Appendix B.1.

Now note that, if $||\eta||_\infty \to \infty$, then there exists $X' \subseteq X$ such that either $E_X \left[ p_0(X) \log \left( \frac{p_0(X)}{p(X)} \right) I_Y \right] \to \infty$ or $E_X \left[ (1 - p_0(X)) \log \left( \frac{1 - p_0(X)}{1 - p(X)} \right) I_Y \right] \to \infty$. Hence, $h(p) \to \infty$ as $||\eta||_\infty \to \infty$. Thus, $h(p)$ is a coercive function.

Since $h(p)$ is continuous and coercive, it follows that $S$ is a compact set.

\[ \square \]

In a very similar manner, the following lemma also holds for Poisson model set up.

**Lemma 2.** $S = \{ \lambda : h(\lambda) \leq \kappa \}$ is a compact set.

**Proof.** Again, recall that continuity of $h(\lambda)$ in $\lambda$ can be shown using the same techniques as in Appendix C.1, and it is easily seen that if $||\eta||_\infty \to \infty$, then $h(\lambda) \to \infty$. Thus, $h(\lambda)$ is continuous and coercive, ensuring that $S$ is compact. \[ \square \]
Using compactness of $S$, in the same way as in Chatterjee and Bhattacharya (2020), condition (S6) of Shalizi can be shown to be equivalent to (6.11) and (6.12) in Theorems 5 and 6 below, corresponding to binary and Poisson cases. In the supplement we show that these equivalent conditions are satisfied in our model setups.

**Theorem 5.** For the binary regression setup, (S6) is equivalent to the following, which holds under Assumptions 1–6:

\[
\sum_{n=1}^{\infty} \int_{S} P \left( \left| \frac{1}{n} \log R_n(p) + h(p) \right| > \kappa - h(\Theta) \right) d\pi(p) < \infty. \tag{6.11}
\]

**Theorem 6.** For the Poisson regression model setup, (S6) is equivalent to the following, which holds under Assumptions 1–5 and 7:

\[
\sum_{n=1}^{\infty} \int_{S} P \left( \left| \frac{1}{n} \log R_n(\lambda) + h(\lambda) \right| > \kappa - h(\Lambda) \right) d\pi(\lambda) < \infty. \tag{6.12}
\]

Assumption (S7) of Shalizi also holds for both the model sets up because of continuity of $h(p)$ and $h(\lambda)$. Hence, all the assumptions (S1)–(S7) stated in Appendix A are satisfied for binary and Poisson regression setups.

Overall, our results lead to the following theorems.

**Theorem 7.** Assume the nonparametric binary regression setup. Then under the Assumptions 1–6,

\[ \lim_{n \to \infty} \pi(A | Y_n) = 0. \tag{6.13} \]

Also, for any measurable set $A$ with $\pi(A) > 0$, if $\beta > 2h(A)$, where $h$ is given by equation (6.1), or if $A \subset \bigcap_{k=n}^{\infty} G_k$ for some $n$, where $G_k$ is given by 6.6, then the followings hold:

(i) \[ \lim_{n \to \infty} \frac{1}{n} \log \left[ \pi(A | Y_n) \right] = -J(A), \tag{6.14} \]

(ii) \[ h(A) > h(\Theta), \pi(A) > 0 \implies \lim_{n \to \infty} \pi(A | Y_n) = 0. \tag{6.15} \]

**Theorem 8.** Assume the nonparametric Poisson regression setup. Then under Assumptions 1–5 and 7,

\[ \lim_{n \to \infty} \pi(A | Y_n) = 0. \tag{6.16} \]

Also, for any measurable set $A$ with $\pi(A) > 0$, if $\beta > 2h(A)$, where $h$ is given by equation (6.3), or if $A \subset \bigcap_{k=n}^{\infty} G_k$ for some $n$, where $G_k$ is given by 6.7, then the followings hold:

(i) \[ \lim_{n \to \infty} \frac{1}{n} \log \left[ \pi(A | Y_n) \right] = -J(A), \tag{6.17} \]

(ii) \[ h(A) > h(\Lambda), \pi(A) > 0 \implies \lim_{n \to \infty} \pi(A | Y_n) = 0. \tag{6.18} \]

### 7 Rate of convergence

Consider a sequence of positive reals $\epsilon_n$ such that $\epsilon_n \to 0$ while $n\epsilon_n \to \infty$ as $n \to \infty$ and the set $N_{\epsilon_n} = \{ p : h(p) \leq h(\Theta) + \epsilon_n \}$. Then the following result of Shalizi holds.
Theorem 9 (Shalizi (2009)). Assume (S1) to (S7) of Appendix A. If for each \( \delta > 0 \),
\[
\tau \left( \mathcal{G}_n \cap N_{\epsilon n}^c, \delta \right) \leq n \tag{7.1}
\]
eventually almost surely, then almost surely the following holds:
\[
\lim_{n \to \infty} (N_{\epsilon n}|Y_n) = 1. \tag{7.2}
\]

To investigate the rate of convergence in our cases (and also for the case of Chatterjee and Bhattacharya (2020)), it has been proved in Chatterjee and Bhattacharya (2020) that \( \epsilon_n \) will be the rate of convergence for \( \epsilon_n \to 0 \), \( n\epsilon_n \to \infty \) as \( n \to \infty \), if we can show that the following hold:
\[
\frac{1}{n} \log \int_{\mathcal{G}_n \cap N_{\epsilon_n}} R_n(p) \, d\pi(p) \leq -h(\Theta) + \epsilon, \tag{7.3}
\]
\[
\frac{1}{n} \log \int_{\mathcal{G}_n \cap N_{\epsilon_n}} R_n(\lambda) \, d\pi(\lambda) \leq -h(\Lambda) + \epsilon, \tag{7.4}
\]
for any \( \epsilon > 0 \) and all \( n \) sufficiently large.

Following similar arguments of Chatterjee and Bhattacharya (2020), we find that the posterior rate of convergence with respect to KL-divergence is just slower than \( n^{-1} \). To put it another way, it is just slower that \( n^{-\frac{1}{2}} \) with respect to Hellinger distance for the model setups we consider. Our results can be formally stated in Theorem 10 for Binary regression and in Theorem 11 for Poisson regression.

Theorem 10. For the nonparametric binary regression setup, under Assumptions 1–6, \( \lim_{n \to \infty} (N_{\epsilon_n}|Y_n) = 1 \) holds almost surely, where \( N_{\epsilon_n} = \{ p : h(p) \leq h(\Theta) + \epsilon_n \} \), \( \epsilon_n \to 0 \), \( n\epsilon_n \to \infty \) as \( n \to \infty \).

Theorem 11. For the nonparametric Poisson regression setup, under Assumptions 1–5 and 7, \( \lim_{n \to \infty} (N_{\epsilon_n}|Y_n) = 1 \) holds almost surely, where \( N_{\epsilon_n} = \{ \lambda : h(\lambda) \leq h(\Lambda) + \epsilon_n \} \), \( \epsilon_n \to 0 \), \( n\epsilon_n \to \infty \) as \( n \to \infty \).

8 Consequences of model misspecification

Suppose that the true function \( \eta_0 \) consists of countable number of discontinuities but has continuous first order partial derivatives at all other points. Then \( \eta_0 \notin \mathcal{C}^c(\mathbb{X}) \). However, there exists some \( \tilde{\eta} \in \mathcal{C}^c(\mathbb{X}) \) such that \( \tilde{\eta}(x) = \eta_0(x) \) for all \( x \in \mathbb{X} \) where \( \eta_0 \) is continuous. Similar to this kind of situation is mentioned in Chatterjee and Bhattacharya (2020). Observe that, if the probability measure \( Q \) of \( X_i \) is dominated by the Lebesgue measure, then from Theorem 1 we have \( h(\Theta) = 0 \). Then the posterior of \( \eta \) concentrates around \( \tilde{\eta} \), which is the same as \( \eta_0 \) except at the countable number of discontinuities of \( \eta_0 \). Corresponding \( \tilde{p} = H(\tilde{\eta}) \) and \( \tilde{\lambda} = H(\tilde{\eta}) \) will also differ from \( p_0 \) and \( \lambda_0 \). If \( p_0 \) and \( \lambda_0 \) are such that \( 0 < h(\Theta) < \infty \) and \( 0 < h(\Lambda) < \infty \) respectively then the posteriors concentrate around the minimizers of \( h(p) \) and \( h(\lambda) \), provided such minimizers exist in \( \Theta \) and \( \Lambda \), respectively.

8.1 Consequences from the subjective Bayesian perspective

Bayesian posterior consistency has two apparently different viewpoints, namely, classical and subjective. Bayesian analysis starts with a prior knowledge, and updates the knowledge given the data, forming the posterior. It is of utmost importance to know whether the updated knowledge becomes more and more accurate and precise as data are collected indefinitely. This requirement is called consistency of the posterior distribution. From the classical Bayesian point of view we should believe in existence of a true model. On the contrary, if we look from the
subjective Bayesian viewpoint, then we need not believe in true models. A subjective Bayesian thinks only in terms of the predictive distribution of future observations. But Blackwell and Dubins (1962), Diaconis and Freedman (1986) have shown that consistency is equivalent to inter subjective agreement, which means that two Bayesians will ultimately have very close posterior predictive distributions.

Let us define the one-step-ahead predictive distribution of $p$ and $\lambda$, one-step-ahead best predictor (which is the best prediction one could make had the true model, $P$, been known) and the posterior predictive distribution (Shalizi (2009)), with the convention that $n = 1$ gives the marginal distribution of the first observation, as follows:

\[(\text{One-step-ahead predictive distribution of } p): F^n_p = F_p(Y_n|Y_1, \ldots, Y_{n-1}),\]
\[(\text{One-step-ahead predictive distribution of } \lambda): F^n_\lambda = F_\lambda(Y_n|Y_1, \ldots, Y_{n-1}),\]
\[(\text{One-step-ahead best predictor}): P^n = P^n(Y_n|Y_1, \ldots, Y_{n-1}),\]
\[(\text{The posterior predictive distribution}): F^n_\pi = \int F^n_p \, d\pi(p|Y_n).\]

With the above definitions, the following results have been proved by Shalizi.

**Theorem 12** (Shalizi (2009)). Let $\rho_H$ and $\rho_{TV}$ be Hellinger and total variation metrics, respectively. Then with probability 1,

$$\limsup_{n \to \infty} \rho_H^2(P^n, F^n_\pi) \leq h(\Theta);$$
$$\limsup_{n \to \infty} \rho_{TV}^2(P^n, F^n_\pi) \leq 4h(\Theta).$$

In our nonparametric setup, $h(\Theta) = 0$ and $h(\Lambda) = 0$ if $\eta_0$ consists of countable number of discontinuities. Hence, from Theorem 12 it is clear that in spite of such misspecification, the posterior predictive distribution does a good job in learning the best possible predictive distribution in terms of the popular Hellinger and the total variation distance. We state our result formally as follows.

**Theorem 13.** Consider the setups of nonparametric binary and Poisson regression. Assume that the truth function $\eta_0$ consists of countable number of discontinuities but has continuous first order partial derivatives at all other points. Then under Assumptions 1–6 (for binary regression) or under Assumptions 1–5 and 7 (for Poisson regression) the following hold:

$$\limsup_{n \to \infty} \rho_H^2(P^n, F^n_\pi) = 0;$$
$$\limsup_{n \to \infty} \rho_{TV}^2(P^n, F^n_\pi) = 0.$$

9 Conclusion and future work

In this paper we attempted to address posterior convergence of nonparametric binary and Poisson regression, along with the rate of convergence, while also allowing for misspecification, using the approach of Shalizi (2009). We also have shown that, even in the case of misspecification, the posterior predictive distribution can be quite accurate asymptotically, which should be a point of interest from subjective Bayesian viewpoint. The asymptotic equipartition property plays a central role here. It is one of the crucial assumptions and yet relatively easy to establish under mild conditions. It actually brings forward the KL property of the posterior, which in turn characterizes the posterior convergence, and also the rate of posterior convergence and misspecification.
Appendix

A Assumptions and theorems of Shalizi

Following Shalizi (2009), let us consider a probability space \((\Omega, \mathcal{F}, P)\), a sequence of random variables \(\{Y_1, Y_2, \ldots\}\) taking values in the measurable space \((\mathfrak{B}, \mathcal{X})\), having infinite-dimensional distribution \(P\). The theoretical development requires no restrictive assumptions on \(P\) such as it being a product measure, Markovian, or exchangeable, thus paving the way for great generality.

Let \(\mathcal{F}_n = \sigma(Y_n)\) denote the natural filtration, that is, the \(\sigma\)-algebra generated by \(Y_n\). Also, let the distributions of the processes adapted to \(\mathcal{F}_n\) be denoted by \(F_\theta\), where \(\theta\) takes values in a measurable space \((\Theta, \mathcal{T})\). Here \(\theta\) denotes the hypothesized probability measure associated with the unknown distribution of \(\{Y_1, Y_2, \ldots\}\) and \(\Theta\) is the set of hypothesized probability measures. In other words, assuming that \(\theta\) is the infinite-dimensional distribution of the stochastic process \(\{Y_1, Y_2, \ldots\}\), \(F_\theta\) denotes the \(n\)-dimensional marginal distribution associated with \(\theta\); \(n\) is suppressed for the ease of notation. For parametric models, the probability measure \(\theta\) corresponds to some probability density with respect to some dominating measure (such as Lebesgue or counting measure) and consists of unknown, but finite number of parameters. For nonparametric models, \(\theta\) is usually associated with infinite number of parameters and may not even have any density with respect to \(\sigma\)-finite measures.

As in Shalizi (2009), we assume that \(P\) and all the \(F_\theta\) are dominated by a common measure with densities \(p\) and \(f_\theta\), respectively. In Shalizi (2009) and in our case, the assumption that \(P \in \Theta\), is not required, so that all possible models are allowed to be misspecified. Indeed, Shalizi (2009) provides an example of such misspecification where the true model \(P\) is not Markov but all the hypothesized models indexed by \(\theta\) are \(k\)-th order stationary binary Markov models, for \(k = 1, 2, \ldots\). As shown in Shalizi (2009), the results of posterior convergence hold even in the case of such misspecification, essentially because the true model can be approximated by the \(k\)-th order Markov models belonging to \(\Theta\).

Given a prior \(\pi\) on \(\theta\), we assume that the posterior distributions \(\pi(\cdot | Y_n)\) are dominated by a common measure for all \(n > 0\).

A.1 Assumptions

(S1) Letting \(f_\theta(Y_n)\) be the likelihood under parameter \(\theta\) and \(f_{\theta_0}(Y_n)\) be the likelihood under the true parameter \(\theta_0\), given the true model \(P\), consider the following likelihood ratio:

\[
R_n(\theta) = \frac{f_\theta(Y_n)}{f_{\theta_0}(Y_n)}.
\]  

(A.1)

Assume that \(R_n(\theta)\) is \(\mathcal{F}_n \times \mathcal{T}\)-measurable for all \(n > 0\).

(S2) For every \(\theta \in \Theta\), the KL divergence rate

\[
h(\theta) = \lim_{n \to \infty} \frac{1}{n} E \left[ \log \left( \frac{f_{\theta_0}(Y_n)}{f_\theta(Y_n)} \right) \right].
\]  

(A.2)

exists (possibly being infinite) and is \(\mathcal{T}\)-measurable. Note that in the iid set-up, \(h(\theta)\) reduces to the KL divergence between the true and the hypothesized model, so that (A.2) may be regarded as a generalized KL divergence measure.

(S3) For each \(\theta \in \Theta\), the generalized or relative asymptotic equipartition property holds, and so, almost surely with respect to \(P\),

\[
\lim_{n \to \infty} \frac{1}{n} \log [R_n(\theta)] = -h(\theta),
\]  

(A.3)
where \( h(\theta) \) is given by (A.2).

Intuitively, the terminology “asymptotic equipartition” refers to dividing up \( \log [R_n(\theta)] \) into \( n \) factors for large \( n \) such that all the factors are asymptotically equal. Again, considering the iid scenario helps clarify this point, as in this case each factor converges to the same KL divergence between the true and the postulated model. With this understanding note that the purpose of condition (S3) is to ensure that relative to the true distribution, the likelihood of each \( \theta \) decreases to zero exponentially fast, with rate being the KL divergence rate (A.3).

(S4) Let \( I = \{ \theta : h(\theta) = \infty \} \). The prior \( \pi \) on \( \theta \) satisfies \( \pi(I) < 1 \). Failure of this assumption entails extreme misspecification of almost all the hypothesized models \( f_\theta \) relative to the true model \( p \). With such extreme misspecification, posterior consistency is not expected to hold.

(S5) There exists a sequence of sets \( G_n \to \Theta \) as \( n \to \infty \) such that:

1. \( h(G_n) \to h(\Theta) \), as \( n \to \infty \).
2. The following inequality holds for some \( \alpha > 0, \beta > 2h(\Theta) \)
   \[ \pi(G_n) \geq 1 - \alpha \exp(-\beta n) ; \]
3. The convergence in (S3) is uniform in \( \theta \) over \( G_n \setminus I \).

The sets \( G_n \) can be loosely interpreted as the sieves. Method of sieves is common to Bayesian non parametric approach, such that the behaviour of the likelihood ratio and the posterior on the sets \( G_n \) essentially carries over to \( \Theta \). This can be anticipated from the first and the second parts of the assumption; the second part ensuring in particular that the parts of \( \Theta \) on which the log likelihood ratio may be ill-behaved have exponentially small prior probabilities. The third part is more of a technical condition that is useful in proving posterior convergence through the sets \( G_n \). For further details, see Shalizi (2009).

For each measurable \( A \subseteq \Theta \), for every \( \delta > 0 \), there exists a random natural number \( \tau(A, \delta) \) such that
\[
\frac{1}{n} \log \left[ \int_A R_n(\theta)\pi(\theta)d\theta \right] \leq \delta + \limsup_{n \to \infty} \frac{1}{n} \log \left[ \int_A R_n(\theta)\pi(\theta)d\theta \right],
\]
for all \( n > \tau(A, \delta) \), provided \( \limsup_{n \to \infty} \frac{1}{n} \log \left[ \int_A R_n(\theta)\pi(\theta)d\theta \right] < \infty \). Regarding this, the following assumption has been made by Shalizi:

(S6) The sets \( G_n \) of (A5) can be chosen such that for every \( \delta > 0 \), the inequality \( n > \tau(G_n, \delta) \) holds almost surely for all sufficiently large \( n \).

To understand the essence of this assumption, note that for almost every data set \( \{Y_1, Y_2, \ldots\} \) there exists \( \tau(G_n, \delta) \) such that equation (A.4) holds with \( A \) replaced by \( G_n \) for all \( n > \tau(G_n, \delta) \). Since \( G_n \) are sets with large enough prior probabilities, the assumption formalizes our expectation that \( R_n(\theta) \) decays fast enough on \( G_n \) so that \( \tau(G_n, \delta) \) is nearly stable in the sense that it is not only finite but also not significantly different for different data sets when \( n \) is large. See Shalizi (2009) for more detailed explanation.

(S7) The sets \( G_n \) of (S5) and (S6) can be chosen such that for any set \( A \) with \( \pi(A) > 0 \),
\[
\lim_{n \to \infty} h(G_n \cap A) = h(A), \quad \text{(A.5)}
\]
Under the above assumptions, Shalizi (2009) proved the following results.
Theorem 14 (Shalizi (2009)). Consider assumptions (S1)–(S7) and any set $A \in T$ with $\pi(A) > 0$ and $h(A) > h(\Theta)$. Then,

$$\lim_{n \to \infty} \pi(A|Y_n) = 0, \text{ almost surely.}$$

The rate of convergence of the log-posterior is given by the following result.

Theorem 15 (Shalizi (2009)). Consider assumptions (S1)–(S7) and any set $A \in T$ with $\pi(A) > 0$. If $\beta > 2h(A)$, where $\beta$ corresponds to assumption (S5), or if $A \subset \cap_{k=n}^{\infty} G_k$ for some $n$, then

$$\lim_{n \to \infty} \frac{1}{n} \log \pi(A|Y_n) = -J(A), \text{ almost surely.}$$
B Verification of (S1) to (S7) for binary regression

B.1 Verification of (S1) for binary regression

Observe that

\[ f_p(Y_n|X_n) = \prod_{i=1}^{n} f(y_i|x_i) = \prod_{i=1}^{n} p(x_i)^{y_i} (1 - p(x_i))^{1-y_i}, \quad (B.1) \]

\[ f_{p_0}(Y_n|X_n) = \prod_{i=1}^{n} f_0(y_i|x_i) = \prod_{i=1}^{n} p_0(x_i)^{y_i} (1 - p_0(x_i))^{1-y_i}. \quad (B.2) \]

Therefore,

\[ \frac{1}{n} \log R_n(p) = \frac{1}{n} \sum_{i=1}^{n} \left\{ (y_i \log \left( \frac{p(x_i)}{p_0(x_i)} \right)) + (1 - y_i) \log \left( \frac{1 - p(x_i)}{1 - p_0(x_i)} \right) \right\}. \quad (B.3) \]

To show measurability of \( R_n(p) \), first note that for any \( a \in \mathbb{R} \),

\[ \left\{ (y_i, \eta) : y_i \log \left( \frac{p(x_i)}{p_0(x_i)} \right) + (1 - y_i) \log \left( \frac{1 - p(x_i)}{1 - p_0(x_i)} \right) < a \right\} = \left\{ \eta : \log \left( \frac{p(x_i)}{p_0(x_i)} \right) < a \right\} \cup \left\{ \eta : \log \left( \frac{1 - p(x_i)}{1 - p_0(x_i)} \right) < a \right\}. \quad (B.4) \]

Note that for given \( p \), there exists \( 0 \leq \epsilon < 1/2 \) such that \( \epsilon < p(x) < 1 - \epsilon \), for all \( x \in \mathcal{X} \). Now consider a sequence \( \tilde{\eta}_j, j = 1, 2, \ldots \) such that \( \| \tilde{\eta}_j - \eta \|_{\infty} \to 0 \), as \( j \to \infty \). Then, with \( \tilde{p}_j(x) = H(\tilde{\eta}_j(x)) \), note that there exists \( j_0 \geq 1 \) such that for \( j \geq j_0 \), \( \epsilon < \tilde{p}_j(x) < 1 - \epsilon \), for all \( x \in \mathcal{X} \). Hence, using the inequality \( 1 - \frac{1}{x} \leq \log x \leq x - 1 \) for \( x > 0 \), we obtain

\[ \left| \log \left( \frac{\tilde{p}_j(x_i)}{p(x_i)} \right) \right| \leq C \| \tilde{p}_j - p \|_{\infty} \quad \text{and} \quad \left| \log \left( \frac{1 - \tilde{p}_j(x_i)}{1 - p(x_i)} \right) \right| \leq C \| \tilde{p}_j - p \|_{\infty}, \quad \text{for some} \ C > 0, \text{for all} \ x \in \mathcal{X}. \]

Hence, for \( j \geq j_0 \),

\[ \left| \log \left( \frac{\tilde{p}_j(x_i)}{p(x_i)} \right) - \log \left( \frac{p(x_i)}{p_0(x_i)} \right) \right| = \left| \log \left( \frac{\tilde{p}_j(x_i)}{p(x_i)} \right) \right| \leq C \| \tilde{p}_j - p \|_{\infty}. \quad (B.5) \]

Now, since \( H \) is continuously differentiable, using Taylor’s series expansion up to the first order we obtain,

\[ \| \tilde{p}_j - p \|_{\infty} = \sup_{x \in \mathcal{X}} |H(\tilde{\eta}_j(x)) - H(\eta(x))| \]

\[ = \sup_{x \in \mathcal{X}} |H'(u(\tilde{\eta}_j(x), \eta(x)))| \| \tilde{\eta}_j - \eta \|_{\infty}, \quad (B.6) \]

where \( u(\tilde{\eta}_j(x), \eta(x)) \) lies between \( \eta(x) \) and \( \tilde{\eta}_j(x) - \eta(x) \). Since \( \| \tilde{\eta}_j - \eta \|_{\infty} \to 0 \), as \( j \to \infty \), it follows from (B.6) that \( \| \tilde{p}_j - p \|_{\infty} \to 0 \), as \( j \to \infty \). This again implies, thanks to (B.5), that

\[ \left| \log \left( \frac{\tilde{p}_j(x_i)}{p(x_i)} \right) - \log \left( \frac{p(x_i)}{p_0(x_i)} \right) \right| \to 0, \text{as} \ j \to \infty. \]

In other words, \( \log \left( \frac{p(x_i)}{p_0(x_i)} \right) \) is continuous in \( \eta \), and hence \( \left\{ \eta : \log \left( \frac{p(x_i)}{p_0(x_i)} \right) < a \right\} \) of (B.4) is measurable. Similarly, \( \log \left( \frac{1 - p(x_i)}{1 - p_0(x_i)} \right) \) is also continuous in \( \eta \), so that \( \left\{ \eta : \log \left( \frac{1 - p(x_i)}{1 - p_0(x_i)} \right) < a \right\} \) is also measurable. Hence, the individual terms in (B.3) are measurable. Since sums of measurable functions are measurable, it follows that \( \log R_n(p) \), and hence \( R_n(p) \), is measurable.
B.2 Verification of (S2) for binary regression

for every \( p \in \Theta \), we need to show that the KL divergence rate

\[
h(p) = \lim_{n \to \infty} \frac{1}{n} E_{p_0} \left[ \log \left\{ \frac{f_{p_0}(Y_n | X_n)}{f_p(Y_n | X_n)} \right\} \right] = \lim_{n \to \infty} \frac{1}{n} E_{p_0} \left[ -\log \{ R_n(p) \} \right].
\]

exists (possibly being infinite) and is \( \mathcal{T} \)-measurable.

Now,

\[
\frac{1}{n} \log R_n(p) = \frac{1}{n} \sum_{i=1}^{n} \left\{ (y_i \log p(x_i)) + (1 - y_i) \log (1 - p(x_i)) \right\} \quad (B.7)
\]

Therefore,

\[
\frac{1}{n} E_{p_0} \left[ -\log \{ R_n(p) \} \right] = \frac{1}{n} \sum_{i=1}^{n} \left\{ (p_0(x_i) \log p_0(x_i)) + (1 - p_0(x_i)) \log (1 - p_0(x_i)) \right\} \quad (B.8)
\]

The last line follows from Assumption 4 and SLLN. Here \( E_X(\cdot) = \int_X \cdot \ dQ \).

Hence,

\[
h(p) = E_X \left( p_0(X) \log \left\{ \frac{p_0(X)}{p(X)} \right\} \right) + E_X \left( (1 - p_0(X)) \log \left\{ \frac{(1 - p_0(X))}{(1 - p(X))} \right\} \right). \quad (B.10)
\]

B.3 Verification of (S3) for binary regression

Here we need to verify the asymptotic equipartition, that is, almost surely with respect to \( P \),

\[
\lim_{n \to \infty} \frac{1}{n} \log \{ R_n(p) \} = -h(p) = \lim_{n \to \infty} \frac{1}{n} E \left[ \log \left\{ \frac{f_p(Y_n | X_n)}{f_{p_0}(Y_n | X_n)} \right\} \right]. \quad (B.11)
\]

Observe that,
\[
\frac{1}{n} \log R_n(p) = \frac{1}{n} \sum_{i=1}^{n} \left\{ (y_i \log p(x_i)) + (1 - y_i) \log (1 - p(x_i)) \right\} \\
-\frac{1}{n} \sum_{i=1}^{n} \left\{ (y_i \log p_0(x_i)) + (1 - y_i) \log (1 - p_0(x_i)) \right\}.
\]

By rearranging the terms we get,

\[
-\frac{1}{n} \log R_n(p) = \frac{1}{n} \sum_{i=1}^{n} \left\{ y_i \log \left( \frac{p_0(x_i)}{p(x_i)} \right) + (1 - y_i) \log \left( \frac{1 - p_0(x_i)}{1 - p(x_i)} \right) \right\}.
\]

Using the inequality \(1 - \frac{1}{x} \leq \log x \leq x - 1\) for \(x > 0\), compactness of \(\mathcal{X}\), and continuity of \(p(x)\) in \(x \in \mathcal{X}\) for given \(p \in \Theta\), \(|\log \left( \frac{p_0(x_i)}{p(x_i)} \right)| \leq C\|p - p_0\|_{\infty}\) and \(|\log \left( \frac{1 - p_0(x_i)}{1 - p(x_i)} \right)| \leq C\|p - p_0\|_{\infty}\), for some \(C > 0\). Hence,

\[
\sum_{i=1}^{\infty} i^{-2} \text{var} \left[ \left\{ y_i \log \left( \frac{p_0(x_i)}{p(x_i)} \right) + (1 - y_i) \log \left( \frac{1 - p_0(x_i)}{1 - p(x_i)} \right) \right\} \right] \\
= \sum_{i=1}^{\infty} i^{-2} p_0(x_i)(1 - p_0(x_i)) \\
\times \left\{ \left[ \log \left( \frac{p_0(x_i)}{p(x_i)} \right) \right]^2 + \left[ \log \left( \frac{1 - p_0(x_i)}{1 - p(x_i)} \right) \right]^2 - 2 \log \left( \frac{p_0(x_i)}{p(x_i)} \right) \times \log \left( \frac{1 - p_0(x_i)}{1 - p(x_i)} \right) \right\} \\
\leq 4C^2\|p_0\|_{\infty}\|p - p_0\|_{\infty}^2 \sum_{i=1}^{\infty} i^{-2} < \infty.
\] (B.13)

Observe that \(y_i\) are observations from independent random variables. Hence by Kolmogorov’s SLLN for independent random variables,

\[
-\frac{1}{n} \log R_n(p) = \frac{1}{n} \sum_{i=1}^{n} \left\{ y_i \log \left( \frac{p_0(x_i)}{p(x_i)} \right) + (1 - y_i) \log \left( \frac{1 - p_0(x_i)}{1 - p(x_i)} \right) \right\} \\
\to \left[ E_X \left( p_0(X) \log \left( \frac{p_0(X)}{p(X)} \right) \right) \right] + E_X \left( (1 - p_0(X)) \log \left( \frac{(1 - p_0(X))}{(1 - p(X))} \right) \right) = h(p),
\]

almost surely, as \(n \to \infty\).

**B.4 Verification of (S4) for binary regression**

If \(I = \{p : h(p) = \infty\}\) then we need to show \(\Pi(I) < 1\). Note that due to compactness of \(\mathcal{X}\) and continuity of \(H\) and \(\eta\), given \(\eta \in \Theta\), \(p\) is bounded away from 0 and 1. Hence,

\(h(p) \leq \|p - p_0\|_{\infty} \times \left( \frac{1}{\inf_{x \in \mathcal{X}} p(x)} + \frac{1}{1 - \sup_{x \in \mathcal{X}} p(x)} \right) < \infty\), almost surely. In other words, (S4) holds.

**B.5 Verification of (S5) for binary regression**

In our model, the parameter space is \(\Theta = C'(\mathcal{X})\). We need to show that there exists a sequence of sets \(\mathcal{G}_n \to \Theta\) as \(n \to \infty\) such that:

1. \(h(\mathcal{G}_n) \to h(\Theta)\), as \(n \to \infty\).
2. The inequality \( \pi(G_n) \geq 1 - \alpha \exp(-\beta n) \) holds for some \( \alpha > 0, \beta > 2h(\Theta) \).

3. The convergence in (S3) is uniform in \( p \) over \( G_n \setminus I \).

We shall work with the following sequence of sieve sets considered in Chatterjee and Bhattacharya (2020): for \( n \geq 1 \),

\[
G_n = \left\{ \eta \in C'(X) : \|\eta\|_\infty \leq \exp((\beta n)^{1/4}), \|\eta'_j\|_\infty \leq \exp((\beta n)^{1/4}); j = 1, 2, \ldots, d \right\}. \tag{B.14}
\]

Then \( G_n \to C'(X) \) as \( n \to \infty \) (Chatterjee and Bhattacharya 2020).

B.5.1 Verification of (S5) (1)

We now verify that \( h(G_n) \to h(\Theta) \), as \( n \to \infty \). Observe that:

\[
h(p) = \left[ E_X \left( p_0(X) \log \left( \frac{p_0(X)}{p(X)} \right) \right) + E_X \left( (1 - p_0(X)) \log \left( \frac{(1 - p_0(X))}{(1 - p(X))} \right) \right) \right]. \tag{B.15}
\]

Recall that \( h(p) \) is continuous in \( p \) and \( p \) is continuous in \( \eta \), which follows from (B.6). Hence, continuity of \( h(p) \), compactness of \( G_n \), along with its non-decreasing nature with respect to \( n \) implies that \( h(G_n) \to h(\Theta) \), as \( n \to \infty \).

B.5.2 Verification of (S5) (2)

\[
\pi(G_n) = \Pi \left( \|\eta\| \leq \exp((\beta n)^{1/4}) \right) \\
- \pi \left( \|\eta'_j\| \leq \exp((\beta n)^{1/4}); j = 1, 2, \ldots, d \right) \\
= \pi \left( \|\eta\| \leq \exp((\beta n)^{1/4}) \right) \\
- \pi \left( \bigcup_{j=1}^{d} \left\{ \|\eta'_j\| \leq \exp((\beta n)^{1/4}) \right\} \right) \\
\geq 1 - \Pi \left( \|\eta\| > \exp((\beta n)^{1/4}) \right) - \sum_{j=1}^{d} \Pi \left( \|\eta'_j\| \leq \exp((\beta n)^{1/4}) \right) \\
\geq 1 - \left( c_\eta + \sum_{j=1}^{d} c_{\eta'_j} \right) \exp(-\beta n).
\]

where the last inequality follows from Assumption 3.

B.5.3 Verification of (S5) (3)

We need to show that uniform convergence in (S3) in \( p \) over \( G_n \setminus I \) holds, where \( I = \{ p : h(p) = \infty \} \) as in subsection B.4. In our case, \( I = \emptyset \). Hence, we need to show uniform convergence in (S3) in \( p \) over \( G_n \). We need to establish that \( G_n \) is compact, but this has already been shown by Chatterjee and Bhattacharya (2020). In a nutshell, Chatterjee and Bhattacharya (2020) proved compactness of \( G_n \) for each \( n \geq 1 \) by showing that \( G_n \) is closed, bounded and equicontinuous and then by using Arzela-Ascoli lemma to imply compactness. It should be noted that boundedness of the partial derivatives as in Assumption 1 is used to show Lipschitz continuity, hence equicontinuity.
Consider $G \in \{G_n : n = 1, 2, \ldots \}$. Now, to show uniform convergence we only need to show the following (see, for example, Chatterjee and Bhattacharya (2020)):

(i) $\frac{1}{n} \log(R_n(p)) + h(p)$ is stochastically equicontinuous almost surely in $p \in G$.

(ii) $\frac{1}{n} \log(R_n(p)) + h(p) \to 0$ for all $p \in G$ as $n \to \infty$.

We have already shown almost sure pointwise convergence of $n^{-1} \log(R_n(p))$ to $-h(p)$ in Appendix B.3. Hence it is enough to verify stochastic equicontinuity of $\frac{1}{n} \log(R_n(p)) + h(p)$ in $G \in \{G_n : n = 1, 2, \ldots \}$. Stochastic equicontinuity usually follows easily if one can prove that the function concerned is almost surely Lipschitz continuous (Chatterjee and Bhattacharya (2020)). Observe that, if we can show that both $\frac{1}{n} \log(R_n(p))$ and $h(p)$ are Lipschitz then this would imply that $\frac{1}{n} \log(R_n(p)) + h(p)$ is Lipschitz (sum of Lipschitz functions is Lipschitz).

We now show that $\frac{1}{n} \log(R_n(p))$ and $h(p)$ are both Lipschitz in $G$. Now,

$$\frac{1}{n} \log R_n(p) = \frac{1}{n} \sum_{i=1}^{n} \left\{ y_i \log \left( \frac{p(x_i)}{p_0(x_i)} \right) + (1 - y_i) \log \left( \frac{1 - p(x_i)}{1 - p_0(x_i)} \right) \right\}. \tag{B.16}$$

Let $p_1, p_2$ correspond to $\eta_1, \eta_2 \in \Theta$. Note that, since $\|\eta\|_\infty \leq \exp(\sqrt{m})$ on $G = G_m$ $(m \geq 1)$, it follows that $0 < \kappa_B \leq p_1(x), p_2(x) \leq 1 - \kappa_B < 1$, for all $x \in X$. Thus, there exists $C > 0$ such that $\left| \log \left( \frac{p_1(x)}{p_2(x)} \right) \right| \leq C\|p_1 - p_2\|_\infty$ and $\left| \log \left( \frac{1 - p_1(x)}{1 - p_2(x)} \right) \right| \leq C\|p_1 - p_2\|_\infty$, for $x \in X$. Hence,

$$\left| \frac{1}{n} \log R_n(p_1) - \frac{1}{n} \log R_n(p_2) \right| = \left| \frac{1}{n} \sum_{i=1}^{n} \left\{ y_i \log \left( \frac{p_1(x_i)}{p_2(x_i)} \right) + (1 - y_i) \log \left( \frac{1 - p_1(x_i)}{1 - p_2(x_i)} \right) \right\} \right| \leq 2C\|p_1 - p_2\|_\infty,$$

showing Lipschitz continuity of $\frac{1}{n} \log R_n(p)$ with respect to $p$ corresponding to $\eta \in G = G_m$. Since $H$ is continuously differentiable, $\eta$ and $\eta'$ are bounded on $G$, with the same bound for all $\eta$, it follows that $p$ is Lipschitz on $G$.

To see that $h(p)$ is also Lipschitz in $G = G_m$, it is enough to note that

$$|h(p_1) - h(p_2)| = \left| E_X \left( p_0(X) \log \left( \frac{p_2(X)}{p_1(X)} \right) \right) + E_X \left( (1 - p_0(X)) \log \left( \frac{1 - p_2(X)}{1 - p_1(X)} \right) \right) \right| \leq 2C\|p_1 - p_2\|_\infty,$$

and the result follows since $p$ is Lipschitz on $G$.

### B.6 Verification of (S6) for binary regression

We need to show:

$$\sum_{n=1}^{\infty} \int_{S'} P \left( \left| \frac{1}{n} \log R_n(p) + h(p) \right| > \kappa - h(\Theta) \right) \, d\sigma(p) < \infty. \tag{B.17}$$
Let us take $\kappa_1 = \kappa - h(\Theta)$. Observe that,

$$\frac{1}{n} \log R_n(p) + h(p)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left\{ y_i \log \left( \frac{p(x_i)}{p_0(x_i)} \right) + (1 - y_i) \log \left( \frac{1 - p(x_i)}{1 - p_0(x_i)} \right) \right\}$$

$$+ \left[ E_X \left( p_0(X) \log \left\{ \frac{p_0(X)}{p(X)} \right\} \right) + E_X \left( (1 - p_0(X)) \log \left\{ \frac{(1 - p_0(X))}{(1 - p(X))} \right\} \right) \right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left\{ y_i \log \left( \frac{p(x_i)}{p_0(x_i)} \right) - E_X \left( p_0(X) \log \left\{ \frac{p(X)}{p_0(X)} \right\} \right) \right\}$$

$$+ \frac{1}{n} \sum_{i=1}^{n} \left\{ (1 - y_i) \log \left( \frac{1 - p(x_i)}{1 - p_0(x_i)} \right) - E_X \left( (1 - p_0(X)) \log \left\{ \frac{(1 - p(X))}{(1 - p_0(X))} \right\} \right) \right\}.$$

It follows that:

$$P \left( \left| \frac{1}{n} \log R_n(p) + h(p) \right| > \kappa_1 \right) > \kappa_1 \right) \leq \frac{1}{\kappa_1}$$

$$+ P \left( \left| \frac{1}{n} \sum_{i=1}^{n} \left\{ y_i \log \left( \frac{p(x_i)}{p_0(x_i)} \right) - E_X \left( p_0(X) \log \left\{ \frac{p(X)}{p_0(X)} \right\} \right) \right\} \right| > \frac{\kappa_1}{2} \right) \leq \frac{1}{\kappa_1}$$

$$+ P \left( \left| \frac{1}{n} \sum_{i=1}^{n} \left\{ (1 - y_i) \log \left( \frac{1 - p(x_i)}{1 - p_0(x_i)} \right) - E_X \left( (1 - p_0(X)) \log \left\{ \frac{(1 - p(X))}{(1 - p_0(X))} \right\} \right) \right\} \right| > \frac{\kappa_1}{2} \right).$$

Since $y_i$ are binary, it follows using the inequalities $1 - \frac{1}{x} \leq \log x \leq x - 1$, for $x > 0$ and Assumptions 5 and 6, that the random variables $V_i = y_i \log \left( \frac{p(x_i)}{p_0(x_i)} \right)$ and $W_i = y_i \log \left( \frac{1 - p(x_i)}{1 - p_0(x_i)} \right)$ are absolutely bounded by $C\|p - p_0\|_{\infty}$, for some $C > 0$. We shall apply Hoeffding’s inequality (Hoeffding (1963)) separately on the two terms of (B.20) involving $V_i$ and $W_i$.

Note that for $\eta \in \mathcal{G}_n$,

$$P \left( \left| \frac{1}{n} \sum_{i=1}^{n} \left\{ y_i \log \left( \frac{p(x_i)}{p_0(x_i)} \right) - E_X \left( p_0(X) \log \left\{ \frac{p(X)}{p_0(X)} \right\} \right) \right\} \right| > \frac{\kappa_1}{2} \right)$$

$$\leq P \left( \left| \frac{1}{n} \sum_{i=1}^{n} \left\{ y_i \log \left( \frac{p(x_i)}{p_0(x_i)} \right) - p_0(x_i) \log \left( \frac{p(x_i)}{p_0(x_i)} \right) \right\} \right| > \frac{\kappa_1}{4} \right)$$

$$+ P \left( \left| \frac{1}{n} \sum_{i=1}^{n} \left\{ p_0(x_i) \log \left( \frac{p_0(x_i)}{p_0(x_i)} \right) - E_X \left( p_0(X) \log \left\{ \frac{p(X)}{p_0(X)} \right\} \right) \right\} \right| > \frac{\kappa_1}{4} \right)$$

$$\leq 4 \exp \left\{ -\frac{\kappa_1^2}{8C^2\|p - p_0\|_{\infty}^2} \right\} \leq 4 \exp \left\{ -\frac{\kappa_1^2}{8C^2L^2\|\eta - \eta_0\|_{\infty}^2} \right\},$$

where $L > 0$ is the Lipschitz constant associated with $H$. Here it is important to note that for $\eta \in \mathcal{G}_n$, $H(\eta)$ is Lipschitz in $\eta$ thanks to continuous differentiability of $H$, and boundedness of $\eta$ and $\eta'$ by the same constant on $\mathcal{G}_n$. Also note that (B.21) holds irrespective of $x_i$; $i = 1, \ldots, n$ being random or non-random (see Chatterjee and Bhattacharya (2020)).
Similarly, for $\eta \in G_n$, 

\[ P \left( \left| \frac{1}{n} \sum_{i=1}^{n} \left( (1 - y_i) \log \left( \frac{1 - p(x_i)}{1 - p_0(x_i)} \right) - E_X \left( (1 - p_0(X)) \log \left( \frac{1 - p(X)}{1 - p_0(X)} \right) \right) \right) \right| > \frac{\kappa_1}{2} \right) \leq 4 \exp \left\{ -\frac{n\kappa_1^2}{8C^2L^2\|\eta - \eta_0\|_\infty^2} \right\}. \]

Now, 

\begin{align*}
\sum_{n=1}^{\infty} \int_{s^c} P \left( \left| \frac{1}{n} \sum_{i=1}^{n} \left( y_i \log \left( \frac{p(x_i)}{p_0(x_i)} \right) - E_X \left( p_0(X) \log \left( \frac{p(X)}{p_0(X)} \right) \right) \right) \right| > \frac{\kappa_1}{2} \right) d\pi(p) \\
\leq \sum_{n=1}^{\infty} \int_{G_n} 4 \exp \left\{ -\frac{n\kappa_1^2}{8C^2L^2\|\eta - \eta_0\|_\infty^2} \right\} d\pi(\eta) + \sum_{\eta \in G_n} \pi(\eta). \tag{B.23}
\end{align*}

and

\begin{align*}
\sum_{n=1}^{\infty} \int_{s^c} P \left( \left| \frac{1}{n} \sum_{i=1}^{n} \left( (1 - y_i) \log \left( \frac{1 - p(x_i)}{1 - p_0(x_i)} \right) - E_X \left( (1 - p_0(X)) \log \left( \frac{1 - p(X)}{1 - p_0(X)} \right) \right) \right) \right| > \frac{\kappa_1}{2} \right) d\pi(p) \\
\leq \sum_{n=1}^{\infty} \int_{G_n} 4 \exp \left\{ -\frac{n\kappa_1^2}{8C^2L^2\|\eta - \eta_0\|_\infty^2} \right\} d\pi(\eta) + \sum_{n=1}^{\infty} \pi(\eta). \tag{B.24}
\end{align*}

Then proceeding in the same way as (S-2.25) – (S-2.30) of Chatterjee and Bhattacharya (2020), and noting that $\sum_{n=1}^{\infty} \pi(\eta) < \infty$, we obtain (B.17).

Hence (S6) holds.

Remark 1. It is important to clarify the role of Assumption 6 here. Note that, we need a lower bound for $\log \left( \frac{p(x)}{p_0(x)} \right)$. For instance, if $H(\eta(x)) = \frac{\exp(\eta(x))}{1 + \exp(\eta(x))}$, then even if $\|\eta\|_\infty \leq \sqrt{\beta n}$ on $G_n$, it holds that $\log \left( \frac{p(x)}{p_0(x)} \right) \geq C - \sqrt{\beta n}$ for all $x \in X$, for all $\eta \in G_n$, for some constant $C$.

In our bounding method using the inequality $\log x \geq 1 - 1/x$ for $x > 0$, we have $\log \left( \frac{p(x)}{p_0(x)} \right) \geq -\|p - p_0\|_\infty \geq -2 \exp(\sqrt{\beta n}) \|p - p_0\|_\infty$. It would then follow that the exponent of the Hoeffding inequality is $O(1)$. This would fail to ensure summability of the corresponding terms involving $V_i$. Thus, we need to ensure that $p(x)$ is bounded away from 0. Similarly, the infinite sum associated with $W_i$ would not be finite unless $1 - p(x)$ is bounded away from 0.

B.7 Verification of (S7) for Binary Regression

This verification follows from the fact that $h(p)$ is continuous. Indeed, for any set $A$ with $\pi(A) > 0$, $G_n \cap A \uparrow A$. It follows from continuity of $h$ that $h \left( (G_n \cap A) \downarrow h(A) \right)$ as $n \to \infty$ and hence (S7) holds.
C Verification of (S1) to (S7) for Poisson regression

C.1 Verification of (S1) for Poisson regression

Observe that
\[ f_\lambda(Y_n|X_n) = \prod_{i=1}^{n} f(y_i|x_i) = \prod_{i=1}^{n} \exp \left( -\lambda(x_i) \right) \left( \frac{\lambda(x_i)}{y_i!} \right)^{y_i}, \]
\[ f_{\lambda_0}(Y_n|X_n) = \prod_{i=1}^{n} f_0(y_i|x_i) = \prod_{i=1}^{n} \exp \left( -\lambda_0(x_i) \right) \left( \frac{\lambda_0(x_i)}{y_i!} \right). \]

Therefore,
\[ R_n(\lambda) = \exp \left( -\sum_{i=1}^{n} [\lambda(x_i) - \lambda_0(x_i)] \right) \prod_{i=1}^{n} \left( \frac{\lambda(x_i)}{\lambda_0(x_i)} \right)^{y_i} \tag{C.1} \]
and,
\[ \frac{1}{n} \log R_n(\lambda) = \left( -\frac{1}{n} \sum_{i=1}^{n} [\lambda(x_i) - \lambda_0(x_i)] \right) + \frac{1}{n} \sum_{i=1}^{n} y_i \log \left( \frac{\lambda(x_i)}{\lambda_0(x_i)} \right). \tag{C.2} \]

Note that for any \( a \in \mathbb{R}, \left\{ (y_i, \eta) : y_i \log \left( \frac{\lambda(x_i)}{\lambda_0(x_i)} \right) < a \right\} = \bigcup_{r=1}^{\infty} \left\{ \eta : r \log \left( \frac{\lambda(x_i)}{\lambda_0(x_i)} \right) < a \right\}. \]

Let \( \tilde{\eta}_j : j = 1, 2, \ldots \) be such that \( \|\tilde{\eta}_j - \eta\|_\infty \to 0 \), as \( j \to \infty \). Then, letting \( \tilde{\lambda}_j(x) = H(\tilde{\eta}_j(x)) \), for all \( x \in \mathbb{X} \), it follows, since \( 0 < C_1 \leq \lambda(x) \leq C_2 < \infty \) on \( \mathbb{X} \), that there exists \( j_0 \geq 1 \) such that for \( j \geq j_0, 0 < C_1 \leq \tilde{\lambda}_j(x) \leq C_2 < \infty \). Hence, using the inequalities \( 1 - \frac{1}{r} \leq \log x \leq x - 1 \) for \( x > 0 \), we obtain
\[ \left| r \log \left( \frac{\tilde{\lambda}_j(x_i)}{\lambda_0(x_i)} \right) - r \left( \frac{\lambda(x_i)}{\lambda_0(x_i)} \right) \right| = r \left| \log \left( \frac{\tilde{\lambda}_j(x_i)}{\lambda_0(x_i)} \right) \right| \leq rC \|\tilde{\lambda}_j - \lambda\|_\infty \to 0, \]
in the same way as in the binary regression, using Taylor’s series expansion up to the first order. Hence, \( r \log \left( \frac{\lambda(x_i)}{\lambda_0(x_i)} \right) \) is continuous in \( \eta \), ensuring measurability of
\[ \left\{ \eta : r \log \left( \frac{\lambda(x_i)}{\lambda_0(x_i)} \right) < a \right\}, \]
and hence of \( \left\{ (y_i, \eta) : y_i \log \left( \frac{\lambda(x_i)}{\lambda_0(x_i)} \right) < a \right\}. \) It follows that \( \frac{1}{n} \sum_{i=1}^{n} y_i \log \left( \frac{\lambda(x_i)}{\lambda_0(x_i)} \right) \) is measurable.

Also, continuity of \( \lambda(x_i) - \lambda_0(x_i) \) with respect to \( \eta \) ensures measurability of \( -\frac{1}{n} \sum_{i=1}^{n} [\lambda(x_i) - \lambda_0(x_i)] \). Thus, \( \frac{1}{n} \log R_n(\lambda), \) and hence \( R_n(\lambda), \) is measurable.

C.2 Verification of (S2) for Poisson regression

For every \( \lambda \in \Lambda \), we need to show that the KL divergence rate
\[ h(\lambda) = \lim_{n \to \infty} \frac{1}{n} E_{\lambda_0} \left[ \log \left( \frac{f_{\lambda_0}(Y_n|X_n)}{f_\lambda(Y_n|X_n)} \right) \right] = \lim_{n \to \infty} \frac{1}{n} E_{\lambda_0} \left[ -\log \left\{ R_n(\lambda) \right\} \right]. \]
exists (possibly being infinite) and is \( T \)-measurable.

Now,
\[ \frac{1}{n} \log R_n(\lambda) = \left( -\frac{1}{n} \sum_{i=1}^{n} [\lambda(x_i) - \lambda_0(x_i)] \right) + \frac{1}{n} \sum_{i=1}^{n} y_i \log \left( \frac{\lambda(x_i)}{\lambda_0(x_i)} \right) \]
Therefore,
\[
\frac{1}{n} E_{\lambda_0} \left[ - \log \{ R_n(\lambda) \} \right] = \left( \frac{1}{n} \sum_{i=1}^{n} [\lambda(x_i) - \lambda_0(x_i)] \right) + \frac{1}{n} \sum_{i=1}^{n} \lambda_0(x_i) \log \left( \frac{\lambda_0(x_i)}{\lambda(x_i)} \right).
\]

\[
\lim_{n \to \infty} \frac{1}{n} E_{\lambda_0} \left[ - \log \{ R_n(\lambda) \} \right] = \lim_{n \to \infty} \left( \frac{1}{n} \sum_{i=1}^{n} [\lambda(x_i) - \lambda_0(x_i)] \right) + \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \lambda_0(x_i) \log \left( \frac{\lambda_0(x_i)}{\lambda(x_i)} \right)
\]
\[
= E_X [\lambda(X) - \lambda_0(X)] + E_X \left[ \lambda_0(X) \log \left( \frac{\lambda_0(X)}{\lambda(X)} \right) \right].
\]

The last line holds due to Assumption 4 and SLLN. Here \( E_X (\cdot) = \int_X \cdot dQ \). In other words,
\[
h(\lambda) = E_X [\lambda(X) - \lambda_0(X)] + E_X \left[ \lambda_0(X) \log \left( \frac{\lambda_0(X)}{\lambda(X)} \right) \right]. \tag{C.3}
\]

### C.3 Verification of (S3) for Poisson regression

Here we need to verify the asymptotic equipartition property, that is, almost surely with respect to the true model \( P \),
\[
\lim_{n \to \infty} \frac{1}{n} \log \{ R_n(\lambda) \} = -h(\lambda) = \lim_{n \to \infty} \frac{1}{n} E \left[ \log \left( \frac{f_{\lambda}(Y_n|X_n)}{f_{\lambda_0}(Y_n|X_n)} \right) \right]. \tag{C.4}
\]

Now,
\[
-\frac{1}{n} \log R_n(\lambda) = \frac{1}{n} \sum_{i=1}^{n} \left\{ [\lambda(x_i) - \lambda_0(x_i)] + y_i \log \left( \frac{\lambda_0(x_i)}{\lambda(x_i)} \right) \right\}.
\]

As before, for given \( \lambda \), there exists \( C > 0 \) such that \( \left| \log \left( \frac{\lambda_0(x_i)}{\lambda(x_i)} \right) \right| \leq C \| \lambda - \lambda_0 \|_\infty \). Hence,
\[
\sum_{i=1}^{\infty} i^{-2} \text{Var} \left[ \left\{ [\lambda(x_i) - \lambda_0(x_i)] + y_i \log \left( \frac{\lambda_0(x_i)}{\lambda(x_i)} \right) \right\} \right]
\]
\[
= \sum_{i=1}^{\infty} i^{-2} \lambda_0(x_i) \left[ \log \left( \frac{\lambda_0(x_i)}{\lambda(x_i)} \right) \right]^2
\]
\[
\leq C^2 \| H(\kappa_0) \| \| \lambda - \lambda_0 \|_\infty^2 \sum_{i=1}^{\infty} i^{-2}
\]
\[
< \infty. \tag{C.5}
\]

Observe that \( y_i \) are observations from independent random variables. Hence from Kolmogorovs SLLN for independent random variables and from Assumption 4, (C.4) holds as \( n \to \infty \).

### C.4 Verification of (S4) for Poisson regression

If \( I = \{ \lambda : h(\lambda) = \infty \} \) then we need to show \( \Pi(I) < 1 \). But this holds in almost the same way as for binary regression. In other words, (S4) holds for Poisson regression.

### C.5 Verification of (S5) for Poisson regression

The parameter space here remains the same as in the binary regression case, that is, \( \Theta = C'(\mathcal{X}) \). We also consider the same sequence \( \mathcal{G}_n \) as in binary regression. We need to verify that
1. \( h(\mathcal{G}_n) \to h(\Lambda) \), as \( n \to \infty \);

2. The inequality \( \pi(\mathcal{G}_n) \geq 1 - \alpha \exp(-\beta n) \) holds for some \( \alpha > 0, \beta > 2h(\Lambda) \);

3. The convergence in (S3) is uniform over \( \mathcal{G}_n \setminus I \).

C.5.1 Verification of (S5) (1)

We now need to verify that \( h(\mathcal{G}_n) \to h(\Lambda) \) as \( n \to \infty \). But this holds in the same way as for binary regression.

C.5.2 Verification of (S5) (2)

Again, this holds in the same way as for binary regression.

C.5.3 Verification of (S5) (3)

Using the same arguments as in the binary regression case, here we only need to show that \( \frac{1}{n} \log(R_n(\lambda)) \) and \( h(\lambda) \) are both Lipschitz.

Recall that

\[
\frac{1}{n} \log R_n(\lambda) = \frac{1}{n} \sum_{i=1}^{n} \left\{ \lambda_0(x_i) - \lambda(x_i) \right\} + y_i \log \left( \frac{\lambda(x_i)}{\lambda_0(x_i)} \right).
\]

For any \( \eta_1, \eta_2 \in \mathcal{G} \), there exists \( C > 0 \) such that

\[
\left| \log \left( \frac{\lambda_1(x)}{\lambda_2(x)} \right) \right| \leq C\|\lambda_1 - \lambda_2\|_{\infty}, \text{ for all } x \in \mathbb{X},
\]

where \( \lambda_1 = H(\eta_1) \) and \( \lambda_2 = H(\eta_2) \). Hence,

\[
\left| \frac{1}{n} \log R_n(\lambda_1) - \frac{1}{n} \log R_n(\lambda_2) \right| \leq \|\lambda_1 - \lambda_2\|_{\infty} \left( 1 + C \times \frac{1}{n} \sum_{i=1}^{n} y_i \right).
\]

Thus, \( \frac{1}{n} \log R_n(\lambda) \) is almost surely Lipschitz with respect to \( \lambda \). Since, by Kolmogorov’s SLLN for independent variables, \( \frac{1}{n} \sum_{i=1}^{n} y_i \xrightarrow{a.s.} E_X(\lambda_0(X)) \) \( < \infty \), as \( n \to \infty \), and since \( \lambda = H(\eta) \) is Lipschitz in \( \eta \in \mathcal{G}_n \) in the same way as in binary regression, the desired stochastic equicontinuity follows. Lipschitz continuity of \( h(\lambda) \) in \( \mathcal{G}_n \) follows using similar techniques.

C.6 Verification of (S6) for Poisson Regression

Since

\[
\sum_{n=1}^{\infty} \int_{\mathcal{G}_n^c} P \left( \left| \frac{1}{n} \log R_n(\lambda) + h(\lambda) \right| > \kappa - h(\Lambda) \right) \, d\pi(\lambda)
\]

\[
\leq \sum_{n=1}^{\infty} \int_{\mathcal{G}_n} P \left( \left| \frac{1}{n} \log R_n(\lambda) + h(\lambda) \right| > \kappa - h(\Lambda) \right) \, d\pi(\lambda)
\]

\[
+ \sum_{n=1}^{\infty} \int_{\mathcal{G}_n^c} P \left( \left| \frac{1}{n} \log R_n(\lambda) + h(\lambda) \right| > \kappa - h(\Lambda) \right) \, d\pi(\lambda)
\]

\[
\leq \sum_{n=1}^{\infty} \int_{\mathcal{G}_n} P \left( \left| \frac{1}{n} \log R_n(\lambda) + h(\lambda) \right| > \kappa - h(\Lambda) \right) \, d\pi(\lambda) + \sum_{n=1}^{\infty} \pi(\mathcal{G}_n^c), \quad (C.6)
\]

and the second term of (C.6) is finite, it is enough to show that the first term of (C.6) is finite.
Let us take \( \kappa_1 = \kappa - h(A) \). Observe that for \( \eta \in \mathcal{G}_n \),

\[
P \left( \frac{1}{n} \log R_n(\lambda) + h(\lambda) \right) > \kappa_1 \right) \nonumber\]

\[
\leq P \left( \frac{1}{n} \sum_{i=1}^{n} \left[ \lambda_0(x_i) \log \left( \frac{\lambda(x_i)}{\lambda_0(x_i)} \right) - E_X \left( \lambda_0(X) \log \left( \frac{\lambda(X)}{\lambda_0(X)} \right) \right) \right] > \kappa_1 \frac{3}{\kappa} \right) \nonumber\]

\[
+ P \left( \frac{1}{n} \sum_{i=1}^{n} \left[ (\lambda_0(x_i) - \lambda(x_i)) - E_X (\lambda_0(X) - \lambda(X)) \right] > \kappa_1 \frac{3}{\kappa} \right) \nonumber\]

\[
+ P \left( \frac{1}{n} \sum_{i=1}^{n} \left[ y_i \log \left( \frac{\lambda(x_i)}{\lambda_0(x_i)} \right) - \lambda_0(x_i) \log \left( \frac{\lambda(x_i)}{\lambda_0(x_i)} \right) \right] > \kappa_1 \frac{3}{\kappa} \right). \tag{C.7} \nonumber\]

Using Hoeffding’s inequality and Lipschitz continuity of \( H \) in \( \mathcal{G}_n \) as in binary regression, we find that (C.7) and (C.8) are bounded above by \( 2 \exp \left( -C_1 \frac{n \kappa^2}{\|\eta - \eta_0\|_\infty^2} \right) \), and \( \exp \left( -C_2 \frac{n \kappa^3}{\|\eta - \eta_0\|_\infty} \right) \), for some \( C_1 > 0 \) and \( C_2 > 0 \). These bounds hold even if the covariates are non-random.

To bound (C.9), we shall first show that the summands are sub-exponential, and then shall apply Bernstein’s inequality (see, for example, Uspensky (1937), Bennett (1962), Massart (2003)). Direct calculation yields

\[
E \left[ \exp \left( t \left( y_i \log \left( \frac{\lambda(x_i)}{\lambda_0(x_i)} \right) - \lambda_0(x_i) \log \left( \frac{\lambda(x_i)}{\lambda_0(x_i)} \right) \right) \right) \right] \\
= \exp \left[ -t \lambda_0(x_i) \log \left( \frac{\lambda(x_i)}{\lambda_0(x_i)} \right) \right] \times \exp \left[ \lambda_0(x_i) \left\{ \exp \left( t \log \left( \frac{\lambda(x_i)}{\lambda_0(x_i)} \right) \right) - 1 \right\} \right]. \tag{C.10} \nonumber\]

The first factor of (C.10) has the following upper bound:

\[
\exp \left[ -t \lambda_0(x_i) \log \left( \frac{\lambda(x_i)}{\lambda_0(x_i)} \right) \right] \leq \exp \left( c_\lambda \|\lambda\|_\infty |t| \right). \tag{C.11} \nonumber\]

A bound for the second factor of (C.10) is given as follows:

\[
\exp \left[ \lambda_0(x_i) \left\{ \exp \left( t \log \left( \frac{\lambda(x_i)}{\lambda_0(x_i)} \right) \right) - 1 \right\} \right] \\
\leq \exp \left[ \|\lambda_0\|_\infty \left( \exp \left( t \frac{\|\lambda - \lambda_0\|_\infty}{\kappa P} \right) - 1 \right) \right] \\
\leq \exp \left[ \|\lambda_0\|_\infty \left( c_\lambda |t| + c_\lambda^2 t^2 \right) \right], \tag{C.12} \nonumber\]

for \( |t| \leq c_\lambda^{-1} \), where \( c_\lambda = C \|\lambda - \lambda_0\|_\infty \), for some \( C > 0 \).

Combining (C.10), (C.11) and (C.12) we see that (C.10) is bounded above by \( \exp \left( c_\lambda^2 t^2 \right) \) provided that

\[
c_\lambda |t| \geq 2 \left( \|\lambda_0\|_\infty - 1 \right) \geq 2 \left( \kappa P - 1 \right). \tag{C.13} \nonumber\]

The rightmost bound of (C.13) is close to zero if \( \kappa P \) is chosen sufficiently small. Now consider the function \( g(t) = \exp \left( c_\lambda^2 t^2 \right) - f(t) \), where \( f(t) \) is given by (C.10). Since \( g(t) \) is continuous in \( t \) and \( g(0) = 0 \) and \( g(t) > 0 \) on \( 2 / \left( \kappa P - 1 \right) \leq |t| \leq c_\lambda^{-1} \), it follows that on the sufficiently small interval \( 0 \leq |t| \leq 2 / \left( \kappa P - 1 \right) \), \( g(t) > 0 \). In other words, (C.10) is bounded above by \( \exp \left( c_\lambda^2 t^2 \right) \) for \( 0 \leq |t| \leq c_\lambda^{-1} \). Thus, \( z_i = y_i \log \left( \frac{\lambda(x_i)}{\lambda_0(x_i)} \right) - \lambda_0(x_i) \log \left( \frac{\lambda(x_i)}{\lambda_0(x_i)} \right) \) are independent sub-exponential variables with parameter \( c_\lambda \).

Bernstein’s inequality, in conjunction with Lipschitz continuity of \( H \) on \( \mathcal{G}_n \), then ensures that (C.9) is bounded above by \( 2 \exp \left( -\frac{n}{2} \min \left\{ \frac{C_1 \kappa^2}{\|\eta - \eta_0\|_\infty}, \frac{C_2 \kappa^3}{\|\eta - \eta_0\|_\infty} \right\} \right) \), for positive constants \( C_1 \) and \( C_2 \).
The rest of the proof of finiteness of (C.6) follows in the same (indeed, simpler) way as Chatterjee and Bhattacharya (2020). Hence (S6) holds.

**Remark 2.** Arguments similar to that of Remark 1 shows that it is essential to have $\lambda$ bounded away from zero.

### C.7 Verification of (S7) for Poisson regression

This verification follows from the fact that $h(\lambda)$ is continuous, similar to binary regression.

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