DYNAMIC AND ELECTROSTATIC MODELING FOR A PIEZOELECTRIC SMART COMPOSITE AND RELATED STABILIZATION RESULTS

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Abstract. A cantilevered piezoelectric smart composite beam, consisting of perfectly bonded elastic, viscoelastic and piezoelectric layers, is considered. The piezoelectric layer is actuated by a voltage source. Both fully dynamic and electrostatic approaches, based on Maxwell’s equations, are used to model the piezoelectric layer. We obtain (i) fully-dynamic and electrostatic Rao-Nakra type models by assuming that the viscoelastic layer has a negligible weight and stiffness, (ii) fully-dynamic and electrostatic Mead-Marcus type models by neglecting the in-plane and rotational inertia terms. Each model is a perturbation of the corresponding classical smart composite beam model. These models are written in the state-space form, the existence and uniqueness of solutions are obtained in appropriate Hilbert spaces. Next, the stabilization problem for each closed-loop system, with a thorough analysis, is investigated for the natural $B^*$—type state feedback controllers. The fully dynamic Rao-Nakra model with four state feedback controllers is shown to be not asymptotically stable for certain choices of material parameters whereas the electrostatic model is exponentially stable with only three state feedback controllers (by the spectral multipliers method). Similarly, the fully dynamic Mead-Marcus model lacks of asymptotic stability for certain solutions whereas the electrostatic model is exponentially stable by only one state feedback controller.

1. Introduction. A piezoelectric smart composite beam consisting of a stiff elastic layer 1, a complaint (viscoelastic) layer 2, and a piezoelectric layer 3 is considered in this paper, see Fig. 1. The piezoelectric layer 3 is also an elastic beam with electrodes at its top and bottom surfaces, insulated at the edges (to prevent fringing effects), and connected to an external electric circuit. (See Figure 1). As the electrodes are subjected to a voltage source, the piezoelectric layer compresses or extends, inducing a bending moment in the composite structure. The electrostatic assumption (due to Maxwell’s equations) is widely used to model the single piezoelectric layer which entirely ignores the dynamic effects for the Maxwell’s magnetic equations, see i.e. [4, 33, 34]. In fact, even though it is minor in comparison to the mechanical, the dynamic electromagnetic effects have a dramatic effect on the control of these materials [22, 38, 39]. Many control approaches are also available to control piezoelectric beams such as feedback, feedforward, sensorless, etc. [7, 8].

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Figure 1. A voltage-actuated piezoelectric smart composite of length $L$ with thicknesses $h_1, h_2, h_3$ for its layers 1, 2, 3, respectively. The longitudinal motions of top and bottom layers 1 and 3 are controlled by $g^1(t), g^3(t), V(t)$, and the bending motions (of the whole composite) are controlled by $M(t), g(t)$.

For the fully-dynamic models, written in the state-space formulation $\dot{\varphi} = A\varphi + Bu(t)$, the $B^*$-type observation for the piezoelectric layer naturally corresponds to the total induced current at its electrodes. It is more physical in terms of practical applications. As well, measuring the total induced current at the electrodes of the piezoelectric layer is easier than measuring displacements or the velocity of the composite at one end of the beam, i.e. see [3, 5, 20].

The models of piezoelectric smart composites in the literature all assume the electrostatic assumption for its piezoelectric layer. These models differ by the assumptions for the viscoelastic layer and the geometry of the composite, i.e. see [36]. The models are either a Mead-Marcus (M-M) type or a Rao-Nakra (R-N) type as obtained in [3, 15], respectively. The M-M model only describes the bending motion, and the R-N model describes bending and longitudinal motions all together. These models reduce to the classical counterparts once the piezoelectric strain is taken to be zero [19, 31]. The active boundary feedback stabilization of the classical R-N model (having no piezoelectric layer) is only investigated for hinged [28] and clamped-free [40] boundary conditions. The exact controllability of the M-M and R-N models are shown for the fully clamped, fully hinged, and clamped-hinged models [12, 29]. The exponential stability in the existence of the passive distributed “shear” damping term is investigated for the R-N and M-M models ([1, 42]).

Asymptotic stabilization of piezoelectric smart composite models are investigated in [3, 15] for various PID-type feedback controllers and a shear-type distributed damping. The exponential stability of the electrostatic M-M and R-N models for clamped-free boundary conditions has been open problems for more than a decade. Recently, exponential stability of the electrostatic R-N model is shown by using four type feedback controllers [24], two for the longitudinal motions and two for the bending motions, by using the compact perturbation argument [35] together with the use of spectral multipliers [28]. Exponential stability with only three controllers is shown by using a tedious spectral-theoretic approach [41]. The fully dynamic R-N model is also shown to be not asymptotically stable for many choices of material
parameters by using the natural $B^*$-type feedback controllers [24]. The charge-actuated electrostatic counterparts are shown to be exponentially stable whereas the current-actuated electrostatic counterparts can only be asymptotically stabilizable [25].

In this paper, we develop a novel modeling strategy to model a cantilevered piezoelectric smart composite with/without the magnetic effects due to the Maxwell’s equations. We mainly use the methodology developed in [11, 24]. We obtain two models:

(i) A fully dynamic Rao-Nakra-type model by assuming that the stiffness and the weight of the middle layer are negligible.

(ii) A fully dynamic Mead-Marcus-type model by assuming that in-plane and rotational inertia terms are negligible.

By discarding the dynamic electromagnetic magnetic effects (electro-magnetic kinetic energy) in these models, electrostatic Rao-Nakra and Mead-Marcus models are simply derived. The voltage control is part of the charge boundary conditions corresponding to the dynamic charge equation. The fully dynamic Mead-Marcus type model is novel in the literature. All models are shown to be well-posed and are written in the state-space formulation.

For each model, natural $B^*$-type state feedback controllers are considered. We summarize our findings as the following:

I. The fully dynamic closed-loop Rao-Nakra-type model is shown to be asymptotically stable for the inertial sliding solutions, see Theorem 5.1. It is simply because the outer layers of the composite are made of different materials, piezoelectric and elastic, and therefore, the top and the bottom layers have different speeds of wave propagation. In fact, if the outer layers are identical piezoelectric beams as in [24], asymptotic stability fails for inertial sliding solutions.

II. In contrast to the result above, the fully dynamic Mead-Marcus-type model is not asymptotically stable for inertial-sliding solutions if the material parameters satisfy a number-theoretical condition, see Theorem 5.6.

III. The electrostatic Rao-Nakra-type model is recently shown to be exponentially stable with four controllers in [24]. This result is improved in Theorem 5.4 by eliminating one redundant controller. The proof uses higher order spectral multipliers as in [30], see Lemma 5.3. In fact, the same result is recently announced in [41] by a different approach, which requires to determine the spectrum of the closed-loop system.

IV. The electrostatic Mead-Marcus-type model is exponentially stable by only one feedback controller. Similarly, the proof mainly uses the spectral multipliers, see Theorem 5.9. Our result rigorously proves the findings of [3] where only the asymptotic stability is claimed without a proof.

Note that modeling and well-posedness results for only the R-N model are briefly announced in [23].

2. Modeling assumptions: Classical sandwich beam theory and electromagnetism. The piezoelectric smart composite beam considered in this paper is consisting of a stiff layer, a compliant layer, and a piezoelectric layer, see Figure 1. The composite occupies the region $\Omega = \Omega_{xy} \times (0, h) := [0, L] \times [-b, b] \times (0, h)$ at equilibrium. The total thickness $h$ is assumed to be small in comparison to the
dimensions of $\Omega_{xy}$. The layers are indexed from 1 to 3 from the stiff layer to the piezoelectric layer, respectively.

Let $0 = z_0 < z_1 < z_2 < z_3 = h$, with

$$h_i = z_i - z_{i-1}, \quad i = 1, 2, 3.$$  

We use the rectangular coordinates $X = (x, y)$ to denote points in $\Omega_{xy}$, and $(X, z)$ to denote points in $\Omega = \Omega^s \cup \Omega^{ve} \cup \Omega^p$, where $\Omega^s, \Omega^{ve}$, and $\Omega^p$ are the reference configurations of the stiff, viscoelastic, and piezoelectric layer, respectively, and they are given by

$$\Omega^s = \Omega_{xy} \times (z_0, z_1), \quad \Omega^{ve} = \Omega_{xy} \times (z_1, z_2), \quad \Omega^p = \Omega_{xy} \times (z_2, z_3).$$

For $(X, z) \in \Omega$, let $U(X, z) = (U_1, U_2, U_3)(X, z)$ denote the displacement vector of the point (from reference configuration). For the beam theory, all displacements are assumed to be independent of $y$–coordinate, and $U_2 \equiv 0$. The transverse displacements is $w(x, y, z) = U_3(x)$ for any $i$ and $x \in [0, L]$. Define $u^i(x, y, z) = U_i(x, 0, z_i) = u_i(x)$ for $i = 0, 1, 2, 3$ and for all $x \in (0, L)$.

Define

$$\vec{\psi} = [\psi^1, \psi^2, \psi^3]^T, \quad \vec{\phi} = [\phi^1, \phi^2, \phi^3]^T, \quad \vec{v} = [v^1, v^2, v^3]^T$$

where

$$\psi^i = \frac{u^i - u^{i-1}}{h_1}, \quad \phi^i = \psi^i + w_x, \quad v^i = \frac{u^{i-1} + u^i}{2}, \quad i = 1, 2, 3.$$  

where $\psi^i$ is the total rotation angles (with negative orientation) of the deformed filament within the $i$th layer in the $x-z$ plane, $\phi^i$ is the (small angle approximation for the) shear angles within each layer, $v^i$ is the longitudinal displacement of the center line of the $i$th layer.

For the middle layer, we apply Mindlin-Timoshenko small displacement assumptions, while for the outer layers Kirchhoff small displacement assumptions are applied. Therefore,

$$\phi^1 = \phi^3 = 0, \quad \psi^1 = \psi^3 = -w_x, \quad \phi^2 = \psi^2 + w_x.$$  

Let $G_2$ be the shear modulus of the viscoelastic layer. Defining $\ddot{\psi} = \frac{v^i - v^{i-1}}{2}$, and

$$\alpha^1 = c_{11}^3, \quad \alpha^2 = c_{12}^2, \quad \alpha^3 = \alpha^2 + \gamma^2 \beta, \quad \alpha^4 = c_3^3,$$

$$\gamma = \gamma_{31}, \quad \gamma_1 = \gamma_{15}, \quad \beta = \frac{1}{\varepsilon_{33}}, \quad \beta_1 = \frac{1}{\varepsilon_{11}},$$

where $c_{ij}^k$ are elastic stiffness coefficients of each layer, and $\gamma_{ij}$ and $\varepsilon_{ij}$, are piezoelectric and permittivity coefficients for the piezoelectric layer. Refer to [27] for sample piezoelectric constants. The displacement field, strains, and the constitutive relationships for each layer are given in Table 1. We follow the dynamic approach in [21, 24] to include the electromagnetic effects for the piezoelectric layer $\Omega$ . The magnetic field $B$ is perpendicular to the $x-z$ plane, and therefore, $B_2(x)$ is the only non-zero component. Assuming $E_1 = D_1 = 0$, and by $\frac{dE_3}{dt} = -\mu \frac{\partial D_3}{\partial z}$, we obtain the total induced current accumulated $[0, x] \in [0, L]$ portion of the piezoelectric beam $B_2 = -\mu \int_0^x D_3(\xi, z, t) d\xi$. This is physical since $B_2(0) = 0$ at the clamped end of the beam and $B_2(L) = -\mu \int_0^L \frac{\partial D_3}{\partial z}(\xi, z, t) d\xi$ at the free end of the beam. Analogously, we define $p = \int_0^x D_3(\xi, t) d\xi$ to be the total charge of the piezoelectric beam at $x \in (0, L)$ so that $p_x = D_3$, and $p(0) = 0$. 

The equations of motion describing the overall “small” vibrations on the composite beam are dictated by the variables $v^1(x,t), v^3(x,t), w(x,t), \phi^2(x,t)$ which correspond to the longitudinal vibrations of Layers 1⃝ and 3⃝, bending of the composite 1⃝-2⃝-3⃝, and shear of Layer 2⃝.

| Layers         | Displacements, Stresses, Strains, Electric fields, and Electric displacements |
|----------------|--------------------------------------------------------------------------------|
| Layer 1⃝ - Elastic | $U_1^1(x,z) = v^1(x) - (z - \hat{z}_1)w_x$, $U_3(x,z) = w(x)$ |
|                | $S_{11} = \frac{\partial v^1}{\partial x} + \frac{1}{2} (w_x)^2 - (z - \hat{z}_1)\frac{\partial w}{\partial x}$, $S_{13} = 0$ |
|                | $T_{11} = \alpha_1 S_{11}$, $T_{13} = T_{12} = T_{23} = 0$ |
| Layer 2⃝ - Viscoelastic | $U_1^2(x,z) = v^2(x) + (z - \hat{z}_2)\psi^2(x)$, $U_3(x,z) = w(x)$ |
|                | $S_{11} = \frac{\partial v^2}{\partial x} - (z - \hat{z}_2)\frac{\partial \psi^2}{\partial x}$, $S_{13} = \frac{1}{2} \beta \phi^2$ |
|                | $T_{11} = \alpha_1^2 S_{11}$, $T_{13} = 2G_2 S_{13}$, $T_{12} = T_{23} = 0$ |
| Layer 3⃝ - Piezoelectric | $U_1^3(x,z) = v^3(x) - (z - \hat{z}_3)w_x$, $U_3(x,z) = w(x)$ |
|                | $S_{11} = \frac{\partial v^3}{\partial x} + \frac{1}{2} (w_x)^2 - (z - \hat{z}_3)\frac{\partial w}{\partial x}$, $S_{13} = 0$ |
|                | $T_{11} = \alpha_3^3 S_{11} - \gamma \beta D_3$, $T_{13} = T_{12} = T_{23} = 0$ |
|                | $E_1 = \beta_1 D_1$, $E_3 = -\gamma \beta S_{11} + \beta D_3$ |

Table 1. Linear constitutive relationships for each layer. $U^i, T_{ij}, S_{ij}, D_i$, and $E_i$ denote displacements, the stress tensor, strain tensor, electrical displacement, and electric field for $i, j = 1, 2, 3$. 

![Diagram](image-url)
Assume that the beam is subject to a distribution of boundary forces \((\tilde{g}^1, \tilde{g}^3, \tilde{g})\) along its edge \(x = L\), see Fig. 1. Now define

\[
\dot{g}^i(x, t) = \int_0^{z_i} \dot{g}(x, z, t) \, dz, \quad g(x, t) = \int_0^h \tilde{g}(x, z, t) \, dz,
\]

\[
m_i = \int_{z_{i-1}}^{z_i} (z - \hat{z}_i) \dot{g}(x, z, t) \, dz, \quad i = 1, 3
\]

to be the external force resultants defined as in [14]. For our model, it is appropriate to assume that \(\dot{g}^1, \dot{g}^3\) are independent of \(z\), Let \(V(t)\) be the voltage applied at the electrodes of the piezoelectric layer. By using Table 1 (with the assumption \(D_1 = 0\)), the Lagrangian for the ACL beam is

\[
L = \int_0^T \left[ K - (P + E) + B + W \right] \, dt
\]

where

\[
K = \frac{1}{2} \int_0^L \left[ \rho_1 h_1 (\dot{v}^1)^2 + \rho_2 h_2 (\dot{v}^2)^2 + \rho_3 h_3 (\dot{v}^3)^2 + \rho_2 h_2 (\dot{\psi}^2)^2 \\
+ (\rho_1 h_1 + \rho_2 h_2) \dot{w}_x^2 + (\rho_1 h_1 + \rho_2 h_2 + \rho_3 h_3) \dot{w}^2 \right] \, dx,
\]

\[
P + E = \frac{1}{2} \int_0^L \left[ \alpha^3 h_3 (v_x^3)^2 + \frac{h_2^2}{12} w_{xx}^2 \right] \, dx,
\]

\[
\psi = \frac{1}{2} \int_0^L \left[ \alpha^2 h_2 (v_x^2)^2 + \frac{h_2^2}{12} \psi_x^2 + G_2 h_2 (\dot{\phi}^2)^2 + \alpha^1 h_1 \left( (v_1^1)^2 + \frac{h_1^2}{12} w_{xx}^2 \right) \right] \, dx,
\]

\[
B = \frac{\mu h}{2} \int_0^L \dot{p}^2 \, dx,
\]

\[
W = \int_0^L [-p_t V(t)] \, dx + g^1 v^1(L) + g^3 v^3(L) + gw(L) - M w_x(L).
\]

Here, \(M = m_1 + m_3\), \(p = \int_0^x D_3(\xi, t) \, d\xi\) is the total electric charge at point \(x\), \(\rho_i\) is the volume density of the \(i\)th layer, and \(K, P + E, B,\) and \(W\) are the kinetic energy, the total stored energy, and the magnetic energy of the beam, and the work done by the external mechanical and electrical forces [24].

### 2.1. Hamilton’s Principle

By using (1)-(2), the variables \(v^2, \phi^2\) and \(\psi^2\) can be written as the functions of state variables as the following

\[
v^2 = \frac{1}{2} (v^1 + v^3) + \frac{h_3 - h_1}{4} w_x, \quad \psi^2 = \frac{1}{h_2} (v^1 + v^3) + \frac{h_1 + h_3}{2h_2} w_x
\]

\[
\phi^2 = \frac{1}{h_2} (-v^1 + v^3) + \frac{h_1 + 2h_2 + h_3}{2h_2} w_x.
\]

Thus, we choose \(w, v^1, v^3\) as the state variables. Let \(H = \frac{h_1 + 2h_2 + h_3}{2}\). Application of Hamilton’s principle, by using forced boundary conditions, i.e. clamped at \(x = 0\), by setting the variation of admissible displacements \(\{v^1, v^3, p, w\}\) of \(L\) to zero yields the following coupled equations of stretching in odd layers, dynamic charge in the
piezoelectric layer, the bending of the whole composite:

\[
\begin{align*}
(\rho_1 h_1 + \frac{\rho_2 h_2}{3}) \ddot{v}^1 + \frac{\rho_2 h_2}{6} \ddot{v}^3 - \frac{1}{6} \alpha^2 h_2 v_{xx}^3 - (\alpha^1 h_1 + \frac{1}{3} \alpha^2 h_2) v_{xx}^1 \\
- \frac{\rho_2 h_2}{12} (2h_1 - h_3) \ddot{w}_{xx} + \frac{\rho_2 h_2}{12} (2h_1 - h_3) w_{xxx} - G_2 \phi^2 = f_1, \\
(\rho_3 h_3 + \frac{\rho_2 h_2}{3}) \ddot{v}^3 + \frac{\rho_2 h_2}{6} \ddot{v}^1 - \frac{1}{6} \alpha^2 h_2 v_{xx}^1 - (\alpha^3 h_3 + \frac{1}{3} \alpha^2 h_2) v_{xx}^3 \\
+ \frac{\rho_2 h_2}{12} (2h_3 - h_1) \ddot{w}_{xx} - \frac{\rho_2 h_2}{12} (2h_3 - h_1) w_{xxx} + G_2 \phi^2 + \gamma h_3 p_{xx} = f^3, \\
\mu h_3 \ddot{p} - \beta h_3 p_{xx} + \gamma h_3 v_{xx}^3 = 0,
\end{align*}
\]

with associated boundary and initial conditions

\[
\begin{align*}
v^1, v^3, w, w_x, p \big|_{x=0} = 0, \\
\frac{1}{6} \alpha^2 h_2 v_{xx}^3 + (\alpha^1 h_1 + \frac{1}{3} \alpha^2 h_2) v_{xx}^1 \\
+ \frac{\rho_2 h_2}{12} (2h_1 - h_3) \ddot{w} - \frac{\rho_2 h_2}{12} (2h_1 - h_3) w_{xx} \big|_{x=L} = g^1(t), \\
\frac{1}{6} \alpha^2 h_2 v_{xx}^3 + (\alpha^3 h_3 + \frac{1}{3} \alpha^2 h_2) v_{xx}^1 - \frac{\rho_2 h_2}{12} (2h_3 - h_1) \ddot{w} \\
+ \frac{\rho_2 h_2}{12} (2h_3 - h_1) w_{xx} - \gamma h_3 p_x \big|_{x=L} = g^3(t), \\
\beta h_3 p_x - \gamma h_3 v_{xx}^3 = -V(t) \big|_{x=L}, \\
(\alpha^1 h_1)^3 + \alpha_3 (h_3)^3 + \alpha_2 h_2 ((h_1)^2 + (h_3)^2 - h_1 h_3) w_{xx} - \frac{\rho_2 h_2}{12} (2h_1 - h_3) v_{xx}^3 \\
+ \frac{\rho_2 h_2}{12} (2h_1 - h_3) v_{xx}^3 - \frac{\rho_2 h_2}{12} (2h_1 - h_3) v_{xx}^3 \\
+ \frac{\rho_2 h_2}{12} (2h_1 - h_3) v_{xx}^3 - \frac{\rho_2 h_2}{12} (2h_1 - h_3) v_{xx}^3 \\
+ \frac{\rho_2 h_2}{12} (2h_1 - h_3) v_{xx}^3 - \frac{\rho_2 h_2}{12} (2h_1 - h_3) v_{xx}^3 \\
+ \frac{\rho_2 h_2}{12} (2h_3 - h_1) \ddot{w} - \frac{\rho_2 h_2}{12} (2h_3 - h_1) \ddot{w} \\
+ \frac{\rho_2 h_2}{12} (2h_3 - h_1) \ddot{w} + \frac{\rho_2 h_2}{12} (2h_3 - h_1) \ddot{w} + G_2 H \phi^2 \big|_{x=L} = g(t), \\
(v^1, v^3, p, w, \dot{v}^1, \dot{v}^3, \ddot{p}, \dot{w}) (x, 0) = (v_0^1, v_0^3, w_0, p_0, v_0^1, v_0^3, p_1, w_1).
\end{align*}
\]

Note that the equations of motion (5)-(6) does not have any distributed damping term. It is simply because the aim of the paper is to investigate the stabilizability of the closed-loop system with only the \(B^*\)--type stabilizing boundary controllers. Presumably, adding a (viscous) distributed damping term in the form of a shear damping or Kelvin-Voigt damping would automatically make the structure asymptotically stable. For practical applications, it is more relevant to consider a shear-type of damping due to the viscoelastic middle layer by replacing the term \(G_2 \phi^2\) by \(\tilde{G}_2 \phi^2\) in (5) - (6) where \(\tilde{G}_2\) is the damping coefficient [11].

3. Fully-dynamic Rao-Nakra (R-N) model. The model obtained above is highly coupled, and it is not very easy to analyze the controllability properties. For this reason, we assume the thin-compliant-layer Rao-Nakra sandwich beam assumptions that the viscoelastic layer is thin and its stiffness negligible. Therefore we work with the perturbed model for \(\rho_2, \alpha^2 \to 0\); as in [11]. This approximation retains the potential energy of shear and transverse kinetic energy so that the model
above reduces to
\[
\begin{cases}
\rho_1 h_1 \ddot{v}^1 - \alpha^1 h_1 v_{x x}^1 - G_2 \phi^2 = f^1, \\
\rho_3 h_3 \ddot{v}^3 - \alpha^3 h_3 v_{x x}^3 + \gamma \beta h_3 p_{x x} + G_2 \phi^2 = f^3, \\
\mu h_3 \ddot{p} - \beta h_3 p_{x x} + \gamma \beta h_3 v_{x x}^3 = 0, \\
m \ddot{w} - K_1 \dddot{v} + K_2 \dddot{w} = -G_2 H \phi^2_x = f, \\
\phi^2 = \frac{1}{h_2} (-v^3 + v^3 + H w_x)
\end{cases}
\]
with the boundary and initial conditions
\[
\begin{cases}
v^1(0) = v^3(0) = p(0) = w(0) = w_x(0) = 0, \alpha^1 h_1 v_x^1(L) = g^1(t), \\
\alpha^3 h_3 v_x^3(L) - \gamma \beta h_3 p_x(L) = g^3(t), \beta h_3 p_x(L) - \gamma \beta h_3 v_{x x}^3(L) = -V(t), \\
K_2 \dddot{w}(L) = -M(t), K_1 \dddot{v} + K_2 \dddot{w}(L) + G_2 H \phi^2(L) = g(t), \\
(v^1, v^3, p, w, v^1, \dot{v}^3, \dot{p}, \dot{w})(x, 0) = (v_0^1, v_0^3, p_0, w_0, v_1^1, v_1^3, p_1, w_1)
\end{cases}
\]
where \(m = \rho_1 h_1 + \rho_2 h_2 + \rho_3 h_3, \ K_1 = \frac{\alpha^1 h_3^3}{12} + \frac{\alpha^3 h_3^3}{12}, \) and \( K_2 = \frac{\alpha^1 h_3^3}{12} + \frac{\alpha^3 h_3^3}{12}. \)

**Semigroup well-posedness:** Define
\[
H^1_L(0, L) = \{ \psi \in H^1(0, L) : \psi(0) = 0 \}, \\
H^2_L(0, L) = \{ \psi \in H^2(0, L) : \psi(0) = \psi_x(0) = 0 \},
\]
and the complex linear spaces
\[
\mathcal{X} = \mathbb{L}^2(0, L), \ V = (H^1_L(0, L))^3 \times H^2_L(0, L), \ H = \mathcal{X}^3 \times H^1_L(0, L), \ \mathcal{H} = V \times H.
\]
The energy associated with (7)-(8) is
\[
E(t) = \frac{1}{2} \int_0^L \left\{ \rho_1 h_1 |v^1|^2 + \rho_3 h_3 |v^3|^2 + \mu h_3 |p|^2 + m |\dot{w}|^2 + \alpha^1 h_1 |v_x^1|^2 + \alpha^3 h_3 |v_x^3|^2 + K_1 |\dddot{w}|^2 + K_2 |w_{x x}^2| - \gamma \beta h_3 v_x p_x - \gamma \beta h_3 v_x \dddot{v} + \beta h_3 p_x |\dddot{v}|^2 + G_2 h_2 |\phi^2|^2 \right\} dx.
\]
This motivates the definition of the inner product on \( \mathcal{H} : \)
\[
\left\langle \begin{bmatrix} u_1 \\ \vdots \\ u_8 \end{bmatrix}, \begin{bmatrix} v_1 \\ \vdots \\ v_8 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix}, \begin{bmatrix} v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}, \begin{bmatrix} \alpha^1 \beta h_1 u_5 \alpha^3 \beta h_3 u_7 \mu h_3 u_7 \mu h_3 \alpha^3 \beta h_3 u_7 \mu h_3 \alpha^3 \beta h_3 u_7 \mu h_3 \alpha^3 \beta h_3 u_7 \mu h_3 \right\rangle
\]
\[
= \int_0^L \left\{ \rho_1 h_1 u_5 \overline{\dddot{v}}_5 + \rho_3 h_3 u_7 \overline{\dddot{v}}_7 + \mu h_3 u_7 \overline{\dddot{v}}_7 + m u_8 \overline{\dddot{v}}_8 + K_1 (u_8)_{xx} \overline{\dddot{v}}_8 \right\}
\]
\[
+ \alpha^1 h_1 (u_1)_{xx} \overline{(\dddot{v})_x} + K_2 (u_4)_{xx} \overline{(\dddot{v})_x}
\]
\[
+ \frac{G_2}{h_2} (-u_1 + u_2 + H(u_4)_x) (-\overline{\dddot{v}}_1 + \overline{\dddot{v}}_2 + H(\overline{\dddot{v}})_{xx})
\]
\[
+ \h_3 \left\{ \begin{bmatrix} \alpha^3 + \gamma^2 \beta \\ -\gamma \beta \\ -\gamma \beta \\ \beta \end{bmatrix}, \begin{bmatrix} u_{2x} \\ u_{3x} \\ v_{2x} \\ v_{3x} \end{bmatrix} \right\} \right\} dx
\]
where \( \langle \cdot, \cdot \rangle_{C^2} \) is the inner product on \( C^2. \) Obviously, \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \) does indeed define an inner product, with the induced energy norm, since
\[
\begin{bmatrix} \alpha^3 + \gamma^2 \beta & -\gamma \beta \\ -\gamma \beta & \beta \end{bmatrix} > 0.
\]
well, \(|-u_1 + u_2 + H(u_4)|_{L^2(0,L)}\) does not violate the coercivity of (9), see (Lemma 2.1, [11]) for the details.

Consider that all external body forces are zero, i.e. \(f^1, f^3, f \equiv 0\). Writing \(\varphi = [v^1, v^3, p, w, \dot{v}^1, \dot{v}^3, \dot{p}, \dot{w}]^T\) and \(F(t) = (g^1(t), g^3(t), V(t), M(t), g(t))\), the control system (7)-(8) can be put into the state-space form

\[
\begin{bmatrix}
\dot{\varphi} + \begin{bmatrix}
0_{4 \times 4} & I_{4 \times 4} \\
M^{-1}A & 0_{4 \times 4}
\end{bmatrix} \varphi = \begin{bmatrix}
0_{4 \times 5} \\
B_0
\end{bmatrix} F(t), \quad \varphi(x,0) = \varphi^0.
\end{bmatrix}
\]

where \(A : \text{Dom}(A) \subset H \to H\) with \(\text{Dom}(A) = \{(\vec{z}, \vec{\psi}) \in V \times V, A\vec{z} \in V'\}\), and the operators \(A : V \to V', \ \mathcal{M} : H^1_0(0,L) \to (H^1_0(0,L))', \ \mathcal{B} \in \mathcal{L}(\mathbb{C}^5, (\text{Dom}(A))'), \ \mathcal{B}_0 \in \mathcal{L}(\mathbb{C}^5, V')\) are

\[
\begin{align*}
\langle A\psi, \tilde{\psi} \rangle_{V', V} &= \langle \psi, \tilde{\psi} \rangle_V, \quad M = [\rho_1 h_1 I \ \rho_3 h_3 I \ \mu h_3 I \ M], \\
\langle \mathcal{M}\psi, \tilde{\psi} \rangle_{(H^1_0(0,L))', (H^1_0(0,L))} &= \int_0^L (m\dot{\psi}\tilde{\psi} + K_1 \psi_x \tilde{\psi}_x) \, dx, \\
B_0 &= \begin{pmatrix}
\delta_L & 0 & 0 & 0 & 0 \\
0 & \delta_L & 0 & 0 & 0 \\
0 & 0 & -\delta_L & 0 & 0 \\
0 & 0 & 0 & (\delta_L)_x & \delta_L
\end{pmatrix}.
\end{align*}
\]

In the above, \(\delta_L = \delta(x - L)\) is the Dirac-Delta distribution, and \(V'\) and \((\text{Dom}(A))'\) are dual spaces of \(V\) and \(\text{Dom}(A)\) pivoted with respect \(H\), respectively. We have the following well-posedness and perturbation theorem:

**Theorem 3.1.** Let \(T > 0\), and \(F \in (L^2(0,T))^5\). For any \(\varphi^0 \in H, \varphi \in C([0,T]; H)\), and there exists a positive constant \(c_1(T)\) such that (10) satisfies

\[
\|\varphi(T)\|_H^2 \leq c_1(T) \left( \|\varphi^0\|_H^2 + \|F\|_{(L^2(0,T))^5}^2 \right).
\]

**Proof.** The proof is provided in [23].

**Theorem 3.2.** For fixed initial data and no applied forces, the solution \((v^1, v^3, p, w) \in H\) of (5)-(6) converges to the solution of \((v^1, v^3, p, w) \in H\) in (7)-(8) as \(\|\rho_2, \alpha^2\|_{\mathbb{R}^2} \to 0\).

**Proof.** The system (7)-(8) is the perturbation of the system (5)-(6). The proof is analogous to the one in [11].

### 3.1 Electrostatic Rao-Nakra (R-N) model.

Note that, if we exclude the magnetic effects by \(\mu \to 0\) in (7)-(8), or \(B \equiv 0\) in (4), the \(p\)-equation in (7)-(8) can be solved for \(p\). Then, the simplified equations of motion are as the following

\[
\begin{align*}
\rho_1 h_1 \ddot{v}^1 - \alpha_1^1 h_1 v_{xx}^1 - G_2 \phi^2 &= 0 \\
\rho_3 h_3 v^3 - \alpha_1^3 h_3 v_{xx}^3 + G_2 \phi^2 &= 0 \\
m\ddot{w} - K_1 w_{xx} + K_2 w_{xxx} - G_2 H \dot{\phi}_x^2 &= 0 \\
\phi^2 &= \frac{1}{h_2^2} (-v^1 + v^3 + H w_x)
\end{align*}
\]
with the boundary and initial conditions
\[
\begin{align*}
v^3(0) = v^3(0) = w(0) = w_x(0) &= 0, \\
\alpha_1 h_1 v^1_1(L) = g^1(t), & \quad \alpha_3 h_3 v^3_1(L) = -\gamma V(t), \\
K_2 w_{xx}(L) = -M(t), & \quad K_1 \dot{w}_x(L) - K_2 w_{xxx}(L) + G_2 H \dot{\phi}^2(L) = g(t), \\
v^1, v^3, w, \dot{v}^1, \dot{v}^3, \dot{w}(x,0) &= (v^1_0, v^3_0, w_0, v^1_1, v^3_1, w_1).
\end{align*}
\]

(12)

where we use (3). We also free \( g^2(t) \equiv 0 \), which is the mechanical strain controller at the tip \( x = L \), since the voltage control \( V(t) \) of the piezoelectric layer is able to control the strains by itself.

**Semigroup Formulation:** Define the complex linear space \( \mathcal{H} = V \times H \) where \( V = (H^1_0(0,L))^2 \times H^1_0(0,L), \quad H = X^2 \times H^1_0(0,L) \). The natural energy associated with (11)-(12) is
\[
E(t) = \frac{1}{2} \int_0^L \left\{ \rho_1 h_1|\dot{v}^1|^2 + \rho_3 h_3|\dot{v}^3|^2 + m|\dot{w}|^2 + \alpha_1 h_1|v^1_x|^2 + \alpha_3 h_3|v^3_x|^2 + K_1|\dot{w}_x|^2 \\
+ K_2|w_{xx}|^2 + G_2 h_2|\dot{\phi}^2|^2 \right\} \, dx.
\]

This motivates definition of the inner product on \( \mathcal{H} \)
\[
\left\langle \begin{bmatrix} u_1 \\ \vdots \\ u_6 \end{bmatrix}, \begin{bmatrix} v_1 \\ \vdots \\ v_6 \end{bmatrix} \right\rangle_{\mathcal{H}} = \left\langle \begin{bmatrix} u_4 \\ u_5 \\ u_6 \end{bmatrix}, \begin{bmatrix} v_4 \\ v_5 \\ v_6 \end{bmatrix} \right\rangle_{V} + \left\langle \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right\rangle_{H}.
\]

Writing \( \varphi = [v^1, v^3, w, \dot{v}^1, \dot{v}^3, \dot{w}]^T \) and \( F(t) = (g^1(t), V(t), M(t), g(t)) \), the control system (11)-(12) can be put into the state-space form
\[
\begin{bmatrix}
\dot{\varphi} \\
\dot{A} \varphi
\end{bmatrix} = \begin{bmatrix}
0_{3 \times 3} & I_{3 \times 3} \\
M^{-1} A & 0_{3 \times 3}
\end{bmatrix} \varphi + \begin{bmatrix}
0_{3 \times 4} \\
B_0
\end{bmatrix} F(t), \quad \varphi(x,0) = \varphi^0.
\]

(13)

where \( \text{Dom}(A) = \{(\xi, \xi) \in V \times V, \xi \in V'\}, \quad A : \text{Dom}(A) \times V \subset \mathcal{H} \rightarrow \mathcal{H}, \quad M = [\rho_1 h_1 I \rho_3 h_3 I \mathcal{M}], \) and the operators \( B \in \mathcal{L}(\mathbb{C}^4, (\text{Dom}(A))^4), \quad B_0 \in \mathcal{L}(\mathbb{C}^4, V') \)

\[
\left\langle A \psi, \tilde{\psi} \right\rangle_{V', V} = \left\langle \psi, \tilde{\psi} \right\rangle_V, \quad B_0 = \begin{bmatrix}
\delta_L & 0 & 0 & 0 \\
0 & \delta_L & 0 & 0 \\
0 & 0 & \delta_L & \delta_L
\end{bmatrix}.
\]

**Theorem 3.3.** Let \( T > 0 \), and \( F \in (L^2(0,T))^4 \in L^2(0,T) \). For any \( \varphi^0 \in \mathcal{H}, \quad \varphi \in C([0,T]; \mathcal{H}) \) and there exists a positive constants \( c_1(T) \) such that (13) satisfies
\[
\|\varphi(T)\|_{\mathcal{H}}^2 \leq c_1(T) \left\{ \|\varphi^0\|_{\mathcal{H}}^2 + \|F\|_{L^2(0,T)}^2 \right\}.
\]

**Proof.** The proof can be found in [23]. \( \square \)
Theorem 3.4. For fixed initial data and no applied forces, the solution \((v^1, v^3, w) \in \mathcal{H}\) of (5)-(6) converges to the solution of \((v^1, v^3, w) \in \mathcal{H}\) in (11)-(12) as \(\| \rho_2, \alpha^2, \mu \| \to 0\).

4. Fully dynamic Mead-Marcus (M-M) model. Another way to simplify (5)-(6) is to assume the Mead-Marcus type sandwich beam assumptions that the longitudinal and rotational inertia terms are negligible in (5)-(6). This type of perturbation is singular, and therefore, solutions do not behave continuously with respect to this perturbation. However, it is noted in [11] that this type of approximation provides a close approximation to the original Rao-Nakra model in the low-frequency range. In this section, we follow the methodology in [10]. By \(\ddot{v}^1, \ddot{v}^3, \ddot{w}_{xx} \to 0\) in (5)-(6), we obtain

\[
\begin{aligned}
-\frac{1}{6} \alpha^2 h_2 v_{xx}^3 &= (\alpha^1 h_1 + \frac{1}{3} \alpha^2 h_2) v_{xx}^1 + \frac{\alpha^2 h_2}{12} (2h_1 - h_3) w_{xxx} - G_2 \phi^2 = f^1, \\
-\frac{1}{6} \alpha^2 h_2 v_{xx}^3 &= (\alpha^3 h_3 + \frac{1}{3} \alpha^2 h_2) v_{xx}^3 - \frac{\alpha^2 h_2}{12} (2h_3 - h_1) w_{xxx} + G_2 \phi^2 + \gamma \beta h_3 p_{xx} = f^3, \\
\mu h_3 \ddot{p} - \beta h_3 p_{xx} + \gamma \beta h_3 v_{xx}^3 &= -V(t) \delta_L, \\
m \ddot{w} - \frac{\alpha^2 h_2}{12} (2h_1 - h_3) v_{xxx}^1 + \frac{\alpha^2 h_2}{12} (2h_3 - h_1) v_{xxx}^3 \\
+ \frac{1}{12} (\alpha^1 h_1 + \alpha^3 h_3 + \alpha^2 h_2 (h_1^2 + h_3^2 - h_1 h_3)) w_{xxxx} - H G_2 \phi_x^2 &= f, \\
\phi^2 &= \frac{1}{h_2} \left( -v^1 + v^3 + Hw_x \right),
\end{aligned}
\]

with the boundary and initial conditions

\[
\begin{aligned}
v^1, v^3, w, w_x, p \mid_{x=0} &= 0, \\
\frac{1}{6} \alpha^2 h_2 v_{xx}^3 + (\alpha^1 h_1 + \frac{1}{3} \alpha^2 h_2) v_{xx}^1 - \frac{\alpha^2 h_2}{12} (2h_1 - h_3) w_{xx} \mid_{x=L} &= g^1(t), \\
\frac{1}{6} \alpha^2 h_2 v_{xx}^3 + (\alpha^3 h_3 + \frac{1}{3} \alpha^2 h_2) v_{xx}^3 + \frac{\alpha^2 h_2}{12} (2h_3 - h_1) w_{xx} \\
- \gamma \beta h_3 p_x \mid_{x=L} &= g^3(t), \\
\beta h_3 p_x (L) - \gamma \beta h_3 v_{xx}^3 (L) \mid_{x=L} &= 0, \\
\alpha^1(h_1^2 + \alpha_3(h_3)^2 + \alpha_2 h_2(h_1)^2 + (h_3)^2 - h_1 h_3) w_{xx} &
\end{aligned}
\]

\[
\begin{aligned}
- \frac{\alpha^2 h_2}{12} (2h_1 - h_3) v_{xx}^1 + \frac{\alpha^2 h_2}{12} (2h_3 - h_1) v_{xx}^3 \mid_{x=L} &= M(t), \\
- \frac{\alpha^1(h_1^2 + \alpha_3(h_3)^2 + \alpha_2 h_2(h_1)^2 + (h_3)^2 - h_1 h_3) w_{xx}}{12} &
\end{aligned}
\]

\[
\begin{aligned}
\frac{\alpha^2 h_2}{12} (2h_1 - h_3) v_{xx}^1 - \frac{\alpha^2 h_2}{12} (2h_3 - h_1) v_{xx}^3 + G_2 H \phi_x^2 \mid_{x=L} &= g(t), \\
(v^1, v^3, p, w, \ddot{v}^1, \ddot{v}^3, \ddot{p}, \dot{w})(x, 0) &= (v_{10}^1, v_{03}^3, p_0, w_0, v_{11}^1, v_{13}^3, p_1, w_1).
\end{aligned}
\]

For the rest of the section, we free all mechanical controllers \(g^1(t), g^3(t), M(t), g(t) \equiv 0\) and just keep voltage controller \(V(t)\). Let \(\zeta = \frac{G_2}{h_2}\), and define the positive constants
Now let

\[
A = \frac{1}{12} \left( \alpha^1 h_1^2 + \alpha^3 h_3^2 + \frac{\alpha^2 h_2^2 \left( 3 \beta^2 h_2 (\alpha^1 h_1^3 + \alpha^3 h_3^3) + 12 \alpha^1 \alpha^3 h_1 h_3 (h_1^2 + h_3^2 - h_1 h_3) \right)}{\alpha^2 (h_2^2 (\alpha^1 h_1^3 + \alpha^2 h_2^3 + 4 \alpha^2 h_2 h_3) + 12 \alpha^1 \alpha^2 h_1 h_2 h_3)} \right),
\]

\[
B_1 = \frac{\alpha^2 h_2^2 (4 \alpha^1 h_1 + \alpha^2 h_2 + 4 \alpha^2 h_3 + 12 \alpha^1 \alpha^2 h_1 h_2 h_3)}{6 \alpha^2 h_2^3 + 12 \alpha^1 h_1},
\]

\[
B_2 = \frac{\alpha^2 h_2^2 (4 \alpha^1 h_1 + \alpha^2 h_2 + 4 \alpha^2 h_3 + 12 \alpha^1 \alpha^2 h_1 h_2 h_3)}{6 \alpha^2 h_2^3 + 12 \alpha^1 h_1},
\]

\[
B_3 = \frac{\alpha^2 h_2^2 (4 \alpha^1 h_1 + \alpha^2 h_2 + 4 \alpha^2 h_3 + 12 \alpha^1 \alpha^2 h_1 h_2 h_3)}{6 \alpha^2 h_2^3 + 12 \alpha^1 h_1},
\]

\[
B_4 = \frac{\alpha^2 h_2^2 (4 \alpha^1 h_1 + \alpha^2 h_2 + 4 \alpha^2 h_3 + 12 \alpha^1 \alpha^2 h_1 h_2 h_3)}{6 \alpha^2 h_2^3 + 12 \alpha^1 h_1},
\]

\[
C = \frac{\alpha^2 h_2^2 (4 \alpha^1 h_1 + \alpha^2 h_2 + 4 \alpha^2 h_3 + 12 \alpha^1 \alpha^2 h_1 h_2 h_3)}{6 \alpha^2 h_2^3 + 12 \alpha^1 h_1}.
\]

Now we multiply the first and second equations in (14) by \( \frac{1}{12} \alpha^2 h_2 + \frac{1}{12} \alpha^3 h_3 \) and \(- \frac{1}{12} \alpha^2 h_2 + \frac{1}{12} \alpha^1 h_1 \), respectively, and add these two equations. An alternate formulation is obtained as the following

\[
\begin{aligned}
mv + A w &- B_1 \gamma \beta h_2 h_3 \phi_x^2 + \gamma \beta B_3 p & x = 0, \\
C \phi_x^2 - \phi_x &+ B_1 w & x = 0, \\
\mu h_2 p &- \beta B_4 p & x = 0, \\
\phi_x^2 &- B_1 w & x = 0,
\end{aligned}
\]

with the boundary and initial conditions

\[
\begin{aligned}
w, w_x, \phi_x^2, p \big|_{x=0} &= 0, \\
Aw &+ \gamma \beta B_3 p \big|_{x=L} = 0, \\
-Aw &+ B_1 \gamma \beta h_2 h_3 \phi_x^2 - \gamma \beta B_3 p \big|_{x=L} = 0, \\
\phi_x^2 &- B_1 w & x = 0, \\
w, p, w_x, p_x \big|_{x=L} &= (w_0, p_0, w^1, p^1).
\end{aligned}
\]

By 16, the boundary conditions can be further simplified to obtain

\[
\begin{aligned}
w, w_x, \phi_x^2, p \big|_{x=0} &= 0, \\
Aw &+ \gamma \beta B_3 p \big|_{x=L} = 0, \\
-Aw &+ B_1 \gamma \beta h_2 h_3 \phi_x^2 - \gamma \beta B_3 p \big|_{x=L} = 0.
\end{aligned}
\]

Now let \( \xi = C \phi \). The elliptic equation for \( \phi_x^2 \) in (17) can be written as

\[
\begin{aligned}
-\phi_x^2 + \xi \phi_x^2 &= -B_1 w & x = B_2 p, \\
\phi_x^2(0) &= \phi_x^2(L) = 0.
\end{aligned}
\]

Since \( a(\phi, \psi) = \int_0^L (\xi \phi_x \psi_x + \phi \psi) \) is a continuous and coercive bilinear form on \( H_0^1(0, L) \), and \( b(\psi) = \int_0^L g \psi \) is a continuous functional on \( H_0^1(0, L) \) for given \( g \in L^2(0, L) \), by the Lax-Milgram theorem, the elliptic equation \( a(\phi, \psi) = b(g, \psi) \) has a unique solution and this solution is \( \phi = P_\xi g \) where the operator \( P_\xi^{-1} \) is defined by

\[
P_\xi^{-1} := (-D_x^2 + \xi I)^{-1}.
\]

It follows from (17) that

\[
\phi_x^2 = -P_\xi(B_1 w & x + B_2 p_x).
\]
Plugging (20) in (17), we obtain
\[
\begin{align*}
\{ m\ddot{w} + Aw_{xxxx} + (B_1^2\gamma \beta_3 h_3\xi P_l w_{xxxx})_x + \gamma \beta_2 h_3(B_1 B_2 \xi P_l p_{xxx})_x + \gamma \beta B_3 p_{xxx} = 0, \\
\mu h_3 \ddot{p} - \beta B_4 p_{xx} - \gamma \beta B_3 w_{xxx} - \gamma \beta h_3 \xi P_l \left(B_1 B_2 w_{xxx} + B_3^2 p_{xx}\right) = -V(t)\delta L.
\end{align*}
\]
with the boundary and initial conditions
\[
\begin{align*}
|w, w_x, p|_{x=0} = 0, & \quad |w_{xx}, p_x|_{x=L} = 0, \\
\left| Aw_{xxxx} + B_1^2\gamma \beta_2 h_3\xi (P_l w_{xxxx}) + B_1 B_2 \xi (P_l p_{xxx}) + \gamma \beta B_3 p_{xxx} \right|_{x=L} = 0, \\
(w, p, \dot{w}, \dot{p})(x, 0) = (w_0, p_0, w^1, p^1).
\end{align*}
\]

**Remark 1.** This model describes the coupling between bending of the whole composite and magnetic charge of the piezoelectric layer. This can be compared to the model obtained in [21] where the equations of motion describe the coupling between the longitudinal motions on the single piezoelectric layer and the magnetic charge equation. This model is novel and is never mathematically analyzed in the literature.

**Lemma 4.1.** Let \( \text{Dom}(D^2_\xi) = \{ w \in H^2(0, L) : w(0) = w_x(L) = 0 \} \). Define the operator \( J = \xi P_\xi - I \). Then \( J \) is continuous, self-adjoint and non-positive on \( L^2(0, L) \). Moreover, for all \( w \in \text{Dom}(P_\xi) \),
\[
Jw = P_\xi D^2_x = (\xi I - D^2_\xi)^{-1} D^2_x w. \tag{21}
\]

**Proof.** Continuity and self-adjointness easily follow from the definition of \( J \). We first prove that \( J \) is a non-positive operator. Let \( u \in L^2(0, L) \). Then \( P_\xi u = (\xi I - D^2_\xi)^{-1} u = s \) implies that \( s \in \text{Dom}(D^2_\xi) \) and \( \xi s - s_{xx} = u \)
\[
\langle Ju, u \rangle_{L^2(0, L)} = \langle (\xi P_\xi - I)u, u \rangle_{L^2(0, L)} = \langle \xi s - s_{xx}, s_{xx} \rangle_{L^2(0, L)} = \xi \| s_{xx} \|^2_{L^2(0, L)} - \| s_{xx} \|^2_{L^2(0, L)} = -\xi \| s_{xx} \|^2_{L^2(0, L)} \leq 0.
\]

To prove (21), let \( P_\xi D^2_x w := v \). Then \( w_{xx} = P_\xi^{-1} v \). Adding and subtracting \( \xi w \) on the left hand side yields \( \xi w - \xi w + w_{xx} = P_\xi^{-1} v \), we obtain \( \xi w - P_\xi^{-1} w = P_\xi^{-1} v \), and therefore \( v = -w + \xi P_\xi w = Jw \).

The equations in (17) can be simplified as
\[
\begin{align*}
m\ddot{w} + Aw_{xxxx} + \gamma \beta B_3 P_{xxx} + \gamma \beta h_3 h_3 \xi (B_1^2 J w_x + B_1 B_2 J p)_x &= 0, \\
\mu h_3 \ddot{p} - \beta B_4 p_{xx} - \gamma \beta B_3 w_{xxx} - \gamma \beta h_3 h_3 \xi P_l \left(B_1 B_2 w_{xxx} + B_3^2 p_{xx}\right) &= -V(t)\delta L. \tag{22}
\end{align*}
\]
with the boundary and initial conditions
\[
\begin{align*}
|w, w_x, p|_{x=0} = 0, & \quad |w_{xx}, p_x|_{x=L} = 0, \\
\left| Aw_{xxxx} + \gamma \beta h_3 h_3 B_1^2 \xi (J w_x) + \gamma \beta h_3 h_3 B_1 B_2 \xi (J p) + \gamma \beta B_3 p_{xxx} \right|_{x=L} = 0, \\
(w, p, \dot{w}, \dot{p})(x, 0) = (w_0, p_0, w^1, p^1).
\end{align*}
\]

**Semigroup well-posedness:** Define \( \mathcal{H} = V \times \mathcal{H} \) where
\[
V = H^2_L(0, L) \times H^1_L(0, L), \quad \mathcal{H} = \mathcal{X}^2 = (L^2(0, L))^2. \tag{24}
\]
The energy associated with (22)-(23) is

\[
E(t) = \frac{1}{2} \int_0^L \left\{ m|\ddot{w}|^2 + \mu h_3 |\dot{p}|^2 + A|w_{xx}|^2 + \beta B_4 |p_x|^2 + \gamma B_3 p_x \dot{w}_{xx}
\right. \\
\left. + \gamma B_3 \ddot{p} w_{xx} - \gamma h_2 h_3 \zeta (J(B_1 w_x + B_2 p))(B_1 \ddot{w}_x + B_2 \ddot{p}) \right\} dx \\
= \frac{1}{2} \int_0^L \left\{ m|\ddot{w}|^2 + \mu h_3 |\dot{p}|^2 - \gamma h_2 h_3 \zeta (J(B_1 w_x + B_2 p))(B_1 \ddot{w}_x + B_2 \ddot{p}) \\
+ \begin{pmatrix} A & \gamma B_3 \\ \gamma B_3 & \beta B_4 \end{pmatrix} \begin{bmatrix} w_{xx} \\ p_x \end{bmatrix}, \begin{bmatrix} w_{xx} \\ p_x \end{bmatrix} \right\} c^2 \right\} dx.
\]

This motivates definition of the inner product on \( \mathcal{H} \)

\[
\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \in \mathcal{H}
\]

\[
= \int_0^L \left\{ m u_3 v_3 + \mu h_3 u_4 v_4 - \gamma h_2 h_3 \zeta (J(B_1 (u_1)_x + B_2 u_2))(B_1 (v_1)_x + B_2 (v_2)) \\
+ \begin{pmatrix} A & \gamma B_3 \\ \gamma B_3 & \beta B_4 \end{pmatrix} \begin{bmatrix} (u_1)_{xx} \\ (u_2)_x \end{bmatrix}, \begin{bmatrix} (v_1)_{xx} \\ (v_2)_x \end{bmatrix} \right\} c^2 \right\} dx.
\]

Here \( \langle \cdot, \cdot \rangle_\mathcal{H} \) defines an inner product with induced energy norm since \( J \) is a non-positive operator, and the matrix

\[
\begin{bmatrix} A & \gamma B_3 \\ \gamma B_3 & \beta B_4 \end{bmatrix}
\]

is positive definite and the determinant is

\[
AB_4 - \gamma^2 B_3^2 = \frac{4}{3} h_5^2 h_3^2 h_3 \alpha_2 \alpha_3^2 + \frac{5}{3} h_4^2 h_5 h_3 \alpha_1^2 \alpha_3^2 + \frac{4}{3} h_3^4 h_3 h_5 \alpha_2 \alpha_1^2 \\
+ 4h_3^4 h_5^2 \alpha_1 \alpha_2 \alpha_3 + \frac{16}{3} h_4^2 h_5 h_3^2 \alpha_1 \alpha_3^2 + \frac{4}{3} h_3^2 h_5 h_3 \alpha_1^2 \alpha_2 \\
+ \frac{1}{3} h_3^2 h_5^2 \alpha_1^2 \alpha_2 \alpha_3 + \frac{4}{3} h_4^2 h_5 h_3 \alpha_1 \alpha_3^2 + h_3^2 h_5 h_3 \alpha_1 \alpha_2 \alpha_3 \left( \frac{4}{3} h_3^2 - h_1 h_3 + 2h_3^2 \right) \\
+ \frac{1}{12} h_3^6 h_3 \alpha_2 \alpha_3 + \frac{1}{4} h_3^6 h_3 \alpha_1 \alpha_2 \alpha_3 + 4h_4^2 h_5 h_3 \alpha_2 \alpha_3^2 + \frac{1}{3} h_1 h_3^2 h_3 \alpha_1 \alpha_3 \alpha_2 \alpha_3 \\
+ \frac{1}{4} h_3^2 h_5^2 \alpha_1^2 \alpha_3 + 4h_4^2 h_5 h_3 \alpha_1 \alpha_3^2 + \frac{1}{3} h_1 h_3^2 h_3 \alpha_1 \alpha_3 \alpha_2 \alpha_3 + 12h_3^2 h_5^2 \alpha_1 \alpha_3 \alpha_3 \\
+ \frac{1}{4} h_3^2 h_5^2 \alpha_1 \alpha_3 \alpha_3 + \frac{1}{8} h_1 h_3^2 h_3 \alpha_1 \alpha_2 \alpha_3 \alpha_3 \\
+ h_3^2 h_5^2 \alpha_1 \alpha_2 \alpha_3 \alpha_3 \left( 20h_3^2 - 12h_1 h_3 + 16h_3^2 \right) + \frac{1}{4} h_3^2 h_5^2 \alpha_1 \alpha_2 \alpha_3 \alpha_3 \\
+ 12h_3^2 h_5^2 \alpha_1 \alpha_3 \alpha_3 + 8h_1 h_3^2 h_3 \alpha_1 \alpha_2 \alpha_3 \alpha_3 + \frac{1}{4} h_3^4 h_3 \alpha_2 \alpha_3 \alpha_3 > 0.
\]

Define the operator \( \mathcal{A} : \text{Dom}(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H} \) with \( \text{Dom}(\mathcal{A}) = \{(z, \dot{z}) \in V \times V, A z \in V'\} \), where \( A = \begin{bmatrix} 0 & I_{2 \times 2} \\ A & 0_{2 \times 2} \end{bmatrix} \) and

\[
A = \begin{bmatrix} -A \frac{D_z}{D_x} - \frac{\gamma h_2 h_3 B_2}{m} D_x J D_x - \frac{\gamma h_2 h_3 B_2}{m} D_x J - \frac{\gamma B_4}{m} D_x \frac{\beta B_4}{\mu h_3} D_x^3 + \frac{\gamma B_4}{m} D_x J - \frac{\gamma B_4}{m} D_x J \\ \frac{\gamma h_2 h_3 B_2}{m} J D_x + \frac{\gamma B_4}{m} D_x^3 + \frac{\beta B_4}{\mu h_3} D_x^2 + \frac{\gamma B_4}{m} J \end{bmatrix},
\]

(25)
The operator boundary conditions (23) yield

\[ \{ \tilde{z} = (z_1, z_2)^T, A \tilde{z} \in H, \ (z_1)_{xx} = (z_2)_{xx} = 0, \ A_{21,xx} + \gamma \beta h_2 h_3 B^T_1 \zeta (J z_1) + \gamma \beta h_2 h_3 B_1 B_2 \zeta (J z_2) + \gamma \beta B_3 z_{2xx} \in H^1_R(0, L) \} \]

where \( H^1_R(0, L) = \{ \varphi \in H^1(0, L) : \varphi(L) = 0 \} \). It is now obvious that \( A \tilde{z} \in H \) as \( z \in \text{Dom}(A) \). Define the control operator \( B \)

\[
B_0 \in \mathcal{L}(\mathbb{C}, V'), \text{ with } B_0 = \begin{bmatrix} 0 \\ -\frac{1}{\mu h_2} \delta_L \end{bmatrix},
\]

\[
B \in \mathcal{L}(\mathbb{C}, \text{Dom}(A)'), \text{ with } B = \begin{bmatrix} 0_{2 \times 1} \\ B_0 \end{bmatrix}.
\]

where \( V' \) is the dual of \( V \) pivoted with respect \( H \).

Writing \( \varphi = [w, p, \dot{w}, \dot{p}]^T \), the control system (22)-(23) with the voltage controller \( V(t) \) can be put into the state-space form

\[
\begin{align*}
\dot{\varphi} = \begin{bmatrix} 0 & I_{2 \times 2} \\ -A & 0 \end{bmatrix} \varphi + \begin{bmatrix} 0 \\ B \end{bmatrix} V(t), \quad \varphi(x, 0) = \varphi^0.
\end{align*}
\]  

(26)

**Lemma 4.2.** The operator \( A \) satisfies \( A^* = -A \) on \( H \), and

\[ \text{Re} \langle A \varphi, \varphi \rangle_H = \text{Re} \langle A^* \varphi, \varphi \rangle_H = 0. \]

Also, \( A \) has a compact resolvent.

**Proof.** Let \( \tilde{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \tilde{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \text{Dom}(A) \). Integration by parts and using the boundary conditions (23) yield

\[
\langle A \tilde{y}, \tilde{z} \rangle_H = \int_0^L \left\{ -A(y_1)_{xxx} \bar{z}_3 - \gamma \beta h_2 h_3 \zeta B^T_1 \zeta (J y_1)_{xx} \bar{z}_3 \\
-\gamma \beta h_2 h_3 B_1 B_2 (J y_2)_x \bar{z}_3 - \gamma \beta B_3 (y_2)_{xxx} \bar{z}_3 + \gamma \beta h_2 h_3 B_1 B_2 J (y_1)_x \bar{z}_4 \\
+ \gamma \beta B_3 (y_1)_{xxx} \bar{z}_4 + \beta B_4 (y_2)_x \bar{z}_4 + \gamma \beta h_2 h_3 B_2 J y_2 \bar{z}_4 \\
+ \gamma \beta h_2 h_3 B_1 J (y_3)_x + B_2 J y_4 (B_1 (\bar{z}_1)_x + B_2 \bar{z}_2) + A(y_1)_{xx} (\bar{z}_1)_{xx} \\
+ \gamma \beta B_3 (y_4) (\bar{z}_1)_{xx} + \beta B_3 (y_3)_{xx} (\bar{z}_2)_x + \beta B_4 (y_4) (\bar{z}_2)_x \right\} dx
\]

\[
= \int_0^L \left\{ -A(y_1)_{xx} (\bar{z}_3)_x + \gamma \beta h_2 h_3 B^T_1 J y_1 (\bar{z}_3)_x + \gamma \beta h_2 h_3 B_1 B_2 J y_2 (\bar{z}_3)_x \\
+ \gamma \beta B_3 (y_2)_{xx} (\bar{z}_3)_x - \gamma \beta h_2 h_3 B_1 B_2 y_1 (J \bar{z}_4)_x - \beta h_2 h_3 B_3 (y_1)_{xx} (\bar{z}_4)_x \\
- \beta B_4 (y_2)_x (\bar{z}_4)_x + \gamma \beta h_2 h_3 B_2 J y_2 \bar{z}_4 \right\} dx
\]

\[
= \langle \tilde{y}, A^* \tilde{z} \rangle_H
\]

\[
= \langle \tilde{y}, -A \tilde{z} \rangle_H.
\]
This shows that $A$ is skew-symmetric. To prove that $A$ is skew-adjoint on $\mathcal{H}$, i.e. $A^* = -A$ on $\mathcal{H}$, it is required to show that for any $v \in \mathcal{H}$ there is $u \in \text{Dom}(A)$ so that $Au = v$, see [37, Proposition 3.7.3]. This is equivalent to solving the system of equations for $u \in \text{Dom}(A)$. Using (3) to simplify the equations leads to

\[
\begin{align*}
 mv_3 &= -A(u_1)_{xxx} - \gamma \beta h_2 h_3 s B_1^2(J(u_1)_x) - \gamma \beta h_2 h_3 s B_1 B_2(J u_2)_x - \gamma \beta B_3 (u_2)_{xxx} \\
 \mu h_3 v_4 &= \gamma \beta h_2 h_3 s B_1 B_2 J(u_1)_x + \gamma \beta h_3 B_3 (u_1)_{xxx} + \beta B_4 (u_2)_{xx} + \gamma \beta h_2 h_3 s B_2^2 (J u_2) \\
 v_1 &= u_3 \\
 v_2 &= u_4.
\end{align*}
\]

Define the bilinear forms $a$ and $c$ by

\[
\begin{align*}
a \left( \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} F \\ G \end{bmatrix} \right) &= -\gamma \beta h_2 h_3 s \langle J(B_1(u_1)_x + B_2 u_2), B_1(v_1)_x + B_2 v_2 \rangle_x \\
&+ \left\langle \begin{bmatrix} A & \gamma \beta B_3 \\ \gamma \beta B_3 & \beta B_4 \end{bmatrix}, \begin{bmatrix} (u_1)_{xx} \\ (u_2)_x \end{bmatrix}, \begin{bmatrix} (v_1)_{xx} \\ (v_2)_x \end{bmatrix} \right\rangle_{\mathcal{X}^2}, \\
c \left( \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} F \\ G \end{bmatrix} \right) &= m \langle u_1, v_1 \rangle_x + \mu h_3 \langle u_2, v_2 \rangle_x.
\end{align*}
\]

Let $(F, G) \in \mathcal{H}_2^2(0, L) \times \mathcal{H}^1(0, L)$. If we multiply (27) by $F$ and (28) by $G$, then integrate by parts, we obtain

\[
a \left( \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} F \\ G \end{bmatrix} \right) + c \left( \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} F \\ G \end{bmatrix} \right) = 0. \tag{29}
\]

The bilinear forms $a$ and $c$ are symmetric, bounded and coercive $H_2^2(0, L)$ and $H_1^1(0, L)$, respectively. Therefore, by Lax- Milgram theorem, there exists a unique pair $(u_1, u_2) \in H_2^2(0, L) \times H_1^1(0, L)$ satisfying (29).

The last step of our proof is to show that $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \text{Dom}(A)$. However, this step is straightforward since (29) is the definition of $\text{Dom}(A)$. To see this let $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in H_2^2(0, L) \times H_1^1(0, L)$, and all boundary conditions.
hold. A simple calculation shows that

\[
\begin{align*}
c \left( A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} F \\ G \end{bmatrix} \right) \\
= c \left( \begin{bmatrix} \frac{1}{m} (-A(u_1))_{xxx} - \gamma \beta h_2 h_3 \xi \left( B_1^2(J(u_1))_x + B_2^2(J(u_2)) \right) \\
\frac{1}{h^3} (\gamma \beta h_2 h_3 \xi \left( B_1 B_2 J(u_1) + B_2^2(J(u_2)) \right)) \\
+ B_1 B_2 (J(u_2)) - \gamma \beta B_3(u_2)_{xxx} \\
\gamma \beta h_3 B_3(u_1)_{xxx} + \beta B_4(u_2)_{xx} \end{bmatrix}, \begin{bmatrix} F \\ G \end{bmatrix} \right)
\end{align*}
\]

\[
= \left( \begin{bmatrix} -A(u_1)_{xxx} - \gamma \beta h_2 h_3 \xi \left( B_1^2(J(u_1))_x \\
\gamma \beta h_2 h_3 \xi \left( B_1 B_2 J(u_1) + B_2^2(J(u_2)) \right) \\
+ B_1 B_2 (J(u_2)) - \gamma \beta B_3(u_2)_{xxx} \\
\gamma \beta h_3 B_3(u_1)_{xxx} + \beta B_4(u_2)_{xx} \end{bmatrix}, \begin{bmatrix} F \\ G \end{bmatrix} \right) \bigg|_{x^2}
\]

and therefore

\[
\begin{align*}
a \left( \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} F \\ G \end{bmatrix} \right) + c \left( \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} F \\ G \end{bmatrix} \right) = 0.
\end{align*}
\]

(30)

Now if we let

\[
\begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} F_1 \\ G_1 \end{bmatrix} \in (C_0^\infty(0, L))^2,
\]

it follows that

\[
a \left( \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} F_1 \\ G_1 \end{bmatrix} \right) + c \left( \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} F_1 \\ G_1 \end{bmatrix} \right) = 0
\]

holds for all \( \begin{bmatrix} F_1 \\ G_1 \end{bmatrix} \in (C_0^\infty(0, L))^2 \). Therefore in \((C_0^\infty(0, L))^2\)' we have

\[
\begin{align*}
A(u_1)_{xxx} + \gamma \beta h_2 h_3 \xi \left[ B_1^2(J(u_1))_x + B_2 B_2(J(u_2))_x \right] + \gamma \beta B_3(u_2)_{xxx} &= 0, \\
\gamma \beta h_2 h_3 \xi \left[ B_1 B_2 J(u_1) + B_2^2(J(u_2)) \right] + \gamma \beta h_3 B_3(u_1)_{xxx} + \beta B_4(u_2)_{xx} &= 0.
\end{align*}
\]

(31)

If we substitute (31) into (29) we obtain that

\[
\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\]

satisfies (30). This together with (31) implies that

\[
\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \text{Dom}(A).
\]

Consider all external volume and surface forces are zero. The only external force acting on the system is purely electrical, i.e. \( V(t) \neq 0 \). We have the following well-posedness theorem for (26).
Theorem 4.3. Let $T > 0$, and $V(t) \in L^2(0, T)$. For any $\varphi^0 \in \mathcal{H}$, $\varphi \in C[[0, T]; \mathcal{H}]$ and there exists a positive constants $c_1(T)$ such that (26) satisfies

$$\|\varphi(T)\|^2_{\mathcal{H}} \leq c_1(T) \left\{ \|\varphi^0\|^2_{\mathcal{H}} + \|V(t)\|^2_{L^2(0, T)} \right\}.$$

Proof. The operator $A : \text{Dom}(A) \to \mathcal{H}$ is the infinitesimal generator of $C_0$–semigroup of contractions by Lümer-Phillips theorem by using Lemma 4.2. The operator $B$ defined above is an admissible control operator. The conclusion follows.

4.1. Electrostatic Mead-Marcus (M-M) model. Notice that if the magnetic effects in (17) are neglected, i.e. $\mu_p \equiv 0$, the last equation can be solved for $p_{xx}$.

Then we obtain the following model

$$\begin{cases}
mw + \tilde{A}w_{xxx} - \beta \gamma h_2 h_3 \bar{B} \phi^2_x = -\frac{\gamma B_3}{B_4} V(t)(\delta_L)_x, \\
\zeta \tilde{C} \phi^2 - \phi^2_{xx} + \bar{B} w_{xxx} = -\frac{\beta B_3}{B_4} V(t) \delta_L,
\end{cases}$$

(32)

with the simplified boundary conditions

$$w(0) = w_x(0) = \phi^2(0) = w_{xx}(L) = 0, \quad \tilde{A}w_{xxx}(L) + \beta \gamma h_2 h_3 \bar{B} \phi^2(L) = g(t).$$

(33)

Here the coefficients $\tilde{A} = A - \frac{\gamma^2 B_2^2}{B_4} > 0, \tilde{B} = B_1 - \frac{\gamma B_3 B_4}{B_4} > 0, \tilde{C} = C + \frac{\gamma h_2 h_3 B_2^2}{B_4}$ are functions of the materials parameters. This model fits in the form of the abstract Mead-Marcus beam model obtained in [11, 12]. The main difference is the boundary conditions.

Using the definitions of $J = P_1 D_x^2$ with $P_1 = (\zeta \tilde{C} I - D_x^2)^{-1}$ in Lemma 4.1, (32)-(33) can be simplified to

$$mw + \tilde{A}w_{xxx} + \gamma \beta h_2 h_3 \tilde{B}^2 (Jw_x)_x = -\frac{\gamma}{B_4} \left[ \zeta \gamma h_2 h_3 \tilde{B}^2 (P_1 \delta_L)_x + B_3(\delta_L)_x \right] V(t)$$

(34)

with the boundary conditions

$$w(0) = w_x(0) = w_{xx}(L) = 0, \quad \tilde{A}w_{xxx}(L) + \gamma \beta h_2 h_3 \tilde{B}^2 Jw_x(L) = g(t).$$

(35)

Semigroup well-posedness: Consider only the case $g(t) \equiv 0$. Define $\mathcal{H} = V \times \mathcal{H} = H^2_0(0, L) \times L^2(0, L)$. The energy associated with (22)-(23) is

$$E(t) = \frac{1}{2} \int_0^L \left\{ m|\bar{w}|^2 + \tilde{A}|w_{xx}|^2 - \gamma \beta h_2 h_3 \tilde{B}^2 Jw_x \bar{w} \right\} dx.$$

This motivates definition of the inner product on $\mathcal{H}$

$$\left\langle \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle_{\mathcal{H}} = \langle u_2, v_2 \rangle_{\mathcal{H}} + \langle u_1, v_1 \rangle_V$$

$$= \int_0^L \left\{ m u_2 v_2 + \tilde{A}(u_2)_{xx}(v_2)_{xx} - \gamma \beta h_2 h_3 \tilde{B}^2 J(u_1)_{x}(\bar{u}_1)_{x} \right\} dx.$$

Define the operator $A : \text{Dom}(A) \subset \mathcal{H} \to \mathcal{H}$ where

$$A = \begin{bmatrix}
0 & I \\
\frac{m}{\nu} (\tilde{A} D_x^4 + \gamma \beta h_2 h_3 \tilde{B}^2 D_x J D_x) & 0
\end{bmatrix}$$

(36)
with
\[
\text{Dom}(A) = \{(z_1, z_2) \in \mathcal{H}, z_2 \in H_x^2(0, L), (z_1)_{xx}(L) = 0, \\
\dot{A}(z_1)_{xxx} + \gamma \beta \kappa h_2 h_3 \tilde{B}^2 J(z_1)_{x} \in H^1(0, L), \\
\dot{A}(z_1)_{xxx}(L) + \gamma \beta \kappa h_2 h_3 \tilde{B}^2 J(z_1)(L) = 0\}.
\]

Define the control operator \(B \in \mathcal{L}(C, \text{Dom}(A)')\) by
\[
B = \begin{bmatrix}
0 \\
\mu_B \left[ \chi h_2 h_3 \tilde{B} \Delta_2 (P_\xi \delta_L)_x + B_3 (\delta_L)_x \right]
\end{bmatrix}
\]
and the dual operator \(B^* \in \mathcal{L}(\mathcal{H}, C)\) is
\[
B^* \Phi = \frac{\gamma}{m B_4} \left[ \chi h_2 h_3 \tilde{B} \Delta_2 (P_\xi \Phi_2)_x(L) + B_3 (\Phi_2)_x(L) \right].
\]

Writing \(\varphi = [w, \dot{w}]^T\), the control system (34)-(35) with the voltage controller \(V(t)\) can be put into the state-space form
\[
\begin{aligned}
\dot{\varphi} &= A \varphi + BV(t), \\
\varphi(x, 0) &= \varphi^0.
\end{aligned}
\]

**Theorem 4.4.** Let \(T > 0\), and \(V(t) \in L^2(0, T)\). For any \(\varphi^0 \in \mathcal{H}, \varphi \in C[[0, T]; \mathcal{H}]\) and there exists a positive constant \(\gamma_0\) such that (37) satisfies
\[
\|\varphi(T)\|^2_\mathcal{H} \leq \gamma_0 \left\{ \|\varphi^0\|^2_\mathcal{H} + \|V\|^2_{L^2(0, T)} \right\}.
\]

**Proof.** The proof follows the steps of Theorem 4.4. \(\square\)

5. **Stabilization results.** In this paper, the top and bottom layers are made of different materials, one is elastic and another one is piezoelectric, in contrast to model in [24]. This causes different speeds of wave propagation at the top and bottom layers. In fact, if both the top and the bottom layers of the composite are piezoelectric, it is shown in [24] that the fully dynamic R-N model (13) with four \(B^*\) -type feedback controllers lacks of asymptotic stability for many choices of material parameters of the piezoelectric layers. These solutions are corresponding to the “bending-free” or “inertial sliding” motions. In contrast to this result, inertial-sliding solutions are asymptotically stable. We also show that the electrostatic R-N model is shown to be exponentially stable with only three feedback controllers in comparison to the four feedback controllers in [24]. We improve that result.

The fully dynamic M-M model has unstable solutions with the \(B^*\) -feedback. This is in line with the result in [24] that the material parameters are sensitive to the stabilization of the system with only one feedback controller. The electrostatic M-M model is asymptotically stable with only one controller. We also mention an exponential stability result for the electrostatic M-M beam model at the end without a proof since the proof is too long and it is beyond the scope of this paper.

5.1. **The fully dynamic Rao-Nakra (R-N) model.** Let \(k_1, k_2, k_3, k_4 \in \mathbb{R}^+\), and choose the state feedback controllers in the fully-dynamic model (10) as the
following
\[
F(t) = \begin{pmatrix} g(t) \\ V(t) \\ M(t) \\ g(t) \end{pmatrix} = KB^* \varphi = \begin{pmatrix} -k_1 \dot{v}_1(L) \\ k_2 p(L) \\ k_3 \dot{w}_x(L) \\ -k_4 \dot{w}(L) \end{pmatrix}
\]
(39)
where \( K = \text{diag}(-k_1, k_2, k_3, -k_4) \), and \( \dot{v}_1(L), \dot{p}(L), \dot{w}_x(L) \) are the velocity of the elastic layer, total induced current accumulated at the electrodes of the piezoelectric layer, angular velocity and velocity of the bending of the composite at the tip \( x = L \), respectively.

Consider the eigenvalue problem \( \mathcal{A} \varphi = \lambda \varphi \) for the inertial sliding solutions, i.e. \( w = 0 \):
\[
\begin{cases}
\alpha_1 h_1 v_{xx}^1 - \frac{Q_2}{h_2} (-v^1 + v^3) = -\tau^2 \rho_1 h_1 v^1, \\
\alpha_3^2 h_3 v_{xx}^3 - \gamma \beta h_3 p_{xx}^3 + \frac{Q_2}{h_2} (-v^1 + v^3) = -\tau^2 \rho_3 h_3 v^3, \\
\beta h_3 p_{xx} - \gamma \beta h_3 v_{xx}^3 = -\tau^2 \mu h_3 p, \\
\frac{Q_2 H}{h_2} (-v^1 + v^3)_x = 0,
\end{cases}
\]
(40)
with the overdetermined boundary conditions
\[
\begin{align*}
|w &= w_x = v_x^i = p_x^i|_{x=0} = |v_x^i = p_x^i|_{x=L} = 0, \\
w(L) = w_x(L) = w_{xx}(L) = w_{xxx}(L) = 0, \quad i = 1, 3.
\end{align*}
\]
(41)
By using the boundary conditions \( v^1(0) = v^3(0) = 0 \), the last equation in (40) implies that \( v^1 = v^3 \). Then (40) reduces to
\[
\begin{cases}
\alpha_1 h_1 v_{xx}^1 = -\tau^2 \rho_1 h_1 v^1, \\
\alpha_3^2 h_3 v_{xx}^3 - \gamma \beta h_3 p_{xx}^3 = -\tau^2 \rho_3 h_3 v^3, \\
\beta h_3 p_{xx} - \gamma \beta h_3 v_{xx}^3 = -\tau^2 \mu h_3 p,
\end{cases}
\]
(42)
By the boundary conditions for \( v^1 \), we obtain \( v^1 = v^3 \equiv 0 \). Finally, by the boundary conditions for \( p \), the equation \( \beta h_3 p_{xx} = -\tau^2 \mu h_3 p \) has only the solution \( p \equiv 0 \). We have the following immediate result:

**Theorem 5.1.** The inertial sliding solutions of the fully dynamic model (10) is strongly stable by the feedback (39).

**Proof.** It can be shown in (40) that \( 0 \in \sigma(\mathcal{A}) \) since \( \lambda = 0 \) corresponds to the trivial solution. Therefore, there are only isolated eigenvalues. There are also no eigenvalues on the imaginary axis, or in other words, the set
\[
\{ z \in \mathcal{H} : \text{Re} \langle \mathcal{A} z, z \rangle_{\mathcal{H}} = -k_1 |\dot{v}(L)|^2 - k_2 |\dot{p}(L)|^2 = 0 \}
\]
(43)
has only the trivial solution by the argument (40)-(42). Therefore, by La Salle’s invariance principle, the system is asymptotically stable.

It is important to note that the asymptotic stability of other solutions is still an open problem due to the strong coupling between bending, stretching, and charge equations.
5.2. Electrostatic Rao-Nakra (R-N) model. Let \( k_1, k_2, k_3, k_4 \in \mathbb{R}^+ \), and choose the state feedback controllers in the model with no magnetic effects (11) as the following

\[
\mathbf{F}(t) = \begin{pmatrix} g^1(t) \\ V(t) \\ M(t) \\ g(t) \end{pmatrix} = KB^* \varphi = \begin{pmatrix} -k_1 \dot{v}^1(L) \\ -k_2 \dot{v}^3(L) \\ k_3 \dot{w}_x(L) \\ -k_4 \dot{w}(L) \end{pmatrix}
\]

(44)

where \( K = \text{diag}(-k_1, -k_2, k_3, -k_4) \), and \( \dot{v}^3(L) \) is the velocity of the piezoelectric layer, at the tip \( x = L \).

The model without magnetic effects is exponentially stable with the feedback (44). We recall the following theorem:

**Theorem 5.2.** [24, Theorem 4.3] Let the feedback (44) be chosen. The solutions \( \varphi(t) = e^{(A+KBB^*)t} \varphi_0 \) for \( t \in \mathbb{R}^+ \) of the closed-loop system

\[
\begin{align*}
\dot{\varphi} &= A \varphi + KBB^* \varphi, \\
\varphi(x, 0) &= \varphi^0.
\end{align*}
\]

(45)

is exponentially stable in \( \mathcal{H} \).

In this paper, we use modified multipliers to reduce the number of controllers to three, i.e. the controller \( g(t) = -k_4 \dot{w}(L) \) may be removed. To achieve this, the following result plays a key role in order to show that there are no eigenvalues on the imaginary axis. Note that the analogous result in [24] requires \( u(L) = 0 \).

**Lemma 5.3.** Let \( \lambda = i \mu \). The eigenvalue problem

\[
\begin{align*}
\alpha_1 h_1 z_{xx}^1 - G_2 \phi^2 &= \lambda^2 \rho_1 h_1 z_{xx}^1, \\
\alpha_3 h_3 z_{xx}^3 + G_2 \phi^2 &= \lambda^2 \rho_3 h_3 z_{xx}^3, \\
-K_2 u_{xxxx} + G_2 H \phi_x^2 &= \lambda^2 (\mu - K_1 u_{xx}), \\
\phi^2 &= \frac{1}{h_2} (-z_1^1 + z_3^3 + Hu_x)
\end{align*}
\]

(46)

with the overdetermined boundary conditions

\[
\begin{align*}
u(0) &= u_x(0) = u_x(L) = u_{xxx}(L) = 0, \\
z(0) &= z^1(L) = z^3(L) = 0, \quad i = 1, 3,
\end{align*}
\]

(47)

has only the trivial solution.

**Proof.** Now multiply the equations in (46) by \( x z_{xx}^1, x z_{xx}^3 \), and \( x \tilde{u}_{xxx} \), respectively, use the boundary conditions (47), integrate by parts on \( (0, L) \), and add them up:

\[
0 = \int_0^L \left\{ -\alpha_1 h_1 |z_{xx}^1|^2 - \alpha_3 h_3 |z_{xx}^3|^2 - K_2 |u_{xxx}|^2 - 3 \rho_1 h_1 \lambda^2 |z_{xx}^1|^2 \\
-3 \rho_3 h_3 \lambda^2 |z_{xx}^3|^2 + 3 \mu \lambda^2 |u_x|^2 + K_1 \lambda^2 |u_{xx}|^2 + 3 G_2 H |\phi_x^2|^2 \\
-G_2 H \phi_x^2 x \phi_{xxx}^2 - K_2 \tilde{u}_{xxx} (x u_{xxx}) - G_2 H \phi_x^2 \tilde{u}_{xxx} - \alpha_3 h_3 z_{xx}^3 (x \tilde{z}_{xx}^3) \\
-\alpha_1 h_1 \tilde{z}_{xx}^1 (x z_{xx}^1) - \rho_1 h_1 \lambda^2 \tilde{z}_{xx}^1 (x z_{xx}^1) - \rho_1 h_1 \lambda^2 \tilde{z}_{xx}^1 (x z_{xx}^1) \\
-\rho_3 h_3 \lambda^2 \tilde{z}_{xxx}^3 (x \tilde{z}_{xxx}^3) + \lambda^2 (\mu - K_1 \tilde{u}_x) (x \tilde{z}_{xxx}^3) \right\} \, dx
\]

(48)

where we use the boundary conditions for \( z_{xx}^1(L) = z_{xx}^3(L) = 0 \) via the differential equations (46) since \( \phi_x^2(L) = z^1(L) = z^3(L) = 0 \).
Now consider the conjugate eigenvalue problem corresponding to (46)-(47):

$$\begin{cases}
\alpha_1 h_1 z_{1xx}^1 - G_2 \bar{\phi}_2^2 = \bar{\lambda}^2 \rho_1 h_1 z^1,
\alpha_1^3 h_3 z_{3xx}^3 + G_2 \bar{\phi}_2^3 = \bar{\lambda}^2 \rho_3 h_3 z^3,
- K_2 \bar{u}_{xxxx} + G_2 H \bar{\phi}_2^2 = \bar{\lambda}^2 (m \bar{u} - K_1 \bar{u}_{xx}),
\bar{\phi}_2^2 = \frac{1}{h_2} (-\bar{z}^1 + \bar{z}^3 + H \bar{u}_x)
\end{cases} \tag{49}$$

with overdetermined boundary conditions

$$\bar{u}(0) = \bar{u}_x(0) = \bar{u}_x(L) = \bar{u}_{xxx}(L) = 0,$n
$$\bar{z}^i(0) = \bar{z}^i(L) = z_{xx}^i(L) = 0, \quad i = 1, 3. \tag{50}$$

Now multiply the equations in (49) by $x_{xx}^1$, $x_{xxx}^3$, and $x_{xxxx}^1$, respectively, integrate by parts on $(0, L)$, and add them up:

$$0 = \int_0^L \left\{ G_2 h_2 z_{xx}^1 \phi_2^2(x \phi_2^2) + G_2 h_2 H \phi_2^2 \bar{u}_{xxx} + K_2 \bar{u}_{xxxx}(x u_{xxx}) ight. $$
$$+ \alpha_1 h_1 z_{xx}^1 (x z_{xx}^1) + \alpha_1^3 h_3 z_{xxx}^3 (x z_{xxx}^3) + \rho_1 h_1 \lambda^2 z^1 (x z_{xx}^1) $$
$$+ \rho_3 h_3 \bar{\lambda}^2 z^3 (x z_{xxx}^3) - \bar{\lambda}^2 (m \bar{u} - K_1 \bar{u}_x) (x u_{xxx}) \right\} \, dx. \tag{51}$$

Since $\lambda = i \mu$, adding (48) and (51) yields

$$0 = \int_0^L \left\{ -\alpha_1 h_1 |z_{xx}^1|^2 - \alpha_1^3 h_3 |z_{xxx}^3|^2 - K_2 |u_{xxx}|^2 - 3 \rho_1 h_1 \lambda^2 |z_x^1|^2 $$
$$- 3 \rho_3 h_3 \lambda^2 |z_x^3|^2 + 3 \mu \lambda^2 |u_x|^2 + K_1 \lambda^2 |u_{xx}|^2 + 3 G_2 H |\phi_2^2|^2 \right\} \, dx. \tag{52}$$

Now multiply the equations in (46) by $3 z_{xx}^1$, $3 z_{xxx}^3$, and $3 \bar{u}_x$, respectively, integrate by parts on $(0, L)$, and add them up:

$$0 = \int_0^L \left\{ 3 \alpha_1 h_1 |z_{xx}^1|^2 + 3 \alpha_1^3 h_3 |z_{xxx}^3|^2 + 3 K_2 |u_{xxx}|^2 - 3 G_2 H |\phi_2^2|^2 $$
$$+ 3 \rho_1 h_1 \lambda^2 |z_x^1|^2 + 3 \rho_3 h_3 \lambda^2 |z_x^3|^2 - 3 \mu \lambda^2 |u_x|^2 - K_1 \lambda^2 |u_{xx}|^2 \right\} \, dx. \tag{53}$$

Finally, adding (52) and (53) yields

$$\int_0^L \left\{ \alpha_1 h_1 |z_{xx}^1|^2 + \alpha_1^3 h_3 |z_{xxx}^3|^2 + K_2 |u_{xxx}|^2 \right\} \, dx = 0. \tag{54}$$

This implies that $z_{xx}^1 = z_{xxx}^3 = u_{xxx} = 0$, and by using the overdetermined boundary conditions (47), we obtain that $z^1 = z^3 = u \equiv 0$. \hfill \Box

Let the controller $g(t)$ be removed in (44). Now the number of feedback controllers is reduced to three:

$$\begin{pmatrix}
g^1(t) \\
V(t) \\
M(t)
\end{pmatrix} = B^* \varphi = \begin{pmatrix}
-k_1 \bar{v}^1(L) \\
-k_2 \bar{v}^3(L) \\
-k_3 \bar{w}_x(L)
\end{pmatrix}. \tag{55}$$

**Theorem 5.4.** Let the feedback (55) be chosen. Then the solutions

$$\varphi(t) = e^{i(A + KBB^*)t} \varphi_0 \text{ for } t \in \mathbb{R}^+ \text{ of the closed-loop system (11)-(12) is exponentially stable in } \mathcal{H}.$$

**Proof.** To prove this, we replace Lemma 4.2 by Lemma 5.3 in the proof of Theorem 4.3 in [24]. The rest of the proof uses the compact perturbation argument the same way as in Theorem 4.3 in [24]. \hfill \Box
5.3. The fully dynamic Mad-Marcus (M-M) model. The model with magnetic effects is a strongly coupled system for bending, shear and charge equations. We consider the bending-free model with the following $B^r$-type feedback controller $V(t) = -k_1 \dot{p}(L)$, $k_1 > 0$:
\[
\begin{align*}
\mu h_3 \ddot{p} - \beta B_4 p_{xx} - \gamma \beta h_2 h_3 \varsigma B_2^2 Jp &= -V(t)\delta_L, \\
p(0) &= 0, \quad \beta B_4 p_x(L) = -k_1 \dot{p}(L), \\
(p, \dot{p})(x, 0) &= (p_0, p_1).
\end{align*}
\] (56)

Let $\mathcal{H} = H_1^1(0, L) \times L^2(0, L)$. The energy associated with (56) is
\[
E(t) = \frac{1}{2} \int_0^L \left\{ \mu h_3 |\ddot{p}|^2 + \beta B_4 |p_x|^2 - \gamma \beta h_2 h_3 \varsigma B_2^2 (Jp) \right\} dx.
\]

This motivates definition of the inner product on $\mathcal{H}$
\[
\left\langle \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle_{\mathcal{H}} = \int_0^L \left\{ \mu h_3 u_1 \bar{v}_2 + \beta B_4 (u_1)_x (\bar{v}_1)_x - \gamma \beta h_2 h_3 \varsigma B_2^2 J(u_1)(\bar{u}_1) \right\} dx.
\]

Define the operator $A : \text{Dom}(A) \subset \mathcal{H} \rightarrow \mathcal{H}$:
\[
A = \begin{bmatrix} \frac{\beta B_4}{\mu h_3} D_x^2 + \frac{\gamma \beta h_2 h_3 \varsigma B_2^2}{p} J & I \\
0 & 0
\end{bmatrix}
\] (57)

where
\[
\text{Dom} A = (H^2(0, L) \cap H_1^1(0, L)) \cap \{ z : \beta B_4 (z_1)_x + \gamma \beta h_2 h_3 \varsigma B_2^2 J z_1 \in \mathcal{H}, \\
\beta B_4 (z_1)_x(L) + k_1 \mu h_3 z_2(L) = 0 \}
\] (58)

**Theorem 5.5.** The operator $A$ defined by (57)-(58) is dissipative in $\mathcal{H}$. Moreover, $A^{-1}$ exists and is compact on $\mathcal{H}$. Therefore, $A$ generates a $C_0$-semigroup of contractions on $\mathcal{H}$ and the spectrum $\sigma(A)$ consists of isolated eigenvalues only.

**Proof.** Let $Y \in \text{Dom}(A)$. Then
\[
\langle AY, Y \rangle = \int_0^L \left[ (\beta B_4(y_1)_x + \gamma \beta h_2 h_3 \varsigma B_2^2 J y_1) \bar{y}_2 \\
+ (\beta B_4(y_2)_x(y_1))_x - \gamma \beta h_2 h_3 \varsigma B_2^2 J(y_2)(\bar{y}_1) \right] dx
\]
\[
= \beta B_4(y_1)_x |\bar{y}_2|_{x=0}^L + \int_0^L [\beta B_4(-y_1)_x(\bar{y}_2)_x + (y_1)_x(\bar{y}_2)_x + \gamma \beta h_2 h_3 \varsigma B_2^2 (J y_1 \bar{y}_2 - J(y_1)(\bar{y}_2))] dx
\]

Therefore, $\text{Re} \langle AY, Y \rangle = -k_1 \mu h_3 |z_2(L)|^2 \leq 0$. Therefore, $A$ generates a $C_0$-semigroup of contractions on $\mathcal{H}$.

Next, we show that $0 \in \sigma(A)$, i.e. $0$ is not an eigenvalue. Let $Z \in \mathcal{H}$. We show that there exists $Y \in \text{Dom}(A)$ such that $AY = Z$:
\[
y_2 = z_1 \in H_1^1(0, L), \\
\beta B_4 (y_1)_x + \gamma \beta h_2 h_3 \varsigma B_2^2 J y_1 = z_2.
\]

Since $J y_1 = P^{-1}_x (y_1)_x$, the second equation can be re-written as
\[
(\beta B_4 I + \beta h_2 h_3 \varsigma B_2^2 P^{-1}_x)(y_1)_x = z_2
\]
where \((\beta B_4 I + \gamma \beta h_2 h_3 B_2^2 P_x^{-1})\) is a positive operator, and therefore, is invertible. Therefore,

\[
(y_1)_{xx} = (\beta B_4 I + \gamma \beta h_2 h_3 B_2^2 P_x^{-1})^{-1} y_2 := \psi \in L^2(0, 1).
\]

By integrating the equation twice we conclude that

\[
y_1 = \int_x^1 (x - \tau) \psi(\tau) d\tau + \int_0^1 \tau \psi(\tau) d\tau - \frac{k_1 \mu h_3 z_1(L)}{\beta B_4} x.
\]

Thus, \(Y \in \text{Dom}(A)\). Since \(0 \in \sigma(A)\), and \(A^{1/2}S^{-1}\) is compact on \(H\), the spectrum \(\sigma(A)\) consists of isolated eigenvalues only.

Let \(\lambda = i\tau\). Consider the eigenvalue problem \(A \varphi = \lambda \varphi\) corresponding to \((56)\):

\[
\begin{align*}
C \phi^2 - \phi_{xx}^2 + B_2 p_{xx} &= 0 \\
-\beta B_4 p_{xx} + \gamma \beta h_2 h_3 B_2 \phi^2 &= \mu h_3 \tau^2 p,
\end{align*}
\]

\((59)\)

Now let \(\Phi = [p \ p_x \ \varphi \ \varphi_x]^T\). The system \((59)\) can be written as

\[
\Phi_x = \tilde{A} \Phi := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\mu h_3 \tau^2}{\beta B_4} & 0 & \frac{\gamma h_2 h_3 B_2}{B_4} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\mu h_3 B_2 \tau^2}{\beta B_4} & 0 & \frac{\gamma h_2 h_3 B_2^2}{B_4} & 0 \end{pmatrix} \Phi.
\]

Now consider the auxiliary eigenvalue problem \(\tilde{A} \tilde{\Phi} = \tilde{\lambda} \tilde{\Phi}\) that has the following characteristic equation:

\[
\tilde{\lambda}^4 + \left( \frac{h_3 \mu \tau^2}{\beta B_4} - \frac{B_2^2 \gamma h_2 h_3}{B_4} - C_3 \right) \tilde{\lambda}^2 - \frac{C_3 h_3 \mu \tau^2}{\beta B_4} = 0.
\]

Let \(\tau^2 > \frac{\beta B_4}{\mu h_3} \left( C_3 + \frac{B_2^2 \gamma h_2 h_3}{B_4} \right)\). There are four complex conjugate eigenvalues \(\tilde{\lambda} = \{ \mp i a_1, \mp i a_2 \}\) where

\[
a_1 = \frac{-\frac{B_2^2 \gamma h_2 h_3}{B_4} - C_3 + h_3 \mu \tau^2 + \sqrt{\left( \frac{h_3 \mu \tau^2}{\beta B_4} - \frac{B_2^2 \gamma h_2 h_3}{B_4} - C_3 \right)^2 + 4 C h_3 \mu \tau^2 \beta B_4}}{\sqrt{2}}
\]

\[
a_2 = \frac{-\frac{B_2^2 \gamma h_2 h_3}{B_4} - C_3 + h_3 \mu \tau^2 - \sqrt{\left( \frac{h_3 \mu \tau^2}{\beta B_4} - \frac{B_2^2 \gamma h_2 h_3}{B_4} - C_3 \right)^2 + 4 C h_3 \mu \tau^2 \beta B_4}}{\sqrt{2}}.
\]

Define \(b_1 := \frac{B_2}{\gamma h_2 h_3 B_4} \left( \frac{\mu h_1 \tau^2}{B_4} - a_2^2 \right), b_2 := \frac{B_2}{\gamma h_2 h_3 B_4} \left( \frac{\mu h_1 \tau^2}{B_4} - a_1^2 \right)\) where \(b_1, b_2 \neq 0\) and \(b_1 - b_2 \neq 0\). By using the boundary conditions at \(x = 0\),

\[
y_1(x) = \frac{a_1 (b_1 C_1 - C_2) \sin(a_2 x) + a_2 (C_2 - b_2 C_1) \sin(a_1 x)}{a_1 a_2 (b_1 - b_2)}
\]

\[
y_2(x) = \frac{a_1 b_2 (b_1 C_1 - C_2) \sin(a_2 x) + a_2 b_1 (C_2 - b_2 C_1) \sin(a_1 x)}{a_1 a_2 (b_1 - b_2)}
\]
where $C_1, C_2$ are arbitrary constants. By using the boundary conditions at $x = L$, the coefficient matrix for $(C_1 C_2)^T$ has the determinant
\[
\beta B_4 \cos (a_1 L) \cos (a_2 L) + \frac{ih_3 K_1 \mu \tau (a_1 b_1 \sin (a_2 L) - a_2 b_2 \sin (a_1 L) \cos (a_2 L))}{a_1 a_2 (b_1 - b_2)} = 0.
\]
Assume $a_1 = \frac{(2n-1)\pi}{2L}, a_2 = \frac{(2m-1)\pi}{2L}$ for some $n, m \in \mathbb{R}^+$ so that
\[
\tau_{nm} = \sqrt{\frac{\beta B_4}{\mu h_3} \left( C_\zeta + \frac{B_2^2 \gamma \chi h_2 h_3}{B_4} \right) + (2n - 1)^2 + (2m - 1)^2}.
\]
Then, the determinant becomes zero. Therefore, we find a non-trivial solution of (59):
\[
\begin{cases}
p_{nm}(x) = \sin(a_{2,m}x) - \sin(a_{1,n}x), \\
\phi_{nm}(x) = b_{2,m} \sin(a_{2,m}x) - b_{1,n} \sin(a_{1,n}x).
\end{cases}
\]
(61)
Note that the condition $a_1 = \frac{(2n-1)\pi}{2L}, a_2 = \frac{(2m-1)\pi}{2L}$ is equivalent to the following condition that material parameters satisfy
\[
(2n - 1)^2(2m - 1)^2 - C\zeta \left[ (2n - 1)^2 + (2m - 1)^2 \right] = \frac{C_\zeta L^2}{\pi^2} \left( C_\zeta + \frac{B_2^2 \gamma \chi h_2 h_3}{B_4} \right).
\]

**Theorem 5.6.** Then the solutions $\{\varphi(t)\}_{t \in \mathbb{R}^+}$ with $V(t) = -k_1 \dot{p}(L)$ of the closed-loop system (56) is NOT strongly stable in $\mathcal{H}$ if $a_1 = \frac{(2n-1)\pi}{2L}, a_2 = \frac{(2m-1)\pi}{2L}$ for some $m, n \in \mathbb{Z},$ and $\tau_{nm}$ is defined by (60).

**Proof.** Consider the eigenvalue problem (59) with (62). We show that there are eigenvalues on the imaginary axis, or in other words, the set
\[
\{ z \in \mathcal{H} : \text{Re} \langle Az, z \rangle_{\mathcal{H}} = -k_1 \mu h_3 |z_2(L)|^2 = 0 \}
\]
has non-trivial solutions. With (62), the eigenvalue problem becomes overdetermined with the extra boundary condition $p(L) = 0$. If $\frac{a_1}{a_2} = \frac{2n-1}{2m-1}$, the nontrivial solution (61) automatically satisfies $p_{nm}(L) = 0$. In other words, the $B^*-$type feedback $V(t) = -k_1 \dot{p}_{nm}(L)$ does not stabilize the system if the material parameters satisfy $\frac{a_1}{a_2} = \frac{2n-1}{2m-1}$ for some $m, n \in \mathbb{Z}$ with $\tau_{nm}$ defined by (60). □

### 5.4. Electrostatic Mead-Marcus (M-M) model.

The electrostatic M-M model (34)-(35) is a continuous perturbation of the classical Euler-Bernoulli model due to the operator $J$ defined in Lemma 4.1. Controlling the Euler-Bernoulli beam through its boundary has been a long standing problem in the PDE control theory, see [6, 13, 16, 17] and the references therein. It is proved that one of the two controllers acting on the boundary is unnecessary to achieve exponential stability.

The only boundary feedback stabilization result for the model (34)-(35) is provided by [40] for a three-layer composite (having no piezoelectric layer) with clamped-free boundary conditions, and only a mechanical controller $g(t)$ is applied at the free end $x = L$, see (15). This type of mechanical boundary control is ruled out for a smart piezoelectric M-M beam since we want to control the overall bending motion of the composite by only an electrical controller $V(t)$ which controls the bending moment at the tip $x = L$, not the transverse shear.

We choose the following $B^*$-type feedback controller
\[
V(t) = -k_1 (P\xi \ddot{w}_x)(L, t), \quad g(t) \equiv 0, \quad k_1 > 0
\]
(63)
where \( P_\xi \dot{w}_x(L) = \frac{1}{B^2} (\dot{w}_x - \dot{\phi}^2)(L) \), and \( \dot{\phi}^2 \) is the velocity of the shear of middle layer. Presumably, this type of feedback is a perturbation of the angular velocity feedback \( \dot{w}_x(L) \) in the Euler-Bernoulli model. The energy of the system dissipates and it satisfies

\[
\frac{dE(t)}{dt} = \gamma V(t) \int_0^L \left[ h_2h_3\xi \ddot{B}(P_\xi \dot{w}_x)(L) + \frac{B_3}{B_4} \dot{w}_x(L) \right] = -\frac{k_2}{B_4} (h_2h_3\xi \ddot{B}B_2P_\xi + B_3I) \dot{w}_x(L) \cdot P_\xi \dot{w}_x(L) \leq 0
\]

where \( h_2h_3\xi \ddot{B}B_2P_\xi + B_3I \) is a non-negative operator. Observe that \( P_\xi \dot{w}_x(L) \cdot \dot{w}_x(L) \) is the total piezoelectric effect due to the coupling of the charge equation to shear and bending at the same time. In fact, this is a damping injection through the shear of the middle layer to control the bending moments at \( x = L \).

Let the operator \( A \) be the same as (36) with the new domain

\[
\text{Dom}(A) = \{(z_1, z_2) \in H, z_2 \in H^2_0(0, L), \quad \tilde{A}(z_1)_{xx} + \gamma \beta h_2h_3\xi \ddot{B}^2 J(z_1)_x \in H^1(0, L), \quad \tilde{A}(z_1)_{xx}(L) + k_1 \frac{B_4}{B_3} (\xi h_2h_3\ddot{B}B_2P_\xi + B_3I) \tilde{z}_2(L) \cdot P_\xi \tilde{z}_2(L) \bigg|_{x=L} = 0, \quad (64)\]

**Theorem 5.7.** The operator \( A \) defined by (36)-(64) is dissipative in \( \mathcal{H} \). Moreover, \( A^{-1} \) exists and is compact on \( \mathcal{H} \). Therefore, \( A \) generates a \( C_0 \)-semigroup of contractions on \( \mathcal{H} \) and the spectrum \( \sigma(A) \) consists of isolated eigenvalues only.

**Proof.** Let \( Y \in \text{Dom}(A) \). Then

\[
\langle Ay, Y \rangle = \int_0^L \left[ (-\tilde{A}(y_1)_{xxxx} - \gamma \beta h_2h_3\ddot{B}^2 D_x J(y_1)_x) \tilde{y}_2 \\
(\tilde{A}(y_2)_{xx}(\tilde{y}_1)_{xx} - \gamma \beta h_2h_3\ddot{B}^2 J(y_2)_x(\tilde{y}_1)_x) \right] \text{ } dx
\]

\[
= + \left( -\tilde{A}(y_1)_{xxxx} - \gamma \beta h_2h_3\ddot{B}^2 D_x J(y_1)_x \right) \tilde{y}_2 \bigg|_{x=0}^L + \tilde{A}(y_1)_{xx}(\tilde{y}_2)_x \bigg|_{x=0}^L + \int_0^L \left[ -\tilde{A}(y_1)_{xx}(\tilde{y}_2)_{xx} + \gamma \beta h_2h_3\ddot{B}^2 J(y_1)_x(\tilde{y}_2)_x \right] dx
\]

\[
+ \int_0^L \left[ \tilde{A}(y_2)_{xx}(\tilde{y}_1)_{xx} - \gamma \beta h_2h_3\ddot{B}^2 J(y_2)_x(\tilde{y}_1)_x \right] dx.
\]

Therefore, \( \text{Re} \langle Ay, Y \rangle = -\frac{k_1 B_4}{B_3} (h_2h_3\ddot{B}B_2P_\xi + B_3I) \tilde{z}_2(L) \cdot P_\xi \tilde{z}_2(L) \leq 0 \). Therefore \( A \) is dissipative. Therefore, if \( A^{-\frac{d}{dt}} \) exists, \( A \) must be densely defined in \( \mathcal{H} \). Therefore, \( A \) generates a \( C_0 \)-semigroup of contractions on \( \mathcal{H} \). Next, we show that \( 0 \in \sigma(A) \), i.e. \( 0 \) is not an eigenvalue. We solve the following problem:

\[
\begin{cases}
\dot{w}_{xxxx} + \gamma \beta h_2h_3\ddot{B}^2 Jw_x = 0, \\
w(0) = w_x(0) = w_x(L) = \dot{w}_{xxxx}(L) + \gamma \beta h_2h_3\ddot{B}^2 Jw_x(L) = 0.
\end{cases}
\]

(65)

Let \( Jw_x := u \). By the definition of \( J = (\xi \ddot{C} I - D_x^2)^{-1} D_x^2 \), (65) is re-written as

\[
\begin{cases}
\dot{w}_{xxxx} - \beta \gamma h_2h_3\ddot{B}u_x = 0, \\
\ddot{C}u - u_{xx} + \ddot{B}w_{xxx} = 0, \\
w(0) = w_x(0) = u(0) = w_x(L) = u_x(L) = \dot{w}_{xxxx}(L) - \beta \gamma h_2h_3\ddot{B}u(L) = 0.
\end{cases}
\]
By using the last boundary condition, we integrate the first equation and plug it in the \( u \)–equation to get \( (\xi + \frac{\beta \gamma h_2 h_3 \beta^2}{A}) u - u_{xx} = 0 \). Since \( \xi + \frac{\beta \gamma h_2 h_3 \beta^2}{A} > 0 \), by the boundary conditions for \( u \), we obtain that \( u \equiv 0 \). This implies that \( w_{xxx} = 0 \).

By the boundary conditions \( w \equiv 0 \). \( \dot{A}w_{xxx} - \beta \gamma h_2 h_3 \beta^2 u = 0 \). Thus, \( 0 \in \sigma(A) \), and \( A^{-1} \) is compact on \( H \). Hence the spectrum \( \sigma(A) \) consists of isolated eigenvalues only.

**Theorem 5.8.** Let the feedback (63) be chosen and \( g(t) \equiv 0 \) in (32). Then the solutions \( \varphi(t) \) for \( t \in \mathbb{R}^+ \) of the closed-loop system (37) is strongly stable in \( H \).

**Proof.** If we can show that there are no eigenvalues on the imaginary axis, or in other words, the set

\[
\{ z \in H : \langle A z, z \rangle_H = -\frac{k_1 \gamma}{B_4} (h_2 h_3 \tilde{B} B_2 P_\xi + B_3 I) \langle z_2 \rangle_x (L) \cdot (P_\xi \tilde{z}_2 \rangle_x (L) = 0 \}
\]

has only the trivial solution, i.e. \( z = 0 \); then by La Salle’s invariance principle, the system is strongly stable. Therefore, proving the asymptotic stability of the (1)-(3) reduces to showing that the following eigenvalue problem \( Az = \lambda z \):

\[
\begin{aligned}
\dot{A}w_{xxx} + \gamma \beta \gamma h_2 h_3 \beta^2 (Jw_x)_x + \lambda^2 w &= 0, \\
w(0) = w_x(0) = w_x(L) = w_{xx}(L) &= 0, \\
\dot{A}w_{xx}(L) + \gamma \beta \gamma h_2 h_3 \beta^2 Jw_x(L) &= (P_\xi w_x)(L) = 0.
\end{aligned}
\]

has only the trivial solution. By using the definition of (34), i.e. \( (Jw_x) = (\xi P_\xi w_x) - w_x \), we obtain that \( (Jw_x)(L) = 0 \) since both terms \( (P_\xi w_x)(L) \) and \( w_x(L) \) are zero by (66).

Let \( \lambda = i\omega \) where \( \omega \in \mathbb{R} \). Then (67) reduces to

\[
\begin{aligned}
\dot{A}w_{xxx} + \gamma \beta \gamma h_2 h_3 \beta^2 (Jw_x)_x - \omega^2 w &= 0, \\
w(0) = w_x(0) = w_x(L) = w_{xx}(L) &= 0, \\
\dot{A}w_{xx}(L) + \gamma \beta \gamma h_2 h_3 \beta^2 Jw_x(L) &= (P_\xi w_x)(L) = 0.
\end{aligned}
\]

Note that the following integrals are true,

\[
\begin{aligned}
\int_0^L xw_{xxx} \bar{w}_{xxx} dx &= -\frac{1}{2} \int_0^L |w_{xxx}|^2 dx, \\
\int_0^L xw \bar{w}_{xx} dx &= \int_0^L \frac{3}{2} \int_0^L |w_x|^2 dx
\end{aligned}
\]

and

\[
\begin{aligned}
\int_0^L x(Jw_x)_x \bar{w}_{xxx} dx &= \int_0^L x((\xi P_\xi - I)w_x)_x \bar{w}_{xxx} dx \\
\int_0^L \xi (P_\xi w_x)_x x \bar{w}_{xxx} dx + \frac{1}{2} \int_0^L |w_{xxx}|^2 dx
\end{aligned}
\]

Let \( z = P_\xi w_x \). Then \( \xi z - \bar{z}_{xx} = w_x \), and therefore

\[
\int_0^L \xi (P_\xi w_x)_x x \bar{w}_{xxx} dx = \int_0^L \xi z_x x(\xi \bar{z}_{xx} - \bar{z}_{xxxx}) = -\frac{1}{2} \int_0^L (\xi^2 |z_x|^2 + \xi |z_{xx}|^2) dx
\]

Multiplying the equation by \( x \bar{w}_{xxx} \) and integrate by parts using the boundary conditions to obtain

\[
\int_0^L (\bar{A}|w_{xxx}|^2 + 3m|w_x|^2 + \xi^2 |z_x|^2 + \xi |z_{xx}|^2) dx = 0.
\]
By using the overdetermined boundary conditions we obtain \( w \equiv 0 \).

We state the following stability theorem and skip the proof since it goes beyond the scope of the paper. The proof uses the same type of frequency domain approach and spectral multipliers used in [26, Theorem 4] where a stronger \( B^* - \) type feedback is chosen \( V(t) = -k_1 [\varsigma h_2 h_3 \tilde{B}_2(P \dot{w}_x(L)) + B_3 \dot{w}_x(L)] \) in comparison to (63).

**Theorem 5.9.** Let the feedback (63) be chosen and \( g(t) \equiv 0 \) in (32). Then the solutions \( \varphi \) for \( t \in \mathbb{R}^+ \) of the closed-loop system (37) is exponentially stable in \( \mathcal{H} \).

Note that our result not only confirms the results in [3] but also improves them since only the asymptotic stability is mentioned in [3] without a proof.

6. **Conclusion and Final Remarks.** In this paper, electrostatic voltage-controlled piezoelectric smart composite beam models are shown to be exponentially stable with the choice of the \( B^* - \) type state feedback, which are all mechanical. This is similar to the charge-actuation case but not the current actuation case where only the asymptotic stability can be achieved [25]. For the fully dynamic R-N model, asymptotic stability can be achieved for “inertial sliding solutions” yet this is not the case for the fully dynamic M-M model. There may still be eigenvalues on the imaginary axis. This implies that one electric controller for the piezoelectric layer may not be enough in general to asymptotically (or exponentially) stabilize larger classes of solutions involving bending motions. This lines up with the results for the charge or current-actuated models [25]. The stabilization results are summarized in Table 2.

Finally, we can conclude that even though the magnetic effects are minor in comparison to the mechanical and electrical effects for a piezoelectric layer, they have dramatic effects in controlling these composites. Note that the stabilizability of fully dynamic R-N and M-M models for energy-space solutions is still an open problem. On the other hand, consideration of a remedial damping injection (by a mechanical feedback controller) to the piezoelectric layer of the fully dynamic models is under consideration. Numerical results confirm that mechanical feedback controllers have a stronger effect to suppress vibrations [27]. Together with the effect of shear damping, the investigation of the optimal decay rates to tune up the damping parameters and the feedback gains is the topic of future research.

The modeling of the three layer composition can be formed into a bimorph energy harvester [9, 32] or a shear mode energy harvester [18] to convert the stabilization problem to an energy harvesting problem. The mathematical analysis provided in this paper will be a perfect foundation for future research on these models.

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Table 2. Stability results for the closed-loop system with the $B^*-$feedback controller corresponding to the control $V(t)$ of the piezoelectric layer.

| Assumption | Model   | $B^*-$measurement for $V(t)$ at $x = L$ | Stability   |
|------------|---------|----------------------------------------|-------------|
| E-static   | Rao-Nakra | Stretching & compressing velocity       | E.S.        |
| F. Dynamic |         | Induced current                         |             |
| E-static   | Mead-Marcus | Angular velocity (bending) + shear      | E. S.       |
| F. Dynamic |         | velocity (middle layer)                 |             |
|            |         | Induced current                         | Not A.S.    |

Different electro-magnetic assumptions for cantilevered R-N and M-M models. Here E.S.=Exponentially Stability for all modes, A.S.= Asymptotically Stability for inertial sliding solutions, Not A.S.=Not Asymptotically Stability for inertial sliding solutions.

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