The Hausdorff- and Nagata-dimension of intrinsic Möbius spaces

Merlin Incerti-Medici

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Abstract

Intrinsic Möbius spaces are spaces with an additional geometric structure that ‘measures tetrahedrons’. They lie strictly between metric spaces and topological spaces. In this paper, we ask the question, whether certain notions of dimension generalize from metric spaces to intrinsic Möbius spaces.

We define the notion of Hausdorff- and Nagata-dimension for intrinsic Möbius spaces and prove that both notions are invariant under Möbius equivalences. We start by generalizing the classical definition of Hausdorff- and Nagata-dimension for metric spaces to quasi-metric spaces. We then prove that all quasi-metrics that induce the same intrinsic Möbius structure have the same Hausdorff- and Nagata-dimension respectively, by proving their invariance under rescaling and involution.

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1 Introduction

It is well known that for any hyperbolic metric space we can define its boundary at infinity and equip it with a visual metric. It is also well known that there is a choice to be made when defining such a metric and that the resulting metric space is not independent of that choice. However, it is known that all these metrics belong to the same Möbius structure, meaning that they all define the same cross-ratio-triple
\[(w, x, y, z) \mapsto \text{crt}_d(wxyz) := (d(w, x)d(y, z) : d(w, y)d(x, z) : d(w, z)d(x, y)) \in \mathbb{R}P^2.\]

It turns out that Möbius structures are a geometric structure in their own right that can exist without the aid of a metric. This has been studied in [B] and [IM] where the notion of generalized Möbius spaces has been introduced. In fact, generalized Möbius spaces lie somewhere between metric spaces and topological spaces. They carry a natural topological structure, which is not necessarily metrizable. Thus, one may wonder which geometric notions that are known on metric spaces can be carried over to generalized Möbius spaces. In this paper, we will give a positive answer to this question in the case of the Hausdorff- and the Nagata-dimension.

Let \(X\) be a set with at least three points. A \textit{semi-metric} \(d\) on \(X\) is a map \(d : X \times X \to \mathbb{R}_{\geq 0}\) such that:

1. For all \(x, y \in X\), \(d(x, y) = d(y, x)\).
2. \(d(x, y) = 0 \iff x = y\).

An \textit{extended semi-metric} on \(X\) is a function \(d : X \times X \to [0, \infty]\) such that there exists exactly one point \(\omega \in X\) such that \(d(x, y) < \infty\) for all \(x, y \in X \setminus \{\omega\}\) and \(d(x, \omega) = \infty\), whenever \(x \in X \setminus \{\omega\}\). If there is such a point \(\omega\), we also say that \(d\) has a \textit{point at infinity} and often denote that point by \(\infty\).

An analogous definition can be made for metrics and quasi-metrics (see below). Throughout the paper, we will frequently work in the extended setting.

Let \(X\) be a set, \(d\) an (extended) semi-metric on \(X\). An \(n\)-tuple is called \textit{non-degenerate}, if it consists of mutually different points. Denote by \(A_4\) the set of \textit{admissible quadruples}, that is, the set of all quadruples \((wxyz) \in X^4\) in which no point of \(X\) appears more than twice. Further, denote

\[\Delta := \{(a : b : c) \in \mathbb{R}P^2|a, b, c > 0\} \quad \overline{\Delta} := \Delta \cup \{(1 : 1 : 0), (1 : 0 : 1), (0 : 1 : 1)\}.\]

The \textit{cross-ratio-triple} of \((X, d)\) introduced above is a map

\[\text{crt}_d : A_4 \to \overline{\Delta}\]

\[(wxyz) \mapsto (d(w, x)d(y, z) : d(w, y)d(z, x) : d(w, z)d(x, y)).\]

If one of the points in the quadruple lies at infinity, appearing infinite distances are said to ‘cancel’, i.e. we define

\[\text{crt}_d(\infty xyz) := (d(y, z) : d(z, x) : d(x, y)),\]

\[\text{crt}_d(\infty\infty yz) := (0 : 1 : 1)\]

and analogously for any permutation of the points in the quadruple (cf. the homomorphism \(\phi\) introduced below). Equivalently, one can denote

\[L_4 := \{(x, y, z) \in \mathbb{R}^3|x + y + z = 0\} \quad \overline{L_4} := L_4 \cup \{(0, \infty, -\infty), (-\infty, 0, \infty), (\infty, -\infty, 0)\}\]

and define a homeomorphism
\[ \Phi : \Delta \rightarrow L_4 \]
\[ (a : b : c) \mapsto \left( \ln \left( \frac{b}{c} \right), \ln \left( \frac{c}{a} \right), \ln \left( \frac{a}{b} \right) \right). \]

We define the topology on \( \overline{L_4} \) to be such that \( \Phi \) extends to a homeomorphism \( \Phi : \overline{\Delta} \rightarrow \overline{L_4} \) by sending
\[ (1 : 1 : 0) \mapsto (\infty, -\infty, 0) \]
\[ (1 : 0 : 1) \mapsto (-\infty, 0, \infty) \]
\[ (0 : 1 : 1) \mapsto (0, \infty, -\infty). \]

The map \( M : \mathcal{A}_4 \rightarrow \overline{L_4} \), which is defined by \( M := \Phi \circ \text{crt}_d \), encodes the same information as the map \( \text{crt}_d \) but is sometimes more convenient for notational reasons.

Before we can generalize the notion of cross-ratio-triple introduced above, we need to introduce one more piece. Consider the triple \( ((12)(34), (13)(42), (14)(23)) \). The symmetric group of four elements \( S_4 \) acts on this triple by permuting the numbers 1-4. Whenever \( \sigma \in S_4 \) acts on the numbers, it induces a permutation on the triple. Define \( \phi(\sigma) \in S_3 \) to be the permutation on the triple induced by the action of \( \sigma \). It is easy to check that \( \phi : S_4 \rightarrow S_3 \) is a group homomorphism. One can interpret the expression \( (12)(34) \) to denote two opposite edges of a tetrahedron whose corners are labeled by the numbers 1-4. In this interpretation, \( \phi \) is the group homomorphism that sends a permutation of the corners to the induced permutation of pairs of opposite edges.

We are now ready to define generalized Möbius structures. Let \( X \) be a set with at least three distinct points. Classically, a Möbius structure is defined as an equivalence class of metrics \( d \) on \( X \) where two metrics are called equivalent if they have the same cross-ratio-triple as defined above. In order to study the geometry of a (not necessarily metric) space with a cross-ratio-triple, we use the following definition that goes back to Sergei Buyalo’s work in \( [B] \).

**Definition 1.** Let \( X \) be a set with at least three points. A map \( M : \mathcal{A}_4 \rightarrow \overline{L_4} \) is called a generalized Möbius structure if and only if it satisfies the following conditions:

1) For all \( P \in \mathcal{A}_4 \) and all \( \pi \in S_4 \), we have
\[ M(\pi P) = sgn(\pi) \varphi(\pi) M(P). \]

2) For \( P \in \mathcal{A}_4 \), \( M(P) \in L_4 \) if and only if \( P \) is non-degenerate.

3) For \( P = (x, x, y, z) \), we have \( M(P) = (0, \infty, -\infty) \).

4) Let \( (x, y, \omega, \alpha, \beta) \) be an admissible 5-tuple \( (x, y, \omega, \alpha, \beta) \) such that \( (\omega, \alpha, \beta) \) is a non-degenerate triple, \( \alpha \neq x \neq \beta \) and \( \alpha \neq y \neq \beta \). Then, there exists some \( \lambda = \lambda(x, y, \omega, \alpha, \beta) \in \mathbb{R} \cup \{\pm \infty\} \) such that
\[ M(\alpha x \omega \beta) + M(\alpha \omega y \beta) - M(\alpha x y \beta) = (\lambda, -\lambda, 0). \]
Moreover, when \((\omega, \alpha, \beta)\) is non-degenerate, \(x \neq \beta\) and \(y \neq \alpha\), the first component of the left-hand-side expression is well-defined. Analogously, the second component of the left-hand-side expression is well-defined when \((\omega, \alpha, \beta)\) is non-degenerate, \(x \neq \alpha\) and \(y \neq \beta\).

The pair \((X, M)\) is called a generalized Möbius space.

Note that the map \(M\) induces a map \(crt := \overline{P}^{-1} \circ M : \mathcal{A}_4 \to \overline{\mathcal{X}}\) which encodes the same information as \(M\). One could define a generalized Möbius structure in terms of \(crt\), however some of the conditions above are more convenient to formulate for \(M\). Given a generalized Möbius space \((X, M)\), we will switch between \(M\) and \(crt\) without further comment.

In [B] (cf. also [PS] or [IM]), it has been shown that any generalized Möbius structure \((X, crt)\) can be written as \((X, crt_d)\) for some semi-metric \(d\) on \(X\). We say \(d\) induces \(crt\). We call two semi-metrics \(d, d'\) Möbius equivalent if they induce the same generalized Möbius structure, i.e. \(crt_d = crt_{d'}\). Given a generalized Möbius structure \(M\), we have that for every triple of mutually different points \(A = (\omega, \alpha, \beta) \in X^3\), there is a semi-metric \(d_A\) that induces \(M\). These semi-metrics are characterized by the fact that for every semi-metric \(d\) that induces \(M\), we have

\[
d_A(x, y) = \frac{d(x, y)}{d(x, \omega)d(\omega, y)} \frac{d(\alpha, \omega)d(\omega, \alpha)}{d(\alpha, \beta)}.\]

As mentioned earlier, a generalized Möbius structure \(crt\) induces a topology on \(X\). A basis of this topology is given by the collection of all open balls \(B_{A,r}(x) := \{y \in X | d_A(x, y) < r\}\) where \(A = (\omega, \alpha, \beta)\) goes over all triples of mutually different points, \(r > 0\) and \(x \neq \omega\). This topology is called Möbius topology. If there exists a metric \(d\) that induces \(crt\), then the Möbius topology and the metric topology coincide (cf. [B] or [IM]).

Given two generalized Möbius spaces \((X, M), (X', M')\), a bijection \(f : X \to X'\) is called a Möbius equivalence if it sends \(M\) to \(M'\), i.e. \(M(wxyz) = M'(f(w)f(x)f(y)f(z))\) for all admissible quadruples \((wxyz)\). Any Möbius equivalence is also a homeomorphism (cf. [IM]).

Since semi-metrics lack certain properties that are useful in the context of generalized Möbius spaces, we want to specialize to quasi-metrics. We briefly recall the definition of a quasi-metric.

**Definition 2.** Let \(X\) be a set, \(K \geq 1\). A \(K\)-quasi-metric on \(X\) is a semi-metric \(d : X \times X \to [0, \infty)\) such that for all \(x, y, z \in X\),

\[
d(x, y) \leq K \max(d(x, z), d(z, y)).\]

We define an extended \(K\)-quasi-metric analogously to the notion of extended semi-metrics above.

The following condition turns out to characterize generalized Möbius structures that are induced by a quasi-metric (see [IM] for more details).

**Definition 3.** Let \((X, crt)\) be a generalized Möbius space. We say that \(crt\) satisfies the (corner)-condition if the closure of \(\text{Im}(crt) \subset \mathbb{RP}^2\) does not contain the points \((1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\), i.e. \(\overline{\text{Im}(crt)} \cap \{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\} = \emptyset\).

It has been shown in [IM] that a generalized Möbius structure \(crt\) satisfies the (corner)-condition if and only if one, and hence all, semi-metrics \(d\) that have a point at infinity and induce \(crt\) are
quasi-metrics. From now on, we will only consider generalized Möbius structures that satisfy the (corner)-condition. We call such \( \text{crt} \) intrinsic Möbius structures and \((X, \text{crt})\) an intrinsic Möbius space. Note that the \( d_A \) become quasi-metrics under this assumption.

We will frequently need the following construction.

**Definition 4.** Let \((X, d)\) be an (extended) quasi-metric space, \( o \in X \setminus \{ \infty \} \). The involution of \( d \) at the point \( o \) is defined to be

\[
d_o(x, y) := \begin{cases} 
0 & x = y, \\
\frac{d(x, y)}{d(x, o)d(y, o)} & x \neq \infty, y \neq \infty, x \neq y, \\
\frac{1}{d(x, o)} & x = \infty, x \neq y, \\
\frac{1}{d(x, y)} & y = \infty, x \neq y.
\end{cases}
\]

Note that we use the convention that \( \frac{\lambda}{0} = \infty \) for all \( \lambda > 0 \).

It is well-known that \( d_o \) is again a quasi-metric (cf. for example Proposition 5.3.6 [BS]).

Note that \( \text{crt}_d = \text{crt}_{d_o} \). It is now clear from the properties of the quasi-metrics \( d_A \) that any two semi-metrics that induce the same generalized Möbius structure are related by finitely many rescalings and involutions. Thus, when we have a notion for quasi-metric spaces that is invariant under rescaling and involution (like the Hausdorff-dimension, as we will see below), then this notion can be defined for intrinsic Möbius spaces by simply choosing a quasi-metric that induces the intrinsic Möbius structure. The invariance under involution and rescaling shows independence of choice.

We will use this procedure to show that there exists a notion of Hausdorff-dimension on intrinsic Möbius spaces. Specifically, we will generalize the definition of Hausdorff-measures from metric spaces to quasi-metric spaces. We will show that there is a singular dimension just like in the metric case, which we will call the Hausdorff-dimension \( \dim_{\text{Haus}}(X, d) \) of a quasi-metric space \((X, d)\). We will then prove the following

**Theorem 1.** Let \((X, M)\) be a intrinsic Möbius space, \( d \) a quasi-metric inducing \( M \). Let \( d' \) be a rescaling or an involution of \( d \).

Then \( \dim_{\text{Haus}}(X, d) = \dim_{\text{Haus}}(X, d') \). In particular, we can define the Hausdorff-dimension of \((X, M)\) to be \( \dim_{\text{Haus}}(X, M) := \dim_{\text{Haus}}(X, d) \). Furthermore, \( \dim_{\text{Haus}}(X, M) \) is invariant under Möbius equivalence.

The Nagata-dimension has been introduced in a note by Assouad (cf. [A]). It is a variation of the asymptotic dimension which is due to Gromov. The Nagata-dimension can be defined as follows:

**Definition 5.** Let \((X, d)\) be a metric space and \( B \) a cover of \( X \). The cover \( B \) is called \( C \)-bounded if for every \( B \in B \), \( \text{diam}(B) \leq C \), i.e. the diameter of every set in the cover \( B \) is bounded by \( C \).

Let \( s > 0 \), \( m \in \mathbb{N} \). We say that a family of subsets \( B \subset \mathcal{P}(X) \) has \( s \)-multiplicity \( \leq m \) if for every set \( U \in X \) with \( \text{diam}(U) \leq s \), there are at most \( m \) elements \( B \in B \) with \( B \cap U \neq \emptyset \).

The Nagata-dimension \( \dim_N(X, d) \) is defined to be the infimum of all \( n \) such that there exists a constant \( c > 0 \) such that for all \( s > 0 \) there exists a \( cs \)-bounded cover of \( X \) with \( s \)-multiplicity \( \leq n + 1 \).
The class of metric spaces that have finite Nagata-dimension includes doubling spaces, metric trees, euclidean buildings, homogeneous or pinched negatively curved Hadamard manifolds and others (cf. [LS]). The Nagata-dimension is, however, not preserved by quasi-isometries.

The Nagata-dimension can immediately be generalized to quasi-metric spaces, simply by replacing the metric in the definition by a quasi-metric. We will prove the following

**Theorem 2.** Let \((X, d)\) be an (extended) quasi-metric space, \(o \in X\). Let \(d'\) be a rescaling or an involution of \(d\). Then \(\dim_N(X, d) = \dim_N(X, d')\).

In particular, given a M"obius structure \(M\) and any two quasi-metrics \(d, d'\) that induce \(M\), we have \(\dim_N(X, d) = \dim_N(X, d')\). Thus we can define \(\dim_N(X, M) := \dim_N(X, d)\) for any quasi-metric \(d\) that induces \(M\). Furthermore, \(\dim_N(X, M)\) is invariant under M"obius equivalence.

These two theorems allow us to generalize the notions of Hausdorff- and Nagata-dimension from metric spaces to intrinsic M"obius spaces. This begs the question whether there are any intrinsic M"obius spaces that are not induced by a metric space, making these generalizations empty generalizations. It turns out that there are such examples and we will state one such example at the end of the paper.

The rest of the paper is organized as follows. In Section 2, we will generalize the Hausdorff-dimension to quasi-metric spaces and prove Theorem 1. We will use it as an illustrating example for our strategy. In Section 3, we will use the same strategy to prove Theorem 2. In Section 4, we summarize our results and provide an example of an intrinsic M"obius space whose M"obius structure cannot be induced by a metric.

The results in section 3 are generalizations of results that are known for metric spaces and are due to Lang-Schlichenmaier and Xiangdong respectively (cf. [LS] and [X]). The upshot of the generalizations presented in this paper is that in considering not only metrics but also quasi-metrics, we can define the Hausdorff- and Nagata-dimension of intrinsic M"obius spaces whose M"obius structure is not induced by a metric.

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## 2 The Hausdorff-dimension

### 2.1 Definition

Consider an intrinsic M"obius space \((X, M)\) and let \(d\) be a - possibly extended - \(K\)-quasi-metric that induces \(M\). Let \(A \subseteq X\) be a subset, \(s \geq 0\) and \(\delta > 0\). We denote by \(B_r(x) := \{y \in X | d(x, y) \leq r\}\) the closed ball of radius \(r\) around \(x\) with respect to \(d\). Furthermore, we say a \(\delta\)-cover of \(A\) is a cover of \(A\) by closed balls \(B_r(x_i)\) such that for all \(i \in I, r_i \leq \delta\). To avoid clustered notation, we will often omit the index set \(I\) and simply speak of the index \(i\). We define

\[
\mu^\delta_{A,d}(A) := \inf \left( \sum_i r_i^s \left| \{B_r(x_i)\} \text{ is a } \delta\text{-cover of } A \setminus \{\infty\} \right. \right).
\]
Clearly, $\mu_{s,d}^{\delta}(A)$ is increasing for any $A \subseteq X$ as $\delta$ goes to zero. We define the $s$-dimensional Hausdorff measure to be the, thus existing, limit

$$\mu_{s,d}^{\delta}(A) := \lim_{\delta \to 0} \mu_{s,d}^{\delta}(A) \in [0, \infty].$$

It is easy to see that $\mu_{s,d}^0(\emptyset) = 0$ and that $\mu_{s,d}^{\delta}$ is monotone and subadditive for countable unions of subsets. Thus, we have defined an outer measure on $X$. This definition is completely analogous to the definition of the $s$-dimensional Hausdorff measure on a metric space, except that we have to exclude the point at infinity, if it belongs to $A$. In preparation to dealing with the point at infinity, we first show that adding or removing one point that does not lie at infinity from the set $A$, does not change $\mu_{s,d}^{\delta}(A)$.

Let $A \subseteq X$ and $p \in X \setminus (A \cup \{\infty\})$. Let $B_{r_i}(x_i)$ be a cover of $A$ by balls of radius $r_i \leq \delta$. By adding one ball $B_{\delta}(p)$, we get a cover of $A \cup \{p\}$. When computing $\mu_{s,d}^{\delta}(A \cup \{p\})$, this cover contributes with the expression

$$\sum_i r_i^s + \delta^s.$$

Taking the infimum over all possible $\delta$-covers, we see that

$$\mu_{s,d}^{\delta}(A) \leq \mu_{s,d}^{\delta}(A \cup \{p\}) \leq \mu_{s,d}^{\delta}(A) + \delta^s.$$

Taking the limit $\delta \to 0$ yields

$$\mu_{s,d}^{\delta}(A) = \mu_{s,d}^{\delta}(A \cup \{p\}).$$

So we see that $\mu_{s,d}^{\delta}$ doesn’t change when we add or remove a point in $X$. We now prove the following lemma.

**Lemma 1.** Let $d$ be a quasi-metric on $X$ and $s \geq 0$. Suppose, $\mu_{s,d}^{\delta}(X) < \infty$. Then for all $t > s$, $\mu_{s,d}^{t}(X) = 0$.

**Proof.** The proof is exactly the same as for the Hausdorff measure of a metric space. Let $\{B_{r_i}(x_i)\}_i$ be a cover of $X \setminus \{\infty\}$ with $r_i \leq \delta$ for all $i$. Then

$$\sum_i r_i^t = \sum_i r_i^s r_i^{t-s} \leq \sum_i r_i^s \delta^{t-s} = \delta^s \sum_i r_i^s.$$

Taking the infimum over all covers with $r_i \leq \delta$, we get

$$\mu_{s,d}^{t}(X) \leq \delta^s \mu_{s,d}^{s}(X).$$

By assumption, $\mu_{s,d}^{\delta}(X) < \infty$ and by construction $\mu_{s,d}^{s}(X) \leq \mu_{s,d}^{\delta}(X) < \infty$. Thus, as we let $\delta \to 0$, we get

$$\mu_{s,d}^{t}(X) \leq 0.$$

Since $\mu_{s,d}^{t}$ is non-negative, this implies that $\mu_{s,d}^{t}(X) = 0$.

Lemma 1 tells us that there is a critical value $c \in [0, \infty]$, such that for all $s < c$, $\mu_{s,d}^{c}(X) = \infty$, while for all $t > c$, $\mu_{s,d}^{t}(X) = 0$.
Definition 6. Let \((X, M)\) be a Möbius space satisfying the (corner)-condition. Choose a quasi-metric \(d\) that induces \(M\) and define the Hausdorff measures \(\mu^s_d\) on \(X\) using \(d\). Define the Hausdorff-dimension of \((X, d)\) to be 
\[
\dim_{Haus}(X, d) := \inf(s \in \mathbb{R} | \mu^s_d(X) = 0).
\]
Furthermore, define the Hausdorff-dimension of \((X, M)\) to be 
\[
\dim_{Haus}(X, M) := \inf(s \in \mathbb{R} | \mu^s_d(M) = 0).
\]

2.2 Well-definedness

Clearly, the Hausdorff-dimension of \((X, M)\) is well-defined if and only if we can show that this definition is independent of the choice of \(d\). Recall from Section 1 that, if \(d\) induces \(M\), then the quasi-metrics \(d_A\) can be expressed as a rescaling and an involution of \(d\). Specifically, for \(A = (\omega, \alpha, \beta)\), we can write
\[
d_A(x, y) = \frac{d(x, y)}{d(\omega, y)} \cdot \frac{d(\alpha, \omega) d(\omega, \beta)}{d(\alpha, \beta)},
\]
where infinite distances cancel each other. Therefore, if we can show that rescaling and taking an involution of \(d\) does not change the Hausdorff-dimension, then we have shown that every quasi-metric \(d\) that induces \(M\) induces the same Hausdorff-dimension, making it well-defined. In order to distinguish between balls for different quasi-metrics in the upcoming discussion, we denote the closed ball of radius \(r\) around \(x\) with respect to a quasi-metric \(d\) by \(B_{d,r}(x)\).

Proof of Theorem 2. Suppose \((X, d)\) has no point at infinity. We can then enlarge \(X\) by an additional point, denoted \(\infty\), and extend the \(K\)-quasi-metric \(d\) to a \(K\)-quasi-metric on \(X \cup \{\infty\}\) by setting \(d(x, \infty) := \infty\) for all \(x \in X\) and \(d(\infty, \infty) := 0\). This yields an extended \(K\)-quasi-metric space with the same Hausdorff-dimension as \((X, d)\) as the Hausdorff-measure on \(X\) doesn’t change. Thus, we can assume without loss of generality that \((X, d)\) has a point at infinity.

We start by looking at rescalings of \(d\). Let \(\lambda > 0\) and consider the quasi-metrics \(d\) and \(\lambda d\). It is easy to see that the Hausdorff-dimension is invariant under bijective bi-Lipschitz maps. Since the identity map from \((X, d)\) to \((X, \lambda d)\) is bi-Lipschitz, this implies that they have the same Hausdorff-dimension.

Now let \(d'\) be the involution of \(d\) at the point \(o \in X\). We need the following lemma.

Lemma 2. Fix a constant \(\epsilon > 0\). For all \(\delta < \frac{1}{K^2}\) and all \(\delta\)-coverings \(\{B_{d,r_i}(x_i)\}_i\) of \(X \setminus \{\infty\}\), there is a subfamily of the collection \(\{B_{d',\frac{2}{K^3}r_i}(x_i)\}_i\) that is a \(\frac{K^3}{2}\)-\(\delta\)-covering of \(X \setminus \{\infty\} \cup B_{d,\epsilon}(o)\).

We will first prove how the lemma implies invariance under involution. Let \(\epsilon = \frac{1}{n}\). Let \(\{B_{d,r_i}(x_i)\}\) be a \(\delta\)-covering of \(X \setminus \{\infty\}\) for \(\delta < \frac{1}{K^3}\). Then we find a \(n^2K^3\)-\(\delta\)-covering of \(X \setminus \{\infty\} \cup B_{d,\epsilon}(o)\) of the form \(B_{d',n^2K^3,r_i}(x_i)\) where \(i\) runs over a subset of the indices of the original \(\delta\)-covering. Thus,
\[
\mu^s_{n^2K^3}(X \setminus \{\infty\} \cup B_{d,\epsilon}(o)) \leq \sum_i (n^2K^3 r_i)^s = (n^2K^3)^s \sum_i r_i^s.
\]

Taking the infimum over all \(\delta\)-coverings of \(X \setminus \{\infty\}\), we get
\[ \mu_{n^2K^3,d',\delta}'(X \setminus \{\infty\} \cup B_{d',\frac{1}{d}}(o)) \leq (n^2K^3)^\delta \mu_{d',\delta}(X). \]

Now suppose, \( \mu_d^\delta(X) = 0 \). Then \( \mu_{d',\delta}(X) = 0 \) for all \( \delta > 0 \) and hence \( \mu_{n^2K^3,d',\delta}'(X \setminus \{\infty\} \cup B_{d',\frac{1}{d}}(o)) = 0 \) for all \( n \in \mathbb{N} \) and all \( 0 < \delta < \frac{1}{n^2K^3} \) (and thus for all \( \delta > 0 \)).

We have seen earlier that adding one point doesn’t change the Hausdorff measure. Therefore, we can add the point \( \infty \) and get

\[ \mu_{n^2K^3,d',\delta}'(X \setminus \{\infty\} \cup B_{d',\frac{1}{d}}(o)) = 0 \]

for all \( 0 < \delta < \frac{1}{n^2K^3} \) and \( n \in \mathbb{N} \). Taking the limit for \( \delta \to 0 \) yields

\[ \mu_{d'}(X \setminus B_{d',\frac{1}{d}}(o)) = 0 \]

for all \( n \in \mathbb{N} \). Using the \( \sigma \)-subadditivity of \( \mu_{d'} \), we see that

\[ \mu_{d'}(X) = \mu_{d'}(X \setminus \{o\}) \leq \sum_{n=1}^{\infty} \mu_{d'}(X \setminus B_{d',\frac{1}{d}}(o)) = \sum_{n=1}^{\infty} 0 = 0. \]

We conclude that if \( \mu_d^\delta(X) = 0 \), then \( \mu_{d'}^\delta(X) = 0 \). Therefore, the Hausdorff-dimension of \((X, d')\) is at most the Hausdorff-dimension of \((X, d)\). Since the involution of \( d' \) at the point \( \infty \) is again \( d \), we get the reversed inequality as well. This implies that the Hausdorff-dimension is invariant under involution and that the Hausdorff-dimension of an intrinsic Möbius space is well-defined.

We are left to prove that the Hausdorff-dimension is invariant under Möbius equivalence. Let \( f : (X, M) \to (X', M') \) be a Möbius equivalence. Consider the quasi-metric \( d_A \) for some triple of mutually different points \( A \in X' \). Since \( f \) is a bijection, \( f(A) \) is a triple of mutually different points in \( X' \). Since \( f \) is a Möbius equivalence, it sends \( d_A \) isometrically to \( d_{f(A)} \), i.e. \( d_A(x, y) = d_{f(A)}(f(x), f(y)) \) (cf. [IM]). Therefore, \( \text{dim}_{\text{Haus}}(X, d_A) = \text{dim}_{\text{Haus}}(X', d_{f(A)}) \) and \( \text{dim}_{\text{Haus}}(X, M) = \text{dim}_{\text{Haus}}(X, M') \). This implies Theorem [1] up to the proof of Lemma [2].

Proof of Lemma [2] Let \( \{B_{d,r}(x_i)\}_i \) be a \( \delta \)-covering of \( X \setminus \{\infty\} \). Recall that

\[ d'(x_i, y) = \frac{d(x_i, y)}{d(x_i, o)d(o, y)} \leq \frac{r_i}{d(x_i, o)d(o, y)}. \]

Let \( B_{d,r}(x_i) \) be a ball such that \( d(x_i, o) > \frac{r_i}{K} \). The collection of balls \( B_{d,r}(x_i) \) with such \( x_i \) covers \( X \setminus \{\infty\} \cup B_{d,e}(o) \), since for all \( y \) with \( d(y, o) > \epsilon \), we find some \( i \) such that \( y \in B_{r}(x_i) \). We then see that

\[ \epsilon < d(y, o) \leq K \max(d(y, x_i), d(x_i, o)). \]

Since \( Kd(y, x_i) \leq K\delta < \frac{1}{K} < \epsilon \), the inequality above implies \( \epsilon < Kd(x_i, o) \) and thus the collection of \( B_{d,r}(x_i) \) with \( d(x_i, o) > \frac{r_i}{K} \) covers \( X \setminus \{\infty\} \cup B_{d,e}(o) \). For similar reasons, we have \( d(y, o) > \frac{r_i}{K} \) for all \( y \in B_{d,r}(x_i) \). Thus, we have for all \( x_i \) with \( d(x_i, o) > \frac{r_i}{K} \) and for all \( y \in B_{d,r}(x_i) \)
\[ d'(x_i, y) \leq \frac{r_i}{d(x_i, o)d(o, y)} \leq \frac{K^3r_i}{\epsilon^2}. \]

This implies that \( B_{d', r_i}(x_i) \subseteq B_{d'}(x_i, o) \) and thus the collection \( \{ B_{d', \frac{K^3r_i}{\epsilon^2}}(x_i) \}_{d(x_i, \omega) > \frac{\epsilon}{2}} \) is a \( \frac{K^3g}{\epsilon^2} \)-covering of \( X \setminus (\{ \infty \} \cup B_{d', o}(o)) \). This proves the lemma.

\( \square \)

**Remark 1.** Consider a metric space \((X, d)\) together with a Borel measure \(\mu\) and let \(Q > 0\). We call \(\mu\) **Ahlfors \(Q\)-regular** if there exists a constant \(C > 0\) such that for every \(R > 0\) and every ball \(B_R\) of radius \(R\), we have

\[ \frac{1}{C}R^Q \leq \mu(B_R) \leq CR^Q. \]

It is a well-known result that, whenever \((X, d)\) admits an Ahlfors \(Q\)-regular measure, the Hausdorff-dimension of \((X, d)\) is equal to \(Q\) (cf. [H]). The same is true for quasi-metric spaces \((X, d)\) and the notion of Hausdorff-dimension for quasi-metric spaces we introduced above. The proof is analogous to the proof in the metric case.

In [LS], Li and Shanmugalingam showed that, whenever a metric space \((X, d)\) admits an Ahlfors \(Q\)-regular measure, there is a metric \(d'\) which is bi-Lipschitz to the quasi-metric obtained by taking the involution of \(d\) at a point \(o \in X\) and \((X, d')\) admits an Ahlfors \(Q\)-regular measure as well. This proves that the Hausdorff-dimension for metric spaces is invariant under the operation of taking the involution of a metric and then taking a specific metric that is bi-Lipschitz to the involution.

It is possible that this approach generalizes to quasi-metric spaces, yielding a quasi-metric notion of Ahlfors \(Q\)-regularity for Borel measures on Möbius spaces and an alternative proof of the invariance of the Hausdorff-dimension under involutions. However, if we want to define Ahlfors \(Q\)-regularity on Möbius spaces by considering quasi-metric spaces first and then proving invariance under rescaling and involution – as we have done here – we are confronted with the fact that the topology induced by one quasi-metric \(d\) does not agree with the Möbius topology of \(M_d\) in general (an example can be found in [S]). Thus, certain measure theoretic constructions that behave nicely for metric spaces may not yield Borel measures in the context of quasi-metric spaces.

### 3 The Nagata-dimension

#### 3.1 Definition

Let \((X, M)\) be an intrinsic Möbius space. Let \(d\) be an (extended) \(K\)-quasi-metric on \(X\) that induces \(M\). If \(d\) has a point at infinity in \(X\), denote that point by \(\infty\). Define the Nagata-dimension of \((X, d)\) as follows:

**Definition 7.** Let \(B\) be a cover of \(X \setminus \{\infty\}\) and \(C > 0\). We say that \(B\) is a **\(C\)-bounded cover of \(X\)**, if every set \(B \in B\) has \(\text{diam}(B) \leq C\).
Let \( s > 0, m \in \mathbb{N} \) and \( B \subset \mathcal{P}(X) \) be a collection of subsets of \( X \). We say that \( B \) has \( s \)-multiplicity \( \leq m \) if every set \( U \subset X \) with \( \text{diam}(U) \leq s \) intersects at most \( m \) elements of \( B \).

The Nagata-dimension \( \dim_N(X, d) \) is defined to be the infimum of all \( n \) for which there exists a constant \( c > 0 \) such that for all \( s > 0 \) there exists a \( cs \)-bounded cover of \( X \) with \( s \)-multiplicity \( \leq n + 1 \).

Note that the Nagata-dimension does not depend at all on whether \( (X, d) \) has a point at infinity or not. This doesn’t pose a problem due to the following proposition, which is a generalization of a result that can be found in [LS] and [L].

**Proposition 1.** Let \( (X, d) \) be a quasi-metric space such that \( X = Y \cup Z \). Then \( \dim_N(X, d) = \max(\dim_N(Y, d), \dim_N(Z, d)) \).

Clearly, the Nagata-dimension of a point is zero and hence \( \dim_N(X, d) = \dim_N(X \setminus \{p\}, d) \) for any point \( p \in X \). The proof of Proposition 1 is a simple generalization of the proof in [LS] (cf. also [L]).

We now need to show that for all Möbius equivalent quasi-metrics \( d, d', \dim_N(X, d) = \dim_N(X, d') \) (Möbius equivalent meaning that the two quasi-metrics induce the same Möbius structure). As in our treatment of the Hausdorff-dimension in Section 2, it is enough to show invariance under rescaling and involution. Once we have proven this, we can define \( \dim_N(X, M) := \dim_N(X, d) \) for any quasi-metric \( d \) that induces \( M \) and get a well-defined notion of Nagata-dimension for Möbius spaces.

### 3.2 Well-definedness

The goal of this section is to prove Theorem 2. If the Nagata-dimension of quasi-metric spaces is invariant under rescaling and involution, it is also invariant under Möbius equivalence by the same argument we used to prove Möbius invariance of the Hausdorff-dimension. Since the Nagata-dimension is invariant under bi-Lipschitz maps, we immediately see that it is invariant under rescaling. Thus, we are left to prove that the Nagata-dimension is invariant under taking involution. This is the content of the following proposition which is a generalization of a result due to Xiangdong (cf. [X]).

**Proposition 2.** Let \( d \) be a \( K \)-quasi-metric on \( X \), \( o \in X \setminus \{\infty\} \). Let \( d_o \) be the involution of \( d \) at the point \( o \). Then

\[
\dim_N(X, d) = \dim_N(X, d_o).
\]

Before we start with the actual proof, which is a generalization of the proof in [X], we will show that, in order to prove proposition 2 it is enough to prove that \( \dim_N(X, d) \geq \dim_N(X, d_o) \). There are two cases to consider. If \( X \) has a point \( \infty \) at infinity with respect to \( d \), an easy computation shows that the involution of \( d_o \) at \( \infty \) is again \( d \), i.e.

\[
d(x, y) = \frac{d_o(x, y)}{d_o(x, \infty)d_o(\infty, y)}.
\]

Therefore, the roles of \( d \) and \( d_o \) in the lemma above are interchangeable and it is sufficient to prove \( \dim_N(X, d) \geq \dim_N(X, d_o) \).
If $X$ has no point at infinity with respect to $d$, we extend $(X, d)$ by adding a point $\infty$ at infinity (cf. Section 2). When doing so, $d$ remains a quasi-metric and we can write $d$ as the involution of $d_0$ at the point $\infty$ as above. If we assume that $\dim_N(X, d) \geq \dim_N(X, d')$ for all quasi-metrics $d, d'$, where $d'$ is an involution of $d$ at a point in $X$, we get

$$\dim_N(X, d) = \dim_N(X \cup \{\infty\}, d)$$
$$\geq \dim_N(X \cup \{\infty\}, d_0)$$
$$= \dim_N(X, d_0)$$
$$= \dim_N(X \cup \{\infty\}, d_0)$$
$$\geq \dim_N(X \cup \{\infty\}, d)$$
$$= \dim_N(X, d).$$

In the third line, we use Proposition 1 and in the last inequality, we use the fact that $d$ on $X \cup \{\infty\}$ can be written as the involution of $d_0$ at the point $\infty$. It is thus sufficient to prove that $\dim_N(X, d) \geq \dim_N(X, d_0)$ in order to prove proposition 2. In particular, this tells us that we may assume that $\dim_N(X, d) < \infty$.

Before we can prove Proposition 2, we need to do some preparations. Specifically, we need to state and generalize several results from [LS].

**Proposition 3.** Let $(X, d)$ be a quasi-metric space with $K$-quasi-metric $d$ and $n \geq 0$ an integer. The following are equivalent:

1. The Nagata-dimension $\dim_N(X, d) \leq n$, i.e. there exists a constant $c_1 > 0$ such that for all $s > 0$, $X \setminus \{\infty\}$ has a $c_1 s$-bounded covering with $s$-multiplicity $\leq n + 1$.
2. There exists a constant $c_2 > 0$ such that for all $s > 0$, $X \setminus \{\infty\}$ admits a $c_2 s$-bounded covering of the form $B = \bigcup_{k=0}^{n} B_k$ where each family $B_k$ has $s$-multiplicity $\leq 1$.

We use Proposition 3 to prove

**Proposition 4.** Let $(X, d)$ be a quasi-metric space with $\dim_N(X) \leq n < \infty$. Then there is a constant $c > 0$ such that for all sufficiently large $r > 1$, there exists a sequence of coverings $B^j$ of $X \setminus \{\infty\}$ with $j \in \mathbb{Z}$, satisfying the following properties:

1. For every $j \in \mathbb{Z}$, we can write $B^j = \bigcup_{k=0}^{n} B_k^j$ where each $B_k^j$ is a $cr^j$-bounded family with $r^j$-multiplicity $\leq 1$.
2. For every $j \in \mathbb{Z}$ and $x \in X$, there exists a set $C \in B^j$ that contains the closed ball $B_{d,r^j}(x)$.
3. For every $k \in \{0, \ldots, n\}$ and every bounded set $B \subset X$, there is a set $C \in B_k := \bigcup_{j \in \mathbb{Z}} B_k^j$ such that $B \subset C$.
4. Whenever $B \in B_k^j$ and $C \in B_k^i$ for some $k$ and $i < j$, then either $B \subset C$ or $d(x, y) > r^i$ for all $x \in B, y \in C$. 

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Proposition 3 is a generalized version of a larger proposition in [LS] which characterizes the definition of the Nagata-dimension by four different conditions. While it would be interesting to check whether the entire proposition can be generalized to quasi-metrics, we only need the characterization presented here for which we provide a more direct proof. Proposition 4 is a direct generalization from [LS] and so is its proof. We leave it as an exercise for the reader to modify the proof as necessary.

Proof of proposition 3. The implication (2) ⇒ (1) is trivial. For (1) ⇒ (2), suppose \( \dim_N(X, d) = n \) and let \( c > 0 \) be the constant in the definition of \( \dim_N(X, d) \). Let \( s > 0 \). We can find a \( cK^{2n} \)-bounded covering \( B \) of \( X \setminus \{ \infty \} \) with \( K^{2n} \)-multiplicity \( \leq n + 1 \). For simplicity, we assume without loss of generality that \( c \geq 1 \). We are going to construct \( n + 1 \) families \( B_i \) of \( cK^{4n} \)-bounded subsets of \( X \) such that \( \bigcup_{i=1}^{n+1} B_i \) is a covering of \( X \) and each \( B_i \) has \( s \)-multiplicity \( \leq 1 \). This proves that \((X, d)\) satisfies (2) with constant \( cK^{1n} \).

Let \( B \in \mathcal{B} \). Define inductively \( N^0B := B \) and \( N^iB \) as the \( s \)-neighbourhood of \( N^{i-1}B \) for \( i > 0 \). Note that \( \text{diam}(N^iB) \leq K^2 \max(s, \text{diam}(N^{i-1}B)) \leq cK^{2n+2i}s \) and the collection of all \( N^iB \) – denoted \( N^i\mathcal{B} \) – has \( K^{2(n-i)} \)-multiplicity \( \leq n + 1 \). Both statements are easily proved by induction. We now define our new covering. Let \( i \in \{1, \ldots, n+1\} \). We define \( B_i \) to be the collection of sets of the following form

\[
A = \bigcap_{j=1}^{i} N^{i-1}B_j \setminus \bigcup_{B \notin \{B_1, \ldots, B_i\}} N^iB
\]

where \( B_1, \ldots, B_i \in \mathcal{B} \) are mutually distinct sets.

Since \( N^i\mathcal{B} \) has \( K^{2(n-i)} \)-multiplicity \( \leq n + 1 \) and \( \{x\} \) is a set of diameter zero, every point \( x \) is contained in at most \( n + 1 \) many elements of \( N^i\mathcal{B} \) for every \( i \geq 0 \). We claim that \( \bigcup_{i=1}^{n+1} B_i \) is a covering of \( X \). We will show this by using induction to prove the following claim for every \( x \in X \) and then show that the induction ends at \( i = n + 1 \).

Claim 1. Let \( x \in X \). Then for all \( i \geq 1 \), either \( x \) is contained in an element of \( B_j \) for some \( j \leq i \), or there are mutually distinct \( B_1, \ldots, B_{i+1} \in \mathcal{B} \) such that \( x \in \bigcap_{j=1}^{i} N^{i-1}B_j \cap N^iB_{i+1} \).

Proof of Claim. Let \( x \in X \). We start the induction at \( i = 1 \). If \( x \) is contained in exactly one element \( B \in \mathcal{B} \) and is not contained in \( N^1C \) for any \( C \in \mathcal{B} \), then \( x \) is contained in an element of \( B_1 \). Suppose this is not the case. Then \( x \) is contained in at least one element \( B \in \mathcal{B} \) (since \( \mathcal{B} \) is a covering) and at least one element \( N^1C \in N^i\mathcal{B} \) such that \( B \neq C \) (since \( x \) is not covered by \( B_1 \)). This concludes the start of the induction.

Suppose now, the claim is true for all \( 1 \leq j \leq i \) for a fixed \( i \geq 1 \). We want to prove the claim for \( i + 1 \). Suppose \( x \) is not contained in \( B_j \) for all \( 1 \leq j \leq i \). Then, by induction-assumption, there exist mutually distinct \( B_1, \ldots, B_{i+1} \) such that \( x \in \bigcap_{j=1}^{i} N^{i-1}B_j \cap N^iB_{i+1} \subset \bigcap_{j=1}^{i+1} N^iB_j \). Suppose, \( x \) is not contained in any element of \( B_{i+1} \); in particular, \( x \notin \bigcap_{j=1}^{i+1} N^iB_j \setminus \bigcap_{B \notin \{B_1, \ldots, B_{i+1}\}} N^{i+1}B \). Therefore, there exists some \( C \in \mathcal{B} \setminus \{B_1, \ldots, B_{i+1}\} \), such that \( x \in N^{i+1}C \). Hence, \( x \in \bigcap_{j=1}^{i+1} N^iB_j \cap N^{i+1}B_{i+2} \) where we denote \( B_{i+2} := C \). This implies that either, \( x \) is contained in an element of \( B_j \) for \( j \leq i+1 \), or we find mutually distinct elements \( B_1, \ldots, B_{i+2} \in \mathcal{B} \) such that \( x \in \bigcap_{j=1}^{i+1} N^iB_j \cap N^{i+1}B_{i+2} \). This proves the claim.

\(\square\)
The claim tells us the following. Starting at \( i = 1 \), we can check for every \( \mathcal{B}_i \), whether it contains \( x \). If \( x \) is not contained in \( \mathcal{B}_j \) for all \( 1 \leq j \leq n+1 \), then we find distinct elements \( B_1, \ldots, B_{n+2} \in \mathcal{B} \) such that \( x \in \bigcap_{j=1}^{n+2} N^{n+1}B_j \). Since \( N^{n+1}\mathcal{B} \) has \( K^{-2}s \)-multiplicity \( \leq n+1 \), this cannot happen. Therefore, \( x \) has to be contained in \( \mathcal{B}_i \) for some \( i \leq n+1 \). This implies that \( \bigcup_{i=1}^{n+1} \mathcal{B}_i \) is a covering of \( X \).

We are left to show that \( \mathcal{B}_1 \) is \( cK^{4n}s \)-bounded and has \( s \)-multiplicity \( \leq 1 \). Let \( A \in \mathcal{B}_1 \). Then \( A \subset N_i^{-1}B \) for some \( B \in \mathcal{B} \) and therefore, \( \text{diam}(A) \leq cK^{2n+2}(i-1)s \leq cK^{4n}s \) for all \( i \leq n+1 \).

For the \( s \)-multiplicity, consider a subset \( U \) with \( \text{diam}(U) \leq s \). Suppose this is not true. Then we find some \( \ell \in \{ a \} \). By construction, \( A \) and \( \overline{\mathcal{B}} \) have the form \( \bigcap_{j=1}^{i} N_i^{-1}B_j \setminus \bigcup_{\ell \notin \{ B_1, \ldots, B_i \}} N_i^\ell B \) and \( \bigcap_{j=1}^{i} N_i^{-1}\overline{\mathcal{B}}_j \setminus \bigcup_{\ell \notin \{ \overline{B}_1, \ldots, \overline{B}_i \}} N_i^\ell B \) respectively. We want to show that \( \{ B_1, \ldots, B_i \} = \{ \overline{B}_1, \ldots, \overline{B}_i \} \). Assume this is not true. Then we find some \( \ell \) such that \( \overline{\mathcal{B}}_\ell \notin \{ B_1, \ldots, B_i \} \). Therefore, \( a \notin N_i^\ell B \).

However, since \( \overline{\mathcal{B}} \in N_i^{-1}\overline{\mathcal{B}} \) and \( d(a, \overline{\mathcal{B}}) \leq s \), \( a \in N_i^\ell B \), a contradiction. Therefore, \( \{ B_1, \ldots, B_i \} = \{ \overline{B}_1, \ldots, \overline{B}_i \} \) and \( A = \overline{\mathcal{B}} \). This implies that \( \mathcal{B}_1 \) has \( s \)-multiplicity \( \leq 1 \) for all \( 1 \leq i \leq n+1 \) which completes the proof of Proposition 3.

\[ \square \]

**Proof of proposition 3.** The proof of Proposition 3 is a generalization of the proof in [X]. To illustrate how the generalizations in all the proofs are executed, we will present the proof. As discussed earlier, we only need to prove that \( \dim_N(X, d) \geq \dim_N(X, d_o) \). Moreover, we can assume \( n := \dim_N(X, d) < \infty \). We start by noticing that \( \dim_N(X, d) = \dim_N(X \setminus \{ \infty, o \}, d) \). Let \( s > 0 \). By Proposition 4, we find \( c \geq 1 \), \( r > 1 \) and a sequence of coverings \( \mathcal{B}^j \) of \( (X \setminus \{ \infty, o \}, d) \), \( j \in \mathbb{Z} \) with properties (i)-(iv) of Proposition 4. Put \( c' := K^tc, c' := 2K^3c^2, \tilde{c} := 10crK^3 \). We will construct a family \( \bigcup_{k=0}^{n} \mathcal{E}_k \) that is \( c' \)-s-bounded with respect to \( d_o \), covers \( X \setminus \{ \infty, o \} \) and each \( \mathcal{E}_k \) has \( s \)-multiplicity \( \leq 1 \). Since adding the two points \( \infty, o \) doesn’t change the Nagata-dimension, this implies that \( \dim_N(X, d) \leq n \).

Let \( a := \inf\{ d(x, o) | x \in X \setminus \{ \infty, o \} \} \) and \( b := \sup\{ d(x, o) | x \in X \setminus \{ \infty, o \} \} \). If \( 0 < a \) and \( b < \infty \), then \( \text{id} : (X \setminus \{ \infty, o \}, d) \rightarrow (X \setminus \{ \infty, o \}, d_o) \) is bi-Lipschitz and hence preserves the Nagata-dimension. This can be seen by considering the following two inequalities.

\[
d_o(x, y) = \frac{d(x, y)}{d(x, o)d(o, y)} \leq \frac{d(x, y)}{a^2}
\]

\[
d(x, y) = d_o(x, y)d(o, y) \leq d_o(x, y)b^2
\]

Thus, we assume from now on that either \( a = 0 \) or \( b = \infty \). Suppose \( a > 0 \) and therefore, \( b = \infty \). Then

\[
d_o(x, y) = \frac{d(x, y)}{d(x, o)d(o, y)} \leq \frac{K}{\min(d(x, o), d(o, y))} \leq \frac{K}{a}
\]

for all \( x, y \in X \setminus \{ \infty, o \} \). Thus, \( (X \setminus \{ \infty, o \}, d_o) \) is bounded by \( \frac{K}{a} \). If \( c' \geq \frac{K}{a} \), then we can cover \( X \setminus \{ \infty, o \} \) by one \( c' \)-s-bounded set which we then choose as our covering.
Now suppose that $0 < s < \frac{K}{cr}$ and $a \geq 0$. Consider the set $A_s := \{ x \in X \mid 0 < d(x, o) \leq \frac{2K}{cr} \}$. Since $a < \frac{K}{cr}$, $A_s$ is non-empty. Note that the complement of $A_s$ is $c's$-bounded with respect to $d_o$, since for all $x, y \in X \setminus A_s$

$$d_o(x, y) \leq \frac{K}{\min(d(x, o), d(o, y))} \leq \frac{Kc's}{2K} < c's.$$

Let $x \in A_s$. By property (ii), for each $j \in \mathbb{Z}$ there exists a $C^j_x \in \mathcal{B}^j$ such that $B_{d,r^j}(x) \subset C^j_x$. Since $\mathcal{B}^j$ is $cr^j$-bounded, $\text{diam}_{d_j}(C^j_x) \leq cr^j$. We analyze the diameter of $C^j_x$ with respect to $d_o$. For any $y, z \in C^j_x$, we have

$$d_o(y, z) = \frac{d(y, z)}{d(y, o)d(o, z)} \leq \frac{cr^j}{d(y, o)d(o, z)}.$$

We now show that $\text{diam}_{d_o}(C^j_x) \to 0$ as $j \to -\infty$. For this, we estimate $d(y, o)$ from below for all $y \in C^j_x$. We know that

$$d(x, o) \leq K \max(d(x, y), d(y, o)) \leq K \max(cr^j, d(y, o)).$$

For $j$ sufficiently small, this inequality takes the form

$$d(x, o) \leq Kd(y, o).$$

Therefore, we see that

$$\text{diam}_{d_o}(C^j_x) \leq \frac{K^2cr^j}{d(x, o)^2} \xrightarrow{j \to -\infty} 0.$$

In particular, this means that for any $x \in A_s$ and $j$ sufficiently small, we find a set $C^j_x \in \mathcal{B}^j$ such that $\text{diam}_{d_o}(C) \leq cs$.

**Lemma 3.** Let $x \in A_s$. Define

$$j(x) := \sup\{ j \in \mathbb{Z} \mid \text{there is a } C \in \mathcal{B}^j : B_{d,r^j}(x) \subset C \text{ and } \text{diam}_{d_o}(C) \leq cs \}.$$

Then $j(x) < \infty$.

The map $j(x)$ points us to the largest $C^j_x$ that such that the family $\{C^j_x\}_{x \in A_s}$ is $cs$-bounded. After removing some redundant elements, this family has $s$-multiplicity $\leq n + 1$ which is what we want.

**Proof of Lemma** Suppose $a = 0$. Then we can find a sequence $x_i \in X \setminus \{\infty, o\}$ such that $d(x_i, o) \to 0$. Then, for all $j \in \mathbb{Z}$ with $r^j > d(x, o)$, we have that $x_i \in B_{d,r^j}(x)$ for sufficiently large $i$. Now we see that

$$d_o(x, x_i) = \frac{d(x, x_i)}{d(x, o)d(o, x_i)} \xrightarrow{i \to \infty} \infty,$$

where we use the fact that $d$ is continuous in each variable with respect to the Möbius topology and thus, $d(x, x_i) \to d(x, o) > 0$ (cf. [IM]). Thus, $\text{diam}_{d_o}(C^j_x) \geq \text{diam}_{d_o}(B_{d,r^j}(x)) = \infty$ and hence $r^{j(x)} \leq d(x, o)$. In particular, $j(x) < \infty$. 

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Now suppose \( a > 0 \) and, consequently, \( b = \infty \). We find a sequence \( x_i \in X \setminus \{ \infty, o \} \) such that 
\( d(x_i, o) \to \infty \). Now choose and fix \( I \in \mathbb{N} \) sufficiently large such that
\[
\frac{d(x, x_I)}{d(o, x_I)} \geq \frac{d(x, x_I)}{K \max(d(x, x_I), d(x, o))} = \frac{1}{K}.
\]
For every \( j \in \mathbb{Z} \) such that \( r^j > d(x, x_I) \), we have \( x_I \in B_{d, r^j}(x) \) and thus,
\[
\text{diam}_{d_a}(C_j^I) \geq \text{diam}_{d_a}(B_{d, r^j}(x)) \geq d_a(x, x_I) = \frac{d(x, x_I)}{d(o, x) d(o, x_I)} \geq \frac{1}{K d(x, o)} \geq \frac{1}{K} \frac{c's}{2K} > \tilde{c}s,
\]
where we use that \( x \in A_s \) for the second-to-last inequality. This implies that \( r^{j(x)} \leq d(x, x_I) \). In particular, \( j(x) < \infty \).

We now define a subfamily of \( \mathcal{B} \) by \( \mathcal{C} := \{ C_x^{j(x)} \mid x \in A_s \} \). Note that we can write \( \mathcal{C} = \bigcup_{k=0}^n \mathcal{C}_k \) where \( \mathcal{C}_k := \mathcal{C} \cap B_k \). Each family \( \mathcal{C}_k \) may contain too many sets to have \( s \)-multiplicity \( \leq 1 \). The next lemma allows us to throw away those elements of \( \mathcal{C} \) that are not needed.

**Lemma 4.** For every \( C \in \mathcal{C} \) there exists a maximal element \( \overline{C} \) of \( \mathcal{C} \) such that \( C \subseteq \overline{C} \).

**Proof.** It is enough to show that there is no infinite strictly increasing sequence in \( \mathcal{C} \). Suppose we have a sequence \( (C_i)_i = (C_{x_i}^{j(x_i)}) \), such that \( C_x^{j(x)} \subseteq C_x^{j(x+1)} \). Without loss of generality, we can assume that there is a \( k \in \{0, \ldots, n\} \) such that \( C_i \in \mathcal{C}_k \) for all \( i \) by passing to a subsequence if necessary. Since \( B_k \) has \( r^j \)-multiplicity \( \leq 1 \), we have \( j(x_i) \neq j(x_{i'}) \) for all \( i \neq i' \). Therefore, we can assume that \( j(x_i) \to \pm \infty \) by again passing to a subsequence if necessary. If \( j(x_i) \to -\infty \), this would imply that the strictly increasing sequence \( C_i \) is contained in a ball of radius \( \epsilon > 0 \) for arbitrarily small \( \epsilon \). This implies that all \( C_i \) are contained in a single point which cannot be as the sequence is strictly increasing. Thus, \( j(x_i) \to \infty \) and \( \bigcap_{i=1}^\infty C_i = X \setminus \{ \infty, o \} \) by properties (ii), (iii) and (iv) and because \( C_i \subseteq B_{d, r^{j(x_i)}}(x_i) \).

On the other hand, \( \text{diam}_{d_a}(C_i) \leq \tilde{c}s \) for all \( i \) and therefore, \( \text{diam}_{d_a}(X \setminus \{ \infty, o \}) \leq \tilde{c}s \). This implies that \( a > 0 \) and \( b = \infty \), since \( (X, d_a) \) is unbounded if \( a = 0 \). Choose a sequence \( (y_j)_j \) in \( X \setminus \{ \infty, o \} \) such that \( d(y_j, o) \to \infty \) and, therefore, \( d(y_j, x) \to \infty \). Then we have
\[
\text{diam}_{d_a}(X \setminus \{ \infty, o \}) \geq d_o(x, y_j) = \frac{d(x, y_j)}{d(o, y_j)} \geq \frac{d(x, y_j)}{d(x, o) K \max(d(o, x), d(x, y_j))} \xrightarrow{j \to \infty} \frac{1}{K d(x, o)} \geq \frac{c's}{2K} > \tilde{c}s
\]
for all \( x \in A_s \). This implies \( \tilde{c}s < \text{diam}_{d_a}(X \setminus \{ \infty, o \}) \leq \tilde{c}s \) which is a contradiction.

Define by \( \mathcal{D} \) the subfamily of \( \mathcal{C} \) consisting of the maximal elements of \( \mathcal{C} \) with respect to inclusion. By Lemma 4, this is still a covering of \( A_s \). Furthermore, define \( \mathcal{D}_k := \mathcal{D} \cap B_k \).

We claim that for every \( k \in \{0, \ldots, n\} \), \( \mathcal{D}_k \) has \( s \)-multiplicity \( \leq 1 \) with respect to \( d_a \). We prove this by proving the following claim.
Claim 2. Let \( C_x^{j(x)}, C_y^{j(y)} \in \mathcal{D}_k \). Then either \( C_x^{j(x)} = C_y^{j(y)} \), or \( d_o(x', y') > s \) for all \( x' \in C_x^{j(x)}, y' \in C_y^{j(y)} \).

Proof of Claim 2. Suppose \( C_x^{j(x)} \neq C_y^{j(y)} \) and suppose, by contradiction, there are \( x' \in C_x^{j(x)}, y' \in C_y^{j(y)} \) such that \( d_o(x', y') \leq s \). Without loss of generality, assume \( j(x) \leq j(y) \). Since \( C_x^{j(x)} \) and \( C_y^{j(y)} \) are distinct and maximal elements of \( \mathcal{C} \subset \mathcal{B} \), property (iv) of Proposition 4 implies that \( d(\xi, \eta) \geq r^{j(x)} \) for all \( \xi \in C_x^{j(x)}, \eta \in C_y^{j(y)} \). In particular, \( d(x', y') \geq r^{j(x)} \).

By definition of \( j(x) \), we know that \( \text{diam}_{d_o}(C_x^{j(x)}) \leq \tilde{c}s \). Let \( z \in C_x^{j(x)} \cup \{y'\} \). Recall that \( d_o \) is a \( K' \)-quasi-metric for \( K' \geq K \). By increasing \( K \) if necessary, we can assume without loss of generality that \( d_o \) is also a \( K \)-quasi-metric. Then

\[
d_o(x, z) \leq K \max(d(x, x'), d(x', z)) \leq K \max(\tilde{c}s, s) = K\tilde{c}s,
\]

since \( \tilde{c} = 10crK^3 > 1 \). On the other hand,

\[
d_o(x, z) = \frac{d(x, z)}{d(x, o)d(o, z)}.
\]

Using this, the definitions of \( c' \) and \( \tilde{c} \) and the fact that \( x \in A_s \), we get

\[
d(x, z) \leq K\tilde{c}s \frac{2K}{sc'}d(o, z) = \frac{1}{K\tilde{c}}d(o, z).
\]

This implies that

\[
d(z, o) \leq K \max(d(z, x), d(x, o)) = K \max \left( \frac{1}{K\tilde{c}}d(o, z), d(x, o) \right) = Kd(x, o).
\]

By putting \( z = x' \) and \( z = y' \) respectively, we conclude that \( d(x', o) \leq Kd(x, o) \) and \( d(y', o) \leq Kd(x, o) \). Since \( \mathcal{B}^j \) satisfies property (ii), there is a \( C' \in \mathcal{B}^{j(x)+1} \) such that \( B_{d,r(x)+1}(x) \subset C' \). Let \( w \in C' \). Then, since \( \mathcal{B}^j \) is \( cr^j \)-bounded,

\[
d(x, w) \leq cr^{j(x)+1}d(x, w) \leq crK \max(d(x', x), d(x', y')) \leq \frac{crK}{K\tilde{c}}d(x', o) \leq \frac{1}{10K}d(x, o).
\]

Therefore,

\[
d(w, o) \leq K \max(d(w, x), d(x, o)) \leq K \max \left( \frac{1}{10K}d(x, o), d(x, o) \right) = Kd(x, o)
\]

and

\[
d(x, o) \leq K \max(d(x, w), d(w, o)) \leq K \max \left( \frac{1}{10K}d(x, o), d(w, o) \right) = Kd(w, o),
\]
as \(d(x, o) > \frac{1}{10} d(x, o)\). Together, this implies that \(\frac{1}{K} d(x, o) \leq d(w, o) \leq K d(x, o)\). Now we can estimate

\[
d_o(x, w) = \frac{d(x, w)}{d(x, o) d(o, w)} \\
\leq \frac{cr d(x, y)}{d(x, o)^2} \\
\leq K \frac{cr d(x', y')}{d(x, o)^2} \\
= K \frac{cr d_o(x', y') d(x, o) d(y', o)}{d(x, o)^2} \\
\leq K \frac{cr s K d(x, o) K d(x, o)}{d(x, o)^2} \\
= cr K^3 s \\
\leq \tilde{c} s
\]

where we used that \(d_o(x', y') \leq s, d(x', o) \leq K d(x, o), d(y', o) \leq K d(x, o)\) and \(\tilde{c} = 10cr K^3\). Thus, \(diam_{d_o}(C') \leq \tilde{c} s\) which contradicts the definition of \(j(x)\). The claim follows.

From the claim, we conclude that every \(\mathcal{D}_k\) has \(s\)-multiplicity \(\leq 1\). Further, \(\bigcup_{k=0}^n \mathcal{D}_k\) is a \(\tilde{c}\)-bounded cover of \(A_s\). Put \(B_s := X \setminus (A_s \cup \{\infty, o\})\). If \(B_s = \emptyset\), we are done. If \(B_s \neq \emptyset\), we estimate its diameter with respect to \(d_o\) as follows. For all \(x, y \in B_s\), we have \(d(x, o) > \frac{2K}{c s}\) and thus

\[
d_o(x, y) = \frac{d(x, y)}{d(x, o) d(o, y)} \leq \frac{K}{\min(d(x, o), d(o, y))} \leq K \frac{c' s}{2K} < c' s.
\]

Hence, \(diam_{d_o}(B_s) < c' s\). Define \(\mathcal{D}^{far}_0 := \{C \in \mathcal{D}_0 | d_o(C, B_s) > s\}\) and \(\mathcal{D}^{close}_0 := \{C \in \mathcal{D}_0 | d_o(C, B_s) \leq s\}\) where \(d_o(A, B) := \inf\{d(a, b) | a \in A, b \in B\}\). We now define \(E := B_s \cup \bigcup_{C \in \mathcal{D}^{close}_0} C\). The diameter of \(E\) is bounded by

\[
diam_{d_o}(E) \leq K^4 \max(s, \tilde{c} s, c' s) = K^4 c' s = c'' s.
\]

Defining \(\mathcal{E}_0 := \mathcal{D}^{far}_0 \cup \{E\}\) and \(\mathcal{E}_k := \mathcal{D}_k\) for \(k \in \{1, \ldots, n\}\), yields a \(c''\)-bounded covering of \(X \setminus \{\infty, o\}\) with respect to \(d_o\). Clearly, \(\mathcal{E}_0\) still has \(s\)-multiplicity \(\leq 1\) and so does \(\mathcal{E}_k\) for every \(k \in \{1, \ldots, n\}\). Thus, we have constructed a \(c''\)-bounded covering of \(X \setminus \{\infty, o\}\) with \(s\)-multiplicity \(\leq n + 1\). This proves that \(dim_N(X, d_o) \leq n = dim_N(X, d)\) which completes the proof of Proposition 2 and Theorem 2.

\[
4\text{ Concluding remarks}
\]

In this paper, we generalized the notion of Hausdorff- and Nagata-dimension to quasi-metric spaces. We then showed that both of these notions are invariant under rescaling and taking involution, which allowed us to define both dimensions for intrinsic Möbius spaces. In fact, by defining both notions for quasi-metrics first and then proving that either dimension is the same for all Möbius equivalent
quasi-metrics we additionally proved that the Hausdorff- and Nagata-dimensions are invariant under Möbius equivalence.

The upshot of this is, that we have well-defined notions of Hausdorff- and Nagata-dimension for intrinsic Möbius spaces that are not induced by metric spaces. In the realm of metric spaces, the invariances of the Nagata-dimension have been known and are due to Lang-Schlichenmaier and Xiangdong. We finish by stating an example of an intrinsic Möbius structure that is not induced by a metric to indicate that our results indeed constitute an improvement. In [IM] it has been shown that for any $K$-quasi-metric $d$, the image of the induced Möbius structure $crt_d$ doesn’t intersect the $\frac{1}{K^2}$-neighbourhood $\{(a:b:1)|a < \frac{1}{K^2}, b < \frac{1}{K^2}\}$. Since every metric is a 2-quasi-metric, any intrinsic Möbius structure whose image intersects the neighbourhood above for $K = 2$ cannot be induced by a metric. In [S], Viktor Schroeder presents a quasi-metric space whose induced Möbius structure sends admissible quadruples into the $\frac{1}{4}$-neighbourhood of $(0:0:1)$. Specifically, $crt_d(0,1,\frac{7}{16},\frac{9}{16}) = (\frac{512}{2165}, (\frac{21}{65})^2 : 1) \approx (0.12 : 0.1 : 1)$. Therefore, we have a new class of intrinsic Möbius spaces for which we have introduced the Hausdorff- and Nagata-dimension.

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