Dynamic inverse wave problems—part I: regularity for the direct problem

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Abstract
For parameter identification problems the Fréchet-derivative of the parameter-to-state map is of particular interest. In many applications, e.g. in seismic tomography, the unknown quantity is modeled as a coefficient in a linear differential equation, therefore computing the derivative of this map involves solving the same equation, but with a different right-hand side. It then remains to show that this right-hand side is regular enough to ensure the existence of a solution. For second-order hyperbolic PDEs with time-dependent parameters the needed results are not as readily available as in the stationary case, especially when working in a variational framework. This complicates for example the reconstruction of a time-dependent density in the wave equation. To overcome this problem we extend the existing regularity results to the time-dependent case.

Keywords: regularity, hyperbolic PDEs, evolution equation, Fréchet-derivative, dynamic inverse problems

1. Introduction

One of the motivations of this work is the identification of the space- and time-dependent mass density $\rho$ from the solution $u$ of the wave equation

$$u'' - \text{div} \frac{\nabla u}{\rho} = f \quad \text{in } [0,T] \times \Omega,$$

$$u = 0 \quad \text{in } [0,T] \times \partial \Omega, \quad u(0) = u_0 \text{ and } u'(0) = u_1.$$  \hspace{1cm} (1.1a)

A common approach is to write this problem as an abstract evolution equation of the form

$$u''(t) + A(t)u(t) = f(t) \quad \text{in } V^*, \quad \text{a.e. in } I,$$  \hspace{1cm} (1.2a)
where \( I := (0, T) \) with \( 0 < T < \infty \), \( V = H^1_0(\Omega) \), \( H = L^2(\Omega) \) and \( A(t) \in \mathcal{L}(V; V^*) \) is given via

\[
\langle A(t) \varphi, \psi \rangle_{V^* \times V} = \int_\Omega \nabla \varphi \cdot \nabla \psi \, dx, \quad \varphi, \psi \in V.
\]

This abstract problem is then analyzed using semigroup theory \([16]\) or variational techniques \([19, 12]\). Either of these tools yield a well-defined solution \( u \in L^2(I; V) \cap H^1(I; H) \cap H^2(I; V^*) \) for a strongly positive \( A \in W^{1,\infty}(I; \mathcal{L}(V; V^*)) \), i.e. \( \rho \in W^{1,\infty}(I; L^\infty(\Omega)) \) with \( \rho(t, x) \geq \rho_0 > 0 \) for almost all \((t, x) \in I \times \Omega\). Since our task is to reconstruct \( \rho \) from \( u \), the parameter-to-state map reads \( F : \rho \mapsto u \) and in order to apply Newton-based algorithms for the inversion of \( F \), knowledge of its Fréchet derivative is crucial. Formal derivation of (1.2a) shows that if this derivative exists, then \( u_h = \partial F(\rho)[h] \) with perturbation \( h \in W^{1,\infty}(I; L^\infty(\Omega)) \) would have to solve

\[
u''(t) - \text{div} \frac{\nabla u_h(t)}{\rho(t)} = -\text{div} \left( h(t) \frac{\nabla u(t)}{\rho(t)^2} \right)
\]

in a weak sense and satisfy homogeneous initial conditions. Without further information about \( u \) we only know that the right-hand side of this equation lies in \( L^2(I; V^*) \), which (in contrast to elliptic or parabolic equations, see e.g. \([1, 5–7]\) and the references therein) is not enough to ensure existence of \( u_h \). The equation would be solvable if the right-hand side was an element of \( H^1(I; V^*) \), which requires \( u \in H^1(I; V) \). This is the reason why we are interested in obtaining higher regularity of the solution \( u \).

A prominent approach to obtain higher regularity is the formal differentiation of the abstract formulation (1.2a) with respect to time, which yields

\[
u'''(t) + A'(t)u(t) + A(t)u'(t) = f'(t),
\]

and then treating this (complemented by suitable initial values) as a new problem for \( v := u' \), see \([18]\). The expression \( A'(t)u(t) \) is regarded as independent of \( v \) and moved to the right-hand side. The resulting equation then reads \( v''(t) + A(t)v(t) = f'(t) - A'(t)u(t) \). Now, to ensure the existence of such a \( v \), one needs that \( t \mapsto A'(t)u(t) \) is an element of either \( L^2(I; H) \) or \( H^1(I; V^*) \). The latter would require \( u \in H^1(I; V) \), which is what we are trying to show in the first place. The former can be fulfilled by assuming \( A'(t) \in \mathcal{L}(V; H) \), which is clearly violated in our example due to the time dependence of \( \rho \). Another possible approach is to first show spatial regularity for \( u \) and then use integration by parts to see that \( A'(t)u(t) \) can indeed be applied to elements of \( H \), i.e. \( u(t) \in \mathcal{D}(A'(t)) \). However, this is not possible without using specific knowledge about the Hilbert spaces \( H \) and \( V \).

Note that in the case of static parameters, one also needs regularity results in order to show Fréchet-differentiability of the forward operator. Following an analogous approach would lead to \( u'''(t) + Au'(t) = f'(t) \) instead of (1.3). This does not involve problematic expressions on the right-hand side.

We overcome all of these problems by regarding \( u \) as dependent on \( v \) by writing \( u(t) = u_0 + \int_0^t v(s) \, ds \). This results in a mixed integral and differential equation for \( v \), which we analyze using common variational techniques. Afterwards we show that \( v \in L^2(I; V) \cap H^1(I; H) \) indeed equals \( u' \) and therefore \( u \in H^1(I; V) \cap H^2(I; H) \). By iterating this process we can get even higher regularity for \( u \).
The analysis is done on the abstract problem (1.2) and is therefore not limited to the wave equation. It can directly be applied to related phenomena like the elastic wave- or Maxwell’s equations. Following this notion we want to include even more general cases of multiple unknown parameters, or a parameter that appears in more than one coefficient of the PDE. Hence we augment (1.2) even further by additional time-dependent operators $B, C, Q$ and try to achieve higher differentiability with respect to time of the solution of

$$\frac{d}{dt} [C(t)u'(t)] + B(t)u'(t) + (A(t) + Q(t))u(t) = f(t) \text{ in } V^*, \text{ a.e. in } I.$$ (1.4)

The operator $Q(t) \in \mathcal{L}(V; H)$ may include asymmetric parts in the PDE that depend only on first-order spatial derivatives, e.g. a transport term $\nabla u(t) \cdot b(t)$. The operators $B, C$ also generate time-dependent zero order terms after differentiating this equation more than once. In contrast to $A$ these expressions can safely be moved to the right-hand side, but we aim to get a single regularity theorem that is applicable to a general equation like (1.4). An example for a parameter appearing in multiple operators would be the equation $u''/\rho - \text{div}(\nabla u/\rho) = f$. This is a more realistic version of our introductory PDE, is also used in [9] and can be formulated as (1.4) as long as we write the left-hand side as $(u'/\rho)' + (\rho'/\rho^2)u' - \text{div}(\nabla u/\rho)$.

To our knowledge, there are only two results in literature that yield satisfactory regularity results. One is stated by [11] where he uses a variational approach, transforms the time interval from $(0, T)$ to $(0, \infty)$ and then solves the weak formulation in the space-time domain using a weaker form of Lax–Milgram. The resulting assumptions regarding the smoothness of the operators are the same, but his theory only applies to homogeneous initial conditions and requires that the first $k - 1$ derivatives of the right-hand side vanish at zero. The other one is from [8], using semigroup theory. The downside of these results is that they are limited to the simpler abstract problem (1.2) instead of (1.4). The needed compatibility conditions agree with ours in this specific setting. The author also introduces additional structure to $V$ and $H$ in order to show an abstract spatial regularity result.

Many articles on parameter identification in hyperbolic PDEs focus on equations with time-independent principal part, e.g. [2, 9, 13] and [10]. Our results give the means to treat these problems in a setting where the parameters are time-dependent. Thus it is now possible to tackle the corresponding dynamic inverse problems, such as waves propagating with a time-dependent wave speed or density. These results also benefit some of today’s other applications like optimization, see e.g. [17] or [14], or solving quasi-linear PDEs with Newton-like methods, where knowledge of the Fréchet derivative is important as well.

This article is organized as follows: we begin by stating an existence and uniqueness result for equation (1.4) and our regularity theorem, which we then aim to prove. For this, we need solvability of an auxiliary problem, which we analyze in section 3. This then allows for a compact proof of our main theorem in section 4.

Note that this article is the first part of a work on dynamic inverse problems. The second part demonstrates the usefulness of the following results in the context of an inverse problem. To be more precise, they are used for the analysis of the reconstruction of time-dependent parameters $A, B, C$ and $Q$ in (1.4).

2. Preliminaries and main theorem

Let $V, H$ denote real Hilbert spaces where $V$ is separable and the embedding $V \hookrightarrow H$ is dense, so that $V \hookrightarrow H \hookrightarrow V^*$ establishes a Gelfand triple. The dual space of $V$ is denoted by $V^*$. Without loss of generality we assume that $\|\cdot\|_H \leq \|\cdot\|_V$. For a Banach space $X$ we denote
by $W^{k,p}(I;X)$ the usual Bochner space of $X$-valued time-dependent functions (see e.g. [19]). Furthermore, $\mathcal{L}(X;Y)$ is the space of linear and continuous operators from $X$ to $Y$, and in the case $X = Y$ we just write $\mathcal{L}(X)$. The inner product of $H$ is denoted by $(\cdot, \cdot)$, and the duality pairing between $V^*$ and $V$ by $(\cdot, \cdot)$.

Taking (1.4) complemented by suitable initial conditions then reads

$$\frac{d}{dt} \left(C(t)u'(t)\right) + B(t)u'(t) + (A(t) + Q(t))u(t) = f(t) \text{ in } V^*, \text{ a.e. in } I, \quad (2.1a)$$
\hspace{1cm}

$$u(0) = u_0 \text{ in } H, \quad (C(\cdot)u'(\cdot))(0) = C(0)u_1 \text{ in } V^*, \quad (2.1b)$$

where $A \in L^\infty(I;\mathcal{L}(V;V^*))$, $B, C \in L^\infty(I;\mathcal{L}(H))$ and $Q \in L^\infty(I;\mathcal{L}(V;H))$. The operators $A$ and $C$ are assumed to be pointwise self-adjoint and strongly positive, i.e. $\langle C(t)\varphi, \varphi \rangle \geq c_0 \|\varphi\|^2_H$ and $\langle A(t)\varphi, \varphi \rangle \geq a_0 \|\varphi\|^2_V$ with $a_0 > 0$ and $c_0 > 0$. In the case that $A$ only fulfills the weaker Garding-inequality $\langle A(t)\varphi, \varphi \rangle \geq a_0 \|\varphi\|^2_V - \lambda \|\varphi\|^2_H$ with $\lambda \in \mathbb{R}$ one may remedy this by replacing $A$ with $A + \lambda I$ and $Q$ with $Q - \lambda I$. For notational simplicity we define the operator

$$C : L^2(I;H) \rightarrow L^2(I;H), \quad (Cu)(t) = C(t)u(t),$$

and analogously $\mathcal{A}, B$ and $Q$. This allows us to write $(Cu)'(t)$ instead of $(C(\cdot)u'(\cdot))'(t)$ and equation (2.1a) becomes $(Cu)' + Bu' + (A + Q)u = f$ in the $L^2(I;V^*)$ sense. We start by stating suitable conditions for the solvability of this problem.

**Lemma 2.1 (Well-posedness).** Provided that $A \in W^{1,\infty}(I;\mathcal{L}(V;V^*))$, $B, C \in W^{1,\infty}(I;\mathcal{L}(H))$, $Q \in W^{1,\infty}(I;\mathcal{L}(V;H))$, $f \in L^2(I;H)$ or $f \in H^1(I;V^*)$, $u_0 \in V$, and $u_1 \in H$ there exists a uniquely determined $u \in L^2(I;V) \cap H^1(I;H)$ with $(Cu)' + Bu' + (A + Q)u = f$ in the $L^2(I;V^*)$ sense.

**Proof.** The proof for the case where the operators are continuously differentiable in time can be found in [4], but the assertion also holds for $W^{1,\infty}$ because all assumptions on the operators only have to be satisfied up to a set of measure zero. The existence of a solution can also be deduced from the following corollary 3.4 by setting $k = 0$ in the corresponding proof, which is correct without any differentiability assumptions on $B$ and $Q$. They arise when showing uniqueness of the solution. If $Q \in L^\infty(I;\mathcal{L}(H))$ then this regularity of $Q$ would not be needed.

We will make frequent use of the following lemma, which can be seen as a generalization of the product rule.

**Lemma 2.2.** Let $X, Y$ be separable Banach spaces and $G \in W^{1,\infty}(I;\mathcal{L}(X;Y^*))$. Given any $u \in H^1(I;X), v \in H^1(I;Y)$ the map $t \mapsto \langle G(t)u(t), v(t) \rangle$ belongs to $W^{1,1}(I)$ and we have

$$\frac{d}{dt} \langle G(t)u(t), v(t) \rangle_{Y^*,Y} = \langle G'(t)u(t), v(t) \rangle_{Y^*,Y} + \langle G(t)u'(t), v(t) \rangle_{Y^*,Y} + \langle G(t)u(t), v'(t) \rangle_{Y^*,Y},$$

which holds for almost all $t \in I$.

**Proof.** We only need to show that the assertion holds for $G \in C^1(I;\mathcal{L}(X;Y^*))$. It is clear that both the right-hand side of (2.2) and the mapping $t \mapsto \langle G(t)u(t), v(t) \rangle_{Y^*,Y}$ belong to $L^1(I)$. Due to dense embeddings there exist $u^\varepsilon \in C^\infty(I;X), v^\varepsilon \in C^\infty(I;Y)$ such that $u^\varepsilon \rightarrow u$ in $H^1(I;X)$ and $v^\varepsilon \rightarrow v$ in $H^1(I;Y)$ when $\varepsilon \rightarrow 0$. Due to the chain rule we have
\[
\frac{d}{dt} \langle G(t)u(t,t), v(t) \rangle = \langle G'(t)u(t,t), v(t) \rangle + \langle G(t)(u^2)(t,t), v(t) \rangle + \langle G(t)u(t,t), (v')^2(t) \rangle
\]
in the classical sense. In particular,
\[
- \int_0^T q'(t) \langle G(t)u(t,t), v(t) \rangle \, dt
= \int_0^T q(t) \left[ \langle G'(t)u(t,t), v(t) \rangle + \langle G(t)(u^2)(t,t), v(t) \rangle + \langle G(t)u(t,t), (v')^2(t) \rangle \right] \, dt. \tag{2.3}
\]
Both sides of this equation converge to the respective terms evaluated at \( u \) and \( v \) when \( \varepsilon \to 0 \).

We follow the usual technique of showing that derivatives of \( u \) are themselves solutions of evolution equations. Formally differentiating (2.1) \( k \) times with respect to \( t \) leads to
\[
f^{(k)} = (Cu^{(k+1)})' + (kC' + B)u^{(k+1)} + \left( A + Q + kB' + \frac{k(k+1)}{2}C'' \right) u^{(k)}
+ \sum_{j=1}^k \left[ \binom{k}{j} \left( A^{(j)} + Q^{(j)} \right) + \binom{k}{j+1} B^{(j+1)}(0) + \binom{k+1}{j+2} C^{(j+2)}(0) \right] u^{(k-j)},
\]
in the sense of \( L^2(I; V^*) \), together with initial values
\[
u^{(k)}(0) = u_k \text{ in } H, \quad (Cu^{(k+1)})(0) = C(0)u_{k+1} \text{ in } V^*, \tag{2.4a}
\]
which are given recursively for \( k \geq 0 \) through
\[
C(0)u_{k+2} = f^{(k)}(0) - ((k+1)C'(0) + B(0))u_{k+1}
- \sum_{j=0}^k \left[ \binom{k}{j} \left( A^{(j)}(0) + Q^{(j)}(0) \right) + \binom{k}{j+1} B^{(j+1)}(0) + \binom{k+1}{j+2} C^{(j+2)}(0) \right] u_{k-j}, \tag{2.5}
\]
starting from already given \( u_0 \) and \( u_1 \). Note that these compatibility conditions resemble those from theorem 30.1 in [18] if we were to choose \( C \) to be the identity in \( L(H) \), \( B = Q = 0 \), and \( A \) as time-independent.

Our intention is to interpret (2.4a) as a second-order evolution equation for \( u^{(k)} \) without moving lower order derivatives of \( u \) to the right-hand side. To this end, for some Banach space \( X \) and \( g \in X \) we introduce the operator \( R_{X,g} : L^2(I; X) \to H^1(I; X) \) via
\[
(R_{X,g}v)(t) := g + \int_0^t v(s) \, ds. \tag{2.6}
\]
In view of an efficient notation, we abbreviate the operator \( R_{X,g} \) by \( R_g \) if not explicitly stated otherwise. We also agree on the definition and notation of the following operator, which denotes the consecutive composition of \( R_{X,\mu_l} \) for \( l = m, \ldots, n \), i.e.
\[
\bigotimes_{l=m}^n R_{X,\mu_l} v := \left\{ \begin{array}{ll}
(R_{X,\mu_m} \circ \cdots \circ R_{X,\mu_1}) v, & m \leq n \\
v, & \text{else}.
\end{array} \right. \tag{2.7}
\]
Writing \( v := u^{(k)} \) now leads to the auxiliary problem
\( f^{(k)} = (Cv')' + (kC' + B)v' + (A + Q)v + \left( kB' + \frac{k(k + 1)}{2} C'' \right) v \\
+ \sum_{j=1}^{k} \left[ \left( \frac{k}{j} \right) (A^{(j)} + Q^{(j)}) + \left( \frac{k}{j+1} \right) B^{(j+1)} + \left( \frac{k+1}{j+2} \right) C^{(j+2)} \right] \bigcap_{l=k-j}^{k-1} R_{k-l} v, \quad (2.8a) \)

equipped with initial values
\[
v(0) = u_k \text{ in } H, \quad (Cv')(0) = C(0)u_{k+1} \text{ in } V'.
\]
(2.8b)

Note that the right-hand side of (2.8a) is nonlinear in \( \psi \) because \( R_{k} \) is affine linear for \( g \neq 0 \). Also, the dependence of (2.8a) and (2.5) on \( C^{(k+2)} \) and \( B^{(k+1)} \) is merely notational, because the coefficients in front of them vanish. It is also worth mentioning that we explicitly wrote the zeroth index of the sum in (2.8a) since we need strong positivity properties from \( A \) and \( C \).

The main assertion of this article is the following regularity result, which we are going to prove by showing that solutions to (2.8) exist and that they have to be equal to \( u' \).

**Theorem 2.3.** Let \( k \in \mathbb{N} \cup \{0\}, \quad l = \max\{1,k\} \). Furthermore, suppose that \( A \in W^{k+1,\infty}(I;L(V;V^*)) \), \( Q \in W^{l,\infty}(I;L(V;H)) \), \( B \in W^{l,\infty}(I;L(H)) \), \( C \in W^{k+1,\infty}(I;L(H)) \), as well as \( f \in H^k(I;H) \) or \( f \in H^{k+1}(I;V^*) \) and \( u_j \in V \) for \( j = 0, \ldots, k \), \( u_{k+1} \in H \). Then the unique solution \( u \) of problem (2.1) lies in \( H^k(I;V) \cap H^{k+1}(I;H) \) with \( Cu' \in L^2(I;V^*) \) and satisfies the energy estimate
\[
\|\psi\|_{W^{k,\infty}(I;V)}^2 + \|\psi^{(k+1)}\|_{L^\infty(I;H)}^2 \leq \Lambda \left( \sum_{j=0}^{k} ||u_j||_V^2 + ||u_{k+1}||_H^2 + ||f||^2 \right),
\]
(2.9)

where \( f \) is measured in either the \( H^k(I;H) \)- or the \( H^{k+1}(I;V^*) \) norm and \( \Lambda = \Lambda(k) \) is a constant depending continuously on \( 1/c_0, 1/a_0, T \) and the operators \( A, B, C, Q \), measured in the spaces above.

We would like to give an example for the application of this regularity result to a partial differential equation. In particular, we are interested in showing what the conditions for the \( u_i \) entail in practice. For simplicity we use the wave equation with a time- and space-dependent coefficient in the divergence term.

**Example 2.4.** Suppose \( \Omega \subset \mathbb{R}^n \) is a bounded domain. The problem
\[
\begin{align*}
\psi(t,x) &= \text{div} (a(t,x)\nabla u(t,x)) = f(t,x) \text{ for all } (t,x) \in I \times \Omega, \\
u &= 0 \text{ on } I \times \partial \Omega, \quad u(0,\cdot) = u_0, \quad u'(0,\cdot) = u_1,
\end{align*}
\]
possesses a unique weak solution \( u \in L^2(I;H_0^1(\Omega)) \cap H^1(I;L^2(\Omega)) \cap H^2(I;L^2(\Omega)) \cap H^1(I;H^{-1}(\Omega)) \) (according to theorem 2.1) if \( u_0 \in H_0^1(\Omega), \quad u_1 \in L^2(\Omega), \quad f \in L^2(I;L^2(\Omega)) \) and \( a \in W^{1,\infty}(I;L^\infty(\Omega)) \), provided that \( a(t,x) \geq a_0 > 0 \) almost everywhere in \( I \times \Omega \). Applying theorem 2.3 to this problem yields \( u \in H^k(I;H_0^1(\Omega)) \cap H^{k+1}(I;L^2(\Omega)) \cap H^{k+2}(I;H^{-1}(\Omega)) \) if \( f \in H^k(I;L^2(\Omega)) \) and \( a \in W^{1,\infty}(I;L^\infty(\Omega)) \). Additionally the initial values defined in (2.5) have to satisfy \( u_i \in H_0^1(\Omega) \) \( (i = 0, \ldots, k) \) and \( u_{k+1} \in L^2(\Omega) \). These can be seen as compatibility conditions and can be fulfilled by either one of the following assumptions:

(i) Homogeneous initial data \( u_0 = u_1 = 0 \) and \( f(0) = 0 \) for \( i = 0, \ldots, k - 1 \). In this case the regularity result is the same as obtained by Lions [11].
(iii) \( f^{(k-j)}(0) \in H_{0}^{j-1}(\Omega), \) \( a^{(k-j)}(0) \in W^{j,\infty}(\Omega) \) for \( j = 1, \ldots, k \) and smooth initial values \( u_0 \in H_{0}^{k+1}(\Omega) \) and \( u_1 \in H_{0}^{k}(\Omega) \).

**Proof (Sketch).** Use integration by parts to see that e.g.

\[
A(0)g = v \mapsto \int_{\Omega} a(0,x) \nabla v(x) \nabla g(x) \, dx = -\text{div}(a(0, \cdot) \nabla g)
\]

(where the last equality is to be understood in the distributional sense) is not only an element of \( H^{-1}(\Omega) \), but also lies in \( H_{0}^{k}(\Omega) \) if \( a(0) \in W^{k+1,\infty}(\Omega) \) and \( g \in H_{0}^{k+2}(\Omega) \). From this we conclude that \( u_j \) lies in \( H_{0}^{k}(\Omega) \) if \( f^{(k-2)}(0) \in H_{0}^{k}(\Omega) \), \( a^{(j)}(0) \in W^{l+1,\infty}(\Omega) \) and \( u_j \in H_{0}^{l+2}(\Omega) \) for all \( j = 0, \ldots, k - 2 \). This is satisfied for \( k \geq 2 \) if \( f^{(k-j)}(0) \in H_{0}^{l+j-2}(\Omega) \), \( a^{(k-j)}(0) \in W^{l+j-1,\infty}(\Omega) \) \( u_j \in H_{0}^{l+2}(\Omega) \), \( u_0 \in H_{0}^{l+k}(\Omega) \) and \( u_1 \in H_{0}^{l+k-1}(\Omega) \) by induction over \( k \geq 2 \), because the assumptions for \( k \) imply the assumptions for all \( j = 0, \ldots, k - 2 \) with smoothness \( l + 2 \). Therefore \( u_j \in H_{0}^{l+2}(\Omega) \) for \( j = 0, \ldots, k - 2 \) holds. The assertion follows by applying this to \( u_k \in H_{0}^{k}(\Omega) \), \( u_{k+1} \in L^{2}(\Omega) \).

3. Well-posedness of the auxiliary problems

We need solvability of the auxiliary problems (2.8) for \( k \in \mathbb{N} \). They can be written in the form

\[
(Cv')' + Bv' + (A + Q)v + \sum_{i=1}^{k} (D_i + E_i)(R_{i}v) = f \text{ in } L^{2}(I; V^*),
\]

(3.1a)

\[
v(0) = v_0 \text{ in } H, \quad (Cv')(0) = v_1 \text{ in } V^*,
\]

(3.1b)

with different \( B, Q, f \) and suitably defined \( D_i, E_i \), where \( D_i(t) \in L(\mathcal{V}; V^*) \) while \( E_i(t) \in L(V; H) \). We analyze this equation and then transfer the results to (2.8). To this end, we start by discretizing (3.1) in space by taking an orthogonal basis \( \{\phi_i\}_{i \in \mathbb{N}} \subset V \) of \( V \) which is also an orthonormal basis in \( H \) and introduce the finite-dimensional subspaces \( V_{m} := \text{span} \{\phi_i; i = 1, \ldots, m\} \). We set \( v_{m} := \sum_{i=1}^{m} a_{i}(t) \phi_{i} \) where \( a_{i} \in W^{2,1}(I) \), so that \( v_{m} \in W^{2,1}(I; V) \). We want \( v_{m} \) to solve (3.1a) in \( L^{2}(I; V_{m}^*) \). Testing this equation with \( \phi_i \) shows that this is equivalent to

\[
\langle f(t), \phi_i \rangle = \sum_{j=1}^{m} \left( (a_j'(t) (C(t)\phi_j, \phi_i))' + \alpha_j(t) (B(t)\phi_j, \phi_i) + \alpha_j(t) ((A(t)\phi_j, \phi_i) + (Q(t)\phi_j, \phi_i)) + \sum_{l=1}^{k} (R_{lj}(t) \phi_j, \phi_i) \right)
\]

(3.2a)

being satisfied for all \( i = 1, \ldots, m \). This \( m \times m \)-system is complemented by the projected initial values from (3.1b), that is

\[
(v_{m}(0), \phi_i) = (v_0, \phi_i), \quad (C(0)v_{m}'(0), \phi_i) = (v_1, \phi_i), \quad \text{for } i = 1, \ldots, m.
\]

(3.2b)

These ordinary differential equations are commonly referred to as *Galerkin equations* and their solvability is the subject of the following lemma.

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Lemma 3.1 (Galerkin solutions). Provided that \( A \in W^{1,\infty}(I; \mathcal{L}(V; V^*)) \), \( C \in W^{1,\infty}(I; \mathcal{L}(H)) \), \( B \in L^\infty(I; \mathcal{L}(H)) \), \( Q \in L^\infty(I; \mathcal{L}(V; H)) \), \( D_i \in W^{1,\infty}(I; \mathcal{L}(V; V^*)) \), \( E_i \in L^\infty(I; \mathcal{L}(V; H)) \) for \( i = 1, \ldots, k \), as well as \( f \in L^2(I; H) \) or \( f \in H^1(I; V^*) \), \( v_0 \in V \) and \( v_1 \in H \) the Galerkin equations \((3.2a)\) and \((3.2b)\) admit one unique solution \( v_m \).

Proof. The idea is to transform equation \((3.2a)\) into a system of linear differential equations in the sense of Caratheodory (see [3], chapter 2, theorem 1.1). To this end, we apply the generalized product rule from lemma 2.2 to \((3.2a)\) to get

\[
\langle f(t), \varphi_i \rangle = \sum_{j=1}^m \left( \alpha_j(t) (C(t) \varphi_j, \varphi_i) + \beta_j(t) \left( (B(t) + C'(t)) \varphi_j, \varphi_i \right) + \gamma_j(t) \left( (A(t) \varphi_j, \varphi_i) \right) \right) + \sum_{l=1}^k (R_{l \in \mathbb{R}; 0} \beta_j)  ( (D_l(t) \varphi_j, \varphi_i) + (E_l(t) \varphi_j, \varphi_i) )
\]

and then introduce the following matrix-valued functions \( M_l \in L^\infty(I; \mathbb{R}^{m \times m}) \) for \( l = -2, \ldots, k \) component-wise via

\[
(M_{-2}(t))_{ij} := (C(t) \varphi_j, \varphi_i), \quad (M_{-1}(t))_{ij} := \left( (C'(t) + B(t)) \varphi_j, \varphi_i \right), \quad (M_0(t))_{ij} := (A(t) \varphi_j, \varphi_i), \quad (M_l(t))_{ij} := (D_l(t) \varphi_j, \varphi_i) + (E_l(t) \varphi_j, \varphi_i)
\]

for \( i, j = 1, \ldots, m \). We also introduce the vector-valued function \( F \in W^{1,\infty}(I; \mathbb{R}^m) \) via \( F_i(t) := (f(t), \varphi_i) \) for \( i = 1 \ldots, m \) and agree on a new set of variables \( \gamma^l := R_l^{i,0} \alpha \) for \( l = 0, \ldots, k \), \( \gamma := (\gamma^l)_{l=0, \ldots, k} \) and \( \beta = \alpha' \). This allows rewriting \((3.3)\) as a \((k+2)m \times (k+2)m\) system of first-order differential equations, i.e.

\[
M_{-2} \beta'(t) = F(t) - M_{-1}(t) \beta(t) - \sum_{l=0}^k M_l(t) \gamma^l(t)
\]

and

\[
\gamma^0(t) = \beta(t)
\]

\[
\gamma^l(t) = \gamma^l(t) \quad \text{for } l = 1, \ldots, k
\]

which is complemented by initial values

\[
M_{-2}(0) \beta(0) = \left( (v_{1i}, \varphi_i) \right)_{i=1, \ldots, m}, \quad \gamma^0(0) = \left( ((v_{0i}, \varphi_i))_{i=1, \ldots, m}, \quad \gamma^l(0) = 0 \quad \text{for } l = 1, \ldots, k.
\]

Here, \( I_m \), \( I_{km} \) and \( I_{(k+1)m} \) represent identity matrices in \( \mathbb{R}^{m \times m} \), \( \mathbb{R}^{km \times km} \) and \( \mathbb{R}^{(k+1)m \times (k+1)m} \), respectively. Since the \( \varphi_i \) form an orthogonal basis for \( V_m \) and \( C(t) \) is invertible due to its coercivity, \( M_{-2} \) and therefore the system \((3.4)\) and the initial conditions can be made explicit. It is then straightforward to show the existence of a unique solution \((\beta, \gamma) \in W^{1,\infty}(I; \mathbb{R}^{(k+2)m})\) following [15] or [3] and hence finding a uniquely determined \( v_m := \sum_{j=1}^m \alpha_j(t) \varphi_j \in W^{2,\infty}(I; V_m) \) that solves \((3.2)\).

\( \square \)

Our goal is now to establish an upper bound for \( v_m \) independent of \( m \in \mathbb{N} \).

Lemma 3.2 (A priori estimate). Provided that \( A \in W^{1,\infty}(I; \mathcal{L}(V; V^*)) \), \( C \in W^{1,\infty}(I; \mathcal{L}(H)) \), \( B \in L^\infty(I; \mathcal{L}(H)) \), \( Q \in L^\infty(I; \mathcal{L}(V; H)) \), \( D_i \in W^{1,\infty}(I; \mathcal{L}(V; V^*)) \), \( E_i \in L^\infty(I; \mathcal{L}(V; H)) \) for \( i = 1, \ldots, k \), as well as \( f \in L^2(I; H) \) or \( f \in H^1(I; V^*) \), \( v_0 \in V \) and \( v_1 \in H \) the Galerkin solution \( v_m \) satisfies the energy estimate...
\[
\text{ess sup}_{t \in I} \left( \|v_m(t)\|^2_V + \|v'_m(t)\|^2_H \right) \leq \Lambda \left( \|v_0\|^2_V + \|v_1\|^2_H + \|f\|^2 \right),
\]

(3.5)

where \(f\) is measured in either the \(L^2(I;H)\)- or the \(H^1(I;V^*)\) norm and \(\Lambda > 0\) is some constant depending continuously on \(\Lambda/\alpha_0, \Lambda/\alpha_0, T\), and the operators \(A, B, C, Q, (D_i)_{i=1,...,k}(E_i)_{i=1,...,k}\) measured in the spaces above.

**Proof.** We take \(v_m \in W^{2,\infty}(I; V_m)\) to be the unique solution to the Galerkin equations from the previous lemma, test (3.1a) with \(v'_m(t) \in V_m\) and see that

\[
\begin{align*}
&\langle (Cv'_m)(t), v'_m(t) \rangle + \langle B(t)v''_m(t), v'_m(t) \rangle + \langle A(t)v_m(t), v'_m(t) \rangle + \langle Q(t)v_m(t), v'_m(t) \rangle + \sum_{i=1}^k \langle D_i(t)(R_0v_m)(t), v'_m(t) \rangle + \sum_{i=1}^k \langle E_i(t)(R_0v_m)(t), v'_m(t) \rangle = \langle f(t), v'_m(t) \rangle \\
&\text{holds for almost all } t \in I. \text{ We make use of the product rule from lemma } 2.2 \text{ and the fact that } A(t) \text{ and } C(t) \text{ are self-adjoint to get} \\
2\langle (Cv'_m)(t), v'_m(t) \rangle = 2\langle C(t)v''_m(t) + C'(t)v'_m(t), v'_m(t) \rangle \\
= \langle C'(t)v'_m(t), v'_m(t) \rangle + \langle (C(t)v'_m(t), v'_m(t))' \rangle, \\
2\langle A(t)v_m(t), v'_m(t) \rangle = \langle (A(t)v_m(t), v_m(t))' \rangle - \langle A'(t)v_m(t), v_m(t) \rangle.
\end{align*}
\]

Together with \(\|v_m(0)\|_V \leq \|v_0\|_V, \|C(0)v'_m(0)\|_H \leq \|v_1\|_H\) this gives

\[
c_0\|v'_m(t)\|_H^2 + a_0\|v_m(t)\|_V^2 \\
\leq \frac{1}{c_0}\|v_1\|_H^2 + \|A\|_{W^{1,\infty}(L^2(V;V^*))} \|v_0\|_V^2 + \int_0^t \left( \langle C(s)v''_m(s), v'_m(s) \rangle + \langle A(s)v_m(s), v'_m(s) \rangle \right) ds \\
= \frac{1}{c_0}\|v_1\|_H^2 + \|A\|_{W^{1,\infty}(L^2(V;V^*))} \|v_0\|_V^2 + 2 \int_0^t \left( \langle Cv'_m(s), v'_m(s) \rangle + \langle A(s)v_m(s), v'_m(s) \rangle \right) ds \\
+ \int_0^t \langle C'(s)v'_m(s), v'_m(s) \rangle + \langle A'(s)v_m(s), v_m(s) \rangle ds \\
\leq \frac{1}{c_0}\|v_1\|_H^2 + \|A\|_{W^{1,\infty}(L^2(V;V^*))} \|v_0\|_V^2 + 2 \int_0^t \langle f(s), v'_m(s) \rangle - \langle Q(s)v_m(s), v'_m(s) \rangle ds \\
- 2 \int_0^t \langle B(s)v''_m(s), v'_m(s) \rangle + \sum_{i=1}^k \langle D_i(s)(R_0v_m)(s), v'_m(s) \rangle + \sum_{i=1}^k \langle E_i(s)(R_0v_m)(s), v'_m(s) \rangle ds \\
+ \int_0^t \langle C'(s)v'_m(s), v'_m(s) \rangle + \langle A'(s)v_m(s), v_m(s) \rangle ds.
\]

(3.6)

Since we are looking to apply Grönwall’s inequality, we have to modify the right-hand side so that it only depends on \(\|v_m(s)\|_V^2\) and \(\|v'_m(s)\|_H^2\). We perform this individually for each summand.
If \( f \in L^2(I; H) \) then
\[
2 \int_0^t \langle f(s), v_m'(s) \rangle \, ds = 2\langle f(t), v_m(t) \rangle - 2\langle f(0), v_m(0) \rangle - 2 \int_0^t \langle f'(s), v_m(s) \rangle \, ds \leq \epsilon \|v_m(t)\|^2_V + \frac{1 + \xi}{t} \|f\|^2_H + \|v_0\|^2_V + \int_0^t \|f'(s)\|^2_V + \|v_m(s)\|^2_V \, ds,
\]
which holds for all \( \epsilon > 0 \).

For the last summand on the right-hand side we immediately see that
\[
\int_0^t \left( C'(s) v_m'(s), v_m'(s) \right) + \left( A'(s) v_m(s), v_m(s) \right) \, ds \leq \|C'\| \int_0^t \|v_m'(s)\|^2_H \, ds + \|A'\| \int_0^t \|v_m(s)\|^2_V \, ds,
\]
the expression with \( B \) can be handled analogously.

For terms including \( D_i \), we have to use integration by parts because in contrast to parabolic equations we cannot expect \( \|v_m'(s)\|_V \) to be bounded during the limit process \( m \to \infty \), which is already apparent on the left-hand side of (3.6). In employing lemma 2.2 we see that
\[
-\int_0^t \langle D_i(s)(R_0 v_m)(s), v_m'(s) \rangle \, ds = -\langle D_i(t)(R_0 v_m)(t), v_m(t) \rangle + \int_0^t \langle D_i'(s)(R_0 v_m)(s), v_m(s) \rangle - \langle D_i(s)(R_0^{-1} v_m)(s), v_m(s) \rangle \, ds \leq \|D_i\|_{L^\infty(I; L^1(V; V^*)^*)} \|R_0 v_m(t)\|_V \|v_m(t)\|_V
\]
\[
+ \int_0^t \|D_i'\| \|R_0 v_m(s)\|_V \|v_m(s)\|_V + \|D_i\| \|R_0^{-1} v_m(s)\|_V \|v_m(s)\|_V \, ds \leq T^{-1} \|D_i\|_{L^\infty(I; L^1(V; V^*)^*)} \|v_m(t)\|_V \int_0^t \|v_m(s)\|_V \, ds
\]
\[
+ (T \|D_i'\| + T^{i-1} \|D_i\|) \int_0^t \|v_m(s)\|^2_V \, ds \leq \epsilon \|v_m(t)\|^2_V + \left( \frac{T^{2i-1} \|D_i\|^2}{4\epsilon} + T \|D_i'\| + T^{i-1} \|D_i\| \right) \int_0^t \|v_m(s)\|^2_V \, ds,
\]
holds again for some arbitrary \( \epsilon > 0 \). Here, we made use of
\[
\int_0^t \| (R_0 v_m)(s) \|_V \, ds \leq T \int_0^t \|v_m(s)\|^2_V \, ds.
\]
The fact that $E_i(t), Q(t) \in \mathcal{L}(V; H)$ allows for

$$-2 \int_0^t \langle Q(s) v_m(s), v_m'(s) \rangle \, ds \leq \|Q\|_{L^\infty(I; \mathcal{L}(V; H))} \int_0^t \|v_m(s)\|_V^2 + \|v_m'(s)\|_H^2 \, ds$$

for $Q$ and

$$-2 \int_0^t \langle E_i(s)(R_0 v_m)(s), v_m'(s) \rangle \, ds \leq \|E_i\|_{L^\infty(I; \mathcal{L}(V; H))} \int_0^t \|v_m(s)\|_V^2 + \|v_m'(s)\|_H^2 \, ds$$

for $E_i$. If we insert all these intermediate computations into (3.6) and choose e.g. $\epsilon = a_0/2$ we arrive at an inequality of the form

$$\|v_m(t)\|_V^2 + \|v_m'(t)\|_H^2 \leq \bar{\Lambda} \left( \|v_0\|_V^2 + \|v_1\|_H^2 + \int_0^t \|v_m(s)\|_V^2 + \|v_m'(s)\|_H^2 \, ds \right),$$

where $f$ is measured in either the $L^2(I; H)$- or the $H^1(I; V^*)$ norm and $\bar{\Lambda}$ is a constant as stated in the assertion. By applying Grönwall’s lemma we conclude

$$\|v_m(t)\|_V^2 + \|v_m'(t)\|_H^2 \leq \bar{\Lambda} \left( \|v_0\|_V^2 + \|v_1\|_H^2 + \|f\|_V^2 \right) \exp(\bar{\Lambda} T)$$

for almost all $t \in I$.

As a consequence of the previous result the norms of $v_m \in V_m$ in $L^2(I; V)$ and $H^1(I; H)$ are bounded uniformly with respect to $m \in \mathbb{N}$. This guarantees existence of a subsequence $v_m$ (also denoted with $m$) converging weakly in $L^2(I; V) \cap H^1(I; H)$, whose weak limit we denote by $v$. The next step is to show that this weak limit solves the auxiliary problem (3.1).

**Lemma 3.3.** Under the assumptions of lemma 3.2, there exists $v \in L^2(I; V) \cap H^1(I; H)$ with $(\mathcal{C} v)' \in L^2(I; V^*)$ that solves (3.1) and also satisfies the energy estimate (3.5).

**Proof.** Energy estimates in $L^2(I; V) \cap H^1(I; H)$ are transferred from $v_m$ to $v$ due to their weak convergence in this space and $\|v\| \leq \liminf \|v_m\|$. We are also able to achieve $L^\infty$-estimates in time, because $(v_m)_{m \in \mathbb{N}}$ is bounded in $L^\infty(I; V) \cong L^1(I; V^*)^*$, and $(v_m')_{m \in \mathbb{N}}$ is bounded in $L^\infty(I; H) \cong L^1(I; H)^*$. Thus, the Banach–Alaoglu theorem states that there are subsequences

$$v_{m_j} \rightharpoonup v \quad \text{in} \quad L^1(I; V^*),$$

$$v_{m_j}' \rightharpoonup v' \quad \text{in} \quad L^1(I; H)$$

when $j \to \infty$. The limits have to be the same as before, thus $v$ fulfills the $L^\infty(I; V) \cap W^{1,\infty}(I; H)$ energy estimate (3.5).

The fact that $v$ satisfies (3.1) can be seen by testing the Galerkin equations with functions in $C^0_c(I; V)$ that are finite linear combinations of the basis functions $(\varphi_i)_{i \in \mathbb{N}} \subset V$. The same holds for the initial values. This approach is the same as for time-independent operators, thus we refer to [12] for the details.
We are now in a position to apply this existence theorem to our auxiliary problem (2.8).

**Corollary 3.4.** Let \( k \in \mathbb{N} \). Suppose \( A \in W^{k+1,\infty}(I; \mathcal{L}(V;V^*)) \), \( Q \in W^{k,\infty}(I; \mathcal{L}(V;H)) \), \( B \in W^{k,\infty}(I; \mathcal{L}(H)) \), \( C \in W^{k+1,\infty}(I; \mathcal{L}(H)) \), \( f \in H^{k+1}(I; V^*) \) or \( f \in H^{k+1}(I; V^*) \), \( u_j \in V \) for \( j = 0, \ldots, k \) and \( u_{k+1} \in H \). Then there exists \( v \in L^2(I; V) \cap H^1(I; H) \) with \( (Cv')' \in L^2(I; V^*) \) that solves (2.8) and also satisfies the energy estimate

\[
\text{ess sup}_{t \in I} \left( \|v(t)\|_V^2 + \|v'(t)\|_{H^1}^2 \right) \leq \Lambda \left( \sum_{j=0}^{k} \|u_j\|_V^2 + \|u_{k+1}\|_H^2 + \|f^{(k)}\|^2 \right)
\]

(3.7)

where \( f^{(k)} \) is measured in either the \( L^2(I;H) \)- or the \( H^1(I; V^*) \) norm and \( \Lambda = \Lambda(k) \) is a constant depending continuously on \( \lambda \), \( \mu \), \( \nu \), \( T \) and the operators \( A, B, C, Q \) measured in the spaces above.

**Proof.** To obtain (3.1) from (2.8), we set

\[
\tilde{B} = B + C', \quad \tilde{Q} = Q + kB' + \frac{k(k+1)}{2} C',
\]

\[
D_j = \binom{k}{j} A^{(j)}, \quad E_j = \binom{k+1}{j+2} C^{(j+2)} + \binom{k}{j+1} B^{(j+1)} + \binom{k}{j} Q^{(j)},
\]

for \( j = 1, \ldots, k \). The operators \( A \) and \( C \) are left as-is, and the initial values are \( u_k \) and \( u_{k+1} \) as defined in (2.5). The right-hand side \( \tilde{f} \) consists of \( f^{(k)} \) and those parts of

\[
\left( \bigotimes_{l=k-j}^{k-1} \mathcal{R}_{u_l} v \right)(t) = \left( \mathcal{R}_{u} v \right)(t) + \sum_{l=k-j}^{k-1} \left( \mathcal{R}_{u} v \right)(t) = \left( \mathcal{R}_{u} v \right)(t) + \sum_{l=k-j}^{k-1} u_l \frac{t^{k-1-l}}{(k-1-l)!}
\]

that are independent of \( v \), i.e.

\[
\tilde{f}(t) = f^{(k)}(t) - \sum_{j=1}^{k-1} (D_j(t) + E_j(t)) \sum_{l=k-j}^{k-1} u_l \frac{t^{k-1-l}}{(k-1-l)!}.
\]

The assertion follows by utilizing lemma 3.3. \( \Box \)

4. Proof of the main theorem

We have seen that solutions to the auxiliary problems (2.8) exist, but have yet to show that they are linked to (2.1). As they were obtained using formal differentiation, it is somewhat intuitive to see if functions obtained by integrating these solutions with respect to time over \( (0,t) \) indeed solve the previous problem. The following lemma resolves the situation.

**Lemma 4.1.** Suppose \( v \) is a solution to problem (2.8) with \( k \in \mathbb{N} \) and \( u_{k-1} \in V \). Then \( R_{u_{k-1}} v \) solves problem (2.8) with \( k \) replaced by \( k - 1 \).

**Proof.** We set \( w = R_{u_{k-1}} v \). Clearly \( w \) possesses the correct initial values since \( w(0) = u_{k-1} \) and \( w'(0) = v(0) = u_k \). By a straightforward (albeit lengthy) computation we show that
 holds true in the $L^2(I; V^*)$ sense. We start by employing the fundamental theorem of calculus to the sum in the preceding equation and compute

$$
\sum_{j=0}^{k-1} \left[ \binom{k-1}{j} (A^{(j)} + Q^{(j)}) + \binom{k-1}{j+1} B^{(j+1)} + \binom{k}{j+2} C^{(j+2)} \right] \left( \sum_{l=k-1-j}^{k-2} R_{n_l}v \right) (t)
$$

$$
= \sum_{j=0}^{k-1} \left[ \binom{k-1}{j} (A^{(j)}(0) + Q^{(j)}(0)) + \binom{k-1}{j+1} B^{(j+1)}(0) + \binom{k}{j+2} C^{(j+2)}(0) \right] u_{k-1-j} + \int_0^t \left( \sum_{j=0}^{k-1} \left[ \binom{k-1}{j} (A^{(j)} + Q^{(j)}) + \binom{k-1}{j+1} B^{(j+1)} \right] \right) + \binom{k}{j+2} C^{(j+2)} \left( \sum_{l=k-1-j}^{k-1} R_{n_l}v \right) (s) \, ds.
$$

Turning to (2.5) shows that the expression before the integral can be rewritten using $C(0)u_{k+1}$. Using lemma 2.2 we are able to recast the integral expression, too, and end up with

$$
\sum_{j=0}^{k-1} \left[ \binom{k-1}{j} (A^{(j)} + Q^{(j)}) + \binom{k-1}{j+1} B^{(j+1)} + \binom{k}{j+2} C^{(j+2)} \right] \left( \sum_{l=k-1-j}^{k-1} R_{n_l}v \right) (t)
$$

$$
= -C(0)u_{k+1} + f^{(k-1)}(0) - (kC'(0) + B(0))u_k + \int_0^t \sum_{j=0}^{k-1} \left[ \binom{k-1}{j} (A^{(j+1)}) + \binom{k-1}{j+1} B^{(j+2)} + \binom{k}{j+2} C^{(j+3)} \right] \left( \sum_{l=k-1-j}^{k-1} R_{n_l}v \right) (s) \, ds
$$

$$
+ \int_0^t \sum_{j=0}^{k-1} \left[ \binom{k-1}{j} (A^{(j)} + Q^{(j)}) + \binom{k-1}{j+1} B^{(j+1)} \right] + \binom{k}{j+2} C^{(j+2)} \left( \sum_{l=k-1-j}^{k-1} R_{n_l}v \right) (s) \, ds.
$$

Shifting the index of the sum in the first integral on the left-hand side to $j + 1$, using an identity for binomial coefficients, and adding a zero then shows

$$
\sum_{j=0}^{k-1} \left[ \binom{k-1}{j} (A^{(j)} + Q^{(j)}) + \binom{k-1}{j+1} B^{(j+1)} + \binom{k}{j+2} C^{(j+2)} \right] \left( \sum_{l=k-1-j}^{k-1} R_{n_l}v \right) (t)
$$

$$
= -C(0)u_{k+1} + f^{(k-1)}(0) - (kC'(0) + B(0))u_k - \int_0^t (kC'' + B')(v)(s) \, ds
$$

$$
+ \int_0^t \sum_{j=0}^{k} \left[ \binom{k}{j} (A^{(j)} + Q^{(j)}) + \binom{k}{j+1} B^{(j+1)} + \binom{k+1}{j+2} C^{(j+2)} \right] \left( \sum_{l=k-j}^{k-1} R_{n_l}v \right) (s) \, ds.
$$
if we combine the previous results. Adding this to the remaining parts of (4.1) lets us conclude
\[
\begin{align*}
&f^{(k-1)} - ((k-1)C' + B)u' - \sum_{j=0}^{k-1} \left[ \binom{k-1}{j} A^j \right] (\mathcal{L}(j) \\
&\quad + Q^{(j)}) + \left( \binom{k}{j+1} B^{(j+1)} + \binom{k}{j+2} C^{(j+2)} \right) \left[ \int_{t-k-1-j}^{t} R_{\omega} \right](t) \\
&= C(0)u_{k+1} + C'(0)u_k + \int_0^t f(s) + (C'v)(s) - ((kC' + B)v)(s) + ((kC'' + B')v)(s) \, ds \\
&\quad - \int_0^t \sum_{j=0}^{k-1} \left[ \binom{k}{j} (A^j(s) + Q^{(j)}(s)) + \left( \binom{k+1}{j+2} C^{(j+2)}(s) \right) \left[ \int_{t-k-j}^{t} R_{\omega} \right](s) \, ds \\
&= C(0)u_{k+1} + C'(0)u_k + \int_0^t (Cv)'(s) + (C'v)(s) \, ds \\
&\quad = C(0)u_{k+1} + C'(0)u_k + \int_0^t (Cv)'(s) \, ds = (Cv)'(t) = (Cw)'(t).
\end{align*}
\]

To prove our main theorem we can use the results from before to make a compact induction argument. For the reader’s convenience we repeat the assertions of theorem 2.3 before proving them.

**Theorem 2.3.** Let \( k \in \mathbb{N} \cup \{0\} \), \( l = \max\{1, k\} \) and suppose that \( A \in W^{k+1,\infty}(I; \mathcal{L}(V; V^*)) \), \( Q \in W^{l,\infty}(I; \mathcal{L}(V; V^*)) \), \( B \in W^{l,\infty}(I; \mathcal{L}(H)) \), \( C \in W^{k+1,\infty}(I; \mathcal{L}(H)) \), as well as \( f \in H^k(I; H) \) or \( f \in H^{k+1}(I; V^*) \), and \( u_j \in V \) for \( j = 0, \ldots, k \) and \( u_{k+1} \in H \). Then the unique solution \( u \) of problem (2.1) lies in \( H^k(I; V) \cap H^{k+1}(I; H) \) with \( (C \alpha^{(k+1)})' \in L^2(I; V^*) \) and satisfies the energy estimate
\[
\|u\|^2_{W^{k,\infty}(I; V)} + \|u^{(k+1)}\|^2_{L^2(I; H)} \leq \Lambda \left( \sum_{j=0}^{k} \|u_j\|^2_V + \|u_{k+1}\|^2_H + \|f\|^2 \right), \tag{2.9}
\]

where \( f \) is measured in either the \( H^k(I; H) \)- or the \( H^{k+1}(I; V^*) \) norm and \( \Lambda = \Lambda(k) \) is a constant depending continuously on \( 1/c_0, 1/\alpha_0, T \) and the operators \( A, B, C, Q \), measured in the spaces above.

**Proof.** Induction over \( k \in \mathbb{N} \cup \{0\} \), where the hypotheses is that the assertion as given in the theorem holds for \( k \) and that \( u^{(k)} \) is the unique solution of (2.8).

The case \( k = 0 \) is covered by theorem 2.1. We assume that the hypotheses holds for \( k - 1 \) and the requirements for \( k \) are met. Using corollary 3.4 we conclude that a solution \( v \in L^2(I; V) \cap H^1(I; H) \) with \( (Cv)' \in L^2(I; V^*) \) to the auxiliary problem (2.8), which also fulfills the estimates from equation (3.7), exists. Due to lemma 4.1 we know that \( R_{u_{k-1}} v \) satisfies (2.8) for \( k - 1 \), which is uniquely solved by \( u^{(k-1)} \). Therefore \( v = (R_{u_{k-1}} v)' = u^{(k)} \) (which is unique) and \( u^{(k)} \) satisfies the estimate (3.7), which we add to the energy estimates for the case \( k - 1 \) to obtain (2.9).

**Remark 4.2.** Note that in the case of \( l = k = 0 \) the existence of a solution to (2.1) still holds, but not its uniqueness.
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