SUBNORMAL CLOSURE OF A HOMOMORPHISM

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ABSTRACT. Let \( \varphi: \Gamma \to G \) be a homomorphism of groups. In this paper we introduce the notion of a subnormal map (the inclusion of a subnormal subgroup into a group being a basic prototype). We then consider factorizations \( \Gamma \xrightarrow{\psi} M \xrightarrow{\pi} G \) of \( \varphi \), with \( n \) a subnormal map. We search for a universal such factorization. When \( \Gamma \) and \( G \) are finite we show that such universal factorization exists: \( \Gamma \to \Gamma_\infty \to G \), where \( \Gamma_\infty \) is a hypercentral extension of the subnormal closure \( C \) of \( \varphi(\Gamma) \) in \( G \) (i.e. the kernel of the extension \( \Gamma_\infty \to C \) is contained in the hypercenter of \( \Gamma_\infty \)). This is closely related to the a relative version of the Bousfield-Kan \( \mathbb{Z} \)-completion tower of a space. The group \( \Gamma_\infty \) is the inverse limit of the normal closures tower of \( \varphi \) introduced by us in a recent paper. We prove several stability and finiteness properties of the tower and its inverse limit \( \Gamma_\infty \).

1. INTRODUCTION

Throughout this note \( \varphi: \Gamma \to G \) is a homomorphism of groups. In a previous paper \cite{FS1} we considered the notion of the free normal closure \( \Gamma^\varphi \) of \( \varphi \), (related to \cite{BH, BHS}) and the free normal closures tower \( \{ \Gamma_{i,\varphi} \}_{i=1}^\infty \) of \( \varphi \) (see equation (1.1) below). In this paper we study the behavior and the properties of the inverse limit \( \Gamma_\infty,\varphi \) of the tower of free normal closures of \( \varphi \). This tower generalizes and connects the quotients of the lower central series \( \Gamma/\gamma_i(\Gamma) \), \( i = 1, 2, 3, \ldots \), gotten here for \( G = 1 \), and the descending series of successive normal closures of a subgroup \( H \) of \( G \) (see \S 2). Notice that in the case \( G = 1 \), the group \( \Gamma_\infty,\varphi \) is the nilpotent completion of \( \Gamma \). Thus for an arbitrary map \( \varphi \), the group \( \Gamma_{\infty,\varphi} \) can be thought of as a relative nilpotent completion associated with a homomorphism rather than with a group. Some of the results here support this point of view.

Let us recall from \cite{FS2} section 4] that the normal closures tower (we often omit the word “free”) associated with a homomorphism \( \varphi \) is a tower of groups as follows:
\[ \Gamma_k \xrightarrow{\varphi_k} \Gamma_{k+1} \]

where \( \varphi_i \) is a normal map (our terminology for a crossed module), and where \( \Gamma_{i+1} = \Gamma_{i+1, \varphi} \) is the (free) normal closure \( \Gamma_{i, \varphi} \) of the map \( \varphi_i : \Gamma_i \to \Gamma_i \), for all \( i \geq 1 \). The group \( \Gamma_\infty = \Gamma_\infty, \varphi \), the stability of the tower \( \{ \Gamma_i, \varphi \} \) and its relation to the groups \( \Gamma, G \) and the map \( \varphi \), are thus the main topics of the present study.

One way to think about the above tower is that it represents an attempt to factor the map \( \varphi \), in a universal way, into a composition of “simpler maps” (i.e. normal maps). This, of course, cannot be done in general, and \( \Gamma_\infty, \varphi \) is a kind of “hybrid” of \( \Gamma \) and \( G \) giving a factorization \( \Gamma \to \Gamma_\infty, \varphi \to G \). Passing to topological spaces via the classifying space construction we get a map \( B\Gamma \to BG \). The present result shows that for finite groups \( \Gamma, G \) this map has a finite “relative” Bousfield-Kan tower of principal fibrations [BK] whose fibres are, in general, neither connected nor nilpotent groups:

\[ B\Gamma_\infty = B\Gamma_n \to B\Gamma_{n-1} \to \cdots \to B\Gamma_2 \to BG; \]

with a terminal term \( B\Gamma_\infty = B\Gamma_n \to BG \) being the universal space, under \( B\Gamma \) for which there is such a tower of principal fibrations. This raises the question of finding such universal decompositions of more general maps of spaces \( X \to Y \). Notice that we have an induced map \( \varphi_\infty : \Gamma \to \Gamma_\infty \); we often ask how far is this map from an isomorphism.

One quick corollary of our main results concerns a map of nilpotent groups:

**Theorem 1.** Let \( \varphi : \Gamma \to G \) be a homomorphism of nilpotent groups, then \( \varphi_\infty : \Gamma \to \Gamma_\infty \) is an isomorphism.

**Theorem 1** is Corollary 2.13(2). This corollary can be viewed as a version for any map of nilpotent groups, of the well known property that any subgroup of a nilpotent group is subnormal, i.e. equal to its subnormal closure.

In this spirit we have for any homomorphism of finite groups:

**Theorem 2.** Let \( \varphi : \Gamma \to G \) be a homomorphism of finite groups. Then \( \varphi_\infty \) induces an isomorphism of the descending central series quotients, \( \Gamma/\gamma_i(\Gamma) \cong \Gamma_\infty/\gamma_i(\Gamma_\infty) \), for all \( i \geq 1 \).

**Theorem 2** is proved in Proposition 2.14. We note that by previous results, this is certainly not true for a general map of groups, e.g., when \( \Gamma \) is a free group with infinitely many generators and \( G = 1 \).

Our next result extends a partial result in [FS1] to the general finite case. An estimate of the size of \( \Gamma_\infty \) is given below in **Theorem 2.11**.
Theorem 3. If \( \varphi : \Gamma \to G \) be a homomorphism of finite groups then \( \Gamma_{\infty,\varphi} \) is a finite group.

We note that even when \( \varphi \) is the trivial map between two finite cyclic groups, the groups \( \Gamma_{i,\varphi} \) grow indefinitely in size with \( i \), but their inverse limit is finite, and is isomorphic to the domain in this case (Theorem 1).

Recall that for a finite group \( G \) and a subgroup \( H \leq G \), the subnormal closure of \( H \) in \( G \) is the smallest subnormal subgroup of \( G \) containing \( H \). Thus Theorem 3 is again an extension to any map of finite groups, of the trivial observation that the subnormal closure of a subgroup of a finite group is well defined and finite. This result complements the dual result proved in [FS1] for the tower of injective normalizers of a map of finite groups.

The next theorem characterizes, for maps of finite groups, the factorization \( \Gamma \to \Gamma_{\infty} \to G \) as a universal one among all subnormal factorizations (see below) the proof is given in §3:

The following is a basic definition:

Definition 4. A subnormal map \( n : M \to G \), is a homomorphism such that there exists a finite series of normal maps \( n_i : M_{i+1} \to M_i \), \( 1 \leq i \leq k \), with \( M_1 = G \), whose composition is \( n \).

\[
\begin{array}{ccccccc}
M_{k+1} & \xrightarrow{n_{k}} & M_k & \xrightarrow{n_{k-1}} & \ldots & \xrightarrow{n_2} & M_2 & \xrightarrow{n_1} & G.
\end{array}
\]

Notice that the image subgroup \( n(M_{k+1}) \) is subnormal in \( G \).

For example, any map of nilpotent groups is a subnormal map.

Theorem 5. If \( \varphi : \Gamma \to G \) be a homomorphism of finite groups then the factorization \( \Gamma \to \Gamma_{\infty} \to G \) is the universal initial factorization of \( \varphi \), among all factorizations \( \Gamma \to S \to G \) of \( \varphi \), with the right map \( S \to G \) being a subnormal map; namely, it maps uniquely to every such factorization via \( \Gamma_{\infty,\varphi} \to S \).

Main tools. Here we sum-up the main technical tools for proving the above results. They consist of showing that the limiting group \( \Gamma_{\infty,\varphi} \) does not change up to a canonical isomorphism, when one changes the domain or range of \( \varphi \) in certain controlled ways.

First we consider changing the range via factoring our map \( \varphi \) through any subnormal map \( M \to G \).

The following theorem is one of our main tools showing we can perform the above mentioned replacement:

Theorem 6. Let \( \varphi : \Gamma \to G \) be a homomorphism and let \( \Gamma \xrightarrow{\psi} M \xrightarrow{n} G \) be a factorization of \( \varphi \) such that \( n : M \to G \) is a subnormal map. Then \( \Gamma_{\infty,\varphi} \cong \Gamma_{\infty,\psi} \).

Theorem 6 is proved right after Proposition 2.4.

The next result shows that one can factor out a certain portion \( K \) of the kernel \( K \) of \( \varphi \) and still obtain that \( \Gamma_{\infty,\varphi} \cong (\Gamma/K)_{\infty,\rho} \), where \( \rho : \Gamma/K \to G \) is the map induced by \( \varphi \).
To define $K$, define the descending series of successive commutators of $K$ with $\Gamma$ by $K_1 = K$, $K_2 = [\Gamma, K]$, and in general $K_{i+1} = [\Gamma, K_i]$. If this series terminates after a finite number of steps, we let $K$ be the final member of this series.

**Proposition 7.** Let $\varphi: \Gamma \to G$ be a homomorphism, and suppose that the descending series of successive commutators of $K = \ker \varphi$ with $\Gamma$ terminates after a finite number of steps. Let $K$ be the terminal member of this series. Then $\Gamma_{\infty, \varphi} \cong (\Gamma/K)_{\infty, \rho}$, where $\rho: \Gamma/K \to G$ is the map induced by $\varphi$.

Proposition 7 is Proposition 2.9(2). Notice that in the notation of Proposition 7, the kernel of $\rho$ is $K/K$ and hence $\ker \rho$ is contained in some member of the ascending central series of $\Gamma/K$. In Example 4.1 we show however that the nilpotency class of $K/K$ cannot be bounded.

## 2. Equivalence of normal closures towers

In this section we prove the results of the introduction, and analyze further the group $\Gamma_{\infty, \varphi}$. Our main tool is to compare the normal closures tower $\{\Gamma_{i, \varphi}\}$ to towers $\{\Lambda_{i, \rho}\}$, for various homomorphism $\rho: \Lambda \to H$, which are, of course, related to $\varphi$.

As noted above, throughout this paper $\varphi: \Gamma \to G$ is a fixed homomorphism of groups. We use the following notation. $K = \ker \varphi$ and $\Gamma_{\varphi}$ is the (free) normal closure of $\varphi$. This, we recall, is the universal factorization $\Gamma \xrightarrow{\varphi} \Gamma_{\varphi} \xrightarrow{\rho} G$ of $\varphi$ with the right map being normal.

As in diagram (1.1), $\{\Gamma_i\} = \{\Gamma_{i, \varphi}\}$ is the normal closures tower of $\varphi$, and $\varphi_i$, $\varphi_i$ are as in diagram (1.1). $\Gamma_{\infty} = \varprojlim \Gamma_i$ and $\varphi_{\infty}: \Gamma \to \Gamma_{\infty}$ is the natural map.

**Terminology 2.1.** Let $k \geq 0$ be an integer, and let $\{H_i\}_{i=k}$ be a decreasing or increasing series of groups. We say that $\{H_i\}$ terminates if there exists an integer $t \geq k$, such that $H_t = H_{t+1} = H_{t+2} = \ldots$. In this case we call $H_t$ the terminal member of the series and say that the series terminates at $H_t$.

Recall from [R, p. 385] the notion of the series of successive normal closures—in the usual sense—of a subgroup $H \leq G$ in $G$. This is the increasing series defined by $C_0 = G$, $C_1 = \langle H^{C_0} \rangle$, and in general $C_{i+1} = \langle H^{C_i} \rangle$.

**Notation 2.2.**

1. The series of successive normal closures of $\varphi(\Gamma)$ in $G$ will be denoted $G = C_0, C_1, C_2, \ldots$. If this series terminates, then we denote by $\mathcal{C}$ its terminal member.

2. Let $K := \ker \varphi$ and define the decreasing series of successive commutators of $K$ with $\Gamma$ by $K_1 = K$, $K_2 = [\Gamma, K]$, and in general $K_{i+1} = [\Gamma, K_i]$. If this series terminates, then we denote by $\mathcal{K}$ its terminal member.

Next $H = \gamma_1(H), \gamma_2(H), \ldots$ denotes the descending central series of the group $H$. If this series terminates, then $\gamma_{\infty}(H)$ denotes the terminal member of this series. Finally, the
ascending central series of $H$ is denoted $1 = Z_0(H), Z_1(H), \ldots$, and the same convention as above applies for the notation $Z_\infty(H)$.

We refer the reader to [FS1, section 2] for the notions of a normal map and of normal morphism, where references to previous work on this subject in given. In [FS1, section 3] the reader will find some basic properties of $\Gamma^\varphi$ and in [FS1, section 4] some basic properties of the normal closures tower of $\varphi$.

Let us start by recalling the naturality of the normal closures tower.

**Lemma 2.3.** Any commutative diagram

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\varphi} & G \\
\downarrow{\mu} & & \downarrow{\eta} \\
\Gamma' & \xrightarrow{\varphi'} & G'
\end{array}
\]

induces a commutative diagram between the towers of normal closures of $\varphi$ and $\varphi'$:

Namely if we set $\Gamma_1 := G$, $\Gamma'_1 = G'$ and $\eta_1 := \eta$, we have:

\[
\begin{array}{ccccccccc}
\Gamma & \xrightarrow{\varphi} & \Delta_1 & \xrightarrow{\varphi_1} & \Delta_1 & \xrightarrow{\varphi_1} & \ldots & \xrightarrow{\varphi_1} & \Delta_1 \\
\downarrow{\mu} & & \downarrow{\eta_1} & & \downarrow{\eta_1} & & \ldots & & \downarrow{\eta_1} \\
\Gamma' & \xrightarrow{\varphi'} & \Delta'_1 & \xrightarrow{\varphi'_1} & \Delta'_1 & \xrightarrow{\varphi'_1} & \ldots & \xrightarrow{\varphi'_1} & \Delta'_1 \\
\end{array}
\]

Thus there is a canonical map $\eta_\infty: \Gamma_\infty \to \Gamma'_\infty$ and equalities: $\varphi_i \circ \eta_i = \mu \circ \varphi'_i$, for all $i \geq 1$.

**Proof.** Take in diagram (2.2) of [FS1], $M' = \Gamma'_2$, $\psi' = \varphi'_2$ and $n' = \varphi'_1$, and use the universality properties of $\Gamma_2$ to obtain $\eta_2$. Then proceed in this manner replacing each time $\Gamma_i$, $\Gamma'_i$, with $\Gamma_{i+1}$, $\Gamma'_{i+1}$ respectively. \[\Box\]

The following Proposition will be used in the proof of Theorem 6 of the introduction. It addresses the replacement of $G$ by the domain of any subnormal map $M \to G$ that factors our map $\varphi$. In that case the map of towers (2.3) is in fact a pro-isomorphism of towers:

**Proposition 2.4.** Let $\Gamma \xrightarrow{\psi} M \xrightarrow{n} G$ be a factorization of $\varphi$ with $n$ a normal map. Then the above commutative diagram extends to a natural commutative diagram in which the upper (resp. lower) row is the normal closures tower of $\Gamma \xrightarrow{\psi} M$ (resp. $\Gamma \xrightarrow{\psi} G$):

\[
\begin{array}{ccccccccc}
\Gamma & \xrightarrow{\Delta_1} & \Delta_1 & \xrightarrow{\psi_1} & \Delta_1 & \xrightarrow{\psi_1} & \ldots & \xrightarrow{\psi_1} & \Delta_1 \\
\downarrow{\mu} & & \downarrow{\mu_1} & & \downarrow{\mu_1} & & \ldots & & \downarrow{\mu_1} \\
\Gamma' & \xrightarrow{\Delta'_1} & \Delta'_1 & \xrightarrow{\psi'_1} & \Delta'_1 & \xrightarrow{\psi'_1} & \ldots & \xrightarrow{\psi'_1} & \Delta'_1 \\
\end{array}
\]

Thus there is a canonical map $\eta_\infty: \Gamma_\infty \to \Gamma'_\infty$ and equalities: $\psi_i \circ \eta_i = \mu \circ \psi'_i$, for all $i \geq 1$.

**Proof.** We need the following lemma which is an obvious analog of the similar situation for two normal subgroup of the same group:
Lemma 2.5. Consider the commutative diagram

\[
\begin{array}{ccc}
M_1 & \xrightarrow{\mu} & M_2 \\
\downarrow{n_1} & & \downarrow{n_2} \\
G & & \\
\end{array}
\]

where \( n_i \) are normal maps, for \( i = 1, 2 \), and \( \mu \) is a normal morphism.

Proof. Define an action of \( M_2 \) on \( M_1 \) by \( m_1^{m_2} = m_1^{m_2(m_2)} \), \( m_i \in M_i \). Then it is immediate that \( \mu \) becomes a normal map with the above normal structure.

As usual let \( \Gamma_1 := G \) and \( \varphi_1 := \varphi \). Consider the normal closures tower corresponding to \( \psi \). So the groups \( \Gamma_i \) in diagram (1.1) are replace by \( \Delta_i \) and the maps \( \varphi_i, \overline{\varphi}_i \) are replaced by \( \psi_i, \overline{\psi}_i \), respectively. Thus \( \Delta_1 = M \) and \( \psi_1 = \psi \). Set \( \mu_1 = n \), thus \( \mu_1 : \Delta_1 \to \Gamma_1 \) is a normal map.

We show that there are normal maps \( \mu_i, \rho_i, i \geq 1 \), as in diagram (2.3) such that

\[
(2.4) \quad \varphi_{i+1} \circ \rho_i = \psi_i \text{ and } \psi_i \circ \mu_i = \varphi_i, \text{ for all } i \geq 1.
\]

Now since \( \Gamma \xrightarrow{\psi_1} \Delta_1 \xrightarrow{\mu_1} \Gamma_1 \) is a factorization of \( \varphi_1 \), by the universality property of \( \Gamma_2 \), there exists a normal morphism \( \rho_1 : \Gamma_2 \to \Delta_1 \) for the lower right triangle, and such that \( \varphi_2 \circ \rho_1 = \psi_1 \). By Lemma 2.5 \( \rho_1 \) is a normal map.

Let now \( i \geq 2 \), and suppose that \( \rho_{i-1} \) and \( \mu_{i-1} \) were defined, they are normal maps, diagram (2.3) is commutative up to the the \((i-1)\)-step, and equation (2.4) holds. By equation (2.4) we have \( \varphi_i \circ \rho_{i-1} = \psi_{i-1} \). By the universality of \( \Delta_i \), and since \( \rho_{i-1} \) is a normal map, there exists a map \( \mu_i : \Delta_i \to \Gamma_i \) such that

(a) \( \psi_i \circ \mu_i = \varphi_i \).

(b) \( \mu_i \circ \rho_{i-1} = \overline{\varphi}_i \), and \( \mu_i \) is a normal morphism for the right triangle.

By Lemma 2.5 \( \mu_i \) is a normal map.

Next, by the universality property of \( \Gamma_{i+1} \), by (a), and since \( \mu_i \) is a normal map, there exists a normal morphism \( \rho_i : \Gamma_{i+1} \to \Delta_i \) for the lower right triangle, this triangle is commutative \((\rho_i \circ \mu_i = \overline{\varphi}_i)\) and \( \varphi_{i+1} \circ \rho_i = \psi_i \). Again by Lemma 2.5 \( \rho_i \) is a normal map. This completes the induction step and the proof of the proposition.

Proof of Theorem 6 Consider the subnormal series of normal maps

\[
M_{k+1} \xrightarrow{n_{k}} M_k \xrightarrow{\cdots} M_2 \xrightarrow{n_1} G = M_1
\]

where \( k \geq 1, M_{k+1} = M, M_1 = G \), and \( n_t \) is a normal map, for all \( 1 \leq t \leq k \). Let \( \psi_t : \Gamma \to M_t \) be the maps defined by \( \psi_{k+1} = \psi \), and for \( 1 \leq t \leq k \), let \( \psi_t = \psi \circ n_k \circ \cdots \circ n_t \) (so \( \psi_1 = \varphi \)).

The universality property of the inverse limit and Proposition 2.4 imply that \( \Gamma_{\infty, \psi_t} \cong \Gamma_{\infty, \psi_{t+1}} \), for all \( 1 \leq t \leq k \), so since \( \Gamma_{\infty, \varphi} = \Gamma_{\infty, \psi_1} \) and \( \Gamma_{\infty, \psi} = \Gamma_{\infty, \psi_{k+1}} \), the theorem holds.

As a corollary to Theorem 6 we get
Proposition 2.6.  (1) Let $H$ be a subnormal subgroup of $G$ containing $\varphi(\Gamma)$. Then $\Gamma_{\infty,\varphi} \cong \Gamma_{\infty,\psi}$, where $\psi: \Gamma \to H$ is the restriction of $\varphi$ in the range;
(2) if the series $\{C_i\}$ terminates, then $\Gamma_{\infty,\varphi} = \Gamma_{\infty,\psi}$, where $\psi: \Gamma \to C$ is the restriction of $\varphi$ in the range.

Proof. Let $n: H \hookrightarrow G$ be the inclusion map. Then $n$ is a subnormal map, and $\Gamma \xrightarrow{\psi} H \xrightarrow{n} G$ is a factorization of $\varphi$. Hence (1) follows from Theorem 6, and (2) is immediate from (1). \qed

We now turn our attention to the kernel $K$.

Lemma 2.7. Let $L \leq \ker c_{\varphi}$ be a normal subgroup of $\Gamma$. Let $\rho: \Gamma/L \to G$ be the homomorphism induced by $\varphi$, and let $\beta: \Gamma/L \to \Gamma^\varphi$ be the homomorphism induced by $c_{\varphi}$. Then $\Gamma^\varphi$ and $(\Gamma/L)^\rho$ are naturally isomorphic.

Proof. Consider the following commutative diagram. We show that $u$ is an isomorphism whose inverse is $v$.

(2.5) \[ \begin{array}{c}
\Gamma \\
\downarrow u \\
\Gamma^\varphi \\
\downarrow c_{\varphi} \\
(\Gamma/L)^\rho \\
\downarrow v \\
\Gamma^\varphi \\
\downarrow c_{\varphi} \\
\Gamma \\
\downarrow \beta \\
G
\end{array} \]

here $\alpha \circ \rho = \varphi$, $\alpha \circ \beta = c_{\varphi}$, and $u$ and $v$ are the unique normal morphisms obtained from the universality properties of $\Gamma^\varphi$ and $(\Gamma/L)^\rho$ respectively.

Notice that $\alpha \circ \beta \circ \overline{\varphi} = c_{\varphi} \circ \overline{\varphi} = \varphi = \alpha \circ \rho$. Since $\alpha$ is surjective we see that $\beta \circ \overline{\varphi} = \rho$. By the universality property of $(\Gamma/L)^\rho$ we get the map $v$. Also, $(\alpha \circ c_{\rho}) \circ \overline{\rho} = \alpha \circ \rho = \varphi$. By the universality property of $\Gamma^\varphi$ we get the map $u$.

Next we have $c_{\varphi} \circ u \circ v = \alpha \circ c_{\rho} \circ v = \alpha \circ \beta = c_{\varphi}$. Also $u \circ v \circ \overline{\varphi} = u \circ \overline{\rho} = \overline{\varphi}$. Hence $u \circ v = 1_{\Gamma^\varphi}$, by the uniqueness in the universality property of $\Gamma^\varphi$.

Further, $\alpha \circ c_{\rho} \circ v \circ u = \alpha \circ \beta \circ u = c_{\varphi} \circ u = \alpha \circ c_{\rho}$. Since $\alpha$ is surjective, $c_{\rho} \circ v \circ u = c_{\rho}$. Also $v \circ u \circ \overline{\rho} = v \circ \overline{\varphi} = \overline{\rho}$. So, as above, $v \circ u$ is the identity map of $(\Gamma/L)^\rho$. \qed

As a corollary we get

Corollary 2.8. Let $L \leq \ker c_{\varphi}$ be a normal subgroup of $\Gamma$, and let $\rho: \Gamma/L \to G$ be the homomorphism induced by $\varphi$. Then $\Gamma/L \xrightarrow{\beta} \Gamma^\varphi \xrightarrow{\overline{\varphi}} G$, where $\beta$ and $\overline{\varphi}$ are as in diagram (2.5), is the universal normal decomposition of $\rho$. 

Proof. This follows from Lemma 2.7 because in diagram (2.5) we may take that $c_\rho = \beta$, $(\Gamma/L)^c = \Gamma^c$ and $\overline{\varphi} = \overline{\psi}$. \hfill \Box

In particular we get Proposition 7 of the introduction as part (2) of the following:

Proposition 2.9. Let $L \leq \Gamma$ be a subgroup with $L \leq \ker \varphi_i$, for all integers $i \geq 1$. Let $\rho: \Gamma/L \to G$ be the map induced by $\varphi$, let $\{(\Gamma/L)_i\} = \{(\Gamma/L)_{i,\rho}\}$ be the normal closures tower of $\rho$, and let $\rho_i, \overline{\rho}_i$ be the maps for the tower $\{(\Gamma/L)_i\}$. Then

(1) there is a natural isomorphism $\Gamma_i \cong (\Gamma/L)_i$, where $\rho_i: \Gamma/L \to (\Gamma/L)_i$ is the map induced by $\varphi_i$, and $\overline{\varphi}_i = \overline{\rho}_i$, for all $i \geq 1$. In particular $\Gamma_{\infty} \cong (\Gamma/L)_{\infty}$;

(2) if the series $\{K_i\}$ terminates at $\mathcal{K}$, then (1) above holds for $L := \mathcal{K}$.

Proof. Part (1) is immediate from Corollary 2.8. For (2) note that $\varphi_{i+1}(\ker \varphi_i) \leq \ker \overline{\varphi}_i \leq Z(\Gamma_{i+1})$. It follows by induction on $i$, that $K_i \leq \ker \varphi_i$, for all $i \geq 1$. This implies that if the series $\{K_i\}$ terminates, then $\mathcal{K} \leq \ker \varphi_i$, for all $i \geq 1$, so (2) follows from (1). \hfill \Box

Combining Propositions 2.9(2) and 2.6(2) we get:

Corollary 2.10. Assume that both the series $\{C_i\}$ and the series $\{K_i\}$ terminate at $\mathcal{C}$ and $\mathcal{K}$ respectively. Then there is an isomorphism $\Gamma_{\infty,\varphi} \cong (\Gamma/\mathcal{K})_{\infty,\psi}$, where $\psi: \Gamma/\mathcal{K} \to \mathcal{C}$, is the map induced by $\varphi$.

Proof. By Proposition 2.9(2), $\Gamma_{\infty,\varphi} = (\Gamma/\mathcal{K})_{\infty,\rho}$, where $\rho: \Gamma/\mathcal{K} \to G$ is the map induced by $\varphi$. Then, by Proposition 2.6(2) (with $\Gamma/\mathcal{K}$ in place of $\Gamma$), $(\Gamma/\mathcal{K})_{\infty,\rho} = (\Gamma/\mathcal{K})_{\infty,\psi}$. \hfill \Box

The following theorem proves in particular the assertion of Theorem 3 of the introduction.

Theorem 2.11. Let $\varphi: \Gamma \to G$ be a homomorphism of finite groups. Let $\rho: \Gamma \to \mathcal{C}$ and let $\psi: \Gamma/\mathcal{K} \to \mathcal{C}$ be the maps induced by $\varphi$. Then

(1) $\Gamma_{\infty,\varphi} \cong \Gamma_{\infty,\rho} = (\Gamma/\mathcal{K})_{\infty,\psi}$;

(2) the normal closures series $\{\Gamma_i,\rho\}$ and the series $\{(\Gamma/\mathcal{K})_{i,\psi}\}$ terminate, hence $\Gamma_{\infty,\varphi} \cong \Gamma_{t,\rho} \cong (\Gamma/\mathcal{K})_{t,\psi}$, where $\Gamma_{t,\rho}$ and $(\Gamma/\mathcal{K})_{t,\psi}$ are the terminal members of the respective series;

(3) $\Gamma_{\infty,\varphi}$ is finite and $|\Gamma_{\infty,\varphi}| \leq |\Gamma/\mathcal{K}| \cdot g(|\mathcal{C}|)$, where $g(1) = 1$, and for any integer $t \geq 2$, $g(t) = t^k$, where $k = \frac{1}{2}(\log_p t + 1)$ and $p$ is the least prime divisor of $t$.

Proof. Part (1) follows from Proposition 2.6(2) and Corollary 2.10. Then, by definition, $\mathcal{C} = \langle \varphi(\Gamma)^\mathcal{C} \rangle$, so parts (2) and (3) follow from [FS1, Theorem 4.1]. \hfill \Box

Furthermore we can use Corollary 2.10 to prove the following lemma, which leads to the proof of Theorem 4 of the introduction.

Lemma 2.12. Assume that $\varphi(\Gamma)$ is subnormal in $G$ and that the series $\{K_i\}$ (see Notation 2.2(2)) terminates. Then $\Gamma_{\infty} = \Gamma/\mathcal{K}$.
Proof. Since \( \varphi(\Gamma) \) is subnormal in \( G \), we have \( \mathcal{C} = \varphi(\Gamma) \). By Proposition 2.6(2), to evaluate \( \Gamma_\infty \), we may assume that \( \varphi \) is surjective. By [FS1 Corollary 3.9(2)], \( \Gamma_2 = \Gamma/[\Gamma, K] \), and \( \varphi_2 : \Gamma \to \Gamma_2 \) is the canonical map. Iterating on [FS1 Corollary 3.9(2)] we see that \( \Gamma_i = \Gamma/K_i \), for all \( i \geq 2 \), thus \( \Gamma_\infty = \Gamma/K \). \( \square \)

Part (2) of the following corollary is Theorem 1 of the introduction.

Corollary 2.13.  
(1) If \( \varphi(\Gamma) \) is subnormal in \( G \) and \( K \) is contained in \( Z_i(\Gamma) \) for some integer \( i \geq 0 \) (which holds if \( \varphi \) is injective), then \( \Gamma_\infty = \Gamma \);
(2) if \( \Gamma \) and \( G \) are nilpotent, then \( \Gamma_\infty = \Gamma \).

Proof. Part (1) is an immediate consequence of Lemma 2.12, since under the hypotheses of (1), \( K = 1 \). Then (2) follows from (1). \( \square \)

We now turn to the nilpotent quotients of \( \Gamma_\infty = \Gamma_{\infty, \varphi} \), and prove Theorem 2 of the introduction.

Proposition 2.14. Let \( k \geq 1 \). The map \( \varphi_{(\infty,k)} : \Gamma/\gamma_k(\Gamma) \to \Gamma_\infty/\gamma_k(\Gamma_\infty) \) induced by the canonical map \( \varphi_\infty : \Gamma \to \Gamma_\infty \) is injective. If \( \Gamma \) and \( G \) are finite, then it induces an isomorphism \( \Gamma/\gamma_\infty(\Gamma) \cong \Gamma_\infty/\gamma_\infty(\Gamma_\infty) \).

Proof. Let \( \psi : \Gamma/\gamma_k(\Gamma) \to G/\gamma_k(G) \) be the map induced by \( \varphi \). By naturality (Lemma 2.3) we have the following commutative diagram

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\varphi_\infty} & \Gamma_\infty \\
\mu \downarrow & & \eta_\infty \downarrow \\
\Gamma/\gamma_k(\Gamma) & \xrightarrow{\cong} & (\Gamma/\gamma_k(\Gamma))_\infty/\psi \\
\psi_\infty \downarrow & & \eta \downarrow \\
G/\gamma_k(G) & \xrightarrow{\psi} & G/\gamma_k(G)
\end{array}
\]

where the map \( \eta_\infty \) is obtained from Lemma 2.3, and where \( \psi_\infty \) is an isomorphism by Corollary 2.13(2). The diagram is commutative since \( \mu \) is surjective. Since \( \psi_\infty \) is an isomorphism, \( \varphi_{(\infty,k)} \) is injective.

Assume that \( \Gamma \) and \( G \) are finite. By Proposition 2.6(2) we may assume that \( G = \mathcal{C} \), so \( G = \langle \varphi(\Gamma)^G \rangle \). Also, by Theorem 2.11(2), the series \( \{ \Gamma_i \} \) terminates, so \( \Gamma_\infty = \Gamma_t \), for some \( t \geq 1 \), and now \( \varphi_\infty = \varphi_t \). By [FS1 Lemma 4.2(1)], \( \Gamma_\infty = \Gamma_t = \langle \varphi_t(\Gamma)^\Gamma_t \rangle \). Hence the conjugates of the image of \( \varphi_t(\Gamma) \) in \( \Gamma_t/\gamma_k(\Gamma_t) \) generates it. By [FS1 Lemma 4.4], the image of \( \varphi_t(\Gamma) \) in \( \Gamma_t/\gamma_k(\Gamma_t) \) equals \( \Gamma_t/\gamma_k(\Gamma_t) \). This shows that \( \varphi_{(\infty,k)} \) is surjective. \( \square \)

3. **Universality of \( \Gamma_\infty \)**

In this section we prove theorem 5 of the introduction.
Proof. To facilitate the discussion, and in view of the theorem we are now proving, we refer to the factorization $\Gamma \to \Gamma_{\infty,\varphi} \to G$ as the *subnormal closure of $\varphi$*. When the maps are understood, we refer to the group $\Gamma_{\infty,\varphi}$ itself as the subnormal closure of $\Gamma$ with respect to $G$. Notice that the subnormal closure is a functor on maps of groups and thus acts on squares as in equation (2.1) of Lemma 2.3 and respects compositions. We use the following

**Notation 3.1.** Given a commutative diagram of group homomorphisms:

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{\varphi} & G \\
\downarrow & & \downarrow \eta \\
\Gamma & \xrightarrow{\psi} & H
\end{array}
$$

we denote by $\bar{\eta}_{\infty}$ the induced map on the subnormal closures:

$$
\bar{\eta}_{\infty} : \Gamma_{\infty,\varphi} \to \Gamma_{\infty,\psi}.
$$

Throughout this proof $\Gamma_{\infty}$ denotes $\Gamma_{\infty,\varphi}$. We begin the proof by noting that by Theorem 2.11, $\Gamma_{\infty}$ is finite. Further, by Theorem 2.11 the map $\rho : \Gamma \to \mathcal{C}$, where $\mathcal{C}$ is as in Notation 2.2(1), satisfies $\Gamma_{\infty} = \Gamma_{t,\rho} = \Gamma_{\infty,\rho}$, for some integer $t \geq 1$. By the same theorem we have a finite tower of normal maps, the first $r$ being normal inclusions leading from the subnormal closure $\mathcal{C}$ to $G$:

$$
\Gamma_{\infty} = \Gamma_{t,\rho} \xrightarrow{\rho_{t-1}} \Gamma_{t-1,\rho} \cdots \xrightarrow{} \Gamma_{2,\rho} \to \mathcal{C} = C_r \hookrightarrow C_{r-1} \hookrightarrow \ldots \to G.
$$

This implies that the canonical map:

$$
l : \Gamma_{\infty} \to G
$$

is a subnormal map, by definition. But by lemma 4.2 of [FS1], the normal closure of the image of the map $\varphi_{\infty} : \Gamma \to \Gamma_{\infty} = \Gamma_{t,\rho}$, is $\Gamma_{\infty}$. Since the tower (3.1) is finite and terminates at $\Gamma_{\infty}$, we have

$$
\Gamma_{\varphi_{\infty}} \to \Gamma_{\infty}
$$

is the trivial extension and we take this map as the identity. Of course this implies that the normal closures tower of $\varphi_{\infty} : \Gamma \to \Gamma_{\infty}$ is constant, so

$$
\bar{l}_{\infty} : \Gamma_{\infty,\varphi_{\infty}} \to \Gamma_{\infty}
$$

is the identity map.

Now we show that the map $l$ is initial among all subnormal factorization maps of $\varphi$. Let $\Gamma \xrightarrow{\psi} S \xrightarrow{s} G$ be a factorization of our map $\varphi$ via a subnormal map $s : S \to G$. We need to show that there is a unique map $\bar{s} : \Gamma_{\infty} \to S$ rendering the following diagram commutative.

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{\varphi_{\infty}} & \Gamma_{\infty} \\
\downarrow & = & \downarrow \bar{s} \\
\Gamma & \xrightarrow{\psi} & S \\
\downarrow & & \downarrow s \\
\Gamma_{\infty} & \xrightarrow{l} & G
\end{array}
$$
To see this we consider the map induced on the subnormal closures by the given subnormal map $s$:

$$
\begin{array}{cccc}
\Gamma & \overset{\Gamma_{\infty,\psi}}{\longrightarrow} & S & \\
\downarrow & \cong & \downarrow & \\
\Gamma & \overset{s_{\infty}}{\longrightarrow} & G & \\
\end{array}
$$

Here $\bar{s}_{\infty}$ is the map induced by naturality of the subnormal closure as in Lemma 2.3, and $\Gamma_{\infty,\psi}$ is the subnormal closure of the map $\psi : \Gamma \to S$. Since $\bar{s}_{\infty}$ is an isomorphism by Proposition 2.4, one gets a well defined map

$$
\tilde{s} = \lambda \circ (\bar{s}_{\infty})^{-1} : \Gamma_{\infty} \to S.
$$

To see that this latter map is unique consider any map $\sigma : \Gamma_{\infty} \to S$ as in diagram 3.2, with $\tilde{s}$ replaced with $\sigma$. This map $\sigma$ induces, by naturality, the following commutative diagram of groups, where the two lower squares do not depend on the choice of $\sigma$:

$$
\begin{array}{cccc}
\Gamma & \overset{\Gamma_{\infty,\psi}}{\longrightarrow} & S & \\
\downarrow & \cong & \downarrow & \\
\Gamma & \overset{s_{\infty}}{\longrightarrow} & G & \\
\end{array}
$$

Now we rewrite the map $\sigma$ in terms of the maps in the given decomposition $\Gamma \to S \to G$ alone: We read in the middle upper square: $\sigma = \sigma_{\infty} \circ \lambda$. But since:

$$
\overline{\sigma_{\infty} \circ s_{\infty}} = (\sigma \circ s)_{\infty} = \bar{l}_{\infty} = \text{id}
$$

It follows that both $\sigma_{\infty}$, being the inverse to $s_{\infty}$, and thus $\sigma$ itself, are determined by $\lambda$—constructed out of $\psi$ and $s$ as claimed.

4. Examples

In this section we give two examples. In both examples we take $G$ to be perfect with $G = \langle \varphi(G) \rangle$. In the first Example 4.1 we assume that $\Gamma$ and $G$ are finite and that $\ker \varphi \leq Z_{\infty}(\Gamma)$, and we show that $\ker \varphi_{\infty}$ can have arbitrarily large nilpotency class. In the second Example 4.4 we show that if $\Gamma$ is perfect, then $\Gamma_{\infty} = \Gamma_{2}$ is the universal $\varphi$-central extension of $G$ (see [FS2]).

Example 4.1. In this example we assume that $\Gamma$ and $G$ are finite and that $\ker \varphi \leq Z_{\infty}(\Gamma)$. The purpose of this example is to show that the nilpotency class of $\ker \varphi_{\infty}$ can be arbitrarily large. We first need the following easy lemma and its corollary.
**Lemma 4.2.** Let $G$ be perfect and set $H := \Gamma_\infty$. Then $H$ is a central product $H = H^{(\infty)} \circ Z_\infty(H)$. Here, $H^{(\infty)}$ is the last term of the derived series of $H$. If the nilpotent residual $\Gamma/\gamma_\infty(\Gamma)$ of $\Gamma$ has nilpotency class $c$, then the nilpotency class of $Z_\infty(H)$ is $\leq c + 1$.

**Proof.** First, by [FS1, Theorem 4.1], $H = \Gamma_\infty$ is finite, because $G = \langle \varphi(\Gamma)^G \rangle$. Set $L = H^{(\infty)}$ and $Z = Z_\infty(H)$. There is surjection $H \rightarrow G$ whose kernel is contained in $Z$, and since $G$ is perfect, we get that $H = L \cdot Z$. Since $L$ is perfect, $Z_\infty(L) = Z(L)$, and of course $[L, Z] \leq Z_\infty(L) = Z(L)$. By the three subgroup lemma we get $[L, Z] = 1$, and so $H = L \circ Z$.

Next, by Proposition 2.14, the nilpotent residuals of $\Gamma$ and $H$ are isomorphic. Clearly $\gamma_\infty(H) = L$ and $\Gamma/\gamma_\infty(\Gamma) \cong H/\gamma_\infty(H) = H/L \cong Z/(Z \cap L)$ is nilpotent of class $c$. Hence, since $Z \cap L \leq Z(Z)$, we see that the nilpotency class of $Z$ is $\leq c + 1$. \hfill $\Box$

**Corollary 4.3.** Let $G$ be perfect, and let $c$ (resp. $d$) be the nilpotency class of the nilpotent residual of $\Gamma$ (resp. of $K := \ker \varphi$). Then $\ker \varphi_\infty$ has nilpotency class $\geq d - c - 1$.

**Proof.** Let $f: \Gamma_\infty \rightarrow G$. Then $\ker f \leq Z_\infty(\Gamma_\infty)$. Since $\varphi_\infty \circ f = \varphi$, we see that $\varphi_\infty(K) \leq \ker f \leq Z_\infty(\Gamma_\infty)$. Now $\ker \varphi_\infty \leq K$. By Lemma 4.2, the nilpotency class of $\varphi_\infty(K) \leq c + 1$. Since $K/\ker \varphi_\infty \cong \varphi_\infty(K)$, and the nilpotency class of $K$ is $d$, the corollary follows. \hfill $\Box$

We now construct examples where $G$ is perfect (in fact simple), $\gamma_\infty(\Gamma) = [\Gamma, \Gamma]$, so one has:

- The nilpotent residual of $\Gamma$ has nilpotency class $c = 1$.
- The nilpotency class $d$ of $\ker \varphi$ is arbitrarily large.

By Corollary 4.3, the nilpotency class of $\ker \varphi_\infty$ is $\geq d - 2$, so it is arbitrarily large.

Let $p$ be a prime, let $n \geq 2$ such that $p$ divides $n$ and let $q$ be a prime power such that $p$ divides $q - 1$. Let $H$ be an image of $SL_n(q)$ such that $Z(H) \cong Z_p$ is cyclic of order $p$. Let $B$ be an elementary abelian group of order $p^n$, and let $\Gamma$ be the wreath product $G = H \wr B$. By [M, section 3, p. 282–283], $Z_\infty(\Gamma)$ is of nilpotency class $(n - 1)p + 1$ (M, Lemma 3.2, p. 283). Also, $\gamma_\infty(\Gamma) = [\Gamma, \Gamma]$ is isomorphic to a direct product of $p^n$ copies of $H$ ([M, Lemma 3.1, p. 283]). Clearly $\Gamma/Z_\infty(\Gamma)$ is isomorphic to $PSL_n(q) \wr B$. But $PSL_n(q) \wr B$ is contained in $G = PSL_{mp^n}(q)$. It follows that there exists a homomorphism $\varphi: \Gamma \rightarrow G$ whose kernel is $Z_\infty(\Gamma)$. Hence $G, \Gamma$ and $\ker \varphi$ have the claimed properties.

**Example 4.4.** Suppose $\Gamma$ is perfect. Then under the above assumptions, $G$ is perfect and $\Gamma_\infty = \Gamma_2$ is a perfect group which is the universal $\varphi$-central extension of $G$.

Indeed, since $G = \langle \varphi(\Gamma)^G \rangle$, and $\varphi(\Gamma)$ is perfect (because $\Gamma$ is), $G$ is perfect. Also, by [FS1, Lemma 3.3], $\Gamma_2$ is a central extension of $G$, and $\Gamma_2$ is generated by $\{\varphi(\Gamma)^g \mid g \in G\}$, so $\Gamma_2$ is perfect. The same argument shows that $\Gamma_3$ is a perfect central extension of $\Gamma_2$. By [CDFS, Prop. 1.8, p. 637], $\Gamma_3$ is a central extension of $G$. It is easy to check now that $\Gamma_2 = \Gamma_3$, by the universal property of $\Gamma_2$, and that the assertion above holds.
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