The standard quantum limit of coherent beam combining

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Abstract

Coherent beam combining refers to the process of generating a bright output beam by merging independent input beams of individually diffusing relative phases by locking them to each other. We report the first quantum mechanical noise limit calculations for coherent beam combining and compare our results to quantum-limited amplification. Our coherent beam combining scheme is based on an optical Fourier transformation which renders the scheme compatible with integrated optics combined with feed-back stabilization of the relative phases. The scheme can be laid out for an arbitrary number of input beams and approaches the shot noise limit for a large number of inputs.

1. Introduction

High power lasers are indispensable for various applications in industrial manufacturing, precision metrology and fundamental research. For many laser systems the achievable output power is limited and cannot be easily scaled up. A way to bypass this limitation is to coherently combine several laser beams requiring interferometric quality of the wave fronts and proper phase relation between the individual beams. Different strategies have been explored for coherent beam combining (for a review see [1]). One such strategy uses e.g. a fan-out grating operated in reverse, letting the phases emitted by each of several laser amplifiers self-lock through using the feedback from a common output coupler [2]. A more recent variant is coherent polarization beam combining [3].

The strive for higher laser powers also pushed the development of high power fiber lasers. Yet, since 2011, when thermal mode instabilities (TMI) were first observed [4], power scaling of fiber lasers came to a hold. TMI cause high power fiber lasers to become unstable above a certain threshold power and lead to chaotic fluctuations of their modal profile. Again, coherent beam combining (CBC) promises power scaling beyond the TMI threshold. Multiple beams—typically split from a common seed laser—are individually amplified by identical fiber amplifiers operating below the TMI threshold and are then recombed, e.g. via constructive interference in a cascade of beam splitters, to generate the high power output beam [5].

Apart from scaling the output power by combining the noisy outputs of amplifiers, CBC can also be applied to input beams with a quantum-limited noise profile. As we will see, the favorable noise scaling of CBC allows to extend the power range of (nearly) quantum noise-limited beams. Currently the power range is limited to the domain of less than 100 Watts, which limits, e.g. the precision of quantum-limited metrology and the efficiency of coherent parametric processes.

Here we aim to elucidate the quantum limits of CBC paying particular attention to the scaling of the output noise as a function of the accuracy of phase locking and the number of combined beams. We derive the quantum mechanical limits of CBC and report on the noise scaling when combining either quantum noise limited or noisy beams, such as those emerging from a linear amplifier.

The paper is organized as follows. In section 2.1 the coherent combination of coherent states is introduced and the impact of a finite phase locking accuracy onto the output quadratures of the combined state are derived. In section 2.2 the standard quantum limit for the variance of the phase locking scheme is derived on the basis of an optical circuit realizing the discrete Fourier transformation and its inverse. Moreover, the quadrature variances of the combined state are derived with respect to the quantum limited phase locking accuracy.
In section 3, we compare the output noise in CBC to that of a quantum limited linear amplifier. In the appendix A.1 we motivate the standard quantum limit for measuring the relative phase fluctuations between two states. Appendix A.2 elucidates the quantum mechanical origin of the noise penalty in quantum-limited amplifiers and finally in appendix A.4 it is demonstrated that the 3 dB limit does not apply to optical signals with fluctuations dominated by classical excess noise rather than by the quantum uncertainty.

2. Quantum-limited coherent combination

In CBC the optical power from several distinct beams, the phase of each of which diffuses independently in time, is interferometrically merged into a single optical mode. Ideally, the output power is the sum of the input powers and thus an integer multiple of the single beam power. In this respect CBC can be seen as an amplifier with discrete steps in the gain. We emphasize, however, that in contrast to linear amplifiers, CBC requires multiple coherent input signals and a priori knowledge on the signals amplitude as a resource.

The potential of CBC schemes lies on the one hand in their capability to reach optical power levels beyond the limits of conventional amplifiers [4]. On the other hand—and this is the property we want to highlight—CBC allows to prepare quantum noise limited outputs asymptotically. In the following sections we will derive the standard quantum limit for the output noise in CBC.

2.1. Formal description of noise in CBC

A CBC setup for the combination of N beams can be envisioned as a symmetric beam splitter with N input- and output ports each.

The interference between the inputs results in an output highly concentrated in a single port if the relative phases of the input states are adjusted to realize the discrete Fourier transform of the input modes as shown in [6].

Let us consider the coherent combination of N input laser beams with the following properties: we assume that the amplitudes are stabilized individually, which can be done, and that they have a finite spectral linewidth. This line width gives us an upper limit for how fast the individual phases \( \psi_k \), \( k \in \{1, 2, \ldots, N\} \) each diffuse in time resulting in a variance, which increases with time.

To achieve a stable output, the phases of the beams are mutually locked to each other. The feedback system providing the locking will necessarily have a certain time constant \( t_0 \). The phase variation, which the feedback system will have to compensate for, is determined by this time constant. By choosing a short enough time constant the relative phase variance of each input channel will be significantly smaller than a radian, \( \text{Var}(\psi_k(t + t_0) - \psi_k(t)) \leq \text{Var}(\psi_k) \ll 1 \). To ease the calculation below and without loss of generality, we assume that the initial phases of the different input channels at time \( t = 0 \) vary by significantly less than one radian, \( \text{Var}(\psi_k(0)) \ll 1 \). Note, that one could define one of the input beams as the phase reference with zero diffusion. The accuracy of the locking of the remaining phases, however, is ultimately limited by quantum noise such that the optical phase at the output will still fluctuate with a certain variance with respect to the phase reference. As a result of the phase fluctuations the output amplitude will also fluctuate. It is the purpose of the following calculation to estimate the lower limit of these fluctuations in phase and amplitude compatible with quantum physics.

The input states are quantum mechanically described by their mode operators \( \hat{a}_k \), which can be represented in a linearized form to single out the contributions of signal amplitude and quantum noise.

\[
\hat{a}_k = \alpha \exp(i \psi_k) + \delta^{(i)} \hat{a}_k. \tag{1}
\]

\( \alpha \) denotes the mean complex amplitude common to all input beams, \( \exp(i \psi_k) \) describes the phase fluctuations due to the (quantum-)limited phase-lock accuracy, and \( \delta^{(i)} \hat{a}_k = \delta \hat{a}_k + i \delta \hat{a}^*_k \), denotes the quantum mechanical Heisenberg uncertainty along the conjugate quadratures \( \hat{x} = (\hat{a} + \hat{a}^*)/2 \) and \( \hat{p} = (\hat{a} - \hat{a}^*)/(2i), [\hat{a}, \hat{a}^*] = 1 \).

The constructively interfering signal in the CBC output port is calculated by adding up the instantaneous amplitudes of the N input beams

\[
\hat{a}_{\text{out}} = \sum_{k=1}^{N} \frac{\hat{a}_k}{\sqrt{N}}. \tag{2}
\]

Note, that the mode operators are rescaled by \( \sqrt{N} \) as the states are split symmetrically onto the N output ports.
Following equations (1) and (2), the output state is given by

$$\hat{a}_{\text{out}} = \sum_{k=1}^{N} \frac{\alpha \exp(i \psi_{k})}{\sqrt{N}} + \frac{\delta^{(2)} \hat{a}_{k}}{\sqrt{N}}$$  \hspace{1cm} (3)

Let us discuss the constituents on the right-hand side of equation (3) in more detail starting with the complex amplitude $\alpha_{\text{out}}$. With the assumptions made above, the input phases will vary by significantly less than one radian, $\text{Var}(\psi_{k}) \ll 1$, which allows for approximating the exponential phase fluctuation term via its Taylor series expansion to second order. The first order term represents a phase shift of the complex amplitude and the second order term describes a reduction of the output amplitude due to phase locking imperfections.

$$\alpha_{\text{out}} = \sum_{k=1}^{N} \frac{\alpha}{\sqrt{N}} \exp(i \psi_{k}) \approx \frac{\alpha}{\sqrt{N}} \sum_{k=1}^{N} \left( 1 + i \psi_{k} - \frac{\psi_{k}^{2}}{2} \right)$$ \hspace{1cm} (4)

Summing over the $N$ independent random phases $\psi_{k}$ with equal individual variances $\text{Var}(\psi_{k}) = D$ yields a statistical distribution with the overall variance $ND$. The distribution of the squared random phases, $\psi_{k}^{2}$, is described by the gamma-distribution $\Gamma(k, \theta)$, with shape $k = N/2$ and scale $\theta = 2D$. For reasonably large $N$, the gamma distribution can be approximated by its mean value fluctuations. Summing over the $N$ inputs and taking the beam splitting factor $1/\sqrt{N}$ into account yields another Gaussian random variable with the same variance as for the input beams. Hence, the quantum noise is preserved in the combining process.

$$\alpha_{\text{out}} = \frac{\alpha}{\sqrt{N}} \left[ N + i \sqrt{ND} - \frac{1}{2} \{ ND \pm \sqrt{2ND} \} \right] \approx \frac{\alpha}{\sqrt{N}} \left( 1 - \frac{D}{2} \right) \pm \alpha \left( \frac{1}{\sqrt{2}} D + i \sqrt{D} \right).$$ \hspace{1cm} (5)

The second term in equation (3) concerns the evolution of the input states’ quantum noise. The quantum fluctuations of coherent states can be modeled as independent Gaussian random variables, such that the variance of the combined fluctuation scales identically to the scaling of the phase fluctuations. Summing over the $N$ inputs and taking the beam splitting factor $1/\sqrt{N}$ into account yields another Gaussian random variable with the same variance as for the input beams. Hence, the quantum noise is preserved in the combining process.

$$\delta^{(2)} \hat{a}_{\text{out}} = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \delta^{(2)} \hat{a}_{k} = \delta^{(2)} \hat{a}.$$ \hspace{1cm} (6)

Combining equations (5) and (6) yields the expression for the coherently combined output

$$\hat{a}_{\text{out}} = \sqrt{N} \alpha \left( 1 - \frac{D}{2} \right) + \delta^{(2)} \hat{a} \pm \alpha \left( \frac{1}{\sqrt{2}} D + i \sqrt{D} \right).$$ \hspace{1cm} (7)

Here we want to emphasize that the reason for the presence of the noise term in equation (7) is phase mismatch between the phase diffusing input channels. Our goal is to reduce the relative phase noise at the input by synchronizing the input phases to minimize the noise of the complex amplitude at the output. When doing so, the overall global phase can still fluctuate in time as described above. To eliminate the noise of the overall global phase we would need some phase reference. We can add such a reference to the discussed system assuming that one specific input channel has small phase variance, e.g. $\text{Var}(\psi_{i}) \approx 0$ apart from quantum uncertainty. This assumption does not influence the calculations which lead to equation (7), because the influence of a single channel on the output is negligible in the limit of large $N$ considered here. But now proper phase synchronization (see below) will lead to reducing both amplitude and phase fluctuations in the output. In the following subsection we will discuss the fundamental quantum limit for the possibility of such a phase synchronization.

### 2.2. Fourier transform based CBC

In this section, we derive the standard quantum limit for the phase locking accuracy in combining $N$ coherent input states. The maximal accuracy with which two optical signals can be phase-locked is limited by the precision in estimating their relative phase. The standard quantum limit for the minimal variance in estimating the relative phase of two coherent states depends solely on their mean photon number $n$

$$\text{Var}_{\text{SQL}}(\psi_{k}) \geq \frac{1}{n}.$$ \hspace{1cm} (8)

To extend this result to the combination of $N$ coherent states, we consider a CBC scheme based on the discrete Fourier transform. A sketch of this scheme is shown in figure 1. The discrete Fourier transform of $N$ coherent input fields $\alpha_{j} = |\alpha| \exp(i\psi_{j}), j \in \{0, 1, \ldots, N-1\}$ yields the outputs
\[ \beta_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \alpha_j e^{-i\Delta \psi_j \frac{k}{N}}. \]  

(9)

The first output port \((k = 0)\) readily provides the coherently combined beam \(\beta_0 = \sqrt{N} \bar{\alpha}\), where \(\bar{\alpha} = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \alpha_j\) is the average input amplitude. The relative optical phases of the input states are encoded as Fourier amplitudes in the beams exiting through the output ports with \(1 \leq k \leq N - 1\). In the next step, we apply the inverse discrete Fourier transform to the corresponding outputs, where the extracted CBC port \((k = 0)\) is left in the vacuum state. In essence, this scheme subtracts the instantaneous average value of the input beams \(\bar{\alpha}\) from the individual inputs. The resulting outputs, hence provide the error signals \(\Delta_j = \alpha_j - \bar{\alpha}, \ j \in \{0, 1, \ldots, N - 1\}\). For small relative phases the error signals \(\Delta_i\) are to a good approximation purely imaginary, \(\alpha_j \approx i|\alpha_i|\), such that the phase difference between the individual states and the average state \(\bar{\alpha}\) is directly encoded in the amplitude of the error signals. It is therefore not required to simultaneously measure both the \(X\) and the \(P\) quadrature and single photon detection is sufficient to extract the required information.

The sensitivity of detecting a phase difference at the output ports of the inverse Fourier transformation is ultimately limited by quantum mechanics.

Due to the stochastic nature of photon measurements in each channel it is not possible to perform perfect phase correction based on a single detector click. Properly designed feedback scheme will decrease the input phase mismatch down to a certain limit after some number of clicks. In this paper, we do not discuss the details of feedback algorithm. But we assume an 'ideal' feedback that improves phase difference after each click of any detector and, potentially, takes into account the possibility of one of the input channels to be the phase reference. This implies that there is no phase correction for this channel, instead the other channels will be subject to a corresponding additional adjustment. For such a scheme we argue that the limit corresponds to an error signal with mean photon number of one when summed over all output ports

\[ n_{\text{err}} = \sum_{j=0}^{N-1} |\Delta_j|^2 = \sum_{j=0}^{N-1} |\alpha \exp(i\psi_j) - \bar{\alpha}|^2 = \sum_{j=0}^{N-1} |\alpha \exp(i\psi_j)|^2 - \frac{1}{N} \sum_{k=0}^{N-1} \alpha \exp(i\psi_k)|^2 \approx n \cdot \left(1 + i\psi_j\right) - \frac{1}{N} \sum_{k=0}^{N-1} \left(1 + i\psi_k\right)^2 = n \cdot \sum_{j=0}^{N-1} \left(\psi_j - \langle \psi \rangle\right)^2, \]  

(10)

By using the Taylor approximation we have arrived at an expression containing the sum over the squared deviations of the individual phases from the instantaneous average phase \(\langle \psi \rangle\). This last line in equation (10) is \(n(N - 1)\) times the unbiased sample variance of the relative input phases

\[ n \sum_{j=0}^{N-1} (\psi_j - \langle \psi \rangle)^2 = n(N - 1)D. \]  

(11)

The requirement of detecting at least one photon on average finally yields the standard quantum limit for the input states’ minimal phase uncertainty

\[ n(N - 1)\text{Var}_{\text{SQL}} = 1 \Rightarrow \text{Var}_{\text{SQL}} = \frac{1}{(N - 1)n}. \]  

(12)

Imperfections in the phase-locking accuracy can be captured by introducing an accuracy factor \(\xi\), where the minimal value \(\xi = 1\) corresponds to the quantum limit: \(D = \xi \text{ Var}_{\text{SQL}} = \frac{\xi}{(N - 1)n}\).

The output noise can be divided into independent fluctuations along the in-phase quadrature \(\Delta x = \alpha \Delta \psi / \sqrt{2}\), corresponding to the amplitude direction, and the out-of-phase quadrature \(\Delta p = i\alpha \Delta \psi\). Using the expression for the quantum limited phase variance and considering the coherent state quadrature variance \(\text{Var}_{\text{coh}} = 1/4\) (see equation (17) in the appendix), the resulting noise variance along the quadratures yields
Here we refer to a noiseless amplifier combination of both pure and noisy input beams. The acceptable phase variance scales quadratically in the number of combined states. Hence, CBC can outperform the output noise variances of a linear amplifier irrespective of the phase locking accuracy, if only the number of combined signals is sufficiently large.

3. Comparison between the quantum limits in linear amplification and CBC

An amplifier can be defined as a device that takes an input signal, increases its power and outputs a signal that is a rescaled version of the input. The amplification gain $G$ is defined as the power ratio between the output and the input signal. In the context of linear phase-insensitive amplifiers, one is often confronted with the famous 3 dB limit [7]. In vague terms, the 3 dB limit states, that the amplification of a signal with high gain $G$ increases the field quadrature variances by a factor 2$G$. The exact equation for the output quadrature variance of a quantum limited linear amplifier is

$$
\text{Var}_{\text{out}}(p) = \text{Var}_{\text{coh}} + n \xi \text{Var}_{\text{SQL}} = \left(1 + \frac{4 \xi}{N \xi - 1}\right) \text{Var}_{\text{coh}},
$$

where $n = |\alpha|^2$. The standard quantum-limit for the output noise in CBC corresponds to $\xi = 1$. The magnitudes of the excess noise variance $\text{Var}^{\text{excess}}_{x,p}$, i.e. excluding the shot noise, are not symmetric along the quadratures but are dominated by phase noise $\text{Var}^{\text{excess}}_{\psi} = 2n(N-1)\text{Var}^{\text{excess}}_{\psi}$. For a large number of input states $N \gg 1$ both variances approach the shot noise level. In this section we discussed the minimum noise resulting from adding $N$ laser beams with fixed quantum noise limited amplitude and individually diffusing phases. In the following section we compare this power enhancement by coherent combining of many unknown laser fields, with the power enhancement by phase-insensitive amplification of a single input beam of quantum limited amplitude and phase noise.

The additional excess noise impedes many quantum optical applications that rely on quantum-limited noise properties and has profound implications e.g. on the maximal distance between repeater stations of optical communication channels [8]. There is the notion of noiseless amplification but it bears some ambiguity. Depending on the context, it may refer to an amplifier preserving the quantum state’s quadrature variances $\text{Var}_{\text{amp}} = \text{Var}_{\text{in}}$. Such an amplifier would apply a coherent displacement along the signal’s exact complex amplitude, which can only be realized if the phase of the input state is known a priori and is unphysical otherwise. Here we refer to a noiseless amplifier as one for which the signal’s quadrature variance scales equally to its mean photon number $\text{Var}_{\text{amp}} = G \text{Var}_{\text{in}}$, such that the noise figure does not change. The noise figure is the ratio of the signal-to-noise ratios $\text{SNR}_{\text{in}}$ and $\text{SNR}_{\text{out}}$. This amplifier model could be realized by a phase-sensitive amplifier, and is also approximately valid for the phase-insensitive amplification of input states with classical noise significantly exceeding the quantum noise limit (see appendix A.4 for details). Note, that the excess noise may be reduced if one succeeds in operating the amplifier close to the saturation regime [9].

Let us finally compare the quantum limits for the noise scaling of CBC and linear amplification in detail. The minimal noise variance of a quantum limited linear amplifier is phase symmetric and proportional to the gain factor $G$ as outlined in equation (14). The noise scaling in CBC is fundamentally different. In adding up multiple input beams, the phase fluctuations from the imperfect phase locking are progressively averaged out as sketched in figure 2.

In stark contrast to the linear amplifier, the output noise is not proportional to the gain factor $G = N$, but scales inversely with the number of combined states (see equation (13)). This statistical noise suppression results in a significantly reduced noise footprint compared to a linear amplifier. Let us analyze how large a relative phase fluctuation $\Delta^2 \psi$ can be tolerated in CBC before the output variance surpasses that of a quantum-limited amplifier $\text{Var}_{\text{out}}^{\text{amp}}(N)$. As the output noise is typically dominated by the fluctuations along the out-of-phase quadrature $p$, we choose the corresponding variance as the conservative reference for the comparison

$$
\text{Var}_{\text{SQL}}^{\text{coh}}(p) = \left(1 + \frac{4 \xi}{N \xi - 1}\right) \text{Var}_{\text{coh}} \Rightarrow \xi = \frac{1}{2}(N - 1)^2.
$$

The acceptable phase variance scales quadratically in the number of combined states. Hence, CBC can outperform the output noise variances of a linear amplifier irrespective of the phase locking accuracy, if only the number of combined signals is sufficiently large.

4. Conclusion

CBC is an enabling technology for the generation of brightest beams with excellent beam quality. Our analysis reveals that CBC also exhibits intriguing properties regarding the quantum limited noise scaling in the combination of both pure and noisy input beams.
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Appendix

A.1. Standard quantum limit for relative phase measurements

CBC is based on constructive interference, such that the noise profile of the combined beam depends crucially on the relative optical phases of the input beams. The precision of phase locking is fundamentally limited by the Heisenberg uncertainty principle

$$\Delta x \Delta p \geq \frac{1}{4}.$$  \hspace{1cm} (16)

For coherent states the quantum uncertainty is distributed symmetrically onto the quadrature axes

$$\Delta^2 x = \Delta^2 p = \text{Var}_{\text{coh}} = \frac{1}{4}.$$ \hspace{1cm} (17)

An overview on the quantum noise limits in optical interferometry can be found in [10–12]. The ultimate phase estimation precision compatible with the physical laws of quantum mechanics is given by the Heisenberg limit is $\Delta \psi_{\text{HL}} = 1/n$ [13] which, however, can only be achieved using exotic quantum states such as N00N states. The optimal precision for estimating the relative phase between coherent states is given by the standard quantum limit $\Delta \psi_{\text{SQL}} = 1/\sqrt{n}$ [14].

A CBC locking scheme satisfying the standard quantum limit for the relative phase accuracy can be realized using a single photon detector and is sketched in figure A1(a). Two coherent beams $|\alpha_1\rangle$ and $|\alpha_2\rangle$ interfere on a symmetric beam splitter and are combined in the output port with constructive interference Information about the relative phase of the coherent states can be extracted from the optical power exiting the beam splitter through the destructive interference port. Ideally, the amplitudes and the signal phases are equal such that the photon counter in this port detects the vacuum state. The error signal in the minus port is depicted in figure A1(b). The minimal phase shift $\Delta \psi$ that can be detected between the input beams is determined by the requirement that at least a single photon is detected

$$\langle \hat{n} (\Delta \psi) \rangle = |\alpha|^2 \langle (1 - \cos(\Delta \psi)) \rangle \approx n \left(1 - \left(1 - \frac{\psi^2}{2}\right)\right) \geq 1 \Rightarrow \Delta^2 \psi_{\text{min}} = \frac{2}{n},$$ \hspace{1cm} (18)

where we used the Taylor approximation for the cosine function as we assume small $\psi$. In this derivation one of the states is modeled as having a fixed phase and all phase fluctuations are attributed to the second state. This can straightforwardly be symmetrized by equally distributing the variance among the two inputs. This yields the standard quantum limit for the minimal detectable phase variance of each of the input states $\Delta \psi_{\text{SQL}} = 1/\sqrt{n}$.
A.2. Noise scaling in linear amplifiers

In the canonical quantization of the electromagnetic field [15], the photon annihilation operator \( \hat{a} \) is identified as the quantum counterpart to the classical amplitude variable \( \alpha \). This suggests the following operator description for the amplified signal \( \hat{a}_{\text{out}} = g \hat{a}_{\text{in}} \). Both, the input \( \hat{a}_{\text{in}} \) and the output \( \hat{a}_{\text{out}} \) refer to optical modes, which for a proper quantum transformation are connected by a unitary transformation. As the commutator bracket is invariant under unitary transformations a necessary condition for the amplifier output \( \hat{a}_{\text{out}} \) is that the canonical commutation relation is preserved \([\hat{a}_{\text{out}}, \hat{a}_{\text{out}}^\dagger] = 1\). From the naive approach, however, we obtain

\[
[\hat{a}^\dagger_{\text{out}}, \hat{a}^\dagger_{\text{in}}] = g [\hat{a}^\dagger_{\text{in}}, \hat{a}^\dagger_{\text{in}}^\dagger] g = g^2 = G, \tag{19}
\]

which violates the canonical commutation relation for any non-trivial gain \( g \neq 1 \). Yet, the commutation relation can be preserved by introducing an additional noise term \( \hat{N} \)

\[
[\hat{a}^\dagger_{\text{out}}, \hat{a}^\dagger_{\text{out}}] = [g \hat{a}^\dagger_{\text{in}} + \hat{N}, g \hat{a}^\dagger_{\text{in}} + \hat{N}] = g^2 [\hat{a}^\dagger_{\text{in}}, \hat{a}^\dagger_{\text{in}}^\dagger] + [\hat{N}, \hat{N}^\dagger] + g [\hat{N}, \hat{a}^\dagger_{\text{in}}] + g [\hat{a}^\dagger_{\text{in}}, \hat{N}^\dagger]. \tag{20}
\]

The amplitude operator of the input signal \( \hat{a}_{\text{in}} \) and the noise term \( \hat{N} \) act on different subsystems—namely the optical field mode and the internal modes of the amplifier, respectively. As the amplifier should not be biased towards a certain input signal, the optical mode \( \hat{a}_{\text{in}} \) and the internal modes of the amplifier \( \hat{N} \) cannot be correlated and must commute. Consequently, in order to preserve the bosonic commutation relation for \( \hat{a}_{\text{out}} \), the commutator bracket of the noise term must satisfy \([\hat{N}, \hat{N}^\dagger] = 1 - g^2\), which allows to express the noise term as an operator of the form

\[
\hat{N} = \sqrt{g^2 - 1} \hat{b}^\dagger + N_{cl} \tag{21}
\]

\( \hat{b} \) denotes a bosonic ancilla mode with vanishing mean amplitude \( \langle \hat{b} \rangle = 0 \). Note, that the commutator equation could in principle also be satisfied if we set \( \hat{N} = \sqrt{1 - g^2} \hat{b} \). For an amplifier, however, we require \( g^2 > 1 \), such that \( [\sqrt{1 - g^2} \hat{b}, (\sqrt{1 - g^2}) \hat{b}] = [\sqrt{1 - g^2} \hat{b}, \hat{b}] = 1 - g^2 \) cannot be satisfied. The additional \( \epsilon \)-number \( N_{cl} \) can be introduced to account for any classical imperfections of the amplifier. For amplifiers working at the quantum limit \( N_{cl} = 0 \). The fundamental input-output relation for the quantum limited phase-insensitive linear amplifier follows as

\[
\hat{a}^\dagger_{\text{out}} = g \hat{a}^\dagger_{\text{in}} + \sqrt{g^2 - 1} \hat{b}^\dagger. \tag{22}
\]

In order to quantify the performance of the amplifier, we calculate the output variance for a coherent input state \( |\alpha\rangle \). Coherent states are symmetric minimum uncertainty states with quadrature uncertainties \( \text{Var}(x) = \text{Var}(p) = 1/4 \). In a phase-insensitive linear amplifier both quadrature variances are increased equally. Therefore, we can restrict the analysis to the variance of the X quadrature \( \text{Var}(x) = \langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2 \)

\[
\text{Var}(x_{\text{out}}) = \left( \frac{\langle \hat{a}^\dagger_{\text{out}} + \hat{a}^\dagger_{\text{out}}^\dagger \rangle}{2} \right)^2 - \left( \frac{\langle \hat{a}^\dagger_{\text{out}} + \hat{a}^\dagger_{\text{out}}^\dagger \rangle}{2} \right)^2. \tag{23}
\]

Applying equation (22), the output variance can be decomposed into three terms \( \text{Var}(x_{\text{out}}) = \Gamma_{\text{sig}}(g) + \Gamma_{\text{noise}}(g) + \Gamma_{\text{cross}}(g) \). The first term describes the rescaling of the input signal’s variance.
\begin{equation}
\Gamma_{\text{t}+\text{t}}(g) = \left( g \frac{\hat{a}_{\text{in}} + \hat{a}_{\text{in}}^\dagger}{2} \right)^2 - \left( g \frac{\hat{a}_{\text{in}} + \hat{a}_{\text{in}}^\dagger}{2} \right)^2 = g^2 \langle \hat{X}_{\text{fin}}^2 \rangle - g^2 \langle \hat{X}_{\text{fin}} \rangle^2 = g^2 \cdot \text{Var}(x_{\text{tin}}). \tag{24}
\end{equation}

Note, that this term is solely dependent on the signal mode $a$ and describes the linear rescaling of both the signal power and the noise with equal gain $g^2$. Even though the modes of the amplifier $b$ are not represented in this term, the variance is already increased. However, in the absence of the additional noise term, the SNR is still preserved. The second term is responsible for the reduction of the SNR. Assuming that the internal mode of the amplifier is initially in the vacuum state and noting that the quadrature uncertainty of the vacuum state is identical to that of a generic coherent state we obtain

\begin{equation}
\Gamma_{\text{noise}}(g) = \left( \sqrt{g^2 - 1} \frac{\hat{b}^\dagger + \hat{b}}{2} \right)^2 - \left( \sqrt{g^2 - 1} \frac{\hat{b}^\dagger + \hat{b}}{2} \right)^2 = (g^2 - 1) \text{Var}(x_{\text{tin}}) = (g^2 - 1) \text{Var}(x_{\text{vac}}). \tag{25}
\end{equation}

The last term is a cross-term between the signal operator and the noise operator which vanishes as we require $\langle \hat{b} \rangle = 0$

\begin{equation}
\Gamma_{\text{cross}}(g) = g \sqrt{g^2 - 1} \left( \langle \hat{a}_{\text{in}} + \hat{a}_{\text{in}}^\dagger \rangle \langle \hat{b}^\dagger + \hat{b} \rangle \right)_{\langle \hat{a}_{\text{in}} \rangle = 0} = 0. \tag{26}
\end{equation}

Note, that the expectation values for the signal and the noise term in the product could be evaluated independently as they are required to be uncorrelated. In sum we arrive at a lower limit for the output variance of the field quadratures

\begin{equation}
\text{Var}(x_{\text{out}}) \geq (2g^2 - 1)\text{Var}(x_{\text{in}}). \tag{27}
\end{equation}

Let us try to get some intuition on the factor $2g^2 - 1$. A simple, conventional amplifier can be composed of a quantum receiver performing a measurement on the coherent input state $|\alpha\rangle$ to derive an estimate $\hat{a}$ of the complex amplitude. Subsequently, a signal source prepares a rescaled coherent state $|g\hat{a}\rangle$ according to the obtained estimate. The origin of the excess noise in this example is founded in the impossibility to precisely determine the complex amplitude $\alpha$. The measurement can, for instance, be performed with a heterodyne detector or a double homodyne detector which both measure the conjugate field quadratures simultaneously. Such measurements come at the price of adding at least one unit of vacuum noise to the measurement outcome [16, 17]. Consequently, the variance of the estimates’ distribution is at least twice the quadrature variance of the coherent state $\hat{V} \geq 2\text{Var}_{\text{coh}}$. Applying the amplification factor to this distribution linearly increases its variance as $g^2 \hat{V} \geq g^2 \text{Var}_{\text{coh}}$. Preparing a coherent state at the phase space coordinate according to the amplified estimator finally adds another unit of shot noise $\text{Var}_{\text{coh}}$ to the distribution, i.e. the Heisenberg uncertainty of the prepared quantum state itself. In sum this yields the output noise of equation (27).

In contrast, the output noise of a quantum-limited amplifier is only $(2g^2 - 1)$-times as high as the coherent input noise. In the regime of low gain, a quantum-limited amplifier offers an appreciable performance gain. However, for high gain both the quantum-limited amplifier and the measure&prepare amplifier asymptotically approach the 3 dB limit.

It is worth noting that the aforementioned noise limit only applies to deterministic and phase-insensitive amplifiers, which successfully amplify each signal state and which operate independent of the signal phase. While these properties indeed apply to the notion of a conventional amplifier, alternative concepts such as phase-sensitive amplifiers and probabilistic amplifiers may offer advantageous noise properties (depending on the targeted application) and have also been demonstrated experimentally. Phase-sensitive amplifiers [18, 19] exhibit a phase dependent gain profile. In its simplest form a phase-sensitive amplifier provides (positive) gain only along one preferred quadrature, while the conjugate quadrature gets squeezed and is attenuated. Such an amplifier can be useful if, for example, the set of input states is aligned only along a single quadrature. An important example is the amplification of states from the binary phase-shift keyed alphabet [20]. Probabilistic amplification has been demonstrated to allow for noiseless amplification of coherent states in various experimental settings [21–24]. All these schemes achieve noiseless amplification via a probabilistic nonlinear interaction and measurement that allows to herald successfully amplified signals. In probabilistic amplifiers the purity of the state is traded off against the success probability.

### A.3. Error signals in the Fourier CBC scheme

In this section we provide a detailed calculation of the error signal amplitudes $\epsilon_l$ at the output of the inverse Fourier transformation. The separation of the coherently combined signal after the initial Fourier transformation as captured by the term $(1 - \delta_{k,0})$ at the end of the first line in equation (28)
The bosonic Heisenberg uncertainty relation for bosonic modes which remains unaltered by the amplification process.

\[ \alpha_i = \sum_{n=k}^{N-1} \alpha_n e^{\frac{\pi}{N} k n} \left( 1 - \delta_{k,0} \right) = \sum_{n=k}^{N-1} \alpha_n \left[ \sum_{k=0}^{N-1} e^{\frac{\pi}{N} k (n-\ell)} - 1 \right] \]

Hence the error signal amplitude at each output port \( \alpha_i \) is exactly the difference between the input amplitude \( \alpha_i \) and the instantaneous mean amplitude \( \bar{\alpha} \).

### A.4. Quantum-limited linear amplification by multiple amplifier stages

It is a common misconception that high gain linear amplification is unconditionally linked to a 3 dB reduction of the SNR. However, this noise penalty only applies to the amplification of pure signals (in the quantum mechanical sense). Let us shed some light on this by considering the amplification of a coherent state \( |\alpha\rangle \) by either a single amplifier with intensity gain \( G \) or alternatively by two consecutive amplifiers with gain \( \sqrt{G} \) each (see figure A2). Clearly, the mean amplitude of the amplified signals is identical in both cases. One could, however, assume that due to the 3 dB noise penalty of linear amplifiers the latter scheme suffers from double the noise power as two individual amplification steps were invoked.

For the single amplifier, the variance of the output state scales with the factor \((2G - 1)\) which yields the famous 3 dB noise penalty for \( G \gg 1 \). At this point, it is crucial to understand how the scaling factor comes about. In fact, the usual representation obscures the physics behind the amplification process, such that it is helpful to rewrite the factor as

\[ \text{Var}_{\text{amp1}} = (2G - 1) \text{Var}_{\text{coh}} = (1 + 2G - 1) \text{Var}_{\text{coh}}. \]  

In this form the scaling of the output state’s variance is divided in two parts. The initial ’1’ represents the variance of the symmetric Heisenberg uncertainty relation for bosonic modes which remains unaltered by the amplification process. The second term \(2(G - 1)\) describes the additional variance due to the scaling of the input variance in the amplification process. The factor of 2 is due to the simultaneous amplification of the two conjugate quadratures \( X \) and \( P \) with non-vanishing commutator \([X, P] = i\). In an illustrative picture, the amplifier performs a measurement on the input state and subsequently prepares an amplified version in the amplification process. The measurement on the non-commuting observables results in the 3 dB penalty, as e.g. characteristic in heterodyne detection.

Clearly, the same transformation also applies in the multi-stage amplifier to the state after the first amplifier, the only difference being the reduced gain of \( \sqrt{G} \). At the second amplifier stage, it is crucial to distinguish between the classical and the quantum mechanical components of the input variance. The bosonic Heisenberg uncertainty component experiences the identical transformation as in the first amplifier

\[ 1 \cdot \text{Var}_{\text{coh}} \rightarrow (1 + 2(G - 1)) \text{Var}_{\text{coh}}, \]

while the classical excess noise from the previous amplification stage does not suffer from the quantum penalty and hence just scales with the classical gain factor \( \sqrt{G} \): \n
\[ (2(\sqrt{G} - 1)) \text{Var}_{\text{coh}} \rightarrow \sqrt{G} (2(\sqrt{G} - 1)) \text{Var}_{\text{coh}} = 2G - 1. \]

Adding up the individual contributions yields the characteristic factor of any quantum-limited linear amplification \((2G - 1)\) proving that the SNR in multi-stage amplifiers is indeed equivalent to just a single amplifier. As a consequence, consecutive high gain amplifications still yields only a 3 dB SNR reduction compared to the input state Note, however, that larger degradation of the SNR occurs if the signal is subject to losses in between the amplifier stages.

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References

[1] Fan T Y 2005 Laser beam combining for high-power, high-radiance sources IEEE J. Sel. Top. Quantum Electron. 11 567–77
[2] Leger J R, Swanson G J and Veldkamp W B 1987 Coherent laser addition using binary phase gratings Appl. Opt. 26 4391
[3] Uberta R, Bratcher A and Tiemann B G 2010 Coherent polarization beam combination IEEE J. Quantum Electron. 46 1191–6
[4] Eidam T, Wirth C, Jauregui C, Stutzki F, Jansen F, Otto H-J, Schmidt O, Schreiber T, Limpert J and Tünnermann A 2011 Experimental observations of the threshold-like onset of mode instabilities in high power fiber amplifiers Opt. Express 19 13218–24
[5] Gaida C, Kienel M, Müller M, Klenke A, Gebhardt M, Stutzki F, Jauregui C, Limpert J and Tünnermann A 2015 Coherent combination of two tm-doped fiber amplifiers Opt. Lett. 40 2301–4
[6] Braunstein S L, CerF N J, Iblisdir S, van Loock P and Massar S 2001 Optimal cloning of coherent states with a linear amplifier and beam splitters Phys. Rev. Lett. 86 4938–41
[7] Caves C M 1982 Quantum limits on noise in linear amplifiers Phys. Rev. D 26 1817–39
[8] Essiambre R and Tkach R W 2012 Capacity trends and limits of optical communication networks Proc. IEEE 100 1035–55
[9] Saraf S, Urbanek K, Byer R L and King P J 2005 Quantum noise measurements in a continuous-wave laser-diode-pumped Nd:YAG saturated amplifier Opt. Lett. 30 1195
[10] Berry D W, Higgins B L, Bartlett S D, Mitchell M W, Pryde G J and Wiseman H M 2009 How to perform the most accurate possible phase measurements Phys. Rev. A 80 052114
[11] Demkovicz-Dobrzański R, Jarzyna M and Kolodyński J 2015 Quantum limits in optical interferometry Progress in Optics vol 60 (Amsterdam: Elsevier) pp 345–435
[12] Dowling J P and Seshadreesan K P 2015 Quantum optical technologies for metrology, sensing, and imaging J. Lightwave Technol. 33 2359–70
[13] Holland M J and Burnett K 1993 Interferometric detection of optical phase shifts at the heisenberg limit Phys. Rev. Lett. 71 1355–8
[14] Caves C M 1981 Quantum-mechanical noise in an interferometer Phys. Rev. D 23 1693–708
[15] Leonhardt U 1997 Measuring the Quantum State of Light (Cambridge: Cambridge University Press)
[16] Arthurs E and Kelly J I 1965 B.S.T.J. briefs: on the simultaneous measurement of a pair of conjugate observables Bell Syst. Tech. J. 44 725–9
[17] Stenholm S 1992 Simultaneous measurement of conjugate variables Ann. Phys. 218 254
[18] Levenson J A, Abram L, Rivera T and Grangier P 1993 Reduction of quantum noise in optical parametric amplification J. Opt. Soc. Am. B 10 2233–8
[19] McKinstrie C J and Radic S 2004 Phase-sensitive amplification in a fiber Opt. Express 12 4973–9
[20] Corcoran B, Olsson S L, Lundstrom C, Karlsson M and Andrekson P 2012 Phase-sensitive optical pre-amplifier implemented in an 80 km dgsuk link National Fiber Optic Engineers Conf. (Optical Society of America) p PDP5A.4
[21] Usuga M A, Müller C R, Wittmann C, Marek P, Filip R, Marquardt C, Leuchs G and Anderssen U L 2010 Noise-powered probabilistic concentration of phase information Nat. Phys. 6 767
[22] Zavatta A, Fiurasek J and Bellini M 2010 A high-fidelity noiseless amplifier for quantum light states Nat. Photon. 5 352
[23] Ferreyrol F, Barbieri M, Blandino R, Fossier S, Tualle-Brouri R and Grangier P 2010 Implementation of a nondeterministic optical noiseless amplifier Phys. Rev. Lett. 104 123603
[24] Xiang G Y, Ralph T C, Lund A P, Walk N and Pryde G J 2010 Heralded noiseless linear amplification and distillation of entanglement Nat. Photon. 4 516
[25] Müller C R, Pfeintinger C, Dirmeier T, Khan I, Vogl U, Marquardt C, Leuchs G, Sánchez-Soto I L, Teo Y S, Hradil Z and Řeháček J 2016 Evading vacuum noise: Wigner projections or husimi samples Phys. Rev. Lett. 117 070801