Escape Dynamics of Many Hard Disks

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Many-particle effects in escapes of hard disks from a square box via a hole are discussed in a viewpoint of dynamical systems. Starting from $N$ disks in the box at the initial time, we calculate the probability $P_n(t)$ for at least $n$ disks to remain inside the box at time $t$ for $n = 1, 2, \ldots, N$. At early times the probabilities $P_n(t)$, $n = 2, 3, \ldots, N - 1$ are described by superpositions of exponential decay functions. On the other hand, after a long time the probability $P_n(t)$ decays in power $\sim t^{-2n}$ for $n \neq 1$, in contrast to the fact that it decays in a different power $\sim t^{-n}$ for the case without any disk-disk collision. Chaotic and non-chaotic properties of the escape systems are discussed by the dynamics of finite time largest Lyapunov exponents, whose decay properties are related with those of the probability $P_n(t)$.

I. INTRODUCTION

The escape of materials from a finite area is known as an essential concept to understand many physical features in a variety of natural phenomena. It describes a wide scale of physical phenomena from a microscopic scale (e.g. $\alpha$ decay of nuclei [12] and light emissions from molecules [3, 5]) to a macroscopic one (e.g. ejections or evaporation of stars in cosmology [6, 7]). It also plays an important role to analyze properties of materials, for example, by means of particle escapes from a quantum dot [8–10] or from an optical potential trap [11–13]. Escape basins of magnetic field lines in plasma [14–16] and transition of states in chemical reactions (as an escape of excited chemical species from a reactant region) [17–18], etc. The concept of escape is also used to calculate transport coefficients in chaotic dynamical systems [19–21], and as a mechanism to produce electric currents as an escape of electrons from a particle reservoir [22–24].

Escapes occur when particles reach a specific region like an escape of holes in a reactant region) [16–18], etc. The outline of this paper is as follows. In Sec. II we introduce our model consisting of many hard disks in a square box with a hole, and discuss escape phenomena, clarifying effects of a finite size of holes [32, 33], weakness of chaos [27], specific orbits causing a power decay [33, 34], etc.

The principal aim of this paper is to discuss many-particle effects in dynamical properties of escape phenomena. As a system consisting of many particles, we use many hard disks in a square box, which have been widely used to investigate statistical and dynamical properties [28, 29]. The FTLLE converges to the finite size largest Lyapunov exponent (FTLLE) $\lambda(t)$ [44, 45], which is introduced as an exponential rate of expansion or contraction of an infinitesimally small initial error at a finite time $t$. The FTLLE converges to the well-known largest Lyapunov exponent in the long time limit, whose positivity means dynamics of the system to be chaotic. We compare quantitatively decay properties of the survival probability $P_n(t)$ and the corresponding FTLLE $\lambda(t)$, and clarify roles of chaotic or non-chaotic dynamics in escape phenomena of many hard disks. Dependences of the survival probabilities and the FTLLEs on the system length and the hole size, etc., are also discussed as scaling properties.

The occurrence of exit events is a universal property in many physical systems, such as the survival probability $\sim t^{-n}$ as the probability for a particle to remain in the initially confined finite area, the escape time $\sim n!$ as the time period for a particle to stay in the finite area, and a velocity when a particle escapes from the finite area $\sim n!$, etc. These quantities decay in time as a feature of escape phenomena in which the materials continue escaping from the initially confined area. Many works have already done to clarify their decay properties in escapes of a single particle by using dynamical theories. It is conjectured, for example, that the survival probability decays exponentially in time for chaotic systems based on an ergodic argument, while it decays in power for non-chaotic systems [30]. This conjecture led to further dynamical studies of escape phenomena, clarifying effects of a finite size of holes [32, 33], weakness of chaos [27], specific orbits causing a power decay [33, 34], etc.
particle system. In Sec. III we discuss decay properties of FTLLEs in escape systems of many hard disks, and investigate connections between properties of the survival probabilities and the FTLLEs. Finally, we give conclusion and remarks on the contents of this paper in Sec. IV.

II. ESCAPE PROPERTIES OF MANY-HARD-DISK SYSTEMS

A. Many hard disks in a square box with a hole

We consider the system consisting of many hard disks inside a square box with a hole. Here, the mass and the radius of the disks are \( m \) and \( r \), respectively, and the length of each side of the box is \( L \), and a hole in the box is put as the region of the length \( r + (h/2) \) in both the sides from a single corner of the box, as shown in Fig. 1 as a schematic illustration. Here, the length \( h/2 \) is the effective length of each side of the hole, where centers of disks can reach. In this system, each disk moves with a velocity, then collides elastically with other disks or walls, or leaves the box via the hole. We assume that any disk does not enter into the box via the hole from its outside, and any disk is removed from the box when it reaches the hole. As an important system parameter we use the particle density \( \rho \equiv N \pi r^2 / L^2 \) of the system at the initial time \( t = 0 \).

FIG. 1: Many hard disks with the radius \( r \) in a square box with length \( L \) of each side. The box has a hole with each length \( r + (h/2) \) of both the sides from a corner of the box. The dotted lines in inner sides of the box are for positions of the center of disks closest to walls of the box.

B. Survival probabilities of many-hard-disk systems

To characterize escape behaviors of \( N \) disks from a square box via a hole, we introduce the probabilities \( P_n(t) \), for which at least \( n \) disks remain inside the box at time \( t \) \( (n = 1, 2, \ldots, N) \). We call this probability \( P_n(t) \) "the \( n \)-particle survival probability", or simply the survival probability, as a generalization of the well-known survival probability discussed in one-particle escape systems [30, 34, 35]. For almost arbitrary initial conditions, except for some specific cases, for example, that disks move in a periodic orbit without reaching the hole, the \( n \)-particle survival probability goes to zero, i.e. \( \lim_{t \to +\infty} P_n(t) = 0 \), in the long time limit \( t \to +\infty \), and escape properties of many-particle systems are characterized by decay properties of \( P_n(t) \).

In Fig. 2 we show graphs of the \( n \)-particle survival probabilities \( P_n(t) \) of \( N \) hard disks as functions of time \( t \) for \( n = 5 \) (the solid line), 4 (the dashed line), 3 (the dotted line), 2 (the dash-dotted line), 1 (the dash-double-dotted line) with \( N = 5 \). Here, we used values of the parameters as \( m = 1 \), \( r = 1/2 \), \( h = 0.1(L - 2r) \) and \( \rho = 0.1 \) (so \( L \approx 6.27 \)). For the survival probabilities shown in Fig. 2 we calculated \( 5 \times 10^8 \) number of ensembles from random initial conditions at \( t = 0 \) in which the disk positions and momenta are distributed into the microcanonical distribution of the system without the hole with the value \( N \) of energy \( E \).

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a long time. We fitted this power decay of $P_1(t)$ to a power function $\alpha_1 t^{-1}$ with the value $\alpha_1 = 86.6$ of the fitting parameter $\alpha_1$. This result is simply explained by the fact that only a single disk exists in the majority of times in the decay of $P_1(t)$ and a single disk in the square box is not chaotic, leading to the power decay $\sim t^{-1}$ of the survival probability [30]. On the other hand, as shown in the inset of Fig. 2 as linear-log plots of $P_n(t)$, the $N$-particle survival probability $P_N(t)$ decays exponentially in time. To clarify this property, we fitted the graph of $P_N(t)$ to an exponential function $\exp(-\alpha_2 t)$ with the value $\alpha_2 = 3.65 \times 10^{-2}$ of the fitting parameter $\alpha_2$, which is almost indistinguishable with the graph of $P_N(t)$ in the inset of Fig. 2. It should be noted that an exponential decay of the survival probability also appears in one-particle chaotic systems [30].

C. Exponential decays of survival probabilities

Now, we proceed to discuss decay properties of the $n$-particle survival probabilities $P_n(t)$ for the middle numbers $n = 2, 3, \cdots, N - 1$. The inset of Fig. 2 suggests that different from the graph of $P_N(t)$, the graphs of $P_n(t)$ for $n = 2, 3, \cdots, N - 1$ do not seem to show simple exponential decays at early times, although $n$ disks for $n = 2, 3, \cdots, N - 1$ can involve disk-disk collisions making the system to be chaotic.

In order to describe decay behaviors of the survival probabilities $P_n(t)$, $n = 2, 3, \cdots, N - 1$ at early times, we introduce the probability density $f_k(\tau)$ of the time $\tau$ for the $k$-th disk escape to occur (i.e. the escape time $\tau$ of $k$ disks), for $k = 1, 2, \cdots, N$. Using the escape-time probability density $f_k(\tau)$ of $k$ disks, the $(N - k + 1)$-particle survival probability $P_{N-k+1}(t)$ for $N - k + 1$ disks to remain inside the box at time $t$ is represented as

$$P_{N-k+1}(t) = 1 - \int_0^t d\tau f_k(\tau), \quad (1)$$

so that we obtain

$$f_k(\tau) = -\frac{dP_{N-k+1}(\tau)}{d\tau}, \quad (2)$$

$k = 1, 2, \cdots, N$. By Eq. (2), for example, if the $N$-particle survival probability $P_N(t)$ decays exponentially in time, i.e. $P_N(t) = \exp(-at)$ with a positive constant $a$ as shown in Fig. 2, then we obtain the escape-time probability density $f_1(\tau) = f_1(\tau; a)$ for the first escaping disk, in which $f_1(\tau; a)$ is given by

$$f_1(\tau; a) = ae^{-a\tau} \quad (3)$$

for $N > 1$. Equation (1) or (2) also leads to the normalization condition $\int_0^\infty d\tau f_k(\tau) = 1$ of the escape-time probability density $f_k(\tau)$ as far as the survival probability $P_{N-k+1}(t)$ goes to zero in the long time limit, i.e. $\lim_{t \to \infty} P_{N-k+1}(t) = 0$, noting the initial condition $P_{N-k+1}(0) = 1$ of the survival probability.

The escape time $\tau$ of $k$ disks is represented as the sum $\tau = \sum_{j=1}^k \tau_j$ of the time-interval $\tau_j \equiv t_j - t_{j-1}$ from the time $t_{j-1}$ of the $(j-1)$-th escape to the time $t_j$ of $j$-th escape of the $k$ disks, defining $t_0 \equiv 0$. For the case that the dynamics of $k$ disks inside the box is chaotic and the probability for each disk to stay inside the box decays exponentially in time, we assume that the probability density of the time $\tau_k$ for $k$-th escaping disk is independent of other times $\tau_j$, $j = k-1, k-2, \cdots, 1$, and the probability density $f_k(\tau)$ of the escape time $\tau$ satisfies the recurrence relation $f_k(\tau) = \int_0^\tau ds f_1(\tau - s)f_{k-1}(s)$ with $f_1(\tau_k) = f_1(\tau_k; a_k)$ and a positive constant $a_k$. Under this assumption, we obtain the probability density $f_k(\tau) = \tilde{f}_k(\tau)$ of the escape time $\tau$ of the $k$ disks with $k \geq 2$, in which $\tilde{f}_k(\tau)$ is given by

$$\tilde{f}_k(\tau) = \int_0^\tau dt_k \tilde{f}_1(\tau - t_k; a_k) \int_0^{t_k} dt_{k-1} \tilde{f}_1(t_k - t_{k-1}; a_{k-1}) \times \int_0^{t_{k-1}} dt_{k-2} \tilde{f}_1(t_{k-1} - t_{k-2}; a_{k-2}) \cdots \int_0^{t_2} dt_2 \tilde{f}_1(t_3 - t_2; a_2) \tilde{f}_1(t_2; a_1) \quad (4)$$

$$= \left( \prod_{j=1}^k a_j \right) \sum_{\{a_j\}} e^{-a_j \tau} \quad (5)$$

in which we assumed the condition $a_j \neq a_k$ for $j \neq k$. From Eqs. (1) and (5) we derive the $(N - k + 1)$-particle survival probability $P_{N-k+1}(t) = \tilde{P}_{N-k+1}(t)$, in which $\tilde{P}_{N-k+1}(t)$ is given by

$$\tilde{P}_{N-k+1}(t) = \sum_{\{a_j\}} \frac{a_1 a_2 \cdots a_{j-1} a_{j+1} a_{j+2} \cdots a_k e^{-a_j t}}{(a_1 - a_j)(a_2 - a_j) \cdots (a_{j-1} - a_j)(a_{j+1} - a_j)(a_{j+2} - a_j) \cdots (a_k - a_j)} \quad (6)$$

for $k = 2, \cdots, N - 1$. The derivation of Eq. (6) from Eqs. (5) and (1), as well as the derivation of Eq. (6) from Eqs.
systems at early times. Here, the system parameters used to obtain these data, other than those shown in Table 1 and the ensemble number $2 \times 10^6$ only for the case of $(\rho, h/(L - 2r)) = (10^{-1}, 10^{-3})$, are the same as those of the system whose survival probabilities are shown in Fig. 2. The factor $1/\bar{a}$ used for the quantities $a_j/\bar{a}$ in this table, comes from the fact that the exponential decay rate of the survival probability in escapes of a single chaotic point particle via a small hole in a two-dimensional space is proportional to $\hbar v_0/S$ with the hole size $h$, the particle speed $v_0$, and the area $S$ for the point particle to move before escaping \cite{30}. Table I suggests that the quantity $a_j/\bar{a}$ for each value of $j$ takes similar values for a wide variety of initial particle densities $\rho$ and hole size ratios $h/(L - 2r)$ like in escapes of a single particle, although the value of $a_j/\bar{a}$ decreases as the index $j$ increases.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
$\rho$ & $L - 2r$ & $h/(L - 2r)$ & $\bar{a}$ & $a_1/\bar{a}$ & $a_2/\bar{a}$ & $a_3/\bar{a}$ \\
\hline
$10^{-1}$ & 5.27 & $10^{-1}$ & $2.69 \times 10^{-2}$ & 1.36 & 0.933 & 0.607 & 0.358 \\
$10^{-2}$ & 18.8 & $10^{-1}$ & $7.52 \times 10^{-3}$ & 1.08 & 0.802 & 0.565 & 0.357 \\
$10^{-3}$ & 61.7 & $10^{-1}$ & $2.29 \times 10^{-3}$ & 1.08 & 0.806 & 0.578 & 0.355 \\
$10^{-1}$ & 5.27 & $10^{-2}$ & $2.69 \times 10^{-3}$ & 1.37 & 0.952 & 0.632 & 0.369 \\
$10^{-1}$ & 5.27 & $10^{-3}$ & $2.69 \times 10^{-4}$ & 1.38 & 0.956 & 0.640 & 0.372 \\
\hline
\end{tabular}
\caption{Decay rates $a_j, j = 1, 2, 3, 4$, divided by $\bar{a} = \hbar v_0/(L - 2r)^2$, for a 5-disk system with various values of the initial particle density $\rho$ and the hole size ratio $h/(L - 2r)$.}
\end{table}

D. Disk-disk collisions and power decays of survival probabilities

Now we proceed to discuss the effects of disk-disk collisions in decays of the $n$-particle survival probability $P_n(t)$ after a long time.

For comparisons in the discussions on effects of disk-disk collisions, we calculate survival probabilities of another type of many-disk systems in which disks can overlap with each other. (This type of disk systems without any disk-disk collision may be regarded as the ones consisting of $N$ point-particles in the square box which has the side length $L - 2r$ and the hole in the region of the length $r$ in both the sides from a single corner of the box.) Figure 4 is the graphs of the $n$-particle survival probabilities $P_n(t)$ of such 5 disks without their collisions for $n_0 = 5$ (the solid line), 4 (the dashed line), 3 (the dotted line), 2 (the dash-dotted line) and 1 (the dash-double-dotted line), as well as the $n$-particle survival probabilities $P_n(t)$ of 5 disks with their collisions for $n_0 = 5$ (the circles), 4 (the triangles), 3 (the squares), 2 (the diamonds) and 1 (the inverted triangles). Here, the survival probabilities $P_n(t)$ are the same as in Fig. 2 and all values of the system parameters for the survival probabilities $P_n'(t)$ are the same as those for $P_n(t)$ except for the conditions in which disks can overlap without any impact in their time-evolutions and the initial velocity distribution of each particle is given by a uni-
It is shown in Fig. 4 that at some early times the survival probabilities $P_{n_0}(t)$, $n_0 = 4, 3, \cdots, 1$ of the system without any disk-disk collision decay faster than the corresponding survival probabilities $P_{n_c}(t)$, $n_c = 4, 3, \cdots, 1$ of the system with disk-disk collisions. In contrast, after a long time the survival probabilities $P_{n_c}(t)$ decay slower than the corresponding survival probabilities $P_{n_0}(t)$. We can also recognize in Fig. 4 that after a long time the survival probabilities $P_{n_0}(t)$ decays in power $\sim t^{-1}$, similar to the survival probability $P_1(t)$.

It is important to note in Fig. 4 that unlike the survival probability $P_1(t)$, decays of the survival probabilities $P_{n_0}(t)$, $n_c = 2, 3, \cdots, N - 1$ show essential differences from those of the corresponding survival probabilities $P_{n_c}(t)$, $n_0 = 2, 3, \cdots, N - 1$ after a long time. However, it is rather unclear to discuss power decays of these survival probabilities $P_{n_c}(t)$ after a long time by their fittings to power functions in Fig. 4. To clarify this point quantitatively, we introduce the function $F_n(t)$ defined by

$$F_n(t) \equiv \frac{d \ln P_n(t)}{d \ln t} = \frac{t}{P_n(t)} \frac{d P_n(t)}{dt}, \quad (7)$$

which takes a constant value if the $n$-particle survival probability $P_n(t)$ decays in power $\sim t^{-1}$. (It may be noted that if the survival probability $P_n(t)$ decays exponentially in time as $P_n(t) = \exp(-at)$ with a positive constant $a$, then the function $F_1(t)$ becomes $-at$, leading to the initial value $F_n(0) = 0$ of the function $F_n(t)$.) In Fig. 5(a) we show the slopes $F_n(t)$ of the quantities $\ln P_n(t)$, and (b) the slopes $F'_n(t)$ of the quantities $\ln P'_n(t)$, as functions of $\ln t$ for $n = 5$ (the solid line), 4 (the dashed line), 3 (the dotted line), 2 (the dash-dotted line) and 1 (the dash-double-dotted line) for 5-disk systems. The open circles in (a) are the points for the slope $F_n(t)$ to cross the line $-(2n - 1)$ for $n = 4, 3$ and 2 and the line $-1/2$ for $n = 1$.

Figure 5 shows that both the 1-particle survival probabilities $P_1(t)$ and $P'_1(t)$ decay in power $\sim t^{-1}$ after a long time, but power decay properties of other $n$-particle survival probabilities $P_n(t)$ and $P'_n(t)$ for $n = 2, 3, \cdots, N - 1$ after a long time are different with each other. For the case with disk-disk collisions, Fig. 5(a) shows that the survival probabilities $P_2(t)$ and $P_3(t)$ decay in power

$\cdots$
\[ P_n(t) \xrightarrow{t \to \infty} \begin{cases} \eta_1 t^{-1} & \text{for } n = 1 \\ \eta_n t^{-2n} & \text{for } n = 2, 3, \cdots \end{cases} \tag{8} \]

with constants \( \eta_n, n = 1, 2, \cdots \), although the graph of \( F_4(t) \) in Fig. 5a does not show clearly such an asymptotic power decay of the survival probability \( P_4(t) \) yet. In contrast, for the case without any disk-disk collision, Fig. 5b shows that the survival probabilities \( P'_n(t) \), \( P''_n(t) \) and \( P''_n(t) \) decays in power \( \sim t^{-4} \), \( \sim t^{-3} \) and \( \sim t^{-4} \), respectively, suggesting their asymptotic decays as

\[ P'_n(t) \xrightarrow{t \to \infty} \eta'_n t^{-n} \quad \text{for } n = 1, 2, \cdots \tag{9} \]

with constants \( \eta'_n, n = 1, 2, \cdots \). The difference between Eqs. (8) and (9) would be regarded as an important effect of disk-disk collisions in decays of the \( n \)-particle survival probabilities.

III. FINITE-TIME LYAPUNOV EXPONENTS OF ESCAPE SYSTEMS

A. Decays of finite-time Lyapunov exponents

The system considered in this paper is chaotic, as far as two colliding hard disks exist inside the box. In order to characterize chaotic dynamics of the system with a dynamical instability, we introduce the finite-time largest Lyapunov exponent (FTLLE) \( \lambda(t) \) at time \( t \), which is defined by

\[
\lambda(t) = \lim_{|\delta \Gamma(0)| \to 0} \frac{1}{t} \ln \frac{|\delta \Gamma(t)|}{|\delta \Gamma(0)|}. \tag{10}
\]

Here, \( \delta \Gamma(t) \) is a small deviation of the phase space vector (consisting of the position vector and the momentum vector) of the hard disks existing inside the box at time \( t \), and the dimension of the vector \( \delta \Gamma(t) \) reduces by four at every time when a disk escapes from the hole. The well-known largest Lyapunov exponent of the system in the non-escape case with \( h = 0 \) is given by \( \lim_{t \to +\infty} \lambda(t) \).

On the other hand, in the case for disks to escape with \( h > 0 \), even if the system is chaotic initially, then we have \( \lim_{t \to +\infty} \lambda(t) = 0 \) for almost any initial condition for which no disk (or only one disk) exists inside the box in the long time limit \( t \to +\infty \). Therefore, we characterize chaotic or non-chaotic properties of the system by a decay behavior of the FTLLE \( \lambda(t) \) at time \( t \).

In Fig. 6 we show an example of time-dependences of the FTLLE \( \lambda(t) \) as circles, as well as the times \( t_n \) for the \( n \)-th escape of disks from the box as a vertical line (\( n = 1, 2, \cdots, 5 \)). Here, we used the 5-disk system whose \( n \)-particle survival probabilities \( P_n(t) \) are shown in Fig. 2. As expected in escape systems, the FTLLE \( \lambda(t) \) decays globally, although it can increase temporarily by disk-disk collisions. It is important to note for contents of this paper that some local decays of \( \lambda(t) \) looks like a straight line in Fig. 6 as a log-log plot of \( \lambda(t) \) as a function of \( t \). To specify this property quantitatively, we fitted the FTLLE \( \lambda(t) \) in the time period \((300, t_1)\) after the \((N-1)\)-th disk escapes at the time \( t = t_2 < 300 \) until the \( N \)-th (i.e. the last) disk escapes at the time \( t = t_1 \), to a power function \( \beta_1 t^{-\beta_2} \) with fitting parameters \( \beta_1 \) and \( \beta_2 \), as shown the straight thin line in Fig. 6. Here, we used the values \( \beta_1 = 128 \) and \( \beta_2 = 0.981 \) for these fitting parameters.

In this time period, there is only one disk inside the box with no disk-disk collision and the system is not chaotic, so this result suggests that the FTLLE decays almost in power \( \sim t^{-1} \) after a long time for non-chaotic cases, while the FTLLE should approach to a nonzero finite value for chaotic cases with disk-disk collisions.

For a comparison, we also show in Fig. 6 an example of time-dependences of the FTLLE \( \lambda(t) \) of the system without any disk-disk collision as triangles. Here, for the calculation of the FTLLE \( \lambda(t) \) we used the same values of system parameters and the initial distribution of \( \delta \Gamma(0) \) as those for the FTLLE \( \lambda(t) \) in Fig. 6 except for absence of the disk-disk collision, and for the FTLLE \( \lambda(t) \) the initial distribution of \( \Gamma(0) \) is taken as same as the system whose \( n \)-particle survival probabilities \( P'_n(t) \) are shown in Fig. 4. Different from the FTLLE \( \lambda(t) \) for the system with disk-disk collisions, the FTLLE \( \lambda(t) \) in this figure shows a smoothly decreasing function of time, except for some abrupt changes of \( \lambda(t) \) at the escape times of disks.
B. Averages and fluctuations of the finite-time largest Lyapunov exponents at escape times

Noting that FTLLEs depend on the initial conditions of the phase space vector $\Gamma(t)$ and its deviation $\delta \Gamma(t)$ at the time $t = 0$, we discuss decay properties of FTLLEs via an ensemble average over an initial distribution of $\Gamma(t)$ and $\delta \Gamma(t)$. In order to discuss such statistical properties of FTLLEs, we introduce the value $\lambda_n$ of a FTLLE $\lambda(t)$ at the time $t = t_n$ when the $(N-n+1)$-th particle escape from the box occurs ($n = 1, 2, \cdots, N$), as shown for the FTLLE $\lambda(t)$ in Fig. 6. To calculate them, we use the initial ensemble in which the initial phase space vector $\Gamma(0)$ is distributed into the micro-canonical distribution with a constant energy $E$, and components of the initial Lyapunov vector $\delta \Gamma(0)$ are chosen as the ones uniformly distributed under the constraint with a constant value of the amplitude $|\delta \Gamma(0)|$. Starting from these initial ensembles of $\lambda(0)$ and $\delta \Gamma(0)$ we calculate distributions of $\lambda_n$ and $t_n$, $n = 1, 2, \cdots, N$.

Figure 7 is the graphs of the local time averages $\langle \ln \lambda_n \rangle_t$ of $\ln \lambda_n$ as functions of local time averages $\langle \ln t_n \rangle_t$ of $\ln t_n$, for $n = 5$ (the solid line), 4 (the dashed line), 3 (the dotted line), 2 (the dash-dotted line) and 1 (the dash-double-dotted line), for a 5-disk system.

![Figure 7](Image)

**FIG. 7:** (Color online) The local time averages $\langle \ln \lambda_n \rangle_t$ of $\ln \lambda_n$ as functions of local time averages $\langle \ln t_n \rangle_t$ of $\ln t_n$, for $n = 5$ (the solid line), 4 (the dashed line), 3 (the dotted line), 2 (the dash-dotted line) and 1 (the dash-double-dotted line), for a 5-disk system.

so that we obtain the local averages $X_n^{(k)} = \sum_{j=1}^{kN_a} X(\lambda_n, t_n)/N_a$, $k = 1, 2, \cdots, \mathcal{N}_\xi/N_a$. To smooth out a graph of the data $X_n^{(k)}$ without reducing their data points we further take the second local average $X_n^{(k)} = \sum_{j=1}^{kN_a} X_n^{(j)}/(2N_a + 1)$ with $N_a = \text{Min}(10^3, (\mathcal{N}_\xi/N_a) - k, k - 1)$. $k = 1, 2, \cdots, \mathcal{N}_\xi/N_a$. In Fig. 7 we show the graphs of the quantities $\langle \ln \lambda_n \rangle_t$ as functions of $\langle \ln t_n \rangle_t$, to discuss the value of power in power decays of the FTLLEs which would be given by the slope of the quantities $\langle \ln \lambda_n \rangle_t$ as functions of $\langle \ln t_n \rangle_t$.

Figure 7 suggests that a local average of the $N$-th FTLLE $\lambda_N$ converges to the largest Lyapunov exponent after a long time without decay, while the other $n$-th FTLLEs $\lambda_n$, $1, 2, \cdots, N - 1$ decay in power after some times because there are quite few disk-disk collisions in orbits taking long times up to the $(N - n + 1)$-th disk escape from the box, namely almost non-chaotic orbits with zero Lyapunov exponents. Therefore, a transition from chaotic orbits to non-chaotic orbits would be characterized by the time starting a decay of local averages of the FTLLEs.

In order to investigate quantitatively power decays of local averages of the $n$-th FTLLEs $\lambda_n = 1, 2, \cdots, N - 1$ after a long time, we calculate the slope $\mathcal{G}_n$ of the local averages $\langle \ln \lambda_n \rangle_t$ of $\ln \lambda_n$ as functions of the local averages $\langle \ln t_n \rangle_t$ of $\ln t_n$. A constant value of the slope $\mathcal{G}_n$ in time means a power decay $\sim t^{-\gamma}$ of the local average of the $n$-th FTLLEs $\lambda_n$. In Fig. 8(a), we plotted the slopes as the local averages $\mathcal{G}_n$ of $\mathcal{G}_n = \left[\langle \ln \lambda_n \rangle_t^{(j+1)} - \langle \ln \lambda_n \rangle_t^{(j)}\right] / \left[\langle \ln t_n \rangle_t^{(j+1)} - \langle \ln t_n \rangle_t^{(j)}\right]$ as a function of local average $\ln t_n$ of $\ln t_n = \left[\langle \ln t_n \rangle_t^{(j+1)} + \langle \ln t_n \rangle_t^{(j)}\right] / 2$, for $n = 5$ (the solid line), 4 (the dashed line), 3 (the dotted line), 2 (the dash-dotted line) and 1 (the dash-double-dotted line), in which $\langle \ln \lambda_n \rangle_t^{(j)}$ and its corresponding quantity $\langle \ln t_n \rangle_t^{(j)}$, $j = 1, 2, \cdots, \mathcal{N}_\xi/N_a$, are the $j$-th values of $\langle \ln \lambda_n \rangle_t$ and $\langle \ln t_n \rangle_t$, respectively, with $\langle \ln t_n \rangle_t^{(1)} < \langle \ln t_n \rangle_t^{(2)} < \cdots < \langle \ln t_n \rangle_t^{(\mathcal{N}_\xi/N_a)}$. Fig. 8(a) with Fig. 7 suggests that the local averages of the $n$-th FTLLEs $\lambda_n$, $n = 1, 2, 3$ decay asymptotically in a power $\sim t^{-\gamma}$, while the one of the $N$-th FTLLEs $\lambda_N$ seems to converge to a positive value as the largest Lyapunov exponent without decaying to zero. For a comparison, in Fig. 8(b) we show the slope $\mathcal{G}_n$ for the system without any disk-disk collision, for $n = 5$ (the solid line), 4 (the dashed line), 3 (the dotted line), 2 (the dash-dotted line) and 1 (the dash-double-dotted line). Here, the slope $\mathcal{G}_n$ were calculated in the same system and the same calculation process as the one for the slope $\mathcal{G}_n$ except for an absence of disk-disk collision and the initial distribution of $\Gamma(0)$ used in the system whose $n$-particle survival probabilities $P_n(t)$ are shown in Fig. 4. For the
case without any disk-disk collision, all of the slopes $\mathcal{G}_n$, $n = 1, 2, \cdots, N$ seem to approach gradually to the value $-1$ as the time go to infinity. In contrast, the slopes $\mathcal{G}_n$, $n = 1, 2, \cdots, N - 1$ for the case with disk-disk collisions, show rapid decays from values around zero to the value $-1$.

The FTLLEs $\lambda_n$, $n = 1, 2, \cdots, N$ as functions of the escape times $t_n$ have large fluctuations, so that it would be meaningful to discuss not only their average properties but also their fluctuations. As a fluctuating feature of the FTLLE $\lambda_n$, in Fig. 9 we show the graphs of the averaged ratios $\langle \Delta \lambda_n^2 \rangle_t / \langle \lambda_n \rangle_t$, divided by the local averages $\langle \lambda_n \rangle_t$, $n = 1, 2, \cdots, N - 1$ increase in time, then seem to reach almost to a constant value as shown for $n = 1, 2$ in Fig. 9 meaning that in this time period variances of the $n$th FTLLE $\lambda_n$ keep to have a similar amplitude to the local average of the FTLLEs which would go to zero as time goes to infinity. These increases of the quantities $\langle \Delta \lambda_n^2 \rangle_t / \langle \lambda_n \rangle_t$ as function of time are supposed to come from non-chaotic orbits which take long times for the $(N - n + 1)$-th disk escape from the box.

C. Relations among transition times in decays of the survival probabilities and the finite-time largest Lyapunov exponents

So far, we have discussed individually decay properties of survival probabilities and FTLLEs of many hard disks in a box with a hole. Now we consider their relations by introducing some times to distinguish those properties of the survival probabilities and the FTLLEs in different time scales.

First, as discussed in Secs. III C and III D, decays of the $n$-particle survival probabilities $P_n(t)$, $n = 1, 2, \cdots, N - 1$ transfer from the superpositions of exponential decays to the power decays for many-hard-disk sys-
Third, we also define the times after which orbits are almost non-chaotic. The times of exponential decays, we introduce the time

\[ \rho, \frac{h}{\rho} = 2 \]

temps. As intermediate times for starting their power decays after their decays expressed by the superpositions of exponential decays, we introduce the time \( t_{n}(\text{sur}) \), \( n = 1, 2, \cdots, N-1 \) as the times when the slope \( F_{n}(t) \) crosses the line \(-1/2\) for \( n = 1 \) and the line \(-2(n - 1)\) for \( n = 2, 3, \cdots, N-1 \), respectively. These times are indicated as those of the open circles in Fig. 10(a).

Second, as discussed in Sec III B, the local averages of FTLLEs \( \lambda_{n} \) at the escape times \( t_{n}, n = 1, 2, \cdots, N-1 \) decay in power \( \sim t^{-1} \) after a long time. These power decays to zero would be caused by non-chaotic orbits of disks which have longer escape times than those of chaotic orbits with frequent disk-disk collisions. We introduce the times \( t_{n}(\text{ave}), n = 1, 2, \cdots, N-1 \) as those for the slopes \( \overline{c}_{n}, n = 1, 2, \cdots, N-1 \) to take the value \(-0.5\), respectively, and we use these times to estimate the times after which orbits are almost non-chaotic. The times \( t_{n}(\text{ave}) \), \( n = 1, 2, \cdots, N-1 \) are indicated as those of the open circles in Fig. 8(a). Third, we also define the times \( t_{n}(\text{flu}), n = 1, 2, \cdots, N-1 \) as the times when the quantities \( (\Delta \lambda_{n}^{2})_{t}/(\lambda_{n}^{2})_{t} \), \( n = 1, 2, \cdots, N-1 \) take their local minima, respectively, as indicated as those of the open circles in Fig. 9.

In order to discuss relations among these times \( t_{n}(\text{sur}), t_{n}(\text{ave}) \) and \( t_{n}(\text{flu}) \), we show in Fig. 10 the points \( \left( \tilde{a}_{n}(\text{sur}), \tilde{a}_{n}(\text{ave}) \right) \), \( n = 2, 3, 4, 5 \), for \( (\rho, h/(L-2r)) = (10^{-1}, 10^{-1}) \) (the closed circles), \( (\rho, h/(L-2r)) = (10^{-1}, 10^{-2}) \) (the closed triangles), \( (\rho, h/(L-2r)) = (10^{-2}, 10^{-1}) \) (the closed squares), and the points \( \left( \tilde{a}_{n}(\text{sur}), \tilde{a}_{n}(\text{flu}) \right) \), \( n = 2, 3, 4, 5 \), for \( (\rho, h/(L-2r)) = (10^{-1}, 10^{-1}) \) (the open circles), \( (\rho, h/(L-2r)) = (10^{-1}, 10^{-2}) \) (the open triangles), \( (\rho, h/(L-2r)) = (10^{-2}, 10^{-1}) \) (the open squares) for 5-disk systems. The index number at the right-low side of each point corresponds to the value \( n \) of each data for its point. The straight line shows the cases in which \( \tilde{a}_{n}(\text{sur}) \) is equal to \( \tilde{a}_{n}(\text{ave}) \) or \( \tilde{a}_{n}(\text{flu}) \).

It is important to note in Fig. 10 that the time \( t_{n}(\text{ave}) \) takes a quite similar value to the time \( t_{n}(\text{ave}) \), \( n = 1, 2, \cdots, N-1 \) for different particle densities \( \rho \) and hole size ratios \( h/(L-2r) \), noting that the points \( \left( \tilde{a}_{n}(\text{sur}), \tilde{a}_{n}(\text{ave}) \right) \) are close on the straight line to indicate the relation \( \tilde{a}_{n}(\text{sur}) = \tilde{a}_{n}(\text{ave}) \) shown in this figure. This result gives a supportive evidence that a transition from a decay as a superposition of exponential decays to a power decay for the \( n \)-particle survival probability \( P_{n}(t) \) corresponds to a transition of the FTLLE \( \lambda_{n} \) to its power decay caused by non-chaotic properties of the system. It should also be emphasized that the points \( \left( t_{n}(\text{sur}), t_{n}(\text{ave}) \right) \) for different hole sizes with \( h/(L-2r) = 10^{-1} \) and \( 10^{-2} \), shown as almost coincidences of the closed circles and the closed triangles in Fig. 10 reminding the scalings of the decay rates \( a_{j} \) by multiplying by \( 1/a_{j} \) shown in Table II for the exponential decay rates of the survival probabilities. On the other hand, we could not find such kinds of equivalence and scaling between the time \( t_{n}(\text{ave}) \) and \( t_{n}(\text{ave}, \text{flu}) \), although \( \tilde{a}_{n}(\text{ave}) \) and \( \tilde{a}_{n}(\text{ave}, \text{flu}) \) have similar orders of magnitudes, as shown in Fig. 11. This figure shows that the quantities \( \tilde{a}_{n}(\text{ave}, \text{flu}), n = 1, 2, \cdots, N-1 \), decrease as the hole size ratio \( h/(L-2r) \) decreases, and they increase as the initial particle density \( \rho \) decreases.

**IV. CONCLUSION AND REMARKS**

In this paper, we have discussed escape properties of many hard disks from a square box via a hole. Starting \( N \) disks inside the box at the initial time \( t = 0 \), we introduced the \( n \)-particle survival probability as the probability for \( n \leq N \) disks to remain inside the box without escaping up to the time \( t \geq 0 \). Escape behaviors of many hard disks is described qualitatively by
decay properties of the $n$-particle survival probabilities $P_n(t)$, $n = 1, 2, \ldots, N$, and we showed that at early times the probabilities $P_n(t)$, $n = 2, 3, \ldots, N$ decay as superpositions of exponential functions, while after a long time the probabilities $P_n(t)$, $n = 1, 2, \ldots, N - 1$ decay in power. As an important effect of disk-disk collisions, it was shown that the power of the power decay for the probability $P'_n(t)/t$ after a long time is given by $-n$ for $n = 1$ and $-2n$ for $n = 2, 3, \ldots$, in contrast to the case without any disk-disk collision in which the power of the power decay for the $n$-particle survival probability $P_n(t)$ after a long time is simply given by $-n$ for $n = 1, 2, \ldots$.

This result means that we can obtain informations on the particle number inside the box and particle-particle interactions from decay behaviors of the $n$-particle survival probabilities. We also discussed scaling properties for exponential decay rates of the probabilities $P_n(t)$ for various hole sizes and initial particle densities.

In order to discuss escape features based on dynamical characteristics of many-particle systems, we introduced the finite time largest Lyapunov exponents (FTLLEs) of hard disks inside a box with a hole. The FTLLE is defined as the exponential rate of expansion or contraction of the absolute magnitude of a small deviation of the phase space vector of the system at a finite time, and it converges to a non-zero finite value as the largest Lyapunov exponent for chaotic system with a dynamical instability in the long time limit. The dynamics of an escape system consisting of many hard disks in a box should be chaotic at early times for disk-disk collisions, but it becomes non-chaotic in the long time limit for only a single (or even zero) disk to exist inside the box after a long time when other disks have already escaped from the box via a hole. In this sense, a transition from a chaotic dynamics to non-chaotic dynamics occurs in escape systems consisting of many hard disks in a box with a hole. In this paper, we discussed this dynamical transition by decay properties of FTLLEs of the escape systems. We introduced the FTLLE $\lambda_n$ of many-hard-disk system in a box with a hole at the time $t = t_n$ when the $(N - n + 1)$-th disk escape occurs. It was shown that a local time average of the FTLLEs $\lambda_n$ decays in power $\sim t^{-1}$ after a long time, suggesting that the transition from a chaotic dynamics to a non-chaotic dynamics in escape systems could be described by the transition from the value 0 of the slope of a local time average of $\ln \lambda_n$ with respect to a local time average of $\ln t_n$ to its value $-1$. We estimated this transition time as the time when this slope of the local time average of $\ln \lambda_n$ takes the intermediate value $-1/2$, and showed that this time may be strongly correlated to the transition time for the $n$-particle survival probability to decay in power. This result would give a quantitative evidence on a connection between the escape features of many-hard-disk systems and their chaotic or non-chaotic dynamical characteristics. It was also shown that these transition times show scaling behaviors on initial particle densities and hole sizes. We also discussed a fluctuation property of the FTLLEs $\lambda_n$ in the sense that a local time variation of $\lambda_n$ divided by a local average of $\lambda_n$ takes a local minimum, then takes a locally almost constant value, as a function of time.

In the escape model discussed in this paper, we put a single hole for disks to escape, which consists of two regions over two sides from a single corner of the box. This hole configuration is chosen so that orbit properties for a disk bouncing between two confronting walls without any collision with other disks, the so-called bouncing ball (or sticky) orbits, is the same for two kinds of the confronting walls. The bouncing ball orbits are known to play an essential role in power decays of survival probabilities in escapes of single-particle billiard models [32, 34]. On the other hand, it would be meaningful to note what happens in particle escapes for different hole configurations. As such an example, in Fig. 11 we show the $n_1$-particle survival probabilities $P'_{n_1}(t)$ of a 5-disk system as functions of time $t$ for $n_1 = 5$ (the circles), 4 (the triangles), 3 (the squares), 2 (the diamonds) and 1 (the inverted triangles) for $n_2 = 5$ (the solid line), 4 (the dashed line), 3 (the dotted line), 2 (the dash-dotted line), 1 (the dash-double-dotted line) for the hole of the length $r + h$ in a single side [for the hole of the length $r + (h/2)$ in both the sides] from a single corner of the box. The main figure and the inset are for graphs of $P'_{n_1}(t)$ and $P_{n_2}(t)$ on log-log plots and linear-log plots, respectively.
Figure 11 shows that the $n$-particle survival probabilities $P_n'(t)$ of the escape system with the hole in the single side of the box decay faster than the $n$-particle survival probabilities $P_n(t)$ of the system with the hole in both the sides of box for $n = 2, 3, \ldots, N$, while the survival probability $P_n'(t)$ decay slower (faster) than the survival probability $P_n(t)$ after a long time (at early times). It should be also noted that the power decay $\sim t^{-1}$ of the survival probability $P_n'(t)$ after a long time looks to be much weaker than that of $P_n(t)$. In general, escape behaviors of disks from the box could depend on where we put a hole, because of not only the properties of periodic orbits but also finite size effects of disks [51]. Another problem involving finite size effects of disks would be an open problem involving finite size effects of disks from the box could depend on where we put a hole, because of not only the properties of periodic orbits but also finite size effects of disks [51]. Another problem involving finite size effects of disks would be an open problem involving finite size effects of disks 

Many interesting features including these points, effects of various particle-particle interactions or many holes and so on, would remain as open problems on escape phenomena of many-particle systems.

**Appendix A: Derivations of Eqs. (5) and (6)**

In this appendix, we derive Eq. (5) from Eqs. (5) and (4), as well as Eq. (6) from Eqs. (11) and (15).

First, we introduce the Laplace transformation

$$f_1(\omega; a) = \int_0^{+\infty} d\tau \, f_1(\tau; a) \exp(-\omega \tau)$$

of the function $f_1(\tau; a) \equiv \frac{a}{\omega + a}$ of $\tau (> 0)$, which is given by

$$\hat{f}_1(\omega; a) = \frac{a}{\omega + a} \tag{A1}$$

for $\omega + a > 0$, by using Eq. (3). Second, using Eq. (A1) and the convolution formula of Laplace transformations, the Laplace transformation $\tilde{f}_k(\omega) = \int_0^{+\infty} d\tau \, \tilde{f}_k(\tau) \exp(-\omega \tau)$ of the function $\tilde{f}_k(\tau)$ for $k \geq 2$ is given by

$$\tilde{f}_k(\omega) = \frac{n}{\omega + a_j}, \tag{A3}$$

in a form of the partial fraction under the assumption $a_j \neq a_k$ for $j \neq k$. Here, $A_j$ is a constant and is given by

$$A_j = \lim_{\omega \rightarrow -a_j} (\omega + a_j) \sum_{l=1}^{k} \frac{A_l}{\omega + a_l} = \lim_{\omega \rightarrow -a_j} (\omega + a_j) \prod_{l=1}^{k} \tilde{f}_l(\omega; a_l)$$

$$= \frac{1}{(a_1 - a_j)(a_2 - a_j) \cdots (a_{j-1} - a_j)(a_{j+1} - a_j)(a_{j+2} - a_j) \cdots (a_k - a_j)}. \tag{A4}$$

From Eq. (A3), the inverse-transformation $\tilde{f}_k(\tau)$ of the function $\tilde{f}_k(\omega)$ is represented as

$$\tilde{f}_k(\tau) = \sum_{j=1}^{k} A_j e^{-a_j \tau}. \tag{A5}$$

By inserting Eq. (A4) into Eq. (A5), we obtain Eq. (5).

Using Eqs. (A2), (A3) and the normalization condition $\int_0^{+\infty} d\tau \, \tilde{f}_1(\tau; a_j) = \tilde{f}_1(0; a_j) = 1$ of the probability density $\tilde{f}_1(\tau; a_j)$, we obtain

$$\int_0^{+\infty} d\tau \, \tilde{f}_k(\tau) = \tilde{f}_k(0) = \sum_{j=1}^{k} \tilde{f}_1(0; a_j)$$

$$= 1 = \sum_{j=1}^{k} \frac{A_j}{a_j}. \tag{A6}$$

which includes the normalization condition of the probability density $\tilde{f}_k(\tau)$. From Eqs. (11), (A5) and (A6) we derive

$$\tilde{P}_{N-k+1}(t) = 1 - \sum_{j=1}^{k} A_j \frac{1 - e^{-a_j t}}{a_j}$$

$$= \sum_{j=1}^{k} \frac{A_j}{a_j} e^{-a_j t} \tag{A7}$$

for the survival probability $P_{N-k+1}(t) = \tilde{P}_{N-k+1}(t)$ in the case of the probability density $f_k(\tau) = \tilde{f}_k(\tau)$ of the escape time $\tau$ of the $k$ disks. By inserting Eq. (A3) into Eq. (A7), we obtain Eq. (6).

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