A Generalization of an Alternating Sum Formula for Finite Coxeter Groups

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Abstract

For $W$ a finite Coxeter group, a formula is found for the size of $W$ equivalence classes of subsets of a base. The proof is a case-by-case analysis using results and tables of Orlik and Solomon. As a corollary we obtain an alternating sum identity which generalizes a well-known identity from the theory of Coxeter groups.

1 Introduction

Let $W$ be a finite Coxeter group and $\Pi$ a set of fundamental roots or base of $W$. Note that $W$ acts on subsets of $\Pi$. For $J, K \subseteq \Pi$, one writes $J \sim K$ if $J = w(K)$ for some $w \in W$. Letting $\lambda(K)$ denote the equivalence class of $K$ under this action, a natural problem is to compute the size $|\lambda(K)|$ of this equivalence class. It suffices to work with irreducible $W$. For the classical types $|\lambda(K)|$ can be computed without much difficulty; for the exceptional types $|\lambda(K)|$ has been essentially computed by Carter [3].

Theorem 1 of this note will give a formula for $|\lambda(K)|$ in terms of the index of the parabolic subgroup $W_K$ in its normalizer $N_W(W_K)$ and the characteristic polynomial $\chi(L^K, t)$ of an upper interval in the intersection lattice $L$ of root hyperplanes corresponding to $W$. This result is of interest in that it gives an expression for $|\lambda(K)|$ in terms of quantities naturally associated to $W$ and $K$. Our proof is a case-by-case analysis; a unified proof for all finite Coxeter groups would be desirable.

Theorem 2 applies Theorem 1 to obtain a generalization of the alternating sum formula

$$\sum_{K \subseteq \Pi} (-1)^{|K|} \frac{|W|}{|W_K|} = 1.$$ 

The above alternating sum formula is discussed in Sections 1.15-1.16 of Humphreys [3]. Topological proofs using the Hopf trace formula are given by Solomon [4] and Steinberg [5]. Chapter 3 of Humphreys gives applications to the invariant theory of Coxeter groups, and Section 6.2 of Carter [2] gives an application to proving the irreducibility of the Steinberg character. We hope that our generalization will have similar interpretations and applications.
2 Notation

We collect some notation which will be indispensable for what follows. Denote the set of root hyperplanes of $W$ by $A$. Let $L$ be the set of intersections of the hyperplanes in $A$, taking $V \in L$. Partially order $L$ by reverse inclusion. Recall that the Moebius function $\mu$ is defined by $\mu(X, X) = 1$ and $\sum_{X \leq Z \leq Y} \mu(Z, Y) = 0$ if $X < Y$ and $\mu(X, Y) = 0$ otherwise. The characteristic polynomial of $L$ is defined as

$$\chi(L, x) = \sum_{X \in L} \mu(V, X) x^{\dim(X)}.$$ 

For $K \subseteq \Pi$, let $Fix(W_K)$ be the fixed space of the parabolic subgroup $W_K$ in its action on $V$. Let $L^{Fix(W_K)}$ denote the poset isomorphic to the segment $\{Y \in L(A) | Y \geq Fix(W_K)\}$.

Let $n$ be the rank of $W$, and let $N_W(W_K)$ be the normalizer of $W_K$ in $W$. As in the introduction, for $J, K \subseteq \Pi$ write $J \sim K$ if $J = w(K)$ for some $w \in W$ and let $\lambda(K)$ denote the equivalence class of $K$ under this action.

3 Main Results

The following lemma of Bergeron, Bergeron, Howlett, and Taylor translates the equivalence relation $\sim$ into a condition about conjugacy of parabolic subgroups.

**Lemma 1** (Bergeron, Bergeron, Howlett, Taylor [1]) If $J, K \subseteq \Pi$, then $J \sim K$ if and only if $W_J$ and $W_K$ are conjugate.

Now we derive an expression for $|\lambda(K)|$ in terms of quantities naturally associated to $W$.

**Theorem 1** Let $W$ be a finite Coxeter group of rank $n$. Then for all $K \subseteq \Pi$,

$$|\lambda(K)| = (-1)^{n-|K|} \frac{|W_K|}{|N_W(W_K)|} \chi(L^{Fix(W_K)}, -1).$$

**Proof:** As both sides of the conjectured equation are multiplicative with respect to direct product of groups, it suffices to prove the theorem for irreducible $W$. This will be done case by case.

For $W$ of type $A_{n-1}$, it is easy to see that $J \sim K$ exactly when $W_J$ and $W_K$ are isomorphic. Given $K$, define equivalence classes called blocks on the set $\{1, \cdots, n\}$ by letting $p \sim q$ if it is possible to transpose $p$ and $q$ using only elements in $W_K$. Let $n_i$ be the number of blocks of size $i$. A subset $J \subseteq \Pi$ such that $W_J$ is isomorphic to $W_K$ arises from any of the $(n - \sum i n_i)!$ permutations of the blocks of $K$. One must then divide by $\prod_i n_i!$ since permuting the blocks of size $i$ amongst themselves leads to the same $J$. Thus,

$$|\lambda(K)| = \frac{(n - \sum i n_i)!}{\prod_i n_i!} = \frac{(n - |K|)!}{\prod_i n_i!}.$$

Since $N_W(W_K)$ is isomorphic to $\prod_i [(A_i)^{n_i} \times A_{n_i}]$, one has that $\frac{|W_K|}{|N_W(W_K)|} = \frac{1}{\prod_i n_i!}$. Proposition 2.1 of Orlik and Solomon [3] states that $\chi(L^{Fix(W_K)}, t) = (t-1) \cdots (t-(n-|K|-1))$. Thus,
the symmetry of the Dynkin diagram which switches $\alpha_k \sim \alpha_1$ and let $\lambda$ be the number of parts of $\lambda$ of size $i$. Arguing as for type $A$,

$$|\lambda(K)| = \frac{(n - j - \sum i n_i)!}{\prod_i n_i^i} = \frac{(n - |K|)!}{\prod_i n_i^i}. $$

On page 12 of Carter [4], it is proved that $|W_K| = \prod_i 2^{n_i}$. Proposition 2.2 of Orlik and Solomon [3] states that $\chi(L^{Fix(W_K)}, t) = (t - 1)(t - 2) \cdots (t - (2(n - |K|)) - 1)$. Therefore we conclude as desired that

$$(-1)^{|K|} \frac{|W_K|}{|N_W(W_K)|} \chi(L^{Fix(W_K)}, -1) = \frac{(n - |K|)!}{\prod_i n_i^i}. $$

Next we proceed to type $D_n$. Take $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ a fundamental set of simple roots with $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = \varepsilon_{n-1} + \varepsilon_n$.

There are two subcases. The first subcase is that $\{\alpha_{n-1}, \alpha_n\} \subseteq K$. Let $W_K$ be isomorphic to $D_j \times A_{\lambda_j - 1} \times A_{\lambda_2 - 1} \times \cdots$ (here we insist that $j + \sum_i \lambda_i = n$) and let $n_i$ be the number of parts of $\lambda$ of size $i$. Arguing as in type $A$ shows that $|\lambda(K)| = 2^{(n - |K|)!}$. The extra factor of 2 comes from the symmetry of the Dynkin diagram which switches $\alpha_{n-1}$ and $\alpha_n$ but leaves the other roots fixed.

On page 11 of Carter [4], it is proved that $|W_K| = \prod_i 2^{n_i}$. Proposition 2.6 of Orlik and Solomon [3] states that $\chi(L^{Fix(W_K)}, t) = (t - 1)(t - 2) \cdots (t - (2(n - |K|)) - 1)$. Thus

$$(-1)^{|K|} \frac{|W_K|}{|N_W(W_K)|} \chi(L^{Fix(W_K)}, -1) = \frac{2(n - |K|)!}{\prod_i n_i^i}. $$

The second subcase is that $\{\alpha_{n-1}, \alpha_n\} \not\subseteq K$. All $J$ such that $J \sim K$ also satisfy $\{\alpha_{n-1}, \alpha_n\} \not\subseteq J$. Given $K$, define equivalence classes called blocks on the set $\{1, \ldots, n\}$ by letting $p \sim q$ if it is possible to transpose $p$ and $q$ up to sign change, using only elements in $W_K$. Let $n_i$ be the number of blocks of size $i$. Observe that

$$|\lambda(K)| = |\{J : J \sim K, J \subseteq \{\alpha_1, \ldots, \alpha_{n-1}\}\}| + |\{J : J \sim K, J \subseteq \{\alpha_1, \ldots, \alpha_{n-2}, \alpha_n\}\}| - |\{J : J \sim K, J \subseteq \{\alpha_1, \ldots, \alpha_{n-2}\}\}|$$

$$= 2^{(n - |K|)!} - (n - |K| - 1)! - (n - |K| - 2)! \prod_{i \geq 1} n_i^i.$$

On page 11 of Carter [4], it is proved that $|W_K| = \prod_i 2^{n_i}$. Proposition 2.6 of Orlik and Solomon [3] states that $\chi(L^{Fix(W_K)}, t) = (t - 1)(t - 2) \cdots (t - (2(n - |K|)) - 3)(t - (n - |K| + r - 1)$, where $r$ is the number of blocks of size greater than one. Clearly $r = n - |K| - n_1$. Therefore,
\[
(-1)^{n-|K|} \frac{|W_K|}{|N_W(W_K)|} \chi(L^{Fix(W_K)}, -1) = \frac{2}{\prod_i 2^{n_i} n_i!} [(n - |K|) - n_1] (n - |K| - 1)! = |\lambda(K)|.
\]

Next we proceed to the cases \(I_2(m)\) and \(G_2\). On page 175 of Orlik and Solomon [6], \(\chi(L^{Fix(W_K)}, t)\) is computed in the following relevant cases:

- If \(n - |K| = 0\) then \(\chi(L^{Fix(W_K)}, t) = 1\).
- If \(n - |K| = 1\) then \(\chi(L^{Fix(W_K)}, t) = t - 1\).
- If \(n - |K| = 2\) then \(\chi(L^{Fix(W_K)}, t) = (t - 1)(t - n)\), where \(n + 1\) is the number of lines of \(L\) contained in \(Fix(W_K)\).

As an application of this computation, we prove the theorem for \(I_2(m)\) with \(m\) even (\(I_2(m)\) with \(m\) odd and \(G_2\) are easy and similar). One checks that \(|\lambda(K)| = 1\) for all \(K\) contained in the two element set \(\Pi = \{\alpha_1, \alpha_2\}\). For \(K = \emptyset\), \(\frac{|W_K|}{|N_W(W_K)|} = \frac{1}{2^m}\) and \(Fix(W_K)\) contains \(m\) lines of \(L\), so the theorem checks. For \(K\) such that \(|K| = 1\), \(\frac{|W_K|}{|N_W(W_K)|} = \frac{1}{2}\) and \(\chi(L^{Fix(W_K)}, t) = t - 1\), so the theorem checks. Finally, for \(K = \Pi\), \(\frac{|W_K|}{|N_W(W_K)|}\) and \(\chi(L^{Fix(W_K)}, t)\) are both equal to 1.

The proof of the theorem for the exceptional cases \(E_6, H_3, H_4, E_6, E_7, E_8\) consists of a finite number of straightforward calculations based on tables 3-8 of Orlik and Solomon [6]. We will work out the details for an example to illustrate what is involved.

For the example, take \(W = E_7\) and \(K\) any subset of \(\Pi\) whose Dynkin diagram has type \((A_1)^2\). As the first column of their Table 7 indicates, all such \(K\) are equivalent under \(\sim\). A glance at the Dynkin diagram of \(E_7\) thus shows that \(|\lambda(K)| = 15\). The third entry in the first row of their Table 7 gives that \(\frac{|W|}{|N_W(W_K)|} = 945\). The end of the third row in their Table 7 gives that \(\chi(L^{Fix(W_K)}, t) = (t - 1)(t - 5)(t - 7)(t - 9)(t - 11)\). This data shows that the theorem checks.

The only minor complications arise in computing \(|\lambda(K)|\) in cases where there are subsets \(J\) with Dynkin diagram isomorphic to \(K\), but such that \(J \not\sim K\). As the tables of Orlik and Solomon (loc. cit.) indicate, this happens only for \(W = E_7\), and the values \(|\lambda(K)|\) for these cases appear on the top of page 279 of their article. \(\Box\)

From its statement, it is not evident that Theorem \([6]\) generalizes any known identities about Coxeter groups. Theorem \([3]\) shows that this is indeed the case.

**Theorem 2** Let \(W\) be a finite Coxeter group of rank \(n\) with base \(\Pi\). Then

\[
\sum_{K \subseteq \Pi} (-1)^{n-|K|} \frac{|W|}{|W_K|} \frac{\chi(L^{Fix(W_K)}, t)}{\chi(L^{Fix(W_K)}, -1)} = t^n.
\]

**Proof:** As is explained on page 274 of Orlik and Solomon [6], elementary properties of Moebius functions yield the identity

\[
\sum_{Y \in L} \chi(L^Y, t) = t^n.
\]

Observe that \(W\) acts on the lattice \(L\). Let \(O(Fix(W_K))\) be the orbit of \(Fix(W_K)\) under this action. Lemma 3.4 of Orlik and Solomon (loc. cit.) states that \(|O(Fix(W_K))| = \frac{|W|}{|N_W(W_K)|}\). For
Y ∈ L(A), Lemma 3.4 of Orlik and Solomon (loc. cit.) shows that Fix(WY) = Y. Since the parabolic subgroup WY is conjugate to a standard parabolic subgroup WK for some K, it follows that Y ∈ O(Fix(WK)) for some K. Then Y ∈ O(Fix(WJ)) for exactly the |λ(K)| many J’s such that J ∼ K. Therefore, the Orlik-Solomon identity becomes

\[ \sum_{K \subseteq \Pi} \frac{|W|}{|NW(W_K)|} \frac{\chi(L^{\text{Fix}}(W_K), t)}{|\lambda(K)|} = t^n. \]

The result follows from the formula for |λ(K)| in Theorem 1. □

Remarks:

• Observe that setting t = -1 in Theorem 2 yields the well-known identity

\[ \sum_{K \subseteq \Pi} (-1)^{|K|} \frac{|W|}{|W_K|} = 1. \]

As noted in the introduction, this identity has a topological proof and arises in the invariant theory of W. It would be desirable to find analogous interpretations for Theorem 2.

• The identity

\[ \sum_{K \subseteq \Pi} (-1)^{|K|} \frac{|W|}{|W_K|} = 1 \]

has two known generalizations. To describe the first, let di(W) be the degrees of a Coxeter group W. Then from Sections 1.11 and 3.15 of Humphreys [5],

\[ \sum_{K \subseteq \Pi} (-1)^{|K|} \frac{n!}{\prod_{i=1}^{n} t^{d_i(W_K)} - 1} = t^n. \]

To describe the second, note that W acts on the left cosets vW_K by left multiplication. Let f_K(w) be the number of left cosets of W_K fixed by w. Let det(w) be the determinant of w in its action on V. Proposition 1.16 of Humphreys [5] states that

\[ \sum_{K \subseteq \Pi} (-1)^{|K|} f_K(w) = det(w). \]

What is the relation of Theorem 2 with these generalizations? Can Theorem 2 be further extended to include them?

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