Embedding of Circulant Networks into Wheels

Stalin Mary\(^1\), Indra Rajasingh\(^1\)\(^*\)

\(^1\)Mathematics Division, School of Advanced Sciences, Vellore Institute of Technology, Chennai – 600127.

E-mail: indrarajasingh@yahoo.com

Abstract - Circulant graphs have been studied over many years because of its vast applications in telecommunication industry. We find the minimum wirelengths of embedding the circulant graph \(G(\pm\{1, 2\})\) and \(G(\pm\{1, 2, ..., j\})\) into variation of wheel graphs on \(n\) vertices in this paper.

Keywords: Embedding, Congestion, Circulant graph, Wheel graph

1. Introduction

Graph is a mathematical model represented by points and lines joining certain pairs of points. These points are addressed as vertices or nodes and the lines are addressed as edges or links. Graphs occur naturally in several real world applications such as transportation network, electronic circuit, social network, communication network etc., [1]. Networks are graphs whose vertices represent modules and lines represent routes that convey or transmit messages. Links also represent transfer of data. An one-one function \(E\) of a graph \(G\) into a graph \(H\) is called an embedding of \(G\) into \(H\) if every edge \((x, y)\) in \(G\) is mapped to a path between \(E(x)\) and \(E(y)\) in \(H\) [2]. Circulant networks have optimal routing capabilities and optimal fault tolerance [3, 4, 5, 6]. This paved way for using circulant networks as telecommunication networks [4]. They are also used in arrangement of binary codes [7]. The undirected circulant graphs are used in ILLIAC type computers [8]. A wheel graph \(W_n\) consists of \(n - 1\) number of vertices in the outer cycle which are all connected to a vertex called its hub. In wheel network, information flows to and from a single person. It is a communication network where the employees get information fast. Very often, circuit layout designs include wheel networks [15]. Gear graphs and Helm graphs are obtained from wheel graphs [8]. The minimum wirelengths of embedding \(G(n; \pm\{1, 2\})\) and \(G(n; \pm\{1, 2, ..., j\})\) into wheels, one-point union of wheels and union of wheels with cut edges are obtained in this paper.

2. Preliminaries

Definition 2.1. Maximum Subgraph Problem [9]: For a given graph \(G\) on \(n\) nodes and a given integer \(k, k \leq n\), finding a subgraph on \(k\) nodes with number of edges greater than or equal to any other subgraph of \(G\) on \(k\) vertices, is called Maximum Subgraph Problem.
**Lemma 2.1** Let \( G \) and \( H \) be finite graphs with \( n \) vertices. An one-one mapping \( f: V(G) \to V(H) \) inducing a one-one mapping \( P_f: E(G) \to \{P_f(u,v): P_f(u,v) \text{ is a path between } f(u) \text{ and } f(v) \text{ in } H \text{ for } (u,v) \in E(G)\} \) is called an embedding of \( G \) into \( H \).

**Definition 2.3.** Let \( f: G \to H \) be an embedding. For \( e \in E(H) \), the number of paths in \( \{P_f(u,v): P_f(u,v) \text{ is a path between } f(u) \text{ and } f(v) \text{ in } H \text{ for } (u,v) \in E(G)\} \) that contain \( e \) is called the congestion on \( e \) with respect to \( f \) and is denoted by \( c_f(e) \). For \( S \subseteq E(H) \), we define \( c_f(S) = \sum_{e \in S} c_f(e) \).

**Definition 2.4.** [12] A subgraph \( K \) of a graph \( G \) is said to be convex if all the shortest paths between any two vertices of \( K \) lie in \( K \). An edge cut \( S \) of \( G \) is said to be a convex cut if \( G \setminus S \) splits into two components, each of which is convex.

**Definition 2.5.** Let \( \mathcal{E}: G \to H \) be an embedding. Then, the expression \( \sum_{e \in E(H)} c_{\mathcal{E}}(e) \) is called the wirelength of embedding \( G \) into \( H \) with respect to \( \mathcal{E} \) and is denoted by \( WL_{\mathcal{E}}(G,H) \). \( \min_{\mathcal{E}} WL_{\mathcal{E}}(G,H) \) is called the wirelength of embedding \( G \) into \( H \) and is denoted by \( WL(G,H) \). The wirelength problem [11] of \( G \) into \( H \) is to determine \( WL(G,H) \).

**Definition 2.6.** **Congestion Lemma** [12] For an embedding \( \mathcal{E} \) of a graph \( G \) into \( H \), let \( S \) be a convex edge cut of \( H \) such that \( H \setminus S \) splits into components \( H_1 \) and \( H_2 \) and let \( G_1 = \mathcal{E}^{-1}(H_1) \) and \( G_2 = \mathcal{E}^{-1}(H_2) \). Suppose \( G_1 \) and \( G_2 \) are maximal subgraphs of \( G \) and \( P_{\mathcal{E}}(u,v) \) with \( u \in G_1 \) and \( v \in G_2 \) contains exactly one edge in \( S \), for every \( (u,v) \in E(G) \), then

\[
c_{\mathcal{E}}(S) = \sum_{v \in \mathcal{E}(G_1)} deg_G(v) - 2|E(G_1)|
\]

\[
= \sum_{v \in \mathcal{E}(G_2)} deg_G(v) - 2|E(G_2)|
\]

**Lemma 2.7.** **r-Partition Lemma** [13] Let \( \mathcal{E}: G \to H \) be an embedding. Let \( E^r(H) \) be the multiset in which each edge of \( H \) is repeated \( r \) times. Let \( \{R_1, R_2, ..., R_t\} \) be a partition of \( E^r(H) \) where each \( R_i \) is an edge cut of \( H \). Then

\[
WL_{\mathcal{E}}(G,H) = \frac{1}{r} \sum_{i=1}^{t} c_{\mathcal{E}}(R_i)
\]

**Definition 2.8.** **Circulant graph** [14] Let \( K \subseteq \left\{1,2, ..., \left\lceil \frac{n}{2} \right\rceil \right\}, n \geq 3 \). A graph \( G \) with vertex set \( V = \{0,1, ..., n-1\} \) and the edge set \( E = \{(i,j): |j - i| \equiv k \text{ mod } n, k \in K\} \) is called a circulant graph and is denoted by \( G(n; \pm K) \).

**Lemma 2.9.** [15] Any set of \( r \) consecutive vertices on the outer cycle of \( G(n; \pm 1) \), \( 1 \leq r \leq n \) induces a subgraph of \( G(n; \pm K) \) with maximum number of edges.

**Lemma 2.10.** [15] A maximum subgraph on \( r \) vertices of \( G(n; \pm K) \), where \( K = \left\{1,2, ..., \left\lceil \frac{n}{2} \right\rceil \right\}, 1 \leq j \leq \left\lceil \frac{n}{2} \right\rceil, 1 \leq r \leq n, n \geq 3 \) has the number of edges as,
\[ I_G(r) = \begin{cases} 
\frac{r(r-1)}{2}; & r \leq j + 1 \\
rf - \frac{j(j+1)}{2}; & j + 1 < r \leq n - j \\
\frac{1}{2}((n-r)^2 + (4j + 1)r - (2j + 1)n); & n - j < r \leq n 
\end{cases} \]

**Definition 2.11. Wheel graph** [16] Consider a cycle on \( n - 1 \) nodes. Take a new node and join it to each of the \( n - 1 \) nodes on the cycle. The resultant graph is called a wheel graph and is denoted by \( W_n \). The new vertex of degree \( n - 1 \) is called the hub of \( W_n \).

3. **Embedding of \( G(n; \pm\{1, 2\}) \) into wheel \( W_n \)**

**Embedding Algorithm \( G(n; \pm\{1, 2\}, W_n) \)**

*Input:* \( G(n; \pm\{1, 2\}), n \geq 5 \) and \( W_n \).

*Algorithm:*

i. The nodes of \( G(n; \pm 1) \) are given labels \( 0, 1, 2, \ldots, n - 1 \) in the clockwise sense.
ii. The hub vertex is given label \( 0 \) and the cycle vertices are given labels \( 1, 2, \ldots, n - 1 \) in the clockwise sense. See Figure 1.

*Output:* An embedding \( E(x) = x \) have minimum wirelength.

**Proof of Correctness:**

Let \( S_i = \{(i, i + 1), (i + 2, i + 3)\}, 0 \leq i \leq n - 2 \), where all the numbers are taken modulo \( n \). Each \( S_i \) is an edge cut of \( W_n \). \( \{S_i\}, 1 \leq i \leq n - 1 \) is a partition of \( E^2(W_n) \). \( W_n \setminus S_i \) has two components \( H_{t_1} \) and \( H_{t_2} \). For \( 0 \leq i \leq n - 1 \), since each \( S_i \) is a cut induced by 4 consecutive labels, \( G_{t_1} \) and \( G_{t_2} \) are also induced by consecutive labels. Hence, they are maximum subgraphs in \( G(n; \pm\{1, 2\}) \). Again, since \( \{S_i\}_{0 \leq i \leq n - 1} \) is a partition of \( E^2(W_n) \), by Congestion Lemma and 2-Partition Lemma, we see that the wirelength is minimum.

![Figure 1. Embedding of \( G(n; \pm\{1, 2\}) \) into wheel \( W_n \).](image)

**Theorem 3.1.** The wirelength of \( G(n; \pm\{1, 2\}) \) into \( W_n \), \( n \geq 5 \) is given by \( WL(G(n; \pm\{1, 2\}), W_n) = 3n - 1 \).
Proof. Let \( G \) be the circulant graph \( G(n; \pm \{1, 2\}) \) and \( H \) be the wheel graph with \( n - 1 \) vertices in the outer cycle. By congestion lemma and 2-partition lemma,

\[
WL(G(n; \pm \{1, 2\}), W_n) = \frac{1}{2} \sum_{i=0}^{n-1} c_e(S_i)
\]

\[
= \frac{n-1}{2} [4(2) - 2(1)]
\]

\[
= \frac{3}{2}(n - 1)
\]

Therefore, the minimum wirelength of embedding \( G(n; \pm \{1, 2\}) \) into \( W_n \), \( n \geq 5 \) is \( 3(n - 1) \).

Proceeding along the same line, we arrive at the following general result.

**Theorem 3.2.** The wirelength of embedding \( G(n; \pm \{1, 2, \ldots, j\}) \) into \( W_n \) is

\[
WL(G(n; \pm \{1, 2, \ldots, j\}), W_n) = (r - 1)(n - 1).
\]

4. Embedding of \( G(n; \pm \{1, 2\}) \) into one-point union of wheels

**Definition 4.1.** Let \( W_{n_1}, W_{n_2}, \ldots, W_{n_k} \) be wheels such that their hubs are merged as one single vertex. The resultant graph is called an one-point union of wheels denoted as \( W_{n_1} \wedge W_{n_2} \wedge \ldots \wedge W_{n_k} \). See Figure 2.

![Figure 2. \( W_5 \wedge W_6 \wedge W_7 \).](image)

**Theorem 4.1.** Let \( G(n; \pm \{1, 2\}) \) be the circulant graph and \( \wedge_{i=1}^{k} W_{n_i} \) be the one-point union of \( k \) wheels and let \( n = n_1 + n_2 + \cdots + n_k + 1 \). Then

\[
WL\left(G, \bigwedge_{i=1}^{k} W_{n_i}\right) = 3(n - k)
\]

**Proof.** Edge cuts similar to those in the Proof of Correctness of the embedding algorithm for \( W_n \), and the 2-partition lemma yeild
\[
WL\left( G, \bigwedge_{i=1}^{k} W_{n_i} \right) = 3(n_1 - 1) + 3(n_2 - 1) + \cdots + 3(n_k - 1)
\]
\[
= 3(n - k)
\]

5. Embedding of \( G(n; \pm\{1, 2\}) \) into union of wheels with cut edges

Definition 5.1. Let \( W_{n_1}, W_{n_2}, \ldots, W_{n_k} \) be wheels with \( v_1, v_2, \ldots, v_k \) as vertices. Let \( e_1, e_2, \ldots, e_{k-1} \) be edges such that \( e_i = (v_i, v_{i+1}), 1 \leq i \leq k - 1 \). See Figure 3. The resultant graph is denoted by

\[
H = \bigwedge_{i=1}^{k} W_{n_i}
\]

![Figure 3. Wheels with cut edges.](image)

Theorem 5.1. Let \( G(n; \pm\{1, 2\}) \) be the circulant graph and let \( H = \bigwedge_{i=1}^{k} W_{n_i} \), where \( n_1 + n_2 + \cdots + n_k = n \). Then, \( WL(G, H) = 3(n - k) + \sum_{i=1}^{k} E(n_i) \), where \( E(n_i) \) is the expression in Theorem 2.10.

6. Conclusion
Finding wirelength of an embedding is an NP-complete problem. Hence this study becomes significant while dealing with standard interconnection networks. In this paper, the circulant network has been considered as the guest graph with a variation of wheel graphs as the host graph.

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