Let $J$ be a strongly stable monomial ideal in $S = K[x_1, \ldots, x_n]$ and let $\mathcal{Mf}(J)$ be the family of all homogeneous ideals $I$ in $S$ such that the set of all terms outside $J$ is a $K$-vector basis of the quotient $S/I$. We show that an ideal $I$ belongs to $\mathcal{Mf}(J)$ if and only if it is generated by a special set of polynomials, the $J$-marked basis of $I$, that in some sense generalizes the notion of reduced Gröbner basis and its constructive capabilities. Indeed, although not every $J$-marked basis is a Gröbner basis with respect to some term order, a sort of normal form modulo $I \in \mathcal{Mf}(J)$ can be computed for every homogeneous polynomial, so that a $J$-marked basis can be characterized by a Buchberger-like criterion. Using $J$-marked bases, we prove that the family $\mathcal{Mf}(J)$ can be endowed, in a very natural way, with a structure of affine scheme that turns out to be homogeneous with respect to a non-standard grading and flat in the origin (the point corresponding to $J$), thanks to properties of $J$-marked bases analogous to those of Gröbner bases about syzygies.

**Introduction**

Let $J$ be any monomial ideal in the polynomial ring $S := K[x_0, \ldots, x_n]$ in $n + 1$ variables endowed so that $x_0 < x_1 < \ldots < x_n$ and let us denote $\mathcal{N}(J)$ the set of terms outside $J$. In this paper we consider the family $\mathcal{Mf}(J)$ of ideals $I$ of $S$ such that $S = I \oplus \langle \mathcal{N}(J) \rangle$ as a $K$-vector space and investigate under which conditions this family is in some natural way an algebraic scheme. If $\mathcal{N}(J)$ is not finite, the family of such ideals can be too large. For instance, if $J = (x_0) \subset K[x_0, x_1]$, the family of all ideals such that $S/I$ is generated by $\mathcal{N}(J) = \{x_1^n : n \in \mathbb{N}\}$ depends on infinitely many parameters. For this reason we restrict ourselves to the homogeneous case.

To study the family $\mathcal{Mf}(J)$ we introduce a set of particular homogeneous polynomials, called $J$-marked set, that becomes a $J$-marked basis when it generates an ideal $I$ that belongs to $\mathcal{Mf}(J)$. If $J$ is strongly stable a $J$-marked basis satisfies most of the good properties of a reduced homogeneous Gröbner basis and, for this reason, we assume that $J$ is strongly stable. However, even under this assumption, a $J$-marked basis does not need to be a Gröbner basis (Example 3.18). We show that a suitable rewriting procedure allows to compute a sort of normal forms and to recognize a $J$-marked basis by a Buchberger-like criterion. This criterion is the tool by which we construct the family $\mathcal{Mf}(J)$ following the line of the computation of a Gröbner stratum, that is the family of all ideals that have $J$ as initial ideal with respect to a fixed term order. In the last years, several authors have been working on Gröbner strata, proving that they have a natural and well defined structure of algebraic schemes, that springs out of a procedure based on Buchberger’s algorithm [4, 11, 16, 18, 19], and that they are homogeneous with respect to a non standard positive grading over $\mathbb{Z}^{n+1}$ [6]. In this context, it is worth also

*Key words and phrases.* Family of schemes, strongly stable ideal, Gröbner basis, flatness.
to recall that in [13] a method is described to compute all liftings of a homogeneous ideal with an approach different from, but close to the method applied to study Gröbner strata.

The paper is organized in the following way. In section 1 we give definitions and basic properties of $J$-marked sets and bases, with several examples. In section 2, in the hypothesis that $J$ is strongly stable, we prove the existence of a sort of normal form, modulo the ideal generated by a $J$-marked set, for every homogeneous polynomial (Theorem 2.2). A consequence is that, if $J$ is strongly stable, a $J$-marked set $G$ is a $J$-marked basis if and only if $J$ and the ideal generated by $G$ share the same Hilbert function (Corollaries 2.3 and 2.4). From now we suppose that $J$ is strongly stable and in section 3 define a total order (Definitions 3.3 and 3.9) on some special polynomials and give an algorithm to compute our normal forms by a rewriting procedure. This computation opens the access to effective methods for $J$-marked bases, as a Buchberger-like criterion (Theorem 3.12) that recognizes when a $J$-marked set is a $J$-marked basis $G$, also allowing to lift syzygies of $J$ to syzygies of $G$.

In section 4 we study the family $\mathcal{Mf}(J)$, computing it by the Buchberger-like criterion and showing that there is a bijective correspondence between the ideals of $\mathcal{Mf}(J)$ and the points of an affine scheme (Theorem 4.11). A possible objection to our construction is that it depends on a procedure of reduction, which is not unique in general. For this reason we show that $\mathcal{Mf}(J)$ has a structure of an affine scheme, that is given by the ideal generated by minors of some matrices and that is homogeneous with respect to a non-standard grading over the additive group $\mathbb{Z}^{n+1}$ (Lemma 4.2 and Theorem 4.3). Moreover, we note that $\mathcal{Mf}(J)$ is flat in $J$ and that the Castelnuovo-Mumford regularity of every ideal $I \in \mathcal{Mf}(J)$ is bounded from above by the Castelnuovo-Mumford regularity of $J$ (Proposition 4.5). In the Appendix, over a field $K$ of characteristic zero, we give an explicit computation of a family $\mathcal{Mf}(J)$ which is scheme-theoretically isomorphic to a locally closed subset of the Hilbert scheme of 8 points in $\mathbb{P}^2$ (see also [12]). We note that it strictly contains the union of all Gröbner strata with $J$ as initial ideal and that it is not isomorphic to an affine space, even though the point corresponding to $J$ is smooth.

We refer to [3, 10, 14] for definitions and results about Gröbner bases and to [20] for definitions and results about Hilbert functions of standard graded algebras.

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1. Generators of a Quotient $S/I$ and Generators of $I$

In this section we investigate relations among generators of a homogeneous ideal $I$ of $S$ and generators of the quotient $S/I$, under some fixed conditions on generators of $S/I$.

For every integer $m \geq 0$, the $K$-vector space of all $m$-degree homogeneous polynomials of $I$ is denoted $I_m$. The initial degree of an ideal $I$ is the integer $\alpha_I := \min\{m \in \mathbb{N} : I_m \neq 0\}$.

We will denote by $x^\alpha = x_0^{\alpha_0} \ldots x_n^{\alpha_n}$ any term in $S$, $|\alpha|$ is its degree, and we say that $x^\alpha$ divides $x^\beta$ (for short $x^\alpha | x^\beta$) if there exists a term $x^\gamma$ such that $x^\beta = x^\alpha x^\gamma$. For every term $x^\alpha \neq 1$ we set $\min(x^\alpha) = \min\{x_i : x_i | x^\alpha\}$ and $\max(x^\alpha) = \max\{x_i : x_i | x^\alpha\}$.

**Definition 1.1.** The support $\text{Supp}(h)$ of a polynomial $h$ is the set of terms that occur in $h$ with non-null coefficients.

If $J$ is a monomial ideal, $B_J$ denotes its (minimal) monomial basis and $\mathcal{N}(J)$ its sous-échelier, that is the set of terms outside $J$. For every polynomial $f$ of $J$, we get $\text{Supp}(f) \cap \mathcal{N}(J) = \emptyset$. 

Definition 1.2. Given a monomial ideal $J$ and an ideal $I$, a $J$-normal form modulo $I$ of a polynomial $h$ is a polynomial $h_0$ such that $h - h_0 \in I$ and $\text{Supp}(h_0) \subseteq \mathcal{N}(J)$.

If $I$ is homogeneous, the $J$-normal form modulo $I$ of a homogeneous polynomial $h$ is supposed to be homogeneous too.

Definition 1.3. A marked polynomial is a polynomial $f \in S$ together with a specified term of $\text{Supp}(f)$ that will be called head term of $f$ and denoted $\text{Ht}(f)$.

Definition 1.4. A finite set $G$ of homogeneous marked polynomials $f_\alpha = x^\alpha - \sum c_{\alpha \gamma} x^\gamma$, with $\text{Ht}(f_\alpha) = x^\alpha$, is called $J$-marked set if the head terms $\text{Ht}(f_\alpha)$ (different two by two) form the monomial basis $B_J$ of a monomial ideal $J$ and every $x^\gamma$ belongs to $\mathcal{N}(J)$, so that $|\text{Supp}(f) \cap J| = 1$. A $J$-marked set $G$ is a $J$-marked basis if $\mathcal{N}(J)$ is a basis of $S/G$ as a $K$-vector space, i.e. $S = (G) \oplus \langle \mathcal{N}(J) \rangle$ as a $K$-vector space.

Remark 1.5. The ideal $(G)$ generated by a $J$-marked basis $G$ has the same Hilbert function of $J$, hence $\dim_K J_m = \dim_K (G)_m$ for every $m \geq 0$, by the definition of $J$-marked basis itself.

Definition 1.6. The family of all homogeneous ideals $I$ such that $\mathcal{N}(J)$ is a basis of the quotient $S/I$ as a $K$-vector space will be denoted $\mathcal{Mf}(J)$ and called $J$-marked family.

Remark 1.7. (1) If $I$ belongs to $\mathcal{Mf}(J)$, then $I$ contains a $J$-marked set.

(2) A $J$-marked family $\mathcal{Mf}(J)$ contains every homogeneous ideal having $J$ as initial ideal with respect to some term order, but it can also contain other ideals, as we will see in Example 3.18.

Proposition 1.8. Let $G$ be a $J$-marked set. The following facts are equivalent:

(i) $G$ is a $J$-marked basis;
(ii) the ideal $(G)$ belongs to $\mathcal{Mf}(J)$;
(iii) every polynomial $h$ of $S$ has a unique $J$-normal form modulo $(G)$.

Proof. It follows by the definition of $J$-marked basis. \hfill \Box

Remark 1.9. A $J$-marked basis is unique for the ideal that it generates, by the unicity of $B_J$ and of the $J$-normal forms of monomials.

In next examples we will see that not every $J$-marked set $G$ is also a $J$-marked basis, even when $(G)$ and $J$ share the same Hilbert function. Moreover, it can happen that a $J$-marked set $G$ is not a $J$-marked basis, although there exists an ideal $I$ containing $G$ but not generated by $G$ such that $\mathcal{N}(J)$ is a $K$-basis for $S/I$.

Example 1.10. (i) In $K[x, y, z]$ let $J = (xy, z^2)$ and $I$ be the ideal generated by $f_1 = xy + yz, f_2 = z^2 + xz$, which form a $J$-marked set. Note that $J$ defines a 0-dimensional subscheme in $\mathbb{P}^2$, while $I$ defines a 1-dimensional subscheme, because it contains the line $x + z = 0$. Therefore, $I$ and $J$ do not have the same Hilbert function, so that ${f_1, f_2}$ is not a $J$-marked basis by Remark 1.5.

(ii) In $K[x, y, z]$, let $J = (xy, z^2)$ and $I$ be the ideal generated by $g_1 = xy + x^2 - yz, g_2 = z^2 + y^2 - xz$, which form a $J$-marked set. Note that $J$ and $I$ have the same Hilbert function because they are both complete intersections of two quadrics. However, $\mathcal{N}(J)$ is not free in $K[x, y, z]/I$ because $z g_1 + y g_2 = x^2 z + y^3 \in I$ is a sum of terms in $\mathcal{N}(J)$. Hence ${g_1, g_2}$ is not a $J$-marked basis.
(iii) In $K[x, y, z]$, let $J = (xy, z^2)$ and $I$ be the ideal generated by $f_1 = xy + yz, f_2 = z^2 + xz, f_3 = xyz$. Both $I$ and $J$ define $0$-dimensional subschemes in $\mathbb{P}^2$ of degree 4. Moreover, $I$ belongs to $\mathcal{Mf}(J)$ because for every $m \geq 2$ the $K$-vector space $U_m = I_m + \langle x^m, y^m, x^{m-1}z, y^{m-1}z \rangle$ is equal to $K[x, y, z]_m$. This is obvious for $m = 2$. Assume $m \geq 3$. Then, $U_m$ contains all the terms $y^{m-i}z^i$, because $y^2z = zf_1 - f_2$ belongs to $J$. Moreover $U_m$ contains all the terms $x^{m-i}y^i$ because $x^2y = xf_1 - f_3 \in I$ and $xy^{m-1} = y^{m-2}f_1 - y^{m-1} \in U_m$. Finally, by induction on $i$, we can see that all the terms $x^iz^{m-i}$ belong to $U_m$. Indeed, as already proved, $z^m$ belongs to $U_m$, hence $x^{i-1}z^{m-i+1} \in U_m$ implies $x^iz^{m-i} = x^{i-1}z^{m-i-1}f_2 - x^{i-1}z^{m-i+1} \in U_m$. However, the $J$-marked set $G = \{f_1, f_2\}$ does not generate $I$ and is not a $J$-marked basis, as shown in (i).

2. Strongly stable ideals $J$ and $J$-marked bases

In this section we show that the properties of $J$-marked sets improve decisively if $J$ is strongly stable.

Recall that a monomial ideal $J$ is strongly stable if and only if, for every $x_0^\alpha \cdots x_n^\alpha$ in $J$, also the term $x_0^\alpha \cdots x_i^{\alpha_i-1} x_j^{\alpha_j+1} \cdots x_n^\alpha$ belongs to $J$, for each $0 \leq i < j \leq n$ with $\alpha_i > 0$, or, equivalently, for every $x_0^{\beta_0} \cdots x_n^{\beta_n}$ in $\mathcal{N}(J)$, also the term $x_0^{\beta_0} \cdots x_h^{\beta_h+1} \cdots x_k^{\beta_k-1} \cdots x_n^{\beta_n}$ belongs to $\mathcal{N}(J)$, for each $0 \leq h < k \leq n$ with $\beta_k > 0$.

A strongly stable ideal is always Borel-fixed, that is fixed under the action of the Borel subgroup of lower-triangular invertibles matrices. If $ch(K) = 0$, also the vice versa holds (e.g. [5]) and [7] guarantees that in generic coordinates the initial ideal of an ideal $I$, with respect to a fixed term order, is a constant Borel-fixed monomial ideal, denoted $gin(I)$ and called the generic initial ideal of $I$.

In [17] a reduction relation $\rightarrow_F$ modulo a given set $\mathcal{F}$ of marked polynomials is defined in the usual sense of Gröbner bases theory and it is proved that, if $\rightarrow_F$ is Noetherian, then there exists an admissible term order $\prec$ on $S$ such that $Ht(f)$ is the $\prec$-leading term of $f$, for all $f \in \mathcal{F}$, being the converse already known [3]. If we take a $J$-marked set $G$, $\rightarrow_G$ can be non-Noetherian, as the following example shows. However, we will see that, if $J$ is a strongly stable ideal and $G$ is a $J$-marked set, every homogeneous polynomial has a $J$-normal form modulo $(G)$.

Example 2.1. Let us consider the $J$-marked set $G = \{f_1 = xy + yz, f_2 = z^2 + xz\}$, where $Ht(f_1) = xy$ and $Ht(f_2) = z^2$. The term $h = xyz$ can be rewritten only by $xyz - zf_1 = -yz^2$ and the term $-yz^2$ can be rewritten only by $-yz^2 + yf_2 = xyz$, which is again the term we wanted to rewrite. Hence, the reduction relation $\rightarrow_G$ is not Noetherian. Observe that in this case $J = (xy, z^2)$ is not strongly stable, but $\rightarrow_G$ can be non-Noetherian also if $J$ is strongly stable, as Example 3.18 will show.

Theorem 2.2. (Existence of $J$-normal forms) Let $G = \{f_\alpha = x^\alpha - \sum c_{\alpha\gamma}x^\gamma : Ht(f_\alpha) = x^\alpha \in B_J\}$ be a $J$-marked set, with $J$ strongly stable. Then, every polynomial of $S$ has a $J$-normal form modulo $(G)$.

Proof. It is sufficient to prove that our assertion holds for the terms, because $G$ is formed by homogeneous polynomials. Let us consider the set $E$ of terms which have not a $J$-normal form modulo $(G)$. Of course $E \cap B_J = \emptyset$. If $E$ is not empty and $x^\beta$ belongs to $E$, then $x^\beta = x^i x^\delta$ for some $x^\delta \in J$. We choose $x^\beta$ so that its degree $m$ is the minimum in $E$ and that, among the
terms of degree $m$ in $E$, $x_i$ is minimal. Let $\sum c_{\beta^i}x^\gamma$ be a $J$-normal form modulo $(G)$ of $x^\delta$, that exists by the minimality of $m$. Thus we can rewrite $x^\delta$ by $\sum c_{\beta^i}x_i^x\gamma$. We claim that all terms $x_i^x\gamma$ do not belong to $E$. On the contrary, if $x_i^x\gamma$ belongs to $E$, then $x_i^x\gamma = x_j^x\delta$ for some $x^\delta$ in $J$. If it were $x_i < x_j$ then, by the strongly stable property and since $x^\gamma$ belongs to $\mathcal{N}(J)$, we would get that $x^\delta = x_i^x\gamma/x_j$ belongs to $\mathcal{N}(J)$, that is impossible. So, we have $x_j < x_i$ and by the minimality of $x_i$ the term $x_i^x\gamma$ has a $J$-normal form modulo $(G)$. This is a contradiction and so $E$ is empty.

**Corollary 2.3.** If $J$ is a strongly stable ideal and $I$ a homogeneous ideal containing a $J$-marked set $G$, then $\mathcal{N}(J)$ generates $S/I$ as a $K$-vector space, so that $\dim_K I_m \geq \dim_K J_m$, for every $m \geq 0$.

**Proof.** By Theorem 2.2, for every polynomial $h$ there exists a polynomial $h_0$ such that $h - h_0$ belongs to $(G) \subseteq I$ and Supp($h_0$) $\subseteq \mathcal{N}(J)$. So, all the elements of $S/I$ are linear combinations of terms of $\mathcal{N}(J)$ and the thesis follows. □

**Corollary 2.4.** Let $J$ be a strongly stable ideal and $G$ be a $J$-marked set. Then, $G$ is a $J$-marked basis if and only if $\dim_K(I)_m \leq \dim_K(J)_m$, for every $m \geq 0$ or, equivalently, $\mathcal{N}(J)$ is free in $S/(G)$.

**Proof.** By Proposition 1.8 $G$ is a $J$-marked basis if and only if every polynomial has a unique $J$-normal form modulo $(G)$. So, it is enough to apply Theorem 2.2 and Corollary 2.3. □

**Corollary 2.5.** Let $J$ be a strongly stable ideal and $I$ be a homogeneous ideal. Then $I$ belongs to $\mathcal{Mf}(J)$ if and only if $I$ has a $J$-marked basis.

**Proof.** If $I$ has a $J$-marked basis then $I$ belongs to $\mathcal{Mf}(J)$ by definition. Vice versa, apply Remark 1.7(1) and Corollary 2.4. □

**Remark 2.6.** Every reduced Gröbner basis of a homogeneous ideal with respect to a graded term order is a $J$-marked basis for some monomial ideal $J$, hence every homogeneous ideal contains a $J$-marked basis. But, unless we are in generic coordinates, not every (homogeneous) ideal contains a $J$-marked basis with $J$ strongly stable, as for example a monomial ideal which is not strongly stable.

Let $G$ be a $J$-marked basis with $J$ strongly stable. Thanks to the existence and the unicity of $J$-normal forms, $G$ can behave like a Gröbner basis in solving problems, as the membership ideal problem in the homogeneous case. Indeed, by the unicity of $J$-normal forms, a polynomial belongs to the ideal $(G)$ if and only if its $J$-normal form modulo $(G)$ is null. But, until now, we have not a computational method to construct $J$-normal forms yet.

In next section, by exploiting the proof of Theorem 2.2, we provide an algorithm which, in the hypothesis that $J$ is strongly stable, reduces every homogeneous polynomial to a $J$-normal form modulo $(G)$ in a finite number of steps, although $G$ is not necessarily Noetherian. This fact allows us also to recognize when a $J$-marked set is a $J$-marked basis by a Buchberger-like criterion and, hence, to develop effective computational aspects of $J$-marked bases.

### 3. Effective methods for $J$-marked bases

Let $I$ be the homogeneous ideal generated by a $J$-marked set $G = \{f_{\alpha} = x^\alpha - \sum c_{\alpha\gamma}x^\gamma : Ht(f_{\alpha}) = x^\delta \in B_J\}$, where $J$ is strongly stable, so that every polynomial has a $J$-normal form modulo $I$, by Theorem 2.2.
In this section we obtain an efficient procedure to compute in a finite number of steps a $J$-normal form modulo $I$ of every homogeneous polynomial. To this aim, we need some more definitions and results.

For every degree $m$, the $K$-vector space $I_m$ formed by the homogeneous polynomials of degree $m$ of $I$ is generated by the set $W_m = \{x^\beta f_\alpha : x^{\beta + \alpha} \text{ has degree } m, f_\alpha \in G\}$, that becomes a set of marked polynomials letting $Ht(x^\beta f_\alpha) = x^{\beta + \alpha}$.

**Lemma 3.1.** Let $x^\beta$ be a term of $J_m \setminus B_J$ and $x_i = \min(x^\beta)$. Then $x^\beta/x_i$ belongs to $J_{m-1}$.

*Proof.* By the hypothesis there exists at least a term of $J_{m-1}$ that divides the given term $x^\beta$. So, let $x_j$ such that $x^\beta/x_j$ belongs to $J_{m-1}$. If $x_j = x_i$, we are done. Otherwise, we get $x^\beta = x_i x_j x^\delta$, for some term $x^\delta$, so that $x_i x^\delta = x^\beta/x_j$ belongs to $J_{m-1}$. By the definition of a strongly stable ideal and since $x_j > x_i$, we obtain that $x^\beta/x_i = x_j x^\delta$ belongs to $J_{m-1}$. □

**Definition 3.2.** For every $m \geq \alpha_J$, we define the following special subset of $W_m$, by induction on $m$. If $m = \alpha_J$ is the initial degree of $J$, we set $V_m := G_m$. For every $m > \alpha_J$, we set $V_m := G_m \cup \{g_\beta : x^\beta \in J_m \setminus G_m\}$, where $g_\beta := x_i g_\delta$ with $x_i = \min(x^\beta)$ and $g_\delta$ the unique polynomial of $V_{m-1}$ with head term $x^\lambda = x^\beta/x_i$.

**Remark 3.3.** By construction, for every element $g_\beta$ of $V_m \subseteq W_m$ there exist $x^\delta$ and $f_\alpha \in G$ such that $g_\beta = x^\delta f_\alpha$ and $x^\delta = 1$ or $\max(x^\delta) \leq \min(x^\alpha)$. In particular, we get $\min(x^\delta) = \min(x^\beta)$. Note that Definition 3.2 makes sense due to Lemma 3.1.

For every integer $m \geq \alpha_J$, we define the following total order $\succeq_m$ on $V_m$.

**Definition 3.4.** For every $f_\alpha, f_\alpha' \in G_m$, we set $f_\alpha \succeq_m f_\alpha'$ if and only if $Ht(f_\alpha) \geq Ht(f_\alpha')$ with respect to a fixed term order $\geq$. For every $g_\beta \in V_m \setminus G_m$ and $f_\alpha \in G_m$, we set $g_\beta \succeq_m f_\alpha$. For every $m > \alpha_J$, given $x_i g_\delta, x_j g_\eta \in V_m \setminus G_m$, where $x_i = \min(x_i x^\lambda)$ and $x_j = \min(x_j x^\delta)$, we set $$x_i g_\delta \succeq_m x_j g_\eta \iff x_i > x_j \text{ or } x_i = x_j \text{ and } g_\delta \succeq_m g_\eta.$$

By the definition of $V_m$ and by well-known properties of a strongly stable ideal, we get the routine VCONSTRUCTOR to compute $V_m$, for every $\alpha_J \leq m \leq s$.

**Lemma 3.5.** With the above notation,

$$x_i g_\delta \in V_m \setminus G_m \text{ and } x^\beta \in \text{Supp}(x_i g_\delta) \setminus \{x_i x^\lambda\} \text{ with } g_\beta \in V_m \Rightarrow x_i g_\delta \succeq_m g_\beta.$$

*Proof.* By induction on $m$, first observe that for $m = \alpha_J$ there is nothing to prove because $V_{\alpha_J} = G_{\alpha_J}$. For $m > \alpha_J$, let $g_\beta = x_j g_\eta \not\in G_m$. If $x_i = x_j$, then $x^\eta$ belongs to $\text{Supp}(g_\eta) \setminus \{x^\lambda\}$ and, by the induction, we have $g_\eta \succeq_{m-1} g_\delta$. Otherwise, note that every term of $\text{Supp}(x_i g_\delta)$ is divided by $x_i$, so $x_j x^\eta = x_i x^\lambda$ and, by Remark 3.3 we get $x_j = \min(x^\beta) = \min(x_i x^\lambda) \leq x_i$. □

**Proposition 3.6.** (Construction of $J$-normal forms) With the above notation, every term $x^\beta \in J_m \setminus G_m$ can be reduced to a $J$-normal form modulo $I$ in a finite number of reduction steps, using only polynomials of $V_m$. Hence, the reduction relation $\xrightarrow{\text{V}}$ is Noetherian in $S_m$.

*Proof.* By definition of $V_m$, every term $x^\beta$ of $J_m$ is the head term of one and only one polynomial $g_\beta$ of $V_m \subseteq W_m$. Hence, we rewrite $x^\beta$ by $g_\beta$ getting a $K$-linear combination of terms belonging to $\text{Supp}(g_\beta) \setminus \{x^\beta\}$. Applying Lemma 3.5 repeatedly, we are done since $V_m$ is a finite set. □
1: procedure \textit{VConstructor}(G,s) \rightarrow V_{\alpha}, \ldots, V_{s}

\textbf{Require:} \ G is a \ J\text{-}marked set ordered with respect to a graduate term order on the head terms, with \ J\ a strongly stable ideal, and \ s \geq \alpha_J.

\textbf{Ensure:} \ V_m\ ordered by \ \succeq_m, \ for \ every \ \alpha_J \leq m \leq s

2: \ \alpha_J := \min\{deg(Ht(f_\alpha))|f_\alpha \in G\}

3: \ V_{\alpha_J} := G_\alpha

4: \textbf{for} \ m = \alpha_J + 1 \ \textbf{to} \ s \ \textbf{do}

5: \ V_m := G_m;

6: \textbf{for} \ i = 0 \ \textbf{to} \ n \ \textbf{do}

7: \textbf{for} \ j = 1 \ \textbf{to} \ |V_{m-1}| \ \textbf{do}

8: \textbf{if} \ i \leq \min(Ht(V_{m-1}[j])) \ \textbf{then}

9: \ V_m = V_m \cup \{x_iV_{m-1}[j]\}

10: \textbf{end if}

11: \textbf{end for}

12: \textbf{end for}

13: \textbf{end for}

14: \textbf{return} V_{\alpha}, \ldots, V_s;

15: \textbf{end procedure}

\textbf{Definition 3.7.} A homogeneous polynomial, with support contained in \ \mathcal{N}(J)\ and in relation by \ \mathcal{V}_m\ to a homogeneous polynomial \ h\ of degree \ m, is denoted \ \bar{h}\ and called \ \mathcal{V}_m\-reduction of \ h.

For every homogeneous polynomial \ h\ of degree \ m, \ \bar{h}\ is a \ J\-normal form modulo \ I. \ Hence, from the procedure described in the proof of Proposition 3.6 we obtain the routine \textit{NormalFormConstructor} that, actually, form a step of a division algorithm with respect to a \ J\-marked set, with \ J\ strongly stable.

1: procedure \textit{NormalFormConstructor}(h,V_m) \rightarrow \bar{h}

\textbf{Require:} \ h\ is a homogeneous polynomial of degree \ m

\textbf{Require:} \ a \ list \ V_m, \ as \ defined \ in \ Definition 3.2, \ and \ ordered \ by \ \succeq_m

\textbf{Ensure:} \ \mathcal{V}_m\-reduction \ \bar{h} \ of \ h

2: \ L := |V_m|;

3: \textbf{for} \ K = 1 \ \textbf{to} \ L \ \textbf{do}

4: \ x^n := Ht(V_m[K]);

5: \ a := \text{coefficient of} \ x^n \ \text{in} \ h;

6: \textbf{if} \ a \neq 0 \ \textbf{then}

7: \ h := h - a \cdot V_m[K];

8: \textbf{end if}

9: \textbf{end for}

10: \textbf{return} h;

11: \textbf{end procedure}

\textbf{Remark 3.8.} There is a strong analogy between the union of the sets \ V_m\ and the so-called \ staggered bases, introduced by [8] and studied also by [15].
Now, we extend to $W_m$ the order $\geq_m$ defined on $V_m$. Recall that, in our setting, a term $x^\delta$ is higher than a term $x^{\delta'}$ with respect to the degree reverse lexicographic term order (for short $x^\delta >_{drl} x^{\delta'}$) if $|\delta| > |\delta'|$ or $|\delta| = |\delta'|$ and the first non null entry of $\delta - \delta'$ is negative.

**Definition 3.9.** Let the polynomials of $G_m$ be ordered as in Definition 3.3 and $x^\delta f_\alpha$, $x^{\delta'} f_{\alpha'}$ be two elements of $W_m$. We set

$$x^\delta f_\alpha \geq_m x^{\delta'} f_{\alpha'} \iff x^\delta >_{drl} x^{\delta'} \quad \text{or} \quad x^\delta = x^{\delta'} \quad \text{and} \quad Ht(f_\alpha) \geq Ht(f_{\alpha'}).$$

**Lemma 3.10.** (i) For every two elements $x^\delta f_\alpha$, $x^{\delta'} f_{\alpha'}$ of $W_m$ we get

$$x^\delta f_\alpha \geq_m x^{\delta'} f_{\alpha'} \Rightarrow \forall x^\eta: \quad x^{\delta+\eta} f_\alpha \geq_m x^{\delta'+\eta} f_{\alpha'},$$

where $m' = |\delta + \eta + \alpha|$.

(ii) Every polynomial $g_\beta \in V_m$ is the minimum with respect to $\leq_m$ of the subset $W_\beta$ of $W_m$ containing all polynomials of $W_m$ with $x^\delta$ as head term.

(iii) $x^\delta f_\alpha \in W_m \setminus G_m$ and $x^{\delta'} \in \text{Supp}(x^\delta x^\alpha) \setminus \{x^\delta x^\alpha\}$ with $g_\beta \in V_m \Rightarrow x^\delta f_\alpha \succ_m g_\beta$.

**Proof.** (i) It follows by the analogous property of every term order.

(ii) The statement holds by construction of $V_m$ and by Remark 3.3. Indeed, by same arguments as before, if $x^\delta f_\alpha$ is any polynomial of $W_\beta$ and $g_\beta = x^{\delta'} f_{\alpha'} \in V_m$, with $\max(x^{\delta'}) \leq \min(x^{\alpha'})$ as in Remark 3.3 then $x_j = \min(x^{\delta'}) = \min(x^{\delta'+\alpha'}) = \min(x^{\delta+\alpha}) \leq \min(x^{\delta})$. If the equality holds, it is enough to observe that $\frac{x^\delta}{x^\delta} f_\alpha \in W_{m-1}$ and $\frac{x^\delta}{x^\delta} f_{\alpha'} \in V_{m-1}$ by construction.

(iii) It is analogous to the proof of Lemma 3.5. If $x^\beta$ belongs to $J_m$ we are done. Otherwise, note that every term of $\text{Supp}(x^\delta f_\alpha)$ is multiple of $x^\delta$, in particular $x^{\delta'+\alpha'} = x^{\delta+\gamma}$ for some $x^\gamma \in N(J)$. Let $x_i = \min(x^{\delta})$ and $x_j = \min(x^{\delta'})$. By Remark 3.3 we get $x_j = \min(x^{\delta'+\alpha'}) = \min(x^{\delta+\gamma}) \leq \min(x^{\delta}) = x_i$. If $x_j = x_i$, then $x^\beta/x_i$ belongs to the support of $\frac{x^\delta}{x^\delta} f_\alpha$ and use induction.

\[\square\]

In Remark 2.6 we have already observed that in generic coordinates every homogeneous ideal has a $J$-marked basis, with $J$ strongly stable. Now, given a strongly stable ideal $J$, we describe a Buchberger-like algorithmic method to check if a $J$-marked set is or not a $J$-marked basis, recovering the well-known notion of $S$-polynomial from the Gröbner bases theory.

**Definition 3.11.** The $S$-polynomial of two elements $f_\alpha$, $f_{\alpha'}$ of a $J$-marked set $G$ is the polynomial $S(f_\alpha, f_{\alpha'}) := x^\delta f_\alpha - x^{\delta'} f_{\alpha'}$, where $x^{\delta+\alpha} = x^{\delta'+\alpha'} = \text{lcm}(x^\alpha, x^{\alpha'})$.

**Theorem 3.12.** (Buchberger-like criterion) Let $J$ be a strongly stable ideal and $I$ the homogeneous ideal generated by a $J$-marked set $G$. With the above notation:

$$I \in \mathcal{Mf}(J) \iff S(f_\alpha, f_{\alpha'}) = 0, \forall f_\alpha, f_{\alpha'} \in G.$$

**Proof.** Recall that $I \in \mathcal{Mf}(J)$ if and only if $G$ is a $J$-marked basis, so that every polynomial has a unique $J$-normal form modulo $I$. Since $S(f_\alpha, f_{\alpha'})$ belongs to $I$ by construction, its $J$-normal form modulo $J$ is null and coincides with $S(f_\alpha, f_{\alpha'})$, by the unicity of $J$-normal forms.

For the converse, by Corollary 2.4 it is enough to show that, for every $m$, the $K$-vector space $I_m$ is generated by the $\dim_K J_m$ elements of $V_m$. More precisely we will show that every polynomial $x^\delta f_\alpha \in W_m$ either belongs to $V_m$ or is a $K$-linear combination of elements of $V_m$ lower than $x^\delta f_\alpha$ itself. We may assume that this fact holds for every polynomial in $W_m$ lower than $x^\delta f_\alpha$. If $x^\delta f_\alpha$ belongs to $V_m$ there is nothing to prove. If $x^\delta f_\alpha$ does not belong
to $V_m$, let $x^\delta f_{\alpha'} = \min(W_{\delta+\alpha}) \in V_m$, so that $x^\delta f_\alpha \succ_m x^\delta f_{\alpha'}$, and consider the polynomial $g = x^\delta f_\alpha - x^\delta f_{\alpha'}$.

If $g$ is the $S$-polynomial $S(f_\alpha, f_{\alpha'})$, then it is a $K$-linear combination $\sum c_i g_{\eta_i}$ of polynomials of $V_m$ because $\overline{S(f_\alpha, f_{\alpha'})} = 0$ by the hypothesis. Moreover, by construction, $x^\delta f_{\alpha'}$ belongs to $V_m$ and, thanks to Lemma 3.10 for all $i$ we have $x^\delta f_\alpha \succ_m g_{\eta_i}$.

If $g$ is not the $S$-polynomial $S(f_\alpha, f_{\alpha'})$, then there exists a term $x^\beta \neq 1$ such that $g = x^\beta S(f_\alpha, f_{\alpha'}) = x^\beta(x^{\eta} f_\alpha - x^{\eta} f_{\alpha'})$, where recall that by the hypothesis $S(f_\alpha, f_{\alpha'})$ is a $K$-linear combination $\sum c_i g_{\eta_i}$ of elements of $V_m - \delta$ lower than $x^\eta f_\alpha$, being again $x^\eta f_\alpha \succ_m x^\eta f_{\alpha'}$. Hence, $x^\delta f_\alpha = x^\delta f_{\alpha'} + \sum c_i x^\beta g_{\eta_i}$, where all polynomials appearing in the right hand are lower than $x^\delta f_{\alpha'}$ with respect to $\succ_m$. So we can apply to them the inductive hypothesis for which either they are elements of $V_m$ or they are $K$-linear combinations of lower elements in $V_m$. This allows us to conclude.

Let $H = (h_1, \ldots, h_t)$ be a syzygy of a $J$-basis $G = \{f_{\alpha_1}, \ldots, f_{\alpha_t}\}$ such that every polynomial $h_i \sum c_i x^\beta$ is homogeneous and every product $h_i f_{\alpha_i}$ has the same degree $m$. A syzygy $M = (m_1, \ldots, m_t)$ of $J$ is homogeneous if, for every $1 \leq i \leq t$, we have $m_i x^{\alpha_i} = c_{ie} x^e$, for a constant term $x^e$ and $c_{ie} \in K$.

**Definition 3.13.** The head term $Ht(H)$ of the syzygy $H$ is the head term of the polynomial $H_{\text{max}} := \max_{i \in \{1, \ldots, t\}} \{x^\beta f_{\alpha_i} : x^\beta \in \text{Supp}(h_i)\}$. If $Ht(H) = x^\eta$, let $H^+ = (h_1^+, \ldots, h_t^+)$ be the $t$-uple such that $h_i^+ = c_i x^\beta$, where $x^\beta x^{\alpha_i} = x^\eta$, i.e. $x^\beta f_{\alpha_i} \in W_{\eta}$. Given a homogeneous syzygy $M$ of $J$, we say that $H$ is a lifting of $M$, or that $M$ lifts to $H$, if $H^+ = M$.

**Corollary 3.14.** Every homogeneous syzygy of $J$ lifts to a syzygy of a $J$-marked basis $G$.

**Proof.** Recall that syzygies of type $(0, \ldots, x^\beta, \ldots, -x^\beta', 0, \ldots)$ form a system of homogeneous generators of syzygies of $B_J = \{\ldots, x^{\alpha}, \ldots, x^{\alpha'} \}$, where $x^{\alpha+\beta} = x^{\beta+\alpha'} = \text{lcm}(x^{\alpha}, x^{\alpha'})$. Thus, apply Theorem 3.12.

Until now we have shown that a $J$-marked basis satisfies the characterizing properties of a Gröbner basis. In the following result we consider a property that does not characterize Gröbner bases, but it is satisfied by Gröbner bases. We show that it is satisfied by $J$-marked bases too, by standard arguments.

**Corollary 3.15.** Let $\{M_1, \ldots, M_t\}$ be a set of homogeneous generators of the module of syzygies of $J$. Then, a set $\{K_1, \ldots, K_t\}$ of liftings of the $M_i$’s generates the module of syzygies of $G$.

**Proof.** First, observe that the module of syzygies of $G = \{f_{\alpha_1}, \ldots, f_{\alpha_t}\}$ is generated by the syzygies $H = (h_1, \ldots, h_t)$ such that every $h_i = \sum c_i x^\beta$ is a homogeneous polynomial and every product $h_i f_{\alpha_i}$ has the same degree $m$. Let $H^+$ the syzygy of $J$, as computed in Definition 3.13. Hence, there exist homogeneous polynomials $q_1, \ldots, q_t$ such that $H^+ = \sum q_i M_i$. Let $H_1 = H - \sum q_i K_i$. By construction we get that $H_{\text{max}}(H_1) \prec_m H_{\text{max}}(H)$, by Lemmas 3.3 and 3.10. Since $\lesssim_m$ is a total order on the finite set $W_m$, we can conclude.

**Remark 3.16.** In the proof of Theorem 3.12 we do not use $V_m$-reductions of all $S$-polynomials $x^\delta f_\alpha - x^\delta f_{\alpha'}$ of elements in $G$, but only of those such that either $x^\delta f_\alpha$ or $x^\delta f_{\alpha'}$ belongs to some $V_m$. Moreover, we can consider the analogous property to that of the improved Buchberger algorithm that only considers $S$-polynomials corresponding to a set of generators for the syzygies.
of $J$. Thus we can improve Corollary 2.24 and say that, in the same hypotheses:

$$I \in \mathcal{Mf}(J) \iff \forall m \leq m_0, \ dim_k I_m = dim_k J_m \iff \forall m \leq m_0, \ dim_k I_m \leq dim_k J_m$$

where $m_0$ is the maximum degree of generators of syzygies of $J$. Hence, to prove that $dim_k I_m = dim_k J_m$ for some $m$ it is sufficient that the $V_m$-reductions of the $S$-polynomials of degree $\leq m$ are null.

**Example 3.17.** Let $J = (x^2, xy, xz, y^3) \subset k[x, y, z]$, where $x > y > z$ and consider a $J$-marked set $G = \{f_{x^2}, f_{xy}, f_{xz}, f_{y^3}\}$. In order to check whether $G$ is a $J$-marked basis it is sufficient to verify if the polynomials $S(f_{x^2}, f_{xy}), S(f_{x^2}, f_{xz}), S(f_{xz}, f_{y^3})$ have $V_m$-reductions null, but it is not necessary to control $S(f_{xz}, f_{y^3})$ because $yzf_{xy}$ is the element of $V_3$ with head term $xy^2z$.

**Example 3.18.** Let $J = (x^3, x^2y, xy^2, y^3)_{\geq 4}$ be a strongly stable ideal in $k[x, y, z]$, with $x > y > z$, and $G = B_j \cup \{f\} \setminus \{xy^2z\}$ a $J$-marked set, where $f = xy^2z - y^4 - x^2z^2$ with $Ht(f) = xy^2z$. We can verify that $G$ is a $J$-marked basis using the Buchberger-like criterion proved in Theorem 3.12. Indeed, the $S$-polynomials non involving $f$ vanish and all the $S$-polynomials involving $f$ are multiple of either $x \cdot (y^4 + x^2z^2)$ or $y \cdot (y^4 + x^2z^2)$. Since the terms $y^4 \cdot x, y^4 \cdot y, x^2z^2 \cdot x, x^2z^2 \cdot y$ belong to $V_5$, all the $S$-polynomials have $V_m$-reductions null. Notice also that, in this case, $G'$ is not Noetherian because, although the $V_7$-reduction of $x^2y^2z^3$ is 0, being $z^2 \cdot x^2y^2z \in V_7$ (while $xzf \notin V_7$), a different choice of reduction gives the loop:

$$x^2y^2z^3 \xrightarrow{f} xy^4z^2 + x^2z^4 \xrightarrow{x^2z^2} xy^4z^2 \xrightarrow{f} y^6z + x^2y^2z^3 \xrightarrow{y^5} x^2y^2z^3.$$  

Moreover, $G$ is not a Gröbner basis with respect to any term order $\prec$. Indeed, $xy^2z^2 \triangleright y^4z$ and $xy^2z^2 \triangleright x^2z^3$ would be in contradiction with the equality $(xy^2z^2)^2 = x^2z^3 \cdot y^4z$.

4. **J-marked families as affine schemes**

In this section $J$ is always supposed strongly stable, so that we can use all results described in the previous sections for $J$-marked bases.

Here we provide the construction of an affine scheme whose points correspond, one to one, to the ideals of the $J$-marked family $\mathcal{Mf}(J)$. Recall that $\mathcal{Mf}(J)$ is the family of all homogeneous ideals $I$ such that $\mathcal{N}(J)$ is a basis for $S/I$ as a $K$-vector space, hence $\mathcal{Mf}(J)$ contains all homogeneous ideals for which $J$ is the initial ideal with respect to a fixed term order. We generalize to any strongly stable ideal $J$ an approach already proposed in literature in case $J$ is considered an initial ideal (e.g. [4, 6, 11, 18, 19]).

For every $x^\alpha \in B_J$, let $F_\alpha := x^\alpha - \sum C_{\alpha \gamma}x^\gamma$, where $x^\gamma$ belongs to $\mathcal{N}(J)_{[\alpha]}$ and the $C_{\alpha \gamma}$’s are new variables. Let $C$ be the set of such new variables and $N := |C|$. The set $G$ of all the polynomials $F_\alpha$ becomes a $J$-marked set letting $Ht(F_\alpha) = x^\alpha$. From $G$ we can obtain the J-marked basis of every ideal $I \in \mathcal{Mf}(J)$ specializing in a unique way the variables $C$ in $K^N$, since every ideal $I \in \mathcal{Mf}(J)$ has a unique J-marked basis (Remark 1.9 and Corollary 2.5). But not every specialization gives rise to an ideal of $\mathcal{Mf}(J)$.

Let $\mathcal{V}_m$ be the analogous for $G$ of $V_m$ for any $G$. Let $H_{\alpha \alpha'}$ be the $\mathcal{V}_m$-reductions of the $S$-polynomials $S(F_\alpha, F_{\alpha'})$ of elements of $G$ and extract their coefficients that are polynomials in $K[C]$. We will denote by $\mathfrak{R}$ the ideal of $K[C]$ generated by these coefficients. Let $\mathfrak{R}'$ be the ideal of $K[C]$ obtained in the same way of $\mathfrak{R}$ but only considering $S$-polynomials $S(F_\alpha, F_{\alpha'}) = x^\delta F_\alpha - x^\delta F_{\alpha'}$ such that $x^\delta F_\alpha$ is minimal among those with head term $x^{\delta+\alpha}$. 
Theorem 4.1. There is a one to one correspondence between the ideals of \( \mathcal{M}(J) \) and the points of the affine scheme in \( K^N \) defined by the ideal \( \mathcal{A} \). Moreover, \( \mathcal{A}' = \mathcal{A} \).

Proof. For the first assertion it is enough to apply Theorem 3.12 observing that a specialization of the variables \( C \) in \( K^N \) gives rise to a \( J \)-marked basis if and only if the values chosen for the variables \( C \) form a point of \( K^N \) on which all polynomials of the ideal \( \mathcal{A} \) vanish.

For the second assertion, first recall that, by Remark 3.16, every \( S \)-polynomial \( x^\delta F_\alpha - x^\delta' F_{\alpha'} \) can be written as the sum \( (x^\delta F_\alpha - x^\delta'' F_{\alpha''}) + (x^\delta'' F_{\alpha''} - x^\delta' F_{\alpha'}) \) of two \( S \)-polynomials, where \( x^\delta'' F_{\alpha''} \) belongs to \( V_m \). Note that, considering the variables \( C \) as parameters, the support of \( x^\delta F_\alpha - x^\delta' F_{\alpha'} \) is contained in the union of the supports of \( x^\delta F_\alpha - x^\delta'' F_{\alpha''} \) and of \( x^\delta'' F_{\alpha''} - x^\delta' F_{\alpha'} \). In particular, the coefficients in \( x^\delta F_\alpha - x^\delta' F_{\alpha'} \), i.e. the generators of \( \mathcal{A} \), are combinations of the coefficients in \( (x^\delta F_\alpha - x^\delta'' F_{\alpha''}) + (x^\delta'' F_{\alpha''} - x^\delta' F_{\alpha'}) \), i.e. of the generators of \( \mathcal{A}' \). \( \Box \)

Now, by exploiting ideas of \([11]\), we show how to obtain \( \mathcal{A} \) in a different way, using the rank of some matrices.

By Corollary 4.11, a specialization \( C \to c \in K^N \) transforms \( G \) in a \( J \)-basis \( G \) if and only if \( \dim_K(G)_m = \dim_K J_m \), for every degree \( m \). Thus, for each \( m \), consider the matrix \( A_m \) whose columns correspond to the terms of degree \( m \) in \( S = K[x_1, \ldots, x_n] \) and whose rows contain the coefficients of the terms in every polynomial of degree \( m \) of type \( x^\beta F_\alpha \). Hence, every entry of the matrix \( A_m \) is \( 1 \), \( 0 \) or one of the variables \( C \). Let \( \mathcal{A} \) be the ideal of \( K[C] \) generated by the minors of order \( \dim_K J_m + 1 \) of \( A_m \), for every \( m \).

Lemma 4.2. The ideal \( \mathcal{A} \) is equal to the ideal \( \mathcal{A}' \).

Proof. Let \( a_m = \dim_K J_m \). We consider in \( A_m \) the \( a_m \times a_m \) submatrix whose columns corresponds to the terms in \( J_m \) and whose rows are given by the polynomials \( x^\beta F_\alpha \) that are minimal with respect to the partial order \( \succ_m \). Up to a permutation of rows and columns, this submatrix is upper-triangular with \( 1 \) on the main diagonal. We may also assume that it corresponds to the first \( a_m \) rows and columns in \( A_m \). Then the ideal \( \mathcal{A} \) is generated by the determinants of \( a_m + 1 \times a_m + 1 \) submatrices containing that above considered. Moreover the Gaussian row-reduction of \( A_m \) with respect to the first \( a_m \) rows is nothing else than the \( V_m \)-reduction of the \( S \)-polynomials of the special type considered defining \( \mathcal{A}' \). \( \Box \)

Definition 4.3. The affine scheme \( S(J) \) defined by the ideal \( \mathcal{A} = \mathcal{A}' = \mathcal{A} \) is called \( J \)-marked scheme.

Theorem 4.4. The \( J \)-marked scheme \( S(J) \) is homogeneous with respect to a non-standard grading \( \lambda \) of \( K[C] \) over the group \( \mathbb{Z}^{n+1} \) given by \( \lambda(C_{\alpha \gamma}) = \alpha - \gamma \).

Proof. To prove that \( \mathcal{M}(J) \) is \( \lambda \)-homogeneous it is sufficient to show that every minor of \( A_m \) is \( \lambda \)-homogeneous. Let us denote by \( C_{\alpha \alpha} \) the coefficient \( (= 1) \) of \( x^\alpha \) in every polynomial \( F_\alpha \): we can apply also to the “symbol” \( C_{\alpha \alpha} \) the definition of \( \lambda \)-degree of the variables \( C_{\alpha \gamma} \), because \( \alpha - \alpha = 0 \) is indeed the \( \lambda \)-degree of the constant 1. In this way, the entry in the row \( x^\beta F_\alpha \) and in the column \( x^\delta \) is \( \pm C_{\alpha \gamma} \) if \( x^\delta = x^\beta x^\gamma \) and is 0 otherwise.

Let us consider the minor of order \( s \) determined in the matrix \( A_m \) by the \( s \) rows corresponding to \( x^\beta F_\alpha \) and by the \( s \) columns corresponding to \( X^\gamma_i, i = 1, \ldots, s \). Every monomial that appears in the computation of such a minor is of type \( \prod_{i=1}^s C_{\alpha_i \gamma_i} \) with \( x^{\delta_i} = x^\beta_i x^{\gamma_i} \). Then its
and only if such a term ordering exists and, in this case, we can isomorphically project \( S \) in the Zariski tangent space at the origin (see [6]). As a consequence of this projection we can does not exist, then in general we cannot embed \( S \) the geometrical features of the affine scheme \( J \) affine space, provided the origin is a smooth point. If for a given ideal \( J \) we can find a suitable term ordering such that \( \text{St}_t(J, \prec) = S(J) \), but there are cases in which \( \bigcup_\prec \text{St}_t(J, \prec) \) is strictly contained in \( S(J) \) (see the Appendix).

The existence of a term order such that \( S(J) = \text{St}_t(J, \prec) \) has interesting consequences on the geometrical features of the affine scheme \( S(J) \). In fact the \( \lambda \)-grading on \( k[C] \) is positive if and only if such a term order exists and, in this case, we can isomorphically project \( S(J) \) in the Zariski tangent space at the origin (see [3]). As a consequence of this projection we can prove, for instance, that the affine scheme \( S(J) \) is connected and that it is isomorphic to an affine space, provided the origin is a smooth point. If for a given ideal \( J \) such a term ordering does not exist, then in general we cannot embed \( S(J) \) in the Zariski tangent space at the origin (see the Appendix). However we do not know examples of Borel ideals \( J \) such that either \( S(J) \) has more than one connected component or \( J \) is smooth and \( S(J) \) is not rational.

Denote \( \text{reg}(I) \) the Castelnuovo-Mumford regularity of a homogeneous ideal \( I \).

**Proposition 4.5.** A \( J \)-marked family \( \mathcal{Mf}(J) \) is flat in the origin. In particular, for every ideal \( I \) in \( \mathcal{Mf}(J) \), we get \( \text{reg}(J) \geq \text{reg}(I) \).

**Proof.** Analogously to what is suggested in [2] and by referring to [11] Corollary, section 3, part I], we know that \( \mathcal{Mf}(J) \) is a flat family at \( J \), i.e. at the point \( C = 0 \), if and only if every syzygy of \( J \) lifts to a syzygy among the polynomials of \( \mathcal{G} \) or, equivalently, the restrictions to \( C = 0 \) of the syzygies of \( \mathcal{G} \) generate the \( S \)-module of syzygies of \( J \). By Corollary [3.14] we know that every syzygy of \( J \) lifts to a syzygy of \( G \), for every specialization of \( C \) in the affine scheme defined by the ideal \( \mathcal{R} \). And this is true thanks to Theorem [3.12] that allows also to lift a syzygy of \( J \) to a syzygy of \( \mathcal{G} \) over the ring \( (K[C]/\mathcal{R})[x_0, \ldots, x_n] \). So, the first assertion holds.

For the second assertion, it is enough to recall that Castelnuovo-Mumford regularity is upper semicontinuous in flat families [9 Theorem 12.8, Chapter III] and that in our case the syzygies of \( J \) lift to syzygies of \( G \) for every specialization of the variables \( C \) in the affine scheme \( S(J) \), i.e. for every ideal \( I \) of \( \mathcal{Mf}(J) \), not only in some neighborhood of \( J \). □

**Remark 4.6.** A given homogeneous ideal \( I \) belongs to \( \mathcal{Mf}(J) \) if and only \( I \) has the same Hilbert function of \( J \) and the affine scheme defined by the ideal of \( K[C] \) generated by \( \mathcal{R} \) and by the coefficients of the \( \mathcal{V}_m \)-reductions of the generators of \( I \) is not empty. Indeed, the ideal \( I \) belongs to \( \mathcal{Mf}(J) \) if and only if it has the same Hilbert function of \( J \) and there exists a specialization \( \bar{C} \) in \( S(J) \) such that every generator of \( I \) belongs to the ideal \( (\bar{C}) \) generated by the polynomials
of $\mathcal{G}$ evaluated on $\mathcal{C}$. The generators of $I$ belong to $(\mathcal{G})$ if and only if their $\mathcal{V}_m$-reductions evaluated on $\mathcal{C}$ become null.

**APPENDIX: AN EXPLICIT COMPUTATION**

Let $J$ be the strongly stable ideal $(x^4, x^3y, x^2y^2, xy^3, x^3z, x^2yz, xy^2z, y^5)$ in $K[z, y, x]$ (where $x > y > z$ and $ch(K) = 0$), already considered in Example 3.18. Note that for every term order $\prec$, we can find in degree 4 a monomial in $J$ lower than a monomial in $\mathcal{N}(J)$, because $xy^2z > x^2z^2$ and $xy^2z > y^4$ would be in contradiction with the equality $(xy^2z)^2 = x^2z^2 \cdot y^4$. Hence, $J_4$ is not a segment (in the usual meaning) with respect to any term order.

The affine scheme $\mathcal{S}(J)$ can be embedded as a locally closed subscheme in the Hilbert scheme of 8 points in the projective plane (see 12), which is irreducibly smooth of dimension 16, and contains all the Gröbner strata $\mathcal{S}_h(J, \prec)$, for every $\prec$, and also some more point, for instance the one corresponding to the ideal $I$ of Example 3.18.

Letting $\mathcal{G} = \{F_1, \ldots, F_8\} \subset K[x, y, z, c_1, \ldots, c_{64}]$, where

\[
\begin{align*}
F_1 &= x^4 + c_1 z^2 x^2 + c_2 y^4 + c_3 z^2 y x + c_4 z^3 y + c_5 z^3 x + c_6 z^2 y^2 + c_7 z^2 y + c_8 z^4, \\
F_2 &= x^3 y + c_9 z^2 x^2 + c_{10} y^4 + c_{11} z^2 y x + c_{12} z^3 y + c_{13} z^3 x + c_{14} z^2 y^2 + c_{15} z^3 y + c_{16} z^4, \\
F_3 &= x^2 y^2 + c_{17} z^2 x^2 + c_{18} y^4 + c_{19} z^2 y x + c_{20} z^3 y + c_{21} z^3 x + c_{22} z^2 y^2 + c_{23} z^3 y + c_{24} z^4, \\
F_4 &= x y^3 + c_{25} z^2 x^2 + c_{26} y^4 + c_{27} z^2 y x + c_{28} z^3 y + c_{29} z^3 x + c_{30} z^2 y^2 + c_{31} z^2 y + c_{32} z^4, \\
F_5 &= x^3 z + c_{33} z^2 x^2 + c_{34} y^4 + c_{35} z^2 y x + c_{36} z^3 y + c_{37} z^3 x + c_{38} z^2 y^2 + c_{39} z^3 y + c_{40} z^4, \\
F_6 &= x^2 y z + c_{41} z^2 x^2 + c_{42} y^4 + c_{43} z^2 y x + c_{44} z^3 y + c_{45} z^3 x + c_{46} z^2 y^2 + c_{47} z^3 y + c_{48} z^4, \\
F_7 &= x y^2 z + c_{49} z^2 x^2 + c_{50} y^4 + c_{51} z^2 y x + c_{52} z^3 y + c_{53} z^3 x + c_{54} z^2 y^2 + c_{55} z^3 y + c_{56} z^4, \\
F_8 &= y^5 + c_{57} z^2 x^2 + c_{58} y^4 + c_{59} z^2 y x + c_{60} z^3 y + c_{61} z^4 x + c_{62} z^3 y^2 + c_{63} z^4 y + c_{64} z^5,
\end{align*}
\]

by Maple 12 we compute the ideal $\mathfrak{R}$' and the following ideal defining the Zariski tangent space $T$ to $\mathcal{S}(J)$ at the origin that has dimension 16

\[
I(T) = \langle c_{64}, c_{63}, c_{61}, c_{56}, c_{55}, c_{53}, c_{48}, c_{47}, c_{46}, c_{45}, c_{44}, c_{40}, c_{39}, c_{38}, c_{37}, c_{36}, c_{32}, c_{31}, c_{30}, c_{29}, c_{28} - c_{54}, c_{27} - c_{52}, c_{26} - c_{51}, c_{25}, c_{24}, c_{23}, c_{22}, c_{21}, c_{20}, c_{19}, c_{18}, c_{17}, c_{16}, c_{15}, c_{14}, c_{13}, c_{12}, c_{11}, c_{10}, c_9, c_8, c_7, c_6, c_5, c_4, c_3, c_2, c_1 \rangle.
\]

In the ideal $\mathfrak{R}$ we eliminate several variables of type $C$ by applying [12 Theorem 5.4] and by substituting variables that appear only in the linear part of some polynomials of $\mathfrak{R}$. It follows that $\mathcal{S}(J)$ can be isomorphically projected on a linear space $T' \cong \mathbb{A}^{19}$ containing $T$. In this embedding, $\mathcal{S}(J)$ is the complete intersection of the following three hypersurfaces in $\mathbb{A}^{19}$ of degrees 4, 4 and 8, respectively:

\[
\begin{align*}
G_1 &= c_{41}^2 c_{49} c_{50} + c_{41} c_{49} c_{50} c_{51} + c_{41} c_{50} c_{51} c_{52} + c_{42} c_{49} c_{50} c_{55} + c_{43} c_{49} c_{50} c_{57} + c_{44} c_{49} c_{50} c_{59} + c_{45} c_{49} c_{50} c_{51} + c_{50} c_{51} c_{57} + c_{50} c_{57} c_{58} - c_{41} c_{49} c_{52} - c_{49} c_{50} c_{53} - c_{49} c_{51} c_{52} - 2 c_{50} c_{52} c_{57} + c_{43} c_{49} c_{50} - c_{41}^2 + c_{41} c_{49} c_{51} - c_{42} c_{49} c_{52} - c_{43} c_{49} c_{54} - c_{49} c_{54} - c_{53}, \\
G_2 &= c_{41} c_{42} c_{49} c_{50} + c_{42} c_{49} c_{50} c_{51} + c_{42} c_{49} c_{50} c_{58} + c_{42} c_{50} c_{57} + c_{43} c_{49} c_{50} + c_{50} c_{57} + c_{50} c_{51} c_{58} + c_{50}^2 c_{58} - c_{42} c_{49} c_{52} - c_{44} c_{49} c_{50} - c_{50}^2 c_{53} - c_{50}^2 c_{60} - 2 c_{50} c_{51} c_{52} - 2 c_{50} c_{52} c_{58} + c_{34} c_{49} - c_{41} c_{42} + c_{42} c_{49} - c_{42} c_{58} + c_{43} c_{50} + c_{50} c_{51} + c_{52} + c_{44}, \\
G_3 &= c_{41} c_{49} c_{50} - c_{41}^2 c_{50} c_{51} + c_{41} c_{49} c_{50} c_{58} - 2 c_{41}^2 c_{49} c_{50} c_{57} + c_{41} c_{42}^2 c_{59} + 2 c_{41} c_{49} c_{50} c_{51} + c_{41}^2 c_{49} c_{50} c_{58} - 2 c_{41} c_{49} c_{50} c_{51} c_{57} - c_{41} c_{49} c_{50} c_{57} + c_{42} c_{49} c_{51} + c_{42} c_{50} c_{58} - c_{43} c_{50} c_{51} - c_{49} c_{50}^2 c_{58} + 2 c_{49} c_{50} c_{51} c_{57} c_{58} + 2 c_{49} c_{50} c_{57} c_{58} - c_{49} c_{50} c_{51} c_{57} - c_{49} c_{50}^2 c_{58} + 2 c_{49} c_{50} c_{51} c_{52} - 2 c_{41} c_{42} c_{49} c_{52}.
\end{align*}
\]
Among the generators of the corresponding Jacobian ideal we have the following minors $D_i$'s obtained by computing derivatives of $G_1, G_2, G_3$ with respect to the sets of variables $A_i$'s, for $1 \leq i \leq 5$:

$$D_1 := -(c_{49}c_{50} - 1)(c_{49}c_{50} - 1)(c_{49}c_{50} + 1), \quad A_1 = \{c_{61}, c_{44}, c_{53}\};$$
$$D_2 := -(c_{49}c_{50} + 1)(c_{49}c_{50} - 1)c_{49}, \quad A_2 = \{c_{53}, c_{44}, c_{62}\};$$
$$D_3 := -c_{50}(2c_{49}c_{50} - 1)(c_{49}c_{50} - 1), \quad A_3 = \{c_{43}, c_{61}, c_{53}\};$$
$$D_4 := c_{49}(c_{49}c_{50} - 1)^2(2c_{49}c_{50} - 1), \quad A_4 = \{c_{43}, c_{61}, c_{44}\};$$
$$D_5 := (c_{49}c_{50} + 1)c_{49}^2(2c_{49}c_{50} - 1), \quad A_5 = \{c_{53}, c_{60}, c_{61}\}.$$

The polynomials $D_i$'s define the empty set, so that $S(J)$ is smooth as we expected and, in particular, $J$ corresponds to a smooth point on $S(J)$. Moreover, $\mathcal{M}(J)$ has dimension 16 but we claim that it cannot be isomorphically projected on $T$. Indeed, note that we can choose a set of 16 variables that is complementary to the tangent space and that do not contain the variables $c_{53}, c_{44}, c_{61}$ which occur in the linear parts of the polynomials $G_i$'s. These variables appear also in other parts of the polynomials and their coefficients are $c_{49}c_{50} + 1$, $c_{49}c_{50} - 1$ and $2c_{49}c_{50} - 1$, respectively. If $\tilde{c} \in T$ is a point of the tangent space on which none of the coefficients vanishes, we obtain a unique point of $\mathcal{M}(J)$ of which $\tilde{c}$ is the projection on $T$. If $\tilde{c} \in T$ is a general point of the tangent space on which one of this coefficient vanishes, one can see that $\tilde{c}$ is not the projection of any point of $\mathcal{M}(J)$. Hence, the projection of $\mathcal{M}(J)$ on $T$ does not coincide with the tangent space $T$, but only with an open set. However, this fact implies that $\mathcal{M}(J)$ is rational, in particular irreducible.

We point out that the variables $c_{49}$ and $c_{50}$, that appear in the coefficients of the variables $c_{53}, c_{44}, c_{61}$, are the coefficients in the polynomial $F_7$ of the two terms $z^2x^2, y_4$ whose behaviour prevents the ideal $J$ from being a segment. Indeed, in this case the affine scheme $\mathcal{M}(J)$ is homogeneous with respect to a non-positive grading.

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