A uniqueness theorem for stationary Kaluza-Klein black holes

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Abstract

We prove a uniqueness theorem for stationary $D$-dimensional Kaluza-Klein black holes with $D-2$ Killing fields, generating the symmetry group $\mathbb{R} \times U(1)^{D-3}$. It is shown that the topology and metric of such black holes is uniquely determined by the angular momenta and certain other invariants consisting of a number of real moduli, as well as integer vectors subject to certain constraints.

1 Introduction

The classic black hole uniqueness theorems state that four dimensional, stationary, asymptotically flat black hole spacetimes are uniquely determined by their mass and angular momentum in the vacuum case, and by their mass, angular momentum, and charge in the Einstein-Maxwell case. The solutions are in fact given by the Kerr metrics in the first case and the Kerr-Newman metrics in the second. This was proven in a series of papers [2, 45, 1, 37, 24, 23, 30]; for a coherent exposition clarifying many important details and providing a set of consistent technical assumptions see [7].

The black hole uniqueness theorem is not true as stated in general spacetime dimensions $D \geq 5$. For example, in $D = 5$ dimensions, there exist asymptotically flat, stationary vacuum black holes with the same mass and angular momenta, but with non-isometric spacetime metrics, and in fact even different topology [10, 13, 43, 11, 4, 15, 12]. One would nevertheless hope that a similar uniqueness theorem still applies if additional invariants (“parameters”) are specified beyond the mass and angular momenta.

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Unfortunately, except in the static case [19, 46], such a classification result is not known, nor is it known what could be the nature of the additional invariants.

In this paper, we consider the special case of stationary vacuum black hole space-times in dimension \( D \geq 4 \) with a compact, non-degenerate, connected horizon, admitting \( D - 3 \) commuting additional Killing fields with closed orbits. The spacetimes that we consider asymptote to a flat Kaluza-Klein space with 1, 2, 3 or 4 large spatial dimensions and a corresponding number of toroidal extra dimensions. We will first show how to associate certain invariants to such a spacetime consisting of a collection of “moduli” \( \{ l_i \in \mathbb{R}_{>0} \} \) and certain generalized “winding numbers” \( \{ a_i \in \mathbb{Z}^{D - 3} \} \). The moduli may be thought of as the length of various rotation “axis” within the spacetime, whereas the winding numbers characterize the nature of the action of the \( D - 3 \) rotational symmetries near a given axis. The collection of these winding numbers uniquely characterizes the topology and symmetry structure of the exterior of the black hole, and we refer to it as the “interval structure” of the manifold. This analysis also implies that the horizon must be topologically the cartesian product a torus of the appropriate dimension and either a 3-sphere, ring \( (S^2 \times S^1) \), or Lens-space \( L(p, q) \).

Our notion of interval structure may be viewed as a generalization of what has been called “weighted orbit space” in the mathematics literature on 4-manifolds with torus action [11, 42], but the latter notion does not involve the moduli \( \{ l_i \} \). Also, in the context of stationary black holes, a similar notion called “rod structure” was first considered by [21, 20]; see [14] for the static case. The main difference between this and our notion is that our winding numbers are found to obey an integrality condition as well as certain other constraints, whereas there were no such constraints in [21, 20]. The latter are a necessary and sufficient condition for the spacetime to have the structure of a smooth manifold with torus action. These topological considerations are described in detail in sec. 3.

We will then prove a uniqueness theorem which states that there can be at most one black hole spacetime with the same angular momenta and interval structure.\(^1\)

Our uniqueness theorem generalizes a theorem proved in a previous paper [28] on asymptotically flat vacuum black holes in \( D = 5 \) dimensions; see also [29] for the Einstein-Maxwell case.

The proof of the theorem proceeds in two steps: First, one reduces the full Einstein equations to equations on the space of symmetry orbits. Because the spacetime is assumed to have a total number of \( D - 2 \) independent commuting Killing fields, the space of symmetry orbits is two-dimensional—in fact it is shown to be a manifold with boundaries and corners homeomorphic to a half-plane. The parameters \( \{ l_i \} \) are essentially the lengths of the various boundary segments. The arguments in the first step are topological in nature, and the only role of Einstein’s equations is to provide additional information about the fundamental group of the manifold via the topological censorship theorem [9]. That information is needed to rule out the presence of conical singularities in the orbit space.\(^2\). Our results in this part may be thought of as a

\(^{1}\)It has been brought to our attention that a conjecture in this direction had been made at the talk [22], see also [21].

\(^{2}\)Here our analysis also fills a gap in our previous paper [28], where the absence of such conical singularities had to be assumed by hand.
generalization of [41, 42] to a higher dimensional situation.

The second step is to cast the reduced Einstein equations on the orbit space into a suitable form. Here, we make use of a formulation due to [34] involving certain potentials. The form of the equations leads to a partial differential equation for a quantity representing the “difference” between any two black hole metrics of the type considered which has been called “Mazur identity” [37]. Using this identity, one can prove the uniqueness theorem. The vectors \( \{a_i\} \) and parameters \( \{l_i\} \) are important to treat the boundary conditions of the differential equation. The arguments in the second step are geometrical/analytical, and involve the use of Einstein’s equations in an essential way. The simpler case of a spherical black hole with trivial interval structure was previously treated by a similar method in [36].

While our uniqueness theorem in higher dimensions is in some ways similar to the corresponding theorem in four dimensions, there are some notable differences. The first, more minor, difference is that higher dimensional black holes are not only classified by the mass and angular momenta, but in addition depend on the interval structure. In \( D = 4 \) the interval structure of a single black hole spacetime is trivial. A more substantial difference is that in \( D = 4 \) dimensions, the additional axial Killing field is in fact guaranteed by the rigidity theorem [24, 38, 16, 44, 6]. While a generalized rigidity theorem can be established in \( D \) dimensions [26, 39, 27], this theorem now only guarantees at least one additional axial Killing field. For the arguments of the present paper to work, we need however \( D - 3 \) commuting axial Killing fields. It does not seem likely that our theorem covers all asymptotically Kaluza-Klein, stationary black hole spacetimes in \( D \) dimensions.

A third difference is that we have not been able so far to establish for which given set of angular momenta and interval structure there actually exists a regular black hole solution. The situation in this regard is in fact unclear even in five asymptotically large dimensions with no small extra dimensions. Here, solutions corresponding to various simple interval structures have been constructed. These include solutions with horizon topology \( S^3, S^2 \times S^1, L(p,q) \), which are the possible topologies allowed by our uniqueness theorem. However, by contrast with the cases \( S^3, S^2 \times S^1 \) [40, 13, 43, 14], the black holes with lens space horizon topology found so far [4, 15] are not regular, and are thus actually not covered by our theorem. The situation is very different in four dimensions. Here the interval structure for single black hole spacetimes only involves the specification of a single parameter (related to the area of the horizon), and a regular black hole solution is known to exist for any choice of this parameter and the angular momentum—the corresponding Kerr solution. The mass, surface gravity, angular velocity of the horizon etc. of the solution can all be expressed in terms of these parameters.

2 Description of the problem, assumptions, notations

Let \((M, g)\) be a \(D\)-dimensional, stationary black hole spacetime satisfying the vacuum Einstein equations, where \(D \geq 4\). The asymptotically timelike Killing field is called \( t \),
so \( \mathcal{L}_x g = 0 \). We assume that \( M \) has \( s+1 \) asymptotically flat large spacetime dimensions and \( D-s-1 \) asymptotically small extra dimensions, where \( s > 0 \). More precisely, we assume that a subset of \( M \) is diffeomorphic to the cartesian product of \( \mathbb{R}^s \) with a ball removed—corresponding to the asymptotic region of the large spatial dimensions—and \( \mathbb{R} \times \mathbb{T}^{D-s-1} \)—corresponding to the time-direction and small dimensions. We will refer to this region as the asymptotic region and call it \( M_\infty \). The metric is required to behave in this region like

\[
g = -d\tau^2 + \sum_{i=1}^{s} dx_i^2 + \sum_{i=s+1}^{D-1} d\varphi_i^2 + O(R^{-\alpha}),
\]

where \( \alpha > 0 \) is some constant, and where \( O(R^{-\alpha}) \) stands for metric components that drop off faster than \( R^{-\alpha} \) in the radial coordinate \( R = \sqrt{x_1^2 + ... + x_s^2} \), with \( k \)-th derivatives dropping off faster than \( R^{-\alpha-k} \). These terms are also required to be independent of the coordinate \( \tau \), which together with \( x_i \) forms the standard cartesian coordinates on \( \mathbb{R}^{s+1} \). The remaining coordinates \( \varphi_i \) are \( 2\pi \)-periodic and parametrize the torus \( \mathbb{T}^{D-s-1} \).

The timelike Killing field is assumed to be equal to \( \partial/\partial \tau \) in \( M_\infty \). We call spacetimes satisfying these properties “asymptotically Kaluza-Klein” spacetimes.

The domain of outer communication is defined by

\[
\langle \langle M \rangle \rangle = I^+ (M_\infty) \cap I^- (M_\infty),
\]

where \( I^\pm \) denote the chronological past/future of a set. The black hole region \( B \) is defined as the complement in \( M \) of the causal past of the asymptotic region, and its boundary \( \partial B = H \) is called the (future) event horizon.

In this paper, we also assume the existence of \( D-3 \) further linearly independent Killing fields, \( \psi_1, ..., \psi_{D-3} \), so that the total number of Killing fields is equal to the number of spacetime dimensions minus two. These are required to mutually commute, to commute with \( t \), and to have periodic orbits which close for the first time after \( 2\pi \). The Killing fields \( \psi_i \) are referred to as “axial” by analogy to the four-dimensional case, even though their zero-sets are generically higher dimensional surfaces rather than “axis” in \( D > 4 \). We also assume that, in the asymptotic region \( M_\infty \), the action of the axial symmetries is conjugate to the standard rotations in the cartesian product of flat Minkowski spacetime \( \mathbb{R}^{s,1} \) times the standard flat torus \( \mathbb{T}^{D-s-1} \). In other words, \( \psi_i = \partial/\partial \varphi_i \) or \( \psi_j = x_{2j-1} \partial x_{2j} - x_{2j} \partial x_{2j-1} \) for \( j = 1, ..., [s/2] \) in \( M_\infty \). The group of isometries is hence \( \mathcal{G} = \mathbb{R} \times \mathcal{T} \), where \( \mathbb{R} \) corresponds to the flow of \( t \), and where \( \mathcal{T} = \mathbb{T}^{D-3} \) corresponds to the commuting flows of the axial Killing fields. Looking at the action of \( \mathcal{G} \) on the asymptotic region, it is evident that an asymptotically Kaluza-Klein spacetime can have at most \( [s/2] + D-s-1 \) commuting axial Killing fields. If this number is \( D-3 \) as we are assuming, then \( s \) can be either 1, 2, 3 or 4. Thus our spacetime is asymptotically the direct product of 2, 3, 4- or 5-dimensional Minkowski spacetime and a \( (D-2), (D-3), (D-4) \)- or \( (D-5) \)-dimensional flat torus.

\( ^3 \)For the axisymmetric spacetimes considered in this paper, we will derive below a stonger asymptotic expansion, see eq. (67).

\( ^4 \)The notation \([x]\) means the largest integer \( n \) such that \( n \leq x \).
We are going to analyze the uniqueness properties of the asymptotically Kaluza-Klein spacetimes just described. Unfortunately, in order to make our arguments in a consistent way, we will have to make certain further technical assumptions about the global nature of \((M,g)\) and the action of the symmetries. Our assumptions are in parallel to those made by Chrusciel and Costa in their study \[7\] of 4-dimensional stationary black holes. The requirements are (a) that \(\langle \langle M \rangle \rangle\) contains an acausal, spacelike, connected hypersurface \(S\) asymptotic to the \(\tau = 0\) surface in the asymptotic region \(M_\infty\), whose closure has as its boundary \(\partial S = \mathcal{H}\) a cross section of the horizon. We assume \(\mathcal{H}\) to be compact and (for simplicity) to be connected. (b) We assume that the orbits of \(t\) are complete. (c) We assume that the horizon is non-degenerate. (d) We assume that \(\langle \langle M \rangle \rangle\) is globally hyperbolic.

For the spacetimes described, one of the following two statements is true: If \(t\) is tangent to the null generators of \(H\) then the spacetime must be static \[47\]. On the other hand, if \(t\) is not tangent to the null generators of \(H\), then the rigidity theorem \[26\] implies that there exists a linear combination

\[
K = t + \Omega_1 \psi_1 + \cdots + \Omega_{D-3} \psi_{D-3}, \quad \Omega_i \in \mathbb{R}
\]

so that the Killing field \(K\) is tangent and normal to the null generators of the horizon \(H\), and

\[
g(K, \psi_i) = 0 \quad \text{on } H.
\]

From \(K\), one may define the surface gravity of the black hole by \(\kappa^2 = \lim_H (\nabla_a f) \nabla^a f / f\),

with \(f = (\nabla^a K^b) \nabla_a K_b\), and it may be shown that \(\kappa\) is constant on \(H\) \[48\]. In fact, the non-degeneracy condition implies \(\kappa > 0\).

In the first case, one can prove that the spacetime is actually unique \[30\], and in fact isometric to the Schwarzschild spacetime when \(D = 4\), for higher dimensions see \[19, 46\]. In this paper, we will be concerned with the second case, and we will give a uniqueness theorem for such spacetimes. Of particular importance for us will be the orbit space \(M = \langle \langle M \rangle \rangle / \mathcal{G}\), so in the next section we will look in detail at this space.

3 Analysis of the orbit space

3.1 Manifolds with torus actions

To begin, we consider a somewhat simpler situation, namely an orientable, smooth, compact connected Riemannian manifold \(\Sigma\) of dimension \(s \geq 3\), with a smooth effective\(^5\) action of the \(N\)-dimensional torus \(T = \mathbb{T}^N\). Thus, we assume that \(\text{Diff}(\Sigma)\) contains a copy of \(T\). Such actions have been analyzed and classified in the case \(s = 4\) in a classic work by Orlik and Raymond \[41, 42\], and—repeating many of their arguments—in \[28\]. Some of our arguments for general \(s\) are in parallel with this case, others are not.

We may equip \(\Sigma\) with a Riemannian metric \(h\), and by averaging \(h\) with the action of \(T\) if necessary, we may assume that \(T\) acts by isometries of \(h\). Later, \(\Sigma\) will be a

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\(^5\)This means that if \(k \cdot x = x\) for all \(x \in \Sigma\), then \(k\) is necessarily the identity. Given an action of the above type, one may always pass to an effective action by taking a quotient of \(T\) if necessary.
spatial slice of our physical spacetime (so that \( s = D - 1 \)) and \( N \) will be taken to be \( D - 3 \), but for the moment this is not relevant.

It will be useful to view the \( N \)-torus as the quotient \( \mathbb{R}^N / \Lambda_N \), where \( \Lambda_N = (2\pi \mathbb{Z})^N \) is the standard \( 2\pi \)-periodic \( N \)-dimensional lattice. Elements \( k \in T \) will consequently be identified with equivalence classes of \( N \)-dimensional vectors, \( k = [\tau_1, \ldots, \tau_N] \in \mathbb{R}^N / \Lambda_N \). The standard basis of \( \Lambda_N \) will be denoted \( b_1, \ldots, b_N \), i.e.,

\[
\begin{align*}
b_i &= (0, \ldots, 0, 2\pi, 0, \ldots, 0),
\end{align*}
\]

where the non-zero entry is in the \( i \)-th position. Various facts about lattices that we will use in this section may be found in the classic monograph \([3]\).

We denote the commuting Killing fields generating the action of \( T \) by \( \psi_i, i = 1, \ldots, N \). The flows of these vector fields are denoted \( F^\tau_i \), and we assume that they are normalized so that the flows are periodic with period \( 2\pi \), so \( F_i^{2\pi}(x) = x \) for any \( x \in \Sigma \), and any \( i \). The action of a group element \( k = [\tau_1, \ldots, \tau_N] \) on a point is abbreviated by

\[
k \cdot x = F_i^{\tau_1} \cdots F_N^{\tau_N}(x). \tag{5}
\]

We also abbreviate the action of \( k \) on a tensor field \( T = T_{a_1 \cdots a_N} b_1 \cdots b_N \) on \( \Sigma \) by \( k \cdot T = \left[ F_i^{\tau_1} \circ \cdots \circ F_N^{\tau_N} \right]^* T \), where the \( * \) denotes the push-forward/pull-back of the tensor field. Because the Killing fields commute, we have in particular \( k \cdot \psi_i = \psi_i \) for any \( k \in T \). If \( \psi_1, \ldots, \psi_N \) are Killing fields as above, then so are \( \hat{\psi}_1, \ldots, \hat{\psi}_N \), where

\[
\hat{\psi}_i = \sum_{i=1}^N A_{ij} \psi_j, \quad A = \pm \begin{pmatrix} A_{11} & \cdots & A_{N1} \\
& \ddots & \\
& & A_{NN} \end{pmatrix} \in SL(N, \mathbb{Z}). \tag{6}
\]

Another way of saying this is that we may conjugate the action of \( T = \mathbb{T}^N \) by the inner automorphism \( \alpha_A([\tau]) = [\tau A^T] \) of \( T \), and the modified Killing fields \( \hat{\psi}_i \) generate the conjugated action. The freedom of choosing different \( 2\pi \)-periodic Killing fields to generate the action of \( T = \mathbb{R}^N / \Lambda_N \) is closely related to possibility of choosing different bases in the lattice \( \Lambda_N \), because any such change of basis is implemented by an integer matrix \( A \) with \( \det A = \pm 1 \).

As is standard, we define the orbit and the isotropy subgroup associated with a point by, respectively

\[
O_x = \{ k \cdot x \mid k \in T \}, \quad I_x = \{ k \in T \mid k \cdot x = x \}. \tag{7}
\]

\( I_x \) is a closed (hence compact) subgroup of \( T \), and \( O_x \) is a smooth manifold that can be identified with \( T / I_x \). Being compact and abelian, \( I_x \) must be isomorphic to \( \mathbb{T}^n \times \prod Z/p_j Z \). A more precise description of the action \( I_x \) in an open neighborhood of \( x \) will be given below. The set of all orbits \( \hat{\Sigma} = \{ O_x \mid x \in \Sigma \} \) is called the factor space and is also written as \( \hat{\Sigma} = \Sigma / T \). It is not a manifold for general group actions.

It will be useful to define the non-negative, symmetric \( N \times N \) Gram matrix of the Killing fields,

\[
f_{ij} = h(\psi_i, \psi_j). \tag{8}
\]

\(^6\)The automorphism property is \( \alpha_A(\alpha_A(kk')) = \alpha_A(k)\alpha_A(k') \) for all \( k, k' \in T \).
It will also be convenient to distinguish points in $\Sigma$ according to the dimension of their orbit. For this, we define
\[
S_r = \{ x \in \Sigma \mid \dim O_x = r \}
\]
\[
= \{ x \in \Sigma \mid \text{rank} [f(x)] = r \}
\]
\[
= \{ x \in \Sigma \mid \dim I_x = n = N - r \}.
\] (9)

Evidently, $n = N - r$ is also equal to the number of independent linear combinations of the Killing fields $\psi_1, \ldots, \psi_N$ that vanish at points of $S_r$. Clearly, we have
\[
\Sigma = \bigcup_{r=0}^{N} S_r.
\] (10)

**Lemma 1.** Let $(\Sigma, h)$ be a Riemannian manifold of dimension $s$, with $N$ mutually commuting Killing fields $\psi_i, i = 1, \ldots, N$. Let $f_{ij}$ be the Gram matrix, and let $x$ be a point such that $\text{rank} [f(x)] = r$. Then it follows that $N - r \leq [(s - r)/2]$.

**Proof:** Let $V_x \subset T_x \Sigma$ be the span of the Killing fields $\psi_i|_x, i = 1, \ldots, N$ at $x$, and let $W_x$ be the orthogonal complement. The assumptions of the lemma mean that the dimension of $V_x$ is $r$, and that there exist $N - r$ linear combinations of $\psi_i|_x, i = 1, \ldots, N$ that vanish. By forming suitable linear combinations of the Killing fields, we may hence assume that $\text{span} \{ \psi_i|_x, i = 1, \ldots, r \} = V_x$, and that $\psi_i|_x = 0, i = r + 1, \ldots, N$. Let $D$ be the derivative operator of $h$, and let $t_i = D\psi_i|_x$, where $i = r + 1, \ldots, N$. Then each $t_i$ is a linear map $t_i : T_x \Sigma \rightarrow T_x \Sigma$. The Killing equation implies that $t_i$ is skew symmetric with respect to the bilinear form $h : T_x \Sigma \times T_x \Sigma \rightarrow \mathbb{R}$, i.e. $h(t_i X, Y) = -h(X, t_i Y)$. Evaluating the $D$-derivative of the commutator $[\psi_i, \psi_j] = 0$ at $x$ for $r < i, j \leq N$ then implies that the corresponding commutator $t_i t_j - t_j t_i = 0$ vanishes, too. Evaluating the derivative of the commutator $[\psi_i, \psi_j] = 0$ at $x$ for $r < i \leq N$ and $0 < j \leq r$ then furthermore shows that $t_i \upharpoonright V_x = V_x$, and consequently $t_i \upharpoonright W_x = W_x$. Now let us choose an orthogonal basis $\{ e_1, \ldots, e_{s-r} \}$ of $W_x$, and use that to identify $t_i, r < i \leq N$ with a linear map $\mathbb{R}^{s-r} \rightarrow \mathbb{R}^{s-r}$. These linear maps must hence skew symmetric, i.e., commuting elements of the Lie-algebra $\mathfrak{o}(s-r,\mathbb{R})$. They must also be linearly independent. Indeed, assume on the contrary that that a non-trivial linear combination $\lambda_1 t_{r+1} + \cdots + \lambda_{N-r} t_N$ vanishes. Then both the Killing field $s = \lambda_1 \psi_{r+1} + \cdots + \lambda_{N-r} \psi_N$, as well as its derivative $D s$ vanish at the point $x$. It is a well-known property of Killing fields (see e.g. [48]) that a Killing field vanishes identically on a connected Riemannian manifold if it vanishes at a point together with its derivative. Hence, the Killing fields $\psi_i, r < i \leq N$ would be linearly dependent, a contradiction. Thus, we conclude that the linear maps $t_i, r < i \leq N$ may be viewed as forming a $(N - r)$-dimensional abelian subalgebra of $\mathfrak{o}(s-r,\mathbb{R})$. Any maximal abelian subalgebra of $\mathfrak{o}(s-r,\mathbb{R})$ has dimension $[(s-r)/2]$, so $N - r \leq [(s-r)/2]$.

In the situation considered later in this section, we have $N = s - 2$ Killing fields. The lemma then implies that the sets $S_r$ are non-empty only for $r = s - 2, s - 3, s - 4$, so we have $\Sigma = S_{s-2} \cup S_{s-3} \cup S_{s-4}$.

Our task will now be to construct, for each orbit $O_x$, an open neighborhood of it and a coordinate system in which we can explicitly understand the action of the
group \( T \). We will then be able to locally take the quotient of this neighborhood by \( T \) and thereby get a local description of the orbit space. By patching the local regions together, we will be able to characterize the manifold structure of the orbit space.

Let \( x \) be an arbitrary but fixed point in \( S_\tau \). Then the dimension of \( O_x \) is \( r \), and the dimension of the isotropy group \( I_x \) is \( n = N - r \). As we have just seen, \( n \) may only take on the values 0, 1, \ldots, \([s-r]/2\]. We first show that if \( x \in S_\tau \), there exists a matrix \( \pm A \in SL(N,\mathbb{Z}) \) such that the vector fields \( \dot{\psi}_i, 0 < i \leq N \) defined as in eq. (6) satisfy \( \dot{\psi}_i|_x = 0, r < i \leq N \) and such that \( \dot{\psi}_i|_x, 0 < i \leq r \) span the tangent space \( T_x O_x \).

We start our discussion with a general lemma.

**Lemma 2.** Let \( \mathcal{L} \subset T = \mathbb{T}^N \) be an \( n \)-dimensional closed subgroup. Then there are matrices of integers \((A_{ij})_{i,j=1}^N\) and \((v_{ij})_{i,j=1}^N\) where \( r = N - n \) and \( \det A = \pm 1 \), with the property that \( \mathcal{L} = \alpha_A(\mathcal{L}_0 \times \mathcal{L}_1) \). Here

\[
\begin{align*}
\mathcal{L}_0 &= \{0_r\} \times \mathbb{R}^{N-r}/\Lambda_{N-r}, \\
\mathcal{L}_1 &= (v^{-1}A_r)/\Lambda_r \times \{0_{N-r}\},
\end{align*}
\]

(11)

(12)

where \( \Lambda_r \) has been identified with the lattice generated by \( b_1, \ldots, b_r \), with origin denoted \( 0_r \), and where \( \Lambda_{N-r} \) has been identified with the lattice generated by \( b_{r+1}, \ldots, b_N \), with origin denoted \( 0_{N-r} \). We have also written \( v^{-1}A_r \) for the lattice of \( \mathbb{R}^r \) generated by \( \sum_{j=1}^r (v^{-1})_{ij}b_j \), where \( i = 1, \ldots, r \). Hence \( \mathcal{L}_0 \) is connected, \( \mathcal{L}_1 \) is finite,

\[
\mathcal{L}_1 \cong \mathbb{Z}^{p_1}_{\alpha_1} \times \cdots \times \mathbb{Z}^{p_M}_{\alpha_M}, \quad |\mathcal{L}_1| = p_1^{\alpha_1} \cdots p_M^{\alpha_M} = |\det(v_{ij})_{i,j=1}^N|,
\]

(13)

with \( p_j > 0 \) prime.

**Proof:** Let us first assume that \( \mathcal{L} \) is also connected. Then \( \mathcal{L} \) is a compact, abelian, connected Lie-group and so must be isomorphic to \( \mathbb{T}^n \). Let \( \beta : \mathbb{T}^n \to \mathcal{L} \) be the isomorphism. We identify \( T = \mathbb{T}^N \) with \( \mathbb{R}^N/\Lambda_N \), where \( \Lambda_N \) is the standard lattice. Similarly, we identify \( \mathbb{T}^n \) with \( \mathbb{R}^n/\Lambda_n \), with \( \Lambda_n = \text{span}_\mathbb{Z}(b_j)_{j=r+1}^N \). Let \( \alpha_j = \beta(b_j) \in \Lambda_n \), where \( i = r + 1, \ldots, N \). If \( \lambda_i \in \mathbb{R} \) are such that

\[
\begin{align*}
\sum \alpha_j &= \lambda_1 a_{i+1} + \cdots + \lambda_r a_N \\
&= \beta(\lambda_1 b_{r+1} + \cdots + \lambda_r b_N) \in \Lambda_N,
\end{align*}
\]

(14)

then it follows that \( \lambda_i \in \mathbb{Z} \). We conclude from [3 Cor. 3, I.2.2] that there are vectors \( a_1, \ldots, a_r \in \Lambda_N \) such that \( a_1, \ldots, a_N \) form a basis of \( \Lambda_N \). We now let \( A \) be the \( N \times N \) matrix of integers such that \( b_i A^T = a_i \) for \( i = 1, \ldots, N \). Then \( \det A = \pm 1 \) because the matrix relates two bases of the lattice \( \Lambda_N \). Since \( \mathcal{L}_0 \) viewed as a subgroup of \( \mathbb{T}^N \) is generated precisely by \( b_{r+1}, \ldots, b_N \), this proves the lemma when \( \mathcal{L} \) is connected.

In the general case, \( \mathcal{L} \) is isomorphic to the cartesian product of a torus and cyclic groups of order given by a prime power, i.e. there is an isomorphism \( \beta : \mathbb{T}^n \times \prod \mathbb{Z}_{p_j}^{\alpha_j} \to \mathcal{L} \). For \( j = 1, \ldots, M \), let \( c_j \) be the image under \( \beta \) of the generator of the \( j \)-th cyclic finite group in the decomposition, projected onto the (real) span of \( a_1, \ldots, a_r \). The vectors \( c_1, \ldots, c_M \) together with \( a_1, \ldots, a_r \) generate an \( r \)-dimensional lattice \( \Gamma_r \). Let
Let \( \gamma_1, \ldots, \gamma_r \) be a basis of the lattice \( \Gamma_r \). It follows from [3, Thm. 1, I.2.2] that there are integers \( v_{ij} \) such that \( v_{ii} > 0, v_{ij} > v_{ji} \) for \( j > i \), and

\[
\begin{align*}
\v_1 & = v_{11} \gamma_1, \\
\v_2 & = v_{21} \gamma_1 + v_{22} \gamma_2, \\
& \vdots \\
\v_r & = v_{r1} \gamma_1 + v_{r2} \gamma_2 + \cdots + v_{rr} \gamma_r. 
\end{align*}
\]

It is evident that \( \mathcal{L} \) is given by the image under \( \alpha_A \) of the cartesian product of the group given by the real multiples of \( \v_{r+1}, \ldots, \v_N \) mod \( \Lambda_N \) and the group of integer multiples of \( \gamma_1, \ldots, \gamma_r \) mod \( \Lambda_N \). The first group is the image under \( \alpha_A \) of \( \mathcal{L}_0 \), while the second is the image of \( \mathcal{L}_1 \). This proves that \( \mathcal{L} = \alpha_A(\mathcal{L}_0 \times \mathcal{L}_1) \).

From the system (15) one sees that the order of \( \mathcal{L}_1 \) is given by

\[
|\mathcal{L}_1| = \prod_{i=1}^{r} v_{ii} = \det(v_{ij})^{r}_{i,j=1}.
\]

On the other hand, \( \alpha_A^{-1} \circ \beta \) is an isomorphism between \( \mathbb{T}^n \times \prod p_{\alpha j}^{\alpha j} \) and \( \mathcal{L}_0 \times \mathcal{L}_1 \). The number of connected components of the first group is given by \( \prod p_{\alpha j}^{\alpha j} = |\prod Z_{p_{\alpha j}}| \), while it is given by \( |\mathcal{L}_1| \) for the second. This finishes the proof of the lemma.

We apply this lemma to the isotropy group \( I_x \subset \mathcal{T} \), and we formulate the intermediate result as another lemma for future reference:

**Lemma 3.** Let \( x \in S_r \). There are integer matrices \( (v_{ij})^{r}_{i,j=1} \) and \( (A_{ij})^{N}_{i,j=1} \) (depending on \( x \)) with \( \det A = \pm 1 \) such that \( I_x = \alpha_A(\mathcal{L}_0 \times \mathcal{L}_1) \), with \( \mathcal{L}_0 \) and \( \mathcal{L}_1 \) the groups given above in eq. (11). Alternatively, we can say that \( I_x \) is generated by the elements

\[
k(\tau_1, \ldots, \tau_N) := \alpha_A \left[ \frac{1}{2\pi} \left( \sum_{i=r+1}^{N} \tau_i b_i + \sum_{i,j=1}^{r} (v^{-1})_{ij} \tau_i b_j \right) \right],
\]

where \( \tau_i \in \mathbb{R} \) for \( r+1 \leq i \leq N \), and \( \tau_i \in 2\pi \mathbb{Z} \) for \( 1 \leq i \leq r \).

If we define \( \hat{\psi}_i = \sum A_{ij} \psi_j \), then lemma 3 implies that \( \hat{\psi}_i|_x = 0 \) for \( i = r+1, \ldots, N \), and \( \hat{\psi}_i|_x \) span \( T_x O_x \) for \( j = 1, \ldots, r \).

We now continue our analysis by inspecting the action of \( I_x \) on the tangent space \( T_x \Sigma \). Let \( k \in I_x \). Then, because \( k \cdot x = x \), this induces a linear map \( k : T_x \Sigma \to T_x \Sigma \) satisfying \( h(k \cdot X, k \cdot Y) = h(X, Y) \) for all \( X, Y \in T_x \Sigma \). In fact, because \( k \cdot \psi_i = \psi_i \) for any of our Killing fields, it follows that \( k \) leaves each vector in the tangent space \( T_x O_x \) invariant. But then it also leaves the orthogonal complement \( W_x \) invariant. Let \( \{e_1, \ldots, e_{s-r} \} \) be an orthogonal basis of \( W_x \). So for every \( k \in I_x \), we get a representing orthogonal matrix \( (k_{ij}), 0 < i, j \leq s-r \) acting on the orthogonal basis by \( k \cdot e_i = \sum k_{ij} e_j \). Because \( \Sigma \) is assumed to be orientable, we have a distinguished non-vanishing rank \( s \) totally anti-symmetric tensor field \( \epsilon \) (determined up to sign by \( \epsilon_{a_1 \ldots a_s} \epsilon_{b_1 \ldots b_s} h^{a_1 b_1} \cdots h^{a_s b_s} = s! \)). This tensor is invariant under the isometries of \( \Sigma \), so in
particular \(k \cdot \epsilon = \epsilon\) at point \(x\), for any \(k \in I_x\). Because \(k \cdot \psi_i\) for any of our Killing fields, this implies that the action of \(k\) on \(W_x\) preserves the orientation, so the matrix \((k_{ij})\) representing this action has determinant \(\det (k_{ij}) = +1\), and \((k_{ij}) \in SO(s-r)\). In particular, \((k_{ij})\) must have an even number of \(-1\) eigenvalues. The matrices \((k_{ij})\) commute for different choices of \(k \in I_x\), and so we may put them simultaneously into Jordan normal form. By making a change of basis of the \(\{e_1, \ldots, e_{s-r}\}\) with an orthogonal element \(g \in O(s-r)\), we may achieve that

\[
k \cdot (e_{2j-1} + ie_{2j}) = e^{i\theta_j}(e_{2j-1} + ie_{2j}), \quad 0 < j \leq [(s-r)/2] \quad \text{if} \ s - r \ \text{even} \quad (17)
\]

together with \(k \cdot e_{s-r} = e_{s-r}\) when \(s - r\) is odd\(^7\). The phases \(\theta_j\) depend on \(k\). For the elements of the isotropy group given by lemma \(^3\) we have in fact

\[
k(0, \ldots, 2\pi, \ldots 0) \cdot (e_{2j-1} + ie_{2j}) = \exp \left( 2\pi i \sum_{m=1}^{r} (v^{-1})_{lm} w_{mj} \right)(e_{2j-1} + ie_{2j}), \quad 0 < j \leq [(s-r)/2] \quad (18)
\]

if \(s - r\) is even together with \(k(0, \ldots, 2\pi, \ldots 0) \cdot e_{s-r} = e_{s-r}\) when \(s - r\) is odd. Here, the \(2\pi\) is in the \(l\)-th slot, with \(l \leq r\). The \(w_{ij}\) are integers, which follows from the fact that the group elements \(k(\sum_j v_{ij}b_j)\) are the identity, by lemma \(^3\). The above formula becomes somewhat more transparent if we note that the elements \(\gamma_j = \sum_{j=1}^{r} (v^{-1})_{ij} b_j\) defined for \(i = 1, \ldots, r\) generate a copy of the isotropy subgroup \(I_x \cong (v^{-1}\Lambda_r)/\Lambda_r \cong \prod_j \mathbb{Z}_{\rho_j}^* \cong \langle \gamma_j \rangle_{\text{mod} \Lambda_r}\), see lemma \(^2\). Thus, we may view the exponential expression in the above formula as a homomorphism

\[
\theta_j : (v^{-1}\Lambda_r)/\Lambda_r \to \mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}, \quad \theta_j(\gamma_j) = e^{2\pi i \sum_{m=1}^{r} (v^{-1})_{km} v_{mj}}. \quad (19)
\]

We also have

\[
k(0, \ldots, \tau_1, \ldots 0) \cdot (e_{2j-1} + ie_{2j}) = \exp(i\tau_j w_{ij})(e_{2j-1} + ie_{2j}), \quad 0 < j \leq [(s-r)/2] \quad (20)
\]

together with \(k(0, \ldots, \tau_1, \ldots 0) \cdot e_{s-r} = e_{s-r}\) when \(s - r\) is odd. Here, the \(\tau_1\) is in the \(l\)-th slot, and \(r + 1 \leq l \leq N\). The \(w_{ij}\) are again integers.

As yet, the basis \(\{e_1, \ldots, e_{s-r}\}\) has only been defined in \(W_x\), but we now wish to define it for any \(W_y\), with \(y \in O_x\). Let

\[
x(\tau_1, \ldots, \tau_r) = k(\tau_1, \ldots, \tau_r, 0, \ldots, 0) \cdot x, \quad 0 \leq \tau_i < 2\pi\quad (21)
\]

where \(k(\tau)\) is as in lemma \(^3\). Note that \(x(\tau)\) is periodic in \(\tau\) with period \(2\pi\) in each component of \(\tau\), and that \(\tau \in [0, 2\pi)^r \to x(\tau) \in O_x\) provide (periodic) coordinates in \(O_x\). We define our basis elements in \(W_{x(\tau)}\) by transporting \(\{e_1, \ldots, e_{s-r}\}\) to \(x(\tau)\) with the group element in eq. \(^2\). We call this basis \(\{e_1(\tau), \ldots, e_{s-r}(\tau)\}\). We note that this is still an orthonormal system, because it was obtained by an isometry between

\(^7\)Here it has been used that \((k_{ij})\) has determinant \(+1\). Otherwise \((k_{ij})\) could also act as a reflection on an odd number of basis vectors.
\( W_x \rightarrow W_x(\tau) \). Note that this basis is not periodic in \( \tau \), by eq. (17). To obtain an orthonormal basis \( \{ \tilde{e}_1(\tau), \ldots, \tilde{e}_{s-r}(\tau) \} \) that is periodic in \( \tau \), we set
\[
\tilde{e}_{2j-1}(\tau) + i\tilde{e}_{2j}(\tau) = \exp \left( -i \sum_{m,l=1}^{r} \tau_l(v^{-1})_{lm} w_{mj} \right) (e_{2j-1}(\tau) + ie_{2j}(\tau)),
\]
for \( 0 < j \leq [(s - r)/2] \), together with \( \tilde{e}_{s-r}(\tau) = e_{s-r}(\tau) \) when \( s - r \) is odd.

In an open neighborhood of \( O_x \), we now define coordinates as follows. First, on \( O_x \), we use the coordinates \( (y_{s-r+1}, \ldots, y_s) \in [0, 2\pi)^r \mapsto x(y_{s-r+1}, \ldots, y_s) \). In a neighborhood of \( O_x \) we use
\[
(y_1, \ldots, y_s) \mapsto \text{Exp}_{(y_{s-r+1}, \ldots, y_s)} \left( \sum_{j=1}^{s-r} y_j \tilde{e}_j(y_{s-r+1}, \ldots, y_s) \right).
\]
Here, “\text{Exp}” is the exponential map for our metric \( h \), i.e., \( (y_1, \ldots, y_{s-r}) \) are Riemannian normal coordinates transverse to \( O_x \). They cover an open neighborhood of \( O_x \). From the construction of the coordinates, the action of the isometry group \( \mathcal{T} \) in these coordinates is described by the following lemma:

**Lemma 4.** Let \( x \in S_r \), let \( (v_{ij}) \) be the matrix and \( k(\tau_1, \ldots, \tau_N) \in I_x \) be as in lemma 3. Then, in terms of the coordinates (23) covering a neighborhood of \( O_x \), the action of \( \mathcal{T} \) is given by
\[
k(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0) \cdot (y_1 + iy_2, \ldots, y_{s-r-1} + iy_{s-r}, y_{s-r+1}, \ldots, y_s) = \left( \exp \left[ i \sum_{l,m=1}^{r} \sigma_l(v^{-1})_{lm} w_{mj} \right] (y_{2j-1} + iy_{2j}) \right)^{(s-r)/2} \cdot (y_{s-r+i} + \sigma_i)_{i=1}^{r}
\]
when \( s - r \) is even. When \( s - r \) is odd, \( y_{s-r} \) remains unchanged. Furthermore,
\[
k(0, \ldots, 0, \sigma_{r+1}, \ldots, \sigma_N) \cdot (y_1 + iy_2, \ldots, y_{s-r-1} + iy_{s-r}, y_{s-r+1}, \ldots, y_s) = \left( \exp \left[ i \sum_{l=r+1}^{N} \sigma_l w_{ij} \right] (y_{2j-1} + iy_{2j}) \right)^{(s-r)/2} \cdot (y_{s-r+i})_{i=1}^{r}
\]
when \( s - r \) is even. When \( s - r \) is odd, \( y_{s-r} \) remains unchanged.

Let \( A \) be the matrix in lemma 4 and let \( \hat{\psi}_i = \sum_j A_{ij} \psi_j \). By lemma 4 the Killing fields \( \hat{\psi}_i \) are related to the coordinate vector fields \( \partial_{y_i} \) as:
\[
\begin{pmatrix}
\hat{\psi}_1 \\
\vdots \\
\hat{\psi}_r \\
\hat{\psi}_{r+1} \\
\vdots \\
\hat{\psi}_N
\end{pmatrix}
= \begin{pmatrix}
v_{11} & \ldots & v_{1r} & w_{11} & \ldots & w_1[(s-r)/2] \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
v_r & \ldots & v_{rr} & w_r & \ldots & w_r[(s-r)/2] \\
0 & \ldots & 0 & w_{r+1} & \ldots & w_{r+1}[(s-r)/2] \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & w_{N1} & \ldots & w_N[(s-r)/2]
\end{pmatrix}
\begin{pmatrix}
\partial_{y_{s-r+1}} \\
\vdots \\
y_1 \partial_{y_2} - y_2 \partial_{y_1} \\
\vdots \\
y_{s-r-1} \partial_{y_{s-r}} - y_{s-r} \partial_{y_{s-r-1}}
\end{pmatrix}
\]
when \( s - r \) is even. When \( s - r \) is odd, there is an analogous expression. Let us denote the \( N \times (r + [(s - r)/2]) \) matrix in this expression as \( C \). When \( N - r = [(s - r)/2] \), \( C \) is a square \( N \times N \) matrix. Furthermore, each of the commuting, locally defined Killing fields \( \partial/\partial y_k \) and \( y_{2j-1} \partial/\partial y_{2j} - y_{2j} \partial/\partial y_{2j-1} \) on the right side of the above equation is periodic, with period precisely \( 2\pi \). Hence, when \( N - r = [(s - r)/2] \), the matrix \( C \) must have determinant \( \pm 1 \). So we get the condition

\[
\det (v_{ij})^r \cdot \det (w_{(r+i)j})^{N-r} = \det C = \pm 1.
\]  

(27)

Because both determinants on the left are integers, we conclude we conclude that they must be \( \pm 1 \). In view of lemma 2, this means \( p_1 = \cdots = p_r = 1 \) and \( \det (w_{(r+i)j})^{N-r} = \pm 1 \). We summarize our findings in another lemma:

**Lemma 5.** Let \( \psi_1, \ldots, \psi_N \) be Killing fields as above, \( x \in S_r \), \( n = N - r = [(s - r)/2] \). Then \( p_1 = \cdots = p_r = 1 \) (see lemma 2), and \( \det (w_{(r+i)j})^n = \pm 1 \). Furthermore, in that case \( I_x \) is connected.

With the help of the above lemmas, we are now ready to analyze the orbit space \( \tilde{\Sigma} \) in the case when \( N = s - 2 \). We first cover \( \Sigma \) by the coordinate systems defined in eq. (23). Within each such coordinate system, we can then separately perform the quotient by \( T \). We need to distinguish the cases \( n = 0, 1, 2 \), where \( n = s - 2 - r \), and where the coordinate system covers a point \( x \in S_r \).

Case 0: For \( n = 0 \) and hence \( r = s - 2 \), the isotropy group \( I_x \) is discrete and is isomorphic to the group generated by the elements \( \gamma_j = \sum_{j=1}^{s-2} (v^{-1})_{ij} b_j \), see lemmas 2, 3. It is also isomorphic to \( \prod_j \mathbb{Z}_{p_j}^{\alpha_j} \). Furthermore, by combining lemmas 3 and 4, the action of these isotropy group in a neighborhood of \( O_x \) can be written as

\[
k(0, \ldots, 2\pi, \ldots, 0) \cdot (y_1 + iy_2, y_3, \ldots, y_s) = \left( \vartheta(\gamma_j)(y_1 + iy_2), y_3, \ldots, y_s \right),
\]

(28)

where we are using the notation introduced in eq. (19) for the homomorphism \( \vartheta : \prod_j \mathbb{Z}_{p_j}^{\alpha_j} \to S^1 \), and where the “\( 2\pi \)” is in the \( j \)-th slot. Consider now the kernel \( \ker \vartheta \). If \( g \) is an element in its kernel, then it is evident from the above formula that the corresponding isometry of \( \Sigma \) acts by the identity both in a full neighborhood of \( O_x \). Consequently, \( g \) must be the identity element of the group, since we are assuming the action to be effective. In particular, \( \vartheta \) is injective. Consider next the image \( \text{ran} \vartheta \). This is a finite subgroup of the circle group \( S^1 \). Hence it is given by \( \text{ran} \vartheta = \{ e^{2\pi ik/q} \mid k = 0, \ldots, q - 1 \} \cong \mathbb{Z}_q \) for some \( q \). It follows from the fact that \( \vartheta \) is injective that

\[
|\text{ran} \vartheta| = q = | \prod_j \mathbb{Z}_{p_j}^{\alpha_j} | = \prod_j p_j^{\alpha_j}.
\]

(29)

Furthermore, it follows that the inverse \( \vartheta^{-1} \) is a well-defined map on \( \mathbb{Z}_q \), which can be viewed as taking values in the isotropy group \( I_x \subset T \).

It follows from the discussion that, within the neighborhood considered, the quotient is modeled upon \( \mathbb{R}^2/\mathbb{Z}_q \), where \( q = \prod p_j^{\alpha_j} = | \det v | \) (see lemma 2, 3), and where the cyclic group of \( q \) elements acts on the coordinates \( y_1 + iy_2 \) by complex phases \( e^{2\pi i/q} \).
Thus, in a neighborhood of \( O_x \), the quotient space is an orbifold \( \mathbb{R}^2/\mathbb{Z}_q \). In particular, we see that the orbits having non-trivial discrete isotropy group must be isolated points in \( \hat{\Sigma} \). These orbits are also called “exceptional orbits”. The other orbits in case (0) have no isotropy group and are called “principal orbits”.

**Case 1:** For \( n = 1 \), lemma 5 applies and \( p_i = 1 \) for all \( i \). We first factor by the group elements \([0, \ldots, 0, \sigma_+, \ldots, \sigma_-] \), see eq. (25), and afterwards by the group elements \([\sigma_1, \ldots, \sigma_{s-3}, 0] \), see eq. (24). Then it is quite clear that the resulting quotient space of our neighborhood of \( O_x \) is locally modeled upon \( \mathbb{R} \times \mathbb{R}_{>0} \). The first factor corresponds to the variable \( y_3 \), while the second factor to the variable \( \sqrt{y_1^2 + y_2^2} \).

**Case 2:** For \( n = 2 \), lemma 5 applies and \( p_i = 1 \) for all \( i \). We first factor by the group elements \([0, \ldots, 0, \sigma_{s-3}, \sigma_{s-2}] \), see eq. (25), and afterwards by the group elements \([\sigma_1, \ldots, \sigma_{s-4}, 0, 0] \), see eq. (24). Then it is quite clear that the resulting quotient space of our neighborhood of \( O_x \) is locally modeled upon \( \mathbb{R}_{>0} \times \mathbb{R}_{>0} \). The first factor corresponds to the variable \( \sqrt{y_2^2 + y_2^2} \), while the second factor to the variable \( \sqrt{y_1^2 + y_2^2} \).

Thus, we have proved the following theorem:

**Theorem 1.** Let \( \Sigma \) be an orientable connected \( s \)-dimensional Riemannian manifold with \( s - 2 \) pairwise commuting Killing fields generating an action of the group \( \mathcal{T} = \mathbb{T}^{s-2} \) by isometries. Then the quotient space \( \hat{\Sigma} = \Sigma/\mathcal{T} \) is an orbifold with conical singularities, boundary segments, and corners. Thus, each point of \( \hat{\Sigma} \) has a neighborhood modeled on a neighborhood of the tip of a cone \( \mathbb{R}^2/\mathbb{Z}_q \), on a half-space \( \mathbb{R} \times \mathbb{R}_{>0} \), or on a corner, \( \mathbb{R}_{>0} \times \mathbb{R}_{>0} \). In the first case, the corresponding isotropy group is finite and \( q \) is given by the order of this group.

Each point of the boundary segments, corners, or orbifold points in \( \Sigma \) is associated with an isotropy group \( I_x \) as in lemma 3. It follows from our discussion in case 1) that, as long as we stay within one boundary segment, the isotropy group does not change. Furthermore, by lemmas 5 and 3, the isotropy group \( I_x \) is connected for points \( x \) associated with boundaries and corners. For \( x \) associated with conical singularities, \( I_x \) is discrete, again by lemmas 5 and 3. It also follows from our discussion of cases 1) and 2) that, for each boundary segment and each corner, the isotropy group is completely characterized by an integer matrix \( A \) of determinant \( \pm 1 \). Furthermore, it follows from our discussion in case 0) that the isotropy group \( I_x \) is characterized by an integer \( q \) and an injective homomorphism \( \vartheta^{-1} : \mathbb{Z}_q \to \mathbb{T}^{s-2} \), whose image is \( I_x \). There is one such matrix \( A \) for each boundary segment one for each corner, and one such \( q, \vartheta^{-1} \) for each conical singularity. The matrices \( A \) are actually not completely characterized by the corresponding isotropy subgroup \( I_x \). In fact, by lemma 2, \( \mathcal{L} = I_x, x \in S_r \) the position of the isotropy subgroup within \( \mathcal{T} \) is uniquely determined by the class \( (N = s - 2) \)

\[
[A] \in \frac{SL(N, \mathbb{Z})}{U(N - r, r; \mathbb{Z})} \quad (30)
\]

where \( U(N - r, r; \mathbb{Z}) \) is the group of block-upper triangular matrices with block sizes \( N - r, r \) with integer entries and determinant \( \pm 1 \). The quotient by such matrices \( U \) takes into account the fact that left-multiplying an \( A \) by such a matrix gives the same isotropy subgroup. When \( N - r = n = 1 \) (corresponding to case 1, and a boundary segment),
the class of \( A \) is determined by the last row \((a_{N1}, \ldots, a_{NN})\) of the matrix \( A \), and we have \( \sum a_{Ni} \psi_i \mid x = 0 \) for each point \( x \) in \( M \) corresponding to the boundary segment under consideration. When \( N - r = n = 2 \) (corresponding to case 2, and a corner), the class of \( A \) is determined by the last two rows \((a_{(N-1)1}, \ldots, a_{(N-1)N}), (a_{1N}, \ldots, a_{NN})\) up to a \( SL(2, \mathbb{Z}) \) transformation acting on each column of the \( N \times 2 \) matrix formed from these. We have \( \sum a_{(N-1)i} \psi_i \mid x = 0 \) and \( \sum a_{Ni} \psi_i \mid x = 0 \) for each point \( x \) in \( \Sigma \) corresponding to the corner under consideration.

If \( \{I_j\} \subset \partial \hat{\Sigma} \) is the collection of boundary segments, and if \( I_{ij} = I_i \cap I_j \) are the corresponding corners, then for each \( I_i \), we have a vector \( \mathbf{a}(I_i) \in \mathbb{Z}^N \) which is the last row of the matrix \( A \) corresponding to that boundary segment. The greatest common divisor (g.c.d.) of the entries of the vector may be assumed to be equal to 1,

\[
g.c.d.\{a_i(I_j) \mid i = 1, \ldots, D - 3\} = 1, \tag{31}
\]

For each corner \( I_{ij} \), the corresponding vectors \( \mathbf{a}(I_i) \) and \( \mathbf{a}(I_j) \) must be such that the \( N \times 2 \) matrix formed from these vectors can be supplemented by \( N - 2 \) rows of integers to an \( SL(N, \mathbb{Z}) \)-matrix, and this introduces a constraint on the pair \( \mathbf{a}(I_i), \mathbf{a}(I_j) \). In the case \( s = 4 \) (i.e., \( N = 2 \)), the constraint at each corner \( I_{ij} \) is simply that \( \det(\mathbf{a}(I_i), \mathbf{a}(I_j)) = \pm 1 \). In the general, the constraint on the vectors adjacent to a corner \( I_{ij} \) can be restated as follows applying \([3, \text{Lemma 2, I.2.3}]\):

**Proposition 1.** Let \( \{I_j\} \subset \partial \hat{\Sigma} \) be the boundary segments. With each boundary segment there is associated a vector \( \mathbf{a}(I_j) \in \mathbb{Z}^{s-2} \) and \( \sum a_i(I_j) \psi_i = 0 \) at the corresponding points of \( \Sigma \). At a corner \( I_{ij} = I_i \cap I_j \), the vectors are subject to the constraint

\[
g.c.d.\{Q_{kl} \mid 1 \leq k < l \leq D - 3\} = 1. \tag{32}
\]

Here, the numbers \( Q_{kl} \in \mathbb{Z} \) are defined by

\[
Q_{kl} = |\det(a_k(I_i) a_k(I_j) a_l(I_i) a_l(I_j))|. \tag{33}
\]

Let \( \{\hat{x}_i\} \subset \hat{\Sigma} \) be the conical singularities. With each one, there is associated a natural number \( q_i > 1 \), specifying the type \( \mathbb{R}^2/\mathbb{Z}_{q_i} \) of the conical singularity, and a homomorphism \( \vartheta^{-1}_i : \mathbb{Z}_{q_i} \to T \), whose image is the discrete isotropy subgroup at \( x_i = \) any point in \( \Sigma \) in the class \( \hat{x}_i \in \hat{\Sigma} \).

**Remarks:**

1. The data consisting of (i) the vectors \( \{\mathbf{a}(I_j)\} \), (ii) the pairs \( \{q_j, \vartheta^{-1}_j\} \), (iii) the orientation of \( \hat{\Sigma} \), (iv) the topological type of \( \hat{\Sigma} \) (genus) has been called the “weighted orbit space” by Orlik and Raymond \([41, 42]\) for the case \( s = 4 \). Our proposition hence may be viewed as a generalization of their analysis to higher dimensions.

2. If the boundary \( \partial \hat{\Sigma} \) is empty, then, as explained in detail in \([41, \text{Sec. 1.3}]\), there are additional invariants associated with the \( T \)-space \( \Sigma \). These may be characterized as obstructions to lift certain cross sections on the boundaries of tubular neighborhoods of the orbifold-type orbits \( \hat{x}_i \) to \( \Sigma \) and may be thought of as a class in the space

\[
H^2 \left( \hat{\Sigma}, \bigcup_{i=1}^m D_i^2; \mathbb{Z}^{s-2} \right) \cong \mathbb{Z}^{s-2} \tag{34}
\]
where each $D_i^2$ is a disk around $x_i$. This class has to be added to the data.

By a similar analysis we can also prove the following theorem on cohomogeneity-1 torus actions:

**Theorem 2.** Let $(\mathcal{H}, \gamma)$ be a connected, orientable, compact Riemannian manifold of dimension $s - 1 > 1$ with an isometry group containing an $(s - 2)$-dimensional torus $\mathcal{T} = \mathbb{T}^{s-2}$. Then the orbit space $\hat{\mathcal{H}} = \mathcal{H}/\mathcal{T}$ is diffeomorphic to a closed interval as a manifold with boundary, or to a circle. In the first case, we have the following possibilities concerning the topology of $\mathcal{H}$:

\[
\mathcal{H} \cong \begin{cases} 
\mathbb{S}^2 \times \mathbb{T}^{s-3} \\
\mathbb{S}^3 \times \mathbb{T}^{s-4} \\
L(p, q) \times \mathbb{T}^{s-4}
\end{cases}
\]

Here $L(p, q)$ is a 3-dimensional Lens space. In the second case, $\mathcal{H} \cong \mathbb{T}^{s-1}$.

**Proof:** Let $\psi_i, i = 1, \ldots, s-2$ be the commuting Killing fields of period $2\pi$ generating the action of $\mathcal{T}$ on $\mathcal{H}$. In the decomposition $\mathcal{H} = \bigcup \mathcal{S}_r$ defined as in eqs. (9), (10), only the sets with $r = s - 1$ and $r = s - 2$ may be non-zero, by lemma II. We consider these cases separately.

**Case 0:** Let $x \in \mathcal{S}_{s-1}$, and let $T_x \mathcal{H} = T_x \mathcal{O}_x \oplus W_x$ be the orthogonal decomposition into vectors tangent to $\mathcal{O}_x$ and those orthogonal to $\mathcal{O}_x$. By assumption, the dimension of $W_x$ is one. If $k \in I_x$ is in the isotropy group, then it leaves $T_x \mathcal{O}_x$ invariant, as $k \cdot \psi_i = \psi_i$ for all $i$. So $k$ acts as $\pm 1$ on $W_x$. But $k$ also preserves the rank $(s - 1)$ anti-symmetric tensor $\epsilon$ compatible with the metric, which exists since $\mathcal{H}$ is orientable. So $k$ acts as $+1$ on $W_x$, and hence as the identity on $T_x \mathcal{H}$. The action of $k$ must hence leave invariant any piecewise smooth geodesic on $(\mathcal{H}, \gamma)$ through $x$, and therefore $k$ must act as the identity on all of $\mathcal{H}$, since this is a connected manifold. Thus, the isotropy group $I_x$ is trivial in case 0. Consequently, near $O_x$, $\hat{\mathcal{H}} = \mathcal{H}/\mathcal{T}$ has the structure of a 1-dimensional manifold, i.e., an open interval.

**Case 1:** Let $x \in \mathcal{S}_{s-2}$. By exactly the same arguments as given above using lemmas III and IV, the action of $\mathcal{T}$ is given near $O_x$ in local coordinates $(y_1, \ldots, y_{s-1})$ by

\[
k(\sigma_1, \ldots, \sigma_{s-2}) \cdot (y_1 + iy_2, y_3, \ldots, y_{s-1}) = \left( \exp \left[ i \sum_{l=1}^{s-2} w_l \sigma_l \right] (y_1 + iy_2, y_3 + \sigma_1, \ldots, y_{s-1} + \sigma_{s-3}) \right).
\]

Here, $\pm A$ is some $SL(s - 2, \mathbb{Z})$ matrix, the numbers $w_l$ are integers, and $w_{s-2} = \pm 1$ (see lemma IV). It is evident from this that $\sqrt{y_1^2 + y_2^2}$ furnishes a coordinate for $\hat{\mathcal{H}}$ in a neighborhood of $O_x$, thus identifying this neighborhood locally with a half-open interval.

Because $\hat{\mathcal{H}}$ can be covered by neighborhoods of the kind described in cases 0 and 1), i.e., open and half open intervals, and because $\hat{\mathcal{H}}$ is compact in a natural topology and connected, it follows that $\hat{\mathcal{H}}$ must be a 1-dimensional connected compact manifold with or without boundaries. In the first case, $\hat{\mathcal{H}}$ is diffeomorphic to a closed interval, in the
second case to a circle. In the first case, the two boundary points of this closed interval correspond to orbits \( O_x \) respectively \( O_y \) in \( \mathcal{H} \) where an integer linear combination \( \sum a_{i,1} \psi_i \) respectively \( \sum a_{i,2} \psi_i \) vanishes. We can redefine our action of \( \mathcal{T} \) using instead the Killing fields \( \hat{\psi}_i = \sum A_{ij} \psi_j \) for some integer matrix \( A \) with \( \det A = \pm 1 \) in such a way that on \( O_x \) we have \( \hat{\psi}_1 = 0 \), while on \( O_y \) we have \( p \hat{\psi}_1 + q \hat{\psi}_2 = 0 \). Consider now the subgroup \( \mathcal{L} \subset \mathcal{T} \) generated by \( \hat{\psi}_3, \ldots, \hat{\psi}_{s-2} \). Clearly, \( \mathcal{L} \) is isomorphic to \( \mathbb{T}^{s-4} \).

It follows from the discussion of the cases 0) and 1) that there are no points in \( \mathcal{H} \) which are fixed under a non-trivial element of \( \mathcal{L} \), so \( \mathcal{H} \cong (\mathcal{H}/\mathcal{L}) \times \mathbb{T}^{s-4} \). Then, \( \mathcal{H}/\mathcal{L} \) is a three-dimensional manifold on which there acts the subgroup of isometries in \( \mathcal{T} \) generated by \( \hat{\psi}_1, \hat{\psi}_2 \). It is not difficult to see, and argued carefully in [28], that \( \mathcal{H}/\mathcal{L} \) is isomorphic to \( S^3 \) if \( (p, q) = (0, 1) \), isomorphic to \( S^2 \times \mathbb{T} \) if \( (p, q) = (1, 0) \), and a Lens-space \( L(p, q) \) otherwise.

In the second case, \( \mathcal{H} \) must be diffeomorphic to the direct product of \( \mathcal{T} \) and a circle, i.e. to \( \mathbb{T}^{s-1} \).

### 3.2 The fundamental group of \( \Sigma \)

In the previous section, we have analyzed oriented \( s \)-dimensional manifolds \( \Sigma \) with an effective action of \( \mathcal{T} = \mathbb{T}^{s-2} \). We showed that the quotient space \( \hat{\Sigma} = \Sigma / \mathcal{T} \) was an orientable 2-manifold with a finite number of conical singularities in the interior, and with boundaries and corners. With each of the conical singularities \( \hat{x}_i \in \hat{\Sigma} \) there was associated an integer \( q_i \in \mathbb{Z} \) and an injective homomorphism \( \vartheta_i^{-1} : \mathbb{Z}_{q_i} \to \mathcal{T} \). These homomorphisms may be written as

\[
\vartheta_i^{-1}(e^{2\pi i/q_j}) = (e^{2\pi i p_{1,j}/q_j}, \ldots, e^{2\pi i p_{s-2,j}/q_j}),
\]

(37)

where \( \text{g.c.d.}\{q_j, \text{g.c.d.}\{p_{1,j}, \ldots, p_{s-2,j}\}\} = 1 \). Furthermore, with each of the boundary intervals \( I_i \subset \partial \Sigma \), there was associated a vector \( a_i = (a_{1,i}, \ldots, a_{s-2,i}) \in \mathbb{Z}^{s-2} \). On a corner, the vectors are subject to the constraint (32), (33). If \( \Sigma \) is compact, then \( \hat{\Sigma} \) is a compact oriented 2-dimensional topological manifold, and hence topologically of the form

\[
\hat{\Sigma} \cong \hat{\Sigma}_g \setminus \bigcup_{j=1}^{d} D_j^2
\]

(38)

where each \( D_j^2 \) is a 2-dimensional disk, and where \( \hat{\Sigma}_g \) is a closed Riemann surface of genus \( g \).

One can show that the manifold \( \Sigma \) with \( \mathcal{T} \)-action is fixed up to equivariant isomorphism by the data consisting of \( \hat{\Sigma} \), \( \{I_i\} \), \( \{\hat{x}_i\} \), \( \{q_i, p_i\} \), \( \{a_i\} \); we will indicate how to prove this in subsection 3.4. Therefore, any topological invariant of \( \Sigma \) must be expressible in terms of these data. It is evident that the fundamental group \( \pi_1(\Sigma) \) should provide a strong invariant for the topology of \( \Sigma \). It is given in the next theorem:

**Theorem 3.** Let \( \Sigma \) be a compact orientable manifold with an effective action of \( \mathcal{T} = \mathbb{T}^{s-2} \)
\[ \pi_1(\Sigma) = \left\{ k_1, \ldots, k_{s-2}, d_1, \ldots, d_c, h_1, \ldots, h_d, m_1, \ldots, m_g, l_1, \ldots, l_g \right\} \]

\[ \left[ m_1, l_1 \right] \cdots \left[ m_g, l_g \right] \cdot d_1 \cdots d_c \cdot h_1 \cdots h_d; \]

\[ \left[ m_i, k_j \right]; \left[ l_i, k_j \right]; \left[ d_i, k_j \right]; \left[ h_i, k_j \right]; \left[ k_i, l_j \right]; \]

\[ k_1^{a_1,1} \cdots k_{s-2}^{a_{s-2},1}; \]

\[ d_1^{p_1,1} \cdots k_{s-2}^{p_{s-2},1}; \]

\[ d_c^{q_c,1} k_1^{p_1,c} \cdots k_{s-2}^{p_{s-2},c} \]  \hspace{1cm} (39)

Here, we are using the usual notation for a finitely generated group in terms of its relations, and \([x, y] = xyx^{-1}y^{-1}\) is the commutator of group elements. Above, \(g\) is the number of handles of \(\Sigma\), \(c\) is the number of conical singularities, \(b\) is the number of intervals \(\left\{ I_i \right\}\), and \(d\) is the number of boundary components in \(\partial \Sigma\) homeomorphic to circles, see eq. (35).

**Proof:** The proof is essentially an application of the Seifert-Van Kampen theorem, which is described e.g. in [35, Chap. 4]. Let \(x \in \Sigma\) be any point with trivial isotropy group, and let \(k_i, i = 1, \ldots, s-2\) be the closed loops obtained by a applying the \(i\)-th generator of \(\pi_1(T)\) (=generator of the \(i\)-th copy of \(T^1\) in \(T^{s-2}\)) to \(x\). Let \(d_i, i = 1, \ldots, c\) be lifts of loops going around the \(i\)-th conical singularity in \(\partial \Sigma\), and let \(h_i, i = 1, \ldots, d\) be lifts of loops going around the \(i\)-th hole of \(\Sigma\) (=boundary component in \(\partial \Sigma\)). We cut out a small disk \(D^2_i\) around each of the conical singularities in \(\Sigma\), we cut out a small neighborhood of the boundary in \(\Sigma\), and we consider the corresponding subset of \(\Sigma\). This subset will have a homotopy group generated by \(k_1, \ldots, k_{s-2}, d_1, \ldots, d_c, h_1, \ldots, h_d\), and generators \(m_1, l_1, \ldots, m_g, l_g\) corresponding to the \(g\) handles of \(\Sigma\). The relations are

\[ \left[ m_1, l_1 \right] \cdots \left[ m_g, l_g \right] \cdot d_1 \cdots d_c \cdot h_1 \cdots h_d; \]

\[ \left[ m_i, k_j \right]; \left[ l_i, k_j \right]; \left[ d_i, k_j \right]; \left[ h_i, k_j \right]; \left[ k_i, l_j \right]. \]  \hspace{1cm} (40)

We now glue back in the neighborhood of the boundary. Since, near the \(i\)-th boundary segment \(I_i\), the generator \(k_1^{a_1,1} \cdots k_{s-2}^{a_{s-2},1}\) shrinks to zero size, we receive the relations

\[ k_1^{a_1,1} \cdots k_{s-2}^{a_{s-2},1}; \cdots; k_1^{a_1,b} \cdots k_{s-2}^{a_{s-2},b} \]  \hspace{1cm} (41)

via the Van Kampen theorem. We finally glue in the disks around the conical singularities, each of which corresponds to a tube \(D^2 \times T^{s-2}\). We must perform the gluing in such a way that the standard action of \(T\) on \(D^2 \times T^{s-2}\) matches up with the action of \(T\) on \(\Sigma\) near the exceptional orbits. This action is characterized by the homomorphism \((37)\) for the \(j\)-th tube; we receive the relations

\[ d_1^{p_1,1} k_1^{p_1,1} \cdots k_{s-2}^{p_{s-2},1}; \cdots; d_c^{q_c,1} k_1^{p_1,c} \cdots k_{s-2}^{p_{s-2},c} \]  \hspace{1cm} (42)

from this operation, again via the Van Kampen theorem. **\Box**

The theorem has an interesting corollary in \(s = 4\) if the action of \(T\) has a fixed point, i.e. when the orbit space has a corner. The vectors associated with the intervals \(I_i; I_{i+1}\) adjacent to the corner, \(a_i, a_{i+1}\), must then satisfy \(\det(a_i; a_{i+1}) = \pm 1\) [see eq. (32)]. This imposes the relation \(k_1 = k_2 = e\) in eq. (39). Then, if \(\pi_1(\Sigma) = 0\), this will imply that \(g = d = 0\), and \(q_1, \ldots, q_c = 0\). In other words, if \(s = 4\), if the action has fixed point,
and if \( \Sigma \) is simply connected, then there are no conical singularities, i.e., exceptional orbits. This was first proved using methods from singular cohomology in [21].

The above theorem has another related corollary which will be relevant below in our application to the structure of black holes. Let \( D^2 \subset \Sigma \) be any disk in the interior of the orbit manifold not intersecting any of the boundaries or conical singularities. Thus, the orbits are all \((s - 2)\)-dimensional tori, with no fixed points. The inverse image of \( D^2 \) in \( \Sigma \) is homeomorphic to \( D^2 \times \mathbb{T}^{s-2} \), with \( \mathcal{T} \) acting on the second factor. Let us denote the generators of \( \pi_1(D^2 \times \mathbb{T}^{s-2}) \) by \( k_1, \ldots, k_{s-2} \), which are the \( s - 2 \) generators of \( \pi_1(\mathbb{T}^{s-2}) = \mathbb{Z}^{s-2} \). Without loss of generality, we may assume that \( k_j \) are the image of the paths generated by the action of the \( j \)-th copy on \( \mathcal{T} = \mathbb{T}^{s-2} \) on a point \( x \in D^2 \times \mathbb{T}^{s-2} \).

From the inclusion \( f : D^2 \times \mathbb{T}^{s-2} \to \Sigma \), we get a corresponding homomorphism \( f_* : \pi_1(D^2 \times \mathbb{T}^{s-2}) \to \pi_1(\Sigma) \). The way we have set things up, we may assume that \( f_*(k_j) = k_j \), using the same notation and assumptions as in the above theorem 3.

**Lemma 6.** If \( f_* : \pi_1(D^2 \times \mathbb{T}^{s-2}) \to \pi_1(\Sigma) \) is surjective, then we have \( g = d = 0, q_1 = \cdots = q_c = 1 \). In other words, \( \Sigma \) is a topologically a disk, and there are no conical singularities.

**Proof:** Using eq. (39) and the formula \( f_*(k_j) = k_j \), we see that \( f_*\pi_1(D^2 \times \mathbb{T}^{s-2}) \) is a normal subgroup of \( \pi_1(\Sigma) \). By assumption, the factor group \( \pi_1(\Sigma)/f_*\pi_1(D^2 \times \mathbb{T}^{s-2}) \) is trivial. From the quotient, the group \( \pi_1(\Sigma) \) [see eq. (39)] receives the additional relations \( k_j = e \) for \( j = 1, \ldots, s - 2 \). This means that the factor group is isomorphic to

\[
\pi_1(\Sigma)/f_*\pi_1(D^2 \times \mathbb{T}^{s-2}) \cong \left\{ d_1, \ldots, d_c, h_1, \ldots, h_d, m_1, \ldots, m_g, l_1, \ldots, l_s \mid d_1 \cdots d_c \cdot h_1 \cdots h_d \cdot d_c^* \cdot \ldots \cdot d_1^* \right\}.
\]

Evidently, this group is non-trivial unless \( g = d = 0, q_1 = \cdots = q_c = 1 \), from which the lemma follows.

### 3.3 The orbit space of the domain of outer communication

We next want to determine the orbit space of a \( D \)-dimensional asymptotically Kaluza-Klein stationary black hole spacetime \((M, g)\) with \( D - 3 \) axial Killing fields \( \psi_i, i = 1, \ldots, D - 3 \) generating an (effective) action of \( \mathcal{T} = \mathbb{T}^{D-3} \). Thus, the total group isometries is \( \mathcal{G} = \mathcal{T} \times \mathbb{R} \), with \( \mathbb{R} \) the additive group generated by the asymptotic timelike Killing field \( t \). We have the following theorem:

**Theorem 4.** Let \((M, g)\) be a stationary, asymptotically Kaluza-Klein, \( D \)-dimensional vacuum black hole spacetime with isometry group \( \mathcal{G} = \mathbb{R} \times \mathcal{T} \), satisfying the technical assumptions stated in sec. 2. Then the orbit space \( \hat{M} = \langle [M] \rangle / \mathcal{G} \) of the domain of outer communication is a 2-dimensional manifold with boundaries and corners homeomorphic to a half-plane. In particular, there are no conical singularities in \( \hat{M} \). The possibilities for the horizon topology are eqs. (35), with \( s = D - 1 \). One of the boundary segments \( I_j \subset \partial \hat{M} \) is the quotient of the horizon \( \hat{H} = H/\mathcal{G} \), while the remaining \( I_j \) correspond to the various “axis”, where \( \sum a_i(I_j)\psi_i = 0 \). The vectors \( a(I_j) \in \mathbb{Z}^{D-3} \) are subject to the constraint (32) on each corner \( I_i \cap I_j \).
Remark: In the statement concerning the horizon topology, eq. (35), we do not mean that the torus factors (such as in $H \cong S^2 \times T^{D-1}$) correspond to the rotations in the extra dimensions near infinity.

Proof: The “structure theorem” 4.3 of [7] states that $\langle \langle M \rangle \rangle$ contains a smooth, spacelike, acausal slice $\Sigma$ whose boundary is a cross section $\mathcal{H}$ of the horizon, which is asymptotic to a $\tau = \text{const.}$ slice in the exterior under the identification of the exterior with (part of) $\mathbb{R}^{s,1} \times T^{D-s-1}$, which is invariant under the action of $T = T^{D-3}$ and which is transversal to the orbits of $t$ represented by the factor $\mathbb{R}$ in $G$. Furthermore, if $F^r$ is the flow of $t$, then $\langle \langle M \rangle \rangle = \cup_t F^r(\Sigma)$. This result will allow us to reduce the proof of thm. 4 to thm. 1.

We first factor $\langle \langle M \rangle \rangle$ by $\mathbb{R}$. We can identify the resulting space with $(\Sigma, h)$, with $h$ the Riemannian metric induced from $g$. This metric is asymptotic to the standard flat metric on $\mathbb{R}^s \times T^{D-s-1}$ ($s = 1, 2, 3$ or $4$) in the exterior region. Evidently, $T$ acts as a group of isometries on $(\Sigma, h)$, and $\Sigma$ contains no points with discrete isotropy group. For definiteness, we focus on the case $s = 4$, the other cases are similar. The action of $T$ is then conjugate in the exterior region to the standard action which acts on $T^{D-5}$ by rotations along the generators, and which acts on $\mathbb{R}^4$ by rotations in the 12- and 34-plane.

We divide $\Sigma$ up into two pieces $\Sigma_0 \cup \Sigma_\infty$. The region $\Sigma_\infty$ is the asymptotic region, and $\Sigma_0$ is the rest. The split can be arranged so that both pieces are separately invariant under the action of $T$. $\Sigma_0$ is a compact manifold with boundary $\partial \Sigma_0$ consisting of $\mathcal{H}$ and of a second boundary component $\cong S^3 \times T^{D-5}$ bordering on $\Sigma_\infty$. The quotient of $\Sigma$ is given by the union of the quotients $\hat{\Sigma}_0 = \Sigma_0/T$ and $\hat{\Sigma}_\infty = \Sigma_\infty/T$. The action of $T$ on the exterior region is conjugate to the action of $T$ on $(\mathbb{R}^4 \setminus \{(x_1, x_2, x_3, x_4) \mid R < r\}) \times T^{D-5}$, where $R$ is the standard radius on $\mathbb{R}^4$. So the quotient is given by $\Sigma_\infty \cong \{(R_1, R_2) \in \mathbb{R}^2 \mid R_1, R_2 > 0, R_1^2 + R_2^2 > r^2\}$, where $R_1$ can be identified with $\sqrt{x_1^2 + x_2^2}$ and $R_2$ with $\sqrt{x_3^2 + x_4^2}$. The boundary components of $\hat{\Sigma}_\infty$ defined by $R_i = 0, i = 1, 2$ correspond to an axis in the spacetime where $\psi_1, \psi_2, \psi_3$ vanish. The quotient $\hat{\Sigma}_0$ can be determined as in thm. 1 but we must now consider a compact manifold $(\Sigma_0, h)$ with boundaries. Near the boundary component $\cong S^3 \times T^{D-5}$ of $\partial \Sigma_0$, the quotient space $\hat{\Sigma}_0$ must look like $\cong \{(R_1, R_2) \in \mathbb{R}^2 \mid R_1, R_2 > 0, R_1^2 + R_2^2 < r^2\}$. Near the horizon boundary component $\mathcal{H}$, we can analyze the quotient space by combining the arguments in thms. 1 and 2. In summary, $\Sigma_0$ is a compact manifold with boundaries, corners and possibly conical singularities in the interior. The quotient $\hat{\mathcal{H}} = \mathcal{H}/T \subset \partial \Sigma_0$ is represented by a boundary segment in the first case described in thm. 2, i.e. when the horizon topologies are as in eq. (35) with $s = D - 1$. It is represented by the boundary of a removed disk from $\Sigma_0$ in the second case described in the thm. 2, i.e. when the horizon topology is $\mathcal{H} \cong T^{D-1}$. The other boundary components of $\Sigma_0$ are line segments corresponding to axis. The quotients $\Sigma_0$ and $\Sigma_\infty$ are glued together along the joint boundary $\{(R_1, R_2) \in \mathbb{R}^2 \mid R_1, R_2 > 0, R_1^2 + R_2^2 = r^2\}$. It is clear that $\Sigma_0$ is oriented and connected. Therefore, it must be a handle body with possibly different boundary components, each homeomorphic to circles, and with conical singularities in the interior. Gluing $\Sigma_\infty$ onto $\Sigma_0$, we thus see that $\Sigma$ is homeomorphic to the connected sum of a half-plane and a handle body $\hat{\Sigma}_g$, with a number of disks removed and with
orbifold points. Therefore, topologically
\[ \hat{\Sigma} \cong (\mathbb{R} \times \mathbb{R}_{>0}) \# \hat{\Sigma}_g \setminus \bigcup_{j=1}^d D_j^2. \] (44)

To rule out the presence of handles, removed disks, and points with conical singularities, we now use the topological censorship theorem for asymptotically Kaluza-Klein spaces [9], see also [17, 18]. This theorem states that any curve \( \gamma \) with endpoints in \( \Sigma_\infty \) can be continuously deformed to a curve entirely within \( \Sigma_\infty \). Furthermore, any closed loop in \( \Sigma_\infty \) is homotopic to a closed loop of in a neighborhood of \( \Sigma_\infty \) of the form \( D^2 \times T \), where \( D^2 \) is a two-dimensional disk that can be identified with a corresponding disk in \( \hat{\Sigma}_\infty \). These facts together imply that if \( f : D^2 \times T \rightarrow \Sigma \) is the embedding map, then \( f_* : \pi_1(D^2 \times T) \rightarrow \pi_1(\Sigma) \) is surjective. If \( \Sigma \) were a compact manifold (without boundary), we could now directly apply lemma 6, and thereby conclude can be no handles, removed disks, nor conical singularities, and that \( \hat{\Sigma} \) would consequently be a disk. In the case at hand, \( \Sigma \) is a manifold with boundary components consisting of \( \mathcal{H} \) and \( S^3 \times T^d \). Nevertheless, using eq. (44), the arguments leading to lemma 6 and \( \mathcal{B} \) can be very easily adapted to this case, or one may alternatively compactify \( \Sigma \) by gluing in appropriate manifolds with boundary \( \mathcal{H} \) and \( S^3 \times T^d \). In either case, we conclude that \( \hat{\Sigma} \) is a homeomorphic to a half-plane, and that there are no conical singularities.

Since \( \hat{M} = \langle \langle M \rangle \rangle / \mathcal{G} \cong \hat{\Sigma} \), this proves the theorem.

3.4 Model spaces, examples

We finally discuss to what extent the structure of the space \( \langle \langle M \rangle \rangle \) as a manifold with \( \mathcal{G} \)-action is determined by the associated data described in thm. 4. As we have seen in the proof of this theorem, the study of \( \langle \langle M \rangle \rangle \) as a manifold with \( \mathcal{G} \)-action essentially boils down to the study of a \( (D−1) \)-dimensional spatial slice \( \Sigma \) with corresponding action of \( T \), and it is hence sufficient from a topological viewpoint to study this situation.

Thus, let us assume that we are given an oriented \( s \)-dimensional manifold \( \Sigma \) with \( T \)-action, with corresponding orbit space \( \hat{\Sigma} \) and decoration data, as described in prop. 1 and thm. 3. For simplicity, let us consider the case that \( \Sigma \) has no boundaries. The general case can be treated quite similarly. Then we can ask whether \( \Sigma \) as a manifold with \( T \)-action is uniquely determined by the orbit space and decoration data. In other words, given another such manifold \( \Sigma' \), does there exist a diffeomorphism \( h : \Sigma \rightarrow \Sigma' \), and an automorphism \( \alpha_A : T \rightarrow T \) such that \( h(k \cdot x) = \alpha_A(k) \cdot h(x) \) for all \( x \in \Sigma, k \in T \)? As shown in the case \( s = 4 \) in [42, Para. 1], the answer to this question is in the affirmative. (In the case that \( \partial \Sigma \neq 0 \), the decoration data must include also the invariant mentioned in remark (2) after prop. 1.) The proof of this theorem really extends straightforwardly to the case of \( \Sigma \) with arbitrary dimension, so we will not describe it here in detail.

A related question is whether for a given \( \hat{\Sigma} \) and given decoration data as described in prop. 1 we can find a corresponding manifold \( \Sigma \) with \( T \)-action described by these data. The question is again in the affirmative, and we now outline how one can construct
such a manifold. Thus, let us assume that we are given (i) an orbit space \( \hat{\Sigma} \) which is an oriented two-dimensional manifold with boundaries, corners, and conical singularities, (ii) vectors \( \{ a(I_j) \} \), one for each component \( I_j \subset \partial \hat{\Sigma} \), satisfying the constraints \( (32) \) (iii) a collection \( \{ q_i, p_j \} \), one for each conical singularity \( \hat{x}_i \in \hat{\Sigma} \), as described in around \( (37) \). We want to construct a corresponding manifold \( \Sigma \) with \( \mathcal{T} \)-action.

For simplicity, let us assume that \( \Sigma \) is a half-plane \( \mathbb{R}_{>0} \times \mathbb{R} \), with finitely many conical singularities in the interior, and with boundary divided into the segments \( I_1, \ldots, I_b \). We first consider the conical singularities in the interior. We may assume that they are all in a disk \( D^2 \subset \mathbb{R}_{>0} \times \mathbb{R} \). We cut out this disk, and we consider \( D^2 \times \mathcal{T} \) with standard action of \( \mathcal{T} \) on the second factor. We cut out from this region \( c \) tubes of the form \( D_i^2 \times \mathcal{T} \), with each \( D_i^2 \) a small disk containing the \( i \)-th of the \( c \) conical singularities. Near the conical singularities, we would like the \( \mathcal{T} \)-action to be described by the homomorphisms \( \vartheta^{-1}_i : \mathbb{Z}_{q_i} \to \mathcal{T} \) given in eq. \( (37) \). A model space for this action is

\[
D_i^2 \times \vartheta^{-1}_i \mathcal{T}, \quad D_i^2 = \{ z \in \mathbb{C} \mid |z - z_i| \leq 1 \},
\]

where \( g \in \mathbb{Z}_{q_i} \subset S^1 \) acts on the disk by multiplication with the complex phase. We glue in these model spaces along the boundaries where we cut out the \( c \) singularities in the interior, and with boundary divided into the segments \( I_1, \ldots, I_b \). We cut out from this region \( c \) tubes of the form \( D_i^2 \times \mathcal{T} \), with each \( D_i^2 \) a small disk containing the \( i \)-th of the \( c \) conical singularities. We now construct a second \( \mathcal{T} \)-space \( \Sigma_1 \) that incorporates the data \( \{ a(I_j) \} \). These data were constructed above by giving, for each orbit, a neighborhood together with a set of coordinates in which the action of \( \mathcal{T} \) was explicitly given. It is intuitively clear that we can turn this around and define \( \Sigma_1 \) to be the collection of these coordinate charts with corresponding \( \mathcal{T} \)-action, and we now briefly explain how this can be done. For simplicity and concreteness, we consider explicitly the case when \( s = \text{dim} \Sigma = 4 \). The construction is well-known in topology and is sometimes called “linear plumbing”, see [25]. We present the construction in such a way that the generalization to general \( s \) should be fairly obvious, details will be given in [8].

The construction of \( \Sigma_1 \) is as follows. Let \( b \geq 2 \) be the number of boundary segments \( \{ I_j \} \). On the boundary \( \partial S^3 \) of the four-dimensional solid ball \( B^4 = \{ y_1^2 + y_2^2 + y_3^2 + y_4^2 < 1 \} \), we consider the disjoint subsets

\[
S_+ := \{(y_1, y_2, y_3, y_4) \in S^3 \mid \sqrt{y_3^2 + y_4^2} < 1/4\},
\]

\[
S_- := \{(y_1, y_2, y_3, y_4) \in S^3 \mid \sqrt{y_1^2 + y_2^2} < 1/4\}.
\]

Both of these subsets are topologically solid tori. We consider the disjoint union of \( b - 1 \) copies of the solid ball \( B^4 \), and on the \( i \)-th copy we define an action of \( \mathcal{T} = \mathbb{T}^2 \) generated by the two \( 2\pi \)-periodic vector fields \( \psi_1, \psi_2 \) given by

\[
\begin{pmatrix}
y_1 \partial_{y_2} - y_2 \partial_{y_1} \\
y_3 \partial_{y_4} - y_4 \partial_{y_3}
\end{pmatrix} = \begin{pmatrix} a_1(I_i) & a_2(I_i) \\
a_1(I_{i+1}) & a_2(I_{i+1}) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.
\]

The consistency condition on the \( i \)-th corner \( (33), (32) \) guarantees that the determinant of the above matrix is \( \pm 1 \). We wish to glue the \( S_+ \)-part of the boundary of the \( i \)-th
copy of the ball $B^4$ to the $S_-$-part of the boundary of the $(i+1)$-th copy in such a way that the actions of $T$ on these copies are compatible. It is not difficult to see that this is achieved if we identify these parts by the maps $f_i : S_- \to S_+$ defined by

$$f_i(y_1, y_2, y_3, y_4) = \left(y_3, y_4, y_1 \sin(n_i \varphi) + y_2 \cos(n_i \varphi), y_1 \cos(n_i \varphi) - y_2 \sin(n_i \varphi)\right),$$  \hspace{1cm} (48)

where $\varphi = \arctan \frac{y_3}{y_4}$ and $n_i = a_1(I_1)a_2(I_{i+2}) - a_2(I_1)a_1(I_{i+2})$, i.e. we have $f_{i*} \psi_1 = \psi_1$ and $f_{i*} \psi_2 = \psi_2$. Thus, for $b > 2$ we define:

$$\Sigma_1 = (\ldots ((B^4 \cup_{f_1} B^4) \cup_{f_2} B^4) \ldots \cup_{f_{b-3}} B^4) \cup_{f_{b-2}} B^4.$$  \hspace{1cm} (49)

For $b = 2$ be define $\Sigma_1 = B^4$. The space $\Sigma_1$ has a 3-dimensional boundary whose structure is determined by the first and last vector $a(I_1)$, and $a(I_b)$. It is either $T^1 \times S^2, S^3$, or a lens space $L(p, q)$, see thm. 2.

We may cut out from $\Sigma_1$ a tube $D^2 \times T$, and glue the boundary obtained in this way onto $\partial \Sigma_0$. The manifold $\Sigma$ obtained in this way is the desired $T$-space $\Sigma$ in the special case considered. The general case may be treated in a similar way, as we will discuss in a future paper [8]. We may call the manifold $\Sigma$ constructed from the decoration data of the orbit space $X(\epsilon, \Sigma, \{a(I_1)\}, \{q_i, p_i\})$, where $\epsilon$ is an orientation, and $\Sigma$ an oriented two-dimensional manifold with boundaries and corners. We give some examples (without conical singularities):

**Example 1:** (From [41]) Let $s = 4$, $\hat{\Sigma} = D^2$, $\partial D^2 = I_1 \cup I_2 \cup I_3$, and consider the data $\{(1, 0), (0, 1), (1, 1)\}$. Then the space $X[D^2, \{(1, 0), (0, 1), (1, 1)\}]$ is the complex projective space $\mathbb{C}P^2 = \mathbb{C}^3/\sim$, where the equivalence relation is $(z_1, z_2, z_3) \sim (\lambda z_1, \lambda z_2, \lambda z_3)$ and the action of $T = T^2$ is $[\tau_1, \tau_2] \cdot (z_1, z_2, z_3)_{\sim} = (e^{i \tau_1} z_1, e^{i \tau_2} z_2, z_3)_{\sim}$. The equivalence $X[D^2, \{(1, 0), (0, 1), (1, 1)\}] \cong \mathbb{C}P^2$ can be seen e.g. by noting that the axis in $\mathbb{C}P^2$ corresponding to the vectors $(1, 0), (0, 1), (1, 1)$ are given by the set of points $(z_1, z_2, z_3)_{\sim} \in \mathbb{C}P^2$ such that, respectively, $z_1 = 0, z_2 = 0, z_3 = 0$.

**Example 2:** Let $s = 4$, $\hat{\Sigma} = D^2$ and consider the data $\{(1, 0), (0, 1), (1, 0), (0, 1)\}$ (four intervals). Then the space $X[D^2, \{(1, 0), (0, 1), (1, 0), (0, 1)\}]$ is $S^2 \times S^2$, with the standard action of $T$. This is easily seen by considering the isotropy groups of the action. In fact, examples 1 and 2 constitute in some sense the most general case in $s = 4$ because one can show that [41, 42], topologically, $\Sigma$ is a connected sum of projective spaces an $S^2 \times S^2$’s in the situation under consideration.

**Example 3:** Let $s = 5$, $\hat{\Sigma} = D^2$ and consider the data $\{(1, 0, 0), (q_1, q_2, p), (0, 1, 0)\}$. The constraints on the corners are fulfilled if we have $\text{g.c.d.}(p, q_1) = 1 = \text{g.c.d.}(p, q_2)$. The corresponding space $X[D^2, \{(1, 0, 0), (q_1, q_2, p), (0, 1, 0)\}]$ is a generalized lens space $L(p; q_1, q_2)$. The generalized lens space is defined as the quotient of $S^5$ (realized as the unit sphere in $\mathbb{C}^3$) by the discrete subgroup of isometries of order $p$ generated by an

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8 If $X, Y$ are sets and $f$ is a map $f : A \subset X \to Y$, then $X \cup_f Y$ is the set defined as the quotient of the disjoint union $X \cup Y$ by the equivalence relation $x \sim y :\iff (x, y) \in \text{graph } f$. 

element $\lambda$ acting as $\lambda \cdot (z_1, z_2, z_3) = (e^{2\pi i/p} z_1, e^{2\pi i q_1/p} z_2, e^{2\pi i q_2/p} z_3)$. The action of $T = \mathbb{T}^3$ on an equivalence class $(z_1, z_2, z_3) \sim \in L(p; q_1, q_2)$ under this action is
\[
[t_1, t_2, t_3] \cdot (z_1, z_2, z_3) \sim = (e^{i t_1/p} z_1, e^{i(t_1 + q_1 t_1)/p} z_2, e^{i(t_2 + q_2 t_1)/p} z_3) \sim .
\]
The axis corresponding to the vectors $(1, 0, 0), (q_1, q_2, p), (0, 1, 0), (1, 1, 0)$ are, respectively, $z_2 = 0, z_2 = z_3 = 0, z_3 = 0$. Note that $\pi_1(L(p; q_1, q_2)) \cong \mathbb{Z}_p$, so for $p \neq 1$ this space is not simply connected.

**Example 4:** Let $s, \hat{\Sigma}$ be as in the previous example, but let the data now be \{(1, 0, 0), (q_1, q_2, p), (0, 1, 0), (1, 1, 0)\}. The constraints on the corners are fulfilled if we have $\text{g.c.d.}(p, q_1) = 1 = \text{g.c.d.}(p, q_2)$. The manifold in question is now topologically (combining the examples 1 and 3)
\[
X[D^2, \{(1, 0, 0), (q_1, q_2, p), (0, 1, 0), (1, 1, 0)\}] \cong L(p; q_1, q_2)\#(\mathbb{C}P^2 \times S^1).
\]

## 4 Stationary vacuum black holes in $D$ dimensions

In the previous section, we looked at the topology of the domain of outer communication $\langle \langle M \rangle \rangle$ and the structure of the orbits of the symmetries. In this section, we investigate the spacetime metric, i.e. the implications of the Einstein equations $R_{ab} = 0$.

These equations imply a set of coupled differential equations for the metric on the two-dimensional factor space $\hat{M}$, described above in thm. 4. To understand these equations in a geometrical way, we note that the projection $\pi : \langle \langle M \rangle \rangle \to \langle \langle M \rangle \rangle/G = M$ (with $G = \mathbb{T}^{D-3} \times \mathbb{R}$ the isometry group) defines a $G$-principal fibre bundle over the interior of $\hat{M}$, because we argued in the previous section that such points correspond to points in the domain of outer communication with trivial isotropy group. At each point $x \in \langle \langle M \rangle \rangle$ in a fibre over $\pi(x)$ in the interior of $\hat{M}$, we may uniquely decompose the tangent space at $x$ into a subspace of vectors tangent to the fibres, and a space $W_x$ of vectors orthogonal to the fibres. Evidently, the distribution of vector spaces $W_x$ is invariant under the group $G$ of symmetries, and hence forms a “horizontal bundle” in the terminology of principal fibre bundles [33]. According to standard results in the theory of principal fibre bundles [33], a horizontal bundle is equivalent to the specification of a $G$-gauge connection $\tilde{D}$ on the factor space, whose curvature we denote by $\tilde{F} = T_I \tilde{F}_\alpha^I dx^\alpha \wedge dx^\beta$, with $T_I, I = 0, \ldots, D - 3$ the generators of the abelian group $G$. Roman indices $\alpha, \beta, \ldots$ take the values 1, 2. The horizontal bundle gives an isomorphism $W_x \to T_{\pi(x)} \hat{M}$ for any $x$, and this isomorphism may be used to uniquely construct a smooth covariant tensor field $\hat{t}_{\alpha\beta\cdots\gamma}$ on the interior of $\hat{M}$ from any smooth $G$-invariant covariant tensor field $t_{ab\cdots c}$ on $M$.

For example, the metric $g_{ab}$ on $\hat{M}$ thereby gives rise to a symmetric tensor $\hat{g}_{\alpha\beta}$ on $\hat{M}$. One can show with a significant amount of labor [3] (see also [7]) that the $D - 2$ dimensional subspaces spanned by the Killing fields at points of $\langle \langle M \rangle \rangle$ corresponding to interior points of $\hat{M}$ always contain a timelike vector. Hence the bilinear form induced from $g_{ab}$ on $W_x$ has signature $(++)$, so $\hat{g}_{\alpha\beta}$ is in fact a Riemannian metric. We let $\hat{D}$ act on ordinary tensors $\hat{t}_{\alpha\beta\cdots\gamma}$ as the connection of $\hat{g}_{\alpha\beta}$, with Ricci tensor denoted $\hat{R}_{\alpha\beta}$. 
By performing the well-known Kaluza-Klein reduction of the metric $g_{ab}$ along the orbits of $G$, we can locally write the Einstein equations as a system of equations on the interior of the factor space $\hat{M}$ in terms of metric $\hat{g}_{\alpha\beta}$, the components $\hat{F}_{\alpha\beta}^I$ of the curvature, and the Gram matrix field $G_{IJ}$

$$G_{IJ} = g(X_I, X_J), \quad X_I = \begin{cases} t & \text{if } I = 0, \\ \psi_i & \text{if } I = i = 1, \ldots, D - 3. \end{cases} \quad (52)$$

The resulting equations are similar in nature to the “Einstein-equations” on $\hat{M}$ for $\hat{g}_{\alpha\beta}$, coupled to the “Maxwell fields” $\hat{F}_{\alpha\beta}^I$ and the “scalar fields” $G_{IJ}$, see [32, 5]. We will not write these equations down here, as we will not need them in this most general form.

In our case, the equations simplify considerably because one can show (see e.g. [7]) that the distribution of horizontal subspaces $W_x$ is locally integrable, i.e., locally tangent to a family of two-dimensional submanifolds. In that case, the connection is flat, $\hat{F}_{\alpha\beta}^I = 0$, and the dimensionally reduced equations may be written as

$$\hat{D}^\alpha (rG^{-1} \hat{D}_\alpha G) = 0, \quad (53)$$

together with

$$\hat{R}_{\alpha\beta} = \hat{D}_\alpha \hat{D}_\beta \log r - \frac{1}{4} \text{Tr} \left( \hat{D}_\alpha G^{-1} \hat{D}_\beta G \right). \quad (54)$$

Greek indices have been raised with $\hat{g}^{\alpha\beta}$. The equations are well-defined a priori only at points in the interior of $\hat{M}$ where the Gram determinant

$$r^2 = -\det G \quad (55)$$
does not vanish. Chrusciel has shown [8] (based on previous work of Carter [2] and also of [7]) that $r^2 > 0$ away from the boundary of $\hat{M}$. The reduced Einstein equations are hence well-defined there. On the other hand, $r$ vanishes on any boundary component $I_j$ of $\hat{M}$ corresponding to an axis, i.e. where a linear combination $\sum a_i(I_j) \psi_i = 0$ vanishes, because the Gram matrix then has a non-trivial kernel. It also vanishes on the segment of $\partial \hat{M}$ corresponding to the horizon $H$, because the span of $X_I, I = 0, \ldots, D - 3$ is tangent to $H$ and hence a null space, with the signature of $G$ consequently being $(0 + + \cdots +)$ there.

Taking the trace of the first reduced Einstein equation (53), one finds that $r$ is a harmonic function on the interior of $\hat{M}$,

$$\hat{D}^\alpha \hat{D}_\alpha r = 0. \quad (56)$$

Since $\hat{M}$ is an (orientable) simply connected 2-dimensional analytic manifold with connected boundary and corners by thm. [1], we may map it analytically to the upper complex half plane $\{ \zeta \in \mathbb{C} \mid \text{Im} \zeta > 0 \}$ by the Riemann mapping theorem. Furthermore, since $r$ is harmonic, we can introduce a harmonic scalar field $z$ conjugate to $r$

$$\hat{D}^\alpha z = \epsilon^{\alpha\beta} \hat{D}_\beta r, \quad (57)$$
where $\hat{\epsilon}_{\alpha\beta}$ is the anti-symmetric tensor on $\hat{M}$ satisfying $\hat{\epsilon}^{\alpha\beta}\hat{\epsilon}_{\alpha\beta} = 2$. Thus both $r, z$ are harmonic functions on $(\hat{M}, \hat{g})$, and $r = 0$ on $\partial\hat{M}$. Combining this with the fact $\hat{M}$ is homeomorphic to a half-plane, one can argue (see e.g. [7, 6.3] or [49]) that $r$ and $z$ are globally defined coordinates, and identify $\hat{M}$ with $\{z + ir \in \mathbb{C} \mid r > 0\}$. In these coordinates, the metric $\hat{g}$ globally takes the form
\[
\hat{g} = e^{2\nu(r,z)}(dr^2 + dz^2).
\] (58)

Since eq. (53) is invariant under conformal rescalings of $\hat{g}_{\alpha\beta}$, and since a 2-dimensional metric is conformally flat, it decouples from eq. (54). In fact, writing the Ricci tensor $\hat{R}_{\alpha\beta}$ of (58) in terms of $\nu$, one sees that eq. (54) equation may be used to determine $\nu$ by a simple integration, see e.g. [21] for details.

The boundary $r = 0$ of $\hat{M}$ consists of several segments according to our classification theorem 4. In the description of $\hat{M}$ as the upper complex half plane $\hat{M} = \{z + ir \in \mathbb{C} \mid r > 0\}$, these are represented by a collection of intervals $\{I_j\}$ of the $z$-axis. The length of the $j$-th interval as measured by the coordinate $z$ is called $l(I_j)$. Because the coordinates $(r, z)$ were canonically defined, the numbers $l(I_j) \geq 0$ are invariantly defined, i.e. are the same for isometric spacetimes. Each segment is either an axis for which there is a vector $\hat{a}(I_j) \in \mathbb{Z}^{D-3}$ such that $\sum_i a_i(I_j)\psi_i = 0$, or it corresponds to the horizon. In that case, we put the corresponding vector to zero, $\hat{a}_H = 0$, because no non-trivial linear combination of the axial Killing fields vanishes in the interior of the corresponding interval $I_H$, see thm. 4. Concerning the length $l_H$ of the horizon segment, we have the following lemma.

**Lemma 7.** The length of the horizon interval satisfies
\[
(2\pi)^{D-3}l_H = \kappa A_H,
\] (59)
where $A_H$ is the area of the horizon cross section $\mathcal{H}$, and where $\kappa > 0$ is the surface gravity.

The proof of lemma 7 is given in appendix A.

We call the collection of real positive numbers $\{l(I_j)\}$ and integer vectors $\{\hat{a}(I_j)\}$ associated with the intervals the “interval structure” of the spacetime. As we explained in the previous section, the collection $\{\hat{a}(I_j)\}$ determines the manifold structure of $\langle \hat{M} \rangle$ and the action of $\mathcal{G}$ on this space up to diffeomorphism. In particular, the vector fields $X_I$ are determined up to diffeomorphism. Furthermore, if we are given $G_{IJ}$ and $\hat{g}$ (i.e., $\nu$) as functions of $r, z$, then we can reconstruct the metric $g$ of the spacetime in the domain of outer communication. In a local coordinate system consisting of $r, z$ and $\xi^I, I = 0, \ldots, D - 3$, such that the Killing fields are given by $X_I = \partial/\partial\xi^I$, the metric locally takes the form
\[
g = e^{2\nu(r,z)}(dr^2 + dz^2) + G_{IJ}(r, z)\, d\xi^I d\xi^J.
\] (60)

For $M = \mathbb{R}^{4,1} \times \mathbb{T}^{D-5}$, the axial symmetries are the rotations in the 12-plane of $\mathbb{R}^{4,1}$ generated by the Killing field $\psi_1$, the rotations in the 34-plane of $\mathbb{R}^{4,1}$ generated by the Killing field $\psi_2$ and the rotations of the $D - 5$ compact extra dimensions generated
by Killing fields $\psi_3, \ldots, \psi_{D-3}$. The coordinates $r, z$ as constructed above are given by $r = R_1 R_2$ and $z = \frac{1}{2}(R_1^2 - R_2^2)$, with $R_1 = \sqrt{x_1^2 + x_3^2}$ and $R_2 = \sqrt{x_3^2 + x_4^2}$, and with $x_i$ the standard spatial Cartesian coordinates of $\mathbb{R}^{4,1}$. The conformal factor is given by $e^{2\nu} = 1/2 \sqrt{r^2 + z^2}$, and the Gram matrix is given by

$$G = \begin{pmatrix} -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \rho(1 - \cos \theta) & 0 & 0 & \cdots & 0 \\ 0 & 0 & \rho(1 + \cos \theta) & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \quad (61)$$

Here, we have introduced the coordinates $\rho, \theta$ which are related to $r, z$ by

$$r = \rho \sin \theta, \quad z = \rho \cos \theta, \quad \rho = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_4^2), \quad \theta = \arctan \frac{2\sqrt{(x_1^2 + x_2^2)(x_3^2 + x_4^2)}}{x_1^2 - x_3^2 + x_2^2 - x_4^2} \quad (63)$$

in terms of the spatial cartesian coordinates $x_i$ of $\mathbb{R}^{4,1}$. For a general $D$-dimensional asymptotically Kaluza-Klein spacetime with asymptotically flat 5-dimensional part, we can determine the asymptotic form of the metric as follows: First, one establishes, using standard results on elliptic equations, that $G$ has a poly-homogeneous asymptotic expansion in powers of $1/\rho$ for large $\rho$ of the form

$$G(\rho, \theta) \sim \sum_{n \geq 0} \rho^{-n} G_n(\rho, \theta), \quad (64)$$

where $G_0$ is the diagonal Gram matrix for $\mathbb{R}^{4,1} \times \mathbb{T}^{D-5}$ given above, and where the other terms $G_n$ represent corrections. The entries of the correction matrices are of the same order in $\rho$ as those of $G_0$, up to possibly additional powers of $\log \rho$. We then insert this ansatz into the first reduced Einstein equation eq. (53). Since the leading part $G_0$ is a solution to the equation, we get from this an equation for the correction matrix elements. At the lowest non-trivial order in $1/\rho$ this equation delivers a decoupled system of second order ordinary differential equations in $\theta$ for the entries of the correction matrix $G_1$. These equations have a unique solution satisfying the boundary conditions arising from the fact that the 11, 12, 13, $\ldots$-components of $G$ must vanish for $\theta = 0$ and sufficiently large $\rho$ (as this represents an axis for $\psi_1$), while the 21, 22, 23, $\ldots$-components must vanish for $\theta = \pi$ and sufficiently large $\rho$ (as this represents an axis for $\psi_2$). Hence, these components must similarly vanish also for $G_1$. We do not give the details of the straightforward but somewhat lengthy calculation but only quote the solution, written in a block-matrix form:

$$G_1 = \begin{pmatrix} 2M & b_1(1 - \cos \theta) & b_2(1 + \cos \theta) & b_i \\ b_1(1 - \cos \theta) & (M - A + \eta)(1 - \cos \theta) & \zeta \sin^2 \theta & (c_i)(1 - \cos \theta) \\ b_2(1 + \cos \theta) & \zeta \sin^2 \theta & (M - A - \eta)(1 + \cos \theta) & (d_i)(1 + \cos \theta) \\ (b_i) & (c_i)(1 - \cos \theta) & (d_i)(1 + \cos \theta) & (h_{ij}) \end{pmatrix}. \quad (63)$$
Here, the quantities $M, A, \zeta, \eta, h_{ij}, b_i, c_i, d_i$ are undetermined real constants and $i, j$ range through $3, \ldots, D - 3$ in this block-matrix. Because we must have $-\det G = r^2$, they are subject to the constraint

$$2A = \sum_{i=3}^{D-3} h_{ii}.$$ 

According to eq. (57), we are still free to change the coordinate $z$ by adding a constant. This will result in adding a constant to $\eta$, and we may thus fix the remaining ambiguity in $z$ by setting $\eta = 0$. We will do this in the following. The asymptotic form of the conformal factor $e^{2\nu}$ can similarly be determined by the second reduced Einstein equation, eq. (54), together with the asymptotic form of the Gram matrix eq. (64). Again, we omit the straightforward but somewhat lengthy calculation and give only the result, which is

$$e^{2\nu} = \frac{1}{2\rho} + \frac{M - A}{4\rho^2} + \ldots,$$

where the dots represent terms that go to zero faster as $\rho \to \infty$. Thus, in a coordinate system $(\tau, \rho, \theta, \varphi_1, \ldots, \varphi_{D-3})$ such that

$$t = \partial/\partial \tau, \quad \psi_i = \partial/\partial \varphi_i, \quad i = 1, \ldots, D - 3,$$

we obtain the following asymptotic form of the metric eq. (60) for large $\rho$:

**Asymptotic form of the metric** for stationary black hole spacetime with $D - 3$ axial Killing fields, behaving as $\mathbb{R}^{4,1} \times T^{D-5}$ near infinity:

$$g = -\left(1 - \frac{2M}{\rho}\right) d\tau^2 + \frac{1}{2\rho} \left(1 + \frac{M - A}{2\rho}\right) (d\rho^2 + \rho^2 d\theta^2)$$

$$+ \rho(1 - \cos \theta) \left(1 + \frac{M - A}{\rho}\right) d\varphi_1^2 + \rho(1 + \cos \theta) \left(1 + \frac{M - A}{\rho}\right) d\varphi_2^2$$

$$+ \sum_{i,j=3}^{D-3} \left(\delta_{ij} + \frac{h_{ij}}{\rho}\right) d\varphi_i d\varphi_j + \frac{2\zeta \sin^2 \theta}{\rho} d\varphi_1 d\varphi_2$$

$$+ \frac{2b_1(1 - \cos \theta)}{\rho} d\varphi_1 d\tau + \frac{2b_2(1 + \cos \theta)}{\rho} d\varphi_2 d\tau + \frac{2}{\rho} \sum_{i=3}^{D-3} b_i d\varphi_i d\tau$$

$$+ \frac{2}{\rho} \sum_{i=3}^{D-3} d_i d\varphi_2 d\varphi_i + \frac{2(1 + \cos \theta)}{\rho} \sum_{i=3}^{D-3} c_i d\varphi_1 d\varphi_i + \ldots,$$ 

where the dots represent terms that are higher order in $1/\rho$. The constants $b_i$ are proportional to the angular momenta of the solutions, both in the asymptotically small and large dimensions. These can be defined e.g. by the Komar expressions

$$J_i = \int_{S^{1} \times T^{D-5}} * d\psi_i,$$
where the integration is over a surface at infinity, and where \(d\psi_i\) denote the 2-forms obtained by taking the exterior differential of \(\psi_i\) after lowering the index. The constant \(M\) is related to the ADM-mass of the solution, see e.g. [31, sec. 3].

For \(M = \mathbb{R}^{3,1} \times T^{D-4}\), the axial symmetries may be taken as the rotations in the 12-plane of \(\mathbb{R}^{3,1}\) and rotations of the \(D - 4\) compact extra dimensions. The functions \(r, z\) are then given by \(r = \sqrt{x_1^2 + x_2^2}\) and \(z = x_3\), with \(x_i\) the standard spatial Cartesian coordinates on \(\mathbb{R}^{3,1}\). The conformal factor is just \(e^{2\nu} = 1\). For a general \(D\) dimensional asymptotically Kaluza-Klein spacetime with asymptotically flat 4-dimensional part, we may again derive an expression for the asymptotic form of the metric as above. The same can also be done for \(M = \mathbb{R}^{2,1} \times T^{D-3}\) and \(M = \mathbb{R}^{1,1} \times T^{D-2}\). Since the analysis is quite similar, we do not give the results here.

5 Uniqueness theorem for stationary black holes with \((D - 3)\) axial symmetries

In the previous two sections, we have analyzed stationary black hole spacetimes that are asymptotically \(\mathbb{R}^{s,1} \times T^{D-s-1}\) where \(s = 1, 2, 3\) or 4, and which have an isometry group \(G = \mathbb{R} \times T^{D-3}\) (with no points in the domain of outer communication whose isotropy group is discrete). We have derived a number of “invariants” associated with such solutions:

- We showed that the orbit space of the domain of outer communication by \(G\) is a half plane \(\hat{M} = \{z + ir \mid r > 0\}\). The boundary of the half-plane is divided into a finite collection of intervals \(\{I_j\}\). With each interval, there is associated its length \(l(I_j) \in \mathbb{R}_{>0}\), and a vector \(a(I_j) \in \mathbb{Z}^{D-3}\) subject to the normalization (31). One of the intervals corresponds to the orbit space \(\hat{H}\) of the horizon and is associated with the zero vector, while the others correspond to an “axis” in spacetime, i.e. points where the linear combination \(\sum_i a_i(I_j) \psi_i = 0\) vanishes. For adjacent intervals \(I_j\) and \(I_{j+1}\) (not including the horizon), there is a compatibility condition stating that the collection of minors \(Q_{kl} \in \mathbb{Z}, 1 \leq k < l \leq D - 3\) given by

\[
Q_{kl} = \left| \text{det} \begin{pmatrix} a_k(I_{j+1}) & a_k(I_j) \\ a_l(I_{j+1}) & a_l(I_j) \end{pmatrix} \right|
\]  

have greatest common divisor g.c.d.\(\{Q_{kl}\} = 1\), see the discussion around (32). The data \(\{l(I_j)\}\) together with \(\{a(I_j)\}\) were called the “interval structure”.

- Because the spacetime is asymptotically Kaluza-Klein, we can define its mass, and the angular momenta \(\{J_i\}\) corresponding to the axial Killing fields, \(i = 1, \ldots, D - 3\). Some of the angular momenta correspond to the large, and some to the small (extra) dimensions.

\(^9\text{For a half infinite interval, this would be } \infty.\)
The asymptotic form of the metric (67) contains additional real parameters \( \{h_{ij}\}, \{c_i\}, \{d_i\}, \zeta \) which are related to the asymptotic metric on the tori generated by the axial Killing fields \( \psi_i, \) \( i = 1, \ldots, D - 3 \) in the region of spacetime near infinity. These numbers are invariantly defined.

The collection of angular velocities \( \{\Omega_i\} \), the surface gravity \( \kappa \), and horizon area.

It is natural to ask the following questions: Is the spacetime \((M, g)\) under consideration uniquely determined by the above data? To what extent can the data be specified independently? The following theorem provides an answer to the first question and a partial answer to the second question.

**Theorem 5.** There can be at most one stationary, asymptotically Kaluza-Klein spacetime \((M, g)\) with \( D - 3 \) axial Killing fields, satisfying the technical assumptions stated in sec. 2, for a given interval structure \( \{a(I_j), l(I_j)\} \) and a given set of angular momenta \( \{J_i\}, \) \( i = 1, \ldots, D - 3 \).

This uniqueness theorem is the main result of this paper. A consequence of the theorem is that the interval structure and angular momenta uniquely determine the other invariants mentioned above, such as e.g. the mass of the spacetime. In \( D = 4 \) with no extra dimensions, the only non-trivial interval structure for a single black hole spacetime is given by the intervals \((-\infty, -z_0), [-z_0, z_0], [z_0, \infty)\). The middle interval corresponds to the horizon, while the half-infinite ones to the axis of the rotational Killing field. The interval vectors \( a(I_j) \) are 1-dimensional integer vectors in this case and hence trivial. For each \( z_0 > 0 \) and for each angular momentum \( J \), there exist a solution given by the appropriate member of the Kerr-family of metrics. Thus, the Kerr metrics exhaust all possible stationary, axially symmetric single black hole spacetimes (satisfying the technical assumptions stated in sec. 2). This is of course just the classical uniqueness theorem for the Kerr-solution [1, 2, 37, 45, 24], see [7] for a rigorous account. The mass \( m \) of the non-extremal Kerr solution characterized by \( z_0, J \) is related to these parameters by \( z_0 = \sqrt{m^2 - J^2/m^2} > 0 \). Hence the uniqueness theorem may be stated equivalently in terms of \( m \) and \( J \), which is more commonly done. Note that the length of the horizon interval, \( l_H = 2z_0 \) tends to zero in the extremal limit, in accordance with lemma [7].

In higher dimensions, one may similarly derive relations between the interval structure and angular momenta on the one side, and the other invariants on the other side for any given solution. Such formulae are provided for the Myers-Perry or black-ring solutions e.g. in [21]. Of course, for most interval structures it is not known whether there actually exists a solution, so in this sense much less is known in higher dimensions than in \( D = 4 \).

**Proof of thm.** For definiteness, we give a proof here for spacetimes asymptotic to \( \mathbb{R}^{4,1} \times T^{D-5} \), the other cases are similar. We will show that the the domains of outer communication of any two spacetimes as in the theorem must be isometric. It then follows from the argument given in [16] based on the characteristic initial value formulation of the Einstein equations that the metrics of the interior of the two black holes must also coincide. (The last step can be avoided if one assumes that the spacetime metric is analytic.)
The key step is to define from the reduced Einstein equations (53) a set of equations which describe the difference between two solutions as described in the theorem. This formulation is due to [37, 34], see also [36], and it involves certain potentials which we define first. We first consider the twist 1-forms
\[ \omega_i = * (\psi_1 \wedge \cdots \wedge \psi_{D-3} \wedge d\psi_i) \quad i = 1, \ldots, D-3, \quad (70) \]
where the Killing fields have been identified with 1-forms via the metric. Using the vacuum field equations and standard identities for Killing fields [48], one shows that these 1-forms are closed, \( d\omega_i = 0 \). Since the Killing fields commute, the twist forms are invariant under \( G \), and so we may define corresponding 1-forms \( \hat{\omega}_i \) on the interior of the factor space \( \hat{M} = \{ z + ir \in \mathbb{C} \mid r > 0 \} \). These 1-forms are again closed. Thus, the “twist potentials”
\[ \chi_i = \int_0^{\hat{x}} \hat{\omega}_i \quad (71) \]
are globally defined on \( \hat{M} \) and independent of the path connecting 0 and the point \( \hat{x} \in \hat{M} \), and \( d\chi_i = \hat{\omega}_i \). The twist potentials and the Gram matrix of the axial Killing fields \( f_{ij} = g(\psi_i, \psi_j) \), satisfy a system of coupled differential equations on \( \hat{M} \) which follow from the reduced Einstein equation (53). They are
\[ 0 = \hat{D}^\alpha \left( r (\det f)^{-1} \hat{D}_\alpha \chi_i + r \hat{D}_\alpha \log \det f \right) \quad (72) \]
\[ 0 = \hat{D}^\alpha \left( r (\det f)^{-1} f^{ij} \hat{D}_\alpha \chi_j \right) \quad (73) \]
\[ 0 = \hat{D}^\alpha \left( r f^{jk} \hat{D}_\alpha f_{ki} + r (\det f)^{-1} f^{jk} \chi_i \hat{D}_\alpha \chi_k \right) \quad (74) \]
\[ 0 = \hat{D}^\alpha \left( - r \hat{D}_\alpha \chi_i + r \chi_i \hat{D}_\alpha \log det f + r (f^{jk} \hat{D}_\alpha f_{ij}) \chi_k + r (\det f)^{-1} \chi_j (\hat{D}_\alpha \chi_j) \chi_i \right). \quad (75) \]
Here we are using the summation convention and \( f^{ij} \) denotes the components of the inverse of the matrix \( f_{ij} \), which is used to raise indices on \( \chi_i \). To verify these equations, it is necessary to use the relations
\[ \hat{D}_\alpha \alpha^i = r (\det f)^{-1} \hat{D}_\alpha \beta^i f^{ij} \hat{D}_\beta \chi_j, \quad (76) \]
as well as
\[ \beta = f^{ij} \alpha_i \alpha_j - (\det f)^{-1} r^2 \quad (77) \]
for the scalar products \( \alpha_i = g(t, \psi_i) \) and \( \beta = g(t, t) \). Again, \( \alpha^i \) means \( f^{ij} \alpha_j \). The above equations can be written in a compact matrix form. For this, one introduces the \((D-2) \times (D-2)\) matrix field \( \Phi \) which is written in an obvious block-matrix notation as
\[ \Phi = \begin{pmatrix} (\det f)^{-1} & - (\det f)^{-1} \chi^T \\ -(\det f)^{-1} \chi & f + (\det f)^{-1} \chi \otimes \chi^T \end{pmatrix}, \quad (78) \]
with \( \chi^T = (\chi_1, \ldots, \chi_{D-3}) \). The matrix \( \Phi \) satisfies \( \Phi^T = \Phi \), \( \det \Phi = 1 \), and is positive semi-definite, being the sum of two positive semi-definite matrices. Hence it may be
written in the form $\Phi = S^T S$ for some matrix $S$ of determinant 1. The equations (72) can be stated in terms of $\Phi$ as

$$\hat{D}^\alpha (r \Phi^{-1} \hat{D}_\alpha \Phi) = 0.$$  \tag{79}$$

Consider now two black hole solutions $(M, g)$ and $(\tilde{M}, \tilde{g})$ as in the statement of the theorem. We denote the corresponding matrices defined as above by $\Phi$ and $\tilde{\Phi}$, and we use the same “tilde” notation to distinguish any other quantities associated with the two solutions. $\langle \langle M \rangle \rangle$ as a manifold with a $G$-action is uniquely determined by the interval structure modulo diffeomorphisms preserving the action of $G$ and similarly for the tilde spacetime. Therefore, since the interval structures are assumed to be the same for both spacetimes, $\langle \langle M \rangle \rangle$ and $\langle \langle \tilde{M} \rangle \rangle$ are isomorphic as manifolds with a $G$ action, and we may hence assume that $\tilde{r} = r$ and $\tilde{z} = z$. Consequently, it is possible to combine the divergence identities (79) for the two solutions into a single identity on the upper complex half plane, called “Mazur identity”. It is given by

$$\hat{D}_\alpha (r \hat{D}^\alpha \sigma) = r \tilde{g}^{\alpha \beta} \text{Tr} \left( \tilde{N}_\alpha^T \tilde{N}_\beta \right),$$ \tag{80}$$

and it can be proven in almost exactly the same way as the identity given in [37]. Here, we have written

$$\sigma = \text{Tr}(\tilde{\Phi} \Phi^{-1} - I), \quad \tilde{N}_\alpha = \tilde{S}^{-1}(\tilde{\Phi}^{-1} \hat{D}_\alpha \tilde{\Phi} - \tilde{\Phi}^{-1} \hat{D}_\alpha \Phi) S,$$ \tag{81}$$

where in turn $S$ and $\tilde{S}$ are matrices such that $\Phi = S^T S$ and $\tilde{\Phi} = \tilde{S}^T \tilde{S}$ hold. The key point about the Mazur identity (80) is that on the left side we have a total divergence, while the term on the right hand side is non-negative. This structure can be exploited in various ways. In this paper, we follow a strategy invented by Weinstein [49, 50], which differs from that originally devised by Mazur.

The basic idea is to view $r, z$ as cylindrical coordinates in an auxiliary space $\mathbb{R}^3$ consisting of the points $x = (r \cos \gamma, r \sin \gamma, z)$, and to view $\sigma$ as a rotationally symmetric function defined on this $\mathbb{R}^3$, minus the $z$-axis. The Mazur identity then gives

$$\Delta \sigma \geq 0 \quad \text{on} \quad \mathbb{R}^3 \setminus \{\text{z-axis}\},$$ \tag{82}$$

where $\Delta$ is the ordinary Laplacian on $\mathbb{R}^3$. As we will show, $\sigma$ is globally bounded on $\mathbb{R}^3$, including at infinity and the $z$-axis. Furthermore, we claim that $\sigma \geq 0$ at any point away from the axis: Writing $F = \tilde{S} S^{-1}$, we have $\sigma = \text{Tr}(F^T F) - (D - 2)$. Now, $F^T F \geq 0$, and $\det F^T F = \det \tilde{\Phi} \det \Phi^{-1} = 1$, so we may bring $F^T F$ into the form $\text{diag}(e^{u_1}, \ldots, e^{u_{D-3}}, e^{-u_1 - \cdots - u_{D-3}})$ by a similarity transformation. Thus, $\sigma$ will be non-negative if and only if

$$\frac{1}{D - 2} (e^{u_1} + \cdots + e^{u_{D-3}} + e^{-u_1 - \cdots - u_{D-3}}) \geq 1,$$ \tag{83}$$

which in turn follows directly because the exponential function is convex. Thus, we are in a position to apply the maximum principle arguments in [50], which imply that
\( \sigma = 0 \) everywhere. As we now see, this implies that the metrics \( g \) and \( \tilde{g} \) are isometric on the domain of outer communication, thus proving the theorem.

First, \( \sigma = 0 \) implies that \( u_1 = \cdots = u_{D-3} = 0 \), and hence that \( \Phi = \tilde{\Phi} \) everywhere in \( \hat{M} \). Therefore, the twist potentials and the Gram matrices of the axial Killing fields are identical for the two solutions, \( \tilde{f}_{ij} = f_{ij} \) and \( \tilde{\chi}_i = \chi_i \). To see that the other scalar products between the Killing fields coincide for the two solutions, let \( \alpha_i = g(t, \psi_i), \beta = g(t, t) \) as above, and define similarly the scalar products \( \tilde{\alpha}_i, \beta \) for the other spacetime. The right side of eq. (76) does not depend upon the conformal factor \( \nu \), so since \( \tilde{\chi}_i = \chi_i \) and \( \tilde{f}_{ij} = f_{ij} \), it also follows that \( \tilde{\alpha}_i = \alpha_i \) up to a constant. That constant has to vanish, since it vanishes at infinity. Furthermore, from eq. (77) we have \( \tilde{\beta} = \beta \). Thus, all scalar products of the Killing fields are equal for the two solutions, \( \tilde{G}_{IJ} = G_{IJ} \) on the entire upper half plane. Viewing now the second reduced Einstein equation (54) as an equation for \( \nu \) respectively \( \tilde{\nu} \), and bearing in mind that \( \nu = \tilde{\nu} \) at infinity, one concludes that \( \tilde{\nu} = \nu \). Thus, summarizing, we have shown that if the boundary integral in the integrated Mazur identity eq. (80) could be shown to vanish, then \( \tilde{G}_{IJ} = G_{IJ} \), \( \tilde{r} = r \), \( \tilde{z} = z \) and \( \tilde{\nu} = \nu \). Since \( \tilde{t} = t, \psi_i = \psi_i \) it follows from eqs. (60) and (58) that \( \tilde{g} = g \) in the domain of outer communication.

It remains to be shown that \( \sigma \) is globally bounded, including at the \( z \)-axis (corresponding to \( \partial \hat{M} \)) and at infinity. It is here that the assumptions of the theorem about the interval structures and angular momenta are needed. We must consider the following separate cases: (1) The parts of \( \partial \hat{M} \) corresponding to a rotation axis of the Killing fields, (2) the part corresponding to the horizon, and (3) infinity.

1. On each segment \( z \in I_j = (z_j, z_{j+1}), r = 0 \) of the boundary \( \partial \hat{M} \) representing an axis, we know that the null spaces of the Gram matrices \( f_{ij} \) and \( \tilde{f}_{ij} \) coincide, because we are assuming that the interval structures of both solutions are identical. Furthermore, from eq. (77), and from the fact that \( \tilde{\omega}_i \) vanishes on any axis by definition, the twist potentials \( \chi_i \) are constant on the \( z \)-axis outside of the segment \( (z_h, z_{h+1}) \) representing the horizon. The difference between the constant value of \( \chi_i \) on the \( z \)-axis left and right to the horizon segment can be calculated as follows:

\[
\chi_i(r = 0, z_h) - \chi_i(r = 0, z_{h+1}) = \int_{z_h}^{z_{h+1}} \tilde{\omega}_i = \frac{1}{(2\pi)^{D-3}} \int_{\mathcal{H}} \ast (d\psi_i) = \frac{1}{(2\pi)^{D-3}} \int_{\mathbb{S}^3 \times T^{D-5}} \ast (d\psi_i) = \frac{1}{(2\pi)^{D-3}} J_i .
\]

The first equality follows from the definition of the twist potentials, the second from the defining formula for the twist 1-forms and the fact that these are invariant under the action of the \( D - 3 \) independent rotation isometries each with period \( 2\pi \) (with \( \mathcal{H} \) a horizon cross section), the third equation follows from Gauss’ theorem and the fact that \( d(\ast d\psi_i) = 0 \) because \( \psi_i \) is a Killing vector on a Ricci-flat manifold, and the last equality follows from the Komar expression for the angular momentum. The analogous expressions hold in the spacetime \( (\bar{M}, \tilde{g}) \).
Because we assume that \( J_i = \tilde{J}_i \), we can add constants to \( \chi_i \), if necessary, so that \( \chi_i = \tilde{\chi}_i \) on the axis, and in fact that \( \chi_i - \tilde{\chi}_i = O(r^2) \) near any axis. One may now analyze the behavior of \( \sigma \) near our boundary segment \( I_j = (z_j, z_{j+1}), r = 0 \), given by

\[
\sigma = -1 + \frac{\det f}{\det \hat{f}} + \frac{f^{ij}(\chi_i - \tilde{\chi}_i)(\chi_j - \tilde{\chi}_j)}{\det f} + f^{ij}(\tilde{f}_{ij} - f_{ij}).
\]  

(84)

Let \( a(I_j) \in \mathbb{Z}^{D-3} \) be the vector generating the kernel of the matrix \( f \) in on our interval \( I_j \). By lemma [2] we can find a matrix \( B \in SL(D - 3, \mathbb{Z}) \) such that \( a(I_j)B^T = (1, 0, \ldots, 0) \). Thus, redefining the axial Killing fields to \( \hat{\psi}_i = \sum_j A_{ij}\psi_j \) and \( A = B^{-1} \) if necessary, we can assume without loss of generality that \( a(I_j) = (1, 0, \ldots, 0) \). By arguments parallel to those in the proof of lemma [7], the matrix \( f \) then takes the following form near \( I_j \):

\[
\hat{f} \sim \begin{pmatrix}
    r^2e^{2\nu} & O(r^2) & \ldots & O(r^2) \\
    O(r^2) & \hat{f}_{22} & \ldots & \hat{f}_{2(D-3)} \\
    \vdots & \vdots & \ddots & \vdots \\
    O(r^2) & \hat{f}_{(D-3)2} & \ldots & \hat{f}_{(D-3)(D-3)}
\end{pmatrix},
\]

(85)

and similarly for the second solution. It follows from this expression and eq. (84) that \( \sigma \) is finite near the interval \( I_j \).

(2) On the horizon segment, the matrices \( f_{ij}, \tilde{f}_{ij} \) are invertible, so \( \sigma \) is finite there.

(3) Near infinity, the matrices \( f_{ij} \) and \( \tilde{f}_{ij} \) both tend to the same limiting matrix, as is evident from our discussion of the asymptotic form of the metric in sec. [4]. Thus, \( f_{ij} - \tilde{f}_{ij} = O(1/\rho) \), where \( \rho = \sqrt{r^2 + z^2} = |x| \). We claim that the twist potentials \( \chi_i \) and \( \tilde{\chi}_i \) also approach the same value near infinity. First, we already know that they are equal on the axis \( r = 0 \) and away from the horizon interval. We now take a path in eq. (71) that stays on the axis \( r = 0 \) until a very large \( |z| \), and then follows a half circle \( \sqrt{r^2 + z^2} = |x| = \text{const.} \) in the asymptotic region. In this region, we can use the asymptotic form of the metric, eq. (67) derived above. In the coordinates \( \rho = |x| \) and \( \tan \theta = r/z \) as in eq. (67), the \( d\theta \)-component of the twist 1-forms are given to leading order in \( 1/\rho \) by

\[
\hat{\omega}_i = \frac{1}{2} \frac{1}{(2\pi)^{3-D}} J_i \begin{cases}
(1 - (-1)^i \cos \theta) \sin \theta d\theta + \ldots & \text{for } i = 1, 2, \\
\sin \theta d\theta + \ldots & \text{for } i = 3, \ldots, D - 3,
\end{cases}
\]

(86)

where dots stand for terms of higher order in \( 1/\rho \), or terms proportional to \( d\rho \), which do not contribute in a line integral as in (71) along a large circle of constant \( \rho \). The asymptotic behavior of the twist potentials is hence

\[
\chi_i = \frac{1}{2} \frac{1}{(2\pi)^{3-D}} J_i \begin{cases}
\cos \theta - (-1)^i \frac{1}{2} \cos^2 \theta - 1 + (-1)^{i+1} \frac{1}{2} + \ldots & \text{for } i = 1, 2, \\
\cos \theta - 1 + \ldots & \text{for } i = 3, \ldots, D - 3,
\end{cases}
\]

(87)
where the dots stand for terms of order $O(1/\rho)$. The same formula holds for the tilde quantities. Therefore, because $J_i = \tilde{J}_i$, it follows that $\chi_i - \tilde{\chi}_i = O(1/\rho)$. Thus, $\sigma$ tends to a zero for $|x| \to \infty$.

Thus we have shown that $\sigma$ remains bounded, including the axis, horizon segment, and tends to zero near infinity. As we have explained, this concludes the proof of the theorem. \qed

6 Conclusions and outlook

In this paper, we have proved a uniqueness theorem for $D$-dimensional stationary, asymptotically Kaluza-Klein black hole spacetimes satisfying the vacuum Einstein equations, allowing a group of isometries $G = \mathbb{R} \times T^{D-3}$. We showed that the solutions are uniquely determined by certain combinatorial data specifying the group action, certain moduli, and the angular momenta. This combinatorial data in particular determines the topology of the spacetime outside the black hole, and the topology of the horizon.

To be able to prove our uniqueness theorem, we also had to make a number technical assumptions. They mainly concern the analyticity of the metric and the causal structure of the spacetime. One feels that it ought to be possible to remove these assumptions, but it is not clear to us how this could be done in practice.

The more unsatisfactory aspect of our analysis is that we have not been able to prove or disprove the existence of smooth black hole solutions associated with more elaborate topological structure/combinatorial data, such as “black lenses” etc. Some partial results have been obtained in the literature on this (see e.g. [4]), but the general situation is still unclear.

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A Proof of lemma 7

Lemma 6: The length of the horizon interval satisfies

$$(2\pi)^{D-3}l_H = \kappa A_H,$$  \hspace{1cm} (88)

where $A_H$ is the area of the horizon cross section $\mathcal{H}$, and where $\kappa > 0$ is the surface gravity.

Proof: We take the horizon to correspond to the interval $z \in (z_1, z_2), r = 0$ on the boundary of the orbit space $\hat{M}$. Let $v = (1, \Omega_1, \ldots, \Omega_{D-3})$. Then by definition $G_{IJ}v^Iv^J = g(K, K)$, where $K$ is the Killing vector $[3]$, which is tangent to the null generators of the horizon $H$, so $G_{IJ}v^Iv^J = 0$ on $H$. It the follows e.g. from the min-max principle that $G_{IJ}v^I = 0$ on the horizon, so $\lim_{r \to 0} G_{IJ}v^J = 0$ in the orbit space.
for $z \in (z_1, z_2)$. As was shown in [21, sec. 3], one can furthermore use the first reduced Einstein equation (53) to show that $\lim_{r \to 0} G_{IJ} v^I/r = 0$ for $z \in (z_1, z_2)$. Let us now choose coordinates $(u, r, \varphi_1, \ldots, \varphi_{D-3})$ near $H$ such that $K = \partial/\partial u, \psi_i = \partial/\partial \varphi_i$. Let us define $X_i$ as $X_1$ above in eq. (52), with $t$ replaced by $K$, and let $\hat{G}_{IJ} = g(X_I, X_J)$. Then the reduced Einstein equations also hold for $\hat{G}$, and furthermore, near $r = 0$ and $z \in (z_1, z_2)$, we have

$$\hat{G} \sim \begin{pmatrix} -r^2 \det f^{-1} & O(r^2) & \ldots & O(r^2) \\ O(r^2) & f_{11} & \ldots & f_{1(D-3)} \\ \vdots & \vdots & \ddots & \vdots \\ O(r^2) & f_{(D-3)1} & \ldots & f_{(D-3)(D-3)} \end{pmatrix},$$

up to terms of higher order in $r$. Here, $z \in (z_1, z_2)$, and $f_{ij}(z)$ is the limit as $r \to 0$ of $g(\psi_i, \psi_j)$. Following [21, sec. 3], the second reduced Einstein equation (54) furthermore gives

$$\partial_u \nu \to 0, \quad \partial_z \nu \to -\frac{1}{2} \partial_z \log \det f, \quad \text{as} \quad r \to 0, z \in (z_1, z_2).$$

We conclude from the last relation that $e^{-2\nu} \to c^2 \det f$ for some constant $c > 0$ as $r \to 0, z \in (z_1, z_2)$. From the form of the metric given in eq. (60) (with $G$ replaced by $\hat{G}$), it follows that, near $H$, we have

$$g \sim e^{2\nu}(dz^2 + dr^2 - c^2 r^2 du^2) + \sum_{i,j=1}^{D-3} f_{ij}(z) d\varphi_i d\varphi_j + 2r^2 \sum_{i=1}^{D-3} O(1) dud\varphi_i$$

$$= e^{2\nu}(dz^2 + dUdV) + \sum_{i,j=1}^{D-3} f_{ij}(z) d\varphi_i d\varphi_j + \frac{1}{c} \sum_{i=1}^{D-3} O(1) (VdU - UdV) d\varphi_i.$$  

The minus sign in front of the $du^2$-term follows from the fact that $K$ is timelike in a neighborhood outside $H$, which in turn follows directly from $\nabla_a (K^a K_b) = \kappa K_b$. In the last line we switched to Kruskal-like coordinates $U, V$ defined by $UV = r^2, U/V = e^{2cu}$. It is apparent in these coordinates that $H$ corresponds to $V = 0$. The restriction of $K = \partial/\partial u$ to $H$ is found to be $cU \partial/\partial U$, from which one concludes in view of the equation $K^a \nabla_a K^b = \kappa K^b$ on $H$ that $c = \kappa$. The lemma may now be proven by calculating the horizon area in the coordinates $z, \varphi_i$ using the above form of the metric. It is

$$A_H = \int_{z_1}^{z_2} dz \left( \prod_i \int_0^{2\pi} d\varphi_i \right) \sqrt{e^{2\nu} \det f} = \frac{1}{\kappa} (2\pi)^{D-3} (z_2 - z_1),$$

from which the lemma follows immediately in view of $l_H = z_2 - z_1$. \hfill \Box

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