DISTRIBUTED BURNING IN TYPE Ia SUPERNOVAE: A STATISTICAL APPROACH

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ABSTRACT

We present a statistical model which shows the influence of turbulence on a thermonuclear flame propagating in C+O white dwarf matter. Based on a Monte Carlo description of turbulence, it provides a method for investigating the physics in the so-called distributed burning regime. Using this method, we perform numerical simulations of turbulent flames and show that in this particular regime the flamelet model for the turbulent flame velocity loses its validity. In fact, at high turbulent intensities, burning in the distributed regime can lead to a deceleration of the turbulent flame and thus induce a competing process to turbulent effects that cause a higher flame speed. It is also shown that in dense C+O matter turbulent heat transport is described adequately by the Peclet number rather than by the Reynolds number, which means that flame propagation is decoupled from small-scale turbulence. Finally, at the onset of our results we argue that the available turbulent energy in an exploding C+O white dwarf is probably too low to make a deflagration-to-detonation transition possible.

Subject headings: methods: statistical — nuclear reactions, nucleosynthesis, abundances — supernovae: general — turbulence

1. INTRODUCTION

The thermonuclear explosion of a C+O Chandrasekhar mass white dwarf, which is believed to be the underlying process of a Type Ia supernova (SN Ia), has been subject to numerous investigations. However, despite the fact that there are different plausible models explaining the history of the explosion (Nomoto, Thielemann, & Yokoi 1984; Woosley & Weaver 1986; Müller & Arnett 1986; Livne 1993; Khokhlov 1991a; Arnett & Livne 1994; Niemeyer & Hillebrandt 1995; Höplich, Khokhlov, & Wheeler 1995; Höplich 1995; Wheeler et al. 1995; Höplich & Khokhlov 1996), many important details still remain unclear. Thermonuclear reactions provide the source of energy which possibly unbinds the white dwarf. Thus, an SN Ia is characterized by the physics of thermonuclear flames that propagate through the star. The physical conditions of the flames vary drastically during the different temporal and spatial stages of this process. A better understanding of these conditions, their interaction with a flame, and finally, their consequences regarding the explosion itself are still important issues from a theoretical point of view.

Here we focus on the interaction between thermonuclear burning and turbulence taking place during the burning process. Turbulence is caused by different kinds of instabilities (like shear instabilities or the Rayleigh-Taylor instability) that occur on certain length scales and time-scales (for an overview, see Niemeyer & Woosley 1997). Rough estimates give a turbulent Reynolds number Re ≈ 10^{14} at an integral scale of Λ ≈ 10^6 cm (Hillebrandt & Niemeyer 1997). Consequently, the Kolmogorov scale l_{K}, i.e., the scale where microscopic dissipation becomes important, is about 10^{-4} cm. This enormous dynamical range makes a direct representation—at least by means of numerical methods—practically impossible. On the other hand, turbulence is a characteristic feature during the explosion process and it must be considered in any realistic model of an SN Ia. The simultaneous coupling to energy generation due to nuclear reactions leads directly to the physics of turbulent combustion, where a few crucial problems, even for terrestrial conditions, still remain unsolved. These problems include the prescription of the effective turbulent flame speed or the existence of different modes in turbulent combustion along with their physical properties. Global features of turbulent combustion can be systematically classified by a small set of dimensionless parameters. This classification, which has already found wide utilization in the chemical combustion community, can be used in the field of SNe Ia in order to emphasize universality in the physics of turbulent combustion.

In this work we study the properties of turbulent flames in a certain state, the so-called distributed flame regime. The physical characterization of the latter is given by the situation where turbulent motions are fast enough to disturb the flame on microscopic scales. Therefore, a locally distributed flame no longer looks like a laminar flame, which is a basic difference to turbulent flames in the flamelet regime. Turbulent burning fronts in an exploding white dwarf are flamelets at high and intermediate densities, but below ~ 5 × 10^7 g cm^{-3} one cannot expect that the flamelet picture is still valid (Niemeyer & Woosley 1997).

Until now, in the context of SNe Ia, burning in the distributed regime has been considered only qualitatively. Therefore, we wish to present first quantitative results regarding this issue. Furthermore, there are good reasons to
believe that this combustion mode plays an important role in the deflagration-to-detonation transition (DDT), which in turn is a promising model, for empirical and theoretical reasons, for the explosive stage of an SN Ia (e.g., Khokhlov 1991b; Niemeyer & Woosley 1997; Niemeyer & Kerstein 1997; Khokhlov et al. 1997).

In order to attempt a representation of turbulent dynamics in the distributed flame regime, we use a new model, formulated in one spatial dimension, which nevertheless provides essential features of three-dimensional homogeneous turbulence. It consists of a statistical description of turbulent mixing and a deterministic evolution of the underlying microphysics. This model allows a systematic investigation of turbulence phenomena. When it is coupled to a nuclear reaction network, it gives first insights of how turbulence interacts with a nuclear reaction network, it gives first insights of how turbulence interacts with nuclear reactions. The resulting method is then used to investigate some properties of turbulent flames in the distributed regime, such as their effective flame speed. Finally, we discuss the relevance of our results to the DDT problem.

2. ONE-DIMENSIONAL TURBULENCE

Since fundamental aspects of turbulence can be recovered from the knowledge of the statistical moments and correlations of the velocity flow, the statistical approach to turbulence is particularly appealing. Therefore, the history of statistical methods in turbulence theory is rather long. In this context we present a novel model of turbulence (Kerstein 1999). It is a stochastic method, realized as a Monte Carlo simulation, which allows computation of statistical properties of the flow velocity and of passive scalars in stationary and decaying homogeneous turbulence. One-dimensional turbulence (ODT) represents many aspects of three-dimensional turbulence, but it is formulated in only one spatial dimension. This model provides the temporal evolution of a characteristic transverse velocity profile \( u(y, t) \) of the turbulent medium, where \( y \) is the spatial location on a finite domain \([0, Y]\) and \( t \) is the elapsed time. This is done in a twofold way: \( u(y, t) \) is subject to a molecular diffusion process and to a random sequence of profile rearrangements representing turbulent eddies. Reflecting the typical behavior of turbulence kinematics, the event rate of these profile rearrangements (so-called eddy mappings) is proportional to a locally averaged shear of the velocity profile \( u \). Given the transverse velocity profile at a certain time, \( u(y, t) \), the mapping which models the action of a turbulent eddy of size \( l \) at the position \( y_0 \) reads

\[
\hat{u}(y, t) = \begin{cases} 
  u(3y - 2y_0, t) & y_0 \leq y \leq y_0 + f_1 l \\
  u(-3y + 4y_0 + 2l, t) & y_0 + f_1 l \leq y \leq y_0 + f_2 l \\
  u(3y - 2y_0 - 2l, t) & y_0 + f_2 l \leq y \leq y_0 + l 
\end{cases}
\]

(1)

where it is \( f_1 = 1 - f_2 = \frac{1}{3} \) and \( \hat{u}(y, t) = u(y, t) \) for all \( y \notin [y_0, y_0 + l] \). This three-valued map represents the typical features of a turbulent vortex, namely, rotation and compression. Each random map defines an eddy timescale, \( \tau(y_0, l, t) \), via

\[
\tau(y_0, l, t) = \frac{l}{2A |u(y_0, t) - u(y_0 + l/2, t)|}.
\]

(2)

Here \( u_0 \) is the boxcar averaged profile of \( u \) over a length scale \( l \); \( A \) is the only model-independent and dimensionless parameter which has to be fixed empirically. Vortical kinetic energy is fed by the kinetic energy of the local shear. Thus, equation (2) can be interpreted as an energy balance. It then is

\[
\frac{1}{2} \rho_d (\pi t^{-1})^2 = \frac{1}{2} \rho_0 (2A |u(y_0, t) - u(y_0 + l/2, t)|)^2
\]

(3)

where \( \rho_d \) is the fluid density. Equation (3) is used to introduce the statistical hypothesis of ODT. It assumes that the occurrence of eddies with size \( l \) and location \( y_0 \) (with respective tolerances \( dl, dy \)) is governed by a Poisson...
\[ \frac{A}{2}(y_0, l, t) dydl \equiv \lambda(y_0, l, t) dydl, \] (4)

where the processes for different values of \( y_0 \) and \( l \) are statistically independent. In this context \( A \) can be viewed as a factor which scales the event rate \( \lambda \). Motivated by the results in Kerstein (1999), we choose \( A = 0.23 \). In addition, the microscopic evolution is modeled by a diffusion process of the kind \( u = v \delta \), with the kinematic viscosity \( v \). The temporal evolution of turbulence, stationary or decaying, depends on the choice of boundary conditions for \( u \). A decaying turbulent intensity is obtained by periodic boundary conditions, whereas the stationary case is given by the choice of jump periodic boundary conditions: \( u(y + Y, t) = u(y) + u_0, u_0 \) at \( u(Y, t = 0) \) and \( u(y, t = 0) \) being strictly monotonic on [0, Y].

The numerical implementation of ODT reproduces many typical features of three-dimensional homogeneous turbulence (Kerstein 1999). For instance, power spectra of the kinetic energy show the self-similar \( k^{-5/3} \) power law within the inertial range down to the scale where the transition to dissipation takes place (cf. Fig. 1). An obvious advantage of this Ansatz is the high spatial resolution of turbulence compared with multidimensional numerical models. In combination with the relatively moderate computational effort, ODT appears to be a useful tool for performing parameter studies in turbulence theory. On the other hand, ODT does not consider any pressure fluctuations (dynamical or external) in the temporal evolution of the velocity profile \( u \). The reason for this artifact is the inherent conservation of kinetic energy in only one spatial velocity component due to equation (3). Note that the pressure gradient term in the Navier-Stokes equations redistributes energy among the different spatial components. No such redistribution is considered here. However, for isobaric flows ODT appears to be an appropriate model of turbulence.

3. TURBULENT FLAMES IN DENSE C + O MATTER

Although the aim of this work is to investigate some properties of turbulent burning in dense C + O matter, we begin this section by recalling some properties of laminar, conductive flames. Since the physics of undisturbed flames inside a white dwarf is relatively well understood, the first reason to discuss these properties is to verify established results, such as those given in the work of Timmes & Woosley (1992). The other reason is that thermal conduction and viscous diffusion will be employed as the underlying microscopical processes for ODT in order to model turbulent burning fronts in white dwarfs.

The state of unburned matter, i.e., density and nuclear composition, uniquely defines the propagation velocity \( s_{\text{lam}} \) of the conductive flame. Timmes & Woosley (1992) calculated the flame velocities as well as their thickness for various fuel compositions and densities. At lower densities, for \( 10^4 \text{ g cm}^{-3} \geq \rho \geq 10^7 \text{ g cm}^{-3} \), the speed of the laminar flame decreases rapidly. This behavior is accompanied by a strong increase of the flame thickness \( \delta_{\text{lam}} \), which is essentially the size of the nuclear reactive zone (Timmes & Woosley 1992; Khokhlov et al. 1997). As already mentioned in the introduction, we are especially interested in densities of the order \( \sim (1-3) \times 10^7 \text{ g cm}^{-3} \). Thus, we set up a conductive flame propagating into unburned matter consisting of half \( ^{12}\text{C} \) and half \( ^{16}\text{O} \) at densities of \( \rho = 1.3 \times 10^7 \text{ g cm}^{-3} \) and \( \rho = 2.3 \times 10^7 \text{ g cm}^{-3} \). This is done by solving the equations for the conservation of mole fractions and enthalpy in planar geometry, viz.,

\[ \frac{dY_i}{dt} = \sum_j -Y_j \lambda_{ij}(t) + Y_i \lambda_{ki}(t), \] (5)

\[ \frac{\partial T}{\partial t} = \frac{1}{\rho c_p} \frac{\partial}{\partial x} \left( \sigma \frac{\partial T}{\partial x} \right) - \frac{1}{\rho} \frac{\partial}{\partial x} \left( \frac{1}{\rho} \left( \frac{\partial}{\partial x} \sigma \right) \right) + \frac{1}{\rho} \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \sigma \right) + \frac{1}{\rho} \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \sigma \right) + S, \] (6)

\[ S = N_A \sum_i \frac{dY_i}{dt} B_i. \] (7)

Here \( S \) denotes the local specific energy generation rate, \( \sigma \) is the thermal conductivity, \( N_A \) the Avogadro number, and \( B_i \) is the nuclear binding energy of the nucleus considered. The nuclear reaction network consists of seven species, viz., \( ^4\text{He}, ^{12}\text{C}, ^{16}\text{O}, ^{20}\text{Ne}, ^{24}\text{Mg}, ^{28}\text{Si}, \) and \( ^{56}\text{Ni} \). Equation (5) already makes use of the fact that the Lewis number, that is, the ratio of heat diffusion to mass diffusion, in white dwarf matter is around \( 10^7 \), and consequently, microscopic transport of the element species can be neglected. Furthermore, in white dwarf matter the laminar flame speed is much smaller than the speed of sound, \( s_{\text{lam}} < 0.01\text{a}_{\text{vo}} \). Therefore, pressure does not change significantly across the flame front.

Finally, the typical timescale for free collapse or free expansion of the star can be estimated by (Fowler & Hoyle 1964)

\[ \tau_h \approx \frac{446}{\rho^7/2} \text{ s}. \] (8)

For a white dwarf of radius \( R_{\text{wd}} = 10^8 \text{ cm} \) and of mass \( M_{\text{wd}} = 1.4 M_{\odot} \), equation (8) gives \( \tau_h \approx 0.02 \text{ s} \). Timmes & Woosley (1992) stated that gravitational influence can be safely dropped since the diffusion timescale of the flame, \( \tau_{\text{diff}} \approx \delta_{\text{lam}}/D \), where \( \delta_{\text{lam}} \) is the flame thickness and \( D \) is the temperature diffusion coefficient, is several orders of magnitude smaller than \( \tau_h \). However, for very low densities (\( \rho \lesssim 10^7 \text{ g cm}^{-3} \)) these two timescales can become comparable and a possible expansion of the star could directly affect the laminar flame. To avoid additional complexity, we assume in this work that the hydrodynamical timescales are much longer than the relevant timescales of the burning front. Thus, we consider the small-scale burning front to not be affected by stellar expansion.

The numerical solution of equations (5) and (6) are given explicitly in time combined with an appropriate equation of state representing ions, blackbody radiation, electrons, and positrons. For \( \rho = 2.3 \times 10^7 \text{ g cm}^{-3} \), Figure 2 shows the propagation of the laminar flame, which is indicated by the moving jump in the temperature profile. In the steady state evolution, the flame moves into the unburned matter at a velocity \( s_{\text{lam}} = 2.1 \times 10^4 \text{ cm s}^{-1} \). The flame thickness \( \delta_{\text{lam}} \) is characterized by the width of the temperature jump caused by carbon burning. It is \( \delta_{\text{lam}} = 1.1 \text{ cm} \). After \(^{12}\text{C} \) destruction, the burned material has a temperature \( 3.2 \times 10^8 \text{ K} \) and a density of \( 1.45 \times 10^7 \text{ g cm}^{-3} \). For these conditions, the subsequent oxygen burning has a destruction timescale of \( \tau_{\text{dest,O}} \approx 0.1 \text{ s} \). That means that as the carbon flame propagates with a steady state velocity \( s_{\text{lam}} \) oxygen burning ignites at a distance of \( 2.1 \times 10^3 \text{ cm} \) behind the flame front. This distance is about 10 times larger than our whole com-
putational domain. Consequently, we are not able to track the whole spatial region where nuclear reactions that are triggered by carbon burning take place. The extension of the reactive zone is even larger if one takes burning of heavier elements, such as silicon, into account. Thus, the resolution of the flame with all its reactions up to nuclear burning is a\*\*\*\*\**\*\*\*\resolution. Statistical equilibrium would require an immense spatial scale of the flame. The obvious step is to incorporate effects and thus can be described by the Kolmogorov theory. The Kolmogorov theory sets the scaling law for $u'$ to be $u' \sim l^{1/3}$. In between these scale bounds nuclear burning is affected by turbulent motion and different kinds of burning regimes are revealed (for an overview, see Niemeyer & Woosley 1997). Here we put our emphasis on the special situation where homogeneous, isotropic turbulence interacts with the microscopic structure of the laminar flame, i.e., where $l_G$ becomes comparable to $\delta_{lam}$. The relation $l_G \approx \delta_{lam}$ marks the transition into the distributed burning regime, where the smallest turbulent vortices can enter the interior of the laminar flame and can carry away reactive material before it is completely burned (Peters 1986; Niemeyer & Woosley 1997). Taking into account the small Prandtl numbers in the dense matter of a white dwarf, we can describe this situation by a constraint on the turbulent Karlovitz number. This number is the ratio of the diffusion timescale and the eddy turnover time at the smallest length scale of turbulent heat transport, which turns out to be the Kolmogorov scale, $l_k$, only in case of $Pr \approx 1$. For Prandtl numbers much smaller than 1, turbulent heat transport is subject to significant diffusion effects already at a scale of $l_k Pr^{-3/4}$. Thus, we have

$$Ka = \frac{u'(l_k)\delta_{lam}}{s_{lam}l_k} Pr^{1/2} \gtrsim 1$$

Using Kolmogorov theory, an equivalent representation of the latter condition reads

$$Ka = \frac{u'(L)\delta_{lam}}{s_{lam}L} Pr^{1/2} Re^{1/2}$$

$$= \left(\frac{u'(L)}{s_{lam}}\right)^2 Re^{-1/2} Pr^{-1/2} \gtrsim 1.$$  

There is a simple relation between the Gibson length and the Karlovitz number as follows

$$l_G \approx \delta_{lam} Ka^{-2}.$$  

Therefore, $l_G < \delta_{lam}$ is equivalent to $Ka > 1$. The actual value of $Ka$ for a given turbulent flame plays an outstanding role in turbulent combustion physics.

We now combine ODT with the structure of a laminar flame to show how certain properties of a nuclear flame in dense degenerate matter of white dwarfs depend on the Karlovitz number. The obvious step is to incorporate equation (6) together with a viscous transport evolution for the transversal velocity profile $u(x, t)$. Then turbulent advection is modeled by the random eddy mappings generated by ODT (cf. eq. [1]). This system reads as

$$\frac{\partial \hat{u}}{\partial t} = \frac{Pr}{\hat{\rho} \rho_c} \frac{\partial}{\partial x} \left( \frac{\sigma}{\alpha} \frac{\partial (\hat{u})}{\partial x} \right),$$

$$\frac{\partial \hat{T}}{\partial t} = \frac{1}{\hat{\rho} c_p} \frac{\partial}{\partial x} \left( \frac{\sigma}{\alpha} \frac{\partial \hat{T}}{\partial x} \right) - \frac{\partial}{\partial t} \hat{\rho} \frac{\partial \hat{u}}{\partial x} + \frac{1}{c_p} \hat{S},$$

where $\hat{u}$, $\hat{\rho}$, $\hat{T}$ are the rearranged profiles of the velocity, the density, and the temperature according to the eddy mapping (cf. eq. [1]).

We choose the amplitude of velocity fluctuations of $u$ to be $\sim 10^{-7}$ $cm$ s$^{-1}$ at $\sim 10^6$ $cm$ and to obey Kolmogorov scaling. These values are motivated through the expected speed of buoyant unstable hot bubbles of size $L \approx 10^6$ $cm$, which become Rayleigh-Taylor unstable and eventually give the main contribution to turbulent energy on large scales. Having fixed this velocity at a certain length scale, we
can use Kolmogorov scaling to estimate \( u'(l) \) on smaller scales \( l \). Thus, the amplitude of turbulent velocity fluctuations in our model is comparable to the expected small-scale velocity fluctuations within an SN Ia.

However, because of computational limitations we cannot consider \( u \) to be the actual fluid velocity of the white dwarf matter. Since its Prandtl number is around \( \Pr \approx 10^{-2} \), the numerical resolution of both the viscous and the heat conductive diffusion scale would cost too much numerical effort even in one spatial dimension. To get out of this dilemma, we use an "artificial" Prandtl number of \( 2 \times 10^{-2} \) and of \( 10^{-1} \). This choice underestimates the inertial range of turbulent motions within an exploding Chandrasekhar mass white dwarf. But, we believe that the physics of distributed burning in this kind of matter is decoupled from scales much smaller than \( l_\text{d} \Pr^{-3/4} \) because turbulent eddies of sizes well below this limit will be smeared out instantly by temperature diffusion. Therefore, we do not essentially change the flame properties in the distributed burning regime by increasing \( \Pr \). Of course, this is valid only for subunity Prandtl numbers; otherwise, one possibly leaves the distributed regime.

The random velocity field \( u \) is generated by stationary ODT; thus \( u' \) becomes constant in time after an initial transient phase. For our studies we perform simulations where \( u' \) (at a length scale \( l = 1 \text{ cm} \)) is equal to or larger than a given value of \( s_{\text{lam}} \). Then we estimate some flame properties such as the effective turbulent flame speed, \( s_T \). So far, analytical results do not predict the function \( s_T(u'; Ka) \) correctly. For instance, experimentally observed bending effects or even complete flame quenching are still mostly unexplained problems (Ronney 1995).

An overview of the various numerical models is shown in Table 1. Each realization is uniquely parametrized by its Reynolds, Karlovitz, and Prandtl numbers. To get the turbulent Reynolds numbers, \( \Re = u'(L) L / \nu \), we have estimated the integral scale \( L \) for each model from the wavenumber \( k_L \) for which the relevant energy power spectrum \( E(k) \) has a maximum. Together with the rms turbulent velocity and the kinematic viscosity, \( \nu = \Pr \rho c_p \sigma^{-1} \), we eventually obtain the Reynolds number, \( \Re \), and the Karlovitz number, \( Ka \).

### Table 1

| Model | Reynolds Number (Re) | Karlovitz Number (Ka) | Prandtl Number (Pr) | Flame Speed \( s_T \) (x10^6 cm s^-1) |
|-------|----------------------|----------------------|---------------------|-----------------------------------|
| A1... | 114                  | 0.06                 | 0.1                 | 2.15                              |
| A2... | 270                  | 0.23                 | 0.1                 | 2.4                               |
| A3... | 452                  | 0.51                 | 0.1                 | 2.7                               |
| A4... | 598                  | 0.78                 | 0.1                 | 3.1                               |
| A5... | 673                  | 0.92                 | 0.1                 | 3.4                               |
| A6... | 820                  | 1.24                 | 0.1                 | 4.2                               |
| A7... | 1285                 | 2.43                 | 0.1                 | 5.0                               |
| A8... | 1493                 | 3.05                 | 0.1                 | 5.3                               |
| B1... | 6182                 | 2.29                 | 0.02                | 4.9                               |
| C1... | 28                   | 0.68                 | 0.1                 | 1.0                               |
| C2... | 119                  | 5.80                 | 0.1                 | 1.8                               |
| C3... | 212                  | 13.9                 | 0.1                 | 2.4                               |
| C4... | 314                  | 25.7                 | 0.1                 | 2.1                               |
| C41...| 323                  | 25.3                 | 0.1                 | 2.2                               |
| C5... | 366                  | 32.4                 | 0.1                 | 1.9                               |
| C51...| 371                  | 32.2                 | 0.1                 | 2.0                               |

Note—\( X(\text{^{12}C}) = X(\text{^{16}O}) = \frac{1}{2} \), \( \rho = 2.3 \times 10^{-7} \text{ g cm}^{-3} \) (A1–A9, B1), \( \rho = 1.3 \times 10^{-7} \text{ g cm}^{-3} \) (C1–C5, C41, C51).

An example of a numerical realization is illustrated in Figure 3, where a laminar and a turbulent flame are shown in a space-time diagram.

Table 1 also shows the measured turbulent flame speeds \( s_T \) as a function of \( Ka \) and \( Pr \). These velocities have been obtained by measuring the slope of the function

\[
\Delta(t) = \int c(x, t) \, dx ,
\]

where \( c(x, t) \) is the reaction-progress variable defined by

\[
\frac{d \rho}{dT} = \frac{T_{\text{un}}}{T_{\text{un}} - T_{\text{b}}} - 1 \frac{T_{\text{un}}}{T_{\text{b}}} .
\]

Thus, \( s_T = d \Delta(t)/dt \) defines the effective flame speed. This value is constant only in the case of laminar flames.
However, in the turbulent case it is subject to fluctuations. These estimates enable us to express the dimensionless turbulent flame speed, \( U_T = \frac{\dot{S}_T}{\dot{S}_{lam}} \), as a function of the dimensionless turbulent fluctuation velocity, \( U = \frac{u}{s_{lam}} \). We should mention that we computed the dimensionless turbulent flame speed using the alternative formula

\[
U_T = \left( \int \rho_s \dot{S}_T(x)dx \left( \int \rho_{lam}(x)\dot{S}_{lam}(x)dx \right)^{-1} \right),
\]

where \( \dot{S}_T \) and \( \dot{S}_{lam} \) denote the turbulent and laminar profile of the energy generation rate. In fact, the estimated values are in good agreement with the ones obtained through the equation \( s_T = \frac{d\Delta(t)}{dt} \).

For our studies we vary the rms turbulent velocity, the Prandtl number, and the density, while the integral length scale \( L \) remains at a constant value.

The results are shown in Figure 4. It clearly demonstrates that the turbulent flame speed at high turbulent strain rates, \( U \gg 1 \), does not follow a law valid in the flamelet regime. At a Karlovitz number greater than 1, i.e., in the distributed burning regime, there is a significant deviation from this law: stronger turbulent intensities do not necessarily lead to a faster burning rate. This effect is more pronounced in the low-density case, where higher turbulent velocities even cause a slight deceleration of the flame speed. This result confirms the intuitive picture that fast flames (flames at higher density) are more resistant to distortion through turbulence than the slower ones (flames at lower density).

As already mentioned in this section, the transport of heat is expected to be decoupled from turbulence on scales smaller than \( l_s \Pr^{-3/4} \). This assertion is numerically confirmed in the realization B1, where we significantly decrease the Kolmogorov length scale (using a Prandtl number of 0.02) while keeping the fluctuation velocity \( u(L) \) at a value used in the realization A7 (\( \Pr = 0.1 \)). The estimated turbulent flame speed \( s_T \) of both models, however, turns out to be nearly the same (see Fig. 4). This behavior suggests that in the case of a small Prandtl number, one should use the Peclet number, \( \text{Pe} = \text{RePr} \), as the governing parameter for the strength of turbulent flame dynamics.

For clarity, Table 2 gives a summary of all the dimensionless quantities that we have used or mentioned in this section.

### 4. DISCUSSION

#### 4.1. Burning in the Distributed Regime

The coupling of ODT as a model of homogeneous turbulence to reactive hydrodynamics gives insight into the physics of distributed burning. First of all, the qualitative physical picture of this burning process is revealed: turbulent vortices enter the reaction zone and carry away burning material to the front and to the back of the flame. As a consequence, one observes a completely different behavior than in the flamelet regime, because unlike the flamelet regime, the distributed regime changes the local flame velocity. (The local flame velocity is the speed at which the flame propagates locally normal to itself.)

A different feature is the existence of localized burning with high burning rates, commonly referred to as a local explosion (Radford & Chan 1995). Turbulent motion forms regions where hot reaction products are mixed with cold unburned material. These regions have a higher temperature than pure fuel and possess comparably short destruction timescales for carbon. Their eventual reignition causes high burning rates, as shown in Figure 5. We find that with higher Karlovitz numbers more of these events

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**TABLE 2**

| Symbol | Name                  | Mathematical Definition | Physical Description                  |
|--------|-----------------------|-------------------------|----------------------------------------|
| Re     | Reynolds number       | \( \frac{u(L)L}{U} \)  | integral scale turbulent diffusivity   |
| Pr     | Prandtl number        | \( \frac{\rho c_p y}{\sigma} \) | viscous diffusivity                    |
| Pe     | Peclet number         | \( \frac{\text{Re Pr}}{\sigma} \) | integral scale turbulent diffusivity   |
| Ka     | Karlovitz number      | \( \frac{u(l_s)^{1/2} \text{Pr}^{1/2}}{s_{lam} \text{Pr}^{1/2}} \) | heat diffusion time scale              |
| Le     | Lewis number          | \( \frac{\sigma}{\rho c_p D_m} \) | smallest turbulent heat transport timescale |

\( D_m \) denotes the mass diffusion coefficient.
occur and their strength (burning rate) is increased. Terrestrial experiments show the generation of blast waves after the occurrence of local explosions. In some experiments the strongest waves developed even into a detonation (Radford & Chan 1995). However, laboratory experiments on turbulent flames deal with much higher expansion factors, $\rho_u/\rho_b$, than those given by flames in white dwarf matter. Thus, we expect the effect of blast waves in the context of SNe Ia to be less effective than in typical chemical flames. However, because our model does not consider pressure waves, the whole effect of local explosions on the burning process cannot be presented here, and we cannot completely exclude the possibility that blast waves possibly enhance the global burning rate by compression and preheating of the surrounding medium. Neglecting such effects in our model possibly means an underestimation of turbulent combustion rate in the distributed regime.

Taking into account the estimates on the turbulent flame speed of Table 1, we obtain relatively good agreement with experimental data presented by Abdel-Gayed, Bradley, & Lawes (1987, hereafter AG87). Their work contains about 1650 experimental results on turbulent flame velocities at various strain rates $U = u'/s_{lam}$, fuels, and equivalence ratios. Their results were obtained under conditions far from those in an exploding Chandrasekhar mass white dwarf, but it is worth verifying if certain physical statements, like the function $U(T(Ka;Re))$, appear to show universality. These experimental results and earlier work (Abdel-Gayed & Bradley 1985) present two important phenomena. First, for a fixed Reynolds number, $U_T$ is not a strong monotonic function of the strain rate $U$. Rather, it reaches a maximum at a certain turbulent intensity and eventually saturates or even decreases for higher values of the strain rate (bending effect). As a second feature, there is flame extinction at a Karlovitz number $K a \gtrsim 7$ for Reynolds numbers $Re > 10^2$. Figure 6 shows several experimental values of the function $U_T$ taken from the work of AG87. These values correspond to the same sample of ratios $u'/s_{lam}$ we use in our numerical calculations.

Our numerical results do show the bending effect for high strain rates, which is in agreement with laboratory experiments. However, they do not show complete flame quenching, i.e., a situation where fuel is not consumed anymore.

This is not surprising because the basic laws of thermodynamics prevent a turbulent (or laminar) flame from complete quenching. The first law of thermodynamics states that the reactive mixture must eventually reach the adiabatic flame temperature. The second law demands that heat must be transported from the burned to the unburned regions. Therefore, there is no way to prevent consumption of unburned material in a system without possible heat losses. And clearly, unlike under laboratory conditions, the flame in an exploding white dwarf is not subject to significant heat losses.

The comparison with our results shows that the flame speeds presented here are lower than those in AG87. We think that one reason for this discrepancy is the different value of the Lewis number in our work and in the data of AG87. It is known that high Lewis number flames are more sensitive to strain than flames with a low value of Le. In fact, analytical results on planar laminar flames show that the flame speed decreases when a velocity field with strain, i.e., a nonvanishing gradient of this velocity field, is invoked parallel to the flame surface. One finds that at higher Lewis numbers the decrease of the flame speed is stronger (Tromans 1981). Therefore, we expect thermonuclear flames in C+O matter to be more sensitive on strain effects, because of their Lewis numbers of approximately $10^7$. Another source of this deviation could be the already aforementioned difference in the expansion factors between laboratory and white dwarf combustion.

Thermonuclear flames in the distributed regime thus experience a twofold effect through turbulence. One is the acceleration due to an enhancement of the heat transport produced by small-scale turbulence. The other is a slowing down due to additional turbulent strain, which can be measured by the Karlovitz number. However, we stress that we have not revealed the actual physical mechanism responsible for the deceleration. This is the task of future work.

4.2. Implications for a Deflagration-to-Detonation Transition

As already stated by many authors, a delayed detonation within the explosion of a Chandrasekhar mass white dwarf is a reasonable model. It is a good fit to experimental constraints for element abundances obtained from observed
spectra. In addition, although it is almost clear that after
ignition the burning process evolves as a deflagration, there
is no fundamental physical reason why a transition to
detonation in the subsequent stages of the explosion should
be impossible. Therefore, DDT models should be taken into
account in supernova theory. In the context of this work, we
discuss the possibility of distributed burning leading to a
detonation.

Based on the Zeldovich gradient mechanism, Khokhlov
et al. (1997) calculated the minimum size, \( L_c \), of a region
where fuel and ashes are mixed and out of which a stable
detonation wave can form. That is, the size of those eddies
whose turnover time equals a typical burning timescale that
separates burned and unburned material. A lower bound of
this scale is given by the length scale, where distributed
burning in a white dwarf can occur. A lower bound of this
scale is given by the density drop quenches the degrad-
ation–flame, whereas the Rayleigh-Taylor instability still
takes place on scales of \( \sim 3 \times 10^6 \) cm. Before the star
recombines again, fuel and ashes are mixed and turbulence
eventually produces a complete mixed region of size of the
order of \( L_c \). Thus, pulsation may lead to a DDT.

Apart from considering pulsations of the star, it is worth
asking whether burning in the distributed regime alone can
lead to a DDT. Niemeyer & Woosley (1997) advocated this
burning mode as a favorite model for DDT. Basically, they
proposed that if turbulent burning is able to form a region
of incompletely burned material with a certain temperature
gradient, then a carbon detonation can occur when two
presumptions are fulfilled. First, the aforementioned region
must have at least a size of the critical radius necessary for a
detonation. (These critical radii could be calculated explicitly
for different densities and nuclear compositions; see
Niemeyer & Woosley 1997.) Second, the temperature
gradient must be shallow enough in order to allow a sonic
phase velocity of nuclear burning. For instance, consider a
mixed region consisting of half carbon and oxygen at a
density of \( 3 \times 10^7 \) g cm\(^{-3} \) and a maximum
of \( 2 \times 10^8 \) K. It then follows that this region
must have a size of at least \( L_c \sim 50 \) m. Furthermore,
temperature differences within it should not exceed \( \sim 10^3 \)
K. Then, a stable detonation wave can form.

Both approaches require a rather shallow temperature
gradient over a region with a sufficient amount of carbon.
Until now, it has not been demonstrated how turbulence
can achieve or lead to such a high level of temperature
homogeneity.

In this context it is interesting to know the maximum
length scale, \( l_{max} \), where distributed burning in a white
dwarf can occur. A lower bound of this scale is given by
the size of those eddies whose turnover time equals a typical
turbulent burning timescale \( \tau_{\text{nuc}} \). That is,
\[
l_{max}/u(l_{max}) = \tau_{\text{nuc}}. \tag{16}
\]

In the case of laminar flames \( \tau_{\text{nuc}} \) is just the diffusion
timescale \( \tau_{\text{lam}} \) equals \( \delta_{\text{lam}}/\text{\textlam} \approx 0.7 \times 10^{-4} \) s. But
what is the value \( \tau_{\text{nuc}} \) in the presence of turbulence? We argue that even in the
 turbulent case the choice of \( \tau_{\text{lam}} \) as the typical burning
timescale is natural. To justify this assumption, we compute
the spatial distribution of destruction timescales across the
flame. In general, this distribution is characterized by a
minimum value \( \tau_{\text{nuc}} \). It corresponds to the location of the
maximum energy generation within the flame. In our
numerical models, we find that \( \tau_{\text{nuc}} \) is independent of the
turbulent intensity, i.e., its value is the same for the laminar
case and for the distributed regime. In fact, \( \tau_{\text{nuc}} \) equals \( \tau_{\text{lam}} \).

Then, \( q_{\text{\textlam}} \text{\textlam} \) equals \( \tau_{\text{\textlam}} \).

Equation (16) leads to the useful expression
\[
l_{max} \approx \delta_{\text{\textlam}} \tau_{\text{\textlam}} \tag{17}
\]

This number can be viewed as the size of the distributed
flame, i.e., an estimate on the size of the interface that
separates burned and unburned material. However, there
are reasons to believe that in reality \( l_{max} \) exceeds the value
given by equation (17). This is because there are turbulent
vortices with bigger size than \( l_{max} \) that can enter the flame
and transport burning material with longer burning times
than \( \tau_{\text{nuc}} \). Hence, we do justice to these considerations and
set the effective value of \( l_{max} \)
\[
l_{max} \approx x \delta_{\text{\textlam}} \tau_{\text{\textlam}} \tag{18}
\]
with \( x > 1 \). The actual value of \( x \) however, i.e., the ability
of turbulence to stretch out the flame brush, has to be esti-
mated by more rigorous and quantitative considerations
(see Lisewski et al. 2000).

If \( l_{max} \) itself reached the critical size \( L_c \) required for a
detonation, then burning in the distributed regime would
become more of a promising model for DDT. But still it
would not be clear if present temperature fluctuations
would be small enough to allow this transition to happen.
Thus, the equality of \( l_{max} \) and \( L_c \) is only a necessary condi-
tion for a DDT. At a density of \( 2.3 \times 10^7 \) g cm\(^{-3} \),
Khokhlov et al. (1997) obtain \( L_c \approx 10^4 \) cm. Now the expected
turbulent velocities at an integral scale of \( L = 10^6 \) cm are
\( \sim (g_{\text{eff}} L)^{1/2} \sim 10^5 \) cm s\(^{-1} \), whereas \( g_{\text{eff}} \sim 10^5 \) cm s\(^{-2} \) is the
effective gravitational acceleration. Hence, at densities \( \rho \approx
2 \times 10^7 \) g cm\(^{-3} \) the corresponding Karlovitz numbers are
only around \( Ka \approx 10 \). According to equation (17), a very
high Karlovitz number \( Ka \approx 10^5 \) is required to obtain
equality of \( l_{max} \) and \( L_c \). Consequently, the enhancement
factor \( x \) in equation (18) would have to be of the order of
\( 10^3 \). This means that the ashes-fuel interface has to be
extended much more in size than given by equation (17).
Also, at a density of \( 10^7 \) g cm\(^{-3} \) where \( l_{\text{\textlam}} \approx 4.7 \times 10^3 \) cm
and \( \delta_{\text{\textlam}} \approx 4.2 \) cm (Timmes & Woosley 1992), the value of \( x \)
still has to be around \( 10^3 \) in order to enable a direct
DDT.

Furthermore, the process of a DDT demands that the
mixing timescale, \( \tau_{\text{\textmix}} \), which can be regarded as the time it
takes turbulence to form a complete mixed region of size \( L_c \),
must be smaller than a typical burning timescale, \( \tau_{\text{nuc}} \), of
the mixed turbulent matter. But, in order to avoid quenching
by expansion, \( \tau_{\text{nuc}} \) in turn must be smaller than the
hydrodynamical time of the star, \( \tau_{\text{\texth}} \) (cf. eq. [8]). These conditions
read as \( \tau_{\text{\textmix}} < \tau_{\text{nuc}} < \tau_{\text{\texth}} \), or as
\[
\frac{L_c}{u(L_c)} < \tau_{\text{nuc}} l_{\text{\textmax}} = \frac{446}{\rho^{1/2}}. \tag{19}
\]

For \( \rho = 2 \times 10^7 \) g cm\(^{-3} \), we have \( \tau_{\text{\textmix}} \approx 5 \) ms while the
hydrodynamical time is \( \tau_{\text{\texth}} = 0.02 \) s. Here the question arises
whether turbulence can form such a region whose burning
time is smaller than \( \tau_{\text{\texth}} \). This problem is addressed in forthcoming
work, where we study—by using ODT—the
burning times on different length scales within the distrib-
uted flame. However, until we have a better understanding
of the large-scale structure (length scales of \( \gtrsim L_c \)) of flames
in the distributed regime, the preliminary result from equa-
tion (17) suggests that Karlovitz numbers in white dwarf
C+O matter at densities around \( 10^7 \) g cm\(^{-3} \) are too small
to form a region of critical size necessary for a DDT.
5. CONCLUSIONS

The aim of this work is to investigate the physics of distributed burning in an exploding white dwarf of Chandrasekhar mass. With ODT as a new model for homogeneous, isotropic turbulence, we are able to study the effective turbulent flame speeds resulting from this kind of burning in dense C + O matter. Our results show that in this regime the local properties of the thermonuclear flame are readily changed; the flame cannot be represented by the flamelet model anymore, and moreover, higher turbulent intensities in general can lead to a lower local flame speed. This behavior has already been observed in laboratory experiments and is introduced the first time in the context of nuclear explosions in Type Ia SNe. We have also found that, in cases where the Gibson length \( l_G \) is smaller than \( \delta_{\text{lam}} \) and where the Prandtl number is much smaller than 1, turbulent burning decouples from turbulent scales smaller than \( l_k \Pr^{-3/4} \). This suggests that the appropriate parameter for turbulent heat transport inside a white dwarf is given by the Peclet number instead of the Reynolds number. Because of the small ratio of viscous momentum to diffusive heat transport, the former is by several orders of magnitude smaller than the Reynolds number: \( \text{Pe} = \Pr \text{Re} \).

In addition, first quantitative considerations demand that a turbulence-induced deflagration-to-detonation transition can happen only deep in the distributed burning regime, i.e., at Karlovitz numbers larger than 10\(^3\). However, our estimates for the turbulent velocity at the integral scale in an exploding white dwarf give values of \( K_\alpha \) that are too small (at least by a factor of 10\(^3\)) to make a direct DDT probable.

Finally, it is worthwhile to remark that we have completely left out questions about the influence of the turbulent spectrum on this issue. We have shown only that a DDT is unlikely when the Kolmogorov theory of turbulence is considered. Niemeyer & Kerstein (1997) proposed that turbulence in an exploding white dwarf is dominated by a potential energy cascade instead of Kolmogorov's kinetic energy cascade. This domination leads to different scaling properties (Bolgiano-Obhukov scaling) of the turbulent velocity. As a consequence, higher turbulent velocities are necessary for the transition to distributed burning. Furthermore, Niemeyer & Woosley give the speed of the turbulent fluctuations, \( v_{\text{RT}} \approx 0.5 \left( \theta_{\text{eff}} L \right)^{1/2} \approx u(L) \), which may not be the upper limit for turbulent velocities at the largest turbulent scale. It is based on the original idea of Kerstein (1996), that the turbulent flame velocity, \( s_T \), is increased by local expansion caused by thermonuclear reactions (active turbulent combustion). At the largest length scale, \( L \), and at a certain temporal stage of the burning process, the speed of the flame may exceed \( v_{\text{RT}} \). Because the relation \( u(L) \approx s_T \) still holds at this point, a significant enhancement of turbulent velocity fluctuations may occur. But it is unclear if the growth rate, \( \dot{s}_T/\tau_s \), is high enough to realize this enhancement within a hydrodynamical time \( \tau_s \). These arguments favor a DDT because their effect is an increase of the turbulent fluctuations on lengths comparable to \( L \). However, both arguments are rather speculative because they have not been confirmed numerically, analytically, or experimentally in the context of SNe Ia yet.

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