On the topological decomposition of the hypersurfaces in projective toric manifolds

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Abstract
In this paper, we want to discuss the topology of the non-singular hypersurface $Y^n$ with complex dimension $n$ in a projective toric manifold $X^{n+1}$. When $n$ is odd, our main results are a decomposition of $Y^n \cong Y' \# s(S^n \times S^n)$ as a connected sum of $s$ copies of $S^n \times S^n$ with a differential manifold $Y'$ such that $b_n(Y') = 0$ or $2$. When $n$ is even and the degree of $Y$ in $X$ is big enough, we find that $Y$ also admits such a decomposition $Y' \# s(S^n \times S^n)$, where $Y'$ satisfy $b_n(Y') - |\text{sign}(Y')| = b_n(X) \pm \text{sign}(H^n(X))$, where $\text{sign}(H^n(X))$ is the signature of a certain bilinear form defined on $H^n(X, \mathbb{Z})$.

1 Introduction

1.1 Projective toric manifold and its hypersurfaces

Definition 1.1. A toric variety is a normal algebraic variety $X$ containing the algebraic torus $(\mathbb{C}^*)^n$ as a Zariski open subset in such a way that the normal action $(\mathbb{C}^*)^n$ on itself extends to an action on $X$.

In this paper, we call $X$ a projective toric manifold if $X$ is a compact, smooth toric variety that admits a holomorphic embedding into a certain $\mathbb{C}P^N$.

The algebraic topology of projective toric manifold has been fully studied by many people and many results can be found in these two classical books [4],[5]. In this paper, what we need are the following two propositions ([5], page 56,101,102).

Proposition 1.2. Let $X$ be a projective toric manifold, then $X$ is simply connected and the odd dimension homology groups of $X$ vanish, i.e. $H_{\text{odd}}(X, \mathbb{Z}) = 0$.

Proposition 1.3. $H_*(X, \mathbb{Z})$ can be generated by the projective toric submanifolds of $X$, i.e. there exist smooth toric submanifolds $\{X_i\}$ with $x_i = [X_i] \in H_*(X, \mathbb{Z})$ such that the homomorphism

$\sum \mathbb{Z}x_i \rightarrow H_*(X, \mathbb{Z})$
is surjective.

Then we introduce the hypersurface of a projective toric manifold. Let $X$ be a projective toric manifold. For any holomorphic embedding $X \hookrightarrow \mathbb{C}P^n$, let $F_1$ be a hyperplane of $\mathbb{C}P^n$, we get a subvariety $i : Y = F_1 \cap X \hookrightarrow X$ of $X$ and $Y$ is called a hypersurface of $X$. By Bertini’s theorem, for a generic hyperplane $F_1$ in $\mathbb{C}P^n$, $Y$ is smooth.

Given such a hypersurface $Y$ of $X \hookrightarrow \mathbb{C}P^n$, we can also construct the smooth hypersurface $i_d : Y_d \hookrightarrow X$ of $X$ with $(i_d)_*(Y_d) = d(i_*[Y])$, $0 < d \in \mathbb{Z}$. Indeed, we can take $Y_d := F_d \cap X$, where $F_d$ is a generic hypersurface of $\mathbb{C}P^n$ with degree $d$ and it is well-known that $Y_d$ is also a smooth hypersurface of $X$.

In this paper, all the hypersurfaces we consider are smooth and when we say a hypersurface $Y_d$, it always means $Y_d$ is a smooth hypersurface.

Similar to the degree of a hypersurface in $\mathbb{C}P^n$, we can define the degree of a (smooth) hypersurface in $X$. Let $\alpha_X$ be the element of $H^2(X, \mathbb{Z})$ such that $\alpha_X \cap [X] = i_*[Y]$. We define the degree of a hypersurface $Y$ in $X$ by

$$
\deg Y := \langle \alpha_Y^{n+1}, [X] \rangle
$$

For the hypersurface $Y_d$, we have relation $\partial \alpha_Y = \alpha_{Y_d}$ and we have $\deg Y_d = d^{n+1} \deg Y$.

### 1.2 Main results

Let $X^{n+1}$ be a projective toric manifold with complex dimension $n + 1$, $n > 2$. Let $i : Y \hookrightarrow X$ be the hypersurface of $X$ with complex dimension $n$ and $i_d : Y_d \hookrightarrow X$ be the hypersurface with $(i_d)_*(Y_d) = d(i_*[Y])$, $0 < d \in \mathbb{Z}$. In this paper, we want to discuss the topological decomposition of the hypersurface $Y_d$. Our main results are:

**Theorem 1.4.** When $n$ is odd, for any integer $d > 0$, we have decomposition:

$$
Y_d \cong Y_d' \# s_d(S^n \times S^n)
$$

where the $n$-th Betti number $b_n(Y_d') = 0$ or 2.

**Theorem 1.5.** When $n$ is even, for sufficiently big $d >> 0$, we have decomposition:

$$
Y_d \cong Y_d' \# s_d(S^n \times S^n)
$$

with $s_d = \frac{b_n(Y_d) - b_n(X) - |\text{sign}(Y_d) - \text{sign}(H^n(X))|}{2}$, here $\text{sign}(Y_d)$ is the classical signature of $Y_d$ and $\text{sign}(H^n(X))$ is the signature of the bilinear form defined by:

$$
H^n(X) \otimes H^n(X) \longrightarrow \mathbb{Z}$$

$$(x, y) \mapsto \langle x \cup y \cup \alpha_{Y_d}, [X] \rangle$$

Furthermore, we have limit estimate:

$$
0 < \lim_{d \to +\infty} \frac{2s_d}{\deg Y_d} = \lim_{d \to +\infty} \frac{2s_d}{d^{n+1} \deg Y} = 1 - \frac{2^{n+1}(2^{n+1} - 1)}{(n + 1)!} \frac{B_{n+2}}{2} < 1
$$

here $B_{n+2}$ is the $\frac{n+2}{2}$-th Bernouli number.
For $Y_d$, we have relations $b_n(Y_d) = b_n(Y_d) - 2s_d$ and $\text{sign}(Y_d) = \text{sign}(Y_d')$. By the limit estimates in proposition 4.4, we know $|\text{sign}(Y_d)| = |\text{sign}(Y_d')|$ tends to $+\infty$ as $d \to +\infty$. From theorem 1.5, we can deduce that

**Corollary 1.6.** When $n$ is even and $d$ is big enough:

$$b_n(Y_d') - |\text{sign}(Y_d')| = b_n(X) \pm |\text{sign}(H^n(X))|$$

**Remark 1.7.** Let $F$ be a nonsingular algebraic hypersurface in complex projective space, in [8], Kulkarni and Wood proved that there is a differentiable connected sum decomposition

$$F = M_k^\sharp(S^n \times S^n)$$

where $b_n(M) = 0$ or $2$ for $n$ odd, and $b_n(M) - |\text{sign}(M)| = b_n(\mathbb{CP}^n) \pm \text{sign}(H^n(\mathbb{CP}^{n+1})) = 0$ or $2$ for $n$ even.

Our theorem is a generalization of their theorem to the case of hypersurfaces in projective toric manifolds.

## 2 Basic idea of removing handles

### 2.1 Geometric point of view

Choose a point $(x, y) \in S^n \times S^n$ and there are two embedded spheres: $S_1 := S^n \times \{y\}$, $S_2 := \{x\} \times S^n \hookrightarrow S^n \times S^n$ with properties:

(1). $S_1$ intersects $S_2$ transversally at one point $(x, y)$.

(2). The normal bundles of $S_1, S_2$ in $S^n \times S^n$ are trivial.

(3). Denote $\eta_1 := S_1 \times D^n \subset S^n \times S^n$ and $\eta_2 := S_2 \times D^n \subset S^n \times S^n$ by the closure of their normal bundles, we see $\eta_1 \cup \eta_2$ is a manifold with boundary $S^{2n-1}$ and

$$S^n \times S^n = (\eta_1 \cup \eta_2) \cup_{S^{2n-1}} D^{2n}$$

Conversely, let $M^{2n}$ be a smooth manifold and $S_1, S_2$ be two embedded n-spheres of $M^{2n}$ with:

(1). $S_1$ intersects $S_2$ transversally at one point.

(2). The normal bundles of $S_1$ and $S_2$ are trivial.

We denote $\xi_1 := S_1 \times D^n$, $\xi_2 := S_2 \times D^n$ by the closure of their normal bundles. Observe that $\eta_1 \cup \eta_2 \cong \xi_1 \cup \xi_2$ and we get:

$$M \cong (M - \xi_1 \cup \xi_2) \cup_{S^{2n-1}} (\eta_1 \cup \eta_2) \cong M^\sharp \# S^n \times S^n$$

where $M' = (M - \xi_1 \cup \xi_2) \cup_{S^{2n-1}} D^{2n}$. This is the basic idea of removing handles from a $2n$-manifold ([12]). Next we want to realize this idea by algebraic topology.

### 2.2 Homological point of view

From the point of view of homology, let $M^{2n}$ be a simply connected smooth closed manifold of dimension $2n$, $n > 2$ and $h : \pi_n(M) \to H_n(M, \mathbb{Z})$ be the Hurewicz map. For every
\( \alpha, \beta \in h(\pi_n(M)) \subset H_n(M, \mathbb{Z}) \) with intersection number \( \alpha \cdot \beta = 1 \), by Whitney’s embedding theory and Whitney’s trick ([10], p142), there are two embedding \( n \)-spheres \( f_\alpha, f_\beta : S^n \hookrightarrow M^{2n} \) with:

(1). The homology elements \( \alpha \) and \( \beta \) are represented by \( f_\alpha, f_\beta \), i.e. \( (f_\alpha)_*[S^n] = \alpha, \ (f_\beta)_*[S^n] = \beta \)

(2). The spheres \( f_\alpha(S^n) \) and \( f_\beta(S^n) \) intersect transversally at only one point.

Following the geometric idea of removing handles, the next question is how to determine the normal bundles. In general, the normal bundles of \( f_\alpha(S^n), \ f_\beta(S^n) \) are not easy to determine.

In this paper, the situation seems relatively simpler: let \( K \subset h(\pi_n(M)) \) be a free Abel group such that each element \( \alpha \in K \) can be represented by an embedded \( n \)-sphere \( f_\alpha : S^n \hookrightarrow M \) with stable trivial normal bundle.

When \( n \) is even, for the embedding \( f_\alpha \) representing \( \alpha \in K \), the normal bundle of \( f_\alpha \) is just determined by the self-intersection number \( \alpha \cdot \alpha \) of \( \alpha \). Indeed, \( \alpha \cdot \alpha = 0 \) if and only if the normal bundle of \( f_\alpha \) is trivial. So, if we could find a free subgroup \( \oplus_{i=1}^s (\mathbb{Z} \alpha_i \oplus \beta_i) \) of \( K \) with intersection matrix \( \oplus_{i=1}^s \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \), topologically, \( M \) admits a decomposition:

\[
M \cong M'^s \ast (S^n \times S^n)
\]

When \( n \) is odd, the intersection number \( \alpha \cdot \alpha \) is always zero and can not determine the normal bundle of \( f_\alpha \), we need two techniques.

**Technique 1:** find a quadratic function \( \psi : K \to \mathbb{Z}_2 \) with:

(1). \( \psi(\alpha + \beta) = \psi(\alpha) + \psi(\beta) + (\alpha \cdot \beta)_2 \), where \( (\alpha \cdot \beta)_2 \in \mathbb{Z}_2 \) is the mod 2 class of the intersection number \( \alpha \cdot \beta \in \mathbb{Z} \), which is also the definition of the quadratic function over \( \mathbb{Z} \).

(2). \( \psi(\alpha) = 0 \) if and only if \( \alpha \) can be represented by an embedded \( n \)-sphere \( f_\alpha \) with trivial normal bundle.

For any free subgroup \( \oplus_{i=0}^s (\mathbb{Z} \alpha'_i \oplus \beta'_i) \) of \( K \) with intersection matrix \( \oplus_{i=0}^s \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \), by the standard results of the quadratic function ([15], p172), we can find a new basis \( \{ \alpha_i, \beta_i \} \) of this subgroup such that the intersection matrix of \( \mathbb{Z} \alpha_0 \oplus \mathbb{Z} \beta_0 \oplus \{ \oplus_{i=1}^s (\mathbb{Z} \alpha_i \oplus \beta_i) \} \) is still \( \oplus_{i=0}^s \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \) and \( \psi(\alpha_i) = \psi(\beta_i) = 0, i \neq 0, \psi(\alpha_0) = \psi(\beta_0) = 0 \) or 1. In this case, although we can not determine the value of \( \psi(\alpha_0) \), at least, we have decomposition:

\[
M \cong M'^s \ast (S^n \times S^n)
\]

In general, the quadratic function \( \psi \) is not always exist on \( K \) and we need the second technique.

**Technique 2:** find an embedded \( n \)-sphere \( g : S^n \hookrightarrow M \) with:

(1). \( g_*[S^n] = 0 \in H_n(M, \mathbb{Z}) \).

(2). The normal bundle \( \eta_g \) of \( g : S^n \hookrightarrow M \) is isomorphic to the tangent bundle \( TS^n \) of \( S^n \).

If we could find such an embedding \( g \), for every element \( \alpha \in K \) which is represented by an embedding \( f_\alpha \), if the stable trivial normal bundle \( \eta_{f_\alpha} \) is not trivial, by Wall’s technique ([15], p167), there exists a new embedding \( f'_\alpha \) with normal bundle \( \eta_{f'_\alpha} \) such that:
Ker

It is known that by an embedding with trivial normal bundle. So, for any free subgroup with intersection matrix

\[ \oplus \]

Function, (cf [the original definition of Kervaire, which is defined by the Arf invariant of a certain quadratic

\[ (1). \]

Proposition 3.2.

For any

\[ \mathbb{R}_q \subset W^{2n+q}, \]

W connected and \( y \in H_{n+1}(W, \mathbb{M}, \mathbb{Z}_2) \), we can find \( N \subset M \) representing \( \partial y \in H_n(M, \mathbb{Z}_2) \), i.e \( [N] = \partial y \) with \( N = \partial V \), here \( i : V \subset W \times [0, 1] \) is a connected submanifold with \( i_*[V] = y \), where \( [V] \in H_{n+1}(V, \partial V, \mathbb{Z}_2) \) is the fundamental class. Furthermore, \( V \) meets \( W \times 0 \) transversally in \( N \subset M \).

3 Odd case

3.1 Wu class, quadratic function, and Kervaire invariant

Given a smooth manifold \((M^n, \partial M)\), the Steenrod operator \( Sq = \sum_{i=0} S q^i : H^*(M, \partial M, \mathbb{Z}_2) \rightarrow H^*(M, \partial M, \mathbb{Z}_2) \) determines a linear form on \( H^*(M, \partial M, \mathbb{Z}_2) \):

\[ H^*(M, \partial M) \rightarrow \mathbb{Z}_2 \]

\[ x \mapsto < Sq(x), [M] > \]

where \([M] \in H_n(M, \partial M, \mathbb{Z}_2)\) is the fundamental class of the Poincaré pair \((M, \partial M)\). Since the cup product induces the isomorphism \( H^*(M, \mathbb{Z}_2) \cong Hom(H^*(M, \partial M, \mathbb{Z}_2), \mathbb{Z}_2) \), there exists a unique element \( v(M) = 1 + v_1(M) + v_2(M) + \cdots \in H^*(M, \mathbb{Z}_2) \) such that for each \( x \in H^*(M, \partial M, \mathbb{Z}_2) \):

\[ < v(M) \cup x, [M] > = < Sq(x), [M] >, \quad < v_1(M) \cup x, [M] > = < Sq^1 x, [M] > \]

Definition 3.1. \( v(M) = \sum_{i=0} v_i(M) \) is called the Wu class of \( M \).

By the definition, we see \( v_i(M) = 0 \iff Sq^i : H^{n-i}(M, \partial M, \mathbb{Z}_2) \rightarrow \mathbb{Z}_2 \) is zero.

In his paper [3], Browder gave a geometric definition of Kervaire invariant, which is equivalent to his original definition of Kervaire invariant in [1]. This geometric definition is very close to the original definition of Kervaire, which is defined by the Arf invariant of a certain quadratic function, (cf [7]). First, it is known in [3] that:

Proposition 3.2. For any \( x \in H_n(M^{2n}, \mathbb{Z}_2) \), we can find an embedded \( N^n \subset M^{2n} \) with \([N] = x\).

Proposition 3.3. If \( M^{2n} \times \mathbb{R}_q \subset W^{2n+q}, \) \( W \) connected and \( y \in H_{n+1}(W, \mathbb{M}, \mathbb{Z}_2) \), we can find \( N \subset M \) representing \( \partial y \in H_n(M, \mathbb{Z}_2) \), i.e \([N] = \partial y \) with \( N = \partial V \), here \( i : V \subset W \times [0, 1] \) is a connected submanifold with \( i_*[V] = y \), where \([V] \in H_{n+1}(V, \partial V, \mathbb{Z}_2) \) is the fundamental class. Furthermore, \( V \) meets \( W \times 0 \) transversally in \( N \subset M \).
We see, in this case, the normal bundle of \( N \) in \( W \times 0 \) has a normal \( q \)-frame \((N \times \mathbb{R}^q \subset M \times \mathbb{R}^q \subset W)\). The obstruction to extending this frame to a normal \( q \)-frame on \( V \subset W \times [0,1] \) lies in \( H^{i+1}(V, N, \pi_i(V_{n+q,q})) \) and \( \pi_i(V_{n+q,q}) = 0, q < n, \pi_n(V_{n+q,q}) = \mathbb{Z}_2 \). We find the last and only one obstruction \( \sigma \in H^{n+1}(V, N, \pi_n(V_{n+q,q})) = \mathbb{Z}_2 \).

**Definition 3.4.** For the element \( x = \partial y \in H_n(M, \mathbb{Z}_2) \), where \( \partial : H_{n+1}(W, M, \mathbb{Z}_2) \to H_n(M, \mathbb{Z}_2) \) and \( y \in H_{n+1}(W, M, \mathbb{Z}_2) \), we define: \( \psi(x) = \langle \sigma, [V] \rangle \), which is denoted briefly by \( \sigma \) for convenience.

We see this definition seems not intrinsic, it depends on the choice of \( N \) and \( V \). We should put some condition to make \( \psi \) well-defined. Browder proved ([3]):

**Proposition 3.5.** The obstruction to extend a \( q \)-frame defines a quadratic form:

\[
\psi : \text{Ker}(H_n(M, \mathbb{Z}_2) \to H_n(W, \mathbb{Z}_2)) \to \mathbb{Z}_2
\]

if and only if \( v_{n+1}(W) = 0 \).

**Proposition 3.6.** For the embedding \( \phi(S^n) \in M^{2n}, n \text{ odd} \), if \( \phi(S^n) \) is nullhomotopic in \( W \), then \( \psi([\phi(S^n)]) = 0 \) if and only if the normal bundle of \( \phi(S^n) \) is trivial.

**Definition 3.7.** If \( \text{Ker}(H_n(M, \mathbb{Z}_2) \to H_n(W, \mathbb{Z}_2)) \) is non-singular under the intersection pair, we define the Kervaire invariant \( k \) by its Arf invariant of the quadratic form \( \psi \).

### 3.2 Proof of the odd case I

Let \( X^{n+1} \) be a projective toric manifold with complex dimension \( n > 2 \), odd, and \( i : Y^n \hookrightarrow X^{n+1} \) be a hypersurface of \( X^{n+1} \).

**Lemma 3.8.** \( H_n(Y, \mathbb{Z}) \) is spherical and every element \( \alpha \in H_n(Y, \mathbb{Z}) \) can be represented by an embedding \( f_\alpha : S^n \hookrightarrow Y \) such that the normal bundle \( \eta_{f_\alpha} \) of \( f_\alpha \) is stable trivial.

**Proof.** First, by Lefschetz’s hyperplane section theorem and Proposition 1.2., we know \( (X, Y) \) is \( n \)-connected and \( H_n(X) = 0 \). We have exact sequence:

\[
\begin{align*}
H_{n+1}(X, Y) & \longrightarrow H_n(Y) \longrightarrow H_n(X) = 0 \\
\pi_{n+1}(X, Y) & \longrightarrow \pi_n(Y) \longrightarrow \pi_n(X)
\end{align*}
\]

From this diagram, we observe that \( h_Y : \pi_n(Y) \to H_n(Y) \) is surjective and for every element \( \alpha \in H_n(Y) \), by the Whitney embedding theorem, we can choose an embedding \( f_\alpha : S^n \hookrightarrow Y \) to represent \( \alpha \) such that \( i \circ f_\alpha \) is nullhomotopic in \( X \), i.e. \( \pi_n(i)[f_\alpha] = 0 \).

Second, we want to show the normal bundle \( \eta_{f_\alpha} \) is stable trivial. We have bundle identity:

\[
TX|S^n = (i \circ f_\alpha)^*TX = TS^n \oplus \eta_{f_\alpha} \oplus \eta_X|S^n
\]

here \( \eta_X \) is the normal bundle of \( i : Y \longrightarrow X \). Since \( \eta_X \) is a complex line bundle, it is known that \( \eta_X \cong i^*L_Y \), where \( L_Y \) is a complex line bundle over \( X \) with Euler class \( e(L_Y) \cap [X] = i_*[Y] \).
Since \( i \circ f_\alpha \) is nullhomotopic, the bundle identity becomes:
\[
\epsilon^{2n+2} = (i \circ f_\alpha)^* TX = TS^n \oplus \eta_{f_\alpha} \oplus (i \circ f_\alpha)^* L_Y = \epsilon^{n+1} \oplus \eta_{f_\alpha}
\]

here \( \epsilon \) is the trivial real 1-bundle.

\[\Box\]

**Proof of the odd case I**: For the complex line bundle \( L_Y \) in the above lemma, consider \( W = D(-L_Y) \), where \(-L_Y\) is the stable inverse bundle of \( L_Y \), i.e. \( L_Y \oplus -L_Y \) is trivial, and \( D(-L_Y) \) is the disk bundle of \(-L_Y\). Then for the embedding: \( Y \hookrightarrow X \hookrightarrow W \), we see the normal bundle of \( Y \) in \( W \) is trivial and we get \( Y \times \mathbb{R}^q \subset W \) for some \( q > 0 \).

Observe that \( \text{Ker}(H_n(Y, \mathbb{Z}_2) \to H_n(W, \mathbb{Z}_2)) = H_n(Y, \mathbb{Z}_2) \) and by proposition 3.5, if the Wu class \( v_{n+1}(W) = 0 \), there is a quadratic function \( \psi' : H_n(Y, \mathbb{Z}_2) \to \mathbb{Z}_2 \) and we also obtain a quadratic function on \( H_n(Y, \mathbb{Z}) \):

\[
\psi : H_n(Y, \mathbb{Z}) \longrightarrow H_n(Y, \mathbb{Z}_2) \xrightarrow{\psi'} \mathbb{Z}_2
\]

Furthermore, by proposition 3.6, we know \( \psi(\alpha) = 0 \) if and only if the normal bundle \( \eta_{f_\alpha} \) is trivial.

Since \( H_n(Y, \mathbb{Z}) \) is unimodular, by technique 2 in subsection 2.2, \( H_n(Y, \mathbb{Z}) \cong \mathbb{Z} \alpha_0 \oplus \mathbb{Z} \beta_0 \oplus \oplus_{i=1}^s (\mathbb{Z} \alpha_i \oplus \beta_i) \), \( s = b_n(Y') - 2 \) with intersection matrix \( \oplus_{i=0}^s \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and \( \psi(\alpha_i) = \psi(\beta_i) = 0, i \neq 0 \). Topologically, we get decomposition:

\[
Y \cong Y' \sharp s(S^n \times S^n)
\]

where \( b_n(Y') = 2 \). If the Kervaire (Arf) invariant \( k \) of \( \psi \) or \( \psi' \) vanishes, we can make \( b_n(Y') = 0 \).

So we finish the proof of the odd case when \( v_{n+1}(W) = 0 \). When \( v_{n+1}(W) \neq 0 \), the quadratic function \( \psi \) is not necessary well-defined and we will use technique 2 to deal with it in next subsection.

### 3.3 Proof of the odd case II

In his paper [3], Browder proved:

**Theorem 3.9** (Browder). Suppose \( M^{2n} \times \mathbb{R}^q \subset W, n \neq 1, 3 \) or 7, \( W \) is 1-connected. \((W, M)\) is \( n \)-connected and suppose \( v_{n+1}(W) \neq 0 \). Then there exists an embedded \( S^n \subset M^{2n} \) and \( U^{n+1} \subset M^{2n} \times \mathbb{R}^{q+1} \) with \( \partial U = S^n \) such that the normal bundle \( \xi \) to \( S^n \) in \( M^{2n} \) is non-trivial, but \( \xi \oplus \epsilon^1 \) is trivial, where \( \epsilon^1 \) is the trivial one dimensional real vector bundle. Hence \( S^n \) is homologically trivial (mod 2) with nontrivial normal bundle.

**Remark 3.10.** It seems we can use this theorem to find the embedding sphere in technique 2. But the shortage is: the embedding sphere \( S^n \) is only mod 2 trivial. We want to add some condition to make it work in integral homology.

**Theorem 3.11.** Under the same hypothesis of Browder’s theorem above, if we further assume that \( H_{n+1}(W, \mathbb{Z}_2) \) is generated by the element \( \{x_i\} \) such that each \( x_i \) can be represented by an oriented closed manifold \( N_i \), i.e \([N_i] = x_i\). Then there exist an embedded \( S^n \subset M^{2n} \) such that \([S^n] = 0 \in H_n(M, \mathbb{Z}) \) and the normal bundle \( \xi \) to \( S^n \) in \( M^{2n} \) is non-trivial but stable trivial.
Proof. We follow Browder’s proof (Step 2 to Step 7 is almost unchanged):

Step 1: Since \(v_{n+1}(W) \neq 0\), we know \(Sq^{n+1} : H^{n+q-1}(W, \partial W, \mathbb{Z}_2) \to \mathbb{Z}_2\) is not zero. By assumption, \(\exists N_i\) such that \(Sq^{n+1}y_i \neq 0\), where \(y_i \cap [W] = [N_i]\).

Step 2: For convenience, we denote \(N_i\) by \(N\) and \(y_i\) by \(y\). Let \(N_0 = N - intD^{n+1}\), we see \(\partial N_0 = S^{n+1}\) and \(N_0\) is homotopic to an \(n\)-complex. Since \((W, M)\) is \(n\)-connected, \(\exists f : N_0 \to M\) such that the diagram is commutative up to homotopy:

\[
\begin{array}{ccc}
N_0 & \xrightarrow{f} & M \\
\downarrow & & \downarrow \\
N & \xrightarrow{\partial} & W
\end{array}
\]

Step 3: Let \(g = f|_{\partial N_0} : S^n \to M\), we see \(g_*[S^n] = 0 \in H_n(M, \mathbb{Z})\). Since \(M\) is 1-connected, by Whitney’s theorem, we can make \(g\) homotopic to an embedding and we still denote it by \(g\). Since \(g\) is nullhomotopic in \(W\), then the normal bundle of \(g(S^n)\) is stable trivial. We wish to show that the normal bundle of this sphere is not trivial.

Step 4: We make the map \(f : N_0 \to M \times \mathbb{R}^q \times [-1, 0]\) homotopic to an embedding \(g_0 : N_0 \to M \times \mathbb{R}^q \times [-1, 0]\) such that \(g_0|_{\partial N_0} = g\). And we extend \(g : S^n \to M\) to \(\tilde{g} : D^{n+1} \subset W \times [0, 1]\) which meets \(W \times 0\) transversally in \(g(S^n)\). Then we get an embedding \(g_1 : N_0 \cup_{S^n} D^{n+1} \cong N \to W \times [-1, 1]\), which is isotopic to the origin \(N \subset W\).

Step 5: \(M \times \mathbb{R}^q \subset W \times 0\) define a \(q\)-frame of the normal bundle of \(g(S^n) \subset M \subset W \times 0\). We know the obstruction \(\sigma \in \pi_{n+1}(V_{n+1,q})\) to extend this \(q\)-frame to \(D^{n+1}\) is zero if and only if the normal bundle of \(g(S^n)\) is trivial.

Step 6: Now assume the normal bundle of \(g(S^n)\) is trivial and we get a \(q\)-frame on \(D^{n+1}\):

\[D^{n+1} \times D^n \times \mathbb{R}^q \subset W \times [0, 1]\]

such that \(D^{n+1} \times D^n \times 0 = \tilde{g}(D)\) and \(S^n \times D^n \times 0\) is the normal bundle of \(g(S^n)\). Let \(V = M \times [-1, 0] \cup_{S^n \times D^n} D^{n+1} \times D^n\), then we get \(V \times \mathbb{R}^q \subset W \times [-1, 1]\), \(g_1(N) \subset intV \times \mathbb{R}^q\).

Step 7: Let \(Y = W \times [-1, 1]/\partial(W \times [-1, 1])\) we get:

\[Y \xrightarrow{a} \Sigma^q V/\partial V \xrightarrow{b} T(\eta_N \oplus \epsilon)\]

where \(\eta_N\) is the normal bundle of \(N\) in \(W\). Let \(U\) be the mod 2 Thom class of \(\eta_N \oplus \epsilon\), we get: \((ba)^*U = \Sigma x \in H^{n+q}(Y, \mathbb{Z}_2)\) and \(\langle Sq^{n+1}(x)\rangle[W] = Sq^{n+1}(\Sigma x)[Y] \neq 0\). Also, \(\langle Sq^{n+1}(b^*U)\rangle[\Sigma^q[V]] \neq 0\) and \(\langle Sq^{n+1}(\Sigma^{-q}(b^*U))\rangle[V] \neq 0\). On the other hand, \(\langle Sq^{n+1}(\Sigma^{-q}(b^*U))\rangle = 0\) since \(\Sigma^{-q}(b^*U)) = 0\) in \(H^n(V, \partial V, \mathbb{Z}_2)\).

Proof of the odd case II: When \(v_{n+1}(W) \neq 0\), in our case \(Y \times \mathbb{R}^q \subset W = D(-L)\), by proposition 1.3, we see \(H_\ast(X, \mathbb{Z})\) is generated by the toric submanifolds which are certainly oriented and \(W = D(-L_Y)\) is the disk bundle over \(X\), whose homology group is also generated by these toric submanifolds. Then all the conditions of theorem 3.11 are satisfied. Thus, there exists an embedding sphere \(g : S^n \to Y\) such that \(g_*[S^n] = 0\) and the normal bundle \(\eta_0 \cong TS^n\).

By the technique 2 in section 2 and lemma 3.8, we have topological decomposition:

\[Y \cong Y' \# s(S^n \times S^n)\]

where \(b_0(Y') = 0\). Then we finish the proof of theorem 1.4.
4 Even case

4.1 Intersection form and signature

Let $X^{n+1}$ be a projective toric manifold with complex dimension $n + 1$, $n > 2$, even, and $i : Y \hookrightarrow X$ be a hypersurface of $X$. Since $n$ is even, the $n$-th homology group $H_n(Y, \mathbb{Z})$ admits a unimodular symmetric intersection form:

$$ H_n(Y, \mathbb{Z}) \otimes H_n(Y, \mathbb{Z}) \longrightarrow \mathbb{Z} $$

Since $(X, Y)$ is $n$-connected and $H_{odd}(X, \mathbb{Z}) = 0$, like the odd case, we have

$$ 0 \longrightarrow H_{n+1}(X, Y) \longrightarrow H_n(Y) \overset{i_*}{\longrightarrow} H_n(X) \longrightarrow 0 $$

The vanishing cycles $\text{Ker}(i_*) \subset H_n(Y, \mathbb{Z})$ is what we mainly concerned, because:

**Lemma 4.1.** Each element $\alpha \in \text{Ker}(i_*)$ can be represented by an embedding $f_\alpha : S^n \hookrightarrow Y$ such that $f_\alpha[S^n] = \alpha$ and the normal bundle $\eta_{f_\alpha}$ of $f_\alpha$ is stable trivial.

**Proof.** Since $\pi_n(X, Y) \cong H_n(X, Y, \mathbb{Z}) \cong \text{Ker}(i_*)$, we see for each element $\alpha \in \text{Ker}(i_*)$, there exists an embedding $f_\alpha$ representing $\alpha$ and $\pi_n(i)(f_\alpha) = 0 \in \pi_n(X)$.

The proof of the stable triviality of the normal bundle $\eta_{f_\alpha}$ is similar to the proof in lemma 3.8. □

When we restrict the intersection form of $H_n(Y, \mathbb{Z})$ on $\text{Ker}(i_*)$, we get:

**Proposition 4.2.** The intersection form on $\text{Ker}(i_*)$ is of type even, i.e. for any $\alpha \in \text{Ker}(i_*)$, $\alpha \cdot \alpha$ is even.

**Proof.** For any $\alpha \in \text{Ker}(i_*)$, by lemma 4.1., we can use an embedding $f_\alpha$ to represent it. It is known that $\alpha \cdot \alpha = < e(\eta_{f_\alpha}), [S^n] >$, where $e(\eta_{f_\alpha})$ is the Euler class of the normal bundle $\eta_{f_\alpha}$. Furthermore, $< e(\eta_{f_\alpha}), [S^n] >$ is even if and only if the $n$-th Stiefel-Whitney class $w_n(\eta_{f_\alpha})$ of $\eta_{f_\alpha}$ is zero and this is just proved in lemma 4.1. □

The intersection pair on $H_n(Y, \mathbb{Z})$ is equivalent to the cup product on $H^n(Y, \mathbb{Z})$

$$ H^n(Y, \mathbb{Z}) \otimes H^n(Y, \mathbb{Z}) \longrightarrow \mathbb{Z} $$

$$(\alpha, \beta) \mapsto < \alpha \cup \beta, [Y] >$$

through the Poincaré duality $PD : H^n(Y, \mathbb{Z}) \longrightarrow H_n(Y, \mathbb{Z})$ and we also have exact sequence:

$$ 0 \longrightarrow H^n(X, \mathbb{Z}) \overset{i^*}{\longrightarrow} H_n(Y, \mathbb{Z}) \longrightarrow H^{n+1}(X, Y) \longrightarrow 0 $$

We see the intersection form $(\text{Ker}(i_*), \cdot)$ is equivalent to $(PD^{-1}(\text{Ker}(i_*)), \cup)$ and the reason why we use the language of cohomology instead of homology is:
Lemma 4.3. \( PD^{-1}(\text{Ker}(i_*)) = (i^*H^n(X))^\perp \)

Proof. For any \( \alpha \in \text{Ker}(i_*) \) and \( \beta \in H^n(X, \mathbb{Z}) \), we have:

\[
< PD^{-1}(\alpha) \cup i^*\beta, [Y] >= < i^*\beta, \alpha > = < \beta, i_*\alpha >= 0
\]

we get \( PD^{-1}(\text{Ker}(i_*)) \subset (i^*H^n(X))^\perp \).

On the other hand, for any \( PD^{-1}(\gamma) \in (i^*H^n(X))^\perp \), we see

\[
< PD^{-1}(\gamma) \cup i^*H^n(X, \mathbb{Z}), [Y] >= < i^*H^n(X, \mathbb{Z}), \gamma >= < H^n(X, \mathbb{Z}), i_*\gamma >= 0
\]

Since \( H_n(X, \mathbb{Z}) \) and \( H^n(X, \mathbb{Z}) \) are free Abel groups, we get \( i_*\gamma = 0 \).

Next, we want to discuss some limit estimates about the \( n \)-th Betti number and the signature of the pair \( (H^n(Y_d, \mathbb{Z}), \cup) \). Recall that \( i_d : Y_d \hookrightarrow X^{n+1} \) is the hypersurface of the toric manifold \( X \) with \( (i_d)_*[Y_d] = d(i_*[Y]) \) and \( \deg Y_d = < \alpha_{Yd}^{n+1}, [X] >= d^{n+1} \deg Y \), where \( \alpha_{Yd} \cap [X] = (i_d)_*[Y_d] \).

We have proposition:

Proposition 4.4. We have limits:

\[
\lim_{d \to +\infty} \frac{b_n(Y_d)}{\deg Y_d} = \lim_{d \to +\infty} \frac{b_n(Y_d)}{d^{n+1} \deg Y} = 1
\]

\[
0 < \lim_{d \to +\infty} \frac{|\text{sign}(Y_d)|}{b_n(Y_d)} = 2^{n+2}(2^{n+2} - 1) \frac{B_{n+2}}{(n+1)!} < 1
\]

Proof. For the first limit, we know the Euler number \( \chi(Y_d) \) of \( Y_d \) equals \( b_n(Y_d) + 2 \sum_{j=1}^{n-1} (-1)^j b_j(X) \) and

\[
\chi(Y_d) = < c_n(Y_d), [Y_d] > = < \frac{c(TX)}{1 + \deg Y}, [X] > = d^{n+1} < \alpha_{Y}^{n+1}, [X] > + O(d^n)
\]

here \( c(TX) \) and \( c_n(Y_d) \) are the Chern classes. We have:

\[
\lim_{d \to +\infty} \frac{\chi(Y_d)}{d^{n+1}} = \lim_{d \to +\infty} \frac{b_n(Y_d)}{d^{n+1} \deg Y} = \deg Y
\]

and we get:

\[
\lim_{d \to +\infty} \frac{b_n(Y_d)}{\deg Y_d} = 1
\]

For the second limit, we have identity:

\[
\text{sign}(Y_d) = < \tanh(\deg Y)L(X), [X] >
\]

where \( L(X) = L_1(X) + L_2(X) + \cdots \) is the \( L \)-class of \( X \) and \( \tanh(\deg Y) = \sum_{j=1}^{+\infty} (-1)^j 2^{2j} (2^2j - 1) \frac{B_{2j}}{(2j)!} (\deg Y)^{2j-1} \). Observe that:

\[
\text{sign}(Y_d) = (-1)^{\frac{n}{2}} 2^{n+2} (2^{n+2} - 1) \frac{B_{n+2}}{(n+2)!} d^{n+1} \deg Y + O(d^n)
\]
we have limit:

\[
\lim_{d \to +\infty} \frac{|\text{sign}(Y_d)|}{b_n(Y_d)} = 2^{n+2}(2^{n+2} - 1) \frac{B_{n+2}}{(n+1)!}
\]

Furthermore, when \( j > 1 \), we see:

\[
1 + \frac{1}{2^{2j}} + \frac{1}{3^{2j}} + \cdots = \frac{B_j(2\pi)^{2j}}{2^{2j}j!} < \frac{\pi^2}{6}
\]

\[
\frac{B_j 2^{2j} (2^{2j} - 1)}{(2j)!} < \frac{\pi^2}{3} \frac{2^{2j} (2^{2j} - 1)}{(2\pi)^{2j}} < \frac{\pi^2}{3} \frac{4^j}{\pi^{2j}} < 1
\]

\[\square\]

**Corollary 4.5.** \( \lim_{d \to +\infty} b_n(Y_d) = +\infty \) and \( \lim_{d \to +\infty} \text{sign}(Y_d) = +\infty \). When \( d \) is big enough, \((H_n(Y_d, \mathbb{Z}), \cdot)\) is indefinite.

### 4.2 Proof of the even case

Let \((H, <, >)\) be a unimodular symmetric bilinear form over \(\mathbb{Z}\) and \(F\) be a nonzero subgroup of \(H\) such that \(H/F\) is free and the map \(F \to \text{Hom}(F, \mathbb{Z})\) induced by \(<, >\) is injective. Denote \(E = F^\perp := \{x \in H | <x, F> = 0\}\), we have:

**Theorem 4.6.** If \(\text{rank} H \geq \text{Max}\{4\text{rank} F, 2\text{rank} F + 5\}\), then \(E\) admits an orthogonal decomposition:

\[(E, <, >) \cong (A, <, >) \oplus (\oplus_{i=1}^s (\mathbb{Z}x_i \oplus \mathbb{Z}y_i, <, >))\]

where the intersection matrix of \(\mathbb{Z}x_i \oplus \mathbb{Z}y_i\) is \[
\begin{pmatrix}
0 & 1 \\
1 & c_i
\end{pmatrix},
\]

\(c_i = 0\) or \(1\). For \((A, <, >)\), there are two possibilities:

1. \((A, <, >)\) is definite and \(\text{rank} A \geq \text{max}\{3\text{rank} F, \text{rank} F + 5\}\)
2. \(\text{rank} A < \text{max}\{3\text{rank} F, \text{rank} F + 5\}\)

We’ll prove the even case first and the proof of this theorem will be given in the next subsection.

**Proof of the even case:**

**Step 1:** For the bilinear symmetric space \((H^n(Y_d, \mathbb{Z}), \cup)\), we know \(PD^{-1}(\text{Ker}(i_d)_*)) = (i^*H^n(X))^\perp\). We want to show the injectivity of the map \(H^n(X, \mathbb{Z}) \to \text{Hom}(H^n(X, \mathbb{Z}), \mathbb{Z})\) induced by the cup product in \(H^n(Y_d, \mathbb{Z})\).

Since \(Y_d\) is the hypersurface of \(X\), the hard Lefschetz theorem ([11]) tell us that the cohomology element \(\alpha_{Y_d}\) representing \(Y_d\) induces an injective map:

\[
\cup \alpha_{Y_d} : H^n(X, \mathbb{Z}) \longrightarrow H^{n+2}(X, \mathbb{Z})
\]

For \(i^*H^n(X, \mathbb{Z}) \subset H^n(Y_d, \mathbb{Z})\), we have diagram:

\[
\begin{array}{ccc}
H^n(X, \mathbb{Z}) & \xrightarrow{\cup \alpha_{Y_d}} & H^{n+2}(X, \mathbb{Z}) & \xrightarrow{\cong} & \text{Hom}(H^n(X, \mathbb{Z}), \mathbb{Z}) \\
\downarrow i_d^* & & & & \\
i_d^* H^n(X, \mathbb{Z}) & \longrightarrow & H^n(Y_d, \mathbb{Z}) & \xrightarrow{\cong} & \text{Hom}(H^n(Y_d, \mathbb{Z}), \mathbb{Z})
\end{array}
\]
indeed, for any \( x, y \in H^n(X, \mathbb{Z}) \), \( x(y) = i^*x \cup i^*y, [Y_d] \rangle = \langle x \cup y \cup \alpha_{Y_d}, [X] \rangle = (x \cup \alpha_{Y_d})(y) \).

Furthermore, we see the restriction of \( (H^n(Y_d, \mathbb{Z}), \cup) \) to \( H^n(X, \mathbb{Z}) \) is just the bilinear form defined in theorem 1.5.

Thus we get a pair \((H_d, \cup) = (H^n(Y_d, \mathbb{Z}), \cup)\) with a free subgroup \( F := i^*H^n(X, \mathbb{Z}) \) such that

1. \( F^\perp = PD^{-1}(\text{Ker}(i_d)_*) = E_d \) with even type (proposition 4.2, lemma 4.3).
2. If \( d \) is big enough, \( \text{rank}\, H_d > \text{Max}\{4\text{rank}\, F, 2\text{rank}\, F + 5\} \) (proposition 4.4)

**Step 2:** By the algebraic decomposition theorem 4.6,

\( (\text{Ker}(i_d)_*, \cdot) \cong (E_d, \cup) \cong A_d \oplus (\bigoplus_{i=1}^{s_d} (\mathbb{Z}x_i \oplus \mathbb{Z}y_i, <, >)) \)

where the intersection matrix of \( \mathbb{Z}x_i \oplus \mathbb{Z}y_i \) is \( \begin{pmatrix} 0 & 1 \\ 1 & c_i \end{pmatrix} \), \( c_i = 0 \) or 1

By proposition 4.2, \( \text{Ker}(i_d)_* \) is of type even, \( c_i \) must be zero. Since \( \lim_{d \to \infty} |\text{sign}\, H^n(Y_d, \mathbb{Z})| = +\infty \), when \( d \) is big enough, the possibility (2) of theorem 4.6 can not happen, and \( A_d \) is definite.

**Step 3:** By the process of removing handles of the even case in section 2, we see

\( \text{Ker}(i_d)_* = A_d \oplus \bigoplus_{i=1}^{s_d} (\mathbb{Z}x_i \oplus \mathbb{Z}y_i) \)

where the intersection matrix of \( \bigoplus_{i=1}^{s_d} (\mathbb{Z}x_i \oplus \mathbb{Z}y_i) \) is \( \bigoplus_{s_d} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and we get:

\[ Y_d \cong Y_d^{s_d} \mathbb{Z}^{s_d}(S^n \times S^n) \]

Since \( A_d \) is definite and \( \text{sign}(\mathbb{Z}x_i \oplus \mathbb{Z}y_i) = 0 \), we get identity \( 2s_d = \text{rank}(\text{Ker}(i_d)_*) - |\text{sign}(\text{Ker}(i_d)_*)| \). Also, since \( \text{Ker}(i_d)_* = (H^n(X, \mathbb{Z}))^\perp \subset H^n(Y_d, \mathbb{Z}) \) and the restriction of \( (H^n(Y_d, \mathbb{Z}), \cup) \) to \( H^n(X, \mathbb{Z}) \) is just the bilinear form defined in theorem 1.5, we get

\[ 2s_d = b_n(Y_d) - b_n(X) - |\text{sign}(Y_d) - \text{sign}(H^n(X))| \]

For the limit estimate, we have:

\[ \lim_{d \to +\infty} \frac{2s_d}{\deg Y_d} = \lim_{d \to +\infty} \frac{2s_d}{b_n(Y_d)} = 1 - \lim_{d \to +\infty} \frac{|\text{sign}(Y_d)|}{b_n(Y_d)} \]

### 4.3 Proof of the algebraic decomposition theorem

In order to prove theorem 4.6., we need some lemmas.

**Lemma 4.7.** Assume \( E \) satisfy \( \text{rank}\, E \geq 3\text{rank}\, F \), we can choose a basis \( \{f_1, f_2, \cdots, f_{r+h}\} \) of \( \text{Hom}(E, \mathbb{Z}) \) such that \( \bigoplus_{i=1}^{r} \mathbb{Z} f_i \to \text{Hom}(E, \mathbb{Z})/E \) is surjective, \( r \leq \text{rank}\, F \), and \( \bigoplus_{j=1}^{h} \mathbb{Z} f_{r+j} \subset E \subset \text{Hom}(E, \mathbb{Z}) \), \( h \geq 2r \).

**Proof.** First, since \( H/F \) is free, we have:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & E & \xrightarrow{<,>} & \text{Hom}(H, \mathbb{Z}) & \longrightarrow & \text{Hom}(F, \mathbb{Z}) & \longrightarrow & 0 \\
\| & & & \downarrow & & & \downarrow & & \\
0 & \longrightarrow & E & \xrightarrow{<,>} & \text{Hom}(E, \mathbb{Z}) & \longrightarrow & \text{Hom}(E, \mathbb{Z})/E & \longrightarrow & 0
\end{array}
\]
exists an indivisible element $x$ of the basis $Z$.

First, by the lemma above, we have a basis $\{x, y\}$ and $F$ make $Z$ such that $\text{Hom}_F$ see $f$ any $\mathbb{E}$ have an element $y$.

Assume $D \subseteq \mathbb{E}$ is definite, we can choose proper $\{c_i \in \mathbb{E}\}$ to make $D'$ indefinite, we can still find an indivisible element $x \in D'$ such that $x, x > 0$, and we can also find $y \in \oplus_{j=1}^{h} \mathbb{E}$ such that $x, y > 1$. So, all we need to do is to prove the next lemma.

**Lemma 4.8.** Assume $E$ is indefinite and $\text{rank}E \geq \text{Max}\{3\text{rank}F, \text{rank}F + 5\}$, we can find two elements $x, y \in E$ with $x, x > 0$, $x, y > 1$, $y, y > 0$ or $1$.

**Proof.** First, by the lemma above, we have a basis $\{f_1, \cdots, f_r, f_{r+1}, \cdots f_{r+h}\}$ of $\text{Hom}(E, \mathbb{E})$ with $Z$ is unimodular. Then we can find $f_{r+i} = f_{r+i} - \sum_{j=1}^{r} a_{ij} f_j$ with $[\sum_{j=1}^{r} a_{ij} f_j] = [f_{r+i}'] \in \text{Hom}(E, \mathbb{E})/E$, we see $f_{r+i} \in \mathbb{E}$.

Thus we obtain a basis $\{f_1 \cdots f_r, f_{r+1}, \cdots f_{r+h}\}$ of $\text{Hom}(E, \mathbb{E})$ such that $\{f_{r+1}, \cdots f_{r+h}\} \subset E \subset \text{Hom}(E, \mathbb{E})$.

Second, when $D$ is indefinite, since $\text{rank}D \geq 5$, it is known from Meyer’s theorem that there exists an indivisible element $x \in D$ such that $x, x > 0$ (8, p255). Then we can also choose an element $y' \in Zf_{r+1} + \cdots + Zf_{r+h} \subset \mathbb{E}$ such that $x, y' > 1$. Let $y = y' - \left\lfloor \frac{y'}{2} \right\rfloor x'$, we have $x, x > 0$, $x, y > 1$, $y, y > 0$ or $1$.

Third, when $D$ happens to be definite, define $D' := Zf_{r+1} - c_1 f_1 + Zf_{r+2} - c_2 f_1 + Z(f_{r+3} - c_3 f_2) + Zf_{r+4} - c_4 f_2 + \cdots + Z(f_{r+1} - c_2r_{r+1} f_r) + Zf_{r+p} - c_2r_{r+p} f_r + \cdots + Z(f_{r+1} - c_n f_n)$. If we can choose proper $\{c_i \in \mathbb{E}\}$ to make $D'$ indefinite, we can still find an indivisible element $x \in D'$ such that $x, x > 0$, and we can also find $y \in \oplus_{j=1}^{h} \mathbb{E}$ such that $x, y > 1$. So, all we need to do is to prove the next lemma.

**Lemma 4.9.** Following lemma 4.8, suppose $D$ is definite, we can choose proper $\{c_i \in \mathbb{E}\}$ to make $D'$ indefinite.

**Proof.** Assume $D$ is positive definite under $\langle, \rangle$. Consider the real space $\mathbb{E} := \mathbb{E} \otimes \mathbb{R}$, $\overline{D} := D \otimes \mathbb{R}$, $\overline{D'} := D' \otimes \mathbb{R}$ and let $\{f_1^*, \cdots, f_{r+h}^*\}$ be the Euclidean orthogonal standard basis of $\mathbb{E}$. Define:

$$F : \mathbb{E} \rightarrow \mathbb{R}$$

$$\sum a_i f_i^* \mapsto \sum a_i a_j < f_i^*, f_j^* >$$

Observe that $F$ is just the the extension of the map $E \rightarrow \mathbb{E}$, $x \mapsto <x, x>$ to $\mathbb{E}$.

Note that $E$ is indefinite and $\mathbb{Q}$-unimodular under $\langle, \rangle$, since by assumption $H$ is unimodular and $F$ is $\mathbb{Q}$-unimodular. Then we can find a $v \in \mathbb{E}$ such that $F(v) < 0$ and the Euclidean norm
$|v| = 1$, i.e. $v = \sum_{i=1}^{r} a_i f_i^* + \sum_{j=1}^{h} b_j f_{r+j}^*$, $\sum a_i^2 + \sum b_j^2 = 1$. Since $D$ is definite, we see $F(D - \{0\}) > 0$ and $(a_1, \ldots, a_r) \neq (0, \ldots, 0)$.

In the Euclidean norm with orthogonal standard basis $\{f_i^*\}$, we have decomposition $E = D' \oplus (D')^\perp$. By calculation, we see $(D')^\perp$ has a standard orthogonal basis:

$$( D')^\perp = \text{span}\{ \frac{f_1^* + c_1 f_{r+1}^* + c_2 f_{r+2}^*}{\sqrt{1 + c_1^2 + c_2^2}}, \frac{f_2^* + c_3 f_{r+3}^* + c_4 f_{r+4}^*}{\sqrt{1 + c_3^2 + c_4^2}}, \ldots, \frac{f_{r-1}^* + c_{2r-3} f_{3r-3}^* + c_{2r-2} f_{3r-2}^*}{\sqrt{1 + c_{2r-3}^2 + c_{2r-2}^2}}, \frac{f_r^* + c_{2r-1} f_{3r-1}^* + c_{2r} f_{3r}^* + \cdots, c_h f_{r+h}^*}{\sqrt{1 + c_{2r-1}^2 + \cdots + c_h^2}} \}$$

For convenience, denote this basis by $\{g_1, g_2, \cdots, g_r\}$.

For the vector $v = \sum_{i=1}^{r} a_i f_i^* + \sum_{j=1}^{h} b_j f_{r+j}^*$, we can decompose $v = v_1 + v_2$ such that $v_1 \in D'$ and $v_2 \in (D')^\perp$. By calculation,

$$v_2 = \frac{a_1 + c_1 b_{r+1} + c_2 b_{r+2}}{\sqrt{1 + c_1^2 + c_2^2}} g_1 + \cdots + \frac{a_{r-1} + c_{2r-3} b_{3r-3} + c_{2r-2} b_{3r-2}}{\sqrt{1 + c_{2r-3}^2 + c_{2r-2}^2}} g_{r-1} + \frac{a_r + c_{2r-1} b_{3r-1} + c_{2r} b_{3r} + \cdots, c_h b_{r+h}}{\sqrt{1 + c_{2r-1}^2 + \cdots + c_h^2}} g_r$$

Since $\sum a_i^2 + \sum b_j^2 = 1$, for $\forall \epsilon > 0$, we can choose proper $c_i \in \mathbb{Z}$ ([8], p256) such that

$$\left| \frac{a_1 + c_1 b_{r+1} + c_2 b_{r+2}}{\sqrt{1 + c_1^2 + c_2^2}} \right| < \frac{\epsilon}{r}, \ldots, \left| \frac{a_{r-1} + c_{2r-3} b_{3r-3} + c_{2r-2} b_{3r-2}}{\sqrt{1 + c_{2r-3}^2 + c_{2r-2}^2}} \right| < \frac{\epsilon}{r}, \left| \frac{a_r + c_{2r-1} b_{3r-1} + c_{2r} b_{3r} + \cdots, c_h b_{r+h}}{\sqrt{1 + c_{2r-1}^2 + \cdots + c_h^2}} \right| < \frac{\epsilon}{r}$$

The function $F$ is continuous and $F(v) < 0$, if the Euclidean norm of $v_2 = v - v_1$ is small enough, then the element $v_1 \in (D')^\perp$ satisfy $F(v_1) < 0$. Furthermore, $D'$ is not negative definite, since $D$ is positive and $\text{rank} D' = \text{rank} D, 2\text{rank} D > \text{rank} E = \text{rank} D + \text{rank} F$, thus we see $D'$ is indefinite.

Proof of theorem 4.6.: We use induction on $\text{rank} H$, since $\text{rank} H \geq \text{Max}\{4\text{rank} F, 2\text{rank} F + 5\}$, we get $\text{rank} E \geq \text{Max}\{3\text{rank} F, \text{rank} F + 5\}$. If $E$ is definite, we’re done. If $E$ is indefinite, then by the lemmas we’ve just proved, there exist two elements $x, y \in E$ such that $\langle x, x \rangle = 0, \langle x, y \rangle = 1, \langle y, y \rangle = 0$ or 1.

We get orthogonal decomposition under $\langle, \rangle$:

$$H = H' \oplus (\mathbb{Z}x \oplus \mathbb{Z}y), \ E = E' \oplus (\mathbb{Z}x \oplus \mathbb{Z}y)$$

Observe that $F \subset H'$ and $E \cap H' = E' = F^\perp \subset H'$, by the induction, we’ve finished our proof.
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