On the determination of the $\text{grad} - \text{div}$ criterion

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Abstract

Grad-div stabilization, adding a term $-\gamma \text{grad} \text{div} u$, has proven to be a useful tool in the simulation of incompressible flows. Such a term requires a choice of the coefficient $\gamma$ and studies have begun appearing with various suggestions for its value. We give an analysis herein that provides a

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restricted range of possible values for the coefficient in 3d turbulent flows away from walls. If $U, L$ denote the large scale velocity and length respectively and $\kappa$ is the signal to noise ratio of the body force, estimates suggest that $\gamma$ should be restricted to the range

$$\frac{\kappa^2}{24} L U \leq \gamma \leq \frac{\kappa^2}{4} Re L U, \text{ mesh independent case},$$

$$\frac{\kappa^2}{24} L U \leq \gamma \leq \frac{\kappa^2}{4} \left( \frac{h}{L} \right)^{-\frac{3}{4}} L U, \text{ mesh dependent case}.$$
independent approach limiting \( \gamma \) to values where the additional dissipation introduced does not disturb statistical equilibrium. Denote time averaging by

\[
\langle \phi \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \phi(t) dt.
\]

For 3d, fully developed, turbulent flows away from walls, it is known that the total energy dissipation rate balances energy input, \( \langle \varepsilon(u) \rangle = \mathcal{O}(U^3/L) \), where the energy dissipation rate (per unit volume) \( \varepsilon(u) \) is

\[
\varepsilon(u) = \frac{1}{|\Omega|} \int_{\Omega} \nu |\nabla u(x, t)|^2 + \gamma |\nabla \cdot u(x, t)|^2 dx
\]

so

\[
\langle \varepsilon \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \varepsilon(u) dt.
\]

This balance is one of the two laws of experimental turbulence, [F95] Ch. 5. Building on [DF02], we analyze the dependence of \( \langle \varepsilon \rangle \) on \( \gamma \) for the simplest system arising when incompressibility is relaxed by \( -\gamma \text{grad div } u \), given by

\[
\begin{align*}
    u_t + \text{div}(u \otimes u) - \frac{1}{2} \left( \nabla \cdot u \right) u &- \nu \Delta u - \gamma \nabla \nabla \cdot u = f(x).
\end{align*}
\]

(1)

This continuum model arises from the common penalty approximation

\[
\nabla \cdot u = 0 \text{ replaced by } \frac{1}{\gamma} \nabla p + \nabla \cdot u = 0 \text{ for } \gamma >> 1
\]

so that

\[
\nabla p = -\gamma \nabla \nabla \cdot u.
\]

Its solutions satisfy the same á priori energy bound as a discrete NSE system with \text{grad} – \text{div} stabilization:

\[
\frac{1}{2} \frac{d}{dt} ||u(t)||^2 + \{ \nu ||\nabla u(x, t)||^2 + \gamma ||\nabla \cdot u(x, t)||^2 \} = (f, u).
\]

The domain \( \Omega = (0, L_{\Omega})^3 \) is a 3d periodic box, \( f(x) \) and \( u(x, 0) \) are periodic, satisfy \( \nabla \cdot u(x, 0) = 0 \), and \( \nabla \cdot f = 0 \)

and have zero mean:

\[
\begin{align*}
    u(x + L_{\Omega}e_j, t) &= u(x, t) \quad (2) \\
    \text{and} \\
    \int_{\Omega} \phi dx &= 0 \text{ for } \phi = u, u_0, f. \quad (3)
\end{align*}
\]

\(^1\)The energy input rate at the large scales is \( U^3/L \). Briefly, the kinetic energy of the large scales scales with dimensions \( U^2 \). The "rate" has dimensions \( 1/\text{time} \). A large scale quantity with this dimensions is formed by \( U/L \) which is the turn over time for the large eddies, i.e., the time it takes a large eddy with velocity \( U \) to travel a distance \( L \). Thus the "rate of energy input" has dimensions \( U^3/L \).
The body force $f(x)$ is assumed smooth so that it inputs energy only into large scales. Recalling $f(x)$ has mean zero, define the \textit{signal to noise ratio} of the body force $\kappa$ by

$$\kappa = \sqrt{\frac{\|f\|^2_{L^2(\Omega)} \Omega}{\frac{1}{|\Omega|} \int_{\Omega} |f(x)|^2 \, dx}}.$$ 

Since $\nabla \cdot u \neq 0$ the nonlinearity is explicitly skew symmetrized by adding $-\frac{1}{2}(\nabla \cdot u)u$.

Let $(\cdot, \cdot), ||\cdot||$ denote the $L^2(\Omega)$ inner product and norm. Let $F, L, U$ denote

$$F = \left(\frac{1}{|\Omega|} \|f\|^2\right)^{\frac{1}{2}},$$

$$L = \min \left\{L_{\Omega}, \frac{F}{\|\nabla f\|_{L^\infty}}, \frac{F}{\left(\frac{1}{|\Omega|} \|\nabla f\|^2\right)^{\frac{1}{2}}} \right\},$$

$$U = \left(\frac{1}{|\Omega|} \|u\|^2\right)^{\frac{1}{2}}.$$ 

Non-dimensionalization in the standard way by

$$t^* = \frac{t}{T}, \quad x^* = \frac{x}{L}, \quad U = \frac{L}{T}, \quad u^* = \frac{u}{U}$$

gives:

$$u^*_t + \text{div}^*(u^* \otimes u^*) - \frac{1}{2}(\nabla^* \cdot u^*)u^* - \frac{\nu}{LU} \Delta^* u^* - \frac{\gamma}{LU} \nabla^* \nabla^* \cdot u^* = \frac{f(x)}{U^2}.$$ 

We recall $\text{Re} = \frac{LU}{\nu}$ and define the non-dimensional parameter

$$\mathcal{R}_\gamma = \frac{LU}{\gamma}.$$ 

\textbf{Theorem 1} Let $u(x,t)$ be a weak solution of \eqref{eq:navier-stokes}. Then,

$$\langle \varepsilon(u) \rangle \leq \left(6 + \text{Re}^{-1} \frac{1}{4} \kappa^2 \mathcal{R}_\gamma \right) \frac{U^3}{L}. \quad (5)$$ 

This estimate gives insight into $\gamma$ by asking \textit{grad} -- \textit{div} dissipation be comparable to (respectively) the pumping rate of energy to small scales by the nonlinearity, $U^3/L$, and to the correction to the asymptotic, $\text{Re} \to \infty$, rate due to energy dissipation in the inertial range, $\text{Re}^{-1} \frac{U^3}{T}$. The cases

$$2 \simeq \kappa^2 \mathcal{R}_\gamma$$

and

$$\text{Re}^{-1} \simeq \kappa^2 \mathcal{R}_\gamma$$
yield

\[
\frac{\kappa^2}{24} \leq \frac{\gamma}{LU} \leq \frac{\kappa^2}{4} Re.
\]

Let \( \eta \approx Re^{-3/4}L \) denote the Kolmogorov microscale so \( Re = (\eta/L)^{-4/3} \). When the model is solved on a spacial mesh with meshwidth \( \eta \ll h \) the smallest scale available is \( O(h) \). Replacing \( \eta \) by \( h \) leads to an estimate of mesh dependence of

\[
\frac{\kappa^2}{24} \leq \frac{\gamma}{LU} \leq \frac{\kappa^2}{4} \left( \frac{h}{L} \right)^{-4/3}.
\]

1.1 Related work

The energy dissipation rate is a fundamental statistic in experimental and theoretical studies of turbulence, e.g., Sreenivasan [S84], Frisch [F95]. In 1968, Saffman [S68], addressing the estimate of energy dissipation rates, \( \langle \varepsilon \rangle \approx U^3/L \), wrote that

"This result is fundamental to an understanding of turbulence and yet still lacks theoretical support." - P.G. Saffman 1968

In 1992 Constantin and Doering [CD92] made a fundamental breakthrough, establishing a direct link between the phenomenology of energy dissipation and that predicted for general weak solutions of shear flows directly from the NSE. This work builds on Busse [B78], Howard [H72] (and others) and has developed in many important directions. It has been extended to shear flows in Childress, Kerswell and Gilbert [CKG01], Kerswell [K08] and Wang [W97]. For flows driven by body forces extensions include Doering and Foias [DF02], Cheskidov, Doering and Petrov [CDP06] (fractal body forces), and [L07] (helicity dissipation). Energy dissipation in models and regularizations studied in [L02], [L07], [LRS10], [LST10].

2 Analysis of the energy dissipation rate

Compared to the NSE case [DF02] the term

\[
-\frac{1}{2}(\nabla \cdot u)u
\]

adds dependence on \( \gamma \) since \( \text{div} u \neq 0 \). A smooth enough solution of (11) satisfies the same \( \alpha \) priori energy bound as a discrete NSE system with \( \text{grad} - \text{div} \) stabilization

\[
\frac{1}{2} \frac{d}{dt} ||u(t)||^2 + \{ \nu ||\nabla u(x,t)||^2 + \gamma ||\nabla \cdot u(x,t)||^2 \} = (f, u).
\]
We thus define weak solutions to the model as follows.

**Definition 2** A weak solution of (1) is a distributional solution satisfying the energy inequality

\[
\frac{1}{2}||u(T)||^2 + \int_0^T \nu ||\nabla u(t)||^2 + \gamma ||\nabla \cdot u(t)||^2 dt \leq \frac{1}{2}||u(0)||^2 + \int_0^T (f, u) dt. \tag{6}
\]

From (2.1) standard differential inequalities establish that

\[
\frac{1}{2}||u(T)||^2 + \frac{1}{T} \int_0^T \varepsilon(u) dt \leq C < \infty, \quad C = C(\text{data}) \text{ independent of } T. \tag{7}
\]

From (2.2) \(\varepsilon\) is well defined and finite and

\[
\frac{1}{T}||u(T)||^2 \to 0 \text{ as } O(T).\]

\(L\) has units of length and satisfies

\[
||\nabla f||_{L^\infty} \leq \frac{F}{L}
\]

and

\[
\frac{1}{|\Omega|} \int_\Omega |\nabla f(x)|^2 dx \leq \frac{F^2}{L^2}. \tag{8}
\]

Dividing (6) by \(1/(T|\Omega|)\) gives

\[
\frac{1}{2T|\Omega|}||u(T)||^2 + \frac{1}{T|\Omega|} \int_0^T \nu ||\nabla u(t)||^2 + \gamma ||\nabla \cdot u(t)||^2 dt \leq \frac{1}{2} \frac{1}{T|\Omega|}||u(0)||^2 + \frac{1}{T|\Omega|} \int_0^T (f, u) dt. \tag{9}
\]

Define

\[
U_T := (\frac{1}{T} \int_0^T \frac{1}{|\Omega|} ||u||^2 dt)^{1/2}.
\]

Given (7) and the definition of \(F\), this is

\[
\frac{1}{T} \int_0^T \varepsilon(u) dt \leq O(\frac{1}{T}) + \frac{1}{T|\Omega|} \int_0^T (f, u) dt \leq O(\frac{1}{T}) + F \sqrt{\frac{1}{T} \int_0^T \frac{1}{|\Omega|} ||u||^2 dt}
\]

\[
\leq O(\frac{1}{T}) + FU_T
\]

6
To estimate $F$, set the test function in the weak form to be $f(x)$ (recall $\nabla \cdot f = 0$). This yields

$$F^2 = \frac{(u(T) - u_0, f)}{T|\Omega|} - \frac{1}{T|\Omega|} \int_0^T (u \otimes u, \nabla f) - \left(\frac{1}{2}(\nabla \cdot u)u, f\right)dt + \frac{1}{T} \int_0^T \nu(\nabla u, \nabla f)dt.$$

Of the four terms on the RHS, by (11) the first is $O(1/T)$. The second and fourth are bounded using Hölders and Young’s inequalities by

second:

$$\left| \frac{1}{T|\Omega|} \int_0^T (u \otimes u, \nabla f)dt \right| \leq \|\nabla f\|_{L^\infty} \frac{3}{T|\Omega|} \int_0^T \|u\|^2 dt \leq 3\frac{F}{L} U^2_T,$$

fourth:

$$\left| \frac{1}{T} \int_0^T \nu \left| \nabla u, \nabla f \right| dt \right| \leq \left( \frac{1}{T} \int_0^T \frac{\nu}{|\Omega|} \left| \nabla u \right|^2 dt \right)^\frac{1}{2} \left( \frac{1}{T} \int_0^T \frac{\nu}{|\Omega|} \left| \nabla f \right|^2 dt \right)^\frac{1}{2} \leq \left( \frac{1}{T} \int_0^T \frac{\nu}{|\Omega|} \left| \nabla u \right|^2 dt \right)^\frac{1}{2} \frac{\sqrt{F}}{L} \leq \frac{1}{2} F \frac{1}{T} \int_0^T \frac{\nu}{|\Omega|} \|\nabla u\|^2 dt + \frac{1}{2} UF \nu.$$

The third term is treated as follows:

third:

$$\left| \frac{1}{T} \int_0^T \frac{1}{|\Omega|} \left(\frac{1}{2}(\nabla \cdot u)u, f\right)dt \right| \leq \frac{1}{2} \|f\|_{L^\infty} \sqrt{\frac{1}{T} \int_0^T \frac{1}{|\Omega|} \left| \nabla \cdot u \right|^2 dt U_T}$$

$$\leq \frac{\gamma}{2} F \frac{1}{T} \int_0^T \frac{1}{|\Omega|} \left| \nabla \cdot u \right|^2 dt + \frac{1}{8\gamma} UF \frac{1}{|\Omega|} \int_0^T |f(x)|^2 dx U_T^2 \leq \frac{\gamma}{2} F \frac{1}{T} \int_0^T \frac{1}{|\Omega|} \left| \nabla \cdot u \right|^2 dt + \frac{1}{8\gamma} \nu^2 U T^2.$$
Inserting this in the RHS of (10) gives
\[
\frac{1}{T} \int_0^T \varepsilon(u) dt \leq \mathcal{O}\left(\frac{1}{T}\right) + FU_T
\]
\[
\leq \mathcal{O}\left(\frac{1}{T}\right) + 3\frac{U^3}{L} + U_T\frac{1}{2U} \int_0^T \varepsilon(u) dt + \frac{\nu U}{2L^2} U_T + \frac{1}{8\gamma} U\kappa^2 U^3_T.
\]
Letting \( T \to \infty \) we have, as claimed, that
\[
\langle \varepsilon(u) \rangle \leq \left(6 + \Re^{-1} + \frac{1}{4} \kappa^2 \Re \gamma \right) \frac{U^3}{L}.
\]

3 Conclusions

The analysis herein suggests the following linkage. Weak imposition of \( \nabla \cdot u = 0 \) at higher Reynolds numbers means explicit skew symmetrization becomes necessary. Since \( \nabla \cdot u \neq 0 \), this leads to a second nonlinear term \(-\frac{1}{2}(\nabla \cdot u)u\). The parameter \( \gamma \) affects the size of \( ||\nabla \cdot u|| \) which affects the rate at which \(-\frac{1}{2}(\nabla \cdot u)u\) pumps energy to smaller scales. This leads to restricting \( \gamma \) by aligning this energy transfer rate with that of the underlying incompressible Navier-Stokes equations. To summarize\(^3\):

\begin{quote}
Compressibility, however so slight
Doubles nonlinearity for skew-symmetry.
Cascades can stop by penalty, however light,
Unless its criterion is chosen with sagacity.
\end{quote}

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