Many-body superradiance and dynamical symmetry breaking in waveguide QED

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The many-body decay of extended collections of two-level systems remains an open problem. Here, we investigate whether an array of emitters coupled to a one-dimensional bath undergoes Dicke superradiance, a process whereby a completely inverted system synchronizes as it decays, generating correlations between emitters via dissipation. This leads to the release of all the energy in the form of a rapid photon burst. We derive the minimal conditions for the burst to happen as a function of the number of emitters, the chirality of the waveguide, and the single-emitter optical depth, both for ordered and disordered ensembles. Many-body superradiance occurs because the initial fluctuation that triggers the emission is amplified throughout the decay process. We show that this avalanche-like behavior leads to a dynamical spontaneous symmetry breaking, where most photons are emitted into either the left- or the right-propagating optical modes, giving rise to an emergent chirality. Superradiant bursts may be a smoking gun for the generation of correlated photon states of exotic quantum statistics. This physics can be explored in diverse setups, ranging from forming close to nanofibers to superconducting qubits coupled to transmission lines.

The decay rate of a single emitter is dictated by its radiative environment [1,3]. This realization contributed to the development of cavity quantum electrodynamics (QED). Here, highly-reflecting mirrors isolate a single optical mode, yielding a localized (or zero-dimensional) reservoir for the emitter, which enhances its decay into the cavity. The case of a one-dimensional (1D) bath is studied in “waveguide QED”, where an atom is interfaced with a propagating optical mode. Recent years have seen tremendous progress in waveguide QED experiments, with platforms including cold atoms coupled to optical nanofibers [14–16], cold atoms [9–11] coupled to a 1D photonic channel, as shown in Fig. 1(a). The guided mode of the waveguide mediates interactions of the emitters’ density matrix in the rotating frame [34,35].

\[
\dot{\rho} = -\frac{i}{\hbar} [\mathcal{H}_L + \mathcal{H}_R, \rho] + \mathcal{L}_g[\rho] + \mathcal{L}_{ng}[\rho]. \tag{1}
\]

Here, the Hamiltonians \(\mathcal{H}_L/R\) allow for the possibility of distinct coupling to left- and right-propagating waveguide modes (at rates \(\Gamma_L/R\) for a single emitter), and read [36]

\[
\mathcal{H}_L = -\frac{i\hbar}{2} \sum_{i<j} e^{ik_{1D}|zi-zj|} \hat{\sigma}^i_{eg} \hat{\sigma}^j_{ge} + \text{H.c.}, \tag{2a}
\]

\[
\mathcal{H}_R = -\frac{i\hbar}{2} \sum_{i>j} e^{ik_{1D}|zi-zj|} \hat{\sigma}^i_{eg} \hat{\sigma}^j_{ge} + \text{H.c.}, \tag{2b}
\]

where \(\hat{\sigma}^i_{ge} = |g_i\rangle \langle e_i|\) is the coherence operator between the ground and excited states of emitter \(i\) at position \(z_i\), \(k_{1D}\) is the photon wavevector, and H.c. stands for Hermitian conjugate. The total decay rate of a single emitter into the waveguide is denoted as \(\Gamma_{1D} = \Gamma_L + \Gamma_R\). The Lindblad operators \(\mathcal{L}_g[\rho]\) and \(\mathcal{L}_{ng}[\rho]\) describe the decay of emitters to guided and non-guided modes.
FIG. 1. Many-body superradiance from an ensemble of emitters coupled to a waveguide. (a) Schematic: an array of $N$ emitters of lattice constant $d$ interact via a 1D bath, which supports propagation of photons of wavevector $k_{1D}$. The single-emitter decay rates into left and right-propagating modes of the waveguide are $\Gamma_L$ and $\Gamma_R$, respectively, and any other parasitic decay is denoted by $\Gamma'$. (b) Rate of emission into the waveguide for an array of $N = 16$ emitters coupled to a bidirectional (solid lines) and a chiral (dotted lines) waveguide with $\Gamma_R = 3 \Gamma_L$ and $\Gamma' = 0$. Dashed lines show the calculation without Hamiltonian contribution (for the bidirectional waveguide) which is significant at late times (inset).

respectively, and read

$$
\mathcal{L}_{\alpha}[\hat{\rho}] = \sum_{i,j=1}^{N} \frac{\Gamma_{ij}^{\alpha}}{2} \left( 2 \hat{\sigma}_i^{\alpha} \hat{\rho} \hat{\sigma}_{eg} - \hat{\rho} \hat{\sigma}_i^{\alpha} \hat{\sigma}_{eg} - \hat{\sigma}_i^{\alpha} \hat{\sigma}_{eg} \hat{\rho} \right),
$$

(3)

where $\Gamma_{ij}^{\alpha} = \Gamma_L e^{ik_{1D}(z_j - z_i)} + \Gamma_R e^{-ik_{1D}(z_j - z_i)}$ and $\Gamma_{ij}^{\alpha} = \Gamma' \delta_{ij}$. We consider that non-guided decay is not collective, either because it represents local parasitic decay or because emitters are far separated and interactions via non-guided modes are negligible.

Emission of photons into the waveguide is correlated due to the shared bath. This is captured by collective jump operators found by diagonalizing the $N \times N$ Hermitian matrix $\Gamma$ of elements $\Gamma_{ij}$ [37, 38]. Photons can only be emitted into the left- or right-propagating modes, and thus $\Gamma$ has only two non-zero eigenvalues. Decay into the waveguide is described in terms of two independent decay channels as

$$
\mathcal{L}_a[\hat{\rho}] = \sum_{\nu = +, -} \Gamma_{\nu} \left( 2 \hat{\sigma}_{\nu} \hat{\rho} \hat{\sigma}_{\nu} - \hat{\rho} \hat{\sigma}_{\nu} \hat{\sigma}_{\nu} - \hat{\sigma}_{\nu} \hat{\sigma}_{\nu} \hat{\rho} \right),
$$

(4)

where $\hat{\sigma}_{\nu}$ are collective jump operators and $\Gamma_{\nu}$ are collective decay rates, found as the eigenvectors and eigenvalues of $\Gamma$ respectively. The $\{+, -\}$ notation indicates that $\hat{\sigma}_{\nu}$ generates a photon in a symmetric (antisymmetric) superposition of left- and right-propagating modes.

A fully inverted initial state, $|\psi(t = 0)\rangle = |e\rangle^\otimes N$, will decay due to vacuum fluctuations, leading to emission into the waveguide at a (normalized) rate

$$
R(t) = \frac{1}{N \Gamma_{1D}} \sum_{\nu = +, -} \Gamma_{\nu} \langle \hat{\theta}_{\nu}^+ \hat{\theta}_{\nu} \rangle.
$$

(5)

For large enough $N$ and $\Gamma' = 0$, a superradiant burst occurs for any lattice constant, as shown in Fig. 1(b) for ordered arrays. Calculations are performed using quantum trajectories [39, 41]. Maximal superradiance occurs in a bidirectional waveguide (i.e., $\Gamma_R = \Gamma_L$), at the so-called “mirror configuration” ($k_{1D}d = n\pi$ with $n \in N$ [42, 43]), as this situation corresponds to that studied by Dicke.

Dissipative dynamics are the main driving mechanism for the burst. The coherent (i.e., Hamiltonian) interactions only contribute well beyond the time of maximum emission. At later times the Hamiltonian plays a significant role, as shown in the inset to Fig. 1(b). For low excitation densities, the Hamiltonian contributes to the trapping of the excitation in dark states that do not decay [39].

As we postulated in prior work [30], the minimal condition for a burst is that the first photon enhances the emission of the second. The same physical insight can be used to derive a condition for a superradiant burst into one particular channel, even if emitters can decay to more than one reservoir (as happens in the presence of finite local loss $\Gamma'$). To do so, we adapt the calculation from Ref. [28] of “directional superradiance” and define a second-order correlation function conditioned on measurement only of the waveguide modes

$$
g^{(2)}(\theta) = \frac{1}{N \Gamma_{1D}} \sum_{\nu = +, -} \sum_{\mu = +, -} \Gamma_{\nu} \Gamma_{\mu} \langle \hat{\theta}_{\nu}^+ \hat{\theta}_{\nu} \hat{\theta}_{\nu}^+ \hat{\theta}_{\nu} \rangle.
$$

(6)

Here, sums over $\pm$ account for waveguide emission, while sums in $i$ account for local decay. The minimal condition for superradiant emission of photons into the waveguide is found by imposing $g^{(2)}(0) > 1$. Importantly, this condition selects the processes in which there is an enhancement of the photon emission rate into the waveguide only (instead of into all possible decay channels). A naive use of the unconditioned correlation function [30] cannot capture that not all light is collected.

For an ensemble of initially-inverted emitters this condition becomes [see Supplementary Information (SI) [44]]

$$
\text{Var} \left( \frac{\langle \Gamma_{\nu} \rangle}{\Gamma_{1D}} \right) > 1 + \frac{\Gamma'}{\Gamma_{1D}},
$$

(7)

where $\text{Var}(\cdot)$ is the variance [44]. This expression is general; it applies to systems with any number of emitters,
in disordered or ordered spatial configurations, and coupled to waveguides with any degree of chirality. A burst will occur if there are only a few dominant decay channels (maximizing the variance), and if collective decay overcomes local loss (represented by the last term of the equation). As emission is constrained to 1D, there are at most two bright channels, while \( N - 2 \) are dark (i.e., of zero decay rate). Therefore, the conditions for a burst are more easily satisfied than for arrays in free space. \([30, 31, 45]\). Physically, the difference is that restrictions of emission into a 1D bath eliminates most of the competition between different imprinted phase patterns, enabling a more robust phase-locking than in free space, where photons can be emitted in all directions.

For ordered arrays of lattice constant \( d \), the two collective decay rates admit the analytical form

\[
\Gamma_{\pm} = \frac{N \Gamma_{1D}}{2} \pm \sqrt{\frac{N^2 (\Gamma_L - \Gamma_R)^2}{4} + \Gamma_L \Gamma_R \sin^2 N k_{1D} d \sin^2 k_{1D} d}. \tag{8}
\]

As shown in Fig. 2(a), the two decay rates are generally distinct and finite, leading to competition between the \( \pm \) channels. Different lattice constants give rise to situations ranging from the Dicke model with a single non-zero collective decay rate to maximum competition, where the decay rates are degenerate due to an emergent chiral translational symmetry. \([41]\). For chiral waveguides there is no degeneracy, as any level of chirality breaks translation symmetry. In this case, the rates are almost independent of the lattice constant, as interference effects are suppressed.

For ordered arrays, the minimal burst condition reads

\[
N \left( \Gamma_L^2 + \Gamma_R^2 \right) \left( \Gamma_{1D}^2 \right) > 2 \Gamma_L \Gamma_R \sin^2 N k_{1D} d \sin^2 k_{1D} d > 2 \Gamma_{1D}' \Gamma_{1D}. \tag{9}
\]

The crossover plot in Fig. 2(b) shows that a large parasitic decay quenches the superradiant burst for small \( N \). However (and regardless of the level of independent decay) a burst is always recovered if the number of emitters is increased beyond a certain threshold.

For disordered systems we obtain the minimal burst condition in terms of single-emitter decay rates by placing a lower bound on the trace of \( \Gamma^2 \), as the eigenvalues do not admit an analytical form. As demonstrated in \([41]\),

\[
\text{Tr}[\Gamma^2] \geq N^2 (\Gamma_L^2 + \Gamma_R^2), \tag{10}
\]

Disordered systems saturate this bound [see Fig 2(c)], while ordered systems may display a burst for lower \( N \) due to interference effects [see Eq. (9) and Fig 2(b)].

Generically, correlations imparted by the jump operators not only produce an accelerated emission of the second photon, but also of the subsequent ones. This "avalanche"-like nature of photon emission implies that an initial fluctuation is amplified throughout the decay process. Shot-to-shot fluctuations between collective jump operators (thus between angles of emission) have been predicted for ensembles in free space. \([40]\). In a 1D bath, however, these fluctuations are more striking as there are only two directions and the fluctuations break mirror symmetry. For instance, if the first photon is measured by a detector to the right, it is very likely that subsequent photons are also detected in that direction. This process gives rise to an emergent chirality even in the case of a bidirectional waveguide. To explore this physics, we unravel \( \mathcal{L}_g[\rho] \) in terms of a different pair of operators,

\[
\hat{\Theta}_{L/R} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} e^{\pm ik_{1D} d i} \hat{g}_i \propto \sqrt{\Gamma_{1D}^+} \hat{\Theta}_+ \pm i \sqrt{\Gamma_{1D}^-} \hat{\Theta}_-, \tag{11}
\]

which describe the emission of photons to the left- and right-propagating modes with rates \( \Gamma_{L/R} \).
is akin to a process of spontaneous symmetry breaking, state \([33, 47, 48]\)). An atom emitted the photon thus preparing a superposition induced by atom-atom interactions or to the photon detector this enhanced chirality to the correlations produced by the decay.

\[
\Gamma \hat{\eta} \quad \text{configuration where } \hat{\eta} \text{ follows the direction of the first. Subsequent jumps further to the right (or left) of the first.}
\]

\[
\text{The direction of the first photon is stochastic due to the uncorrelated initial state, with probability depending only on the relative decay rate of each operator, i.e., } P_{L/R} = \Gamma_{L/R}/\Gamma_{1D}. \text{ Emergent chirality is already evident at the level of two emissions. The detection probabilities for the second photon to be the same as or different to the first one are related to the correlation functions in Eq. (12),}
\]

\[
\tilde{g}^{(2)}(0) = \frac{2\Gamma_L \Gamma_R \tilde{g}^{(2)}_{LL}(0) + \Gamma_R^2 + \Gamma_L^2 \tilde{g}^{(2)}_{RR}(0)}{\Gamma_{1D}^2},
\]

where \(\tilde{g}^{(2)}_{\alpha\beta}(0) = \langle \hat{\eta}_\alpha^\dagger \hat{\eta}_\beta^\dagger \hat{\eta}_\beta \hat{\eta}_\alpha \rangle / \langle \hat{\eta}_\alpha^\dagger \hat{\eta}_\alpha \rangle \langle \hat{\eta}_\beta^\dagger \hat{\eta}_\beta \rangle\) and we have used \(\tilde{g}^{(2)}_{LL}(0) = \tilde{g}^{(2)}_{RR}(0)\) and \(\tilde{g}^{(2)}_{LR}(0) = \tilde{g}^{(2)}_{RL}(0)\).

The direction of the first photon is stochastic due to the uncorrelated initial state, with probability depending only on the relative decay rate of each operator, i.e., \(P_{L/R} = \Gamma_{L/R}/\Gamma_{1D}\). Emergent chirality is already evident at the level of two emissions. The detection probabilities for the second photon to be the same as or different to the first one are related to the correlation functions in Eq. 12,

\[
\begin{align*}
\tilde{g}^{(2)}_{LL}(0) &= 2 - \frac{2}{N}, \quad (13a) \\
\tilde{g}^{(2)}_{LR}(0) &= 1 + \frac{1}{N^2} \frac{\sin^2 N k_{1D} d}{\sin^2 k_{1D} d}. \quad (13b)
\end{align*}
\]

For large \(N\), the second photon is twice as likely to follow the direction of the first. Subsequent jumps further enhance the chirality (except in the mirror configuration where \(\hat{\eta}_{L/R}\) are identical). One can attribute this enhanced chirality to the correlations produced by atom-atom interactions or to the photon detection (far-field measurement is unable to distinguish which atom emitted the photon thus preparing a superposition state \(33, 47, 48\)).

For a bidirectional waveguide, this emergent chirality is akin to a process of spontaneous symmetry breaking, where mirror symmetry is broken dynamically. A large superradiant burst implies that, for a single realization, most photons are emitted in one direction. Of course, the symmetry is recovered when averaged over realizations, as the first photon is randomly emitted into either direction.

We characterize this behavior by counting the photons emitted in both directions and computing the “photon imbalance”, as shown in Fig. 3(a). For a single quantum trajectory – in which atoms evolve from \(|e\rangle^{\otimes N}\) to \(|g\rangle^{\otimes N}\), via both coherent evolution with a non-Hermitian Hamiltonian and decay by the action of the jump operators – we define the photon imbalance as the difference in the number of times that the two jump operators act. This corresponds to counting the total number of photons emitted to the right \((N_R)\) and left \((N_L)\) with imbalance

\[
I = N_R - N_L.
\]

\(I\) must be \(-N, -N+2, ..., N\), as the number of emitted photons is fixed (i.e., \(N_R + N_L = N\)). The set of possible photon imbalances \(I\) has a probability distribution, \(P(I)\). The imbalance distribution depends on the lattice constant and the degree of intrinsic (single-emitter) chirality of the waveguide.

In the mirror configuration, as the left and right operators are identical, the normalized probability of emitting a photon in each direction reduces to approximately \(P_{L/R}\) at any stage of the decay. Hence, the photon imbalance roughly follows the binomial distribution

\[
P(I) \approx \frac{N!}{(N+1-I)! (N+1+I)!} \left( \frac{\Gamma_R}{\Gamma_{1D}} \right)^{N+I} \left( \frac{\Gamma_L}{\Gamma_{1D}} \right)^{N-I}. \quad (15)
\]

As shown in Fig. 3(b), this analytical expression is in good agreement with the results obtained by numerical
evolution using quantum trajectories. Minor discrepancies originate from the action of the Hamiltonian in between jumps (only for a chiral waveguide) and noise from the finite number of trajectories.

Away from the mirror configuration, the repeated action of a jump operator enhances its probability of acting again, thus amplifying the initial fluctuation and either breaking mirror symmetry (for a bidirectional waveguide) or collectively enhancing chirality (for a chiral one). Repeated action leads to emission that finishes at early times because it produces strongly enhanced photon emission, so the time between jumps is small and Hamiltonian evolution is negligible. This is shown in Fig. 3(c). For a bidirectional waveguide, almost all photons are emitted in one direction. A mildly-chiral waveguide becomes almost perfectly chiral.

On average, photons are not radiated as fast as in Dicke’s scenario as both operators will act, and emission will be relatively less enhanced. Hamiltonian evolution becomes relevant for the imbalance statistics at later times [49] by scrambling the states and reducing enhancement. As shown in Fig. 3(d), for the bidirectional waveguide, it gives rise to an almost-flat imbalance distribution and for the chiral waveguide, the enhancement of the chirality is reduced. Nonetheless, the probability of detecting all photons in a single direction is much greater than the probability predicted by the binomial distribution for independent emission. This resembles a recent prediction for multilevel atoms in a cavity, where there is a higher probability of large imbalances between ground state populations compared to single-atom predictions [50].

In conclusion, we have established a condition for enhanced emission into a preferential channel when emitters decay to more than one reservoir. We have found the minimal conditions for the emission of a superradiant burst into a 1D bath and determined that the burst should be observable in different experimental setups, such as superconducting qubits coupled to transmission lines and atoms coupled to nanofibers. Many-body superradiance gives rise to an emergent chirality in the system, with large amounts of photons being emitted in one direction. As shown in [41], large photon imbalances disappear for $\Gamma^\prime \sim \Gamma_{1D}$. Nevertheless, pronounced imbalances should be observable in state-of-the-art experimental setups with superconducting qubits, where $\Gamma^\prime \simeq 0.01\Gamma_{1D}$ [15], and quantum dots, where $\Gamma^\prime \simeq 0.1\Gamma_{1D}$ [13].

An interesting avenue for future research is to investigate the quantum state of photons produced via many-body decay. The mirror configuration produces multiphoton states with similar metrological properties to Fock states [51, 55]. However, in this configuration, photons need to be recombined into a single pulse, as they are emitted in both directions. This issue should be partially overcome at different lattice constants or in chiral waveguides. However, in these cases, dynamics may populate dark states, which are prevalent at low excitation densities [7, 19, 56, 57], trapping the last few photons in the pulse. Moreover, the direction of emission is initially random. However, stimulated emission may overcome this problem [58, 59]. Another promising line of inquiry involves the possibility of using measurement and feedback control on the output light to access entangled dark states.

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1. Calculation of the conditional statistics

A necessary condition for observing a burst in a detector placed at the end of the waveguide is that the emission of photons into the waveguide increases at $t = 0$. This condition can be written as $\tilde{g}^{(2)}(0) > 1$, where the second-order correlation function reads

$$\tilde{g}^{(2)}(0) = \left( \sum_{\nu=+,+} \sum_{\mu=+,-} \Gamma_\nu \Gamma_\mu \langle \hat{\rho}^\dagger_\nu \hat{\rho}^\dagger_\mu \hat{\rho}_\nu \hat{\rho}_\mu \rangle \right) = 1 + \sum_{\nu=+,+} \sum_{\mu=+,-} \Gamma_\nu^2 - N \Gamma_{1D} (2 \Gamma_{1D} + \Gamma') \frac{N^2 \Gamma_{1D} (\Gamma_{1D} + \Gamma')}{N^2 \Gamma_{1D} (\Gamma_{1D} + \Gamma')}. \quad (16)$$

The condition $\tilde{g}^{(2)}(0) > 1$ implies $\text{Tr} [\Gamma^2] > N \Gamma_{1D} (2 \Gamma_{1D} + \Gamma')$. The variance of $\{ \Gamma_\nu \}$

$$\text{Var} \left( \left\{ \frac{\Gamma_\nu}{\Gamma_{1D}} \right\} \right) = \frac{1}{N \Gamma_{1D}^2} \sum_{\nu=+,+} (\Gamma_\nu^2 - \Gamma_{1D}^2), \quad (17)$$

can be used to rewrite this expression as Eq. (7) of the main text.

1.1. Minimal burst condition for arbitrary spatial configurations

Even when the analytical expression of the eigenvalues is not known, one can obtain a condition that guarantees a burst by finding a lower bound for $\sum_\nu \Gamma_\nu^2 \equiv \text{Tr} [\Gamma^2]$. The trace of $\Gamma^2$ for an arbitrary spatial configuration is readily calculated as

$$\text{Tr}[\Gamma^2] = \sum_{i,j=1}^N \Gamma_{ij}^2 \Gamma_{ji}^2 = N^2 (\Gamma_R^2 + \Gamma_L^2) + 2 \Gamma_R \Gamma_L \sum_{i,j=1}^N \cos(2k_{1D} (z_i - z_j))$$

$$= N^2 (\Gamma_R^2 + \Gamma_L^2) + 2 \Gamma_R \Gamma_L \left[ \left( \sum_{i=1}^N \cos(2k_{1D} z_i) \right)^2 + \left( \sum_{i=1}^N \sin(2k_{1D} z_i) \right)^2 \right] \geq N^2 (\Gamma_R^2 + \Gamma_L^2). \quad (18)$$

Thus, choosing a set of parameters that satisfies $N (\Gamma_R^2 + \Gamma_L^2) > \Gamma_{1D} (2 \Gamma_{1D} + \Gamma')$, i.e., Eq. (10) in the main text, guarantees superradiance regardless of the arrangement of the emitters. If the emitters form an ordered array with lattice constant $k_{1D} d = n \pi/ N$ [i.e., at the degeneracy points in Fig. 2(a)], the bound is saturated. This can be intuitively understood from the fact that $\text{Tr} [\Gamma^2]$ is related to the variance of $\{ \Gamma_\nu \}$ which is minimized at degeneracy.

1.2. Enhancement of the emission of the third photon

To confirm that $\tilde{g}^{(2)}(0) > 1$ is a sufficient condition to predict a burst, we compute the conditional third order correlation function, which reads

$$\tilde{g}^{(3)}(0) = \sum_{\nu=+,+} \sum_{\mu=+,+} \Gamma_\nu \Gamma_\mu \Gamma_\chi \langle \hat{\rho}^\dagger_\nu \hat{\rho}^\dagger_\mu \hat{\rho}^\dagger_\chi \hat{\rho}_\mu \hat{\rho}_\nu \hat{\rho}_\chi \rangle$$

$$= 1 - \frac{6 \Gamma_{1D}^2 + 8 \Gamma' \Gamma_{1D} + 3 \Gamma'^2}{N (\Gamma_{1D} + \Gamma')^2} + \frac{12 \Gamma_{1D}^2 + 8 \Gamma' \Gamma_{1D} + 2 \Gamma'^2}{N^2 (\Gamma_{1D} + \Gamma')^2} \sum_{\nu=+,+} \Gamma_\nu^3 \Gamma_{1D} + 2 (N - 2) \Gamma' \sum_{\nu=+,+} \Gamma_\nu + \frac{(3 N \Gamma_{1D} - 12 \Gamma_{1D} + 2 (N - 2) \Gamma') \sum_{\nu=+,+} \Gamma_\nu^3}{N^3 \Gamma_{1D} (\Gamma_{1D} + \Gamma')^2}. \quad (20)$$
Below we prove by contradiction that there is no combination of parameters such that \( \tilde{g}^{(2)}(0) \leq 1 \) while \( \tilde{g}^{(3)}(0) > 1 \), so the emission of the third photon is never enhanced if the second was not. The inequality \( \tilde{g}^{(2)}(0) \leq 1 \) implies

\[
\text{Tr} [\Gamma^2] < N\Gamma_{1D}(2\Gamma_{1D} + \Gamma').
\]

Combining the above equation with \( \tilde{g}^{(3)}(0) > 1 \) yields

\[
2\text{Tr} [\Gamma^3] > N^2\Gamma_{1D}\Gamma' (\Gamma_{1D} + \Gamma') + 2N\Gamma_{1D}(6\Gamma_{1D}^2 + 6\Gamma_{1D}\Gamma' + \Gamma'^2).
\]

After some algebraic manipulations, one finds

\[
\begin{align*}
\text{Tr}[\Gamma^2] &= \sum_{i,j=1}^{N} \Gamma_{ij} \Gamma_{ji} = N^2(\Gamma_L^2 + \Gamma_R^2) + 2\Gamma_L \Gamma_R \sum_{i,j=1}^{N} \cos [2k_{1D}(z_i - z_j)], \quad (23a) \\
\text{Tr}[\Gamma^3] &= \sum_{i,j,k=1}^{N} \Gamma_{ij} \Gamma_{jk} \Gamma_{ki} = N^3(\Gamma_L^3 + \Gamma_R^3) + 3N\Gamma_{1D}\Gamma_L \Gamma_R \sum_{i,j=1}^{N} \cos [2k_{1D}(z_i - z_j)]. \quad (23b)
\end{align*}
\]

Therefore, the condition for the third emission to be the first one that is enhanced reduces to

\[
F(N) \equiv N^2 + BN + C < 0,
\]

with

\[
\begin{align*}
B &= \left( \frac{\Gamma'}{\Gamma_{1D}} \right)^2 - \frac{2\Gamma'}{\Gamma_{1D}} - 6, \quad (25a) \\
C &= 2 \left( \frac{\Gamma'}{\Gamma_{1D}} \right)^2 + 12 + \frac{12\Gamma'}{\Gamma_{1D}}. \quad (25b)
\end{align*}
\]

While \( C > 0 \) for all possible values of the ratio \( \Gamma_{1D}/\Gamma' \), \( B > 0 \) only if \( \Gamma'/\Gamma_{1D} > 1 + \sqrt{7} \). Thus, if \( \Gamma'/\Gamma_{1D} > 1 + \sqrt{7} \), the inequality in Eq. (24) is never satisfied. If \( \Gamma'/\Gamma_{1D} < 1 + \sqrt{7} \), the discriminant of \( F(N) \) is negative, so \( F(N) \) does not have real roots and hence it is never negative. We then conclude that no combination of parameters yield \( \tilde{g}^{(2)}(0) \leq 1 \) while \( \tilde{g}^{(3)}(0) > 1 \), so \( \tilde{g}^{(2)}(0) > 1 \) is a sufficient condition to predict a burst.

### 1.3. Enhancement of emission into all channels

The condition to have emission enhancement into all possible channels is calculated by imposing \[30\]

\[
\text{Var} \left( \frac{\Gamma_{1D}}{\Gamma_0} \right) > 1,
\]

with \( \tilde{\Gamma}_1 = \Gamma_+ + \Gamma' \), \( \tilde{\Gamma}_2 = \Gamma_+ - \Gamma' \), \( \tilde{\Gamma}_{\geq 2} = \Gamma' \) and \( \Gamma_0 = \Gamma_{1D} + \Gamma' \). After simplifying one finds

\[
\frac{N^2 \left( \Gamma_L - \Gamma_R \right)^2}{2} + 2\Gamma_L \Gamma_R \frac{\sin^2 Nk_{1D}d}{\sin^2 k_{1D}d} + \frac{\Gamma_{1D}^2}{2} (N^2 - 2N) > N(\Gamma_{1D} + \Gamma')^2,
\]

which agrees with conditions found in other works \[60\]. However, note that Eq. (27) is a condition for overall superradiant emission, as opposed to Eq. (7).

### 2. Emergent translational symmetry at the degeneracy points

Consider a finite and ordered array of \( N \) emitters whose interactions are described through the matrix of coefficients \( \Gamma_{ij} \). If the interactions satisfy periodic or antiperiodic boundary conditions (i.e. if \( \Gamma_{i,N+1} = \pm \Gamma_{i,1} \)), the system becomes translationally invariant since one can identify particle 1 with particle \( N + 1 \). Periodic boundary conditions (PBC) emerge for bidirectional waveguides at lattice constants \( k_{1D}d = 2n\pi/N \), with \( n \in \mathbb{N} \). Antiperiodic boundary conditions (APBC) are achieved for \( k_{1D}d = (2n+1)\pi/N \). Therefore, at \( k_{1D}d = n\pi/N \), the eigenvectors of \( \Gamma \) must obey
Bloch’s theorem. The jump operators thus take the form of annihilation operators for spin waves with momentum \( k \) in the first Brillouin zone, i.e.,

\[
\hat{S}_k = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{-ikdj} \hat{\sigma}_j^e.
\]  

(28)

The only two jump operators with finite decay rates are those that match the wavevector of the guided mode, i.e., \( k = \pm k_{1D} \). These are the left and right jump operators defined in Eq. (11) of the main text. Since the system is mirror-symmetric, the two jump operators must have identical decay rates. Thus, the two non-zero eigenvalues of the \( \Gamma \) matrix become degenerate at \( k_{1D}d = n\pi/N \).

We can also demonstrate that the two non-zero eigenvalues are degenerate at \( k_{1D}d = n\pi/N \) by employing the analytical form of the collective jump operators. For the case of a bidirectional waveguide they read

\[
\hat{O}_\pm = \sqrt{\frac{\Gamma_{1D}}{\Gamma_\pm}} \sum_{i=1}^{N} \mathcal{F}_\pm \left[ k_{1D}d \left( \frac{N+1}{2} - i \right) \right] \hat{\sigma}_i^e,
\]  

(29)

with \( \mathcal{F}_{(+)}[\cdot] = \cos[\cdot](\sin[\cdot]) \). Notice that \( \hat{O}_\pm \) differ by a \( \pi/2 \) phase shift. Due to the emergent translational symmetry at \( k_{1D}d = n\pi/N \), \( \hat{O}_\pm \) must have the same decay rates at these points.

### 3. Channel competition and burst condition

The degree of competition between the channels \( \pm \) is dictated by the collective decay rates, \( \Gamma_\pm \). The situation of minimal competition is that of a single bright channel. There are two scenarios that realize this:

- A perfectly-chiral waveguide, for which \( \Gamma_+ = N\Gamma_{1D} \) and \( \Gamma_- = 0 \). While there is only one jump operator for any lattice constant, this situation is not exactly that studied by Dicke, due to the non-zero coherent evolution.

- Mirror configuration for waveguides of arbitrary chirality. For bidirectional waveguides, the coherent interactions mediated by the Hamiltonian vanish, and the system reduces to the Dicke model.

The burst condition in Eq. (9) reduces to \( (N-2)\Gamma_{1D} > \Gamma’ \) for any of these two situations. Notice that the condition for an array with any lattice constant and \( N \gg 1 \) coupled to a bidirectional waveguide reduces to \( (N-4)\Gamma_{1D} > 2\Gamma’ \). In the limit of strong parasitic decay, constructive interference for \( k_{1D}d = n\pi \) makes a superradiant burst possible with fewer emitters than is needed for most other spatial configurations.

The scenario of maximal competition between decay channels occurs whenever their rates are identical. This happens only for bidirectional waveguides at the degeneracy points discussed in the last section. As seen in Fig. 2 of the main text, any level of chirality breaks the translation symmetry and opens a gap in the decay spectrum. In chiral waveguides and away from the “mirror configuration”, the decay rates barely oscillate with \( k_{1D}d \) (as the sinusoidal term does not scale with \( N \) and can be ignored) and saturate to \( \Gamma_+ \simeq N\max(\Gamma_L, \Gamma_R) \) and \( \Gamma_- \simeq N\min(\Gamma_L, \Gamma_R) \).

### 4. Chirality induced by jumps

Directional correlation functions [Eq. (12) of the main text] can be defined for more than two jumps, and should show emergent chirality. Below, we ignore Hamiltonian evolution, since we want to emphasize the effect of jumps on directional emission. The probability that the \( n \)th photon is emitted in the \( \alpha \) direction (\( \alpha = L, R \)) if the previous \( n-1 \) photons were emitted to the left is proportional to

\[
\tilde{g}_{L\alpha}^{(n)}(0) = \frac{\langle \hat{\sigma}^{n-1}_L \hat{\sigma}_L \hat{O}_\alpha \hat{O}_{\alpha} \rangle}{\langle \hat{\sigma}^n_L \hat{\sigma}_L \rangle^{n-1} \langle \hat{O}_\alpha \hat{O}_{\alpha} \rangle}.
\]  

(30)
FIG. 4. Ratio of probabilities of detecting a photon to the left or right after \( n - 1 \) photons have been emitted to the left, versus photon number. The full evolution is shown in purple, and the red line depicts the analytical ratio obtained by only considering jumps.

Using Eq. (11), and simplifying, we find

\[
\tilde{g}_{LL}^{(n)}(0) = \frac{1}{N^n} \sum_{\sigma_1 < \sigma_2 < \ldots < \sigma_n} n! \left( \sum_{m=1}^{n} e^{+ik_1d \sigma_m} \right)^2 \frac{n!N!}{N^n(N-n)!} \quad \text{(31a)}
\]

\[
\tilde{g}_{LR}^{(n)}(0) = \frac{(n-1)!}{N^n} \sum_{\sigma_1 < \sigma_2 < \ldots < \sigma_n} \left( \sum_{m=1}^{n} e^{-2ik_1d \sigma_m} \right)^2 \frac{(n-1)!N!}{(N-n)!N^n(N-1)} \left( N - n + \frac{n-1}{N} \frac{\sin^2 Nk_{1D}d}{\sin^2 k_{1D}d} \right) \quad \text{(31b)}
\]

Here, the sum over \( \sigma_1, \sigma_2, \ldots, \sigma_n \) accounts for all the possible final states after the \( n \) detection events, i.e., all possible choices of \( n \) emitters to be the ones that emitted a photon and decayed to the ground state. For each final state, we need to coherently sum the contribution of all the quantum paths that lead to it. In the case of \( \tilde{g}_{LL}^{(n)}(0) \), the maximum constructive interference between paths is achieved since all \( n! \) paths have the same phase. On the other hand, the quantum paths for each final state in \( \tilde{g}_{LR}^{(n)}(0) \) do not have the same phase and hence do not achieve maximum constructive interference.

The ratio \( r(n) = \tilde{g}_{LL}^{(n)}(0)/\tilde{g}_{LR}^{(n)}(0) \) is a measure of the emergent chirality after the \( n \)th detection event in the absence of Hamiltonian dynamics. From Eqs. (31a) and (31b),

\[
r(n) = \frac{nN(N-1)\sin^2 k_{1D}d}{(N-n)N\sin^2 k_{1D}d + (n-1)\sin^2 Nk_{1D}d} \quad \text{(32)}
\]

Fig. 4 shows \( r(n) \) with and without Hamiltonian evolution. For the former, the probabilities are calculated numerically using quantum trajectories. As noted in the main text, Hamiltonian evolution scrambles the emitters’ phases and suppresses chirality enhancement.

For many atoms \( N \gg 1 \) that are not in the mirror configuration (thus \( \sin^2 k_{1D}d \approx \sin^2 Nk_{1D}d \)), the ratio reduces to \( r(n) \approx n \) for \( n \ll N \), both for the full and the approximated dynamics. This means that if \( n \) photons have been emitted into one direction, the next photon is \( n \) times more likely to be emitted in that same direction.

5. Imbalance statistics with large parasitic decay

Here we investigate in more detail the effects of parasitic decay in the photon imbalance probability distributions shown in Fig. 3. Figures 5 and 6 show the probability distributions for three different ratios of \( \Gamma'/\Gamma_{1D} \) for late and early times, respectively. The possible values for \( I \) are now \( \{-N, -N+1, \ldots, N\} \) since \( N_L \) and \( N_R \) can take any value as long as \( N_R + N_L \leq N \). The enhancement in the probability of maximum imbalance disappears for \( \Gamma' \sim \Gamma_{1D} \).
but survives parasitic decays achievable in state-of-the-art experimental setups with superconducting qubits, where
\( \Gamma' \approx 0.01 \Gamma_{1D} \) [13], and quantum dots, where \( \Gamma' \approx 0.1 \Gamma_{1D} \) [13]. If \( \Gamma' \ll \Gamma_{1D} \), correlations are washed out, and there is a very suppressed probability of large imbalances, as shown in Fig. 7.

**FIG. 5.** Imbalance statistics on the emission direction for 16 emitters with (a) \( \Gamma' = 0.01 \Gamma_{1D} \), (b) and (c) \( \Gamma' = \Gamma_{1D} \) for \( \Gamma_{1Dt} = 100 \). For each subplot, the top and bottom panels show respectively the probability distribution for a bidirectional and a chiral waveguide (\( \Gamma_R = 3 \Gamma_L \)).

**FIG. 6.** Imbalance statistics on the emission direction for the same conditions of Fig. 5 but for \( \Gamma_{1D}t = 1 \).

**FIG. 7.** Imbalance statistics on the emission direction for 16 emitters with large parasitic decay (\( \Gamma' = 10 \Gamma_{1D} \)) emitting into a (a) bidirectional and (b) a chiral waveguide (\( \Gamma_R = 3 \Gamma_L \)).
6. Details on numerical evolution

Here we give more details on the numerical evolution and the choice of parameters. The number of trajectories used for Figs. 1 and 3 (on the main text), and SI Fig. 1 is detailed in Table I.

| TABLE I. Number of trajectories used for Figs. 1 and 3 of the main text, and SI Fig. 1 |
|-----------------------------------------|-----------------|-----------------|
| Fig. 1                                  | Bidirectional   | Chiral          |
|                                         | 2,000           | 2,000           |
| Fig. 3(b)                               | 12,000          | 12,000          |
| Fig. 3(c) and Fig. 3(d)                 | 20,000          | 13,000          |
| SI Fig. 1                               | 5,000           | 8,000           |

The calculations for Figs. 2(b) and 2(c) are performed in the constrained Hilbert space composed of states with at most two atoms in $|g\rangle$. A burst is predicted if the emission rate at the final time $N\Gamma_{1D}t_{\text{fin}} = 10^{-5}$ is larger than at $t = 0$.

The probability in Fig. 2(c) is computed with 100 random configurations. The numerical boundary for each $N$ is defined as the minimum $\Gamma_{1D}/\Gamma'$ for which a burst occurs in every configuration. If $k_{1D}z_{\text{max}} \ll 2\pi$, all the emitters are approximately coupling to the waveguide at the same point, and the system is effectively in the mirror configuration. To make sure we are just considering truly disordered configurations the emitter positions are chosen at random from $[0, z_{\text{max}}]$ with $k_{1D}z_{\text{max}} \gg 2\pi$. In Figs. 3(c) and (d) $\pi$ over an irrational number was chosen to avoid arrays that are commensurable with the photon wavelength $\lambda_{1D}$, like at the degeneracy points, and show the behavior of a generic array.