This paper presents a robust synthesis algorithm for uncertain linear time-varying (LTV) systems on finite horizons. The uncertain system is described as an interconnection of a known LTV system and a perturbation. The input-output behavior of the perturbation is specified by time-domain Integral Quadratic Constraints (IQCs). The objective is to synthesize a controller to minimize the worst-case performance. This leads to a non-convex optimization. The proposed approach alternates between an LTV synthesis step and an IQC analysis step. Both induced $L_2$ and terminal Euclidean norm penalties on output are considered for finite horizon performance. The proposed algorithm ensures that the robust performance is non-increasing at each iteration step. The effectiveness of this method is demonstrated on a two-link robot arm example.

**Keywords** robust control, linear time-varying systems, integral quadratic constraints

1 Introduction

This paper considers robust synthesis for uncertain linear time-varying (LTV) systems on finite horizons. This problem is motivated by systems that have finite-time performance requirements and for which model uncertainty is a significant factor. Examples of such systems include aircraft landings [1], missile interceptors [2] and space-launch or reentry systems [3, 4, 5]. Robust synthesis algorithms provide formal guarantees of closed-loop stability and performance over a range of parametric and dynamic uncertainties. Many of these existing algorithms, e.g. $\mu$-synthesis [6, 7, 8] have been developed for uncertain linear time-invariant (LTI) system and infinite-horizon robustness metrics. This enables the use of frequency-domain techniques. In contrast, this paper is developed for uncertain finite-horizon, LTV systems using time-domain techniques.

The specific formulation in this paper uses an uncertain system described by an interconnection of a known LTV system and a perturbation. The input-output behavior of the perturbation is described by time-domain Integral Quadratic Constraints (IQCs) [9]. The performance objective is specified by an induced gain from $L_2$ input disturbances to a mixture of an $L_2$ and terminal Euclidean norm on the output. The objective is to synthesize a controller to minimize the worst-case performance over all allowable uncertainties. This worst-case performance can be used to robustly bound the state at the end of the finite horizon in the presence of external disturbances and model uncertainty.
This robust synthesis problem leads, in general, to a non-convex optimization. The proposed algorithm, presented in Section 4, iterates between a nominal synthesis step and robustness analysis step. The nominal synthesis step relies on existing results for finite horizon $H_{\infty}$ synthesis [1, 10, 11, 12, 13]. In particular, the two coupled Riccati Differential Equation (RDE) approach [14, 15, 16] is used as these results allow for Euclidean norm penalties at the final time. The robustness analysis step uses the IQC framework introduced in [9]. This framework has been extended in [17] to assess robustness of uncertain LTV systems on finite horizons. The approach presented in [17] will be used in this paper for the robustness analysis. Finally, a scaled plant construction is required to link the nominal synthesis and robustness analysis steps. A MATLAB implementation of the proposed algorithm including the example in this paper are available in the LTVTools [18] toolbox.

The proposed method is analogous to the existing DK iteration method for uncertain LTI systems on infinite horizons. The algorithm in this paper generalizes this method to uncertain LTV systems on finite horizons. Similar extensions have been made in [19, 20] for Linear Parameter-Varying (LPV) systems. Two other closely related works are [21] and [22]. The work in [21] considers an extension of the Glover-McFarlane loop-shaping method to LTV systems. This leads to a robust stabilization problem with single full block uncertainty. The work in [22] provides convex synthesis conditions for robust performance of uncertain LTV systems. However, this work assumes special structure on uncertainty structure and this structure is used to convexify the synthesis optimization. The algorithm proposed in this paper considers a more general robust performance formulation than in [21] and [22].

This paper builds on our initial work in [23]. There are three main distinctions in this paper. First, we use the dynamic IQC multipliers for the proposed finite horizon synthesis method, whereas the prior work [23] used the memoryless IQCs and related classes of uncertainties. Second, we use a time-varying IQC factorization to construct a scaled plant. This step ensures that the worst-case gain at each iteration is monotonically non-increasing. Finally, this paper provides all details and proofs regarding the proposed approach. The effectiveness is demonstrated using a nonlinear robot arm example.

Notation: Let $\mathbb{R}^{n \times m}$ and $\mathbb{S}^n$ denote the sets of $n$-by-$m$ real matrices and $n$-by-$n$ real, symmetric matrices. The finite-horizon $L_2[0,T]$ norm of a (Lebesgue integrable) signal $v : [0, T] \to \mathbb{R}^n$ is $\|v\|_{2,[0,T]} := \left( \int_0^T v(t)^T v(t) \, dt \right)^{1/2}$. If $\|v\|_{2,[0,T]} < \infty$ then $v \in L_2[0,T]$. $\mathbb{R}L_{\infty}$ is the set of rational functions with real coefficients that are proper and have no poles on the imaginary axis. $\mathbb{RH}_{\infty} \subset \mathbb{R}L_{\infty}$ contains functions that are analytic in the closed right-half of the complex plane. The notation $G^\sim$ denotes the adjoint of the dynamical system $G$, which is formally defined in [13].

2 Preliminaries

2.1 Nominal Performance

Consider an LTV system $H$ defined on the horizon $[0, T]$:

$$
\begin{align*}
\dot{x}(t) &= A(t) x(t) + B(t) d(t) \\
e(t) &= C(t) x(t) + D(t) d(t)
\end{align*}
$$

where $x \in \mathbb{R}^{n_x}$ is a state, $d \in \mathbb{R}^{n_d}$ is the input, and $e \in \mathbb{R}^{n_e}$ is the output. The state matrices $A : [0, T] \to \mathbb{R}^{n_x \times n_x}$, $B : [0, T] \to \mathbb{R}^{n_x \times n_d}$, $C : [0, T] \to \mathbb{R}^{n_e \times n_x}$, and $D : [0, T] \to \mathbb{R}^{n_e \times n_d}$ are piecewise-continuous (bounded) real matrix valued functions of time. It is assumed throughout that $T < \infty$. Thus $d \in L_2[0, T]$ implies $x$ and $e$ are in $L_2[0, T]$ for any initial condition $x(0)$ (Chapter 3 of [13]). Explicit time dependence of the state matrices will be omitted when it is clear from the context. The performance of $H$ will be assessed in terms of an induced gain with two components. First partition the output as follows:

$$
\begin{bmatrix}
e_1(t) \\
e_E(t)
\end{bmatrix} =
\begin{bmatrix}
C_f(t) \\
C_E(t)
\end{bmatrix} x(t) +
\begin{bmatrix}
D_f(t) \\
0
\end{bmatrix} d(t)
$$
where \( e_I \in \mathbb{R}^{n_I} \) and \( e_E \in \mathbb{R}^{n_E} \) with \( n_e = n_E + n_I \). The generalized performance metric of \( H \) is then defined as,

\[
\|H\|_{[0,T]} := \sup_{a \neq 0 \in L_2[0,T]} \left[ \frac{\|e_E(T)\|_2^2 + \|e_I\|_2^2}{\|d\|_2^2} \right]^{1/2}
\]

This defines an induced gain from the input \( d \) to a mixture of an \( L_2 \) and terminal Euclidean norm on the output \( e \). Note that if \( n_E = 0 \) then there is no terminal Euclidean norm penalty on the output. This case corresponds to the standard, finite-horizon \( L_2 \) gain of \( H \). Similarly, if \( n_I = 0 \) then there is no \( L_2 \) penalty on the output. This case corresponds to a finite-horizon \( L_2 \)-to-Euclidean gain. This can be used to bound the terminal output \( e_E(T) \) resulting from an \( L_2 \) disturbance input. Zero feed-through from \( d \) to \( e_E \) ensures that the Euclidean penalty is well-defined at time \( t = T \). The next theorem states an equivalence between a bound on this performance metric \( \|H\|_{[0,T]} \) and the existence of a solution to a related RDE (Theorem 3.7.4 of [13]).

**Theorem 1.** Consider an LTV system (1) with \( \gamma \) > 0 given. Let \( Q : [0,T] \rightarrow S^{n_x}, S : [0,T] \rightarrow \mathbb{R}^{n_x \times n_d}, R : [0,T] \rightarrow S^{n_d}, \) and \( F \in \mathbb{R}^{n_x \times n_d} \) be defined as follows*.

\[
Q := C_I^T C_I, \quad S := C_I^T D_I, \quad R := D_I^T D_I - \gamma^2 I_{n_d}, \quad F := C_E(T)^T C_E(T)
\]

The following statements are equivalent:

1. \( \|H\|_{[0,T]} < \gamma \)

2. \( R(t) \prec 0 \) for all \( t \in [0,T] \). Moreover, there exists a differentiable function \( P : [0,T] \rightarrow S^{n_x} \) such that \( P(T) = F \) and

\[
\dot{P} + A^T P + PA + Q - (PB + S)R^{-1}(PB + S)^T = 0
\]

This is a Riccati Differential Equation (RDE).

The nominal performance \( \|H\|_{[0,T]} < \gamma \) is achieved if the associated RDE solution exists on \( [0,T] \) when integrated backward from \( P(T) = F \). The assumption \( R(t) \prec 0 \) ensures \( R(t) \) is invertible and hence the RDE is well-defined \( \forall t \in [0,T] \). Thus, the solution of the RDE exists on \( [0,T] \) unless it grows unbounded. The smallest bound on \( \gamma \) is obtained using bisection.

**2.2 Nominal Synthesis**

This subsection provides conditions to synthesize a controller that is optimal with respect to the nominal performance metric introduced in the previous subsection. Consider the feedback interconnection shown in Figure 1.

---

*If \( n_I = 0 \) then \( Q = 0_{n_x} \), \( S = 0_{n_x \times n_d} \), and \( R = -\gamma^2 I_{n_d} \). Similarly, if \( n_E = 0 \) then \( F = 0_{n_x} \).
The LTV system $G$ defined on $[0, T]$ is given by:

$$
\begin{bmatrix}
\dot{x}(t) \\
e_I(t) \\
e_E(t) \\
y(t)
\end{bmatrix} =
\begin{bmatrix}
A(t) & B_d(t) & B_u(t) \\
C_I(t) & 0 & D_{lu}(t) \\
C_E(t) & 0 & 0 \\
C_y(t) & D_{yd}(t) & 0
\end{bmatrix}
\begin{bmatrix}
x(t) \\
d(t) \\
u(t)
\end{bmatrix}
$$

(2)

where $d \in \mathbb{R}^{n_d}$ is the generalized disturbance, $u \in \mathbb{R}^{n_u}$ is the control input and $y \in \mathbb{R}^{n_y}$ is the measured output. This plant structure also assumes no feedthrough from $d$ to $e_E$. This is required to ensure that the nominal performance metric is well-posed. In addition, the standard $H_\infty$ synthesis framework imposes additional structure on the matrices relating $d$ to $e_I$ and $d$ to $y$. This is done to simplify notation and is obtained via standard loop transformations under some minor technical assumptions (Chapter 17 of [24]). This leads to the following additional structure on the plant matrices:

$$
C_I := \begin{bmatrix} 0 \\ C_1 \end{bmatrix}, \quad D_{lu} := \begin{bmatrix} I_{n_u} \\ 0 \end{bmatrix}, \quad D_{yd} := \begin{bmatrix} 0 \\ I_{n_y} \end{bmatrix}
$$

The nominal synthesis problem is to find a causal linear time-varying controller $K : \mathcal{L}_2^{n_y}[0, T] \rightarrow \mathcal{L}_2^{n_y}[0, T]$ to optimize the closed-loop nominal performance, i.e.:

$$
\inf_{K} \| \mathcal{F}_I(G, K) \|_{[0, T]}
$$

As noted previously, if $n_E = 0$ then the nominal performance metric is the (finite-horizon) induced $L_2$ gain. In this case, the synthesis problem is equivalent to the existing finite-horizon $H_\infty$ problem as considered in [10, 13]. The theorem below states the necessary and sufficient conditions for existence of a $\gamma$-suboptimal controller for the nominal performance metric (with $n_E$ not necessarily equal to zero). Theorem 2 is a special case of results presented in [14, 16].

**Theorem 2.** Consider an LTV system (2) with $\gamma > 0$ given. Let $B, \hat{C}, \hat{R}$ and $\hat{\hat{R}}$ be defined as follows.

$$
B := \begin{bmatrix} B_d & B_u \end{bmatrix}, \quad \hat{R} := \text{diag}\{-\gamma^2 I_{n_d}, I_{n_u}\}, \quad \hat{C} := \begin{bmatrix} C_I^T & C_y^T \end{bmatrix}^T, \quad \hat{\hat{R}} := \text{diag}\{-\gamma^2 I_{n_1}, I_{n_y}\}
$$

1. There exists an admissible output feedback controller $K$ such that $\| \mathcal{F}_I(G, K) \|_{[0, T]} < \gamma$ if and only if the following three conditions hold:

   (a) There exists a differentiable function $X : [0, T] \rightarrow \mathbb{S}^{n_y}$ such that $X(T) = C_E(T)^T C_E(T)$,

$$
\dot{X} + A^T X + X A - X B \hat{R}^{-1} B^T X + C_I^T C_I = 0
$$

   (b) There exists a differentiable function $Y : [0, T] \rightarrow \mathbb{S}^{n_y}$ such that $Y(0) = 0$,

$$
-\dot{Y} + AY + Y A^T - Y \hat{C}^T \hat{\hat{R}}^{-1} \hat{\hat{C}} Y + B_d B_d^T = 0
$$

   (c) $X(t)$ and $Y(t)$ satisfy the following point-wise in time spectral radius condition,

$$
\rho(X(t)Y(t)) < \gamma^2, \quad \forall t \in [0, T]
$$

(3)

2. If the conditions above are met, then the closed loop performance $\| \mathcal{F}_I(G, K) \|_{[0, T]} < \gamma$ is achieved by the following central controller:

$$
\begin{align*}
\hat{x}(t) &= A_K(t) \hat{x}(t) + B_K(t) y(t) \\
u(t) &= C_K(t) \hat{x}(t)
\end{align*}
$$
where

\[
\begin{align*}
Z &:= (I - \gamma^{-2}XY)^{-1} \\
A_K &:= A + \gamma^{-2}BdB_d^TX - ZYCy^TC_y - BuB_u^TX \\
B_K &:= ZYCy^T \\
C_K &:= -B_u^TX
\end{align*}
\]

For a given \( \gamma > 0 \), the RDEs associated with \( X \) and \( Y \) are integrated backward and forward in time, respectively. If solution to both RDEs exist then the spectral radius coupling condition (3) is checked. If all three conditions are satisfied then the central controller achieves a closed-loop performance of \( \gamma \). The smallest possible value of \( \gamma \) is obtained using bisection. The results in [14, 16] also consider the effect of uncertain initial conditions.

3 Robust Performance

3.1 Uncertain LTV Systems

An uncertain, time-varying system \( F_u(N, \Delta) \) is shown in Figure 2. This consists of an interconnection of a known finite horizon LTV system \( N \) and a perturbation \( \Delta \). This perturbation represents block-structured uncertainties and/or nonlinearities. The term “uncertainty” is used for simplicity when referring to \( \Delta \). It is assumed throughout that the interconnection \( F_u(N, \Delta) \) is well-posed. A formal definition for well-posedness is given in [24, 9]. The LTV system \( N \) is described by the following state-space model:

\[
\begin{bmatrix}
\dot{x}_N(t) \\
v(t) \\
e_1(t) \\
e_E(t)
\end{bmatrix}
=
\begin{bmatrix}
A_N(t) & B_w(t) & B_d(t) \\
C_v(t) & D_{vw}(t) & D_{vd}(t) \\
C_{I_1}(t) & D_{Iw}(t) & D_{Id}(t) \\
C_E(t) & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_N(t) \\
v(t) \\
w(t) \\
d(t)
\end{bmatrix}
\] (4)

In addition to notations defined earlier \( v \in \mathbb{R}^{n_v} \) and \( w \in \mathbb{R}^{n_w} \) are signals associated with the uncertainty \( \Delta \). The state vector is denoted as \( x_N \in \mathbb{R}^{n_N} \) to refer to the states of system \( N \).

3.2 Worst-Case Gain

The robust performance of the uncertain system \( F_u(N, \Delta) \) is assessed using the worst-case gain as defined below.

**Definition 1.** Let an LTV system \( N \) be given by (4) and uncertainty \( \Delta : L_2^{n_v}[0, T] \rightarrow L_2^{n_w}[0, T] \) be in some set \( S \). Assume the interconnection \( F_u(N, \Delta) \) is well-posed. The worst-case gain is then defined as:

\[
\gamma_{wc} := \sup_{\Delta \in S} \| F_u(N, \Delta) \|_{[0, T]}
\]
The worst-case gain is the largest induced gain of the uncertain time-varying system over all uncertainties \( \Delta \) in set \( S \). This is difficult to compute directly as it involves an optimization over the entire uncertainty set. Instead, we focus on computing an upper bound on the worst-case gain using dissipation inequalities and IQC conditions.

### 3.3 Integral Quadratic Constraints (IQC)

IQC [9] are used to describe the input-output behavior of \( \Delta \). A time-domain formulation is used here for the analysis of the uncertain time-varying system. This formulation is based on the graphical interpretation as shown in Figure 3. Time domain IQCs, as used in this paper, are defined for \( \Delta \) by specifying a filter \( \Psi \) and a finite-horizon constraint on the filter output \( z \).

**Example 1.** Let \( S \) denote the set of LTI uncertainties \( \Delta \in \mathbb{RH}_\infty \) with \( \| \Delta \|_\infty \leq 1 \). Let \( (\Psi, M) \) be defined as follows:

\[
\Psi := \begin{bmatrix} \Psi_{11} & 0 \\ 0 & \Psi_{11} \end{bmatrix} \quad \text{with} \quad \Psi \in \mathbb{RH}_{\infty}^{n_\Psi \times 1},
\]

\[
M := \begin{bmatrix} M_{11} & 0 \\ 0 & -M_{11} \end{bmatrix} \quad \text{with} \quad M \in \mathbb{S}^{n_\Psi} \quad \text{and} \quad M_{11} > 0.
\]

It is shown in Appendix II of [25] that the pair \( (\Psi, M) \) defines a valid time domain IQC for \( \Delta \) over any \( T < \infty \) i.e. \( S \subseteq \mathcal{I}(\Psi, M) \).

**Example 2.** Let \( S \) be the set of LTV parametric uncertainties \( \delta(t) \in \mathbb{R} \) with a given norm-bound \( \beta(t) \), i.e. \( w(t) = \delta(t) \cdot v(t), |\delta(t)| \leq \beta(t), \forall t \in [0, T] \). Let \( n_v = n_w = n \) and \( M_{11} : [0, T] \to \mathbb{S}^n \) be piecewise continuous with
\( M_{11}(t) > 0, \forall t \in [0, T] \). Then \( \Delta \) satisfies the IQC defined by the time-varying matrix:

\[
M(t) := \begin{bmatrix} \beta(t)^2 M_{11}(t) & 0 \\ 0 & -M_{11}(t) \end{bmatrix}
\]  

(8)

and a static filter \( \Psi := I_{2n} \), i.e. \( S \subseteq \mathcal{I} (\Psi, M) \).

A library of IQCs is provided in [9, 26] for various types of perturbations. Most IQCs are for bounded, causal operators with multipliers \( \Pi \) specified in the frequency domain. Under mild assumptions, a valid time-domain IQC \( \mathcal{I} (\Psi, M) \) can be constructed from \( \Pi \) via a \( J \)-spectral factorization [27].

### 3.4 Dissipation Inequality Condition

Consider an extended system as shown in Figure 4. This interconnection includes the IQC filter \( \Psi \) but the uncertainty \( \Delta \) has been removed. The precise relation \( w = \Delta(v) \) is replaced, for the analysis, by the constraint on the filter output \( z \).

![Figure 4: Analysis Interconnection](image)

The extended system of \( N \) (Equation 4) and \( \Psi \) (Equation 5) is governed by the following state space model:

\[
\begin{bmatrix}
\dot{x}(t) \\
z(t) \\
e_I(t) \\
e_E(t)
\end{bmatrix} = \begin{bmatrix} A(t) & B(t) & C_z(t) & D_z(t) \\
0 & C_I(t) & 0 & D_I(t) \\
e_I(t) & C_E(t) & 0 & D_{Ie}(t)
\end{bmatrix} \begin{bmatrix} x(t) \\
w(t) \\
d(t)
\end{bmatrix}
\]  

(9)

The extended state vector is \( x := [x_N \ x_\psi] \in \mathbb{R}^n \) where \( n := n_N + n_\psi \). The state-space matrices are given by:

\[
A := \begin{bmatrix} A_N & 0 \\
B_{I_0} C_\psi & A_\psi \end{bmatrix}, B := \begin{bmatrix} B_w \\
B_{I_0} D_{wz} + B_{I_0} D_{wd} \\
B_{I_0} D_{we} \end{bmatrix}
\]

\[
C_z := \begin{bmatrix} D_{I_0} C_w \\
C_I \end{bmatrix}, C_I := \begin{bmatrix} C_I \\
0 \end{bmatrix}, C_E := \begin{bmatrix} C_E \\
0 \end{bmatrix}
\]

\[
D_z := \begin{bmatrix} D_{I_0} D_{wz} + D_{I_0} D_{we} \\
D_{I_0} D_{we} \end{bmatrix}, D_I = \begin{bmatrix} D_{I_0 w} & D_{I_0 d} \end{bmatrix}
\]

The following differential linear matrix inequality (DLMI) is used to compute an upper bound on the worst-case gain of \( \mathcal{F}_u(N, \Delta) \).

\[
\begin{bmatrix}
P + A^T P + PA & PB \\
B^T P & 0
\end{bmatrix} + \begin{bmatrix} Q & S \\
ST & R
\end{bmatrix} + \begin{bmatrix} C_z^T & D_z^T \\
D_z^T & M \end{bmatrix} \leq -\epsilon I
\]  

(10)

This inequality depends on the IQC matrix \( M \). It is compactly denoted as \( \text{DLMI}_{\text{Rob}}(P, M, \gamma^2, t) \leq -\epsilon I \). This notation emphasizes that the constraint is a DLMI in \( (P, M, \gamma^2) \) for fixed \( N, \Psi \) and \( (Q, S, R, F) \).
states a sufficient DLMI condition to bound the generalized (robust) induced performance measure of \( F_u(N, \Delta) \). The proof uses IQCs [9] and a standard dissipation argument [28, 29, 30].

**Theorem 3.** Consider an LTI system \( N \) given by (4) and let \( \Delta : L^2_{\infty}[0,T] \rightarrow L^2_{\infty}[0,T] \) be an operator. Assume \( F_u(N, \Delta) \) is well-posed and \( \Delta \in \mathcal{I}(\Psi, M) \). Let \( Q : [0, T] \rightarrow S^n, S : [0, T] \rightarrow R^{n \times (n_u + n_d)} \), and \( F : [0, T] \rightarrow S^{(n_u + n_d)} \), and \( F \in \mathbb{R}^{n \times n} \) be defined as follows.

\[
Q := C^T \Pi C, \quad S := C^T \Pi D, \quad R := D^T \Pi D - \gamma^2 \text{diag}\{0_{n_u}, I_{n_d}\}, \quad F := C_E(T)^T C_E(T)
\]

(11)

If there exists \( \epsilon > 0, \gamma > 0 \) and a differentiable function \( P : [0, T] \rightarrow S^n \) such that \( P(T) \geq F \) and,

\[
\text{DLMI}_{\text{Rob}}(P, M, \gamma^2, t) \leq -\epsilon I \quad \forall t \in [0, T]
\]

(12)

then \( \|F_u(N, \Delta)\|_{[0,T]} < \gamma \).

**Proof.** Let \( d \in L^2[0,T] \) and \( x_N(0) = 0 \) be given. By well-posedness, \( F_u(N, \Delta) \) has a unique solution \( (x_N, v, w, e_I, e_E) \). Define \( x := \begin{bmatrix} z_N \end{bmatrix} \). Then \( (x, z, e_I, e_E) \) are a solution of the extended system (9) with inputs \((w, d)\) and initial condition \( x(0) = 0 \). Moreover, \( z \) satisfies the IQC defined by \((\Psi, M)\). Define a storage function by \( V(x,t) := x^T P(t) x \). Left and right multiply the DLMI (10) by \([x^T, w^T, d^T]\) and its transpose to show that \( V \) satisfies the following dissipation inequality for all \( t \in [0,T] \):

\[
\dot{V} + \begin{bmatrix} x \\ \frac{w}{d} \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ \frac{w}{d} \end{bmatrix} + \frac{z}{d}^T M z \leq -\epsilon d^T d.
\]

(13)

Use the choices for \((Q, S, R)\) to rewrite the second term as \( e_I^T e_I - \gamma^2 d^T d \). Integrate over \([0,T]\) to obtain:

\[
x(T)^T P(T) x(T) + \int_0^T z(t)^T M(t) z(t) dt + \|e_I\|^2_{2,[0,T]} \leq (\gamma^2 - \epsilon) \|d\|^2_{2,[0,T]}.
\]

Apply \( P(T) \geq F = C_E(T)^T C_E(T) \) and \( \Delta \in \mathcal{I}(\Psi, M) \) to conclude:

\[
\|e_E(T)\|^2_2 + \|e_I\|^2_{2,[0,T]} \leq (\gamma^2 - \epsilon) \|d\|^2_{2,[0,T]}.
\]

(14)

This inequality implies \( \|F_u(N, \Delta)\|_{[0,T]} < \gamma \). \( \square \)

### 3.5 Computational Approach

Numerical implementation using IQCs often involve a fixed choice of \( \Psi \) and optimization subject to convex constraints on \( M \). Two examples are provided as follows.

**Example 3.** Consider an LTI uncertainty \( \Delta \in \mathbb{R}^{H_{\infty}} \) with \( \|\Delta\|_{\infty} \leq 1 \). By Example 1, \( \Delta \) satisfies any IQC \((\Psi, M)\) with \( \Psi := \begin{bmatrix} \Psi_{11} & 0 \\ 0 & \Psi_{11} \end{bmatrix}, M := \begin{bmatrix} M_{11} \\ M_{11} \end{bmatrix}, \) and \( M_{11} \succ 0 \). A typical choice for \( \Psi_{11} \) is:

\[
\Psi_{11} := \begin{bmatrix} 1, \frac{1}{(s+p)}, \ldots, \frac{1}{(s+p)^q} \end{bmatrix}^T \text{ with } p > 0
\]

(15)

The analysis is performed by selecting \((p, q)\) to obtain (fixed) \( \Psi \) and optimizing over the convex constraint \( M_{11} \succ 0 \). The results depend on the choice of \((p, q)\). Larger values of \( q \) represent a richer class of IQCs and hence yield less conservative results but with increasing computational cost. Note that the IQC filter \( \Psi \) is not square in general with \( n_z = 2(q + 1) \) outputs.

**Example 4.** Conic combinations of multiple IQCs can be incorporated in analysis. Let \((\Psi_i, M_i)\) with \( i = 1, 2, \ldots, N \) define \( N \) valid IQCs for \( \Delta \). Hence \( \int_0^T z_i^T M z_i dt \geq 0 \) where \( z_i \) is the output \( \Psi_i \) driven by \( v \) and \( w = \Delta(v) \). The
multiple constraints can be multiplied by $\lambda_i \geq 0$ and combined to yield:

$$\int_0^T \sum_{i=1}^N \lambda_i z_i^T M_i z_i \, dt \geq 0$$  \hfill (16)$$

Thus a valid time-domain IQC for $\Delta$ is given by

$$\Psi := \begin{bmatrix} \Psi_1 \\ \vdots \\ \Psi_N \end{bmatrix} \text{ and } M(\lambda) := \begin{bmatrix} \lambda_1 M_1 \\ \vdots \\ \lambda_N M_N \end{bmatrix}$$  \hfill (17)$$

The analysis optimizes over $\lambda$ given selected $(\Psi_i, M_i)$.

An iterative algorithm given in [17] is used in this paper to compute the smallest upper bound on the worst-case gain. It combines the DLMI formulation in the Theorem 3 with a related Riccati Differential Equation (RDE). The algorithm returns the upper bound $\bar{\gamma}_{\text{wc}}$ along with the decision variables $P$ and $M$.

4 Robust Synthesis

4.1 Problem Formulation

An uncertain feedback interconnection is shown in Figure 5 where $G$ is an LTV system on $[0, T]$ and $\Delta$ is assumed to lie in some set $\mathcal{S}$ that is described by valid time domain IQCs.

![Figure 5: Uncertain Feedback Interconnection $\mathcal{F}_u(\mathcal{F}_I(G, K), \Delta)$](image)

The finite horizon robust synthesis problem is to synthesize a controller which minimizes the impact of both worst-case disturbances and worst-case uncertainties, i.e.:

$$\inf_K \sup_{\Delta \in \mathcal{S}} \| \mathcal{F}_u(\mathcal{F}_I(G, K), \Delta) \|_{[0, T]}$$  \hfill (18)$$

Let the LTV system $G$ defined on $[0, T]$ be given as:

$$\begin{bmatrix} \dot{x}_G(t) \\ v(t) \\ e_I(t) \\ e_E(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A_G(t) & B_w(t) & B_d(t) & B_u(t) \\ C_v(t) & D_{vw}(t) & D_{vd}(t) & D_{vu}(t) \\ C_I(t) & D_{lw}(t) & D_{ld}(t) & D_{lu}(t) \\ C_E(t) & 0 & 0 & 0 \\ C_y(t) & D_{gw}(t) & D_{gd}(t) & D_{yu}(t) \end{bmatrix} \begin{bmatrix} x_G(t) \\ w(t) \\ d(t) \\ u(t) \end{bmatrix}$$  \hfill (19)$$
where $x_G \in \mathbb{R}^{n_G}$ is the state. This plant structure has no feedthrough from $d$ to $e_E$ for well-posedness. The synthesis problem (18) involves the worst-case gain computed over the entire uncertainty set. As noted earlier, instead we focus on minimizing worst-case gain upper bounds. In other words, we define IQCs $I(\Psi, M)$ such that $S \subseteq I(\Psi, M)$ and maximize over $\Delta \in I(\Psi, M)$ in Equation (18). The goal is to design a linear time-varying controller $K : L_2^{n_G}[0, T] \rightarrow L_2^{n_Y}[0, T]$ to minimize the worst-case gain upper bound on $F_n(F(G, K), \Delta)$. This leads to a non-convex synthesis problem and involves solving for the controller as well as for IQC multipliers.

The approach taken here is to decompose the synthesis into two subproblems. First, solve a nominal synthesis problem (on a specially constructed scaled plant) to obtain $K$. Second, solve an IQC analysis problem to compute the worst-case gain upper bound. These subproblems can be solved iteratively, similar to coordinate descent, to get a reasonable sub-optimal solution. The proposed algorithm utilizes this approach to obtain a finite horizon sub-optimal controller. As with DK synthesis, there are no guarantees that the coordinate-wise iteration will lead to a local optima let alone a global optima. However, it is a useful heuristic that will enable the robust synthesis to extended naturally from LTI to LTV systems. The following assumption is made for the structure of IQC matrix $M$ and filter $\Psi$.

**Assumption 1.** The IQC decision variables $M : [0, T] \rightarrow \mathbb{S}^{n_2}$ for a specified IQC filter $\Psi : L_2^{(n_u+n_w)}[0, T] \rightarrow L_2^{n_2}[0, T]$ are assumed to have the following block diagonal structure

$$M(t) := \begin{bmatrix} M_v(t) & 0 \\ 0 & -M_w(t) \end{bmatrix}, \quad \Psi := \begin{bmatrix} \Psi_v & 0 \\ 0 & \Psi_w \end{bmatrix}$$

with constraints $M_v(t) \succ 0$ and $M_w(t) \succ 0$, $\forall t \in [0, T]$. Moreover, $\Psi$ has a feedthrough matrix $D_{\Psi}(t) := \begin{bmatrix} D_{\Psi v}(t) & D_{\Psi w}(t) \end{bmatrix} \in \mathbb{R}^{n_2 \times (n_u+n_w)}$ with full column rank $\forall t \in [0, T]$.

This block diagonal assumption is made to simplify the notation. More general IQC multipliers are considered for (infinite-horizon) synthesis in [8]. As discussed in Example 3, the IQC filter $\Psi$, is typically pre-specified by a collection of basis functions. In this case, the worst-case gain condition in Theorem 3 is a differential LMI in the variables $M_v$, $M_w$, $P$, and $\gamma^2$. The filter $\Psi$ is, in general, non-square with $n_z \neq n_v + n_w$. The proposed synthesis method requires a non-unique factorization such that resulting factor is invertible square system i.e. $n_z = n_v + n_w$. The finite horizon factorization (Lemma 2 in Appendix B) can be used to construct square invertible systems $U_v$ and $U_w$ such that,

$$\begin{align*}
\Psi_v M_v \Psi_v &= U_v^* U_v \\
\Psi_w M_w \Psi_v &= U_w^* U_w
\end{align*}$$

The assumption that feedthrough matrix $D_{\Psi}(t)$ has full column rank is required for the existence of such factorization. This factorization is used in the proposed synthesis algorithm below to construct a scaled plant.

### 4.2 Algorithm

A high-level overview of the proposed iterative method is given in Algorithm 1. The uncertain finite horizon system is $F_n(F(G, K), \Delta)$ with $G$ given by Equation (19) and $\Delta$ specified by uncertainty set $I(\Psi, M)$. The robust synthesis algorithm is specified to run a given maximum number of iterations $N_{syn}$. It is initialized with scalings $U_v^{(0)} := I_{n_v}$ and $U_w^{(0)} := I_{n_w}$. There is also an initial performance scaling set to $\gamma_s^{(0)} := 1$.

The beginning of each iteration involves the construction of a scaled plant $G_{scl}$ as shown in Figure 6. This step is described further in the next subsection. For now, it is sufficient to note that $G_{scl} = G$ on the first iteration due to the initialization choices. The next step is to perform finite horizon nominal synthesis on the scaled plant. This step is performed using the synthesis results described previously in Section 2.2. This yields a controller $K^{(i)}$ and the achievable closed-loop performance $\gamma_s^{(i)}$. Each iteration concludes with an IQC analysis on the uncertain closed-loop of $N := F_n(G, K^{(i)})$ and $\Delta$ as shown in Figure 5. This closed-loop uses the original (unscaled) plant $G$ and the controller $K^{(i)}$ obtained from the nominal synthesis step. The worst-case gain upper bound $\gamma_a^{(i)}$ is computed using the algorithm in [17] as summarized in Section 3. This iterative algorithm requires additional initialization including
Algorithm 1 Finite Horizon Robust Synthesis

1: Given: G
2: Initialize: \( N_{\text{syn}}, U_v^{(0)} := I_{n_v}, U_w^{(0)} := I_{n_w}, \gamma_a^{(0)} := 1 \)
3: for \( i = 1 : N_{\text{syn}} \) do
4: Scaled Plant Construction (Section 4.3): Construct a scaled plant \( G_{\text{scl}}^{(i)} \) using \( G, U_v^{(i-1)}, U_w^{(i-1)}, \gamma_a^{(i-1)} \).
   Output: \( G_{\text{scl}}^{(i)} \)
5: Nominal LTV Synthesis (Section 2.2): Perform nominal controller synthesis on the scaled plant \( G_{\text{scl}}^{(i)} \).
   Output: \( K^{(i)}, \gamma_s^{(i)} \)
6: IQC Analysis (Section 3): Choose the basis functions for \( \Psi \) and perform worst-case gain iterations on \( F_u(N^{(i)}, \Delta) \) using iterative algorithm presented in [17] where \( N^{(i)} := F_l(G, K^{(i)}) \) denotes the closed loop LTV system. Perform finite horizon factorization using the same \( \Psi \) and computed decision variables \( M^{(i)} \) to compute the uncertainty channel scalings \( U_v^{(i)} \) and \( U_w^{(i)} \).
   Output: \( P^{(i)}, M^{(i)}, \gamma_a^{(i)}, U_v^{(i)}, U_w^{(i)} \)
7: end for

number of analysis iterations \( N_{\text{iter}} \), stopping tolerance \( \text{tol} \), DLM time grid \( t_{\text{DLMI}} \) and spline basis function time grid \( t_{\text{sp}} \), which are not included in Algorithm 1. All subsequent iterations require the construction of a scaled plant using the IQC results. The construction of this scaled plant links together the nominal synthesis and IQC analysis steps. It is described further in Section 4.3. Algorithm 1 terminates after \( N_{\text{syn}} \) iterations. More sophisticated stopping conditions can be employed. For example, the iterations could be terminated if no significant improvement in worst-case gain is achieved. The algorithm returns the controller of order \( n_K \) that achieves the best (smallest) bound on the worst-case gain, where \( n_K = n_G + n_\psi \).

4.3 Construction of a Scaled Plant

The scaled open loop plant \( G_{\text{scl}}^{(i)} \) is constructed as shown in Figure 6 by scaling the performance channels and uncertainty channels of original open loop plant \( G \) using \( U_v^{(i-1)}, U_w^{(i-1)} \) and \( \gamma_a^{(i-1)} \) obtained from the previous iteration. This scaling ensures appropriate normalization of the performance and uncertainty channels. This is a key step which integrates the nominal synthesis and worst-case gain problem. To simplify the notation, the superscripts \((i-1)\) will be dropped in the remainder of this subsection.

Let \( N := F_l(G, K) \) be the closed-loop (without uncertainty). For a given IQC filter \( \Psi \) an extended system \( N_{\text{ext}} \) similar to Figure 4 can be constructed. The next lemma gives a formal statement connecting robust performance of the extended system \( N_{\text{ext}} \) to nominal performance of scaled system \( N_{\text{scl}} \) as shown in Figure 7.
Lemma 1. Let $\epsilon > 0$, $\gamma > 0$, $M_v(t) > 0$, $M_w(t) > 0$ and a differentiable function $P : [0, T] \rightarrow \mathbb{S}^n$ such that $P(T) \succeq F$ be given with the choice of $(Q, S, R, F)$ as in Equation (11). The following statements are equivalent.

1. $\text{DLMIRob}(P, M, \gamma^2, t) \leq -\epsilon I \forall t \in [0, T]$

2. $\|N_{scl}\|_{[0,T]} \leq 1 - \epsilon$

Proof. A proof of this lemma is given in Appendix A. It uses a time-varying factorization of $(\Psi, M)$ to construct $U_v$ and $U_w$.

The above lemma states that extended system given by Equation (9) satisfies the robust performance condition (12) if and only if the scaled system has nominal performance less than 1.

4.4 Main Theorem

The plant $G^{(1)}_{scl} = G$ for the robust synthesis may include the uncertainty and performance channel design weights as in standard robust control workflow [24, 31]. These weights can be static, dynamic and/or time-varying depending on the requirement. Typically, multiple design iterations are performed to tune these weights and yield an acceptable trade-off between robustness and performance. Note that the first nominal LTV synthesis step in Algorithm 1 may not yield a finite performance $\gamma^{(1)}_{s}$. For example, if the uncertainty level is too high then the RDEs for nominal synthesis of $G^{(1)}_{scl} = G$ may not have a solution on $[0, T]$ for any finite $\gamma^{(1)}_{s}$. However, in this case, finite performance can be achieved by reducing the uncertainty level and restarting the iteration. The main theorem is presented next with a technical assumption that the first nominal synthesis step yields a finite performance.

Theorem 4. If the first nominal synthesis step yields a finite performance $\gamma^{(1)}_{s}$ then all the subsequent iterations are well-posed at each step and worst-case gain is non-increasing, i.e.

$$\gamma^{(i+1)}_{a} \leq \gamma^{(i)}_{a} \quad \forall i \geq 1$$

Proof. The first iteration ($i = 1$) is different from the subsequent one. Due to initialization choices $G^{(1)}_{scl} = G$. The synthesis step is performed with no modifications and yields a controller $K^{(1)}$ that guarantees the closed loop performance of $\gamma^{(1)}_{s}$. By assumption, we have $\gamma^{(1)}_{s} < \infty$. The IQC analysis step performed on the closed loop $N^{(1)} := F_t(G, K^{(1)})$ uncertain plant then achieves a finite horizon worst-case gain upper bound of $\gamma^{(1)}_{a} < \infty$. Thus, the first iteration is well-posed.

All subsequent iterations ($i > 1$) begin with the iteration count update in the for loop. The IQC analysis step from previous iteration shows that there exists $(P^{(i-1)}, M^{(i-1)}, \gamma^{(i-1)}_{a})$ for a chosen $\Psi$ that satisfies DLMIR (10). This implies that the finite horizon factorization exists and multipliers $U_v^{(i-1)}$ and $U_w^{(i-1)}$ can be obtained using Lemma 2 in Appendix B. Using these multipliers and worst-case gain $\gamma^{(i-1)}_{a}$, scaled plant similar to Figure 7 can be constructed. By Lemma 1, this scaled plant satisfies nominal performance $< 1$. Removing the controller yields the scaled open-loop plant $G^{(i)}_{scl}$. Thus, the construction of a scaled open-loop plant as shown in Figure 6 is well-defined. The synthesis step performed on $G^{(i)}_{scl}$ optimizes over all time-varying finite horizon controllers to yield a new controller $K^{(i)}$ that guarantees performance $\gamma^{(i)}_{s} < 1$. This new controller $K^{(i)}$ yields better nominal performance than the previous.
controller \(K^{(i-1)}\) when used with the unscaled plant \(G\). Thus, the closed loop \(N^{(i)} := F_u(G, K^{(i)})\) must satisfy the nominal performance \(< 1\) when using \(\gamma_i^{(i-1)}\). Lemma 1 can be used backwards in the next analysis step of \(N^{(i)}\). Specifically, the closed loop with unscaled plant \(G\) and \(K^{(i)}\) satisfies the DLMI analysis condition with \((P^{(i-1)}, M^{(i-1)}, \gamma_i^{(i-1)})\). Further, analysis step on \(N^{(i)} := F_l(G, K^{(i)})\) optimizes over all feasible \(P\) and \(M\). This yields a worst-case gain \(\gamma_a^{(i)}\) no greater than the previous step \(\gamma_i^{(i-1)}\). Thus \(\forall i \geq 1\), we have \(\gamma_i^{(i+1)} \leq \gamma_i^{(i)}\).

5 Example

Consider an example of a two link robot arm as shown in the Figure 8. The mass and moment of inertia of the \(i^{th}\) link are denoted by \(m_i\) and \(I_i\). The robot properties are \(m_1 = 3kg, m_2 = 2kg, l_1 = l_2 = 0.3m, r_1 = r_2 = 0.15m, I_1 = 0.09kg \cdot m^2,\) and \(I_2 = 0.06kg \cdot m^2\). The nonlinear equations of motion [32] for the robot are given by:

\[
\begin{bmatrix}
\alpha + 2\beta \cos(\theta_2) & \delta + \beta \cos(\theta_2) \\
\delta + \beta \cos(\theta_2) & \delta
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2
\end{bmatrix}
+
\begin{bmatrix}
-\beta \sin(\theta_2) \dot{\theta}_2 & -\beta \sin(\theta_2) (\dot{\theta}_1 + \dot{\theta}_2) \\
\beta \sin(\theta_2) \dot{\theta}_1 & 0
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\theta_2
\end{bmatrix}
=
\begin{bmatrix}
\tau_1 \\
\tau_2
\end{bmatrix}
\]

with

\[
\alpha := I_1 + I_2 + m_1 l_1^2 + m_2 (l_1^2 + r_2^2) = 0.4425 kg \cdot m^2
\]

\[
\beta := m_2 l_1 r_2 = 0.09 kg \cdot m^2
\]

\[
\delta := I_2 + m_2 r_2^2 = 0.105 kg \cdot m^2.
\]

The state and input are \(\eta := [\theta_1 \ \theta_2 \ \dot{\theta}_1 \ \dot{\theta}_2]^T\) and \(\tau := [\tau_1 \ \tau_2]^T\), where \(\tau_i\) is the torque applied to the base of link \(i\).

A trajectory \(\bar{\eta}\) of duration 5 second was selected for the tip of the arm to follow. This trajectory is shown as a solid black line in Figure 9. An equivalent trajectory in polar coordinates is also shown in Figure 10. The equilibrium input torque \(\bar{\tau}\) can be computed using inverse kinematics. The robot should track this trajectory in the presence of small torque disturbances \(d\). The input torque vector is \(\tau = \bar{\tau} + u + d\) where \(u\) is an additional control torque to reject the disturbances. The nonlinear dynamics (21) are linearized around the trajectory \((\bar{\eta}, \bar{\tau})\) to obtain an LTV system \(H\):

\[
\dot{x}(t) = A(t)x(t) + B(t)(u(t) + d(t))
\]

where \(x(t) := \eta(t) - \bar{\eta}(t)\) is the deviation from the equilibrium trajectory. An uncertain output feedback weighted interconnection of \(H\) is shown in the Figure 11. Let \(\delta \theta := \begin{bmatrix} \delta \theta_1 \\ \delta \theta_2 \end{bmatrix}\) represent first-order perturbations in angular positions, which is the output of interest \(e_E\). The measurement is also \(\delta \theta\) but corrupted by noise \(n = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}\) and is fed to the controller as \(y = \delta \theta + n\). The controller generates a commanded torque \(u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\) is corrupted by input disturbance \(d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}\). The second control channel gets further corrupted by LTI input uncertainty \(\Delta\). The plant input perturbation \(\Delta\) is a SISO LTI system with \(\|\Delta\|_{\infty} \leq \beta\) where uncertainty level \(\beta = 0.8\). This corresponds to the uncertainty set as
The synthesis objective is to minimize the closed-loop, worst-case gain from the generalized disturbance \( \tilde{d} := [\tilde{d}_1 \ \tilde{d}_2 \ \tilde{n}_1 \ \tilde{n}_2] \) to the generalized error \( \tilde{e} := [\tilde{e}_E] \). The weighted control effort \( \tilde{u} \) is penalized in an \( L_2 \) sense while \( \tilde{e}_E \) is penalized with a terminal Euclidean norm at \( T = 5 \) second. Let \( I_2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \). The following (constant) design weights are chosen for the performance channels:

\[
W_d = 0.1 \ I_2, \quad W_n = 0.01 \ I_2, \quad W_u = 0.5 \ I_2, \quad W_E = I_2
\]

The design weight associated with the uncertainty channels are not considered in this example, however, in general, the weights \( W_v \) and \( W_w \) can also be used for the respective uncertainty channels. As noted earlier, these design weights can be dynamic and/or time-varying. Let \( \tilde{G} \) denote this weighted design interconnection for robust synthesis. It can be expressed in state-space form as in Equation (19). Algorithm 1 is run with \( N_{syn} = 7 \) iterations. No significant improvement is obtained after \( 7 \)th iteration. The IQC analysis step is performed based on the approach in [17] and using parameterization similar to Example 3 with \( p = 10, \ q = 1, \ tol = 5 \times 10^{-3}, \ N_{iter} = 10, \ t_{DLMI} = 20 \) and \( \tau_{sp} \) as 10 evenly spaced grid points on the horizon \( [0, 5] \) seconds.

Let \( K_{0.8} \) denote the controller obtained at the end of the robust synthesis algorithm. This controller achieves the closed-loop worst-case performance of \( \gamma_{wc} = 0.126 \). It took 11.7 hours to complete the 7 iterations on a standard desktop computer with 3 GHz Core i7 processor. In addition, a nominal synthesis with \( \Delta = 0 \) was performed using the approach in Section 2.2. This controller, denoted as \( K_0 \), achieves a closed-loop nominal performance of \( \gamma_0 = 0.085 \). It took 49.8 seconds to perform this nominal synthesis. The corresponding uncertain closed-loops with the nominal and robust controllers are denoted by \( \tilde{T}_0 := \mathcal{F}_u(\tilde{G}(K_0), \Delta) \) and \( \tilde{T}_{0.8} := \mathcal{F}_u(\tilde{G}(K_{0.8}), \Delta) \). Figure 12 shows the worst-case performance versus the uncertainty level \( \beta \) for the uncertain closed-loops with these two controllers. The curve for \( \tilde{T}_0 \) (blue circles) has \( \gamma = 0.085 \) at \( \beta = 0 \) as reported above. The curve for \( \tilde{T}_{0.8} \) (red squares) has \( \gamma = 0.126 \).
at $\beta = 0.8$ as also reported above. This figure reveals the typical trade-off between performance and robustness. The nominal controller $K_0$ achieves better nominal performance ($\beta = 0$) than $K_{0.8}$. However, $K_{0.8}$ is more robust to higher levels of uncertainties.

![Figure 12: Worst-Case Gain Comparison](image)

As noted earlier, the primary design goal was to tightly bound the states at the final time $T = 5$ seconds. To study this further consider the impact of disturbance $d$ on the Euclidean outputs $e_E$. Let $G$ denote the unweighted plant which has the same inputs/outputs as the weighted plant $\tilde{G}$ but with all weights set to identity. Further, let the respective uncertain interconnection using $G$ be denoted as $T_0 := F_u(F_l(G, K_0), \Delta)$ and $T_{0.8} := F_u(F_l(G, K_{0.8}), \Delta)$. Nominal analysis was performed for both the $T_0(d \rightarrow e_E)$ and $T_{0.8}(d \rightarrow e_E)$ interconnections using bisection. The algorithm gives both upper and lower bounds on the nominal performance. The upper bounds are obtained as 0.65 and 0.766 respectively, which are shown as blue and red disk in Figure 13 at the final time. The corresponding lower bounds are obtained as 0.647 and 0.763. The worst-case disturbance $\|d\|_2[0,T] \leq 0.5$ for both interconnections are computed by solving the two point boundary value problem as presented in [33]. These specific bad disturbances pushes the state trajectory (dashed line) as far as the computed lower bound in the LTV simulation.

![Figure 13: Nominal Analysis ($\beta = 0$)](image)

![Figure 14: Robust Analysis ($\beta = 0.8$)](image)
A worst-case terminal Euclidean norm bound is computed for both the interconnections at the uncertainty level $\beta = 0.8$. The corresponding upper bound using the algorithm in [17] was obtained as 1.071 and 0.926, respectively. This shows approximately a 13.5% reduction in Euclidean norm bound. As a graphical illustration, these bounds are depicted in Figure 14 as a disk at the final time $T = 5$ seconds. The bound accounts for all the disturbances $d$ that satisfy $\|d\|_{2,[0,T]} \leq 0.5$ and all the LTI uncertainties $\Delta$ with norm bound $\beta = 0.8$.

To obtain a reasonable lower bound on the worst-case gain, first 100 uncertainties are sampled randomly as first order linear time-invariant (LTI) systems with at most size 0.8. Then, uncertainty block $\Delta$ was replaced with each of sampled uncertainties and nominal LTV analysis was performed from $d$ to $e$ for both the interconnections. Worst-case uncertainties are then obtained after maximizing performance over the sample space. Let, the specific bad perturbation that yields to the poor performance for both $T_0(d\rightarrow e)$ and $T_{0.8}(d\rightarrow e)$ be denoted as $\Delta_{wc1}$ and $\Delta_{wc2}$ respectively.

\[ \Delta_{wc1} = -0.8s + 12.18 \quad \frac{s}{s + 15.23}, \quad \Delta_{wc2} = -0.8s + 25.89 \quad \frac{s}{s + 32.36}, \]

The worst-case gain lower bound corresponding to these perturbations are obtained as 1.001 and 0.903 respectively. It is evident that a combination of worst-case disturbance (scaled to have size 0.5) and uncertainty (of size 0.8) pushes the states of the closed loop system (dashed line) as far as the lower bound of the worst-case gain. Overall, these simple comparison results show a typical robustness and performance trade-off. The nominal controller performs best at no uncertainty whereas the robust controller performs better at modeled uncertainty level.

6 Conclusions

This paper proposed an iterative algorithm to design a controller that bounds the worst-case gain of an uncertain LTV system on a finite horizon. The gain included both an induced $L_2$ and terminal Euclidean norm penalty on the output. Dynamic IQCs were used to describe the input output behavior of the uncertainty. The proposed method was explained using the two-link robot arm example.

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A Proof of Lemma 1

Lemma 1. Let $\epsilon > 0$, $\gamma > 0$, $M_b(t) > 0$, $M_w(t) > 0$ and a differentiable function $P : [0,T] \rightarrow \mathbb{S}^n$ such that $P(T) \geq F$ be given with the choice of $(Q,S,R,F)$ as in Equation (11). The following statements are equivalent.

1. $DLMI_{Rob}(P,M,\gamma^2,t) \leq -\epsilon I \quad \forall t \in [0,T]$

2. $\|N_{scl}\|_{[0,T]} \leq 1 - \epsilon$

Proof: (1 $\Rightarrow$ 2) This proof is presented in two parts. First, we show that $DLMI_{Rob}(P,M,\gamma^2,t) \leq -\epsilon I \quad \forall t \in [0,T]$ can equivalently be written as an inequality with only single IQC. Next, we show that the state-space realization of the extended system $N_{ext}$ and the scaled system $N_{scl}$ are indeed the same, which allow us to rewrite the robust performance DLMI as a nominal performance DLMI for $N_{scl}$. Integrating the related dissipation inequality completes the proof. To begin, define a storage function $V(x,t) := x^TP(t)x$. Left and right multiply the DLMI (10) by
\[ [x^T, w^T, d^T] \text{ and its transpose to show that } V \text{ satisfies the following dissipation inequality for all } t \in [0, T] \]

\[
\dot{V} + \begin{bmatrix} x \end{bmatrix}^T \begin{bmatrix} Q & S \\ ST & R \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + z^T Mz \leq -\epsilon d^T d.
\] (22)

where \( x = \begin{bmatrix} z \n_x \end{bmatrix} \in \mathbb{R}^n \) is the state of the extended system as shown in Figure 4. Consider the outputs of the IQC filter \( \Psi = \begin{bmatrix} 0 & 0 \end{bmatrix} \) be partitioned as \( z \) := \( \begin{bmatrix} z_w \end{bmatrix} \). Let \( \Psi_v \) have the following state-space representation with state \( x_v \), input \( v \), and output \( z_w \):

\[
\begin{align*}
\dot{x}_v(t) &= A_1 x_v(t) + B_1 v(t) \\
z_v(t) &= C_1 x_v(t) + D_1 v(t)
\end{align*}
\] (23)

A similar state-space expression also holds for \( \Psi_w \) with matrices \( (A_2, B_2, C_2, D_2) \), state \( x_w \), input \( w \), and output \( z_w \).

Thus the term \( z^T Mz \) in (22) can be expressed as:

\[
z^T Mz = z^T v M_v z_v - z^T w M_w z_w
\]

\[
= \begin{bmatrix} x_v \\ v \end{bmatrix}^T C_1^T M_v \begin{bmatrix} C_1 & D_1 \end{bmatrix} \begin{bmatrix} x_v \\ v \end{bmatrix} - \begin{bmatrix} x_w \\ w \end{bmatrix}^T C_2^T M_w \begin{bmatrix} 0 & D_2 \end{bmatrix} \begin{bmatrix} x_w \\ w \end{bmatrix}
\] (24)

First, consider only the terms involving \( v \) and define the quadratic storage matrices as:

\[
\begin{bmatrix} Q_v & S_v \\ S_v^T & R_v \end{bmatrix} := \begin{bmatrix} C_1^T \\ D_1^T \end{bmatrix} M_v \begin{bmatrix} C_1 & D_1 \end{bmatrix}
\] (25)

By Lemma 2 in Appendix B, the condition \( M_v(t) > 0, \forall t \in [0, T] \) implies that there exists \( X_v : [0, T] \to \mathbb{S}^{n_v} \) such that:

\[
\dot{X}_v + A_1^T X_v + X_v A_1 + Q_v - (X_v B_1 + S_v) R_v^{-1} (X_v B_1 + S_v)^T = 0, \quad X_v(T) = 0
\] (26)

Moreover, Lemma 2 in Appendix B also implies that there exists a spectral factor \( U_v \) with a state-space realization as \( (A_1, B_1, \tilde{C}_1, \tilde{D}_1) \) with \( \tilde{C}_1 := W_v^{-1} B_1^T \begin{bmatrix} X_v & S_v \end{bmatrix} \), \( \tilde{D}_1 := W_v \) and \( R_v = W_v^T W_v \). Note that \( x_v \) is the state and \( \tilde{v} \) is the output of the spectral factor \( U_v \). The RDE (26) can be re-written in terms of the state matrices of \( U_v \) as:

\[
Q_v = -\dot{X}_v - A_1^T X_v - X_v A_1 + \tilde{C}_1^T \tilde{C}_1
\] (27)

substitute \( Q_v \) in (25) using the ARE and \( S_v^T = \tilde{D}_1^T \tilde{C}_1 - B_1^T X_v \) to obtain the following expression:

\[
z_v^T M_v z_v = - x_v \dot{x}_v x_v - (A_1 x_v + B_1 \tilde{v})^T X_v x_v - x_v^T X_v (A_1 x_v + B_1 \tilde{v}) + (\tilde{C}_1 x_v + \tilde{D}_1 \tilde{v})^T (\tilde{C}_1 x_v + \tilde{D}_1 \tilde{v})
\] (28)

This can be simplified to the following expression:

\[
z_v^T M_v z_v = - x_v^T \dot{x}_v x_v - x_v^T X_v x_v - x_v^T X_v \dot{x}_v + \tilde{v}^T \tilde{v}
\]

\[
= - \frac{d}{dt} (x_v^T X_v x_v) + \tilde{v}^T \tilde{v}
\] (29)

Similarly, with \( M_w(t) > 0, \forall t \in [0, T] \) the spectral factor \( U_w \) can be obtained with a state-space realization \( (A_2, B_2, \tilde{C}_2, \tilde{D}_2) \), states \( x_w \) and outputs \( \tilde{w} \). The following expression holds:

\[
z_w^T M_w z_w = - \frac{d}{dt} (x_w^T X_w x_w) + \tilde{w}^T \tilde{w}
\] (30)
where \( X_w : [0,T] \to \mathbb{S}^{n_{xw}} \) is a solution to a related RDE with respective quadratic storage matrices and the boundary condition \( X_w(T) = 0 \). Subtract Equation (30) from (29) to get the left hand side of the Equation (24) as follows:

\[
\int_0^T \dot{z}^T M z \, dt = -d \left[ x_\psi^T X x_\psi \right] + 2 \dot{\tilde{v}}^T \tilde{w} - \tilde{w}^T \tilde{w}
\]

where \( x_\psi = \begin{bmatrix} \frac{\dot{x}_w}{x_w} \end{bmatrix} \), \( X(t) := \begin{bmatrix} X_w(t) & 0 \\ 0 & -X_w(t) \end{bmatrix} \) and \( X(T) = 0 \). Let the modified matrix \( \hat{P}(t) \) be defined as follows:

\[
\hat{P}(t) := P(t) - \begin{bmatrix} 0 & 0 \\ 0 & X(t) \end{bmatrix}
\]

This yields a modified storage function \( \tilde{V}(x,t) := x^T \hat{P}(t)x \). The modified storage function has the form:

\[
\tilde{V}(x,t) = V(x,t) - \frac{1}{2} x_w^T X_w x_w + \int_0^t x_w^T X_\psi x_\psi \, dt
\]

where the second and third term can be interpreted as hidden energy stored in the IQC multiplier. With modified storage function \( \tilde{V} \) the dissipation inequality (22) can be recast as,

\[
\dot{\tilde{V}} + \begin{bmatrix} x_w \\ \varphi \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x_w \\ \varphi \end{bmatrix} + \begin{bmatrix} \dot{\varphi} \\ \dot{\varphi} \end{bmatrix} + J_{n_{\psi},n_w} \begin{bmatrix} \dot{\varphi} \\ \dot{\varphi} \end{bmatrix} \leq -\epsilon d^T d
\]

This dissipation inequality is equivalent to (22) with a single IQC \( \mathcal{I}(U, J_{n_{\psi},n_w}) \) where \( U := \begin{bmatrix} U_\psi & 0 \end{bmatrix} \). Next, we show that \( \mathcal{I}(U, J_{n_{\psi},n_w}) \) is a valid time-domain IQC. To see this, define \( V_\psi(x_\psi(t),t) := x_\psi(t)^T X(t)x_\psi(t) \), \( \tilde{z}(t) := \begin{bmatrix} \dot{\varphi}(t) \\ \dot{\varphi}(t) \end{bmatrix} \) and integrate Equation (31) both sides from 0 to \( T \) to obtain:

\[
\int_0^T \dot{\tilde{z}}^T J_{n_{\psi},n_w} \tilde{z} \, dt = V_\psi(x_\psi(T),T) + V_\psi(x_\psi(0),0) - \int_0^T \dot{\tilde{z}}^T J_{n_{\psi},n_w} \tilde{z} \, dt
\]

Note that \( V_\psi(x_\psi(0),0) = 0 \) because \( x_\psi(0) = 0 \) and \( V_\psi(x_\psi(T),T) = x_\psi(T)^T X(T)x_\psi(T) = 0 \) due to the boundary condition \( X(T) = 0 \). The time-varying factorization RDE. Thus if \( \int_0^T \dot{\tilde{z}}^T J_{n_{\psi},n_w} \tilde{z} \, dt \geq 0 \) then we have \( \int_0^T \dot{\tilde{z}}^T J_{n_{\psi},n_w} \tilde{z} \, dt \geq 0 \). Finally, note that \( \hat{P}(t) \) satisfies the same boundary condition as \( P(t) \) i.e. \( \hat{P}(T) \geq \hat{F} \) because of the boundary condition \( X(T) = 0 \). Thus, \( \tilde{V}(x,t) \) is a valid storage function \( \forall t \in [0,T] \). Let the extended system \( N \) with spectral factor \( U \) be written in partitioned form as:

\[
\begin{bmatrix} \dot{x} \\ \dot{\varphi} \\ \dot{\varphi} \\ e_t \\ e_E \end{bmatrix} = \begin{bmatrix} A & B_w & B_d \\ C_\varphi & D_{uw} & D_{vd} \\ C_\varphi & D_{uw} & 0 \\ C_t & D_{tw} & D_{td} \\ C_E & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ d \end{bmatrix}
\]

where \( x = \begin{bmatrix} x_N \\ x_\psi \\ x_w \end{bmatrix} \in \mathbb{R}^n \) and state-space matrices:

\[
A := \begin{bmatrix} A_N & 0 & 0 \\ B_1 C_{v_} & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix}, B_w := \begin{bmatrix} B_w \\ B_1 D_{uw} \\ 0 \end{bmatrix}, B_d := \begin{bmatrix} B_d \\ B_1 D_{vd} \end{bmatrix}
\]

\[
C_\varphi := \begin{bmatrix} \hat{D}_1 C_{v_} & \hat{C}_1 \\ 0 & 0 \end{bmatrix}, C_t := \begin{bmatrix} C_t & 0 & 0 \\ 0 & 0 \end{bmatrix}, C_E := \begin{bmatrix} C_E & 0 & 0 \\ \hat{D}_1 D_{vw} & \hat{D}_1 D_{vd} \end{bmatrix}
\]

\[
D_{uw} := \hat{D}_2, D_{tw} := D_{tw}, D_{td} := D_{td}
\]
Using the choice of \((Q,S,R)\) from Equation (11), the following partitioned DLMI is equivalent to the dissipation inequality (34) for state-space realization of (36).

\[
\begin{bmatrix}
\dot{\tilde{P}} + A^T \tilde{P} + \tilde{P} A & \tilde{P} B_w & \tilde{P} B_d \\
B_w^T \tilde{P} & 0_{n_w} & 0 \\
B_d^T \tilde{P} & 0 & -\gamma^2 I_{n_d}
\end{bmatrix}
+ \begin{bmatrix}
C_{\tilde{v}}^T \\
C_{\tilde{v}}^T \\
C_{\tilde{v}}^T
\end{bmatrix}
\begin{bmatrix}
C_{\tilde{t}} & D_{\tilde{w}} & D_{\tilde{v}}
\end{bmatrix}
\begin{bmatrix}
\tilde{P} \\
D_{\tilde{w}}^T & D_{\tilde{v}}^T & D_{\tilde{v}}^T
\end{bmatrix}
\leq -\epsilon I
\]

(37)

The condition \(M_w(t) > 0, \forall t \in [0,T]\) is sufficient to ensure that \(D_{\tilde{w} \tilde{w}} := \tilde{D}_2\) is nonsingular. The output equation for \(\tilde{w}\) can be written as: \(\tilde{w} = D_{\tilde{w} \tilde{w}}^{-1}(\tilde{w} - C_{\tilde{w}} x)\). Use this relation to substitute for \(\tilde{w}\) in Equation (36). This gives the following scaled system:

\[
\begin{bmatrix}
\dot{\tilde{x}} \\
\dot{\tilde{v}} \\
e_I \\
e_E
\end{bmatrix}
= \begin{bmatrix}
A & B_w & B_d \\
0 & C_{\tilde{v}} & D_{\tilde{w} \tilde{w}} \\
0 & C_{\tilde{t}} & D_{\tilde{v} \tilde{w}} \\
0 & 0 & 0
\end{bmatrix}
L
\begin{bmatrix}
\tilde{x} \\
\tilde{w} \\
d
\end{bmatrix}
\]

(38)

where the nonsingular time-varying matrix \(L\) is defined as:

\[
L := \begin{bmatrix}
I_n & 0 & 0 \\
-D_{\tilde{w} \tilde{w}} C_{\tilde{v}} & D_{\tilde{w} \tilde{w}}^{-1} & 0 \\
0 & 0 & I_{n_d}
\end{bmatrix}
\]

(39)

Equation (38) can be rewritten as follows:

\[
\begin{bmatrix}
\dot{\tilde{x}} \\
\dot{\tilde{v}} \\
e_I \\
e_E
\end{bmatrix}
= \begin{bmatrix}
\tilde{A} & B_{\tilde{w}} & B_{\tilde{d}} \\
0 & C_{\tilde{v}} & D_{\tilde{w} \tilde{w}} \\
0 & C_{\tilde{t}} & D_{\tilde{v} \tilde{w}} \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{x} \\
\tilde{w} \\
d
\end{bmatrix}
\]

(40)

where the updated state-space matrices are defined as:

\[
\begin{align*}
\tilde{A} & := \begin{bmatrix}
A_N & 0 & -B_w \tilde{D}_2^{-1} C_2 \\
B_1 C_v & A_1 & -B_1 D_{vw} \tilde{D}_2^{-1} C_2 \\
0 & 0 & A_2 - B_2 \tilde{D}_2^{-1} C_2 
\end{bmatrix},
B_{\tilde{w}} & := \begin{bmatrix}
B_w \tilde{D}_2^{-1} \\
B_1 D_{vw} \tilde{D}_2^{-1} \\
B_2 \tilde{D}_2^{-1}
\end{bmatrix} \\
\tilde{C}_v & := \begin{bmatrix}
\tilde{D}_1 C_v \\
\tilde{C}_1 \\
-\tilde{D}_1 D_{vw} \tilde{D}_2^{-1} C_2
\end{bmatrix},
\tilde{C}_t & := \begin{bmatrix}
C_I & 0 & -D_{1w} \tilde{D}_2^{-1} C_2 
\end{bmatrix} \\
D_{\tilde{w} \tilde{w}} & := \tilde{D}_1 D_{vw} \tilde{D}_2^{-1},
D_{\tilde{w} \tilde{v}} & := D_{1w} \tilde{D}_2^{-1}
\end{align*}
\]

Note that the following state-space matrices of the inverse system of \(U_w\) shows up in above representation.

\[
U_w^{-1} := \begin{bmatrix}
A_2 - B_2 \tilde{D}_2^{-1} C_2 & B_2 \tilde{D}_2^{-1} \\
-D_{2} C_2 & D_2^{-1}
\end{bmatrix}
\]

(41)

Let scaled signal \(\tilde{d} := \gamma \hat{d}\) and state-space matrices \((A_{scl}, B_{scl}, C_{scl}, D_{scl})\) be defined as follows:

\[
A_{scl} := \tilde{A},
B_{scl} := B_{\tilde{w}} \gamma^{-1} B_d,
C_{scl} := \begin{bmatrix}
\tilde{C}_v \\
\tilde{C}_t
\end{bmatrix},
D_{scl} := \begin{bmatrix}
\tilde{D}_1 D_{vw} \tilde{D}_2^{-1} & \gamma^{-1} \tilde{D}_1 D_{vd} \\
D_{1w} \tilde{D}_2^{-1} & \gamma^{-1} D_{1d}
\end{bmatrix}
\]

(42)
It is readily verified that, with above definition, the scaled plant \(N_{scl}\) has a state-space realization as follows:

\[
\begin{bmatrix}
\dot{x} \\
\dot{v} \\
\dot{e}_l \\
e_E
\end{bmatrix} =
\begin{bmatrix}
A_{scl} & B_{scl} \\
C_{scl} & D_{scl}
\end{bmatrix}
\begin{bmatrix}
A \\
B \\
C \\
0
\end{bmatrix}
\begin{bmatrix}
x \\
\dot{w} \\
e_t
\end{bmatrix}
\] (43)

Perform the congruence transformation by multiplying the DLMI (37) on the left/right by \(L^T/L\) to get:

\[
\begin{bmatrix}
\dot{\hat{P}} + \hat{A}^T\hat{P} + \hat{P}\hat{A} & \hat{PB}_{\omega} & \hat{PB}_d \\
B_{\omega}^T\hat{P} & 0_{n_\omega} & 0 \\
B_d^T\hat{P} & 0 & -\gamma^2 I_{n_d}
\end{bmatrix}
\begin{bmatrix}
\hat{c}_I \\
D_{I\omega} \\
D_{Id}
\end{bmatrix}
\begin{bmatrix}
\hat{c}_I \\
D_{I\omega} \\
D_{Id}
\end{bmatrix}
\begin{bmatrix}
\hat{x} \\
\hat{w} \\
e_t
\end{bmatrix}
\leq -\epsilon I
\] (44)

This DLMI can also be written in more compact notation using the state matrices of \(N_{scl}\). Multiply inequality (44) left and right by \([x^T, \hat{w}^T, d^T]\) and its transpose to show that \(\hat{V}(x(t), t) = x(t)^T\hat{P}(t)x(t)\) satisfies the following dissipation inequality \(\forall t \in [0, T]\):

\[
\dot{\hat{v}} + e_t^T\hat{e}_l - \gamma^2 \hat{d}^Td + \hat{v}^T\hat{v} - \hat{\omega}^T\hat{\omega} \leq -\epsilon \hat{d}^Td
\] (45)

Define \(\hat{d} := \gamma d, \hat{\epsilon} := \epsilon \gamma^{-2}\) and combine the inputs \(\hat{w}, \hat{d}\) together to rewrite the inequality (45) as follows:

\[
\dot{\hat{v}} + e_t^T\hat{e}_l - \gamma^2 \hat{d}^Td + \dot{\hat{v}}\hat{v} - \hat{\omega}^T\hat{\omega} \leq -\hat{\epsilon} \hat{d}^T\hat{d}
\] (46)

Integrate over \([0, T]\) to obtain the following dissipation inequality:

\[
\hat{V}(x(T), T) - \hat{V}(x(0), 0) + \|e_{e_t}(0, T)\| - \|
\|e_{e_t}(0, T)\| - \|
\] (47)

Note that \(\hat{V}(x(0), 0) = 0\) as \(x(0) = 0\) and \(\hat{V}(x(T), T) = x(T)^T\hat{P}(T)x(T)\) with boundary condition \(\hat{P}(T) \geq F\) as shown earlier. Apply this boundary condition in inequality (47) with the definition of \(F\) from Equation (11) to conclude:

\[
\|e_{e_t}(T)\|^2 + \|e_{e_t}(0, T)\|^2 \leq \|e_{e_t}(0, T)\|^2 - \hat{\epsilon} \|\hat{d}\|^2
\] (48)

Divide both sides by \(\|e_{e_t}(0, T)\|^2 < \infty\) and define \(\hat{\epsilon} := \frac{\|e_{e_t}(0, T)\|^2}{\|e_{e_t}(0, T)\|^2}\) to show that \(\|N_{scl}\| \leq 1 - \hat{\epsilon}\). This proof can be worked backwards to prove \((2 \Rightarrow 1)\).

**Remark 1.** *If the IQC decision variables \(M\) and the state-space matrices of \(\Psi\) are constant on the given time horizon, then for sufficiently large time horizon \(T\), the RDE solution for the finite horizon factorization converges to that of the steady state Algebraic Riccati Equation (ARE). As a result, the state-space realization of the finite horizon factorization converges to the infinite horizon LTI spectral factorization. However, the ARE solution \(X\) for infinite horizon spectral factorization is sign indefinite, thus it fails to satisfy the terminal boundary condition on \(\hat{P}\). Thus, it is important to note in the above proof that in order to satisfy the boundary condition on storage function, one must use finite horizon factorization.*

**B  Finite Horizon Factorization**

For infinite-horizon LTI systems, spectral factorization results are found in standard robust control textbooks [24, 31, 34]. The following lemma provides a time-varying finite horizon generalization of this result.
Lemma 2. Consider an LTV system $\Psi : L^n_T[0, T] \rightarrow L^n_T[0, T]$ be given with state-space realization as follows:

\[
\begin{align*}
\dot{x}(t) &= A(t) x(t) + B(t) d(t) \\
\epsilon(t) &= C(t) x(t) + D(t) d(t)
\end{align*}
\] (49)

with $x \in \mathbb{R}^n$, $\epsilon \in \mathbb{R}^n$, $d \in \mathbb{R}^n$ and $D(t)$ is full column rank $\forall t \in [0, T]$. Let $M : [0, T] \rightarrow \mathbb{S}^n$ be a given piecewise continuous matrix valued function with $M(t) > 0, \forall t \in [0, T]$. Let $Q : [0, T] \rightarrow \mathbb{S}^n$, $S : [0, T] \rightarrow \mathbb{R}^{n\times(n_d)}$, $R : [0, T] \rightarrow \mathbb{S}^n$ be defined as follows.

\[
Q := C^TMC, \quad S := C^TMD, \quad R := D^TMD
\] (50)

with $R(t) > 0, \forall t \in [0, T]$. The following statements hold.

1. There exist a differentiable function $X : [0, T] \rightarrow \mathbb{S}^n$ such that $X(T) = 0$ and

\[
\dot{X} + A^T X + X A + Q - (XB + S)R^{-1}(XB + S)^T = 0
\] (51)

2. $\Phi := \Psi^{-1}M\Psi$ has a finite horizon factorization $\Phi = U^{-1}U$ where $U$ is square invertible LTV system defined on $[0, T]$ with the following state-space realization:

\[
U = \begin{bmatrix} A \\ W^{-T}(B^T \times S^{T}) \\ B \end{bmatrix}
\] (52)

where $R(t) = W(t)^TW(t), \forall t \in [0, T]$.

Proof. Since $R(t) > 0, \forall t \in [0, T]$, RDE does not have a finite escape time and thus always have a bounded unique solution regardless of the boundary condition (Corollary 2.3 of [35], Theorem 8 in [36]). Further, it can be verified that the time-varying state-space realization of $\Psi^{-1}M\Psi$ is related to that of a system $U^{-1}U$ by a similarity transformation matrix $\begin{bmatrix} X(t) & 0 \end{bmatrix}$.

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