BLOWUPS IN TAME MONOMIAL IDEALS

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Abstract. We study blowups of affine $n$-space with center an arbitrary monomial ideal and call monomial ideals that render smooth blowups tame ideals. We give a combinatorial criterion to decide whether the blowup is smooth and apply this criterion to discuss a smoothing procedure proposed by Rosenberg, monomial building sets and permutohedra.

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1. INTRODUCTION

In this paper we consider the question what happens when we choose a nonregular closed subscheme as center of a blowup and in particular we pose the question when the blowup of the affine space $\mathbb{A}^n_K$ in a nonreduced subscheme defined by a monomial ideal is a smooth scheme. The base field $K$ will almost exclusively be taken to have characteristic zero and moreover, we sometimes need $K$ to be algebraically closed. By a blowup of an algebraic variety $X$ in a closed subscheme $Z \subseteq X$ we mean an algebraic variety $\tilde{X}$ together with a proper birational map $\pi : \tilde{X} \to X$ that is an isomorphism outside $Z$. The closed subscheme $Z$ is then called the center of the blowup. When the center $Z$ is defined by an ideal $I$, the blowup in $Z$ is also called the blowup in $I$.

We give an example of a surface where a resolution is obtained with one blowup in a nonregular center (see [1]):

Example 1. Let $X \subseteq \mathbb{A}^3_K$ be the surface defined by $x^2 - y^3z^3 = 0$ (see fig. 1). The equation defining $X$ is invariant under the permutation of $y$ and $z$. The singular locus of $X$ is the coordinate cross $Z = V(x, yz)$. The traditional approach, as for example explained in [2][3][4], is to first blow up in the origin or in one of the axes. If we want to resolve $X$ by one blowup we have to use a singular center. As one easily computes, the blowup of $\mathbb{A}^3_K$ with center the reduced ideal $(x, yz)$ is singular. Hence the blowup of $X$ is embedded in a singular ambient scheme. One cannot speak of an “embedded resolution” of $X$. However, the blowup in the nonradical ideal $I = (x, y^2z, yz^2)(x, yz)(x, y)(x, y)^2(x, y)^2$ resolves $X$ and the ambient space is...
Figure 1. Image of the surface defined by \(x^2 - y^3z^3 = 0\).

smooth. To achieve normal crossings one can modify the ideal \(I\) or perform further blowups. See [5] for a detailed description of similar centers.

In this paper we focus on monomial ideals in \(K[x_1, \ldots, x_n]\) and blowups of \(\mathbb{A}^n\) in these ideals. In general, blowing up \(\mathbb{A}^n\) in a nonregular subscheme produces singularities. We call a monomial ideal that renders a smooth blowup \(\mathbb{A}^n\) a tame ideal. The restriction to monomial ideals guarantees that the blowup \(\mathbb{A}^n\) is a toric variety, so that we can reduce the question of tameness to combinatorics. The smooth affine toric varieties take a special simple form; they are a direct product of an affine space and a torus. Hence monomial ideals provide a convenient testing ground to study blowups in nonregular subschemes.

Theorem 9 gives a necessary and sufficient condition for a monomial ideal to be tame; the criterion uses the structure of the Newton polyhedron associated to the monomial ideal. We discuss several criteria for blowups in products of monomial ideals to be tame. We apply these criteria to three known constructions: (i) Rosenberg smoothing, (ii) blowing up in building sets of arrangements of linear subspaces and (iii) permutohedral blowups, that is, blowups in monomial centers that are invariant under any permutation of the coordinates.

Rosenberg [6] considers monomial ideals whose zero sets are unions of coordinate axes. These ideals are not tame. In [6] two constructions are given to modify such an ideal \(I\) such that the zero set is unchanged and the modified ideal is tame. The two constructions are intersecting and multiplying the ideal \(I\) with a suitably chosen ideal.

A set of subspaces of a vector space \(V\) that is closed under taking sums is called an arrangement of linear subspaces in \(V\). A building set of an arrangement is a subset of the arrangement such that any element \(U\) of the arrangement can be written as a direct sum of elements of the building set that are maximal with respect to inclusion in \(U\). De Concini and Procesi [7] have shown that one can construct so-called wonderful models of subspace arrangements by a sequence of blowups with the elements of a building set as centers. MacPherson and Procesi [8] and Li [9] generalized the construction of De Concini and Procesi to wonderful conical compactifications and to arrangements of subvarieties of a smooth variety, respectively. We consider the case of linear subspaces that are defined by monomial coordinate ideals and provide an elementary proof of the result of De Concini and Procesi in this particular setting.

The group \(S_n\) acts in a natural way on \(\mathbb{A}^n\) by permuting the coordinates. We consider subschemes that are invariant under \(S_n\) and can be written as unions of coordinate subspaces. In general, such subschemes are nonregular. We show that blowing up \(\mathbb{A}^n\) in a certain class of monomial ideals that are invariant under \(S_n\) and whose zero set is a union of coordinate subspaces results in a smooth toric variety \(\mathbb{A}^n\). This class of ideals is related to permutohedra (see e.g. [10]) and therefore these ideals are called permutohedral ideals.

The outline of the paper is as follows. In Section 2 we give a brief overview of blowups with the focus on monomial ideals, mainly to fix the notation and the setting. In Section 3 we state and prove the smoothness criterion, which is then applied to discuss the Rosenberg
smoothing procedure in Section 4. Criteria for the tameness of products of monomial ideals and in particular of products of coordinate ideals is studied in Section 5. We apply these criteria to building sets in Section 6 and to permutohedral ideals in Section 7.

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2. Blowups of \( \mathbb{A}^n \) in monomial ideals

We work with a smooth affine scheme \( W \cong \mathbb{A}^n_K \cong \text{Spec}(K[x_1, \ldots, x_n]) \) over a field \( K \). When the field \( K \) is irrelevant, we write \( \mathbb{A}^n \) instead of \( \mathbb{A}^n_K \). For an ideal \( I \) we write \( V(I) \) for its zero set. To keep the notation simple we write \( K[x] \) occasionally for \( K[x_1, \ldots, x_n] \) and for \( a = (a_1, \ldots, a_n) \in \mathbb{Z}^n \) we use \( x^a = x_1^{a_1} \cdots x_n^{a_n} \). Furthermore, the standard basis vectors in \( \mathbb{R}^n \) are denoted \( e_1, \ldots, e_n \). Below we shortly discuss the construction of the blowup of \( \mathbb{A}^n \) to fix the notation and to provide the setting in which we work. We also give some definitions that will be used frequently. For more complete discussions of and introductions to blowups and resolution of singularities we refer to \[11, 12, 13, 14\].

A blowup of \( \mathbb{A}^n \) is a scheme \( \mathring{\mathbb{A}}^n \) together with a projection \( \pi : \mathring{\mathbb{A}}^n \to \mathbb{A}^n \), the associated blowup map. Each blowup is completely determined by a closed subscheme \( Z \subset \mathbb{A}^n \), which is called the center of the blowup. Equivalently, the ideal \( I \) that defines \( Z \) determines the blowup completely. The center \( Z \) is the locus of points above which \( \pi \) is not an isomorphism.

We also say that we blow up the associated coordinate ring \( K[x_1, \ldots, x_n] \) with center \( I \). The blowup map \( \pi : \mathring{\mathbb{A}}^n \to \mathbb{A}^n \) is constructed as follows: since \( \mathbb{A}^n \) is Noetherian, \( I \) is finitely generated, say, \( I = (g_1, \ldots, g_k) \subset K[x_1, \ldots, x_n] \). The blowup \( \mathring{\mathbb{A}}^n \) of \( \mathbb{A}^n \) with center \( Z = V(I) \) is defined as the Zariski-closure of the graph of the map

\[
\sigma : \mathring{\mathbb{A}}^n \setminus Z \to \mathbb{P}^{k-1}, \quad p \mapsto (g_1(p) : \ldots : g_k(p)).
\]

Thus the blowup \( \mathring{\mathbb{A}}^n \) lives in \( \mathbb{A}^n \times \mathbb{P}^{k-1} \). The projection \( \pi : \mathring{\mathbb{A}}^n \to \mathbb{A}^n \) on the first factor is the blowup map, and the preimage \( \pi^{-1}(Z) \) of the center \( Z \) is called the exceptional locus. The blowup \( \mathring{\mathbb{A}}^n \) can be covered by \( k \) affine charts, each corresponding to a generator \( g_i \) of \( I \). The coordinate ring of the \( i \)-th chart is

\[
K[x_1, \ldots, x_n, g_1/g_i, \ldots, g_k/g_i].
\]

A smooth center defined by monomials is a coordinate subspace \( Z = V(x_1, \ldots, x_k) \subset \mathbb{A}^n \). The coordinate ring of the \( i \)-th affine chart of the blowup then reduces to

\[
K[x_1/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_k/x_i, x_{k+1}, \ldots, x_n].
\]

The associated blowup map \( \pi_{x_i} : \mathring{\mathbb{A}}^n \to \mathbb{A}^n \) is given in the \( i \)-th chart by

\[
x_j \mapsto \begin{cases} x_{i,j}, & \text{if } j \neq i \text{ and } j \leq k, \\ x_j, & \text{otherwise}. \end{cases}
\]

More generally, let \( \mathcal{I} \) be any monomial ideal in \( K[x_1, \ldots, x_n] \) and let \( \pi : \mathring{\mathbb{A}}^n \to \mathbb{A}^n \) be the blowup of \( \mathbb{A}^n \) with center \( \mathcal{I} \). Then \( \mathring{\mathbb{A}}^n \) is a toric variety, which is not necessarily normal and could be singular. Let \( \{x^a : a \in A\} \), with \( A \) a finite set in \( \mathbb{N}^n \), be a set of generators of \( \mathcal{I} \). The blowup \( \mathring{\mathbb{A}}^n \) of the affine space will be covered with the affine toric varieties \( U_a \) given by

\[
U_a := \text{Spec}(K[x_1, \ldots, x_n][x^{a'-n}, a' \in A \setminus \{a\}]).
\]

If \( U_a \) is smooth and \( K = \mathbb{C} \) then \( U_a \) is isomorphic to \( \mathbb{C}^{n-k} \times (\mathbb{C}^*)^k \) for some \( k \), see for example \[15\] chapter 2, \[16\]. We call \( U_a \) the \( a \)-chart of \( \mathring{\mathbb{A}}^n \).
Using monomial ideals has the advantage that we can apply techniques from convex geometry and toric geometry and that we can use combinatorial arguments. In the paragraphs below we recall some basic notions of convex geometry in order to introduce the notion of the ideal tangent cone and discuss some of its properties. Most of our definitions can be found in standard text books such as [15, 16, 17, 18, 19].

We call a subset \( C \subseteq \mathbb{R}^n \) a polyhedral cone if there exist vectors \( v_1, \ldots, v_l \in \mathbb{R}^n \), such that \( C = \{ \sum_{i=1}^l \lambda_i v_i : \lambda_1, \ldots, \lambda_l \in \mathbb{R}_+ \} \). We then say that \( C \) is generated by \( v_1, \ldots, v_l \) and write \( C = \mathbb{R}_+(v_1, \ldots, v_l) \). For the cone generated by the standard basis vectors of \( \mathbb{R}^n \) we write \( \mathbb{R}^n_+ \).

A polyhedral cone is called rational if one can find generators \( v_i \) with integer coordinates. We then call a generator \( v_i \in \mathbb{Z}^n \) primitive if its coordinates are relatively prime. For any cone \( C \) we define the associated lattice cone to be \( C \cap \mathbb{Z}^n \). If \( C \) is a rational polyhedral cone then \( C \cap \mathbb{Z}^n \) is finitely generated over \( \mathbb{Z} \) by Gordan’s lemma (see for example [15]). For any subset \( N \subseteq \mathbb{R}^n \) we write \( \text{conv}(N) \) for the convex hull of \( N \). The positive convex hull of a subset \( M \subseteq \mathbb{R}^n \) is the convex hull of the Minkowski sum of \( M \) with \( \mathbb{R}^n_+ \), written \( \text{conv}(M + \mathbb{R}^n_+) \). A polytope is the convex hull of a finite set of points. The Minkowski sum of two sets \( P \) and \( Q \) is the set consisting of all points \( p + q \) where \( p \) and \( q \) run over \( P \) and \( Q \), respectively. A polyhedron is the Minkowski sum of a polytope and a polyhedral cone.

**Remark 1.** We make an observation that will be used frequently in the sequel. Let \( a \) and \( b \) be two different points in the intersection of two polyhedra \( P \) and \( Q \). Then \( a + b \) is not a vertex of \( P + Q \). Indeed, \( 2a \) and \( 2b \) are in \( P + Q \) and \( a + b = \frac{1}{2}(2a) + \frac{1}{2}(2b) \).

**Definition 1.** Let \( P = \text{conv}(M) + \mathbb{R}_+(v_1, \ldots, v_l) \) with \( M \subseteq \mathbb{R}^n \) finite, be an unbounded polyhedron. If for a vertex \( m \) of \( \text{conv}(M) \) the ray \( m + \mathbb{R}_+v_i \) is an edge of \( P \) we call \( m + \mathbb{R}_+v_i \) the infinitely far vertex of \( P \) in direction \( v_i \).

The support of a monomial ideal \( \mathcal{I} \) is the set of all exponent vectors of monomials in \( \mathcal{I} \),

\[
\text{supp}(\mathcal{I}) = \{ a \in \mathbb{N}^n : x^a \in \mathcal{I} \}.
\]

Its convex hull \( \text{conv}(\text{supp}(\mathcal{I})) \) is called the Newton polyhedron of \( \mathcal{I} \), which is denoted by \( N(\mathcal{I}) \). Equivalently, for a finite set of generators \( x^a, a \in A, \) of \( \mathcal{I} \) the Newton polyhedron is defined as the positive convex hull \( \text{conv}(A + \mathbb{R}^n_+) \). It is important to note that \( N(\mathcal{I}) \cap \mathbb{Z}^n \supset \text{supp}(\mathcal{I}) \) but that the inclusion can be proper. For example, the ideals \((x,y)^2\) and \((x^2,y^2)\) in \( K[x,y] \) have the same Newton polyhedron, but not the same support.

**Remark 2.** A monomial ideal \( I \) is integrally closed if and only if all lattice points of the Newton polyhedron of \( I \) are in the support of \( I \), see for example [20, 21]. It is a well-known theorem from Zariski [22] that any integrally closed ideal is complete. Furthermore, Zariski proves that complete ideals in \( K[x,y] \) can be written as a product of simple complete ideals, which then correspond to the exceptional divisors of blowing up in the simple complete ideals; see for example [23, 24, 25, 26] for more on complete ideals and a full list of references.

The following definition is of crucial importance for the sequel:

**Definition 2.** For a set of vectors \( v_1, \ldots, v_l \in \mathbb{R}^n \) the set \( \{ \lambda_1 v_1 + \ldots + \lambda_l v_l : \lambda_i \in \mathbb{N} \} = N(v_1 : 1 \leq i \leq l) \) is called the \( \mathbb{N} \)-span of the \( v_i \). For two \( \mathbb{N} \)-spans \( C_1 = N(v_1, \ldots, v_n) \) and \( C_2 = N(w_1, \ldots, w_r) \) we define the sum \( C_1 + C_2 \) to be the \( \mathbb{N} \)-span \( N(v_1, \ldots, a_n w_1, \ldots, w_r) \); any element of \( C_1 + C_2 \) is a sum \( x + y \) with \( x \in C_1 \) and \( y \in C_2 \). We write \( \Sigma_n \) for \( N(v_1, \ldots, e_n) \). For a monomial ideal \( \mathcal{I} = (x^a : a \in A) \) a subset \( A \subseteq \mathbb{N}^n \) we define

\[
IT_n(\mathcal{I}) := N(a' - a : a' \neq a \in A) + \Sigma_n,
\]

the ideal tangent cone of the monomial \( x^a \). We call \( IT_n(\mathcal{I}) \) pointed if \( IT_n(\mathcal{I}) \cap (-IT_n(\mathcal{I})) = \{0\} \). A minimal set of generators of \( IT_n(\mathcal{I}) \) is a finite set \( S \) of vectors in \( IT_n(\mathcal{I}) \) that generate \( IT_n(\mathcal{I}) \) and no element in \( S \) is an \( \mathbb{N} \)-linear combination of the other elements in \( S \). We call an element of the minimal generating system a minimal generator of \( IT_n(\mathcal{I}) \).
We remark that the ideal tangent cone is in general not a polyhedral lattice cone. For a point \( a \) in the support of \( I \), we define the real tangent cone in \( a \) to be the polyhedral cone \( T_a(I) \) generated by all vectors \( p - a \) where \( p \) lies in \( N(I) \). The real tangent cone \( T_a(I) \cap \Delta^n \) contains \( T_a(I) \), and the inclusion can be proper.

**Lemma 3.** Let \( N(v_1, \ldots, v_l) = IT_a(I) \) be the ideal tangent cone of \( N \) in a point \( a \). If the ideal tangent cone is pointed it has a unique minimal set of generators.

**Proof.** Assume \( IT_a(I) \) has two different minimal sets of generators \( v_1, \ldots, v_l \) and \( w_1, \ldots, w_m \), and assume \( v_1 \notin \{ w_1, \ldots, w_m \} \). Then there are \( \alpha_{ij} \in \mathbb{N} \) and \( \beta_i \in \mathbb{N} \) such that

\[
\sum_i \alpha_{ij} v_i = \sum_j \beta_j w_j.
\]

Hence we have \( v_1 = \sum_i \beta_{ij} v_i \). The coefficient \( \sum_j \alpha_{ij} \beta_j \) must be greater or equal 1 because otherwise \( v_1 \notin \mathbb{N}(v_2, \ldots, v_l) \), which contradicts that \( v_1, \ldots, v_l \) form a minimal system of generators. If \( \sum_j \alpha_{ij} \beta_j > 1 \), \( 0 \) would be a nontrivial \( \mathbb{N} \)-linear combination of the \( v_i \), contradicting \( IT_a(I) \) to be pointed. Hence the only possibility is \( \alpha_{ik} = 1 \) and \( \beta_k = 1 \) for some \( k \) and zero otherwise. But then \( v_1 = w_k \), which contradicts that \( v_1 \notin \{ w_1, \ldots, w_m \} \). \( \square \)

**Lemma 4.** Let \( N \) be the Newton polyhedron of a monomial ideal \( I \) in \( K[x_1, \ldots, x_n] \) and \( a \) be in \( \text{supp}(I) \). Then the ideal tangent cone \( IT_a(I) \) is pointed if and only if \( a \) is a vertex.

**Proof.** First assume \( a \) is a vertex. Suppose \( IT_a(I) = N(a_1, \ldots, a_m) \), with \( a_i \in \mathbb{Z}^n \). The vectors \( a_1 \) are vectors of the form \( b_i - a \) where the \( b_i \) are some (rather special) points of \( N \), not equal to \( a \) and the standard basis vectors \( e_1, \ldots, e_n \). If \( IT_a(I) \) is not pointed, then we can write \( v = \sum_i v_i a_i \) with \( v_i \in \mathbb{N} \) not all zero. But then \( a = \sum_i \mu_i b_i \) with \( \mu_i = v_i/\sum_i v_i \) is a convex linear combination of other points in \( N \) and \( a \) is not a vertex.

Conversely, if \( a \) is not a vertex, then \( a \) is a \( \mathbb{Q} \)-linear combination of some vectors \( b_i \) of \( N \): \( a = \sum_i \lambda_i b_i \) with \( \lambda_i \in \mathbb{Q} \), positive and \( \sum_i \lambda_i = 1 \). Multiplying with a common denominator \( d \in \mathbb{N} \) we get: \( da = \sum_i \mu_i b_i = 0 \) and \( \sum_i \mu_i = d \). The \( a - b_i \) generate the tangent cone and \( \sum_i \mu_i (a - \mu_i b_i) \) is a nontrivial \( \mathbb{N} \)-linear combination representing zero and thus \( IT_a(I) \) is not pointed. \( \square \)

By definition of the support of an ideal \( I = (x^\mu : a \in A) \), \( \text{supp}(I) \) consists of all \( c \in \mathbb{Z}^n \) such that \( x^c \in I \). But then \( c \) must be of the form \( a + m_1 e_1 + \ldots + m_n e_n \) for some \( a \in A \) and some nonnegative integers \( m_1, \ldots, m_n \). It follows that for a fixed \( b \in A \) we have

\[
N(a - b : a \in \text{supp}(I)) = N(a' - b : a' \in A) + \sum_n.
\]

**Definition 5.** Let \( I \subset K[x_1, \ldots, x_n] \) be a monomial ideal. We call a pointed ideal tangent cone \( IT_a(I) \) simplicial if the minimal system of generators of \( IT_a(I) \) consists of exactly \( n \) vectors.

Since the standard basis vectors are contained in each ideal tangent cone, the following lemma is obvious:

**Lemma 6.** Let \( I = (x^\mu : a \in A) \) be an arbitrary monomial ideal and \( N = N(I) \) the associated Newton polyhedron. If the ideal tangent cone of \( a \in N \) is simplicial then the set of minimal generators form a basis of \( \Delta^n \).

Let \( IT_a(I) = N(v_1, \ldots, v_l) \) with \( l \geq n \) and let \( K[IT_a(I)] = K[x^{v_1}, \ldots, x^{v_l}] \) be the monomial algebra generated by \( IT_a(I) \). It can be seen easily that \( K[IT_a(I)] \) is the coordinate ring of the \( a \)-chart \( U_a \) of \( \Delta^n \): The exponents \( a' - a \), with \( a' \in A \setminus a \), of the generators \( x^{a' - a} \) of the coordinate ring \( K[x^{a'} : a' \in A \setminus a] \) of the \( a \)-chart of the blowup correspond in the Newton polyhedron \( N \) to the vectors pointing from \( a \) to \( a' \). The generators \( x^{a' - a} : a' \in A \setminus a \) correspond in the Newton polyhedron \( N \) to the \( n \) standard basis vectors. These generators make up the infinitely far vertices \( a + \mathbb{R}_+ e_i \). We have the inclusion \( K[IT_a(I)] \subseteq K[x^{a' - a}] \). Hence \( U_a \) contains a torus \( (K^*)^n \) as a dense subset.
Now consider the monoid homomorphism $\pi : \mathbb{N}^l \to \mathbb{Z}^n$, $b \mapsto \sum_{i=1}^l b_i v_i$. The image of $\pi$ is $IT_{\mathbb{A}}(\mathcal{F})$ and we get a homomorphism of monomial algebras
\begin{equation}
\hat{\pi} : K[t_1, \ldots, t_l] \to K[x, x^{-1}], \quad t_i \mapsto x^{v_i}.
\end{equation}
This construction yields the explicit description of the $\mathbb{A}$-chart of $\mathbb{A}^n$ as a toric variety. The following lemma can be found in standard textbooks, like [15] [16].

**Lemma 7.** Let $v_1, \ldots, v_l$ in $\mathbb{Z}^n$. The kernel of the map $K[t_1, \ldots, t_l] \to K[x, x^{-1}]$ defined by $t_i \mapsto x^{v_i}$ is generated by binomials of the form $t^\alpha - t^\beta$ for some $\alpha, \beta \in \mathbb{N}^n$.

**Proof.** Let $f \in K[t_1, \ldots, t_l]$ be in the kernel. We expand $f$ as a sum of monomials $f = \sum_{\alpha} c_\alpha t^\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_l)$ is a multi-index. Since $f$ maps to zero, we get $\sum_{\alpha} c_\alpha x^{\alpha \cdot \nu} = 0$, where $\alpha \cdot \nu = \sum_{i=1}^l \alpha_i v_i \in \mathbb{Z}^n$. Hence for all $\nu \in \mathbb{Z}^n$
\begin{equation}
\sum_{\alpha \cdot \nu = \nu} c_\alpha = 0.
\end{equation}
Thus if for some $\alpha$ the coefficient $c_\alpha$ is nonzero, there is another $\beta$ with $\alpha \cdot \nu = \beta \cdot \nu$ and $c_\beta \neq 0$. Then we consider $f' = f - c_\alpha (t^\alpha - t^\beta)$, which has less monomial terms than $f$. Hence the proof is completed by induction on the number of monomial terms. \(\square\)

### 3. A Smoothness Criterion

The main result of this section is Theorem [9] which contains a smoothness criterion and characterizes the ideal tangent cones of the smooth affine open charts $U_\mathbb{A}$. The result is the starting point for our further explorations.

The open affine charts $U_\mathbb{A}$ of the blowup $\mathbb{A}^n$ are by their construction toric varieties. Requiring that $U_\mathbb{A}$ is smooth singles out a unique affine toric variety.

**Lemma 8.** Let $\mathcal{F} = (x^a : a \in A)$ be a monomial ideal in $K[x]$, $N$ the associated Newton polyhedron and $\pi : \mathbb{A}^n \to \mathbb{A}^n$ the blowup of $\mathbb{A}$ in the center $\mathcal{F}$. If $a$ is a vertex of $N$ such that the $\mathbb{A}$-chart $U_\mathbb{A}$ is smooth, then the $\mathbb{A}$-chart is isomorphic to $\mathbb{A}^n$.

**Proof.** By construction $U_\mathbb{A}$ is an $n$-dimensional affine toric variety, and since it is smooth, we have $U_\mathbb{A} \cong \mathbb{A}^{n-k} \times (k^*)^k$ for some $k$, see e.g. [15]. But if $k$ is nonzero the ideal tangent cone of the vertex $a$ is not pointed. \(\square\)

**Theorem 9.** Let $\mathcal{F} = (x^a : a \in A)$ be a monomial ideal in $K[x]$, $N$ the associated Newton polyhedron and $\pi : \mathbb{A}^n \to \mathbb{A}^n$ the blowup of $\mathbb{A}$ in the center $\mathcal{F}$. Then we have:

(i) When $a \in A$ is not a vertex of $N$, then the $\mathbb{A}$-chart is already covered by the affine charts of $\mathbb{A}^n$ corresponding to the vertices of $N$.

(ii) When $a$ is a vertex of $N$ then the $\mathbb{A}$-chart is smooth if and only if the ideal tangent cone $IT_{\mathbb{A}}(\mathcal{F})$ is simplicial.

(iii) When $a$ is a vertex of $N$ and the ideal tangent cone is simplicial, then each minimal generator of $IT_{\mathbb{A}}(\mathcal{F})$ is primitive and $IT_{\mathbb{A}}(\mathcal{F}) = T_{\mathbb{A}}(N) \cap \mathbb{Z}^n$.

(iv) When $a$ is a point of $\text{supp}(I)$ such that all its neighboring lattice points are in $\text{supp}(\mathcal{F})$, then the $\mathbb{A}$-chart is isomorphic to $(k^*)^n$.

**Proof.** (i): If $a \in N$ is not a vertex, $a$ is contained in the convex hull of some vertices $a_1, \ldots, a_m$ of $N$. We thus can write $a = \sum_{i=1}^m \lambda_i a_i$ where the $\lambda_i$ are nonzero positive rational numbers with $\sum_{i=1}^m \lambda_i = 1$. The vectors $a_i - a$ are in the ideal tangent cone and
\begin{equation}
\sum_{i=1}^m \lambda_i (a_i - a) = 0.
\end{equation}
Thus by multiplying with a common denominator of the \( \lambda_i \) we can write \( 0 = \sum_{i=1}^{m} \alpha_i(a_i - a) \) with \( \alpha_i \in \mathbb{N} \) and \( \alpha_i \geq 1 \) for all \( i \). Thus \( IT_a(I) \) is not pointed and for any \( j \) we have

\[
a - a_j = \sum_{i \neq j} \alpha_i(a_i - a) + (\alpha_j - 1)(a_j - a) \in IT_a(I).
\]

But then it follows that \( IT_{a_1}(I) \subset IT_{a}(I) \) for all \( a_1 \), since for any \( a' \neq a \) we have

\[
a' - a_i = (a' - a) + (a - a_i) \in IT_a(I).
\]

Therefore \( K[IT_a(I)] \subset K[IT_a(I)] \) and thus \( \text{Spec}(K[IT_a(I)]) \subset \text{Spec}(K[IT_a(I)]) \).

\[(i): \text{ The ideal tangent cone } IT_a(I) \text{ is generated by all standard basis vectors } e_i, \text{ for } 1 \leq i \leq n \text{ of } \mathbb{Z}^n \text{ and the vectors } a' - a \text{ where } a' \text{ runs over all vertices of } N, \text{ except } a. \]

Together we denote these generators by \( b_i, 1 \leq i \leq M. \) The ideal tangent cone of \( a \) has a unique minimal generating set \( S \). Since we can try to eliminate any of the \( b_i \) to get a minimal generating set, we have \( S \subset \{b_1, \ldots, b_M\} \).

First assume \( U_a \) is smooth. By Lemma 8 we know that \( U_a \) is isomorphic to \( \mathbb{A}^n \) and thus \( K[IT_a(I)] \cong K[y_1, \ldots, y_n] \). Hence \( S \) contains exactly \( n \) elements and we may assume \( S = \{b_1, \ldots, b_n\} \). Any \( b_i \in S \) can be expressed as an \( \mathbb{N} \)-linear combination of the \( b_i \) with \( 1 \leq k \leq n \). In particular, the \( \mathbb{N} \)-span of the set \( S \) contains the basis vectors \( e_i \) of \( \mathbb{Z}^n \). Hence \( S \) is a basis for \( \mathbb{Z}^n \) and thus \( IT_a(I) \) is simplicial. Conversely, assume the set \( S \) contains precisely \( n \) linearly independent elements. Then \( S \) is a basis for \( \mathbb{Z}^n \). The coordinate ring of \( U_a \) is \( K[x^{b_i}: 1 \leq i \leq n] \) and we have a map \( K[z_1, \ldots, z_n] \to K[x^{b_i}: 1 \leq i \leq M] \) sending \( z_i \) to \( x^{b_i} \). This map is clearly surjective. By Lemma 7 we know that the kernel is generated by binomials. Hence suppose \( f = z^n - z^n \) is a generator of the kernel, then we must have \( \sum_{i=1}^{n} u_i b_i - \sum_{i=1}^{n} u_i b_i = 0 \). Since the \( b_i \) are a basis, we must have \( u_i = u_i \) and thus \( f = 0 \). Therefore the coordinate ring of \( U_a \) is \( K[z_1, \ldots, z_n] \) and \( U_a \cong \mathbb{K}^n \).

\[(ii): \text{ Suppose that } b_1 \text{ is primitive for some positive integer } r \geq 1. \text{ Then the vectors } \frac{1}{r}b_1, b_2, \ldots, b_n \text{ constitute a basis of } \mathbb{Z}^n. \text{ However the matrix that relates this basis to the basis } S \text{ has determinant } 1/r. \text{ Hence we must have } r = 1. \text{ Furthermore, we clearly have } IT_a(I) \subset T_a(N) \cap \mathbb{Z}^n. \text{ Suppose } p \in T_a(N) \cap \mathbb{Z}^n. \text{ Then } p \text{ is an } \mathbb{R}_{\geq 0} \text{-linear combination and a } \mathbb{Z} \text{-linear combination of the vectors } b_i \text{ in } S. \text{ Since the } b_i \text{ are a basis, the coefficients in the expansions coincide and are thus in } \mathbb{N} = \mathbb{Z} \cap \mathbb{R}_{\geq 0}. \text{ Thus } p \in IT_a(I). \]

\[(iv): \text{ When } a \text{ is in each lattice direction surrounded by points of } \text{supp}(I), \text{ then the ideal tangent cone of } a \text{ is all of } \mathbb{Z}^n. \text{ Thus the coordinate ring of } U_a \text{ is } K[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}], \text{ which proves the claim.} \]

\[\square\]

**Definition 10.** We call a monomial ideal \( \mathcal{I}(x^a: a \in A) \) in \( K[x] \) tame if the blowup \( \widetilde{\mathbb{A}}^n \) of \( \mathbb{A}^n \) is smooth. The corresponding Newton polyhedron \( N(\mathcal{I}) \) is called tame if \( \mathcal{I} \) is tame.

### 4. Blowup in the Rosenberg ideal

In \([6]\) Rosenberg constructs nonreduced monomial ideals \( \mathcal{R} \) in \( K[x] \) whose zero set is a union of coordinate axes in \( \mathbb{A}^n \) and such that the blowup of \( \mathbb{A}^n \) with center \( \mathcal{R} \) is smooth. The constructed ideals are invariant under any permutation of the coordinates of \( \mathbb{A}^n \) that leaves the zero set invariant. Rosenberg also generalizes the construction to closed subschemes \( Z \) of \( \mathbb{A}^n \) that are unions of coordinate subspaces of dimension \( 1 \leq r < n \). The idea of the construction is as follows: Consider the reduced ideal \( \mathcal{I} \) of \( Z \). The blowup of \( \mathbb{A}^n \) with center \( \mathcal{I} \) will in general be singular; from some vertices in the Newton polyhedron \( N(\mathcal{I}) \) emanate too many edges, so that the associated ideal tangent cones are not simplicial. One can transform \( \mathcal{I} \) by multiplication or intersection with another monomial ideal such that the blowup becomes smooth and the radical ideal \( \mathcal{I} \) is unchanged. We study the effect on \( N(\mathcal{I}) \) of this method. For a sheaf-theoretic interpretation and further details also see \([6]\).
Let $1 < s < n$ and let $\mathcal{I}$ be the reduced ideal of the first $s$ coordinate axes:

$$(15) \quad \mathcal{I} = \bigcap_{i=1}^{s} \langle x_1, \ldots, \hat{x_i}, \ldots, x_n \rangle.$$

The ideal $\mathcal{I}$ is generated by the monomials $x_i x_j$ with $1 \leq i < j \leq s$ and the monomials $x_i$ with $s < i$. We now briefly show that the blowup with center $\mathcal{I}$ is singular: Consider the $x_n$-chart. Here $u_{jk} = \frac{x_j x_k}{x_n}$ with $j \neq k \leq s$ are minimal generators. These generators satisfy relations of the form

$$(16) \quad u_{jk} : x_n - x_j x_k.$$

The chart expression is then

$$(17) \quad \text{Spec} \left( K[x, u_{jk}, \frac{x_l}{x_n} : l > s]/(u_{jk} x_n - x_j x_k : j < k \leq s) \right),$$

which is clearly singular.

The idea to smooth $N(\mathcal{I})$ is to “pull apart” a vertex $v$ of $N(\mathcal{I})$, from which more than $n$ edges emanate. Each new vertex should inherit some edges from $v$ but the corresponding ideal tangent cone should have exactly $n$ minimal generators. Rosenberg proposes two possibilities to smooth nonsimplicial ideal tangent cones. The first method consists in slicing the distracting vertex off $N(\mathcal{I})$, see fig. 2. This slicing corresponds to intersecting $\mathcal{I}$ with another ideal.

The second method is to stretch the Newton polyhedron, that is, one substitutes the polyhedron $N(\mathcal{I})$ with the Minkowski sum of $N(\mathcal{I})$ with another suitable Newton polyhedron, see fig. 3. This stretching corresponds to multiplying $\mathcal{I}$ with another ideal.

To leave the zero set unchanged, the radical ideal of $\mathcal{I}$ has to be preserved. For both smoothing methods, we use a power of the maximal ideal $m$, where $m = \langle x_1, \ldots, x_n \rangle$ is the reduced ideal of the origin of $\mathbb{A}^n$. The choice of $m$ as smoothing ideal is in no way unique! We could choose any ideal $\mathcal{J}$ such that the blow up of $\mathbb{A}^n$ with center $\mathcal{I} \cap \mathcal{J}$ or $\mathcal{I} \cdot \mathcal{J}$ is smooth and such that $\sqrt{\mathcal{I} \cap m^k} = \sqrt{\mathcal{I} \cdot m^k} = \mathcal{I}$. But using $m$ ensures the zero set is unchanged as we have $m \supset \mathcal{I}$ from which it follows that

$$(18) \quad \sqrt{\mathcal{I} \cap m^k} = \mathcal{I} \cap m = \mathcal{I}.$$
Proof. From the definition of the ideal tangent cone it follows that 

\[ a \mathcal{IT} K \]

Let \( \text{Lemma 12} \).

Let \( \text{Lemma 13} \).

The product are sums of ideal tangent cones of the factors of the product. Nonreduced ideals for a sequence of blowups. First we show that the ideal tangent cones of a \( I \) are tame. Li [9] has shown that a sequence of blowups in ideals \( I \) are tame. We define \( \hat{R} := I \cap m^3 \) and observe that the ideal \( \hat{R} \) is invariant under each permutation of the coordinates of \( k^n \) that leaves \( I \) invariant.

**Proposition 11.** Let \( I \) be the reduced ideal of the first \( s \) coordinate axes of \( k^n \) defined in eqn. (12). Then the ideal \( \hat{R} = I \cap m^3 \) is tame.

**Proof.** We show the smoothness of \( \hat{R}^n \) with Theorem [9]. The Newton polyhedron \( N(\hat{R}) \) has the set of vertices

\[
\{2e_i + e_j : 1 \leq i \leq s, 1 \leq j \leq n, i \neq j\} \cup \{3e_i : s < i \leq n\}.
\]

It can easily be computed that the remaining generators of \( \hat{R} \) correspond to interior points of \( N(\hat{R}) \) or lie in faces of the Newton polyhedron. For example, for \( 2e_j + e_i \) with \( 1 \leq i \leq s < j \leq n \) we have

\[
2e_j + e_i = \frac{1}{2}(2e_i + e_j) + \frac{1}{2}(3e_j).
\]

Consider the ideal tangent cone \( \mathcal{IT}_{2e_i + e_j}(\hat{R}) \) for \( 1 \leq i \leq s \) and \( 1 \leq j \leq n \). The associated edge vectors are the \( e_i \) for the infinitely far vertices, \( e_j - e_i \) and \( e_k - e_j \) with \( k \neq \{i, j\} \) for the adjacent vertices. There are exactly \( n \) minimal generators and the ideal tangent cone is simplicial.

For a vertex \( 3e_i \) with \( s < i \leq n \) the ideal tangent cone is generated by \( e_i \) (infinitely far vertex) and the vectors \( e_k - e_i \) for \( k \neq i \) (adjacent vertices). Thus \( \mathcal{IT}_{3e_i}(\hat{R}) \) is simplicial. The blowup of \( k^n \) with center \( \hat{R} \) is therefore smooth.

\[ \square \]

5. Blowups in products of ideals

In this section we apply the smoothness criterion [9] to investigate which products of monomial ideals are tame. Li [9] has shown that a sequence of blowups in ideals \( I_1, \ldots, I_r \) is the same as the single blowup in the product \( I_1 \cdot I_2 \cdots I_r \) of ideals. Bodnár [27] computes the nonreduced ideals for a sequence of blowups. First we show that the ideal tangent cones of a product are sums of ideal tangent cones of the factors of the product.

**Lemma 12.** Let \( I = (x^a : a \in A) \) and \( J = (x^b : b \in B) \) with \( A, B \subseteq \mathbb{N}^n \) be two ideals in \( K[x] \). Then we have \( N(I) \cdot J) = N(I) + N(J) \).

**Proof.** See [19 Lemma 2.2].

**Lemma 13.** Let \( I_1 = (x^a : a \in A) \) and \( I_2 = (x^b : b \in B) \) be two monomial ideals in \( K[x] \) with associated Newton polyhedra \( P = N(I_1) \) and \( Q = N(I_2) \). Then \( \mathcal{IT}_{a + b}(I_1 \cdot I_2) = \mathcal{IT}_a(I_1) + \mathcal{IT}_b(I_2) \).

**Proof.** From the definition of the ideal tangent cone it follows that

\[
\mathcal{IT}_a(I_1) = N(a' - a : a' \in A \setminus a) + \Sigma_n,
\]

\[
\mathcal{IT}_b(I_2) = N(b' - b : b' \in B \setminus b) + \Sigma_n.
\]

\[ \square \]

**Figure 4.** The Newton polyhedra of \( I \cap m^2 \) (left) and \( \hat{R} \) (right).
In $P + Q$ holds
\[
IT_{a+b}(\mathcal{I}) = N(a' + b - (a+b), a + b' - (a+b), a' + b' - (a+b)) + \Sigma_n
\]
\[
= N(a' - a, b' - b : a' \in A \setminus a, b' \in B \setminus b) + \Sigma_n.
\]
Hence the assertion follows. \hfill \Box

One easily shows that a monomial ideal has a unique minimal set of generators. From now on we assume that monomial ideals are generated by a minimal set of generators.

**Definition 14.** If $\mathcal{I}(x^a : a \in A)$ is a monomial ideal with $A$ minimal, then we call $A$ the cloud of $\mathcal{I}$. Let $\mathcal{I}$ and $\mathcal{J}$ be two monomial ideals in $K[x]$ with clouds $A$ and $B$. We say that the clouds $A$ and $B$ are transverse if the affine hulls of $A$ and $B$ are transverse.

If $A$ and $B$ are two transverse clouds, then we can perform a $\mathbb{Z}$-linear transformation to achieve that $A \subseteq N^s \times 0^{n-s}$ and $B \subseteq 0^s \times N^{n-s}$ for some integer $s$.

**Lemma 15.** Let $\mathcal{I} = (x^a : a \in A) \subseteq K[x]$ be a monomial ideal with $A \subseteq N^s \times 0^{n-s}$. Then the ideal $\mathcal{I}$ in $K[x]$ is tame if and only if the ideal $\mathcal{I} := \mathcal{I} \cap K[x_1, \ldots, x_s]$ is tame.

**Proof.** The cloud of $\mathcal{I}$ is given by points $a = \tilde{a} \times 0^s$ where $\tilde{a}$ is an element of the cloud of $\tilde{\mathcal{I}}$. Hence we have $N(\mathcal{I}) \times \mathbb{R}^{n-s} = N(\tilde{\mathcal{I}})$. The minimal generators of $IT_{a}(\mathcal{I})$ for a point $a \in N(\mathcal{I})$ are thus of the form $b_1, \ldots, b_s, e_{s+1}, \ldots, e_n$. Hence $IT_{a}(\mathcal{I})$ and $IT_{\tilde{a}}(\tilde{\mathcal{I}})$ are simplicial if and only if $t = s$. \hfill \Box

**Lemma 16.** Let $P \subseteq \mathbb{R}^s$ and $Q \subseteq \mathbb{R}^t$ be polyhedra. Let $e$ be a vertex of $P$ and $f$ be a vertex of $Q$. Then $(e, f)$ is a vertex of $P \times Q$ in $\mathbb{R}^{s+t}$.

**Proof.** Because $e$ is a vertex of $P$ there exist a vector $a \in \mathbb{R}^s$ and a real number $\alpha$ with $a \cdot e = \alpha$ and $a \cdot p < \alpha$ for all $p \in P \setminus e$. Here $\cdot$ denotes the Euclidean scalar product. Similarly, there exists $b \in \mathbb{R}^t$ and $\beta \in \mathbb{R}$ such that $b \cdot f = \beta$ and $b \cdot q < \beta$ for all $q \in Q \setminus f$. Denoting $c = (a, b) \in \mathbb{R}^{s+t}$ we have
\[
(e, f) \cdot c = a \cdot e + b \cdot f = a + \beta,
\]
and for all $(p, q) \in P \times Q \setminus (e, f)$
\[
(p, q) \cdot c = a \cdot p + b \cdot q < a + \beta.
\]
\hfill \Box

**Lemma 17.** Let $\mathcal{I}_1 = (x^a : a \in A)$ and $\mathcal{I}_2 = (x^b : b \in B)$ be two monomial ideals in $K[x]$ with transverse clouds $A, B \subseteq \mathbb{N}^n$ and associated Newton polyhedra $P$ resp. $Q$. Then the following hold:

(i) If $A \subseteq N^s \times 0^{n-s}, B \subseteq 0^s \times N^{n-s}$ and $\tilde{P} = P \cap \mathbb{R}^s, \tilde{Q} = Q \cap \mathbb{R}^{n-s}$, then $N(\mathcal{I}_1 \cdot \mathcal{I}_2)$ is $\tilde{P} \times \tilde{Q}$.

(ii) If $\mathcal{I}_1$ and $\mathcal{I}_2$ are tame, then $\mathcal{I}_1 \cdot \mathcal{I}_2$ is tame.

**Proof.** For (i) we can suppose that $A \cup B$ span $\mathbb{N}^n$ (Lemma 15). With Lemma 12 follows
\[
N(\mathcal{I}_1 \cdot \mathcal{I}_2) = P + Q = \tilde{P} \oplus \tilde{Q} \cong \tilde{P} \times \tilde{Q}.
\]
Now we show (ii): We may assume the conditions in (i) hold and thus using Lemma 14. we have $\mathcal{I}_1 \subset K[x_1, \ldots, x_s]$ and $\mathcal{I}_2 \subset K[x_{s+1}, \ldots, x_n]$. If $\tilde{a}$ is a vertex of $\tilde{P}$ resp. $\tilde{b}$ a vertex of $\tilde{Q}$ then Lemma 13 yields that $(\tilde{a}, \tilde{b})$ is a vertex of $N(\mathcal{I}_1 \cdot \mathcal{I}_2)$. Conversely each vertex of $N(\mathcal{I}_1 \cdot \mathcal{I}_2)$ is a sum of two vertices $\tilde{a} \times 0^s \in \tilde{P}$ and $0^{n-s} \times \tilde{b} \in \tilde{Q}$. Hence the set of vertices of $N(\mathcal{I}_1 \cdot \mathcal{I}_2)$ is $\{a + b : a \text{ a vertex of } P, b \text{ a vertex of } Q\}$. Using Lemma 13 and the transversality we have $IT_{a+b}(\mathcal{I}_1 \cdot \mathcal{I}_2) = IT_{a}(\mathcal{I}_1) \oplus IT_{b}(\mathcal{I}_2)$, so that the minimal generators are of the form $(v, 0)$ with $v$ a minimal generator for $IT_{a}(\mathcal{I}_1)$ and $(0, w)$ with $w$ a minimal generator for $IT_{b}(\mathcal{I}_2)$. Thus there are precisely $s + (n-s) = n$ minimal generators. \hfill \Box

**Proposition 18.** Let $\mathcal{I}_i$ for $i = 1, \ldots, k$ be monomial ideals in $K[x]$ with pairwise transverse clouds. Then $\mathcal{I} := \prod_{i=1}^k \mathcal{I}_i$ is tame if each $\mathcal{I}_i$ is tame.
Proof. Let $\mathcal{I} = (x^a : a \in A)$ be ideals in $K[x]$ with pairwise transverse clouds. We use induction over $k$: For $k = 1$ the assertion is trivial. For the induction step $k$ to $k+1$ let the first $k$ clouds be contained in $\mathbb{N}^n \times 0^{n-s}$ for an $s \leq n$. This implies $A_{k+1} \subseteq 0^n \times \mathbb{N}^{n-s}$. By the induction assumption, the product ideal $\prod_{i=1}^k \mathcal{I}_i$ is tame. Because the cloud of $\mathcal{I}$ is $A = A_{k+1} \cup \bigcup_{i=1}^k A_i$ and $A_{k+1}$, and $A_{k+1}$ are transverse one can use Lemma 17 to conclude $\mathcal{I}$ is tame. □

Example 2. Let $\mathcal{I}_1 = (x, y)$ and $\mathcal{I}_2 = (x^2, zw, w^3)$ be ideals in $K[x, y, z, w]$; $\mathcal{I}_1$ is the reduced ideal of the $zw$-plane in $A^4$ and the nonreduced ideal $\mathcal{I}_2$ defines the $xy$-plane in $A^4$. One easily computes that both ideals are tame. Using Proposition 18 one finds that the blowup of $A^4$ with center
\[(24) \quad \mathcal{I}_1 \cdot \mathcal{I}_2 = (xz^2, yz^2, xzw, yzw, xw^3, yw^3),\]
is smooth. The zero set of $\mathcal{I}_1 \cdot \mathcal{I}_2$ is the union of the $zw$- and the $xy$-plane.

Example 3. Let $\mathcal{I}_1 = (x, y)$ and $\mathcal{I}_2 = (x, z)$ be the reduced ideals of the $z$- resp. $y$-axis in affine three-dimensional space with coordinates $x, y, z$. Both ideals are tame but the clouds are not transverse and the blowup $\mathbb{A}^3$ in the product $\mathcal{I}_1 \cdot \mathcal{I}_2 = (x^2, xy, xz, yz)$, a nonreduced structure on the coordinate cross, is not smooth. To see this, consider the $yz$-chart: The ideal tangent cone $IT_{\mathcal{I}_2 \cup \mathcal{I}_3}(\mathcal{I}_1 \cdot \mathcal{I}_2)$ has the minimal system of generators $e_1 - e_2, e_1 - e_3, e_2, e_3$. Hence $IT_{\mathcal{I}_2 \cup \mathcal{I}_3}(\mathcal{I}_1 \cdot \mathcal{I}_2)$ is not simplicial. In the $yz$-chart $\mathbb{A}^3 = \text{Spec}(K[x, y, z, w]/(xz - yw))$ has an isolated (simple toric) singularity.

Let $\mathcal{I} = (x^a : a \in A)$ be a monomial ideal in $K[x_1, \ldots, x_n]$. A direct application of Remark 1 implies that the vertices of $N(\mathcal{I}^\alpha)$ are of the form $2a$ for some vertex $a$ of $N(\mathcal{I})$. Since $IT_{\mathcal{I}^\alpha}(\mathcal{I}^\alpha) = IT_{\mathcal{I}}(\mathcal{I})$, either both are pointed or both are not pointed, which shows that $a$ is a vertex of $N(\mathcal{I})$ if and only if $2a$ is vertex of $N(\mathcal{I}^\alpha)$. Furthermore, $IT_{\mathcal{I}^\alpha}(\mathcal{I}^\alpha)$ is simplicial if and only if $IT_{\mathcal{I}}(\mathcal{I})$ is simplicial and thus $\mathcal{I}$ is tame if and only if $\mathcal{I}^\alpha$ is tame. This can also be proven by more general methods for homogeneous ideals, as for example [28, ex. 5.10 (b),II], [29, ex. III-15]. We have an easy generalization:

Lemma 19. Let $\mathcal{I}_1, \ldots, \mathcal{I}_k$ be monomial ideals in $K[x_1, \ldots, x_n]$ and let $\alpha_1, \ldots, \alpha_k$ be some positive integers. Then $\mathcal{I} = \mathcal{I}_1 \cdot \cdots \cdot \mathcal{I}_k$ is smooth if and only if $\mathcal{I} = \mathcal{I}_1^{\alpha_1} \cdot \cdots \cdot \mathcal{I}_k^{\alpha_k}$ is smooth.

Proof. Any vertex of $\mathcal{I}$ is of the form $p = \sum \alpha_i a_i$ for $a_i$ a vertex of $N(\mathcal{I}_i)$. The ideal tangent cone of $q = \sum \alpha_i a_i$ in the cloud of $\mathcal{I}$ is the same as the ideal tangent cone of $p$ in the cloud of $\mathcal{I}$. In particular, if one is pointed, then so is the other, and thus $p$ is a vertex precisely when $q$ is a vertex. Furthermore, if $IT_p(\mathcal{I})$ is simplicial, then so is $IT_q(\mathcal{I})$.

5.1. Coordinate ideals. A coordinate ideal in $K[x]$ is an ideal generated by a subset of the $x_1, \ldots, x_n$. In this section we study products of coordinate ideals. Let $\mathcal{I}$ be a coordinate ideal in $K[x]$. In the Newton polyhedron $N(\mathcal{I}) = \text{conv}(\{ e_i : i \in I \}) + \mathbb{R}^n_+$ the ideal tangent cone of a vertex is of the form
\[(25) \quad IT_{e_i}(\mathcal{I}) = n(e_j - e_i : j \in I \setminus i) + \Sigma_n.\]

There are $n$ minimal generators among these, and thus $IT_{e_i}(\mathcal{I})$ is simplicial. For a coordinate ideal $\mathcal{I} = (x_i : i \in I) \subset K[x_1, \ldots, x_n]$ we identify the cloud $A = \{ e_i : i \in I \}$ with the subset $I$ of $\{ 1, 2, \ldots, n \}$ by identifying $i \in I$ with $e_i \in A$. For this reason we call $I$ the cloud of $\mathcal{I}$. Two coordinate ideals $\mathcal{I} = (x_i : i \in I)$ and $\mathcal{J} = (x_j : j \in J)$ have transverse clouds if and only if $I \cap J = \emptyset$. By Lemma 17 the product of coordinate ideals with transverse clouds is tame.

Let $\mathcal{I} = \prod_{i=1}^n \mathcal{I}_i$ be a product of coordinate ideals $\mathcal{I}_i = (x_j : j \in I_i)$ with $I_i \subseteq \{ 1, \ldots, n \}$. Then each monomial generator of $\mathcal{I}$ is a product of $s$ monomials $x_j$. All generators of $\mathcal{I}$ thus look like $x_{j_1} \cdots x_{j_s}$, with $j_i \in I_i$. From this we see that each vertex $v$ of the Newton
polyhedron is a sum of \( s \) standard basis vectors. Hence all vertices lie on the affine hyperplane 
\[
H_s := \{ v \in \mathbb{R}^n : v \cdot 1 = s \}.
\]
Here \( 1 \) denotes the vector \((1, \ldots, 1)\).

**Lemma 20.** Let \( \mathcal{I} \) be a product of \( s \) coordinate ideals \( \mathcal{I}_i = (x_j : j \in I_i) \) in \( K[\mathbf{x}] \). Each minimal generator of the ideal tangent cone of the Newton polyhedron \( N(\mathcal{I}) \) in a point \( \mathbf{a} = \sum k \alpha_k e_k \) with \( \sum k \alpha_k = s \) and \( \mathbf{a} \) an element of the cloud of \( \mathcal{I} \), is of the form \( e_j \) resp. \( e_j - e_k \) for \( k \) with \( \alpha_k \neq 0 \) and some \( j \in \{1, \ldots, n\} \). At least one standard basis vector \( e_j \) is a minimal generator of \( IT_\mathbf{a}(\mathcal{I}) \).

**Proof.** The form of the minimal generators follows from equation (26) with Lemma 18. Since \( \mathbb{R}(e_i - e_j : i \neq j) \) is \((n-1)\)-dimensional and the minimal generators form a basis of \( \mathbb{R}^n \), not all minimal generators can be of the form \( e_i - e_j \).

**Definition 21.** For \( v = (v_1, \ldots, v_n) \in \mathbb{Z}^n \) we define the norm \( |v| = \sum_{i=1}^n v_i \).

Note that on ideal tangent cones of coordinate ideals and products of them \(|.|\) is nonnegative. The minimal generators are of norm 1 and 0. We say a minimal generator points in the direction \( i \) if it is \( e_i \) or of the form \( e_j - e_i \) for some \( j \). Since all standard basis vectors are in the ideal tangent cone we need at least a minimal generator in the direction \( i \) for all \( i \). By the pigeon hole principle, the ideal tangent cone is not simplicial if for some \( i \) there are at least two minimal generators pointing in direction \( i \). The only way this can happen is that \( e_i - e_k \) and \( e_i - e_l \) are minimal generators for two different \( k \) and \( l \). Hence, only norm zero minimal generators obstruct simplicity for products of coordinate ideals.

**Proposition 22.** Let \( \mathcal{I} = \prod_{i=1}^s \mathcal{I}_i \) be a product of coordinate ideals \( \mathcal{I}_i = (x_k : k \in I_i) \) and let \( \mathcal{J} \) be a coordinate ideal containing \( \mathcal{I} \). If \( \mathcal{I} \) is tame, then so is \( \mathcal{I} \cdot \mathcal{J} \).

**Proof.** By Lemma 15 we may assume \( \mathcal{J} \) contains all \( x_m \) not in \( I = \bigcup I_i \). Any vertex of \( N(\mathcal{I}) \) is of the form \( \mathbf{p} = \sum_{i=1}^s e_k_i \) with \( k_i \in I_i \). By Lemma 20 the minimal generators of \( IT_\mathbf{p}(\mathcal{I}) \) are either of the form \( e_{l_i} - e_{k_i} \) for some \( l_i \in I_i \) or of the form \( e_l \) and by assumption there are precisely \( n \) of them. We claim that the minimal generators of \( IT_{\mathbf{p} + e_j}(\mathcal{I} \cdot \mathcal{J}) \) can be obtained from the minimal generators of \( IT_\mathbf{p}(\mathcal{I}) \) as follows: all minimal generators of the form \( e_l \) for \( l \neq j \) are replaced by \( e_l - e_j \) and all others are left unchanged. Since by Lemma 13 we have

\[
IT_{e_j}(\mathcal{J}) = \mathbb{N}(e_j, e_m \mid j - e_j : m \neq j), \quad \text{and} \quad IT_{\mathbf{p} + e_j}(\mathcal{I} \cdot \mathcal{J}) = IT_\mathbf{p}(\mathcal{I}) + IT_{e_j}(\mathcal{J}),
\]

all new generators are in \( IT_{\mathbf{p} + e_j}(\mathcal{I} \cdot \mathcal{J}) \). There are still \( n \) minimal generators and thus it suffices to prove the claim.

We first prove that \( e_j \) is a minimal generator of \( IT_\mathbf{p}(\mathcal{I}) \) and hence a new minimal generator. If \( j \notin I \), then in the expansion of \( e_j \) in minimal generators no \( e_{l_i} - e_{k_i} \) can occur and hence \( e_j \) is a minimal generator. If \( j \in I \), say \( j \in I_1 \), then we must have \( j = k_1 \). Indeed, if not, then by Remark 1 we see that \( e_{k_1} + e_j \) is not a vertex of \( N(\mathcal{I}_1 \cdot \mathcal{J}) \), but then \( \mathbf{p} + e_j \) is not a vertex of \( N(\mathcal{I} \cdot \mathcal{J}) \). If \( e_{k_1} \) is not minimal in \( IT_\mathbf{p}(\mathcal{I}) \), then \( e_{k_1} - e_{k_1} \) must be minimal for some \( i \) but then \( j = k_1 \in I_1 \) and thus \( j = k_1 \) so that \( e_{k_1} - e_{k_1} = 0 \). Hence \( e_j = e_{k_1} \) is minimal.

As \( e_j \) is a new minimal generator, \( IT_\mathbf{p}(\mathcal{I}) \) is in the span of the new minimal generators. Hence we only have to show that all vectors \( e_{a_i} - e_{j} \) are generated by the new minimal generators. The expansion of \( e_{a_i} \) in the minimal generators of \( IT_\mathbf{p}(\mathcal{I}) \) contains precisely one minimal generator of the form \( e_j \) since else the norm would not equal 1. If \( l = j \) we substitute it from the expansion and if \( l \neq j \) we substitute \( e_l \) with \( e_l - e_j \). In both cases we are done.

**Corollary 23.** Consider two coordinate ideals \( \mathcal{I} = (x_i : i \in I) \) and \( \mathcal{J} = (x_j : j \in J) \) in \( K[\mathbf{x}] \) with \( I \subseteq J \). Then \( \mathcal{I} \cdot \mathcal{J} \) is tame.

**Proposition 24.** Let \( \mathcal{I} \) and \( \mathcal{J} \) be two coordinate ideals in \( K[x_1, \ldots, x_n] \) with clouds \( I \) and \( J \), respectively. Then the product \( \mathcal{I} \cdot \mathcal{J} \) is tame if and only if either \( I \cap J = \emptyset \) or one of the clouds is contained in the other.
Proof. We may assume $I \cup J = \{1, \ldots, n\}$. By Lemma \[\text{[17]}\] and Corollary \[\text{[23]}\] the if-part of the proof is done. For the only if-part we remark that for two subsets $I, J$ of $\{1, \ldots, n\}$ there are three possibilities: either (i) they are disjoint, or (ii) one is contained in the other, or (iii) $I \setminus J$, $J \setminus I$ and $I \cap J$ are not empty. We will show that in the third case there is a nonsimplicial vertex in the Newton polyhedron.

We may assume $1 \in I \setminus J$, $2 \in J \setminus I$. By Lemmas \[\text{[13]}\] and \[\text{[20]}\] the ideal tangent cone of $e_1 + e_2$, which is a vertex, is given by

\begin{align*}
\text{IT}_{e_1 + e_2}(\mathcal{J} \cdot \mathcal{J}) = \text{n}(e_r - e_1, e_s - e_2, e_1, e_2 : r \in I \setminus \{1\}, s \in J \setminus \{2\}).
\end{align*}

Choose $c \in I \cup J$, then we claim that $v_1 = e_c - e_1$ and $v_2 = e_c - e_2$ are both minimal and thus $\text{IT}_{e_1 + e_2}(\mathcal{J} \cdot \mathcal{J})$ is not simplicial. If $v_1$ would not be minimal we can subtract some generator and the result is still in $\text{IT}_{e_1 + e_2}(\mathcal{J} \cdot \mathcal{J})$. Since $|v_1| = 0$ we cannot subtract $e_1$ or $e_2$. Since only the first and the second component of elements of $\text{IT}_{e_1 + e_2}(\mathcal{J} \cdot \mathcal{J})$ can be negative we can only subtract $v_2$ to obtain $e_2 - e_1$. But $|e_2 - e_1| = 0$ and the only generator with positive second entry is $e_2$; thus $e_2 - e_1$ is not in $\text{IT}_{e_1 + e_2}(\mathcal{J} \cdot \mathcal{J})$. Thus $v_1$ is minimal and a similar argument shows $v_2$ is minimal.

For three coordinate ideals we can determine explicitly the conditions that have to be satisfied in order that the product is tame. We already know that when all three have disjoint clouds, that the product is smooth. Also, in the case where one cloud is contained in another and the third cloud is disjoint from these, then the product is tame. The remaining possibilities are dealt with below.

Lemma 25. Let $\mathcal{J}$, $\mathcal{J}$ and $\mathcal{K}$ be three monomial coordinate ideals in $K[x_1, \ldots, x_n]$ with clouds $I$, $J$ and $K$ respectively. Suppose that $I \cup J = K$, then the product $\mathcal{J} \cdot \mathcal{J} \cdot \mathcal{K}$ is tame.

Proof. If $I \subset J$, then $J = K$ and the product is tame by Corollary \[\text{[23]}\]. Hence we assume $I \nsubseteq J$ and $J \nsubseteq I$. Let $v$ be a vertex of $N = N(\mathcal{J} \cdot \mathcal{J} \cdot \mathcal{K})$. Then $v = v(I) + v(J) + v(K)$, where $\nu(I)$, $\nu(J)$ and $\nu(K)$ are vertices of the Newton polyhedra of $\mathcal{J}$, $\mathcal{J}$ and $\mathcal{K}$, respectively. By Remark \[\text{[1]}\] we need that $\nu(I) = \nu(K)$ or $\nu(J) = \nu(K)$; if $\nu(K) \in I$, then $\nu(K) + v(I) = \frac{1}{2}(2\nu(K)) + \frac{1}{2}(2\nu(I))$. Hence we may assume $\nu(K) = \nu(I)$. But then it follows that $\nu(J) = \nu(I)$ is in $\text{IT}_{\nu}(\mathcal{J} \cdot \mathcal{J} \cdot \mathcal{K})$. If $\text{IT}_{\nu}(\mathcal{J} \cdot \mathcal{J} \cdot \mathcal{K})$ is not simplicial, there is an index $1 \leq i \leq n$ such that there are two minimal generators $e_i - e_k$ and $e_i - e_l$ for two different indices $1 \leq k, l \leq n$. But the only possible choices $k$ and $l$ are $\nu(I)$ and $\nu(J)$, in case of which $e_i - \nu(I) = e_i - \nu(J) + \nu(J) - \nu(I)$. Hence no $i$ can exist for which $e_i - \nu(I)$ and $e_i - \nu(J)$ are minimal generators and $\text{IT}_{\nu}(\mathcal{J} \cdot \mathcal{J} \cdot \mathcal{K})$ is simplicial.

Lemma 26. Let $\mathcal{J}$, $\mathcal{J}$ and $\mathcal{K}$ be three monomial coordinate ideals in $K[x_1, \ldots, x_n]$ with clouds $I$, $J$ and $K$, respectively. When the inclusions $I \subset J \cup K$, $J \subset K \cup I$ and $K \subset I \cup J$ hold, then the product $\mathcal{J} \cdot \mathcal{J} \cdot \mathcal{K}$ is tame.

Proof. If $I \subset J$, then it follows that $K \subset J$ and thus $J = K \cup I$, in which case the product is tame by Lemma \[\text{[24]}\]. Hence for the remainder we assume that no cloud is contained in another cloud.

Let $v$ be a vertex of the Newton polyhedron of $\mathcal{J} \cdot \mathcal{J} \cdot \mathcal{K}$. Then $v = v(I) + v(J) + v(K)$, where $\nu(I)$, $\nu(J)$ and $\nu(K)$ denote vertices of the Newton polyhedra of $\mathcal{J}$, $\mathcal{J}$ and $\mathcal{K}$, respectively. The ideal tangent cone of $\mathcal{J} \cdot \mathcal{J} \cdot \mathcal{K}$ at $v$ is the sum

\begin{align*}
\text{IT}_{\nu}(\mathcal{J} \cdot \mathcal{J} \cdot \mathcal{K}) &= \text{IT}_{\nu(I)}(\mathcal{J}) + \text{IT}_{\nu(J)}(\mathcal{J}) + \text{IT}_{\nu(K)}(\mathcal{K}).
\end{align*}

First assume that $\nu(I)$, $\nu(J)$ and $\nu(K)$ are distinct. We may assume $\nu(I)$ is contained in $J$. Then $\nu(I) - \nu(J) \in \text{IT}_{\nu(I)}(\mathcal{J})$ and thus $\nu(I) - \nu(J) \in \text{IT}_{\nu}(\mathcal{J} \cdot \mathcal{J} \cdot \mathcal{K})$. If $\nu(J) \in I$ then also $\nu(J) - \nu(I) \in \text{IT}_{\nu}(\mathcal{J} \cdot \mathcal{J} \cdot \mathcal{K})$ and $v$ is not a vertex. Hence we need $\nu(J) \subset K \setminus I$ and by the same reasoning $\nu(K) \in J \setminus I$. But then the vectors $\nu(J) - \nu(K)$ and $\nu(K) - \nu(I)$ and thus their sum $\nu(J) - \nu(I)$ are contained in $\text{IT}_{\nu}(\mathcal{J} \cdot \mathcal{J} \cdot \mathcal{K})$ and $v$ is not a vertex. Hence $\nu(I)$, $\nu(J)$ and $\nu(K)$ cannot be distinct.
We may assume $\nu(I) = \nu(J)$ and $\nu(K) \in I$, so that $\nu(K) - \nu(I)$ is in $IT_\nu(\mathcal{I} \cdot \mathcal{J} \cdot \mathcal{K})$. If $IT_\nu(\mathcal{I} \cdot \mathcal{J} \cdot \mathcal{K})$ is not simplicial then there must be two minimal generators of the form $w - \nu(I)$ and $w - \nu(K)$ for some $w \in \mathbb{Z}_n^*$, but $w - \nu(I) = w - \nu(K) + \nu(K) - \nu(I)$. Hence $IT_\nu(\mathcal{I} \cdot \mathcal{J} \cdot \mathcal{K})$ is simplicial. 

Explicit examples show that the above treated criteria are exhaustive: for any three subsets $I$, $J$, and $K$ of $\{1, \ldots, n\}$ not satisfying any of the criteria

(i) $I \subset J$ and $K \cap \emptyset$,
(ii) $I \subset J \cap K \neq \emptyset$,
(iii) $I \subset J \cup K$ and $I \subset J \cup K$,
(iv) $K = I \cup J$,

one can find examples of three ideals such that the product is not tame.

We close this section with a result that will play a role when we discuss permutohedral blowups and provides a smoothing procedure in Proposition 29.

**Lemma 27.** Let $\mathcal{I}_1, \ldots, \mathcal{I}_N$ be coordinate ideals in $K[x_1, \ldots, x_n]$. Then the ideal $\mathcal{I} = \prod_{i<j}(\mathcal{I}_i + \mathcal{I}_j)$ is smooth.

**Proof.** We write $I_i$ for the cloud of $\mathcal{I}_i$, $P_i$ for the Newton polyhedron of $\mathcal{I}_i$ and $P$ for the Newton polyhedron of $\mathcal{I}$. We introduce the set $\Omega$ of unordered pairs $(i, j)$, where $i$ and $j$ are distinct integers running from $1$ to $N$.

We fix a vertex $a$ of $P$: $a$ is of the form $\sum_{i<j} e_{k_{ij}}$, where $k_{ij}$ is in the cloud of $\mathcal{I}_i + \mathcal{I}_j$, and hence in the cloud of $I_i$ of the cloud of $I_j$ (or in both). The ideal tangent cone $IT_a(\mathcal{I})$ is the sum of the cones $IT_{e_{k_{ij}}}(\mathcal{I}_i + \mathcal{I}_j)$, which is generated by all vectors of the form $e_m - e_{k_{ij}}$, where $m$ runs over all elements in $I_i \cup I_j$, and all standard basis vectors.

We construct a function $w : \Omega \rightarrow \{1, \ldots, N\}$ as follows: for $(i, j)$ we put $w((i, j)) = i$ if $k_{ij}$ is in $I_i$ and $w((i, j)) = j$ if $k_{ij}$ is in $I_j$. We then define a function $\nu : W \rightarrow \{e_1, \ldots, e_n\}$ as follows: we assign $(i, j, w_{ij}) \in W$ the basis vector $e_{k_{ij}}$ from the expansion of $a$. We now claim that when there are different $k$ and $l$ such that $(1, k, 1)$ and $(1, l, 1)$ are in $W$, then $\nu(1, k, 1) = \nu(1, l, 1)$. Suppose $e_r = \nu(1, k, 1)$ and $e_s = \nu(1, l, 1)$ are different, then $e_r + e_s = \frac{1}{2}(e_l) + \frac{1}{2}(e_k)$ is not a vertex of $(P_l \cup P_k) + (P_l \cup P_k)$, which is the Newton polyhedron of $(\mathcal{I}_l + \mathcal{I}_k)(\mathcal{I}_l + \mathcal{I}_k)$. But then $a = e_r + e_s + \cdots$ cannot be a vertex of $\mathcal{I}$. We can write for $a$

$$a = \sum_{k \neq 1} \nu(1, k, 1) + \sum_{k \neq 2} \nu(2, k, 2) + \cdots + \sum_{k \neq N} \nu(N, k, N).$$

When $w((i, j)) = i$ and $\nu((i, j)) = e_r$, then the ideal tangent cone $IT_a(\mathcal{I})$ contains generators $e_m - e_r$, where $m$ runs over all points in $I_i \cup I_j$. Conversely, when $e_m - e_r$ is a minimal generator in $IT_a(\mathcal{I})$, then $e_r = \nu((i, j))$ for some $i, j$.

Assume that $IT_a(\mathcal{I})$ is not smooth. Then there is a direction $i$ for which there are too many minimal generators. We may assume that $e_1 - e_1$ and $e_i - e_2$ are two minimal generators. Thus $1$ and $2$ are in the image of $w$. Then there are $k, k', l, l'$ such that $e_1 = \nu(k, k, l)$ and $e_2 = \nu(k', l', k')$ and $k \neq k'$. We may assume $w(k, k') = k$ so that $\nu(k, k', k) = e_1$. Both $e_1$ and $e_2$ are in the Newton polyhedron of $\mathcal{I}_k + \mathcal{I}_k$ and $e_2 - e_1$ is in the ideal tangent cone of $a$, which contradicts that both $e_1 - e_1$ and $e_1 - e_2$ are minimal generators.

**Corollary 28.** Let $\mathcal{I}$ be the monomial ideal from Lemma 27 and $\mathcal{I} = \prod_{i=1}^N \mathcal{I}_i$. Then $\mathcal{I} \cdot \mathcal{J}$ is tame.

**Proof.** Take the ideal $\mathcal{I}_{k+1} = (0)$ as the $(k + 1)$th factor in the product. With Lemma 27 it then follows that $\mathcal{I} \cdot \mathcal{J} = \prod_{1 \leq i < j \leq k+1}(\mathcal{I}_i + \mathcal{I}_j)$ is tame.
Example 4. Consider the monomial ideals $\mathcal{I}_i = (x_i)$ in $K[x_1, \ldots, x_n]$. By Lemma 24 the ideal $\prod_{i<j}(x_i, x_j)$ is tame. The ideal $\mathcal{I}$ is symmetric under the action of the symmetric group on $\{1, \ldots, n\}$. In Section 7 we will discuss a wider class of such symmetric ideals and prove that they are tame.

Given any set of coordinate ideals $\mathcal{I}_1, \ldots, \mathcal{I}_k$ in $K[x]$, one easily sees that $V(\mathcal{I}_i + \mathcal{I}_j) \subset V(\prod_{i=1}^k \mathcal{I}_i)$. It follows that we have

$$V\left(\prod_{i=1}^k \mathcal{I}_i \cdot \prod_{1 \leq i < j \leq k} (\mathcal{I}_i + \mathcal{I}_j)\right) = V\left(\prod_{i=1}^k \mathcal{I}_i\right).$$

Hence we arrive at the following smoothing procedure:

**Proposition 29.** Suppose that the product $\prod_{i=1}^k \mathcal{I}_i$ of coordinate ideals $\mathcal{I}_1, \ldots, \mathcal{I}_k$ in $K[x]$ is not tame, then the ideal

$$\prod_{i=1}^k \mathcal{I}_i \cdot \prod_{1 \leq i < j \leq k} (\mathcal{I}_i + \mathcal{I}_j)$$

has the same zero set as $\prod_{i=1}^k \mathcal{I}_i$ and is tame.

6. **Blowups in monomial Building Sets**

Fulton and MacPherson study the problem of the compactification of the complement of configuration spaces [30]. They consider configuration spaces $F(n, X)$ of smooth algebraic varieties $X$, where $F(n, X)$ denotes the space of $n$-tuples of mutually distinct points in $X$:

$$F(n, X) = \{(x_1, \ldots, x_n) \in X^n : x_i \neq x_j \text{ for } i \neq j\}.$$

One can construct a compactification $X[n]$ of $F(n, X)$ via a sequence of blowups such that the complement of the original configuration space is a normal crossings divisor. De Concini and Procesi [7] have introduced wonderful models of finite families of linear subspaces of a vector space $X$. A wonderful model for such a subspace arrangement is constructed so that the complement of the arrangement remains unchanged and the subspaces are substituted by a normal crossings divisor. In order to achieve normal crossings, De Concini and Procesi introduced the notion of a building set of an arrangement of linear subspaces.

Later MacPherson and Procesi [8] generalized wonderful models to conic varieties over $\mathbb{C}$. They use conical stratifications and generalize the notion of building sets. Special compactifications have also been studied by Hu [31] and Ulyanov [32]. Li [9] has generalized wonderful models to compactifications of subspace arrangements of a smooth variety. In [9, Thm 1.3] it is shown that the blowup in a product of ideals is smooth if they form a building set. Wonderful compactifications are also studied by combinatorial means by Feichtner [33].

In the following we will introduce the notion of building sets, like in [7]. We use this notion for linear subspaces when we look at zero sets of coordinate ideals $\mathcal{I} = (x_i, i \in I)$ in $K[x]$ with $I \subseteq \{1, \ldots, n\}$, which yield only linear (coordinate) subspaces. With the help of building sets we find a large class of tame product ideals: In Proposition 22 we show that the blowup with center a product $\prod \mathcal{I}_i$ of ideals $\mathcal{I}_i = (x_j : j \in I_i)$ is smooth if $\mathcal{G} = \{\mathcal{I}_i\}$ form a building set. However, in Section 7 we provide a large class of a tame products of coordinate ideals, which do not come from building sets.

Consider a vector space $V \cong K^n$. An arrangement of subspaces is a finite set $\mathcal{C}$ of nonzero subspaces of $V$ closed under taking sums. For $U \in \mathcal{C}$ we call $U_1, \ldots, U_k \in \mathcal{C}$, with $k \geq 2$, a decomposition of $U$ if

$$U = U_1 \oplus \cdots \oplus U_k,$$

and for each subspace $A \subseteq U$ in $\mathcal{C}$ also each $A \cap U_i$ is an element of $\mathcal{C}$ and

$$A = (A \cap U_1) \oplus \cdots \oplus (A \cap U_k).$$
If a subspace \(U\) does not allow a decomposition in \(C\) we call \(U\) irreducible. Given nonzero subspaces \(V_1, \ldots, V_k\), then the arrangement generated by \(V_1, \ldots, V_k\) is the smallest set of subspaces containing \(V_1, \ldots, V_k\) and is closed under taking sums.

**Example 5.** (1) Let \(V = K^4\) with the standard basis \(e_1, \ldots, e_4\). Let \(C = \{U_1, \ldots, U_7\}\) be an arrangement with \(U_1 = K\langle e_1, e_2, e_3 \rangle, U_2 = K\langle e_1 \rangle, U_3 = K\langle e_2 \rangle, U_4 = K\langle e_3 \rangle, U_5 = K\langle e_1, e_2 \rangle, U_6 = K\langle e_2, e_3 \rangle\) and \(U_7 = K\langle e_1, e_3 \rangle\). The irreducible elements of \(C\) are \(U_2, U_3, U_4\). Then \(U_1 = U_5 \oplus U_4\) and \(U_1 = U_2 \oplus U_3 \oplus U_4\) are decompositions of \(U_1\), since \(U_6 = (U_6 \cap U_4) \oplus (U_6 \cap U_5)\) and \(U_7 = (U_7 \cap U_4) \oplus (U_7 \cap U_5)\) are direct sums of elements in \(C\).

(2) Now consider \(C' = C\{U_3\}\). The direct sum \(U_1 = U_4 \oplus U_5\) is not a decomposition since \(U_6 \not\subseteq U_1\) in \(C'\), but \(U_6 \cap U_5 = K\langle e_2 \rangle \not\in C'\).

**Definition 30.** Let \(G\) be a nonempty set of nonzero subspaces of \(V\). Let \(C\) be the arrangement generated by \(G\). We call \(G\) a building set if the following conditions are satisfied:

(a) The irreducible elements of \(C\) are contained in \(G\).

(b) If \(A, B\) are in \(G\) and if \(A + B\) is not a direct sum, then \(A + B\) is already in \(G\).

Equivalently, \(G\) is a building set of the arrangement \(C\) if and only if each element \(e \in C\) is the direct sum \(C = \bigoplus_{i=1}^{k} G_i\) of the set of maximal elements \(G_1, \ldots, G_k\) in \(G\) that are contained in \(C\) (see [2, thm 2.3]).

Note that the decomposition of \(C = \bigoplus_{i=1}^{k} G_i\) with the \(G_i\) maximal with respect to inclusion is unique. Moreover, for all \(A\) in \(G\), \(A \subseteq C\) there exists an \(i\) such that \(A \subseteq G_i\).

**Example 6.** Let \(C\) be an arrangement of subspaces in \(V\).

1. \(C\) is a building set. This is clear because one can write each \(C \in C\) as a “direct sum” of its maximal subspace \(C\).

2. The set of irreducible elements of \(C\) is a building set.

3. Let \(V = K^3\) with basis \(e_1, e_2, e_3\) and let \(G = \{G_1, G_2, G_3\} = \{K\langle e_1, e_2 \rangle, K\langle e_1, e_3 \rangle, K\langle e_2, e_3 \rangle\}\). Then \(G\) is not a building set, for

\[
K^3 = K\langle e_1, e_2, e_3 \rangle = G_1 + G_2 = G_1 + G_3 = G_2 + G_3
\]

is not a direct sum of maximal elements of \(G\).

However, if we consider \(G' := G \cup \{K\langle e_1, e_2, e_3 \rangle\}\), then all sums are direct because \(G'\) is the set of irreducible elements of \(C\).

In the following we identify a coordinate ideal \((x_i : i \in I)\) in \(K[x]\) with the linear subspace \(K\langle e_i, i \in I \rangle\) in \(V = K^n\). We then call a set \(C\) an arrangement of ideals if the corresponding linear subspaces form an arrangement in \(V\). We call a set \(G\) a building set of ideals if the corresponding subspaces form a building set in \(V\). The sum of two elements \(\mathcal{J} + \mathcal{J}\) of an arrangement \(C\) is direct if and only if \(\mathcal{J}\) and \(\mathcal{J}\) have transverse clouds.

**Remark 3.** One can define building sets and arrangements dually. Then instead of taking sums of subspaces one takes intersections of subspaces. Let \(C\) be an arrangement in \(V\). Then \(\mathcal{P} = \{V\perp : V \in C\}\) is an arrangement in the sense of [4, 8] since \(U^\perp \cap W^\perp = (U \cap W)^\perp\). In particular, for coordinate ideals \(\mathcal{J}\), we have \(\mathcal{J}^\perp = V(\mathcal{J})\). For two ideals \(\mathcal{J} = (x_i : i \in I), \mathcal{J} = (x_j : j \in J)\) we have

\[
V(\mathcal{J} + \mathcal{J}) = V(x_k : k \in I \cup J) = V(\mathcal{J}) \cap V(\mathcal{J})
\]

One can also generalize the notions of building sets to arrangements of subvarieties of a smooth variety, see [6].

**Definition 31.** Let \(G\) be a building set of monomial ideals in \(K[x_1, \ldots, x_n]\). For any \(\mathcal{J} \in G\) we can write \(\mathcal{J} = (x_a : a \in A)\) for some subset \(A \subseteq \{1, 2, \ldots, n\}\). We define the building set of sets associated to \(G\) to be the set of all such subsets \(A\).
For any building set of monomial ideals $G$ with associated building set of sets $G$ we generally write $\mathcal{I}_A$ for the ideal $(x_a : a \in A)$ for any $A \in G$. By the defining properties of building sets of ideals we see that $G$ satisfies that for all sets $A$ and $B$ in $G$ with nonempty intersection also their union is in $G$. Conversely, given any set of subsets of $\{1, 2, \ldots, n\}$ satisfying this property, we can obtain a building set of monomial ideals $\mathcal{I}_A$, where $A$ runs over the elements of $G$, for the arrangement that they generate.

**Example 7** (Mickey Mouse example). We consider an arrangement of subsets of $\{1, 2, \ldots, n\}$ generated by $A$, $B$ and $C$ where $A \cap C = \emptyset$, $A \cap B \neq \emptyset$ and $B \cap C \neq \emptyset$, see figure 5. The smallest building set is $G = \{A, B, C, A \cup B, B \cup C, A \cup B \cup C\}$. The biggest is given by $G' = G \cup \{A \cup C\}$.

![Figure 5. Three sets generating an arrangement.](image)

Let us denote for any $D \in G$ the corresponding ideal $\mathcal{I}_D = (x_d : d \in D)$. We write $\mathcal{G}$ for the set of ideals defined by $G$: $\mathcal{G} = \{\mathcal{I}_D : D \in G\}$. We will show that the blowup in the building set $\mathcal{G}$ is smooth; that is, the ideal $\prod_{D \in G} \mathcal{I}_D$ is smooth.

The vertices of the Newton polyhedron of the product $\mathcal{I} = \prod_{D \in G} \mathcal{I}_D$ are of the form $v = \sum_{D \in G} \nu(D)$, where $\nu(D) = e_d$ for some $d \in D$. The function $\nu$ chooses a vertex for each element of the building set. By Remark [1] we know that $\nu(A \cup B) = \nu(A)$ or $\nu(A \cup B) = \nu(B)$, since else $v$ would not be a vertex. The ideal tangent cone of $v$ is given by the sum of the cones

$$ IT_{\nu(D)}(\mathcal{I}_D) = N(e_d - e_{\nu(D)})$, $\Sigma, $$

where $D$ runs over all elements of $G$ and where $\Sigma$ is the set of all basis vectors $\{e_1, \ldots, e_n\}$. We may assume $\nu(A \cup B) = \nu(A)$ and we first additionally assume that $\nu(A)$, $\nu(B)$ and $\nu(C)$ are distinct. There are four possibilities:

(i) $\nu(B \cup C) = \nu(B)$ and $\nu(A \cup B \cup C) = \nu(A)$.
(ii) $\nu(B \cup C) = \nu(B)$ and $\nu(A \cup B \cup C) = \nu(B)$.
(iii) $\nu(B \cup C) = \nu(C)$ and $\nu(A \cup B \cup C) = \nu(A)$.
(iv) $\nu(B \cup C) = \nu(C)$ and $\nu(A \cup B \cup C) = \nu(C)$.

In case (i) we see that all generators of the form $e_i - \nu(A)$ for $i \notin A$, cannot be minimal since if $i \in B \setminus A$ we have $e_i - \nu(A) = e_i - \nu(B) + \nu(B) - \nu(A)$ and similarly for $i \in C$. Hence no $i \in \{1, 2, \ldots, N\}$ exists with two minimal generators of the form $e_i - e_k$ for some $1 \leq k \leq n$. In case (ii) we see that both $\nu(A) - \nu(B)$ and $\nu(B) - \nu(A)$ are in the ideal tangent cone, and thus in this case $v$ is not a vertex. Case (iii) is similar to the first case; we can write for any $b \in B$: $e_b - \nu(A) = e_b - \nu(C) + \nu(C) - \nu(A)$, $e_b - \nu(C) = e_b - \nu(B) + \nu(B) - \nu(C)$ for any $c \in C$ we can write $e_c - \nu(A) = e_c - \nu(C) + \nu(C) - \nu(A)$. Hence also in this case, the ideal tangent cone is simplicial. For the last case we have for any $a \in A$: $e_a - \nu(C) = e_a - \nu(A) + \nu(A) - \nu(A)$. For
any \( b \in B \) we have \( e_b - \nu(A) = e_b - \nu(B) + \nu(B) - \nu(A) \) and \( e_b - \nu(C) = e_b - \nu(B) + \nu(B) - \nu(C) \).

Hence also in this case we cannot have too many minimal generators.

The case that \( \nu(A) \), \( \nu(B) \) and \( \nu(C) \) are not all distinct uses similar arguments and is dealt with easily.

**Proposition 32.** Let \( \mathcal{G} \) be a building set of monomial ideals in \( K[x_1, \ldots, x_n] \). Then the ideal \( \prod_{\mathcal{I} \in \mathcal{G}} \mathcal{I} \) is tame.

**Proof.** Let \( G \) be the associated building set of sets associated to \( \mathcal{G} \). For any \( \mathcal{I} \in \mathcal{G} \) with \( \mathcal{I} = (x_\alpha : \alpha \in A) \) we also write \( \mathcal{I}_A \); we thus label the ideals in \( \mathcal{G} \) by their associated subsets of \( \{1, 2, \ldots, n\} \) in \( G \). We thus have \( \mathcal{I}_{AB} = \mathcal{I}_A + \mathcal{I}_B \).

Fix a vertex \( \mathbf{v} \) of the Newton polyhedron of \( \prod_{\mathcal{I} \in \mathcal{G}} \mathcal{I} \). Then \( \mathbf{v} \) is sum \( \sum_{\mathcal{I} \in \mathcal{G}} \nu(A) \) and \( \nu \) has the following property: for \( A, B \in G \) with \( A \cup B \) in \( G \), we have \( \nu(A \cup B) = \nu(A) \) or \( \nu(A \cup B) = \nu(B) \). The ideal tangent cone of \( \mathbf{v} \) is the sum

\[
IT_\mathbf{v} \left( \prod_{\mathcal{I} \in \mathcal{G}} \mathcal{I} \right) = \sum_{\mathcal{I} \in \mathcal{G}} IT_{\nu(A)}(\mathcal{I}_A).
\]

Suppose that \( IT_\mathbf{v} \left( \prod_{\mathcal{I} \in \mathcal{G}} \mathcal{I} \right) \) is not simplicial. Then for some index \( a \) there are at least two minimal generators \( e_a = e_b \) and \( e_a - e_c \), with \( 1 \leq b < c \leq n \). It follows that we must have \( b = \nu(B) \) and \( c = \nu(C) \) for some \( B \) and \( C \) in \( G \) and furthermore \( a \) lies in \( B \) and \( C \). But then \( B \cap C \neq \emptyset \) and thus \( B \cup C \in G \). We may assume \( e_b = \nu(B \cup C) \). Then \( e_a - e_b \in IT_{\nu(B)}(\mathcal{I}_{B \cup C}) \) and therefore \( e_c - e_b \) lies in \( IT_\mathbf{v} \left( \prod_{\mathcal{I} \in \mathcal{G}} \mathcal{I} \right) \). But then \( e_a - e_b = e_a - e_c + e_c - e_b \), contradicting that \( e_a - e_b \) is minimal. \( \square \)

7. Permutohedral ideals

In this section we prove that the so-called permutohedral ideals are tame. The permutohedral ideal \( \mathcal{I}_{n,k} \) is the ideal in \( K[x] \) defined by

\[
\mathcal{I}_{n,k} = \prod_{i_1 < \ldots < i_k} (x_{i_1}, \ldots, x_{i_k}), \text{ all } i_j \in \{1, \ldots, n\}.
\]

Obviously, the factors in eqn. (33) do not form a building set. Thus the permutohedral ideals form a class of tame ideals that do not stem from building sets.

**Definition 33.** Let \( p_1, \ldots, p_n \) be real numbers. The permutohedron \( P(p_1, \ldots, p_n) \) is the convex polytope in \( \mathbb{R}^n \) defined as the convex hull of all permutations of the vector \( (p_1, \ldots, p_n) \):

\[
P(p_1, \ldots, p_n) = \text{conv}((p_{\sigma(1)}, \ldots, p_{\sigma(n)}) : \sigma \in S_n),
\]

where \( S_n \) is the symmetric group. \( P(p_1, \ldots, p_n) \) lies in the affine hyperplane \( H = \{ x \in \mathbb{R}^n : \sum_{i=1}^n x_i = \sum_{i=1}^n p_i \} \).

We consider the permutohedron \( \pi_{n,k} = P(\binom{n-1}{k-1}, \binom{n-2}{k-1}, \ldots, \binom{k-1}{k-1}, 1, 0, \ldots, 0) \). It is easy to see that \( \pi_{n,k} \) has \( \binom{n}{k-1} \) vertices whose nonzero entries are pairwise distinct and that \( \pi_{n,k} \) lies in the affine hyperplane \( H = \{ x \in \mathbb{R}^n : \sum_{i=1}^n x_i = \binom{n}{k-1} \} \).

**Lemma 34.** The vertices of the Newton polyhedron of \( \mathcal{I}_{n,k} \) are equal to the vertices of \( \pi_{n,k} \).

**Proof.** We will prove that the vertices of \( \pi_{n,k} \) correspond to the vertices of the convex hull of the cloud \( I \) of \( \mathcal{I}_{n,k} \). This will prove the lemma as any vertex of \( N(\mathcal{I}_{n,k}) \) lies in the cloud \( I \). We denote the convex hull of \( I \) by \( W \).

First let \( \mathbf{p} = ((\binom{n-1}{k-1}), (\binom{n-2}{k-1}), \ldots, (\binom{k-1}{k-1}), 1, 0^{k-1}) \) be a vertex of \( \pi_{n,k} \). We show that \( \mathbf{p} \) is also a vertex of \( W \). Expanding \( \binom{n}{k-1} \) we see \( \mathbf{p} \in W \). We consider the hyperplane \( H_1 = \{ x \in \mathbb{R}^n : x_1 = \binom{n}{k-1} \} \) and define \( W \cap H_1 = W_1 \). Then \( W_1 \) is nonempty since \( \mathbf{p} \in W \). For all \( \mathbf{q} \in W, q_1 \leq \binom{n}{k-1} \); this follows from the construction of \( W \) as the sum of the Newton
polyhedra $N((x_1, \ldots, x_n))$. Indeed, there are exactly $\binom{n-1}{k-1}$ factors that contain $x_1$. Hence the maximal entry in the $e_i$-direction is $\binom{n-1}{k-1}$. Thus $W_1$ is a face of $W$. Then consider $H_2 = \{x \in \mathbb{R}^n : x_2 = \binom{n-2}{k-1}\}$ and set $W_2 = W_1 \cap H_2$. Similar to above we see $H_2$ lies on one side of $W_2$; in the factors of $\mathcal{I}_{n,k}$ we have chosen $\binom{n-1}{k-1}$ times $x_1$, and of the remaining factors $\binom{n-2}{k-1}$ contain $x_2$. Hence for any $q \in W$, $q_2$ can at most be $\binom{n-2}{k-1}$. Continuing this way we find that $W_{n-k} \cap H_{n-k+1} = p$ with $H_{n-k+1} = \{x \in \mathbb{R}^n : x_{n-k+1} = 1\}$. Hence $p$ is also a vertex of $W$.

We call a vertex $p$ of $\pi_{n,k}$ a max-vector. Fix any point $v$ in $W$ that is not a max-vector. We will show that $v$ can be written as a convex combination of other points contained in $W$, showing that $v$ is not a vertex.

We start with the case $k = 2$. For $k = 2$ a max-vector is a permutation of $(n-1, \ldots, 1, 0)$. The point $v$ corresponds to a monomial appearing in the expansion of $(x_1, x_2)(x_1, x_2) \cdots (x_{n-1}, x_n)$. Such a monomial arises by choosing from each pair $(x_i, x_j)$ the $x_i$ or the $x_j$. For generic $v$ there are many different possible choices that all contribute to $v$. We translate this choosing of terms from each factor into different graphs: Consider $n$ vertices labeled with the numbers $1, \ldots, n$. To codify the choice we direct an edge from $i$ to $j$ if in the factor $(x_i, x_j)$ the monomial $x_i$ was chosen. Thus for each way of choosing one out of each $(x_i, x_j)$ we get a graph, which is a complete directed graph with $n$ vertices. For any given graph associated to $v$, $v_i$ is the number of outgoing edges from vertex $i$. For a max-vector there is only one graph, which is a tree, see for example the (unique) graph corresponding to $(4, 3, 2, 1, 0)$ for $n = 5$ below:

![Diagram](https://example.com/diagram.png)

If we flip the orientation of an edge $i \rightarrow j$ in a graph associated to $v$ we choose $x_j$ instead of $x_i$ in the factor $(x_i, x_j)$. The resulting graph corresponds to the graph of the point $v' = (v_1, \ldots, v_i - 1, \ldots, v_j + 1, \ldots, v_n)$. Clearly $v' \in W$.

Claim: If $v \in W$ is not a max-vector, then any of its associated graphs contains a cycle.

Proof of Claim: Let $G$ be any of the graphs associated to $v$. We may assume $v_1 \geq v_2 \geq \ldots \geq v_n$. If $v$ is not a max-vector, there is a $j$ with $v_j < n - j$. It follows that no vertex from $1$ to $j - 1$ can be contained in a cycle, since $1$ only has outgoing edges, $2$ has one incoming edge from $1$ and for the rest outgoing, and so on, till we get to $j$. Thus we can restrict the search for cycles to the subgraph of $G$ with vertices $j, \ldots, n$ and may assume that $v_1 < n - 1$.

We can find a cycle with the following procedure: since $v_1 < n - 1$, the vertex $1$ has at least one incoming edge from a vertex $l$. We go from $1$ to $l$. Since $v_l \leq v_1 < n - 1$ also this vertex will have at least one incoming edge. We choose such an incoming edge and go to the next vertex. Since the graph has a finite number of vertices, we will meet a certain vertex for a second time after a finite number of steps. Hence we will find a cycle by retracing a part of our path. This proves the claim.

For $n = 5$ we depicted a graph associated to the vector $(4, 3, 2, 1, 0)$, containing several cycles:
We may assume that the cycle is \( C = 1 \rightarrow 2 \rightarrow \ldots \rightarrow n \). If we flip the orientation of the edge \( 1 \rightarrow 2 \) we see that \( \mathbf{v}^{(1)} = (v_1 - 1, v_2 + 1, v_3, \ldots, v_n) \) is in \( W \). Flipping the edges \( j \rightarrow j + 1 \) we get that the vectors \( \mathbf{v}^{(2)} = (v_1, v_2 - 1, v_3 + 1, v_4, \ldots, v_n), \ldots, \mathbf{v}^{(n)} = (v_{1+1}, v_2, v_3, v_4, \ldots, v_n, v_{n-1}) \) all lie in \( W \). It is easy to see that

\[
\mathbf{v} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{v}^{(n)}.
\]

Hence \( \mathbf{v} \) is contained in \( \text{conv}(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(n)}) \) and \( \mathbf{v} \) is not a vertex of \( N \).

The general case \( k \geq 3 \) is similar: to each factor \( (x_{i_1}, \ldots, x_{i_k}) \) corresponds a \( k \)-simplex with vertices \( i_1, \ldots, i_k \). For example, for \( n = 4 \) and \( k = 2 \) one considers the faces of the tetrahedron.

Like above, to \( \mathbf{v} \) correspond choices of some \( x_{i_j} \) from each factor \( (x_{i_1}, \ldots, x_{i_k}) \). If we choose \( x_{i_j} \) in a simplex \( (x_{i_1}, \ldots, x_{i_k}) \), we mark the vertex \( i_j \) as the only “outgoing” vertex and the other \( k - 1 \) vertices as the “incoming” vertices of the \( k \)-simplex. We search for cycles of vertices on the \( k \)-simplices in the following way: start with the \( k \)-simplex \((1, \ldots, k)\) where we mark the outgoing vertex. Suppose 1 is the outgoing vertex. Then we can leave through 1 to another \( k \)-simplex if 1 is incoming there. We can find a chain of vertices (and \( k \)-simplices) representing any point \( \mathbf{v} \) in \( N \). First let \( \mathbf{v} \) be the lexicographically ordered max-vector \( (\binom{n-1}{k-1}, \binom{n-2}{k-1}, \ldots, 1, 0, \ldots, 0) \). Then 1 is an outgoing vertex in all \( \binom{n-1}{k-1} \) simplices containing it. Since 1 is nowhere incoming we cannot find a cycle containing 1. The vertex 2 is outgoing for \( \binom{n-2}{k-2} \) simplices and incoming for \( \binom{n-2}{k-2} \). But 2 is incoming only from simplices with outgoing 1. Thus 2 can also not be part of a cycle. Continuing this argument we see that no vertex 1, \ldots, \( n - k + 1 \) can be part of a cycle. But the remaining vertices are only incoming and are also not in a cycle.

If \( \mathbf{v} \in W \) is not a max-vector we can assume that \( v_1 = \binom{n-1}{k-1}, \ldots, v_{i-1} = \binom{n-i+1}{k-1}, v_i < \binom{n-i}{k-1} \). Then all vertices 1, \ldots, \( i - 1 \) are not part of a cycle. But \( i \) is at least incoming for one simplex. By a similar method as for \( k = 2 \) we can find a cycle \( i_1 \rightarrow i_2 \rightarrow \ldots \rightarrow i_m \) of length \( m \) of vertices on the \( k \)-simplices for \( i_j \geq i \) and find \( m \) points \( \mathbf{v}^{(i)} \) in \( W \) such that \( \mathbf{v} \) lies in their convex hull. As an illustration of the method, we depicted the 2-simplices of the tetrahedron and marked the outgoing vertices with an asterisk:
The graph corresponds to the vector \((2, 2, 0, 0)\), which is not a max-vector. We can find a cycle \(1 \rightarrow 2 \rightarrow 1\) using the 2-simplices \((1, 2, 3)\) and \((1, 2, 4)\).

**Remark 4.** The permutation polynomial \(p_{n,k}\) is defined by

\[
p_{n,k} = \prod_{1 \leq i_1 < \ldots < i_k \leq n} (x_{i_1} + \ldots + x_{i_k}).
\]

When we expand \(p_{n,k} = \sum_a c_a x^n\) in monomials, then the vectors \(a \in \mathbb{N}^n\) with \(c_a\) nonzero define a finite set \(M\). The vertices of the convex hull of \(M\) correspond to the max-vectors introduced in the proof of Lemma \([34]\). The max-vectors are precisely those \(a \in M\) with \(c_a = 1\).

**Proposition 35.** Each ideal \(\mathcal{I}_{n,k}\) is tame.

**Proof.** Let \(I\) the cloud of \(\mathcal{I}_{n,k}\). By Lemma \([34]\) the vertices of \(I\) are all permutations of \(b = ((n-1 \choose k-1), 1, 0, 0, \ldots)\). We first show that the elements \(e_i, e_{i+1} - e_i\) for \(1 \leq i \leq n - k\) and \(e_n - e_{n-k+1} - e_{n-k+1} \) for \(2 \leq j \leq k\) are all in \(IT_b(\mathcal{I}_{n,k})\).

From Lemma \([20]\) we know that the minimal generators of \(IT_b(\mathcal{I}_{n,k})\) are of the form \(e_i\) or \(e_i - e_j\). There cannot be a vector \(e_i - e_j\), \(j \geq 2\) in \(IT_b(\mathcal{I}_{n,k})\) because this generator would stem from a point \(b + (e_1 - e_j) = ((n-1 \choose k-1) + 1, \ldots, 0)\) in \(I\). But this contradicts \(x_i \leq b_1\) for all \(x\) in \(I\). But then \(e_1\) is a minimal generator of \(IT_b(\mathcal{I}_{n,k})\).

For the vectors \(e_i - e_{i-1}, i = 2, \ldots, n - k + 2\) consider the vector \(a = b + e_i - e_{i-1}\); \(a\) has the same coordinates as \(b\) except for \(a_1 = b_{i-1} - 1\) and \(a_i = b_i + 1\). The vector \(b\) is a max-vector and \(b_1 > b_2 > b_3 > \ldots\). Hence in some factor of the product defining \(\mathcal{I}_{n,k}\) there exist \(x_i\) and \(x_{i-1}\) and we have chosen \(x_{i-1}\) in this factor. Choosing \(x_i\) instead of \(x_{i-1}\) we see \(a \in I\). Hence \(e_i - e_{i-1}, i = 2, \ldots, n - k + 2\) are in \(IT_b(\mathcal{I}_{n,k})\).

Practically the same argument applies to conclude that the vectors \(e_n - e_{n-k+j} - e_{n-k+1}\) are in \(IT_b(\mathcal{I}_{n,k})\).

From the above we see that all vectors \(e_i - e_j\) with \(i > j\) and \(j \leq n - k + 2\) and \(e_i - e_j\) are in \(IT_b(\mathcal{I}_{n,k})\). If \(v \in IT_b(\mathcal{I}_{n,k})\), then \(v_l \geq 0\) for all \(l > n - k + 1\), since \(b_l = 0\) for all \(l \geq n - k + 1\). It follows that if \(e_i - e_j\) and \(e_i - e_j'\), with \(j > j'\) are two minimal generators, then \(j, j' \leq n - k + 1\) and since \(IT_b(\mathcal{I}_{n,k})\) is pointed, it follows that \(j > j'\). Then \(e_i - e_j + e_j - e_j' = e_i - e_j'\) shows that \(e_i - e_j\) is not minimal. Hence \(IT_b(\mathcal{I}_{n,k})\) is simplicial.

**Remark 5.** The proof for \(\mathcal{I}_{n,2}\) can be done by applying Lemma \([27]\) to the ideals \(\mathcal{I}_i = (x_i)\); also see Example \([3]\).

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