Pointwise controllability as limit of internal controllability for the beam equation

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Abstract

This work is devoted to prove the pointwise controllability of the Bernoulli-Euler beam equation. It is obtained as a limit of internal controllability of the same type of equation. Our approach is based on the techniques used in [4].

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1 Introduction

In this paper, we are interested in the passage from internal exact controllability of beam equation to pointwise exact controllability. We consider the following initial and boundary value problem

\[ \frac{\partial^2 u}{\partial t^2}(x, t) + \frac{\partial^4 u}{\partial x^4}(x, t) = g_n(x, t), \quad 0 < x < 1, \quad t > 0, \]  
\[ u(0, t) = \frac{\partial u}{\partial x}(1, t) = \frac{\partial^2 u}{\partial x^2}(0, t) = \frac{\partial^3 u}{\partial x^3}(1, t) = 0, \quad t > 0, \]  
\[ u(x, 0) = u^0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u^1(x), \quad 0 < x < 1, \]  

where \( g_n, \ u^0, \ u^1 \) are in suitable spaces with \( \text{supp}(g_n) = [\xi, \xi + \frac{1}{n}], \) \( n \in \mathbb{N}^* \) and \( \xi \in (0, 1). \) Here \( u \) denotes the transverse displacement of the beam, we suppose that the length of the beam is equal to 1 and the control depends on a parameter \( n \in \mathbb{N}^*. \) Recall that this model describes the transversal vibrations of the Bernoulli-Euler beam.
The problem of internal exact controllability was studied by Haraux [7], Jaffard [9] and Lions [10]. The pointwise exact controllability for a strategic point was studied by Haraux and Jaffard [8] and Lions [10]. However, the convergence of the internal exact controllability of equation (1.1)-(1.3) to the pointwise exact controllability has apparently not yet been studied.

The aim of this paper is to describe what happens when $n$ tends to infinity, we can’t hope to get a pointwise control for the limit problem for any strategic point in $(0,1)$, we use the same techniques introduced in [4].

Our purpose in this paper is to prove the pointwise controllability of the Bernoulli-Euler beam equation. It is obtained as a limit of internal controllability of the same type of equation. Our approach based on the techniques used in [4]. This result can be proved by the standard HUM method (Hilbert uniqueness method) by J.L. Lions [10]. As $n$ tends to infinity, we obtain the solution of an exact internal controllability problem which converges towards to the solution of an exact pointwise controllability problem.

The plan of the paper is as follows. In section 2 we show the regularity of weak solutions of problem (1.1)-(1.3) for a strategic point in $(0,1)$ and we study the behavior of these solutions in an interval of length $\frac{1}{n}$. The exact controllability results are given in section 3. In section 4 we prove an inverse inequality which will give us the estimates on the internal controls in the case of a strategic point. Finally, in section 5 as $n$ tends to infinity we prove that the pointwise exact controllability problem is obtained as limit of exact internal controllability problem of the beam equation.

2 Estimation and regularity results near a point

Now introducing the Hilbert spaces

$$V = \{ u \in H^2(0,1), \ u(0) = 0, \frac{du}{dx}(1) = 0 \}.$$  

$V'$ is the dual space of $V$ with respect to the pivot space $L^2(0,1)$, where the duality is in the sense of $L^2(0,1)$. And

$$D(\partial_x^4) = \left\{ u \in H^4(0,1), \ u(0) = \frac{du}{dx}(1) = 0, \frac{d^2u}{dx^2}(0) = \frac{d^3u}{dx^3}(1) = 0 \right\}.$$  

Consider two given functions $(u^0, u^1)$ in $L^2(0,1) \times V'$, $g_n \in L^2(0,T, L^2(0,1))$ and supp $(g_n) = [\xi, \xi + \frac{1}{n}]$, \(\xi \in (0,1)\) and we will take $\frac{1}{n} < 1 - \xi$. Let $u$ be the solution of (1.1)-(1.3).

**Proposition 2.1.** Assume $g_n \in L^2(0,T, L^2(0,1))$ and $(u^0, u^1)$ in $L^2(0,1) \times V'$. Then for any $T > 0$, problem (1.1)-(1.3) admits a unique solution

$$u \in C \left(0,T,L^2(0,1)\right) \cap C^1 \left(0,T,V'\right).$$

Moreover,

$$u(\xi,t) \in L^2(0,T),$$

and there exists a constant $C > 0$ (independent on $n$ and $T$), such that

$$n \int_{\xi}^{\xi + \frac{1}{n}} \int_0^T |u(x,t)|^2 \ dx \ dt \leq C \left( \|g_n\|_{L^2(0,T,L^2(0,1))}^2 + \|u^0\|_{L^2(0,1)}^2 + \|u^1\|_{V'}^2 \right),$$

for all $n > 0$. 

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Proof. In order to prove \((2.5)\) we put

\[
u^0(x) = \sum_{m=0}^{\infty} a_m \sin \left( \frac{2m+1}{2} \pi x \right), \quad u^1(x) = \sum_{m=0}^{\infty} b_m \sin \left( \frac{2m+1}{2} \pi x \right),
\]

and

\[
g_n(x, t) = \sum_{m=0}^{\infty} g_m(t) \sin \left( \frac{2m+1}{2} \pi x \right),
\]

with \((a_m), \left( \frac{b_m}{(2m+1)^2} \right) \in L^2(\mathbb{R})\) and for \(t\) fixed \((g_m(t)) \in l^2(\mathbb{R})\).

The solution of \((1.1)-(1.3)\) is given by

\[
u(x, t) = \sum_{m=0}^{\infty} \left\{ a_m \cos \left[ \frac{(2m+1)}{2} \frac{\pi}{\xi} \right] t + \frac{b_m}{(2m+1)^2} \sin \left[ \frac{(2m+1)}{2} \frac{\pi}{\xi} \right] t \right\} + \frac{1}{(2m+1)^2} \int_{0}^{t} \sin \left[ \frac{(2m+1)}{2} \frac{\pi}{\xi} \right] (t-s) g_m(s) ds \sin \left( \frac{2m+1}{2} \pi x \right)
\]

Which implies that

\[
u(\xi, t) = \sum_{m=0}^{\infty} \left\{ a_m \cos \left[ \frac{(2m+1)}{2} \frac{\pi}{\xi} \right] t + \frac{b_m}{(2m+1)^2} \sin \left[ \frac{(2m+1)}{2} \frac{\pi}{\xi} \right] t \right\} + \frac{1}{(2m+1)^2} \int_{0}^{t} \sin \left[ \frac{(2m+1)}{2} \frac{\pi}{\xi} \right] (t-s) g_m(s) ds \sin \left( \frac{2m+1}{2} \pi \xi \right).
\]

We see that

\[
\| u^0 \|^2_{L^2(0, 1)} + \| u^1 \|^2_{V'} = \frac{1}{2} \sum_{m=0}^{\infty} \left[ a_m + \frac{b_m^2}{(2m+1)^2} \right],
\]

and

\[
\| g_n \|^2_{L^2(0, 1)} = \frac{1}{2} \sum_{m=0}^{\infty} g_m^2(t).
\]

Integrating \((2.6)\) over \((0, 1)\), we get

\[
\int_{0}^{1} u^2(x, t) dx = \frac{1}{2} \sum_{m=0}^{\infty} \left\{ a_m^2 \cos^2 \left[ \frac{(2m+1)}{2} \frac{\pi}{\xi} \right] t + \frac{b_m^2}{(2m+1)^2} \sin^2 \left[ \frac{(2m+1)}{2} \frac{\pi}{\xi} \right] t \right\} + \frac{1}{(2m+1)^2} \left( \int_{0}^{t} \sin \left[ \frac{(2m+1)}{2} \frac{\pi}{\xi} \right] (t-s) g_m(s) ds \right)^2
\]

\[
+ 2a_m \frac{b_m}{(2m+1)^2} \cos \left[ \frac{(2m+1)}{2} \frac{\pi}{\xi} \right] t \sin \left[ \frac{(2m+1)}{2} \frac{\pi}{\xi} \right] t
\]

\[
+ 2 \frac{b_m}{(2m+1)^2} \sin \left[ \frac{(2m+1)}{2} \frac{\pi}{\xi} \right] \int_{0}^{t} \sin \left[ \frac{(2m+1)}{2} \frac{\pi}{\xi} \right] (t-s) g_m(s) ds
\]

\[
+ 2a_m \frac{b_m}{(2m+1)^2} \cos \left[ \frac{(2m+1)}{2} \frac{\pi}{\xi} \right] t \int_{0}^{t} \sin \left[ \frac{(2m+1)}{2} \frac{\pi}{\xi} \right] (t-s) g_m(s) ds \right\}
\]

(2.7)
we shall estimate the third term of right hand side of (2.7), we used Hölder inequality, we get
\[
\sum_{m=0}^{\infty} \left( \frac{1}{(2m+1)^2} \right)^{\frac{1}{8}} \left( \int_0^t \sin \left[ \left( \frac{2m+1}{2} \right)^2 (t-s) \right] g_m(s) \, ds \right) \leq \sum_{m=0}^{\infty} \int_0^t \sin^2 \left[ \left( \frac{2m+1}{2} \right)^2 (t-s) \right] ds \int_0^t g_m^2(s) \, ds
\]
\[
\leq C(T) \sum_{m=0}^{\infty} \int_0^T g_m^2(t) \, dt.
\]
(2.8)

By Young’s inequality, we get
\[
\sum_{m=0}^{\infty} 2a_m \left( \frac{1}{(2m+1)\pi} \right)^4 \cos \left[ \left( \frac{2m+1}{2} \pi \right)^2 t \right] \sin \left[ \left( \frac{2m+1}{2} \pi \right)^2 (t-s) \right] g_m(s) \, ds
\]
\[
\leq C(T) \sum_{m=0}^{\infty} \left( \frac{1}{(2m+1)\pi} \right)^4 \left( \int_0^T g_m^2(t) \, dt \right)^{\frac{1}{2}}
\]
\[
\leq c_2 \sum_{m=0}^{\infty} \left( \frac{1}{(2m+1)\pi} \right)^6 + C(T) \sum_{m=0}^{\infty} \int_0^T g_m^2(t) \, dt.
\]
(2.10)

and using Young’s and Hölder inequalities, we have
\[
\sum_{m=0}^{\infty} 2a_m \left( \frac{1}{(2m+1)\pi} \right)^4 \cos \left[ \left( \frac{2m+1}{2} \pi \right)^2 t \right] \sin \left[ \left( \frac{2m+1}{2} \pi \right)^2 (t-s) \right] g_m(s) \, ds
\]
\[
\leq c_3 \sum_{m=0}^{\infty} a_m^2 + C(T) \sum_{m=0}^{\infty} \int_0^T g_m^2(t) \, dt.
\]
(2.11)

Integrating (2.7) in (0, T) and using (2.8)-(2.11), we obtain from (2.7) that
\[
\int_0^T \int_0^T u^2(x,t) \, dt \, dx \leq \frac{C(T)}{2} \sum_{m=0}^{\infty} \left\{ a_m^2 + \frac{b_m^2}{(2m+1)^2} \right\} + \int_0^T g_m^2(t) \, dt.
\]

Then
\[
\int_{\xi}^{\xi+\frac{1}{n}} \int_0^T u^2(x,t) \, dt \, dx \leq \frac{C(T)}{n} \left( \| u_0 \|^2_{L^2(0,1)} + \| u_1 \|^2_{L^2(0,T,L^2(0,1))} \right).
\]

This completes the proof of proposition 2.1.

\[\square\]

3 Internal exact controllability of the beams equation

We consider now the following homogenous problem
\[
\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^4 \phi}{\partial x^4}(x,t) = 0, \quad 0 < x < 1, \quad t > 0,
\]
(3.12)
\[
\begin{align*}
\phi(0,t) = \frac{\partial \phi}{\partial x}(0,t) &= \frac{\partial^2 \phi}{\partial x^2}(0,t) = \frac{\partial^3 \phi}{\partial x^3}(0,t) = 0, \\
\phi(x,0) = \phi^0(x), \quad \frac{\partial \phi}{\partial t}(x,0) = \phi^1(x), \quad 0 < x < 1.
\end{align*}
\]
where \((\phi^0, \phi^1) \in L^2(0,1) \times V'.

**Lemma 3.1.** Let \(\xi \in (0,1)\), then for any natural integer \(m\) we have

\[
\inf_{m \geq 0} \int_{\xi}^{\xi + \frac{1}{n}} \sin^2 \left( \frac{2m + 1}{2} \pi x \right) \, dx \geq \frac{1}{2n} - \frac{1}{\pi} \sin \left( \frac{\pi}{2n} \right) \geq c_{\pi, m} \left( \frac{1}{n^3} \right).
\]

**Proof.** For \(m \geq 0\), it is sufficient to note that for any \(x \in (0,1)\), we have

\[
\begin{align*}
\int_{\xi}^{\xi + \frac{1}{n}} \sin^2 \left( \frac{2m + 1}{2} \pi x \right) \, dx &= \frac{1}{2m + 1} \sum_{k=0}^{2m} \int_{(2m+1)\xi + \frac{k}{n}}^{(2m+1)\xi + \frac{k+1}{n}} \sin^2 \left( \frac{\pi y}{2} \right) \, dy \\
&= \frac{1}{2m + 1} \int_{(2m+1)\xi}^{(2m+1)(\xi + \frac{1}{n})} \sin^2 \left( \frac{\pi y}{2} \right) \, dy \\
&= \frac{1}{2m + 1} \int_{(2m+1)\xi}^{(2m+1)(\xi + \frac{1}{n})} \left( \frac{1}{2} - \frac{1}{2} \cos \pi y \right) \, dy \\
&= \frac{1}{2n} - \frac{1}{\pi(2m+1)} \sin \left( \frac{\pi}{2n} (2m+1) \right) \cos \left( \left( \xi + \frac{1}{2n} \right)(2m+1) \right),
\end{align*}
\]

it is clear that

\[
\sin \left( \frac{\pi}{2n} (2m+1) \right) = \frac{\pi}{2n} (2m+1) + c_{\pi, m} \left( \frac{1}{n^3} \right).
\]

Therefore, we get

\[
\begin{align*}
\int_{\xi}^{\xi + \frac{1}{n}} \sin^2 \left( \frac{2m + 1}{2} \pi x \right) \, dx &\geq \frac{1}{2n} - \frac{1}{\pi} \left[ \frac{\pi}{2n} + o \left( \frac{1}{n^3} \right) \right] \\
&\geq c_{\pi, m} \left( \frac{1}{n^3} \right).
\end{align*}
\]

The proof of lemma 3.1 is now completed. \(\square\)

The previous lemma is an essential tool to show the following Proposition.

**Proposition 3.2.** Let \(T \geq 2\), then we have the following. For almost all \(\xi \in (0,1)\) the solution \(\phi\) of (3.12)-(3.14) satisfies

\[
\begin{align*}
\int_0^T \int_{\xi}^{\xi + \frac{1}{n}} |\phi(x,t)|^2 \, dx \, dt &\geq c_{\pi, n} (\|u^0\|^2_{L^2(0,1)} + \|u^1\|^2_{V'}) \\
\forall (u^0, u^1) &\in L^2(0,1) \times V',
\end{align*}
\]

where \(c_{\pi, n} = c_{\pi} o \left( \frac{1}{n^3} \right) \).

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Proof. The solution of \((3.12)-(3.14)\) is given by

\[
\phi(x, t) = \sum_{m=0}^{\infty} \left\{ a_m \cos \left( \frac{2m+1}{2} \pi^2 t \right) + \frac{b_m}{(2m+1)^2 \pi^4} \sin \left( \frac{2m+1}{2} \pi^2 t \right) \right\} \sin \left( \frac{2m+1}{2} \pi x \right).
\]

A simple calculation shows that

\[
\int_{0}^{2} \left| \phi(x, t) \right|^2 \, dt = \sum_{m=0}^{\infty} \left\{ a_m^2 + \frac{b_m^2}{(2m+1)^2 \pi^4} \right\} \sin^2 \left( \frac{2m+1}{2} \pi x \right),
\]

we have

\[
\int_{\xi}^{\xi + \frac{1}{n}} \int_{0}^{T} \left| \phi(x, t) \right|^2 \, dx \, dt \geq \sum_{m=0}^{\infty} \left\{ a_m^2 + \frac{b_m^2}{(2m+1)^2 \pi^4} \right\} \inf_{m \geq 0} \int_{\xi}^{\xi + \frac{1}{n}} \sin^2 \left( \frac{2m+1}{2} \pi x \right) \, dx,
\]

for every \(T \geq 2\), we get

\[
\int_{\xi}^{\xi + \frac{1}{n}} \int_{0}^{T} \left| \phi(x, t) \right|^2 \, dx \, dt \geq \sum_{m=0}^{\infty} \left\{ a_m^2 + \frac{b_m^2}{(2m+1)^2 \pi^4} \right\} \inf_{m \geq 0} \int_{\xi}^{\xi + \frac{1}{n}} \sin^2 \left( \frac{2m+1}{2} \pi x \right) \, dx.
\]

Consequently, by lemma 3.1 and using \((3.15)\) and \((3.18)\), we obtain \((3.16)\). This achieve the proof of proposition 3.2.

Let \((y^0, y^1) \in V \times L^2(0, 1)\) and \(\psi_n\) be the solution of

\[
\frac{\partial^2 \psi_n}{\partial t^2} (x, t) + \frac{\partial^4 \psi_n}{\partial x^4} (x, t) = \chi_n(x) \tilde{\phi}_n(x, t), \quad 0 < x < 1, \quad t > 0,
\]

\[
\psi_n(0, t) = \frac{\partial \psi_n}{\partial x}(1, t) = \frac{\partial^2 \psi_n}{\partial x^2}(0, t) = \frac{\partial^3 \psi_n}{\partial x^3}(1, t) = 0,
\]

\[
\psi_n(x, 0) = y^0(x), \quad \frac{\partial \psi_n}{\partial t}(x, 0) = y^1(x), \quad 0 < x < 1,
\]

\[
\psi_n(x, T) = \frac{\partial \psi_n}{\partial x}(x, T) = 0,
\]

where \(\chi_n\) is the characteristic function of \((\xi, \xi + \frac{1}{n})\) and \(\tilde{\phi}_n(x, t) = n \phi(x, t)\), \(\phi\) is the solution of \((3.12)-(3.14)\).

Lemma 3.3. We suppose that \((y^0, y^1) \in V \times L^2(0, 1)\), then

\[
\left( y^0, \tilde{\phi}_n \right) - \left( y^1, \tilde{\phi}_n \right) = \int_{\xi}^{\xi + \frac{1}{n}} \int_{0}^{T} \left| \tilde{\phi}_n(x, t) \right|^2 \, dx \, dt.
\]

4 An inverse inequality

In this section we suppose that the point \(\xi\) is strategic, that’s

\[
\sin \left( \frac{2m+1}{2} \pi \xi \right) \neq 0, \quad \forall m \in \mathbb{N}.
\]

The quantity

\[
\left( \int_{0}^{T} \phi^2(\xi, t) \, dt \right)^{1/2}
\]
where \( \phi \) is a solution of (3.12)-(3.14) defines a norm on the space \( \mathcal{D}(0, 1) \times \mathcal{D}(0, 1) \) and the initial data \( \phi^0 \) and \( \phi^1 \) are given by

\[
\phi^0 = \sum_{m=0}^{\infty} a_m \sin \left( \frac{2m+1}{2} \pi x \right), \quad \phi^1 = \sum_{m=0}^{\infty} \frac{b_m}{(2m+1)^2 \pi^2} \sin \left( \frac{2m+1}{2} \pi x \right).
\]

Let \( F \) be a real Hilbert space

\[
(\phi^0, \phi^1) \in F \iff \sum_{m=0}^{\infty} \left\{ a_m^2 + \frac{b_m^2}{(2m+1)^2 \pi^2} \right\} \sin^2 \left( \frac{2m+1}{2} \pi \xi \right) < \infty.
\]

We denote by \( F \) the completion of \( \mathcal{D}(0, 1) \times \mathcal{D}(0, 1) \) for this norm and we denote by \( \| \cdot \|_F \) the following quantity:

\[
\| \phi \|_F = \left( \int_0^T \phi^2(\xi, t) \, dt \right)^{1/2}.
\]

Therefore

\[
L^2(0, 1) \times V' \subset F.
\]

If

\[
y^0(x) = \sum_{m=0}^{\infty} a_m \sin \left( \frac{2m+1}{2} \pi x \right), \quad y^1(x) = \sum_{m=0}^{\infty} \frac{b_m}{(2m+1)^2 \pi^2} \sin \left( \frac{2m+1}{2} \pi x \right),
\]

and therefore, its dual

\[
(y^0, y^1) \in F' \iff \sum_{m=0}^{\infty} \left\{ a_m^2 + \frac{b_m^2}{(2m+1)^2 \pi^2} \right\} \sin^2 \left( \frac{2m+1}{2} \pi \xi \right) < \infty.
\]

**Remark 4.1.** If \( \xi \in (0, 1) \) satisfying (4.23), then there exists a constant \( C > 0 \) such that

\[
\left| \sin \left( \frac{2m+1}{2} \pi \xi \right) \right| \geq C, \quad \forall m \in \mathbb{N}
\]

and therefore, \( L^2(0, 1) \times V' = F \).

For the proof, see [1], [2].

The main result of this section is the following:

**Theorem 4.2.** For \( T \geq 2 \), there exists \( c > 0 \), such that for \( (\phi^0, \phi^1) \in F \), the solution \( \phi \) of (3.12)-(3.14) satisfies

\[
\| (\phi^0, \phi^1) \|_F^2 \leq c \left( n \int_0^T \int_0^T \phi^2(x, t) \, dx \, dt \right).
\]

**Proof.** For \( T \geq 2 \), using (3.17), we have

\[
\int_0^T \int_\xi^{\xi+\frac{\pi}{2}} \phi^2(x, t) \, dx \, dt \geq \int_0^2 \phi^2(x, t) \, dx \, dt = \sum_{m=0}^{\infty} \left\{ a_m^2 + \frac{b_m^2}{(2m+1)^2 \pi^2} \right\} \int_\xi^{\xi+\frac{\pi}{2}} \sin^2 \left( \frac{2m+1}{2} \pi x \right) \, dx.
\]

Now, we have to prove that there exists \( c > 0 \) independent on \( n \) such that for every integer \( m \in \mathbb{N} \), we have

\[
n \int_\xi^{\xi+\frac{\pi}{2}} \sin^2 \left( \frac{2m+1}{2} \pi x \right) \, dx \geq c \sin^2 \left( \frac{2m+1}{2} \pi \xi \right).
\]
For $b \geq 0$, $t \geq 0$, we set

$$I(b, t) = \int_0^1 \sin^2(\pi(b + tz)) \, dz.$$ 

As

$$n \int_0^{\xi + \frac{b}{2}} \sin^2 \left( \frac{2m + 1}{2} \pi x \right) \, dx = I \left( \frac{2m + 1}{2}, \frac{2m + 1}{2n} \xi \right),$$

it is sufficient to prove that there exists $c > 0$ such that

$$\forall t \geq 0, \quad I(b, t) \geq c \sin^2(\pi b). \quad (4.25)$$

We have the formula

$$\forall t \geq 0, \quad I(b, t) = \frac{1}{2} \left(1 - \frac{\sin(2\pi(b + t)) - \sin(2\pi b)}{2\pi t}\right)$$

$$= \frac{1}{2} \left(1 - \frac{\sin(2\pi b)[\cos(2\pi t) - 1] + \sin(2\pi t)\cos(2\pi b)}{2\pi t}\right)$$

$$= \frac{1}{2} \left(1 - \frac{-2\sin(2\pi b)\sin^2(\pi t) + 2\cos(\pi t)\sin(\pi t)\cos(2\pi b)}{2\pi t}\right)$$

$$= \frac{1}{2} \left(1 - \frac{\sin(\pi t)}{\pi t}\cos(2\pi b + t)\right).$$

If $t \geq \frac{1}{2}$, then $I(b, t) \geq \frac{1}{2}(1 - \frac{b}{2})$

If $t < \frac{1}{2}$, we distinguish two cases:

**Case 1.** $|b + t| \leq p$, $(p + 1)$. It is then enough to consider the case $p = 0$ and as $\sin(\pi \cdot)$ is concave on $[0, 1]$, we obtain

$$\forall z \in (0, 1), \quad \sin((1 - z)b + (b + t)z\pi) = \sin((1 - z)\pi b + z\pi(t + b))$$

$$\geq (1 - z)\sin(\pi b) + z\sin(\pi(t + b))$$

$$\geq (1 - z)\sin(\pi b).$$

Then

$$\forall z \in (0, 1), \quad |\sin(\pi(b + tz))| \geq (1 - z)|\sin(\pi b)|.$$ 

Hence

$$I(b, t) \geq \sin^2(\pi b) \int_0^1 (1 - z)^2 \, dz$$

$$\geq \frac{1}{3} \sin^2(\pi b).$$

**Case 2.** $0 \leq b \leq b + t \leq p + 1$. It is enough here to consider the case $p = 1$ and we write $1 = b + z_0t$ with $z_0 \in (0, 1)$. We have

$$|\sin(\pi(b + tz))| \geq (1 - z)|\sin(\pi b)| \quad \text{for} \quad z \leq z_0,$$

and

$$|\sin(\pi(b + tz))| \geq z|\sin(\pi(b + t))| \quad \text{for} \quad z > z_0.$$ 

Now, if $z_0 \geq \frac{1}{2}$, we find

$$I(b, t) \geq \sin^2(\pi b) \int_0^{z_0} (1 - z)^2 \, dz + \sin^2(\pi b) \int_{z_0}^1 (1 - z)^2 \, dz$$

$$\geq \sin^2(\pi b) \int_0^{z_0} (1 - z)^2 \, dz$$

$$\geq \frac{7}{24} \sin^2(\pi b).$$
If \( z_0 < \frac{1}{2} \), then
\[
\begin{align*}
b + t - 1 &= b + t - (b + z_0 t) = 1 - 2z_0 t + t - b \\
&= 1 - b + t(1 - 2z_0) \\
&> 1 - b
\end{align*}
\]
and
\[
sin(\pi(b + t - 1)) = -sin(\pi(b + t))
\]
and we have
\[
I(b, t) \geq sin^2(\pi(b + t - 1)) \int_{z_0}^1 z^2 \, dz + sin^2(\pi(b + t - 1)) \int_{z_0}^1 z^2 \, dz
\]
\[
\geq sin^2(\pi(b + t - 1)) \int_{z_0}^1 z^2 \, dz
\]
\[
\geq \frac{1}{3}(1 - z_0^3) sin^2(\pi(1 - b))
\]
\[
\geq \frac{7}{24} sin^2(\pi b).
\]
The proof of the theorem 3.2 is complete.

5 Estimates on the controls

For \( T \geq 2 \) and \( \frac{1}{n} \phi_n(x, t) = \phi(x, t) \) where \( \phi \) is the solution of (3.12), we have

**Theorem 5.1.**

1. If \( (y^0, y^1) \in V \times L^2(0, 1) \), we have
\[
\|\tilde{\phi}_n^0\|_{L^2(0, 1)} + \|\tilde{\phi}_n^1\|_{V'} = o(n^3),
\]
and
\[
\int_{\xi}^{\xi + \frac{1}{n}} \int_0^T \left| \tilde{\phi}_n(x, t) \right|^2 \, dx \, dt = o(n^3).
\]

2. If \( \xi \) is strategic and \( (y^0, y^1) \in F' \), we have
\[
\| (\tilde{\phi}_n^0, \tilde{\phi}_n^1) \|_F = o(n),
\]
and
\[
\int_{\xi}^{\xi + \frac{1}{n}} \int_0^T \left| \tilde{\phi}_n(x, t) \right|^2 \, dx \, dt = o(n).
\]

**Proof.** 1. Applying Hölder and Young’s inequalities. Hence, we see from (3.16) and using lemma 3.3, we have
\[
\left( \|\tilde{\phi}_n^0\|_{L^2(0, 1)} + \|\tilde{\phi}_n^1\|_{V'} \right)^2 \leq c \left( \|\phi_n^0\|_{L^2(0, 1)} + \|\phi_n^1\|_{V'} \right)^2
\]
\[
\leq cn^2 \left( \|\phi_0^0\|_{L^2(0, 1)} + \|\phi_0^1\|_{V'} \right)
\]
\[
\leq cn^3 \int_{\xi}^{\xi + \frac{1}{n}} \int_0^T \left| \tilde{\phi}_n(x, t) \right|^2 \, dx \, dt
\]
\[
\leq cn^3 \|\phi_0^0\|_{L^2(0, 1)} \|y^0\|_{L^2(0, 1)} + \|\phi_0^1\|_{V'} \|y^0\|_{V'}
\]
\[
\leq cn^3 (\|\phi_0^0\|_{L^2(0, 1)} + \|\phi_0^1\|_{V'})
\]

\[9\]
2. When the point $\xi$ is strategic and the initial data $(y^0, y^1) \in F'$. Hence, we see from (4.24), that
\[
\| (\phi_n^0, \phi_n^1) \|_{F'}^2 \leq cn \int_0^{\xi + \frac{1}{n}} \int_0^T \phi_n(x, t) \, dx \, dt
\]
\[
\leq cn\| (\phi_n^0, \phi_n^1) \|_{F'} \| (y^0, y^1) \|_{F'}
\]
\[
\leq cn\| (\phi_n^0, \phi_n^1) \|_{F'}.
\]

The proof of Theorem 5.1 is now complete. \(\square\)

6 Controllability limit as $n \to \infty$

We study here the possibility of divergence of the solution of the controllability problems defined by (3.20)-(3.22). This convergence depends on the nature of the point $\xi$ and on the space of the initial data $y^0$ and $y^1$.

If the point (4.23) holds, we have the following theorem.

**Theorem 6.1.** Suppose that $T \geq 2$, if $\xi$ checks (4.23), $y^0$ and $y^1$ belong to $F'$.
Then, the solution of (3.20)-(3.22) converges for the weak* topology of $L^\infty(0, T, V)$ to the solution of the following pointwise system
\[
\begin{align*}
\frac{\partial^2 \psi}{\partial t^2}(x, t) + \frac{\partial^4 \psi}{\partial x^4}(x, t) &= v(t)\delta_x, \quad 0 < x < 1, \quad t > 0, \\
\psi(0, t) &= \frac{\partial \psi}{\partial x}(1, t) = \frac{\partial^2 \psi}{\partial x^2}(0, t) = \frac{\partial^3 \psi}{\partial x^3}(1, t) = 0, \\
\psi(x, 0) &= y^0(x), \quad \frac{\partial \psi}{\partial t}(x, 0) = y^1(x), \quad 0 < x < 1, \\
\psi(x, T) &= \frac{\partial \psi}{\partial t}(x, T) = 0, \quad 0 < x < 1,
\end{align*}
\] (6.31)

where $v \in L^2(0, T)$ and $\phi(\xi, t) + \frac{1}{\delta_x} \frac{\partial \phi}{\partial x}(\xi, t)$ converges for the weak* topology to $v(t)$ in $H^{-1}(0, T)$.

**Proof.** Multiplying (1.1) by $\psi_n(x, t)$ and integrating by parts on $(0, T) \times (0, 1)$, we have
\[
\forall (u^0, u^1, g_n) \in L^2(0, 1) \times V' \times L^2(0, T, L^2(0, 1)),
\]
\[
\int_0^T \int_0^1 g_n(x, t)\psi_n(x, t) \, dx \, dt
\]
\[
= \int_0^T \int_{\xi - \frac{1}{n}}^{\xi + \frac{1}{n}} \phi_n(x, t)u(x, t) \, dx \, dt - \int_0^1 y^0(x)u^1(x) \, dx + \int_0^1 y^1(x)u^0(x) \, dx 
\]
\[
= n \int_0^T \int_{\xi - \frac{1}{n}}^{\xi + \frac{1}{n}} \phi(x, t)u(x, t) \, dx \, dt - \int_0^1 y^0(x)u^1(x) \, dx + \int_0^1 y^1(x)u^0(x) \, dx.
\] (6.35)

Now, we prove that $(\psi_n)$ and $(g_n)$ are bounded in $L^\infty(0, T, L^2(0, 1))$.

Define
\[
K_n : L^2(0, 1) \times V' \times L^2(0, T, L^2(0, 1)) \to \mathbb{R}
\]
\[
(u^0, u^1, g_n) \mapsto n \int_0^T \int_{\xi - \frac{1}{n}}^{\xi + \frac{1}{n}} \phi(x, t)u(x, t) \, dx \, dt.
\]
Using Hölder inequality, we have
\[ |K_n(u^0, u^1, g_n)|^2 \leq \left( n \int_0^T \int_{\xi}^{\xi+\frac{1}{n}} |\phi(x, t)|^2 \, dx \, dt \right) \left( n \int_0^T \int_{\xi}^{\xi+\frac{1}{n}} |u(x, t)|^2 \, dx \, dt \right). \] (6.36)

Replacing (2.5) in (6.36) and from (5.29), we have
\[ |K_n(u^0, u^1, g_n)|^2 \leq c \left( \|u^0\|_{L^2(0,1)}^2 + \|u^1\|_{V'}^2 + \|g_n\|_{L^2(0,T,L^2(0,1))}^2 \right), \]
which proves that the linear forms \( K_n \) are bounded in \( L^2(0,1) \times V' \times L^\infty(0,T,L^2(0,1)) \).

Therefore, \((\psi_n)_n\) and \((g_n)_n\) are bounded in \( L^\infty(0,T,L^2(0,1)) \) after extraction of a subsequence of \((\psi_n)_n\) and \((g_n)_n\), such that
\[ \psi_n \rightharpoonup \psi \text{ weakly* in } L^\infty(0,T,L^2(0,1)), \]
and
\[ g_n \rightharpoonup g \text{ weakly* in } L^\infty(0,T,L^2(0,1)). \]

The limit of \( K_n \) is given in the following lemma.

**Lemma 6.2.** The linear forms \( K_n \) converge in \( L^2(0,1) \times V \times L^\infty(0,T,L^2(0,1)) \) weakly* to the \( K \) defined by
\[ K(u^0, u^1, g) = \int_0^T v(t)u(\xi, t) \, dt, \] (6.37)
where \( v \in L^2(0,T) \) and
\[ \phi(\xi, t) + \frac{1}{2n} \frac{\partial \phi}{\partial x}(\xi, t) \rightharpoonup v(t) \text{ weakly* in } H^{-1}(0,T). \]

In order to prove the previous lemma, we need the following result.

**Lemma 6.3.** Let \((\phi^0, \phi^1) \in L^2(0,1) \times V' \) and the solution \( \phi(x,t) \) of the problem (3.12)-(3.14) satisfies
\[ \int_0^T \int_{\xi}^{\xi+\frac{1}{n}} |\phi(x,t)|^2 \, dx \, dt = o\left(\frac{1}{n}\right). \] (6.38)

Then, after extraction of a subsequence
\[ \phi(\xi, t) + \frac{1}{2n} \frac{\partial \phi}{\partial x}(\xi, t) \rightharpoonup v(t) \text{ weakly* in } H^{-1}(0,T), \]
where \( v \in L^2(0,T) \).

**Proof of lemma 6.3.** In order to prove lemma 6.3 we suppose that \( w = \left( \frac{\partial^4}{\partial x^4} \right)^{-1} u \) such that \( w \) is the solution of
\[ \frac{\partial^2 w}{\partial t^2}(x,t) + \frac{\partial^4 w}{\partial x^4}(x,t) = f_n(x,t), \quad 0 < x < 1, \quad t > 0, \] (6.39)
\[ w(0,t) = \frac{\partial w}{\partial x}(1,t) = \frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^3 w}{\partial x^3}(1,t) = 0, \] (6.40)
\[ w(x,0) = w^0(x), \quad \frac{\partial w}{\partial t}(x,0) = w^1(x), \quad 0 < x < 1. \] (6.41)
with initial data

\[
\begin{align*}
  w^0 &= \left( \frac{\partial^4}{\partial x^4} \right)^{-1} u^0 \in \mathcal{D}(\partial_x^4) \\
  w^1 &= \left( \frac{\partial^4}{\partial x^4} \right)^{-1} u^1 \in V \\
  f_n &= \left( \frac{\partial^4}{\partial x^4} \right)^{-1} g_n \in L^2(0,T;\mathcal{D}(\partial_x^4)).
\end{align*}
\]

(6.42)

The trace regularity for (6.39)-(6.41) is given in the theorem below.

**Theorem 6.4.** Suppose that \( f_n \in L^2(0,T;\mathcal{D}(\partial_x^4)) \) and \((w^0, w^1) \in \mathcal{D}(\partial_x^4) \times V\) the solution \( w \) of (6.39)-(6.41) verifies

\[
\frac{\partial^4 w}{\partial x^4}(\xi,t) \in L^2(0,T),
\]

(6.43)

and the mapping

\[
L^2(0,T;\mathcal{D}(\partial_x^4)) \times \mathcal{D}(\partial_x^4) \times V \to L^2(0,T)
\]

\[
(f_n, w^0, w^1) \mapsto \frac{\partial^4 w}{\partial x^4}(\xi,t),
\]

(6.44)

is linear and continuous.

Furthermore, we have

\[
\frac{1}{n} \int_0^T \left| \frac{\partial^4 w}{\partial x^4}(x,t) \right|^2 \, dx \, dt \leq C \left( \| f_n \|^2_{L^2(0,T;\mathcal{D}(\partial_x^4)))} + \| w^0 \|^2_{\mathcal{D}(\partial_x^4)} + \| w^1 \|^2_V \right).
\]

(6.45)

**Proof of Theorem 6.4.** The proof of (6.43) and (6.45) can be done by using obvious adaptations of the proof of (2.4) and (2.5), so it is omitted.

\[ \square \]

From (6.24) and (6.38) it follows that \( \phi(\xi,t) \) is bounded in \( F \), after extraction of a subsequence, \( \phi(\xi,t) \) converges in \( L^2(0,T) \) weakly.

On the other hand, from (3.10), (5.28) and (5.29) we have

\[
\| \phi^0 \|^2_{L^2(0,1)} + \| \phi^1 \|^2_{V'} = o(n).
\]

Using (6.43) and (6.44) we can easily prove that the mapping

\[
(\phi^0, \phi^1) \in L^2(0,1) \times V' \to \frac{\partial \phi}{\partial x}(\xi,t) \in H^{-1}(0,T),
\]

is linear and continuous.

Furthermore, we have

\[
\left\| \frac{\partial \phi}{\partial x}(\xi,t) \right\|_{H^{-1}(0,T)} = o(n).
\]

Now, we prove that \( v \in L^2(0,T) \) that is

\[
\forall u \in \mathcal{D}(0,T), \quad |(v, u)|_{\mathcal{D}' \times \mathcal{D}} \leq c \| u \|_{L^2(0,T)}.
\]

We define the following functions

\[
\Phi(x,t) = \int_0^t \phi(x, \tau) \, d\tau - \left( \frac{\partial^4}{\partial x^4} \right)^{-1} \phi^1(x),
\]

and

\[
S_n(x,t) = \int_0^t \Phi(x, \tau) \, d\tau - \left( \frac{\partial^4}{\partial x^4} \right)^{-1} \phi^0(x).
\]

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The functions \( \Phi \) and \( S_n \) are solutions of (3.12)-(3.14) with initial data in \( V \times L^2(0,1) \) and \( \mathcal{D}(\partial_x^2) \times V \)

\[
\| \Phi^0 \|_V + \| \Phi^1 \|_{L^2(0,1)} = o(n),
\]

and

\[
\| S_n^0 \|_{\mathcal{D}(\partial_x^2)} + \| S_n^1 \|_V = o(n).
\]

For \( u \in \mathcal{D}(0,T) \), we have

\[
n \int_0^T \int_\xi^\xi+\frac{1}{n} \phi(x,t)u(t) \, dx \, dt = n \int_0^T \int_\xi^\xi+\frac{1}{n} S_n(x,t) \frac{\partial^2 u}{\partial t^2}(t) \, dx
\]

\[
= \int_0^T \left( S_n(\xi,t) + \frac{1}{2n} \frac{\partial S_n}{\partial x}(\xi,t) \right) \frac{\partial^2 u}{\partial t^2}(t) \, dt
\]

\[
+ n \int_0^T \int_\xi^\xi+\frac{1}{n} \frac{\partial^2 u}{\partial t^2}(t) \int_\xi^y \frac{\partial^2 S_n}{\partial z^2}(z,t) \, dz \, dy \, dx \, dt.
\]

Then

\[
\left( S_n(\xi,t) + \frac{1}{2n} \frac{\partial S_n}{\partial x}(\xi,t) \right) \frac{\partial^2 u}{\partial t^2}(t) dx = n \int_0^T \int_\xi^\xi+\frac{1}{n} \phi(x,t)u(t) \, dx \, dt - R_n,
\]

where

\[
R_n = n \int_0^T \int_\xi^\xi+\frac{1}{n} \frac{\partial^2 u}{\partial t^2}(t) \int_\xi^y \frac{\partial^2 S_n}{\partial z^2}(z,t) \, dz \, dy \, dx \, dt,
\]

then, we prove that

\[
\lim_{n \to \infty} R_n = 0.
\]

Using Hölder’s inequality, we have

\[
|R_n| \leq n \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^2(0,T)} \left( \int_0^T \frac{1}{n} \int_\xi^\xi+\frac{1}{n} (x - \xi) \int_\xi^y \left( \frac{\partial^2 S_n}{\partial z^2}(z,t) \right)^2 \, dz \, dy \, dx \, dt \right)^{1/2}
\]

\[
\leq \frac{1}{\sqrt{8\sqrt{n}}} \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^2(0,T)} \left\| (S_n^0, S_n^1) \right\|_{\mathcal{D}(\partial_x^2) \times V}.
\]

Thus

\[
\lim_{n \to \infty} R_n = 0.
\]

Integrating by part, we get

\[
\left( S_n(\xi,t) + \frac{1}{2n} \frac{\partial S_n}{\partial x}(\xi,t) \frac{\partial^2 u}{\partial t^2}(t) \right) \left\| \phi(t) \right\|_{\mathcal{D}' \times \mathcal{D}} = \left( \phi(\xi,t) + \frac{1}{2n} \frac{\partial \phi}{\partial x}(\xi,t) \right) \left\| u(t) \right\|_{\mathcal{D}'}.
\]

Then

\[
\left| \phi(\xi,t) + \frac{1}{2n} \frac{\partial \phi}{\partial x}(\xi,t) \right| \leq c \| u \|_{L^2(0,1)} + |R_n|.
\]

Passing to the limit as \( n \) tends to infinity, we obtain

\[
\left| (v, u) \right| \leq c \| u \|_{L^2(0,T)}
\]

which proves that \( v \) belongs to \( L^2(0, T) \). The proof of lemma 6.3 is now complete. \( \square \)
Proof of lemma 6.2. Passing to the limit in (6.35), we have

\[ \forall (u^0, u^1, g) \in L^2(0, 1) \times V' \times L^2(0, T, L^2(0, 1)), \]
\[ \int_0^T \int_0^1 g(x, t) \psi(x, t) \, dx \, dt = \int_0^T v(t) u(\xi, t) \, dt - \int_0^1 y^0(x) u^1(x) \, dx + \int_0^1 y^1(x) u^0(x) \, dx, \]

where \( u \) is the solution of (6.31) and \( \psi \) is the solution of (6.30). Since the linear form \( K \) defined in (4.31) is meaningful on \( L^2(0, 1) \times V' \times L^2(0, T, L^2(0, 1)) \), it is sufficient to prove that \( (K_n)_n \) converges to \( K \) on a dense subspace of \( L^2(0, 1) \times V' \times L^2(0, T, L^2(0, 1)) \) and, for example, we consider \( (u^0, u^1, g_n) \in L^2(0, 1) \times V' \times L^2(0, T, L^2(0, 1)) \) and, for example, we consider \( (u^0, u^1, g_n) \in L^2(0, 1) \times V' \times L^2(0, T, L^2(0, 1)) \)

\[ L_n : L^2(0, T, D(\partial_x^2)) \to \mathbb{R} \]
\[ u \mapsto n \int_0^T \int_\xi^{\xi + \frac{1}{n}} \phi(x, t) u(x, t) \, dx \, dt, \]

are defined and bounded on \( L^2(0, T, D(\partial_x^2)) \).

They converge for the weak topology \( L^2(0, T, (D(\partial_x^2))') \) to an element \( L \) of \( L^2(0, T, (D(\partial_x^2))') \).

In order to determine \( L \), we write for \( u \in D(0, T; C^\infty(0, 1)) \):

\[ L_n(u) = n \int_0^T \int_\xi^{\xi + \frac{1}{n}} \phi(x, t) u(\xi, t) \, dx \, dt + n \int_0^T \int_\xi^{\xi + \frac{1}{n}} \phi(x, t) \left( \int_\xi^x \frac{\partial u}{\partial y}(y, t) \, dy \right) \, dx \, dt. \]

We have already seen that

\[ \lim_{n \to \infty} n \int_0^T \int_\xi^{\xi + \frac{1}{n}} \phi(x, t) u(\xi, t) \, dx \, dt = \int_0^T v(t) u(\xi, t) \, dt. \]

On the other hand, it is easy to prove, using Hölder inequality, that for every \( u_n \in D(0, T; C^\infty(0, 1)) \), we have

\[ \lim_{n \to \infty} n \int_0^T \int_\xi^{\xi + \frac{1}{n}} \phi(x, t) \left( \int_\xi^x \frac{\partial u}{\partial y}(y, t) \, dy \right) \, dx \, dt = 0. \]

This completes the proof.

Remark 6.5. By the same method we can obtain the pointwise controllability of the Kirchhoff beam equation

\[ \frac{\partial^2 u}{\partial t^2}(x, t) - \frac{\partial^4 u}{\partial t^2 \partial x^2}(x, t) + \frac{\partial^4 u}{\partial x^4}(x, t) = v(t) \delta_x, \quad 0 < x < 1, \quad t > 0, \]
\[ u(0, t) = \frac{\partial u}{\partial x}(1, t) = \frac{\partial^2 u}{\partial x^2}(0, t) = \frac{\partial^3 u}{\partial x^3}(1, t) = 0, \]
\[ u(x, 0) = u^0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u^1(x), \quad 0 < x < 1, \]

as a limit of internal exact controllability of

\[ \frac{\partial^2 u}{\partial t^2}(x, t) - \frac{\partial^4 u}{\partial t^2 \partial x^2}(x, t) + \frac{\partial^4 u}{\partial x^4}(x, t) = g_n(x, t), \quad 0 < x < 1, \quad t > 0, \]
\[ u(0, t) = \frac{\partial u}{\partial x}(1, t) = \frac{\partial^2 u}{\partial x^2}(0, t) = \frac{\partial^3 u}{\partial x^3}(1, t) = 0, \]
\[ u(x, 0) = u^0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u^1(x), \quad 0 < x < 1, \]
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