Optimal bounds for arithmetic-geometric and Toader means in terms of generalized logarithmic mean

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Abstract

In this paper, we find the greatest values $\alpha_1, \alpha_2$ and the smallest values $\beta_1, \beta_2$ such that the double inequalities $L_{\alpha_1}(a, b) < AG(a, b) < L_{\alpha_2}(a, b)$ and $L_{\beta_1}(a, b) < T(a, b) < L_{\beta_2}(a, b)$ hold for all $a, b > 0$ with $a \neq b$, where $AG(a, b)$, $T(a, b)$ and $L_p(a, b)$ are the arithmetic-geometric, Toader and generalized logarithmic means of two positive numbers $a$ and $b$, respectively.

MSC: 26E60

Keywords: arithmetic-geometric mean; Toader mean; logarithmic mean

1 Introduction

For $p \in \mathbb{R}$, the $p$th generalized logarithmic mean $L_p(a, b)$ [1] of two positive numbers $a$ and $b$ is defined by

$$L_p(a, b) = \begin{cases} \left( \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{1/p}, & a \neq b, p \neq -1, p \neq 0, \\ \left( \frac{b^p - a^p}{p(b-a)} \right)^{1/p}, & a \neq b, p = 0, \\ \frac{b-a}{\log b - \log a}, & a \neq b, p = -1, \\ a, & a = b. \end{cases} \quad (1.1)$$

It is well known that $L_p(a, b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. Many remarkable inequalities for the generalized logarithmic mean can be found in the literature [2–17].

The classical arithmetic-geometric mean $AG(a, b)$ of two positive numbers $a$ and $b$ is defined by starting with $a_0 = a$, $b_0 = b$ and then iterating

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n} \quad (1.2)$$

for $n \in \mathbb{N}$ until two sequences $\{a_n\}$ and $\{b_n\}$ converge to the same number.

The well-known Gauss identity [18] shows that

$$AG(1, r)K\left(\sqrt{1-r^2}\right) = \frac{\pi}{2} \quad (1.3)$$

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for \( r \in (0, 1) \), where \( K(r) = \int_{0}^{\pi/2} (1 - r^2 \sin^2 \theta)^{-1/2} d\theta \), \( r \in [0, 1) \), is the complete elliptic integral of the first kind.

In [19], the Toader mean \( T(a, b) \) of two positive numbers \( a \) and \( b \) was given by

\[
T(a, b) = \frac{2}{\pi} \int_{0}^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \, d\theta
\]

\[
= \begin{cases} 
\frac{2a \mathcal{E}(\sqrt{1-(b/a)^2})}{\pi}, & a > b, \\
\frac{2b \mathcal{E}(\sqrt{1-(a/b)^2})}{\pi}, & a < b, \\
a, & a = b,
\end{cases}
\]

(1.4)

where \( \mathcal{E}(r) = \int_{0}^{\pi/2} (1 - r^2 \sin^2 \theta)^{1/2} d\theta \), \( r \in [0, 1) \) is the complete elliptic integral of the second kind.

Recently, the bounds for the arithmetic-geometric mean \( \text{AG}(a, b) \) and Toader mean \( T(a, b) \) have attracted the attention of many mathematicians. The double inequality

\[
L_{-1}(a, b) = L(a, b) < \text{AG}(a, b) < L^{2/3}(a^{3/2}, b^{3/2})
\]

(1.5)

holds for all \( a, b > 0 \) with \( a \neq b \). The left inequality of (1.5) was first proposed by Carlson and Vuorinen [20] and also was proved by different methods in [21–23]. Vamanamurthy and Vuorinen [24] proved that \( \text{AG}(a, b) < (\pi/2) L(a, b) \) for all \( a, b > 0 \) with \( a \neq b \). The second inequality of (1.5) was proved by Borwein and Borwein [25] and Yang [23].

Vuorinen [26] conjectured that

\[
M_{3/2}(a, b) < T(a, b)
\]

(1.6)

for all \( a, b > 0 \) with \( a \neq b \), where \( M_p(a, b) = [(a^p + b^p)/2]^{1/p} \) (\( p \neq 0 \)) and \( M_0(a, b) = \sqrt{ab} \) is the power mean of order \( p \). This conjecture was proved by Qiu and Shen [27] and Barnard et al. [28].

In [29], Alzer and Qi presented a best possible upper power mean bound for the Toader mean as follows:

\[
T(a, b) < M_{\log 2/\log(\pi/2)}(a, b)
\]

(1.7)

for all \( a, b > 0 \) with \( a \neq b \).

In [30–32], the authors proved that

\[
\tilde{L}_0(a, b) < T(a, b) < \tilde{L}_{1/4}(a, b),
\]

(1.8)

\[
\tilde{S}_{\sqrt{3}/4}(a, b) < T(a, b) < \tilde{S}_{1/2}(a, b)
\]

(1.9)

for all \( a, b > 0 \) with \( a \neq b \), where \( \tilde{L}_p(a, b) = (a^{p+1} + b^{p+1})/(a^p + b^p) \) denotes the \( p \)th Lehmer mean and \( \tilde{S}_p(a, b) \) is the generalized Seiffert mean given by \( \tilde{S}_p(a, b) = p(a - b)/\arctan[2p(a - b)/(a + b)] \) (\( 0 < p \leq 1, a \neq b \)). \( \tilde{S}_0(a, b) = (a + b)/2(a \neq b) \) and \( \tilde{S}_p(a, a) = a \).

Very recently, Chu and Wang [33] proved that

\[
S_{p_1}(a, b) < \text{AG}(a, b) < S_{p_2}(a, b),
\]

(1.10)
\[ S_{p_2}(a,b) < T(a,b) < S_{q_2}(a,b) \]  \hspace{1cm} (1.11)

for all \( a,b > 0 \) with \( a \neq b \) if and only if \( p_1 \leq 1/2, q_1 \geq 1 \) and \( p_2 \leq 1, q_2 \geq 3/2 \). Here the \( p \)th Gini mean of two positive numbers \( a \) and \( b \) is defined by

\[
S_p(a,b) = \begin{cases} 
\left( \frac{a^{p-1} + b^{p-1}}{a+b} \right)^{1/(p-2)}, & p \neq 2, \\
(a^p b^p)^{1/(a+b)}, & p = 2.
\end{cases} \tag{1.12}
\]

The main purpose of this paper is to find the greatest values \( \alpha_1, \alpha_2 \) and the smallest values \( \beta_1, \beta_2 \) such that the double inequalities

\[
L_{\alpha_1}(a,b) < AG(a,b) < L_{\beta_1}(a,b) \quad \text{and} \quad L_{\alpha_2}(a,b) < T(a,b) < L_{\beta_2}(a,b)
\]

hold for all \( a,b > 0 \) with \( a \neq b \) and give some new bounds for the complete elliptic integrals.

### 2 Basic knowledge and lemmas

In order to prove our main results, we need several formulas and lemmas, which we present in this section.

For \( r \in (0,1) \) and \( r' = \sqrt{1-r^2} \), the well-known complete elliptic integrals of the first and second kinds are defined by

\[
\begin{align*}
K &= K(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 \theta)^{-1/2} \, d\theta, \\
K' &= K'(r) = K(r'), \\
K(0) &= \pi/2, \quad K(1) = +\infty,
\end{align*}
\]

and

\[
\begin{align*}
E &= E(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 \theta)^{1/2} \, d\theta, \\
E' &= E'(r) = E(r'), \\
E(0) &= \pi/2, \quad E(1) = 1,
\end{align*}
\]

respectively, and the following formulas were presented in [18], Appendix E, pp.474-475:

\[
\begin{align*}
\frac{dK}{dr} &= \frac{E - r^2 K}{r^2}, & \frac{dE}{dr} &= \frac{E - K}{r}, \\
\frac{d(E - r^2 K)}{dr} &= rK, & \frac{d(K - E)}{dr} &= \frac{rE}{r^2}, \\
E \left( \frac{2\sqrt{r}}{1+r} \right) &= \frac{2E - r^2 K}{1+r}. \tag{2.1}
\end{align*}
\]

In what follows, four special values \( E(\sqrt{2}/2), K(\sqrt{2}/2) \) and \( E(0.9), K(0.9) \) will be used. By numerical computations, these are given by

\[
\begin{align*}
E(\sqrt{2}/2) &= 1.35064 \cdots, & K(\sqrt{2}/2) &= 1.85407 \cdots, \tag{2.3} \\
E(0.9) &= 1.1717 \cdots, & K(0.9) &= 2.28055 \cdots. \tag{2.4}
\end{align*}
\]
**Lemma 2.1** (See [18], Theorem 1.25) For $-\infty < a < b < \infty$, let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and be differentiable on $(a, b)$, let $g'(x) \neq 0$ on $(a, b)$. If $f'(x)/g'(x)$ is increasing (decreasing) on $(a, b)$, then so are

$$f(x) - f(a) \frac{f(x) - f(b)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$ 

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

**Lemma 2.2** (1) The function $r \to (\mathcal{E} - r^2 \mathcal{K})/r^2$ is strictly increasing from $(0,1)$ onto $(\pi/4, 1)$;

(2) The function $r \to 2\mathcal{E} - r^2 \mathcal{K}$ is increasing and log-convex from $(0,1)$ onto $(\pi/2, 2)$;

(3) The function $\mathcal{K}/\log(e^2/r')$ is strictly increasing from $(0,1)$ onto $(\pi/4, 1)$;

(4) The function $(\mathcal{K} - \mathcal{E})/r^2$ is strictly increasing on $(0,1)$; in particular, $\mathcal{K} - \mathcal{E} > (\pi/4)r^2$ for all $r \in (0,1)$.

**Proof** Parts (1) and (2) follow from [18], Theorem 3.21(1) and Exercise 3.43(13).

**Lemma 2.3** The equation

$$(1 + p)^{1/p} = \frac{\pi}{2}$$

has a unique solution $p = p_0 = 3.15295 \cdots$.

**Proof** Let

$$\phi(p) = \begin{cases} (1 + p)^{1/p} - \pi/2, & p \in (-1, 0) \cup (0, +\infty), \\ e - \pi/2, & p = 0. \end{cases}$$

It is easy to verify that the function $\phi$ is continuous and strictly decreasing from $(-1, +\infty)$ onto $(1, +\infty)$. Therefore, Lemma 2.3 easily follows from the continuity and monotonicity of $\phi$ together with the facts that $\phi(3.15295) = 6.14999 \times 10^{-7}$ and $\phi(3.15296) = -4.35155 \times 10^{-7}$.

**Lemma 2.4** The function

$$f(r) = \frac{2(2\mathcal{E} - r^2 \mathcal{K})/\pi - 1 - r^2/4}{r^4}$$

is strictly increasing from $(0,1)$ onto $(1/64, 4/\pi - 5/4)$.

**Proof** Let $f_1(r) = 2(2\mathcal{E} - r^2 \mathcal{K})/\pi - 1 - r^2/4$ and $f_2(r) = r^4$, then $f_1(0) = f_2(0) = 0$ and $f(r) = f_1(r)/f_2(r)$.

A simple calculation yields

$$\frac{f'_2(r)}{f_2(r)} = \frac{4(\mathcal{E} - r^2 \mathcal{K}) - \pi r^2}{8\pi r^4} \leq \frac{f_2(r)}{f'_2(r)}, \quad (2.5)$$

$$f_2(0) = f'_2(0), \quad (2.6)$$
\[
\frac{f_3'(r)}{f_3(r)} = \frac{2K - \pi}{16\pi r^2} \triangleq f_3(r) + f_3'(r), \quad (2.7)
\]
\[
f_3(0) = \tilde{f}_3(0), \quad (2.8)
\]
\[
f_3'(r) = \frac{1}{16\pi} \left( E - r^2 K \right) \cdot \frac{1}{r^2}. \quad (2.9)
\]

Following from Lemma 2.2(1) and (2.9) together with the monotonicity of \(1/r^2\), we clearly see that \(f_3'(r)/f_3(r)\) is strictly increasing on \((0,1)\). Equations (2.5)-(2.8) and Lemma 2.1 lead to the conclusion that \(f(r)\) is strictly increasing on \((0,1)\).

Therefore, Lemma 2.4 follows from the monotonicity of \(f(r)\) together with the facts that \(f(0^+) = 1/64\) and \(f(1^-) = 4/\pi - 5/4\).

\[
\]

The following double inequalities can be obtained from Lemma 2.4 immediately.

**Corollary 2.5 Inequalities**

\[
1 + \frac{r^2}{4} + \frac{r^4}{64} < \frac{2}{\pi} \left( 2E - r^2 K \right) < 1 + \frac{r^2}{4} + \left( \frac{4}{\pi} - \frac{5}{4} \right) r^4
\]

hold for \(0 < r < 1\).

**Lemma 2.6** The inequality

\[
\left[ \frac{(1 + r)^{7/2} - (1 - r)^{7/2}}{7r} \right]^{2/5} < 1 + \frac{r^2}{4}
\]

holds for \(0 < r < 1\).

**Proof** In order to prove inequality (2.10), it suffices to prove that

\[
g(r) = \left[ (1 + r)^{7/2} - (1 - r)^{7/2} \right]^2 - 49r^2 \left( 1 + \frac{r^2}{4} \right)^5
\]

\[
= (1 + r)^7 + (1 - r)^7 - 49r^2 \left( 1 + \frac{r^2}{4} \right)^5 - 2(1 - r^{7/2})
\]

\[
g_1(r) = g_2(r) = 0
\]

for \(0 < r < 1\), where

\[
g_1(r) = (1 + r)^7 + (1 - r)^7 - 49r^2 \left( 1 + \frac{r^2}{4} \right)^5,
\]

\[
g_2(r) = 2(1 - r^{7/2}).
\]

Observe that

\[
g_1'(r) = -\frac{7r}{256} \left[ 32(4 - 5r^2)^2 + 2,848r^4 + 2,240r^6 + 350r^8 + 21r^{10} \right] < 0, \quad (2.12)
\]

\[
g_1(0.56) = 0.0755 \cdots > 0, \quad g_1(0.57) = -0.00966 \cdots < 0, \quad (2.13)
\]
we conclude, from \((2.12)\) and \((2.13)\), that there exists \(r_0 \in (0.56, 0.57)\) such that \(g_1(r) > 0\) for \(r \in (0, r_0)\) and \(g_1(r) < 0\) for \(r \in (r_0, 1)\).

In order to prove \((2.11)\), we divide it into two cases.

Case A \(r \in [r_0, 1)\). In this case, we clearly see that \(g_1(r) \leq 0\) and \(g_2(r) > 0\). This implies that \(g(r) = g_1(r) - g_2(r) < 0\).

Case B \(r \in (0, r_0)\). In this case, \(g_1(r) > 0\). Let \(g_3(r) = 2 - 7r^2 + \frac{35}{4}r^4 - 6r^6\), the difference between \(g_1(r)\) and \(g_3(r)\) yields

\[
g_1(r) - g_3(r) = -\frac{r^6(10,880 + 7,840r^2 + 980r^4 + 49r^6)}{1,024} < 0. \tag{2.14}
\]

We know from \((2.14)\) that \(g_3(r) > g_1(r) > 0\). Moreover,

\[
g_3^2(r) - g_2^2(r) = \frac{-r^6}{16} \left( \frac{1}{2 - 4r^2} \left( 4 + \frac{r^2}{2} \right) \right) \left( \frac{1}{16r^2 - \frac{5}{8}} \right + \frac{27}{4} \right) < 0,
\]

this in conjunction with \(g_3(r) > 0\) implies that

\[
g_3(r) - g_2(r) < 0. \tag{2.15}
\]

Therefore, we clearly see that \(g(r) = [g_1(r) - g_3(r)] + [g_3(r) - g_2(r)] < 0\) from \((2.14)\) and \((2.15)\).

**Lemma 2.7** Let \(\eta(r) = [(1 + r)^{p_0 + 1} - (1 - r)^{p_0 + 1}] / r\) and \(\omega(r) = [(1 - r)^{p_0} - (1 - p_0)^{p_0} - (1 - p_0)r] / r^2\), then the functions \(\eta(r)\) and \(\omega(r)\) both are strictly increasing on \((0, 1)\).

**Proof** We assume that

\[
\eta_1(r) = (1 + r)^{p_0 + 1} - (1 - r)^{p_0 + 1}, \quad \eta_2(r) = r,
\]

\[
\omega_1(r) = (1 - r)^{p_0} - (1 + r)^{p_0} - (1 - p_0)r, \quad \omega_2(r) = r^2,
\]

then \(\eta(r) = \eta_1(r) / \eta_2(r)\) and \(\omega(r) = \omega_1(r) / \omega_2(r)\).

A simple calculation yields

\[
\eta_1(0) = \eta_2(0) = \omega_1(0) = \omega_2(0) = 0, \tag{2.16}
\]

\[
\frac{\eta_1'(r)}{\eta_2'(r)} = \frac{(1 + p_0)(1 + r)^{p_0} - (1 - r)^{p_0}}{1}, \tag{2.17}
\]

\[
\frac{\omega_1'(r)}{\omega_2'(r)} = \frac{p_0(p_0 + 1)(1 + r)^{p_0 - 1} - (1 - r)^{p_0 - 1}}{2}. \tag{2.18}
\]

Lemma 2.1 and \((2.16)\)-\((2.18)\) lead to the conclusion that \(\eta(r)\) and \(\omega(r)\) are strictly increasing on \((0, 1)\).

**Lemma 2.8** Let

\[
\phi_p(r) = \frac{2}{\pi} \left( 2E - r^2K \right) - \left[ \frac{(1 + r)^{p_0 + 1} - (1 - r)^{p_0 + 1}}{2(p + 1)r} \right]^{1/p},
\]

then \(\phi_p(r) > 0\) for \(0 < r < 1\) if and only if \(p \leq 5/2\); \(\phi_p(r) < 0\) for \(0 < r < 1\) if and only if \(p \geq p_0\).
Proof. It is well known that $L_p(a,b)$ is strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$, then $\phi_p(r)$ is strictly decreasing with respect to $p \in \mathbb{R}$. In order to prove Lemma 2.8, we divide it into three cases.

Case 1 $p = 5/2$.

From Corollary 2.5 and Lemma 2.6, we clearly see that

$$
\phi_{5/2}(r) = \frac{2}{\pi} (2E - r^2 K) - \left( \frac{1}{7r} \right)^{2/5} \left( \frac{(1 + r)^{7/2} - (1 - r)^{7/2}}{7r} \right) > 1 + \frac{r^2}{4} + \frac{r^4}{64} - \left( \frac{1}{7r} \right)^{2/5} \left( \frac{(1 + r)^{7/2} - (1 - r)^{7/2}}{7r} \right) > 0
$$

for $0 < r < 1$.

Case 2 $p = p_0$.

We divide it into two subcases.

Subcase A $\phi_{p_0}(r) < 0$ for $r \in (0, 0.9)$.

Since $\phi_p(r)$ is strictly decreasing with respect to $p \in \mathbb{R}$, we clearly see that $\phi_{p_0}(r) < \phi_3(r)$. It suffices to prove that $\phi_3(r) < 0$ for $r \in (0, 0.9)$.

For $r \in (0, \sqrt{2}/2]$, it follows from Corollary 2.5 that

$$
\phi_3(r) = \frac{2}{\pi} (2E - r^2 K) - \left( \frac{1}{8r} \right)^{1/3} \left( \frac{(1 + r)^4 - (1 - r)^4}{8r} \right)^{1/3} < 1 + \frac{r^2}{4} + \left( \frac{4}{\pi} - \frac{5}{4} \right) r^4 - \left( 1 + \frac{r^2}{3} - \frac{r^4}{9} \right)^3
$$

$$
= -\frac{r^2}{12} \left[ 1 - \left( \frac{48}{\pi} - \frac{41}{3} \right) r^2 \right] \leq -\frac{r^2}{12} \left[ 1 - \frac{1}{2} \left( \frac{48}{\pi} - \frac{41}{3} \right) \right] = -\frac{(47\pi - 144)^2 r^2}{72\pi} < 0,
$$

where the first inequality easily follows from

$$
\frac{(1 + r)^4 - (1 - r)^4}{8r} - \left( 1 + \frac{r^2}{3} - \frac{r^4}{9} \right)^3 = \frac{r^6}{729} \left[ 126 + 9(1 - r^4) + r^6 \right] > 0.
$$

For $r \in (\sqrt{2}/2, 0.9)$, taking the derivative of $\phi_3(r)$ yields

$$
\phi_{3}'(r) = \frac{2(E - r^2 K)}{\pi r} - \frac{2r}{3(1 + r^3)^{2/3}} = \mu_1(r) + \mu_2(r), \quad (2.19)
$$

where

$$
\mu_1(r) = \frac{2(E - r^2 K)}{\pi r} - \frac{r}{2}, \quad \mu_2(r) = \frac{r}{2} - \frac{2r}{3(1 + r^3)^{2/3}}.
$$
From Lemma 2.2(4), we clearly see that

\[ \frac{d\mu_1(r)}{dr} = \frac{2}{\pi r^2} \left(K - E - \frac{\pi}{4} r^2\right) > 0 \quad (2.20) \]

for \( r \in (0, 1) \) and

\[ \frac{d\mu_2(r)}{dr} = \frac{9(1 + r^2)^{5/3} - 12 + 4r^2}{18(1 + r^2)^{5/3}} > 0 \]

\[ \frac{19r^2 - 3}{18(1 + r^2)^{5/3}} > 0 \quad (2.21) \]

for \( r \in (\sqrt{2}/2, 0.9) \). Equations (2.19)-(2.21) lead to the conclusion that \( \phi_3'(r) \) is strictly increasing on \( (\sqrt{2}/2, 0.9) \). This in conjunction with (2.3) implies that

\[ \phi_3'(r) > \phi_3'(\sqrt{2}/2) = 0.02163 \cdots > 0 \quad (2.22) \]

for \( r \in (\sqrt{2}/2, 9/10) \). Therefore, from (2.22) we clearly see that \( \phi_3(r) \) is strictly increasing on \( (\sqrt{2}/2, 0.9) \). This in conjunction with (2.4) yields \( \phi_3(r) < \phi_3(0.9) = -0.002687 \cdots < 0 \) for \( r \in (\sqrt{2}/2, 0.9) \).

**Subcase B \( \phi_{p_0}(r) < 0 \) for \( r \in [0.9, 1) \).**

For \( 0.9 \leq r < 1 \), taking the derivation of \( \phi_{p_0}(r) \) yields

\[ \phi_{p_0}'(r) = \frac{2(\mathcal{E} - r^2 \mathcal{K})}{\pi r} - \frac{\omega(r)}{p_0(p_0 + 1)2^{p_0} \eta(r)^{1-1/p_0}}, \quad (2.23) \]

where \( \omega(r) \) and \( \eta(r) \) are defined as in Lemma 2.7. From Lemma 2.2(1), we clearly see that

\[ (\mathcal{E} - r^2 \mathcal{K})/r \]

is strictly increasing on \( (0, 1) \). Lemma 2.7 and (2.4), (2.20) lead to the conclusion that

\[ \phi_{p_0}'(r) \geq \frac{2(\mathcal{E}(0.9) - (1 - 0.9^2)\mathcal{K}(0.9))}{0.9\pi} - \frac{\omega(1)}{p_0(p_0 + 1)2^{p_0} \eta(0.9)^{1-1/p_0}} \]

\[ = 0.522306 \cdots - 0.46787 \cdots = 0.054436 \cdots > 0 \]

for \( 0.9 \leq r < 1 \).

Therefore, it follows from the monotonicity of \( \phi_{p_0}'(r) \) on \( (9/10, 1) \) that \( \phi_{p_0}(r) < \phi_{p_0}(1) = 2[2/\pi - 1/(1 + p_0)^{1/p_0}] = 0 \) for \( 0 < r < 1 \).

**Case 3 \( 5/2 < p < p_0 \).**

Taking the Taylor series of \( \phi_p(r) \) at \( r = 0 \) yields

\[ \phi_p(r) = \left(\frac{5}{12} - \frac{p}{6}\right)r^2 + \frac{(149 - 144p + 24p^2 + 16p^3)r^4}{2880} + o(r^4). \quad (2.24) \]

From (2.24) we clearly see that there exists a sufficiently small \( \delta_1 > 0 \) such that \( \phi_p(r) < 0 \) for \( r \in (0, \delta_1) \) if \( p > 5/2 \). If \( p < p_0 \), then \( \phi_p(1) = 2[2/\pi - 1/(1 + p)^{1/p}] > 0 \). By the continuity of \( \phi_p(r) \) with respect to \( r \), there exists a sufficiently small \( \delta_2 > 0 \) such that \( \phi_p(r) > 0 \) for \( r \in (\delta_2, 1) \).
3 Main results

**Theorem 3.1** Inequality \( L_{-1}(a, b) < AG(a, b) < L_{-1/2}(a, b) \) holds for all \( a, b > 0 \) with \( a \neq b \), where \( L_{-1}(a, b) \) and \( L_{-1/2}(a, b) \) are the best possible lower and upper generalized logarithmic mean bounds for the arithmetic-geometric mean \( AG(a, b) \), respectively.

*Proof* Firstly, from (1.5) we clearly see that \( L_{-1}(a, b) < AG(a, b) \) for all \( a, b > 0 \) with \( a \neq b \).

Next, we prove that \( AG(a, b) < L_{-1/2}(a, b) \) for all \( a, b > 0 \) with \( a \neq b \). Since \( AG(a, b) \) and \( L_p(a, b) \) are symmetric and homogeneous of degree 1, without loss of generality, it suffices to give an assumption that \( a = 1 > b \). Let \( t = b \in (0, 1) \), \( r = (1 - t)/(1 + t) \), then (1.1) and (1.3) lead to

\[
AG(a, b) - L_{-1/2}(a, b) = \frac{\pi}{2K(\sqrt{1 - t^2})} - \left[ \frac{1 - t}{2(1 - \sqrt{1 - t^2})} \right]^2
\]

\[
= \frac{1}{1 + r} \left[ \frac{\pi}{2K} - \left( \frac{r}{\sqrt{1 + r} - \sqrt{1 - r}} \right)^2 \right]
\]

\[
= \frac{h(r)}{2(1 + r)K(r)}, \tag{3.1}
\]

where

\[
h(r) = \pi - (1 + r')K(r).
\]

We can rewrite \( h(r) \) as

\[
h(r) = \pi - \lambda(r') \cdot \frac{K}{\log(e^2/r')} \cdot \tag{3.2}
\]

where \( \lambda(r') = (1 + r') \log(e^2/r') \).

A simple calculation yields

\[
\lambda'(r') = 1 - \frac{1}{r'} - \log r', \tag{3.3}
\]

\[
\lambda'(1) = 0, \tag{3.4}
\]

\[
\lambda''(r') = \frac{1 - r'}{r'} > 0. \tag{3.5}
\]

Equations (3.3)-(3.5) lead to the conclusion that \( \lambda(r') \) is strictly decreasing on \((0, 1)\) with respect to \( r' \). Moreover, the function \( r' = \sqrt{1 - r^2} \) is strictly decreasing on \((0, 1)\). Hence the function \( \lambda(r') \) is strictly increasing on \((0, 1)\) with respect to \( r \). It follows from (3.2) and Lemma 2.2(3) that \( h(r) \) is strictly decreasing on \((0, 1)\). This implies that \( h(r) < 0 \) for \( 0 < r < 1 \) together with \( h(0) = 0 \).

Therefore, \( AG(a, b) < L_{-1/2}(a, b) \) for all \( a, b > 0 \) with \( a \neq b \) follows from (3.1) and \( h(r) < 0 \).

Finally, we prove that \( L_{-1}(a, b) \) and \( L_{-1/2}(a, b) \) are the best possible lower and upper generalized logarithmic mean bounds for the arithmetic-geometric mean \( AG(a, b) \).

For any \( 0 < \varepsilon < 1/2 \) and \( 0 < x < 1 \), it follows from (1.1) and (1.3) that

\[
\lim_{x \to 0^+} \left[ AG(1, x) - L_{-1, \varepsilon}(1, x) \right] = \lim_{x \to 0^+} \left\{ \frac{\pi}{2K(\sqrt{1 - x^2})} - \left[ \frac{1 - x^\varepsilon}{\varepsilon(1 - x)} \right]^{1/\varepsilon} \right\}
\]

\[
= -\frac{1}{1+\varepsilon} \tag{3.6}
\]
and making use of the Taylor expansion as \( x \to 0 \), one has

\[
AG(1, 1 - x) - L_{-1/2, \varepsilon}(1, 1 - x) = \frac{\pi}{2K(\sqrt{2x - x^2})} - \left[ 1 - (1 - x)^{1/2 - \varepsilon} \right]^{-1/2 - \varepsilon} = \left[ 1 - \frac{x}{2} - \frac{x^2}{16} + o(x^2) \right] - \left[ 1 - \frac{x}{2} - \frac{3 + 2\varepsilon}{48} x^2 + o(x^2) \right] = \frac{\varepsilon}{24} x^2 + o(x^2).
\]

Equations (3.6) and (3.7) imply that for any \( 0 < \varepsilon < 1/2 \) there exist \( \delta_1 = \delta_1(\varepsilon) \in (0, 1) \) and \( \delta_2 = \delta_2(\varepsilon) \in (0, 1) \) such that \( AG(1, x) < L_{-1/2, \varepsilon}(1, x) \) for \( x \in (0, \delta_1) \) and \( AG(1, 1 - x) > L_{-1/2, \varepsilon}(1, 1 - x) \) for \( x \in (0, \delta_2) \). \( \square \)

**Theorem 3.2** Inequality \( L_{5/2}(a, b) < T(a, b) < L_{p_0}(a, b) \) holds for all \( a, b > 0 \) with \( a \neq b \), where \( p_0 \) is defined as in Lemma 2.3 and \( L_{5/2}(a, b), L_{p_0}(a, b) \) are the best possible lower and upper generalized logarithmic mean bounds for the Toader mean \( T(a, b) \), respectively.

**Proof** From (1.1) and (1.4) we clearly see that both \( T(a, b) \) and \( L_p(a, b) \) are symmetric and homogeneous of degree 1. Without loss of generality, we assume that \( a = 1 > b \). Let \( t = b \in (0, 1), r = (1 - t)/(1 + t) \), then from (1.1) and (1.4) together with (2.2) we have

\[
T(a, b) - L_p(a, b) = \frac{2}{\pi} \mathcal{E}\left( \sqrt{1 - t^2} \right) - \left[ \frac{1 - r^{p+1}}{(p + 1)(1 - t)} \right]^{1/p} = \frac{2}{\pi} \mathcal{E}\left( \frac{2\sqrt{r}}{1 + r} \right) - \frac{1}{1 + r} \left[ \frac{(1 + r)^{p+1} - (1 - r)^{p+1}}{2(p + 1)r} \right]^{1/p} = \frac{1}{1 + r} \left[ \frac{2}{\pi} \mathcal{E} - r^2 \mathcal{K} \right] - \left( \frac{(1 + r)^{p+1} - (1 - r)^{p+1}}{2(p + 1)r} \right)^{1/p} = \frac{\phi_p(r)}{(1 + r)} \]

where \( \phi_p(r) \) is defined as in Lemma 2.8.

Therefore, Theorem 3.2 follows from (3.8) and Lemma 2.8. \( \square \)

**4 Corollaries and remarks**

From Theorem 3.1 we get a lower bound for the complete elliptic integral of the first kind \( \mathcal{K}(r) \) as follows.

**Corollary 4.1** Inequality

\[
\mathcal{K}(r) > \frac{2\pi [1 + \sqrt{1 - r^2} - 2(1 - r^2)^{1/4}]}{(1 - \sqrt{1 - r^2})^2}
\]

holds for all \( r \in (0, 1) \).

**Remark 4.1** We define \( H(r) = 2\pi [1 + \sqrt{1 - r^2} - 2(1 - r^2)^{1/4}]/(1 - \sqrt{1 - r^2})^2 \). Computational and numerical experiments show that the lower bound in (4.1) can be regarded as an approximation of \( \mathcal{K}(r) \) for some \( r \in (0, 1) \), refer to Table 1 for numerical values.
Table 1 Comparison of $K(r)$ with $H(r)$ for some $r \in (0, 1)$

| $r$  | $K(r)$                  | $H(r)$                  |
|------|-------------------------|-------------------------|
| 0.1  | 1.5747455615...         | 1.5747455614...         |
| 0.2  | 1.5868678474...         | 1.5868678471...         |
| 0.3  | 1.608048619...          | 1.608048612...          |
| 0.4  | 1.63999986...           | 1.63999977...           |
| 0.5  | 1.68575035...           | 1.68574965...           |
| 0.6  | 1.75075380...           | 1.75074958...           |
| 0.7  | 1.84569400...           | 1.84567106...           |
| 0.8  | 1.99530278...           | 1.99517293...           |

Theorem 3.2 enables us to give new bounds for the complete elliptic integrals of the second kind $E(r)$.

**Corollary 4.2 Inequality**

\[ \frac{\pi}{2} \left[ \frac{2(1-(1-r^2)^{3/4})}{7(1-\sqrt{1-r^2})} \right]^{2/5} < E(r) < \frac{\pi}{2} \left[ \frac{1-(1-r^2)^{(p_0+1)/2}}{(p_0+1)(1-\sqrt{1-r^2})} \right]^{1/p_0} \]  

(4.2)

holds for all $r \in (0, 1)$, where $p_0 = 3.15295...$ is defined as in Lemma 2.3.

**Competing interests**

The authors declare that they have no competing interests.

**Authors’ contributions**

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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