On the Landau–Ginzburg description of $(A_1^{(1)})^\oplus N$ invariants

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Abstract

We search for a Landau–Ginzburg interpretation of non-diagonal modular invariants of tensor products of minimal $n = 2$ superconformal models, looking in particular at automorphism invariants and at some exceptional cases. For the former we find a simple description as Landau–Ginzburg orbifolds, which reproduce the correct chiral rings as well as the spectra of various Gepner–type models and orbifolds thereof. On the other hand, we are able to prove for one of the exceptional cases that this conformal field theory cannot be described by an orbifold of a Landau–Ginzburg model with respect to a manifest linear symmetry of its potential.
1 Introduction

Compactifications of the heterotic string to four space-time dimensions have been analysed not only because of their prospects for model building (see e.g. [1, 3]), but also from a less phenomenological point of view, as they provide a connection between otherwise rather distinct areas of research such as Calabi–Yau manifolds [4], Landau–Ginzburg theories [5, 6], and exactly solvable \( n = 2 \) superconformal two-dimensional field theories [7]. An issue that is of relevance to both the phenomenological and the more mathematical aspects is to know to which extent the sets of compactified string models obtained via the various approaches overlap or even are contained in each other.

More specifically, one may ask whether all models which arise from ‘compactification’ on tensor products of minimal \( n = 2 \) superconformal field theories – the so-called Gepner–type [8] models – possess a Landau–Ginzburg interpretation. It is this question that we are going to address in the present paper. A comparison between the space of Landau–Ginzburg theories and the one of Gepner–type models is desirable for the following reasons. On the one hand, in the Landau–Ginzburg framework it is a lot easier than with conformal field theory methods to compute spectra of massless (gauge non-singlet) string modes, and hence to search for interesting models by scanning large classes of string compactifications. Thus if all Gepner–type models possessed a Landau–Ginzburg interpretation, it would not be necessary to scan them separately. On the other hand, after having found a Landau–Ginzburg theory of interest, one could employ the conformal field theory machinery to obtain more detailed information about the model, e.g. compute Yukawa couplings with the help of operator product expansions. In addition, if the Landau–Ginzburg description of some Gepner–type model is in terms of an orbifold of a Fermat–type Landau–Ginzburg potential, it is not even necessary to treat this model independently in the conformal field theory framework, but rather one may describe it as the corresponding orbifold of a Gepner–type model which involves diagonal modular invariants only.

More generally, it is always convenient to have two different descriptions of a particular model. Looking at the model from different points of view, one can gain new insight in its structure, and possibly even in the nature of the methods on which the descriptions are based.

So far the issue raised above has been analysed only for models which employ the standard \( A-D-E \)–type modular invariants of \( A_1^{(1)} \). The result was that all these models possess a simple Landau–Ginzburg description: for \( A \)–type (diagonal) invariants as well as for the \( E_6 \) and \( E_8 \) invariants one is dealing with Fermat–type potentials [5, 6], and for \( D \)–type invariants with \( \mathbb{Z}_2 \) orbifolds thereof [11], while the \( E_7 \) invariant corresponds to a non-Fermat type potential. In this paper we extend the analysis to modular invariants of \( (A_1^{(1)})^{\oplus N} \) with \( N \geq 2 \), which cannot be written as products of \( A-D-E \)–type invariants. Such invariants have been described in [12, 14].

As it turns out, there exist large classes of such invariants for which we can identify a corresponding Landau–Ginzburg orbifold. However, we are also able to find theories which definitely cannot be described by Landau–Ginzburg theories or their orbifolds with respect to manifest linear symmetries of their potentials. Our paper is organized as follows. In section 2 we gather

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1 Here the qualification ‘interesting’ may be taken in the sense that the model possesses a phenomenologically promising low-energy limit [3], but it may concern different aspects as well. For example, in the context of ‘mirror symmetry’ [9, 10] one may look for the mirror partner of a particular model.
the background information about \( n = 2 \) superconformal models and about Landau–Ginzburg theories which is needed below. Afterwards we present, in section 3, invariants that cannot be expressed in the Landau–Ginzburg framework. Section 4 deals with invariants that we are able to identify with Landau–Ginzburg orbifolds. At the end of that section we present a few spectra of compactified string theories which employ these invariants. Finally, some open questions are mentioned in section 5.

2 Minimal \( n = 2 \) superconformal models and Landau–Ginzburg theories

To compare Gepner–type and Landau–Ginzburg–type string compactifications, one may proceed via the identification of massless string modes, or more specifically, of the numbers \( n_{27} \) and \( n_{\bar{27}} \) of massless modes transforming in the two inequivalent 27-dimensional representations of the \( E_6 \) part of the gauge group. In Calabi–Yau language, \( n_{27} \) and \( n_{\bar{27}} \) correspond to the Hodge numbers \( h_{1,2} \) and \( h_{1,1} \), respectively, so that

\[
\chi = 2(n_{27} - n_{\bar{27}})
\]

is the Euler number of the manifold.

However, the comparison actually need not be made for the full compactified string models; rather, it already should be performed for the individual \( n = 2 \) superconformal minimal models which are the building blocks of the Gepner–type compactifications. The objects of interest are then the members of the (chiral, chiral) ring \([3.4]\) In conformal field theory terms, these are the primary conformal fields that are annihilated by the modes \( G^{+}_{1/2} \) and \( \bar{G}^{+}_{1/2} \) of the supercurrents, the chiral primary fields. They may also be characterized by the fact that their charges \( Q \) and \( \bar{Q} \) with respect to the \( U(1) \) currents \( J \) and \( \bar{J} \) contained in the holomorphic and antiholomorphic \( n = 2 \) algebras are related to their conformal dimensions \( \Delta \) and \( \bar{\Delta} \) by

\[
Q = 2\Delta, \quad \bar{Q} = 2\bar{\Delta}.
\]  

(1)

Below we briefly describe how chiral primary fields are realized in minimal models and in Landau–Ginzburg theories.

Consider first the \( n = 2 \) superconformal minimal models. They are conveniently described in terms of the coset construction

\[
(A_1^{(1)})_k \oplus (u_1)_2 / (u_1)_{k+2}.
\]

Correspondingly any \( N \)-fold tensor product of minimal models can be characterized by a set \( (k_1, k_2, ..., k_N) \) of positive integers (the levels of the corresponding affine algebras \( A_1^{(1)} \)) together with some modular–invariant combination of \( (A_1^{(1)})^{\otimes N} \) characters, and the conformal central charge is given by

\[
c = \sum_{i=1}^{N} 3k_i / (k_i + 2).
\]

The primary fields of each minimal model are of the form \( \phi_{l,q,s}^{l,q,s} \) where the quantum numbers \( l, q, \) and \( s \) are integers referring to the highest weights of the holomorphic symmetry algebras \( (A_1^{(1)})_k \), \( (u_1)_{k+2} \), and \( (u_1)_2 \), respectively, and analogously for the antiholomorphic part. In particular, the label \( l \) is constrained by \( 0 \leq l \leq k \); also, fields \( \phi_{l,q,s}^{l,q,s} \) that are related by certain recursion relations have to be identified. The conformal dimensions and \( U(1) \) charges are simple expressions in these quantum numbers, and hence the requirement of chirality is easily imposed, leading in the Neveu–Schwarz sector to the condition

\[
q = l, \quad s = 0 \quad \text{or} \quad q = -l - 2, \quad s = -2
\]

(2)

\textsuperscript{2} We could, instead, work just as well with the Ramond ground states.
on the holomorphic quantum numbers, and to an analogous restriction on the antiholomorphic ones. Because of the field identifications, for one set of the quantum numbers, say the holomorphic ones, one can always restrict to the first of these solutions. If the modular invariant chosen for the affine \((A_1^{(1)})_k\) algebra is the diagonal one, then one obviously needs \(l = \bar{l}\), and hence the chiral primary fields must obey

\[
q = \bar{q} = l = \bar{l}, \quad s = \bar{s} = 0.
\]

Moreover, as long as the \(u_i\) invariants are the diagonal ones, which implies that \(q = \bar{q} \pmod{2k+4}\) and \(s = \bar{s} \pmod{4}\), the condition (2) still implies (3), no matter which affine modular invariant is chosen; analogously, one still has \(q_i = \bar{q}_i = l_i = \bar{l}_i, s_i = \bar{s}_i = 0\) for all \(i = 1, 2, \ldots, N\), even if one is dealing with a \((A_1^{(1)})^{\oplus N}\) invariant of the non-direct product type, which is the type of invariants that we consider in this paper.

Now we come to the description of Landau–Ginzburg theories [3]. Such a theory is defined by the (super)perturbative \(W\) of a supersymmetric two-dimensional lagrangian field theory describing scalar superfields \(\phi_i (i = 1, 2, \ldots, I);\) the potential is believed not to be renormalized owing to \(n = 2\) superconformal invariance. A large class of conformal field theories can be described by such Landau–Ginzburg models and their orbifolds.

If the potential \(W(\phi_i)\) is quasi-homogeneous, i.e. satisfies

\[
W(\lambda^{n_i} \phi_i) = \lambda^d W(\phi_i)
\]

for some integers \(n_i\) and \(d\), then the scaling dimensions and \(U(1)\) charges of the superfields \(\phi_i\) are given by \(\Delta_i = Q_i/2\) and \(Q_i = n_i/d\), and the conformal central charge is \(c = 3 \sum_{i=1}^I (1 - 2Q_i)\). The equations of motion identify the partial derivatives \(\partial_i W = \partial W / \partial \phi_i\) of the potential as descendant fields. As a consequence, the chiral ring is isomorphic with the local algebra of the potential, i.e. with the factor ring of the polynomial ring in the fields \(\phi_i\) with respect to the ideal generated by the gradients \(\partial_i W\).

In order that this ring be finite, \(W\) needs to have an isolated critical point [13] at \(\phi = 0\). This implies, in particular, that for any field \(\phi_i, i = 1, \ldots, I\), the potential must contain a monomial of the form \(\phi_i^n\) or of the form \(\bar{\phi}_i^{s_i} \phi_j\). In the latter case we say that \(\phi_i\) points at \(\phi_j\), and we refer to the sum of these \(I\) monomials as to a skeleton of the isolated singularity. We call the set of all non-degenerate polynomials that are quasi-homogeneous with charges \(Q_i = n_i/d\) a configuration \(C_{\{n_1, \ldots, n_I\}}[d]\). Each skeleton already determines a unique configuration and also has only a finite group of phase symmetries. Additional monomials in \(W\), which are required for non-degeneracy if more than one fields point at a particular further field [14], can only reduce this symmetry. On the other hand, a given configuration may accommodate several different skeletons.

We will try to identify a conformal field theory with a particular orbifold of a Landau–Ginzburg model by comparing the chiral rings. The charge degeneracies in this ring are conveniently summarized by the so-called Poincaré polynomial \(P(t, \bar{t})\), which is the polynomial in \(t^{1/d}\) and \(\bar{t}^{1/d}\) defined as the sum of \(t^Q \bar{t}^{\bar{Q}}\) carried out over all chiral primary fields, \(P(t, \bar{t}) = \text{tr}_{\{c, c\}} t^{J_c} \bar{t}^{J_c}\). We will see that this information is already sufficient, in all the cases we consider, to identify the model or disprove the existence of a Landau–Ginzburg representation (under our general assumption of considering only manifest symmetries). If a particular orbifold has the correct Poincaré polynomial, we may further check the identification by considering symmetries, which provide selection rules for the operator product expansions. We will do this by calculating the
massless modes of Gepner–type models that employ the modular invariant under investigation, and of orbifolds thereof. In all cases where the Poincaré polynomial works out correctly we also find that the results for the numbers $n_{27}$ and $n_{\overline{27}}$ of generations and antigenerations calculated in both frameworks agree.\footnote{Special care, however, is necessary to correctly identify the action of symmetry groups in the case of orbifolds (see section 4.2 below).} This is rather non-trivial, as the projection onto integer charges (and with respect to further symmetries) leads to additional twisted sectors, whose contribution to the spectrum does not correspond to any of the chiral states of the original conformal field theory.

For untwisted theories, the Poincaré polynomial is given by $P(t, \bar{t}) \equiv P(t\bar{t})$ with

$$P(t) = \prod_{i=1}^{I} \frac{1 - t^{1-Q_i}}{1 - t^{Q_i}}. \quad (5)$$

For orbifolds, however, this formula is of limited use, as we need to project onto invariant states. In particular, the unique chiral primary field with left–right charges $(c/3, c/3)$ transforms with the inverse determinant squared of the matrix describing the transformation. If this state should survive, as is the case in all examples considered in this paper, this imposes the restriction $\det g = \pm 1$ on the relevant symmetry groups. Abelian symmetry groups can further be assumed to be diagonalized, i.e. to act as phase symmetries. They are a direct product of cyclic groups of order $\mathcal{O}$, which we denote by

$$\mathbb{Z}_{\mathcal{O}}(p_1, p_2, \ldots, p_I), \quad (6)$$

where the integers $p_i$ in parentheses indicate that the $i^{th}$ field transforms with $\exp(2\pi i p_i/\mathcal{O})$ under the generator of the group.

In addition to the projected untwisted states the spectrum of the orbifold contains twisted sectors. Their respective (Ramond ground states and) chiral rings only get contributions from the untwisted fields. The left–right charges of the twisted (Neveu–Schwarz) vacua $|h\rangle$, $\sum_{\theta_i^h > 0} (\frac{1}{2} - Q_i \pm (\theta_i^h - \frac{1}{2})) \quad (7)$

with $\theta_i^h$ defined by $hX_i = \exp(2\pi i \theta_i^h)X_i$ and $0 \leq \theta_i^h < 1$, and their transformation under a group element $g$,

$$g|h\rangle = (-1)^{K_g K_h} \varepsilon(g, h)(\det g|_h)(\det g)^{-1}|h\rangle, \quad (8)$$

have been determined by Intriligator and Vafa \[17, 18\]. In (8) $\det g|_h$ is the determinant for the action of $g$ on those fields that are invariant under $h$, and $\varepsilon(g, h)$ are discrete torsions, satisfying certain consistency conditions \[19, 18\]. The integer $K_g$ fixes the sign of the action of $g$ in the Ramond sector; for our purposes we can set $(-1)^{K_g} = \det g|_h^{-1}$ \[18\].

The first step in finding a Landau–Ginzburg representation with conformal central charge $c$ which possesses some prescribed Poincaré polynomial is to enumerate all non-degenerate configurations with the correct central charge, i.e. all sets $\Psi_{(n_1, \ldots, n_I)}[d]$ of charges $Q_i = n_i/d$ such that there is a non-degenerate polynomial in $I$ variables which is quasi-homogeneous with respect to these charges and for which $c/3 = \sum_i (1 - 2Q_i)$. As all non-degenerate configurations with $c = 9$ have been enumerated \[20, 21\], this procedure is straightforward once we know a single non-degenerate polynomial with $c' = 9 - c$, i.e. with $\sum_j (1 - 2Q_j^d) = 3 - c/3$. Having thus obtained
a list of candidate configurations, we can check for each candidate whether the Landau–Ginzburg theory or any of its orbifolds gives rise to the correct Poincaré polynomial; once a candidate with the right Poincaré polynomial is found, we can further check for the chiral ring and for string spectra. If a candidate passes all these tests, we consider it as the Landau–Ginzburg equivalent of the conformal field theory. In fact, in all cases where we succeed in finding a Landau–Ginzburg description, it is in terms of an orbifold of a Fermat–type potential. As a consequence it would be possible, although very tedious, to complete the check of this identification by calculating the full operator product algebra of the chiral primaries.

Let us stress that we only consider orbifolds with respect to manifest linear symmetries of the potential. Of course, there could be additional symmetries of the conformal field theory, such as the non-linear transformation that permutes a $\mathbb{Z}_2$-orbifold of $X^{2a} + U^2$ and a $D$ invariant $Y^a + YV^2$, or even the continuous symmetries of the torus obtained, for example, from the potential $X^3 + Y^3 + Z^3$ by modding the diagonal $\mathbb{Z}_3$. Unfortunately, one does not have a handle on such symmetries within the usual computational framework of Landau–Ginzburg models.

### 3 Invariants without Landau–Ginzburg interpretation

For generic values of $N$ and of the levels $k_i$ there is a large variety of $(A_1^{(1)})^{\otimes N}$ invariants $Z^{(k_1,\ldots,k_N)}$ which are not of the direct product form $\prod_{j=1}^N Z^{(k_j)}$ \cite{12,13}. Among these, there are many infinite series of invariants analogous to the $A$ and $D$ series of $A_1^{(1)}$ invariants, as well as a small number of exceptional invariants (similar to the $E$–type invariants of $A_1^{(1)}$) that occur at isolated values of the levels $k_i$. As it turns out, not all of these exceptional modular invariants of products of minimal models can be described by orbifolds of Landau–Ginzburg models, at least not by orbifolds with respect to manifest linear symmetries of the potential.

We start our discussion of $(A_1^{(1)})^{\otimes N}$ invariants of non-direct product form with two such exceptional invariants. Several of them can be obtained via conformal embeddings. Among these, there are the well-known $E_6$– and $E_8$–type invariants of $A_1^{(1)}$, but also a few others such as \cite{12} the following invariant of $A_1^{(1)} \oplus A_1^{(1)}$ for $k_1 = 3$, $k_2 = 8$:

$$Z^{(3,8)} = |0,0 \oplus 2,4 \oplus 0,8|^2 \oplus |0,4 \oplus 2,2 \oplus 2,6|^2 \oplus |1,2 \oplus 1,6 \oplus 3,4|^2 \oplus |1,4 \oplus 3,0 \oplus 3,8|^2.$$  

(9)

There also exist exceptional modular invariants which cannot be obtained from conformal embeddings. Some invariants of this type have been found in \cite{14}; among them there is the following $A_1^{(1)} \oplus A_1^{(1)}$ invariant at levels $k_1 = k_2 = 8$:

$$Z^{(8,8)} = |0,0 \oplus 8,8 \oplus 8,0|^2 \oplus |2,2 \oplus 2,6 \oplus 6,2 \oplus 6,6|^2 \oplus |0,2 \oplus 0,6 \oplus 8,2 \oplus 8,6 \oplus 2,0 \oplus 2,8 \oplus 6,0 \oplus 6,8 \oplus (4,4)^* \oplus \text{c.c.}]$$

(10)

$$\oplus |2,4 \oplus 6,4 \oplus 4,2 \oplus 4,6|^2 \oplus |0,4 \oplus 8,4 \oplus 4,0 \oplus 4,8|^2 \oplus 2 |4,4|^2.$$  

We will now prove that the exceptional modular invariant (9) at levels $(k_1,k_2) = (3,8)$ cannot be described by a Landau–Ginzburg orbifold. This model has central charge $c/3 = 7/5$ and Poincaré polynomial

$$P(t) = 1 + 2t^{2/5} + 3t^{3/5} + 3t^{4/5} + 2t + t^{7/5}.$$  

(11)
As discussed in the previous section, to enumerate all non-degenerate configurations with \( \sum_j (1 - 2Q_j) = c/3 = 7/5 \), we only need to make use of a single non-degenerate polynomial with \( \sum_j (1 - 2Q_j') = 3 - 7/5 = 8/5 \), such as \( X^{10} + Y^{10} \), which corresponds to the configuration \( \mathcal{C}_{(1,1)}[10] \).

Searching the list \([20]\) of configurations for entries containing \( \mathcal{C}_{(1,1)}[10] \), and eliminating those for which the part with \( c/3 = 7/5 \) is degenerate, we find the following 10 candidates: \( \mathcal{C}_{(1,2)}[10], \mathcal{C}_{(1,5)}[20], \mathcal{C}_{(2,7)}[30], \mathcal{C}_{(5,7)}[40], \mathcal{C}_{(1,1,2)}[5], \mathcal{C}_{(5,5,2)}[15], \mathcal{C}_{(3,4,5)}[15], \mathcal{C}_{(1,4,7)}[15], \mathcal{C}_{(2,5,9)}[20], \) and \( \mathcal{C}_{(4,7,9)}[25] \). Obviously, without further orbifoldizing none of these configurations possesses \( \text{(II)} \) as its Poincaré polynomial.

Thus as a next step we have to check whether any of the orbifolds of a model belonging to the above configurations can reproduce the correct Poincaré polynomial. Generically, configurations involving several fields of equal weight may be difficult to handle, owing to the presence of nonabelian symmetries. Fortunately, although among the configurations just listed there are two cases with two fields of equal weight, none of them can accommodate a non-degenerate polynomial with a nonabelian symmetry group. To see this, note that any such quasi-homogeneous polynomial would have to be of the form \( \sum P_i(X,Y)Z^q \), with a non-vanishing linear term \( P(d-n_1)/n_2 = \alpha X + \beta Y \). By a change of variables we can set \( \beta = 0 \). Then any linear symmetry respecting quasi-homogeneity has to be diagonal and the symmetry group must thus be abelian. We can therefore restrict our considerations to phase symmetries in all 10 cases.

A necessary ingredient for obtaining the Poincaré polynomial \( \text{(II)} \) is to use only twists by symmetries with determinant \( \pm 1 \), as the unique chiral primary field of highest charge \( c/3 = 7/5 \) transforms with the inverse determinant of the twist squared. This very restriction, on the other hand, implies in most cases that the untwisted sector contains invariant states with undesirable charges.

For the configuration \( \mathcal{C}_{(1,2)}[10] \), for example, any non-degenerate polynomial has to contain \( P_1 = X^{10} + Y^5 \) or \( P_2 = X^8Y + Y^5 \) (the coefficients have been rescaled to unity). These two polynomials represent the points of maximal symmetry in the moduli space of the configuration. The symmetries we have to consider are thus generated by \( \mathbb{Z}_2(1,0) \) and \( \mathbb{Z}_5(1,4) \). In the first case, with polynomial \( P_1 \), we can disregard the \( \mathbb{Z}_2 \), as it would only bring us to the \( D \) invariant, i.e. to the configuration \( \mathcal{C}_{(1,1,2)}[5] \) to be considered below. Then the projection onto states invariant under the \( \mathbb{Z}_5 \) group keeps the field \( XY \), which has charge 3/10 and hence should not belong to the chiral ring. In the second case, on the other hand, we only have the \( \mathbb{Z}_2 \) symmetry, which leaves the field \( Y \) with charge 1/5 invariant. In the same way one can check that any orbifold (with respect to a symmetry satisfying \( \det = \pm 1 \)) of the configurations \( \mathcal{C}_{(1,5)}[20], \mathcal{C}_{(2,7)}[30], \mathcal{C}_{(3,4,5)}[15], \mathcal{C}_{(1,4,7)}[15], \mathcal{C}_{(2,5,9)}[20], \) and \( \mathcal{C}_{(4,7,9)}[25] \) has an invariant chiral field of charge 1/5, whereas for \( \mathcal{C}_{(5,7)}[40] \) there is an invariant field with charge 3/10, namely \( (XY)^2 \).

To exclude the configuration \( \mathcal{C}_{(4,7,9)}[25] \), we need to go one step further. In this case the non-degenerate polynomial with maximal symmetry is \( X^4Z + Y^3X + Z^2Y \) and has the symmetry \( \mathbb{Z}_5(2,1,2) \) with determinant 1. From the untwisted sector we thus get \( P_\text{u}(t) = 1 + t^{3/5} + t^{4/5} + t^{7/5} \), which is not yet in contradiction with \( \text{(II)} \). The twisted sectors, however, contribute four states with asymmetric charges \( (Q, \bar{Q}) = (1/5, 6/5) \) or \( (6/5, 1/5) \), which again are not present in the chiral ring of the modular invariant we want to describe.

Finally, we are left with the two configurations \( \mathcal{C}_{(1,1,2)}[5] \) and \( \mathcal{C}_{(5,5,2)}[15] \), which are a little more tedious, as there is a larger number of polynomials with maximal symmetry. In the first
configuration $X_1$ and $X_2$ can point at any variable, whereas in the second configuration $X_1$ and $X_2$ can point at each other. As a consequence, we get 9 and 4 different skeletons, respectively. (If two fields point at the same additional field we need, in fact, additional monomials for non-degeneracy [16], which further restricts the symmetry.) All resulting orbifolds, however, can be excluded as candidates for describing the modular invariant (9) with the same arguments as above.

The modular invariant $Z^{(8,8)}$ described in (10) above can be analysed in a similar manner. The Poincaré polynomial reads

$$P^{(8,8)}(t) = 1 + 3t^{2/5} + 2t^{3/5} + 6t^{4/5} + 2t + 3t^{6/5} + t^{8/5}. \quad (12)$$

There are now 20 non-degenerate configurations that have the correct central charge. Ten of these correspond to $A$-$D$-$E$-type potentials, namely $C_{(1,1)}[10]$, $C_{(1,2,4)}[10]$, $C_{(1,1,2,2)}[5]$, $C_{(1,5)}[30]$, $C_{(1,10,10)}[30]$, $C_{(2,5,14)}[30]$, $C_{(1,5,5,7)}[15]$, $C_{(5,6,10)}[30]$, $C_{(3,5,5,5)}[15]$, and $C_{(4,5,5)}[20]$, while the remaining ones cannot be obtained from tensor products of minimal models: $C_{(1,2)}[15]$, $C_{(1,4)}[25]$, $C_{(2,5)}[35]$, $C_{(4,5)}[45]$, $C_{(2,7,19)}[40]$, $C_{(2,16,17)}[50]$, $C_{(3,19,20)}[60]$, $C_{(7,10,25)}[60]$, $C_{(10,16,37)}[90]$, and $C_{(3,4,5,6)}[15]$. All but the first three of these configurations lead to undesirable states in the untwisted sector or to asymmetric chiral states in the twisted sectors.

The first three models are orbifolds of one another, so it is sufficient to consider only one of them. The starting point that is closest to giving the correct Poincaré polynomial is $C_{(1,1,2,2)}[5]$. Here we have calculated all orbifolds with respect to a subgroup of the nonabelian group generated by the $\mathbb{Z}_5$ with determinant 1, the two $\mathbb{Z}_2$’s which flip the signs of the fields with charge 2/5, and the permutation symmetry of $X_1^5 + X_1 X_2^3$ and $X_2^5 + X_2 X_1^3$. We find that no choice of torsion between the generators of order two leads to the correct Poincaré polynomial. In particular, it is not possible to generate two states of charge 3/5 in the twisted sectors. Thus it also appears very unlikely that this invariant can be represented as a Landau–Ginzburg orbifold. We have, however, not checked whether there is a point in the moduli space of $C_{(1,1)}[10]$ or of $C_{(1,1,2,2)}[5]$ which has a nonabelian linear symmetry that is not a combination of phase symmetries and permutations.

Instead of considering the chiral ring, we could also use the above orbifolds in combination with additional minimal models to construct string vacua and calculate the numbers $n_{27}$ and $n_{27}^\ell$ of $E_6$ representations. The corresponding numbers for the invariants (3) and (4) have been calculated in [12] and [14], respectively. In the case of the modular invariant (3), we have checked that none of the orbifolds is able to reproduce these numbers in all cases, which is another proof that this invariant cannot be described by an orbifold of a non-degenerate Landau–Ginzburg model with respect to a manifest symmetry of its potential.

4 Automorphism invariants and the associated Landau–Ginzburg orbifolds

4.1 Infinite series of automorphism invariants

We now turn our attention to a class of modular invariants of $(A_1^{(1)})^{\otimes N}$ for which we are able to identify an associated Landau–Ginzburg orbifold. The invariants to be described in this section
are all automorphism invariants, which means that each of them is due to some fusion rule automorphism of the diagonal invariant. Such an invariant associates to any \( N \)-tuple \( \vec{l} \equiv (l_1, l_2, \ldots, l_N) \) of \( A_1 \) quantum numbers \( l_i \) \((0 \leq l_i \leq k_i)\) precisely one \( N \)-tuple \( \vec{I} \equiv (I_1, I_2, \ldots, I_N) \) such that the map
\[
\rho : l_i \mapsto I_i = \rho(l_i) \quad \text{for all } i = 1, 2, \ldots, N
\]
is an automorphism of the fusion rules, i.e. such that the fusion rule coefficients \( N_{\vec{l}, \vec{r}} \), satisfy
\[
N_{\rho(\vec{l}), \rho(\vec{r})} = N_{\vec{l}, \vec{r}}.
\]
Since the invariant tensor describing the diagonal invariant is the unit matrix \( \delta_{\vec{l}, \vec{r}} \), the tensor for the automorphism invariant reads
\[
(Z^{(k)})_{\vec{l}, \vec{r}} = \delta_{\vec{l}, \rho(\vec{r})}.
\]

We will mainly analyse certain infinite series of invariants that are present for any \( N \) provided that \( k_i \in 2\mathbb{Z} \) for \( i = 1, 2, \ldots, N \). To describe the relevant automorphism \( \rho \) explicitly, let us consider first the case \( k_i \in 4\mathbb{Z} \) for \( i = 1, 2, \ldots, N \). Denote by \( I_0 \) and \( I_1 \) the index sets of the even and odd \( l_i \) values, respectively, i.e. \( l_i \in 2\mathbb{Z} + s \) for \( i \in I_s \) and \( s = 0, 1 \) satisfying \( I_0 \cup I_1 = \{1, 2, \ldots, N\} \) and \( I_0 \cap I_1 = \emptyset \), and set \( N_s := |I_s| \). Also allow for \( s > 1 \) by identifying \( I_s \equiv I_{s \mod 2} \). Finally set
\[
\sigma_i(l_i) := k_i - l_i.
\]
With these conventions, the automorphism \( \rho \) reads
\[
\rho(l_i) = \begin{cases} 
  l_i & \text{for } i \in I_{N_1}, \\
  \sigma_i(l_i) & \text{for } i \in I_{N_1+1}.
\end{cases}
\]
That this indeed defines an automorphism of the fusion rules may be seen as follows: the fusion rule coefficients are given by \( N_{\vec{l}, \vec{r}} = \prod_{i=1}^N N_{l_i, r_i} \), where \( N_{l_i, r_i} \) are the fusion rules of the \( i \)-th minimal model. The latter satisfy the selection rule
\[
N_{l, l', l''} = 0 \quad \text{for } l + l' + l'' \in 2\mathbb{Z} + 1,
\]
from which it follows that \( N_{\vec{l}, \vec{r}} \) and \( N_{\rho(\vec{l}), \rho(\vec{r})} \) both vanish, except for
\[
N_{\rho(\vec{l}), \rho(\vec{r})} = \prod_{i=1}^N N_{\sigma_i(l_i), \sigma_i(r_i)} = \prod_{i=1}^N N_{\sigma_i(s_i), \sigma_i(t_i)}
\]
with
\[
r_i + s_i + t_i = 0 \text{ or } 2 \quad \text{for all } i = 1, 2, \ldots, N.
\]
The automorphism property (14) then follows from the identity
\[
N_{k-l, k-l', l''} = N_{l, l', l''},
\]
which is obeyed by the fusion rules of the minimal models.

The fact that the map \( (l_i) \mapsto (\bar{l}_i) \) is a fusion rule automorphism ensures invariance under the modular transformation \( S : \tau \mapsto -1/\tau \); thus in order to check invariance under the full modular
group, one merely needs to verify invariance under $T: \tau \mapsto \tau + 1$, which on primary fields acts
diagonally by the phase $\exp(2\pi i(\Delta - \bar{\Delta}))$. In the situation at hand, $T$ invariance is an immediate
consequence of the property $\Delta_{k-l} - \Delta_l = k/4 - l/2$ of conformal dimensions of $A_1^{(1)}$ primaries.

Namely, this property implies that $\Delta_{\rho(l_i)} - \Delta_l \equiv \sum_{i=1}^{N}(\Delta_{\rho(l_i)} - \Delta_{l_i})$ is an integer iff
the number of odd $l_i$ that satisfy $\rho(l_i) = \sigma_i(l_i)$ is even for any tuple $\vec{l}$, as is indeed fulfilled for the map (17).

Let us remark that only for $N = 2$ is the invariant defined by (17) ‘fundamental’ in the sense [22] that it cannot be obtained by forming sums and/or products of the
invariant tensors corresponding to simpler invariants; in that case this invariant was found in [13, 14]. Also note
the structural difference between even and odd values of $N$ which shows up when describing the left–right symmetric primary fields: for these, either $I_1 = \emptyset$, or else $N_0 \in 2\mathbb{Z} + N + 1$ and
$l_i = k_i/2$ for all $i \in I_0$. Thus in particular for odd $N$ the tuples $\vec{l}$ which give $I_0 = \emptyset$ provide
left–right symmetric fields, whereas for even $N$ at least one of the $l_i$ corresponding to a left–right symmetric field must be even.

The automorphism $\rho$ can be described in a similar manner if some of the levels $k_i$ obey $k_i \in 4\mathbb{Z} + 2$. For brevity we describe only the case where $k_i \in 4\mathbb{Z}$ for $i = 1, 2, ..., N - 1$ and
$k_N \in 4\mathbb{Z} + 2$. In this situation the automorphism reads

$$
\rho(l_i) = \begin{cases} l_i & \text{for } i = 1, 2, ..., N - 1 \text{ and } l_i + l_N \in 2\mathbb{Z}, \\
\sigma_i(l_i) & \text{for } i = 1, 2, ..., N - 1 \text{ and } l_i + l_N \in 2\mathbb{Z} + 1,
\end{cases}
\rho(l_N) = \begin{cases} l_N & \text{for } N_1 + l_N \in 2\mathbb{Z} + 1, \\
\sigma_N(l_N) & \text{for } N_1 + l_N \in 2\mathbb{Z}.
\end{cases}
$$

That this is a fusion–rule automorphism follows by the same reasoning as above. $T$ invariance
now requires that the number of those $l_i$, $i = 1, 2, ..., N - 1$, for which $\rho(l_i) = \sigma_i(l_i)$, be even
if $l_N$ is odd, and odd if $l_N$ is even, and this condition is met by (22). Also note that although
the description (22) of the invariant at first sight looks rather different from the corresponding
formula (17) above, it leads to the same structure of the set of left–right symmetric primaries.

### 4.2 Landau–Ginzburg interpretation

We want to identify an infinite series of Landau–Ginzburg theories. This implies that we must
start from the Fermat–type potential

$$
W = \sum_{i=1}^{N} X_i^{k_i+2},
$$

as this is the only infinite series of Landau–Ginzburg potentials matching the central charges
(apart from potentials employing the respective $D$ invariants, but these are related to (22) by an
orbifold construction and thus cannot give anything new). It is then not useful either to consider
the Poincaré polynomial, as we can keep track of the charges of chiral fields as a function of the
levels and thus should identify these fields individually. In order to see the structure of the chiral
ring corresponding to the invariant (17), observe that $\rho(l_i) = \sigma(l_i)$ iff the number of odd $l_j$ with
$j \neq i$ is odd. Thus fields with all $l_i$ even are in the chiral ring. If at least one $l_i$ is odd, then
left–right symmetry implies that there is an odd number of odd \( l_j \), and that all even \( l_j \) must be equal to \( k_j/2 \).

Thus from the chiral ring of the potential (23) we need to project out all fields \( \prod_i X^i \) with an even number of odd \( l_i \). The only symmetries with \( \det = \pm 1 \) that we have at our disposal for the whole series of potentials are the \( \mathbb{Z}_2 \)'s that invert the sign of a number of fields. The subgroup with \( \det = 1 \) may be described as being generated by symmetries of the form

\[
\mathbb{Z}_2^{(i)} = \mathbb{Z}_2(0, \ldots, 1, 0, \ldots, 0),
\]

with the 1’s at the \( i^{th} \) and \((i + 1)^{st}\) position. For \( N \) odd this is exactly the symmetry group we need for the projection in the untwisted sector, as states with all numbers \( l_i \) odd should survive. The twisted sectors come from transformations flipping the signs of an even number of fields. According to (7) this implies a contribution \( 1/2 - Q_i \) to the charge of the twisted vacuum for each twisted field \( X_i \). On the other hand, setting all torsions \( \varepsilon(g, h) = 1 \) in (8), the projection onto invariant states keeps the odd powers of the untwisted fields. As \( 1/2 - Q_i \) is exactly the charge of a chiral field with \( \rho(l_i) = \sigma(l_i) \), we get complete agreement for the charge degeneracies in the chiral rings of the automorphism invariant (17) and the \( (\mathbb{Z}_2)^{N-1} \) orbifold of (23) with respect to the group generated by (24).

For even \( N \) the states with all \( l_i \) odd should be projected out. Accordingly, we need to supplement the projection by transformations with determinant \(-1\) on the fields \( X_i, \ i \leq N \). A negative determinant, however, can always be avoided by letting the respective transformations act on an additional trivial field \( X_{N+1} \), contributing a term \( X_{N+1}^2 \) to the potential. In this way we can reduce the case of \( N \) even to the previous case with \( k_{N+1} = 0 \). The fields with all \( l_i \) odd no longer contribute to the chiral ring, as \( X_{N+1} \) is a descendant field. As above, one can check that the complete chiral ring works out correctly for this \( (\mathbb{Z}_2)^N \) orbifold. Note that in these considerations we only need to require that \( k_i \in 2\mathbb{Z} \) for \( i = 1, 2, \ldots, N \), but not necessarily \( k_i \in 4\mathbb{Z} \); correspondingly, the Landau–Ginzburg interpretation found above only applies to invariants of the type (17), but to those of type (23) etc. as well.

Finally, let us mention that special care is necessary for identifying orbifolds with even order in the above models. We illustrate this for the case \( N = 2 \) with the non-diagonal invariant (17) of \((A_1^{(i)})_{k_1} \oplus (A_1^{(i)})_{k_2}\), which is described by a \( (\mathbb{Z}_2)^2 \) orbifold of \( X_1^{k_1+2} + X_2^{k_2+2} + X_3^2 \). Owing to the projection, the full original \( \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \) symmetry is no longer present in the Landau–Ginzburg representation. Instead, however, we have gained a new \( (\mathbb{Z}_2)^2 \) symmetry that acts on the twisted sectors. Now consider, for example, the \( \mathbb{Z}_2 \) generator \( g_1 \) of \((A_1^{(i)})_{k_1}\), which acts non-trivially on the fields with \( l_1 \) odd and \( l_2 = k_2/2 \) in the chiral ring. These fields correspond to the sector twisted by the transformation \( t_2 \), which acts non-trivially on \( X_2 \) and \( X_3 \). If we now want to orbifoldize a model with this invariant by a symmetry \( \sigma_1 \) that acts like \( g_1 \) in the first factor of the tensor product, then we need to make \( \sigma_1 \) act non-trivially on the twisted vacuum \( |t_2\rangle \). This can be achieved by introducing a discrete torsion \( \varepsilon(\sigma_1, t_2) = -1 \) (compare formula (3)). Another possibility, which keeps the way open for a geometric interpretation of the string vacuum, would be to simulate the effect of this torsion by introducing two additional trivial fields and letting \( \sigma_1 \) act on one and \( t_2 \) act on both of them (such that all determinants remain positive) \(^4\).

---

\(^4\) In fact, within the Landau–Ginzburg framework, any \( \mathbb{Z}_2 \) torsion between two generators can be simulated by introducing three additional trivial fields, with the two generators acting on different pairs of these fields.
4.3 An exceptional automorphism invariant

As a last example we consider an exceptional invariant that is again of the automorphism type, but now the automorphism is not with respect to \((A_1^{(1)})^\otimes N\) fusion rules, but rather with respect to the fusion rules of an extended chiral algebra. The invariant arises for \(N = 2\) and \(k_1 = 3, k_2 = 28\); it reads \([14]\)

\[
Z^{(3,28)} = |0,\bar{0}|^2 \oplus |1,\bar{6}|^2 \oplus |2,\bar{6}|^2 \oplus |3,\bar{0}|^2 \oplus \left[(0,\bar{6}) \otimes (2,\bar{0})^* \oplus (1,\bar{0}) \otimes (3,\bar{6})^* \oplus \mathrm{c.c.}\right],
\]

where

\[
\bar{0} \equiv 0 \oplus 10 \oplus 18 \oplus 28, \quad \bar{6} \equiv 6 \oplus 12 \oplus 16 \oplus 22.
\]

The extension of the chiral algebra corresponds to the conformal embedding of \((A_1^{(1)})_{28}\) in \((G_2^{(1)})_1\), as in the case of the exceptional \(E_8\) type \(A_1^{(1)}\) invariant.

In order to find a Landau–Ginzburg orbifold that corresponds to the invariant \([24]\), we start by writing down the corresponding Poincaré polynomial. It reads

\[
P^{(3,28)}(t) = (1 + t^{1/3})(1 + t^{2/5} + 4t^{3/5} + t^{4/5} + t^{6/5}).
\]

Let us express this Poincaré polynomial in terms of the variables \(x = t^{1/5}, y = u^6 = t^{6/30}\), and \(z = u^{10} = t^{10/30}\); this is suggested by the Landau–Ginzburg potential \(Y^5 + Z^3\) for the exceptional invariant of the level–28 theory, whose building blocks are used in \(Z^{(3,28)}\). In these variables the Poincaré polynomial becomes

\[
P^{(3,28)}(t) = (1 + x^3)(1 + u^{10} + u^{18} + u^{28}) + (x + x^2)(u^6 + u^{12} + u^{16} + u^{22})
\]

\[= (1 + z)[(1 + x^3)(1 + y^3) + (x + x^2)(y + y^2)].
\]

We thus start from the potential

\[
X^5 + Y^5 + Z^3
\]

and try to use a twist in the \((X,Y)\) sector to get the second factor in the expression \([27]\) for the Poincaré polynomial. To avoid a contribution \(t^{1/5}\) to the polynomial, we need to eliminate the fields \(X\) and \(Y\) and any linear combination thereof from the untwisted sector. As the determinant of the twist should be \(\pm 1\), we have to use the \(\mathbb{Z}_5\) which acts as \((X,Y,Z) \mapsto (\lambda X, \lambda^3 Y, Z)\) with \(\lambda^5 = 1\). From the untwisted sector we now get exactly the even powers of \(t^{1/5}\) on the right–hand side of \([27]\), whereas the four twisted sectors contribute \(t^{3/5}\) each. Thus the \(\mathbb{Z}_5(1,4,0)\) orbifold of \(X^5 + Y^5 + Z^3\) reproduces the correct Poincaré polynomial for the automorphism invariant \([24]\).

4.4 Some string spectra

We have verified that the Gepner–type models that employ the invariants described in subsections 4.1 and 4.3 and the string compactifications obtained from the corresponding Landau–Ginzburg orbifolds possess identical spectra of massless (\(E_6\) non-singlet) fields, thus further confirming the identification of the respective theories. In the case of the invariant \([23]\), as well as for \([14]\) with \(N = 2\) and \(k_1, k_2 \in 4\mathbb{Z}\), these spectra have already been listed in \([14]\). The spectra for a few other models employing invariants of the type \([15]\) are listed in table 1, namely for some models with
Table 1: Spectra of models based on exceptional \((A_1^{(1)})^\oplus N\) invariants

| #  | invariant     | \(n_{27}\) | \(n_{27}^{-}\) | \(n_1\) | \(n_g\) | \(\chi\) |
|-----|---------------|-------------|----------------|-------|-------|-------|
| 50  | 1 2 4 4-10    | 43          | 7              | 232   | 4     | -72   |
| 127 | 2 4 12-82     | 51          | 57             | 447   | 3     | 12    |
| 130 | 2 4 16-34     | 66          | 24             | 353   | 3     | -84   |
| 131 | 2 4 18-28     | 36          | 42             | 327   | 3     | 12    |
| 133 | 2 4-22 22     | 74          | 14             | 341   | 3     | -120  |
| 136 | 2 5 12-26     | 59          | 23             | 327   | 3     | -72   |
| 138 | 2 6 8-38      | 38          | 38             | 327   | 3     | 0     |
| 138 | 2 6-8 38      | 38          | 38             | 327   | 3     | 0     |
| 143 | 2 8 8-18      | 66          | 12             | 315   | 3     | -108  |
| 144 | 2 8 10-13     | 18          | 42             | 247   | 3     | 48    |
| 160 | 4 6 4-22      | 41          | 17             | 243   | 3     | -48   |
| 163 | 4 4-10 10     | 62          | 8              | 263   | 3     | -108  |
| 17  | 1 1 1 4-4-4   | 51          | 3              | 213   | 5     | -96   |
| 54  | 1 4 4-4-4     | 37          | 7              | 200   | 4     | -60   |
| 59  | 2 2 4-4-4     | 12          | 30             | 215   | 5     | 36    |
| 95  | 1 8-16-88     | 55          | 61             | 447   | 3     | 12    |
| 98  | 1 8-28-28     | 95          | 17             | 419   | 3     | -156  |
| 111 | 1 12-12-40    | 60          | 30             | 345   | 3     | -60   |
| 117 | 1 16-16-16    | 101         | 11             | 401   | 3     | -180  |
| 153 | 3 4-8-28      | 39          | 27             | 277   | 3     | -24   |
| 158 | 3 8-8-8       | 67          | 7              | 267   | 3     | -120  |
| 159 | 5 4-4-40      | 38          | 32             | 299   | 3     | -12   |
| 161 | 7 4-4-16      | 50          | 14             | 261   | 3     | -72   |
| 162 | 13 4-4-8      | 29          | 23             | 223   | 3     | -12   |
| 54  | 1 4-4-4-4     | 37          | 7              | 200   | 4     | -60   |
Table 2: Spectra of orbifolds of the model # 127 with invariant 24-12-82

| twist     | $n_{27}$ | $n_{27}$ | $n_{1}$ | $n_{g}$ | $\chi$ |
|-----------|----------|----------|---------|---------|--------|
| 1         | 42       | 48       | 357     | 3       | 12     |
| $\mathbb{Z}_2$ (0,1,1,0) | 49       | 49       | 375     | 3       | 0      |
| $\mathbb{Z}_2$ (1,0,1,0) | 33       | 39       | 325     | 3       | 12     |
| $\mathbb{Z}_2$ (1,1,0,0) | 39       | 33       | 325     | 3       | $-12$  |
| $\mathbb{Z}_4$ (1,0,0,3) | 30       | 96       | 479     | 3       | 132    |
| $\mathbb{Z}_4$ (1,0,2,1) | 76       | 40       | 440     | 4       | $-72$  |
| $\mathbb{Z}_4$ (1,2,0,1) | 58       | 34       | 371     | 3       | $-48$  |

$N = 2$ and $k_1 \in 4\mathbb{Z}$, $k_2 \in 4\mathbb{Z} + 2$, for all possible $c = 9$ models with $N = 3$ and $k_1, k_2, k_3 \in 4\mathbb{Z}$, and for the single $c = 9$ model with $N = 4$ and $k_1, k_2, k_3, k_4 \in 4\mathbb{Z}$.

The notation in table 1 is as follows. The first column contains the number which in [23] has been associated to the relevant combination $(k_1, k_2, \ldots)$ of levels of the affine algebras. In the second column we give the chosen invariant, with $k$ standing for the $A$ type invariant at level $k$, and with $k_1$-$k_2$ denoting the invariant (15) with $N = 2$ and levels $k_1, k_2$, etc. The next four columns contain the spectrum, i.e. the numbers $n_{27}, n_{\overline{27}}$ and $n_{1}$ of massless matter fields in the $27, \overline{27}$ and singlet representations of $E_6$, respectively, as well as the number $n_{g}$ of gauge bosons that are present in addition to those of $E_6$. The number in the last column is the Euler number $\chi = 2(n_{\overline{27}} - n_{27})$.

As already mentioned, we checked that the Landau–Ginzburg results agree with all spectra given in [14] whenever we have a Landau–Ginzburg interpretation of the respective invariant. We have done so not only for the models themselves, but for further orbifolded versions of them as well. Note that this requires the use of appropriate discrete torsions, as explained at the end of section 4.2. Actually some of the orbifold results have been reproduced incorrectly in table 3 of [14]. Therefore we also list, in table 2, the correct spectra for these theories. The model in question is the one numbered as # 127, with invariant 24-12-82. The notation used for the modded out symmetry is as in formula (6).

5 Outlook

In this paper we have described a recipe of how to search systematically for the Landau–Ginzburg interpretation of any given modular invariant for tensor products of minimal $n = 2$ superconformal models. We have applied this procedure in a case-by-case analysis to various invariants for which an associated Landau–Ginzburg orbifold could be identified. On the other hand, we were also able to use the method to prove that the particular invariant (4) cannot be described in terms of a Landau–Ginzburg orbifold (with respect to a manifest linear symmetry).

Of course it would be desirable to understand at a more fundamental level why in some cases
such a correspondence exists while in other cases it does not. In this context we note that all invariants described in section 4 are of the automorphism type; thus maybe at least all invariants of this particular type possess a Landau–Ginzburg interpretation. There is, however, no obvious connection between the \( \mathbb{Z}_2 \) group corresponding to the fusion rule automorphism and the \( \mathbb{Z}_2 \) symmetries \( \mathbb{Z}_2 \) modded out in the orbifold construction.

Let us also mention that in the special situations we are considering the possible dependence on the moduli of potentials with \( c \geq 3 \) does not pose any problem for a unique identification of a Landau–Ginzburg theory, as these moduli are fixed by discrete symmetries.

In a sense, the automorphism invariant \((17)\) is a generalization of the \( D \) invariant of minimal models. It is thus tempting to look for a non-linear transformation like the one that relates the \( \mathbb{Z}_2 \) orbifold representation of that invariant to an (untwisted) Landau–Ginzburg model. For \( N = 2 \) and \( W = X^{2a} + Y^{2b} + Z^2 \), for example, \( X' = X^2 \), \( Y' = Y^2 \) and \( Z' = Z/(XY) \) indeed provides a transformation with the required \( (\mathbb{Z}_2)^2 \) identification and with constant determinant. The resulting potential \( (X')^a + (Y')^b + X'Y'(Z')^2 \), however, is degenerate. Although in some cases the configuration determined by this polynomial is non-degenerate, the spectra calculated from a regularizing deformation of such polynomials turn out wrong.

Our findings corroborate with recent indications that the space of Landau–Ginzburg orbifolds is not mirror–symmetric \( [24] \) to diminish the hope that Landau–Ginzburg models are of much use for a classification of \( n = 2 \) superconformal field theories. Still, it appears worth while to apply our ideas to other constructions such as \( n = 2 \) coset models \( [25] \). In any case, a Landau–Ginzburg representation, if it exists, provides an extremely efficient computational framework. In addition, one may get an idea of the nature of the limitations of Landau–Ginzburg models.
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