FINITE DIMENSIONAL GLOBAL ATTRACTOR FOR A CLASS OF TWO-COUPLED NONLINEAR FRACTIONAL SCHRÖDINGER EQUATIONS

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Abstract. In the current issue, we consider a general class of two coupled weakly dissipative fractional Schrödinger-type equations. We will prove that the asymptotic dynamics of the solutions for such NLS system will be described by the existence of a regular compact global attractor in the phase space that has finite fractal dimension.

1. Introduction. The nonlinear Schrödinger equation is a prototypal dispersive nonlinear partial differential equation that has been derived in many areas of physics and analysed mathematically for over 40 years. The study of systems of coupled nonlinear Schrödinger type equations in mathematical and physical aspects are of comparative interest and has attracted much attention for many researchers as they appear in different sides of applications to various branches of physics. There has been a vast amount of literature involving coupled NLS equations over the years, but recently there has been additional interest, mainly due to the developments in nonlinear optics and condensed matter physics.

In nonlinear optics, coupled nonlinear Schrödinger equations provides a “canonical” description for the propagation of light along birefringent optical fiber [25], [39] and [40]. In hydrodynamics, a coupling nonlinear Schrödinger system was considered to study the modulation instability of gravity waves in fluids in great depths [8]. As an asymptotic model governing the evolution of two orthogonal pulse envelopes in birefringent optical fibers, coupled NLS systems appear either in [27] or in [28] and can be derived from the governing equation of atmospheric gravity waves when studying collision interactions of envelopes Rossby solitons in a barotropic atmosphere [33]. Similar coupled systems arise also when studying either Feshbach resonances in atomic Bose-Einstein condensate systems [35] or atmospheric gravity waves [26]. Subsequently, with the appearance of memory materials, a great attention has been focused on the study of problems involving the fractional space derivative.

Lately, interest in dissipative systems increased even more because of intensive elaboration of strange attractors. For a given dissipative dynamical system, the first

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question that arises is determining the existence of a bounded (or even compact) attracting set, the **global attractor** that contains much of the relevant information about the flow, on which one may reduce the qualitative study of the system. We refer the reader to [34], [30], [9], [4] and [29] for general frameworks of this theory.

In the current paper, we investigate the asymptotic behaviour of solutions for an infinite dimensional dynamical system generated by a generalized class of two coupled nonlinear fractional dissipative Schrödinger equations that state as follows

\begin{align}
\begin{align*}
\frac{\partial u}{\partial t} - i(-\Delta)^{\alpha/2} u + i \left( |u|^2 + |v|^2 \right) u + 2i\beta \nabla v + \gamma u &= f \\
\frac{\partial v}{\partial t} - i(-\Delta)^{\alpha/2} v + i \left( |u|^2 + |v|^2 \right) v + i\beta u^2 + \gamma v &= g
\end{align*}
\end{align}

supplemented by the following initial conditions

\begin{align}
\begin{align*}
u(0) = u_0, \quad v(0) = v_0
\end{align*}
\end{align} 

where \( \alpha \in (1, 2], \beta \in \mathbb{R} \) and \( \gamma > 0 \) is the damping parameter. The unknown \( u = u(t, x) \) and \( v = v(t, x) \) map \( \mathbb{R}_+ \times \mathbb{R} \) into \( \mathbb{C} \). At \( t = 0 \), the initial conditions \( u_0 \) and \( v_0 \) belong to the fractional Sobolev space \( H^{\frac{\alpha}{2}}(\mathbb{R}) \) that will be specified in the sequel. The functions \( f \) and \( g \), that belong to \( L^2(\mathbb{R}) \), are given source terms that are independent of time.

The system (1) arises in the study of Feshbach resonances in atomic Bose-Einstein condensate systems (see [35]). Related systems describing the propagation of light along birefringent optical fibers (see [20]) have also generated considerable interest. It has also proved to be particularly valuable from the point of view of nonlinear optics (see [6]) and when studying large-scale Rossby waves [33]. In the special case when \( \beta = 0 \) and \( \alpha = 2 \), the resulting system is none other then the Manakov system (see [27]).

Mathematically, the study of fractional NLS equations has attracted a lot of attention among researchers and the prototypical example is the following cubic fractional NLS, initially introduced by N. Laskin [21, 22], that reads

\begin{align}
\begin{align*}
u_t - i(-\Delta)^{\alpha/2} u + i |u|^2 u = 0
\end{align*}
\end{align}

This equation was recently studied by several authors among which stand out B. Guo and Z. Huo [17] and Y. Hong and Y. Sire [19] regarding the question of the initial value problem in Sobolev spaces. Taking into account, in some physical context, an external forcing term and some damping effects, a similar equation was considered in [15] in which the issues of existence and regularity of the global attractor were addressed.

Now let us go back to the matter at hand. In its conservative case when \( \gamma = f = g = 0 \), the study of existence and uniqueness of global smooth solutions to the periodic boundary value problem for (1) was established in [18] (see also [20]) by using the Galerkin method. However, existence and stability of standing waves was considered in [32] for related systems of fractional NLS like equations. While in the dissipative case where, in some physical contexts, external forcing terms and some damping effects have been taken into account, it should be cited that G. Li and C. Zhu have considered the following system

\begin{align}
\begin{align*}
u_t - i(-\Delta)^{\alpha/2} u + i (|u|^2 + |v|^2) u + \gamma u &= f \\
v_t - i(-\Delta)^{\alpha/2} v + i (|u|^2 + |v|^2) v + \delta v_x + \gamma v &= g
\end{align*}
\end{align}

in a one dimensional bounded domain. They have established in [23] the existence of a compact global attractor with finite fractal dimension and their strategy rely
on the work of J. Ghidaglia [13]. In [7], M. Cheng studied the asymptotic dynamics for N-coupled Fractional dissipative NLS like system with power type linearities and he proved the well posedness of the initial value problem as well as the existence of a compact global attractor in the phase space.

2. Mathematical background.

2.1. Notations and main results. Before giving the main results and the layout of this article, we introduce briefly some definitions and notations.

To begin with and for a given $p \in [1, +\infty]$, the usual Lebesgue spaces will be denoted $L^p = L^p(\mathbb{R})$. We recall that in what follows we use the notation $(\cdot, \cdot)$ for the usual scalar product in $L^2(\mathbb{R}) = L^2$ defined by

$$(u, v) = \Re \int_{\mathbb{R}} u(x) \overline{v(x)} \, dx.$$ 

Our convention for the one dimensional space Fourier transform is

$$\mathcal{F}(u)(\xi) = \int_{\mathbb{R}} u(x) e^{-ix\xi} \, dx.$$ 

For a given fractional exponent $s \in (0, 1)$, we recall (see [11] and [31]) that $(-\Delta)^s$ is considered as the homogeneous fractional pseudo-differential operator defined, for $u \in S(\mathbb{R})$, by

$$(-\Delta)^s u(x) = C_s \begin{array}{c}
p.v. \int_{\mathbb{R}^2} \frac{u(x) - u(y)}{|x - y|^{1+2s}} \, dy = \mathcal{F}^{-1}\left(|\xi|^{2s} \mathcal{F}(u)\right)(x),
\end{array}$$

where $C_s > 0$, $S(\mathbb{R})$ denotes the Schwarz class and “p.v.” for principal value.

The fractional Sobolev space $H^s(\mathbb{R}) = H^s$ defined as follows

$$H^s(\mathbb{R}) = \left\{ u \in L^2(\mathbb{R}) \text{ such that } \int_{\mathbb{R}} (1 + |\xi|^{2s}) |\mathcal{F}(u)(\xi)|^2 \, d\xi < +\infty \right\}$$

as an intermediary Banach space between $L^2$ and $H^1(\mathbb{R}) = H^1$ is a Hilbert space endowed, via the Fourier transform approach, with the norm

$$||u||_{H^s} = \left(||u||_{L^2}^2 + ||(-\Delta)^{\frac{s}{2}} u||_{L^2}^2\right)^{\frac{1}{2}}$$

and the associate scalar product denoted by

$$(u, v)_{H^s} = (u, v) + (\langle -\Delta \rangle^{\frac{s}{2}} u, \langle -\Delta \rangle^{\frac{s}{2}} v), \quad u, v \in H^s.$$ 

In order to remove any ambiguity in what follows and for the sake of simplicity, we will extensively use $L^2$ to denote the Hilbert space $(L^2(\mathbb{R}))^2$ endowed with the norm

$$||(u, v)||_{L^2} = \left(||u||_{L^2}^2 + ||v||_{L^2}^2\right)^{\frac{1}{2}}.$$ 

Similarly, the Hilbert spaces $H^s = (H^s(\mathbb{R}))^2$, $s \geq 0$, are endowed with the norm

$$||(u, v)||_{H^s} = ||(u, v)||_{L^2}^2 + ||(-\Delta)^{\frac{s}{2}} (u, v)||_{L^2}^2.$$ 

The first result established in this paper states as follows:

**Theorem 2.1.** Let $\alpha \in (1, 2]$ and $f, g \in L^2$. Then the problem (1)–(2) is well posed in $H^\frac{\alpha}{2}$ and defines an infinite dimensional dissipative dynamical system that possesses a compact global attractor $\mathcal{A}_\alpha$ in $H^\frac{\alpha}{2}$.
Now having obtained the existence of the global attractor, the question arises if it has special regularity properties or if it has finite-dimensional character. This will our second main result

**Theorem 2.2.** Let \( \alpha \in (1, 2] \) and \( f, g \in L^2 \). Then the global attractor \( \mathcal{A}_\alpha \), associated to the dynamical system generated by (1) − (2) is a compact subset of \( H^{\frac{\alpha}{2}} \) with finite fractal dimension in \( \mathbb{H}^{\frac{\alpha}{2}} \).

This article is organized as follows: in the subsection below, we establish some helpful mathematical tools that play a crucial role in what follows. In section 3 we prove the well-posedness of the problem (1) − (2) as well as the existence of a compact global attractor in \( \mathbb{H}^{\frac{\alpha}{2}} \). The regularity of the global attractor \( \mathcal{A}_\alpha \) for (1) − (2) will be discussed in Section 4. Finally, the section 5 we be dedicated to establish the finite fractal dimension of \( \mathcal{A}_\alpha \) using the new idea recently introduced by the author in [2].

In the end of this subsection it should be noted that throughout this article, the constants \( C \)s are numerical positive constants that vary from one line to another and \( A \lesssim B \) means the existence of \( C > 0 \) such that \( A \leq CB \).

2.2. Preliminary results. To begin with, we shall briefly introduce some tools from functional analysis that will be used extensively in the sequel.

We recall a fractional Gagliardo-Nirenberg type inequality (see [12] for instance)

**Lemma 2.3.** Let \( \alpha > 1 \). Then for every \( p \in [2, +\infty] \), there exists \( C = C(\alpha, p) > 0 \) such that

\[
\|u\|_{L^p(\mathbb{R})} \leq C\|u\|_{L^2}^{1-\frac{1}{p}(\frac{1}{2} - \frac{\alpha}{p})} \left\|(-\Delta)^{\frac{\alpha}{4}} u\right\|_{L^2}^{\frac{1}{2} - \frac{1}{p}}, \quad \forall u \in H^{\frac{\alpha}{2}}.
\]

**Proof of Lemma 2.3.** For the sake of completeness, we give a simple proof. Using the Hausdorff-Young inequality and the Hölder inequality, it leads that for \( p > 2 \)

\[
\|u\|_{L^p} \lesssim \|\mathcal{F}(u)\|_{L^{p'}}
\]

where \( p' \) denotes the conjugate exponent of \( p \).

Hence

\[
\|u\|_{L^p} \leq C(\alpha, p) \left(\|u\|_{L^2} + \left\|(-\Delta)^{\frac{\alpha}{4}} u\right\|_{L^2}\right).
\]

Replacing \( u \) by \( x \mapsto u(\lambda x) \), \( \lambda \in \mathbb{R} \) in (4), it leads that

\[
\|u\|_{L^p} \leq C(\alpha, p) \left(\frac{1}{\lambda^2} - \frac{1}{p'}\right)\|u\|_{L^2} + \lambda^{\frac{\alpha}{4} - (\frac{1}{2} - \frac{1}{p'})}\left\|(-\Delta)^{\frac{\alpha}{4}} u\right\|_{L^2}.
\]

Minimizing the right hand side of the previous equation with respect to \( \lambda \) achieves the proof of the lemma. \( \square \)

For later use we recall the following result

**Lemma 2.4.** Let \( s \in [\frac{1}{2}, 1] \). Then there exists \( C > 0 \) such that for all \( p \in [2, +\infty) \),

\[
\|u\|_{L^p} \leq C \sqrt{p}\|u\|_{H^s}, \quad \forall u \in H^s.
\]

**Proof of Lemma 2.4.** Thanks to Theorem 8.5 in [24], the proof follows by using interpolation argument between \( H^{\frac{\alpha}{2}} \) and \( H^1 \). We refer the reader to [3] for a simple proof. \( \square \)
We give now a commutator estimate that states as follows

**Lemma 2.5.** Let \( u \in \mathcal{S}(\mathbb{R}) \) and \( v \in H^\alpha \). Then there exists \( C > 0 \) such that

\[
\|((-\Delta)^{\frac{\alpha}{2}}(uv) - u(-\Delta)^{\frac{\alpha}{2}}v)\|_{L^2} \leq C \|v\|_{L^2} \|\xi|^{\alpha} \mathcal{F}(w)\|_{L^1}.
\]

**Proof of Lemma 2.5.** Thanks to the Fourier transform, one has

\[
\|\mathcal{F}((-\Delta)^{\frac{\alpha}{2}}(uv))(\xi) - \mathcal{F}(u(-\Delta)^{\frac{\alpha}{2}}v)(\xi)\| \\
\leq \int_\mathbb{R} |\xi - \eta|^{\alpha} - |\xi|^{\alpha} |\mathcal{F}(u)(\eta)||\mathcal{F}(v)(\xi - \eta)| \, d\eta \\
\lesssim \int_\mathbb{R} |\eta|^{\alpha} |\mathcal{F}(u)(\eta)||\mathcal{F}(v)(\xi - \eta)| \, d\eta
\]

and the desired estimate yields thanks to the Minkowski inequality and the Plancherel Theorem. \(\square\)

3. Well posedness of the problem and existence of the global attractor.

3.1. The initial value problem.

**Proposition 3.1.** Let \((u_0, v_0) \in H^{\frac{\alpha}{2}}\). Then the problem \((1) - (2)\) has a unique solution

\[
(u, v) \in \mathcal{C}_b([0, +\infty), H^{\frac{\alpha}{2}}) \cap \mathcal{C}_b^1([0, +\infty), H^{-\frac{\alpha}{2}})
\]

and the maps \(S_\alpha(t) : (u_0, v_0) \mapsto (u(t), v(t))\) are continuous on \(H^{\frac{\alpha}{2}}\) in the sense of \(H^{-\frac{\alpha}{2}}\) the space of continuous bounded functions which take values in \(H^{\frac{\alpha}{2}}\) with \(H^{-\frac{\alpha}{2}}\) stands for \((H^{-\frac{\alpha}{2}})^2\).

**Proof of Proposition 3.1.** The proof is very standard and then briefly sketch. We proceed into three steps:

**First step:** A local in time solution.

Let \( T > 0 \). Denoting \( U = \begin{pmatrix} u \\ v \end{pmatrix} \), the system \((1)\) is equivalently written as follows

\[
U_t - i(-\Delta)^{\frac{\alpha}{2}}U + i|U|^2U + i\beta MU + \gamma U = F
\]

where \(|U|^2 = |u|^2 + |v|^2\), \(M = \begin{pmatrix} 0 & 2\pi \\ u & 0 \end{pmatrix}\) and \(F = \begin{pmatrix} f \\ g \end{pmatrix}\).

Since the Hilbert space \(H^{\frac{\alpha}{2}}\) is an algebra (as to Lemma 2.3), the mapping \(\Psi : U \mapsto |U|^2U + \beta MU\) is locally lipschitz on bounded subsets of \(H^{\frac{\alpha}{2}}\), \(\alpha > 1\). This ensures, by a standard fixed point argument on its Duhamel’s formula, the existence of a local in time solution \(U\) in \(\mathcal{C}([0, T^*), H^{\frac{\alpha}{2}})\) for the problem \((1) - (2)\). Moreover, either \(T^* = +\infty\) or \(\|U(t)\|_{H^{\frac{\alpha}{2}}} \rightarrow +\infty\).

**Second step:** The solution is global in time.

To begin with, we establish the following a priori estimate that reads

**Lemma 3.2.** Let \(u_0, v_0 \in H^{\frac{\alpha}{2}}\). Then there exists \(C > 0\) depending only on \(\alpha, \beta, \|f\|_{L^2}\) and \(\|g\|_{L^2}\) such that

\[
\sup_{t \in [0, T^*)} (|u(t)|_{H^{\frac{\alpha}{2}}}^2 + |v(t)|_{H^{\frac{\alpha}{2}}}^2) \leq \frac{3}{2} e^{-\gamma t} \left( |u_0|_{H^{\frac{\alpha}{2}}}^2 + |u_0|_{H^{\frac{\alpha}{2}}}^2 \right) + C.
\]
Proof of Lemma 3.2. On the one hand, the scalar products of (1a) by \(u\) and (1b) by \(v\), lead to

\[
\frac{d}{dt} \left( \|u(t)\|_{L^2}^2 + 2\|v(t)\|_{L^2}^2 \right) + \gamma \left( \|u(t)\|_{L^2} + 2\|v(t)\|_{L^2} \right) = (f, u) + 2(g, v). 
\]

Hence, applying the Cauchy-Schwarz and Young’s inequalities, it can be deduced by the Gronwall’s Lemma that

\[
\|u(t)\|_{L^2}^2 + 2\|v(t)\|_{L^2}^2 \leq (\|u_0\|_{L^2}^2 + 2\|v_0\|_{L^2}^2)e^{-\gamma t} + \frac{\|f\|_{L^2}^2 + \|g\|_{L^2}^2}{2\gamma^2}(1 - e^{-\gamma t}). \tag{6}
\]

On the other hand, the scalar products of (1a) by \(-i(u_t + \gamma u)\) and (1b) by \(-i(v_t + \gamma v)\) lead to the following energy equation that reads

\[
\frac{1}{2} \frac{d}{dt} J(u(t), v(t)) + \gamma J(u(t), v(t)) = K(u(t), v(t)). \tag{7}
\]

where

\[
J(u, v) = \left\| (-\Delta)^{\frac{3}{4}} u(t) \right\|_{L^2}^2 + \left\| (-\Delta)^{\frac{3}{4}} v(t) \right\|_{L^2}^2 - \frac{1}{2} \int_{\mathbb{R}} \left( \|u\|^2 + |v|^2 \right) dx 
- 2\beta(u^2, v) - 2 [if, u] + (ig, v) \tag{8}
\]

\[
K(u, v) = \frac{\gamma}{2} \int_{\mathbb{R}} \left( \|u\|^2 + |v|^2 \right) dx + \gamma \beta(u^2, v) - \gamma [if, u] + (ig, v) \tag{9}
\]

Consequently, thanks to the previous inequality, Lemma 2.3, the Young’s inequality and (6) one easily obtain the existence of a non negative real constant \(C\) that depends only on \(\|f\|_{L^2}, \|g\|_{L^2}, \beta\) and \(\gamma\) such that

\[
\begin{cases}
\frac{1}{2} \left( \left\| (-\Delta)^{\frac{3}{4}} u(t) \right\|_{L^2}^2 + \left\| (-\Delta)^{\frac{3}{4}} v(t) \right\|_{L^2}^2 \right) - C \leq J(u(t), v(t)) \tag{10a}
\end{cases}
\]

\[
J(u(t), v(t)) \leq \frac{3}{2} \left( \left\| (-\Delta)^{\frac{3}{4}} u(t) \right\|_{L^2}^2 + \left\| (-\Delta)^{\frac{3}{4}} v(t) \right\|_{L^2}^2 \right) + C \tag{10b}
\]

and

\[
K(u(t), v(t)) \leq \frac{\gamma}{4} \left( \left\| (-\Delta)^{\frac{3}{4}} u(t) \right\|_{L^2}^2 + \left\| (-\Delta)^{\frac{3}{4}} v(t) \right\|_{L^2}^2 \right) + C. \tag{11}
\]

Gathering (10), (11) and (7) we obtain

\[
\frac{d}{dt} J(u(t), v(t)) + \gamma J(u(t), v(t)) \leq C.
\]

This concludes the proof thanks to the Gronwall Lemma, (10) and (6).

Lemma 3.2 ensure that \(T^* = +\infty\) and then a global in time solution is obtained.

**Third step:** Continuity of the semigroup on \(\mathbb{H}^{\frac{3}{4}}\).

This step is dedicated to prove that the semigroup \((S_\alpha(t))_{t \in \mathbb{R}_+}\) is a strongly continuous family of maps from \(\mathbb{H}^{\frac{3}{4}}\) into itself. To do this we firstly establish the following statement

**Lemma 3.3.** The semi-group \((S_\alpha(t))_{t \in \mathbb{R}_+}\) is continuous on bounded subsets of \(\mathbb{H}^{\frac{3}{4}}\) for the strong topology of \(L^2\).

**Proof of Lemma 3.3.** Let \((\phi_n, \psi_n)_n\) be a bounded sequence in \(\mathbb{H}^{\frac{3}{4}}\) that converges towards \((u_0, v_0) \in \mathbb{H}^{\frac{3}{4}}\) for the strong topology of \(L^2\). We denote

\[
(z_n(t), z_n^*(t)) = S_\alpha(t)(\phi_n, \psi_n) - S_\alpha(t)(u_0, v_0) = (u_n(t) - u(t), v_n(t) - v(t))
\]
the difference of two solutions of (1) – (2) issued respectively from \((\phi_n, \psi_n)\) and \((u_0, v_0)\). Then \((z_n(t), z'_n(t))\) satisfies

\[
\begin{aligned}
(z_n)_{t} - i(-\Delta)^\frac{\alpha}{2} z_n + \gamma z_n + 2i\beta(n_0 v - n_0 v) = i(|u|^2 + |v|^2)u - i(|u_n|^2 + |v_n|^2)u_n \\
(z'_n)_{t} - i(-\Delta)^\frac{\alpha}{2} z'_n + \gamma z'_n + 2i\beta(u_n - u)^2 = i(|u|^2 + |v|^2)v - i(|u_n|^2 + |v_n|^2)v_n
\end{aligned}
\]

Hence, it can be easily deduced that

\[
\frac{1}{2} \frac{d}{dt} \|z(t), z'(t)\|_{L^2}^2 + \gamma \|z(t), z'(t)\|_{L^2}^2 \leq C_0 \int_{\mathbb{R}} \left( |u|^2 + |v|^2 + |u_n|^2 + |v_n|^2 + |u| + |v| \right) (|z_n|^2 + |z'_n|^2) \, dx
\]

(12)

Now we use an argument due to M. Vladimirov [36]. We consider \(p \in (2, +\infty)\). Thanks to the Hölder inequality

\[
\begin{aligned}
\int_{\mathbb{R}} \left( |u_n|^2 + |v_n|^2 + |u|^2 + |v|^2 + |u_n| + |v_n| + |u| + |v| \right) (|z_n|^2 + |z'_n|^2) \, dx \\
\leq \left( |u_n|^2_{L^{2p}} + |v_n|^2_{L^{2p}} + |u|^2_{L^{2p}} + |v|^2_{L^{2p}} \right) \left( |z_n|_{L^4}^2 + |z'_n|_{L^4}^2 \right) \|z_n, z'_n\|_{L^2}^{2p-2} \\
+ \left( |u_n|_{L^{2p}} + |v_n|_{L^{2p}} + |u|_{L^{2p}} + |v|_{L^{2p}} \right) \left( |z_n|_{L^4}^2 + |z'_n|_{L^4}^2 \right) \|z_n, z'_n\|_{L^2}^{2p-2}
\end{aligned}
\]

Since \((u_n, v_n)\) and \((u, v)\) remain uniformly bounded in \(H^{\frac{\alpha}{2}}\) (Lemma 3.2), it may be deduced, in accordance with (12), Lemma 2.3 and Lemma 2.4, that

\[
\frac{d}{dt} \|z(t), z'(t)\|_{L^2}^2 \leq C_0 C_1^{\frac{1}{p}} p \|z(t), z'(t)\|_{L^2}^{2(1 - \frac{1}{2p})},
\]

which is equivalent to

\[
\frac{d}{dt} \|z(t), z'(t)\|_{L^2}^2 \leq C_\alpha C_1^{\frac{1}{p}}.
\]

Integrating (13) on \([0, T]\) for a chosen \(T > 0\) such that \(C_\alpha T < 1\), leads to

\[
\sup_{t \in [0, T]} \|S_\alpha(t)(\phi_n, \psi_n) - S_\alpha(t)(u_0, v_0)\|_{L^2} \leq \left( C_\alpha C_1^{\frac{1}{p}} T + \|(\phi_n, \psi_n) - (u_0, v_0)\|_{L^2}^{\frac{1}{p}} \right)^{\frac{1}{p}}
\]

(14)

Therefore

\[
\limsup_{n \to +\infty} \sup_{t \in [0, T]} \|S_\alpha(t)(\phi_n, \psi_n) - S_\alpha(t)(u_0, v_0)\|_{L^2} \leq C_1 \left( C_\alpha T \right)^{\frac{1}{p}}
\]

(15)

and then the continuity of \(S_\alpha(t)\) on bounded subsets of \(H^{\frac{\alpha}{2}}\) for the strong topology of \(L^2\) is established by letting \(p \to +\infty\).

We now apply the famous J. Ball’s argument (see [5] and [37]). Let \((u_n, v_n)\) be a sequence that converges in \(H^{\frac{\alpha}{2}}\) towards \((u_0, v_0) \in H^{\frac{\alpha}{2}}\). Thanks to the energy equation (7) one has on the one hand

\[
J(S_\alpha(t)(\phi_n, \psi_n)) = J(\phi_n, \psi_n)e^{-2\gamma t} + 2 \int_0^t e^{-2\gamma(t-s)} K(S_\alpha(s)(\phi_n, \psi_n)) \, ds
\]

(16)

and in the other hand,

\[
J(S_\alpha(t)(u_0, v_0)) = J(u_0, v_0)e^{-2\gamma t} + \int_0^t e^{-2\gamma(t-s)} K(S_\alpha(s)(u_0, v_0)) \, ds,
\]

(17)
where $J$ and $K$ are defined by (8) and (9).

Remark that due to Lemma 3.3 and Lemma 2.3, the semi-group $(S_{\alpha}(t))_{t \in \mathbb{R}^+}$ is continuous on bounded subsets of $\mathbb{H}^2$ for the strong topology of $L^p$, $p \in [2, +\infty)$.

Hence,

$$S_{\alpha}(t)(\phi_n, \psi_n) \xrightarrow{n \to +\infty} S_{\alpha}(t)(u_0, v_0) \quad \text{strongly in } L^p, \quad p \in [2, +\infty).$$

(18)

Thanks to the Lebesgue dominated convergence Theorem, the estimates (18), (16) and (17) allow us to obtain that

$$\lim_{n \to +\infty} J(S_{\alpha}(t)(\phi_n, \psi_n)) = J(S_{\alpha}(t)(u_0, v_0))$$

from which we deduce, thanks to (10), that

$$\|(-\Delta)^{\frac{p}{2}} S_{\alpha}(t)(\phi_n, \psi_n)\|_{L^2} \to \|(-\Delta)^{\frac{p}{2}} S_{\alpha}(t)(u_0, v_0)\|_{L^2} \quad \text{as } n \to +\infty.$$

Hence, the desired result and the third step are therefore concluded as well as the proof of the current proposition. \[\square\]

3.2. A compact global attractor for the semigroup $(S_{\alpha}(t))_{t \in \mathbb{R}^+}$. For the sake of completeness and clarity we recall the definition of the global attractor (see for instance [9] and [34]).

**Definition 3.4.** Let $(\mathcal{H}, (S(t))_{t \geq 0})$ be a continuous dynamical system where $\mathcal{H}$ denotes the phase space (complete metric space). We say that a nonempty subset $\mathcal{A}$ of $\mathcal{H}$ is a global or universal attractor for the semigroup $(S(t))_{t \geq 0}$ if and only if $\mathcal{A}$ enjoys the following properties:

1. $\mathcal{A}$ is compact.
2. $\mathcal{A}$ is invariant i.e: $S(t)\mathcal{A} = \mathcal{A}, \forall t \geq 0$.
3. $\mathcal{A}$ attracts the bounded sets of $\mathcal{H}$; i.e: $d_h(S(t)B, \mathcal{A}) \to 0$ as $t \to +\infty, \forall B$ bounded set in $\mathcal{H}$ with $d_h$ stands for the Hausdorff semi-distance.

**Proposition 3.5.** The semigroup $(S_{\alpha}(t))_{t \in \mathbb{R}^+}$ possesses a bounded absorbing ball $\mathcal{B}_{\alpha}$ in $\mathbb{H}^2$, i.e: for any bounded subset $B \subseteq \mathbb{H}^2$ there exists $t(B) > 0$ such that

$$S_{\alpha}(t)B \subseteq \mathcal{B}_{\alpha}, \quad \forall t \geq t(B).$$

**Proof of Proposition 3.5.** The proof is a mere consequence of Lemma 3.2. \[\square\]

We claim now to prove the existence of the global attractor $\mathcal{A}_{\alpha}$. To do this, one only has, thanks to Theorem 1.1 and Remark 1.4 in [34] and Theorem 5.1 in [9], to prove the asymptotic compactness of the semi-group $(S_{\alpha}(t))_{t \in \mathbb{R}^+}$, that is

**Proposition 3.6.** The semi-group $(S_{\alpha}(t))_{t \in \mathbb{R}^+}$ is asymptotically compact in $\mathbb{H}^2$ i.e., for every bounded sequence $(u_k, v_k)_k$ in $\mathbb{H}^2$ and every sequence $t_k \to +\infty$, $(S_{\alpha}(t_k)(u_k, v_k))_k$ is relatively compact in $\mathbb{H}^2$.

**Proof of Proposition 3.6.** To begin with, we show that the semigroup $(S_{\alpha}(t))_{t \in \mathbb{R}^+}$ is asymptotically compact in $L^2$ on bounded subsets of $\mathbb{H}^2$.

**Lemma 3.7.** For every bounded sequence $(u_j, v_j)_j$ in $\mathbb{H}^2$ and every nonnegative sequence $t_j \to +\infty$, $(S_{\alpha}(t_j)(u_j, v_j))_j$ is relatively compact in $L^2$. 

Proof of Lemma 3.7. Let \((u_0, v_0) \in \mathbb{H}^{\frac{\alpha}{2}}\) and \((u(t), v(t)) = S_\alpha(t)(u_0, v_0)\). Consider a smooth cut-off function \(\theta\) such that \(\theta(x) = 1\) if \(|x| \leq 1\) and \(\theta(x) = 0\) if \(|x| \geq 2\), \(x \in \mathbb{R}\). For a given \(r > 0\), we set

\[
\theta_r(x) = \theta \left( \frac{x}{r} \right) \quad \text{and} \quad \left\{ \begin{array}{l}
U_r(t) = \int_{\mathbb{R}} |u(t)|^2 (1 - \theta_r(x))^2 \, dx \\
V_r(t) = \int_{\mathbb{R}} |u(t)|^2 (1 - \theta_r(x))^2 \, dx
\end{array} \right.
\]

Making the scalar products of Multiplying (1a) and (1b) by \((1 - \theta_R)\) then making the scalar products of the first resulting equation by \(u(1 - \theta_R)\) and the second one by \(v(1 - \theta_R)\) lead to

\[
\frac{1}{2} \frac{d}{dt} (U_r(t) + 2V_r(t)) + \gamma (U_r(t) + 2V_r(t))
\]

\[
= \left( i(1 - \theta_r)(-\Delta)^{\frac{\alpha}{2}} u_r(1 - \theta_r)u \right) + 2 \left( i(1 - \theta_r)(-\Delta)^{\frac{\alpha}{2}} v, (1 - \theta_r)v \right) + (1 - \theta_r) f_r, (1 - \theta_r) u + 2((1 - \theta_r) g_r, (1 - \theta_r) v) .
\]

Thanks to the Cauchy-Schwarz inequality, one has

\[
\frac{d}{dt} (U_r(t) + 2V_r(t)) + \gamma (U_r(t) + 2V_r(t)) \leq \left\| (1 - \theta_r)(-\Delta)^{\frac{\alpha}{2}} u, v \right\|_{L^2} \left\| (1 - \theta_r)(u, 2v) \right\|_{L^2}
\]

\[
\leq \left\| (1 - \theta_r)(-\Delta)^{\frac{\alpha}{2}} (u, 2v) \right\|_{L^2} \left\| (1 - \theta_r)(u, 2v) \right\|_{L^2} \leq C \left\| (u, 2v) \right\|_{L^2} \left\| (1 - \theta_r)(u, 2v) \right\|_{L^2} \leq C \left\| (u, 2v) \right\|_{L^2} .
\]

Using the Lemma 2.5,

\[
\left\| (1 - \theta_r)(-\Delta)^{\frac{\alpha}{2}} (u, 2v) \right\|_{L^2} \leq \left\| \theta_r(-\Delta)^{\frac{\alpha}{2}} (u, 2v) \right\|_{L^2} \leq \left\| (u, 2v) \right\|_{L^2} \left\| \theta_r \right\|_{L^1(\mathbb{R})} \leq C \left\| (u, 2v) \right\|_{L^2} .
\]

Hence

\[
\left\| (1 - \theta_r)(-\Delta)^{\frac{\alpha}{2}} (u, 2v) \right\|_{L^2} \leq \left\| \theta_r(-\Delta)^{\frac{\alpha}{2}} (u, 2v) \right\|_{L^2} \leq \left\| (u, 2v) \right\|_{L^2} \leq C \left\| (u, 2v) \right\|_{L^2} \leq \frac{1}{\gamma}. \quad (20)
\]

Gathering (19) and (20) then applying the Young inequality and the Gronwall Lemma lead to

\[
U_r(t) + 2V_r(t) \leq \left[ \left\| u_0 \right\|_{L^2(\mathbb{R})}^2 \right] e^{-\gamma t} + C \left( \left\| f \right\|_{L^2(\mathbb{R})} + \left\| g \right\|_{L^2(\mathbb{R})} \right) (1 - e^{-\gamma t}). \quad (21)
\]

Consider now a bounded sequence \((u_j, v_j)\) in \(\mathbb{H}^{\frac{\alpha}{2}}\) and \(t_j \to +\infty\). We split \(S_\alpha(t_j)(u_j, v_j)\) as follows:

\[
S_\alpha(t_j)(u_j, v_j) = \theta \alpha S_\alpha(t_j)(u_j, v_j) + (1 - \theta_r) S_\alpha(t_j)(u_j, v_j) = W_j(t_j) + V_j(t_j)
\]

We deduce from (21) that for a given \(\epsilon > 0\), there exist \(r_0 \geq 0\) such that

\[
\left\| V_j(t_j) \right\|_{L^2} \leq \epsilon, \quad \forall j \geq j_0 .
\]

Moreover, \((W_j(t_j))_j\) remains trapped in a compact set of \(L^2\) which ensure by classical argument that \((S_\alpha(t_j)(u_j, v_j))_j\) is relatively compact in \(L^2\) and the proof is complete.
Next, by the use of the well known John Ball’s argument in identical manner as in the third step of the proof of Proposition 3.1 thanks to Lemma 3.3 and Lemma 3.7, the proof of the Proposition 3.6 is therefore achieved.

4. Regularity of the global attractor. Following the strategy in [15], we consider a solution \((u(t), v(t))\) of (1)-(2) which takes values in \(\mathbb{H}^{2,2}\). For the sake of simplicity, we may assume that \((u(t), v(t))\) remains into the \(\mathbb{H}^{2,2}\)-absorbing set, \(\mathcal{B}_\alpha\), for \(t \geq 0\).

The first part of our second main result, is stated as follows

**Theorem 4.1.** The global attractor \(\mathcal{A}_\alpha\) associated to the semi-group \((\mathcal{S}_\alpha(t))_{t \in \mathbb{R}^+}\) is in fact a compact subset of \(\mathbb{H}^\alpha\).

4.1. The auxiliary problem. We introduce, for a given level \(N > 0\), the orthogonal projector \(P_N\) acting in \(L^2(\mathbb{R})\) by setting

\[
P_N(u) = \int_{\mathbb{R}} \chi \left(\frac{\xi}{N}\right) \mathcal{F}(u)(\xi)e^{i\xi d}d\xi = \mathcal{F}^{-1} \left(\chi \left(\frac{\xi}{N}\right) \mathcal{F}(u)(\xi)\right).
\]

where \(\chi\) is the characteristic function of the interval \([-1, 1]\). Actually, \(P_N\) is the projector onto the low-frequencies modes of a given function, at level \(N\). Clearly, \((-\Delta)^{\frac{3}{2}} P_N = P_N (-\Delta)^{\frac{3}{2}}\). Moreover, \(P_N\) and \(Q_N = Id - P_N\) are bounded operators from \(H^s, s \geq 0\), into itself and satisfy the following statements that read

**Lemma 4.2.** Let \(0 \leq s_1 \leq s_2\). Then there exists \(C > 0\), that does not depends on \(N\), such that

\[
\left\|(-\Delta)^{\frac{3}{2}} P_N(u)\right\|_{L^2} \leq C N^{s_2-s_1} \left\|(-\Delta)^{\frac{3}{2}} P_N(u)\right\|_{L^2}, \quad \forall u \in H^{s_1}.
\]

\[
\left\|(-\Delta)^{\frac{3}{2}} Q_N(u)\right\|_{L^2} \leq C N^{s_2-s_1} \left\|(-\Delta)^{\frac{3}{2}} Q_N(u)\right\|_{L^2}, \quad \forall u \in H^{s_2}.
\]

Moreover, for \(1 < p < +\infty\), \(P_N\) extends to a bounded operator from \(L^p(\mathbb{R})\) into itself whose norm does not depends on \(N\).

**Proof of Lemma 4.2.** The first part follows merely from the very definition of \(P_N\) and some well known properties from Fourier analysis (see [16]). While for the second part the proof is classical and we refer the reader to [38] for more details.

To highlight the regularity of the global attractor \(\mathcal{A}_\alpha\), the strategy consists on splitting the solution as

\((u(t), v(t)) = (P_N(u(t)), P_N(v(t)) + (Q_N(u(t)), Q_N(v(t)))\)

and then, thanks to Lemma 4.2, the regularity of \((u, v)\) depends only on the regularity of \((Q_N(u), Q_N(v))\). Thus, we shall focus on the long-time behavior of \((Q_N(u), Q_N(v))\), solution for \(Q_N(1)\) supplemented with initial data \((Q_N(u_0), Q_N(v_0))\), that will be approximated by a more regular function \((z_u, z_v)\) solving the auxiliary system that reads

\[
\begin{align*}
(z_u)_t - i(-\Delta)^{\frac{3}{2}} z_u + \gamma z_u + 2i\beta \left[ (P_N(u) + z_u) (P_N(v) + z_v) \right] \\
+ iQ_N \left[ \left| (P_N(u) + z_u)^2 + (P_N(v) + z_v) \right| (P_N(u) + z_u) \right] = Q_N(f) \quad (23a) \\
(z_v)_t - i(-\Delta)^{\frac{3}{2}} z_v + \gamma z_v + i\beta Q_N \left( (P_N(u) + z_u)^2 \right) \\
+ iQ_N \left[ \left| (P_N(u) + z_u)^2 + (P_N(v) + z_v) \right| (P_N(v) + z_v) \right] = Q_N(g) \quad (23b) \\
z_u(0) = 0, \quad z_v(0) = 0. \quad (23c)
\end{align*}
\]
Now, we state the main result of this subsection.

**Proposition 4.3.** There exists $N_0 > 0$ large enough depending only on $\gamma$, $\beta$, $\alpha$, $\|f\|_{L^2}$ and $\|g\|_{L^2}$ such that for any $N \geq N_0$, the problem (23) has a unique local in time solution $(z_u, z_v) \in \mathcal{C}(\mathbb{R}^+, \mathbb{H}^\alpha)$ that remains uniformly bounded in $\mathbb{H}^\frac{\alpha}{2}$.

**Proof of Proposition 4.3.** The proof is standard and then sketched for the sake of conciseness. The existence of a local in time solution $(z_u, z_v)$ in $\mathcal{C}([0, T^*), \mathbb{H}^\alpha)$ for the problem (23) is standard and obtained by a fixed point argument thanks to Lemma 2.3 which ensure that $\mathbb{H}^\alpha$ is an algebra.

Taking the inner products of (23a) by $-i((z_u)_t + \gamma z_u)$ and (23b) by $-i((z_v)_t + \gamma z_v)$, then summing up the resultant equations we obtain that

\[
\frac{1}{2} \frac{d}{dt} \Phi(z_u(t), z_v(t)) + \gamma \Phi(z_u(t), z_v(t)) = \Psi(z_u(t), z_v(t))
\]

where we set

\[
\Phi(z_u, z_v) = \frac{1}{\gamma} \Phi(z_u, z_v) + \gamma \Phi(z_u, z_v) - 2 \beta \left( (P_N(u) + z_u)^2, (P_N(v) + z_v) \right)
\]

\[
-2 \left( (P_N(u) + z_u)^2, (P_N(v) + z_v) \right) dt
\]

\[
\Psi(z_u, z_v) = \frac{1}{\gamma} \int \Phi(z_u, z_v) d\gamma - \beta \left( (P_N(u) + z_u)^2, (P_N(v) + z_v) \right)
\]

\[
- \beta \left( (P_N(u) + z_u)^2, (P_N(v) + z_v) \right) dt
\]

For the purpose of establishing an upper and a lower bounds for $\Phi(z_u, z_v)$, it should be noted that due to Lemma 2.3 and Lemma 4.2 we have

\[
\|h\|_{L^2} \leq \frac{1}{\sqrt{2}} \|\Phi(h)\|_{L^2}
\]

\[
\|h\|_{L^4} \leq \frac{1}{\sqrt{2}} \|\Phi(h)\|_{L^4}
\]

where $h$ denotes either $z_u$ or $z_v$. Now we need the following result

**Lemma 4.4.** There exists $C > 0$ depending only on $\gamma$, $\beta$, $\alpha$, $\|f\|_{L^2}$ and $\|g\|_{L^2}$ such that

\[
\|P_N(u)\|_{H^\frac{\alpha}{2}} + \|P_N(v)\|_{H^\frac{\alpha}{2}} + \|P_N(u_t)\|_{H^{-\frac{\alpha}{2}}} + \|P_N(v_t)\|_{H^{-\frac{\alpha}{2}}} \leq C
\]

**Proof of Lemma 4.4.** Thanks to Lemma 4.2, $P_N$ is uniformly (in $N$) bounded from $H^\frac{\alpha}{2}$ into itself. Moreover, in accordance with (1), $(P_N(u), P_N(v))$ satisfies

\[
P_N(u_t) = i(-\frac{\alpha}{2}) P_N(u) - \gamma P_N(u) + P_N(f) - iP_N \left( (|u|^2 + |v|^2)u \right) - 2i\beta P_N (\bar{w}v)
\]

\[
P_N(v_t) = i(-\frac{\alpha}{2}) P_N(v) - \gamma P_N(v) + P_N(g) - iP_N \left( (|u|^2 + |v|^2)v \right) - i\beta P_N (u\bar{w})
\]

from which, and the fact that $H^\frac{\alpha}{2}$ is an algebra (Lemma 2.3), we deduce that either $P_N(u_t)$ or $P_N(v_t)$ remain uniformly bounded in $H^{-\frac{\alpha}{2}}$ and the proof is then completed.
Thanks to the Cauchy-Schwarz inequality, (27), (28) and Lemma 4.4 we obtain, for \( N \) large enough, the existence of \( C_0, C_1 > 0 \) that do not depend on \( N \) such that

\[
\frac{1}{2} \left\| (-\Delta)^{\frac{1}{2}} (z_u, z_v) \right\|_{L^2}^2 - C_0 \left( \frac{\left\| (-\Delta)^{\frac{1}{2}} (z_u, z_v) \right\|_{L^2}^4}{N^{2\alpha - 1}} \right) - C_1 \leq \Phi(z_u, z_v) \tag{30a}
\]

\[
\Phi(z_u, z_v) \leq \frac{3}{2} \left\| (-\Delta)^{\frac{1}{2}} (z_u, z_v) \right\|_{L^2}^2 + C_0 \left( \frac{\left\| (-\Delta)^{\frac{1}{2}} (z_u, z_v) \right\|_{L^2}^4}{N^{2\alpha - 1}} \right) + C_1 \tag{30b}
\]

Now we derive an upper bound for \( \Psi(z_u, z_v) \). Thanks again to Lemma 2.3 and Lemma 4.2, denoting \( h \) either \( z_u \) or \( z_v \), we have

\[
\left\| h \right\|_{L^2} \leq \frac{1}{N^{\alpha - 1}} \left\| (-\Delta)^{\frac{1}{2}} h \right\|_{L^2}^2 . \tag{31}
\]

Independently,

\[
\left\| \left( \left\| P_N(u) + z_u \right\|^2 + \left\| P_N(v) + z_v \right\|^2 \right) (P_N(u) + z_u), P_N(u) \right\|
+ \left\| \left( \left\| P_N(u) + z_u \right\|^2 + \left\| P_N(v) + z_v \right\|^2 \right) (P_N(v) + z_v), P_N(v) \right\|
\leq \left\| \left( \left\| P_N(u) + z_u, P_N(v) + z_v \right\|^2 \right) (P_N(u) + z_u, P_N(v) + z_v) \right\|_{H^{\frac{1}{2}}}
\times \left( \left\| P_N(u) \right\|_{H^{-\frac{1}{2}}} + \left\| P_N(v) \right\|_{H^{-\frac{1}{2}}} \right)
\lesssim \left( \left\| P_N(u) + z_u \right\|_{L^\infty} + \left\| P_N(v) + z_v \right\|_{L^\infty} \right) \left\| (P_N(u) + z_u, P_N(v) + z_v) \right\|_{H^{\frac{1}{2}}}
\times \left( \left\| P_N(u) \right\|_{H^{-\frac{1}{2}}} + \left\| P_N(v) \right\|_{H^{-\frac{1}{2}}} \right)
\lesssim \left( 1 + \left\| (-\Delta)^{\frac{1}{2}} (z_u, z_v) \right\|_{L^2} \right) \left( 1 + \left\| (-\Delta)^{\frac{1}{2}} (z_u, z_v) \right\|_{L^2} \right) . \tag{32}
\]

Gathering (27), (28) and (32) it may be deduced, in accordance with the Young inequality, that

\[
\Psi(z_u(t), z_v(t)) \leq C_0 + \frac{\gamma}{4} \left\| (-\Delta)^{\frac{1}{2}} (z_u, z_v) \right\|_{L^2}^2 + \frac{C_1}{N^{\alpha - 1}} \left\| (-\Delta)^{\frac{1}{2}} (z_u, z_v) \right\|_{L^2}^3
+ \frac{C_2}{N^{2\alpha - 1}} \left\| (-\Delta)^{\frac{1}{2}} (z_u, z_v) \right\|_{L^2}^4 . \tag{33}
\]

In accordance with (24), (30) and (33), the Gronwall Lemma implies that for \( N \) chosen large enough

\[
\Phi(z_u(t), z_v(t)) \leq C_0 + \frac{C_1}{N^{\alpha - 1}} \int_0^t e^{-\gamma(t-s)} \left\| (-\Delta)^{\frac{1}{2}} (z_u(s), z_v(s)) \right\|_{L^2}^3 ds
+ \frac{C_2}{N^{2\alpha - 1}} \int_0^t e^{-\gamma(t-s)} \left\| (-\Delta)^{\frac{1}{2}} (z_u(s), z_v(s)) \right\|_{L^2}^4 ds
\]

Hence, thanks again to (30), we deduce by classical arguments that

\[
\sup_{s \in [0, T^*]} \left( \left\| (-\Delta)^{\frac{1}{2}} z_u(s) \right\|_{L^2}^2 + \left\| (-\Delta)^{\frac{1}{2}} z_v(s) \right\|_{L^2}^2 \right) \leq C
\]

which concludes the proof of the current proposition. \( \square \)

We now prove that \( (z_u, z_v) \) remains bounded in \( H^\alpha \), with an upper bound that depends on \( N \).
Proposition 4.5. There exist a real $K > 0$ and $N_0 > 0$ large enough that depend only on $\gamma$, $\beta$, $\alpha$, $\|f\|_{L^2}$ and $\|g\|_{L^2}$ such that for any $N \geq N_0$, the following estimate holds true
\[
\sup_{t \in \mathbb{R}^+} (\|z_u(t)\|_{H^\infty} + \|z_v(t)\|_{H^\infty}) \leq K N^{\frac{2}{\alpha}}.
\]

Proof of Proposition 4.5. In order to prove that $((\Delta)^{\alpha} z_u, (\Delta)^{\alpha} z_v)$ is bounded in $L^2$ we will prove equivalently that $(Y_u, Y_v) = ((z_u)_t, (z_v)_t)$ is bounded in $L^2$. To do this we will establish some a priori estimates on $\|(Y_u, Y_v)\|_{L^2}$ which can be proved rigorously using some smooth approximation argument. For the sake of simplicity we denote $\Lambda_u = \mathcal{P}_N(u) + z_u$ and $\Lambda_v = \mathcal{P}_N(v) + z_v$.

First we differentiate the system (23) with respect to $t$. Taking the scalar product of the first resulting equation by $-i((Y_u)_{t} + \gamma Y_v)$ and the second one by $-i((Y_v)_{t} + \gamma Y_u)$ lead, by mere computations, to
\[
\frac{1}{2} \frac{d}{dt} \mathcal{T}(Y_u(t), Y_v(t)) + \gamma \mathcal{T}(Y_u(t), Y_v(t)) = \Theta(Y_u(t), Y_v(t))
\]
where
\[
\mathcal{T}(Y) = \|((\Delta)^{\frac{\alpha}{2}} (Y_u, Y_v))\|_{L^2}^2 - 4\beta (\Lambda_u Y_u, Y_v) - 2\beta (Y_u^2, Y_v) - 4\beta (\Lambda_u Y_v, \Lambda_v Y_v) - 2\beta (Y_u^2, Y_v) - 4\beta (\Lambda_u Y_v, \Lambda_v Y_v, P_N(u_t), P_N(v_t)) - 4 (\mathcal{R}(\Lambda_u Y_u, \Lambda_v Y_v, P_N(u_t), P_N(v_t)))
\]
and
\[
\Theta(Y) = -2\beta \gamma (\Lambda_u Y_u + \Lambda_u Y_v, P_N(u_t), P_N(v_t)) - 2\beta (Y_u (P_N(v_t) + Y_v, P_N(u_t))) - 2 \gamma (\mathcal{R}(\Lambda_u Y_u, \Lambda_v Y_v, P_N(u_t), P_N(v_t))) - (\Lambda_u^2 + |\Lambda_u|^2) Y_v Y_u Y_u (\Lambda_u) + \mathcal{R}(\Lambda_u Y_u, \Lambda_v Y_v, P_N(u_t), P_N(v_t)) - 2\beta (Y_u (P_N(u_t) + Y_u, P_N(v_t))) - 2 \gamma (\mathcal{R}(\Lambda_u Y_u, \Lambda_v Y_v, P_N(u_t), P_N(v_t))) - 2 \gamma (\mathcal{R}(\Lambda_u Y_u, \Lambda_v Y_v, P_N(u_t), P_N(v_t))) - 2 \gamma (\mathcal{R}(\Lambda_u Y_u, \Lambda_v Y_v, P_N(u_t), P_N(v_t))) - 2 \gamma (\mathcal{R}(\Lambda_u Y_u, \Lambda_v Y_v, P_N(u_t), P_N(v_t)))
\]
First of all, thanks to Lemma 2.3, Lemma 4.2 and since $(\Lambda_u, \Lambda_v)$ is uniformly bounded in $H^{\frac{\alpha}{2}}$, we have on the one hand
\[
\|((\Lambda_u Y_u, Y_v)) + (Y_u^2, \Lambda_v)\| \leq (\|\Lambda_u\|_{L^\infty} + \|\Lambda_v\|_{L^\infty}) (\|Y_u\|_{L^2}^2 + \|Y_v\|_{L^2}^2) \leq \frac{1}{N^\alpha} \|(\Delta)^{\frac{\alpha}{2}} (Y_u, Y_v)\|_{L^2}^2.
\]
On the other hand
\[
\left(\left|\langle \Lambda, \Lambda \rangle \right|^2, \left|\left(Y_u, Y_v\right)\right|^2\right) - 2 \left|\left(\Re(\overline{\Lambda}u), \Re(\overline{\Lambda}v)\right)\right|^2
\lesssim \left\|\left|\Lambda_u\right|_2 + \left|\Lambda_v\right|_2\right\|_2^2 \left(\left|\left|Y_u\right|_2 + \left|Y_v\right|_2\right|^2\right)
\lesssim \frac{1}{N^{\alpha - \frac{2}{2}}} \left|\left|\left(-\Delta\right)^{\frac{3}{2}}(Y_u, Y_v)\right|\right|^2.
\] (38)

Moreover,
\[
\left|\left|\left(\left|\Lambda\right|^2 + \left|\Lambda\right|, \overline{Y}uP_N(u) + \overline{Y}vP_N(v)\right)\right| + \left|\left|\left(\Re(\overline{\Lambda}u + \Lambda v), \overline{\Lambda}uP_N(u) + \overline{\Lambda}vP_N(v)\right)\right| + \left|\left|\left(\overline{Y}u\Lambda_v, P_N(v)\right)\right|\right|
\lesssim \left|\left|\left(-\Delta\right)^{\frac{3}{2}}(Y_u, Y_v)\right|\right|^2.
\] (39)

As a result of (37), (38), (39) and (35) it follows, for \(N\) large enough, that
\[
\frac{1}{2} \left|\left|\left(-\Delta\right)^{\frac{3}{2}}(Y_u, Y_v)\right|\right|^2 - C \leq \gamma(Y_u, Y_v) \leq \frac{3}{2} \left|\left|\left(-\Delta\right)^{\frac{3}{2}}(Y_u, Y_v)\right|\right|^2 + C
\] (40)

where \(C > 0\) does not depend on \(N\).

We will now proceed to determinate an upper bound for \(\Theta(Y_u, Y_v)\).

To begin with, thanks to Lemma 4.4 and Proposition 4.3 one easily obtain the existence of \(K_0 > 0\) depending only on \(\left|\left|f\right|\right|_2, \left|\left|g\right|\right|_2, \gamma, \beta\) and \(\alpha\) such that
\[
\sup_{t \in \mathbb{R}^+} \left(\left|\left|\Lambda_u\right|_2 - \frac{2}{\alpha}ight| + \left|\left|\Lambda_v\right|_2 - \frac{2}{\alpha}\right|\right) \leq K_0.
\] (41)

Now by the use of (41), Lemma 2.3, Lemma 4.2 and in accordance with Proposition 4.3, we have
\[
\left|\left|\left(\Re(\overline{\Lambda}uY_u + \overline{\Lambda}vY_v), \overline{Y}u(\Lambda_u) + \overline{Y}v(\Lambda_v)\right)\right|\right|
\lesssim \left|\left|\left|Y_u\right|^2 + Y_uY_v(\Lambda_u + \Lambda_v)\right|\right|_{H^{\frac{1}{2}}} \left(\left|\left|\left|\Lambda_u\right|_2 - \frac{2}{\alpha}\right| + \left|\left|\left|\Lambda_v\right|_2 - \frac{2}{\alpha}\right|\right|_{H^{\frac{1}{2}}}\right)
\lesssim \left(\left|\left|Y_u\right|_{L^\infty} + \left|\left|Y_v\right|_{L^\infty}\right|\right|_{H^{\frac{1}{2}}} \left(\left|\left|\left|\left|Y_u\right|\right|_2 - \frac{2}{\alpha}\right| + \left|\left|\left|\left|Y_v\right|\right|_2 - \frac{2}{\alpha}\right|\right|_{H^{\frac{1}{2}}}\right)
\lesssim \frac{1}{N^{\alpha - \frac{2}{2}}} \left|\left|\left(-\Delta\right)^{\frac{3}{2}}(Y_u, Y_v)\right|\right|^2.
\] (42)

By similar arguments we obtain
\[
\left|\left|\left(Y_u^2, (\Lambda_u)\right)\right| + \left|\left|Y_u^2, P_N(u)\right|\right| + \left|\left|Y_vP_N(v)\right|\right|
\lesssim \frac{1}{N^{\alpha - \frac{2}{2}}} \left|\left|\left(-\Delta\right)^{\frac{3}{2}}(Y_u, Y_v)\right|\right|^2.
\] (43)
 Independently, recalling that \((\Lambda_u)_t = Y_u + P_N(u_t)\) and \((\Lambda_v)_t = Y_v + P_N(v_t)\) we easily check that

\[
(\Re(\overline{\Lambda_u} P_N(u_t) + \overline{\Lambda_v} P_N(v_t)), Y_u(\Lambda_u)_t + \overline{Y_v}(\Lambda_v)_t) + (Y_v P_N(\overline{u_t}), (\Lambda_u)_t)
\]

\[
+ (\Re(\overline{\Lambda_u} Y_u + \overline{\Lambda_v} Y_v), P_N(\overline{\pi}) (\Lambda_u)_t + P_N(\overline{\pi})(\Lambda_v)_t)
\]

\[
+ (\Re(\overline{\Lambda_u}(\Lambda_u)_t + \overline{\Lambda_v}(\Lambda_v)_t), Y_u P_N(u_t) + \overline{Y_v} P_N(v_t))
\]

\[= (\overline{\Lambda_u} P_N(u_t) + \overline{\Lambda_v} P_N(v_t), |Y_u|^2 + |Y_v|^2) + (Y_u Y_v, P_N(u_t)) \tag{44}
\]

\[
+ 2(\Re(\overline{\Lambda_u} P_N(u_t) + \overline{\Lambda_v} P_N(v_t)), Y_u P_N(u_t) + \overline{Y_v} P_N(v_t))
\]

\[
+ (\overline{\Lambda_u} Y_u + \overline{\Lambda_v} Y_v, |P_N(\overline{u_t})|^2 + |P_N(\overline{\pi})|^2) + (Y_v, P_N(u_t)^2) .
\]

Due to Lemma 2.3 and Lemma 4.2 one has

\[
|||(\Re(\overline{\Lambda_u} P_N(u_t) + \overline{\Lambda_v} P_N(v_t)), Y_u P_N(u_t) + \overline{Y_v} P_N(v_t))|||
\]

\[
+ ||(\overline{\Lambda_u} Y_u + \overline{\Lambda_v} Y_v, P_N(\overline{\pi}) (\Lambda_u)_t + P_N(\overline{\pi})(\Lambda_v)_t)|| + ||(Y_v, P_N(u_t))||^2
\]

\[\lesssim ||Y_u||_{L^\infty} + ||Y_v||_{L^\infty} + ||\Lambda_u||_{L^\infty} + ||\Lambda_v||_{L^\infty} ||(P_N(u_t), P_N(v_t))||_{L^2}^2
\]

\[\lesssim N^{\frac{a+1}{2}} \frac{\|(-\Delta)^{\frac{a}{2}} (Y_u, Y_v)\|_{L^2}}{N^{\frac{a+1}{2}}} . \tag{45}
\]

Hence, in the light of (42) and (44) and (45), we deduce that

\[
(\Re(\overline{\Lambda_u} P_N(u_t) + \overline{\Lambda_v} P_N(v_t)), Y_u(\Lambda_u)_t + \overline{Y_v}(\Lambda_v)_t)
\]

\[
+ (\Re(\overline{\Lambda_u} Y_u + \overline{\Lambda_v} Y_v), P_N(\overline{\pi}) (\Lambda_u)_t + P_N(\overline{\pi})(\Lambda_v)_t) + (Y_v P_N(u_t), P_N(u_t))
\]

\[\leq N^{\frac{a+1}{2}} \frac{\|(-\Delta)^{\frac{a}{2}} (Y_u, Y_v)\|_{L^2}}{N^{\frac{a+1}{2}}} . \tag{46}
\]

Now it only remains to bound the terms containing either \(P_N(u_t)\) or \(P_N(v_t)\). For that purpose, observe that \(P_N(u_t)\) and \(P_N(v_t)\) satisfy the following system that reads

\[
\begin{align*}
P_N(u_t) &= -i(\Delta)^{\frac{a}{2}} P_N(u_t) - \gamma P_N(u_t) - 2i\beta [P_N(\overline{\pi} v) + P_N(\overline{\pi} v)]
\end{align*}
\]

\[
- i P_N \left( |u|^2 + |v|^2 \right) u_t - 2i P_N \left( |\Re(u \overline{\pi}) + \Re(v \overline{\pi})| \right) u_t
\]

\[
P_N(v_t) = -i(\Delta)^{\frac{a}{2}} P_N(v_t) - \gamma P_N(v_t) - 2i\beta P_N(u_t)
\]

\[
- i P_N \left( |u|^2 + |v|^2 \right) v_t - 2i P_N \left( |\Re(u \overline{\pi}) + \Re(v \overline{\pi})| \right) v_t .
\]

Consequently, due to the fact that \(H^{\frac{a}{2}}\) is an algebra, either

\[
P_N \left( |u|^2 + |v|^2 \right) u_t , P_N \left( |u|^2 + |v|^2 \right) v_t \quad \text{and} \quad [P_N(\overline{\pi} v) + P_N(\overline{\pi} v)]
\]

or \(P_N \left( |\Re(u \overline{\pi}) + \Re(v \overline{\pi})| \right) u_t \), \(P_N \left( |\Re(u \overline{\pi}) + \Re(v \overline{\pi})| \right) v_t \) and \(P_N(\overline{u} u_t)\) are uniformly bounded in \(H^{-\frac{a}{2}}\). Hence, due to Lemma 4.2 and Lemma 4.4

\[
||P_N(u_t)||_{H^{-\frac{a}{2}}} + ||P_N(v_t)||_{H^{-\frac{a}{2}}} \lesssim N^\alpha . \tag{48}
\]
This enables us, in accordance with (36), to have
\[
\left| (\overline{\Lambda}u Y_u + \overline{\Lambda}v Y_v, P_N(u_t)) \right| + \left| (\overline{\Lambda}u Y_u, P_N(v_t)) \right|
\]
\[
+ \left| (\Re(\overline{\Lambda}u Y_u + \overline{\Lambda}v Y_v), \overline{\Lambda}u P_N(u_t) + \overline{\Lambda}v P_N(v_t)) \right| 
\]
\[
\lesssim \left| (\overline{\Lambda}u + \Lambda_v)^2 (Y_u, Y_v) \right| H^{\frac{3}{2}} \left( ||P_N(u_t)||_{H^{-\frac{3}{2}}} + ||P_N(v_t)||_{H^{-\frac{3}{2}}} \right)
\]
\[
\lesssim N^{\alpha} \left| (\overline{\Lambda}u, Y_v) \right|_{L^2}. \tag{49}
\]
Gathering (37), (39), (42), (46) and (49), it may be deduced in accordance with the
Young inequality that for \(N\) large enough
\[
\Theta(Y_u, Y_v) \leq \frac{\gamma}{4} \left| (\overline{\Delta} Y_u, Y_v) \right|_{L^2}^2 + CN^{2\alpha}
\]
This, with (40), leads to
\[
\frac{d}{dt} \Upsilon(Y_u(t), Y_v(t)) + \gamma \Upsilon(Y_u(t), Y_v(t)) \leq C N^{2\alpha}
\]
which completes the proof thanks to the Gronwall Lemma, again (40) and Lemma 4.2.

4.2. Proof of Theorem 4.1. To begin with, we proceed to the large time comparison between
\[
(u, v) = (P_N(u) + Q_N(u), P_N(v) + Q_N(v))
\]
and
\[
(\Lambda_u, \Lambda_v) = (P_N(u) + z_u, P_N(v) + z_v)
\]
where \((z_u, z_v)\) stands for the unique solution of the auxiliary system (23).

**Proposition 4.6.** There exist \(N_0, C > 0\) depending only on \(\gamma, \beta, \alpha, \|f\|_{L^2}\) and
\(\|g\|_{L^2}\) such that for any \(N \geq N_0\) and for all \(t \geq 0\),
\[
||u(t) - \Lambda_u(t)||_{H^{\frac{3}{2}}}^2 + ||v(t) - \Lambda_v(t)||_{H^{\frac{3}{2}}}^2 \leq C e^{-\gamma t}.
\]

**Proof of Proposition 4.6.** The proof is classical and then details are omitted. For the sake of simplicity we shall denote
\[
\begin{cases}
\Theta_u(t) = u(t) - \Lambda_u(t) = Q_N(u(t)) - z_u(t) \\
\Theta_v(t) = v(t) - \Lambda_v(t) = Q_N(v(t)) - z_v(t).
\end{cases}
\]
Then, by mere computations, \((\Theta_u, \Theta_v)\) satisfies the following equation that reads
\[
\frac{1}{2} \frac{d}{dt} \Xi(\Theta_u(t), \Theta_v(t)) + \gamma \Xi(\Theta_u(t), \Theta_v(t)) = \Gamma(\Theta_u(t), \Theta_v(t)) \tag{51}
\]
where we set
\[
\Xi(\Theta_u, \Theta_v) = ||(\overline{\Delta} \hat{\Theta})(\Theta_u, \Theta_v)||_{L^2}^2 - 2 \left( ||\Re(\overline{\Lambda}u \Theta_u)||_{L^2}^2 + ||\Re(\overline{\Lambda}v \Theta_v)||_{L^2}^2 \right)
\]
\[
- (||u||^2 + ||v||^2, ||\Theta_u||^2 + ||\Theta_v||^2) - 2\beta (\Theta_u \Theta_v) - 4\beta (\Theta_u \Lambda_u, \Theta_v)
\]
\[
\Gamma(\Theta_u, \Theta_v) = \gamma (||\Theta_u||^2 + ||\Theta_v||^2, \overline{\Lambda}u \Theta_u + \overline{\Lambda}v \Theta_v) + \gamma \beta (\Theta_u^2, \Theta_v)
\]
\[
+ (||\Theta_u||^2 + ||\Theta_v||^2, \overline{\Lambda}u \Theta_u + \overline{\Lambda}v \Theta_v) - 2\beta (\overline{\Theta}_u \Theta_v, (\Lambda_u)_t)
\]
\[
- (||\Theta_u||^2 + ||\Theta_v||^2, \overline{\Lambda}u(\Theta_u)_t + \overline{\Lambda}v(\Theta_v)_t) + \beta (\Theta_u^2, (\Theta_v)_t)
\]
\[
- 2(\Re(\overline{\Lambda}u \Theta_u + \overline{\Lambda}v \Theta_v), \overline{\Theta}_u (\Lambda_u)_t + \overline{\Theta}_v (\Lambda_v)_t) - \beta (\Theta_u^2, v_t). \tag{53}
\]
Thanks to Lemma 4.4 and Proposition 4.3, it may be deduced that $u_t$, $v_t$, $(\Lambda u)_t$ and $(\Lambda v)_t$ remain uniformly bounded in $H^{-\frac{\alpha}{2}}$. Then applying Lemma 2.3 and Lemma 4.2 by means of which, using similar computations as (37), (42) and (43), we conclude on the one hand that for $N$ chosen large enough
\[
\frac{1}{2} \left| \left| (\Delta)^{\frac{\alpha}{4}} (\Theta u, \Theta v) \right| \right|_{L^2}^2 \leq \Xi(\Theta u, \Theta v) \leq \frac{3}{2} \left| \left| (\Delta)^{\frac{\alpha}{4}} (\Theta u, \Theta v) \right| \right|_{L^2}^2
\] (54)
and on the other hand
\[
\Gamma(\Theta u, \Theta v) \lesssim \left| \left| (\Delta)^{\frac{\alpha}{4}} (\Theta u, \Theta v) \right| \right|_{L^2}^2 \leq \frac{2}{3} \left| \left| (\Delta)^{\frac{\alpha}{4}} (\Theta u, \Theta v) \right| \right|_{L^2}^2 .
\] (55)
Gathering (54), (55) and (51) completes the proof thanks to the Gronwall Lemma.

Propositions 4.6, 4.3 and 4.5 enable us to prove, in identical manner as in [1] or [14], that $\mathcal{A}_\alpha$ is a bounded subset of $H^\alpha$. The proof of the compactness of the global attractor $\mathcal{A}_\alpha$ in $H^\alpha$, based on the famous J. Ball’s argument, is standard and similar to that in [14] to which we refer the reader. We omit it for the sake of conciseness and the proof of Theorem 4.1 is therefore completed.

5. Fractal dimension of the global attractor. Firstly, for the sake of completeness we start by recalling the definition of the fractal dimension.

**Definition 5.1.** The fractal dimension of a compact subset $\mathcal{M}$ of a metric space $\mathcal{H}$ is defined by
\[
d_f(\mathcal{M}) = \lim_{\epsilon \to 0} \frac{\ln N(\mathcal{M}, \epsilon)}{\ln (\frac{1}{\epsilon})},
\]
where $N(\mathcal{M}, \epsilon)$ denotes the minimal number of closed balls of the radius $\epsilon$ which cover the set $\mathcal{M}$.

We now state the main result of the current section

**Theorem 5.2.** Let $f, g \in L^2$. Then the compact global attractor $\mathcal{A}_\alpha$ possesses a finite fractal dimension in $H^\alpha$.

5.1. Some helpful tools. In order to prove Theorem 5.2 we begin by recalling a merely deduced result from that given in [10].

**Theorem 5.3.** Let $\mathcal{X}$ be a Banach space and $M$ be a bounded closed set in $\mathcal{X}$. Assume that there exists a mapping $S : M \to \mathcal{X}$ such that
1. $M \subseteq S(M)$. Moreover, $S$ is Lipschitz on $M$, i.e, there exists $L > 0$ such that for all $u_1, u_2 \in M$,
\[
\|S(u_1) - S(u_2)\|_\mathcal{X} \leq L \|u_1 - u_2\|_\mathcal{X}.
\]
2. There exists a compact semi-norm $\|\cdot\|_\mathcal{Y}$ on $\mathcal{X}$ (i.e: $\mathcal{X} \to \mathcal{Y}$ is compact) such that for all $u_1, u_2 \in M$,
\[
\|S(u_1) - S(u_2)\|_\mathcal{X} \leq \delta \|u_1 - u_2\|_\mathcal{X} + K \|u_1 - u_2\|_\mathcal{Y} + \|S(u_1) - S(u_2)\|_\mathcal{Y}.
\]
where $0 < \delta < 1$ and $K > 0$ are constants.

Then $M$ is a compact subset of $\mathcal{X}$ and has a finite fractal dimension.

Now, we state an important result (see [2]) that will enables us to apply Theorem 5.3.
Lemma 5.4. Let $\mathcal{C}$ be a compact subset of $L^2(\mathbb{R})$. Then for every $\epsilon > 0$ there exists $R = R(\epsilon) > 0$ such that for all $\vartheta \in \mathcal{C}$ and for all $u \in H^\frac{1}{2}$ the following estimate holds
\[
\int_{\mathbb{R}} |\vartheta| |u|^2 \, dx \leq \epsilon \|u\|^2_{H^\frac{1}{2}} + R \|u\|^2_{L^2([-R,R])}.
\]

Proof of Lemma 5.4. To begin with, for arbitrary $R > 0$ we write
\[
\int_{\mathbb{R}} |\vartheta| |u|^2 \, dx = \int_{[-R,R] \cap \{|\vartheta| \leq R\}} |\vartheta| |u|^2 \, dx
+ \int_{[-R,R] \cap \{|\vartheta| > R\}} |\vartheta| |u|^2 \, dx + \int_{\{|x| > R\}} |\vartheta| |u|^2 \, dx
\]
we obtain by applying the Hölder inequality that
\[
\int_{\mathbb{R}} |\vartheta| |u|^2 \, dx \leq R \|u\|^2_{L^2([-R,R])} + \left( \|\vartheta\|_{L^2(\{|\vartheta| > R\})} + \|\vartheta\|_{L^2(\{|x| > R\})} \right) \|u\|^2_{L^2(\mathbb{R})}
\]
by means of which and the continuous embedding of $H^\frac{1}{2}$ in $L^4$ (Lemma 2.3) we deduce that
\[
\int_{\mathbb{R}} |\vartheta| |u|^2 \, dx \leq R \|u\|^2_{L^2([-R,R])} + C \left( \|\vartheta\|_{L^2(\{|\vartheta| > R\})} + \|\vartheta\|_{L^2(\{|x| > R\})} \right) \|u\|^2_{H^\frac{1}{2}}.
\]
Knowing that $\|\vartheta\|_{L^2(\{|\vartheta| > R\})} + \|\vartheta\|_{L^2(\{|x| > R\})} \to 0$ as $R \to +\infty$, we deduce that for every $\epsilon > 0$ there exists $R = R(\vartheta, \epsilon) > 0$ satisfying
\[
\int_{\mathbb{R}} |\vartheta| |u|^2 \, dx \leq \epsilon \|u\|^2_{H^\frac{1}{2}} + R \|u\|^2_{L^2([-R,R])}
\]
(56)
Since $\mathcal{C}$ is precompact in $L^2(\mathbb{R})$ then there exist finitely many points $\vartheta_1, \ldots, \vartheta_n \in \mathcal{C}$ such that $\mathcal{C} \subseteq \bigcup_{k=1}^n B_k(\vartheta_k, \epsilon)$ where
\[
B_k(\vartheta_k, \epsilon) = \{ \vartheta \in L^2(\mathbb{R}) \text{ such that } \|\vartheta - \vartheta_k\|_{L^2(\mathbb{R})} \leq \epsilon \}.
\]
Moreover, for a fixed $1 \leq k \leq n$ and for every $\epsilon > 0$, we obtain from (56) that there exists $R_k = R_k(\vartheta_k, \epsilon) > 0$ such that for all $\vartheta \in B_k(\vartheta_k, \epsilon)$ we have
\[
\int_{\mathbb{R}} |\vartheta| |u|^2 \, dx \leq \int_{\mathbb{R}} |\vartheta_k| |u|^2 \, dx + \int_{\mathbb{R}} |\vartheta - \vartheta_k| |u|^2 \, dx
\]
\[
\lesssim R_k \|u\|^2_{L^2([-R_k,R_k])} + \epsilon \|u\|^2_{H^\frac{1}{2}}.
\]
Choosing $R = \max_{1 \leq k \leq n} R_k$ achieves the proof of the lemma. \qed

5.2. Proof of Theorem 5.2. In order to check the assumptions in Theorem 5.3, we use a recent new idea introduced in [2] that will allow us to establish the following result that states as follows

Proposition 5.5. There exist $t^* > 0$ and $R^* > 0$ that depend on $\gamma$, $\beta$, $\alpha$ and $\|f\|_{L^2}$ and $\|g\|_{L^2}$ such that for all $(u_0, v_0), (\varphi_0, \psi_0) \in \mathcal{A}_\alpha$,
\[
\|(S_\alpha(t^*)u_0, v_0) - S_\alpha(t^*)(\varphi_0, \psi_0)\|^2_{H^\frac{1}{2}}
\]
\[
\leq \frac{1}{2} \|(u_0 - \varphi_0, v_0 - \psi_0)\|^2_{H^\frac{1}{2}} + L \|(u_0 - \varphi_0, v_0 - \psi_0)\|^2_{L^2([-2R^*, 2R^*])},
\]
where $L > 0$ is a non negative real constant depending only on $t^*$ and $R^*$.
proof of Proposition 5.5. Firstly, let \((u_0, v_0), (\varphi_0, \psi_0) \in \mathcal{A}_\alpha\) and we denote for the sake of simplicity \(S(t)(u_0, v_0) = (u(t), v(t))\), \(S(t)(\varphi_0, \psi_0) = (\varphi(t), \psi(t))\) and finally
\[(Z(t), Y(t)) = S(t)(u_0, v_0) - S(t)(\varphi_0, \psi_0) = (u(t) - \varphi(t), v(t) - \psi(t)).\]

Then \((Z(t), Y(t))\) satisfies the following system
\[
\begin{cases}
Z_t - i(-\Delta)^{\frac{\alpha}{2}} Z + i(|u|^2 + |v|^2) Z + 2i\beta (\overline{v}Z + \overline{\psi} Y) \\
+ i\varphi \Re \left[(\overline{u} + \overline{\varphi})Z + (\overline{v} + \overline{\psi}) Y\right] + \gamma Z = 0 \quad \text{(58a)} \\
Y_t - i(-\Delta)^{\frac{\alpha}{2}} Y + i(|u|^2 + |v|^2) Y + i\beta(u + \varphi) Z \\
+ i\psi \Re \left[(\overline{v} + \overline{\psi}) Y + (\overline{u} + \overline{\varphi}) Z\right] + \gamma Y = 0 \quad \text{(58b)}
\end{cases}
\]

\[(0, 0) = (u_0 - \varphi_0, v_0 - \psi_0) \in \mathcal{A}_\alpha. \quad \text{(58c)}
\]

Lemma 5.6. There exist \(R_0 > 0\) and \(C_1, C_2 > 0\) depending only on \(\gamma, \beta, \alpha\) and \(\mathcal{A}_\alpha\) such that
\[
\|(Z(t), Y(t))\|_{L^2}^2 \leq C_1 \|(Z_0, Y_0)\|_{L^2}^2 e^{-\gamma t} + C_2 R_0 \int_0^t e^{-\gamma (t-s)} \|(Z(s), Y(s))\|_{L^2}^2 ds. \quad \text{(59)}
\]

proof of Lemma 5.6. On the one hand, taking the scalar products of \((58a)\) by \(Z\) and \((58b)\) by \(Y\) then summing the resultant equations we obtain
\[
\frac{1}{2} \frac{d}{dt} \|(Z, Y)\|_{L^2}^2 + \gamma \|(Z, Y)\|_{L^2}^2 = \left( (\overline{u} + \varphi) Z + (\overline{v} + \psi) Y, \Im m(\varphi Z + \psi Y) \right)
+ \beta \Im m \int_Z Z^2 \overline{Y} dx + 2 \beta \Im m \int_Z \overline{v} Z^2 dx. \quad \text{(60)}
\]

On the other hand, the scalar products of \((58a)\) by \(-i(Z_t + \gamma Z)\) and \((58b)\) by \(-i(Y_t + \gamma Y)\) allow us to have
\[
\frac{1}{2} \frac{d}{dt} J(Z(t), Y(t)) + \gamma J(Z(t), Y(t)) = K(Z(t), Y(t)) \quad \text{(61)}
\]
where
\[
J(Z(t), Y(t)) = \left\|(-\Delta)^{\frac{\alpha}{2}} (Z, Y)\right\|_{L^2}^2 - 2\beta(Z, \overline{v}) - 4\beta(\varphi Z, Y)
- \left( |u|^2 + |v|^2, |Z|^2 + |Y|^2 \right) - 2 \left( \left\|\Re(\overline{\varphi} Z)\right\|_{L^2}^2 + \left\|\Re(\overline{\psi} Y)\right\|_{L^2}^2 \right) \quad \text{(62)}
\]
and
\[
K(Z(t), Y(t)) = \gamma \left( |Z|^2 + |Y|^2, \overline{\varphi} Z + \overline{\psi} Y \right) + \left( |Z|^2 + |Y|^2, \varphi Z_t + \psi Y_t \right)
- \left( |Z|^2 + |Y|^2, \overline{u} u_t + \overline{v} v_t \right) - 2\beta(\overline{Y} Z, \varphi_t) + \beta(Z, Y_t)
- 2 \left( \Re(\varphi Z + \overline{\psi} Y), \overline{Z} Z_t + \overline{Y} Y_t \right) - \beta(Z, \overline{u}_t) + \gamma \beta(Z, Y_t). \quad \text{(63)}
\]

since \(u, v, \varphi \) and \(\psi\) are uniformly bounded in \(H^\frac{\alpha}{2}\), we infer from Lemma 2.3 the existence of \(C_0 > 1\) depending only on \(\gamma, \beta, \alpha\) and \(\mathcal{A}_\alpha\) such that
\[
\left| \left( (\overline{u} + \varphi) Z + (\overline{v} + \psi) Y, \Im m(\varphi Z + \psi Y) \right) \right|
+ \left| \int_Z Z^2 \overline{Y} dx \right| + \left| \int_Z \overline{v} Z^2 dx \right| \leq C_0 \|(Z, Y)\|_{L^2}^2. \quad \text{(64)}
\]
we easily obtain the existence of

\[ L \]

Thanks to Theorem 4.1 and Lemma 2.3, Lemma 5.7.

\[ R \]

and on the other hand

we obtain, in accordance with (61), that

\[ \frac{1}{2} \frac{d}{dt} \Psi(Z(t), Y(t)) + \gamma \Psi(Z(t), Y(t)) = \Phi(Z(t), Y(t)) \] (67)

Thanks to the estimate (64), one has on the one hand

\[ \| (Z(t), Y(t)) \|_{L^2}^2 \leq \Psi(Z(t), Y(t)) \leq (1 + C_0) \| (Z(t), Y(t)) \|_{L^2}^2 \] (68)

and on the other hand

\[ \Phi(Z) \leq \int_R \left( |u|^2 + |v|^2 + |\varphi|^2 + |\psi|^2 \right) (|Z|^2 + |Y|^2) \, dx 
+ \int_R \left( |u_t| + |v_t| + |\varphi_t| + |\psi_t| \right) (|Z|^2 + |Y|^2) \, dx. \] (69)

Thanks to Theorem 4.1 and Lemma 2.3,

\[ \text{either } \{ |u|^2 + |v|^2, (u, v) \in A \} \text{ and } \{ |\varphi|^2 + |\psi|^2, (\varphi, \psi) \in A \}, \]
\[ \text{or } \{ |u_t| + |v_t|, (u, v) \in A \} \text{ and } \{ |\varphi_t| + |\psi_t|, (\varphi, \psi) \in A \} \]

are compact subsets of \( L^2 \). Hence, using the key Lemma 5.4 and in accordance with

\[ \| \Phi(Z, Y) \| \leq \frac{\gamma}{2} \| (Z, Y) \|_{L^2}^2 + C R_0 \| (Z, Y) \|_{L^2([-R_0, R_0])}^2. \]

This implies, thanks to (68), that

\[ \Phi(Z, Y) \leq \frac{\gamma}{2} \Psi(Z, Y) + C R_0 \| (Z, Y) \|_{L^2([-R_0, R_0])}^2. \] (70)

Gathering (67) and (70) achieves the proof of the lemma thanks again to (68) and the Gronwall Lemma.

\[ \square \]

**Lemma 5.7.** Let \( R > 0 \). Then there exist \( K, c_1 \) and \( c_2 > 0 \) depending only on \( \gamma, \alpha, \beta \) and \( \mathcal{A}_\alpha \) such that

\[ \| (Z(t), Y(t)) \|_{L^2([-R, R])}^2 \leq \| (Z_0, Y_0) \|_{L^2([-2R, 2R])}^2 e^{c_1 t} + K \frac{R_0}{R^{2\alpha}} e^{c_2 t} \| (Z_0, Y_0) \|_{L^2([-R_0, R_0])}^2 \] (71)

where \( R_0 \) denotes the fixed nonnegative reel given by Lemma 5.6.

**proof of Lemma 5.7.** Let \( \theta \) be a cut-off smooth function such that \( \theta(x) = 1 \) if \( |x| \leq 1 \) and \( \theta(x) = 0 \) if \( |x| \geq 2, x \in \mathbb{R} \). Now consider for a given \( R > 0 \),

\[ \theta_R : x \mapsto \theta \left( \frac{x}{R} \right), x \in \mathbb{R}. \]
We multiply (58a) and (58b) by \( \theta_R \) then we make the scalar products of the first resultant equation by \( \theta_R Z \) and the second one by \( \theta_R Y \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \left( \| \theta_R Z \|_{L^2}^2 + \| \theta_R Z \|_{L^2}^2 + \| \theta_R Y \|_{L^2}^2 \right) + \gamma \left( \| \theta_R Y \|_{L^2}^2 + \| \theta_R Y \|_{L^2}^2 \right) \\
= \left( (u + \varphi) \theta_R Z + (v + \psi) \theta_R Y, \theta_R \mathbb{R} m [\varphi Z + \psi Y] \right) \\
+ \left( \theta_R (-\Delta)^{\frac{\alpha}{2}} Z, \theta_R Z \right) + \left( \theta_R (-\Delta)^{\frac{\alpha}{2}} Y, \theta_R Y \right) \\
+ 2 \beta m \int_{\mathbb{R}} \theta_R^2 Z^2 dx + \beta \arg \int_{\mathbb{R}} \theta_R^2 Z^2 Y dx.
\]

Since \( u, v, \varphi \) and \( \psi \) are uniformly bounded in \( H^{\frac{\alpha}{2}} \) which is continuously embedded in \( L^\infty \), we infer from (72) that

\[
\frac{1}{2} \frac{d}{dt} \left( \| \theta_R Z \|_{L^2}^2 + \| \theta_R Z \|_{L^2}^2 + \| \theta_R Y \|_{L^2}^2 \right) + \gamma \left( \| \theta_R Y \|_{L^2}^2 + \| \theta_R Y \|_{L^2}^2 \right) \\
\lesssim \left( \| \theta_R Y \|_{L^2}^2 + \| \theta_R Y \|_{L^2}^2 \right) + \| \theta_R (-\Delta)^{\frac{\alpha}{2}} Z - (-\Delta)^{\frac{\alpha}{2}} (\theta_R Y) \|_{L^2} \| \theta_R Z \|_{L^2} \| \theta_R Y \|_{L^2} \| \theta_R Y \|_{L^2} \\
+ \| \theta_R (-\Delta)^{\frac{\alpha}{2}} Y - (-\Delta)^{\frac{\alpha}{2}} (\theta_R Y) \|_{L^2} \| \theta_R Y \|_{L^2} \| \theta_R Y \|_{L^2} \\
\end{align}

Knowing that \( \| \xi^\alpha \mathcal{F}(\theta_R) \|_{L^1(\mathbb{R})} \lesssim \frac{1}{R^{\alpha}} \), we deduce from Lemma 2.5

\[
\frac{1}{2} \frac{d}{dt} \left( \| \theta_R Z \|_{L^2}^2 + \| \theta_R Z \|_{L^2}^2 + \| \theta_R Y \|_{L^2}^2 \right) + \gamma \left( \| \theta_R Y \|_{L^2}^2 + \| \theta_R Y \|_{L^2}^2 \right) \\
\leq C_1 \left( \| \theta_R Z \|_{L^2}^2 + \| \theta_R Y \|_{L^2}^2 + \| \theta_R Y \|_{L^2}^2 \right) + \frac{C_2}{R^\alpha} \left( \| Z \|_{L^2} \| \theta_R Z \|_{L^2} + \| Y \|_{L^2} \| \theta_R Y \|_{L^2} \right) .
\]

Hence, thanks to the Young inequality, we obtain from (74) that

\[
\frac{d}{dt} \left( \| \theta_R Z \|_{L^2}^2 + \| \theta_R Z \|_{L^2}^2 \right) \lesssim \left( \| \theta_R Z \|_{L^2}^2 + \| \theta_R Y \|_{L^2}^2 \right) + \frac{1}{R^{2\alpha}} \left( \| Z \|_{H^{\frac{\alpha}{2}}}^2 + \| Y \|_{H^{\frac{\alpha}{2}}}^2 \right) .
\]

Independently, we infer from the scalar products of (58a) by \( Z \) and (58b) by \( Y \) that

\[
\frac{1}{2} \frac{d}{dt} \left( \| Z \|_{L^2}^2 + \| Y \|_{L^2}^2 \right) + \gamma \left( \| Z \|_{L^2}^2 + \| Y \|_{L^2}^2 \right) \lesssim \left( \| Z \|_{L^2}^2 + \| Y \|_{L^2}^2 \right) .
\]

This leads, thanks to Gronwall’s Lemma and in accordance with Lemma 5.6, to

\[
\| (Z(t), Y(t)) \|_{H^{\frac{\alpha}{2}}}^2 \leq C_1 \| (Z_0, Y_0) \|_{H^{\frac{\alpha}{2}}}^2 e^{-\gamma t} + C_2 R_0 e^{C t} \| (Z_0, Y_0) \|_{L^2}^2 .
\]

Gathering (75) and (76) achieves the proof thanks to the Gronwall Lemma.

By means of Lemma 5.6 and Lemma 5.7, we can deduce that

\[
\| (Z(t), Y(t)) \|_{H^{\frac{\alpha}{2}}}^2 \leq \left( C_1 e^{-\gamma t} + C_2 \frac{R_0}{R^{2\alpha}} e^{C t} \right) \| (Z_0, Y_0) \|_{H^{\frac{\alpha}{2}}}^2 \\
+ C_3 R_0 e^{C t} \| (Z_0, Y_0) \|_{L^2}^2 \| (Z_0, Y_0) \|_{L^2}^2 .
\]

Recalling that \( R_0 \) is given by Lemma 5.6, we choose \( t^* > 0 \) such that \( C_1 e^{-\gamma t^*} \leq \frac{1}{4} \) then \( R^* > R_0 \) large enough such that

\[
C_2 \frac{R_0}{(R^*)^{2\alpha}} e^{C t^*} \leq \frac{1}{4}
\]

achieves the proof of the current proposition as well as the proof of Theorem 5.2 by applying theorem 5.3 with \( S = S_\alpha(t^*), M = \mathcal{A}_\alpha \) an \( \mathcal{Y} = L^2([-2R^*, 2R^*]). \)
Remark 5.8. It is important to emphasize that the choice of the one-dimensional domain, $\mathbb{R}$, has played a crucial role in the study of the system (1). Therefore, addressing this issue in a domain with boundary seems to be more difficult since the Fourier characterization of the fractional derivative is no longer applicable. Moreover in higher dimension (for instance in $\mathbb{R}^2$), in addition to the fact that $H^\alpha$ ($\alpha \in (1, 2]$) is not an algebra, the loss of derivatives in the Strichartz estimates for $(-\Delta)^{\alpha/2}$ (see [19] and [17]) prevents the use of the same idea as in [14] when looking at the regularity issue, which makes the study of (1) more complicated and remains an open question.

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