Flux Periodicities and Quantum Hair on Holographic Superconductors

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Superconductors in a cylindrical geometry respond periodically to a cylinder-threading magnetic flux, with the period changing from $hc/2e$ to $hc/e$ depending on whether the Aharonov-Bohm effects are suppressed or not. We show that Holographic Superconductors present a similar phenomenon, and that the different periodicities follow from classical no-hair theorems. We also give the Ginzburg-Landau description of the period-doubling phenomenon.

**Introduction:** The study of superconductors (SCs) in multiply connected geometries has provided important applications as well as theoretical insights. The simplest configuration consists of a superconducting hollow cylinder of negligible width threaded by a magnetic flux $\Phi_B$. Two classic effects in this setup are i) the Aharonov-Bohm (AB) interference between charged quanta winding around the cylinder. This phenomenon exhibits a periodic dependence (of, say, the partition function) on the flux $\Phi_B$ with a period given by the flux quantum $\Phi_0 = h/e$ [19] where $e$ is the fundamental charge unit; ii) the fact that thermodynamic quantities such as the critical temperature $T_c$ acquire a periodicity in the flux with period $\Delta \Phi_B = \Phi_0/2$ [1], which represents a hallmark of pairing in the SC. For short, we shall refer to this as the ‘Little-Parks’ (LP) period. Recently, considerable theoretical work has discussed how the LP periodicity should return to the quantum bound $\Phi_0$ for cylinder radius $R$ comparable to the coherence length $\xi$ [2–4]. An appropriate statement of the LP result, then, is that the two $h/2e$ sub-periods are actually degenerate for $R \gg \xi$. In this Letter, we describe the LP degeneracy and breaking thereof occurring in Holographic Superconductors (HSC) [5]. As we shall see, the dual LP effect emerges from Black Hole no-hair theorems.

**Discrete Gauge Symmetries.** Superconductivity is perhaps the simplest realization of a discrete gauge symmetry. Indeed, in a system with two fields $\xi$ and $\phi$ (either fundamental or composite) with $U(1)$ gauge charges $e$ and $g \equiv N e$ respectively with $N$ an integer, whenever the high charge field $\phi$ condenses, then a $Z_N$ gauge subgroup is unbroken and realized nontrivially by $\xi$. In pairing-based SCs, $N = 2$ and the discrete charge amounts to the total number of electrons in the SC being even or odd. This charge is in principle measurable e.g. by AB interference of magnetic flux tubes lasing the SC [6–8].

Once the SC presents a nontrivial topology, a complete description of the system may require non-local gauge invariant objects, such as the Wilson line $W = \exp(i \oint a_{\mu}/h)$ where $a_{\mu}$ is the gauge potential and the integral runs over a non-contractible circle. For a cylinder of radius $R$ the integral in the circular direction $\chi$ gives $W(\alpha_{\chi}) = \exp(2\pi i R \alpha_{\chi}/h)$, a measure of the magnetic flux within the cylinder, $\exp(2\pi i \Phi_B/\Phi_0)$. Another gauge invariant non-local object is the winding number (or ‘fluidic’) $M \equiv \oint dx^a \partial_a \theta/2\pi$ where $\theta$ is the phase of the condensing field, $\phi = \rho e^{i\theta}$, the integral taken on the same path. $M$ parametrizes the momentum carried by the condensate in the $\chi$ direction and satisfies the quantization condition $M = m \in \mathbb{Z}$, stemming from the single-valuedness of $\phi$’s ‘wavefunction’. This winding number is well defined whenever the $U(1)$ is Higgsed. If a $Z_N$ group is left unbroken then $m$ is effectively defined modulo $N$. Therefore, in general there are $N$ distinct fluidic configurations $\phi = \rho e^{m\theta}/R$ for $\rho \neq 0$. The LP period implies that the $N$ fluidics are degenerate for large enough $R$.

**(Uplifting) The LP Degeneracy in Ginzburg-Landau.** Let us now review the LP effect in terms of the Ginzburg-Landau (GL) theory. The behaviour of the GL order parameter $\phi_{GL}$ in thermodynamic equilibrium is obtained by minimizing an effective Lagrangian, which close enough to the transition is approximated by

$$\mathcal{L}[\phi_{GL}] = |D_\mu \phi_{GL}|^2 - M^2 |\phi_{GL}|^2 - b|\phi_{GL}|^4$$

with $D_\mu \phi_{GL} = (\partial_\mu - ig a_\mu) \phi_{GL}$ (hereafter $h = 1$), we omit the kinetic term for $a_\mu$ and $M, b$ are constants. Treating $a_\mu$ as an external chemical potential $\mu$ and including the external flux by the minimal coupling, one sees that the effective mass-squared of $\phi_{GL}$ on the fluxoid state $m$ is

$$M_{\text{eff}}^2 = M^2 + g^2 [-\mu^2 + (a_{\chi} - m/gR)^2]$$

Provided the chemical potential is large enough then $M_{\text{eff}}^2 < 0$ and $\phi_{GL}$ condenses. Since the different $m$ sectors have identical properties except for a discrete shift of $a_{\chi}$, the condensate forms in consecutive fluxoid channels and the periodicity of the SC phase transition is granted, with period $\Delta a_{\chi} = 1/gR$ (the equivalent of the LP period $\Phi_0/2$). This simple picture directly leads to the array of parabolas in the $\mu$-$a_{\chi}$ phase diagram typical of the LP effect, see Fig. 1 (a). One realizes that the LP effect holds because $a_{\chi}$ and $m$ enter in $\mathcal{L}$ only via covariant derivatives, $|D_\chi \phi_{GL}| = \rho |m/R - g a_{\chi}|$. Hence, a key ingredient behind the LP degeneracy is the locality of $\mathcal{L}$ with respect to $a_{\chi}$ and $m$ [20].

At quantum level, though, one does not expect $a_{\chi}$ to enter in $\mathcal{L}$ via local operators only: nothing prevents e.g. the coefficients $M$ and $b$ in (1) to acquire a direct dependence on $W$ and $m$ (which we shall call ‘non-local’ dependence for short). A related and more familiar phenomenon is the Casimir effect – the dependence on $R$...
of the vacuum energy (a tacit \( \phi_{\text{GL}} \)-independent additive term in (1)). In the presence of a cylinder-threading flux, the vacuum energy depends on \( W \) as well \cite{9}, giving an ‘Aharonov-Bohm’ Casimir effect. Similarly, one expects the vacuum energy to be suppressed by \( R \). The coherence length \( \xi_0 \) plays the role of \( M^{-1} \) of the pair for \( R \to \infty \), the AB-like effects can only be sizeable for \( R \) close to \( \xi_0 \). The physical effect leading to \( m \)-dependence is subtler. It seems to require a coupling in the Lagrangian involving the fundamental charges and the condensate’s phase \( \theta \) such as \( L_{\text{AB}} \equiv j^\mu \partial_\mu \theta \), which is gauge-invariant if \( j_\mu \) is a conserved current (say of the electrons). This coupling is allowed in a Higgs phase and yields another AB-like effect if \( \theta \) has winding \( m \neq 0 \) \cite{10}.

We shall not attempt here to obtain the \( W,m \)-dependence from first principles (see e.g. \cite{2-4} for microscopic derivations of equivalent effects). Instead, let us discuss its impact on the phase transition. It is instructive to consider a local GL model with constant \( M, b \) and gradually turn on a non-local dependence on \( W,m \). We illustrate the resulting \( \mu-\alpha_\chi \) phase diagrams in Fig 1. Starting from a representative local model in (a), introducing a non-local dependence deforms the phase diagram in various ways: the SC/normal critical lines for different \( m \) channels become non-degenerate, and the transitions between different fluxoid sectors bend around. Thus, the non-local dependence uplifts the degeneracy amongst different fluxoid sectors. The order parameter is also affected in an interesting way. It is easy to show even beyond the GL approximation that if parity is preserved and the coefficients in the Lagrangian \( M, b, \ldots \) are allowed to depend on \( W \) but not on \( m \), then the order parameter \( \phi_{\text{GL}} \) is continuous across the transitions between different \( m \)-domains. Conversely, a discontinuous \( \phi_{\text{GL}} \) in these transitions is a signature of \( m \)-dependence (and hence of the coupling \( L_{\text{AB}} \)) on quite general grounds. As we shall see, the confining HSC exhibits this feature.

Holographic Superconductors. The gauge/gravity duality relates strongly coupled gauge theories with gravity in higher dimensions, and has become a powerful method to study strongly interacting systems. To apply it to SCs \cite{5} one devises a 2+1 Conformal Field Theory (CFT) with a large number of ‘colours’ \( \mathcal{N} \) and a \( U(1) \) symmetry. Appropriate quantization \cite{11} leads to an emergent gauge boson— a gauged \( U(1) \). Superconductivity occurs if a \( U(1) \)-charged operator \( O \) condenses. This is dual to 3+1 Anti-de-Sitter (AdS) gravity with a gauge and a charged scalar field (dual to \( O \)). Fixing the chemical potential \( \mu \), at high temperature the ground state is the AdS Reissner-Nordstrom charged Black Brane (BB), which is dual to a conductor. Instead, at low temperature the BB develops scalar hair, and thus becomes dual to a SC \cite{5}.

An interesting development of this model consists in compactifying one spatial dimension, corresponding to the cylindrical geometry considered here. With appropriate boundary conditions on the circle, the theory becomes confining even in supersymmetric frameworks \cite{12}. Indeed, with anti-periodic fermions and periodic bosons (which we shall assume from now on), at temperatures below \( 1/2\pi R \) the ground state becomes the so-called AdS Soliton \cite{12,13}, a horizon-less solution with zero entropy. For \( T > 1/2\pi R \) the BB (with its order \( \mathcal{N}_c^2 \) entropy) dominates, allowing for a dual interpretation as a (de)confinement transition \cite{12}. Additionally, the Soliton exhibits certain quantum effects. E.g. its negative energy relative to AdS is dual to the CFT Casimir energy \cite{13}.

In \cite{14} (see also \cite{15}), the ‘electrical’ response of the Soliton was studied by introducing a gauge field and a charged scalar. It was concluded that i) the confining/deconfining phases are insulating/conducting, the transition between them occurring at \( \mu \) and/or \( T \) around \( 1/R \); ii) both phases can exhibit SC behaviour: below \( T \sim 1/R \) and increasing \( \mu \), one finds first a confined (Soliton) SC and then (for \( \mu \gtrsim 1/R \)) a deconfined (BB) SC. Since the BB SC exhibits a conformal behaviour, the SC/normal critical chemical potential is \( \mu_c \sim T/g \) for large \( g \) (the charge of \( O \)) \cite{14}. By conformality also, the zero-temperature coherence length must be \( \xi_0 \sim 1/g \mu < 1/g \mu_c < R \) in the BB SC. In the Soliton SC, instead, conformality is broken in the infrared. There is a mass gap of order \( 1/R \) so one expects \( \xi_0 \) to be of order \( R \) (as confirmed numerically). From the previous discussion, then, one expects larger AB-effects (and the uplifting of LP degeneracy) in the Soliton SC than in the BB SC. In the Holographic setup, we will find an interesting additional suppression in the BB phase.

Model. In the gravity picture of the HSC, the gravitational degrees of freedom are coupled to a \( U(1) \) gauge field \( A_\alpha \) and a complex scalar \( \Phi \) according to the action

\[
S = \int d^4x \sqrt{-G} \left\{ (\mathcal{R} - \Lambda) / 16\pi G_N - \partial_\alpha \Phi \partial^\alpha \Phi / 4 - |D_\alpha \Phi|^2 / L^2 \right\},
\]

where \( G_N \) is the Newton constant, \( G \) is the determinant of the spacetime metric \( g_{\alpha\beta} \), the cosmological constant \( \Lambda \) determines the AdS radius \( L \) as \( \Lambda = -6/L^2 \) and we omit a potential for the scalar for simplicity. We use coordinates \( (t, z, \chi, y) \) with \( z \) the holographic direction.
such that the AdS-boundary sits at \( z = 0 \). We consider finite temperature solutions with a compact spatial direction (\( \chi \)) of asymptotic radius \( R \), and a noncompact one \( y \). We also assume the ‘probe limit’ (\( g \to \infty \) with \( g_A \) fixed) where the gravitational backreaction of \( A_\alpha \), \( \Phi \) is negligible \([5, 14, 15]\), so \( A_\alpha \), \( \Phi \) become the only dynamical fields on a fixed spacetime. The relevant background metrics then are

\[
\text{d}s^2 = (L^2/z^2) \left[ f_1(z) \text{d}t^2 - d\chi^2 - f(z) - f_\chi(z) \text{d}\chi^2 - \text{d}y^2 \right]
\]

with \( \{f_1, f_\chi\} = \{f(z), 1\} \) for the neutral BB (AdS Soliton) where \( f(z) = 1 - (z/z_0)^3 \) with \( z_0 = 3/4\pi T \) (3R/2). To study superconductivity in a fluxoid \( m \) we assume an ansatz \( \Phi = \psi(z)e^{im\chi/R} \), \( A_\chi = A_\chi(z) \), \( A_t = A_t(z) \), and solve numerically the equations of motion.

The standard dual CFT interpretation starts from expressing the asymptotic behaviour of the gauge field near \( z = 0 \) as \( A_\mu \to a_\mu + J_\mu z \). One then identifies \( J_\mu \) as the conserved \( U(1) \) current carried by the CFT and \( a_\mu \) as the conjugate ‘external’ gauge potential which couples to it. Similarly, from the boundary behavior \( |\Phi| \to e + \langle O \rangle z^3/3 \) one identifies \( O \) as the condensing operator that breaks the \( U(1) \) symmetry, and \( c \) as its source term (which we set to zero to realize spontaneous symmetry breaking).

Since we want to study the response to an external Wilson line \( a_\chi \) and to a chemical potential \( \mu \equiv a_\mu \), the other boundary conditions at \( z = 0 \) are \( A_t = \mu \) and \( A_\chi = a_\chi \).

**Holographic LP Degeneracy.** We start with the BB background (\( T > 1/2\pi R \)). At the BB horizon (\( z = z_0 \)) we have to impose the regularity conditions \([5, 11]\)

\[
A_t = 3\psi' + z_0(gA_\chi - m/R)^2\psi
= 3z_0^2A_\chi' + 2g^2\psi^2(gA_\chi - m/R) = 0,
\]

primes standing for \( z \)-derivatives. With these boundary conditions the numerical integration of the field equations is straightforward. Our results, summarized in Fig. 2, were obtained using the Comsol package. The phase diagram in the \( \mu-a_\chi \) plane (specifically, the SC/normal critical line) is clearly periodic in \( a_\chi \) with period \( 1/gR \) indeed a dual LP effect. This is a consequence of the equations of motion and boundary conditions depending only on local gauge-invariants such as \( [D_\chi \Phi] \). The effective theory for the order parameter \( \langle O \rangle \), then, exhibits no non-local dependence on \( W,m \). In particular, the order parameter \( \langle O \rangle \) is continuous in the transitions between different \( m \) channels. In brief, the expected Aharonov-Bohm effects end up entirely suppressed in the BB SC \([21]\). Of course, a ‘no-hair’ theorem lies behind this suppression.

We shall assume that the charge of \( \Phi \) is an integer multiple of the smallest charge in the theory, \( g = Nc \). In the CFT dual one can think that \( O \) is a composite operator made of \( N \) singlet fields with \( U(1) \) charge \( c \). Since these do not participate in the solutions, \( e \) (or \( N \)) is a free parameter which solely determines the realized \( Z_N \) symmetry and the range of \( a_\chi \). We leave this as an unspecified parameter, since the LP effect implies the degeneracy of the \( N \) fluxoid sectors, whichever is \( N \).

**Holographic Lifting of the LP Degeneracy.** We now turn to the Soliton (\( T < 1/2\pi R \)), which does display a Casimir energy \([13]\) so it is expected to exhibit AB effects as well. The only change with respect to the previous case is that there is now no horizon but instead the \( \chi \) direction closes smoothly at \( z = z_0 \). The regularity conditions there are \([16]\) \( A_\chi = 0 \), \( 3z_0A_\chi' + 2g^2A_\chi\psi^2 = 0 \) and

\[
0 = 3\psi' - g^2z_0A_\chi^2\psi \text{ for } m = 0, \psi = 0 \text{ for } m \neq 0.
\]

In addition, the same ansatz for \( \Phi, A_\chi \), now leads to vortex solutions akin to those of \([11, 17]\) but having their core at the infrared tip of the holographic direction \( z \).

Our results for the numerical integration of the equations are summarized in Fig. 3. As in the BB phase, for large enough \( a_\chi \), the condensation is favored in the \( m \neq 0 \) sectors \([22]\). However, now the domains occupied by different sectors do not appear regularly – the \( 1 \) \( \rightarrow \) \( 0 \)-dependence: AB effects are indeed present. In the gravity theory, this arises through the boundary conditions (2), which depend directly on \( m \) and \( A_\chi \). Let us also stress that the order parameter \( \langle O \rangle \) exhibits jumps in the transitions between different \( m \) sectors, a manifestation of the coupling \( L_{1-g} \).

**Little-Parks and Quantum hair.** HSCs suggest an interplay between BH physics and superconductivity. The mere existence of the BB SC is a consequence of the

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**FIG. 2:** Phase diagram for the BB SC at \( T = 1/\pi R, g\mu_\star R = 10.1 \), manifestly showing a LP effect. Thick solid blue lines separate the SC and normal phases. Dashed green lines are transitions between different fluxoid sectors. Thin solid blue lines are existence lines for different fluxoid condensates. Inset: behaviour of the order parameter \( \langle O \rangle \) for \( \mu = 1.03\mu_\star \). Presented in units of \( R \).

**FIG. 3:** Phase diagram for the Soliton SC, showing no LP degeneracy. We have \( g\mu_\star R = 1.81 \). Line coding as in Fig.2. Inset: form of \( \langle O \rangle \) for \( \mu = 2.1\mu_\star \). Presented in units of \( R \).
The fact that the AdS space evades the no-hair (or uniqueness) theorems: not only regular AdS BHs with scalar hair exist, they even become the ground state at low temperatures. Another well known departure from the spirit of the no-hair theorems is given by the so-called Quantum Hair (QH) – the ‘charges’ which are measurable quantum-mechanically but not classically. A sharp example is given by the discrete gauge symmetries discussed above: the charge defined by the unbroken $Z_N$ gauge group is measurable at long distances, yet it is associated to a massive gauge field. Then, BHs must be able to carry this charge without classically affecting the spacetime [7, 8]. Note that even if this extends the kinds of hairs supported by BHs, these still comply with a classical-uniqueness theorem: classically, BH solutions are insensitive to the amount of QH (see footnote 22).

The connection between quantum hair and the LP effect becomes apparent once one realizes that the defining property of QH – classical undetectability – implies the (classical) degeneracy between different discrete charge sectors. This suggests that the LP effect results from a magnetic form of discrete charge acting as QH. To be more precise, let us split the spatial gauge field as $a_\chi \equiv m'/gR + \bar{a}_\chi$ with $m'$ an integer and $\bar{a}_\chi$ the non-integer part of $a_\chi$ modulo $1/gR$. In the BH phase all quantities depend only on $\bar{a}_\chi$ and on $m - m'$. Therefore for every choice of $\bar{a}_\chi$, the configurations $m = m' = k$ with $k = 0, \ldots, N-1$ are degenerate classically. These configurations are a magnetic counterpart of discrete gauge charge: there is an $N$-fold of them and the winding number $m$ is locked to the ‘magnetic flux’ $m'$. Since this is a discrete gauge charge it is a QH, and classically it must leave the BB solutions unaffected even including the backreaction (since both the stress tensor and boundary conditions depend only on local operators). Hence, in the BB SC the different magnetic sectors are classically degenerate, and the short ‘LP’ period $\Delta a_\chi = 1/gR$ follows. Instead, the Soliton SC is horizon-free and does not obey classical-uniqueness. Indeed, the Soliton is classically sensitive to QH: the closing of the spatial circle at $z = z_0$ enforces Wilson line-dependent boundary conditions and yields the fundamental $1/eR$ periodicity [23].

The discussion so far treated AdS gravity classically, which is valid for small curvatures. Quantum corrections in the gravity theory will generically spoil the LP periodicity, even if by a small amount. Again, the simplest quantum effect is the AB effect from fundamental charges winding around the circle, which exhibits the fundamental period $1/eR$ and need not be exponentially suppressed [24]. According to the AdS/CFT dictionary, quantum effects in the gravity side map to $1/N$ corrections in the large $N$ expansion of the CFT. One infers that the AB effect is $1/N$-suppressed in the CFT deconfined plasma phase. By contrast, the classical sensitivity to QH exhibited by the Soliton means that AB effects are unsuppressed in the confining phase. In CFT terms, this can be understood from i) the large $N$ is a classical limit [18] and ii) the deconfined plasma state should have a classical counterpart (in which quantum effects do die out with $1/N$) while the confining state should not. According to this interpretation, the BH classical uniqueness theorem implies a dual ‘theorem’: the deconfined plasma states have a classical limit at large $N$.

As a final remark, one may speculate that the properties found here for the (de)confining SCs may extend beyond the applicability of the large $N$ limit. If so, it is tempting to infer that the lifting of the LP effect should be more (un)suppressed in (un)conventional SCs.

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[19] We use units where the speed of light is $c = 1$.
[20] The $1/eR$ period persists for $L$ beyond the GL form (1) provided only local gauge invariant operators enter.
[21] The Aharonov-Bohm effects would vanish even for $R < \xi_0$ if one chose periodic fermions on the circle (since these are compatible only with the BB solution).
[22] Multi-center solutions could have lower energy than the single vortex for $|m| > 1$. The $m = 2$ lines in Fig. 3, then, represent ‘upper’ bounds on the actual lines.
[23] It seems that a ‘proof’ of classical uniqueness in our context follows by noting that i) classical sensitivity to QH can arise only from a collapsing spatial circle (CSC); and ii) no regular BB solutions with CSCs exist.
[24] Uplifting of the electric QH proceeds via AB-interference of virtual magnetic flux tubes, so it is exponentially suppressed [8]. The instantons that capture this effect are almost identical to the vortices on the Soliton.