THE COMPRESSION METHOD AND APPLICATIONS

T. AGAMA

Abstract. In this paper we introduce and develop the method of compression of points in space. We introduce the notion of the mass, the rank, the entropy, the cover and the energy of compression. We leverage this method to prove some class of inequalities related to Diophantine equations. In particular, we show that for each \( L < n - 1 \) and for each \( K > n - 1 \), there exist some \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) with \( x_i \neq x_j \) for all \( 1 \leq i < j \leq n \) such that

\[
\frac{1}{K^n} \ll \prod_{j=1}^{n} \frac{1}{x_j} \ll \frac{\log(n)}{nL^{n-1}}
\]

and that for each \( L > n - 1 \) there exist some \((x_1, x_2, \ldots, x_n) \) with \( x_i \neq x_j \) for all \( 1 \leq i < j \leq n \) and some \( s \geq 2 \) such that

\[
\sum_{j=1}^{n} \frac{1}{x_j^s} \gg s \frac{n}{L^{s-1}}.
\]

1. Introduction

The Erdős-Straus conjecture is the assertion that for each \( n \in \mathbb{N} \) for \( n \geq 3 \) there exist some \( x_1, x_2, x_3 \in \mathbb{N} \) such that

\[
\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = \frac{4}{n}.
\]

More formally the conjecture states

**Conjecture 1.1.** For each \( n \geq 3 \), does there exist some \( x_1, x_2, x_3 \in \mathbb{N} \) such that

\[
\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = \frac{4}{n}.
\]

Despite its apparent simplicity, the problem still remain unresolved. However there has been some noteworthy partial results. For instance it is shown in [2] that the number of solutions to the Erdős-Straus Conjecture is bounded polylogarithmically on average. The problem is also studied extensively in [3] and [4]. The Erdős-Straus conjecture can also be rephrased as a problem of an inequality. That is to say, the conjecture can be restated as saying that for all \( n \geq 3 \) the inequality holds

\[
\frac{3}{n} \leq \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \leq \frac{3}{n}
\]

for \( c_1 = c_2 = \frac{4}{3} \) for some \( x_1, x_2, x_3 \in \mathbb{N}^3 \). Motivated by this version of the problem, we introduce the method of compression. This method comes somewhat close to

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addressing this problem and its variants. Using this method, we managed to show that

**Theorem 1.1.** For each \( L \in \mathbb{N} \) with \( L > n - 1 \) there exist some \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n \) with \( x_i \neq x_j \) for all \( 1 \leq i < j \leq n \) such that

\[
\frac{c_1}{L} \leq \sum_{j=1}^{n} \frac{1}{x_j} \leq \frac{c_2}{L}
\]

for some \( c_1, c_2 > 1 \). In particular, for each \( L \geq 3 \) there exist some \((x_1, x_2, x_3) \in \mathbb{N}^3 \) with \( x_1 \neq x_2, x_2 \neq x_3 \) and \( x_3 \neq x_1 \) such that

\[
\frac{c_1}{L} \leq \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \leq \frac{c_2}{L}
\]

for some \( c_1, c_2 > 1 \).

Perhaps more general is the result

**Theorem 1.2.** For each \( L > n - 1 \) there exist some \((x_1, x_2, \ldots, x_n) \) with \( x_i \neq x_j \) for all \( 1 \leq i < j \leq n \) and some \( s \geq 2 \) such that

\[
\sum_{j=1}^{n} \frac{1}{x_j^s} \gg s \frac{n}{L^{s-1}}.
\]

**Theorem 1.3.** For each \( L < n - 1 \) and for all \( s \geq 2 \), there exist some \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n \) with \( x_i \neq x_j \) for \( 1 \leq i < j \leq n \) such that

\[
\sum_{j=1}^{n} \frac{1}{x_j^s} \ll \log^s \left( \frac{n}{L} \right).
\]

### 2. Compression

**Definition 2.1.** By the compression of scale \( m > 0 \) on \( \mathbb{R}^n \) we mean the map \( \mathcal{V}: \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that

\[
\mathcal{V}_m[(x_1, x_2, \ldots, x_n)] = \left( \frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n} \right)
\]

for \( n \geq 2 \) and with \( x_i \neq 0 \) for all \( i = 1, \ldots, n \).

**Remark 2.2.** The notion of compression is in some way the process of re scaling points in \( \mathbb{R}^n \) for \( n \geq 2 \). Thus it is important to notice that a compression pushes points very close to the origin away from the origin by certain scale and similarly draws points away from the origin close to the origin.

**Proposition 2.1.** A compression of scale \( m > 0 \) with \( \mathcal{V}_m: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a bijective map.

**Proof.** Suppose \( \mathcal{V}_m[(x_1, x_2, \ldots, x_n)] = \mathcal{V}_m[(y_1, y_2, \ldots, y_n)] \), then it follows that

\[
\left( \frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n} \right) = \left( \frac{m}{y_1}, \frac{m}{y_2}, \ldots, \frac{m}{y_n} \right).
\]

It follows that \( x_i = y_i \) for each \( i = 1, 2, \ldots, n \). Surjectivity follows by definition of the map. Thus the map is bijective. \( \square \)
3. The mass of compression

Definition 3.1. By the mass of a compression of scale $m$ we mean the map $\mathcal{M} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) = \sum_{i=1}^{n} \frac{m}{x_i}.$$ 

Remark 3.2. Next we prove upper and lower bounding the mass of the compression of scale $0 < m$.

Proposition 3.1. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$, then the estimates holds

$$m \log \left(1 - \frac{n-1}{\sup(x_j)}\right) \ll \mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) \ll m \log \left(1 + \frac{n-1}{\inf(x_j)}\right)$$

for $n \geq 2$.

Proof. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_j \geq 1$. Then it follows that

$$\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) = m \sum_{j=1}^{n} \frac{1}{x_j} \leq m \sum_{k=0}^{n-1} \frac{1}{\inf(x_j) + k}$$

and the upper estimate follows by the estimate for this sum. The lower estimate also follows by noting the lower bound

$$\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) = m \sum_{j=1}^{n} \frac{1}{x_j} \geq m \sum_{k=0}^{n-1} \frac{1}{\sup(x_j) - k}.$$ 

The estimates obtained for the mass of compression is quite suggestive. It restricts the entries of any of our choice of tuple to be distinct. After a little heuristics, it can be seen the left estimate for the mass of compression tends to be almost flawed if we allow for tuples with at least two similar entries. Thus in building this Theory, and with all the results we will obtained, we will enforce that the entries of any choice of tuple is distinct.

3.1. Application of mass of compression. In this section we apply the notion of the mass of compression to the Erdős-Straus conjecture.

Theorem 3.3. There exist some $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ for each $n \geq 2$ with $x_j \geq 1$ such that

$$m \frac{n}{L_1} \ll \mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) \ll m \frac{n}{L_2}$$

for some $L_1, L_2 \in \mathbb{N}$. 

Proof. First choose \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) such that \(\sup(x_j) > \inf(x_j) > n - 1\) for \(j = 1, \ldots, n\). Then from Proposition 3.1, we have the upper bound
\[
\mathcal{M}(\mathbb{V}_m([x_1, x_2, \ldots, x_n])) \ll m \log \left( 1 + \frac{n - 1}{\inf(x_j)} \right) = m \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left( \frac{n - 1}{\inf(x_j)} \right)^k \ll m \frac{n}{\inf(x_j)}.
\]
The lower bound also follows by noting that
\[
\mathcal{M}(\mathbb{V}_m([x_1, x_2, \ldots, x_n])) \gg m \log \left( 1 - \frac{n - 1}{\sup(x_j)} \right)^{-1} = m \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{n - 1}{\sup(x_j)} \right)^k \gg m \frac{n}{\sup(x_j)}
\]
and the inequality follows by taking \(\sup(x_j) = L_1\) and \(\inf(x_j) = L_2\). \(\square\)

Theorem 3.3 is redolent of the Edos-Strauss conjecture. Indeed It can be considered as a weaker version of the conjecture. It is quite implicit from Theorem 3.3 that there are infinitely many points in \(\mathbb{N}^n\) that satisfy the inequality with finitely many such exceptions. Therefore in the opposite direction we can assert that there are infinitely many \(L_1, L_2 \in \mathbb{N}\) that satisfies the inequality. We state a consequence of the result in Theorem 3.3 to shed light on this assertion.

Corollary 3.1. For each \(L \in \mathbb{N}\) with \(L > n - 1\) there exist some \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) with \(x_i \neq x_j\) for all \(1 \leq i < j \leq n\) such that
\[
\frac{n}{L} \ll \frac{1}{x_j} \ll \frac{n}{L}.
\]
In particular, for each \(L \geq 3\) there exist some \((x_1, x_2, x_3) \in \mathbb{N}^3\) with \(x_1 \neq x_2, x_2 \neq x_3\) and \(x_1 \neq x_3\) such that
\[
\frac{3}{L} \ll \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \ll \frac{3}{L}.
\]

Proof. First choose \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) with \(x_i \neq x_j\) for all \(1 \leq i < j \leq n\) such that \(\sup(x_j) > \inf(x_j) > n - 1\). By taking \(K = \sup(x_j)\) and \(L = \inf(x_j)\) for any such points, it follows that
\[
\frac{n}{L} \ll \frac{1}{x_j} \ll \frac{n}{K} \ll \frac{n}{L}.
\]
The special case follows by taking \(n = 3\). \(\square\)

It is important to recognize that the condition \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) with \(x_i \neq x_j\) all \(1 \leq i < j \leq n\) in the statement of the result is not only a quantifier but a requirement; otherwise, the estimate for the mass of compression will be flawed.
completely. To wit, suppose that we take \( x_1 = x_2 = \ldots = x_n \), then it will follow that \( \inf(x_j) = \sup(x_j) \), in which case the mass of compression of scale \( m \) satisfies

\[
\sum_{k=0}^{n-1} \frac{1}{\inf(x_j) - k} \leq M(\mathcal{V}_m([x_1, x_2, \ldots, x_n])) \leq \sum_{k=0}^{n-1} \frac{1}{\inf(x_j) + k}
\]

and it is easy to notice that this inequality is absurd. By extension one could also try to equalize the sub-sequence on the bases of assigning the supremum and the Infimum and obtain an estimate but that would also contradict the mass of compression inequality after a slight reassignment of the sub-sequence. Thus it is important for the estimates to make any good sense to ensure that any tuple \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) must satisfy \( x_i \neq x_j \) for all \( 1 \leq i < j \leq n \). Thus our Theory will be built on this assumption, that any tuple we use has to have distinct entry. Since all other statistic will eventually depend on the mass of compression, this assumption will be highly upheld.

**Remark 3.4.** The result can be interpreted as saying that for each \( L \geq 3 \) there exist some \((x_1, x_2, x_3) \in \mathbb{N}^3\) such that

\[
c_1 \frac{3}{L} \leq \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \leq c_2 \frac{3}{L}
\]

for some constants \( c_1, c_2 > 1 \). The Erdős-Straus conjecture will follow if we can take \( c_1 = c_2 = \frac{4}{3} \). Investigating the scale of these constants is the motivation for this Theory and will be developed in the following sequel.

**Theorem 3.5.** For each \( K > n - 1 \) and for each \( L < n - 1 \), there exist some \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) with \( x_i \neq x_j \) for all \( 1 \leq i < j \leq n \) such that

\[
\frac{n}{K} \ll \sum_{j=1}^{n} \frac{1}{x_j} \ll \log \left( \frac{n}{L} \right).
\]

**Proof.** Let us choose \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) with \( x_i \neq x_j \) for all \( 1 \leq i < j \leq n \) such that \( \inf(x_j) < n - 1 \) and \( \sup(x_j) > n - 1 \). Then we set \( L = \inf(x_j) \) and \( K = \sup(x_j) \), then the result follows from the estimate in Theorem 3.1.

**Remark 3.6.** Next we expose one consequence of Theorem 3.5.

**Corollary 3.2.** For each \( K > 2 \), there exist some \((x_1, x_2, x_3) \in \mathbb{N}^3\) with \( x_i \neq x_j \) for all \( 1 \leq i < j \leq 3 \) such that

\[
c_1 \frac{3}{K} \leq \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \leq c_2 \log 3
\]

for some \( c_1, c_2 > 1 \).

### 4. The rank of compression

In this section we introduce the notion of the rank of compression. We launch the following language in that regard.

**Definition 4.1.** Let \((x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\) for \( n \geq 2 \) then by the rank of compression, denoted \( \mathcal{R} \), we mean the expression

\[
\mathcal{R}(x_1, x_2, \ldots, x_n) = \left\| \left( \frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n} \right) \right\|.
\]
Remark 4.2. It is important to notice that the rank of a compression of scale $m > 0$ is basically the distance of the image of points under compression from the origin. Next we relate the rank of compression of scale $m > 0$ with the mass of a certain compression of scale 1.

**Proposition 4.1.** Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, then we have

$$ \mathcal{R} \circ \mathcal{V}_m[(x_1, x_2, \ldots, x_n)]^2 = m^2 \mathcal{M} \circ \mathcal{V}_1 \left[ \left( x_1^2, x_2^2, \ldots, x_n^2 \right) \right]. $$

**Proof.** The result follows from definition 4.1 and definition 3.1.

**Remark 4.3.** Next we prove upper and lower bounding the rank of compression of scale $m > 0$ in the following result. We leverage pretty much the estimates for the mass of compression of scale $m > 0$.

**Theorem 4.4.** Let $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$, then we have

$$ m \sqrt{\log \left( 1 - \frac{n-1}{\sup(x_j^2)} \right)^{-1}} \ll \mathcal{R} \circ \mathcal{V}_m[(x_1, x_2, \ldots, x_n)] \ll m \sqrt{\log \left( 1 + \frac{n-1}{\inf(x_j^2)} \right)}.$$

**Proof.** The result follows by leveraging Proposition 4.1 and Proposition 3.1.

4.1. **Application of rank of compression.** In this section we expose one consequence of the rank of compression. We apply this to estimate the second moment unit sum of the Erdős Type problem. We state this more formally in the following result.

**Theorem 4.5.** For each $L > \sqrt{n - 1}$, there exist some $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ such that

$$ \frac{n}{L^2} \ll \sum_{j=1}^n \frac{1}{x_j^2} \ll \frac{n}{L^2}.$$

In particular for each $L \geq 2$, there exist some $(x_1, x_2, x_3) \in \mathbb{N}^3$ with $x_1 \neq x_2$, $x_2 \neq x_3$ and $x_1 \neq x_3$ and some constant $c_1, c_2 > 1$ such that

$$ c_1 \frac{3}{L^2} \leq \frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} \leq c_2 \frac{3}{L^2}. $$

**Proof.** Let us choose $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ in Theorem 4.4 such that $L = \inf(x_j)$ with $L^2 > n - 1$. Then the inequality follows immediately. The special case follows by taking $n = 3$.

**Remark 4.6.** Next we present a second moment variant inequality of the unit sum of positive integers in the following statement.

**Corollary 4.1.** For each $L \geq 3$, there exist some $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{N}^5$ with $x_i \neq x_j$ for all $1 \leq i < j \leq 5$ and some constant $c_1, c_2 > 1$ such that

$$ c_1 \frac{5}{L^2} \leq \frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} + \frac{1}{x_4^2} + \frac{1}{x_5^2} \leq c_2 \frac{5}{L^2}. $$
5. The entropy of compression

In this section we launch the notion of the entropy of compression. Intuitively, one could think of this concept as a criteria assigning a weight to the image of points under compression. We provide some quite modest estimates of this statistic and exploit some applications, in the context of some Diophantine problems.

**Definition 5.1.** Let \((x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\) with \(x_i \neq 0, 1\) for all \(i = 1, 2, \ldots, n\). By the entropy of a compression of scale \(m > 0\) we mean the map \(E : \mathbb{R}^n \rightarrow \mathbb{R}\) such that

\[
E(V_m[(x_1, x_2, \ldots, x_n)]) = \prod_{i=1}^{n} \frac{m}{x_i}.
\]

**Remark 5.2.** Next we relate the mass of a compression to the entropy of compression and deduce reasonable good bounds for our further studies. We could in fact be economical with the bounds but they are okay for our needs.

**Proposition 5.1.** For all \(n \geq 2\), we have

\[
M(V_m[(x_1, x_2, \ldots, x_n)]) = mM \left( V_1 \left[ \left( \prod_{i \neq 1} \frac{1}{x_i}, \prod_{i \neq 2} \frac{1}{x_i}, \ldots, \prod_{i \neq n} \frac{1}{x_i} \right) \right] \right) \times E(V_1[(x_1, x_2, \ldots, x_n)]).
\]

**Proof.** By Definition 5.1 we have

\[
M(V_m[(x_1, x_2, \ldots, x_n)]) = \sum_{i=1}^{n} \frac{m}{x_i} \sum_{\sigma:[1,n] \rightarrow [1,n]} \prod_{\sigma(i) \neq \sigma(j)}^{n-1} x_{\sigma(i)}^{i \in [1,n]} \prod_{\sigma(j) \neq \sigma(i)}^{n-1} x_{\sigma(j)}^{i \in [1,n]}
\]

The result follows immediately from this relation. \(\Box\)

**Proposition 5.2.** Let \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\), then we have

\[
\log \left(1 - \frac{n-1}{n \sup(x_j)^{n-1}}\right) \leq E(V_1[(x_1, x_2, \ldots, x_n)]) \leq \log \left(1 + \frac{n-1}{n \inf(x_j)^{n-1}}\right).
\]

**Proof.** The result follows by using the relation in Proposition 5.1 and leveraging the bounds in Proposition 3.1 and noting that

\[
M \left( V_1 \left[ \left( \prod_{i \neq 1} \frac{1}{x_i}, \prod_{i \neq 2} \frac{1}{x_i}, \ldots, \prod_{i \neq n} \frac{1}{x_i} \right) \right] \right) \leq n \sup(x_j)^{n-1}
\]

and

\[
M \left( V_1 \left[ \left( \prod_{i \neq 1} \frac{1}{x_i}, \prod_{i \neq 2} \frac{1}{x_i}, \ldots, \prod_{i \neq n} \frac{1}{x_i} \right) \right] \right) \geq n \inf(x_j)^{n-1}.
\]

\(\Box\)
5.1. **Applications of the entropy of compression.** In this section we lay down one striking and a stunning consequence of the entropy of compression. One could think of these applications as analogues of the Erdős type result for the unit sums of triples of the form \((x_1, x_2, x_3)\). We state two consequences of these estimates in the following sequel.

**Theorem 5.3.** For each \(L > n - 1\), there exist some \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) with \(x_i \neq x_j\) for all \(1 \leq i < j \leq n\) such that

\[
\frac{1}{L^n} \ll \prod_{i=1}^{n} \frac{1}{x_i} \ll \frac{1}{L^n}.
\]

**Proof.** Let us choose \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) with \(x_i \neq x_j\) for all \(1 \leq i < j \leq n\) such that \(L > n - 1\) with \(\inf(x_j) = L\), then the result follows immediately in Proposition 5.2.

Theorem 5.3 tells us that for some tuple \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) with \(x_i \neq x_j\) for all \(1 \leq i < j \leq n\) there must exist some constant \(c_1, c_2 > 1\) such that we have the inequality

\[
\frac{c_1}{L^n} \leq \prod_{j=1}^{n} \frac{1}{x_j} \leq \frac{c_2}{L^n}.
\]

Next we present a second application of the estimates of the entropy of compression in the following sequel.

**Theorem 5.4.** For each \(L < n - 1\) and for each \(K > n - 1\), there exist some \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) with \(x_i \neq x_j\) for all \(1 \leq i < j \leq n\) such that

\[
\frac{1}{K^n} \ll \prod_{j=1}^{n} \frac{1}{x_j} \ll \frac{\log(K)}{nL^{n-1}}.
\]

**Proof.** Let us choose a tuple \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) with \(x_i \neq x_j\) for all \(1 \leq i < j \leq n\) such that \(\sup(x_j) = K > n - 1\) and \(L = \inf(x_j) < n - 1\), then the result follows immediately.

**Corollary 5.1.** For each \(L < 4\) and for each \(K > 4\), there exist some \((x_1, x_2, x_3, x_4, x_5) \in \mathbb{N}^5\) with \(x_i \neq x_j\) for all \(1 \leq i < j \leq 5\) and some constant \(c_1, c_2 > 1\) such that

\[
\frac{c_1}{K^5} \leq \frac{1}{x_1} \times \frac{1}{x_2} \times \frac{1}{x_3} \times \frac{1}{x_4} \times \frac{1}{x_5} \leq \frac{c_2 \log 5}{5L^4}.
\]

**Proof.** The result follows by taking \(n = 5\) in Theorem 5.3.

6. **Compression gap**

In this section we introduce the notion of the gap of compression. We investigate this concept in-depth and in relation to the already introduced concepts.

**Definition 6.1.** Let \((x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\) with \(x_i \neq 0, 1\) for all \(i = 1, 2, \ldots, n\). Then by the gap of compression of scale \(m > 0\), denoted \(G \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]\), we mean the expression

\[
G \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \left\| \left( x_1 - \frac{m}{x_1}, x_2 - \frac{m}{x_2}, \ldots, x_n - \frac{m}{x_n} \right) \right\|
\]
The gap of compression is a definitive measure of the chasm between points and their image points under compression. We can estimate this chasm by relating the compression gap to the mass of an expansion in the following ways.

**Proposition 6.1.** Let \((x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\) for \(n \geq 2\) with \(x_j \neq 0, 1\) for \(j = 1, \ldots, n\), then we have
\[
G \circ V_m[(x_1, x_2, \ldots, x_n)]^2 = M \circ V_1 \left[ \left( \frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2} \right) \right] + m^2 M \circ V_1[(x_1^2, \ldots, x_n^2)] - 2mn.
\]

**Proof.** The result follows by using using Definition 6.1 and Definition 3.1.

**Remark 6.2.** We are now ready to provide an estimate for the gap of compression.

**Theorem 6.3.** Let \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) for \(n \geq 2\), then we have
\[
G \circ V_m[(x_1, x_2, \ldots, x_n)]^2 \ll n\sup(x_j^2) + m^2 \log \left( 1 + \frac{n - 1}{\inf(x_j)^2} \right) - 2mn
\]
and
\[
G \circ V_m[(x_1, x_2, \ldots, x_n)]^2 \gg n\inf(x_j^2) + m^2 \log \left( 1 - \frac{n - 1}{\sup(x_j^2)} \right)^{-1} - 2mn.
\]

**Proof.** The result follows by exploiting Proposition 3.1 in Proposition 10.1 and noting that
\[
n\inf(x_j^2) \leq M \circ V_1 \left[ \left( \frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2} \right) \right] \leq n\sup(x_j^2).
\]

### 6.1. Application of the compression gap

In this section we give one striking application of the notion of the gap of compression. It applies to all points \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\).

**Theorem 6.4.** Let \(n \leq m\), then for each \(L > \sqrt{n - 1}\) there exist some \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) with \(x_i \neq x_j\) for all \(1 \leq i < j \leq n\) such that
\[
\frac{m^2 n}{L} \ll \left\| \left( x_1 - \frac{m}{x_1}, x_2 - \frac{m}{x_2}, \ldots, x_n - \frac{m}{x_n} \right) \right\| \ll \frac{m^2}{L}.
\]

**Proof.** For \(m \geq n\), choose \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) with \(x_i \neq x_j\) for all \(1 \leq i < j \leq n\) such that \(\inf(x_j) > \sqrt{n - 1}\) and set \(\sup(x_j) = K\) and \(L = \inf(x_j)\). Then the result follows from the estimate in Theorem 6.3.

**Theorem 6.5.** Let \(m \geq n\). For each \(L < \sqrt{n - 1}\) and each \(K > \sqrt{n - 1}\) there exist some \((x_1, x_2, \ldots, x_n)\) with \(x_i \neq x_j\) for all \(1 \leq i < j \leq n\) such that
\[
\frac{m\sqrt{n}}{K} \ll \left\| \left( x_1 - \frac{m}{x_1}, x_2 - \frac{m}{x_2}, \ldots, x_n - \frac{m}{x_n} \right) \right\| \ll m\sqrt{\log \left( \frac{n}{L} \right)}.
\]

**Proof.** Let \(m \geq n\). Then in Theorem 6.3 choose \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) with \(x_i \neq x_j\) for all \(1 \leq i < j \leq n\) such that \(\inf(x_j) < \sqrt{n - 1}\) and \(\sup(x_j) > \sqrt{n - 1}\) and set \(\inf(x_j) = L\) and \(\sup(x_j) = K\). Then the result follows immediately.
7. The energy of compression

In this section we introduce the notion of the energy of compression. We launch more formally the following language.

Definition 7.1. Let \( (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) with \( x_i \neq 0, 1 \) for all \( i = 1, 2, \ldots, n \) for \( n \geq 2 \), then by the energy dissipated under compression on \( (x_1, x_2, \ldots, x_n) \), denoted \( E \), we mean the expression

\[
E = \sum_{m=1}^{n} \mathcal{V}_m[(x_1, x_2, \ldots, x_n)] = \mathcal{G} \circ \mathcal{V}_m[(x_1, x_2, \ldots, x_n)] \times \mathcal{E}(\mathcal{V}_m[(x_1, x_2, \ldots, x_n)]).
\]

Remark 7.2. Given that we have obtained upper and lower bounds for the compression gap and the entropy of any points under compression, we can certainly get control on the energy dissipated under compression in the following proposition.

Proposition 7.1. Let \( (x_1, x_2, \ldots, x_n) \in \mathbb{N}^n \), then we have

\[
E \circ \mathcal{V}_1[(x_1, x_2, \ldots, x_n)] \leq \frac{1}{(\text{Inf}(x_j))^{n-1} \sqrt{n}} \log \left( 1 + \frac{n - 1}{\text{Inf}(x_j)} \right)
\]

and

\[
E \circ \mathcal{V}_1[(x_1, x_2, \ldots, x_n)] \geq \frac{1}{\sqrt{n}^{n-1} \sup(x_j)^{n-1} \log \left( 1 - \frac{n - 1}{\sup(x_j)} \right)}^{-1}.
\]

Proof. The result follows by plugging the estimate in 6.3 and 5.2 into definition 7.1.

7.1. Applications of the energy of compression. In this section we give some consequences of the notion of the energy of compression.

Theorem 7.3. For each \( K > n - 1 \) and for each \( L < n - 1 \), there exist some \( (x_1, x_2, \ldots, x_n) \in \mathbb{N}^n \) with \( x_i \neq x_j \) for all \( 1 \leq i < j \leq n \) such that

\[
\sqrt{n} \leq \left\| \left( x_1 - \frac{1}{x_1}, x_2 - \frac{1}{x_2}, \ldots, x_n - \frac{1}{x_n} \right) \right\| \leq \frac{\log \left( \frac{L}{K} \right)}{L^{n-1} \sqrt{n}}.
\]

Proof. First choose \( (x_1, x_2, \ldots, x_n) \in \mathbb{N}^n \) with \( x_i \neq x_j \) for all \( 1 \leq i < j \leq n \) such that \( \text{Inf}(x_j) < n - 1 \) and \( \text{sup}(x_j) > n - 1 \). Now set \( K = \text{sup}(x_j) \) and \( \text{Inf}(x_j) = L \), then the result follows by exploiting the estimates in Proposition 7.1.

Corollary 7.1. For each \( K \geq 5 \) and for each \( L < 4 \), there exist some \( (x_1, x_2, x_3, x_4, x_5) \in \mathbb{N}^5 \) with \( x_i \neq x_j \) for all \( 1 \leq i < j \leq 5 \) such that

\[
\sqrt{\frac{5}{K^5}} \leq \left\| \left( x_1 - \frac{1}{x_1}, x_2 - \frac{1}{x_2}, x_3 - \frac{1}{x_3}, x_4 - \frac{1}{x_4}, x_5 - \frac{1}{x_5} \right) \right\| \leq \frac{\log \left( \frac{L}{K} \right)}{L^{4} \sqrt{5}}.
\]

Proof. The result follows by taking \( n = 5 \) in Theorem 5.3.

8. The cover of compression

In this section we introduce the notion of the cover of compression. The cover of compression is basically the s-fold direct product of compression on points in space. We launch the following language in that regard.
**Definition 8.1.** Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq 0, 1$ for all $i = 1, 2, \ldots, n$. Then by the $s$-fold cover of compression on the point, we mean the direct product

$$\otimes_{m=1}^{s} V_m([x_1, x_2, \ldots, x_n])$$

**Remark 8.2.** Next we show that we can get control on the mass of the $s$-fold cover of any compression by the $s$ powers of the mass of the $s$th compression.

**Proposition 8.1.** Let $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ for $n \geq 2$, then we have the estimate

$$M \circ \otimes_{m=1}^{s} V_m([x_1, x_2, \ldots, x_n]) \gg s! \left[ \log^s \left( 1 - \frac{n - 1}{\sup(x_j)} \right)^{-1} - s \frac{n}{\inf(x_j)^{s-1}} \right]$$

and

$$M \circ \otimes_{m=1}^{s} V_m([x_1, x_2, \ldots, x_n]) \ll s! \left[ \log^s \left( 1 + \frac{n - 1}{\inf(x_j)} \right) - s \frac{n}{\sup(x_j)^{s-1}} \right].$$

**Proof.** First we notice that by an application of Stirling formula we have

$$M \circ \otimes_{m=1}^{s} V_m([x_1, x_2, \ldots, x_n]) = s! \sum_{j=1}^{n} \frac{1}{x_j^s}$$

$$= s! \left[ \left( \sum_{j=1}^{n} \frac{1}{x_j} \right)^s - s \sum_{j=1}^{n} \prod_{1 \leq i \leq n, i \neq j} \frac{1}{x_i} \right]$$

The estimate follows by plugging the upper bound in Proposition 3.1 into this estimate and noting that

$$\frac{n}{\sup(x_j)^{s-1}} \leq \sum_{j=1}^{n} \prod_{1 \leq i \leq n, i \neq j} \frac{1}{x_i} \leq \frac{n}{\inf(x_j)^{s-1}}.$$  \[\square\]

### 8.1. Application of the cover of compression.

In this section we present some consequences of the cover of compression. We provide two applications in the context of a Diophantine problem. We generalize the result in Theorem 3.3 at the compromise of some slightly worst implicit constants.

**Theorem 8.3.** For each $L > n - 1$ there exist some $(x_1, x_2, \ldots, x_n)$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ and some $s \geq 2$ such that

$$\sum_{j=1}^{n} \frac{1}{x_j^s} \gg s \frac{n}{L^{s-1}}.$$  

**Proof.** First choose $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ such that $\inf(x_j) = L > n - 1$ and $K = \sup(x_j)$. By choosing $s \geq 2$ such that $n^{s-1} < L$ then the inequality follows from the estimates in Proposition 8.1.  \[\square\]

Theorem 8.3 can be thought of as a one-sided generalization of Theorem 3.3.
Theorem 8.4. For each \( L < n - 1 \) and for all \( s \geq 2 \), there exist some \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) with \( x_i \neq x_j \) for \( 1 \leq i < j \leq n \) such that

\[
\sum_{j=1}^{n} \frac{1}{x_j^s} \ll \log^s \left( \frac{n}{L} \right).
\]

Proof. Let us choose \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) with \( x_i \neq x_j \) for \( 1 \leq i < j \leq n \) such that \( \inf(x_j) < n - 1 \) and \( \sup(x_j) > n - 1 \). Again set \( \inf(x_j) = L \) and \( \sup(x_j) = K \), then the result follows from the estimates in Proposition 8.1. \( \square \)

Corollary 8.1. For each \( L < 4 \) and for all \( s \geq 2 \), there exist some \((x_1, x_2, x_3, x_4, x_5) \in \mathbb{N}^5\) such that

\[
\frac{1}{x_1^s} + \frac{1}{x_2^s} + \frac{1}{x_3^s} + \frac{1}{x_4^s} + \frac{1}{x_5^s} \ll \log^s(5/L).
\]

9. The measure and cost of compression

In this section we introduce the notion of the measure and the cost of compression. We launch the following languages.

Definition 9.1. Let \((x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\) with \( x_i \neq 0, 1 \) for all \( i = 1, 2, \ldots, n \) for \( n \geq 2 \). Then by the measure of compression on \((x_1, x_2, \ldots, x_n)\), denoted \( \mathcal{N} \), we mean the expression

\[
\mathcal{N} \circ \mathcal{V}_1[(x_1, x_2, \ldots, x_n)] = \left| \mathcal{E}(\mathcal{V}_m[(x_1, x_2, \ldots, x_n)]) - \mathcal{E}(\mathcal{V}_m\left[\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)\right]) \right|.
\]

The corresponding cost of compression, denoted \( \mathcal{C} \) is given

\[
\mathcal{C} \circ \mathcal{V}_1[(x_1, x_2, \ldots, x_n)] = \mathcal{N} \circ \mathcal{V}_m[(x_1, x_2, \ldots, x_n)] \times \mathcal{G} \circ \mathcal{V}_m[(x_1, x_2, \ldots, x_n)].
\]

Next we estimate from below and above the measure and the cost of compression in the following sequel. We leverage the estimates established thus far to provide these estimates.

Proposition 9.1. Let \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\), then the following estimates remain valid

\[
\mathcal{N} \circ \mathcal{V}_1[(x_1, x_2, \ldots, x_n)] \ll \sup(x_j)^n
\]

and

\[
\mathcal{N} \circ \mathcal{V}_1[(x_1, x_2, \ldots, x_n)] \gg \inf(x_j)^n.
\]

Proof. The result follows by exploiting the estimates in Theorem 5.2 in definition 9.1. \( \square \)

Proposition 9.2. Let \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\), then we have

\[
\mathcal{C} \circ \mathcal{V}_1[(x_1, x_2, \ldots, x_n)] \ll \sup(x_j)^{n+1} \sqrt{n}
\]

and

\[
\mathcal{C} \circ \mathcal{V}_1[(x_1, x_2, \ldots, x_n)] \gg \inf(x_j)^{n+1} \sqrt{n}.
\]

Proof. The result follows by leveraging various estimates developed. \( \square \)
10. The ball induced by compression

In this section we introduce the notion of the ball induced by a point \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) under compression of a given scale. We launch more formally the following language.

**Definition 10.1.** Let \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) with \(x_i \neq x_j\) for all \(1 \leq i < j \leq n\). Then by the ball induced by \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) under compression of scale \(m > 0\), denoted \(B_{\frac{1}{2}G \circ V_m}[(x_1, x_2, \ldots, x_n)]\) we mean the inequality

\[
\left| \vec{y} - \frac{1}{2} \left( x_1 + \frac{m}{x_1}, x_2 + \frac{m}{x_2}, \ldots, x_n + \frac{m}{x_n} \right) \right| < \frac{1}{2} G \circ V_m[(x_1, x_2, \ldots, x_n)].
\]

A point \(\vec{z} = (z_1, z_2, \ldots, z_n) \in B_{\frac{1}{2}G \circ V_m}[(x_1, x_2, \ldots, x_n)]\) if it satisfies the inequality.

**Remark 10.2.** Next we prove that smaller balls induced by points should essentially be covered by the bigger balls in which they are embedded. We state and prove this statement in the following result.

In the geometry of balls induced under compression of scale \(m > 0\), we assume implicitly that \(0 < m \leq 1\).

For simplicity we will on occasion choose to write the ball induced by the point \(\vec{x} = (x_1, x_2, \ldots, x_n)\) under compression as

\[
B_{\frac{1}{2}G \circ V_m} \vec{x}.
\]

We adopt this notation to save enough work space in many circumstances. We first prove a preparatory result in the following sequel. We find the following estimates useful.

**Proposition 10.1.** Let \((x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\) for \(n \geq 2\) with \(x_j \neq 0, 1\) for \(j = 1, \ldots, n\), then we have

\[
G \circ V_m[(x_1, x_2, \ldots, x_n)]^2 = M \circ V_1 \left[ \left( \frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2} \right) \right] + m^2 M \circ V_1[(x_1^2, \ldots, x_n^2)] - 2mn.
\]

In particular, we have the estimate

\[
G \circ V_m[(x_1, x_2, \ldots, x_n)]^2 = M \circ V_1 \left[ \left( \frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2} \right) \right] - 2mn + O \left( m^2 M \circ V_1[(x_1^2, \ldots, x_n^2)] \right)
\]

for \(\vec{x} \in \mathbb{N}^n\).

Proposition 10.1 offers us an extremely useful identity. It allows us to pass from the gap of compression on points to the relative distance to the origin. It tells us that points under compression with a large gap must be far away from the origin than points with a relatively smaller gap under compression. That is to say, the inequality

\[
G \circ V_m[\vec{x}] < G \circ V_m[\vec{y}]
\]

if and only if \(||\vec{x}|| < ||\vec{y}||\) for \(\vec{x}, \vec{y} \in \mathbb{N}^n\). This important transference principle will be mostly put to use in obtaining our results.
Lemma 10.3 (Compression estimate). Let \((x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) for \(n \geq 2\), then we have
\[
\mathcal{G} \circ \mathcal{V}_m[(x_1, x_2, \ldots, x_n)]^2 \ll n \sup(x_j^2) + m^2 \log \left(1 + \frac{n-1}{n \inf(x_j^2)}\right) - 2mn
\]
and
\[
\mathcal{G} \circ \mathcal{V}_m[(x_1, x_2, \ldots, x_n)]^2 \gg n \inf(x_j^2) + m^2 \log \left(1 - \frac{n-1}{n \sup(x_j^2)}\right)^{-1} - 2mn.
\]

Theorem 10.4. Let \(z = (z_1, z_2, \ldots, z_n) \in \mathbb{N}^n\) with \(z_i \neq z_j\) for all \(1 \leq i < j \leq n\). Then \(z \in B_{\mathcal{G} \circ \mathcal{V}_m}[\mathcal{G}(\mathcal{Y})]\) if and only if
\[
\mathcal{G} \circ \mathcal{V}_m[z] \leq \mathcal{G} \circ \mathcal{V}_m[\mathcal{Y}].
\]

Proof. Let \(z \in B_{\mathcal{G} \circ \mathcal{V}_m}[\mathcal{G}(\mathcal{Y})]\) for \(z = (z_1, z_2, \ldots, z_n) \in \mathbb{N}^n\) with \(z_i \neq z_j\) for all \(1 \leq i < j \leq n\), then it follows that \(||\mathcal{Y}|| > ||z||\). Suppose on the contrary that
\[
\mathcal{G} \circ \mathcal{V}_m[z] \geq \mathcal{G} \circ \mathcal{V}_m[\mathcal{Y}],
\]
then it follows that \(||\mathcal{Y}|| \leq ||z||\), which is absurd. Conversely, suppose
\[
\mathcal{G} \circ \mathcal{V}_m[z] < \mathcal{G} \circ \mathcal{V}_m[\mathcal{Y}]
\]
then it follows from Proposition 10.1 that \(||z|| \leq ||\mathcal{Y}||\). It follows that
\[
||z - \frac{1}{2} \left(y_1 + \frac{m}{y_1}, \ldots, y_n + \frac{m}{y_n}\right)|| < \left|| \mathcal{Y} - \frac{1}{2} \left(y_1 + \frac{m}{y_1}, \ldots, y_n + \frac{m}{y_n}\right)\right|| = \frac{1}{2} \mathcal{G} \circ \mathcal{V}_m[\mathcal{Y}].
\]
This certainly implies \(z \in B_{\mathcal{G} \circ \mathcal{V}_m}[\mathcal{G}(\mathcal{Y})]\) and the proof of the theorem is complete. \(\square\)

It is very crucial to recognize that the slightly annoying restriction of the underlying point should not in anyway limit the scope of generality of this result. For the lattice points \(z\) and \(\mathcal{Y}\) can be taken in \(\mathbb{R}^n\) except for those with some zero entry and the result can be adapted by exploiting the estimates in Proposition 10.1.

Theorem 10.5. Let \(x = (x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) with \(x_i \neq x_j\) for all \(1 \leq i < j \leq n\). If \(\mathcal{Y} \in B_{\mathcal{G} \circ \mathcal{V}_m}[x]\) then
\[
B_{\mathcal{G} \circ \mathcal{V}_m}[\mathcal{G}(\mathcal{Y})] \subseteq B_{\mathcal{G} \circ \mathcal{V}_m}[x].
\]

Proof. First let \(\mathcal{Y} \in B_{\mathcal{G} \circ \mathcal{V}_m}[x]\) and suppose for the sake of contradiction that
\[
B_{\mathcal{G} \circ \mathcal{V}_m}[\mathcal{G}(\mathcal{Y})] \not\subseteq B_{\mathcal{G} \circ \mathcal{V}_m}[x].
\]
Then there must exist some \(z \in B_{\mathcal{G} \circ \mathcal{V}_m}[\mathcal{G}(\mathcal{Y})]\) such that \(z \not\in B_{\mathcal{G} \circ \mathcal{V}_m}[x]\). It follows from Theorem 10.4 that
\[
\mathcal{G} \circ \mathcal{V}_m[z] \geq \mathcal{G} \circ \mathcal{V}_m[x].
\]
It follows that
\[
\mathcal{G} \circ \mathcal{V}_m[\mathcal{Y}] > \mathcal{G} \circ \mathcal{V}_m[z]
\]
\[
\geq \mathcal{G} \circ \mathcal{V}_m[x]
\]
\[
> \mathcal{G} \circ \mathcal{V}_m[\mathcal{Y}].
\]
which is absurd, thereby ending the proof.

**Remark 10.6.** Theorem 10.5 tells us that points confined in certain balls induced under compression should by necessity have their induced ball under compression covered by these balls in which they are contained.

### 10.1. Interior points and the limit points of balls induced under compression.

In this section we launch the notion of an interior and the limit point of balls induced under compression. We study this notion in depth and explore some connections.

**Definition 10.7.** Let \( \vec{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{N}^n \) with \( y_i \neq y_j \) for all \( 1 \leq i < j \leq n \). Then a point \( \vec{x} \in B_{\frac{1}{2}V_0V_m}[\vec{y}] \) is an interior point if

\[
B_{\frac{1}{2}V_0V_m}[\vec{x}] \subseteq B_{\frac{1}{2}V_0V_m}[\vec{y}]
\]

for most \( \vec{x} \in B_{\frac{1}{2}V_0V_m}[\vec{y}] \). An interior point \( \vec{x} \) is then said to be a limit point if

\[
B_{\frac{1}{2}V_0V_m}[\vec{x}] \subseteq B_{\frac{1}{2}V_0V_m}[\vec{y}]
\]

for all \( \vec{x} \in B_{\frac{1}{2}V_0V_m}[\vec{y}] \).

**Remark 10.8.** Next we prove that there must exist an interior and limit point in any ball induced by points under compression of any scale in any dimension.

**Theorem 10.9.** Let \( \vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{N}^n \) with \( x_i \neq x_j \) for all \( 1 \leq i < j \leq n \). Then the ball \( B_{\frac{1}{2}V_0V_m}[\vec{x}] \) contains an interior point and a limit point.

**Proof.** Let \( \vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{N}^n \) with \( x_i \neq x_j \) for all \( 1 \leq i < j \leq n \) and suppose on the contrary that \( B_{\frac{1}{2}V_0V_m}[\vec{x}] \) contains no limit point. Then pick

\[
\vec{z}_1 \in B_{\frac{1}{2}V_0V_m}[\vec{x}]
\]

Then by Theorem 10.5, it follows that

\[
B_{\frac{1}{2}V_0V_m}[\vec{z}_1] \subset B_{\frac{1}{2}V_0V_m}[\vec{x}]
\]

with \( G \circ V_m[\vec{z}_1] < G \circ V_m[\vec{x}] \). Again pick \( \vec{z}_2 \in B_{\frac{1}{2}V_0V_m}[\vec{z}_1] \). Then by employing Theorem 10.5, it follows that

\[
B_{\frac{1}{2}V_0V_m}[\vec{z}_2] \subset B_{\frac{1}{2}V_0V_m}[\vec{z}_1]
\]

with \( G \circ V_m[\vec{z}_2] < G \circ V_m[\vec{z}_1] \). By continuing the argument in this manner we obtain the infinite descending sequence of the gap of compression

\[
G \circ V_m[\vec{x}] > G \circ V_m[\vec{z}_1] > G \circ V_m[\vec{z}_2] > \cdots
\]

thereby ending the proof of the theorem.

**Proposition 10.2.** The point \( \vec{x} = (x_1, x_2, \ldots, x_n) \) with \( x_i = 1 \) for each \( 1 \leq i \leq n \) is the limit point of the ball \( B_{\frac{1}{2}G_0V_1}[\vec{y}] \) for any \( \vec{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \) with \( y_i > 1 \) for each \( 1 \leq i \leq n \).

**Proof.** Applying the compression \( V_1 : \mathbb{R}^n \rightarrow \mathbb{R}^n \) on the point \( \vec{x} = (x_1, x_2, \ldots, x_n) \) with \( x_i = 1 \) for each \( 1 \leq i \leq n \), we obtain \( V_1[\vec{x}] = (1, 1, \ldots, 1) \) so that \( G \circ V_1[\vec{x}] = 0 \) and the corresponding ball induced under compression \( B_{\frac{1}{2}G_0V_1}[\vec{x}] \) contains only the point \( \vec{x} \). It follows by Definition 10.9 the point \( \vec{x} \) must be the limit point of the ball \( B_{\frac{1}{2}G_0V_1}[\vec{x}] \). It follows that

\[
B_{\frac{1}{2}G_0V_1}[\vec{x}] \subseteq B_{\frac{1}{2}G_0V_1}[\vec{y}]
\]
Definition 10.10. Study this notion in depth and explore some possible connections.

The notion of admissible points of balls induced by points under compression. We hold for some \( \vec{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \) with \( y_i > 1 \) for each \( 1 \leq i \leq n \), then there must exists some point \( \vec{z} \in B_{\frac{1}{2}G \circ V_1}[\vec{x}] \) such that \( \vec{z} \notin B_{\frac{1}{2}G \circ V_1}[\vec{y}] \). Since \( \vec{x} \) is the only point in the ball \( B_{\frac{1}{2}G \circ V_1}[\vec{x}] \), it follows that \( \vec{x} \notin B_{\frac{1}{2}G \circ V_1}[\vec{y}] \).

Appealing to Theorem 10.4, we have the corresponding inequality of compression gaps
\[
G \circ V_1[\vec{x}] > G \circ V_1[\vec{y}]
\]
so that by appealing to Proposition 10.1 and the ensuing remarks, we have the inequality of their corresponding distance relative to the origin
\[
||\vec{x}|| > ||\vec{y}||.
\]
This is a contradiction, since by our earlier assumption \( \vec{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \) with \( y_i > 1 \) for each \( 1 \leq i \leq n \). Thus the point \( \vec{x} = (x_1, x_2, \ldots, x_n) \) with \( x_i = 1 \) for each \( 1 \leq i \leq n \) must be the limit point of any ball of the form
\[
B_{\frac{1}{2}G \circ V_1}[\vec{y}]
\]
for any \( \vec{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \) with \( y_i > 1 \) for each \( 1 \leq i \leq n \).

10.2. Admissible points of balls induced under compression. We launch the notion of admissible points of balls induced by points under compression. We study this notion in depth and explore some possible connections.

Definition 10.10. Let \( \vec{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{N}^n \) with \( y_i \neq y_j \) for all \( 1 \leq i < j \leq n \). Then \( \vec{y} \) is said to be an admissible point of the ball \( B_{\frac{1}{2}G \circ V_m}[\vec{x}] \) if
\[
||\vec{y} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, \ldots, x_n + \frac{m}{x_n}\right)|| = \frac{1}{2}G \circ V_m[\vec{x}].
\]

Remark 10.11. It is important to notice that the notion of admissible points of balls induced by points under compression encompasses points on the ball. These points in geometrical terms basically sit on the outer of the induced ball. Next we show that all balls can in principle be generated by their admissible points.

Theorem 10.12. The point \( \vec{y} \in B_{\frac{1}{2}G \circ V_m}[\vec{x}] \) is admissible if and only if
\[
B_{\frac{1}{2}G \circ V_m}[\vec{y}] = B_{\frac{1}{2}G \circ V_m}[\vec{x}]
\]
and \( G \circ V_m[\vec{y}] = G \circ V_m[\vec{x}] \).

Proof. First let \( \vec{y} \in B_{\frac{1}{2}G \circ V_m}[\vec{x}] \) be admissible and suppose on the contrary that
\[
B_{\frac{1}{2}G \circ V_m}[\vec{y}] \neq B_{\frac{1}{2}G \circ V_m}[\vec{x}].
\]
Then there exists some \( \vec{z} \in B_{\frac{1}{2}G \circ V_m}[\vec{x}] \) such that
\[
\vec{z} \notin B_{\frac{1}{2}G \circ V_m}[\vec{y}].
\]
Applying Theorem 10.4, we obtain the inequality
\[
G \circ V_m[\vec{y}] \leq G \circ V_m[\vec{z}] < G \circ V_m[\vec{x}].
\]
It follows from Proposition 10.1 that \(|\vec{x}| < |\vec{y}|\) or \(|\vec{y}| < |\vec{x}|\). By joining this points to the origin by a straight line, this contradicts the fact that the point \(\vec{y}\) is an admissible point of the ball \(B_{1\frac{1}{2}G\circ V_m}[\vec{x}]\) is an admissible point. The latter equality of compression gaps follows from the requirement that the balls are indistinguishable. Conversely, suppose

\[
B_{1\frac{1}{2}G\circ V_m}[\vec{y}] = B_{1\frac{1}{2}G\circ V_m}[\vec{x}]
\]

and \(G \circ V_m[\vec{y}] = G \circ V_m[\vec{x}].\) Then it follows that the point \(\vec{y}\) lives on the outer of the two indistinguishable balls and so must satisfy the equality

\[
\left\|\vec{z} - \frac{1}{2} \left(\frac{m}{y_1}, \ldots, \frac{m}{y_n}\right)\right\| = \left\|\vec{z} - \frac{1}{2} \left(\frac{m}{x_1}, \ldots, \frac{m}{x_n}\right)\right\| = \frac{1}{2} G \circ V_m[\vec{x}].
\]

It follows that

\[
\frac{1}{2} G \circ V_m[\vec{x}] = \left\|\vec{y} - \frac{1}{2} \left(\frac{m}{x_1}, \ldots, \frac{m}{x_n}\right)\right\|
\]

and \(\vec{y}\) is indeed admissible, thereby ending the proof. \(\square\)

10.3. **The dilation of the ball induced by compression.** In this section we introduce the notion of the dilation of balls induced by points under compression. We study this in relation to other concepts of compression.

**Definition 10.13.** Let \(\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) and \(V_m : \mathbb{R}^n \rightarrow \mathbb{R}^n\) be a compression of scale \(m\). Then by the dilation of the induced ball \(B_{1\frac{1}{2}G\circ V_m}[\vec{x}]\) by a scale factor of \(t > 0\), we mean the map

\[
B_{1\frac{1}{2}G\circ V_m}[\vec{x}] \rightarrow B_{1\frac{1}{2}G\circ V_m}[\vec{x}] = B_{1\frac{1}{2}G\circ V_m}[t\vec{x}].
\]

**Remark 10.14.** Next we show that we can in practice embed all balls in their positive dilation.

**Proposition 10.3.** Let \(\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\). For all \(t > 1\), we have

\[
B_{1\frac{1}{2}G\circ V_m}[\vec{x}] \subset B_{1\frac{1}{2}G\circ V_m}[\vec{x}] = B_{1\frac{1}{2}G\circ V_m}[t\vec{x}].
\]

**Proof.** First let \(\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{N}^n\) and take \(t > 1\). Suppose

\[
B_{1\frac{1}{2}G\circ V_m}[\vec{x}] \notin B_{1\frac{1}{2}G\circ V_m}[\vec{x}] = B_{1\frac{1}{2}G\circ V_m}[t\vec{x}].
\]

Then it follows that there exist some \(\vec{z} \in B_{1\frac{1}{2}G\circ V_m}[\vec{x}]\) such that \(\vec{z} \notin B_{1\frac{1}{2}G\circ V_m}[\vec{x}] = B_{1\frac{1}{2}G\circ V_m}[t\vec{x}].\) By Theorem 10.3. It follows that

\[
G \circ V_m[\vec{x}] > G \circ V_m[\vec{x}]
\]

\[
\geq G \circ V_m[t\vec{x}]
\]

\[
> tG \circ V_m[\vec{x}].
\]

This is absurd since \(t > 1\), and the proof is complete. \(\square\)
The result in Proposition 10.3 can be thought of as an analogue of most embedding theorems. It tells us for the most part we can in principle cover all balls of various sizes by their dilates. Next we show that dilation of balls and their sub-balls still preserves an embedding in the ball. We formalize this assertion in the following proposition.

**Proposition 10.4.** Let \( \vec{y} \in \mathbb{N}^n \) with \( \vec{y} \in B_{1/2^G \circ V_m} [\vec{x}] \). Then for any \( t > 1 \), we have

\[
B_{1/2^G \circ V_m} [\vec{y}] \subseteq B_{1/2^G \circ V_m} [\vec{x}].
\]

**Proof.** First suppose \( \vec{y} \in \mathbb{N}^n \) with \( \vec{y} \in B_{1/2^G \circ V_m} [\vec{x}] \). Then by Theorem 10.5 it follows that

\[
B_{1/2^G \circ V_m} [\vec{y}] \subseteq B_{1/2^G \circ V_m} [\vec{x}]
\]

and it follows from Proposition 10.1 that \( ||\vec{y}|| < ||\vec{x}|| \). Now suppose on the contrary that \( B_{1/2^G \circ V_m} [\vec{y}] \not\subseteq B_{1/2^G \circ V_m} [\vec{x}] \).

Then it follows that there exist some \( \vec{z} \in \mathbb{N}^n \) with \( \vec{z} \in B_{1/2^G \circ V_m} [\vec{y}] \) such that \( \vec{z} \notin B_{1/2^G \circ V_m} [\vec{x}] \). By appealing to Theorem 10.4, it follows that

\[
G \circ V_m [t\vec{y}] > G \circ V_m [\vec{z}]
\]

\[
\geq G \circ V_m [t\vec{x}].
\]

This certainly implies \( ||t\vec{x}|| < ||t\vec{y}|| \) for \( t > 1 \) by appealing to Proposition 10.1. This is a contradiction, and the proof of the Proposition is complete. \( \square \)

### 10.4. The order of points in the ball induced under compression.

In this section we introduce the notion of the order of points contained in balls induced under compression on points in \( \mathbb{N}^n \). We launch the following formal language.

**Definition 10.15.** Let \( \vec{y} = (y_1, y_2, \ldots , y_n) \in \mathbb{N}^n \) with \( \vec{y} \in B_{1/2^G \circ V_m} [\vec{x}] \). Then we say the point \( \vec{y} \) is of order \( t > 0 \) in the ball if \( \vec{x} \parallel \vec{y} \) and there exist some \( t > 0 \) such that

\[
B_{1/2^G \circ V_m} [t\vec{y}] = B_{1/2^G \circ V_m} [\vec{x}].
\]

Otherwise we say the point \( \vec{y} \) is free in the ball.

**Remark 10.16.** Next we show that the existence of order of points in a ball induced by points under compression is mostly in continuum. We formalize this claim in the following proposition.

**Proposition 10.5.** Let \( \vec{x}, \vec{y}, \vec{z} \in \mathbb{N}^n \) with \( \vec{y} \in B_{1/2^G \circ V_m} [\vec{x}] \) and \( B_{1/2^G \circ V_m} [\vec{x}] \subseteq B_{1/2^G \circ V_m} [\vec{z}] \). If the point \( \vec{y} \in B_{1/2^G \circ V_m} [\vec{x}] \) is of order \( t > 1 \) and the point \( \vec{x} \in B_{1/2^G \circ V_m} [\vec{z}] \) is of order \( s > 1 \). Then the point

\[
\vec{y} \in B_{1/2^G \circ V_m} [\vec{z}]
\]

is of order \( st > 1 \).
Proof. First suppose \( x, y, \bar{z} \in \mathbb{N}^n \) with \( y \in B_{G \circ V_m}([x]) \) and \( B_{G \circ V_m}([x]) \subset B_{G \circ V_m}([\bar{z}]) \). Then by Theorem 10.5, we have the following chains of ball embedding
\[
B_{G \circ V_m}([y]) \subset B_{G \circ V_m}([x]) \subset B_{G \circ V_m}([\bar{z}]).
\]
Since \( y \in B_{G \circ V_m}([x]) \) is of order \( t > 1 \), it follows that
\[
B_{G \circ V_m}([y]) = B_{G \circ V_m}([t y]) = B_{G \circ V_m}([x]).
\]
and by appealing to Theorem 10.4, \( G \circ V_m([t y]) = G \circ V_m([x]) \) and it follows that \( ||t y|| = ||x|| \), by Proposition 10.1. Again the point \( x \in B_{G \circ V_m}([\bar{z}]) \) is of order \( s > 1 \) and it follows that
\[
B_{G \circ V_m}([x]) = B_{G \circ V_m}([s x]) = B_{G \circ V_m}([\bar{z}]).
\]
By appealing to Theorem 10.3, it follows that \( G \circ V_m([s x]) = G \circ V_m([\bar{z}] \) and \( ||s x|| = ||\bar{z}|| \). By combining the two relations, we have
\[
st||y|| = ||\bar{z}||
\]
It follows that \( sty = \bar{z} \) and the result follows immediately. \( \square \)

11. Application to the Erdős-Ulam problem

In this section we apply the topology to the Erdős-Ulam problem in the following sequel. We first launch the following preparatory results.

Lemma 11.1. Let \( x \in \mathbb{N}^n \) with \( m \in \mathbb{Q} \). Then \( G \circ V_m([x]) \times G \circ V_m([x]) \in \mathbb{Q} \). That is, the square of compression gap induced on the point \( y \in \mathbb{N}^n \) is always rational.

Proof. Suppose \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{N}^n \) and let \( m \in \mathbb{Q} \), then by invoking Proposition 10.1, we have
\[
G \circ V_m([x]) \times G \circ V_m([x]) = M \circ V_1 \left[ \left( \frac{1}{x_1}, \ldots, \frac{1}{x_n} \right) \right] + m^2 M \circ V_1 [(x_1^2, \ldots, x_n^2)] = 2mn.
\]
The result follows since
\[
M \circ V_1 \left[ \left( \frac{1}{x_1}, \ldots, \frac{1}{x_n} \right) \right], m^2 M \circ V_1 [(x_1^2, \ldots, x_n^2)] \in \mathbb{Q}
\]
thereby proving the Lemma. \( \square \)

Lemma 11.2. Let \( x \in \mathbb{N}^n \) with \( G \circ V_m([x]) > 1 \), then
\[
B_{G \circ V_m([x])}^2 G \circ V_m([x]) \subset B_{G \circ V_m([G \circ V_m([x])])} G \circ V_m([x])^2.
\]

Proof. Suppose \( x \in \mathbb{N}^n \) and let \( G \circ V_m([x]) > 1 \). First, we notice that the two balls so constructed
\[
B_{G \circ V_m([x])}^2 G \circ V_m([x])^2 \quad \text{and} \quad B_{G \circ V_m([G \circ V_m([x])])} G \circ V_m([x])^2
\]
are centered at the same point. Thus it suffices to show that
\[
(G \circ V_m([x]))^2 \leq G \circ V_m(G \circ V_m([x])^2).
\]
Now let us set \( t = G \circ V_m(t x) > 1 \). Then we obtain
\[
G \circ V_m([tx]) > t(G \circ V_m([x])
\]
and the result follows by substitution. □

Remark 11.3. We are now ready to prove the Erdős-Ulam conjecture. We assemble the tools we have developed thus far to solve the problem.

11.1. Proof of the Erdős-Ulam conjecture. In this section we assemble the tools we have developed thus far to solve the Erdős-Ulam problem. We provide a positive solution to the problem as espoused in the following result.

Theorem 11.4. There exists a dense set of points in \( \mathbb{R}^2 \) at rational distances from each other.

Proof. Pick arbitrarily \( \vec{x} \in \mathbb{N}^2 \) and apply the compression \( V_m[\vec{x}] \) for \( m \in \mathbb{N} \). Consider the ball induced under compression \( B_{G \circ V_m[\vec{x}]^2} \). Now dilate the ball with the scale factor \( t = G \circ V_m[\vec{x}] > 1 \), then by Lemma 12.3 we obtain the embedding of balls

\[
B_{G \circ V_m[\vec{x}]^2} \subset B_{G \circ V_m[\vec{x}]^2} \subset B_{G \circ V_m[\vec{x}]^2} \subset \cdots \subset B_{G \circ V_m[\vec{x}]^2}.
\]

Let us now consider the inner ball, centered at the same point as the outer ball, but of rational radius by Lemma 11.1.

For each admissible point \( \vec{z} \) of \( B_{G \circ V_m[\vec{x}]^2} \) we join with a line to the admissible point exactly opposite. These two points are at rational distances

\[
\frac{1}{2}(G \circ V_m[\vec{x}])^2 + \frac{1}{2}(G \circ V_m[\vec{x}])^2 = (G \circ V_m[\vec{x}])^2
\]

from each other. We remark that the point \( \vec{z} \in \mathbb{R}^2 \) is an arbitrary admissible point and are dense on the ball. Since there exist dense set of points on circles of this form at rational distance from each other there are arbitrarily and infinitely many rational distance chords at all directions and lines sufficiently close to each other and of rational distances joining admissible points of the ball

\[
B_{G \circ V_m[\vec{x}]^2} \subset B_{G \circ V_m[\vec{x}]^2} \subset B_{G \circ V_m[\vec{x}]^2} \subset \cdots \subset B_{G \circ V_m[\vec{x}]^2}.
\]

We construct sequence of embedding of balls in the following manner

\[
B_{\frac{1}{2n}(G \circ V_m[\vec{x}])^2} \subset B_{\frac{1}{2n}(G \circ V_m[\vec{x}])^2} \subset \cdots \subset B_{\frac{1}{2n}(G \circ V_m[\vec{x}])^2} \subset B_{\frac{1}{2n}(G \circ V_m[\vec{x}])^2}.
\]

for \( n \geq 2 \). The upshot is concentric balls all centered at the same point with successively smaller radius

\[
\frac{1}{2n}(G \circ V_m[\vec{x}])^2
\]

for \( n \geq 2 \). We remark that the lines drawn joining points on the bigger ball will also join points on the smaller balls at rational distance. The distance of points on different balls on the same line are also at rational distance from each other. That is, if \( s_1 \in B_{\frac{1}{2n}(G \circ V_m[\vec{x}])^2} \) and \( s_2 \in B_{\frac{1}{2n}(G \circ V_m[\vec{x}])^2} \) and \( s_1 \) and \( s_2 \) sit on the same line, then they must be of rational distance

\[
\frac{1}{2}(G \circ V_m[\vec{x}])^2 - \frac{1}{4}(G \circ V_m[\vec{x}])^2 = \frac{1}{2}(G \circ V_m[\vec{x}])^2.
\]
by Lemma [11]. In general, the radius of the annular region of successive balls so constructed is rational given by

$$\frac{1}{2n} (G \circ V_m[\vec{x}])^2 - \frac{1}{2(n+1)} (G \circ V_m[\vec{x}])^2 = \frac{1}{2n(n+1)} (G \circ V_m[\vec{x}])^2$$

for \( n \in \mathbb{N} \) for all \( n \geq 1 \). Again we construct sequence of embedding of balls centered at the same point as before below

$$B_{\frac{1 + 2n}{4(n+1)}(G \circ V_m[\vec{x}])^2} \subseteq \cdots \subseteq B_{\frac{1}{2n} (G \circ V_m[\vec{x}])^2} \subseteq [G \circ V_m[\vec{x}]]^2$$

for \( n \in \mathbb{N} \) with \( n \geq 2 \). Admissible points of each of these balls are at rational distances away from the admissible point exactly opposite. That is, they are

$$\frac{1 + 2n}{2n(n+1)}$$

for \( n \geq 1 \). It is not difficult to see that we can embed this sequence of ball embedding into the a priori sequence of ball embedding. By carrying out the argument in this manner repeatedly, we then generate a dense set of points \( s_n \in \mathbb{R}^2 \) as admissible points of infinitely many embedding that are at rational distance from each point on same line as radii of the annular regions induced by concentric balls. Now for the largest ball so constructed, let us locate the center and chop into sectors of equal area, so that each sector subtends an angle of 120° at the center. We remark that this configuration is propagated on all the concentric balls constructed, so that it suffices to analyse the situation in only one ball. Let us choose arbitrarily a ball contained in any of the constructed embedding

$$B_{\{(G \circ V_m[\vec{x}])^2 \}} [G \circ V_m[\vec{x}]]^2.$$ 

Next let us pick an admissible point \( \vec{y} \in B_{\{(G \circ V_m[\vec{x}])^2 \}} [G \circ V_m[\vec{x}]]^2 \) and on one of the sectors constructed and join to the admissible point on a different sector by a straight line. Let us locate dense set of admissible points of embedded concentric balls that are contained in the ball \( B_{\{(G \circ V_m[\vec{x}])^2 \}} [G \circ V_m[\vec{x}]]^2 \) and on the same line and at rational distance with the admissible point \( \vec{y} \) and join them to corresponding admissible points on the different sector - as before - with a straight line so that it is parallel with the chord above. It follows from this construction a portrait of piled up isosceles trapezoid and one isosceles triangle with one vertex as the center of the ball. This is the consequence of chopping the sector of the bigger ball by mutually parallel lines.

Let us now consider the isosceles triangle produced with one vertex at the center and denote the length of the lateral sides to be \( T \) units, where \( T \) is rational. Then it follows that the base length - which is also the length of the shorter side of a parallel side of the next trapezoid is given by

$$2T\sqrt{3} \text{ units}.$$ 

Next we construct the intersecting diagonals in each of the isosceles trapezoid. It is important to note that the diagonals of of equal length and their intersections produce two equilateral triangles below and above and inside the trapezoid, with the base of the triangle becoming the base of the smaller triangle induced. Let \( K \) units denotes the length of the longer lateral side of the isosceles triangle whose base is the longest chord in the ball \( B_{\{(G \circ V_m[\vec{x}])^2 \}} [G \circ V_m[\vec{x}]]^2 \) of rational length by
virtue of our construction, we obtain the length of each diagonal as

$$2T\sqrt{3} + 2S\sqrt{3} = (2T + 2S)\sqrt{3} \text{ units.}$$

Next let us apply the reduction map by a scale factor $\sqrt{3}$ to the parallel sides of the trapezoid given as

$$\mathcal{R}_{\sqrt{3}} : \text{Trap} \rightarrow \text{Trap}_{\sqrt{3}}$$

while keeping the length of the legs unchanged. Then the length of the diagonals of the new trapezoid are now rational and is given by

$$|\text{Diag}_{\text{Trap}}| = (2T + 2S) \text{ units}$$

since $S, T \in \mathbb{R}$. By shrinking the old isosceles trapezoid by a fixed rational scale factor and throwing away vertices of the old isosceles trapezoid to obtain another new isosceles trapezoid covered by the a priori old one and subsequently applying the reduction map to the parallel sides

$$\mathcal{R}_{\sqrt{3}} : \text{Trap} \rightarrow \text{Trap}_{\sqrt{3}}$$

while keeping the length of the legs unchanged, we obtain a new isosceles trapezoid covered by the a priori newly reduced version of scale factor $\sqrt{3}$. Next let us throw away the reduced isosceles trapezoid of the old one of rational scale factor. By repeating the process, we generate a dense set of points as vertices of the new isosceles trapezoid which are at rational distance from each other and remaining vertices of all other reduced isosceles trapezoid of scale factor $\sqrt{3}$ from the construction and covering this new one. That is each vertex of the new isosceles trapezoid is at rational distance from the remaining vertices of the isosceles and all other isosceles trapezoid in the sector. This completes the proof, since the radius of the ball is determined by the point $\vec{x} \in \mathbb{N}^2$ under compression and this point can be chosen arbitrarily in space and the trapezoid in our construction are dense in the sector by virtue of our construction. That is, we can cover the entire plane with this construction by arbitrarily taking points far away from the origin. $\square$

12. Application to the function modeling an $l$-step self avoiding walks

12.1. Compression lines. In this section we study the notion of lines induced under compression of a given scale and the associated geometry. We first launch the following language.

**Definition 12.1.** Let $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_1 \neq 0, 1$ for $1 \leq i \leq n$. Then by the line $L_{\vec{x}, \mathcal{V}_m[\vec{x}]}$ produced under compression $\mathcal{V}_m : \mathbb{R}^n \rightarrow \mathbb{R}^n$ we mean the line joining the points $\vec{x}$ and $\mathcal{V}_m[\vec{x}]$ given by

$$\vec{r} = \vec{x} + \lambda(\vec{x} - \mathcal{V}_m[\vec{x}])$$

where $\lambda \in \mathbb{R}$.

**Remark 12.2.** In striving for the simplest possible notation and to save enough work space, we will choose instead to write the line produced under compression $\mathcal{V}_m : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $L_{\mathcal{V}_m[\vec{x}]}$. Next we show that the lines produced under compression of two distinct points not on the same line of compression cannot intersect at the corresponding points and their images under compression.
Lemma 12.3. Let \( \vec{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \) with \( \vec{a} \neq \vec{x} \) and \( a_i, x_j \neq 0, 1 \) for \( 1 \leq i, j \leq n \). If the point \( \vec{a} \) lies on the corresponding line \( L_{\mathbb{R}}[\vec{x}] \), then \( \mathbb{V}_m[\vec{a}] \) also lies on the same line.

Proof. Pick arbitrarily a point \( \vec{a} \) on the line \( L_{\mathbb{R}}[\vec{x}] \) produced under compression for any \( \vec{x} \in \mathbb{R}^n \). Suppose on the contrary that \( \mathbb{V}_m[\vec{a}] \) cannot live on the same line as \( \vec{a} \). Then \( \mathbb{V}_m[\vec{a}] \) must be away from the line \( L_{\mathbb{R}}[\vec{x}] \). Produce the compression line \( L_{\mathbb{R}}[\vec{a}] \) by joining the point \( \vec{a} \) to the point \( \mathbb{V}_m[\vec{a}] \) by a straight line. Then it follows from Proposition 10.1

\[ \mathcal{G} \circ \mathbb{V}_m[\vec{x}] > \mathcal{G} \circ \mathbb{V}_m[\vec{a}] \cdot \]

Again pick a point \( \vec{c} \) on the line \( L_{\mathbb{R}}[\vec{a}] \), then under the assumption it follows that the point \( \mathbb{V}_m[\vec{c}] \) must be away from the line. Produce the compression line \( L_{\mathbb{R}}[\vec{c}] \) by joining the points \( \vec{c} \) to \( \mathbb{V}_m[\vec{c}] \). Then by Proposition 10.1 we obtain the following decreasing sequence of lengths of distinct lines

\[ \mathcal{G} \circ \mathbb{V}_m[\vec{x}] > \mathcal{G} \circ \mathbb{V}_m[\vec{a}] > \mathcal{G} \circ \mathbb{V}_m[\vec{c}] . \]

By repeating this argument, we obtain an infinite descending sequence of lengths of distinct lines

\[ \mathcal{G} \circ \mathbb{V}_m[\vec{x}] > \mathcal{G} \circ \mathbb{V}_m[\vec{a}] > \cdots > \mathcal{G} \circ \mathbb{V}_m[\vec{a}] > \cdots . \]

This proves the Lemma. \( \square \)

It is important to point out that Lemma 12.3 is the ultimate tool we need to show that certain function is indeed a function modeling \( l \)-step self avoiding walk. We first launch such a function as an outgrowth of the notion of compression. Before that we launch our second Lemma. One could think of this result as an extension of Lemma 12.3.

Lemma 12.4. Let \( \vec{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \) and \( \vec{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n \) be points with identical configurations with \( \vec{a} \neq \vec{b} \) and \( a_i, b_j \neq 0, 1 \) for \( 1 \leq i, j \leq n \). If the corresponding lines \( L_{\mathbb{R}}[\vec{a}] : r_1 = \vec{a} + \lambda(\vec{a} - \mathbb{V}_m[\vec{a}]) \) and \( L_{\mathbb{R}}[\vec{b}] : r_2 = \vec{b} + \mu(\vec{b} - \mathbb{V}_m[\vec{b}]) \) for \( \mu, \lambda \in \mathbb{R} \) intersect, then

\[ \vec{a} - \mathbb{V}_m[\vec{a}] \parallel \vec{b} - \mathbb{V}_m[\vec{b}] . \]

Proof. First consider the points \( \vec{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \) and \( \vec{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n \) with \( \vec{a} \neq \vec{b} \) and \( a_i, b_j \neq 0, 1 \) for \( 1 \leq i, j \leq n \) with corresponding lines \( L_{\mathbb{R}}[\vec{a}] : r_1 = \vec{a} + \lambda(\vec{a} - \mathbb{V}_m[\vec{a}]) \) and \( L_{\mathbb{R}}[\vec{b}] : r_2 = \vec{b} + \mu(\vec{b} - \mathbb{V}_m[\vec{b}]) \) for \( \mu, \lambda \in \mathbb{R} \). Suppose they intersect at the point \( \vec{s} \), then it follows that the point \( \mathbb{V}_m[\vec{s}] \) lies on the lines \( L_{\mathbb{R}}[\vec{a}] : r_1 = \vec{a} + \lambda(\vec{a} - \mathbb{V}_m[\vec{a}]) \) and \( L_{\mathbb{R}}[\vec{b}] : r_2 = \vec{b} + \mu(\vec{b} - \mathbb{V}_m[\vec{b}]) \) and the result follows immediately. \( \square \)

Lemma 12.3 combined with Lemma 12.4 tells us that the line produced by compression on points with certain configuration away from other lines of compression are not intersecting. We leverage this principle to show that a certain function indeed models a self-avoiding walk.

Remark 12.5. Next we show that the lines produced under compression and their corresponding lines under translation are non-intersecting.
Proposition 12.1. Let $L_{V_m}[\vec{x}]$ and $L_{V_m}[\vec{y}]$ be two distinct lines under compression. Then the corresponding lines $L_{\Gamma \circ V_m}[\vec{x}]$ and $L_{\Gamma \circ V_m}[\vec{y}]$ for a fixed $\vec{a} \in \mathbb{R}^n$ are distinct and non-intersecting.

Proof. Suppose the lines $L_{\Gamma \circ V_m}[\vec{x}]$ and $L_{\Gamma \circ V_m}[\vec{y}]$ for a fixed $\vec{a} \in \mathbb{R}^n$ intersect and let $\vec{s}$ be their point of intersection. Then it follows that there exist some $1 \geq k_1, k_2 > 0$ such that $\Gamma_{k_1 \vec{a}} \circ V_m[\vec{x}] = \vec{s}$ and $\Gamma_{k_2 \vec{a}} \circ V_m[\vec{y}] = \vec{s}$. Then we can write

$$V_m[\vec{x}] = \Gamma_{k_1 \vec{a}}^{-1} \circ \Gamma_{k_2 \vec{a}} \circ V_m[\vec{y}]$$

$$= \Gamma_{(k_2 - k_1) \vec{a}} \circ V_m[\vec{y}].$$

It follows that either the point $V_m[\vec{x}]$ lies on the line $L_{\Gamma \circ V_m}[\vec{y}]$ or the point $V_m[\vec{y}]$ lies on the line $L_{\Gamma \circ V_m}[\vec{x}]$. Without loss of generality, let us let the point $V_m[\vec{x}]$ lie on the line $L_{\Gamma \circ V_m}[\vec{y}]$. Under the underlying assumption, the following equations hold

$$\Gamma_{k_1 \vec{a}} \circ V_m[\vec{x}] = V_m[\vec{x}] \quad \text{and} \quad \Gamma_{k_2 \vec{a}} \circ V_m[\vec{y}] = V_m[\vec{y}].$$

This is absurd since the lines $L_{V_m}[\vec{x}]$ and $L_{V_m}[\vec{y}]$ are distinct.

Proposition 12.2. Compression balls are non-overlapping.

Proof. Pick arbitrarily points $\vec{x}, \vec{y} \in \mathbb{R}^n$ with $x_i, y_i \neq 0$ for all $i = 1, \ldots, n$ such that $\vec{x} \neq \vec{y}$ with $||\vec{x}|| \neq ||\vec{y}||$. Then it follows that $||\vec{x}|| < ||\vec{y}||$ or $||\vec{x}|| > ||\vec{y}||$. Without loss of generality, let us assume that $||\vec{x}|| < ||\vec{y}||$, then it follows from Proposition 10.1

$$G \circ V_m[\vec{x}] < G \circ V_m[\vec{y}]$$

or

$$G \circ V_m[\vec{y}] < G \circ V_m[\vec{x}].$$

By appealing to Theorem 10.4 and Theorem 10.5 it follows that

$$B_{\frac{1}{2}G \circ V_m}[\vec{x}][\vec{x}] \subset B_{\frac{1}{2}G \circ V_m}[\vec{y}][\vec{y}]$$

or

$$B_{\frac{1}{2}G \circ V_m}[\vec{y}][\vec{y}] \subset B_{\frac{1}{2}G \circ V_m}[\vec{x}][\vec{x}].$$

This completes the proof since the points $\vec{x}$ and $\vec{y}$ with $||\vec{x}|| \neq ||\vec{y}||$ were chosen arbitrarily.

Next we show that there must exist some point in a bigger ball whose induced ball under compression has admissible points way off a certain line in the underlying ball. We find the following Lemma useful.

Lemma 12.6. The point $\frac{\vec{y} + V_m[\vec{y}]}{2}$ with $y_i \in B_{\frac{1}{2}G \circ V_m}[\vec{x}][\vec{x}]$ for all $i \in \mathbb{N}$ is on the line $L_{V_m}[\vec{x}]$ if and only if the limits point $\vec{z}$ of the ball $B_{\frac{1}{2}G \circ V_m}[\vec{x}][\vec{x}]$ is on the line $L_{V_m}[\vec{x}]$.

Proof. Let $\frac{\vec{y} + V_m[\vec{y}]}{2}$ for $i \in \mathbb{N}$ with $y_i \in B_{\frac{1}{2}G \circ V_m}[\vec{x}][\vec{x}]$ be on the line $L_{V_m}[\vec{x}]$ with $\vec{x} \neq \vec{y}_i$ for all $i \in \mathbb{N}$. Then by Lemma 12.3 it follows that $V_m[\vec{y}_i] \in B_{\frac{1}{2}G \circ V_m}[\vec{x}][\vec{x}]$ with $G \circ V_m[\vec{y}_i] < G \circ V_m[\vec{x}]$. Let us now construct the ball induced by compression on this point given by $B_{\frac{1}{2}G \circ V_m}[\vec{x}][\vec{y}_i]$ and by Proposition 10.5

$$B_{\frac{1}{2}G \circ V_m}[\vec{y}_i][\vec{y}_i] \subset B_{\frac{1}{2}G \circ V_m}[\vec{x}][\vec{x}]$$
The needle function

Proposition 13.1.

Proof. Consider the ball $B_{\frac{1}{2}R} \circ \mathbb{V}_m[\tilde{y}]$ and suppose on the contrary that for any point $\tilde{z} \in B_{\frac{1}{2}R} \circ \mathbb{V}_m[\tilde{x}]$ the corresponding induced ball under compression $B_{\frac{1}{2}R} \circ \mathbb{V}_m[\tilde{z}]$ intersects the compression line $L_{\mathbb{V}_m}[\tilde{z}]$. Then by Lemma 12.6 the limit point of the ball $B_{\frac{1}{2}R} \circ \mathbb{V}_m[\tilde{z}]$ is on the line $L_{\mathbb{V}_m}[\tilde{z}]$. It follows from Lemma 12.6 the point $\frac{\tilde{z} + \mathbb{V}_m[\tilde{z}]}{2}$ must lie on the line $L_{\mathbb{V}_m}[\tilde{z}]$. This is a contradiction since the point $\tilde{z}$ is an arbitrary point in the ball $B_{\frac{1}{2}R} \circ \mathbb{V}_m[\tilde{x}]$ and so is the point $\frac{\tilde{z} + \mathbb{V}_m[\tilde{z}]}{2}$. This completes the proof of the theorem.

Theorem 12.7.

There exist some $\tilde{z} \in B_{\frac{1}{2}R} \circ \mathbb{V}_m[\tilde{x}]$ such that admissible points of the induced ball $B_{\frac{1}{2}R} \circ \mathbb{V}_m[\tilde{z}]$ are not on the line $L_{\mathbb{V}_m}[\tilde{z}]$.

Proof. Consider the ball $B_{\frac{1}{2}R} \circ \mathbb{V}_m[\tilde{x}]$ and suppose on the contrary that for any point $\tilde{z} \in B_{\frac{1}{2}R} \circ \mathbb{V}_m[\tilde{x}]$ the corresponding induced ball under compression $B_{\frac{1}{2}R} \circ \mathbb{V}_m[\tilde{z}]$ intersects the compression line $L_{\mathbb{V}_m}[\tilde{z}]$. Then by Lemma 12.6 the limit point of the ball $B_{\frac{1}{2}R} \circ \mathbb{V}_m[\tilde{z}]$ is on the line $L_{\mathbb{V}_m}[\tilde{z}]$. It follows from Lemma 12.6 the point $\frac{\tilde{z} + \mathbb{V}_m[\tilde{z}]}{2}$ must lie on the line $L_{\mathbb{V}_m}[\tilde{z}]$. This is a contradiction since the point $\tilde{z}$ is an arbitrary point in the ball $B_{\frac{1}{2}R} \circ \mathbb{V}_m[\tilde{x}]$ and so is the point $\frac{\tilde{z} + \mathbb{V}_m[\tilde{z}]}{2}$. This completes the proof of the theorem.

13. The needle function

In this section we introduce and study the needle function. We combine the geometry of lines under compression and the geometry of balls under compression to prove that this function is a function modeling an $l$-step self avoiding walk.

Definition 13.1. By the needle function of scale $m$ and translation factor $\tilde{a}$, we mean the composite map

$$\Gamma_{\tilde{a}} \circ \mathbb{V}_m : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

such that for any $\tilde{x} \in \mathbb{R}^n$

$$\Gamma_{\tilde{a}} \circ \mathbb{V}_m[\tilde{x}] = \tilde{y}$$

where $\tilde{x} = (x_1, x_2, \ldots, x_n)$ with $x_i \neq 0, 1$ for $1 \leq i \leq n$ and $\Gamma_{\tilde{a}}[\tilde{x}] = (x_1 + a_1, x_2 + a_2, \ldots, x_n + a_n)$.

Proposition 13.1. The needle function $\Gamma_{\tilde{a}} \circ \mathbb{V}_m : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a bijective map of order 2.

Proof. We remark that the translation with translation factor $\tilde{a}$ for a fixed $\tilde{a}$ given by $\Gamma_{\tilde{a}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a bijective map. The result follows since the composite of bijective maps is still bijective.

Theorem 13.2. The map $(\Gamma_{\tilde{a}_1} \circ \mathbb{V}_m) \circ \cdots \circ (\Gamma_{\tilde{a}_k} \circ \mathbb{V}_m) : \mathbb{R}^n \rightarrow \mathbb{R}^n$, where

$$(\Gamma_{\tilde{a}_1} \circ \mathbb{V}_m) \circ \cdots \circ (\Gamma_{\tilde{a}_k} \circ \mathbb{V}_m)$$

is the $k$-fold needle function with mixed translation factors $\tilde{a}_1, \ldots, \tilde{a}_k \in \mathbb{R}^n$, is a function modeling an $l$-step self avoiding walk.
Proof. Pick arbitrarily a point \( \bar{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{N}^n \) for \( n \geq 2 \) with \( x_i \neq 0, 1 \) for \( 1 \leq i \leq n \) and apply the compression \( \mathbb{V}_m[\bar{x}] \) and construct the ball \( \mathcal{B}_{\frac{1}{2}} \mathbb{V}_m[\bar{x}] \). Now choose a point \( \bar{a} \in \mathcal{B}_{\frac{1}{2}} \mathbb{V}_m[\bar{x}] \) and construct the ball \( \mathcal{B}_{\frac{1}{2}} \mathbb{V}_m[\bar{a}] \) so that admissible points do not sit on the compression line \( L_{\mathbb{V}_m[\bar{x}]} \). Let us now join the point \( \mathbb{V}_m[\bar{x}] \) to the closest admissible point \( \bar{t} \) of \( \mathcal{B}_{\frac{1}{2}} \mathbb{V}_m[\bar{a}] \) on the line \( L_{\mathbb{V}_m[u]} \) under a suitable translation vector \( \bar{a} \neq \bar{t} \). Let us now traverse the line produced under compression to the line produced by translation of the point \( \Gamma_{\bar{a}}(\mathbb{V}_m[\bar{x}]) \) with the starting point \( \bar{x} \) to \( \mathbb{V}_m[\bar{x}] \) and from \( \mathbb{V}_m[\bar{x}] \) to \( \Gamma_{\bar{a}}(\mathbb{V}_m[\bar{x}]) = \bar{t} \) and finally from \( \bar{t} \) to \( \mathbb{V}_m[\bar{t}] \). The upshot is a 3-step self avoiding walk. Again we choose a point \( \bar{s} \in \mathcal{B}_{\frac{1}{2}} \mathbb{V}_m[\bar{a}] \) so that the ball \( \mathcal{B}_{\frac{1}{2}} \mathbb{V}_m[\bar{s}] \) satisfies the relation

\[
\mathcal{B}_{\frac{1}{2}} \mathbb{V}_m[\bar{s}] \subset \mathcal{B}_{\frac{1}{2}} \mathbb{V}_m[\bar{a}]
\]

and with the property that admissible points of the inner ball are not allowed to sit on the compression line \( L_{\mathbb{V}_m[\bar{a}]} \). We then join the point \( \mathbb{V}_m[\bar{t}] \) to the closest admissible point of the ball \( \mathcal{B}_{\frac{1}{2}} \mathbb{V}_m[\bar{s}] \) on the line \( L_{\mathbb{V}_m[\bar{a}]} \) under a suitable translation factor \( \bar{a} \neq \bar{t} \). By traversing all these lines starting from the point \( \bar{x} \) to \( \mathbb{V}_m[\bar{x}] \), \( \bar{z} = \mathbb{V}_m[\bar{z}] \) to \( \Gamma_{\bar{a}}(\bar{z}) \), \( \Gamma_{\bar{a}}(\bar{z}) \) to \( \mathbb{V}_m(\bar{z}) \) and finally from \( \mathbb{V}_m \circ \Gamma_{\bar{a}}(\bar{z}) \) to \( \Gamma_{\bar{a}} \circ \mathbb{V}_m \circ \Gamma_{\bar{a}}(\bar{z}) \), we obtain a 4-step self avoiding walk. By continuing this argument \( \frac{1}{n} \) number of times, we produce an \( l \)-step self avoiding walk. This completes the proof. \( \square \)

We remark that we can certainly do more than this by estimating the total length of the self-avoiding walk modeled by this function in the following result.

**Theorem 13.3.** The total length of the \( l \)-step self-avoiding walk modeled by the needle function \( (\Gamma_{\bar{a}} \circ \mathbb{V}_m) \circ \cdots \circ (\Gamma_{\bar{a}} \circ \mathbb{V}_m) : \mathbb{R}^n \rightarrow \mathbb{R}^n \) for \( \bar{a} \in \mathbb{R}^n \) with \( i = 1, 2, \ldots, \frac{l}{2} \) is of order

\[
\ll \frac{l}{2} \sqrt{n} \left( \max \left\{ \sup(x_{jk}) \right\}_{1 \leq j \leq \frac{l}{2}, 1 \leq k \leq n} + \max \left\{ \sup(a_{jk}) \right\}_{1 \leq j \leq \frac{l}{2}, 1 \leq k \leq n} \right)
\]

and at least

\[
\gg \frac{l}{2} \sqrt{n} \left( \min \left\{ \inf(x_{jk}) \right\}_{1 \leq j \leq \frac{l}{2}, 1 \leq k \leq n} + \min \left\{ \inf(a_{jk}) \right\}_{1 \leq j \leq \frac{l}{2}, 1 \leq k \leq n} \right)
\]

Proof. We note that the total length of the \( l \)-step self avoiding walk modeled by the needle function is given by the expression

\[
\sum_{i=1}^{\frac{l}{2}} \mathcal{G} \circ \mathbb{V}_m[\bar{x}_i] + \sum_{i=1}^{\frac{l}{2}} ||\bar{a}_i||
\]

and the result follows by applying the estimates in Lemma 10.3. \( \square \)

14. A combinatorial interpretation

In this section we provide a combinatorial twist of the main result in this paper. We reformulate Theorem 13.2 in the language of graphs. We launch the following language:
Definition 14.1 (Compression graphs). By a compression graph \( G \) of order \( k > 1 \) induced by \( \vec{x}_1 = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n \) with \( u_i \neq 0, 1 \) for all \( 1 \leq i \leq n \), we mean the pair \((V, E)\) where \( V \) is the vertex set

\[
V := \{ \vec{x}_1, V_m[\vec{x}_1], \Gamma_{\vec{x}_1}(V_m[\vec{x}_1]) = \vec{x}_2, V_m[\vec{x}_2], \ldots, \Gamma_{\vec{x}_k}(V_m[\vec{x}_{k-1}]) = \vec{x}_k, V_m[\vec{x}_k] \}
\]

and \( E \) the set of edges

\[
E := \{ L_{\vec{x}_1, V_m[\vec{x}_1]}, L_{V_m[\vec{x}_1], \vec{x}_2}, \ldots, L_{\vec{x}_k, V_m[\vec{x}_k]} \}.
\]

We now state a graph-theoretic version of Theorem 13.2.

Theorem 14.2 (A combinatorial version). There exists a compression graph of order \( l + 1 \) with \( l \in \mathbb{N} \) and whose edges are paths.

Proof. Pick \( \vec{x}_1 = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n \) very far away from the origin with the property that \( u_i > 1 \) and for all \( 1 \leq i \leq n \). Next we apply the compression \( V_1 \) on \( \vec{x}_1 \) and obtain a point \( V_1[\vec{x}_1] \in \mathbb{R}^n \). Let us join the point \( \vec{x}_1 \) to the point \( V_1[\vec{x}_1] \) by a straight line so that by traversing this line starting from \( \vec{x}_1 \) to \( V_1[\vec{x}_1] \), we have the path \( L_{\vec{x}_1, V_1[\vec{x}_1]} \). Next we pick a point \( \vec{x}_2 = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n \setminus B_{\vec{x}_1} \Gamma_{\vec{x}_1} \setminus V_1[\vec{x}_1] \) with \( v_i > 1 \) for all \( 1 \leq i \leq n \) such that no admissible point of the induced ball of the compression on \( \vec{x}_2 \) namely

\[
B_{\vec{x}_1} \cup \Gamma_{\vec{x}_1} \setminus V_1[\vec{x}_1]
\]
sit on the compression line \( L_{\vec{x}_1, V_1[\vec{x}_1]} \). Next apply the compression \( V_1 \) on \( \vec{x}_2 \) and we obtain the point \( V_1[\vec{x}_2] \). Next let us join the point \( \vec{x}_2 \) to the point \( V_1[\vec{x}_2] \) by a straight line. Let us now apply the translation \( \Gamma_{\vec{x}_1} \) under some suitable translation vector to join the point \( V_1[\vec{x}_1] \) to exactly one the points \( \vec{x}_2, V_1[\vec{x}_2] \) whose distance is minimal. Without loss of generality let us assume that the point \( \vec{x}_2 \) is closer to \( V_1[\vec{x}_1] \) than \( V_1[\vec{x}_2] \) to \( V_1[\vec{x}_1] \) and apply the translation

\[
\Gamma_{\vec{x}_1}(V_1[\vec{x}_1]) = \vec{x}_2.
\]

Next we join the point \( V_1[\vec{x}_1] \) to the point \( \vec{x}_2 \) by a straight line \( L_{V_1[\vec{x}_1], \vec{x}_1} \) and the point \( \vec{x}_2 \) to the point \( V_1[\vec{x}_2] \) by the straight line \( L_{\vec{x}_2, V_1[\vec{x}_2]} \). By traversing the line \( L_{\vec{x}_1, V_1[\vec{x}_1]} \) starting from \( \vec{x}_1 \) to \( V_1[\vec{x}_1] \) and the line \( L_{V_1[\vec{x}_1], \vec{x}_2} \) continuing from \( V_1[\vec{x}_1] \) to \( \vec{x}_2 \) and the line \( L_{\vec{x}_2, V_1[\vec{x}_2]} \) continuing from \( \vec{x}_2 \) to \( V_1[\vec{x}_2] \), we obtain a path induced by four vertices. By repeating this argument and in the sense of proof of Theorem 13.2 we obtain a compression graph \( G = (V, E) \) with vertex and edge set

\[
V := \{ \vec{x}_1, V_m[\vec{x}_1], \Gamma_{\vec{x}_1}(V_m[\vec{x}_1]) = \vec{x}_2, V_m[\vec{x}_2], \ldots, \Gamma_{\vec{x}_k}(V_m[\vec{x}_{k-1}]) = \vec{x}_{k+1}, V_m[\vec{x}_{k+1}] \}
\]

and \( E \) the set of edges

\[
E := \{ L_{\vec{x}_1, V_m[\vec{x}_1]}, L_{V_m[\vec{x}_1], \vec{x}_2}, \ldots, L_{\vec{x}_k, V_m[\vec{x}_k]} \}.
\]

where the points in set \( V \) are the vertices and each line in \( E \) are the edges of the graph \( G \), with the edges being a path. \( \Box \)
15. Application to the Erdős unit distance problem

Erdős posed in 1946 the problem of counting the number of unit distances that can be determined by a set of \( n \) points in the plane. It is known (see [9]) that the number of unit distances that can be determined by \( n \) points in the plane is lower bounded by

\[
n^{1 + \frac{1}{\log \log n}}.
\]

Erdős asks if the upper bound for the number of unit distances that can be determined by \( n \) points in the plane can also be a function of this form. In other words, the problem asks if the lower bound of Erdős is the best possible. What is known currently is the upper bound (see [10]) of the form

\[
n^{\frac{4}{3}}
\]
due to Spencer, Szemerédi and Trotter.

**Definition 15.1** (Translation of balls). Let \( \vec{x} \in \mathbb{R}^k \) and \( B_{\frac{1}{2}G \circ V_m}|x_1| \vec{x} \) be the ball induced under compression. Then we denote the map

\[
T_{\vec{v}} : B_{\frac{1}{2}G \circ V_m}|x_1| \vec{x} \rightarrow B_{\frac{1}{2}G \circ V_m}|x_1| \vec{x} + \vec{v}
\]
as the translation of the ball by the vector \( \vec{v} \in \mathbb{R}^k \), so that for any \( \vec{y} \in B_{\frac{1}{2}G \circ V_m}|x_1| \vec{x} \)

\[
\vec{y} + \vec{v} \in B_{\frac{1}{2}G \circ V_m}|x_1| \vec{x} + \vec{v}.
\]

**Theorem 15.2.** Let \( E \subset \mathbb{R}^2 \) be a set of \( n \) points in general position and \( I = \{ ||x_j - x_t|| : x_t, x_j \in E \subset \mathbb{R}^2, ||x_j - x_t|| = 1, 1 \leq t, j \leq n \} \), then we have

\[
\#I \ll n^{1 + o(1)}.
\]

**Proof.** First pick a point \( \vec{x}_j \in \mathbb{R}^2 \), set \( G \circ V_1[\vec{x}_j] = 1 \) and apply the compression \( V_1 \) on \( \vec{x}_j \). Next construct the ball induced under compression

\[
B_{\frac{1}{2}G \circ V_1}|x_1| \vec{x}_j.
\]

We remark that the ball so constructed is a ball of radius \( \frac{1}{2}G \circ V_1[\vec{x}_j] = \frac{1}{4} \), so that for any admissible point \( \vec{x}_k \neq \vec{x}_j \) with \( \vec{x}_k \in B_{\frac{1}{2}G \circ V_1}|x_1| \vec{x}_j \) there must exists the admissible point \( V_1[\vec{x}_k] \) such that

\[
||\vec{x}_k - V_1[\vec{x}_k]|| = 1
\]
so that any such \( \frac{n}{2} \) pairs of admissible points determines at least \( \frac{n}{2} \) unit distances. Now for any \( n \) such admissible points on the ball and by virtue of the restriction (15.1)

\[
G \circ V_1[\vec{x}_j] = 1
\]
we make the optimal assignment

\[
\max_{1 \leq j \leq n} \sup_{1 \leq s \leq 2} (x_{js}) = n^{o(1)},
\]
since points \( \vec{x}_t \) far away from the origin with \( x_{ts} \) for \( 1 \leq s \leq 2 \) must have large compression gaps by virtue of Lemma ???. In particular, the point \( \vec{x}_t \) must be such that \( x_{ts} = 1 + \epsilon \) with \( 1 \leq s \leq 2 \) for any small \( \epsilon > 0 \) in order to satisfy the
requirement in (15.1). The number of unit distances induced by \( n \) admissible points on the ball so constructed is at most

\[
1 = \sum_{1 \leq j \leq n} \sum_{x_j \in \mathbb{R}^2} \mathcal{G} \circ \mathcal{V}_1[\vec{x}_j] \quad \text{subject to} \quad \max_{1 \leq j \leq n} \sup_{1 \leq s \leq 2} (x_{js}) = n^{o(1)}
\]

\[
\ll 2 \sum_{1 \leq j \leq n} \sup_{1 \leq s \leq 2} (x_{js})
\]

\[
= n^{o(1)} \sum_{1 \leq j \leq n} 1 \ll 2 \ n^{1+o(1)}.
\]

Now for any set of \( n \) points in general position in the plane \( \mathbb{R}^2 \), let us apply the translation with a fixed vector \( \vec{v} \in \mathbb{R}^2 \):

\[
T_{\vec{v}} : \mathcal{B}_{\frac{1}{2}} \mathcal{G} \circ \mathcal{V}_1[\vec{x}_j] \rightarrow \mathcal{B}_{\frac{1}{2}} \mathcal{G} \circ \mathcal{V}_1[\vec{x}_j]
\]

so that the new ball \( \mathcal{B}_{\frac{1}{2}} \mathcal{G} \circ \mathcal{V}_1[\vec{x}_j] \) now lives in the smallest region containing all the \( n \) points in general position. We remark that this new ball is still of radius \( \frac{1}{2} \) but contains points - including admissible points - all of which are translates of points in the previous ball \( \mathcal{B}_{\frac{1}{2}} \mathcal{G} \circ \mathcal{V}_1[\vec{x}_j] \) by a fixed vector \( \vec{v} \in \mathbb{R}^2 \). We remark that the unit distances are all preserved so that the number of unit distances determined by the \( n \) points in general position is upper bounded by

\[
\ll 2 \ n^{1+o(1)}
\]

thereby ending the proof. \( \square \)

16. Application to the Erdős-Anning type theorem in higher dimensions

The well-known Erdős-Anning Theorem is the assertion that infinite number of points in the plane \( \mathbb{R}^2 \) have can mutual integer distances only if all the points lie on the straight line. The theorem was first proved by Paul Erdős and Norman H. Anning \([11]\). In this paper we obtain a quantitative lower bound for the number of points with mutual integer distances in any finite subset of an infinite set of points on the same line in the space \( \mathbb{R}^n \). In particular for any finite subset \( S \subset \mathbb{N}^2 \) of an infinite set of points on the same line in the plane \( \mathbb{R}^2 \) the number of points on the shortest line with mutual integer distances containing points in \( S \) must satisfy the lower bound

\[
\gg 2 \sqrt{2} |S| \sum_{k \leq \max_{x \in S} \mathcal{G} \circ \mathcal{V}_1[\vec{x}] \mathcal{G} \circ \mathcal{V}_1[\vec{x}]} \sum_{k \leq \max_{x \in S} \mathcal{G} \circ \mathcal{V}_1[\vec{x}] \mathcal{G} \circ \mathcal{V}_1[\vec{x}]} \frac{1}{k}
\]

where \( \mathcal{G} \circ \mathcal{V}_1[\vec{x}] \) is the compression gap of the compression induced on \( \vec{x} \in \mathbb{N}^2 \).
Proof. Under the main assumption with \( G \circ V_m[\bar{x}], G \circ V_m[\bar{a}] \in \mathbb{N} \) then appealing to Lemma 12.3, we have the inequality
\[
G \circ V_m[\bar{x}] > G \circ V_m[\bar{a}]
\]
and the line \( L_{V_m[\bar{x}]} \) is only a segment of the line \( L_{V_m[\bar{a}]} \) by virtue of the estimate in Proposition 10.1 so that \( ||\bar{x} - \bar{a}||, ||V_m[\bar{x}] - V_m[\bar{a}]||, ||\bar{x} - V_m[\bar{a}]||, ||\bar{a} - V_m[\bar{x}]|| \in \mathbb{N} \). This completes the proof of the proposition. □

Theorem 16.1. Let \( \mathcal{R} \subset \mathbb{R}^n \) be an infinite set of collinear points and \( \mathcal{S} \subset \mathcal{R} \) be an arbitrary and finite set with \( \mathcal{S} \subset \mathbb{N}^n \). Then the number of points with mutual integer distances on the shortest line containing points in \( \mathcal{S} \) satisfies the lower bound
\[
\geq n \sqrt{n} |\mathcal{S} \cap B_{\frac{1}{2}} G \circ V_1[\bar{x}]| \sum_{k \leq \max_{x \in \mathcal{S} \cap \mathcal{B}} G \circ V_1[x]} 1 \frac{1}{k}
\]
Proof. Let us pick arbitrarily the lattice point \( \bar{x} \in \mathbb{N}^n \) and apply the compression \( V_1[\bar{x}] \). Next construct the line induced by compression \( L_{V_1[\bar{x}]} \) and the ball
\[
B_{\frac{1}{2}} G \circ V_1[\bar{x}]
\]
containing the line so that the end points of the line are admissible points of the ball. Next cover all the points on the line by the set \( \mathcal{R} \). Let us choose \( \mathcal{S} \subset \mathcal{R} \) to be the set of all lattice points on the line of compression induced and in the ball constructed. Now, the number of points with mutual integer distances on the line of compression \( L_{V_1[\bar{x}]} \) can be lower bounded by virtue of Lemma 12.3 by the sum
\[
\sum_{x \in \mathcal{S} \cap \mathcal{B}_{\frac{1}{2}} G \circ V_1[x]} 1 = \sum_{x \in \mathcal{S} \cap \mathcal{B}_{\frac{1}{2}} G \circ V_1[x]} \frac{G \circ V_1[x]}{k}
\]
\[
\geq \sqrt{n} \min_{x \in \mathcal{S} \cap \mathcal{B}_{\frac{1}{2}} G \circ V_1[x]} \inf_{j=1}^n \frac{n x_j}{k}
\]
\[
geq \sqrt{n} |\mathcal{S} \cap \mathcal{B}_{\frac{1}{2}} G \circ V_1[x]| \sum_{k \leq \max_{x \in \mathcal{S} \cap \mathcal{B}} \frac{1}{G \circ V_1[x]}} 1 \frac{1}{k}
\]
thereby establishing the lower bound. \qed

**Corollary 16.1.** There are infinitely many collinear points with mutual integer distances on any line in $\mathbb{R}^n$ for all $n \geq 2$.

17. Application to counting integral points in a circle and a grid

The Gauss circle problem is a problem that seeks to count the number of integral points in a circle centered at the origin and of radius $r$. It is fairly easy to see that the area of a circle of radius $r > 0$ gives a fairly good approximation for the number of such integral points in the circle, since on average each unit square in the circle contains at least an integral point. In particular, by denoting $N(r)$ to be the number of integral points in a circle of radius $r$, then the following elementary estimate is well-known

$$N(r) = \pi r^2 + |E(r)|$$

where $|E(r)|$ is the error term. The real and the main problem in this area is to obtain a reasonably good estimate for the error term. In fact, it is conjectured that

$$|E(r)| \ll r^{1/2 + \epsilon}$$

for $\epsilon > 0$. The first fundamental progress was made by Gauss [14], where it is shown that

$$|E(r)| \leq 2\pi r \sqrt{2}.$$ 

G.H. Hardy and Edmund Landau almost independently obtained a lower bound [12] by showing that

$$|E(r)| \neq o(r^{1/2} (\log r)^{1/4}).$$

The current best upper bound (see [13]) is given by

$$|E(r)| \ll r^{16/21}.$$ 

In this paper we study a variant of this problem in the region between a general $k$ dimensional grid $2r \times 2r \times \cdots \times 2r$ ($k$ times) and the largest sphere contained in the grid. In particular, we obtain the following lower bound for the number of integral points in this region

**Theorem 17.1.** Let $\mathcal{N}_{r,k}$ denotes the number of integral points in the region bounded by the $2r \times 2r \times \cdots \times 2r$ ($k$ times) grid covering a sphere of radius $r$ and a sphere of radius $r$. Then $\mathcal{N}_{r,k}$ satisfies the lower bound

$$\mathcal{N}_{r,k} \gg r^{k-\delta} \times \frac{1}{\sqrt{k}}$$

for some small $\delta > 0$.

**Proof.** Pick arbitrarily a point $(x_1, x_2, \ldots, x_k) = \vec{x} \in \mathbb{R}^k$ with $x_i > 1$ for $1 \leq i \leq k$ and $x_i \neq x_j$ for $i \neq j$ such that $G \circ \mathcal{V}_m[\vec{x}] = 2r$. This ensures the ball induced under compression is of radius $r$. Next we apply the compression of fixed scale $m \leq 1$, given by $\mathcal{V}_m[\vec{x}]$ and construct the ball induced by the compression given by

$$E_{\frac{1}{2}} G \circ \mathcal{V}_m[\vec{x}]$$

with radius $\left(\frac{G \circ \mathcal{V}_m[\vec{x}]}{2}\right) = r$. By appealing to Theorem [10, 12] admissible points $\vec{y}_i \in \mathbb{R}^k$ ($\vec{y}_i \neq \vec{x}$) of the ball of compression induced must satisfy the condition $G \circ
\[ \forall_m [\bar{x}_l] = 2r. \] Also by appealing to Theorem 10.4 points \( \bar{x}_l \notin B_{2r} \mathcal{G} \circ \forall_m [\bar{x}] \) must satisfy the inequality

\[ \mathcal{G} \circ \forall_m [\bar{x}_l] \geq \mathcal{G} \circ \forall_m [\bar{x}] = 2r. \]

In particular points in \( \bar{x}_l \notin B_{2r} \mathcal{G} \circ \forall_m [\bar{x}] \) contained in the \( 2r \times 2r \times \cdots \times 2r \) grid that covers this ball must satisfy for their coordinates

\[ \max_{\bar{x}_l \in (2r)^k} \sup(x_{l_i}) = (2r)^{1+\delta} \]

for some small \( \delta > 0 \) so that \( \mathcal{G} \circ \forall_m [\bar{x}_l] \geq 2r. \) The number of integral points contained in the region between the \( 2r \times 2r \times \cdots \times 2r \) grid covering the ball and the ball is lower bounded by

\[ N_{r,k} = \sum_{\bar{x}_l \in (2r)^k \subset N} 1 \]

\[ \geq \sum_{\bar{x}_l \in (2r)^k \subset N} \frac{2r}{\mathcal{G} \circ \forall_m [\bar{x}]_{\geq 2r}} \]

\[ \gg \sum_{\bar{x}_l \in (2r)^k \subset N} \frac{2r}{\sqrt{k} \sup(x_{l_i})} \]

\[ = 2r \sum_{\bar{x}_l \in (2r)^k \subset N} \frac{1}{\sqrt{k} \sup(x_{l_i})} \]

\[ \geq \frac{2r}{\sqrt{k}} \sum_{\bar{x}_l \in (2r)^k \subset N} \frac{1}{\max_{\bar{x}_l \in (2r)^k} \sup(x_{l_i})} \]

\[ = \frac{2r}{(2r)^{1+\delta} \sqrt{k}} \sum_{\bar{x}_l \in (2r)^k \subset N} 1 \]

\[ \gg \frac{r}{r^{1+\delta} \sqrt{k}} \times r^k \]

and the lower bound follows. \( \square \)

**Remark 17.2.** We now apply the method to obtain a lower bound for the number of lattice points in \( k \)-dimensional sphere of radius \( r > 0. \)

**Theorem 17.3.** Let \( N_k(r) \) denotes the number of integral points in the \( k \)-dimensional sphere of radius \( r > 0. \) Then \( N_k(r) \) satisfies the lower bound

\[ N_k(r) \gg \sqrt{k} \times r^{k-1+o(1)}. \]

**Proof.** Pick arbitrarily a point \( (x_1, x_2, \ldots, x_k) = \bar{x} \in \mathbb{R}^k \) with \( x_i > 1 \) for \( 1 \leq i \leq k \) and \( x_i \neq x_j \) for \( i \neq j \) such that \( \mathcal{G} \circ \forall_m [\bar{x}] = 2r. \) This ensures the ball induced under compression is of radius \( r. \) Next we apply the compression of fixed scale \( m \leq 1, \) given by \( \forall_m [\bar{x}] \) and construct the ball induced by the compression given by

\[ B_{2r} \mathcal{G} \circ \forall_m [\bar{x}] \]

with radius \( (\mathcal{G} \circ \forall_m [\bar{x}]) = r. \) By appealing to Theorem 10.12 admissible points \( \bar{x}_l \in \mathbb{R}^k (\bar{x}_l \neq \bar{x}) \) of the ball of compression induced must satisfy the condition \( \mathcal{G} \circ \)
\[ \forall m[\vec{x}_l] = 2r. \] Also by appealing to Theorem 10.4 points \( \vec{x}_l \in B_{2r}G \circ V_m[\vec{x}] \) must satisfy the inequality
\[ G \circ \forall m[\vec{x}_l] < G \circ \forall m[\vec{x}] = 2r. \]
For points \( \vec{x}_l \in B_{2r}G \circ V_m[\vec{x}] \) contained in the \( 2r \times 2r \times \cdots \times 2r \) \((k \text{ times})\) grid that covers this ball we make the assignment
\[ \min_{\vec{x}_l \in (2r)^k} \inf_{i=1}^k (x_{l_i}) = r^{o(1)} \]
as \( r \to \infty \). The number of integral points in the largest ball contained in the \( 2r \times 2r \times \cdots \times 2r \) \((k \text{ times})\) grid is
\[
N_k(r) = \sum_{\vec{x}_l \in (2r)^k \subset \mathbb{N}^k} 1 \geq \sum_{\vec{x}_l \in (2r)^k \subset \mathbb{N}^k} \frac{G \circ \forall m[\vec{x}_l]}{2r} \geq \sum_{\vec{x}_l \in (2r)^k \subset \mathbb{N}^k} \frac{\sqrt{k} \inf(x_{l_i})}{2r} \geq \frac{\sqrt{k}}{2r} \sum_{\vec{x}_l \in (2r)^k \subset \mathbb{N}^k} \min_{\vec{x}_l \in (2r)^k} \inf(x_{l_i}) \geq \frac{\min_{\vec{x}_l \in (2r)^k} \inf(x_{l_i})}{2r} \frac{k}{\sqrt{k}} \sum_{\vec{x}_l \in (2r)^k \subset \mathbb{N}^k} 1 \geq \frac{\min_{\vec{x}_l \in (2r)^k} \inf(x_{l_i})}{2r} \frac{k}{\sqrt{k}} r^k
\]
and the lower bound follows by our choice
\[ \min_{\vec{x}_l \in (2r)^k} \inf(x_{l_i}) = r^{o(1)} \]
as \( r \to \infty \). \[ \square \]

17.1. Application to counting the number of integral points on the boundary of a \( k \)-dimensional sphere.

**Theorem 17.4.** Let \( \mathcal{N}_{r,k} \) denotes the number of integral points on the boundary of a \( k \)-dimensional sphere of radius \( r \). Then \( \mathcal{N}_{r,k} \) satisfies the lower bound
\[
\mathcal{N}_{r,k} \gg r^{k-1} \sqrt{k}.
\]

**Proof.** Pick arbitrarily a point \( (x_1, x_2, \ldots, x_k) = \vec{x} \in \mathbb{R}^k \) with \( x_i > 1 \) for \( 1 \leq i \leq k \) and \( x_i \neq x_j \) for \( i \neq j \) such that \( G \circ \forall m[\vec{x}] = 2r \). This ensures the ball induced under
compression is of radius $r$. Next we apply the compression of fixed scale $m \leq 1$, given by $V_m[x]$ and construct the ball induced by the compression given by

$$B_{\frac{G \circ V_m[x]}{2}}$$

with radius $\frac{(G \circ V_m[x])}{2} = r$. We remark that this ball is exactly covered by the $k$-dimensional box $2r \times 2r \times \cdots \times 2r$ ($k$ times). By appealing to Theorem 10.12 admissible points $\bar{x}_l \in \mathbb{R}^k$ ($\bar{x}_l \neq \bar{x}$) of the ball of compression induced must satisfy the condition $G \circ V_m[\bar{x}_l] = 2r$. The number of integral points on the boundary of the $k$-dimensional sphere is lower bounded by

$$N_{r,k} = \sum_{\bar{x}_l \in [2r]^k \cap N^k} 1$$

$$\geq \sum_{\bar{x}_l \in [2r]^k \cap N^k} \frac{G \circ V_m[\bar{x}_l]}{2r}$$

$$\gg \sum_{\bar{x}_l \in [2r]^k \cap N^k} \frac{\sqrt{k} \inf(x_l)}{2r}$$

$$\geq \frac{\sqrt{k}}{2r} \sum_{\bar{x}_l \in [2r]^k \cap N^k} 1$$

$$\geq \frac{\sqrt{k}}{2r} \times [2r]^k$$

and the lower bound follows. \qed

17.2. **Application to counting the number of integral points in the annular region induced by spheres in $\mathbb{R}^k$.**

**Theorem 17.5.** Let $N_{R,r,k}$ denotes the number of integral points in the annular region induced by two $k$ dimensional spheres of radii $r$ and $R$ with $R > r$. Then $N_{R,r,k}$ satisfies the lower bound

$$N_{R,r,k} \gg (R^{k-1} - \frac{s^{k+1}}{k}) \sqrt{k},$$

for $k > \frac{\log R}{\log R - \log r}$.

**Proof.** Pick arbitrarily a point $(x_1, x_2, \ldots, x_k) = \bar{x} \in \mathbb{R}^k$ with $x_i > 1$ for $1 \leq i \leq k$ and $x_i \neq x_j$ for $i \neq j$ such that $G \circ V_m[\bar{x}] = 2R$. This ensures the ball induced under compression is of radius $R$. Next we apply the compression of fixed scale $m \leq 1$, given by $V_m[\bar{x}]$ and construct the ball induced by the compression given by

$$B_{\frac{G \circ V_m[\bar{x}]}{2}}$$

with radius $\frac{(G \circ V_m[\bar{x}])}{2} = R$. By appealing to Theorem 10.12 admissible points $\bar{x}_l \in \mathbb{R}^k$ ($\bar{x}_l \neq \bar{x}$) of the ball of compression induced must satisfy the condition $G \circ V_m[\bar{x}_l] = 2R$. Also by appealing to Theorem 10.4 points $\bar{x}_l \notin B_{\frac{G \circ V_m[\bar{x}]}{2}}$ must satisfy the inequality

$$G \circ V_m[\bar{x}_l] \geq G \circ V_m[\bar{x}] = 2R.$$
Similarly, by appealing to Theorem 10.4, points $\vec{x}_l \in B_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[x]}[\vec{x}]$ must satisfy the inequality

$$\mathcal{G} \circ \mathbb{V}_m[\vec{x}_l] < 2R.$$ 

Next let us construct the $2R \times 2R \times \cdots \times 2R$ $(k \text{ times})$ grid so that the ball $B_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[x]}[\vec{x}]$ is the largest it contains. Now, by appealing to Theorem 10.4, let us pick a point $\vec{x}_u \in B_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[x]}[\vec{x}]$ such that $\mathcal{G} \circ \mathbb{V}_m[\vec{x}] = 2r$. This ensures the ball induced under compression is of radius $r$. Next we apply the compression of fixed scale $m \leq 1$, given by $\mathbb{V}_m[\vec{x}]$ and construct the ball induced by the compression given by

$$B_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[x_u]}[\vec{x}_u]$$

with radius $(\mathcal{G} \circ \mathbb{V}_m[\vec{x}] ) = r$. Also by appealing to Theorem 10.4, points $\vec{x}_t \notin B_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[x]}[\vec{x}]$ must satisfy the inequality

$$\mathcal{G} \circ \mathbb{V}_m[\vec{x}_t] \geq \mathcal{G} \circ \mathbb{V}_m[\vec{x}_u] = 2r$$

so that we can choose

$$\max_{\vec{x}_l \in [2r]^k} \sup (x_{l_i})_{i=1}^k = (2r)^{1+\delta}$$

for some small $\delta > 0$. Next we construct the $2r \times 2r \times \cdots \times 2r$ $(k \text{ times})$ grid so that the ball $B_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[x_u]}[\vec{x}_u]$ is the largest ball in this grid. The number of integral points contained in the annular region induced by the ball of radii $r$ and $R$ is given by the identity

$$N_{R,r,k} = \sum_{\vec{x}_l \in [2R]^k \subset \mathbb{N}^k \atop \mathcal{G} \circ \mathbb{V}_m[\vec{x}_l] < 2R} 1 - \sum_{\vec{x}_l \in [2r]^k \subset \mathbb{N}^k \atop \mathcal{G} \circ \mathbb{V}_m[\vec{x}_l] \leq 2r} 1$$

Next we lower bound the first sum and upper bound the second sum. For the first sum, we obtain the lower bound

$$\sum_{\vec{x}_l \in [2R]^k \subset \mathbb{N}^k \atop \mathcal{G} \circ \mathbb{V}_m[\vec{x}_l] < 2R} 1 \geq \sum_{\vec{x}_l \in [2R]^k \subset \mathbb{N}^k} \frac{\mathcal{G} \circ \mathbb{V}_m[\vec{x}_l]}{2R}$$

$$= \sum_{\vec{x}_l \in [2R]^k \subset \mathbb{N}^k} \frac{\mathcal{G} \circ \mathbb{V}_m[\vec{x}_l]}{2R}$$

$$\gg \sum_{\vec{x}_l \in [2R]^k \subset \mathbb{N}^k} \frac{\sqrt{k} \inf(x_{l_i})_{i=1}^k}{2R}$$

$$= \frac{\sqrt{k}}{2R} \sum_{\vec{x}_l \in [2R]^k \subset \mathbb{N}^k} 1$$

$$= \frac{\sqrt{k}}{2R} \times |2R|^k.$$
Similarly for the upper bound of the second sum, we have

\[
\sum_{\vec{x}_l \in \{2r\}^k \subset N^k} \frac{2r}{G \circ V_m[\vec{x}_l]} \leq \sum_{\vec{x}_l \in \{2r\}^k \subset N^k} 2r \ \inf_{i=1}^k x_l^i \leq \sum_{\vec{x}_l \in \{2r\}^k \subset N^k} \frac{2r}{\sqrt{k}} \leq \frac{2r}{\sqrt{k}} \sum_{\vec{x}_l \in \{2r\}^k \subset N^k} 1 = \frac{2r}{\sqrt{k}} \sum_{\vec{x}_l \in \{2r\}^k \subset N^k} 1 \ll \frac{r}{\sqrt{k}} \times |2r|^k
\]

and the lower bound follows by combining the estimates of both sums. \( \square \)

18. Application to the general distance problem in \( \mathbb{R}^k \)

**Theorem 18.1.** Let \( D_{n,d} \) denotes the number of \( d \)-unit distances \( (d > 0) \) that can be formed from a set of \( n \) points in \( \mathbb{R}^k \). Then the lower bound holds

\[
D_{n,d} \gg \frac{n \sqrt{k}}{d}.
\]

**Proof.** Pick arbitrarily a point \((x_1, x_2, \ldots, x_k) = \vec{x} \in \mathbb{R}^k \) with \( x_i > 1 \) for \( 1 \leq i \leq k \) and \( x_i \neq x_j \) for \( i \neq j \) such that \( G \circ V_m[\vec{x}] = d \) for a fixed \( d > 0 \). This ensures the ball induced under compression is of radius \( \frac{d}{2} \). Next we apply the compression of fixed scale \( m \leq 1 \), given by \( V_m[\vec{x}] \) and construct the ball induced by the compression given by

\[
B_{\frac{d}{2} G \circ V_m[\vec{x}]}
\]

with radius \( \frac{(G \circ V_m[\vec{x}])}{2} = \frac{d}{2} \). By appealing to Theorem 10.12 admissible points \( \vec{x}_l \in \mathbb{R}^k \) \((\vec{x}_l \neq \vec{x})\) of the ball of compression induced must satisfy the condition \( G \circ V_m[\vec{x}_l] = d \). Next we count the number of \( d \)-unit distances formed by a set of \( n \) points in \( \mathbb{R}^k \) by counting pairs of admissible points \((\vec{x}_l, \vec{x}_h)\) on the ball \( B_{\frac{d}{2} G \circ V_m[\vec{x}] ol[\vec{x}] ol[\vec{x}]} \).
such that $\mathbb{V}_m[\tilde{x}_l] = \tilde{x}_h$ so that the number of $d$-unit distances is lower bounded by

$$D_{n,d} = \sum_{1 \leq l \leq \frac{n}{d}} 1$$

$$= \sum_{1 \leq l \leq \frac{n}{d}} \frac{\mathcal{G} \circ \mathbb{V}_m[\tilde{x}_l]}{d}$$

$$\gg \sum_{1 \leq l \leq \frac{n}{d}} \frac{\sqrt{k} \inf(x_i)}{d}$$

$$\geq \frac{\sqrt{k}}{d} \sum_{1 \leq l \leq \frac{n}{d}} 1$$

$$= \frac{n \sqrt{k}}{2d}$$

and the lower bound follows. \[\square\]

19. Application to counting the average number of integer powered distances in $\mathbb{R}^k$

**Theorem 19.1.** Let $D_{n,d^r}$ denotes the number of $d^r$-unit distances ($d > 0$) that can be formed from a set of $n$ points in $\mathbb{R}^k$ for a fixed $r > 1$. Then the lower bound holds

$$\sum_{1 \leq d \leq t} D_{n,d^r} \gg n \sqrt{k} \log t$$

for a fixed $t > 1$.

**Proof.** Pick arbitrarily a point $(x_1, x_2, \ldots, x_k) = \tilde{x} \in \mathbb{R}^k$ with $x_i > 1$ for $1 \leq i \leq k$ and $x_i \neq x_j$ for $i \neq j$ such that $\mathcal{G} \circ \mathbb{V}_m[\tilde{x}] = d^r$ for a fixed $d > 0$ and $r > 1$. This ensures the ball induced under compression is of radius $\frac{d^r}{\sqrt{k}}$. Next we apply the compression of fixed scale $m \leq 1$, given by $\mathbb{V}_m[\tilde{x}]$ and construct the ball induced by the compression given by

$$B_{\frac{d^r}{\sqrt{k}}} \mathcal{G} \circ \mathbb{V}_m[\tilde{x}]$$

with radius $\frac{(\mathcal{G} \circ \mathbb{V}_m[\tilde{x}])}{2} = \frac{d^r}{\sqrt{k}}$. By appealing to Theorem 10.12 admissible points $\tilde{x}_l \in \mathbb{R}^k$ ($\tilde{x}_l \neq \tilde{x}$) of the ball of compression induced must satisfy the condition $\mathcal{G} \circ \mathbb{V}_m[\tilde{x}_l] = d^r$. Next we count the number of $d^r$-unit distances formed by a set of $n$ points in $\mathbb{R}^k$ by counting pairs of admissible points $(\tilde{x}_l, \tilde{x}_h)$ on the ball $B_{\frac{d^r}{\sqrt{k}}} \mathcal{G} \circ \mathbb{V}_m[\tilde{x}]$ such that $\mathbb{V}_m[\tilde{x}_l] = \tilde{x}_h$ so that the average number of $d^r$-unit distances for $1 \leq d \leq t$
with fixed \( t, r > 1 \) is lower bounded by

\[
\sum_{1 \leq d \leq t} D_{n,d^r} = \sum_{1 \leq d \leq t} \sum_{1 \leq l \leq \frac{n}{d}} 1_{\bar{x}_l \in \mathbb{R}^n} G \circ V_m[\bar{x}_l] = \sum_{1 \leq d \leq t} \sum_{1 \leq l \leq \frac{n}{d}} \frac{\sqrt{G \circ V_m[\bar{x}_l]}}{d} \geq \sum_{1 \leq d \leq t} \sum_{1 \leq l \leq \frac{n}{d}} \frac{\sqrt{k} \sqrt{\text{Inf}(x_l)\frac{n}{d}}}{d} \geq \sum_{1 \leq d \leq t} \frac{n \sqrt{k}}{2d} \sum_{1 \leq l \leq \frac{n}{d}} 1 \geq \frac{n \sqrt{k}}{2} \sum_{1 \leq d \leq t} \frac{1}{d}
\]

and the lower bound follows. \( \square \)

20. Application to the Ehrhart volume conjecture

The Ehrhart volume conjecture is the assertion that any convex body \( K \) in \( \mathbb{R}^n \) with a single lattice point in it’s interior as barycenter must have volume satisfying the upper bound

\[
\text{Vol}(K) \leq \frac{(n + 1)^n}{n!}.
\]

The conjecture has only been proven for various special cases in very specific settings. For instance, Ehrhart proved the conjecture in the two dimensional case and for simplices \([16]\). The conjecture has also been settled for a large class of rational polytopes \([15]\). In this paper, we study the Ehrhart volume conjecture. We show that the claimed inequality fails for some convex bodies, providing a counter example to the Ehrhart volume conjecture. The main idea that goes into the disprove pertains to a certain construction of a ball in \( \mathbb{R}^n \) and the realization that after some little tweak of the internal structure, the ball satisfies the requirements of the conjecture but has too much volume, at least a volume beyond that postulated by Ehrhart. In particular, we prove the following lower bound

**Theorem 20.1.** Let \( \text{Vol}(K) \) denotes the volume of a ball in \( \mathbb{R}^n \) with only one lattice points in it’s interior as its center of mass. Then \( \text{Vol}(K) \) satisfies the lower bound

\[
\text{Vol}(K) \gg \frac{n^n}{\sqrt{n}}.
\]

**Proof.** Pick arbitrarily a point \((x_1, x_2, \ldots, x_n) = \bar{x} \in \mathbb{R}^n\) with \( x_i > 1 \) for \( 1 \leq i \leq n \) and \( x_i \neq x_j \) for \( i \neq j \) such that \( G \circ V_m[\bar{x}] = n \). This ensures the ball induced under
compression is of radius $\frac{n}{2}$. Next we apply the compression of fixed scale $m \leq 1$, given by $\mathcal{V}_m[\vec{x}]$ and construct the ball induced by the compression given by

$$K := B_{\frac{1}{2} \mathcal{G} \mathcal{V}_m[\vec{x}]}$$

with radius $\left(\mathcal{G} \mathcal{V}_m[\vec{x}]\right)_{\frac{1}{2}} = \frac{n}{2}$. By appealing to Theorem 10.12 admissible points $\vec{x}_l \in \mathbb{R}^k$ ($\vec{x}_l \neq \vec{x}$) of the ball of compression induced must satisfy the condition $\mathcal{G} \mathcal{V}_m[\vec{x}_l] = n$. Also by appealing to Theorem 10.4 points $\vec{x}_l \in B_{\frac{1}{2} \mathcal{G} \mathcal{V}_m[\vec{x}]}$ must satisfy the inequality $\mathcal{G} \mathcal{V}_m[\vec{x}_l] < \mathcal{G} \mathcal{V}_m[\vec{x}] = n$.

The number of integral points in the largest ball contained in the $n \times n \times \cdots \times n$ ($n$ times) grid that shares admissible points on both sides with the grid is

$$N_n(n) = \sum_{\vec{x}_l \in \mathbb{R}^n} \frac{1}{\mathcal{G} \mathcal{V}_m[\vec{x}_l]} \geq \sum_{\vec{x}_l \in \mathbb{R}^n} \frac{\sqrt{n} \inf(x_{l_i})}{n} \geq \frac{1}{n} \sum_{\vec{x}_l \in \mathbb{R}^n} \sqrt{n} \inf(x_{l_i}) \geq \frac{\sqrt{n}}{n} \sum_{\vec{x}_l \in \mathbb{R}^n} \min_{1 \leq i \leq n} \inf(x_{l_i}) \geq \frac{\min_{\vec{x}_l \in \mathbb{R}^n} \inf(x_{l_i})}{n} \sum_{1 \leq i \leq n} \frac{1}{n} \sum_{\vec{x}_l \in \mathbb{R}^n} \sqrt{n} \geq \sqrt{n} \times n^n.$$

We note that the number of lattice points $N_n(n)$ in the ball $K := B_{\frac{1}{2} \mathcal{G} \mathcal{V}_m[\vec{x}]}$ and the volume $\text{Vol}(K)$ satisfies the asymptotic relation $N_n(n) \sim \text{Vol}(K)$ so that by removing all sub-grid of the grid $n \times n \times \cdots \times n$ ($n$ times) contained in the ball $K := B_{\frac{1}{2} \mathcal{G} \mathcal{V}_m[\vec{x}]}$ except the sub-grid $\frac{n}{2} \times \frac{n}{2} \times \cdots \times \frac{n}{2}$ ($n$ times), we see that we are left with only one lattice point as the center of the ball. This completes the construction.

\[\square\]

21. Final remarks

The method of compression could be a potentially useful and as well powerful tool for resolving the Erdős-Straus conjecture. It can also find its place as a toolbox for quite a good number of Diophantine problem. The theory as it stands is still open to further development, which we do not pursue in this current version. One area that could be tapped is to investigate the geometry of compression. That is, to analyze the topology and the geometry of this concept.
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Department of Mathematics, African Institute for Mathematical science, Ghana
Email address: theophilus@aims.edu.gh/emperordagama@yahoo.com