1. Introduction

Universality of one dimensional dynamical system was discovered by Feigenbaum and independently by Coullet and Tresser in the mid 1970’s. Moreover, the universality of the higher dimensional maps is conjectured by Coullet and Tresser in [CT]. The similar universality properties are expected in higher dimensional maps which are strongly dissipative and close to the one dimensional maps. In particular, renormalizable maps with period doubling type are interesting in higher dimension. Universality of two dimensional strongly dissipative infinitely renormalizable Hénon-like maps was introduced in [CLM]. The Cantor attractor of Hénon-like map is the counterpart of that of one dimensional maps but it has typically unbounded geometry. The same geometric properties are common in certain classes of the sectionally dissipative three dimensional Hénon-like family in [Nam1, Nam2, Nam3]. There exists the universal expression of Jacobian determinant of infinitely renormalizable three
dimensional Hénon-like maps but it does not imply the universal expression. However, *unbounded geometry* of Cantor attractor were generalized in the special invariant subspace of infinitely renormalizable maps in \[\text{Nam1, Nam3}\].

This paper is about the generalization of Hénon renormalization in arbitrary finite dimension. In the viewpoint of a perturbation of one dimensional map in higher dimension, Higher dimensional renormalizable Hénon-like map is a counterpart of the perturbation of Misiurewicz maps in higher dimension which appear in \[\text{WY}\]. A certain invariant class of infinitely renormalizable three dimensional maps is generalized and every results in \[\text{Nam1, Nam3}\] are extended in arbitrary dimension. For instance, the following results are extended in arbitrary dimension.

— Universality of Jacobian determinant of renormalized maps.
— Existence of single invariant surfaces under certain conditions.
— Existence of \(C^r\) renormalizable two dimensional Hénon-like map with invariant \(C^r\) surface.
— Unbounded geometry of Cantor attractor.

Hénon-like map \(F\) from the hypercube \(B\) to \(\mathbb{R}^{m+2}\) is defined as follows

\[
F: (x, y, z) \mapsto (f(x) - \varepsilon(x, y, z), x, \delta(x, y, z))
\]

where \(f(x)\) is a unimodal map, \(z = (z_1, z_2, \ldots, z_m)\) and \(\delta = (\delta^1, \delta^2, \ldots, \delta^m)\) is a map from \(B \subset \mathbb{R}^{m+2}\) to \(\mathbb{R}^m\). Let us assume that \(F\) has two hyperbolic fixed points, \(\beta_0\) which has positive eigenvalues and \(\beta_1\) which has both positive and negative eigenvalues. In this paper, we assume that both \(\|\varepsilon\|\) and \(\|\delta\|\) are bounded above by \(O(\bar{\varepsilon})\) where \(\bar{\varepsilon}\) is a small enough positive number. Since \(\|\delta\|\) is sufficiently small, each fixed point has only one expanding direction. We assume that the product of any two different eigenvalues is strictly less than one at fixed points of \(F\), namely, *sectionally dissipative* at fixed points. Hénon-like map is called *renormalizable* if \(W^u(\beta_0)\) intersects \(W^s(\beta_1)\) at the orbit of a single point. Thus \(F\) has one dimensional unstable manifold and codimension one stable manifold at fixed points.

**Theorem 1.1** (Universality of \(\text{Jac} R^n F\)). Let \(F\) be the \(m+2\) dimensional infinitely renormalizable Hénon-like map. Then

\[
\text{Jac} R^n F = b^n a(x)(1 + O(\rho^n))
\]

where \(b = b_F\) is the average Jacobian of \(F\), \(a(x)\) is the universal function for \(\rho \in (0, 1)\).

Let the Hénon-like map with the condition \(\partial_z \varepsilon \equiv 0\) for all \(1 \leq j \leq m\) be the *toy model map*, say \(F_{\text{mod}}\) as follows

\[
F_{\text{mod}}(x, y, z) = (f(x) - \varepsilon(x, y), x, \delta(x, y, z)).
\]

If \(F_{\text{mod}}\) is infinitely renormalizable, then \(n^{th}\) renormalization of \(F_{\text{mod}}\) contains two dimensional renormalized map with universality

\[
\pi_{xy} \circ R^n F_{\text{mod}}(x, y, z) = R^n F_{\text{2d}}(x, y) = (f_n(x) + b_1^n a(x) y (1 + O(\rho^n)), x)
\]

\(^1\)The claim for counterpart might require renormalizable higher dimensional Hénon-like map has *rank one* attractor. Renormalizable two dimensional Hénon-like maps have one dimensional global attracting set in \[\text{LM}\] and by slight modifying proof can show that it is true for higher dimensional Hénon-like maps. But we would not deal with this fact in this paper.

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where \( b_1 \) is the average Jacobian of the two dimensional Hénon-like map for some \( 0 < \rho < 1 \). If \( \| \delta \| \ll b_1 \), then there exists the continuous invariant plane field over the critical Cantor set under the \( DF_{\text{mod}} \). Moreover, \( F \) is a small perturbation of the model map, that is, \( \varepsilon(x, y, z) = \varepsilon_{2d}(x, y) + \tilde{\varepsilon}(x, y, z) \) and \( \max\{ \| \partial_j \varepsilon \| \mid 1 \leq j \leq m \} \), then it also has continuous invariant plane field. Furthermore, there exists a single surface \( Q \) invariant under \( F \). Additionally if \( F \) is infinitely renormalizable, then there exists an invariant surface \( Q_n \) under \( R^n F \) for each \( n \in \mathbb{N} \) as the graph of \( C^r \) map \( \xi = (\xi^1, \xi^2, \ldots, \xi^m) \) from \( xy \)-plane to \( z \)-axis for all \( 1 \leq j \leq m \) for \( 2 \leq r < \infty \) (Lemma 7.6). Then two dimensional \( C^r \) Hénon-like map is defined as follows

\[
(1.1) \quad F_{2d, \xi}(x, y) = (f(x) - \varepsilon(x, y, \xi), x)
\]

where \( Q \equiv \text{graph}(\xi) \) is a \( C^r \) invariant surface under \( F \). Then universality theorem of infinitely renormalizable \( C^r \) Hénon-like maps are obtained.

**Theorem 1.2** (Universality of \( C^r \) Hénon-like maps with invariant single surfaces for \( 2 \leq r < \infty \)). Let Hénon-like map \( F_{2d, \xi} \) be the \( C^r \) map for some \( 2 \leq r < \infty \) which is defined in (1.1). Suppose that \( F_{2d, \xi} \) is infinitely renormalizable. Then

\[
R^n F_{2d, \xi} = (f_n(x) - b_{1,2d} a(x) y(1 + O(\rho^n)), x)
\]

where \( b_{1,2d} \) is the average Jacobian of \( F_{2d, \xi} \) and \( a(x) \) is the universal function for some \( 0 < \rho < 1 \).

Cantor attractor of \( C^r \) Hénon-like map also has the geometric properties which are the same as that of Cantor attractor for analytic maps. In particular, the critical Cantor set has unbounded geometry (Theorem 9.2). Moreover, all of these dynamical properties of Cantor attractor are generalized in higher dimensional Hénon-like map \( F \) through its invariant surfaces.

### 1.1. Notations.

For the given map \( F \), we denote the set \( A \) is related to \( F \) to be \( A(F) \) or \( A_F \). \( F \) can be omitted if there is no confusion without \( F \). The domain of the function \( F \) is denoted to \( \text{Dom}(F) \) and the image of the set \( B \) under a function \( F \) is denoted to be \( \text{Im}(F) \). \( F|_A \) is called the restriction \( F \) on \( A \) where \( A \subset \text{Dom}(F) \). If \( F(B) \subset B \), then we call \( B \) is an (forward) invariant set under \( F \).

Let the projection from \( \mathbb{R}^{m+2} \) to its \( x \)-axis, \( y \)-axis and \( z \)-axis be \( \pi_x, \pi_y \) and \( \pi_z \) respectively. Moreover, let the projection from \( \mathbb{R}^3 \) to \( xy \)-plane be \( \pi_{xy} \) and so on. The derivative of the map \( f \) is expressed as \( Df \). The chain rule implies that \( D(f \circ g)(w) = Df \circ g(w) \cdot Dg(w) \).

In this paper, the boldfaced letter means the condensed expression with \( m \) coordinates. For example,

\[
z = (z_1, z_2, \ldots, z_m), \quad \delta = (\delta^1, \delta^2, \ldots, \delta^m) \quad \text{and} \quad \xi = (\xi^1, \xi^2, \ldots, \xi^m).
\]

The dot product of two objects presented boldfaced letters, say \( \mathbf{A} \) and \( \mathbf{B} \) means the inner product of them. Denote it by \( \mathbf{A} \cdot \mathbf{B} \). Let the set distance \( \text{dist}_{\text{min}}(R, S) \) be the minimal distance between two sets, \( R \) and \( S \)

\[
\text{dist}_{\text{min}}(R, S) = \inf \{ \text{dist}(r, s) \text{ for all } r \in R \text{ and } s \in S \}.
\]

Denote the set of periodic points of \( F \) to be \( \text{Per}_F \). The orbit of the point \( w \) under the map \( f \) is denoted to be \( \text{Orb}(w, f) \). Denote the (complete) orbit of \( w \) to be \( \text{Orb}(w) \) unless the map
is emphasized or is ambiguous on the context in the related description. $A = O(B)$ means that there exists a positive number $C$ such that $A \leq CB$. Moreover, $A \asymp B$ means that there exists a positive number $C$ which satisfies $\frac{1}{C} B \leq A \leq CB$.

2. Hénon renormalization in higher dimension

2.1. Hénon-like maps in higher dimension. Let $B_{2d}$ be the domain of two dimensional Hénon-like map and it is the square region with the center origin. Let $B$ be the box domain which is a thickened domain of two dimensional Hénon-like map, that is, $B = B_{2d} \times [-c, c]^m$ for some $c > 0$ and a fixed positive number $m$. Let us define the $m+2$ dimensional Hénon-like map on the hypercube $B$ as follows

$$F(x, y, z) = (f(x) - \varepsilon(x, y, z), x, \delta(x, y, z))$$

where $f : I^x \to I^x$ is a unimodal map, $z = (z_1, z_2, \ldots, z_m)$ and $\delta = (\delta^1, \delta^2, \ldots, \delta^m)$ is the map from $B$ to $\mathbb{R}^m$. For simplicity, let us assume that the length of each side of $B$ is same. Denote the domain, $B = I^x \times I^v$ where $I^x$ is the line parallel to $x$-axis and $I^v = I^y \times I^z$ where $I^y$ is the line parallel to $y$-axis and $I^z$ is the hypercube $[-c, c]^m$.

Remark 2.1. On the following sections, some objects defined on the two dimensional space has the subscript $2d$. For example, $B_{2d}$ is the square domain of the two dimensional Hénon-like map and $F_{2d}$ is the two dimensional Hénon-like map defined on $B_{2d}$. However, same notation without any index indicates the $m+2$ dimensional object. For instance, $F$ and $B$ are the higher dimensional Hénon-like map and its box domain respectively.

Observe that the image of the codimension one hyperplane, $\{x = C\}$ under $F$ is contained in the codimension one hyperplane, $\{y = C\}$.

![Figure 2.1. Image of $\{x = \text{const.}\}$ under higher dimensional Hénon-like map](image)

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In this paper, higher dimensional analytic Hénon-like maps in (2.1) have the following properties.

— $F$ is orientation preserving map.
— $f$ has non flat unique critical point in $I^x$.
— Both $||\varepsilon|| \leq \varepsilon$ and $||\delta|| \leq \delta$ with sufficiently small positive numbers $\varepsilon$ and $\delta$.
— $F$ has only two fixed points one of which, say $\beta_0$, has only positive eigenvalues.
— $F$ is sectionally dissipative at fixed points, that is, any product of two eigenvalues of each fixed point has the absolute value strictly less than one.
— The fixed points, $\beta_0$ and $\beta_1$ have codimension one stable manifold and one dimensional unstable manifold.

The orientation preserving higher dimensional Hénon-like map is called renormalizable if $W^u(\beta_0)$ and $W^s(\beta_1)$ intersects in the single orbit of a point. Observe that sectionally dissipativeness imply that each fixed point has condimension one stable manifold and one dimensional unstable manifold.

Let $p_0 \in \text{Orb}_{\mathbb{Z}}(w)$ be the point which is farthest point from $\beta_1$ along the component of $W^s_{\text{loc}}(\beta_1)$ which contains $\beta_1$. Let $p_k = F^k(p_0)$ for each $k \in \mathbb{Z}$. Then $p_0$ and the forward orbit of $p_0$ are on $W^s_{\text{loc}}(\beta_1)$. The local stable manifold of $p_{-n}$, $W^s_{\text{loc}}(p_{-n})$ where $n \leq 0$ is pairwise disjoint component of $W^s(\beta_1)$ and $W^s_{\text{loc}}(p_{-n})$ converges to $W^s(\beta_1)$ because $p_{-n}$ converges to $\beta_0$ as $n \to +\infty$. Let $M_{-n}$ be the connected component of $W^s(\beta_1)$ which contains $p_{-n}$, say $W^s_{\text{loc}}(p_{-n})$, to be for every $n \geq 0$. For instance, $M_0$ denotes $W^s_{\text{loc}}(\beta_1)$. Moreover, we can define $M_1$ as the component of $W^s(\beta_1)$ whose image under $F$ is contained in $M_{-1}$ such that it does not have any point of $\text{Orb}_{\mathbb{Z}}(w)$. $M_1$ is on the opposite side of $M_{-1}$ from $M_0$. We may assume that $M_1$ is a curve connecting the up and down sides of the square domain $B$ inside.

Then we can easily check the curves $[p_0, p_1]_{\beta_0}^u$ and $[p_1, p_2]_{\beta_0}^u$ which are parts of $W^u(\beta_0)$ does not intersect $M_1$ and $M_{-1}$ respectively when $F$ is renormalizable.

### 2.2. Renormalization of $m + 2$ dimensional Hénon-like maps.

The analytic definition of period doubling renormalization of Hénon-like map requires conjugation which is not just dilation because $F^2$ is not Hénon-like map, that is, the image of the hyperplane, \{x = C\} in $B$ under $F^2$ is not the part of the hyperplane, \{y = C\}. Thus we need non-linear coordinate change map for renormalization. Define the horizontal-like diffeomorphism as follows

$$H(x, y, z) = (f(x) - \varepsilon(x, y, z), y, z - \delta(y, f^{-1}(y), 0)).$$

The inverse of $H$, $H^{-1}$ is as follows

$$H^{-1}(x, y, z) \equiv (\phi^{-1}(w), y, z + \delta(y, f^{-1}(y), 0))$$

where $\phi^{-1}$ is the straightening map satisfying $\phi^{-1} \circ H(w) = x$ for $w = (x, y, z)$.

Let us define $\text{Dom}(H)$ as the region enclosed by hypersurfaces, \{f(x) - \varepsilon(x, y, z) = \text{const.}\}, \{y = \text{const.}\} and \{z_j = \text{const.}\} for $1 \leq j \leq m$ such that the image of this region under $H$ is $V \times I^r$. Let $U_U$ be the space of unimodal maps defined on the set $U$. 

Proposition 2.1. Let $F(w) = (f(x) - \varepsilon(w), x, \delta(w))$ be a higher dimensional Hénon-like map and $H$ be the horizontal-like diffeomorphism defined in (2.2) where $\|\varepsilon\| \leq \bar{\varepsilon}$ and $\|\delta\| \leq \tilde{\delta}$ with small enough positive numbers $\bar{\varepsilon}$ and $\tilde{\delta}$. Suppose that the unimodal map $f$ is renormalizable. Then there exists a unimodal map $f_1 \in \mathcal{U}_V$ such that $\|f_1 - f^2\|_V < C\bar{\varepsilon}$ and the map $H \circ F^2 \circ H^{-1}$ is the Hénon-like map $(x, y, z) \mapsto (f_1(x) - \varepsilon_1(x, y, z), x, \delta_1(x, y, z))$ with the norm, $\|\varepsilon_1\| = O(\bar{\varepsilon}^2 + \varepsilon \bar{\delta})$ and $\|\delta_1\| = O(\bar{\varepsilon} + \bar{\delta}^2)$.

Proof. By the straightforward calculation, $H \circ F^2 \circ H^{-1}$ is

$$(f(f(x) - \varepsilon \circ F \circ H^{-1}(w)) - \varepsilon \circ F^2 \circ H^{-1}(w), x, \delta \circ F \circ H^{-1}(w) - \delta(x, f^{-1}(x), 0))$$

Thus the first coordinate function of $H \circ F^2 \circ H^{-1}$ is

$$f(f(x) - \varepsilon \circ F \circ H^{-1}(w)) - \varepsilon \circ F^2 \circ H^{-1}(w).$$

By the linear approximation, we obtain the estimation of first coordinate function.

$$f(f(x)) - \varepsilon \circ F \circ H^{-1}(w) - \varepsilon \circ F^2 \circ H^{-1}(w)$$

$$= f^2(x) - f'(f(x)) \cdot \varepsilon \circ F \circ H^{-1}(w) - \varepsilon(f(x), x, 0) \cdot \partial_x \varepsilon \circ (f(x), x, 0) \cdot \varepsilon \circ F \circ H^{-1}(w)$$

$$+ \sum_{j=1}^m \partial_{x_j} \varepsilon \circ (f(x), x, 0) \cdot \delta^j \circ F \circ H^{-1}(w) + h.o.t.$$
\[+ \sum_{j=1}^{m} \partial_{z_j} \varepsilon \circ (x, f^{-1}(x), 0) \cdot \delta^i \circ H^{-1}(w)\]

\[- \sum_{j=1}^{m} \partial_{z_j} \varepsilon \circ (f(x), x, 0) \cdot \delta^i \circ F \circ H^{-1}(w) + h.o.t.\]

Let \(v(x) = \varepsilon(x, f^{-1}(x), 0)\). Then the unimodal map, \(f_1(x)\) of the first component of \(H \circ F^2 \circ H^{-1}\) is the following

\[f^2(x) - v \circ f(x) - [f'(f(x)) - \partial_x \varepsilon \circ (f(x), x, 0)] \cdot v(x).\]

Thus \(\|f_1(x) - f^2(x)\| = O(\|\varepsilon\|)\). Moreover, the norm of \(\varepsilon_1(w)\) is \(O(\|\varepsilon\|^2 + \varepsilon \| \delta \|)\). Let us estimate the the from third to \((m + 2)^{th}\) coordinate maps of \(H \circ F^2 \circ H^{-1}\). Recall \(\delta_1(w) = \delta \circ F \circ H^{-1}(w) - \delta(x, f^{-1}(x), 0)\). The estimation of each coordinate map of \(\delta_1\) is

\[\delta^i \circ F \circ H^{-1}(w) - \delta^i(x, f^{-1}(x), 0)\]

\[= \delta^i(x, \phi^{-1}(w), \delta \circ H^{-1}(w)) - \delta^i(x, f^{-1}(x), 0)\]

\[= \partial_y \delta^i \circ (x, f^{-1}(x), 0) \cdot (\phi^{-1}(w) - f^{-1}(x)) + \sum_{j=1}^{m} \partial_{z_j} \delta^i \circ (x, f^{-1}(x), 0) \cdot \delta^j \circ H^{-1}(w) + h.o.t.\]

\[= \partial_y \delta^i \circ (x, f^{-1}(x), 0) \cdot (f^{-1})(x) \cdot \varepsilon \circ H^{-1}(w)\]

\[+ \sum_{j=1}^{m} \partial_{z_j} \delta^i \circ (x, f^{-1}(x), 0) \cdot \delta^j \circ H^{-1}(w) + h.o.t.\]

for \(1 \leq i \leq m\). Then \(\|\delta_1^i\| = O(\|\varepsilon\| \|\delta\| + \|\delta\|^2)\) for all \(1 \leq i \leq m\). Hence, \(\|\delta_1\|\) is also \(O(\|\varepsilon\| \|\delta\| + \|\delta\|^2)\).

Define pre-renormalization of \(F\) as \(PRF \equiv H \circ F^2 \circ H^{-1}\) on \(H(B^1_n)\). Define \(H(B^1_n)\) as \(\text{Dom}(PRF)\) unless any other statements are specified. Since \(H(B^1_n)\) is the hypercube, domain \(B\) is recovered as the image of \(H(B^1_n)\) under the appropriate linear expanding map \(\Lambda(x, y, z) = (sx, sy, sz)\) for some \(s < -1\). Thus we see that \(\text{Dom}(PRF) = \Lambda^{-1}(B)\).

**Definition 2.1** (Renormalization). Let \(V\) be the (minimal) closed subinterval of \(I^x\) such that \(V \times I^y\) is invariant under \(H \circ F^2 \circ H^{-1}\) and let \(s: V \to I\) be the orientation reversing affine rescaling. With the rescaling map \(\Lambda(x, y, z) = (sx, sy, sz)\), the renormalization of the \(m + 2\) dimensional Hénon-like map is defined as \(\Lambda \circ H \circ F^2 \circ H^{-1} \circ \Lambda^{-1}\) on the domain \(B \equiv I^x \times I^y\)

\[RF = \Lambda \circ H \circ F^2 \circ H^{-1} \circ \Lambda^{-1}.\]

If \(RF\) is also renormalizable, we can define the second renormalization of \(F\) as the renormalization of \(RF\). Then if \(F\) is \(n\) times renormalizable, then the \(n^{th}\) renormalization is defined successively

\[R^n F = \Lambda_{n-1} \circ H_{n-1} \circ (R^{n-1} F)^2 \circ H_{n-1}^{-1} \circ \Lambda_{n-1}^{-1}.\]

where \(R^{n-1} F\) is the \((n - 1)^{th}\) renormalization of \(F\) for \(n \geq 1\).
Let the set of infinitely renormalizable higher dimensional Hénon-like map be $\mathcal{I}(\varepsilon)$ where $\max\{\|\varepsilon\|, \|\delta\|\} \leq \bar{\varepsilon}$ for small enough positive number $\bar{\varepsilon}$. We call the higher dimensional Hénon-like map just Hénon-like map unless it makes any confusion in the following sections.

3. Critical Cantor set

The minimal attracting set for two dimensional infinitely renormalizable Hénon-like map $F$ is the Cantor set on which $F$ acts as the dyadic adding machine. The topological construction of the invariant Cantor set of higher dimensional Hénon-like map is exactly same as that of two-dimensional Hénon-like map (Corollary ?? below). Thus we use the same definitions and notions of the two dimensional case in this section.

Denote $\Psi_1^w$ by $\psi_1^w \equiv H^{-1} \circ \Lambda^{-1}$. Thus it is the non-linear scaling map which conjugates $F^2$ to $RF$ on $\Psi_1^v(B)$. Denote $\Psi_c^1$ by $\psi_c^1 \equiv F \circ \psi_v$. The subscript $v$ and $c$ are associated to the maps with the critical value and the critical point respectively. Similarly, let $\psi_v^2$ and $\psi_c^2$ be the non linear scaling maps conjugating $RF$ to $R^2F$. Let

$$
\Psi_{vv}^n = \psi_v^1 \circ \psi_v^2, \quad \Psi_{vc}^n = \psi_c^1 \circ \psi_v^2, \quad \Psi_{vc}^n = \psi_v^1 \circ \psi_c^2, \ldots
$$

and so on. Successively we can define the non-linear scaling map of the $n^{th}$ level for any $n \in \mathbb{N}$ as follows

$$
\Psi_w^n = \psi_{w_1}^1 \circ \cdots \circ \psi_{w_n}^n, \quad w = (w_1, \ldots, w_n) \in \{v, c\}^n
$$

where $w = (w_1, \ldots, w_n)$ is the word of length $n$ and $W^n = \{v, c\}^n$ is the $n$-fold Cartesian product of $\{v, c\}$.

Lemma 3.1. Let the Hénon-like map, $F$ be in $\mathcal{I}(\varepsilon)$. Then the derivative of the map $\Psi_w^n$ is exponentially shrinking for $n \in \mathbb{N}$ with $\sigma$, that is, $\|D\Psi_w^n\| \leq C\sigma^n$ for every words $w \in W^n$ where $C > 0$ depends only on $B$ and $\bar{\varepsilon}$.

Proof. The identical equation $H \circ H^{-1} = \text{id}$ implies that $^2$

$$
f \circ \phi^{-1}(w) - \varepsilon \circ H^{-1}(w) = x.
$$

Thus we have $\phi^{-1}(w) = f^{-1}(x + \varepsilon \circ H(w))$. Recall that

$$
\varepsilon \circ H^{-1}(w) = \varepsilon(\phi^{-1}(w), y, z + \delta(y, f^{-1}(y), 0)).
$$

Then by the chain rule, each partial derivatives of $\phi^{-1}$ is as follows

---

$^2$ The first coordinate map of $H^{-1}(w)$, $\phi^{-1}(x, y, z)$ is not the inverse function of the same function $\phi(w)$. However, $\phi^{-1}(w)$ is a $\varepsilon-$perturbation of $f^{-1}(x)$ as follows

$$
f \circ \phi^{-1}(w) - \varepsilon \circ H^{-1}(w) = x.
$$
for 1 \leq i \leq m. Then
\[
\partial_x \phi^{-1}(w) = \frac{(f^{-1})'(x + \epsilon \circ H^{-1}(w))}{1 - (f^{-1})'(x + \epsilon \circ H^{-1}(w)) \cdot \partial_x \epsilon \circ H^{-1}(w)}
\]
for 1 \leq i \leq m. Let us estimate \|D\phi^{-1}\|. The above equation (3.1) implies that \|\partial_x \phi^{-1}\| \asymp \| (f^{-1})' \| and furthermore, \phi^{-1}(x, y_0, z_0) \asymp f^{-1}(x) for every \((y_0, z_0) \in I^c\). The fact that \phi^{-1}: \Lambda^{-1}(B) \to \pi_x(B^1)\) implies that the domain of \(f^{-1}\) is \(\pi_x(\Lambda^{-1}(B))\). Then \| (f^{-1})' \| is away from one and \|\partial_y \phi^{-1}\| and \|\partial_z \phi^{-1}\| are \(O(\epsilon)\) for \(i = 1, 2, \ldots, m\) by the equation (3.1).

The norm of derivatives of \(\phi_n^{-1}(w)\) for each \(n\) has the same upper bounds because \(f_n \to f^*\) exponentially fast and \(\|\partial_y \phi_n^{-1}\|\) and \(\|\partial_z \phi_n^{-1}\|\) for \(i = 1, 2, \ldots, m\) are bounded by \(O(\epsilon^{2n})\). The dilation, \(\Lambda^{-1}\) contracts by the factor \(\rho(1 + O(\rho^n))\) where \(\rho = \text{dist}(F, F^*)\). The above estimations imply that \(\|D\phi_n^{-1}\|\) and \(\|D(F \circ H_n^{-1})\|\) are uniformly bounded and the upper bounds are independent of \(n\). Thus \(\|\psi_n^w\| \leq C\sigma\) for \(w = v, c\). Hence, the composition of \(\psi_n^w\) for \(k = 1, 2, \ldots, n\) contracts by the factor \(C\sigma^n\), that is, \(\|D\psi_n^w\| \leq C\sigma^n\) for some \(C > 0\).

Let \(B^1_v \equiv B^1_v(F)\) and \(B^1_c \equiv B^1_c(F)\). Thus by the definitions on Section 2.2 \(B^1_v(F) = \psi^1_v(B)\) and \(B^1_c(F) = F \circ \psi^1_c(B)\). Moreover, \(F(B^1_v) \subset B^1_v\). If the Hénon-like map \(F\) is \(n\) times renormalizable, we can define \(B^n_v(R^n F)\) and \(B^n_c(R^n F)\) as \(\psi^{n+1}_v(B)\) and \(F_n \circ \psi^{n+1}_c(B)\) respectively for each \(n \geq 1\). We call the regions \(B^n_w \equiv B^n_w(F) = \Psi^n_w(B)\) pieces of the \(n^{th}\) level or \(n^{th}\) generation where \(w \in W^n\). Moreover, \(W^n\) can be an additive group under the following correspondence from \(W^n\) to the numbers with base 2 of mod \(2^n\)
\[
w \mapsto \sum_{k=0}^{n-1} w_{k+1}2^k \pmod{2^n}
\]
where the symbols \(v\) and \(c\) are corresponding to 0 and 1 respectively.

**Corollary 3.2.** The diameter of each piece shrinks exponentially fast for each \(n \geq 1\), that is, \(\text{diam}(B^n_w) \leq C\sigma^n\) for all \(w \in W^n\) where the constant \(C > 0\) depends only on \(B\) and \(\epsilon\).

The construction of the critical Cantor set for higher dimensional Hénon-like map in \(I(\epsilon)\) is the same as that of two dimensional Hénon-like map. See Section 5 in [CLM]. Define the
critical Cantor set of the infinitely renormalizable Hénon-like map $F$ as follows

$$\mathcal{O} \equiv \mathcal{O}_F = \bigcap_{n=1}^{\infty} \bigcup_{w \in W^n} B_w^n.$$ 

Infinitely renormalizable Hénon-like map, $F$ acts as the dyadic adding machine on the above invariant Cantor set. For detailed construction of the dyadic group as Cantor set, see [?].

4. Average Jacobian

Let us consider the average Jacobian of the infinitely renormalizable map $F$. The definition and properties of average Jacobian of higher dimensional Hénon-like maps is the same as those of two dimensional ones. For the sake of completeness, we describe Lemma and Theorem in this section below. Let the Jacobian determinant of $F$ at $w$ be $\text{Jac} F(w)$. Thus

$$\log \left| \frac{\text{Jac} F(y)}{\text{Jac} F(z)} \right| \leq C \quad \text{for any } y,z \in B$$

by some constant $C$ which is not depending on $y$ or $z$. The diameter of the domain $B_w^n$ converges to zero exponentially fast by Lemma 3.1. It implies the following lemma.

**Lemma 4.1 (Distortion Lemma).** There exist a constant $C$ and the positive number $\rho < 1$ satisfying the following estimate

$$\log \left| \frac{\text{Jac} F^k(y)}{\text{Jac} F^k(z)} \right| \leq C\rho^n \quad \text{for any } y,z \in B_w^n$$

where $k = 1, 2, 2^2, \ldots, 2^n$.

Existence of the unique invariant probability measure, say $\mu$, on $\mathcal{O}_F$ enable us to define the average Jacobian.

$$b_F \equiv b = \exp \int_{\mathcal{O}_F} \log \text{Jac} F \, d\mu.$$ 

On each level $n$, the measure $\mu$ on $\mathcal{O}_F$ satisfies that $\mu(B_{w_n}^n \cap \mathcal{O}_F) = 1/2^n$ for every word $w_n$ of length $n$.

**Corollary 4.2.** For any piece of $B_{w_n}^n$ on the level $n$ and any point $w \in B_{w_n}^n$,

$$\text{Jac} F^{2^n}(w) = b^{2^n}(1 + O(\rho^n))$$

where $b$ is the average Jacobian of $F$ for some positive $\rho < 1$.

**Proof.** Since

$$\int_{B_{w_n}^n} \log \text{Jac} F^{2^n} \, d\mu = \int_{\mathcal{O}_F} \log \text{Jac} F \, d\mu = \log b,$$

there exists a point $\eta \in B_{w_n}^n$ such that $\log \text{Jac} F^{2^n}(\eta) = \frac{\log b}{\mu(B_{w_n}^n)} = 2^n \log b$.

For any $w \in B_{w_n}^n$, $\log \text{Jac} F^{2^n}(z) \leq C\rho^n + \log \text{Jac} F^{2^n}(\eta)$, and $O(\rho^n) = \log(1 + O(\rho^n))$ for a fixed constant $\rho$. Then

$$\log \text{Jac} F^{2^n}(w) = \log(1 + O(\rho^n)) + \log \text{Jac} F^{2^n}(\eta)$$

and

$$\log \text{Jac} F^{2^n}(\eta) = \log b$$

for some $b$. Thus

$$\log \text{Jac} F^{2^n}(w) \leq (C\rho^n + \log b) + (\log(1 + O(\rho^n)) + \log b) = (C\rho^n + \log(b^{2^n})) + (\log(1 + O(\rho^n)) + \log b)$$

and

$$\log \text{Jac} F^{2^n}(w) = (C\rho^n + \log(b^{2^n})) + (\log(1 + O(\rho^n)) + \log b).$$

Therefore

$$\log \text{Jac} F^{2^n}(w) = (C\rho^n + \log(b^{2^n})) + (\log(1 + O(\rho^n)) + \log b) = (C\rho^n + \log(b^{2^n})) + (\log(1 + O(\rho^n)) + \log b).$$

The result follows.
\[ = \log \left[ (1 + O(\rho^n)) \cdot b^{2n} \right] \]

Hence, \( \text{Jac} F^{2n}(w) = b^{2n}(1 + O(\rho^n)) \).

Since \( F \) is the \( m + 2 \) dimensional map, it has Lyapunov exponents \( \chi_0, \chi_1, \ldots, \chi_{m+2} \). Let \( \chi_0 \) be the maximal one. Since \( F \) is ergodic with respect to the invariant finite measure \( \mu \) on the critical Cantor set, we get the following inequality for any Lyapunov exponent \( \chi \)

\[ |\mu| \chi(x) \leq \int_{\mathcal{O}_F} \log \|DF(x)\| \, d\mu(x) \]

where \( |\mu| \) is the total mass of \( \mu \) on \( \mathcal{O}_F \).

**Theorem 4.3.** The maximal Lyapunov exponent of \( F \) on \( \mathcal{O}_F \) is 0.

**Proof.** Let \( \mu_n \) be \( 2^n \mu|_{B_n^w} \), an invariant measure under \( F^{2n} \) and let \( \nu_n \) be the (unique) invariant measure on \( R^n F|_{\mathcal{O}_{R^n F}} \). Then

\[ 2^n \chi_0(F, \mu) = \chi_0(F^{2n}|_{B_n^w}, \mu_n) = \chi_0(R^n F, \nu_n) \leq \int_{B_n^w} \log \|D(R^n F)\| \, d\nu_n \leq C \]

for every \( n \in \mathbb{N} \), where \( C \) is a constant independent of \( n \). The last inequality comes from the uniformly bounded \( C^1 \) norm of derivative of \( R^n F \). Then the maximal Lyapunov exponent \( \chi_0 \leq 0 \). If \( \chi_0 < 0 \), then the support of \( \mu \) contains some periodic cycles by Pesin’s theory. But \( \mathcal{O}_F \) does not contain any periodic cycle because \( F \) acts on \( \mathcal{O}_F \) as a dyadic adding machine. Hence, \( \chi_0 = 0 \).

\[ \square \]

### 5. Universal expression of Jacobian determinant

The universality of average Jacobian is involved with the asymptotic behavior of the non-linear scaling map \( \Psi^n_{v_n} \), between the renormalized map \( F_n \equiv R^n F \) and \( F^{2n} \) for each \( n \in \mathbb{N} \). \( \Psi^n_{v_n} \) conjugate \( F^{2n} \) to \( F_n \). Thus using chain rule and Corollary 4.2, \( \text{Jac} F_n \) is the product of the average Jacobian of \( F^{2n} \) and the ratio of the \( \text{Jac} \Psi^n_{v_n} \) at \( w \) and \( F_n(w) \) as follows

\[ \text{Jac} F_n(w) = \text{Jac} F^{2n}(\Psi^n_{v_n}(w)) \frac{\text{Jac} \Psi^n_{v_n}(w)}{\text{Jac} \Psi^n_{v_n}(F_n(w))} \]

\[ = b^{2n} \frac{\text{Jac} \Psi^n_{v_n}(w)}{\text{Jac} \Psi^n_{v_n}(F_n(w))} (1 + O(\rho^n)). \]

Denote the word, \( v^n \) by \( v \). Then the universality of Jacobian of \( \Psi^n_{v_n} \) implies the universality of \( \text{Jac} F_n \) in Theorem 5.10 below. The asymptotic of non-linear part of \( \Psi^n_{v_n} \) is essential to the universal expression of \( \text{Jac} \Psi^n_{v_n} \).
5.1. Asymptotic of $\Psi^n_k$ for fixed $k^{th}$ level. For every infinitely renormalizable Hénon-like map $F$, we have a well defined tip

\begin{equation}
\{\tau\} \equiv \{\tau_F\} = \bigcap_{n \geq 0} B_{\nu_n}^n
\end{equation}

where the pieces $B_{\nu_n}^n$ are defined as $\Psi^n(B(R^n F))$. The tip of the renormalized map, $R^k F$ is denoted by $\tau_k = \tau(R^k F)$ for each $k \in \mathbb{N}$. Since every $B_{\nu_n}^n(F)$ contains $\tau_F$, let us condense the notation $\Psi^n_v$ into $\Psi^n_{\text{tip}}$. Moreover, in order to simplify the notation and calculations, we would let the tip move to the origin as a fixed point of each $\Psi^n_v(R^k F)$ for every $k \in \mathbb{N}$ by conjugation of the appropriate translations. Let us define $\Psi^{k+1}_k$ in this section.

\begin{equation}
\Psi_k(w) \equiv \Psi^{k+1}_k(w) = \Psi^{k+1}_v(w + \tau_{k+1}) - \tau_k
\end{equation}

Let the derivative of the map defined $\Psi_k$ on (5.3) at the origin be $D_k \equiv D^{k+1}_k$.

\begin{equation}
D^{k+1}_k \equiv D_k = D\Psi^{k+1}_k(0) = D(\Psi^{1}_v(R^k F))(\tau_{k+1})
= D(T_k \circ \Psi^{1}_v(R^k F) \circ T^{-1}_{k+1})(0)
\end{equation}

where $T_j : w \mapsto w - \tau_j$ for $j = k, k+1$. Then we can decompose $D_k$ into the matrix of which diagonal entries are 1s and the diagonal matrix as follows

\begin{equation}
\begin{pmatrix}
1 & t_k & u_1^1 & \ldots & u^m_k \\
1 & u_1^1 & \ldots & u^m_k \\
d_k^1 & 1 & \ldots & \\
d_k^m & 1
\end{pmatrix}
\begin{pmatrix}
\alpha_k \\
\sigma_k \\
\sigma_k \\
\sigma_k
\end{pmatrix}
= 
\begin{pmatrix}
\alpha_k & \sigma_k t_k & \sigma_k u_1^1 & \ldots & \sigma_k u^m_k \\
\sigma_k \\
\sigma_k d_k^1 \\
\vdots \\
\sigma_k d_k^m
\end{pmatrix}
\begin{pmatrix}
\sigma_k \\
\sigma_k \\
\sigma_k d_k^1 \\
\vdots \\
\sigma_k d_k^m
\end{pmatrix}
\begin{pmatrix}
\sigma_k \\
\sigma_k \\
\sigma_k \\
\sigma_k
\end{pmatrix}
\end{equation}

where $\text{Id}_{m \times m}$ is the $m \times m$ identity matrix. We condense the expression of $D_k$ using the boldfaced letters. Let $m$ dimensional vector $(d_1^k d_2^k \ldots d^m_k)^T$ be $d_k$ and $(u_1^k u_2^k \ldots u^m_k)$ be $u_k$ where $T$ is the transpose of the matrix. Recall that $\sigma_k = -\sigma (1 + O(\rho^k))$. Decompose $\Psi^{k+1}_k$ into the linear and non-linear parts. Then

\begin{equation}
D_k = 
\begin{pmatrix}
1 & t_k & u_k \\
1 & u_k \\
d_k & 1
\end{pmatrix}
\begin{pmatrix}
\alpha_k \\
\sigma_k \\
\sigma_k \cdot \text{Id}_{m \times m}
\end{pmatrix}
= 
\begin{pmatrix}
\alpha_k & \sigma_k t_k & \sigma_k u_k \\
\sigma_k \\
\sigma_k d_k
\end{pmatrix}
\begin{pmatrix}
\sigma_k \\
\sigma_k \\
\sigma_k \cdot \text{Id}_{m \times m}
\end{pmatrix}
\begin{pmatrix}
\sigma_k \\
\sigma_k \\
\sigma_k
\end{pmatrix}
\end{equation}

\begin{equation}
\Psi^{k+1}_k \equiv \Psi_k(w) = D_k \circ (\text{id} + s_k)(w)
\end{equation}

where $w = (x, y, z)$ and $s_k(w) = (s_k(w), 0, r^1_k(y), r^2_k(y), \ldots, r^m_k(y)) = O(|w|^2)$ near the origin. Denote $(r^1_k(y), r^2_k(y), \ldots, r^m_k(y))$ by $r_k$. Comparing the derivative of $H^{-1}_k \circ \Lambda^{-1}_k$ at

\footnote{If we need to distinguish the scaling maps, $\Psi^n_k$ around tip from its composition with translations, then we use the notation, $\Psi^n_{k, \text{tip}}$}
the tip and $D_k$ and Corollary 3.2 we obtain the following estimations

$$t_k = \partial_y \phi_k^{-1}(\tau_{k+1}) = \partial_x \phi_k^{-1}(\tau_{k+1}) \cdot \left[ \partial_y \varepsilon_k(\tau_k) + \sum_{j=1}^{m} \partial_{z_j} \varepsilon_k(\tau_k) \cdot d^j_k \right]$$

(5.7) $u^i_k = \partial_{z_i} \phi_k^{-1}(\tau_{k+1}) = \partial_x \phi_k^{-1}(\tau_{k+1}) \cdot \partial_{z_i} \varepsilon_k(\tau_k)$

and $d^i_k = \frac{d}{dy} \delta^i_k(\pi_y(\tau_{k+1}), f_k^{-1}(\pi_y(\tau_{k+1})), 0)$

where $\phi_k^{-1}(w) = \pi_x H_k^{-1}(w)$ for $1 \leq i \leq m$. Thus the norm of each element of $D_k$, $|t_k|$, $\|u^i_k\|$ and $\|d^i_k\|$ is bounded by $O(\varepsilon^{2k})$. Since $\|\partial_\tau \phi_k^{-1}\|$ at the tip exponentially converges to $\sigma$ as $k \to \infty$, $\alpha_k = \sigma^2 \left( 1 + O(\rho^k) \right)$ for some $\rho \in (0, 1)$.

Lemma 5.1. Let $s_k$ be the function defined in (3.6). For each $k \in \mathbb{N}$,

1. $|\partial_x s_k| = O(1)$, $|\partial_y s_k| = O(\varepsilon^{2k})$, $|\partial_z s_k| = O(\varepsilon^{2k})$
2. $|\partial_x^2 s_k| = O(1)$, $|\partial_x^2 s_k| = O(\varepsilon^{2k})$, $|\partial_{xy} s_k| = O(\varepsilon^{2k})$
3. $\partial_{zz} s_k = O(\varepsilon^{2k})$, $\partial_{xx} s_k = O(\varepsilon^{2k})$, $\partial_{zz} s_k = O(\varepsilon^{2k})$
4. $\|r_k(y)\| = O(\varepsilon^{2k})$, $\|r_k(y)\| = O(\varepsilon^{2k})$, $\|r_k(y)\| = O(\varepsilon^{2k})$

for $1 \leq i, j \leq m$.

Proof. The map $\Psi_k$ has the two expressions, $D_k \circ (id + s_k)(w)$ and $T_k \circ H_k^{-1} \circ \Lambda_k \circ T_{k+1}^{-1}$, that is,

$$\Psi_k = D_k \circ (id + s_k)(w) = T_k \circ H_k^{-1} \circ \Lambda_k \circ T_{k+1}^{-1} = H_k^{-1} \circ \Lambda_k^{-1}(w + \tau_{k+1}) - \tau_k$$

Recall that

$$D_k \circ (id + s_k)(w) = \begin{pmatrix} \alpha_k & \sigma_k t_k & \sigma_k u^1_k & \cdots & \sigma_k u^m_k \\ \sigma_k & \cdots & \cdots & \cdots & \cdots \\ \sigma_k d^1_k & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_k d^m_k & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} x + s(w) \\ y \\ z_1 + r^1_k(y) \\ \vdots \\ z_m + r^m_k(y) \end{pmatrix} = \begin{pmatrix} \alpha_k & \sigma_k t_k & \sigma_k u_k \\ \sigma_k & \cdots & \cdots & \cdots & \cdots \\ \sigma_k d_k & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} x + s(w) \\ y \\ z + r_k(y) \end{pmatrix}$$

In order to obtain the asymptotic of the non-linear part of $\Psi_k$, we need to compare the first and $z_i$ coordinates for each $1 \leq i \leq m$ of above two expressions of $\Psi_k$. Let $\tau_k = (\tau^1_k, \tau^2_k, \tau^3_k, \ldots, \tau^{2m}_k)$ $= (\tau^1_k, \tau^2_k, \tau^3_k)$ for each $k \geq 1$. Firstly, let us compare the $z_i$ coordinates of two expression of $\Psi_k$.

$$\sigma_k(d_k^i y + z_i + r^i_k(y)) = \pi_{z_i}(H_k^{-1} \circ \Lambda_k^{-1}(w + \tau_{k+1}) - \tau_k)$$

$$= \sigma_k(z_i + \tau_{k+1}) + \delta^i_k(\sigma_k(y + \tau_k^i), f_k^{-1}(\sigma_k(y + \tau_k^i)), 0) - \tau_k^i$$

Thus we have the following equation

$$\sigma_k r^i_k(y) = -\sigma_k d_k^i y + \delta^i_k(\sigma_k(y + \tau_k^i), f_k^{-1}(\sigma_k(y + \tau_k^i)), 0) + \sigma_k \tau_k^i - \tau_k^i.$$
Then $|r^i_k(y)| \leq C (|d^i_k y| + \|\delta_k\|_{C^0})$ for some $C > 0$ and for all $1 \leq i \leq m$. The domain is bounded and $\|\delta_k\| = o(\varepsilon_k^2)$. Then we obtain $\|r_k(y)\| = O(\varepsilon_k^2)$. Moreover,

$$(r_k^i)'(y) = -d^i_k + \frac{d}{dy} \delta^i_k (\sigma_k(y + \tau^y_k), f_k^{-1}(\sigma_k(y + \tau^y_k), 0))$$

Thus $|r_k^i(y)|$ is controlled by $\|\delta\|_{C^1}$ for all $1 \leq i \leq m$. Similarly, the second derivative $|r_k^{i''}(y)|$ is also controlled by $\|\delta\|_{C^2}$ for all $1 \leq i \leq m$. Then $\|r_k^i(y)\| = O(\varepsilon_k^2)$ and $\|r_k^{i''}(y)\| = O(\varepsilon_k^2)$.

Secondly, compare first coordinates using (5.4) and (5.6). Thus

$$(5.8) \quad \alpha_k x + \alpha_k \cdot s_k(w) + \sigma_k t_k y + \sigma_k u_k \cdot (z + r_k(y)) = \phi_k^{-1}(\sigma_k w + \sigma_k \tau_k) - \pi_x(\tau_k).$$

It implies the following equations

$$(5.9) \quad \alpha_k \cdot \partial_x s_k = \sigma_k \cdot \partial_x \phi_k^{-1} - \alpha_k$$

$$\alpha_k \cdot \partial_y s_k = \sigma_k \cdot \partial_y \phi_k^{-1} - \sigma_k t_k - \sigma_k \cdot u_k \cdot r_k(y)$$

$$\alpha_k \cdot \partial_z s_k = \sigma_k \cdot \partial_z \phi_k^{-1} - \sigma_k u_k^i$$

for $1 \leq i \leq m$. Then by the equation (5.4), $\|\partial_x \phi_k^{-1}\| = O(1)$, $\|\partial_y \phi_k^{-1}\| = O(\varepsilon_k^2)$ and $\|\partial_z \phi_k^{-1}\| = O(\varepsilon_k^2)$ for all $1 \leq i \leq m$. By the equation (5.7), $|t_k|$ and $\|u_k\|$ is $O(\varepsilon_k^2)$. Hence, $\|\partial_x s_k\| = O(1)$, $\|\partial_y s_k\| = O(\varepsilon_k^2)$ and $\|\partial_z s_k\| = O(\varepsilon_k^2)$ for all $1 \leq i \leq m$. By the above equation (5.9), each second partial derivatives of $s_k$ are comparable with the second partial derivatives of $\phi_k^{-1}$ over the same variables because $\|r_k^i(y)\| = O(\varepsilon_k^2)$.

Let us estimate some second partial derivatives of $\phi_k^{-1}$. Recall that

$$\phi_k^{-1}(w) = f_k^{-1}(x + \varepsilon_k \circ H_k^{-1}(w))$$

$$\varepsilon_k \circ H_k^{-1}(w) = \varepsilon_k(\phi_k^{-1}(w), y, z + \delta_k(y, f^{-1}(y), 0)).$$

Thus

$$\partial_x \phi_k^{-1}(w) = (f_k^{-1})'(x + \varepsilon_k \circ H_k^{-1}(w)) \cdot \left[ 1 + \partial_x (\varepsilon_k \circ H_k^{-1}(w)) \right]$$

$$\partial_x (\varepsilon_k \circ H_k^{-1}(w)) = \partial_x \varepsilon_k \circ H_k^{-1}(w) \cdot \partial_x \phi_k^{-1}(w)$$

$$\partial_{xx} (\varepsilon_k \circ H_k^{-1}(w)) = \partial_x (\varepsilon_k \circ H_k^{-1}(w)) \cdot \partial_x \phi_k^{-1}(w) + \partial_x \varepsilon_k \circ H_k^{-1}(w) \cdot \partial_{xx} \phi_k^{-1}(w)$$

$$= \partial_x \varepsilon_k \circ H_k^{-1}(w) \cdot \left[ \partial_x \phi_k^{-1}(w) \right]^2 + \partial_x \varepsilon_k \circ H_k^{-1}(w) \cdot \partial_{xx} \phi_k^{-1}(w).$$

Moreover, $\|\varepsilon_k\|_{C^2}$ and $\|\delta_k\|_{C^2}$ bounds the norm of every second derivatives of $\|\phi_k^{-1}\|$ except $\|\partial_{xx} \phi_k^{-1}\|$. Let us estimate $\partial_{xx} \phi_k^{-1}(w)$

$$\partial_{xx} \phi_k^{-1}(w) = (f_k^{-1})''(x + \varepsilon_k \circ H_k^{-1}(w)) \cdot \left[ 1 + \partial_x (\varepsilon_k \circ H_k^{-1}(w)) \right]$$

$$+ (f_k^{-1})'(x + \varepsilon_k \circ H_k^{-1}(w)) \cdot \partial_{xx} (\varepsilon_k \circ H_k^{-1}(w)).$$

Recall that $\|\varepsilon_k\|_{C^2}$ and $\|\delta_k\|_{C^2}$ are $O(\varepsilon_k^2)$. Since both $\|(f^{-1})''\|$ and $\|(f^{-1})'\|$ are $O(1)$, so is $\|\partial_{xx} \phi_k^{-1}\|$. Any other second derivative of $\|\phi_k^{-1}\|$ is bounded by $O(\varepsilon_k^2)$. For example, the following expression of $\partial_{yy} \phi_k^{-1}$

$$\partial_{yy} \phi_k^{-1}(w) = (f_k^{-1})''(x + \varepsilon_k \circ H_k^{-1}(w)) \cdot \left[ 1 + \partial_x (\varepsilon_k \circ H_k^{-1}(w)) \right]$$

$$+ (f_k^{-1})'(x + \varepsilon_k \circ H_k^{-1}(w)) \cdot \partial_{yy} (\varepsilon_k \circ H_k^{-1}(w)).$$
5.2. Estimation of non linear part of \( S^n_k \). We consider the behavior of the non linear scaling map from \( k^{th} \) level to \( n^{th} \) level. Let

\[
\Psi^n_k = \Psi_k \circ \cdots \circ \Psi_{n-1}, \quad B^n_k = \text{Im} \Psi^n_k
\]

By Lemma \[3.1\],

\[
\text{diam}(B^n_k) = O(\sigma^{n-k}) \quad \text{for} \quad k < n
\]

Then combining Lemma \[3.1\] and Lemma \[5.1\], we have the following corollary.

**Corollary 5.2.** For all points \( w = (x, y, z) \in B^n_k \) and where \( k < n \), we have

\[
|\partial_x s_k(w)| = O(\sigma^{n-k}) \quad \text{and} \quad |\partial_y s_k(w)| = O(\varepsilon^2 \sigma^{n-k}) \quad \text{and} \quad |\partial_z s_k(w)| = O(\varepsilon^2 \sigma^{n-k})
\]

\[
\|r^n_k(y)\| = O(\varepsilon^2 \sigma^{n-k}) \quad \text{and} \quad \|r'^n_k(y)\| = O(\varepsilon^2 \sigma^{n-k})
\]

**Proof.** By definition, \( s_k(w) \) is quadratic and higher order terms at the tip, \( r_k \). Similarly, \( r'_k(y) \) only contains quadratic and higher order terms at the tip. Then use Taylor’s expansion and upper bounds of \( \text{diam}(B^n_k) \) is \( O(\sigma^{n-k}) \).

Since the origin is the fixed point of each \( \Psi_j \) and \( D_j \) is \( \Psi_j(0) \) for every \( k \leq j \leq n \), we can let the derivative of \( \Psi^n_k \) at the origin be the composition of consecutive \( D_i \)'s for \( k \leq i \leq n-1 \).

\[
D^n_k = D_k \circ D_{k+1} \circ \cdots \circ D_{n-1}
\]

We can decompose \( D^n_k \) into two matrices, the matrix whose diagonal entries are ones and the diagonal matrix by reshuffling.

**Remark 5.1.** The notations \( t_{n+1,n}, u_{n+1,n} \) and \( d_{n+1,n} \) are simplified as \( t_n, u_n \) and \( d_n \) like the notations used in \[5.1\]. Moreover, \( \alpha_{n+1,n}, \sigma_{n+1,n} \) are abbreviated as \( \alpha_n, \sigma_n \) respectively. Thus \( \alpha_n = \sigma^2(1 + O(\rho^n)) \), \( \sigma_n = -\sigma(1 + O(\rho^n)) \). Using the similar abbreviation, \( D_n \) denote \( D^n_{n+1} \) and \( s_n \) is the \( s^n_{n+1} \).

**Lemma 5.3.** The derivative of \( \Psi^n_k \) at the origin, \( D^n_k \) is decomposed into the dilation and non dilation parts as follows

\[
D^n_k = \begin{pmatrix}
1 & t_{n,k} & u_{n,k} \\
1 & 1 & d_{n,k}
\end{pmatrix}
\begin{pmatrix}
\alpha_{n,k} & \sigma_{n,k} \\
\sigma_{n,k} & \sigma_{n,k} \cdot \text{Id}_{m \times m}
\end{pmatrix}
\]

Moreover, \( \alpha_{n,k} = (\sigma^2)^{n-k}(1 + O(\rho^k)) \) and \( \sigma_{n,k} = (\sigma)^{n-k}(1 + O(\rho^k)) \) for some \( \rho \in (0, 1) \).
Each \( t_{n,k}, u_{n,k} \) and \( d_{n,k} \) are comparable with the \( t_{k+1,k}, u_{k+1,k} \) and \( d_{k+1,k} \) respectively and converges to the numbers \( t_{*,k}, u_{*,k} \) and \( d_{*,k} \) super exponentially fast as \( n \to \infty \).

Proof. Using the definition of each derivatives of \( \Psi_j \) on the equation (5.4) at the fixed point zero, we obtain

\[
D_k^n = \prod_{j=k}^{n-1} D_j = \prod_{j=k}^{n-1} \begin{pmatrix} \alpha_j & \sigma_j t_j & \sigma_j u_j \\ \sigma_j & \sigma_j & \sigma_j \end{pmatrix}.
\]

By the straightforward calculation, we have following matrix,

\[
D_k^n = \begin{pmatrix} \prod_{j=k}^{n-1} \alpha_j & T_{n,k} & U_{n,k} \\ \prod_{j=k}^{n-1} \sigma_j & n-1 & \prod_{j=k}^{n-1} \sigma_j \cdot \text{Id}_{m \times m} \\ \prod_{j=k}^{n-1} \sigma_j \cdot m_{j} & \prod_{j=k}^{n-1} \sigma_j \cdot m_{j} & \prod_{j=k}^{n-1} \sigma_j \cdot \text{Id}_{m \times m} \end{pmatrix}
\]

where

\[
U_{n,k} = \sigma_k \sigma_{k+1} \sigma_{k+2} \cdots \sigma_{n-2} \sigma_{n-1} u_k \\
+ \alpha_k \sigma_k \sigma_{k+1} \sigma_{k+2} \cdots \sigma_{n-3} \sigma_{n-2} \sigma_{n-1} u_{k+1} \\
+ \alpha_k \sigma_k \sigma_{k+1} \sigma_{k+2} \cdots \sigma_{n-3} \sigma_{n-2} \sigma_{n-1} u_{k+2} \\
\vdots \\
+ \alpha_k \sigma_k \sigma_{k+1} \sigma_{k+2} \cdots \sigma_{n-3} \sigma_{n-2} \sigma_{n-1} u_{n-1}
\]

\[
T_{n,k} = \sigma_k \sigma_{k+1} \sigma_{k+2} \cdots \sigma_{n-3} \sigma_{n-2} \sigma_{n-1} \left[ u_k \cdot \left( d_{k+1} + d_{k+2} + d_{k+3} + \cdots + d_{n-1} \right) + t_k \right] \\
+ \alpha_k \sigma_k \sigma_{k+1} \sigma_{k+2} \cdots \sigma_{n-3} \sigma_{n-2} \sigma_{n-1} \left[ u_{k+1} \cdot \left( d_{k+2} + d_{k+3} + \cdots + d_{n-1} \right) + t_{k+1} \right] \\
+ \alpha_k \sigma_k \sigma_{k+1} \sigma_{k+2} \cdots \sigma_{n-3} \sigma_{n-2} \sigma_{n-1} \left[ u_{k+2} \cdot \left( d_{k+3} + \cdots + d_{n-1} \right) + t_{k+2} \right] \\
\vdots \\
+ \alpha_k \sigma_k \sigma_{k+1} \sigma_{k+2} \cdots \sigma_{n-3} \sigma_{n-2} \sigma_{n-1} \left[ u_{n-2} \cdot d_{n-1} + t_{n-2} \right] \\
+ \alpha_k \sigma_k \sigma_{k+1} \sigma_{k+2} \cdots \sigma_{n-3} \sigma_{n-2} \sigma_{n-1} \cdot t_{n-1}.
\]

Moreover,

\[
\sigma_{n,k} = \prod_{j=k}^{n-1} \sigma_j = \prod_{j=k}^{n-1} (-\sigma) (1 + O(\rho^j)) = (-\sigma)^{n-k} (1 + O(\rho^k))
\]

(5.11)

\[
\alpha_{n,k} = \prod_{j=k}^{n-1} \alpha_j = \prod_{j=k}^{n-1} \sigma^2 (1 + O(\rho^j)) = \sigma^{2(n-k)} (1 + O(\rho^k))
\]

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By the definition of $d_{n,k}$ and (5.11), each components of the diffeomorphic part and the scaling part are separated

$$d_{n,k} = \sum_{j=k}^{n-1} d_j$$

(5.12)

$$u_{n,k} = \sum_{j=k}^{n-2} (-\sigma)^{j-k} u_j (1 + O(\rho^k))$$

$$t_{n,k} = \sum_{j=k}^{n-1} (-\sigma)^{j-k} \left[ u_j \cdot \sum_{i=j}^{n-2} d_{i+1} + t_j + t_{n-1} \right] (1 + O(\rho^k)).$$

Since $d_j = O(\tilde{\varepsilon}^{2^j})$, $u_j = O(\tilde{\varepsilon}^{2^j})$ and $|t_j| = O(\tilde{\varepsilon}^{2^j})$ for each $j \in \mathbb{N}$, each terms of the series in (5.12) shrink super exponentially fast. Then the sum $d_{n,k}$, $u_{n,k}$ and $t_{n,k}$ are comparable with the first terms of each series. Moreover, $d_{n,k}$, $u_{n,k}$ and $t_{n,k}$ converge to $d_{*,k}$, $u_{*,k}$ and $t_{*,k}$ as $n \to \infty$ super exponentially fast respectively.

After reshuffling of $\Psi^n_k$, we can factor out $D^n_k$ from the map $\Psi^n_k$. Then we have

$$\Psi^n = D^n_k \circ (\text{id} + S^n_k)$$

where $S^n_k = (S^n_k(w), 0, R_{n,k}(y)) = O(|w|^2)$ near the origin. When we calculate directly the composition, $H_1^{-1} \circ \Lambda_1^{-1} \circ \cdots \circ H_{n-1}^{-1} \circ \Lambda_{n-1}^{-1}$. Observe that the map

$$R_{n,k} = (R^1_{n,k}, R^2_{n,k}, \ldots, R^m_{n,k})$$

depends only on $y$.

**Proposition 5.4.** The third coordinate of $S^n_k$, $R_{n,k}(y)$ has the following norm estimations.

$$\|R_{n,k}\| = O(\tilde{\varepsilon}^{2^k}), \quad \|(R_{n,k})'\| = O(\tilde{\varepsilon}^{2^k} \sigma^{n-k}) \quad \text{and} \quad \|(R_{n,k})''\| = O(\tilde{\varepsilon}^{2^k} \sigma^{2(n-k)})$$

for all $k < n$.

**Proof.** The proof comes from the recursive formula between each partial derivatives of $S^n_k$ and $S^n_{k+1}$. So before proving this lemma we need some intermediate calculations. For a point $w = (x, y, z) \in B$, let

$$w^n_{k+1} = \begin{pmatrix} x^n_{k+1} \\ y^n_{k+1} \\ z^n_{k+1} \end{pmatrix} = \Psi^n_{k+1}(w) \in B^n_{k+1}$$

By the equation (5.13), we have

$$\begin{pmatrix} x^n_{k+1} \\ y^n_{k+1} \\ z^n_{k+1} \end{pmatrix} = \begin{pmatrix} \alpha_{n,k+1} & \sigma_{n,k+1} \cdot t_{n,k+1} & \sigma_{n,k+1} \cdot u_{n,k+1} \\ \sigma_{n,k+1} & \sigma_{n,k+1} \cdot d_{n,k+1} & \sigma_{n,k+1} \cdot \text{Id}_{m \times m} \\ \sigma_{n,k+1} \cdot d_{n,k+1} & \text{Id}_{m \times m} & \text{Id}_{m \times m} \end{pmatrix} \begin{pmatrix} x + S^n_{k+1}(w) \\ y \\ z + R_{n,k+1}(y) \end{pmatrix}.$$
\[ x_{k+1}^n = \alpha_{n,k+1}(x + S_{k+1}^n(w)) + \sigma_{n,k+1} t_{n,k+1} \cdot y + \sigma_{n,k+1} u_{n,k+1} \cdot (z + R_{n,k+1}(y)) \]
(5.14)  
\[ y_{k+1}^n = \sigma_{n,k+1} \cdot y \]
\[ z_{k+1}^n = \sigma_{n,k+1} d_{n,k+1} \cdot y + \sigma_{n,k+1} (z + R_{n,k+1}(y)). \]

For any fixed \( k < n \), the recursive formula of \( \Psi_k^n \) is
\[ D_k^n \circ (id + S_k^n) = \Psi_k^n = \Psi_k \circ \Psi_{k+1}^n = D_k \circ (id + s_k) \circ \Psi_{k+1}^n \]
(5.15)
Thus \[ \Psi_k^n(w) = D_k^n \circ (id + S_{k+1}^n)(w) + D_k \circ s_k(w_{k+1}^n) \]
and note that
\[ D_k \circ s_k(w_{k+1}^n) = \begin{pmatrix} \alpha_k & \sigma_k t_k & \sigma_k u_k \\ \sigma_k & \sigma_k & \sigma_k \cdot \text{Id}_{m \times m} \\ \sigma_k d_k & \sigma_k \cdot \text{Id}_{m \times m} \end{pmatrix} \begin{pmatrix} s_k(w_{k+1}^n) \\ 0 \\ r_k(y_{k+1}^n) \end{pmatrix}. \]

Moreover, the first partial derivatives of each coordinate are as follows
\[ \frac{\partial x_{k+1}^n}{\partial x} = \alpha_{n,k+1} \left(1 + \frac{\partial S_{k+1}^n}{\partial x}(w)\right) \]
\[ \frac{\partial x_{k+1}^n}{\partial y} = \alpha_{n,k+1} \frac{\partial S_{k+1}^n}{\partial y}(w) + \sigma_{n,k+1} t_{n,k+1} + \sigma_{n,k+1} u_{n,k+1} \cdot (R_{n,k+1})'(y) \]
\[ \frac{\partial x_{k+1}^n}{\partial z_i} = \alpha_{n,k+1} \frac{\partial S_{k+1}^n}{\partial z_i}(w) + \sigma_{n,k+1} u_{i,n,k+1} \]
(5.16)
\[ \frac{\partial y_{k+1}^n}{\partial y} = \frac{\partial z_{k+1}^n}{\partial z_i} = \sigma_{n,k+1} \]
\[ \frac{\partial z_{k+1}^n}{\partial y} = \sigma_{n,k+1} d_{n,k+1} + \sigma_{n,k+1} \cdot (R_{n,k+1})'(y) \]
\[ \frac{\partial y_{k+1}^n}{\partial x} = \frac{\partial y_{k+1}^n}{\partial z_i} = \frac{\partial z_{k+1}^n}{\partial x} = 0 \]

for every \( 1 \leq i \leq m \). In order to estimate of \( R_{n,k}(y) \), compare the third coordinates of the functions in (5.14). Recall \( \sigma^{-1} = \lambda \). Then
\[ z_k^n = \sigma_{n,k} d_{n,k} \cdot y + \sigma_{n,k}(z + R_{n,k}(y)) \]
\[ = \sigma_{n,k} d_{n,k} \cdot y + \sigma_{n,k}(z + R_{n,k+1}(y)) + \sigma_k \cdot r_k(y_{k+1}^n) \]

Then
\[ R_{n,k}(y) = R_{n,k+1}(y) + \sigma_{n,k}^{-1} \cdot \sigma_k \cdot r_k(y_{k+1}^n) \]

where \( \sigma_{n,k}^{-1} \cdot \sigma_k \) is \((-\lambda)^{n-k-1}(1 + O(\rho))\). By the equation (5.16), the recursive relation between \( R_k^n(y), R_{k+1}^n(y) \) and the bounds of \( r_k(y_{k+1}^n) \), we obtain the following formulas
\[ R_{n,k}(y) = R_{n,k+1}(y) + O\left((-\lambda)^{n-k-1}r_k(y_{k+1}^n)\right) \]
\[ (R_{n,k})'(y) = (R_{n,k+1})'(y) + O(r_k'(y_{k+1}^n)) \]
and \[ (R_{n,k})''(y) = (R_{n,k+1})''(y) + O(\sigma^{n-k} \cdot r_k''(y_{k+1}^n)). \]

Hence, by the equation (5.14) and the chain rule
\[ \|R_{n,k}\| \leq \|R_{n,k+1}\| + K_0 \varepsilon^{2k} \]
\[ \|(R_{n,k})'\| \leq \|(R_{n,k+1})'\| + K_1 \varepsilon^{2k} \sigma^{n-k} \]
\[ \|(R_{n,k})''\| \leq \|(R_{n,k+1})''\| + K_2 \varepsilon^{2k} \sigma^{2(n-k)} \]

for all \( k < n \). Then,
\[ \|R_{n,k}\| = O(\varepsilon^{2k}), \quad \|(R_{n,k})'\| = O(\varepsilon^k \sigma^{n-k}) \quad \text{and} \quad \|(R_{n,k})''\| = O(\varepsilon^{2k} \sigma^{2(n-k)}) \]
for all \( k < n \). \hfill \square

**Lemma 5.5.** For \( k < n \), we have

1. \( |\partial_z S^n_{k}| = O(1), \quad |\partial_y S^n_{k}| = O(\varepsilon^k), \quad |\partial_z S^n_{k}| = O(\varepsilon^k) \)
2. \( |\partial^2_{xy} S^n_{k}| = O(\varepsilon^k \sigma^{n-k}), \quad |\partial^2_{xz} S^n_{k}| = O(\varepsilon^k \sigma^{n-k}) \)
3. \( |\partial^2_{yz} S^n_{k}| = O(\varepsilon^k), \quad |\partial^2_{zz} S^n_{k}| = O(\varepsilon^k) \).

for every \( 1 \leq i, j \leq m \).

**Proof.** Compare the first coordinates of \( \Psi_n^k \) in (5.15). Thus
\[ x_k^n = \alpha_{n,k}(x + S^n_k(w)) + \sigma_{n,k} t_{n,k} \cdot y + \sigma_{n,k} u_n \cdot (z + R_{n,k}(y)) \]
\[ = \alpha_{n,k}(x + S^n_{k+1}(w)) + \sigma_{n,k} t_{n,k} \cdot y + \sigma_{n,k} u_n \cdot (z + R_{n,k+1}(y)) + \alpha_k \cdot s_k(w_{k+1}^n) \]
\[ + \sigma_k u_k \cdot r_k(y_{k+1}^n). \]

Then we obtain the recursive formula for \( S^n_k(w) \) as follows
\[ S^n_k(w) = S^n_{k+1}(w) + \alpha_{n,k} \cdot s_k(w_{k+1}^n) + \alpha_{n,k} \cdot \sigma_{n,k} u_n \cdot (R_{n,k+1}(y) - R_{n,k}(y)) \]
\[ + \alpha_{n,k} \cdot \sigma_k u_k \cdot r_k(y_{k+1}^n). \]

Let us take the first partial derivatives of each side of above equation and use (5.16). Then we can have the recursive formulas of each first partial derivatives of \( S^n_k(w) \). Let us take the coordinate expression of \( w_{k+1}^n \) as \( (x_{k+1}^n, y_{k+1}^n, z_{k+1}^n, z_{k+1}^n, \ldots, z_{k+1}^n m) \). Then
\[ \frac{\partial S^n_k}{\partial x} = \left(1 + \frac{\partial s_k}{\partial x_{k+1}^n}\right) + \frac{\partial s_k}{\partial x_{k+1}^n} \]
\[ \frac{\partial S^n_k}{\partial y} = \left(1 + \frac{\partial s_k}{\partial x_{k+1}^n}\right) \frac{\partial S^n_{k+1}}{\partial y} + K_1 \lambda^{n-1-k} \left[(t_{n,k+1} + u_{n,k+1} \cdot (R_{n,k+1})'(y)) \frac{\partial s_k}{\partial x_{k+1}^n} \right] \]
\[ + \sum_{j=1}^{m} \left(d_{n,k+1} + (R_{n,k+1})'(y) \right) \frac{\partial s_k}{\partial(z_{k+1}^n)_j} \]
\[ + K_1 \lambda^{n-k-1} u_{n,k} \cdot \left( (R_{n,k+1})'(y) - (R_{n,k})'(y) \right) + K_2 \lambda^{n-k} u_k \cdot r_k'(y_{n+1}) \]

\[
\frac{\partial S_k^n}{\partial z_i} = \left( 1 + \frac{\partial s_k}{\partial x_{k+1}^n} \right) \frac{\partial S_{k+1}^n}{\partial z_i} + K_1 \lambda^{n-k-1} \left[ u_{n,k+1}^i \frac{\partial s_k}{\partial x_{k+1}^n} + \frac{\partial s_k}{\partial (z_{k+1}^n)} \right]
\]

where \( \alpha_{n-k}^{-1} \alpha_k \sigma_{n,k+1} = K_1(-\lambda)^{n-k-1} \) and \( \alpha_{n-k}^{-1} \sigma_{n,k+1} = K_2(-\lambda)^{n-k+1} \) for each \( 1 \leq i \leq m \). By Corollary 5.2 and Proposition 5.4, \( |\partial s_k/\partial x_{k+1}^n| \) is \( O(\sigma^{n-k}) \) and \( |\partial s_k/\partial y_{n,k+1}| \) and \( |\partial s_k/\partial (z_{k+1}^n)| \) is \( O(\varepsilon^{2k} \sigma^{n-k}) \) for all \( 1 \leq i \leq m \). Moreover, \( |t_{n,k}|, \|u_{n,k}\| \) and \( \|d_{n,k}\| \) are \( O(\varepsilon^{2k}) \).

With all these facts, the bounds of partial derivatives of \( S_k^n \) are as follows

\[
\left| \frac{\partial S_k^n}{\partial x} \right| \leq (1 + O(\rho^{n-k})) \left| \frac{\partial S_{k+1}^n}{\partial x} \right| + C \sigma^{n-k}
\]

\[
\left| \frac{\partial S_k^n}{\partial y} \right| \leq (1 + O(\rho^{n-k})) \left| \frac{\partial S_{k+1}^n}{\partial y} \right| + C \varepsilon^{2k}
\]

\[
\left| \frac{\partial S_k^n}{\partial z_i} \right| \leq (1 + O(\rho^{n-k})) \left| \frac{\partial S_{k+1}^n}{\partial z_i} \right| + C \varepsilon^{2k}
\]

for all \( 1 \leq i \leq m \), for some constant \( C > 0 \) and \( \rho \in (0,1) \). Hence, using above recursive formulas we have

\[
\left| \frac{\partial S_k^n}{\partial x} \right| = O(\sigma), \quad \left| \frac{\partial S_k^n}{\partial y} \right| = O(\varepsilon^{2k}) \quad \text{and} \quad \left| \frac{\partial S_k^n}{\partial z} \right| = O(\varepsilon^{2k})
\]

for all \( k < n \). The second partial derivatives of \( S_k^n \) are as follows by the chain rule

\[
\frac{\partial^2 S_k^n}{\partial x z_i} = \left( 1 + \frac{\partial s_k}{\partial x_{k+1}^n} \right) \frac{\partial^2 S_{k+1}^n}{\partial x z_i} + \alpha_{n,k+1} \left( 1 + \frac{\partial s_k}{\partial x_{k+1}^n} \right) \left( \frac{\partial^2 s_k}{\partial (x_{k+1}^n)^2} \frac{\partial S_{k+1}^n}{\partial y} \right) + \sigma_{n,k+1} \left( 1 + \frac{\partial s_k}{\partial x_{k+1}^n} \right) \left[ t_{n,k+1} + u_{n,k+1} \cdot (R_{n,k+1})'(y) \right] \frac{\partial^2 s_k}{\partial (x_{k+1}^n)^2} + \frac{\partial^2 s_k}{\partial x_{k+1}^n y_{n,k+1}^n} \frac{\partial^2 S_{k+1}^n}{\partial x_{k+1}^n z_{k+1}^n}
\]

\[
\frac{\partial^2 S_k^n}{\partial y z_i} = \left( 1 + \frac{\partial s_k}{\partial x_{k+1}^n} \right) \frac{\partial^2 S_{k+1}^n}{\partial y z_i} + \alpha_{n,k+1} \left( 1 + \frac{\partial s_k}{\partial x_{k+1}^n} \right) \left( \frac{\partial^2 s_k}{\partial (x_{k+1}^n)^2} \frac{\partial S_{k+1}^n}{\partial z_i} \right) + \sigma_{n,k+1} \left( 1 + \frac{\partial s_k}{\partial x_{k+1}^n} \right) \left[ u_{n,k+1}^{i,i} \frac{\partial^2 s_k}{\partial (x_{k+1}^n)^2} + \frac{\partial^2 s_k}{\partial x_{k+1}^n (z_{k+1}^n)} \right] + \sigma_{n,k+1} \left( t_{n,k+1} + u_{n,k+1} \cdot (R_{n,k+1})'(y) \right) \left( \frac{\partial^2 s_k}{\partial y} + K_1(-\lambda)^{n-k-1} u_{n,k+1} \right) \frac{\partial^2 S_{k+1}^n}{\partial (x_{k+1}^n)^2}
\]
\[ + \left( \sigma_{n,k+1} \frac{\partial S_{n,k+1}}{\partial z_i} + K_4 u_{n,k+1}^i \right) \frac{\partial^2 s_k}{\partial x_{n,k+1} \partial y_{n,k+1}} \]
\[ + \left[ \sigma_{n,k+1} \frac{\partial S_{n,k+1}}{\partial y} + K_4 \left( t_{n,k+1} + u_{n,k+1} \cdot (R_{n,k+1})'(y) \right) \right] \frac{\partial^2 s_k}{\partial x_{n,k+1}^n (z_{n,k+1})^i_j} \]
\[ + \left( \sigma_{n,k+1} \frac{\partial S_{n,k+1}}{\partial z_i} + K_4 u_{n,k+1}^i \right) \sum_{j=1}^{m} \left( d_{n,k+1}^j + (R_{n,k+1})'(y) \right) \cdot \frac{\partial^2 s_k}{\partial x_{n,k+1}^n (z_{n,k+1})^i_j} \]
\[ \frac{\partial^2 S_k^n}{\partial z_i z_j} = \left( 1 + \frac{\partial s_k^n}{\partial x_{k+1}^n} \right) \frac{\partial^2 S_{k+1}^n}{\partial z_i z_j} + \left[ \alpha_{n,k+1} \frac{\partial S_{k+1}^n}{\partial z_i} + \frac{\partial S_{k+1}^n}{\partial z_j} + \sigma_{n,k+1} u_{n,k+1}^j \frac{\partial S_{n,k+1}}{\partial z_i} \right] \frac{\partial^2 s_k}{\partial x_{n,k+1}^n (z_{n,k+1})^j} \]
\[ + \left[ \sigma_{n,k+1} \frac{\partial S_{n,k+1}}{\partial z_i} + K_4 u_{n,k+1}^i \right] \frac{\partial^2 s_k}{\partial x_{n,k+1}^n (z_{n,k+1})^i} \quad \frac{\partial^2 s_k}{\partial x_{n,k+1}^n (z_{n,k+1})^i} + K_4 \frac{\partial^2 s_k}{\partial (z_{n,k+1})^i (z_{n,k+1})^i} \]

where \( K_4 = \alpha_{n,k} \alpha_k \sigma_{n,k+1} = O(1) \) for every \( 1 \leq i, j \leq m \).

By Lemma 5.9, Corollary 5.2, and Proposition 5.4, the bounds of \( |\partial^2 s_k/\partial (x_{n,k+1})^2| \) is \( O(\sigma^{n-k}) \) and \( |\partial^2 s_k/\partial u v| \) is \( O(\varepsilon^k \sigma^{n-k}) \) where \( u, v = x_{n+1}^n, y_{n+1}^n, (z_{n+1}^n)_1, \ldots, (z_{n+1}^n)_m \) but both \( u \) and \( v \) are not \( x_{n+1}^n \) simultaneously. The upper bounds of the norm of the first and the second partial derivatives of \( s_k \) and the estimation of \( |t_{n,k}|, \|u_{n,k}\| \) and \( \|d_{n,k}\| \) imply the bounds of norm of second partial derivatives of \( S_k^n \) as follows.

\[ |\partial^2 S_k^n|_{xy} \leq (1 + O(\rho^{n-k})) \left| \frac{\partial^2 S_{k+1}^n}{\partial x \partial y} \right| + C \varepsilon^2 k \sigma^{n-k} \]
\[ |\partial^2 S_k^n|_{xz} \leq (1 + O(\rho^{n-k})) \left| \frac{\partial^2 S_{k+1}^n}{\partial x \partial z} \right| + C \varepsilon^2 k \sigma^{n-k} \]
\[ |\partial^2 S_k^n|_{yz} \leq (1 + O(\rho^{n-k})) \left| \frac{\partial^2 S_{k+1}^n}{\partial y \partial z} \right| + C \varepsilon^2 k \]
\[ |\partial^2 S_k^n|_{z_i z_j} \leq (1 + O(\rho^{n-k})) \left| \frac{\partial^2 S_{k+1}^n}{\partial z_i \partial z_j} \right| + C \varepsilon^2 k \]

Hence, \( |\partial^2 S_k^n|_{xy} = O(\varepsilon^2 k \sigma^{n-k}), |\partial^2 S_k^n|_{xz} = O(\varepsilon^2 k \sigma^{n-k}), |\partial^2 S_k^n|_{yz} = O(\varepsilon^2 k), \) and \( |\partial^2 S_k^n|_{z_i z_j} = O(\varepsilon^2 k) \) for every \( 1 \leq i, j \leq m \).

5.3. Universal properties of coordinate change map, \( \Psi_k^n \). On the following Lemma 5.8 we would show that the non-linear part of the coordinate change map \( id + S(x, y, z) \) is a small perturbation of the one-dimensional universal function. The content of this section is to rephrase some parts of Section 7 in [CLM].

Recall the one dimensional map \( f_s : I \to I \) is the fixed point of the (periodic doubling)
renormalization operator of the unimodal maps, namely, $Rf_s = f_s$. Let the critical point of $f_s$ be $c_s$ and $I = [-1,1]$. Also assume that $f_s(c_s) = 1$ and $f^2_s(c_s) = -1$. Let us take the intervals $J^*_c = [-1, f^4_s(c_s)]$ and $J^*_v = f_s(J^*_v) = [f^3_s(c_s), 1]$. Then these intervals are the smallest invariant intervals under $f^2_s$ around the critical point and the critical value respectively. Observe that the critical point $c_s$ is in $J^*_c$ and $f_s(J^*_c) = J^*_v$. Let the onto map $s: J^*_c \to I$ be the orientation reversing affine rescaling. Thus $s \circ f_s: J^*_v \to [-1,1]$ is an expanding diffeomorphism. We can consider the inverse contraction

$$g_s: I \to J^*_v, \quad g_s = f_s^{-1} \circ s^{-1}$$

where $f_s^{-1}$ is the branch of the inverse function which maps $J^*_c$ onto $J^*_v$. The map $g_s$ is called the presentation function and it has the unique fixed point at 1. By the definition of $g_s$ implies that

$$f_s^2|_{J^*_v} = g_s \circ f_s \circ (g_s)^{-1}$$

Then by the appropriate rescaling of the presentation function, $g_s$, we can define the renormalization at the critical value, $R^n_{f_s}$. Inductively we can define $g^n_s$ on the smallest interval $J^*_v(n)$ containing the critical value 1 with period $2^n$. Let $G^n_s: I \to I$ be the diffeomorphism of the rescaled map of $g^n_s$.

Then the fact that $g_s$ is the contraction implies the existence of the limit.

$$u_s = \lim_{n \to \infty} G^n_s: I \to I$$

and the convergence is exponentially fast in $C^3$ topology. Moreover, we see the following lemmas in [CLM].

**Lemma 5.6** (Lemma 7.1 in [CLM]). For every $n \geq 1$

1. $J^*_v(n) = g^n_s(I)$
2. $R^n_{f_s} f_s = G^n_s \circ f_s \circ (G^n_s)^{-1}$
3. $u_s \circ f_s = f^* \circ u_s$

**Lemma 5.7** (Lemma 7.3 in [CLM]). Assume that there is the sequence of smooth functions $g_k: I \to I, \quad k = 1, 2, \ldots n$ such that $\|g_k - g_s\|_{C^3} \leq C \rho^k$ where the $g_s = \lim_{k \to \infty} g_k$ for some constant $C > 0$ and $\rho \in (0,1)$. Let $g^n_k = g_k \circ \cdots \circ g_n$ and let $G^n_k = a^n_k \circ g^n_k: I \to I$, where $a^n_k$ is the affine rescaling of $\Im g^n_k$ to $I$. Then $\|G^n_k - G^n_{s-k}\|_{C^1} \leq C_1 \rho^{n-k}$, where $C_1$ depends only on $\rho$ and $C$.

Let us normalize the functions $u_s$ and $g_s$ which have the fixed point at the origin and the derivatives at the origin is 1. Let

$$v_s(x) = \frac{u_s(x + 1) - 1}{u'_s(1)}$$

Abusing notation, we denote the normalized function of $g_s(x)$ to be also the $g_s(x)$ in the following lemma.

**Lemma 5.8.** There exists the positive constant $\rho < 1$ such that for all $k < n$ and for every $y \in I^y$ and $z \in I^z$

$$| \id + S^n_k(\cdot, y, z) - v_s(\cdot) | = O(\bar{\varepsilon}^{2k} y + \bar{\varepsilon}^{2k} \sum_j z_j + \rho^{n-k})$$
Proof. The map \( \text{id} + S_k^n(\cdot, y, z) \) is the normalized map of \( \Psi_k^n \) such that the derivative at the origin is the identity map, and \( v_* \) is also the normalized map of \( u_* \), which is the conjugation of the renormalization fixed point at the critical point and the critical value in Lemma 5.6. Thus the normalized map, \( \text{id} + S_k^n \) conjugation of the renormalization fixed point at the critical point and the critical value in \( v \) converges to the same function \( F \) of \( n \) enough to prove the whole lemma. By Lemma 5.5 we have

\[
\partial_y S_k^n = O(\varepsilon^k), \quad \partial_z S_k^n = O(\varepsilon^k)
\]

for all \( 1 \leq i \leq m \) and moreover,

\[
\partial_{xy}^2 S_k^n = O(\varepsilon^k \sigma^{n-k}), \quad \partial_{xz}^2 S_k^n = O(\varepsilon^k \sigma^{n-k})
\]

for all \( 1 \leq i \leq m \). Thus the proof of asymptotic along the section parallel to \( x-\)axis is enough to prove the whole lemma. By Lemma 5.6, we obtain

\[
\text{dist}_{C^3}(\text{id} + s_k(\cdot, 0, 0), g_\sigma(\cdot)) = O(\rho^k)
\]

and by Lemma 5.7 we obtain

\[
\text{(5.17)} \quad \text{dist}_{C^1}(\text{id} + S_k^n(\cdot, 0, 0), G^{n-k}_n(\cdot)) = O(\rho^{n-k}).
\]

Since the \( G^{n-k}_n \rightarrow v_* \) exponentially fast, we have the exponential convergence of the function \( \text{id} + S_k^n(\cdot, 0, 0) \) to \( v_*(\cdot) \). Hence, the above asymptotic and the exponential convergence at the origin prove the first part of the lemma. Furthermore, \( C^1 \) convergence of (5.17) implies that

\[
\left| 1 + \partial_x S_k^n(\cdot, 0, 0) - v'_*(\cdot) \right| = O(\rho^{n-k})
\]

where \( \rho \in (0, 1) \).

5.4. Estimation of the quadratic part of \( S_k^n \) for \( n \). We estimate the asymptotic of \( S_k^n \) using the estimation of the partial derivatives and recursive formulas. Then it implies the estimation of the asymptotic of the non-linear part of \( \Psi_k^n \) for \( n \). In order to simplify notations, we would treat the case \( k = 0 \) and consider the behavior of \( S_0^n \) instead of \( S_k^n \).

In this section, let the variable \( y \) be \( z_0 \) if we express the quadratic sum of \( y \) and \( z_j \) variables to simplify the notations.

Lemma 5.9. The following asymptotic is true

\[
\left| \left[ x + S_0^n(x, y, z) \right] - \left[ v_*(x) + \sum_{0 \leq i, j \leq m} a_{F, ij} z_i z_j \right] \right| = O(\rho^n)
\]

where constants \( |a_{F, ij}| \) are \( O(\varepsilon) \) for all \( 0 \leq i, j \leq m \) and for some \( \rho \in (0, 1) \).

Proof. For any fixed \( k \geq 0 \), the recursive formula for \( n > k \) comes from the \( \Psi_k^{n+1} = \Psi_k^n \circ \Psi_k^{n+1} \).

Thus

\[
S_k^{n+1}(w) = s_n(w) + D^{-1}_n \circ S_k^n \circ D_n \circ (\text{id} + s_n)(w).
\]

Let \( k = 0 \) for simplicity, and compare each coordinates of the both sides. Then
By the direct calculation, we obtain the following equation

\[
(S_{n+1}^0(w), 0, R_{n+1,0}(y))
\]

\[
= (s_n(w), 0, r_n(y)) + \left( \begin{array}{c}
\alpha_n^{-1} (-t_n + u_n \cdot d_n) \\
\sigma_n^{-1} \\
-\sigma_n^{-1} d_n
\end{array} \right)
\left( \begin{array}{c}
\alpha_n^{-1} u_n \\
\sigma_n^{-1} \cdot \text{Id}_{m \times m}
\end{array} \right)
\left( \begin{array}{c}
S_n^0(w) \\
0 \\
R_{n,0}(y)
\end{array} \right)
\]

\[
= \left( \begin{array}{c}
\alpha_n \sigma_n t_n \\
\sigma_n \\
\sigma_n d_n
\end{array} \right)
\left( \begin{array}{c}
\alpha_n \sigma_n u_n \\
\sigma_n y \\
\sigma_n d_n \sigma_n \cdot \text{Id}_{m \times m}
\end{array} \right)
\left( \begin{array}{c}
x + s_n(w) \\
y \\
z + r_n(y)
\end{array} \right).
\]

Firstly, let us compare from third to \( m + 2 \) coordinates of each side of the above equation. Using the Taylor’s expansion and Lemma 5.1, we obtain

\[
R_{n+1,0}(y) = r_n(y) + \frac{1}{\sigma_n} R_{n,0}(\sigma_n y)
\]

Recall the coordinate expression of \( R_{n,0}(y) = (R_{n,0}^1(y), R_{n,0}^2(y), \ldots, R_{n,0}^m(y)) \). Then we have the following form of \( R_{n,0}^j(y) \).

\[
R_{n,0}^j(y) = a_n^j y^2 + A_n^j(y) y^3
\]

\[
R_{n+1,0}^j(y) = \frac{1}{\sigma_n} \left( a_n^j \cdot (\sigma_n y)^2 + A_n^j(\sigma_n y) \cdot (\sigma_n y)^3 \right) + c_n^j y^2 + O(\varepsilon^{2n} y^3).
\]

for each \( 1 \leq j \leq m \). Thus \( a_{n+1}^j = \sigma_n a_n^j + c_n^j \) and \( \|A_{n+1}^j\| \leq \|\sigma_n\| \|A_n^j\| + O(\varepsilon^{2n}) \). for all \( 1 \leq j \leq m \). Hence, \( A_n^j \to 0 \) and \( a_n^j \to 0 \) exponentially fast as \( n \to \infty \).

Moreover, the second part of Lemma 5.8 implies the following equation

\[
(5.19) \quad \| x + S_n^0(x, y, z) - [v_*(x) + S_n^0(0, y, z)] \| = O(\rho^n).
\]

Secondly, compare the first coordinates of \( (5.18) \) at \( (0, y, z) \)

\[
S_{n+1}^0(0, y, z) = s_n(0, y, z)
\]
implies that and higher order terms, equation (5.19) show the asymptotic of where 

\[ S \]

Hence, 

\[ a = \] 

Taylor’s expansion of 

\[ a \]

It implies that 

\[ \text{The recursive formula for} \quad S_{0}^{n+1}(0, y, z) \implies \] that 

\[ S_{0}^{n}(0, y, z) \] 

\[ + \frac{1}{\alpha_n} \sum_{0 \leq i, j \leq m} a_{n, ij}(\sigma_n z_i)(\sigma_n z_j) + A_n(\sigma_n y, \sigma_n z) \cdot \left( \sum_{0 \leq i, j, k \leq m} (\sigma_n z_i)(\sigma_n z_j)(\sigma_n z_k) \right) \] 

\[ + \sum_{0 \leq i, j \leq m} e_{n, ij} \cdot \left( \sum_{0 \leq i, j, k \leq m} z_i z_j z_k \right). \]

Hence, 

\[ a_{n+1, ij} = \frac{\sigma^2}{\alpha_n} a_{n, ij} + \sum_{0 \leq i, j \leq m} e_{n, ij} \quad \text{for} \quad 0 \leq i, j \leq m \quad \text{and moreover,} \]

\[ \|A_{n+1}\| \leq \|A_n\| \cdot \frac{|\sigma_n|^3}{|\alpha_n|} + O(\bar{\varepsilon}^2). \]

It implies that 

\[ a_{n, ij} \to a_{F, ij} \quad \text{for} \quad 0 \leq i, j \leq m \quad \text{and} \quad \|A_n\| \to 0 \] exponentially fast as \( n \to \infty \). 

The exponential convergence of \( S_{0}^{n}(0, y, z) \) to the quadratic function of \( y \) and \( z \) and the equation (5.19) show the asymptotic of \( S_{0}^{n}(x, y, z) \). 

\[ \square \]
5.5. Universality of $\text{Jac} R^n F$. Let the $n^{th}$ renormalized map of $F$ be $R^n F \equiv F_n = (f_n - \varepsilon_n, x, \delta_n)$. Recall that $\Psi_{\text{tip}}^n \equiv \Psi_{\nu}^n$ from $n^{th}$ level to $0^{th}$ level and the tip $\tau_F$ is contained in $\in B_{\nu}^n$ for all $n \in \mathbb{N}$. Thus $\Psi_{\text{tip}}^n$ is the original coordinate change rather than the normalized function $\Psi_{\nu}^n$ conjugated by translations $T_n$.

Recall the equation (5.1) again

$$\text{Jac} F_n(w) = \text{Jac} F^{2^n}(\Psi_{\text{tip}}^n(w)) \frac{\text{Jac} \Psi_{\text{tip}}^n(w)}{\text{Jac} \Psi_{\text{tip}}^n(F_n w)} = b^{2^n} \frac{\text{Jac} \Psi_{\text{tip}}^n(w)}{\text{Jac} \Psi_{\text{tip}}^n(F_n w)}(1 + O(\rho^n)).$$

**Theorem 5.10** (Universal expression of Jacobian determinant). For the function $F \in \mathcal{I}(\bar{\varepsilon})$ for sufficiently small $\bar{\varepsilon} > 0$, we obtain that

$$\text{Jac} F_n = b^n a(x) (1 + O(\rho^n))$$

where $b$ is the average Jacobian of $F$ and $a(x)$ is the universal positive function for some $\rho \in (0, 1)$.

**Proof.** For the higher dimensional Hénon-like map, Lemma 5.8 and Lemma 5.9 are essentially the same as for the two dimensional Hénon-like maps. Then the proof of theorem is also the same as two dimensional maps. See Universality theorem in [CLM]. \[\square\]

6. Toy model Hénon-like map in higher dimension

Let higher dimensional Hénon-like map satisfying $\varepsilon(w) = \varepsilon(x, y)$, that is, $\partial_i \varepsilon \equiv 0$ for all $1 \leq i \leq m$ be the toy model map. Denote the toy model map by $F_{\text{mod}}$. Then the projected map $\pi_{xy} \circ F_{\text{mod}} = F^{2d}_2$ is exactly two dimensional Hénon-like map. Let the horizontal-like diffeomorphism of $F_{\text{mod}}$ be $H_{\text{mod}}$. Thus we see that $\pi_{xy} \circ H_{\text{mod}} = H^{2d}_2$. Then we obtain that $\pi_{xy} \circ R F_{\text{mod}} = R F^{2d}_2$.

**Proposition 6.1.** Let $F_{\text{mod}} = (f(x) - \varepsilon(x, y), x, \delta(w))$ be a toy model map in $\mathcal{I}(\bar{\varepsilon})$. Then $n^{th}$ renormalized map $R^n F_{\text{mod}}$ is a toy model map, that is,

$$\pi_{xy} \circ R^n F_{\text{mod}} = R^n F^{2d}_2$$

for every $n \in \mathbb{N}$. Moreover, $\varepsilon_n$ is of the following form

$$\varepsilon_n(w) = b_1^n a(x) y(1 + O(\rho^n))$$

where $b_1$ is the average Jacobian of the two dimensional map, $F_2 = \pi_{xy} \circ F_{\text{mod}}$ and $a(x)$ is the non vanishing diffeomorphism on $\pi_x(B)$. 26
6.1. Tangent bundle splitting under $DF_{\text{mod}}$. Let $DF$ and $D\delta$ be the Fréchet derivative of $F$ and $\delta$ respectively. For the given point $w = (x, y, z)$, let us denote $w_i = (x_i, y_i, z_i) = F^i(x, y, z)$. The (Fréchet) derivative of $F$ has the block matrix form

$$DF_{\text{mod}} = \begin{pmatrix} DF_{2d} & 0 \\ \partial_x \delta & \partial_y \delta \\ \partial_z \delta & \partial_z \delta \end{pmatrix}$$

where $\partial_z \delta$ is the $m \times m$ block matrix

$$\begin{pmatrix} \partial_{z_1} \delta^1 & \partial_{z_2} \delta^1 & \cdots & \partial_{z_m} \delta^1 \\ \partial_{z_1} \delta^2 & \partial_{z_2} \delta^2 & \cdots & \partial_{z_m} \delta^2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{z_1} \delta^m & \partial_{z_2} \delta^m & \cdots & \partial_{z_m} \delta^m \end{pmatrix}$$

The Lemmas in this section using block matrices are due to the same notions in [Nam1] because the construction of invariant cone field could be applied to the object of any finite dimension. In the below we use the same notations in [Nam1].

$$DF_{\text{mod}}(x, y, z) = \begin{pmatrix} A(w) & 0 \\ C(w) & D(w) \end{pmatrix} \equiv \begin{pmatrix} A_1 & 0 \\ C_1 & D_1 \end{pmatrix}.$$  \hspace{1cm} (6.1)

where $A_w = DF_{2d}(x, y)$, $0 = (0)$, $C_w = (\partial_x \delta(w) \partial_y \delta(w))$ and $D_w = \partial_z \delta(w)$. Since we assume that $F_{\text{mod}}$ and $F_{2d}$ are diffeomorphisms, $DF_{\text{mod}}$ and $A_w$ are invertible. It implies that $D_w$ is invertible at each $w$. Let $w_N$ be $F^N(w)$ and the derivative of the $N^{th}$ iterated map $F_{\text{mod}}^N$ be $DF_{\text{mod}}^N$. Denote $DF_{\text{mod}}^N$ as the block matrix form as follows

$$DF_{\text{mod}}^N(x, y, z) = \begin{pmatrix} A_N(w) & 0 \\ C_N(w) & D_N(w) \end{pmatrix} \equiv \begin{pmatrix} A_N & 0 \\ C_N & D_N \end{pmatrix}.$$  \hspace{1cm} (6.2)

Then for each $N \geq 1$,

$$\begin{pmatrix} A_N & 0 \\ C_N & D_N \end{pmatrix} = \begin{pmatrix} A_1(w_{N-1}) \\ C_1(w_{N-1}) \end{pmatrix} \begin{pmatrix} 0 \\ D_1(w_{N-1}) \end{pmatrix} \equiv \begin{pmatrix} A_{N-1} & 0 \\ C_{N-1} & D_{N-1} \end{pmatrix}.$$

Let $A_0 \equiv 1$, $C_0 \equiv 1$, $D_0 \equiv 1$ and $w = w_0$ for notational compatibility. Then by direct calculations, we obtain

$$A_N = A_1(w_{N-1}) A_{N-1} = \prod_{i=0}^{N-1} A_1(w_{N-i-1})$$  \hspace{1cm} (6.3)

$$D_N = D_1(w_{N-1}) D_{N-1} = \prod_{i=0}^{N-1} D_1(w_{N-i-1})$$

$$C_N = C_1(w_{N-1}) A_{N-1} + D_1(w_{N-1}) C_{N-1}$$

$$= \sum_{i=0}^{N-1} D_1(w_{N-1-i}) C_1(w_{N-1-i}) A_{N-1-i}.$$
matrix expressions, $[DF_{\text{mod}}^N(w)]^{-1}$ is
\begin{equation}
(6.4) \quad DF_{\text{mod}}^{-N} = \begin{pmatrix}
A_N^{-1} & 0 \\
-D_N^{-1}C_NA_N^{-1} & D_N^{-1}
\end{pmatrix}
\end{equation}

at the point, $F^N(w)$.

Let the cone at $w$ with some positive number $\gamma$ to be
\begin{equation}
(6.5) \quad C(\gamma)_w = \{ u + v \mid u \in \mathbb{R}^2 \times \{0\}, v \in \{0\} \times \mathbb{R}^m \text{ and } \frac{1}{\gamma} \|u\| > \|v\| \}.
\end{equation}

The cone field over a given compact invariant set $\Gamma$ is the union of the cones at every points in $\Gamma$
\begin{equation}
(6.6) \quad C(\gamma) = \bigcup_{w \in \Gamma} C(\gamma)_w.
\end{equation}

Let $\|DF\|$ be the operator norm of $DF$. The minimum expansion rate (or the strongest contraction rate) of $DF$ is defined by the equation, $\|DF^{-1}\| = \frac{1}{m(DF)}$.

**Lemma 6.2.** Let $A_N, 0, C_N$ and $D_N$ be components of $DF_{\text{mod}}^N$ defined on (6.2). Suppose that $\|D_1\| < m(A_1)$. Then $\|C_NA_N^{-1}\| < \kappa$ for some $\kappa > 0$ independent of $N$.

**Proof.** See Lemma 7.2 in [Nam1].

**Lemma 6.3.** Let $F_{\text{mod}} \in I(\bar{\varepsilon})$ with small enough $\bar{\varepsilon} > 0$. Suppose that $\|D_1\| \leq \frac{4}{2} \cdot m(A_1)$ for some $\rho \in (0, 1)$. Let $C(\gamma)$ over the given compact invariant set $\Gamma$ be the cone field which is defined on (6.6) with cones in (6.5). Then $C(\gamma)$ is invariant under $DF_{\text{mod}}^{-1}$ for all sufficiently small $\gamma > 0$. More precisely, any given invariant compact set $\Gamma$ has the dominated splitting.

**Proof.** See Lemma 7.3 in [Nam1].

**Remark 6.1.** In Lemma 6.2 components of matrix form, $A_1, D_1, A_N$ and $D_N$ depends on each point $w \in \Gamma$. Then $\frac{\|D_1(w)\|}{m(A_1(w))} \leq \frac{1}{2} \rho_w$ for some positive $\rho_w < 1$. Then the actual assumption is that the set of $\rho_w > 0$ for $w \in \Gamma$ is totally bounded above by the number less than $\frac{1}{2}$. However, since $\Gamma$ is compact, $\{ \rho_w \mid w \in \Gamma \}$ is the precompact set. Then $\rho$ can be chosen as the supremum of $\{ \rho_w \mid w \in \Gamma \}$. Then $\kappa$ in Lemma 6.2 is independent of $w \in \Gamma$. Thus $\kappa$ is also uniform number independent of $w$. Moreover, the cone field $C(\gamma)$ in Lemma 6.3 is contracted in uniform rate by $DF^{-1}$.

\footnote{The operator norm is defined on the linear operator at each point. For example, $\|DF_w\| = \sup_{\|v\|=1} \{\|DF_wv\|\}$

The value $\|DF\|$ is defines as $\sup_{w \in B} \|DF_w\|$.}
6.2. Tangent bundle splitting under a small perturbation of toy model map. The existence of the invariant cone field under $DF_{\text{mod}}$ is still true when a small perturbation of $DF_{\text{mod}}$ is chosen. Let us consider the block diagonal matrix form of $DF$. Let the following map be a perturbation of the toy model map, $F_{\text{mod}}(w) = (f(x) - \varepsilon_2 d(x, y), x, \delta(w))$

\begin{equation}
F(w) = (f(x) - \varepsilon_2 d(x, y) - \tilde{\varepsilon}(w), x, \delta(w))
\end{equation}

where $\varepsilon(w) = \varepsilon_2 d(x, y) + \tilde{\varepsilon}(w)$. Thus $\partial_z \varepsilon(w) = \partial_z \tilde{\varepsilon}(w)$.

\begin{equation}
DF = \begin{pmatrix}
D\tilde{F}_{2d} & \partial_z \varepsilon \\
\partial_x \delta & \partial_y \delta & \partial_z \partial \delta
\end{pmatrix} = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\end{equation}

where $D\tilde{F}_{2d} = \begin{pmatrix}
f'(x) - \partial_x \varepsilon(w) & -\partial_y \varepsilon(w) \\
1 & 0
\end{pmatrix}$ and $\partial_z \varepsilon$ is the row vector $(\partial_{z_1} \varepsilon \ \partial_{z_2} \varepsilon \ \ldots \ \partial_{z_m} \varepsilon)$.

If $B \equiv 0$, then $F$ is $F_{\text{mod}}$.

**Lemma 6.4.** Let $F$ be a perturbation of the toy model map $F_{\text{mod}}$ defined in (6.7) and $A$, $B$, $C$ and $D$ are components of block matrix form of $DF$ defined in (6.8). Suppose that $\|D_1\| \leq \frac{\rho_1}{2} \cdot m(A_1)$ for some $\rho_1 \in (0, 1)$. Suppose also that $\|B\| \|C\| \leq \rho_0 \cdot m(A) \cdot m(D)$ where $\rho_0 < \frac{\kappa}{2}$ for sufficiently small $\gamma > 0$. Then the cone field $C(\gamma)$ defined on (6.6) is invariant under $DF^{-1}$.

**Proof.** See Lemma 7.4 in [Nam1].

Then the tangent space $T_\Gamma B$ of $DF$ has the splitting of the invariant subbundles $E^1 \oplus E^2$ such that

1. $T_\Gamma B = E^1 \oplus E^2$.
2. Both $E^1$ and $E^2$ are invariant under $DF$.
3. $\|DF^n|_{E^1(x)}\| \|DF^{-n}|_{E^2(F^{-n}(x))}\| \leq C \mu^n$ for some $C > 0$ and $0 < \mu < 1$ and $n \geq 1$.

Thus $T_\Gamma B$ is dominated over the compact invariant set $\Gamma$. Moreover, the dominated splitting implies that invariant sections $w \mapsto E^1(w)$ and $w \mapsto E^2(w)$ are continuous by Theorem 1.2 in [?]. Then the maps, $w \mapsto E^i(w)$ for $i = 1, 2$ are continuous.

---

If the bounded operator $T$ has $\|T\| < 1$, then $\text{Id} - T$ is invertible. Moreover,

$$(\text{Id} - T)^{-1} = \sum_{n=0}^{\infty} T^n$$

Since, $\|T^n\| \leq \|T\|^n$ for every $n \in \mathbb{N}$, $\|(\text{Id} - T)^{-1}\| \leq \frac{1}{1 - \|T\|}$. Equivalently, we get the lower bound of the minimum expansion rate, $m(\text{Id} - T) \geq 1 - \|T\|$.
7. Single invariant surfaces

7.1. Invariant surfaces and two dimensional ambient space. Let us consider dominated splitting of $DF$ over the given compact set. One of the subbundle, for instance, $E^{ss}$ is uniformly contracted under $DF|_{E^{ss}}$. If there exists an invariant submanifold whose codimension is at least one such that this manifold is tangent to invariant tangent subbundle and it contains the invariant compact set, then we expect the dynamics is reduced in this invariant manifold.

**Definition 7.1.** A $C^r$ submanifold $Q$ which contains $\Gamma$ is locally invariant under $f$ if there exists a neighborhood $U$ of $\Gamma$ in $Q$ such that $f(U) \subset Q$.

The necessary and sufficient condition for the existence of these submanifolds, see [CP] and its references.

**Theorem 7.1 ([CP]).** Let $\Gamma$ be an invariant compact set with a dominated splitting $T\Gamma M = E^1 \oplus E^2$ such that $E^1$ is uniformly contracted. Then $\Gamma$ is contained in a locally invariant submanifold tangent to $E^2$ if and only if the strong stable leaves for the bundle $E^1$ intersect the set $\Gamma$ at only one point.

Moreover, the existence of invariant submanifold is robust under $C^1$ perturbation.

**Proposition 7.2 ([CP]).** Let $\Gamma$ be an invariant compact set with a dominated splitting $E^1 \oplus E^2$ such that $E^1$ is uniformly contracted. If $\Gamma$ is contained in a locally invariant submanifold tangent to $E^2$, then the same holds for any diffeomorphism $C^1$ close to $f$ and any compact set $\Gamma'$ contained in a small neighborhood of $\Gamma$.

Let us consider the toy model map which is infinitely renormalizable. Recall that $DF_{\text{mod}}$ be $(\begin{array}{cc} A & 0 \\ C & B \end{array})$ where $A$ is the derivative of $\pi_{xy} \circ F_{\text{mod}}$. For a given invariant compact set, let us assume that $\Gamma$ has the dominated splitting $T\Gamma M = E^{ss} \oplus E^{pu}$ where $DF|_{E^{ss}}$ is $D$, $DF|_{E^{pu}}$ is $A$ and $\|D\|\|A^{-1}\| < \rho < 1$. Let $W^{ss}(w)$ be the strong stable manifold which is tangent to $E^{ss}(w)$ for each $w \in \Gamma$. A pseudo unstable manifold, $W^{pu}(w)$ is defined similarly. Observe that $W^{pu}(w)$ is two dimensional manifold at every point $w \in \Gamma$.

**Lemma 7.3** (Lemma A.1 and Lemma A.2 in [Nam3]). Let $F$ be an infinitely renormalizable Hénon-like map. Then the critical Cantor set, $\mathcal{O}_F$ is the set of accumulation points of $\text{Per}_F$. Moreover, $W^s(w) \cap \text{Per}_F = \{w\}$ for every point $w \in \text{Per}_F$.

The toy model map has invariant set of codimension two hyperplane under $F$

\[
\bigcup_{(x,y) \in I\times I} \{(x, y, z) \mid z \in I^z\}
\]

where $I^z = I^{z_1} \times I^{z_2} \times \cdots \times I^{z_m}$. Moreover, any vector which perpendicular to $xy$–plane is invariant under $DF_{\text{mod}}$ up to the size. Then by Theorem 7.1 and Lemma 7.1 implies the following lemma.

**Lemma 7.4.** Let $F_{\text{mod}}$ be the toy model map in $I(\varepsilon)$. Suppose that

\[
\sup_{w \in \text{Per}_F} \frac{\|D_1(w)\|}{m(A_1(w))} \leq \frac{1}{2}
\]

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where $A$ and $D$ are the block matrix in (6.1). Then there exists a locally invariant $C^1$ single surface $Q$ which contains $\text{Per}_{F_{\text{mod}}}$.

The surface $Q$ meets transversally and uniquely strong stable manifold, $W^{ss}$ at each $w \in \text{Per}_{F_{\text{mod}}}$. 

Proof. Lemma 6.3 implies the dominated splitting over $\text{Per}_{F_{\text{mod}}}$. This dominated splitting implies that the hyperplanes in (7.1) at each point $w$ of $\text{Per}_{F_{\text{mod}}}$ is the strong stable manifold at $w$. Transversal intersection of invariant cone fields implies that the surface $Q$ tangent to $E^{pu}$ over $\text{Per}_{F_{\text{mod}}}$ meets transversally each $W^{ss}(w)$. We may assume that $Q$ is locally invariant by Theorem 7.1. Let us show the uniqueness of the intersection point. Suppose that $w$ and $w'$ are points in $Q \cap W^{ss}(w)$. If $w \neq w'$, then $w' \not\in \text{Per}_{F_{\text{mod}}}$ by Lemma 7.3. Take a small neighborhood $U$ of $w'$ in the invariant surface $Q$. Then $U$ converges to the neighborhood of $\text{Per}_{F_{\text{mod}}}$ as $n \to \infty$ by Inclination Lemma. Thus $Q$ cannot be a submanifold of the ambient space because it accumulates itself. This contradicts to Theorem 7.1. Hence, $w$ is the unique intersection point. □

Let $F_{\text{mod}} \in \mathcal{I}(\xi)$ which is sectionally dissipative at fixed points. Recall that the invariant plane field, $E^{pu}$ over $\text{Per}_{F_{\text{mod}}}$ is two dimensional. Thus $E^{pu}$ contains the unstable direction at every periodic points. Then the invariant surface $Q$ tangent to $E^{pu}$ contains the set

$$A \equiv \mathcal{O} \cup \bigcup_{n \geq 0} W^u(\text{Orb}(q_n))$$

where each $q_n$ is the periodic point with period $2^n$. The set $A$ is called the global attracting set.

**7.2. Invariant surfaces containing $\text{Per}$ as the graph of $C^r$ map.** Recall $b_1$ be the average Jacobian of $F_{2d} \equiv \pi_{xy} \circ F_{\text{mod}}$. Suppose that $\|\partial_z \delta\| \leq \rho \cdot m(F_{2d})$ for some $\rho < 1$ with dominated splitting over $\text{Per}_{F_{\text{mod}}}$. Denote $\text{Per}_{F_{\text{mod}}}$ by $\Gamma$. Then there exists a locally invariant single surface which contains $A_{F_{\text{mod}}}$. The set of $m$–dimensional hyperplanes which are perpendicular to $xy$–plane

$$\bigcup_{(x, y) \in \pi_{xy}(B)} \{ (x, y, z) \mid z \in I^z \}$$

is invariant under $F_{\text{mod}}$. Since tangent subbundle $E^{ss}$ for $DF_{\text{mod}}$ is constant and strong stable manifold is unique at each point in $\Gamma$, the above set contains the strong stable foliation over $\Gamma$. The angle between each tangent spaces $E^{ss}_w$ and $E^{pu}_w$ is (uniformly) positive. Thus the maximal angle between $E^{pu}$ and $T\mathbb{R}^2$ is less than $\frac{\pi}{2}$.

**Remark 7.1.** If $T_1 B = E^{ss} \oplus E^{pu}$ is $r$–dominated splitting, then $Q$ which is invariant single surface tangent to $E^{pu}$ is a $C^r$ surface. Moreover, since the strong stable manifolds at each point is the set of perpendicular lines to $xy$–plane, $Q$ is the graph of $C^r$ function from a region in $I^x \times I^y$ to $I^z$.

We may assume that invariant surfaces tangent to the invariant plane field has the neighborhood, say also $Q$, of the tip, $\tau_{F_{\text{mod}}}$ in the given invariant single surface which satisfies the following properties by Lemma 7.4.

1. $Q$ is contractible.

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(2) $Q$ contains $\tau_{F_{\text{mod}}}$ in its interior and is locally invariant under $F^{2^N}$ for big enough $N \in \mathbb{N}$.

(3) Topological closure of $Q$ is the graph of $C^r$ map from a neighborhood of $\tau(\pi_{xy} \circ F_{\text{mod}})$ in $xy$–plane to $I^z$.

By $C^1$ robustness of the existence of single invariant surfaces, if $F$ be a sufficiently small perturbation of $F_{\text{mod}}$, then there exist invariant surfaces each of which is the graph of $C^r$ map from a region in the $xy$–plane to $I^z$.

There exists an invariant surface $Q$ under $F$ on $\pi_{xy}(B_0^n)$ as the graph of the $C^r$ function $\xi$ with $\|D\xi\| \leq C\varepsilon$ only if there exists the dominated splitting in the previous subsection. Then the image of $Q$ under the map $(\Psi^n_{\text{tip}})^{-1}$ is also an invariant surface under $R^nF$ for every big enough $n \in \mathbb{N}$ (See Lemma 7.6 below). The existence is based on the global implicit function theorem in [ZG].

**Theorem 7.5 (Theorem 1 in [ZG]).** Let $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is continuous mapping and it is continuously differentiable for second variable $u \in \mathbb{R}^m$. Suppose that

$$
\left| \left[ \frac{\partial f(x,u)}{\partial u} \right]_{ii} \right| - \sum_{i \neq j} \left| \frac{\partial f(x,u)}{\partial u} \right|_{ij} \geq d > 0
$$

for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$ and $i = 1, 2, \ldots, m$ where $\left[ \frac{\partial f(x,u)}{\partial u} \right]_{ij}$ is the ij entry of the Jacobian matrix of $f$ over the second variable $u$. Then there exists the unique mapping $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that $f(x,g(x)) = 0$. Moreover, $g$ is continuous. If $f$ is continuously differentiable, then so is $g$.

**Proposition 7.6.** Let $F \in I(\varepsilon)$ and $Q$ be an invariant surface under $F$, which is the graph of $C^r$ function $\xi = (\xi^1, \xi^2, \ldots, \xi^m)$ on $\pi_{xy}(B^n_{\text{tip}})$ such that $\|D\xi\| \leq C_0\varepsilon$ for some $C_0 > 0$. Then $Q_n \equiv (\Psi^n_{\text{tip}})^{-1}(Q)$ is an invariant surface under $R^nF$ which is the graph of $\xi_n = (\xi^1_n, \xi^2_n, \ldots, \xi^m_n) : \pi_{xy}(B(R^nF)) \rightarrow \pi_x(B(R^nF))$ such that

$$
\xi_n(x,y) = cy(1 + O(\rho^n))
$$

where $c = (c_1, c_2, \ldots, c_m)$ for some constants $c_i$ for $1 \leq i \leq m$.

**Proof.** The $n^{th}$ renormalization of $F$, $R^nF$ is $(\Psi^n_{\text{tip}})^{-1} \circ F^{2^n} \circ \Psi^n_{\text{tip}}$. Thus $Q_n \equiv (\Psi^n_{\text{tip}})^{-1}(Q)$ is an invariant surface under $R^nF$. Let us choose a point $w' = (x', y', z') \in Q \cap B^n_{\text{tip}}$ where $B^n_{\text{tip}} \equiv \Psi^n_{\text{tip}}(B(R^nF))$ and $z' = \xi(x', y')$ where $\xi(x', y') = (\xi^1(x', y'), \xi^2(x', y'), \ldots, \xi^m(x', y'))$. Thus

$$
\text{graph}(\xi) = (x', y', \xi(x', y')) = (x', y', z').
$$

Moreover, let $(\Psi^n_{\text{tip}})^{-1}(x', y', z') = (x, y, z) \in Q_n$. Thus by equation (5.14), each coordinates of $\Psi^n_{\text{tip}}$ as follows

\begin{align}
(7.2) & \quad x' = \alpha_{n,0}(x + S^n_0(w)) + \sigma_{n,0} t_{n,0} \cdot y + \sigma_{n,0} u_{n,0} \cdot (z + R_{n,0}(y)) \\
(7.3) & \quad y' = \sigma_{n,0} \cdot y \\
(7.4) & \quad z' = \sigma_{n,0} d_{n,0} \cdot y + \sigma_{n,0} (z + R_{n,0}(y))
\end{align}
where \( w = (x, y, z) \). Let us show that \( Q_n \) is the graph of a well defined function \( \xi_n = (\xi_1^n, \xi_2^n, \ldots, \xi_m^n) \) from \( \pi_{xy}(B(R^nF)) \) to \( \pi_{y}(B(R^nF)) \), that is, \( z_i = \xi_n(x, y) \) for \( 1 \leq i \leq m \). By the equations (7.3) and (7.4), we see that

\[
\sigma_{n,0} \cdot z = z' - \sigma_{n,0} d_{n,0} \cdot y - \sigma_{n,0} R_{n,0}(y)
\]

\[
= \xi(x', y') - \sigma_{n,0} d_{n,0} \cdot y - \sigma_{n,0} R_{n,0}(y)
\]

\[
= \xi \circ (\alpha_{n,0}(x + S_{n,0}^n(w)) + \sigma_{n,0} t_{n,0} \cdot y + \sigma_{n,0} u_{n,0} \cdot (z + R_{n,0}(y)), \sigma_{n,0} y)
\]

\[
- \sigma_{n,0} d_{n,0} \cdot y - \sigma_{n,0} R_{n,0}(y).
\]

(7.5)

Let us show the existence of the solution of (7.3) for \( z \). Define the function \( G_n \) from \( B \) to \( \pi_{z}(B) \). Each coordinate function \( G_n^i \) of \( G_n \) is

\[
G_n^i(x, y, z) = \xi^i \circ (\alpha_{n,0}(x + S_{n,0}^n(w)) + \sigma_{n,0} t_{n,0} \cdot y + \sigma_{n,0} u_{n,0} \cdot (z + R_{n,0}(y)), \sigma_{n,0} y)
\]

\[
- \sigma_{n,0} d_{n,0} \cdot y - \sigma_{n,0} R_{n,0}(y) - \sigma_{n,0} \cdot z_i.
\]

where \( z = (z_1, z_2, \ldots, z_m) \) for \( 1 \leq i \leq m \). Then the partial derivative of \( G_n^i \) over \( z_j \) is as follows

\[
\partial_{z_j} G_n^i(x, y, z) = \partial_{z_j} \xi^i \circ (\alpha_{n,0}(x + S_{n,0}^n(w)) + \sigma_{n,0} t_{n,0} \cdot y + \sigma_{n,0} u_{n,0} \cdot (z + R_{n,0}(y)), \sigma_{n,0} y)
\]

\[
\cdot [\alpha_{n,0} \cdot \partial_{z_j} S_{n,0}^n(w) + u_{n,0}^i \sigma_{n,0}]
\]

for \( i \neq j \). Moreover, if \( i = j \), then

\[
\partial_{z_i} G_n^i(x, y, z) = \partial_{z_i} \xi^i \circ (\alpha_{n,0}(x + S_{n,0}^n(w)) + \sigma_{n,0} t_{n,0} \cdot y + \sigma_{n,0} u_{n,0} \cdot (z + R_{n,0}(y)), \sigma_{n,0} y)
\]

\[
\cdot [\alpha_{n,0} \cdot \partial_{z_i} S_{n,0}^n(w) + u_{n,0}^i \sigma_{n,0}] - \sigma_{n,0}
\]

Recall that \( \alpha_{n,0} = \sigma_{2n}(1 + O(\rho^n)), \sigma_{n,0} = (-\sigma)^n(1 + O(\rho^n)), \| \partial_{z_i} S_{n,0}^n \| = O(\bar{\varepsilon}) \) and \( |u_{n,0}^j| = O(\bar{\varepsilon}) \) for all \( 1 \leq j \leq m \). Then for \( m \geq 2 \),

\[
\| \partial_{z_j} G_n^i \| \leq \| \partial_{z_i} \xi^i \| \cdot [\sigma^{2n} C_0 \bar{\varepsilon} + \sigma^n C_1 \bar{\varepsilon}]
\]

for \( i \neq j \) and for some positive numbers, \( C_0 \) and \( C_1 \). Recall that \( \| u_{n,0} \| = \| u_{1,0} \| \) by the equation (5.12) and \( |u_{n,0}^i| = |\partial_{z_i} \bar{\varepsilon}| \) for each \( 1 \leq j \leq m \). Thus if small enough \( \| \partial_{z} \bar{\varepsilon} \| \) is chosen for a small perturbation of the toy model map, then we may assume that

\[
\| u_{n,0} \| \cdot |\sigma_{n,0}| < \frac{1}{4 C \sigma^n}
\]

where the constant \( C \geq \max_{1 \leq i \leq m} \| \partial_{z_i} \xi^i \| \). Then we may also assume that

\[
\sum_{i \neq j} \| \partial_{z_j} G_n^i \| \leq \frac{1}{4} \sigma^n
\]

However,

\[
\min_{w \in B(R^nF)} |\partial_{z_i} G_n^i(w)| \geq | - \| \partial_{z_i} \xi^i \| \cdot [\sigma^{2n} C_0 \bar{\varepsilon} + \sigma^n C_1 \bar{\varepsilon}] + |\sigma_{n,0}| \geq \frac{1}{2} \sigma^n
\]

Let us consider the Jacobian matrix \( \left( \frac{\partial}{\partial_{z_j}} G^i(x, y, z) \right)_{ij} \) of \( z \) variables. Then the sum of absolute value of diagonal elements dominates the sum of all other elements for every big
enough \( n \in \mathbb{N} \). Then applying Theorem 7.5 to the map \( G_n \) for every sufficiently big \( n \in \mathbb{N} \), there exists the \( C^r \) map \( \xi_n \) from \( \pi_{xy}(B(R^m)) \) to \( \mathbb{R}^m \). Furthermore, since each surface \( Q_n \) is contractible, the function \( \xi_n(x,y) \) is defined globally by the \( C^r \) continuation of coordinate charts.

Let us calculate the bounds of the norm \( \|D\xi_n(x,y)\| \). By (7.3) and (7.4) with chain rule, we obtain the following equations

\[
\partial_x \xi \cdot \frac{\partial x^i}{\partial x} = \sigma_{n,0} \cdot \partial_x \xi
\]

\[
\partial_x \xi \cdot \frac{\partial x^i}{\partial y} + \partial_y \xi \cdot \sigma_{n,0} = \sigma_{n,0} \cdot d_{n,0} + \sigma_{n,0} \cdot \partial_y \xi_n + \sigma_{n,0} \cdot (R_{n,0})'(y).
\]

By the equation (7.6), each partial derivatives of \( \xi_n \) as follows

\[
\frac{\partial \xi_n}{\partial x} = \frac{1}{\sigma_{n,0}} \cdot \partial_x \xi \cdot \left[ \alpha_{n,0} \left( 1 + \partial_x S_0^n(w) \right) + \sigma_{n,0} \sum_{j=1}^m \sum_{n,j=0}^m \frac{\partial \xi_j}{\partial x} \right]
\]

\[
\frac{\partial \xi_n}{\partial y} = \frac{1}{\sigma_{n,0}} \cdot \partial_x \xi \cdot \left[ \alpha_{n,0} \partial_y S_0^n(w) + \sigma_{n,0} t_{n,0} + \sigma_{n,0} \sum_{j=1}^m \sum_{n,j=0}^m \frac{\partial \xi_j}{\partial y} + (R_{n,0})'(y) \right]
\]

\[
+ \partial_y \xi - d_{n,0} - (R_{n,0})'(y).
\]

Recall that \( \sigma_{n,0} \approx (-\sigma)^n \), \( \alpha_{n,0} \approx \sigma^{2n} \) for each \( n \in N \). Thus

\[
\left\| \frac{\partial \xi_n}{\partial x} \right\| \leq \|D\xi\| C_0 \sigma^n \leq C \overline{\sigma} \sigma^n
\]

for some \( C_0 > 0 \). Recall also that \( \|\partial_y S_0^n(w)\| \leq C_3 \overline{\sigma} \) for some \( C_3 > 0 \) by Proposition 5.4. and moreover the values \( t_{n,0}, u_{n,0}, \) and \( d_{n,0} \) respectively as \( n \to \infty \) by Lemma 5.3. Furthermore each partial derivatives \( \partial_x \xi(w) \) and \( \partial_y \xi(w) \) to the value of each partial derivatives at the origin exponentially fast as \( n \to \infty \). We can show that \( (R_{n,0})'(y) \) converges to zero exponentially fast by the estimation of \( R_{n,0} \) in Lemma 5.9. With all of these facts we obtain that

\[
\xi_n(x,y) = cy(1 + O(p^n))
\]

where \( c = \frac{t_{*,0} \cdot \partial_x \xi(\tau_F) + \partial_y \xi(\tau_F)}{1 - u_{*,0} \cdot \partial_x \xi(\tau_F)} \). In other words, each coordinate of \( c \) is

\[
c_i = \frac{t_{*,0} \cdot \partial_x \xi^i(\tau_F) + \partial_y \xi^i(\tau_F) - d_{*,0}^i}{1 - u_{*,0} \cdot \partial_x \xi(\tau_F)}
\]

where \( c = (c_1, c_2, \ldots, c_m) \) for \( 1 \leq i \leq m \). \( \square \)
8. Universality of conjugated two dimensional $C^r$ Hénon-like map

8.1. Renormalization of conjugated two dimensional Hénon-like map. Let $F \in \mathcal{I}(\varepsilon)$ be a small perturbation of toy model map $F_{\text{mod}} \in \mathcal{I}(\varepsilon)$. Let $Q_n$ and $Q_k$ be invariant surfaces under $R^m F$ and $R^k F$ respectively and assume that $k < n$. Then by Lemma 7.6, $\Psi^n_k$ is the coordinate change map between $R^k F^{2^{n-k}}$ and $R^m F$ from level $n$ to $k$ such that $\Psi^n_k(Q_n) \subset Q_k$. Let us define $C^r$ two dimensional Hénon-like map $2d F_{n, \xi}$ on level $n$ as follows

\begin{equation}
2d F_{n, \xi} \equiv \pi_{xy} \circ R^m F \circ (\pi_{xy})^{-1}
\end{equation}

where the map $(\pi_{xy})^{-1} : (x, y) \mapsto (x, y, \xi_n(x, y))$ is a $C^r$ diffeomorphism on the domain of two dimensional map, $\pi_{xy}(B)$. In particular, the map $F_{2d, \xi}$ is defined as follows

\begin{equation}
F_{2d, \xi}(x, y) = (f(x) - \varepsilon(x, y, \xi), x)
\end{equation}

where $\text{graph}(\xi)$ is a $C^r$ invariant surface under the $m + 2$ dimensional map $F : (x, y, z) \mapsto (f(x) - \varepsilon(x, y, z), x, \delta(x, y, z))$. Let us assume that $2 \leq r < \infty$. By Lemma 7.6, the invariant surfaces, $Q_n$ and $Q_k$ are the graph of $C^r$ maps $\xi_n(x, y)$ and $\xi_k(x, y)$ respectively. Then we can apply techniques for two dimensional conjugated map in three dimension to the maps in $m + 2$ dimension. All results in this and following section are the same as those of two dimensional Hénon-like maps by invariant surfaces in three dimension. See Section 4 and Section 5 in [Nam3].

The map $2d \Psi^n_{k, \xi_{\text{tip}}}$ is defined as the map which satisfies the following commutative diagram

\begin{equation}
\begin{array}{ccc}
(Q_n, \tau_n) & \xrightarrow{\Psi^n_{k, \text{v,tip}}} & (Q_k, \tau_k) \\
\pi_{xy, n} & \downarrow \pi_{xy, k} & \downarrow \pi_{xy, k} \\
(2d B_n, \tau_{2d, n}) & \xrightarrow{2d \Psi^n_{k, \xi_{\text{tip}}}} & (2d B_k, \tau_{2d, k})
\end{array}
\end{equation}

where $Q_n$ and $Q_k$ are invariant $C^r$ surfaces with $2 \leq r < \infty$ of $R^m F$ and $R^k F$ respectively and $\pi_{xy, n}$ and $\pi_{xy, k}$ are the inverse of the graph maps, $(x, y) \mapsto (x, y, \xi_n)$ and $(x, y) \mapsto (x, y, \xi_k)$ respectively.

Using translations $T_k : w \mapsto w - \tau_k$ and $T_n : w \mapsto w - \tau_n$, we can let the tip move to the origin as the fixed point of the new coordinate change map, $\Psi^n_k := T_k \circ \Psi^n_{k, \text{tip}} \circ T_n^{-1}$ defined on Section 5.1. Thus due to the above commutative diagram, the corresponding tips in $2d B_j$ for $j = k, n$ is changed to the origin. Let $\pi_{xy} \circ T_j$ be $T_{2d, j}$ for $j = k, n$. This origin is also the fixed point of the map $2d \Psi^n_{k, \xi} := T_{2d, k} \circ 2d \Psi^n_{k, \xi_{\text{tip}}} \circ T_{2d, n}^{-1}$ where $T_{2d, j} = \pi_{xy} \circ T_j$ with $j = k, n$. By the direct calculation, we obtain the expression of the map $2d \Psi^n_{k, \xi}$ as follows

\begin{equation}
2d \Psi^n_{k, \xi} = \pi_{xy, k} \circ \Psi^n_k(x, y, \xi_n)
\end{equation}
by Proposition 8.1 as follows.

Recall the definitions of the horizontal-like diffeomorphism $F_k$, two dimensional map (8.3) holds. By the definition of the map $\Phi$, $\psi$ holds. Moreover, the map defined on the equation (8.3) with $n = k + 1$, $2d \Psi_{k, \xi}^{k+1}$ preserves each horizontal line and is the inverse of the horizontal map (8.3).

If $F \in \mathcal{I}(\varepsilon)$ has the invariant surfaces as the graph from $I^x \times I^y$ to $I^x$ on every level, then $2d \Psi_{k, \xi}^{k+1}$ is the conjugation between $(2d F_k, \xi)^2$ and $2d F_{k+1, \xi}$ for each $k \in \mathbb{N}$. Then the two dimensional map $F_{2d, \xi}$ is called the formally infinitely renormalizable map with $C^r$ conjugation. Moreover, the map defined on the equation (8.3) with $n = k + 1$, $2d \Psi_{k, \xi}^{k+1}$ preserves each horizontal line and is the inverse of the horizontal map

\[(x, y) \mapsto (f_k(x) - \varepsilon_k(x, y, \xi_k), y) \circ (\sigma_k x, \sigma_k y)\]

by Proposition 8.1 as follows.

**Proposition 8.1.** Let the coordinate change map $2d \Psi_{k, \xi}^{k+1}$ between $(2d F_k, \xi)^2$ and $2d F_{k+1, \xi}$ be $2d \Psi_{k, \xi}^{k+1}$ which is defined on (8.3) as the conjugation. Then

\[2d \Psi_{k, \xi}^{k+1} = H_{k, \xi}^{-1} \circ \Lambda_{k}^{-1}\]

for every $k \in \mathbb{N}$ where $H_{k, \xi}(x, y) = (f_k(x) - \varepsilon_k(x, y, \xi_k), y)$ and $\Lambda_{k}^{-1}(x, y) = (\sigma_k x, \sigma_k y)$.

**Proof.** Recall the definitions of the horizontal-like diffeomorphism $H_k$ and its inverse, $H_k^{-1}$ as follows

\[
H_k(w) = (f_k(x) - \varepsilon_k(w), y, z - \delta_k(y, f_k^{-1}(y)), 0))
\]

\[
H_k^{-1}(w) = (\phi_k^{-1}(w), y, z + \delta_k(y, f_k^{-1}(y)), 0)).
\]

Observe that $H_k \circ H_k^{-1} = id$ and $f_k \circ \phi_k^{-1}(w) - \varepsilon_k \circ H_k^{-1}(w) = x$ for all points $w \in \Lambda_k^{-1}(B)$. Then if we choose the set $\sigma_k \cdot \text{graph}(\xi_{k+1}) \subset \Lambda_k^{-1}(B)$, then the similar identical equation holds. By the definition of the map $2d \Psi_{k, \xi}^n$, the following equation holds
Hence, \( H_2 \) successively. Two dimensional map \( \Psi \) is involved with the fact that \( H_2 \) is renormalizable.

Proposition 8.1 enable us to define the renormalization of the analytic \( \text{Hénon-like} \) maps. Let us calculate \( H_\cdot, \xi \circ 2d_\cdot \Psi^{k+1} \). The second coordinate function of it is just \( \sigma_k y \). The first coordinate function is as follows

\[
    2d_\cdot \Psi^{k+1} = \pi_{xy} \circ \Psi^{k+1} \circ (\pi_{xy})^{-1}
\]

(8.5)

\[
    = \pi_{xy} \circ \Psi_k^{k+1}(x, y, \xi_k)
    = \pi_{xy} \circ H_k^{-1} \circ \Lambda_k^{-1}(x, y, \xi_k)
    = \pi_{xy} \circ H_k^{-1}(\sigma_k x, \sigma_k y, \sigma_k \xi_k)
\]

(*) \[
    = \pi_{xy}((\phi_1^{-1}(\sigma_k x, \sigma_k y, \sigma_k \xi_k), \sigma_k y, \xi_k((\phi_1^{-1}(\sigma_k y)))
    = (\phi_1^{-1}(\sigma_k x, \sigma_k y, \sigma_k \xi_k), \sigma_k y)
\]

In the above equation, (*) is involved with the fact that \( H_k^{-1} \circ \Lambda_k^{-1}(\text{graph}(\xi_k)) \subset \text{graph}(\xi) \). Let us calculate \( H_k, \xi \circ 2d_\cdot \Psi^{k+1} \). The second coordinate function of it is just \( \sigma_k y \). The first coordinate function is as follows

\[
    f_k \circ \phi_1^{-1}(\sigma_k x, \sigma_k y, \sigma_k \xi_k) - \varepsilon_k((\phi_1^{-1}(\sigma_k x, \sigma_k y, \sigma_k \xi_k), \sigma_k y, \xi_k((\phi_1^{-1}(\sigma_k y))))
    (\sigma_k x).
\]

Hence, \( H_k, \xi \circ 2d_\cdot \Psi^{k+1} \) is realized to be \( (\sigma_k x, \sigma_k y) \). However, \( H_k, \xi \circ (H^{-1}_k(x, y) \circ \Lambda_k^{-1}(x, y)) = (\sigma_k x, \sigma_k y) \). Therefore, by the uniqueness of the inverse map of \( H_k, \xi(x, y) \),

\[
    2d_\cdot \Psi^{k+1} = H_k^{-1} \circ \Lambda_k^{-1}.
\]

\[\square\]

Proposition 8.1 enable us to define the renormalization of the two dimensional \( C^r \) \( \text{Hénon-like} \) maps as the extension of renormalization of the analytic \( \text{Hénon-like} \) maps.

**Definition 8.1.** Let \( F : (x, y) \mapsto (f(x) - \varepsilon(x, y), x) \) be a \( C^r \) \( \text{Hénon-like} \) map with \( r \geq 2 \). If \( F \) is renormalizable, then \( RF \), the renormalization of \( F \) is defined as follows

\[
    RF = (\Lambda \circ H) \circ F^2 \circ (H^{-1} \circ \Lambda^{-1})
\]

where \( H(x, y) = (f(x) - \varepsilon(x, y), y) \). Define the linear scaling map \( \Lambda(x, y) = (sx, sy) \) if \( s : J \mapsto I \) is the orientation reversing affine scaling and \( J \) is minimal such that \( J \times I \) is invariant under \( H \circ F^2 \circ H^{-1} \).

If \( F \) is renormalizable \( n \) times, then the above definition can be applied to \( R^n F \) for \( 1 \leq k \leq n \) successively. Two dimensional map \( 2dF_{n, \xi} \) with the \( C^r \) function \( \xi_n \) is the same as \( R^n_{2d, \xi} \) by Lemma 8.1 and the above definition. Thus if the maps \( 2dF_{n, \xi} \) are defined on every \( n \in \mathbb{N} \), then the map \( 2dF_{n, \xi} \) is realized to be \( R^n_{2d, \xi} \) and it is called the \( n \)th renormalization of \( F_{2d, \xi} \).

**8.2. Universality of two dimensional \( \text{Hénon-like} \) maps.** Recall that \( Q \) is an \( F \) invariant surface which is tangent to \( E^{pu} \) over the critical Cantor set \( \mathcal{O}_F \). The critical Cantor set restricted to any invariant surface \( Q \), say \( \mathcal{O}_{F|Q} \), is the same as \( \mathcal{O}_F \). The ergodic measure on \( \mathcal{O}_{F_{2d, \xi}} \) is defined as the push forward measure \( \mu \) on \( \mathcal{O}_F \) by the map \( \pi_{xy} \). In particular, it is defined as follows

\[
    \mu_{2d, \xi}(\pi_{xy}(\mathcal{O}_F \cap B^n_{w})) = \mu_{2d, \xi}(\pi_{xy}(\mathcal{O}_F) \cap \pi_{xy}(B^n_{w})) = \frac{1}{2^n}.
\]

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Since $\pi_{xy}(O_F)$ is independent of $\xi$, $\mu_{2d,\xi}$ is also independent of $\xi$. Then we suppress $\xi$ in the notation of the measure $\mu_{2d}$. Let us define the average Jacobian of $F_{2d,\xi}$

$$b_{2d} = \exp \int_{O_{F_{2d}}} \log \text{Jac} F_{2d,\xi} \, d\mu_{2d}.$$ 

Observe that this average Jacobian is independent of the surface map $\xi$.

**Lemma 8.2.** Let $F \in \mathcal{F}(\xi)$ with a sufficiently small perturbation of toy model map satisfying $\|\partial_x \delta\| \ll b_{2d}$. Suppose that there exist $R^n F$ invariant $C^r$ surfaces each of which, say $Q_n$, is tangent to $E^{pu}$ over the critical Cantor set for $2 \leq r < \infty$. Suppose also that $Q_n = \text{graph}(\xi_n)$ where $\xi_n$ is $C^r$ map from $I^x \times I^y$ to $I^x$. Let $R^n F_{2d,\xi}$ be $\pi_{xy} \circ F_n | Q_n \circ (\pi_{xy})^{-1}$ for each $n \geq 1$. Then

$$\text{Jac} R^n F_{2d,\xi} = b_{2d}^{2n} a(x)(1 + O(\rho^n))$$

where $b_{2d}$ is the average Jacobian of $F_{2d,\xi}$ and $a(x)$ is the universal function of $x$ for some positive $\rho < 1$.

**Proof.** By the distortion Lemma 4.1 and Corollary 4.2, we obtain

$$\text{Jac} F_{2d,\xi}^{2n} = b_{2d}^{2n}(1 + O(\rho^n)).$$

Moreover, the chain rule implies that

$$\text{Jac} R^n F_{2d,\xi} = b_{2d}^{2n} \frac{\text{Jac}_{2d} \Psi_0^{n,\xi}(w)}{\text{Jac}_{2d} \Psi_0^{n,\xi}(R^n F_{2d,\xi}(w))}(1 + O(\rho^n))$$

where $w = (x, y, z)$. After letting the tip on every level move to the origin by the appropriate linear map, the equation (8.4) implies that

$$\text{Jac}_{2d} \Psi_0^{n,\xi} = \sigma_{n,0} (\alpha_{n,0} \cdot \partial_x (x + S_0^n(x, y, \xi_n)) + \sigma_{n,0} u_{n,0} \cdot \partial_x \xi_n).$$

Then in order to have the universal expression of the Jacobian, we need the asymptotic of the following maps

$$\partial_x (x + S_0^n(x, y, \xi_n)) \text{ and } \frac{\sigma_{n,0}}{\alpha_{n,0}} \partial_x \xi_n.$$

By Lemma 5.9,

$$x + S_0^n(x, y, \xi_n) = v_*(x) + a_F y^2 + \sum_{j=1}^m a_{F,j} y \cdot \xi_n^j + \sum_{1 \leq i \leq j \leq m} a_{F,ij} \xi_n^i \cdot \xi_n^j + O(\rho^n)$$

with $C^1$ convergence where $v_*(x)$ is the universal function. Thus

$$\partial_x (x + S_0^n(x, y, \xi_n)) = v'_*(x) + \sum_{j=1}^m a_{F,j} y \cdot \partial_x \xi_n^j + \sum_{1 \leq i \leq j \leq m} a_{F,ij} (\partial_x \xi_n^i \cdot \xi_n^j + \xi_n^i \cdot \partial_x \xi_n^j) + O(\rho^n).$$

By Lemma 7.6 we see $\|\partial_x \xi_n\| \leq C \varepsilon \sigma^n$. Then

$$\partial_x (x + S_0^n(x, y, \xi_n)) = v'_*(x) + O(\rho^n).$$

By the equation (7.6) in Lemma 7.6

---

6Every matrix norm is greater than equal to the spectral radius of any given matrix.
Let Hénon-like map where \((\bar{C} > 0) \text{ for } 1 \leq i \leq m\). Moreover, \((\bar{x}, \bar{y})\) converges to the origin \((0, 0)\) as \(n \to \infty\) exponentially fast by Corollary 3.2.

\[
\text{diam}(2d\Psi^n_{0,\xi}(B)) \leq \text{diam}(\Psi^n_{0}(B)) \leq C\sigma^n
\]

for some \(C > 0\). In addition to the exponential convergence of \(\partial_x \xi(\bar{x}, \bar{y})\) to \(\partial_x \xi(0, 0)\), \(u_{n,0}\) converges to \(u_{*,0}\) super exponentially fast. Then,

\[
\frac{\sigma_{n,0}}{\alpha_{n,0}} \frac{\partial \xi}{\partial x} = \frac{\partial_x \xi(\bar{x}, \bar{y})}{1 - u_{n,0} \cdot \partial_x \xi(\bar{x}, \bar{y})} (1 + \partial_x S^n_0(x, y, \xi_n))
\]

Thus we obtain that

\[
\frac{\sigma_{n,0}}{\alpha_{n,0}} \frac{\partial \xi}{\partial x} = \frac{1}{1 - u_{n,0} \cdot \partial_x \xi(\bar{x}, \bar{y})} \left( 1 + \partial_x S^n_0(x, y, \xi_n) \right)
\]

where \((\bar{x}, \bar{y}) \in B(F_{2d,\xi})\) for \(1 \leq i \leq m\). Moreover, \((\bar{x}, \bar{y})\) converges to the origin \((0, 0)\) as \(n \to \infty\) exponentially fast by Corollary 3.2.

\[
\text{diam}(2d\Psi^n_{0,\xi}(B)) \leq \text{diam}(\Psi^n_{0}(B)) \leq C\sigma^n
\]

for some \(C > 0\). In addition to the exponential convergence of \(\partial_x \xi(\bar{x}, \bar{y})\) to \(\partial_x \xi(0, 0)\), \(u_{n,0}\) converges to \(u_{*,0}\) super exponentially fast. Then,

\[
\frac{\sigma_{n,0}}{\alpha_{n,0}} u_{n,0} \cdot \partial_x \xi_n = \frac{u_{*,0} \cdot \partial_x \xi}{1 - u_{*,0} \cdot \partial_x \xi} v'(x) + O(\rho^n).
\]

Let \((x', y') = w' = R^n F_{2d,\xi}(w)\). Then we obtain

\[
\frac{\text{Jac}_{2d}\Psi^n_{0,\xi}(w)}{\text{Jac}_{2d}\Psi^n_{0,\xi}(w')} = \frac{1 + \partial_x(S^n_0, \xi(\bar{x}, \bar{y})) + \frac{\sigma_{n,0}}{\alpha_{n,0}} u_{n,0} \cdot \partial_x \xi(x, y)}{1 + \partial_x(S^n_0, \xi(\bar{x}, \bar{y})) + \frac{\sigma_{n,0}}{\alpha_{n,0}} u_{n,0} \cdot \partial_x \xi(x', y')}
\]

where \(S^n_0(x, y, \xi_n) = S^n_0(\xi(x, y))\). The translation does not affect Jacobian determinant and each translation from tip to the origin converges to the map \(w \mapsto \tau_\infty\) exponentially fast where \(\tau_\infty\) is the tip of two dimensional degenerate map \(F_s(x, y) = (f_s(x), x)\) which is the renormalization fixed point. Then by the similar calculation in Theorem 5.10 the equation (8.9) converges to the following universal function exponentially fast.

\[
\lim_{n \to \infty} \frac{\text{Jac}_{2d}\Psi^n_{0,\xi}(w)}{\text{Jac}_{2d}\Psi^n_{0,\xi}(w')} = \frac{v'(x - \pi_x(\tau_\infty)) + \frac{u_{*,0} \cdot \partial_x \xi(\pi_{xy}(\tau_F))}{1 - u_{*,0} \cdot \partial_x \xi(\pi_{xy}(\tau_F))} v'(x - \pi_x(\tau_\infty))}{v'(f_s(x) - \pi_y(\tau_\infty)) + \frac{u_{*,0} \cdot \partial_x \xi(\pi_{xy}(\tau_F))}{1 - u_{*,0} \cdot \partial_x \xi(\pi_{xy}(\tau_F))} v'(f_s(x) - \pi_y(\tau_\infty))}
\]

\[
= \frac{v'(x - \pi_x(\tau_\infty))}{v'(f_s(x) - \pi_y(\tau_\infty))} \equiv a(x).
\]

\[
\text{Theorem 8.3} \quad \text{(Universality of } C^r \text{ Hénon-like maps with } C^r \text{ conjugation for } 2 \leq r < \infty\).
\]

Let Hénon-like map \(F_{2d,\xi}\) be the \(C^r\) map with \(2 \leq r < \infty\) which is defined on (8.2). Suppose that \(F_{2d,\xi}\) is infinitely renormalizable. Then

\[
R^n F_{2d,\xi}(x, y) = (f_n(x) - b_{2d} a(x) y (1 + O(\rho^n)), x)
\]

where \(b_{2d}\) is the average Jacobian of \(F_{2d,\xi}\) and \(a(x)\) is the universal function for some \(0 < \rho < 1\).

\[
\text{Proof.} \quad \text{By the smooth conjugation of two dimensional map and } F_n|_{Q_n}, \text{ we see that}
\]

\[
R^n F_{2d,\xi}(x, y) = (f_n(x) - \varepsilon_n(x, y, \xi_n), x)
\]

Let \(\varepsilon_n(x, y, \xi_n)\) be \(\varepsilon_n, \xi_n(x, y)\). Then the Jacobian of \(R^n F_{2d,\xi}\) is \(\partial_y \varepsilon_n, \xi_n(x, y)\). By Lemma
\[ \partial_y \varepsilon_n, \xi_n(x, y) = b_{2d}^{n} a(x) (1 + O(\rho^n)). \]

Then

\[ \varepsilon_n, \xi_n(x, y) = b_{2d}^{n} a(x) y (1 + O(\rho^n)) + U_n(x). \]

The map \( U_n(x) \) which depends only on the variable \( x \) can be incorporated to \( f_n(x) \).

\[ \square \]

**Theorem 8.4.** Let \( R^k F \in \mathcal{I}(\varepsilon^{2^k}) \) be the map with invariant surfaces \( Q_k \equiv \text{graph}(\xi_k) \) tangent to \( E^{mu} \) over the critical Cantor set. Then the coordinate change map \( 2d \Psi^n_{k, \xi} \) is as follows

\[ 2d \Psi^n_{k, \xi} = (\alpha_{n, k} (x + 2d S^n_k(w)) + \sigma_{n, k} \cdot 2d t_{n, k} \cdot y, \sigma_{n, k} y) \]

where \( x + 2d S^n_k(w) \) has the asymptotic

\[ x + 2d S^n_k(w) = v_\ast(x) + a_{F, k} y^2 + O(\rho^{n-k}) \]

where \( |a_{F, k}| = O(\varepsilon^{2^k}) \).

**Proof.** By Proposition 8.1, the coordinate change map \( 2d \Psi^n_{k, \xi} \) is the composition of the inverse of horizontal diffeomorphisms with linear scaling maps as follows

\[ H_{k, \xi}^{-1} \circ \Lambda_{k}^{-1} \circ H_{k+1, \xi}^{-1} \circ \Lambda_{k+1}^{-1} \circ \cdots \circ H_{n, \xi}^{-1} \circ \Lambda_{n}^{-1}. \]

Then after reshuffling non-linear and linear parts separately by the direct calculations and letting the tip move to the origin by the appropriate translations on each levels, the coordinate change map is of the form (8.12). However, the calculation in Section 7.2 in [CLM] can be used because analyticity is not required for any calculation of recursive formulas. Thus we have the following estimation

\[ x + 2d S^n_k(x, y) = v_\ast(x) + a_{F, k} y^2 + O(\rho^{n-k}) \]

where \( |a_{F, k}| = O(\varepsilon^{2^k}) \). Alternatively, let us choose the equation (8.3)

\[ 2d \Psi^n_{k, \xi} = (\alpha_{n, k} (x + S^n_k(x, y)) + \sigma_{n, k} t_{n, k} \cdot y + \sigma_{n, k} u_{n, k} \cdot (\xi_n + R_{n, k}(y)), \sigma_{n, k} y) \]

where \( S^n_{k, \xi}(x, y) = S^n_k(x, y, \xi_n(x, y)) \). By Proposition 7.6, the map \( \xi_n \) is

\[ \xi_n(x, y) = cy + \eta(y) + O(\rho^n) \]

where \( \eta = (\eta_1, \eta_2, \ldots, \eta_m) \) is quadratic or higher order terms with \( \|\eta\|_{C^1} \leq C_0 \sigma^{n-k} \) for some \( C_0 > 0 \). Recall that \( u_{n, k} \) converges to \( u_{s, k} \) super exponentially fast and \( \|R_{n, k}\| \leq C_1 \sigma^{n-k} \) for some \( C_1 > 0 \). Recall also that \( \alpha_{n, k} = \sigma^{2(n-k)}(1 + O(\rho^n)) \) and \( \sigma_{n, k} = (-\sigma)^{n-k}(1 + O(\rho^n)) \). Hence, we define each terms of \( 2d \Psi^n_{k, \xi} \) appropriately

\[ 2d S^n_k(x, y) = S^n_k(x, y) + \frac{\alpha_{n, k}}{\sigma_{n, k}} u_{n, k} \cdot [\xi_n(x, y) - cy + R_{n, k}(y)] \]

\[ 2d t_{n, k} = t_{n, k} + u_{n, k} \cdot c \]

as desired. \[ \square \]
9. Unbounded geometry of critical Cantor set

The unbounded geometry of the Cantor set, $O_F$ of small perturbation of the toy model map, $F \in \mathcal{I}(\bar{\epsilon})$ with $b_1 \gg \|\partial_z \delta\|$ is involved with that of the map on the invariant surfaces of each level, $F|_Q$. Since the $C^r$ conjugation preserves this property, the fact that the Cantor set of $C^r$ Hénon-like map $F_{2d}$ has the unbounded geometry is sufficient to show the same property of $O_F$.

Recall that the minimal distance between two boxes $B_1, B_2$ is the infimum of the distance between all points of each boxes, $\text{dist}_{\min}(B_1, B_2)$.

**Definition 9.1.** $F \in \mathcal{I}(\bar{\epsilon})$ has bounded geometry if

$$\text{dist}_{\min}(B_{w^{n+1}}^n, B_{w^c}^{n+1}) \simeq \text{diam}(B_{w^v}^{n+1}) \quad \text{for } \nu \in \{v, c\}$$

$$\text{diam}(B_w^n) \simeq \text{diam}(B_{w^v}^{n+1}) \quad \text{for } \nu \in \{v, c\}$$

for all $w \in W^n$ and for all $n \geq 0$.

By the definition of each $B_w^n$, if $F$ does not have bounded geometry, then we call $O_F$ has bounded geometry. Otherwise, we call $O_F$ has unbounded geometry.

Let $F_{2d}$ be an infinitely renormalizable two dimensional Hénon-like map and $b_1$ be the average Jacobian of $F_{2d}$. Then the unbounded geometry of the Cantor set depends on Universality theorem and the asymptotic of the tilt, $-t_k \simeq b_1^k$ but it does not depend on the analyticity of the map. Moreover, unbounded geometry holds if we choose $n > k$ such that $b_{2d} \simeq \sigma^{n-k}$ for every sufficiently large $k$. This is true on the parameter space of $b_{2d}$ almost everywhere with respect to Lebesgue measure.

**Theorem 9.1 ([HLM]).** The given any $0 < A_0 < A_1$, $0 < \sigma < 1$ and any $p \geq 2$, the set of parameters $b \in [0, 1]$ for which there are infinitely many $0 < k < n$ satisfying

$$A_0 < \frac{b^{\sigma^k}}{\sigma^{n-k}} < A_1$$

is a dense $G_\delta$ set with full Lebesgue measure.

**Theorem 9.2.** Let $m + 2$ dimensional Hénon-like map, $F \in \mathcal{I}(\bar{\epsilon})$ be a small perturbation of the toy model map with $\|\partial_z \delta\| \ll b_1$ where $b_1$ is the average Jacobian of $F_{2d}$. Let $F_{b_1}$ be the parametrized $m + 2$ dimensional Hénon-like map for $b_1 \in [b_0, b_*]$ where $\|\partial_z \delta\| \ll b_0 < b_*$. Then there exists $G_\delta$ subset $S$ with full Lebesgue measure of $[b_0, b_*]$ such that the critical Cantor set $O_{F_{b_1}}$ has unbounded geometry.

**Proof.** The comparison of minimal distances between two adjacent boxes and the diameter of each boxes for every level. Two dimensional invariant surface under $F_{b_1}$ enable us to apply the proof of two dimensional Hénon-like maps. See the proof of Theorem 6.3 in [Nam3] for unbounded geometry of the critical Cantor set. \qed
References

[CLM] A. de Carvalho, M. Lyubich, M. Martens, Renormalization in the Hénon family, I : Universality but non-rigidity. J. Stat. Phys. 121 No. 516, (2005), 611-669.

[CP] S. Crovisier, E. P. Pujals, Essential hyperbolicity and homoclinic bifurcations : A dichotomy phenomenon / mechanism for diffeomorphisms. IMPA preprint (2010)

[CT] P. Coullet, C. Tresser. Itération d'endomorphismes et groupe de renormalisation. J. Phys. Colloque C 539, C5(1978), 25-28.

[HLM] P. Hazard, M. Lyubich, M. Martens. Renormalizable Hénon-like map and unbounded geometry. Nonlinearity, 25(2):397-420, 2012.

[LM] M. Lyubich, M. Martens, Renormalization in the Hénon family, II : The heteroclinic web. Invent. math., (2011), 186, 115-189.

[Nam1] Y. Nam, Invariant space under Hénon renormalization, preprint (2014), available at http://arxiv.org/abs/1408.4619v3

[Nam2] Y. Nam, Renormalization of $C^r$ Hénon-map, preprint (2014), available at http://arxiv.org/abs/1412.8337

[Nam3] Y. Nam, Renormalization of three dimensional Hénon-map I, preprint (2014), available at http://arxiv.org/abs/1408.4289

[WY] Q. Wang and L.-S. Young, Towards a theory of rank one attractors, Annals of Math., 167 (2008), 349-480.

[ZG] W. Zhang, S. S. Ge, A global implicit function theorem without initial point and its applications to control of non-affine systems of high dimensions. J. of math. analysis and applications 313 (2006), 251-261

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