FINDING NORMAL BINARY FLOATING-POINT FACTORS IN CONSTANT TIME

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ABSTRACT. Solving the floating-point equation $x \otimes y = z$, where $x$, $y$ and $z$ belong to floating-point intervals, is a common task in automated reasoning for which no efficient algorithm is known in general. We show that it can be solved by computing a constant number of floating-point factors, and give a constant-time algorithm for computing successive normal floating-point factors of normal floating-point numbers in radix 2. This leads to a constant-time procedure for solving the given equation.

1. INTRODUCTION

Floating-point arithmetic has long been the prevailing system used for numerical computing, allowing efficient and precise calculation on modern hardware. Whilst it closely approximates real arithmetic, it does not always satisfy its laws. These sporadic exceptions often confuse programmers and complicate mathematical reasoning to the point that deciding even the simplest statements can take excessive amounts of time. Our interest stems from the area of formal software verification, where automated reasoning is used to prove that programs satisfy their specifications. Although modern verification systems can reason about floating-point arithmetic, it has much less support than integer or bit-vector arithmetic. This makes proving correctness far more difficult and drastically limits practical applications of formal methods. Our work seeks to fill these gaps.

Denoting floating-point multiplication by $\otimes$, we study the following problem:

Problem A. Suppose $x \otimes y = z$ where $x$, $y$ and $z$ are bounded. Can those bounds be tightened, and if so, what are the tightest bounds?

We can give a simplified example without getting into the details of floating-point arithmetic:

Example 1.1. For the sake of exposition, we will use decimal numbers with two decimal places, rounding downward the results of arithmetic operations. For example, since $0.50 \cdot 1.01 = 0.505$, we have $0.50 \otimes 1.01 = 0.50$, where $\otimes$ is rounded multiplication.

Let $x$ and $y$ be two-place decimal numbers, and suppose $x \otimes y = 5.00$ where $2.20 \leq x \leq 2.50$ and $1.00 \leq y \leq 2.50$. These bounds are clearly quite loose. For instance, the lower bound on $y$ is too small: if $x = 2.50$ and $y = 1.00$, then $x \otimes y = 2.50$, which is much less than the desired 5.00.

First, we construct an equivalent condition that uses real multiplication instead of floating-point multiplication. This can be accomplished with a simple transformation. Since the numbers that round downward to 5.00 are exactly those in...
[5.00, 5.01), it follows that \( x \otimes y = 5.00 \) is equivalent to \( 5.00 \leq xy < 5.01 \). By getting rid of the \( \otimes \) operator, we can now reason using real arithmetic:

1. Multiplying \( x \) and \( y \) and combining their bounds, we get \( 2.20 \leq xy \leq 6.25 \).
2. This is worse than our existing bounds on \( xy \), so we cannot improve them.
3. Dividing the bounds on \( xy \) by \( y \), we obtain \( 5.00/2.50 \leq x < 5.01/1.00 \).
4. This also does not improve our existing bounds on \( x \).
5. Dividing by \( x \) instead, we obtain \( 5.00/2.50 \leq y < 5.01/2.20 \).
6. We can thus improve our bounds on \( y \) to \( 2.00 \leq y < 2.27 \).

If we were allowed the full range of the reals, we would now be finished, stating that \( 5.00 \leq xy < 5.01 \) where \( 2.20 \leq x \leq 2.50 \) and \( 2.00 \leq y < 2.27 \). We can easily show that these bounds are tight for real multiplication:

- If \( x = 2.20 \) and \( y < 2.27 \), then \( xy < 5.01 \). If \( y \geq 2.27 \) instead, then \( xy \geq 5.01 \).
- If \( x = 2.50 \) and \( y = 2.00 \), then \( xy = 5.00 \).

Having used each of the bounds on \( x \), \( y \) and \( xy \), we know that this set of bounds is tight on its own. However, we also require that \( x \) and \( y \) are decimal numbers with two decimal places. Thus, since \( y < 2.27 \), we can further conclude that \( y \leq 2.27 \). Since the bounds all have two decimal places now, we would ideally like to be done. If we check, though, we find that the bounds are no longer tight: if \( x = 2.20 \) and \( y = 2.27 \), then \( xy = 4.994 \), which is too small, and therefore \( x \geq 2.21 \).

We can continue tightening even further. If \( x = 2.21 \) instead and \( y = 2.27 \), then \( xy = 5.0167 \), which is too large. Therefore \( y \leq 2.26 \), and so on. From this point, division does not help us as much as it did earlier. Since the product of the bounds on \( x \) and \( y \) is so close to the optimal result, we can easily be forced to iterate over billions of values. We therefore aim to devise an algorithm to solve this problem efficiently. We consider the following to be the primary contributions of this paper:

- An algorithm to compute, in constant time, the next normal floating-point factor of any normal floating-point number in radix 2.
  - This algorithm is also partially applicable in higher bases. We do not assume a particular radix in any of our results.
- When the above algorithm is applicable, a constant-time procedure for finding optimal interval bounds on variables of floating-point constraints of the form \( x \otimes y = z \).

In addition, we believe the following secondary contributions are of interest:

\(^1\)The overline denotes repeating digits.
• A novel demonstration of structure in the rounding errors of floating-point products.
• A characterization of upward and downward floating-point rounding in terms of the remainder of floored division.
• A floating-point predicate for deciding, in any base, whether a number is floating-point factor of another floating-point number.
• Results on solving certain kinds of interval constraints over integer remainders.

1.1. Related work. Although floating-point arithmetic has existed for a long time, work on solving floating-point equations and inequalities has been sporadic and has mostly been concerned with binary floating-point arithmetic. Michel [6] developed partial solutions for floating-point equations of the form $x \circ y = z$ where $\circ$ is one of rounded addition, subtraction, multiplication, or division, as well as a complete solution for rounded square root. Ziv et al. [8] showed that, under mild conditions, if $z$ can take at least 2 different values, then the equation $x \otimes y = z$ has at least one solution in binary floating-point arithmetic. However, their work does not treat other bases, nor the case where $z$ is fixed. Bagnara et al. [2] gave bounds on $x$ and $y$ in terms of $z$ for binary floating-point arithmetic. At present, there are no known efficient algorithms for computing the solution set of $x \otimes y = z$ in general. Indeed, in terms of similar efforts, the only basic operation which has been solved (apart from square root) is addition [1, 4].

1.2. Outline. In Section 2 we introduce our notation and develop the formalisms used in this paper. We also give an account of the basic properties of floating-point arithmetic that we will need. Due to the substantial amount of notation, a glossary is supplied for reference in Appendix B. In Section 3 we relate optimal intervals, the division-based algorithm and floating-point factors, and simplify the problem by narrowing down the conditions under which the division-based algorithm does not produce an optimal result. In Section 4 we then develop conditions for numbers being factors based on the rounding error in the product. Then, in Section 5 we show that we can control that rounding error under certain conditions and thus directly find floating-point factors. Combining these in Section 6 we give an algorithm to compute such factors efficiently. Appendix A contains some ancillary proofs used in the results of Section 6. In Section 7 we conclude and discuss open problems.

2. Preliminaries

Following convention, we denote the integers by $\mathbb{Z}$, the reals by $\mathbb{R}$, and the extended reals by $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$, and the same sets excluding zero by $\mathbb{Z}^*$, $\mathbb{R}^*$ and $\overline{\mathbb{R}}^*$. We write $f[X]$ for the image of a set $X$ under a function $f$, and $f^{-1}[X]$ for the preimage of $X$ under $f$. To simplify exposition, we will denote the images of $X \times Y$ and $X \times \{y\}$ under a binary operator $\circ$ simply by $X \circ Y$ and $X \circ y$, respectively. We will use such notation most frequently in the context of intervals. We also discuss the size of intervals in terms of their width on the extended real line. To this end, we define a notion of diameter for subsets of the extended reals:

**Definition 2.1 (Set diameter).** For any $X \subseteq \overline{\mathbb{R}}$, we define
\[
\text{diam } X = \sup \{d(x, y) \mid x, y \in X\} \cup \{0\},
\]
where $d : \mathbb{R}^2 \to \mathbb{R}$ is a function such that

$$d(x, y) = \begin{cases} 
0 & \text{if } x = y, \\
|x - y| & \text{otherwise}.
\end{cases}$$

The diameter satisfies the following properties for all sets $X \subseteq \mathbb{R}$:

1. $\text{diam } \emptyset = 0$ (diameter of empty set)
2. For all $x \in \mathbb{R}$, $\text{diam } \{x\} = 0$ (diameter of singleton)
3. If $\inf X < \sup X$, then $\text{diam } X = \sup X - \inf X$
4. For all $a \in \mathbb{R}^*$, $\text{diam } aX = |a| \text{diam } X$ (absolute homogeneity)
5. For all $b \in \mathbb{R}$, $\text{diam } (X + b) = \text{diam } X$ (translation invariance)

Importantly, the diameter of an interval is equal to the distance between its endpoints:

**Lemma 2.2.** For all $a, b \in \mathbb{R}$, if $a < b$, then

$$\text{diam } [a, b] = \text{diam } (a, b) = \text{diam } [a, b] = \text{diam } (a, b) = b - a.$$

Additionally, if two intervals overlap, then the larger one contains at least one of the endpoints of the smaller one:

**Lemma 2.3.** Let $a, b \in \mathbb{R}$ and let $I \subseteq \mathbb{R}$ be an interval. If $[a, b] \cap I$ is nonempty and $\text{diam } I > b - a$, then either $a \in I$ or $b \in I$.

With basic terms now defined, we turn our attention to floating-point arithmetic. To the best of the author’s knowledge, there is no well-established formalism for floating-point theory, although there are a few conventions. We shall begin with an informal description of floating-point arithmetic. Readers interested in a deeper introduction may wish to consult a reference such as Muller et al. [7] or Goldberg [5].

A finite floating-point number is usually written in the style of scientific notation as $\pm d_1.d_2d_3 \cdots d_p \times \beta^e$, where each digit $d_i$ lies in $\{0, \ldots, \beta - 1\}$. The $\pm d_1.d_2d_3 \cdots d_p$ part is the significand, the number of digits $p$ is the precision, $\beta$ is the base (commonly 2 or 10), and $e$ is the exponent. For example, with precision $p = 3$ and base $\beta = 2$, we can write the decimal number 0.75 as either $1.10 \times 2^{-1}$ or $0.11 \times 2^1$. Of the two representations, the former is preferred, as it is normalized—that is, it has no leading zeros—and is therefore unique.

With a higher precision we can represent more numbers. For instance, we need $p \geq 4$ to represent the binary number 1.101 exactly. Regardless of precision, however, there are still limits imposed by the choice of base. Just as $\frac{1}{2}$ lacks a finite decimal expansion, for any given base there are numbers that cannot be represented. Notably, the decimal 0.1 is $0.0001100$ in binary, and thus it has no representation in any binary floating-point system. As an example, with precision $p = 3$ and base $\beta = 2$, we take $1.10 \times 2^{-1}$ or $0.11 \times 2^1$. Of the two representations, the former is preferred, as it is normalized—that is, it has no leading zeros—and is therefore unique.

Consequently, the sum, product or ratio of two precision-$p$ base-$\beta$ floating-point numbers is rarely itself a precision-$p$ base-$\beta$ floating-point number and thus computing with them typically involves accepting some degree of rounding error. In ordinary usage, we minimize rounding error by rounding to nearest: that is, by choosing the floating-point number closest to the real number we are rounding. Other rounding modes such as rounding upward, downward, or towards zero (also
known as truncation) are typically only used in situations where the direction of the rounding is more important than the accuracy of a single operation. For instance, interval arithmetic avoids underestimation by rounding upper bounds upward and lower bounds downward.

2.1. Definitions and terminology. We now formally define the various special terms that will be used throughout the rest of the paper, and give their basic properties. Appendix B contains a summary of the definitions introduced here. We begin with a construction of the floating-point numbers.

Definition 2.4 (Floating-point numbers). A floating-point format is a quadruple \((\beta, p, e_{\text{min}}, e_{\text{max}})\) such that \(\beta \geq 2\), \(p \geq 1\), and \(e_{\text{min}} \leq e_{\text{max}}\), where \(\beta\) is the base, \(p\) is the precision, \(e_{\text{min}}\) is the minimum exponent, and \(e_{\text{max}}\) is the maximum exponent. For convenience, we also define the minimum quantum exponent \(q_{\text{min}} = e_{\text{min}} - p + 1\) and the maximum quantum exponent \(q_{\text{max}} = e_{\text{max}} - p + 1\). Given these quantities, we define the set \(\mathbb{F}^*\) of finite nonzero floating-point numbers as follows:

\[
\mathbb{F}^* = \{ M \cdot \beta^{e-p+1} | M, e \in \mathbb{Z}, 0 < |M| < \beta^p, e_{\text{min}} \leq e \leq e_{\text{max}} \}.
\]

From here, we define the set \(\mathbb{F}\) of finite floating-point numbers and the set \(\overline{\mathbb{F}}\) of floating-point numbers:

\[
\mathbb{F} = \mathbb{F}^* \cup \{0\},
\]

\[
\overline{\mathbb{F}} = \mathbb{F} \cup \{-\infty, +\infty\}.
\]

To avoid repetition, from this point forward we will assume that the sets of floating-point numbers correspond to some floating-point format \((\beta, p, e_{\text{min}}, e_{\text{max}})\).

Lemma 2.5. \(\mathbb{F}^*, \mathbb{F}\) and \(\overline{\mathbb{F}}\) are each closed under negation.

Furthermore, the greatest finite floating-point number and the least positive floating-point number have simple expressions:

\[
\max \mathbb{F}^* = \max \mathbb{F} = (\beta^p - 1)\beta^{e_{\text{max}}}
\]

\[
\min \overline{\mathbb{F}} \cap (0, +\infty] = \beta^{q_{\text{min}}}
\]

We further categorize the elements of \(\mathbb{F}^*\) as follows:

Definition 2.6 (Normal and subnormal numbers). A nonzero finite floating-point number \(x \in \mathbb{F}^*\) is normal if \(|x| \geq \beta^{e_{\text{min}}}\), or subnormal otherwise.

The normal numbers are so called because they each have a normalized representation in scientific notation, restricting the exponent to \(\{e_{\text{min}}, \ldots, e_{\text{max}}\}\). By contrast, a subnormal number can only be written without leading zeros by choosing an exponent below \(e_{\text{min}}\). Note that we do not consider zero to be normal or subnormal.

A number written in scientific notation has an exponent and significand. Similarly, each finite floating-point number has an associated exponent and significand determined by the floating-point format. We will now define these concepts and generalize them to the finite reals.

\[2\text{In IEEE 754 parlance, the “quantum” of a floating-point number is the value of a unit in the last digit of its significand, often referred to as the “unit in last place”. This is necessarily an integer power of }\beta, \text{ and the quantum exponent is the exponent of that power.}\]
Definition 2.7 (Exponent and significand). The functions $E : \mathbb{R} \to \mathbb{Z}$ and $Q : \mathbb{R} \to \mathbb{Z}$ are defined as follows:

$$
E(x) = \begin{cases} 
\lfloor \log_\beta |x| \rfloor, & |x| \geq \beta^{e_{\min}}, \\
e_{\min}, & |x| < \beta^{e_{\min}}.
\end{cases}
$$

$$
Q(x) = E(x) - p + 1.
$$

Given $x \in \mathbb{R}$, the integers $E(x)$ and $Q(x)$ are the exponent and quantum exponent of $x$, respectively. The significand of $x$ is the number $x \beta^{-E(x)}$. If $x \beta^{-Q(x)}$ is an integer, then it is the integral significand of $x$.

To justify these names, we establish that these indeed compute the exponent of a number written as a product of a significand and a power of $\beta$, respecting the minimum exponent:

Lemma 2.8 (Exponent law). Let $m \in \mathbb{R}$ and $e \in \mathbb{Z}$ such that $|m| < \beta$ and $e \geq e_{\min}$. If either $|m| \geq 1$ or $e = e_{\min}$, then $E(m \beta^e) = e$. Otherwise, $E(m \beta^e) < e$.

Proof. First, note that if $|m \beta^e| \geq \beta^{e_{\min}}$, then

$$
E(m \beta^e) = \lfloor \log_\beta |m \beta^e| \rfloor = \lfloor \log_\beta |m| + e \rfloor = \lfloor \log_\beta |m| \rfloor + e.
$$

We now proceed by cases:

- Suppose $|m| \geq 1$. Then $|m \beta^e| \geq \beta^{e_{\min}}$ and $\log_\beta |m| = 0$, and thus $E(m \beta^e) = e$.
- Suppose $e = e_{\min}$ and $|m| < 1$. Then $|m \beta^e| < \beta^{e_{\min}}$ and so we have $E(m \beta^e) = e_{\min} = e$ immediately by the definition of $E$.
- Suppose $e > e_{\min}$ and $|m| < 1$. If $|m \beta^e| \geq \beta^{e_{\min}}$, then $\log_\beta |m| < 0$ and hence $E(m \beta^e) = \lfloor \log_\beta |m| \rfloor + e < e$. If $|m \beta^e| < \beta^{e_{\min}}$ instead, then $E(m \beta^e) = e_{\min} < e$.

Since the result holds in all cases, we are finished. \qed

Corollary 2.8.1. Let $m \in \mathbb{R}$ and $e \in \mathbb{Z}$. If $|m| < \beta$ and $e \geq e_{\min}$, then $E(m \beta^e) \leq e$.

Corollary 2.8.2. Let $M \in \mathbb{R}$ and $q \in \mathbb{Z}$ such that $|M| < \beta^p$ and $q \geq q_{\min}$. If either $|M| \geq \beta^{p-1}$ or $q = q_{\min}$, then $Q(M \beta^q) = q$. Otherwise, $Q(M \beta^q) < q$.

Corollary 2.8.3. Let $M \in \mathbb{R}$ and $q \in \mathbb{Z}$. If $|M| < \beta^p$ and $q \geq q_{\min}$, then $Q(M \beta^q) \leq q$.

Expanding on the above, the function $E$ also satisfies the following properties for all $x, y \in \mathbb{R}$:

(8) \quad $E(x) = E(-x)$ \quad (evenness)

(9) \quad if $|x| \leq |y|$, then $E(x) \leq E(y)$ \quad (piecewise monotonicity)

(10) \quad if $x \in \mathbb{F}$, then $e_{\min} \leq E(x) \leq e_{\max}$ \quad (exponent bound)

(11) \quad |x \beta^{-E(x)}| < \beta \quad (significand bound)

(12) \quad 1 \leq |x \beta^{-E(x)}| \quad \text{if and only if } \beta^{e_{\min}} \leq |x| \quad (normality)
From the above properties, the definition of $Q$ immediately implies the following:

\begin{align*}
(13) & \quad Q(x) = Q(-x) \quad \text{(evenness)} \\
(14) & \quad \text{if } |x| \leq |y|, \text{ then } Q(x) \leq Q(y) \quad \text{(piecewise monotonically)} \\
(15) & \quad \text{if } x \in \mathbb{F}, \text{ then } q_{\min} \leq Q(x) \leq q_{\max} \quad \text{(quantum exponent bound)} \\
(16) & \quad |x\beta^{-Q(x)}| < \beta^p \quad \text{(integral significand bound)} \\
(17) & \quad \beta^{p-1} \leq |x\beta^{-Q(x)}| \text{ if and only if } \beta^{q_{\min}} \leq |x| \quad \text{(normality)}
\end{align*}

Additionally, the following fact establishes an important connection between integral significands and the finite floating-point numbers:

\begin{align*}
(18) & \quad \text{if } x \in \mathbb{F}, \text{ then } x\beta^{-Q(x)} \in \mathbb{Z} \quad \text{(integrality)}
\end{align*}

In other words, every finite floating-point number has an integral significand.

**Definition 2.9** (Unit in first place). The function $\text{ufp} : \mathbb{R} \to \mathbb{R}$ is defined as follows:

$$
\text{ufp}(x) = \begin{cases} 
\beta^{\lfloor \log_\beta |x| \rfloor}, & x \neq 0, \\
0, & x = 0.
\end{cases}
$$

For any $x \in \mathbb{R}$, we refer to $\text{ufp}(x)$ as the unit in first place of $x$.

For all $x, y \in \mathbb{R}$ and $n \in \mathbb{Z}$, the following statements hold:

\begin{align*}
(19) & \quad \text{ufp}(x) \leq |x| \quad \text{(lower bound)} \\
(20) & \quad \text{if } x \neq 0, \text{ then } \text{ufp}(x) \leq |x| < \beta \text{ufp}(x) \quad \text{(bounds)} \\
(21) & \quad \text{ufp}(x) = |x| \text{ iff } x = 0 \text{ or } |x| = \beta^k \text{ for some } k \in \mathbb{Z} \quad \text{(exactness)} \\
(22) & \quad \text{ufp}(x\beta^n) = \text{ufp}(x)\beta^n \quad \text{(scaling)} \\
(23) & \quad \text{ufp}(x)\text{ufp}(y) = \text{ufp}(x\text{ufp}(y)) \quad \text{(product law)} \\
(24) & \quad \text{if } y \neq 0, \text{ then } \text{ufp}(x)/\text{ufp}(y) = \text{ufp}(x/\text{ufp}(y)) \quad \text{(quotient law)}
\end{align*}

There is also a close relationship between $\text{ufp}$ and $E$. In particular, the base-$\beta$ logarithm of the unit in first place is the exponent (if exponents were unbounded):

\begin{align*}
(25) & \quad \text{if } |x| \geq \beta^{q_{\min}}, \text{ then } \text{ufp}(x) = \beta^{E(x)} \quad \text{(bounds)} \\
(26) & \quad \text{if } |x| < \beta^{q_{\min}}, \text{ then } \text{ufp}(x) < \beta^{E(x)}
\end{align*}

Additionally, we have the following lemma on the exponents of quotients:

**Lemma 2.10.** Let $x, z \in \mathbb{R}^*$ and let $m_x = x\beta^{-E(x)}$ and $m_z = z\beta^{-E(z)}$. If $\beta^{q_{\min}} \leq |z/x|$, then $E(z/x) = E(z) - E(x) + \log_\beta \text{ufp}(m_z/m_x)$.

**Proof.** Suppose $\beta^{q_{\min}} \leq |z/x|$. Then,

$$
\beta^{E(z/x)} = \text{ufp}(z/x)
= \text{ufp}\left(x\beta^{E(z)}m_x\beta^{-E(z)}\right)
= \text{ufp}(m_z/m_x)\beta^{E(z)-E(x)}.
$$

Taking logarithms, we obtain the desired result. \hfill \square

**Corollary 2.10.1.** Let $x, z \in \mathbb{R}^*$ and let $m_x = x\beta^{-E(x)}$ and $m_z = z\beta^{-E(z)}$. If $\beta^{q_{\min}} \leq |z/x|$, then

\begin{enumerate}
\item If $1 \leq |m_z/m_x| < \beta$, then $E(z/x) = E(z) - E(x)$.
\end{enumerate}
(2) If $|m_x| < |m_z|$, then $E(z/x) \leq E(z) - E(x) - 1$.

Since $\mathbb{F}$ is a finite subset of $\mathbb{R}$, there is a well-defined notion of successor and predecessor for its elements:

**Definition 2.11** (Successor and predecessor). The successor $x^+$ of a floating-point number $x \in \mathbb{F}$ is the least floating-point number greater than $x$, or $+\infty$ if none exists. Similarly, the predecessor $x^-$ is the greatest floating-point number less than $x$, or $-\infty$ if none exists.

The predecessor and successor satisfy the following for all $x \in \mathbb{F}$:

\begin{align*}
(27) & \quad x^- \leq x \leq x^+ & \text{(lower/upper bound)} \\
(28) & \quad -(x^+) = (-x)^- & \text{(duality)} \\
(29) & \quad \text{if } x < +\infty, \text{ then } x^+ > x & \text{(strictness)} \\
(30) & \quad \text{if } -\infty < x, \text{ then } x^- < x \\
(31) & \quad \text{if } -\infty < x < +\infty, \text{ then } x^{+-} = x^- = x & \text{(cancellation)}
\end{align*}

The successor and predecessor can also be expressed arithmetically in terms of the quantum thus:

**Lemma 2.12.** For all $x \in \mathbb{F}$,

$$x^+ = \begin{cases} 
\min \mathbb{F} & \text{if } x = -\infty, \\
\min x + \beta^Q(x)-1 & \text{if } x < -\beta^{e_{\text{min}}} \text{ and } x = -\beta^E(x), \\
+\infty & \text{if } x \geq \max \mathbb{F}, \\
x + \beta^Q(x) & \text{otherwise.}
\end{cases}$$

**Proof.** That $(-\infty)^+ = \min \mathbb{F}$ and $(\max \mathbb{F})^+ = (+\infty)^+ = +\infty$ follows trivially from the definitions of floating-point successor and $\mathbb{F}$. Let $x \in \mathbb{F}$ such that $\min \mathbb{F} \leq x < \max \mathbb{F}$. Let $M = x\beta^{-Q(x)}$ and $x' = x + \beta^Q(x)$. Then $x' = (M + 1)\beta^Q(x)$. Since $|M| < \beta^p$, it follows that $-\beta^p < M + 1 \leq \beta^p$. Therefore, if $M + 1 < \beta^p$, we have $x' \in \mathbb{F}$ by definition. Hence, since $x < \max \mathbb{F} = (\beta^p - 1)\beta^{e_{\text{max}}}$, if $E(x) = e_{\text{max}}$, then $M < \beta^p - 1$ and thus $x' \in \mathbb{F}$. If $M + 1 = \beta^p$ and $E(x) < e_{\text{max}}$ instead, then $x' = \beta^{E(x)+1}$ and hence $x' \in \mathbb{F}$ again. Therefore $x' \in \mathbb{F}$ in all cases, and since $x < x'$, we have $x < x^+ \leq x'$ by the definition of successor.

If $x \geq 0$, then $|x| < |x^+|$ and hence $E(x) \leq E(x^+)$. If $|x| < \beta^{e_{\text{min}}-1}$, then $E(x) = e_{\text{min}} \leq E(x^+)$. If $x \leq -\beta^{e_{\text{min}}+1}$ and $x \neq -\beta^E(x)$, then $x < -\beta^E(x)$, so $x < x^+ \leq -\beta^E(x)$ and hence $E(x^+) = E(x)$.

Now, let $M_+ = x^+\beta^{-Q(x^+)}$ and $k = Q(x^+) - Q(x)$.

Suppose $E(x) \leq E(x^+)$. Then $k \geq 0$ and hence $M_+\beta^k$ is an integer. Since $x < x^+ \leq x'$, dividing by $\beta^Q(x)$ gives $M < M_+\beta^k \leq M + 1$. Then $M_+\beta^k = M + 1$ and therefore $x^+ = x'$.

Suppose $E(x) > E(x^+)$. Then $x = -\beta^E(x)$ and $x \leq -\beta^{e_{\text{min}}+1}$. Therefore $M = -\beta^{p-1}$. Let $b = -((\beta^p - 1)\beta^{Q(x)+1})$. Then $b \in \mathbb{F}$. Since $x < b$, we have $x < x^+ \leq b$. Therefore $Q(x^+) = Q(x) - 1$. Hence $M_+ \leq -((\beta^p - 1)$, and since $|M_+| < \beta^p$, we have $M_+ = -(\beta^p - 1)$, so $M_+ = b = x + \beta^Q(x)^{-1}$.

$\square$
Corollary 2.12.1. For all $x \in \mathbb{F}$,

\[
x^- = \begin{cases} 
-\infty & \text{if } x \leq \min \mathbb{F}, \\
 x - \beta Q(x) - 1 & \text{if } \beta e_{\min} < x < +\infty \text{ and } x = \beta E(x), \\
\max \mathbb{F} & \text{if } x = +\infty, \\
 x - \beta Q(x) & \text{otherwise}.
\end{cases}
\]

Having defined the basic structure of $\mathbb{F}$, we now turn to the definition of the computational operations involving the floating-point numbers.

**Definition 2.13 (Rounding).** A rounding function is any function $f : \mathbb{R} \rightarrow \mathbb{F}$.

The downward rounding function $\text{RD} : \mathbb{R} \rightarrow \mathbb{F}$ and the upward rounding function $\text{RU} : \mathbb{R} \rightarrow \mathbb{F}$ are defined as follows:

$$
\text{RD}(x) = \max \{ y \in \mathbb{F} \mid x \leq y \},
$$

$$
\text{RU}(x) = \min \{ y \in \mathbb{F} \mid x \geq y \}.
$$

A rounding function $f$ is **faithful** if and only if $f(x) \in \{ \text{RD}(x), \text{RU}(x) \}$ for all $x \in \mathbb{R}$.

It is easy to show that both RD and RU are nondecreasing functions. In addition, they satisfy the following properties for all $x \in \mathbb{R}$:

(32) $\text{RD}(x) \leq x \leq \text{RU}(x)$ (lower/upper bound)

(33) $\text{RU}(-x) = -\text{RD}(x)$ (duality)

(34) $x \in \mathbb{F}$ if and only if $\text{RU}(x) = \text{RD}(x)$ (exactness)

In addition, the following gap property holds for all $x \in \mathbb{R}$:

(35) $\text{RU}(x) - \text{RD}(x) = \begin{cases} 
\infty & \text{if } |x| > \max \mathbb{F}, \\
0 & \text{if } x \in \mathbb{F}, \\
\beta Q(x) & \text{otherwise}.
\end{cases}$

We also give a partial definition of **rounding to nearest**, which is the default rounding mode used in practice:

**Definition 2.14 (Rounding to nearest).** A rounding function $f$ rounds to nearest if $|f(x) - x| = \min_{y \in \mathbb{F}} |y - x|$ for all $x \in [\min \mathbb{F}, \max \mathbb{F}]$.

We denote by RN an arbitrary nondecreasing and faithful rounding to nearest. This definition does not give a tie-breaking rule for when $x$ is the exact midpoint between two floating-point numbers. It also does not fully specify how to round values outside $[\min \mathbb{F}, \max \mathbb{F}]$. We leave these unspecified since IEEE 754 specifies two different round-to-nearest modes with different rules in these cases. In particular, even though $+\infty$ and $-\infty$ are infinitely far apart from any finite number, according to IEEE 754, they may still result from rounding finite values to nearest. More specifically, it states that the rounding to nearest of any $x$ such that $|x| \geq \beta^{e_{\max}}(\beta - \beta^{1-p}/2)$ is the infinity with the same sign as $x$. Note that this bound is the exact midpoint of $\max \mathbb{F}$ and $\beta^{e_{\max}+1}$.

---

3Also known as monotonic, (weakly) increasing, (weakly) order-preserving, or isotone.

4The two tie-breaking rules used in IEEE 754 are “ties to even” (choose the result with an even integral significand) and “ties away from zero” (choose the larger magnitude result). The former is the default and must be supported; the second one need only be available for decimal systems.
We shall denote by \(\text{fl}\) an arbitrary nondecreasing faithful rounding function. This condition is satisfied by most practical rounding modes, including RD, RU, and rounding to nearest as defined above. The following properties hold for all \(x \in \mathbb{R}\):

\[
\begin{align*}
(36) & \quad x \in \mathbb{F} \text{ if and only if } \text{fl}(x) = x \quad \text{(exactness)} \\
(37) & \quad \text{fl}(\text{fl}(x)) = \text{fl}(x) \quad \text{(idempotence)}
\end{align*}
\]

Throughout this paper, we will frequently use the preimages of floating-point intervals under rounding. The following results are basic and will be used frequently:

**Lemma 2.15.** Let \(X \subseteq \mathbb{F}\). If \(X\) is a nonempty floating-point interval, then \(\text{fl}^{-1}[X]\) is an extended real interval such that \(\text{fl}^{-1}[X] \supseteq [\min X, \max X]\).

*Proof.* Suppose \(X\) is a nonempty floating-point interval, and suppose to the contrary that \(\text{fl}^{-1}[X]\) is not an interval. Then there are some \(a, b \in \text{fl}^{-1}[X]\) and \(y \in \mathbb{F}\) such that \(a < y < b\) and \(y \notin \text{fl}^{-1}[X]\). Since \(\text{fl}\) is nondecreasing, it follows that \(\text{fl}(a) \leq \text{fl}(y) \leq \text{fl}(b)\). However, since \(\text{fl}(a), \text{fl}(b) \in X\) where \(X\) is a floating-point interval, we thus have \(\text{fl}(y) \in X\). Contradiction! Hence \(\text{fl}^{-1}[X]\) is an interval.

Now, since \(\text{fl}\) is the identity over the floating-point numbers, \(\text{fl}(x) = x\) for all \(x \in X\) and hence \(\text{fl}^{-1}[X] \supseteq X\). Since \(\mathbb{F}\) is finite, \(X\) is also finite, and hence \([\min X, \max X]\) is the smallest interval containing \(X\) as a subset. Therefore, since \(\text{fl}^{-1}[X]\) is an interval, it follows that \(\text{fl}^{-1}[X] \supseteq [\min X, \max X]\). \(\square\)

Specifically, we have the following for preimages of intervals under RU and RD:

**Lemma 2.16.** For all nonempty floating-point intervals \(I \subseteq \mathbb{F}\),

- \(\text{RD}^{-1}[I] = [\min I, \max I] \cup (\max I, (\max I)^+)\),
- \(\text{RU}^{-1}[I] = ([\min I], \min I) \cup [\min I, \max I]\).

**Corollary 2.16.1.** For all \(x \in \mathbb{F}\), \(\text{RD}^{-1}[[x]] = \{x\} \cup (x^+, x)\) and \(\text{RU}^{-1}[[x]] = (x^-, x) \cup \{x\}\).

*Remark.* We can usually write the above as half-open intervals, but we must use caution if infinities are included. For instance, \(\text{RD}^{-1}[[+\infty]] = \{+\infty\}\), but \([+\infty, (+\infty)^+] = [+\infty, +\infty) = \emptyset\).

Similarly, the following holds for the preimages of intervals under RN:

**Lemma 2.17.** For all nonempty floating-point intervals \(I \subseteq \mathbb{F}\), we have \(\text{RN}^{-1}[I] \supseteq L \cup [\min I, \max I] \cup R\), where

\[
L = \begin{cases} 
\emptyset & \text{if } \min I \leq \min \mathbb{F} \text{ or } \min I = +\infty, \\
((\min I)^{-} + \min I)/2, \min I & \text{otherwise.}
\end{cases}
\]

\[
R = \begin{cases} 
\emptyset & \text{if } \max I \geq \max \mathbb{F} \text{ or } \max I = -\infty, \\
(\max I, (\max I + (\max I)^+)/2) & \text{otherwise.}
\end{cases}
\]

*Remark.* Since the definition of RN does not specify what rounds to \(+\infty\) or \(-\infty\), we can only say with certainty that \(\text{RN}^{-1}[[+\infty]] \supseteq \{+\infty\}\) and that \(\text{RN}^{-1}[[\max \mathbb{F}]] \supseteq ((\max \mathbb{F})^+ - \max \mathbb{F})/2, \max \mathbb{F}\).
To ensure the generality of our results, we define floating-point multiplication in terms of $\text{fl}$ rather than fixing a specific rounding function:

**Definition 2.18** (Multiplication). The floating-point multiplication operator $\otimes$ is defined by $x \otimes y = \text{fl}(xy)$ for all $x, y \in \mathbb{F}$ such that the product $xy$ is defined.

Note that the extended real product is defined if and only if the factors do not include both zero and an infinity. In particular, the product of an extended real with a nonzero (finite) real number is always defined. As a consequence, however, $\mathbb{R}$ is not closed under multiplication, and so neither is $\mathbb{F}$. In fact, neither $\mathbb{F}^*$ nor $\mathbb{F}$ are necessarily closed under floating-point multiplication, since it is possible for a sufficiently large product to round to infinity.

However, since $\text{fl}$ is nondecreasing, we can still carry over some important ordering properties from the reals.

**Lemma 2.19.** Let $a \in \mathbb{F}^*$ and $x, y \in \mathbb{F}$ such that $x \leq y$. If $a > 0$, then $a \otimes x \leq a \otimes y$. If $a < 0$, then $a \otimes y \leq a \otimes x$.

Since $\mathbb{F}$ is a subset of $\mathbb{R}$, it is totally ordered by the usual order relation. As such, there is a simple definition of an interval over $\mathbb{F}$:

**Definition 2.20** (Floating-point interval). A floating-point interval is any set $X \cap \mathbb{F}$ where $X$ is an extended real interval.

With the exception of $[-\infty, +\infty] \cap \mathbb{F}$, due to the gaps between floating-point numbers, every floating-point interval has many representations in terms of extended real intervals. However, since $\mathbb{F}$ is finite, each of its nonempty subsets has a minimum and a maximum, making it trivial to find the smallest interval containing it. For all $a, b \in \mathbb{R}$, we have the following:

\begin{align*}
(38) & \quad \text{if } a, b \in \mathbb{F} \text{ and } a \leq b, \text{ then } a, b \in [a, b] \cap \mathbb{F} \quad \text{(tightness)} \\
(39) & \quad [a, b] \cap \mathbb{F} = [\text{RU}(a), \text{RD}(b)] \cap \mathbb{F} \quad \text{(rounding)}
\end{align*}

That is, if the bounds are in $\mathbb{F}$, they are necessarily the tightest possible bounds, and intersecting an interval with $\mathbb{F}$ corresponds to rounding its bounds “inward”.

Having defined the necessary terminology and structures, we can now properly discuss the problem.

2.2. **Objective.** To phrase the problem formally, we shall describe the bounds on the variables using sets instead—specifically, intervals. Thus, if we consider $x, y$ and $z$ to be more generally drawn from some sets $X$, $Y$ and $Z$, we are interested in finding the least and the greatest values of $x$ (or $y$ or $z$) such that $x \otimes y = z$ is satisfiable. Or, in other words, if we consider the set $V = \{(x, y, z) \mid x \in X, y \in Y, z \in Z, x \otimes y = z\}$, what is the smallest three-dimensional “box” containing $V$?

We can state the problem precisely thus:

**Problem A** (restatement). Let $X, Y, Z$ be nonempty floating-point intervals, and let $X', Y', Z'$ be sets such that

\begin{align*}
X' & = \{x \in X \mid \exists y \in Y \exists z \in Z (x \otimes y = z)\}, \\
Y' & = \{y \in Y \mid \exists x \in X \exists z \in Z (x \otimes y = z)\}, \\
Z' & = \{z \in Z \mid \exists x \in X \exists y \in Y (x \otimes y = z)\}.
\end{align*}

Given the endpoints of $X$, $Y$ and $Z$, how can we compute the minimum and maximum of each of $X'$, $Y'$ and $Z'$ in constant time?
Having formally defined the problem, we will now simplify it.

3. Simplifying the problem

As shown in Example 1.1, we can use real division to correctly improve the bounds on the factors of a floating-point product. Although it did not find optimal bounds immediately, the division-based approach shown in the example is still essential to solving the problem efficiently. Therefore, we shall first informally discuss how to generalize it to arbitrary combinations of bounds.

3.1. The division-based algorithm, informally. In the example at the start, we reasoned in terms of inequalities. This was convenient for exposition, but it makes expressing a general algorithm difficult. Instead, we will now discuss in terms of sets. Let \( X, Y, Z \subseteq \mathbb{F} \) be nonempty floating-point intervals and let \( x \in X \).

Suppose we have \( x \otimes y \in Z \) for some \( y \in Y \). First, note that since floating-point multiplication is exactly rounded, we can simplify by observing that \( x \otimes y \in Z \) if and only if \( xy \in \text{fl}^{-1}[Z] \). This brings us two benefits. Firstly, we recover the use of division: if \( y \) is nonzero and finite, then \( xy \in \text{fl}^{-1}[Z] \) if and only if \( x \in \text{fl}^{-1}[Z]/y \).

Secondly, since \( Z \) is a floating-point interval, it follows that \( \text{fl}^{-1}[Z] \) is an (extended real) interval, and thus we can make use of some established results in interval arithmetic. For the sake of simplicity, we will assume all of \( Y \) to be nonzero and finite. Given this, we can eliminate the quantifier by noting that \( x \in \text{fl}^{-1}[Z]/Y \) for some \( y \in Y \) if and only if \( x \in \text{fl}^{-1}[Z]/\Box Y \). It is at this point that our method runs into trouble: since \( Y \) is a floating-point interval, it has many gaps, and so we cannot use real interval arithmetic to efficiently compute the set of quotients. In Example 1.1, we worked with inequalities instead, so we did not see the gaps in \( Y \). Translated in terms of sets, what we did corresponds to enlarging the set of denominators \( Y \) to the extended real interval \( \Box Y = [\min Y, \max Y] \). Since \( \Box Y \) is a superset of \( Y \), it follows that \( \text{fl}^{-1}[Z]/\Box Y \) is a superset of \( \text{fl}^{-1}[Z]/Y \), and so this substitution is sound, though it may not produce an optimal result. Importantly, however, the quotient of two intervals is much easier to compute, as we saw in the example. Furthermore, it has a simple representation, since it must itself be an interval (or a union of two disjoint intervals if we permit \( Y \) to contain zero).

By computing \( \text{fl}^{-1}[Z]/\Box Y \), we essentially solve a relaxation of Problem A. What remains, then, is to shrink this set so that it contains exactly the values of \( x \) such that \( x \otimes y \in Z \) holds for some \( y \in Y \). In Example 1.1 this corresponds to the further refinements we performed. However, the naive approach we used needed many iterations to converge to the optimal result. Thus, our focus for this paper will be to develop an efficient refinement method.

3.2. Sufficient and necessary conditions. In order to simplify the problem, we will need to precisely determine the conditions under which the division-based algorithm produces optimal results. In the second half of Example 1.1 we encountered certain combinations of values for which the equation \( x \otimes y = z \) has no solutions over the floating-point numbers. Unlike with the reals, there is no guarantee that we can find a value for \( x \) satisfying the equality for any given \( y \) and \( z \). The existence of a solution is necessary for any optimal bound, however, and so the following definition will be useful:

**Definition 3.1** (Floating-point factors). A floating-point number \( x \) is a floating-point factor of a floating-point number \( z \) if and only if \( x \otimes y = z \) for some \( y \in \mathbb{F} \).
Given a set \( Z \subseteq \mathbb{F} \), we say that \( x \) is feasible for \( Z \) if and only if \( z \) is a floating-point factor of some \( z \in Z \). The set of all floating-point factors of the members of \( Z \) is denoted \( \text{Feas}(Z) \).

For the sake of brevity, we will typically simply write “factor” when it is clear from context that we are discussing floating-point factors. It is worth noting that sets of factors satisfy the following closure property:

**Lemma 3.2.** For any \( Z \subseteq \mathbb{F} \), the set \( \text{Feas}(Z) \) is closed under negation.

**Proof.** Let \( Z \subseteq \mathbb{F} \). If \( \text{Feas}(Z) \) is empty, it is trivially closed under negation. Suppose \( \text{Feas}(Z) \) is nonempty instead, and let \( x \in \text{Feas}(Z) \). Then there is some \( y \in \mathbb{F} \) such that \( x \otimes y \in Z \). Therefore,

\[
-x \otimes -y = fl(-x \cdot -y) = fl(xy) = x \otimes y \in Z,
\]

and so \(-x \in \text{Feas}(Z)\). \( \square \)

**Corollary 3.2.1.** For any \( x \in \mathbb{F} \) and \( Z \subseteq \mathbb{F} \), we have

\[
\max \{ y \in \text{Feas}(Z) \mid y \leq x \} = -\min \{ y \in \text{Feas}(Z) \mid y \geq -x \}.
\]

**Remark.** The above corollary implies that we only need one procedure to compute both the minimum and maximum of \([a, b] \cap \text{Feas}(Z)\) for any \( a, b \in \mathbb{F} \).

The following two lemmas are crucial. In the first lemma, we give an equivalent condition that is easier to work with. Then, in the second lemma, we relate it to the set of quotients from the division-based algorithm.

**Lemma 3.3.** Let \( x \in \mathbb{F} \), and let \( Y, Z \subseteq \mathbb{F} \) be floating-point intervals. Then there exists \( y \in Y \) such that \( x \otimes y \in Z \) if and only if \( x \) is feasible for \( Z \) and \( x \otimes s \geq \min Z \) and \( x \otimes t \leq \max Z \) for some \( s, t \in Y \).

**Proof.** If \( x \otimes y \in Z \) for some \( y \in Y \), then \( x \in \text{Feas}(Z) \) and \( \min Z \leq x \otimes y \leq \max Z \) trivially. Suppose instead that \( x \) is feasible for \( Z \), and that \( x \otimes s \geq \min Z \) and \( x \otimes t \leq \max Z \) for some \( s, t \in Y \). Then there is some \( w \in \mathbb{F} \) such that \( x \otimes w \in Z \). If \( w \in Y \), then the result follows immediately. Since \( Y \) is a floating-point interval, if instead \( w \notin Y \), then either \( w < \min Y \) or \( w > \max Y \). For the following, note that \( \min Z \leq x \otimes w \leq \max Z \). We shall divide the problem into four cases, depending on the sign of \( x \) and the region \( w \) lies in:

1. Suppose \( w < \min Y \). Then \( w < s \) and \( w < t \).
   (a) Suppose \( x \geq 0 \). Multiplying, we obtain \( x \otimes w \leq x \otimes t \), and thus \( \min Z \leq x \otimes t \leq \max Z \). Since \( Z \) is a floating-point interval, it follows that \( x \otimes t \in Z \).
   (b) Suppose \( x < 0 \). Then, we can multiply \( w < x \) by \( x \) to obtain \( x \otimes w \geq x \otimes s \), and thus \( \max Z \geq x \otimes s \geq \min Z \). Since \( Z \) is a floating-point interval, we have \( x \otimes s \in Z \).

2. Suppose \( w > \max Y \). Then \( w > s \) and \( w > t \).
   (a) Suppose \( x \geq 0 \). Then \( x \otimes w \geq x \otimes s \), and thus \( x \otimes s \in Z \) by the reasoning of case (1)(b).
   (b) Suppose \( x < 0 \). Then \( x \otimes w \leq x \otimes t \), and hence \( x \otimes w \in Z \) by case (1)(a).
Since we have either $x \otimes s \in Z$ or $x \otimes t \in Z$ in all cases, the result follows. \hfill \Box

**Lemma 3.4.** Let $x \in \mathbb{F}$, and let $Y, Z \subseteq \mathbb{F}$ where $Y$ is a floating-point interval. If $\text{fl}(xy) \in Z$ for some $y \in [\min Y, \max Y]$, then $x \otimes s \geq \min Y$ and $x \otimes t \leq \max Z$ for some $s, t \in Y$.

**Proof.** If $Y$ is empty, the result holds trivially. Suppose instead that $Y$ is nonempty and $\text{fl}(xy) \in Z$ for some $y \in [\min Y, \max Y]$. Since $\min Y$ and $\max Y$ are floating-point numbers and $\min Y \leq y \leq \max Y$, by the monotonicity of rounding, we have $\min Y \leq \text{RD}(y) \leq \max Y$ and $\min Y \leq \text{RU}(y) \leq \max Y$. Since $Y$ is a floating-point interval, it thus follows that $\text{RD}(y) \in Y$ and $\text{RU}(y) \in Y$. We shall first deal with the special cases of $x$ being zero or infinite.

Suppose $x = 0$. Since $xy$ is defined by assumption, $y$ must be finite, and hence $xy = 0$. Thus $\text{fl}(xy) = 0$. Since $y$ is finite, at least one of either $\text{RD}(y)$ or $\text{RD}(y)$ is finite, and thus either $x \otimes \text{RD}(y) = 0$ or $x \otimes \text{RU}(y) = 0$. Therefore either $x \otimes \text{RD}(y) = 0$ or $x \otimes \text{RU}(y) = 0$. Since $\text{fl}(xy) = 0$ and $\min Z \leq \text{fl}(xy) \leq \max Z$, the result follows.

Suppose $x = +\infty$ or $x = -\infty$ instead. Since $xy$ is defined by assumption, $y$ must be nonzero and hence either $xy = +\infty$ or $xy = -\infty$. Therefore $\text{fl}(xy) = xy$. Since $y$ is nonzero, at least one of $\text{RU}(y)$ or $\text{RD}(y)$ is nonzero. If $\text{RD}(y)$ is nonzero, then it has the same sign as $y$ and hence $xy = x \text{RD}(y)$. Similarly, if $\text{RU}(y)$ is nonzero, then it has the same sign as $y$ and hence $xy = x \text{RU}(y)$. Therefore either $x \otimes \text{RD}(y) = \text{fl}(xy)$ or $x \otimes \text{RU}(y) = \text{fl}(xy)$, and hence the result.

We now handle the remaining case of $x$ being finite and nonzero. Suppose $x \in \mathbb{F}^*$ instead. Then multiplication by $x$ is always defined. Note that $\min Y \leq y \leq \max Y$ and $\min Z \leq \text{fl}(xy) \leq \max Z$. We shall consider each sign of $x$ separately:

- Suppose $x$ is positive. Multiplying by $x$, it follows that $x \min Y \leq xy \leq x \max Y$.
  
  and by the monotonicity of rounding, we have $x \otimes \min Y \leq \text{fl}(xy) \leq x \otimes \max Y$.

  Combining these bounds with our previous bounds on $\text{fl}(xy)$, we obtain $x \otimes \min Y \leq \text{fl}(xy) \leq \max Z$, $\min Z \leq \text{fl}(xy) \leq x \otimes \max Y$, and hence the result holds.

- Suppose $x$ is negative. Then, similarly, we have $x \otimes \min Y \geq \text{fl}(xy) \geq x \otimes \max Y$.

  Therefore, since $\text{fl}(xy) \in Z$, it follows that $x \otimes \min Y \geq \text{fl}(xy) \geq \min Z$, $\max Z \geq \text{fl}(xy) \geq x \otimes \max Y$,

  and thus we obtain the result.

Since we have proved the result for all possible cases, we are finished. \hfill \Box

**Remark.** The statement here considers $Y$ more generally, but it somewhat obscures the connection with the set of quotients described earlier. Note that if $Y \subseteq \mathbb{F}^*$—that is, all of $Y$ is finite and nonzero—then $\text{fl}(xy) \in Z$ for some $y \in \Box Y$ if and only if $x \in \text{fl}^{-1}[Z] / \Box Y$, where $\Box Y = [\min Y, \max Y]$. This distinction is important,
since $0/0$ and $\infty/\infty$ are undefined. For an example where this equivalence does not hold in general, consider $x \in \mathbb{F}$, $Y = \{0\}$ and $Z = \{0\}$.

Combining the previous two lemmas yields the desired result immediately:

**Lemma 3.5.** Let $x \in \mathbb{F}$, and let $Y, Z \subseteq \mathbb{F}$ be floating-point intervals. Then there exists $y \in Y$ such that $x \otimes y \in Z$ if and only if $x$ is feasible for $Z$ and $\text{fl}(xa) \in Z$ for some $a \in [\min Y, \max Y]$.

This brings us to our first major result. The following theorem breaks the problem of computing optimal bounds into two simpler problems: computing bounds using the division-based algorithm, and then shrinking those bounds to the nearest feasible values.

**Theorem 3.6.** Let $X, Y, Z \subseteq \mathbb{F}$ be nonempty floating-point intervals, and let $R$ and $X'$ be sets such that

$$R = \{ x \in \mathbb{F} | \exists y \in [\min Y, \max Y] (\text{fl}(xy) \in Z) \},$$

$$X' = \{ x \in X | \exists y \in Y (x \otimes y \in Z) \}.$$

Then $X' = [\min X, \max X] \cap R \cap \text{Feas}(Z)$.

**Proof.** Let $x \in \mathbb{F}$. Since $X' \subseteq X \subseteq \mathbb{F}$ by definition, by Lemma 3.5 we have $x \in X'$ if and only if $x \in X$, $x \in \text{Feas}(Z)$ and $x \in R$. Therefore,

$$X' = X \cap R \cap \text{Feas}(Z).$$

Since $X$ is a floating-point interval, it follows that $X = [\min X, \max X] \cap \mathbb{F}$. Since $\text{Feas}(Z) \subseteq \mathbb{F}$, we then have

$$X \cap \text{Feas}(Z) = [\min X, \max X] \cap \text{Feas}(Z),$$

and hence the result. \qed

**Remark.** Note that $R$ is the set of quotients $R = \text{fl}^{-1}[Z]/[\min Y, \max Y]$ whenever $Y \subseteq \mathbb{F}^*$. Also note that the division-based algorithm yields the set $X \cap R$, and so intersecting it with $\text{Feas}(Z)$ suffices to compute optimal bounds.

Theorem 3.6 shows that we can solve Problem A by using the classical division-based algorithm to compute the set of quotients (which is simple for almost any choice of $\text{fl}$) and then intersecting that set with the set of factors. As such, the remainder of this paper will predominantly focus on solving this new problem. More specifically:

**Problem B.** Let $Z \subseteq \mathbb{F}$ be a floating point interval, and let $x \in \mathbb{F}$. What is the least number feasible for $Z$ that is no less than $x$? What is the greatest number feasible for $Z$ that is no greater than $x$?

Note that Theorem 3.6 also highlights that the division-based algorithm alone is sufficient for optimality if and only if the bounds it outputs are feasible. That is, if we consider some sets $X, Z \subseteq \mathbb{F}$ and some set of quotients $Q$, if $\min X \cap Q$ is feasible for $Z$, then $\min X \cap Q \cap \text{Feas}(Z) = \min X \cap Q$. Therefore, the difficult case is when the bounds produced by the division-based algorithm are infeasible.
3.3. Formalizing the division-based algorithm. Before going into detail on computing feasible bounds, we shall give a brief formal overview of the division-based algorithm, and how to use it to compute the set of quotients. Given a fixed denominator, the following result characterizes the generalized set of quotients used in the division-based algorithm, as well as in Lemma 3.4 and Theorem 3.6.

**Lemma 3.7.** Let $x \in \mathbb{R}$ and $Z \subseteq \mathbb{F}$. Then,

$$\{y \in \mathbb{R} \mid \text{fl}(xy) \in Z\} = \begin{cases} \text{fl}^{-1}[Z]/x & \text{if } x \in \mathbb{R}^*, \\ \mathbb{R} & \text{if } x = 0 \text{ and } 0 \in Z, \\ \mathbb{R}^* & \text{if } x \in (-\infty, +\infty) \text{ and } (-\infty, +\infty) \subseteq Z, \\ (0, +\infty) & \text{if } x \in (-\infty, +\infty) \text{ and } x \in Z \text{ and } -x \notin Z, \\ [-\infty, 0) & \text{if } x \in (-\infty, +\infty) \text{ and } x \notin Z \text{ and } -x \in Z, \\ \emptyset & \text{otherwise.} \end{cases}$$

With this lemma, we can compute the full set of quotients by taking the union over all possible denominators. This is trivial for all of the special cases, so we shall instead focus on the most important case of nonzero finite denominators. The next lemma follows immediately from Lemma 3.7.

**Lemma 3.8.** Let $X \subseteq \mathbb{R}^*$ and $Z \subseteq \mathbb{F}$. Then,

$$\{y \in \mathbb{R} \mid \exists x \in X (\text{fl}(xy) \in Z)\} = \text{fl}^{-1}[Z]/X.$$

Since we are working with intervals in this problem, the above lemma allows us to use basic interval arithmetic to compute the set of quotients. However, some care must be taken, since fl$^{-1}[Z]$ is not a closed interval for all $Z \subseteq \mathbb{F}$.

**Lemma 3.9.** Let $X \subseteq \mathbb{F}$ and $Z \subseteq \mathbb{F}$ be nonempty floating-point intervals. If $0 \notin X$, then $\text{fl}^{-1}[Z]/[\min X, \max X]$ is an interval.

**Proof.** Suppose $0 \notin X$ and let $\Box X = [\min X, \max X]$. Then, since $X$ is a floating-point interval, we also have $0 \notin \Box X$, and so $\frac{1}{\Box X} = [1/\max X, 1/\min X]$. Since $Z$ is a floating-point interval, it follows that $\text{fl}^{-1}[Z]$ is an interval. Therefore $\text{fl}^{-1}[Z]/\Box X = \text{fl}^{-1}[Z] \cdot \frac{1}{\Box X}$ is a product of intervals, and is hence itself an interval. \hfill \Box

Note that since the set of quotients computed in the above lemma is an interval, intersecting it with the floating-point numbers produces a floating-point interval.

3.4. Narrowing down infeasibility. Due to Theorem 3.6, we would like to devise a method to find the factors nearest any given non-factor. To that end, we now identify some more practical conditions for feasibility. The following lemma gives a simple and direct test for feasibility, and also shows that floating-point multiplication and division are closely related:

**Lemma 3.10.** Let $x \in \mathbb{F}^*$ and let $Z \subseteq \mathbb{F}$ be a floating-point interval. Let $z \in Z$. Then $x$ is feasible for $Z$ if and only if either $x \odot \text{RD}(z/x) \in Z$ or $x \odot \text{RU}(z/x) \in Z$.

**Proof.** If either $x \odot \text{RD}(z/x) \in Z$ or $x \odot \text{RU}(z/x) \in Z$, then $x$ is trivially feasible for $Z$. For the other half, suppose $x$ is feasible for $Z$. Then $x \odot y \in Z$ for some $y \in \mathbb{F}$. Now, let $Y = \text{fl}^{-1}[Z]/x$. Then $y \in Y$ and $z/x \in Y$, and since $Z$ is a floating-point interval, it follows that $Y$ is an interval. Therefore, if $y \leq z/x$, then $y \leq \text{RD}(z/x) \leq z/x$, and so $\text{RD}(z/x) \in Y$. If $y > z/x$ instead, then $y \geq \text{RU}(z/x) \geq \text{fl}^{-1}[Z]/x$. \hfill \Box
Let \( x \in \mathbb{F}^* \) and \( z \in \mathbb{F} \). Then \( x \) is a floating-point factor of \( z \) if and only if either \( x \otimes \text{RD}(z/x) = z \) or \( x \otimes \text{RU}(z/x) = z \).

Additionally, it is worth mentioning that the above serves as a manual proof of a general statement of the automatically derived “invertibility condition” for multiplication of Brain et al. [3].

Although it is clear that if the set of products is large, there will be many solutions, it is unclear how large exactly it needs to be. The following results give a more precise idea of how its diameter relates to the set of solutions. In the following lemma, we describe the minimum diameter needed to ensure feasibility in all cases.

**Lemma 3.11.** Let \( Z \subseteq \mathbb{F} \) be a floating-point interval, and let \( x \in \mathbb{F}^* \) and \( z \in \mathbb{F} \). If \( |z/x| \leq \max \mathbb{F} \) and \( \text{diam } \text{fl}^{-1}[Z] > |x|\beta^{Q(z/x)} \), then \( x \) is feasible for \( Z \).

**Proof.** If \( z/x \in \mathbb{F} \), then \( x \otimes (z/x) = z \), and hence \( x \) is feasible for \( Z \). Suppose \( z/x \notin \mathbb{F} \) instead, and suppose \( |z/x| \leq \max \mathbb{F} \) and \( \text{diam } \text{fl}^{-1}[Z] > |x|\beta^{Q(z/x)} \). First, note that \( \text{RD}(z/x) \leq z/x \leq \text{RU}(z/x) \), and hence \( z \) lies between \( x \text{RD}(z/x) \) and \( x \text{RU}(z/x) \). More precisely, let \( I = x \cdot [\text{RD}(z/x), \text{RU}(z/x)] \). Then \( z \in I \), and hence \( I \cap \text{fl}^{-1}[Z] \) is nonempty. Since \( Z \) is a floating-point interval, \( \text{fl}^{-1}[Z] \) is an interval.

We shall now show that \( \text{fl}^{-1}[Z] \) is strictly wider than \( I \):

\[
\text{diam } I = |x|(|\text{RU}(z/x) - \text{RD}(z/x)) = |x|\beta^{Q(z/x)}
\]

\[
< \text{diam } \text{fl}^{-1}[Z].
\]

Therefore, since \( I \) and \( \text{fl}^{-1}[Z] \) are overlapping intervals, by Lemma 2.11, it follows that either \( \min I \in \text{fl}^{-1}[Z] \) or \( \max I \in \text{fl}^{-1}[Z] \). Hence either \( x \otimes \text{RU}(z/x) \in \mathbb{Z} \) or \( x \otimes \text{RD}(z/x) \in \mathbb{Z} \), and so \( x \) is feasible for \( Z \). \( \square \)

The next lemma builds on the previous result to give a width independent of the denominator under which virtually all values are feasible.

**Lemma 3.12.** Let \( Z \subseteq \mathbb{F} \) be a floating-point interval, and let \( x \in \mathbb{F}^* \) and \( z \in \mathbb{F} \). If \( \beta^{\text{min}} \leq |z/x| \leq \max \mathbb{F} \) and \( \text{diam } \text{fl}^{-1}[Z] \geq \beta^{Q(z)+1} \), then \( x \) is feasible for \( Z \).

**Proof.** Suppose \( \beta^{\text{min}} \leq |z/x| \leq \max \mathbb{F} \) and \( \text{diam } \text{fl}^{-1}[Z] \geq \beta^{Q(z)+1} \). Then,

\[
|x|\beta^{Q(z/x)} = |x|\beta^{E(z/x) - p + 1}
\]

\[
\leq |x||z/x|\beta^{1-p}
\]

\[
= |z|\beta^{1-p}
\]

\[
< \beta^{E(z)+1}\beta^{1-p}
\]

\[
= \beta^{Q(z)+1}
\]

\[
\leq \text{diam } \text{fl}^{-1}[Z].
\]

Therefore, by Lemma 3.11, \( x \) is feasible for \( Z \). \( \square \)

**Remark.** For any ordinary rounding function (e.g., RD, RU, or RN), the requirement that \( \text{diam } \text{fl}^{-1}[Z] \geq \beta^{Q(z)+1} \) roughly corresponds to \( Z \) containing at least \( \beta \) floating-point numbers with the same exponent as \( z \) (cf. Ziv et al. [8]).
The next lemma provides an even stronger bound, under the condition that the significand of the numerator is no greater in magnitude than the significand of the denominator:

**Lemma 3.13.** Let $Z \subseteq \mathbb{F}$ be a floating-point interval, and let $x \in \mathbb{F}^*$ and $z \in Z$. Let $m_x = x\beta^{-E(x)}$ and $m_z = z\beta^{-E(z)}$. If $\text{diam} \mathbb{F}^{-1}[Z] \geq \beta^{Q(z)}$, $|m_z| \leq |m_x|$, and $\beta^{e_{\min}} \leq |z/x| \leq \max \mathbb{F}$, then $x$ is feasible for $Z$.

**Proof.** Suppose $\beta^{e_{\min}} \leq |z/x| \leq \max \mathbb{F}$. If $|m_z| = |m_x|$, then $|z/x| = \beta^{E(z) - E(x)}$ and hence $z/x \in \mathbb{F}$, so $x$ is trivially feasible for $Z$. Suppose $|m_z| < |m_x|$ instead, and also suppose $\text{diam} \mathbb{F}^{-1}[Z] \geq \beta^{Q(z)}$. Then, by Lemma 2.10 it follows that $E(z/x) = E(z) - E(x) - 1$. Thus,

$$|x|\beta^{Q(z/x)} = |x|\beta^{E(z/x) - p + 1} \leq |x|\beta^{E(z) - p} = |m_x|\beta^{Q(z) - 1} < \beta^{Q(z)} \leq \text{diam} \mathbb{F}^{-1}[Z].$$

Therefore, by Lemma 3.11 $x$ is feasible for $Z$. □

**Remark.** Any “ordinary” rounding function almost always satisfies the condition on the diameter on the preimage. Specifically, we always have $\text{diam} \mathbb{F}^{-1} \{(z)\} = \beta^{Q(z)}$ unless $|z|$ is normal and a power of $\beta$. This can be seen in Figure 1; note the different diameters of the preimages of 1 under rounding.

Note that Lemmas 3.11 and 3.13 are both easier to apply when using standard rounding modes, or more specifically, when the preimage under rounding of any floating-point number is not too narrow. Although the usual rounding functions are both faithful and monotonic, and these are indeed very useful properties, they are not enough on their own. To illustrate, let us construct an ill-behaved rounding function which is still both faithful and monotonic:
Example 3.14. Let $f : \mathbb{R} \to \mathbb{F}$ be a function such that

$$ f(x) = \begin{cases} 
    \text{RD}(x) & \text{if } x \leq \max \mathbb{F}, \\
    +\infty & \text{otherwise}.
\end{cases} $$

Since RD is nondecreasing and faithful, it is easy to see that $f$ is also nondecreasing and faithful. However, the only real that it rounds to $\max \mathbb{F}$ is $\max \mathbb{F}$ itself. That is, we have $f^{-1}(\{\max \mathbb{F}\}) = \{\max \mathbb{F}\}$, and so the preimage under rounding has zero diameter. Further, given $x, y \in \mathbb{F}$, we have $x \otimes y = \max \mathbb{F}$ if and only if $xy = \max \mathbb{F}$ exactly. This means that finding the floating-point factors of $\max \mathbb{F}$ requires integer factorization. Specifically, since $\max \mathbb{F} = (\beta^p - 1)\beta^{q_{\text{max}}}$, the floating-point factors of $\max \mathbb{F}$ are exactly the floating-point numbers with a nonnegative exponent whose integral significands divide $\beta^p - 1$.

As demonstrated above, faithfulness and monotonicity are not enough to ensure that a rounding function has a tractable preimage. We therefore restrict our attention to a better-behaved class of rounding functions.

Definition 3.15 (Regular rounding). A rounding function $f$ is regular if it is faithful, monotonic, and $\text{diam } f^{-1}(\{x\}) \geq \beta^{Q(x) - k}$ where $k = 1$ if $|x\beta^{-E(x)}| = 1$ or $k = 0$ otherwise for all $x \in \mathbb{F}^*$.

Remark. Note that we do not require the preimages under rounding of $0$, $+\infty$, and $-\infty$ to have a minimum diameter. This is because they already have every nonzero floating-point number as a floating-point factor, and so we do not need their preimages at all.

Given this definition, the following is then straightforward (albeit tedious) to prove:

Lemma 3.16. RD and RU are regular rounding functions.

However, RN is not necessarily regular, since it is unclear what rounding to nearest means for numbers outside $[\min \mathbb{F}, \max \mathbb{F}]$ and thus our definition does not fully specify the preimages of $\min \mathbb{F}$ and $\max \mathbb{F}$ under RN. For instance, we could choose RN such that $RN^{-1}(\{x\}) = ([x^- + x]/2, x]$ when $x = \max \mathbb{F}$. This is clearly not regular, since $\text{diam } RN^{-1}(\{x\}) = \beta^{q_{\text{max}}}/2$. However, this inconvenience is not as important as it may seem. As mentioned earlier, IEEE 754-style rounding to nearest specifies that a number rounds to an infinity if and only if its magnitude is no less than $(\max \mathbb{F} + \beta^{q_{\text{max}} + 1})/2$. This in turn means that the preimages of $\min \mathbb{F}$ and $\max \mathbb{F}$ under rounding have a diameter of $\beta^{q_{\text{max}}}$, as required for regularity. In particular, the preimages of $\min \mathbb{F}$ and $\max \mathbb{F}$ under any regular rounding to nearest are at least as large as those under IEEE 754-style rounding:

Lemma 3.17. RN is a regular rounding function if and only if $RN^{-1}(\{+\infty\}) \subseteq I$ and $RN^{-1}(\{-\infty\}) \subseteq -I$ where $I = ([\max \mathbb{F} + \beta^{q_{\text{max}} + 1}]/2, +\infty]$.

Therefore, since IEEE 754-style systems are virtually the only ones in use, we do not consider regularity to be an especially onerous requirement.

Now, under the assumption of regular rounding, we can now substantially refine the conclusion of Lemma 3.13.

Lemma 3.18. Let $x \in \mathbb{F}^*$ and $z \in \mathbb{F}$, and let $m_x = x\beta^{-E(x)}$ and $m_z = z\beta^{-E(z)}$. If $fl$ is regular and $\beta^{e_{\text{min}}} \leq |z/x| \leq \max \mathbb{F}$ and $|m_z| \leq |m_x|$, then either $|m_z| = 1$ or $x$ is a floating-point factor of $z$. 

Proof. Suppose \( f_l \) is regular and \( \beta \min \leq |z/x| \leq \max F \). Suppose also that \( |m_z| \neq 1 \). Then, by the definition of regular rounding, we have \( \text{diam} f_l^{-1}\{\{z\}\} \geq \beta Q(z) \). Therefore by Lemma 3.13, it follows that \( x \) is feasible for \( \{z\} \), and hence the result.

Remark. Note that here we show that \( x \) is a factor of \( z \), whereas Lemmas 3.11 and 3.13 only prove that it is feasible for some superset \( Z \).

Combining the previous results, we obtain a much clearer picture of what it means when a number is infeasible:

**Theorem 3.19.** Let \( Z \subseteq F \) be a floating-point interval, and let \( x \in F^* \) and \( z \in Z \). Let \( m_x = x\beta^{-E(x)} \) and \( m_z = z\beta^{-E(z)} \). If \( f_l \) is regular and \( |z/x| \leq \max F \) and \( x \) is not feasible for \( Z \), then either

\[
\begin{align*}
(1) & \quad |z/x| < \beta \min, \\
(2) & \quad 1 = |m_z| < |m_x| \text{ and diam } f_l^{-1}[Z] < \beta Q(z), \text{ or} \\
(3) & \quad |m_x| < |m_z| \text{ and diam } f_l^{-1}[Z] < \beta Q(z)+1.
\end{align*}
\]

Proof. Follows immediately from Lemmas 3.11, 3.13 and 3.18.

Remark. When rounding the quotient produces a subnormal number, as in case (1), the loss of significant digits can drastically magnify the error and make \( x \) infeasible unless \( Z \) is exceptionally wide. This makes subnormal quotients difficult to handle in general, since the difference between \( z \) and \( x \otimes f_l(z/x) \) may be as large as \( x \) itself.

Since we assume \( f_l \) is regular, case (2) is the only instance where \( x \) is infeasible and \( |m_z| \leq |m_x| \). Although it is a very constrained special case, it shall take some work to dispense with, as we will see later. The most common cause of infeasibility is case (3). In this instance, we have \( |m_x| \leq |m_z| \), but \( Z \) is too narrow to provide a direct solution, likely due to containing fewer than \( \beta \) numbers. Note that this means that having more than one number in \( Z \) usually suffices for feasibility when \( \beta = 2 \).

Theorem 3.19 gives a straightforward classification of cases where infeasibility can occur. Although we are not able to give a full treatment of the subnormal quotient case in this paper, we will present an efficient solution for the other two cases. In the next section, we develop criteria for feasibility that are more amenable to a computational solution.

## 4. Quantifying the Error

Roughly speaking, according to Lemma 3.10, a finite nonzero floating-point number is feasible for some set \( Z \) if and only if we can undo the division using multiplication. In this section, we will study the rounding error in this “round-trip” and derive bounds that will allow us to find floating-point factors efficiently.

### 4.1. The mod operator.

We shall make use of a generalized notion of the remainder of a division. Given numerator \( n \), denominator \( d \), and quotient \( q \), a remainder \( r \) satisfies \( n = qd + r \) and \( |r| < |d| \). These quantities are traditionally integers, but we generalise the operation to apply to real numbers. Fixing \( n \) and \( d \), we shall determine the remainder \( r \) by choosing the quotient \( q = \lfloor n/d \rfloor \). Thus, we define the binary operator \( \text{mod} \) as follows:
Definition 4.1 (Remainder of floored division). For any numerator \( x \in \mathbb{R} \) and denominator \( y \in \mathbb{R}^* \), we define \((x \text{ mod } y) = x - y\lfloor x/y \rfloor\).

As the notation suggests, there is a close connection with modular arithmetic, which the next two lemmas will illustrate. Firstly, the remainder is periodic in the numerator, with a period equal to the denominator. That is, we can add or subtract integer multiples of the denominator to the numerator without affecting the result:

Lemma 4.2. For any \( x \in \mathbb{R} \), \( y \in \mathbb{R}^* \), and \( n \in \mathbb{Z} \), \((x + ny \text{ mod } y) = (x \text{ mod } y)\).

Proof. Follows directly by a straightforward computation. \(\square\)

Secondly, if we restrict ourselves wholly to the integers and fix the denominator, equality of remainders is equivalent to congruence:

Lemma 4.3. For all \( a, b \in \mathbb{Z} \), \( n \in \mathbb{Z}^* \), we have \((a \text{ mod } n) = (b \text{ mod } n)\) if and only if \(a \equiv b \pmod{|n|}\).

Proof. If \((a \text{ mod } n) = (b \text{ mod } n)\), then \(a - n\lfloor a/n \rfloor = b - n\lfloor b/n \rfloor\), and hence \(a \equiv b \pmod{|n|}\). Suppose instead that \(a \equiv b \pmod{|n|}\). Then there exist \(p, q, r \in \mathbb{Z}\) such that \(a = pn + r\) and \(b = qn + r\). As \((pn + r \text{ mod } n) = (qn + r \text{ mod } n) = (r \text{ mod } n)\), the result follows. \(\square\)

As might be expected from being so closely related to congruence, the mod operator has many useful properties, though we shall only need a few. The following properties hold for all \( x \in \mathbb{R} \) and \( a, y \in \mathbb{R}^* \):

\[
\begin{align*}
(40) & \quad |(x \text{ mod } y)| < |y| \quad \text{(upper bound)} \\
(41) & \quad (ax \text{ mod } ay) = a(x \text{ mod } y) \quad \text{(distributive law)} \\
(42) & \quad \text{if } x, y \in \mathbb{Z}, \text{ then } (x \text{ mod } y) \in \mathbb{Z} \quad \text{(integrality)} \\
(43) & \quad \text{if } (x \text{ mod } y) \neq 0, \text{ then } (-x \text{ mod } y) = y - (x \text{ mod } y) \quad \text{(complement)}
\end{align*}
\]

It is also worth noting that the remainder (when nonzero) has the same sign as the denominator:

\[
(44) \quad y(x \text{ mod } y) \geq 0
\]

Finally, the following results simplify some common cases of the numerator:

\[
\begin{align*}
(45) & \quad (0 \text{ mod } y) = 0 \\
(46) & \quad (x \text{ mod } y) = 0 \text{ if and only if } x = ny \text{ for some } n \in \mathbb{Z} \\
(47) & \quad (x \text{ mod } y) = x \text{ if and only if } 0 \leq x/y < 1 \\
(48) & \quad (x \text{ mod } y) = x \text{ if and only if } xy \geq 0 \text{ and } |x| < |y|
\end{align*}
\]

The proofs of these statements are simple and follow easily from the definition of mod. With these results, we can now formulate the rounding error in RD and RU in terms of mod.

4.2. Rounding error. When rounding a real number, we are often forced to approximate it due to the limitations of the number system. For systems as complicated as the floating-point numbers, the error in this approximation—the rounding error—can seem unpredictable. The notion is simple: given a rounding function \( f \), the rounding error of some \( x \in \mathbb{R} \) with respect to \( f \) is just \( f(x) - x \). However, it
is difficult to tell immediately from this definition what values the rounding error can take and how it varies with x.

We are principally interested in studying rounding error with respect to fl. This is helpful, since we have defined fl to be a faithful (and nondecreasing) rounding function, and thus fl(x) is always either RD(x) or RU(x). Therefore, the rounding error with respect to fl must either be the rounding error with respect to RD or with respect to RU. We shall begin by deriving arithmetic expressions for RD(x) and RU(x).

**Lemma 4.4.** Let $x \in \mathbb{R}$. If $|x| \leq \max \mathbb{F}$, then $RD(x) = x - (x \mod \beta^Q(x))$ and $RU(x) = x + (-x \mod \beta^Q(x))$.

**Proof.** Suppose $|x| \leq \max \mathbb{F}$, and let $q = Q(x)$ and $M = x\beta^{-q}$. Then $q_{\text{min}} \leq q \leq q_{\text{max}}$, and thus if $|\lfloor M \rfloor| < \beta^p$, then we have $|M|\beta^{q} \in \mathbb{F}$ by definition. In particular, if $q = q_{\text{max}}$, then $|x| \leq \max \mathbb{F} = (\beta^p - 1)\beta^q$, and multiplying by $\beta^{-q}$ and taking the floor we obtain $|\lfloor M \rfloor| \leq \beta^p - 1$ and so $|M|\beta^{q} \in \mathbb{F}$. If $|M| = -\beta^p$ and $q < q_{\text{max}}$ instead, then $|M|\beta^{q} = \beta^{q-1}\beta^{q+1}$, and so again $|M|\beta^{q} \in \mathbb{F}$. Since $|M| < \beta^p$, we have $-\beta^p \leq |M| < \beta^p$, and we have thus shown that $|M|\beta^{q} \in \mathbb{F}$ in all cases.

Now, since $|M| < M$, multiplying by $\beta^q$, we obtain $|M|\beta^{q} \leq x$. Then, since $|M|\beta^{q}$ is a floating-point number, by the monotonicity of rounding, we have

$$RD(|M|\beta^{q}) = |M|\beta^{q} \leq RD(x).$$

We shall now prove the other direction of this inequality. We begin by showing that $E(x) \leq E(RD(x))$:

- If $x \leq 0$, we have $|x| \leq |RD(x)|$ and thus $E(x) \leq E(RD(x))$.
- If $0 < x < \beta^e_{\text{min}}$, then $0 \leq RD(x) < \beta^e_{\text{min}}$ and thus $E(x) = E(RD(x)) = e_{\text{min}}$ by definition.
- If $x \geq \beta^e_{\text{min}}$ instead, then $\beta^{E(x)} \leq x$, so $RD(\beta^{E(x)}) = \beta^{E(x)} \leq RD(x)$ and thus $E(x) \leq E(RD(x))$.

Now, let $\hat{q} = Q(RD(x))$ and $\hat{M} = RD(x)\beta^{-\hat{q}}$, and let $k = E(RD(x)) - E(x)$. Clearly, $k$ is a positive integer and $k = \hat{q} - q$. Therefore, since $\hat{M}$ is the integral significand of RD(x), we have

$$RD(x)\beta^{-q} = RD(x)\beta^{-(\hat{q} - k)} = \hat{M}\beta^k,$$

and so it follows that $RD(x)\beta^{-q}$ is also an integer. Since RD(x) $\leq x$, multiplying by $\beta^{-q}$, we obtain $RD(x)\beta^{-q} \leq M$. Taking the floor and multiplying by $\beta^q$ we then obtain $|RD(x)\beta^{-q}|\beta^q = RD(x) \leq |M|\beta^q$, and so $RD(x) = |M|\beta^q$. Therefore,

$$x - (x \mod \beta^Q(x)) = \beta^Q(x)\lfloor x\beta^{-Q(x)} \rfloor = RD(x).$$

Then, since $RU(x) = -RD(-x)$, the result follows immediately. \hfill \square

### 4.3. Round-trip error in terms of mod.

For any given floating-point numerator $z$ and denominator $x$, any error in attempting to “round-trip” multiplication using division stems from rounding the quotient $z/x$. This is easy to show: since fl is assumed to be faithful, if $z/x$ is also a floating-point number, then $x \otimes fl(z/x) = x$ exactly. Consequently, any instance where $x \otimes fl(z/x) \neq x$ implies that $z/x$ is not exactly representable. This is the common case, since the quotient of two floating-point numbers rarely fits into the available number of digits. However, the product is also rounded, so an inexact quotient may still be close enough. For instance, if we round downward, it is sufficient that the product lies in $[z, z^+]$. A number of
Figure 2. Graphs of exact products of floating-point divisors $x$ and rounded quotients $RD(z/x)$ and $RU(z/x)$ where $(\beta, p, e_{\min}, e_{\max}) = (10, 2, -1, 1)$ and $z = 2.5$. The x-axis shows all positive finite floating-point numbers, spaced equally.

such examples can be seen in Figure 2 as well as a few instances where the product has no error (note that the graph is centered on $z$). Other patterns are visible as well, such as periodic behavior and a reduction in error whenever the significand of $x$ is greater than the significand of $z$.

To identify which divisors produce quotients with sufficiently small errors, we shall first need to derive more tractable expressions for the product of the divisor and the rounded quotient:

**Lemma 4.5.** Let $x \in \mathbb{R}^+$ and $z \in \mathbb{R}$. If $|z/x| \leq \max F$, then $x \ RD(z/x) = z - (z \ mod \ x\beta^{Q(z/x)})$ and $x \ RU(z/x) = z + (-z \ mod \ x\beta^{Q(z/x)})$. 
Proof. Suppose $|z/x| \leq \max \mathbb{F}$. Then, by Lemma 4.4 and the distributivity of multiplication over mod,

$$x \text{RD}(z/x) = x(z/x - (z/x \mod \beta^Q(z/x))) = z - x(z/x \mod \beta^Q(z/x)) = z - (z \mod x\beta^Q(z/x)).$$

The proof for $x \text{RU}(z/x)$ is identical, mutatis mutandis. □

Remark. Note that $x\beta^Q(z/x)$ also appears in the bound in Lemma 3.11.

Lemma 4.6. Let $Z \subseteq \mathbb{F}$, and let $x \in \mathbb{F}^*$ and $z \in Z$. Let $I = \text{fl}^{-1}[Z] - z$. If $|z/x| \leq \max \mathbb{F}$, then

1. $x \otimes \text{RD}(z/x) \in Z$ if and only if $- (z \mod x\beta^Q(z/x)) \in I$, and
2. $x \otimes \text{RU}(z/x) \in Z$ if and only if $(-z \mod x\beta^Q(z/x)) \in I$.

Proof. Suppose $|z/x| \leq \max \mathbb{F}$, then $z \in \mathbb{F}$, and thus Lemma 4.5 applies. Now, since $x \in \mathbb{F}^*$ and $z$ is finite, for any $y \in \mathbb{F}$, we have $x \otimes y \in \mathbb{F}$ if and only if $xy - z \in \text{fl}^{-1}[Z] - z$, so the result follows directly from the equalities of Lemma 4.5. □

Of particular note is that Lemma 4.5 makes the effect of the quotient’s rounding error obvious by directly giving an error term. This is useful, but it is inconvenient that both the numerator and denominator of the quotient appear on the right of the mod. In order to make this more tractable, we shall need to separate them.

Lemma 4.7. Let $x \in \mathbb{R}^*$ and $z \in \mathbb{R}$, and let $m_x = x\beta^{-E(x)}$. Then $x\beta^Q(z/x) = m_x\beta^Q(z)+k$, where $k = E(z/x) - (E(z) - E(x))$.

Proof. By the definition of $Q$ and $E$,

$$Q(z) + k = Q(z) + E(z/x) - E(z) + E(x) = E(z/x) + E(x) - p + 1 = Q(z/x) + E(x).$$

Therefore, by the definition of $m_x$,

$$m_x\beta^{Q(z)+k} = m_x\beta^{Q(z)}\beta^{E(z/x)} = m_x\beta^{E(x)}\beta^Q(z/x) = x\beta^Q(z/x),$$

and so the result follows. □

Remark. From Lemma 2.10 if $|z/x| \geq \beta^{e_{\min}}$, then $k = \log_\beta \text{ufp}(m_z/m_x)$, where $m_z = z\beta^{-E(z)}$. In that case, we have $k = 0$ if and only if $1 \leq |m_z/m_x| < \beta$. Note that if $1 \leq |m_z| \leq |z| < \beta$, then $1 \leq |m_z/m_x| < \beta$. Additionally, $|m_z| \geq 1$ if and only if $|x| \geq \beta^{e_{\min}}$.

With the above result, we can express the error due to the quotient in terms of integer remainder. The problem of finding factors then becomes that of finding integer divisors such that this remainder (and, consequently, the error) is small enough to be negligible.
Lemma 4.8. Let $Z \subseteq \mathbb{F}$, and let $x \in \mathbb{F}^*$ and $z \in Z$. Let $M_x = x \beta^{-Q(x)}$ and $M_z = z \beta^{-Q(z)}$. Let $I = (\lfloor z^{-1} | Z | - z \rfloor) \beta^{p-1-Q(z)}$ and $k = E(z/x) - (E(z) - E(x))$. If $|z/x| \leq \max \mathbb{F}$, then

1. $x \otimes RD(z/x) \in Z$ if and only if $-(M_z \beta^{p-1} \mod M_z \beta^k) \in I$, and
2. $x \otimes RU(z/x) \in Z$ if and only if $-(M_z \beta^{p-1} \mod M_z \beta^k) \in I$.

Proof. Let $I' = \lfloor z^{-1} | Z | - z \rfloor$ and $m_x = x \beta^{-E(x)}$. By Lemma 4.7 we have $x \beta^{Q(z)/x} = m_x \beta^{Q(z) + k}$. Factoring using the distributivity of multiplication over mod, we obtain

\[(z \mod m_x \beta^{Q(z) + k}) = (M_z \beta^{p-1} \mod M_z \beta^k) \beta^{Q(z) + 1 - p},\]

\[(-z \mod m_x \beta^{Q(z) + k}) = (-M_z \beta^{p-1} \mod M_z \beta^k) \beta^{Q(z) + 1 - p}.\]

Therefore, since $I = I' \beta^{p-1-Q(z)}$,

1. $-(z \mod m_x \beta^{Q(z) + k}) \in I'$ if and only if $-(M_z \beta^{p-1} \mod M_z \beta^k) \in I$, and
2. $(-z \mod m_x \beta^{Q(z) + k}) \in I'$ if and only if $(-M_z \beta^{p-1} \mod M_z \beta^k) \in I$.

Hence, if $|z/x| \leq \max \mathbb{F}$, then the result holds by Lemma 4.6. \qed

Lemmas 3.10 and 4.8 give us all the tools we need to transform this into an integer optimization problem. However, the conditions involved are verbose; to avoid excessive repetition, we introduce the notion of “plausibility”.

Definition 4.9 (Plausibility). Let $Z \subseteq \mathbb{F}$, and let $z \in Z$ be finite. A nonzero integer $M$ is $z$-plausible for $Z$ if and only if either $-(M_z \beta^{p-1} \mod M \beta^k) \in I$ or $(-M_z \beta^{p-1} \mod M \beta^k) \in I$, where $M_z = z \beta^{-Q(z)}$, $I = (\lfloor z^{-1} | Z | - z \rfloor) \beta^{p-1-Q(z)}$, and $k = \log_\beta \text{ufp}(M_z/M)$. The set of such integers is $\text{Plaus}_Z(z)$.

Remark. The language and conditions here are almost identical with Lemma 4.8. However, note that the $k$ here differs, in that it effectively assumes that $|z/x| \geq \beta^{-\text{emin}}$. This is for the sake of simplicity, as we cannot handle the case of $|z/x| < \beta^{-\text{emin}}$ at this time.

We now show the connection between plausibility and feasibility:

Lemma 4.10. Let $Z \subseteq \mathbb{F}$, and let $x \in \mathbb{F}^*$, $z \in Z$ and $M, q \in \mathbb{Z}$. If $x = M \beta^q$ and $\beta^{-\text{emin}} \leq |z/x| \leq \max \mathbb{F}$, then $x$ is feasible for $Z$ if and only if $M$ is $z$-plausible for $Z$.

Proof. Suppose $x = M \beta^q$ and $\beta^{-\text{emin}} \leq |z/x| \leq \max \mathbb{F}$, and let $M_x = x \beta^{-Q(x)}$. Since $x$ is nonzero by assumption, $M$ is also nonzero. Since $|z/x| \leq \max \mathbb{F}$, $z$ is finite. Let $k_1 = E(z/x) - (E(z) - E(x))$ and $k_2 = \log_\beta \text{ufp}(M_z/M)$. Then, by Lemma 2.10 we have $k_1 = \log_\beta \text{ufp}(M_z/M_x)$. Now,

\[M_x = x \beta^{-Q(x)} = M \beta^q - Q(x),\]

and therefore

\[k_1 = \log_\beta \text{ufp}(M_z/M_x) = \log_\beta \text{ufp}\left(\frac{M_z}{M \beta^{q-Q(x)}}\right) = \log_\beta(\text{ufp}(M_z/M) \beta^{Q(x)-q}) = k_2 + Q(x) - q.\]

And hence,

\[M_x \beta^{k_1} = M \beta^{q-Q(x)} \cdot \beta^{k_2+Q(x)-q} = M \beta^{k_2}.\]
Therefore, by Lemma 4.18 and the definition of plausibility, \( x \) is feasible for \( Z \) if and only if \( M \) is \( z \)-plausible for \( Z \).

\[ \square \]

**Remark.** Plausibility roughly corresponds to feasibility if the exponents were unbounded: if \( e_{\text{min}} = -\infty \) and \( e_{\text{max}} = +\infty \), then \( \sup E = +\infty \) and \( \beta_{\text{min}} = 0 \), and hence the condition in the lemma is trivially satisfied.

The next lemma provides some values that can serve as bounds on the minimum or maximum of a subset of plausible integers.

**Lemma 4.11.** Let \( Z \subseteq \mathbb{F} \) and let \( z \in Z \). Let \( M_z = z_\beta^{-Q(z)} \). If \( z \in \mathbb{F}^* \), then \((-1)^a M_z^b \beta^c \) is \( z \)-plausible for \( Z \) for all nonnegative \( a, b, c \in Z \) such that \( b \leq 1 \).

**Proof.** Suppose \( z \in \mathbb{F}^* \) and let \( a, b, c \) be nonnegative integers such that \( b \leq 1 \). Let \( M = (-1)^a M_z^b \beta^c \) and \( k = \log_\beta \text{ufp}(M_z/M) \). Then,

\[
k = \log_\beta \text{ufp}(M_z/M)
= \log_\beta \text{ufp} \left( \frac{M_z}{(-1)^a M_z^b \beta^c} \right)
= \log_\beta \text{ufp}(M_z^{1-b}) - c,
\]

and therefore \( M \beta^k = (-1)^a M_z^b \text{ufp}(M_z^{1-b}) = (-1)^a M_z^b \beta^d \) for some integer \( d \).

Hence, if \( b = 1 \), then \((-M_z \beta^{p-1} \mod M \beta^k) = (-M_z \beta^{p-1} \mod (-1)^a M_z) = 0 \).

Suppose \( b = 0 \) instead. Then \( M \beta^k = \lfloor (-1)^a \text{ufp}(M_z) \rfloor = (-1)^a \beta^d \). Since \( |M_z| < \beta^p \), we have \( d \leq p - 1 \). Therefore \(-M_z \beta^{p-1} \mod M \beta^k = 0 \) again. Since \( 0 \in I \) trivially, the result holds by the definition of plausibility.

\[ \square \]

**Corollary 4.11.1.** Let \( Z \subseteq \mathbb{F} \) and let \( z \in Z \). If \( z \in \mathbb{F}^* \), then \( \text{Plaus}_Z(z) \) is not bounded from above or below.

Finally, we arrive at the first of the two most important lemmas in this section. We divide the problem into two cases, as per Theorem 3.19. We first consider the lower half of the problem; that is, when the magnitude of the denominator’s significand is no greater than the numerator’s.

**Lemma 4.12.** Let \( Z \subseteq \mathbb{F} \) be a floating-point interval and let \( x \in \mathbb{F}^* \) and \( z \in Z \).

Let \( M_x = x_\beta^{-Q(x)} \). Let \( M_b = \min \{ M \in \text{Plaus}_Z(z) \mid M \geq M_x \} \). Let \( A = \{ y \in \text{Feas}(Z) \mid y \geq x \} \). If \( \beta_{\text{min}} \leq |z/x| \leq \max \mathbb{F} \) and \( 1 \leq |m_x| \leq |m_z| \), then \( A \) is nonempty and \( \min A = M_b \beta^{Q(x)} \).

**Proof.** Suppose \( \beta_{\text{min}} \leq |z/x| \leq \max \mathbb{F} \) and \( 1 \leq |m_x| \leq |m_z| \). Let \( b = M_b \beta^{Q(x)} \).

We shall begin by showing that \( A \) is nonempty and \( \min A \leq b \). Since \( M_x \leq M_b \) by definition, multiplying by \( \beta^{Q(x)} \) gives \( x \leq b \). Therefore, it suffices to prove that \( b \) is feasible for \( Z \). By Lemma 4.11, \( |M_z| \) and \( -\beta^{p-1} \) are \( z \)-plausible for \( Z \). Therefore, if \( x > 0 \), then \( 0 < M_x \leq |M_z| \) and thus \( M_x \leq M_b \leq |M_z| \) by the definition of \( M_b \).

If \( x < 0 \) instead, then \( M_x \leq M_b \leq -\beta^{p-1} < 0 \) similarly. Therefore \( x \) and \( b \) have the same sign and \( \beta^{p-1} \leq |M_b| \leq |M_z| < \beta^p \) in all cases. Since \( M_b \) is an integer, we thus have \( b = M_b \beta^{Q(x)} \in \mathbb{F}^* \) by definition. Multiplying by \( \beta^{Q(x)} \), we obtain \( \beta^{E(x)} \leq |b| \leq |m_z| \beta^{E(x)} \) and thus \( E(b) = E(x) \). Now, since \( |b| \geq \beta^{E(x)} \), taking the
reciprocal and multiplying by $|z|$, we obtain
\[
|z/b| \leq |m_z|\beta^{E(z)−E(x)} = |M_z|\beta^{Q(z/x)} \leq (\beta^p − 1)\beta^{\epsilon_{\min}} = \max F.
\]
Similarly, since $|b| \leq |m_z|\beta^{E(x)}$, we have
\[
|z/b| \geq \beta^{E(z)−E(x)} = \beta^{E(z/x)} \geq \beta^{\epsilon_{\min}}.
\]
Therefore, since $b = M_b\beta^{Q(z/x)}$ and $\beta^{\epsilon_{\min}} \leq |z/b| \leq \max F$, we have $b \in \text{Plaus}_Z(z)$, by Lemma 4.10 we have $b \in \text{Feas}(Z)$. Since $x \leq b$, we have $b \in A$ and thus $A \leq b$.

Now, let $a = \min A$. To show the equality, we now prove that $b \leq a$. Since $x$ and $b$ have the same sign and $x \leq a$, we have either $0 < x \leq a < b$ or $x \leq a < b < 0$. Therefore, since $E(x) = E(b)$, we have $E(a) = E(x)$. Since $z/a$ lies between $z/x$ and $z/b$ and both $\beta^{\epsilon_{\min}} \leq |z/x| \leq \max F$ and $\beta^{\epsilon_{\min}} \leq |z/b| \leq \max F$, we have $\beta^{\epsilon_{\min}} \leq |z/a| \leq \max F$. Now, let $M_a = a\beta^{−Q(x)}$. Since $E(a) = E(x)$, $M_a$ is an integer. Then, since $a = M_a\beta^{Q(x)} \in \text{Feas}(Z) \cap \mathbb{F}^*$ and $\beta^{\epsilon_{\min}} \leq |z/a| \leq \max F$, by Lemma 4.10 we have $M_a \in \text{Plaus}_Z(z)$. Since $x \leq a$, we have $M_x \leq M_a$ and thus $M_a \geq M_b$ and so $a \geq b$. Therefore $a = b$.

We now consider the upper half, where instead the magnitude of the denominator's significand exceeds the numerator's. Note that in the lemma's case (2), we do not yet determine the exact value. This requires some finesse, and we shall handle it when we combine these two lemmas.

**Lemma 4.13.** Let $Z \subseteq \mathbb{F}$ be a floating-point interval and let $x \in \mathbb{F}^*$ and $z \in Z$. Let $M_x = x\beta^{−Q(x)}$. Let $M_b = \min \{M \in \text{Plaus}_Z(z) \mid M \geq M_x\}$. Let $A = \{y \in \text{Feas}(Z) \mid y \geq x\}$. If $\beta^{\epsilon_{\min}} \leq |z/x| \leq \max F$ and $1 \leq |m_z| < |m_x|$, then

1. If $M_b = \beta^p$ and $E(x) \geq \epsilon_{\max}$, then $A$ is empty.
2. If $M_b = −|m_z|$ and $E(z)−E(x) > \epsilon_{\max}$, then either $A$ is empty or min $A \geq M_b\beta^{Q(x)}$.
3. Otherwise, $A$ is nonempty and min $A = M_b\beta^{Q(x)}$.

**Proof.** Suppose $\beta^{\epsilon_{\min}} \leq |z/x| \leq \max F$ and $1 \leq |m_z| < |m_x|$. Then $\beta^{−1} < |m_z/m_x| < 1$, and thus by Lemma 4.4.10 we have $E(z/x) = E(z) − E(x) − 1$.

We shall first show that either $A$ is empty or min $A \geq b$. Suppose to the contrary that $A$ is nonempty and min $A < b$. Let $a = \min A$ and $M_a = a\beta^{−E(x)}$. Then, since $x \leq a$ by definition, we have $x < a < b$ and so $M_x < M_a < M_b$.

Suppose $x > 0$. Then $0 < x \leq a < b$. Then $E(x) \leq E(a)$ and thus $M_a$ is an integer. By Lemma 4.4.10 we have $\beta^p \in \text{Plaus}_Z(z)$, and since $M_a < \beta^p$, we have $M_b \leq \beta^p$. Therefore $M_x \leq M_a < \beta^p$, and hence $x \leq a < \beta^{E(x)+1}$. Taking the reciprocal and multiplying by $|z|$, we obtain $|z/\beta^{E(x)+1}| < |z/a| \leq |z/x|$. Since $|z/x| \leq \max F$ and
\[
|z/\beta^{E(x)+1}| = |m_z|\beta^{E(z)−E(x)−1} = |m_z|\beta^{E(z/x)} \geq \beta^{E(z/x)} \geq \beta^{\epsilon_{\min}},
\]
it follows that $\beta^{\epsilon_{\min}} < |z/a| \leq \max F$. Therefore, by Lemma 4.10 $M_a$ is $z$-plausible for $Z$. Therefore $M_b \leq M_a$, which is a contradiction.
Suppose $x < 0$ instead. By Lemma 4.11, we have $-|M_z| \in \text{Plaus}_Z(z)$, and since $M_z \leq |M_z|$, we thus have $M_z \leq |M_z|$ by definition. Multiplying by $\beta(x)$, we obtain

$$x \leq a < b \leq -|m_z|\beta(x) \leq -\beta(x) < 0,$$

and thus $E(x) = E(b) = E(a)$. Therefore $M_a$ is the integral significand of $a$. Therefore, since $M_a < M_b \leq |M_z|$, we have $|M_a| > |M_z|$ and thus $|M_z| \leq |M_a| - 1$. Dividing by $|M_a|$ and multiplying by $\beta$, we obtain

$$\left|\frac{M_z\beta}{M_a}\right| \leq \beta - \frac{\beta}{|M_a|} < \beta - 1.$$ 

Therefore,

$$|z/a| = \frac{|m_z\beta(x)|}{|m_a\beta(x)|} = \frac{|m_z/m_a|\beta(x)}{\beta(x)} = \frac{|m_z|m_a}{|m_a|} = \frac{M_z\beta}{M_a} \leq \beta - \beta - 1 < \beta - 1.$$ 

Now, since $x \leq a < b < 0$, we also have $|z/x| \leq |z/a|$ and hence also $\beta^{\min} \leq |z/a|$. Therefore, by Lemma 4.10, $M_a$ is z-plausible for $Z$. Therefore $M_b \leq M_a$, which is again a contradiction, so either $A$ is empty or $b \leq a$. In particular, this means that the second result holds trivially.

We now demonstrate the first result by showing that if $A$ is nonempty and $E(x) \geq e_{\max}$, then $M_b < \beta$. Suppose $A$ is nonempty and $E(x) \geq e_{\max}$. We shall first show that $a \leq \max F$. Let $c \in \{+\infty, -\infty\}$. If $c \in Z$, then $x \otimes (c/x) = c \in Z$, so $x \in \text{Feas}(Z)$ and hence $a = x \leq \max F$. If instead neither $+\infty$ nor $-\infty$ are in $Z$, then $+\infty$ is not feasible for $Z$, so $a \leq \max F$. Therefore $0 < x < b \leq a \leq \max F$ in all cases, and since $E(x) = E(\max F) = e_{\max}$, dividing by $\beta^{\max}$ gives $M_b \leq M_a \leq \beta - 1$, as desired.

Finally, we proceed to the third result. We first show that the conditions imply that $b \in F^*$ and $\beta^{\min} \leq |z/b| \leq \max F$.

If $M_z < 0$, then $-\beta < M_z \leq M_b \leq -|M_z|$ and thus $0 < |M_b| < \beta$, so $b \in F^*$ immediately. Suppose $M_z > 0$. Then $0 < M_z \leq M_b \leq \beta$. If $M_b \neq \beta$, then $0 < |M_b| < \beta$ and hence $b \in F^*$ again. Otherwise, if $M_b = \beta$ and $E(x) < e_{\max}$, then $b = \beta(x) \leq \beta^{\max} = e_{\max}$ and thus $b \in F^*$. Therefore, if either $M_b \neq \beta$ or $E(x) < e_{\max}$ then $b \in F^*$.

If $x > 0$, then $0 < x < b \leq \beta(x) + 1$ and thus $|z/\beta(x) + 1| \leq |z/b| \leq |z/x|$, and since $|z/\beta(x) + 1| \geq \beta^{\min}$, we have $\beta^{\min} \leq |z/b| \leq \max F$.

Suppose $x \leq 0$. Then $x \leq b \leq -|M_z|\beta^{(x)}$ and hence $|z/x| \leq |z/b| \leq \beta^{(x)} - E(x)$. Therefore, if $E(x) - E(x) \leq e_{\max}$, then $\beta^{\min} \leq |z/x| \leq \max F$. If $M_b \neq -|M_z|$ instead, then $M_z \leq M_b < -|M_z|$, and therefore $\beta^{\min} \leq |z/x| \leq |z/b| < \max F$. 


Therefore, if either \( M_b \neq \beta^p \) or \( E(x) < e_{\text{max}} \), and either \( M_b = -|M_z| \) or \( E(z/x) < e_{\text{max}} \), then both \( b \in \mathbb{F}^* \) and \( \beta^\text{min} \leq |z/b| < \max \mathbb{F} \). Therefore, by Lemma 4.10, we have \( b \in \text{Feas}(Z) \), so \( a \leq b \), and hence \( a = b \). \( \square \)

Before we can combine Lemmas 4.12 and 4.13, however, we need the following result in order to give a complete solution. The following lemma essentially states that if a quotient is too large to be feasible, larger quotients must also be infeasible.

**Lemma 4.14.** Let \( Z \subseteq \mathbb{F} \) be a floating-point interval, \( x, y \in \mathbb{F}^* \), and \( z \in Z \). If \( |z/x| > \max \mathbb{F} \) and \( x \) is not feasible for \( Z \) and \( |y| \leq |x| \), then \( y \) is not feasible for \( Z \).

**Proof.** Suppose \( |z/x| > \max \mathbb{F} \) and \( |y| \leq |x| \). Clearly, \( z \) is nonzero. We proceed by contraposition. Suppose \( y \) is feasible for \( Z \). We shall first prove the result for positive \( x \) and \( y \), and then generalize to all cases. Note that since \( y \) is feasible for \( Z \), by Lemma 4.10, we have \( \{ y \otimes \text{RD}(z/y) \} \subseteq Z \) or \( \{ y \otimes \text{RU}(z/y) \} \subseteq Z \).

Suppose \( z > 0 \). Now, note that since \( |z/x| > \max \mathbb{F} \) and \( y < x \), we have \( \text{RD}(z/y) = \text{RD}(z/x) \) or \( \text{RU}(z/y) = \text{RU}(z/x) \). Therefore, if \( y \otimes \text{RD}(z/y) = +\infty \in Z \), then \( x \otimes +\infty = +\infty \in Z \) also, so \( x \) is feasible for \( Z \). If \( y \otimes \text{RD}(z/y) \in Z \) instead, then since \( y \leq x \), multiplying gives

\[
\min Z \leq y \otimes \text{RD}(z/y) \leq x \otimes \text{RD}(z/y) = x \otimes \text{RD}(z/x) \leq z \leq \max Z,
\]

and since \( Z \) is a floating-point interval, it follows that \( x \otimes \text{RD}(z/x) \in Z \). Therefore \( x \) is feasible for \( Z \).

Suppose \( z < 0 \) instead. Then, we similarly have \( \text{RD}(z/y) = -\infty \) and \( \text{RU}(z/y) = \text{RU}(z/x) = \min \mathbb{F} \). Hence, if \( y \otimes \text{RD}(z/y) = -\infty \in Z \), then \( x \otimes -\infty = -\infty \in Z \), and so \( x \) is feasible for \( Z \). If \( y \otimes \text{RU}(z/y) \in Z \) instead, then since \( y \leq x \), multiplying gives

\[
\max Z \geq y \otimes \text{RU}(z/y) \geq x \otimes \text{RU}(z/y) = x \otimes \text{RU}(z/x) \geq z \geq \min Z,
\]

and since \( Z \) is a floating-point interval, it follows that \( x \otimes \text{RU}(z/x) \in Z \). Therefore \( x \) is feasible for \( Z \).

We have thus shown the result for positive \( x \) and \( y \). If either \( x \) or \( y \) is negative instead, then by Lemma 4.12, \( |y| \) is feasible for \( Z \), and by the above result, so is \( |x| \). Therefore, applying Lemma 4.12 again, we find that \( x \) is feasible for \( Z \) and hence the result holds in all cases. \( \square \)

We now give the main result of this section. The following theorem reduces the problem to an optimization problem over plausible integers.

**Theorem 4.15.** Let \( Z \subseteq \mathbb{F} \) be a floating point interval. Let \( x \in \mathbb{F}^* \) and let \( z \in Z \). Let \( A = \{ y \in \text{Feas}(Z) \mid y \geq x \} \) and let \( M_b = \min \{ M \in \text{Plaus}(z) \mid M \geq M_z \} \) and let \( M_w = \min \{ M \in \text{Plaus}(z) \mid M \geq |M_z| \} \), where \( M_x = x \beta^{-Q(x)} \). If \( x \) and \( z \) are normal and \( \beta^\text{min} \leq |z/x| \leq \max \mathbb{F} \), then either \( A \) is empty or \( min A = M_b \beta^{Q(x)} \) or \( min A = M_w \beta^{Q(x)} \).

**Proof.** Let \( m_z = z \beta^{-E(z)} \) and \( m_x = x \beta^{-Q(x)} \), and suppose \( x \) and \( z \) are normal and \( \beta^\text{min} \leq |z/x| \leq \max \mathbb{F} \). Then \( 1 \leq m_x \) and \( 1 \leq m_z \). If \( |m_x| < |m_z| \), then by Lemma 4.12 \( A \) is nonempty and \( min A = M_b \beta^{Q(x)} \).

Suppose \( |m_z| < |m_x| \) instead and let \( b = M_b \beta^{Q(x)} \). Suppose also that \( A \) is nonempty and \( min A \neq b \). Then, by Lemma 4.13 we have \( M_b = -|M_z| \) and
\[ E(z) - E(x) > \epsilon_{\text{max}} \text{ and min } A > b. \] Then \( b = -|M_z| \beta^{Q(z)} \) and so \( b \in \mathbb{F}^* \) by definition and
\[
z/b = -\beta^{E(z) - E(x)} \leq -\beta^{\epsilon_{\text{max}} + 1} < \min \mathbb{F}.
\]

Therefore, by Lemma 4.14, no floating-point numbers in \([b, 0)\) are feasible for \( Z \). Since \( x \leq b < \min A \), we also have that no floating-point numbers are feasible for \( Z \) in \([x, b)\). Furthermore, 0 is also not feasible for \( Z \), as otherwise we would have since \( b \otimes 0 = 0 \in Z \). Therefore no floating-point numbers are feasible for \( Z \) in \([x, 0]\). Hence, by Lemma 3.2, no floating-point numbers are feasible for \( Z \) in \([x, -x]\). Therefore, \( |A| > |x| \), and hence \( \min A = \min \{ y \in \text{Feas}(Z) \mid y \geq |x| \} \). Therefore, applying Lemma 4.13 again, since \( A \) is nonempty by assumption and \( M_z \) is positive, the only remaining possibility is \( \min A = M_z \beta^{Q(x)} \), and so the result follows. \( \square \)

**Remark.** Note that \( M_b = M_z \) whenever \( x \) is positive. Additionally, since the feasible numbers are closed under negation by Lemma 3.2, any procedure for finding the least feasible numbers given a lower bound can also be used to find greatest feasible number given an upper bound.

With this result, we now focus on developing a method to efficiently find the smallest plausible integers, given a lower bound. Whether or not an integer is plausible depends on the rounding function, but in all cases, it is a matter of finding numbers such that an integer remainder lies within a certain interval:

**Theorem 4.16.** Let \( M_x \in Z \) and \( Z \subseteq \mathbb{F} \) be a floating-point interval. Let \( z \in Z \) be normal and let \( M_z = z \beta^{-Q(z)} \). Let \( I = (\lfloor -1 |Z| - z) \beta^{p - 1 - Q(z) - k} \) and
\[
A = \{ M \in \mathbb{Z}^* \mid -(M_z \beta^{-k} \cdot \beta^{p - 1} \mod M) \in I \lor (-M_z \beta^{-k} \cdot \beta^{p - 1} \mod M) \in I \},
\]
where \( k = \log_\beta \text{ufp}(M_z/M_z) \). If \( \beta^{p - 1} \leq |M_z| < \beta^p \), then
\[
\min \{ M \in A \mid M \geq M_x \} = \min \{ M \in \text{Plaus}_Z(z) \mid M \geq M_x \}.
\]

**Proof.** Suppose \( \beta^{p - 1} \leq |M_z| < \beta^p \), and let \( M_b = \min \{ M \in \text{Plaus}_Z(z) \mid M \geq M_x \} \) and \( k_b = \log_\beta \text{ufp}(M_z/M_b) \). We first show that \( M_b \in A \). Since \( z \) is normal, we have \( \beta^{p - 1} \leq |M_z| < \beta^p \). By the definition of plausibility, we have either \(-M_z \beta^{p - 1} \mod M_b \beta^{k_b}) \notin I\beta^k \) or \((-M_z \beta^{p - 1} \mod M_b \beta^{k_b}) \notin I\beta^k \). Thus, by the distributivity of multiplication over mod, it suffices to show that \( k = k_b \) to prove that \( M_b \in A \). We now proceed by cases:

1. Suppose \( |M_x| \leq |M_z| \). Then \( 1 \leq |M_z/M_x| < \beta \), so \( k = 0 \).
   (a) By Lemma 4.11 we have \( |M_z| \in \text{Plaus}_Z(z) \). Therefore, if \( M_x > 0 \), then \( 0 < M_x \leq M_x \leq |M_z| \), so \( 1 \leq |M_z/M_x| \leq |M_z/M_x| < \beta \) and hence \( k_b = k = 0 \).
   (b) By Lemma 4.11 we have \( -\beta^{p - 1} \in \text{Plaus}_Z(z) \). Therefore, if \( M_x < 0 \), then \( -|M_x| \leq M_x \leq M_b \leq -\beta^{-p - 1} \), and hence \( k_b = k = 0 \) again similarly.

2. Suppose \( |M_x| > |M_z| \) instead. Then \( \beta^{-1} \leq |M_z/M_x| < 1 \), and so \( k = -1 \).
   (a) By Lemma 4.11 we have \( \beta^p \in \text{Plaus}_Z(z) \). Therefore, if \( M_x > 0 \), then \( |M_x| < M_x \leq M_x \leq |M_z|, \) so \( \beta^{-1} \leq |M_z/M_x| \leq |M_z/M_x| < \beta^{-1} \) and hence \( k_b = k = -1 \).
   (b) By Lemma 4.11 we have \(-|M_z| \in \text{Plaus}_Z(z) \). Therefore, if \( M_x < 0 \) and \( M_b \neq -|M_z| \), then \( M_x \leq M_b < -|M_z| \), so \( \beta^{-1} \leq |M_z/M_x| \leq |M_z/M_x| < \beta^{-1} \), and hence \( k_b = k = -1 \).
   (c) If \( M_b = -|M_z| \) instead, then \( k_b = 0 > k \), but \( -|M_z| \in A \) trivially.
Therefore, if $M_b \neq -|M_z|$, then $k = k_b$ and so $M_b \in A$ in all cases.

We now proceed in the other direction. Let $M_a = \min \{ M \in A \mid M \geq M_z \}$ and $k_a = \log_2 \text{ufp}(M_z/M_a)$. Then $M_x \leq M_a \leq M_b$, and since $M_x$ and $M_b$ are nonzero and have the same sign, we have $|M_z|/M_b \leq |M_z|/M_a \leq |M_z|/M_x$. Therefore $k_a$ lies between $k$ and $k_b$. If $M_b \neq -|M_z|$, then $k_a = k_b = k$ and hence $M_a \in \text{Plaus}_Z(z)$ by definition. If $M_a = M_b = -|M_z|$, then $M_a \in \text{Plaus}_Z(z)$ trivially. Suppose $M_a < -|M_z|$ instead. Then, similarly to case (2)(b), we have $k = k_a = -1$. Therefore, by the definitions of $A$ and plausibility, the distributivity of multiplication over mod implies that $M_a \in \text{Plaus}(Z)$. Hence $M_a \in \text{Plaus}(Z)$ in all cases, so $M_b \leq M_a$, and therefore $M_a = M_b$ as desired.

Remark. According to Theorem 3.19 if $\text{fl}$ is regular and $|M_z| < |M_x|$, but $x$ is infeasible for $Z$, it follows that $|M_z| = \beta^{p-1}$. Since $k < 0$ if and only if $|M_z| < |M_x|$, under those conditions we therefore have

$$A = \{ M \in Z^* \mid -(\beta^{p-1} \beta^p \mod M) \in I \lor (-(\beta^{p-1} \beta^p \mod M) \in I) \},$$

and hence $M_z \beta^{-k}$ is always an integer. Note that this is also analogous to interpreting the significand of $z$ as being $\beta$. Thus, we can transform every case into one where $|M_z| \leq |M_x|$. This means that numerator of the remainder always takes the form $N \cdot \beta^{p-1}$ for some integer $N$, which will be useful for finding a solution.

Although mod can seem unpredictable, these results make our work considerably easier. We shall next study these remainders and devise an algorithm to solve this problem.

5. **Minimizing remainders of a constant divided by a variable**

In the previous section, we reduced the problem of finding factors to an optimization problem over remainders where the dividend is constant but the divisor is not. The underlying problem is not specific to the floating-point numbers, and we can state it generally as follows:

**Problem C.** Given a rational $x$ and an interval $I$ containing zero, what are the integers $n$ such that $(x \mod n) \in I$? Specifically, given a fixed integer $m$, what is the least (or greatest) such $n$ where $n \geq m$ (or $n \leq m$)?

Naively, this problem can be solved by exhaustive search on the divisors in at most approximately $|x|$ trials, but this is clearly impractical. We do not have a provably efficient solution to the general problem, and it may well be the case that one does not exist at all.\(^5\) However, the specific case we are interested in can be solved in a constant number of arithmetic operations.

To better grasp the problem, let us first observe how the remainder varies with the divisor. Plotting $(-1000 \mod n)$ in Figure 2 we see parabolas near $n = 30$, as well as between $n = 20$ and $n = 25$. After $n = 40$, a pattern of “stepped” parabolas emerges, with an increasing number of “steps” later on. Finally, the parabolas start disappearing entirely around $n = 60$ and give way to a linear pattern. As one might expect from rounding being expressible in terms of remainder, these patterns are also visible in Figure 2.

---

\(^5\)For instance, if $x$ is a positive integer and $I = [0, 0]$, it is equivalent to factorization.
Figure 3. Graphs of \((-1000 \mod n\) and \([-1000/n]\), respectively above and below the x-axis. The upper graph follows a parabola whenever the lower graph behaves linearly (e.g. around \(n = 30\)). Similarly, linear segments in the upper graph correspond to constant segments in the lower (e.g. near \(n = 80\)).

This behavior may seem an unusual coincidence at first, but it occurs for any choice of numerator. Algebraically, the reason is remarkably simple. By the definition of the mod operator, we have

\[
(-1000 \mod n) = -1000 - n\lfloor -1000/n \rfloor.
\]

Thus, if \([-1000/n]\) is constant over a certain interval, then \((-1000 \mod n\) is linear in \(n\) over that interval. If \([-1000/n]\) changes linearly, then \((-1000 \mod n\) is instead quadratic in \(n\) over that interval. This relationship can be clearly seen in Figure 3 comparing the plots above and below the x-axis. The “stepped parabolas” correspond to values of \(n\) where the slope of \(-1000/n\ is close to linear, but less than 1, and so \([-1000/n]\ does not change at every step.

More formally, consider integers \(a, n, i\) such that \(n\ and \(n + i\ are nonzero, and let \(q = \lfloor a/n \rfloor:\)
Constant quotient: If \( \lfloor \frac{a}{n + i} \rfloor = q \), then
\[
(a \mod n + i) = a - (n + i)q
= (a \mod n) - iq.
\]

Linear quotient: If there is some integer \( d \) such that \( \lfloor \frac{a}{n + i} \rfloor = q + di \), then
\[
(a \mod n + i) = a - (n + i)(q + di)
= (a \mod n) - (di^2 + (q + nd)i).
\]

Note that the above still holds if we permit the variables to take arbitrary real values. In this way we can also find parameters corresponding to each parabola inside the “stepped” segments.

In our work, we will focus on linear and quadratic segments. These are especially useful, since we already have the tools to work with them. The following observations are key:

(1) Given a linear or quadratic segment, we can directly check it for a satisfying divisor by solving a linear or quadratic equation, respectively.
(2) The graph of remainders can always be partitioned into such segments.
(3) Therefore, we can search for a solution one segment at a time, rather than one divisor at a time.

With a good (that is, small) partition, we can shrink the search space significantly. However, this is still an exhaustive search, so proving the absence of a solution requires checking every segment. Furthermore, it is not clear that it is necessarily possible to construct a partition that is asymptotically smaller than magnitude of the dividend.

For the specific case we are interested in, however, the number of segments does not matter: for a suitable partition, we can show that the remainder at either the start or the end of any segment always satisfies the bounds. More specifically, according to Theorem 4.16, we are interested in remainders of the form \((-ab \mod n)\) or \((ab \mod n)\) for nonzero integers \(a, b, n\). We now study the solution space under these special conditions.

5.1. **Conditions for a solution via extrapolation.** Given nonzero integers \(a\) and \(b\), we construct an appropriate partition using a function \(f : \mathbb{R}^* \rightarrow \mathbb{R}\) such that \(f(x) = (-x^2 - |ab| \mod x)\). Clearly, we have \(f(n) = (-|ab| \mod n)\) for all nonzero integers \(n\). Since the numerator is a quadratic itself, the interval between each pair of consecutive roots of \(f\) is a quadratic segment (Figure 4 bottom). Now, note the following:

(1) A segment contains solutions if and only if the value of least magnitude within it satisfies the bounds.
(2) In each segment, the value of least magnitude is either at the start or the end of the segment.
(3) The start and the end of each segment lie between two consecutive roots of \(f\), and are obtained by rounding those roots.

---

6Specifically, it corresponds to the second case with \(d = 1/w\), where \(w\) is the number of integers \(m\) satisfying \(k = \lfloor a/m \rfloor\). We can see this below the x-axis in Figure 4 where \(w\) is the “width” of a level (e.g. around \(n = 80\), we have \(w = 7\)).

7For instance, we can trivially assign each pair of consecutive divisors its own segment.
Figure 4. Graphs of $(-1000 \mod x)$ and $(-x^2 - 1000 \mod x)$. The dots indicate points where $x$ is an integer.

Therefore, it suffices to prove that if $f(x) = 0$, then either $(-ab \mod \lfloor x \rfloor)$ or $(-ab \mod \lceil x \rceil)$ lies within the bounds. Note that a linear mapping does not suffice: although it fits better when $x$ is large (this can be seen in Figure 4), it does not provide sufficiently strong bounds when $x$ is small. The next result further explains the behavior of real extensions of remainders. In particular, we see that a quadratic extension makes quadratic segments have constant quotients.

Lemma 5.1. Let $I$ be an interval not containing zero and let $f$ be a real-valued function continuous over $I$. If $(f(x) \mod x) \neq 0$ for all $x \in I$, then $\lfloor f(x)/x \rfloor = \lfloor f(y)/y \rfloor$ for all $x, y \in I$.

Proof. We proceed by contraposition. Suppose $\lfloor f(x)/x \rfloor \neq \lfloor f(y)/y \rfloor$ for some $x, y \in I$. Without loss of generality, we assume that $\lfloor f(x)/x \rfloor < \lfloor f(y)/y \rfloor$. Let $q : I \to \mathbb{R}$ be a function such that $q(t) = f(t)/t$. Then, since $f$ is continuous over $I$, so is $q$. 


If $|q(y)| \leq q(x)$, then $|q(y)| \leq |q(x)|$ by monotonicity, which is a contradiction. Therefore $q(x) < |q(y)| \leq q(y)$, and hence by the intermediate value theorem there is some $w \in [x, y] \subseteq I$ such that $q(w) = |q(y)|$. Thus $f(w)$ is an integer multiple of $w$, and so $(f(w) \mod w) = 0$, as desired. \hfill \square

**Corollary 5.1.1.** Let $I$ be an interval not containing zero and let $f$ be a real-valued function continuous over $I$. If $(f(x) \mod x) \neq 0$ for all $x \in I$, then $(f(x) \mod x) = f(x) - xq$ for all $x \in I$, where $q$ is an integer such that $q = [f(y)/y]$ for all $y \in I$.

The following lemma allows us to express the remainders of interest as quadratics.

**Lemma 5.2.** Let $a, b \in \mathbb{Z}$ and let $f, g, h : \mathbb{R}^* \to \mathbb{R}$ be functions such that

\[
\begin{align*}
    f(t) &= (-t^2 - |ab| \mod t), \\
    g(t) &= (-ab \mod t), \\
    h(t) &= -(ab \mod t).
\end{align*}
\]

Let $I$ be an interval not containing zero. Let $q = \lfloor (-x^2 - |ab|)/x \rfloor$ for some $x \in I$. If $f(t) \neq 0$ for all $t \in I$, then for all $n \in I \cap \mathbb{Z}$,

\[
\begin{align*}
    f(n) &= -n^2 - qn - |ab|, \\
    g(n) &= \begin{cases}
        f(n) & \text{if } ab \geq 0, \\
        n - f(n) & \text{otherwise.}
    \end{cases} \\
    h(n) &= g(n) - n.
\end{align*}
\]

**Proof.** Suppose $f(t) \neq 0$ for all $t \in I$ and let $n \in I \cap \mathbb{Z}$. Then, by Lemma 5.1 we immediately have $f(n) = -n^2 - qn - |ab|$. Since $n$ is an integer, we also have $f(n) = (-|ab| \mod n)$. Thus, since $f(n)$ is nonzero, it follows that $n$ does not divide $ab$. Therefore $g(n)$ and $h(n)$ are nonzero, and hence

\[
g(n) - n = -(n - (-ab \mod n)) = -(ab \mod n) = h(n).
\]

If $ab \geq 0$, then $f(n) = (-ab \mod n) = g(n)$. If $ab < 0$, then $f(n) = (ab \mod n) = -h(n) = n - g(n)$, and thus $g(n) = n - f(n)$ as desired. \hfill \square

We now proceed to demonstrate the conditions under which the roots of the quadratic mapping provide a solution. We first derive a simpler expression for the quadratic passing through a given root.

**Lemma 5.3.** Let $a, b, \delta \in \mathbb{R}$ and let $x \in \mathbb{R}^*$. Then $-(x + \delta)^2 + (x + \delta)k - ab = -\delta^2 - (x - \frac{ab}{2})\delta$ where $k = \frac{x^2 + ab}{2}$.

**Remark.** Note that $(-x^2 - ab \mod x) = 0$ if and only if $\frac{-x^2 - ab}{x}$ is an integer.

We now bound the magnitude of that quadratic in terms of the difference between the factors of the dividend.

**Lemma 5.4.** Let $a, b, x \in \mathbb{R}$. If $1 < a + 1 \leq |x| \leq b - 1$, then $|x - ab/x| < b - a - 1$.

**Proof.** Let $f : \mathbb{R}^* \to \mathbb{R}$ be a function such that $f(t) = t - \frac{ab}{t}$ and suppose $1 < a + 1 \leq |x| \leq b - 1$. Since $a$ and $b$ are positive, we have $ab > 0$ and so $f$ is strictly
increasing over \(-\infty, 0\) and over \((0, +\infty)\). Hence \(f(a + 1) \leq f(|x|) \leq f(b - 1)\). Now, since \(b - 1\) and \(a\) are positive, we have

\[
(b - a - 1) - f(b - 1) = (b - a - 1) - \left( b - 1 - \frac{ab}{b - 1} \right)
= \frac{ab}{b - 1} - a
= \frac{a}{b - 1}
> 0,
\]
and hence \(b - a - 1 > f(b - 1)\). Similarly, since \(b\) and \(a + 1\) are positive,

\[
-(b - a - 1) - f(a + 1) = -(b - a - 1) - \left( a + 1 - \frac{ab}{a + 1} \right)
= \frac{ab}{a + 1} - b
= \frac{b}{a + 1}
< 0,
\]
and so \(-(b - a - 1) < f(a + 1)\). Therefore, \(|f(|x|)| < b - a - 1\). If \(x > 0\), then \(|x| = x\) and the result follows immediately. If \(x < 0\) instead, we have

\[
|f(|x|)| = |f(-x)| = |f(x)| = |f(x)|,
\]
and so the result follows.

\textbf{Corollary 5.4.1.} Let \(a, b, x, \delta \in \mathbb{R}\). If \(1 < a + 1 \leq |x| \leq b - 1\), then

\[
-\delta^2 - \left( x - \frac{ab}{x} \right) \delta \leq |\delta|(b - a - 1) + \delta^2.
\]

\textit{Remark.} We assume that \(a + 1 \leq x \leq b - 1\) in order to obtain the necessary bound. When \(a\) and \(b\) are integers, if \(x = a\) or \(x = b\), then \((-ab \mod x) = 0\), which lies within the bounding interval by assumption.

We now further refine the bound above under the assumption that the factors of the dividend are within a factor of two of each other. This always holds when \(\beta = 2\), which is the most common case by far.

\textbf{Lemma 5.5.} Let \(a, b \in \mathbb{R}\) and \(\delta \in (-1, 1)\). If \(b \leq 2a\), then \(|\delta|(b - a - 1) + \delta^2 \leq |\delta|a\).

\textit{Proof.} Suppose \(b \leq 2a\). Then \(b - a - 1 \leq a - 1\), and since \(|\delta| < 1\), we have \(\delta^2 \leq |\delta|\). Therefore, multiplying by \(|\delta|\) and adding \(\delta^2\),

\[
|\delta|(b - a - 1) + \delta^2 \leq |\delta|(a - 1) + \delta^2
= |\delta|a + \delta^2 - |\delta|
\leq |\delta|a,
\]
and hence the result.

\textbf{Lemma 5.6.} Let \(x \in \mathbb{R}\) and \(\delta \in (0, 1)\). If \(\delta x\) and \((\delta - 1)(x + 1)\) are integers, then they are both nonzero and have opposite signs.
Proof. We proceed by cases. Note that the first two cases do not depend on \( \delta x \) and \((\delta - 1)(x + 1)\) being integers.

- Suppose \( \delta x > 0 \). Since \( \delta \) is positive, it must be that \( x \) is also positive. Therefore, since \( \delta - 1 \) is negative, it follows that \((\delta - 1)(x + 1)\) is negative, and hence the result follows.

- Suppose \((\delta - 1)(x + 1) > 0 \). Since \( \delta - 1 \) is negative, we have \( x + 1 < 0 \) and \( x < -1 \). Therefore \( \delta x < -\delta < 0 \), and hence the result.

For the remaining cases, we proceed by contraposition:

- Suppose \( \delta x = 0 \). Since \( \delta \) is positive, we must have \( x = 0 \) and hence \((\delta - 1)(x + 1) = \delta - 1 \), which is not an integer, so the result follows.

- Suppose \((\delta - 1)(x + 1) = 0 \). Since \( \delta - 1 \) is negative, we must have \( x = -1 \) and hence \( \delta x = -\delta \), which is not an integer, and the result follows.

- Suppose both \( \delta x < 0 \) and \((\delta - 1)(x + 1) < 0 \). Since \( \delta \) is positive and \( \delta - 1 \) is negative, it follows that \( x \) is negative and \( x + 1 \) is positive and thus \(-1 < x < 0 \). Multiplying by \( \delta \), we obtain \(-\delta < \delta x < 0 \), and therefore \( \delta x \) is not an integer. Hence the result follows.

Since we have demonstrated the result in all cases, the proof is finished.

We now show that rounding a root of the quadratic extension gives a solution for a sufficiently wide interval.

**Lemma 5.7.** Let \( a, b \in \mathbb{Z} \) and let \( x \in \mathbb{R}^* \). Let \( I \) be an interval containing zero. If \( 0 < a + 1 \leq |x| \leq b - 1 \), \((-x^2 - ab \mod x) = 0 \), \( b \leq 2a \) and \( \text{diam} \, I \geq a \), then for some \( \hat{x}_1, \hat{x}_2 \in \{[x], [\bar{x}]\} \),

- either \((-ab \mod \hat{x}_1) \in I \) or \(-(ab \mod \hat{x}_1) \in I \), and
- either \((ab \mod \hat{x}_2) \in I \) or \(-(ab \mod \hat{x}_2) \in I \).

**Proof.** Suppose \( 0 < a + 1 \leq |x| \leq b - 1 \), \((-x^2 - ab \mod x) = 0 \), \( b \leq 2a \) and \( \text{diam} \, I \geq a \). Since \( a \) is an integer, we have \( 1 \leq a + 1 \leq |x| \), and thus \( |x| \) and \([x]\) are nonzero and have the same sign as \( x \). If \( x \in \mathbb{Z} \), then \( x = |x| = [x] \), so

\[
(-ab \mod |x|) = (-ab \mod x) = (-x^2 - ab \mod x) = 0 \in I,
\]

and hence the result. Suppose \( x \notin \mathbb{Z} \) instead, and let \( f : \mathbb{R} \to \mathbb{R} \) be a function such that \( f(t) = -t^2 + kt - ab \) where \( k = \frac{x^2 + ab}{2} \). Then, since \((-x^2 - ab \mod x) = 0 \), it follows that \( k \) is an integer and thus \( f(t) \) is an integer whenever \( t \) is an integer. Now, let \( \delta = |x| - x \) and \( y = \frac{ab}{x} - x - \delta \). Then, by Lemma 5.3

\[
f([x]) = -[x]^2 + k[x] - ab
\]

\[
= -(x + \delta)^2 + k(x + \delta) - ab
\]

\[
= -\delta^2 - \left(x - \frac{ab}{x}\right) \delta
\]

\[
= \delta \left(\frac{ab}{x} - x - \delta\right)
\]

\[
= \delta y.
\]
Since \( x \) is not an integer, it follows that \( |x| - |x| = 1 \) and hence \( |x| = x + \delta - 1 \). Thus, by Lemma 5.3 again,

\[
f([x]) = -|x|^2 + k|x| - ab
\]

\[
= -(x + (\delta - 1))^2 + k(x + (\delta - 1)) - ab
\]

\[
= -(\delta - 1)^2 - \left(x - \frac{ab}{x}\right)(\delta - 1)
\]

\[
= (\delta - 1) \left(\frac{ab}{x} - x - (\delta - 1)\right)
\]

\[
= (\delta - 1)(y + 1).
\]

Therefore,

\[
f([x]) - f([x]) = \delta y - (\delta - 1)(y + 1)
\]

\[
= y + 1 - \delta
\]

\[
= ab/x - x + 1 - 2\delta,
\]

and thus by Lemma 5.4,

\[
|f([x]) - f([x])| \leq |ab/x - x| + |1 - 2\delta| \leq |ab/x - x| + 1 < b - a \leq a.
\]

Now, let \( c = \min R \) and \( d = \max R \) where \( R = \{f([x]), f([x])\} \). Then \( \text{diam} [c, d] = d - c < b - a \). Since \( f([x]) \) and \( f([x]) \) are both integers and \( \delta \in (0, 1) \), it follows by Lemma 5.6 that \( \delta y \) and \( (\delta - 1)(y + 1) \) are nonzero and have opposite signs and hence \( c < 0 < d \). Thus both \([c, d] \) and \( I \) contain zero, and since \( \text{diam} [c, d] < a \leq \text{diam} I \), we have either \( c \in I \) or \( d \in I \) by Lemma 5.3 and therefore either \( f([x]) \in I \) or \( f([x]) \in I \). Similarly, since \([-d, -c] \) also contains zero and has the same diameter as \([c, d] \), we also have either \( f([x]) \in I \) or \( -f([x]) \in I \). Now, since \( |\delta| < 1 \) and \( |\delta - 1| < 1 \), it follows by Lemmas 5.4 and 5.5 that \( \delta y \) and \( (\delta - 1)(y + 1) \) lie strictly between \(-a \) and \( a \) and hence \( |f([x])| < a < ||[x]|| \) and \( |f([x])| < a < ||[x]|| \).

- Suppose \( f([x]) \) and \( x \) have the same sign. Then \(-f([x])\) also has the same sign, and \(-f([x])\) and \( f([x]) \) have the opposite sign. Therefore,

\[
f([x]) = (f([x]) \mod |x|) = (-ab \mod |x|),
\]

\[
-f([x]) = (-f([x]) \mod |x|) = (ab \mod |x|),
\]

and thus

\[
f([x]) = -(ab \mod |x|),
\]

\[
-f([x]) = (ab \mod |x|).
\]

Hence either \((ab \mod |x|) \in I \) or \(-(ab \mod |x|) \in I \), and also either \((ab \mod |x|) \in I \) or \(-(ab \mod |x|) \in I \).

- Suppose instead \( f([x]) \) and \( x \) have opposite signs. Then \( f([x]) \) and \(-f([x])\) have the same sign as \( x \), and hence \(-f([x]) \) has the opposite sign of \( x \). Therefore,

\[
f([x]) = (f([x]) \mod |x|) = (-ab \mod |x|),
\]

\[
-f([x]) = (-f([x]) \mod |x|) = (ab \mod |x|),
\]
and thus
\[ f([x]) = -(ab \mod [x]), \]
\[ f([x]) = -(ab \mod [x]). \]

Hence either \((-ab \mod [x]) \in I\) or \(-(ab \mod [x]) \in I\), and also either \((ab \mod [x]) \in I\) or \(-(ab \mod [x]) \in I\). Therefore, for some \(\hat{x}_1, \hat{x}_2 \in \{[x], [x]\}\), we have either \((-ab \mod \hat{x}_1) \in I\) or \(-(ab \mod \hat{x}_1) \in I\), and also either \((ab \mod \hat{x}_2) \in I\) or \(-(ab \mod \hat{x}_2) \in I\).

With the above result and Theorem 5.2, we can finally show that, for regular rounding functions, roots of the quadratic extension give a bound on plausible integers. With the help of Lemma 5.2, this lets us compute plausible integers efficiently in base 2.

**Theorem 5.8.** Let \(M_z \in \mathbb{Z}^*\). Let \(Z \subseteq \mathbb{R}\) and let \(z \in Z\) be normal. Let \(M_z = z\beta^{-Q(z)}\) and \(M'_z = M_z\beta^{-k}\) where \(k = \log_\beta \text{upf}(M_z/M_x)\). Let \(A\) be a set such that
\[ A = \{M \in \mathbb{Z}^* \mid M \geq M_z \land (-(M'_z\beta^{-1} \mod M) \in I \lor (-M'_z\beta^{-1} \mod M) \in I)\}, \]
where \(I = (\beta^{-1}[z] - z)\beta^{-1-Q(z)-k}\). Let \(f : \mathbb{R}^* \to \mathbb{R}\) be a function such that \(f(t) = |t - |M'_z|\beta^{-1} \mod t|\). Let \(r \in \mathbb{R}^*\). If \(f\) is regular, \(f(r) = 0\), \(r \geq M_z\), and \(\beta^{-p} \leq |M_z| < \beta^p\) and \(|M'_z| \leq 2\beta^{-p-1}\), then \(\min A \leq |r|\).

**Proof.** Suppose the conditions hold. If \(|M_z| = |M'_z|\), then \(M'_z = M_z\) and hence \((-M'_z\beta^{-1} \mod M_z) = 0 \in I\), so \(\min A = M_x\). Suppose \(|M_z| \neq |M'_z|\) instead. We first prove that \(k \in \{-1, 0\}\) and that \(M'_z\) is an integer such that \(|M_z| < |M'_z|\).

- Suppose \(|M_z| < |M'_z|\). Then \(1 < |M_z/M_x| < \beta\), and hence \(k = 0\). Thus \(M'_z = M_z\) and \(|M_z| < |M'_z| = |M'_z|\).
- Suppose \(|M_z| > |M'_z|\) instead. Then \(1/\beta < |M_z/M_x| < 1\), and hence \(k = -1\). Since \(z\) is normal, we have \(|M_z| \geq \beta^{-p-1}\) and thus \(|I| \geq \beta^p > \beta^{M_z}\).

Therefore \(M'_z\) is an integer and \(|M_z| < |M'_z|\). Next, we show that if \(r\) has a different sign to \(M_x\) or \(|r| \notin [\beta^{-p-1} + 1, |M'_z| - 1]\), then \(\min A \leq |r|\):

- Suppose \(M_x > 0\). Then \(M_x \leq |M'_z|\) and \((-M'_z\beta^{-1} \mod |M'_z|) = 0 \in I\), we have \(|M'_z| \in A\) and therefore \(M_x \leq \min A \leq |M'_z|\). Since \(r \geq M_x\) by assumption, \(r\) is also positive and thus has the same sign. Now, suppose \(|r| \notin [\beta^{-p-1} + 1, |M'_z| - 1]\). Then either \(r < \beta^{-p-1} + 1\) or \(r > \beta^{-p-1}\). If \(r > |M'_z| - 1\), then \([r] \geq |M'_z| \geq \min A\). Suppose \(r < \beta^{-p-1} + 1\) instead. Then \(\beta^{-p-1} \leq M_x \leq r < \beta^{-p-1} + 1\), and so \(M_x = \beta^{-p}\) since \(M_x\) is an integer. Therefore, since \((-M'_z\beta^{-1} \mod \beta^{-p-1}) = 0 \in I\), it follows that \(\min A = M_x = \beta^{-p} \leq r \leq |r|\).
- Suppose \(M_x < 0\) instead. Then \(M_x \leq -\beta^{-p-1}\), and since \((-M'_z\beta^{-p-1} \mod -\beta^{-p}) = 0 \in I\), we have hence \(M_x \leq \min A \leq -\beta^{-p-1}\) by definition. If \(r\) has a different sign to \(M_x\), then it is positive, and hence \(\min A < 0 < r \leq |r|\). Suppose \(r\) instead has the same sign as \(M_x\) and suppose \(|r| \notin [\beta^{-p-1} + 1, |M'_z| - 1]\). Then either \(r < -|M'_z| + 1\) or \(r > -\beta^{-p-1} - 1\). If \(r > -\beta^{-p-1} - 1\), then \([r] \geq -\beta^{-p-1} \geq \min A\). If \(r < -|M'_z| + 1\) instead, then \(-|M'_z| \leq M_x \leq r < -|M'_z| + 1\), so \(M_x = -|M'_z| \in A\) and thus \(\min A = M_x \leq r \leq |r|\).

Suppose henceforth that \(r\) and \(M_x\) have the same sign and that \(\beta^{-p-1} + 1 \leq |r| \leq |M'_z| - 1\). We now show that \(\text{diam } I \geq \beta^{-p}\). First, note that if \(|M_z| \neq \beta^{-p-1}\), then
Therefore, \( \text{diam } f^{-1}([z]) \geq \beta^Q(z) \geq \beta^Q(z) + k \). If \( |M_x| = \beta^{p-1} \) instead, then \( |M_x| < |M_x| \), so \( k = -1 \) and thus, since \( \text{fl} \) is regular, we have \( \text{diam } f^{-1}([z]) \geq \beta^Q(z) - 1 = \beta^Q(z) + k \). Therefore,

\[
\text{diam } I = \beta^{p-1-Q(z)-k} \cdot \text{diam}(f^{-1}[Z] - z) = \beta^{p-1-Q(z)-k} \cdot \text{diam } f^{-1}[Z] \\
\geq \beta^{p-1-Q(z)-k} \cdot \text{diam } f^{-1}([z]) \geq \beta^p - 1.
\]

We can now apply Lemma 5.7 to complete the proof. Let \( a = \beta^{p-1} \) and \( b = |M_x| \).

Suppose \( M_x' > 0 \). By Lemma 5.7 we have either \((-ab \mod \hat{r}) \in I \) or \((-ab \mod \hat{r}) \in I \) for some \( \hat{r} \in \{[r], [\hat{r}]\} \). Thus, since \( ab = \beta^{p-1} M_x' \) and \( M_x \leq |r| \leq \hat{r} \), we have \( \hat{r} \in A \) and thus \( \min A \leq \hat{r} \leq |r| \).

Suppose \( M_x' < 0 \) instead. Then, by Lemma 5.7, we have either \((-ab \mod \hat{r}) \in I \) or \((ab \mod \hat{r}) \in I \) for some \( \hat{r} \in \{[r], [\hat{r}]\} \). Since \( ab = -\beta^{p-1} M_x' \), it follows that either \(-\beta^{p-1} M_x' \mod \hat{r} \in I \) or \(-\beta^{p-1} M_x' \mod \hat{r} \in I \). Therefore \( \min A \leq \hat{r} \leq |r| \) again similarly, and hence the result follows.

**Remark.** By Theorem 4.11, we have \( \min A = \min \{M \in \text{Plaus}_2(z) \mid M \geq M_x\} \).

Thus, this result gives us a solution to our Problem A via Theorem 4.15.

If \( \beta = 2 \), then \( 2 \beta^{p-1} = \beta^p \) and thus \( |M_x| \leq \beta^{p-1} \) holds whenever either \( |M_x| \leq |M_x| \) or \( |M_x| = \beta^{p-1} \). Since we assume \( \text{fl} \) is regular, we do not need to concern ourselves with the case of \( |M_x| < |M_x| \), as Lemma 4.15 implies that \( x \) is a floating-point factor of \( z \) whenever \( \beta^{\text{min}} \leq |z/x| \leq \max \mathbb{F} \) where \( x = M_x \beta^n \) for some integer \( q \) such that \( q_{\text{min}} \leq q \leq q_{\text{max}} \).

If \( r \) is the least root of \( f \) greater than \( M_x \), then \( f \) is nonzero over \( (M_x, r) \). Thus, according to Lemma 5.2, the restrictions of \( M \mapsto -(M_x' \beta^{p-1} \text{ mod } M) \) and \( M \mapsto (-M_x' \beta^{p-1} \text{ mod } M) \) to \( (M_x, r) \cap \mathbb{Z} \) are quadratics. Thus, if \( |r| \) is not a solution, then we necessarily have \( \min A = [r] \). Otherwise, \( \min A \leq |r| \), and since \( I \) is an interval, we can thus compute \( \min A \) by solving these quadratics for the interval bounds, rounding the solutions, and taking the minimum.

As discussed above, combining the bound given by Theorem 5.9 with Lemma 5.2 allows us to solve Problem A for normal numbers in binary floating-point arithmetic. We next turn to assembling our results into an algorithm.

6. Algorithms

Theorem 5.6 states that the classical division-based algorithm can be used to produce optimal bounds for floating-point arithmetic if we can devise some way of finding the feasible floating-point numbers nearest any given infeasible number. If the bounds and their quotients are normal numbers, Theorem 4.12 states that this can be done by finding the two nearest plausible integral significands. Finally, for regular rounding functions in base 2 (and other bases under certain conditions), Theorem 5.8 allows us to efficiently find the smallest plausible integral significand greater than or equal to a given integer. Putting these together, we can derive an algorithm for efficiently computing optimal bounds on the factors of a binary floating-point multiplication. Algorithms 1 to 8 form a complete pseudocode implementation of the results of this paper. They are annotated with the properties they satisfy and justifications for them. The proofs of correctness for Algorithms 4
to [6] require some additional minor results which are included in Appendix A. We thus obtain the following:

**Theorem 6.1.** Algorithms [1] to [6] are correct. If arithmetic operations take time $O(1)$, then these algorithms have $O(1)$ time complexity.

By using Theorem 3.6 to combine Algorithm [1] with the division-based algorithm, we obtain an efficient method of computing optimal interval bounds when $\beta = 2$ in most cases. However, note that if the division-based algorithm produces subnormal bounds or bounds whose quotients are subnormal, Algorithm [1] is not guaranteed to return a feasible number. Similarly, in higher bases, we are not guaranteed an optimal result. However, it is possible to show that Algorithm [1] still returns a lower bound, although it is not clear how many iterations it would take to find the true minimum (or, in the worst case, prove infeasibility).

**Algorithm 1** Finding the next feasible number.

**Require:** $\text{fl}$ regular; normal floating-point number $x$; floating-point interval $Z$; normal $z \in Z$ such that $\beta^{e_{\min}} \leq |z/x| \leq \max F$ and either $\beta = 2$ or $1 < |z\beta^{-E(z)}| \leq 2$

**Ensure:** returns $\inf \text{Feas}(Z) \cap [x, +\infty]$

1: function $\text{NextFeasible}(x, z, Z)$
2: if $x \in \text{Feas}(Z)$ then $\triangleright$ constant time using Lemma 3.10
3: return $x$
4: end if
5: $M_x := x \beta^{-Q(x)} \triangleright$ by Theorem 3.19, either $|M_z| = \beta^{p-1}$ or $|M_x| < |M_z|$
6: $M_b := \text{NextPlausible}(M_x, z, Z) \triangleright$ by Theorem 3.15, if a solution exists, it is either $M_b \beta^{Q(x)}$ or $M_c \beta^{Q(x)}$, where $M_x \leq M_b \leq M_c$
7: if $M_b \beta^{Q(x)} \in \text{Feas}(Z)$ then
8: return $M_b \beta^{Q(x)}$
9: end if
10: $M_c := \text{NextPlausible}(|M_x|, z, Z)$
11: if $M_c \beta^{Q(x)} \in \text{Feas}(Z)$ then
12: return $M_c \beta^{Q(x)}$
13: end if
14: return $+\infty \triangleright$ Feas$(Z) \cap [x, +\infty]$ is empty by Theorem 3.15
15: end function

7. Conclusion

In this paper, we have rigorously presented the first step towards an efficient solution to the problem of computing optimal interval bounds on the variables of the floating-point constraint $x \otimes y = z$. The algorithms presented here are sound and complete for binary floating-point arithmetic involving normal numbers, running in constant time in the word-RAM model. This has immediate implications for floating-point decision procedures based on interval constraint solving. Additionally, we have found new structure in floating-point products and their rounding errors, which we hope will be of further use in the study of floating-point theory.

As suggested earlier, a number of questions remain open in this topic:
Algorithm 2 NextPlausible

Require: \( \text{fl regular}; \) integer \( M_x \) such that \( \beta^{p-1} \leq |M_x| < \beta^p \); floating-point interval \( Z; \) \( z \in Z \) such that either \( |M_x| \leq |M_z| \leq 2\beta^{p-1} \) or \( \beta|M_z| = \beta^p \leq 2\beta^{p-1} \)

Ensure: returns \( \min \{M \in \text{Plaus}_{2\beta}(Z) \mid M \geq M_x\} \)

1: function NextPlausible\((M_x, z, Z)\)
2: \( M_z := z\beta^{-Q(z)} \)
3: \( k := \log_{\beta} \text{ufp}(M_z/M_x) \)
4: \( I := (\lceil M_z \rceil - z)\beta^{-Q(z) - k} \)
5: return NextDivisorInBounds\((\beta^{p-1}, M_z\beta^{-k}, M_x, I)\) \( \triangleright \) by Theorems 4.16 and 5.8
6: end function

Algorithm 3 NextDivisorInBounds

Require: integers \( a, b, n \) such that \( 0 < |a| \leq |n| \leq |b| \leq |2a| \); interval \( I \) containing zero such that \( \text{diam} I \geq |a| \)

Ensure: returns \( \min \{m \in Z \mid m \geq n \land ((-ab \mod m) \notin I \lor -(ab \mod m) \notin I)\} \)

1: function NextDivisorInBounds\((a, b, n, I)\)
2: if \((-ab \mod n) \notin I \lor -(ab \mod n) \notin I\) then
3: return \( n \)
4: end if
5: \( r := \text{NextQuadraticModLinearRootFloor}(-1, -|ab|, n) \)
6: if \((-ab \mod r) \notin I \lor -(ab \mod r) \notin I\) then
7: return \( r + 1 \) \( \triangleright \) by Lemma 5.7, since there are no solutions in \([n, r]\)
8: end if
9: \( q := (-n^2 - |ab|)/n \) \( \triangleright \) solution is in \([n, r]\); find minimum using Lemma 5.2
10: if \( ab > 0 \) then
11: \( c := \text{NextQuadraticPointWithinBounds}(-1, [q], -ab, n, I) \)
12: \( d := \text{NextQuadraticPointWithinBounds}(-1, [q], -ab, n, I) \)
13: else
14: \( c := \text{NextQuadraticPointWithinBounds}(1, [q], -ab, n, I) \)
15: \( d := \text{NextQuadraticPointWithinBounds}(1, [q], -ab, n, I) \)
16: end if
17: return \( \min \{c, d\} \)
18: end function

Algorithm 4 NextQuadraticModLinearRootFloor

Require: nonzero integers \( a, c, n \)

Ensure: returns \( \min \{|x| \mid x \in [n, +\infty) \land (ax^2 + c \mod x) = 0\} \)

1: function NextQuadraticModLinearRootFloor\((a, c, n)\)
2: \( q := (an^2 + c)/n \)
3: \( R_1 := \text{QuadraticRootsFloor}(a, -[q], c) \)
4: \( R_2 := \text{QuadraticRootsFloor}(a, -[q], c) \)
5: return \( \min \{m \in R_1 \cup R_2 \mid m \geq n\} \) \( \triangleright \) by Lemma A.3
6: end function
Algorithm 5 NextQuadraticPointWithinBounds

Require: integers $a, b, c, n$ where $a \neq 0$; interval $I$
Ensure: returns $\inf \{ m \in \mathbb{Z} \mid m \geq n \land am^2 + bm + c \in I \}$

1: function NextQuadraticPointWithinBounds($a, b, c, n, I$)
2: if $an^2 + bn + c \in I$ then
3:    return $n$
4: end if
5: $I' := I \cap \mathbb{Z}$
6: if $I' = \emptyset$ then
7:    return $+\infty$
8: end if
9: $R_1 := $ QuadraticRootsFloor($a, b, c - \max I'$)
10: $R_2 := $ QuadraticRootsFloor($a, b, c - \min I'$)
11: $R := R_1 \cup R_2$
12: $R := \{ r + 1 \mid r \in R \}$ $\triangleright$ solutions are ceilings of roots per Lemma A.2
13: return $\min \{ m \in R \cup R \mid m \geq n \land am^2 + bm + c \in I \} \cup \{ +\infty \}$
14: end function

Algorithm 6 QuadraticRootsFloor

Require: integers $a, b, c$ where $a \neq 0$
Ensure: returns $\{ \lfloor x \rfloor \mid x \in \mathbb{R} \land ax^2 + bx + c = 0 \}$

1: function QuadraticRootsFloor($a, b, c$)
2: if $a < 0$ then
3:    $a := -a$
4:    $b := -b$
5:    $c := -c$
6: end if
7: $\Delta := b^2 - 4ac$
8: if $\Delta < 0$ then
9:    return $\emptyset$
10: end if
11: $r_1 := \left\lfloor \frac{-b + \sqrt{\Delta}}{2a} \right\rfloor$ $\triangleright$ by applying Lemma A.1 to the quadratic formula
12: $r_2 := \left\lfloor \frac{-b + \sqrt{\Delta}}{2a} \right\rfloor$
13: return $\{ r_1, r_2 \}$
14: end function

- What is the optimal time complexity for computing feasible numbers in general (i.e. in all bases and with subnormals)? Can we asymptotically outperform brute force search? Does an efficient algorithm exist for this task?
- Although the algorithms in this paper are relatively brief, the work to develop them is much longer. Is there a shorter derivation of this solution?
- The results of this paper rely on a mixture of integer and floating-point reasoning. Can these results be stated naturally using only floating-point arithmetic? If so, is there a version of this algorithm which uses only floating-point arithmetic?
Separately, Section 3 contains results which may be useful for solving mod constraints beyond the scope of this paper. Can they be applied more broadly?

Appendix A. Additional proofs

This section contains some additional results used to show the correctness of Algorithms 4 to 6.

Lemma A.1. Let \( x \in \mathbb{R} \) and \( n \in \mathbb{Z} \). If \( n > 0 \), then \( \lfloor x/n \rfloor = \lfloor x\rfloor/n \).

Proof. Suppose \( n > 0 \) and let \( \delta_1 = x - \lfloor x \rfloor \) and \( \delta_2 = (\lfloor x \rfloor \mod n) \). Since \( n \) is a positive integer and \( \lfloor x \rfloor \) is an integer, we have \( 0 \leq \delta_2 \leq n - 1 \). Hence \( 0 \leq \delta_1 + \delta_2 < n \) and thus \( (\lfloor \delta_1 + \delta_2 \rfloor)/n = 0 \). By the definition of mod, we also have \( (\lfloor x - \delta_2 \rfloor)/n = \lfloor x/n \rfloor \) and thus

\[
\lfloor x/n \rfloor = \frac{\lfloor x \rfloor + \delta_1 + \delta_2 - \delta_2}{n} = \frac{\lfloor x \rfloor/n + (\delta_1 + \delta_2)/n}{n} = \lfloor x \rfloor/n + \lfloor (\delta_1 + \delta_2)/n \rfloor = \lfloor x/n \rfloor.
\]

\( \square \)

Lemma A.2. Let \( f \) be a continuous real function and let \( n \in \mathbb{Z} \) and \( a, b \in \mathbb{R} \). Let \( M = \{ m \in \mathbb{Z} \mid m \geq n \wedge f(m) \in [a,b]\} \). If \( M \) is nonempty, then either \( \min M = n \) or \( \min M = \lfloor x \rfloor \) for some \( x \in \mathbb{R} \) such that either \( f(x) = a \) or \( f(x) = b \).

Proof. If \( f(n) \in [a,b] \), the result follows trivially. Suppose instead that \( M \) is nonempty and \( m > n \) where \( m = \min M \). Then \( a \leq f(m) \leq b \) and either \( f(m-1) < a \) or \( f(m-1) > b \) by assumption. Suppose \( f(m-1) < a \). Then, by the intermediate value theorem, there is some \( x \in (m-1, m] \) such that \( f(x) = a \). Since \( \lfloor x \rfloor = m \), the result follows. Suppose \( f(m-1) > b \) instead. Then, similarly, there is some \( x \in (m-1, m] \) such that \( f(x) = b \). Since \( \lfloor x \rfloor = m \) again, we are finished. \( \square \)

Lemma A.3. Let \( a, c \in \mathbb{R}^+ \) and let \( q : \mathbb{R}^+ \to \mathbb{R} \) be a function such that \( q(t) = (at^2 + c)/t \). For all \( x \in \mathbb{R}^+ \), there exists some \( y \in [x, +\infty) \) such that \( q \) is continuous over \([x, y]\) and \( q(y) \in \{ [q(x)], [q(x)] \} \).

Proof. In the following, note that \( q \) is continuous over its domain and \( q(t) = at + c/t \) for all \( t \in \mathbb{R}^+ \). We proceed by cases.

Let \( x \in \mathbb{R}^+ \) and suppose \( x \) is positive. Suppose \( a \) is also positive. Then,

\[
\lim_{t \to \infty} q(t) = \lim_{t \to \infty} at + \lim_{t \to \infty} c/t = \infty + 0 = +\infty,
\]

and thus \( q \) grows without bound. Since it is also continuous over \((0, +\infty)\), it therefore attains \([q(x)]\) at some \( y \in [x, +\infty) \), and hence the result follows.

Suppose \( a \) is negative instead. Then \( \lim_{t \to \infty} q(t) = -\infty \) similarly, and hence \( q \) has no lower bound. Since it is continuous over \((0, +\infty)\), we thus have \( q(y) = [q(x)] \) for some \( y \in [x, +\infty) \), and hence the result follows.

Now, suppose \( x \) is negative instead. Suppose \( c \) is positive. Then,

\[
\lim_{t \to 0^-} q(t) = \lim_{t \to 0^-} at + \lim_{t \to 0^-} c/t = 0 - \infty = -\infty,
\]

and thus \( q \) grows without bound. Since it is also continuous over \((0, +\infty)\), it therefore attains \([q(x)]\) at some \( y \in [x, +\infty) \), and hence the result follows.

Suppose \( a \) is negative instead. Then \( \lim_{t \to 0^-} q(t) = +\infty \) similarly, and hence \( q \) has no upper bound. Since it is continuous over \((0, +\infty)\), we thus have \( q(y) = [q(x)] \) for some \( y \in [x, +\infty) \), and hence the result follows.
and therefore \( q \) has no lower bound over \((-\infty, 0)\). Since it is also continuous over \((-\infty, 0)\), we thus have \( q(y) = [q(x)] \) for some \( y \in [x, 0) \), as desired.

Suppose \( c \) is negative instead. Then \( \lim_{t \to 0^-} q(t) = +\infty \), and hence \( q \) has no upper bound over \((-\infty, 0)\). Since \( q \) is continuous over \((-\infty, 0)\), it follows that \( q(y) = [q(x)] \) for some \( y \in [x, 0) \). Therefore the result holds in all cases, and we are done. \( \square \)

**Lemma A.4.** Let \( a, c \in \mathbb{R}^+ \) and let \( q : \mathbb{R}^+ \to \mathbb{R} \) be a function such that \( q(t) = (at^2 + c)/t \). Let \( x \in \mathbb{R}^+ \) and \( R = \{ t \in [x, +\infty) \mid q(t) \in \mathbb{Z} \} \). Then \( R \) has a minimum, and either \( q(r) = [q(x)] \) or \( q(r) = [q(x)] \) where \( r = \min R \).

**Proof.** Let \( S \) be a set such that
\[
S = \{ t \in [x, +\infty) \mid q(t) \in \{ [q(x)], [q(x)] \} \}.
\]
Clearly, \( S \subseteq R \). For all \( t \in \mathbb{R}^+ \) and \( u \in \mathbb{R} \), we have \( q(t) = u \) if and only if \( at^2 - ct + c = 0 \). Therefore, for any particular \( u \), there are at most 2 choices of \( t \) that satisfy the equation, and hence \( S \) has at most 4 elements. By Lemma A.3, there is some \( y \in [x, +\infty) \) such that \( q \) is continuous over \([x, y]\) and \( q(y) \in \{ [q(x)], [q(x)] \} \), and so \( y \in S \). Therefore \( S \) is nonempty and finite, and hence has a minimum.

Now, let \( s = \min S \). Since \( S \subseteq R \), we thus have \( s \in R \). Let \( t \in R \cap [x, s] \). Then \( q(t) \) is an integer by definition, and hence either \( q(t) \geq [q(x)] \) or \( q(t) \leq [q(x)] \). Therefore, since \( q \) is continuous over \([x, y]\) and thus also continuous over \([x, t]\), the intermediate value theorem implies that either \( q(u) = [q(x)] \) or \( q(u) = [q(x)] \) for some \( u \in [x, t] \). Therefore \( u \in S \), and hence \( s \leq u \) by definition. Since \( u \leq t \leq s \), we thus have \( s = t = u \) and hence \( R \cap [x, s] = \{ s \} \), so \( \min R = s \). By the definition of \( S \), we have either \( q(s) = [q(x)] \) or \( q(s) = [q(x)] \), so we are done. \( \square \)

**Corollary A.4.1.** Let \( a, c \in \mathbb{R}^+ \) and let \( q : \mathbb{R}^+ \to \mathbb{R} \) be a function such that \( q(t) = (at^2 + c)/t \). For all \( x \in \mathbb{R}^+ \),
\[
\min \{ t \in [x, +\infty) \mid (at^2 + c \mod t) = 0 \} = \min \{ t \in [x, +\infty) \mid q(t) \in \{ [q(x)], [q(x)] \} \}.
\]

**APPENDIX B. Glossary**

**B.1. List of terms.**

- **diameter:** the greatest distance between any pair of a set’s elements
- **exponent:** the integer \( e \) in \( \pm d_1.d_2\ldots d_p \times \beta^e \)
- **faithful:** a rounding function is faithful if it always rounds to either the nearest number above or below (i.e. does not jump over floating-point numbers)
- **feasible:** \( x \) is feasible for \( Z \) if \( x \) is a floating-point factor of some \( z \in Z \)
- **floating-point factor:** \( x \) is a floating-point factor of \( z \) if \( x \otimes y = z \) for some floating-point \( y \)
- **floating-point interval:** a set of consecutive floating-point numbers
- **floating-point number:** see Definition 2.4, roughly, a number of the form \( \pm d_1.d_2\ldots d_p \times \beta^e \)
- **integral significand:** the significand multiplied by \( \beta^{e-1} \), making it an integer
- **normal:** a floating-point number which can be written without leading zeros with an exponent between \( \epsilon_{\min} \) and \( \epsilon_{\max} \); equivalently, a floating-point number with magnitude no less than \( \beta^{\epsilon_{\min}} \)
- **plausible:** see Definition 4.3, roughly, the integer \( M \) is \( z \)-plausible for \( Z \) if \( M \) is feasible for \( Z \) in the hypothetical case that exponents were unbounded
predecessor: the number immediately before
quantum: the place value of a 1 in the last place of a floating-point number
quantum exponent: the exponent of the quantum (the quantum is always a power of the base)
regular: see Definition 6.13 roughly, a rounding function under which no floating-point number has an especially small preimage
rounding function: a function from \( \mathbb{R} \) to \( \mathbb{F} \)
significand: the \( \pm d_1.d_2 \ldots d_p \) part of \( \pm d_1.d_2 \ldots d_p \times \beta^e \)
subnormal: nonzero, but smaller than any normal number
successor: the number immediately after
unit in first place: the place value of a 1 in the first digit of a number written in base-\( \beta \) scientific notation; equally, \( \beta^e \) where \( e \) is the exponent of the number

B.2. List of symbols.

- \( m \) significand
- \( M \) integral significand
- \( e \) exponent
- \( q \) quantum exponent
- \( E(x) \) exponent of \( x \)
- \( Q(x) \) quantum exponent of \( x \)
- \( x^- \) predecessor of \( x \)
- \( x^+ \) successor of \( x \)
- \( \otimes \) rounded multiplication: \( x \otimes y = \text{fl}(xy) \)
- \( \text{fl} \) any nondecreasing and faithful rounding function
- \( \text{RD}(x) \) \( x \) rounded downward (i.e. to the nearest number below)
- \( \text{RU}(x) \) \( x \) rounded upward (i.e. to nearest number above)
- \( \text{RN}(x) \) \( x \) rounded to the nearest number; no rule is specified for breaking rounding ties
- \( \text{fl}^{-1}[Z] \) preimage of \( Z \) under rounding; the set of extended real numbers that round to some \( z \in Z \)
- \( \text{Feas}(Z) \) set of all floating-point numbers feasible for \( Z \)
- \( \text{Plaus}_Z(z) \) set of integers \( z \)-plausible for \( Z \)
- \( \text{ufp}(x) \) unit in first place of \( x \)
- \( \text{diam } X \) diameter of the set \( X \)
- \( (x \mod y) \) remainder of floored division of \( x \) by \( y \)
- \( \mathbb{Z} \) integers
- \( \mathbb{Z}^* \) nonzero integers
- \( \mathbb{R} \) real numbers
- \( \mathbb{R}^* \) nonzero real numbers
- \( \overline{\mathbb{R}} \) real numbers with \( +\infty \) and \( -\infty \)
- \( \mathbb{F}^* \) nonzero real numbers with \( +\infty \) and \( -\infty \)
- \( \mathbb{F} \) nonzero finite floating-point numbers
- \( \mathbb{F}^* \) finite floating-point numbers
- \( \overline{\mathbb{F}} \) finite floating-point numbers with \( +\infty \) and \( -\infty \)
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