ON THE MINIMAL IDEMPOTENTS OF
TWISTED GROUP ALGEBRAS OF CYCLIC 2-GROUPS

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Abstract. For a field $K$ of the second kind with respect to 2 and of characteristic different from 2, we consider the decomposition of the binomials $x^{2^n} - a$ into a product of irreducible factors over $K$ and find the explicit form of the minimal idempotents of the twisted group algebra $K^t\langle g \rangle$ of a cyclic 2-group $\langle g \rangle$ over $K$.

Introduction

The starting point of a lot of investigations on twisted group algebras is to find the minimal idempotents of the twisted group algebra $K^tG$ where $G$ is a cyclic group and $K$ is a field. When $\langle g \rangle$ is a cyclic $p$-group, $p$ is an odd prime and $K$ is a field of characteristic different from $p$, Nachev and Mollov [3] have found the explicit form of the minimal idempotents of $K^t\langle g \rangle$. For $p = 2$ additional difficulties arise which

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are connected with the decomposition of the polynomial $x^{2n} - a$ into irreducible factors over the field $K$. The purpose of the present paper is to find the explicit form of the minimal idempotents of $K^t\langle g \rangle$ when $\langle g \rangle$ is a cyclic 2-group and $K$ is a field of the second kind with respect to 2 (and of characteristic different from 2). We shall mention that the semisimplicity of the twisted group algebra $K^t\langle g \rangle$ in this case is a well known fact (see e.g. [1] or [4]).

The paper is organized as follows. In Section 1 we give some notation, definitions and preliminary results. Section 2 deals with the decomposition of an arbitrary polynomial $x^{2n} - a$ in irreducible factors over a field $K$ of the second kind with respect to 2. In Section 3 we find the explicit form of the minimal idempotents of the twisted group algebra $K^t\langle g \rangle$ of an arbitrary cyclic 2-group $\langle g \rangle$ over a field $K$ of the second kind with respect to 2.

1. Notation, Definitions and Preliminary Results

Let $p$ be a prime, let $K$ be a field of characteristic different from $p$ and let $\bar{K}$ be the algebraic closure of $K$. We denote by $\varepsilon_n$ a $p^n$-th primitive root of 1 in $\bar{K}$. The field $K$ is called a field of the first kind with respect to $p$ [5, p. 684], if $K(\varepsilon_i) \neq K(\varepsilon_2)$ for some $i > 2$. Otherwise $K$ is called a field of the second kind with respect to $p$. An equivalent definition is the following. If the Sylow $p$-subgroup $K(\varepsilon_2)_p$ of the multiplicative group $K(\varepsilon_2)^*$ is cyclic, then $K$ is of the first kind with respect to $p$ and if $K(\varepsilon_2)_p$ is the quasicyclic group $\mathbb{Z}(p^\infty)$ then $K$ is a field of the second kind with respect to $p$. Typical examples of fields of the first and of the second kind with respect to any prime $p$ are $\mathbb{Q}$ and $\mathbb{R}$, respectively. Let $K^*$ be the multiplicative group of $K$ and let $G$
be a multipliucative group. A twisted group algebra \( K^t G \) of \( G \) over \( K \) [5, p. 13] is an associative \( K \)-algebra with basis \( \{ \bar{x} \mid x \in G \} \) and with multiplication defined on the basis by
\[
\bar{x}\bar{y} = \gamma(x, y)\bar{y}\bar{x}, \gamma(x, y) \in K^*.
\]
We denote by \( KG \) the ordinary group algebra of \( G \) over \( K \). It is well known that if \( G = \langle g \rangle \) is a cyclic group then \( K^t \langle g \rangle \) is a commutative algebra. If the group \( \langle g \rangle \) is of order \( n \) then \( \bar{g}^n = a \) for some \( a \in K^* \). Obviously this equality determines the twisted group algebra \( K^t \langle g \rangle \). We define
\[
K^n = \{ a^n \mid a \in K \}, \ n \in \mathbb{N}.
\]
Clearly \( K^n \) is closed with respect to the multiplication and especially \( K^p^n \supseteq K^{p+n} \).

The following definition is well known [6, §44, p. 142].

**Definition 1.1.** Let \( \alpha \) belong to the finite extension \( L \) of the field \( K \) and let \( f(x) = x^n + a_1 x^{n-1} + \ldots + a_n \) be the minimal polynomial of \( \alpha \) over \( K \). The element
\[
N(\alpha) = (-1)^n a_n^{(L:K)/n}
\]
is called the norm of \( \alpha \) (in \( L \) over \( K \)).

Clearly \( N(\alpha) = (\alpha_1 \ldots \alpha_n)^{(L:K)/n} \), where \( \alpha_1, \ldots, \alpha_n \) are the zeros of the polynomial \( f(x) \), i.e. all the conjugate elements of \( \alpha \) over \( K \). It is well known that \( N(\alpha\beta) = N(\alpha)N(\beta) \) for \( \alpha, \beta \in L \) and if \( a \in K \) then \( N(a) = a^{(L:K)} \).

In order to see that some special binomials are indecomposable over the field \( K \) we shall use the following theorem.

**Theorem 1.2** [2, Theorem 16.6, p. 225]. The binomial \( x^n - a \), \( a \in K \), is irreducible over \( K \) if and only if \( a \in K^p \) for all primes \( p \) dividing \( n \) and \( a \not\in -4K^4 \) whenever \( 4|n \).
Allover in this paper the base field $K$ will be of the second kind with respect to 2 and of characteristic different from 2. If $a \in K$ and $n$ is a fixed positive integer, then we denote by $H_n(a)$ the greatest integer $s$ in the interval $[0, n]$ such that $a \in K(\varepsilon_2)^{2^s}$. As usually we assume that 0 and 1 are the trivial idempotents of an algebra.

2. Decomposition of Special Binomials over a Field in Irreducible Factors

If it is not explicitly stated, in this section we assume that $K \neq K(\varepsilon_2)$, i.e. $-1 \notin K^2$.

Lemma 2.1. For every $n \geq 2$ the only conjugated over $K$ element $\bar{\varepsilon}_n$ of $\varepsilon_n$ is $\varepsilon_n^{-1}$.

Proof. Since $\varepsilon_n$ is a root of the equation $x^{2^2n-1} + 1 = 0$ over $K$, we obtain that $\bar{\varepsilon}_n$ is a root of the same equation. Hence $\bar{\varepsilon}_n \in \langle \varepsilon_n \rangle$ and $\varepsilon_n \bar{\varepsilon}_n \in K \cap \langle \varepsilon_n \rangle = \{-1, 1\}$, i.e. $\varepsilon_n = t\varepsilon_n^{-1}$, where $t \in \{-1, 1\}$. Besides

$$\varepsilon_n + \varepsilon_n^{-1} = (\varepsilon_{n+1} + t\varepsilon_{n+1}^{-1})^2 - 2t = (\varepsilon_{n+1} + \varepsilon_{n+1})^2 - 2t \in K.$$ 

Therefore $\varepsilon_n + \varepsilon_n^{-1} \in K$ and $\varepsilon_n \varepsilon_n^{-1} = 1 \in K$, which shows that $\bar{\varepsilon}_n = \varepsilon_n^{-1}$.

Lemma 2.2. Let $\alpha \in K(\varepsilon_2)$. Then the following conditions are equivalent.

(i) $\alpha \in K$;
(ii) $\bar{\alpha} = \alpha$, where $\bar{\alpha}$ is the conjugated of $\alpha$;
(iii) $N(\alpha) = \alpha^2$.

Proof. The equivalence of (i) and (ii) is obvious. The equivalence of (ii) and (iii) follows from the equality $\alpha^2 = N(\alpha) = \alpha\bar{\alpha}$.

Lemma 2.3. The polynomial $f(x) = x^{2^n} - a$, $a \in K$, $n \geq 2$, is irreducible over $K$ if and only if $a \notin K^2 \cup (-K^4)$. 

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Really since \[ 4 = (\varepsilon_3 + \varepsilon_3^{-1})^4 = (\varepsilon_3 + \varepsilon_3) \in K^4, \] then \[ 4K^4 = K^4. \]

Then the proof of the lemma follows from Theorem 1.2.

**Lemma 2.4.** Let \( \alpha \in K(\varepsilon_2) \). Then \( \alpha^{2^n} \in K \) if and only if \( \alpha = a\varepsilon_{n+1}^{l}, a \in K, l \in \mathbb{Z}. \)

**Proof.** Let \( \alpha^{2^n} \in K \). By Lemma 2.2 we see that \( (N(\alpha))^{2^n} = N(\alpha^{2^n}) = \alpha^{2^{n+1}}. \) Therefore we obtain \( N(\alpha) = \alpha^{2\varepsilon_n^{-l}} = (\alpha\varepsilon_{n+1}^{-l})^2 \) for some \( l \in \mathbb{Z} \). On the other hand, by Lemma 2.1 we have that \( N(\varepsilon_{n+1}^{-l}) = 1. \) Hence \( N(\alpha\varepsilon_{n+1}^{-l}) = N(\alpha) = (\alpha\varepsilon_{n+1}^{-l})^2 \) and by Lemma 2.2 it follows that \( \alpha\varepsilon_{n+1}^{-l} \in K \), i.e. \( \alpha = a\varepsilon_{n+1}^{l}. \) The inverse statement of the lemma is obvious.

**Lemma 2.5.** For every \( n \in \mathbb{N} \) it holds

\[ K \cap K(\varepsilon_2)^{2^n} = K^{2^n} \cup (-K^{2^n}). \]

**Proof.** Let \( u \in K \cap K(\varepsilon_2)^{2^n}. \) Then \( u = \alpha^{2^n}, \alpha \in K(\varepsilon_2) \) and, by Lemma 2.4, \( \alpha = b\varepsilon_{n+1}^{l}, b \in K, l \in \mathbb{Z}. \) Therefore

\[ u = \alpha^{2^n} = b^{2^n}(-1)^l \in K^{2^n} \cup (-K^{2^n}), \]

i.e.

\[ K \cap K(\varepsilon_2)^{2^n} \subseteq K^{2^n} \cup (-K^{2^n}). \]

The opposite inclusion is obvious.

**Lemma 2.6.** The equality \( K^{2^n} \cap (-K^{2^n}) = 0 \) holds for arbitrary \( m, n \in \mathbb{N}. \)

**Proof.** The equality \( K^{2} \cap (-K^{2}) = 0 \) follows from the fact that \( \varepsilon_2 \notin K. \) Moreover \( K^{2^n} \cap (-K^{2^n}) \subseteq K^{2} \cap (-K^{2}) = 0. \)

**Lemma 2.7.** The polynomial \( f(x) = x^{2^n} - \alpha, \alpha \in K(\varepsilon_2), n \in \mathbb{N}, \)

is irreducible over \( K(\varepsilon_2) \) if and only if \( \alpha \notin K(\varepsilon_2)^2. \)
For $n \geq 2$ the proof follows from Lemma 2.3 applied to $K(\varepsilon_2)$ bearing in mind that $-K(\varepsilon_2)^4 \subseteq K(\varepsilon_2)^2$.

For $n \geq 1$ the proof follows from Theorem 1.2 applied to the field $K(\varepsilon_2)$.

Let $a \in K^*$ and $n \in \mathbb{N}$. We denote by $H_n(a)$ the greatest integer $s$ in the interval $[0, n]$ such that $a \in K(\varepsilon_2)^{2^s}$. Lemma 2.5 gives immediately that the integer $H_n(a)$ coincides with the greatest integer $s \in [0, n]$ such that $a \in K^{2^s} \cup (-K^{2^s})$. If $a \in K^{2^s}$, then we call the element $a$ of the first kind and if $a \in -K^{2^s}$, then we call $a$ an element of the second kind. Clearly the integer $H_n(a)$ always exists and is uniquely determined, by Lemma 2.6 the kind of $a$ is also completely determined. We call the integer $H_n(a)$ the $n$-height of $a$ in $K^*$.

**Theorem 2.8.** Let $K$ be a field of the second kind with respect to $2$ and of characteristic different from $2$, let $f(x) = x^{2^n} - a$ be a polynomial over $K$, $a \neq 0$, $n \in \mathbb{N}$, and let $H_n(a) = s$. Then

1) If $K = K(\varepsilon_2)$, then

\[ f(x) = \prod_{i=0}^{2^s-1} (x^{2^{n-s}} - b^{i_s} \varepsilon), \quad a = b^{2^s}, \quad b \in K. \]

and the factors of (1) are irreducible polynomials over $K$.

2) If $K \neq K(\varepsilon_2)$ and 2.1) $s = 0$ or 2.2) $s = 1$ and the element $a$ is of the second kind, then $f(x)$ is irreducible over $K$.

In the other cases $f(x)$ is decomposed in irreducible factors over $K$ in the following way.

3) If $K \neq K(\varepsilon_2)$, $s \geq 1$, and $a$ is of the first kind, then

\[ f(x) = (x^{2^{n-s}} - b)(x^{2^{n-s}} + b) \prod_{i=1}^{2^{s-1}-1} \left[ x^{2^{n-s}+1} - (\varepsilon_s^i + \varepsilon_s^{-i})bx^{2^{n-s}} + b^2 \right], \]

\[ a = b^{2^s}, \quad b \in K. \]
4) If $K \neq K(\varepsilon_2)$, $s \geq 2$, and $a$ is of the second kind, then

$$f(x) = \prod_{i=0}^{2^{s-1}-1} [x^{2^{n-s+1}} - (\varepsilon_s^i + \varepsilon_{i-1}^s)\varepsilon_{s+1}bx^{2^n-s} + b^2], \ a = -b^{2^s},$$

$b \in K$.

Proof. 1) Let $K = K(\varepsilon_2)$. Obviously (1) is a decomposition of $f(x)$ over $K$. If $s < n$, then $b\varepsilon_i \not\in K^2$, $-K^4 \subseteq K^2$ and by Lemma 2.7 the factors of (1) are irreducible polynomials over $K$. The case $s = n$ is trivial.

2) Let $K \neq K(\varepsilon_2)$ and 2.1) $s = 0$ or 2.2) $s = 1$ and $a$ is of the second kind.

If $n = 1$, then in the case 2.1) we have $a \not\in K^2$ and in the case 2.2) by Lemma 2.6 we have again $a \not\in K^2$. Now Theorem 1.2 gives the irreducibility of $f(x)$ over $K$.

Let $n \geq 2$. If $s = 0$, then $a \not\in K(\varepsilon_2)^2 \supset (-K^4)$, i.e. $a \not\in K^2 \cup (-K^4)$. Therefore, by Lemma 2.3, the polynomial $f(x)$ is irreducible over $K$. Let $s = 1$ and the element $a$ is of the second kind. Then, by the definition of $a$, it follows that $a \in (-K^2) \setminus (-K^4)$ and, by Lemma 2.6, $a \not\in K^2 \cup (-K^4)$. Thus, again by Lemma 2.3, $f(x)$ is irreducible over $K$.

3) Let $K \neq K(\varepsilon_2)$, $s \geq 1$, and let $a$ be of the first kind. Then $a = b^{2^s}$, $b \in K$. We obtain the decomposition (1) of the polynomial $f(x)$ over $K(\varepsilon_2)$. We shall show that all the factors of $f(x)$ in (1) are irreducible over $K(\varepsilon_2)$. Really, if $s < n$, then by the definition of $H_n(a)$ we obtain that $b \not\in K(\varepsilon_2)^2$ and, since $\varepsilon_s^i = \varepsilon_{s+1}^{2^i} \in K(\varepsilon_2)^2$, it follows that $b\varepsilon_i \not\in K(\varepsilon_2)^2$. Now, by Lemma 2.7 the factors in (1) are irreducible over $K(\varepsilon_2)$. For $s = n$ the factors of (1) are of the first degree and also are irreducible over $K(\varepsilon_2)$. Grouping and multiplying the conjugated over $K$ factors in (1), by Lemma 2.1, we obtain that the factors of (2) are with coefficients from $K$. Their irreducibility over $K$ follows from the irreducibility of the factors of (1) over $K(\varepsilon_2)$.
4) Let \( K \neq K(\varepsilon_2) \), \( s \geq 2 \), and let \( a \) be of the second kind. Then \( a = -b^{2^s} \), \( b \in K \), and we obtain the following decomposition of the polynomial \( f(x) \) over \( K(\varepsilon_2) \)

\[
f(x) = \prod_{i=0}^{2^s-1} (x^{2^{n-s}} - b\varepsilon_{s+1}\varepsilon_s^i).
\]

As in the case 3) we see that (4) is a decomposition of \( f(x) \) in irreducible factors over \( K(\varepsilon_2) \) and (4), in view of Lemma 2.1, gives the decomposition (3) of \( f(x) \) in irreducible factors over \( K \).

3. Minimal Idempotents of Twisted Group Algebras of Cyclic 2-Groups

**Theorem 3.1.** Let \( K \) be a field of the second kind with respect to 2 and of characteristic different from 2 and let \( \langle g \rangle \) be a cyclic group of order \( 2^n \). Let the twisted group algebra \( K^t\langle g \rangle \) be defined by the equality \( \bar{g}^{2^n} = a, a \in K^* \), and let \( H_n(a) = s \). Then the minimal idempotents \( e_i \) and \( f_k \) of \( K^t\langle g \rangle \) are the following.

1) If \( K = K(\varepsilon_2) \) then

\[
e_i = \frac{1}{2^s} \sum_{j=0}^{2^s-1} \varepsilon_s^{-ij} b^{-j} \bar{g}^{2^{n-s}j}, \quad i = 0, 1, \ldots, 2^s - 1,
\]

where \( b^{2^s} = a, b \in K \).

2) If \( K \neq K(\varepsilon_2) \) and 2.1) \( s = 0 \) or 2.2) \( s = 1 \) and the element \( a \) is of the second kind then the only minimal idempotent of \( K^t\langle g \rangle \) is the unity.

3) If \( K \neq K(\varepsilon_2) \), \( s \geq 1 \), and \( a \) is of the first kind then

\[
f_k = \frac{1}{2^s} \sum_{j=0}^{2^s-1} \delta_{kj} b^{-j} \bar{g}^{2^{n-s}j}, \quad k = 1, 2; \quad \delta_{1j} = 1, \quad \delta_{2j} = (-1)^j,
\]

\[
e_i = \frac{1}{2^s} \sum_{j=0}^{2^s-1} (\varepsilon_s^{ij} + \varepsilon_s^{-ij}) b^{-j} \bar{g}^{2^{n-s}j}, \quad i = 1, 2, \ldots, 2^{s-1} - 1,
\]
where $b^{2^s} = a$, $b \in K$.

4) If $K \neq K(\varepsilon_2)$, $s \geq 2$, and $a$ is of the second kind then

\[ e_i = \frac{1}{2^n} \sum_{j=0}^{2^s-1} (\varepsilon_s^j + \varepsilon_s^{-j})\varepsilon_s^j b^{-j} g 2^{n-s-j}, \quad i = 0, 1, \ldots, 2^s-1 - 1, \]

where $-b^{2^s} = a$, $b \in K$.

Proof. Let $L$ be the splitting field of the polynomial $f(x) = x^{2^n} - a$ over $K$. It is well known, that the minimal idempotents of the group algebra $L(g)$ are

\[ e_\beta = \frac{1}{2^n} \sum_{i=0}^{2^n-1} \beta^{-i} g^i, \]

where $\beta$ runs on the zeros of the polynomial $x^{2^n} - 1$. Since $L$ is a splitting field of the polynomial $f(x)$, then there exists $\gamma \in L$, such that $\gamma^{2^n} = a$. Then the equality $\bar{g}^{2^n} = a$ implies $(\gamma^{-1}\bar{g})^{2^n} = 1$ and therefore the cyclic group $\langle \gamma^{-1}\bar{g} \rangle$ is a group basis of the twisted group algebra $L^t(g)$, i.e. $L^t(g)$ coincides with the group algebra $L(\gamma^{-1}\bar{g})$. Hence the minimal idempotents of $L^t(g)$ are obtained from (4) by replacing $g$ with $\gamma^{-1}\bar{g}$. So we obtain that the minimal idempotents of $L^t(g)$ will be of the form

\[ e_\beta = \frac{1}{2^n} \sum_{i=0}^{2^n-1} (\beta\gamma)^{-i} \bar{g}^i, \]

When $\beta$ runs on the zeros of $x^{2^n} - 1$, $\beta\gamma$ will run on the zeros of $f(x)$. Therefore we can set $\beta\gamma = \alpha$ and the minimal idempotents of the twisted group algebra $L^t(g)$ will be of the form

\[ e_\alpha = \frac{1}{2^n} \sum_{j=0}^{2^n-1} \alpha^{-j} \bar{g}^j, \]

where $\alpha$ runs on the zeros of $f(x)$. From here we shall obtain the minimal idempotents of $K(\varepsilon_2)^t(g)$ summing the minimal idempotents $e_\alpha$, where
α runs on the zeros of an arbitrary fixed irreducible factor of \( f(x) \) over \( K(\varepsilon_2) \). By Theorem 2.8 the irreducible factors of \( f(x) \) over \( K(\varepsilon_2) \) are of the form \( \varphi_i(x) = x^{2^n-s} - \lambda \varepsilon_i^s, i = 0,1,\ldots,2^s - 1 \), where \( \lambda^{2^s} = a, \lambda \in K(\varepsilon_2) \). Let \( \mu^{2^n-s} = \lambda, \mu \in L \). The zeros of \( \varphi_i(x) \) are \( \mu \varepsilon_i^n \varepsilon_{n-s}^r, r = 0,1,\ldots,2^n-s-1 \), and the idempotents \( e_i \) of \( K(\varepsilon_2)^t\langle g \rangle \) are

\[
e_i = \frac{1}{2^n} \sum_{j=0}^{2^n-1} \left( \sum_{r=0}^{2^n-s-1} \varepsilon_i^{-rj} \right) \mu^{-j} \varepsilon_i^{-ij} \bar{g}^j.
\]

For a fixed \( j \) not divisible by \( 2^n-s \) the sum in the brackets is equal to zero. Therefore if we replace \( j \) by \( j2^n-s, j = 0,1,\ldots,2^n-s-1 \), we shall obtain

\[
(5) \quad e_i = \frac{1}{2^s} \sum_{j=0}^{2^s-1} \varepsilon_i^{-ij} \lambda^{-j} \bar{g}^j 2^{n-s}, i = 0,1,\ldots,2^s-1.
\]

Now in the case 1) for \( K = K(\varepsilon_2) \) we assume \( \lambda = b \in K \) and obtain the formula (1). The case 2) is clear because in this case \( f(x) \) is irreducible over \( K \). In the case 3) we have again \( \lambda = b \in K \) and \( e_0 = f_1 \in K^tG \), \( e_{2^n-1} = f_2 \in K^tG \). The other idempotents in this case are obtained from (5) summing the pairs of conjugates over \( K \). In this way the formula (2) is completely established. In the case 4) by Lemma 2.4 we have \( \lambda = b\varepsilon_{s+1}, b \in K \). The minimal idempotents in this case are also obtained from (5) summing the pairs of conjugates over \( K \) and this gives the formula (3).

As a consequence of Theorem 3.1 one can obtain the minimal idempotents of the factor-algebra \( K[x]/I \), where \( I \) is the ideal of the algebra \( K[x] \) generated by the polynomial \( x^{2^n} - a, a \in K^* \). They are obtained from the idempotents from Theorem 3.1 assuming that \( \bar{g} = x + I \).

We shall note that when \( \langle g \rangle \) is a cyclic 2-group and \( K \) is a field of the first kind with respect to 2 then the problem of finding the explicit form of the minimal idempotents of the twisted group algebra \( K^t\langle g \rangle \) is
open because of the serious problems that arise with the decompositions of the polynomials $x^{2^n} - a$ into irreducible factors over the field $K$.

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