Interior design of a two-dimensional semiclassical black hole

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We look into the inner structure of a two-dimensional dilatonic evaporating black hole. We establish and employ the homogenous approximation for the black-hole interior. The field equations admit two types of singularities, and their local asymptotic structure is investigated. One of these singularities is found to develop, as a spacelike singularity, inside the black hole. We then study the internal structure of the evaporating black hole from the horizon to the singularity.

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INTRODUCTION

The discovery of Hawking radiation and the black-hole (BH) evaporation phenomenon\(^1\) raised several outstanding open questions and intriguing puzzles. One such problem which attracted a lot of attention is the information puzzle: Simple thought experiments suggest that in the process of black-hole formation and evaporation, a pure initial quantum state will evolve into a mixed state, and consequently part of the information encoded in the initial state will get lost.

Apparently this problem of information loss is intimately related to another conceptual problem in black-hole physics: The formation of a spacetime singularity inside the BH. Pictorially speaking, a portion of the initial information propagates to the singularity and disappears there. It is widely anticipated, however, that the formation of spacetime singularities is a mere artifact of the classical (and semiclassical) theory, but Quantum Gravity will eventually resolve the black-hole singularities, and presumably solve the information puzzle as well.

In 1992, Callan, Giddings, Harvey, and Strominger (CGHS)\(^2\) proposed a simplified framework for investigating black-hole formation and evaporation. They introduced a two-dimensional toy-model with gravity and matter fields coupled to a dilaton scalar field. At the classical level this model admits a one-parameter family of static black-hole solutions (parameterized by their mass). When semiclassical corrections are added, the two-dimensional black hole emits thermal radiation and consequently evaporates. Based on the trace anomaly, CGHS provided simple explicit expressions for the semiclassical contribution to the energy-momentum tensor. Consequence the semiclassical dynamics may be formulated as a closed (constrained) system of second-order partial differential equations. It thus provides a simple framework for exploring various aspects of black-hole evaporation, and particularly the information puzzle\(^3\).

Originally it was hoped\(^2\) that the CGHS evaporating BH will be free of any singularities. However, Russo, Susskind and Thorlacius\(^4\) soon found that a singularity inevitably develops inside the CGHS BH, at a certain value of the dilaton filed. Thus, the CGHS formalism does not resolve the information puzzle at the semiclassical level. One may still hope, however, that when the model is fully quantized the singularity (and the information puzzle) will be resolved. This approach was pursued by Ashtekar, Taveras, and Varadarajan\(^5\), who formulated the quantum-field analog of the two-dimensional CGHS model. In this set-up, the original CGHS variables are replaced by quantum operators. Spacetime evolution is then described by a system of operator partial differential equations. The exact solution to these operator equations is not known. Nevertheless, some approximate solutions were constructed\(^6\), illuminating certain aspects of the problem of black-hole evaporation, and suggesting that the singularity will indeed be resolved in the quantized theory.

The above discussion signifies the inevitable spacetime singularity as a key feature of the CGHS semiclassical BH spacetime. However, the detailed structure and properties of this singularity have not been investigated so far to the best of our knowledge. The main purpose of this manuscript is to present such a detailed analysis of the structure of that singularity. The motivation for this investigation is obvious from the discussion above: First, since this singularity plays such a crucial role in the information-loss puzzle, it will be useful to better understand its properties. For example, one would like to know how strong this singularity is, and what are the prospects for extending semiclassical spacetime beyond it. Second, understanding the asymptotic behavior of the various fields at the singularity may provide a useful starting point for exploring how quantum treatments (like that of Ref.\(^6\) for example) may resolve the singularity.

Although the main objective of this paper is the asymptotic behavior near the singularity, we also analyze here the internal structure of the evaporating semiclassical BH in the entire range from the horizon to the singularity. We do this by constructing approximate solutions in various domains of the BH interior, and then matching these solutions at their respective overlap regions. Understanding the entire BH internal structure is interesting on its own right, but is also important for full determination of the singularity structure: When the
latter is derived by a purely local analysis, one obtains a family of local asymptotic solutions which depend on certain free parameters (or free functions, depending on the context). The value of these parameters in an actual BH solution needs to be determined by matching to initial conditions (e.g., at the horizon). The full internal solution, from the horizon to the singularity, is required for obtaining the right values of these parameters.

As long as the evaporating black hole is macroscopic, the semiclassical effects are very weak in a local dynamical sense. This allows one to employ the adiabatic approximation. Namely, any local region of spacetime may be well approximated by a certain classical CGHS BH solution (with a certain mass parameter). The classical BH solutions are static outside the horizon but homogeneous inside it, implying that the interior of a macroscopic evaporating BH is (locally) approximately homogeneous. This homogeneous approximation is a key element in our investigation. It greatly simplifies the analysis, as the field equations now reduce to ordinary differential equations.

In Sec. II we present the CGHS action and field equations, and reformulate them in new variables. Sec. III is devoted to the homogenous approximation. We explore its main properties and its domain of validity. In Sec. IV we analyze the singularities of the homogenous field equations, and obtain the local asymptotic behavior of the solution near these singularities. In Sec. V we construct an approximate solution for the fields inside the evaporating BH, in the entire range from the horizon to the spacelike singularity.

In Sec. VI we present simple effective Lagrangian and Hamiltonian which yields the local asymptotic dynamics near the spacelike singularity. The construction of such effective Lagrangian and Hamiltonian is motivated by the idea, that perhaps it will be possible to employ simple quantum-mechanical considerations to resolve the singularity (and thereby to explore the possible extension of spacetime beyond it). Although such a quantum-mechanical treatment is beyond the scope of the present paper, we nevertheless take here the preparatory step (carried out entirely within the semiclassical framework) of constructing the effective near-singularity Lagrangian and Hamiltonian. Finally, in the last section we summarize our main results and discuss their significance.

**THE MODEL AND FIELD EQUATIONS**

We begin by presenting the two-dimensional semiclassical CGHS model. Beside the dilaton $\phi$, this model contains a large number $N \gg 1$ of identical scalar matter fields $f_i$, and a cosmological constant $\lambda^2$. We express the metric in double-null coordinates $u, v$ (the “conformal gauge”), namely $ds^2 = -e^{2\phi}dudv$. The action then reads

$$\frac{1}{8} \int d^2\sigma [e^{-2\phi}(-2\rho_{uv} + 4\phi_{,u} \phi_{,v} - \lambda^2 e^{2\phi}) - \frac{1}{2} \sum_{i=1}^{N} f_{,iu} f_{,iv} + \frac{N}{12} \rho_{,u} \rho_{,v}].$$

(1)

The last term in the action expresses the semiclassical effects. The Einstein equations take the form:

$$T_{uv} = e^{-2\phi} (2\phi_{,uv} - 4\phi_{,u} \phi_{,v} - \lambda^2 e^{2\phi}) - \frac{N}{12} \rho_{uv} = 0$$

and

$$T_{uu} = e^{-2\phi} (4\rho_{,u} \phi_{,u} - 2\phi_{,uu}) + \frac{1}{2} \sum_{i=1}^{N} f_{,iu} f_{,iu}$$

$$- \frac{N}{12} (\rho_{,u} \rho_{,u} - \rho_{uu} - \tilde{z}_u(u)) = 0,$$

$$T_{vv} = e^{-2\phi} (4\rho_{,v} \phi_{,v} - 2\phi_{,vv}) + \frac{1}{2} \sum_{i=1}^{N} f_{,iv} f_{,iv}$$

$$- \frac{N}{12} (\rho_{,v} \rho_{,v} - \rho_{vv} - \tilde{z}_v(v)) = 0,$$

where $\tilde{z}_u(u)$ and $\tilde{z}_v(v)$ encode the information about the initial quantum state, which in turn determines the semi-classical fluxes. The dilaton and matter equations are:

$$- 4\phi_{,uv} + 4\phi_{,u} \phi_{,v} + 2\rho_{uv} + \lambda^2 e^{2\phi} = 0,$$

$$\partial_u \phi, f_i = 0.$$  

Throughout this paper we set $\lambda = 1$. This choice is equivalent to the change of variable $\rho' = \rho + \ln(\lambda)$, which does not affect the field equations otherwise. We also set $f_i = 0$, as we are dealing here with the evaporation rather than formation of the BH.

Following Ref. [1], we define new variables: $R \equiv e^{-2\phi}$, $S \equiv 2(\rho - \phi)$. A straightforward substitution of the new variables in Eqs. (2) yields:

$$R_{uv} = -e^S - K \rho_{uv},$$

$$S_{uv} = K \rho_{uv} / R,$$

where $\rho = \frac{1}{2}(S - \ln R)$ is to be substituted, and $K \equiv N/12$ expresses the magnitude of the quantum effects. The constraint equations become:

$$R_{uw} - R_{uw} S_{uw} + \hat{T}_{uw} = 0,$$

where hereafter $u$ stands for either $u$ or $v$. The semi-classical energy fluxes $\hat{T}_{uw}$ along both null directions are given by:

$$\hat{T}_{uw} = K \left[ \rho_{uw} - \rho^2_{uw} + \tilde{z}_u(w) \right].$$

It is useful to re-express the system of evolution equations (8) in its standard form, in which $R_{uw}$ and $S_{uw}$ are explicitly given in terms of lower-order derivatives:

$$R_{uw} = -e^S (2R - K) - R_{uu} R_{uv} \frac{K}{2R(2R - K)},$$

$$S_{uw} = e^S \frac{K}{2R(R - K)} + R_{uu} R_{uv} \frac{K}{2R(2R - K)}.$$
This form makes it obvious that the evolution equations become singular when $R = K$. This singularity, which in the original variables takes place at $\phi = -\frac{1}{2}\ln(\frac{M}{K})$, was already noticed previously. Below we shall explore in some detail the homogeneous variant of this singularity.

The semi-classical equations reduce to the classical ones by setting $K = 0$:

$$
R_{uv} = -e^S, \\
S_{uv} = 0,
$$

and the constraint equations:

$$
R_{uvw} - R_{uw} S_{vw} = 0.
$$

This set of equations admits a one-parameter family of solutions (up to gauge transformations), which is the dilatonic two-dimensional analog of the standard Schwarzschild solution. Each member of this family describes a black-hole space-time, which (like its standard four-dimensional counterpart) is static outside the BH and homogeneous inside it. Throughout this paper we shall simply refer to this class of solutions as the "Schwarzschild solution" (despite a slight abuse of standard terminology). We shall focus on the internal part of the BH. In Eddington-like double-null coordinates the internal solution takes the form

$$
R = -e^{\nu + u} + M, \\
S = v + u,
$$

where $M$ is the (dilatonic, 2-dimensional) Schwarzschild mass parameter.

**HOMOGENEOUS SET-UP**

**Justification and domain of validity**

We shall now restrict our attention to the homogeneous solutions of the semiclassical field equations inside the evaporating BH, namely, solutions which only depend on $t$, where we define

$$
t \equiv v + u, \\
x \equiv v - u.
$$

We first need to discuss the justification for this homogeneous approximation and its domain of validity. The classical interior solution is obviously homogeneous as it depends on $v + u$ solely. In an evaporating BH, however, the semiclassical effects spoil the exact homogeneity, as manifested by the drift in the BH mass. Yet, as long as the BH is macroscopic ($M >> K$), this drift is very slow. Indeed it undoubtedly has a dramatic effect over the long evaporation time-scale—the BH eventually disappears (or at least becomes microscopic) after all. Yet, from a local point of view this drift is of negligibly small rate. Thus, over a typical dynamical time/length scale (say, $\Delta v, \Delta u$ of order $\sim 10$), the change in $M$ is $\Delta M << M$ and may be neglected, which naturally leads to the homogeneous approximation.

The local, relative magnitude of the semiclassical terms is of order $K/R$, as can be seen in the evolution equations. As long as the BH is macroscopic ($M >> K$), and as long as we are dealing with the portion $R >> K$ of the BH space-time, we may apply the adiabatic point of view to the evaporation process: Namely, in each region of space-time the solution is approximately Schwarzschild, with a local effective mass parameter $M$ which slowly drifts in $v$ and/or $u$. We shall refer to this slowly varying function $M(v, u)$ as the effective mass of the evaporating BH. (In fact, as long as the interior of the BH is considered, the effective mass depends only on $v$.)

The adiabatic approximation, wherever applicable, automatically implies approximate homogeneity inside the BH—simply because the Schwarzschild interior solution is homogeneous.

It should be emphasized, however, that the adiabatic approximation does not apply in the small-$R$ region near the singularity, where $R$ becomes comparable to $K$. In this region there are strong local semiclassical effects, and the local solution is very different from Schwarzschild (as demonstrated in the next section). We argue, however, that the homogeneous approximation is still valid in this small-$R$ domain. Generally speaking, this follows from causality: Since the small-$R$ region is located at the causal future of the moderate-$R$ interior region, it simply inherits the approximate homogeneity of the latter.

To be more specific, let us pick a point inside the evaporating BH, in a region where the effective mass is still macroscopic, $M(v) \equiv M_0 >> K$. This point is picked at a certain $R$ value, $R = R_0$, which is $>> K$—say $R_0 = M_0/2$ (recall that the horizon is located at $R \approx M$). Let us denote the coordinates of this point by $(v_0, u_0)$, and let $t_0 \equiv v_0 + u_0, x_0 \equiv v_0 - u_0$. Let $\Sigma$ denote an initial surface which is the line $t = t_0$ restricted to the range $x_0 \leq x \leq x_0 + \Delta x$, with $\Delta x$ taken to be $\sim 10$ (say). This construction is illustrated in Fig.1. We shall now consider two cases, namely two slightly different solutions: In case (i), we specify on $\Sigma$ initial data which exactly correspond to the classical Schwarzschild solution with mass $M(v_0) \equiv M_0$. Since these data are independent of $x$ by construction, the evolving solution will be precisely homogeneous, throughout the domain of dependence $D_+(\Sigma)$. Provided that $\Delta x$ is taken to be large enough, $D_+(\Sigma)$ will reach the small-$R$ space-like singularity (more precisely, the $R = K$ singularity which we explore in the next section). To this end $\Delta x$ only need to exceed the span of $t$ from $t_0$ to the singularity (of the homogeneous solution). This span is of order 1 (for instance, for $R_0 = M_0/2$ this $t$-span is $\ln(1/2)$ in the classical solution, and approximately the same number in the semiclassical solution). Hence taking $\Delta x \sim 10$.
In case (ii), we specify on $\Sigma$ the initial data which correspond to the actual evaporating BH solution. Since both $M_0$ and $R_0$ are $\gg K$, the adiabatic approximation applies, and the solution in the neighborhood of $(v_0, u_0)$ is approximately Schwarzschild. Therefore, we may regard the actual initial data (ii) as the homogeneous, Schwarzschild data of case (i) plus a small perturbation. This initial perturbation will evolve into a small perturbation throughout the interior of $D_+(\Sigma)$. The relative magnitude of this perturbation will be controlled by the small parameter $K/M_0$, and for sufficiently small value of this parameter we may neglect the perturbation at the leading order.

The smallness of the perturbations in the interior of $D_+(\Sigma)$ is then guaranteed by standard stability theorems for nonlinear hyperbolic systems. One may be concerned, however, about the effect of the perturbation in the immediate neighborhood of the small-$R$ singularity, and in particular on the very structure of the latter. A close examination, which is beyond the scope of the present paper, reveals that this singularity is locally stable to inhomogeneities. (That is, an inhomogeneous variant of the $R = K$ singularity exists, and this inhomogeneous singularity is generic, in the sense that it depends on four arbitrary functions of $x$.)

We conclude that in the macroscopic-mass domain ($M_0 \gg K$) the homogeneous solution provides a good approximation not only in the domain $R > R_0$, but also up to the small-$R$ singularity. This was also verified numerically by directly integrating the field equations with initial data corresponding to a BH which forms by a collapsing null shell and subsequently evaporates. In particular it was numerically verified that the structure of the evolving spacelike singularity is well described by the homogeneous $R = K$ singularity (described in the next section).

In the above construction we considered for concreteness an initial hypersurface $\Sigma$ located at $R = M_0/2$. We could have started instead at any other $R = R_0$ initial value which satisfied $K << R_0 < M_0$. The evolving homogeneous solution depends very weakly on $R_0$, because of the local smallness of the semiclassical corrections at $R > R_0$. For concreteness, and in order to avoid the arbitrariness associated with the extra parameter $R_0$, in the analysis below we shall consider the homogeneous solution obtained at the limit $R_0 \to M_0$. The solution obtained in this way may be thought of as the (horizon-regular) semiclassical counterpart of the homogeneous interior Schwarzschild solution. It carries (besides $K$) a single parameter $M_0$, representing the BH’s remaining mass at the epoch of interest. In the rest of the paper we shall simply refer to this parameter as $M$ for brevity.

Homogeneous field equations

In the homogeneous framework the evolution equations become

$$\ddot{R} = -e^{S} \left( \frac{2R-K}{2(R-K)} \right) - \dot{R}^2 \frac{K}{2(R-K)^2},$$

$$\ddot{S} = e^S \frac{K}{2R(R-K)} + \dot{R}^2 \frac{K}{2R^2(R-K)},$$

where an over-dot denotes differentiation with respect to $t$. The constraint equation now reads

$$\ddot{R} - \dot{R} \dot{S} + \ddot{T} = 0,$$

where $\ddot{T} = \ddot{T}_{vv} = \ddot{T}_{uu}$ is given by

$$\ddot{T} = K[\ddot{\rho} + \dot{\rho}^2 + \ddot{\tau}],$$

and $\ddot{\tau} \equiv \ddot{z}_v = \ddot{z}_u$. Note that the equalities $\ddot{T}_{vv} = \ddot{T}_{uu}$ and $\ddot{z}_v = \ddot{z}_u$ are direct consequences of the homogeneous set-up. The latter equality also implies that $\ddot{\tau}$ must be a constant. (By homogeneity $\ddot{\tau}$ could at most depend on $t$, but then $d\ddot{\tau}/dt = \ddot{\tau}_v = \ddot{\tau}_u = 0.$)
The Schwarzschild solution inside the BH may be expressed in the explicitly homogeneous form
\[ R = -e^t + M, \quad S = t. \]
In particular it satisfies the relations
\[ \dot{R} = -e^S = R - M, \]
which will be useful in the analysis below. Note also the classical homogeneous constraint equation \( \ddot{R} - \dot{R} \dot{S} = 0. \)

**SINGULARITIES**

The semiclassical equations are singular at the points \( R = 0 \) and \( R = K \), as can be seen from Eqs. (12,13). In this section we will look at the asymptotic behavior near these two special \( R \) values.

Since we set the initial conditions for the homogeneous solution at \( R >> K \), the first singularity to be encountered is \( R = K \). As it turns out, this singularity is characterized by the divergence of \( \dot{R} \) and \( \dot{S} \), while \( R \) and \( S \) are finite.

We therefore assume that in the right-hand side of Eqs. (12,13) we can neglect the terms proportional to \( e^S \) compared to those \( \propto R^2 \), obtaining at leading order
\[ \dot{R} = -\frac{R^2}{2(K-R)}, \]
\[ \dot{S} = -\frac{R^2}{2K(R-K)}. \]
Eq. (18) constitutes a closed equation for \( R \), which is easily solved:
\[ R(t) = K + B |t - t_0|^{\frac{2}{3}}. \]
Then Eq. (19) is solved to yield
\[ S(t) = -\frac{B}{K} |t - t_0|^{\frac{4}{3}} + (t - t_0)B_2 + B_3. \]

The solution depends on four free parameters \( B, B_2, B_3, t_0 \) as required, hence this is a locally-generic asymptotic solution. We see that the variables \( R, S \) are continuous at \( R = K \), but \( \dot{R} \) diverges as \( |t - t_0|^{-1/3} \) (and the same for \( \dot{S} \)). This behavior, which is numerically demonstrated in Fig. 2, justifies our preassumption that \( e^S \) is indeed negligible compare to \( R^2 \).

In the second singular point \( R = 0 \), we expand Eqs. (12,13) near \( R = 0 \) under the same assumption as before, namely that the terms proportional to \( e^S \) may be ignored (this requires that \( e^S << R^2/R, \) which will again be justified a posteriori). We obtain
\[ \ddot{R} = -\frac{R^2}{2K}. \]
\[ \ddot{S} = -\frac{R^2}{2R^2}. \]

FIG. 2: A numerical plot of \( R \) (dashed) and of \( \dot{R} \) (solid) near the \( R = K \) singularity, for \( K = 1 \) and \( M = 10^2 \). While \( R \) is bounded, the divergence of \( \dot{R} \) at the singularity is clearly seen.

These equations are easily solved:
\[ R(t) = C_1 |t - t_0|^2 \]
and
\[ S(t) = C_2 + C_3 |t - t_0| + 2 \ln |t - t_0| \]
where \( C_1, C_2, C_3, t_0 \) are four arbitrary parameters. We find that the present situation differs from the \( R = K \) singularity, because now \( R \) (like \( R \)) is finite and \( S \) (like \( \dot{S} \)) diverges. Nevertheless, our preassumption \( e^S << R^2/R \) is still justified, because \( e^S \) vanishes as \( (t - t_0)^2 \) whereas \( R^2/R \) approaches a constant.

It is quite surprising to find that although the evolution equations are singular at \( R = 0 \), the evolving solution for both \( R \) and \( e^S \) is regular there. In particular, the metric function \( \rho = (S - \ln R)/2 \) is finite at \( R = 0 \). Note, however, that the dilaton \( \phi \) diverges there.

Our homogeneous solution starts at \( R = R_0 >> K \), and \( R \) decreases until the \( R = K \) singularity is reached. The divergence of \( \dot{R} \) and \( \dot{S} \) at \( R = K \) results in an inability to provide a unique prediction for the evolution of the fields beyond that point. Therefore the \( R = 0 \) singularity and its neighborhood are beyond our domain of prediction. In the rest of this paper we shall concentrate on the \( R = K \) singularity, as well as on the global behavior in the domain \( K < R < M \).

**CONSTRUCTING THE INTERIOR HOMOGENEOUS SOLUTION**

So far, we dealt with the local structure of the fields near the singularity. Our ultimate goal, however, is to understand the global behavior of the fields inside the BH. To this end, we need to follow the evolution of \( R(t) \) and \( S(t) \) from the initial hypersurface (say at \( R = M \)) up to the singularity at \( R = K \). We shall do this by designing a couple of analytical approximations, the union of which cover the entire domain \( K < R < M \). We shall also augment the analytic approximations by direct numerical integration of the homogeneous evolution...
Setting the initial conditions

The required initial conditions for the homogeneous solution are the four functions $R, \dot{R}, S, \dot{S}$, all set at a certain initial moment $t_0$. The basic strategy of setting the initial data follows from the discussion in section a. Suppose that we want to explore the interior of the evaporating BH at a stage (i.e. $v$ value) where its remaining mass is $M$ (with $M >> K$). Then we take the initial conditions to be those corresponding to the classical internal Schwarzschild solution with the same mass parameter $M$, at a hypersurface $R = \text{const} \equiv R_0$, at the limit $R_0 \to M$. From Eq. (10) this amounts to setting $R = M$, $\dot{R} = 0$, $\dot{S} = 1$, and $S = t_0 \to -\infty$. In actual numerical implementations we pick a large negative value of $t_0$ (say, $t_0 = -30$) and set $S = t_0$. Note that in such a horizon-limit setup of initial conditions the choice $\dot{R} = 0$ is crucial for regularity. Any other choice of initial $\dot{R} = 0$ at $t \to -\infty$ will lead to a solution which lacks a regular horizon, and is hence inappropriate for approximating the spacetime of an evaporating BH.

For this initial-value setup, the constraint equation (14) implies that $\ddot{T}$ vanishes at the horizon limit $t \to -\infty$. Eq. (15) then yields $\dot{z} = 1/4$. (15)

Approximate solutions in the different regimes

We need to evolve the fields $R, S$ from the initial hypersurface, which we set to be at $R \to M$, up to the singularity at $R = K$ (recall we assume $M >> K$). We notice three important domains in this overall range: (i) the macroscopic domain $R >> K$. (ii) very close to the $R = K$ singularity, and (iii) the domain $K << R << M$

Note that (iii) is an extension of (ii), and it overlaps with (i). We shall now discuss the approximate solution in each of these domains.

The macroscopic (or classical) domain

It is easy to see that in the domain $R >> K$ the quantum contribution to the right-hand side of Eq. (12) is negligible compared to the classical contribution ($K = 0$). The same applies to Eq. (13). Therefore, in this domain the semiclassical solution may be well approximated by the classical solution (16). (For numerical verification see Fig. 4 below.)

Near the singularity

This domain is characterized by the inequality $R - K << K$. The approximate solution in this region was constructed in the previous section, Eqs. (20),(21). This solution is characterized by four parameters, of which the most significant one is $B$.

The small-$R$ approximation ($R << M$)

We seek an intermediate approximate solution, in the domain where $R$ is << $M$, yet it is not quite close to $K$ (say, $R \sim 3K$). Formulating the desired approximation, that will apply both near the singularity and in the portion $R << M$ of the classical domain, requires understanding the common basis of these two approximations. Consider the expression given in the right-hand side of Eq. (12) for $\dot{R}$. In the near-singularity approximation the first term in the right-hand side (which is $\propto e^S$) is negligible compared to the second one since $\dot{R}$ diverges, whereas $S(t)$ is bounded as seen in Eq. (21). One observes, however, that this term remains unimportant even in the portion $R << M$ of the classical domain. In this regime, both $\dot{R}$ and $-\dot{S}$ are given approximately by $R - M \cong -M$, see Eq. (17), hence $e^S/R^2$ scales as $1/M$. Therefore, if we fix $R$ and increase $M$, the first term in the right-hand side of Eq. (12) becomes negligible—just like in the near-singularity approximation. Omitting this term we obtain:

$$\dot{R} = -\dot{R}^2 \frac{K}{2R(R - K)}.$$ (26)

We shall refer to this approximation as the small-$R$ approximation. Note that the near-singularity approximation may be obtained from the small-$R$ approximation by substituting $R \cong K$, as may be seen by comparing Eqs. (26) and (13).

Solving Eq. (26) for $\dot{R}$ we obtain

$$\dot{R} = A \sqrt{\frac{R}{R - K}},$$ (27)

where $A$ is a free parameter. One more simple integration yields an expression for $t(r)$, which we do not need however.

In a particular evaporating-BH space-time, the parameter $A$ in Eq. (27) is to be determined from the effective mass parameter $M$. This can be done by matching the small-$R$ approximation and the classical approximation in their overlap domain $K << R << M$. Eqs. (27) and (17) respectively yield $\dot{R} \cong A$ and $\dot{R} \cong -M$ in this domain, implying

$$A \cong -M.$$ (28)

The small-$R$ approximation was established above by justifying it in both edges of its domain of validity.
(namely at $R \cong K$ and at $K << R << M$). We still need to demonstrate its validity in the region in between, where $R$ is of order a few times $K$. To this end we note from Eqs. (24, 28), that for fixed $R$ and $K$, $\dot{R}$ scales as $M$ (just like $e^S$). Therefore, in the right-hand side of Eq. (12) the second term scales as $M^2$, whereas the first term scales as $M$ and can therefore be neglected at the large-$M$ limit ($M >> K$). This justifies the small-$R$ approximation a posteriori, in its entire domain $K < R << M$.

The validity of the small-$R$ approximation, and its effectiveness compared to the near-singularity approximation, is demonstrated numerically in Fig. 3.

![FIG. 3: Plots of $R(t)$ for the full numerical solution (solid black), the near-singularity approximation (dotted black) and the small-$R$ approximation (dashed gray). All plots are for $K = 1$ and $M = 10^2$. (a) The domain where $K < R < 8K$. The large deviation of the near-singularity approximation is apparent. (b) A zoom at the domain closer to the singularity, where $K < R < 1.4K$.](image)

Finally, we can now determine the near-singularity coefficient $B$ in Eq. (29), as a function of the effective mass $M$. Differentiating Eq. (20) yields $\dot{R} = -\frac{3}{2}B^{3/2}\sqrt{1/(R-K)}$. Substituting Eq. (28) and $R \cong K << M$ in Eq. (27) yields $\dot{R} = -M\sqrt{K/(R-K)}$. Matching the two last expressions for $\dot{R}$, we obtain

$$B = \left(\frac{3}{2}\sqrt{KM}\right)^2. \tag{29}$$

This analytic expression for $B$ (valid at the leading order in $K/M$) may be verified numerically by evaluating $\dot{R}/\sqrt{R-K}$ at the limit where $R \rightarrow K$. For $K/M$ values of 0.01, 0.003, and 0.001, we numerically find the fractional deviation of numerical $B$ from (29) to be 0.016, 0.006, and 0.002, respectively—suggesting that this fractional deviation may scale as $K/M$.

**Global approximate solution**

A global approximate homogenous solution in the range $K < R < M$ can be derived by merging the classical approximation $\dot{R} \cong R - M$ and the small-$R$ approximation $\dot{R} \cong -M\sqrt{R-K}$ to obtain a single approximate expression:

$$\dot{R} \cong (R - M)\sqrt{\frac{R}{R-K}}. \tag{30}$$

Substituting $R >> K$ in Eq. (30) yields the classical approximation. Substituting $R << M$ yields the small-$R$ approximation (recall $A = -M$). Since the union of the domains $R >> K$ and $R << M$ covers the entire range of integration $K < R < M$ (recall that we assume $M >> K$ throughout), the approximation (30) is valid everywhere. We shall refer to it as the large-mass approximation.

A comparison of the large-mass approximation and the full numerical solution, in the entire domain $R < M$, is presented in Fig. 4.

![FIG. 4: Plots of $R(a)$ and $\dot{R}(b)$ for the full numerical solution (dashed black), for the global large-mass approximation (solid gray), and for the classical Schwarzschild solution (dotted black), for $K = 1$ and $M = 10^3$. The breakdown of the classical approximation for $\dot{R}$ at small $R$ is evident. The validity of the large-mass approximation throughout the entire domain $K < R < M$ is clearly demonstrated.](image)
The occurrence of a spacelike singularity at $R = K$ makes it difficult (if not impossible) to explore the evolution of physics—and spacetime—beyond $R = K$ by semiclassical methods. In Ref. 1, it was suggested that quantum evolution will still be regular even at the (would-be) semiclassical singularity. This motivates one to study the evolution of $R$ and $S$ in the neighborhood of the singularity in a quantum-mechanical framework. One of the simplest options would be to address the quantum problem at the "mini-superspace" level: Since at the semiclassical level the homogeneous solution provides a good approximation to the actual spacetime evolution (as long as $M >> K$), one may drop all spatial derivatives and analyze the simpler problem in which $R$ and $S(t)$ depend on $t$ solely. Furthermore, since the critical stage of evolution is the transition across the $R = K$ singularity, it may be sufficient (at least as a first step) to analyze the problem at the leading order in $R - K$.

This problem of quantum evolution is certainly beyond the scope of the present paper. Nevertheless, we do take here the preparatory steps which are to be implemented already at the semiclassical level: Namely, the formulation of an effective Lagrangian (and subsequently an effective Hamiltonian) that describes the evolution of $R(t)$ and $S(t)$ near $R = K$. We therefore seek a Lagrangian $L(R, S, $\dot{R}$, $\dot{S}$)$ that will recover the evolution equations near $R = K$.

Since Eq. (13) constitutes a closed equation of motion for $\dot{R}(t)$, we start by constructing an effective Lagrangian $L(R, \dot{R})$ for this equation. This turns out to be

$$L_R = \frac{1}{2}(R - K)\dot{R}^2,$$

as one can easily verify by applying to it the Euler-Lagrange equation. The momentum conjugate to $R$ is

$$P_R = (R - K)\dot{R}. \tag{32}$$

This yields the effective Hamiltonian for the $R$-field:

$$H_R = \frac{1}{2}\frac{P_R^2}{R - K} \tag{33}$$

Next we treat the combined system of $R$ and $S$ (still in the homogeneous framework and at the leading order in $R - K$). To this end we find it convenient to replace $S$ by the new variable $Z = R + KS$. From Eqs. (18,19) it follows that $\dot{Z} = 0$. Hence $Z(t)$ corresponds to a free one-dimensional motion, with $L_Z = \frac{1}{2}\dot{Z}^2$, conjugate momentum $P_Z = \dot{Z}$, and Hamiltonian $H_Z = \frac{1}{2}P_Z^2$. The overall near-singularity Lagrangian and Hamiltonian are $L = L_R + L_Z$ and $H = H_R + H_Z$.

**EFFECTIVE LAGRANGIAN AND HAMILTONIAN FOR THE NEAR-SINGULARITY REGION**

In this manuscript we primarily discussed two closely related issues. The first is the structure and main properties of the spacelike singularity which forms inside a CGHS evaporating BH. The second is the structure of the fields (spacetime and dilaton) in the entire range of the BH interior, from the horizon up to the singularity.

As a first step we established the homogeneous approximation, a central tool which allowed us to analyze the above issues by means of ordinary differential equations. This approximation is valid as long as the BH mass is macroscopic.

We find that the field equations admit two singular points: at $R = 0$ and $R = K$. Using the homogeneous approximation, we wrote down the asymptotic form of the field equations near each of these singularities [Eqs. (18,19) and (22,23)] and solved them exactly, obtaining the asymptotic solutions (20,21) and (24,25) at the two singularities. In the scenario of BH formation, $R$ starts at large values and then decreases, so the $R = K$ singularity will be reached first. It is therefore not clear whether the $R = 0$ singularity will ever form. Hence, in the rest of our analysis we concentrate on the $R = K$ singularity.

Next we explored the evolution of the fields inside the BH from the horizon up to the singularity. By combining several approximations, we composed a global approximation for $R$ as a function of $t$, given in Eq. (30). This expression can be further integrated in a straightforward manner to yield an expression for $R(t)$ (though in a functionally-implicit form). It should also be possible to use this expression in order to construct an approximate global solution for $S(t)$, but this requires some technical work which is beyond the scope of this manuscript.

The overall approximate expression (30) for $\dot{R}(R)$ allows us to determine the parameter $B$ of Eq. (20), which was left as a free parameter by the local analysis near $R = K$. This parameter is found to depend on the BH’s effective mass $M$, through Eq. (29).

Our analysis made an extensive use of the local approximate homogeneity which characterizes the $R = K$ singularity. One should bare in mind, however, that the overall, large-scale, structure of the singularity is inhomogeneous after all. The parameter $B$, which characterizes the local singularity structure, slowly drifts along the $R = K$ singularity line of Fig. 1. This is explicitly seen by substituting $M = M(\nu)$ in Eq. (29). (A similar drift may also apply to the other local parameters $B_2, B_3, t_0$.) This qualitative picture holds as long as the remaining BH mass is macroscopic, namely $M(\nu) >> K$.

From the asymptotic solution (20,21), one immediately observes that $R$ and $S$ are continuous at $R = K$, yet their derivatives diverge there. The same applies to $\rho$. Thus, the metric tensor is continuous and non-singular, but its derivative diverges. The $R = K$ singularity may be clas-
sified as deformationally-weak \cite{9, 10}: An extended body will only experience a finite tidal deformation as it approaches the singularity.

We briefly discuss here the possibility of extending semiclassical spacetime beyond the \( R = K \) singularity. From the discussion above it is obvious that a differentiable extended spacetime is ruled out, because differentiability already breaks down on the approach to \( R = K \) from the past. On the other hand, continuous extensions do exist. In fact, there is a two-parameter family of such continuous extensions (characterized by the values of \( B \) and \( B_2 \) beyond \( R = K \)). Thus, in our view, the problem of extending the semiclassical solution beyond the singularity is primarily a problem of ambiguity: A priori it is not clear which of the infinite possible extensions will be chosen by Nature (if any). We expect that a fully-quantized treatment (like \cite{5} for example) should provide the answer to the extension problem: First, it will only experience a finite tidal deformation as it approaches the singularity .

Finally we constructed an effective Lagrangian and Hamiltonian near the \( R = K \) singularity. Such an effective Hamiltonian may be a useful input for future quantum-mechanical treatments of the behavior near the singularity.

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[11] In the CGHS model the evaporation proceeds in a constant rate \( K/4 \) (at the leading order); hence, \( M(v) = M_{init} - (K/4)(v - v_{init}) \), where \( M_{init} \) and \( v_{init} \) respectively denote the initial mass and the moment \( v \) of collapse.
[12] The main effect of the inhomogeneity is to slowly drift the parameters of the homogeneous solution [e.g. the parameters \( B \) and \( t_0 \) in Eq. \cite{6}] in the spatial direction along the singular line. The local temporal structure is not affected at the leading order. This was verified analytically by expanding the inhomogeneous singular solution off the moment of singularity \( t_0(x) \) (at which \( R = K \)) up to seven orders in \( t - t_0(x) \) \cite{7}.
[13] Note that the above argument which establishes the homogeneous approximation does not directly apply to the case \( R_0 = M_0 \), because the latter hypersurface in Schwarzschild is null rather than spacelike. Nevertheless, the limiting solution obtained as \( R_0 \to M_0 \) is well defined and is perfectly regular at the horizon.
[14] The homogeneous evolution equations \cite{12, 13} admit a generic class of solutions in which \( \tilde{R} \) approaches a negative constant as \( t \to -\infty \). However in this class obviously \( \tilde{R} \to \infty \) at that limit, and the curvature diverges as well.
[15] This is the unique \( \tilde{z} = \tilde{z}_u = \tilde{z}_v \) value in a precisely homogeneous solution with a regular horizon. We point out, however, that the actual evaporating-BH solution is not precisely homogeneous, and one finds that at the horizon limit \( \tilde{z}_u = 1/4 \) and \( \tilde{z}_v = 0 \). This difference between \( \tilde{z}_u \) and \( \tilde{z}_v \) may be regarded as a measure for the deviation of the actual evaporating-BH spacetime from the precisely homogeneous solution. Simple analytical arguments suggest that the effect of this change in \( \tilde{z} \) by \( 1/4 \) will be small for large \( M/K \) (and vanish at the macroscopic limit \( K/M \to 0 \)). We explored this effect numerically (within the homogeneous framework), for \( K/M \) ranging between 0.01 and 0.001, and verified that this is indeed the case. As a quantitative measure for this effect, one can look at the change in the actual \( B \) parameter at the \( \tilde{R} \approx K \) singularity, induced by such a change of \( 1/4 \) in \( \tilde{z} \). Our numerical results suggest that the fractional change in \( B \) scales as \( K/M \).
[16] The effective Hamiltonian \( H_R \) may also be obtained in a more direct manner as follows: One starts from the original Lagrangian density \cite{11}, omits all spatial derivatives (namely, replace \( \partial_v \) and \( \partial_\phi \) by a t-derivative), omit the \( f_i \) contribution, and transform \( \phi, \rho \) to \( R, S \), to obtain a Lagrangian \( L(R, S, \dot{R}, \dot{S}) \). Then one constructs from \( L \) the conjugate momenta \( P_R, P_S \) and the Hamiltonian \( H(R, S, P_R, P_S) \). One then expands \( H \) in powers of \( R - K \). The effective Hamiltonian \( H_R \) is obtained (up to a multiplicative constant) as the leading order in this expansion.
[17] The specific form of \( S(t) \) is less relevant here than that of \( R(t) \), since the singularity is determined by the value of \( R \) but insensitive to \( S \).