THE HODGE–POINCARÉ POLYNOMIAL OF THE MODULI SPACES OF STABLE VECTOR BUNDLES OVER AN ALGEBRAIC CURVE

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Abstract. Let X be a nonsingular complex projective variety that is acted on by a reductive group G and such that $X^{ss} \neq X^{o}(0) \neq \emptyset$. We give formulae for the Hodge–Poincaré series of the quotient $X^{o}(0)/G$. We use these computations to obtain the corresponding formulae for the Hodge–Poincaré polynomial of the moduli space of properly stable vector bundles when the rank and the degree are not coprime. We compute explicitly the case in which the rank equals 2 and the degree is even.

1. Introduction and statement of results

Let $\mathcal{M}(n, d)$ be the moduli space of semistable vector bundles of rank $n$ and degree $d$. The cohomology of $\mathcal{M}(n, d)$ has been of great interest to a large number of mathematicians for the last forty years. If we denote by $\mathcal{M}_{o}(n, d)$ the moduli space of (properly) stable vector bundles, it is not difficult to see that when $(n, d) = 1$ then $\mathcal{M}(n, d) \cong \mathcal{M}_{o}(n, d)$.

The first results on the cohomology of $\mathcal{M}(n, d)$ are due to P. E. Newstead who computed the Betti numbers of $\mathcal{M}(2, 1) \cong \mathcal{M}_{o}(2, 1)$ from the results obtained in his paper [23]. From these results Harder observed that the Betti numbers of $\mathcal{M}(2, 1)$ can be also computed by arithmetic methods and the Weil conjectures [11]. The latter method was generalized by Harder and Narasimhan to obtain the Betti numbers of $\mathcal{M}(n, d)$ when $(n, d) = 1$ (see [12]).

Another way to carry out these computations was introduced by Atiyah and Bott in their seminal paper [1]. This is based on the definition of a stratification in the infinite dimensional space of all possible holomorphic structures on a fixed $C^{\infty}$ bundle of rank $n$ and degree $d$. The stratification turns out to be equivariantly perfect with respect to the action of the gauge group that is acting on the space of all possible holomorphic structures. Then, the equivariant Morse inequalities deduced from that allow us to obtain an inductive formula for the equivariant Betti numbers of the stratum classifying the semistable points. From this, when $(n, d) = 1$ they obtain a formula for the Betti numbers of $\mathcal{M}(n, d)$, by representing this moduli as a geometric invariant theory (GIT) quotient of the space of all possible holomorphic structures by the action of the group of complex automorphisms or complexified gauge group.

There is still another way of doing this that is described in [13]. Except for the fact that Atiyah and Bott’s method for computing the Betti numbers works with infinite dimensional spaces and groups, the method of [13] can be regarded as a generalization of that of [1].

Note that there are several ways of representing $\mathcal{M}(n, d)$ as a geometric invariant theory quotient, in this case, as the quotient of a nonsingular projective algebraic variety by the action of an algebraic reductive group. One can look at the cohomology of the GIT quotient $X//G$ of a nonsingular projective algebraic variety $X$ acted on by an algebraic reductive group $G$. Let $X^{ss}$ and $X^{o}(0)$ denote the set of semistable and properly stable points for the action of $G$. When $X^{o}(0) = X^{ss}$ (note that for $\mathcal{M}(n, d)$ this corresponds to $\mathcal{M}(n, d) \cong \mathcal{M}_{o}(n, d)$) or what is the same, when $(n, d) = 1$, in [13] the Poincaré polynomial of $X//G$ is computed bearing in mind that there is a natural identification between the cohomology of $X//G$ with rational coefficients and the equivariant cohomology of $X^{ss}$ with rational coefficients.

1.1. Actually, it is proved that there is a stratification $\{S_{\beta} : \beta \in B\}$ of $X$ by nonsingular $G$-invariant locally closed subvarieties $S_{\beta}$ satisfying the following properties:

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(i) $X^{ss}$ coincides with the unique open stratum $S_0$.
(ii) The equivariant Morse inequalities are equalities, that is

$$P^G_t(X) = P^G_t(X^{ss}) + \sum_{\beta \neq 0} t^{2\text{codim}S_\beta} P^G_t(S_\beta).$$

If $H^*_G(Y)$ denotes the G-equivariant cohomology ring of $Y$, then $P^G_t(Y) = \sum_i t^i \dim H^i_G(Y)$ is the equivariant Poincaré series of $Y$. Here $\text{codim}S_\beta$ denotes the complex codimension of $S_\beta$ in $X$. In general one has to take care of the possibility of the strata being not connected but for our purposes we don’t need to consider this.

(iii) If $\beta \neq 0$ there is a proper nonsingular subvariety $Z_\beta$ of $X$ invariant under the action of a reductive subgroup $\text{Stab}_\beta$ of $G$ such that

$$H^*_G(S_\beta) \cong H^*_G(Z_\beta),$$

where $Z_\beta^{ss}$ is the semistable stratum of $Z_\beta$ under the action of $\text{Stab}_\beta$ appropriately linearized.

Under the hypothesis of $X_0^s = X^{ss}$ one has that $P^G_t(X^{ss}) = P_t(X/\!/G)$ where $P_t$ is the usual Poincaré polynomial. Then from the identity in (ii) one may obtain a formula that computes the Betti numbers of $X/\!/G$ when $X_0^s = X^{ss}$ (see [13] 8.1).

The stratification $\{S_\beta : \beta \in B\}$ can be defined either using the moment map and symplectic geometry or algebraically. The indexing set $B$ is going to be given by a finite set of points in a positive Weyl chamber of the Lie algebra of a maximal compact torus $T$ of $G$.

Our research is focused on the cohomology of the moduli space of properly stable vector bundles $\mathcal{M}_0^s(n, d)$ when $(n, d) \neq 1$. If we represent $\mathcal{M}_0^s(n, d)$ as a GIT quotient $X/\!/G$, the condition $(n, d) \neq 1$ implies that $X^{ss} \neq X_0^s \neq \emptyset$ or what is the same, there are semistable points in $X$ that are not properly stable.

When $X^{ss} \neq X_0^s \neq \emptyset$ it could happen that $X/\!/G$ may have serious singularities. In [14] F. Kirwan shows a way of blowing up the variety $X$ along a sequence of nonsingular subvarieties to obtain a variety $\bar{X}$ with a linear action of $G$ such that $\bar{X}^{ss} = X_0^s$. Then, $\bar{X}/\!/G$ can be regarded as a “partial” resolution of singularities of $X/\!/G$ in the sense that the most serious singularities of $X/\!/G$ have been resolved. This may be used to compute the Betti numbers of $\bar{X}/\!/G$ in terms of those of $X/\!/G$ and the dimensions of the rational intersection homology groups of $X/\!/G$ in terms of Betti number of the partial desingularisations $\bar{X}/\!/G$. In [16] these techniques are applied to the case of the moduli space of semistable vector bundles of rank $n$ and degree $d$ when $(n, d) \neq 1$.

From this resolution of singularities, in [13] the stratification $\{S_\beta : \beta \in B\}$ is refined to obtain a stratification of the set $X^{ss}$ of semistable points, so that the set $X_0^s$ of properly stable points is an open stratum. This new stratification is not equivariantly perfect hence one may not expect to obtain nice formulae for the Betti numbers of the categorical quotient $X_0^s/\!/G$.

In this paper we use Deligne’s extension of Hodge theory together with the previous refined stratification in order to study the Hodge–Poincaré series of a nonsingular projective variety that is acted on by a reductive group and such that $X^{ss} \neq X_0^s \neq \emptyset$. This can be carried out because of the good properties of the Hodge–Deligne series and its relationship with the Hodge–Poincaré series given by Poincaré duality. We obtain formulae for these series. After that we use these computations to obtain corresponding formulae for the Hodge–Poincaré series of the moduli space of stable vector bundles when the rank and the degree are not coprime. Finally, we compute explicitly the Hodge–Poincaré polynomial of the moduli space of (properly) stable vector bundles when the rank is 2 and the degree is even. Using Poincaré duality one may obtain the Hodge–Deligne polynomial of $\mathcal{M}_0^s(2, d)$ when $d$ is even. The latter was first computed in [21]. The Hodge–Poincaré polynomial is given in the following Theorem.

**Theorem.** The Hodge–Poincaré polynomial of $\mathcal{M}_0^s(2, d)$ for $d$ even is given by

$$HP(\mathcal{M}_0^s(2, d))(u, v) = \frac{1}{2(1 - uv)(1 - u^2v^2)} \left[ 2(1 + u)^g(1 + v)^g(1 + u^2v)^g(1 + uv)^g - (uv)^{g-1}(1 + u)^{2g}(1 + v)^{2g}(2 - (uv)^g)^{g-1} + (uv)^{g+1} - (uv)^{2g-2}(1 - u^2)^g(1 - v^2)^g(1 - uv)^g \right].$$

The layout of the paper is as follows. In Sections 2 we give an account of the results on Deligne’s extension of Hodge theory, that we shall need throughout this paper. In Section 3 we introduce the Morse stratification $\{S_\beta\}_{\beta \in B}$ of $X$ and its refined stratification. In Section 4, from the previous stratifications, we obtain formulae for
the Hodge–Poincaré series of the geometric quotient $X^s_{(0)}/G$. This formulae are adapted in Section 5 to obtain the corresponding ones to $\mathcal{M}^s_{(0)}(n,d)$ when $(n,d) \neq 1$. Finally, in Section 6 we compute explicitly the Hodge–Poincaré polynomial of $\mathcal{M}^s_{(0)}(2,d)$ when $d$ is even.

2. Hodge Theory

We use Deligne’s extension of Hodge theory which applies to varieties which are not necessarily compact, projective or smooth (see [5], [6] and [7]). We start by giving a review of the notions of pure Hodge structure, mixed Hodge structure, Hodge–Deligne and Hodge–Poincaré polynomials.

Definition 2.1. A pure Hodge structure of weight $m$ is given by a finite dimensional $\mathbb{Q}$-vector space $H_Q$ and a finite decreasing filtration $F^p$ of $H = H_Q \otimes \mathbb{C}$

$$H \supset \ldots \supset F^p \supset \ldots \supset (0),$$

called the Hodge filtration, such that $H = F^p \otimes F^{-m-p+1}$ for all $p$. When $p + q = m$, if we set $H^{p,q} = F^p \cap F^q$, the condition $H = F^p \otimes F^{-m-p+1}$ for all $p$ implies an equivalent definition for a pure Hodge structure. That is, a decomposition

$$H = \bigoplus_{p+q=m} H^{p,q}$$

satisfying that $H^{p,q} = H^{q,p}$, where $H^{q,p}$ is the complex conjugate of $H^{p,q}$.

Definition 2.2. A mixed Hodge structure consists of a finite dimensional $\mathbb{Q}$-vector space $H_Q$, an increasing filtration $W_l$ of $H_Q$, called the weight filtration

$$\ldots \supset W_l \subset \ldots \subset H_Q,$$

and the Hodge filtration $F^p$ of $H = H_Q \otimes \mathbb{C}$, where the filtrations $F^p Gr_l^W$ induced by $F^p$ on

$$Gr_l^W = (W_l H_Q / W_{l-1} H_Q) \otimes \mathbb{C} = W_l H / W_{l-1} H$$
give a pure Hodge structure of weight $l$. Here $F^p Gr_l^W$ is given by $(W_l H \cap F^p + W_{l-1} H) / W_{l-1} H$.

Associated to the Hodge filtration and the weight filtration we can consider the quotients $Gr_l^W = W_l / W_{l-1}$ of Definition 2.2 and for the Hodge filtration $Gr_l^W$ of $H^p = F^p Gr_l^W / F^{p+1} Gr_l^W$.

Definition 2.3. The Hodge numbers of $H$ are

$$h^{p,q}(H) = \dim Gr_p^F Gr_l^W H.$$ 

2.4. A morphism of type $(r,r)$ between mixed Hodge structures, $H_Q$ with filtrations $W_m$ and $F^p$, and $H'_Q$ with $W'_l$ and $F'^p$, is given by a linear map

$$L : H_Q \rightarrow H'_Q$$
satisfying $L(W_m) \subset W'_m \otimes \mathbb{C}$ and $L(F^p) \subset F'^{p+r}$. Any such morphism is then strict in the sense that $L(F^p) = F'^{p+r} \cap \text{Im}(L)$, and the same for the weight filtration.

Definition 2.5. A morphism of type $(0,0)$ between mixed Hodge structures, is called a morphism of mixed Hodge structures.

Deligne proved that the usual cohomology groups $H^k(X, \mathbb{Q})$ and those with compact support, we denote the latter by $H^k_c(X)$, of a complex variety $X$ which may be singular and not projective, carry a mixed Hodge structure (see [5], [6] and [7]). From these

Definition 2.6. For any complex algebraic variety $X$, we define its Hodge–Deligne polynomial (or virtual Hodge polynomial) as (see [3])

$$H(X)(u, v) = \sum_{p,q,k} (-1)^{p+q+k} h^{p,q}(H^k_c(X)) u^p v^q \in \mathbb{Z}[u,v].$$

We define its Hodge–Poincaré polynomial as

$$HP(X)(u, v) = \sum_{p,q,k} (-1)^{p+q+k} h^{p,q}(H^k(X)) u^p v^q.$$

Danilov and Khovanskiï (3) observed that $H(X)(u, v)$ coincides with the classical Hodge polynomial when $X$ is smooth and projective.
Remark 2.7. When our algebraic variety $X$ is smooth, Poincaré duality gives us the following functional identity relating Hodge–Deligne and Hodge–Poincaré polynomials

\[ \mathcal{H}(X)(u,v) = (uv)^{\dim_{\mathbb{C}} X} \cdot H^P(X)(u^{-1}, v^{-1}) \]

where $\dim_{\mathbb{C}} X$ denotes the complex dimension of $X$.

Theorem 2.8 \cite[Theorem 2.2]{20}. Let $X$ be a complex variety. Suppose that $X$ is a finite disjoint union $X = \bigcup X_i$, where $X_i$ are algebraic subvarieties. Then

\[ \mathcal{H}(X)(u,v) = \sum_i \mathcal{H}(X_i)(u,v). \]

Another result from \cite{20} that will be useful for our computations is

Lemma 2.9 \cite[Lemma 2.3]{20}. Suppose that $\pi : X \rightarrow Y$ is an algebraic fiber bundle with fiber $F$ which is locally trivial in the Zariski topology, then

\[ \mathcal{H}(X)(u,v) = \mathcal{H}(F)(u,v) \cdot \mathcal{H}(Y)(u,v). \]

If $X$ is an algebraic variety acted on by a group $G$, consider $EG \rightarrow BG$ a universal classifying bundle for $G$, where $BG = EG/G$ is the classifying space of $E$ and $EG$ is the total space of $G$. We form the space $X \times_G EG$ which is defined to be the quotient space of $X \times EG$ by the equivalence relation $(x, e \cdot g) \sim (g \cdot x, e)$. Then, the equivariant cohomology ring of $X$ is the following

\[ H^*_G(X) = H^*(X \times_G EG). \]

Although $EG$ and $BG$ are not finite-dimensional manifolds, there are natural Hodge structures on their cohomology. This is trivial in the case of $EG$. Deligne proved that there is a pure Hodge structure on $H^*(BG)$ and that $H^{p,q}(H^*(BG)) = 0$ for $p \neq q$ (see \cite[§9]{7}). We may regard $EG$ and $BG$ as increasing unions of finite-dimensional varieties $(EG)_m$ and $(BG)_m$ for $m \geq 1$ such that $G$ acts freely on $(EG)_m$ with $(EG)_m/G = (BG)_m$ and the inclusions of $(EG)_m$ and $(BG)_m$ in $EG$ and $BG$ respectively induce isomorphisms of cohomology in degree less than $m$ which preserve the Hodge structures. In the same way $X \times_G EG$ is the union of finite-dimensional varieties whose natural mixed Hodge structures induce a natural mixed Hodge structure on $H^*(X \times_G EG)$. Using that we have the following

Definition 2.10. For any complex algebraic variety $X$ acted on by an algebraic group $G$, we define its equivariant Hodge–Poincaré polynomial as

\[ HP_G(X)(u,v) = \sum_{p,q,k} (-1)^{p+q+k} h_{G}^{p,q,k}(X) u^p v^q, \]

here $h_{G}^{p,q,k}(X) = h^{p,q}(H^*(X \times G EG))$.

2.11. Suppose now that $G$ is connected. The relationship between cohomology and equivariant cohomology is accounted for by a Leray spectral sequence for the fibration

\[ X \times_G EG \rightarrow BG \]

whose fiber is $X$. The $E_2$-term of this spectral sequence is given by $E_2^{p,q} = H^p(X) \otimes H^q(BG)$ which abuts to $H^{p+q}_G(X)$. This spectral sequence preserves Hodge structures.

If $X$ is a nonsingular projective variety that is acted on linearly by a connected complex reductive group $G$, one has that the fibration \cite{20} is cohomologically trivial over $\mathbb{Q}$ (see \cite[Theorem 5.8]{13}). Then

\[ H^*_G(X) \cong H^*(X) \otimes H^*(BG). \]

This isomorphism is actually an isomorphism of mixed Hodge structures \cite[8.2.10]{7}.

We have another fibration, that is

\[ X \times_G EG \rightarrow X/G \]

with fiber $EG$. When $G$ acts freely on $X$, that is the stabilizer of every point is trivial, then it induces the isomorphism

\[ H^*(X \times_G EG) \cong H^*(X/G). \]
Remark 2.12. Let $GL(n)$ and $SL(n)$ be the general linear group and the special linear group respectively. In this paper we only consider these groups with complex coefficients. The Hodge–Poincaré series of $BGL(n)$ and $BSL(n)$ are given by get

\begin{equation}
HP(BGL(n))(u, v) = \prod_{1 \leq k \leq N} (1 - u_k v_k)^{-1} \quad \text{and} \quad HP(BSL(n))(u, v) = \prod_{2 \leq k \leq N} (1 - u_k v_k)^{-1}.
\end{equation}

3. Stratifications

Let $X$ be a nonsingular complex projective variety in $\mathbb{P}^n$ and $G$ a reductive group that acts linearly on $X$. Assume that $X$ is embedded in $\mathbb{P}^n$ by a line bundle $L$ which is the restriction of the hyperplane bundle $H$ on $\mathbb{P}^n$ to $X$. In order to obtain a good quotient of $X$ by $G$ in the sense of GIT, we need to restrict ourselves to the set of semistable points of $X$ by the action of $G$. We denote the set of semistable points by $X^{ss}$ and the good quotient or GIT quotient by $X^{ss}/G = X/G$ together with the quotient map $X^{ss} \to X/G$. Regarding the set of stable points, throughout this paper it will be convenient to use Mumford’s original definition of properly stable points, nowadays called stable. A point $x \in X$ is properly stable if $\dim \mathcal{O}(x) = \dim G$ and there exists an invariant homogeneous polynomial $f$ of degree $\geq 1$ such that $f(x) \neq 0$ and the action of $G$ on $X_f$ is closed. We denote by $X^{ss}(0)_0$ the set of properly stable points of $X$ by $G$ and $X^{ss}(0)$ the geometric quotient of $X$ by $G$. We also denote $X^{ss}(i)$ the set of points of $X$ that satisfy the same properties as the properly stable points but $\dim \mathcal{O}(x) = \dim G - i$.

In this section we describe a couple of stratifications. The first one was introduced in Paragraph 1.1. This requires the hypothesis of $X^{ss} = X^{ss}(0)$. This stratification turns out to be equivariantly perfect, which implies that one may obtain inductive formulae for computing the Betti numbers and Poincaré polynomials of $X/G$ when $G$ is connected.

For the latter, we stratify the set $X^{ss}$ in such a way that the set of properly stable points, $X^{ss}(0)$ is an open stratum. Unfortunately, this stratification is not equivariantly perfect, then one does not expect to obtain nice formulae for computing the Betti numbers and Poincaré polynomials of $X^{ss}(0)/G$ when $G$ is connected. But this stratification may be used to compute the Hodge–Poincaré polynomials of the properly stable part as we will do later on in this paper.

3.1. The general construction. When $X^{ss} = X^{ss}(0)$ there exists a stratification $\{S_\beta : \beta \in \mathcal{B}\}$ of $X$ by nonsingular $G$-invariant locally closed subvarieties $S_\beta$ satisfying properties (i), (ii) and (iii) of Paragraph 1.1. This stratification can be defined either using symplectic geometry and the moment map, or algebraically.

When $X$ is a nonsingular complex projective variety in $\mathbb{P}^n$ and $G$ a reductive group that acts linearly on $X$, this stratification is defined as follows (see [13] for details). Assume that $G$ acts on $X$ via a rational representation $\rho : G \to GL(n + 1)$. It is known that $G$ is reductive if and only if it is the complexification of any maximal compact subgroup $K$. One can choose coordinates so that $\rho$ restricts to an unitary representation of $K$, $\rho_K : K \to U(n + 1)$. Let $T$ be a maximal compact torus of $K$, and let $t$ be its Lie algebra. The maximal torus $T$ acts on $X$ via a morphism $T \to U(n + 1)$, after conjugating this morphism by an element of $U(n + 1)$, we may assume that $T$ acts via

\begin{equation}
t \mapsto \text{diag}(\alpha_0(t), \ldots, \alpha_n(t)),
\end{equation}

where $\alpha_j$ are the characters of $T$. We choose an inner product on $t$, invariant under the action of the Weyl group, and use this to identify $t$ and its dual, $t^\ast$. Under this inner product, we identify the characters $\alpha_j$ with points in $t^\ast$, these are the weights of $T$. By abuse of notation, we denote the weights by $\alpha_j$. Let $W := \{\alpha_0, \ldots, \alpha_n\}$ the set of weights for the action of $T$. Then, the indexing set $\mathcal{B}$ is defined as follows. An element $\beta \in t^\ast$ belongs to $\mathcal{B}$ if and only if $\beta$ is the closest point to 0 of the convex hull, Conv$S$, of some nonempty subset $S$ of $W$. Then, if $\beta \in \mathcal{B}$, $\beta$ is the closest point to 0 of the convex hull

\begin{equation}
\text{Conv}\{\alpha_i \in W \text{ such that } \alpha_i, \beta = \|\beta\|^2\},
\end{equation}

where $\cdot$ is the inner product and $\|\cdot\|$ its associated norm. If we choose a positive Weyl chamber of $t^\ast$, let $t^\ast_+$, the indexing set $\mathcal{B}$ can be then identified with a finite set of points in $t^\ast_+$.

Regarding the varieties $Z_\beta$ of Paragraph 1.1(iii), we define

\begin{equation}
Z_\beta := \{(x_0 : \ldots : x_n) \in X \text{ such that } x_i = 0 \text{ if } \alpha_i, \beta \neq \|\beta\|^2\}
\end{equation}

and

\begin{equation}
Y_\beta := \{(x_0 : \ldots : x_n) \in X : x_i = 0 \text{ if } \alpha_i \cdot \beta < \|\beta\|^2 \text{ and } \exists x_i \neq 0 \text{ with } \alpha_i \cdot \beta = \|\beta\|^2\}.
\end{equation}
The variety $Z_\beta$ is a proper closed subvariety of $X$ and $Y_\beta$ is a locally closed subvariety. There is a retraction $p_\beta : Y_\beta \to Z_\beta$ defined by

$$
p_\beta(x_0 : \ldots : x_n) = (x_0' : \ldots : x'_n)
$$

such that $x'_i = x_i$ if $\alpha_i, \beta = \|\beta\|^2$ and $x'_i = 0$ otherwise. Moreover, the subvarieties $Z_\beta$ and $Y_\beta$ are nonsingular and $p_\beta$ is a locally trivial fibration whose fibre at any point is isomorphic to $\mathbb{C}^{m_\beta}$ for some $m_\beta \geq 0$ (see [13] 3.2).

Let $\text{Stab}\beta$ be the stabiliser of $\beta$ under the adjoint action of $G$. Let $Z_\beta^s$ be the set of semistable points of $Z_\beta$ with respect to the action of $\text{Stab}\beta$ properly linearised. As we will see in this paper, this linearisation corresponds to a modification of the moment map used to define the strata (for details see [13]). Let $Y_\beta^s := p_\beta^{-1}(Z_\beta^s)$. Then, the restriction of $p_\beta$ to $Y_\beta^s$

$$p_\beta : Y_\beta^s \to Z_\beta^s
$$

is a locally trivial fibration whose fibre is $\mathbb{C}^{m_\beta}$ for some $m_\beta \geq 0$. Let $B$ be the Borel subgroup of $G$ associated to the choice of positive Weyl chamber $t_+$ and let $P_\beta$ be the parabolic subgroup $B\text{Stab}\beta$, then in [13] it is proved that $Y_\beta^s$ are $P_\beta$-invariant and that

$$S_\beta \cong G \times_{p_\beta} Y_\beta^s.
$$

An element $g \in G$ belongs to $P_\beta$ if and only if $\lim_{t \to \infty} \exp(-it\beta)g \exp(it\beta)$ is an element of $G$. This limit defines a surjection $q_\beta : P_\beta \to \text{Stab}\beta$ which is actually a retraction. From [15] and the fact that $p_\beta$ and $q_\beta$ are retractions, one deduces that

$$H^*_G(S_\beta) \cong H^*_{\text{Stab}\beta}(Z_\beta^s)
$$

which is property (iii) of Paragraph [13]. Moreover, the stratification $S_\beta \subseteq B$ satisfies that $S_0 = X^s$ and is equivariantly perfect (see properties (i) and (ii) of Paragraph [13]). One has that $X = \bigcup_{\beta \in B} S_\beta$, and there is a partial order on $B$ such that

$$S_\beta \subseteq S_\gamma \cup \bigcup_{\gamma > \beta} S_\gamma,
$$

where $\gamma > \beta$ if $\|\gamma\| > \|\beta\|$.

### 3.2. A stratification for the set of semistable points.

When $X^s \neq X^s_0 \neq \emptyset$ the GIT quotient $X//G$ may have serious singularities. For the moduli space of semistable vector bundles, $M(n,d)$, the problem of finding natural desingularisations has been studied by Seshadri [23]. Narasimhan and Ramanan [22], and Kirwan [10]. The latter is an application of the method described in [14] that works for a smooth complex projective variety $X$ acted on by a reductive group $G$ and such that $X^s \neq X^s_0 \neq \emptyset$. In this paper we use Kirwan’s method which allows us to define a stratification of $X^s$ such that $X^s_0$ is an open stratum. This method consists of blowing up $X$ along a sequence of smooth $G$-invariant subvarieties to obtain a $G$-invariant morphism $\pi : \tilde{X}^s \to X^s$, where $\tilde{X}^s$ is a projective variety acted on linearly by $G$ properly lifted, and such that $\tilde{X}^s = \tilde{X}^s_0$ with respect to the induced action. The induced birational morphism $\tilde{X}//G \to X//G$ can be regarded as a “partial desingularisation” of the GIT quotient $X//G$ in the sense that the more serious singularities of $X//G$ have been resolved. The only singularities of $\tilde{X}//G$ are finite quotient singularities (for more details see [14]).

### 3.1. This desingularisation process is based on the fact that there exist semistable points that are not properly stable if and only if there exists a non-trivial connected reductive subgroup of $G$ fixing a semistable point. Let $r_1 > 0$ be the maximal dimension of a reductive subgroup of $G$ fixing a point of $X^s$ and let $\mathcal{R}(r_1)$ be a set of representatives of conjugacy classes of all connected reductive subgroups $R_1$ of dimension $r_1$ in $G$ such that

$$Z_{R_1}^s := \{x \in X^s \text{ such that } R_1 \text{ fixes } x\}
$$

is non-empty. We consider

$$\bigcup_{R_1 \in \mathcal{R}(r_1)} GZ_{R_1}^s
$$

where $GZ_{R_1}^s := \{gx \text{ such that } g \in G \text{ and } x \in Z_{R_1}^s\}$. These sets are non-singular closed subvarieties of $X^s$. In the first step we blow up $X^s$ along the subvariety $\bigcup_{R_1 \in \mathcal{R}(r_1)} GZ_{R_1}^s$. In [14] 8.3 it is proved that the blow up of $X^s$ along the subvariety $\bigcup_{R_1 \in \mathcal{R}(r_1)} GZ_{R_1}^s$ is the same as the result of blowing up $X^s$ along each $GZ_{R_1}^s$ for $R_1 \in \mathcal{R}(r_1)$ one at a time. Let $X_{(R_1)}$ be the blown up variety along $\bigcup_{R_1 \in \mathcal{R}(r_1)} GZ_{R_1}^s$ and $E_1$ be the exceptional divisor. The action of $G$ on $X^s$ lifts to an action on $X_{(R_1)}$ with respect to $\pi_1^* L^{2k} \otimes \mathcal{O}(-E_1)$ where $\pi_1 : X_{(R_1)} \to X^s$ and $k$ is any integer. When $k$ is large enough the set $X_{(R_1)}$ with respect to the lifted action is independent of $k$. 
3.2. In [14] 7.17, the set $X^{ss}_{(R_l)}$ is characterized by the following properties:

(a) The complement of $X^{ss}_{(R_l)}$ in $X_{(R_l)}$, i.e. $X_{(R_l)} \setminus X^{ss}_{(R_l)}$, is the proper transform of the subset $\phi^{-1}(\phi(GZ^{ss}_{R_l}))$ of $X^{ss}$ where $\phi : X^{ss} \to X/G$ is the quotient map;

(b) No point of $X^{ss}_{(R_l)}$ is fixed by a reductive subgroup of $G$ of dimension at least $r_1$, and a point in $E^{ss}_1 = X^{ss}_{(R_l)} \cap E_1$ is fixed by a reductive subgroup of dimension less than $r_1$ in $G$ if and only if it belongs to the proper transform of the subvariety $Z^{ss}_{R_l}$ of $X^{ss}_{(R_l)}$.

If we repeat this process for $X^{ss}_{(R_l)}$ and so on, after at most $r_1 - 1$ steps we obtain a $G$-invariant morphism $\pi : \tilde{X}^{ss} \to X^{ss}$, where $\tilde{X}^{ss}$ is a projective variety acted on linearly by $G$ properly lifted, and such that $\tilde{X}^{ss} = \tilde{X}_{(0)}$.

This is equivalent to constructing a sequence of varieties

$$X^{ss}_{(R_0)} = X^{ss}, X^{ss}_{(R_1)}, \ldots, X^{ss}_{(R_\tau)} = \tilde{X}^{ss}$$

where $R_1, \ldots, R_\tau$ are connected reductive subgroups of $G$ with

$$r_1 = \dim R_1 \geq \dim R_2 \geq \ldots \geq \dim R_\tau \geq 1,$

and if $1 \leq l \leq \tau$ then $X_{(R_l)}$ is the blow up of $X^{ss}_{(R_{l-1})}$ along its closed nonsingular subvarieties $GZ^{ss}_{R_l}$. It is satisfied that $GZ^{ss}_{R_l} \cong G \times N_l Z^{ss}_{R_l}$, where $N_l$ is the normaliser of $R_l$ in $G$. Similarly, $\tilde{X}^{ss}/G = \tilde{X}^{ss}/\Gamma$ can be obtained from $X^{ss}/G$ by blowing up along the proper transforms of the images $Z_{R_l} / (N/R)$ in $X^{ss}$ of the subvarieties $GZ^{ss}_{R_l}$ in decreasing order of $\dim R$. Note that

$$GZ^{ss}_{R_l}/G \cong Z_{R_l} / (N/R).$$

As in Subsection 3.3 for each $1 \leq l \leq \tau$ we have a $G$-equivariant stratification

$$\{S_{\beta,l} : (\beta, l) \in B_l \times \{l\}\}$$

of $X_{R_l}$ by nonsingular $G$-invariant locally closed subvarieties such that one of the strata, indexed by $(0, l) \in B_l \times \{l\}$, coincides with the open subset $X^{ss}_{(R_l)}$ of $X_{(R_l)}$. There is a partial ordering in $B_l$ given by $\gamma > \beta$ if $\|\gamma\| > \|\beta\|$. Then, $0$ is its minimal element, and if $\beta \in B_l$ then the closure in $X_l$ of the stratum $S_{\beta,l}$ satisfies

$$\overline{S_{\beta,l}} \subseteq \bigcup_{\gamma \in B_l, \gamma > \beta} S_{\gamma,l}.$$

If $\beta \in B_l$ and $\beta \neq 0$ then the stratum $S_{\beta,l}$ retracts $G$-equivariantly onto its transverse intersection with the exceptional divisor $E_l$ for the blow-up $X_{(R_l)} \to X^{ss}_{(R_{l-1})}$. This exceptional divisor is isomorphic to the projective bundle $\mathbb{P}(N_l)$ over $G\tilde{Z}^{ss}_{R_l}$, where $\tilde{Z}^{ss}_{R_l}$ is the proper transform of $Z^{ss}_{R_l}$ in $X^{ss}_{(R_{l-1})}$ and $N_l$ is the normal bundle to $G\tilde{Z}^{ss}_{R_l}$ in $X^{ss}_{(R_{l-1})}$.

3.3. Now, let

$$\pi_l : E_l \to G\tilde{Z}^{ss}_{R_l}$$

be the projection obtained from the restriction of $\pi_l : X_{(R_l)} \to X^{ss}_{(R_{l-1})}$ to $E_l$. It is satisfied that this restriction is a locally trivial fibration whose fibre is isomorphic to $\mathbb{P}(N_l)$. It is proved in [14] that the stratification $\{S_{\beta,l} : \beta \in B_l\}$ is determined by the action of $S_l$ on the fibres of $N_l$ over $G\tilde{Z}^{ss}_{R_l}$. More precisely, in Lemma 7.9 of [14] it is showed that when $x \in \tilde{Z}^{ss}_{R_l}$ the intersection of $S_{\beta,l}$ with the fibre $\pi_l^{-1}(x) = \mathbb{P}(N_l,x)$ of $\pi_l$ at $x$ is the union of those strata indexed by points in the adjoint orbit $\text{Ad}(G) \beta$ in the stratification of $\mathbb{P}(N_l,x)$ induced by the representation $\rho_l$ of $R_l$ on the normal $N_l,x$ to $G\tilde{Z}^{ss}_{R_l}$ at $x$. Let $B(\rho_l)$ be the indexing set corresponding to $\rho_l$, if each $\text{Ad}(G)$-orbit meets $B(\rho_l)$ in at most one point then one may assume that

$$B_l = B(\rho_l).$$

Moreover, each stratum $S_{\beta,l}$ retracts onto its intersection with the exceptional divisor $E_l$, and if $B_l = B(\rho_l)$ then this intersection retracts onto

$$G \times_{N_l \cap \text{Stab} \beta} Z^{ss}_{\beta,l} \cap \pi_l^{-1}(\tilde{Z}^{ss}_{R_l}),$$

where

$$\pi_l : Z^{ss}_{\beta,l} \cap \pi_l^{-1}(\tilde{Z}^{ss}_{R_l}) \to \tilde{Z}^{ss}_{R_l},$$

is a fibration with fibre $Z^{ss}_{\beta,l}(\rho_l)$ (see [14] 7.16). The variety $Z^{ss}_{\beta,l}(\rho_l)$ is defined as $Z^{ss}_{\beta}$ but with respect to the induced action by $\rho_l$ of $R_l$ on $\mathbb{P}(N_l)$. If $\beta$ is a maximal element of $B(\rho_l)$ with respect to the partial order, then
\[ Z_{\beta,l}^{ss}(\rho_l) = Z_{\beta,l}(\rho_l) \] which is a nonsingular projective variety. Moreover, by [13] 7.11 and [13] 6.18, the codimension of \( S_{\beta,l} \) in \( X(R_l) \) is given by
\[
d(\beta, l) := \text{codim} S_{\beta,l} = n(\beta, l) - \dim R_l / B\text{Stab}\beta,
\]
where \( n(\beta, l) \) is the number of weights \( \alpha \) of the representation \( \rho_l \) such that \( \alpha, \beta < \| \beta \|^2 \).

Then, there is a stratification \( \{ \Sigma_\gamma \}_\gamma \) of \( X^{ss} \) (see [13]) indexed by
\[
\Gamma = \{ R_1 \} \sqcup \{ R_1 \} \times \{ B_1 \} \times \{ 0 \} \sqcup \ldots \sqcup \{ R_\tau \} \times \{ B_\tau \} \times \{ 0 \} \sqcup \{ 0 \}
\]
defined as follows. We take as the highest stratum \( \Sigma_{R_\tau} \) the nonsingular closed subvariety \( GZ_{R_\tau}^{ss} \) whose complement in \( X^{ss} \) can be naturally identified with \( X(R_\tau) \setminus E_1 \). We have \( GZ_{R_\tau}^{ss} \cong G \times_{N_1} Z_{R_\tau}^{ss} \) where \( N_1 \) is the normaliser of \( R_1 \) in \( G \), and \( Z_{R_\tau}^{ss} \) is equal to the set of semi-stable points for the action of \( N_1 \), or equivalently for the induced action of \( N_1 / R_1 \), on \( Z_{R_\tau} \), which is a union of connected components of the fixed point set of \( R_1 \) in \( X \). Moreover since \( R_1 \) has maximal dimension among those reductive groups of \( G \) with fixed points in \( X^{ss} \), we have \( Z_{R_\tau}^{ss} = Z_{R_\tau} \), where \( Z_{R_\tau} \) denotes, for each \( l \), the set of properly stable points for the action of \( N_1 / R_1 \) on \( Z_{R_\tau} \) for \( 1 \leq l \leq \tau \).

**Remark 3.4.** Note that \( x \in Z_{R_1} \) is properly stable for the action of \( N_1 / R_1 \) if and only if \( x \in (Z_{R_i})^{ss}_{(r_i)} \) for the action of \( N_1 \) on \( Z_{R_i} \), where \( r_1 = \dim R_1 \). Moreover since \( GZ_{R_i}^{ss} \cong G \times_{N_1} Z_{R_i}^{ss} \) and is closed in \( X^{ss} \), we also have that \( x \in Z_{R_i} \) is properly stable for the action of \( N_1 / R_1 \) if and only if \( x \in (GZ_{R_i})^{ss}_{(r_i)} \) for the action of \( G \) on \( GZ_{R_i} \).

We take as our next strata the nonsingular locally closed subvarieties
\[
\Sigma_{\beta,1} = S_{\beta,1} \setminus E_1,
\]
for \( \beta \in B_1 \) with \( \beta \neq 0 \), of \( X(R_1) \setminus E_1 = X^{ss} \setminus GZ_{R_1}^{ss} \), whose complement in \( X(R_1) \setminus E_1 \) is just \( X^{ss}_{(R_1)} \setminus E_1 = X^{ss}_{(R_1)} \setminus E_1^{ss} \) where \( E_1^{ss} = X^{ss}_{(R_1)} \cap E_1 \), and then we take the intersection of \( X^{ss}_{(R_1)} \setminus E_1^{ss} \subseteq X^{ss} \setminus GZ_{R_1}^{ss} \) with \( GZ_{R_1}^{ss} \subseteq X^{ss} \).

**Remark 3.5.** In [17] and [18] it is claimed that this intersection is \( GZ_{R_2}^{ss} \), where \( Z_{R_2}^{ss} \) is the set of properly stable points for the action of \( N_2 / R_2 \) on \( Z_{R_2} \). This is not necessarily true; in [19] it is explained that instead one needs to \( Z_{R_2}^{ss} \) to be the intersection of \( Z_{R_2} \) with \( GZ_{R_2}^{ss} \) defined for the action of \( G \) on \( GZ_{R_2} \) where \( r_2 = \dim R_2 \). This is an open subset of the set of properly stable points for the action of \( N_2 / R_2 \) on \( Z_{R_2} \), which is the intersection of \( Z_{R_2} \) with \( (GZ_{R_2})^{ss}_{(r_2)} \) defined for the action of \( N_2 \) (not \( G \)) on \( GZ_{R_2} \).

The next strata are the nonsingular locally closed subvarieties
\[
\Sigma_{\beta,2} = S_{\beta,2} \setminus (E_2 \cup \tilde{E}_1),
\]
for \( \beta \in B_2 \) with \( \beta \neq 0 \), of \( X(R_2) \setminus (E_2 \cup \tilde{E}_1) \), whose complement in \( X(R_2) \setminus (E_2 \cup \tilde{E}_1) \) is \( X^{ss}_{(R_1)} \setminus (E_2 \cup \tilde{E}_1) \). The stratum after these is \( GZ_{R_3}^{ss} \), where \( Z_{R_3} \) is the intersection of \( Z_{R_3} \) with \( (GZ_{R_3})^{ss}_{(r_3)} \) defined for the action of \( G \) on \( GZ_{R_3} \) where \( r_3 = \dim R_3 \).

Then, in general we have two different types of strata. For each \( 1 \leq l \leq \tau \), either
\[
\Sigma_{R_l} = GZ_{R_l},
\]
where \( Z_{R_l} \) is the intersection of \( Z_{R_l} \) with \( (GZ_{R_l})^{ss}_{(r_l)} \) defined for the action of \( G \) on \( GZ_{R_l} \) where \( r_l = \dim R_l \). Or
\[
\Sigma_{\beta,l} = S_{\beta,l} \setminus (E_1 \cup \tilde{E}_{l-1} \cup \ldots \cup \tilde{E}_1),
\]
for \( \beta \in B_l \) with \( \beta \neq 0 \), of \( X(R_l) \setminus (E_l \cup \tilde{E}_{l-1} \cup \ldots \cup \tilde{E}_1) \), whose complement in \( X(R_l) \setminus (E_l \cup \tilde{E}_{l-1} \cup \ldots \cup \tilde{E}_1) \) is \( X^{ss}_{(R_l)} \setminus (E_l \cup \tilde{E}_{l-1} \cup \ldots \cup \tilde{E}_1) \). We take \( \Sigma_0 = X_0 \) as our final stratum, which is the unique one indexed by 0. This is actually the unique open stratum for the stratification.

We have a partial order induced at \( \Gamma \), this is given by the partial orders of \( B_i \) for all \( i \) together with the ordering in the expression [13]. For this partial ordering, the maximal element is \( R_1 \) and the minimal element is 0, and the closure of each stratum \( \Sigma_\gamma \) indexed by \( \gamma \in \Gamma \) in \( X^{ss} \), satisfies
\[
\bigcup_{\gamma \geq \gamma} \Sigma_\gamma.
\]
3.2.1. Explicit description. In this subsection we give an explicit description of the strata we have already defined. As we have seen, there are two different types of strata. For each $1 \leq l \leq \tau$, either
\[
\Sigma_{R_l} = GZ_{R_l},
\]
in this case $GZ_{R_l} \cong G \times_{N_l} Z_{R_l}^\ss$. Or
\[
\Sigma_{\beta,l} = S_{\beta,l} \setminus (E_l \cup \tilde{E}_{l-1} \cup \ldots \cup \tilde{E}_1),
\]
for $\beta \in B_l$ with $\beta \neq 0$. For the latter, we have seen in Subsection 3.3 that
\[
S_{\beta,l} = G Y_{\beta,l}^\ss \cong G \times_{P_\beta} Y_{\beta,l}^\ss,
\]
where $Y_{\beta,l}^\ss$ fibres over $Z_{\beta,l}^\ss$ with fibre $\mathbb{C}^{m_{\beta,l}}$ for some $m_{\beta,l} > 0$, and $P_\beta$ is the parabolic subgroup $B_{\mathrm{Stab}(\beta)}$ of $G$.

By Lemmas 7.6 and 7.11 of [14]
\[
S_{\beta,l} \cap E_l = G(Y_{\beta,l}^\ss \cap E_l) \cong G \times_{P_\beta} (Y_{\beta,l}^\ss \cap E_l)
\]
where $Y_{\beta,l}^\ss \cap E_l$ fibres over $Z_{\beta,l}^\ss$ with fibre $\mathbb{C}^{m_{\beta,l}-1}$. Thus
\[
S_{\beta,l} \setminus E_l \cong G \times_{P_\beta} (Y_{\beta,l}^\ss \setminus E_l)
\]
where $Y_{\beta,l}^\ss \setminus E_l$ fibres over $Z_{\beta,l}^\ss$ with fibre $\mathbb{C}^{m_{\beta,l}-1} \times (\mathbb{C} \setminus \{0\})$.

In [18] it is proved that we can replace the indexing set $\mathcal{B}_l \setminus \{0\}$, whose elements correspond to the $G$-adjoint orbits $\Ad(G)\beta$ of elements of the indexing set for the stratification of $\mathcal{P}(N_{l,\beta})$ induced by the representation $\rho_l$ (see Paragraph 3.3), by the set of $N_l$-adjoint orbits $\Ad(N_l)\beta$. If $q_\beta : P_\beta \to \mathrm{Stab}(\beta)$ is the projection, then
\[
\Sigma_{\beta,l} = G Y_{\beta,l}^\ss = G \times_{P_\beta} Y_{\beta,l}^\ss
\]
where
\[
Y_{\beta,l}^\ss = Y_{\beta,l}^\ss \setminus (E_l \cup \tilde{E}_{l-1} \cup \ldots \cup \tilde{E}_1) \to \mathrm{Stab}\beta \times_{N_l \cap \mathrm{Stab}\beta} (Z_{\beta,l}^\ss \cap \pi_l^{-1}(Z_{R_l}^\ss))
\]
is a fibration with fibre $\mathbb{C}^{m_{\beta,l}-1} \times (\mathbb{C} \setminus \{0\})$, and
\[
Z_{\beta,l}^\ss \cap \pi_l^{-1}(Z_{R_l}^\ss) \to Z_{R_l}^\ss
\]
is a fibration with fibre $Z_{\beta,l}^\ss(\rho_l)$. We set
\[
Y_{\beta,l}^\ss = Y_{\beta,l}^\ss \setminus (E_l \cup \tilde{E}_{l-1} \cup \ldots \cup \tilde{E}_1) \cap p_\beta^{-1}(Z_{\beta,l}^\ss \cap \pi_l^{-1}(Z_{R_l}^\ss)).
\]
It is proved in [18] that $Y_{\beta,l}^\ss \cong P_\beta \times_{Q_\beta} Y_{\beta,l}^\ss$, where $Q_{\beta,l} = q_\beta^{-1}(N_l \cap \mathrm{Stab}\beta)$. Then
\[
\Sigma_\gamma = \Sigma_{\beta,l} \cong G \times_{Q_{\beta,l}} Y_{\beta,l}^\ss
\]
where
\[
p_\beta : Y_{\beta,l}^\ss \to Z_{\beta,l}^\ss \cap \pi_l^{-1}(Z_{R_l}^\ss)
\]
is a fibration with fibre $\mathbb{C}^{m_{\beta,l}-1} \times (\mathbb{C} \setminus \{0\})$ for a certain $m_{\beta,l} > 0$.

4. Cohomological formulae

In this section we study the equivariant Hodge–Poincaré series of the set of properly stable points of a complex projective variety $X$ equipped with a linear action of a complex reductive group $G$ and such that $X^\ss(0)$ is nonempty and there are properly stable points in $X^\ss$ that are not semistable. When $G$ is connected these formulae allow us to compute the Hodge–Poincaré series of the geometric quotient $X^\ss(0)/G$.

To do that, we consider the stratification $\{\Sigma_\gamma\}$ of $X^\ss$ indexed by
\[
\Gamma = \{R_1\} \cup \{R_1\} \times \{B_1 \setminus \{0\}\} \cup \ldots \cup \{R_\tau\} \cup \{B_\tau \setminus \{0\}\} \cup \{0\}
\]
where $\Sigma_0 = X^\ss(0)$ is an open stratum, that we introduced in Subsection 3.2. Since $X^\ss(0)$ is an open set of $X^\ss$, both have the same dimension. We have the following identity for the equivariant Hodge-Poincaré series of $X^\ss(0)$.

**Proposition 4.1.** With the previous notation, we have that
\[
HP_G(X^\ss(0))(u,v) = HP_G(X^\ss)(u,v) - \sum_{\gamma \in \Gamma} (uv)^{\lambda(\gamma)} HP_G(\Sigma_\gamma)(u,v),
\]
where $\lambda(\gamma)$ is the complex codimension of $\Sigma_\gamma$ in $X^\ss$. 
Proof. For each $\gamma \in \Gamma$ regard $EG$ as an increasing union of smooth finite-dimensional varieties $(EG)_m$ where $G$ acts freely on $(EG)_m$. Let

$$\Sigma_\gamma \times_G EG = \bigcup_{m \geq 0}(\Sigma_\gamma \times_G EG)_m$$

where $(\Sigma_\gamma \times_G EG)_m = \Sigma_\gamma \times_G (EG)_m$, and such that the immersion $(\Sigma_\gamma \times_G EG)_m \hookrightarrow \Sigma_\gamma \times_G EG$ induces isomorphisms in cohomology in degrees less than or equal to $m$. From this decomposition one obtains the following decomposition

$$X^{ss} \times_G EG = \bigsqcup_{\gamma \in \Gamma} (\Sigma_\gamma \times_G EG)_m = \bigcup_{m \geq 0} (\Sigma_\gamma \times_G EG)_m = \bigcup_{m \geq 0} (X^{ss} \times_G EG)_m.$$  

Then, for each $m$ we have that $(X^{ss} \times_G EG)_m = \bigsqcup_{\gamma \in \Gamma} (\Sigma_\gamma \times_G EG)_m$ where, because of our choices, $(X^{ss} \times_G EG)_m$ and $(\Sigma_\gamma \times_G EG)_m$ are smooth finite-dimensional varieties for every $m$ and $\gamma$. Using identity (I) we obtain

$$HP((X^{ss} \times_G EG)_m)(u, v) = (uv)^{\dim<(X^{ss} \times_G EG)_m>_{\Sigma_\gamma}} \mathcal{H}((X^{ss} \times_G EG)_m)(u^{-1}, v^{-1}).$$

Applying first Theorem 2.8 and then (1) we get

$$HP((X^{ss} \times_G EG)_m)(u, v) = \sum_{\gamma \in \Gamma}(uv)^{\dim<(X^{ss} \times_G EG)_m>_{\Sigma_\gamma}} HP((\Sigma_\gamma \times_G EG)_m)(u, v).$$

Note that $\dim\mathcal{C}(X^{ss} \times_G EG)_m - \dim\mathcal{C}(\Sigma_\gamma \times_G EG)_m$ equals $\dim\mathcal{C}X^{ss} + \dim\mathcal{C}(EG)_m - \dim\mathcal{C}G - \dim\mathcal{C}\Sigma_\gamma - \dim\mathcal{C}(EG)_m + \dim\mathcal{C}G$, which is independent of $m$ and is actually the codimension of $\Sigma_\gamma$ in $X^{ss}$. We name it $\lambda(\gamma)$.

Since the immersion $(X^{ss} \times_G EG)_m$ in $X^{ss} \times_G EG$ induces isomorphisms in cohomology in degrees less than or equal to $m$, and the same is true for $\Sigma_\gamma$, from the previous identity we have the following

$$\sum_{j=1}^{m} \sum_{p,q,j} (-1)^{p+q+j}h_{p,q,j}^G(X^{ss})u^pv^q + \sum_{j>m} \sum_{p,q,j} (-1)^{p+q+j}h_{p,q,j}^G(H^j((X^{ss} \times_G EG)_m))u^pv^q =$$

$$= \sum_{\gamma \in \Gamma} \sum_{j=1}^{m} \sum_{p,q,j} (-1)^{p+q+j}h_{p,q,j}^G(\Sigma_\gamma)u^{p+\lambda(\gamma)}v^{q+\lambda(\gamma)} + \sum_{\gamma \in \Gamma} \sum_{j>m} \sum_{p,q,j} (-1)^{p+q+j}h_{p,q,j}^G(H^j((\Sigma_\gamma \times_G EG)_m))u^{p+\lambda(\gamma)}v^{q+\lambda(\gamma)}.$$

Now, since this identity is independent of $m$ and $X^*_b$ and $X^{ss}$ have the same dimension, we conclude. $\square$

We shall need the following Lemma for future computations.

**Lemma 4.2.** Let $Y \rightarrow Z$ be a locally trivial fibration in the Zariski topology with fibre $F$, and such that it is compatible with respect to the action of the group $G$ that acts on $Y$ and $Z$ respectively. Then

$$HP_G(Y)(u, v) = HP_G(Z)(u, v) \cdot HP(F)(u, v).$$

**Proof.** Regarding $Y \times_G EG$ and $Z \times_G EG$ as increasing unions of smooth finite-dimensional varieties, for each $m$ we have that the fibre $Y \rightarrow Z$ induces a new fibration $Y \times_G (EG)_m \rightarrow Z \times_G (EG)_m$ with fibre $F$. Now, applying (1) and Lemma 2.8 and bearing in mind that the dimension of $Y$ is equal to the sum of the dimension of $Z$ and the dimension of $F$, we obtain

$$HP(Y \times_G (EG)_m)(u, v) = HP(Z \times_G (EG)_m)(u, v) \cdot HP(F)(u, v).$$

Since this identity is independent of $m$, we finish the proof of the Lemma. $\square$

Following Subsection 3.2 we have that the strata $\Sigma_\gamma$ for $\gamma \neq 0$ fall into two classes. Either $\gamma = R_l$ for some $l \in \{1, \ldots, \tau\}$, in which case

$$\Sigma_{R_l} = GZ_{R_l}^*,$$

or else $\gamma = (R_l, \beta)$ where $\beta \in B_l \setminus \{0\}$ for some $l \in \{1, \ldots, \tau\}$ and the stratum $\Sigma_\gamma$ is

$$\Sigma_\gamma = \Sigma_{\beta, l} = S_{\beta, l}(E_l \cup \tilde{E}_{l-1} \cup \ldots \cup \tilde{E}_1).$$

In the first case, $\Sigma_{R_l} = GZ_{R_l}^* \cong G \times_{N_l} Z_{R_l}^*$, where $N_l$ is the normaliser of $R_l$. Then $H^*_G(\Sigma_{R_l}) \cong H^*_N(Z_{R_l}^*)$ which is an isomorphism of mixed Hodge structures, hence induces the following identity

$$HP_G(\Sigma_{R_l})(u, v) = HP_N(Z_{R_l}^*)(u, v).$$

(21)
In the latter, we recall \( \Sigma_{\beta,l} = GY_{\beta,l}^E = G \times P_\beta Y_{\beta,l}^E \) (see (15), (16) and (17)) that

\[
Y_{\beta,l}^E = Y_{\beta,l}^{ss}(E_l \cup \hat{E}_l \cup \ldots \cup \hat{E}_1) \rightarrow \text{Stab}_\beta \times N_{l\cap \text{Stab}_\beta} (Z_{\beta,l}^{ss} \cap \pi_1^{-1}(Z_{R_l}^*))
\]

is a fibration with fibre \( \mathbb{C}^{m,\beta,l-1} \times (\mathbb{C} \setminus \{0\}) \) for some \( m_{\beta,l} > 0 \), and

\[
Z_{\beta,l}^{ss} \cap \pi_1^{-1}(Z_{R_l}^*) \rightarrow Z_{R_l}^*
\]
is a fibration with fibre \( Z_{\beta}^{ss}(\rho_l) \). From (22) we get the following fibration

\[
G \times \text{Stab}_\beta Y_{\beta,l}^E \rightarrow G \times N_{l\cap \text{Stab}_\beta} (Z_{\beta,l}^{ss} \cap \pi_1^{-1}(Z_{R_l}^*))
\]
whose fibre is \( \mathbb{C}^{m,\beta,l-1} \times (\mathbb{C} \setminus \{0\}) \). Using next Lemma 4.2 we obtain

\[
HP_{\text{Stab}_\beta}(Y_{\beta,l}^E)(u,v) = HP(C^{m,\beta,l-1} \times \mathbb{C} \setminus \{0\})(u,v) \cdot HP_{N_{l\cap \text{Stab}_\beta}}(Z_{\beta,l}^{ss} \cap \pi_1^{-1}(Z_{R_l}^*)) (u,v).
\]

Now, identity (11) tells us that

\[
HP(C^{m,\beta,l-1} \times \mathbb{C} \setminus \{0\})(u,v) = (uv)^{m_{\beta,l}} H(C^{m,\beta,l-1} \times \mathbb{C} \setminus \{0\})(u^{-1}, v^{-1}) =
= (uv)^{m_{\beta,l}} \cdot (uv)^{-m_{\beta,l+1}} ((uv)^{-1} - 1) = 1 - uv,
\]
then (24) is

\[
HP_{\text{Stab}_\beta}(Y_{\beta,l}^E)(u,v) = (1 - uv) \cdot HP_{N_{l\cap \text{Stab}_\beta}}(Z_{\beta,l}^{ss} \cap \pi_1^{-1}(Z_{R_l}^*)) (u,v).
\]
We also have that \( q_{\beta} : P_{\beta} \rightarrow \text{Stab}_\beta \) is a retraction, which implies the following isomorphism of equivariant cohomology

\[
H^*_{\text{Stab}_\beta}(Y_{\beta,l}^E) \cong H^*_{P_{\beta}}(Y_{\beta,l}^E).
\]
Moreover, from \( \Sigma_{\beta,l} = GY_{\beta,l}^E = G \times P_\beta Y_{\beta,l}^E \) we get that \( H^*_{P_{\beta}}(Y_{\beta,l}^E) \cong H^*_G(\Sigma_{\beta,l}) \). These are isomorphisms of mixed Hodge structures, hence identity (25) gives us

\[
HP_G(\Sigma_{\beta,l})(u,v) = (1 - uv) \cdot HP_{N_{l\cap \text{Stab}_\beta}}(Z_{\beta,l}^{ss} \cap \pi_1^{-1}(Z_{R_l}^*)) (u,v).
\]
Then, we have the following

**Proposition 4.3.** When \( X \) is a nonsingular projective variety acted on linearly by a reductive group \( G \), the equivariant Hodge–Poincaré series of \( X_{(0)}^* \) are given by

\[
HP_G(X_{(0)}^*)(u,v) = HP_G(X^{ss})(u,v) - \sum_{l=1}^{\tau} (uv)^{\lambda(R_l)} HP_{N_l}(Z_{R_l}^*)(u,v) -
- \sum_{l=1}^{\tau} \sum_{\beta \in B_l \setminus \{0\}} (uv)^{\lambda(\beta,l)} (1 - uv) \cdot HP_{N_{l\cap \text{Stab}_\beta}}(Z_{\beta,l}^{ss} \cap \pi_1^{-1}(Z_{R_l}^*)) (u,v).
\]

Moreover, the codimension of the strata are given by the following formulae:

\[
\lambda(R_l) = \text{codim} \Sigma_{R_l} = z(\beta)
\]
where \( z(\beta) + 1 = \dim Z_{\beta,l}(\rho_l) \). And

\[
\lambda(\beta,l) = \text{codim} \Sigma_{(\beta,l)} = \dim Q_{\beta,l} - \dim N_l - m_{\beta,l}.
\]

**Proof.** Identity (27) follows from Proposition 4.1 and identities (21) and (26). Regarding the codimension of the strata, for \( \Sigma_{R_l} = GZ_{R_l}^* \) one has that

\[
\dim GZ_{R_l}^* = \dim G - \dim N_l + \dim Z_{R_l}^*
\]
where \( N_l \) is the normaliser of \( R_l \). Now, \( Z_{R_l}^* \subseteq \tilde{Z}_{R_l}^{ss} \) is an open subset, therefore they have the same dimension. Moreover, one has a locally trivial fibration \( E_l \rightarrow G\tilde{Z}_{R_l}^{ss} \) whose fibre is a projective space of dimension \( z(\beta) \), then we obtain the first formula.

For the later, from (18) and (19) we get that

\[
\dim \Sigma_{(\beta,l)} = \dim G - \dim Q_{\beta,l} + m_{\beta,l} + \dim \tilde{Z}_{R_l}^{ss} + z(\beta).
\]
Since \( z(\beta) = \dim X^{ss} - \dim G + \dim N_l - \dim \tilde{Z}_{R_l}^{ss} \), then \( \text{codim} \Sigma_{(\beta,l)} = \dim Q_{\beta,l} - \dim N_l - m_{\beta,l} \). □
Remark 4.4. When $G$ is connected and acts freely on $X$, from paragraph 2.11 we know that $H^*_G(X^*_s) \cong H^*(X^*_s/G)$ then identity (27) gives us the Hodge–Poincaré series of the quotient $X^*_s/G$. Regarding the codimension of the strata, for those strata $S_{\beta,i}$ that can be described as $S_{\beta,i} \setminus (E_l \cup \bar{E}_{l-1} \cup \ldots \cup \bar{E}_1)$, then its codimension equals that of $S_{\beta,i}$. The latter is given in (13).

4.5. Our goal is to obtain an explicit formula for the equivariant Hodge–Poincaré series of $X^*_s$ from identity (27). We have fixed a maximal torus $T$, and let $W$ be the set of weights for the action of $T$ on $G$. We have already seen that if $\beta \in B$, then $\beta$ is the closest point to 0 of

\[
\text{Conv}\{\alpha \in W : \alpha \cdot \beta = \|\beta\|^2\} = \text{Conv}\{\alpha \in W : (\alpha - \beta), \beta = 0\}.
\]

We have that $T$ is a maximal torus of $\text{Stab}\beta$. We can define a $\beta$-sequence of length $q$ (see 13 §5 for details) as a sequence $\beta = (\beta_1, \ldots, \beta_q)$ of $q$ nonzero elements of $t$ satisfying that for each $1 \leq j \leq q$

(a) $\beta_j$ is the closest point to 0 of the convex hull

\[
\text{Conv}\{\alpha - \beta_1 - \ldots - \beta_{j-1} \text{ such that } \alpha \in W \text{ and } \alpha \cdot \beta_k = \|\beta_k\|^2 \text{ for } 1 \leq k \leq j\}.
\]

(b) $\beta_j$ lies in the unique Weyl chamber containing $t$, of the subgroup $\bigcap_{1 \leq i \leq j} \text{Stab}\beta_i$.

Moreover, a sequence $\beta = (\beta_1, \ldots, \beta_q)$ of $q$ nonzero elements of $t$ with $q > 1$ is a $\beta$-sequence if and only if $\beta_1 \in B \setminus \{0\}$ and the sequence $\beta' = (\beta_2, \ldots, \beta_q)$ is a $\beta$-sequence for the action of $\text{Stab}\beta_1$ on $Z_{\beta_1}$. Now, for each $\beta$-sequence $\beta = (\beta_1, \ldots, \beta_q)$, let $T_\beta$ be the subtorus of $T$ generated by the set of weights $\{\beta_1, \ldots, \beta_q\}$, and let $Z_\beta$ be the union of the connected components of the fixed points set of $T_\beta$ on $X$.

4.6. Throughout this paper we have assumed that $X$ is a nonsingular projective variety that is acted on by a reductive group $G$, and such that $X^{ss} \neq X^*_s \neq \emptyset$. In the blow up procedure we obtain varieties $X_{(i)}$ for $i = 1, \ldots, \tau$ and one may consider Morse stratifications $\{S_{\beta,i}\}_{\beta \in B \setminus \{0\}}$ on each $X_{(i)}$ satisfying the properties of Paragraph 4.1. The groups $R_i$ are connected reductive subgroups of $G$ that fix semistable points of $X$. Let $B_\beta$ be the set of $\beta$-sequences defined for the stratification $\{S_{\beta,i}\}_{\beta \in B \setminus \{0\}}$.

In Paragraph 3.3 we saw that the stratification $\{S_{\beta,i}\}_{\beta \in B \setminus \{0\}}$ is determined by the action of $R_i$ on the fibres of $N_i$ over $G\hat{Z}_{R_i}^{ss}$. Let $\rho_i$ be the representation of $R_i$ on the normal $N_i,x$ to $G\hat{Z}_{R_i}^{ss}$ at $x$. Let $B(\rho_i)$ be the set of $\beta$-sequences for the representation $\rho_i$.

For every $\beta$-sequence $\beta = (\beta_1, \ldots, \beta_q) \in B(\rho_i)$ we define the varieties $Z^{ss}_{\beta,i}(\rho_i)$ and $Z_{\beta}(\rho_i)$ as in the previous section but with respect to $\rho_i$. Let $z(\beta,i)$ be its dimension, i.e., $z(\beta,i) + 1$ is the number, counting multiplicities, of weights $\alpha$ such that $\alpha \cdot \beta_j = \|\beta_j\|^2$ for $j = 1, \ldots, q$.

Let $d(\beta,i)$ be the sum over $j = 1, \ldots, q$ of the codimension in $Z_{\beta,j-1,i}$ of the corresponding stratum in $Z_{\beta,j-1,i}$ to $\beta_j$. If for every index $j = 1, \ldots, q$ we denote by $e_j$ the number, counting multiplicities, of weights $\alpha$ such that $\alpha \cdot \beta_j = \|\beta_j\|^2$ for $k = 1, \ldots, j - 1$ and $\alpha \cdot \beta_j < \|\beta_j\|^2$, then $d(\beta,i)$ is given by

\[
d(\beta,i) = \sum_{j=1}^{q} e_j - \frac{1}{2} \dim \text{Stab}(\beta_1, \ldots, \beta_{j-1})/\text{Stab}(\beta_1, \ldots, \beta_j),
\]

where $\text{Stab}_\beta = \text{Stab}\beta_1 \cap \ldots \cap \text{Stab}\beta_q$. Let $q(\beta)$ be the length of the $\beta$-sequence $\beta$. Now, let

\[
w(\beta, R_i, G) = \prod_{j=1}^{q} w(\beta_j, R_i \cap \text{Stab}(\beta_1, \ldots, \beta_{j-1}), \text{Stab}(\beta_1, \ldots, \beta_j))
\]

where $w(\beta, R_i, G')$ is the number of adjoint $R_i$-orbits contained in the orbit of $\text{Ad}(G')\beta$.

Theorem 4.7. The $G$-equivariant Hodge–Poincaré series of $X^*_s$ are given by

\[
HP_G(X^*_s)\cdot (u,v) = HP_G(X^{ss})\cdot (u,v) - \sum_{l=1}^{\tau} (uv)^{\lambda(R_l)} HP_{N_l}(Z^{ss}_{R_l})\cdot (u,v) + \sum_{l=1}^{\tau} \sum_{0 \neq \beta \in B(\rho_l)} (-1)^d(\beta)(uv)^{d(\beta)}(1 - (uv)^{z(\beta)+1})w^{-1}(\beta, R_l, G) \cdot HP_{N_l \cap \text{Stab}_\beta}(Z^{ss}_{R_l})\cdot (u,v).
\]

Moreover, when $G$ acts freely on $X^*_s$ one obtains that $HP(X^*_s\cdot (u,v)) = HP_G(X^*_s\cdot (u,v))$, then this formula gives the Hodge–Poincaré series of the geometric quotient $X^*_s\cdot (u,v)/G$. 
Proof. From Proposition 4.8 we have that when $X$ is a nonsingular projective variety acted on linearly by a reductive group $G$, the equivariant Hodge–Poincaré series of $X^*_0$ are given by

$$
HP_G(X^*_0)(u, v) = HP_G(X^ss)(u, v) - \sum_{l=1}^{\tau} \sum_{l=1}^{\tau} (uv)^{\lambda(R_l)} HP_{\overline{N}_l}(Z_{R_l}^*)(u, v) - \sum_{l=1}^{\tau} \sum_{\beta \in \mathcal{B}_l \setminus \{0\}} (uv)^{\lambda(\beta, l)} (1 - uv) \cdot HP_{\overline{N}_l(\beta, l)^{\text{Stab}}}(Z_{\beta, l}^* \cap \pi_t^{-1}(Z_{R_l}^*))(u, v),
$$

the codimension of the strata are given by $\lambda(R_l) = \text{codim} \Sigma_{R_l} = z(\beta)$, where $z(\beta) + 1 = \dim Z_{\beta, l}(\rho_l)$. And $\lambda(\beta, l) = \text{codim} \Sigma_{(\beta, l)}$. For every $l = 1, \ldots, \tau$, using induction over the length of the $\beta$-sequences in $\mathcal{B}_l$ we obtain the following formula

$$
HP_{\overline{N}_l(\beta, l)^{\text{Stab}}}(Z_{\beta, l}^* \cap \pi_t^{-1}(Z_{R_l}^*))(u, v) = HP_{\overline{N}_l(\beta, l)^{\text{Stab}}}(Z_{\beta, l} \cap \pi_t^{-1}(Z_{R_l}^*)) (u, v) - \sum_{l=1}^{\tau} \sum_{\beta' \in B_l \setminus \{0\}} (-1)^q (uv)^{d(\beta', l)} \cdot HP_{\overline{N}_l(\beta', l)^{\text{Stab}}}(Z_{\beta', l} \cap \pi_t^{-1}(Z_{R_l}^*))(u, v),
$$

where $\beta'$ are $\beta$-sequences of length $q - 1$ in $B_l$. And $d(\beta', l)$ is given by (30). Note that for $\beta$-sequences of length 1, that is, an element $\beta$, one has that $d(\beta, l) = \lambda(\beta, l)$. Now, combining (32) and (33) we obtain the following formula

$$
HP_G(X^*_0)(u, v) = HP_G(X^ss)(u, v) - \sum_{l=1}^{\tau} \sum_{\beta \in \mathcal{B}_l \setminus \{0\}} (uv)^{\lambda(R_l)} HP_{\overline{N}_l}(Z_{R_l}^*)(u, v) + \sum_{l=1}^{\tau} \sum_{\beta \in \mathcal{B}_l \setminus \{0\}} (uv)^{\lambda(R_l)} HP_{\overline{N}_l(\beta, l)^{\text{Stab}}}(Z_{\beta, l} \cap \pi_t^{-1}(Z_{R_l}^*))(u, v).
$$

By 7.6 and 7.9, each $\beta$-sequence $\beta$ of the Morse stratification in each $X_{(R_l)}$ corresponds to $w(\beta, R_l, G)$ $\beta$-sequences for the stratification associated to the representation $\rho_l$. Moreover, the fibration (17) restricted to $Z_{\beta, l} \cap \pi_t^{-1}(Z_{R_l}^*) \to Z_{R_l}$ is a fibration with fibre $Z_{\beta, l}(\rho_l)$ which is a projective space of dimension $z(\beta)$. Then, the associated equivariant spectral sequence degenerates by Deligne’s criterion. Hence, from (34) we get

$$
HP_G(X^*_0)(u, v) = HP_G(X^ss)(u, v) - \sum_{l=1}^{\tau} \sum_{\beta \in \mathcal{B}_l \setminus \{0\}} (uv)^{\lambda(R_l)} HP_{\overline{N}_l}(Z_{R_l}^*)(u, v) + \sum_{l=1}^{\tau} \sum_{\beta \in \mathcal{B}_l \setminus \{0\}} (uv)^{\lambda(R_l)} HP_{\overline{N}_l(\beta, l)^{\text{Stab}}}(Z_{\beta, l} \cap \pi_t^{-1}(Z_{R_l}^*))(u, v).
$$

By Remark 4.4 we conclude the proof of the Theorem. □

For a given $l \in \{1, \ldots, \tau\}$, in order to compute $HP_{\overline{N}_l}(Z_{R_l}^*)$ we shall need the following slightly different version of Lemma 1.21 of [10]. To make the notation simpler to the eye, we set $N_l = N$ and $R_l = R$.

**Lemma 4.8.** $H^*(Z_{R_l}^*)$ is the invariant part of $H^*_{N_0}(Z_{R_l}^*)$ under the action of the finite group $\pi_0N = N/N_0$, and

$$
H^*_{N_0}(Z_{R_l}^*) = H^*(BR) \otimes H^*_{N_0/R}(Z_{R_l}^*).
$$

For a given $l \in \{1, \ldots, \tau\}$, when the index $\beta$ is maximal with respect to the partial order of $\mathcal{B}_l \setminus \{0\}$ one may compute the equivariant Hodge–Poincaré polynomial of $\Sigma_{\beta, l}$ from that of the varieties $Z_{R_l}^*$.

**Lemma 4.9.** For a given $l \in \{1, \ldots, \tau\}$, when the index $\beta$ is maximal with respect to the partial order given in $\mathcal{B}_l \setminus \{0\}$ the equivariant Hodge–Poincaré polynomial of $\Sigma_{\beta, l}$ is given by

$$
HP_G(\Sigma_{\beta})(u, v) = HP_{\overline{N}_l(\beta)^{\text{Stab}}}(Z_{R_l}^*)(u, v) \cdot (1 - (u \cdot v)^{z(\beta)+1}),
$$

where $z(\beta) + 1 = \dim Z_{\beta, l}(\rho_l)$.

Moreover, for $l = 1$ one has that

$$
HP_G(\Sigma_{\gamma})(u, v) = HP_G(S_{\beta, l})(u, v) \cdot (1 - uv).
$$
Proof. For the first statement, from the description immediately after Lemma 4.2 one has that (see (26))

\[ HP_G(\Sigma_{\beta,1})(u, v) = (1 - uv) \cdot HP_{N_1 \cap \text{Stab}\beta}(Z_{\beta,1}^{ss} \cap \pi_1^{-1}(Z_R^s))(u, v) \]

where

\[ (35) \]

\[ Z_{\beta,1}^{ss} \cap \pi_1^{-1}(Z_R^s) \rightarrow Z_R^s \]

is a fibration with fibre \( Z_{\beta,1}^{ss}(\rho_l) \). When \( \beta \) is a maximal element of \( B_1 \setminus \{0\} \) for the given partial order, we get that \( Z_{\beta,1}^{ss}(\rho_l) = Z_{\beta,1}(\rho_l) \) and this is actually a nonsingular projective variety. Then the spectral sequence associated to the fibration degenerates by Deligne’s criterion and we obtain the following isomorphism of equivariant cohomology

\[ (36) \]

\[ H_{N_1 \cap \text{Stab}\beta}^*(Z_{\beta,1}^{ss} \cap \pi_1^{-1}(Z_R^s)) \cong H_{N_1 \cap \text{Stab}\beta}^*(Z_R^s) \otimes H^*(\mathbb{P}^{z(\beta)}), \]

where \( z(\beta) + 1 = \dim Z_{\beta,1}(\rho_l) \). Now, (36) is an isomorphism of mixed Hodge structures, so it induces the following identity of equivariant Hodge-Poincaré polynomials

\[ (37) \]

\[ HP_{N_1 \cap \text{Stab}\beta}(Z_{\beta,1}^{ss} \cap \pi_1^{-1}(Z_R^s))(u, v) = HP_{N_1 \cap \text{Stab}\beta}(Z_R^s)(u, v) \cdot \frac{1 - (u \cdot v)^{z(\beta)+1}}{1 - u \cdot v}, \]

which completes the proof of the first statement.

Regarding the second part of the Lemma, we have already seen that the stratification \( \{ S_{\beta,1} \}_{\beta \in B_1 \setminus \{0\}} \) satisfies that each \( S_{\beta,1} \) retracts onto its intersection with the exceptional divisor and if \( B(R_1) = B(\rho_l) \), then it retracts onto

\[ (38) \]

\[ G \times_{N_1 \cap \text{Stab}\beta} Z_{\beta,1}^{ss} \cap \pi_1^{-1}(Z_R^s) \]

where

\[ (39) \]

\[ \pi_1 : Z_{\beta,1}^{ss} \cap \pi_1^{-1}(Z_R^s) \rightarrow Z_R^{ss} \]

is a fibration with fibre \( Z_{\beta,1}^{ss}(\rho_l) \). Since \( \beta \) is maximal, then \( Z_{\beta,1}^{ss}(\rho_l) = Z_{\beta,1}(\rho_l) \) and this is a nonsingular projective variety. For \( l = 1 \) we have that \( R_1 \) has maximum dimension among the reductive subgroups of \( G \) fixing a semistable point, then \( Z_{R_1}^{ss} = Z_{R_1} \), and in this particular case \( Z_{R_1}^{ss} = Z_{R_1} \). Hence, from identities (38) and (39) we get the following identity

\[ H_G^*(S_{\beta,1}) = H_{N_1 \cap \text{Stab}\beta}^*(Z_{\beta,1}^{ss} \cap \pi_1^{-1}(Z_R^s)) \cong H_{N_1 \cap \text{Stab}\beta}^*(Z_R^s) \otimes H^*(\mathbb{P}^{z(\beta)}) \]

which implies

\[ HP_G(S_{\beta,1})(u, v) = HP_{N_1 \cap \text{Stab}\beta}(Z_R^s)(u, v) \cdot \frac{1 - (u \cdot v)^{z(\beta)+1}}{1 - u \cdot v}, \]

comparing the latter with (26) and (37), we finish the proof of the Lemma. \( \square \)

5. Cohomological Formulæ for the Moduli Space of Stable Vector Bundles When the Rank and the Degree Are Not Coprime

Let \( \mathcal{M}(n, d) \) be the moduli space of semistable vector bundles of rank \( n \) and degree \( d \) over an algebraic curve \( X \) of genus \( g \). A vector bundle \( E \) is semistable (respectively properly stable) if for every proper subbundle \( F \) of \( E \) it is satisfied that \( \mu(F) \leq \mu(E) \) (resp. \( \mu(F) < \mu(E) \)) where \( \mu(E) = \deg(E) / \text{rank}(E) \) is the slope of the vector bundle \( E \). Let \( \mathcal{M}^s(0)(n, d) \) be the subset of \( \mathcal{M}(n, d) \) consisting of properly stable vector bundles. It is well known that when the rank and the degree are coprime \( \mathcal{M}(n, d) = \mathcal{M}^s(0)(n, d) \). In this paper we are interested in the case in which \( (n, d) \neq 1 \). Under this hypothesis we have that \( \mathcal{M}^s(0)(n, d) \) is an open subset of \( \mathcal{M}(n, d) \), hence they have the same dimension. One also has that

\[ \dim \mathcal{M}(n, d) = n^2(g - 1) + 1. \]

In this Section we obtain formulæ for the Hodge–Poincaré polynomial of \( \mathcal{M}^s(0)(n, d) \) when \( (n, d) \neq 1 \). We first need to understand the corresponding stratification \( \{ \Sigma_\gamma \}_{\gamma \in \Gamma} \) for \( \mathcal{M}(n, d) \).

There are several ways of representing \( \mathcal{M}(n, d) \) as a geometric invariant theory quotient. Following [24] §3, we know that if \( E \) is a semistable bundle of rank \( n \) and degree \( d \), satisfying that \( d > n(2g - 1) \), then

(i) \( E \) is generated by its sections;

(ii) \( H^1(X, E) = 0 \).
Using Riemann-Roch, property (ii) implies that
\begin{equation}
\dim H^0(X, E) = d + n(1 - g),
\end{equation}
where \( g \) is the genus of \( X \). Let \( p = d + n(1 - g) \). Then it follows that there is a holomorphic map \( h \) from \( X \) to the Grassmannian \( \text{Gr}(n, p) \) of \( n \)-dimensional quotients of \( \mathbb{C}^p \) such that the pull-back \( h^* T \) of the tautological bundle \( T \) on \( \text{Gr}(n, p) \) is isomorphic to \( E \).

5.1. Let us define \( \mathcal{R}(n, d) \) to be the subset of \( \text{Hol}_d(X, \text{Gr}(n, p)) := \{ \text{holomorphic maps from } X \text{ to } \text{Gr}(n, p) \} \) consisting of those holomorphic maps \( h \) such that \( E_h = h^* T \) has degree \( d \) and the map of sections \( \mathbb{C}^p \to H^0(X, E_h) \) induced from the quotient bundle map \( \mathbb{C}^p \to X \to E_h \) is an isomorphism. Tensoring by a line bundle of degree \( l \) gives an isomorphism of \( \mathcal{M}(n, d) \) with \( \mathcal{M}(n, d + n l) \) for any \( l \in \mathbb{Z} \). Then, we may assume \( d >> 0 \) in which case \( \mathcal{R}(n, d) \) is a non-singular quasi-projective variety. Moreover, if \( d > n(2g - 1) \) there is a quotient \( \mathcal{E} \) of the trivial bundle of rank \( p \) over \( \mathcal{R}(n, d) \times X \) with the following properties

(a) \( \mathcal{E} \) has the local universal property for families of bundles over \( X \) of rank \( n \) and degree \( d \) satisfying (i) and (ii).

(b) If \( h \in \mathcal{R}(n, d) \) then the restriction of \( \mathcal{E} \) to \( \{ h \} \times X \) is the pull-back \( E_h \) of the tautological bundle \( T \) on \( \text{Gr}(n, p) \).

(c) If \( h \) and \( g \) lie in \( \mathcal{R}(n, d) \) then \( E_h \) and \( E_g \) are isomorphic as bundles over \( X \) if and only if \( h \) and \( g \) lie in the same orbit of the natural action of \( \text{GL}(p) \) on \( R(n, d) \).

(d) If \( h \in \mathcal{R}(n, d) \) then the stabilizer of \( h \) in \( \text{GL}(p) \) is isomorphic to the group \( \text{Aut}(E_h) \) of complex analytic automorphisms of \( E_h \).

Moreover, if \( N \) is any large enough integer then \( R(n, d) \) can be embedded as a quasi-projective subvariety of the product \( \text{Gr}(n, p)^N \). This embedding gives us a linearisation of the action of \( \text{SL}(p) \) on \( R(n, d) \). If \( N \) and \( d \) are large enough then the following condition is satisfied

(e) The point \( h \in R(n, d) \) is semistable in the sense of GIT theory for the linear action of \( \text{SL}(p) \) on \( R(n, d) \) if and only if \( E_h \) is a semistable bundle. If \( h \) and \( g \) lie in \( R(n, d)^{ss} \) then they represent the same point of \( \mathcal{R}(n, d)/\text{SL}(p) \) if and only if \( \text{grad}E_h \cong \text{grad}E_g \), where \( \text{grad}E_h \) is the canonical graded object associated to every Jordan–Hölder filtration of \( E_h \).

From these properties one obtains that there is a natural identification of the moduli space of semistable vector bundles, \( \mathcal{M}(n, d) \), and the GIT quotient \( \mathcal{R}(n, d)/\text{SL}(p) \) (see [24] for details). Note that \( \mathcal{R}(n, d) \) is only a quasi-projective variety, this would not affect the desingularization process since the closure \( \overline{\mathcal{R}(n, d)} \) of \( \mathcal{R}(n, d) \) embedded in \( \text{Gr}(n, p)^N \) contains no more semistable points than \( \mathcal{R}(n, d) \) does (see [16], just before Section 3, for more details).

Once we have described \( \mathcal{M}(n, d) \) as a GIT quotient, in order to find the strata \( \Sigma_{\gamma} \) for \( \gamma \in \Gamma \), we need to understand how to blow up \( \mathcal{M}(n, d) \) to obtain a variety \( \tilde{\mathcal{M}}(n, d) \) such that the properly stable points are the same as the semistable ones with respect to the action of \( \text{SL}(p) \) properly linearised. We need to blow up \( \mathcal{M}(n, d) \) along a sequence of subvarieties of the form \( Z_{R}/(N/R) \) determined by a conjugacy class \( R \) of non-trivial connected subgroups of stabilizers of semistable points. Or what is the same, blow up \( \mathcal{R}(n, d)^{ss} \) along varieties \( \text{SL}(p)Z_R^{ss} \) where \( Z_R^{ss} := \{ h \in \mathcal{R}(n, d)^{ss} \text{ such that } E_h \text{ is fixed by } R \} \neq 0 \) in decreasing order of \( \dim R \) (see Subsection 3.2).

Since the central one parameter subgroup \( \mathbb{C}^* \) of \( \text{GL}(p) \) acts trivially on \( \mathcal{R}(n, d) \), finding stabilizers in \( \text{GL}(p) \) is essentially equivalent to finding stabilisers in \( \text{SL}(p) \). From Paragraph 5.1(d) we know how to find such a subgroup \( R \) of \( \text{GL}(p) \), this is always the connected component of the automorphism group \( \text{Aut}E \) of a semistable vector bundle \( E \).

Let \( E \) be a semistable bundle. Let \( \text{grad}E = m_1 E_1 + \ldots + m_s E_s \) be the graded object associated to each of its Jordan–Hölder filtrations, then the \( E_i \) are all properly stable bundles satisfying that \( \mu(E_i) = \mu(E) \) for all \( i \) and \( E_i \not\cong E_j \) for all \( i \neq j \). One has that \( \dim \text{Aut}E \leq \dim \prod_{1 \leq i \leq s} \text{GL}(m_i) \) with equality if and only if \( E \cong \text{grad}E \). If \( E \cong \text{grad}E \) then
\begin{equation}
\text{Aut}E \cong \prod_{1 \leq i \leq s} \text{GL}(m_i).
\end{equation}

In order to construct the partial desingularisation of \( \mathcal{M}(n, d) \), we need to find the semistable vector bundles \( E \) of rank \( n \) and degree \( d \) for which \( \dim \text{Aut}E \) is maximal. Assume that \( (n, d) = \gamma \not= 1 \) where \( n = nm' \) and \( d = md' \) satisfying that \( (n', d') = 1 \). The bundles whose dimension of automorphisms is maximal are those \( E \) of the form
\[ E = E'^{\oplus m} \]
where $E'$ is a properly stable vector bundle of rank $n'$ and degree $d'$. Then, the first step in the construction of $\tilde{R}(n, d)^{ss}$ is to blow up $R(n, d)^{ss}$ along $GL(p)Z^{ss}_{GL(m)} := \{ h \in R(n, d)^{ss} \text{ such that } E_h \cong E'^{\otimes m} \text{ for } E' \in \mathcal{M}^s(n', d') \}$, where $\mathcal{M}^s(n', d')$ is a moduli space of properly stable bundles of rank $n'$ and degree $d'$, and $p = d+n(1-g)$ (see (31)) and $m = n/n'$. Let $R_1(n, d)$ be the blow up and $R_1(n, d)^{ss}$ be the semistable stratum after that. From Paragraph 5.1 (a), $\tilde{R}(1, d)/R_1(1, d)^{ss}$ is isomorphic to $\phi^{-1}(\phi(GL(p)Z^{ss}_{GL(m)}))$ where $\phi : R(n, d)^{ss} \to R(n, d)//SL(p)$ is the quotient map. From Paragraph 5.1 (e) we have that $\phi(GL(p)Z^{ss}_{GL(m)}) = \{ E \in M(n, d) \text{ such that grad}E \cong E'^{\otimes m}\}$, then $R_1(n, d)/R_1(n, d)^{ss}$ corresponds to the set $\{ h \in R(n, d) \text{ such that grad}E_h \cong E'^{\otimes m}\}$. Then, the first stratum is $\Sigma_{GL(m)} = GL(p)Z^{ss}_{GL(m)} = GL(p)Z^{ss}_{GL(m)}$ and $\cup_{\beta \in \tilde{R}_1 \setminus \{0\}} S_{\beta,1}$ is going to be the set $\{ h \in R(n, d) \text{ such that grad}E_h \cong E'^{\otimes m}\}$. The strata $\{\Sigma_{\beta,1}\}_{\beta \in \tilde{R}_1 \setminus \{0\}}$ are given by $\Sigma_{\beta,1} = S_{\beta,1} \setminus \mathbb{P}(N_1)$ where $N_1$ is the normal bundle corresponding to the first step in the blow up. Before analyzing the normal bundles let us explain how the blow up would work.

5.2. Following the previous analysis, one gets that conjugacy classes of connected reductive subgroups of dimension less than or equal to $n^2$ in $GL(p)$ which stabilize some point $h \in R(n, d)^{ss}$ correspond to unordered sequences $(m_1, n_1), \ldots, (m_s, n_s)$ of pairs of positive integers satisfying the following conditions (see [10] §3 and [17] §5):

(i) $\sum_{1 \leq j \leq s} m_j n_j = n$;
(ii) $\sum_{1 \leq j \leq s} m_j^2 \leq m^2$ and;
(iii) $n$ divides $n_j d$ for each $j$.

A representative $R$ of the conjugacy class corresponding to $(m_1, n_1), \ldots, (m_s, n_s)$ is given by the image of $\prod_{1 \leq j \leq s} GL(m_j)$ in $GL(p)$ given by some fixed isomorphism of $\prod_{1 \leq j \leq s} \mathbb{C}^{m_j} \otimes \mathbb{C}^{p_j}$ with $\mathbb{C}^p$, where $p_j = d_j + n_j(1-g) = n_j p/n$ and $d_j = n_j d/n$. Moreover, if $N$ is the normaliser of $\tilde{R}$ in $GL(p)$, then its connected component of the identity is given by

$$N_0 \cong \prod_{1 \leq j \leq s} (GL(m_j) \times GL(p_j))/\mathbb{C}^*,$$

where $\mathbb{C}^*$ is the diagonal central one parameter subgroup of $GL(m_j) \times GL(p_j)$, and $\pi_0(N) = N/N_0$ is the product

$$\prod_{j \geq 0, k \geq 0} S(\mathbb{Z}\{i : m_i = j \text{ and } n_i = k\})$$

where $S(n)$ is the symmetric group of permutations of a set of $n$ elements.

Then, at the $k$-stage of the blow up, there is a sequence $(m_1, n_1), \ldots, (m_s, n_s)$ satisfying (i), (ii) and (iii) above, and such that a representative $R_k$ of the corresponding conjugacy class is given by

$$R_k = \prod_{1 \leq j \leq s} GL(m_j)$$

embedded in $GL(p)$ as before. The variety $GL(p)Z^{ss}_{R_k}$ is identified with the set $\{ h \in R(n, d)^{ss} : E_h \cong m_1 E_1 \oplus \ldots \oplus m_s E_s \}$ where $E_j$ are semistable vector bundles of rank $n_j$ and degree $d_j = n_j d/n$. To obtain $R_{k+1}(n, d)$ we need to blow up $R_k(n, d)^{ss}$ along the proper transform of the variety $GL(p)Z^{ss}_{R_k}$ of $R(n, d)^{ss}$, then $R_{k+1}(n, d)\setminus R_{k+1}(n, d)^{ss}$ will be the set of those $h \in R(n,d)$ such that grad$E_h \cong \text{grad}(m_1 E_1 \oplus \ldots \oplus m_s E_s)$. Moreover, bearing in mind that $Z^{ss}_{R_k}$ is the set of properly stable points of $Z_{R_k}$ with respect to the action of $N_k/R_k$, where $N_k$ is the normaliser of $R_k$ in $GL(p)$, the stratum $\Sigma_{R_k} = GL(p)Z^{ss}_{R_k}$ is given by the set of $h \in R(n, d)^{ss}$ such that

$$E_h \cong m_1 E_1 \oplus \ldots \oplus m_s E_s$$

where $E_j$ are non-isomorphic properly stable vector bundles of rank $n_j$ and degree $d_j = n_j d/n$.

In [10] §7 is explained how to compute the normal bundle $N_k$ at a point $h$ of $GL(p)Z^{ss}_{R_k}$. Let $GL(p)\hat{Z}^{ss}_{R_k}$ be the proper transform of $GL(p)Z^{ss}_{R_k}$ in $R_k(n, d)^{ss}$. Then, the normal bundle to $GL(p)\hat{Z}^{ss}_{R_k}$ at a point $h$ in $R_k(n, d)^{ss}$ such that

$$E_h \cong m_1 E_1 \oplus \ldots \oplus m_s E_s$$
where $E_j$ are non-isomorphic properly stable vector bundles of rank $n_j$ and degree $d_j = n_jd/n$, is identified with $H^1(\text{End}_{\mathbb{P}^1}^j E_h)$, where $\text{End}_{\mathbb{P}^1}^j E_h = \text{End}E_h/\text{End}_{\mathbb{P}^1} E_h$. Here $\text{End} E_h$ is the vector bundle of holomorphic endomorphisms of $E_h$ and $\text{End}_{\mathbb{P}^1} E_h$ is the subbundle of $\text{End}E_h$ consisting of those endomorphisms that preserve the decomposition (46) (see [14] for details). One has that
\[
\text{End}_{\mathbb{P}^1}^j E_h \cong \bigoplus_{i,j}(m_i m_j - \delta_i^j)\text{Hom}(E_i, E_j)
\]
where $\delta_i^j$ is the Kronecker delta. Then
\[
(47) \quad H^1(\text{End}_{\mathbb{P}^1}^j E_h) \cong \bigoplus_{i,j=1}^s \mathbb{C}^{m_i m_j - \delta_i^j} \otimes H^1(E_i^* \otimes E_j).
\]
Since every morphism between two properly stable vector bundles of the same slope is either zero or an isomorphism, we have that
\[
\dim H^0(\text{End}_{\mathbb{P}^1}^j E_h) = \sum_{1 \leq j \leq s} (m_j^2 - 1).
\]
Then, using Riemann-Roch
\[
(48) \quad \dim H^1(\text{End}_{\mathbb{P}^1}^j E_h) = (g - 1)(n^2 - \sum_{1 \leq j \leq s} n_j^2) + \sum_{1 \leq j \leq s} (m_j^2 - 1),
\]
note that this dimension coincides with the codimension of $\Sigma_{R_k} = GL(p)Z_{GL(m)}^s$ in $\mathcal{R}(n, d)^{ss}$, that is $\lambda(R_k)$ (see (28)). Regarding the weights of the representation of $R_k$ on the normal (47), from (17) §5 one has that these are of the form $\xi = \eta - \eta'$ where $\eta$ and $\eta'$ are weights of the standard representation of $R_k$ on $\oplus_{i=1}^s \mathbb{C}^{m_i}$.

5.3. We have already seen that the highest stratum $\Sigma_{GL(m)} = GL(p)Z_{GL(m)}^s$ corresponds to elements $h \in \mathcal{R}(n, d)^{ss}$ such that $E_h \cong E^{n \oplus m}$ where $E' \in M^*_0(n', d')$. Then, $\Sigma_{GL(m)}$ may be identified with the product $\prod_m \mathcal{R}(n', d')^s_{(0)}$. We denote $GL(m)$ by $R_1$. In the second step $R_2 = GL(m - 1) \times \mathbb{C}^*$ and $\Sigma_{GL(m-1) \times \mathbb{C}^*} = GL(p)Z_{GL(m-1) \times \mathbb{C}^*}$ consists of points $h \in \mathcal{R}(n, d)^{ss}$ such that $E_h \cong E^{n \oplus (m-1)} \oplus E^m$ where $E'$ and $E^m$ are properly stable bundles of the same rank and degree but non-isomorphic. Then
\[
\Sigma_{GL(m-1) \times \mathbb{C}^*} \cong \prod_{m} \mathcal{R}(n', d')^s_{(0)} \setminus \bar{\Delta}_j
\]
where $\bar{\Delta}_j := \{(h_1, \ldots, h_n) \in \prod_m \mathcal{R}(n', d')^s_{(0)} \text{ such that } h_i = h_j \text{ for some } i \neq j \}$. Hence, if we choose an index $j \leq n$ we have that $R_j = GL(m - j) \times (\mathbb{C}^*)^j$ and
\[
\Sigma_{R_j} \cong \prod_{m} \mathcal{R}(n', d')^s_{(0)} \setminus \bar{\Delta}_j
\]
where $\bar{\Delta}_j := \{(h_1, \ldots, h_n) \in \prod_m \mathcal{R}(n', d')^s_{(0)} \text{ such that there exist indexes } i_1, \ldots, i_\alpha \text{ with } i_\alpha \neq i_\beta \text{ for all } \alpha \neq \beta \text{ and such that } h_{i_1} = \ldots = h_{i_\alpha} \}$. This description applies in general as follows. Let $R_k$ be a representative of the conjugacy class corresponding to a sequence $(m_1, n_1), \ldots, (m_s, n_s)$ as in Paragraph 5.2, then
\[
R_k = \prod_{1 \leq j \leq s} GL(m_j)
\]
properly embedded in $GL(p)$. Assume also that $n_i \neq n_j$ for all $i \neq j$. Then, the stratum $\Sigma_{R_k}$ consists of elements $h \in \mathcal{R}(n, d)^{ss}$ such that
\[
E_h \cong m_1 E_1 \oplus \ldots \oplus m_s E_s
\]
where $E_j$ are non-isomorphic properly stable vector bundles of rank $n_j$ and degree $d_j = n_jd/n$. Hence
\[
(49) \quad \Sigma_{R_k} \cong \prod_{m_1} \mathcal{R}(n_1, d_1)^s_{(0)} \times \ldots \times \prod_{m_s} \mathcal{R}(n_s, d_s)^s_{(0)}.
\]
Consider now an index $j$ such that in the $(k + j)$-step of the blow up, the representative $R_{k+j}$ of the corresponding conjugacy class has dimension
\[
m_1^2 + \ldots + m_s^2 \geq \dim R_{k+j} \geq m_1 + \ldots + m_s,
\]
there is no loss of generality in assuming that
\[
R_{k+j} \cong GL(m_1 - j_1) \times (\mathbb{C}^*)^{j_1} \times \ldots \times GL(m_s - j_s) \times (\mathbb{C}^*)^{j_s}.
\]
then \( \Sigma_{R_{k+j}} \) consists of elements \( h \in \mathcal{R}(n,d) \) such that
\[
E_h \cong (m_1 - j_1)E_1 \oplus \bigoplus_{i=1}^{j_1} E^1_i \oplus \ldots \oplus (m_s - j_s)E^s + \bigoplus_{i=1}^{j_s} E^s_i
\]
where all the bundles \( E^\alpha, E^s \) for all indexes \( \tau_\alpha = 1, \ldots, j_\alpha \) and \( \alpha = 1, \ldots, s \) are non-isomorphic properly stable bundles satisfying that \( \text{rank}(E^\tau_\alpha) = \text{rank}(E^s) = n_\alpha \) and \( \text{deg}(E^\tau_\alpha) = \text{deg}(E^s) = n_\alpha d/n \). Then
\[
\Sigma_{R_{k+j}} \cong \prod_{m_1} \mathcal{R}(n_1, d_1)_{0} \times \ldots \times \prod_{m_s} \mathcal{R}(n_s, d_s)_{0} \setminus (\Delta^1_{j_1} \sqcup \ldots \sqcup \Delta^s_{j_s}).
\]
Here, \( \Delta^\alpha_{j_\alpha} := \{ (h_1, \ldots, h_{m_\alpha}) \in \prod_{m_\alpha} \mathcal{R}(n_\alpha, d_\alpha)_{0} \) such that \( h_{i_1} = \ldots = h_{i_{j_\alpha}} \) for some partition \( \{i_1, \ldots, i_{j_\alpha}\} \) of \( \{1, \ldots, m_\alpha\} \) for \( \alpha = 1, \ldots, s \).

Now, in (1) we saw that \( \mathbb{G}Z_{R_{i}}^{\ast}/G \cong \mathbb{Z}_{R_{i}}/(N_i/R_i) \). Then, one has that
\[
\Sigma_{R_i}/G = \mathbb{G}Z_{R_i}^{\ast}/G \cong \mathbb{Z}_{R_i}^{\ast}/(N_i/R_i).
\]

Bearing in mind (50) we have that
\[
\mathbb{Z}_{R_{k+j}}^{\ast}/(N_{k+j}/R_{k+j}) = \Sigma_{R_{k+j}}/G \cong
\]
\[
\cong \left( \prod_{m_1} \mathcal{M}(n_1, d_1)_{0} \times \prod_{m_s} \mathcal{M}(n_s, d_s)_{0} \setminus (\Delta^1_{j_1} \sqcup \ldots \sqcup \Delta^s_{j_s}) \right)/\pi_0(N_{k+j}),
\]
where \( \Delta^\alpha_{j_\alpha} := \{ (E_1, \ldots, E_{m_\alpha}) \in \prod_{m_\alpha} \mathcal{M}(n_\alpha, d_\alpha)_{0} \) such that \( E_{i_1} = \ldots = E_{i_{j_\alpha}} \) for some partition \( \{i_1, \ldots, i_{j_\alpha}\} \) of \( \{1, \ldots, m_\alpha\} \) for \( \alpha = 1, \ldots, s \). The quotient \( \pi_0(N_{k+j}) = N_{k+j}/(N_{k+j})_0 \) acts by permuting the factors, here \( (N_{k+j})_0 \) is the connected component of the identity of \( N_{k+j} \). Moreover
\[
\mathbb{Z}_{R_{k+j}}^{\ast}/((N_{k+j})_0/R_{k+j}) \cong \prod_{m_1} \mathcal{M}(n_1, d_1)_{0} \times \prod_{m_s} \mathcal{M}(n_s, d_s)_{0} \setminus (\Delta^1_{j_1} \sqcup \ldots \sqcup \Delta^s_{j_s}).
\]

Hence, bearing in mind that \( \Delta^1_{j_1} \sqcup \ldots \sqcup \Delta^s_{j_s} \) is a disjoint union of non-singular varieties, from (1) and Theorem 28 the Hodge–Poincaré polynomial of \( \mathbb{Z}_{R_{k+j}}^{\ast}/((N_{k+j})_0/R_{k+j}) \) is given by
\[
HP(\mathbb{Z}_{R_{k+j}}^{\ast}/((N_{k+j})_0/R_{k+j}))(u,v) = \prod_{i=1}^{s} IP(\mathcal{M}(n_i, d_i)_{0})(u,v)]^{m_i} - \sum_{i=1}^{s} (uv)^{\lambda_i} IP(\Delta^i_{j_1})(u,v)
\]
where \( \lambda_i \) is the codimension of \( \Delta^i_{j_1} \) in \( \prod_{m_1} \mathcal{M}(n_1, d_1)_{0} \times \ldots \times \prod_{m_s} \mathcal{M}(n_s, d_s)_{0} \).

5.4. Regarding the equivariant Hodge–Poincare polynomial of \( \Sigma_R \) with respect to \( GL(p) \) for certain \( R \), since \( GL(p)Z_R \cong GL(p) \times XZ_R \) then \( H_{GL(p)}(\Sigma_R) \cong H^\ast_N(Z_R) \) which is an isomorphism of mixed Hodge structures, it is enough to compute the equivariant Hodge–Poincaré polynomial of \( Z_R^{\ast} \) with respect to \( N \). From Lemma 4.3, \( H^\ast_N(Z_R) \) is the invariant part of \( H^\ast_N(Z_R) \) under the action of the finite group \( \pi_0 N = N/N_0 \), and
\[
H^\ast_N(Z_R) \cong H^\ast(\mathbb{R}, BR) \otimes H^\ast_{N_0}(Z_R).
\]

Note that for a given \( R = \prod_{i=1}^{s} GL(m_i) \) that corresponds to a sequence of pairs \( (m_1, n_1), \ldots, (m_s, n_s) \) satisfying the properties of Paragraph 5.2 one has that
\[
N_0/R \cong \prod_{i=1}^{s} GL(p_i)/\mathbb{C}^\ast
\]
where \( p_i = d_i + (1-g)n_i = n_i n/p \). Moreover, it is known that \( GL(p_i) \) acts on \( \mathcal{R}^s_0(n_i, d_i) \) in such a way that \( \mathbb{C}^\ast \) acts trivially and \( SL(p_i) \) acts freely, then one deduces that the action of \( N_0/R \) on \( Z_R \) is free. This implies that
\[
H^\ast_{N_0/R}(Z_R) \cong H^\ast(Z_R)/(N_0/R).
\]
The Hodge–Poincaré polynomials of \( BR \) and \( Z_R/(N_0/R) \) can be computed using identities (3) and (5) respectively.

For future computations we need to understand the equivariant cohomology group \( H^\ast_S(Z_R^{\ast}) \) for a subgroup \( S \) of \( N \) such that \( N_0 \subseteq RS \). From the latter one gets that
\[
N_0/R \cong S_0/R \cap S_0
\]
where $N_0$ and $S_0$ are the connected components of the identity of $N$ and $S$ respectively. Then $H^*_S(Z^*_R)$ is the invariant part of $H^*_S(Z^*_R)$ under the action of the finite group $\pi_0S = S/S_0$, induced by the natural map $\pi_0S \to \pi_0N$, and

$$H^*_S(Z^*_R) \cong H^*(B(R \cap S_0)) \otimes H^*(Z^*_R/(N_0/R)).$$

where $Z^*_R/(N_0/R)$ is given by (53).

Using Paragraph 4.6, let $\mathcal{B}(p)$ be the set of $\beta$-sequences for the representation $\rho_t$ of $R_t$ on the corresponding normal bundle, and for each $\beta$-sequence $\beta$, let $q(\beta), d(\beta), z(\beta), w(\beta, R_t, GL(p))$ be the positive integers defined in that paragraph. Then, we have the following theorem.

**Theorem 5.5.** The equivariant Hodge–Poincaré series of $\mathcal{R}^*_S(n,d)$ with respect to $GL(p)$ for $(n,d) \neq 1$ is given by

$$HP_{GL}(\mathcal{R}^*_S(n,d))(u,v) = HP_{GL}(\mathcal{R}^*_S(n,d))(u,v) - \sum_{l \geq 1} (uv)^{l(R_l)}HP_{N_l}(Z^*_R)(u,v) +$$

$$+ \sum_{l=1}^\infty \sum_{0 \neq \beta \in \mathcal{B}(p)} (-1)^{q(\beta)}(uv)^{d(\beta)}(1 - (uv)^{z(\beta) + 1})w^{-1}(\beta, R_l, G) \cdot HP_{N_l \cap Stab(\beta)}(Z^*_R/(N_0/R_l))(u,v).$$

Moreover, $HP(M^*_S(n,d))(u,v) = (1 - uv) \cdot HP_{GL}(\mathcal{R}^*_S(n,d))(u,v)$ and $H^*_N(Stab(\mathcal{R}^*_S(n,d))(u,v) is the invariant part

$$\bigotimes_{1 \leq j \leq s} H^*(B(GL(m_j) \cap Stab(\beta))) \otimes H^*(Z^*_R/(N_0/R_l))$$

under the action of the finite group $\pi_0(N_l \cap Stab(\beta)) = (N_l \cap Stab(\beta))/(N_l \cap Stab(\beta)_{0}$, induced by the natural map $\pi_0(N_l \cap Stab(\beta)) \to \pi_0N_l$, and $Z^*_R/(N_0/R_l)$ is given by (53).}

**Proof.** The identity comes directly from Theorem 4.7. The central one-parameter subgroup $C^*$ of $GL(p)$ acts trivially on $\mathcal{R}^*_S(n,d)$ and such that $SL(p)$ acts freely, then

$$H^*_S(GL)(\mathcal{R}^*_S(n,d)) \cong H^*_S(GL)(\mathcal{R}^*_S(n,d)) \otimes H^*(BC^*)$$

and

$$HP(M^*_S(n,d))(u,v) = HP_{SL}(\mathcal{R}^*_S(n,d))(u,v) = (1 - uv) \cdot HP_{GL}(\mathcal{R}^*_S(n,d))(u,v).$$

For every $\beta \in \mathcal{B}(p)$ one has that $(N_0/R_l \subseteq R_l \cap (N_l \cap Stab(\beta))$. Moreover, if $R_l = \prod_{j=1}^g GL(m_j)$ then $R_l \cap Stab(\beta) = \prod_{j=1}^g (GL(m_j) \cap Stab(\beta))$. Then, from (53) we conclude that $H^*_N(Stab(\beta))(u,v)$ is the invariant part

$$\bigotimes_{1 \leq j \leq s} H^*(B(GL(m_j) \cap Stab(\beta))) \otimes H^*(Z^*_R/(N_0/R_l))$$

under the action of the finite group $\pi_0(N_l \cap Stab(\beta)) = (N_l \cap Stab(\beta))/(N_l \cap Stab(\beta)_{0}$, induced by the natural map $\pi_0(N_l \cap Stab(\beta)) \to \pi_0N_l$, and $Z^*_R/(N_0/R_l)$ is given by (53).

**5.6.** Regarding the equivariant Hodge–Poincaré polynomial of $\mathcal{R}(n,d)^ss$ with respect to $GL(p)$, this was computed by Earl and Kirwan in [3]. Every semistable vector bundle $E$ of rank $n$ and degree $d$ has a strictly ascending canonical filtration

$$0 = F_0 \subset F_1 \subset \ldots \subset F_P = E$$

such that the quotients $Q_j = E_j/E_{j-1}$ are semistable and the slopes $\mu(Q_j) = \deg(Q_j)/\text{rank}(Q_j) = d_j'/n_j'$ satisfy that $\mu(Q_j) > \mu(Q_{j+1})$ for every $j$. The $P$-tuple $\overline{\mu} = (\mu(Q_1), \ldots, \mu(Q_P))$ is called the type of $E$. Let $\overline{d}_{\overline{\mu}} = (d/n, \ldots, d/n)$ and

$$d_{\overline{\mu}} = \sum_{1 \leq i < j \leq P} n_i'd_j' - n_j'd_i' + n_i'n_j'(g - 1).$$

Then, in Theorem 1 of [3] it is proved, among other results, that $HP_{GL}(\mathcal{R}(n,d)^ss)(u,v)$ is given by the inductive formula

$$HP_{GL}(\mathcal{R}(n,d)^ss)(u,v) =$$

$$= \prod_{j=1}^n (1 + u^j v^{j-1})^g (1 + u^{j-1} v^j)^g \prod_{l=1}^{n-1} (1 - w^l)^2 - \sum_{\overline{\mu} \neq \overline{\mu}_0} (uv)^{d_{\overline{\mu}}} \prod_{1 \leq j \leq P} HP_{GL}(\mathcal{R}(n_j', d_j')^ss)(u,v).$$
This formula is valid for both the cases in which \((n, d) = 1\) and \((n, d) \neq 1\). For future computations we need to know \(HPG{\text{GL}}(p)(R(2, 0))^{ss}(u, v)\), this is given by (see [5] §3 equation (23), noting the misprint in this equation)

\[
HPG{\text{GL}}(p)(R(2, 0))^{ss}(u, v) = \frac{(1 + u)^{g(1 + v)^{g(1 + u^2v^2)^{g(1 + uv)^{g + 1}(1 + u)^{2g}(1 + v)^{2g}}} - (1 - u^2v^2)^{g - (1 - uv)^{g}}}}{(1 - u^2v^2)(1 - uv)^{g}}.
\]

Moreover, \(HPG{\text{GL}}(p)(R(n, d))^{ss}(u, v)\) does not change if we replace \(d\) by \(d + n \cdot z\) for any \(z \in \mathbb{Z}\) (see [5] proof of Theorem 1). Then, [58] gives also the equivariant Hodge–Poincaré polynomial of \(R(2, d)^{ss}\) with respect to \(GL(p)\) for \(d\) even.

6. Explicit computations for rank 2 vector bundles with even degree

In this section we compute explicitly \(HP(M^*_n(2, d))(u, v)\) for \(d\) even. Using Poincaré duality, from the Hodge–Poincaré polynomial one may obtain the Hodge–Deligne polynomial of \(M^*_n(2, d)\) for \(d\) even. The latter was first computed in [21]. Assuming \(d \gg 0\) from Paragraph [5.1] we know that \(M(2, d) \cong R(2, d)/SL(p)\) where \(p = d + 2(1 - g)\) and \(M^*_n(2, d) \cong R^*_n(2, d)/SL(p)\). Moreover, from (57)

\[
HP(M^*_n(2, d))(u, v) = HPSL(p)(R^*_n(2, d))(u, v) = (1 - uv) \cdot HPG{\text{GL}}(p)(R^*_n(2, d))(u, v).
\]

We need to understand the stratification \(\{\Sigma_{\gamma}\}_{\gamma \in \Gamma}\) of \(R^{ss}(2, d)\) such that \(\Sigma_{0} = R^*_n(2, d)\). To do that we blow up \(R^{ss}(2, d)\) along the subvarieties \(GL(p)\bar{Z}_{R}^{ss}\) where \(R\) is a representative of the conjugacy class of all connected reductive subgroups of dimension \(\dim R\) and \(\bar{Z}_{R}^{ss}\) is the proper transform of \(Z_{R}^{ss} := \{h \in R^{ss}(2, d)\) such that \(h\) is fixed by \(R\}\) in decreasing order of dimension of \(R\). The blow up is done in two steps and the indexing set is

\[
\Gamma = \{R_1 \cup \{R_1\} \times \{B_1\} \cup \{R_2\} \cup \{R_2\} \times \{B_2\} \cup \{0\}\},
\]

where \(R_1 = GL(2)\) and \(R_2 = T = GL(1) \times GL(1)\) which is the maximal torus \(T\) of \(GL(2)\).

6.1. The highest stratum is \(\Sigma_{R_1} = \Sigma_{GL(2)} = GZ_{GL(2)}\). Since \(R_1 = GL(2)\) has maximum dimension among those reductive subgroups of \(GL(p)\) with fixed points in \(R^{ss}(2, d)\) then \(Z_{GL(2)} = Z_{GL(2)}^{ss}\) (see Subsection 5.2). The conjugacy class \(R_1 = GL(2)\) is embedded in \(GL(p)\) using a fixed isomorphism \(\mathbb{C}^{2} \otimes \mathbb{C}^{1 - g + d/2} \cong \mathbb{C}^{p}\) and letting \(GL(2)\) act on the first factor. The normaliser of \(GL(2)\) in \(GL(p)\) is

\[
N(GL(2)) = (GL(2) \times GL(1) \times GL(1))/\mathbb{C}^{*}
\]

and note that its connected component, \(N_{0}(GL(2)) = N(GL(2))\).

Then, \(Z_{GL(2)}^{ss}\) is the subvariety of \(R(2, d)^{ss}\) consisting of all \(h \in R(2, d)^{ss}\) fixed by \(GL(2)\). Hence

\[
\Sigma_{GL(2)} = GZ_{GL(2)}^{ss} = GL(p)Z_{GL(2)}^{ss} = GL(p) \times \Sigma_{GL(2)} = Z_{GL(2)}^{ss}
\]

is the subvariety of \(R(2, d)^{ss}\) of those \(h\) such that \(E_h \cong L \oplus L\) for some \(L \in Jac^{d/2}\). Bearing in mind the previous isomorphism one gets that

\[
H^{*}_{GL}(\Sigma_{GL(2)}) \cong H^{*}_{GL}(GL(p)Z_{GL(2)}^{ss}) \cong H^{*}_{N(GL(2))(Z_{GL(2)}^{ss})}.
\]

Moreover, from Lemma [18] and [53] one obtains

\[
H^{*}_{GL}(\Sigma_{GL(2)}) \cong H^{*}(BGL(2)) \otimes H^{*}(N(GL(2))/GL(2))Z_{GL(2)}^{ss} \cong H^{*}(BGL(2)) \otimes H^{*}(N(GL(2))/GL(2)) \cong H^{*}(BGL(2)) \otimes H^{*}(Jac^{d/2}).
\]

These are isomorphisms of pure Hodge structures, so using [53] it induces the following identity of Hodge-Poincaré polynomials

\[
HPGL(\Sigma_{GL(2)})(u, v) = HPGL(GL(p)Z_{GL(2)}^{ss})(u, v) = HPN(GL(2))(Z_{GL(2)}^{ss})(u, v) = HPBGL(2)(u, v) \cdot HP(Jac^{d/2})(u, v) = \frac{(1 + u)^{g(1 + v)^{g}}}{(1 - uv)(1 - u^2v^2)}.
\]

The codimension of \(\Sigma_{GL(2)}\) in \(R(2, d)^{ss}\) can be computed from [18], this is

\[
\lambda(GL(2)) := \text{codim}(\Sigma_{GL(2)}) = 3g.
\]
6.2. To obtain \( \sum_{\beta \in B_1 \setminus \{0\}} (uv)^{\lambda(\beta,1)} HP_{GL(p)}(\Sigma_{\beta,1})(u,v) \), we first need to investigate the stratification \( \{S_{\beta,1}\}_{\beta \in B_1} \) of the variety \( R_1(2,d) \) -i.e., the variety obtained as a result of the first blow up- since \( \Sigma_{\beta,1} \cong S_{\beta,1} \setminus E_1 \) where \( E_1 \) is the exceptional divisor. In order to understand the index set \( B_1 \) from Paragraph 3.3 we need to look at the representation of \( SL(2) \) on the normal \( H^1(End'_B E_h) \) to \( GL(p)Z_{GL(2)} \) at a point \( h \in R(2,d) \) such that \( E_h = L \oplus L \) with \( L \in \text{Jac}^{d/2} \). The normal \( H^1(End'_B E_h) \) is equal to

\[
H^1(O) \otimes \text{Lie}(SL(2))
\]

where \( \text{Lie}(SL(2)) \) is the Lie algebra of \( SL(2) \). Now, \( SL(2) \) acts trivially on \( H^1(O) \) and by conjugation on \( \text{Lie}(SL(2)) \), so the weights of the representation are \( 2, 0 \) and \(-2\) each of them with multiplicity \( g \). This implies that there is only one weight lying in the positive Weyl chamber, so there is only one unstable stratum to be removed in the blow up procedure. Let \( S_{\beta,1} \) be the unique unstable stratum. From Paragraph 3.2 (a) we have that \( S_{\beta,1} \) consists of elements \( h \in R(2,d) \) such that \( \text{grad} E_h = L \oplus L \) with \( L \in \text{Jac}^{d/2} \); then \( \Sigma_{\beta,1} \cong S_{\beta,1} \setminus \mathbb{P}(H^1(End'_B E_h)) \) consist of elements \( h \in R(2,d) \) such that the vector bundle \( E_h \) is the middle term of a non-split extension

\[
0 \to L \to E_h \to L \to 0
\]

with \( L \in \text{Jac}^{d/2} \). The index \( \beta \) is maximum with respect to the partial order in \( B_1 \setminus \{0\} \). From Lemma 4.3 we have that

\[
HP_{GL(p)}(\Sigma_{\beta,1})(u,v) = HP_{N(GL(2)) \cap \text{Stab}_p}(Z_{GL(2)}^g)(u,v) \cdot (1 - (uv)^{z(\beta)+1}).
\]

Here, \( z(\beta) = g - 1 \) and \( \text{Stab}_p \) is the stabiliser of \( \beta \in \mathfrak{t}_+ \), where \( \mathfrak{t} \) is the Lie algebra of the maximal torus \( T \) of \( GL(2) \), under the adjoint action of \( GL(p) \). One has that \( \text{Stab}_p = N_0(T) = (T \times GL(p/2))/\mathbb{C}^* \), hence

\[
N(GL(2)) \cap \text{Stab}_p \cong (T \times GL(p/2))/\mathbb{C}^*.
\]

From Theorem 5.5 one has that \( H^*_N(GL(2)) \cong \text{Stab}_p(Z_{GL(2)}^g) \) is the invariant of

\[
H^*(BT) \otimes H^*(Z_{GL(2)}^g)/(N_0(GL(2))/GL(2))
\]

under the action of \( (T \times GL(p/2))/\mathbb{C}^* \)/\( (T \times GL(p/2))/\mathbb{C}^* \) \( = \text{Id} \). Moreover, bearing in mind identity \( 55 \) one has that \( Z_{GL(2)}^g/(N_0(GL(2))/GL(2) \cong \text{Jac}^{d/2} \), hence

\[
H^*_N(GL(2)) \cong H^*(BT) \otimes H^*(\text{Jac}^{d/2})
\]

which is an isomorphism of pure Hodge structures, then

\[
HP_{GL(p)}(\Sigma_{\beta,1})(u,v) = (1 - (uv)^g) \cdot HP(BT)(u,v) \cdot HP((Z_{GL(2)}^g)/(N_0(GL(2))/GL(2)))(u,v) =
\]

\[
= (1 - (uv)^g) \cdot \frac{1}{(1 - uv)^2} \cdot (1 + u)^g(1 + v)^g = \frac{(1 - u^g v^g)(1 + u)^g(1 + v)^g}{(1 - uv)^2}.
\]

Regarding the codimension of \( \Sigma_{\beta,1} \) in \( R^{ss}(2,d) \), in Remark 4.4 we saw that this coincides with the codimension of \( S_{\beta,1} \). The latter is given by \( 13 \), which in this case is

\[
\lambda(\beta,1) \triangleq \text{codim} \Sigma_{\beta,1} = \text{codim} S_{\beta,1} = 2g - \text{dim} GL(2)/B = 2g - 1
\]

here \( B \) is the Borel subgroup of \( GL(2) \) of upper triangular matrices. Hence, one obtains that

\[
\sum_{\beta \in B_1 \setminus \{0\}} (uv)^{\lambda(\beta,1)} HP_{GL(p)}(\Sigma_{\beta,1})(u,v) = (uv)^{2g-1} \cdot \frac{(1 - u^g v^g)(1 + u)^g(1 + v)^g}{(1 - uv)^2}.
\]

6.3. In the second blow up we consider \( R_2 = T = GL(1) \times GL(1) \) which is the maximal torus \( T \) of \( GL(2) \) consisting of all diagonal matrices, embedded in \( GL(p) \) using the above embedding of \( GL(2) \) in \( GL(p) \). Then

\[
N(T) = (N^T \times GL(p/2))/\mathbb{C}^*
\]

where \( N^T \) is the normaliser of \( T \) in \( GL(2) \). Now

\[
\Sigma_T \cong GL(p)Z^T_T \cong GL(p) \times_{N(T)} Z^T_T
\]

is the subvariety of \( R^{ss}(2,d) \) consisting of all \( h \) such that \( E_h \cong L_1 \oplus L_2 \) where \( L_1 \not\cong L_2 \) and \( L_1 \in \text{Jac}^{d/2} \). Bearing in mind \( 67 \), the polynomial \( HP_{GL(p)}(\Sigma_T)(u,v) \) is the same as \( HP_{N(T)}(Z^T_{\Sigma}) \)(u,v). The latter is computed in the following lemma.
Lemma 6.4.

\[ HP_{N(T)}(Z_\mathbb{T})(u,v) = (1 - uv)^{-1}(1 - u^2v^2)^{-1}\left(\frac{1}{2}(1 + u)^{2g}(1 + v)^{2g}(1 + uv) + \right. \]
\[ \left. + \frac{1}{2}(1 - u^2)^g(1 - v^2)^g(1 - uv) - (uv)^g(1 + u)^g(1 + v)^g. \right\] \]

Proof. By Lemma 4.4, we know that \( H^*_{N(T)}(Z_\mathbb{T}) \) is the invariant part of \( H^*(BT) \otimes H^*(Z_\mathbb{T}/(N_0(T)/T)) \) under the action of the finite group \( \pi_0(N(T)) = N(T)/N_0(T) = N^T/T \cong \mathbb{Z}/2 \). Moreover, by \( \mathcal{53} \) one has that \( Z_\mathbb{T}/(N_0(T)/T) \cong \text{Jac}^{d/2} \times \text{Jac}^{d/2} \setminus \Delta \) where \( \text{Jac}^{d/2} \) is the Jacobian of degree \( d/2 \) and \( \Delta \) is the diagonal of \( \text{Jac}^{d/2} \times \text{Jac}^{d/2} \).

Then, the Hodge–Poincaré polynomial \( HP_{N(T)}(Z_\mathbb{T})(u,v) \) is given by

\[ HP^+(BT)(u,v)HP^+(\text{Jac}^{d/2} \times \text{Jac}^{d/2} \setminus \Delta)(u,v) + HP^-(BT)(u,v)HP^-(\text{Jac}^{d/2} \times \text{Jac}^{d/2} \setminus \Delta)(u,v) \]

where the subscripts + and − correspond to the corresponding eigenspaces of eigenvalues +1 and −1 for the action of \( \mathbb{Z}/2 \) in both \( H^*(BT) \) and \( H^*(\text{Jac}^{d/2} \times \text{Jac}^{d/2} \setminus \Delta) \) respectively. Moreover, the Hodge–Poincaré polynomial of \( \text{Jac}^{d/2} \times \text{Jac}^{d/2} \setminus \Delta \) is given by \( \mathcal{54} \). The codimension of \( \Delta \cong \text{Jac}^{d/2} \) in \( \text{Jac}^{d/2} \times \text{Jac}^{d/2} \) is \( g \), then

\[ HP(\text{Jac}^{d/2} \times \text{Jac}^{d/2} \setminus \Delta)(u,v) = HP(\text{Jac}^{d/2} \times \text{Jac}^{d/2})(u,v) - (uv)^g HP(\text{Jac}^{d/2})(u,v), \]

hence

\[ HP^+(\text{Jac}^{d/2} \times \text{Jac}^{d/2} \setminus \Delta)(u,v) = HP^+(\text{Jac}^{d/2} \times \text{Jac}^{d/2})(u,v) - (uv)^g HP^+(\text{Jac}^{d/2})(u,v), \]

and the same is satisfied for \( HP^-(\text{Jac}^{d/2} \times \text{Jac}^{d/2} \setminus \Delta)(u,v) \). We have that \( H^*(BT) \cong H^*(BGL(1)) \otimes H^*(BGL(1)) \) and \( H^*(\text{Jac}^{d/2} \times \text{Jac}^{d/2}) \cong H^*(\text{Jac}^{d/2} \otimes H^*(\text{Jac}^{d/2}) \). These are isomorphisms of mixed Hodge structures, actually these are isomorphisms of pure Hodge structures. The action of the non-trivial element of \( \mathbb{Z}/2 \) on

\[ H^p,q(\text{Jac}^{d/2} \times \text{Jac}^{d/2}) \cong H^p(\text{Jac}^{d/2}) \otimes H^q(\text{Jac}^{d/2}) \cong \bigoplus_{p_1 + p_2 = p, q_1 + q_2 = q} H^{p_1,q_1}(\text{Jac}^{d/2}) \otimes H^{p_2,q_2}(\text{Jac}^{d/2}) \]

is given by \( a \otimes b \in H^{p_1,q_1}(\text{Jac}^{d/2}) \otimes H^{p_2,q_2}(\text{Jac}^{d/2}) \) goes to \((-1)(p_1+q_1)(p_2+q_2))b \otimes a \). Analogously for \( H^*(BT) \). Note that \( \mathbb{Z}/2 \) acts trivially on the diagonal.

One has that

\[ HP^+(\text{Jac}^{d/2} \times \text{Jac}^{d/2})(u,v) = \sum_{p,q} (-1)^{p+q} \dim \text{Sym}(H^{p,q}(\text{Jac}^{d/2} \times \text{Jac}^{d/2}))u^pv^q \]

where \( \text{Sym} \) denotes the symmetric part, and \( HP^-(\text{Jac}^{d/2} \times \text{Jac}^{d/2})(u,v) = \sum_{p,q} (-1)^{p+q} \dim \text{Ant}(H^{p,q}(\text{Jac}^{d/2} \times \text{Jac}^{d/2}))u^pv^q \) where \( \text{Ant} \) denotes the antisymmetric part. The same is satisfied for \( BT \).

Now, when \( (p_1,q_1) \neq (p_2,q_2) \) one has that \( \dim \text{Sym}(H^{p,q}(\text{Jac}^{d/2} \times \text{Jac}^{d/2})) = \dim \text{Ant}(H^{p,q}(\text{Jac}^{d/2} \times \text{Jac}^{d/2})) = \frac{1}{2} \dim (H^{p,q}(\text{Jac}^{d/2} \times \text{Jac}^{d/2})) \). When \( (p_1,q_1) = (p_2,q_2) \) elements of the form \( a \otimes a \) for \( a \in H^{p_1,q_1}(\text{Jac}^{d/2}) \) also need to be considered. Then, when \( p_1 + q_1 \) is even these elements are symmetric, and when \( p_1 + q_1 \) is odd are antisymmetric. Hence, it is satisfied that

\[ \dim \text{Sym}(H^{2p_1,2q_1}(\text{Jac}^{d/2} \times \text{Jac}^{d/2})) - \dim \text{Ant}(H^{2p_1,2q_1}(\text{Jac}^{d/2} \times \text{Jac}^{d/2})) = (-1)^{p_1+q_1} \dim H^{p_1,q_1}(\text{Jac}^{d/2}), \]

and

\[ \dim \text{Sym}(H^{2p_1,2q_1}(\text{Jac}^{d/2} \times \text{Jac}^{d/2})) + \dim \text{Ant}(H^{2p_1,2q_1}(\text{Jac}^{d/2} \times \text{Jac}^{d/2})) = \dim H^{2p_1,2q_1}(\text{Jac}^{d/2} \times \text{Jac}^{d/2}). \]
Then \[ H^+(\text{Jac}^{d/2} \times \text{Jac}^{d/2})(u, v) = \sum_{p, q} (-1)^{p+q} \dim \text{Sym}(H^{p,q}(\text{Jac}^{d/2} \times \text{Jac}^{d/2}))u^pv^q = \]
\[ = \sum_{(p_1, q_1) \neq (p_2, q_2)} (-1)^{p_1+q_1+p_2+q_2} \frac{1}{2} \dim (H^{p_1,q_1}(\text{Jac}^{d/2}) \otimes H^{p_2,q_2}(\text{Jac}^{d/2}))u^{p_1+q_1+v_2+q_2} + \]
\[ + \sum_{2p_1 = p, 2q_1 = q} \frac{1}{2} \left( \dim H^{p_1,q_1}(\text{Jac}^{d/2}) \right)^2 u^{p_1+v_2+q_2} + (-1)^{p_1+q_1} \dim H^{p_1,q_1}(\text{Jac}^{d/2})u^{2p_1, v^{2q_1}} = \]
\[ = \frac{1}{2} [ HP(\text{Jac}^{d/2})(u, v)]^2 + \frac{1}{2} HP(\text{Jac}^{d/2})(-u^2, -v^2). \]

Bearing in mind that \( \mathbb{Z}/2 \) acts trivially on the diagonal, one gets that
\[ HP^+(\text{Jac}^{d/2} \times \text{Jac}^{d/2} \setminus \Delta)(u, v) = \]
\[ = \frac{1}{2} [ HP(\text{Jac}^{d/2})(u, v)]^2 + \frac{1}{2} HP(\text{Jac}^{d/2})(-u^2, -v^2) - (uv)^n HP(\text{Jac}^{d/2})(u, v) = \]
\[ = \frac{1}{2} (1 + u)^{2g}(1 + v)^{2g} + \frac{1}{2} (1 - u^2)^g(1 - v^2)^g - (uv)^n(1 + u)^g(1 + v)^g. \]

Analogously
\[ HP^-(\text{Jac}^{d/2} \times \text{Jac}^{d/2} \setminus \Delta)(u, v) = \]
\[ = \frac{1}{2} [ HP(\text{Jac}^{d/2})(u, v)]^2 - \frac{1}{2} HP(\text{Jac}^{d/2})(-u^2, -v^2) = \]
\[ = \frac{1}{2} (1 + u)^{2g}(1 + v)^{2g} - \frac{1}{2} (1 - u^2)^g(1 - v^2)^g. \]

Repeating the previous argument for \( H^*(BT) \) but taking into account that \( H^{p,q}(BT) \neq 0 \) only when \( p = q \), we obtain
\[ HP^+(BT)(u, v) = \frac{1}{(1 - uv)(1 - u^2v^2)} \quad \text{and} \quad HP^-(BT)(u, v) = \frac{uv}{(1 - uv)(1 - u^2v^2)}. \]

Finally, from (68), (69), (70), and (71) we conclude. □

The codimension of \( \Sigma_T \) in \( \mathcal{R}(2, d)^{ss} \) can be computed from (48), this is
\[ \lambda(T) := \text{codim} \Sigma_T = 2g - 2. \]

6.5. Regarding \( \sum_{\beta \in B_2 \setminus \{0\}} (uv)^{\lambda(\beta, 2)} HP_{GL(p)}(\Sigma_{\beta, 2})(u, v) \) we have to look at the representation of \( T \) on the normal \( H^1(\text{End}_{\beta} E) \) to \( GL(p)\mathcal{Z}_T^* \) at a point \( h \in \mathcal{R}^{ss}(2, d) \) such that \( E_h = L_1 \oplus L_2 \) where \( L_1 \not\cong L_2 \) and \( L_i \in \text{Jac}^{d/2} \). Here
\[ H^1(\text{End}_{\beta} E) = H^1(\text{Hom}(L_1, L_2)) \oplus H^1(\text{Hom}(L_2, L_1)), \]
a diagonal matrix \( \text{diag}(t_1, t_2) \) acts as multiplication by \( t_1t_2^{-1} \) on the first factor and by \( t_2t_1^{-1} \) on the second. Then the weights of the representation of \( T \cap SL(2) \) are 2 and \(-2\) each with multiplicity \( g - 1 \). This implies that again there is only one unstable stratum \( S_{\beta, 2} \) to be removed. Hence the index \( \beta \) of this stratum is maximum for the given partial order of \( B_2 \). From Paragraph 6.3(a) we know that this stratum is the proper transform of the set of all \( h \in \mathcal{R}(2, d)^{ss} \) such that \( \text{grad} E_h \cong L_1 \oplus L_2 \) where \( L_i \in \text{Jac}^{d/2} \). Hence \( \Sigma_{\beta, 2} = S_{\beta, 2} \setminus P(\text{Hom}(L_1, L_2)) \oplus H^1(\text{Hom}(L_2, L_1)) \) consists of \( h \in \mathcal{R}(2, d)^{ss} \) such that \( E_h \) is the middle term of a non-split extension
\[ 0 \to L_1 \to E_h \to L_2 \to 0 \]
with \( L_1 \not\cong L_2 \) and \( L_i \in \text{Jac}^{d/2} \). Again, from Lemma 4.9 we have that
\[ HP_{GL(p)}(\Sigma_{\beta, 2})(u, v) = HP_{N(T) \cap \text{Stab} \beta}(Z_T^*)(u, v) \cdot (1 - (uv)^{z(\beta)+1}). \]

Here, \( z(\beta) = g - 2 \) and \( N(T) \cap \text{Stab} \beta \cong (T \times GL(p/2))/C^* \). From Theorem 5.5 one has that \( H^*_N(T) \cap \text{Stab} \beta(Z_T^*) \) is the invariant of
\[ H^*(BT) \otimes H^*(Z_T^*/(N_0(T)/T)) \]
under the action of \((T\times GL(p/2))/\mathbb{C}^* \backslash (T\times GL(p/2))/\mathbb{C}^*\) = Id. Moreover, by (53) we have that \(Z_T/(N_0(T)/T) \cong \text{Jac}^{d/2} \times \text{Jac}^{d/2} \backslash \Delta\) where \(\Delta\) is the diagonal in \(\text{Jac}^{d/2} \times \text{Jac}^{d/2}\). The Hodge–Poincaré polynomial of \(\text{Jac}^{d/2} \times \text{Jac}^{d/2} \backslash \Delta\) is given by (54). Bearing in mind that the codimension of \(\Delta\) in \(\text{Jac}^{d/2} \times \text{Jac}^{d/2}\) is \(g\), one obtains

\[
H_{\text{GL}(p)}(\Sigma_{\beta,2})(u,v) = (1 - (uv)^{-1}) \cdot H_P(\text{BT})(u,v) \cdot H_P(Z_T/(N_0(T)/T)(u,v) =
\]

\[
= (1 - (uv)^{-1})(1 + u)^{2g}(1 + v)^{2g} - (uv)^g(1 + u)^{g/(1 + v)^{g}}(1 - uv)^{-2}.
\]

Regarding the codimension of \(\Sigma, \beta\), this polynomial was obtained independently by Muñoz et al. (see (54) and (68)), which in this case is

\[
\lambda(\beta, 1) := \text{codim} \Sigma, \beta = \text{codim} S, \beta = g - 1 - \text{dim} T/B = g - 1
\]

here \(B\) is a Borel subgroup of \(T\). Hence, one obtains that

\[
\sum_{\beta \in B_1 \{0\}} (uv)^{\lambda(\beta, 2)} H_{\text{GL}(p)}(\Sigma_{\beta, 2})(u,v) = (1 - (uv)^{-1} - (uv)^{2g-2}) \left( (1 + u)^{2g}(1 + v)^{2g} - (uv)^g(1 + u)^g(1 + v)^g \right) / (1 - uv)^2.
\]

From the previous analysis we obtain the following Theorem.

**Theorem 6.6.** The Hodge–Poincaré polynomial of \(\mathcal{M}^s_{(0)}(2, d)\) for \(d\) even is given by

\[
H_P(\mathcal{M}^s_{(0)}(2, d))(u,v) = \frac{1}{2(1 - uv)(1 - u^2v^2)} \left[ (1 + u)^g(1 + v)^g(1 + u^2v)^g(1 + uv^2)^g - (uv)^g(1 + u)^g(1 + v)^g - (uv)^g(1 + u)^g(1 + v)^g - (uv)^g(1 + u)^g(1 + v)^g - (uv)^g(1 + u)^g(1 + v)^g - (uv)^g(1 + u)^g(1 + v)^g - (uv)^g(1 + u)^g(1 + v)^g - (uv)^g(1 + u)^g(1 + v)^g - (uv)^g(1 + u)^g(1 + v)^g - \right].
\]

**Proof.** From Proposition 4.3, Lemma 6.4 and identities (58), (62), (63), (66), (72), and (75) we have that

\[
H_{\text{GL}(p)}(\mathcal{R}^s_{(0)}(2, d))(u,v) = \frac{1}{2(1 - uv)^2(1 - u^2v^2)} \left[ (1 + u)^g(1 + v)^g(1 + u^2v)^g(1 + uv^2)^g - (uv)^g(1 + u)^g(1 + v)^g - (uv)^g(1 + u)^g(1 + v)^g - (uv)^g(1 + u)^g(1 + v)^g - (uv)^g(1 + u)^g(1 + v)^g - (uv)^g(1 + u)^g(1 + v)^g - (uv)^g(1 + u)^g(1 + v)^g - (uv)^g(1 + u)^g(1 + v)^g - (uv)^g(1 + u)^g(1 + v)^g - (uv)^g(1 + u)^g(1 + v)^g - \right].
\]

Bearing in mind identity (59), we conclude. \(\square\)

**Remark 6.7.** Since \(\mathcal{M}^s_{(0)}(2, d)\) is a smooth variety, from identities (1) and (40) we have that the Hodge–Deligne polynomial of \(\mathcal{M}^s_{(0)}(2, d)\) is given by

\[
H(\mathcal{M}^s_{(0)}(2, d))(u,v) = (uv)^{4g-3} \cdot H_P(\mathcal{M}^s_{(0)}(2, d))(u^{-1}, v^{-1}),
\]

then

\[
H(\mathcal{M}^s_{(0)}(2, d))(u,v) = \frac{1}{2(1 - uv)(1 - u^2v^2)} \left[ (1 + u)^g(1 + v)^g(1 + u^2v)^g(1 + uv^2)^g - (1 + u)^g(1 + v)^g(1 + u^2v)^g(1 + uv^2)^g - (1 + u)^g(1 + v)^g(1 + u^2v)^g(1 + uv^2)^g - \right].
\]

this polynomial was obtained independently by Muñoz et al. (see (21), Theorem 5.2) using the relation of \(\mathcal{M}^s_{(0)}(2, d)\) with certain moduli spaces of triples.

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