THE GAUSS MAP OF A HARMONIC SUBMERSION WITH TOTALLY GEODESIC FIBERS

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Abstract. In this paper, we investigate submersions \( f : S^3 \rightarrow S^2 \) with totally geodesic fibers. We prove that \( f \) is harmonic if and only its Gauss map is a harmonic unit vector field. This result can be viewed as an analogue of the classical theorem of Ruh and Vilms [18] about the harmonicity of the Gauss map of a minimal submanifold in the euclidean space. Using the harmonicity of the Gauss map and the maximum principle, we prove that a harmonic submersion \( f : S^3 \rightarrow S^2 \) with totally geodesic fibers is a Hopf fibration. As a consequence, we derive that a harmonic unit vector field on \( S^3 \) with totally geodesic leaves is a Hopf vector field.

1. Introduction

One of the most important objects in submanifold theory is the Gauss map. In codimension one, the Gauss map associates to every point of the hypersurface its oriented unit normal vector. This concept can be generalized also to higher codimensional oriented submanifolds. Let \( f : M^m \rightarrow \mathbb{R}^n \) be an isometric immersion of an \( m \)-dimensional oriented Riemannian manifold \( M^m \) into the euclidean space. The image \( df(T_x M) \) of the tangent space of \( M^m \) at the point \( x \), can be taken after a suitable parallel displacement in \( \mathbb{R}^n \), into a point \( \nu(x) \) of the oriented Grassmann space \( G_m(\mathbb{R}^n) \) of \( m \)-dimensional oriented subspaces of \( \mathbb{R}^n \). The map \( \nu : M^m \rightarrow G_m(\mathbb{R}^n) \) defined in this way, is called the generalized Gauss map of the submanifold. In 1970, Ruh and Vilms [18] proved that if the immersion \( f \) is minimal, then its generalized Gauss map is harmonic. Let us mention here that according to a result of Chern [5], the Gauss map of a minimal surface in the euclidean space is anti-holomorphic.

There is an analogue Gauss map also for submersions between manifolds. Following the terminology introduced by Baird in [3], the Gauss map of a submersion \( f : M^m \rightarrow N^n \) between two oriented Riemannian manifolds, is the mapping \( \mathcal{G} : M^m \rightarrow G_{m-n}(M^m) \) which associates to each point \( x \in M^m \) the tangent plane to the fiber of \( f \) passing through \( x \). Here, \( G_{m-n}(M^m) \) denotes the Grassmann bundle over \( M^m \), whose fiber at each point \( x \in M^m \) is the Grassmannian of oriented \((m-n)\)-planes in \( T_x M \). If \( n = m - 1 \), then the kernel of \( f \) is generated by a unit vector field. Hence, the Gauss map \( \mathcal{G} \) may be regarded as a section of the unit tangent bundle \( UM \) of \( M^m \), i.e. a unit vector field of \( M^m \). Throughout this paper we will equip the unit tangent bundle \( UM \) of the Riemannian manifold \((M^m, g)\) with its standard Sasaki metric \( g_S \) induced from \( g \).

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Our first result is an analogue of the classical theorem of Ruh and Vilms [18] in the case of harmonic submersions with totally geodesic fibers between unit euclidean spheres. More precisely, we prove the following theorem.

**Theorem A.** Let \( f : V \subset S^3 \to S^2 \) be a submersion with totally geodesic fibers defined in a saturated neighbourhood \( V \) of \( S^3 \). Then \( f \) is harmonic map if and only if its Gauss map \( \mathcal{G} : V \to US^3 \) is harmonic unit vector field, i.e. it satisfies the equation

\[
\Delta \mathcal{G} + |\nabla \mathcal{G}|^2 \mathcal{G} = 0,
\]

where \( \Delta \) is the rough Laplacian operator.

A harmonic unit vector field in the sphere is a critical point of the energy functional with respect to variations through nearby unit vector fields. Therefore, being harmonic unit vector field is less restrictive than that of being harmonic map with values in the unit tangent bundle of the sphere. It turns out that these notions are equivalent, if the unit vector field is additionally divergence free and it has totally geodesic integral curves; see [15, 23, 24].

According to a beautiful result of Gluck and Warner [13], any great circle fibration of \( S^3 \) generates a graphical surface in \( S^2 \times S^2 \). Generically, at least locally, the converse is also true; see Theorem 2.7. In particular, the fibration is smooth and globally defined in \( S^3 \) if and only if the graphical surface is generated by a smooth strictly length decreasing map. Therefore, there is a huge class of maps with totally geodesic fibers. Moreover, all these maps are homotopically non-trivial. The idea to obtain this duality is to consider the map which assigns each great circle of the foliation to the 2-plane of \( \mathbb{R}^4 \) containing the circle. The fact that the space of oriented 2-planes in \( \mathbb{R}^4 \) is diffeomorphic to the product of 2-spheres plays an important role. In our analysis, we strongly use the geometric set up of Gluck and Warner [13] and the fact that \( \mathbb{R}^4 \) admits a quaternionic structure.

Next we turn our attention to the classification of harmonic maps \( f : S^3 \to S^2 \). The most typical example of a harmonic map between spheres is the Hopf fibration. Specifically, the Hopf fibration is the map described by the action \( S^1 \hookrightarrow S^3 \to \mathbb{C}P^1 \). The complex projective space \( \mathbb{C}P^1 \) equipped with its standard Fubini-Study metric is isometric with the sphere of radius \( 1/2 \). It turns out that the Hopf fibration is a Riemannian submersion with totally geodesic fibers. Composing it with an isometry of \( S^3 \) and the homothety from the sphere of radius \( 1/2 \) into the unit sphere \( S^2 \), we obtain again a harmonic submersion with totally geodesic fibers. All these maps are called Hopf fibrations. According to a conjecture of Eells, any harmonic map \( f : S^3 \to S^2 \) must be weakly conformal; see [2, Note 10.4.1] or [22]. So, a question is whether a harmonic map \( f : S^3 \to S^2 \) can be written as the composition of a Hopf fibration with a conformal map from \( S^2 \) to \( S^2 \). In general, Eells’ conjecture is still open. We give an affirmative answer if \( f \) is a harmonic submersion with totally geodesic fibers.

**Theorem B.** Let \( f : S^3 \to S^2 \) be a globally defined harmonic submersion with totally geodesic fibers. Then \( f \) is a Hopf fibration.

It turns out that the vector field \( \zeta \) generating the fibers of a Hopf fibration can be written in the form \( J\nu \), where \( J \) is a complex structure of \( \mathbb{R}^4 \) and \( \nu \) the unit normal of \( S^3 \). For this reason, any vector field of this type is called Hopf vector field. It is not known if there exists globally defined harmonic unit vector fields on \( S^3 \) other than the Hopf vector fields.
As an immediate consequence of Theorem A and B, we obtain the following result which generalises a previous one in [15].

**Theorem C.** A harmonic unit vector field on $S^3$ with geodesic leaves is a Hopf vector field.

The main ingredient in the proof of the uniqueness theorems is the second fundamental form $\varphi$ of the horizontal distribution of the submersion. It turns out that, along the fibers of $f$ the tensor $\varphi$ satisfies a Ricatti type ODE and on the horizontal distribution it satisfies a Codazzi type system of PDEs; see Section 2. Using the fact that the Gauss map of the submersion is a harmonic unit vector field, we arrive at the conclusion that $(\text{trace } \varphi)^2$ is a subharmonic function. Then, from the maximum principle and the Bernstein type theorem for strictly length decreasing minimal maps in [20, Theorem A], we obtain that $\varphi$ is an orthogonal complex structure. This leads us to the conclusion that $f$ is a Hopf fibration.

## 2. Fibrations with totally geodesic fibers

### 2.1. Great circle fibrations and the Ricatti equation

Let $V$ be an open and convex subset of $S^3$ and $f : V \to S^2$ be a submersion with totally geodesic fibers. Denote by $\zeta$ the unit vector field generating the fibers, by $V = \ker df = \text{span}\{\zeta\}$ the vertical line bundle and by $\mathcal{H} = V^\perp$ the horizontal plane bundle. Consider the tensor $\varphi : \mathcal{H} \to \mathcal{H}$, given by

$$\varphi(v) = -\nabla_v \zeta,$$

for all $v \in \mathcal{H}$, where $\nabla$ stands for the standard Levi-Civita connection of $S^3$. The tensor $\varphi$ is the (formal) second fundamental form of $\mathcal{H}$. It is well-known that $\varphi$ satisfies the equations

$$\nabla^\mathcal{H}_v \varphi = \varphi^2 + I \quad \text{and} \quad (\nabla^\mathcal{H}_v \varphi)w - (\nabla^\mathcal{H}_w \varphi)v = 0,$$

(2.1)

for any pair of vector fields $v, w$ on $\mathcal{H}$; see for example [2, page 313]. Denote by $J$ the complex structure of $\mathcal{H}$ and by $\{\alpha_1 = \zeta, \alpha_2, \alpha_3 = J\alpha_2\}$ a local orthonormal frame on $V$. Moreover, denote by $\varphi_{ij} = \langle \varphi(\alpha_i), \alpha_j \rangle$, $i, j \in \{2, 3\}$, the components of $\varphi$ with respect to the orthonormal frame $\{\alpha_2, \alpha_3\}$. Then, trace $\varphi = \varphi_{22} + \varphi_{33} = -\text{div}(\zeta)$. From the Ricatti equation in (2.1), we obtain the following:

**Lemma 2.1.** Let $\gamma$ be an integral curve of $\zeta$ and $\{\alpha_2, \alpha_3\}$ is a parallel orthonormal frame along $\gamma^* \mathcal{H}$. Then:

(a) The components $\varphi_{ij}$, $i, j \in \{2, 3\}$, of $\varphi$ with respect to this frame, satisfy the ODEs

$$\begin{cases}
\zeta(\varphi_{22}) &= 1 + \varphi_{22}^2 + \varphi_{23} \varphi_{32}, \\
\zeta(\varphi_{33}) &= 1 + \varphi_{33}^2 + \varphi_{23} \varphi_{32}, \\
\zeta(\varphi_{23}) &= \varphi_{23}(\varphi_{22} + \varphi_{33}), \\
\zeta(\varphi_{32}) &= \varphi_{32}(\varphi_{22} + \varphi_{33}).
\end{cases}$$

(2.2)

(b) The functions trace $\varphi$, trace $(\varphi \circ J)$ and det $\varphi$ satisfy the ODEs

$$\begin{cases}
\zeta(\text{trace}(\varphi \circ J)) &= \text{trace}(\varphi)(\text{trace}(\varphi \circ J)), \\
\zeta(1 + \text{det } \varphi) &= \text{trace}(\varphi)(1 + \text{det } \varphi), \\
\zeta(\text{trace } \varphi) &= \text{trace}(\varphi)^2 - 2(1 + \text{det } \varphi) + 4.
\end{cases}$$

(2.3)

and

$$\zeta\{(\text{trace } \varphi)^2 - 4 \text{det } \varphi\} = 2(\text{trace } \varphi)\{(\text{trace } \varphi)^2 - 4 \text{det } \varphi\}.$$ 

(2.4)
For sake of completeness, let us include here a proof of the following elementary fact.

**Lemma 2.2.** Let $f : \mathbb{S}^3 \to \mathbb{S}^2$ be a smooth globally defined submersion with totally geodesic fibers. Then, there exists a smooth globally defined unit vector field generating the vertical distribution $\mathcal{V}$. Moreover, the horizontal distribution $\mathcal{H}$ possesses a (unique up to sign) complex structure $J$.

**Proof.** Let $x$ be an arbitrary point in $\mathbb{S}^3$. It is clear that there exists an open sufficiently small geodesic ball $B(x, r_x)$ where we can define a smooth unit vector field $\zeta_x \in \mathcal{V}$. Let $B(y, r_y)$ be a geodesic ball around another point $y \in \mathbb{S}^3$ and $\zeta_y$ a smooth unit vector field defined on $B(y, r_y)$ generating the vertical distribution. If $B_{r_x}(x) \cap B_{r_y}(y) \neq \emptyset$, then clearly we can arrange that $\zeta_x = \zeta_y$ in the intersection. Therefore, we can extend $\zeta_x$ in the union $B_{r_x}(x) \cup B_{r_y}(y)$. In particular, following this procedure we may extend $\zeta_x$ along any curve emanating from the point $x$. We claim that this procedure leads to a smooth globally defined unit vector field on the vertical distribution. It suffices to show that if $\gamma_1, \gamma_2 : [0, 1] \to \mathbb{S}^3$ are two arbitrary curves joining the points $x, y \in \mathbb{S}^3$ and $\zeta$ is a unit vector at $\gamma_x$, then the extension of $\zeta$ at $y$ along $\gamma_1$ is the same with the extension of $\zeta$ at $y$ along $\gamma_2$. Indeed! Since $\mathbb{S}^3$ is simply connected, there exists a homotopy $H : [0, 1] \times [0, 1] \to \mathbb{S}^3$ between $\gamma_1$ and $\gamma_2$. Denote by $\gamma_s : [0, 1] \to \mathbb{S}^3$ the curve given by $\gamma_s(t) = H(s, t)$ and with $\zeta_{\gamma_s(t)} \in \mathcal{V}_{\gamma_s(t)}$ the unit vector field along $\gamma_s$ arising from the extension of $\zeta_x$. Define the set

$$\mathcal{A} = \{ s \in [0, 1] : \zeta_{\gamma_s(1)} = \zeta_{\gamma_0(1)} \}.$$

Since $0 \in \mathcal{A}$, the set $\mathcal{A}$ is non-empty. The proof will be concluded if we show that

$$s_0 = \sup \mathcal{A} = 1 \in \mathcal{A}.$$

Let $B_t = B(\gamma_{s_0}(t), r(\gamma_{s_0}(t)))$ be geodesic balls along $\gamma_{s_0}$, where we can define unit vectors with the desired property. The collection $\{B_t\}_{t \in [0, 1]}$ form an open covering of $\gamma_{s_0}([0, 1])$. Because of compactness, there exists a finite covering of $\gamma_{s_0}([0, 1])$ by $\{B_{t_j}\}_{j \in \{1, \ldots, n(s_0)\}}$. Moreover, there exist a positive number $\varepsilon > 0$ such that, for any $t \in [0, 1],$

$$V_t = B(\gamma_{s_0}(t), \varepsilon) \subset \bigcup_{j=1}^{n(s_0)} B_{t_j}.$$

Denote by $d$ the distance function on $\mathbb{S}^3$. Observe that, if $\gamma : [0, 1] \to \mathbb{S}^3$ is a curve joining the points $x$ and $y$ with

$$d(\gamma(t), \gamma_{s_0}(t)) < \varepsilon,$$

for all $t \in [0, 1]$, then the extension of $\zeta$ at $y$ along $\gamma$ leads to $\zeta_{\gamma_{s_0}(1)}$. On the other hand, since the homotopy $H$ is a uniformly continuous map, there exist $\delta > 0$ such that for all $s \in [0, 1]$ with $|s - s_0| < \delta$ we have

$$d(\gamma_s(t), \gamma_{s_0}(t)) = d(H(s, t), H(s_0, t)) < \varepsilon.$$

Because $s_0 = \sup \mathcal{A}$, there exist $s_1 \in \mathcal{A}$ with the property $s_0 - \delta < s_1 \leq s_0$. Thus, $|s_1 - s_0| < \delta$ and so $d(\gamma_{s_1}(t), \gamma_{s_0}(t)) < \varepsilon$, for all $t \in [0, 1]$. Hence, $\zeta_{\gamma_{s_1}(1)} = \zeta_{\gamma_{s_0}(1)}$. Since $s_1 \in \mathcal{A}$, we deduce that $\zeta_{\gamma_{s_1}(1)} = \zeta_{\gamma_{s_0}(1)}$ from where it follows that $s_0 \in \mathcal{A}$. It remain to show that $s_0 = 1$. Let us suppose to the contrary that $s_0 < 1$. Consider a point $s_2 \in (s_0, 1)$ such that $s_0 < s_2 < s_0 + \delta$. This would imply that $d(\gamma_{s_2}(t), \gamma_{s_0}(t)) < \varepsilon$, for all $t \in [0, 1]$. Consequently, $\zeta_{\gamma_{s_2}(1)} = \zeta_{\gamma_{s_0}(1)} = \zeta_{\gamma_{s_0}(1)}$ which implies that $s_2 \in \mathcal{A}$. This contradicts the fact that $s_0 = \sup \mathcal{A}$. Hence $s_0 = 1$. That $\mathcal{H}$ admits a complex structure is obvious now. This completes the proof of lemma.
2.2. Great circle fibrations and length decreasing maps. Due to an impressive result of Gluck and Warner [13], there is a relation between great circle fibrations of the 3-sphere and length decreasing maps between two dimensional euclidean spheres. This construction is obtained as follows: Associate at each point \( x \in S^3 \) the 2-dimensional subspace of \( \mathbb{R}^4 \) spanned by the great circle of the fibrations passing through \( x \). In this way we obtain a map with values in the Grassmann space \( \mathbb{G}_2(\mathbb{R}^4) \simeq S^2 \times S^2 \) of oriented 2-planes is \( \mathbb{R}^4 \). It turns out that this particular map is the graph of a strictly length decreasing map. The converse is also true, i.e. any strictly length decreasing map between 2-dimensional euclidean unit spheres, gives rise to a great circle foliation of the 3-sphere. To make the paper self-contained, let us describe here this duality following our exposition.

Denote by \( \Lambda^2(\mathbb{R}^4) \) the dual space of all alternative multilinear forms of degree 2. Elements of \( \Lambda^2(\mathbb{R}^4) \) are called 2-vectors. As a matter of fact, for given vectors \( v_1 \) and \( v_2 \) on \( \mathbb{R}^4 \), the exterior product \( v_1 \wedge v_2 \) is the linear map which on an alternating form \( \Omega \) of degree 2 takes the value

\[
(v_1 \wedge v_2)(\Omega) = \Omega(v_1, v_2).
\]

The exterior product is linear in each variable separately. Interchanging two elements the sign of the product changes and if two variables are the same the exterior product vanishes. A 2-vector \( \omega \) is called simple or decomposable if it can be written as a single wedge product of vectors, that is

\[
\omega = v_1 \wedge v_2.
\]

Note that there are 2-vectors which are not simple. One can verify that the exterior product \( v_1 \wedge v_2 \) is zero if and only if the vectors are linearly dependent. Moreover, if \( \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\} \) consists a basis for \( \mathbb{R}^4 \), then the collection \( \{\varepsilon_i \wedge \varepsilon_j : 1 \leq i < j \leq 4\} \) consists a basis of \( \Lambda^2(\mathbb{R}^4) \). Therefore, the dimension of the vector space of 2-vectors is 6.

We can equip \( \Lambda^2(\mathbb{R}^4) \) with a natural inner product \( \langle \cdot, \cdot \rangle \). Indeed, define

\[
(v_1 \wedge v_2, w_1 \wedge w_2) = \langle v_1, v_1 \rangle \langle v_2, w_2 \rangle - \langle v_1, w_2 \rangle \langle v_2, v_1 \rangle,
\]

on simple 2-vectors and then extend linearly. Note that, if \( \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\} \) is an orthonormal basis of \( \mathbb{R}^4 \) then, the 2-vectors \( \{\varepsilon_i \wedge \varepsilon_j : 1 \leq i < j \leq 4\} \) consists an orthonormal basis for the exterior power \( \Lambda^2(\mathbb{R}^4) \).

Each simple vector represents a unique 2-dimensional subspace of \( \mathbb{R}^4 \). Moreover, if \( \omega_1 \) and \( \omega_2 \) are simple vectors representing the same subspace, then there exists a non-zero real number \( \lambda \) such that \( \omega_1 = \lambda \omega_2 \). Therefore, there is an obvious equivalence relation on the space of simple 2-vectors such that the space of equivalence classes is to an one-to-one correspondence with the space of 2-dimensional subspaces of \( \mathbb{R}^4 \). Additionally, we can consider another relation on the set of non-zero simple 2-vectors: \( \omega_1 \) and \( \omega_2 \) are called equivalent if and only if

\[
\omega_1 = \lambda \omega_2
\]

for some positive number \( \lambda \). Denote by \( [\omega] \) the class containing all simple 2-vectors that are equivalent to \( \omega \). The equivalence classes now obtained are called oriented 2-dimensional subspaces of \( \mathbb{R}^4 \) and the space \( \mathbb{G}_2(\mathbb{R}^4) \) of all equivalence classes is called Grassmann space of oriented 2-planes is \( \mathbb{R}^4 \). Consequently, a plane \( \Pi \) in \( \mathbb{R}^4 \) can be associated with the equivalence class of the 2-vector \( \omega = v_1 \wedge v_2 \), where \( \{v_1, v_2\} \) is an orthonormal basis of \( \Pi \).
There exists a natural linear endomorphism $*$ of $\Lambda^2(\mathbb{R}^4)$ which maps a 2-plane $\Pi$ in $\mathbb{R}^4$ into its orthogonal complements $\Pi^\perp$. Specifically, if $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ is the standard orthonormal basis of $\mathbb{R}^4$ and

$$\omega = \alpha_{12}\varepsilon_1 \wedge \varepsilon_2 + \alpha_{13}\varepsilon_1 \wedge \varepsilon_3 + \alpha_{14}\varepsilon_1 \wedge \varepsilon_4 + \alpha_{23}\varepsilon_2 \wedge \varepsilon_3 + \alpha_{24}\varepsilon_2 \wedge \varepsilon_4 + \alpha_{34}\varepsilon_3 \wedge \varepsilon_4,$$

we define

$$\ast \omega = \alpha_{34}\varepsilon_1 \wedge \varepsilon_2 - \alpha_{24}\varepsilon_1 \wedge \varepsilon_3 + \alpha_{23}\varepsilon_1 \wedge \varepsilon_4 - \alpha_{14}\varepsilon_2 \wedge \varepsilon_3 + \alpha_{13}\varepsilon_2 \wedge \varepsilon_4 + \alpha_{12}\varepsilon_3 \wedge \varepsilon_4.$$

The operator $\ast$ is called Hodge star operator. Let us mention here that $\ast$ actually is an isometry and it satisfies

$$\ast^2 = \ast \circ \ast = I.$$

Using elementary arguments, one can show that a non-zero 2-vector $\omega$ is simple if and only if $\omega \wedge \omega = 0$ or, equivalently, if and only if $(\omega, \ast \omega) = 0$. Hence, we may represent the space of oriented two planes in $\mathbb{R}^4$ in the form

$$\mathbb{G}_2(\mathbb{R}^4) = \{[\omega] : \omega \in \Lambda^2(\mathbb{R}^4), \|\omega\|_{\Lambda^2(\mathbb{R}^4)} = 1 \text{ and } (\omega, \ast \omega) = 0\}.$$

The Hodge star operator has eigenvalues $+1$ and $-1$, both of multiplicity 3. In particular, the corresponding eigenspaces of the $\ast$ are 3-dimensional and they are given by

$$E_- = \{\omega \in \Lambda^2(\mathbb{R}^4) : \ast \omega = -\omega\} \quad \text{and} \quad E_+ = \{\omega \in \Lambda^2(\mathbb{R}^4) : \ast \omega = \omega\}.$$

The eigenspaces $E_-$ and $E_+$ are mutually perpendicular and

$$\Lambda^2(\mathbb{R}^4) = E_- \oplus E_+.$$

As a matter of fact, any element $\omega \in \Lambda^2(\mathbb{R}^4)$ can be uniquely written in the form

$$\omega = \frac{\omega - \ast \omega}{2} \oplus \frac{\omega + \ast \omega}{2}, \quad (2.5)$$

where the first term in the right hand side of (2.5) belongs to $E_-$ and the second to $E_+$. Moreover, if $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ is the standard basis of $\mathbb{R}^4$, then the collection

$$\left\{\frac{\varepsilon_1 \wedge \varepsilon_2 + \varepsilon_3 \wedge \varepsilon_4}{2}, \frac{\varepsilon_1 \wedge \varepsilon_3 - \varepsilon_2 \wedge \varepsilon_4}{2}, \frac{\varepsilon_1 \wedge \varepsilon_4 + \varepsilon_2 \wedge \varepsilon_3}{2}\right\},$$

forms an orthogonal basis of $E_+$ and

$$\left\{\frac{\varepsilon_1 \wedge \varepsilon_2 - \varepsilon_3 \wedge \varepsilon_4}{2}, \frac{\varepsilon_1 \wedge \varepsilon_3 + \varepsilon_2 \wedge \varepsilon_4}{2}, \frac{\varepsilon_1 \wedge \varepsilon_4 - \varepsilon_2 \wedge \varepsilon_3}{2}\right\},$$

forms an orthogonal basis of $E_-$. Consider now the euclidean spheres

$$\mathbb{S}^2_- = \{\omega \in E_- : \|\omega\|_{\Lambda^2(\mathbb{R}^4)} = 1/\sqrt{2}\} \quad \text{and} \quad \mathbb{S}^2_+ = \{\omega \in E_+ : \|\omega\|_{\Lambda^2(\mathbb{R}^4)} = 1/\sqrt{2}\}.$$

**Lemma 2.3.** The Grassmann space $\mathbb{G}_2(\mathbb{R}^4)$ can be identified with the direct product $\mathbb{S}^2_- \times \mathbb{S}^2_+$.

**Proof.** Let $\Pi = [\omega] \in \mathbb{G}_2(\mathbb{R}^4)$ and assume that the representative $\omega$ is chosen to have the form $\omega = v_1 \wedge v_2$, where $\{v_1, v_2\}$ is an orthonormal basis of the under consideration plane. Observe that, if $\{w_1, w_2\}$ is another orthonormal frame of $\Pi$ which belongs in the same orientation with $\{v_1, v_2\}$, then $v_1 \wedge v_2 = w_1 \wedge w_2$. Now, the map

$$[\omega] \mapsto \frac{\omega - \ast \omega}{2} \oplus \frac{\omega + \ast \omega}{2}, \quad (2.6)$$

is well defined and gives rise to a bijection between $\mathbb{G}_2(\mathbb{R}^4)$ and $\mathbb{S}^2_- \times \mathbb{S}^2_+$. This completes the proof. \qed
Suppose now that $V$ is a saturated open neighbourhood of $S^2$ such that $f(V) \subset S^2_\pm$ is simply connected. Define the maps $h_\pm : f(V) \subset S^2_\pm \to S^2_\pm$ given by
\[
h_\pm(f(x)) = \frac{x \wedge \zeta(x) \pm *(x \wedge \zeta(x))}{2},
\]
where $x \in V$. We give now an alternative proof of a result due to Gluck and Warner \[13\].

**Lemma 2.4.** The following statements hold true:

(a) The maps $h_\pm$ are well defined and smooth.

(b) The map $G = (h_-, h_+)$ gives rise to a strictly length decreasing graphical map.

(c) If $h_-$ is a diffeomorphism, then the singular values $\mu_1$ and $\mu_2$ of $h_+ \circ h_-^{-1}$ are related with the second fundamental form $\varphi$ by
\[
\mu_1 \circ f = \frac{\sqrt{(1 - \det \varphi)^2 + (\operatorname{trace} \varphi)^2} - \sqrt{|\varphi|^2 - 2 \det \varphi}}{1 + \det \varphi + \operatorname{trace} (\varphi \circ J)},
\]
and
\[
\mu_2 \circ f = \frac{\sqrt{(1 - \det \varphi)^2 + (\operatorname{trace} \varphi)^2} + \sqrt{|\varphi|^2 - 2 \det \varphi}}{1 + \det \varphi + \operatorname{trace} (\varphi \circ J)}.
\]

**Proof.** (a) Consider points $x, y \in V$ such that $f(x) = f(y)$. Then, $x$ and $y$ belongs in the same circle of the foliation from where we deduce that $x \wedge \zeta(x) = y \wedge \zeta(y)$. Therefore, $h_\pm(f(x)) = h_\pm(f(y))$ and $h_\pm$ is well defined. Smoothness of the maps $h_\pm$ is clear.

(b) Let us compute now the differentials of $h_\pm \circ f$. Fix a point $x \in V$ and consider a local orthonormal basis $\{\alpha_1 = \zeta, \alpha_2, \alpha_3\}$ of $T_xV$. Then,
\[
*(x \wedge \alpha_1) = \alpha_2 \wedge \alpha_3, \quad *(x \wedge \alpha_3) = \alpha_1 \wedge \alpha_2, \quad *(\alpha_1 \wedge \alpha_3) = -x \wedge \alpha_2,
\]
\[
*(x \wedge \alpha_2) = -\alpha_1 \wedge \alpha_3, \quad *(\alpha_1 \wedge \alpha_2) = x \wedge \alpha_3, \quad *(\alpha_2 \wedge \alpha_3) = x \wedge \alpha_1.
\]
Differentiating (2.7) with respect to $\alpha_2$ and $\alpha_3$, we get that
\[
d(h_- \circ f)(\alpha_2) = (1 - \varphi_{23}) x \wedge \alpha_3 - \alpha_1 \wedge \alpha_2 - \varphi_{22} x \wedge \alpha_2 + \alpha_1 \wedge \alpha_3, \quad (2.8)
\]
\[
d(h_- \circ f)(\alpha_3) = -\varphi_{33} x \wedge \alpha_3 - \alpha_1 \wedge \alpha_2 - (1 + \varphi_{32}) x \wedge \alpha_2 + \alpha_1 \wedge \alpha_3, \quad (2.9)
\]
\[
d(h_+ \circ f)(\alpha_2) = -(1 + \varphi_{23}) x \wedge \alpha_3 + \alpha_1 \wedge \alpha_2 - \varphi_{22} x \wedge \alpha_2 - \alpha_1 \wedge \alpha_3, \quad (2.10)
\]
\[
d(h_+ \circ f)(\alpha_3) = -\varphi_{33} x \wedge \alpha_3 + \alpha_1 \wedge \alpha_2 + (1 - \varphi_{32}) x \wedge \alpha_2 - \alpha_1 \wedge \alpha_3. \quad (2.11)
\]
Observe that the vectors
\[
\left\{ \frac{x \wedge \alpha_3 - \alpha_1 \wedge \alpha_2}{\sqrt{2}}, \frac{x \wedge \alpha_2 + \alpha_1 \wedge \alpha_3}{\sqrt{2}} \right\}
\]
form an orthonormal basis of the tangent space at $h_-(f(x))$ of $S^2_- \subset E_-$ and
\[
\left\{ \frac{x \wedge \alpha_3 + \alpha_1 \wedge \alpha_2}{\sqrt{2}}, \frac{x \wedge \alpha_2 - \alpha_1 \wedge \alpha_3}{\sqrt{2}} \right\}
\]
form an orthonormal basis of the tangent space at $h_+(f(x))$ of $S^2_+ \subset E_+$. 
We claim now that either
\[ D_- = \det \begin{bmatrix} 1 - \varphi_{23} & -\varphi_{33} \\ -\varphi_{22} & 1 - \varphi_{32} \end{bmatrix} = -1 - \det \varphi + \varphi_{23} - \varphi_{32} = -1 - \det \varphi - \text{trace}(\varphi \circ J) \neq 0, \]
everywhere on \( V \), or
\[ D_+ = \det \begin{bmatrix} -1 - \varphi_{23} & -\varphi_{33} \\ -\varphi_{22} & 1 - \varphi_{32} \end{bmatrix} = -1 - \det \varphi + \varphi_{32} - \varphi_{23} = -1 - \det \varphi + \text{trace}(\varphi \circ J) \neq 0, \]
everywhere in \( V \). Indeed! From the last two equations of (2.3) it follows that \( 1 + \det \varphi \) is strictly positive on \( V \). Moreover, from the first equation of (2.3) and the first equation of (2.2) it follows that \( \varphi_{23} - \varphi_{32} \) is nowhere zero. Consequently, either \( \varphi_{23} - \varphi_{32} \) is strictly negative and \( D_- \) is strictly negative in \( V \), or \( \varphi_{23} - \varphi_{32} \) is strictly positive and \( D_+ \) is strictly negative in \( U \). Since \( f(V) \) is simply connected, we deduce that one of the maps \( h_{\pm} \) is a diffeomorphism and \( G \) is graphical.

Without loss of generality, assume that \( h_- \) is a diffeomorphism. Then, \( \varphi_{23} - \varphi_{32} < 0 \) everywhere in \( V \). In this situation, we will show that \( h_+ \circ h_-^{-1} \) is strictly length decreasing, i.e.
\[ |d(h_+ \circ h_-^{-1})(df(a))| < |df(a)|, \]
or, equivalently,
\[ |d(h_+ \circ f)(\alpha)|^2 < |d(h_- \circ f)(\alpha)|^2, \] (2.12)
for any vector \( \alpha \in \mathcal{H} \). Indeed! If \( \alpha = \kappa_1 \alpha_2 + \kappa_2 \alpha_3 \), then from (2.8), (2.9), (2.10) and (2.11), we get that (2.12) holds if and only if
\[ \varphi_{23} \kappa_1^2 + (\varphi_{33} - \varphi_{22}) \kappa_1 \kappa_2 - \varphi_{32} \kappa_2^2 < 0, \] (2.13)
for any \( \kappa_1, \kappa_2 \in \mathbb{R} \). On the other hand, (2.13) holds for any \( \kappa_1, \kappa_2 \), if and only if the matrix
\[ A = \begin{bmatrix} \varphi_{23} & \frac{1}{2}(\varphi_{33} - \varphi_{22}) \\ \frac{1}{2}(\varphi_{33} - \varphi_{22}) & -\varphi_{32} \end{bmatrix} \]
has negative eigenvalues or, equivalently, if and only if
\[ \text{trace } A = \varphi_{23} - \varphi_{32} < 0 \quad \text{and} \quad 4 \det A = -(\text{trace } \varphi)^2 + 4 \det \varphi > 0. \]
The validity of the first condition is clear. Suppose now to the contrary, that there is a point \( x_0 \in V \), where
\[ (\text{trace } \varphi)^2(x_0) - 4 \det \varphi(x_0) \geq 0. \]
From (2.4) it follows that the same inequality holds along the integral curve \( \gamma \) of \( \alpha_1 \) passing through \( x_0 \). From the third identity of (2.3), we obtain that along \( \gamma \) it holds
\[ \alpha_1(\text{trace } \varphi) = (\text{trace } \varphi)^2 - 2 \det \varphi + 2 = (\text{trace } \varphi)^2 - 4 \det \varphi + 2(\det \varphi + 1) \geq 2(\det \varphi + 1) > 0, \]
which leads to a contradiction. Therefore, \( h_+ \circ h_-^{-1} \) is a strictly length decreasing map.

(c) To compute the singular values of the map \( h_+ \circ h_-^{-1} \) we proceed as follows. Fix a point \( x_0 \in V \) and suppose that \( \{\alpha_1 = \zeta, \alpha_2, \alpha_3\} \in T_x V \) and \( \beta_2, \beta_3 \in T_{f(x)} S^2 \) are orthonormal basis of the singular decomposition of \( f \). Then, from (2.8), (2.9), (2.10) and (2.11), we get the expressions for the singular values of \( h_+ \circ h_-^{-1} \). This completes the proof. \( \square \)
Let us see now the converse. Suppose that $G : V \subset S^2_+ \to S^2_+ \times S^2_+ \simeq G_2 (\mathbb{R}^4)$ is the graph of a smooth map $g : V \subset S^2_+ \to S^2_+$, where $V$ is an open domain of $S^2$. Then, for any $x \in V$, the element $G(x) = x \oplus g(x)$ describes a plane in $\mathbb{R}^4$ and $G(x) \cap S^3$ gives rise to a great circle of $S^3$. In order to explicitly describe the plane generated by $G(x)$, we will use the quaternionic structure of $\mathbb{R}^4$. Recall that, as a vector space the quaternionics are

$$\mathbb{H} = \{ a_0 + a_1 i + a_2 j + a_3 k : a_0, a_1, a_2, a_3 \in \mathbb{R}^4 \}.$$  

They become an associative algebra with 1 as the multiplicative unit via

$$i^2 = j^2 = k^2 = -1, \quad i \cdot j = -j \cdot i = k, \quad j \cdot k = -k \cdot j = i, \quad k \cdot i = -i \cdot k = j.$$  

We denote by $\text{Re} \mathbb{H}$ the one dimensional linear subspace spanned by the element 1 and $\text{Im} \mathbb{H}$ the orthogonal complement of $\text{Re} \mathbb{H}$. Hence, an element $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$ can be described by its real part $\text{Re}(x) = x_0$ and its imaginary part $\text{Im}(x) = x_1 i + x_2 j + x_3 k$. Moreover, the conjugate $\overline{x}$ of $x$ is defined to be the quaternionic number

$$\overline{x} = x_0 - x_1 i - x_2 j - x_3 k.$$  

The euclidean inner product and the norm on $\mathbb{R}^4 \simeq \mathbb{H}$ can be equivalently written in the form

$$\langle x, y \rangle = \text{Re}(x \cdot \overline{y}) = \text{Re}(\overline{x} \cdot y) \quad \text{and} \quad |x|^2 = x \cdot \overline{x} = \overline{x} \cdot x,$$

for any $x, y \in \mathbb{H}$. Moreover, the standard outer product in $\mathbb{R}^3$ can be regarded as the map $\times : \text{Im} \mathbb{H} \times \text{Im} \mathbb{H} \to \text{Im} \mathbb{H}$ given by

$$x \times y = \text{Im}(x \cdot y),$$

for any $x, y \in \text{Im} \mathbb{H}$.

Let us collect in the following lemma the most important properties of the quaternionic multiplications; for more details see [14, page 186].

**Lemma 2.5.** The following identities hold:

(a) For any $x, y, z \in \mathbb{H}$, we have $\langle z \cdot x, z \cdot y \rangle = \langle x, y \rangle |z|^2 = \langle x, z \cdot y \cdot z \rangle$.

(b) For any $x \in \text{Im} \mathbb{H}$, we have $x^2 = -|x|^2$.

(c) For any $x, y \in \mathbb{H}$, we have $\overline{x \cdot y} = \overline{y} \cdot \overline{x}$.

(d) For any $x, y \in \text{Im} \mathbb{H}$, we have $x \cdot y + y \cdot x = -2\langle x, y \rangle$; hence orthogonal imaginaries anti-commute.

Consider now the unit sphere $S^3 \subset \mathbb{H}$ as the subset of quaternions of length 1. Moreover, we consider $S^2_+ \subset \text{Im} \mathbb{H} \subset \mathbb{H}$ as the subset of pure imaginary quaternions with length $\sqrt{2}$. Under these considerations, the space $E_-$ is spanned by the vectors

$$\left\{ \frac{1 \wedge i - j \wedge k}{2}, \frac{1 \wedge j + i \wedge k}{2}, \frac{1 \wedge k - i \wedge j}{2} \right\},$$

and $E_+$ by the vectors

$$\left\{ \frac{1 \wedge i + j \wedge k}{2}, \frac{1 \wedge j - i \wedge k}{2}, \frac{1 \wedge k + i \wedge j}{2} \right\}.$$  

By straightforward elementary computations, we obtain the following lemma.
Lemma 2.6. Let \( x \in V \). Then the following facts hold:

(a) If \( g(0) \neq -x \), then the 2-plane \( G(x) = x \oplus g(x) \) is generated by the orthonormal vectors

\[
\xi(x) = -\frac{x + g(x)}{|x + g(x)|} \in \text{Im} \mathbb{H} \quad \& \quad \eta(x) = \sqrt{2} g(x) \cdot \xi(x) = \frac{1 - 2g(x) \cdot x}{\sqrt{2} |x + g(x)|} \in \mathbb{H}.
\]

(b) If \( g(x) = -x \), then the 2-plane \( G(x) = x \oplus (-x) \) is generated by orthonormal vectors which have one of the following forms:

\[
\begin{align*}
\xi_1(x) &= \frac{x \times i}{|x \times i|} \in \text{Im} \mathbb{H} \quad \& \quad \eta_1(x) = -\sqrt{2} x \cdot \xi_1(x) = -\frac{\sqrt{2} x \cdot (x \times i)}{|x \times i|} \in \text{Im} \mathbb{H}, \text{ if } x_1 \neq \pm 1, \\
\xi_2(x) &= \frac{x \times j}{|x \times j|} \in \text{Im} \mathbb{H} \quad \& \quad \eta_2(x) = -\sqrt{2} x \cdot \xi_2(x) = -\frac{\sqrt{2} x \cdot (x \times j)}{|x \times j|} \in \text{Im} \mathbb{H}, \text{ if } x_2 \neq \pm 1, \\
\xi_3(x) &= \frac{x \times k}{|x \times k|} \in \text{Im} \mathbb{H} \quad \& \quad \eta_3(x) = -\sqrt{2} x \cdot \xi_3(x) = -\frac{\sqrt{2} x \cdot (x \times k)}{|x \times k|} \in \text{Im} \mathbb{H}, \text{ if } x_3 \neq \pm 1.
\end{align*}
\]

Now we give the classification of Gluck and Warner [13] of the great circle fibrations of the 3-sphere following our approach.

Theorem 2.7. Suppose that \( G : V \subset S^2_+ \rightarrow S^2_+ \times S^2_+ \simeq G_2(\mathbb{R}^4) \) is the graph of a smooth map \( g : V \subset S^2_+ \rightarrow S^2_+ \), where \( V \) is a path connected domain of the sphere. The following statements hold:

(a) Let \( x_0 \) be a point in \( V \). Then, the circle \( G(x) \cap S^3 \) can be represented by

\[
S(x_0, t) = \cos t \xi(x_0) + \sin t \eta(x_0),
\]

where \( \xi \) and \( \eta \) are the vectors obtained in Lemma 2.6. In particular:

(a1) If \( g(x_0) \neq \pm x_0 \), there exist an open neighbourhood \( U_{x_0} \subset V \) and a positive number \( \varepsilon_{x_0} > 0 \), such that the map \( S : U_{x_0} \times (-\varepsilon_{x_0}, \varepsilon_{x_0}) \rightarrow S^3 \) given by

\[
S(x, t) = \cos t \xi(x) + \sin t \eta(x) = \left( \cos t + \sqrt{2} \sin t g(x) \right) \cdot \xi(x),
\]

is a diffeomorphism.

(a2) If \( g(x) = x \) in an open set \( U_{x_0} \) around \( x_0 \), then \( S : U_{x_0} \times (-\pi/2, \pi/2) \rightarrow S^3 \) given by

\[
S(x, t) = \cos t \xi(x) + \sin t \eta(x) = \frac{-\cos t x + \sqrt{2} \sin t}{\sqrt{2}},
\]

is a diffeomorphism. Geometrically, the map \( S \) describes the projection from the poles \( \pm 1 \in S^3 \subset \mathbb{H} \) onto the equator \( S^2 = S^3 \cap \text{Im} \mathbb{H} \).

(a3) If \( g(x) = -x \) in an open set \( U \) around \( x_0 \) which does not contain the point \( \pm k \in \mathbb{H} \), then there exists a sufficiently small open neighbourhood \( U_{x_0} \subset U \) and a positive number \( \varepsilon_{x_0} > 0 \) such that the map \( S : U_{x_0} \times (-\varepsilon_{x_0}, \varepsilon_{x_0}) \rightarrow S^3 \) given by

\[
S(x, t) = \cos t \xi(x) + \sin t \eta(x) = \frac{\cos t x \times k - \sqrt{2} \sin t x \cdot (x \times k)}{|x \times k|},
\]

is a diffeomorphism. In particular, the image of \( S \) lies within \( S^2 = S^3 \cap \text{Im} \mathbb{H} \).
In each of the cases \((a_1), (a_2)\) and \((a_3)\), the map \(f : S(U_{x_0} \times (-\varepsilon_{x_0}, \varepsilon_{x_0})) \subset S^3 \to U_{x_0}\) given by
\[
f \left( \cos t \xi(x) + \sin t \eta(x) \right) = (\xi(x), g(x), \xi(x)) = x,
\]
is a submersion with totally geodesic fibers, which are generated by the unit vector field \(\zeta = dS_{x_0}(\partial_t)\).

(b) The great circle foliation
\[
\mathcal{F} = \bigcup_{x \in U_{x_0}} G(x) \cap S^3
\]
is smooth if and only if \(g : U_{x_0} \subset S^2 \to S^2_+\) is strictly length decreasing.

(c) If \(g : S^2_\epsilon \to S^2_\epsilon\) is strictly length decreasing, then the maps \(f\) given in (a) generate a globally defined submersion from \(S^3\) onto \(S^2_\epsilon\) with totally geodesic fibers. Moreover, \(f\) is a Hopf fibration if and only if \(g\) is constant map.

The proof of Theorem 2.7 follows by direct computations and using Lemma 2.2, 2.5 and 2.6.

2.3. Applications. Let us collect some immediate applications of the classification theorem of the great circle fibrations of the 3-sphere.

**Corollary 2.8** (Gluck [10]). Let \(f : S^3 \to S^2\) be a submersion with totally geodesic fibers. The dual 1-form associated with the unit vector field which generates the fibers of \(f\), gives rise to a contact structure of \(S^3\).

*Proof.* Let us denote with \(\zeta\) a unit vector field which generates the fibers of \(f\) and with \(\omega\) its associated 1-form, i.e. the form given by \(\omega(\alpha) = \langle \zeta, \alpha \rangle\), for all tangent vectors \(\alpha\). Recall that \(\eta\) is a contact form in \(S^3\) if and only if \(\omega \wedge d\omega \neq 0\). Let now \(\{\alpha_1 = \zeta, \alpha_2, \alpha_3\}\) be a local orthonormal frame and \(\varphi\) the tensor introduced in the second section. We compute
\[
(\omega \wedge d\omega)(\alpha_1, \alpha_2, \alpha_3) = d\omega(\alpha_2, \alpha_3) = \alpha_2(\omega(\alpha_3)) - \alpha_3(\omega(\alpha_2)) - \omega([\alpha_2, \alpha_3])
\]
\[
= \langle \nabla_{\alpha_2} \alpha_1, \alpha_3 \rangle - \langle \nabla_{\alpha_3} \alpha_1, \alpha_2 \rangle = -\varphi_{23} + \varphi_{32}
\]
\[
= \text{trace}(\varphi \circ J),
\]
where \(J\) stands for the complex structure of \(\mathcal{H}\). Recall that in the proof of Lemma 2.4(c) we proved that \(\text{trace}(\varphi \circ J)\) is nowhere zero. Hence, \(\omega\) is a contact form. This completes the proof.

**Corollary 2.9** (Gluck & Gu [11]). Suppose that \(f : S^3 \to S^2\) is a submersion with totally geodesic fibers. If the unit vector field \(\alpha_1\) which generates the fibers of \(f\) is divergence free, then \(f\) is a Hopf fibration.

*Proof.* Since \(\alpha_1\) is divergence free, then trace \(\varphi = 0\). Then, from Lemma 2.1(b) it follows that det \(\varphi = 1\). Hence, from Lemma 2.4(c) it follows that the singular values of \(g\) are equal. Consequently, \(g\) is a conformal map. This implies that the graph of \(g\) is a minimal surface in \(S^2_\epsilon \times S^2_{\epsilon^2}\); see for example [2, Proposition 4.5.3], [7] or [17]. On the other hand, \(g\) is strictly length decreasing. From [20, Theorem A] it follows that \(g\) must be constant. Consequently, from Theorem 2.7(c) we deduce that \(f\) is a Hopf fibration. This completes the proof.

**Corollary 2.10** (Heller [16]). Let \(f : S^3 \to S^2\) be a weakly conformal submersion with totally geodesic fibers. Then \(f\) is a Hopf fibration.
Remark 2.11. In [16] Heller proves a more general result. More precisely, he shows that up to conformal transformations of $S^2$ and $S^3$, every conformal fibration of $S^3$ by circles (not necessarily great circles) is the Hopf fibration.

\section*{Corollary 2.12 (Escobales [8])} Suppose that $f : V \subset S^3 \to S^2$ is a submersion with totally geodesic fibers and equal constant singular values defined in an open neighbourhood $V$ of $S^3$. Then $f$ is a Hopf fibration.

\textbf{Proof.} By assumption, the singular values of $f$ are $0, \lambda, \lambda$, where $\lambda$ is a non-zero constant. From the formulas (2.15), we deduce that

\[ \varphi_{22} = \varphi_{33} = 0 \quad \text{and} \quad \varphi_{23} + \varphi_{32} = 0. \]

From Lemma 2.1(b) it follows that $\det \varphi = 1$, which implies that

\[ \varphi_{23} = -1 \quad \text{and} \quad \varphi_{32} = 1. \]

Now from Lemma 2.4(c) the singular values of $g$ are zero, which implies that $g$ is constant. Hence $f$ is a Hopf fibration. This completes the proof. \qed
3. Harmonicity of the Gauss map

Let us assume now that \( g : V \subset S^2 \rightarrow S^2 \) is strictly length decreasing, where \( V \) is an open saturated set. Fix a point \( x_0 \in S^2 \). Then from Theorem 2.7, it follows that there exists a sufficiently small neighbourhood \( U_{x_0} \) around the point \( x_0 \) where the map \( \xi \) is a diffeomorphism. Hence, by setting \( y = \xi(x) \) and \( h = \sqrt{2} g \circ \xi^{-1} \), we may reparametrise the foliation by \( \vartheta : \xi(U_{x_0}) \times [-\pi, \pi] \rightarrow S^3 \) given by

\[
\vartheta(y, t) = \cos t \, y + \sin t \, h(y) \cdot y.
\]

Observe that \( h \) a smooth map from an open domain of the unit 2-sphere \( S^3 \cap \text{Im} \mathbb{H} \) with values in the same sphere. According to Theorem 2.7, we have that

\[
\alpha_1(\vartheta(y, t)) = h(y) \cdot \vartheta(y, t) \quad \text{and} \quad f(\vartheta(y, t)) = \overline{\alpha} \cdot h(y) \cdot y.
\]

On the other hand, one can readily check that for any \( (y, t) \) it holds

\[
f(\vartheta(y, t)) = \vartheta(y, t) \cdot h(y) \cdot \vartheta(y, t),
\]

Hence, for \( p = \vartheta(y, t) \) we deduce the unit vector field \( \alpha_1 \) generating the leaves of the foliation is related with \( f \) via the equations

\[
\alpha_1(p) = p \cdot f(p) \quad \text{and} \quad f(p) = \overline{p} \cdot \alpha_1(p), \quad (3.1)
\]

for any \( p \in V \subset S^3 \).

We would like to relate the Hessian of the vector field \( \alpha_1 \) with the Hessian of the map \( f \). We need the following auxiliary lemma.

**Lemma 3.1.** Let \( f : V \subset S^3 \rightarrow S^2 \) be a submersion with totally geodesic fibers defined in an open saturated neighbourhood \( V \) of \( S^3 \). Then,

\[
\langle df(\alpha_i), df(\alpha_j) \rangle = \langle \alpha_i, \alpha_j \rangle + \langle \varphi(\alpha_i), \varphi(\alpha_j) \rangle - \langle \varphi(\alpha_i), J\alpha_j \rangle - \langle \varphi(\alpha_j), J\alpha_i \rangle, \quad (3.2)
\]

where \( i, j \in \{2, 3\} \), \( \alpha_2, \alpha_3 \in \mathcal{H} \) and \( J \) is the complex structure of \( \mathcal{H} \).

**Proof.** Observe at first that any tangent vector \( \alpha \in T_pS^3 \) satisfies the equations

\[
\overline{\alpha} = -\overline{p} \cdot \alpha \cdot \overline{p} \quad \text{and} \quad \alpha = -p \cdot \overline{\alpha} \cdot p. \quad (3.3)
\]

Indeed! Consider a curve \( \sigma : (-\varepsilon, \varepsilon) \rightarrow S^3 \) such that \( \sigma(0) = p \) and \( \sigma'(0) = \alpha \). Differentiating the expression \( \sigma \cdot \overline{\sigma} = 1 \), and estimating at \( t = 0 \), we get the first identity of (3.3). The second follows immediately from the first. Differentiating the second equation of (3.1) with respect to \( \alpha_i, i, j \in \{2, 3\} \), we get

\[
df(\alpha_i) = \overline{\alpha} \cdot \alpha + \overline{p} \cdot D\alpha_i \alpha_1 = \overline{\alpha} \cdot \alpha + \overline{p} \cdot \nabla_\alpha \alpha_1 = \overline{\alpha} \cdot \alpha - \overline{p} \cdot \varphi(\alpha_i), \quad (3.4)
\]

where \( D \) is the standard connection of \( \mathbb{H} = \mathbb{R}^4 \). Using the identities of Lemma 2.5 and (3.3), we obtain

\[
\langle df(\alpha_i), df(\alpha_j) \rangle = \langle \overline{\alpha} \cdot \alpha - \overline{p} \cdot \varphi(\alpha_i), \overline{\alpha} \cdot \alpha - \overline{p} \cdot \varphi(\alpha_j) \rangle \quad (3.5)
\]

\[
= \langle \alpha_i, \alpha_j \rangle + \langle \varphi(\alpha_i), \varphi(\alpha_j) \rangle + \langle \alpha_i \cdot \overline{p} \cdot \alpha_1, \varphi(\alpha_j) \rangle + \langle \alpha_j \cdot \overline{p} \cdot \alpha_1, \varphi(\alpha_i) \rangle.
\]

For any \( p \in S^3 \) the vectors \( \{\alpha_1, \alpha_2, \alpha_3, p\} \) forms a basis of \( \mathbb{H} \). We decompose the vector \( \alpha_2 \cdot \overline{p} \cdot \alpha_3 \) in terms of \( \alpha_1, \alpha_2, \alpha_3 \) and \( p \). We easily observe that

\[
\langle \alpha_2 \cdot \overline{p} \cdot \alpha_3, \alpha_2 \rangle = \langle \alpha_2 \cdot \overline{p} \cdot \alpha_3, \alpha_3 \rangle = 0.
\]
Furthermore,
\[ \langle \alpha_2 \cdot \overline{p} \cdot \alpha_3, p \rangle = \langle \alpha_2 \cdot \overline{p} \cdot \alpha_3 \cdot \overline{p}, 1 \rangle = -\langle \alpha_2 \cdot \overline{\alpha}_3, 1 \rangle = 0, \] (3.6)
since \( \langle \alpha_2, \alpha_3 \rangle = 0 = \text{Re}(\alpha_2 \cdot \overline{\alpha}_3) \). Hence, \( \alpha_2 \cdot \overline{p} \cdot \alpha_3 = \pm \alpha_1 \). Choosing positive orientation, we get
\[
\begin{align*}
\alpha_1 \cdot \overline{p} \cdot \alpha_2 &= -\alpha_2 \cdot \overline{p} \cdot \alpha_1 = \alpha_3, \\
\alpha_3 \cdot \overline{p} \cdot \alpha_1 &= -\alpha_1 \cdot \overline{p} \cdot \alpha_3 = \alpha_2, \\
\alpha_2 \cdot \overline{p} \cdot \alpha_3 &= -\alpha_3 \cdot \overline{p} \cdot \alpha_2 = \alpha_1.
\end{align*}
\] (3.7)
Using (3.3), we obtain
\[ \alpha_1 \cdot \overline{p} \cdot \alpha_1 = -\alpha_1 \cdot \overline{\alpha}_1 \cdot p = -p, \quad \alpha_2 \cdot \overline{p} \cdot \alpha_2 = \alpha_3 \cdot \overline{p} \cdot \alpha_3 = -p. \] (3.8)
From (3.5) and the equations (3.7), we easily get (3.2). This completes the proof. \( \square \)

**Proposition 3.2.** Let \( f : V \subset S^3 \to S^2 \) be submersion with totally geodesic fibers defined in an open saturated neighbourhood \( V \) of the unit sphere \( S^3 \). Denote by \( \alpha_1 \) the unit vector field tangent to the fibers of \( f \). Then the Hessian of \( \alpha_1 \), at a point \( p \in S^3 \), satisfies the formula
\[ \nabla^2_{\alpha_i, \alpha_j} \alpha_1 = \nabla_{\alpha_i} \nabla_{\alpha_j} \alpha_1 - \nabla_{\alpha_i, \alpha_j} \alpha_1 = p \cdot B_f(\alpha_i, \alpha_j) - \langle \nabla_{\alpha_i} \alpha_1, \nabla_{\alpha_j} \alpha_1 \rangle \alpha_1, \] (3.9)
where \( i, j \in \{2, 3\} \) and \( \{\alpha_2, \alpha_3\} \) a local orthonormal frame of \( \mathcal{H} \). In particular,
\[ \Delta \alpha_1 + |\nabla \alpha_1|^2 \alpha_1 = p \cdot \tau_f, \]
where here \( \Delta \) stands for the rough Laplacian operator and \( \tau_f \) for the tension field of \( f \).

**Proof.** Differentiating the identity
\[ \alpha_1(p) = p \cdot f(p) \]
with respect to \( \alpha_j \), we get
\[ D_{\alpha_j} \alpha_1 = \alpha_j \cdot f + p \cdot df(\alpha_j). \] (3.10)
Differentiating (3.10) with respect to \( \alpha_i \), we have
\[ D_{\alpha_i} D_{\alpha_j} \alpha_1 = \nabla_{\alpha_i} \alpha_j \cdot f - \langle \alpha_i, \alpha_j \rangle \alpha_1 - \langle df(\alpha_i), df(\alpha_j) \rangle \alpha_1 + \alpha_i \cdot df(\alpha_j) + \alpha_j \cdot df(\alpha_i) + p \cdot \nabla^I_{\alpha_i} df(\alpha_j). \] (3.11)
Using (3.10) and (3.11), we deduce
\[
\begin{align*}
\nabla^2_{\alpha_i, \alpha_j} \alpha_1 &= p \cdot B_f(\alpha_i, \alpha_j) - \left( \langle \alpha_i, \alpha_j \rangle + \langle df(\alpha_i), df(\alpha_j) \rangle \right) \alpha_1 \\
&\quad + \alpha_i \cdot df(\alpha_j) + \alpha_j \cdot df(\alpha_i) - \left( \langle \alpha_i, \varphi(\alpha_j) \rangle + \langle \varphi(\alpha_i), \alpha_j \rangle \right) p.
\end{align*}
\] (3.12)
From the formulas (3.7)-(3.8), we get
\[
\begin{align*}
\alpha_i \cdot \overline{p} \cdot \varphi(\alpha_j) &= -\langle \varphi(\alpha_j), \alpha_i \rangle p + \langle \varphi(\alpha_j), J\alpha_i \rangle \alpha_1, \\
\alpha_j \cdot \overline{p} \cdot \varphi(\alpha_i) &= -\langle \varphi(\alpha_i), \alpha_j \rangle p + \langle \varphi(\alpha_i), J\alpha_j \rangle \alpha_1.
\end{align*}
\] (3.13)
Combining Lemma 2.5, (3.3), (3.4) and (3.13), we obtain
\[
\begin{align*}
\alpha_i \cdot df(\alpha_j) + \alpha_j \cdot df(\alpha_i) &= -\left( \alpha_i \cdot \overline{p} \cdot \alpha_j \cdot \overline{p} + \alpha_j \cdot \overline{p} \cdot \alpha_i \cdot \overline{p} \right) \alpha_1 \\
&\quad - \alpha_i \cdot \overline{p} \cdot \varphi(\alpha_j) - \alpha_j \cdot \overline{p} \cdot \varphi(\alpha_i) \\
&= 2 \langle \alpha_i, \alpha_j \rangle \alpha_1 + \left( \langle \varphi(\alpha_j), \alpha_i \rangle + \langle \varphi(\alpha_i), \alpha_j \rangle \right) p \\
&\quad - \left( \langle \varphi(\alpha_j), J\alpha_i \rangle + \langle \varphi(\alpha_i), J\alpha_j \rangle \right) \alpha_1.
\end{align*}
\] (3.14)
Substituting (3.14) in (3.12) and using (3.2), we obtain the desired formula (3.9). This completes the proof of the proposition. \( \square \)
As a direct consequence of Proposition 3.2 we derive our first main result of the paper.

**Theorem A.** Let \( f : V \subset S^3 \rightarrow S^2 \) be a submersion with totally geodesic fibers, defined in an open saturated neighbourhood \( V \) of \( S^3 \). Then \( f \) is a harmonic map if and only if its Gauss map \( G : V \rightarrow US^3 \) is a harmonic unit vector field.

4. **Uniqueness of the Hopf fibration**

Now let us state and prove the our second main theorem.

**Theorem B.** Let \( \varphi : S^3 \rightarrow S^2 \) be a globally defined harmonic submersion with totally geodesic fibers. Then \( \varphi \) is a Hopf fibration.

**Proof.** Consider a local orthonormal frame \( \{ \alpha_1, \alpha_2, \alpha_3 \} \) such that \( \alpha_1 \in \mathcal{V} \) and \( \alpha_2, \alpha_3 \in \mathcal{H} \). Let us introduce the functions \( v = \text{trace} \varphi = \varphi_{22} + \varphi_{33} \) and \( u = \text{trace}(\varphi \circ J) = \varphi_{32} - \varphi_{23} \). From Lemma 2.2, the functions \( v \) and \( u \) are smooth and globally defined on \( S^3 \). Since \( f \) is harmonic with totally geodesic fibers, from Theorem A we have that

\[
\Delta \alpha_1 + |\nabla \alpha_1|^2 \alpha_1 = 0.
\]

From the equations

\[
\langle \Delta \alpha_1, \alpha_2 \rangle = 0 = \langle \Delta \alpha_1, \alpha_3 \rangle
\]

we obtain

\[
(\varphi_{22} - \varphi_{33})\langle \nabla_{\alpha_2} \alpha_3, \alpha_2 \rangle + (\varphi_{23} + \varphi_{32})\langle \nabla_{\alpha_2} \alpha_2, \alpha_3 \rangle = \alpha_2(\varphi_{22}) + \alpha_3(\varphi_{32})
\]

and

\[
(\varphi_{33} - \varphi_{22})\langle \nabla_{\alpha_2} \alpha_2, \alpha_3 \rangle + (\varphi_{23} + \varphi_{32})\langle \nabla_{\alpha_3} \alpha_3, \alpha_2 \rangle = \alpha_2(\varphi_{23}) + \alpha_3(\varphi_{33}).
\]

Since

\[
(\nabla^H_{\alpha_2} \varphi)\alpha_3 = (\nabla^H_{\alpha_3} \varphi)\alpha_2,
\]

we obtain the following system of PDEs

\[
(\varphi_{22} - \varphi_{33})\langle \nabla_{\alpha_2} \alpha_2, \alpha_3 \rangle - (\varphi_{23} + \varphi_{32})\langle \nabla_{\alpha_3} \alpha_3, \alpha_2 \rangle = \alpha_3(\varphi_{22}) - \alpha_2(\varphi_{32})
\]

and

\[
(\varphi_{23} + \varphi_{32})\langle \nabla_{\alpha_2} \alpha_2, \alpha_3 \rangle + (\varphi_{22} - \varphi_{33})\langle \nabla_{\alpha_3} \alpha_2, \alpha_3 \rangle = \alpha_3(\varphi_{23}) - \alpha_2(\varphi_{33}).
\]

Substracting (4.4) from (4.1), adding (4.2) and (4.3) and using (2.3), we deduce that the functions \( u \) and \( v \) satisfy the system of differential equations

\[
\alpha_1(v) = v^2 - 2(1 + \det \varphi) + 4, \quad \alpha_1(u) = uv, \quad \alpha_2(u) = \alpha_3(v) \quad \text{and} \quad \alpha_3(u) = -\alpha_2(v).
\]

(4.5)

From (4.5) and (2.3), we deduce that

\[
\Delta v = \alpha_1 \alpha_1(v) + \alpha_2 \alpha_2(v) + \alpha_3 \alpha_3(v) - \nabla_{\alpha_2} \alpha_2(v) - \nabla_{\alpha_3} \alpha_3(v)
\]

\[
= 2v \alpha_1(v) - 2v(1 + \det \varphi) - [\alpha_2, \alpha_3]u - v \alpha_1(v)
\]

\[
+ \langle \alpha_2, \nabla_{\alpha_3} \alpha_3 \rangle \alpha_2(u) - \langle \alpha_3, \nabla_{\alpha_2} \alpha_2 \rangle \alpha_3(u)
\]

\[
= u^2 v + v \alpha_1(v) - 2v(1 + \det \varphi) = v(u^2 + v^2 - 4 \det \varphi)
\]

\[
= v(|\varphi|^2 - 2 \det \varphi).
\]

Since \(|\varphi|^2 - 2 \det \varphi \geq 0\), we obtain that

\[
\Delta v^2 = 2v \Delta v + 2|\nabla v|^2 = 2v^2(|\varphi|^2 - 2 \det \varphi) + 2|\nabla v|^2 \geq 0.
\]
From the maximum principle it follows that the function \( v \) is constant. On the other hand
\[
v = \varphi_{22} + \varphi_{33} = -\text{div}(\alpha_1)
\]
and by the Stokes’ Theorem we conclude that \( v = \text{trace} \varphi = 0 \). Moreover, from the first equation of (4.5) we deduce that \( \det \varphi = 1 \). From Lemma 2.4(c) it follows that the associated graph in \( \mathbb{S}^2 \times \mathbb{S}^2 \) is generated by a conformal and strictly length decreasing map \( g \). Note that conformality implies minimality; see [7]. Then, from [20, Theorem A], it follows that \( g \) must be constant. From Theorem 2.7(c) we deduce that \( f \) is a Hopf fibration. \( \square \)

**Theorem C.** A harmonic unit vector field on \( \mathbb{S}^3 \) with geodesic leaves is a Hopf vector field.

**Proof.** Let \( \zeta \) be a harmonic unit vector field globally defined on \( \mathbb{S}^3 \) with geodesic leaves. We define the following equivalence relation \( \sim \): we say that \( x, y \in \mathbb{S}^3 \) are equivalent if and only if they lie in an integral curve of \( \zeta \). Then \( \mathbb{S}^3/\sim \) is diffeomorphic to \( \mathbb{S}^2 \) and the quotient map \( f : \mathbb{S}^3 \to \mathbb{S}^2 \) is a smooth submersion whose fibers are the integral curves of \( \zeta \). The proof follows immediately from Theorem A and B. \( \square \)

**Remark 4.1.** Let us conclude the paper with some comments and final remarks.

(a) Let \( f : \mathbb{S}^3 \to \mathbb{S}^2 \) be a submersion with totally geodesic fibers that are generated by the vector field \( \alpha_1 \). Then, as in Lemma 2.4 the function \( \text{trace}(\varphi \circ J) \) is nowhere zero. So we may assume that \( \text{trace}(\varphi \circ J) > 0 \) at any point of \( \mathbb{S}^3 \). Consider now an orthonormal basis \( \{\alpha_1, \alpha_2, \alpha_3 = J\alpha_2\} \) at a fixed point \( x \in \mathbb{S}^3 \) such that \( \{\alpha_2, \alpha_3\} \) forms the non-zero singular directions of \( df \) at \( x \). Then, from (3.2), we obtain that the non-zero singular values \( \lambda_2, \lambda_3 \) of \( df \) at \( x \) satisfy
\[
\lambda_2^2 = \varphi_{22}^2 + (\varphi_{23} - 1)^2 > 1 \quad \text{and} \quad \lambda_3^2 = \varphi_{33}^2 + (\varphi_{32} + 1)^2 > 1,
\]
since \( \varphi_{23} < 0 < \varphi_{32} \). Moreover, by a direct computation, we see that
\[
|df| = |\varphi - J|
\]
at the point \( x \). In particular, for a Hopf fibration we have \( \lambda_2 = 2 = \lambda_3 \) and \( \varphi = -J \).

(b) Let \( f : \mathbb{S}^3 \to \mathbb{S}^2 \) be a smooth map and \( \lambda_1 = 0 \leq \lambda_2 \leq \lambda_3 \) be its singular values. The 2-dilation \( \text{Dil}_2(f) = \sup \lambda_2 \lambda_3 \) of \( f \) measures how much \( f \) contract the areas of surfaces \( \Sigma \subset \mathbb{S}^3 \). In [1], it is shown that if \( \text{Dil}_2(f) < 2 \) then \( f \) is null-homotopic. Since, any submersion with totally geodesic fibers is homotopically non-trivial, it follows that the 2-dilation of any such map is greater or equal than 2.

(c) The notion of a Hopf vector field can be defined in any sphere \( \mathbb{S}^{2n-1}, n > 1 \), using the standard complex structure of \( \mathbb{R}^{2n} \). It turns out that a Hopf vector field in \( \mathbb{S}^{2n-1} \) is harmonic unit vector field and it satisfies an equation of the form
\[
\Delta \zeta + |\nabla \zeta|^2 \zeta = 0.
\]
Moreover, it is known that such vector fields are minimisers of the volume functional
\[
\text{Vol}(\eta) = \int \sqrt{\det(I + (\nabla \eta)^T \circ (\nabla \eta))} \, \text{dvol}_{\mathbb{S}^{2n-1}},
\]
where \( \text{dvol}_{\mathbb{S}^{2n-1}} \) is the volume form of \( \mathbb{S}^{2n-1} \). In dimension 3, the Hopf vector fields are the only minimisers of the above functional; for more results in this direction we refer to [4,9,12]. However, in higher dimensions not so much is known.
(d) An interesting problem is to classify minimal maps $f : S^3 \to S^2$ with totally geodesic fibers. The map $f$ is harmonic when we equip $S^3$ with the graphical metric $g = g_{S^3} + f^*g_{S^2}$, where $g_{S^3}$ is the metric of $S^3$ and $g_{S^2}$ is the metric of $S^2$. It seems that the methods developed in the present paper cannot be easily used, since in the structure equations (2.1) the curvature operator of the graphical metric shows up; for more information regarding this problem we refer to [17].

(e) In [19] it is shown that the graphical mean curvature flow (GMCF) will smoothly deform any strictly length decreasing map $h : S^2 \to S^2$ into a constant map. Moreover, this process will preserve the strict length decreasing property. Hence, GMCF will generate a smooth deformation of a submersion with totally geodesic fibers into a Hopf fibration.

(f) An analogue problem in submanifold theory is to classify minimal $m$-dimensional submanifolds in $S^n$ with index of relative nullity at least $m-2$. Such submanifolds are foliated by $(m-2)$-dimensional totally geodesic spheres. Under the assumption of completeness, it turns out that any such submanifold is either totally geodesic or has dimension three. In the latter case there are plenty of examples, even compact ones. For more details, we refer to [6, 21].

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