COUNTEREXAMPLES FOR LOCAL ISOMETRIC EMBEDDING

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1. Introduction

In this paper, we construct metrics on 2-manifold which cannot be even locally isometrically embedded in the Euclidean space $\mathbb{R}^3$. By isometric embedding of $(M^2, g)$ with $g = \sum_{i,j=1}^{2} g_{ij} dx_i dx_j$ in $\mathbb{R}^3$, we mean there exists a surface in $\mathbb{R}^3$ with the induced metric equaling $g$, namely, the three coordinate functions $(X(x_1, x_2), Y(x_1, x_2), Z(x_1, x_2))$ defined on $M^2$ satisfy

$$dX^2 + dY^2 + dZ^2 = \sum_{i,j=1}^{2} g_{ij} dx_i dx_j.$$

To be precise, we state the results in the following

**Theorem 1.1.** There exists a smooth metric $g$ in $B_1 \subset \mathbb{R}^2$ with Gaussian curvature $K_g \leq 0$ such that there is no $C^3$ isometric embedding of $(B_r(0), g)$ in $\mathbb{R}^3$ for any $r > 0$.

**Theorem 1.2.** There exists a smooth metric $g$ in $B_1 \subset \mathbb{R}^2$ with Gaussian curvature $K_g(0) = 0$ and $K_g(x) < 0$ for $x \neq 0$ such that there is no $C^{3,\alpha}$ isometric embedding of $(B_r(0), g)$ in $\mathbb{R}^3$ for any $r > 0$ and $\alpha > 0$.

Pogorelov [P2] constructed a simple $C^{2,1}$ metric $g$ in $B_1 \subset \mathbb{R}^2$ with sign-changing Gaussian curvature such that $(B_r, g)$ cannot be realized as a $C^2$ surface in $\mathbb{R}^3$ for any $r > 0$. Recently the first author [N] gave a $C^\infty$ metric $g$ on $B_1$ with no smooth isometric embedding of $(B_r, g)$ in $\mathbb{R}^3$ for any $r > 0$. The sign of the Gaussian curvature $K_g$ also changes.

On the positive side, when the sign of $K_g$ for any smooth metric $g$ does not change, the local smooth isometric embedding was settled by Pogorelov [P1], Nirenberg [Ni], and Hartman and Winter [HW2]. When $K_g \geq 0$ for the $C^k$ metric with $k \geq 10$, there is a $C^{k-6}$ isometric embedding of $(B_{r_k}, g)$ in $\mathbb{R}^3$, this was done by Lin [L1]. When $K_g$ changes sign cleanly, namely, $K_g(0) = 0, \nabla g(0) \neq 0$ for a $C^k$ metric $g$, Lin [L2] showed that there exists a $C^{k-3}$ isometric embedding in $\mathbb{R}^3$ for $(B_{r_k}, g)$ with $k \geq 6$. When $K_g \leq 0$ and $\nabla^2 K_g(0) \neq 0$ for the smooth metric $g$, there is a local smooth isometric

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embedding of \( g \) in \( \mathbb{R}^3 \), see Iwasaki [I]. When \( K_g = -x_1^{2m} \tilde{K}(x) \) with \( \tilde{K}(0) > 0 \) for the smooth metric \( g \), the same local isometric embedding also holds, see Hong [H]. Recently, Han, Hong, and Lin [HHL] showed that the local isometric embedding exists under the assumption \( K_g \leq 0 \) with a certain non-degeneracy of the gradient of \( K_g \), or \( K_g \leq 0 \) with finite order vanishing.

If one allows higher dimensional ambient space, say \( \mathbb{R}^4 \), Poznyak [Po1] proved that any smooth metric \( g \) on \( M^2 \) can be locally smoothly isometrically embedded in \( \mathbb{R}^4 \). In fact, any \( C^k \) metric on n-manifold \( M^n \) has a \( C^k \) global isometric embedding in \( \mathbb{R}^{N_n} \) with \( N_n \) large for \( 3 \leq k \leq \infty \). This is the work by Nash [Na2].

If we start with an analytic metric \( g \) on \( M^n \), one always has a local analytic isometric embedding of \((M^n, g)\) in \( \mathbb{R}^{n(n+1)/2} \). This was proved by Janet [J], Cartan [C] very earlier on, and initiated by Schläfli in 1873!

Lastly, any \( C^0 \) metric \( g \) on a compact n-manifold \( M^n \) which can be differentially embedded in \( \mathbb{R}^{n+1} \) has a \( C^1 \) isometric embedding in \( \mathbb{R}^{n+1} \), see Nash [Na1] and Kuiper [K].

For general description and further results on isometric embedding problem, we refer to [GR], [P2] and [Y].

The heuristic idea of the construction is to arrange the metric \( g \) in \( B_1 \) so that the second fundamental form of any isometric embedded surface in \( \mathbb{R}^3 \), \( \Pi \iota \) vanishes at one point, where \( \iota : (B_1, g) \to \mathbb{R}^3 \) is the isometric embedding which is supposed to exist. Further we force \( \Pi \iota \) to vanish along the boundary of a small domain \( \Omega \) near the center of \( B_1 \), where the Gaussian curvature \( K_g < 0 \) (in \( \Omega \)). By the maximal principle, one cannot have a saddle surface with vanishing second fundamental form along the boundary. So \((\Omega, g)\) cannot be realized in \( \mathbb{R}^3 \). We repeat the construction near the center of \( B_1 \) at every scale so that \((B_1, g)\) is not isometrically embeddable in \( \mathbb{R}^3 \) near the center.

The way to force \( \Pi \iota \) to vanish at one point, say \( o \), is the following. We modify the flat metric \( g_0 = dx^2 \) in \( \mathbb{R}^2 \) only over certain region \( \Lambda \) slightly away from the center \( o \) to a new one \( g \) so that, for a segment \( A_1 A_2 \) with \( A_1, A_2 \in \partial \Lambda \), the length of \( A_1 A_2 \) under \( g \) is shorter than the one of the geodesic \( A_1 A_2 \) under the flat \( g_0 \), and \( K_g \leq 0 \) in a subregion \( \Lambda_s \) containing \( A_1 A_2 \). Because of \( \det \Pi(i(0)) = 0 \), we only need to deal with the other principle curvature. Suppose the second one \( \kappa_2 \neq 0 \), say \( \kappa_2 < 0 \). We show that there is a flat concave cylinder \( \Sigma \) near \( i(B_1) \), which is isometric to \((B_1, g_0)\) provided the embedding \( i \) is \( C^3 \) (This assertion for \( C^2 \) embedding case remains unclear to us). Now \( i(A_1 A_2) \) supported on the saddle surface \( i(\Lambda_s) \) can only stay above the concave cylinder \( \Sigma \). Then the length of \( i(A_1 A_2) \) is longer than the one of the projection of \( i(A_1 A_2) \) down to the flat \( \Sigma \), call it \( P \circ i(A_1 A_2) \).

We know the length of \( P \circ i(A_1 A_2) \) under \( g_0 \) is equal to or longer than that of the geodesic \( A_1 A_2 \) under \( g_0 \). But we start from \( A_1 A_2 \) with shorter length under \( g \) than under \( g_0 \). This contradiction shows that \( \Pi \iota(i(0)) \) vanishes.

Inevitably, \( K_g \) is positive somewhere in \( \Lambda \) if \( \Lambda \) is surrounded by flat region with metric \( dx^2 \). We add “tails” extending to the boundary \( \partial B_1 \) for the
modifying regions \( \Lambda \), modify the metric on the tails, then we have the \( \eta \) with \( K_\eta \leq 0 \) in \( B_1 \). It turns out that we cannot work with a segment in the construction, we go with a minimal tree connecting three points on \( \partial \Lambda \) for each \( \Lambda \), see section 2 for details.

Now that we have a non-isometrically embeddable metric (with nonpositive Gaussian curvature), the nearby metrics are almost non-isometrically embeddable. Based on this observation, we construct a non-isometrically embeddable metric with negative Gaussian curvature except for one point in section 3.

2. Metric with nonpositive curvature

Recall any three segments in \( \mathbb{R}^2 \) with equal angles \( \frac{2\pi}{3} \) at the common vertex form a minimal tree \( T \), namely, the length of \( T \) is less than that of any arcs connecting the other three vertices.

**Lemma 2.1.** Let \( u = -\text{Im} e^{\log^2 z} = -e^{\log^2 r - \theta^2} \sin (2\theta \log r) \), \( 0 < \theta < 2\pi \).

Then there exists a large integer \( K \) such that

\[
\int_T u ds < 0,
\]

where the minimal tree \( T = AA_1 \cup AA_2 \cup AA_3 \) with \( A = (-e^{-K}, 0) \), \( A_2 = (-1, 0) \), \( A_1, A_2 \in \partial B_1 \), \( \angle A_1 AA_2 = \angle A_2 AA_3 = \frac{2\pi}{3} \). Moreover, \( u_r < 0 \) for \( r = 1 \).

**Proof.** Set \( \Omega_u = B_1 \cap \text{Sector} A_1 AA_2, \Omega_l = B_1 \cap \text{Sector} A_2 AA_3, \widehat{A_1 A_2} = \partial \Omega_u \cap \partial B_1, \widehat{A_2 A_3} = \partial \Omega_l \cap \partial B_1 \). Let the angle from \( A_1 A \) to \( x \) be \( \varphi \), or \( \varphi(x) = \angle A_1 Ax \), then \( 0 \leq \varphi(x) \leq \frac{4\pi}{3} \) for \( x \in \Omega_u \cup \Omega_l \).

We apply Green formula to harmonic functions \( u \) and \( \varphi \) in \( \Omega_u \) and \( \Omega_l \),

\[
\int_{\partial \Omega_u} u \varphi \gamma ds = \int_{\partial \Omega_u} \varphi u \gamma ds
\]

\[
\int_{\partial \Omega_l} u \left( \varphi - \frac{4}{3} \pi \right) \gamma ds = \int_{\partial \Omega_l} \left( \varphi - \frac{4}{3} \pi \right) u \gamma ds,
\]

where \( \gamma \) is the outward unit normal of the integral domain. We then have

\[
\int_{AA_1} -uds + \int_{AA_2} uds = \int_{A_1 A_2} \varphi u_r ds + \int_{AA_2} \frac{2}{3} \pi u_\theta ds
\]

\[
\int_{AA_2} -uds + \int_{AA_3} uds = \int_{A_2 A_3} \left( \varphi - \frac{4}{3} \pi \right) u_r ds + \int_{AA_2} \frac{2}{3} \pi u_\theta ds.
\]

It follows that

\[
\int_{AA_1 \cup AA_3} uds = 2 \int_{AA_2} uds + \int_{A_1 A_2} -\varphi u_r ds + \int_{A_2 A_3} \left( \varphi - \frac{4}{3} \pi \right) u_r ds
\]

\[
= 2 \int_{AA_2} uds + \int_{A_1 A_2} \varphi e^{-\theta^2} 2\theta ds + \int_{A_2 A_3} \left( \frac{4}{3} \pi - \varphi \right) e^{-\theta^2} 2\theta ds.
\]
On the other hand,
\[
\int_{AA_2} u ds = \int_{e^{-K}}^{e^0} -e^{(\log^2 r - \pi^2)} \sin (2\pi \log r) \, dr
\]
\[
= \frac{1}{2\pi e^2} \int_{-2\pi K}^{0} -e^{\left( \frac{r^2}{4\pi^2} + \frac{4}{\pi^2} \right)} \sin t \, dt.
\]
We choose large enough integer \( K \) so that \( \int_{AA_2} u ds < 0 \) and
\[
2 \int_{AA_2} u ds + \int_{A_1 A_2} \varphi e^{-\theta^2} 2\theta ds + \int_{A_2 A_3} \left( \frac{4}{3\pi} - \varphi \right) e^{-\theta^2} 2\theta ds < 0.
\]
Therefore
\[
\int_{T} u ds < 0.
\]

**Remark.** By applying Green formula to the above harmonic function \( u \) and linear functions, one sees that \( \int_{T} u ds > 0 \) for any segment \( \Gamma \subset \Omega_u \cup \Omega_l \), connecting two boundary points on \( \partial B_1 \).

**Lemma 2.2.** There exists a function \( v \in C^\infty_0 (B_{1.1}) \) satisfying
\[
v = 0 \quad \text{in} \quad \{(x_1, x_2) | x_1 < 0.9\} \setminus B_1
\]
\[
\Delta v \geq 0 \quad \text{in} \quad B_1
\]
\[
\int_{T} v ds < 0
\]
where the minimal tree \( T = CC_1 \cup CA_2 \cup CC_3 \) with \( A_2 = (-1, 0) \), \( C = \left( -\frac{1}{10} e^{-K} - 0.8, 0 \right) \), \( C_1, C_3 \in \partial B_1 \) and \( \angle C_1 CA_2 = \angle A_2 CC_3 = \frac{2}{3}\pi \). Moreover \( T \subset \{(x_1, x_2) | x_1 < -0.1\} \).

**Proof.** Set \( D = (-e^{-2K}, 0) \), \( D_1, D_2 \in \partial B_1 \) with \( \angle D_1 DA_2 = \angle A_2 DD_3 = \frac{2}{3}\pi \), and \( D_4 = (20, x_2 (D_3)) \), \( D_5 = (20, x_2 (D_1)) \). Set \( \Omega_p = \text{Pentagon} D_1 DD_3 D_4 D_5 \).

Let \( w \) satisfy
\[
\Delta w = 0 \quad \text{in} \quad \Omega_p
\]
\[
w = u \quad \text{on} \quad D_1 D \cup D_3 D
\]
\[
w = 0 \quad \text{on} \quad D_1 D_5 \cup D_3 D_4
\]
\[
w = N \quad \text{on} \quad D_4 D_5
\]
\[
w = u \quad \text{in} \quad B_1 \setminus \text{Sector} D_1 DD_3,
\]
where \( u \) is the one in Lemma 2.1.

We choose large enough \( N \) so that \( w_\gamma > u_\gamma \) on \( D_1 D \cup D_3 D \) and \( w_\gamma > 0 \) on \( D_1 D_5 \cup D_3 D_4 \), where \( \gamma \) is the inward unit normal of \( \partial \Omega_p \) this time. (If one insists, we can smooth off \( \partial \Omega_p \).)
Next we mollify \( w \) by the usual (radially symmetric) mollifier \( \rho_\delta \in C_0^\infty (B_\delta) \) with \( 0 < \delta < e^{-2K} \) to be determined later. We see that the smooth function \( w * \rho_\delta \) satisfies
\[
\begin{align*}
\triangle w * \rho_\delta (x) & \geq 0 \quad \text{for } x_1 \leq 19.9 \\
w * \rho_\delta (x) & = u \quad \text{for } x \text{ inside } \Omega_i = B_1 \setminus \text{Sector}D_1DD_3 \text{ and } \delta \text{ away from } \partial \Omega_i \\
w * \rho_\delta (x) & = 0 \quad \text{for } x \text{ outside } \Omega_o = (B_1 \setminus \text{Sector}D_1DD_3) \cup \Omega_p \text{ and } \delta \text{ away from } \partial \Omega_o.
\end{align*}
\]
Finally, set \( C_0 = (-0.8, 0) \) and
\[
v (x) = w * \rho_\delta (10 (x - C_0)).
\]
By making \( \delta \) even smaller yet positive if necessary so that \( \int_T v ds < 0 \), we obtain the desired function \( v \) in the above lemma.

**Corollary 2.1.** Let \( v \) be the function in Lemma 2.2. There exists a family of smooth metrics in \( \mathbb{R}^2 \)
\[
g_\delta = e^{2\delta v} dx^2 \quad \text{for } 0 < \delta < \delta_0
\]
such that
\[
\begin{align*}
g_\delta & = dx^2 \quad \text{in } \{(x_1, x_2) | x_1 < 0.9\} \setminus B_1 \\
K_{g_\delta} & \leq 0 \quad \text{in } B_1 \\
L (T, g_\delta) & < L (T, dx^2),
\end{align*}
\]
where \( L (T, g) \) is the length of the minimal tree \( T \) from Lemma 2.2 in metric \( g \).

**Proof.** We only prove the last two inequalities. One has
\[
K_{g_\delta} = -e^{-2\delta v} \triangle (\delta v) \leq 0 \quad \text{in } B_1.
\]
Also
\[ L(T, g_\delta) = \int_T e^{\delta v} \, ds \]
\[ \frac{dL}{d\delta} \bigg|_{\delta=0} = \int_T v \, ds < 0. \]
Thus there exists \( \delta_0 \) such that \( L(T, g_\delta) < L(T, dx^2) \) for \( 0 < \delta < \delta_0 \). \( \square \)

Let \( \psi \in C^1([-1, 1]) \) satisfy \( 0 \leq \psi \leq 1 \) and \( \psi(\pm 1) = 0 \). Set
\[ \gamma = \{(x_1, x_2) \mid x_1 = \psi(x_2), \, |x_2| \leq 1\}, \quad Q = \{(x_1, x_2) \mid 0 < x_1 < \psi(x_2), \, |x_2| \leq 1\} \]
\[ \Pi = [0, 2] \times [-2, 2] \subset R^2, \quad F = \Pi \setminus Q. \]

**Lemma 2.3.** Let \( f \in C^3(F) \). Assume the graph \( \Sigma \) of \( f \) is flat or \( \det D^2 f = 0 \) and \( D^2 f \neq 0 \) in \( F \). Also assume a unit \( C^1 \) continuous eigenvector \( V_0 \) for the zero eigenvalue of \( D^2 f \) is transversal to \( \gamma \). For any \( 0 < \tau < 1 \), there exists \( \varepsilon > 0 \) so that if \( \|D^2 f - \begin{bmatrix} 0 & 0 \\ -\tau & 0 \end{bmatrix}\| \leq \varepsilon \tau \), one can extend \( f \) to \( \Pi \) with the graph of the extension being flat and concave.

**Proof.** We take the \( C^2 \) Legendre coordinate system on \( F \subset \Pi \) (cf. [HW1]).

\[
\begin{align*}
\begin{cases}
    t = x_1 \\
    s = f_2(x_1, x_2).
\end{cases}
\end{align*}
\]
Notice that the graph of \( f, \Sigma \) is flat, or \( \det D^2 f = 0 \), it follows that \( \{(x_1, x_2) \mid f_2(x_1, x_2) = s = \text{const}\} \) is a straight segment in \( \mathbb{R}^2 \) and \( x_1(t, s) \) (\( \|V_0\) is independent of \( t \). Also \( \frac{\partial f}{\partial t}(x(t, s)) \) is independent of \( t \). Hence we can represent a portion \( \Sigma^p \) of the graph \( \Sigma \) in the ruling form
\[ (x_1, x_2, x_3)(t, s) = h(t, s) = c(s) + t\delta(s) = (t, x_2(t, s), f(t, x_2(t, s))), \]
where \( c(s), \delta(s) \in C^2 \) and \( s \in S = [f_2(2, 2), f_2(2, -2)], \quad t \leq 2 \).

We may assume \( \nabla f(2, 0) = 0 \). If \( \varepsilon \) is chosen small enough, then \( \delta(s) (\|V_0\) is close to \( (1, 0, 0) \) in \( C^1 \) norm. Take \( \varepsilon \) small, then
\[ \{(x_1, x_2, f(x_1, x_2)) \mid ((x_1, x_2) \in \gamma)\} \subset \partial \Sigma^p. \]
Set \( U = \{(t, s) \mid -1 \leq t \leq 2, \, s \in S\} \). Take \( \varepsilon \) small so that \( \|\delta(s) - (1, 0, 0)\|_{C^1} \) small, then \( (t, s) \in U \) is a \( C^2 \) coordinate system for \( \Pi \).

Now \( \Sigma^\varepsilon = h(U) \) is a \( C^2 \), flat, concave graph over a domain \( \Omega \) in \( \mathbb{R}^2 \) with \( \Pi \subset \Omega \). Indeed, the normal of \( \Sigma^\varepsilon \) is
\[ N = \frac{h_t \times h_s}{\|h_t \times h_s\|}. \]
We know
\[ h_t = \left(1, \frac{-f_2}{f_2}, f_1 + f_2 \frac{f_2}{f_2}\right) \xrightarrow{\varepsilon \to 0} (1, 0, 0) \]
\[ h_s = \left(0, \frac{1}{f_2}, \frac{f_2}{f_2}\right) \xrightarrow{\varepsilon \to 0} \left(0, -\frac{1}{\tau}, -\frac{8}{\tau}\right), \]
then \( h_t \times h_s \xrightarrow{\varepsilon \to 0} (0, \frac{\varepsilon}{t}, \frac{1}{t}) \). So \( \Sigma^e \) is a \( C^2 \) graph if we choose \( \varepsilon \) small enough.

Next, the second fundamental form of \( \Sigma^e \) is

\[
\begin{aligned}
H &= \begin{bmatrix}
\langle h_{tt}, N \rangle & \langle h_{ts}, N \rangle \\
\langle h_{st}, N \rangle & \langle h_{ss}, N \rangle 
\end{bmatrix} \\
&= \frac{1}{\|h_t \times h_s\|} \begin{bmatrix}
0 & 0 \\
0 & \langle c'' + t\delta'', \delta \times (c' + t\delta') \rangle 
\end{bmatrix}
\end{aligned}
\]

and the Gaussian curvature

\[ K_g = 0. \]

Finally, the nonzero principle curvature of \( \Sigma^e \)

\[ \kappa = \left[ \frac{\tau^3}{(1 + s^2)\varepsilon} + o(\varepsilon) \right] \langle c'' + t\delta'', \delta \times (c' + t\delta') \rangle. \]

On the other hand, from the graph representation of \( \Sigma^p, \kappa \xrightarrow{\varepsilon \to 0} -\tau/(1 + s^2)^{3/2} \).

So for \( t \) in a certain range close to 2, say \( t \in [1, 2] \), the quadratic function in terms of \( t \),

\[ \langle c'' + t\delta'', \delta \times (c' + t\delta') \rangle = a_0 + a_1 t + a_2 t^2 \]

is close to \(-1/\tau^2\) as \( \varepsilon \to 0 \). It follows that \( a_0 + a_1 t + a_2 t^2 \) is still close to \(-1/\tau^2\) for \( t \in [-1, 2] \), if we choose \( \varepsilon \) small enough. So \( \Sigma^e \) is concave.

Lemma 2.4. Let \( f \) be the extended function in Lemma 2.3, let \( w \in C^2 (\Pi) \) satisfy \( w = f \) on \( F \), det \( D^2 w \leq 0 \) in \( \Pi \), and \( \left\| D^2 w - \begin{bmatrix} 0 & 0 \\ 0 & -\tau \end{bmatrix} \right\|_C \leq \varepsilon \tau \).

Then

\[ f \leq w \text{ in } \Pi. \]

Proof. Suppose there is a point \( x' = (x'_1, x'_2) \in M \) such that \( w(x') < f(x'). \)

We know \( x'_2 \in (-1, 1) \). For simplicity, we may assume

\[ f(x') - w(x') = \sup_{x_2 \in [-1, 1]} \left[ f(x'_1, x_2) - w(x'_1, x_2) \right]. \]

Then \( f_2(x') = w_2(x') \). It follows that the two tangent lines \( l_f, l_w \) to \( f \) and \( w \) at \( x' \) in the plane \( \{(x_1, x_2, x_3) | x_1 = x'_1 \} \) are parallel. Since \( w(x'_1, \cdot) \) is concave, \( l_w \) is above \( w \).

Let \( T \subset \mathbb{R}^3 \) be the tangent plane to the graph \( \Sigma_f \) of \( f \) at \( (x', f(x')) \). Let \( R = T \cap \Sigma_f \). Then \( R \) is a segment (ruling) transversal to \( l_f \). Let \( (x^0, z^0) \) be the extended function in Lemma 2.3, let \( \langle c'' + t\delta'', \delta \times (c' + t\delta') \rangle = a_0 + a_1 t + a_2 t^2 \)

\[ \langle c'' + t\delta'', \delta \times (c' + t\delta') \rangle = a_0 + a_1 t + a_2 t^2 \]

is close to \(-1/\tau^2\) as \( \varepsilon \to 0 \). It follows that \( a_0 + a_1 t + a_2 t^2 \) is still close to \(-1/\tau^2\) for \( t \in [-1, 2] \), if we choose \( \varepsilon \) small enough. So \( \Sigma^e \) is concave.

Let \( m(x) \) be the linear function with graph as the plane \( E \) through \( l_w \) and \( l_0 \). Let \( V = \{(x_1, x_2) | |x'_1| < x_1 < 2, |x_2| < 2 \} \). Because \( \Sigma_w \) is a ruling surface on \( F \), then

\[ w(x) \leq m(x) \text{ on } \partial V. \]
Note that det $D^2w \leq 0$, by the maximum principle,
$$w(x) \leq m(x) \quad \text{in} \quad V.$$ 
On the other hand, there is $(x^*, w(x^*)) \in R$ with $x^* \in V$ such that
$$w(x^*) > m(x^*).$$
This contradiction completes the proof of the above lemma. \hfill \Box

Let $r$ be a rotation in $\mathbb{R}^2$ through an angle $1^\circ$. Let $v$ be the function in Lemma 2.2, set
$$w(x) = \sum_{i=1}^{360} v \left( r^i (1000x) - (360, 0) \right).$$
Pick two sequences $z_n \in \mathbb{R}^2$ and $\rho_n > 0$ such that
$$z_n \to 0 \quad \text{as} \quad n \to +\infty$$
$$B_{\rho_n}(z_n) \cap B_{\rho_k}(z_k) = \emptyset \quad \text{for} \quad n \neq k.$$ 
Take another sequence $\delta_n > 0$ going to 0 fast enough so that the smooth metric $g_{II}$ in $\mathbb{R}^2$ satisfying
$$g_{II} = e^{2\delta_n w(z_n+x/\rho_n)} dx^2 \quad \text{in} \quad B_{\rho_n}(z_n)$$
$$g_{II} = dx^2 \quad \text{otherwise.}$$

**Remark.** Certainly our $v$ is only smooth in $B_{1,1}(0)$, that leaves the function $w$ nonsmooth, even undefined near the corresponding tails. At this stage, we do not need any information on the metric $g_{II}$ near those tails (Figure 1 and 3). We can make a smooth extension of $v$ to $\mathbb{R}^2$ with $v \in C^\infty_0(B_2)$ if one insists. Then the Gaussian curvature of $g$ would be positive near the transition region. In the proof of Theorem 1.1, we will extend the tails to the boundary, make $v$ a smooth subharmonic function inside the unit ball. Then the Gaussian curvature would be nonpositive in the unit ball.

**Proposition 2.1.** Let $i$ be a $C^3$ isometric embedding
$$i : (B_r(0), g_{II}) \to \mathbb{R}^3$$
for some $r > 0$. Then the second fundamental form of $i(B_r(0))$ vanishes at $i(0)$, or $II(i(0)) = 0$.

**Proof.** We may assume $i(B_r)$ is the graph $\Sigma_w$ of a function $x_3 = w(x_1, x_2)$ and $w(0) = 0$, $\nabla w(0) = 0$. Then $II(i(0)) = D^2w(0)$ and det $D^2w(0) = 0$. Suppose
$$D^2w(0) \neq 0.$$ 
Let $P_3$ be the projection from $\mathbb{R}^3$ to $x_1, x_2$ plane. Set $J(x) = P_3(i(x))$. We may assume DJ is the identity map on the tangent space $\mathbb{R}^2$ at 0, and
$$D^2w(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & -\tau \end{bmatrix}.$$
For a sufficiently large $n$, $B_{\rho_n} (z_n) \subset B_r$ and
\[ g_{11} = e^{2\delta_n v (r^{180}(1000(z_n+x/\rho_n))-(360,0))} \] in the $179^\circ$ to $181^\circ$ section of the ball $B_{\rho_n} (z_n)$.

In order to simply the presentation, we work with the metric $g_{\delta_n} = e^{2\delta_n v(x)} dx^2$ as in the Corollary 2.1. Let $\Sigma^e$ be the flat, concave extension of $i \left( B_2 \setminus B_1 \right)$ by Lemma 2.3, where $B_\rho^c = \{ (x_1, x_2) \mid x_1 < 0 \} \cap B_\rho$. Note that we may consider the graph $x_3 = w_\varepsilon (x) = w (\varepsilon x)$ for small $\varepsilon$, then
\[ \left\| D^2 w_\varepsilon - \begin{bmatrix} 0 & 0 \\ 0 & -\varepsilon^2 \tau \end{bmatrix} \right\|_{C^1} \leq \varepsilon^3, \]
make the extension, then scale back.

Since $i \left( B_2^c \right)$ is negatively curved, or $\det D^2 w \leq 0$ and concave, we apply Lemma 2.4 to conclude that $i \left( B_2^c \right)$ is above $\Sigma^e$.

Let $P$ be the normal projection of points $p$ above $\Sigma^e$ down to $\Sigma^e$, that is $[p - P(p)] \perp \Sigma^e$. By concavity of $\Sigma^e$, we have
\[ \text{Length} \left( (T, g_{\delta_n}) \right) = \text{Length} \left( (T, g_{\Sigma^e}) \right) \geq \text{Length} \left( (P (i (T)), g_{\Sigma^e}) \right). \]
Where $g_{\Sigma^e}$ and $g_{\Sigma^e}$ is the induced metrics on $\Sigma^e$ and $\Sigma^e$.

Note that $P \circ i (C_1) = i (C_1), P \circ i (C_3) = i (C_3), P \circ i (A_2) = i (A_2)$, there is an isometry $i_0 : \Sigma^e \rightarrow (\mathbb{R}^2, dx^2)$ such that $i_0 \circ P \circ i (C_1) = C_1, i_0 \circ P \circ i (C_3) = C_3, i_0 \circ P \circ i (A_2) = A_2$. Apply Corollary 2.1, we have
\[ \text{Length} \left( (P (i (T)), g_{\Sigma^e}) \right) = \text{Length} \left( (i_0 \circ P \circ i (T), dx^2) \right) > \text{Length} \left( (T, g_{\delta_n}) \right). \]

Thus we arrive at
\[ \text{Length} \left( (T, g_{\delta_n}) \right) > \text{Length} \left( (T, g_{\delta_n}) \right). \]
This contradiction finishes the proof of the above proposition.  

Now we give the constructive proof of Theorem 1.1.

**Proof.** Step1. Let $\tilde{k}$ be a smooth function in $\mathbb{R}^2$ satisfying
\[ \tilde{k} < 0 \quad \text{in} \quad B^n = B_{2^{-n}} (2^{-n}, 0), \quad n = 1, 2, 3, \cdots \]
\[ \tilde{k} = 0 \quad \text{otherwise}. \]

Let $u_1$ be a smooth solution of
\[ \triangle u_1 = \tilde{k}. \]

Then the Gaussian curvature of the metric $g_1 = e^{2u_1} dx^2$ satisfies
\[ K_{g_1} = -e^{-2u_1} \triangle u_1 < 0 \quad \text{in} \quad B^n \]
\[ K_{g_1} = 0 \quad \text{otherwise}. \]
Step 2. Choose a sequence $z_{n,k}$ outside each $B^n$ and $\{(x_1, x_2) | x_2 = 0\}$ such that

$$\lim_{k \to \infty} z_{n,k} \in \partial B^n$$

$$\partial B^n \subset \{z_{n,k}\}^\infty_{k=1}.$$ 

For each $z_{n,k}$, choose a simply connected thin tail $T_{n,k}$ with $T_{n,k}$ connecting $z_{n,k}$ and the boundary $\partial B_1$ such that

- $z_{n,k} \in T_{n,k}$
- $\partial T_{n,k} \cap \partial B_1 = \text{a piece of arc with positive length}$
- $T_{n,k} \subset \mathbb{R}_+^2 = \{(x_1, x_2) | x_2 > 0\}$ for $x_2(z_{n,k}) > 0$
- $T_{n,k} \subset \mathbb{R}_-^2 = \{(x_1, x_2) | x_2 < 0\}$ for $x_2(z_{n,k}) < 0$
- $T_{n,k} \cap T_{m,j} = \emptyset$ for $(n,k) \neq (m,j)$.

![Figure 2. Tails extending to the boundary.](image)

We modify the metric $g_1 = e^{2u_1} dx^2$ over each tail $T_{n,k}$. But we proceed with the tails in the upper and lower half planes separately.

Since $K_{g_1} \equiv 0$ in the simply connected domain $\mathbb{R}_+^2 \setminus \cup_{n=1}^\infty B^n$. We represent $g_1 = dy_+^2$ in $\mathbb{R}_+^2 \setminus \cup_{n=1}^\infty B^n$ by a different coordinate system $y_+$. Over each $T_{n,k} \subset \mathbb{R}_+^2$, we plant a metric

$$g_2 = e^{2V_{n,k}} dy_+^2 \quad \text{in} \quad x^{-1}(T_{n,k}),$$

where $V_{n,k}$ is similar to the one in the construction before Proposition 2.1, but the 360 disjoint sub-tails extend to the boundary $x^{-1}(\partial B_1)$ within
We know $V_{n,k} = 0$ in $x^{-1}(B_1 \setminus T_{n,k})$. With $V_{n,k} = N_{n,k}$ chosen large enough on $x^{-1}(\partial B_1)$ intersection with the $x$ pre-image of the 360 sub-tails, we make

$$\Delta V_{n,k} \geq 0 \text{ in } x^{-1}(B_1).$$

We modify the metric $g_1 = e^{2u_1} dx^2$ over the tails in the lower half plane $\mathbb{R}^2_-$ with different coordinate system in the same way.

So far, we obtain a new metric $g_2 = e^{2u_2} dx^2$ in $B_1$ (which may not be smooth). We modify $g_2$ over the tails one last time.

Let

$$g_3 = e^{2\epsilon_{n,k} V_{n,k}} dy^2_\pm \text{ in } x^{-1}(T_{n,k}) \text{ for } T_{n,k} \subset \mathbb{R}^2_+$$

$$g_3 = e^{2\epsilon_{n,k} V_{n,k}} dy^2_- \text{ in } x^{-1}(T_{n,k}) \text{ for } T_{n,k} \subset \mathbb{R}^2_-.$$

By choosing $\epsilon_{n,k} > 0$, $\epsilon_{n,k} \to 0$ sufficiently fast for $k \to \infty$, we can assure $g_3 = e^{2u_3} dx^2$ is a smooth metric with $K_{g_3} \leq 0$ in $B_1$.

Step 3. Suppose there is an isometric embedding $i : (B_r, g) \to \mathbb{R}^3$ for some $r > 0$. Then there is $n_*$ such that

$$B^{n_*} \subset B_r.$$

Applying Proposition 2.1, we have

$$II \circ i = 0 \text{ on } \partial B^{n_*}.$$

We may assume $i(B_r)$ is represented as a graph $x_3 = f(x_1, x_2)$ with $\nabla f(0, 0) = 0$. Also we may assume the projection of $i(B^{n_*})$ down to $x_1, x_2$ plane is a
domain \( \Omega \). Then
\[
\det D^2 f = K_g \left( 1 + |\nabla f|^2 \right)^2 < 0 \quad \text{in} \quad \Omega
\]
\[
D^2 f = 0 \quad \text{on} \quad \partial \Omega.
\]
From \( D^2 f = 0 \) on \( \partial \Omega \), it follows that \( \nabla f = \text{const.} \) on \( \partial \Omega \) and \( f \) coincides with a linear function on \( \partial \Omega \). After subtracting the linear function from \( f \), we may further assume \( f = 0 \) on \( \partial \Omega \). We still have \( \det D^2 f < 0 \) in \( \Omega \). From the maximum principle, we see that \( f \equiv 0 \) in \( \Omega \). This contradiction finishes the proof of Theorem 1.1.

3. Metric with negative curvature except for one point

Relying on the metric constructed in Section 2, we construct a smooth metric \( g \) in \( B_1 \) with negative Gaussian curvature except for one point, namely, \( K_g (x) < 0 \) for \( x \neq 0 \), such that the surface \( (B_1, g) \) is not \( C^{3,\alpha} \) isometrically embeddable in \( \mathbb{R}^3 \) even locally near 0.

For any surface \( (\Omega, g) \), we define the \( C^{3,\alpha} \) isometric embedding norm by
\[
\| (\Omega, g) \|_E = \inf \left\{ \| II (i (\Omega)) \|_{C^{1,\alpha}} \mid \text{\( C^{3,\alpha} \) isometric embedding} \right\}.
\]

Now we give a constructive proof of Theorem 1.2.

**Proof.** Let the annulus \( A^n = B_{1/n} \setminus B_{1/(n+1)} \subset \mathbb{R}^2 \). We construct a metric \( g = e^{2u_0} dx^2 \) on \( B_1 \) such that a non-isometrically embeddable metric \( g \) as in Theorem 1.1 is planted (not just cut and pasted) over each annulus \( A^n \).

![Figure 4. Non-embeddable metric in each annulus.](image)
Set
\[ \tilde{\varphi}_n(r) = \begin{cases} 
  e^{-r^2/2} & r = |x| > \frac{1}{n} \\
  0 & 0 \leq r \leq \frac{1}{n} 
\end{cases} \]

We choose \( \mu_1 > 0, \mu_2 > 0, \cdots, \mu_n > 0, \cdots \) such that \( \varphi_n = \mu_n \tilde{\varphi}_n \) satisfies that \( \sum_{n=1}^{\infty} \varphi_n \) is smooth and even \( \sum_{n=1}^{\infty} \epsilon_n \varphi_n \) is smooth for \( (\epsilon_1, \epsilon_2, \cdots) \in \ell_+^\infty \).

For \( \epsilon = (\epsilon_1, \epsilon_2, \cdots) \in \ell_+^\infty \), that is \( \epsilon_1 > 0, \epsilon_2 > 0, \cdots \) and \( \|\epsilon\|_\infty = \max \epsilon_m < +\infty \), set
\[ \Phi_\epsilon = \sum_{m=1}^{\infty} \epsilon_m \varphi_m \]
\[ g_\epsilon = e^{2(\epsilon_0 + \epsilon)dx^2} \]

By the construction, \( (A^n, e^{2u_0dx^2}) \) is not \( C^3 \) isometrically embeddable in \( \mathbb{R}^3 \) for any \( n \), then we have the following.

There exists \( 0 < \eta_1 \) such that \( \|(A^1, g_{\Phi_\epsilon})\|_E \geq 1 \) for \( \epsilon \in \ell_+^\infty \) with \( \|\epsilon\|_\infty \leq \eta_1 \).

Next there exists \( 0 < \eta_2 < \eta_1 \) such that \( \|(A^m, g_{\Phi_\epsilon})\|_E \geq m \) for \( m = 1, 2 \) and \( \epsilon = (\eta_1, \epsilon_2, \epsilon_3, \cdots) \in \ell_+^\infty \) with \( \|\epsilon\|_\infty \leq \eta_2 \).

Inductively there exists \( 0 < \eta_k < \eta_{k-1} \) such that \( \|(A^m, g_{\Phi_\epsilon})\|_E \geq m \) for \( m = 1, 2, \cdots, k \) and with \( \epsilon = (\eta_1, \eta_2, \cdots, \eta_k, \epsilon_{k+1}, \epsilon_{k+2}, \cdots) \in \ell_+^\infty \) with \( \|\epsilon\|_\infty \leq \eta_k \).

Finally let \( \Psi = \sum_{m=1}^{\infty} \eta_m \varphi_m, g = g_{\Phi} \). We see that
\[ \|(A^m, g)\|_E \geq m \text{ for } m = 1, 2, 3, \cdots \]
\[ K_g(x) < 0 \text{ for } x \neq 0 \text{ and } K_g(0) = 0. \]

It follows that there is no \( C^{3, \alpha} \) isometric embedding of \( (B_r(0), g) \) in \( \mathbb{R}^3 \) for any \( r > 0, \alpha > 0 \).

\[ \square \]

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