Index defects in the theory of spectral boundary value problems

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Abstract. We study index defects in spectral boundary value problems for elliptic operators. Explicit analytic expressions for index defects in various situations are given. The corresponding topological indices are computed as homotopy invariants of the principal symbol.

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Introduction

The classical Hirzebruch formula

\[
\text{sign } M = \int_M \hat{L}(p_1, p_2, \ldots, p_k)
\]

(0.1)

expresses the signature of a closed oriented \( 4k \)-dimensional manifold in terms of its Pontryagin characteristic classes. From the viewpoint of elliptic operator theory, formula (0.1) expresses the index of a specific elliptic operator (later called the

\[\text{2000 Mathematics Subject Classification.} \text{ Primary 58J20, 58J32; Secondary 58J28, 46L85.}\]

The work was partially supported by the Russian Foundation for Basic Research under grants Nos. 02-01-00118, 02-01-06515, and 00-01-00161, by Arbeitsgruppe Partielle Differentialgleichungen und Komplexe Analysis, Institut für Mathematik, Universität Potsdam and DFG.
Hirzebruch operator) via stable homotopy invariants of its principal symbol. (For the Hirzebruch operator, these invariants coincide with the Pontryagin classes of the manifold.)

Unfortunately, formula (0.1) has no immediate analog for manifolds with boundary: there are examples showing that the signature of such a manifold cannot be expressed in terms of Pontryagin classes.

In 1973, Hirzebruch [Hir73] considered a class of manifolds with boundary (arising from algebraic-geometric considerations on Hilbert modular varieties) that have naturally defined relative Pontryagin classes. Although the right-hand side of (0.1) makes sense in this case, the equality in (0.1) fails. The difference between the right- and left-hand sides was called the signature defect, and the problem was to compute it, i.e., find a function \( f \) of the boundary of the manifold such that the difference

\[
\text{sign } M - f(\partial M)
\]

can be expressed via Pontryagin classes of the manifold, or, in the language of elliptic theory, via the principal symbol of the Hirzebruch operator.

Hirzebruch conjectured a formula for \( f(\partial M) \) and proved it in a number of examples. A complete solution of the signature defect problem was given later by Atiyah–Donnelly–Singer [ADS83] and Müller [Müll84].

The aim of the present survey is to describe index defects for some natural classes of general elliptic operators on manifolds with boundary. We consider only boundary value problems. Note, however, that index defects also occur in completely different situations, e.g., for elliptic operators in pseudodifferential subspaces of Sobolev spaces (rather than in Sobolev spaces themselves) on compact closed manifolds (see [SS99, SS00b]). Here we deal with two classes of operators important in applications, namely, operators satisfying Gilkey’s parity condition [Gil89a] and operators on \( \mathbb{Z}_n \)-manifolds in the sense of Freed and Melrose [FM92], and give explicit index defect formulas in both cases.

0.1. The classical theory: the Atiyah–Singer and Atiyah–Bott index formulas.

The Atiyah–Singer formula on closed manifolds. Let \( D \) be an elliptic operator, say, in Sobolev spaces on a closed manifold \( M \). It is well known that \( D \) is Fredholm. The celebrated Atiyah–Singer theorem [AS68] gives a topological formula for the index \( \text{ind } D \) in terms of the principal symbol \( \sigma(D) \). By applying the difference construction to the principal symbol, one obtains an element

\[ [\sigma(D)] \in K_c(T^*M) \]

in the \( K \)-group with compact supports of the cotangent bundle \( T^*M \). The Atiyah–Singer formula reads

\[
\text{ind } D = \text{ind}_t[\sigma(D)],
\]

where \( \text{ind}_t[\sigma(D)] \) is a functional of the principal symbol of the operator which can be written out in closed form. In other words, the Atiyah–Singer formula expresses
an analytic invariant of the operator (the index) in terms of topological invariants of the principal symbol.

The Atiyah–Bott index formula for boundary value problems. If the boundary $\partial M$ is not empty, then the operator $D$ is no longer Fredholm (one can show that it always has an infinite-dimensional kernel), and one should equip it with boundary conditions to obtain a well-posed problem. The classical boundary conditions are most natural.

A classical boundary value problem is a system of equations of the form

$$
\begin{cases}
Du = f, \\
B(u|_{\partial M}) = g,
\end{cases}
$$

where $u$ and $f$ are functions on $M$ and $g$ is a function on $\partial M$. The operator $B$ in the boundary condition is a differential operator; it is applied to the restriction of the unknown function to the boundary.

The ellipticity condition for problem (0.3) (see [Hör85]) can be stated in terms of the principal symbols $\sigma(D)$ of the operator and $\sigma(B)$ of the boundary condition. Atiyah and Bott [AB64] showed that the index theory of classical boundary value problems is similar to that of elliptic operators on closed manifolds. Namely, problem (0.3) defines a difference element

$$
[\sigma(D,B)] \in K_{c}(T^*(M \setminus \partial M)),
$$

where $T^*(M \setminus \partial M)$ is the cotangent bundle over the interior of $M$, and the index of the corresponding Fredholm operator is given by the formula

$$
(0.4) \quad \text{ind}(D,B) = \text{ind}_{c}[\sigma(D,B)],
$$

similar to (0.2).

However, the theory of classical boundary value problems has an essential drawback. For some operators, there are no well-posed classical boundary conditions at all!

Atiyah and Bott showed that the obstruction to the existence of well-posed boundary conditions is of topological nature and computed it. The obstruction proves to be nonzero for most geometric operators: the Dirac operator, the Hirzebruch operator, and the Cauchy–Riemann operator. In other words, there are no Fredholm classical boundary value problems for these operators.

It is still possible to sidestep the obstruction and, in particular, equip the above-mentioned operators with well-posed boundary conditions. To this end, one has to consider a more general class of boundary value problems, namely, so-called problems in subspaces, which are described in the next subsection.

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1To simplify the presentation, we consider first-order operators and occasionally speak of functions instead of sections of vector bundles.

2That is, defining a Fredholm problem for the original differential expression $D$ in suitable spaces.
0.2. Boundary value problems in subspaces. Spectral problems. A boundary value problem in subspaces is a boundary value problem of the form
\[
\begin{cases}
Du = f, \\
B (u|_{\partial M}) = g, \quad g \in \text{Im } P,
\end{cases}
\]
where the right-hand side $g$ of the boundary condition lies in the range $\text{Im } P \subset C^\infty (\partial M, G)$ of a pseudodifferential projection operator
\[
P : C^\infty (\partial M, G) \to C^\infty (\partial M, G)
\]
in the function space on the boundary. This class of boundary value problems was introduced in [SS98] and further studied in [NSSS98, SSS01, SSS99b]. In particular, it was shown that the ellipticity condition can be stated in terms of the principal symbols of $D$, $B$, and $P$, just as in the classical case. However, from the topological point of view these problems are opposite to classical boundary value problems.

The two most important differences are as follows.

I. There exists a Fredholm boundary value problem in subspaces for an arbitrary elliptic operator. An example is given by the spectral Atiyah–Patodi–Singer boundary value problem [APS75]
\[
\begin{cases}
Du = f, \\
\Pi_+ (A) u|_{\partial M} = g, \quad g \in \text{Im } \Pi_+ (A).
\end{cases}
\]
Here $D$ is a first-order operator assumed to have the form
\[
D|_{U_{\partial M}} \simeq \frac{\partial}{\partial t} + A
\]
in a collar neighborhood $U_{\partial M}$ of the boundary, where $A$ is an elliptic self-adjoint operator called the tangential operator of $D$, and $\Pi_+ (A)$ is the spectral projection of $A$ on $\mathbb{R}_+$, i.e., the orthogonal projection on the subspace spanned by eigenvectors of $A$ with nonnegative eigenvalues. Topologically, problem (0.6) can in essence be viewed as the general case of a problem in subspaces, since an arbitrary boundary value problem can be reduced to a spectral problem by a stable homotopy (see [SS99, SSS99b]). Therefore, for simplicity we consider only spectral problems (0.6) for operators $D$ satisfying (0.7). By $\text{ind} (D, \Pi_+ (A))$ we denote the index of problem (0.6).

II. The index of a boundary value problem in subspaces is not determined by the principal symbol of the operator $D$. To illustrate this, consider a deformation of lower-order terms of $D$ such that some eigenvalue of the tangential operator

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3 Atiyah, Patodi, and Singer used only homogeneous boundary conditions. However, problem (0.6) is equivalent to the corresponding homogeneous problem as far as the solvability and the index problem are concerned.

4 But not analytically!
changes its sign. At this point, the spectral projection \( \Pi_+ (A) \) experiences a jump, so that the index of the problem may change. On the other hand, the index remains constant as long as the deformation produces continuously varying spectral projections. Let us give a simple example.

Consider the zero-order deformation

\[ D_\tau = D + \tau \chi(t) \]

of the Cauchy–Riemann operator

\[ D = \frac{\partial}{\partial t} + i \frac{\partial}{\partial \varphi} \]

on the cylinder \( S^1 \times [0, 1] \), where \( \chi(t) \) is a smooth function such that \( \chi(0) = 1 \) and \( \chi(1) = 0 \). The tangential operator of the family \( D_\tau \) depends on \( \tau \) only on one of the bases of the cylinder, namely, on \( S^1 \times \{0\} \), where it has the form

\[ A_\tau = i \frac{\partial}{\partial \varphi} + \tau. \]

The eigenvalues of \( A_\tau \) are given by the formula \( \tau + 2\pi n, n \in \mathbb{Z} \). As \( \tau \) passes through zero, one of the eigenvalues changes its sign, so that the spectral projection undergoes a jump. The index is also discontinuous:

\[
\text{ind}(D_\tau, \Pi_+(A_\tau)) = \begin{cases} 
-2, & -2\pi < \tau < 0, \\
-1, & \tau = 0, \\
0, & 0 < \tau < 2\pi.
\end{cases}
\]

This example makes it clear that one cannot obtain an index formula similar to (0.4) for spectral boundary value problems; in other words, a topological computation of the index in this case is impossible in principle.

### 0.3. The index and the index defect for spectral problems

The aim of this survey is to show that in many cases of interest one obtains a homotopy invariant of the principal symbol of the operator by adding some analytic invariant to the index of a spectral boundary value problem.

The correction term is naturally called an index defect of the problem, since it is this term that restores the homotopy invariance of the index. It is natural to require that the correction term be determined solely by the structure of the operator in a neighborhood of the boundary, for on closed manifolds the analytic index itself is homotopy invariant and zero can be taken for the correction term.

Therefore, we introduce the following statement of the index defect problem.

**The index defect problem for spectral boundary value problems.** Construct a functional def-ind\( (D) \) of elliptic operators \( D \) on a manifold with boundary such that

1. The sum \( \text{ind}(D, \Pi_+(A)) + \text{def-ind \( D \)) \) is a homotopy invariant of \( D \).
2. def-ind\( (D) \) is determined solely by the tangential operator \( A \).

\[ \text{def-ind \( D \)) \text{ is determined solely by the tangential operator } A. \]

\[ \text{def-ind}(D) \text{ is determined solely by the tangential operator } A. \]

\[ \text{It readily follows that the sum is a homotopy invariant of the principal symbol of } D. \]
A functional with these properties will be called an index defect. Conditions (1) and (2) imply that the index defect is determined by the spectral projection and is its homotopy invariant.

Needless to say, to obtain an actual defect formula (which is our main problem), we should compute the homotopy invariant in (1) topologically, i.e., express it in the form

\[ \text{ind}(D, \Pi_+(A)) + \text{def-ind } D = \text{ind}_t(\sigma(D)), \]

where \( \text{ind}_t(\sigma(D)) \) is a functional on the set of homotopy classes of elliptic principal symbols.

In the remaining part of the introduction, we explain the main methods that can be used to define index defects and describe approaches to the proof of the corresponding index defect formulas (0.8). However, prior to proceeding to these topics, we consider the following phenomenon of utmost importance.

The obstruction to index defect formulas. The desired index defect formula (0.8) can be viewed as a decomposition of the index of a spectral boundary value problem into a finite-dimensional contribution of the principal symbol and an (infinite-dimensional) contribution of the tangential operator:

\[ \text{ind}(D, \Pi_+(A)) = f_1(\sigma(D)) + f_2(A), \]

where the functional \( f_1 \) is a homotopy invariant of the principal symbol.

It turns out that there is no index decomposition of the form (0.9) on the set of all elliptic operators (see [SSS99a]). Therefore, index defect formulas and decompositions of the form (0.9) can be sought only in some subsets of the space of elliptic operators on a manifold with boundary.

The obstruction to the existence of decompositions (0.9) was computed in [SSS99a]. It is the one-dimensional cohomology class of the space of elliptic operators whose value on a cycle is equal to the Atiyah–Patodi–Singer spectral flow of the corresponding family of tangential operators. There exists a decomposition (0.9) on a subspace \( \Sigma \) of the space of elliptic operators if and only if the restriction of this cohomology class to \( \Sigma \) is trivial.

The cited result, unfortunately, proves only the existence of a decomposition (0.9) and does not give a satisfactory formula for an index defect. To study index defects, one has to use other methods.

0.4. Approaches to the definition of index defects. Let us briefly describe two methods useful in defining index defects.

A. The geometric index formula of Atiyah–Patodi–Singer. In 1975, Atiyah–Patodi–Singer [APS76a] obtained the formula

\[ \text{ind}(D, \Pi_+(A)) = \int_M a(D) - \eta(A) \]

for the index of spectral boundary value problems, where the density \( a(D) \) is determined by the coefficients of \( D \), just as in the case of closed manifolds. The
new contribution to the index is given by the spectral \(\eta\)-invariant \(\eta(A)\) of the
tangential operator \(A\).

This formula is often called a geometric index formula, since for geometric
operators (the Hirzebruch, Dirac, Todd and Euler operators) the integrand on the
right-hand side is determined by the metric and coincides with the local Atiyah–
Singer density in the case of closed manifolds (i.e., with the \(L\)-form for the Hirze-
bruch operator, the \(A\)-form for the Dirac operator, etc.).

Unfortunately, the Atiyah–Patodi–Singer formula does not define an index
defect, since neither of the terms on the right-hand side is a homotopy invariant
of \(D\). However, formula (0.10) can be used to define index defects as follows.

We have already pointed out that the index of the spectral boundary value
problem experiences jumps under homotopies of \(D\). However, the sum
\[(0.11) \quad \text{ind}(D, \Pi_+ (A)) + \eta(A)\]
varies smoothly by the Atiyah–Patodi–Singer theorem. Moreover, the sum (0.11) is
homotopy invariant if and only if for an arbitrary homotopy \(D_\tau\) (with parameter \(\tau\))
in our class of operators the derivative
\[(0.12) \quad \frac{d}{d\tau} \int_M a(D_\tau)\]
is zero. Since the density \(a(D_\tau)\) is given by a closed-form expression involving the
coefficients of \(D_\tau\), one can use a detailed analysis of \(a(D_\tau)\) to construct classes of
operators for which the derivative is zero and hence the \(\eta\)-invariant of the tangential
operator is the desired index defect.

B. Operator algebras. Another method for defining index defects relies on
\(K\)-theory of operator algebras.

As was mentioned already, any index defect is determined by the spectral
projection \(\Pi_+ (A)\) and is its homotopy invariant. For simplicity, consider the class
of matrix projections
\[(0.13) \quad \Pi_+ (A) : C^\infty (\partial M, \mathbb{C}^N) \to C^\infty (\partial M, \mathbb{C}^N)\]
and assume that the matrix entries lie in some algebra \(\mathcal{A}\) of operators on the
boundary.

Then index defects admit an alternative description as homotopy invariants
of projections.

The homotopy classes of projections (0.13) with entries in \(\mathcal{A}\) (for arbitrary \(N\))
generate the \(K\)-group \(K_0 (\mathcal{A})\) of the algebra \(\mathcal{A}\) (e.g., see [Bla98]). An index defect
functional defines a homomorphism
\[d : K_0 (\mathcal{A}) \to \mathbb{R}\]
of the \(K\)-group into real numbers.

Moreover, a simple computation shows that the homotopy invariance of the
sum \(\text{ind}(D, \Pi_+ (A)) + d(\Pi_+ (A))\) is equivalent to either of the following two con-
ditions for the functional \(d\) (provided that \(\mathcal{A}\) contains the ideal of finite rank
operators).
The functional $d$ is a dimension type invariant of projections. More precisely, for two arbitrary projections $P_1, P_2$, $\text{Im} P_1 \subset \text{Im} P_2$, $\dim \text{Im}(P_2 - P_1) < \infty$, differing by a finite rank projection $P_2 - P_1$, one has
$$d(P_2) - d(P_1) = \dim \text{Im}(P_2 - P_1).$$

The functional $d$ defines a commutative diagram
$$K_0(K) = \mathbb{Z}$$
$$K_0(A) \xrightarrow{d} \mathbb{R},$$
where the diagonal arrow is the natural embedding $\mathbb{Z} \subset \mathbb{R}$.

0.5. Examples. Now we consider some explicit implementations of the methods. Each example starts with a description of the class of operators for which the index defect is to be considered.

**Example 0.1. Problems in even subspaces [SS99].** Consider operators $D$ such that the symbol of the tangential operator is even in the momentum variables $\xi$:
$$\sigma(A)(x, \xi) = \sigma(A)(x, -\xi).$$
This condition singles out the subalgebra of operators with even principal symbol in the algebra of pseudodifferential operators on the boundary. In this subalgebra, consider the subalgebra $A$ of zero-order operators. If $\partial M$ is even-dimensional, then the vertical arrow in diagram (0.14) is a monomorphism, which implies the existence of the desired functional $d$. One can also prove that $d$ is unique under some natural conditions.

Thus, there is a well-defined homotopy invariant $d(P) \in \mathbb{R}$ on the set of pseudodifferential projections $P$ with even principal symbol. In contrast with the index of elliptic operators, this functional is not integer; it can take arbitrary rational values whose denominators are powers of 2 (dyadic values) see [SS00a].

Thus, the value of the dimension functional $d$ on the spectral projection can be taken as the index defect for operators $D$ whose tangential operator $A$ has an even principal symbol. In other words, the sum
$$\text{ind}(D, \Pi_+(A)) + d(\Pi_+(A))$$
is a homotopy invariant of $D$. This poses a problem of computing this invariant in terms of the principal symbol of $D$. In the next subsection, we explain the main ideas underlying the computation of this invariant.

**Example 0.2. Spectral problems on $\mathbb{Z}_n$-manifolds [SS01].** Consider a manifold $M$ whose boundary is represented as the total space of a covering
$$\pi : \partial M \rightarrow X$$
over a smooth base $X$. Geometrically, such a manifold can be viewed as a smooth model of the singular space $\tilde{M}^\pi$ (known as a $\mathbb{Z}_n$-manifold, where $n$ is the number of sheets of the covering) obtained by identifying the points in each fiber of $\pi$. A neighborhood of a singular point looks like an open book with $n$ sheets (see Fig. 1), and $X \subset \tilde{M}^\pi$ is the edge, where the sheets meet.

On $M$ we consider elliptic operators whose tangential operator $A$ is the lift by $\pi$ of some operator $A_0$ on the base of the covering. (The lift is well defined, since $\pi$ is a local diffeomorphism.)

For a trivial covering, this condition guarantees that the index of the corresponding spectral boundary value problem viewed as the mod $n$-residue

$$\text{mod } n \cdot \text{ind } D \in \mathbb{Z}_n$$

is a homotopy invariant of $D$. Freed and Melrose \cite{FM92} obtained the mod $n$-index formula

$$\text{mod } n \cdot \text{ind } D = \text{ind}_t [\sigma (D)],$$

where

$$[\sigma (D)] \in K_c (\tilde{T}^*\tilde{M}^\pi)$$

is the difference element defined by the principal symbol of $D$ in the $K$-group of the singular space $\tilde{T}^*\tilde{M}^\pi$ obtained from the cotangent bundle $T^*M$ by identifying the points in each fiber of the covering $\partial T^*M \to T^*X \times \mathbb{R}$, by analogy with the definition of $\tilde{M}^\pi$.

However, the mod $n$-index is not an invariant of the principal symbol if the covering is nontrivial. The index defect in this case turns out to be given by the difference

$$\text{def-ind}(D) \overset{\text{def}}{=} \eta (A) - n\eta (A_0) \in \mathbb{R}/n\mathbb{Z}.$$
An expression of this type is known as the relative Atiyah–Patodi–Singer $\eta$-invariant $[\text{APS76a}]$ of an operator $A_0$ with coefficients in a flat bundle. The index defect problem in this case is to compute the homotopy invariant

$$\text{inv} D = \text{mod } n \cdot \text{ind } D + \text{def-ind } D \in \mathbb{R}/n\mathbb{Z}$$

as a residue modulo $n$.

0.6. Approaches to index defect formulas. To state and prove an index defect formula, one can apply all methods that are useful in the proof of ordinary index formulas. We content ourselves with describing only two approaches.

**Method 1. Homotopy classification.**

Roughly speaking, the method consists of two steps.

1. First, one carries out the homotopy classification of elliptic operators $D$ to be considered, or, more technically, computes the group of stable homotopy classes of these operators.

2. Second, one finds a generating set of this group, so that the proof of an index defect formula is reduced to its verification for the generators.

This scheme goes back to the first proof of the Atiyah–Singer formula $[\text{AS63}]$, where elliptic operators are classified (modulo stable homotopies) by elements of the $K$-group $K_c(T^*M)$ of the cotangent bundle at step (1), the Hirzebruch operator with coefficients in various vector bundles gives a rational generating set of the $K$-group (on an orientable even-dimensional manifold) at step (2), and finally cobordism theory is used to compare the analytic and topological index of these geometric operators.

**Method 2. Poincaré duality.**

Another method for proving index formulas is based on Poincaré duality in $K$-theory. Let us illustrate this method using the classical Atiyah–Singer theorem as an example.

On a closed manifold, there is an index homomorphism

$$\text{ind}_a : K_c(T^*M) \rightarrow \mathbb{Z},$$

which takes each element of $K$-theory to the (analytic) index of the corresponding elliptic operator. On the other hand, Poincaré duality in $K$-theory gives the pairing

$$\langle \cdot, \cdot \rangle : K_c(T^*M) \times K(M) \rightarrow \mathbb{Z},$$

$$(x, y) \mapsto p_t(xy),$$

which is nonsingular on the free parts of the groups. Here

$$p_t : K_c(T^*M) \rightarrow \mathbb{Z}$$

is the direct image mapping induced by the projection $p : M \rightarrow pt$ to a one-point space. It follows that the homomorphism (0.15) can be represented as the pairing with some element $y \in K(M)$; i.e.,

$$\text{ind } D = \langle [\sigma(D)], y \rangle$$
for all elliptic operators $D$, where $y$ is uniquely determined by $M$ modulo torsion. Therefore, to obtain an index formula, it suffices to compute the element $y$. In these terms, the Atiyah–Singer formula states that one can take $y = 1 \in K(M)$, the element corresponding to the trivial line bundle.

Let us show how one can apply these methods to find and prove index defect formulas.

**Continuation of Example 0.1** (an index defect formula in even subspaces).

We consider operators $D$ with even principal symbol of the tangential operator on an even-dimensional manifold $M$. The homotopy classification of the corresponding spectral boundary value problems turns out to be isomorphic (modulo 2-torsion) to that of classical boundary value problems, i.e., to the group $K_c(T^*(M \setminus \partial M))$. Therefore, to obtain a topological formula for the homotopy invariant

$$\text{inv} D = \text{ind}(D, \Pi_+ (A)) + d(\Pi_+ (A)),$$

it suffices to generalize the Atiyah–Bott topological index (0.4) to boundary value problems in even subspaces. Such a generalization was obtained in [SS99]. We point out that the topological index in this formula proves to be a half-integer, and a topological consequence of this formula is the half-integrality of the index defect. The index defect formula has a number of applications. For example, it enabled the authors [SS02] to solve Gilkey’s problem on the nontriviality of $\eta$-invariants of even-order operators on odd-dimensional manifolds.

**Continuation of Example 0.2** (an index defect formula on $\mathbb{Z}_n$-manifolds).

For a manifold whose boundary is an $n$-sheeted covering, the sum

$$\text{inv} D \overset{\text{def}}{=} \text{mod } n \cdot \text{ind } D + \text{def-ind } D$$

can be viewed as a homomorphism

$$\text{inv} : K_c(T^*M) \longrightarrow \mathbb{R}/n\mathbb{Z},$$

(0.17)

The two main differences between (0.17) and (0.15) are as follows:

1. now we use the group $\mathbb{R}/n\mathbb{Z}$ rather than $\mathbb{R}$;
2. the space $T^*M$ has singularities.

To take account of (1), one should replace the classical Poincaré duality (0.16) by “Poincaré duality with coefficients,” i.e., *Pontryagin duality*

$$(0.18) \quad \langle , \rangle : K_c(T^*M) \times K(M, \mathbb{R}/\mathbb{Z}) \longrightarrow \mathbb{R}/\mathbb{Z}.$$

(See [SS02]; here $K(M, \mathbb{R}/\mathbb{Z})$ is the $K$-group with coefficients in the compact group $\mathbb{R}/\mathbb{Z}$.)

To tackle (2), one needs an extension of Poincaré duality to singular manifolds like $\overline{M}^*$. We point out that duality in the classical sense may not be valid on a manifold with singularities. However, the desired duality can be obtained in the
framework of Connes’ noncommutative geometry \cite{Con94}. A detailed exposition is given in Appendix C, and now we describe only the main ideas involved.

To a singular $\mathbb{Z}_n$-manifold $\overline{M}$, one assigns a noncommutative $C^*$-algebra $A_{M,\pi}$, which should be regarded as the “function algebra” on $\overline{M}$. Now Poincaré duality gives a pairing

$$\langle \cdot, \cdot \rangle : K_c(T^*M) \times K_0(A_{M,\pi}) \to \mathbb{Z},$$

where the second group is the $K$-group of the $C^*$-algebra $A_{M,\pi}$. As a special case of this construction for $\partial M = \emptyset$, one obtains the pairing (0.16), since in this case the algebra $A_{M,\pi}$ coincides with the algebra of continuous functions on $M$ and the $K$-group of the algebra of continuous functions on a space coincides with the $K$-group of the space.

Applying the constructions (0.18) and (0.19) to the mapping (0.17), one finds

$$\text{inv} D = \langle [\sigma(D)], y \rangle \in \mathbb{Q}/n\mathbb{Z},$$

just as in the case of a closed manifold, for some element

$$y \in K_0(A_{M,\pi}, \mathbb{Q}/n\mathbb{Z})$$

depending only on the manifold. The index defect theorem for $\mathbb{Z}_n$-manifolds in the authors’ paper \cite{SS01} gives an explicit formula for this element.

Acknowledgements. The authors wish to express their keen gratitude to Professor B.-W. Schulze of Potsdam University, where the paper was written, for his kind hospitality. We also thank V. Nazaikinskii, who read a preliminary version of the paper and made a number of important remarks.

The contents of the paper are as follows. In the first section, we define spectral boundary value problems and prove a theorem on index decompositions. Section 2 deals with index defects in even subspaces. In Section 3, we study index defects on $\mathbb{Z}_n$-manifolds. The paper concludes with three appendices, the first dealing with the Atiyah–Patodi–Singer $\eta$-invariant and the remaining two with Poincaré duality in $K$-theory on smooth manifolds and $\mathbb{Z}_n$-manifolds.

1. Spectral boundary value problems and their index

1.1. Atiyah–Patodi–Singer spectral boundary value problems. We start by introducing some notation. Let $M$ be a smooth compact manifold with boundary and

$$D : C^\infty(M, E) \to C^\infty(M, F)$$

an elliptic first-order differential operator on $M$ acting in the spaces of sections of vector bundles $E, F \in \text{Vect}(M)$.

We choose some collar neighborhood

$$U_{\partial M} \simeq \partial M \times [0, 1)$$
of the boundary \( \partial M \). The normal coordinate on the half-open interval \([0, 1)\) is denoted by \( t \). Then \( D \) can be represented in this neighborhood in the form

\[
D|_{U_{\partial M}} \simeq \frac{\partial}{\partial t} + A(t)
\]

(here \( \simeq \) stands for equality up to a vector bundle isomorphism) for a smooth family \( A(t) \) of elliptic first-order differential operators on \( \partial M \). This representation can be obtained as follows. In the collar neighborhood of the boundary, we take some isomorphisms \( \pi^* (E|_{\partial M}) \simeq E|_{\partial M \times [0,1)} \) and \( \pi^* (F|_{\partial M}) \simeq F|_{\partial M \times [0,1)} \), where \( \pi : \partial M \times [0,1) \to \partial M \) is the natural projection. Then \( D \) is isomorphic in \( U_{\partial M} \) to an operator in the spaces

\[
C^\infty(\partial M \times [0,1), \pi^* (E|_{\partial M})) \longrightarrow C^\infty(\partial M \times [0,1), \pi^* (F|_{\partial M}))
\]

where the operator \( \partial/\partial t \) is well defined, and we obtain a decomposition

\[
D|_{U_{\partial M}} \simeq \Gamma(t) \frac{\partial}{\partial t} + A'(t),
\]

where \( \Gamma(t) \) is a vector bundle homomorphism. By ellipticity, \( \Gamma(t) \) is an isomorphism, and we arrive at (1.1).

For simplicity, we also assume that for small \( t \) the family \( A(t) \) is independent of \( t \) and coincides with a given self-adjoint operator

\[
A : C^\infty(\partial M, E|_{\partial M}) \longrightarrow C^\infty(\partial M, E|_{\partial M})
\]
on the boundary. The operator \( A \) is called the tangential operator of \( D \).

**Definition 1.1.** The Atiyah–Patodi–Singer spectral boundary value problem for an elliptic operator \( D \) is the boundary value problem

\[
\begin{aligned}
Du &= f, \\
\Pi_+(A)u|_{\partial M} &= g, \quad g \in \text{Im } \Pi_+(A),
\end{aligned}
\]

where \( \Pi_+(A) \) is the nonnegative spectral projection of \( A \):

\[
\Pi_+(A)e_\lambda = \begin{cases} e_\lambda, & \lambda \geq 0, \\
0, & \lambda < 0,
\end{cases}
\]

for any eigenvector \( e_\lambda \) of \( A \) with eigenvalue \( \lambda \).

For an arbitrary elliptic operator \( D \), the spectral boundary value problem defines a bounded Fredholm operator in the spaces

\[
(D, \Pi_+(A)) : H^s(M, E) \longrightarrow H^{s-1}(M, F) \oplus \text{Im } \Pi_+(A), \quad s > 1/2,
\]

where \( \text{Im } \Pi_+(A) \) is the closure of the range of \( \Pi_+(A) \) in \( H^{s-1/2}(\partial M, E|_{\partial M}) \) (see [APS73]).

As usual, the index \( \text{ind}(D, \Pi_+(A)) \) of the spectral boundary value problem is independent of the Sobolev smoothness exponent \( s \) and can be computed in spaces of smooth functions.
Note that, in contrast with the index of elliptic operators on closed manifolds, the index of the Atiyah–Patodi–Singer problem is not invariant under homotopies of $D$. Indeed, consider a smooth homotopy

$$D_{\tau} : C^{\infty}(M, E) \rightarrow C^{\infty}(M, F), \quad \tau \in [0, 1],$$

of elliptic operators, i.e.

an elliptic operator family with coefficients smoothly depending on $\tau$. Suppose that an eigenvalue of the tangential operator $A_{\tau}$ changes its sign at some point $\tau = \tau'$. Then the corresponding spectral projection experiences a jump, and consequently, the spaces in which the operator acts change discontinuously. This intuitive argument is stated in Theorem 1.4 below in terms of the spectral flow.

1.2. The spectral flow. Consider a smooth family $\{A_{\tau}\}_{\tau \in [0, 1]}$ of elliptic self-adjoint operators on a closed manifold and assume that the operators at the endpoints $t = 0$ and $t = 1$ are invertible.

**Definition 1.2.** The spectral flow $sf\{A_{\tau}\}_{\tau \in [0, 1]}$ of the family $\{A_{\tau}\}_{\tau \in [0, 1]}$ is the net number of eigenvalues of $A_{\tau}$ that change their sign from minus to plus as the parameter $\tau$ increases from 0 to 1 (see Fig. 3).

Unfortunately, this definition does not make sense for general families. The spectral flow of an arbitrary family $\{A_{\tau}\}$ can be defined by different methods (see [Phi96], [Mel93], [Sal93], [BBW93], [DZ98], [NSS99], and other papers). For example, one can slightly deform the straight line $\lambda = 0$ in the $(\lambda, \tau)$-plane in a way such that the spectral curves of the operators $A_{\tau}$ intersect the perturbed line transversally. Then the spectral flow can be defined as the intersection number of the perturbed line with the graph of the spectrum of the family. The desired perturbation can be constructed explicitly as follows (e.g., see [Mel93]).

A continuous family of elliptic self-adjoint operators has the following property. For an arbitrary $\tau' \in [0, 1]$, there exists a number $\lambda_{\tau'}$ that is not an eigenvalue of $A_{\tau}$ in an $\varepsilon$-neighborhood of $\tau'$. This follows from the fact that the spectrum of $A_{\tau}$ is discrete.

Now we choose a finite partition

$$(1.4) \quad 0 = \tau_0 < \tau_1 < \ldots < \tau_N = 1$$

doing the interval $[0, 1]$ and numbers $\{\lambda_i\}_{i=0,N-1}$, referred to as weights, such that $\lambda_i$ is not an eigenvalue of the family $A_{\tau}$ on the interval $[\tau_i, \tau_{i+1}]$. We also assume that $\lambda_0 = \lambda_N = 0$.

**Definition 1.3.** The spectral flow of the family $\{A_{\tau}\}_{\tau \in [0, 1]}$ is the number

$$(1.5) \quad sf\{A_{\tau}\}_{\tau \in [0, 1]} = \sum_{i=1}^{N-1} \text{ind}(\Pi_{\lambda_i}(A_{\tau_i}), \Pi_{\lambda_{i-1}}(A_{\tau_i})),$$

where $\Pi_{\lambda}(A)$ is the spectral projection of a self-adjoint operator $A$ corresponding to eigenvalues greater than or equal to $\lambda$ and

$$\text{ind}(\Pi_{\lambda}(A), \Pi_{\mu}(A)) = \text{sgn}(\mu - \lambda) \dim \Lambda_{\lambda,\mu},$$
is the relative index of two projections. Here \( \Lambda_{\lambda, \mu} \) is the spectral subspace of \( A \) corresponding to eigenvalues in the interval \([\min \{ \lambda, \mu \}, \max \{ \lambda, \mu \}]\).

One can show that the spectral flow (1.5) is well defined, i.e., is independent of the choice of the partition \( \{ \tau_i \} \) and the weights \( \{ \lambda_i \} \).

Let us now return to our original problem and consider a homotopy \( D_\tau, \tau \in [0, 1] \), of elliptic operators on a manifold with boundary. The corresponding family of tangential operators will be denoted by \( \{ A_\tau \} \). We shall now give a formula for the difference of indices at the endpoints of the homotopy. It turns out that the difference is equal to the spectral flow of the family of tangential operators.

**Theorem 1.4.** (The spectral flow theorem.) One has

\[
\text{ind}(D_0, \Pi_+ (A_0)) - \text{ind}(D_1, \Pi_+ (A_1)) = \text{sf}\{A_\tau\}_{\tau \in [0,1]}. 
\]

**Remark 1.5.** This result does not contradict the homotopy invariance of the index, since we change not only the operator \( D \), but also the spaces where the index is computed.

**Proof.** 1. One can readily obtain (1.6) if the family \( A_\tau \) is invertible for all \( \tau \). Indeed, in this case the right-hand side of (1.6) is zero by the definition of the spectral flow. Let us verify that the indices on the left-hand side are equal. To this end, we note that the family of nonnegative spectral projections is smooth. Consider the Cauchy problem

\[
U_\tau = \left[ \Pi_+ (A_\tau), \Pi_+ (A_\tau) \right] U_\tau, \quad U_0 = \text{Id}. 
\]

One can readily verify that the solution \( U_\tau \) is a unitary elliptic operator specifying an isomorphism

\[
U_\tau : \text{Im} \Pi_+ (A_0) \longrightarrow \text{Im} \Pi_+ (A_\tau)
\]
of the subspaces defined by the pseudodifferential projections \( \Pi_+ (A_0) \) and \( \Pi_+ (A_\tau) \). The composition

\[
\begin{pmatrix}
1 & 0 \\
0 & U_\tau^{-1}
\end{pmatrix}
(D_\tau, \Pi_+ (A_\tau))^t : C^\infty (M, E) \rightarrow C^\infty (M, F) \oplus \text{Im} \Pi_+ (A_0)
\]

has the same index as the original problem \((D_\tau, \Pi_+ (A_\tau))\). On the other hand, the composition acts in spaces independent of \( \tau \). Thus, by the standard index invariance, its index does not change.

2. Now consider the general case in which the operators of the tangential family \( A_\tau \) may be noninvertible. To this end, we choose some partition (1.4). Using the argument in the previous part of the proof, on the first interval \([\tau_0, \tau_1]\) we obtain

\[
\text{ind}(D_0, \Pi_{\lambda_0} (A_{\tau_0})) = \text{ind}(D_\tau, \Pi_{\lambda_0} (A_\tau)), \quad \tau \in [\tau_0, \tau_1].
\]

Considering this equation for \( \tau = \tau_1 \) and replacing the projection \( \Pi_{\lambda_0} (A_{\tau_1}) \) by \( \Pi_{\lambda_1} (A_{\tau_1}) \) (they differ by a finite-dimensional projection), we obtain

\[
\text{ind}(D_0, \Pi_{\lambda_0} (A_{\tau_0})) = \text{ind}(D_{\tau_1}, \Pi_{\lambda_1} (A_{\tau_1})) + \text{ind}(\Pi_{\lambda_1} (A_{\tau_1}), \Pi_{\lambda_0} (A_{\tau_1})).
\]

A similar modification of \( \text{ind}(D_{\tau_1}, \Pi_{\lambda_1} (A_{\tau_1})) \) at \( \tau_2 \) gives

\[
\text{ind}(D_0, \Pi_{\lambda_0} (A_{\tau_0})) = \text{ind}(D_{\tau_2}, \Pi_{\lambda_2} (A_{\tau_2}))
\]

\[
+ \text{ind}(\Pi_{\lambda_1} (A_{\tau_1}), \Pi_{\lambda_0} (A_{\tau_1})) + \text{ind}(\Pi_{\lambda_2} (A_{\tau_2}), \Pi_{\lambda_1} (A_{\tau_2})).
\]

Proceeding similarly at the subsequent points \( \tau_3, \tau_4, \ldots \), we obtain the desired equation

\[
\text{ind}(D_0, \Pi_+ (A_0)) = \text{ind}(D_1, \Pi_+ (A_1)) + sf\{A_\tau\}_{\tau \in [0,1]}.
\]

\[\Box\]

1.3. A theorem on index decompositions. It follows from the spectral flow theorem in the previous subsection that the index of the spectral boundary value problem \((D, \Pi_+ (A))\) is uniquely determined by the principal symbol \( \sigma (D) \) and the tangential operator \( A \). There arises a natural question: Is it possible to decompose the index of the spectral boundary value problem into the sum

\[
(1.7) \quad \text{ind} (D, \Pi_+ (A)) = f_1 (\sigma (D)) + f_2 (A)
\]

of a homotopy invariant \( f_1 (\sigma (D)) \) of the principal symbol of the operator and a functional \( f_2 (A) \) of the tangential operator? If this representation is possible, how to find it?

We shall sometimes refer to Eq. (1.7) as an “index decomposition.”

Remark 1.6. The functional \( f_2 \) is not a homotopy invariant of the tangential operator in general. Indeed, the set of operators with a given principal symbol \( \sigma (D) \) contains operators with arbitrary index. Thus, \( f_2 \) can take infinitely many values. A more precise analysis shows that \( f_2 \) is a homotopy invariant of the corresponding spectral projection.

\[\text{To justify this and subsequent index computations, one uses the fact that the relative index ind}(P, Q) \text{ of projections coincides with the index of the Fredholm operator } Q : \text{Im } P \rightarrow \text{Im } Q.\]
It turns out that there is an obstruction to the index decomposition. Indeed, suppose that (1.7) is valid. Consider a homotopy \( D_\tau \) of elliptic operators such that the homotopy of their tangential operators \( A_\tau \) is periodic: \( A_0 = A_1 \). We claim that in this case the indices of the spectral boundary value problems for \( D_0 \) and \( D_1 \) are equal. Indeed,

\[
f_1(\sigma(D_0)) = f_1(\sigma(D_1)),
\]
since the symbols are homotopic, and

\[
f_2(A_0) = f_2(A_1),
\]
since the tangential operators \( A_0 \) and \( A_1 \) coincide by assumption.

On the other hand, by virtue of the spectral flow theorem, the difference of indices at the endpoints of the homotopy is equal to the spectral flow of the periodic family of tangential operators. Thus, we obtain the following result.

**Proposition 1.7.** If the index decomposition is valid, then for an arbitrary homotopy of tangential operators \( A_\tau, \tau \in S^1 \), one has

\[
(1.8) \quad \text{sf}\{A_\tau\}_{\tau \in S^1} = 0.
\]

It is well known that there exist periodic families of elliptic self-adjoint operators with nontrivial spectral flow (1.8). (Simple examples can be found in \([Sav99]\).) Therefore, the index decomposition (1.7) does not exist on the space of all elliptic operators.

In other words, to achieve (1.7), one has to consider subspaces rather than the entire space of elliptic operators. Using this idea, one can prove a result similar to Proposition 1.7, where one considers only homotopies of tangential operators within some given class of operators.

Namely, let \( \Sigma \) be a subspace of the space of all elliptic Hermitian symbols acting in the restriction of the bundle \( E \) to the boundary. In the space \( \text{Ell}(M, E, F) \) of elliptic operators on \( M \) acting between the spaces of sections of vector bundles \( E \) and \( F \), we consider the subspace \( \text{Op}(\Sigma) \) of operators such that the symbols of the corresponding tangential operators belong to \( \Sigma \):

\[
\text{Op}(\Sigma) = \left\{ D \in \text{Ell}(M, E, F) \mid \sigma(A) \in \Sigma \right\}.
\]

We assume that \( \Sigma \) is nondegenerate in the following sense: the natural mapping \( \text{Op}(\Sigma) \to \Sigma \) taking an elliptic operator on the manifold with boundary to the principal symbol of its tangential operator is surjective. In other words, every element of \( \Sigma \) can be realized as the symbol of the tangential operator for some elliptic operator on \( M \).

**Definition 1.8.** The class \( \text{Op}(\Sigma) \) admits an index decomposition if there exist two functionals

\[
f_{1,2} : \text{Op}(\Sigma) \longrightarrow \mathbb{R}
\]
such that
(1) the first functional is a homotopy invariant of the principal symbol of the operator, i.e., \( f_1(D) = f_1(\sigma(D)) \);
(2) the second functional is determined by the tangential operator, i.e., \( f_2(D) = f_2(A) \);
(3) for \( D \in \text{Op}(\Sigma) \), one has
\[
\text{ind}(D, \Pi_+(A)) = f_1(\sigma(D)) + f_2(A).
\]

We shall state a necessary and sufficient condition for the existence of a decomposition (1.7) in terms of the following condition on the class \( \Sigma \) of symbols.

**Definition 1.9.** The class \( \text{Op}(\Sigma) \) is said to be admissible if for an arbitrary periodic family \( \{A_t\}_{t \in S^1} \) of elliptic self-adjoint operators on \( \partial M \) one has
\[
\text{sf}\{A_t\}_{t \in S^1} = 0
\]
provided that \( \sigma(A_t) \in \Sigma \) for all \( \tau \).

**Theorem 1.10.** (The index decomposition theorem.) The class \( \text{Op}(\Sigma) \) admits an index decomposition if and only if it is admissible.

**Proof.** Necessity can be proved by analogy with Proposition 1.7. The proof of sufficiency can be found in [SSS99a]. \( \square \)

The admissibility condition can be verified effectively. Indeed, the principal symbol of an elliptic self-adjoint operator \( A \) on a manifold \( X \) (in our case, \( X = \partial M \)) defines an element
\[
[\sigma(A)] = [\text{Im} \Pi_+ \sigma(A)] \in K^0(S^*X)
\]
in the \( K \)-group, where \( \text{Im} \Pi_+ \sigma(A) \in \text{Vect}(S^*X) \) is the subbundle generated by the positive spectral subspaces of the principal symbol \( \sigma(A) \) on the cosphere bundle \( S^*X \) (with respect to some Riemannian metric). Then the spectral flow of a periodic family \( A = \{A_t\}_{t \in S^1} \) of elliptic self-adjoint operators can be computed by the Atiyah–Patodi–Singer formula [APS76b]

\[
(1.9) \quad \text{sf}\{A_t\}_{t \in S^1} = \langle \text{ch}[\sigma(A)] \cup \text{Td}(T^*X \otimes \mathbb{C}) , [S^*X \times S^1] \rangle.
\]

Here \( \text{ch}[\sigma(A)] \in H^{ev}(S^*X \times S^1) \) is the Chern character of the element
\[
[\sigma(A)] = [\text{Im} \Pi_+ \sigma(A)] \in K^0(S^*X \times S^1)
\]
defined by the principal symbol of the family, \( \text{Td} \) is the Todd class, and \( \langle , \rangle \) is the pairing between homology and cohomology.

**Remark 1.11.** The obstruction to the index decomposition given in Theorem 1.10 has the following cohomological interpretation. Note that the spectral flow of a periodic family of tangential operators with symbols in \( \Sigma \) defines a homomorphism
\[
\text{sf} : \pi_1(\Sigma) \longrightarrow \mathbb{Z},
\]
\[
\{\sigma(A_t)\}_{t \in S^1} \longmapsto \langle \text{ch}[\sigma(A)] \cup \text{Td}(T^*X \otimes \mathbb{C}) , [S^*X \times S^1] \rangle,
\]
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of the fundamental group into integers. It vanishes on commutators. Therefore, by the well-known isomorphism $H_1(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$, the spectral flow defines a cohomology class

$$[sf] \in H^1(\Sigma, \mathbb{R}).$$

Now the admissibility condition is equivalent to the vanishing of this cohomology class.

Theorem 1.10 shows that if the integral (1.9) is zero for an arbitrary periodic homotopy in some class $\Sigma$, then the corresponding class of spectral boundary value problems is admissible and the index admits a decomposition.

Now consider examples in which this condition is satisfied.

1.4. Examples. Consider the antipodal involution

$$\alpha : S^* \partial M \rightarrow S^* \partial M, \quad \alpha(x, \xi) = (x, -\xi)$$

of the cosphere bundle of the boundary $\partial M$. For a vector bundle $E \in \text{Vect}(M)$, we consider elliptic Hermitian symbols

$$a : \pi^* E|_{\partial M} \rightarrow \pi^* E|_{\partial M}, \quad \pi : S^* \partial M \rightarrow \partial M,$

invariant under the involution

$$a(x, -\xi) = a(x, \xi).$$

(1.10)

In this case, the contributions to (1.9) from antipodal points $\pm \xi$ in the integral over the cosphere $S^* \partial M$ corresponding to any $x \in \partial M$ cancel provided that $\partial M$ is odd-dimensional. Here we have used the fact that $\alpha$ preserves (or reverses) the orientation of $S^* \partial M$ depending on the parity of the dimension of $\partial M$. Thus, Eq. (1.9) is satisfied in the case of an even-dimensional manifold $M$ for operators whose tangential operators have even principal symbols (see Eq. (1.10)), and the index decomposition for the corresponding spectral boundary value problems is possible. In the following section, we obtain the index defect formula for this case.

2. Odd subspaces. For an odd-dimensional manifold $M$, the antipodal involution $\alpha$ preserves the orientation of $S^* \partial M$. In this case, one should consider odd symbols $a$ antiinvariant under $\alpha$:

$$a(x, -\xi) = -a(x, \xi).$$

A computation shows that the contributions of antipodal points to (1.9) cancel modulo a form lifted from the base of the cosphere bundle. Therefore, the integral is zero, and this class of boundary value problems has an index decomposition. This “odd” case is largely analogous to the even case. Some new phenomena appear in this case. We touch on this theory only briefly at the end of Subsection 2.2. For a detailed exposition, we refer the reader to SS00b.

3. Manifolds whose boundary is a covering. Suppose that the boundary is a smooth $n$-sheeted covering

$$\pi : \partial M \rightarrow X.$$
We consider the class of operators adapted to this structure in the sense that their tangential operators are lifted from the base of the covering. The lift is defined by the local diffeomorphism $\pi$.

Let us compute the spectral flow of a periodic family of tangential operators $\{A_{\tau}\}_{\tau \in \mathbb{S}^1}$. By assumption, this family is the lift of some family $\{\tilde{A}_{\tau}\}_{\tau \in \mathbb{S}^1}$ of elliptic self-adjoint operators from the base $X$. Since formula (1.9) is local, we obtain

$$\text{sf}\{A_{\tau}\}_{\tau \in \mathbb{S}^1} = n\text{sf}\{\tilde{A}_{\tau}\}_{\tau \in \mathbb{S}^1} \in n\mathbb{Z}.$$  

This is zero as a residue modulo $n$. Therefore, the assumption of Theorem 1.10 is satisfied modulo $n$, and the index of the corresponding boundary value problems as a residue modulo $n$ admits the desired decomposition. The index defect in this situation will be studied in Section 3.

2. Index defects for problems with parity conditions

In this section, we describe index defects for spectral boundary value problems in even subspaces (see Example 0.1 in the introduction). The methods of $K$-theory of operator algebras are very effective in this case. In the framework of this approach, the index defect appears naturally as a dimension type functional of subspaces determined by spectral projections.

In this section, we first define a dimension-type functional of subspaces and then prove the defect formula.

2.1. The dimension functional for even subspaces.

**Definition 2.1.** A pseudodifferential operator

$$P : C^\infty(X, E) \to C^\infty(X, E)$$

is said to be even if its principal symbol $\sigma(P)$ is invariant under the antipodal involution:

$$\sigma(P) = \alpha^*\sigma(P), \quad \alpha : S^*X \to S^*X, \quad \alpha(x, \xi) = (x, -\xi).$$

**Proposition 2.2.** Let $A$ be an even-order elliptic self-adjoint differential operator. Then the spectral projection $\Pi_+(A)$ is even.

**Proof.** The principal symbol of a differential operator of order $n$ has the property

$$\alpha^*\sigma(A) = (-1)^n \sigma(A).$$

Since

$$\Pi_+(A) = \frac{A + |A|}{2|A|},$$

(for an invertible $A$), we see that the principal symbol of $\Pi_+(A)$ is given by

$$\sigma(\Pi_+(A)) = \Pi_+(\sigma(A)).$$

---

7 We use a theorem similar to Theorem 1.10, where one considers the index modulo $n$ and the spectral flow modulo $n$; this result can be proved by the same method.
where \( \Pi_+ (\sigma (A)) \) is the orthogonal projection on the nonnegative spectral bundle of the symbol \( \sigma (A) \). (Here we use the following result due to Seeley \[\text{See67}\]: the principal symbol of a function of a self-adjoint operator is equal to the same function of the symbol.)

The last formula gives the desired equality

\[
\sigma (\Pi_+ (A)) = \sigma^* (\Pi_+ (A))
\]

for even-order operators. \[\Box\]

We denote the set of even pseudodifferential projections on \( X \) by \( \widehat{\text{Even}} (X) \), and the Grothendieck group of the semigroup of homotopy classes of even projections will be denoted by \( K(\widehat{\text{Even}} (X)) \). Let us give an alternative description of this group.

Consider the algebra \( \Psi_{ev} (X) \) of scalar even pseudodifferential operators of order zero. The Grothendieck group of the semigroup of homotopy classes of projections in matrix algebras over \( \Psi_{ev} (X) \) is denoted by \( K_0 (\Psi_{ev} (X)) \) and called the \textit{even} \( K \)-group of the algebra \( \Psi_{ev} (X) \) (e.g., see \[\text{Bla98}\]).

\textbf{Lemma 2.3.} One has

\[ K_0 (\Psi_{ev} (X)) \cong K(\widehat{\text{Even}} (X)). \]

\textbf{Proof.} The proof is immediate from the definitions of the Grothendieck group \( K(\widehat{\text{Even}} (X)) \) and the \( K \)-group of an algebra. \[\Box\]

Let us compute the \( K \)-group of \( \Psi_{ev} (X) \).

\textbf{Theorem 2.4.} If \( X \) is odd-dimensional, then there is an isomorphism

\[
(\mathbb{Z} \oplus K(X)) \otimes \mathbb{Z} \left[ \frac{1}{2} \right] \to K_0 (\Psi_{ev} (X)) \otimes \mathbb{Z} \left[ \frac{1}{2} \right].
\]

Here the mapping takes an integer \( k \in \mathbb{Z} \) to a projection of rank \( k \) and a vector bundle \( E \in \text{Vect}(X) \) to a projection defining \( E \) as a subbundle of some trivial bundle. By \( \mathbb{Z} [1/2] \) we denote the ring of dyadic rationals.

\textbf{Proof.} 1. Let \( \overline{\Psi}_{ev} (X) \) be the \( C^* \)-algebra obtained as closure of \( \Psi_{ev} (X) \) in the \( L^2 \)-norm. The closure does not change the \( K \)-group:

\[ K_0 (\Psi_{ev} (X)) \cong K_0 (\overline{\Psi}_{ev} (X)). \]

2. The algebra \( \overline{\Psi}_{ev} (X) \) contains the ideal \( K \) of compact operators, and one has the exact sequence of algebras

\[
0 \to K \to \overline{\Psi}_{ev} (X) \to C(P^* X) \to 0.
\]

Here the projection takes each operator to its principal symbol. We also treat even symbols as continuous functions on the projectivization \( P^* X = S^* X / \mathbb{Z}_2 \) of the cosphere bundle. The sequence is well defined by virtue of the well-known norm estimates

\[
\inf_{K \in K} \| D + K \|_{L^2 (X) \to L^2 (X)} = \sup_{(x, \xi) \in S^* X} | \sigma (D)(x, \xi) |.
\]
Furthermore, the sequence (2.3) induces the sequence
\[ \rightarrow K_1(C(P^*X)) \xrightarrow{\text{ind}} K_0(K) \xrightarrow{} K_0(\Psi_{ev}(X)) \xrightarrow{\text{smbl}} K_0(C(P^*X)) \rightarrow K_1(K) \rightarrow \]
of K-groups, which is in our case reduced to
\[ (2.4) \quad K^1(P^*X) \xrightarrow{\text{ind}} \mathbb{Z} \xrightarrow{} K_0(\Psi_{ev}(X)) \xrightarrow{\text{smbl}} K_0(C(P^*X)) \rightarrow K^1(K) \rightarrow 0. \]
Here we have substituted the well-known computations \( K_0(K) = \mathbb{Z} \) and \( K_1(K) = 0 \) and replaced the K-groups of the function algebra \( C(P^*X) \) by the topological K-groups of \( P^*X \).

Let us describe the mappings in (2.4). The first mapping takes an elliptic even symbol to the index of the corresponding operator. The second mapping takes a positive integer \( k \) to a projection of rank \( k \). (Such a projection is even, since its symbol is zero.) Finally, the mapping smbl takes a pseudodifferential projection \( P \) to the range \( \text{Im} \sigma(P) \in \text{Vect}(P^*X) \) of its principal symbol treated as a vector bundle over \( P^*X \).

For the existence of a functional \( d \) making the diagram
\[ \begin{array}{ccc}
K_0(K) = \mathbb{Z} & \xrightarrow{d} & \mathbb{R} \\
\downarrow & & \downarrow \\
K_0(\Psi_{ev}(X)) & \xrightarrow{d} & \mathbb{R}
\end{array} \]
commute, it is necessary that the vertical arrow be a monomorphism or, equivalently, the index mapping in (2.4) be zero. This condition is satisfied if \( X \) is odd-dimensional. Indeed, it is well known (e.g., see [Pal65]) that the index of operators with even principal symbol is zero on such manifolds. Therefore, the sequence can finally be rewritten as
\[ (2.6) \quad 0 \rightarrow \mathbb{Z} \rightarrow K_0(\Psi_{ev}(X)) \rightarrow K_0(C(P^*X)) \rightarrow 0. \]

3. Let us slightly simplify this sequence further. To this end, we note that the natural projection \( p : P^*X \rightarrow X \) for an odd-dimensional \( X \) induces an isomorphism in K-theory modulo 2-torsion. More precisely, the following is valid.

**Proposition 2.5.** [Gil89b] The projection \( P^*X \rightarrow X \) induces an isomorphism
\[ p^* : K^*(X) \otimes \mathbb{Z} \left[ \frac{1}{2} \right] \rightarrow K^*(P^*X) \otimes \mathbb{Z} \left[ \frac{1}{2} \right]. \]

**Sketch of Proof.** One first verifies the statement over a point \( x \in X \): here \( K^1(\mathbb{R}^{2n}) = 0 \) and \( K^0(\mathbb{R}^{2n}) = \mathbb{Z}_{2^n} \), which shows that
\[ p^* : K^* \{ \{ x \} \} \otimes \mathbb{Z} \left[ \frac{1}{2} \right] \rightarrow K^* (P^*X) \otimes \mathbb{Z} \left[ \frac{1}{2} \right] \]
is an isomorphism. Now the isomorphism for the entire space can be proved by the Mayer–Vietoris principle ([BT82]). \( \square \)
Taking a tensor product of the sequence (2.4) by the ring of dyadic rationals (which does not violate the exactness), we obtain the exact sequence

\[ 0 \rightarrow \mathbb{Z} \left\lfloor \frac{1}{2} \right\rfloor \rightarrow K_0(\mathbb{F}_{ev}(X)) \otimes \mathbb{Z} \left\lfloor \frac{1}{2} \right\rfloor \rightarrow K(X) \otimes \mathbb{Z} \left\lfloor \frac{1}{2} \right\rfloor \rightarrow 0. \]

4. This exact sequence has a splitting mapping

\[ K^0(X) \otimes \mathbb{Z} \left\lfloor \frac{1}{2} \right\rfloor \rightarrow K(\mathbb{F}_{ev}(X)) \otimes \mathbb{Z} \left\lfloor \frac{1}{2} \right\rfloor \]

taking a vector bundle to a projection on the space of its sections in \( C^\infty(X, \mathbb{C}^N) \) for sufficiently large \( N \).

5. The splitting obviously gives the desired isomorphism (2.2). \( \square \)

The first component of the isomorphism (2.2) will be called the \textit{dimension functional} for even pseudodifferential projections. It is useful to restate Theorem 2.4 in the following way.

**Theorem 2.6.** \([SS99]\) (A theorem on the dimension functional.) \textit{On an odd-dimensional manifold} \( X \), \textit{there exists a unique group homomorphism}

\[ d : K(\mathbb{F}_{ev}(X)) \rightarrow \mathbb{Z} \left\lfloor \frac{1}{2} \right\rfloor \]

\textit{with the property}

\[ d(P) = \text{rank} P \]

\textit{for a finite rank projection} \( P \) \textit{under the normalization}

\[ d(P_F) = 0, \]

\textit{for all projections} \( P_F : C^\infty(X, E) \rightarrow C^\infty(X, E) \) \textit{on the space of sections of a subbundle} \( F \subset E \). \textit{Moreover, for an arbitrary even projection} \( P \) \textit{and for a sufficiently large} \( N \) \textit{the projection} \( 2^N P \) \textit{is homotopic to the direct sum of a projection on the space of sections of a subbundle and a finite rank projection.}

**2.2. The index defect formula.** Let us apply the dimension functional \( d \) of even projections to the theory of spectral boundary value problems.  

1. **The dimension \( d \) as an index defect.** We shall consider elliptic operators \( D \) with even tangential operator \( A \). In this case, the spectral projection \( \Pi_+(A) \) is even as well (see Proposition 2.2).

**Remark 2.7.** Clearly, this condition cannot be satisfied for first-order differential operators. Thus, in this subsection we consider the more general class of operators that are pseudodifferential far from the boundary and have the form (\[\text{[1]}\]) with a first-order pseudodifferential operator \( A \) on \( \partial M \) near the boundary. For this class of operators, the spectral boundary value problems are also well defined and have the Fredholm property. Operators of this form are considered in detail in \([Hör83]\).
Lemma 2.8. The sum
\[ \text{ind}(D, \Pi_+(A)) + d(\Pi_+(A)) \]
is a homotopy invariant of the operator \( D \).

Proof. Consider a homotopy \( D_\tau, \tau \in [0, 1] \), and the corresponding homotopy \( A_\tau \) of tangential operators. By the homotopy invariance of the dimension functional and property (2.7), it follows that the variation of the dimension is equal to the spectral flow
\[ d(\Pi_+(A_1)) - d(\Pi_+(A_0)) = \text{sf}\{A_\tau\}_{\tau \in [0,1]} . \]
(The equality can be obtained by analogy with the proof of the theorem on the spectral flow with the use of a partition of the interval \([0, 1]\) and some choice of weights on the intervals of the partition.) A similar formula holds for the variation of the index:
\[ \text{ind}(D_1, \Pi_+(A_1)) - \text{ind}(D_0, \Pi_+(A_0)) = -\text{sf}\{A_\tau\}_{\tau \in [0,1]} . \]
Combining the two expressions, we obtain the desired homotopy invariance of the sum indicated in the lemma. □

This homotopy invariant will be denoted by
\[ \text{inv} D \overset{\text{def}}{=} \text{ind}(D, \Pi_+(A)) + d(\Pi_+(A)). \]
Let us give a topological formula for this analytic invariant. This can be done by generalizing the Atiyah–Singer topological index to the case of spectral boundary value problems in even subspaces.

2. The topological index. It turns out that the principal symbol \( \sigma(D) \) has a natural extension to the double
\[ 2M = M \cup \partial M \]
of the manifold \( M \). The double is obtained by gluing two copies of \( M \) along the common boundary.

To construct the desired extension, we take the symbol \( \sigma(D) \) on the first copy of \( M \) and the symbol \( \alpha^* \sigma(D) \) on the second copy. Here
\[ \alpha : S^* M \to S^* M \]
is the antipodal involution. Then in a neighborhood of the boundary the symbols \( \sigma(D) \) and \( \alpha^* \sigma(D) \) have the form
\[ i\tau + a(x, \xi) \quad \text{and} \quad -i\tau + a(x, \xi), \]
respectively. They are taken into one another by the coordinate transformation
\[ x \to x, \quad t \to -t \]
in a neighborhood of the boundary. Therefore, taken together, they define an elliptic symbol \( \sigma(D) \cup \alpha^* \sigma(D) \) on the double of \( M \). The difference element of this symbol will be denoted by
\[ [\sigma(D) \cup \alpha^* \sigma(D)] \in K_c(T^* 2M). \]
Then the topological index of $D$ is defined as half the Atiyah–Singer topological index of this element on the double:

$$\text{ind}_t[\sigma(D)] \stackrel{\text{def}}{=} \frac{1}{2} \text{ind}_t[\sigma(D) \cup \alpha^* \sigma(D)].$$

3. The index defect formula.

**Theorem 2.9.** [SS99] Let $D$ be an elliptic operator on an even-dimensional manifold with even tangential operator $A$. Then

$$\text{inv} D = \text{ind}_t[\sigma(D)].$$

**Proof.** For sufficiently large $N$, Theorem 2.6 in the previous subsection gives a homotopy $\pi_\tau, \tau \in [0, 1]$, of the direct sum of $2^N$ copies of the symbol $\sigma(\Pi_+(A))$ to a projection on some subbundle $E_0 \subset 2^N E|_{\partial M}$. One can lift this homotopy of projections to a homotopy $\sigma(D_\tau)$ of elliptic symbols with the properties

$$\sigma(D_0) = 2^N \sigma(D), \quad \sigma(\Pi_+(A_\tau)) = \pi_\tau.$$ 

By the homotopy invariance of both sides of Eq. (2.8), it suffices to prove the equality only for the symbol $\sigma(D_1)$ obtained at the end of the homotopy. This symbol in a neighborhood of the boundary depends only on the absolute value of the covector $\xi$. Thus, one can consider an elliptic operator $D_1$ with this symbol and the corresponding spectral boundary value problem $(D_1, \Pi_+)$ such that the spectral projection $\Pi_+$ is actually a projection on the space of sections of the subbundle $E_0$. The spectral boundary value problem in this case is a classical boundary value problem; the index defect (i.e. the dimension functional) is zero, and Eq. (2.8) follows from the Atiyah–Bott formula. \[\square\]

**Remark 2.10.** (On the dimension functional of odd subspaces.) A similar functional was constructed in [SS00b] on the space of “odd projections” with principal symbols satisfying

$$\alpha^* \sigma(P) = 1 - \sigma(P).$$

Such projections arise as spectral projections of odd-order differential operators (cf. Proposition 2.2). However, the methods of $K$-theory cannot be applied directly to odd projections, since odd symbols do not form an algebra. Moreover, the geometry of odd projections differs from the geometry of even ones: for example, an odd projection on a manifold of dimension $2k$ can act only in the space of sections of a vector bundle whose dimension is a multiple of $2k - 1$ (see [SS00b]).

In the cited paper, we constructed a dimension functional and proved the defect formula. Let us only mention that the topological index in the odd case can be obtained if on the double $2M$ one considers the symbol $\sigma(D) \cup \alpha^* \sigma(D)^{-1}$.

**Remark 2.11.** One can show [SS99b] that the value $d(\Pi_+(A))$ of the dimension functional and the element $[\sigma(D) \cup \alpha^* \sigma(D)]$ form a complete system of stable homotopy invariants of the spectral boundary value problem $(D, \Pi_+(A))$, i.e., classify these problems modulo stable homotopy.
2.3. The dimension functional and the $\eta$-invariant. The dimension functional for even pseudodifferential projections was constructed in Theorem 2.6. More precisely, the theorem claims the existence of the functional. In this subsection, we address the question of an explicit analytic expression for this functional. It turns out that such a description can be given in terms of the spectral Atiyah–Patodi–Singer $\eta$-invariant. The reader can find some details about the $\eta$-invariant in Appendix A.

Gilkey [Gil89a] noted that the $\eta$-invariant is rigid within the class of differential operators satisfying the parity condition

\begin{equation}
\text{ord} A + \dim X \equiv 1 \pmod{2},
\end{equation}

which relates the order of the operator to the dimension of the manifold. Rigidity is understood in the sense that the fractional part of the $\eta$-invariant is not only spectrally invariant but also homotopy invariant.

Condition (2.9) coincides with the conditions under which there exists a functional $d$ of the corresponding spectral projections. (Recall that in the previous subsections this functional was considered for even projections on odd-dimensional manifolds and odd projections on even-dimensional manifolds.) This is not a mere coincidence. In fact, the $\eta$-invariant of an arbitrary elliptic self-adjoint operator $A$ coincides with the value of the dimension functional on the spectral projection $\Pi_+(A)$.

**Theorem 2.12.** [SS99, SS00b] If an elliptic self-adjoint differential operator satisfies the parity condition (2.9), then

\[ \eta (A) = d (\Pi_+(A)). \]

**Proof.** For simplicity, we consider only operators of even order $2l$.

1. It was shown in Theorem 2.6 that the dimension $d$ of even projections can be defined as the unique homomorphism

\[ d : K (\hat{\textrm{Even}} (X)) \to \mathbb{R} \]

with the following two properties: (a) $d (P) = \text{rank} P$ for a finite rank projection $P$, (b) $d (P_F) = 0$ for projections on the spaces of sections of vector bundles $F \in \text{Vect}(X)$. Therefore, to prove the theorem it suffices to show that the $\eta$-invariant defines a similar homomorphism and enjoys the same properties.

2. Thus, we should consider the $\eta$-invariant for general even pseudodifferential operators. Unfortunately, the rigidity of the $\eta$-invariant is lost in this class, since lower-order terms of the operator contribute to the $\eta$-invariant. However, the invariance still holds (see Appendix A) in the class of $\mathbb{R}^*$-invariant operators, i.e., operators with complete symbol having an asymptotic expansion

\[ \text{smbl} A \sim a_{2l} (x, \xi) + a_{2l-1} (x, \xi) + a_{2l-2} (x, \xi) + \ldots, \]

where each homogeneous term $a_k (x, \xi)$ of order $k$ has the following parity with respect to the momentum variables $\xi$:

\[ a_k (x, -\xi) = (-1)^k a_k (x, \xi). \]
For $\mathbb{R}_+$-invariant elliptic self-adjoint operators, the spectral projection $\Pi_+ (A)$ is either even or odd according to the parity of the order of the operator.

Let us verify property (a) for the $\eta$-invariant (as a homotopy invariant of the spectral projection). For a nonnegative Laplacian $\Delta$ with $k$-dimensional kernel, we obtain

$$\dim \text{Im} \Pi_+ (-\Delta^i) = k.$$ 

On the other hand, the $\eta$-invariant can be expressed in terms of the $\zeta$-invariant as

$$\eta (-\Delta^i) = k - \zeta (\Delta^i).$$

But the $\zeta$-invariant of the differential operator $\Delta^i$ is zero in odd dimensions (see the remark following Theorem 4.7 in Appendix A). This proves property (a):

$$\eta (-\Delta^i) = \dim \text{Im} \Pi_+ (-\Delta^i) = d(\Pi_+ (-\Delta^i)).$$

Similarly, one obtains (b):

$$\eta (\Delta^i) = 0, \quad \text{Im} \Pi_+ (\Delta^i) = C^\infty (M, E).$$

By the characterization of the dimension functional, this proves that the $\eta$-invariant of an operator $A$ coincides with the dimension $d$ of the corresponding spectral projection. □

3. Index defects on twisted $\mathbb{Z}_n$-manifolds

In this section, we consider another geometric situation in which index defects naturally arise. This situation comes from elliptic theory on manifolds with singularities. More precisely, we consider so-called twisted $\mathbb{Z}_n$-manifolds. Such manifolds look like a book with $n$ sheets near the singularity; the singular set is the edge where the sheets meet.

The index theorem was stated and proved by Freed and Melrose [FM92] for the special case in which a neighborhood of the singular set consists of $n$ sheets globally. Note that in this case it is natural to view the index as a residue modulo $n$, and the index theorem computes this index-residue.

We consider general twisted $\mathbb{Z}_n$-manifolds. In this case, the index-residue is no longer a homotopy invariant and an index defect arises. Intuitively, the proof of the index defect formula follows the scheme explained in Subsection 0.6, so here we give only the main steps of the proof. We first define a special element in the $K$-theory of a $\mathbb{Z}_n$-manifold in Subsection 3.4. Then the topological index is defined as the pairing with this element. The last subsection contains an application of the defect formula.

3.1. Twisted $\mathbb{Z}_n$-manifolds and elliptic operators.

1. Twisted $\mathbb{Z}_n$-manifolds

Definition 3.1. A twisted $\mathbb{Z}_n$-manifold, where $n$ is a positive integer, is a smooth compact manifold $M$ with boundary $\partial M$ equipped with the structure of the total space of an $n$-sheeted covering

$$\pi : \partial M \to X$$

(3.1)
over a smooth base $X$ (see Fig. 3).

Geometrically, a twisted $\mathbb{Z}_n$-manifold $(M, \pi)$ naturally defines the singular space

$$\mathcal{M}^\pi = M / \sim,$$

obtained by the identification of points in each fiber of the covering (see Fig. 1); the corresponding equivalence relation $\sim$ is defined as

$$x \sim y \iff x = y \text{ or } \{x, y \in \partial M \text{ and } \pi(x) = \pi(y)\}.$$

Sullivan [Sul70] introduced the notion of a $\mathbb{Z}_n$-manifold. These manifolds correspond to the structure of a trivial covering $\pi$. One of the motivations showing the interest in such manifolds is that (in the orientable case) the manifold $\mathcal{M}^\pi$ has a fundamental class in homology with coefficients in $\mathbb{Z}_n$:

$$[\mathcal{M}^\pi] \in H_m(\mathcal{M}^\pi, \mathbb{Z}_n), \quad m = \dim M.$$  

These manifolds with singularities were also used to give a geometric realization of bordism theory with coefficients in the group $\mathbb{Z}_n$. Further developments in this direction can be found, e.g., in [Bot92].

For brevity, we frequently omit the word “twisted” in what follows.

We point out that while most technical constructions of the theory deal with the smooth model $(M, \pi)$, the most important results, e.g., the defect formula in Subsection 3.3, are stated in terms of the singular space $\mathcal{M}^\pi$ itself.

2. Natural mappings associated with coverings. Let us recall some natural mappings induced by the covering (3.3). First, one has the direct image mapping

$$\pi! : \text{Vect}(\partial M) \longrightarrow \text{Vect}(X)$$
taking a vector bundle $E \in \text{Vect}(\partial M)$ to the bundle
\[ \pi_! E \in \text{Vect}(X), \quad (\pi_! E)_x = C^{\infty}(\pi^{-1}(x), E), \quad x \in X. \]
This definition leads to a natural isomorphism
\[ \beta_E : C^{\infty}(\partial M, E) \xrightarrow{\cong} C^{\infty}(X, \pi_! E) \]
of section spaces on the total space and the base of the covering. For example, scalar functions on the total space correspond to sections of the bundle $\pi_! 1 \in \text{Vect}(X)$ on the base. (Here 1 stands for the trivial line bundle.)

This isomorphism enables one to transfer operators acting on $\partial M$ to $X$ and vice versa. More precisely, for an operator $P : C^{\infty}(\partial M, E) \longrightarrow C^{\infty}(\partial M, E)$ on $\partial M$, by $\pi_! P$ we denote its direct image given by the formula
\[ \pi_! P = \beta_E P \beta^{-1}_E : C^{\infty}(X, \pi_! E) \longrightarrow C^{\infty}(X, \pi_! E). \]
One also has the inverse image
\[ \pi_! P' = \beta^{-1}_E P' \beta_E : C^{\infty}(\partial M, E) \longrightarrow C^{\infty}(\partial M, E) \]
of operators $P'$ on $X$.

On twisted $\mathbb{Z}_n$-manifolds, we consider the following class of operators adapted to the structure of the singularity.

3. The class of operators. For a pair $(M, \pi)$, we consider elliptic differential operators
\[ D : C^{\infty}(M, E) \longrightarrow C^{\infty}(M, F), \]
which in a neighborhood of the boundary are lifted from the base of the covering. More precisely, we suppose that they satisfy the following assumption.

Assumption 1. The restrictions of $E$ and $F$ to the boundary are lifted from the base:
\[ E|_{\partial M} \simeq \pi^* E_0, \quad F|_{\partial M} \simeq \pi^* F_0, \quad E_0, F_0 \in \text{Vect}(X); \]
moreover, the corresponding isomorphisms are given. For some operator
\[ D_0 : C^{\infty}(X \times [0, 1], E_0) \longrightarrow C^{\infty}(X \times [0, 1], F_0) \]
on the cylinder with base $X$, the direct image $(\pi \times 1)_* D$ in a neighborhood of the boundary can be inserted in the commutative diagram
\[ C^{\infty}(X \times [0, 1], (\pi \times 1)_* E) \xrightarrow{(\pi \times 1)_* D} C^{\infty}(X \times [0, 1], (\pi \times 1)_* F) \]
\[ \simeq C^{\infty}(X \times [0, 1], E_0 \otimes \pi_1) \xrightarrow{D_0 \otimes \pi_1} C^{\infty}(X \times [0, 1], F_0 \otimes \pi_1). \]
Here we suppose that we have chosen a diffeomorphism
\[ U_{\partial M} \simeq [0, 1) \times \partial M \]
of a collar neighborhood of the boundary and extended the projection \( \pi \) to it as \( \pi \times 1 \). Then \( D_0 \otimes 1_{\pi1} \) is the operator \( D_0 \) with coefficients in the flat bundle \( \pi1 \) (e.g., see [APS76b]).

We also assume for simplicity that the operator \( D_0 \) in a neighborhood of the base of the cylinder \( X \times [0, 1] \) has the form (1.1), i.e.,

\[
D_0|_{X \times [0, \varepsilon)} = \Gamma \left( \frac{\partial}{\partial t} + A_0 \right)
\]

where \( \Gamma \) is some vector bundle isomorphism and the tangential operator \( A_0 \) is a first-order elliptic self-adjoint operator on \( X \).

It follows from this assumption that in a neighborhood of the boundary the operator \( D \) has the form

\[
\frac{\partial}{\partial t} + \pi^! (A_0 \otimes 1_{\pi1})
\]

(up to a vector bundle isomorphism); i.e., the tangential operator \( A \) is equal to \( \pi^! (A_0 \otimes 1_{\pi1}) \).

We note that the classical operators satisfy Assumption 1 in an appropriate geometric setting. For example, the Hirzebruch operator on an oriented even-dimensional Riemannian manifold \( M \) satisfies the assumption if (1) in a collar neighborhood of the boundary we take a metric pulled back from the cylinder \( X \times [0, 1] \); (2) the covering \( \pi : \partial M \to X \) is oriented. Similar statements hold also for the Dirac and Todd operators; we leave them to the reader.

4. The difference element. Assumption 1 is closely related to the above-discussed manifolds with singularities. Indeed, note that the total space \( T^* M \) of the cotangent bundle is a (noncompact) \( \mathbb{Z}_n \)-manifold. It follows that the principal symbol of an elliptic operator \( D \) defines a difference element

\[
[\sigma(D)] \in K_c(T^*M^\pi)
\]

in the \( K \)-group, since the commutative diagram (3.4) shows that the restriction of \( \sigma(D) \) to the boundary is isomorphic to a symbol lifted from the base of the covering \( T^* M|_{\partial M} \to T^* X \times \mathbb{R} \).

3.2. The Freed–Melrose index theorem modulo \( n \). This subsection deals with the index theory on \( \mathbb{Z}_n \)-manifolds corresponding to trivial coverings \( \pi \); i.e., the boundary of the corresponding smooth manifold \( M \) is a disjoint union of \( n \) copies of the base of the covering.

1. The index modulo \( n \). It turns out that in this case operators satisfying Assumption 1 have a nontrivial homotopy invariant. Namely, let

\[
\text{mod } n \text{-ind } D \in \mathbb{Z}_n
\]

be the index of the spectral boundary value problem \( (D, \Pi_+(A)) \) treated as a residue modulo \( n \).

Proposition 3.2. The index-residue mod \( n \)-ind \( D \) is a homotopy invariant of the operator \( D \).
Proof. Consider a homotopy \( \{ D_t \}_{t \in [0,1]} \). By the spectral flow theorem,
\[
\text{ind}(D_1, \Pi_+(A_1)) - \text{ind}(D_0, \Pi_+(A_0)) = -\text{sf} \{ A_t \}_{t \in [0,1]}.
\]
On the other hand, by assumption, the family \( A_t \) of tangential operators is the direct sum of \( n \) copies of the family \( A_{0,t} \) of tangential operators on \( X \). Therefore,
\[
\text{sf} \{ A_t \}_{t \in [0,1]} = n \text{sf} \{ A_{0,t} \}_{t \in [0,1]} \equiv 0 \pmod{n}.
\]
This shows that the spectral flow vanishes as a residue modulo \( n \) and proves the homotopy invariance of the index-residue. \( \square \)

This homotopy invariant index residue was computed in terms of the principal symbol by Freed and Melrose. In Subsection 3.5, we obtain a more general formula, and now we briefly recall the Freed–Melrose formula, just to make the exposition complete.

2. The Freed–Melrose theorem. Consider the category of \( \mathbb{Z}_n \)-manifolds with morphisms given by embeddings that take the boundary to the boundary and the fibers of the coverings to the fibers. The direct image mapping in \( K \)-theory extends to this category. More precisely, for an embedding \( f \) of a pair \((M, \pi)\) in \((N, \pi_N)\) there is a direct image mapping
\[
f! : K_c(T^*M) \rightarrow K_c(T^*M_N^*).
\]
On the other hand, there is a universal space in which one can embed an arbitrary \( \mathbb{Z}_n \)-manifold corresponding to a trivial covering. The universal space can be defined as follows. From \( \mathbb{R}^L \) we cut away the union of \( n \) disjoint discs. We denote the resulting manifold with boundary by \( M_n \). It can be viewed as a \( \mathbb{Z}_n \)-manifold, since its boundary consists of \( n \) diffeomorphic spheres. (The diffeomorphisms are given by translations.) One can readily compute the \( K \)-group of the cotangent bundle of this space:
\[
K_c(T^*M_n^*) \simeq \mathbb{Z}_n.
\]
Freed and Melrose proved the following index theorem.

Theorem 3.3. [FM92] One has
\[
\text{mod } n \cdot \text{ind} D = f! [\sigma(D)],
\]
where the direct image
\[
f! : K_c(T^*M) \rightarrow K_c(T^*M_k^*) \simeq \mathbb{Z}_n
\]
is induced by an embedding
\[
f : M \rightarrow M_n.
\]

The proof models the \( K \)-theoretic proof of the Atiyah–Singer index theorem based on embeddings. The cornerstone of the proof is the statement that the
analytic index is preserved under the direct image mapping, that is, the diagram

\[
\begin{array}{ccc}
K_c(T^*M) & \xrightarrow{f_*} & K_c(T^*N) \\
\mod n\text{-ind} & & \mod n\text{-ind} \\
\downarrow & & \downarrow \\
\mathbb{Z}_n & & \mathbb{Z}_n
\end{array}
\]

commutes for an embedding of \( M \) in \( N \).

### 3.3. The index defect problem on twisted \( \mathbb{Z}_n \)-manifolds.

In contrast to the case of \( \mathbb{Z}_n \)-manifolds corresponding to trivial coverings, considered in the previous subsection, the index modulo \( n \) is no longer a homotopy invariant if the covering is nontrivial. By way of illustration, consider the following simple example.

On the cylinder \( S^1 \times [T_1, T_2] \) with coordinates \((\varphi, t)\), consider the scalar elliptic operator

\[
D = \frac{\partial}{\partial t} + (-i \frac{\partial}{\partial \varphi} + t).
\]

Clearly, the tangential operator is lifted from the base of any of the coverings \( \pi_n : S^1 \to S^1, \varphi \mapsto n\varphi \).

On the other hand, if \( T_1 \) or \( T_2 \) passes through zero, then the index of the corresponding spectral boundary value problem jumps by 1 and is not constant as a residue modulo \( n \) for any \( n \neq 1 \).

#### 1. The index defect.

The example shows that the index-residue is not homotopy invariant and cannot be computed topologically. The following theorem gives the index defect in this situation.

**Theorem 3.4.** [SS01] On a twisted \( \mathbb{Z}_n \)-manifold, the difference

\[
\eta(A) - n\eta(A_0) \in \mathbb{R}/n\mathbb{Z}
\]

is an index defect. Here \( n \) is the number of sheets of the covering, and \( \eta(A) \) and \( \eta(A_0) \) are the spectral Atiyah–Patodi–Singer \( \eta \)-invariants of the tangential operators \( A \) and \( A_0 \), respectively. In other words, the sum

\[
(3.6) \mod n\text{-ind} (D, \Pi_+(A)) + \eta(A) - n\eta(A_0) \in \mathbb{R}/n\mathbb{Z},
\]

is a homotopy invariant of \( D \).

**Proof.** Consider a homotopy \( D_t \) of operators. The corresponding homotopy of tangential operators will be denoted by \( A_t \) and the homotopy of operators on the base \( X \) by \( A_{0,t} \).

As \( t \) varies, the index \( \text{ind}(D_t, \Pi_+(A_t)) \) can jump. The geometric Atiyah–Patodi–Singer index formula (see Appendix A) shows that the sum

\[
\text{ind}(D_t, \Pi_+(A_t)) + \eta(A_t)
\]

is a smooth function of the parameter. Furthermore, the derivative of the sum with respect to \( t \) is a local invariant; namely, it is equal to the integral over the manifold of an expression determined by the complete symbol of the family \( A_t \).
of tangential operators. On the other hand, the complete symbols of $A_t$ and $A_{0,t}$ locally coincide by Assumption 1. Therefore, the jumps in the sum

$$\text{ind} (D_t, \Pi_+(A_t)) + \eta(A_t) - n\eta(A_{0,t})$$

of three terms come only from the third term. These jumps, however, are multiples of $n$. Hence, reducing (3.7) modulo $n$, we obtain a quantity independent of $t$. This proves the theorem. □

2. A relation to known invariants. We denote the homotopy invariant constructed in the theorem by

$$\text{inv} D \overset{\text{def}}{=} \text{mod } n - \text{ind} (D, \Pi_+(A)) + \eta(A) - n\eta(A_0) \in \mathbb{R}/n\mathbb{Z}.$$ 

In some special cases, it can be reduced to the following invariants.

(1) For a trivial covering, we have an isomorphism

$$A \simeq A_0 \oplus A_0 \oplus \ldots \oplus A_0 \mod n \text{ copies.}$$

Hence $\eta(A) = n\eta(A_0)$, and $\text{inv} D$ is thereby reduced to the Freed–Melrose \text{mod } n\text{-index} \cite{FM92} \mod n\text{-ind}(D, \Pi_+) \in \mathbb{Z}/n \subset \mathbb{R}/n\mathbb{Z}$.

(2) On the other hand, if we consider only the fractional part $\{ \}$ of (3.6), then the index vanishes and we obtain the Atiyah–Patodi–Singer relative $\eta$-invariant \cite{APS76a, APS76b} \{\eta(A_0 \otimes 1_{\pi!}) - n\eta(A_0)\} \in \mathbb{R}/\mathbb{Z}$ of the operator $A_0$ with coefficients in the flat bundle $\pi! \in \text{Vect}(X)$.

3. The invariant $\text{inv} D$ as an obstruction. Let us give an interpretation of the invariant $\text{inv} D$ as an obstruction.

Suppose that the manifold $M$ itself is the total space of a covering $\tilde{\pi}$ over some base $Y$ and the induced covering at the boundary coincides with $\pi$:

$$\begin{array}{ccc}
\partial M & \subset & M \\
\pi \downarrow & & \downarrow \tilde{\pi} \\
X & \subset & Y.
\end{array}$$

**Proposition 3.5.** If an operator

$$D : C^\infty(M, E) \rightarrow C^\infty(M, F)$$

satisfying Assumption 1 is the lift of some operator $D_0$ from $Y$, then

$$\text{inv} D = 0.$$
Proof. By the Atiyah–Patodi–Singer formula, the sum
\[ \text{ind} (D, \Pi_{+} (A)) + \eta (A) \]
can be expressed as the integral over the manifold of an expression depending on the complete symbol of \( D \). On the other hand, the operators \( D \) and \( D_0 \) on the total space and on the base locally coincide. Thus, we obtain
\[ \text{ind} (D, \Pi_{+} (A)) + \eta (A) = n \left( \text{ind} (D_0, \Pi_{+} (A_0)) + \eta (A_0) \right). \]
Transposing the term \( n \eta (A_0) \) to the left-hand side, we obtain \( \text{inv} D = 0 \).

Remark 3.6. We point out that an arbitrary operator \( D \) is not induced by some elliptic operator \( D_0 \) on the base in general. Proposition 3.5 gives a necessary condition for the existence of such operators: if \( D_0 \) exists, then \( \text{inv} D = 0 \).

The principal symbol of \( D \) defines an element in the \( K \)-group of the space \( T^* M \). To associate some topological index with this element, in the following subsection we use Poincaré duality on manifolds with singularities of the above-described type. For the reader’s convenience, the survey contains Appendices B and C, where we discuss Poincaré duality on smooth manifolds and on singular \( \mathbb{Z}_n \)-manifolds.

3.4. The element of \( K \)-theory defined by a manifold whose boundary is a covering. In this subsection, we show that a pair \((M, \pi)\) consisting of a compact manifold \( M \) and a covering \( \pi : \partial M \to X \) defines a special element in the \( K \)-group
\[ K_0 (\mathcal{A}_M, \pi, \mathbb{Q}/n\mathbb{Z}) \]
with coefficients in the group \( \mathbb{Q}/n\mathbb{Z} \). Here \( \mathcal{A}_M, \pi \) is the \( C^* \)-algebra of the twisted \( \mathbb{Z}_n \)-manifold. This algebra is defined in Appendix C. To define our element, we first need a geometric description of elements of the \( K \)-group. The coefficient group \( \mathbb{Q}/n\mathbb{Z} \) looks slightly complicated, and hence we first consider a geometric realization of the \( K \)-groups with finite coefficient group \( \mathbb{Z}_n \). After this, we return to the complicated coefficients.

1. \( K \)-theory with coefficients \( \mathbb{Z}_n \). There are several equivalent approaches to the definition of these \( K \)-groups (e.g., see [Bla98]). We use the definition coming from topological \( K \)-theory. Namely, the group \( \mathbb{Z}_n \) has the corresponding Moore space. It is the 2-dimensional complex \( M_n \) obtained from the unit disk by the identification of points on its boundary under the natural action of the group \( \mathbb{Z}_n \):
\[ M_n = \left\{ z \in \mathbb{C} \mid |z| \leq 1 \right\} / \left\{ e^{i\varphi} \sim e^{i\varphi + 2\pi k/n} \right\}. \]
In particular, for \( n = 2 \) this space is the real projective plane \( \mathbb{R}P^2 \).

The \( K \)-group with coefficients \( \mathbb{Z}_n \) for an algebra \( \mathcal{A} \) can be defined as
\[ K_0 (\mathcal{A}, \mathbb{Z}_n) = K_0 (\widetilde{C}_0 (M_n, \mathcal{A})). \]
where \( \tilde{C}_0(M_n, \mathcal{A}) \) is the algebra of \( \mathcal{A} \)-valued continuous functions on the Moore space vanishing at a marked point \( pt \in M_n \). This definition is similar to the definition of the topological \( K \)-groups with coefficients of a space \( Y \):

\[
K^0(Y, \mathbb{Z}_n) = K^0(Y \times M_n, Y \times pt).
\]

**Proposition 3.7.** The group \( K_0(\tilde{C}_0(M_n, \mathcal{A}_{M,\pi})) \) is isomorphic to the group of stable homotopy classes of triples \((E', F', \sigma')\), where \( E', F' \in \text{Vect}(M \times M_n) \) and

\[
\sigma' : \pi_1 E'|_{\partial M} \longrightarrow \pi_1 F'|_{\partial M}
\]

is an isomorphism over the product \( X \times M_n \). Here trivial triples are understood as triples induced by a vector bundle isomorphism defined on \( M \times M_n \).

This result is a generalization of Lemma 6.10 to families. The proof is similar to that of the lemma.

The triples introduced in this proposition can sometimes be written out starting from explicit geometric data on the \( \mathbb{Z}_n \)-manifold with the use of the following proposition.

**Proposition 3.8.** A triple \((E, F, \sigma)\), where

\[
(3.10) \quad E \in \text{Vect}(M), \ F \in \text{Vect}(X), \ \pi_1 (E|_{\partial M}) \cong kF,
\]

and \( \sigma \) is an isomorphism, defines an element of \( K_0(\mathcal{A}_{M,\pi}, \mathbb{Z}_k) \).

**Proof.** On the Moore space \( M_k \), we take a line bundle \( \varepsilon \) such that \([\varepsilon]^{-1} \in \tilde{K}(M_k) \ cong \mathbb{Z}_k\) is the generator of the reduced group. We also choose a trivialization \( \rho : k\varepsilon \rightarrow \mathbb{C}^k \).

Now consider the triple \((E, F, \sigma)\). With it we associate the following element (in the sense of Proposition 3.7):

\[
[E \otimes \varepsilon, E, \sigma'] \in K_0(\tilde{C}_0(M_k, \mathcal{A}_{M,\pi}))
\]

where the isomorphism \( \sigma' \) is defined as the composition

\[
(3.11) \quad \pi_1 E|_{\partial M} \otimes \varepsilon \xrightarrow{\sigma^{-1}} kF \otimes \varepsilon \cong F \otimes k\varepsilon \xrightarrow{1 \otimes \rho} F \otimes \mathbb{C}^k \cong k\varepsilon \xrightarrow{\sigma^{-1}} \pi_1 E|_{\partial M}.
\]

\[\square\]

2. **\( K \)-theory with coefficients \( \mathbb{Q}/n\mathbb{Z} \).** Now we treat \( \mathbb{Q}/n\mathbb{Z} \) as the direct limit of finite groups \( \mathbb{Z}_{nN} \) corresponding to the embeddings

\[
\mathbb{Z}_{nN} \subset \mathbb{Q}/n\mathbb{Z}, \ \ x \mapsto x/N.
\]

Then the \( K \)-groups with coefficients \( \mathbb{Q}/n\mathbb{Z} \) can also be defined as the direct limit

\[
K_0(\mathcal{A}_{M,\pi}, \mathbb{Q}/n\mathbb{Z}) = \lim_{\longrightarrow} K_0(\mathcal{A}_{M,\pi}, \mathbb{Z}_{nN})
\]

3. **The \( K \)-theory element of a \( \mathbb{Z}_n \)-manifold.** The bundle \( \pi_1 \in \text{Vect}(X) \) is flat, and therefore, for large \( N \) there exists a trivialization

\[
N\pi_1 \cong \mathbb{C}^N.
\]
Then the triple \([C^N, C, \alpha]\) defines an element of \(K_0(A_M, \pi, \mathbb{Z}_N)\) by virtue of Proposition \ref{prop:triple-definition}. By letting \(N \to \infty\), we obtain an element of the \(K\)-group with coefficients \(\mathbb{Q}/n\mathbb{Z}\).

Unfortunately, this construction is ambiguous in the choice of the trivialization \(\alpha\), and different trivializations give different elements.

It turns out that there is a canonical choice of \(\alpha\). Namely, for the covering \(\pi: \partial M \to X\) consider the mapping \(f: X \to BS_n\) to the classifying space \(BS_n\) of the permutation group on \(n\) elements. (Here we treat the bundle \(\pi\) as an associated bundle of the principal \(S_n\)-bundle; then \(f\) is the classifying mapping for this principal bundle.) Moreover, \(\pi! 1 \in \text{Vect}(X)\) is the pullback of the universal bundle \(\gamma_n \in \text{Vect}_n(BS_n)\) over the classifying space. Assume that the range of \(f\) is contained in a finite skeleton \((BS_n)_{N'}\) of the classifying space. The space \(BS_n\) enjoys the Mittag-Leffler condition (see \cite{Ati61}): for given \(N'\), there exists an \(L \geq 0\) such that
\[
\text{Im} \left[ K^1((BS_n)_{N+L}) \to K^1((BS_n)_N) \right] = \text{Im} \left[ K^1((BS_n)_{N+M+L}) \to K^1((BS_n)_N) \right]
\]
for all nonnegative \(M\).

Now let us choose an \(N\) such that the sum of \(N\) copies of the restriction of the universal bundle \(\gamma_n\) to \((BS_n)_{N'+L}\) is trivial. We take some trivialization \(N\gamma_n \simeq \mathbb{C}^{nN}\). Then on \(M\) we choose the induced trivialization
\[
\alpha = f^* \alpha'.
\]
One can show that for this special trivialization the \(K\)-theory element defined by the triple \((C^N, C, \alpha)\) is independent of the choice of \(\alpha'\) and \(f\). We denote this element by
\[
[\tilde{\pi}1] \in K_0(A_M, \mathbb{Q}/n\mathbb{Z}).
\]

3.5. The index defect formula.

1. The general formula.

\begin{thm} \textbf{SS01} \end{thm}
For an elliptic operator \(D\) on a twisted \(\mathbb{Z}_n\)-manifold \((M, \pi)\), one has
\[
\text{inv} D = \langle [\sigma(D)], [\tilde{\pi}1] \rangle,
\]
where \(\langle , \rangle\) is Poincaré duality with coefficients:
\[
\langle , \rangle : K_0(T^*M^\vee) \times K_0(A_M, \mathbb{Q}/n\mathbb{Z}) \to \mathbb{Q}/n\mathbb{Z}.
\]

\textbf{Sketch of proof.} It is well known in index theory how to compute fractional homotopy invariants (see \cite{APS76}). The idea is to express the fractional invariant as an index of some family of elliptic operators. Then it suffices to apply the Atiyah–Singer formula for families. Let us use this idea in the present situation.
1. Consider the family

\[ D^* \oplus (D \otimes 1_\varepsilon) : C^\infty (M, F \oplus E \otimes \varepsilon) \longrightarrow C^\infty (M, E \oplus F \otimes \varepsilon) \]

of first-order elliptic operators on \( M \) parametrized by the Moore space \( M_{nN} \). (The number \( N \) will be chosen later on.) Here \( D^* \) is the adjoint operator, and \( D \otimes 1_\varepsilon \) stands for operator \( D \) with coefficients in the bundle \( \varepsilon \in \text{Vect}(M_{nN}) \). On the other hand, consider the direct sum of \( N \) copies of this family. It turns out that the sum admits well-posed classical boundary conditions. Indeed, for sufficiently large \( N \) the direct sum of \( N \) copies of the flat bundle \( \pi 1 \) is trivial. Let us choose some trivialization

\[ \gamma \in \mathbb{C}^{nN}. \]

Then we obtain an isomorphism

\[ \pi_1 (N E|_{\partial M}) \simeq \pi_1 N \otimes E_0 \overset{\alpha \otimes 1}{\longrightarrow} \mathbb{C}^{nN} \otimes E_0 \]
on the base of the covering and a similar isomorphism

\[ \pi_1 (N E|_{\partial M}) \otimes \varepsilon \simeq \pi_1 N \otimes E_0 \otimes \varepsilon \overset{\alpha \otimes 1}{\longrightarrow} \mathbb{C}^{nN} \otimes E_0 \otimes \varepsilon \simeq nN \varepsilon \otimes E_0 \overset{\rho \otimes 1}{\longrightarrow} \mathbb{C}^{nN} \otimes E_0. \]

The corresponding isomorphisms of the spaces of sections are denoted by

\[ B_1 = \alpha \otimes 1 : C^\infty (X, \pi_1 (N E|_{\partial M})) \longrightarrow C^\infty (X, \mathbb{C}^{nN} \otimes E_0), \]

\[ B_2 = (\rho \otimes 1) (\alpha \otimes 1) : C^\infty (X, \pi_1 (N E|_{\partial M}) \otimes \varepsilon) \longrightarrow C^\infty (X, \mathbb{C}^{nN} \otimes E_0). \]

In this notation, we define the following family of boundary value problems:

\[ \begin{cases} 
ND^* u = f_1, \\
N (D \otimes 1_\varepsilon) v = f_2, \\
B_1 \beta E u|_{\partial M} + B_2 \beta E v|_{\partial M} = g, \\
g \in C^\infty (X, \mathbb{C}^{nN} \otimes E_0). 
\end{cases} \]

One can show that this family consists of elliptic boundary value problems. By \( \Phi_\alpha(D) \) we denote the family \( \{3.15\} \) for some given trivialization \( \alpha \).

2. By virtue of the embedding \( \mathbb{Z}_{nN} \subset \mathbb{Q}/n\mathbb{Z} \), we can treat the index-residue \( \text{ind} \Phi_\alpha(D) \in \mathbb{K}(M_n) \simeq \mathbb{Z}_{nN} \) of the family as a fractional rational number. The main step in the proof of the defect formula is the following result.

**Lemma 3.10.** One has

\[ \text{inv} D = \text{ind} \Phi_\alpha(D) \in \mathbb{R}/n\mathbb{Z} \]

(provided the trivialization \( \alpha \) is chosen as in Subsection 3.4).

**Proof of the Lemma.** There is a linear ellipticity-preserving homotopy between \( \{3.15\} \) and the boundary value problem

\[ \begin{cases} 
ND^* u = f_1, \\
N (D \otimes 1_\varepsilon) v = f_2, \\
B_1 \beta E \Pi_-(A) u|_{\partial M} + B_2 \beta E \Pi_+(A) v|_{\partial M} = g, \\
g \in C^\infty (X, \mathbb{C}^{nN} \otimes E_0) 
\end{cases} \]
for the same operator $ND^* \oplus N(D \otimes 1_\varepsilon)$. By the Agranovich–Dynin formula \[ \text{AD62} \] the index of the family $\Phi_\alpha(D)$ is the sum
\[
\text{(3.16)} \quad \text{ind } \Phi_\alpha(D) = N \text{ind}(D, \Pi_\alpha(A))([\varepsilon] - 1)
\]
\[ + \text{ind} \left( N \text{Im } \Pi_- (A) \oplus N \text{Im } \Pi_+ (A) \otimes \varepsilon B_1 + B_2 \right) C^\infty \left( X, \mathbb{C}^{nN} \otimes E_0 \right) \]

of the index of the family of spectral boundary value problems for $ND^*$ and $N(D \otimes 1_\varepsilon)$ and the index of an operator family on the boundary. Let us compute the index of that family.

To this end, we decompose the space of right-hand sides of the family into the direct sum
\[
C^\infty \left( X, \mathbb{C}^{nN} \otimes E_0 \right) \simeq nN \text{Im } \Pi_- (A_0) \oplus nN \varepsilon \otimes \text{Im } \Pi_+ (A_0) .
\]
Here the isomorphism is defined by the formula
\[
nN \text{Im } \Pi_- (A_0) \oplus nN \varepsilon \otimes \text{Im } \Pi_+ (A_0) \simeq \left(\varepsilon B_1 + B_2\right) C^\infty \left( X, \mathbb{C}^{nN} \otimes E_0 \right) .
\]

This permits us to express the index of the family $B_1 + B_2$ in Eq. (3.16) as
\[
\text{ind} \left( N \text{Im } \Pi_+ (A) \overset{\Pi_+(A_0)}{\longrightarrow} nN \text{Im } \Pi_+ (A_0) \right) (|\varepsilon| - 1) \in \tilde{K} (M_{nN}) .
\]

Finally, pushing down the space $\text{Im } \Pi_+ (A)$ to the base of the covering, we reduce the index to the form
\[
\text{ind} \left( N \text{Im } \Pi_+ (\pi_1 A) \overset{\Pi_+(A_0)}{\longrightarrow} nN \text{Im } \Pi_+ (A_0) \right) (|\varepsilon| - 1) .
\]

The index of an elliptic operator (not a family!) in this formula can be expressed in terms of $\eta$-invariants by the Atiyah–Patodi–Singer “index formula for flat bundles” \[ \text{APS76b} \]
\[
\text{(3.17)} \quad \text{ind} \left( N \text{Im } \Pi_+ (\pi_1 A) \overset{\Pi_+(A_0)}{\longrightarrow} nN \text{Im } \Pi_+ (A_0) \right) = N \eta (A) - nN \eta (A_0) + \left( [\sigma (A_0)], [\pi_1] \right) ,
\]

where the brackets stand for the pairing
\[
\langle , \rangle : K^1_c (T^* X) \times K^1 (X, \mathbb{Q}) \longrightarrow \mathbb{Q}
\]

of the difference element of the elliptic self-adjoint operator $A_0$
\[
[\sigma (A_0)] \in K^1 (T^* X)
\]

and the element $[\pi_1] \in K^1 (X, \mathbb{Q})$ corresponding to the trivialized flat bundle $N\pi_1$. (More details on this formula can be found in \[ \text{Gil95} \].)

Substituting (3.17) into Eq. (3.16) and transposing the $\eta$-invariants to the left-hand side, we obtain
\[
\text{inv } D = \text{ind } \Phi_\alpha (D) + \left( [\sigma (A_0)], [\pi_1] \right) .
\]

2) It remains to show that for a special choice of the trivialization (3.14) the last term in (3.17) is zero.
Indeed, consider the classifying mapping
\[ f : X \to (BS)_{N'} \]
The computation of the pairing (B.18) can be moved to the classifying space:
\begin{equation}
\langle [\sigma(A_0)], [\pi_1] \rangle = \langle f_! [\sigma(A_0)], [\gamma] \rangle, \quad [\pi_1] = f^* [\gamma] \in K^1(X, \mathbb{Q}),
\end{equation}
where \([\gamma_n] \in K^1((BS_n)_{N'}) \otimes \mathbb{Q}\) is the element defined by the trivialized flat bundle \(N\gamma_n\). The embedding \((BS_n)_{N'} \subset (BS_n)_{N'+L'}\) induces the commutative diagram
\[
\begin{array}{ccc}
K^1_c(T^*(BS_n)_{N'}) \times K^1((BS_n)_{N'+L'}, \mathbb{Q}) & \to & K^1_c(T^*(BS_n)_{N'}, \mathbb{Q}) \\
\downarrow & & \downarrow \\
K^1_c(T^*(BS_n)_{N'+L'}) \times K^1((BS_n)_{N'+L'}, \mathbb{Q}) & \to & \mathbb{Q}.
\end{array}
\]
It follows from this diagram that the pairing (3.19) gives zero, since the range of the mapping
\[
K^1_c(T^*(BS_n)_{N'}) \to K^1_c(T^*(BS_n)_{N'+L'})
\]
is contained in the torsion subgroup.

Therefore, the expression for \(\text{ind}(D)\) is reduced to the desired relation
\[
\text{ind}(D) = \text{inv}D.
\]
This proves Lemma 3.10.

3. To complete the proof of the theorem, it suffices to relate the families index of \(\Phi_{\alpha}(D)\) to the Poincaré duality pairing. In the index theory of classical boundary value problems, there is a well-known operation of “order reduction,” which reduces elliptic boundary value problems to zero-order operators (see [Hör85] or [SSS99]). This operation preserves the index. Applying this operation to the family \(\Phi_{\alpha}(D)\), one can show that the result is a family of admissible operators in the sense of Subsection 6.2 of Appendix C. A computation shows that this family of admissible operators coincides with the family corresponding to the product of the elements
\[
[\sigma(D)] \in K_c(T^*M^\tau), \quad [\bar{\pi}_1] \in K_0(A_{M,\pi}, \mathbb{Q}/n\mathbb{Z}).
\]
Since the Poincaré duality pairing of two elements is defined as the index, we obtain the desired formula for the index of the problem as the Poincaré pairing with coefficients:
\[
\text{ind } \Phi_{\alpha}(D) = \langle [\sigma(D)], [\bar{\pi}_1] \rangle.
\]
Together with the equality in Lemma 3.10, this completes the proof of the theorem.

Remark 3.11. Using the results of Appendix C, one can compute Poincaré duality topologically for a regular covering \(\pi\). In this case, we obtain an index defect formula in topological terms.
2. The index defect in the $G$-equivariant case. In a number of cases, the invariant $\text{inv}$ can be computed effectively with the use of Lefschetz theory. Suppose we are given an action of a finite group $G$ on $M$ such that the action is free on the boundary $\partial M$. As a covering $\pi$, we take the natural projection to the quotient space:

$$\pi : \partial M \to \partial M/G.$$ 

Note that we do not require that the action be free in the interior of $M$.

Consider a $G$-equivariant elliptic operator $D$ on $M$. By $L(D,g) \in \mathbb{C}$, $g \in G$, we denote the usual contribution to the Lefschetz formula (see [Don78]) of the fixed point set of an element $g$.

**Proposition 3.12.** One has

$$\text{inv} D \equiv - \sum_{g \neq e} L(D,g) \pmod{n}. \tag{3.20}$$

**Proof.** Consider the equivariant index $\text{ind}_g (D,\Pi_+)$ of the spectral boundary value problem and the equivariant $\eta$-function (see [Don78]) of the tangential operator $A$ on the boundary.

By $(D,\Pi_+)^G$ and $A^G$ we denote the restrictions of the corresponding operators to the spaces of $G$-invariant sections. Clearly, $A^G$ is equivalent to $A_0$ on the base of the covering. On the other hand, one can express the usual invariants in terms of their equivariant counterparts:

$$\text{ind} (D,\Pi_+)^G = \frac{1}{|G|} \sum_{g \in G} \text{ind}_g (D,\Pi_+), \quad \eta (A^G) = \frac{1}{|G|} \sum_{g \in G} \eta (A,g).$$

This is easy to check with the use of character theory. By virtue of these expressions, we can rewrite $\text{inv} D$ as

$$\text{inv} D = \text{ind}_e (D,\Pi_+) - \sum_{g \neq e} \eta (A,g).$$

The $\eta$-invariants here can be expressed by the equivariant Atiyah–Patodi–Singer formula (see [Don78]):

$$-\eta (A,g) = \text{ind}_g (D,\Pi_+) - L(D,g).$$

Thus, we obtain

$$\text{inv} D = |G| \text{ind} (D,\Pi_+)^G - \sum_{g \neq e} L(D,g).$$

This gives the desired equation \((3.20)\). \hfill \Box

3.6. An application to $\eta$-invariants. The index defect formula enables one to express the fractional part of the $\eta$-invariant in the following situation.

Let $M$ be an even-dimensional spin manifold such that its boundary is a covering with spin structure coinciding with that induced from the base. Let us also choose a vector bundle $E \in \text{Vect} (M)$ that at the boundary is the lift of some bundle $E_0 \in \text{Vect}(X)$. In a collar neighborhood of the boundary, we take a product
metric on $M$ that is the lift of some metric from the base. Finally, we choose a similar connection in $E$.

**Proposition 3.13.** The Dirac operator $D_M$ on $M$ with coefficients in $E$ satisfies the assumptions of Theorem 3.9, and one has the formula

$$\{\eta(D_X \otimes 1_{E_0})\} = \frac{1}{n} \left( \int_M \tilde{A}(M) \operatorname{ch}E - \left( [\sigma(D_M)], \left[ \tilde{\pi} \right] \right) \right) \in \mathbb{R}/\mathbb{Z},$$

for the fractional part of the $\eta$-invariant of the self-adjoint Dirac operator $D_X$ with coefficients in the bundle $E_0$ on the base of the covering.

**Proof.** This formula follows from the defect formula, where the index of the spectral boundary value problem is expanded by the Atiyah–Patodi–Singer formula

$$\operatorname{ind}(D_M, \Pi) = \int_M A(M) \operatorname{ch}E - \eta(D_{\partial M}).$$

$\square$

4. Appendix A. The Atiyah–Patodi–Singer $\eta$-invariant

In this appendix, we give a brief overview of the spectral $\eta$-invariant. Most of the results were proved in the original paper [APS76b], and there is also a very stimulating exposition in [BBW93]. Therefore, here we either omit the proofs or only indicate the main idea.

4.1. The geometric index formula and the $\eta$-invariant. Atiyah–Patodi–Singer [APS75] gave a formula for the index of spectral boundary value problems for geometric first-order operators. Namely, using the heat equation method [ABP73], they obtained the formula

$$\operatorname{ind}(D, \Pi_+(A)) = \int_X a(D) - \eta(A)$$

for the index of the spectral boundary value problem on a manifold $X$ for an operator having the form (1.1) near the boundary. The first contribution is defined by the constant term $a(D)$ in the local asymptotic expansion of the heat kernel

$$\operatorname{tr}(e^{-tD^*D}(x,x)) - \operatorname{tr}(e^{-tD D^*}(x,x))$$

as $t \to 0$. This term is determined just as in the case of operators on closed manifolds as some algebraic expression in the coefficients of the operator. The new feature of the spectral boundary value problem is the so-called $\eta$-invariant of the tangential operator $A$. Let us recall its definition.

Let $A$ be an elliptic self-adjoint operator of some positive order on a closed manifold $M$. The $\eta$-function of $A$ is defined by the formula

$$\eta(A, s) = \sum_{\lambda_j \in \operatorname{Spec} A, \lambda_j \neq 0} \operatorname{sgn}\lambda_j |\lambda_j|^{-s} \equiv \operatorname{Tr} \left( A \left( A^2 \right)^{-s/2-1/2} \right).$$
It is analytic in the half-space $\text{Re } s > \dim M/\text{ord } D$ (where the series is absolutely convergent). This spectral function is a generalization of the $\zeta$-function
\[ \zeta(A, s) = \sum_{\lambda_j \in \text{Spec } A} |\lambda_j|^{-s} \]
of positive definite elliptic operators. By analogy with the $\zeta$-invariant
\[ \zeta(A) = \frac{1}{2} \zeta(A, 0) , \]
it is natural to introduce the following definition.

**Definition 4.1.** The $\eta$-invariant of $A$ is the number
\begin{equation}
\eta(A) = \frac{1}{2} \left( \eta(A, 0) + \dim \ker A \right) \in \mathbb{R}.
\end{equation}

Of course, for this definition to make sense, it is necessary to have an analytic extension of the $\eta$-function to the point $s = 0$. Analytic methods show that the $\eta$-function extends meromorphically to the entire complex plane, possibly with a simple pole at the origin. Atiyah–Patodi–Singer [APS76b] for odd-dimensional manifolds and Gilkey [Gil81] for even-dimensional manifolds proved, using global topological methods, that the residue at the point $s = 0$ is nevertheless zero. Thus, the $\eta$-function is holomorphic at $s = 0$ and the $\eta$-invariant is well defined.

The $\eta$-invariant is by definition only a spectral invariant, and it can vary for a deformation of the operator. Consider an example.

**Example 4.2.** On the circle of length $2\pi$ with coordinate $\varphi$, consider the operator
\[ A_t = -i \frac{d}{d\varphi} + t. \]
Here $t$ is some real constant. Let us compute the $\eta$-invariant. Since the spectrum is given by the lattice $t + \mathbb{Z}$ (with period one), it follows that the $\eta$-invariant is a periodic function of the parameter $t$. Thus, we can suppose that $0 < t < 1$. Collecting the eigenvalues in pairs, we can rewrite the $\eta$-function as
\[ \eta(A_t, s) = \sum_{n \geq 1} \left[ (n + t)^{-s} - (n - t)^{-s} \right] + t^{-s}. \]
Let us show that this series absolutely converges on the real line for $s > 0$ and compute the limit as $s \to +0$. By the Taylor formula, we have
\[ \left[ (n + t)^{-s} - (n - t)^{-s} \right] = -2tsn^{-s} + O \left( \frac{s}{n^{2+s}} \right). \]
Thus, as $s \to +0$ we obtain
\[ \sum_{n>1} \left[ (n + t)^{-s} - (n - t)^{-s} \right] \sim -2ts \sum_{n \geq 1} n^{-s} \sim -2ts \int_1^\infty x^{-s-1}dx = -2t, \]
and for the $\eta$-invariant we have
\[ \eta(A_t) = \frac{\eta(A_t, 0) + \dim \ker A_t}{2} = \frac{1}{2} - \{t\}, \]
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where \{ \} \in [0, 1) is the fractional part of a number. Hence, for a smooth family \( A_t \) of elliptic operators the corresponding family of \( \eta \)-invariants is only piecewise smooth. In addition, the jumps (which are integral!) occur for parameter values such that some eigenvalue changes its sign. Let us also mention the half-integer parameter values \( t \in \mathbb{Z} + 1/2 \), where the spectrum of \( A_t \) is symmetric with respect to the origin. For these values, the \( \eta \)-function vanishes identically. In this case, one says that the operator \( A_t \) has a spectral symmetry, and the \( \eta \)-invariant is regarded as a measure of spectral asymmetry of the operator.

4.2. The derivative of the \( \eta \)-invariant. It turns out that the \( \eta \)-invariant is a piecewise smooth function of the parameter in the general case as well. More precisely, the following result is valid.

**Proposition 4.3.** For a smooth family \( \{ A_t \}_{t \in [0, 1]} \) of elliptic self-adjoint operators, the following assertions hold.

1. If the operator \( A_{t_0} \) is invertible, then the function \( \eta (A_t) \) is smooth in a neighborhood of \( t_0 \) and its derivative can be expressed in terms of the derivative of the \( \zeta \)-function of the auxiliary family

   \[
   B_{t,t_0} = |A_{t_0}| + (t - t_0) \left( \frac{d}{d\tau} A_\tau \right) \bigg|_{\tau = t_0}.
   \]

   Namely,

   \[
   \frac{d}{dt} \eta (A_t) \bigg|_{t=t_0} = \frac{d}{dt} \zeta (B_{t,t_0}) \bigg|_{t=t_0}.
   \]

2. In the general case, \( \eta (A_t) \) is piecewise smooth and can be decomposed in the form

   \[
   \eta (A_{t'}) - \eta (A_0) = \int_{t_0}^{t'} \omega (t_0) dt_0 + sf (A_{t'}),
   \]

   as the sum of a smooth function of the parameter and a piecewise constant function \( sf \) called the spectral flow (see Subsection 1.2). Here

   \[
   B_{t,t_0} = |A_{t_0}| + P_{\ker A_{t_0}} + (t - t_0) \left( \frac{d}{d\tau} A_\tau \right) \bigg|_{\tau = t_0},
   \]

   and \( P_{\ker A_{t_0}} \) is the orthogonal projection on the kernel of \( A_{t_0} \).

**Corollary 4.4.** The fractional part

\[
\{ \eta (A_t) \} \in \mathbb{R}/\mathbb{Z} \cong S^1
\]

of the \( \eta \)-invariant is a smooth function of the parameter \( t \) for a smooth family \( A_t \).

In the general case, the \( \eta \)-invariant is not homotopy invariant and can take arbitrary real values.
4.3. The homotopy invariance of the $\eta$-invariant. It turns out, however, that in some special operator classes the $\eta$-invariant possesses homotopy invariance. To this end, it is necessary that the two components in the decomposition (4.3) vanish. The easiest way to eliminate the second component, i.e. the spectral flow, is to consider only the fractional part \( \{ \eta (A) \} \) of the $\eta$-invariant. To obtain the vanishing of the second component, it is convenient to use the formula for the derivative on the left-hand side in Eq. (4.2). Seeley \[ See67 \] proved (see also \[ Agr94 \] and \[ GS95 \]) that the value of the $\zeta$-function at the origin can be expressed via the principal symbol of the operator. Explicitly, for a positive self-adjoint operator $A$ with principal symbol having the asymptotic expansion

\[
\sigma (A) \sim a_m + a_{m-1} + a_{m-2} + \ldots ,
\]

the $\zeta$-invariant is computed by the following procedure. Let us define the symbols $b_{-m-j}$, $j \geq 0$, by the recursion relations

\[
(4.4) \quad b_{-m-j} (x, \xi, \lambda) \left( a_m (x, \xi) - \lambda \right)
+ \sum_{k+l+|\alpha|=j, l>0} \frac{1}{\alpha!} (-i\partial_\xi)^\alpha b_{-m-k} (x, \xi, \lambda) (-i\partial_x)^\alpha a_{m-l} (x, \xi) = 0.
\]

The symbols in (4.4) depend on the coordinates $x$, momenta $\xi$, and additionally on the parameter $\lambda$. In this notation, the $\zeta$-invariant is given by

\[
(4.5) \quad 2\zeta (A) = \frac{1}{(2\pi)^{\dim M} \ord A} \int_{S^*M} \int_0^\infty b_{-\dim M - \ord A} (x, \xi, -\lambda) d\lambda.
\]

Note the following properties of this formula.

1. (Locality.) For two locally isomorphic operators $A$ and $A'$, their $\zeta$-invariants coincide:

\[ \zeta (A) = \zeta (A') . \]

(Operators are said to be \textit{locally isomorphic} if their complete symbols coincide in a neighborhood of every point of the manifold in some coordinate system for some trivialization of vector bundles.)

2. (Homogeneity.) The terms $b_j$ are positively homogeneous functions:

\[ b_j (x, t\xi, t^m \lambda) = (-1)^j b_j (x, \xi, \lambda) , \quad t > 0 . \]

These properties enable one to find classes of operators where the derivative of the $\eta$-invariant in Eq. (4.2) is zero. Let us introduce some of the known classes.

First, we define a class of pseudodifferential operators that generalize differential operators.

\textbf{Definition 4.5.} A classical pseudodifferential operator $A$ with complete symbol

\[ \sigma (A) \sim a_m + a_{m-1} + a_{m-2} + \ldots \]
is said to be $\mathbb{R}_+\text{-invariant}$ if the components of its complete symbol are homogeneous functions
\[ a_j (x, t\xi) = t^j a_j (x, \xi), \quad t \in \mathbb{R}_+, \]
with respect to the group of nonzero real numbers $\mathbb{R}_+$.

To define the second class of operators, recall that a flat bundle $\gamma \in \text{Vect}(M)$ is a vector bundle with locally constant (i.e. constant on connected subsets) transition functions. For an operator
\[ A : C^\infty (M, E) \to C^\infty (M, F), \]
one can define an \textit{operator with coefficients in the flat bundle}, denoted by
\[ A \otimes 1_\gamma : C^\infty (M, E \otimes \gamma) \to C^\infty (M, F \otimes \gamma). \]
This is locally isomorphic to the direct sum of $\dim \gamma$ copies of $A$. One can globally define this operator by gluing the local complete symbols of $A$ with the use of a partition of unity. We also require that the transition functions of a flat bundle be unitary.

Example 4.6. On the circle $\mathbb{S}^1$ with coordinate $\varphi$, consider the vector bundle $\gamma$ with the transition function $e^{2\pi i t}$. Then the operator
\[ -i \frac{d}{d\varphi} + t \]
from Example 4.2 is locally isomorphic to $-i \frac{d}{d\varphi} \otimes 1_\gamma$. The isomorphism
\[ e^{-t\varphi} \left( -i \frac{d}{d\varphi} \right) e^{it\varphi} = -i \frac{d}{d\varphi} + t \]
is given by the trivialization $e^{it\varphi}$ of $\gamma$.

**Theorem 4.7.** [APS76b, Gil89a] The fractional part of the $\eta$-invariant is homotopy invariant in the following two classes of elliptic self-adjoint operators:

1. \textit{the class of direct sums}
\[ A \otimes 1_\gamma \oplus (-\dim \gamma A) \]
with a given flat bundle $\gamma \in \text{Vect}(X)$;

2. \textit{the class of $\mathbb{R}_+$-invariant operators} if the following parity condition is satisfied:
\[ (\text{4.6}) \quad \dim A + \text{ord} M \equiv 1 \pmod{2}. \]

**Proof.** 1) The main idea of the proof is to use the locality of the $\zeta$-invariant. More precisely, for the fractional part of $\eta$ one has
\[ \{ \eta (A \otimes 1_\gamma \oplus (-nA)) \} = \{ \eta (A \otimes 1_\gamma) \} - \{ n \eta (A) \}, \quad n = \dim \gamma. \]
Then for a smooth homotopy $A_t$ this gives (see Eq. (4.2))
\[ \frac{d}{dt} \{ \eta (A_t) \} = \frac{d}{dt} \zeta (B_t), \quad B_t = |A_{t_0}| + (t - t_0) \left. \left( \frac{d}{d\tau} A_\tau \right) \right|_{\tau = t_0}. \]
A similar formula
\[ \frac{d}{dt} \{ \eta (A_t \otimes 1_\gamma) \} = \frac{d}{dt} \zeta (B'_t) \]
is valid for the derivative of the operators with coefficients in the flat bundle. Recall that \( A \otimes 1_\gamma \) and \( nA \) are locally isomorphic. Therefore, the positive definite operators \( B_t \) and \( B'_t \) are locally isomorphic as well. Therefore, the locality of the \( \zeta \)-invariant gives the desired relation
\[ \frac{d}{dt} \{ \eta (A_t \otimes 1_\gamma) \} = \frac{d}{dt} \{ n\eta (A_t) \} . \]
This proves the homotopy invariance.

2) Consider first the case of even-order operators. It is easy to see that for a homotopy \( A_t \) the corresponding positive definite operators \( B_t \) (see Eq. (4.2)) are also \( \mathbb{R}^* \)-invariant. By induction, this gives the \( \mathbb{R}^* \)-homogeneity
\[ b_j (x, -\xi, \lambda, t) = (-1)^j b_j (x, \xi, \lambda, t) \]
of the coefficients corresponding to \( B_t \) (see the recursion relations (4.4)). Hence,
\[ \frac{d}{dt} \{ \eta (A_t) \} = \frac{d}{dt} \zeta (B_t) = \text{Const} \int_{S^* M} \left( \int_0^\infty b_{-\dim M - \ord A} (x, \xi, -\lambda, t) \, d\lambda \right) . \]
Using the homogeneity (4.7) and the assumption that the manifold is odd-dimensional, we see that the integrand \( b_{-\dim M - \ord A} (x, \xi, -\lambda, t) \) is an odd function on the sphere \( S^* x M \). Therefore, the integral is zero, and we obtain the desired relation
\[ \frac{d}{dt} \{ \eta (A_t) \} = 0. \]

For odd-order operators, one obtains the different homogeneity
\[ b_j (x, -\xi, \lambda, -t) = (-1)^{j+1} b_j (x, \xi, \lambda, t) . \]
Substituting these homogeneous functions into the expression for the \( \zeta \)-invariant, we obtain
\[ \frac{d}{dt} \zeta (B_t) = \frac{d}{dt} \zeta (B_{-t}) . \]
This gives the desired homotopy invariance of the fractional part of the \( \eta \)-invariant:
\[ \frac{d}{dt} \{ \eta (A_t) \} = \frac{d}{dt} \zeta (B_t) = 0. \]

\[ \square \]

Remark 4.8. Note that in the proof we also obtained the vanishing of both the derivative and the \( \zeta \)-invariant itself for \( \mathbb{R}^* \)-invariant even-order operators on odd-dimensional manifolds.
5. Appendix B. Elliptic operators and Poincaré duality. Smooth theory

In this appendix, we show that Poincaré duality in $K$-theory on smooth manifolds can naturally be described in terms of elliptic operators. We consider both closed manifolds and manifolds with boundary. Using the Poincaré isomorphism, we construct Poincaré duality as a nonsingular pairing. In this context, the Atiyah–Singer index theorem can be used to make the pairing effectively computable. The topics covered in this appendix can also be found in the recent book [HR00]. Our approach is closer to differential equations.

5.1. The Poincaré isomorphism on a closed manifold.

1. Atiyah’s generalized elliptic operators. It is well known that for a sufficiently nice topological space $X$ (e.g., a finite CW-complex) there is a pairing

\begin{equation}
H_i(X, \mathbb{Z}) \times H^i(X, \mathbb{Z}) \rightarrow \mathbb{Z}
\end{equation}

of homology and cohomology groups. (The pairing is nondegenerate on the free parts of the groups.) A similar pairing can be constructed in $K$-theory on a smooth closed manifold $M$. Namely, we consider the pairing

\begin{equation}
\langle \cdot, \cdot \rangle : K_c^0(T^*M) \times K^0(M) \rightarrow \mathbb{Z}
\end{equation}

taking the difference element $[\sigma(D)] \in K_c^0(T^*M)$ of an elliptic operator

\[ D : C^\infty(M, E) \rightarrow C^\infty(M, F) \]

and a vector bundle $G \in \text{Vect}(M)$ to the index of $D$ with coefficients in $G$. The pairing (5.2) is nondegenerate on the free parts of the groups. (This can be proved by using the Atiyah–Singer index formula and by passing with the use of the Chern character to cohomology.) Comparing (5.1) and (5.2), we can make a guess that the $K$-homology groups of $M$ can be defined in terms of topological $K$-theory:

\begin{equation}
K_0(M) \overset{\text{def}}{=} K_c^0(T^*M).
\end{equation}

Unfortunately, this definition has a significant drawback, since the right-hand side of the formula does not make sense for a singular space $M$. Nevertheless, the right-hand side of (5.3) can be defined for an arbitrary compact space provided that we interpret $K_c^0(T^*M)$ as the group of stable homotopy classes of elliptic operators on $M$.

Atiyah [Ati69] suggested the following abstract notion of an elliptic operator.

Definition 5.1. A generalized elliptic operator over a compact space $X$ is a triple $(D, H_1, H_2)$, where

\[ D : H_1 \rightarrow H_2 \]

\[ \sigma(D) \otimes 1_G : \pi^*(E \otimes G) \rightarrow \pi^*(F \otimes G). \]

Any operator with this symbol is denoted by $D \otimes 1_G$. 

\[ 8 \text{ Recall that an operator with coefficients in a vector bundle has the principal symbol} \]

\[ \sigma(D) \otimes 1_G : \pi^*(E \otimes G) \rightarrow \pi^*(F \otimes G). \]
is a Fredholm operator acting in the Hilbert spaces $H_1, H_2$, these spaces are modules over the $C^*$-algebra $C(X)$ of continuous complex-valued functions on $X$, and $D$ almost commutes with the module structure; i.e.,

$$[D, f] \in \mathcal{K}$$

for all $f \in C(X)$, where $\mathcal{K}$ is the space of compact operators from $H_1$ to $H_2$.

Note that the compactness of the commutator in this definition originates from the fundamental property of differential operators on manifolds: for a smooth function $f$, the order of the commutator $[D, f]$ is at most the order of $D$ minus one.

Atiyah showed that generalized elliptic operators on a compact space $X$ define elements in the $K$-homology group $K_0(X)$ of the space, where $K_0$ is the generalized homology theory dual to topological $K$-theory. Atiyah also conjectured that the $K$-homology groups can be defined solely in terms of generalized elliptic operators. This conjecture was proved by Kasparov [Kas73] and independently by Brown–Douglas–Fillmore [BDF77]. It turned out more convenient to work with a certain modification of Atiyah’s original definition.

We start from the description of the even group $K_0(X)$. (The odd groups $K_1(X)$ will be introduced later.)

2. Even cycles and the group $K_0(X)$. This group is generated by so-called even cycles.

Definition 5.2. An even cycle in the $K$-homology of a space $X$ is a pair $(F,H)$ given by a $\mathbb{Z}_2$-graded Hilbert space

$$H = H_0 \oplus H_1,$$

where the components $H_{0,1}$ are $*$-modules over the $C^*$-algebra $C(X)$, and an odd (with respect to the grading) bounded operator

$$F : H \to H.$$

The operator and the module structure have the properties

$$(5.4) \quad f(F - F^*) \sim 0, \quad f(F^2 - 1) \sim 0, \quad [F, f] \sim 0 \quad \text{for all } f \in C(X),$$

where $\sim$ means equality modulo compact operators.

The even $K$-homology group $K_0(X)$ can be obtained by introducing some equivalence relation on the cycles. The simplest one is the stable operator homotopy. (See [Bla98], chapter VIII, where one can find a number of other equivalence relations on cycles. These relations are pairwise equivalent.) Let us define this relation.

Two cycles are said to be isomorphic if the corresponding $C(X)$-modules $H$ are isomorphic and the operators $F$ coincide under this isomorphism. Two cycles are homotopic if they become isomorphic after some homotopy of the operators $F$. A trivial cycle is a cycle for which all relations in (5.4) are satisfied exactly.
Finally, two cycles are \textit{stably operator homotopic} (for short, stably homotopic) if they become homotopic after adding some trivial cycles to each of them.

One can show that the set of stable homotopy classes of even cycles is an abelian group with respect to the direct sum. This group is denoted by $K_0(X)$. It is called the \textit{$K$-homology group} of space $X$.

\textbf{Remark 5.3.} The $K$-homology groups behave covariantly for continuous mappings. Consider a continuous mapping $f : X \to Y$. Then a cycle $(F,H)$ over $X$ can be treated as a cycle over $Y$ if the $C(Y)$-module structure on $H$ is defined as the composition of the induced mapping $f^* : C(Y) \to C(X)$ with the original $C(X)$-module structure on $H$. This mapping of cycles induces a homomorphism

$f : K_0(X) \to K_0(Y)$.

After these definitions, it is almost a tautology to say that ordinary elliptic operators on closed manifolds define $K$-homology elements. However, because of its importance, we describe the corresponding construction in detail.

Namely, consider an elliptic pseudodifferential operator $D : C^\infty(M,E) \to C^\infty(M,F)$ acting on sections of some bundles $E$ and $F$ on a closed manifold $M$. Then its partial isomorphism part

$D' = (1 + D^*D)^{-m/2}D$

in the polar decomposition defines a Fredholm operator

$D' : L^2(M,E) \to L^2(M,F)$,

where both spaces are $C(M)$-modules. (The module structure is the pointwise product of functions.) Moreover, $D'$ commutes with the module structure up to compact operators. For smooth functions, this follows from the composition formulas for pseudodifferential operators. Then the general case follows by continuity.

We define the matrix operator

$F = \begin{pmatrix} 0 & D^* \\ D' & 0 \end{pmatrix}$.

Then $F$ is a self-adjoint odd operator on the naturally $\mathbb{Z}_2$-graded $C(M)$-module $H = L^2(M,E) \oplus L^2(M,F)$. Thus, the pair $(F,H)$ is an even cycle. The corresponding element in $K$-homology is denoted by

(5.5) $[D] \in K_0(M)$.

\textbf{3. The quantization mapping.} This construction can be interpreted in quite a different way. Namely, the principal symbol of $D$

$\sigma(D) : \pi^*E \to \pi^*F, \quad \pi : T^*M \to M$,

is a vector bundle isomorphism over $T^*M$ except for the zero section. The corresponding difference element in $K$-theory with compact supports is denoted by
Considering two elements (5.5) and (5.6) together, one can readily show that there is a well-defined mapping
\[(5.7) \quad Q: K^0_c(T^*M) \rightarrow K_0(M), \quad [\sigma(D)] \mapsto [D],\]
which sends symbols to the corresponding operators. This mapping is naturally called the \textit{quantization mapping}. It is well defined on homotopy classes, since operators with the same symbol are homotopic.

4. Odd cycles. We have defined the group \(K^0_0(X)\) in terms of even cycles. The odd group \(K^1_0(X)\) is generated by odd cycles.

\textbf{Definition 5.4.} An \textit{odd cycle} in the \(K\)-homology of a space \(X\) is given by a Hilbert space \(H\) that is a *-module over \(C(X)\) and a bounded operator
\[F: H \rightarrow H\]
such that
\[(5.8) \quad f(F - F^*) \sim 0, \quad f(F^2 - 1) \sim 0, \quad [F, f] \sim 0\]
for an arbitrary function \(f\).

The only difference between odd and even cycles is in the \(\mathbb{Z}_2\)-grading structure.

The set of stable homotopy classes of odd cycles on \(X\) is denoted by \(K^1_1(X)\). Let us show how odd cycles arise from elliptic self-adjoint operators on closed manifolds. Consider an elliptic self-adjoint operator
\[A : C^\infty(M, E) \rightarrow C^\infty(M, F).\]
We can extend it to the \(L^2\)-spaces of sections (if the order of \(A\) is zero). This gives the odd cycle \(((1 + A^2)^{-m/2} A, L^2(M, E))\). The corresponding \(K\)-homology class is denoted by
\[[A] \in K^1_1(M).\]
On the other hand, the principal symbol of \(A\) defines a difference element
\[[\sigma(A)] \in K^1_c(T^*M).\]
Let us recall its definition (see \[\text{APS76b}\]). The principal symbol of an elliptic self-adjoint operator is an invertible Hermitian endomorphism of a bundle over the cosphere bundle \(S^*M\). Hence, over \(S^*M\) we can consider the vector bundle denoted by \(\text{Im} \sigma(\Pi_+(A))\) and generated at a point \((x, \xi) \in S^*M\) by the eigenvectors of the symbol \(\sigma(A)(x, \xi)\) corresponding to positive eigenvalues. Then the difference element is defined by the formula
\[[\sigma(A)] = \partial[\text{Im} \Pi_+(A)] \in K^1_c(T^*M),\]
where
\[\partial : K(S^*M) \rightarrow K^1_c(T^*M)\]
is the coboundary mapping in the $K$-theory of the pair $S^*M \subset B^*M$. (Here $B^*M$ is the bundle of unit balls in $T^*M$ with respect to some Riemannian metric, and the cosphere bundle $S^*M$ is realized as its boundary.)

By analogy with the even case, there is a well-defined quantization mapping

$$Q : K_*^c(T^*M) \rightarrow K_1(M), \quad [\sigma(A)] \mapsto [A].$$

This is well defined, since the coboundary $\partial$ induces an isomorphism

$$K(S^*M)/K(M) \cong K_1^c(T^*M).$$

A classical theorem (see [Kas88]) claims that the quantization mapping is an isomorphism. This isomorphism is called the Poincaré isomorphism.

**Theorem 5.5.** (The Poincaré isomorphism.) The quantization mappings given by (5.7) and (5.9) define isomorphisms

$$Q : K_*^c(T^*M) \rightarrow K_*(M),$$

where the index $*$ is either 0 or 1.

The proof can be found in [Kas88].

### 5.2. Duality and the topological index

As was already mentioned, $K$-homology is dual to topological $K$-theory. The most important manifestation of this duality is the pairing

$$\langle \cdot, \cdot \rangle : K_i(X) \times K^j(X) \rightarrow \mathbb{Z}, \quad i = 0, 1,$$

which we define in this subsection. The pairing (5.10) will be defined as the composition of the product

$$K_i(X) \times K^j(X) \rightarrow K_0(X)$$

with the mapping

$$p_0 : K_0(X) \rightarrow K_0(pt) = \mathbb{Z}$$

induced by the projection of $X$ into a one-point space.

Let us start by saying that $p_0$ takes a Fredholm operator to its analytic index. Indeed, an even cycle over a point is just a Fredholm operator, and Fredholm operators have only one stable homotopy invariant, namely, the index. This shows that $K_0(pt) = \mathbb{Z}$.

It remains to define the product (5.11). Let us consider it for $i = j = 0$.

The formula for the product (e.g., see [Ati69]) for a general element of the group $K_0(X)$ mimics the construction of an operator with coefficients in a vector bundle. Namely, this mapping takes a cycle $(F, H)$ and a vector bundle $G$ to the even cycle

$$(F \otimes 1_{\mathbb{C}^N}, 1_H \otimes P(H \otimes \mathbb{C}^N)),$$

where $P : \mathbb{C}^N \rightarrow \mathbb{C}^N$ is some projection over $X$ defining $G$ and

$$1_H \otimes P(H \otimes \mathbb{C}^N) \subset H \otimes \mathbb{C}^N$$
is the range of the projection $1_H \otimes P$. Thus, the pairing $\langle [D], [G]\rangle$ of an elliptic operator $D$ with a bundle $G$ can be computed by applying the Atiyah–Singer formula to $D \otimes 1_G$:

$$\langle [D], [G]\rangle = \text{ind}_t(D \otimes 1_G).$$

Let us also mention that the product of odd groups

$$K_1(X) \times K^1(X) \rightarrow K_0(X)$$

is defined in the spirit of the theory of Toeplitz type operators (see [BD82]).

Comparing the constructions of the present section with the de Rham theory, one can make the following glossary of similar terms:

| de Rham theory | Elliptic theory |
|---------------|----------------|
| cohomology $H^*(M)$ | topological $K$-theory $K^*(M)$ |
| homology $H_*(M)$ | $K$-homology $K_*(M)$ |
| cocycle $\omega$, $d\omega = 0$ | vector bundle $E$ |
| cycle $\gamma$, $\partial\gamma = 0$ | elliptic operator $D$ |
| integral $\int_\gamma \omega$ | index $\text{ind}(D \otimes 1_E)$ |

5.3. Poincaré duality on manifolds with boundary. Absolute and relative cycles. Let us now consider duality for manifolds with boundary.

Let $M$ be a compact smooth manifold with boundary $\partial M$. In this case, one also has Poincaré duality, frequently called Poincaré–Lefschetz duality. In (co)homology (for oriented $M$), the duality amounts to two group isomorphisms

$$H^i(M) \rightarrow H_{n-i}(M, \partial M), \quad H^i(M, \partial M) \rightarrow H_{n-i}(M), \quad n = \dim M.$$ 

Thus, duality relates the ordinary groups to the so-called relative groups.

Let us define similar isomorphisms in $K$-theory. For a manifold $M$ with boundary, one has two natural $K$-homology groups: the ordinary group $K_*(M)$ and the so-called relative $K$-homology group $K_*(M, \partial M)$. The relative groups are defined as follows.

It is clear that in Definition 5.2 we have used only the algebra $C(M)$ of functions on $M$. At the same time, a manifold with boundary has another natural $C^*$-algebra, namely, the algebra $C_0(M \setminus \partial M)$ of functions vanishing at the boundary. We denote the group generated by cycles over $C_0(M \setminus \partial M)$ by $K_*(M, \partial M)$ and call it the relative $K$-homology of the manifold.

Let us show how elements of these groups arise from elliptic operators on manifolds with boundary. On the analogy with homology theory, we refer to cycles over the algebra $C(M)$ as absolute cycles and cycles over $C_0(M \setminus \partial M)$ as relative cycles.

We first describe the ordinary (absolute) cycles.

1. Elliptic operators and absolute cycles. In this case, we consider elliptic operators $D$ induced near the boundary by vector bundle isomorphisms. Just as
in the case of closed manifolds, such operators almost commute with functions $f \in C(M)$:

$$[D, f] \sim 0.$$ 

Therefore, the construction of the previous subsection can be carried out word for word in this case; i.e., $D$ defines a $K$-homology class

$$[D] \in K_\bullet(M).$$

On the other hand, the principal symbol of such an operator is an isomorphism on $T^* M$ everywhere near the boundary. Hence, the operator $D$ has a difference element

$$[\sigma(D)] \in K^*_\bullet(T^* (M \setminus \partial M)).$$

**Remark 5.6.** This class of zero-order operators induced in a neighborhood of the boundary by vector bundle isomorphisms naturally arises in index theory of classical boundary value problems. Namely, the classical procedure of order reduction (see [Hör85] or [SSS99b]) reduces a boundary value problem for a differential operator to a pseudodifferential operator of order zero of precisely the form considered here. Moreover, the reduction preserves the index.

### 2. Elliptic operators and relative cycles

Relative cycles arise as follows. An elliptic operator $D$ of order one on $M$ defines an element

$$[D] \in K_\bullet(M, \partial M),$$

which can be constructed as follows. Consider an embedding $M \subset \widetilde{M}$ of $M$ in some closed manifold $\widetilde{M}$ of the same dimension. Let $\widetilde{D}$ be an arbitrary extension of $D$ to this closed manifold as an elliptic operator. On $\widetilde{M}$ we consider the zero-order operator

$$\widetilde{F} = \left(1 + \widetilde{D}^* \widetilde{D}\right)^{-1/2} \widetilde{D}.$$ 

The restriction of this operator to $M$ is defined as

$$F = i^* \widetilde{F} i_\ast : L^2(M, E) \to L^2(M, F),$$

where $i_\ast : L^2(M) \to L^2(\widetilde{M})$ is the extension by zero and $i^* : L^2(\widetilde{M}) \to L^2(M)$ is the restriction operator.

If $D$ is symmetric, then one can readily verify that $F$ satisfies the properties

$$F - F^* \sim 0, \quad f (F^2 - 1) \sim 0, \quad [F, f] \sim 0$$

for a function $f \in C_0(M \setminus \partial M)$ vanishing on the boundary. These relations show that $F$ defines an element in $K_1(M, \partial M)$.

If $D$ is not symmetric, then instead of the operator $F$ one considers the corresponding matrix operator as in Subsection 5.3. In this case, the pair $(F, H)$ defines an element of the group $K_0(M, \partial M)$. 
3. The quantization mapping and the Poincaré isomorphism. For elliptic operators defining absolute cycles on manifolds with boundary, we obtain two elements

\[ [\sigma(D)] \in K^*_c(T^* (M \backslash \partial M)), \quad [D] \in K_* (M). \]

Similarly, operators corresponding to relative cycles also define a pair of elements

\[ [\sigma(D)] \in K^*_c(T^* M), \quad [D] \in K_* (M, \partial M). \]

(Here \([\sigma(D)]\) is the Atiyah–Singer difference element of \(D\).) One can show that the corresponding quantization mappings

\[ Q: K^*_c(T^* (M \backslash \partial M)) \rightarrow K_* (M), \]

\[ K^*_c(T^* M) \rightarrow K_* (M, \partial M), \]

\[ [\sigma(D)] \mapsto [D] \]

are well defined. It was proved in [Kas88] that both quantization mappings are isomorphisms. They are called the Poincaré isomorphisms in \(K\)-theory on manifolds with boundary.

4. The exact sequence of a pair in \(K\)-homology. It is well known that in \(K\)-homology for an embedding \(\partial M \subset M\) there is a six-term exact sequence

\[ \begin{array}{c}
K_0 (\partial M) \\
\downarrow \\
K_0 (M) \\
\downarrow \\
K_1 (M, \partial M) \\
\downarrow \\
K_0 (M, \partial M) \\
\downarrow \\
K_1 (M) \\
\downarrow \\
K_1 (\partial M)
\end{array} \]

It follows from the results of [BDT89] that this sequence is isomorphic to the exact sequence of the pair \(T^* M|_{\partial M} \subset T^* M\) in topological \(K\)-theory:

\[ \begin{array}{c}
K^*_c (T^* \partial M) \\
\downarrow \\
K^*_c (T^* (M \backslash \partial M)) \\
\downarrow \\
K^*_c (T^* M) \\
\downarrow \\
K^*_c (T^* (M \backslash \partial M)) \\
\downarrow \\
K^*_c (T^* \partial M)
\end{array} \]

The isomorphism can be obtained by applying the quantization mappings term by term to the sequence (5.15).
6. Appendix C. Poincaré duality on \( \mathbb{Z}_n \)-manifolds

In this section, we construct Poincaré duality for \( \mathbb{Z}_n \)-manifolds (see [SS01]). Unfortunately, these manifolds with singularities have no duality in the framework of the usual topological \( K \)-groups. Quite remarkably, however, the duality can be restored if one applies the approach of noncommutative geometry [Con94] to this problem and states duality in terms of \( K \)-theory of some noncommutative algebras of functions on the corresponding spaces. We show that the approach of noncommutative geometry, at least for \( \mathbb{Z}_n \)-manifolds, can be completely described in terms of elliptic operator theory.

Before we proceed to the description of duality for \( \mathbb{Z}_n \)-manifolds, we recall the main features of the passage from topological \( K \)-theory to \( K \)-theory of algebras. This passage goes as follows.

1. The topological group \( K^0(X) \) of a compact space \( X \) can be identified with the Grothendieck group of homotopy classes of projections in matrix algebras over \( C(X) \). Clearly, this definition makes sense for an arbitrary unital \( C^* \)-algebra \( A \). The corresponding group is denoted by \( K_0(A) \). Similarly, the odd group \( K^1(X) \) is identified with the group of stable homotopy classes of unitary operators in matrix algebras over \( C(X) \). The similar group for a \( C^* \)-algebra \( A \) is denoted by \( K_1(A) \). Summarizing, the new groups are reduced in the commutative case to the topological \( K \)-groups

\[
K^*(X) \simeq K_*(C(X)).
\]

(2) \( K \)-homology. It is clear from Definition 5.2 that the definition of the \( K \)-homology group of a space uses only the \( C^* \)-algebra of functions on it. Therefore, the same definition for an arbitrary \( C^* \)-algebra \( A \) gives two groups \( K^0(A) \) and \( K^1(A) \). These groups are generated, respectively, by even and odd cycles over \( A \).

**Remark 6.1.** We note the change of variance of the functors (and the corresponding change in the position of indices). For example, an analog of the topological \( K \)-groups is the group \( K_* (A) \), which is a covariant functor with respect to algebra homomorphisms. Similarly, the \( K \)-homology of spaces translates into the \( K \)-cohomology (a contravariant functor) of algebras.

For a nonunital algebra \( A \), the \( K_0 \)-group is defined as the kernel of the natural surjective mapping

\[
K_0(A) \overset{\text{def}}{=} \ker \left( K_0(A^+) \to K_0(\mathbb{C}) \right),
\]

where \( A^+ = A \oplus \mathbb{C} \) is the algebra with attached unit. The mapping is induced by the algebra homomorphism \( A \oplus \mathbb{C} \to \mathbb{C} \). The \( K_1 \)-group is defined slightly simpler:

\[
K_0(A) \overset{\text{def}}{=} K_1(A^+).
\]

The construction is similar to the definition of topological \( K \)-theory for locally compact spaces (see [Ati89]).
6.1. Relative cycles. The $C^*$-algebra of a $\mathbb{Z}_n$-manifold. Since a $\mathbb{Z}_n$-manifold for $n = 1$ is just a smooth manifold with boundary, it is natural to construct duality separately for relative and absolute cycles.

Consider a twisted $\mathbb{Z}_n$-manifold $(M, \pi)$ (see Definition 3.1), or a $\mathbb{Z}_n$-manifold for short. On $M$, we take some elliptic operator

$$D : C^\infty(M, E) \to C^\infty(M, F)$$

satisfying Assumption 1. Its principal symbol defines a difference element

$$(6.1) \quad [\sigma(D)] \in K_c(T^*M_\pi)$$

of the topological $K$-group of the $\mathbb{Z}_n$-manifold $(T^*M, \pi)$. Note that (6.1) includes additional information (compared with the image of $[\sigma(D)]$ in $K_c(T^*M)$) related to the structure of the space $T^*M$. Let us define a similar element in $K$-homology.

Note that by assumption, the operator $D$ almost commutes both with functions $f \in C_0 (M \setminus \partial M)$ and with transpositions of the sheets of $\pi$ in a neighborhood of the boundary.

By $A_{M,\pi}$ we denote the $C^*$-subalgebra of operators in $L^2(M)$ generated by multiplications by functions $f \in C_0 (M \setminus \partial M)$ and transposition operators in a neighborhood of the boundary. If we take a neighborhood of the boundary where transpositions of the sheets are allowed and denote the complement of this neighborhood by $M'$, then the algebra can be explicitly described in the form

$$A_{M,\pi} = \left\{ (u, v) \mid u \in C_0 (M'), v \in C_0 (X \times (0, 1], \text{End}_{\pi|1}) \right\}$$

as a subalgebra in the direct sum $C_0 (M') \oplus C_0 (X \times (0, 1], \text{End}_{\pi|1})$. Here $\text{End}_{\pi|1}$ is the vector bundle of endomorphisms of $\pi|1 \in \text{Vect}(X)$.

**Remark 6.2.** The algebra $A_{M,\pi}$ can also be defined as the $C^*$-algebra of the etale-groupoid $BN94$ corresponding to the extension of the equivalence relation (3.2) to the neighborhood of the boundary.

Under Assumption 1, one can prove the following result.

**Lemma 6.3.** The spaces $L^2(M, E)$ and $L^2(M, F)$ have a natural module structure over the algebra $A_{M,\pi}$ such that $D$ almost commutes with this module structure. Thus, $D$ defines a class $[D] \in K^*(A_{M,\pi})$.

**Proof.** By Assumption 1, in a neighborhood of the boundary one has vector bundle isomorphisms

$$E \simeq \pi^* E_0, \quad E_0 \in \text{Vect}(X \times [0, 1]).$$

In a similar way, for the spaces of sections we have

$$C(\partial M \times [0, 1], E) \simeq C(X \times [0, 1], \pi|1 \otimes E_0).$$

The latter space has a natural $C_0 (X \times (0, 1], \text{End}_{\pi|1})$-module structure. This module structure can be glued with the $C(M)$-module structure on the entire manifold, thus defining the desired module structure over the algebra $A_{M,\pi}$. A
similar construction applies to $F$. The desired almost commutativity of $D$ with the elements of $A_{M,\pi}$ follows from the equivariance of $D$ with respect to transpositions of sheets of the covering. □

6.2. Absolute cycles. Nonlocal operators. Clearly, the relative cycles on a $\mathbb{Z}_n$-manifold were obtained in the previous subsection by restricting the class of operators to those almost commuting with the algebra $A_{M,\pi}$. In this subsection, we enlarge the class of absolute cycles on $M$ requiring only that the operators in question almost commute with elements of the algebra of continuous functions on the singular space $\overline{M}$ but not of the larger algebra $\mathcal{C}(M) \supset \mathcal{C}(\overline{M})$.

1. Definition of the class of operators. Needless to say, ordinary absolute cycles on $M$ (represented by elliptic operators of order zero induced by vector bundle isomorphisms near the boundary) are cycles for the algebra $\mathcal{C}(\overline{M})$ as well. However, there are also more general cycles on the space $\overline{M}$. To find them, let us replace $\mathcal{C}(\overline{M})$ by a homotopic algebra of functions on $M$ that are lifted from the base of the covering in a given neighborhood of the boundary. We denote this algebra by the same symbol. Here is an “heuristic derivation” of relative cycles.

Consider some operator $D$ on $M$. In a neighborhood of the boundary, its direct image $(\pi \times 1)_! D$ acts on the base of the covering $\pi \times 1 : \partial M \times [0,1) \to X \times [0,1)$. The spaces in which the operator acts have the usual $C(X)$-module structure (determined by the pointwise product of functions). Therefore, $D$ almost commutes with functions in $\mathcal{C}(\overline{M})$ if (a) it is a pseudodifferential operator far from the boundary and (b) its direct image in a neighborhood of the boundary is a pseudodifferential operator on the base. Thus, we arrive at the following definition.

**Definition 6.4.** An *admissible operator* on $M$ is an operator of the form

$$D = \psi' D' \varphi' + \psi'' (\pi_1^1 D'') \varphi'',$$

where

$$D' : \mathcal{C}^\infty(M, E) \to \mathcal{C}^\infty(M, F)$$

is a pseudodifferential operator of order zero on $M$,

$$D'' : \mathcal{C}^\infty(X \times [0,1], (\pi \times 1)_! E) \to \mathcal{C}^\infty(X \times [0,1], (\pi \times 1)_! F)$$

is a pseudodifferential operator on $X \times [0,1]$, the cutoff functions $\varphi'$ and $\psi'$ vanish in a neighborhood of the boundary, and $\varphi''$ and $\psi''$ vanish in a neighborhood of the subset $X \times \{1\} \subset X \times [0,1]$. Finally, the operator $D''$ is induced by a vector bundle isomorphism in the neighborhood $X \times [0,1/2) \subset X \times [0,1]$ on the cylinder.

**Remark 6.5.** Inverse images like $\pi_1^1 D''$ are in general *nonlocal* operators. By way of illustration, consider the trivial covering

$$\partial M = X \sqcup X \sqcup \ldots \sqcup X \rightarrow X$$

n copies
of \( n \) copies of \( X \). On \( \partial M \), we choose a trivial bundle \( E = \mathbb{C}^n \); the direct image of a differential operator on the total space is always a diagonal operator

\[
\pi^!P = \text{diag} \left( P\big|_{X_1}, \ldots, P\big|_{X_n} \right) : C^\infty(X, \mathbb{C}^n) \to C^\infty(X, \mathbb{C}^n),
\]

whereas the inverse image of an operator on the base is a matrix operator of general form. Therefore, near the boundary we admit usual pseudodifferential operators as well as operators corresponding to transpositions of values of a function on the sheets of the covering. In a sense, admissible operators are obtained as an extension of the algebra of pseudodifferential operators by the nonlocal operators corresponding to elements of \( \mathcal{A}_M, \pi \).

2. Symbols of admissible operators. One can readily define the symbol of an admissible operator and state the ellipticity condition. Namely, the symbol of an admissible operator \( D \) is a pair \((\sigma_M, \sigma_X)\) of usual symbols, where

\[
\sigma_M : \pi^* \big|_{M'} : p^*_M E \to p^*_M F, \quad p_M : S^* M \to M,
\]

is a symbol on the part of the manifold where the operator is pseudodifferential and

\[
\sigma_X : p^*_X (\pi E) \mid_{X \times [0,1]} \to p^*_X (\pi^! F) \mid_{X \times [0,1]}, \quad p_X : S^* (X \times [0,1]) \to X \times [0,1],
\]

is a symbol on the cylinder \( X \times [0,1] \). The symbols satisfy the compatibility condition

\[
\pi^! (\sigma_M \mid_{\partial M'}) = \sigma_X.
\]

An operator is said to be elliptic if both components of its symbol are invertible. An elliptic operator is Fredholm in appropriate Sobolev spaces.

Summarizing, we see that admissible elliptic operators \( D \) on a \( \mathbb{Z}_n \)-manifold define absolute cycles

\[
[D] \in K_* \left( \overline{M'} \right).
\]

Let us show that the symbol of an elliptic admissible operator defines an element in \( K \)-theory. In other words, let us define an analog of the Atiyah–Singer difference construction for admissible elliptic operators.

3. The difference construction for admissible operators. Let us cut \( M \) into two parts

\[
M' = M \setminus \{ \partial M \times [0,1] \} \quad \text{and} \quad \partial M \times [0,1].
\]

Then the symbol \( \sigma(D) \) of an admissible elliptic operator \( D \) is naturally represented as a pair \((\sigma_M, \sigma_X)\). Each symbol in this pair has the corresponding difference element

\[
[\sigma_M] \in K^0_c (T^* M'), \quad [\sigma_X] \in K^0_c (T^* (X \times (0,1))).
\]

In the latter case, we use the difference construction for an absolute cycle with symbol \( \sigma_X \) of order zero.

However, a single element of some topological \( K \)-group cannot be constructed from these data, since the manifolds \( T^* M' \) and \( T^* (X \times (0,1)) \) cannot be glued
smoothly (their boundaries may be nondiffeomorphic). Nevertheless, the pasting can be done if we glue the algebras of these manifolds rather than the manifolds themselves.

Let \( A_{T^*M,\pi} \) be the \( C^* \)-algebra of the \( \mathbb{Z}_n \)-manifold \((T^*M,\pi)\). It is the subalgebra

\[
A_{T^*M,\pi} \subset C_0(T^*M') \oplus C_0(T^*(X \times (0,1]), \text{End} p^*\pi_1),
\]
defined by the compatibility condition

\[
A_{T^*M,\pi} = \{ u \oplus v \mid \beta u|_{\partial M'}, \beta^{-1} = v|_{t=1} \}
\]
on the boundaries.

The difference construction for admissible operators is a mapping

\[
(6.3) \quad \chi : \text{Ell} (M,\pi) \longrightarrow K_0 (A_{T^*M,\pi})
\]
of the group \( \text{Ell} (M,\pi) \) of stable homotopy classes of elliptic admissible operators on \( M \) into the \( K_0 \)-group of the algebra \( A_{T^*M,\pi} \).

Let us explicitly describe the element \( \chi [D] \) corresponding to an elliptic admissible operator

\[
D : C^\infty (M,E) \longrightarrow C^\infty (M,F)
\]
with symbol \( \sigma(D) = (\sigma_M,\sigma_X) \). By stabilization, one can always assume that the bundle \( F \) is trivial: \( F = \mathbb{C}^k \). Now let us choose some embeddings of \( E \) and \( \mathbb{C}^k \) in trivial vector bundles and some projections

\[
P_E, P_{\mathbb{C}^k} : \mathbb{C}^{N+L} \longrightarrow \mathbb{C}^{N+L}
\]
defining \( E \) and \( \mathbb{C}^k \) as

\[
E \simeq \text{Im} P_E \subset \mathbb{C}^N \oplus 0, \quad \mathbb{C}^k \simeq \text{Im} P_{\mathbb{C}^k} \subset 0 \oplus \mathbb{C}^L.
\]

Let \( P_{\pi;E}, P_{\pi;\mathbb{C}^k} \) be the direct images of these projections in a neighborhood of the boundary. Clearly, these projections define the direct images of the corresponding subbundles.

Then the difference construction of \( D \) is defined by the formula

\[
(6.4) \quad \chi [D] = [P_1 \oplus P_2] - [P_{\mathbb{C}^k} \oplus P_{\pi;\mathbb{C}^k}],
\]
where the projection \( P_1 \) over \( T^*M' \) is defined by

\[
P_E \cos^2 |\xi| + P_{\mathbb{C}^k} \sin^2 |\xi| + (\sigma_M^{-1} P_{\mathbb{C}^k} + \sigma_M P_E) \sin |\xi| \cos |\xi|, \quad |\xi| \leq \pi/2,
\]
\[
P_{\mathbb{C}^k}, \quad |\xi| > \pi/2
\]
(we assume that the symbol \( \sigma_M \) is homogeneous of order zero in the covariable \( \xi \)), and the projection \( P_2 \) over \( T^*(X \times [0,1]) \) is defined by one of the expressions

\[
P_{\pi;E} \cos^2 |\xi| + P_{\pi;\mathbb{C}^k} \sin^2 |\xi| + 1/2 (\sigma_X^{-1} P_{\pi;\mathbb{C}^k} + \sigma_X P_{\pi;E}) \sin 2 |\xi|,
\]
\[
P_{\pi;E} \cos^2 \varphi + P_{\pi;\mathbb{C}^k} \sin^2 \varphi + 1/2 (\sigma_X^{-1} P_{\pi;\mathbb{C}^k} + \sigma_X P_{\pi;E}) \sin 2 \varphi,
\]
\[
P_{\pi;\mathbb{C}^k}.
\]
Here the first case is taken for the parameter values $x' \in X \times [1/2, 1], |\xi| \leq \pi/2$, the second case for $x' \in X \times [0, 1/2], |\xi| < \pi t$, and the third expression applies to the remaining points. Here for brevity we write

$$\varphi = |\xi| + \pi/2 (1 - 2t).$$

**Remark 6.6.** Geometrically, these projections define a subbundle that coincides over the zero section (i.e. for $|\xi| = 0$) with $E \subset \mathbb{C}^{N+L}$, coincides for $|\xi| \geq \pi/2$ with the orthogonal subbundle $\mathbb{C}^k \subset \mathbb{C}^{N+L}$, and is a rotation of one of the subbundles towards the other via the isomorphisms defined by the elliptic symbol $\sigma(D)$ at the intermediate points.

**Proposition 6.7.** The difference construction (6.3) is a well-defined group isomorphism.

**Proof.** The difference construction $\chi$ preserves the equivalence relations on the groups $\text{Ell}^0(M, \pi)$ and $K(\mathcal{A}_{T^*M, \pi})$. This can be proved by observing that a homotopy of operators gives a homotopy of symbols and hence a homotopy of the corresponding projections $P_{1,2}$. Furthermore, $\chi[D]$ is independent of the choice of embeddings in a trivial bundle, since all such embeddings are pairwise homotopic. The proof of the fact that $\chi$ defines a one-to-one mapping can be obtained along the same lines. $\square$

**6.3. The Poincaré isomorphism.** Using the difference construction of the previous subsection, we define quantization mappings on $\mathbb{Z}_n$-manifolds: one for absolute cycles $K_*(\mathcal{A}_{T^*M, \pi}) \to K_*(\mathcal{M}^\pi)$, corresponding to the theory of admissible nonlocal operators, and another for relative cycles $K_*^{T^*}(\mathcal{M}^\pi) \to K_*^{T^*}(\mathcal{M}),$ corresponding to the operators discussed in Section 3 and Subsection 6.1.

**Theorem 6.8.** (The Poincaré isomorphism.) The quantization mappings on $\mathbb{Z}_n$-manifolds are isomorphisms.

**Proof.** 1) The algebra $\mathcal{A}_{T^*M, \pi}$ has the ideal

$$I = C_0(T^*(X \times (0,1)), \text{End}^*\pi; 1)$$

with quotient $\mathcal{A}_{T^*M, \pi}/I \simeq C_0(T^*M)$. Consider the exact sequence

$$K_*(\mathcal{T}^*X) \to K_0(\mathcal{A}_{T^*M, \pi}) \to K_c(T^*M) \to K_c^1(T^*X) \to$$

of this pair. (To obtain this sequence, we use the natural isomorphisms $K_*(C_0(Y, \text{End}G)) \simeq K_*(C_0(Y)) \simeq K_*(Y)$ for a vector bundle $G \in \text{Vect}(Y)$.) Now consider the commutative diagram

$$\begin{array}{cccccccc}
K_*(\mathcal{T}^*X) & \to & K_*(\mathcal{A}_{T^*M, \pi}) & \to & K_c^* (T^*M) & \to & K_{c+1}^* (T^*X) & \to \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
K_*(\mathcal{M}^\pi) & \to & K_*(M, \partial M) & \to & K_{*+1}(X)
\end{array}$$
The second row is the exact sequence of the pair $X \subset \overline{M}$ in $K$-homology. The vertical arrows (except for the second) are Poincaré isomorphisms on closed manifolds and manifolds with boundary. Thus, by the 5-lemma, the mapping

$$K_* (\mathcal{M}_{T^* M, \pi}) \to K_* (\overline{M'})$$

is an isomorphism as well.

2) The proof of the second isomorphism can be carried out in a similar way. In this case, the following diagram is relevant:

$$
\begin{aligned}
&\downarrow \downarrow \downarrow \downarrow \\
&K_1 (X) \leftarrow K_0 (A_{M, \pi}) \leftarrow K_0 (M) \leftarrow K_0 (X) \leftarrow \\
&\downarrow \downarrow \downarrow \downarrow \\
&K_1 (\mathcal{M}_{T^* M, \pi}) \leftarrow K_0 (\mathcal{M}_{T^* M, \pi}) \leftarrow K_0 (\mathcal{M}_{T^* M, \pi}) \leftarrow K_0 (\mathcal{M}_{T^* M, \pi}) \leftarrow \\
&\downarrow \downarrow \downarrow \downarrow \\
&K_1 (T^* X) \leftarrow K_0 (\mathcal{M}_{T^* M, \pi}) \leftarrow \mathbb{R} \times T^* X \subset \overline{T^* M'}.
\end{aligned}
$$

(Here the first row is the exact sequence of the pair $\mathbb{R} \times T^* X \subset \overline{T^* M'}$.)

6.4. Poincaré duality.

1. Definition of the pairing. Let us now define Poincaré–Lefschetz duality on $\mathbb{Z}_n$-manifolds following the same scheme as in the case of smooth manifolds. Since we have relative and absolute cycles, two dualities are expected:

\begin{align}
(6.6) & \quad K^i_c (T^* \mathcal{M}_{T^* M, \pi}) \times K_i (A_{M, \pi}) \to \mathbb{Z} \\
\text{and} & \\
(6.7) & \quad K_i (A_{T^* M, \pi}) \times K^i (\overline{M'}) \to \mathbb{Z}.
\end{align}

To save space, we consider only the first duality. The second can be considered in a similar way.

Let us define the pairing (6.6) as follows: we act on the first argument by the Poincaré isomorphism

$$Q : K^i_c (T^* \mathcal{M}_{T^* M, \pi}) \to K^* (A_{M, \pi})$$

and then apply the index pairing

$$K^* (A_{M, \pi}) \times K_* (A_{M, \pi}) \to \mathbb{Z}$$

of $K$-groups of opposite variance.

**Theorem 6.9.** (Poincaré duality.) On a $\mathbb{Z}_n$-manifold, the pairings

\begin{align}
(6.8) & \quad K^i_c (T^* \mathcal{M}_{T^* M, \pi}) \times K_i (A_{M, \pi}) \to \mathbb{Z}, \quad i = 0, 1,
\end{align}

are nonsingular on the free parts of the groups.

**Proof.** Fixing the first argument of the pairing, we obtain a mapping

$$K^i_c (T^* \mathcal{M}_{T^* M, \pi}) \otimes \mathbb{Q} \to K^i_c (A_{M, \pi}).$$
where we write $G' = \text{Hom}(G, \mathbb{Q})$. This mapping occurs in the commutative diagram

$$K_c^1(T^*X, \mathbb{Q}) \leftarrow K_c^0(T^*\overline{M}^\pi, \mathbb{Q}) \leftarrow K_c^0(T^*(M \setminus \partial M), \mathbb{Q}) \leftarrow K_c^0(T^*X, \mathbb{Q})$$

All vertical arrows except for the second are isomorphisms (by Poincaré duality on smooth manifolds). Thus, by the 5-lemma the remaining mapping is also an isomorphism. Hence, (6.8) is nondegenerate in the first factor.

The nondegeneracy in the second factor can be proved in a similar way. □

2. An application to spin$^c$-manifolds. Consider a pair $(M, \pi)$, where $M$ has a spin$^c$-structure induced over the boundary by a spin$^c$-structure on the base $X$ of the covering $\pi$. Then the group $K_0^c(T^*\overline{M}^\pi)$ is a free $K_0^c(M)$-module with one generator (where $n = \dim M$). A generator is given by the difference element

$$[\sigma(D)] \in K_c^n(T^*\overline{M}^\pi)$$

of the Dirac operator on $M$. (This can be proved by analogy with the case of closed manifolds; e.g., see [LM89].) Therefore, one can define the Poincaré duality pairing

$$K_0^c(M) \times K_*\left(\mathcal{A}_M, \pi\right) \to \mathbb{Z}$$

using the composition $K_0^c(M) \to K_c^c(T^*\overline{M}^\pi)$. It follows from Theorem 6.9 that this pairing is nonsingular on the free parts of the groups.

3. Computation of the pairing. Our aim is to find a computable formula for the Poincaré duality pairing. To be definite, we consider only the case $i = 0$ in (6.6). To this end, we start from an explicit geometric realization of the groups. Let us first give a realization of the group $K_0^c(\mathcal{A}_M, \pi)$.

**Lemma 6.10.** The group $K_0^c(\mathcal{A}_M, \pi)$ is isomorphic to the group of stable homotopy classes of triples

$$(E, F, \sigma), \quad E, F \in \text{Vect}(M), \quad \sigma : \pi_1 E|_{\partial M} \to \pi_1 F|_{\partial M},$$

where $\sigma$ is a vector bundle isomorphism. Here trivial triples are those induced by a global vector bundle isomorphism over $M$.

**Proof.** This lemma is similar to Proposition 6.7, both give a topological realization of the $K_0$-group of the $C^*$-algebra of the $\mathbb{Z}_n$-manifolds. Hence, a triple $(E, C^k, \sigma)$ (note that an arbitrary triple can be reduced to this form) defines the element

$$[P_{E} \oplus P_2] - [P_{C^k} \oplus P_{\pi(C^k)}] \in K_0^c(\mathcal{A}_M, \pi)$$

of the $K_0^c$-group, where the projection $P_2$ over $X \times [0, 1]$ is defined as

$$P_2 = P_{\pi,E} \cos^2 \varphi + P_{\pi,C^k} \sin^2 \varphi + (\sigma^{-1}(x) P_{\pi,C^k} + \sigma(x) P_{\pi,E}) \frac{\sin 2\varphi}{2}, \quad \varphi = \pi \frac{t}{2},$$
and $P_E$ and $P_{C_k} \subset \mathbb{C}^N$ are some projections on subbundles isomorphic to $E$ and $\mathbb{C}^k$, respectively. One must also assume that these subbundles are mutually orthogonal.

The proof of the fact that this construction gives a well-defined isomorphism with $K_0 (A_{M,\pi})$ can be carried out by analogy with the proof of Proposition 6.7.

This realization enables one to define the product

$$K_0^0 (T^*M^\pi) \times K_0 (A_{M,\pi}) \longrightarrow K_0 (A_{T^*M,\pi})$$

going geometrically in terms of operators with coefficients in vector bundles. More precisely, for two elements $[\sigma] \in K_0^0 (T^*M^\pi)$, and $[E, F, \sigma'] \in K_0 (A_{M,\pi})$ we consider the symbol

$$(6.9) \quad \sigma \otimes 1_E \oplus \sigma^{-1} \otimes 1_F$$
on $M$. The direct image of the restriction of this symbol to the boundary is

$$\pi_! (\sigma \otimes 1_E \oplus \sigma^{-1} \otimes 1_F |_{\partial M}) = \pi_! \sigma \otimes 1_{\pi_! E |_{\partial M}} \oplus \pi_! \sigma^{-1} \otimes 1_{\pi_! F |_{\partial M}}$$

$$\simeq (\pi_! \sigma \oplus \pi_! \sigma^{-1}) \otimes 1_{\pi_! E |_{\partial M}}.$$

The last isomorphism is induced by the vector bundle isomorphism

$$\pi_! E |_{\partial M} \simeq \pi_! F |_{\partial M}.$$

There is a standard homotopy of the symbol $(\pi_! \sigma \oplus \pi_! \sigma^{-1}) \otimes 1_{\pi_! E |_{\partial M}}$ to the identity:

$$\begin{pmatrix} \pi_! \sigma & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \tau & \sin \tau \\ -\sin \tau & \cos \tau \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \pi_! \sigma^{-1} \end{pmatrix} \begin{pmatrix} \cos \tau & -\sin \tau \\ \sin \tau & \cos \tau \end{pmatrix}, \quad \tau \in [0, \pi/2].$$

Therefore, we have constructed an extension of the symbol (6.9) to an elliptic symbol of some admissible operator on $M$. Finally, the desired product

$$[\sigma] \times [E, F, \sigma'] \in K_0 (A_{T^*M,\pi})$$
can be defined as the difference element of this symbol. One can prove that the pairing

$$(6.10) \quad \langle , \rangle : K_0^0 (T^*M^\pi) \times K_0 (A_{M,\pi}) \longrightarrow K_0 (A_{T^*M,\pi}) \xrightarrow{\text{ind}} \mathbb{Z}$$
defined as a composition of this product with the index mapping coincides with Poincaré duality defined earlier.
6.5. A topological index for $\mathbb{Z}_n$-manifolds. Clearly, definition (6.10) of duality in the previous subsection still contains one component defined analytically. This is the index mapping

$$\text{ind} : K_0(A^*_T M, \pi) \longrightarrow \mathbb{Z}.$$  

A topological formula for it was obtained in [SS01]. Let us briefly recall this result.

We suppose that the following geometric condition is satisfied: there is a free action of a finite group $G$ on $\partial M$ by diffeomorphisms such that $\pi$ is the projection to the quotient space $\partial M/G$.

Consider two pairs $(M, \pi)$ and $(M', \pi')$.

**Definition 6.11.** An embedding $f$ of the pair $(M, \pi)$ in $(M', \pi')$ is an embedding of manifolds with boundary such that $f : M \rightarrow M'$, $f(\partial M) \subset \partial M'$ and the restriction of $f$ to the boundary is a $G$-equivariant mapping.

An embedding induces the direct image mapping $f_* : K_*(A^*_T M, \pi) \longrightarrow K_*(A^*_T M', \pi')$ of the $K$-groups.

For these $\mathbb{Z}_n$-manifolds, one can give a universal space in which an arbitrary $\mathbb{Z}_n$-manifold can be embedded, the embedding being unique up to homotopy. To this end, by $\pi_N : EG_N \longrightarrow BG_N$ we denote the $N$-universal bundle for the group $G$ such that the spaces $EG_N$ and $BG_N$ are smooth compact manifolds without boundary. (For the existence of such models, e.g., see [LM98].)

**Proposition 6.12.** For a sufficiently large $N$, there exists an embedding $f$ of $(M, \pi)$ in $(EG_N \times [0, \infty), \pi_N)$. The embedding is unique up to homotopy.

**Sketch of Proof.** By the $N$-universality of the covering $\pi_N$, there exists an equivariant mapping $\partial M \rightarrow EG_N$. If the dimension of the space $EG_N$ is sufficiently large, then a general position argument shows that slightly deforming this mapping one obtains a smooth embedding. Then, by virtue of the $N$-connectedness of $EG_N$, this extends to a mapping $M \rightarrow EG_N$. In turn, by a small deformation outside the boundary $\partial M$, this mapping can be made an embedding globally. □

The index theorem can be stated with the use of an embedding in the universal space. To this end, we introduce the $K$-group of the (infinite-dimensional) classifying space for $\mathbb{Z}_n$-manifolds as the direct limit

$$K_*(A^*_T EG \times [0, \infty), \pi_\infty) = \lim_{\longrightarrow} K_*(A^*_T EG_N \times [0, \infty), \pi_N)$$

of the $K$-groups corresponding to the filtration of $EG$ by skeletons.

**Theorem 6.13.** [SS01] The even $K$-group of the classifying space is isomorphic to $\mathbb{Z}$, and the index can be computed in terms of the direct image mapping:

$$\text{ind} D = f_!(\sigma(D)),$$

$$f_!(\sigma(D)) \in K_0(A^*_T EG \times [0, \infty), \pi_\infty) \simeq \mathbb{Z}.$$
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