Method for investigating rarefied gas flow around a cylindrical surface at arbitrary Knudsen numbers

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Abstract. A moment method for solving the linearized kinetic Boltzmann equation for arbitrary Knudsen numbers is presented. The isothermal flow of a rarefied gas around a cylindrical surface (the limiting cylindrical Couette problem) is investigated. The moments of the collision integral are calculated for the hard sphere model. The moment of resistance force acting per unit length of the surface, the profile of the gas flow velocity in the transient regime, and the gas velocity on the surface are calculated.

1. Introduction

The analysis of boundary value problems in the kinetic theory of inhomogeneous gases at arbitrary Knudsen numbers is relevant due to its fundamental scientific significance and a wide area of practical applicability. When solving the kinetic Boltzmann equation by analytical methods, serious difficulties of a mathematical nature arise [1, 2]. This circumstance does not allow taking into account a sufficient number of moments in the distribution function of gas molecules. The problem of isothermal flow around a cylindrical or spherical surface is one of the simplest and allows one to evaluate the effectiveness of a particular method. In [3], the problem of the rotation of a sphere was solved by the Liz method [4] in a one-time approximation (the discontinuity of the distribution function doubles the order of the system of moment equations). This led to a significant error already in the zone of moderately low Knudsen numbers: the coefficient of isothermal slip has an overestimated value $C_m=2$. In [5], the Liz method took into account two moments of the two-stream distribution function and solved a system of fourth-order moment equations. This made it possible to trace the transition of the distribution function to the Chapman - Enskog function and obtain the value $C_m=1.108$. However, a non-physical feature appeared: almost the entire angular momentum is concentrated in the tapering cone of influence of the sphere as it moves away from its surface. In these articles, the model Boltzmann equation in the form of Bhatnagar - Gross - Krook was considered [2]. In this paper, we use the exact Boltzmann collision integral. The problem of the rotation of a sphere was also the subject of a numerical study in [6]. In articles [7, 8], the problem of heat transfer in the cases of spherically symmetric and radially symmetric problems was analyzed by the Liz method.

2. Statement of the problem

We consider a rarefied gas bounded by a solid cylindrical surface of radius $R$. The surface rotates around the axis of symmetry with a constant angular velocity $\omega$. The gas is considered isothermal with
temperature $T$. The linear velocity of the surface points is much less than the thermal velocity of gas molecules and the linearization condition is satisfied:

$$ g \omega R \ll 1; \quad g = \left(\frac{m}{2kT}\right)^{1/2} $$

$m$ is the mass of gas molecules, $k$ is the Boltzmann constant. Let us introduce a cylindrical coordinate system $r, \varphi, z$. The $Z$ axis is directed along the rotation axis along the angular velocity vector.

The gas motion above the surface is described by the stationary kinetic Boltzmann equation [1,2]:

$$ \mathbf{v} \nabla f = J(f) $$

(1)

Here $\mathbf{v}$ is the velocity of molecules, $f$ is the distribution function, $J$ is the collision integral. Let us introduce the notation:

- $n$ is the concentration of molecules,
- $c = g v$ is the dimensionless molecular velocity,
- $\lambda$ is the mean free path of gas molecules,
- $\eta = \lambda n (2mkT/\pi)^{1/2}$ is the viscosity coefficient [2].

We will look for the distribution function in the form:

$$ f(r, c) = f_0(1 + \Phi); \quad \Phi = \psi + a_1(r) c_\varphi + a_2(r) c c_\varphi; \quad \psi = 2c_\varphi g\omega R \exp\left(-\alpha r / \lambda\right) \chi(\zeta). $$

(2)

$\alpha$ is the distribution parameter, $\chi(\zeta)$ is the Heaviside function, which takes into account the effect of the surface on the velocity distribution of molecules; $\zeta = \frac{c_\varphi}{c} \left[1 - \left(R/r^2\right)^{1/2}\right]$. Substitution of expression (2) into equation (1) leads to the linearized equation

$$ c \mathbf{v} \nabla \Phi = -n I(\Phi) $$

(3)

Here $I(\Phi)$ is the linearized collision operator [1,2].

3. Moment equations and their solution

The moment equations are obtained by multiplying both sides of equation (3) by $c_\varphi \exp(-c^2)$ and $c c_\varphi \exp(-c^2)$ with subsequent integration over the velocity space. Let’s introduce a dimensionless radial coordinate $x = r/R$ and denote $\varepsilon = k n^{-1} = R/\lambda$ - the reciprocal Knudsen number. Dimensionless moment equations for the functions to be determined $a_1(x), a_2(x)$ have the following form:

$$ \frac{da_2}{dx} + \frac{2a_2}{x} = \frac{2}{\sqrt{\pi}} g\omega R x^{-2} \exp(-\alpha \varepsilon x); $$

$$ \frac{da_1}{dx} - \frac{a_1}{x} + \frac{4}{\sqrt{\pi}} \Omega x a_2 = 2g\omega R x \exp(-\alpha \varepsilon x) \left[\frac{1}{x} \arcsin \frac{1}{x} + \frac{1}{\pi} \left(\frac{2}{x^2} - \frac{1}{x}\right) \sqrt{1 - x^2} - \Omega_0\right] $$

The equations include the moments of the collision operator - integral brackets (bracket integrals) [1]:

$$ \left[F_1(c), F_2(c)\right] = \pi^{-1} \lambda n \int \exp(-c^2) F_1(c) I(F_2(c)) dc $$

In these designations $\Omega_0 = [c_\varphi \chi, c c_\varphi]$ and $\Omega_1 = [c c_\varphi, c c c_\varphi]$. For gas molecules that interact as hard spheres, the integral bracket of continuous functions is found by direct calculation: $\Omega_1 = \pi/8 = 0.3927$. 

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The difference from the value of 0.4 given in [1] is explained by the difference in the determination of the viscosity coefficient through the mean free path. The integral bracket of a discontinuous function is calculated by expanding into a series in Hermite polynomials. At the same time, the hypothesis put forward in [7,8] about a simple connection of such integral brackets with those that arise in problems with plane geometry (zero Knudsen numbers) and can be calculated directly was indirectly confirmed:

$$\Omega_0 = 0.5\Omega_{\text{plane}}x^4; \quad \Omega_{\text{plane}} = \frac{3\pi + 8}{32}\sqrt{\frac{2}{\pi}} = 0.43447$$

The solution of the system of moment equations, decreasing at infinity, has the form ($A$ is the constant of integration to be determined):

$$a_2(x) = -\frac{2}{\sqrt{\pi}} g\omega Rx^{-2}\exp(-\alpha\varepsilon x) + Ax^{-2}$$

$$a_1(x) = g\omega R\left[\frac{8}{\pi}\Omega_0E_0(x) + \frac{4}{\sqrt{\pi}}\Omega_0E_0(x) - \frac{2\alpha}{\pi}D(x) + \frac{2\alpha}{\pi}F(x)\right] + \frac{2}{\sqrt{\pi}}\Omega_0\varepsilon Ax^{-1}$$

Here the following notation of integral functions is introduced:

$$E_0(x) = \int_{-\infty}^{x} s^{-n}\exp(-\alpha\varepsilon s)ds;$$

$$D(x) = \int_{-\infty}^{x} \arcsin\left(\frac{1}{s}\right)\exp(-\alpha\varepsilon s)ds;$$

$$F_0(x) = \int_{-\infty}^{x} s^{-n}\sqrt{1-s^{-2}}\exp(-\alpha\varepsilon s)ds;$$

$$F(x) = F_0(x) - 2F_0(x).$$

### 4. Boundary conditions on the surface and determination of distribution parameters

The constant $A$ is found from the diffuse boundary condition corresponding to the complete accommodation of the tangential component of the momentum of the gas molecules [1,2]. For $r = R$

$$c_r f^+ = c_r f_w$$

where $f^+$ is the distribution function of gas molecules reflected from the surface, $f_w = f_0(1 + 2c_r g\omega R)$. The introduction of the $c_r$ factor is due to the need to go over to the moment form of recording [9]. The moment boundary condition is obtained by multiplying (4) by $c_r$ and integrating over the half-space $c_r > 0$:

$$a_1(1) + \frac{\sqrt{\pi}}{2}a_2(1) = 2g\omega R(1 - \exp(-\alpha\varepsilon))$$

Hence, the constant of integration $A$ is found:

$$A = \frac{4}{\sqrt{\pi}} g\omega R\frac{2 - \exp(-\alpha\varepsilon) + Q(\varepsilon)}{\varepsilon + 2}$$

Here

$$Q(\varepsilon) = \frac{8}{\pi}\Omega_0E_0(1) - \frac{4}{\sqrt{\pi}}\Omega_0E_0(1) + \frac{2\alpha}{\pi}\left[D(1) - F(1)\right]$$
The distribution parameter $\alpha$ is found from the condition: the function $\psi$ must satisfy the linearized kinetic equation

$$e\nabla \psi = -nI(\psi)$$

Multiplying both sides of this equation by $c\cdot c\cdot \nabla$ and integrating over the velocity, we find:

$$\alpha = 4\Omega_i/\sqrt{\pi} = \sqrt{2}\left(\frac{3}{8} + \frac{1}{\pi}\right) = 0.98049$$

5. **Moment of resistance force**

The component of the stress tensor on the surface is equal to

$$\sigma_{\rho\rho} = -m\int v_{\rho} v_{\rho} f dv = -\frac{1}{2}nkTA$$

The moment of resistance force acting per unit length of the surface:

$$M = -2\pi R^2 \sigma_{\rho\rho} = -\pi nkTR^2A$$

In the limiting case of hydrodynamic mode ($Kn \to 0, \epsilon \to \infty$) the known result is obtained [10]:

$$M_0 = -4\pi\eta\omega R^2$$

In the limiting case of the free molecular mode ($Kn \to \infty, \epsilon \to 0$)

$$M_{fm} = -(2\pi nkT)^{1/2}n\omega R^3$$

This result can be obtained directly using the distribution function

$$f = f_0 \left(1 + 2c_{\rho}g\omega R\chi(\xi)\right)$$

For moderately small Knudsen numbers, linearization leads to the formula

$$M = -\frac{4\pi\eta\omega R^2}{1 + 2Kn}$$

Solution of the Navier-Stokes equations with slip boundary condition (viscous slip mode) leads to the relation

$$M = -\frac{4\pi\eta\omega R^2}{1 + 2C_m Kn},$$

where $C_m$ is the coefficient of isothermal slip. Thus, the method presented in this work gives the value $C_m = 1$. In [11], a method was developed for the exact solution of the kinetic equation in the form of the BGK model, and the coefficient $C_m = 1.015064$ was calculated.

In [12,13], the method of discrete the ordinate found the value $C_m = 1.016191$. In [14,15], based on a numerical analysis of the complete Boltzmann equation, $C_m = 1.11414$ was calculated.

In the near free molecular mode, we have:

$$M = M_{fm} \frac{1 + 1.84\epsilon}{1 + 0.5\epsilon}$$

In [7, 8], the free molecular regime is determined not only by the Knudsen number, but by the more complex condition $\epsilon \ln \epsilon \ll 1$. The method proposed in this work does not have this drawback.

Figure1 shows the dependence of the reduced braking torque $M(1+4Kn)/M_0$ on the value $(1+Kn)^{-1}$. The choice of axes is due to the need to detail the graph.
6. Velocity profile of the flow around the cylindrical surface

Based on the obtained distribution function, the dimensionless velocity of the rarefied gas flow was calculated

$$U = \left( \frac{nR}{v_f} \right)^{-1} \int v_f f dv$$

The $U(x)$ dependence at $Kn = 1$ is shown in figure 2.

$$U = \frac{1}{\pi} \left( x \arcsin \frac{1}{x} - \sqrt{x^2 - 1} \right) + \frac{1}{2} a_1(x)$$

For comparison, here are also given the dependences of the gas flow rate in the hydrodynamic

$$U_0 = \frac{1}{x}$$

and free molecular modes

$$U_{fm} = \frac{1}{\pi} \left( \arcsin \frac{1}{x} - \sqrt{x^2 - 1} \right)$$

Figure 3 shows the dependence of the gas velocity on the surface on the reciprocal Knudsen number. The presence of a minimum is confirmed by calculations for the Poiseuille flow by numerical and variational methods [2].

Figure 1. Dependence of the reduced braking torque.  
Figure 2. $U(x)$ dependence graph.  
Figure 3. Dependence of the gas velocity on the surface on the reciprocal Knudsen number.

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