Explicit Determination in $\mathbb{R}^N$ of $(N - 1)$-Dimensional Area Minimizing Surfaces with Arbitrary Boundaries

Harold R. Parks 1 · Jon T. Pitts 2

Received: 20 October 2017 / Published online: 6 February 2019
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Abstract
Let $N \geq 3$ be an integer and $B$ be a smooth, compact, oriented, $(N - 2)$-dimensional boundary in $\mathbb{R}^N$. In 1960, Federer and Fleming (Ann Math 72:458–520, 1960) proved that there is an $(N - 1)$-dimensional integral current spanning surface of least area. The proof was by compactness methods and non-constructive. Thus, it is a question of long standing whether there is a numerical algorithm that will closely approximate the area-minimizing surface. The principal result of this paper is an algorithm that solves this problem—with the proviso that since one cannot guarantee the uniqueness of the area-minimizing surface with a particular given boundary, one must be willing to alter the boundary slightly, but by no more than a small amount that can be limited in advance. Our algorithm is currently theoretical rather than practical. Specifically, given a neighborhood $U$ around $B$ in $\mathbb{R}^N$ and a tolerance $\epsilon > 0$, we prove that one can explicitly compute in finite time an $(N - 1)$-dimensional integral current $T$ with the following approximation requirements:

1. $\text{spt}(\partial T) \subset U$.
2. $B$ and $\partial T$ are within distance $\epsilon$ in the Hausdorff distance.
3. $B$ and $\partial T$ are within distance $\epsilon$ in the flat norm distance.
4. $\mathcal{M}(T) < \epsilon + \inf\{\mathcal{M}(S) : \partial S = B\}$.
5. Every area-minimizing current $R$ with $\partial R = \partial T$ is within flat norm distance $\epsilon$ of $T$.

Keywords
Area-minimizing surfaces · Mass-minimizing currents · Codimension one surfaces · Flat norm approximation · Multigrid approximation · Finite time algorithm

Mathematics Subject Classification 49Q15 · 49Q20 · 49Q05
1 Introduction

In this paper, we will follow the notation and terminology of Federer [5] except as otherwise noted. Fix a positive integer \( N \geq 3 \). In 1960, Federer and Fleming [7] proved that for any smooth, compact, \((N - 2)\)-dimensional, oriented boundary in \( \mathbb{R}^N \), there is an \((N - 1)\)-dimensional spanning surface of least area. The proof was by compactness methods and non-constructive. Thus, it is a question of long standing whether there is a numerical algorithm that will closely approximate the area minimizing surface, and in particular, whether there is a numerical algorithm that can produce a surface guaranteed to be near to the area minimizing surface in both area and location, where the accuracy of the approximation is specified in advance. The principal result of this paper is an algorithm that solves this problem—with the proviso that since one cannot guarantee the uniqueness of the area-minimizing surface with a particular given boundary, one must be willing to alter the boundary slightly, but by no more than a small amount that can be limited in advance.

Our algorithm can be applied to a very general boundary. The sought after area-minimizing surface is not restricted in its topology nor is it restricted in its singularity structure. (The definitive regularity results for such codimension one minimizing surfaces were proved by H. Federer in [6].) Since we have in hand no upper bound on the number of steps our algorithm will require for any particular given boundary and any particular required accuracy, our algorithm is currently more of theoretical than practical interest. Obtaining such an upper bound on the number of steps required would be a significant advance. Our result is the following:

**Theorem 1.1** (Main result) Given a smooth \((N - 2)\)-dimensional integral boundary \( B \), neighborhood \( U \) around \( B \), and \( \epsilon > 0 \), we will compute in finite time an integral current \( T \) that we can guarantee satisfies the following requirements:

1. \( \text{spt}(\partial T) \subset U \).
2. \( \text{dist}_H[\text{spt}(\partial T), \text{spt}(B)] < \epsilon \), where \( \text{dist}_H[\cdot, \cdot] \) is Hausdorff distance.
3. \( \mathcal{F}(\partial T - B) < \epsilon \).
4. \( \mathcal{M}(T) < \epsilon + \inf\{\mathcal{M}(S) : \partial S = B\} \).
5. Every area-minimizing current \( R \) with \( \partial R = \partial T \) is within flat norm distance \( \epsilon \) of \( T \).

**Remark 1.2**

1. Our main result above produces a surface, \( T \), that has its boundary close to, but not equal to, the given boundary, \( B \). The maximum amount by which \( \partial T \) may differ from \( B \) is set in advance, but it must be positive. The surface, \( T \), is guaranteed to be near, both in location and area, to any area-minimizing surface with boundary equal to \( \partial T \) (not \( B \)). If we knew in advance that the boundary, \( B \), determines a unique area-minimizing surface, then our procedure could be modified—indeed simplified—so that the \( T \) it produces is near, both in location and area, to the area-minimizing surface with boundary \( B \). While Morgan [8] has shown that, for a generic boundary, the area-minimizing surface is unique, there are only a few situations in which a unique area-minimizing surface can be guaranteed a priori. Thus, unfortunately, we were not able to capitalize on Morgan’s work.
2. The essentially unavoidable possibility that \( B \) does not determine a unique area-minimizing surface, combined with the metamathematical work of Beeson in [2],
has led us to conclude that it is reasonable—perhaps necessary—to require $T$ to approximate an area-minimizing surface with the same boundary as $T$, rather than requiring $T$ to approximate an area-minimizing surface with boundary $B$. The metamathematical result we refer to is described in the introductory paragraph of [2] as follows: “if a problem has been solved constructively, then the solution depends continuously on all parameters of the problem.” In another paper, [1], Beeson showed that in some instances the area-minimizing surface determined by a smooth curve does not vary continuously as the boundary curve varies. Beeson’s example relies on work of Nitsche in [9]. The curve studied, while carefully chosen, is not particularly exotic: Its parameterization is, for $0 \leq \theta \leq 2\pi$,

$$
\begin{align*}
x &= a \cos \theta - \frac{1}{3} a^3 \cos 3\theta, \\
y &= -a \sin \theta - \frac{1}{3} a^3 \sin 3\theta, \\
z &= a^2 \cos 2\theta,
\end{align*}
$$

(as [9, Fig. 2] shows, the curve roughly resembles the seam on a baseball). The particular value of $a$ is not crucial as long as it is chosen from a certain interval of values for which the curve bounds an unstable portion of Enneper’s minimal surface (Beeson specifies $a = 1.7$). While Beeson’s context is not the same as ours, his work raises the possibility that it might be impossible to replace our Theorem 1.1(5) with the statement.

$$(5') \text{An area-minimizing current} R \text{ with } \partial R = B \text{ is within flat norm distance } \epsilon \text{ of } T.$$  

**Remark 1.3 (The algorithm)** Our algorithm relies in a fundamental way on the approximation results in John Sullivan’s thesis. The main result in that thesis is an approximation theorem that allows any hypersurface to be approximated by a polyhedral hypersurface constructed from the finite set of facets of an appropriate cell complex. Three things about that approximation theorem are especially important to us: (1) The approximating surface is near in the flat norm to the surface being approximated. (2) The area of the approximating surface is not much more than the area of the surface being approximated. (3) Crucially, if a tolerance for nearness and for area increase is specified, then the cell complex, and thus the corresponding space of polyhedral hypersurfaces, that is needed for the approximation can be determined in advance.

A simplified description of our algorithm is the following:

(1) Applying Sullivan’s approximation theorem using decreasing tolerances and corresponding spaces of polyhedral hypersurfaces, we obtain a sequence of “trial surfaces” having areas approaching the absolute minimum for hypersurfaces spanning the given boundary. When restricted to a space of polyhedral hypersurfaces as in Sullivan’s approximation theorem, the problem of finding a surface of smallest area having a given boundary is a linear programming problem.
Even though any convergent subsequence of the sequence of trial surfaces must converge to an area-minimizing surface with the given boundary, any individual trial surface is, as likely as not, relatively far away from any area-minimizing surface. The new device we employ is the construction of an “auxiliary surface.” To form the auxiliary surface, we use a new boundary curve that is the boundary of a portion of the trial surface large enough that the new boundary meets the conditions (1–3) in Theorem 1.1. Again we apply Sullivan’s approximation theorem using the appropriate tolerances and spaces of polyhedral hypersurfaces, to obtain an auxiliary surface that nearly minimizes area among surfaces relatively far from that large portion of the trial surface.

When restricted to a space of polyhedral hypersurfaces as in Sullivan’s approximation theorem, the problem of finding a surface of smallest area having a given boundary while maintaining at least a certain flat norm distance from a particular surface with that same boundary is again a linear programming problem. This is because the difference of two polyhedral hypersurfaces with the same boundary is the boundary of a unique top-dimensional polyhedral region. Requiring that top-dimensional region to have at least a certain volume is a linear constraint. The feasible region is nonempty, because a closed polyhedral surface enclosing extra volume is always available, if needed.

If the area of the auxiliary surface exceeds, by a sufficiently large amount, the area of the portion of the trial surface used to define the new boundary, then we can show that the portion of the trial surface has the properties (4–5) required by Theorem 1.1—that is, we have reached a “stopping condition.” If on the other hand the area of the auxiliary surface does not exceed the area of the portion of the trial surface by that aforementioned sufficiently large amount, then the algorithm proceeds to the next trial surface.

In the body of the paper, we assume that an $\epsilon > 0$ is specified at the outset and based on that $\epsilon$, we make precise the definitions of the terms nearly minimizes, relatively far, and sufficiently large that were used informally above.

Example 1.4 Next, we schematically illustrate how our algorithm might be applied to the simple example of two circles of the same radius, one above the other in parallel planes. We know that the area-minimizing surface is either the two discs as in Fig. 1a, or the catenoid as in Fig. 1b, or in the critical configuration both are area minimizing. Let us suppose we are in the critical configuration in which the two discs and the catenoid are both area minimizing.

Assume we have a space of polyhedral surfaces, $\mathcal{P}$, that can be used to give a good approximation of any surface, in area and location. In Fig. 2a, we have a surface in $\mathcal{P}$ that nearly minimizes area, this surface is the trial surface of Remark 1.3(1). The trial surface is two topological discs, but they are not flat. In Fig. 2b, a large portion of the trial surface defines the new boundary. Let us call this large portion of the trial surface the modified trial surface. The auxiliary surface [as in Remark 1.3(2)] must be in $\mathcal{P}$, have the new boundary as its boundary, stay at least a certain distance away from the modified trial surface, and minimize area among surfaces in $\mathcal{P}$ that have the new boundary as their boundary and that stay away from the modified trial surface.
We might imagine that the auxiliary surface, in trying to minimize area, would be drawn as close as it can get to the discs while not violating its constraint, as in Fig. 3a. On the other hand, since the auxiliary surface must stay away from the trial surface, it might jump very far away from the discs and instead be near the catenoid, as in Fig. 3b.

Continuing our “thought experiment,” we want to see why the auxiliary surface having significantly more area than the modified trial surface tells us that the modified trial surface is near every area-minimizing surface that has the same boundary as the trial surface.

Suppose to the contrary that the auxiliary surface has significantly more area than the modified trial surface and suppose that there exists an area-minimizing surface, call it $X$, that has the same boundary as the modified trial surface, but $X$ is far away from the modified trial surface, rather than near to it. We know that $X$ can be approximated by a polyhedral surface $X' \in \mathcal{P}$ that is not far away from $X$ and that has not much more area than $X$. Since $X$ is far away from the modified trial surface and $X'$ is not far from $X$, we see that $X'$ is far away from the modified trial surface. Thus, $X'$ is an
element of the set of polyhedral surfaces from which the auxiliary surface has been selected. On the other hand, \( X \) is area minimizing, so it has no more area than the modified trial surface. Since \( X' \) has not much more area than \( X \), we conclude that \( X' \) has not much more area than the modified trial surface. Hence \( X' \) has less area than the surface that was chosen to be the auxiliary surface, and that is a contradiction.

If the area of the auxiliary surface does not sufficiently exceed the area of the modified trial surface, then we must make our approximation tolerances smaller and proceed to a new trial surface that has area nearer the minimum possible. In proving the theorem we show that eventually the area of the auxiliary surface will exceed that area of the modified trial surface by a large enough amount that we can conclude that any area-minimizing surface with the same boundary as the modified trial surface must be near to the trial surface.

**Remark 1.5** One might notice that in the preceding example choosing a large portion of the trial surface to define the new boundary has the effect of “tipping the scales” in the direction of that surface. Since we are required to keep any new boundary curve near to the original boundary, the scale tipping is limited. Of course, the auxiliary surface may still have area that is too small for the procedure to terminate. The crucial fact is that eventually the effect of choosing the new boundary will be sufficient for the procedure to terminate. Since the given boundary \( B \) can be quite general, we have no way to predict a priori how many area-minimizing surfaces and how many stable, locally area-minimizing surfaces \( B \) might determine. We also have no way to predict how close together any pair of those locally or globally area-minimizing surfaces might be. So we have no way to predict, or even give an upper bound on, how many iterations the procedure will require to terminate.

In the previous papers [10,11], the theoretical basis was developed for computing approximations to area-minimizing surfaces by numerically approximating functions of least gradient. Those papers required that the given boundary for which an area-minimizing spanning surface was sought must lie on the surface of a convex set. An important feature of the results in those papers was that one could be certain, at least in principle, of when sufficient computation had been done to guarantee any desired accuracy of the approximation in the sense of Hausdorff distance. Uniqueness of the area minimizer was assumed in [10], but not in [11]. Consequently, [11] required, as we do here, that one be willing to change the boundary slightly, but, as here, one sets, in advance, a limit on how much the boundary will be changed.

The method described in [10,11] was implemented numerically in [12]. The results reported there and later results in [13] showed that, in practice, the method gives much better approximations than the theorems of [10,11] guarantee.

The requirement of [10,11] that the boundary lie on the surface of a convex set is often not met. In [14,15], we modified the least gradient method to allow general boundaries, and we implemented the modified method numerically. The results of the implementation were excellent, but nonetheless we were not fully satisfied, because we had no theorem that would guarantee that a surface produced using our method would be close in location to an area-minimizing surface.

Another approach to handling general boundaries relies on motion by mean curvature which was studied extensively by Brakke in [3]. Given a boundary and an
Fig. 4 Area-minimizing surface with a spiral boundary

initial spanning surface, one allows motion by mean curvature to reduce the area of the surface until a stable surface is found. Brakke’s Surface Evolver described in [4] allows this process—and others—to be implemented. The results can be compelling. In Fig. 4a we see the result of applying the least gradient method to a spiral boundary curve. In Fig. 4b we see the result of applying the Surface Evolver to the surface in Fig. 4a. The resemblance to the soap film shown in Fig. 4c is striking.

There is no guarantee that the surface obtained using the Surface Evolver will be close in location to a globally area-minimizing surface. The process could get “hung up” on a local area minimum. Indeed, if the initial surface has the wrong topology or is too near to a stable minimal surface that is not area minimizing, the surface obtained may not even have area close to the minimum area. In the worst case scenario, finding the right starting surface may be very difficult.

A third approach to handling general boundaries is provided by J. Sullivan’s thesis. The main result in his thesis is the approximation theorem described above in Remark 1.3. The crucial fact about Sullivan’s approximation theorem is that the accuracy of the approximation required determines in advance the cell complex that is required. Thus, if one wishes to estimate from below the minimum area required to span a given boundary, one can restrict attention to polyhedral hypersurfaces built up of the facets of a specific cell complex. Finding the smallest such polyhedral surface is a linear programming problem. That smallest polyhedral surface might not be near to any area-minimizing surface, but its area cannot exceed the area of any polyhedral surface that approximates an area-minimizing surface with the given boundary. Since the approximation increases area by at most a known amount, we obtain the desired lower bound on the area of the area-minimizing surface.

We note that, in his thesis, Sullivan uses families of equally spaced parallel planes in various ways. He uses many such families of closely spaced planes to form the multigrids for his approximation theorem. Additionally, in the proof of the cubical deformation theorem [17, Theorem 4.3], n orthogonal families in n-space are positioned so as to control the (m − 1)-dimensional area of the intersection with an m-dimensional surface. The idea of positioning a family of equally spaced parallel planes to control the (m − 1)-dimensional area of the intersection with an m-dimensional surface goes back at least to Reifenberg, who used it in his proof of [16, Lemma 8],
and—indeed, at about the same time—to Federer and W. Fleming, who used it in their proof of the deformation theorem in [7, Section 5].

We thank the referee for his or her helpful comments. This paper is dedicated to the memory of our thesis advisor and friend Frederick J. Almgren, Jr.

2 The Algorithm

The Approximation Theorem obtained by Federer and Fleming tells us that any integral current can be approximated arbitrarily well by an integral polyhedral chain. Consequently, given a smooth, compact, embedded, \((N - 2)\)-dimensional boundary in \(\mathbb{R}^N\), an area-minimizing surface spanning the given boundary can be obtained as the limit of integral polyhedral chains obtained by minimizing mass in an increasing family of finite dimensional subspaces of the vector space of \((N - 1)\)-dimensional polyhedral chains, \(\mathcal{P}_{N-1}(\mathbb{R}^N)\). As a computational method, the obvious shortcoming of such an approach is that, if one has in mind a desired level of accuracy of approximation, there is no way to know whether one has achieved it. What is lacking is a priori information on which finite dimension subspace of \(\mathcal{P}_{N-1}(\mathbb{R}^N)\) is required to obtain the desired accuracy of approximation.

In his thesis [17], John Sullivan has addressed this lack of a priori information. Sullivan’s approximation is carried out using an appropriate cell complex obtained by slicing space with equally spaced parallel planes in each of many directions, a structure that he calls a “multigrid.”

**Definition 2.1** A multigrid in \(\mathbb{R}^N\) is the set of chains generated by a finite family of convex polyhedra in \(\mathbb{R}^N\) and by their vertices, edges, and faces. In our implementation, we need to include only the \((N - 1)\)-dimensional faces and \((N - 2)\)-faces.

Sullivan’s approximation result is the following:

**Theorem 2.2** ([17, Theorem 6.1]) Given \(\epsilon\) and an \((N - 1)\)-current \(T\), we can pick a multigrid \(C\) such that \(T\) has a good approximation \(S\), which is a chain in \(C\), is flat close to \(T\), and has not much more mass, \(\mathbb{M}(S) \leq (1 + \epsilon)\mathbb{M}(T)\). In fact the choice of \(C\) can be made merely knowing \(\epsilon\) and bounds on \(\mathbb{M}(T)\) and on the mass of its boundary.

Using this last approximation result, Sullivan obtains the next result (which we paraphrase) regarding an algorithm for approximating the minimum area that is required to span a given boundary cycle.

**Theorem 2.3** ([17, Corollary 6.2]) Given any boundary cycle in \(\mathbb{R}^N\), with some a priori lower bound on the area of a possible area-minimizing surface, a surface with no more than \(1 + \epsilon\) times the true minimum area can be found by solving a linear programming problem.

In the statement of Theorem 2.3, Sullivan focuses on the approximation of the minimum area. But we note that in Theorem 2.2, the approximating surface also approximates the given boundary; a fact that is important in our work. By making use
of the top-dimensional polyhedra in a sequence of finer and finer multigrids, we are able to obtain an algorithm that not only approximates the minimum area, but that also approximates both the area and the location (in the sense of the $\mathcal{F}$-norm) of an area minimizer with boundary nearly equal to the given boundary. This algorithm is the first to accomplish that goal.

**Theorem 2.4** Let $B \in \mathbb{R}^{N-2}$ with $\partial B = 0$ and smooth support be given. Let $\epsilon > 0$ be given. Let an open set, $U$, with $\text{spt}(B) \subset U$ be given. Then there is a computation requiring finitely many multigrid minimizations that results in a $T$ guaranteed to satisfy the following requirements:

1. $\text{spt}(\partial T) \subset U$,
2. $\text{dist}_H[\text{spt}(\partial T), \text{spt}(B)] < \epsilon$,
3. $B = S + \partial T$ with $\text{spt}(S) \subset U$ and $\mathcal{M}(S) < \epsilon$,
4. $\mathcal{M}(T) < \epsilon + \inf\{\mathcal{M}(S) : \partial S = B\}$,
5. every area-minimizing current $R$ with $\partial R = \partial T$ is within $\mathcal{F}$-distance $\epsilon$ of $T$.

**Proof** Let $B \in \mathbb{R}^{N-2}$ with $\partial B = 0$ and smooth support be given. Let $\epsilon > 0$ be given. Let the open set $U$ with $\text{spt}(B) \subset U$ be given.

For each $0 < r$, set

$$I(r) = \{x : \text{dist}(x, \text{spt} B) < r\}, \quad O(r) = \{x : \text{dist}(x, \text{spt} B) \geq r\}.$$ 

Let $0 < \epsilon_i$, $i = 1, 2, \ldots$, be a decreasing sequence with limit 0. Choose $\epsilon_1$ so that

- $\epsilon_1 < \epsilon/4$,
- $\text{Clos}[I(2\epsilon_1)] \subset U$,
- $\|R[I(\epsilon_1)]\| < \epsilon/3$ holds for any mass minimizer with $\partial R = B$, which we can do by Proposition 5.6 of [17].

For each $i$, use Sullivan’s approximation method (Theorem 2.2) to form a multigrid $G(i)$ such that for any mass minimizer $R$ with $\partial R = B$ there exists $\hat{R} \in G(i) \cap \mathcal{P}_{N-1}$ such that

- there exists $S$ with $B = S + \partial \hat{R}$, $\text{dist}_H[\text{spt}(S), \text{spt}(B)] < \epsilon_i$, and $\mathcal{M}(S) < \epsilon_i$,
- $\text{dist}_H[\text{spt}(\partial \hat{R}), \text{spt}(B)] < \epsilon_i$,
- $\text{spt}(\partial \hat{R}) \subset U$,
- $\mathcal{M}(\hat{R}) \leq \mathcal{M}(R) + \epsilon_i$.

Choose the multigrids $G_1 \subset G_2 \subset G_3 \subset \cdots$.

For each $i$, let $T(i) \subset G(i) \cap \mathcal{P}_{N-1}$ be the set of currents, $T$, satisfying

- there exists $S$ with $B = S + \partial T$, $\text{dist}_H[\text{spt}(S), \text{spt}(B)] < \epsilon_i$, and $\mathcal{M}(S) < \epsilon_i$,
- $\text{dist}_H[\text{spt}(\partial T), \text{spt}(B)] < \epsilon_i$,
- $\text{spt}(\partial T) \subset U$.

Using an appropriate algorithm, obtain $T_i \in T(i)$ such that

$$\mathcal{M}(T_i) \leq \epsilon_i + \inf\{\mathcal{M}(T) : T \in T(i)\}.$$ 

(We are solving a linear programming problem. We are also not requiring the exact solution; only that we be within $\epsilon_i$ of the minimum value of the objective function.)
Claim 1 If \( \mu \) denotes the mass of any mass minimizer \( R \) with \( \partial R = B \), then
\[
\mu - \epsilon_i \leq \mathcal{M}(T_i) \leq \mu + 2\epsilon_i \tag{1}
\]
holds for each \( i \), and the limit of any \( \mathcal{F} \)-convergent subsequence of \( \{T_i\}_{i=1}^{\infty} \) is a mass minimizer with boundary equal to \( B \).

Proof of Claim Let \( R \) be a mass minimizer with \( \partial R = B \).

Since \( T_i \in T(i) \), there exists \( S_i \) with \( B = \partial S_i + \partial T_i = \partial(S_i + T_i) \) and \( \mathcal{M}(S_i) < \epsilon_i \).

Thus, we have
\[
\mu = \mathcal{M}(R) \leq \mathcal{M}(S_i + T_i) \leq \mathcal{M}(S_i) + \mathcal{M}(T_i) \leq \epsilon_i + \mathcal{M}(T_i),
\]
giving us the left-hand inequality in (1).

We have chosen the multigrid \( G(i) \) so that for any mass minimizer \( R \) with \( \partial R = B \) there exists \( \hat{R} \in T(i) \) such that
- there exists \( S \) with \( B = \partial S + \partial \hat{R}, \ dist_H[\ spt(S), \ spt(B) ] < \epsilon_i, \) and \( \mathcal{M}(S) < \epsilon_i \),
- \( \ dist_H[\ spt(\partial \hat{R}), \ spt(B) ] < \epsilon_i, \)
- \( spt(\partial \hat{R}) \subset U, \)
- \( \mathcal{M}(\hat{R}) \leq \mathcal{M}(R) + \epsilon_i \).

Then, \( \hat{R} \) satisfies the conditions for membership in \( T(i) \). By the choice of \( T_i \), we conclude that
\[
\mathcal{M}(T_i) \leq \epsilon_i + \mathcal{M}(\hat{R}) \leq 2\epsilon_i + \mathcal{M}(R) = 2\epsilon_i + \mu,
\]
giving us the right-hand inequality in (1).

Now, let \( T^* \) be the limit of any \( \mathcal{F} \)-convergent subsequence of \( \{T_i\}_{i=1}^{\infty} \). Passing to that subsequence, but without changing notation, we suppose \( T_i \to T^* \). Letting \( S_i \) be as above, we have \( B = \partial(S_i + T_i) \) and \( \mathcal{M}(S_i) \to 0 \). So \( B = \partial T^* \). By the lower semicontinuity of mass, \( \mathcal{M}(T^*) \leq \lim_{i \to \infty} \mathcal{M}(T_i) = \mu \). Thus \( T^* \) is a mass minimizer with boundary \( B \).

Claim 1 has been proved. \( \square \)

Claim 2 For infinitely many \( i \), we have
\[
\mathcal{M}[T_i \mid I(\epsilon_1)] \leq \epsilon/2.
\]

Proof of Claim Suppose Claim 2 were false. Then, there would be but finitely many elements in
\[
J = \{ i : \mathcal{M}[T_i \mid I(\epsilon_1)] \leq \epsilon/2 \}.
\]
Set \( i_0 = 1 + \max J \). Then,
\[
\mathcal{M}[T_i \mid I(\epsilon_1)] > \epsilon/2.
\]
holds for all \( i \geq i_0 \). Since
\[
\mathbb{M}[T_i] = \mathbb{M}[T_i \setminus I(\epsilon_1)] + \mathbb{M}[T_i \setminus O(\epsilon_1)]
\]
we have
\[
\mathbb{M}[T_i \setminus O(\epsilon_1)] = \mathbb{M}[T_i] - \mathbb{M}[T_i \setminus I(\epsilon_1)] < \mathbb{M}[T_i] - \epsilon/2.
\]
So
\[
\lim_{i \to \infty} \mathbb{M}[T_i \setminus O(\epsilon_1)] \leq \mu - \epsilon/2,
\]
where as in Claim 1, \( \mu \) denotes the mass of any minimizer with boundary \( B \).

Passing to an \( \mathcal{F} \)-convergent subsequence, but without changing notation, we may suppose \( T_i \) converges to a mass minimizer \( R \) with \( \partial R = B \). By the lower semicontinuity of mass,
\[
\|R\|[O(\epsilon_1)] \leq \mu - \epsilon/2,
\]
holds. Since \( \mathbb{M}[R] = \mu \), we have
\[
\|R\|[I(\epsilon_1)] \geq \epsilon/2,
\]
contradicting the requirement in the definition of \( \epsilon_1 \) that \( \|R\|[I(\epsilon_1)] < \epsilon/3 \) hold.

Claim 2 has been proved. \( \square \)

Let \( \mathcal{K} \) be a closed set disjoint from \( I(\epsilon_1/2) \), containing \( O(\epsilon_1) \), and having a polyhedral boundary. For each \( i = 1, 2, \ldots \), set
\[
T'_i = T_i \setminus \mathcal{K} \quad \text{and} \quad B_i = \partial T'_i.
\]

For each \( i \), use Sullivan’s approximation method (Theorem 2.2) to form a multigrid \( \mathcal{G}'(i) \), with \( \mathcal{G}(i) \subset \mathcal{G}'(i) \) and \( T'_i \in \mathcal{G}'(i) \), such that for any mass minimizer \( R \) with \( \partial R = B_i \) there exists \( \hat{R} \in \mathcal{G}(i) \cap \mathcal{P}_{N-1} \) such that
- there exists \( S \in \mathcal{G}'(i) \cap \mathcal{P}_{N-1} \) with \( B_i = \partial S + \partial \hat{R} \), \( \mathbb{M}(S) < \epsilon_i \), and \( \text{dist}_H[\text{spt}(S), \text{spt}(B_i)] < \epsilon_i \),
- \( \text{dist}_H[\text{spt}(\partial \hat{R}), \text{spt}(B_i)] < \epsilon_i \),
- \( \text{dist}_H(\partial \hat{R}) \subset U \),
- \( \mathbb{M}(\hat{R}) \leq \mathbb{M}(R) + \epsilon_i \),
- \( \hat{R} - R = X + \partial Y \) for some \( X \) and \( Y \) with \( \mathbb{M}(X) + \mathbb{M}(Y) \leq \epsilon_i \).

Choose the multigrids so that \( \mathcal{G}'(1) \subset \mathcal{G}'(2) \subset \mathcal{G}'(3) \subset \cdots \).

For each \( i \), let \( T'(i) \subset \mathcal{G}'(i) \cap \mathcal{P}_{N-1} \) be the set of currents, \( T \), satisfying
- there exists \( S \in \mathcal{G}'(i) \cap \mathcal{P}_{N-1} \) with \( B_i = \partial S + \partial T \), \( \mathbb{M}(S) < \epsilon_i \), and \( \text{dist}_H[\text{spt}(S), \text{spt}(B_i)] < \epsilon_i \),

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• dist$_{H}[\text{spt}(\partial T), \text{spt}(B_i)] < \epsilon_i$,
• spt$(\partial T) \subset U$.

For each $i$, let $Q(i) \subset T'(i)$ be the set of currents, $Q$, satisfying
• $\mathcal{M}(W) \geq \epsilon / 2$, where $\partial W = T'_i - S - Q$ where $S$ is as in the first condition for membership of $Q$ in $T'(i)$.

Notice that if $B_i = \partial S + \partial Q$, then $W$ satisfying $\partial W = T'_i - S - Q$ is unique and $W \in G'(i) \cap \mathcal{P}_N$.

Using an appropriate algorithm, obtain $Q_i \in Q(i)$ such that

$$\mathcal{M}(Q_i) \leq \epsilon_i + \inf \{ \mathcal{M}(Q) : Q \in Q(i) \}.$$  

(We are solving a linear programming problem. We are also not requiring the exact solution, only that we be within $\epsilon_i$ of the minimum value of the objective function.)

**Stopping Conditions**

(C1) $\mathcal{M}(Q_i) \geq \mathcal{M}(T'_i) + 3 \epsilon_i$
(C2) $\mathcal{M}(T \upharpoonright \mathbb{R}^N \setminus \mathcal{K}) \leq \epsilon / 2$

**Claim 3** If for some $i_0$, the stopping conditions are satisfied, then $T'_{i_0}$ is the desired approximation. That is,
• $B = \partial S + \partial T'_{i_0}$ with spt$(S) \subset U$ and $\mathcal{M}(S) < \epsilon$,
• dist$_{H}[\text{spt}(\partial T'_{i_0}), \text{spt}(B)] < \epsilon$,
• spt$(\partial T'_{i_0}) \subset U$,
• $\mathcal{M}(T'_{i_0}) < \epsilon + \inf \{ \mathcal{M}(S) : \partial S = B \}$,
• every mass minimizing current $R$ with $\partial R = \partial T'_{i_0} = B_{i_0}$ is within $\mathcal{F}$-distance $\epsilon$ of $T'_{i_0}$.

**Proof of Claim** By the choice of $\epsilon_1$, it is immediate that
• dist$_{H}[\text{spt}(\partial T'_{i_0}), \text{spt}(B)] < \epsilon$,
• spt$(\partial T'_{i_0}) \subset U$

hold.

Since $T_{i_0} \in T(i_0)$, there exists $S_1$ with

$$B = \partial S_1 + \partial T_{i_0}, \text{ dist}_H[\text{spt}(S_1), \text{spt}(B)] < \epsilon_{i_0}, \text{ and } \mathcal{M}(S_1) < \epsilon_{i_0}.$$  

So

$$B = \partial S_1 + \partial T_{i_0}$$
$$= \partial S_1 + \partial \left( T_{i_0} \upharpoonright \mathbb{R}^N \setminus \mathcal{K} + T_{i_0} \upharpoonright \mathcal{K} \right)$$
$$= \partial \left( S_1 + T_{i_0} \upharpoonright \mathbb{R}^N \setminus \mathcal{K} \right) + \partial T'_{i_0}.$$
We have
\[ \text{spt}(S_1 + T_{i_0} \mathbb{R}^N \setminus \mathcal{K}) \subset U \]
and
\[ M\left(S_1 + T_{i_0} \mathbb{R}^N \setminus \mathcal{K}\right) \leq M(S_1) + M(T_{i_0} \mathbb{R}^N \setminus \mathcal{K}) \leq \epsilon_{i_0} + \epsilon_i/2 \leq \epsilon, \]
where we have used the stopping condition (C2).

The right-hand inequality in (1) gives us
\[ M(T_{i_0}') < \epsilon + \inf \{ M(S) : \partial S = B \}. \]

Suppose \( R \) is a minimizer with \( \partial R = B_{i_0} \). Let \( \widehat{R} \) be such that
\[ \begin{aligned}
&\bullet \text{there exists } S_2 \text{ with } B_{i_0} = \partial S_2 + \partial \widehat{R}, \text{dist}_H[\text{spt}(S_2), \text{spt}(B_{i_0})] < \epsilon_{i_0}, \text{and } M(S_2) < \epsilon_{i_0}, \\
&\bullet \text{dist}_H[\text{spt}(\partial \widehat{R}), \text{spt}(B_{i_0})] < \epsilon_{i_0}, \\
&\bullet \text{spt}(\partial \widehat{R}) \subset U, \\
&\bullet \text{dist}_H[\text{spt}(\partial \widehat{R}), \text{spt}(B_{i_0})] < \epsilon_{i_0}, \\
&\bullet \widehat{R} - R = X + \partial Y \text{ for some } X \text{ and } Y \text{ with } M(X) + M(Y) \leq \epsilon_{i_0}. 
\end{aligned} \]

Notice that the first three conditions above tell us that \( \widehat{R} \in T'(i_0) \).

Next, note that since \( R \) is a mass minimizer with \( \partial R = \partial T_{i_0}' \), we have
\[ M(R) \leq M(T_{i_0}'). \]

Thus
\[ M(\widehat{R}) \leq M(R) + \epsilon_{i_0} \leq M(T_{i_0}') + \epsilon_{i_0} \]
holds. If it were the case that \( \widehat{R} \in Q(i_0) \), then the choice of \( Q_{i_0} \) would give us
\[ M(Q_{i_0}) \leq \epsilon_{i_0} + M(\widehat{R}) \leq M(T_{i_0}') + 2 \epsilon_{i_0}, \]
contradicting the stopping condition (C1). We conclude that \( \widehat{R} \in T'(i_0) \setminus Q(i_0) \).

Now, let \( W \) satisfy \( \partial W = T_{i_0}' - S_2 - \widehat{R} \) with \( S_2 \) as above. Because \( \widehat{R} \notin Q(i_0) \), we have
\[ M(W) < \epsilon/2. \]

We also have \( \widehat{R} - R = X + \partial Y \) for some \( X \) and \( Y \) with
\[ M(X) + M(Y) \leq \epsilon_{i_0}. \]
Consequently, we see that

\[ T_{t_0}' - R = S_2 + X + \partial Y + \partial W, \]

with

\[ M(S_2) + M(X) + M(Y) + M(W) \leq 2 \varepsilon_{t_0} + \varepsilon/2 \leq \varepsilon. \]

That is, we have \( \mathcal{F}(T_{t_0}' - R) \leq \varepsilon \).

Claim 3 has been proved. \( \square \)

**Claim 4** For some \( i \), the stopping conditions will be satisfied.

**Proof of Claim** Applying Claim 2, we pass to a subsequence (without changing notation) for which the stopping condition (C2) holds for all \( i \).

Arguing by contradiction, we suppose that

\[ M(Q_i) < M(T'_i) + 3 \varepsilon_i \]

holds for every \( i \).

Since \( T_i \in \mathcal{T}(i) \), there exists \( S_i \) with \( B = \partial S_i + \partial T_i \) and \( M(S_i) < \varepsilon_i \). Since \( Q_i \in \mathcal{Q}(i) \), there exists \( S'_i \) with \( \partial T'_i = B_i = \partial S'_i + \partial Q_i \) and \( M(S'_i) < \varepsilon_i \).

Set

\[ P_i = S_i + T_i \setminus \mathbb{R}^N \setminus \mathcal{K} + S'_i + Q_i. \]

We have

\[
\partial P_i = \partial S_i + \partial [T_i \setminus \mathbb{R}^N \setminus \mathcal{K}] + \partial S'_i + \partial Q_i \\
= \partial S_i + \partial [T_i \setminus \mathbb{R}^N \setminus \mathcal{K}] + \partial T'_i \\
= \partial S_i + \partial T_i = B
\]

and

\[
M(P_i) \leq M(S_i) + M[T_i \setminus \mathbb{R}^N \setminus \mathcal{K}] + M(S'_i) + M(Q_i) \\
\leq 2 \varepsilon_i + M[T_i \setminus \mathbb{R}^N \setminus \mathcal{K}] + M(T'_i) + 3 \varepsilon_i = M(T_i) + 5 \varepsilon_i.
\]

We may pass to a subsequence, again without changing notation, such that \( P_i \) converges to \( P^* \) and \( S_i + T_i \) converges \( T^* \). By the lower semicontinuity of mass and the right-hand inequality in (1), we see that both \( P^* \) and \( T^* \) are mass minimizers with boundary \( B \). By construction, \( P^* \) and \( T^* \) are equal in \( I(\varepsilon_1/2) \). By the regularity theory of mass minimizers, the singular set of a minimizer cannot disconnect the surface. We have \( P^* = T^* \).

The fact that \( P^* = T^* \) tells us that \( \mathcal{F}[(S_i + T_i) - P_i] \to 0 \), so we can write \( (S_i + T_i) - P_i = X_i + \partial Y_i \) with \( M(X_i) + M(Y_i) \to 0 \). Then, applying the isoperimetric inequality to \( X_i \), we see that we can write \( (S_i + T_i) - P_i = \partial Z_i \) with \( M(Z_i) \to 0 \).
On the other hand, observe that
\[(S_i + T_i) - P_i = T_i' - Q_i - S_i'.\]
By the definition of \(Q(i)\), we have \(T_i' - Q_i - S_i' = \partial W_i\) with \(M(W_i) \geq \epsilon/2\). This last inequality contradicts \(M(Z_i) \to 0\), because \(W_i\) and \(Z_i\) are \(N\)-dimensional integral currents in \(\mathbb{R}^N\) having the same boundary, so in fact, they are equal.

Claim 4 has been proved.

**Conclusion** Once the sequence \(\epsilon_i\) satisfying the required conditions has been chosen, the algorithm proceeds as follows:

(A1) Set \(i = 1\).
(A2) Compute \(T_i\).
(A3) If the condition \(M(T_i \mid \mathbb{R}^N \setminus K) \leq \epsilon/2\) is satisfied, then advance to step (A4).
Otherwise, increment \(i\) and go to step (A2).
(A4) Compute \(Q_i\).
(A5) If the condition \(M(Q_i) \geq M(T_i \mid K) + 3\epsilon_i\) is satisfied, then return \(T_i'\) and terminate the algorithm. Otherwise, increment \(i\) and go to step (A2).

Claim 4 guarantees that the algorithm terminates after finitely many steps, while Claim 3 guarantees that the returned value \(T_i'\) is the desired approximation.

**References**

1. Beeson, M.J.: Non-continuous dependence of surfaces of least area on the boundary curve. Pac. J. Math. 70(1), 11–17 (1977)
2. Beeson, M.J.: Principles of continuous choice and continuity of functions in formal systems for constructive mathematics. Ann. Math. Logic 12(3), 249–322 (1977)
3. Brakke, K.A.: The Motion of a Surface by Its Mean Curvature. Mathematical Notes, vol. 20. Princeton University Press, Princeton (1978)
4. Brakke, K.A.: The surface evolver. Exp. Math. 1, 141–165 (1992)
5. Federer, H.: Geometric Measure Theory. Die Grundlehren Der Mathematischen Wissenschaften, vol. 153. Springer, New York (1969)
6. Federer, H.: The singular set of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimensions. Bull. Am. Math. Soc. (N.S.) 76(4), 767–771 (1970)
7. Federer, H., Fleming, W.H.: Normal and integral currents. Ann. Math. 72, 458–520 (1960)
8. Morgan, F.: Generic uniqueness for hypersurfaces minimizing the integral of an elliptic integrand with constant coefficients. Indiana Univ. Math. J. 30(1), 29–45 (1981)
9. Nitsche, J.C.C.: Contours bounding at least three solutions of Plateau’s problem. Arch. Ration. Mech. 30, 1–11 (1968)
10. Parks, H.R.: Explicit determination of area minimizing hypersurfaces. Duke Math. J. 44, 519–534 (1977)
11. Parks, H.R.: Explicit determination of area minimizing hypersurfaces, II. Mem. Am. Math. Soc. 342, 1–90 (1986)
12. Parks, H.R.: Numerical approximation of parametric area-minimizing hypersurfaces. SIAM J. Sci. Stat. Comput. 13, 499–511 (1992)
13. Parks, H.R.: Numerical approximation of parametric area-minimizing hypersurfaces. In: Davis, H.T., Nitsche, J.C.C. (eds.) Statistical Thermodynamics and Differential Geometry of Microstructured Materials (Minneapolis, MN, 1991) (New York). Springer, New York, pp. 99–116 (1993)
14. Parks, H.R., Pitts, J.T.: The least-gradient method for computing area minimizing hypersurfaces spanning arbitrary boundaries. J. Comput. Appl. Math. 66, 401–409 (1996)
15. Parks, H.R., Pitts, J.T.: Computing least area hypersurfaces spanning arbitrary boundaries. SIAM J. Sci. Comput. 18, 886–917 (1997)
16. Reifenberg, E.R.: Solution of the Plateau problem for $m$-dimensional surfaces of varying topological type. Acta Math. 104, 1–92 (1960)
17. Sullivan, John: A crystalline approximation theorem for hypersurfaces, Ph.D. thesis, Princeton University, (1990)

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Affiliations

Harold R. Parks$^1$ · Jon T. Pitts$^2$

Jon T. Pitts
jpitts@math.tamu.edu

1 Department of Mathematics, Oregon State University, Corvallis, OR 97331, USA
2 Department of Mathematics, Texas A&M University, College Station, TX 77843, USA