Fractional Branes and $\mathcal{N}=1$ Gauge Theories

M. Bertolini $^a$, P. Di Vecchia $^a$, G. Ferretti $^b$ and R. Marotta $^c$

$^a$ NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

$^b$ Institute for Theoretical Physics - Göteborg University and Chalmers University of Technology, 412 96 Göteborg, Sweden

$^c$ Dipartimento di Scienze Fisiche, Università di Napoli and INFN, Sezione di Napoli Via Cintia - Complesso Universitario M. Sant’ Angelo I-80126 Napoli, Italy

Abstract

We discuss fractional D3-branes on the orbifold $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$. We study the open and the closed string spectrum on this orbifold. The corresponding $\mathcal{N}=1$ theory on the brane has, generically, a $U(N_1) \times U(N_2) \times U(N_3) \times U(N_4)$ gauge group with matter in the bifundamental. In particular, when only one type of brane is present, one obtains pure $\mathcal{N}=1$ Yang-Mills. We study the coupling of the branes to the bulk fields and present the corresponding supergravity solution, valid at large distances. By using a probe analysis, we are able to obtain the Wilsonian β-function for those gauge theories that possess some chiral multiplet. Although, due to the lack of moduli, the probe technique is not directly applicable to the case of pure $\mathcal{N}=1$ Yang-Mills, we point out that the same formula gives the correct result also for this case.
1 Introduction

Fractional branes [1, 2, 3, 4] provide a useful way to construct gauge theories with reduced supersymmetry in string theory. In particular, for theories on orbifold, from the asymptotic behaviour of the fields belonging to the twisted sector, one can read off various quantities of the gauge theory. For instance, one was able to obtain the perturbative $\beta$-function of pure $\mathcal{N} = 2$ Yang–Mills theory from this perspective [5].

The basic idea involved in this type of computations is that of probing the gauge theory living on a stack of branes by putting a single brane of similar kind next to them. The picture emerging from the gauge theory is that one has higgsed the original theory by giving a v.e.v. to the matter multiplet describing the position of the probe brane. The gauge theory on the brane probe becomes free and its coupling constant stops running. By reading off such frozen coupling one obtains an algebraic relation for the running couplings at the moment of the breaking and can thus reconstruct the $\beta$-functions. The determination of the coupling on the probe is obtained by looking at the Born–Infeld action and it is essentially given by the (pull-back of) the twisted fields obtained from supergravity [6].

The above technique is related to the so called gauge/gravity correspondence although, when discussing non conformal theories, one is generically unable (with noticeable exceptions [5]) to obtain a singularity free gravity dual. In particular, fractional branes will always give rise to singular gravity backgrounds but one has nevertheless been able to obtain perturbative $\mathcal{N} = 2$ $\beta$-functions in this way [3, 4, 10] and it is not inconceivable that instanton corrections may also be computed.

In this paper we generalize these techniques to $\mathcal{N} = 1$ gauge theories. In order to use the probe analysis, we need a theory with some chiral multiplets, and the orbifold $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ provides one of the simplest examples. There are four different types of fractional D3-branes in such orbifold [11], none of which is free to move separately. However, we will see that they can move in pairs along certain orbifold directions due to the presence of chiral
multiplets and this allows us to construct a bound state probe and to derive a certain linear combination of $\beta$-functions. The bound state probe moving in one of the three complex directions is nothing but a fractional brane of the $\mathbb{C}^2/\mathbb{Z}_2$ theory of [5] with the scalars describing its position in relation to the orbifold point. Thus, when we are probing $\mathcal{N} = 1$ theories, the low energy theory on the probe is a (free) $U(1)$ $\mathcal{N} = 2$ theory.

The computation of the $\beta$-function proceeds in a way that is very similar to that of [5]. The boundary state technique allows one to obtain the (properly normalized) coupling of the brane to the twisted fields. Thus, the brane acts as a source for such fields and their logarithmic behaviour is re-interpreted, via the Born–Infeld action, as the running coupling. Because of the need for chiral multiplets in the bifundamental, the gauge group under study contains at least two simple factors, each one coming with its own coupling constant. Since the gauge symmetry breaking in the $\mathcal{N} = 1$ case is slightly more involved than that for $\mathcal{N} = 2$, the probe analysis allows one to obtain only a linear combination of the $\beta$-functions for these couplings. However, we show that the formulae obtained give the correct answer even in the case of pure $\mathcal{N} = 1$ Yang–Mills with only one $U(N)$ gauge group although, strictly speaking, the probe analysis is not applicable there. We believe that it should be possible to justify this result with a refinement of the probe analysis.

It is perhaps worth emphasizing that the results for the $\beta$-functions presented here are valid near the gaussian UV fixed point, where the anomalous dimensions vanish. This is justified here because we start with a renormalizable theory which admits a continuum limit. This should not be confused with the case discussed in [12, 7], where the results for the $\beta$-functions concern the theory at a non-gaussian IR fixed point, and contain some non perturbative information encoded in the anomalous dimension of the fields via the exact relation of [13].

The paper is organized as follows. In section 2, we discuss the open and closed string spectrum for fractional branes at the orbifold $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ and, using the technique of boundary state, we compute the coupling of the various bulk fields to the brane and construct the boundary action. In section 3, we obtain the gravity solution corresponding to the most generic combination of fractional branes. The part of the solution that is relevant to the computation of the $\beta$-functions is the behaviour of the twisted fields. Although the solution for the metric is singular (as it was in the $\mathcal{N} = 2$ case), the probe analysis is justified at large distances. In section 4, we will use these results to obtain the $\beta$-functions of the gauge theory. We will also justify the probe analysis from the gauge theory point of view by studying the higgsing of the theory. Moreover, noticing that brane probes become tensionless before reaching the singularity, we also conclude that the short-distance region of spacetime is out of reach of the supergravity solution while the singularity is excised. This implies that we cannot probe the IR of the gauge theory, but this is not relevant in computing the perturbative $\beta$-function. Finally, we will show
that our formulæ also correctly reproduce the \( \beta \)-function for the pure \( \mathcal{N} = 1 \) theory when extrapolated outside of the regime of validity of the probe analysis.

2 Regular and fractional branes on \( \mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \)

In this section we consider both regular and fractional D3-branes of type IIB string theory on the orbifold \( \mathbb{R}^{1,3} \times \mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \), we study the spectrum of the massless open string states having their end-points attached to them and, after constructing the boundary state encoding their properties, we determine their boundary action and the large distance behaviour of the classical solution corresponding to them.

The group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) has four elements: the identity \( e \), the generators of the two \( \mathbb{Z}_2 \) that we denote with \( h_1 \) and \( h_2 \) and their product, denoted by \( h_3 = h_1 h_2 \). By taking the orbifold directions to be along \( x^4, \ldots, x^9 \) and introducing complex coordinates \( (z_1, z_2, z_3) \in \mathbb{C}^3 \) defined by:

\[
z_1 = x^4 + i x^5, \quad z_2 = x^6 + i x^7, \quad z_3 = x^8 + i x^9
\]

the action of the orbifold group on the complex coordinates \( z_i \) \( (i = 1, 2, 3) \) can be defined as:

| \( e \) | \( h_1 \) | \( h_2 \) | \( h_3 \) |
|-----|-----|-----|-----|
| \( z_1 \) | \( z_1 \) | \( -z_2 \) | \( -z_1 \) |
| \( z_2 \) | \( -z_2 \) | \( z_3 \) | \( -z_1 \) |
| \( z_3 \) | \( -z_3 \) | \( z_2 \) | \( z_1 \) |

Table 1: The action of the orbifold generators on \( \mathbb{C}^3 \).

We are interested in studying D3-branes which are transverse to the orbifold, namely with world-volume directions \( x^\alpha \), with \( \alpha = 0, 1, 2, 3 \).

The low energy closed string sector of the orbifold \( \mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \) consists of an untwisted sector and three twisted sectors, corresponding to zero modes of supergravity fields dimensionally reduced on the three exceptional vanishing two-cycles \( C_i \) \( (i = 1, 2, 3) \) characterizing the orbifold, each of them embedded in one of the three four-dimensional subspaces of \( \mathbb{C}^3 \) \[14\]. The three anti self-dual 2-form \( \omega^j_2 \), dual to the cycles \( C_i \), are then completely independent and normalized as:

\[
\int_{C_i} \omega^j_2 = \delta^j_i , \quad \int \omega^3_2 \wedge \omega^3_2 = -\frac{1}{4} , \quad *_4 \omega^3_2 = -\omega^3_2
\]

(2.2)
where the index \( \ast_4 \) indicates the dual in the four-dimensional space in which the two-cycle is embedded.

Let us start by analyzing regular branes, namely those D-branes which are free to move in the full transverse space. In order to study the open string spectrum living on a regular D3-brane, it is convenient to consider the covering space where together with the original brane there are also its three images. If the D3-brane is located at an arbitrary point of the transverse six-dimensional space, the open strings stretched between the brane and its images or between two of its images correspond in general to massive states. But massless states appear when we put the D3-brane at the orbifold fixed point \( z_1 = z_2 = z_3 = 0 \). The generic open string state is the product of a Chan-Paton factor consisting of a \( 4 \times 4 \) matrix describing the open strings attached to the D3-brane and its images and of an oscillator part. In particular a massless state of the NS sector has the following form:

\[
\lambda \otimes \psi^M_{-1/2}|0, k >
\]

where \( \lambda \) denotes the Chan-Paton factor and \( M = 0, 1, ..., 9 \). The action of the generators of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) on the Chan-Paton factors is defined as

\[
\gamma(h) \lambda \gamma^{-1}(h) = \lambda' \quad \text{for} \quad h = e, h_1, h_2, h_3
\]

It is convenient to choose the matrices \( \gamma(h) \) to be:

\[
\begin{align*}
\gamma(e) &= 1 \otimes 1 , \\
\gamma(h_1) &= \sigma_3 \otimes 1 , \\
\gamma(h_2) &= 1 \otimes \sigma_3 , \\
\gamma(h_3) &= \sigma_3 \otimes \sigma_3
\end{align*}
\]

where with \( \otimes \) we denote the usual tensorial product.

Not all states in eq. (2.3) are allowed in an orbifold theory. The only allowed states are those that are left invariant under the combined action of the orbifold group on the oscillators and the Chan-Paton factors. In particular, if \( M = \alpha \), the oscillator part is left invariant by the action of the orbifold group. Therefore, we must require that also the Chan-Paton part be left invariant by the action of \( \gamma(h) \), namely \( \lambda' = \lambda \) in eq. (2.4). On the other hand, since \( \psi_{-1/2}^4|0, k > \) can be either even or odd under the action of the orbifold group, the states surviving the orbifold projection are those with Chan-Paton factors that are respectively even or odd under the action of the orbifold group\(^1\). By a careful analysis one finds the following bosonic spectrum of massless states:

**Vectors:**
\[
\lambda \times \psi_{-1/2}^4|k > \quad \lambda = \{ 1 \otimes 1 , \ \sigma_3 \otimes 1 , \ 1 \otimes \sigma_3 , \ \sigma_3 \otimes \sigma_3 \}
\]

**Scalars:**
\[
\begin{align*}
\lambda \times \psi_{-1/2}^4|k > & \quad \lambda = \{ 1 \otimes i\sigma_2 , \ 1 \otimes \sigma_1 , \ \sigma_3 \otimes i\sigma_2 , \ \sigma_3 \otimes \sigma_1 \} \\
\lambda \times \psi_{-1/2}^6|k > & \quad \lambda = \{ i\sigma_2 \otimes 1 , \ \sigma_1 \otimes 1 , \ i\sigma_2 \otimes \sigma_3 , \ \sigma_1 \otimes \sigma_3 \} \\
\lambda \times \psi_{-1/2}^8|k > & \quad \lambda = \{ i\sigma_2 \otimes \sigma_1 , \ i\sigma_2 \otimes i\sigma_2 , \ \sigma_1 \otimes i\sigma_2 , \ \sigma_1 \otimes \sigma_1 \}
\end{align*}
\]

\(^1\)Remember that world-sheet supersymmetry requires that the orbifold group acts in the same way on the bosonic and fermionic coordinates.
Including also the fermionic spectrum obtained from the Ramond sector one obtains 4 \( \mathcal{N} = 1 \) gauge and 12 chiral multiplets. The gauge theory living on a regular D3-brane that, as we have seen, is described by \( 4 \times 4 \) Chan-Paton factors, has thus gauge group \( U(1)_1 \times U(1)_2 \times U(1)_3 \times U(1)_4 \) and 12 chiral multiplets. It is convenient to use a basis in the space of the \( 4 \times 4 \) Chan-Paton factors where each diagonal entry corresponds to one of the \( U(1) \) factors:

\[
A_\alpha = \begin{pmatrix}
A_1^\alpha & 0 & 0 & 0 \\
0 & A_2^\alpha & 0 & 0 \\
0 & 0 & A_3^\alpha & 0 \\
0 & 0 & 0 & A_4^\alpha 
\end{pmatrix}
\tag{2.6}
\]

A similar structure holds for the 12 chiral multiplets, which, in the above gauge field basis, can be organized in three \( 4 \times 4 \) matrices given by

\[
\Phi_1 = \begin{pmatrix}
0 & a_1 & 0 & 0 \\
b_1 & 0 & 0 & 0 \\
0 & 0 & 0 & c_1 \\
0 & 0 & d_1 & 0 
\end{pmatrix}, \quad \Phi_2 = \begin{pmatrix}
0 & 0 & a_2 & 0 \\
0 & 0 & 0 & b_2 \\
c_2 & 0 & 0 & 0 \\
0 & d_2 & 0 & 0 
\end{pmatrix}, \quad \Phi_3 = \begin{pmatrix}
0 & 0 & 0 & a_3 \\
0 & 0 & b_3 & 0 \\
0 & c_3 & 0 & 0 \\
d_3 & 0 & 0 & 0 
\end{pmatrix}
\tag{2.7}
\]

where \( a_i, \ldots, d_i \) are each a chiral multiplet and we picked the same complex structure as in eq.(2.1).

The superpotential can be written as \( W = \text{tr}(\Phi_1 [\Phi_2, \Phi_3]) \) and can be easily generalized to the non abelian case. To avoid confusion, it is worth noticing that, contrary to the quartic superpotential of [12], this superpotential is renormalizable in the UV. As remarked in the introduction, our results are valid in that region, where we can use the perturbative expression for the \( \beta \)-functions with all the anomalous dimensions \( \gamma \approx 0 \).

Looking at the \( U(1) \) charges of these multiplets we see that each of them is charged with respect to two gauge fields. This means that the chiral multiplets transform in the bifundamental of any given couple \( I, J \) (\( I, J = 1, 2, 3, 4 \)) of gauge groups and that each chiral multiplet contributes to two different \( U(1)_I \). As a consequence there are 6 chiral multiplets charged under a given gauge group. If, instead of only one, we have \( N \) regular D3-branes, then the gauge theory living on them is a supersymmetric \( \mathcal{N} = 1 \) gauge theory with gauge group \( U_1(N) \times U_2(N) \times U_3(N) \times U_4(N) \) and 12 chiral multiplets, transforming according to the fundamental (anti-fundamental) representation of a given gauge group and carrying a flavor index in the fundamental (anti-fundamental) of one of the other 3 gauge groups. This theory, as expected, is conformal. Indeed for any of the four gauge groups the (Wilsonian) \( \beta \)-function reads:

\[
\beta_I = \frac{g_I^2}{(4\pi)^2} \left[ -\frac{11}{3} N + \frac{2}{3} N + \frac{1}{6} 6N + \frac{1}{3} 6N \right] = 0
\tag{2.8}
\]

\( \text{Gauge multiplet} \quad \text{Chiral multiplets} \)
In conclusion we have seen that a regular brane is described by Chan-Paton factors transforming according to a $4 \times 4$ representation of the discrete orbifold group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and that the gauge theory living on it is a conformal invariant theory.

It is known that a discrete abelian group, as the orbifold group $\mathbb{Z}_2 \times \mathbb{Z}_2$, has only one-dimensional irreducible representations. This means that a regular D3-brane must be decomposable into more elementary objects, the fractional D3-branes, whose Chan-Paton factors are indeed just numbers and transform irreducibly under the orbifold group. The fact that the representation of the orbifold group is reducible can be directly seen from the explicit expression for the regular representation given in eq. (2.5) where all the $4 \times 4$ matrices are diagonal. From them one can extract the four one-dimensional irreducible representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$ corresponding to the four types of fractional branes. They read as follows:

\begin{align*}
\gamma_1(e) &= +1 & \gamma_1(h_1) &= +1 & \gamma_1(h_2) &= +1 & \gamma_1(h_3) &= +1 \\
\gamma_2(e) &= +1 & \gamma_2(h_1) &= +1 & \gamma_2(h_2) &= -1 & \gamma_2(h_3) &= -1 \\
\gamma_3(e) &= +1 & \gamma_3(h_1) &= -1 & \gamma_3(h_2) &= +1 & \gamma_3(h_3) &= -1 \\
\gamma_4(e) &= +1 & \gamma_4(h_1) &= -1 & \gamma_4(h_2) &= -1 & \gamma_4(h_3) &= +1 
\end{align*}

The first column is related to the coupling of the fractional branes to the untwisted sector, while the other three columns correspond to the coupling of the fractional branes to the three twisted sectors (with charge according to the corresponding $\pm$ sign). As already noticed, these three twisted sectors are related to the three shrinking cycles $\mathcal{C}_i$ which are located at the orbifold fixed point. Since, according to the above table, fractional branes couple to all twisted sectors, a given fractional D3-brane, whose transverse space coincides with the orbifold directions, is stuck at the orbifold fixed point and there is no way of moving it. The same is true if we consider a bound state of an odd number of fractional D3-brane types. On the other hand, if we consider a bound state of an even number of fractional D3-brane types, it is possible to move it along some directions of the six-dimensional transverse space. Indeed, from the above table, one can easily see that a bound state of four fractional D3-branes, corresponding to a regular D3-brane, can be moved over the entire six-dimensional transverse space because the twisted charges cancel and there is no coupling with any twisted sector. For a bound state of two fractional D3-brane types, the charge under two twisted sectors cancels and the bound state is charged only under one twisted sector. Hence, the bound state is free to move along the two-dimensional space orthogonal to the four directions on which the one twisted sector is stuck. This observation will be relevant in section 4, when we will discuss D-brane probes.

In terms of constituent fractional branes it is now easy to understand the structure of the gauge group of $N$ regular D-branes as a product of the four $U(N)$ gauge groups we
have found before. Each of the $U(N)$ gauge fields corresponds to the massless open strings having their end-points on one of the four fractional D3-branes constituting the regular D3-brane. In particular, one could have a more general gauge group $U(N_1) \times U(N_2) \times U(N_3) \times U(N_4)$ if we had considered a more general bound state of fractional D3-branes consisting of $N_f$ fractional branes of each kind.

From the previous picture it follows that the gauge theory living on a bound state made of two fractional D3-branes of different kind has gauge group $U(1) \times U(1)$. As already noticed, such a D3-brane can now move in the two-dimensional subspace of the entire six-dimensional transverse space orthogonal to the four-dimensional space corresponding to the twisted sector to which the bound state is coupled to. The chiral multiplets are charged under both gauge fields, and represent the motion of a pair of fractional branes along one of the direction $z_i$. For instance, from eq. (2.7) we can see that $a_1, b_1$ are two chiral fields charged with respect to $A^1_\alpha$ and $A^2_\alpha$, so a pair of fractional branes of type 1 and 2 can move along the first complex direction. The same holds for a couple of fractional branes of type 3 and 4.

In string theory one can determine the properties of the fractional Dp-branes of the orbifold $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ by computing the vacuum energy $Z$ of the open strings stretched between two of them that is given by:

$$Z = \int_0^\infty ds \frac{T_{\text{NS-R}}}{T} \left[ \left( \frac{1 + (-1)^F}{2} \right) \left( e + h_1 + h_2 + h_3 \right) e^{-2\pi s(L_0-a)} \right]$$

(2.13)

where the first term under the trace performs the GSO projection, the second term the orbifold projection in the case of our orbifold, while $a = 1/2$ in the NS sector and $a = 0$ in the Ramond sector. The properties of the fractional D3-branes are easily studied by performing in eq. (2.13) the modular transformation $s \rightarrow t = 1/s$ that brings us to the closed string channel, and by rewriting $Z$ as a matrix element between two boundary states with the insertion of closed string propagator. Since the closed string theory living on our orbifold has an untwisted sector together with three twisted sectors we have to consider a boundary state for each of the previous four sectors. They are related to the four terms that appear in the second bracket under the trace in eq. (2.13). When one takes the $e$ inside the bracket corresponding in the closed string channel to the untwisted sector, one gets $1/4$ of the contribution of the open strings stretched between two D3-branes in flat space. This means that the boundary state corresponding to the untwisted sector is equal to the one in flat space apart from an additional normalization factor $1/2$. The other three terms in the second bracket in eq. (2.13), when rewritten in the closed string channel, will determine the boundary states corresponding to the three twisted sectors.

Let us consider for instance the term denoted with $h_1$ in the second bracket in eq. (2.13). Since $h_1$ (see table 1) acts by changing sign to $z_2$ and $z_3$, but leaving $z_1$ invariant, the
twisted boundary state that we will obtain will be equal to the one corresponding to the twisted sector of the orbifold $C^2/Z_2$, where $C^2$ is spanned by $z_2$ and $z_3$, with an additional normalization factor $1/\sqrt{2}$ with respect to the case of the orbifold $C^2/Z_2$. The same is true for the other two terms $h_2$ and $h_3$ where $C^2$ is spanned respectively by $z_1$ and $z_3$ and by $z_1$ and $z_2$. In conclusion in the case of the orbifold $C^3/(Z_2 \times Z_2)$ we can just take the boundary states as given in [13, 14] for the orbifold $C^2/Z_2$ and multiply them with the additional normalization factor $1/\sqrt{2}$.

The previous considerations allow us to write almost immediately the coupling of the fractional D3-branes of the orbifold $C^3/(Z_2 \times Z_2)$ from those published in eqs. (7) and (8) of [3] corresponding to the orbifold $C^2/Z_2$. The supergravity fields a fractional D3-brane couples to are the metric and the RR 4-form potential in the untwisted sector and 3 scalars $b_i$ and 3 4-form potential $A^i_4$ in the twisted sector. The 3 scalars and the 3 4-form potentials correspond to the dimensional reduction of the Kalb-Ramond 2-form potential $B_2$ and of the 6-form potential $C_6$ on the three cycles $C_i$, respectively (with charges according to eqs. (2.9)-(2.12)). In doing that one should remember that for our orbifold the background values of the $B_2$-fluxes are \[ b_0 \equiv \int_{C_i} B_2 = 4\pi^2 \alpha' \frac{1}{2} \]
for any $i$, and $b_i \equiv \int_{C_i} B_2 = b_0 + \bar{b}_i$, where $\bar{b}_i$ are the fluctuations of the fluxes around $b_0$.

The couplings of the untwisted fields with a fractional D3-brane are given by:

\[ \langle B|h \rangle = - \frac{T_3}{4} \alpha' \frac{1}{2} \tilde{b}_i V_4 \quad , \quad \langle B|C_4 \rangle = \frac{T_3}{4 \kappa} C_{0123} V_4 \]

where $T_3 = \sqrt{\pi}$ is the normalization of the boundary state which is related to the brane tension in units of the gravitational coupling constant $\kappa = 8\pi^7 (\alpha')^2 g_s$ [18], $V_4$ is the (infinite) world-volume of the D3-brane, and the index $\alpha$ labels the longitudinal directions.

The couplings of the twisted fields with a fractional D3-brane are given by:

\[ \langle B|\bar{b}_i \rangle = - \frac{T_3}{4 \kappa} \frac{1}{2\pi^2 \alpha'} \bar{b}_i V_4 \quad , \quad \langle B|A^i_4 \rangle = \frac{T_3}{4 \kappa} \frac{1}{2\pi^2 \alpha'} A^i_{0123} V_4 \]

From the previous couplings one can deduce the boundary action of a fractional D3-brane of any kind. For a brane of type 1, which has positive charge with respect to any twisted sector, the boundary action is given by:

\[ S_1 = - \frac{T_3}{4} \int d^4 x \sqrt{-\det G_{\alpha\beta}} \left[ 1 + \frac{1}{2\pi^2 \alpha'} \sum_{i=1}^{3} \bar{b}_i \right] + \]

\[ + \frac{T_3}{4} \int \left[ C_4 \left( 1 + \frac{1}{2\pi^2 \alpha'} \sum_{i=1}^{3} \bar{b}_i \right) + \frac{1}{2\pi^2 \alpha'} \sum_{i=1}^{3} A^i_4 \right] \]

(2.17)
where $\tau_3 = \frac{T_3}{\kappa}$. For the other types of fractional D3-branes the boundary action has the same structure, the only difference are the signs in the coupling to the twisted sectors which can be found in eqs. (2.9)-(2.12).

From the above couplings one can also compute the large distance behaviour of the various fields. For the metric one gets:

$$ds^2 \simeq \left(1 - \frac{Q}{2r^4}\right) \eta_{\alpha\beta} dx^\alpha dx^\beta + \left(1 + \frac{Q}{2r^4}\right) \delta_{lm} dx^l dx^m + ...$$  \hspace{1cm} (2.18)

where $\alpha, \beta = 0, ..., 3$; $l, m = 4, ..., 9$; $r = \sqrt{x^l x^m \delta_{lm}}$ and $Q = \pi g_s (\alpha')^2$ while for the untwisted 4-form potential one gets

$$C_4 \simeq -\frac{Q}{r^4} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 + ...$$  \hspace{1cm} (2.19)

The asymptotic behaviour of the twisted fields is instead given by

$$\tilde{b}_i \simeq K \log (\rho_i/\epsilon) + ...$$  \hspace{1cm} (2.20)

$$A^i_4 \simeq K \log (\rho_i/\epsilon) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 + ...$$  \hspace{1cm} (2.21)

where $\rho_i = |z_i|$, $\epsilon$ is a regulator and

$$K = \frac{T_3}{4\kappa} \frac{1}{2\pi^2 \alpha'} \frac{2\kappa^2 4}{2\pi} = 4\pi g_s \alpha'$$  \hspace{1cm} (2.22)

By considering fractional branes of type 2, 3 and 4 one gets similar results, the only difference being the sign of the coupling to the twisted fields, according to eqs. (2.10)-(2.12).

3 Classical solution for fractional D3-branes

In this section we derive the supergravity solution describing a bound state of $N_1$ fractional D3-branes of type 1, $N_2$ of type 2, $N_3$ of type 3 and $N_4$ of type 4 in the orbifold theory we are considering. Let us start from the action of type IIB supergravity in ten dimensions:

$$S_{IIB} = \frac{1}{2\kappa^2} \left\{ \int d^{10} x \sqrt{-\det G} \mathcal{R} - \frac{1}{2} \int \left[ d\phi \wedge *d\phi + e^{-\phi} H_3 \wedge *H_3 + e^{2\phi} F_1 \wedge *F_1 \\
+ e^{\phi} \tilde{F}_3 \wedge *\tilde{F}_3 + \frac{1}{2} \tilde{F}_5 \wedge *\tilde{F}_5 - C_4 \wedge H_3 \wedge F_3 \right] \right\}$$  \hspace{1cm} (3.1)

where

$$H_3 = dB_2 \ , \ F_1 = dC_2 \ , \ F_3 = dC_2 \ , \ F_5 = dC_4$$  \hspace{1cm} (3.2)
are, respectively the field strengths of the Kalb-Ramond two form and the 0-,2- and 4-form RR potentials, and

\[ \tilde{F}_3 = F_3 + C_0 \wedge H_3 , \quad \tilde{F}_5 = F_5 + C_2 \wedge H_3 \]  

(3.3)

As usual the self-duality constraint \( *\tilde{F}_5 = \tilde{F}_5 \) has to be implemented on shell.

Since we are interested in computing the classical solution of fractional D3-branes in the orbifold \( \mathbb{C}^3 / (\mathbb{Z}_2 \times \mathbb{Z}_2) \), it is convenient to introduce the complex fields:

\[ \tau = C_0 + ie^{-\phi} \quad \text{and} \quad G_3 = dC_2 + \tau dB_2 \]  

(3.4)

and the standard D3-brane ansatz for the untwisted fields \( G_{MN} \) and \( \tilde{F}_5 \), namely:

\[ ds^2 = H^{-1/2} \eta_{\alpha \beta} dx^\alpha dx^\beta + H^{1/2} \delta_{lm} dx^l dx^m \]  

(3.5)

\[ \tilde{F}_5 = dH^{-1} \wedge V_4 + *(dH^{-1} \wedge V_4) \]  

(3.6)

In terms of the complex fields in eq.(3.4), the equations of motion for the axion and dilaton become:

\[ d * d\tau + ie^\phi d\tau \wedge *d\tau + \frac{i}{2} G_3 \wedge *G_3 = 0 \]  

(3.7)

where, since the source we are interested in, namely fractional D3-branes, does not couple to the dilaton and the axion, there is not any source term in the right hand side of the above equation. Hence, requiring the above equation to be solved by constant dilaton and axion, one gets back the constraint \( G_3 \wedge *G_3 = 0 \). Noticing that

\[ *G_3 = -H^{-1} \hat{*}_6 G_3 \wedge V_4 \]  

(3.8)

where \( \hat{*}_6 \) depends only on the 6 transverse directions to the D3-brane and we have extracted all the warp factors, one can solve the constraint imposed by the scalar equation by requiring:

\[ \hat{*}_6 G_3 = -iG_3 \]  

(3.9)

The constant \( i \) on the left hand side, instead, has been fixed by observing that \( \hat{*}_6 \hat{*}_6 G_3 = -G_3 \). Eq.(3.9) seems a general condition satisfied by any classical solution generated by fractional branes living both on orbifold and conifold geometry and it is related to the supersymmetry properties of the system \([19, 20, 21]\).

The equations of motion for the two 2-forms can be grouped together in the following equation:

\[ d * G_3 + d\tau \wedge \left[ ie^\phi * G_3 + *H_3 \right] - i\tilde{F}_5 \wedge G_3 = -2i\kappa^2 \left[ \frac{\delta L_b}{\delta B_2} - \tau \frac{\delta L_b}{\delta C_2} \right] \]  

(3.10)
where $\mathcal{L}_b$ is the Lagrangian density of the given source. We now solve eq. (3.10) by using the ansatz for the untwisted fields given in eqs. (3.5) and (3.6), with constant dilaton and axion, as already noticed. By plugging then eq. (3.9) in eq. (3.10) we get:

$$H^{-1}d \ast_6 G_3 \wedge V_4 = 2i\kappa^2 \left[ \frac{\delta \mathcal{L}_b}{\delta B_2} - \tau \frac{\delta \mathcal{L}_b}{\delta C_2} \right]$$

(3.11)

In order to solve the previous equation we have to write an ansatz for the twisted fields, too. In this orbifold, as discussed in the previous section, there are four kinds of fractional branes, each of them coupled with all the twisted fields as it emerges from the linear couplings dictated by the boundary state. We want to find a classical background generated by a bound state made of all four different kinds of fractional branes. A natural ansatz compatible with the large distance behaviour of the twisted fields given in eqs. (2.20) and (2.21) is:

$$G_3 = d \gamma_i \wedge \omega^j_2$$

(3.12)

with $i = 1, 2, 3$, $\gamma_i = c_i + ib_i$ and where $c_i = \int_{C_i} C_2$ are the Hodge duals of the 4-form potentials the fractional D3-branes actually couple to. Since the twisted fields $\gamma_i$ are obtained by reducing $G_3$ along the cycles $C_i$, they can only depend on the coordinates $z_i$. In particular, inserting the above ansatz in eq. (3.9) one gets that $\gamma_i$ are analytic functions of $z_i$. Moreover, by plugging the ansatz (3.12) in eq. (3.11) we get, for each twisted component, the following equation:

$$\delta^{rs} \partial_r \partial_s \gamma_i - 2\pi i K f_i(N_I) \delta(x^{2i+2}) \delta(x^{2i+3}) = 0$$

(3.13)

where $r, s \in \{2i + 2, 2i + 3\}$, $K$ is defined in eq. (2.22) while the functions $f_i(N_I)$ depend on numbers $N_I$ of fractional branes of the four different types and are given by:

$$f_1(N_I) = N_1 + N_2 - N_3 - N_4$$
$$f_2(N_I) = N_1 - N_2 + N_3 - N_4$$
$$f_3(N_I) = N_1 - N_2 - N_3 + N_4$$

(3.14)

The different signs in the previous expressions are due to the signs appearing in the irreducible representations given in eqs. (2.9)-(2.12), each of them corresponding to a fractional brane of a given type.

One can easily see that the analytic solution of eq. (3.13) is:

$$\gamma_i = iK \left[ \frac{\pi}{2g_s} + f_i(N_I) \log(z_i/\epsilon) \right]$$

(3.15)

where the background value given in eq. (2.14) has been introduced. Let us now consider the field equation for the untwisted 4-form $C_4$ which in this case looks like:

$$d \ast \tilde{F}_5 - \frac{i}{2} G_3 \wedge \tilde{G}_3 + 2\kappa^2 \frac{\delta \mathcal{L}_b}{\delta C_4} = 0$$

(3.16)
and which determines the warp factor $H$. Inserting in this equation the Ansatz (3.3), (3.6) and (3.12), we get:

$$\delta^{lm}\partial_l\partial_m H + \frac{1}{4} \sum_i |\partial z_i|_i^2 \delta_i^3 (x) + 4\pi^3 Q f_0(N_I) \delta(x^4) \ldots \delta(x^9) = 0 \quad (3.17)$$

where $f_0(N_I) = N_1 + N_2 + N_3 + N_4$ and $Q$ is defined before eq.(2.19). The last equation is a generalization of the corresponding one for fractional D3-branes in the orbifold $\mathbb{C}^2/\mathbb{Z}_2$. In this case we have three, instead of one, terms depending on the twisted fields, and by plugging the result (3.15) in eq.(3.17), the solution will be just a triple copy of the solution found in [3]:

$$H(r, z_i) = 1 + f_0(N_I) \frac{Q}{r^4} + \frac{K^2}{4 r^4} \sum_i f_i (N_I)^2 \left[ \log \left( \frac{r^4}{\epsilon^2 (r^2 - \rho_i^2)} \right) - 1 + \frac{\rho_i^2}{r^2 - \rho_i^2} \right] \quad (3.18)$$

with $\rho_i = |z_i|$. One can finally check that our solution also satisfies the equation of motion for the metric. As a consistency check one can verify that the above solution does reproduce the expected asymptotic behaviour (2.18)-(2.21).

From eq.(3.18) one can see that the metric has a singularity of repulson type [22]. This is quite a general feature of supergravity solutions generated by non-conformal sources. The singularity shows up because of the presence of the $K$-dependent term in the function $H$, which is related to the coupling to the twisted fields. This coupling is absent in the case of regular branes, which, as discussed in section 3, are conformal. In $\mathcal{N} = 2$ theories these singularities are often cured by an enhançon mechanism [23], while for the $\mathcal{N} = 1$ conifold case discussed in [7] is the deformation of the conifold which gives back a singularity free solution. As we will show in the next section, in the present case it seems that an enhançon phenomenon is at hand, too, in agreement with observations recently made in [24]. The enhançon is a scale where new light string degrees of freedom become relevant, due to fractional D-strings becoming tensionless, in type IIB. This makes the supergravity action one has started with, unable to describe the physics at scales smaller than the enhançon. In principle, by including these extra degrees of freedom in the low energy action one should get back an enhançon free and singularity free solution, as discussed recently in [23]. This is what we expect to be the case in the $\mathcal{N} = 1$ situation we are discussing, too, although we will not address this problem here. In the next section, by doing a probe analysis we show that fractional brane probes become tensionless at the enhançon. This excises the unwanted singularity, and is enough for our present purpose, but at the same time limits the validity of our solution to distances bigger than the enhançon radius. It would be interesting to understand the relation between this $\mathcal{N} = 1$ version of the enhançon phenomenon, with the pure supergravity analysis performed in [7].
4 Non-conformal $\mathcal{N} = 1$ SYM and brane probe

According to the discussion in section 2, we will now consider the low energy dynamics of a bound state of, say, $N_1$ branes of type 1 and $N_2$ branes of type 2 (all what follows can be applied to any couple of fractional brane types). The supergravity background we are going to probe is then the one discussed in section 3 with $N_3 = N_4 = 0$. Making a probe analysis we will see that supergravity can predict the Wilsonian $\beta$-function of the corresponding gauge theory [27] (for a review on the probe technique we recommend [28]).

Before doing that, let us fix once for all the relation between the gauge couplings of the four possible gauge groups and the fluxes of the NS-NS 2-form $B_2$ along the three shrinking spheres. This can be easily done by considering, for any given type of fractional brane, the gauge kinetic term arising from the DBI action when a $U(1)$ $F_{\alpha\beta}$ field is switched-on on the world-volume. Let us consider, for instance, a brane of type 1. Its DBI action, which we have derived in section 2, can be equivalently written as

$$S = -\frac{T_3}{2\kappa} \int d^4x \sqrt{-\text{det}(G + 2\pi\alpha' F)_{\alpha\beta}} \left( \int_{\mathcal{C}_1} \hat{B}_2 + \int_{\mathcal{C}_2} \hat{B}_2 + \int_{\mathcal{C}_3} \hat{B}_2 - 1 \right)$$

(4.1)

Expanding the above action up to the quadratic terms in the gauge field one gets (recall the generators are normalized as $\text{Tr}(T^a T^b) = 1/2 \delta_{ab}$):

$$S = -\frac{T_3}{4\kappa} (2\pi\alpha')^2 \int d^4x \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \left( \int_{\mathcal{C}_1} \hat{B}_2 + \int_{\mathcal{C}_2} \hat{B}_2 + \int_{\mathcal{C}_3} \hat{B}_2 - 1 \right) + ...$$

(4.2)

By substituting $\frac{T_3}{4\kappa} (2\pi\alpha')^2 = 1/(8\pi g_s)$ one finally gets for the gauge coupling:

$$\frac{1}{g_1^2} = \frac{1}{8\pi g_s} \left( \int_{\mathcal{C}_1} \hat{B}_2 + \int_{\mathcal{C}_2} \hat{B}_2 + \int_{\mathcal{C}_3} \hat{B}_2 - 1 \right)$$

(4.3)

Repeating the above reasoning for all other kinds of fractional branes one ends up with the following set of relations:

$$\frac{1}{g_1^2} = \frac{1}{8\pi g_s} \left( \int_{\mathcal{C}_1} \hat{B}_2 + \int_{\mathcal{C}_2} \hat{B}_2 + \int_{\mathcal{C}_3} \hat{B}_2 - 1 \right)$$

(4.4)

$$\frac{1}{g_2^2} = \frac{1}{8\pi g_s} \left( 1 + \int_{\mathcal{C}_1} \hat{B}_2 - \int_{\mathcal{C}_2} \hat{B}_2 - \int_{\mathcal{C}_3} \hat{B}_2 \right)$$

(4.5)

$$\frac{1}{g_3^2} = \frac{1}{8\pi g_s} \left( 1 - \int_{\mathcal{C}_1} \hat{B}_2 + \int_{\mathcal{C}_2} \hat{B}_2 - \int_{\mathcal{C}_3} \hat{B}_2 \right)$$

(4.6)

$$\frac{1}{g_4^2} = \frac{1}{8\pi g_s} \left( 1 - \int_{\mathcal{C}_1} \hat{B}_2 - \int_{\mathcal{C}_2} \hat{B}_2 + \int_{\mathcal{C}_3} \hat{B}_2 \right)$$

(4.7)

Footnote 2: For the sake of simplicity in this formula (and subsequent ones) we have introduced the dimensionless field $\hat{B}_2$ which is related to $B_2$ as $\hat{B}_2 = (4\pi^2\alpha')^{-1} B_2$. 

13
These are the master formulæ relating gauge theory parameters (left hand side) with supergravity fluxes (right hand side). One should remember, however (as already discussed in the Introduction), that the probe analysis, in relating the energy scale of the gauge theory to some transverse length in the supergravity solution (for spherically symmetric solutions $\Lambda = (2\pi\alpha')^{-1} r$ [24]), implies that there should be some scalar field acquiring a v.e.v. in the effective gauge theory one is describing. This is because transverse directions are seen as scalar fields on the D-brane. Therefore, while the above formulæ are indeed correct, they cannot provide a probe analysis prediction for the pure $\mathcal{N} = 1$ super Yang-Mills theory since, in that case, there are no scalars to relate the energy with. For this reason one should use composite probes to test the supergravity background. We will reconsider the case of the pure $\mathcal{N} = 1$ theory at the end of the section.

Given the above general formulæ, let us now come back to the analysis of the bound state we want to probe. The effective gauge theory living on $N_1$ branes of type 1 and $N_2$ branes of type 2 is a $\mathcal{N} = 1$ super Yang-Mills with gauge group $U(N_1) \times U(N_2)$. The diagonal $U(1)$ factor is free and the relative $U(1)$ factor is subleading to first order in $1/N_1$ and $1/N_2$. Here we will be mainly concerned with the running of the couplings $g_1$ and $g_2$ for the semi-simple factor $SU(N_1) \times SU(N_2)$. In addition to the gauge multiplets, we have two chiral multiplets, one transforming in the $(N_1, \bar{N}_2)$ and the other in the $(\bar{N}_1, N_2)$. The two (Wilsonian) $\beta$-functions are

\[
\beta(g_1) = \frac{g_1^3}{(4\pi)^2} \left[ \frac{11}{3} N_1 + \frac{2}{3} N_1 + \frac{1}{6} 2N_2 + \frac{1}{3} 2N_2 \right] = - \frac{g_1^3}{(4\pi)^2} (3N_1 - N_2) \quad (4.8)
\]

\[
\beta(g_2) = \frac{g_2^3}{(4\pi)^2} \left[ \frac{11}{3} N_2 + \frac{2}{3} N_2 + \frac{1}{6} 2N_1 + \frac{1}{3} 2N_1 \right] = - \frac{g_2^3}{(4\pi)^2} (3N_2 - N_1) \quad (4.9)
\]

and the corresponding gauge couplings are

\[
\frac{1}{g_1^2} = \frac{1}{g_0^2} \left( 1 + \frac{g_0^2}{4\pi^2} \frac{3N_1 - N_2}{2} \log \mu \right) \quad (4.11)
\]

\[
\frac{1}{g_2^2} = \frac{1}{g_0^2} \left( 1 + \frac{g_0^2}{4\pi^2} \frac{3N_2 - N_1}{2} \log \mu \right) \quad (4.12)
\]

where $g_0$ is the bare gauge coupling which can be assumed to be the same for both groups without loss of generality, since it drops out of the $\beta$-function. To obtain the above gauge quantities from supergravity by probe analysis, one has to start from a bound state of $N_1 + 1$ and $N_2 + 1$ fractional branes of type 1 and 2, respectively, corresponding to $U(N_1 + 1) \times U(N_2 + 1)$ gauge theory.
We have seen that it is impossible to move a single brane from the orbifold point while it is possible to move a pair of branes of type 1 and 2 ("the probe"). This corresponds to "Higgsing" the gauge theory in a certain way. We shall now discuss this phenomenon in detail from the gauge theory point of view and show how one can relate the (frozen) coupling that one reads on the probe using the supergravity analysis to the (running) couplings of the $SU(N_1 + 1) \times SU(N_2 + 1)$ theory at the gauge symmetry breaking point.

In order to avoid unnecessary complications, we shall consider the $U(N_1 + 1) \times U(N_2 + 1)$ theory and neglect the contribution from the relative $U(1)$ field, which is subleading to first order in $1/N_1$ and $1/N_2$. The breaking

$$U(N_1 + 1) \times U(N_2 + 1) \rightarrow U(1)' \times U(N_1) \times U(N_2) \quad (4.13)$$

where the group $U(1)'$ refers to the gauge field on the probe, is accomplished by giving the following v.e.v. to the scalar components of the chiral multiplet $\Phi$:

$$a_1 = b_1^T = \begin{pmatrix} v & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad (4.14)$$

where $a_1$ is a $N_1 \times N_2$ matrix. We can assume that $v$ is real without loss of generality. Equation (4.14) represents a classical flat direction and corresponds to moving a pair of branes of type 1 and 2 away from the orbifold point in the $z_1$ direction.

If we write the gauge fields $A_1$ and $A_2$ corresponding to $U(N_1 + 1)$ and $U(N_2 + 1)$ respectively as square matrices it is clear that the gauge bosons that become massive are those in the first row and first column. More explicitly, if we write

$$A_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} A_1^0 & W_1^1 & \cdots & W_1^{N_1} \\ W_1^{1*} & \vdots & \ddots & \vdots \\ W_1^{N_1*} \\ \end{pmatrix}, \quad A_2 = -\frac{1}{\sqrt{2}} \begin{pmatrix} A_2^0 & W_2^1 & \cdots & W_2^{N_2} \\ W_2^{1*} & \vdots & \ddots & \vdots \\ W_2^{N_2*} \\ \end{pmatrix} \quad (4.15)$$

we see that all $W$ fields become massive and so does the linear combination $\propto g_1 A_1^0 + g_2 A_2^0$, whereas the linear combination $\propto g_2 A_1^0 - g_1 A_2^0$ remains massless. In the same way, all the chiral fields in the first rows and columns of $a_1$ and $b_1$ are "eaten" by the massive gauge multiplets except for one linear combination representing the motion of the probe. This chiral multiplet is not charged with respect to $g_2 A_1^0 - g_1 A_2^0$ and thus the theory on the probe becomes free and its coupling constant $g$ stops running. Thus, by reading off the

---

3In the following we shall always suppress the Lorentz index $\alpha$ on the gauge field.
value of such coupling in the IR we obtain information about the value of the couplings $g_1$ and $g_2$ at the breaking point.

To obtain the exact formula, we need to normalize the fields on the brane. The correct normalization is:

$$Z = \frac{1}{\sqrt{g_1^2 + g_2^2}} \left( g_1 A_1^0 + g_2 A_2^0 \right) \quad \text{and} \quad \gamma = \frac{1}{\sqrt{g_1^2 + g_2^2}} \left( g_2 A_1^0 - g_1 A_2^0 \right)$$ (4.16)

where $Z$ and $\gamma$ are the massive and massless bosons respectively. The relation between the probe coupling constant $g$ with $g_1$ and $g_2$ becomes thus:

$$\frac{1}{g^2} = \frac{1}{g_1^2} + \frac{1}{g_2^2} + \frac{2}{g_0^2} - \frac{1}{4\pi^2} (N_1 + N_2) \log \mu$$ (4.17)

where, in doing the last step, we have used eqs. (4.11) and (4.12).

Let us now consider the (probe) action for a bound state of a fractional brane of type 1 and a fractional brane of type 2. The former is

$$S_1 = -\frac{T_3}{2\kappa} \left[ \int d^4 x \sqrt{-\det G_{\alpha\beta}} \left( \int_{C_1} \hat{B}_2 + \int_{C_2} \hat{B}_2 + \int_{C_3} \hat{B}_2 - 1 \right) \right] +$$

$$+ \frac{T_3}{2\kappa} \left[ C_4 \left( \int_{C_1} \hat{B}_2 + \int_{C_2} \hat{B}_2 + \int_{C_3} \hat{B}_2 - 1 \right) - \int \sum_{i=1}^3 \hat{A}_i^4 \right]$$ (4.18)

while the latter is

$$S_2 = -\frac{T_3}{2\kappa} \left[ \int d^4 x \sqrt{-\det G_{\alpha\beta}} \left( 1 + \int_{C_1} \hat{B}_2 - \int_{C_2} \hat{B}_2 - \int_{C_3} \hat{B}_2 \right) \right] +$$

$$+ \frac{T_3}{2\kappa} \left[ \int C_4 \left( 1 + \int_{C_1} \hat{B}_2 - \int_{C_2} \hat{B}_2 - \int_{C_3} \hat{B}_2 \right) + \int \left( \hat{A}_1^4 - \hat{A}_2^4 - \hat{A}_3^4 \right) \right]$$ (4.19)

By summing them up one obtains

$$S_{1+2} = -\frac{T_3}{2\kappa} \left[ \int d^4 x \sqrt{-\det G_{\alpha\beta}} \left( 2 \hat{B}_2 + \int C_4 \int_{C_1} 2 \hat{B}_2 - \int 2 \hat{A}_1^4 \right) \right]$$ (4.20)

where the coupling to the last two twisted sectors has cancelled and the twisted fields left depend on $z_1$, only. The system is then free to move in the $x_4, x_5$ plane and probe computations are allowed, as anticipated.

In order to find the effective gauge coupling describing the low energy dynamics of the probe, one can simply repeat a reasoning similar to that at the beginning of the section. Indeed, repeating the DBI action expansion for the gauge field kinetic term previously described, one gets for the probe gauge coupling $g$:

$$\frac{1}{g^2} = \frac{2}{8\pi g_s} \int_{C_1} \hat{B}_2 = \frac{1}{8\pi g_s} + \frac{1}{4\pi^2} (N_1 + N_2) \log \mu_1$$ (4.21)

\footnote{We use for the R-R twisted fields $A_{[4]}^i$ the same convention we have introduced for the 2-form $B_2$.}
where we have inserted the classical solution given in eq. (2.20). From the above expression we get the corresponding (Wilsonian) \( \beta \)-function to be:

\[
\beta = \mu_1 \frac{\partial g}{\partial \mu_1} = -\frac{(8\pi g_s)^{1/2}}{\left[1 + \frac{8\pi g_s}{16\pi^2} \frac{N_1 + N_2}{2} \log \mu\right]^{3/2}} \frac{g_0^2(N_1 + N_2)}{16\pi^2} = -\frac{2(N_1 + N_2)}{(4\pi^2)} g^3
\]

This is the correct result expected from eqs. (4.4) and (4.5) which indeed imply that:

\[
\frac{1}{g_1^2} + \frac{1}{g_2^2} = \frac{2}{8\pi g_s} \int_{c_1} \hat{B}_{(2)} = \frac{1}{g^2}
\]

This coincides with the result from gauge theory found in eq. (4.17)\(^!\) As one can see, the supergravity prediction, eq. (4.21), is in precise (numerical) agreement with the above equation and eqs. (4.11) and (4.12). Notice that in eq. (4.17) one has \( g_0^2 = 16\pi g_s \), this being consistent with eqs. (4.11)-(4.12) and (4.4)-(4.5).

As anticipated in the previous section, an enhançon phenomenon seems at hand here. Indeed the probe becomes tensionless at a distance \( \hat{\rho}_1 \), the enhançon, given by:

\[
\hat{\rho}_1 = \epsilon e^{-\pi/2(N_1 + N_2)g_s}
\]

This excises the unwanted repulsion singularity from the solution, since it indicates the appearance of new light degrees of freedom which are expected to become relevant at the enhançon scale and to affect the low energy physics. At the same time, the vanishing of the probe at \( \hat{\rho}_1 \) makes the geometry at distances \( \rho < \hat{\rho}_1 \) out of reach. All gauge theory information are then confined to the perturbative region, as it is usually the case for situations in which an enhançon locus shows up in the geometry. Supergravity alone, at least using probe techniques, seems not able to give information on the non-perturbative region of the gauge theory. To go further one should include more states, as recently discussed in [25].

Eq. (4.23) gives the correct gauge theory prediction for any value of \( N_1, N_2 \). Since we are working with a perturbatively renormalizable theory in the UV, where the anomalous dimensions \( \gamma \) are small, it is possible for the two \( \beta \)-functions to be both UV-free (it is sufficient that \( 1/3N_1 < N_2 < 3N_1 \)). Also, one can make one of the gauge group conformal and the other running by choosing \( N_1 = 3N_2 \). In this way \( \beta(g_2) = 0 \) and one directly gets the \( \beta \)-function for \( g_1 \) from that of \( g \). This has to be compared to the case discussed in [29, 4, 27], which deals with the IR behaviour of the theory away from the gaussian fixed point, where \( \gamma \approx -1/2 \). In that case, for any choice of \( N_1, N_2 \), the two couplings will run in opposite directions.

As we have already discussed, we cannot really probe the pure \( \mathcal{N} = 1 \) super Yang-Mills with the probe technique. Nevertheless, let us notice the following fact. Consider, for instance, a supergravity background with just one kind of fractional D3-branes, i.e. \( N_1 \)
branes of type 1, which are described at low energy by pure $\mathcal{N} = 1$ super Yang-Mills with
gauge group $U(N_1)$. Plugging the value for the $B_2$-fluxes dictated by the corresponding
supergravity solution in formula (4.3) with $\rho_1 = \rho_2 = \rho_3 \equiv \rho$, one gets precisely the gauge
coupling and the Wilsonian $\beta$-function of $\mathcal{N} = 1$ pure super Yang-Mills, namely:
\[
\frac{1}{g^2_1} = \frac{1}{8\pi g_s} \left( \int_{C_1} \hat{B}_2 + \int_{C_2} \hat{B}_2 + \int_{C_3} \hat{B}_2 - 1 \right) = \frac{1}{g^2_0} \left( 1 + \frac{g^2_0}{4\pi^2} \frac{3N_1}{2} \log \mu \right) \quad (4.25)
\]
and
\[
\beta(g_1) = -\frac{3N_1}{(4\pi)^2} g^3_1 \quad (4.26)
\]
Physically this probe analysis cannot really be done since the probe does not have any
moduli associated to it, and is stuck at the orbifold fixed point. This is of course a region
which is out of reach of the supergravity solution, because corresponds to distances smaller
then the repulsion singularity. It is however worth noticing how the matching holds in this
case, too. Perhaps this result can be justified by a refinement of the probe analysis.

Acknowledgments

We thank M. Billò for participating in the early stage of this work and for many dis-
cussions, and M. Frau, E. Imeroni, A. Lerda, E. Lozano-Tellechea, B.E.W. Nilsson, F.
Roose, R. Russo, P. Salomonson, F. Sannino and D. Tsimpis for useful discussions. G.F.
and R.M. would like to thank NORDITA for the kind hospitality. This work is partially
supported by the EC RTN contracts HPRN-CT-2000-00131 and HPRN-CT-2000-00122.
M.B. is supported by a EC Marie Curie Postdoc Fellowship under contract number HPMF-
CT-2000-00847.

References

[1] M. R. Douglas, Enhanced gauge symmetry in M(atrix) theory, JHEP 07 (1997) 004,
hep-th/9612126.

[2] M. Douglas, B. Greene and D. Morrison, Orbifold resolution by D-branes, Nucl. Phys.
B506 (1997) 84, hep-th/9704151.

[3] D. Diaconescu, M. R. Douglas and J. Gomis, Fractional branes and wrapped branes,
JHEP 02 (1998) 013, hep-th/9712230.

[4] M. Billò, B. Craps and F. Roose, Orbifold boundary states from Cardy’s condition,
JHEP 01 (2001) 038, hep-th/0011060.
[5] M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda, R. Marotta and I. Pesando, Fractional D-branes and their gauge duals, JHEP 02 (2001) 014, hep-th/0011077.

[6] I. R. Klebanov and N. A. Nekrasov, Gravity duals of fractional branes and logarithmic RG flow, Nucl. Phys. B574 (2000) 263, hep-th/9911096.

[7] I.R. Klebanov and M.J. Strassler, Supergravity and a Confining Gauge Theory: Duality Cascades and $\chi$SB-Resolution of Naked Singularities, JHEP 0008 (2000) 052, hep-th/0007191.

[8] J. Polchinski, N = 2 gauge-gravity duals, Int. J. Mod. Phys. A16 (2001) 707, hep-th/0011193.

[9] M. Billò, L. Gallot and A. Liccardo, Classical geometry and gauge duals for fractional branes on ALE spaces, Nucl. Phys. B614 (2001) 254, hep-th/0105258.

[10] M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda and R. Marotta, N=2 gauge theories on systems of fractional D3/D7 branes, in press on Nucl. Phys. B, hep-th/0107057.

[11] D. Diaconescu and J. Gomis, Fractional branes and boundary states in orbifold theories, JHEP 10 (2000) 001, hep-th/9906242.

[12] I. R. Klebanov and E. Witten, Superconformal field theory on threebranes at a Calabi-Yau singularity, Nucl.Phys. B536 (1998) 199, hep-th/9807080.

[13] M. A. Shifman and A. I. Vainshtein, Solution Of The Anomaly Puzzle In Susy Gauge Theories And The Wilson Operator Expansion, Nucl.Phys. B277 (1986) 456 [Sov.Phys. JETP 64 (1986) 428].

[14] D.R. Morrison and M.R. Plesser, Non-Spherical Horizons, I, Adv. Theor. Math. Phys. 3 (1999) 1, hep-th/9810201; M.R. Douglas and B. Fiol, D-branes and discrete torsion. II, hep-th/9903031.

[15] P. Di Vecchia and A. Liccardo, D-branes in string theories I, hep-th/9912151; D-branes in string theories II, hep-th/9912273.

[16] M. Frau, A. Liccardo and R. Musto, The geometry of fractional branes, Nucl. Phys. B602 (2001) 39, hep-th/0012038.

[17] P. Aspinwall, Enhanced gauge symmetries and K3 surfaces, Phys. Lett. B357 (1995) 329, hep-th/9507012.

[18] P. Di Vecchia, M. Frau, I. Pesando, S. Sciuto, A. Lerda and R. Russo, Classical p-branes from boundary state, Nucl. Phys. B507 (1997) 259, hep-th/9707068.
[19] M. Grana and J. Polchinski, *Supersymmetric Three-Form Flux Perturbations on AdS₅*, Phys. Rev. D63 (2001) 026001, hep-th/0009211.

[20] S. Gubser, *Supersymmetry and F-theory realization of the deformed conifold with three-form flux*, hep-th/0010010.

[21] M. Cvetic, H. Lu and C.N. Pope, *Brane Resolution Through Transgression*, Nucl. Phys. B600 (2001) 103, hep-th/0011023.

[22] K. Behrndt, *About a class of exact string backgrounds*, Nucl.Phys. B455 (1995) 188, hep-th/9506106. M. Cvetic and D. Youm, *Singular BPS saturated states and enhanced symmetries of four-dimensional N=4 supersymmetric string vacua*, Phys. Lett. B359 (1995) 87, hep-th/9507160; R. Kallosh and A. Linde, *Exact supersymmetric massive and massless white holes*, Phys. Rev. D52 (1995) 7137, hep-th/9507022.

[23] C.V. Johnson, A.W. Peet and J. Polchinski, *Gauge theory and the excision of repulson singularities*, Phys. Rev. D61 (2000) 086001, hep-th/9911161.

[24] P. Merlatti, *The enhancon mechanism for fractional branes*, hep-th/0108016.

[25] M. Wijnholt and S. Zhukov, *Inside an enhancon: Monopoles and dual Yang-Mills theory*, hep-th/0110103.

[26] A.W. Peet and J. Polchinski, *UV/IR relations in AdS dynamics*, Phys.Rev. D59 (1999) 065011, hep-th/9809022.

[27] C.P. Herzog, I.R. Klebanov and P. Ouyang, *Remarks on the warped deformed conifold*, hep-th/0108101.

[28] C.V. Johnson, *D-brane Primer*, hep-th/0007170.

[29] I.R. Klebanov and A.A. Tseytlin, *Gravity Duals of Supersymmetric SU(N) x SU(N+M) Gauge Theories*, Nucl. Phys. B578 (2000) 123, hep-th/0002159.