Computationally Efficient Change Point Detection for High-Dimensional Regression

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Abstract. Large-scale sequential data is often exposed to some degree of inhomogeneity in the form of sudden changes in the parameters of the data-generating process. We consider the problem of detecting such structural changes in a high-dimensional regression setting. We propose a joint estimator of the number and the locations of the change points and of the parameters in the corresponding segments. The estimator can be computed using dynamic programming or, as we emphasize here, it can be approximated using a binary search algorithm with \(O(n \log(n) \text{Lasso}(n))\) computational operations while still enjoying essentially the same theoretical properties; here \(\text{Lasso}(n)\) denotes the computational cost of computing the Lasso for sample size \(n\). We establish oracle inequalities for the estimator as well as for its binary search approximation, covering also the case with a large (asymptotically growing) number of change points. We evaluate the performance of the proposed estimation algorithms on simulated data and apply the methodology to real data.

1. Introduction

Much progress and work has been done in the last decade on methodology and theory of high-dimensional data, and we refer to [13, 6] for some overview. The vast majority of the focus has been on regression or classification with homogeneous data from a model with the same high-dimensional parameter for all the samples. Such a homogeneity assumption is not realistic for some datasets, in particular for large-scale data where sample size and the dimensionality are large. Some work addressing the issue of heterogeneous data in high-dimensional settings include factor models [23, 7, 12], mixture regression models [26], change point regression models [22] or “maximin” worst case analysis [24].

Date: January 15, 2016.

This article was produced as part of the activities of FAPESP Research, Innovation and Dissemination Center for Neuromathematics (grant #2013/07699-0, S.Paulo Research Foundation). F.L. was partially supported by a FAPESP’s fellowship (grant #2014/00947-0) and CNPq’s fellowship (grant #233216/2014-6).
We consider here a change point, high-dimensional regression model. We propose a joint estimator, using regularization with $\ell_1$-norms of the parameters in different segments, for the number and the locations of the change points and for the parameters of each corresponding segment. We establish an oracle inequality and consistency for the number of change points, implying near optimal convergence rates for the underlying regression parameters. Our analysis includes the case where the number of change points can be large (and asymptotically growing).

Our estimator can be computed using dynamic programming. To markedly speed up computational time for large-scale data, we can use a computationally efficient binary search algorithm, having computational cost of the order $O(n \log(n)\text{Lasso}(n))$, to approximate the estimator [19, 16]; here Lasso$(n)$ denotes the cost to compute the Lasso for sample size $n$. A main result of our paper establishes that the binary search algorithm essentially enjoys the same theoretical properties as the original estimator. We thus provide a strong justification for using binary search in change point detection in large-scale regression problems.

We evaluate the performance of the estimation algorithms by means of simulations and we also show the utility of our approach for real data. Our work is related to the one in [22] and we will outline the differences in Section 1.1.

The problem of change point detection has been studied already by e.g. Page [25] and since the early 1980s there has been an explosion of contributions (see [18] and references therein). Change point models cover a wide range of applications, from e.g. econometrics [1, 9] to genomics [5, 4, 10]. In most of the literature, “change point detection” deals with the problem of finding the piecewise constant means in univariate or multivariate data, see for example [14] which contains many references. There are also some works studying changes in the parameters of autoregressive models [8] or on network data [21, 3]. A vast list of contributions on the change point detection problem can be found in the recent review paper [18] or in the repository [20]. However, change point models for high-dimensional regression or classification where the number of parameters can be much larger than sample size have not been considered very much.

1.1. Related work and our contribution. We propose a joint estimator of the change points and the parameters for each segment in a high-dimensional linear model, even in the case where the number of segments is unknown. To the best of our knowledge, there is only the independently developed work [22] which is related to our study in the sense of considering a similar motivation and high-dimensional model. In [21], an undirected Gaussian network model is considered which can be
broken into single regressions: the mathematical analysis is not treating the high-dimensional case, and the proposed approach is based on a total-variation, Fused Lasso type penalty with a corresponding approximate computational optimization only. As described next, our results cover multiple, high-dimensional change point regression models with corresponding theoretical guarantees of a computationally efficient algorithm.

In [22], a high-dimensional linear model with one potential change point for two different high-dimensional regression parameters is considered. We address here the situation with multiple change points, with a possibly growing number thereof as the sample size increases. In particular, we face here also the issue if efficient computation (as mentioned in the next paragraph) as well as the problem of determining the number of change points. We use the Lasso, similarly as in [22], for each segment arising from the change points and we then minimize an overall penalized residual sum of squares. We prove an oracle inequality for the penalized residual sum of squares procedure using a sum of $\ell_1$-norm penalties. The result implies near optimal convergence rates for the parameters and in addition, we obtain directly a consistent estimator for the possibly growing number of change points, without the need to do some additional model selection in the spirit of e.g. BIC.

Furthermore and especially important for large-scale data, and since we are considering multiple and possibly very many change points, we focus on the computational task as well whereas the case with one change point as in [22] is computationally very easy. While a dynamic programming algorithm works in general, we prove that a much more efficient binary search algorithm is consistent and has (essentially) the same rates in the oracle inequality as mentioned above. Of course, binary segmentation algorithms are not new, see for example [16], but the derivation of a theoretical consistency guarantee in the high-dimensional change point problem as considered here is entirely novel.

2. Change point model and estimation

Consider a sequence of independent observations $\{(Y_i, X_i)\}_{i=1}^n$ with $p$-dimensional covariates $X_i \in \mathbb{R}^p$ and univariate response $Y_i \in \mathbb{R}$. Assume $\{X_i\}_{i=1}^n$ are i.i.d. with covariance matrix $\Sigma$ and $\{Y_i\}_{i=1}^n$ are given by

$$Y_i = X_i^T \beta^{(i)} + \epsilon_i \quad (i = 1, \ldots, n),$$

where $\epsilon_1, \ldots, \epsilon_n$ are i.i.d., independent of $X_1, \ldots, X_n$, and $\{\beta^{(i)}\}_{i=1}^n$ are piecewise constant. The i.i.d. assumption for $\{X_i\}_{i=1}^n$ is made for notational simplicity but can be relaxed to an i.i.d. assumption within each segment where $\{\beta^{(i)}\}_{i=1}^n$ is constant.
That is, we assume there exists a \((k_0 + 1)\)-dimensional vector \(\alpha^0 = (\alpha^0_0, \ldots, \alpha^0_{k_0})\) satisfying
\[
0 = \alpha^0_0 < \alpha^0_1 < \ldots < \alpha^0_{k_0} = 1
\]
and \(k_0\) real vectors \(\beta^0(1), \ldots, \beta^0(k_0)\) in \(\mathbb{R}^p\) such that
\[
\beta^{(i)} = \sum_{j=1}^{k_0} \beta^0(j) \mathbf{1}\{i/n \in (\alpha^0_{j-1}, \alpha^0_{j}]\}.
\]
This means that the sequence \((X_1, Y_1), \ldots, (X_n, Y_n)\) is independent but only piece-wise identically distributed, with change points at the elements of \(\alpha^0\). To simplify notation here and in the sequel we assume, without loss of generality, that \(\alpha^0_j n \in \mathbb{N}\) for all \(j = 1, \ldots, k_0\).

Sometimes we will use matrix notation for the equations in (2.1). Given an interval \((u, v] \subset [0, 1]\) such that \(un, vn \in \mathbb{N}\) we will denote by \(Y_{(u,v]}\) the vector \((Y_{un+1}, \ldots, Y_{vn})^T\) and by \(\epsilon_{(u,v]}\) the vector \((\epsilon_{un+1}, \ldots, \epsilon_{vn})^T\). Analogously, \(X_{(u,v]}\) will denote the \((v - u)n \times p\) matrix \((X^{(1)}_{(u,v]}, \ldots, X^{(p)}_{(u,v]})\). Then the model in (2.1) can be written as
\[
Y_{(\alpha^0_{j-1}, \alpha^0_{j})} = X_{(\alpha^0_{j-1}, \alpha^0_{j})} \beta^0(j) + \epsilon_{(\alpha^0_{j-1}, \alpha^0_{j})}
\]
for \(j = 1, \ldots, k_0\).

We propose a joint estimator for the change points and the regression parameters in the model given by (2.1), (2.2) and (2.3), without assuming a known upper bound on the number of segments. Given a vector \(\alpha = (\alpha_0, \ldots, \alpha_k)\) satisfying
\[
0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1
\]
we denote by \(\ell(\alpha)\) the number of positive components, that is \(\ell(\alpha) = k\). This value also corresponds to the number of segments in the model. For any \(j = 1, \ldots, \ell(\alpha)\) we denote by \(I_j(\alpha)\) the \(j\)-th interval in \(\alpha\) and by \(r_j(\alpha)\) its length; that is \(I_j(\alpha) = (\alpha_{j-1}, \alpha_j]\) and \(r_j(\alpha) = \alpha_j - \alpha_{j-1}\). We will denote by \(r(\alpha)\) the smallest size of such intervals defined by
\[
r(\alpha) = \min_{j=1, \ldots, \ell(\alpha)} \{r_j(\alpha)\}.
\]

In the sequel we will denote by \(\| \cdot \|_r\) the \(r\)-norm in \(\mathbb{R}^p\). Given tuning parameters \(\lambda > 0, \gamma > 0\) and \(\delta > 0\), and see below for a discussion, we define the joint lasso estimator of the change point parameter \(\alpha^0\) and the coefficients \(\beta^0(1), \ldots, \beta^0(\ell(\alpha))\)
for the $\ell(\alpha^0)$-segments by

$$
\hat{\alpha} = \arg \min_k \arg \min_{\alpha: \ell(\alpha)=k} \left\{ \sum_{j=1}^k L_n(I_j(\alpha), \hat{\beta}(j)) + \gamma k \right\}
$$

$$
\hat{\beta}(j) = \arg \min_{\beta} \left\{ L_n(I_j(\alpha), \beta) + \lambda \sqrt{r_j(\alpha)} \|\beta\|_1 \right\}, \quad j = 1, \ldots, \ell(\alpha),
$$

where the loss function $L_n$ is given by

$$
L_n(I_j(\alpha), \beta) = \frac{\|Y_{I_j(\alpha)} - X_{I_j(\alpha)}\beta\|_2^2}{n}
$$

and the minimization in (2.7) is over the set of all vectors $\alpha = (\alpha_0, \ldots, \alpha_k)$ satisfying $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$ and $r(\alpha) \geq \delta$. The role of $\delta$ is to ensure that within each segment (between two consecutive candidates of change points) there are sufficiently many samples ensuring a reasonable accuracy of the corresponding estimated regression parameter. We sometimes refer to (2.7) as the global estimator which is contrasted with a computationally more efficient version in Section 2.2.

Note that we do not impose an upper bound for $k$, but the condition on the minimal spacing $r(\alpha) \geq \delta$ implies that $k \leq 1/\delta$.

We propose a cross-validation scheme for ordered data, as outlined in Section 5, to choose the tuning parameters $\lambda$ for regularizing with respect to high-dimensionality and sparsity and $\gamma$ for regularizing the number of segments. The ideal value of the parameter $\delta$ is related to the density of the true underlying change points: in practice, it should be chosen reasonably small such as $\delta = 0.1$ while from a theoretical viewpoint, one needs $O(\sqrt{\log(p)/n}) \leq \delta < r(\alpha^0) - O(\sqrt{\log(p)/n})$, i.e., smaller than the minimal distance between the true change points, but it cannot be chosen too small (not too fast convergence to zero asymptotically) for consistent estimation of the change points and the parameters, as described in the theoretical results in Section 8.

We relate the global estimator by considering the Lasso [27] for the sub-interval $(u, v]$ with $un, vn \in \mathbb{N}$, $vn - un \geq 1$ with parameter $\lambda/\sqrt{\max(v - u, \delta)}$. It is given by

$$
\hat{\beta}_{(u,v]} = \arg \min_{\beta} \left\{ \frac{\|Y_{(u,v]} - X_{(u,v]}\beta\|_2^2}{(v - u)n} + \frac{\lambda \|\beta\|_1}{\sqrt{\max(v - u, \delta)}} \right\}.
$$

Observe that the estimator $\hat{\beta}(j)$ in (2.8) equals

$$
\hat{\beta}(j) = \arg \min_{\beta} \left\{ \frac{L_n(I_j(\alpha), \beta)}{r_j(\alpha)} + \frac{\lambda}{\sqrt{r_j(\alpha)}} \|\beta\|_1 \right\},
$$

and therefore, as $r(\alpha) \geq \delta$, $\hat{\beta}(j)$ is equal to the Lasso estimator in (2.10) with $(u, v] = I_j(\alpha)$; that is $\hat{\beta}(j) = \hat{\beta}_{I_j(\alpha)}$. To compute $\hat{\beta}(j)$ we can use, for example, the
R-package \texttt{glmnet} [14], and for computing the vector $\hat{\alpha}$ in (2.7) we can use dynamic programming as described next.

2.1. Exact dynamic programming algorithm. We present first a dynamic programming approach, known for a long time [17, cf.], to compute the estimator in (2.7). It computes the optimum in (2.7) and the estimates in (2.8), at the computational cost of $O(n^2\text{Lasso}(n))$ operations where Lasso($n$) is the cost to compute the Lasso estimator for a sample of size $n$ (see also [2] and references therein).

Let $F_k(v)$ denote the minimum value of the function in (2.7) when considering only the sample $(Y_{(0,v]}, X_{(0,v]})$ and vectors $\alpha$ of size $\ell(\alpha) = k$; that is

$$F_k(v) = \min_{\alpha:\ell(\alpha) = k} \left\{ \sum_{j=1}^{k} L_n(I_j(\alpha), \hat{\beta}_j(\alpha)) + \gamma k \right\}.$$  

It is easy to see that the optimal $(k+1)$-dimensional vector $\alpha$ corresponding to $F_k(1)$ consists of $k - 1$ optimal change points over $(Y_{(0,\alpha_{k-1}]}, X_{(0,\alpha_{k-1}]})$ and a single segment over $(Y_{(\alpha_{k-1},1]}, X_{(\alpha_{k-1},1]})$, where $\alpha_{k-1}$ is the rightmost change point proportion. Moreover, the $k - 1$ segments over $(Y_{(0,\alpha_{k-1}]}$, $X_{(0,\alpha_{k-1}]}$) must minimize the function (2.7) for the sample $(Y_{(0,\alpha_{k-1}]}$, $X_{(0,\alpha_{k-1}]}$), leading to $F_{k-1}(\alpha_{k-1})$. In this way, the dynamic programming recursion is computed for any $v \in V_n = \{i/n: i = 1, \ldots, n\}$ by

$$F_1(v) = \begin{cases} L_n((0,v], \hat{\beta}_{(0,v]}) + \gamma, & \text{if } v \geq \delta; \\ +\infty, & \text{if } v < \delta. \end{cases}$$

$$F_k(v) = \min_{u \in V_n, u < v} \{ F_{k-1}(u) + L_n((u,v], \hat{\beta}_{(u,v]}) + \gamma \}, \quad k = 2, \ldots, k_{\text{max}},$$

where $k_{\text{max}}$ is an upper bound on $k$ (in our case $k_{\text{max}} = 1/\delta$). The estimator $\hat{\alpha}$ in (2.7) is computed by tabulating $F_1(v)$ for all $v \in V_n$ and then by computing $F_2(v)$ for all $v \in V_n$ and so on up to $F_{k_{\text{max}}}(v)$. The optimal value of $k$ is obtained by the equation

$$\hat{k} = \arg \min_{k=1,\ldots,k_{\text{max}}} \{ F_k(1) \}$$

and the vector $\hat{\alpha} = (0, \hat{\alpha}_1, \ldots, \hat{\alpha}_{k_{\text{max}}}, 1)$ is given by

$$\hat{\alpha}_{j-1} = \arg \min_{u \in V_n, u < \hat{\alpha}_j} \{ F_{j-1}(u) + L_n((u,\hat{\alpha}_j], \hat{\beta}_{(u,\hat{\alpha}_j]}) + \gamma \}, \quad j = 2, \ldots, \hat{k}.$$  

2.2. Binary Segmentation algorithm. Here we describe an efficient Binary Segmentation algorithm [29, 16, cf.] to approximate the estimator given by (2.7), with computational cost of the order $O(n \log(n)\text{Lasso}(n))$, where Lasso($n$) is the cost to compute the Lasso estimator for a sample of size $n$. The algorithm will not compute the global estimator defined in (2.7), but we will nevertheless provide in Section
3.2 theoretical guarantees for the algorithm which are the same as for the global estimator.

For \( u, v \in V_n = \{i/n : i = 1, \ldots, n\} \) denote by

\[
H(u, v) = \begin{cases} 
L_n((u, v], \hat{\beta}(u, v)) + \gamma, & \text{if } (v - u)n \geq 1; \\
0, & \text{otherwise}
\end{cases}
\]

and define

\[
h(u, v) = \arg \min_{s \in \{u\} \cup [u + \delta, v - \delta]} \{H(u, s) + H(s, v)\}.
\]

The idea of the Binary Segmentation algorithm is to compute the best single change point for the interval \((0, 1]\) (given by \(h(0, 1) \neq 0\)) and then to iterate this criterion on both segments separated by this point, until no more change points are found (due to the penalty in the objective function). We can describe this algorithm by using a binary tree structure \(T\) with nodes labeled by sub-intervals of the form \((u, v] \in V_n^2\) such that \((v - u)n \geq 1\). The steps of the algorithm are then given by:

1. Initialize \(T\) to the tree with a single root node labeled by \((0, 1]\).
2. For each terminal node \((u, v]\) in \(T\) compute \(s = h(u, v)\). If \(s > u\) add to \(T\) the additional nodes \((u, s]\) and \((s, v]\) as descendants of node \((u, v]\).
3. Repeat 2. until no more nodes can be added to \(T\).

The set of terminal nodes in \(T\), denoted by \(T^0\), will produce the estimated change point vector \(\hat{\alpha}_{bs}\), by picking up the extremes in these intervals; that is

\[
\hat{\alpha}_{bs} = \bigcup_{(u, v] \in T^0} \{u, v\}.
\]

3. Theoretical properties

In this section we present the main theoretical results for the global estimator in (2.7), which can be computed with dynamic programming, as well as for the binary segmentation algorithm. In the sequel we denote by \(S(\beta)\) the support of a parameter vector \(\beta\), given by \(S(\beta) = \{i: \beta_i \neq 0\}\). Our assumptions are as follows.

**Assumption 1.** There exists \(K_X < \infty\) such that

\[
\|X_i\|_\infty \leq K_X
\]

and \(\mathbb{E}(X_i) = 0\) for all \(i\).

**Assumption 2.** There exists \(\sigma^2 < \infty\) such that

\[
\mathbb{E}(\epsilon_i^2) \leq \sigma^2
\]

and \(\mathbb{E}(\epsilon_i) = 0\) for all \(i\).
Assumption 3 (compatibility condition [28]). The covariance matrix $\Sigma$ is positive definite and the compatibility condition holds for $\Sigma$ and the set $S_\ast = \bigcup_{j=1}^{k_0} S(\beta^0(j))$, with constant $\phi_\ast > 0$. That is, for all $\beta \in \mathbb{R}^p$ that satisfy $\|\beta_{S_\ast}\|_1 \leq 3\|\beta_{S_\ast}\|_1$ it holds that

$$
(3.1) \quad \|\beta_{S_\ast}\|_1^2 \leq \frac{(\beta^T \Sigma \beta)_{S_\ast}}{\phi_\ast^2},
$$

where $s_\ast$ is the cardinality of $S_\ast$, see also [9, Ch.6.2.2].

We note that the compatibility constant $\phi_\ast^2$ is always lower-bounded by the minimal eigenvalue of $\Sigma$.

For any $0 \leq i \leq j < k \leq k_0$ denote by

$$
\gamma(i,j,k) = \frac{\alpha_j^0 - \alpha_j^{0-1}}{\alpha_k^0 - \alpha_i^{0-1}}.
$$

We assume the following condition on the vectors $\beta^0(1), \ldots, \beta^0(k_0)$ to guarantee the identifiability of the model parameters.

Assumption 4 (identifiability). If $k_0 > 1$ there exists a constant $m_\ast > 0$ such that

$$
\min_{1 \leq i \leq j < k \leq k_0} \frac{\|\sum_{r=i}^{j} \gamma(i,r,j)\beta^0(r) - \sum_{r=j+1}^{k} \gamma(j+1,r,k)\beta^0(r)\|_1}{s_\ast} \geq m_\ast.
$$

Remark 1. Observe that in the case $k_0 = 2$ the first condition amounts to say that $\|\beta^0(1) - \beta^0(2)\|_1 \geq m_\ast s_\ast$.

We will denote by $M_\ast$ the minimal upper bound such that

$$
\max_{1 \leq j \leq k_0} \|\beta^0(j)\|_\infty \leq M_\ast, \quad \text{and if } k_0 > 1 \text{ also: } \max_{1 < j \leq k_0} \|\beta^0(j-1) - \beta^0(j)\|_\infty \leq M_\ast.
$$

Given $K_X, \phi_\ast, M_\ast$ and $m_\ast$ specified by Assumptions 1-4 we define the constants

$$
d_\ast = \begin{cases} 
m_\ast^2 \phi_\ast^2 & \text{if } k_0 > 1 \\
+\infty & \text{if } k_0 = 1, \end{cases}
$$

and

$$
c_\ast = \left( \frac{K_X M_\ast}{d_\ast} + \frac{\sqrt{8}}{\phi_\ast} \right)^2.
$$
3.1. Global estimator with dynamic programming. For the global estimator in \((2.7)\) computed by dynamic programming, we present here a finite-sample result. The corresponding constants are not of main interest, and an asymptotic interpretation presented afterwards leads to simpler statements.

**Theorem 3.1.** Suppose Assumptions 1-4 hold. Given \(t > 0\), let \(\lambda, \delta, s_\ast\) and \(\gamma\) satisfy:

1. \(\delta + \lambda \sqrt{\delta}/d_\ast < r(\alpha^0)\),
2. \(\lambda/\sqrt{\delta} < M_\ast \phi^2_\ast/24\) and \(\lambda \sqrt{\delta} \geq \lambda_0\), with \(\lambda_0 = 40t\sigma K_\lambda \sqrt{\log(np)/n}\),
3. \(s_\ast < \frac{\lambda_1}{4c_\ast}\) with \(\lambda_1 = 10t K_\lambda^2 \sqrt{\log(np)/n}\),
4. \(\gamma > 6c_\ast \lambda^2 s_\ast\) and \(\gamma + 2\lambda \sqrt{\delta M_\ast} s_\ast < 4d_\ast M_\ast s_\ast \delta\).

Then, with probability at least \(1 - 2/t^2\) we have that

1. \(\ell(\hat{\alpha}) = k_0\),
2. \(\|\hat{\alpha} - \alpha^0\|_1 \leq \lambda \sqrt{\delta}/d_\ast\),
3. \(\sum_{j=1}^{k_0} \left\| \mathbf{X}_{I_j(\hat{\alpha})}(\hat{\beta}^{(j)} - \beta^0(j)) \right\|^2_2/n + \lambda \sqrt{r_j(\hat{\alpha})}\|\hat{\beta}^{(j)} - \beta^0(j)\|_1 \right\| \leq 4c_\ast k_0 \lambda^2 s_\ast\).

**Asymptotic interpretation.** For simplifying the discussion, assume that \(p \gg n\), \(K_\lambda = O(1), \sigma = O(1)\) and that \(\phi^2_\ast, M_\ast, c_\ast, d_\ast\) are all behaving like \(\propto O(1)\) (bounded away from zero and bounded above by a fixed constant). We then distinguish two cases, namely where \(r(\alpha^0) \propto \delta \propto O(1)\) and where \(r(\alpha^0) \propto \delta = o(1)\).

For \(r(\alpha^0) \propto O(1)\) (bounded away from zero), saying that the change points are well separated and there are only finitely many of them, we obtain \(\lambda \propto \sqrt{\log(p)/n}\), \(\lambda_1 \propto \sqrt{\log(p)/n}\) and we thus require that the sparsity \(s_\ast = O(\sqrt{n/\log(p)})\). This is a rather standard assumption for establishing an oracle inequality with \(\ell_1\)-norm control over the estimated parameter (as in statement (3)), see [6, Th.6.2-6.3]. We then obtain the following convergence rates which are analogous as for the Lasso in a standard high-dimensional sparse linear model:

\[
\|\hat{\alpha} - \alpha^0\|_1 = O_P(\sqrt{\log(p)/n}),
\sum_{j=1}^{k_0} \left\| \mathbf{X}_{I_j(\hat{\alpha})}(\hat{\beta}^{(j)} - \beta^0(j)) \right\|^2_2/n = O_P(s_\ast \log(p)/n),
\sum_{j=1}^{k_0} \|\hat{\beta}^{(j)} - \beta^0(j)\|_1 = O_P(s_\ast \sqrt{\log(p)/n}).
\]

For \(r(\alpha^0) \propto \delta = o(1)\), the conditions require that \(\delta^{-1} = O(\sqrt{n/\log(p)})\), i.e., \(\delta\) cannot converge faster to zero than \(\sqrt{\log(p)/n}\). In this regime where the change
points can be $O(\sqrt{\log(p)/n})$-dense and where there can be a growing number thereof, we obtain the results for “the minimal within segments sample size” $\delta n$. That is, $\lambda \asymp O(\sqrt{\log(p)/\delta n})$, and we require again that the sparsity $s_* = O(\sqrt{n/\log(p)})$. The convergence rates become

$$
\|\hat{\delta} - \alpha^0\|_1 = O_P(\sqrt{\log(p)/n}) \quad \text{(independent of $\delta$)},
$$

$$
\sum_{j=1}^{k_0} \|X_{I_j(\hat{\alpha})}(\hat{\beta}(j) - \beta^0(j))\|^2/n = O_P(s_*k_0 \log(p)/(\delta n)),
$$

$$
\sum_{j=1}^{k_0} \sqrt{r_j(\hat{\alpha})}\|\hat{\beta}(j) - \beta^0(j)\|_1 = O_P(s_*k_0 \sqrt{\log(p)/(\delta n)}).
$$

One can further distinguish whether $k_0$ would grow or not, with maximal growth rate of the order $O(\delta^{-1}) = O(\sqrt{n/\log(p)})$. A most extreme case happens when all the change points are equally dense with $k_0 \asymp O(\delta^{-1})$. For the expression $k_0 \sqrt{\log(p)/(\delta n)}$ to converge to zero we need that $\delta^{-1} = o((n/\log(p))^{1/3})$, that is a somewhat less dense regime, and the sparsity then needs to be of the order $s_* = o(\sqrt{\delta^3n/\log(p)})$ to imply that $s_*k_0 \sqrt{\log(p)/(\delta n)} = o(1))$. We summarize the asymptotic interpretations in Table 1.

| regime            | $\delta \asymp r(\alpha^0)$ | $\lambda$ | $k_0$         | $s_*$         |
|-------------------|-----------------------------|-----------|---------------|---------------|
| non-dense         | $> O(1)$                    | $O(\sqrt{\log(p)/n})$ | $O(1)$        | $o(\sqrt{\frac{n}{\log(p)}})$ |
| dense, finite $k_0$ | $\gg O(\sqrt{\log(p)/n})$ | $O(\sqrt{\log(p)/\delta n})$ | $O(1)$        | $o(\sqrt{\frac{\delta n}{\log(p)}})$ |
| equi-dense        | $\gg O(\log(p)^{1/3})$     | $O(\sqrt{\log(p)/\delta n})$ | $o((\frac{n}{\log(p)})^{1/3})$ | $o(\sqrt{\frac{\delta^3n}{\log(p)}})$ |

**Table 1.** Different asymptotic regimes such that $s_*k_0 \lambda = o(1)$ (which ensures convergence to zero for $\sum_{j=1}^{k_0} \sqrt{r_j(\hat{\alpha})}\|\hat{\beta}(j) - \beta^0(j)\|_1$).

3.2. **Binary Segmentation algorithm.** For the binary segmentation algorithm we obtain a similar result as for the global estimator.

**Theorem 3.2.** Suppose Assumptions 1-4 hold. Given $t > 0$, let $\lambda, \delta, s_*$ and $\gamma$ satisfy the conditions (1)-(4) in Theorem 3.1. Then, with probability at least $1 - 2/t^2$ we have that

1. $\ell(\hat{\alpha}^{bs}) = k_0$,
2. $\|\hat{\alpha}^{bs} - \alpha^0\|_1 \leq \lambda \sqrt{\delta/d_*}$ and
3. $\sum_{j=1}^{k_0} \left( \|X_{I_j(\hat{\alpha}^{bs})}(\hat{\beta}(j) - \beta^0(j))\|^2/n + \lambda \sqrt{r_j(\hat{\alpha}^{bs})}\|\hat{\beta}(j) - \beta^0(j)\|_1 \right) \leq 4c_*k_0\lambda^2 s_*$. 
We note that the conditions and statements in Theorem 3.2 for the binary segmentation algorithm are the same as for the global estimator in Theorem 3.1.

4. Simulation study

We evaluate here the performance of the global change point estimator computed with the dynamic programming algorithm (DPA) and of the binary segmentation algorithm (BSA). In the simulations we considered a two segments model, with $\alpha^0 = (0, 0.5, 1)$, $\beta^0(1) = (1, 1, 0, \ldots, 0)$, $\beta^0(2) = (0, \ldots, 0, 1, 1)$, and a three segments model with $\alpha^0 = (0, 0.3, 0.7, 1)$, $\beta^0(1) = (1, 1, 0, \ldots, 0)$, $\beta^0(2) = (0, \ldots, 0, 1, 1)$ and $\beta^0(3) = \beta^0(1)$. For both cases we use the standard deviation of the error $\sigma = 1$ and $X \sim N(0, \Sigma)$, for different structures of $\Sigma$:

1. $\Sigma_{ij} = 1_{\{i = j\}}$ for all $i, j$ (the identity matrix);
2. $\Sigma_{ij} = 0.8 |i - j|$ for all $i, j$ (Toeplitz matrix);
3. $\Sigma_{ij} = 1 - 0.8 \cdot 1_{\{i \neq j\}}$ for all $i, j$ (equi-correlation).

We consider a range of sample sizes and taking as number of covariates $p = 2n$. For all the simulation results, we always used the tuning parameters values $\delta = 0.25$, $\lambda = \sqrt{\log(p)/(\delta n)}$ and $\gamma = 0.25 \lambda$ without further fine-tuning (for both algorithms). For the computations we used the R software and the package glmnet \cite{15} to fit the parameters in each segment.

The results of the methods are shown in Figures 1-3. For each sample size we construct boxplots of the first change point fraction $\hat{\alpha}_1$ for 100 replications (when $\ell(\hat{\alpha}) = 1$ the first change point was treated as missing value). We also computed the proportion of $\ell(\hat{\alpha})$ in the 100 replications, to illustrate the performance in estimating the number of segments. As can be seen from Figures 1-3, the performances of the exact dynamic programming algorithm (DPA) and the binary segmentation algorithm (BS) are similar for larger sample size $n$. For small sample size $n$, the DPA method is superior to the BS algorithm in the three segments model and they both perform well in the two segments model. But the computational times of the algorithms are very different, as illustrated in Figure 4, where we show the mean time on 100 runs of each algorithm for each sample size. As expected, the BS algorithm scales much better with respect to sample size $n$.

5. Application to real data

We consider the “communities and crime data” (by M. Redmond) from the UCI Machine Learning Repository \hspace{1cm} \url{http://archive.ics.uci.edu/ml/datasets/Communities+and+Crime+Unnormalized#}. It comprises information from different
Figure 1. First estimated change point fraction $\hat{\alpha}_1$ and number of estimated segments $\ell(\hat{\alpha})$, as a function of sample size $n$ and $p = 2n$. Model (1) for covariance structure $\Sigma_{ij} = 1_{\{i=j\}}$ for all $i,j$. Top: global estimator (2.7) using DPA; Bottom: BS-algorithm. Left two panels: two segments model with $\alpha^0 = (0, 0.5, 1)$; Right two panels: three segments model with $\alpha^0 = (0, 0.3, 0.7, 1)$. The barplots correspond to the relative frequencies that the algorithm gave a estimated single segment model (blue), a two segments model (magenta), a three segments model (yellow) or a four or more segments model (green).

Communities in the U.S. and combines socio-economic data, from the 1990 US Census and the 1990 US Law Enforcement Management and Administrative Statistics Survey, and crime data from the 1995 US FBI Uniform Crime Report.

Besides specific information to identify the community (name, state, etc.) the dataset comprises 125 predictive variables (population, mean people per household, etc.) and 18 crime indices (number of murders per 100K population, number of violent crimes per 100K population, etc.). After removing all communities with missing values, we obtained a dataset with $n = 319$ communities and $p = 125$ covariates. We selected as response of interest the (scaled) number of murders per 100K population in 1995. We assigned to each community a number identifying its region in the following way: 1-South, 2-West, 3-Midwest, 4-Northeast (these regions are defined by the United States Census Bureau) and then ordered the sample by regions (with the original order from the dataset within every region).

As a cross-validation procedure, we selected a sub-sample of 160 communities with indices $\{2i - 1 : i = 1, \ldots, 160\}$ and a test sample comprising the communities with
Figure 2. As in Figure 1 but with model (2) for covariance matrix $\Sigma_{ij} = 0.8^{|i-j|}$ for all $i, j$. Top: global estimator (2.7) using DPA; Bottom: BS-algorithm. Left two panels: two segments model; Right two panels: three segments model. The barplots correspond to the relative frequencies that the algorithm gave a estimated single segment model (blue), a two segments model (magenta), a three segments model (yellow) or a four or more segments model (green).

indices $\{2i: i = 1, \ldots, 159\}$. For a fixed $\delta = 0.1$, $\lambda \in [0.001, 2]$ and $k \in \{1, \ldots, 10\}$ we computed the estimated $\alpha$ vector with $\ell(\alpha) = k$ given by the exact dynamic programming algorithm over the training dataset (i.e., we used the equivalent tuning parameter $k$ instead of $\gamma$) and we then computed the residual sum of squares over the test dataset. The results are summarized in Figure 5: the DPA (on top) attains the minimum at $\lambda = 0.051$ and $k = 2$; the BSA attains the minimum at $\lambda = 0.073$ and $k = 4$. We see that a one segment model is clearly out-performed with $k \geq 2$, with both algorithms DPA and BSA. We also see that the residual sum of squares curves for $k = 2$ or $k = 3$ are essentially the same for both DPA and BSA. Thus $k = 2$ or $k = 3$ almost leads to a minimum for the BSA, implying that $k \in \{2, 3\}$ seems plausible for both methods. This finding makes sense: if we assume that the data is homogeneous within each region, there would be at most 4 segments.

6. Conclusions

Large-scale data is often exposed to heterogeneity: we consider here the problem of detecting structural changes in the regression parameter of a high-dimensional linear model. We propose a regularized residual sum of squares estimator, mainly
Figure 3. As in Figure 1 but with model (3) for covariance matrix $\Sigma_{ij} = 1 - 0.8 \cdot 1_{i \neq j}$ for all $i, j$. Top: global estimator (2.7) using DPA; Bottom: BS-algorithm. Left two panels: two segments model; Right two panels: three segments model. The barplots correspond to the relative frequencies that the algorithm gave a estimated single segment model (blue), a two segments model (magenta), a three segments model (yellow) or a four or more segments model (green).

using $\ell_1$-norm regularization. The estimator can be either computed by dynamic programming or, as mainly advocated in this work, it can be greedily approximated by a computationally efficient scheme using recursive binary segmentation (BS algorithm). Despite that the BS algorithm will not compute the regularized residual sum of squares, we prove here the same theoretical properties for both methods: namely, the consistency for the true number of segments (which is allowed to grow asymptotically) and an oracle inequality implying a fast convergence rate for prediction and parameter estimation. Thus, the computationally much more efficient BS algorithm has the same theoretical guarantees as the estimator based on a global optimum of the regularized residual sum of squares. We illustrate the methods on simulated as well as on a real dataset.

Appendix A. Lasso estimator on a sub-interval

In this section we present non-asymptotic oracle inequalities for the estimators $\hat{\beta}_{(u,v)}$ in (2.10) that will be essential to derive Theorem 3.1. Given $k \in \mathbb{N}$ we denote by $I^m$ the set of intervals

$$I^m = \{(u, v] \subset (0, 1]: (v - u)n \geq m \text{ and } un, vn \in \mathbb{N}\}.$$
Figure 4. Average computation time of the Dynamic Programming algorithm (DPA), giving the exact global minimum of (2.7) and the Binary Segmentation algorithm (BSA), providing an approximation to the global minimum as a function of sample size. Left panel: one change-point model with $\alpha^0 = (0, 0.5, 1)$; right panel: two change-point model with $\alpha^0 = (0, 0.3, 0.7, 1)$.

We can view the set $I^m$ as the collection of all possible sub-intervals of the set \{1, \ldots, n\} with at least $m$ observations.

Given an interval $(u, v] \in I^1$ we define the oracle $\beta^*_{{(u,v]}}$ by

$$
\beta^*_{{(u,v]}} = \arg\min_{\beta} \|Y_{(u,v]} - X_{(u,v]}\beta\|^2_{L^2(P)}
$$

$$
= \arg\min_{\beta} \mathbb{E}\|Y_{(u,v]} - X_{(u,v]}\beta\|^2_2.
$$

(A.1)

As $\beta^*_{{(u,v]}}$ is the minimizer of the above expression we have that the vector $X_{(u,v]}\beta^*_{{(u,v]}}$ represents the best approximation to $Y_{(u,v]}$ in the linear subspace generated by the columns of $X_{(u,v]}$, with the inner product inherited from the $L^2(P)$ space.

For any $(u, v] \in I^1$, define

$$
\epsilon^*_{{(u,v]}} = Y_{(u,v]} - X_{(u,v]}\beta^*_{{(u,v]}},
$$

(A.2)

and let the set $\mathcal{T}_0$ be given by

$$
\mathcal{T}_0 = \left\{ \max_{(u, v] \in I^1} \max_{1 \leq j \leq p} 2|\epsilon^*_{{(u,v]}}^T X^{(j)}_{(u,v]}|/n \leq \lambda_0 \right\}.
$$

(A.3)

Now define the set $\mathcal{T}_1$ by

$$
\mathcal{T}_1 = \left\{ \max_{(u, v] \in I^1} \|\hat{\Sigma}_{(u,v]} - (v - u)\Sigma\|_\infty \leq \lambda_1 \right\},
$$

(A.4)
Figure 5. Residual sum of squares computed on the test sample of the “communities and crime” dataset for different values of $\lambda \in [0.001, 2]$ (only the range $[0.001, 0.1]$ is shown) and number of segments between $k \in \{1, \ldots, 10\}$ (different lines). Top: Dynamic programming algorithm (DPA); bottom: Binary search algorithm (BSA). For DPA (top), the minimum is attained at $\lambda = 0.051$ for a model with 2 segments, and for BSA (bottom) at $\lambda = 0.073$ for a model with 4 segments. For $k \in \{2, 3\}$, both algorithms DPA and BSA lead to essentially the same curves of the residual sum of squares as a function of $\lambda$. 
where

\[ \hat{\Sigma}_{(u,v)} = \frac{X_{(u,v)}^T X_{(u,v)}}{n}. \]

The following theorem shows oracle inequalities for the estimator (2.10) on the sub-interval \((u, v) \subset (0, 1]\).

**Theorem A.1.** If Assumption 3 holds then on the set \(T_0 \cap T_1\), with \(2\lambda_0 \leq \lambda_\sqrt{\delta}\) and \(s_0\lambda_1 \leq \frac{\phi^2_1}{32}\) we have that

\[
\|X_{(u,v)}(\hat{\beta}_{(u,v)} - \beta^*_s)\|^2_2/n + \lambda\sqrt{\max(v - u, \delta)}\|\hat{\beta}_{(u,v)} - \beta^*_s\|_1 \leq \frac{8\lambda^2 \max(v - u, \delta)s_0}{(v - u)\phi^2_1}.
\]

for all \((u, v) \in I^1\).

**Remark 2.** Observe that the bound on Theorem A.1 is uniform on the set \(I^{\delta n}\) with

\[
\max_{(u,v) \in I^{\delta n}} \left(\|X_{(u,v)}(\hat{\beta}_{(u,v)} - \beta^*_s)\|^2_2/n + \lambda\sqrt{v - u}\|\hat{\beta}_{(u,v)} - \beta^*_s\|_1\right) \leq \frac{8\lambda^2 s_0}{\phi^2_1}.
\]

**Corollary A.2.** Suppose Assumptions 1-3 hold. Given \(t > 0\) and \(\delta > 0\), suppose the regularization parameter \(\lambda\) satisfies

\[
\lambda \geq 40t\sigma K_X \sqrt{\frac{\log(np)}{\delta n}}.
\]

Then if

\[ s_0 < \frac{\lambda_1^{-1}\phi^2_1}{32}, \quad \text{with} \quad \lambda_1 = 10tK^2_X \sqrt{\frac{\log(np)}{n}}\]

we have, with probability at least \(1 - 2/t^2\), that

\[
\max_{(u,v) \in I^{\delta n}} \left(\|X_{(u,v)}(\hat{\beta}_{(u,v)} - \beta^*_s)\|^2_2/n + \lambda\sqrt{v - u}\|\hat{\beta}_{(u,v)} - \beta^*_s\|_1\right) \leq \frac{8\lambda^2 s_0}{\phi^2_1}.
\]

**Appendix B. Proofs**

In this section we present the proofs of the theoretical results in this paper. In the first subsection we prove the oracle inequalities for the Lasso estimator on a subinterval, stated in Theorem A.1 and Corollary A.2. In the second subsection we prove the consistency of the change point estimators, stated in Theorems 3.1 and 3.2.
B.1. Oracle inequalities for the Lasso estimator. We first prove a result about the compatibility condition.

**Lemma B.1.** Suppose Assumption 3 holds. Then on \( \mathcal{F}_1 \), if \( \lambda_1 \) satisfies \( s_* \lambda_1 \leq \frac{\phi_*^2}{32} \), with \( s_* \) the cardinality of \( S_* \), we have that for all \( (u,v) \in I^1 \) and all \( \beta \in \mathbb{R}^p \) that satisfy \( \|\beta_{S_*}\|_1 \leq 3\|\beta_{S_*}\|_1 \) it holds that

\[
\|\beta_{S_*}\|_1^2 \leq \frac{2(\beta^T \hat{\Sigma}_{(u,v)} \beta) s_*}{(v-u) \phi_*^2} .
\]

**Proof.** First note that by Assumption 3, for any \( (u,v) \in I^1 \) we have

\[
\|\beta_{S_*}\|_1^2 \leq \frac{(\beta^T (v-u) \Sigma \beta) s_*}{(v-u) \phi_*^2}
\]

for all \( \beta \in \mathbb{R}^p \) that satisfy \( \|\beta_{S_*}\|_1 \leq 3\|\beta_{S_*}\|_1 \). Therefore the matrix \( (v-u) \Sigma \) satisfies the compatibility condition for the set \( S_* \) with constant \( \sqrt{(v-u) \phi_*} \). Now, by [6] Corollary 6.8 we have that if \( s_* \lambda_1 \leq \frac{\phi_*^2}{32} \), the compatibility condition also holds for the set \( S_* \) and the matrix \( \hat{\Sigma}_{(u,v)} \) instead of \( (v-u) \Sigma \), with \( \phi_*^2 \geq (v-u) \phi_*^2 / 2 \). That means that for all \( \beta \in \mathbb{R}^p \) that satisfy \( \|\beta_{S_*}\|_1 \leq 3\|\beta_{S_*}\|_1 \) it holds that

\[
\|\beta_{S_*}\|_1^2 \leq \frac{(\beta^T \hat{\Sigma}_{(u,v)} \beta) s_*}{\phi_*^2} \leq \frac{2(\beta^T \hat{\Sigma}_{(u,v)} \beta) s_*}{(v-u) \phi_*^2} .
\]

□

We now prove a basic lemma that can be derived straightforward from [6, Lemma 6.3].

**Lemma B.2.** On \( \mathcal{F}_0 \) with \( 2\lambda_0 \leq \lambda \sqrt{\delta} \) we have that

\[
2\|X_{(u,v)}(\hat{\beta}_{(u,v)} - \beta^*_{(u,v)})\|^2/n + \lambda \sqrt{\max(v-u, \delta)} \|\hat{\beta}_{(u,v), S_*}\|_1 \leq 3\lambda \sqrt{\max(v-u, \delta)} \|\hat{\beta}_{(u,v), S_*} - \beta^*_{(u,v), S_*}\|_1
\]

for all \( (u,v) \in I^1 \).

**Proof.** Fix a interval \( (u,v) \in I^1 \) and denote by \( \tilde{\lambda} = \lambda \sqrt{\max(v-u, \delta)} \). The Basic Inequality in [6, Lemma 6.1], derived directly from the definition (2.10) gives

\[
\|X_{(u,v)}(\hat{\beta}_{(u,v)} - \beta^*_{(u,v)})\|^2/n + \tilde{\lambda} \|\hat{\beta}_{(u,v)}\|_1 \leq 2(\epsilon^T_{(u,v)} X_{(u,v)} (\hat{\beta}_{(u,v)} - \beta^*_{(u,v)}) / n + \tilde{\lambda} \|\beta^*_{(u,v)}\|_1 .
\]

Now, on \( \mathcal{F}_0 \) and using \( \tilde{\lambda} \geq 2\lambda_0 \) we have

\[
2\|X_{(u,v)}(\hat{\beta}_{(u,v)} - \beta^*_{(u,v)})\|^2/n + 2\tilde{\lambda} \|\hat{\beta}_{(u,v)}\|_1 \leq \tilde{\lambda} \|\hat{\beta}_{(u,v)} - \beta^*_{(u,v)}\|_1 + 2\tilde{\lambda} \|\beta^*_{(u,v)}\|_1 .
\]

(B.1)
By using the triangle inequality we obtain
\[
\|\tilde{\beta}_{(u,v)}\|_1 = \|\tilde{\beta}_{(u,v),S_\cdot}\|_1 + \|\tilde{\beta}_{(u,v),S_1}\|_1 \\
\geq \|\beta^*_{(u,v),S_\cdot}\|_1 - \|\tilde{\beta}_{(u,v),S_\cdot} - \beta^*_{(u,v),S_\cdot}\|_1 + \|\tilde{\beta}_{(u,v),S_1}\|_1.
\]

On the other hand we also have that
\[
\|\tilde{\beta}_{(u,v)} - \beta^*_{(u,v)}\|_1 = \|\tilde{\beta}_{(u,v),S_\cdot} - \beta^*_{(u,v),S_\cdot}\|_1 + \|\tilde{\beta}_{(u,v),S_1}\|_1.
\]

By plugin-in these last expressions in (B.1) we finish the proof of Lemma B.2.

We now prove the main result in Appendix A, given by Theorem A.1 and Corollary A.2.

**Proof of Theorem A.1.** The proof follows the same lines of reasoning as Theorem 6.1 in [6]. As before, fix a interval \((u, v) \in I^1\) and denote by \(\tilde{\lambda} = \lambda \sqrt{\max(v - u, \delta)}\). Then on \(\mathcal{B}_0 \cap \mathcal{B}_1\), if \(2\lambda_0 \leq \lambda \sqrt{\delta} \leq \tilde{\lambda}\) we have, by Lemma B.2 that
\[
2\|X_{(u,v)}(\tilde{\beta}_{(u,v)} - \beta^*_{(u,v)})\|_2^2/n + \tilde{\lambda}\|\tilde{\beta}_{(u,v)} - \beta^*_{(u,v)}\|_1 \\
= 2\|X_{(u,v)}(\tilde{\beta}_{(u,v)} - \beta^*_{(u,v)})\|_2^2/n + \tilde{\lambda}\|\tilde{\beta}_{(u,v),S_\cdot} - \beta^*_{(u,v),S_\cdot}\|_1 \\
+ \tilde{\lambda}\|\tilde{\beta}_{(u,v),S_\cdot} - \beta^*_{(u,v),S_\cdot}\|_1 \\
\leq 4\tilde{\lambda}\|\tilde{\beta}_{(u,v),S_\cdot} - \beta^*_{(u,v),S_\cdot}\|_1.
\]

Now, if \(s_\cdot\lambda_1 \leq \frac{\tilde{\lambda}^2}{2}\) then by Lemma B.1 and the inequality \(4ab \leq a^2 + b^2\) we can bound above the right hand side of the last expression by
\[
4\tilde{\lambda}\sqrt{2s_\cdot}\|X_{(u,v)}(\tilde{\beta}_{(u,v)} - \beta^*_{(u,v)})\|_2/\sqrt{(v - u)n} \phi_* \\
\leq \|X_{(u,v)}(\tilde{\beta}_{(u,v)} - \beta^*_{(u,v)})\|_2^2/n + 4\frac{\tilde{\lambda}^22s_\cdot}{(v - u)\phi_*^2}
\]
and this concludes the proof.

**Proof of Corollary A.2.** The result follows by combining the result in Theorem A.1 with Lemmas C.3 and C.4.

**B.2. Proofs of Theorems 3.1 and 3.2.** In this subsection we present the proof of Theorems 3.1 and 3.2 and all the auxiliary results.

We need some extra notation. Given the values \(u \leq \eta \leq v\) and vectors \(\beta, \beta^{(1)}\) and \(\beta^{(2)} \in \mathbb{R}^p\) we can write
\[
\|Y_{(u,v)} - X_{(u,v)}\beta\|_2^2 = \|Y_{(u,\eta)} - X_{(u,\eta)}\beta^{(1)}\|_2^2 + \|Y_{(\eta,v)} - X_{(\eta,v)}\beta^{(2)}\|_2^2 \\
+ \|D_{(u,\eta)}(\beta, \beta^{(1)})\|_2^2 - 2\epsilon^T_{(u,\eta)}(\beta^{(1)})D_{(u,\eta)}(\beta, \beta^{(1)}) \\
+ \|D_{(\eta,v)}(\beta, \beta^{(2)})\|_2^2 - 2\epsilon^T_{(\eta,v)}(\beta^{(2)})D_{(\eta,v)}(\beta, \beta^{(2)})
\]
(B.2)
where $D_{(u,\eta)}(\beta, \beta^{(1)}) = X_{(u,\eta)}(\beta - \beta^{(1)})$, $D_{(\eta,v)}(\beta, \beta^{(2)}) = X_{(\eta,v)}(\beta - \beta^{(2)})$, $\hat{\epsilon}_{(u,\eta)}(\beta^{(1)}) = Y_{(u,\eta)} - X_{(u,\eta)}\beta^{(1)}$ and $\hat{\epsilon}_{(\eta,v)}(\beta^{(2)}) = Y_{(\eta,v)} - X_{(\eta,v)}(\beta^{(2)})$.

We can now prove the following result.

**Lemma B.3.** Suppose $k_0 > 1$ and that Assumptions 1-4 hold. Then on $T_0 \cap T_1$, if $u < \alpha_j^0 < v$ for some $j = 1, \ldots, k_0 - 1$ and $s_\lambda \leq \frac{\phi}{8}$ we have

$$\frac{\|D_{(u,\alpha_j^0)}(\beta^{(u,\alpha_j^0)}, \beta^{(u,\alpha_j^0)})\|^2}{n} + \frac{\|D_{(\alpha_j^0, v)}(\beta^{(\alpha_j^0, v)}, \beta^{(\alpha_j^0, v)})\|^2}{n} \geq \frac{\min(\alpha_j^0 - u, v - \alpha_j^0)m_\lambda^2\phi^2}{8}.$$

**Proof.** Let $j = 1, \ldots, k_0 - 1$ and $u < \alpha_j^0 < v$ such that $(u, \alpha_j^0), (\alpha_j^0, v) \in I^1$. Denote by $\eta = \alpha_j^0$. By definition we have

$$\|D_{(u,\eta)}(\beta^{(u,\eta)}, \beta^0(j - 1))\|^2 = \|X_{(u,\eta)}(\beta^{(u,\eta)} - \beta^0(j - 1))\|^2$$

and a similar expression for $\|D_{(\eta,v)}(\beta^{(\eta,v)}, \beta^0(j))\|^2$. By Assumptions 3 and Lemma B.1 we have

$$\frac{\|X_{(u,\eta)}(\beta^{(u,\eta)} - \beta^{(u,\eta)})\|^2}{n} + \frac{\|X_{(\eta,v)}(\beta^{(\eta,v)} - \beta^{(\eta,v)})\|^2}{n} \geq \frac{(\eta - u)\|\beta^{(u,\eta)} - \beta^{(u,\eta)}\|^2\phi^2}{2s_\lambda} + \frac{(v - u)\|\beta^{(\eta,v)} - \beta^{(\eta,v)}\|^2\phi^2}{2s_\lambda},$$

Now observe that

$$(v - u)\beta^{(u,\eta)} = (\eta - u)\beta^{(u,\eta)} + (v - \eta)\beta^{(\eta,v)}$$

then

$$\beta^{(u,\eta)} - \beta^{(u,\eta)} = \left(\frac{v - \eta}{v - u}\right)(\beta^{(\eta,v)} - \beta^{(u,\eta)})$$

and by Assumption 4 and Lemma C.2 we have

$$\|\beta^{(u,\eta)} - \beta^{(u,\eta)}\|_1 \geq \frac{\eta - u}{v - u} \|\beta^{(\eta,v)} - \beta^{(u,\eta)}\|_1 \geq \frac{(v - u)m_\lambda s_\lambda}{(v - u)}.$$

Similarly we obtain

$$\|\beta^{(u,\eta)} - \beta^{(\eta,v)}\|_1 \geq \frac{\eta - u}{v - u} \|\beta^{(\eta,v)} - \beta^{(u,\eta)}\|_1 \geq \frac{(\eta - u)m_\lambda s_\lambda}{(v - u)}.$$

Then

$$\frac{(\eta - u)\|\beta^{(u,\eta)} - \beta^{(u,\eta)}\|^2\phi^2}{2s_\lambda} + \frac{(v - u)\|\beta^{(\eta,v)} - \beta^{(u,\eta)}\|^2\phi^2}{2s_\lambda} \geq \frac{(\eta - u)(v - \eta)\phi^2}{2(v - u)s_\lambda}$$

and as $\max(\eta - u, v - \eta)/(v - u) \geq 1/2$ this concludes the proof. □
Lemma B.4. For any interval \((u, \eta) \in I^1\) and any \(\beta \in \mathbb{R}^p\) we have
\[
2|\hat{\epsilon}^T_{(u, \eta)}(\hat{\beta}_{(u, \eta)})D_{(u, \eta)}(\beta, \hat{\beta}_{(u, \eta)})|/n \leq \lambda\sqrt{\eta-u}\|\beta - \hat{\beta}_{(u, \eta)}\|_1.
\]
Additionally, on \(\mathcal{T}_0\), if \(2\lambda_0 \leq \lambda\sqrt{\delta}\) we have
\[
2|\hat{\epsilon}^T_{(u, \eta)}(\beta^*_0, \eta)D_{(u, \eta)}(\beta, \beta^*_0, \eta)|/n \leq \frac{\lambda\sqrt{\delta}}{2}\|\beta - \beta^*_0, \eta\|_1.
\]

Proof. Note that
\[
\hat{\epsilon}^T_{(u, \eta)}(\hat{\beta}_{(u, \eta)})D_{(u, \eta)}(\beta, \hat{\beta}_{(u, \eta)}) = (Y_{(u, \eta)} - X_{(u, \eta)}\hat{\beta}_{(u, \eta)})^TX_{(u, \eta)}(\beta - \hat{\beta}_{(u, \eta)})
= (X^T_{(u, \eta)}(Y_{(u, \eta)} - X_{(u, \eta)}\hat{\beta}_{(u, \eta)}))^T(\beta - \hat{\beta}_{(u, \eta)}).
\]
By [6, Lemma 2.1] we have that as \(\hat{\beta}_{(u, \eta)}\) is the solution of (2.10) then
\[
\|2(X^T_{(u, \eta)}(Y_{(u, \eta)} - X_{(u, \eta)}\hat{\beta}_{(u, \eta)})/n\|_\infty \leq \frac{\lambda(\eta-u)}{\sqrt{\max(\eta-u, \delta)}} \leq \lambda\sqrt{\eta-u}.
\]
Therefore
\[
2|\hat{\epsilon}^T_{(u, \eta)}D_{(u, \eta)}(\beta)|/n \leq \|2(X^T_{(u, \eta)}(Y_{(u, \eta)} - X_{(u, \eta)}\hat{\beta}_{(u, \eta)})/n\|_\infty\|\beta - \hat{\beta}_{(u, \eta)}\|_1
\leq \lambda\sqrt{\eta-u}\|\beta - \hat{\beta}_{(u, \eta)}\|_1.
\]

The bound for \(2|\hat{\epsilon}^T_{(u, \eta)}(\beta^*_0, \eta)D_{(u, \eta)}(\beta, \beta^*_0, \eta)|/n\) is obtained analogously, by noting that on \(\mathcal{T}_0\), if \(2\lambda_0 \leq \lambda\sqrt{\delta}\) we have
\[
\|2(X^T_{(u, \eta)}(Y_{(u, \eta)} - X_{(u, \eta)}\beta^*_0, \eta)/n\|_\infty = \max_{j=1,\ldots,p} \|2\hat{\epsilon}^T_{(u, \eta)}X^{(j)}_{(u, \eta)}\| \leq \lambda_0 \leq \frac{\lambda\sqrt{\delta}}{2}.
\]

Lemma B.5. Suppose Assumptions 1-4 hold and let
\[
(u, v) \subset (\alpha^0_{j-1} - \lambda\sqrt{\delta}/d_\ast, \alpha^0_j + \lambda\sqrt{\delta}/d_\ast) \cap (0, 1]
\]
for some \(j = 1, \ldots, k_0\), with \((u, v) \in I^n\) and \(\lambda\sqrt{\delta} < r(\alpha^0)d_\ast\). Then on \(\mathcal{T}_0 \cap \mathcal{T}_1\), if \(s_\ast \lambda_1 \leq \frac{\phi^2}{32}\) and \(2\lambda_0 \leq \lambda\sqrt{\delta}\) we have
\[
\|X_{(u, v)}(\beta^0(j) - \hat{\beta}_{(u, v)})\|_2^2/n + \lambda\sqrt{v-u}\|\beta^0(j) - \hat{\beta}_{(u, v)}\|_1 \leq c(r)\lambda^2s_\ast,
\]
where
\[
c(r) = \left(\frac{rK_XM_\ast}{d_\ast} + \frac{\sqrt{\delta}}{\phi_\ast}\right)^2, \quad r = 1\{u < \alpha^0_{j-1}\} + 1\{\alpha^0_j < v\}.
\]

Note that Lemma B.5 is taking the bias \(\|\beta^*_0, (u, v) - \beta^0(j)\|_1\) into account, as pointed out in the proof.
Proof. Observe that
\[
\|X_{(u,v)}(\beta^0(j) - \hat{\beta}_{(u,v)})\|^2/n + \lambda\sqrt{v-u}\|\beta^0(j) - \hat{\beta}_{(u,v)}\|_1 \\
\leq ||X_{(u,v)}(\beta^0(j) - \beta^*_0(u,v))||^2/n + 2||X_{(u,v)}(\beta^0(j) - \beta^*_0(u,v))||2||X_{(u,v)}(\beta^*_0(u,v) - \hat{\beta}_{(u,v)})||2/n \\
+ ||X_{(u,v)}(\beta^*_0(u,v) - \hat{\beta}_{(u,v)})||^2/n + \lambda\sqrt{v-u}\|\beta^*_0(u,v) - \hat{\beta}_{(u,v)}\|_1 \\
+ \lambda\sqrt{v-u}\|\beta^0(j) - \beta^*_0(u,v)\|_1 .
\]
By Theorem [A.1] we obtain that
\[
\|X_{(u,v)}(\beta^*_0(u,v) - \hat{\beta}_{(u,v)})\|^2/n + \lambda\sqrt{v-u}\|\beta^*_0(u,v) - \hat{\beta}_{(u,v)}\|_1 \leq \frac{8\lambda^2s^*}{\phi^2_s}.
\]
On the other hand, if \(\lambda\sqrt{v} < r(\alpha^0)d_s\), we have
\[
\|\beta^0(j) - \beta^*_0(u,v)\|_\infty \leq \frac{\max(\alpha^0_j - u, 0)}{(v-u)}\|\beta^0(j) - \beta^0(j-1)\|_\infty \\
+ \frac{\max(v - \alpha^0_{j-1}, 0)}{(v-u)}\|\beta^0(j-1) - \beta^0(j)\|_\infty \\
\leq \frac{rM_s\lambda}{d_s\sqrt{v-u}}.
\]
Note that this inequality shows in particular that when \(u\) is at distance at most \(d_s\) of \(\alpha^0_{j-1}\) and \(v\) is at distance at most \(d_s\) of \(\alpha^0_j\) then the “bias” between \(\beta^0(j)\) and \(\beta^*_0(u,v)\), measured by \(\|\beta^0(j) - \beta^*_0(u,v)\|_1\), is of order \(dMs_s\). Then, by using this bound we also obtain that
\[
\|X_{(u,v)}(\beta^0(j) - \beta^*_0(u,v))\|^2/n + 2||X_{(u,v)}(\beta^0(j) - \beta^*_0(u,v))||2||X_{(u,v)}(\beta^*_0(u,v) - \hat{\beta}_{(u,v)})||2/n \\
\leq (v-u)s_sK_X^2\|\beta^0(j) - \beta^*_0(u,v)\|_\infty^2 \\
+ 2\sqrt{(v-u)s_sK_X\|\beta^0(j) - \beta^*_0(u,v)\|_\infty}\|X_{(u,v)}(\beta^*_0(u,v) - \hat{\beta}_{(u,v)})||2/\sqrt{n} \\
\leq \frac{r^2K_X^2M_s^2\lambda^2s^*}{d_s^2} + \frac{2\sqrt{8s_s}rK_XM_s\lambda^2}{d_s\phi^2_s}.
\]
By summing all the above bounds we obtain
\[
\|X_{(u,v)}(\beta^0(j) - \hat{\beta}_{(u,v)})\|^2/n + \lambda\sqrt{v-u}\|\beta^0(j) - \hat{\beta}_{(u,v)}\|_1 \leq c(r)\lambda^2s^* ,
\]
where
\[
c(r) = \left(\frac{rK_X^2M_s}{d_s} + \frac{\sqrt{8s_s}}{\phi^2_s}\right)^2.
\]
Now we can prove the main results in Section 3 and 4.

Proof of Theorem 3.2. We begin by proving that points 1-3 hold on \(\mathcal{T}_0 \cap \mathcal{T}_1\) if the conditions of the theorem are satisfied. Then the probability lower bound follows by combining this fact with Lemmas C.3 and C.4.
To simplify notation, given a vector $\alpha$ as in (2.5), with $r(\alpha) \geq \delta$, let us denote by $G(\alpha)$ the value of the function in (2.7) corresponding to the vector $\alpha$; that is

$$G(\alpha) = \sum_{j=1}^{\ell(\alpha)} L(I_j(\alpha), \hat{\beta}(j)) + \ell(\alpha)\gamma,$$

where $\hat{\beta}(j)$ is given by (2.8) and $L = L_n$. By the identity in (2.11) we have that

$$G(\alpha) = \sum_{j=1}^{\ell(\alpha)} L(I_j(\alpha), \hat{\beta}_{I_j(\alpha)}) + \ell(\alpha)\gamma,$$

where $\hat{\beta}_{I_j(\alpha)}$ is the Lasso estimator for the interval $I_j(\alpha)$ given by (2.10). In the sequel we will also need the function $G(\alpha)$ defined on vectors $\alpha$ such that $r(\alpha) < \delta$; in these cases we consider the “extended” version (B.4), because $\hat{\beta}_{I_j(\alpha)}$ is defined in (2.10) even if $r_j(\alpha) < \delta$.

For any $j = 1, \ldots, k_0$ denote by $B(\alpha_0^j, \lambda\sqrt{\delta}/d_*)$ the ball of center $\alpha_0^j$ and radius $\lambda\sqrt{\delta}/d_*$. First we will show that on $T_0 \cap T_1$, if the conditions of the theorem are satisfied we must have $\ell(\hat{\alpha}) = k_0$ and $\|\hat{\alpha} - \alpha_0^j\|_1 \leq \lambda\sqrt{\delta}/d_*$, by showing that

$$\hat{\alpha} \cap B(\alpha_0^j, \lambda\sqrt{\delta}/d_*) = \hat{\alpha}_j$$

for all $j = 1, \ldots, k_0$. To show this, we will prove that if $\hat{\alpha}$ does not satisfy (B.5) then there exists another ordered vector $\alpha = (\alpha_0, \ldots, \alpha_k)$ such that $\alpha_0 = 0$, $\alpha_k = 1$, $r(\alpha) \geq \delta$ and satisfying

$$G(\alpha) < G(\hat{\alpha})$$

which contradicts the fact that $\hat{\alpha}$ minimizes (2.7). So, suppose that (B.5) does not hold, we distinguish two possible cases:

(a) There exists some $i$, $1 \leq i \leq \hat{k} - 1$, such that $\{\hat{\alpha}_{i-1}, \hat{\alpha}_i, \hat{\alpha}_{i+1}\} \subset (\alpha_0^i - \lambda\sqrt{\delta}/d_*, \alpha_0^i + \lambda\sqrt{\delta}/d_*) \cap (0, 1]$ for some $j$, $1 \leq j \leq k_0$.

(b) $\hat{\alpha} \cap B(\alpha_0^j, \lambda\sqrt{\delta}/d_*) = \emptyset$ for some $j = 1, \ldots, k_0 - 1$.

In the case (a) define

$$\alpha = (\hat{\alpha}_0, \ldots, \hat{\alpha}_{i-1}, \hat{\alpha}_i, \hat{\alpha}_{i+1}, \ldots, \hat{\alpha}_{\ell(\hat{\alpha})})$$

so that $\ell(\alpha) = \ell(\hat{\alpha}) - 1$. Denote by $J_1$ and $J_2$ the intervals

$$J_1 = (\hat{\alpha}_{i-1}, \hat{\alpha}_i], \quad J_2 = (\hat{\alpha}_i, \hat{\alpha}_{i+1}]$$

and let $J$ denote their union $J = (\alpha_{i-1}, \alpha_{i+1}]$. Then we obtain

$$G(\alpha) - G(\hat{\alpha}) = L(J, \hat{\beta}_J) - L(J_1, \hat{\beta}_{J_1}) - L(J_2, \hat{\beta}_{J_2}) - \gamma.$$
By the definition (2.8) we have that
\[ L(J, \hat{\beta}_J) \leq L(J, \beta^0(j)) + \lambda \sqrt{|J|} \| \beta^0(j) - \hat{\beta}_J \|_1 \]
and by the equality (B.2) with \( \beta = \beta^0(j) \), \( \beta^{(1)} = \hat{\beta}_{J_1} \), \( \beta^{(2)} = \hat{\beta}_{J_2} \) and \( \eta = \hat{\alpha}_i \) we have that
\[ L(J, \beta^0(j)) = L(J_1, \hat{\beta}_{J_1}) + L(J_2, \hat{\beta}_{J_2}) + \frac{\| D_{J_1}(\beta^0(j), \hat{\beta}_{J_1}) \|_2^2}{n} - \frac{2 \epsilon_{J_1}(\hat{\beta}_{J_1}) D_{J_1}(\beta^0(j), \hat{\beta}_{J_1})}{n} + \frac{\| D_{J_2}(\beta^0(j), \hat{\beta}_{J_2}) \|_2^2}{n} - \frac{2 \epsilon_{J_2}(\hat{\beta}_{J_2}) D_{J_2}(\beta^0(j), \hat{\beta}_{J_2})}{n} \]
Then by Lemmas B.3 and B.5 we have that
\[ L(J, \beta^0(j)) - L(J_1, \hat{\beta}_{J_1}) - L(J_2, \hat{\beta}_{J_2}) \]
\[ \leq \| X_{J_1}(\beta^0(j) - \hat{\beta}_{J_1}) \|_2^2/n + \lambda \sqrt{|J_1|} \| \beta^0(j) - \hat{\beta}_{J_1} \|_1 \]
\[ + \| X_{J_2}(\beta^0(j) - \hat{\beta}_{J_2}) \|_2^2/n + \lambda \sqrt{|J_2|} \| \beta^0(j) - \hat{\beta}_{J_2} \|_1 \]
\[ (B.7) \]
\[ \leq 2c_s \lambda^2 s_* \]
Also by Lemma B.5 we have that
\[ \lambda \sqrt{|J|} \| \beta^0(j) - \hat{\beta}_J \|_1 \leq c(2) \lambda^2 s_* \leq 4c_s \lambda^2 s_* \]
therefore
\[ G(\alpha) - G(\hat{\alpha}) \leq 6c_s \lambda^2 s_* - \gamma \]
and if
\[ \gamma > 6c_s \lambda^2 s_* \]
we obtain \( G(\alpha) < G(\hat{\alpha}) \) which is a contradiction.
In case (b), let \( j_1 \) be such that \( \hat{\alpha} \cap B(\alpha^0_{j_1}, \sqrt{\delta}/d_*) = \emptyset \). We have \( 1 \leq j_1 \leq k_0 - 1 \). We now distinguish two possible sub-cases: (b1) \( \hat{\alpha} \cap B(\alpha^0_{j_1}, \delta) = \emptyset \); and (b2) \( \hat{\alpha} \cap B(\alpha^0_{j_1}, \delta) \neq \emptyset \). In case (b1) we define \( \alpha = \hat{\alpha} \cup \{ \alpha^0_{j_1} \} \) and then \( \alpha \) is a valid candidate vector for the minimization (2.7) because \( r(\alpha) \geq \delta \). Denote by \( J_1 \) and \( J_2 \) the intervals in \( \alpha \) that contain (as an extreme) the point \( \alpha^0_{j_1} \); that is
\[ J_1 = (\alpha_{r_i-1} \cup \alpha^0_{j_1}], \quad J_2 = (\alpha^0_{j_1} \cup \alpha_{r_i+1}], \quad \alpha_{r_i} = \alpha^0_{j_1} \]
and let \( J \) denote their union \( J = (\alpha_{r_i-1}, \alpha_{r_i+1}] \). We have that
\[ (B.8) \]
\[ G(\hat{\alpha}) - G(\alpha) = L(J, \hat{\beta}_J) - L(J_1, \hat{\beta}_{J_1}) - L(J_2, \hat{\beta}_{J_2}) - \gamma. \]
By the equality (B.2) (with \( \eta = v, \beta = \hat{\beta}_J \) and \( \beta^{(1)} = \beta^*_j \), Lemma B.4 and Theorem A.1 we obtain
\[ |L(J, \hat{\beta}_J) - L(J, \beta^*_j)| \leq \frac{8\lambda^2 s_*}{\delta_*^2} \]
and the same applies to the intervals $J_1$ and $J_2$. Then, one more time by the equality (B.2) (with $\beta = \beta_1^*$, $\beta^{(1)} = \beta_{j_1}^*$ and $\beta^{(2)} = \beta_{j_2}^*$), Lemmas B.3 and B.4 and Theorem A.1 we have that

$$L(J, \hat{\beta}_J) - L(J_1, \hat{\beta}_{J_1}) - L(J_2, \hat{\beta}_{J_2}) \geq L(J, \beta_1^*) - L(J_1, \beta_{j_1}^*) - L(J_2, \beta_{j_2}^*) - \frac{24\lambda^2 s_*}{\phi_s^2} \geq \frac{\delta m_2 \phi_s^2 s_*}{8} - \lambda \sqrt{\delta} M_s s_* - \frac{24\lambda^2 s_*}{\phi_s^2}$$

and therefore, as $\lambda < \sqrt{\delta} M_s \phi_s^2 / 24$ we obtain

$$G(\hat{\alpha}) - G(\alpha) \geq \frac{\delta m_2 \phi_s^2 s_*}{8} - 2\lambda \sqrt{\delta} M_s s_* - \gamma.$$ 

In this way, if

$$\gamma + 2\lambda \sqrt{\delta} M_s s_* < \frac{m_2 \phi_s^2 s_*}{8} \delta = 4d_* M_s s_* \delta$$

then (B.6) is satisfied, contradicting the fact that $\hat{\alpha}$ is the minimizer of (2.7).

For case (b2), a more elaborated argument is necessary, because if we add some of the points $\alpha_{j_1}^0$ to $\alpha$ we obtain vectors with intervals of length smaller than $\delta$. Then we need to add some points and to remove others in order to obtain a good candidate vector. Define the vector $\alpha^{(1)} = \hat{\alpha} \cup \{\alpha_{j_1}^0\}$. As before denote by $J_1$ and $J_2$ the intervals in $\alpha^{(1)}$ that contain (as an extreme) the point $\alpha_{j_1}^0$; that is

$$J_1 = (\alpha_{r_{i-1}}^{(1)}, \alpha_{j_1}^0], \quad J_2 = (\alpha_{j_1}^0, \alpha_{r_{i+1}}^{(1)}], \quad \alpha_{r_{i}}^{(1)} = \alpha_{j_1}^0,$$

and let $J$ denote their union $J = (\alpha_{r_{i-1}}^{(1)}, \alpha_{r_{i+1}}^{(1)}]$. By using the extended definition of $G$ in (B.4) we have that

(B.9) $$G(\hat{\alpha}) - G(\alpha^{(1)}) = L(J, \hat{\beta}_J) - L(J_1, \hat{\beta}_{J_1}) - L(J_2, \hat{\beta}_{J_2}) - \gamma.$$ 

If $|J_1| < \delta$, by the condition $\delta + \lambda \sqrt{\delta}/d_* < r(\alpha^0)$ we must have $r_i \geq 2$ and there must exist an interval $K_1 = (\alpha_{r_{i-2}}^{(1)}, \alpha_{r_{i-1}}^{(1)}]$ in $\alpha^{(1)}$ (adjacent to $J_i$ to the left), see Figure 6. Similarly for the interval $J_2$, if $|J_2| < \delta$ then we must have $r_i \leq \ell(\alpha^{(1)} - 2$ and there must exist an interval $K_2 = (\alpha_{r_{i+1}}^{(1)}, \alpha_{r_{i+2}}^{(1)}]$ in $\alpha^{(1)}$ (adjacent to $J_2$ to the right). To take only one case from now on we assume $|J_1| < \delta$ and $|J_2| \geq \delta$, the other possibilities can be handled in a similar way.

Now, we will construct a vector $\alpha$ obtained from $\alpha^{(1)}$ by removing the component $\alpha_{r_{i-1}}^{(1)}$, that is $\alpha = \alpha^{(1)} \setminus \{\alpha_{r_{i-1}}^{(1)}\}$. In this case, by the definition of the intervals $J_1$, $J_2$, $K_1$, $K_2$ and taking $I = K_1 \cup J_1$ (see Figure 6) we have that

$$G(\alpha^{(1)}) - G(\alpha) = L(K_1, \hat{\beta}_{K_1}) + L(J_1, \hat{\beta}_{J_1}) + \gamma - L(I, \hat{\beta}_I).$$
Therefore
\[ G(\hat{\alpha}) - G(\alpha) = G(\hat{\alpha}) - G(\alpha^{(1)}) + G(\alpha^{(1)}) - G(\alpha) \]
\[ = L(J, \hat{\beta}_J) - L(J_2, \hat{\beta}_{J_2}) + L(K_1, \hat{\beta}_{K_1}) - L(I, \hat{\beta}_I). \]

By using the same arguments and in case (b1), with the observation that \( |I| \geq \delta \) implies
\[
\sqrt{|I|} \| \beta^*_{K_1} - \hat{\beta}_I \|_1 \leq \sqrt{|I|} \| \beta^*_{J} - \hat{\beta}_I \|_1 + \sqrt{|I|} \| \beta^*_{K_1} - \beta^*_{J} \|_1
\]
\[
\leq \sqrt{|I|} \| \beta^*_{J} - \hat{\beta}_I \|_1 + \sqrt{|I|} \| \beta^*_{I} - \hat{\beta}_I \|_1 \leq \sqrt{|I|} \| \beta^*_{I} - \hat{\beta}_I \|_1 + \sqrt{\delta M} s_*
\]

we have that
\[
L(J, \hat{\beta}_J) - L(J_2, \hat{\beta}_{J_2}) + L(K_1, \hat{\beta}_{K_1}) - L(I, \hat{\beta}_I)
\geq L(J, \beta^*_J) - L(J_2, \beta^*_J) + L(K_1, \beta^*_K) - \frac{24 \lambda^2 s_*}{\phi^2_*}
\]
\[
- \lambda \sqrt{|I|} \| \beta^*_{K_1} - \beta^*_{I} \|_1
\]
\[
= L(J, \beta^*_J) - L(J_2, \beta^*_J) - L(J_1, \beta^*_J) - \frac{24 \lambda^2 s_*}{\phi^2_*}
\]
\[
- \lambda \sqrt{|I|} \| \beta^*_{J} - \hat{\beta}_I \|_1 - \lambda \sqrt{\delta M} s_*
\geq \frac{\lambda \sqrt{\delta m_* \phi^2_*}}{8 d_*} - 2 \lambda \sqrt{\delta M} \phi^2_* - \frac{32 \lambda^2}{\phi^2_*}
\]
\[
= 2 \lambda \sqrt{\delta M} \phi^2_* - \frac{32 \lambda^2}{\phi^2_*}.
\]

By the condition
\[
\lambda < \frac{\sqrt{\delta M} \phi^2_*}{24}
\]
we obtain
\[
G(\hat{\alpha}) - G(\alpha) > 0,
\]
contradicting the fact that \( \hat{\alpha} \) minimizes (2.7). The last point in the theorem can be derived directly from Lemma B.5 and \( \|\hat{\alpha} - \alpha^0\|_1 \leq \lambda\sqrt{\delta}/d_* \).

\[ \square \]

**Proof of Theorem 3.2.** First we will show that under the conditions of Theorem 3.1 on \( \mathcal{T}_0 \cap \mathcal{T}_1 \) we have that \( h(0,1) = 0 \) if \( k_0 = 1 \) or \( h(0,1) \) is at most at distance \( \lambda\sqrt{\delta}/d_* \) of some of the values in \( (\alpha^0_1, \ldots, \alpha^0_{k_0-1}) \) if \( k_0 > 1 \). This fact can be derived straightforward from the proof of Theorem 3.1, as the objective functions coincide for 1 or 2 segments; that is

\[ G((0,u,1]) = H(0,u) + H(u,1) \text{ for all } u \in [0,1], \]

where \( G \) is given by (B.4) and \( H \) is defined in (2.13).

So first suppose \( k_0 = 1 \) and \( \alpha^0 = (0,1) \). Then by the same arguments used in the proof of case (a) in Theorem 3.1 we have that for \( \alpha^0(u) = (0,u,1) \) we must have

\[ G(\alpha^0) < \min_{u \in (\delta,1-\delta)} G(\alpha(u)) \]

and therefore \( h(0,1) = 0 \). Now suppose \( k_0 > 1 \) and that \( h(0,1) \notin \bigcup_{j=1}^{k_0-1} \mathcal{B}(\alpha^0_j, \lambda\sqrt{\delta}/d_*) \), define

\[ \alpha^{(0)} = (0,h(0,1),1] \]
\[ \alpha^{(1)} = \alpha^{(0)} \cup \{\alpha^0_j\} \]
\[ \alpha^{(2)} = \alpha^{(1)} \setminus \{h(0,1)\}. \]

If \( h(0,1) = 0 \) (meaning that \( \ell(\hat{\alpha}_{bs}) = 1 \)) we can apply the arguments of case (b1) in Theorem 3.1 obtaining that \( G(\alpha^{(0)}) - G(\alpha^{(1)}) > 0 \). On the other hand, if \( h(0,1) \in [\delta, 1-\delta] \) we can apply the same argument of case (b2) in Theorem 3.1 obtaining

\[ G(\alpha^{(0)}) - G(\alpha^{(2)}) = G(\alpha^{(0)}) - G(\alpha^{(1)}) + G(\alpha^{(1)}) - G(\alpha^{(2)}) > 0. \]

In both cases we contradict the fact that \( h(0,1) \) minimizes (2.14). So, if \( k_0 > 1 \) we must have \( h(0,1) \in \bigcup_{j=1}^{k_0-1} \mathcal{B}(\alpha^0_j, \lambda\sqrt{\delta}/d_*) \), with \( j = 1, \ldots, k_0 - 1 \). Now we can replicate the same argument above on each one of the sub-intervals \( (0,h(0,1)] \) and \( (h(0,1),1] \) provided that \( \delta \leq r(\alpha^0) - 2\lambda\sqrt{\delta}/d_* \).

\[ \square \]

**APPENDIX C. AUXILIARY RESULTS**

Given an interval \( (u,v) \subset [0,1] \) denote by

\[ \gamma_{(u,v)}(\alpha^0_j) = |(u,v) \cap (\alpha^0_{j-1},\alpha^0_j)|/(v-u), \quad j = 1, \ldots, k_0, \]

where \( |(u,v) \cap (\alpha^0_{j-1},\alpha^0_j)| \) equals the length of the interval \( (u,v) \cap (\alpha^0_{j-1},\alpha^0_j) \).
Lemma C.1. If $\Sigma$ is positive definite, for any interval $(u, v) \in I_0^n$ we have that
\[
\beta_{(u,v)}^* = \sum_{j=1}^{k_0} \gamma_{(u,v)}(\alpha_j^0) \beta^0(j).
\]

Proof. Observe that for $i \in (\alpha_{j-1}^0, \alpha_j^0] \cap (un, vn]$ we have
\[
E[Y_i - X_i^T \beta]^2 = E[X_i^T (\beta^0(j) - \beta) + \epsilon_i]^2
= E[X_i^T (\beta^0(j) - \beta)]^2 + E(\epsilon_i^2).
\]
Therefore
\[
\beta_{(u,v)}^* = \arg\min_{\beta} \sum_{i=un+1}^{vn} E[X_i^T (\beta^0(i) - \beta)]^2
= \arg\min_{\beta} \sum_{j=1}^{k_0} [(u, v) \cap (\alpha_{j-1}^0, \alpha_j^0]] (\beta^0(j) - \beta)^T \Sigma (\beta^0(j) - \beta)
= \arg\min_{\beta} \beta^T \Sigma \beta - 2\beta^T \Sigma \left( \sum_{j=1}^{k_0} \gamma_{(u,v)}(\alpha_j^0) \beta^0(j) \right)
= \arg\min_{\beta} (\beta - \tilde{\beta})^T \Sigma (\beta - \tilde{\beta}),
\]
where
\[
\tilde{\beta} = \sum_{j=1}^{k_0} \gamma_{(u,v)}(\alpha_j^0) \beta^0(j).
\]
If $\Sigma$ is positive definite we have that the minimizer is $\beta_{(u,v)}^* = \tilde{\beta}$ and this concludes the proof of Proposition C.1. \qed

We now prove a basic result about the constant $m_*$ defined by Assumption 4.

Lemma C.2. If Assumption 4 holds then
\[
\inf_{j=1, \ldots, k_0-1} \inf_{(u, \alpha_j^0], (\alpha_j^0, v) \in I_0^n} \|\beta_{(u,\alpha_j^0)}^* - \beta_{(\alpha_j^0, v)}^*\|_1 \geq m_* s_*.
\]

Proof. As the $\ell_1$-norm is a sum over the different coordinates, we will minimize over each coordinate separately. So, fix $j = 1, \ldots, k_0$ and $i = 1, \ldots, p$; we will show that for any $y_i \in \mathbb{R}$ (fixed), the minimizer of
\[
|y_i - (\beta_{(\alpha_j^0, v)})_i|
\]
over the set $\{v: \alpha_j^0 \leq v \leq 1\}$ is one of the $(\beta_{(\alpha_j^0, v)})_i$, with $k = j + 1, \ldots, k_0$. But this is equivalent to the following optimization problem
\[
\text{Minimize: } \sum_{k=j}^{k_0-1} \max_{v-kj} (v - \alpha_k^0, 0) \frac{(y_i - \beta_i^0(k + 1))}{(v - \alpha_j^0)}
\]
\[
\text{Subject to: } \alpha_j^0 \leq v \leq \alpha_{k_0}^0 = 1,
\]
where the objective function is continuous and linear on each of the intervals $(\alpha_k^0, \alpha_{k+1}^0]$ for $k = j, \ldots, k_0 - 1$. Therefore the solution must be attained at one of the “vertices” 
$\{\alpha_k^0 : k = j + 1, \ldots, k_0\}$. The same result can be obtained fixing $v$ and minimizing over $u < \alpha_j^0$, then the statement of the lemma follows. $\square$

**Lemma C.3.** Suppose Assumptions 1 and 2 hold. Then for all $t > 0$ and

$$
\lambda_0 = 14t\sigma K_X \sqrt{\frac{\log(n^2p)}{n}}
$$

we have

$$
\mathbb{P}(\mathcal{F}_0) \geq 1 - 1/t^2.
$$

**Proof.** For any $i = 1, \ldots, n$ define the vector $Z_i \in \mathbb{R}^d$, with $d = pn(n - 1)$ as

$$(Z_i)_{j,(u,v)} = \begin{cases} 
\epsilon^*_{{(u,v),d}} X^{(j)}_i - \mathbb{E}(\epsilon^*_{{(u,v),d}} X^{(j)}_i); & \text{if } i/n \in (u,v], \\
0; & \text{c.c.} 
\end{cases}
$$

We have that $Z_1, \ldots, Z_n$ are independent, with $\mathbb{E}(Z_i) = 0$ for all $i$. By Assumptions 1 and 2 we also have that $\mathbb{E}\|Z_i\|_\infty^2 \leq 4\sigma^2 K_X^2$ for all $i$. Denote by $S_n = \sum_{i=1}^n Z_i$. By Markov’s inequality we have that

$$
\mathbb{P}\left(2\|n^{-1}S_n\|_\infty > \lambda_0\right) \leq \frac{4\mathbb{E}\|n^{-1/2}S_n\|_\infty^2}{n\lambda_0^2}.
$$

Now, by [11, Corollary 2.3] and Assumptions 1 and 2 we have that

$$
\mathbb{E}\|n^{-1/2}S_n\|_\infty^2 \leq n^{-1}(2e \log d - e) \sum_{i=1}^n \|Z_i\|_\infty^2 
\leq 8\epsilon K_X^2 \sigma^2 \log d
$$

therefore

$$
\mathbb{P}\left(2\|n^{-1}S_n\|_\infty > \lambda_0\right) \leq \frac{32\epsilon K_X^2 \sigma^2 \log(n^2p)}{n\lambda_0^2}.
$$

Moreover, as $\epsilon^*_{{(u,v)}}$ is orthogonal to $X^{(j)}_{(u,v)}$ in the $L^2(P)$ space for all $j = 1, \ldots, p$ and all $(u, v) \in I_0^n$ then

$$
2\|n^{-1}S_n\|_\infty = \max_{(u,v) \in I_0^n} \max_{1 \leq j \leq p} 2|\epsilon^*_{{(u,v),d}} X^{(j)}_i|/n.
$$

and this concludes the proof. $\square$

Now let $\mathcal{F}_1$ be given by

$$
\mathcal{F}_1 = \left\{ \max_{(u,v) \in I_0^n} \|\hat{\Sigma}_{(u,v)} - (v - u)\Sigma\|_\infty \leq \lambda_1 \right\},
$$

where

$$
\hat{\Sigma}_{(u,v)} = X^{T}_{(u,v)} X_{(u,v)}/n.
$$
Lemma C.4. If Assumption 1 holds then for all $t > 0$ and 
\[ \lambda_1 = 10tK^2 \sqrt{\frac{2\log(np)}{n}} \] 
we have 
\[ \mathbb{P}(\mathcal{F}_t) \geq 1 - 1/t^2. \]

Proof. For any $i = 1, \ldots, n$ define the vector $W_i \in \mathbb{R}^d$, with $d = p^2n(n-1)$ as 
\[ (W_i)_{j,l,(u,v)} = \begin{cases} 
X_{(u,v)}^{(j)}X_{(u,v)}^{(l)} - \mathbb{E}(X_{(u,v)}^{(j)}X_{(u,v)}^{(l)}) ; & \text{if } (u,v) \in I_0^n, \ i/n \in (u,v), \\
0 ; & \text{c.c.}
\end{cases} \]

We have that $W_1, \ldots, W_n$ are independent, with $\mathbb{E}(W_i) = 0$ for all $i$. By Assumption 1 we also have that $\mathbb{E}\|W_i\|_\infty^2 \leq 4K_X^4$ for all $i$. Denote by $S_n = \sum_{i=1}^n W_i$. By Markov’s inequality we have that 
\[ \mathbb{P}\left(\|n^{-1}S_n\|_\infty > \lambda_1\right) \leq \frac{\mathbb{E}\|n^{-1/2}S_n\|_\infty^2}{n\lambda_1^2}. \]

Now, by [11, Corollary 2.3] we have that 
\[ \mathbb{E}\|n^{-1/2}S_n\|_\infty^2 \leq n^{-1}(2e \log d - e) \sum_{i=1}^n \|W_i\|_\infty^2 \]
\[ \leq 8eK_X^4 \log d \]

therefore 
\[ \mathbb{P}\left(\|n^{-1}S_n\|_\infty > \lambda_1\right) \leq \frac{32eK_X^4 \log(n^2p^2)}{n\lambda_1^2}. \]

The proof finishes by noting that 
\[ \|n^{-1}S_n\|_\infty = \max_{(u,v) \in I_0^n} \max_{1 \leq j,l \leq p} \left| X_{(u,v)}^{(j)}X_{(u,v)}^{(l)}/n - (v-u)\Sigma_{j,l} \right| \]

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