IRREGULAR EGUCHI-HANSON METRICS AND THEIR
SOLITON ANALOGUES

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Abstract. We verify the extension to the zero section of momentum construction of Kähler-Einstein metrics and Kähler-Ricci solitons on the total space of positive rational powers of the canonical line bundle of toric Fano manifolds with possibly irregular Sasaki-Einstein metrics. More precisely, we show the metric extends to the associated unit circle bundle as a transversely Kähler-Einstein (Sasakian eta-Einstein) metric squashed in the Reeb flow direction.

1. Introduction

The Eguchi-Hanson metric ([10], 1979) is a complete Ricci-flat Kähler metric on the canonical line bundle of $\mathbb{CP}^1$, also expressed as a gravitational instanton. Its holonomy group is $SU(2) = Sp(1)$, and this gives a hyperkähler structure. Around the same period Calabi ([2], 1979) constructed independently a hyperkähler metric on the cotangent bundle of $\mathbb{CP}^m$ for $m \geq 1$. Calabi’s method was to reduce obtaining a Kähler potential with good property to an ordinary differential equations when there is a large group of symmetries. This method, now called the Calabi ansatz, was applied later in many ways by many other mathematicians, typically the momentum construction of Hwang-Singer [20] and Feldman-Ilmanen-Knopf [11]. In our papers [13], [15], [18], we took up the works of Hwang-Singer and Feldman-Ilmanen-Knopf to combine their ideas with the existence of Sasaki-Einstein metrics on toric Sasaki manifolds [17]. Among other things we tried to show the existence of a complete Ricci-flat Kähler metric on the canonical line bundle $K_M$ of a toric Fano manifold $M$ in [13], [15], and complete Kähler-Ricci solitons on some positive rational powers of $K_M$ in [18]. However we left open the issue of extension to the zero section of the line bundles when the Reeb vector field is irregular. In this paper we verify the extension to the zero section of momentum construction of Kähler-Einstein metrics and Kähler-Ricci solitons on the total space of some positive rational powers of canonical line bundle of toric Fano manifolds with possibly irregular Sasaki-Einstein metrics. Our results are described as follows.

Theorem 1.1. Let $M$ be a toric Fano manifold and $L$ a holomorphic line bundle over $M$ such that $K_M = L^\otimes p$ for some positive integer $p$. Then for an integer $k \geq p$, there exists a complete Kähler-Einstein metric $\omega_\varphi$ on the total space of $L^\otimes k$ with $\rho_{\omega_\varphi} = (\frac{2p}{k} - 2)\omega_\varphi$ where $\varphi$ is the profile of the momentum construction starting from the Kähler cone metric of a possibly irregular Sasakian $\eta$-Einstein (transversely Kähler-Einstein) metric $\omega$ on the $U(1)$-bundle of $L$. When restricted
to the $U(1)$-bundle the resulting metric is the starting $\eta$-Einstein metric squashed by a constant given by (18) in the Reeb flow direction.

In particular, this construction gives a complete Calabi-Yau metric on the total space of $K_M$, with $k=p$ in this case.

As indicated in the statement, this theorem uses the momentum construction of Hwang-Singer [20]. In the meantime after [13], the existence of Calabi-Yau metrics on crepant resolutions of Calabi-Yau cones have been obtained by gluing method by [25], [20], [19], [8], [9] by the method of the seminal work by Joyce [21]. Their results say that a Calabi-Yau metric exists for each K"ahler class in the 2nd cohomology class with compact support.

**Theorem 1.2.** Let $M$ be a toric Fano manifold and $K_M = L^\otimes p$, $p \in \mathbb{Z}^+$, then the total space of $L^\otimes k$ admits a complete expanding K"ahler-Ricci soliton if $k > p$, a complete steady K"ahler-Ricci soliton if $k = p$ and a complete shrinking K"ahler-Ricci soliton if $k < p$. These solitons are obtained by momentum construction, and when restricted to the unit circle bundle the resulting metric is the starting $\eta$-Einstein metric squashed by the positive number given by (18) in the Reeb flow direction.

In this soliton case we also use Hwang-Singer’s momentum construction while existence results using gluing method have been known by [24], [5] for expanding solitons and by [6] for steady solitons. For uniqueness there are works by [7], [23] and [3].

The difference of the arguments between our earlier works [13], [15], [18] and the present paper is as follows. In the earlier papers we tried to describe the metric near the zero section using the complex coordinate along the complexified Reeb flow, and since it is irregular in general it appeared difficult to use it for the description. In the present paper we describe the behavior of the metric on the level set of the radial coordinate $r$, which is the Sasakian $\eta$-Einstein manifold before starting the momentum construction, along the flow of the Euler vector field $r\partial/\partial r$, and see the metric converges as $r \to 0$ to a Sasakian $\eta$-Einstein metric squashed by a positive constant in the (real) Reeb flow direction.

After this introduction, in section 2, we recall basic facts about Sasakian geometry, the volume minimization of Martelli-Sparks-Yau [22] and the existence of Sasaki-Einstein metrics on toric Sasaki manifolds [17]. In section 3 we set up momentum construction for complete K"ahler-Einstein metrics and prove Theorem 1.1. In section 4 we study the soliton case and prove Theorem 1.2.

2. Preliminaries on Sasakian geometry.

In this section we briefly review on Sasakian geometry. The reader is referred to [1], [22], [4], [14] and [15] for more detail and related topics. A Sasakian manifold is by definition a Riemannian manifold $(S, g)$ whose Riemannian cone manifold $(C(S), \overline{g})$ with $C(S) \cong S \times \mathbb{R}^+$ and $\overline{g} = dr^2 + r^2 g$ is a K"ahler manifold, where $r$ is the standard coordinate on $\mathbb{R}^+$. From this definition $S$ has odd-dimension $\dim_{\mathbb{R}} S = 2m + 1$, and thus $\dim_{\mathbb{C}} C(S) = m + 1$. $S$ is always identified with the real hypersurface $\{r = 1\}$ in $C(S)$ together with the Riemannian submanifold structure and other various structures naturally equipped for a Sasakian manifold which are described below. Algebraically, $C(S)$ is an affine algebraic variety, and the apex of the cone is the origin which is the unique singularity.
To study the differential geometry of a Sasakian manifold, it is important to notice that the Kähler form $\omega$ of the cone $C(S)$ is expressed using the radial function $r$ as

$$\omega = i\partial\bar{\partial}r^2.$$ 

Thus the geometry of the Kähler cone is determined only from $r$ and the complex structure, denoted $J$. Since $S = \{r = 1\}$ the Sasakian geometry of $S$ is also determined only by $r$ and $J$. One can convince oneself of this fact and the facts described below if one examines the standard example of the unit sphere $\{r = 1\}$ in $\mathbb{C}^{m+1}$ with

$$r^2 = (|z_0|^2 + \cdots + |z_m|^2).$$

Putting $\tilde{\xi} = J(r \frac{\partial}{\partial r})$, $\tilde{\xi} - iJ\tilde{\xi}$ defines a holomorphic vector field on $C(S)$. The restriction of $\tilde{\xi}$ to $S = \{r = 1\}$, which is tangent to $S$, is called the Reeb vector field of $S$ and denoted by $\xi$. The Reeb vector field $\xi$ is a Killing vector field on $S$, and generates a 1-dimensional foliation $\mathcal{F}_\xi$, called the Reeb foliation on $S$. It is also possible to consider the flow of isometries generated by $\xi$, called the Reeb flow which we denote also by $\mathcal{F}_\xi$. The closure of the Reeb flow is a toral subgroup in the isometry group of $S$. If the dimension of the toral group is equal to (resp. greater than) 1 the Sasakian manifold is said to be quasi-regular (resp. irregular), and if the Sasakian manifold is quasi-regular and the $S^1$-action is free it is said to be regular. In the classical Sasakian geometry, the standard normalization of the metric is chosen so that the length of $\xi$ is 1. Below we see some clumsy constants e.g. $[2.1]$, $[2.2]$, but they come from this normalization. We will keep this normalization since it is natural as far as we adopt the above definition of Sasakian manifolds.

Let $\eta$ be the dual 1-form to $\xi$ using the Riemannian metric $g$, i.e. $\eta = g(\xi, \cdot)$. Then $\eta$ can be expressed as

$$\eta = (i(\partial - \bar{\partial}) \log r)|_{r=1} = (2d^c \log r)|_{r=1}.$$ 

Then $d\eta$ is non-degenerate on $D := \text{Ker } \eta$ and thus $S$ becomes a contact manifold with the contact form $\eta$. $D$ is called the contact bundle. The Reeb vector field $\xi$ satisfies

$$i(\xi)\eta = 1 \quad \text{and} \quad i(\xi)d\eta = 0$$

where $i(\xi)$ denotes the inner product, which are often used as the defining properties of the Reeb vector field for contact manifolds. The local orbit spaces of $\mathcal{F}_\xi$ admits a well-defined Kähler structure, and the pull-back of the local Kähler forms to $S$ are glued together to give a global 2-form

$$\omega^T = \frac{1}{2} d\eta = d(d^c \log r)|_{r=1} = (dd^c \log r)|_{r=1}$$

on $S$, which we call the transverse Kähler form. We call the collection of Kähler structures on local leaf spaces of $\mathcal{F}_\xi$ the transverse Kähler structure. A smooth differential form $\alpha$ on $S$ is said to be basic if

$$i(\xi)\alpha = 0 \quad \text{and} \quad L_\xi \alpha = 0$$

where $L_\xi$ denotes the Lie derivative by $\xi$. For example, the transverse Kähler form $\omega^T$ is a basic 2-form. The basic forms are lifted from differential forms on local orbit spaces of the Reeb flow, and preserved by the exterior derivative $d$ which decomposes into $d = \partial B + \bar{\partial} B$. We can define basic cohomology groups using $d$ and basic Dolbeault cohomology groups using $\bar{\partial} B$. We also have the transverse Chern-Weil theory and can define basic Chern classes for complex vector bundles.
with basic transition functions. As in the Kähler case the basic first Chern class $c_1^{B}$ of the Reeb foliation is represented by the $1/2\pi$ times the transverse Ricci form $\rho^T$:

$$\rho^T = -i\partial_B\overline{\partial}_B \log \det(g^T)$$

where

$$\omega^T = i g^T_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

and $z^1, \ldots, z^m$ are local holomorphic coordinates on the local orbit space. A Sasakian manifold $(S, g)$ is called a Sasaki-Einstein manifold if $g$ is an Einstein metric.

**Fact 2.1** (c.f. [1]). Let $(S, g)$ be a $(2m + 1)$-dimensional Sasaki manifold. The following three conditions are equivalent.

(a) $(S, g)$ is a Sasaki-Einstein manifold. The Einstein constant is necessarily $2m$:

$$\text{Ric}_g = 2mg$$

where $\text{Ric}_g$ denotes the Ricci curvature of $g$.

(b) $(C(S), \overline{\gamma})$ is a Ricci-flat Kähler manifold.

(c) The local orbit spaces of the Reeb flow have transverse Kähler-Einstein metrics with Einstein constant $2m + 2$:

$$\rho^T = (2m + 2)\omega^T.$$ 

One may compare Fact 2.1 with (3) - (6) below.

In the previous paragraph we defined the contact form $\eta$ and the transverse Kähler form $\omega$ on $S$. For the purpose of momentum construction of this paper, it is more convenient to consider them to be lifted to the Kähler cone $C(S)$ as

(1) $\eta = 2d^c \log r$

and

(2) $\omega^T = \frac{1}{2} d\eta = dd^c \log r$.

A moment thought shows the transverse Ricci form $\rho^T$ also lifts to $C(S)$. If $S$ is a Sasaki-Einstein manifold then by (c) of Fact 2.1 we have $c_1^B > 0$, i.e. $c_1^B$ is represented by a positive basic $(1,1)$-form. Moreover under the natural homomorphism $H^2_B(F_{\tilde{\xi}}) \to H^2(S)$ of basic cohomology group $H^2_B(F_{\tilde{\xi}})$ to ordinary de Rham cohomology group $H^2(S)$, the basic first Chern class $c_1^B$ is sent to the ordinary first Chern class $c_1(D)$, but by (2)

$$c_1(D) = (2m + 2)[\omega^T] = (m + 1)[d\eta] = 0.$$ 

Conversely if $c_1^B > 0$ and $c_1(D) = 0$ then $c_1^B = \tau[d\eta]$ for some positive constant $\tau$.

A Sasakian manifold $(S, g)$ is said to be toric if the Kähler cone manifold $(C(S), \overline{\gamma})$ is toric, namely if $(m + 1)$-dimensional torus $T^{m+1}$ acts on $(C(S), \overline{\gamma})$ effectively as holomorphic isometries. Then $T^{m+1}$ preserves $\xi$ because $T^{m+1}$ preserves $r$ and the complex structure $J$, and the flow generated by $\xi$ is contained in $T^{m+1}$. The condition of $c_1(D) = 0$ is a $Q$-Gorenstein property of the singularity of the Kähler cone $C(S)$ (c.f. [4]).
Theorem 2.2 \([17]\). A compact toric Sasaki manifold \(S\) with \(\mathbb{Q}\)-Gorenstein Kähler cone \(C(S)\) admits a possibly irregular Sasaki-Einstein metric by deforming the Sasaki structure by varying the Reeb vector field and then performing a transverse Kähler deformation.

The proof of Theorem 2.2 is outlined as follows. Fixing a Reeb vector field determines a fixed transverse holomorphic structure. By an idea due to Martelli-Sparks-Yau \([22]\) we deform the transverse holomorphic structure by deforming the Reeb vector field. We consider the volume functional on the space of Reeb vector fields by picking arbitrary transverse Kähler structure which is in fact independent of the choice of transverse Kähler structure, and find it is a convex proper function to obtain a unique critical point. Then we see the relevant obstruction \([12]\) vanishes. For this transverse holomorphic structure of the critical Reeb vector field we can solve the Monge-Ampère equation by changing the transverse Kähler metric, and get a transverse Kähler-Einstein metric. By Fact 2.1 we obtain a Sasaki-Einstein metric. The critical Reeb vector field is possibly irregular.

A Sasaki metric \(g\) is said to be \(\eta\)-Einstein if there exist constants \(\lambda\) and \(\nu\) such that

\[
\text{Ric}_g = \lambda g + \nu \eta \otimes \eta.
\]

By elementary computations in Sasakian geometry we always have \(\text{Ric}_g(\xi, \xi) = 2m\) on any Sasaki manifolds. This implies that

\[
\lambda + \nu = 2m.
\]

In particular, \(\lambda = 2m\) and \(\nu = 0\) for a Sasaki-Einstein metric.

Let \(\text{Ric}^T\) denote the Ricci curvature of the local orbit space of \(F_\xi\). Then again elementary computations show

\[
\text{Ric}_g = \text{Ric}^T - 2g^T + 2m\eta \otimes \eta
\]

and that the condition of \(\eta\)-Einstein metric is equivalent to

\[
\text{Ric}^T = (\lambda + 2)g^T.
\]

Given a Sasaki manifold with the Kähler cone metric \(\overline{g} = dr^2 + r^2g\), we transform the Sasakian structure by deforming \(r\) into \(r' = ra\) for positive constant \(a\). This transformation is called the \(D\)-homothetic transformation. Then the new Sasaki structure has

\[
\eta' = d \log ra = a \eta, \quad \xi' = \frac{1}{a} \xi,
\]

\[
g' = ag^T + a \eta \otimes a \eta = ag + (a^2 - a) \eta \otimes \eta.
\]

Suppose that \(g\) is \(\eta\)-Einstein with \(\text{Ric}_g = \lambda g + \nu \eta \otimes \eta\). Since the Ricci curvature of a Kähler manifold is invariant under homotheties we have \(\text{Ric}^T = \text{Ric}^{T'}\). From this and \(\text{Ric}_{g'}(\xi', \xi') = 2m\) we have

\[
\text{Ric}_{g'} = \text{Ric}^{T'} - 2g^{T'} + 2m\eta' \otimes \eta' = \lambda g^T + 2g^T - 2ag^T + 2m\eta' \otimes \eta'.
\]

This shows that \(g'\) is \(\eta\)-Einstein with

\[
\lambda' + 2 = \frac{\lambda + 2}{a}.
\]
In summary, under the $D$-homothetic transformation of an $\eta$-Einstein metric $g$ with
\begin{align}
\rho' T &= \rho T, \quad \omega' T = a \omega T, \quad \rho' T = (\lambda' + 2)\omega' T = \frac{\lambda + 2}{a} \omega T,
\end{align}
and thus, for any positive constants $\kappa$ and $\kappa'$, a transverse Kähler-Einstein metric with Einstein constant $\kappa$ can be transformed by a $D$-homothetic transformation to a transverse Kähler-Einstein metric with Einstein constant $\kappa'$. In particular, if we are given a Sasaki-Einstein metric $g$ with $\lambda = 2m$, we may obtain by $D$-homothetic transformation an $\eta$-Einstein metric $g'$ with arbitrary $\lambda' + 2 > 0$. Conversely, if we have an $\eta$-Einstein metric with $\lambda + 2 > 0$ then we obtain a Sasaki-Einstein metric with $\lambda' = 2m$ by $D$-homothetic transformation.

3. Momentum construction for Sasakian $\eta$-Einstein manifolds

Based on the arguments on $D$-homothetic transformation in the previous section we start with a Sasakian $\eta$-Einstein manifold $(S, g)$ with
\begin{align}
\text{Ric}_g &= \lambda g + \nu \eta \otimes \eta \\
\text{and Kähler cone metric on } C(S) \\
\bar{g} &= dr^2 + r^2 g,
\end{align}
Let $\omega^T = \frac{1}{2} d\eta$ be the transverse Kähler form which gives positive Kähler-Einstein metrics on local leaf spaces with
\begin{align}
\rho^T &= \kappa \omega^T,
\end{align}
where we have set
\begin{align}
\kappa := \lambda + 2.
\end{align}
As we work on $C(S)$ we lift $\eta$ on $S$ to $C(S)$ by (1), and use the same notation $\eta$ for the lifted one to $C(S)$. Then $\omega^T$ is also lifted to $C(S)$ by (2), and again use the same notation $\omega^T$ for the lifted one to $C(S)$. The momentum construction (or Calabi ansatz) searches for a Kähler form on $C(S)$ of the form
\begin{align}
\omega &= \omega^T + i \partial \bar{\partial} F(t)
\end{align}
where $t = \log r$ and $F$ is a smooth function of one variable on $(t_1, t_2) \subset (-\infty, \infty)$.

We set
\begin{align}
\tau &= F'(t), \\
\varphi(\tau) &= F''(t).
\end{align}
Since we require $\omega$ to be a positive form and
\begin{align}
i \partial \bar{\partial} F(t) &= i F''(t) \partial t \wedge \bar{\partial} t + i F'(t) \partial \bar{\partial} t \\
&= i \varphi(\tau) \partial t \wedge \bar{\partial} t + \tau \omega^T.
\end{align}
then we must have $\varphi(\tau) > 0$. We also require that the image of $F'$ is an open interval $(0, b)$ with $b \leq \infty$, i.e.
\begin{align}
\lim_{t \to t_1} F'(t) = 0, \quad \lim_{t \to t_2} F'(t) = b.
\end{align}
It follows from $\varphi(\tau) > 0$ that $F'$ is a diffeomorphism from $(t_1, t_2)$ to $(0, b)$, and consider $F'$ as a coordinate change from $t$ to $\tau$. We will set up an ODE to solve constant scalar curvature or Kähler-Ricci soliton equations in terms of $\varphi(\tau)$ with the new coordinate $\tau$. In [20], $\varphi(\tau)$ is called the profile of the Calabi ansatz (10).
Given a positive function $\varphi > 0$ on $(0, b)$ such that
\[
\lim_{\tau \to 0^+} \int_{\tau_0}^{\tau} \frac{dx}{\varphi(x)} = t_1, \quad \lim_{\tau \to b^-} \int_{\tau_0}^{b} \frac{dx}{\varphi(x)} = t_2
\]
we can recover the Calabi ansatz as follows. Fix $\tau_0$ and introduce a function $\tau(t)$ by
(11) $\quad t = \int_{\tau_0}^{\tau(t)} \frac{dx}{\varphi(x)}$.
and then $F(t)$ by
\[
F(t) = \int_{\tau_0}^{\tau(t)} \frac{xdx}{\varphi(x)}.
\]
Put
(12) $\omega_\varphi := \omega^T + dd^c F(t) = (1 + \tau) \omega^T + \varphi(\tau) i\partial t \wedge \overline{\partial} t = (1 + \tau) \omega^T + \varphi(\tau)^{-1} i\partial \tau \wedge \overline{\partial} \tau$.
As we assume $\varphi > 0$ on $(0, b)$, $\omega_\varphi$ defines a Kähler form and have recovered Calabi ansatz.

Next we compute the Ricci form $\rho_\varphi$ and the scalar curvature $\sigma_\varphi$ of $\omega_\varphi$. If we choose $z^0$ to be the coordinate along the holomorphic Reeb flow, then it is easy to check that
\[
dz^0 \wedge d\overline{z}^0 = \frac{dr}{r} \wedge \eta.
\]
Using this one can compute the volume form as
\[
\omega_\varphi^{m+1} = (1 + \tau)^m (m + 1) \varphi(\tau)^{\frac{i}{2}} dz^0 \wedge d\overline{z}^0 \wedge (\omega^T)^m.
\]
The Ricci form can be computed as
\[
\rho_\varphi = \rho^T - i\partial \overline{\partial} \log((1 + \tau)^m \varphi(\tau)) = \kappa \omega^T - i\partial \overline{\partial} \log((1 + \tau)^m \varphi(\tau)).
\]
Using
(13) $\quad dd^c u(\tau) = u'(\tau) \varphi(\tau) dd^c t + \frac{1}{\varphi} (u' \varphi) d\tau \wedge d\overline{\tau}$
for any smooth function $u$ of $\tau$, one computes
(14) $\quad \rho_\varphi = (\kappa - \frac{m \varphi + (1 + \tau) \varphi'}{1 + \tau}) \omega^T - ((\frac{m \varphi}{1 + \tau})' + \varphi'') \varphi dt \wedge d\overline{\tau}.
\]
From this and (12) we see that $\rho_\varphi = \alpha \omega_\varphi$ if and only if
(15) $\quad \kappa - \frac{m \varphi + (1 + \tau) \varphi'}{1 + \tau} = \alpha (1 + \tau),$
(16) $\quad -((\frac{m \varphi}{1 + \tau})' + \varphi'') = \alpha.$
But (16) follows from (15).

Now we consider the situation under which this paper considers. Let $(L, h)$ be a negative Hermitian line bundle over a Kähler manifold such that the Kähler form $\omega_M$ is equal to $\frac{1}{2\pi} \partial \overline{\partial} \log h$. Let $S$ be the unit circle bundle with the induced Sasakian
structure and the radial function \( r_0 = h(z, z)^{1/2} \) on its Kähler cone \( C(S) \cong \{ z \neq 0 \mid z \in L \} \). We consider the total space of \( L \) a resolution of \( C(S) \).

**Definition 3.1.** Consider the unit circle bundle \( S \) for \( (L, h) \) as above. We say that a Sasakian structure on \( S \) given by a Kähler cone metric \( dr^2 + r^2 \eta \) is bundle-adapted if there exists a smooth function \( \psi \) on \( S \) such that \( r \) with

\[
(17) \quad r^2 = r_0^2 \exp \psi
\]
gives a radial function on the Kähler cone of a Sasakian structure.

As recalled in Theorem 2.2, it is shown in [17] that on a toric Sasaki manifold \( S \) with positive transverse first Chern class and with \( \mathbb{Q} \)-Gorenstein Kähler cone \( C(S) \), we can find a Sasaki-Einstein metric by varying the Reeb vector field. This change of Reeb vector field results in a change of radial function to the form (17). The following proposition can be applied for this reason when \( L \rightarrow M \) is a positive rational power of the canonical line bundle over toric Fano manifold \( M \).

**Proposition 3.2.** Let \( \omega_{\varphi} \) be the Kähler form obtained by the momentum construction as above starting from a compact Sasaki manifold \( S \) with a bundle-adapted \( \eta \)-Einstein metric \( \eta \), and with \( (t_1, t_2) = (-\infty, \infty) \) and suppose that the profile \( \varphi \) is defined on \( (0, b) = (0, \infty) \), thus \( \varphi(0) = 0 \). Then \( \omega_{\varphi} \) defines a complete metric, has a noncompact end towards \( \tau = \infty \) and extends to a smooth metric on the total space of the line bundle up to the zero section if and only if \( \varphi \) grows at most quadratically as \( \tau \rightarrow \infty \) and \( \varphi'(0) = 2 \). When restricted to the unit circle bundle \( S \) the resulting metric is the starting \( \eta \)-Einstein metric squashed by \( 2e^{\varphi} \) given by (18) below in the Reeb flow direction.

**Proof.** By Proposition 3.2 in [15], \( \varphi \) must grow at most quadratically as \( \tau \rightarrow \infty \). Now let us consider (12) when \( \tau \rightarrow 0 \). Obviously \( (1 + \tau) \omega_{\varphi} > 0 \) for \( \tau \geq 0 \). The second term on the right hand side of (12) is computed as

\[
\varphi(\tau) dt \wedge d\tau = \varphi(\tau) id log r \wedge d^c log r = \frac{\varphi(\tau)}{r^2} idr \wedge d^c r.
\]

We wish to find the condition for \( \lim_{\tau \rightarrow 0} \varphi(\tau)/r^2 \) to exist and be non-zero. Suppose that

\[
\varphi(\tau) = a_1 \tau + O(\tau^2).
\]

Since \( t = log r \), \( \tau = F'(t) \) and \( \varphi(\tau) = F''(t) \) we have

\[
\frac{dt}{d\tau} = \varphi(\tau) = a_1 \tau + O(\tau^2).
\]

Thus

\[
\lim_{\tau \rightarrow 0} \frac{\varphi(\tau)}{r^2} = \lim_{t \rightarrow -\infty} \frac{\varphi'(t) d\tau}{2\tau d\tau} = \frac{a_1}{2} \lim_{\tau \rightarrow 0} \frac{\varphi(\tau)}{r^2}.
\]

Therefore if \( \lim_{\tau \rightarrow 0} \varphi(\tau)/r^2 \) exists and is non-zero then \( a_1 = 2 \), i.e. \( \varphi'(0) = 2 \). Conversely if \( \varphi'(0) = 2 \) then we have

\[
\frac{dt}{d\tau} = \varphi(\tau) = 2\tau + O(\tau^2) = 2\tau + O(\tau)
\]

where \( \alpha(\tau) \) is a function of \( \tau \) real analytic near \( \tau = 0 \) with \( \alpha(0) = 1 \). We then have

\[
\frac{d\tau}{\tau \alpha(\tau)} = 2dt
\]
and from this
\[ \log \tau + \beta(\tau) = c_0 + 2t \]
where \( \beta(\tau) \) is a real analytic function of \( \tau \) with \( \beta(0) = 0 \) and \( c_0 \) is a constant. From this we have
\[ \tau = e^{-\beta(\tau)} e^{c_0 + 2t} = r^2 e^{c_0 - \beta(\tau)}. \]
Thus we obtain
\[ \lim_{\tau \to 0} \frac{\varphi(\tau)}{r^2} = \lim_{\tau \to 0} \frac{2\tau + O(\tau^2)}{r^2} = 2e^{c_0}. \]
Note that \( i dr \wedge d^c r \) is the Kähler form of \( dr^2 + (dr \circ J)^2 \) and that by the bundle-adapted condition (17) we have
\[ dr = e^\psi dr_0 \]
at \( r = 0 \) which is nondegenerate. Thus \( \omega_\varphi \) converges as \( r \to 0 \) to define a Kähler form on \( L \) along the zero section, and when restricted to \( S \) it is the starting \( \eta \)-Einstein metric squashed by \( 2e^{c_0} \) in the Reeb flow direction. This completes the proof of Proposition 3.2.

In principle one can compute \( 2e^{c_0} \) in (18) for each case when the ODE is solved, see Example 3.4.

Let us return to the general situation of (15) and require \( \varphi(0) = 0 \) and \( \varphi'(0) = 2 \).

Then
\[ \kappa - 2 = \alpha. \]
If \( c \) is the scalar curvature of \( \omega_\varphi \), which is constant, then
\[ \alpha = \frac{c}{m + 1}. \]
Since \( \kappa = \lambda + 2 \) we have \( \lambda = \alpha \). By the remarks after (8), this \( \eta \)-Einstein metric is obtained from a Sasaki-Einstein metric by D-homothetic transformation as long as \( \alpha + 2 > 0 \).

Finally, the ODE (15) is equivalent to
\[ (\varphi(1 + \tau)^m)' = (\alpha + 2)(1 + \tau)^m - \alpha(1 + \tau)^m + 1. \]
Using \( \varphi(0) = 0 \) we obtain the solution
\[ \varphi(\tau) = \frac{\alpha + 2}{m + 1} \left( 1 + \tau - \frac{1}{(1 + \tau)^m} \right) - \frac{\alpha}{m + 2} \left( (1 + \tau)^2 - \frac{1}{(1 + \tau)^m} \right). \]

**Theorem 3.3.** Let \( L \) be a holomorphic line bundle over a compact Kähler manifold \( M \) such that \( K_M = L^\otimes p \) for some positive integer \( p \). Let \( k \geq p \) be an integer and suppose that the total space \( S \) of the \( U(1) \)-bundle associated with \( L^\otimes k \) has a bundle-adapted Sasakian \( \eta \)-Einstein metric with
\[ \rho^T = \frac{2p}{k} \omega^T. \]
Then there exists a complete Kähler-Einstein metric \( \omega_\varphi \) on the total space of \( L^\otimes k \) with
\[ \rho_{\omega_\varphi} = \frac{2p}{k} \omega_\varphi \]
where \( \varphi \) is the profile of the momentum construction. This metric is Ricci-flat when \( k = p \), that is \( L^\otimes k = K_M \). When restricted to the \( U(1) \)-bundle the resulting metric
\(\omega_\varphi\) is the starting \(\eta\)-Einstein metric squashed by \(2e^{c_0}\) given by (18) below in the Reeb flow direction.

Proof. We assume that \(S\) admits a bundle-adapted Sasakian \(\eta\)-Einstein metric with (21). Thus in (9), (3) and (19)
\[
\kappa = \frac{2p}{k}, \quad \alpha = \frac{2p}{k} - 2.
\]
The solution \(\varphi\) with (20) is positive for \(\tau > 0\) since \(k \geq p\), grows at most quadratically as \(\tau \to \infty\), and satisfies \(\varphi(0) = 0\) and \(\varphi'(0) = 2\). Hence by Proposition 3.2 \(\omega_\varphi\) gives a complete metric on the total space of \(L^\otimes k\) such that, when restricted to the \(U(1)\)-bundle the resulting metric \(\omega_\varphi\) is the starting \(\eta\)-Einstein metric squashed by \(2e^{c_0}\) given by (18) below in the Reeb flow direction. \(\square\)

Proof of Theorem 1.1. On the standard Sasakian structure on the total space of \(U(1)\)-bundle we have
\[
[\rho^T]_B = \frac{2p}{k}[\omega^T]_B
\]
as a basic cohomology, which is of course the cohomology of the base manifold \(M\). By D-homothetic transformation (7) with
\[
a = \frac{p}{k(m+1)}
\]
we obtain a Sasakian structure \(g'\) with
\[
[\rho'^T]_B = (2m + 2)[\omega'^T]_B.
\]
Since the standard Sasakian structure is toric, the volume minimizing argument (c.f. section 8 of [17]) gives a Sasaki-Einstein metric with
\[
\rho^T = (2m + 2)\omega^T.
\]
Then using the D-homothetic transformation with \(a = \frac{k(m+1)}{p}\) we obtain an \(\eta\)-Einstein metric with
\[
\rho'^T = \frac{2p}{k}\omega'^T
\]
on the total space of \(U(1)\)-bundle of \(L^\otimes k\). With this \(\eta\)-Einstein metric we can apply Theorem 3.3 and obtain Theorem 1.1. \(\square\)

Example 3.4. For the Calabi-Yau case \(k = p\) in Theorem 1.1 we have \(2e^{c_0} = \frac{1}{m+1}\) and the irregular Eguchi-Hanson metric restricted to the \(U(1)\)-bundle is the Sasakian \(\eta\)-Einstein metric squashed by \(\frac{1}{m+1}\) in the Reeb flow direction.

Proof.
\[
\varphi(\tau) = \frac{2}{m+1}(1 + \tau) - \frac{1}{(1 + \tau)^m} = \frac{2}{m+1} \frac{(1 + \tau)^{m+1} - 1}{(1 + \tau)^m}.
\]
Take \(\tau_0 = 2^{1/(m+1)} - 1\). Then \(\tau(t)\) is obtained from (11) as
\[
t = \frac{1}{2} \log((\tau(t) + 1)^{m+1} - 1).
\]
Since \(t = \log r\), we have
\[
\tau(t) = (r^2 + 1)^{1/(m+1)} - 1,
\]
and

\[ 2e^{c_0} = \lim_{\tau \to 0} \frac{\varphi(\tau)}{\tau^2} = \lim_{\tau \to 0} \frac{(\tau^2 + 1)^{\frac{1}{m+1}}}{\tau^2} = \frac{1}{m + 1}. \]

\[ \square \]

4. Soliton analogues

In this section we consider the case when the momentum construction \((12)\) on \(C(S)\) satisfies the Kähler-Ricci soliton equation

\[ (22) \rho \varphi - \alpha \omega \varphi = -i \widehat{\partial \partial} Q(t) \]

where \(Q(t)\) is a smooth function of \(t = \log r\) whose gradient is a holomorphic vector field. Then by Lemma 4.1 in [18], \(Q(t)\) is necessarily of the form

\[ Q = \mu \tau + c \]

where \(c\) is a constant. Using \((13)\) one computes

\[ \partial \partial Q = \frac{dQ}{d\tau} \varphi(\tau) dd^c t + (\frac{dQ}{d\tau} \varphi(\tau))' \varphi dt \wedge d^c t = \mu \varphi(\tau) \omega T + (\mu \varphi(\tau))' \varphi(\tau) dt \wedge d^c t. \]

Comparing this with \((12)\) and \((14)\) we obtain

\[ (23) \quad \kappa - \frac{m \varphi + (1 + \tau) \varphi'}{1 + \tau} = \alpha (1 + \tau) - \mu \varphi(\tau) \]

and

\[ (24) \quad -\left( \frac{m \varphi}{1 + \tau} + \varphi' \right)' = \alpha - (\mu \varphi(\tau))'. \]

But \((24)\) follows from \((23)\). If we require \(\varphi(0) = 0\) and \(\varphi(0) = 2\) it follows from \((23)\) that

\[ \kappa - 2 = \alpha, \]

and \((23)\) becomes

\[ (25) \quad \varphi' + \left( \frac{m}{1 + \tau} - \mu \right) \varphi + (\kappa - 2) \tau - 2 = 0. \]

In general a solution to the ODE \(y' + p(x)y = q(x)\) is given by

\[ (26) \quad y = e^{- \int p(x) dx} \left( \int q(x) e^{\int p(x) dx} dx + C \right). \]

It follows from \((26)\) that the solution to \((25)\) is given by

\[ (27) \quad \varphi(\tau) = \nu e^{p(1+\tau)} \left( \frac{\kappa - 2}{(1 + \tau)^m} + \sum_{j=0}^{m} \frac{(m + 1)!}{j!} \mu^{j}(1 + \tau)^{j-m} \right) \]

for some constant \(\nu\). But by the requirement \(\varphi(0) = 0\), \(\nu\) is determined by

\[ (28) \quad \nu = e^{-\mu} \left( \frac{-\kappa + 2}{\mu} + \sum_{j=0}^{m} \frac{(m + 1)!}{j!} \mu^{j} \right) =: \nu(\kappa, \mu). \]
Thus the solution (27) becomes

$$\varphi(\tau) = \left(\frac{-\kappa + 2}{\mu} + \frac{-\kappa + 2 + \frac{m}{m+2}}{\mu^{m+2}} \sum_{j=0}^{m} \frac{(m+1)!}{j!} \mu^j \right) \frac{e^{\mu \tau}}{(1 + \tau)^m} + \frac{(\kappa - 2)(1 + \tau)}{\mu} + \frac{\kappa - 2 - \frac{\mu}{m+1}}{\mu^{m+2}} \sum_{j=0}^{m} \frac{(m+1)!}{j!} \mu^j (1 + \tau)^{j-m} \tag{29}$$

Let $M$ be a Fano manifold of dimension $m$, and $L \to M$ be a negative line bundle with $K_M = L^p$, $p \in \mathbb{Z}^+$. Take $k \in \mathbb{Z}^+$. Let $S$ be the $U(1)$-bundle associated with $L^k$, which is a regular Sasakian manifold with the Kähler cone $C(S)$ biholomorphic to $L^k$ minus the zero section. We assume that $S$ admits a possibly irregular Sasakian $\eta$-Einstein metric which is bundle-adapted in the sense of Definition 3.1. When $M$ is toric this is indeed the case as we saw in the proof of Theorem 1.1.

Let $\kappa = \frac{2p}{k}$ and $\omega$ be the $\eta$-Einstein Sasaki metric such that

$$\rho^T = \kappa \omega^T$$

where $\omega^T$ and $\rho^T$ are respectively the transverse Kähler form and its transverse Ricci form. In this set-up we start the momentum construction for the Kähler-Ricci soliton (22), and following the subsequent computations we obtain the solution (29) requiring $\varphi(0) = 0$ and $\varphi'(0) = 2$. But we have not specify the region of the variable $\tau$ yet.

First we consider the case $k \geq p$. Then of course $\kappa \leq 2$ and $\alpha \leq 0$. In this case we take $\mu < 0$, and take the region of the variable $\tau$ to be $[0, \infty)$.

**Claim 4.1.** $\varphi > 0$ on $(0, \infty)$.

**Proof.** Suppose $\varphi(a) = 0$ for $a > 0$. Then by (25), $\varphi'(a) = 2 + (2 - \kappa)a > 0$. Thus $a$ is the only positive zero of $\varphi$ and $\varphi'(a) > 0$. But this can not happen since $\varphi(0) = 0$ and $\varphi'(0) = 2$. \qed

**Claim 4.2.** $F'$ maps $(-\infty, \infty)$ diffeomorphically onto $(0, \infty)$.

**Proof.** Since $F''(t) = \varphi(\tau) > 0$ by Claim 4.1, $F'$ maps its domain diffeomorphically onto its image. From $\varphi(0) = 0$, $\varphi'(0) = 2$ and (11), we see that $t \to -\infty$ as $\tau \to 0$.

On the other hand, as we take $\mu < 0$, $\varphi(\tau)$ grows linearly as $\tau \to \infty$, and we see from (11) that $t \to \infty$ as $\tau \to \infty$. Hence the domain and the range of $F'$ are respectively $(-\infty, \infty)$ and $(0, \infty)$.

**Claim 4.3.** The metric $\omega_\varphi$ defines a complete metric on $L^k$. When restricted to the unit circle bundle $S$ the resulting metric is the starting $\eta$-Einstein metric squashed by $2e^{\nu_0}$ given by (18) below in the Reeb flow direction.

**Proof.** Since $\varphi(\tau)$ grows linearly as $\tau \to \infty$, $\varphi(0) = 0$ and $\varphi'(0) = 2$ this claim follows from Proposition 3.2. \qed

Next we turn to the case $k < p$. Then $\kappa = \frac{2p}{k} > 2$ and $\alpha = \kappa - 2 > 0$.

**Claim 4.4.** In (25) we can take some positive $\mu$ so that $\nu(\kappa, \mu) = 0$, and with the choice of $\mu$ as in Claim 4.4 the solution $\varphi(\tau)$ is expressed as

$$\varphi(\tau) = \frac{(\kappa - 2)(1 + \tau)}{\mu} + \frac{\kappa - 2 - \frac{\mu}{m+1}}{\mu^{m+2}} \sum_{j=0}^{m} \frac{(m+1)!}{j!} \mu^j (1 + \tau)^{j-m}. \tag{30}$$
Proof. The leading order term as $\mu \to 0$ is $(-\kappa + 2)/\mu^{m+2}$, and thus $\nu(\kappa, \mu) \to -\infty$. On the other hand the leading order term as $\mu \to \infty$ is $2/\mu$, and thus $\nu(\kappa, \mu) \to \infty$. Thus we can find a positive $\mu$ such that $\nu(\kappa, \mu) = 0$. (This choice of $\mu$ is actually unique since the coefficients of the monomials inside $\nu(\kappa, \mu)$ changes sign only once when arranged from lower to higher. This argument is due to [11].) □

Claim 4.5. $\varphi(\tau) > 0$ for all $\tau > 0$.

Proof. Since $\varphi(0) = 0$ and $\varphi'(0) = 2 > 0$ we must have

$$\frac{\kappa - 2 - \frac{2\mu}{m+1}}{\mu^{m+2}} < 0.$$ 

As $\varphi'(0) > 0$ we have $\varphi(\tau) > 0$ for small $\tau > 0$. Then as $\tau$ gets bigger the right hand side gets bigger because of the signs of the coefficients of the first and the second term. □

Claim 4.6. $F'$ maps $(-\infty, \infty)$ diffeomorphically onto $(0, \infty)$.

Proof. Since $F''(t) = \varphi'(t) > 0$ by Claim 4.1 $F'$ maps its domain diffeomorphically onto its image. From $\varphi(0) = 0$, $\varphi'(0) = 2$ and [11], we see that $t \to -\infty$ as $\tau \to 0$. On the orther hand, since $\kappa > 2$ and $\mu > 0$, $\varphi(\tau)$ grows linearly as $\tau \to \infty$, and we see from [11] that $t \to \infty$ as $\tau \to \infty$. Hence the domain and the range of $F'$ are respectively $(-\infty, \infty)$ and $(0, \infty)$. □

Claim 4.7. The metric $\omega_{\varphi}$ defines a complete metric on $L^{\otimes k}$. When restricted to the unit circle bundle $S$ the resulting metric is the starting $\eta$-Einstein metric squashed by $2e^{c_0}$ given by (18) below in the Reeb flow direction.

Proof. Since $\varphi(\tau)$ grows linearly as $\tau \to \infty$, $\varphi(0) = 0$ and $\varphi'(0) = 2$ this claim follows from Proposition [3.2] □

Thus we have proved

Theorem 4.8. Let $M$ be a Fano manifold of dimension $m$, and $L \to M$ be a negative line bundle with $K_M = L^{\otimes p}$, $p \in \mathbb{Z}^+$. Take $k \in \mathbb{Z}^+$. Assume that the $U(1)$-bundle $S$ associated with $L^{\otimes k}$ admits a possibly irregular Sasakian $\eta$-Einstein metric which is bundle-adapted in the sense of Definition [3.7]. Then the total space of $L^{\otimes k}$ admits a complete expanding Kähler-Ricci soliton if $k > p$, a complete steady Kähler-Ricci soliton if $k = p$, and a complete shrinking Kähler-Ricci soliton if $k < p$. When restricted to the unit circle bundle $S$ the resulting metric is the starting $\eta$-Einstein metric squashed by the positive number given by (18) in the Reeb flow direction.

The proof of Theorem 1.2 is obtained from Theorem 4.8 in the similar way as the proof of Theorem 1.1

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