LOCAL RIGIDITY AND BIFURCATION FOR YAMABE-TYPE PROBLEMS IN WARPED PRODUCTS

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ABSTRACT. Our aim in this paper is to study local rigidity for metrics defined on a compact manifold $M$ with boundary satisfying constant scalar curvature on $M$ and constant mean curvature on $\partial M$. We present some geometrical hypotheses ensuring local rigidity for both, the general Riemannian and the warped metric case. These conditions arise from the study of a spectral problem which is not included within the classical problems (Neumann, Steklov,….) that we call “mixed eigenvalue problem”. Finally, we apply our previous results for the spatial slice of the de Sitter and Anti-de Sitter Schwarzschild spacetimes.

1. Introduction

The classical Yamabe problem consists in showing that every Riemannian compact manifold, without boundary, admits a conformally related metric with constant scalar curvature. In dimension two, such a problem follows from the Uniformization Theorem of Riemann surfaces. For higher dimensions, the problem was formulated by Yamabe in [17], where the first steps towards a proof of the existence of a solution were given. After the combined efforts of Trudinger [22], Aubin [4] and Schoen [20], the Yamabe problem was completely solved. It is important to highlight that constant scalar curvature metrics can be characterized variationally as critical points of the Hilbert-Einstein functional in conformal classes. It is known that the minimum of this functional in a conformal class is unique, see [2]. However, in many cases a rich variety of constant scalar curvature metrics arise as critical points that are not necessarily minimizers, and for this reason, it is very interesting to find conformal classes where the Yamabe problem has multiple solutions. Among others, bifurcation techniques can be applied in order to obtain multiplicity results. The existence of multiple solutions for the Yamabe problem in different settings has long been studied in the literature, see for instance [1, 5, 6, 15].

Now, if the compact manifold considered in the Yamabe Problem has a nonempty boundary, several possible boundary conditions can be studied. For instance, from the point of view of conformal geometry, a geometrical condition would have to involve the mean curvature of the boundary. This important observation led Escobar in 1992 (see [8]) to study the problem of finding smooth metrics with constant scalar curvature and minimal boundary inside a given
conformal class. In a natural continuation of the latter, the same author in [9] analyzed the conformal deformation of a metric to a scalar flat metric with constant mean curvature on the boundary generalizing the famous Riemann Mapping Theorem to higher dimensions. Subsequently, he also obtained results for the Yamabe problem with boundary under mixed constraints (see [10]):

\[(1) \quad aV + bA = 1\]

where \(V\) and \(A\) are the volume of manifold and the area of its boundary, respectively, when \(a \geq 0\) and \(b\) are real numbers. More precisely, he studied a problem of existence of conformal metrics with constant scalar curvature \(R(g)\) in the manifold and constant mean curvature \(H(g)\) on the boundary, which are related by

\[(n - 1)bR(g) = 2nH(g)a.\]

The solutions of this problem are critical points of the functional formulated by Gibbons-Hawking and York [12, 18] (which we will refer to as the GHY-functional) restricted to the constraint (1). The existence of such critical points was proved by Escobar for manifolds of nonpositive type and for almost any manifold of positive type if \(b\) is sufficiently small. In [13] and [14], Han and Li showed the existence of solutions when \(a > 0\), the manifold is of positive type and either is locally conformally flat with umbilic boundary, or \(n \geq 5\) and the boundary has a nonumbilic point.

Taking advantage of the variational approach, our aim in this paper is to study the rigidity of families of metrics defined on a fixed compact manifold \(M\) with boundary \(\partial M\) and satisfying both, having constant scalar curvature on \(M\) and constant mean curvature on \(\partial M\). Roughly speaking, a family \(\mathcal{F}\) of metrics is (locally) rigid if given another metric \(g\) solution of the Yamabe problem and sufficiently close to some element on \(\mathcal{F}\), then \(g\) belongs to \(\mathcal{F}\).

For our results, we will impose conditions to ensure the non-singularity of the second variation of the GHY-functional under the constraint (1), which will ensure rigidity as a consequence of the Implicit Function Theorem (see [15, Appendix A]). Such a conditions will be obtained by making lower estimates of the first eigenvalue of a particular kind of spectral problem, that we have called mixed eigenvalue problem (see Section 3 for a general background on this problem).

Additionally, we will show how our results are applicable to General Relativity. To this aim, we will show how it is possible to study metric rigidity of the spatial slices of the Anti-de Sitter and de Sitter Schwarzschild spacetimes (see Remark 4.9), which are classical models for black holes. On the one hand, we will show that any Anti-de Sitter Schwarzschild model has a rigid region under metric variations close to the event horizon which covers the black hole. Moreover, under some mild hypotheses, such a rigidity result extends to all the space. On the other hand, we will show how the remaining cases can be analyzed using this approach. Particularly, we will study in detail the so-called Schwarzschild spacetime, showing that such a space is, at least numerically, stable.

This work is organized as follows. In Section 2 besides fixing the notation, we include some preliminaries on bifurcation theory, as well as the criterion that we will use to prove local rigidity. In Section 3 we discuss a general background about the spectrum of the mixed eigenvalue problem, including some classical results as the Courant’s nodal Theorem and the Rayleigh characterization for eigenvalues (which will be necessary in the sequel). Moreover, we present some results for the particular case of warped spaces.
In Section 4, we first introduce the variational setting for the Yamabe problem with boundary, as described before. Then, we obtain our main results about the stability of the solutions of the Yamabe problem, first for the general Riemannian case (Theorem 4.8), and then for the particular case of warped spaces (Theorem 4.10). Additionally, in Section 4.4 we include the study of the spatial fiber of the Anti-de Sitter and de Sitter Schwarzschild spaces. Finally, we have included an Appendix (Section 5) where we compute the first and second variation of the GHY-functional. This computation has been performed by several authors (see [3] for instance), but we include it here for the sake of completeness.

2. Preliminaries

Along this section, we will state the basic elements, results and notation that we are going to use in the rest of the paper. Let $(M^n, g)$ be an arbitrary $n$-dimensional Riemannian compact manifold with non-empty boundary $\partial M$ and $n \geq 3$. We will assume that the boundary $\partial M$ is an $(n-1)$-differential manifold (i.e., a hypersurface) formed by different connected components. We will also assume that such a components are grouped in two disjoint sets $\partial M = \Sigma_1 \cup \Sigma_2$, which will have particular properties (see for instance Figure 1).

For a given system of local coordinates $x_1, \ldots, x_n$ around a point $p$, the metric $g$ and the volume element will be represented in the following form:

$$g = g_{ij} \, dx^i dx^j, \quad dv(g) = \sqrt{\det(g_{ij})} \, dx^1 \wedge \cdots \wedge dx^n,$$

with indexes varying from $1 \leq i, j \leq n$. When $p$ belongs to $\partial M$, we will assume that the coordinate $x_n$ is normal to the hypersurface $\partial M$ and pointing inward. In particular, the area element restricted to the boundary will take the following form:

$$d\sigma(g) = \sqrt{\det(g_{\alpha\beta})} \, dx^1 \wedge \cdots \wedge dx^{n-1}$$

Figure 1. Visual interpretation of our setting.
Finally, the scalar curvature on \((M, g)\) will be denoted by \(R(g)\), while the mean curvature on \(\Sigma_k\) will be denoted by \(H(g_k)\) for \(k = 1, 2\).

2.1. Bifurcation theory on metric variations. Let us introduce some basic framework about bifurcation theory that we will need along the rest of the paper. We refer the reader to \([15, 21]\) and the references therein for general background. We will denote by \(S^k(M)\), with \(k \geq 2\), the space of all symmetric \((0, 2)\)-tensors of class \(C^k\) defined on \(M\). In order to define a norm on such a space, consider an auxiliary Riemannian metric \(g_A\) which defines both, a connection \(\nabla_A\) and an inner product on the space of tensors \(\langle \cdot, \cdot \rangle_A\). Then, for a symmetric tensor \(T \in S^k(M)\), we define the norm \(||T||_{C^k}\) given by

\[
||T||_{C^k} = \max_{j=1,...,k} \left[ \max_{p \in M} ||\nabla_A^{(j)} T(p)||_A \right].
\]

Among the symmetric \((0, 2)\)-tensors, we consider the subspace \(\mathcal{M}\) of Riemannian metrics on \(M\), which has the structure of an open cone of \(S^k(M)\). Recall that, for any \(g \in \mathcal{M}\), the tangent space \(T_g \mathcal{M}\) is naturally identified with \(S^k(M)\).

Let us consider an one-parameter family \(\{g_\lambda\}_{\lambda \in \lambda_0} \subset \mathcal{M}\), where \(\lambda_0\) is an arbitrary open interval on \(\mathbb{R}\). We will assume that, for all \(\lambda\), the metric \(g_\lambda\) has:

(a) constant scalar curvature \(R(g_\lambda)\) on \(M\),
(b) constantly zero mean curvature in \(\Sigma_1\) (i.e., \(\Sigma_1\) a minimal hypersurface) and
(c) constant mean curvature on \(\Sigma_2\).

In this setting, \(\lambda_0\) is said to be a point of bifurcation if there exists a sequence \(\{\lambda_\nu\}_n \subset I\) and a sequence \(\{g_n\}_n \subset \mathcal{M}\) satisfying:

(1) \(\lim_{\lambda_\nu \to \lambda_0} \lambda_\nu = \lambda_0\) and \(\lim_{n \to \infty} g_\nu = g_\lambda\),
(2) For all \(n\), \(g_n\) determines constant scalar curvature on \(M\) and makes \(\Sigma_1\) a minimal hypersurface and \(\Sigma_2\) a CMC-hypersurface. Moreover, the scalar curvature on \(M\) and the constant mean curvature on \(\Sigma_2\) are \(R(g_\lambda)\) and \(H(g_\lambda)\) resp.,
(3) \(g_n \neq g_\lambda\) for all \(n\).

If \(\lambda_0\) is not a bifurcation point, we will say that the family \(\{g_\lambda\}_{\lambda_0}\) is locally rigid at \(\lambda_0\).

Along this paper, we will focus our attention on the concept of local rigidity, even so our studies open the door for future works on the existence of bifurcation points. The criterion for the rigidity will be given by the Implicit Function Theorem (in its version for bundles, see \([15, \text{Appendix A}]\)), and so we will need some variational approach for metrics satisfying previous conditions (a), (b) and (c). That is, we will show that, associated to such a family \(\{g_\lambda\}_{\lambda_0}\), there exists a path of \(C^k\)-functionals (with \(k \geq 2\)) \(\mathcal{F}^k : \mathcal{M} \to \mathbb{R}\) with the property that, given any \(\lambda \in I\), a metric \(g\) in the conformal class of \(g_\lambda\) has constant scalar curvature \(R(g) = R(g_\lambda)\) and makes \(\Sigma_1\) minimal and \(\Sigma_2\) of constant mean curvature with \(H(g_\lambda) = H((g_\lambda)_2)\) if and only if \(d\mathcal{F}^k(g) = 0\). Then, in order to apply the Implicit Function Theorem, we have to show that the points \(g_0\) are non-degenerate critical points for such a functional. In particular, the desired non-degeneracy will follow by showing that all the eigenvalues of the associated Jacobi operator (which is diagonalizable under our conditions) are positive, which lead us to a mixed eigenvalue problem.

\(^1\)As a convention, we will always assume that, when we work in coordinates, the indexes \(i, j\) will vary between \(1, \cdots, n\), while the indexes \(\alpha, \beta\) will vary in \(1, \cdots, n - 1\).
3. Spectrum of the mixed eigenvalue problem

Our aim in this section is to give a general background of the so-called mixed eigenvalue problems, giving the basic properties that we will require in this paper. As we are going to see this kind of problems, that cannot be consider a Neumann problem nor a Steklov one but a mix of them, can be described in the following way

\[
\begin{cases}
\Delta_g f = (G + \beta) f & \text{in } M, \\
- \frac{\partial}{\partial n} f = (J + \beta) f & \text{on } \partial M
\end{cases}
\]

where \( \Delta_g \) is the Laplace-Beltrami operator; \( G : M \to \mathbb{R} \) and \( J : \partial M \to \mathbb{R} \) are functions; \( \partial/\partial n \) denotes the inward normal vector; and \( f \in H^1(M) \).

The first natural question to answer is: to what extent the classical procedures for the study of, say, Dirichlet problems are applicable (or, at least, adaptable) to the mixed one. For this, as a first step, it is essential to describe the mixed problems variationally. Let us define both, \( \mathcal{D} \) and \( \mathcal{E} \) bilinear forms in the following way:

\[
\mathcal{D}(\varphi, \psi) = \int_M \left[ g(\text{grad}(\varphi), \text{grad}(\psi)) - G\varphi\psi \right] dv(g) - \int_{\partial M} J\varphi\psi d\sigma(g)
\]

\[
\mathcal{E}(\varphi, \psi) = \int_M \varphi\psi dv(g) + \int_{\partial M} \varphi\psi d\sigma(g).
\]

Then, for a fixed \( \beta \in \mathbb{R} \), we define the functional

\[
\mathcal{F}(\varphi) = \mathcal{D}(\varphi) - \beta \mathcal{E}(\varphi)
\]

where \( \mathcal{D}(\varphi) := \mathcal{D}(\varphi, \varphi) \) and \( \mathcal{E}(\varphi) := \mathcal{E}(\varphi, \varphi) \); and observe that the first variation of such a functional becomes:

\[
\delta \mathcal{F}_\varphi(\psi) = 2 \left( \mathcal{D}(\varphi, \psi) - \beta \mathcal{E}(\varphi, \psi) \right).
\]

The first Green identity leads us to:

\[
\mathcal{D}(\varphi, \psi) - \beta \mathcal{E}(\varphi, \psi) = \int_M \psi \left( \Delta_g \varphi - G\varphi - \beta \varphi \right) dv(g) + \int_{\partial M} \psi \left( -\partial_n \varphi - J\varphi - \beta \varphi \right) d\sigma(g).
\]

In conclusion, \( \varphi \) is a critical point for the functional \( \mathcal{F} \) if, and only if, it is a solution for the mixed eigenvalue problem. By using this variational approach, we can re-obtain some classical and well-known results for the mixed case. For instance, the Courant’s nodal Theorem follows directly (see [7, Page 452]) as well as the classical characterization for the eigenvalues, as obtained by Courant [7] and Rayleigh [19]. In fact, it follows that the eigenvalues are determined by a sequence \( \{\beta_i\}_{i \in \mathbb{N}} \), repeated according to their multiplicity, and such that \( \lim_{i \to \infty} \beta_i = \infty \). Moreover,

\[
\beta_n = \min_{\varphi \in \{\varphi_1, \ldots, \varphi_{n-1}\}^\perp} \frac{\mathcal{D}(\varphi)}{\mathcal{E}(\varphi)}
\]

where each \( \varphi_i \) (with \( 1 \leq i \leq n - 1 \)) is the eigenfunction associated to \( \beta_i \) and

\[\text{Here, } H^1(M) \text{ is the Sobolev space of } L^2 \text{ functions on } M \text{ with first derivatives in } L^2.\]
\{\varphi_1, \ldots, \varphi_{n-1}\}^\perp = \{\varphi \in \mathcal{H}^1(M) : \mathcal{E}(\varphi, \varphi_i) = 0, \forall i = 1, \ldots, n-1\}.

For convenience, when \(n = 1\) the minimum is taken on the whole \(\mathcal{H}^1(M)\).

About such an eigenvalue, it is also interesting to remark that the first Green identity leads us to the following identity:

\[
\int_M |\text{grad}(\varphi)|^2_g dv(g) = \int_M \varphi \Delta_g \varphi dv(g) - \int_{\partial M} \varphi \partial_n \varphi d\sigma(g)
= \int_M (\overline{\beta} + G) \varphi^2 dv(g) + \int_{\partial M} (\overline{\beta} + J) \varphi^2 d\sigma(g)
= \overline{\beta} \left( \int_M \varphi^2 dv(g) + \int_{\partial M} \varphi^2 d\sigma(g) \right)
+ \int_M G \varphi^2 dv(g) + \int_{\partial M} J \varphi^2 d\sigma(g).
\]

So, if we assume the following normalization

(3) \[
\int_M \varphi^2 dv(g) + \int_{\partial M} \varphi^2 d\sigma(g) = \langle \langle \varphi, \varphi \rangle \rangle = 1,
\]

any eigenvalue \(\overline{\beta}\) will satisfy

(4) \[
\overline{\beta} = \int_M |\text{grad}(\varphi)|^2_g dv(g) - \int_M G \varphi^2 dv(g) - \int_{\partial M} J \varphi^2 d\sigma(g).
\]

In particular, we can obtain easily the following simple lower estimate for the eigenvalues

**Proposition 3.1.** Assume that \(G\) is constant on \(M\) and \(J\) is constant on each connected component of \(\partial M\). Then, the eigenvalues \(\beta\) satisfies

\[\overline{\beta} \geq -(G + \max(J))\]

where \(\max(J)\) denotes the maximal value of \(J\) on \(\partial M\).

**Proof:** From (4), we have that:

\[
\overline{\beta} = \int_M |\text{grad}(\varphi)|^2_g dv(g) - \int_M G \varphi^2 dv(g) - \int_{\partial M} J \varphi^2 d\sigma(g) \geq
\geq -G \int_M \varphi^2 dv(g) - \max(J) \int_{\partial M} \varphi^2 d\sigma(g).
\]

Using the normalization in (3), we have that both previous integrals are less than or equal to 1, so we get to

\[\overline{\beta} \geq -(G + \max(J)).\]
3.1. The mixed eigenvalue problem for warped metric spaces. In this section we will be specially interested in the warped product spaces, that is, we will consider an $n$-dimensional Riemannian space $(M,g)$ where

$$M = (r_1, r_2) \times P,$$

$$g = dr^2 + \alpha^2(r) g^P,$$

$(P,g^P)$ is an $(n-1)$-dimensional closed Riemannian manifold and $\alpha$ is a $C^k$ function with $k \geq 2$. Along this section, we will also assume that previous functions $G$ and $J$ only depends on the parameter $r$. With these assumptions, the mixed eigenvalue problem can be reduced to a Sturm-Liouville problem type with boundary conditions. In fact, when $g$ is a warped metric, the Laplace-Beltrami operator splits as

$$\triangle_g \varphi = -\frac{1}{\alpha^{n-1}} \partial_r (\alpha^{n-1} \partial_r \varphi) + \frac{1}{\alpha^2} \triangle_P \varphi$$

where $\triangle_P$ denotes the Laplace-Beltrami operator on $(P,g^P)$. So, from the first equation on (2), we arrive to the following problem

$$-\frac{1}{\alpha^{n-1}} \partial_r (\alpha^{n-1} \partial_r \varphi) + \frac{1}{\alpha^2} \beta \varphi = (G + \beta) \varphi.$$  \hspace{1cm} (5)

Now, notice that if we use the method of separation of variables, given a solution $\varphi : M \to \mathbb{R}$ of the above problem, it can be written as follows

$$\varphi(r,x) = \varphi_I(r) \varphi_P(x)$$

where $\varphi_I : (r_1, r_2) \to \mathbb{R}$ and $\varphi_P : P \to \mathbb{R}$. Moreover, if we assume that $\varphi_P \equiv \varphi_i^P$ is a non-zero eigenfunction for $\triangle_P$ associated to an eigenvalue $\beta_i$ then $\varphi$ becomes

$$-\frac{1}{\alpha^{n-1}} \partial_r (\alpha^{n-1} \partial_r \varphi_I) + \frac{1}{\alpha^2} \beta_i \varphi_I = (G + \beta) \varphi_I$$

which is a Sturm-Liouville equation on the interval $(r_1, r_2)$. In conclusion, joining previous equation with the initial conditions on $r_1$ and $r_2$, we obtain, for each eigenvalue $\beta_i$ of $\triangle_P$, the following problem

$$-\frac{1}{\alpha^{n-1}} \partial_r (\alpha^{n-1} \partial_r \varphi_I) + \frac{1}{\alpha^2} \beta_i \varphi_I = (G + \beta_i) \varphi_I$$

where

$$-\varphi_I(r_1) = (J(r_1) + \beta_i) \varphi_I(r_1),$$

$$\varphi_I(r_2) = (J(r_2) + \beta_i) \varphi_I(r_2).$$

It is well-known that, for each $i$, these problems admit a sequence of eigenvalues $\{\beta_j^i\}_{j \in \mathbb{N}}$ with

$$\beta_1^i < \beta_2^i \leq \cdots \leq \beta_j^i \leq \cdots$$

Therefore, if we denote by $(\varphi^i_j)_j$ the eigenfunction associated to $\beta_j^i$, we obtain that $\varphi_j = (\varphi^i_j)_j(\varphi_P)_i$ is an eigenfunction for the mixed eigenvalue problem (2) with eigenvalue $\beta_j$. Moreover, taking into account previous construction for the eigenfunctions and the dimension of $\mathcal{H}^1(I \times P)$, we deduce that all the eigenfunctions are obtained by this process, i.e., we have proved:

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By notation, the dot will denote derivative of a real function.
Theorem 3.2. Let \( \varphi \) be an eigenfunction of \((2)\) with associated eigenvalue \( \beta \). Then, \( \varphi \) can be written as the product of two functions \( \varphi_R : (r_1, r_2) \rightarrow \mathbb{R} \) and \( \varphi_P : P \rightarrow \mathbb{R} \) where \( \varphi_P \) is an eigenfunction of \( \triangle_P \) with associated eigenvalue \( \beta \); and \( \varphi_R \) is solution of the following problem:

\[
-\frac{1}{\alpha r} \partial_r \left( \alpha^{n-1} \partial_r \varphi_R \right) + \frac{1}{\alpha} \beta \varphi_R = (G + \beta) \varphi_R, \\
- \varphi_R(r_1) = (J(r_1) + \beta) \varphi_R(r_1), \\
\varphi_R(r_2) = (J(r_2) + \beta) \varphi_R(r_2).
\]

(6)

The rest of the section is devoted to obtain some additional information about the eigenvalues of this mixed problem on the warped case. As before, let us denote by \( \{\beta_i\} \) the ordered sequence of eigenvalues of the Laplacian \( \triangle_P \); and by \( \beta_1 \) the first eigenvalue of the problem \((6)\) with \( \beta = \beta_i \). Then, the Rayleigh quotient applied to the Sturm-Liouville problem \((6)\) reads:

\[
\overline{\beta}_1 = \min_{\varphi \in H^1((r_1, r_2))} \frac{-\alpha^{n-1} \varphi_R \varphi_R |_{r_1}^{r_2} + \int_{r_1}^{r_2} \alpha^{n-1} \left[(\varphi_R)^2 + (\alpha^{-2} \beta_i - G) \varphi_R^2\right] dr}{\int_{r_1}^{r_2} \alpha^{n-1} \varphi_R^2 dr}.
\]

(7)

With this characterization, we are able to prove the following two results:

Lemma 3.3. For all \( i \in \mathbb{N} \),

\[
\beta_i \leq \overline{\beta}_1^{i+1}.
\]

Proof: For a fixed function \( \varphi_R \), the function on the right of \((7)\) is non-decreasing respect to \( \beta_i \). Therefore,

\[
\overline{\beta}_1 = \min_{\varphi \in H^1((r_1, r_2))} \frac{-\alpha^{n-1} \varphi_R \varphi_R |_{r_1}^{r_2} + \int_{r_1}^{r_2} \alpha^{n-1} \left[(\varphi_R)^2 + (\alpha^{-2} \beta_i - G) \varphi_R^2\right] dr}{\int_{r_1}^{r_2} \alpha^{n-1} \varphi_R^2 dr} \leq \min_{\varphi \in H^1((r_1, r_2))} \frac{-\alpha^{n-1} \varphi_R \varphi_R |_{r_1}^{r_2} + \int_{r_1}^{r_2} \alpha^{n-1} \left[(\varphi_R)^2 + (\alpha^{-2} \beta_{i+1} - G) \varphi_R^2\right] dr}{\int_{r_1}^{r_2} \alpha^{n-1} \varphi_R^2 dr} = \overline{\beta}_1^{i+1}.
\]

Proposition 3.4. Consider the mixed eigenvalue problem \((2)\) for a warped metric. If it simultaneously satisfies \( J \leq 0 \) and \( \alpha^2(r) G \leq \beta_1 \) for all \( r \), being at least one of the two inequalities strict, then all the eigenvalues of such a problem are positive.

Proof: Let \( \{\overline{\beta}_j\}_{i,j} \in \mathbb{N} \) be the sequence of all the eigenvalues of the mixed eigenvalue problem. By construction, we know that \( \overline{\beta}_j \leq \overline{\beta}_j+1 \) and, from previous Lemma, that \( \overline{\beta}_1 \leq \overline{\beta}_1^{i+1} \). Joining both inequalities together, we deduce that \( \overline{\beta}_1 \) is the first eigenvalue of the mixed eigenvalue problem, and so, the one we want to estimate. From \((7)\), we deduce that:
\[ \beta_1 = \min_{\varphi_R \in H^1(r_1, r_2)} \left( -\alpha^{n-1} \varphi_R \varphi_R \bigg|^{r_2}_{r_1} + \int_{r_1}^{r_2} \alpha^{n-1} \left[ (\varphi_R)^2 + \left( \alpha^{-2} \beta_1 - G \right) \varphi_R^2 \right] dr \right) \]

and then, by assuming that \( \varphi_R \) achieves such a minimum and using the boundary conditions (6), we arrive to

\[ \beta_1 \left( \int_{r_1}^{r_2} \alpha^{n-1} \varphi_R^2 dr + \alpha^{n-1} \left( \varphi_R^2(r_2) + \varphi_R^2(r_1) \right) \right) = -\alpha^{n-1} \left( \varphi_R^2(r_2)J(r_2) + \varphi_R^2(r_1)J(r_1) \right) + \int_{r_1}^{r_2} \alpha^{n-1} \left[ (\varphi_R)^2 + \left( \alpha^{-2} \beta_1 - G \right) \varphi_R^2 \right] dr. \]

As we can see, the right term of the above equality is (strictly) positive under our hypothesis. Therefore \( \beta_1 > 0 \), as desired.

\[ \Box \]

4. RIGIDITY UNDER METRIC VARIATIONS

4.1. Stating the variational problem. We will consider the variational approach first presented by York [18], Gibbons and Hawking [12], which has been further studied by Araujo in [3].

For this, we study a functional whose critical points are metrics with constant scalar curvature on the manifold \( M \) and where the boundary \( \partial M \) is composed by a minimal hypersurface \( \Sigma_1 \) and a constant mean curvature (CMC for short) hypersurface \( \Sigma_2 \).

In order to obtain such a functional, let us start by considering the so-called Gibbons-Hawking-York functional, GHY-functional for short (also known as the total scalar curvature plus total mean curvature functional):

\[ F : M \to \mathbb{R}, \quad g \mapsto F(g) = \int_M R(g) \, dv(g) + 2 \int_{\partial M} H(g) \, d\sigma(g), \]

where \( R(g) \) is the scalar curvature in \( M \) and \( H(g) \) is the mean curvature on \( \partial M \). The first variation of such a functional reads as follow (see Section 5.1, Equation (32) in the Appendix for details)

\[ \delta F_g(h) = -\int_M \left( R_{ij} - \frac{1}{2} g_{ij} R(g) \right) h^{ij} \, dv(g) \]
\[ - \int_{\partial M} \left( I I_{\alpha\beta} - H(g) g_{\alpha\beta} \right) h^{\alpha\beta} \, d\sigma(g), \]

where \( R_{ij}, h^{ij} \) and \( I I_{\alpha\beta} \) are the components for the Ricci tensor, the tensor \( h \) and the second fundamental form respectively. From here, we deduce the following result

**Proposition 4.1.** A Riemannian metric \( g \) is a critical point for the functional \( F \) if, and only if, the Riemannian manifold \( (M, g) \) is Ricci flat and \( \partial M \) is totally geodesic.
Therefore, the GHY-functional defined in the whole set of Riemannian metrics does not determine the variational problem of our interest. In order to solve this, we will restrict the domain of the functional to a particular subset of Riemannian metrics. For the sake of clearness, such restriction will be performed in two steps. In the first step, we will consider Riemannian metrics satisfying the constraint

\[ C_{a,b}(g) = 1, \]

where:

\[ C_{a,b}(g) = a \int_M dv(g) + b \int_{\Sigma_2} d\sigma(g_2), \]

with \( a, b \) real numbers, \( a \geq 0 \); and \( g_2 \) the metric \( g \) restricted to \( \Sigma_2 \).

**Remark 4.2.**

(i) It is clear that the above constraint was derived from (1), since in our problem only \( \Sigma_2 \) must have constant mean curvature, and not all the boundary. In fact, as it is known, \( \int_M dv(g) \) determines the volume of \( M \) while \( \int_{\Sigma_2} d\sigma(g_2) \) determines the area of \( \Sigma_2 \).

(ii) Note that if \( a = 0 \) (resp. \( b = 0 \)) then \( b = 1/\int_{\Sigma_2} d\sigma(g_2) > 0 \) (resp. \( a = 1/\int_M dv(g) > 0 \)). Normalizing, it is not restrictive to assume that if \( a = 0 \) then \( b = 1 \) or, in other case, if \( b = 0 \) then \( a = 1 \).

Let us denote by \( M_{a,b} \) the space of metrics under the following constraint

\[ M_{a,b} := \{ g \in M : C_{a,b}(g) = 1 \}. \]

In order to study the critical points of the GHY-functional restricted to \( M_{a,b} \), the Lagrange multipliers method leads us to the study of the following functional

\[ F^\lambda(g) = F(g) - \lambda(C_{a,b}(g) - 1), \]

for some \( \lambda \in \mathbb{R} \).

Now, taking into account the first variation of \( F \) (see (8)) and

\[ \delta C_{a,b}(t) = \frac{a}{2} \int_M g^{ij} h_{ij} dv(g) + \frac{b}{2} \int_{\Sigma_2} g^{\alpha\beta} h_{\alpha\beta} d\sigma(g_2), \]

we deduce that the first variation of \( F^\lambda \) takes the following form

\[ \delta (F^\lambda)_g(h) = \int_M \left( R_{ij} - \frac{(R(g) + \lambda a)}{2} g_{ij} \right) h^{ij} dv(g) + \int_{\Sigma_1} (\mathbb{I}_{\alpha\beta} - H(g_1) g_{\alpha\beta}) h^{\alpha\beta} d\sigma(g_1) + \int_{\Sigma_2} \left( \mathbb{I}_{\alpha\beta} - \frac{(2H(g_2) + \lambda b)}{2} g_{\alpha\beta} \right) h^{\alpha\beta} d\sigma(g_2) \]

where \( g_k \) is the metric \( g \) restricted to \( \Sigma_k \). Then, by a classical argument involving the first Bianchi identity, we obtain the following result:
**Proposition 4.3.** A Riemannian metric $g$ on $M$ is critical for $F$ restricted to the space $\mathcal{M}_{a,b}$ if, and only if, $g$ is Einstein, $\Sigma_1$ is totally geodesic, $\Sigma_2$ is totally umbilical with constant mean curvature and, if $R(g)$ denotes the scalar curvature and $H(g_2)$ the mean curvature in $\Sigma_2$, then

$$ (n - 1) b R(g) = 2 n a H(g_2). $$

**Remark 4.4.** From previous relation (10) and Remark 4.2, we have that $a = 0$ if and only if $b = 1$ and $R(g) = 0$. Analogously, we deduce that $b = 0$ if and only if $a = 1$ and $H(g_2) = 0$.

In the second step, we restrict the domain of $F$ to the space of conformal metrics of $g$ which also lie in $\mathcal{M}_{a,b}$, that is, we will consider the subset of Riemannian metrics

$$ \text{Conf}_{a,b}(g) := \mathcal{M}_{a,b} \cap \text{Conf}(g) $$

where $\text{Conf}(g)$ denotes the $\mathcal{H}^1(M)$-conformal metrics of $g$, i.e.

$$ \text{Conf}(g) = \{ fg : f \in \mathcal{H}^1(M) \text{ and } f > 0 \}. $$

**Remark 4.5.** Recall that our aim is to use the fiber bundle version of the Implicit Theorem, and so, a Banach space as $\mathcal{H}^1$ will be enough for our purposes. However, if we try to determine bifurcation points, we have to deal with the fiber bundle version of the classical result given by Smoller and Wasserman for bifurcation [21]. This result requires additional technical conditions as Fredholmness, Palais-Smale, among others, and so, the Sobolev space has not the required regularity. In such a case, it is necessary to restrict even more the conformal class to the Hölder space $C^{k,\alpha}$ with $k \geq 0$ and $0 < \alpha \leq 1$ (see [15] for detailed studies).

The set $\text{Conf}_{a,b}(g)$ is a smooth submanifold of $\text{Conf}(g)$ because it is the set of regular points of the map $C_{a,b}(g)$, being its tangent space $T_g(\text{Conf}_{a,b}(g))$ identified with the following set

$$ \mathcal{H}^1_{a,b}(M) = \left\{ f \in \mathcal{H}^1(M) : f > 0 \text{ and } \frac{a n}{2} \int_M f dv(g) + \frac{b(n - 1)}{2} \int_{\Sigma_2} f ds(g_2) = 0 \right\}. $$

Thus, under this last restriction, the first variation of the functional $F^\lambda$ becomes (see (33), (34))

$$ \delta(F^\lambda)_g(f) = \int_M \left( R(g) - n \frac{(R(g) + \lambda a)}{2} \right) f dv(g) + $$

$$ - \int_{\Sigma_1} (n - 2) H(g_1) f ds(g_1) + $$

$$ - \int_{\Sigma_2} \left( (n - 2) H(g_2) - (n - 1) \frac{\lambda b}{2} \right) f ds(g_2) $$

where $f \in \mathcal{H}^1_{a,b}(M)$. Therefore, we finally arrive to

**Proposition 4.6.** A Riemannian metric $g$ is a critical point of the functional $F$ restricted to the space $\text{Conf}_{a,b}$ if, and only if, the scalar curvature $R(g)$ is constant on $M$, $\Sigma_1$ is a minimal hypersurface ($H(g_1) = 0$) and $\Sigma_2$ has constant mean curvature $H(g_2)$. Under these assumptions, $R(g)$ and $H(g_2)$ are related by (10).
Remark 4.7. For prescribed \( R(g) \) and \( H(g_2) \), the values of \( a \) and \( b \) are determined by the equations \( C_{a,b}(g) = 1 \) and \( (10) \) (see also Remark 4.2). Moreover, if \( g \) is one of the critical points described in previous proposition, it will be also critical for the functional \( F^\lambda \) defined in \( (9) \) with \( \lambda \) solving one of the following equations (if \( a \neq 0 \neq b \), both equations define the same \( \lambda \) assuming \( (10) \)):

\[
\begin{align*}
\delta \lambda &= H(g_2) \frac{2(n-2)}{n-1}, \\
\alpha \lambda &= R(g) \frac{n-2}{n}.
\end{align*}
\]

As we can see, this last restriction lead us to a variational problem whose critical points satisfy the desired properties. Let us now recall the explicit expression for the second variation in this case. For this, assume that \( g \) is a critical point for \( F^\lambda \) and some \( \lambda \). Taking into account the relation between \( \lambda, R(g) \) and \( H(g_2) \), we deduce that the quadratic form associated to the second variation of \( F^\lambda \) takes the following form on \( H^1_{a,b}(M) \) (see (35), (36) and the Appendix for details):

\[
\delta^2(F^\lambda)_{g}(f,k) = \frac{(n-2)(n-1)}{2} \left( \int_M \left[ (\text{grad}(f))^2 - \frac{R(g)}{n-1} f^2 \right] dv(g) - \int_{\partial M} \frac{H(g)}{n-1} f^2 d\sigma(g) \right)
\]

where \( |\text{grad}(f)|^2_g \) denotes the squared norm of the gradient of \( f \) computed with respect to the norm induced by \( g \), \( \Delta_g \) is the Laplace-Beltrami operator with nonnegative spectrum calculated with respect to the metric \( g \) and \( \partial_n \) is the inward normal derivative (recall that, for the second equality, we have used the first Green identity). Then, we can recover the expression for the second variation and, even more, we can describe it in terms of Fredholm operators by using the following inner product on the space \( L^2(M) \cap L^2(\partial M) \)

\[
\langle \langle k, f \rangle \rangle := \int_M kf \ dv(g) + \int_{\partial M} kf \ d\sigma(g).
\]

In particular, the second variation takes the following form

\[
\delta^2(F^\lambda)_{g}(f,k) = \frac{(n-2)(n-1)}{2} \langle \langle J_g(f), k \rangle \rangle
\]

where \( J_g \) is a linear elliptic operator given by

\[
J_g|_M = \Delta_g - \frac{R(g)}{(n-1)}, \quad J_g|_{\partial M} = -\partial_n - \frac{H(g)}{(n-1)}.
\]

\[\text{i.e.,} \quad \Delta_g = -\text{div}_g(\text{grad}).\]
Observe that, as a simple consequence of the second Green identity, this operator $J_g$ is self-adjoint relatively to the $L^2$-inner product. Namely:

$$
\langle\langle J_g(f), k \rangle\rangle = \int_M \left( k \Delta_g f - \frac{R(g)}{(n-1)} k f \right) dv(g) - \int_{\partial M} \left( k \partial_n f + \frac{H(g)}{(n-1)} k f \right) d\sigma(g)
$$

$$
= \int_M \left( f \Delta_g k - \frac{R(g)}{(n-1)} k f \right) dv(g) - \int_{\partial M} \left( f \partial_n k + \frac{H(g)}{(n-1)} k f \right) d\sigma(g)
$$

$$
= \langle\langle f, J_g(k) \rangle\rangle.
$$

In conclusion, $J_g$ coincide with the Jacobi operator associated to $\delta^2(F^\lambda)_g$. This means that, for our studies on bifurcation, we have to study the spectrum of $J_g$ which lead us to the following mixed eigenvalue problem

$$
\begin{cases}
\Delta_g f - \frac{R(g)}{(n-1)} f = \overline{\mu} f & \text{in } M, \\
-\partial_n f - \frac{H(g)}{(n-1)} f = \overline{\mu} f & \text{on } \partial M
\end{cases}
$$

with $f \in H^1_{a,b}(M)$.

### 4.2. General rigidity results under metric variations

The lower bound for eigenvalues proved in Proposition 3.1 (see Section 3) allows us to give a general result for rigidity associated to metric variations by giving a very simple and geometrical argument to ensure rigidity.

**Theorem 4.8.** Let $M$ be a compact manifold with boundary $\partial M = \Sigma_1 \cup \Sigma_2$. Consider $\{g_\lambda\}_{\lambda \in I}$ a family of metrics which are critical points for the functional $F^\lambda$ restricted to $\text{Conf}_{a,b}$ for some $a = a(\lambda)$ and $b = b(\lambda)$; that is, metrics with constant scalar curvature on $M$ and making $\Sigma_1$ and $\Sigma_2$ a minimal and a CMC hypersurface respectively (recall Proposition 4.6). If it is satisfied that $R(g_\lambda) + H((g_\lambda)_2) < 0$ for some $\lambda \in I$, then the family $\{g_\lambda\}_{\lambda \in I}$ is locally rigid at $\lambda_*$.

**Proof:** Let us study the spectrum of $J_{g_\lambda}$ for $\lambda = \lambda_*$. As we have pointed out in the previous section, the spectrum of such a linear map is related with the mixed eigenvalue problem detailed on (13) with $g = g_\lambda$. From Proposition 3.1 we have that any eigenvalue $\overline{\mu}$ for such a problem satisfies:

$$
\overline{\mu} \geq - (R(g_{\lambda_*}) + H((g_{\lambda_*})_2)) > 0,
$$

where the last inequality is obtained by hypothesis. Therefore, all the eigenvalues for the Jacobi operator are positive and $g_{\lambda_*}$ is a non-degenerate critical point. Then, the result follows from the fiber bundle version of the Implicit Function Theorem (see [15, Appendix A]).

### 4.3. Metric rigidity in warped product spaces

Let us now consider

$$
(M^n, g) = ((r_1, r_2) \times P^{n-1}, dr^2 + \alpha(r)^2 g_P)
$$

an $n$-dimensional warped product space with constant scalar curvature and an $(n-1)$-dimensional closed (compact without boundary) manifold $P$. Moreover, we will assume along this section that $\Sigma_1 = \{r_1\} \times P$ is a minimal hypersurface and that $r_2 \in \mathbb{R} \cup \{\infty\}$ (the case with $\{r_2\} \times P$ minimal is analogous).
Now, let us define an one-parameter family of metrics \( \{ g_\gamma \}_{\gamma \in (r_1, r_2)} \) which are critical points for the functional \( F^\lambda \) for some suitable \( \lambda \). Such a family will be naturally identified with a family of open subsets of \( M \) as follows: Let \( \gamma \in (r_1, r_2) \) and consider the open set \( \Omega_\gamma = (r_1, \gamma) \times P \) whose boundary is composed by a fixed set \( \Sigma_1 = \{ r_1 \} \times P \) which is a minimal hypersurface; and other set \( \Sigma_2 = \{ \gamma \} \times P \) which is a hypersurface with constant mean curvature given by

\[
H(\gamma) = -(n - 1) \frac{\dot{\alpha}(\gamma)}{\alpha(\gamma)}
\]

(recall that, as \( \Sigma_1 \) is minimal, previous equation implies that \( \dot{\alpha}(r_1) = 0 \)). By a standard procedure, for a fixed \( r_0 \in (r_1, r_2) \) we can define a diffeomorphism \( \Psi_\gamma : \Omega_\gamma \to \Omega_{r_0} \) preserving the orientation of \( \partial / \partial r \) (for \( \gamma = r_0 \), such a diffeomorphism is just the identity). Then, we define a family of metrics \( \{ g_\gamma \}_{\gamma \in (r_1, r_2)} \) on \( \Omega_{r_0} \) where

\[
g_\gamma = \Psi_\gamma^*(g).
\]

**Remark 4.9.** As it is clear from construction, both \( (\Omega_{r_0}, g_\gamma) \) and \( (\Omega_\gamma, g) \) are isometric under \( \Psi_\gamma \), and so, they share the same geometrical properties. So that, we can make the computations with the latter, which will be simpler in practical cases. Moreover, this approach shows the independence on \( r_0 \) of our results.

Finally, observe that in [11] the rigidity of the family of open sets \( \{ \Omega_\gamma \} \) under hypersurface variations was studied. This previous identification let us go one step further on these studies for rigidity of the family \( \{ \Omega_\gamma \} \), as it allows us to interpret the rigidity results obtained in this section as a sort of rigidity under metric variations of the family \( \{ \Omega_\gamma \} \).

Recall that \( (\Omega_\gamma, g) \) has constant scalar curvature (by hypothesis) and its boundary is the union of a minimal hypersurface \( \Sigma_1 = \{ r_1 \} \times P \) and a CMC hypersurface \( \Sigma_2 = \{ \gamma \} \times P \). Then, previous remark let us ensure that \( g_\gamma \) is a critical point for the functional \( F(\lambda(\cdot)) \), being \( \lambda : (r_1, r_2) \to \mathbb{R} \) a function given implicitly on [12] (recall that \( a \) and \( b \) are determined by \( R(g_\gamma) \) and \( H(g_\gamma) \)), see Remark 4.7. So, we can study the rigid character of the family \( \{ g_\gamma \}_{\gamma \in (r_1, r_2)} \).

Let us recall how the mixed eigenvalue problem [13] is expressed in the warped case (see Section 3.1 for details) for an open set of the form \( \Omega_\gamma \). From Theorem 3.2 we know that the eigenfunctions \( f \) are obtained as a product of two functions \( f_\gamma : (r_1, \gamma) \to \mathbb{R} \) and \( f_P : P \to \mathbb{R} \), being the latter an eigenfunction of the Laplacian \( \triangle_P \) on \( (P, g_P) \) associated to an eigenvalue \( \mu_P \); and the former a solution of the following Sturm-Liouville problem

\[
- \frac{1}{\alpha^{n-1}} \partial_r (\alpha^{n-1} \partial_r f_\mathbb{R}) + \frac{\mu}{\alpha^2} f_\mathbb{R} = \left( \frac{R(g)}{n-1} + \mu \right) f_\mathbb{R},
\]

\[
- \dot{f_\mathbb{R}}(r_1) = \pi f_\mathbb{R}(r_1),
\]

\[
\dot{f_\mathbb{R}}(\gamma) = \left( \frac{H(\gamma)}{n-1} + \mu \right) f_\mathbb{R}(\gamma)
\]

where \( R(g) \) is constant by hypothesis and \( H(\gamma) \) is given by [14]. Moreover, \( f \) and \( f_\mathbb{R} \) are related to the same eigenvalue. Then, we obtain the following rigidity result

**Theorem 4.10.** Consider \( (\mathbb{R}^n, g) = ((r_1, r_2) \times P^{n-1}, dr^2 + \alpha(r)^2 g_P) \) a warped product space where \( P \) is a closed manifold, the scalar curvature \( R(g) \) (i.e. \( R(g_\gamma) \) for all \( \gamma \)) is constant and
\\{r_1 \times P\} is a minimal surface. For a fixed \(r_0\), consider the family of metrics \(\{g_\gamma\}\) on \(\Omega_{r_0}\) as defined on (13). Assume that, for some \(\gamma_* \in (r_1, r_2)\), \(R(g)\) and \(\alpha\) satisfy the following relations

\[
R(g) \leq \frac{\mu_1}{\alpha^2(\gamma)}, \quad \alpha(r) \geq 0 \quad \text{for} \quad r \in (r_1, \gamma),
\]

where \(\mu_1\) is the first nonzero eigenvalue of the Laplacian on \((P, g^P)\), and one of previous inequalities is strict. Then, the family \(\{g_\gamma\}\) is locally rigid at \(\gamma_*\).

**Proof:** Our aim is to apply Proposition 3.4 to the mixed problem (16). For this, we have to show that the inequalities \(H(\gamma) \leq 0\) and \(\alpha^2(r) R(g) \leq \mu_1\) for all \(r \in (r_1, \gamma_*)\) hold, being at least one of them strict. For the former, recall the expression of the mean curvature in (14) and the second inequality on (17). For the latter, observe that the function \(\alpha\) is non-decreasing, and so

\[
\frac{\mu_1}{\alpha^2(r)} \geq \frac{\mu_1}{\alpha^2(\gamma_*)}
\]

for all \(r \in (r_1, \gamma_*)\). Therefore, the latter inequality follows from the condition for the scalar curvature on (17). Finally, one of such an inequalities should be strict, as happen with the hypothesis.

In conclusion, Proposition 3.4 ensures that the eigenvalues for the mixed problem (16) are always rigid.

The previous result has a very simple and nice consequence on product spaces, where the warping function is constant (we will normalize the constant for simplicity).

**Corollary 4.11.** If the warping function \(\alpha \equiv 1\), i.e., the warped product space \((M, g)\) is just a product space, condition (17) reads as

\[
R(g^P) \leq \mu_1
\]

where \(\mu_1\) is the first nonzero eigenvalue of the Laplacian and \(R(g^P)\) denotes the scalar curvature, both in the Riemannian manifold \((P, g^P)\). In particular, if \(R(g^P)\) is negative, the family of metrics \(\{g_\gamma\}\) is always rigid.

**Proof:** The result follows from previous theorem and the fact that product spaces satisfies both, that all the slices \(\{r\} \times P\) are minimal (in particular, for \(\{r_1\} \times P\)) and that \(R(g) = R(g^P)\).

**Example 4.12.** By Theorem 4.10, the following pseudo-hyperbolic space

\[
(0, r_2) \times S^3, dr^2 + \cosh^2(r)g^{S^3}
\]

has a rigid family \(\{g_\gamma\}\) of metrics. In fact, \(\Sigma_1 = \{0\} \times S^3\) is a minimal hypersurface and \(H(\gamma) = -3 \tanh(\gamma) < 0\) for all \(\gamma \in (0, r_2)\) and \(R(g) = -12 \tanh^2(r) < 0\) and \(\mu_1 = 3 > 0\).

**Example 4.13.** Let us consider \(\mathbb{H}^{n-1}\) the \((n-1)\)-dimensional hyperbolic space and \(\Gamma\) a cocompact group of isometries defined on it. The space \(P = \mathbb{H}/\Gamma\) is a compact manifold with constant scalar curvature \(R(g^P) = -1\). Therefore, Corollary 4.14 ensures that, for the product space

\[
((r_1, r_2) \times P, dr^2 + g^P)
\]
the family of metrics \( \{g_\gamma\}_{\gamma \in (r_1, r_2)} \) is (globally) rigid.

**Remark 4.14.** Recall that our mixed problem (13) was not considered in the entire space \( \mathcal{H}^1(M) \), but in the subset \( \mathcal{H}^1_{a,b}(M) \). However, for general values of \( a \) and \( b \), the restriction of the spectral problem for such a subset will not give us, a priori, practical information for the estimate of the first eigenvalue. For this reason, for our results (Theorems 4.8 and 4.10), we have preferred to obtain lower estimates of such an eigenvalue considering the entire space \( \mathcal{H}^1(M) \), which will be enough for our purposes.

For particular cases, as the Schwarzschild case in next section, the restriction to \( \mathcal{H}^1_{a,b}(M) \) will give us valuable information which yields better lower estimates for the first eigenvalue (see Proposition 4.18).

4.4. **An application to the Schwarzschild models.** Let us show now how these simple and geometrical conditions allow us to give some remarkable information about the rigidity of the spatial fiber of the de Sitter and Anti-de Sitter Schwarzschild spacetimes under metric variations. For this, let us begin by recalling some of the basic facts on these spaces.

Consider a 3-dimensional Riemannian manifold \((M, g_{K,E})\) where \(M = I \times S^2\),

\[
g_{K,E} = \psi_{K,E}(r)^{-2} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

where \(K > 0\) and \(E\) are constants, \(\psi_{K,E}(r) = \sqrt{1 - 2K/r + Er^2}\) and \(I = [\hat{r}, r_E)\) is a maximal connected and open interval where \(\psi_{K,E}\) is well defined (i.e., its radicand is nonnegative). Particularly, we are going to be interested in taking \(r_E = 0\) when \(E < 0\) and \(r_E = \infty\) otherwise (with \(\hat{r} > 0\) in this second case). In case \(E = 0\), we have the classical spatial fiber of the Schwarzschild spacetime. For \(E\) negative (resp., positive) we have the spatial fiber of the so-called de Sitter (resp., Anti-de Sitter) Schwarzschild spacetime.

These models can be expressed as a warped metric space under the following change of variable:

\[
ds = -\psi_{K,E}(r)^{-1} dr,
\]

and so, we can naturally define a family of metrics as in Section 4.3. Moreover, reasoning as in Remark 4.9, it is straightforward to deduce that such a family of metrics is biunivocally related with the family of open sets \(\Omega_\gamma := (\hat{r}, \gamma) \times S^2\). Finally observe that the metrics associated to these open sets satisfy our desired properties, i.e., the scalar curvature \(R(g_{K,E}) = -6E\) is constant and \(\partial \Omega_\gamma\) is formed by two components \((\hat{r}) \times S^2\) and \((\gamma) \times S^2\), where the first one is minimal and the second one is a surface with constant mean curvature

\[
H(\gamma) := 2 \sqrt{1 - \frac{2K}{\gamma} + E\gamma^2}
\]

So, we are in conditions to apply our results. In particular, we need to make the following computation in order to apply Theorem 4.8

\[
R(g_{K,E}) + H(\gamma) = -6E + 2\frac{\psi_{K,E}(\gamma)}{\gamma} = 2 \left(1 - \frac{2K}{\gamma} + E\gamma^2 + 3E\gamma\right)^{p_{K,E}(\gamma)},
\]

where \(p_{K,E}(x) = x - 2K + Ex^3(1 - 9E)\) is a polynomial of degree three.
Observe that for any positive value of $E$, $R(g_{K,E}) + H(\hat{r}) = -6E < 0$, and so, there exists some $\delta > 0$ in such a way that $R(g_{K,E}) + H(\gamma) < 0$ for all $\gamma \in (\hat{r}, \hat{r} + \delta)$. So, we deduce the following result

**Corollary 4.15.** Consider $(I \times S^2, g_{K,E})$ a Riemannian manifold with $K, E > 0$, $I = [\hat{r}, \infty)$ and $g_{K,E}$ as in (18). Then, there exists $\delta > 0$ in such a way the family of open sets $\{\Omega_\gamma\}_{\gamma \in (\hat{r}, \hat{r} + \delta)}$ is rigid (in the sense described in Remark 4.9).

**Proof:** From previous computations, we have deduced that there exists $\delta$ such that $R(g_{K,E}) + H(\gamma) < 0$ for all $\gamma \in (\hat{r}, \hat{r} + \delta)$. Then, the result follows from Theorem 4.8. $\square$

Of course, previous result is not optimal in the sense that the family of open sets could be rigid “far away” from $\hat{r}$. For instance, observe that in (19), the first term is always positive and so, the sign of $R(g_{K,E}) + H(\gamma)$ is determined by the sign of the third order polynomial $p_{K,E}$. In particular, if $E > 1/9$, the leading term of such a polynomial is negative, and so, for $\gamma$ big enough, $p_{K,E}$ is negative. In conclusion, we obtain the following result.

**Corollary 4.16.** For $E > \frac{1}{9}$, there exists $\gamma_0$ in such a way that the family of open subsets $\{\Omega_\gamma\}_{\gamma \in (\gamma_0, \infty)}$ is rigid.

As a final step, let us show that there are also conditions to ensure the rigidity of the family of open sets on the entire interval where they are defined. For that, we only need to give conditions for $K$ and $E$ in order to ensure that the polynomial $p_{K,E}$ is always negative. A simple condition for that, together with previous results, are summarized in the following theorem:

**Theorem 4.17.** Consider $((\hat{r}, \infty) \times S^2, g_{K,E})$ an Anti-de Sitter Schwarzschild model, i.e., with $K, E > 0$; and consider the family of open sets $\{\Omega_\gamma\}_{\gamma \in (\hat{r}, \infty)}$ with $\Omega_\gamma = (\hat{r}, \gamma) \times S^2$. Then:

(i) There exists $\delta > 0$ in such a way the family $\{\Omega_\gamma\}_{\gamma \in (\hat{r}, \hat{r} + \delta)}$ is rigid.

(ii) If $E > 1/9$, there exists $\gamma_0 \geq \hat{r}$ such that the family $\{\Omega_\gamma\}_{\gamma \in (\gamma_0, \infty)}$ is rigid.

(iii) If $E > 1/9$ and satisfies the following relation with $K$

$$
\frac{1}{\sqrt{3E(9E - 1)}} < 3K
$$

then the family $\{\Omega_\gamma\}_{\gamma \in (\hat{r}, \infty)}$ is rigid.

**Proof:** (i) and (ii) have been proved before, so we only need to prove (iii). Our aim is to show that under condition (20) the polynomial $p_{K,E}$ is always negative when $\gamma > 0$ and so the result will follow from (19) and Theorem 4.8. By simple computations, we deduce that $p_{K,E}$ attains a local maximum at

$$
\gamma_0 = \frac{1}{\sqrt{3E(9E - 1)}},
$$

Moreover, from (20) we obtain that

$$p_{K,E}(\gamma_0) = \frac{2}{3} \frac{1}{\sqrt{3E(9E - 1)}} - 2K < 0$$

Therefore, it easily follows that the third order polynomial $p_{K,E}$ is always negative under (20) and the result follows. $\square$

---

5Recall that $\hat{r}$ is a zero of $\psi_{K,E}$. 

---
Observe that previous estimates are not enough when $E \leq 0$ (and so, the scalar curvature is positive). On these cases, a sharper estimate is needed and we have to recall that the eigenfunctions should belong, not to the entire $\mathcal{H}^1(M)$ (as we have considered on previous estimates), but to the space $\mathcal{H}^1_{a,b}(M)$ (see [11]). In this sense, let us consider the particular case of the Schwarzschild spacetime (i.e., when $E = 0$). First, we need to specialize the computations made on warped metrics for this case (see Section 3.1 for details) with the metric

$$g_0 = \left(1 - \frac{2K}{r}\right) dr^2 + r^2 g^S$$

and bearing in mind that, in this case, the scalar curvature is zero and the mean curvature $(2/r)\sqrt{1 - \frac{2K}{r}}$, we have that Equation (16) becomes

$$\partial_r \left(r^2 \sqrt{1 - \frac{2K}{r}} \partial_r f_R \right) - \frac{\mu_i}{\sqrt{1 - \frac{2K}{r}}} f_R = -\bar{\mu} \left(\frac{r^2}{\sqrt{1 - \frac{2K}{r}}}\right) f_R,$$

(22)

$$-\dot{f}_R(2K) = \bar{\mu} f_R(2K),$$

$$\dot{f}_R(\gamma) = \left(\frac{\sqrt{1 - \frac{2K}{\gamma}}}{\gamma} + \bar{\mu}\right) f_R(\gamma).$$

where $\mu_i = i(i + 1)$ is an eigenvalue for the Laplacian on the sphere. Moreover, we also know that the eigenfunction $f$ associated to such an eigenvalue is the product of two functions $f_R : (r_1, r_2) \to \mathbb{R}$ and $f_S^2 : S^2 \to \mathbb{R}$, where the former is the eigenfunction associated to $\bar{\mu}$ on previous Sturm-Liouville problem and the latter is the eigenfunction associated to $\mu_i$.

Now recall that, as we pointed out when we introduced the mixed eigenvalue problem in (13), we do not have to consider the whole space $\mathcal{H}^1(M)$, but the subset $\mathcal{H}^1_{a,b}(M)$. Moreover, as the scalar curvature is zero and the mean curvature in $\Sigma_2$ is not zero, by Remark 4.4 we know that $a = 0$ and $b = 1$. By definition, a function $f \in \mathcal{H}^1_{a,b}(M)$ should satisfy:

$$\int_{(\gamma) \times S^2} f d\sigma(g_0) = 0.$$

If such a function is an eigenfunction, i.e., $f = f_R f_S^2$, we arrive to

$$\gamma^2 f_R(\gamma) \int_{S^2} f_S^2 d\sigma(g^{S^2}) = 0$$

and two options arise: either $f_R(\gamma) = 0$, and so, $f_R$ satisfies the first equation in (22) with $f_R(\gamma) = 0$ and $\dot{f}_R(\gamma) = 0$ (recall the last equation on (22)). Then, from the uniqueness of the solution, $f_R \equiv 0$, a contradiction. Otherwise, we have that

$$\int_{S^2} f_S^2 d\sigma(g^{S^2}) = 0.$$

As $f_S^2$ is an eigenfunction for the Laplacian on the sphere associated to eigenvalue $i(i + 1)$ for some $i$, the Courant’s nodal Theorem ensures that $i \neq 0$ ($f_S^2$ cannot be associated to the zero eigenvalue of the Laplacian). Otherwise, the function $f_S^2$ could be considered always positive, and so, the integral condition is not satisfied.
In conclusion, we have proved the following proposition.

**Proposition 4.18.** If \( f \in H^1_{0,1}(M) \) is an eigenfunction for the mixed eigenvalue problem \((13)\) in the Schwarzschild case, its associated eigenvalue is also an eigenvalue for \((22)\) with \( i \neq 0 \).

So, from Lemma 3.3, in order to obtain a rigidity result, we need to estimate the first eigenvalue of \((22)\) for \( i = 1 \), which will be the lowest eigenvalue for our problem. By a very simple numerical approach, we can see that the evolution of the first eigenvalue regarding \( \gamma \) follows the graph in Figure 2 and so, it is always positive. For this reason, we state the following:

**Conjecture 4.19.** Let \((M^3,g_0)\) be the spatial fiber of Schwarzschild spacetime defined in \((21)\). Then, for a fixed \( r_0 \), the family of metrics \( \{g_\gamma\}_{\gamma \in (2K,\infty)} \) for \( \Omega_{r_0} \) defined on \((15)\) is rigid. In fact, taking into account the identification between \((\Omega_{r_0}, g_\gamma)\) and \((\Omega_\gamma, g_0)\) described in Remark 4.3, such a rigidity can be interpreted as the rigidity of the family \( \{\Omega_\gamma\}_{\gamma \in (2K,\infty)} \) under metric variations.

**Figure 2.** Numerical approximation of the first eigenvalue of the Sturm-Liouville problem \((22)\) with \( \mu_1 = \mu_1 = 2 \); for different values of \( \gamma \). The graph shows that the first eigenvalue is always positive and it has the X-axis as an asymptote.

5. **Appendix**

5.1. **First and second variation of GHY-functional.** In this subsection we will include all the basics computations needed in order to compute the first and second variation of the functional \( \mathcal{F}^\lambda \). Such computations have been already appeared elsewhere (see for instance \([16, 3]\) ), but we will include them here for the sake of completeness.

Even if we are interested only in conformal variations, the computation of the first variation of the different elements conforming \( \mathcal{F}^\lambda \) will be done for a more general family of variations.
Let $S^k(M)$ the space of all symmetric $(0, 2)$ tensors of class $C^k(M)$ with $k \geq 2$. We consider a metric variation $g : M^k(M) \times (-\epsilon, \epsilon) \to M$ defined by:

$$g(h, t) = g + th$$

where $M^k(M)$ is the open cone of $S^k(M)$ consisting of all Riemannian metrics on $M$ such that for all $g \in M^k(M)$ the tangent space $T_gM^k(M)$ is identified with the Banach space $S^k(M)$. By compactness, for $|t|$ sufficiently small, $g(h, t)$ is in $M^k(M)$.

**Convention 5.1.** Henceforth, all the elements associated to the metric $g(h, t)$ will be denoted as functions of $t$, assuming that the metric variation $h$ is fixed from the beginning. So, elements as the metric itself, the scalar and mean curvature (among others) will be denoted by $g(t), R(t)$ and $H(t)$ respectively.

For the first variation of $C_{a,b}$, just recall that

$$\delta dv(t) = \frac{1}{2} g^{ij} h_{ij} dv(g),$$

and so

$$\delta C_{a,b}(t) = \frac{a}{2} \int_M g^{ij} h_{ij} dv(g) + \frac{b}{2} \int_{\Sigma^2} g^{\alpha\beta} h_{\alpha\beta} d\sigma(g_2).$$

Now, we will focus on the first variation of the GHY-functional (recall the definition on (8)). From basic computations, we have that

$$\delta F(t) = \int_M \delta (R(t)dv(t)) + \int_{\partial M} \delta (H(t)d\sigma(t))$$

$$= \int_M (\delta R(t))d\sigma(g) + R(g)(\delta dv(t)) +$$

$$+ \int_{\partial M} (\delta H(t))d\sigma(g) + H(g)(\delta d\sigma(t))$$

and so, we need to compute the first variation of both, the scalar curvature and the mean curvature. For the first one, let us consider a point $p \in M$ and a local normal coordinate system $(x_i)$ centred on $p$. Recall that, in such coordinates, at the point $p$ we have:

$$g_{ij} = \delta_{ij}, \quad \partial_k g_{ij} = 0 \quad \text{and} \quad \nabla_i = \partial_i$$

for $1 \leq i, j, k \leq n$, where $\delta_{ij}$ represents the Kronecker delta, that is, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. In particular, the variation of the Christoffel symbols becomes\(^6\)

$$\delta \Gamma^l_{ij}(t) = \frac{1}{2} g^{kl} (\nabla_j h_{il} + \nabla_i h_{jl} - \nabla_l h_{ij}).$$

\(^6\)Observe that the variation of the Levi-Civita connection $\delta \nabla$ is a tensor (in spite of what happens with $\nabla$), and so, a special coordinate system can be considered for its computation.
From the definition of the scalar curvature, we have that:

\[
\delta R(t) = (\delta g^{ij}(t)) R_{ij} + g^{ij}(\delta R_{ij}(t)).
\]

In one hand, one easily obtain that

\[
\delta g_{ij}(t) = h_{ij} \quad \text{and} \quad \delta g^{ij}(t) = -h^{ij}.
\]

On the other hand, the variation of the Ricci tensor has the following expression

\[
\delta R_{ij}(t) = \nabla_i \Gamma_{lj}^i(t) - \nabla_j \delta \Gamma_{li}^j(t) + \delta \left( \Gamma_{lj}^i u - \Gamma_{li}^j u \right) R_{ij}(t) = \nabla_i \left( \nabla_j h^{ij} - \nabla_i h^i_l \right)
\]

where we have used that \(\delta \left( \Gamma_{lj}^i u - \Gamma_{li}^j u \right) R_{ij}(t) = 0\), as the Christoffel symbols vanish for \(t = 0\); the formulae (26) and the fact that \(g^{lm} (\nabla_l h_{jm} - \nabla_m h_{jl}) = 0\) for \(1 \leq l, m \leq n\). In particular, the variation of the scalar curvature takes the following form:

(27) \[
\delta R(t) = -h^{ij} R_{ij} + \nabla_i \left( \nabla_j h^{ij} - \nabla_i h^i_l \right).
\]

Summarizing, the first two elements on the right of (25) bearing in mind (23) and (27), we obtain:

\[
\int_M (\delta R(t)) dv(g) + R(g)(\delta dv(t)) = \int_M \left( -h^{ij} R_{ij} + \frac{1}{2} g^{ij} h_{ij} R(g) \right) dv(g) + \int_M \nabla_i \left( \nabla_j h^{ij} - \nabla_i h^i_l \right) dv(g)
\]

or, by using the Divergence theorem,

(28) \[
\delta \int_M R(t) dv(t) = -\int_M \left( h^{ij} R_{ij} - \frac{1}{2} g^{ij} h_{ij} R(g) \right) dv(g)
\]

where \(N\) denotes the inward normal vector to the hypersurface \(\partial M\).

Next, we need to compute the second term in (25). As the point \(p \in \partial M\), let us consider a different coordinate system. Take \((x_1, \ldots, x_{n-1})\) a normal coordinate system associated to the boundary \(\partial M\), endowed with the metric induced by \(g\), and denote by \(\gamma(x_1, \ldots, x_n)\) the geodesic starting at \((x_1, \ldots, x_{n-1})\) with direction \(N\). Observe that \((x_1, \ldots, x_n) := \gamma(x_1, \ldots, x_{n-1})(x_n)\) defines a coordinate system for \(M\) around \(p\) in which the metric \(g\) takes the following form:

\[
g = dx_n^2 + g_{\alpha\beta} dx^\alpha dx^\beta.
\]

Moreover, as \((x_\alpha)\) is a normal coordinate system for \(\partial M\) around \(p\) and \(N = \partial_n\),

\[
\Gamma^\alpha_{\alpha\beta} = \mathbb{I}_{\alpha\beta}.
\]
Now, we are in conditions to compute the first variation for the mean curvature. Recall that, from definition,

\[ H(t) = g^{\alpha\beta}(t)I_{\alpha\beta}(t), \]

and so,

\[ (29) \quad \delta H(t) = (\delta g^{\alpha\beta}(t))I_{\alpha\beta} + g^{\alpha\beta}(\delta I_{\alpha\beta}(t)). \]

For the variation of the second fundamental form, define \( \nu : \partial M \times (-\epsilon, \epsilon) \to (T\partial M)^\perp \) an unitary inward normal vector of the tangent space for the metric \( g(t) \). In particular, it is not restrictive to assume that, for \( \nu(0) = N = \partial_n \). Then,

\[ \delta I_{\alpha\beta}(t) = \delta g_{ij}(t)\nu^i(t)\Gamma^{j}_{\alpha\beta} = h_{mn}\Gamma^{n}_{\alpha\beta} + (\delta \nu(t))\nabla^m h_{\alpha\beta} + \delta g_{\alpha\beta}. \]

Now, taking into account that \( g_{ij}(t)\nu^i(t)\nu^j(t) = 1 \) and \( g_{\alpha\beta}(t)\nu^\alpha(t) = 0 \) for all \( t \), we obtain that:

\[ (\delta \nu(t))^m = -\frac{1}{2}h_{mn}, \]

which, together with \( (26) \), yields the following expression for \( (30) \):

\[ \delta I_{\alpha\beta}(t) = \frac{1}{2}h_{mn}\Gamma^{n}_{\alpha\beta} + \frac{1}{2}(\nabla_\alpha h_{\beta\mu} + \nabla_\beta h_{\alpha\mu} - \nabla_\mu h_{\alpha\beta}). \]

In conclusion, \( (29) \) becomes

\[ (31) \quad \delta H(t) = -h^{\alpha\beta}I_{\alpha\beta} + \frac{1}{2}h_{nn}g^{\alpha\beta}\Gamma^{n}_{\alpha\beta} + \frac{1}{2}g^{\alpha\beta}(\nabla_\alpha h_{\beta\mu} + \nabla_\beta h_{\alpha\mu} - \nabla_\mu h_{\alpha\beta}) \]

and so, using \( (31) \) and the fact that \( \delta d\sigma(t) = (1/2)g^{\alpha\beta}h_{\alpha\beta}d\sigma(g) \) we get

\[
\delta \int_{\partial M} H(t)d\sigma(t) = \int_{\partial M} \left( -h^{\alpha\beta}I_{\alpha\beta} + \frac{1}{2}h_{nn}g^{\alpha\beta}\Gamma^{n}_{\alpha\beta} + \frac{1}{2}g^{\alpha\beta}(\nabla_\alpha h_{\beta\mu} + \nabla_\beta h_{\alpha\mu} - \nabla_\mu h_{\alpha\beta}) \right) d\sigma(g)
\]

\[ = \int_{\partial M} \left( -I_{\alpha\beta} - \frac{1}{2}H(g)g_{\alpha\beta} \right) h^{\alpha\beta} + \frac{1}{2}h_{nn}H(g) \] \( d\sigma(g) \)

\[ + \int_{\partial M} \left( \nabla^\alpha h_{\alpha\beta} - \frac{1}{2}\nabla^\alpha h_{\alpha\beta} \right) d\sigma(g). \]

Then, using this identity, \( (28) \) and the fact that in our coordinates \( \nu = \partial_n \), we deduce that:

\[
\delta F = \delta \int_{M} R(t)dv(t) + 2\delta \int_{\partial M} H(t)d\sigma(t)
\]

\[ = -\int_{M} \left( h^{ij}R_{ij} - \frac{1}{2}h^{ij}g_{ij}R(g) \right) dv(g)
\]

\[ + \int_{\partial M} \left[ -2I_{\alpha\beta} - H(g)g_{\alpha\beta} \right] h^{\alpha\beta} + h_{nn}H(g) \] \( d\sigma(g) \)

Finally, taking into account that,

\[ \nabla^\alpha h_{\alpha\beta} = D_\alpha h_n^\alpha + h^{\alpha\beta}I_{\alpha\beta} - h_{nn}H(g), \]
where $D$ denotes the induced connection on $\partial M$, and the fact that
\[
\int_{\partial M} D_\alpha h^\alpha_n d\sigma(g) = 0
\]
from the Divergence theorem (recall that $\partial M$ is closed), we finally obtain the following expression for the first variation of the GHY-functional
\[
\delta F_g(h) = -\int_M \left( R_{ij} - \frac{1}{2} g_{ij} R(g) \right) h^{ij} dv(g)
- \int_{\partial M} (\Pi_{\alpha\beta} - H(g) g_{\alpha\beta}) h^{\alpha\beta} d\sigma(g).
\]

Now, we will focus on the computation of the second variation. For this case, and for simplicity, only conformal variations will be considered (for the general case, see [3]). So, let us assume now that $h = fg$ for a positive function $f \in \mathcal{H}^1(M)$. With this assumption, observe that (24) and (25) become
\[
\delta C_{a,b}(fg) = \frac{an}{2} \int f dv(g) + \frac{b(n-1)}{2} \int_{\Sigma} f d\sigma(g),
\]
and
\[
\delta F(fg) = \frac{n-2}{2} \int_M R(g) f dv(g) + (n-2) \int_{\partial M} H(g) f d\sigma(g)
\]
respectively. Therefore, the second variation of the former is given by
\[
\delta^2 (C_{a,b})_{fg} = \frac{n^2 a}{4} \int_M f^2 dv(g) + \frac{(n-1)^2 b}{4} \int_{\Sigma} f^2 d\sigma(g),
\]
where we have used that
\[
\delta dv(fg) = \frac{n}{2} f dv(g), \quad \delta d\sigma(fg) = \frac{n-1}{2} f d\sigma(g).
\]
For the second variation of the latter, observe that both, (27) and (31) become
\[
\delta R(fg) = -f R(g) + (n-1) \Delta_g f, \quad \delta H(fg) = -\frac{1}{2} f H(g) - \frac{n-1}{2} \partial_n f,
\]
where $\Delta_g = -\text{div}_g(\text{grad}(\cdot))$ is the Laplace-Beltrami operator.

Finally, the second variation of the GHY-functional for conformal variations has the following form
\[
\delta^2 (F)_{fg} = \frac{n-2}{2} \int_M f \left( R(g) \left( \frac{n-2}{2} \right) f + (n-1) \Delta_g f \right) dv(g)
+ (n-2) \int_{\partial M} f \left( \left( \frac{n-2}{2} \right) H(g) f - \left( \frac{n-1}{2} \right) \partial_n f \right) d\sigma(g).
\]
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