NON-BRANCHNG TREE-DECOMPOSITIONS

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Abstract

We prove that if a graph has a tree-decomposition of width at most \( w \), then it has a tree-decomposition of width at most \( w \) with certain desirable properties. We will use this result in a subsequent paper to show that every 2-connected graph of large path-width has a minor isomorphic to either a large tree with a vertex attached to every vertex of the tree or a large outerplanar graph.

1 Introduction

All graphs in this paper are finite and simple; that is, they have no loops or parallel edges. Paths and cycles have no “repeated” vertices or edges. A graph \( H \) is a minor of a graph \( G \) if \( H \) can be obtained from a subgraph of \( G \) by contracting edges. An \( H \) minor is a minor isomorphic to \( H \). A tree-decomposition of a graph \( G \) is a pair \((T, X)\), where \( T \) is a tree and \( X \) is a family \( (X_t : t \in V(T)) \) such that:

(W1) \( \bigcup_{t \in V(T)} X_t = V(G) \), and for every edge of \( G \) with ends \( u \) and \( v \) there exists \( t \in V(T) \) such that \( u, v \in X_t \), and

(W2) if \( t_1, t_2, t_3 \in V(T) \) and \( t_2 \) lies on the path in \( T \) between \( t_1 \) and \( t_3 \), then \( X_{t_1} \cap X_{t_3} \subseteq X_{t_2} \).

The width of a tree-decomposition \((T, X)\) is \( \max\{|X_t| - 1 : t \in V(T)\} \). The tree-width of a graph \( G \) is the least width of a tree-decomposition of \( G \). A path-decomposition of \( G \) is a tree-decomposition \((T, X)\) of \( G \) where \( T \) is a path. The path-width of \( G \) is the least width of a path-decomposition of \( G \). Robertson and Seymour [8] proved the following:

Theorem 1.1. For every planar graph \( H \) there exists an integer \( n = n(H) \) such that every graph of tree-width at least \( n \) has an \( H \) minor.

Robertson and Seymour [7] also proved an analogous result for path-width:

Theorem 1.2. For every forest \( F \), there exists an integer \( p = p(F) \) such that every graph of path-width at least \( p \) has an \( F \) minor.

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Bienstock, Robertson, Seymour and the second author [2] gave a simpler proof of Theorem 1.2 and improved the value of \( p \) to \(|V(F)| - 1\), which is best possible, because \( K_k \) has path-width \( k - 1 \) and does not have any forest minor on \( k + 1 \) vertices. A yet simpler proof of Theorem 1.2 was found by Diestel [5].

Motivated by the possibility of extending Theorem 1.2 to matroids Seymour [4, Open Problem 2.1] asked if there was a generalization of Theorem 1.2 for 2-connected graphs with forests replaced by the two families of graphs mentioned in the abstract. In [3] we answer Seymour’s question in the affirmative:

**Theorem 1.3.** Let \( P \) be a graph with a vertex \( v \) such that \( P \setminus v \) is a forest, and let \( Q \) be an outerplanar graph. Then there exists a number \( p = p(P, Q) \) such that every 2-connected graph of path-width at least \( p \) has a \( P \) or \( Q \) minor.

Theorem 1.3 is a generalization of Theorem 1.2. To deduce Theorem 1.2 from Theorem 1.3, given a graph \( G \), we may assume that \( G \) is connected, because the path-width of a graph is equal to the maximum path-width of its components. We add one vertex and make it adjacent to every vertex of \( G \). Then the new graph is 2-connected, and by Theorem 1.3 it has a \( P \) or \( Q \) minor. By choosing suitable \( P \) and \( Q \), we can get an \( F \) minor in \( G \).

Our strategy to prove Theorem 1.3 is as follows. Let \( G \) be a 2-connected graph of large path-width. We may assume that the tree-width of \( G \) is bounded, for otherwise \( G \) has a minor isomorphic to both \( P \) and \( Q \) by Theorem 1.1. So let \((T, X)\) be a tree-decomposition of \( G \) of bounded width. Since the path-width of \( G \) is large, it follows by a simple argument [3, Lemma 4.1] that the path-width of \( T \) is large, and hence it has a subgraph \( T' \) isomorphic to a subdivision of a large binary tree by Theorem 1.2. It now seems plausible that we could use \( T' \) and properties (W3) and (W4) of tree-decompositions, introduced below, which we can assume by [6, 9], to show the desired conclusion. But there is a catch: for instance, a long cycle has a tree-decomposition \((T, X)\) satisfying (W3) and (W4) (and, in fact, the minimality condition used in their proof, as well as that of Bellenbaum and Diestel [1]) such that \( T \) has a subgraph isomorphic to a large binary tree. And yet it feels that this is the “wrong” tree-decomposition and that the “right” tree-decomposition is one where \( T \) is a path. The main result of this paper, Theorem 2.4 below, deals with converting these “branching” tree-decompositions into “non-branching” ones without increasing their width.

The paper is organized as follows. In the next section we review known results about tree-decompositions and state our main result, Theorem 2.4. In Section 3 we introduce a linear quasi-order on the class of finite trees and prove a key lemma—Lemma 3.5. In Section 4 we prove Theorem 2.4, which we restate as Theorem 4.8.

### 2 LINKED TREE-DECOMPOSITIONS

In this section we review properties of tree-decompositions established in [6, 9], and state our main result. The proof of the following easy lemma can be found, for instance, in [9].
Lemma 2.1. Let \((T, Y)\) be a tree-decomposition of a graph \(G\), and let \(H\) be a connected subgraph of \(G\) such that \(V(H) \cap Y_t \neq \emptyset \neq V(H) \cap Y_{t_2}\), where \(t_1, t_2 \in V(T)\). Then \(V(H) \cap Y_t \neq \emptyset\) for every \(t \in V(T)\) on the path between \(t_1\) and \(t_2\) in \(T\).

A tree-decomposition \((T, Y)\) of a graph \(G\) is said to be linked if

(W3) for every two vertices \(t_1, t_2\) of \(T\) and every positive integer \(k\), either there are \(k\) disjoint paths in \(G\) between \(Y_{t_1}\) and \(Y_{t_2}\), or there is a vertex \(t\) of \(T\) on the path between \(t_1\) and \(t_2\) such that \(|Y_t| < k\).

It is worth noting that, by Lemma 2.1, the two alternatives in (W3) are mutually exclusive. The following is proved in [9].

Lemma 2.2. If a graph \(G\) admits a tree-decomposition of width at most \(w\), where \(w\) is some integer, then \(G\) admits a linked tree-decomposition of width at most \(w\).

Let \((T, Y)\) be a tree-decomposition of a graph \(G\), let \(t_0 \in V(T)\), and let \(B\) be a component of \(T \setminus t_0\). We say that a vertex \(v \in Y_{t_0}\) is \(B\)-tied if \(v \in Y_t\) for some \(t \in V(B)\). We say that a path \(P\) in \(G\) is \(B\)-confined if \(|V(P)| \geq 3\) and every internal vertex of \(P\) belongs to \(\bigcup_{t \in V(B)} Y_t - Y_{t_0}\). We wish to consider the following three properties of \((T, Y)\):

(W4) if \(t, t'\) are distinct vertices of \(T\), then \(Y_t \neq Y_{t'}\),

(W5) if \(t_0 \in V(T)\) and \(B\) is a component of \(T \setminus t_0\), then \(\bigcup_{t \in V(B)} Y_t - Y_{t_0} \neq \emptyset\),

(W6) if \(t_0 \in V(T)\), \(B\) is a component of \(T \setminus t_0\), and \(u, v\) are \(B\)-tied vertices in \(Y_{t_0}\), then there is a \(B\)-confined path in \(G\) between \(u\) and \(v\).

The following strengthening of Lemma 2.2 is proved in [6].

Lemma 2.3. If a graph \(G\) has a tree-decomposition of width at most \(w\), where \(w\) is some integer, then it has a tree-decomposition of width at most \(w\) satisfying (W1)–(W6).

We need one more condition, which we now introduce. Let \(T\) be a tree. If \(t_1, t_2 \in V(T)\), then by \(t_1 T t_2\) we denote the vertex-set of the unique path in \(T\) with ends \(t_1\) and \(t_2\). A triad in \(T\) is a triple \(t_1, t_2, t_3\) of vertices of \(T\) such that there exists a vertex \(t\) of \(T\), called the center, such that \(t_1, t_2, t_3\) belong to different components of \(T \setminus t\). Let \((T, W)\) be a tree-decomposition of a graph \(G\), and let \(t_1, t_2, t_3\) be a triad in \(T\) with center \(t_0\). The torso of \((T, W)\) at \(t_1, t_2, t_3\) is the subgraph of \(G\) induced by the set \(\bigcup W_t\), the union taken over all vertices \(t \in V(T)\) such that either \(t \in \{t_1, t_2, t_3\}\), or for all \(i \in \{1, 2, 3\}\), the vertex \(t\) belongs to the component of \(T \setminus t_i\) containing \(t_0\). We say that the triad \(t_1, t_2, t_3\) is \(W\)-separable if, letting \(X = W_{t_1} \cap W_{t_2} \cap W_{t_3}\), the graph obtained from the torso of \((T, W)\) at \(t_1, t_2, t_3\) by deleting \(X\) can be partitioned into three disjoint non-null graphs \(H_1, H_2, H_3\) in such a way that for all distinct \(i, j \in \{1, 2, 3\}\) and all \(t \in t_j T t_0\), \(|V(H_i) \cap W_t| \geq |V(H_j) \cap W_t| = |W_t| - X|/2 \geq 1\). (Let us remark that this condition implies that \(|W_{t_1}| = |W_{t_2}| = |W_{t_3}|\) and \(V(H_i) \cap W_t = \emptyset\) for \(i = 1, 2, 3\).) The last property of a tree-decomposition \((T, W)\) that we wish to consider is
(W7) if \( t_1, t_2, t_3 \) is a \( W \)-separable triad in \( T \) with center \( t \), then there exists an integer \( i \in \{1, 2, 3\} \) with \( W_{t_i} \cap W_1 - (W_{t_i} \cap W_{t_2} \cap W_{t_3}) \neq \emptyset \).

The following is our main result.

**Theorem 2.4.** If a graph \( G \) has a tree-decomposition of width at most \( w \), where \( w \) is some integer, then it has a tree-decomposition of width at most \( w \) satisfying (W1)–(W7).

## 3 A QUASI-ORDER ON TREES

A quasi-ordered set is a pair \((Q, \leq)\), where \( Q \) is a set and \( \leq \) is a quasi-order, that is, a reflexive and transitive relation on \( Q \). If \( q, q' \in Q \) we define \( q < q' \) to mean that \( q \leq q' \) and \( q' \not\leq q \). We say that \( q, q' \) are \( \leq \)-equivalent if \( q \leq q' \leq q \). We say that \((Q, \leq)\) is a linear quasi-order if for every two elements \( q, q' \in Q \) either \( q \leq q' \) or \( q' \leq q \) or both. Let \((Q, \leq)\) be a linear quasi-order. If \( A, B \subseteq Q \) we say that \( B \leq \)-dominates \( A \) if the elements of \( A \) can be listed as \( a_1 \geq a_2 \geq \cdots \geq a_k \) and the elements of \( B \) can be listed as \( b_1 \geq b_2 \geq \cdots \geq b_l \), and there exists an integer \( p \) with \( 1 \leq p \leq \min\{k, l\} \) such that \( a_i \leq b_i \leq a_i \) for all \( i = 1, 2, \ldots, p \), and either \( p < \min\{k, l\} \) and \( a_{p+1} < b_{p+1}, \) or \( p = k \) and \( k \leq l \).

**Lemma 3.1.** If \((Q, \leq)\) is a linear quasi-order, then \( \leq \)-domination is a linear quasi-order on the set of subsets of \( Q \).

**Proof.** It is obvious that \( \leq \)-domination is reflexive. Assume that \( B \leq \)-dominates \( A \) and \( C \leq \)-dominates \( B \). Assume that the elements of \( A \) can be listed as \( a_1 \geq a_2 \geq \cdots \geq a_k \), the elements of \( B \) can be listed as \( b_1 \geq b_2 \geq \cdots \geq b_i \), and the elements of \( C \) can be listed as \( c_1 \geq c_2 \geq \cdots \geq c_m \). By definition, there exists an integer \( p_1 \) with \( 1 \leq p_1 \leq \min\{k, l\} \) such that \( a_i \leq b_i \leq a_i \) for all \( i = 1, 2, \ldots, p_1 \), and either \( p_1 < \min\{k, l\} \) and \( a_{p_1+1} < b_{p_1+1}, \) or \( p_1 = k \leq l \); and there exists an integer \( p_2 \) with \( 1 \leq p_2 \leq \min\{l, m\} \) such that \( b_i \leq c_i \leq b_i \) for all \( i = 1, 2, \ldots, p_2 \), and either \( p_2 < \min\{l, m\} \) and \( b_{p_2+1} < c_{p_2+1}, \) or \( p_2 = l \leq m \). Let \( p = \min\{p_1, p_2\} \). Then \( a_i \leq c_i \leq a_i \) for all \( i = 1, 2, \ldots, p \). If either \( p_1 < \min\{k, l\} \) and \( a_{p_1+1} < b_{p_1+1}, \) or \( p_2 < \min\{l, m\} \) and \( b_{p_2+1} < c_{p_2+1}, \) then \( p < \min\{k, m\} \) and \( a_{p+1} < c_{p+1}. \) If \( p_1 = k \leq l \) and \( p_2 = l \leq m \), then \( p = k \leq m \). Therefore, \( C \leq \)-dominates \( A \), and so \( \leq \)-domination is transitive.

Now let \( A, B \) be as above, and let \( p \) be the maximum integer such that \( p \leq \min\{k, l\} \) and \( a_i \leq b_i \leq a_i \) for all \( i = 1, 2, \ldots, p \). Then if \( p < \min\{k, l\} \), then \( A \leq \)-dominates \( B \) if \( a_{p+1} > b_{p+1} \) and \( B \leq \)-dominates \( A \) if \( a_{p+1} < b_{p+1} \). If \( p = \min\{k, l\} \) then \( A \leq \)-dominates \( B \) if \( k \geq l \) and \( B \leq \)-dominates \( A \) if \( k \leq l \). Hence, \( \leq \)-domination is linear.

We say that \( B \) strictly \( \leq \)-dominates \( A \) if \( B \leq \)-dominates \( A \) in such a way that the numberings and integer \( p \) can be chosen in such a way that either \( p < \min\{k, l\} \), or \( p = k \) and \( k < l \).

**Lemma 3.2.** Let \((Q, \leq)\) be a linear quasi-order, let \( A, B \subseteq Q \), and let \( B \leq \)-dominate \( A \). Then \( B \) strictly \( \leq \)-dominates \( A \) if and only if \( A \) does not \( \leq \)-dominate \( B \).
Proof. Let $p$ be as in the definition of $B \leq$-dominates $A$. Then $p < \min\{k, l\}$ and $a_{p+1} < b_{p+1}$, or $p = k \leq l$. Assume $B$ strictly $\leq$-dominates $A$. If $p < \min\{k, l\}$ then $a_{p+1} < b_{p+1}$, so $A$ does not $\leq$-dominate $B$. If $p = k < l$ then $A$ also does not $\leq$-dominate $B$. Conversely, if $A$ does not $\leq$-dominate $B$, then $p < \min\{k, l\}$ or $k < l$, so $B$ strictly $\leq$-dominates $A$. 

Let $G$ be a graph and let $P$ be a subgraph of $G$. By a $P$-bridge of $G$ we mean a subgraph $J$ of $G$ such that either

- $J$ is isomorphic to the complete graph on two vertices with $V(J) \subseteq V(P)$ and $E(J) \cap E(P) = \emptyset$, or
- $J$ consists of a component of $G - V(P)$ together with all edges from that component to $P$.

We now define a linear quasi-order $\leq$ on the class of finite trees as follows. Let $n \geq 1$ be an integer, and suppose that $T \leq T'$ has been defined for all trees $T$ on fewer than $n$ vertices. Let $T$ be a tree on $n$ vertices, and let $T'$ be an arbitrary tree. We define $T \leq T'$ if either $|V(T)| < |V(T')|$, or $|V(T)| = |V(T')|$ and for every maximal path $P'$ of $T'$ there exists a maximal path $P$ of $T$ such that the set of $P'$-bridges of $T'$ $\leq$-dominates the set of $P$-bridges of $T$. It follows from Lemma 3.3 below that $\leq$ is indeed a linear quasi-order; in particular, it is well-defined.

If $T, T'$ are trees, $P$ is a path in $T$ and $P'$ is a path in $T'$ we define $(T, P) \leq (T', P')$ if either $|V(T)| < |V(T')|$, or $|V(T)| = |V(T')|$ and the set of $P'$-bridges of $T'$ $\leq$-dominates the set of $P$-bridges of $T$.

Lemma 3.3. (i) For every tree $T$ there exists a maximal path $P(T)$ in $T$ such that $(T, P(T)) \leq (T, P)$ for every maximal path $P$ in $T$.
(ii) For every two trees $T, T'$, we have $T \leq T'$ if and only if $(T, P(T)) \leq (T', P(T'))$.
(iii) The ordering $\leq$ is a linear quasi-order on the class of finite trees.

Proof. We prove all three statements simultaneously by induction. Let $n \geq 1$ be an integer, assume inductively that all three statements have been proven for trees on fewer than $n$ vertices, and let $T$ be a tree on $n$ vertices.

(i) Statement (i) clearly holds for one-vertex trees, and so we may assume that $n \geq 2$. Let $\mathcal{B}$ be the set of all $P$-bridges of $T$ for all maximal paths $P$ of $T$. Then every member of $\mathcal{B}$ has fewer than $n$ vertices, and hence $\mathcal{B}$ is a linear quasi-order by $\leq$ by the induction hypothesis applied to (iii). By Lemma 3.1 the set of subsets of $\mathcal{B}$ is linearly quasi-ordered by $\leq$-domination. It follows that there exists a maximal path $P(T)$ in $T$ such that the set of $P(T)$-bridges of $T$ is minimal under $\leq$-domination.

(ii) The statement is obvious when $|V(T)| \neq |V(T')|$, so assume $n = |V(T)| = |V(T')|$, and let $\mathcal{B}$ be the set of all $P$-bridges of $T$ for all maximal paths $P$ of $T$ and the set of all $P'$-bridges of $T'$ for all maximal paths $P'$ of $T'$. Then as in (i) the subsets of $\mathcal{B}$ are linearly quasi-ordered by $\leq$-domination. If $T \leq T'$, then by definition there exists a maximal path $P$ of $T$ such that $(T, P) \leq (T', P(T'))$. Hence $(T, P(T)) \leq (T', P(T'))$ follows from (i). If $(T, P(T)) \leq (T', P(T'))$, then by (i) $(T, P(T)) \leq (T', P')$ for every maximal path $P'$.
Lemma 3.4. This implies, by induction on $P$-

bridges of $T$. We have ($T$ is a maximal path that contains $T$). Let $t$ be a path in $T$, and let $t'$ be a new vertex so that $t'$ strictly $\leq$-dominates the set of $P$-
bridges of $T$, then $T < T'$.

Proof. We have $(T, P) \leq (T', P')$ and $(T', P') \not\leq (T, P)$ by Lemma 3.2. Let $P_1$ be a maximal path that contains $P$; then $(T, P_1) \leq (T, P)$. Therefore, $(T, P_1) \leq (T', P')$ and $(T', P') \not\leq (T, P_1)$. By Lemma 3.3(i), $(T, P(T)) \leq (T, P_1) \leq (T', P')$ and $(T', P') \not\leq (T, P(T))$. By Lemma 3.3(ii), $T \leq T'$ and $T' \not\leq T$. Therefore, $T < T'$.

By a rank we mean a class of $\leq$-equivalent trees. If $r$ is a rank we say that $T$ has rank $r$ or that the rank of $T$ is $r$ if $T \in r$. The class of all ranks will be denoted by $\mathcal{R}$.

Let $T$ be a tree, and let $t$ be a vertex of $T$. By a spine-decomposition of $T$ relative to $t$ we mean a sequence $(T_0, T_1, P_1, \ldots, T_l, P_l)$ such that

(i) $T_0 = T$,

(ii) for $i = 0, 1, \ldots, l$, $P_i$ is a spine of $T_i$, and

(iii) for $i = 1, 2, \ldots, l$, $t \notin V(P_{i-1})$ and $T_i$ is the $P_{i-1}$-

bridge of $T_{i-1}$ containing $t$.

Lemma 3.5. Let $T$ be a tree, let $t$ be a vertex of $T$ of degree three with neighbors $t'_1, t'_2, t'_3$, and let $(T_0, T_1, P_1, \ldots, T_l, P_l)$ be a spine-

decomposition of $T$ relative to $t$ with $t \in V(P_l)$. Then exactly two of $t'_1, t'_2, t'_3$ belong to $V(P_l)$, say $t'_1$ and $t'_2$. Let $r_3, r'_3$ be adjacent vertices of $T$ such that $r_3, r'_3, t'_3, t$ occur on a path of $T$ in the order listed. Thus possibly $t'_3 = r'_3$, but $t'_3 \neq r_3$. Let $T'$ be obtained from $T$ by subdividing the edge $r_3r'_3$ twice (let $r''_3, r'''_3$ be the new vertices so that $r'_3, r''_3, r'''_3, r_3$ occur on a path of $T'$ in the order listed), deleting the edge $tt'_1$, contracting the edges $tt'_2$ and $tt'_3$ and adding an edge joining $t'_1$ and $r'''_3$. Then $T'$ has strictly smaller rank than $T$.

Proof. Let $T'_0 = T'$ and for $i = 1, 2, \ldots, l$, let $T'_i$ be the $P_{i-1}$-

bridge of $T'_{i-1}$ containing $r''_3$. Let $P'$ be the unique maximal path in $T'$ with $V(P') = \{t, t'_2 \cup r'_3 \subseteq V(P')$. From the definition of a spine-

decomposition and the fact that $t'_3 \notin V(P_l)$ we deduce that $r_3 \in V(T_i)$ for all $i = 0, 1, \ldots, l$. It follows that $r_3 \in V(T'_i)$ and $|V(T_i)| = |V(T'_i)|$ for all $i = 0, 1, \ldots, l$. The $P_l$-

bridge of $T_i$ that contains $r_3$ is replaced by $P'$-

bridges of $T'_i$ with smaller cardinalities. Other $P_l$-

bridges of $T_i$ are unchanged in $T'$. Therefore, the set of $P_l$-

bridges of $T_i$ strictly $\leq$-dominates the set of $P'$-

bridges of $T'_i$, and hence $T'_i < T_i$ by Lemma 3.3. This implies, by induction on $l - i$ using Lemma 3.4 that $T'_i < T_i$ for all $i = 0, 1, \ldots, l$; that is, $T'$ has smaller rank than $T$. □
4 A THEOREM ABOUT TREE-DECOMPOSITIONS

Let \((T, Y)\) be a tree-decomposition of a graph \(G\), let \(n\) be an integer, and let \(r\) be a rank. By an \((n, r)\)-cell in \((T, Y)\) we mean any component of the restriction of \(T\) to \(\{v \in V(T) : |Y_v| \geq n\}\) that has rank at least \(r\). Let us remark that if \(K\) is an \((n, r)\)-cell in \((T, Y)\) and \(r \geq r'\), then \(K\) is an \((n, r')\)-cell as well. The size of a tree-decomposition \((T, Y)\) is the family of numbers

\[
(a_{n, r} : n \geq 0, r \in \mathcal{R}),
\]

where \(a_{n, r}\) is the number of \((n, r)\)-cells in \((T, Y)\). Sizes are ordered lexicographically; that is, if

\[
(b_{n, r} : n \geq 0, r \in \mathcal{R})
\]

is the size of another tree-decomposition \((R, Z)\) of the graph \(G\), we say that (2) is smaller than (1) if there are an integer \(n \geq 0\) and a rank \(r \in \mathcal{R}\) such that \(a_{n, r} > b_{n, r}\) and \(a_{n', r'} = b_{n', r'}\) whenever either \(n' > n\), or \(n' = n\) and \(r' > r\).

**Lemma 4.1.** The relation “to be smaller than” is a well–ordering on the set of sizes of tree–decompositions of \(G\).

**Proof.** Since this ordering is clearly linear, it is enough to show that it is well–founded. Suppose for a contradiction that \(\{(a_{n, r}^{(i)} : n \geq 0, r \in \mathcal{R})\}_{i=1}^{\infty}\) is a strictly decreasing sequence of sizes, and for \(i = 1, 2, \ldots\), let \(n_i, r_i\) be such that \(a_{n_i, r_i}^{(i)} > a_{n_i, r_i}^{(i+1)}\) and \(a_{n_i, r_i}^{(i)} = a_{n_i, r_i}^{(i+1)}\) for \((n, r)\) such that either \(n > n_i\), or \(n = n_i\) and \(r > r_i\). Since \(a_{n, r}^{(1)} = 0\) for all \(r \in \mathcal{R}\) and all \(n \geq |V(G)|\), we may assume (by taking a suitable subsequence) that \(n_1 = n_2 = \cdots\), and that \(r_1 \leq r_2 \leq r_3 \leq \cdots\). Since clearly \(a_{n, r}^{(i)} \geq a_{n, r}^{(i)}\) for all \(n \geq 0\), all \(r \leq r'\) and all \(i = 1, 2, \ldots\), we have

\[
a_{n_1, r_1}^{(1)} > a_{n_2, r_1}^{(2)} \geq a_{n_2, r_2}^{(2)} > a_{n_3, r_2}^{(3)} \geq a_{n_3, r_3}^{(3)} > \cdots,
\]

a contradiction. \(\square\)

We say that a tree-decomposition \((T, W)\) of a graph \(G\) is minimal if there is no tree-decomposition of \(G\) of smaller size.

**Lemma 4.2.** Let \(w\) be an integer, and let \(G\) be a graph of tree-width at most \(w\). Then a minimal tree-decomposition of \(G\) exists, and every minimal tree-decomposition of \(G\) has width at most \(w\).

**Proof.** The existence of a minimal tree-decomposition follows from Lemma 4.1. If \(G\) has a tree-decomposition of width at most \(w\), then every minimal tree-decomposition has width at most \(w\), as desired. \(\square\)

**Theorem 4.3.** Let \((T, W)\) be a minimal tree-decomposition of a graph \(G\). Then \((T, W)\) satisfies (W1)–(W6).

**Proof.** That \((T, W)\) satisfies (W3) is shown in [9], and that it satisfies (W4), (W5) and (W6) is shown in [10]. Let us remark that [6] and [9] use a slightly different definition of minimality, but the proofs are adequate, because a minimal tree-decomposition in our sense is minimal in the sense of [6] and [9] as well. \(\square\)
Lemma 4.4. Let \((T, W)\) be a minimal tree-decomposition of a graph \(G\). Then for every edge \(tt' \in E(T)\) either \(W_t \subseteq W_{t'}\) or \(W_{t'} \subseteq W_t\).

Proof. Assume for a contradiction that there exists an edge \(tt' \in E(T)\) such that \(W_t \not\subseteq W_{t'}\) and \(W_{t'} \not\subseteq W_t\). Let \(R\) be obtained from \(T\) by subdividing the edge \(tt'\) and let \(t''\) be the new vertex. Let \(Y_t = W_t \cap W_{t'}\) and \(Y_{t'} = W_{t'}\) for all \(r \in V(T)\), and let \(Y = (Y_r : r \in V(R))\). Then \((R, Y)\) is a tree-decomposition of \(G\) of smaller size than \((T, W)\), contrary to the minimality of \((T, W)\). \(\square\)

Lemma 4.5. Let \((T, W)\) be a minimal tree-decomposition of a graph \(G\), let \(t \in V(T)\), let \(X \subseteq W_t\), let \(B\) be a component of \(T \setminus t\), let \(t'\) be the neighbor of \(t\) in \(B\), let \(Y = \bigcup_{r \in V(B)} W_r\), and let \(H\) be the subgraph of \(G\) induced by \(Y \cup W_t\). If \(H \setminus X = H_1 \cup H_2\), where \(V(H_1) \cap V(H_2) = \emptyset\) and both of \(V(H_1), V(H_2)\) intersect \(W_t\), then either \(W_t - X \subseteq W_t \cap V(H_1)\) or \(W_{t'} - X \subseteq W_t \cap V(H_2)\).

Proof. We first prove the following claim.

Claim 4.5.1. Either \(W_t \cap W_{t'} - X \subseteq V(H_1)\) or \(W_t \cap W_{t'} - X \subseteq V(H_2)\).

To prove the claim suppose for a contradiction that there exist vertices \(v_1 \in W_t \cap W_{t'} \cap V(H_1)\) and \(v_2 \in W_t \cap W_{t'} \cap V(H_2)\). Thus both \(v_1\) and \(v_2\) are \(B\)-tied, and so by (W6), which \((T, W)\) satisfies by Theorem 4.3, there exists a \(B\)-confined path \(Q\) with ends \(v_1\) and \(v_2\). Since \(Q\) is \(B\)-confined, it is a subgraph of \(H \setminus X\), contrary to the fact that \(V(H_1) \cap V(H_2) = \emptyset\) and \(H_1 \cup H_2 = H \setminus X\). This proves Claim 4.5.1.

Since both of \(V(H_1), V(H_2)\) intersect \(W_t\), Claim 4.5.1 implies that \(W_t \not\subset W_{t'}\), and hence \(W_{t'} \subseteq W_t\) by Lemma 4.4. By another application of Claim 4.5.1 we deduce that either \(W_{t'} - X \subseteq W_t \cap V(H_1)\) or \(W_{t'} - X \subseteq W_t \cap V(H_2)\), as desired. \(\square\)

Lemma 4.6. Let \(k \geq 1\) be an integer, let \((T, W)\) be a minimal tree-decomposition of a graph \(G\), let \(t_1, t_2 \in V(T)\), let \(X = W_{t_1} \cap W_{t_2}\), let \(H\) be the subgraph of \(G\) induced by \(\bigcup W_i\), the union taken over all vertices \(t \in V(T)\) such that \(t \in \{t_1, t_2\}\), or for \(i = 1, 2\) the vertex \(t\) belongs to the component of \(T \setminus t_i\) containing \(t_{3-i}\), let \(H \setminus X = H_1 \cup H_2\), where \(V(H_1) \cap V(H_2) = \emptyset\), and assume that \(|W_{t_i} \cap V(H_j)| = k\) and \(|W_{t_i} \cap V(H_i)| \geq k\) for all \(i, j \in \{1, 2\}\) and all \(t \in t_1 T t_2\). Then there exists an integer \(i \in \{1, 2\}\) such that \(W_{t_i} \cap V(H_i) = W_{t'} \cap V(H_i)\) and this set has cardinality \(k\).

Proof. We begin with the following claim.

Claim 4.6.1. For every \(t \in t_1 T t_2\) either \(|W_t \cap V(H_1)| = k\) or \(|W_t \cap V(H_2)| = k\).

To prove the claim let \(R\) be the subtree of \(T\) induced by vertices \(r \in V(T)\) such that either \(r \in \{t_1, t_2\}\) or \(r\) belongs to the component of \(T \setminus \{t_1, t_2\}\) that contains neighbors of both \(t_1\) and \(t_2\), let \(R_1, R_2\) be two isomorphic copies of \(R\), and for \(r \in V(R)\) let \(r_1\) and \(r_2\) denote the copies of \(r\) in \(R_1\) and \(R_2\), respectively. Assume for a contradiction that there is \(t_0 \in t_1 T t_2\) such that \(|W_{t_0} \cap V(H_i)| > k\) for all \(i \in \{1, 2\}\), and choose such a vertex with \(t_0 \in V(R)\) and \(|W_{t_0}|\) maximum. We construct a new tree-decomposition \((T', W')\) as
follows. The tree $T'$ is obtained from the disjoint union of $T \setminus (V(R) - \{t_1, t_2\})$, $R_1$ and $R_2$ by identifying $t_1$ with $(t_1)_1$, $(t_2)_1$ with $(t_1)_2$ and $(t_2)_2$ with $t_2$ (here $(t_1)_2$ denotes the copy of $t_1$ in $R_2$ and similarly for the other three quantities). The family $W' = (W'_t : t \in V(T'))$ is defined as follows:

$$W'_t = \begin{cases} W_t & \text{if } t \in V(T) - V(R) \\ (W_r \cap V(H_1)) \cup (W_{t_1} \cap V(H_2) \cup X) & \text{if } t = r_1 \text{ for } r \in t_1 T t_2 \\ (W_r \cap V(H_2)) \cup (W_{t_2} \cap V(H_1) \cup X) & \text{if } t = r_2 \text{ for } r \in t_1 T t_2 \\ W_r \cap V(H_1) & \text{if } t = r_1 \text{ for } r \in V(R) - t_1 T t_2 \\ W_r \cap V(H_2) & \text{if } t = r_2 \text{ for } r \in V(R) - t_1 T t_2 \end{cases}$$

Please note that the value of $W'_t$ is the same for $t = (t_2)_1$ and $t = (t_1)_2$, and hence $W'$ is well-defined. Since no edge of $G$ has one end in $V(H_1)$ and the other end in $V(H_2)$, it follows that $(T', W')$ is a tree-decomposition of $G$.

We claim that the size of $(T', W')$ is smaller than the size of $(T, W)$. Indeed, let $n_0 = |W_{t_0}|$, and let $Z = \{ t \in V(T') : |W'_t| \geq n_0 \}$. Then $n_0 > 2k + |X|$. We define a mapping $f : Z \to V(T)$ by $f(t) = t$ for $t \in Z - V(R_1) - V(R_2)$, $f(r_1) = r$ for $r \in V(R)$ such that $r_1 \in Z$ and $f(r_2) = r$ for $r \in V(R)$ such that $r_2 \in Z$. We remark that the vertex obtained by identifying $(t_2)_1$ with $(t_1)_2$ does not belong to $Z$, and hence there is no ambiguity. Then $Z$ and $f$ have the following properties:

- $|W_{f(t)}| \geq |W'_t|$ for every $t \in Z$,
- for $r \in V(R)$, at most one of $r_1, r_2$ belongs to $Z$, and
- $(t_0)_1, (t_0)_2 \notin Z$

These properties follow from the assumptions that $|W_{t_i} \cap V(H_j)| = k$ and $|W_{t_i} \cap V(H_j)| \geq k$ for all $i, j \in \{1, 2\}$ and all $t \in t_1 T t_2$. (To see the second property assume for a contradiction that for some $r \in V(R)$ both $r_1$ and $r_2$ belong to $Z$. Then $n_0 = |W_{t_0}| \geq |W_{f(r_1)}| \geq |W'_t| \geq n_0$, by the maximality of $|W_{t_0}|$ and the first property, and so equality holds throughout, contrary to the construction.) It follows from the first two properties that $f$ maps injectively $(n, r)$-cells in $(T', W')$ to $(n, r)$-cells in $(T, W)$ for all $n \geq n_0$ and all ranks $r$. On the other hand, the third property implies that, letting $r_1$ denote the rank of one-vertex trees, no $(n_0, r_1)$-cell in $(T', W')$ is mapped onto the $(n_0, r_1)$-cell in $(T, W)$ with vertex-set $\{t_0\}$. Thus the size of $(T', W')$ is smaller than the size of $(T, W)$, contrary to the minimality of $(T, W)$. This proves Claim 4.6.1.

Now let $t, t' \in t_1 T t_2$ be adjacent. By Lemma 4.4, we may assume that $W_t \subseteq W_{t'}$. Then $W_t \cap V(H_1) \subseteq W_{t'} \cap V(H_1)$ and $W_t \cap V(H_2) \subseteq W_{t'} \cap V(H_2)$. By Claim 4.6.1, we may assume that $|W_{t'} \cap V(H_1)| = k$. Given that $|W_t \cap V(H_1)| \geq k$ we have $W_t \cap V(H_1) = W_{t'} \cap V(H_1)$ and this set has cardinality $k$, as desired. \qed

**Lemma 4.7.** Let $(T', W')$ be a minimal tree-decomposition of a graph $G$, let $t_1, t_2, t_3$ be a $W$-separable triad in $T$ with center $t_0$, and let $X, H, H_1, H_2$ and $H_3$ be as in the definition of $W$-separable triad. Let $k = |W_{t_1} - X|/2$ and for $i = 1, 2, 3$ let $t'_i$ denote the neighbor of $t_0$.
in the component of $T \setminus t_0$ containing $t_i$. Then for all distinct $i, j \in \{1, 2, 3\}$, $V(H_i) \cap W_{t_j'} = V(H_i) \cap W_{t_0}$, and this set has cardinality $k$.

Proof. Let $X_3 = \bigcup W_t$, the union taken over all $t \in V(T)$ that do not belong to the component of $T \setminus t_3$ containing $t_0$. Since $|W_{t_0} \cap V(H_1)| \geq k$ and $|W_{t_0} \cap V(H_2)| \geq k$ by the definition of $W$-separable triad, by Lemma 4.6 applied to $t_1, t_2, H_3$ and the subgraph of $G$ induced by $V(H_1) \cup V(H_2) \cup X_3$ we deduce that $V(H_3) \cap W_{t_0} = V(H_3) \cap W_{t_1'} = V(H_3) \cap W_{t_2'}$, and this set has cardinality $k$. Similarly we deduce that $V(H_2) \cap W_{t_0} = V(H_2) \cap W_{t_1'} = V(H_2) \cap W_{t_2'}$ and $V(H_1) \cap W_{t_0} = V(H_1) \cap W_{t_1'} = V(H_1) \cap W_{t_2'}$, and that the latter two sets also have cardinality $k$.

We are finally ready to prove Theorem 2.3, which, by Lemma 4.2 is implied by the following theorem.

Theorem 4.8. Let $(T, W)$ be a minimal tree-decomposition of a graph $G$. Then $(T, W)$ satisfies (W1)–(W7).

Proof. That $(T, W)$ satisfies (W1)–(W6) follows from Theorem 4.3. Thus it remains to show that $(T, W)$ satisfies (W7). Suppose for a contradiction that $(T, W)$ does not satisfy (W7), and let $t_1, t_2, t_3$ be a $W$-separable triad in $T$ with center $t_0$ such that $W_{t_0} \cap V(P_i) \subseteq X$ for every $i = 1, 2, 3$, where $X = W_{t_1} \cap W_{t_2} \cap W_{t_3}$. Let $H, H_1, H_2$ and $H_3$ be as in the definition of $W$-separable triad, and for $i \in \{1, 2, 3\}$ let $t_i'$ denote the neighbor of $t_0$ in the component of $T \setminus t_0$ containing $t_i$.

Let $n := |W_{t_1}|$, let $k := |W_{t_1} - X|/2$, let $r_1$ denote the rank of 1-vertex trees, and let $T_0$ denote the $(n, r_1)$-cell containing $t_0$. By the definition of $W$-separable triad we have $|W_{t_i'}| \geq n$ for all $i \in \{1, 2, 3\}$, and hence the degree of $t_0$ in $T_0$ is at least three and by Lemmas 4.7 and 4.8 it is at most three.

Let $(T_0, P_0, T_1, P_1, \ldots, T_i, P_i)$ be a spine-decomposition of $T_0$ relative to $t_0$ with $t_0 \in V(P_i)$. Since $P_i$ is a maximal path in $T_i$ we may assume that $t_i', t_0' \in V(P_i)$ and $t_0' \notin V(P_i)$.

It follows from Lemma 4.7 that $W_{t_3} \cap W_{t_3'} = X$. By Lemma 4.6 applied to $t_3$ and $t_3'$ and its neighbor in $t_3T_{t_3'}$ we deduce that there exists a vertex $r_3 \in t_3T_{t_3'} - \{t_3'\}$ such that either $V(H_1) \cap W_{t_3'} = V(H_1) \cap W_r$ for every $r \in t_3T_{t_3'}$, or $V(H_2) \cap W_{t_3'} = V(H_2) \cap W_r$ for every $r \in t_3T_{t_3'}$. Without loss of generality we may assume the latter. We may choose $r_3$ to be as close to $t_3$ as possible. The fact that $W_{t_3} \cap W_{t_3'} = X$ implies that $r_3 \neq t_3$. By another application of Lemma 4.6 this time to $t_3, t_3', r_3$ and the neighbor of $r_3$ in $r_3T_{t_3'}$, we deduce that $|V(H_1) \cap W_{r_3}| = |V(H_2) \cap W_{r_3}| = k$.

Let $r_3'$ be the neighbor of $r_3$ in $r_3T_{t_0}$ and let the tree $T''$ be defined as follows: for every component $B$ of $T \setminus t_0T_{t_3'}$ not containing $t_1, t_2$ or $t_3$ let $r(B)r'(B)$ denote the edge connecting $B$ to $t_0T_{t_3'}$, where $r(B) \in V(B)$ and $r'(B) \in t_0T_{t_3'}$. By Lemma 4.5 there exists an integer $i \in \{1, 2, 3\}$ such that $W_{r(B)} \subseteq W_{r'(B)} \cap V(H_i)$. Let us mention in passing that this, the choice of $r_2$ and Lemma 4.7 imply that for every such component $B$, every $(n, r_1)$-cell is either a subgraph of $B$ or is disjoint from $B$. The tree $T''$ is obtained from $T$ by, for every such component $B$ for which either $i = 2$, or $i = 3$ and $r'(B) = t_0$, deleting the edge $r(B)r'(B)$ and adding the edge $t_1'r(B)$; and for every such component $B$ for which $i = 1$ and $r'(B) = t_0$ deleting the edge $r(B)r'(B)$ and adding the
edge \( t'_2 r (B) \). Since \( W_{v'(B)} \cap (V(H_2) \cup V(H_3)) \subseteq W_{t'_2} \) by the choice of \( r_3 \) and Lemma 4.7 and \( W_{v'(B)} \cap V(H_1) \subseteq W_{t'_2} \) by Lemma 4.7 it follows that \((T'', W)\) is a tree-decomposition of \(G\).

Let \( T' \) be defined as in Lemma 3.5 starting from the tree \( T'' \), let \( t'_0 \) be the vertex that resulted from contracting the edges \( t_0 t'_2 \) and \( t_0 t'_3 \), and let \( W' = (W'_t \mid t \in V(T')) \) be defined by

\[
W'_t = \begin{cases} 
W_t & \text{if } t \in V(T') - r''' t'_0 \\
W_{r_3} \cup (V(H_3) \cap W_{t_0}) & \text{if } t = r'' \\
(W_{r_3} - V(H_2)) \cup (V(H_3) \cap W_{t_0}) & \text{if } t = r'' \\
W_{t'_2} & \text{if } t = t'_0 \\
(W_t - V(H_2)) \cup (V(H_3) \cap W_{t_0}) & \text{if } t \in r''' t'_0 - \{ t'_0 \}
\end{cases}
\]

We claim that \((T', W')\) is a tree decomposition of \(G\). Indeed, since \( V(H_2) \cap W_r \subseteq W_{t_0} \) for all \( r \in r''' t'_0 \) it follows that \((T', W')\) satisfies (W1).

To show that \((T', W')\) satisfies (W2) let \( v \in V(G) \), let \( Z = \{ t \in V(T) : v \in W_t \} \), and let \( Z' = \{ t \in V(T') : v \in W'_t \} \). It suffices to show that \( Z' \) induces a connected subset of \( T' \), for this is easily seen to be equivalent to (W2). To that end assume first that \( v \not\in W_{t'_1} = W_{t'_2} = W_{t_0} \cap (V(H_2) \cup V(H_3)) \). It follows that, since \( Z \) induces a subtree of \( T \), that \( Z' \) induces a subtree of \( T' \). We assume next that \( v \in W_{t_0} \cap V(H_2) \). The choice of \( T'' \) and the definition of \( W' \) imply that no vertex in the component of \( T'' - r''' \) containing \( t'_0 \) belongs to \( Z' \). Again, it follows that \( Z' \) induces a subtree of \( T' \). Finally, let \( v \in W_{t_0} \cap V(H_3) \). Then \( t'_1 T' t'_0 \subseteq Z' \), and it again follows that \( Z' \) induces a subtree of \( T' \). This proves our claim that \((T', W')\) is a tree-decomposition.

We claim that the size of \((T', W')\) is smaller than the size of \((T, W)\). Let \( r \) denote the rank of \( T_0 \), and let \( T'_0 \) denote the \((n, r_1)\)-cell in \((T', W')\) containing \( t'_0 \). First, by the passing remark made a few paragraphs ago, for every integer \( m \geq n \) and every rank \( s \), to every \((m, s)\)-cell in \((T', W')\) other than \( T'_0 \) there corresponds a unique \((m, s)\)-cell in \((T, W)\). (To the \((n + 1, r_1)\)-cell in \((T', W')\) with vertex-set \( \{ r''' \} \) there corresponds the \((n + 1, r_1)\)-cell in \((T, W)\) with vertex-set \( \{ t_0 \} \).) Second, by Lemma 3.5 the rank of \( T_0 \) is strictly larger than the rank of \( T'_0 \). Thus no \((n, r)\)-cell in \((T', W')\) corresponds to \( T_0 \). It follows that \((T', W')\) is a tree-decomposition of \(G\) of smaller size, contrary to the minimality of \((T, W)\). \( \square \)

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