Tachyon Back Reaction

on $d = 2$ String Black Hole

S. Kalyana Rama

School of Mathematics, Trinity College, Dublin 2, Ireland.
email: kalyan@maths.tcd.ie

ABSTRACT. We describe a static solution for $d = 2$ critical string theory including the tachyon $T$ but with its potential $V(T)$ set to zero. This solution thus incorporates tachyon back reaction and, when $T = 0$, reduces to the black hole solution. When $T \neq 0$ we find that (1) the Schwarzschild horizon of the above black hole splits into two, resembling Reissner-Nordstrom horizons and (2) the curvature scalar develops new singularities at the horizons. We show that these features will persist even when $V(T)$ is nonzero. We present a time dependent extension of our static solution and discuss some possible methods for removing the above singularities.
1. Introduction

It is a challenging task to resolve the puzzles of gravitating systems such as the nature of singularities, the end effect of Hawking radiation, and information loss(?) inside the black hole. Recently there has been a renewed effort [1, 2] to solve these problems in the simpler context of two dimensional ($d = 2$) systems, using the string inspired toy models for quantum gravity, the inspiration having come from the discovery of black holes in $d = 2$ critical strings [3, 4, 5].

The $d = 2$ black hole for graviton-dilaton system was discovered in an SL(2, R)/U(1) gauged Wess-Zumino-Witten model [3]; as a solution of $\mathcal{O}(\alpha')$ $\beta$-function equations [4, 7] for critical string theory with $d = 2$ target space-time [3]; and in other forms [3]. Similar toy models in $d = 2$ space-time for quantum gravity including matter were studied first by [1] and then by many others [2] with a view of solving various puzzles of gravitating systems in this simpler context. Most of these models have mainly studied the graviton-dilaton system.

For a more complete story, one should also include tachyon, the only remaining low energy degree of freedom for $d = 2$ strings (and which is a massless excitation for $d = 2$). Though important, its inclusion results in non linear equations which are solved only in a few asymptotic cases [8, 2]. The solution of $\beta$-function equations for $d = 2$ strings including tachyons is not known. However, solving for tachyon in the $d = 2$ string black hole background leads to singular behaviour [3, 4]. Thus it is important to understand the back reaction of tachyons for $d = 2$ critical strings, especially because of its importance as a model for $d = 2$ quantum gravity.

In this work we describe a static solution, recently announced in [9], of the $\beta$-function equations for the low energy $d = 2$ critical string theory including tachyon $T$ but not its potential $V(T)$. This solution incorporates tachyon back reaction. The solution has, beside the black hole mass parameter, a new parameter $\epsilon$ which is a measure of tachyon strength. We find that: (1) The Schwarzschild horizon of the previous black hole splits into two, resembling Reissner-Nordstrom horizons. However, the solution cannot be analytically continued in between the two “horizons”. (2) The curvature scalar $R$ has a point singularity, as for $\epsilon = 0$, but when $\epsilon > 0$ develops new singularities at the horizons. Hence this solution is not a black hole solution in the usual
sense, though for $\epsilon = 0$ it is. Though we needed to set $V = 0$ to obtain an explicit solution, we show later that the features of this solution will persist even when $V$ is correctly taken into account. Thus this shows that when tachyon back reaction is included, the $d = 2$ string does not admit any static solution which can be interpreted as a black hole in the usual sense. We also present a time dependent extension of our solution which, however, does not alter the features of the static solution, and then discuss critically some possible methods of removing the singularities.

In section 2 we present $\beta$-function equations in various forms and obtain a static solution of these equations in section 3 assuming that the tachyon potential vanishes. We discuss the features of this solution in section 4 and show, in section 5, that these features will persist even when the tachyon potential is nonzero. In section 6 we present a time dependent extension of our static solution and then conclude with a critical discussion of possible methods of removing the singularities, at least the new ones.

2. $\beta$-function Equations

In this section we give $\beta$-function equations in various forms for the low energy $d = 2$ critical string theory including graviton ($G_{\mu\nu}$), dilaton ($\phi$), and tachyon ($T$) fields. We consider mostly the “static” case. The sigma model action of the $d = 2$ critical string theory for $G_{\mu\nu}$, $\phi$, and $T$, in a notation similar to that of [4], is given by

$$S_{\text{sigma}} = \frac{1}{8\pi\alpha'} \int d^2 \bar{x} \sqrt{g}(G_{\mu\nu} \nabla x^\mu \nabla x^\nu + \alpha' R\phi + 2T)$$

where $\bar{x}$ are the world sheet coordinates and $x^\mu$, $\mu = 0, 1$ are the target space coordinates. The conformal invariance of the above action requires the following $\beta$-function equations to be satisfied:

$$R_{\mu\nu} + \nabla_\mu \nabla_\nu \phi + \nabla_\mu T \nabla_\nu T = 0$$
$$R + (\nabla \phi)^2 + 2\nabla^2 \phi + (\nabla T)^2 + 4\gamma K = 0$$
$$\nabla^2 T + \nabla \phi \nabla T - 2\gamma K_T = 0$$

(1)

where $\gamma = \frac{2}{\alpha'}$, $K = 1 + \frac{V}{4\gamma}$, $V = \gamma T^2 + \mathcal{O}(T^3)$ is the tachyon potential and
$K_T \equiv \frac{dK}{dT}$. These equations also follow from the target space effective action

$$S = \int d^2x \sqrt{G} e^\phi (R - (\nabla \phi)^2 + (\nabla T)^2 + 4\gamma K) . \quad (2)$$

Note from equation (2) that the dilaton field $e^{-\frac{\phi}{2}}$ acts as a coupling constant.

Consider now the target space conformal gauge $ds^2 = e^\sigma du dv$ where $u = x^0 + x^1$ and $v = x^0 - x^1$. In this gauge the only nonzero connection ($\Gamma^\kappa_{\mu \nu}$) and curvature ($R_{\mu \nu}$) components are

$$\Gamma^u_{uu} = \partial_u \sigma , \quad \Gamma^v_{vv} = \partial_v \sigma ; \quad R_{uv} = \partial_u \partial_v \sigma \quad (3)$$

and equations (4) become

$$\begin{align*}
\partial^2 \phi - \partial_u \sigma \partial_u \phi + (\partial_u T)^2 &= 0 \\
\partial^2 \phi - \partial_v \sigma \partial_v \phi + (\partial_v T)^2 &= 0 \\
\partial_u \partial_v \sigma + \partial_u \partial_\nu \phi + \partial_u T \partial_v T &= 0 \\
\partial_u \partial_v \phi + \partial_u \phi \partial_v \phi + \gamma Ke^\sigma &= 0 \\
2\partial_u \partial_v T + \partial_u \phi \partial_v T + \partial_v \phi \partial_u T - \gamma K_T e^\sigma &= 0 . \quad (4)
\end{align*}$$

Defining new coordinates $\xi = uv$, $\chi = u/v$, and, for convenience $e^\Sigma = \gamma \xi e^\sigma$, equations (4) can be written as

$$\begin{align*}
(\phi_{11} - \Sigma_1 \phi_1 + T_{11}^2) + (\phi_{00} - \Sigma_0 \phi_0 + T_{00}^2) &= 0 \\
(\Sigma_{11} + \phi_{11} + T_{11}^2) - (\Sigma_{00} + \phi_{00} + T_{00}^2) &= 0 \\
2\phi_{01} - \Sigma_1 \phi_0 - \Sigma_0 \phi_1 + 2T_0 T_1 &= 0 \\
(T_{11} + \phi_1 T_1 - \frac{1}{2} e^\Sigma K_T) - (T_{00} + \phi_0 T_0) &= 0 \\
(\phi_{11} + \phi_1^2 + e^\Sigma K) - (\phi_{00} + \phi_0^2) &= 0 \quad (5)
\end{align*}$$

where $(\cdots)_0 = (\chi \frac{d}{d\chi})(\cdots)$ and $(\cdots)_1 = (\xi \frac{d}{d\xi})(\cdots)$. In the following we first specialise to the static case where the fields depend only on $\xi$. Then equations (4) become

$$\begin{align*}
\Sigma_{11} + \phi_1 \Sigma_1 &= \Sigma_{11} + \phi_{11} + T_{11}^2 = 0 \\
T_{11} + \phi_1 T_1 - \frac{1}{2} e^\Sigma K_T &= \phi_{11} + \phi_1^2 + e^\Sigma K = 0 . \quad (6)
\end{align*}$$
To proceed further, we first note that the fields can be expected to evolve depending on the local metric given by $\Sigma$ and hence look for their solutions in terms of $X(\xi) \equiv \Sigma_1$. Letting $F(X) = \phi_1, \ (\cdots)' = \frac{d}{dX}(\cdots)$, and noting that $(\cdots)_1 = (\cdots)'X_1$, equations (6) give

$$X_1 + XF = 0, \quad T_1^2 - XF(1 + F') = 0$$
$$XFF'' + (XF' - F)(1 + F') + e^\Sigma T_1 K_T(XF)^{-1} = 0. \quad (7)$$

The curvature scalar is given by $R = -4\gamma e^{-\Sigma} XF$. Dividing the last equation above by $XF(1 + F')$ and changing the independent variable from $\xi$ to $X$ the $\beta$-function equations for the static case can be written as

$$F\Sigma' + 1 = 0$$
$$X \phi' + 1 = 0$$
$$XFT'' - (1 + F') = 0$$
$$\frac{F''}{1 + F'} + \frac{F'}{F} - \frac{1}{X} - \frac{T'KTe^\Sigma}{XF(1 + F')} = 0 \quad (8)$$

where $X$ and $\xi$ are related by $X_1 = -XF$. Equations (8) can be simplified further, for example, as follows. Let

$$L' = \frac{T'KTe^\Sigma}{XF(1 + F')} \quad (9)$$

In this case the last equation in (8) can be integrated to give

$$F(1 + F') = a^2 X e^L \quad (10)$$

where $a^2$ is a constant. This immediately gives $T_1^2 = a^2 X^2 e^L$ from which we obtain

$$T' = -a F^{-1} e^{\frac{L}{2}} \quad (11)$$

Substituting these expressions in (9) and replacing $L$ by a new field $H = ae^{\frac{L}{2}}$ the $\beta$-function equations become

$$F\Sigma' + 1 = 0$$
$$FT' + H = 0$$
$$1 + F' - \frac{XH}{F} = 0$$
$$H' + \frac{KTe^\Sigma}{2X^2F} = 0 \quad (12)$$
and
\[ XF + F^2 - X^2 H^2 + Ke^\Sigma = 0 \]  \hspace{1cm} \text{(13)}

The above equation forms an integral of the set of differential equations (12).

We present one more version of the \( \beta \)-function equations which appears simpler but is just as hard to solve as other versions. However, this form will be useful later. Note that equations (12) are invariant under the scaling
\[ X \to \lambda X \]
\[ F \to \lambda F \]
\[ \Sigma \to \Sigma + 2 \ln \lambda \]  \hspace{1cm} \text{(14)}

where \( \lambda \) is a constant. Hence define a set of scale invariant variables [10]
\[ x = -X^2 e^{-\Sigma}, \ f = \frac{F}{X}, \ s = x \Sigma'. \]  \hspace{1cm} \text{(15)}

In terms \( x \) and \( f \) equations (12) and (13) become
\[ sf + 1 = 0 \]
\[ x(1 + 2f) \frac{dT}{dx} + H = 0 \]
\[ x(1 + 2f) \frac{df}{dx} + \frac{K}{x} = 0 \]
\[ x(1 + 2f) \frac{dH}{dx} - \frac{K_T}{2x} = 0 \]  \hspace{1cm} \text{(16)}

and
\[ f + f^2 - H^2 - \frac{K}{x} = 0 \]  \hspace{1cm} \text{(17)}

The curvature scalar is given by \( R = 4\gamma xf \).

We now consider the \( \beta \)-function equations (1) in the Schwarzschild gauge
\[ ds^2 = -G(dx^0)^2 + G^{-1}(dx^1)^2. \]  For the static case we have \( G = G(x_1) \). Equations (1) then become
\[ G'' + G'\phi' = 0 \]
\[ \phi'' + T'' = 0 \]
\[ G\phi'' + \phi'(G' + G\phi') + 4\gamma K = 0 \]
\[ GT'' + T'(G' + G\phi') - 2\gamma K_T = 0 \]  \hspace{1cm} \text{(18)}
where in equations (18) and (19) \((\cdot \cdot \cdot)’\) denotes \(\frac{d}{d\xi}\)\((\cdot \cdot \cdot)\) and \(\gamma, K,\) and \(K_T\) are as defined before. These equations appear to have no connection with equations (3) in the conformal gauge. However, if we define \(G\) and \(\psi\) by
\[
G = 4\gamma e^{-G} , \quad \psi' = \phi' - G',
\] (19)
then equations for \(G\), \(\psi\), and \(T\) become exactly the same as equations (3) for \(\Sigma\), \(\phi\), and \(T\) respectively. Thus this implies that restricting the fields \(\Sigma\), \(\phi\), and \(T\) to depend only on \(\xi\) corresponds to the static case. Henceforth, in this paper we will work only in the conformal gauge.

Finally, let us briefly discuss the residual transformations which preserve both the conformal gauge \(ds^2 = e^{\sigma} du dv\) and the static nature of the fields. Such transformations are given by
\[
u = \tilde{u}^b, \quad v = \tilde{v}^b
\] (20)
where \(b\) is an arbitrary constant. Under transformations (20) we have
\[
\begin{align*}
\xi &= \tilde{\xi}^b \\
(\cdot \cdot \cdot)_1 &= b(\cdot \cdot \cdot)_1 \\
\tilde{\sigma} &= \sigma + (b - 1) \ln \tilde{\xi} + 2 \ln b \\
\tilde{\Sigma} &= \Sigma + 2 \ln b \\
\tilde{X} &= bX \\
\tilde{F} &= bF
\end{align*}
\] (21)
where \((\cdot \cdot \cdot)_1\) denotes \((\xi \frac{d}{d\xi})\)(\cdot \cdot \cdot) as before. Also equations (3) remain invariant under these transformations. Thus, for example, one may scale both \(X\) and \(F\) by an arbitrary constant using the residual conformal gauge transformations (20).

3. Static Solutions

We now consider the solutions of \(\beta\)-function equations described in last section. First let \(T = 0\). In this case, the general solution of equations (1) is given by the two dimensional black hole solution presented in [4] which corresponds to the static case. This solution can be written as follows. \(T = 0\)
implies that $H = K_T = 0$ in equation (12). We then obtain $F = 1 - X$ where, using transformations (20), the integration constant is set equal to 1. Equation (13) then gives $e^{2x} = X - 1$. From $X_1 = -XF$ it follows that $X = \alpha(\xi + \alpha)^{-1}$ where $\alpha$ is a constant related to the black hole mass parameter $a$ of ref. [4] by $a = -\gamma\alpha$ (see section 4 also). This constitutes the general solution of equation (11) when $T = 0$.

However, by setting $T = 0$ we loose all information about tachyon back reaction. It is important to incorporate this back reaction particularly in view of the hints of instability found in various works [3, 4, 8] when tachyon is included. The main difficulty when $T \neq 0$ is that the resulting equations are hard to solve explicitly. In this section we present a static solution when $T \neq 0$ thus including the tachyon back reaction.

Notice that equations (13) can be solved explicitly if one assumes that the tachyon potential vanishes, i.e., $V(T) = 0$. The solution obtained in this approximation incorporates tachyon back reaction and exhibits new features which, we show later, will persist even when $V(T) \neq 0$.

When $V = 0$, $K = 1$ and (12) implies that $H = \text{constant}$. Let $H^2 = \epsilon(1 + \epsilon)$ where $\epsilon \geq 0$ (equivalently $\epsilon$ can be $\leq -1$) so that $T_1^2 = X^2H^2 \geq 0$. We then have $F(1 + F') = \epsilon(1 + \epsilon)X$. Note that this equation is invariant under $X \rightarrow \lambda X$ and $F \rightarrow \lambda F$ where $\lambda$ is a constant. This invariance suggests the substitution $X = e^s$, $F = e^s\tilde{F}$. Integrating the resulting equation gives

\[ (F - \epsilon X)'(F + (1 + \epsilon)X)^{1+\epsilon} = \text{constant}. \tag{22} \]

Note that for $\epsilon = 0$ (hence $T = 0$) in the above expression we get the black hole solution of [4].

In principle, this forms the complete solution. However it is difficult to understand this solution — to integrate $X_1 = -XF$, to understand how the various fields $\Sigma$, $\phi$, $T$, and the curvature scalar $R$ behave, etc. Therefore we choose a different parametrisation for the above solution. Let

\[ F - \epsilon X = lB\tau^{-1}; \quad F + (1 + \epsilon)X = B\tau^\delta^{-1} \tag{23} \]

where $\tau \geq 0$ is a new parameter, $l = \pm 1$, $\delta = (1 + 2\epsilon)(1 + \epsilon)^{-1}$, $\epsilon \geq 0$, $B = A(1 + 2\epsilon)$ and $A$ is a constant. Note that for $l = -1$ the constant in equation (22) need not be real and hence our choice of $l$ constitutes minimal analytic continuation. Equations (23) give

\[ X = A\tau^{-1}(\tau^\delta - l); \quad F = A\tau^{-1}(\epsilon\tau^\delta + l(1 + \epsilon)) \tag{24} \]

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from which it follows that \((1 + \epsilon)\tau \dot{X} = F\) where \(\ddot{\cdot} = \frac{d(\cdot)}{d\tau}\). Equations (12) or (13) then give

\[
\begin{align*}
\dot{\phi} &= -X^{-1} \dot{X} \\
\dot{\Sigma} &= -(1 + \epsilon)\tau^{-1} \\
\dot{T} &= -\sqrt{\delta - 1} \tau^{-1} \\
\tau_1 &= -(1 + \epsilon)\tau X 
\end{align*}
\]

which can be integrated to obtain

\[
\begin{align*}
e^\phi &= \beta_0 \tau (\tau^\delta - l)^{-1} \\
T &= -\sqrt{\delta - 1} \ln \tau \\
e^\Sigma &= -lB^2 \tau^{-\frac{1}{1+\epsilon}} \\
\int_0^\tau d\tau (\tau^\delta - l)^{-1} &= A(1 + \epsilon) \ln \left( \frac{\alpha}{m \xi} \right) 
\end{align*}
\]

where \(\beta_0\) and \(\alpha\) are constants and \(m = \pm 1\) is the sign of \(\xi\). The curvature scalar \(R\) is given by

\[
R = 4\gamma (1 + 2\epsilon)^{-2} \tau^{-\delta}(\tau^\delta - l)(\epsilon \tau^\delta + l(1 + \epsilon)) 
\]

Equations (26) and (27) form the solution of equations (8) when \(V = 0\). Using the transformations (20) we can set \(B = 1\). If \(\epsilon = 0\) and \(\tau\) is eliminated, the above expressions reduce to those given in the beginning of this section.

For future use we also write down the expressions for \(x\) and \(f\) defined in (15). For \(\epsilon = 0\) we have

\[
x = \frac{X^2}{1 - X}, \quad f = \frac{1 - X}{X}
\]

where \(X = \alpha(\xi + \alpha)^{-1}\). For \(\epsilon \neq 0\) we get from (26)

\[
x = l(1 + 2\epsilon)^{-2} \tau^{-\delta}(\tau^\delta - l)^2 \\
f = \frac{\epsilon \tau^\delta + l(1 + \epsilon)}{\tau^\delta - l} 
\]
4. Features of the static solutions

Let us now consider in detail the static solution given by equations (26) and (27). The $\tau$-integration in (26) cannot be done in a closed invertible form except when $\epsilon = 0$ or $\infty$. However to understand the solution and its salient features it is not necessary to do this integration. The integrand $(\tau^\delta - l)^{-1}$ is regular for $\epsilon \geq 0$ and is well defined for $\tau \geq 0$. We only need from the equation relating $\tau$ and $\xi$ the value of $\xi$ for a given $\tau$. Since the integrand varies monotonically with respect to $\epsilon$, this information is easy to obtain qualitatively for generic $\epsilon$.

$\epsilon = 0$

This gives the solution obtained in [4]. In this case we get from (26)

$$\ln(\tau - l) = \ln \left( \frac{\alpha}{m\xi} \right)$$

(30)

where $\tau \geq 0$, $\alpha$ is a constant and we have set $B (= A) = 1$. Using the expression for $e^\Sigma$ in (26) and its definition $e^\Sigma \equiv \gamma \xi e^\sigma$ where $\gamma = -\frac{2}{\alpha}$ we see that $l = \text{sign}(\xi)$ so that $e^\sigma \geq 0$. We first take $l = m = 1$ and write

$$\tau - 1 = \frac{\alpha}{\xi}.$$  

(31)

For $\epsilon = 0$ the above equation can be considered to be valid for $-\infty \leq \tau \leq \infty$. We then obtain, with $\beta_0$ a constant,

$$e^\phi = \frac{\beta_0}{\alpha}(\xi + \alpha)$$

$$e^{-\sigma} = -\gamma(\xi + \alpha)$$

(32)

which is the solution of [4] if we set $a = -\gamma \alpha = \beta_0$ where $a$ is the black hole mass parameter of [4]. This solution is asymptotic to flat space time at $\xi = \infty$, has a horizon at $\xi = 0$, and a singularity at $\xi = -\alpha$ where the curvature scalar $R$ becomes divergent. For more details see [4].

However, in obtaining the above solution we let $\tau$ take negative values also. This is alright for $\epsilon = 0$. But when $\epsilon > 0$, $\tau^\delta$ in the integrand is not
well defined for $\tau < 0$. Therefore we always restrict $\tau$ to be $\geq 0$. We then proceed as follows. Let $\xi \geq 0$. Then $l = m = 1$ so that the metric is well defined as explained above. The range $\infty \geq \xi \geq 0$ outside the horizon then corresponds to $1 \leq \tau \leq \infty$. We call this Branch I. When $\xi \leq 0$ we have $l = m = -1$ and the range $-\alpha \geq \xi \geq 0$ inside the horizon corresponds to $0 \leq \tau \leq \infty$. We call this Branch II. These two branches describe the solution given above when $\epsilon = 0$. We assume that these are also the relevant branches when $\epsilon > 0$.

We now make a few comments on the mass of the black hole, $M_{bh}$, which can be calculated using the behaviour of the metric in the asymptotically flat region following the methods of Witten [3] or those of, for example, [12]. We find it convenient to use the formula given by T. Tada and S. Uehara in [12] which, in our notation, reads as

$$M_{bh} = (-\gamma)^{\frac{1}{2}} e^\phi (e^{-\Sigma_2 \phi_1^2} + 1)$$

where $(\cdots)_1 = (\xi \frac{d}{d\xi}) (\cdots)$ as before. Using (32) we obtain

$$M_{bh} = (-\gamma)^{\frac{1}{2}} \beta_0.$$  \hspace{1cm} (34)

Note further that $M_{bh}$ as given in (33) is conserved, i.e.,

$$M_1 = 0$$ \hspace{1cm} (35)

as can be easily verified.

(ii) $\epsilon > 0$

This case includes the tachyon back reaction. Consider branch I, i.e., $\tau \geq 1$ and $l = m = 1$. The integral $\int_0^\tau d\tau (\tau^d - 1)^{-1}$ tends to $-\infty$ as $\tau \to 1_+$ and tends to a finite positive semidefinite value ($= A(1 + \epsilon) \ln \frac{a}{\alpha}$, by definition) as $\tau \to \infty$. Thus we have $\infty \geq \xi \geq \xi_+ > 0$ as $1 \leq \tau \leq \infty$. Similarly for branch II, the integral $\int_0^\tau d\tau (\tau^d + 1)^{-1}$ tends to a finite positive semidefinite value ($= A(1 + \epsilon) \ln \frac{a}{\alpha}$, by definition) as $\tau \to \infty$. It then follows that $-\alpha \leq \xi \leq -\xi_- < 0$ as $0 \leq \tau \leq \infty$. Moreover $\xi_\pm \to 0$ as $\epsilon \to 0$. This implies that the horizon which was located at $\xi = 0$ when $\epsilon = 0$ now splits into two located at $\xi = \pm \xi_\pm$. Thus the horizon resembles Schwarzschild horizon for $\epsilon = 0$ and Reissner-Nordstrom horizon for $\epsilon > 0$. 

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That $\tau = \infty$ does indeed correspond to the location of horizon can also be seen by the zeroes of the metric $G_{\mu\nu}$ in Schwarzschild coordinates $(r, t)$ defined by $d\xi = 2(-\gamma)^{1/2}\xi e^{-\Sigma}dr$ and $d\chi = 2(-\gamma)^{1/2}\chi dt$. The metric is given by

$$ds^2 = e^\Sigma dt^2 - e^{-\Sigma}dr^2 \quad (36)$$

where $e^\Sigma = \gamma\xi e^\sigma$ and $\gamma = -\frac{2}{\sigma^2}$. Equation (26) implies that $e^\Sigma \to 0$ as $\tau \to \infty$ for any $\epsilon \geq 0$. Hence $\tau = \infty$ corresponds to a horizon. Note also that in Schwarzschild coordinates the fields are indeed static, i.e., independent of time coordinate $t$ and that the asymptotically flat region is given by $r \to \infty$.

Though for $\epsilon > 0$ the metric components $G_{\mu\nu}$ have the right zeroes and poles at $\tau \to \infty$, it is not really a horizon in the usual sense because $G_{\mu\nu}$ cannot be made regular by transforming coordinates. Thus $\tau = \infty$ is not just a coordinate singularity. This can be seen by evaluating the curvature scalar $R$ at $\tau = \infty$. We see from equation (27) that $R \to \infty$ as $\tau \to \infty$ with a strength proportional to $\epsilon$ for small values of $\epsilon$. Thus these new horizons have curvature singularities. Also, unlike in the case of Reissner-Nordström black hole, the above solutions cannot be extended analytically into the region $-\xi_- \leq \xi \leq \xi_+$, retaining their correct limiting behaviour as $\epsilon \to 0$. Hence our solution is not really a black hole solution, though for $\epsilon = 0$ it is.

The curvature scalar $R$ has another singularity in branch II at $\tau = 0$ as can be seen from equation (27). This is a singularity inside the inner horizon and is present even when $\epsilon = 0$, i.e., $T = 0$. This is the singularity present in the blackhole solutions of [3, 4].

Note further that at $\tau = \infty$ the dilaton field $e^\phi \to \tau^{-\frac{1}{1+\epsilon}} (\to 0)$. Since $e^{-\frac{\phi}{2}}$ acts as a coupling constant in equation (3), it follows that one is dealing with a strong coupling regime near the horizon when $\epsilon > 0$, i.e., when tachyon back reaction is included.

Consider now the “black hole” mass $M_{bh}$ calculated from the behaviour of the fields in our solution in the asymptotically flat region. It can be easily seen that in the asymptotically flat region (branch I, $\tau \to 1_+$) the tachyon field is negligible. (This is true even when the tachyon potential $V \neq 0$. See section 5 and, for example, [5]). Therefore, equation (33) can still be applied with negligible error to calculate $M_{bh}$ from the fields in the asymptotically flat region. We obtain

$$M_{bh} = \frac{(-\gamma)^{1/2}\beta_0}{1 + 2\epsilon} \quad (37)$$
Thus the behaviour of the fields in the asymptotically flat region is not giving any indication of the new singularities that arise as a result of the tachyon back reaction. However, the formula (33) cannot be relied upon since it is not conserved when one uses the equations of motion (3) which include tachyon also. That is,

\[(M_{bh})_1 \neq 0.\] (38)

It is not clear if any conserved quantity exists when tachyons are included. We could not find any, except the trivial ones given in the equations of motion (3) or their linear combinations.

5. Effect of tachyon potential, \(V(T)\)

In this section we consider the tachyon potential, \(V\). It was necessary to assume that \(V = 0\) in deriving our solution. We show below that even when \(V \neq 0\) the features we found in our solution will persist.

It is convenient to view the tachyon equation in (3) as an equation for an (anti)damped oscillator where the couplings to graviton and dilaton provide the damping force and the tachyon potential \(V\) provides the restoring force. This would mean that as one moves in from the asymptotically flat region (branch I, \(\tau \rightarrow 1_+\)) towards the horizon (\(\tau \rightarrow \infty\)), the tachyon will interact with gravitational field acquiring a large kinetic energy. Then the tachyon potential can be neglected in comparison to its kinetic energy. That this is actually what happens can be seen by calculating the kinetic \((\nabla T)^2\) and potential \(V\) energy terms in the action (3) (or, equivalently, the damping \((\phi_1 T_1)\) and the potential \(e^\Sigma K_T\) terms in the tachyon equation in (3)). Using the expressions \((\nabla T)^2 = 4\gamma e^{-\sigma} T^2_1\), \(T = -\sqrt{\delta - 1} \ln \tau\), \(\phi_1 T_1 = \sqrt{\epsilon(1 + \epsilon)} XF\), and \(e^\Sigma K_T = e^\Sigma (\frac{T}{2} + \mathcal{O}(T^2))\) and the solution given in equation (26) one sees that tachyon kinetic energy indeed dominates its potential energy away from the asymptotically flat region. Hence, neglecting tachyon potential is a valid approximation far inside from the asymptotically flat region (branch I, \(\tau >> 1\)).

Asymptotically, however, the kinetic and potential energy of the tachyon are of the same order and hence \(V\) cannot be neglected\(^1\). But it is reasonable

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\(^1\)Indeed, in the asymptotically flat region \(\tau \rightarrow 1_+\), \(T_V = \sqrt{T}\) where \(T_V\) is the correct tachyon solution including its potential \(V(T) = \gamma T^2\). This relation can be worked out...
to expect that the correct asymptotically flat solution when \( V \neq 0 \) can be matched at some point to (26) which becomes more and more valid as one nears the horizon. That this is likely to be the case can be seen by performing one iterative calculation when \( V \neq 0 \) as follows. Near the asymptotically flat region, let \( \tau = 1 + y \) where \( y > 0 \) is small. Denoting \( \frac{d}{d\tau}(\cdots) = \frac{d}{dy}(\cdots) \), the dilaton and tachyon equations in (3) become

\[
\ddot{\phi} + \tau^{-1} \dot{\phi} + \left(1 + \frac{T^2}{4}\right) \frac{e^\Sigma}{\tau_1^2} = 0
\]

\[
\ddot{T} + \tau^{-1} \dot{T} - \frac{Te^\Sigma}{4\tau_1^2} = 0
\]  

(39)

where we have taken \( V(T) = \gamma T^2 \). Note that \( \tau^{-1} \dot{T} \) and \( -\frac{Te^\Sigma}{4\tau_1^2} \) act like effective damping and restoring forces respectively in the tachyon equation. For \( y \to 0 \) the solutions (26) become, with \( B = 1 \),

\[
\tau_1 = -y + \cdots
\]

\[
e^\Sigma = -1 + \frac{y}{1 + \epsilon} + \cdots
\]

\[
\phi = -\ln y + \text{constant} + \frac{2 + \epsilon}{2(1 + \epsilon)} y + \cdots
\]  

(40)

where \( \cdots \) denote higher order terms in \( y \). The restoring force is given by

\[
-\frac{Te^\Sigma}{4\tau_1^2} = \frac{T}{4y^2} (1 - y + \cdots).
\]  

(41)

It follows from the tachyon equation in (39) that

\[
T = a_0 \sqrt{y} + \cdots
\]  

(42)

where \( a_0 \) is a constant. Note that the above solution for the tachyon with its potential included is indeed the square root of the tachyon solution given in easily by substituting the expressions for graviton and dilaton that follow from equation (26) into the tachyon equation given in (6) and solving for tachyon near \( \tau \to 1_+ \) as we will show. Or, it can also be seen easily from a calculation in “linear dilaton vacuum” similar to that of de Alwis and Lykken in ref. [8].
equation (26) when \( V = 0 \). Substituting this value of tachyon in the dilaton equation in (39) one obtains the correction to the restoring force given by

\[
- \frac{T e^\Sigma}{4\tau^2} = \frac{T}{4y^2}(1 - y - a_0 y + \cdots).
\]

(43)

What the above expression implies is that the tachyon back reaction reduces the restoring force as one moves inside away from the asymptotically flat region. Thus it indicates that the restoring force (and hence the tachyon potential) is less and less important as one moves inside and that it can eventually be neglected, thus validating our assumption. Furthermore, the potential \( V(T) = \gamma T^2 \) itself neglects \( \mathcal{O}(T^3) \) corrections given, e.g. as in [13], by

\[
V = \gamma(T^2 - \frac{T^3}{24} + \mathcal{O}(T^4)).
\]

(44)

Thus the correction term implies that the potential is less steep, suggesting that the potential energy can eventually be neglected (compared to the kinetic energy). Also, various recent works [8, 2, 3, 4] find instability of the \( d = 2 \) string black hole solution when tachyon is included. Since our solution can be viewed as arising from a similar instability, it seems that neglecting tachyon potential is reasonable and that the qualitative features of our solution will persist even when \( V(T) \) is properly taken into account.

We now show by a perturbative analysis near the horizon when \( V \neq 0 \) that the horizon still develops a curvature singularity. This is seen by an analysis of equations (16).

We first show that \( x \to 0 \) (\( \infty \)) in the asymptotically flat region (near horizon). Recall that \( x = -X^2 e^{-\Sigma} \) where \( X = \Sigma_1 \) and note that \((\cdots)_1 = \frac{1}{2}(\cdots)_1 = \frac{1}{2}(\cdots)_1 = \frac{1}{2}(\cdots)_1 \) in \((r, t)\) coordinates described in section 4. Since the metric in \((r, t)\) coordinates is given by equation (38), it follows that in the asymptotically flat region \( \Sigma \to 0 \) and also \( X \to 0 \). Hence, \( x \to 0 \) there. Similarly near the horizon \( e^\Sigma \to 0 \) by definition and hence \( \Sigma \to -\infty \). Since \( X = \Sigma_1 \), it follows that \( X \neq 0 \) and therefore \( x \to \infty \) near the horizon. This behaviour of \( x \) tending to zero in the asymptotically flat region and to infinity near the horizon remains true irrespective of the nature of tachyon potential and can be seen explicitly from equations (28) and (29) for special cases \( T = 0 \) and \( V = 0 \) respectively.

Consider the \( \beta \)-function equations in the form given in [16] and (17). Let us call the last three equations in (16) as \( T, f, \) and \( H \) equation respectively.
To find the solution near \( x = 0 \) or \( \infty \) one first starts with an ansatz for the tachyon field and then solves \( f, H, \) and \( T \) equations, in that order, up to leading corrections. The equation (17) will be used to fix constants. The ansatz is considered correct if the solution of \( T \) equation gives the same form for \( T \), to the leading order, as originally assumed.

Consider first \( x \to 0 \) and the ansatz \( T = a_0 x^n \) for the tachyon. Let \( n > 0 \) so that \( T \) will not be divergent in the asymptotically flat region \( x \to 0 \). After solving \( f, H, \) and \( T \) equations one gets, for \( V = \gamma T^2 \),

\[
\begin{align*}
    f + f^2 &= x^{-1} + f_0^2 - \frac{1}{4} + \cdots \\
    H &= -\frac{a_0}{2} x^{-\frac{1}{4}} + H_0 + \cdots \\
    T &= a_0 x^{\frac{1}{4}} - H_0 \sqrt{x} + \cdots .
\end{align*}
\]

(45)

Here and in the following \( a_0, f_0, \) and \( H_0 \) are constants. For \( V = 0 \) one gets \( H = H_0 + \cdots \) and \( T = a_0 \sqrt{x} + \cdots \) and \( f \) as above.

Next consider the ansatz \( T = a_0 x^n \ln x \) for the tachyon as \( x \to 0 \) (with \( n > 0 \) for the same reasons as above). After solving \( f, H, \) and \( T \) equations one gets, for \( V = \gamma T^2 \),

\[
\begin{align*}
    f + f^2 &= x^{-1} + f_0^2 - \frac{1}{4} + \cdots \\
    H &= -\frac{a_0}{2} x^{-\frac{1}{4}} (\ln x + 4) + \cdots \\
    T &= a_0 x^{\frac{1}{4}} \ln x + \cdots .
\end{align*}
\]

(46)

The above ansatz is not valid for \( V = 0 \). In the above equations \( \cdots \) denote the subleading terms in powers of \( x \) which can be obtained by performing more iterations. Note that for both the ansatzes above, the curvature scalar \( R (= 4 \gamma x f) \) tends to zero as \( x \to 0 \) as expected and the tachyon behaviour agrees with that found in [8].

We now consider the region near horizon where \( x \to \infty \) and \( V \neq 0 \). We would like to know how the solution behaves near the horizon, i.e., \( x \to \infty \), and in particular, whether the curvature scalar \( R \) is divergent or not when \( V \neq 0 \). Let us first consider the ansatz \( T = a_0 x^n + \cdots \), where \( \cdots \) denote subleading terms. The exponent \( n \) can be positive or negative and \( a_0 \) is a constant. After solving \( f, H, \) and \( T \) equations one gets \( n \leq \frac{1}{2} \) and

\[
T \approx \text{constant} + \ln x + x^{n-1}
\]

(47)
schematically, omitting the coefficients. This is in contradiction to the original ansatz which, therefore, is incorrect. Next we take the ansatz \( T = a_0 \ln x + \cdots \). Solving \( f, H, \) and \( T \) equations we get

\[
\begin{align*}
    f + f^2 &= f_0^2 - \frac{1}{4} + \frac{a_0^2}{4} x^{-1} (\ln^2 x + 2 \ln x + 2) - x^{-1} + \cdots \\
   H &= -2a_0 f_0 - \frac{a_0}{8f_0} x^{-1} (\ln x + 1) + \cdots \\
   T &= a_0 \ln x + x^{-1} \left[ \frac{a_0^3}{16f_0^2} (\ln^2 x + 4 \ln x + 6) - \frac{a_0}{16f_0^2} (\ln x + 2) + \frac{a_0}{2f_0^2} \right] + \cdots \quad (48)
\end{align*}
\]

where \( \cdots \) denote \( O(x^{-2} \ln^* x) \) terms for some integer \( * \). Note that this solution of \( T \) agrees to the leading order with our starting ansatz which, therefore, is correct. We also get from equation (17) that

\[
f_0^2 - \frac{1}{4} = 4a_0^2 f_0^2 \quad (49)
\]

which implies that \( f + f^2 \), and hence \( f \), is nonzero as long as \( a_0 \), and hence, \( T \) is nonzero. Note that the solution given in (26) for \( V = 0 \) satisfies the above equation, as it should. One can continue the above iteration to obtain the subleading terms of \( O(x^{-2} \ln^* x) \) but they are not necessary for our purpose.

If the tachyon is not identically zero, \( i.e. a_0 \neq 0 \), then as \( x \to \infty \), \( f \) approaches a nonzero constant and hence the curvature scalar given by \( R = 4\gamma x f \) diverges near the horizon where \( x \to \infty \). Note also that the leading logarithmic behaviour of \( T \) near the horizon when \( V \neq 0 \) agrees with our solution (26) obtained assuming \( V = 0 \).

Thus the curvature scalar \( R \) is divergent at the horizon even when the tachyon potential \( V(T) \neq 0 \). Hence it follows that when tachyon back reaction is included, the \( d = 2 \) string does not admit any static solution which can be interpreted as a black hole in the usual sense.

6. Discussions

In this section we first briefly present a time dependent extension of our solutions and then discuss possible methods of removing the singularities
found in previous sections. Consider the ansatz for the fields \( \Sigma, \phi, \) and \( T \)
\[
\Sigma(\xi, \chi) = \Sigma(\xi) + \tilde{\Sigma}(\chi) \\
\phi(\xi, \chi) = \chi(\xi) + \tilde{\phi}(\chi) \\
T(\xi, \chi) = T(\xi) + \tilde{T}(\chi) \\
\]
which is the nonlinear analog of separation of variables. The variables \( \xi \) and \( \chi \) are defined in section 2. The equations for the \( \xi \) dependent fields are the same as in (6) with \( e^\Sigma \) understood as \( e^{\Sigma(\xi, \chi)} \) and those for the \( \chi \) dependent fields are given by
\[
\begin{align*}
\Sigma_{00} + \phi_0 \Sigma_0 &= 0 \\
\Sigma_{00} + \phi_{00} + T_0^2 &= 0 \\
\Sigma_1 \phi_0 + \Sigma_0 \phi_1 - 2T_0 T_1 &= 0 \\
T_{00} + \phi_0 T_0 &= 0 \\
\phi_{00} + \phi_0^2 &= 0
\end{align*}
\]
where we have omitted the tildes over the \( \chi \) dependent fields. As before \((\cdots)_0 \equiv (\chi \frac{d}{d\chi})(\cdots)\) and \((\cdots)_1 \equiv (\xi \frac{d}{d\xi})(\cdots)\). Note that the above equations are the same as those given in (6) with \( \gamma = 0 \) and \( \xi \) replaced by \( \chi \). From the first, fourth, and fifth equations above it immediately follows that
\[
\Sigma_0 = a_1 \phi_0, \quad T_0 = a_2 \phi_0
\]
where \( a_1 \) and \( a_2 \) are constants. The second equation in (51) gives
\[
a_2^2 = 1 + a_1
\]
while the third equation implies, using (26) for \( \Sigma(\xi), \phi(\xi), \) and \( T(\xi) \), that
\[
a_1 = 0, \quad a_2^2 = (4\epsilon(1 + \epsilon))^{-1}.
\]
The splitting of the \( \beta \)-function equations (5) into (6) and (51) and the use of the solutions (26) of equations (6), which involve \( e^\Sigma \) terms, is justified \textit{a posteriori} since \( a_1 = 0 \) and hence \( \Sigma = \Sigma(\xi) \). The equation for \( \phi(\chi) \) can be easily solved and we get
\[
\begin{align*}
\phi(\chi) &= \ln(b_1 \ln A\chi) \\
T(\chi) &= \ln(b_2 \ln A\chi)
\end{align*}
\]
where \( A, b_1, \) and \( b_2 \) are constants. Note however that this time dependent extension does not alter any of the features of the previous static solution (26). Nor is it likely to represent the general solution of the \( \beta \)-function equations (3).

We now discuss some possible interaction terms that may remove the singularities found in previous sections, at least the new ones:

(1) Higher order \( \alpha' \) corrections: Tseytlin [14] had shown that black hole solutions of [3, 4, 5] survive these corrections. Very likely, the solution given here will also survive these corrections since the tachyon can be thought of as an (anti)damped oscillator gaining energy by gravitational interactions — so that it would have grown strong before one reaches the region of strong curvature where \( \alpha' \) corrections are deemed important. Thus these corrections may not remove the singularities.

(2) Higher massive modes: For similar reasons as above higher massive modes \( A_n \) are of no help. In fact, taking their effective action as given in [13] with zero potential, it is easily seen that \( A_n \)'s have solutions similar to that of tachyon \( T \) and hence do not remove the singularities.

(3) Antisymmetric tensor, \( \mathbf{H} \) (indices on \( \mathbf{H} \) suppressed): This field is not there for \( d = 2 \) space-time. However, for \( d = 2 \) toy models of a \( D \) dimensional space-time, as considered in [1, 2] for example, the resulting equations that include quantum effects will be similar to the \( \beta \)-function equations for \( d = 2 \) strings. \( \mathbf{H} \) field interactions present in such cases may possibly remove the singularities. Moreover, \( \mathbf{H} \) fields invariably arise in any space-time obtained from string theory, so it is natural to include them.

(4) Supersymmetry: This symmetry introduces fermions which may provide enough repulsive force to avoid the formation of the singularities. On the other hand, the fermions might instead form attractive condensates and not remove the singularities.

(5) Other approaches: It will also be interesting to find an analog of our solution, which includes tachyon back reaction, in the context of gauged Wess-Zumino-Witten models such as considered in [3] or in the context of matrix model where tachyon equation in the black hole background has been found recently [15]. Perhaps in these contexts one may get a better insight into the singularities and methods of avoiding them.

The above (and other) possibilities are worth pursuing. It is important to resolve this problem of new singularities — to understand them better and to remove them if possible. This issue is particularly relevant for the string
inspired toy models of $d = 2$ quantum gravity that may answer the puzzles of $d = 4$ space-time. The removal of the singularities seen here might also suggest new interactions that could be important.

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Notes Added:
(1) N. E. Mavromatos has kindly brought ref. [16] to our attention where time dependent perturbations of the black hole solution are considered perturbatively in the asymptotically flat region.

Notes Added in Proof:
(1) In ref. [17] Mann et al. consider $1 + 1$ dimensional black hole formation in various models, for the first time to our knowledge by two sided gravitational collapse. For the string inspired models they find that black hole formation requires dilaton surface charges, interpreted as arising from a tachyon potential.

We comment on two recent papers [18, 19] which appeared after our work:

(2) In ref. [18] Marcus and Oz calculate tachyon effects in $d = 2$ string black hole and find none of the features presented here. This is not surprising since they work with the tachyon equation in black hole background, thus not taking into account the effects of backreacted graviton and dilaton on tachyon. Therefore tachyon back reaction is not completely accounted for in their work.

(3) In ref. [19] Peet et al. point out that besides the singular solution described in this paper, the $d = 2$ string admits another solution which is regular at the horizon. It can be obtained by taking the ansatz $T = \text{const.} + \ldots$ for the tachyon near the horizon in equations (16) and (17). This solution is trivial (that is, $T = \text{const}.$) when the tachyon potential $V = 0$ and has infinite energy when $V \neq 0$ (see [19]). These authors further argue that a singular configuration cannot be dynamically formed starting from a regular one. We find their argument inconclusive and will address this issue elsewhere [20].
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