ABSTRACT

In this work we introduce a convolution operation over the tangent bundle of Riemannian manifolds exploiting the Connection Laplacian operator. We use this convolution operation to define tangent bundle filters and tangent bundle neural networks (TNNs), novel continuous architectures operating on tangent bundle signals, i.e. vector fields over manifolds. We discretize TNNs both in space and time domains, showing that their discrete counterpart is a principled variant of the recently introduced Sheaf Neural Networks. We formally prove that this discrete architecture converges to the underlying continuous TNN. We numerically evaluate the effectiveness of the proposed architecture on a denoising task of a tangent vector field over the unit 2-sphere.

Index Terms— Geometric Deep Learning, Tangent Bundle Signal Processing, Tangent Bundle Neural Networks, Cellular Sheaves

1. INTRODUCTION

The success of deep learning is mostly the success of Convolutional Neural Networks (CNNs) [1]. CNNs have achieved impressive performance in a wide range of applications showing good generalization ability. Based on shift operators in the space domain, one (but not the only one) key attribute is that the convolutional filters satisfy the property of shift equivariance. Nowadays, data defined on irregular (non-Euclidean) domains are pervasive, with applications ranging from detection and recommendation in social networks processing [2], to resource allocations over wireless networks [3], or point clouds for shape segmentation [4], just to name a few. For this reason, the notions of shifts in CNNs have been adapted to convolutional architectures on graphs (GNNs) [5, 6] as well as to the standard convolution if the manifold is the real line. We introduce tangent bundle convolutional filters to process tangent bundle signals (i.e. vector fields over manifolds), we define a frequency representation for them and, by cascading layers consisting of tangent bundle filterbanks and nonlinearities, we introduce Tangent Bundle Neural Networks (TNNs). We then discretize the TNNs in the space domain by sampling points on the manifold and building a cellular sheaf [26] representing a legit approximation of both the manifold and its tangent bundle [24]. We formally prove that the discretized architecture over the cellular sheaf converges to the underlying TNN as the number of sampled points increases. Moreover, we further discretize the architecture in the time domain by sampling the filter impulse function in discrete and finite time steps, showing that space-time discretized TNNs are a principled variant of the very recently introduced Sheaf Neural Networks [23, 27, 28], discrete architectures operating on cellular sheaves and generalizing graph neural networks. Finally, we numerically evaluate the performance of TNNs on a denoising task of a tangent vector field of the unit 2-sphere.

Contributions. In this work we define a convolution operation over the tangent bundles of Riemannian manifolds with the Connection Laplacian operator. Our definition is consistent, i.e. it reduces to manifold convolution [19] in the one-dimensional bundle case, and to the standard convolution if the manifold is the real line. We introduce tangent bundle convolutional filters to process tangent bundle signals (i.e. vector fields over manifolds), we define a frequency representation for them and, by cascading layers consisting of tangent bundle filterbanks and nonlinearities, we introduce Tangent Bundle Neural Networks (TNNs). We then discretize the TNNs in the space domain by sampling points on the manifold and building a cellular sheaf [26] representing a legit approximation of both the manifold and its tangent bundle [24]. We formally prove that the discretized architecture over the cellular sheaf converges to the underlying TNN as the number of sampled points increases. Moreover, we further discretize the architecture in the time domain by sampling the filter impulse function in discrete and finite time steps, showing that space-time discretized TNNs are a principled variant of the very recently introduced Sheaf Neural Networks [23, 27, 28], discrete architectures operating on cellular sheaves and generalizing graph neural networks. Finally, we numerically evaluate the performance of TNNs on a denoising task of a tangent vector field of the unit 2-sphere.

Paper Outline. The paper is organized as follows. We start with some preliminary concepts in Section 2. We define the tangent bundle convolution and filters in Section 3, and Tangent Bundle Neural Networks (TNNs) in Section 4. In Section 5, we discretize TNNs in space and time domains, showing that discretized TNNs are Sheaf Neural Networks and proving the convergence result. Numerical results are in Section 6 and conclusions are in Section 7.
2. PRELIMINARY DEFINITIONS

Manifolds and Tangent Bundles. We consider a compact and smooth $d$-dimensional manifold $M$ isometrically embedded in $\mathbb{R}^p$. Each point $x \in M$ is endowed with a $d$-dimensional tangent (vector) space $T_x M \cong \mathbb{R}^d$, $v \in T_x M$ is said to be a tangent vector at $x$ and can be seen as the velocity vector of a curve over $M$ passing through the point $x$ (formal definitions can be found in [29]). The disjoint union of the tangent spaces is called the tangent bundle $T M = \bigsqcup_{x \in M} T_x M$. The embedding induces a Riemann structure on $M$; in particular, each tangent space $T_x M$ is endowed with an inner product, called Riemann metric, given, for each $v, w \in T_x M$, by

$$
(v, w)_{T_x M} = iv \cdot iw,
$$
(1)

where $iv \in T_x \mathbb{R}^p$ is the embedding of $v \in T_x M$ in $T_x \mathbb{R}^p \subset \mathbb{R}^p$ (the d-dimensional subspace of $\mathbb{R}^p$ which is the embedding of $T_x M$ in $\mathbb{R}^p$), with $i: T M \to T_x \mathbb{R}^p$ being an injective linear mapping referred to as differential [29], and $i$ is the dot product. The Riemann metric induces also a probability measure $\mu$ over the manifold.

Tangent Bundle Signals. A tangent bundle signal is a vector field over the manifold, thus mapping $F : M \to T M$ that associates to each point of the manifold a vector in the corresponding tangent space. An inner product for tangent bundle signals $F$ and $G$ is

$$
\langle F, G \rangle_{T M} = \int_M \langle F(x), G(x) \rangle_{T_x M} d\mu(x),
$$
(2)

and the induced norm is $||F||_{T M}^2 = \langle F, F \rangle_{T M}$. We denote with $L^2(T M)$ the Hilbert Space of finite energy (w.r.t. $||\cdot||_{T M}$) tangent bundle signals. In the following we denote $\langle \cdot, \cdot \rangle_{T M}$ with $\langle \cdot, \cdot \rangle$ when there is no risk of confusion.

Connection Laplacian. The Connection Laplacian is a (second-order) operator $\Delta : L^2(T M) \to L^2(T M)$, given by the trace of the second covariant derivative defined for this work via the Levi-Civita connection [24]. The connection Laplacian $\Delta$ has some desirable properties: it is negative semidefinite, self-adjoint and elliptic. The Connection Laplacian characterizes the heat diffusion equation

$$
\frac{\partial U(x, t)}{\partial t} - \Delta U(x, t) = 0,
$$
(3)

where $U : M \times \mathbb{R}_+^+ \to \mathbb{R}$ and $U(\cdot, t) \in L^2(T M) \forall t \in \mathbb{R}_+^+$ (see [21] for a simple interpretation of (3)). With initial condition set as $U(x, 0) = F(x)$, the solution of (3) is given by

$$
U(x, t) = e^{-t\Delta} F(x),
$$
(4)

which provides a way to construct tangent bundle convolution, as explained in the following section. The Connection Laplacian $\Delta$ has a negative spectrum $\{-\lambda, \phi_i\}_{i=1}^\infty$ of eigenvalues $\lambda$ and corresponding eigenvector fields $\phi_i$ satisfying

$$
\Delta \phi_i = -\lambda_i \phi_i,
$$
(5)

with $0 < \lambda_1 \leq \lambda_2 \leq \ldots$. The $\lambda$'s and the $\phi$'s can be interpreted as the canonical frequencies and oscillation modes of $T M$.

3. TANGENT BUNDLE CONVOLUTIONAL FILTERS

In this section we define the tangent bundle convolutions of a filter impulse response $\hat{h}$ and a tangent bundle signal $F$.

Definition 1. (Tangent Bundle Filter) Let $\hat{h} : \mathbb{R}^+ \to \mathbb{R}$ and let $F \in L^2(T M)$ be a tangent bundle signal. The manifold filter with impulse response $\hat{h}$, denoted with $h$, is given by

$$
G(x) = h F(x) := (\hat{h} * T M F) = \int_0^\infty \hat{h}(t) U(x, t) dt,
$$
(6)

where $U(x, t)$ is the solution of the heat equation in (3) with $U(x, 0) = F(x)$. Injecting (4) in (6), we obtain

$$
G(x) = h F(x) = \int_0^\infty \bar{h}(t) e^{-t\Delta} F(x) dt = h(\Delta) F(x),
$$
(7)

The convolution in Definition 1 is consistent, i.e. it generalizes the manifold convolution [19] and the standard convolution in Euclidean domains (see Appendix A.4 in [30]). The frequency representation $\mathcal{F} h \in \mathbb{R}$ of $F$ can be obtained by projecting $F$ onto the $\phi_i$ basis

$$
[\mathcal{F} h]_i = \langle \mathcal{F} h, \phi_i \rangle = \int_M (F(x), \phi_i(x))_{T_x M} d\mu(x)
$$
(8)

Definition 2. (Bandlimited Tangent Bundle Signals) A tangent bundle signal is said to be $\lambda_M$-bandlimited with $\lambda_M > 0$ if $[\mathcal{F} h]_i = 0$ for all $i$ such that $\lambda_i > \lambda_M$.

Proposition 1. Given a tangent bundle signal $F$ and a tangent bundle filter $h(\Delta)$ as in Definition 1, the frequency representation of the filtered signal $G = h(\Delta) F$ is given by

$$
[\mathcal{G} h]_i = \int_0^\infty \bar{h}(t) e^{-t\lambda_i} dt.
$$
(9)

Proof. See Appendix A.1 in [30].

Definition 3. (Frequency Response) The frequency response $\hat{h}(\lambda)$ of the filter $h(\Delta)$ is defined as

$$
\hat{h}(\lambda) = \int_0^\infty \bar{h}(t) e^{-t\lambda} dt.
$$
(10)

This leads to $[\mathcal{G} h]_i = \hat{h}(\lambda_i) [\mathcal{F} F]_i$, meaning that the tangent bundle filter is point-wise in the frequency domain. Therefore, we can write the frequency representation of the tangent bundle filter as

$$
G = h(\Delta) F = \sum_{i=1}^\infty \hat{h}(\lambda_i) [\mathcal{F} F]_i \phi_i.
$$
(11)

We note that the frequency response of the tangent bundle filter generalizes the frequency response of a standard time filter as well as a graph filter [31].

4. TANGENT BUNDLE NEURAL NETWORKS

We define a layer of a Tangent Bundle Neural Network (TNN) as a bank of tangent bundle filters followed by a pointwise non-linearity. In this setting, pointwise informally means "pointwise in the ambient space". We introduce the notion of differential-preserving non-linearity to formalize this concept.

Definition 4. (Differential-preserving Non-Linearity) Denote with $U_x \subset T_x \mathbb{R}^p$ the image of the injective differential $i$ in $x$. A mapping $\sigma : L^2(T M) \to L^2(T M)$ is a differential-preserving non-linearity if it can be written as $\sigma(\mathcal{F} x) = \tilde{\sigma}(x) \circ \mathcal{F} x$, where $\tilde{\sigma} : U_x \to \mathbb{R}$ is a point-wise non-linearity in the usual (Euclidean) sense. Furthermore, we assume that $\sigma = \tilde{\sigma}$ for all $x \in M$. Thus, the $l$-th layer of a TNN with $F^l$ input signals $\{F^l_{q=1} \}_{q=1}^l$, $F_{l+1}$ output signals $\{F^l_{q=1} \}_{q=1}^{l+1}$, and point-wise non linearity $\sigma(\cdot)$ is written as

$$
F^l_{q=1+1}(x) = \sigma \left( \sum_{q=1}^l \mathcal{h}_q(\Delta)^{l-q} F^l_q(x) \right), \quad u = 1, \ldots, n_{l+1}.
$$
(12)

A TNN of depth $L$ with input signals $\{F^0\}_{q=1}^l$ is built as the stack of $L$ layers defined in (12), where $F^0_{q=1} = F^0_q$. To globally represent the TNN, we collect all the filter impulse responses in a function set $\mathcal{H} = \{h^q \}_{l=1, \ldots, q}$, and we describe the TNN $u$-th output as a mapping $F^l_u = \Psi_u(\mathcal{H}, \Delta, \{F^q\}_{q=1}^l)$ to enhance that it is parameterized by filters $\mathcal{H}$ and Connection Laplacian $\Delta$. 

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5. Discretization in Space and Time

Tangent Bundle Filters and Tangent Bundle Neural Networks operate on tangent bundle signals, thus they are continuous architectures that cannot be directly implemented in practice. Here we provide a principled way of discretizing them both in time and space domains.

Discretization in the Space Domain. The manifold $\mathcal{M}$, the tangent bundle $T\mathcal{M}$, and the Connection Laplacian $\Delta$ can be approximated starting from a set of sampled points (point-cloud). Knowing the coordinates of the sampled points, it is indeed possible to build a specific (orthogonal) cellular sheaf over an undirected geometric graph (see Appendix A.3 in [30]).

Let $\mathcal{M}$ be a cellular sheaf $T\mathcal{M}_n$ following the Vector Diffusion Maps procedure whose details are listed in [24]. In particular, we build a geometric graph $\mathcal{M}_n$, with weights for nodes $i$ and $j$ set as

$$w_{i,j} = \exp \left( \frac{||x_i - x_j||^2}{\epsilon} \right) \left[0 < ||x_i - x_j||^2 \leq \sqrt{c}\right],$$

where $\epsilon$ controls the chosen Gaussian Kernel. We then assign to each node an orthogonal transformation $O_i \in \mathbb{R}^{p \times d}$ computed via a local PCA procedure, that is an approximation of a basis of the tangent space $T_{x_i} \mathcal{M}$, where $d$ is an estimate of $d$ obtained from the same procedure. At this point, an approximation of the transport operator [29] from $T_{x_i} \mathcal{M}$ to $T_{x_j} \mathcal{M}$ is also needed. In the discrete domain, this translates to associating a matrix to each edge of the above graph (the restriction maps of the sheaf). For $\epsilon$ small enough, $T_{x_i} \mathcal{M}$ and $T_{x_j} \mathcal{M}$ are close, meaning that the column spaces of $O_i$ and $O_j$ are similar. If they were coinciding, then the matrices $O_i$ and $O_j$ would have been the same up to an orthogonal transformation $O_{i,j}$ satisfying $O_{i,j} = O_i^T O_j$. However, the subspaces are not coinciding due to curvature. For this season, the transport operator approximation $O_{i,j}$ is defined as the closest orthogonal matrix [24] to $O_{i,j}$, and it is computed as $O_{i,j} = M_{i,j} V_{i,j}^T \in \mathbb{R}^{d \times d}$, where $M_{i,j}$ and $V_{i,j}$ are the SVD of $O_{i,j} = M_{i,j} \Sigma_{i,j} V_{i,j}^T$. We now build a block matrix $S \in \mathbb{R}^{n \times n \times d}$ and a diagonal block matrix $D \in \mathbb{R}^{n \times n \times d}$ with $d \times d$ blocks defined as

$$S_{i,j} = w_{i,j} \tilde{D}_i^{-1} O_{i,j} \tilde{D}_j^{-1}, \quad D_{i,i} = \text{ndeg}(i) I_d,$$

where $\tilde{D}_i = \text{deg}(i)I_d$, $\text{deg}(i) = \sum_j w_{i,j}$ is the degree of node $i$, and $\text{ndeg}(i) = \sum_j w_{i,j}/\text{deg}(i)$.

Finally, we define the (normalized) Sheaf Laplacian as the following matrix

$$\Delta_n = \epsilon^{-1} \left( D^{-1} S - I \right) \in \mathbb{R}^{n \times n \times d},$$

which is the approximated Connection Laplacian of the discretized manifold. A sheaf $T\mathcal{M}_n$ with this (orthogonal) structure is also said to be a discrete $O(\tilde{d})$ -- bundle version and represents a discretized version of $T\mathcal{M}$. We introduce a linear sampling operator $\Omega^X_n : \mathcal{L}^2(T\mathcal{M}) \to \mathcal{L}^2(T\mathcal{M}_n)$ to discretize a tangent bundle signal $F$ as a sheaf signal $f_n \in \mathbb{R}^{n \times d}$ (a 0-cochain of the sheaf) such that

$$f_n = \Omega^X_n F,$$

$$f_n(x_i) = \left[ f_n \right]_{((i-1)d+1):(i+1)d} = O_i^T iF(x_i).$$

We are now in the position of plugging the discretized operator and signal in the definition of tangent bundle filter in (7), obtaining

$$g_n = \int_0^\infty \hat{h}(t)e^{\Delta_n t} f_n dt = h(\Delta_n) f_n \in \mathbb{R}^{n \times d}.$$
\(e^{\Delta n}\) as a sheaf shift operator. At this point, by replacing the filter \(h_{l}^{u}(\Delta n)\) in (19) with (22), we obtain the following architecture:

\[
f_{n,l+1}^{u} = \sigma \left( \sum_{q=1}^{K} \sum_{k=1}^{l} h_{l}^{u,k} (e^{\Delta n})^{k} f_{n,l}^{q} \right), \quad u = 1, ..., F_{l+1},
\]

(23)

that we refer to as discretized space-time tangent bundle neural network (DD-TNN), which can be seen as a principled variant of the recently proposed Sheaf Neural Networks [23, 27, 28], with \(e^{\Delta n}\) as (sheaf) shift operator with order \(K\) diffusion. The layer in (23) can be rewritten in matrix form by introducing the matrices \(X_{n,l} = \{f_{n,l}^{u}\}^{u=1}_{u=F_{l+1}} \in \mathbb{R}^{n \times F_{l+1}}\), and \(H_{l,k} = \{h_{l}^{u,k}\}^{u=1}_{u=F_{l+1}} \in \mathbb{R}^{F_{l+1} \times F_{l+1}}\) as

\[
X_{n,l+1} = \sigma \left( \sum_{k=1}^{K} (e^{\Delta n})^{k} X_{n,l} H_{l,k} \right) \in \mathbb{R}^{n \times F_{l+1}},
\]

(24)

where the filter weights \(\{H_{l,k}\}_{k,l}\) are learnable parameters. We have completed the process of building TNNs from cellular sheaves and back. Manifolds and their Tangent Bundles can be seen as the limits of graphs and cellular sheaves on them, making TNNs also a tool for analyzing large graphs with vector data.

6. NUMERICAL RESULTS

We assess the consistency of the proposed framework by designing a denoising task\(^{1}\). We work on the unit 2-sphere \((M = S_2)\) and its tangent bundle. In particular, we uniformly sample the sphere on \(n\) points \(\mathcal{X} = \{x_1, ..., x_n\}\), and we compute the corresponding cellular sheaf \(T_{M_{xy}}\), Sheaf Laplacian \(\Delta n\) and signal sampler \(\Omega_{n}^{\text{T}}\) as explained in Section 5 (also obtaining \(d = 2\)). We consider the tangent vector field over the sphere given by

\[
iF(x, y, z) = (y, x, 0) \in \mathbb{R}^{3},
\]

(25)

depicted in Fig. 1 for a realization of \(\mathcal{X}\) with \(n = 200\). At this point, we add AWGN with variance \(\tau^2\) to \(iF\) obtaining a noisy field \(\tilde{iF}\), then we use \(\Omega_{n}^{\text{T}}\) to sample it, obtaining \(\tilde{f}_{n} \in \mathbb{R}^{2n}\). We test the performance of the TNN architecture (implemented with a DD-TNN as in (23)) by evaluating its ability of denoising \(\tilde{f}_{n}\). We exploit a one layer architecture with 1 output feature (the denoised signal), and 5 filter taps. We train the architecture to minimize the MSE \(\frac{1}{n} \| \tilde{f}_{n} - f_{n,1} \|^{2}\) between the noisy signal \(\tilde{f}_{n}\) and the output of the network \(f_{n,1}\) via the ADAM optimizer [32], with hyperparameters set to obtain the best results. We compare our architecture with a 1-layer Manifold Neural Network (MNN) architecture (implemented via a GNN as explained in [19]), to make the comparison fair, \(\tilde{iF}\) evaluated on \(\mathcal{X}\) is given as input to the MNN, organizing it in a matrix \(F_{n} \in \mathbb{R}^{n \times 3}\). We train the MNN to minimize the MSE \(\frac{1}{n} \| \tilde{F}_{n} - F_{n,1} \|^{2}\), where

\[\| \cdot \|_{F}\text{ is the Frobenius Norm and } F_{n,1} \text{ is the network output. It is easy to see that the “two” MSEs used for TNN and MNN are completely equivalent due to the orthogonality of the projection matrices } O_{n}.\]

In Table 1 we evaluate TNNs and MNNs for two different sample sizes (\(n = 200\) and \(n = 800\)), for three different noise standard deviation (\(\tau = 10^{-2}, \tau = 5 \cdot 10^{-2}\) and \(\tau = 1 \cdot 10^{-1}\)), showing the (again equivalent) MSEs \(\frac{1}{n} \| \tilde{f}_{n} - f_{n,1} \|^{2}\) and \(\frac{1}{n} \| \tilde{F}_{n} - F_{n,1} \|^{2}\), where \(\tilde{f}_{n}\) is the sampling via \(\Omega_{n}^{\text{T}}\) of the clean field and \(F_{n,1}\) is the matrix collecting the clean field evaluated on \(\mathcal{X}\). The results are averaged over 5 sampling realizations and 5 noise realizations per each of them. As the reader can notice from Table 1, TNNs always perform better than MNNs, due to their “bundle-awareness”. Moreover, the mean performance remains stable as the number of points decreases, but the variances increase, meaning that having more sampling points (thus a better estimation of the Connection Laplacian) results in a more stable decision of the network. A real-world instance of this synthetic task could be denoising of Earth wind fields.

Table 1: MSE on the denoising task

| \(\tau = 10^{-2}\) | \(\tau = 5 \cdot 10^{-2}\) | \(\tau = 1 \cdot 10^{-1}\) |
|-----------------|-----------------|-----------------|
| \(n = 200\) DD-TNN | \(2 \cdot 10^{-4} \pm 1.6 \cdot 10^{-5}\) | \(4.9 \cdot 10^{-3} \pm 2.4 \cdot 10^{-4}\) | \(1.9 \cdot 10^{-2} \pm 1.3 \cdot 10^{-3}\) |
| MNN | \(2.9 \cdot 10^{-4} \pm 1.5 \cdot 10^{-5}\) | \(7 \cdot 10^{-3} \pm 2.8 \cdot 10^{-4}\) | \(2.9 \cdot 10^{-2} \pm 1.5 \cdot 10^{-3}\) |
| \(n = 800\) DD-TNN | \(2 \cdot 10^{-4} \pm 5.7 \cdot 10^{-6}\) | \(5 \cdot 10^{-3} \pm 1.2 \cdot 10^{-4}\) | \(1.9 \cdot 10^{-2} \pm 4.6 \cdot 10^{-4}\) |
| MNN | \(2.8 \cdot 10^{-4} \pm 8.7 \cdot 10^{-6}\) | \(7.3 \cdot 10^{-3} \pm 1.7 \cdot 10^{-4}\) | \(2.9 \cdot 10^{-2} \pm 6.9 \cdot 10^{-4}\) |

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\(1\)https://github.com/clabat9/Tangent-Bundle-Neural-Networks

Fig. 1: Visualization of the embedded tangent vector field \(\tilde{iF}\)

7. CONCLUSIONS

In this work we introduced Tangent Bundle Filters and Tangent Bundle Neural Networks (TNNs), novel continuous architectures operating on tangent bundle signals, i.e. manifold vector fields. We made TNNs implementable by discretization in space and time domains, showing that their discrete counterpart is a principled variant of Sheaf Neural Networks. The results of this preliminary work, in addition to the introduction of a novel tool for processing manifold vector fields, could lead to a deeper understanding of topological neural architectures in terms of transferability and stability, with the opportunity of designing proper signal processing frameworks on tangent bundles and cellular sheaves. We plan to investigate these problems as well as applying TNNs to real-world complex tasks.
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