1D Lieb–Liniger Bose Gas as Non-Relativistic Limit of the Sinh–Gordon Model

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The repulsive Lieb–Liniger model can be obtained as the non-relativistic limit of the Sinh–Gordon model: all physical quantities of the latter model (S-matrix, Lagrangian and operators) can be put in correspondence with those of the former. We use this mapping, together with the Thermodynamical Bethe Ansatz equations and the exact form factors of the Sinh–Gordon model, to set up a compact and general formalism for computing the expectation values of the Lieb–Liniger model both at zero and finite temperatures. The computation of one-point correlators is thoroughly detailed and when possible compared with known results in the literature.

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I. INTRODUCTION

The physics of one-dimensional interacting bosons is well captured by the Lieb–Liniger (LL) model [1]. Despite its deceptive simplicity, this model has become a paradigmatic example of quantum integrable systems since it proved to have a remarkable richness [2]: its Bethe Ansatz equations, for instance, can be explicitly derived and used to study its equilibrium properties at zero and finite temperatures [3]. The explicit analysis of the weak to strong coupling crossover of this model has also set a precise benchmark for approximate many-body techniques [1, 2]. The efforts done for the computation of the correlation functions of the LL model have also greatly stimulated the development of new and general formalisms, such as the quantum inverse scattering method [4] or the bosonization approach [5].

Nowadays a renewed interest in the LL model has been triggered by its accurate experimental realization [6–9]: in quasi one-dimensional traps, the excitations in the transverse directions are effectively frozen and, moreover, the coupling of the ultracold bosons to the external environment can be made very weak [10]. These recent experimental advances have opened new perspectives in the field of strongly correlated quantum systems: in such a highly controllable set-up it is, in fact, possible to thoroughly investigate problems of general nature concerning quantum extended systems, such as the dynamics of integrable systems in the presence of small non-integrable perturbations [11] (e.g., three-body interactions and/or a weak external trapping potential), the issue of thermalization in quantum integrable and non-integrable systems [12] and the behavior of various susceptibilities and response functions.

The key quantities to answer all these questions are the correlation functions of the LL model. Despite the integrability of the model, their explicit computation turned out to be an interesting theoretical challenge. For this reason many different approaches have been developed over the years to tackle different aspects of this difficult problem: a partial list includes bosonization (which gives the correct long-distance behavior of correlation functions) [5, 13, 14], quantum Monte Carlo simulations [15], algebraic Bethe Ansatz [16], analytical-numerical methods based on the exact Bethe Ansatz solution [17, 18], Bogoliubov weak- [19] and strong-coupling methods [20], renormalization group [21], numerical results using stochastic wave-functions [22] and imaginary time simulations [23, 24]. Exact results based on the Yang–Yang equations and the Hellmann–Feynman theorem are presently available for local two-body correlations [20, 25], while the local three-body correlations were determined at zero temperature in [26].

A new method was proposed recently to compute expectation values in the LL model [27]: it exploits a different route from all the previous approaches for it is based on an exact mapping between the non-relativistic LL model and the relativistic integrable Sinh–Gordon (sh-G) model. This proposal not only provides a remarkable simplification of the problem but applies equally well both at zero and finite temperatures. In a nutshell, the logical steps on which the method is based are the following:

1. Due to the relativistic invariance and quantum integrability of the sh-G model, it is possible to set up functional equations [28, 29] for the matrix elements of its local operators on the asymptotic states – known as form factors – and to find their exact solutions [30, 31].

2. The finite temperature and finite density effects of the sh-G model can also be controlled by solving the Thermodynamical Bethe Ansatz (TBA) equations [32, 33].
3. In a proper non-relativistic limit of the sh-G model, all quantities of this theory — \( S \)-matrix, Lagrangian, form factors, TBA equations and so on — reduce to those of the LL model and therefore can be used to establish an explicit mapping between the two models. In particular, from the exact and known expressions of the form factors of the sh-G model we can explicitly obtain the matrix elements of the operators of the LL model we are interested in.

4. To actually compute the LL correlation functions we have to take into account another aspect of the problem, that is, that the LL correlation functions refer to the ground state of the gas at a finite density and at a finite temperature, while those of the sh-G model refer to the vacuum state (i.e. the state without any particles). This apparent difficulty can be, however, readily overcome by using the LeClair–Mussardo formalism \[34\] that, as a matter of fact, is based on the same quantities mentioned above, i.e., the form factors and the TBA equations.

In this paper we provide the details of the mapping between the sh-G and the LL models first established in \[27\], presenting additional results and focusing values. The paper is organized as follows: in Section II we introduce the LL and sh-G models and we recall their TBA equations. In Section III we discuss the form factors of the sh-G model and their correspondence with the operators of the theory. In Section IV we explore in detail the non-relativistic regime of the sh-G model and we explain how the LL model emerges in this limit: the mapping between the two models is discussed at the level of their \( S \)-matrices, Lagrangians and TBA equations. In Section V we describe the method of calculating local correlation functions in the LL model, providing a detailed derivation of the main result, formula (100). Then we present the explicit computation of LL one-point correlators at zero and finite temperatures, together with their comparison with known exact and approximate results available in literature. Our conclusion and outlooks are given in Section VI. Supplementary material is presented in the Appendices, where we also list the non-relativistic limit of the sh-G form factors for the first few cases.

II. THE MODELS

In this section we recall the main properties of the LL and sh-G models and we also discuss the expressions of their free energy and \( T = 0 \) ground-state energy obtained by the TBA equations.

A. Lieb–Liniger Hamiltonian

The LL Hamiltonian describes \( N \) non-relativistic bosons of mass \( m \) in one dimension, interacting via a two-body repulsive \( \delta \)-potential:

\[
H = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2\lambda \sum_{i<j} \delta(x_i - x_j).
\]  

(1)

For cold atomic gases the quantity \( \lambda > 0 \) in the Hamiltonian \[1\] can be determined in terms of the parameters of the three-dimensional Bose gas in the quasi-one-dimensional limit \[10\]. The effective coupling constant of the LL model is given by the dimensionless quantity

\[
\gamma = \frac{2m\lambda}{\hbar^2 n},
\]

(2)

where \( n = N/L \) is the density of the gas (\( L \) is the length of the system). The limit \( \gamma \ll 1 \) is the weak coupling limit and in this regime it is known that the Bogoliubov approximation — obtained by linearizing the Gross–Pitaevskii equation — gives a good estimate of the ground-state energy of the system \[1\]. For large \( \gamma \) one approaches the Tonks–Girardeau limit \[35\] and recently it has become an interesting question, also from the experimental point of view, to study the crossover between the two regimes \[2\]. In the LL model temperatures are usually expressed in units of the quantum degeneracy temperature

\[
k_B T_D = \frac{\hbar^2 n^2}{2m},
\]

(3)

in the following we use the scaled temperature \( \tau = T/T_D \).
In second quantized formalism the non-relativistic field theory describing bosons interacting via a δ-potential is defined by the Hamiltonian

$$
\mathcal{H} = \int dx \left( \frac{\hbar^2}{2m} \frac{\partial \psi^\dagger}{\partial x} \frac{\partial \psi}{\partial x} + \lambda \psi^\dagger \psi \right),
$$

where the complex Bose field \( \psi(x, t) \) satisfies the canonical commutation relations

$$
[\psi(x, t), \psi^\dagger(x', t)] = \delta(x - x'), \quad [\psi(x, t), \psi(x', t)] = 0.
$$

The Lagrangian density associated to the field theory Hamiltonian \( \mathcal{H} \) is

$$
\mathcal{L} = i \frac{\hbar}{2} \left( \psi^\dagger \frac{\partial \psi}{\partial t} - \frac{\partial \psi^\dagger}{\partial t} \psi \right) - \frac{\hbar^2}{2m} \frac{\partial \psi^\dagger}{\partial x} \frac{\partial \psi}{\partial x} - \lambda \psi^\dagger \psi \psi.
$$

Restricting to the subspace of the Hilbert space where the number of particles \( N \) is fixed, the equation of motion for the field \( \psi \) translates into the eigenvalue equation of the LL many-body Hamiltonian [1]:

$$
H \chi_N(x_1, \ldots, x_N) = E_N \chi_N(x_1, \ldots, x_N).
$$

As shown by Lieb and Liniger in their original paper [1], the eigenvalue problem (7) can be solved in terms of a Bethe Ansatz. Let us recall the main steps because they will lead us to the definition of the \( S \)-matrix of the LL model, a basic quantity of its dynamics and of our following discussion. One can easily see that Eq. (7) is just a free Schrödinger equation in the domain where the coordinates of the particles are all distinct. However, we have to enforce the usual boundary conditions for a δ-function potential, that is, the discontinuity of the gradient of the wave function when two coordinates coincide

$$
\frac{\hbar^2}{2m} \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k} \right) \chi_N|_{x_j = x_k + \epsilon} = \lambda \chi_N|_{x_j = x_k}.
$$

If we denote by \( R_1 \) the subset of the configuration space where \( x_1 < x_2 < \cdots < x_N \), the solution of the equations in \( R_1 \) is given by the Bethe wave function

$$
\chi_N(x_1, x_2, \ldots, x_N) = \sum_P a(P) e^{i \sum_{j = 1}^N P(k_j)x_j},
$$

where \( \sum_P \) denotes a sum over permutations of the momenta \( \{k_1, \ldots, k_n\} \) that characterize the state. For configurations outside \( R_1 \) the solution is easily obtained using the symmetry of \( \chi_N \) with respect to the \( x_i \). The coefficients in the sum are related by the boundary conditions [5]. For the permutations \( P : (k, l, k_{\alpha_1}, \ldots, k_{\alpha_N}) \) and \( Q : (l, k, k_{\alpha_1}, \ldots, k_{\alpha_N}) \) the relation between the corresponding coefficients is

$$
a(Q) = \frac{k - l - i \frac{2m}{\hbar} \lambda}{k - l + i \frac{2m}{\hbar} \lambda} a(P).
$$

Hence the wave function gets multiplied by the factor \( a(Q)/a(P) \) whenever two particles with momenta \( p_1 = k \) and \( p_2 = l \) are exchanged. This exchange is equivalent to a scattering process of the two particles and therefore the two-body \( S \)-matrix of the Lieb–Liniger model is expressed by

$$
S_{\text{LL}}(p, \lambda) = \frac{p - i \frac{2m}{\hbar} \lambda}{p + i \frac{2m}{\hbar} \lambda},
$$

where \( p = p_1 - p_2 \) is the momentum difference.

Given the integrability of the model, all its physical properties can be essentially derived from its two-body \( S \)-matrix [11]. This quantity, for instance, determines the thermodynamics of the model, as shown originally by Yang and Yang [3]. In the limit \( N \to \infty \), \( L \to \infty \) with the density \( n \) fixed, the discrete energy levels of the system get encoded in an energy level density function \( \tilde{\rho}(p) \) and in the density \( \tilde{\rho}^{(1)}(p) \) of the occupied levels. Notice that we are going to use a tilde “ for the quantities in the non-relativistic LL TBA, while the corresponding quantities in the TBA for the sh-G model will be later denoted without this tilde. The ratio between the two densities \( \tilde{\rho} \) and \( \tilde{\rho}^{(1)} \) defines the pseudo-energy \( \tilde{\varepsilon}(p) \) through the relation

$$
\frac{\tilde{\rho}(p)}{\tilde{\rho}^{(1)}(p)} = 1 + e^{\tilde{\varepsilon}(p)},
$$

(12)
and this quantity, together with the densities, satisfies the coupled set of integral equations

\[ 2\pi \tilde{\rho}(p) = \frac{1}{\hbar} + \int_{-\infty}^{\infty} dp' \tilde{\phi}(p - p') \tilde{\rho}^{(r)}(p') , \]  

\[ \tilde{\varepsilon}(p) = -\frac{\tilde{\mu}}{k_B T} + \frac{p^2}{2mk_B T} - \int_{-\infty}^{\infty} dp' \frac{1}{2\pi} \tilde{\phi}(p - p') \log \left( 1 + e^{-\tilde{\varepsilon}(p')} \right) , \]  

\[ n = \int_{-\infty}^{\infty} \tilde{\rho}^{(r)}(p) dp , \]

where \( \tilde{\mu} \) is the chemical potential, \( T \) is the temperature and \( k_B \) is the Boltzmann constant. The kernel \( \tilde{\phi}(p) \) that drives all the integral equations entirely follows from the S-matrix \[ \tilde{\phi}(p) = -i \frac{\partial}{\partial p} \log S_{LL}(p) = \frac{4\hbar m \lambda}{\hbar^2 p^2 + 4m^2 \lambda^2} . \]

Once the TBA integral equations (13) are solved, the ground state energy \( \tilde{E} \) and the free energy \( \tilde{F} \) of the system are expressed as

\[ \frac{\tilde{E}}{\mathcal{L}} = \int_{-\infty}^{\infty} dp \frac{p^2}{2m} \tilde{\rho}^{(r)}(p) , \]  

\[ \frac{\tilde{F}}{\mathcal{L}} = \tilde{\mu} n - \frac{k_B T}{2\pi \hbar} \int_{-\infty}^{\infty} dp \log \left( 1 + e^{-\tilde{\varepsilon}(p)} \right) . \]

At zero temperature the energy level density gets a compact support, i.e. it is different from zero only on an interval (which we denote by \([-B, B]\)) and, correspondingly, the TBA equations simplify as

\[ 2\pi \tilde{\rho}^{(r)}(p) = \frac{1}{\hbar} + \int_{-B}^{B} dp' \tilde{\phi}(p - p') \tilde{\rho}^{(r)}(p') , \]  

\[ \tilde{\varepsilon}_0(p) = -\tilde{\mu} + \frac{p^2}{2m} + \int_{-B}^{B} dp' \frac{1}{2\pi} \tilde{\phi}(p - p') \tilde{\varepsilon}_0(p') , \]

where \( \tilde{\varepsilon}_0(p) = \lim_{T \to 0} k_B T \tilde{\varepsilon}(p) \) and the boundary value \( B \) is determined by the normalization condition

\[ n = \int_{-B}^{B} \tilde{\rho}^{(r)}(p) dp . \]

The TBA equations of the sh-G model will be described in the next section where the physical meaning of the pseudo-energy will also be discussed. In Section \[ IV C \] we show that the TBA equations and the excitation spectrum of the LL model can be obtained in a proper limit from those of the sh-G model.

### B. The Sinh–Gordon model

The sh-G model is an integrable relativistically invariant field theory in 1 + 1 dimensions defined by the Lagrangian density

\[ \mathcal{L} = \frac{1}{2} \left( \frac{\partial \phi}{c \partial t} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 - \frac{m^2 c^2}{g^2 \hbar^2} (\cosh(g\phi) - 1) , \]
where $\phi = \phi(x, t)$ is a real scalar field, $m_0$ is a mass scale and $c$ is the speed of light. The parameter $m_0$ is related to the physical (renormalized) mass $M$ of the particle by

$$m_0^2 = M^2 \frac{\pi \alpha}{\sin(\pi \alpha)}.$$  \hspace{1cm} (19)

The explicit presence of the speed of light $c$ will help us in studying later the non-relativistic limit of this theory (see Section IV). Despite the relativistic nature of the sh-G model, its integrability (supported by the existence of an infinite number of conservation laws) implies the absence of particle production processes and that its $n$-particle scattering amplitudes are purely elastic. Moreover, they factorize into $n(n - 1)/2$ two-body $S$-matrices. The energy $E$ and the momentum $P$ of a particle can be written as

$$E = M c^2 \cosh \theta,$$

$$P = M c \sinh \theta,$$

where $\theta$ is the rapidity. In terms of the particle rapidities, the two-body $S$-matrix is given by [37]:

$$S_{\text{sh-G}}(\theta, \alpha) = \frac{\sinh \theta - i \sin(\alpha \pi)}{\sinh \theta + i \sin(\alpha \pi)},$$  \hspace{1cm} (20)

where $\theta$ is the rapidity difference and $\alpha$ is the dimensionless renormalized coupling constant

$$\alpha = \frac{\hbar c g^2}{8 \pi + \hbar c g^2}.$$  \hspace{1cm} (21)

As in the LL model, the two-body $S$-matrix of the sh-G model fully encodes its physical properties, in particular its thermodynamics. Its derivation is quite similar to the one of the LL model, the only difference being the relativistic kinematics [32, 33]. Let us briefly discuss the TBA equations of the sh-G model both at finite temperature and at finite particle density $n$. Note initially that, although the sh-G model is a relativistic theory, its quantum integrability implies the conservation of the number of particles and therefore it makes sense to associate a wave-function to the $N$-particle state. Therefore, the starting point of the TBA approach is the quantization of the rapidities of the $N$-particle state on an interval $L$ with periodic boundary conditions, given by

$$McL \sinh \theta_i + \hbar \sum_{j \neq i}^N \chi(\theta_i - \theta_j) = 2 \pi N_i \hbar,$$  \hspace{1cm} (22)

where the $N_i$ are (positive or negative) integers and $\chi(\theta) = -i \log S_{\text{sh-G}}(\theta)$ is the phase shift of the two-body scattering process of the sh-G model. In the thermodynamic limit $(N \to \infty, L \to \infty, N/L = n = \text{fixed})$ we introduce for the left hand side of Eqs. (22) the quantity

$$J(\theta) = M c \sinh \theta + 2 \pi \hbar \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \chi(\theta - \theta') \rho^{(i)}(\theta'),$$  \hspace{1cm} (23)

and by differentiating it equations (22) turn into the integral equation

$$\rho(\theta) = \frac{M c}{2 \pi \hbar} \cosh \theta + \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \varphi(\theta - \theta') \rho^{(i)}(\theta').$$  \hspace{1cm} (24)

In the equations above

$$\rho(\theta) = \frac{1}{2 \pi \hbar} \frac{\partial}{\partial \theta} J(\theta)$$  \hspace{1cm} (25)

is the density of states, $\rho^{(i)}(\theta)$ is the density of the occupied states (both per unit length) and $\varphi(\theta)$ is the derivative of the phase shift

$$\varphi(\theta) = \frac{\partial \chi(\theta)}{\partial \theta} = -i \frac{\partial}{\partial \theta} \log S_{\text{sh-G}}(\theta).$$  \hspace{1cm} (26)

The pseudo-energy $\varepsilon(\theta)$ is introduced as in [12]

$$\frac{\rho(\theta)}{\rho^{(i)}(\theta)} = 1 + e^{\varepsilon(\theta)}.$$  \hspace{1cm} (27)
By minimizing the free energy

\[ F[\rho, \rho^{(r)}] = E - TS = \]

\[ L \int M \cosh \theta \rho^{(r)}(\theta) d\theta - LT \int \left[ \rho \log \rho - \rho^{(r)} \log \rho^{(r)} - (\rho - \rho^{(r)}) \log(\rho - \rho^{(r)}) \right] d\theta \quad (28) \]

with respect to the densities \( \rho(\theta) \) and \( \rho^{(r)}(\theta) \), with the constraint \( \rho \), one arrives at the TBA equation for the pseudo-energy

\[ \varepsilon(\theta) = \frac{M c^2}{k_B T} \cosh \theta - \frac{\mu}{k_B T} - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \varphi(\theta - \theta') \log \left( 1 + e^{-\varepsilon(\theta')} \right). \quad (29) \]

Once this integral equation for \( \varepsilon(\theta) \) has been solved, the densities \( \rho(\theta) \) and \( \rho^{(r)}(\theta) \) are extracted from equations \( \rf{24} \) and \( \rf{27} \) while the chemical potential is fixed by the constraint

\[ n = \int_{-\infty}^{\infty} d\theta \rho^{(r)}(\theta). \quad (30) \]

Any other thermodynamic quantity can be calculated from the free energy (per unit length) of the system determined from the above minimum principle and finally expressed by the formula

\[ f = \frac{F}{L} = -\frac{k_B T}{2\pi \hbar} \int_{-\infty}^{\infty} d\theta M c \cosh \theta \log \left( 1 + e^{-\varepsilon(\theta)} \right) + \mu n. \quad (31) \]

Let us consider now the \( T \to 0 \) limit of the TBA equations. Due to the non-zero chemical potential, we can assume that the function \( \varepsilon(\theta) \) changes sign at the rapidity values \(-\theta^* \) and \( \theta^* \) so that it is negative on the interval \( I \equiv (-\theta^*, \theta^*) \), zero at the limiting points and positive everywhere else. For \( T \to 0 \), \( \varepsilon(\theta) \) becomes largely negative on the interval \( I \) and largely positive outside. Then from Eq. \( \rf{27} \) we see that on the interval \( (-\theta^*, \theta^*) \) we have \( \rho^{(r)} = \rho \) (and a filling fraction equal to 1), while outside \( \rho^{(r)} = 0 \) (and a filling fraction equal to 0). The boundary value \( \theta^* \) is determined by the condition

\[ n = \int_{-\theta^*}^{\theta^*} \rho^{(r)}(p) dp. \quad (32) \]

For \(-\theta^* < \theta < \theta^* \) equations \( \rf{24} \) and \( \rf{29} \) become

\[ \rho^{(r)}(\theta) = \frac{M c}{2\pi \hbar} \cosh \theta + \int_{-\theta^*}^{\theta^*} \frac{d\theta'}{2\pi} \varphi(\theta - \theta') \rho^{(r)}(\theta'), \quad (33a) \]

\[ \varepsilon_0(\theta) = M c^2 \cosh \theta - \mu + \int_{-\theta^*}^{\theta^*} \frac{d\theta'}{2\pi} \varphi(\theta - \theta') \varepsilon_0(\theta'), \quad (33b) \]

where \( \varepsilon_0(\theta) = \lim_{T \to 0} k_B T \varepsilon(\theta) \).

Let us pause here to comment on the physical meaning of \( \varepsilon(\theta) \). If we raise one of the numbers \( \mathcal{N}_i \) in the Bethe equation that corresponds to a rapidity \( \theta_i \) to a larger value \( \mathcal{N}'_i \), which will correspond to a rapidity \( \theta' \), then the change in the energy of the system is

\[ \Delta E = k_B T (\varepsilon(\theta') - \varepsilon(\theta)). \quad (34) \]

This shows that \( \varepsilon(\theta) \) describes the energy of the excitations over the ground state, and it gives the dressed energy of the quasi-particles (hence the name pseudo-energy). If we demand that the ratio of the densities take the usual form

\[ \frac{\rho^{(r)}(\theta)}{\rho(\theta)} = \frac{1}{e^{(E(\theta) - \mu)/k_B T} + 1} \]

then the excitation energy is fixed to be

\[ E(\theta) = k_B T \varepsilon(\theta) + \mu. \quad (36) \]
III. FORM FACTOR EXPANSION FOR ONE-POINT CORRELATORS

Expectation values in an integrable relativistic field theory are conveniently expressed in terms of the so-called form factors.

The form factors we are going to use in this article are the ones related to a quantum field theory, which in principle are different from the quantities that share the same name in the Bethe ansatz solution of integrable models. In the latter context they are defined as matrix elements of operators between exact Bethe ansatz states, while here we use instead the basis in the Hilbert space consisting of multiparticle scattering states. However, these quantities turn out to be closely related [38]. In this section, for the sake of completeness, we first review the definition and the main properties of the form factors in a general relativistic field theory. Since our final goal is to treat the sh-G model which is a theory with a single type of gapped excitations and multi-particle states without any bound states, we focus our attention on theories of this type. (A well-known example of a theory with different types of excitations and bound states is the Sine–Gordon model [39].) Later we present the explicit expressions of the form factors of the sh-G model.

A. Basic properties of form factors

Consider a local operator $O(x,t)$. Using the translation operator $U = e^{-ip\cdot x}$, where $t = x^0$ and $x = x^1$, we can always shift this operator to the origin $O(x,t) = U^\dagger O(0,0)U$. The matrix elements of $O(0,0)$ between the vacuum and a set of $n$-particle asymptotic states are called the $n$-particle form factors of this operator (see Fig. 1)

$$F_n^{O}(\theta_1, \theta_2, \ldots, \theta_n) = \langle 0 | O(0,0) | \theta_1, \ldots, \theta_n \rangle. \quad (37)$$

For an operator of spin $s$, relativistic invariance implies that under a simultaneous shift in the rapidities $\theta_{ij} = \theta_i - \theta_j$. A generic matrix element of the operator $O(0,0)$ can be expressed in terms of its form factors by using the translation operator and the crossing symmetry, which is implemented by an analytic continuation in the rapidity variables

$$\langle \theta_1, \ldots, \theta_n | O(0,0) | \beta_1, \ldots, \beta_m \rangle = e^{s\Lambda} F_n^{O}(\theta_1, \ldots, \theta_n). \quad (38)$$

This equation indicates that the form factors of a scalar operator depend only on the differences of rapidities, $\theta_{ij} = \theta_i - \theta_j$. A generic matrix element of the operator $O(0,0)$ can be expressed in terms of its form factors by using the translation operator and the crossing symmetry, which is implemented by an analytic continuation in the rapidity variables

$$\langle \theta_1, \ldots, \theta_n | O(0,0) | \beta_1, \ldots, \beta_m \rangle = F_n^{O}(\theta_1 + \Lambda, \ldots, \theta_n + \Lambda) = e^{s\Lambda} F_n^{O}(\theta_1, \ldots, \theta_n). \quad (38)$$

(If $\beta_i = \theta_j$ for some $i$ and $j$, this formula gets modified by contact terms.) Hence, the knowledge of all form factors of an operator is equivalent to the knowledge of the operator itself, (i.e., how it acts on any state of the theory).

The form factors satisfy a set of functional and recursive equations, which for integrable models makes it possible to find in many cases their explicit expressions (for a review, see [28, 30]). For a scalar operator the functional equations (known as Watson equations [29]) come from unitarity and crossing symmetry and their explicit expressions are

$$F_n^{O}(\theta_1, \ldots, \theta_i, \theta_{i+1}, \ldots, \theta_n) = S(\theta_i - \theta_{i+1}) F_n^{O}(\theta_1, \ldots, \theta_{i+1}, \theta_i, \ldots, \theta_n), \quad (40a)$$

$$F_n^{O}(\theta_1 + 2\pi i, \ldots, \theta_n) = \prod_{i=2}^{n} S(\theta_i - \theta_1) F_n^{O}(\theta_1, \ldots, \theta_n). \quad (40b)$$
Their graphical representations are given in Fig. 2.

The recursive equations, on the other hand, come from the pole structure of one-particle intermediate states. The form factors of integrable theories have, in general, two kinds of simple poles in the strip $0 < \text{Im } \theta_{ij} < 2\pi$ and except for these singularities they are analytic in the strip. The first kind of poles corresponds to kinematical singularities at $\theta_{ij} = i\pi$ and their residues give rise to a set of recursive equations between the $n$-particle and the $n+2$-particle form factors (see Fig. 3)

$$-i \text{Res}_{\theta = \theta} F_{n+2}(\theta + i\pi, \theta, \theta_1, \ldots, \theta_n) = \left(1 - \prod_{i=1}^{n} S(\theta - \theta_i)\right) F_n(\theta_1, \ldots, \theta_n).$$

The second kind of poles is instead related to the bound states of the theory. Since there are no bound states in the sh-G model, there are no such poles in the form factors of this theory and we do not need to write here the corresponding residue equations.

The general solution of the Watson equations (40) can be written as

$$F_n(\theta_1, \ldots, \theta_n) = K_n(\theta_1, \ldots, \theta_n) \prod_{i<j} F_{\text{min}}(\theta_{ij}),$$

where the factors $K_n(\theta_1, \ldots, \theta_n)$ are completely symmetric and $2\pi i$-periodic functions in all $\theta_i$, and $F_{\text{min}}(\theta)$ is an analytic function in $0 \leq \text{Im } \theta \leq \pi$ (without zeros and poles in this strip) which tends to a constant value for large values of $\theta$. This function satisfies the equations

$$F_{\text{min}}(\theta) = S(\theta) F_{\text{min}}(-\theta),$$

$$F_{\text{min}}(i\pi - \theta) = F_{\text{min}}(i\pi + \theta),$$

and its role is to take care of the monodromy properties of the form factors as ruled by the Watson equations. The equations (43) and the analyticity requirement of $F_{\text{min}}$ are able to fix this function up to normalization, as we will see explicitly for the sh-G model. The factors $K_n(\theta_1, \ldots, \theta_n)$ in (42) must contain all the expected kinematical poles. In addition, they must fulfill the proper asymptotic behavior. This yields the following final parametrization of the generic $n$-particle form factor

$$F_n(\theta_1, \ldots, \theta_n) = H_n Q_n(x_1, \ldots, x_n) \prod_{i<j} \frac{F_{\text{min}}(\theta_{ij})}{x_i + x_j}.$$
where $H_n$ is a normalization factor, $x_i = e^{\theta}$, and $Q_n(x_1, \ldots, x_n)$ is a symmetric polynomial. In view of Eq. (38), the polynomial $Q_n$ of a scalar operator has the total degree equal to the degree of the polynomial $\prod_{i<j}(x_i + x_j)$ in the denominator, $n(n-1)/2$. The actual expression of the polynomials $Q_n$ can be determined by solving the recursive equations (44). To this aim, it is convenient to make use of the basis given by the elementary symmetric polynomials $\sigma_k^{(n)}$ of the $n$ variables $x_i$ defined by

$$\prod_{i=1}^{n}(x + x_i) = \sum_{k=1}^{n} x^{n-k} \sigma_k^{(n)}(x_1, \ldots, x_n),$$

or explicitly

$$\sigma_k^{(n)} = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}.$$  \hspace{1cm} (45)

It is worth remarking that there is a one-to-one correspondence between the infinite set of form factors $\{F_n(\theta_1, \ldots, \theta_n), \ n \in \mathbb{N}\}$ which are the solutions of the functional and recursive equations, and the operator content of a relativistic field theory (see [11]). Let us see how this correspondence is realized in the sh-G model by solving the form factor equations of this theory.

**B. Form factors of the Sinh–Gordon model**

For the sh-G model the minimal form factor $F_{\min}(\theta)$, solution of equations (43), is [30]

$$F_{\min}(\theta) = N \exp \left\{ \int_{0}^{\infty} \frac{dt}{t} \sinh \left( \frac{\theta}{2} \right) \sinh \left( \frac{\theta(1-\alpha)}{2} \right) \sin^2 \left( \frac{t\theta}{2\pi} \right) \right\},$$

where $\hat{\theta} = i\pi - \theta$ and the normalization constant $N = F_{\min}(i\pi)$ is chosen to be

$$N = \exp \left\{ \int_{0}^{\infty} \frac{dt}{\sinh(t) \cosh \left( \frac{\theta}{2} \right)} \right\} = \frac{1}{\cos \left( \frac{\pi \alpha}{2} \right)} \exp \left\{ -\frac{1}{\pi} \int_{0}^{\pi \alpha} \frac{dt}{\sin(t)} \right\}.$$  \hspace{1cm} (46)

In addition to the functional equations (43), $F_{\min}(\theta)$ also satisfies

$$F_{\min}(i\pi + \theta)F_{\min}(\theta) = \frac{\sinh \theta}{\sinh \theta + \sinh(i\pi \alpha)}. \hspace{1cm} (47)$$

With the choice

$$H_{2n+1} = H_1 \left( \frac{4 \sin(\pi \alpha)}{N} \right)^n,$$

$$H_{2n} = H_2 \left( \frac{4 \sin(\pi \alpha)}{N} \right)^{n-1},$$

the recursive equations for the polynomials $Q_n$ entering (44) can be written as

$$(-1)^n Q_{n+2}(-x, x, x_1, \ldots, x_n) = x D_n(x; x_1, \ldots, x_n) Q_n(x_1, \ldots, x_n),$$

where the functions $D_n$ are given by

$$D_n(x; x_1, \ldots, x_n) = \sum_{k=1}^{n} \sum_{m=1,3,5,\ldots}^{k} (-1)^{(k+1)[m]} x^{2(n-k)+m} \sigma_k^{(n)} \sigma_{k-m}^{(n)}.$$  \hspace{1cm} (48)

In this formula $\sigma_k^{(n)}$ are the elementary symmetric polynomials (46) while

$$[k] = \frac{\sin(k \pi \alpha)}{\sin(\pi \alpha)}.$$  \hspace{1cm} (49)
As shown in [31], a solution of the recursive equations (49) is given by the class of symmetric polynomials

\[ Q_n(k) = \det M_n(k) , \]

where \( M_n(k) \) is a \((n - 1) \times (n - 1)\) matrix with elements

\[ [M_n(k)]_{ij} = \sigma_{2i-j}^{(n)} [i - j + k] . \]

The corresponding form factors are the matrix elements of a continuous family of operators identified with the exponential fields \( e^{k g \phi} \) [31, 42]. With the normalization given by \( H_n(k) = \left( \frac{4 \sin(\pi \alpha)}{N} \right)^{n/2} [k] \), the explicit form of all form factors of these operators is then

\[ F_n(k) = \langle 0 | e^{k g \phi} | \theta_1, \theta_2, \ldots, \theta_n \rangle = [k] \left( \frac{4 \sin(\pi \alpha)}{N} \right)^{\frac{n}{2}} \det M_n(k) \prod_{i<j}^{n} \frac{F_{\min} (\theta_i - \theta_j)}{x_i + x_j} . \]

So, for instance, the one and two-particle form factors are given by

\[ \langle 0 | e^{k g \phi} | \theta \rangle = \frac{2}{\sqrt{N}} \frac{\sin(k \pi \alpha)}{\sqrt{\sin(\pi \alpha)}} , \]

\[ \langle 0 | e^{k g \phi} | \theta_1, \theta_2 \rangle = \frac{4}{N} \frac{\sin^2 (k \pi \alpha)}{\sin(\pi \alpha)} F_{\min} (\theta_1 - \theta_2) . \]

From now on we will concentrate our attention on the form factors of the even powers of the field \( \phi \) because we will need only these operators for the future computation of the expectation values of the LL model (only even operators can have non-zero expectation values). It is useful to express the operator content of the theory in terms of a class of particular operators, denoted by \( : \phi^k : \), which start creating \( n \) particles out of the vacuum only when \( n \geq k \):

\[ F_{: \phi^k :} (\theta_1, \ldots, \theta_n) = 0 \quad \text{if} \quad n < k . \]

In their form factors for \( n = k \) the polynomial term \( Q_{2k}(x_1 \ldots, x_{2k}) \) is equal to the polynomial \( \prod_{i<j}^{2k} (x_i + x_j) \) of the denominator and they cancel each other, giving

\[ F_{: \phi^k :} (\theta_1, \ldots, \theta_k) = 2^k k! \left( \frac{\pi^2 \alpha^2}{N g^2 \sin(\pi \alpha)} \right)^{\frac{k}{2}} \prod_{i<j}^{k} F_{\min} (\theta_{ij}) . \]

In view of the recursive equations (41), the absence of kinematical poles in \( F_{: \phi^k :} (\theta_1, \ldots, \theta_k) \) obviously implies the vanishing values [57]. To compute the form factors of these operators when \( n > k \), we can take advantage of the knowledge of the form factors (54) of the exponential operators. Let us denote by \( \phi^m \) the operator whose form factors \( \tilde{F}^m \) are obtained by extracting the \( \mathcal{O}(k^m) \) term in the expansion of \( F_n(k) \). In view of equations (57) and (58) we have

\[ \tilde{F}^m_n = F_{: \phi^k :} + \sum_{l=2,4, \ldots}^{k-2} A_l^k F_{: \phi^l :} , \]

which implies a mixing among the operators \( : \phi^k : \)

\[ \tilde{\phi}^2 = : \phi^2 : , \]

\[ \tilde{\phi}^4 = : \phi^4 : + A_2^4 : \phi^2 : , \]

\[ \vdots \]

\[ \tilde{\phi}^k = : \phi^k : + \sum_{l=2,4, \ldots}^{k-2} A_l^k : \phi^l : . \]

In Appendix A we discuss how to compute iteratively the coefficients \( A_l^k \).
C. LeClair–Mussardo formalism

At equilibrium the expectation value of a local operator $O(x,t)$ at temperature $T$ and at finite density $n$ is given by

$$\langle O \rangle_{T,n} = \frac{\text{Tr} \left( e^{-\frac{H_{\pi N}}{\pi T}} O \right)}{\text{Tr} \left( e^{-\frac{H_{\pi N}}{\pi T}} \right)}. \quad (61)$$

For translation invariance at equilibrium $\langle O \rangle_{T,n}$ is independent of $x$ and $t$. If we specify this formula to an integrable quantum field theory and we use the basis of multiparticle scattering states, we have

$$\langle O \rangle_{T,n} = \frac{1}{Z_{T,n}} \sum_{k=0}^{\infty} \frac{1}{k!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_k}{2\pi} \left( \prod_{i=1}^{k} e^{-\frac{M \sinh \theta_i - \mu}{\pi T}} \right) \langle \theta_k, \ldots, \theta_1 | O(0,0) | \theta_1, \ldots, \theta_k \rangle, \quad (62)$$

where $Z_{T,n} = \text{Tr} \left( e^{-\frac{H_{\pi N}}{\pi T}} \right)$. As shown in [34], this expression can be neatly written as

$$\langle O \rangle_{T,n} = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_k}{2\pi} \left( \prod_{i=1}^{k} \frac{1}{1 + e^{i\varepsilon(\theta_i)}} \right) \langle \theta_k, \ldots, \theta_1 | O(0,0) | \theta_1, \ldots, \theta_k \rangle_{\text{conn}}, \quad (63)$$

where $\varepsilon(\theta)$ is the pseudo-energy, solution of the Thermodynamical Bethe Ansatz equation of the model of interest, while the connected form factor is defined as [43]

$$\langle \theta_k, \ldots, \theta_1 | O'_{\theta_1}, \ldots, O'_{\theta_k} \rangle_{\text{conn}} = \mathcal{F} \left( \lim_{\eta_1/\eta_2 \to 0} \langle 0 | O | \theta_{1,\pi}, \ldots, \theta_{k,\pi}, \theta_k - i\pi + i\eta, \ldots, \theta_1 - i\pi + i\eta \rangle \right), \quad (64)$$

where $\mathcal{F}$ in front of the expression means taking its finite part, that is, omitting all the terms of the form $\eta/\eta_2$ where $p$ is a positive integer. In Appendix B we give an explicit example for the calculation of the connected limit. In this formulation the $\mu$- and $T$-dependence of the right hand side of (63) is hidden in $\varepsilon(\theta)$ that, for the sh-G model, satisfies the $\mu$- and $T$-dependent equation (29). Expression (63) was checked in various cases [44, 45] and was compared with the direct evaluation of the expectation value (61) using finite volume regularization [46].

Notice that for the sh-G model, in view of the functional relation (48) the connected limit (64) for the product of the $F_{\text{min}}(\theta_{ij})$ in the form factors of $O$ simply becomes

$$\prod_{i<j}^{2k} F_{\text{min}}(\theta_{ij}) \rightarrow (F_{\text{min}}(i\pi))^k \prod_{i<j}^{k} \frac{\sinh \theta_{ij}}{\sinh \theta_{ij} + \sinh(i\pi \alpha)} \frac{\sinh \theta_{ji}}{\sinh \theta_{ji} + \sinh(i\pi \alpha)} = \Lambda^k \prod_{i<j}^{k} \frac{\sinh^2 \theta_{ij}}{\sinh^2 \theta_{ij} + \sinh^2(i\pi \alpha)}, \quad (65)$$

This means that it is not necessary to employ the explicit form (47) of $F_{\text{min}}(\theta)$ to calculate the connected form factors and, for their actual determination, we only have to take the connected limit of the rest of the form factor formula. In particular, the connected form factor $F_{2k,\text{conn}}^{2^k}$ can be calculated from the explicit formula [58] for $F_{2k}^{2^k}$. Since it only depends on the rapidities through the $F_{\text{min}}$ factors, using (65) we can immediately write down the connected form factor:

$$F_{2k,\text{conn}}^{2^k} = 2^{2k}(2k)! \left( \frac{\pi^2 \alpha^2}{g^2 \sin(\pi \alpha)} \right)^k \prod_{i<j}^{k} \frac{\sinh^2 \theta_{ij}}{\sinh^2 \theta_{ij} + \sinh^2(i\pi \alpha)}. \quad (66)$$

IV. THE DOUBLE LIMIT OF THE SINH–GORDON MODEL

In this section we analyze in detail the mapping between the sh-G and the LL models. We show that it is possible to obtain the LL model from the sh-G model by taking the non-relativistic limit simultaneously with the limit $g \to 0$, where $g$ is the sh-G coupling constant. In particular, we show how this mapping is realized at the level of the $S$-matrix, the Lagrangian densities and the Thermodynamical Bethe Ansatz equations.
A. Double limit of the two-particle S-matrix

Let us consider the exact S-matrix of the sh-G model

\[ S_{sh-G}(\theta, \alpha) = \frac{\sinh \theta - i \sin(\alpha \pi)}{\sinh \theta + i \sin(\alpha \pi)}, \]  

and let us take its non-relativistic limit accompanied by a simultaneous limit of the coupling constant \( g \) toward smaller values such that

\[ c \to \infty, \quad g c = \text{fixed}. \]  

The resulting expression

\[ S(\theta, \alpha) \to \frac{p \hbar}{M c} - \frac{i \hbar g^2 c}{\frac{p}{M c} + \frac{\hbar}{2} g^2 c}, \]  

coincides with the LL S-matrix \(^{11}\) once we set the sh-G and LL masses equal, \( M = m \), and

\[ \lambda \equiv \frac{\hbar^2 c^2}{16} g^2. \]  

Hence the S-matrices of the two models coincide in this double limit. It is worth noticing that the resulting coupling \( \lambda \) of the LL model does not need to be small and therefore we shall be able to study the LL model at arbitrarily large values of its coupling. To use this correspondence between the two models to calculate correlation functions in the LL model, we need to establish the relation between the operators of these two theories. For this reason in the next section we show how to perform the limit \(^{68}\) on the fields and the Hamiltonians.

B. Non-relativistic limit at the Lagrangian level

Consider the sh-G Lagrangian density

\[ \mathcal{L} = \frac{1}{2} \left( \frac{\partial \phi}{c \partial t} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 - m_0^2 c^2 \left( \cosh(g \phi) - 1 \right). \]  

To study its non-relativistic limit, it is convenient to write initially the real scalar field in the form \(^{47–50}\)

\[ \psi(x, t) = \sqrt{\frac{\hbar}{2m_0}} \left( e^{-i \frac{m_0 c^2}{\hbar} t} + i \frac{m_0 c^2}{\hbar} \right) \]  

Substituting this expression into the Lagrangian \(^{11}\) and taking the limit \( c \to \infty \), we can discard all the oscillating terms, that is, terms containing factors \( e^{\pm \frac{m_0 c^2}{\hbar} t} \) with \( n \) non-vanishing positive or negative integers. These terms, in fact, oscillate very rapidly in this limit and average to zero when integrated over any small but finite time interval. In more detail, the relativistic canonical momentum can be written as

\[ \Pi(x, t) = \frac{1}{c^2} \phi(x, t) = \]  

\[ \sqrt{\frac{\hbar^2}{2m_0 c^2}} \left[ \left( \psi(x, t) - i \frac{m_0 c^2}{\hbar} \psi(x, t) \right) e^{-i \frac{m_0 c^2}{\hbar} t} + \left( \psi^\dagger(x, t) + i \frac{m_0 c^2}{\hbar} \psi^\dagger(x, t) \right) e^{i \frac{m_0 c^2}{\hbar} t} \right] = \]  

\[ = -i \sqrt{\frac{m_0}{2}} \left( \psi(x, t) e^{-i \frac{m_0 c^2}{\hbar} t} - \psi^\dagger(x, t) e^{i \frac{m_0 c^2}{\hbar} t} \right) + \mathcal{O} \left( \frac{1}{c^2} \right). \]  

This allows us to express \( \psi \) and \( \psi^\dagger \) in terms of \( \phi \) and \( \Pi \) (up to order \( \mathcal{O} \left( \frac{1}{c^2} \right) \)):

\[ \psi(x, t) = e^{i \frac{m_0 c^2}{\hbar} t} \left( \frac{1}{\sqrt{2m_0}} \phi(x, t) + i \frac{\sqrt{2m_0}}{\hbar} \Pi(x, t) \right), \]  

\[ \psi^\dagger(x, t) = e^{-i \frac{m_0 c^2}{\hbar} t} \left( \frac{1}{\sqrt{2m_0}} \phi(x, t) - i \frac{\sqrt{2m_0}}{\hbar} \Pi(x, t) \right). \]
It is easy to show that the commutation relation
\[ [\phi(x, t), \Pi(x', t)] = i\hbar \delta(x - x') \] (75)
implies the following commutation relation for the non-relativistic operators
\[ [\psi(x, t), \psi^\dagger(x', t)] = \delta(x - x') . \] (76)

Turning to the Lagrangian density, the kinetic term \( K \) of (71) becomes
\[ K \rightarrow \frac{\hbar^2}{2m_0c^2} \frac{\partial \psi^\dagger \partial \psi}{\partial t} - \frac{\hbar^2}{2m_0} \nabla \psi^\dagger \nabla \psi + \frac{i}{2} \left( \psi^\dagger \frac{\partial \psi}{\partial t} - \frac{\partial \psi^\dagger}{\partial t} \psi \right) + \frac{1}{2} m_0c^2 \psi^\dagger \psi . \] (77)

Expanding the formula (19) for \( m_0 \) in the combined limit (70) we obtain
\[ m_0^2 = M^2 + \frac{2M^2\lambda^2}{3\hbar^2c^2} + \mathcal{O}\left(\frac{1}{c^4}\right) . \] (78)

From the second term of (77) we see again that the physical masses in the two models should be equal
\( m = M \) and then the first term is of order \( 1/c^2 \) and can be dropped in the limit.

Let us now turn our attention to the interaction term \( \cosh(g\phi) \) in the Lagrangian (71), which is equivalent to an infinite series in terms of even powers of the field \( \phi \). Expressing \( \phi \) in terms of the new fields \( \psi \) and \( \psi^\dagger \), we can use the binomial formula to expand each power \( \phi^{2k} \) in terms of these fields. Taking into account that the oscillating terms should be dropped, only the symmetric “middle term” of the binomial expansion survives from each power. Collecting the combinatorial factors from the different expansions of the powers, we arrive at the following series:

\[ U(\phi) = \frac{m_0^2c^2}{g^2\hbar^2} (\cosh(g\phi) - 1) = \frac{m_0^2c^2}{g^2\hbar^2} \sum_{n=1}^{\infty} \frac{1}{(2n)!} (g\phi)^{2n} \rightarrow \] \[ \rightarrow \frac{m_0^2c^2}{g^2\hbar^2} \sum_{n=1}^{\infty} \frac{1}{(2n)!} \left( \frac{\hbar^2}{2m_0} \right)^n (2n)! g^{2n} \psi^\dagger \psi^n = \sum_{n=1}^{\infty} \frac{1}{(n!)^2} g^{2n} \frac{m_0^2c^2}{2m_0} (\hbar^2 g^2)^n \psi^\dagger \psi^n . \] (79)

The \( n = 1 \) term of the series,
\[ \frac{mc^2}{2} \psi^\dagger \psi , \] (80)
exactly cancels the last term of (77). The \( n = 2 \) term becomes
\[ \frac{\hbar^2c^2g^2}{16} \psi^\dagger \psi^\dagger \psi \psi^\dagger \rightarrow \lambda \psi^\dagger \psi^\dagger \psi^\dagger \psi^\dagger , \] (81)
which is just the interaction term in the LL Lagrangian. The rest of the series can be organized as
\[ \sum_{n=3}^{\infty} \left[ \frac{1}{2n(n!)^2} m_0^{n-2} (\hbar^2 g^2)^{n-1} \psi^\dagger \psi^n + \ldots \right] = \sum_{n=3}^{\infty} \frac{2^{3n-4}}{(n!)^2} \frac{\lambda^{n-1}}{(mc^2)^{n-2}} \psi^\dagger \psi^n + \ldots , \] (82)
where the dots indicate possible higher order terms in \( 1/c \). If we now take the limit (68) all the terms in this series vanish because \( \lambda \) is fixed while \( c \rightarrow \infty \).

In summary, in the double scaling limit the Lagrangian density (71) of the sh-G model becomes the Lagrangian density (6) of the LL model
\[ \mathcal{L} \rightarrow \mathcal{L}' = -\frac{\hbar^2}{2m} \nabla \psi^\dagger \nabla \psi + \frac{i}{2} \left( \psi^\dagger \frac{\partial \psi}{\partial t} - \frac{\partial \psi^\dagger}{\partial t} \psi \right) - \lambda \psi^\dagger \psi^\dagger \psi \psi . \] (83)
So by keeping the coefficient of the \( \psi^\dagger \) term fixed, which is the actual constraint enforced by the double limit (68), all the higher order terms go to zero and we are left with the non-relativistic LL Hamiltonian.
C. The non-relativistic limit of the Sinh–Gordon TBA equations

To study the non-relativistic limit of the TBA equations of Section II B it is convenient to make the coordinate change (using from now that \( m = M \))

\[
p = mc \sinh \theta, \quad dp = mc \cosh \theta \, d\theta.
\]

Using

\[
\int_{-\infty}^{\infty} d\theta \rho^{(\pm)}(\theta) = \frac{N}{L} = \int_{-\infty}^{\infty} dp \tilde{\rho}^{(\pm)}(p),
\]

this implies

\[
\tilde{\rho}^{(\pm)}(p) = \frac{1}{mc \cosh(\theta(p))} \rho^{(\pm)}(\theta(p)) \approx \frac{1}{mc} \rho^{(\pm)}\left(\frac{p}{mc}\right)
\]

and

\[
\tilde{\rho}(p) \approx \frac{1}{mc} \rho\left(\frac{p}{mc}\right).
\]

For the sh-G model

\[
\chi(\theta) = -2 \arctan\left(\frac{\sin(\alpha \pi)}{\sinh(\theta)}\right) \implies \varphi(\theta) = \frac{2 \sin(\alpha \pi) \cosh(\theta)}{\sinh^{2}(\theta) + \sin^{2}(\alpha \pi)},
\]

and in the double limit the kernel \( \varphi(\theta) \) becomes

\[
\varphi(\theta) \rightarrow mc \varphi(p) = mc \frac{4\hbar m \lambda}{\hbar^2 p^2 + 4m^2 \lambda^2}.
\]

Therefore the TBA equations transform into equations

\[
\begin{align*}
2\pi \tilde{\rho}(p) &= \frac{1}{\hbar} + \int_{-\infty}^{\infty} dp' \tilde{\varphi}(p - p') \tilde{\rho}^{(\pm)}(p') , \\
\tilde{\varepsilon}(p) &= -\frac{\tilde{\mu}}{k_B T} + \frac{p^2}{2mk_B T} - \int_{-\infty}^{\infty} dp' \tilde{\varphi}(p - p') \log \left(1 + e^{-\tilde{\varepsilon}(p')}\right),
\end{align*}
\]

and \( \tilde{\rho}/\rho^{(\pm)} = 1 + e^{\tilde{\varepsilon}} \), where

\[
\tilde{\varepsilon}(p) = \varepsilon\left(\frac{p}{mc}\right), \\
\tilde{\mu} = \mu - mc^2.
\]

Observe that it is correct to take the small \( p \) and \( \theta \) limit in the integrands even though the integrals are extended to arbitrarily large momenta: as a matter of fact, the integrals have a finite support because their integrands vanish asymptotically very fast. The expressions for the energies become

\[
\begin{align*}
\frac{\tilde{E}}{L} &= \frac{E - N mc^2}{L} = \int_{-\infty}^{\infty} dp \frac{p^2}{2m} \tilde{\rho}^{(\pm)}(p), \\
\frac{\tilde{F}}{L} &= \frac{F - N mc^2}{L} = \tilde{\mu} n - \frac{k_B T}{2\pi \hbar} \int_{-\infty}^{\infty} dp \log \left(1 + e^{-\varepsilon(p)}\right),
\end{align*}
\]
which coincide with equations (15). Similarly, the limit of the $T = 0$ equations (33) is given by

$$2\pi \hat{\rho}^{(c)}(p) = \frac{1}{\hbar} + \int_{-B}^{B} dp' \hat{\varphi}(p - p') \hat{\rho}^{(c)}(p'),$$

(94a)

$$\hat{\varepsilon}_0(p) = -\bar{\mu} + \frac{p^2}{2m} + \int_{-B}^{B} \frac{dp'}{2\pi} \hat{\varphi}(p - p') \hat{\varepsilon}_0(p'),$$

(94b)

once again in agreement with equations (16).

We saw at the end of Section II B that the pseudo-energy describes the dressed energy of the excitations of the system, which is given by

$$E(\theta) = k_B T \varepsilon(\theta) + \mu$$

(95)

for the sh-G model and by

$$\tilde{E}(\theta) = k_B T \tilde{\varepsilon}(p) + \tilde{\mu}$$

(96)

for the LL model. It is worth observing the different behaviors of the excitation energies in the two models. As it can be seen from the TBA equation (29), the sh-G energy has a gap $\tilde{M}$ that implies that the correlation functions decay exponentially. On the contrary, the LL excitation energy starts as $p^2/2m$ for small momenta (see Eq. (90b)), implying a power-law decay for the correlation functions. Our double limit takes care of this difference automatically and thus it will give correct results for the LL model.

V. LOCAL CORRELATORS FOR THE LIEB–LINIGER MODEL

In this section we calculate LL one-point correlation functions at fixed particle density $n$ and temperature $T$ by applying the formulas of Sections II and III, in connection with the double limit presented in Section IV B. The fields are taken at the same position and time: since our system is taken at equilibrium and translationally invariant, their correlators are obviously space and time independent. We focus our attention on the local $k$-particle correlation functions $g_k$ defined as

$$\langle \psi^\dagger_k \psi^k \rangle = n^k g_k(\gamma, \tau),$$

(97)

where $\gamma$ and $\tau$ are given in (2) and (3). These local correlators play an important role in experiments with ultracold bosons since the pair correlations are responsible for the rates of inelastic collisional processes. Furthermore, the low-temperature recombination rate for a Bose gas is proportional to the local three-body correlation function [51] and measurements of the three-body recombination rate can be used to determine the local correlations and as a tool for distinguishing condensed and non-condensed phases [52]. For $g_2(\gamma, \tau)$ and $g_3(\gamma, \tau = 0)$ exact results are available [20, 25, 26], whereas for the others the asymptotic behavior of the correlators in the regimes of small and large coupling or temperature was computed in [20, 25]. We will present a comparison with the exact and the approximate results present in the literature, showing the improvement that our method brings in the computation of these correlators.

Let us start with the correspondence between the sh-G and the LL operators

$$\langle \hat{\phi}^{2k} \rangle \longrightarrow \left(\frac{\hbar^2}{2m}\right)^k \frac{2k}{k} \langle \psi^\dagger_k \psi^k \rangle,$$

(98)

which can be established along the limit procedure described in Section IV B. To compute the expectation values of these operators at finite density $n$ of the Bose gas and at finite temperature, we need to employ the results of Section III so that

$$\langle \psi^\dagger_k \psi^k \rangle = \lim_{T \to 0} \left(\frac{z}{k}\right)^{-1} \left(\frac{\hbar^2}{2m}\right)^{-k} \sum_{l=k}^{\infty} \frac{1}{l!} \int_{-\pi}^{\pi} d\theta_1 \cdots \int_{-\pi}^{\pi} d\theta_l \ f(\theta_1) \cdots f(\theta_l) \ F_{2l}(\theta_1, \ldots, \theta_l)_{\text{conn}},$$

(99)

where $f(\theta) = 1/(1 + e^{\varepsilon(\theta)})$ are the filling fractions and the notation “lim” denotes the double limit (68). Note that in Eq. (99) the terms with $l < k$ are zero and therefore the first non-zero term in the series is
a \( k \)-fold integral. Now, similarly to what happens for the TBA equations, the filling fractions effectively cut off the integrands at large values of the rapidities, so we can exchange the order of the limit and the integrals, arriving at a fully non-relativistic formula

\[
\langle \psi^k \phi^k \rangle = \left( \frac{2k}{k} \right)^{-1} \left( \frac{h^2}{2m} \right)^{-k} \sum_{l=k}^{\infty} \frac{1}{l!} \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} f(p_1) \cdots \int_{-\infty}^{\infty} \frac{dp_l}{2\pi} f(p_l) \tilde{F}_{2l;\phi^{2k}; (p_1, \ldots, p_l)_{\text{conn}}} .
\]

(100)

Here \( f(p) = 1/(1 + e^{\tilde{z}(p)} ) \) where \( \tilde{z}(p) \) is the solution of the non-relativistic TBA equations \([90, 85]\) and

\[
\tilde{F}_{2l;\phi^{2k}; (p_i)_{\text{conn}}} = \lim_{\{\theta_i = \frac{p_i}{mc}\}_{\text{conn}}} \left( \frac{1}{mc} \right)^l \tilde{F}_{2l;\phi^{2k}; (p_i)_{\text{conn}}} .
\]

(101)

are the double limit of the connected form factors. We go through the steps of the calculation of a specific form factor and we list the explicit expressions of the first few of them in Appendix B.

A first check of the validity of formula (100) is provided by the correlator \( \langle \psi^1(x,t) \psi(x,t) \rangle \). With the explicit connected form factors of :\( \phi^2 : \) the first terms read

\[
\langle \psi^1 \psi \rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi} f(p) \frac{1}{\hbar} + \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} \int_{-\infty}^{\infty} \frac{dp_2}{2\pi} f(p_1) f(p_2) \frac{1}{\hbar} \tilde{\varphi}(p_{12}) \nonumber
\]

\[
+ \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} \int_{-\infty}^{\infty} \frac{dp_2}{2\pi} \int_{-\infty}^{\infty} \frac{dp_3}{2\pi} f(p_1) f(p_2) f(p_3) \frac{1}{\hbar} \tilde{\varphi}(p_{12}) \tilde{\varphi}(p_{23}) + \ldots ,
\]

(102)

where we use the notation \( p_{ij} = p_i - p_j \) and \( \tilde{\varphi}(p) \) is the scattering phase shift in the Lieb–Liniger model \([14]\). The pattern in (102) persists for the further multiple integrals and one easily recognizes that the right hand side of (102) is nothing else but the recursive expansion of

\[
n = \int_{-\infty}^{\infty} dp \tilde{\rho}^{(i)}(p),
\]

(103)

where \( \tilde{\rho}^{(i)}(p) \) is the iterative solution of the integral equation \([90a]\)

\[
f^{-1}(p) \tilde{\rho}^{(i)}(p) = \frac{1}{2\pi \hbar} + \int_{-\infty}^{\infty} \frac{dp'}{2\pi} \tilde{\varphi}(p - p') \tilde{\rho}^{(i)}(p').
\]

(104)

In this way we successfully recover the identity \( \langle \psi^1 \psi \rangle = n \). While the result may appear obvious, it is worth stressing that it was obtained by taking the double scaling limit of the sh-G form factor expansion and employing the LeClair–Mussardo formalism, so it provides an important check of the method.

A. Correlators at \( T = 0 \)

At zero temperature, similarly to the TBA equations, the formula (100) for the expectation values admits a simpler expression

\[
\langle \psi^k \phi^k \rangle = \left( \frac{2k}{k} \right)^{-1} \left( \frac{h^2}{2m} \right)^{-k} \sum_{l=k}^{B} \frac{1}{l!} \int_{-B}^{B} \frac{dp_1}{2\pi} \cdots \int_{-B}^{B} \frac{dp_l}{2\pi} \tilde{F}_{2l;\phi^{2k}; (p_1, \ldots, p_l)_{\text{conn}}} ,
\]

(105)

where \( B \) is the Fermi momentum determined by the set of TBA equations \([94]\) together with

\[
n = \int_{-B}^{B} \tilde{\rho}^{(i)}(p) dp .
\]

(106)
The equations become more transparent by introducing the dimensionless quantities

\[ k \equiv \frac{p}{B}, \quad \nu(k) \equiv \hbar\rho^{(r)}(Bk), \quad \beta \equiv \frac{2m}{B} \frac{\lambda}{\hbar} = \frac{\ln \gamma}{B}, \]

where in the last expression we used the definition of the LL parameter \( \gamma \) of Eq. (2). In terms of these new variables, equations (106,94a) become

\[
1 = \frac{\gamma}{\beta} \int_{-1}^{1} \nu(k) \, dk, \tag{108a}
\]

\[
\nu(k) = \frac{1}{2\pi} + \int_{-1}^{1} \frac{dk'}{2\pi} \frac{2\beta}{(k-k')^2 + \beta^2} \nu(k'), \tag{108b}
\]

while the (non-relativistic) ground state energy is given by

\[
\tilde{E} = \int_{-B}^{B} dp \tilde{\rho}^{(r)}(p) p^2 = \frac{\hbar^2}{2m} n^3 \left( \frac{\gamma}{\beta} \right) \int_{-1}^{1} dk \nu(k) k^2 = \frac{\hbar^2}{2m} n^3 e(\gamma). \tag{109}
\]

The strong coupling expansion of (108a) is obtained by plugging into it the iterative solution of (108b) and expanding the integrals in \( \beta^{-1} \)

\[
1 = \left( \frac{\gamma}{\beta} \right) \left\{ \frac{1}{\pi} + \frac{2}{\pi^2} \beta^{-1} \left( 1 - \frac{2}{3} \beta^{-2} + \frac{16}{15} \beta^{-4} + \ldots \right) \\
+ \frac{4}{\pi^3} \beta^{-2} \left( 1 - \frac{4}{3} \beta^{-2} + \frac{8}{3} \beta^{-4} + \ldots \right) + \frac{8}{\pi^4} \beta^{-3} \left( 1 - 2 \beta^{-2} + \ldots \right) + \ldots \right\}. \tag{110}
\]

This provides a series expansion relation between \( \beta \) and \( \gamma \)

\[
\gamma = \pi \beta - 2 + \frac{4}{3\beta^2} + \ldots \quad \Leftrightarrow \quad \beta = \frac{1}{\pi} \left( \gamma + 2 - \frac{4\pi^2}{3\gamma^2} + \ldots \right), \tag{111}
\]

which is equivalent to a Fermi momentum

\[
B = \hbar n \pi \left( 1 - \frac{2}{\gamma} + \frac{4}{3\gamma^2} + \ldots \right). \tag{112}
\]

Using the formulas above, we can now derive the leading order behavior in \( \gamma^{-1} \) of the general correlator \( g_k(\gamma) \). It is easy to see that the leading order comes from the first non-zero integral in the series (105) where the integrand is the double limit of the connected form factor (66). Taking the double limit, for the first term of \( \langle \psi^k \psi^k \rangle \) we get

\[
\langle \psi^k \psi^k \rangle = \left( \frac{\hbar^2}{2m} \right)^k \frac{2k}{k!} \int_{-B}^{B} \frac{dp_1}{2\pi} \ldots \int_{-B}^{B} \frac{dp_k}{2\pi} \frac{2^{2k}(2k)!}{(2\pi)^k} \left( \frac{\hbar}{8m} \right)^k \prod_{i<j} \frac{\hbar^2 p_{ij}^2}{2m^2 \lambda^2} + \ldots \\
= \left( \frac{B}{\hbar} \right)^k \frac{k!}{(2\pi)^k} \prod_{i<j} \frac{1}{k_{ij}^2 + \beta^2} + \cdots \tag{113}
\]

For the leading order behavior of these quantities, we need to keep only the leading order term \( O(\beta^{-n(n-1)}) \) of the integrand and substitute from (112) and (111) \( B = \hbar n \pi \) and \( \beta = \gamma/\pi \). The final result is

\[
g_k = \frac{k!}{2^k} \left( \frac{\pi}{\gamma} \right)^{k(k-1)} I_n + \ldots, \tag{114}
\]
where

\[ I_n = \int_{-1}^{1} dk_1 \ldots \int_{-1}^{1} dk_k \prod_{i<j} k_{ij}^2. \] (115)

So, in this way we recover the expression obtained in [20] by using completely different methods. Let us now discuss in more detail the results for \( g_1 \), \( g_2 \) and \( g_3 \).

1. Expectation value \( g_1 \)

We already demonstrated that our series expansion sums up to the exact value \( g_1 = 1 \), here we show how convergent the series is. The actual computation consists of the following steps. First we solve numerically the integral equation (108b), second we obtain the \( \beta(\gamma) \) function from (108a), and finally, we integrate numerically the dimensionless forms of the integrals on the right hand side of (105).

Carrying out these steps for \( g_1 = \langle \psi^{\dagger} \psi \rangle / n \), we obtain the plot shown in Fig. 4 for \( g_1 \) as a function of \( \gamma \). The series (105) is nicely saturated by the first few terms for sufficiently large values of \( \gamma \) (one should keep in mind that \( \gamma = 0 \) is a singular point of the LL model and therefore one cannot expect \textit{a priori} any fast convergence nearby). In fact, this plot shows that the exact value \( g_1 = 1 \) is rapidly approached by just the first terms of (105). It is clear that including more terms in the series, (i.e., employing higher particle form factors) extends the fast convergence toward smaller values of \( \gamma \). Notice, however, that the convergence of the series is always remarkably fast for all \( \gamma \geq 1.5 \), where the exact value is obtained within a 5% accuracy just using its first four terms.

2. Expectation value \( g_2(\gamma) \)

Let us continue our discussion with the calculation of the correlation function \( \langle \psi^{\dagger} \psi^{\dagger} \psi \psi \rangle = n^2 g_2 \). This correlator is the expectation value of the interaction term in the Lieb–Liniger Hamiltonian (4), thus it can be exactly determined [20, 25] via the Hellmann–Feynman theorem [53].

\[
\langle \psi^{\dagger} \psi^{\dagger} \psi \psi \rangle = \frac{1}{L} \left( \frac{dH}{d\lambda} \right) = \frac{d}{d\lambda} \left( \frac{\tilde{E}}{L} \right) \quad \Rightarrow \quad g_2 = \frac{de(\gamma)}{d\gamma},
\] (116)

where \( e(\gamma) \) was defined in (109). Our result can be compared with this expression, providing a good possibility to check again the correctness of the approach. We have

\[
\langle \psi^{\dagger} \psi^{\dagger} \psi \psi \rangle = \int_{-B}^{B} \frac{dp_1}{2\pi} \int_{-B}^{B} \frac{dp_2}{2\pi} \frac{1}{2m\lambda h} \tilde{\varphi}(p_{12})p_{12}^2 + \int_{-B}^{B} \frac{dp_1}{2\pi} \int_{-B}^{B} \frac{dp_2}{2\pi} \int_{-B}^{B} \frac{dp_3}{2\pi} \frac{1}{2m\lambda h} \tilde{\varphi}(p_{12})\tilde{\varphi}(p_{23})p_{13}^2 + \ldots
\] (117)
FIG. 5: Plot of $g_2$ as a function of $\gamma$ at $T = 0$ using form factors up to $n = 4, 6$ and $8$ particles, respectively with green dot-dashed, blue dashed and red dotted lines. The exact value is given by the solid line whereas the dot-dot-dashed line below, indicated by the arrow, corresponds to the strong coupling expansion (119).

In terms of the dimensionless quantities the expansion in $\beta$ up to the four-integral term gives

$$\langle \psi^\dagger \psi^\dagger \psi \psi \rangle = n^2 \frac{\gamma^2}{\beta^2} \left\{ \frac{4}{3\pi^2} \beta^{-2} \left( 1 - \frac{8}{5} \beta^{-2} + \frac{24}{7} \beta^{-4} + \cdots \right) + \frac{8}{3\pi^2} \beta^{-3} \left( 1 - \frac{8}{5} \beta^{-2} + \frac{332}{105} \beta^{-4} + \cdots \right) + \frac{16}{3\pi^2} \beta^{-4} \left( 1 - \frac{34}{15} \beta^{-2} + \cdots \right) + \cdots \right\}. \ (118)$$

Substituting the relation (111) we obtain the strong coupling expansion

$$g_2 = \frac{4 \pi^2}{3 \gamma^2} \left( 1 - \frac{6}{\gamma} + (24 - \frac{8}{5} \pi^2) \frac{1}{\gamma^2} \right) + O(\gamma^{-5}). \ (119)$$

The leading order behavior agrees with (114) but it is worth noticing that we also obtained subleading terms in $\gamma^{-1}$. We notice that this result can be obtained using the Hellmann–Feynman theorem and the expansion of the ground-state energy given in [54].

The plot of $g_2(\gamma)$ – obtained by numerical integration and using the integral equations (108) for $\beta(\gamma)$ – is drawn in Fig. 5. As for the previous example, we see that by increasing the number of form factors employed in our series, our result rapidly converges to the exact value. The discrepancy between the exact value and the one obtained with four integrals is less than 3% for $\gamma > 2$. Expression (119) is also plotted in Fig. 5 to show that the determination of $g_2$ (at finite $\gamma$) obtained from the first terms of Eq. (105) is much closer to the exact result. This is because we solve the TBA equations (108) with an arbitrary precision and every term of our series contains infinitely many powers of $\gamma$.

3. Expectation value $g_3(\gamma)$

As a final example let us discuss $g_3$, a quantity known exactly up to now only at $T = 0$ [26]. An analysis similar to the previous cases reveals that

$$\langle \psi^\dagger \psi^\dagger \psi \psi \psi \rangle = n^3 \frac{\gamma^3}{\beta^3} \left\{ \frac{16}{15\pi^3} \beta^{-6} \left( 1 - \frac{144}{35} \beta^{-2} + \cdots \right) + \frac{32}{15\pi^4} \beta^{-7} (1 + \cdots) + \cdots \right\}. \ (120)$$

Trading $\beta$ for $\gamma$ using (111) we arrive at

$$g_3 = \frac{16 \pi^6}{15 \gamma^6} \left( 1 - \frac{16}{\gamma} \right) + O(\gamma^{-8}). \ (121)$$

Here the leading order term is the asymptotic result (114), but as in the previous example, we also obtained the next order in the large $\gamma$ expansion.
The logarithmic plot of $g_3$ using the form factor expansion up to $n = 6$ and 8 particles (one or two terms from the series) is shown in Fig. 6 together with the exact result of [25]. As in the previous examples, this plot shows a nice convergent pattern toward the exact value. The leading order (114) in the large $\gamma$ expansion is also plotted in Fig. 6 to show that in this domain of $\gamma$ this result largely differs from the exact value. The subleading term, given in (121) provides an improvement for larger $\gamma$, however, the result is still quite far from the one obtained with our method.

B. Correlators at finite temperature

To obtain the expectation value at finite values of the temperature $T$ we have to employ the formula (100) which contains non-trivial filling fractions and we need to solve the whole set of TBA equations (85-90). It proves to be useful to introduce now a different set of dimensionless quantities

$$q \equiv \frac{p}{nh\gamma}, \quad \alpha \equiv \frac{\bar{\mu}}{k_BT}, \quad g(q) \equiv \frac{\hbar}{\alpha} \tilde{\rho}(nh\gamma q),$$

which satisfy

$$\tilde{\varepsilon}(q) = -\alpha + \frac{q^2\gamma^2}{\tau} - \int_{-\infty}^{\infty} \frac{dq'}{2\pi} \frac{2}{(q-q')^2 + 1} \log \left( 1 + e^{-\tilde{\varepsilon}(q')} \right),$$

$$g(q) = \frac{1}{2\pi\alpha} + \int_{-\infty}^{\infty} dq' \frac{2}{(q-q')^2 + 1} \frac{g(q')}{1 + e^{\tilde{\varepsilon}(q')}},$$

and

$$\frac{1}{\alpha\gamma} = \int_{-\infty}^{\infty} \frac{g(q)}{1 + e^{\tilde{\varepsilon}(q)}} dq.$$
FIG. 7: Deviations $1 - g_1$ from the exact result $(g_1 = 1)$ as a function of the scaled temperature $\tau$ for a fixed value of $\gamma = 7$. Inset: $1 - g_1$ vs $\gamma$ at $\tau = 1$. In both figures form factors are used up to $n = 4$ (green dot-dashed), $6$ (blue dashed) and $8$ (red dotted) particles.

large while the first one implies that $\alpha$ is a large negative number. This means that, even for $q$ close to zero, the convolution term is small and the leading order $\bar{\varepsilon}(q)$ is given by

$$\bar{\varepsilon}(q) = -\alpha + \frac{q^2 \gamma^2}{\tau},$$

so we can make an expansion in the small parameter $\exp(-\alpha + q^2 \gamma^2 / \tau)$ [95]. At the leading order $g(q) = 1/(2\pi \alpha)$ and this implies

$$\int_{-\infty}^{\infty} dq \frac{d}{2\pi} e^{-\frac{q^2 \gamma^2}{\tau}} = \frac{1}{\gamma},$$

so we arrive at the $\gamma$-independent result

$$e^\alpha = \sqrt{\frac{4\pi}{\tau}}.$$  

We see that for $\tau \gg 1$ $\alpha$ is indeed a large negative quantity.

Due to the condition $\gamma \gg 1$, we can again restrict ourselves to the large $\gamma$ limit of the first non-zero term in the series (100). Substituting the Boltzmann filling fraction $f(\theta) = e^{-\bar{\varepsilon}}$ given above, we arrive at

$$g_k(\gamma, \tau) = \left( \frac{\tau}{\gamma^2} \right)^{\frac{k(k-1)}{2}} J_k,$$  

where

$$J_k = \frac{k!}{\pi^{k/2}} \int dx_1 \ldots dx_k e^{-\sum_{i=1}^k x_i^2} \prod_{i<j}^k (x_i - x_j)^2 = \frac{B_k}{2^{k(k-1)/2}}$$

with $B_{k+1} = (k+1)\Gamma(k+2)B_k$, $B_1 = 1$. This is exactly the result found in [20]. However, as one can check numerically, the filling fraction comes close to a Boltzmann distribution only for such extreme parameter values as $\gamma \sim 1000$ and $\tau \sim 10000$. In Figs. 8 and 9 one can see how large the difference is between this leading order approximation and our result for $\tau = 10$. The improvement achieved in the determination of this quantity by the method proposed in this paper may have an important experimental relevance.

Let us turn now to the numerical results obtained by exactly solving the TBA equations (123), substituting $\bar{\varepsilon}$ in the formula (100) and numerically integrating the first terms in the series. We consider separately the computation of $g_1$, $g_2$ and $g_3$. 

1. Expectation value \( g_1 \)

To test the reliability of our expansion at finite temperature we computed the deviations from the exact result for the trivial expectation value \( g_1 = 1 \). The results are plotted in Fig. 7, showing that the precision does not decrease with increasing temperature: even using only three terms of the expansion (i.e. summing up to \( n = 6 \) particles) the error is \( \lesssim 1\% \) in the range of temperature between \( \tau = 0 \) and \( \tau = 15 \). In the inset of Fig. 7 we plot the deviation from the exact result as a function of the LL parameter \( \gamma \) at a fixed temperature.

2. Expectation value \( g_2(\gamma, \tau) \)

For \( T > 0 \) the Hellmann–Feynman theorem gives

\[
\langle \psi^\dagger \psi^\dagger \psi \psi \rangle = \frac{d}{d\lambda} \left( \frac{\tilde{F}}{L} \right),
\]

where the free energy can be calculated from the TBA approach (123b). In dimensionless variables

\[
g_2(\gamma, \tau) = \tau \frac{d}{d\gamma} \left( \alpha - \gamma \int_{-\infty}^{\infty} \frac{dq}{2\pi} \log(1 + e^{-\tilde{\epsilon}(q)}) \right).
\]

We derive now a simpler expression for this. The trick is to substitute for \( 1/2\pi \) under the integral the rest of Eq. (123b), then using the associativity of the convolution by an even function and finally use the derivative of Eq. (123a) with respect to \( \gamma \). Many terms drop out and we are left with

\[
g_2 = 2\gamma^2 \int_{-\infty}^{\infty} dq \frac{\alpha g(q)}{1 + e^{\tilde{\epsilon}(q)}} q^2 - \tau \int_{-\infty}^{\infty} \frac{dq}{2\pi} \log(1 + e^{-\tilde{\epsilon}(q)}).
\]

The advantage of this expression is that it is enough to solve the TBA equations for the value of \( \gamma \) we are interested in instead of evaluating the free energy for several \( \gamma \) and then differentiating it numerically.

Our evaluations of \( g_2 \) at \( \tau = 1 \) and \( \tau = 10 \) based on the form factor expansion are shown in Fig. 8 together with the exact result (131) and the leading order result (127). The convergence of our series is basically as good as it was for \( T = 0 \) and it is clear that the asymptotic formula fails, especially for \( \gamma < 10 \).
3. Expectation value $g_3(\gamma, \tau)$

In the case of $g_2$ and of $g_3$ at $T = 0$ we checked our results using exact formulae. We learned that on the one hand our result for $g_3$ reaches the same accuracy as $g_2$ for slightly higher values of $\gamma$, on the other hand going to finite temperature does not spoil the precision of our results. Thus we are confident about the reliability of our results for $g_3$ at $T > 0$, at least for not too small values of $\gamma$. We emphasize that neither exact results nor approximations of precision comparable to ours exist in this case, which renders our evaluation a new result.

Fig. 9 shows $g_3$ as a function of $\gamma$ at fixed temperatures $\tau = 1$ and $\tau = 10$. In the figure the asymptotic result (127), valid for large temperature and coupling, is also plotted: one can see that even for $\tau = 10$ the result (127) does not give the exact asymptotic behavior (which is only reached for very large values of the scaled temperature $\tau$). Fig. 10 shows instead $g_3$ as a function of $\tau$ at fixed values of $\gamma = 7$ and $\gamma = 15$. The asymptotic formula (127) is different from our result by a factor of $\sim 10$ at $\gamma = 7$ and $\tau = 10$.

VI. CONCLUSIONS

In this paper we have shown that the Lieb–Liniger model, describing one-dimensional interacting bosons, can be obtained as a non-relativistic limit of an integrable relativistic field theory, the Sinh–Gordon model. In this limit, the $S$-matrix, the Lagrangian and the Thermodynamical Bethe Ansatz equations of the sh-G model reduce to those of the LL model. We have also shown that the pseudo-energies of the sh-G TBA (which are actually the energies of the excitations above the vacuum) become massless modes in the non-relativistic limit, in agreement with the hydro-dynamical description of the LL model given by bosonization.

The mapping between the two models proved to provide an efficient method to compute expectation values in the LL model by using the form factor expansion of the expectation values of the relativistic counterpart. The main advantage of using the form factors of the relativistic integrable sh-G model is that, as for any relativistic quantum field theory, its form factors obey a set of stringent constraints that permits the determination of their exact expressions. Moreover, the quantum integrability of the relativistic model allowed us to employ the rich collection of results valid for these systems, like the TBA and the LeClair–Mussardo formalism.

Using these two formalisms of the relativistic sh-G model (form factors and TBA), we computed the expectation values of the LL model. The method works equally well at $T = 0$ and $T \neq 0$ where the series expansion presents a remarkable convergence behavior for finite values of the LL parameter $\gamma$. The computation of one-point correlators was presented in detail, as well as the comparison with the known results available in the literature. In particular, we have determined the expectation value $g_3(\gamma, \tau)$ at finite temperature, for which there had been only asymptotic analytic results in the literature. This quantity is related to the recombination rate of the atomic gas and thus to the lifetime of the experiments.

It would be interesting to analyze the possibility of extending our method to other non-relativistic strongly correlated systems, identifying their relativistic counterparts. An equally important direction...
FIG. 10: $g_3$ vs the scaled temperature $\tau$ for $\gamma = 7$ and $\gamma = 15$. The blue dashed and the red dotted lines refer to $n = 6$ and 8 particles, respectively; the purple dot-dot-dashed lines show the asymptotic result (127).

would be to see how the methods presented here can be used to compute space- and time-dependent two-point functions.

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Appendix A: Determination of the coefficients $A_k^l$

To find the coefficients $A_k^l$ at level $k$ in Eq. (59) we can proceed in the following way. If we already know the form factors $F_i^{\phi^j}$: $\phi^j$: for $j < k$, we can extract the $A_k^l$‘s one by one from $\tilde{F}_k^{\phi^j}$:

$$A_k^2 = \tilde{F}_k^2 / F_2^{\phi^2}$$ (A1a)
$$A_k^4 = \frac{\tilde{F}_k^4 - A_k^2 F_2^{\phi^2}}{F_2^{\phi^4}}$$ (A1b)
$$\vdots$$
$$A_k^l = \frac{\tilde{F}_k^l - \sum_{j=2,4,\ldots}^{k-2} A_k^l F_2^{\phi^j}}{F_2^{\phi^k}}$$ (A1c)

Once these coefficients are known, all the form factors $\phi^k$: including the ones with $n > k$ particles can be obtained from $\tilde{F}_n^k$. These will be needed at higher levels. For example

$$F_{10}^{\phi^6} = \tilde{F}_{10}^6 - A_4^6 \tilde{F}_{10}^{\phi^4} - A_2^6 \tilde{F}_{10}^{\phi^2} = \tilde{F}_{10}^6 - A_4^6 \left( \tilde{F}_{10}^4 - A_2^4 \tilde{F}_{10}^2 \right) - A_2^6 \tilde{F}_{10}^2.$$

As an example we give the rules for the “operator mixing” (60) explicitly at the first four even levels:

$$\tilde{\phi}^2 = :\phi^2:,$$ (A3a)
$$\tilde{\phi}^4 = :\phi^4: - 4 \frac{\pi^2 \alpha^2}{g^2} :\phi^2:,$$ (A3b)
$$\tilde{\phi}^6 = :\phi^6: - 20 \frac{\pi^2 \alpha^2}{g^2} :\phi^4: + 16 \frac{\pi^4 \alpha^4}{g^4} :\phi^2:,$$ (A3c)
$$\tilde{\phi}^8 = :\phi^8: - 56 \frac{\pi^2 \alpha^2}{g^2} :\phi^6: + 336 \frac{\pi^4 \alpha^4}{g^4} :\phi^4: - 64 \frac{\pi^6 \alpha^6}{g^6} :\phi^2:.$$. (A3d)
Appendix B: Explicit formulas for the connected form factors

In this Appendix we explicitly go through the steps required for the calculation of the limit of the form factors \( F_{2i}^{\phi^2} \) \((\{p_i\})_{\text{conn}}\) that enter into the formula (100), and then we list the first few of them.

We consider explicitly only the simplest non-trivial case, \( F_{4i}^{\phi^2}: (p_1, p_2)_{\text{conn}}\). We start from the four-particle form factor of the exponential operator (54):

\[
F_4(k) = \langle 0| e^{ik\phi} | \theta_1, \theta_2, \theta_3, \theta_4 \rangle = [k] \left( \frac{4 \sin(\pi \alpha)}{N} \right)^2 \left( [k]^3 \sigma_1 \sigma_2 \sigma_3 - [k-1][k][k+1](\sigma_3^2 - \sigma_1^2 \sigma_4) \right) \prod_{i<j} F_{\min}(\theta_i - \theta_j) \frac{F_{\min}(\theta_i - \theta_j)}{x_i + x_j}, \tag{B1}
\]

where the elementary symmetric polynomials (46) are to be understood as \( \sigma_k = \sigma_k^{(4)} \) and we recall that \( x_i = \exp(\theta_i) \) and \( [k] = \sin(k \pi \alpha) / \sin(\pi \alpha) \). The form factor of \( \phi^2 \) is given by the \( \mathcal{O}(k^2) \) term in the Taylor-expansion in \( k \) (for \( \phi^2 \)): there is no mixing (60):

\[
F_{4i}^{\phi^2}(\theta_1, \theta_2, \theta_3, \theta_4) = - \frac{32 \pi^2 \alpha^2}{N^2 g^2} \left( \sigma_3^2 - \sigma_1^2 \sigma_4 \right) \prod_{i<j} F_{\min}(\theta_i - \theta_j) \frac{F_{\min}(\theta_i - \theta_j)}{e^{\theta_i} + e^{\theta_j}}. \tag{B2}
\]

To perform the connected limit (64), we first recall that the product of the minimal form factors in this limit is given by (65). For the rest of the formula we write \( x_3 = -x_2 + i\eta_2, \ x_4 = -x_1 + i\eta_1 \) and we expand the polynomials in the numerator and denominator to obtain the finite part:

\[
\mathcal{F} \left[ \lim_{\eta_1 \to 0, \eta_2 \to 0} \frac{x_2(x_1^2 + x_3^2) \eta_1^2 + 4x_1^2 x_3^2 \eta_1 \eta_2 + x_1^2(x_1^2 + x_3^2) \eta_2^2 + O(\eta^3)}{(x_1 - 2x_1 x_3^2 + x_2^4) \eta_1 \eta_2 + O(\eta^4)} \right]
= \mathcal{F} \left[ \frac{4x_1^2 x_3^2}{(x_1^2 - x_3^2)^2} \eta_1 + \frac{x_1^2(x_1^2 + x_3^2) \eta_2}{(x_1^2 - x_3^2)^2} \eta_1 + O \left( \frac{\eta_1^2}{\eta_2}, \frac{\eta_2^2}{\eta_1} \right) \right] = \frac{4x_1^2 x_3^2}{(x_1^2 - x_3^2)^2} = \frac{1}{\sinh^2 \theta_{12}}. \tag{B3}
\]

Collecting all the terms we obtain

\[
F_{4i}^{\phi^2}: (\theta_1, \theta_2)_{\text{conn}} = \frac{32 \pi^2 \alpha^2}{N^2 g^2} \frac{1}{\sinh^2 \theta_{12}} = \frac{32 \pi^2 \alpha^2}{g^2} \frac{1}{\sinh^2 \theta_{12} + \sinh^2(\pi \alpha)}. \tag{B4}
\]

We note that another way to arrive at this result is to calculate first the connected form factors of \( \exp(k\phi_0) \) and then to extract the \( \mathcal{O}(k^4) \) term to obtain the connected form factors of \( \phi^2 \).

Finally, after substituting \( \theta_1 \to p_i/mc \) and using the definition of \( \alpha \) (21), we can perform the double limit (68-70). We list below the result together with the first few non-relativistic form factors obtained in this way. We use the notation \( p_{ij} = p_i - p_j \), while \( \Sigma_{\rho} \) denotes a sum over permutations of \( \{i, j, k\} = \{1, 2, 3\} \).

\[
\hat{F}_{2i}^{\phi^2}_{\text{conn}} = \frac{1}{\hbar}, \tag{B5a}
\]

\[
\hat{F}_{4i}^{\phi^2}: (p_1, p_2)_{\text{conn}} = \frac{8m\lambda}{4m^2 \lambda^2 + \hbar^2 p_{12}^2} = \frac{2}{\hbar} \hat{\phi}(p_{12}), \tag{B5b}
\]

\[
\hat{F}_{6i}^{\phi^2}: (p_1, p_2, p_3)_{\text{conn}} = \frac{32hm^2 \lambda^2 (12m^2 \lambda^2 + \hbar^2 p_{12}^2 + p_{13}^2 + p_{23}^2)}{(4m^2 \lambda^2 + \hbar^2 p_{12}^2)(4m^2 \lambda^2 + \hbar^2 p_{13}^2)(4m^2 \lambda^2 + \hbar^2 p_{23}^2)} = \frac{1}{\hbar} \sum_{p} \hat{\phi}(p_{ij}) \hat{\phi}(p_{jk}). \tag{B5c}
\]

\[
\hat{F}_{4i}^{\phi^4}: (p_1, p_2)_{\text{conn}} = \frac{1}{m\hbar} \hat{\phi}(p_{12}) p_{12}^2, \tag{B6a}
\]

\[
\hat{F}_{6i}^{\phi^4}: (p_1, p_2, p_3)_{\text{conn}} = \frac{8hm\lambda}{(4m^2 \lambda^2 + \hbar^2 p_{12}^2)(4m^2 \lambda^2 + \hbar^2 p_{13}^2)(4m^2 \lambda^2 + \hbar^2 p_{23}^2)} = \frac{1}{2m\hbar} \sum_{p} \hat{\phi}(p_{ij}) \hat{\phi}(p_{jk}) p_{12}^2. \tag{B6b}
\]
\[ \hat{F}_6^{\alpha\beta}(p_1, p_2)_{\text{conn}} = 36\hbar^3 \frac{p_1^2 p_2^2 p_3^2}{(4m^2\lambda^2 + \hbar^2 p_{12}^2)(4m^2\lambda^2 + \hbar^2 p_{23}^2)(4m^2\lambda^2 + \hbar^2 p_{13}^2)} = \frac{9}{16m^3\lambda^3} \hat{\varphi}(p_{12})\hat{\varphi}(p_{23})\hat{\varphi}(p_{13})p_{12}^2 p_{23}^2 p_{13}^2. \]
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[55] This is reminiscent of the small temperature expansion of the relativistic TBA equations where the leading term, $m \cosh(\theta)/(k_B T)$, automatically dominates for small $T$. 