Entanglement as Internal Constraint

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Abstract

Our investigation aims to study the specific role played by entanglement in the quantum computation process, by elaborating an entangled spin model developed within the ‘hidden measurement approach’ to quantum mechanics. We show that an arbitrary tensor product state for the entity consisting of two entangled qubits can be described in a complete way by a specific internal constraint between the ray and density states of the two qubits. For the individual qubits we use a sphere model representation, which is a generalization of the Bloch or Pauli representation, where also the collapse and noncollapse measurements are represented. We identify a parameter \( r \in [0,1] \), arising from the Schmidt diagonal decomposition, that is a measure of the amount of entanglement, such that for \( r = 0 \) the system is in the singlet state with ‘maximal’ entanglement, and for \( r = 1 \) the system is in a pure product state.

1 Introduction

In quantum computation the concepts of quantum superposition states and quantum entanglement are crucial. We want to study quantum entanglement in the most simple case, namely a system consisting of two entangled spin \( \frac{1}{2} \). The quantum entity consisting of two entangled spin \( \frac{1}{2} \) is described in the tensor product of the two dimensional complex Hilbert spaces that describe the single spins. Let us refer to the first spin as the ‘left spin’ and to the second spin as the ‘right spin’. It is well known that for the two spins being in the singlet state, the typical EPR correlations are encountered, meaning that if the left spin collapses in a certain direction under the influence of a measurement, then the right spin collapses in the opposite direction.

Our aim is to study in detail the entanglement for an arbitrary tensor product state that is not necessarily the singlet state, by making use of the sphere model representation for the spin of a spin \( \frac{1}{2} \) particle that was developed in Brussels within the ‘hidden measurement approach’ to quantum mechanics \[1, 2, 3, 5, 6, 7, 8, 10\]. We do this by introducing ‘constraint functions’ that describe the behavior of the state of one of the spins if measurements are executed on the other spin.

We will consider two types of measurements: (1) noncollapse measurements, of which the action on a mixture of states is described by Luders’s formula, and (2) collapse measurements, of which the action is described by Von Neumann’s formula. We will show that (1) an arbitrary noncollapse measurement on one of the two spins does not provoke any change in the partial trace density matrix of the other spin, i.e., the spins behave as separated entities for noncollapse measurements; (2) an arbitrary collapse measurement on one spin provokes a rotation and a stretching on the other spin, which can be described in detail by means of the sphere model.

2 The Sphere Model

In the sphere model representation a ray state of the spin is represented by a point of a sphere with radius 1 in the three dimensional real space \( \mathbb{R}^3 \), such that the direction of the point towards the origin of the sphere coincides with the direction of the spin in three dimensional space as measured for example by a Stern-Gerlach apparatus.
Let us denote the point \((r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)\) of \(\mathbb{R}^3\) by the vector \(u(r, \theta, \phi)\). The ray state
\[
|\theta \phi\rangle = (\cos \frac{\theta}{2} e^{i \frac{\phi}{2}}, \sin \frac{\theta}{2} e^{-i \frac{\phi}{2}})
\]
vector of \(\mathbb{C}^2\), is then represented by
\[
u(1, \theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)
\]
vector of \(\mathbb{R}^3\), and point of the sphere with radius 1 and center in \((0, 0, 0)\). We remark that this part of our sphere model is nothing else but the well known Poincaré representation of \(\mathbb{C}^2\). A density state of the spin is represented by an interior point of the sphere, which is a convex linear combination of points of the surface of the sphere, in such a way that the weights of the convex combination coincide with the weights of the statistical mixture that corresponds with the density state. It is not difficult to calculate the density state \(D(r, \theta, \phi)\) that corresponds with an arbitrary interior point, \(u(r, \theta, \phi)\), \(r \in [0, 1]\), \(\theta \in [0, \pi]\), \(\phi \in [0, 2\pi]\), of the sphere.

To do this we remark that also a ray state has a density representation, where in this case the density matrix is the orthogonal projection on the ray. This means that the density matrix representing the ray state \(|\theta \phi\rangle\) is given by:
\[
D(1, \theta, \phi) = |\theta \phi\rangle \langle \theta \phi| = \begin{pmatrix}
\cos^2 \frac{\theta}{2} e^{i \phi} & \cos \frac{\theta}{2} \sin \frac{\theta}{2} e^{-i \phi} \\
\cos \frac{\theta}{2} \sin \frac{\theta}{2} e^{i \phi} & \sin^2 \frac{\theta}{2}
\end{pmatrix}
\]
\[
= \frac{1}{2} \begin{pmatrix}
1 + \cos \theta & \sin \theta e^{-i \phi} \\
\sin \theta e^{i \phi} & 1 - \cos \theta
\end{pmatrix}
\]
\[
(3)
\]
The ray state orthogonal to \(|\theta \phi\rangle\) is \(|\pi - \theta, \phi + \pi\rangle\), and this state is represented by the point \(-u(1, \theta, \phi)\) of the sphere, corresponding to the opposite spin direction. We have:
\[
D(1, \pi - \theta, \phi + \pi) = |\pi - \theta, \phi + \pi\rangle \langle \pi - \theta, \phi + \pi| = \frac{1}{2} \begin{pmatrix}
1 - \cos \theta & -\sin \theta e^{-i \phi} \\
-\sin \theta e^{i \phi} & 1 + \cos \theta
\end{pmatrix}
\]
\[
(4)
\]
To find the general representation for \(u(r, \theta, \phi)\) we remark that the center of the sphere, hence the point \(u(0, \theta, \phi) = (0, 0, 0)\), can be written as the convex combination \(\frac{1}{2}u(1, \theta, \phi) + \frac{1}{2}u(1, \pi - \theta, \pi + \phi)\). This means that the density matrix that represent the center of the sphere is given by:
\[
D(0, \theta, \phi) = \frac{1}{2}D(1, \theta, \phi) + \frac{1}{2}D(1, \pi - \theta, \pi + \phi) = \begin{pmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{2}
\end{pmatrix}
\]
\[
(7)
\]
We further have that:
\[
u(r, \theta, \phi) = ru(1, \theta, \phi) + (1 - r)u(0, \theta, \phi)
\]
\[
(8)
\]
and hence:
\[
D(r, \theta, \phi) = rD(1, \theta, \phi) + (1 - r)D(0, \theta, \phi)
\]
\[
(9)
\]
\[
= \frac{1}{2} \begin{pmatrix}
1 + r \cos \theta & r \sin \theta e^{-i \phi} \\
r \sin \theta e^{i \phi} & 1 - r \cos \theta
\end{pmatrix}
\]
\[
(10)
\]
which gives us the representation of a general density state \(D(r, \theta, \phi)\) by means of the interior point \(u(r, \theta, \phi)\) of the sphere (see Figure 1).

We can see that the part of the sphere model that we developed in Brussels that relates to the representation of the density and ray states of the spin, is the Bloch representation. Additionally to this state part of the representation, we however also developed a representation for the measurements in our sphere model. Before we explain the way in which measurements are represented, let us identify some of the special points of this sphere representation. We have seen already that the center of the sphere, hence the point \(u(0, \theta, \phi)\) represents that density state
\[
D(0, \theta, \phi) = \begin{pmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{2}
\end{pmatrix}
\]
\[
(11)
\]
Figure 1: Representation of a general density state $D(r, \theta, \phi)$ by means of the interior point $u(r, \theta, \phi)$ of the sphere.

The North pole of the sphere, hence the point $u(1, 0, \phi)$, represents the state

$$D(1, 0, \phi) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$  \hspace{1cm} (12)$$

which is the orthogonal projector on the first canonical base vector $(1, 0)$ of $C^2$, while the South of the sphere, hence the point $u(1, \pi, \phi)$, represents the state

$$D(1, \pi, \phi) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$  \hspace{1cm} (13)$$

which is the orthogonal projector on the second canonical base vector $(0, 1)$ of $C^2$. An arbitrary point of the straight line connecting the North pole with the South pole of the sphere, hence $u(r, 0, \phi)$ or $u(r, \pi, \phi)$, represents the density states

$$D(r, 0, \phi) = \frac{1}{2} \left( \begin{array}{ccc} 1 + r & 0 \\ 0 & 1 - r \end{array} \right)$$  \hspace{1cm} (14)$$

and

$$D(r, \pi, \phi) = \frac{1}{2} \left( \begin{array}{ccc} 1 - r & 0 \\ 0 & 1 + r \end{array} \right)$$  \hspace{1cm} (15)$$

Without loss of generality we can demonstrate the effect of a measurement by considering states that are on the straight line connecting the North and the South pole of the sphere. So, suppose that the spin is in density state $D(r, 0, 0)$, and that a measurement of the spin is executed with a Stern Gerlach apparatus in the direction $u(1, \theta, \phi)$. Quantum mechanics prescribes the way, by means of Luder’s formula, in which we calculate the density state of the spin after this measurement.

$$D = P(\theta, \phi) D(r, 0, 0) P(\theta, \phi) + (1 - P(\theta, \phi)) D(r, 0, 0) (1 - P(\theta, \phi))$$  \hspace{1cm} (16)$$

where $P(\theta, \phi)$ is the projector on the ray state $|\theta\phi\rangle$. We know that $P(\theta, \phi) = D(1, \theta, \phi)$. If we make this calculation we find

$$D = \frac{1}{2} \left( \begin{array}{ccc} 1 + r \cos^2 \theta & r \sin \theta \cos \theta e^{-i\phi} \\ r \sin \theta \cos \theta e^{i\phi} & 1 - r \cos^2 \theta \end{array} \right)$$  \hspace{1cm} (17)$$

Let us see which point of the sphere corresponds with this density state. To do this, let us first suppose that $\theta \in [0, \frac{\pi}{2}]$. In this case we can introduce

$$r' = r \cos \theta$$  \hspace{1cm} (18)$$

and we have

$$D = \frac{1}{2} \left( \begin{array}{ccc} 1 + r' \cos \theta & r' \sin \theta e^{-i\phi} \\ r' \sin \theta e^{i\phi} & 1 - r' \cos \theta \end{array} \right) = D(r', \theta, \phi)$$  \hspace{1cm} (19)$$
This means that for $\theta \in [0, \frac{\pi}{2}]$ we have that $D(r, 0, 0)$ transform into $D(r \cos \theta, \theta, \phi)$, if a measurement with a Stern Gerlach in direction $(\theta, \phi)$ is executed. Consider now the case where $\theta \in [\frac{\pi}{2}, \pi]$. We can put then

$$r' = r \cos (\pi - \theta) \tag{20}$$
$$\theta' = \pi - \theta \tag{21}$$
$$\phi' = \phi + \pi \tag{22}$$

and find

$$D = \frac{1}{2} \left( 1 + r' \cos \theta' e^{-i\phi'} \begin{array}{c} r' \sin \theta' e^{i\phi'} \\ 1 - r' \cos \theta' \end{array} \right) = D(r', \theta', \phi') \tag{23}$$

which means that for $\theta \in [\frac{\pi}{2}, \pi]$ the density state $D(r, 0, 0)$ transforms into $D(r', \theta', \phi')$, if a measurement with Stern Gerlach in direction $(\theta, \phi)$ is executed. If we consider the sphere we can see easily that in both cases the point $u(r, 0, 0)$ is transformed into the point $$(u(r, 0, 0) \cdot u(1, \theta, \phi)) u(1, \theta, \phi) \tag{24}$$

where $u(r, 0, 0) \cdot u(1, \theta, \phi)$ is the scalar product in $\mathbb{R}^3$ of the vectors $u(r, 0, 0)$ and $u(1, \theta, \phi)$. This means that we have identified a very simple mechanics to describe the quantum measurement effect in our sphere model. The effect is just an ordinary orthogonal projection on the direction of the Stern Gerlach apparatus of the point that represents that ray or density state of the spin in the sphere model (see Figure 2).

![Figure 2: Effect of the measurement on a single spin $\frac{1}{2}$](image)

Let us formulate the general case. Suppose that we have a spin state represented by the point $u(s, \alpha, \beta)$ and we perform a measurement with a Stern Gerlach apparatus in direction $(\theta, \phi)$. We denote the orthogonal projection on the straight line with direction $(\theta, \phi)$ in $\mathbb{R}^3$ by $E(\theta, \phi)$. Then the new state after a quantum mechanical measurement with Stern Gerlach in direction $(\theta, \phi)$, when the state of the spin before the measurement is represented in the sphere model by the point $u(s, \alpha, \beta)$, is given by

$$E(\theta, \phi) u(s, \alpha, \beta) \tag{25}$$

and we have

$$E(\theta, \phi) u(s, \alpha, \beta) = u(s \cos \theta, \theta, \phi) \text{ if } |\alpha - \theta| \in [0, \frac{\pi}{2}] \tag{26}$$
$$E(\theta, \phi) u(s, \alpha, \beta) = u(s \cos(\pi - \theta), \pi - \theta, \phi + \pi) \text{ if } |\alpha - \theta| \in [\frac{\pi}{2}, \pi] \tag{27}$$

It is possible to give a nice geometrical presentation of how the spin state changes under the influence of measurements in different directions (see Figure 3).
Consider a little sphere inside the big sphere of the model, such that the North pole of the little sphere is in the point \( u(s, \alpha, \beta) \), the point that represents the spin state, and the South pole is in the center of the big sphere. Consider now a straight line with direction \( (\theta, \phi) \) through the center of the big sphere, representing the direction of the Stern Gerlach apparatus. The point where this line cuts the little sphere is the point where the spin state will be transformed to under influence of the measurement. This also means that the points of the little sphere are the points that represent the states where under arbitrary angles for the measurement the spin state can be transformed to.

3 Entangled Spins

The entity consisting of two entangled spin \( \frac{1}{2} \) is described by means of the tensorproduct \( C_2^1 \otimes C_2^2 \), where \( C_1 \) and \( C_2 \) are two copies of \( C \), that we label with indices 1 and 2 with the sole purpose of identifying them. This means that the ray states of this entangled spin \( \frac{1}{2} \) entity are described by the rays of \( C_2^1 \otimes C_2^2 \) and the density states by the density matrices of \( C_2^1 \otimes C_2^2 \).

3.1 The Constraint Functions

Suppose that we consider an arbitrary unit vector \( \psi \in C_2^1 \otimes C_2^2 \). Then it is always possible to write \( \psi \) as the sum of product vectors

\[
\psi = \sum_{ij} \lambda_{ij} e_1^i \otimes e_2^j
\]

where \( \lambda_{ij} \in \mathbb{C} \), and \( \{e_1^i\} \) and \( \{e_2^j\} \) are a bases respectively of \( C_2^1 \) and \( C_2^2 \).

Let us consider a measurement on the first spin. This measurement provokes the first spin to collapse with a certain probability into a spin state described by a unit vector \( x_1 \in C_2^1 \). The state \( \psi \) of the entangled spins is transformed in the state

\[
(P_{x_1} \otimes I)(\psi)
\]

where \( P_{x_1} \) is the orthogonal projector of \( C_2^1 \) on \( x_1 \), and \( I \) is the unit operator of \( C_2^2 \). The result is that the entangled spins end up in a product state that is the following:

\[
(P_{x_1} \otimes I)(\psi) = \sum_{ij} \lambda_{ij} (P_{x_1} \otimes I)(e_1^i \otimes e_2^j)
\]

\[
= \sum_{ij} \lambda_{ij} \langle x_1, e_1^i \rangle x_1 \otimes e_2^j
\]

\[
= x_1 \otimes \sum_{ij} \lambda_{ij} \langle x_1, e_1^i \rangle e_2^j
\]
This means that as a consequence of the spin measurement on the first spin, making its state collapse in the state $x_1$, the spin state of the second spin collapses to the state

$$\sum_{ij} \lambda_{ij} \langle x_1, e^i_1 \rangle e^j_2$$

In an analogous way we can show that if a measurement is performed on the second spin that makes its state collapse to the state described by the unit vector $x_2 \in \mathbb{C}^2_2$, the state of the first spin collapses to the state described by the vector

$$\sum_{ij} \lambda_{ij} \langle x_2, e^j_2 \rangle e^i_1$$

**Definition 1 (Constraint Functions)** Let us consider the functions $F_{12}(\psi)$ and $F_{21}(\psi)$ defined in the following way

$$F_{12}(\psi) : \mathbb{C}^2_1 \rightarrow \mathbb{C}^2_2 : x_1 \mapsto \sum_{ij} \lambda_{ij} \langle x_1, e^i_1 \rangle e^j_2$$

$$F_{21}(\psi) : \mathbb{C}^2_2 \rightarrow \mathbb{C}^2_1 : x_2 \mapsto \sum_{ij} \lambda_{ij} \langle x_2, e^j_2 \rangle e^i_1$$

We call $F_{12}(\psi)$ and $F_{21}(\psi)$ the constraint functions related to $\psi$.

These constraint functions map the unit vectors describing the state where the entangled spin collapses to by a measurement on one of the spins to the vector describing the state that the other spin collapses to under influence of the entanglement correlation. A detailed study of the constraint functions can give us a complete picture of how the entanglement correlation works as an internal constraint. Before we arrive at this complete picture, let us proof some properties of the constraint functions that we need to derive the picture.

**Proposition 1** The constraint functions are canonically defined

Proof: Indeed, consider other bases $\{f^k_1\}$ of $\mathbb{C}^2_1$, and $\{f^l_2\}$ of $\mathbb{C}^2_2$, such that

$$\psi = \sum_{kl} \mu_{kl} f^k_1 \otimes f^l_2$$

We have

$$f^k_1 = \sum_i a^k_i e^i_1$$

$$f^l_2 = \sum_j b^l_j e^j_2$$

and hence

$$\psi = \sum_{klij} \mu_{kl} a^k_i b^l_j e^i_1 \otimes e^j_2$$

From this follows that

$$\lambda_{ij} = \sum_{kl} \mu_{kl} a^k_i b^l_j$$

Hence we have

$$\sum_{kl} \mu_{kl} \langle x_1, f^k_1 \rangle f^l_2 = \sum_{klij} \mu_{kl} a^k_i b^l_j \langle x_1, e^i_1 \rangle e^j_2$$

$$= \sum_{ij} \lambda_{ij} \langle x_1, e^i_1 \rangle e^j_2$$

$$= F_{12}(\psi)(x_1)$$

which proves that the definition of $F_{12}(\psi)$ does not depend on the chosen bases. In an analogous way we prove that $F_{21}(\psi)$ is canonically defined. □
Proposition 2  The constraint functions are conjugate linear

Proof: Consider $x_1, y_1 \in \mathbb{C}_1^2$ and $\lambda \in \mathbb{C}$. We have

$$F_{12}(\psi)(x_1 + \lambda y_1) = \sum_{ij} \lambda_{ij} \langle x_1 + \lambda y_1, e_1^i \rangle e_2^j$$

$$= \sum_{ij} \lambda_{ij} \langle x_1, e_1^i \rangle + \lambda^* \langle y_1, e_1^i \rangle e_2^j$$

$$= \sum_{ij} \lambda_{ij} \langle x_1, e_1^i \rangle e_2^j + \lambda^* \sum_{ij} \lambda_{ij} \langle y_1, e_1^i \rangle e_2^j$$

$$= F_{12}(\psi)(x_1) + \lambda^* F_{12}(\psi)(y_1)$$

The conjugate linearity of $F_{21}(\psi)$ is proven in an analogous way.

Let us calculate $F_{21}(\psi) \circ F_{12}(\psi)$ and $F_{12}(\psi) \circ F_{21}(\psi)$.

Proposition 3 We have

$$D_1(\psi) = F_{21}(\psi) \circ F_{12}(\psi)$$

$$D_2(\psi) = F_{12}(\psi) \circ F_{21}(\psi)$$

where $D_1(\psi)$ is the partial trace density matrix to $\mathbb{C}_1^2$ and $D_2(\psi)$ is the partial trace density matrix to $\mathbb{C}_2^2$.

Proof: Let us first calculate $D_1(\psi)$ directly. We have

$$|\psi\rangle \langle \psi| = \sum_{ijkl} \lambda_{ij} \lambda^*_{kl} |e_1^i \rangle \langle e_1^j| \otimes |e_2^j \rangle \langle e_2^i|$$

and hence

$$D_1(\psi) = \sum_{ijkl} \lambda_{ij} \lambda^*_{kl} \langle e_2^j | e_2^l | e_1^i \rangle \langle e_1^j|$$

$$= \sum_{ijkl} \lambda_{ij} \lambda^*_{kl} \delta_{jl} |e_1^i \rangle \langle e_1^j|$$

$$= \sum_{ijk} \lambda_{ij} \lambda^*_{kj} |e_1^i \rangle \langle e_1^j|$$

This means that

$$D_1(\psi)(x_1) = \sum_{ij} \lambda_{ij} \lambda^*_{kj} (e_1^i, x_1) e_1^j$$

Let us now calculate $F_{21}(\psi) \circ F_{12}(\psi)(x_1)$. We have

$$F_{21}(\psi) \circ F_{12}(\psi)(x_1) = \sum_{kl} \lambda_{kl} (F_{12}(\psi)(x_1), e_2^l) e_1^k$$

$$= \sum_{kl} \lambda_{kl} \lambda^*_{ij} (e_1^i, x_1) \langle e_2^l, e_2^k | e_1^j \rangle$$

$$= \sum_{kl} \lambda_{kl} \lambda^*_{ij} (e_1^i, x_1) \delta_{lj} e_1^k$$

$$= \sum_{kl} \lambda_{kl} \lambda^*_{ij} (e_1^i, x_1) e_1^k$$

This proves that

$$F_{21}(\psi) \circ F_{12}(\psi) = D_1(\psi)$$

In an analogous way we prove

$$F_{12}(\psi) \circ F_{21}(\psi) = D_2(\psi)$$
**Proposition 4** The constraint functions are related in the following way. For \( x_1 \in \mathbb{C}_1^2 \) and \( x_2 \in \mathbb{C}_2^2 \) we have

\[
\langle F_{12}(\psi)(x_1), x_2 \rangle = \langle x_1, F_{21}(\psi)(x_2) \rangle^* \tag{62}
\]

Proof: We have

\[
\langle F_{12}(\psi)(x_1), x_2 \rangle = \sum_{ij} \lambda_{ij} \langle x_1, e_i^1 \rangle_e^2, x_2 \rangle = \sum_{ij} \lambda_{ij}^* \langle x_1, e_i^1 \rangle_e^2, x_2 \rangle \tag{63}
\]

and

\[
\langle x_1, F_{21}(\psi)(x_2) \rangle = \sum_{ij} \lambda_{ij} \langle x_2, e_j^2 \rangle_e^1, x_1 \rangle \tag{64}
\]

To derive a complete view of how the entanglement between the two spins works as an internal constraint, let us derive the way in which the Schmidt diagonal form is related to the constraint functions.

### 3.2 The Schmidt Diagonal Form

It is always possible to choose a base in \( \mathbb{C}_1^2 \) and a base in \( \mathbb{C}_2^2 \) such that \( \psi \) becomes very simple. This special form for \( \psi \) is often called the Schmidt diagonalization form. Let us explain how this works. Since \( D_1(\psi) \) is a density matrix, it is of the form

\[
D_1(\psi) = \frac{1}{2} \begin{pmatrix}
1 + r \cos \theta & r \sin \theta e^{-i \phi} \\
r \sin \theta e^{i \phi} & 1 - r \cos \theta
\end{pmatrix} \tag{65}
\]

We choose the base

\[
x_1^1 = \left( \cos \frac{\theta}{2} e^{-i \frac{\phi}{2}}, \sin \frac{\theta}{2} e^{i \frac{\phi}{2}} \right) \tag{66}
\]

\[
x_1^2 = \left( -i \sin \frac{\theta}{2} e^{-i \frac{\phi}{2}}, i \cos \frac{\theta}{2} e^{i \frac{\phi}{2}} \right) \tag{67}
\]

then in this new base, we have

\[
D_1(\psi) = \frac{1}{2} \begin{pmatrix}
1 + r & 0 \\
0 & 1 - r
\end{pmatrix} \tag{68}
\]

Define now

\[
x_2^1 = \frac{\sqrt{2}}{\sqrt{1 + r}} F_{12}(\psi)(x_1^1) \tag{69}
\]

\[
x_2^2 = \frac{\sqrt{2}}{\sqrt{1 - r}} F_{12}(\psi)(x_1^2) \tag{70}
\]

We have then:

\[
\|x_2^1\|^2 = \frac{2}{1 + r} \langle F_{12}(\psi)(x_1^1), F_{12}(\psi)(x_1^1) \rangle \tag{71}
\]

\[
= \frac{2}{1 + r} (x_1^1, F_{21}(\psi) \circ F_{12}(\psi)(x_1^1))^* \tag{72}
\]

\[
= \frac{2}{1 + r} (x_1^1, 1 + r x_1^1)^* \tag{73}
\]

\[
= 1 \tag{74}
\]

and

\[
D_2(\psi)(x_2^1) = \frac{\sqrt{2}}{\sqrt{1 + r}} F_{12}(\psi) \circ F_{21}(\psi) \circ F_{12}(\psi)(x_1^1) \tag{75}
\]
\[
\begin{align*}
F_{12}(\psi)D_1(\psi)(x_1^1) &= \sqrt{\frac{2}{1+r}}F_{12}(\psi)(x_1^1) \\
&= \frac{1}{\sqrt{2}}F_{12}(\psi)(x_1^1) \\
&= \frac{1+r}{2}x_2^1
\end{align*}
\]

Similarly, one can show that
\[
\|x_2^2\|^2 = 1
\]

and
\[
D_2(\psi)(x_2^2) = \frac{1-r}{2}x_2^2
\]

Hence this shows that \(x_1^1\), respectively \(x_2^2\), is a normalized eigenvector of \(D_2(\psi)\) with eigenvalue \(\frac{1+r}{2}\), respectively \(\frac{1-r}{2}\). From this follows that \(D_2(\psi)\) has the form
\[
D_2(\psi) = \frac{1}{2} \begin{pmatrix} 1+r & 0 \\ 0 & 1-r \end{pmatrix}
\]
in the base \(x_1^1, x_2^2\). Let us write now \(\psi\) in the base \(\{x_1^1 \otimes x_2^1, x_1^1 \otimes x_2^2, x_1^1 \otimes x_2^2, x_1^1 \otimes x_2^2\}\) of \(C_1^2 \otimes C_2^2\), hence
\[
\psi = ax_1^1 \otimes x_2^1 + bx_1^1 \otimes x_2^2 + cx_1^2 \otimes x_2^1 + dx_1^2 \otimes x_2^2
\]

We have then
\[
\begin{align*}
F_{12}(\psi)(x_1^1) &= ax_2^1 + bx_2^2 = \frac{\sqrt{1+r}}{\sqrt{2}}x_2^1 \\
F_{12}(\psi)(x_1^2) &= cx_2^1 + dx_2^2 = \frac{\sqrt{1-r}}{\sqrt{2}}x_2^2
\end{align*}
\]

which shows that
\[
\begin{align*}
a &= \frac{\sqrt{1+r}}{\sqrt{2}} \\
b &= 0 \\
c &= 0 \\
d &= \frac{\sqrt{1-r}}{\sqrt{2}}
\end{align*}
\]

and hence
\[
\psi = \frac{\sqrt{1+r}}{\sqrt{2}}x_1^1 \otimes x_2^1 + \frac{\sqrt{1-r}}{\sqrt{2}}x_1^2 \otimes x_2^2
\]

which is the Schmidt diagonal form of \(\psi\) adapted to our sphere model of the spin \(\frac{1}{2}\).

### 3.3 Non collapse measurement

Let us now see what the result of a non collapse measurement is on the density state, using Luder’s formula. Let us write the state in the Schmidt diagonalization form:
\[
|\psi\rangle = \frac{\sqrt{1+r}}{\sqrt{2}}x_1^1 \otimes x_2^1 + \frac{\sqrt{1-r}}{\sqrt{2}}x_1^2 \otimes x_2^2
\]
and choose coordinates such that \( x_1^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, x_1^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) in \( \mathbb{C}_1^2 \) and \( x_2^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, x_2^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) in \( \mathbb{C}_2^2 \).

The density state \( \rho(\psi) \) corresponding with the pure state \( |\psi\rangle \) is given by:

\[
\rho(\psi) = |\psi\rangle \langle \psi |
\]

\[
= \frac{1 + r}{2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{\sqrt{1 - r^2}}{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
+ \frac{\sqrt{1 - r^2}}{2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]

(91)

The projector operator for a measurement along direction \((\theta, \phi)\) is given by \( P(\theta, \phi) = D(1, \theta, \phi) \), i.e.,

\[
P(\theta, \phi) = D(1, \theta, \phi) = \frac{1}{2} \begin{pmatrix} 1 + \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & 1 - \cos \theta \end{pmatrix}
\]

(92)

and its orthogonal by \( 1 - P(\theta, \phi) \), i.e.,

\[
1 - P(\theta, \phi) = D(1, \pi - \theta, \phi + \pi) = \frac{1}{2} \begin{pmatrix} 1 - \cos \theta & -\sin \theta e^{-i\phi} \\ -\sin \theta e^{i\phi} & 1 + \cos \theta \end{pmatrix}
\]

(93)

To obtain the density state \( \rho'(\psi) \) after a non collapse measurement we use Luder’s formula, with the following result:

\[
\rho'(\psi) = (P(\theta, \phi) \otimes \mathbf{1}) \rho(\psi) (P(\theta, \phi) \otimes \mathbf{1}) + ((1 - P(\theta, \phi)) \otimes \mathbf{1}) \rho(\psi) ((1 - P(\theta, \phi)) \otimes \mathbf{1})
\]

(94)

\[
= \frac{1 + r}{4} \begin{pmatrix} 1 + \cos^2 \theta & \cos \theta \sin \theta e^{-i\phi} \\ \cos \theta \sin \theta e^{i\phi} & 1 - \cos^2 \theta \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
+ \frac{\sqrt{1 - r^2}}{4} e^{i\phi} \begin{pmatrix} \cos \theta \sin \theta & \sin^2 \theta e^{-i\phi} \\ \sin^2 \theta e^{i\phi} & -\cos \theta \sin \theta \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
+ \frac{\sqrt{1 - r^2}}{4} e^{-i\phi} \begin{pmatrix} \cos \theta \sin \theta & \sin^2 \theta e^{-i\phi} \\ \sin^2 \theta e^{i\phi} & -\cos \theta \sin \theta \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
+ \frac{1 - r}{4} \begin{pmatrix} 1 - \cos^2 \theta & -\cos \theta \sin \theta e^{-i\phi} \\ -\cos \theta \sin \theta e^{i\phi} & 1 + \cos^2 \theta \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]

(95)

From this, we can calculate \( D_1(\psi) \), i.e., the partial trace density matrix to \( \mathbb{C}_1^2 \) and we obtain

\[
D_1(\psi) = \frac{1 + r}{4} \begin{pmatrix} 1 + \cos^2 \theta & \cos \theta \sin \theta e^{-i\phi} \\ \cos \theta \sin \theta e^{i\phi} & 1 - \cos^2 \theta \end{pmatrix}
+ \frac{1 - r}{4} \begin{pmatrix} 1 - \cos^2 \theta & -\cos \theta \sin \theta e^{-i\phi} \\ -\cos \theta \sin \theta e^{i\phi} & 1 + \cos^2 \theta \end{pmatrix}
= \frac{1}{2} \begin{pmatrix} 1 + r \cos^2 \theta & r \sin \theta \cos \theta e^{-i\phi} \\ r \sin \theta \cos \theta e^{i\phi} & 1 - r \cos^2 \theta \end{pmatrix}
\]

(96)

(97)

This is the same density matrix as we found for a measurement on a single spin \( \frac{1}{2} \).

Also, we can calculate \( D_2(\psi) \), i.e., the partial trace density matrix to \( \mathbb{C}_2^2 \) and we find:

\[
D_2(\psi) = \frac{1}{2} \begin{pmatrix} 1 + r & 0 \\ 0 & 1 - r \end{pmatrix}
\]

(98)

(99)

(100)

(101)

(102)

(103)

(104)

which is independent of \((\theta, \phi)\). This means that a noncollapse measurement on one spin does not provoke any change in the partial trace density matrix of the other spin: the spins behave as separated entities for noncollapse measurements.
3.4 Collapse measurement

Let us now study the effect of a non collapse measurement using the constraint functions. Again, the state \( \psi \) is written in the Schmidt diagonalization form:

\[
|\psi\rangle = \frac{1 + r}{\sqrt{2}} x_1^1 \otimes x_1^2 + \frac{1 - r}{\sqrt{2}} x_1^1 \otimes x_2^2
\]

with \( r \in [0, 1] \), and we use \( \{x_1^1, x_1^2\} \), respectively \( \{x_2^1, x_2^2\} \), as basis for \( \mathbb{C}_1^2 \), respectively \( \mathbb{C}_2^2 \). These two orthonormal basis are related by the following expressions:

\[
x_1^1 = \frac{\sqrt{2}}{\sqrt{1 + r}} F_{12}(\psi)(x_1^1)
\]

\[
x_1^2 = \frac{\sqrt{2}}{\sqrt{1 - r}} F_{12}(\psi)(x_1^2)
\]

which in the sphere representation means that the north pole of the first sphere is mapped onto the north pole of the second sphere, and the south pole of the first sphere is mapped to the south pole of the second sphere (in the bases \( \{x_1^1, x_1^2\} \) and \( \{x_2^1, x_2^2\} \)).

Let us now study the mapping \( F_{12}(\psi) \) for the other states. From (106) and (107) it follows immediately that \( F_{12}(\psi) \) does not preserve the norm. Let us calculate the norm of \( F_{12}(\psi)(z) \) for an arbitrary vector \( z = \psi(\theta, \phi) \):

\[
\| F_{12}(\psi)(z) \|^2 = \langle F_{12}(\psi)(z), F_{12}(\psi)(z) \rangle = \frac{1 + r}{2} \cos^2 \frac{\theta}{2} + \frac{1 - r}{2} \sin^2 \frac{\theta}{2}
\]

\[
= \frac{1}{2} (1 + r \cos \theta)
\]

If we consider for a moment the angle \( \theta \) as a variable, we see that the square of the norm varies between \( \frac{1 + r}{2} \) and \( \frac{1 - r}{2} \), depending on the value of \( \theta \). For \( \theta = 0 \), and hence the state represented by the north pole of the sphere, we have:

\[
\| F_{12}(\psi)(z) \|^2 = \frac{1 + r}{2}
\]

and for \( \theta = \pi \), and hence the state represented by the south pole of the sphere, we have:

\[
\| F_{12}(\psi)(z) \|^2 = \frac{1 - r}{2}
\]

Not only the norm, but also orthogonality is in general not conserved by \( F_{12}(\psi) \). Let us consider for example an orthonormal base \( \{\psi_u = \psi(\theta, \phi), \psi_{-u} = \psi(\pi - \theta, \phi + \pi)\} \). We can use the conjugate linearity of \( F_{12}(\psi) \) to obtain:

\[
F_{12}(\psi)(\psi_u) = \sqrt{\frac{1 + r}{2}} \cos \frac{\theta}{2} e^{i\hat{\phi}} x_1^1 + \sqrt{\frac{1 - r}{2}} \sin \frac{\theta}{2} e^{-i\hat{\phi}} x_1^2
\]

\[
F_{12}(\psi)(\psi_{-u}) = \sqrt{\frac{1 + r}{2}} i \sin \frac{\theta}{2} e^{i\hat{\phi}} x_1^1 - i \sqrt{\frac{1 - r}{2}} \cos \frac{\theta}{2} e^{-i\hat{\phi}} x_1^2
\]

Therefore, for \( 0 \neq \theta \neq \pi \) we find that orthogonal states are mapped onto orthogonal states iff:

\[
\langle F_{12}(\psi)(\psi_u), F_{12}(\psi)(\psi_{-u}) \rangle = \frac{1 + r}{2} i \cos \frac{\theta}{2} \sin \frac{\theta}{2} - \frac{1 - r}{2} i \cos \frac{\theta}{2} \sin \frac{\theta}{2} = 0
\]

\[
\iff \frac{1 + r}{2} = \frac{1 - r}{2} = 0
\]

\[
\iff r = 0
\]
Translated on the sphere this gives that diametrical opposite points are mapped to diametrical opposite points only in the special case $r = 0$, (except the north and south pole which are always mapped onto the north and south pole of the second sphere).

We consider now the following situation. Take vector $\psi_{v_1} = \psi(\theta_{v_1}, \phi_{v_1})$ representing the point $v_1(\theta_{v_1}, \phi_{v_1})$ on the sphere. Consider now:

$$\psi_{v_2} = \frac{1}{\|F_{12}(\psi)(\psi_{v_1})\|} F_{12}(\psi)(\psi_{v_1})$$

(117)

which is the normalized vector. This means that there are $\theta_{v_2}$ and $\phi_{v_2}$ such that:

$$\psi_{v_2} = \psi(\theta_{v_2}, \phi_{v_2})$$

(118)

We want to find out where the corresponding point $v_2(\theta_{v_2}, \phi_{v_2})$ lies on the sphere. Therefore we compare the inproduct of $\psi_{v_2}$ with $x_2^1$ with the inproduct of $\psi_{v_1}$ with $x_1^1$. We have:

$$\langle \psi_{v_2}, x_2^1 \rangle = \frac{1}{\|F_{12}(\psi)(\psi_{v_1})\|} \sqrt{2} \sqrt{1 + r} \langle F_{12}(\psi)(\psi_{v_1}), F_{12}(\psi)(x_1^1) \rangle$$

(119)

$$= \frac{1}{\|F_{12}(\psi)(\psi_{v_1})\|} \sqrt{2} \sqrt{1 + r} \langle \psi_{v_1}, D_1(x_1^1) \rangle^*$$

(120)

$$= \frac{1}{\|F_{12}(\psi)(\psi_{v_1})\|} \sqrt{1 + r} \langle \psi_{v_1}, x_1^1 \rangle^*$$

(121)

$$= \sqrt{\frac{1 + r}{1 + r \cos \theta_{v_1}}} \cdot \langle \psi_{v_1}, x_1^1 \rangle^*$$

(122)

Only in the case when $r = 0$ (i.e., the singlet state) we have that the inproducts are equal (and consequently, antipodal points on the sphere are mapped to antipodal points, as mentioned before).

An interesting case is when $\theta_{v_1} = \frac{\pi}{2}$. Then we find:

$$\langle \psi_{v_2}, x_2^1 \rangle = \sqrt{1 + r} \cdot \langle \psi_{v_1}, x_1^1 \rangle^*$$

(123)

and

$$\langle \psi_{v_1}, x_1^1 \rangle^* = \frac{1}{\sqrt{2}} e^{-\frac{\theta_{v_1}}{2}}$$

(124)

To see what this gives on the sphere, we use the following formula:

$$\frac{1 + u(\theta', \phi') \cdot u(\theta, \phi)}{2} = |\langle \psi(\theta', \phi'), \psi(\theta, \phi) \rangle|^2$$

(125)

for $\psi(\theta', \phi') = \psi_{v_2}$ (hence $u(\theta', \phi') = v_2(\theta_{v_2}, \phi_{v_2})$) and $\psi(\theta, \phi) = x_2^1$. So we get:

$$\frac{1 + v_2(\theta_{v_2}, \phi_{v_2}) \cdot u(\theta, \phi)}{2} = \frac{1 + r}{2}$$

(126)

and as a consequence:

$$v_2(\theta_{v_2}, \phi_{v_2}) \cdot u(\theta, \phi) = r$$

(127)

This means that on the sphere, the elements of the equator are mapped onto a cone that makes an angle $\beta$ with the north south axis of the second sphere, such that:

$$\cos \beta = r$$

(128)

And indeed, only for $r = 0$ this is again an equator (and hence conserving the angle between the elements of the equator and the north pole). For $r \in [0, 1]$ we get a cone with an angle $0 < \beta < \frac{\pi}{2}$, which means that the equator has ‘raised’ to the north. For $r$ approaching 1 the sphere is stretched more and more to the north pole of the second sphere. Remember that in this limit case the superposition state becomes a
product state, and this fits with the fact that for product states indeed the map $F_{12}(\psi)$ maps the first element of the product to the second.

To see the general scheme we use

$$
\langle \psi_{v_2}, x^1_2 \rangle = \sqrt{\frac{1 + r}{1 + r \cos \theta_{v_1}}} \cdot \langle \psi_{v_1}, x^1_1 \rangle^* \tag{129}
$$

in the relation (123) to obtain:

$$
\frac{1 + \nu_2(\theta_{v_2}, \phi_{v_2}) \cdot u(\theta, \phi)}{2} = \frac{1 + r \cos \theta_{v_1}}{2} \cos^2 \frac{\theta_{v_1}}{2} \tag{130}
$$

$$
= \frac{1 + r}{1 + r \cos \theta_{v_1}} \frac{1 + \cos \theta_{v_1}}{2} \tag{131}
$$

which yields

$$
\nu_2(\theta_{v_2}, \phi_{v_2}) \cdot u(\theta, \phi) = \frac{r + \cos \theta_{v_2}}{1 + r \cos \theta_{v_1}} \tag{132}
$$

From formula (132) it follows that straight lines through the center of the left sphere are mapped onto straight lines through the point $u(r, 0, 0)$ along the north south axis in the second sphere, which gives a nice geometrical representation of this ‘stretching’ on the second sphere (see Figure 4). This also shows that indeed only for $r = 0$ antipodal points of the first sphere are mapped onto antipodal points of the second sphere.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Straight lines through the center of the left sphere are mapped onto straight lines through the point $u(r, 0, 0)$ along the north south axis in the second sphere.}
\end{figure}

### 4 Conclusions

We have studied the quantum entity consisting of two entangled spin $\frac{1}{2}$ which in standard quantum mechanics is described in the tensor product of the two dimensional complex Hilbert spaces that describe the single spins. We have introduced ‘constraint functions’ that describe the behavior of the state of one of the spins if measurements are executed on the other spin. By making use of the sphere model representation for the spin $\frac{1}{2}$ ‘s that was developed in Brussels, we studied in detail the entanglement for an arbitrary tensor product state, which is not necessarily the singlet state.

We considered two types of measurements: (1) noncollapse measurements, of which the action on a mixture of states is described by Luder’s formula, and (2) collapse measurements, of which the action is described by Von Neumann’s formula. Our result is that (1) an arbitrary noncollapse measurement on one spin does not provoke any change in the partial trace density matrix of the other spin: the spins behave as separated entities for noncollapse measurements; (2) an arbitrary collapse measurement on one spin provokes a rotation and a ‘stretching’ on the other spin, which gives a nice geometrical representation of
how entanglement works as an internal constraint. We conclude by remarking that our study is a further elaboration of earlier studies of the entanglement influence as constraint, more specifically to be found in [4, 9].

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