Elementary functions in thermodynamic Bethe ansatz*

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Abstract

Some years ago, Fendley found an explicit solution to the thermodynamic Bethe ansatz (TBA) equation for an $\mathcal{N} = 2$ supersymmetric theory in 2D with a specific $F$-term. Motivated by this, we seek explicit solutions for other super-potential cases utilizing the idea from the ODE/IM correspondence. We find that the TBA equations, corresponding to a wider class of super-potentials, admit solutions in terms of elementary functions such as modified Bessel functions and confluent hyper-geometric series.

Keywords: thermodynamic bethe ansatz, ODE/IM correspondence, supersymmetry

1. Introduction

The thermodynamic Bethe ansatz (TBA) is one of the most efficient tools in the field of integrable systems [1]. Once input data such as factorized $S$-matrices [2, 3], special patterns of Bethe ansatz roots (string hypothesis) [4, 5], or the fusion relations [6, 7] are given, it provides finitely or infinitely many coupled integrals equations as output. These equations make quantitative analyses possible in integrable 1 + 1D quantum field theories of finite size [2] or 1D quantum systems at finite temperatures [1]. The numerical analysis provides physical quantities such as the specific heat or the magnetic susceptibility for the whole range of temperature [9], or flow of the $g$ function with change in the system size [8]. On the other hand, some limited information, such as the central charge, is available analytically from the TBA equations. This is due to the fact that the nonlinearity of the TBA equations defies explicit solutions in most cases.

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Some years ago, Fendley [10] obtained a rare example: an explicit solution in the massless limit of an integrable $\mathcal{N} = 2$ supersymmetric theory in 2D. There is a deep structure behind the model which connects the solution to the (massive) TBA equation and the solution to the Painlevé III (PIII) equation. The proof in [10] relies on the heavy machinery in the solution to the PIII equation which has been developed in [11–13]. In particular, it utilizes the Tracy–Widom representation of the PIII solution, valid for a massive theory in general, while the explicit solution in terms of elementary function is possible only in the massless case. Then one may wonder if any simpler derivation is possible for the result in [10], as the massless theory possesses a larger symmetry and thus offers a simpler structure.

In this communication, we shall argue that the ordinary differential equation/integrable model (ODE/IM) correspondence provides a much simpler explanation of the solution. This program was actually suggested in [10]. We shall make it concrete. The Stokes multiplier $\tau$, associated with a simple ODE with an irregular singularity at infinity, turns out to provide the solution in [10]. This may sound odd as there seems to be no relation between the original problem and the ODE, thus there is no reason to consider a specific ODE. There is, however, a relation. We take the super-potential corresponding to Fendley’s solution. From the potential, we construct a function, which solves an ODE. At this stage, the Stokes multiplier is a trivial constant. We then ‘deform’ the ODE by a weak gauge field (or small angular momentum). Remarkably, the first nontrivial response of the Stokes multiplier to the gauge field reproduces the solution in [10].

This immediately leads to a generalization. There exists a list of relevant super-potentials [17–19] for the Landau–Ginzburg description of super-conformal theories. The corresponding TBA equations for perturbed cases are partially derived in [14, 15]. Thus, starting from one of the available potentials, we can construct an ODE and evaluate the first nontrivial response to the weak gauge field. The resultant Stokes multipliers are then transformed automatically into $Y$-functions. We shall show that these $Y$-functions solve TBA equations in perturbed $\mathcal{N} = 2$ minimal theories with SU(2)$_k$ and with SU(3)$_1$ chiral rings, which generalize Fendley’s solution for SU(2)$_k$.

This paper is organized as follows. Section 2 is devoted to a short review of the ODE/IM correspondence. Fendley’s solution is re-derived from the Stokes multiplier associated with a special ODE in section 3. In section 4, we apply the working hypothesis obtained in the previous section to the TBA equations for perturbed $\mathcal{N} = 2$ minimal theories in 2D with SU(2)$_k$ and SU($n)_1$ chiral rings. We demonstrate the applications of the exact solutions in section 5. Section 6 is devoted to a summary and future problems.

2. The ODE/IM correspondence

We summarize results from the ODE/IM correspondence which are relevant in the following discussions. For details, see [16].

We consider a simple ODE of $n$th order in the complex plane $x \in \mathbb{C}$,

$$\left((-)^{n-1}\frac{d^n}{dx^n} + \left(x^\alpha - E\right)\right)\psi(x,E) = 0,$$

(1)

where $\alpha \in \mathbb{R}_{3-1}$.

Since it has the irregular singularity at $\infty$, we conveniently divide the complex plane into sectors. Let $S_j$ be a sector in the complex plane,
The sector $S_0$ thus includes the positive real axis.

Let $\phi(x, E)$ be a solution to (1) which decays exponentially as $x$ tends to $\infty$ inside $S_0$, 
\[
\frac{d^p \phi(x, E)}{dx^p} \sim (-1)^p x^{(1-p+2p+i\pi)} \exp\left(-\frac{x^{\alpha+1}}{\alpha+1}\right), \quad x \in S_0
\]
for $p \in \mathbb{Z}_{\geq 0}$.

The crucial observation in [21] is the ‘discrete rotational symmetry’ of (1): the invariance under the simultaneous transformations
\[
\Omega E \rightarrow E \Omega
\]
We then introduce
\[
\phi_j = q^{(1-p+2p+i\pi)} \phi(\Omega x, \Omega E).
\]

Thanks to the discrete rotational symmetry, any $\phi_j (j \in \mathbb{Z})$ is a solution to (1). The set $\{\phi_j\}_{j=0}^{n}$ forms the fundamental system of solutions (FSS) in $S_j$. To see this, we introduce the Wronskian matrix
\[
\Phi_{j_1 \cdots j_n}(x, E) = \begin{vmatrix}
\phi_{j_1} & \cdots & \phi_{j_n} \\
\vdots & \ddots & \vdots \\
\phi_{j_{m-1}} & \cdots & \phi_{j_{m-1}}
\end{vmatrix},
\]
where $m \leq n$. In particular, when suffixes $\{j\}$ are consecutive integers, e.g., $j_k = j + k - 1$, we write simply $\Phi_{j_1 \cdots j_n}(x, E)$ and their determinants $W_{j_1 \cdots j_n}(x, E)$ (we drop the $x$ dependence when $m = n$). By using the asymptotic form (2), one can check that $W_{j_1}(E) = 1$, hence the set $\{\phi_j, \cdots, \phi_{j+n-1}\}$ is linearly independent.

We are interested in the relation among the FSSs in different sectors. Let us start from the relation between $S_0$ and $S_1$. Two Wronskian matrices $\Phi_0^{(n)}$ and $\Phi_1^{(n)}$ are simply connected by
\[
\Phi_1^{(n)} = \Phi_{j_1 \cdots j_n}^{(n)}(E) \Phi_0^{(n)}
\]
where
\[
\mathcal{M}^{(n)}(E) = \begin{pmatrix}
t_{1}^{(1)}(E) & 1 & 0 & \cdots & 0 \\
-t_{1}^{(2)}(E) & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
(-1)^{n-1} t_{1}^{(n)}(E) & 0 & 0 & \cdots & 0
\end{pmatrix}.
\]

The entries $t_{1}^{(n)}(E)$ are called the Stokes multipliers. The discrete rotational symmetry then results in
\[
\Phi_{j}^{(n)} = \Phi_{j+n}^{(n)} \mathcal{M}_{j}^{(n)}(E \Omega)\Omega
\]

The first observation of the ODE/IM correspondence is that this linear relation, evaluated at the origin, can be identified with Baxter’s TQ relation [22] for $n = 2$. That is, let
Then one presents

\[ \tau_1 = \phi_0/\phi_1 + \phi_2/\phi_1 \text{ equivalently as} \]

\[ T_1(E) = q^{2z_1}(E \Omega^{-2}) + q^{z_2}(E \Omega^{-2}) \]

This is known as the dressed vacuum form (DVF) in integrable models.

Suppose that wave functions are given in advance. Then, thanks to the normalization of \( \psi \), the Stokes multipliers are represented by wave functions, e.g.,

\[ \tau_1^{(1)}(E) = W^{(n)}_{0,2,...,n}(E). \]

Let us introduce more generally

\[ \tau^{(a)}_m(E) = W^{(n)}_{0,...,a-1,a+m,...,a+m-1}(E). \]

Some of them appear in the connection problem between \( S_0 \) and \( S_m \) [35].

The identity among Wronskians implies that

\[ \tau^{(a)}_m(E) = \tau^{(a+1)}_m(E) + \tau^{(a-1)}_m(E) \]

where \( 1 \leq a \leq n-1 \), \( m \in \mathbb{Z}_{>1} \). \( \tau^{(a)}_m = \tau^{(0)}_m = 1 \) and \( \tau^{(a)}_0 = 1 \).

After a suitable shift of the parameters and a change of variables \( \tau \rightarrow E \), one arrives at the SU(\( n \)) T-system [29].

\[ T^{(a)}_m(v + i)T^{(a)}_m(v - i) = T^{(a+1)}_m(v)T^{(a-1)}_m(v) + T^{(a)}_{m+1}(v)T^{(a)}_{m-1}(v). \]

The conditions \( T^{(0)}_m(v) = T^{(a)}_m(v) = 1 \) are again imposed. By employing the further transformation [28, 29],

\[ Y^{(a)}_m(v) = \frac{\frac{T^{(a)}_m(v)T^{(a)}_{m+1}(v)}{T^{(a+1)}_m(v)T^{(a-1)}_m(v)}}{\frac{\frac{T^{(a)}_m(v)T^{(a)}_{m+1}(v)}{T^{(a+1)}_m(v)T^{(a-1)}_m(v)}}}. \]

one obtains the SU(\( n \)) Y-system,

\[ Y^{(a)}_m(v + i)Y^{(a)}_m(v - i) = \frac{1 + Y^{(a)}_{m-1}(v)}{1 + Y^{(a)}_{m-1}(v)} \frac{1 + Y^{(a+1)}_{m-1}(v)}{1 + Y^{(a+1)}_{m-1}(v)} \frac{1 + Y^{(a)}_{m+1}(v)}{1 + Y^{(a)}_{m+1}(v)} \frac{1 + Y^{(a-1)}_{m+1}(v)}{1 + Y^{(a-1)}_{m+1}(v)}. \]

The Y-system for ADE scattering models was originally introduced in [30].

It is well known under the assumption of the analytic properties on \( Y^{(a)}_m \) that the above algebraic equations can be transformed into the TBA equations. This also manifests the ODE/IM correspondence.

We can also formulate the problem on the positive real axis. To simplify notations, let us concentrate on the case \( n = 2 \) (the radial Schrödinger problem),

\[ \left( -\frac{d^2}{dx^2} + (x^{2\alpha} - E) + \frac{\ell(\ell + 1)}{x^2} \right) \psi(x, \ell) = 0. \]

This is also regarded as the introduction of a gauge field when rewriting it as

\[ \left( -\frac{d}{dx} - \frac{\ell}{x} \right) \frac{d}{dx} + \frac{\ell}{x} \left( x^{2\alpha} - E \right) \psi(x, \ell) = 0. \]

While in the absence of the gauge field the \( Q \)-function is directly related to the value of the wave function at the origin, as in (8), this is no longer the case in the presence of the gauge
field. It is however shown in [21, 23] that the $Q$-function appears naturally if one considers the connection problem of the FSS near the origin and the FSS at large $x$. Denote two solutions near the origin,

$$\chi^{\pm}(x, E, \ell) \sim \frac{1}{\sqrt{2\ell + 1}}x^{\mp(\ell + \frac{1}{2})}$$

and set more generally, analogously to (4),

$$\chi_j^{\pm}(x, E, \ell) = q^{j\ell}\chi^{\pm}(x, \Omega_j^2 E, \ell).$$

The $x \to 0$ behavior implies that

$$\chi_j^{\pm}(x, E, \ell) = q^{j\ell}(\ell + \frac{1}{2})\chi^{\pm}(x, E, \ell).$$

The ‘radial’ connection relation is given by

$$\phi(x, E, \ell) = D^- (E, \ell)\chi^{-}(x, E, \ell) + D^+(E, \ell)\chi^+(x, E, \ell),$$

or equivalently

$$\phi_j(x, E, \ell) = D^- (E\Omega_j^2, \ell)\chi_j^{-}(x, E, \ell) + D^+(E\Omega_j^2, \ell)\chi_j^+(x, E, \ell).$$

The connection relations among $\phi_j(x, E, \ell)$ assume the same form, e.g. (5). One then derives the DVF in the radial problem as

$$T_j(E, \ell) = q^{j\ell}\frac{D^+(E\Omega_j^2, \ell)}{D^\mp(E, \ell)}.$$

By comparing with (9), one concludes that $D^\pm$ generalizes $Q^\pm$ for the nonzero $\ell$ case. They recover (9) by putting $\ell = 0$. In terms of $D^\mp$, the Wronskian representation of the generalized Stokes multipliers (10) is given by

$$T_j(E, \ell) = q^{-(j+1)(\ell + \frac{1}{2})}D^+(E\Omega_j^{j+1}, \ell)D^- (E\Omega_j^{-j-1}, \ell) - q^{-(j+1)(\ell + \frac{1}{2})}D^- (E\Omega_j^{j+1}, \ell)D^+(E\Omega_j^{-j-1}, \ell).$$

This is to be identified with the quantum Wronskian relation [24], except for a difference in normalization as discussed in [21].

### 3. Revisiting Fendley’s solution

In [14, 15], a class of integrable $\mathcal{N} = 2$ supersymmetric theories in 2D, described by Landau–Ginzburg actions, has been analyzed. For models with spontaneously broken $\mathbb{Z}_n$ symmetry a set of TBA equations has been proposed. Especially in the latter paper, direct relations of the solution to TBA equations and solutions to PIII or to affine Toda equations are argued.

2 We change the sign of $D^+$ from [21].
When the super-potential is given by \( W(X) = \frac{X^3}{3} - X \), the explicit TBA equations read

\[
A(\theta, \mu) = 2u(\theta, \mu) - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \ln \left( 1 + B(\theta', \mu)^2 \right),
\]

\[
B(\theta, \mu) = -\int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{e^{-\hat{A}(\theta', \mu)}}{\cosh(\theta - \theta')},
\]

(21)

In the above, \( u(\theta, \mu) = \mu \cosh \theta \) and \( \mu \) corresponds to a physical mass. It reduces to \( e^{\theta}/2 \) in the massless limit [31].

Fendley found the following explicit solution [10] for the massless case:

\[
e^{-\hat{A}(\theta)} = -2\pi i \frac{d}{dz} (\text{Ai}(z))^2,
\]

\[
B(\theta) = 2\pi i \frac{d}{dz} \text{Ai} \left( z e^{i\frac{\pi}{3}} \right) \text{Ai} \left( z e^{-i\frac{\pi}{3}} \right)
\]

(22)

where \( z = (3\theta/4)^{2/3} \).

We shall re-derive the solution from the ODE side, starting from

\[
\left( -\frac{d^2}{dx^2} + (x - E) \right) \psi(x, E) = 0
\]

(23)

which is the case \((n, \alpha) = (2, \frac{1}{2})\) in (1). It follows from (3) that

\[
q = e^{\frac{2\pi i}{3}}, \quad \Omega = e^{-\frac{2\pi i}{3}}.
\]

It is well known that the Airy function solves this equation. Respecting the leading asymptotic from (2), the desired solution of (23) is given by

\[
\phi(x, E) = \sqrt{\frac{2\pi}{i}} \text{Ai}(x - E).
\]

(24)

This immediately gives\(^3\) \( Q^\pm \) in (8)

\[
\begin{aligned}
Q^-(E) &= \sqrt{\frac{2\pi}{i}} \text{Ai}(-E), \\
Q^+(E) &= -\sqrt{\frac{2\pi}{i}} \frac{d}{dE} \text{Ai}(-E).
\end{aligned}
\]

(25)

We remark that the ODE (23) is not totally independent of the original problem. Although in the \( \mathcal{N} = 2 \) symmetric theory the argument of the super-potential \( W(X) \) is a super-field \( X \), we allow for a usual variable in \( W(x) \). Then the solution \( \phi(x, E) \) has a well known integral representation,

\[
\phi(x, E) = \int e^{(x-E)^{2/3}W(\frac{z(x-E)}{2})} dz.
\]

(26)

The contour must be chosen so as to reproduce the asymptotic behavior (2) of \( \phi \).

\(^3\) Actually, the role played by the Airy function in \( \mathcal{N} = 2 \) SUSY theory, especially its relation to \( Q^\pm \), was first noted in [20], independently from [10], exactly in the context of the ODE/IM correspondence.
This is equivalent to the three terms relation for the Airy function, and it leads to the conclusion $\tau_1 = T_1 = 1$. We can easily check this by using the DVF (9) and (24). By choosing the upper index in (9) we have

$$T_1(E) = e^{-\frac{2}{3}i} \frac{Ai(E e^{\frac{2}{3}i} x)}{Ai(E)} + e^{\frac{2}{3}i} \frac{Ai(E e^{-\frac{2}{3}i} x)}{Ai(E)} = -e^{\frac{2}{3}i} \frac{Ai(E e^{\frac{2}{3}i} x)}{Ai(E)} - e^{-\frac{2}{3}i} \frac{Ai(E e^{-\frac{2}{3}i} x)}{Ai(E)}.$$  

Thus $T_1 = 1$ thanks to (27).

Now the $T$-system is trivially represented as

$$T_1^2 = 1, \quad T_2 = 0. \quad (28)$$

This simply gives a trivial solution of the TBA, $Y_1 = 0$, which is far from Fendley’s solution.

We then ‘deform’ the ODE by the nonzero angular momentum term as suggested in [10],

$$\left( -\frac{d}{dx} - \frac{\ell}{x} \left( \frac{d}{dx} + \frac{\ell}{x} \right) + x - E \right) \psi(x, \ell) = 0. \quad (29)$$

Below we shall argue that this replacement leads to the desired $T, Y$-system and to the TBA.

The Stokes multiplier has the form (19)

$$T_i(E, \ell) = \xi^i D^z \left( E \Omega^{-\frac{1}{2}}, \ell \right) D^{\frac{1}{2}} \left( E \Omega^\ell, \ell \right), \quad (30)$$

where $h = -2\ell \pi$. We assume that the following limit exists:

$$\lim_{\ell \to 0} \frac{1}{\sqrt{2\ell + 1}} D^z(E, \ell) = Qz(E). \quad (32)$$

The quantum Wronskian relation is then rewritten with $\xi$ as

$$T_j(E, \ell) = \xi^{-j+1} D^+ \left( E \Omega^{j+1}, \ell \right) D^{-} \left( E \Omega^{j+1}, \ell \right) - \xi^{j+1} D^{-} \left( E \Omega^{j+1}, \ell \right) D^+ \left( E \Omega^{j+1}, \ell \right). \quad (33)$$

When $q$ is at a root of unity, the SU(2) $T$-system (13) closes among finite elements [25]. In the present case, this is due to a simple relation,

$$T_i(E, \ell) = \xi^3 + \xi^{-3} + T_i(E, \ell). \quad (34)$$

Then one ends up with

$$T_1(E \Omega, \ell) T_1(E \Omega^{-1}, \ell) = 1 + T_3(E, \ell),$$

$$T_3(E \Omega, \ell) T_3(E \Omega^{-1}, \ell) = 1 + T_3(E, \ell) T_3(E, \ell) \equiv \left( \xi^3 + T_3(E, \ell) \right) \left( \xi^{-3} + T_3(E, \ell) \right). \quad (35)$$

In the following, we derive (21) from the above truncated $T$-system as the first nontrivial equation in the expansion of $h$. Then we shall show that a similar expansion of the quantum Wronskian relation (20) yields Fendley’s solution (22).
In this example, the $T$-system is identified with the $Y$-system. This is achieved by introducing

$$Y_i(\theta, \ell) = T_i(E, \ell), \quad Y_i(\theta, \ell) = T_2(E, \ell).$$  \quad (36)

Here the parameter $\theta$ is related to $E$ by

$$E = E_0 e^{\pm \theta},$$  \quad (37)

and the constant $E_0$ will be determined later.

The $Y$-system is represented by new variables as

$$y_i(\theta + \frac{\pi}{2}, \ell) + y_i(\theta - \frac{\pi}{2}, \ell) = y_i(\theta),$$  \quad (38)

$$y_i(\theta + \frac{\pi}{2}, \ell) y_i(\theta - \frac{\pi}{2}, \ell) = y_i(\theta)^2 + 1.$$  \quad (39)

Next we consider the expansion in $h$. The solution (28), strictly at $h = 0$, suggests the expansions

$$y_i(\theta, \ell) = 1 + h y_i(\theta) + O(h^2), \quad y_i(\theta, \ell) = h y_i(\theta) + O(h^2).$$  \quad (40)

The first nontrivial equation in the expansion of (38) is $O(h)$, while it is $O(h^2)$ for (39),

$$y_i(\theta) = -B(\theta), \quad y_i(\theta) = e^{-A(\theta)}$$  \quad (41)

and assume that $y_1$ and $y_2$ are analytic and nonzero in the strip $\Im \theta \in [-\pi/2, \pi/2]$. We also assume that the right-hand sides of (40) are analytic and nonzero in the narrow strip including the real axis of $\theta$. These assumptions are justified by the solution in (49), a posteriori. One then obtains

$$A(\theta) = m_A e^{\theta} + C_A - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta')} \ln\left(1 + B(\theta')^2\right),$$

$$B(\theta) = m_B e^{\theta} + C_B - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta')} e^{-A(\theta')}$$  \quad (43)

where ‘mass terms’ are introduced to take account of the zero mode in the Fourier transformation.

Without loss of generality we can always choose $m_A = 1$ by tuning the origin of $\theta$ (or a redefinition of $E_0$). It will be later shown that $m_B = 0$. The integration constants $C_A, C_B$ are found to be zero. This can be verified from the asymptotic values $e^{-A(-\infty)} = 2/\sqrt{3}, B(-\infty) = -1/\sqrt{3}$.

We have a remark. The quantum sine–Gordon model has $\mathcal{N} = 2$ supersymmetry at a special coupling constant. Fendley et al [14] utilized this and started from the TBA for the generic quantum sine–Gordon model. Then they took a similar limit in the above and derived (43). Here the initial point is different: we start from the ODE.

The above observation concludes that the expansion of $T_{1,2}$ in $h$ yields the desired TBA equations (if $m_B = 0$).
We then use (33) to obtain explicit solutions, given the data

\[
\lim_{\ell \to 0} \frac{1}{\sqrt{2\ell + 1}} D^-(E, \ell) = Q^-(E) \approx \frac{2\pi}{\sqrt{i}} \text{Ai}(-E),
\]

\[
\lim_{\ell \to 0} \frac{1}{\sqrt{2\ell + 1}} D^+(E, \ell) = Q^+(E) \approx -\frac{2\pi}{\sqrt{i}} \frac{d}{dE} \text{Ai}(-E).
\]  

(44)

This looks hopeless at first sight, as the expansion of the right-hand side of (33) contains derivatives of \(D^z\) at \(h = 0\) which are unknown to us. By fortunate cancellations of derivative terms, we nevertheless find it possible. First consider the case \(j = 0\) in (33) where \(T_0 = 1\). The \(O(h^0)\) and the \(O(h^1)\) equations read respectively

\[
e^{-\bar{\xi} Q^+ (E\Omega)Q^- (E\Omega^{-1})} - e^{\bar{\xi} Q^- (E\Omega)Q^+ (E\Omega^{-1})} = 1,
\]  

\[
e^{-\bar{\xi} Q^- (E\Omega)Q^+ (E\Omega^{-1})} = \frac{i}{3} \left( e^{\bar{\xi} Q^- (E\Omega)Q^+ (E\Omega^{-1})} + e^{-\bar{\xi} Q^+ (E\Omega)Q^- (E\Omega^{-1})} \right).
\]  

(45)

Next consider \(j = 1\) in (33). The \(O(h^0)\) term on the right-hand side is found to be unity using (45) (replacing \(E\) by \(-E\)), while the \(O(h^1)\) terms contain derivative terms of \(h\). We find that these derivative terms can be completely rewritten in terms of \(Q^2\) thanks to (46). Altogether, one obtains

\[
T_1(E) = 1 - i \left( e^{\bar{\xi} Q^- (-E\Omega)Q^+ (-E\Omega^{-1})} + e^{-\bar{\xi} Q^+ (-E\Omega)Q^- (-E\Omega^{-1})} \right) + O(h^2).
\]  

(47)

Third, take \(j = 2\) in (33). It is simplified, as \(\Omega^2 = -1\),

\[
T_2(E) = -\bar{\xi}^3 D^+ (E\Omega^3, \ell)D^- (E\Omega^{-3}, \ell) = -2i\bar{h}Q^+ (-E)Q^- (-E) + O(h^2).
\]  

(48)

Then from equations (36), (40), (42), (47) and (48) we conclude

\[
B(\theta) = -i \left( e^{\bar{\xi} Q^- (-Ee^{-\bar{\xi}})Q^+ (-Ee^{\bar{\xi}})} + e^{-\bar{\xi} Q^+ (-Ee^{-\bar{\xi}})Q^- (-Ee^{\bar{\xi}})} \right)
\]

\[
= 2\pi \frac{d}{dE} \text{Ai}(Ee^{\bar{\xi}}) \text{Ai}(Ee^{\bar{\xi}}),
\]

\[
e^{-A(\theta)} = -2iQ^+ (-E)Q^- (-E) = -2\pi \frac{d}{dE} \text{Ai}(E)^2.
\]  

(49)

where we use (44).

Finally, let us check \(m_B = 0\) in (43) and evaluate \(E_0\) in (37). This is done by evaluating the left-hand side of (43) in the limit \(\theta \to \infty\) or, equivalently, \(E \to \infty\). The convolution terms do not contribute since the integration kernel becomes exponentially small. One easily evaluates the asymptotic behavior \(\Re E \gg 1\) from (49),

\[
A(\theta) \sim \frac{4}{3} E^{2/3}, \quad B(\theta) \sim -\frac{1}{4} E^{-2/3} + O(E^{-1}).
\]  

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Using these in the second equation of (43), we conclude that \( m_B = 0 \), while the first equation, with the convention \( m_A = 1 \), leads to

\[
\frac{4}{3} E^2 = e^{\theta} \quad \text{or} \quad E = \left( \frac{3}{4} e^{\theta} \right)^{\frac{1}{2}}.
\]

By identifying \( z \) with \( E \) in (22), we thus conclude that Fendley’s solution is successfully recovered from the ODE.

4. Generalizations

Let us summarize our findings so far. The input is the super-potential \( W(x) \). Once this is fixed, we construct a wave function (26) which solves a simple ODE (23). We then ‘deform’ the ODE by the angular momentum term as in (29). This makes the associated Stokes multipliers nontrivial. Then the first nontrivial response of the Stokes multipliers with respect to the small angular momentum yields Fendley’s solution.

Since there exists a list of super-potentials for \( \mathcal{N} = 2 \) supersymmetric theories in 2D [17–19], one naturally wonders if the above procedure works starting from other super-potentials. Below we shall discuss the super-potential of the types \( SU(2)_k \) and \( SU(3)_1 \), which provide affirmative evidence for this expectation.

4.1. Exact solution: \( SU(2)_k \)

In this case, the relevant super-potential takes the form

\[
W_k(x = e^{i\theta} + e^{-i\theta}) = \frac{2}{k + 2} \cos(k + 2)\theta.
\]

More explicitly,

\[
W_1(x) = \frac{x^3}{3} - x, \quad W_2(x) = \frac{1}{4} (x^4 - 4x^2 + 2),
\]

\[
W_3(x) = \frac{1}{5} (x^5 - 5x^3 + 5x), \quad W_4(x) = \frac{1}{6} (x^6 - 6x^4 + 9x^2 - 2)
\]

and so on.

According to the above strategy we first construct a wave function,

\[
\phi^{(k)}(x, E) = \int_c e^{(x-E)^{k+2}} W_k(z(x-E)^{-1/2}) dz.
\]  

The contour should meet the requirement that \( \phi^{(k)}(x, E) \) is exponentially decreasing on the real axis of \( x \).

Note that (26) is contained as the \( k = 1 \) case. We immediately see that \( \phi^{(k)}(x, E) \) satisfies an ODE,

\[
\left( -\frac{d^2}{dx^2} + (x - E)^k \right) \phi^{(k)}(x, E) = 0.
\]
As before, we interpret this as the \( (n, \alpha) = (2, \frac{1}{2}) \) case of a generalized ODE,

\[
\left( -\frac{d^2}{dx^2} + (x^{2n} - E)^2 \right) \phi^{(k)}(x, E) = 0,
\]

with vanishing boundary condition for \( \Re x \gg 1 \).

This has been proposed to be the ODE for the spin \( \frac{k}{2} \) SU(2) case with \( q = e^{i\pi/(2k+1)} \) \cite{26, 27}.

We have, analogously to (25),

\[
Q^-(E) = \frac{2E_i}{(k+2)\pi} K_{\frac{k+1}{2}} \left( \frac{2}{k+2} (-E)^{\frac{k+1}{2}} \right),
\]

\[
Q^+(E) = -\frac{d}{dE} Q^-(E)
\]

where \( K_{\nu} \) stands for the modified Bessel function.

We then include the angular momentum term with the effect

\[
Q^-(E) \rightarrow D^{-}(E, \ell), \quad Q^+(E) \rightarrow D^{+}(E, \ell).
\]

The Stokes multiplier takes the same form as (30), while the parameters take different values,

\[
\Omega = e^{-\frac{\pi i}{\pi}}, \quad \xi = e^{\frac{\pi i}{\pi}}.
\]

The SU(2) T-system remains valid, while (34) is replaced by

\[
T_{k+2}(E, \ell) = \xi^{k+2} + \xi^{-(k+2)} + T_{k}(E, \ell).
\]

The transformation from the T-system to the Y-system is accomplished by \cite{28}

\[
Y_j(\theta, \ell) = T_{j-1}(E, \ell)T_{j+1}(E, \ell) \quad (1 \leq j \leq k),
\]

\[
Y_{k}(\theta, \ell) = T_{k}(E, \ell),
\]

\[
E = E_{0}^{(k)} e^{\frac{\pi i}{\pi} \theta}
\]

where \( T_{0} \) is set to be unity. The coefficient \( E_{0}^{(k)} \) will be determined later.

We obtain as a result

\[
Y_{j}^{(\ell)}(\theta + \frac{\pi i}{2}, \ell)Y_{j}^{(\ell)}(\theta + \frac{\pi i}{2}, \ell) = \left( 1 + Y_{j-1}(\theta, \ell) \right) \left( 1 + Y_{j+1}(\theta, \ell) \right) \quad (1 \leq j \leq k - 1),
\]

\[
Y_{k}^{(\ell)}(\theta + \frac{\pi i}{2}, \ell)Y_{k}^{(\ell)}(\theta + \frac{\pi i}{2}, \ell) = \left( 1 + Y_{k-1}(\theta, \ell) \right) \left( 1 + \xi^{k+2} \xi^{-(k+2)} Y_{k}(\theta, \ell) \right) \left( 1 + \xi^{-(k+2)} Y_{k}(\theta, \ell) \right),
\]

\[
Y_{j}^{(\ell)}(\theta + \frac{\pi i}{2}, \ell)Y_{j}^{(\ell)}(\theta + \frac{\pi i}{2}, \ell) = \left( 1 + \xi^{k+2} Y_{j}(\theta, \ell) \right) \left( 1 + \xi^{-(k+2)} Y_{j}(\theta, \ell) \right)
\]

where we set \( Y_{0} = 0 \).

By strictly setting \( h = -2\ell \pi = 0 \), we obtain constant solutions for \( t_{0,j} = T_{j}(E, 0) \)

\[
t_{0,j} = \frac{\sin \left( \frac{(j+1)\pi}{2} \right)}{\sin \left( \frac{\pi}{2} \right)} \quad 1 \leq j \leq k + 1;
\]

in particular, \( t_{0,k+1} = 0 \). Then \( Y_{j}(\theta, 0) \) is determined by (56). We assume the expansion around \( h = 0 \),
\[ T_j(\theta, \ell) = t_{0,j} + h t_{1,j}(\theta) + O(h^2) \quad (1 \leq j \leq k + 1), \]
\[ Y_j(\theta, \ell) = y_{0,j} + h y_{1,j}(\theta) + O(h^2) \quad (1 \leq j \leq k), \]
\[ Y_0(\theta, \ell) = 1 + h y_{1,0}(\theta) + O(h^2). \]  

(59)

Note that \( y_{0,j} \neq 0 \) if \( j < k \) and \( Y_j(\theta, \ell) = h v_j(\theta) \).

This leads to an important consequence. Although the original \( Y \)-system (58) consists of \( k + 1 \) equations among \( k + 1 \) \( Y \)-functions, the first nontrivial relations close only among \( y_{1,k}(\theta) \) and \( y_{1,1}(\theta) \),

\[ y_{1,k}(\theta + \pi i/2) y_{1,k}(\theta - \pi i/2) = (t_{0,k-1})^2 \left( y_{1,1}(\theta)^2 + 1 \right), \]
\[ y_{1,1}(\theta + \pi i/2) + y_{1,1}(\theta - \pi i/2) = y_{1,k}(\theta). \]

They are very similar to (41) by identifying \( y_{1,1} = y_1 \). Consequently, one obtains the analogous TBA

\[ A(\theta) = m_A e^{\theta} - \ln 2 \cos \frac{\pi}{k + 2} - \int \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta')} \ln \left( 1 + B(\theta')^2 \right), \]
\[ B(\theta) = m_B e^{\theta} - \int \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta')} e^{-A(\theta')} \]  

where

\[ y_{1,1}(\theta) = -B(\theta), \quad y_{1,k}(\theta) = e^{-A(\theta)}. \]  

(61)

The integration constants are fixed by the asymptotic values,

\[ e^{-A(\infty)} = 2 \cot \frac{\pi}{k + 2}, \quad B(-\infty) = -\cot \frac{\pi}{k + 2}. \]

As before we choose \( E_0^{(s)} \) in (57) such that \( m_A = 1 \). Below we shall argue that \( m_B = 0 \). Then the resultant TBA agrees with the result in [15] (for \( \Theta = \pi/k + 2 \)). We again remark that (60) was derived in [15] by taking a limit from the TBA for the \( \mathcal{N} = 0 \) sine–Gordon model at a special coupling constant which consists of \( k + 1 \) integral equations.

Now we are in a position to derive the explicit solutions for \( A(\theta) \) and \( B(\theta) \), or equivalently \( y_{1,k}(\theta) \) and \( y_{1,1}(\theta) \). Thanks to (56) and (59), one immediately finds

\[ y_{1,1}(\theta) = t_{1,k}(E), \quad y_{1,k}(\theta) = t_{0,k-1} t_{1,k+1}(E). \]  

(62)

The right-hand side of the above equations can be evaluated through the quantum Wronskian relations (33) with \( j = 0 \), \( k \) and \( k + 1 \). Note that \( \Omega \) and \( \xi \) are given in (55). After simple manipulations, we obtain

\[ y_{1,k}(\theta) = -2i t_{0,k-1} Q^-(-\Omega) Q^+(-\Omega), \]
\[ y_{1,1}(\theta) = -i \left( e^{i\pi/3} Q^-(\Omega) Q^+(\Omega^{-1}) + e^{-i\pi/3} Q^-(\Omega) Q^+(\Omega^{-1}) \right). \]

By the use of (53), solutions are represented explicitly. Let

\[ A_{11}^{(k)}(E) = \frac{1}{\pi} \sqrt{\frac{E}{k + 2}} K_{\pi} \left( \frac{2}{k + 2} E^{\pi/2} \right) \]  

(63)

which reduces to the Airy function \( \text{Ai} \) if \( k = 1 \).
Then one finds solutions which generalize (49) naturally,
\[ B(\theta) = 2\pi \frac{d}{dE} A^{(k)}(E\Omega) \Gamma^{(k)}(E\Omega^{-1}), \]
\[ e^{-A(\theta)} = -4\pi \cos \frac{\pi}{k + 2} \frac{d}{dE} A^{(k)}(E)^2. \]  

(64)

Then we substitute the above explicit solution into (60) and take the limit \( \theta \to \infty \). From the known asymptotic behavior
\[ A^{(k)}(E) \sim \frac{E^{-\frac{2}{k}} e^{-\frac{\pi}{2k^2}}}{2\sqrt{\pi}}, \]
one easily checks that \( m_B = 0, m_A = 1 \) and
\[ E = E_0^{(k)} e^{\frac{\pi}{2k^2}}, \quad E_0^{(k)} = \left( \frac{k + 2}{4} \right)^{\frac{3}{2k^2}}. \]

We assume \( k \in \mathbb{N} \) in deriving TBA (60). Once it is obtained, however, \( k \) enters as a mere parameter. One can take, for example, \( k \in \mathbb{R}_{>0} \). We have checked numerically that (64) still satisfies (60) in this case.

4.2. Exact solution: SU(3)_1

Next consider the perturbation of the SU(3)_1 type. The super-potential reads
\[ W(z, x) = \frac{z^4}{4} - xz. \]

According to our working hypothesis, we introduce
\[ \phi(x, E) = \int_{c} e^{W(z, x - E)} dz \]
which satisfies the third order ODE,
\[ \left( \frac{d^3}{dx^3} + x - E \right) \phi(x, E) = 0. \]

(65)

As always, we choose \( C \) such that the asymptotic behavior of \( \phi \) agrees with (2).

We regard this as a special case of (1), with \( n = 3 \) and \( \alpha = 1/3 \). The analysis in [32, 33] shows that the ODE is related to CFT with \( A^{(3)}_2 \) symmetry.

The explicit solution to (65) with the desired property (2) is given by Meijer’s G-function or a linear combination of confluent hypergeometric series as discussed in appendix B. Here we do not specify its explicit form but use a symbol \( \varphi \):
\[ \varphi(x - E) = \phi(x, E). \]

(66)

We define, for later use,
\[ Q^{[0]}(E) = \varphi(-E), \quad Q^{[1]}(E) = -\frac{d}{dE} \varphi(-E), \]
\[ Q^{[2]}(E) = \frac{1}{2} \frac{d^2}{dE^2} \varphi(-E). \]

(67)
As before, we consider a ‘radial’ ODE [34]

\[ (D(\gamma_2 - 2)D(\gamma_1 - 1)D(\gamma_0) + x - E)\psi(x, E, g) = 0. \] (68)

The operator \( D(g) \) is defined by

\[ D(g) := \frac{d}{dx} - \frac{g}{x}. \]

The parameters \( g_i \) are constrained by

\[ g_0 + g_1 + g_2 = 3. \]

We denote by \( \phi(x, E, g) \) the solution which behaves as (2) in \( S_0 \), where \( g \) stands for \( \{g_0, g_1, g_2\} \).

We further introduce

\[ \chi_j(x, E, g) = q^j\phi(q^{-j}x, \Omega^3E, g) \] (69)

where \( q = e^{2\pi i/(3\alpha + 3)} \) and \( \Omega = q^{-a} \). Then any \( \phi_j(x, E, g), j \in \mathbb{Z} \) is also a solution to the ODE.

The connection relations among \( \{\phi_j\} \) remain formally the same as (5), although the components in (6) now possess a dependence on \( g \).

We denote by \( \{\chi^{[i]}\} \) another FSS of (68) near the origin characterized by the behavior for \( x \to 0 \),

\[ \chi^{[i]}(x, E, g) \sim \mathcal{N}_g x^i \quad (i = 0, 1, 2), \quad \mathcal{N}_g = \left( \prod_{0 \leq j < k \leq 2} (g_i - g_j)^{-1} \right). \]

The ordering \( \Im g_0 < \Im g_1 < \Im g_2 \) is assumed from now on. The normalization factor \( \mathcal{N}_g \) is chosen so that the Wronskian determinant of \( \{\chi^{[0]}, \chi^{[1]}, \chi^{[2]}\} \) is unity. Following (69), we set

\[ \chi_j^{[i]}(x, E, g) = q^j\chi^{[i]}(q^{-j}x, \Omega^3E, g) = q^j\chi^{[i]}(q^{-j}x, q^{-j}E, g). \]

It can be shown that, as in the case of SU(2) [21],

\[ \chi_j^{[i]}(x, E, g) = q^{j(1-i)}\chi^{[i]}(x, E, g). \]

We set, analogously to (18),

\[ \phi(x, E, g) = \sum_{i=0}^{2} D^{[i]}(E, g)\chi^{[i]}(x, E, g) \]

or slightly more generally

\[ \phi_j(x, E, g) = \sum_{i=0}^{2} D^{[i]}(E, g)\chi_j^{[i]}(x, E, g). \] (70)

When \( \{g_0, g_1, g_2\} = \{0, 1, 2\} \) the original ODE (65) is recovered. This implies the limit

\[ \lim_{\{g_0, g_1, g_2\} \to \{0, 1, 2\}} \mathcal{N}_g D^{[i]}(E, g) = Q^{[i]}(E). \] (71)

---

4 We are now considering \( a = \frac{1}{3} \) thus explicitly \( q = \Omega^{-3} = e^{i\pi/2} \).
The generalized Stokes multipliers
\[ \tau^{(1)}_m(E, g) = W^{(3)}_{0,1,m+2}(E, g), \quad \tau^{(2)}_m(E, g) = W^{(3)}_{0,1,m+2}(E, g) \] (72)
are expressible in terms of \( D^{(1)}(E, g) \), using (70) in the above,
\[ \tau^{(1)}_m(E, g) = \sum_{\sigma} \text{sgn} \alpha q_{(m+1)}(1-\alpha) q_{(m+2)}(1-\alpha) \times D^{(1)}_{\sigma_1}(E, g) D^{(1)}_{\sigma_2}(E, g), \]
\[ \tau^{(2)}_m(E, g) = \sum_{\sigma} \text{sgn} \alpha q_{(m+1)}(1-\alpha) q_{(m+2)}(1-\alpha) \times D^{(1)}_{\sigma_1}(E, g) D^{(1)}_{\sigma_2}(E, g). \] (73)
where \( \sigma \) signifies the permutation of \{0, 1, 2\}.

They also have the DVF representations [34]. The explicit forms are given in appendix A for \( \tau^{(1)}_1 \) and \( \tau^{(2)}_1 \). The results there suggest that it is convenient to set
\[ T^{(a)}_m(E) = \tau^{(a)}_m(Eq^{\frac{a}{a}} + i, g) \] (74)
and
\[ E = E_0 e^{i\theta}. \] (75)
The \( g \) dependence is dropped from \( T \). The SU(3) \( T \)-system is then recovered,
\[ T^{(a)}_{m}(\theta + \frac{\pi}{3},i) T^{(a)}_{m}(\theta - \frac{\pi}{3},i) = T^{(a+1)}_{m}(\theta) T^{(a-1)}_{m}(\theta) + T^{(a)}_{m-1}(\theta) T^{(a)}_{m+1}(\theta) \] (76)
where \( a = 1, 2 \) and \( m \geq 1 \). We have set \( T^{(0)}_{m} = T^{(3)}_{m} = 1 \) and \( T^{(a)}_{0} = 1 \) and used \( q = e^{i\theta} \). We perform a transformation similar to (14) [29],
\[ Y^{(a)}_{m}(\theta) = \frac{T^{(a)}_{m-1}(\theta) T^{(a)}_{m+1}(\theta)}{T^{(a-1)}_{m}(\theta) T^{(a+1)}_{m}(\theta)}. \] (77)
This yields the SU(3) \( Y \)-system
\[ Y^{(a)}_{m}(\theta + \frac{\pi}{3}) Y^{(a)}_{m}(\theta + \frac{\pi}{3}) = \frac{1 + Y^{(a)}_{m-1}(\theta) + Y^{(a)}_{m+1}(\theta)}{1 + Y^{(a-1)}_{m}(\theta) + Y^{(a+1)}_{m}(\theta)}. \] (78)
where \( a = 1, 2, m \geq 1 \), and set \( (Y^{(0)}_{m})^{-1} = (Y^{(3)}_{m})^{-1} = 0, Y^{(a)}_{m} = 0. \)

Note that, in contrast to the SU(2) case, the truncation of the \( Y \)-system (78) to a finite set does not occur.\(^5\)

From now on, we consider the special choice of \( g_n \)
\[ g_0 = \frac{h}{2\pi}, \quad g_1 = 1 + \frac{h}{2\pi}, \quad g_2 = 2 - \frac{h}{\pi} \] (79)
and take the limit \( h \to 0. \)

\(^5\) There is, however, an elaborate way to introduce a set of nonlinear equations which truncate among finite elements [36]. We however do not adopt this approach here.
We can easily check (for $a=1, 2$, $k \in \mathbb{Z}_{\geq 0}$) that
\[
\lim_{h \to 0} T^{(a)}_{4k} = \lim_{h \to 0} T^{(a)}_{4k+1} = 1, \quad \lim_{h \to 0} T^{(a)}_{4k+2} = \lim_{h \to 0} T^{(a)}_{4k+3} = 0. \tag{80}
\]
This motivates us to assume the expansions,
\[
T^{(a)}_m(\theta) = 1 + h t^{(a)}_{1,m}(\theta) + h^2 t^{(a)}_{2,m}(\theta) + O\left( h^3 \right) \quad m = 4k, 4k + 1, \tag{81}
\]
\[
T^{(a)}_m(\theta) = h t^{(a)}_{1,m}(\theta) + h^2 t^{(a)}_{2,m}(\theta) + O\left( h^3 \right) \quad m = 4k + 2, 4k + 3.
\]

Similarly consider the expansions of the $Y$-function,
\[
Y^{(a)}_m(\theta) = y^{(a)}_{0,m} + h y^{(a)}_{1,m}(\theta) + h^2 y^{(a)}_{2,m}(\theta) + O\left( h^3 \right). \tag{82}
\]

By using the $k = 1$ case of (81) in (77) we obtain
\[
Y^{(a)}_1(\theta) = h t^{(a)}_{1,1}(\theta) + h^2 \left( -i t^{(a)}_{1,1}(\theta) t^{(a)}_{1,2}(\theta) + t^{(a)}_{2,2}(\theta) \right) + O\left( h^3 \right) \tag{83}
\]
for $(a, a) = (1, 2)$ or $(2, 1)$. Substituting these into (78), one deduces expansions for other $Y$-functions,
\[
Y^{(a)}_2(\theta) = -1 + h y^{(a)}_{1,2}(\theta) + h^2 y^{(a)}_{2,2}(\theta) + O\left( h^3 \right),
\]
\[
Y^{(a)}_3(\theta) = -1 + h y^{(a)}_{1,3}(\theta) + h^2 y^{(a)}_{2,2}(\theta) + O\left( h^3 \right) \quad (a = 1, 2) \tag{84}
\]
and so on. There are complex expressions of $y^{(a)}_{j,m}$ in terms of $t^{(a)}_{j,m}$, which we shall omit.

The first nontrivial relations of the case $m = 1$ in (78) exist at $O(h^2)$,
\[
y^{(a)}_{1,1} \left( \frac{\pi}{3} + 1 \right), y^{(a)}_{1,1} \left( \frac{\pi}{3} - 1 \right) = y^{(a)}_{1,1}(\theta) y^{(a)}_{2,1}(\theta) \tag{85}
\]
where $(a, a) = (1, 2)$ or $(2, 1)$.

Next consider the $m = 2$ case. The $O(h^3)$ equations require
\[
y^{(a)}_{1,3}(\theta) = -y^{(a)}_{1,2}(\theta) \tag{86}
\]
while the $O(h^4)$ equations yield
\[
y^{(a)}_{1,2} \left( \frac{\pi}{3} + 1 \right) + y^{(a)}_{1,2} \left( \frac{\pi}{3} - 1 \right) = -y^{(a)}_{1,1}(\theta) + y^{(a)}_{1,2}(\theta) + \frac{y^{(a)}_{2,3}(\theta) + y^{(a)}_{2,2}(\theta)}{y^{(a)}_{1,2}(\theta)}. \tag{87}
\]

This again significantly differs from the SU(2) case. The equations do not close among $y^{(a)}_{1,m}$, i.e. the first order coefficients in $h$. To determine the last terms in (87), one must consider equations containing $y^{(a)}_{3,m}$ and so on. This leads to an infinite hierarchy of equations.

There are, however, amazing cancellations. In appendix C, it will be shown that as a direct consequence of (73), the last terms in (87) are simplified drastically,
\[
\frac{y^{(a)}_{2,3}(\theta) + y^{(a)}_{2,2}(\theta)}{y^{(a)}_{1,2}(\theta)} = \begin{cases} 3i & a = 1, \\ -3i & a = 2. \end{cases} \tag{88}
\]

Thus equations (85) and (87) provide the closed relations among $y^{(a)}_{1,m}$ $(a = 1, 2)$.

The result can be neatly written down by introducing
\[
y^{(a)}_{1,1}(\theta) = e^{-A^{(a)}(\theta)}, \quad y^{(1)}_{1,2}(\theta) = B_0(\theta) + i, \quad y^{(2)}_{1,2}(\theta) = B_0(\theta) - i. \tag{89}
\]
Then we have
\[
e^{-A_1(\theta+\frac{\pi}{4})-A_1(\theta-\frac{\pi}{4})} + A_2(\theta) = B_0(\theta) + i,
\]
\[
e^{-A_2(\theta+\frac{\pi}{4})-A_2(\theta-\frac{\pi}{4})} + A_1(\theta) = B_0(\theta) - i,
\]
\[
B_0\left(\theta + \frac{\pi}{3}\right) + B_0\left(\theta - \frac{\pi}{3}\right) - B_0(\theta) = -e^{-A_1(\theta)},
\]
\[
B_0\left(\theta + \frac{\pi}{3}\right) + B_0\left(\theta - \frac{\pi}{3}\right) - B_0(\theta) = -e^{-A_2(\theta)}.
\]

Under suitable assumptions on analyticity, we obtain the following TBA:

\[
A_r(\theta) = m_r e^\theta - \sum_{\ell=0,1} \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \Phi_{r,\ell}(\theta - \theta') \ln(ia_\ell + B_\ell(\theta')) \quad (r = 1, 2),
\]
\[
B_\ell(\theta) = -\sum_{r=1,2} \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \Phi_{r,\ell}(\theta - \theta') e^{-A_r(\theta')}, \quad (\ell = 0, 0)
\]

(90)

where \(a_0 = -a_0 = 1\) and

\[
\Phi_{1,0}(\theta) = \Phi_{2,0}(\theta) = \frac{\sin \frac{\pi}{3}}{\cosh \theta - \cos \frac{\pi}{3}},
\]

\[
\Phi_{1,0}(\theta) = \Phi_{2,0}(\theta) = \frac{\sin \frac{\pi}{3}}{\cosh \theta + \cos \frac{\pi}{3}}.
\]

We have used the limiting values

\[
e^{-A_1(-\infty)} = \frac{3}{2\sqrt{2}} e^{\frac{\pi}{4}}, \quad e^{-A_2(-\infty)} = \frac{3}{2\sqrt{2}} e^{-\frac{\pi}{4}},
\]

\[
B_0(-\infty) = \frac{3 + i}{4}, \quad B_0(-\infty) = \frac{3 - i}{4}
\]

to fix the integration constants. The mass coefficients \(m_r\) can be set to unity with proper choice of \(E_0\) in (75).

Next let us discuss the solutions in terms of \(\varphi\) in (66). Our strategy is similar to the SU(2) case. Expand the Wronskian relations (72) in powers of \(h\). Use the fact \(i^{(1)} = 1\) to replace the derivatives of \(D_{ij}\) by \(D_{ij}^{(1)}\) taking (71) into account. Then use (67) and represent the result by \(\varphi\).

There is, of course, no guarantee that all derivatives can be rewritten by this trick. We however found that, parallel to the SU(2) case, this replacement can be made successfully.

As the analogous argument is presented for (88) in appendix C, we shall omit details and write down the final results,

\[
e^{-A_1(\theta)} = i^{(1)}_{12}(\theta) = 3\omega^3 w_E \left[ \varphi \left( E\omega^{-1} \right), \varphi \left( E\omega^3 \right) \right] \frac{d^2}{dE^2} \varphi \left( E\omega^{-1} \right),
\]

\[
e^{-A_2(\theta)} = i^{(1)}_{12}(\theta) = 3\omega^{-3} w_E \left[ \varphi \left( E\omega^{-1} \right), \varphi \left( E\omega^3 \right) \right] \frac{d^2}{dE^2} \varphi \left( E\omega \right),
\]

(91)

where \(\omega = e^{\frac{\pi}{4}}\) and \(w_E[f, g] = f \frac{d}{dx} g - g \frac{d}{dx} f\).
We use (C.97) in appendix C to write down the solutions for $B_0$ and $B_0^\circ$,

$$B_0(\theta) = y_{1,2}^{(1)}(\theta) + i = 3\omega^{-1}w_E \left[ \varphi(E\omega^{-3}), \varphi(E\omega^{-5}) \right]^\frac{d^2}{dE^2} \varphi(E\omega) + i,$$

$$B_0(\theta) = y_{1,2}^{(1)}(\theta) - i = -3\omega^{-1}w_E \left[ \varphi(E\omega), \varphi(E\omega^5) \right]^\frac{d^2}{dE^2} \varphi(E\omega^{-3}) - i. \quad (92)$$

By using the $n=3$ case of (2) in (91) and (92), the $\theta \to \infty$ asymptotic forms of $A$, $B_0$ and $B_0^\circ$ are easily derived. Substituting these into (90), we can fix $E_0$ in (75) (with $m_r = 1$)

$$E_0 = \left( \frac{4}{3\sqrt{3}} \right)^\frac{1}{2}.$$

5. Applications

5.1. Analytic evaluation of the Cecotti–Fendley–Intriligator–Vafa index

We have been able to derive some explicit solutions for the $\mathcal{N}=2$ TBA. This may open up the possibility to investigate these systems in a quantitative manner. As an application of the above results, we discuss the analytic evaluation of the Cecotti–Fendley–Intriligator–Vafa (CFIV) index $Q_{\text{CFIV}}$ [15] for the SU(2)$_k$ case in the massless limit. The CFIV index is defined by

$$\text{Tr}(-1)^F e^{-\beta H}$$

where $F$ denotes the Fermion number. This is reformulated in the TBA framework, and it is expressed as

$$Q_{\text{CFIV}}(\mu) = \mu \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh \theta e^{-A(\theta, \mu)}$$

where $\mu = m\beta$ and $m$ is the mass. From the viewpoint of $\mu^*$ geometry, $\mu$ corresponds to the radial coordinate and $Q_{\text{CFIV}}(\mu)$ is expressible in terms of the solution to the PIII equation. We are interested in the massless limit. In this case $Q_{\text{CFIV}}$ is a constant and is given by

$$Q_{\text{CFIV}} = 2 \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} e^{-A(\theta)}$$

where the factor 2 takes account of the contributions from the left and right edges. By change of integration variables from $\theta$ to $E$, we have

$$Q_{\text{CFIV}} = 2 \int_0^\infty dE e^{-A(E)}$$

$$= 16\pi \cos \frac{\pi}{k + 2} \int_0^{\infty} dE e^{\frac{k}{2}A^{(3)}(E)} \frac{d}{dE} A^{(3)}(E).$$

Using the recursion relation for the modified Bessel function, $\frac{d}{dE} A^{(3)}(E)$ is given by a sum of three terms. We introduce the notation

$$G_{\sigma}(p, q, \ell) = \int_0^\infty dE e^{2n+\ell+1} K_{\frac{E+1}{2}}(\frac{E+1}{n+1}) K_{\frac{E+1}{2}}(\frac{E+1}{n+1}).$$
Then $Q_{\text{CFIV}}$ is expressed as

$$Q_{\text{CFIV}} = \frac{8}{\pi (k + 2)} \left( G_{2\frac{1}{2}} \left( \frac{1}{2}, \frac{1}{2}, k - 1 \right) - G_{2\frac{1}{2}} \left( \frac{1}{2}, -\frac{k + 1}{2}, 0 \right) - G_{2\frac{1}{2}} \left( \frac{1}{2}, \frac{k + 3}{2}, 0 \right) \right).$$

On the other hand, $G_n(p, q, \ell)$ has already been evaluated in [39],

$$G_n(p, q, \ell) = \left( \frac{2(n + 1)!}{4^2 \Gamma \left( \frac{2 + \ell}{n + 1} \right) \Gamma \left( 1 + \frac{\ell}{2(n + 1)} \right) \Gamma \left( 1 + \frac{-p + q + \ell}{2(n + 1)} \right) \Gamma \left( 1 + \frac{-p - q + \ell}{2(n + 1)} \right) \right)^{2(n + 1)}$$

Substituting these results, we find that the resultant $Q_{\text{CFIV}}$ is simplified considerably,

$$Q_{\text{CFIV}} = \frac{k}{k + 2}$$

which agrees with the known result in [15].

5.2. Sub-leading perturbations

In [19], sub-leading perturbation potentials are also discussed, for example,

$$W(z, t) = z^6 - \frac{t^2}{2}$$

which is simply rewritten as $t^{3/2} W(z^2/t^{1/2})/2$. From the point of view of functional integrals over super-fields, it is argued that $Q_{\text{CFIV}}$ must be twice that for $W(z)$. Let us interpret this in terms of an ODE.

According to our working hypothesis, consider

$$\psi(t) = \int_C e^{W(z, t) dz}.$$ 

This satisfies the third order ODE

$$\frac{d^3}{dt^3} \psi(t) = \frac{1}{8} \left( \psi(t) + 2 t \frac{d}{dt} \psi(t) \right)$$

or in terms of a rescaled variable $x = t/(2)^{3/2}$

$$\frac{d^3}{dx^3} \phi(x) - 4x \frac{d}{dx} \phi(x) - 2\phi(x) = 0$$

where $\phi(x) = \psi(2^{3/2} x)$.

It is well known that Ai$(x)$, Bi$(x)$ and Ai$(x)$Bi$(x)$ are solutions to this ODE. Now we shift $x \to x - E$ and take $\phi_0^2$, $\phi_1^2$ and $\phi_j \phi_{j+1}$ as FSS in $S_j$, where $\phi_j$ is defined in (4). Indeed, it is easily checked that

$$W \left[ \phi_0^2, \phi_1^2, \phi_0 \phi_1 \right] = 2 \left( W \left[ \phi_0, \phi_1 \right] \right)^2$$

where $W$ denotes the Wronskian determinant. Thus the three functions are linearly independent. The connection between $S_0$ and $S_{-1}$ is easily solved. It follows from the three term relation of the Ai function that
With nonzero angular momentum, this may be modified as
\[ T^2\phi^2 = \phi_0^2 + \phi_1^2 + 2\phi_0\phi_{-1} = \phi_0^2 + \phi_2^2 + 2\phi_0\phi_2. \]
The Stokes multiplier is thus squared, \( T = \tau_1^2 \). Since \( Q_{\text{CFIV}} \) is essentially the logarithm of the Stokes multiplier, this means that \( Q_{\text{CFIV}} \) should be doubled, in agreement with the argument in [19].

More generally, for a perturbation potential
\[ W_m(z, t) = \frac{z^m}{2m} - \frac{t}{2}c^2, \]
the associated function
\[ \psi_m(t) = \int e^{W_m(z, t)}dz \]
is found to satisfy a special case of so\((m + 1)\) ODE in the classification of [27],
\[ \frac{d^m}{dt^m}\psi_m(t) = (-1)^m\sqrt{t}\frac{d}{dt}\psi_m(t). \]
This is again consistent with the observation in [19].

5.3. Eigenvalues of conserved quantities

The formula for the vacuum expectation values of conserved quantities \( I_{2m-1} \) in CFT based on \( U_q(sl_2) \) symmetry is conjectured in [24]. In the present framework, the result is translated to
\[ \int e^{\alpha\theta} \ln(1 + Y_k(\theta)) \frac{d\theta}{2\pi} \propto I_m \]
for \( k = 1 \) and \( m = 2n - 1 \). The analytic evaluation of the above is difficult except for the \( m = 1 \) case, to which the dilogarithm technique can be applied. This is due to the fact that the analytic expression of \( Y_k \) is not available in general.

Although all \( I_m \) become null strictly at \( h = 0 \), we expect that the left-hand side brings conserved quantities order by order in \( h \). Indeed, the first order quantity in \( h \) for \( m = 1 \) agrees with the CFIV index. Let us assume that this conjecture is valid for arbitrary \( k \) and \( m \) and evaluate
\[ \tilde{I}_m := \int e^{\alpha\theta} e^{-A(\theta)} \frac{d\theta}{2\pi}. \]
Since we have the explicit solution to \( e^{-A(\theta)} \), the evaluation of \( \tilde{I}_m \) is immediate.

The actual calculation parallels that for the CFIV index. Thanks to (93), we find
\[ \tilde{I}_m = \frac{2^{2(m-1)} \Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{m}{2} + \frac{1}{k+2}\right) \Gamma\left(\frac{m}{2} + 1 - \frac{1}{k+2}\right)}{\Gamma(m) \Gamma\left(\frac{1}{2} + \frac{1}{k+2}\right) \Gamma\left(\frac{1}{2} - \frac{1}{k+2}\right)}. \]

6. Summary and conclusion

In this report, we propose a hypothesis that the combination of the wave functions associated with \( F \)-term potentials yields the \( Y \)-functions of the corresponding massless TBA equations. This has been successfully demonstrated for nontrivial examples. As applications, expectation values of conserved quantities, beyond the CFIV index, are conjectured by using the explicit solutions.
Many questions obviously remain open. The unexpected cancellation and the truncation of TBA equations, observed for the SU(3) case, may be generic for SU(n ≥ 3), which needs a proof. Moreover, we need to understand the intrinsic reason why such an amazing cancellation should occur.

The super-potentials for the super-conformal theories, except for A-type, contain multiple variables \([17, 18]\). It is not clear how to extend the observation in this report, especially how to define ‘wave functions’, in these cases.

The argument given in this paper is restricted to the massless case, in which ODE equations help us to find solutions. The original problem is concerned, however, with the generally massive quantum field theory: the TBA, or the CFIV index, is developed for the analysis of such a case. We note the recent progress in the ODE/IM correspondence towards massive deformation \([40–43]\). Hopefully this will help us to analyze the massive TBA through auxiliary linear problems associated with integrable partial differential equations.

Ultimately, the reason why our hypothesis works well remains a mystery. This is the most serious problem at the present moment. We hope to come back to this in the near future.

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Appendix A. Dressed vacuum form: SU(3)

We write down (72) in such a way that the connection to integrable systems is obvious. For this purpose, we introduce

\[
D^{(2)}(E, g) = \prod_{0 \leq i < j \leq 2} (g_j - g_i) \left( D^{[0]}(E)D^{[1]}(E\Omega)^3 q^{-\xi_i} - D^{[1]}(E)D^{[0]}(E\Omega)^3 q^{-\xi_i} \right).
\]

By using the elementary formula on matrix determinants \([34]\), we can derive

\[
\tau_1^{[1]}(E) = \xi_0 \frac{D^{[0]}(E)}{D^{[0]}(E\Omega^2)} + \xi_1 \frac{D^{[0]}(E\Omega^3)}{D^{[0]}(E\Omega^2)} \frac{D^{[2]}(E\Omega)}{D^{[2]}(E\Omega^2)} + \xi_2 \frac{D^{[2]}(E\Omega^6)}{D^{[2]}(E\Omega^2)},
\]

where we set

\[
\xi_0 = q^6, \quad \xi_1 = q^5, \quad \xi_2 = q^4.
\]

Further, use \(\Omega^3 = q^{-1}\) and write

\[Q(E) = D^{[0]}(E), \quad Q(E) = D^{[2]}(E\Omega^2), \quad T_1^{[1]}(E) = \tau_1^{[1]}(E\Omega^2).
\]

then we obtain the expression

\[
T_1^{[1]}(E) = \xi_0 \frac{Q(E\Omega^2)}{Q(E\Omega^2)} + \xi_1 \frac{Q(E\Omega^2)}{Q(E\Omega^2)} \frac{Q(E\Omega^2)}{Q(E\Omega^2)} + \xi_2 \frac{Q(E\Omega^2)}{Q(E\Omega^2)}.
\]

This agrees with the known DVF for the integral model with \(A_2^{[1]}\) symmetry.
Similarly, by setting \( T_i^{(2)}(E) = t_i^{(2)}(Eq^2) \) one finds

\[
T_i^{(2)}(E) = \xi_0^2 \frac{\tilde{Q}(Eq^4)}{Q(Eq^2)} + \xi_1^2 \frac{\tilde{Q}(Eq^{-2})}{Q(Eq^2)} + \xi_2^2 \frac{Q(Eq^{-4})}{Q(Eq^{-2})}
\]

where

\[
\xi_0 = (\xi_2)^{-1}, \quad \xi_1 = (\xi_1)^{-1}, \quad \xi_2 = (\xi_0)^{-1}.
\]

**Appendix B. Solutions to the third order ODE**

There are elementary functions that solve the third order equations, but are not widely known, in contrast to the second order case. It is shown, however, in [37] that the solution \( \varphi(x) \) to (65) with the desired asymptotic behavior (2) is given by Meijer’s G-function:

\[
\varphi(x) = c \left( G_{0,0}^{0,0} \left( \begin{array}{c} x^4 \\ 0, \frac{3}{4}, \frac{1}{2} \end{array} \right) \right)
\]

for \( 0 \leq \arg(x^4/4^2) \leq 6\pi \). The normalization constant \( c \) should be determined so as to be consistent with (2). To verify this, one requires several machineries.

In this appendix, we take a formal but simple approach to represent the solution \( \varphi(x) \): we try to represent it with a familiar object, a limit of the generalized hypergeometric series \( {}_3F_2 \).

The following argument generalizes that given for the Airy function in its relation to finite size lattice models [38].

The generalized hypergeometric series \( {}_3F_2(a_1, a_2, a_3; b_1, b_2; \xi) \) satisfies the following differential equation:

\[
\delta \left( \delta + b_1 - 1 \right) \left( \delta + b_2 - 1 \right) - z \left( \delta + a_1 \right) \left( \delta + a_2 \right) \left( \delta + a_3 \right) {}_3F_2 = 0,
\]

\[
\delta := \frac{d}{dz}.
\]

In the following we adopt abbreviations \( a := (a_1, a_2, a_3) \), \( b := (b_1, b_2) \) and \( {}_3F_2(a; b; \xi) \).

Consider the limit

\[
\lim_{N \to \infty} {}_3F_2 \left( \begin{array}{c} a \\ b \end{array} \right) \left( \begin{array}{c} x^L \\ L^2 N \left( N^2 - 1 \right) \end{array} \right).
\]

It is easy to check that it satisfies (65) where \( L = 4, \sigma = -\frac{1}{4} \) and

\[
a = (-N, -N + \sigma, -N + 2\sigma), \quad b = (1 + \sigma, 1 + 2\sigma).
\]

The parameter \( N \) corresponds to the lattice size for the SU(2) case [38].

The desired solution, \( \varphi \), should however satisfy the asymptotic behavior (2). In order to accomplish this, we first keep \( N \) a large but finite positive integer and use the fact that two other independent solutions to (B.94) are

\[\text{To meet (65), we rotate } x \to e^{\pi/4} x.\]
\( \zeta^{-\sigma} F_2(a'; b'|\zeta') \) and \( \zeta^{-2\sigma} F_2(a^*; b^*|\zeta) \)

where \( a' = a - \sigma(1, 1, 1) \), \( b' = (1 - \sigma, 1 + \sigma) \) and \( a^* = a - 2\sigma(1, 1, 1) \), \( b^* = (1 - \sigma, 1 - 2\sigma) \).

Let a subsidiary variable \( z^L = \zeta \) and a subsidiary function \( \Psi_{N, \sigma} \) be

\[
\Psi_{N, \sigma}(x) = \frac{z^{L\sigma+1}}{(1 - z^L)^N} F_2(a; b|z^L).
\]

The ODE for \( \Psi_{N, \sigma} \) reads

\[
\begin{align*}
\zeta^3 \frac{d^3 \Psi_{N, \sigma}}{dz^3} &- L^2 3N(N+1)z^L \frac{d \Psi_{N, \sigma}}{dz} \\
&+ L^3 N(N^2-1)z^L \Psi_{N, \sigma} + 3L^2 N(N+1)z^L \Psi_{N, \sigma} = 0,
\end{align*}
\]

(B.95)

where \( \sigma L = -1 \) is used.

We select a solution which vanishes at the regular singular point \( z = 1 \). Let us consider the linear combination of three independent solutions

\[
\Psi_{N, \sigma} := \frac{1}{(1 - z^L)^W} (\alpha_1 \psi_1 + \alpha_2 \psi_2 + \alpha_3 \psi_3)
\]

where

\[
\begin{align*}
\psi_1 &= z^{L\sigma+1} F_2(a; b|z^L), \\
\psi_2 &= z F_2(a'; b'|z^L), \\
\psi_3 &= z^{-L\sigma+1} F_2(a^*; b^*|z^L).
\end{align*}
\]

The characteristic exponents at \( z = 1 \) of (B.95) are \( -N, -N + 1, 2N + 2 \); thus it suffices to require

\[
(\alpha_1 \psi_1 + \alpha_2 \psi_2 + \alpha_3 \psi_3) \bigg|_{z=1} = 0, \quad \frac{d}{dz} (\alpha_1 \psi_1 + \alpha_2 \psi_2 + \alpha_3 \psi_3) \bigg|_{z=1} = 0.
\]

These fix \( \alpha_i \) (but for an overall factor),

\[
\begin{align*}
\alpha_1(N) &= c \frac{W[\psi_2, \psi_3]}{W[\psi_1, \psi_2, \psi_3]}, \\
\alpha_2(N) &= -c \frac{W[\psi_1, \psi_3]}{W[\psi_1, \psi_2, \psi_3]}, \quad \alpha_3(N) = c \frac{W[\psi_1, \psi_2]}{W[\psi_1, \psi_2, \psi_3]}
\end{align*}
\]

where \( W \) denotes the Wronskian determinant evaluated at \( z = 1 \).

We assume the following limit exists:

\[
\alpha_i = \lim_{N \to \infty} \left( \frac{1}{4^3 N(N^2 - 1)} \right)^{\frac{i}{2}} \alpha_i(N)
\]

although the actual evaluation of the limit may not be simple.

In the scaled variables \( x \), the regular singular point is relocated at \( x_0 = (4^3 N(N^2 - 1))^{\frac{1}{2}} \).

By an argument similar to that for the SU(2) case, we can show that \( \Psi_{N, \sigma} \) is a decreasing
function in \(0 < x < x_0\). In the ‘scaling limit’ \(N \to \infty\), \(x_0\) also goes to infinity. Thus we conclude that \(\varphi = \lim_{N \to \infty} \Psi_{n,\alpha}\) is the desired decaying function, which is written as follows:

\[
\varphi(x) = \alpha_1 \phi_{1/2} \left( \frac{3}{4} \left\| \frac{1}{4} \right. \right) + \alpha_2 x \phi_{3/2} \left( \frac{3}{4} \left\| \frac{3}{4} \right. \right) + \alpha_3 x^2 \phi_{5/2} \left( \frac{3}{4} \left\| \frac{3}{4} \right. \right)
\]

**Appendix C. The cancellation of terms**

Using the expansions (81) in definitions of (77), we obtain the expressions for \(y^{(\alpha)}_{jm}(\theta)\) in terms of \(t^{(\alpha)}_{jm}(\theta)\). In particular, we are interested in \(y^{(1)}_{2,2}(\theta) + y^{(2)}_{2,2}(\theta)\) and \(y^{(1)}_{2,3}(\theta) + y^{(2)}_{2,3}(\theta)\). Since \(T^{(\alpha)}_m\) satisfies the T-system, \(t^{(\alpha)}_{jm}(\theta)\) are not necessarily independent: there are some relations, e.g.,

\[
t^{(1)}_{1,3}(\theta) = -t^{(2)}_{1,2}(\theta), \quad t^{(2)}_{1,3}(\theta) = -t^{(1)}_{1,2}(\theta),
\]

\[
t^{(1)}_{1,1}(\theta) = -t^{(2)}_{1,2}(\theta), \quad t^{(2)}_{1,1}(\theta) = -t^{(1)}_{1,2}(\theta),
\]

\[
t^{(1)}_{2,3}(\theta) = t^{(1)}_{1,2}(\theta + \frac{\pi}{3}) + t^{(1)}_{1,2}(\theta - \frac{\pi}{3}) + t^{(1)}_{1,2}(\theta) t^{(2)}_{1,2}(\theta) - t^{(1)}_{2,2}(\theta),
\]

\[
t^{(2)}_{2,3}(\theta) = t^{(2)}_{1,2}(\theta + \frac{\pi}{3}) + t^{(2)}_{1,2}(\theta - \frac{\pi}{3}) + t^{(2)}_{1,2}(\theta) t^{(1)}_{1,2}(\theta) - t^{(1)}_{2,2}(\theta)
\]

and so on. Using these relations, we find relatively simple expressions,

\[
y^{(1)}_{2,3}(\theta) + y^{(2)}_{2,3}(\theta) = \left( t^{(1)}_{1,3}(\theta) \right)^2 - \left( \frac{t^{(2)}_{1,2}(\theta + \frac{\pi}{3}) t^{(2)}_{1,2}(\theta - \frac{\pi}{3})}{t^{(1)}_{1,2}(\theta)} \right)^2 - t^{(1)}_{2,3}(\theta) - t^{(2)}_{2,3}(\theta),
\]

\[
y^{(1)}_{2,2}(\theta) + y^{(2)}_{2,2}(\theta) = \left( t^{(1)}_{1,1}(\theta) \right)^2 - \left( \frac{t^{(2)}_{1,2}(\theta + \frac{\pi}{3}) t^{(2)}_{1,2}(\theta - \frac{\pi}{3})}{t^{(1)}_{1,2}(\theta)} \right)^2 - t^{(1)}_{2,1}(\theta) - t^{(2)}_{2,1}(\theta).
\]

They however still show that we need \(O(h^2)\) terms \(t^{(\alpha)}_{jm}(\theta)\) in the expansions of \(T^{(\alpha)}_m\). Note that

\[
y^{(2)}_{1,2}(\theta) = \frac{t^{(2)}_{1,2}(\theta + i\frac{\pi}{3}) t^{(2)}_{1,2}(\theta - i\frac{\pi}{3})}{t^{(1)}_{1,2}(\theta)} \quad \text{and} \quad y^{(1)}_{1,2}(\theta) = \frac{t^{(1)}_{1,2}(\theta + i\frac{\pi}{3}) t^{(1)}_{1,2}(\theta - i\frac{\pi}{3})}{t^{(2)}_{1,2}(\theta)}
\]

(C.96)

also appear in the last term of (87).

The key idea is to employ the expansions of \(D^{(0)}(E, g)\),

\[
D^{(i)}(E, g) = d^{(i)}_{0}(E) + h d^{(i)}_{1}(E) + h^2 d^{(i)}_{2}(E) + O(h^3), \quad (i = 0, 1, 2).
\]

Although \(d^{(i)}_{0}(E)\) is denoted by \(D^{(i)}(E)\) in the main body, we use \(d^{(i)}_{0}(E)\) in the appendix for uniformity. Substituting this into (73), taking account of the shift in (74) and (75), we are able to derive the expressions of \(t^{(\alpha)}_{jm}(\theta)\) in terms of \(d^{(i)}_{0}\). The point is that the latter has a smaller number of independent elements (for fixed order of expansions in \(h\)) and a smaller number of relations.
The obvious constraint is $\tau_0^{(1)}(E, g) = 1$. We write it as

$$\tau_0^{(1)}(E, g) = 1 = \tau_{0,0}^{(1)}(E) + h\tau_{1,0}^{(1)}(E) + h^2\tau_{2,0}^{(1)}(E) + O(\hbar^3)$$

or

$$\tau_0^{(1)}(E) = 1, \quad \tau_1^{(1)}(E) = 0, \quad \tau_{2,0}^{(1)}(E) = 0\ldots.$$

The constraints above are written in terms of sums of triple products of $d_j^{[i]}$. By comparing the expressions in terms of $d_j^{[i]}$, we find

$$t_{1,1}^{(2)}(\theta) + \frac{t_{1,2}^{(0)}(\theta + i\frac{\pi}{3})t_{1,2}^{(0)}(\theta - i\frac{\pi}{3})}{t_{1,2}^{(0)}(\theta)} = -i\tau_{0,0}^{(1)}(\exp i\frac{\pi}{2}) = -i,$$

$$t_{1,1}^{(1)}(\theta) + \frac{t_{1,2}^{(1)}(\theta + i\frac{\pi}{3})t_{1,2}^{(1)}(\theta - i\frac{\pi}{3})}{t_{1,2}^{(1)}(\theta)} = i\tau_{0,0}^{(1)}(\exp i\frac{\pi}{2}) = i.$$

There are also unexpected relations,

$$t_{1,1}^{(2)}(\theta) - \frac{t_{1,2}^{(2)}(\theta + i\frac{\pi}{3})t_{1,2}^{(2)}(\theta - i\frac{\pi}{3})}{t_{1,2}^{(2)}(\theta)} = -i\tau_{0,0}^{(1)}(\exp i\frac{\pi}{2}) + i$$

$$= 6\left(-d_0^{[0]}(\exp\omega^2)d_1^{[1]}(\exp\omega^3) + id_0^{[0]}(\exp\omega^3)d_1^{[1]}(\exp\omega^7)\right)d_0^{[2]}(\exp\omega^3),$$

$$t_{1,1}^{(1)}(\theta) - \frac{t_{1,2}^{(1)}(\theta + i\frac{\pi}{3})t_{1,2}^{(1)}(\theta - i\frac{\pi}{3})}{t_{1,2}^{(1)}(\theta)} = -i\tau_{0,0}^{(1)}(\exp i\frac{\pi}{2})$$

$$= t_{1,1}^{(1)}(\theta) - \frac{t_{1,2}^{(1)}(\theta + i\frac{\pi}{3})t_{1,2}^{(1)}(\theta - i\frac{\pi}{3})}{t_{1,2}^{(1)}(\theta)} - i$$

$$= 6\left(id_0^{[0]}(\exp\omega^5)d_0^{[1]}(\exp\omega) + d_0^{[0]}(\exp\omega)d_0^{[1]}(\exp\omega^5)\right)d_0^{[2]}(\exp\omega^5).$$

where $\omega = \exp i\frac{\pi}{2}$. 
We also find
\[
\begin{align*}
t_{12}^{(1)}(\theta) + t_{24}^{(1)}(\theta) &= 1, \\
t_{22}^{(1)}(\theta) + t_{44}^{(1)}(\theta) + \tau_{0,0}^{(1)}(-\Theta\omega) - 2\tau_{2,2}^{(1)}(-\Theta\omega) \\
&= 3\left(d_{0,0}^{(0)}(\Theta\omega)d_{1,1}^{(1)}(\Theta\omega) + id_{0,0}^{(0)}(\Theta\omega)d_{1,1}^{(1)}(\Theta\omega)\right)d_{0,0}^{(2)}(-\Theta\omega), \\
t_{11}^{(1)}(\theta) + t_{22}^{(1)}(\theta) &= 1, \\
t_{12}^{(2)}(\theta) + t_{24}^{(2)}(\theta) + \tau_{0,0}^{(2)}(-\Theta\omega^5) - 2\tau_{2,2}^{(2)}(-\Theta\omega^5) \\
&= 3\left(d_{0,0}^{(0)}(-\Theta\omega^5)d_{1,1}^{(1)}(-\Theta\omega) - id_{0,0}^{(0)}(-\Theta\omega)d_{1,1}^{(1)}(-\Theta\omega^5)\right)d_{0,0}^{(2)}(\Theta\omega^5).
\end{align*}
\]

Using these results, the sums of our interest simplify considerably,
\[
\begin{align*}
y_{12}^{(1)}(\theta) + y_{24}^{(1)}(\theta) &= 9\left(d_{0,0}^{(0)}(\Theta\omega)d_{1,1}^{(1)}(\Theta\omega) + id_{0,0}^{(0)}(\Theta\omega)d_{1,1}^{(1)}(\Theta\omega)\right)d_{0,0}^{(2)}(-\Theta\omega), \\
y_{22}^{(1)}(\theta) + y_{44}^{(1)}(\theta) &= 9\left(id_{0,0}^{(0)}(-\Theta\omega)d_{1,1}^{(1)}(-\Theta\omega) - d_{0,0}^{(0)}(-\Theta\omega^5)d_{1,1}^{(1)}(-\Theta\omega)\right)d_{0,0}^{(2)}(\Theta\omega^5).
\end{align*}
\]

The right-hand sides of (C.96) are also derived from the above results,
\[
\begin{align*}
y_{12}^{(2)}(\theta) &= 3\left(d_{0,0}^{(0)}(\Theta\omega)d_{1,1}^{(1)}(\Theta\omega) - id_{0,0}^{(0)}(\Theta\omega)d_{1,1}^{(1)}(\Theta\omega)\right)d_{0,0}^{(2)}(-\Theta\omega), \\
y_{11}^{(2)}(\theta) &= -3\left(d_{0,0}^{(0)}(-\Theta\omega)d_{1,1}^{(1)}(-\Theta\omega) + id_{0,0}^{(0)}(-\Theta\omega^5)d_{1,1}^{(1)}(-\Theta\omega)\right)d_{0,0}^{(2)}(\Theta\omega^5). \quad (C.97)
\end{align*}
\]

Thereby, we conclude the validity of (88).

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