Nonlinearly charged dilatonic black holes and their Brans-Dicke counterpart: Energy dependent spacetime

S. H. Hendi¹,² and M. S. Talezadeh³

Abstract Regarding the wide applications of dilaton gravity in the presence of electrodynamics, we introduce a suitable Lagrangian for the coupling of dilaton with gauge field. There are various Lagrangians which show the coupling between scalar fields and electrodynamics with correct special situations. In this paper, taking into account conformal transformation of Brans-Dicke theory with an electrodynamics Lagrangian, we show that how scalar field should couple with electrodynamics in dilaton gravity. In other words, in order to introduce a correct Lagrangian of dilaton gravity, one should check at least two requirements: compatibility with Brans-Dicke theory and appropriate special situations. Finally, we apply the mentioned method to obtain analytical solutions of dilaton-Born-Infeld and Brans-Dicke-Born-Infeld theories with energy dependent spacetime.

1 Introduction

Einstein gravity of general relativity is one of the successful theories of 20th century. Although this theory can explain the dynamics of our solar system well enough, it cannot describe the high curvature regimes. In other words, one may use its generalization for describing the particle motion near the black hole horizon. Another problem of Einstein gravity is its inaccurate result for the accelerated expansion of the universe. In addition, Einstein gravity does not accommodate either Mach’s principle or Dirac’s large-number hypothesis. Between various modifications of Einstein gravity, Brans and Dicke were pioneers in studying the alternative theories known as Brans-Dicke (BD) theory [1]. BD theory is a straightforward way to disentangle further degrees of freedom which are not present in the usual Hilbert-Einstein action. Regarding the four dimensional stationary BD solution, one finds it is just the Kerr solution with a constant scalar field [2]. Beside the vacuum solutions, Cai and Myung showed that the 4-dimensional charged black hole solution of BD theory is just the Reissner–Nordström (RN) solution with a trivial scalar field [3,4,5,6]. This is because the stress energy tensor of Maxwell field is traceless only in 4-dimension, and therefore, the action of Maxwell field is not invariant under the conformal transformation in higher dimensions. In order to find the differences between BD theory and Einstein gravity, one may regard higher dimensions. As an example it was found that higher dimensional charged black holes in BD theory would be the RN solution with a non-trivial scalar field. Classical and quantum aspects black holes and gravitational collapse in Brans-Dicke (BD) theory have been investigated in literature [7,8,9,10].

Another distinguishing mark of BD theory is the existence of a conformal mapping between this theory and Einstein gravity. A conformal mapping is a diffeomorphism between two metrizable bundles.

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if the pulled back bundle is conformally equivalent to the original one. The conformal transformation technique is one of the strong and useful mathematical tools for investigating the scalar-tensor theories of gravity \cite{11,12,13}. In other words, using conformal transformation, one can represent gravity in two conformally related frames: the Jordan or string frame (scalar field is non-minimally coupled to the metric tensor) and the Einstein frame (scalar field is minimally coupled to the metric tensor) \cite{14,15,16}. Investigation of these frames enables one to explore the relation between different physical theories and hence generate the new exact solutions \cite{17,18}. In addition, it was shown that the higher-order theories of gravity can be related to the non-minimally coupled version of scalar tensor gravity \cite{19}. In general, equipped with the conformal transformations enables one to investigate gravitational theories with more details since one can exchange, via these transformations, a model of gravity to another where the relations become more easier to manipulate. Although there are many studies related to conformal transformation between the Einstein and Jordan frames, there was not a user-friendly rule for presenting mathematical properties of this transformation. Furthermore, the standard coupling between dilaton and gauge fields has not been explored with a systematic details. These are our main motivations to present conformal transformation for obtaining standard gauge-gravity Lagrangians with a methodical way. In this work, we obtain a new criterion for conformal consistency of dilaton gravity coupled with electrodynamics.

The main goal of the present work is finding how an arbitrary Lagrangian of electrodynamics can couple with dilaton field in Einstein frame. In this regard, we show that most of previous introduced Lagrangians are not consistent with any BD counterpart. In order to obtain original solutions, we focus on the challenging Lagrangian, dilatonic Born-Infeld theory, in an energy dependent spacetime, so-called gravity’s rainbow. The motivation of considering gravity’s rainbow comes from Horava-Lifshitz gravity \cite{20,21}, in which help us to regard different scaling of space and time for type IIA and IIB string theory \cite{22,23}. AdS/CFT correspondence \cite{24,25,26} and dilaton black objects \cite{27,28,29,30}. The relation between the Horava-Lifshitz gravity and gravity’s rainbow was investigated in Ref. \cite{31}. Gravity’s rainbow is based on the modification of usual energy-momentum dispersion relation in the UV limit. It is worthwhile to mention that such modification has also been regarded in different subjects, such as: discrete spacetime \cite{32}, spin-network in loop quantum gravity (LQG) \cite{33}, spacetime foam \cite{34}, non-commutative geometry \cite{35,36} and ghost condensation \cite{37}.

Motivated by the recent results mentioned above, we obtain the correct Lagrangian of gauge-dilaton coupling with corresponding field equations. We also introduce the correct action and field equations of some interesting models of nonlinear electrodynamics in dilaton gravity. Finally, we consider an energy dependent spacetime to obtain black hole solutions of BD-BI and dilatonic BI theories.

## 2 Field equations and conformal transformations

The action of \((n+1)\)-dimensional BD theory with a scalar field \(\Phi\) and a self-interacting potential \(V(\Phi)\) in the presence of a matter field can be written as

\[
I = -\frac{1}{16\pi} \int d^{n+1}x \sqrt{-g} \left[ \Phi R + \mathcal{L}(\Phi, \nabla \Phi) + \mathcal{L}_m(\Psi) \right],
\]

in which \(\mathcal{L}(\Phi, \nabla \Phi)\) is an arbitrary Lagrangian of scalar field and its first derivative and \(\mathcal{L}_m\) is Lagrangian of pure matter field with field strength \(\Psi\). The standard choice for the Lagrangian of scalar field is

\[
\mathcal{L}(\Phi, \nabla \Phi) = -\frac{\omega}{\Phi} (\nabla \Phi)^2 - V(\Phi),
\]

where the first and second expressions are, respectively, characteristic as kinetic and potential terms in string frame. Now, we would like to obtain electrodynamic Lagrangian coupled with scalar field in Einstein frame. Indeed, via the conformal transformation \cite{3,4,5,6} the BD theory can be transformed into the Einstein-dilaton gravity with a minimally coupled scalar dilaton field. We should note that the suitable conformal transformation comes from the fact that \(\sqrt{-\Phi R}\) term in Jordan frame should transform to \(\sqrt{-\bar{g}} \bar{R}\) in Einstein frame. Suitable conformal transformation can be written as

\[
\bar{g}_{\mu\nu} = \Phi^{2/(n-1)} g_{\mu\nu},
\]

\[
\bar{\Phi} = \frac{n - 3}{4\alpha} \ln \Phi,
\]

where the first and second expressions are, respectively, characteristic as kinetic and potential terms in string frame. Now, we would like to obtain electrodynamic Lagrangian coupled with scalar field in Einstein frame. Indeed, via the conformal transformation \cite{3,4,5,6} the BD theory can be transformed into the Einstein-dilaton gravity with a minimally coupled scalar dilaton field. We should note that the suitable conformal transformation comes from the fact that \(\sqrt{-\Phi R}\) term in Jordan frame should transform to \(\sqrt{-\bar{g}} \bar{R}\) in Einstein frame. Suitable conformal transformation can be written as

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\bar{\Phi} = \frac{n - 3}{4\alpha} \ln \Phi,
\]
where
\[ \alpha = \frac{(n-3)((n-1)\omega + n)^{-1/2}}{2}. \] (4)

It was shown that all functions and quantities in Jordan frame \((g_{\mu\nu}, \Phi \text{ and } \Psi)\) can be transformed into Einstein frame \((\bar{g}_{\mu\nu}, \bar{\Phi} \text{ and } \bar{\Psi})\). Applying the mentioned conformal transformation on the BD action \(L\), one can obtain corresponding action in Einstein frame

\[ \bar{I} = -\frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x \sqrt{-\bar{g}} \left\{ \mathcal{R} + \bar{L}(\bar{\Phi}, \bar{\nabla}\bar{\Phi}) + \bar{L}(\bar{\Psi}, \bar{\Phi}) \right\}, \] (5)

where \(\mathcal{R}\) and \(\nabla\) are, respectively, the Ricci scalar and covariant derivative corresponding to the metric \(\bar{g}_{\mu\nu}\), and \(\bar{L}(\bar{\Phi}, \bar{\nabla}\bar{\Phi})\) is (transformed) Lagrangian of scalar field and \(\bar{L}(\bar{\Psi}, \bar{\Phi})\) is Lagrangian of matter field which is coupled with scalar field after transformation. Considering Eq. (4), one obtains

\[ \bar{L}(\bar{\Phi}, \bar{\nabla}\bar{\Phi}) = -\frac{4}{n-1}(\bar{\nabla}\bar{\Phi})^2 - \bar{V}(\bar{\Phi}), \] (6)

where the first and second expressions are, respectively, corresponding to the kinetic and potential terms in Einstein frame. It is easy to find \(\bar{V}(\bar{\Phi})\) is

\[ \bar{V}(\bar{\Phi}) = \Phi^{-(n+1)/(n-1)}V(\Phi). \] (7)

Now, we are in a position to choose a special matter field. We consider general form Lagrangian of electrodynamics as a matter field. Thus, we can write

\[ \bar{L}_m(\bar{\Phi}) = \mathcal{L}(\mathcal{F}), \] (8)

where \(\mathcal{L}(\mathcal{F})\) is an arbitrary Lagrangian of electrodynamics in which \(\mathcal{F} = F_{ab}F^{ab}\) is the Maxwell invariant and \(F_{ab} = \partial_a A_b - \partial_b A_a\) is the electromagnetic field tensor with the gauge potential \(A_a\). For various explicit forms of (linear and nonlinear) \(\mathcal{L}(\mathcal{F})\), we refer the reader to [38,39,40].

The total Lagrangian of BD theory in the presence of electromagnetic field is

\[ I_G = -\frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x \sqrt{-g} \left\{ \Phi R - \frac{\omega}{\Phi}(\nabla\Phi)^2 - V(\Phi) + \mathcal{L}(\mathcal{F}) \right\}, \] (9)

where we can obtain its corresponding equations of motion with the following explicit forms

\[ G_{\mu\nu} = \frac{\omega}{\Phi^2} \left( \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} (\nabla\Phi)^2 \right) - \frac{V(\Phi)}{2\Phi} g_{\mu\nu} + \frac{1}{\Phi} (\nabla_\mu \nabla_\nu \Phi - g_{\mu\nu} \nabla^2 \Phi) \]
\[ + \frac{2}{\Phi} \left( \frac{1}{4} g_{\mu\nu} \mathcal{L}(\mathcal{F}) - F_{\mu\lambda} F_\nu^\lambda \mathcal{L}_\mathcal{F} \right), \] (10)

\[ \nabla^2 \Phi = \frac{(n+1) \mathcal{L}(\mathcal{F}) - 4 \mathcal{L}_\mathcal{F}}{2((n-1)\omega + n)} + \frac{1}{2((n-1)\omega + n)} \left( (n-1)\Phi \frac{dV(\Phi)}{d\Phi} - (n+1) V(\Phi) \right), \]
\[ \nabla_\mu (\mathcal{L}_\mathcal{F} F^\mu{}_{\nu}) = 0, \] (11)

where \(G_{\mu\nu}\) and \(\nabla_\mu\) are, respectively, the Einstein tensor and covariant derivative of manifold \(\mathcal{M}\) with metric \(g_{\mu\nu}\) and \(\mathcal{L}_\mathcal{F} = \frac{d\mathcal{L}(\mathcal{F})}{d\Phi}\). Now, we would like to obtain electrodynamic Lagrangian coupled with scalar field in Einstein frame. Indeed, via the conformal transformation [34,15,13] the BD theory can be transformed into the Einstein-dilaton gravity with a minimally coupled scalar dilaton field. Applying the mentioned conformal transformation, we obtain

\[ \bar{I}_G = -\frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x \sqrt{-\bar{g}} \left\{ \bar{\mathcal{R}} - \frac{4}{n-1} (\bar{\nabla}\bar{\Phi})^2 - \bar{V}(\bar{\Phi}) + \bar{L}(\bar{\mathcal{F}}, \bar{\Phi}) \right\}. \] (13)

One can obtain the equations of motion by varying this action (13) with respect to \(\bar{g}_{\mu\nu}, \bar{\Phi}\) and \(\bar{F}_{\mu\nu}\)

\[ \bar{R}_{\mu\nu} = \frac{4}{n-1} \left( \nabla_\mu \bar{\Phi} \nabla_\nu \bar{\Phi} + \frac{1}{4} \bar{V} \bar{g}_{\mu\nu} \right) - 2 A e^{m\Phi} B e^{k\Phi} \bar{L}_\mathcal{F} F_{\mu\eta} F_{\nu}^\eta \]
\[ + \frac{A}{n-1} e^{m\Phi} \left[ 2 \bar{Y} \bar{L}_\mathcal{F} - \bar{L}(\bar{\mathcal{Y}}) \right] \bar{g}_{\mu\nu}, \] (14)
\[ \nabla^2 \phi = \frac{n-1}{8} \frac{\partial \tilde{V}}{\partial \phi} - A \alpha (\frac{n-1}{8})^e^{\alpha \phi} \left[ m \tilde{L}(\tilde{Y}) + k \tilde{Y} \tilde{L}_Y \right], \tag{15} \]

\[ \left( e^{(m+k)\alpha \phi} \tilde{L}_Y \tilde{F}^{\mu \nu} \right) = 0 \tag{16} \]

where \( \tilde{L}_Y = \frac{\partial \tilde{L}(\tilde{Y})}{\partial \tilde{Y}} \) and we use the following familiar notation

\[ \tilde{L}(\tilde{F}, \tilde{\Phi}) = A e^{\alpha \tilde{Y}} \tilde{L}(\tilde{Y}), \quad \tilde{Y} = B e^{\alpha \tilde{\Phi}} \tilde{F}, \tag{17} \]

\[ \tilde{\Phi} = \frac{\Phi - 4}{(n-1)(n-3)} e^{\alpha \tilde{Y}} = 1, \quad \tilde{\Phi} = \frac{\Phi}{(n+1)/(n-1)} e^{\alpha \tilde{F}} = 1. \]

Using Eq. (3), one finds

\[ k = \frac{16}{(n-1)(n-3)}, \tag{19} \]

\[ m = -\frac{4(n+1)}{(n-1)(n-3)}. \tag{20} \]

Taking into account the mentioned calculations, we find that considering an arbitrary nonlinear electrodynamics model in Jordan frame (BD theory) leads to a corresponding \( \tilde{L}(\tilde{F}, \tilde{\Phi}) \) in Einstein frame (dilaton gravity) with the following form

\[ \tilde{L}(\tilde{F}, \tilde{\Phi}) = A e^{-\frac{4(n+1)\alpha}{(n-1)(n-3)}} \tilde{L}(\tilde{Y}), \tag{21} \]

\[ \tilde{Y} = B e^{\frac{16\alpha}{(n-1)(n-3)}} \tilde{F}. \tag{22} \]

### 3 Case Studies

In this section, we consider BD action with various models of linear and nonlinear electrodynamics and try to obtain the correct Lagrangian of gauge-dilaton coupling in the Einstein frame by using the mentioned prescription.

#### 3.1 Linear Maxwell source:

Maxwell electrodynamics is the most familiar theory between other abelian gauge fields. It is a linear theory with excellent consequences in the classical physics. Thus, as the first example, we consider the Lagrangian of linear Maxwell field

\[ \mathcal{L}(\mathcal{F}) = -\mathcal{F}, \tag{23} \]

and therefore \( \tilde{L}(\tilde{Y}) = \tilde{Y} \) with \( A = 1 \) and \( B = -1 \). Taking into account Eqs. (21) and (22), we find

\[ \tilde{L}(\tilde{F}, \tilde{\Phi}) = -e^{-\frac{4(n+1)\alpha}{(n-1)(n-3)}} e^{\frac{16\alpha}{(n-1)(n-3)}} \tilde{F} = -e^{\frac{-4\alpha}{(n-1)(n-3)}} \tilde{F}, \tag{24} \]

which is coupled Lagrangian of dilaton-Maxwell theory in which introduced in Ref. [3,4].
3.2 Born-Infeld theory

Nonlinear electromagnetic theories have been considered in the context of superstring theory. In other words, it was shown that the Born-Infeld (BI) type Lagrangian may be regarded as all order loop corrections to gravity [41,42]. In addition, one may find that the dynamics of D-branes are related to the BI action [43]. Considering BI type electrodynamics coupled to the gravitational fields leads to black hole solutions with interesting properties. Here, we apply the obtained consequences to the case of Born-Infeld theory [44,45,46,47,48,49]

\[ \mathcal{L}(F) = 4\beta^2 \left( 1 - \sqrt{1 + \frac{F}{2\beta^2}} \right). \]  

(25)

Considering Eqs. (21) and (22), and using \( \bar{L}(\bar{Y}) = 1 - \sqrt{1 + \bar{Y}} \) with \( A = 4\beta^2 \) and \( B = \frac{1}{2\beta^2} \), we obtain the following lagrangian for the coupling Lagrangian of dilaton-Born-Infeld theory

\[ \bar{L}(\bar{F}, \bar{\Phi}) = 4\beta^2 e^{-\frac{4(n+1)\bar{\Phi}}{(n-1)(n-3)}} \left[ 1 - \sqrt{1 + \frac{e^{\frac{16\bar{\Phi}}{(n-1)(n-3)}} \bar{F}}{2\beta^2}} \right]. \]  

(26)

We should note that the mentioned Lagrangian is the same as that in Ref. [5]. It is worthwhile to mention that although the Lagrangian presented in Ref. [50,51,52] has correct Maxwell limit for \( \beta \to \infty \), is not consistent with conformal transformation and Jordan frame (this point was indicated in Ref. [5]). As it was presented in Ref. [5], the correct field equations in which arisen from the variation principle of action can be written as

\[ R_{\mu\nu} = \frac{4}{n-1} \left( \bar{\nabla}_{\mu} \bar{\Phi} \bar{\nabla}_{\nu} \bar{\Phi} + \frac{1}{4} \bar{V}(\bar{\Phi}) g_{\mu\nu} \right) - \frac{1}{n-1} \bar{L}(\bar{F}, \bar{\Phi}) g_{\mu\nu} \]

\[ + \frac{2e^{-\frac{4\bar{\Phi}}{(n-1)(n-3)}}}{\sqrt{1 + \bar{Y}}} \left( \bar{T}^{\mu\eta} \bar{T}^{\eta\nu} - \frac{\bar{F}}{n-1} g_{\mu\nu} \right), \]  

(27)

\[ \bar{\nabla}^2 \bar{\Phi} = \frac{n-1}{8} \frac{\partial \bar{V}(\bar{\Phi})}{\partial \bar{\Phi}} + \frac{\alpha}{2(n-3)} \left( (n+1)\bar{L}(\bar{F}, \bar{\Phi}) + \frac{4e^{-\frac{4\bar{\Phi}}{(n-1)(n-3)}} \bar{F}}{\sqrt{1 + \bar{Y}}} \right), \]  

(28)

\[ \bar{\nabla}_{\mu} \left( \frac{e^{-\frac{4\bar{\Phi}}{(n-1)(n-3)}} \bar{F}}{\sqrt{1 + \bar{Y}} \bar{T}^{\mu\nu}} \right) = 0, \]  

(29)

where

\[ \bar{Y} = \frac{e^{\frac{16\bar{\Phi}}{(n-1)(n-3)}} \bar{F}}{2\beta^2}. \]  

(30)

3.3 Power Maxwell invariant (PMI) source:

One of the special classes of nonlinear electrodynamics is Power Maxwell Invariant (PMI) theory [38, 53, 54, 55, 56, 57, 58, 59, 60]. The PMI theory has an interesting result which distinguishes this nonlinear theory from others; this theory enjoys conformal invariancy when the power of Maxwell invariant is a quarter of spacetime dimensions (\( \text{power} = \text{dimensions}/4 \)). It is worth mentioning that the idea is to take advantages of the conformal symmetry to construct the analogues of the 4 dimensional Reissner-Nordström solutions with an inverse square law for the electric field of the point-like charges in arbitrary dimensions.

Now, we take into account the Lagrangian of nonlinear PMI model with the following explicit form [38, 53, 54, 55, 56, 57, 58, 59, 60]

\[ \mathcal{L}(F) = (-F)^s, \]  

(31)
and thus one finds \( \hat{L}(\hat{Y}) = (\hat{Y})^{s} \) with \( A = 1 \) and \( B = -1 \). Taking into account Eqs. (21) and (22), we find

\[
\hat{L}(\hat{F}, \hat{\Phi}) = e^{\frac{4(n+1)\Phi}{(n-1)(n-3)}} \left( -e^{\frac{4(n+1)\Phi}{(n-1)(n-3)}} \hat{F} \right)^{s},
\]

which is coupled Lagrangian of dilaton-PMI theory. One can apply the variation principle to the corresponding action to obtain the equation of motions with the following explicit forms

\[
\bar{R}_{\mu\nu} = \frac{4}{n-1} \left( \nabla_{\mu} \hat{\Phi} \nabla_{\nu} \hat{\Phi} + \frac{1}{4} \hat{V}(\hat{\Phi}) g_{\mu\nu} \right) - \left( \frac{\nabla}{n-1} \right)^{4(n+1)\Phi/((n-1)(n-3))} g_{\mu\nu} + 2s \bar{Y}^{s-1} e^{\frac{4s\bar{Y}}{n-1}} \left( F_{\mu\eta} F_{\nu}^{\eta} - \frac{F_{\mu\nu}}{n-1} \right),
\]

\[
\nabla^{2} \hat{\Phi} = \frac{n-1}{8} \frac{\partial \hat{V}(\hat{\Phi})}{\partial \hat{\Phi}} + \frac{\alpha(n-4s+1)}{2(n-3)} e^{-\frac{4(n+1)\Phi}{(n-1)(n-3)}} \bar{Y}^{s},
\]

\[
\nabla_{\mu} \left( e^{\frac{4s\bar{Y}}{n-1}} \left( -F_{\mu\nu} \right)^{s-1} F_{\nu}^{\mu} \right) = 0,
\]

where

\[
\bar{Y} = -e^{\frac{4s\bar{Y}}{n-1}} \frac{8\beta^{2}}{\bar{Y}}.
\]

### 3.4 Logarithmic theory

Soleng theory is one of the nonlinear electrodynamics with logarithmic form [61]. It is also remarkable that this theory is one of the BI-type theories and enjoys most of the BI properties such as the absence of shock waves, birefringence phenomena and respecting an electric-magnetic duality [62,63,64]. In addition, like BI electrodynamics, the self-energy of the point-like charges is finite in this theory. The explicit form of Soleng Lagrangian is

\[
L(\bar{F}) = -8\beta^{2} \ln \left( 1 + \frac{\bar{F}}{8\beta^{2}} \right),
\]

where we set \( \hat{L}(\hat{Y}) = \ln (1 + \hat{Y}) \) with \( A = -8\beta^{2} \) and \( B = \frac{1}{8\beta^{2}} \). Applying the mentioned function with Eqs. (21) and (22), we obtain

\[
\hat{L}(\hat{F}, \hat{\Phi}) = -8\beta^{2} e^{-\frac{4(n+1)\Phi}{(n-1)(n-3)}} \ln \left( 1 + \frac{e^{\frac{4(n+1)\Phi}{(n-1)(n-3)}} \hat{F}}{8\beta^{2}} \right),
\]

which is coupled Lagrangian of dilaton-Soleng theory. Varying the corresponding action leads to the following field equations

\[
\bar{R}_{\mu\nu} = \frac{4}{n-1} \left( \nabla_{\mu} \hat{\Phi} \nabla_{\nu} \hat{\Phi} + \frac{1}{4} \hat{V}(\hat{\Phi}) g_{\mu\nu} \right) + \frac{8\beta^{2} \ln(1 + \bar{Y})}{n-1} e^{-\frac{4(n+1)\Phi}{(n-1)(n-3)}} g_{\mu\nu} + \frac{2e^{-4s\bar{Y}/(n-1)}}{1 + \bar{Y}} \left( F_{\mu\eta} F_{\nu}^{\eta} - \frac{F_{\mu\nu}}{n-1} \right),
\]

\[
\nabla^{2} \hat{\Phi} = \frac{n-1}{8} \frac{\partial \hat{V}(\hat{\Phi})}{\partial \hat{\Phi}} - \frac{\alpha}{2(n-3)} \left[ (n + 1) \hat{L}(\hat{F}, \hat{\Phi}) + 4e^{-\frac{4s\bar{Y}}{1 + \bar{Y}}} \hat{F} \right],
\]

\[
\nabla_{\mu} \left( e^{-\frac{4s\bar{Y}}{1 + \bar{Y}}} \left( -F_{\mu\nu} \right)^{s-1} F_{\nu}^{\mu} \right) = 0,
\]

where

\[
\bar{Y} = \frac{e^{\frac{4s\bar{Y}}{1 + \bar{Y}}}}{8\beta^{2}}.
\]
3.5 Exponential theory

In addition to BI and logarithmic types for the nonlinear abelian gauge field, very recently one of the present authors proposed an exponential form of nonlinear electrodynamics [39,40]. However, unlike BI and Soleng theories, the Exponential form of electrodynamics cannot deal with the problem of self-energy completely, its electric field singularity is much weaker than in the Einstein-Maxwell theory.

Here, we consider exponential form of nonlinear electrodynamics [39,40] and apply our prescription to obtain its correct coupling with dilaton field in the Einstein frame

\[ L(F) = \beta^2 \left( \exp\left( -\frac{F}{\beta^2} \right) - 1 \right). \]  

Regarding this model, we find \( \bar{L}(\bar{Y}) = \exp(\bar{Y}) - 1 \) with \( A = \beta^2 \) and \( B = -\frac{1}{\beta^2} \). Taking into account Eqs. (21) and (22), one can find

\[ \bar{L}(\bar{F}, \bar{\Phi}) = \beta^2 \exp\left( -\frac{16 \alpha \Phi}{\beta^2} \right) \left[ (n+1) \bar{L}(\bar{F}, \bar{\Phi}) + 4 \exp(\bar{Y}) F \right], \]

which is coupled Lagrangian of dilaton gravity with the nonlinear electrodynamics proposed by Hendi [39,40]. It is worthwhile to mention that, although in the special case \( \alpha \to 0 \) the Lagrangian of Ref. [65] reduces to the correspondence Lagrangian of [39,40], it is not consistent with BD theory. Whereas proposed Lagrangian of Eq. (44) has both requirement conditions, suitable special limits and consistency with BD theory. Using the variation principle, one finds

\[ \bar{R}_{\mu\nu} = \frac{4}{n-1} \left( \nabla_\mu \Phi \nabla_\nu \Phi + \frac{1}{4} \bar{V}(\Phi) \bar{g}_{\mu\nu} \right) + \beta^2 \left[ \exp(\bar{Y}) - 1 \right] \frac{1}{n-1} \exp\left( -\frac{16 \alpha \Phi}{\beta^2} \right) \bar{g}_{\mu\nu} 
+ 2 e^{-4 \alpha \Phi/(n-1)} \exp(\bar{Y}) \left( \bar{F}_{\mu\eta} \bar{F}^\eta_{\nu} - \frac{\bar{F}^2}{n-1} \bar{g}_{\mu\nu} \right), \]

\[ \nabla^2 \Phi = \frac{n-1}{8} \frac{\partial \bar{V}(\Phi)}{\partial \Phi} + \frac{\alpha}{2(n-3)} \left[ (n+1) \bar{L}(\bar{F}, \bar{\Phi}) + 4 e^{-4 \alpha \Phi/(n-1)} \exp(\bar{Y}) F \right], \]

where

\[ \bar{Y} = -\frac{16 \alpha \Phi}{\beta^2}. \]

4 BD versus dilaton solutions

Having the suitable field equations, we are in a position to take into account the field equations to obtain suitable metric for the spacetime. Solving the field equations of dilaton gravity in Einstein frame is not hard and one can obtain consistent functions for \( \bar{g}_{\mu\nu}, \bar{F}_{\mu\nu} \) and \( \bar{\Phi} \), depending the initial conditions of spacetime, such as spherical or cylindrical symmetry.

By assuming the \( (\bar{g}_{\mu\nu}, \bar{F}_{\mu\nu}, \bar{\Phi}) \) as solutions of Eqs. (14)-(16) with potential \( \bar{V}(\Phi) \) and comparing Eqs. (11b)-(12) with Eqs. (14)-(16) we find the solutions of Eqs. (11b)-(12) with potential \( V(\Phi) \) can be written as

\[ [g_{\mu\nu}, F_{\mu\nu}, \Phi] = \left[ \exp\left( -\frac{8 \alpha \Phi}{(n-1)(n-3)} \right) \bar{g}_{\mu\nu}, \bar{F}_{\mu\nu}, \exp\left( \frac{4 \alpha \bar{\Phi}}{n-3} \right) \right]. \]
5 Implementation of second case to an energy dependent Ricci-flat spacetime

Now, we are in a position to apply previous results to a typical spacetime. In order to enrich the physical properties of the solutions, we consider \((n + 1)\)-dimensional energy dependent spacetime with flat boundary. In other words, we want to obtain black hole solutions of BI-Dilaton gravity’s rainbow and their BD-BI counterpart. We start with the following metric

\[
ds^2 = -\frac{N(r)}{f^2(\varepsilon)}dt^2 + \frac{1}{g^2(\varepsilon)} \left( \frac{dr^2}{N(r)} + r^2 R(r)^2 d\Omega_{n-1}^2 \right),
\]

where \(r^2 d\Omega_{n-1}^2 = r^2 \sum_{i=1}^{n-1} d\theta_i^2\) is an \((n-1)\)-dimensional Euclidean space, \(N(r)\) is the metric functions and also \(f(\varepsilon)\) and \(g(\varepsilon)\) are rainbow functions. We should introduce a suitable potential to solve the field equations, \((27)\) and \((29)\), simultaneously. Taking into account Eq. \((29)\) with metric \((50)\), we find the following differential equation

\[
\left[ \frac{4\alpha \overline{\Phi}}{(n-1)} - \left( \frac{R'}{R} + \frac{1}{r} \right) \right] F_{tr}^2 e^{\frac{4\alpha \overline{\Phi}}{(n-1)}} - n\beta^2 \left[ \frac{4\alpha \overline{\Phi}}{(n-1)^2} - (n-1)^2 \frac{R'}{R} - \frac{1}{r} + \frac{1}{r(n-1)R F_{tr}} \right] = 0,
\]

where its solution is for the electric field is

\[
F_{tr}(r) = E(r) = \frac{q \exp \left( \frac{4\alpha \overline{\Phi}}{n-1} \right)}{[rR(r)]^{n-1}} \sqrt{1 + \frac{f^2(\varepsilon)g^2(\varepsilon)q^2 \exp \left( \frac{4\alpha \overline{\Phi}}{n-1} \right)}{\beta^2 [rR(r)]^{2(n-1)}}}.
\]

Regarding a suitable ansatz \(R(r) = \exp \left( \frac{4\alpha \overline{\Phi}}{n-1} \right)\), one finds the difference between \(tt\) and \(rr\) components of Eq. \((27)\) as

\[
\overline{\Phi}'' + 2\overline{\Phi} \left( \frac{1}{\alpha(n-1)} \overline{\Phi} + \frac{1}{r} \right) = 0
\]

with the following solution for the dilaton field

\[
\overline{\Phi}(r) = \frac{(n-1)\gamma}{2\alpha} \ln \left( \frac{b}{r} \right).
\]

Now, we should find the metric function \(N(r)\). To do so, we should set a suitable potential, \(\nabla(\overline{\Phi})\). For dilatonic Maxwell Lagrangian, suitable potential is \(\nabla(\overline{\Phi}) = 2\Lambda e^{\frac{4\alpha \overline{\Phi}}{n-1}}\), in which motivates us to consider the following potential

\[
\nabla(\overline{\Phi}) = 2\Lambda e^{\frac{4\alpha \overline{\Phi}}{n-1}} + \frac{W(r)}{\beta^2},
\]

where for the Maxwell limit \((\beta \to \infty)\), second term vanishes. In order to obtain metric functions and unknown \(W(r)\), we regard \(\theta_i \theta_i\) component of Eq. \((27)\) with Eq. \((28)\) to obtain, respectively,

\[
\left( \frac{R'}{R} - \frac{1}{r} \right) \frac{N'}{N} - \frac{(n-2)R''}{R^2} \frac{2(n-1)R'}{rR} - \frac{(n-2)}{r^2} - \frac{\nabla(\overline{\Phi})}{(n-1)Ng^2(\varepsilon)} + \frac{4\beta^2 e^{-\frac{4\alpha \overline{\Phi}}{(n-1)Ng^2(\varepsilon)}}}{(n-1)Ng^2(\varepsilon)} \left[ 1 - \frac{1}{\sqrt{1 - \frac{4\alpha \overline{\Phi}}{(n-1)Ng^2(\varepsilon)} f^2(\varepsilon)g^2(\varepsilon)F_{tr}^2}} \right] = 0
\]

\[
\overline{\Phi}'' + \overline{\Phi} \left[ (n-1) \left( \frac{1}{r} + \frac{R'}{R} + \frac{N'}{N} \right) + \frac{4\alpha f(\varepsilon)F_{tr}}{(n-3)(rR)(n-1)} - \frac{n-1}{8Ng^2(\varepsilon)} \left( \frac{d}{d\overline{\Phi}} \nabla(\overline{\Phi}) \right) - \right.
\]
\[
\frac{2(n + 1)\alpha^2\epsilon}{(n - 3)Ng^2(\epsilon)} \left[ 1 - \frac{1}{\sqrt{1 - \frac{16n^4}{\beta^2(\epsilon)g^2(\epsilon)F^2_n}}} \right] = 0. \quad (57)
\]

It is a matter of calculation to show that the following functions satisfy the field equations
\[
W(r) = \frac{4q(n - 1)\beta^2f^2(\epsilon)g^2(\epsilon)R}{(\alpha^2 + 1)q^n b^n} \int \frac{F_{ir}}{r^{n(1-\gamma)} - 1} \, dr + \frac{4\beta^4}{R^{2n+1}q} \left( 1 - \frac{F_{ir}R^{n-3}r^{n-1}}{q} \right) - \frac{4q\beta^2f^2(\epsilon)g^2(\epsilon)F_{ir}}{r^n} \left( \frac{b}{r} \right)^{-(n-1)\gamma},
\]
\[
N(r) = \left( \frac{(1 + \alpha^2)r^2}{(n - 1)} \right) \frac{2A}{g^2(\epsilon)(\alpha^2 - n)} - \frac{m}{r^{n-1}(1-\gamma)} - \frac{4(1 + \alpha^2)q^2(\epsilon)g^2(\epsilon)r^{2n(n-2)}}{(n - \alpha^2)r^{2n-2}} \left( \frac{1}{2(n-1)}F_1(\eta) - \frac{1}{\alpha^2 + n - 2}F_2(\eta) \right),
\]
where
\[
R(r) = \left( \frac{b}{r} \right)^\gamma,
\]
\[
F_{ir} = \frac{q}{r^{n-1}} \frac{(\frac{b}{r})^{2n-2(n-1)}}{\sqrt{1 + \frac{f^2(\epsilon)g^2(\epsilon)q^2}{\beta^2r^{2n-2(n-1)}}}},
\]
\[
F_1(\eta) = 2F_1 \left( \frac{1}{2} \frac{(n - 3)\gamma}{\alpha^2 + n - 2} \right) \left[ 1 + \frac{(n - 3)\gamma}{\alpha^2 + n - 2} \right], -\eta),
\]
\[
F_2(\eta) = 2F_1 \left( \frac{1}{2} \frac{(n - 3)\gamma}{2(n-1)} \right) \left[ 1 + \frac{(n - 3)\gamma}{2(n-1)} \right], -\eta),
\]
\[
\eta = \frac{g^2(\epsilon)f^2(\epsilon)q^2(\epsilon)\Gamma(n-5)/(n-3)}{\beta^2r^{2n-1}},
\]
\[
\gamma = \frac{\alpha^2 + n - 2}{2\alpha^2 + n - 3}.
\]

It is notable that \( b \) and \( m \) are integration constants, \( \gamma = \alpha^2 / (1 + \alpha^2) \) and \( 2F_1([a, b], [c], z) \) is the hypergeometric function. It is worthwhile to mention that obtained functions satisfy all field equations, simultaneously. In addition, one can find that for large values of the nonlinearity parameter, \( \beta \),

\[
W(r)_{\text{large } \beta} = \frac{g^4(\epsilon)f^4(\epsilon)q^4\alpha^4(\frac{5}{2})^{2(n^2+11n+11)}/(n-3)\gamma}{2(n^2+3)r^{10n(n-1)+2(1-\gamma)(2n+13n-17)}} + \frac{g^8(\epsilon)f^8(\epsilon)q^8\alpha^2(\frac{5}{2})^{2(n^2+17n+16)}/(n-3)\gamma}{(3n^2+4\alpha^2-9)r^{2n^2+4n+13}/(n-3)^\beta} + O(1/\beta^4),
\]

which confirms that the nonvanishing term of Eq. (51) is the first term for \( \beta \to \infty \). In addition, we can conclude that the second term of Eq. (55) \( (\frac{W(r)}{\beta^4}) \) vanishes in this limit (\( \beta \to \infty \)), as we expected. In other words, for \( \beta \to \infty \), the Maxwell solutions (and also their corresponding potential \( (\nabla \times \mathbf{A} = 2\epsilon \frac{4\alpha^2}{n-1} \sqrt{\mathbf{A}}) \)) will be recovered.

Now, we can obtain black hole solutions of BD-BI gravity’s rainbow. We consider the following Ricci-flat energy dependent spacetime
\[
ds^2 = -\frac{A(r)}{f^2(\epsilon)} \, dt^2 + \frac{1}{g^2(\epsilon)B(r)} \left( dr^2 + r^2H(r)^2d\Omega_{n-1}^2 \right).
\]

(62)
Using the conformal transformation, we find that the following functions satisfy all BD-BI field equations, simultaneously,

\[
A(r) = \left( \frac{b}{r} \right)^{-\frac{4\gamma}{n-3}} N(r), \\
B(r) = \left( \frac{b}{r} \right)^{\frac{4\gamma}{n-3}} N(r), \\
H(r) = \left( \frac{b}{r} \right)^{\frac{\gamma(n-5)}{(n-3)}} N(r), \\
\Phi(r) = \left( \frac{b}{r} \right)^{\frac{2\gamma(n-1)}{(n-3)}}.
\]

where \( V(\Phi) \) can be calculated from Eq. (7).

6 Conclusions

In this paper, we discussed about the conformal relation of scalar tensor gravity in the Einstein and Jordan frames. The main goal of this paper was presenting a formal method to obtain the correct coupling between scalar field and electrodynamics. We introduced a general method for all linear and nonlinear electrodynamics. We obtained the correct Lagrangian of dilaton-electrodynamics and related field equations as well. We also obtained the Lagrangian of known theory as the cases study and found that our method are valid for various models of electrodynamics.

We should note that in order to introduce a correct Lagrangian of dilaton gravity coupled with electrodynamics, we should applied two requirements: compatibility with BD theory and appropriate special situations. We also reported that some of dilatonic black hole papers used the inconsistent Lagrangian and did not care for the BD consistency.

We also, applied proposed method to the case of BI nonlinear electrodynamics and obtained a suitable potential for consistency of field equations. We achieved analytical solutions of both dilaton-BI and BD-BI theories in gravity’s rainbow and found the effects of rainbow functions in metric functions as well as other fields. We left geometrical and thermodynamical properties of the solutions for an independent work.

In this paper, we restricted ourselves to BD theory. It is worthwhile to extend BD theory to a general form of scalar tensor gravity in both Einstein and Jordan frames. We left these issues for the forthcoming work.

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References

1. C. Brans and R. Dicke, Phys. Rev. 124, 925 (1961).
2. S. W. Hawking, Commun. Math. Phys. 25, 167 (1972).
3. R. G. Cai and Y. S. Myung, Phys. Rev. D 56, 3466 (1997).
4. M. H. Dehghani, J. Pakravan and S. H. Hendi, Phys. Rev. D 74, 104014 (2006).
5. S. H. Hendi, J. Math. Phys. 49, 082501 (2008).
6. S. H. Hendi and R. Katebi, Eur. Phys. J. C 72, 2235 (2012).
7. M. A. Scheid, S. L. Shapiro, and S. A. Teukolsky, Phys. Rev. D 51, 4208 (1995).
8. M. A. Scheid, S. L. Shapiro, and S. A. Teukolsky, Phys. Rev. D 51, 4236 (1995).
9. G. Kang, Phys. Rev. D 54, 7483 (1996).
10. H. P. de Oliveira and E. S. Cheb-Terrab, Class. Quantum Grav. 13, 425 (1996).
11. P. Jordan, Zeit. Phys. 157, 112 (1959);
12. S. W. Hawking and G. F. R. Ellis, The Large Scale Structure of Space-time (Cambridge University Press, 1999);
13. Y. Fujii and K. I. Maeda, The Scalar-Tensor Theory of Gravitation, (Cambridge University Press, 2003);
14. D. Wands, Class. Quantum Gravit. 11, 269 (1994);
15. D. I. Kaiser, Phys. Rev. D 81, 084044 (2010);
16. M. P. Dabrowski, J. Garecki and D. B. Blaschke, Annalen Phys. (Berlin) 18, 13 (2009).
17. J. D. Bekenstein, Ann. Phys. 82, 535, (1974);
18. J. P. Abreu, P. Crawford, J. P. Mimiso, Class. Quant. Grav. 11, 1919, (1994).
19. S. Capozziello, M. De Laurentis, Phys. Rept. 509, 167 (2011).
20. P. Horava, Phys. Rev. D 79, 084008 (2009).
21. P. Horava, Phys. Rev. Lett. 102, 161301 (2009).
22. R. Gregory, S. L. Parameswaran, G. Tasinato and I. Zavala, JHEP 12, 047 (2010).
23. P. Burda, R. Gregory and S. Ross, JHEP 11, 073 (2014).
24. S. S. Gubser and A. Nellore, Phys. Rev. D 80, 105007 (2009).
25. R. Gambini and J. Pullin, Phys. Rev. D 59, 124021 (1999).
26. G. Bertoldi, B. A. Burrington and A. W. Peet, Phys. Rev. D 82, 106013 (2010).
27. J. Tarrio and S. Vandoren, JHEP 09, 017 (2011).
28. G. ’t Hooft, Class. Quantum Gravit. 13, 1023 (1996).
29. S. H. Hendi, Mir Faizal, B. Eslam Panah and S. Panahiyan, [arXiv:1508.00234].
30. J. D. Bekenstein, Ann. Phys. 82, 535, (1974).
31. J. P. Abreu, P. Crawford, J. P. Mimiso, Class. Quant. Gravit. 11, 1919, (1994).
32. S. M. Carroll, J. A. Harvey, V. A. Kostelecky, C. D. Lane and T. Okamoto, Phys. Rev. Lett. 87, 141601 (2001).
33. S. H. Hendi, Phys. Rev. D 82, 064040 (2010).
34. S. H. Hendi, JHEP 03, 065 (2012).
35. S. H. Hendi, Ann. Phys. (N.Y.) 333, 282 (2013).
36. E. S. Fradkin and A. A. Tseytlin, Phys. Lett. B 163, 123 (1985).
37. R. R. Metsaev, M. A. Rakhmanov and A. A. Tseytlin, Phys. Lett. B 193, 207 (1987).
38. R. G. Leigh, Mod. Phys. Lett. A 4, 2767 (1989).
39. M. Born and L. Infeld, Proc. R. Soc. London A 143, 410 (1934).
40. M. Born and L. Infeld, Proc. R. Soc. London A 144, 425 (1934).
41. M. H. Dehghani and H. R. Rastegar-Sedehi, Phys. Rev. D 74, 124018 (2006).
42. D. L. Wiltshire, Phys. Rev. D 38, 2445 (1988).
43. M. H. Dehghani and H. R. Rastegar-Sedehi, Phys. Rev. D 79, 044012 (2009).
44. M. H. Dehghani and H. R. Rastegar-Sedehi, JCAP, 02, 020 (2007).
45. A. Sheykhi, Phys. Lett. B 662, 7 (2008).
46. M. Hassaine and C. Martinez, Phys. Rev. D 75, 027502 (2007).
47. M. H. Dehghani and H. R. Rastegar-Sedehi, Gen. Relativ. Gravit. 41, 1355 (2009).
48. S. H. Hendi, Phys. Lett. B 677, 123 (2009).
49. M. H. Dehghani and H. R. Rastegar-Sedehi, JCAP, 02, 020 (2007).
50. M. H. Dehghani, M. M. H. Dehghani and H. R. Rastegar-Sedehi, Gen. Relativ. Gravit. 41, 1355 (2009).
51. S. H. Hendi, Phys. Lett. B 677, 123 (2009).
52. H. Maeda, M. Hassaine and C. Martinez, Phys. Rev. D 79, 044012 (2009).
53. S. H. Hendi and R. Gregory and S. Ross, JHEP 11, 073 (2014).
54. M. Hassaine and C. Martinez, Class. Quantum Gravit. 25, 195023 (2008).
55. S. H. Hendi and H. R. Rastegar-Sedehi, Gen. Relativ. Gravit. 41, 1355 (2009).
56. S. H. Hendi, Phys. Lett. B 677, 123 (2009).
57. H. Maeda, M. Hassaine and C. Martinez, Phys. Rev. D 79, 044012 (2009).
58. S. H. Hendi and R. Gregory and S. Ross, JHEP 11, 073 (2014).
59. S. H. Hendi, Phys. Lett. B 690, 220 (2010).
60. S. H. Hendi, Phys. Lett. B 690, 220 (2010).
61. H. H. Soleng, Phys. Rev. D 52, 6178 (1995).
62. G. Boillat, J. Math. Phys. 11, 941 (1970).
63. G. Boillat, J. Math. Phys. 11, 1482 (1970).
64. G. W. Gibbons and D. A. Rasheed, Nucl. Phys. B 454, 185 (1995).
65. A. Sheykhi and S. Hajkhalili, Phys. Rev. D 89, 104019 (2014).