BAHADUR EFFICIENCIES OF THE EPPS–PULLEY TEST FOR NORMALITY

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The test for normality suggested by Epps and Pulley is a serious competitor to tests based on the empirical distribution function. In contrast to the latter procedures, it has been generalized to obtain a genuine affine invariant and universally consistent test for normality in any dimension. We obtain approximate Bahadur efficiencies for the test of Epps and Pulley, thus complementing recent results of Milošević et al. For certain values of a tuning parameter that is inherent in the Epps–Pulley test, this test outperforms each of its competitors considered by Milošević et al., over the whole range of six close alternatives to normality. Bibliography: 21 titles.

1. Introduction

The purpose of this article is to derive Bahadur efficiencies for the test of normality proposed by Epps and Pulley [9], thus complementing recent results of Milošević et al. [15], who confined their study to tests of normality based on the empirical distribution function. To be specific, assume that $X_1, X_2, \ldots$ is a sequence of independent and identically distributed (i.i.d.) copies of a random variable $X$ that has an absolutely continuous distribution with respect to Lebesgue measure. To test the hypothesis $H_0$ that the distribution of $X$ is some unspecified non-degenerate normal distribution, Epps and Pulley [9] proposed to use the test statistic

$$T_{n, \beta} = n \int_{-\infty}^{\infty} \left| \psi_n(t) - e^{-t^2/2} \right|^2 \varphi_\beta(t) \, dt.$$ 

Here, $\psi_n(t) = n^{-1} \sum_{j=1}^{n} \exp(itY_{n,j})$ is the empirical characteristic function of the so-called scaled residuals $Y_{n,1}, \ldots, Y_{n,n}$, where $Y_{n,j} = S_n^{-1}(X_j - \bar{X}_n)$, $j = 1, \ldots, n$, and $\bar{X}_n = n^{-1} \sum_{j=1}^{n} X_j$, $S_n^2 = n^{-1} \sum_{j=1}^{n} (X_j - \bar{X}_n)^2$ are the sample mean and the sample variance of $X_1, \ldots, X_n$, respectively, and $\beta > 0$ is a so-called tuning parameter. Moreover,

$$\varphi_\beta(t) = \frac{1}{\beta \sqrt{2\pi}} \exp\left(-\frac{t^2}{2\beta^2}\right), \quad t \in \mathbb{R},$$

is the density of the centred normal distribution with variance $\beta^2$. A closed-form expression of $T_{n, \beta}$ that is amenable to computational purposes is

$$T_{n, \beta} = \frac{1}{n} \sum_{j,k=1}^{n} \exp\left( -\frac{\beta^2}{2} (Y_{n,j} - Y_{n,k})^2 \right) - \frac{2}{\sqrt{1+\beta^2}} \sum_{j=1}^{n} \exp\left( -\frac{\beta^2 Y_{n,j}^2}{2(1+\beta^2)} \right) + \frac{n}{\sqrt{1+2\beta^2}}. \quad (1.1)$$

Epps and Pulley did not obtain either the limit null distribution of $T_{n, \beta}$ as $n \to \infty$ or the consistency of a test for normality that rejects $H_0$ for large values of $T_{n, \beta}$. Their procedure, however, turned out to be a serious competitor to the classical tests of Shapiro–Wilk, Shapiro–Francia and Anderson–Darling in simulation studies (see [3]). In the special case $\beta = 1,$...
Baringhaus and Henze [5] generalized the approach of Epps and Pulley to obtain a genuine
test of multivariate normality, and they derived the limit null distribution of \( T_{n,1} \).
Moreover, they proved the consistency of the test of Epps and Pulley against each alternative to normality
having a finite second moment. The latter restriction was removed by S. Csörgő [6]. By an
approach different from that adopted in [5], Henze and Wagner [12] obtained both the limit
null distribution and the limit distribution of \( T_{n,\beta} \) under contiguous alternatives to normality.
Under fixed alternatives to normality, the limit distribution of \( T_{n,\beta} \) is normal, as elaborated
by [4] in much greater generality for weighted \( L^2 \)-statistics. For more information on \( T_{n,\beta} \),
especially on the role of the tuning parameter \( \beta \), see [7, Sec. 2.2].

Note that \( T_{n,\beta} \) is invariant with respect to affine transformations \( X_j \mapsto aX_j + b \), where
\( a, b \in \mathbb{R} \) and \( a \neq 0 \). Hence, under \( H_0 \), both the finite-sample and the asymptotic distribution
of \( T_{n,\beta} \) do not depend on the parameters \( \mu \) and \( \sigma^2 \) of the underlying normal distribution
\( N(\mu, \sigma^2) \). Under \( H_0 \), we thus assume that \( \mu = 0 \) and \( \sigma^2 = 1 \). The rest of the paper unfolds as
follows: in Sec. 2, we revisit the notion of approximate Bahadur efficiency. Sections 3 and 4
deal with stochastic limits and local Bahadur slopes, and Sec. 5 tackles an eigenvalue problem
connected with the limit null distribution of the test statistic. The final Sec. 6 contains results
regarding local approximate Bahadur efficiencies of the Epps–Pulley test for the six close
alternatives considered in [15] and a wide spectrum of values of the tuning parameter \( \beta \).

2. Approximate Bahadur efficiency

There are several options to compare different tests for the same testing problem as the
sample size \( n \) tends to infinity (see [16]). One of these options is asymptotic efficiency due
to Bahadur (see [1]). This notion of asymptotic efficiency requires knowledge of the large
deviation function of the test statistic. Apart from the notable exception given in [18], such
knowledge, however, is hitherto not available for statistics that contain estimated parameters,
such as \( T_{n,\beta} \) given in (1.1). To circumvent this drawback, one usually employs the so-called
approximate Bahadur efficiency, which only requires results on the tail behavior of the limit
distribution of the test statistic under the null hypothesis. To be more specific with respect to
the title of this paper, let \( X, X_1, X_2, \ldots \) be a sequence of i.i.d. random variables, where the
distribution of \( X \) depends on a real-valued parameter \( \vartheta \in \Theta \), where \( \Theta \) denotes the parameter
space, and only the case \( \vartheta = 0 \) corresponds to the case that the distribution of \( X \) is standard
normal. Assume that \( (S_n)_{n \geq 1} \), where \( S_n = S_n(X_1, \ldots, X_n) \), is a sequence of test statistics of
the hypothesis \( H_0 : \vartheta = 0 \) against the alternative \( H_1 : \vartheta \in \Theta \setminus \{0\} \). Furthermore, assume
that rejection of \( H_0 \) is for large values of \( S_n \). The sequence \( (S_n) \) is called a standard sequence
if the following conditions hold (see, e.g., [16, p. 10], or [8, p. 3427]):

- There is a continuous distribution function \( G \) such that, for \( \vartheta = 0 \),
  \[
  \lim_{n \to \infty} P_0(S_n \leq x) = G(x), \quad x \in \mathbb{R}. \tag{2.1}
  \]
- There is a constant \( a_S \), \( 0 < a_S < \infty \), such that
  \[
  \log(1 - G(x)) = -\frac{a_S x^2}{2}(1 + o(1)) \quad \text{as} \quad x \to \infty. \tag{2.2}
  \]
- There is a real-valued function \( b_S(\vartheta) \) on \( \Theta \setminus \{0\} \), with \( 0 < b_S(\vartheta) < \infty \), such that, for
each \( \vartheta \in \Theta \setminus \{0\} \),
  \[
  \frac{S_n}{\sqrt{n}} \xrightarrow{P_\vartheta} b_S(\vartheta). \tag{2.3}
  \]
Then the so-called approximate Bahadur slope
\[
c^*_S(\vartheta) = a_S \cdot b^2_S(\vartheta), \quad \vartheta \in \Theta \setminus \{0\},
\]
is a measure of approximate Bahadur efficiency. Usually, it is true that $c_S^*(\vartheta) \sim \ell(S) \cdot \vartheta^2$ as $\vartheta \to 0$. In this case, $\ell(S)$ is called the local (approximate) index of the sequence $(S_n)$. We see that the sequence $(S_n)$, where $S_n := \sqrt{T_{n,\beta}}$, is a standard sequence. To this end, we derive the stochastic limit of $T_{n,\beta}/n$ for a general alternative in Sec. 3. In Sec. 4, we specialize this stochastic limit for local alternatives, and we derive the local index for the Epps–Pulley test statistic.

3. Stochastic limit of $T_{n,\beta}/n$

To calculate the asymptotic Bahadur efficiency of the test of Epps and Pulley, we need the following result.

**Theorem 3.1.** Assume that $\mathbb{E}(X^2) < \infty$. Then

$$\frac{T_{n,\beta}}{n} \xrightarrow{p} \mathbb{E} \left[ \exp \left( -\frac{\beta^2(Y_1 - Y_2)^2}{2} \right) \right] - \frac{2}{\sqrt{1 + \beta^2}} \mathbb{E} \left[ \exp \left( -\frac{\beta^2 Y_1^2}{2(1 + \beta^2)} \right) \right] + \frac{1}{\sqrt{1 + 2\beta^2}}$$

Here, $\xrightarrow{p}$ denotes convergence in probability, and $Y_j = (X_j - \mu)/\sigma$, $j \geq 1$, where $\mu = \mathbb{E}(X)$ and $\sigma^2 = \mathbb{V}(X)$.

**Proof.** From (1.1), we have

$$\frac{T_{n,\beta}}{n} = \frac{1}{n^2} \sum_{j,k=1}^n \exp \left( -\frac{\beta^2}{2} \left( \frac{X_j - X_k}{S_n} \right)^2 \right)$$

$$= \frac{2}{\sqrt{1 + \beta^2}} \frac{1}{n} \sum_{j=1}^n \exp \left( -\frac{\beta^2}{2(1 + \beta^2)} \left( \frac{X_j - \bar{X}_n}{S_n} \right)^2 \right) + \frac{1}{\sqrt{1 + 2\beta^2}}$$

By symmetry, it follows that

$$\mathbb{E}(A_{n,1}) = \frac{1}{n} + \frac{n - 1}{n} \cdot \mathbb{E} \left[ \exp \left( -\frac{\beta^2}{2} \left( \frac{X_1 - X_2}{S_n} \right)^2 \right) \right],$$

$$\mathbb{E}(A_{n,2}) = \mathbb{E} \left[ \exp \left( -\frac{\beta^2}{2(1 + \beta^2)} \left( \frac{X_1 - \bar{X}_n}{S_n} \right)^2 \right) \right].$$

Since $\bar{X}_n \to \mu$ and $S_n \to \sigma$ almost surely as $n \to \infty$ by the strong law of large numbers, it follows from Lebesgue’s dominated convergence theorem that

$$\lim_{n \to \infty} \mathbb{E}(A_{n,1}) = \mathbb{E} \left[ \exp \left( -\frac{\beta^2(Y_1 - Y_2)^2}{2} \right) \right],$$

$$\lim_{n \to \infty} \mathbb{E}(A_{n,2}) = \mathbb{E} \left[ \exp \left( -\frac{\beta^2 Y_1^2}{2(1 + \beta^2)} \right) \right],$$

and thus the expectation of $T_{n,\beta}/n$ converges to the stochastic limit figuring in Theorem 3.1. Likewise, the variance of $T_{n,\beta}/n$ converges to zero. □

4. Local Bahadur slopes

As in [15], we now assume that $\mathcal{G} = \{G(x; \vartheta)\}$ is a family of distribution functions (DF’s) with densities $g(x; \vartheta)$ such that $\vartheta = 0$ corresponds to the standard normal DF $\Phi$ and density $\varphi$, and each of the distributions for $\vartheta \neq 0$ is nonnormal. Moreover, we assume that the regularity assumptions WD in [17] are satisfied. If $X, X_1, X_2, \ldots$ are i.i.d. random variables with DF
$G(\cdot; \vartheta)$, we have to consider the stochastic limit figuring in Theorem 3.1 as a function of $\vartheta$ and expand this function at $\vartheta = 0$. To this end, let

$$\gamma = \frac{\beta^2}{2}, \quad \delta = \frac{\beta^2}{2(1 + \beta^2)}.$$  

(4.1)

Then, putting

$$\mu(\vartheta) = \int xg(x; \vartheta) \, dx,$$

$$\sigma^2(\vartheta) = \int x^2g(x; \vartheta) \, dx - \mu^2(\vartheta),$$

Theorem 3.1 yields

$$\frac{T_{n, \beta}}{n} \overset{p}{\longrightarrow} b_{T_{n, \beta}}(\vartheta),$$

where $\overset{p}{\longrightarrow}$ denotes convergence in probability under the true parameter $\vartheta$, and

$$b_{T_{n, \beta}}(\vartheta) = \int \int \exp \left(-\frac{\gamma(x - y)^2}{\sigma^2(\vartheta)}\right) g(x; \vartheta)g(y; \vartheta) \, dx \, dy$$

$$- \frac{2}{\sqrt{1 + \beta^2}} \int \exp \left(-\frac{\delta(x - \mu(\vartheta))^2}{\sigma^2(\vartheta)}\right) g(x; \vartheta) \, dx + \frac{1}{\sqrt{1 + \beta^2}}.$$  

(4.2)

Here and in what follows, each unspecified integral is over $\mathbb{R}$.

Notice that $b_{T_{n, \beta}}(0) = 0$. We have to find the quadratic (first nonvanishing) term in the Taylor expansion of $b_{T_{n, \beta}}$ around zero, i.e., we look for some (local index) $\Delta_{n} > 0$ such that

$$b_{T_{n, \beta}}(\vartheta) = \Delta_{n} \vartheta^2 + O(\vartheta^3) \quad \text{as} \quad \vartheta \to 0.$$

Writing $g_0'(x; \vartheta), g_0''(x; \vartheta)$ for derivatives of $g(x; \vartheta)$ with respect to $\vartheta$, we have

$$g(x; \vartheta) = \varphi(x) + \vartheta \cdot g_0'(x; 0) + \frac{\vartheta^2}{2} g_0''(x; 0) + O(\vartheta^3)$$

and thus, since $\mu(0) = 0$ and $\sigma^2(0) = 1$,

$$\mu(\vartheta) = \vartheta \int xg_0'(x; 0) \, dx + \frac{\vartheta^2}{2} \int xg_0''(x; 0) \, dx + O(\vartheta^3),$$

$$\sigma^2(\vartheta) = 1 + \vartheta \int x^2g_0'(x; 0) \, dx + \frac{\vartheta^2}{2} \int x^2g_0''(x; 0) \, dx - \mu(\vartheta)^2 + O(\vartheta^3).$$

Consequently, putting

$$\mu_1 := \mu'(0), \quad \mu_2 := \mu''(0), \quad \sigma_1 := (\sigma^2)'(0), \quad \sigma_2 := (\sigma^2)''(0)$$  

(4.4)

for the sake of brevity, it follows that

$$\mu_1 = \int xg_0'(x; 0) \, dx, \quad \mu_2 = \int xg_0''(x; 0) \, dx,$$

$$\sigma_1 = \int x^2g_0'(x; 0) \, dx, \quad \sigma_2 = \int x^2g_0''(x; 0) \, dx - 2\mu'(0)^2.$$

To tackle the integral that figures in (4.2), notice that

$$g(x; \vartheta)g(y; \vartheta) = \varphi(x)\varphi(y) + \vartheta [g_0'(x; 0)\varphi(y) + g_0'(y; 0)\varphi(x)]$$

$$+ \vartheta^2 \left[ \frac{1}{2} g_0''(x; 0)\varphi(y) + \frac{1}{2} g_0''(y; 0)\varphi(x) + g_0'(x; 0)g_0'(y; 0) \right] + O(\vartheta^3).$$
Moreover, it follows from a geometric series expansion that
\[
\frac{1}{\sigma^2(\vartheta)} = 1 - \vartheta \sigma_1 + \vartheta^2 \left[ \sigma_1^2 - \frac{\sigma_2^2}{2} \right] + O(\vartheta^3). \tag{4.5}
\]

From an expansion of the exponential function, we thus obtain
\[
\exp \left( -\frac{\gamma(x - y)^2}{\sigma^2(\vartheta)} \right) = e^{-\gamma(x-y)^2} \left[ 1 + \vartheta \sigma_1 \gamma(x - y)^2 \right. \\
&\left. + \vartheta^2 \left\{ \frac{1}{2} \sigma_1^2 \gamma^2(x - y)^4 - \left( \sigma_1^2 - \frac{\sigma_2^2}{2} \right) \gamma(x - y)^2 \right\} \right] + O(\vartheta^3).
\]
Using
\[
\int \int e^{-\gamma(x-y)^2} (x-y)^{2k} \varphi(x) \varphi(y) \, dx \, dy = \frac{4^k \Gamma(k + 1/2)}{\sqrt{\pi} (4 \gamma + 1)^{k+1/2}}, \quad k = 0, 1, 2,
\]
\[
\int e^{-\gamma(x-y)^2} \varphi(x) \, dx = \frac{1}{\sqrt{1 + \beta^2}} \cdot e^{\delta y^2}
\]
\[
\int e^{-\gamma(x-y)^2} (x-y)^2 \varphi(y) \, dy = e^{-\delta x^2} \cdot \frac{x^2 + \beta^2 + 1}{(1 + \beta^2)^{5/2}},
\]
and putting
\[
D_0 = \int \int e^{-\gamma(x-y)^2} g_0'(x;0)g_0'(y;0) \, dx \, dy, \tag{4.6}
\]
\[
J_{1,k} = \int e^{-\delta x^2} x^k g_0'(x;0) \, dx, \quad k = 0, 1, 2; \quad J_2 = \int e^{-\delta x^2} g_0''(x;0) \, dx, \tag{4.7}
\]
some algebra gives
\[
\int \int \exp \left( -\frac{\gamma(x - \vartheta)^2}{\sigma^2(\vartheta)} \right) g(x;\vartheta)g(y;\vartheta) \, dx \, dy = \frac{1}{\sqrt{1 + 2\beta^2}} + \vartheta \left\{ \frac{2J_{1,0}}{\sqrt{1 + \beta^2}} + \frac{2\sigma_1 \gamma}{(1 + 2\beta^2)^{3/2}} \right\} \\
+ \vartheta^2 \left\{ \frac{J_2}{\sqrt{1 + \beta^2}} + D_0 + \frac{2\sigma_1 \gamma (J_{1,1} + (\beta^2 + 1)J_{1,0})}{(1 + \beta^2)^{5/2}} + \frac{\beta^2 ((2\beta^2 + 1)\sigma_2 - (\beta^2 + 2)\sigma_1^2)}{2(1 + 2\beta^2)^{5/2}} \right\}. \tag{4.8}
\]
As for the integral figuring in (4.3), we use (4.5). Neglecting the terms that are of order \(O(\vartheta^3)\), straightforward but tedious calculations give
\[
\exp \left( -\frac{\delta(x - \mu(\vartheta))^2}{\sigma^2(\vartheta)} \right) = e^{-\delta x^2} \cdot \left\{ 1 + \delta \vartheta U(x) + \frac{\delta \vartheta^2}{2} V(x) \right\} + O(\vartheta^3),
\]
where, recalling (4.4),
\[
U(x) = \sigma_1 x^2 + 2\mu_1 x, \\
V(x) = \delta \sigma_1^2 x^4 + 4\delta \mu_1 \sigma_1 x^3 + (4\delta \mu_1^2 - 2\sigma_1^2 + \sigma_2) x^2 - (4\mu_1 \sigma_1 - 2\mu_2) x - 2\mu_1^2.
\]
Thus,
\[
\int \exp \left( -\frac{\delta(x - \mu(\vartheta))^2}{\sigma^2(\vartheta)} \right) g(x; \vartheta) \, dx = \int e^{-\delta x^2} \left( 1 + \delta \vartheta U(x) + \frac{\delta \vartheta^2}{2} V(x) \right) \varphi(x) \, dx \\
+ \vartheta \int e^{-\delta x^2} \left( 1 + \delta \vartheta U(x) + \frac{\delta \vartheta^2}{2} V(x) \right) g_\vartheta'(x; 0) \, dx \\
+ \frac{\vartheta^2}{2} \int e^{-\delta x^2} \left( 1 + \delta \vartheta U(x) + \frac{\delta \vartheta^2}{2} V(x) \right) g_\vartheta''(x; 0) \, dx \\
+ O(\vartheta^3) = I_1(\vartheta) + \vartheta I_2(\vartheta) + \frac{\vartheta^2}{2} I_3(\vartheta) + O(\vartheta^3).
\]

We have
\[
I_1(\vartheta) = \frac{1}{(1 + 2\delta)^{1/2}} + \vartheta \cdot \frac{\delta \sigma_1}{(1 + 2\delta)^{3/2}} \\
+ \vartheta^2 \left[ \frac{3\delta^2 \sigma_1^2}{2(1 + 2\delta)^{5/2}} + \frac{\delta(4\delta \mu_1^2 - 2\sigma_1^2 + \sigma_2)}{2(1 + 2\delta)^{3/2}} - \frac{\delta \mu_1^2}{(1 + 2\delta)^{1/2}} \right].
\]

Furthermore,
\[
I_2(\vartheta) = \int e^{\delta x^2} g_\vartheta(x; 0) \, dx + \delta \vartheta \int e^{-\delta x^2} (\sigma_1^2 x^2 + 2\mu_1 x) g_\vartheta'(x; 0) \, dx + O(\vartheta^2),
\]
\[
I_3(\vartheta) = \int e^{-\delta x^2} g_\vartheta''(x; 0) \, dx + O(\vartheta).
\]

Recalling (4.7), we thus obtain, apart from the term which is \(O(\vartheta^3)\),
\[
\int \exp \left( -\frac{\delta(x - \mu(\vartheta))^2}{\sigma^2(\vartheta)} \right) g(x; \vartheta) \, dx = \frac{1}{(1 + 2\delta)^{1/2}} + \vartheta \left[ \frac{\delta \sigma_1}{(1 + 2\delta)^{3/2}} + J_{1,0} \right] \\
+ \vartheta^2 \left[ \frac{J_2}{2} + \delta \sigma_1^2 J_{1,2} + 2\delta \mu_1 J_{1,1} - \frac{\delta ((\delta + \frac{1}{2}) (\sigma_2 - 2\mu_2^2) - (\delta + 1) \sigma_1^2)}{(2\delta + 1)^{5/2}} \right].
\] (4.9)

Upon combining (4.8) and (4.9) and recalling (4.1), \(b_{T_\beta}(\vartheta)\) figuring in (4.2), (4.3) takes the form
\[
b_{T_\beta}(\vartheta) = \Delta_\beta \vartheta^2 + O(\vartheta^3) \text{ as } \vartheta \to 0,
\]
where
\[
\Delta_\beta = D_0 + \frac{\beta^2}{(2\beta^2 + 1)^{5/2}} \left( \left( J_{1,0} - J_{1,2} \right) \sigma_1 - 2J_{1,1} \mu_1 \right) \beta^2 + J_{1,0} \sigma_1 - 2J_{1,1} \mu_1 \\
+ \frac{\beta^2}{(2\beta^2 + 1)^{5/2}} \left( \left( 2\mu_1^2 + \frac{3}{4} \sigma_1^2 \right) \beta^2 + \mu_1^2 \right),
\]
and \(D_0\) and \(J_{1,0}, J_{1,1}, J_{1,2}\) are defined in (4.6) and (4.7), respectively.

5. Approximations to solutions of the eigenvalue problem

We now turn to conditions (2.1) and (2.2). The limit null distribution of \(T_{n,\beta}\) as \(n \to \infty\), is given by the distribution of
\[
T_\beta := \int_{-\infty}^{\infty} Z^2(t) \varphi_{\beta}(t) \, dt.
\]
Here, $Z$ is a centred Gaussian random element of the Fréchet space of continuous real-valued functions having covariance kernel $K(s, t) = \mathbb{E}[Z(s)Z(t)]$, where

$$K(s, t) = \exp \left(-\frac{(s-t)^2}{2}\right) - \left(1 + st + \frac{(st)^2}{2}\right) \exp \left(-\frac{s^2 + t^2}{2}\right), \quad s, t \in \mathbb{R} \quad (5.1)$$

(see Theorem 2.1 and Theorem 2.2 of [12]). In fact, $Z$ may also be regarded as a Gaussian random element of the separable Hilbert space $L^2$ of (equivalence classes of) functions that are square integrable with respect to $\varphi_\beta(t) \, dt$. The distribution of $T_\beta$ is that of $\sum_{j=1}^{\infty} \lambda_j(\beta) N_j^2$, where $N_1, N_2, \ldots$ is a sequence of i.i.d. standard normal random variables, and $\lambda_1(\beta), \lambda_2(\beta), \ldots$ is a sequence of positive eigenvalues of the integral operator $\mathcal{K}$ on $L^2$ defined by

$$\mathcal{K}: L^2 \to L^2, \quad f \mapsto \mathcal{K}f(s) = \int_{-\infty}^{\infty} K(s, t) f(t) \varphi_\beta(t) \, dt, \quad s \in \mathbb{R}. \quad (5.4)$$

Since $S_n$ figuring in (2.1) equals $\sqrt{n} \cdot G$, the function $G$ is the distribution function of $\tilde{Z} := \left(\sum_{j=1}^{\infty} \lambda_j(\beta) N_j^2\right)^{1/2}$. From [21], we thus have

$$\log(1 - G(x)) = \log \mathbb{P}(\tilde{Z} > x) = \log \mathbb{P}(\tilde{Z}^2 > x^2) \sim -\frac{x^2}{2\lambda_1(\beta)} \quad \text{as} \quad x \to \infty,$$

where $\lambda_1(\beta)$ denotes the largest eigenvalue. Hence, the approximate Bahadur slope of the Epps–Pulley test statistic is given by

$$e^*_{T_\beta}(\vartheta) = \frac{b_{T_\beta}(\vartheta)}{\lambda_1(\beta)}. \quad (5.2)$$

Thus, one has to tackle the so-called eigenvalue problem, i.e., to find positive values $\lambda$ and functions $f$ such that $\mathcal{K}f = \lambda f$ or, in other words, to solve the integral equation

$$\int_{-\infty}^{\infty} K(s, t) f(t) \varphi_\beta(t) \, dt = \lambda f(s), \quad s \in \mathbb{R}. \quad (5.3)$$

Since explicit solutions of such integral equations are only available in exceptional cases (for nonclassical goodness-of-fit test statistics, see [10] and [11]), we employ a stochastic approximation method. This method is related to the quadrature method in the classical literature on numerical mathematics (see [2, Chap. 3]), and which can also be found in machine learning theory (see [20]).

For the approximation of spectra of Hilbert–Schmidt operators, see [13]. To be specific, let $Y$ be a random variable having density $\varphi_\beta$. Then (5.3) reads

$$\lambda f(s) = \mathbb{E}[K(s, Y)f(Y)], \quad s \in \mathbb{R}. \quad (5.4)$$

An empirical counterpart to (5.4) emerges if we let $y_1, y_2, \ldots, y_N$, $N \in \mathbb{N}$, be independent realizations of $Y$. An approximation of the expected value in (5.4) is then

$$\mathbb{E}[K(s, Y)f(Y)] \approx \frac{1}{N} \sum_{j=1}^{N} K(s, y_j)f(y_j), \quad s \in \mathbb{R}. \quad (5.5)$$

If we evaluate (5.5) at the points $y_1, \ldots, y_n$, the result is

$$\lambda f(y_i) = \frac{1}{N} \sum_{j=1}^{N} K(y_i, y_j)f(y_j), \quad i = 1, \ldots, N, \quad (5.6)$$
which is a system of $N$ linear equations. Writing $v = (f(y_1), \ldots, f(y_N)) \in \mathbb{R}^N$ and $\tilde{K} = (K(y_i, y_j)/N)_{i,j=1,\ldots,N} \in \mathbb{R}^{N \times N}$, we can rewrite (5.6) according to 

$$\tilde{K}v = \lambda v$$

in matrix form, from which the (approximated) eigenvalues $\lambda_1, \ldots, \lambda_N$ can be computed explicitly. Note that for each eigenvalue $\lambda_j$ we have an eigenvector $v_j \in \mathbb{R}^N$, the components of which are the (approximated) values of the eigenfunctions $f_j$ computed at $y_1, \ldots, y_N$.

The simulation of eigenvalues was performed in the statistical computing language R (see [19]). We chose $N = 1000$, as parameters for the simulation and we considered the tuning parameters $\beta \in \{0.25, 0.5, 0.75, 1, 2, 3, 5, 10\}$. Each entry in Table 1 stands for the mean of 10 simulation runs.

| $\lambda$ \(\beta\) | 0.25 | 0.5  | 0.75 | 1    | 2    | 3    | 5    | 10   |
|-----------------------|------|------|------|------|------|------|------|------|
| $\lambda_1$           | 0.00040 | 0.01065 | 0.03829 | 0.07507 | 0.15207 | 0.16149 | 0.13552 | 0.08791 |
| $\lambda_2$           | 0.00003 | 0.00304 | 0.01735 | 0.04454 | 0.12921 | 0.14577 | 0.12606 | 0.08178 |
| $\lambda_3$           | 0.00000 | 0.00021 | 0.00720 | 0.00417 | 0.03964 | 0.07676 | 0.08703 | 0.06879 |
| $\lambda_4$           | 0.00000 | 0.00004 | 0.00076 | 0.00417 | 0.03966 | 0.06642 | 0.07997 | 0.06459 |
| $\lambda_5$           | 0.00000 | 0.00000 | 0.00011 | 0.00098 | 0.01692 | 0.03755 | 0.05678 | 0.05518 |

Table 1. Approximate first five eigenvalues of $\mathcal{K}$ for different weight functions $\varphi_\beta$, each entry is the mean of 10 simulation runs.

6. Alternatives

As in [15], we consider the following close alternatives:

- a Lehmann alternative with density $g_1(x; \vartheta) = (1 + \vartheta)\Phi^{\vartheta}(x)\varphi(x)$;
- the first Ley–Paindaveine alternative with density (see [14]) $g_2(x; \vartheta) = \varphi(x)e^{-\vartheta(1-\Phi(x))}(1 + \vartheta\Phi(x))$;
- the second Ley–Paindaveine alternative with density (see [14]) $g_3(x; \vartheta) = \varphi(x)(1 - \vartheta \pi \cos(\pi \Phi(x)))$;
- a contamination alternative (with $N(\mu, \sigma^2)$ for several pairs $(\mu, \sigma^2) \neq (0, 1)$) with density $g_4^{[\mu, \sigma^2]}(x; \vartheta) = (1 - \vartheta)\varphi(x) + \frac{\vartheta}{\sigma}\varphi(\frac{x - \mu}{\sigma})$.

As in [15], we computed the local (as $\vartheta \to 0$) relative approximate Bahadur efficiencies with respect to the likelihood ratio test (LRT). The LRT is the best test regarding exact Bahadur efficiency, and it is often used as a benchmark test. Table 2 displays the local approximate Bahadur efficiencies of $T_{n,\beta}$ with respect to the LRT, for each of the six alternatives considered in [15], and for $\beta \in \{0.25, 0.5, 0.75, 1, 2, 4, 5, 10\}$. A comparison with Table 1 of [15] shows that the Epps–Pulley test with $\beta = 0.5$ dominates the Kolmogorov–Smirnov test for each of the six alternatives, and for $\beta = 1, \beta = 2$ and $\beta = 3$, it outperforms the tests of Cramér–von Mises, the Watson variation of this test, and the Watson–Darling variation of the Kolmogorov–Smirnov test, respectively. If $\beta = 0.75$, the Epps–Pulley test dominates the Anderson–Darling test for each of the alternatives with the exception of the final contamination alternative. As a conclusion, the test of Epps and Pulley should receive more attention as a test for normality.
| Alt. \( \beta \) | 0.25 | 0.5 | 0.75 | 1 | 2 | 3 | 5 | 10 |
|----------------|-------|-----|------|---|---|---|---|----|
| Lehmann        | 0.996 | 0.895 | 0.854 | 0.743 | 0.514 | 0.406 | 0.328 | 0.267 |
| 1st Ley–Paindaveine | 0.947 | 0.944 | 0.998 | 0.937 | 0.745 | 0.612 | 0.507 | 0.417 |
| 2nd Ley–Paindaveine | 0.824 | 0.872 | 0.986 | 0.981 | 0.881 | 0.754 | 0.641 | 0.533 |
| Contamination with N(1,1) | 0.760 | 0.649 | 0.592 | 0.499 | 0.328 | 0.255 | 0.205 | 0.166 |
| Contamination with N(0.5,1) | 0.945 | 0.824 | 0.766 | 0.654 | 0.438 | 0.343 | 0.276 | 0.224 |
| Contamination with N(0,0.5) | 0.084 | 0.267 | 0.474 | 0.587 | 0.675 | 0.606 | 0.526 | 0.442 |

Table 2. Approximate local Bahadur efficiency of \( T_{n,\beta} \) with respect to the LRT.

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