Idempotents of $2 \times 2$ matrix rings over rings of formal power series

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ABSTRACT
Let $A_1, \ldots, A_s$ be unitary commutative rings which do not have non-trivial idempotents and let $A = A_1 \oplus \cdots \oplus A_s$ be their direct sum. We describe all idempotents in the $2 \times 2$ matrix ring $M_2(A[[X]])$ over the ring $A[[X]]$ of formal power series with coefficients in $A$ and in an arbitrary set of variables $X$. We apply this result to the matrix ring $M_2(\mathbb{Z}_n[[X]])$ over the ring $\mathbb{Z}_n[[X]]$ where $\mathbb{Z}_n \cong \mathbb{Z}/n\mathbb{Z}$ for an arbitrary positive integer $n$ greater than 1. Our proof is elementary and uses only the Cayley-Hamilton theorem (for $2 \times 2$ matrices only) and, in the special case $A = \mathbb{Z}_n$, the Chinese remainder theorem and the Euler-Fermat theorem.

1. Introduction
In this paper we consider non-trivial unitary associative rings only in the sense that $0 \neq 1$. An element $a$ in a ring $A$ is called an idempotent if $a^2 = a$. The idempotents were introduced in Ring Theory by Benjamin Peirce in his seminal book [1] (published lithographically in 1870 in a small number of copies for distribution among his friends and posthumously in its journal form by his son Charles Sanders Peirce in 1881). There Peirce established the decomposition of rings with idempotents known nowadays as the Peirce decomposition, see [1, Proposition 41]. Already 150 years the study of idempotents is among the main topics in Ring Theory and its applications. For example the search in the database of Mathematical Reviews gives more than 2700 publications with the word ‘idempotents’ in the title. For different aspects of the study of idempotents in the classical spirit see for example Ánh, Birkenmeier and van Wyk [2] and the references there.

It is well-known that the idempotents of the $d \times d$ matrix algebra $M_d(F)$ over a field $F$ coincide with the diagonalizable matrices with eigenvalues equal to 0 and 1. In 1946 Foster [3] described the commutative rings $A$ with the property that the idempotents in $M_d(A)$ are diagonalizable for all $d$. In 1966 Steger [4] showed that important classes of rings have this property. Among them are polynomial rings in one variable over a principal ideal ring (also with zero divisors) and polynomial rings in two variables over a $\pi$-regular ring with finitely
many idempotents. (The ring $A$ is $\pi$-regular if for any $a \in A$ there exists an $n$ such that $a^n = a^n a^n$.). The results of Foster and Steger were generalized also for matrices over non-commutative rings, see Song and Guo [5]. In 1976 Quillen [6] and Suslin [7] confirmed into the affirmative the Serre conjecture [8] and proved that when $A$ is a principal ideal domain and $X$ is a finite set of variables, then every finitely generated projective $A[X]$-module is free. Claudio Procesi drew my attention to the fact that the Quillen-Suslin theorem implies that for a principal ideal domain $A$, $|X| < \infty$, and for all $d$ every idempotent matrix in $M_d(A[X])$ is diagonalizable with eigenvalues equal to 0 and 1. In [9] Gómez-Torrecillas, Kutas, Lobillo and Navarro presented an algorithm for computing a primitive idempotent of a central simple algebra over the field $\mathbb{F}_q(x)$ of rational functions over the finite field $\mathbb{F}_q$ with applications to coding theory. In 1967 J.A. Erdos [10] proved that every singular matrix over a field is a product of idempotent matrices. See also the recent preprint of Nguyen [11] and the references there for further developments in this direction. We shall also mention the paper by Anh, Birkenmeier and van Wyk [12] where the authors mimic the behaviour of idempotents in matrix rings in a more general setup and the following three papers which are related to the present project: by Birkenmeier, Kim and Park [13] and Kanwar, Leroy and Matczuk [14] for relations between the idempotents of $A$, $A[X]$ and $A[[X]]$ where $X$ is a finite set of variables and by Isham and Monroe [15] for the properties of the idempotents in $\mathbb{Z}_n$ where $\mathbb{Z}_n \cong \mathbb{Z}/n\mathbb{Z}$ is the ring of residue classes modulo $n$.

It is a natural problem to describe explicitly the idempotents of the matrix rings $M_d(A)$ and $M_d(A[x])$, respectively, over a commutative ring $A$ and over the polynomial ring $A[x]$ if the idempotents of $A$ are known. Most of the explicit results in this direction are for the $2 \times 2$ matrix ring $M_2(\mathbb{Z}_n[x])$ for positive integers $n$ with a small number of prime factors. For $p$ prime Kanwar et al. [16] described the idempotents of $M_2(\mathbb{Z}_p[x])$, $M_2(\mathbb{Z}_{2p}[x])$ ($p$ odd) and $M_2(\mathbb{Z}_{3p}[x])$ ($p > 3$). Balmaceda and Datu [17] found the idempotents in $M_2(\mathbb{Z}_{pq}[x])$ and $M_2(\mathbb{Z}_{p^2}[x])$, where $p$ and $q$ are any primes. Finally, Mittal [18] described the idempotents in $M_2(\mathbb{Z}_{pqr}[x])$ for three pairwise different primes greater than 3. Mittal raised also the question to find the idempotents in $M_2(\mathbb{Z}_n[x])$ for any square-free positive integer $n$.

The main step in [16] is the description of the idempotents in $M_2(A[x])$ where $A$ is a commutative ring without non-trivial (i.e. different from 0 and 1) idempotents. Since this holds for $A = \mathbb{Z}_p$ for $p$ prime, the authors apply the Chinese remainder theorem and the Euler-Fermat theorem to handle the cases $A = \mathbb{Z}_{2p}, \mathbb{Z}_{3p}$. A similar approach was applied in [17] and [18].

In the present paper we simplify the ideas in [16–18] and give an explicit presentation of the idempotents of $M_2(A[[X]])$ where $A$ is a direct sum of a finite number of commutative rings without non-trivial idempotents and $A[[X]]$ is the ring of formal power series in an arbitrary (also infinite) set of commuting variables. As a consequence we describe the idempotents of $M_2(\mathbb{Z}_n[X])$ when $n$ is an arbitrary positive integer greater than 1. Our proofs are very transparent and use well-known elementary arguments only. They are based on the Cayley-Hamilton theorem (for $2 \times 2$ matrices only) and, as in [16–18], on the Chinese remainder theorem and the Euler-Fermat theorem.

2. The main result

We assume that $X$ is an arbitrary set of commuting variables and for a non-trivial commutative ring $A$ we consider the ring $A[[X]]$ of formal power series in $X$. In order to
make the exposition self-contained we include also the proofs of some well-known facts on idempotents.

The following lemma is a special case of [13, Proposition 2.5] and [14, Lemma 1].

**Lemma 2.1:** Let $A$ be a commutative ring without non-trivial idempotents. Then the ring $A[[X]]$ also has only trivial idempotents.

**Proof:** Let $a(X) = a_0 + a_1 + a_2 + \cdots \in A[[X]]$ be an idempotent, where $a_i$ is the homogeneous component of degree $i$ of $a(X)$. Let $a(X) \notin A$ and $a_1 = \cdots = a_{k-1} = 0$, $a_k \neq 0$. Since $a^2(X) = a(X)$, comparing the homogeneous components of $a(X)$ and $a^2(X)$ we obtain that $a_0^2 = a_0$ in $A$ and $2a_0a_k = a_k$. Since $A$ has trivial idempotents only, we have that either $a_0 = 1$ or $a_0 = 0$. Both cases are impossible: If $a_0 = 1$, then $2a_k = a_k$ and $a_k = 0$; if $a_0 = 0$, then again $a_k = 0$ which is a contradiction. Therefore the idempotent $a(X)$ belongs to $A$ and hence is trivial. \[\blacksquare\]

The following proposition is a special case of [16, Proposition 3.2] but in virtue of Lemma 2.1 both statements are equivalent. We give a simpler straightforward proof.

**Proposition 2.2:** Let $A$ be a commutative ring without non-trivial idempotents. Then all idempotents in $M_2(A)$ are

$$a = \begin{pmatrix} \alpha & \beta \\ \gamma & 1 - \alpha \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad 0_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\alpha, \beta, \gamma \in A$ and $\alpha(1 - \alpha) = \beta \gamma$.

**Proof:** Let

$$a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(A), \quad \alpha, \beta, \gamma, \delta \in A,$

be an idempotent. By the Cayley-Hamilton theorem we have

$$a^2 - \text{tr}(a)a + \text{det}(a)I_2 = 0_2.$$

Subtracting the equality $a^2 - a = 0_2$ we obtain that

$$(\text{tr}(a) - 1)a = \text{det}(a)I_2.$$

The determinant $\text{det}(a)$ of $a$ is an idempotent of $A$ because the equality $a^2 = a$ implies $\text{det}(a) = \text{det}(a^2) = \text{det}(a)$. Hence $\text{det}(a) = 1$ or $\text{det}(a) = 0$. First, let $\text{det}(a) = 1$. Comparing the entries of the matrices in the equality $(\text{tr}(a) - 1)a = I_2$ we obtain that

$$(\text{tr}(a) - 1)\alpha = (\text{tr}(a) - 1)\delta = 1, \quad (\text{tr}(a) - 1)\beta = (\text{tr}(a) - 1)\gamma = 0.$$ 

Hence $(\text{tr}(a) - 1)\alpha = (\text{tr}(a) - 1)\delta = 1$, $(\text{tr}(a) - 1)\beta = (\text{tr}(a) - 1)\gamma = 0$. 

Hence $(\text{tr}(a) - 1)$ is invertible in $A$. This implies that $\beta = \gamma = 0$ and $\alpha = \delta \neq 0$. Hence $\alpha I_2 = a = a^2 = a^2 I_2$ and $\alpha$ is an idempotent. Therefore $\alpha = 1$ and $a = I_2$. Now, let $\text{det}(a) = 0$. Hence $(\text{tr}(a) - 1)a = 0_2$ and

$$(\alpha + \delta - 1)\alpha = (\alpha + \delta - 1)\delta = 0,$$
\((-\alpha + \delta - 1)\)^2 = (\alpha + \delta - 1)\alpha + (\alpha + \delta - 1)\delta - (\alpha + \delta - 1) = -(\alpha + \delta - 1).

We obtain that \(- (\alpha + \delta - 1)\) is an idempotent and is equal to 1 or 0. The former case implies that \(-a = 0_2\) and \(a = 0_2\). In the latter case \(\delta = 1 - \alpha\) and \(a\) has the desired form (1). Since \(\det(a) = \alpha\delta - \beta\gamma = 0\) we obtain the restriction \(\alpha(1 - \alpha) = \beta\gamma\). A direct verification shows that all matrices of this form are idempotents.

**Corollary 2.3:** Let \(A_1, \ldots, A_s\) be commutative rings without non-trivial idempotents and \(A = A_1 \oplus \cdots \oplus A_s\) be their direct sum. Then all idempotents \(a(X) \in M_2(A[[X]])\) in the \(2 \times 2\) matrix ring with entries from \(A[[X]]\) are obtained by the following procedure. We split the set of indices \(\{1, \ldots, s\}\) in three parts

\[ P = \{p_1, \ldots, p_k\}, \quad Q = \{q_1, \ldots, q_l\}, \quad R = \{r_1, \ldots, r_m\} \]

and present \(A\) in the form \(A = A_P \oplus A_Q \oplus A_R\), where

\[ A_P = \bigoplus_{p \in P} A_p, \quad A_Q = \bigoplus_{q \in Q} A_q, \quad A_R = \bigoplus_{r \in R} A_r. \]

We choose power series \(\alpha(X), \beta(X), \gamma(X) \in A_p[[X]]\) such that \(\alpha(X)(1 - \alpha(X)) = \beta(X)\gamma(X)\). Then \(a(X) = (a_p(X), I_2, 0_2)\), where \(I_2 \in M_2(A_Q[[X]])\), \(0_2 \in M_2(A_R[[X]])\) and

\[ a_p(X) = \begin{pmatrix} \alpha(X) & \beta(X) \\ \gamma(X) & 1 - \alpha(X) \end{pmatrix} \in M_2(A_P[[X]]). \tag{2} \]

**Proof:** Since each \(A_i\), \(i = 1, \ldots, s\), does not have non-trivial idempotents, and by Lemma 2.1 the same holds for the rings of power series \(A_i[[X]]\). Applying Proposition 2.2 we obtain that the idempotents in \(M_2(A_i[[X]])\) are of the form

\[ a_i(X) = \begin{pmatrix} \alpha_i(X) & \beta_i(X) \\ \gamma_i(X) & 1 - \alpha_i(X) \end{pmatrix}, \quad \alpha_i(X)(1 - \alpha_i(X)) = \beta_i(X)\gamma_i(X), \tag{3} \]

where \(\alpha_i(X), \beta_i(X), \gamma_i(X) \in A_i[[X]]\), or \(a_i(X) = I_2\), or \(a_i(X) = 0_2\). We present the set \(\{1, \ldots, s\}\) as a disjoint union of three subsets \(P, Q\) and \(R\), where \(p \in P\) if \(a_p(X)\) is of the form (3), \(q \in Q\) if \(a_q(X) = I_2^{(q)}\) (the identity matrix in \(M_2(A_q[[X]])\)) and \(r \in R\) if \(a_r(t) = 0_2^{(r)}\) (the zero matrix in \(M_2(A_r[[X]])\)). Let

\[ a(X) = (a_{p_1}(X), \ldots, a_{p_k}(X)) \in A_{P_1}[[X]] \oplus \cdots \oplus A_{P_k}[[X]]. \]

Since \(A_P[[X]] = A_{P_1}[[X]] \oplus \cdots \oplus A_{P_k}[[X]]\), we obtain that \(a(X)\) has the form (2) and

\[ \alpha(X) = (\alpha_{p_1}(X), \ldots, \alpha_{p_k}(X)), \]
\[ \beta(X) = (\beta_{p_1}(X), \ldots, \beta_{p_k}(X)), \]
\[ \gamma(X) = (\gamma_{p_1}(X), \ldots, \gamma_{p_k}(X)) \]

satisfy the relation \(\alpha(X)(1 - \alpha(X)) = \beta(X)\gamma(X)\) because the coordinate power series \(\alpha_p(X), \beta_p(X), \gamma_p(X)\) satisfy \(\alpha_p(X)(1 - \alpha_p(X)) = \beta_p(X)\gamma_p(X)\) for all \(p = p_1, \ldots, p_k\). Obviously \((I_2^{(q_1)}, \ldots, I_2^{(q_l)})\) equals the identity matrix in \(M_2(A_Q[[X]])\) and \((0_2^{(r_1)}, \ldots, 0_2^{(r_m)})\) is the zero matrix in \(M_2(A_R[[X]])\) which completes the proof.
We shall apply Corollary 2.3 for $A = \mathbb{Z}_n$, $n > 1$. We need the following well-known fact. We include the proof for self-completeness of the exposition.

**Lemma 2.4:** Let $p$ be a prime and $d > 0$. Then all the idempotents of the ring $\mathbb{Z}_{pd}$ are trivial.

**Proof:** Let $\alpha \in \mathbb{Z}$ be such that its image $\overline{\alpha}$ in $\mathbb{Z}_{pd}$ is an idempotent. Hence $\alpha^2 - \alpha \equiv 0 \pmod{p^d}$ and $p^d$ divides $\alpha(\alpha - 1)$. Since $\alpha$ and $\alpha - 1$ are coprime, we have that either $p^d$ divides $\alpha$ and $\overline{\alpha} = 0$ in $\mathbb{Z}_{pd}$, or $p^d$ divides $\alpha - 1$ and $\overline{\alpha} = 1$ in $\mathbb{Z}_{pd}$. □

The following theorem was the main motivation to start the present project.

**Theorem 2.5:** Let $n > 1$ be a positive integer. Then all idempotents $a(X)$ in $M_2(\mathbb{Z}_n[[X]])$ are obtained in the following way. We present $n$ as a product $n = PQR$ of three pairwise coprime positive integers $P$, $Q$, $R$. If $P > 1$ we choose three power series $\alpha(X), \beta(X), \gamma(X) \in \mathbb{Z}[[X]]$ such that $\alpha(X)(1 - \alpha(X)) \equiv \beta(X)\gamma(X) \pmod{P}$. Then modulo $n$

$$a(X) \equiv \left(\frac{\overline{\alpha}(X)}{\gamma(X)} \quad \frac{\overline{\beta}(X)}{1 - \overline{\alpha}(X)}\right),$$

where:

(i) If $P, Q, R > 1$, then

$$\overline{\alpha}(X) \equiv (\alpha(X) + (1 - \alpha(X))P^{\varphi(Q)})(1 - (PQ)^{\varphi(R)}),$$

$$\overline{\beta}(X) \equiv \beta(X)(1 - P^{\varphi(Q)}^{\varphi(R)}), \quad \overline{\gamma}(X) \equiv \gamma(X)(1 - P^{\varphi(Q)}^{\varphi(R)});$$

(ii) If $P, Q > 1, R = 1$, then

$$\overline{\alpha}(X) \equiv \alpha(X) + (1 - \alpha(X))P^{\varphi(Q)}, \quad \overline{\beta}(X) \equiv \beta(X)(1 - P^{\varphi(Q)}),$$

$$\overline{\gamma}(X) \equiv \gamma(X)(1 - P^{\varphi(Q)});$$

(iii) If $P, R > 1, Q = 1$, then

$$\overline{\alpha}(X) \equiv \alpha(X)(1 - P^{\varphi(R)}), \quad \overline{\beta}(X) \equiv \beta(X)(1 - P^{\varphi(R)}), \quad \overline{\gamma}(X) \equiv \gamma(X)(1 - P^{\varphi(R)});$$

(iv) If $P = 1, Q, R > 1$, then $\overline{\alpha}(X) \equiv 1 - Q^{\varphi(R)}, \overline{\beta}(X) \equiv \overline{\gamma}(X) \equiv 0$;

(v) If $P > 1, Q = R = 1$, then $\overline{\alpha}(X) \equiv \alpha(X), \overline{\beta}(X) \equiv \beta(X), \overline{\gamma}(X) \equiv \gamma(X)$;

(vi) If $P = R = 1, Q > 1$, then $a(X) \equiv I_2$;

(vii) If $P = Q = 1, R > 1$, then $a(X) \equiv 0_2$,

and $\varphi$ is the Euler totient function.

**Proof:** As in [16, p. 151], if $n = \prod p^d$, where $p$ are the prime divisors of $n$, we present the ring $\mathbb{Z}_n$ as the direct sum of the rings $\mathbb{Z}_{pd}$. If $a(X) \in M_2(\mathbb{Z}_n[[X]])$ is an idempotent, by Lemma 2.4 we can apply Corollary 2.3. We divide the prime divisors of $n$ in three groups $\{p_1, \ldots, p_k\}, \{q_1, \ldots, q_l\}$ and $\{r_1, \ldots, r_m\}$ depending on the form of the projection of $a(X)$ in $M_2(\mathbb{Z}_{pd}[[X]])$: $p \in \{p_1, \ldots, p_k\}$ if $a(X)$ is of the form (3) and $p \in$
\{q_1, \ldots, q_l\} or \( p \in \{r_1, \ldots, r_m\} \) if the projection is, respectively, the identity matrix and the zero matrix. Let \( P = p_1^{d_1} \cdots p_k^{d_k}, Q = q_1^{e_1} \cdots q_l^{e_l}, R = r_1^{f_1} \cdots r_m^{f_m} \). The image of \( a(X) \) in \( M_2(\mathbb{Z}_P[[X]]) \cong M_2(\mathbb{Z}_{p_1^{d_1}}[[X]]) \oplus \cdots \oplus M_2(\mathbb{Z}_{p_k^{d_k}}[[X]]) \) is of the form (3). If we choose the images of \( \alpha(X), \beta(X), \gamma(X) \) modulo \( p_i^{d_i} \), we can find their images modulo \( P \) using the Chinese remainder theorem. The images of \( a(X) \) in \( M_2(\mathbb{Z}_Q[[X]]) \cong M_2(\mathbb{Z}_{q_1^{e_1}}[[X]]) \oplus \cdots \oplus M_2(\mathbb{Z}_{q_l^{e_l}}[[X]]) \) and \( M_2(\mathbb{Z}_R[[X]]) \cong M_2(\mathbb{Z}_{r_1^{f_1}}[[X]]) \oplus \cdots \oplus M_2(\mathbb{Z}_{r_m^{f_m}}[[X]]) \) are, respectively, equal to the identity matrix and the zero matrix. Since \( P, Q \) and \( R \) are pairwise coprime, it is sufficient to check whether the forms of \( a(X) \) given in the cases (i) – (vii) in the theorem satisfy the required conditions modulo \( P, Q \) and \( R \). We shall check this for \( \overline{a}(X) \) in the case (i) only. The other cases are handled similarly. Since \( \varphi(Q), \varphi(R) \geq 1 \), obviously

\[
\overline{a}(X) \equiv (\alpha(X) + (1 - \alpha(X))P^{\varphi(Q)})(1 - (PQ)^{\varphi(R)} \equiv \alpha(X) \pmod{P}. 
\]

By the Euler-Fermat theorem \( P^{\varphi(Q)} \equiv 1 \pmod{Q} \). Hence

\[
\overline{a}(X) \equiv (\alpha(X) + (1 - \alpha(X)) \equiv 1 \pmod{Q},
\]

and in a similar way we establish that \( \overline{a}(X) \equiv 0 \pmod{R} \). 

\section*{Acknowledgments}

The author is very grateful to Claudio Procesi for the useful remarks and to the anonymous referee for the careful reading and the valuable suggestions for improving the exposition.

\section*{Disclosure statement}

No potential conflict of interest was reported by the author(s).

\section*{Funding}

The present project was partially supported by Grant KP-06 N 32/1 of 07.12.2019 ‘Groups and Rings – Theory and Applications’ of the Bulgarian National Science Fund.

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