On Path-Pairability of Cartesian Product of Complete Bipartite Graphs

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March 18, 2014

Abstract

We study path-pairability of Cartesian product of graphs and prove that the Cartesian product of the complete bipartite graph $K_{m,m}$ with itself is path-pairable. The result improves the known bound on the minimal value of the maximum degree of a path-pairable graph. Further results about path-pairability of graph products are presented.

Introduction

In this paper we discuss graph theoretic concepts emerging from a practical networking problem introduced by Csaba, Faudree, Gyárfás and Lehel in [4], [6] and [7]. A graph $G$ on at least $2k$ vertices is called $k$-path-pairable if for any pair of disjoint sets of (pairwise different) vertices $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$ of $G$ there exist $k$ edge-disjoint $x_i - y_i$ paths joining the vertices. The path-parability number of a graph $G$ is the largest positive integer $k$ for which $G$ is $k$-path-pairable and it is denoted by $pp(G)$. A graph on exactly $2k$ vertices is called path-pairable if it is $k$-path-pairable. The motivation of setting edge-disjoint paths between certain pairs of nodes naturally arose in the study of communication networks. There are various reasons to measure the capability of the network by its path-pairability number, that is, the maximum number of pairs of users for which the network can provide separated communication channels without data collision. The initial problem and its graph theoretical model is discussed in [4].

Path-pairability is closely related to several other concepts such as linkedness and weak-linkedness. A graph $G$ is $k$-linked/weakly $k$-linked if for every ordered set of $2k$ vertices $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$ there exist vertex-disjoint/edge-disjoint paths $P_1, \ldots, P_k$ such that each $P_i$ is an $s_i, t_i$-path. We wish to highlight that in case of weak-linkedness repetition of the vertices is allowed while it is forbidden in case of path-pairability. Note that in case of linkedness, the two conventions lead to the same concept. By definition, weakly $k$-linked graphs are $k$-path-pairable.

Though path-pairability is considered as a special variant of weak-linkedness, the two concepts differ in several respects. One particular difference is their relation to connectivity. Weakly $k$-linked graphs are necessarily $k$-edge-connected. While relation to connectivity and edge-connectivity has been the main research area in the study of linkedness and weak linkedness, path-pairability turned out to be unrelated to these concepts. There exist $k$-path-
pairable graphs that are not even 2-edge connected. One illustrative example is the star graph $K_{1,n}$ which is $k$-path pairable for $k \leq \left\lfloor \frac{n}{2} \right\rfloor$.

A central problem in the study of path-pairability has been the necessary minimal conditions concerning the maximum degree $\Delta(G)$. While for arbitrary fixed values of $k$ there exist 3-regular $k$-path-pairable graphs [5], it was proven by Faudree [6] that a path-pairable graph with maximum degree $\Delta$ has at most $2\Delta^2$ vertices. In other words, Faudree’s result placed a lower bound of $O\left(\frac{\log n}{\log \log n}\right)$ on the maximum degree of a path-pairable graph on $n$ vertices. This bound is assumed to be asymptotically sharp, though path-pairable graphs with maximum degree of the right order of magnitude have yet to be explored. The most interesting and promising candidate is the $d$-dimensional hypercube $Q_d$ on $n = 2^d$ vertices with $\Delta(Q_d) = d = \log n$. Although it is known that $Q_n$ is not path-pairable for even values of $d$ ([2]), the question is open for odd dimensional hypercubes if $d \geq 5$ ($Q_1$ and $Q_3$ are both path-pairable).

**Conjecture 1** ([4]). The $(2k+1)$-dimensional hypercube $Q_{2k+1}$ is path-pairable for all $k \in \mathbb{N}$.

The best known constructions have maximum degree of $O(\sqrt{n})$ and are obtained by taking Cartesian Product of complete graphs. The *Cartesian product* of graphs $G$ and $H$ is the graph $G \square H$ with vertices $V(G \square H) = V(G) \times V(H)$, and $(x,u)(y,v)$ is an edge if $x = y$ and $uv \in E(H)$ or $xy \in E(G)$ and $u = v$. The Cartesian product of graphs has been extensively studied in the past decades. It gave rise to important classes of graphs; for example, the $n$-dimensional grid can be considered as Cartesian product of lower dimensional grids. Hypercubes are well known members of this family with similar recursive structures: the Cartesian product of $m$-dimensional and $n$-dimensional hypercubes is an $(m+n)$-dimensional one. The study of graph products leads to various deep structural problems such as invariance and inheritance of graph parameters: connections between several parameters of products and their factors have been investigated. We mention a couple of relevant results with no claim of being exhaustive. Chiue and Shieh [1] proved that Cartesian product of a $k$-connected and an $l$-connected graph is $(k+l)$-connected. Similar result for edge connectivity was proved by Xu and Yang [11]. Inheritance of linkedness has been investigated by Mészáros [10] who proved that the Cartesian product of an $a$-linked graph $G$ and a $b$-linked graph $H$ is $(a+b-1)$-linked, given that the graphs are sufficiently large in terms of $a$ and $b$. We mention that the technique presented in [10] proves similar theorems for weak-linkedness and parametrized path-parability, that is, the Cartesian product of an $a$-path-pairable graph $G$ and a $b$-path-pairable graph $H$ is $(a+b-1)$-path-pairable. While the provided lower bounds on linkedness and weak-linkedness of the product graphs are known to be sharp for certain graphs, in case the path-pairability, the exact connection has yet to be explored. In fact, little is known about inheritance of path-pairability in product graphs. Kubicka, Kubicki and Lehel [9] investigated path-pairability of complete grid graphs, that is, the Cartesian product of complete graphs, and proved that the two-dimensional complete grid $K_a \times K_b$ of size $n = ab$ is path-pairable. For $a$ equals $b$ that gives examples of path-pairable graphs with maximum degree $\Delta = 2a - 2 \approx 2\sqrt{n}$.

In this paper our main objective is to provide additional examples of path-pairable graphs arisen as Cartesian products. We prove that the Cartesian product of the complete bipartite graph $K_{m,m}$ with itself is path-pairable. Our result improves the upper bound on $\Delta(G)$ to $\sqrt{n}$. 

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Theorem 1. The product graph $K_{m,m} \square K_{m,m}$ is path-pairable for even values of $m$ if $m \geq 104$.

We follow the notation of [8]. For the sake of completeness we recall definitions of the mainly used concepts. A $G$-layer $G_x (x \in V(H))$ of a Cartesian product $G \square H$ is the subgraph induced by the set of vertices $\{(u,x) : u \in V(G)\}$. An $H$-layer is defined analogously. $G$ and $H$ layers are often referred to as rows and columns, respectively. We call edges of $G \square H$ lying in $G$-layers vertical while edges lying in $H$-layers are called horizontal. Unless it is misleading we also use the notation $G_z = G_x$ and $H_z = H_y$ for layers corresponding to $z = (x,y) \in G \square H$.

We also refer the reader to [8] for further details on product graphs. For a comprehensive survey of results concerning path-pairability, we refer to [3] and [7].

Proof of Theorem 1

Let us denote the two classes of the bipartite graph $K_{m,m}$ by $A_1$ and $A_2$. We introduce further notation for certain sets of the vertices in the product graph $G = K_{m,m} \square K_{m,m}$ as follows: $A_{11} = A_1 \square A_1$, $A_{12} = A_1 \square A_2$, $A_{21} = A_2 \square A_1$ and $A_{22} = A_2 \square A_2$. We will refer to these sets as classes of $G$. We also set a cyclic order of the four classes clockwise. References to next class and previous class are translated in accordance with that given cyclic order. We label the $m^2$ elements of each class by an $(u,v)$ pair where $u = 1, \ldots , m$ and $v = 1, \ldots , m$.

Given a pairing of the vertices, we carry out the linking in three phases named: swarming, line-up and final match. For a pair of terminals of $G$ we first ship them to the same class (swarming), then send them forward to the same row/column of the next class (line-up). Finally, we join the to paths by their newly established ends with a single vertex of the next class (final match). Note that during the phases terminals of different pairs might temporarily share vertices but will eventually get sorted to their partners at the end of the final match phase.

Swarming

In this phase we ship one terminal of each pair to the class of its partner. If a pair lies with both vertices within a class, they simply skip the swarming phase. A terminal $(u,v)$, belonging to class $A_{11}$ and heading to $A_{12}$, shall follow the path $(u,v) \rightarrow (u+1,v)$, where $(u+1,v)$ denotes the appropriate vertex of $A_{12}$ and addition is calculated modulo $m$. Similarly, we send $u,v$ to $A_{21}$ via the path $(u,v) \rightarrow (u,v+1)$. Should $(u,v)$ travel to $A_{22}$, we allocate it the path $(u,v) \rightarrow (u+1,v) \rightarrow (u+1,v+2)$ where $(u+1,v)$ belongs to $A_{12}$ and $(u+1,v+2)$ belongs to $A_{22}$. Terminals belonging to other classes will be shipped by the same rules, increasing the appropriate coordinate by 1 at the first step and increasing the other one by 2 in the second step, if applicable. Travelling via paths of length two is always carried out clockwise.

One can easily verify that the above arrangement of paths assures that if $m \geq 5$, no edge is being utilized twice during the swarming phase. We now choose the travelling terminal for each pair such that at the end of the swarming phase every class hosts exactly $\frac{m^2}{2}$ pairs. Starting with an arbitrary election, we can assume without loss of generality that $A_{11}$ hosts most pairs and that at least one terminal $x \in A_{11}$ received its pair $y$ from a class hosting less than $\frac{m^2}{2}$ pairs. Sending $x$ to the class of $y$, instead, balances the distribution of the pairs. Repetition of the previous step leads to an equal distribution.
We define $G'$ with $V(G') = V(G)$ with a new edge set $E(G')$ by deleting those edges from $E(G)$ we used in the swarming phase. Observe that by the given shipping method, every vertex of $G$ hosts at most 5 terminals and uses at most 8 of its edges, that is, the minimal degree of $G'$ is at least $m - 8$. We continue the linking in $G'$.

**Line-up**

We ship each pair of terminals to the next class such that terminals shipped by a horizontal edge shall share the same column of the new class while vertically shipped terminals will arrive in the same row. For every pair there are at least $m - 16$ available columns/rows in the next class. Our intention is to pair up the pairs with the rows/columns such that every one of them will contain $\frac{m}{2}$ pairs. We recall a straightforward corollary of Hall’s theorem.

**Lemma 1.** A bipartite graph $G = (A, B, E)$ with vertex classes of size $n$ whose minimum degree is at least $\frac{n}{2}$ contains a perfect matching.

We define the following bipartite graph $G = (A, B, E)$ as follows: represent each pair of terminals hosted in $A_{11}$ by a vertex in $A$ while each column of $A_{12}$ is represented by $\frac{m^2}{2}$ independent vertices in $B$. Certainly, $|A| = |B| = \frac{m^2}{2}$. We connect two vertices of $A$ and $B$ by an edge if both terminals of the corresponding pair have horizontal edges to the corresponding column of $A_{12}$. Easy to see that the graph has minimum degree at least $\frac{m^2}{2} - 16m$, hence, by Lemma 1, it contains a perfect matching for $n \geq 64$.

Observe, that if two pairs of terminals sharing a vertex of a class $C$ are distributed to the same vertical layer of the next class $C'$, at least one of the terminals will no be able to land there. We need to guarantee a matching between the pairs and the layers of $C'$ without such a collision. Recall that each vertex of $C$ hosts at most 5 terminals, hence each pair of terminals has at most 8 additional pairs to collide with. Consider a perfect matching for which the number of above collisions is minimal. Let $(x, y)$ and $(x', y')$ colliding pairs of terminals being sent to layer $L$ of $C'$. We may assume $x$ and $x'$ share the same vertex of $C$. We want to find a pair $(u, v)$ sent to a layer $L' \neq L$ of $E$ such that

i) $(x, y)$ can be sent from $C$ to $L'$ (instead of $L$) during the line-up without causing further collision,

ii) $(u, v)$ can be sent from $C$ to $L$ (instead of $L'$) during the line-up without causing further collision.

The pair $(x, y)$ can be initially sent to $m - 16$ layers of $C'$, at most 8 of which might contains terminals that initially shared vertex with $(x, y)$ in $C$. In order to avoid further collisions we exclude these layers, leaving us at least $m - 24$ choices of $L'$. We also want to exclude layers that already received terminals from the vertex of $x$ or $y$, yielding at most 8 additional excluded layers, that is, at least $m - 32$ choices of $L'$ and so $(m - 32) \cdot \frac{m}{2}$ choices for $(u, v)$. We want to choose $(u, v)$ such that it initially did not share vertex in $C$ with any terminal currently stationed in $L$ and that $u$ and $v$ still can be moved (having withdrawn from $L'$) from $C$ to $L$ (the corresponding edges have not been used yet). For the first constraint, recall that $L$ contains $\frac{m}{2}$ pairs, every one of which shares vertex with at most 8 additional terminals. There
are at most $4m$ additional terminals that initially cannot be sent to $L$ because the appropriate edges had already been used during the first phase.

Now assume that the appropriate edge that would channel $u$ or $v$ to $L$ has already been used. It can either occur if another terminal was sent from that particular vertex of $C$ to $L$ during the line-up or if the edges was used during the swarming phase. The first conditions means that $(u, v)$ collides with the other pair of terminals that was sent to $L$, hence $(u, v)$ is one of the above listed $4m$ pairs. In the remaining case the missing edge is one of those at most $8 \cdot \frac{n}{2} = 4m$ edges the complete layer $L$ used up during the swarming. The mentioned edges have at most $4m$ endpoints in $C$ and at most $5 \cdot 4m = 20m$ pairs of terminals corresponding to them.

Overall, it means the if $(m - 32) \cdot \frac{m}{2} > 24m$ (that is, $m > 56$), one can find an appropriate $(u, v)$. Swapping the positions of $(u, v)$ and $(x, y)$ we reduced the number of collisions, contradicting our assumption.

We repeat the same procedure for the remaining three classes. It can be easily verified that no edge is used more than once. We define $G''$ by the deletion of the used edges the same way we obtained $G'$. We proceed in $G''$ to the final match.

**Final match**

For a row/column filled with $\frac{m^2}{2}$ pairs of terminals we assign every pair a vertex of the appropriate row/column of the next class being adjacent to both terminals (see Figure). Note that during the first two phases each vertex has used at most 13 of its edges. We use Lemma 1 to find the appropriate assignment. Let $A$ form the set in which every pair of terminals of a certain row/column is represented by a vertex. The set $B$ is formed by any $\frac{m^2}{2}$ vertices of the appropriate column/row of the next class. We connect vertices by edges if both terminals of the pair are adjacent to the appropriate vertex in the next class. Our bipartite graph has two classes of size $\frac{m^2}{2}$ and minimum degree $\frac{m^2}{2} - 26$. If $m \geq 104$, the required matching is provided by Lemma 1. That completes the proof.

![Figure 1: Line-up and final match phases.](image)

**Corollary 1.** There exists a path pairable graph $G$ on $n$ vertices with $\Delta(G) = \sqrt{n}$ for infinitely many values of $n$. 
Note that with a detailed analysis of our presented technique it can be proved that the product graph $K_{a,b} \square K_{c,d}$ is path-pairable if $\frac{\max(a,b,c,d)}{\min(a,b,c,d)} < 2$ and $a, b, c, d$ are large enough (in terms of the previous ratio). Path-pairability of $K_{a,b} \square K_{c,d}$ in the general case is still subject to further investigation.

**Question 1.** For which values of $a, b, c, d \in \mathbb{Z}^+ (a \leq b, c \leq d)$ is the product graph $K_{a,b} \square K_{c,d}$ path-pairable?

We mention that the product of two star graphs is not only non-path-pairable but in fact it can be proved that $pp(K_{1,b} \square K_{1,d}) \leq \lceil \frac{b+d}{2} \rceil$. It indicates that, while the result of Kubicka, Kubicki and Lehel may suggest a certain multiplicative inheritance of path-pairability, such correspondence does not hold in general.

**Proposition 1.** The Cartesian product $K_{1,b} \square K_{1,d}$ is at most $\lceil \frac{b+d}{2} \rceil$-path-pairable.

**Proof.** Let $C$ and $R$ denote the sets of vertices of degree two in an arbitrary column and an arbitrary row not containing the unique vertex of degree $(a + b)$ (denoted by $z_{a+b}$) and let $x$ be an additional vertex of degree two. We denote the unique vertex of the intersection $C \cap R$ by $y$. We place terminals in $C \cup R \cup \{x\}$ such that $x$ and $y$ form a pair, as well as the unique vertices of degree $(a + 1)$ and $(b + 1)$ (denoted by $z_{a+1}$ and $z_{b+1}$) form another. Observe that paths that join the above two pairs both use either the edge between $z_{a+1}$ and $z_{a+b}$ or between $z_{b+1}$ and $z_{a+b}$, hence the pairing cannot be achieved.

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