A note on perfect simulation for exponential random graph models

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Abstract

In this paper we propose a perfect simulation algorithm for the Exponential Random Graph Model, based on the Coupling From The Past method of Propp & Wilson [1990]. We use a Glauber dynamics to construct the Markov Chain and we prove the monotonicity of the ERGM for a subset of the parametric space. We also obtain an upper bound on the running time of the algorithm that depends on the mixing time of the Markov chain.

Keywords: exponential random graph, perfect simulation, coupling from the past, MCMC, Glauber dynamics

1 Introduction

In recent years there has been an increasing interest in the study of probabilistic models defined on graphs. One of the main reasons for this interest is the flexibility of these models, making them suitable for the description of real datasets, for instance in social networks [Newman et al., 2002] and neural networks [Cerqueira et al., 2017]. The interactions in a real network can be represented through a graph, where the interacting objects are represented by the vertices of the graph and the interactions between these objects are identified with the edges of the graph.

In this context, the Exponential Random Graph Model (ERGM) has been vastly explored to model real networks [Robins et al., 2007]. Inspired on the Gibbs distributions of Statistical Physics, one of the reasons of the popularity of ERGM can be attributed to the specific definition of the probability distribution that takes into account appealing properties of the graphs such as number of edges, triangles, stars, etc.
In terms of statistical inference, learning the parameters of an ERGM is not a simple task. Classical inferential methods such as Monte Carlo maximum likelihood (Geyer & Thompson, 1992) or pseudolikelihood (Strauss & Ikeda, 1990) have been proposed, but exact calculation of the estimators is computationally infeasible except for small graphs. In the general case, estimation is made by means of a Markov Chain Monte Carlo (MCMC) procedure (Snijders, 2002). It consists in building a Markov chain whose stationary distribution is the target law. The classical approach is to simulate a “long enough” random path, so as to get an approximate sample. But this procedure suffers from a “burn-in” time that is difficult to estimate in practice and can be exponentially long, as shown in Bhamidi et al. (2011). To overcome this difficulty, Butts (2015) proposed a novel algorithm called bound sampler. The simulation time of the bound sampler is fixed and depends only on the number of vertices of the graph. Despite being approximate methods for sampling from the ERGM, the bound sampler is recommended when the mixing time of the chain is long (Butts, 2015).

In order to get a sample of the stationary distribution of the chain, an alternative approach introduced by Propp & Wilson (1996) is called Coupling From The Past (CFTP). This method yields a value distributed under the exact target distribution, and is thus called perfect (also known as exact) simulation. Like MCMC, it does not require computing explicitly the normalising constant; but it does not require any burn-in phase either, nor any detection of convergence to the target distribution.

In this work we present a perfect simulation algorithm for the ERGM. In our approach, we adapt a version of the CFTP algorithm, which was originally developed for spin systems (Propp & Wilson, 1996) to construct a perfect simulation algorithm using a Markov chain based on the Glauber dynamics. For a specific family of distributions based on simple statistics we prove that they satisfy the monotonicity property and then the CFTP algorithm is computationally efficient to generate a sample from the target distribution. Finally, and most importantly, we prove that the mean running time of the perfect simulation algorithm is at most as large (up to logarithmic terms) than the mixing time of the Markov chain. As a consequence, we argue CFTP is superior to MCMC for the simulation of the ERGM.

2 Definitions

2.1 Exponential random graph model

A graph is a pair \( g = (V,E) \), where \( V = \{1, 2, \ldots, N\} \) is a finite set of vertices and \( E \subseteq V \times V \) is a set of edges that connect pairs of vertices. In this work we consider undirected graphs; i.e, graphs for which the set \( E \) satisfies the following properties:
1. For all \( i \in V \), \((i, i) \notin E \)

2. If \((i, j) \in E \), then \((j, i) \in E \).

Let \( G_N \) be the collection of all undirected graphs with set of vertices \( V \). A graph \((V, E) \in G_N \) can be identified with a symmetric binary matrix \( x \in \mathcal{M}_N(\{0, 1\}) \) such that \( x(i, j) = 1 \) if and only if \((i, j) \in E \).

We will thus denote by \( x \) a graph, which is completely determined by the values \( x(i, j) \), for \( 1 \leq i < j \leq N \). The empty graph \( x(0) \) is given by \( x(0)(i, j) = 0 \), for \( 1 \leq i < j \leq N \), and the the complete graph \( x(1) \) is given by \( x(1)(i, j) = 1 \), for \( 1 \leq i < j \leq N \).

In the ERGM model, the probability of selecting a graph in \( G_N \) is defined by its structure: its number of edges, of triangles, etc.

To define subgraphs counts in a graph, let \( V_m \) be the set of all possible permutations of \( m \) distinct elements of \( V \). For \( v_m \in V_m \), define \( x(\mathcal{V}_m) \) as the subgraph of \( x \) induced by \( v_m \). For a graph \( g \) with \( m \) vertices, we say \( x(\mathcal{V}_m) \) contains \( g \) (and we write \( x(\mathcal{V}_m) \succeq g \)) if \( g(i, j) = 1 \) implies \( x(\mathcal{V}_m)(i, j) = 1 \). Then, for \( x \in G_N \) and \( g \in G_m \), \( m \leq N \), we define the number of subgraphs \( g \) in \( x \) by the counter

\[
N_g(x) = \sum_{v_m \in V_m} \mathbf{1}\{x(\mathcal{V}_m) \succeq g\}.
\]

(1)

Let \( g_1, \ldots, g_s \) be fixed graphs, where \( g_i \) has \( m_i \) vertices, \( m_i \leq N \) and \( \beta \in \mathbb{R}^s \) a vector of parameters. By convention we set \( g_1 \) as the graph with only two vertices and one edge. Following [Chatterjee et al. (2013)] and [Bhamidi et al. (2011)] we define the probability of graph \( x \) by

\[
p_N(x|\beta) = \frac{1}{Z_N(\beta)} \exp\left(\sum_{i=1}^{s} \beta_i \frac{N_{g_i}(x)}{N^{m_i-2}}\right).
\]

(2)

Observe that the set \( G_N \) is equipped with a partial order given by

\[
x \preceq y \quad \text{if and only if} \quad x(i, j) = 1 \quad \text{implies} \quad y(i, j) = 1.
\]

(3)

Moreover, this partial ordering has a maximal element \( x^{(1)} \) (the complete graph) and a minimal element \( x^{(0)} \) (the empty graph). Considering this partial order, we say \( p_N(\cdot|\beta) \) is monotone if the conditional distribution of \( x(i, j) = 1 \) given \( x_{-ij} \) is a monotone increasing function. That is, \( p_N(\cdot|\beta) \) is monotone if, and only if \( x \preceq y \) implies

\[
p_N(x(i, j) = 1|\beta, x_{-ij}) \leq p_N(y(i, j) = 1|\beta, y_{-ij}).
\]

(4)
Let $E = \{(i, j) : i, j \in V \text{ and } i < j\}$ be the set of possible edges in a graph with set of vertices $V$. For any graph $x \in G_N$, a pair of vertices $(i, j) \in E$ and $a \in \{0, 1\}$ we define the modified graph $x^a_{ij} \in G_N$ given by

$$x^a_{ij}(k, l) = \begin{cases} x(k, l), & \text{if } (k, l) \neq (i, j); \\ a, & \text{if } (k, l) = (i, j). \end{cases}$$

Then, Inequality (4) is equivalent to

$$\frac{p_N(x^1_{ij} | \beta)}{p_N(x^0_{ij} | \beta)} \leq \frac{p_N(y^1_{ij} | \beta)}{p_N(y^0_{ij} | \beta)}. \quad (5)$$

Moreover, if this inequality holds for all $x \preceq y$ then the distribution is monotone.

Monotonicity plays a very important role in the development of efficient perfect simulation algorithms. The following proposition gives a sufficient condition on the parameter vector $\beta$ under which the corresponding ERGM distribution is monotone.

**Proposition 1.** Consider the ERGM given by (2). If $\beta_i \geq 0$ for all $i \geq 2$ then the distribution $p_N(\cdot ; \beta)$ is monotone.

The short proof of Proposition 1 is given in Section 6.

### 3 Coupling From The Past

In this section we recall the construction of a Markov chain with stationary distribution $p_N(\cdot ; \beta)$ using a local update algorithm called Glauber dynamics. This construction and the monotonicity property given by Proposition 1 are the keys to develop a perfect simulation algorithms for the ERGM.

#### 3.1 Glauber dynamics

In the Glauber dynamics the probability of a transition from graph $x$ to graph $x^a_{ij}$ is given by

$$p(x_{ij} = a | \beta, x_{-ij}) = \frac{p_N(x^a_{ij} | \beta)}{p_N(x^0_{ij} | \beta) + p_N(x^1_{ij} | \beta)} \quad (6)$$

and any other (non-local) transition is forbidden.

For the ERGM, this can be written

$$p(x_{ij} = 1 | \beta, x_{-ij}) = \frac{1}{1 + \exp \left\{-2\beta_1 + \sum_{k=2}^{s} \frac{\beta_k}{N_{mk-2}} \left(N_{g_k}(x^0_{ij}) - N_{g_k}(x^1_{ij}) \right) \right\}} \quad (7)$$

and

$$p(x_{ij} = 0 | \beta, x_{-ij}) = 1 - p(x_{ij} = 1 | \beta, x_{-ij}).$$
It can be seen easily that the Markov chain with the Glauber dynamics has an invariable distribution equal to \( p_N(\cdot; \beta) \). It can be implemented with the help of the local update function \( \phi: \mathcal{G}_N \times \mathbf{E} \times [0, 1] \to \mathcal{G}_N \) defined as

\[
\phi(\mathbf{x}, (i, j), u) = \begin{cases} 
\mathbf{x}_{ij}^0, & \text{if } u \leq p(\mathbf{x}_{ij} = 0 | \beta; \mathbf{x}_{-ij}); \\
\mathbf{x}_{ij}^1, & \text{otherwise.}
\end{cases}
\]

A transition in the Markov chain from graph \( \mathbf{x} \) to graph \( \mathbf{y} \) is obtained by the random function

\[
\mathbf{y} = \phi(\mathbf{x}, \mathbf{e}, \mathbf{u}), \quad \text{with } \mathbf{e} \sim \text{Uniform}(\mathbf{E}), \ \mathbf{u} \sim \text{Uniform}(0, 1).
\]

For random vectors \( \mathbf{e}_{-n}^{-1} = (e_{-n}, e_{-n+1}, \ldots, e_{-1}) \) and \( \mathbf{u}_{-n}^{-1} = (u_{-n}, u_{-n+1}, \ldots, u_{-1}) \), with \( e_{-k} \in \mathbf{E} \) and \( u_{-k} \in [0, 1] \), we define the random map \( F_{-n}^0 \) given by the following induction:

\[
F_{-n}^0(\mathbf{x}, \mathbf{e}_{-n}^{-1}, \mathbf{u}_{-n}^{-1}) = \phi(\mathbf{x}, e_{-1}, u_{-1}), \quad \text{and}
\]

\[
F_{-n}^0(\mathbf{x}, \mathbf{e}_{-n}^{-1}, \mathbf{u}_{-n}^{-1}) = F_{-n+1}^0(F_{-n}^0(\mathbf{x}, \mathbf{e}_{-n}^{-1}, \mathbf{u}_{-n}^{-1}), e_{-n+1}, u_{-n+1}), \quad \text{for } n \geq 2.
\]

The CFTP protocol of [Propp & Wilson 1996] relies on the following elementary observation. If, for some time \( -n \) and for some uniformly distributed \( \mathbf{e}_{-n}^{-1} = (e_{-n}, e_{-n+1}, \ldots, e_{-1}) \) and \( \mathbf{u}_{-n}^{-1} = (u_{-n}, u_{-n+1}, \ldots, u_{-1}) \), the mapping \( F_{-n}^0(\cdot, \mathbf{e}_{-n}^{-1}, \mathbf{u}_{-n}^{-1}) \) is constant, that is if it takes the same value at time 0 on all possible graphs \( \mathbf{x} \), then this constant value is readily seen to be a sample from the invariant measure of the Markov chain. Moreover, if \( F_{-n} \) is constant for some positive \( n \), then it is also constant for all larger values of \( n \). Thus, it suffices to find a value of \( n \) large enough so as to obtain a constant map, and to return the constant value of this map. For example, one may try some arbitrary value \( n_1 \), check if the map \( F_{-n_1}^0(\cdot, \mathbf{e}_{-n_1}^{-1}, \mathbf{u}_{-n_1}^{-1}) \) is constant, try \( n_2 = 2n_1 \) otherwise, and so on...

Since the state space has the huge size \( |\mathcal{G}_N| = 2^{\frac{N(N-1)}{2}} \), it would be computationally intractable to compute the value of \( F_{-n}^0(\mathbf{x}, \mathbf{e}_{-n}^{-1}, \mathbf{u}_{-n}^{-1}) \) for all \( \mathbf{x} \in \mathcal{G}_N \) in order to determine if they coincide.

This is where the monotonicity property helps: it is sufficient to inspect the maximal and minimal elements of \( \mathcal{G}_N \). If they both yield the same value, then all other initial states will also coincide with them, and the mapping will be constant.

To apply the monotone CFTP algorithm, we use the partial order defined by \( \preceq \) and the extremal elements \( \mathbf{x}^{(0)} \) and \( \mathbf{x}^{(1)} \). The resulting procedure, which simulates the ERGM using the CFTP protocol, is described in Algorithm 1.

Let

\[
T_{N}^{\text{stop}} = \min\{n > 0 : F_{-n}^0(\mathbf{x}^{(0)}, \mathbf{e}_{-n}^{-1}, \mathbf{u}_{-n}^{-1}) = F_{-n}^0(\mathbf{x}^{(1)}, \mathbf{e}_{-n}^{-1}, \mathbf{u}_{-n}^{-1})\}
\]

be the stopping time of Algorithm 1. Proposition 2 below guarantees that the law of the output of Algorithm 1 is the target distribution \( p_N(\cdot; \beta) \). The proof of Proposition 2 follows directly from Theorem 2 in [Propp & Wilson 1996] and is omitted.
Algorithm 1 CFTP for ERGM

1: Input: $E$, $\phi$
2: Output: $x$
3: $\text{upper}(i, j) \leftarrow 1$, for all $(i, j) \in E$
4: $\text{lower}(i, j) \leftarrow 0$, for all $(i, j) \in E$
5: $n \leftarrow 0$
6: while $\text{upper} \neq \text{lower}$ do
7: $n \leftarrow n + 1$
8: $\text{upper}(i, j) \leftarrow 1$, for all $(i, j) \in E$
9: $\text{lower}(i, j) \leftarrow 0$, for all $(i, j) \in E$
10: Choose a pair of vertices $e_{-n}$ randomly on $E$
11: Simulate $u_{-n}$ with distribution uniform on $[0, 1]$
12: $\text{upper} \leftarrow F_0^{n} \cdot \text{upper}, e_{-1}^{n}, u_{-1}^{n}$
13: $\text{lower} \leftarrow F_0^{n} \cdot \text{lower}, e_{-1}^{n}, u_{-1}^{n}$
14: end while
15: $x \leftarrow \text{upper}$
16: Return: $n$, $x$

Proposition 2. Suppose that $\mathbb{P}(T^{\text{stop}}_N < \infty) = 1$. Then the graph $x$ returned by Algorithm 1 has law $p_N(\cdot; \beta)$.

4 Convergence Speed: Why CFTP is Better

By the construction of the perfect simulation algorithm it is expected that the stopping time $T^{\text{stop}}_N$ is related to the mixing time of the chain. In Theorem below we provide an upper bound on this quantity.

Given vectors $e^n_i = (e_1, e_2, \ldots, e_n)$ and $u^n_i = (u_1, u_2, \ldots, u_n)$ define the forward random map $F_0^n(x, e^n_i, u^n_i)$ as

$$
F_0^1(x, e^n_1, u^n_1) = \phi(x, e_1, u_1);
F_0^n(x, e^n_i, u^n_i) = F_0^n(F_0^{n-1}(x, e^{n-1}_i, u^{n-1}_i), e^n_i, u^n_i), \quad \text{for } n \geq 2.
$$

For $x \in G_N$, let the Markov chain $\{Y^x_n\}_{n \in \mathbb{N}}$ be given by

$$
Y^x_0 = x
Y^x_n = F_0^n(x, e^n_i, u^n_i), \quad n \geq 1
$$

and denote by $p^x_n$ the distribution of this Markov chain at time $n$. Now define

$$
\bar{d}(n) = \max_{x, z \in G_N} \| p^x_n - p^z_n \|_{TV},
$$

6
where \( \| \cdot \|_{TV} \) stands for the total variation distance.

Let \( T_{mix}^N \) be given by

\[
T_{mix}^N = \min\{ n > 0 : d(n) \leq \frac{1}{e} \}.
\] (10)

The following theorem shows that the expected value of \( T_{stop}^N \) is upper bounded by \( T_{mix}^N \), up to a explicit multiplicative constant that depends on the number of vertices of the graph.

**Theorem 1.** Let \( T_{mix}^N \) be the mixing time defined by (10) for the forward map (9). Then

\[
\mathbb{E}[T_{stop}^N] \leq 2 \left( \log \left( \frac{N(N-1)}{2} \right) + 1 \right) T_{mix}^N.
\] (11)

The proof of this relation is inspired from an argument given in Propp & Wilson (1996), and is presented in Section 6. The upper bound in Theorem 1 could be also combined with other results on the mixing time of the chain to obtain an estimate of the expected value of \( T_{stop}^N \), when they are available. As an example, we cite the results obtained by Bhamidi et al. (2011), where they present a study of the mixing time of the Markov chain constructed using the Glauber dynamics for ERGM as we consider here.

They show that for models where \( \beta \) belongs to the high temperature regime the mixing time of the chain is \( \Theta(N^2 \log N) \). On the other hand, for models under the low temperature regime, the mixing time is exponentially slow.

Observe that the mixing time \( T_{mix}^N \) is directly related to the “burn-in” time in MCMC, where the non-stationary forward Markov chain defined by (9) approaches the invariant distribution. This can be observed in practice in Figure 1, the convergence time for both MCMC and CFTP are of the same order of magnitude, as suggested by Theorem 1.

In this figure we compare some statistics of the graphs obtained by MCMC at different sample sizes of the forward Markov chain and by CFTP at its convergence time. The model is defined by (2), where \( g_2 \) is chosen as the graph with 3 vertices and 2 edges, with parameters \((\beta_1, \beta_2) = (-1.1, 0.4)\), and the mean proportion of edges in the graphs is approximately \( p = 0.86 \). We observe in Figure 1a that MCMC remains a long time strongly dependent on the initial value of the chain when this value is chosen in a region of low probability. On the other hand, when the initial value is chosen appropriately, then MCMC and CFTP have comparable performances, as shown in Figure 1b.
(a) Initial state for MCMC chosen as Erdős-Rényi graph with parameter 0.1.

(b) Initial state for MCMC chosen as Erdős-Rényi graph with parameter 0.8.

Figure 1: Boxplots of number of edges (left) and number of 2-stars (right) for the ERGM with $N = 80$ and parameter $(\beta_1, \beta_2) = (-1.1, 0.4)$. The MCMC algorithm was run for $n = 80,000$ time steps (respectively $n = 100,000$ (b) and $n = 130,000$ (c)) and two different initial states. The results for the CFTP algorithm at the time of convergence are shown in (d), with mean 130.500 time steps.

5 Discussion

In this paper we proposed a perfect simulation algorithm for the Exponential Random Graph Model, based on the Coupling From The Past method of Propp & Wilson [1996]. The justification of the correctness of the CFTP algorithm is generally restricted to the monotonicity property of the dynamics, given by Proposition 1, and to prove that $T_N^{\text{stop}}$ is almost-surely finite, but with no clue about the expected time to convergence. Here, in
contrast, we prove a much stronger result: not only do we upper-bound the expectation of the waiting time, but we show that CFTP compares favorably with standard MCMC forward simulation: the waiting time is (up to a logarithmic factor) not larger than the time one has to wait before a run of the chain approaches the stationary distribution. Moreover, CFTP has the advantage of providing a draw from the exact stationary distribution, in contrast to MCMC which may keep for a long time a strong dependence on the initial state of the chain, as shown in the simulation example of Figure 1. We thus argue that for these models the CFTP algorithm is a better tool than forward MCMC simulation in order to get a sample from the Exponential Random Graph Model.

6 Proofs

Proof of Proposition 1. Let \( x, z \in G_N \) such that \( x \preceq z \). For a pair of vertices \((i, j)\), we define a new count that considers only the subgraphs of \( x \) that contain the vertices \( i \) and \( j \), that is

\[
N_{g_k}(x, (i, j)) = \sum_{v_{mk} \in V_{mk} \atop i, j \in v_{mk}} 1\{x(v_{mk}) \succeq g_k\} \quad (12)
\]

In the same way, we have that

\[
0 \leq N_{g_k}(z_1^{ij}) \leq N_{g_k}(z_0^{ij}). \quad (13)
\]

Since \( x \preceq z \) we have that \( N_{g_k}(x_1^{ij}, (i, j)) \leq N_{g_k}(z_1^{ij}, (i, j)) \). For \( \beta_k \geq 0, k = 2, \ldots, s \), we have that

\[
\sum_{k=2}^{s} \frac{\beta_k}{N_{mk}^{s-2}} \left(N_{g_k}(x_1^{ij}, (i, j)) - N_{g_k}(x_0^{ij}) - N_{g_k}(z_1^{ij}, (i, j)) + N_{g_k}(z_0^{ij})\right) = \sum_{k=2}^{s} \frac{\beta_k}{N_{mk}^{s-2}} (N_{g_k}(x_1^{ij}, (i, j)) - N_{g_k}(z_1^{ij}, (i, j))) \leq 0 \quad (14)
\]

Finally,

\[
\frac{p_N(y_1^{ij} | \theta) p_N(z_0^{ij} | \theta)}{p_N(y_0^{ij} | \theta) p_N(z_1^{ij} | \theta)} = \exp \left\{ \sum_{k=2}^{s} \frac{\beta_k}{N_{mk}^{s-2}} (N_{g_k}(x_1^{ij}, (i, j)) - N_{g_k}(z_1^{ij}, (i, j))) \right\} \leq 1
\]

and this concludes the proof.

Proof of Theorem 1. Define the stopping time of the forward algorithm by

\[
\overline{T}_N^{\text{stop}} = \min\{n > 0 : F_0^n(x_0, e_1^n, u_1^n) = F_0^n(x_1, e_1^n, u_1^n)\}
\]
To prove the theorem claim we use the random variable $\tilde{T}_{N}^{\text{stop}}$, since $\tilde{T}_{N}^{\text{stop}}$ and $T_{N}^{\text{stop}}$ have the same probability distribution. In fact,

$$\mathbb{P}(T_{N}^{\text{stop}} > n) = \mathbb{P}(F_{-n}(x^{(0)}, e_{-n}^{-1}, u_{-n}^{-1}) \neq F_{-n}(x^{(1)}, e_{-n}^{-1}, u_{-n}^{-1})) = \mathbb{P}(F_{0}^{n}(x^{(0)}, e_{n}^{n}, u_{n}^{n}) = F_{0}^{n}(x^{(1)}, e_{n}^{n}, u_{n}^{n}) = \mathbb{P}(\tilde{T}_{N}^{\text{stop}} > n).$$  \hfill(15)

Let $\{Y_{n}^{1}\}_{n \in \mathbb{N}}$ and $\{Y_{n}^{0}\}_{n \in \mathbb{N}}$ be the Markov chains obtained by [8] with initial states given by $x^{(0)}$ and $x^{(1)}$, respectively. Define $l(y)$ as the length of the longest increasing chain such that the top element is $y$. In the case $Y_{n}^{0} = Y_{1}^{1}$ we have $l(Y_{n}^{0}) = l(Y_{n}^{1})$. Otherwise, if $Y_{n}^{0} \neq Y_{n}^{1}$ we have that $Y_{n}^{0}$ has at least one different edge of $Y_{n}^{1}$, since our algorithm use a local update. Then, $l(Y_{n}^{0}) + 1 \leq l(Y_{n}^{1})$. So, we have that

$$\mathbb{P}(\tilde{T}_{N}^{\text{stop}} > n) = \mathbb{P}(Y_{n}^{0} \neq Y_{1}^{1}) = \mathbb{P}(l(Y_{n}^{0}) + 1 \leq l(Y_{n}^{1}))$$

$$\leq \mathbb{E}[l(Y_{n}^{1}) - l(Y_{n}^{0})] = |\mathbb{E}_{p_{n}^{0}}[l(Y)] - \mathbb{E}_{p_{n}^{1}}[l(Y)]|$$

$$\leq \|p_{n}^{1} - p_{n}^{0}\|_{\text{TV}} \left[\max_{x \in \mathbb{G}_{N}} l(x) - \min_{x \in \mathbb{G}_{N}} l(x)\right] \leq \bar{d}(n) \max_{x \in \mathbb{G}_{N}} l(x)$$  \hfill(16)

Since the update function of the transitions of the Markov chain given by [8] updates only one edge at each step, we have that $\max_{x \in \mathbb{G}_{N}} l(x) = l(x^{(1)})$ and $l(x^{(1)})$ is the length of the longest increasing chain started on $x^{(0)}$ with top element is $x^{(1)}$, that is $l(x^{(1)}) = \frac{N(N-1)}{2}$. Thus

$$\bar{d}(n) \geq \frac{2}{N(N-1)} \mathbb{P}(\tau_{n}^{*} > n).$$

Since $\mathbb{P}(\tilde{T}_{N}^{\text{stop}} > n)$ is submultiplicative (Theorem 6 in [Propp & Wilson 1996]) we have that

$$\mathbb{E}[\tilde{T}_{N}^{\text{stop}}] \leq \sum_{i=1}^{\infty} n \mathbb{P}(\tilde{T}_{N}^{\text{stop}} > in) \leq \sum_{i=1}^{\infty} n \mathbb{P}(\tilde{T}_{N}^{\text{stop}} > n)]^{i} = \frac{n}{\mathbb{P}(\tilde{T}_{N}^{\text{stop}} \leq n)}. $$  \hfill(17)

Then, (16) and (17) imply that

$$\mathbb{E}[\tilde{T}_{N}^{\text{stop}}] \leq \frac{n}{1 - \bar{d}(n) \frac{N(N-1)}{2}}. \hfill(18)$$

Set $n = \left(\log \left(\frac{N(N-1)}{2}\right) + 1\right) T_{N}^{\text{mix}}$. Using the submultiplicative property of $\bar{d}(n)$ (Lemma 4.12 in [?]) we have that

$$\bar{d}(n) \leq [\bar{d}(T_{N}^{\text{mix}})]^{\log \left(\frac{N(N-1)}{2}\right) + 1} \leq \frac{2}{cN(N-1)}$$  \hfill(19)

and by (18) and (19) we conclude that

$$\mathbb{E}[\tilde{T}_{N}^{\text{stop}}] \leq \left(\log \left(\frac{N(N-1)}{2}\right) + 1\right) T_{N}^{\text{mix}} \leq 2 \left(\log \left(\frac{N(N-1)}{2}\right) + 1\right) T_{N}^{\text{mix}}. \quad \square$$
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