Magnetization plateau in the spin ladder with the four-spin exchange

Tōru Sakai and Yasumasa Hasegawa
Faculty of Science, Himeji Institute of Technology, Kamigori, Ako-gun, Hyogo 678-1297, Japan
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The magnetization process of the $S=1/2$ antiferromagnetic spin ladder with the four-spin cyclic exchange interaction at $T = 0$ is studied by the exact diagonalization of finite clusters and size scaling analyses. It is found that a magnetization plateau appears at half the saturation value if the ratio of the four- and two-spin exchange coupling constants $J_4$ is larger than the critical value $J_{c4} = 0.05 \pm 0.04$. The phase transition with respect to $J_4$ at $J_{c4}$ is revealed to be the Kosterlitz-Thouless-type.

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The field-induced spin gap is one of recent interesting topics on the one-dimensional (1D) quantum spin systems. The gap can be detected as a plateau of the magnetization curve at low temperatures. The appearance of such a magnetization plateau was theoretically predicted in several systems: the anisotropic $S = \frac{3}{2}$ antiferromagnetic chain \[\text{(1)}\], the $S = 1$ bond-alternating chain \[\text{(2)}\], the $S = \frac{1}{2}$ ferromagnetic-ferromagnetic-antiferromagnetic chain \[\text{(3)}\], the frustrated bond-alternating chain \[\text{(4)}\], the three-leg ladder \[\text{(5)}\], and the frustrated two-leg ladder \[\text{(6)}\]. In particular the two-leg ladder attracts a great interest in the context of the superconductivity in a doped system \[\text{(7)}\]. The standard $S = \frac{1}{2}$ uniform antiferromagnetic spin ladder has the spin gap of the lowest excitation from the nonmagnetic ground state (GS), which leads to a plateau at the magnetization $m = 0$ \[\text{(8)}\]. A strong coupling approach \[\text{(9)}\] indicated an additional plateau at half the saturated magnetization due to the next-nearest antiferromagnetic coupling which yields the frustration. In this paper we show another possibility of the magnetization plateau in the two-leg spin ladder, which is induced by a four-spin exchange interaction.

A four-spin exchange interaction described by a product of two-spin exchanges in a spin ladder was investigated by a field theoretical approach \[\text{(10)}\]. It indicated the possibility of a different type of massive phase from the Haldane phase \[\text{(11)}\] in the nonmagnetic GS, but the state in a strong magnetic field was not discussed. On the other hand a mean field analysis \[\text{(12)}\] suggested that the $S = \frac{1}{2}$ triangular lattice antiferromagnet would have a magnetization plateau at half the saturated magnetization, if there exists a four-spin cyclic exchange interaction. It was verified by the exact diagonalization \[\text{(13)}\]. The recent experiments revealed that such multiple-spin exchange interactions are realized in the two-dimensional (2D) solid $^3\text{He}$ \[\text{(14,17)}\] and the 2D Wigner solid of electrons formed in a Si inversion layer \[\text{(18)}\], as well as the bcc $^3\text{He}$ \[\text{(19)}\]. In order to test the possibility of a similar magnetization plateau in 1D quantum spin systems, we consider the $S = \frac{1}{2}$ uniform antiferromagnetic spin ladder with the four-spin cyclic exchange at every plaquette. The magnetization process of the system is described by

The Hamiltonian

\[
\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_Z, \\
\mathcal{H}_0 = \sum_j (S_{1,j} \cdot S_{1,j+1} + S_{2,j} \cdot S_{2,j+1} + S_{1,j} \cdot S_{2,j}) + J_4 \sum_j (P_{4,j} + P_{4,j}^{-1}), \\
\mathcal{H}_Z = -H \sum_j (S_{1,j}^z + S_{2,j}^z),
\]

where $P_{4,j}$ is the cyclic permutation operator which exchanges the four spins around the $j$-th plaquette as $S_{1,j} \rightarrow S_{1,j+1} \rightarrow S_{2,j+1} \rightarrow S_{2,j} \rightarrow S_{1,j}$, $J_4$ is the strength of the four-spin exchange and $H$ is the applied magnetic field normalized by the two-spin exchange coupling constant. We assume $J_4$ is positive, as it is in the solid $^3\text{He}$. This system subjected to the periodic boundary condition is studied by the exact diagonalization of the finite clusters and the size scaling of the low-lying energy spectra. For $L \times 2$-spin systems, the lowest energy of $\mathcal{H}_0$ in the subspace where $\sum_j (S_{1,j}^z + S_{2,j}^z) = M$ is denoted as $E(L,M)$. Using Lanczos’ algorithm, we calculated $E(L,M)$ ($M = 0, 1, 2, \ldots, L$) for even-$L$ systems up to $L = 16$. The macroscopic magnetization is defined as $m = M/2$.

The nonmagnetic GS of the system \[\text{(1)}\] with $J_4 = 0$ is in a massive phase equivalent to the Haldane phase of the $S = 1$ antiferromagnetic chain and the low-lying excitation has a finite energy gap for $m = 0$. On the other hand, the magnetic GS is always gapless \[\text{(20,21)}\] except for the saturation. Thus the magnetization curve has a plateau at $m = 0$, while no other plateau appear up to $m = 1$, as far as $J_4 = 0$. The four-spin exchange, however, is expected to induce a plateau at $m = \frac{1}{2}$, because the interaction stabilizes the ‘uuud’ state, mentioned in Ref. \[\text{(14)}\], of the four spins around every plaquette within a mean field argument. We concentrate on the plateau at $m = \frac{1}{2}$, rather than the nonmagnetic GS.

The magnetic excitation gap giving $\delta M = \pm 1$ of the $L \times 2$-spin systems described by the total Hamiltonian $\mathcal{H}$ is given by

\[
\delta M = \left| \frac{E(L,M+1) - E(L,M)}{2} \right|.
\]
\[ \Delta_{\pm} = E(L, M \pm 1) - E(L, M) \pm H. \] (2)

For the gapless system in the thermodynamic limit, the conformal field theory [22] (CFT) predicts the asymptotic form of the size dependence of the gap as \( \Delta_+ \sim O(1/L) \) with fixed \( m = M/L \). When \( H_+ \) and \( H_- \) are defined as
\[
E(L, M + 1) - E(L, M) \to H_+ \quad (L \to \infty),
E(L, M) - E(L, M - 1) \to H_- \quad (L \to \infty),
\] (3)
\( H_+ \) and \( H_- \) has the same value and it gives the magnetic field \( H \) for the magnetization \( m \) in the thermodynamic limit. In contrast to the gapless case, if the system has a finite gap even in the infinite length limit, \( \Delta_+ \) and \( \Delta_- \) are still finite for \( L \to \infty \). It leads to the difference between \( H_+ \) and \( H_- \) and a plateau appears for \( H_- < H < H_+ \) at \( m = M/L \) in the magnetization curve at \( T = 0 \).

The sum \( \Delta \equiv \Delta_+ + \Delta_- \) is a good order parameter to investigate the plateau-nonplateau transition with the finite-size scaling [3], because \( \Delta \) corresponds to the length of the plateau in the magnetization curve in the thermodynamic limit. The scaled gap \( L\Delta \) of finite systems \( (L = 8 \sim 16) \) at \( m = 1/2 \) is plotted versus \( J_4 \) in Fig. 1. For \( J_4 > 0.2 \) the scaled gap obviously increases with increasing \( L \), which means that a finite gap exists in the thermodynamic limit. For small \( J_4 \) around the region \( 0 < J_4 < 0.1 \), the scaled gap looks almost independent of \( L \). It implies that the system is gapless at a finite region of the parameter \( J_4 \), which is reminiscent of the Kosterlitz-Thouless (KT) transition [23]. According to our precise analysis, the \( L\Delta \) curves for \( L, \) and \( L + 2 \) have an intersection in the region \( 0 < J_4 < 0.1 \) for each \( L \). Thus the critical point \( J_{kc} \) can be estimated by the phenomenological renormalization group (PRG) equation [24]
\[
(L + 2)\Delta_{L+2}(J_4) = L\Delta_L(J_4).
\] (4)

We define \( J_{kc,L,L+2} \) as the \( L \)-dependent fixed point of \( J_{kc} \) and it is extrapolated to the thermodynamic limit. \( J_{kc,L,L+2} \) is plotted versus \( 1/(L + 1) \) as solid circles in Fig. 1. Although the convergence of \( J_{kc,L,L+2} \) with increasing \( L \) is not good, the least square fitting of the form \( J_{kc,L,L+2} \sim J_{kc} + A/(L + 1) \) gives the extrapolated result \( J_{kc} = 0.05 \pm 0.01 \) as the dashed line in Fig. 1. To test the precision of the value, we did the same analysis using the gap for \( \delta M = \pm 2 \) instead of \( \Delta_+ \) as solid squares shown in Fig. 1 where the fixed point can be obtained only for \( L \geq 10 \). It gave \( J_{kc} = 0.01 \pm 0.01 \) which is not well coincide with the above result, which implies that the available system size is not enough to determine \( J_{kc} \) with the fitting of \( 1/(L + 1) \). (Such a difficulty of the precise decision of the critical point by PRG is sometimes due to the logarithmic size correction in the case of the KT transition. [27]) Assuming that the system is gapless for \( J_4 = 0 \), we conclude \( J_{kc} = 0.05 \pm 0.04 \) within the present analysis.

We present the GS magnetization curve in the thermodynamic limit for \( J_4 = 0 \) and 0.1. In the latter case the magnetization plateau should appear at \( m = \frac{1}{2} \) in contrast to the former, as discussed above. Note that the four-spin exchange interaction reduces the spin gap just above the nonmagnetic GS. According to our present analysis based on PRG, the gap for \( m = 0 \) vanishes at a critical value \( J_{kc} \), which should be distinguished from \( J_{kc} \) for \( m = \frac{1}{2} \), and the nonmagnetic GS will belong to a different massive phase from the Haldane phase for \( J_4 > J_{kc} \). The critical value \( J_{kc} \), however, is obviously larger than 0.1. Thus even in the case of \( J_4 = 0.1 \) the spin gap due to the Haldane mechanism still exists for \( m = 0 \) and we can use the same method to give the magnetization curve as used for the \( S = 1 \) antiferromagnetic chain [25].

For \( J_4 = 0.1 \) the left hand sides of the form [1] calculated for \( m = 0, \frac{1}{2}, 1, \frac{3}{2}, \frac{5}{2} \) and \( \frac{7}{4} \) are plotted versus \( 1/L \)
in Fig. 3. It shows $H_+ = H_-$ except for $m = 0$ and $\frac{1}{2}$. Thus we take the mean value of the two for the magnetic field $H$ for each $m$.

![Graph](image)

FIG. 3. $E(L, M+1) - E(L, M)$ and $E(L, M) - E(L, M-1)$ plotted versus $1/L$ with fixed $m$ for $J_4 = 0.1$. The dashed curves are guides to the eye. The extrapolated points for $H_L$ and obtain $H_P$ in Fig. 3. Thus we use the Shanks’ transformation [29] to get $H_{c1} = 0.15 \pm 0.03$, $H_+ = 1.84 \pm 0.06$ and $H_- = 1.99 \pm 0.09$, respectively.

Since the nonmagnetic GS is massive for $J_4 = 0$ and $0.1$, the size correction of $H_+$ decays faster than $\frac{1}{2}$ as shown in Fig. 3. Thus we use the Shanks’ transformation [29]

$$P_n = \frac{(P_{n-1} - P_{n+1})}{(P_{n-1} + P_{n+1} - 2P_n)}$$

twice for the sequence $E(L, 1) - E(L, 0)$ for $L = 6, 8, 10, 12, 14$, and obtain $H_{c1} = 0.503 \pm 0.003$ and $0.15 \pm 0.03$ for $J_4 = 0$ and $0.1$, respectively. The saturation field $H_{c2}$ is given by the $L$-independent quantity $E(L, L) - E(L, L - 1)$. Obviously different and the size correction decays faster than $1/L$, as shown in Fig. 3 which is consistent with a finite gap. Then we estimate $H_+$ and $H_-$ by the Shanks’ transformation and get $H_+ = 1.99 \pm 0.09$ and $H_- = 1.84 \pm 0.06$. For $J_4 = 0$ $H_+$ and $H_-$ correspond even at $m = 1/2$. We present the results for $J_4 = 0$ and $0.1$ in Fig. 3, where we also used the values of $H$ for $m = 1/6, 3/8, 5/8, 5/6$ and $7/8$ which are estimated by the same method as mentioned above. The curve has a plateau at $m = 1/2$ ($H_- < H < H_+$) for $J_4 = 0.1$.

Our present PRG analysis shows that the the gap does not behave as $\Delta \sim (J_4 - J_{4c})^\nu$. If we define the size-dependent exponent $\nu_L$, it diverges as $L$ increases. Instead, if the gap behaves like

$$\Delta \sim \frac{a}{(J_4 - J_{4c})^\sigma},$$

as in the case of universality class of KT transitions ($\sigma = \frac{1}{4}$), the Roomany-Wyld approximation for the Callen-Symanzik $\beta$-function [20], which is defined as

$$\beta_{L,L+2}(J_4) = 1 + \log \left( \frac{\Delta_{L+2}(J_4)}{\Delta_L(J_4)} \right) / \log \left( \frac{L+2}{L} \right),$$

should have the form

$$\beta_{L,L+2}(J_4) \sim (J_4 - J_{4c,L+2})^{1+\sigma}.$$  

(7)

Fitting the form (7) to the calculated function (8) for each $L$, $\sigma$ is estimated as follows: $\sigma_{10,12} = 0.38 \pm 0.10$, $\sigma_{12,14} = 0.43 \pm 0.10$ and $\sigma_{14,16} = 0.49 \pm 0.10$. The results are consistent with $\sigma = \frac{1}{4}$. Thus we conclude the critical behavior near $J_{4c}$ for $m = \frac{1}{2}$ is characterized by the universality class of the KT transition.

Furthermore we estimate the central charge $c$ of CFT, using the asymptotic form of the GS energy per site

$$\frac{1}{L} E(L, M) \sim c(m) - \frac{\pi}{6} \nu_s \frac{1}{L^2} (L \to \infty),$$

(8)

where $\nu_s$ is the sound velocity which is the gradient of the dispersion curve at the origin. The result shown in Fig. 3 suggests $c = 1$ with only a few percent errors for $m = \frac{1}{2}$ and $0 \leq J_4 \leq 0.1$. It also supports the KT transition.

The critical exponent $\eta$, associated with the spin correlation function in the leg direction like $(S_0^x S_n^x) \sim (-1)^r r^{-\eta}$, can be estimated by the form of the gap

$$\Delta_+ \sim \nu_s \eta L \quad (L \to \infty).$$

The estimated $\eta$, shown in Fig. 3 seems close to $\frac{1}{4}$ around the critical point $J_{4c}$, rather than $\frac{1}{2}$ which is expected for the KT transition. We think there is a possible jump from $\eta = \frac{1}{4}$ to $\eta = \frac{1}{2}$ at $J_{4c}$ because the elementary excitations is expected to behave like the free Fermion systems ($\eta = \frac{1}{4}$) at the edge of the plateau for $J_4 > J_{4c}$. The present small cluster analysis could not detect such a discontinuity. Another exponent $\eta^f$ defined as $(S_0^x S_n^y) \sim \cos(k_F r) r^{-\eta^f}$ can also be estimated from the $L$-dependence of the soft mode.
gap with the momentum $2k_F = 2\pi m$. We checked the validity of the relation $\eta \eta^2 = 1$ around $J_{4c}$, which is consistent with the Luttinger liquid theory leading to $\eta = \frac{1}{2}$ in the free Fermion case.

The spin gap at $m = 0$ has already been observed in several real ladder compounds, for example $\text{Cu}_2(\text{C}_2\text{H}_{12}\text{N}_2)_2\text{Cl}_4$ \cite{20} and $\text{La}_6\text{Ca}_8\text{Cu}_{24}\text{O}_{41}$ \cite{31}. The magnetization plateau, however, has not been detected at any finite magnetization. We hope some new ladder materials with the field-induced spin gap will be discovered in the near future.

In summary the finite cluster calculation and size scaling study showed that the $S = \frac{1}{2}$ antiferromagnetic spin ladder with the four-spin cyclic exchange interaction at every plaquette has the magnetization plateau at $m = \frac{1}{2}$ for $J_4 > J_{4c} = 0.05 \pm 0.04$ and the phase transition with respect to $J_4$ belongs to the KT universality class.

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**FIG. 5.** Estimated central charge $c$ and exponent $\eta$ around $J_{4c}$. The result indicates $c = 1$ which is consistent with the KT transition. $\eta$ is close to $\frac{1}{2}$ rather than $\frac{1}{4}$.

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