S′DARBOUX COORDINATES AND WKB APPROXIMATIONS IN DEFORMATION QUANTIZATION

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Abstract. We introduce a method for calculating the joint spectra of functions which comprise a quantum integrable system under a deformation quantization star product. The main result involves a construction by formal power series in \( \hbar \) of s′Darboux coordinates, a concept we introduce for the star product analogue of Darboux normal coordinates. We will find that underlying the quantum integrable system is a set of s′Darboux coordinates which in turn give rise to number operators for the system. We present an explicit correction to the lowest order Bohr-Sommerfeld or EBK quantization rule.

1. Introduction

Let us describe a path from canonical quantization to deformation quantization. \( T^\ast(\mathbb{R}) \) is a symplectic manifold with natural coordinates \((x, p)\) and Poisson bracket is \( \{x, p\} = 1 \). To quantize, we promote \((x, p)\) to operators \((\hat{x}, \hat{p})\) on \( L^2(\mathbb{R}) \) with the canonical quantization relation \( [\hat{x}, \hat{p}] = i\hbar \). One such choice has \( \hat{p} = -i\hbar \partial/\partial x \). We may now build operators out of the generators \((\hat{x}, \hat{p})\). Classical mechanics is left behind—only the names of operators \((\hat{x}, \hat{p})\) betray their classical origin.

However, it is possible to recover the original classical structure by using a symbol correspondence, which associates to each operator \( \hat{A} \) its symbol \( A \), a function on classical phase space taking values in power series in \( \hbar \). The operator product \( \hat{A} \hat{B} \) becomes a non-commutative product of functions \( A \ast B \), called a star product. When \( A \) and \( B \) are suitably well-behaved, a star product may be developed as a power series in \( \hbar \) of bi-differential operators. All symbol correspondences are such that the associated star commutator has \( \{A, B\} \ast = A \ast B - B \ast A = i\hbar \{A, B\} + O(\hbar^2) \)—the Poisson bracket has reappeared.

Having thus returned to the classical phase space, it is natural to ask whether the Hilbert space \( L^2(\mathbb{R}) \) was needed at all. So began the so-called deformation quantization program of [1]. On an arbitrary symplectic manifold, not necessary a cotangent bundle, they sought to create a star product from scratch, assuming only that

- the star product is associative;
- the star product is the ordinary product at lowest order, i.e. \( A \ast B = A \cdot B + O(\hbar) \);
- the star commutator is the Poisson bracket at lowest order, i.e. \( \{A, B\} \ast = i\hbar \{A, B\} + O(\hbar^3) \)

Since the inception of deformation quantization, investigators have affirmed the existence of a star product. We are particularly attracted to the construction of Fedosov [2], who builds a star product from the symplectic form and a symplectic connection.
In view of these successes, one might be tempted to conclude that the deformation quantization program is complete. We disagree. As a practical matter, many calculations that are straightforward in traditional Hilbert space quantum mechanics are not yet possible in star product quantum mechanics. In particular, we lack systematic methods for computing spectra. ¹

This paper is a step in that direction. In certain types of quantum integrable systems, we will show that the joint spectra can be developed as a formal power series in $\hbar$.

The organization is as follows. In Section 3, we will introduce s’Darboux coordinates, the star product analogue of Darboux normal coordinates, and show how s’Darboux coordinates may be developed from Darboux coordinates. In Sections 4 and 5, we will review Dirac’s creation and annihilation operator method and define quantum integrable systems. In Section 6, we will show that creation and annihilation operators underlie some quantum integrable systems and develop methods for determining these operators. In Section 7, we will demonstrate these techniques with the Moyal star product and show that they lead to higher order EBK quantization rules. Finally, in Section 8, we will generalize this result to the Fedosov star product.

2. DEFINITIONS AND CONVENTIONS

Symbol correspondence. Operators will wear hats, e.g. $\hat{A}$, and we denote the corresponding symbol by $A = s(\hat{A})$. When the symbol $A$ has a power series of the form $A = a + \hbar A_1 + \ldots$ we call the first term, or principal symbol, $\pi A$. When we need to name a principal symbol, we will try to use the corresponding lower case letter, as above where $a = \pi A$.

Lists. On our $2M$ dimensional symplectic manifold, we shall often work with lists of $M$ or $2M$ functions. When we refer to a specific item in a list, we shall attach an index, e.g. $H^i$, but when we refer to the items collectively, we will drop the index, e.g. $H$.

Other notations will be developed as needed.

3. FROM DARBOUX TO S’DARBOUX COORDINATES

On a $2M$ dimensional symplectic manifold $\mathcal{M}$, Darboux normal coordinates are coordinates $z^a$ satisfying

$$\{z^a, z^b\} = J^{ab},$$

where $J^{ab}$ is the constant matrix

$$J^{ab} = \begin{pmatrix} 0 & I_{M \times M} \\ -I_{M \times M} & 0 \end{pmatrix}.$$

A collection of $2M$ symbols $Z^i$ are said to be s’Darboux coordinates if they have

$$\{Z^a, Z^b\}_\star = i\hbar J^{ab},$$

¹The present form of this paper is a somewhat rushed presentation of results which should have been published much earlier. I present them now to establish whatever priority they may have. Based on a preliminary survey of the literature, I believe the results are new. However, I am aware that many others have also pursued work in this direction. In a future version intended for official publication, I fully intend to give due credit to earlier work—hoping, of course, that I do not find the exact result elsewhere!
that is, modulo the \( \hbar \), they satisfy the Darboux coordinate algebra under the star bracket.

In this section we prove

**Theorem 1.** For every set of \( 2M \) Darboux coordinates \( z^i \in C^\infty(\mathcal{M}) \) there exist s’Darboux coordinates \( Z^i \in C^\infty(\mathcal{M})[[\hbar]] \) such that \( z^i = \pi Z^i \). The \( Z^i \) are not unique; instead, given s’Darboux coordinates \( Z^i \) there are other sets \( Z'^i = Z^i + \hbar^n X^i \) with \( \pi X^i \in \{ \gamma, z^i \} \) for some \( \gamma \in C^\infty(\mathcal{M}) \).

The proof is by induction; we construct \( Z^i \) order by order in \( \hbar \). Already,

\[
\{ z^i, z^j \}_* = i\hbar J^{ij} + O(\hbar^2)
\]

since \( z^i \) are Darboux coordinates. We next assume that

\[
\{ Z^i, Z^j \}_* = i\hbar J^{ij} + O(\hbar^{n+2}).
\]

To complete the induction, we need to show there exist \( X^i \in C^\infty(\mathcal{M})[[\hbar]] \) so that

\[
\{ Z^i + \hbar^{n+1} X^i, Z^j + \hbar^{n+1} X^j \}_* = i\hbar J^{ij} + O(\hbar^{n+3}).
\]

(We then take \( Z^i \to Z^i + \hbar^{n+1} X^i \).) Rewriting the left side of Eq. (3.6), we have

\[
\{ Z^i + \hbar^{n+1} X^i, Z^j + \hbar^{n+1} X^j \}_* = \{ Z^i, Z^j \}_* + \hbar^{n+1} \left( \{ X^i, Z^j \}_* + \{ Z^i, X^j \}_* \right) + O(\hbar^{2n+3})
\]

\[
= \{ Z^i, Z^j \}_* + i\hbar^{n+2} \left( \{ \pi X^i, Z^j \} + \{ Z^i, \pi X^j \} \right) + O(\hbar^{n+3})
\]

If Eq. (3.6) is to hold, the terms containing \( \pi X^i \) must cancel off the \( O(\hbar^{n+2}) \) part of \( \{ Z^i, Z^j \}_* \).

The condition can be expressed more geometrically. First define coordinates \( z_i = J_{ij} z^j \) using the inverse \( J_{ij} \) of \( J^{ij} \). Then define a two form,

\[
\tilde{\omega}_n = \frac{1}{i\hbar} \{ Z^i, Z^j \}_* dz_i \wedge dz_j
\]

and a one form

\[
\theta = X^i dz_i.
\]

Then, since

\[
d\theta = J^{ik} \frac{\partial X^i}{\partial z^k} dz_j \wedge dz_i = \{ z^j, X^i \} dz_j \wedge dz_i,
\]

Eq. (3.6) requires that

\[
\tilde{\omega} + \frac{\hbar^{n+1}}{2} d\theta = \omega + O(\hbar^{n+2}),
\]

where \( \omega = J^{ij} dz_i \wedge dz_j \) is the usual symplectic two form. Taking the exterior derivative, we obtain the condition

\[
d\omega = \frac{1}{i\hbar} \{ z^k, \{ Z^i, Z^j \}_* \} dz_k \wedge dz_i \wedge dz_j = O(\hbar^{n+2}).
\]

Now, if the right side of this expression had not a Poisson bracket but a \( \star \) bracket, and if also \( z^k \) were \( Z^k \), then it would vanish by the Jacobi identity. To investigate whether these changes leave it unchanged at this order, we form the difference

\[
\{ z^k, \{ Z^i, Z^j \}_* \} - \frac{1}{i\hbar} \{ Z^k, \{ Z^i, Z^j \}_* \} = \{ z^k, \{ Z^i, Z^j \}_* \} - \{ Z^k, \{ Z^i, Z^j \}_* \} + O(\hbar^{n+3}) = \hbar^{n+3},
\]
where we have used the induction hypothesis of Eq. 3.5. Using this calculation, we have that
\[
\tilde{\omega} = \frac{1}{i\hbar} \{z^k, \{Z^i, Z^j\}_s\} dz_k \wedge dz_i \wedge dz_j
\]
(3.14)
\[
= \frac{1}{(i\hbar)^2} (Z^k, \{Z^i, Z^j\}_s) dz_k \wedge dz_i \wedge dz_j + O(h^{n+2})
\]
\[
= O(h^{n+2}),
\]
where, as planned, we have used the Jacobi identity. Now, because we have assumed that the Darboux coordinates \(z^i\) are \(C^\infty\), we have also assumed that the cohomology of \(M\) is trivial. Thus, \(\tilde{\omega}\) is not only closed but exact at \(O(h^{n+2})\), and there exists a \(\theta\) making Eq. (3.11) true.

Of course, \(\theta\) is defined only up to an exact one-form; adding \(d\gamma\) to \(\theta\) adds \(h^{n+1}\{z^i, \gamma\}\) to \(Z^i\).

4. Digression: Dirac algebra

Suppose we have a set of \(M\) operators \(\hat{A}^i\) which act on the Hilbert space \(L^2(\mathbb{R}^M)\) and satisfy the algebra
\[
[\hat{A}^i, \hat{A}^j] = \hbar \delta_{ij}
\]
\[
[\hat{A}^i, \hat{A}^j] = 0.
\]
(4.1)
The \(M\) operators \(\hat{N}^i = \hat{A}^i \hat{A}^i\) are positive definite and commute. Dirac showed the \(\hat{N}^i\) have simultaneous eigenspaces \(V_n\) labeled by non-negative integers \(n^i\): for \(\psi \in V_n\), \(\hat{N}^i \psi = \hbar n^i \psi\).

We can attempt to duplicate this construction using our s’Darboux symbols \(Z^i\). We create
\[
A^i = \frac{1}{\sqrt{2}} (Z^i - iZ^{i+M})
\]
for \(i = 1, \ldots, M\). These symbols satisfy Dirac’s algebra under the star bracket:
\[
\{\bar{A}^i, A^j\}_s = \hbar \delta_{ij}
\]
\[
\{A^i, A^j\}_s = 0.
\]
(4.3)
We can also define \(M\) commuting symbols \(N^i = \bar{A}^i \star A^i\). 3

In order to conclude that the spectra of the \(N^i\) are bounded from below, Dirac’s original argument makes use of the positive definiteness of the \(\hat{N}^i\). In this case, because we do not have a Hilbert space, we cannot complete Dirac’s argument. Instead, we will simply postulate the following: if a set of \(M\) symbols \(A^i \in C^\infty(M)[[\hbar]]\) satisfy Eq. (4.3), then the spectrum of \(\hbar^{-1}N^i\) consists of all non-negative integers. 4

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2One could also work on a chart on \(M\). We would then have a theorem about local s’Darboux coordinates on a manifold with arbitrary cohomology.

3There is no sum here. However, given a basis \(\sigma^a\) for the Lie algebra \(u(M)\), we may define \(\sigma^a \bar{A}^i \star A^j\), a representation of \(u(M)\) under the \(\star\) product. The \(N^i\) are a maximally commuting subalgebra.

4Readers are encouraged to explore if this postulate is necessary.
5. QUANTUM INTEGRABLE SYSTEMS

In classical mechanics on a 2M dimensional symplectic manifold, an integrable system is a collection of M Poisson commuting, functionally independent functions \( h^i \). We will define a quantum integrable system as a collection of M star commuting, functionally independent functions \( H^i \in C^\infty(M)[[\hbar]] \).

Because \( \{ H^i, H^j \}_\star = 0 \) implies \( \{ \pi H^i, \pi H^j \} = 0 \), we may associate to a quantum integrable system \( H^i \) the classical integrable system \( h^i = \pi H^i \). Treatment of the quantum problem may begin only after understanding the underlying classical mechanics.

In this paper, we will concentrate on the case where the classical evolution takes place on tori. To be precise, we have the usual energy-momentum map \( h: M \rightarrow \mathbb{R}^M : p \mapsto (h^1, \ldots, h^M) \). For values \( E \) in the interior of \( hM \), we will assume that \( h^{-1}(E) \) is always a single M-torus. That is, the \( h^i \) give rise to a global set of action-angle variables \( (\theta^1, \ldots, \theta^M, I^1, \ldots, I^M) \) satisfying

\[
\begin{align*}
\{ \theta^i, \theta^j \} &= 0, \\
\{ I^i, I^j \} &= 0, \\
\{ \theta^i, I^j \} &= \delta_{ij},
\end{align*}
\]

and functions \( f^i: \mathbb{R}^M \rightarrow \mathbb{R} \) which express the \( h^i \) in terms of the \( I^i \):

\[
h^i = f^i \circ I. 
\]

In view of Eq. (5.1), it is tempting to regard the \( (\theta, I) \) as Darboux coordinates. This we should not do, however, as the \( \theta^i \) are not \( C^\infty \). Nevertheless, we may create Darboux coordinates from this set by defining

\[
\begin{align*}
z^i &= \sqrt{2I^i} \cos(\theta^i), \\
z^{i+M} &= -\sqrt{2I^i} \sin(\theta^i);
\end{align*}
\]

it is straightforward to verify \( \{ z^i, z^j \} = J^{ij} \). In doing so we must assume that the basis contours used to define the \( I^i \) have been chosen so that \( I^i \geq 0 \) everywhere. In other work, my collaborators will argue that such contours do exist for this type of integrable system. In addition, we shall assume the \( \theta^i \) can be and indeed are chosen so that the \( z^i \) are \( C^\infty \).

Previously, we proved Darboux coordinates \( z^i \) can be quantized to s’Darboux symbols \( Z^i \) satisfying \( z^i = \pi Z^i \). In Eq. (4.2) and following, we showed how to create number operators \( N^i \) from the \( Z^i \). Next, we will show that the \( Z^i \) can be made so that \( \{ H^i, N^j \}_\star = 0 \); when this condition holds, we shall say the \( N^i \) are good number operators for \( H^i \) and that the \( Z^i \) are compatible with \( H^i \).

6. EXISTENCE OF GOOD NUMBER OPERATORS

In this section, we prove two theorems. The first concerns “approximately good” number operators. It will help prove the second, our main result.

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5In \( I^i \), we have an exception to using the lower case for classical functions.

6The angle coordinates \( \theta^i \) are, of course, not unique. Taking \( \theta^i \rightarrow \theta^i + \{ \theta^i, \gamma \} \) for some function \( \gamma \) gives new angle coordinates.

7Already, it is possible to construct quantum integrable systems from a classical one: from \( h^i \) find \( (I, \theta) \), form \( z^i \), quantize \( z^i \) to \( Z^i \), form number symbols \( N^i \), and then define \( H^i = h^i \delta N \). Because the \( Z^i \) are not unique, neither are the resulting \( H^i \).
We first introduce some new notation. Let $f$ be an analytic function with Taylor series $f(x) = \sum c_n x^n$. Define $f \circ A = \sum c_n A^n$; that is, for each term $x^n$, we substitute the $n$-fold star product of $A$. If $A$ is the image of $\hat{A}$ under a symbol correspondence, then $f \circ A = s(f(\hat{A}))$. Thus the \( \circ \) notation gives us a way to talk about the symbol of a function of an operator in the context of deformation quantization, where there is neither a Hilbert space nor operators on it.

For the purposes of this section, the only fact we need to know about \( \circ \) is that for commuting functions $N^i, F \circ N = F \circ N + O(h)$. This fact follows immediately from the deformation quantization postulates.

The two theorems can now be stated as

**Theorem 2.** Suppose $H^i$ and $N^i$ satisfy \( \{H^i, H^j\}_\ast = 0, \{N^i, N^j\}_\ast = 0, \{N^i, H^j\}_\ast = O(h^{n+2}) \), and there are $f^i$ such that $\pi H^i = f^i \circ \pi N$. Then there exist $F^i \in C^\infty(\mathbb{R}^M)[[h]]$ and $G^i \in C^\infty(\mathcal{M})[[h]]$ such that $H^i = F^i \circ N + h^{n+1}G^i$. Furthermore, $\pi G^i = \{G, h^i\} + g^i \circ 1$, for some $G, g^i \in C^\infty(\mathcal{M})$.

**Theorem 3.** There exist $s'$Darboux symbols $Z^i$ compatible with the quantum integrable system $H^i$.

Before proving these we remark that nothing in Theorem 2 requires that the $N^i$ are number symbols. However, if they are, the spectrum of $H^i = F^i \circ N + O(h^n)$ is just $F^i \circ (hn) + O(h^n)$, where $n_i$ are non-negative integers.

6.1. **Proof of Theorem 2.** We prove the theorem by induction, constructing $F^i$ order by order. We begin with $F^i = f^i$. Then

\[
F^i \circ N = f^i \circ N + O(h) = f^i \circ \pi N + O(\hbar) = H^i + O(h).
\]

We next assume that $F^i$ is such that

\[
F^i \circ N = H^i + O(h^{s+1}).
\]

To complete the induction, we need to show there exist $X^i \in C^\infty(\mathbb{R}^M)$ such that

\[
(F^i + h^{s+1}X^i) \circ N = H^i + O(h^{s+2})
\]

whenever $s \leq n - 1$. (We then take $F^i \to F^i + h^{s+1}X^i$.) Working with the left side of Eq. (6.3), the condition becomes

\[
F^i \circ N + h^{s+1}X^i \circ \pi N = H^i + O(h^{s+2}).
\]

Because $X^i \circ \pi N$ depends only on the $N^i$, $X^i$ may be used to enforce this equation only when the other terms also depend only on the $N^i$. To verify this, we compute

\[
\{\pi N^j, F^i \circ N - H^i\} = \frac{1}{ih}\{N^j, F^i \circ N - H^i\}_\ast + O(h^{s+2})
\]

\[
= \frac{1}{ih} (0 + O(h^{n+2})) + O(h^{s+2})
\]

\[
= O(h^{n+1}) + O(h^{s+2}).
\]

This is of $O(h^{s+2})$ whenever $s \leq n - 1$. At the final stage of the induction process, we have $s = n - 1$ and

\[
H^i = F^i \circ N + h^{n+1}G^i,
\]

for some $G^i$. 
This concludes the first part of theorem 2. For the second part, we compute

$$0 = \{H^i, H^j\}_*$$

(6.7)

$$= \{F^i \circ N + \hbar^{n+1} G^i, F^j \circ N + \hbar^{n+1} G^j\}_*$$

$$= \hbar^{n+1}(\{F^i \circ N, G^j\}_* + \{G^i, F^j \circ N\}_*) + O(\hbar^{2n+2})$$

$$= i\hbar^{n+2}(\{f^i \circ \pi N, G^j\} + \{\pi G^i, f^j \circ \pi N\}) + O(\hbar^{n+3}).$$

Thus, since $$\{\pi H^i, \pi G^j\} + \{\pi G^i, \pi H^j\} = 0$$, we have $$\pi G^j = \{G, \pi H^j\} + g^j \circ \pi I$$ for some functions $$G$$ and $$g^j$$. Note that $$G$$ is not determined uniquely in this decomposition—we could add a function of the $$\pi N^i$$. With this result, equation (6.6) becomes

$$H^i = F^i \circ N + \hbar^{n+1}(\{G, \pi H^i\} + g^j \circ \pi N) + O(\hbar^{n+2}).$$

This completes theorem 2.

6.2. Proof of Theorem 3. We will construct the symbols $$Z^i$$ order by order in $$\hbar$$, taking care that the associated number symbols $$N^i$$ commute with the $$H^i$$ at the current order. We assume that $$Z^i$$ are s’Darboux symbols, but that $$\{\tilde{A}^i * A^i, H^j\}_* = O(\hbar^{n+2})$$. For the induction step, we will create new d’Darboux symbols

$$Z^i = Z^i + \hbar^{n+1} g^i,$$

(6.9)

so that the associated $$N^i = \tilde{A}^i * A^i$$ satisfy $$\{\tilde{A}^i * A^i, H^j\}_* = O(\hbar^{n+3})$$.

As discussed before, the the freedom in the $$Z^i$$, allows us to take $$\pi g^i = \{\gamma, z^i\}$$, with $$\gamma$$ a real function. This modifies the number operators at order $$\hbar^{n+1}$$. First,

$$A^i = \frac{1}{\sqrt{2}} (Z^i - iZ^{i+1}) = A^i + \hbar^{n+1}\{\gamma, \pi A^i\} + O(\hbar^{n+2}).$$

Then

$$N^i = (\tilde{A}^i + \hbar^{n+1}\{\gamma, \pi \tilde{A}^i\} + O(\hbar^{n+2})) \ast (A^i + \hbar^{n+1}\{\gamma, \pi A^i\} + O(\hbar^{n+2}))$$

$$= N^i + \hbar^{n+1}(\pi \tilde{A}^i \{\gamma, A^i\} + \pi A^i \{\gamma, \pi \tilde{A}^i\}) + O(\hbar^{n+2})$$

$$= N^i + \hbar^{n+1}\{\gamma, I^i\} + O(\hbar^{n+2})$$

(6.11)

where we have used that $$\pi \tilde{A}^i \{\gamma, A^i\} = I^i$$.

We want to choose $$\gamma$$ so that $$\{N^i, H^j\}_* = O(\hbar^{n+3})$$. Computing, we have

$$\{N^i, H^j\}_* = \{N^i + \hbar^{n+1}\{\gamma, I^i\}, s(F^j \circ \tilde{N}) + \hbar^{n+1}(\{G, h^j\} + g^j \circ I)\}_* + O(\hbar^{n+3})$$

$$= i\hbar^{n+2}\{\gamma, I^i\}, h^j\} + i\hbar^{n+2}\{I^i, \{G, h^j\} + g^j \circ I\} + O(\hbar^{n+3})$$

$$= i\hbar^{n+2}\{\{\gamma, I^i\}, h^j\} + i\hbar^{n+2}\{I^i, \{G, h^j\}\} + O(\hbar^{n+3}).$$

(6.12)

We have used the decomposition of theorem 2. Choosing $$\gamma = G$$ and using the Jacobi identity, we finish theorem 3.

Remark. $$\gamma$$ is determined only up to a function of the actions, and thus the $$A^i$$ are not determined uniquely. To what does this freedom correspond on the quantum level? Consider the following transformation of the $$\tilde{A}^i$$:

$$\tilde{A}^i \rightarrow e^{-i\hat{G} \circ \hat{N}/\hbar} \tilde{A}^i e^{i\hat{G} \circ \hat{N}/\hbar},$$

(6.13)
where $G \in C^\infty(\mathbb{R}^M)[[\hbar]]$. This transformation leaves the $\hat{N}^i$ and the $\hat{A}$ algebra invariant. On the level of wavefunctions, this transformation of the creation operators amounts to an $\hat{N}^i$ dependent rephasing.

Using the commutation relations, the transformation can also be written

$$
\hat{A}^i \to \hat{A}^i e^{\theta \hat{G}^i / \hbar},
$$

with $\hat{G}^i = G \circ (\hat{N} + \hbar/2) - G \circ (\hat{N} + \hbar/2 - \hbar e_i)$. At lowest order in $\hbar$, we have $s(\hat{G}^i)/\hbar = (\partial_i G) \circ I = \{\theta^i, G \circ I\}$, and thus $\hat{A}^i \to \hat{A}^i e^{\theta^i G^i}$. This is just the classical freedom in the origin of the angle variables. The indeterminacy of $\gamma$ is the manifestation in Eq. (6.13) at higher order in $\hbar$.

Of course, Eq. (6.14) shows the $N^i$ symbols themselves are uniquely determined at each order. Later, when we work with the Moyal star product, we will see directly that the $O(\hbar^2)$ part of $N^i$ is uniquely determined from the $H^i$.

7. Example: Quantization of Moyal Symbols

In this section, we will specialize to the Moyal star product and show how our prior analysis leads to higher order Bohr-Sommerfeld quantization rules for some quantum integrable systems of Weyl symbols.

The assumptions are now as follows: we have a set of $M$ symbols $H^i$ which Moyal commute and whose principal symbols $h^i = \pi H^i$ give rise to a global set of action angle variables $(I, \theta)$. We suppose that the series for $H^i$ are even in $\hbar$. Further, we suppose that the functions $z^i = \sqrt{2T^i} \cos \theta^i$ and $z^{i+M} = -\sqrt{2T^i} \sin \theta^i$ are $C^\infty$ and are Darboux coordinates.

To achieve a quantization rule beyond the usual WKB order, we need to

- extend the $z^i$ to symbols $Z^i$ satisfying $\pi Z^i = z^i$ and $\{Z^i, Z^j\}_\star = i\hbar J^{ij} + O(\hbar^3)$.
- modify the $Z^i$ so that the associated number operators $N^i$ satisfy $\{H^i, N^j\}_\star = O(\hbar^3)$.

This is the work of section and respectively. The first problem will require some techniques. Before developing them, we must review the Moyal star product.

7.1. The Moyal star product. The Moyal star product is a star product on $T^*(\mathbb{R}^M)$ phase space, with the standard Poisson bracket

$$
\{f, g\} = J^{ij} \partial_i f \partial_j g = \partial_i f \partial^j g, \tag{7.1}
$$

Here, the partial derivatives $\partial_i$ are with respect to the standard $(x, p)$ coordinates on $T^*(\mathbb{R}^M)$, and $J$ is as before. The star product, for $f, g \in C^\infty(\mathbb{R}^{2M})[[\hbar]]$, is given by the Moyal formula

$$
f \star g = \sum_{n=0}^{\infty} \frac{1}{N!} \left( \frac{i\hbar}{2} \right)^n \{f, g\}_n \tag{7.2}
$$

where

$$
\begin{align*}
\{f, g\}_0 &= f \cdot g \\
\{f, g\}_1 &= \{f, g\} = \partial_i f \partial^i g \\
\{f, g\}_2 &= \partial_i \partial_j f \partial^{ij} \partial^{j} g
\end{align*} \tag{7.3}
$$

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8We use $J$ to raise and lower indices. For example $T^i = J^{ij} T_j$ and $T_i = J_{ij} T^j$. 
S’Darboux Coordinates and WKB Approximations in Deformation Quantization

and so on. The Moyal commutator $\{f, g\}_\hbar = f \star g - g \star f$ is an odd series in $\hbar$ and because of this, as we shall see, we can develop expansions for $Z^i, N^i$, etc as even series. Accordingly, the results from Sections 3 and 6 will differ somewhat.

The notation of Eq. (7.3) is not always convenient. To avoid proliferating indices, we will resume a diagrammatic notation introduced earlier in [5]. We define $f \to g = \{f, g\}$, and, for more complicated diagrams, we exhibit an example conversion of an arrow to more explicit notation:

\[
\begin{align*}
D & \quad = \quad \partial_i D A \quad \partial^i B \quad C \\
A & \quad \quad \quad B \quad \quad C
\end{align*}
\]

7.2. Construction of Moyal s’Darboux Coordinates. We need symbols $Z^i = z^i + h^2 Z^i_2 + \ldots$ satisfying

\[
\{Z^i, Z^j\}_\hbar = i\hbar J^{ij} + O(\hbar^5).
\]

We have found such $Z^i_2$. First define \footnote{The fact that the $\Gamma^{abc}$ are the connection coefficients for the flat symplectic connection in $z^i$ coordinates is not central to the derivation, but it is highly suggestive, especially in view of Fedosov’s employ \footnote{Despite the known non-uniqueness of the $Z_2$, one wonders if this construction is canonical in some sense.} of a symplectic connection in constructing his star product. For now, we simply note that $\Gamma^{abc}$ is completely symmetric in its three indices.}

\[
\Gamma^{abc} = z^a \quad z^b \quad z^c.
\]

Then

\[
Z^i_2 = \frac{1}{48} \Gamma^{abc} z^i \to \Gamma^{abc}
\]

satisfies Eq. (7.5). \footnote{Despite the known non-uniqueness of the $Z_2$, one wonders if this construction is canonical in some sense.}

The key to verifying this will be the following identity, which we present without motivation:

\[
\{z^d, \Gamma^{abc}\} = z^b
\]
The proof is as follows. Because the bracket \( \{z^b, z^d\} \) is constant, we have
\[
0 = z^a \rightarrow (z^b \rightarrow z^d) \rightarrow z^c
\]
\[
= z^a \rightarrow z^b \rightarrow z^d \rightarrow z^c + z^a \rightarrow z^d \rightarrow z^b \rightarrow z^c
\]
\[
+ z^a \rightarrow z^b \rightarrow z^c + z^a \rightarrow z^d \rightarrow z^c
\]
\[
= z^d \rightarrow (z^a \rightarrow z^b \rightarrow z^c) + z^b \rightarrow z^a \rightarrow z^d \rightarrow z^c.
\]

With rearrangement, the identity follows.

A resolution of the arrow,
\[
(7.9) \rightarrow = \rightarrow z^a z_a \rightarrow,
\]
will also prove useful. At order \( \hbar^3 \) in \( \{Z^i, Z^j\} \), we have
\[
(7.10) \quad \hbar^3 \left( \{z^i, Z_2^j\} - i \leftrightarrow j \right) - \frac{1}{24} \{z^i, z^j\}_3.
\]
According to Eq. (7.5), this must vanish. Working with the \( Z_2^i \) terms, we have
\[
\{z^i, Z_2^j\} - i \leftrightarrow j = \frac{1}{48} \{z^i, \Gamma_{abc}\{z^j, \Gamma^{abc}\}\} - i \leftrightarrow j
\]
\[
= \frac{1}{48} \left( \{z^i, \Gamma_{abc}\}\{z^j, \Gamma^{abc}\} + \Gamma_{abc}\{z^i, \{z^j, \Gamma^{abc}\}\} \right) - i \leftrightarrow j
\]
\[
= \frac{1}{48} \{z^i, \Gamma_{abc}\}\{z^j, \Gamma^{abc}\} - i \leftrightarrow j
\]
\[
= \frac{1}{24} \{z^i, \Gamma_{abc}\}\{z^j, \Gamma^{abc}\}
\]
\[
(7.11) = \frac{1}{24} \{z^i, z^j\}_3
\]

We have used the Jacobi identity, the identity (7.8), and the resolution of the arrow. Equation (7.5) is verified.

We now have an explicit form for the first two terms in \( Z^i = z^i + \hbar^2 Z_2^i + \ldots \). According to Theorem 1, the remaining terms in the \( Z^i \) series can be completed so that the \( Z^i \) are s’Darboux coordinates. To make the forthcoming analysis clearer, we shall now assume this has been done, although none of the results depend on the specific form of \( Z_4^i \), etc.
7.3. **Good number operators.** Associated to our $Z^i$ are number operators $N^i$. They are easily expressed in terms of $A^i = a^i + \hbar^2 A_2^i + O(\hbar^3)$.

\[
N^i = \hat{A}^i \star A^i
\]

\[
= \hat{a}^i a^i + i\hbar \{ \hat{a}^i, a^i \} - \frac{\hbar^2}{8} (\hat{a}^i, a^i)_2 + \hbar^2 (\hat{A}_2^i a^i + a^i \hat{A}_2^i) + O(\hbar^3)
\]

\[
= I^i - \frac{\hbar}{2} - \frac{\hbar^2}{8} (\hat{a}^i, a^i)_2 - \frac{\hbar^2}{48} \Gamma_{abc} I^i \rightarrow \Gamma^{abc} + O(\hbar^3)
\]

The last two terms define

\[
N_2^i = -\frac{1}{8} (\hat{a}^i, a^i)_2 - \frac{1}{48} \Gamma_{abc} I^i \rightarrow \Gamma^{abc}.
\]

The appearance of an $O(\hbar)$ term \(^{11}\) means that will be more convenient to develop $F^i$ as $H^i = F^i \tilde{\delta} (N^i + \hbar/2)$.

Already, we have \(\{N^i, H^j\}_* = O(\hbar^3)\). We would like to modify $Z^i$ so that $\{N^i, H^j\}_* = O(\hbar^3)$. Taking $F^i = f^i$, \(^{12}\) we have $H^i - F^i \tilde{\delta} (N + \hbar/2) = O(\hbar^3)$.

By the second part of Theorem 2, $H^i = f^i \tilde{\delta} (N + \hbar/2) + \hbar^2 G^i$ for $G^i$ in which $\pi G^i = \{G, h^i\} + g^i \circ I$. The \{G, h^i\} term in $\pi G^i$ may be isolated subtracting the angle independent terms from $H^i - f^i \tilde{\delta} (N + \hbar/2)$.

Defining a complement to the angle average by $\langle f \rangle = f - \langle f \rangle$, we have $\{G, h^i\} = \langle H^i - f^i \tilde{\delta} (N + \hbar/2) \rangle$.

Using a more accurate expansion of $\tilde{\delta}$, \(^{[6]}\) we have

\[
H^i = f^i \tilde{\delta} (N + \hbar/2) = h^i - \frac{\hbar^2}{16} (I^j, I^k)_2 (\partial_j \partial_k f^i) \circ I
\]

\[
- \frac{\hbar^2}{24} (I^j \rightarrow I^k \leftarrow I^l) (\partial_j \partial_k \partial_l f^i) \circ I + \hbar^2 N_2^i (\partial_j f^i) \circ I + O(\hbar^4)
\]

\[
= \hbar^i + \hbar^2 K_2^i + \hbar^2 \Omega_{ij} N_2^j + O(\hbar^4),
\]

where, for convenience, we have defined the frequency matrix $\Omega_{ij} = (\partial_j f^i) \circ I$ and

\[
K_2^i = \frac{\hbar^2}{16} (I^j, I^k)_2 (\partial_j \partial_k f^i) \circ I + \frac{\hbar^2}{24} I^j \rightarrow I^k \leftarrow I^l (\partial_j \partial_k \partial_l f^i) \circ I.
\]

We now have

\[
\{G, h^i\} = \frac{\partial G}{\partial \theta^k} \Omega_{ik} = \langle H^i_2 - K_2^i \rangle \Omega_{ij} N_2^j.
\]

and may solve for $\partial G/\partial \theta^k$:

\[
\frac{\partial G}{\partial \theta^i} = \langle \Omega_{ik}^{-1} (H^k_2 - K_2^k) \rangle \langle - \rangle N_2^j \rangle.
\]

\(^{11}\)This term is, of course, related to the Maslov index.

\(^{12}\)Recall that $f^i$ is $h^i$ expressed as a function of its action variables, i.e. $f^i \circ I = h^i$. 
As in Theorem 4 we define $Z^i = z^i + h^2 \{ \gamma, z^i \} + O(h^4)$, and choose $\gamma = G$, so that $\{ A^0 \ast A^0, H^1 \}_* = O(h^5)$. The new number operators are given by Eq. (6.11):

$$N^i = n^i + h^2 N^i_2 + O(h^4)$$

(7.18)

$$= \dagger f + h^2 \partial_G + h^2 \f \frac{\partial G}{\partial \theta}$$

$$= \dagger f + h^2 \bar{N}^i_2 + \f \partial_G (H^2_2 - K^2_2) + h^2 \ partial G \partial \theta$$

$$= \dagger f + h^2 \bar{N}^i_2 + \f \partial_G (H^2_2 - K^2_2)$$

Note that $\langle N^i_2 \rangle = \bar{N}^i_2$.

Finally, we can obtain the quantization condition. We require

$$H^{i} = h^{i} + h^{2} H^{2}_{2} + O(h^{4}) = F^{i} \delta (N^{i} + h^{2}/2) + O(h^{4})$$

(7.19)

$$= f^{i} \delta (N^{i} + h^{2}/2) + h^{2} F^{2} + O(h^{4})$$

$$= h^{i} + h^{2} K^{i}_{2} + h^{2} \Omega_{ij} N^{j}_{2} + h^{2} + h^{2} F^{i} \circ I + O(h^{4})$$

By the work of Theorem 2 we know this equation is satisfied by taking

$$F^{2} \circ I = \langle H^{2}_{2} - K^{i}_{2} - \Omega_{ij} N^{j}_{2} \rangle$$

(7.20)

$$= \langle H^{2}_{2} - K^{i}_{2} - \Omega_{ij} N^{j}_{2} \rangle.$$  

$F = f^{i} + h^{2} F^{2}$ is the second order Bohr-Sommerfeld quantization rule for quantum integrable systems of Moyal symbols. Equation (7.20) contains two important terms: the $K^{i}_{2}$ comes from knowing the symbol of a function of an operator, while the $N^{i}_{2}$ term comes directly from the $s'$Darboux construction.

8. Construction of s'Darboux coordinates in the Fedosov star product

Let $\mathcal{M}$ be a manifold with symplectic form $\omega$ and a torsionless symplectic connection $\nabla$, i.e. one with $\nabla \omega = 0$. From $\omega, \nabla$, Fedosov constructs a star product. 2 To summarize his result, given any $f \in C^{\infty}(\mathcal{M})$ and Darboux local coordinates $z^{i}$, there are functions $f^{(n)}_{i_{1} \ldots i_{n}}$ which are completely symmetric in the numbered indices 14 such that Fedosov's star product is

(8.1)

$$f \star g = f^{(0)} g^{(0)} + \f h \frac{i}{2} f^{(1)}_{11} \omega^{i_{j} i_{j}} g^{(1)}_{j_{j}} + \ldots + \frac{1}{n!} \left( \frac{i \hbar}{2} \right)^{n} f^{(n)}_{i_{1} \ldots i_{n}} \omega^{i_{1} j_{1} \ldots i_{n} j_{n}} g^{(n)}_{j_{1} \ldots j_{n}} + \ldots$$

where $\omega^{ij}$ is the Poisson tensor with respect to the $z^{i}$. For early $f^{(n)}$, the results are

$$f^{(0)} = f$$

$$f^{(1)}_{11} = \nabla_{11} f$$

$$f^{(2)}_{i_{1} i_{2}} = \nabla_{11} \nabla_{i_{2}} f$$

$$f^{(3)}_{i_{1} i_{2} i_{3}} = \nabla_{11} \nabla_{12} \nabla_{13} f - R_{11223} f \nabla_{j} f,$$

13Here we have one extra power of $\hbar$.

14For this section, we establish a convention that implicit any expression containing numbered indices is a symmetrization over those indices.
where the derivatives are taken with respect to the \( x^i \) coordinates, and where \( R_{i j k}^l = \partial_i \omega^l_{jk} - \partial_j \omega^l_{ik} + \partial_k \omega^l_{ij} - \partial^l \omega_{ijk} \) is the Riemann curvature tensor in those coordinates.\(^{15}\)

Given Darboux coordinates \( z^a \), we wish to construct s’Darboux symbols \( Z^a = z^a + \hbar^2 Z^a \). It is natural to guess that \( Z^a \) is as in Eq. (7.7), but with \( \Gamma \) the new, possibly curved, symplectic connection written in the \( z^a \) coordinates. We shall verify this.

First, we need to prove a generalization of Eq. (7.8):

\[
\nabla_a \nabla_1 \nabla_2 \nabla_3 z^d - R_{a_1 a_2 a_3} = -\partial^d \Gamma_{a_1 a_2 a_3}
\]

where \( \Gamma^{abc} \) are the coefficients of the symplectic connection in \( z^a \) coordinates.

To prove this, we need only the fact that when \( \Gamma^{abc} \) for a symplectic connection is written in Darboux coordinates, it is completely symmetric in its indices.\(^{2}\) Let \( \partial_a = \partial/\partial z^a \). First, we have

\[
\begin{align*}
\nabla_a \nabla_1 \nabla_2 \nabla_3 z^d &= \nabla_a \nabla_2 \delta^d_{a_3} \\
&= -\nabla_a \Gamma^{d}_{a_3 a_2} \\
&= -\partial_a \Gamma^{d}_{a_3 a_2} + \Gamma^{\sigma}_{a_1 a_3} \Gamma^{d}_{\sigma a_2} + \Gamma^{\sigma}_{a_1 a_2} \Gamma^{d}_{\sigma a_3} \\
&= -\partial_a \Gamma^{d}_{a_3 a_2} + 2\Gamma^{\sigma}_{a_1 a_3} \Gamma^{d}_{\sigma a_2} 
\end{align*}
\]

where, in the last step, we have used the implicit symmetrization on numbered indices and the symmetry of \( \Gamma^{abc} \). Using the same property of \( \Gamma^{abc} \), we also have

\[
R_{a_1 a_2 a_3} = -\partial_a \Gamma^{d}_{a_3 a_2} + \partial^d \Gamma_{a_1 a_2 a_3} + \Gamma^{\eta}_{a_1 a_3} \Gamma^{d}_{\eta a_2} + \Gamma^{\eta}_{a_1 a_2} \Gamma^{d}_{\eta a_3} \\
= -\partial_a \Gamma^{d}_{a_3 a_2} + \partial^d \Gamma_{a_1 a_2 a_3} + 2\Gamma^{\eta}_{a_1 a_3} \Gamma^{d}_{\eta a_2}.
\]

Subtracting these two results, we obtain Eq. (8.3). By a calculation very similar to the one in section 7.2, we can show that

\[
Z^d = z^d + \frac{\hbar^2}{48} \Gamma_{abc} \{ z^d, \Gamma^{abc} \}
\]

satisfy \( \{ Z^i, Z^j \} = i\hbar J^{ij} + O(\hbar^3) \).

Following our previous development, we obtain the EBK formula in the Fedosov quantization by replacing the \( \Gamma \) in Eq. (7.20) with this new \( \Gamma \), and by changing the ordinary derivatives to covariant derivatives.

9. Conclusion

In this paper, we have developed a method for calculating spectra of functions which comprise a quantum integrable system under a star product.

The methods developed here may have other applications. A future, more official, presentation may also include thoughts about

- star product quantization rules on cohomologically non-trivial manifolds;
- star Lie algebras from Poisson Lie algebras and applications to the semiclassics of spin;
- Heisenberg evolution of s’Darboux coordinates.

\(^{15}\)It seems Fedosov’s own expression for \( f^{(3)} \) contains an error. He has 1/4 as the coefficient of \( R \); we think it is just 1.
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