A Knapsack Intersection Hierarchy Applied to All-or-Nothing Flow in Trees

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Abstract. We introduce a natural knapsack intersection hierarchy for strengthening linear programming relaxations of packing integer programs, i.e., $\max\{w^T x : x \in P \cap \{0,1\}^n\}$ where $P = \{x \in [0,1]^n : Ax \leq b\}$ and $A, b, w \geq 0$. The $t$th level $P^t$ corresponds to adding cuts associated with the integer hull of the intersection of any $t$ knapsack constraints (rows of the constraint matrix). This model captures the maximum possible strength of “t-row cuts”, an approach often used by solvers for small $t$. If $A$ is $m \times n$, then $P^m$ is the integer hull of $P$ and $P^1$ corresponds to adding cuts for each associated single-row knapsack problem. Thus, even separating over $P^3$ is NP-hard. However, for fixed $t$ and any $\epsilon > 0$, results of Pritchard imply there is a polytime $(1 + \epsilon)$-approximation for $P^t$. We then investigate the hierarchy’s strength in the context of the well-studied all-or-nothing flow problem in trees (also called unsplittable flow on trees). For this problem, we show that the integrality gap of $P^t$ is $O(n/t)$ and give examples where the gap is $\Omega(n/t)$. We then examine the stronger formulation $P^t_{\text{rank}}$ where all rank constraints are added. For $P^t_{\text{rank}}$, our best lower bound drops to $\Omega(1/c)$ at level $t = n^c$ for any $c > 0$. Moreover, on a well-known class of “bad instances” due to Frieze and Gao, we show that we can achieve this gap; hence a constant integrality gap for these instances is obtained at level $n^c$.

1 Introduction

In this paper we study linear relaxations for packing integer programs (PIP). A PIP is described by a 0-1 optimization problem $\max\{w^T x : x \in \{0,1\}^n, Ax \leq b\}$, where $w \in \mathbb{Z}_+^n$, $A \in \mathbb{Z}_+^{m \times n}$, and $b \in \mathbb{Z}_+^m$. These integer programs capture well-known problems such as 0-1 knapsack, matroid optimization, maximum stable set, demand matching and all-or-nothing flow in trees (also called unsplittable flow on trees). PIPs are also called 0-1 multidimensional knapsack problems in the case where $m$ is fixed. We introduce a hierarchy of strengthened PIP formulations where level $t$ is defined by adding cuts associated with the integer hulls of all intersections of $t$ constraints. This knapsack intersection hierarchy is inspired by successful computational approaches; in the case of a single constraint it corresponds to the cuts added in the pioneering work of Crowder, Johnson, and Padberg [CJP83]. We evaluate the strength of this hierarchy applied to the well-studied “all-or-nothing flow” problem in trees (ANF-Tree). This problem generalizes weighted matching but has no known polytime $O(1)$-approximation. In this section we formally define the hierarchy and in the next section we discuss our results for ANF-Tree.

PIPs generally have a (one or more) natural linear relaxation $P := \{x \in [0,1]^n : Ax \leq b\}$, where $A, b \geq 0$. The discrete problem of interest is to optimize over the integer hull $P_I := \text{conv}(P \cap \{0,1\}^n)$. Computational solution strategies for PIPs often use some form of branch and cut method, and one of the most effective approaches is to rely on cuts for the knapsack polytopes associated with individual constraints [CJP83]. Let $a^j$ be row $j$ of $A$ and $b^j$ be element $j$ in $b$. For each $j \in [m]$, let $K^j$ denote the polytope $\{x \in [0,1]^n : \sum_i a^j_i x_i \leq b^j\}$. The knapsack cuts for $K^j$ are the inequalities which are valid for the integer hull $K_I(j) := \text{conv}(K^j \cap \{0,1\}^n)$, i.e., the knapsack polytope for constraint $j$. On each iteration of a branch and cut approach (e.g., see [Sch98]), one has a feasible—but fractional—solution $\bar{x}$ for a current relaxation $P'$ of $P_I$. In [CJP83], they generate knapsack cuts for some constraint. That is, for some $j$, they find a valid inequality $c^T x \leq d$ for $K^j$ for which $c^T \bar{x} > d$. Adding such inequalities to $P'$ gives a tighter formulation for $P_I$ on which to recurse.

This approach has also been extended to multi-row cuts. This can be set up in various ways, e.g.: (1) by aggregating multiple constraints to form a single inequality and then generating cuts for the associated
knapsack [DLTW14, Xav17, DM18] and (2) by considering cuts associated with the integer hull of the intersection of several knapsack polytopes [LW08, KPP04]. The latter set-up is potentially stronger in the following sense: there are instances where adding all cuts of type (2) defines the integer hull but adding (any number of) cuts of type (1) does not.

We discuss a framework to measure the strength of cuts in the latter setting. For some \( S \subseteq [m] \), we denote the intersection of the fractional knapsack polytopes associated with constraints in \( S \); we define \( K(S) := \cap_{j \in S} K(j) \). We consider a relaxation where all cuts are added for the associated integer hull \( K_t(S) \). We then define a knapsack intersection hierarchy for \( P_t \) as follows. For each \( t \in [m] \), define

\[
P^t := \bigcap_{|S|=t} K_t(S).
\]

In other words, \( P^t \) is obtained from \( P \) by adding, for each \( S \subseteq [m] \) with \( |S| = t \), all valid inequalities for \( K_t(S) \). Clearly, \( P^{t+1} \subseteq P^t \) and \( P^m = P_t \), so we have the following hierarchy:

\[
P \supseteq P^1 \supseteq \ldots \supseteq P^{m-1} \supseteq P^m = P_t.
\]

Separating over \( K_t(S) \) is NP-Hard given that, even for \( t = 1 \), 0-1 knapsack is a special case, Hence it is already NP-Hard to separate over \( P^1 \), the first level of the hierarchy; this fate is shared by a different hierarchy, since the Chvátal closure of a polyhedron is NP-hard to separate [Eis99]. To mitigate this, we show that results of Pritchard [Pri10] lead to a tractable formulation, that is, one that is polynomially sized, but approximate, when \( t \) is constant.

**Theorem 1.** For \( 0 < \epsilon \leq 1 \), there is an approximate formulation for \( P^t \) of size \( O(n^3 \epsilon^{-1} t^4 + 1) \) for which the value of an optimal solution is at most a \((1/(1-\epsilon))\)-factor larger than the optimal solution to \( P^t \).

**Corollary 1.** For fixed \( t \) there is a PTAS for \( \max\{w^T x : x \in P^t\} \).

We defer the proofs to [Section 2]. We now discuss the impact of the knapsack hierarchy on formulations for ANF-Tree.

### 1.1 All-or-Nothing Flow in Trees

The all-or-nothing flow problem [CMS07] is defined for a multiflow problem whose input is a supply graph \( G \) and demand graph \( H \). \( G \) and \( H \) may also be endowed with edge capacities \( u_e : e \in E(G) \) and demands \( d(f) : f \in E(H) \). We call \( E(H) \) the requests and a subset \( R \) of requests is routable if there is a (fractional) multiflow which routes the requests in \( R \) using \( G \)'s capacity. The problem is “all-or-nothing” in the sense that if \( f \in R \), then we must route the whole \( d(f) \) units of demand. An instance is said to satisfy the no-bottleneck-assumption (NBA) if \( d(f) \leq u_e \) for every request \( f \) and supply edge \( e \). ANF-Tree is the special case of all-or-nothing flow where the supply graph \( G \) is a tree.

When the NBA holds, there is a polylog approximation in general graphs [CKS13] and a 48-approximation in trees [CMS07]. Without the NBA, however, the natural LP has a super-constant integrality gap. The first theoretical progress for the non-NBA setting was a quasi-PTAS when the supply graph is a path [BCE06].

A sequence of papers has ultimately yielded a constant-factor approximation (and integrality gap) for paths, the best of which is an \( O(1 + \frac{1}{t+\epsilon}) \)-approximation [GMW20], and an LP with integrality gap \( 7 + \epsilon \) [BSW14]. For trees, however, the strongest result is an \( O(\log^2 n) \)-approximation [CEK08, FG15, ACEW16]. It remains an open question whether ANF-Tree has an \( O(1) \)-approximation (or even an \( O(\log n) \)-approximation), and whether ANF-Path has a PTAS.

For trees we use the following notation. An instance \( I = (T, R) \) of ANF-Tree consists of an undirected capacitated tree \( T = (V, E, u) \) and a set of requests \( R \), defined as follows. \( V \) is the set of vertices and each edge \( e \in E \) has some positive capacity \( u_e \). Each request \( r \in R \) imposes some non-negative demand \( d_r \) on all edges along the unique simple path \( P_r \) between \( s_r \in V \) and \( t_r \in V \). A request may also have a profit \( w_r \). We assume that \( s_r \neq t_r \) for all \( r \in R \). For each edge \( e \), let \( R_e = \{ r \in R : e \in P_r \} \). We denote \( k = |R| \) and

\[\] A well-known example for the Chvátal rank actually shows that one may need an unbounded number of rounds of aggregated cuts in order to obtain the integer hull (e.g., see Section 23 in [Sch98]).
A subset $S \subseteq R$ of requests is feasible or routable if, for each edge $e \in E$, the total demand of all requests $r \in S \cap R_e$ is at most the capacity $u_e$. The goal is to select a feasible subset $S \subseteq R$ which maximizes the profit $\sum_{r \in S} w_r$. We formalize this with the following IP.

$$\begin{align*}
\text{max} & \quad w^T x \\
\text{such that} & \quad \sum_{r \in R_e} d_e x_r \leq u_e \quad \forall e \in E \\
& \quad x_r \in \{0, 1\}^k \quad \forall r \in R
\end{align*}$$

ANF-IP

The natural LP relaxation ANF-LP is defined by replacing $x \in \{0, 1\}^k$ with $x \in [0, 1]^k$; ANF-Path is defined similarly.

One approach for strengthening ANF-LP is to add rank constraints [CEK09, FG15]. For $S \subseteq R$ its rank is defined as $\text{rank}(S) := \max\{|T| : T \subseteq S \text{ and } T \text{ is feasible}\}$, and the rank constraint is then $\sum_{i \in S} x_i \leq \text{rank}(S)$. Adding all such inequalities to ANF-LP defines the Rank-LP. We denote by $P_{\text{rank}}$ the polytope obtained from adding all rank constraints to an ANF-Tree relaxation $P$. Rank-LP is NP-Hard to separate, but it can be $O(1)$-approximated [CEK09, FG15].

We summarize the known results for these general ANF-Tree formulations.

**Theorem 2.**

1. The integrality gap of ANF-LP is $\Omega(k)$ [CCGK02].
2. For ANF-Path, the integrality gap of Rank-LP is $O(\log k)$ and the best known lower bound is $\Omega(1)$ [CEK09].
3. For ANF-Tree, the integrality gap of Rank-LP is $O(\log^2 k)$ and the best known lower bound is $\Omega(\sqrt{\log k})$ [FG15].

1.2 Knapsack Hierarchy and Strengthening ANF-Tree Relaxations

In the rest of the paper, $P$ refers to the feasible region of ANF-LP and hence $P_{\text{rank}}$ refers to same for Rank-LP. The first result shows the general dependence of the integrality gap for $P^t$ on $k$ and $t$; this is similar to the Sherali-Adams hierarchy [CEK09] (although the proofs are not similar). The upper bound part is proved in Lemma 1 and the lower bound is proved in Lemma 5 - see Section 3.

**Theorem 3.** For ANF-Tree, the integrality gap of $P^t$ is $O(k/t)$ and there are instances where it is $\Omega(k/t)$.

We now focus on the Friggstad-Gao instances (definition in Section 3). These instances established the $\Omega(\sqrt{\log k})$ ANF-Tree lower bound in Theorem 2; this is significant since it established a super-constant lower bound even when all rank constraints are added. Furthermore, these are single-sink instances, i.e., all requests share one common endpoint. We establish the following lower bounds for the knapsack hierarchy on the Friggstad-Gao instances.

**Theorem 4.** For constant $t$, the integrality gap of $P_{\text{rank}}^t$ is $\Omega(\sqrt{\log k})$. For any $c > 0$, the integrality gap of both $P_{\text{rank}}^{kc}$ and $P^t_{\text{rank}}$ is $\Omega(1/c)$.

Interestingly, the proof of this lower bound depends on an upper bound proof. Namely, we define a colouring problem for which we upper bound the chromatic number (see Section 4.2). We can also show that our analysis of $P_{\text{rank}}^t$ on the Friggstad-Gao instances is tight in the following sense.

**Theorem 5.** On the class of Friggstad-Gao instances, for any $c > 0$, the integrality gap of both $P^{kc}$ and $P_{\text{rank}}^{kc}$ is $O(1/c)$.

It remains an intriguing question whether this constant integrality gap of $O(1/c)$ holds for general tree instances, even in the single-sink scenario.
1.3 Related Work

The use of hierarchies for integer programs dates back to the notion of Chvátal rank \[Chv73\]. The \textit{Chvátal closure} of a polyhedron \( P \) is the polyhedron \( P' \subseteq P \) which is defined by the system of Chvátal-Gomory cutting planes obtainable from \( P \). If we denote \( P^1_C = P' \) then the hierarchy is generated by \( P^{t+1}_C = (P^t_C)' \). Chvátal proved that \( P \supseteq P^1_C \supseteq \ldots \supseteq P^{t-1}_C \supseteq P^0_C = P \). As discussed, it is NP-hard to separate over \( P^1 \), both in the Chvátal hierarchy and the knapsack hierarchy considered in this paper. Other hierarchies have since been introduced and widely studied, such as the hierarchy defined by the split closure \[CKS90\] and hierarchies introduced by Lovász-Schrijver \[LS91\], Sherali-Adams \[SA90\], Parillo \[Par03\], and Lasserre \[Las01\]. These other hierarchies also have that the integer hull is obtained after \( n \) rounds of the hierarchy, where \( n \) is the number of variables. In that sense, the knapsack hierarchy is different since \( P^m = P_1 \) where \( m \) is the number of constraints. For ANF-tree formulations, however the number of variables equals \( k \), the number of requests. Moreover, for ANF-Tree instances one may show that \( m \leq 4k \) (see Appendix A.3 in \[FG15\]). Hence the knapsack hierarchy is equal to the integer hull at level \( O(k) \) for ANF-Tree.

We summarize the existing work on the effectiveness of classical hierarchies on ANF-Tree. Friggstad and Gao showed that the Lovász-Schrijver hierarchy is ineffective at reducing the integrality gap of ANF-Tree after 2 rounds and amounts to adding the rank one constraints \[FG15\]. Additionally, Chekuri, Ene, and Korula prove that after applying \( t \) rounds of the Sherali-Adams hierarchy to ANF-LP, the integrality gap is \( \Omega(k/t) \) \[CEK99\], matching the result for our hierarchy. For the case of 0-1 knapsack, Karlin, Mathieu, and Nguyen show that \( t^2 \) rounds of Lasserre reduce the integrality gap to \( t/(t-1) \) \[KMN10\]. We are not aware of any work done on whether this would generalize to ANF-Tree.

In the remainder of the paper we introduce the well-known “bad gap instances” for ANF: in particular, the so-called staircase and Friggstad-Gao instances (in Section 3). We then prove our results (in Sections 2, 4 and 5), and discuss future work (in Section 6).

2 Preliminary Proofs

We begin by showing \[Theorem 1\], establishing tractability of our hierarchy for constant \( t \).

\textit{Proof (Theorem 1).} We use a result by Pritchard which gives a \((1-\epsilon)\)-approximate extended formulation for \( K_f(S) \) with size \( O(n^{1+t^2\epsilon^{-1}}) \) \[Pri10\]. For \( 0 < \epsilon \leq 1 \), denote the projection of this extended formulation onto \( \mathbb{R}^n \) by \( K_f(S) \). Furthermore, denote by \( P^t \) the polytope \( \bigcap_{|S|=t} K_f(S) \). Since \( K_f(S) \) is a \textit{polyhedral approximation}, we have \((1-\epsilon)K_f(S) \subseteq K_f(S) \subseteq K_f(S) \). It follows that

\[
(1-\epsilon)P^t = (1-\epsilon) \bigcap_{|S|=t} K_f(S) = \bigcup_{|S|=t} (1-\epsilon)K_f(S) \subseteq \bigcap_{|S|=t} K_f(S) = P^t
\]

and

\[
P^t = \bigcap_{|S|=t} K_f(S) \subseteq \bigcup_{|S|=t} K_f(S) = P^t.
\]

Therefore, \( P^t \) is a polyhedral \((1-\epsilon)\)-approximate extended formulation for \( P^t \). There are \( \binom{n}{t} \) \( O(n^t) \) \( S \) with \(|S|=t \), so since each \( K_f(S) \) has size \( O(n^{1+t^2\epsilon^{-1}}) \), \( P^t \) has size \( O(n^{t^2\epsilon^{-1}}) \cdot O(n^t) = O(n^{t^3\epsilon^{-1}+t+1}) \) as desired. \( \Box \)

The proof of the upper bound in \[Theorem 3\] uses a strategy which appears again in the proof of \[Theorem 5\]: pick some \( x \in P^t \) and partition the requests into sets \( S_1, \ldots, S_q \) such that the profit of each \( x_{S_i} \) (i.e., the vector \( x \) but with elements not in \( S_i \) set to zero) can easily be bounded, thus establishing a bound on the profit of \( x_{S_1} + \cdots + x_{S_q} = x \).

\textbf{Lemma 1.} For ANF-Tree, the integrality gap of \( P^t \) is \( O(k/t) \).

\textbf{Proof.} Let \( \text{OPT} \) be the optimal value of \( \max \{ w^T x : x \in P_1 \} \). Consider some set \( S \subseteq R \) with \(|S| \leq t/4 \). It can be shown that for any ANF-Tree instance there exists an equivalent instance (in the sense of having the
same integer hull) with \( m \leq 4k \) (see Appendix A.3 in [FG15]), so we assume w.l.o.g. that \( m \leq 4k \). So, if the problem is reduced to contain only the requests in \( S \), then at most \( t \) edges are needed to define the integer hull. Let \( T \) be this set of at most \( t \) edges. Then, we must have \( \max\{w^T x : x \in K_I(T), x_{R|S} = 0\} \leq \text{OPT} \) because any such \( x \) is in \( P_t \).

We can arbitrarily partition the requests into \( q := O(\frac{k}{4t}) \) such sets \( S_1, \ldots, S_q \) (i.e., with \( S_i \subseteq R \) and \( |S_i| \leq t/4 \)) with corresponding edge sets \( T_1, \ldots, T_q \) (i.e., where \( |T_i| \leq t \) and \( T_i \) defines the integer hull for requests \( S_i \)). Then \( \max\{w^T x : x \in P^k\} \leq \sum_{i=1}^{q} \max\{w^T x : x \in K_I(T_{i}), x_{R|S_i} = 0\} \leq O(\frac{k}{4t}\text{OPT}) \), so the integrality gap is \( O(k/t) \) as desired. \( \square \)

3 ANF-Tree Preliminaries

3.1 Staircase Instances

For \( k \geq 1 \), we define the staircase\(^4\) ANF-Path instance \( S^k = (T, R) \) as follows. Let \( T \) be a path graph on \( k + 1 \) vertices, that is, \( V = \{1, \ldots, k + 1\} \) and \( E = \{(1, 2), (2, 3), \ldots, (k, k + 1)\} \). We refer to vertex \( 1 \) as the root or \( r \). For each \( i = 1, \ldots, k \), define \( u_{i,i+1} = 2^{i-1} \) and create a request \( i \) with \( s_i = i \), \( t_i = r \), \( d_i = 2^{i-1} \), and \( w_i = 1 \). See Fig. 1 for an illustration. These instances were first described by Chakrabarti, Chekuri, Gupta, and Kumar [CCGK02].

3.2 Friggstad-Gao Instances

In this section, we describe the family of Friggstad-Gao ANF-Tree instances, which were introduced in [FG15].

We define the tree \( T^h_{FG} \) with height \( h \geq 2 \) as follows. There is a root vertex \( r \) which has a single child \( v_1 \). Apart from \( r \) and the leaves (which are in level \( h \)), all vertices have \( 2^{h-1} \) children. We denote the set of vertices with distance \( \ell \) from \( r \) by \( \text{level}_\ell \), that is, \( \text{level}_0 = \{r\} \), \( \text{level}_1 = \{v_1\} \), and for \( \ell \in [h] \), \( |\text{level}_\ell| = 2^{h-1}(\ell-1) \).

For each edge \( e = uv \) with \( u \in \text{level}_{\ell-1} \) and \( v \in \text{level}_{\ell} \), define \( u_v = 2^{h(\ell-1)} \). For all \( \ell \geq 1 \) and each vertex \( v \in \text{level}_\ell \), create a request associated with \( v \) with \( s_v = v \), \( t_v = r \), demand \( d_v = 2^{h(\ell+1)} - 2^{h(\ell-\ell)} \), and profit \( w_v = 2^{-(h-1)(\ell-1)} \). See Fig. 2 for an example. This defines a single-sink instance since every request terminates at \( r \). Moreover, since the profit of each request in any level is the inverse of the number of requests in that level, the total profit of requests in any level is exactly \( 1 \). A simple calculation shows that the number of requests (and equivalently the number of edges) in levels \( 0 \) through \( \ell \) is

\[
n(\ell) = \sum_{i=0}^{\ell-1} (2^{h-1})^i = \frac{2^{(h-1)(\ell-1)} - 1}{2^{h-1} - 1} = \Theta \left( 2^{(h-1)(\ell-1)} \right).
\]

\(^4\) In the literature, this instance is referred to as a staircase because of a common way of visualizing ANF-Path instances where the capacity is plotted above the vertices on the Y axis.
Thus, \( h = \Theta(\sqrt{\log k}) \) where \( k = n(h) \) is the total number of requests/edges, and for any \( \ell \) we have \( \ell = \Theta \left( \frac{\log n(\ell)}{h} \right) \).

The following lemmas establish some fundamental properties of these instances. We use \( T^{<v} \) to denote the requests in the subtree of \( T \) rooted at \( v \) with \( v \) itself removed.

**Lemma 2.** For any edge \( e = uv \) where \( u \in \text{level}_{\ell-1} \) and \( v \in \text{level}_\ell \) for some \( \ell \), the set of requests in \( T^{<v} \) is routable on \( e \). That is, \( 1_{T^{<v}} \in K_I(e) \).

**Proof.** In any level \( \ell' > \ell \), \( 2^{h(h-\ell'+1)} \) is an upper bound for the demand \( 2^{h(h-\ell')} - 2^{h(h-\ell')} \) of the requests. The number of vertices in \( \text{level}_k \) that are also in the subtree below \( e \) is \( 2^{(h-1)(\ell'-\ell)} \). Thus, the demand on \( e \) from routing all requests in \( \text{level}_\ell \) is at most \( 2^{(h-1)(\ell'-\ell)}2^{h(h-\ell'+1)} = 2^{h^2-h\ell'+\ell+1} \). Therefore, summing over all \( \ell' > \ell \), we have

\[
\sum_{\ell'=\ell+1}^{h} 2^{h^2-h\ell'+\ell+1} = 2^{h^2-h\ell+\ell+1} \sum_{\ell'=\ell+1}^{h} 2^{-\ell'} = 2^{h^2-h\ell+\ell+1}(2^{-\ell} - 2^{-h}) \leq 2^{h(h-\ell+1)} = u_e. \]

**Lemma 3.** Let \( r \) be the root of the tree \( T^{h}_{FG} \) and \( P \) be any path from \( r \) to a leaf. Then the demands for the requests of \( P \) form a routable set.

**Proof.** The requests associated with level \( \ell \) have demand \( 2^{h(h-\ell+1)} - 2^{h(h-\ell)} \), so the total demand of such a path is

\[
\sum_{\ell=1}^{h} 2^{h(h-\ell+1)} - 2^{h(h-\ell)} = 2^{h^2} - 2^{h(h-1)} + 2^{h(h-1)} - 2^{h(h-2)} + \ldots + 2^{2h} - 2^{h} + 2^{h} - 2^{0} = 2^{h^2} - 1.
\]

This is less than \( 2^{h^2} \), the capacity of the topmost edge. A similar argument shows that no other edges are violated by leveraging the self-similar structure of the tree.

**Lemma 4.** The vector \( \frac{1}{2} \) is in \( K_I(e) \) for every edge \( e \).

**Proof.** Consider any edge \( e = uv \) where \( u \in \text{level}_{\ell-1} \) and \( v \in \text{level}_\ell \). The only requests which route on \( e \) are those in the subtree rooted at \( v \). Therefore it is sufficient to show that \( b := \frac{1}{2}(1_{\{v\}} + 1_{T^{<v}}) \in K_I(v) \). Note that \( 1_{\{v\}} \) is in \( K_I(e) \) since \( d_v = 2^{h(h-\ell'+1)} - 2^{h(h-\ell)} < 2^{h(h-\ell+1)} = u_e \), and by **Lemma 2** we have that \( 1_{T^{<v}} \in K_I(e) \). It follows that any convex combination of these vectors, and hence \( b \), lies in \( K_I(e) \).
4 Integrality Gap Lower Bound

In Section 4.1, we prove a lower bound of $\Omega(k/t)$ on the integrality gap of $P^t$, matching the upper bound shown in Lemma 1 and thus proving Theorem 3. However, this lower bound does not hold for $P^t_{\text{rank}}$. To resolve this case, we show in Section 4.2 that on the Friggstad-Gao tree instances, for any $c > 0$ the integrality gap is reduced to $\Omega(1/c)$ for both $P^k$ and $P^k_{\text{rank}}$, despite that $P_{\text{rank}}$ has integrality gap $\Omega(\sqrt{\log k})$ for these instances.

In the following we assume that all requests are routable on their own, i.e., for each $r \in R$ and $e \in P_r$, $d_r \leq u_e$. We also assume that it is impossible to route all requests together, as the optimal solution would then be trivial.

4.1 Path Instances

For path instances, it is known that the integrality gap of $P_{\text{rank}}$ is $O(\log k)$ and it is conjectured to be $O(1)$ CEK09. However, the ANF-LP has an integrality gap of $\Omega(k)$, which is evidenced by the staircase instances $S^k$ CCGK02. We now prove the upper bound from Theorem 3 by showing that the integrality gap of $P^t$ is $\Omega(k/t)$.

Lemma 5. The integrality gap of $P^t$ is $\Omega(k/t)$.

Proof. Let $t > 1$. We show that $\frac{1}{t+1} \in P^t$ for instances $S^k$, as defined in Section 3.1. Let $S \subseteq E(S^k)$ with $|S| = t$. For each edge $(i, i+1) \in S$, request $i$ is feasible alone. All other requests are feasible together without violating this edge’s capacity, because any other request $j$ which routes on $(i, i+1)$ has demand $2^j$, edge $(i, i+1)$ has capacity $2^t$, and $\sum_{j=0}^{t-1} 2^j < 2^t$. These feasible sets define a partition of $R(S^k)$ into $t+1$ sets: a set for each of the requests with the same indices as the $t$ edges of $S$ and a set of all other requests. Since all of these sets are feasible, the indicator vector for each of these sets lies in $K_1(S)$. Since these sets partition $R(S^k)$, the vector $\frac{1}{t+1}$ is a convex combination of these sets, and hence $\frac{1}{t+1} \in K_1(S)$. Since this holds for every such $S$, we have $\frac{1}{t+1} \in P^t$ and its total profit is $\Omega(k/t)$, thus establishing the integrality gap.

4.2 Tree Instances

In this section, we prove Theorem 4, which gives a lower bound on the integrality gap of $P^t$ on instances $T := T_{FG}$. Recall that Lemma 4 establishes $1/2 \in P^t$ by proving that for each edge $e$, the 1/2 vector can be written as a convex combination of (incidence vectors of) two sets, each of which is routable on $e$. We generalize this to any value of $t$ by showing that for $1/c \in P^t$ for sufficiently small $c$, and thus the integrality gap is $\Omega(\sqrt{\log k/c})$.

Let $S \subseteq E(T)$. We call a set $X \subseteq R(T)$ $S$-routable if for each $S \subseteq R(T)$, $\sum_{i \in X \cap R(e)} d_e \leq u_e$. Our key structural result gives a condition when we can express a vector $1/c$ as a convex combination of $S$-routable sets.

We cast this convex combination question as a question of colouring the set of all requests. For $S \subseteq E(T)$, we define the $S$-chromatic number, denoted by $\chi(S)$, to be the minimum value $c$ such that $R(T)$ can be partitioned into $c$ sets, each of which is $S$-routable. Given such a partition, the vector $1/c$ is trivially a convex combination of the indicator vectors of the $S$-routable sets in the partition. Thus, if we can show that $\chi(S) \leq c$ for every $|S| = t$, we have guaranteed that $1/c \in P^t$. Hence, the integrality gap established by Friggstad and Gao decreases by at most a factor of $c/2$ for $P^t$, since the result of Friggstad and Gao is associated with the feasible vector $1/2 \in P^0$. In fact, the following holds even if we start with the stronger formulation $P_{\text{rank}}$; we explain why at the end of this section.

Observation 1. The integrality gap of $P^t$ is $\Omega(h/c)$, where $c(t) := \max \{ \chi(S) : S \subseteq R, |S| = t \}$.

Theorem 4 follows from the following proposition which the rest of this section is dedicated to proving.

Proposition 1. If $|S| \leq 2^{h(c-1)}$, then $\chi(S) \leq c + 1$.
layered colour class $X$ can be added to colour class 1. Let $X$ colour class 3, and so on up to colour class $c$.

Proof (Theorem 4). For constant $t$ and $S \subseteq R$ with $|S| = t$, $\chi(S) \leq 2$ for sufficiently large $h$. Thus, the integrality gap of $P^t$ is $\Omega(h) = \Omega(\sqrt{\log k})$.

Now consider some $d > 0$ and let $t = k^d$. Let $S \subseteq R$ with $|S| = t$. Then, if $|S| \leq 2^{h(c-1)}$, we have $c = \Omega(\log(k^d)/h)$. Hence, $\chi(S) = \Omega(\log(k^d)/h)$, so by Observation 1, the integrality gap is $\Omega(h^2/\log(k^d)) = \Omega(1/d)$.

To establish that this lower bound holds even when all rank inequalities are added, we use Theorem 5 from [FG13] which proves that $x/9$ satisfies all rank constraints if $x$ satisfies all valid constraints of the form $x_i + x_j \leq 1$; these are trivially satisfied by the vector $1/c$ for $c \geq 2$. \hfill \Box

Our proof of Proposition 1 is based on the following colouring result. The tree $T'$ plays the role of a subtree essentially induced by the edges from some set $S$ with $|S| = t$.

Lemma 6. Let $T'$ be a subtree of $T$ rooted at some vertex $v$. If each level of $T'$ has at most $2^{h(c-1)}$ vertices, then $V(T')$ can be partitioned into at most $c$ sets which are $E(T')$-routable.

Proof. We prove this by induction using a stronger induction hypothesis. Specifically, not only does the colouring exist but we may use the following special type of colouring. We define layers $L_k$ of $T'$ inductively where $L_1 = \{v\}$. For each $k \geq 1$, $L_{k+1}$ consists of the children of the requests in layer $L_k$ which are contained in $T'$. Then for each $i = 1, 2, \ldots, c$ we claim that $X_i = L_i \cup L_{i+c} \cup L_{i+2c} \cup \ldots$ is $E(T')$-routable. Hence, $X_1, X_2, \ldots, X_c$ is a valid $c$-colouring which we call layered. We claim that a layered colouring always exists for any such subtree $T'$. The base case is a single-vertex tree which is trivially true for any $c \geq 1$.

Now consider the children of $v$ in $T'$. Call these $v_1, v_2, \ldots, v_p$ and let $T_i$ be the subtrees of $T'$ associated with each $v_i$. By induction, each $T_i$ has a layered colouring which uses at most $c$ colours. Assume we have such a colouring and without loss of generality that each $v_i$ has colour class 2, the next layer below that has colour class 3, and so on up to colour class $c$, after which the next layer has colour class 1. We show that $v$ can be added to colour class 1. Let $X_1$ denote the union of the colour classes $i$ which occur for the $T_j$. Each layered colour class $X_i$ is $E(T_i)$-routable and thus is also $E(T')$-routable. Hence, it only remains to show that $X_1 \cup \{v\}$ is also $E(T')$-routable. Note that $X_1 \cup \{v\}$ consists of layers $L_1 \cup L_{c+1} \cup L_{2c+1} \cup \ldots \cup L_{qc+1}$ of $T'$ for some choice of $q$. Recall that Lemma 3 asserts that the requests along any path from $v$ to the leaves of $T$ is routable. We show that for all $i$, the total demand of requests of $L_{ic+1}$ is at most the demand of a single request in $L_{(i-1)c+2}$, so the total demand from requests in $X_1$ on any edge is at most the demand from

Fig. 3. A diagram to aid with understanding the proof of Lemma 6. The key observation is illustrated by the arrows on the left: the total demand of every request in the box at the tail of an arrow is at most the demand of a single request at the tip of that arrow.
routing a path from $v$ to a leaf, and thus is routable. See Fig. 3 for a visual depiction of this. Suppose this is not the case. By the self similarity of the tree, we can assume that the demand of a request in $L_{ic+1}$ is

$$2^{h-(i-1)c+1} - 2^{h-(i-1)c+1} = (2^h - 1)2^{h-(i-1)c}$$

and the demand of a request in $L_{(i-1)c+2}$ is

$$2^{h-((i-1)c+2)} - 2^{h-((i-1)c+2)} = (2^h - 1)2^{h-(i-1)c-2}.$$ 

Then, we have

$$|L_{ic+1}| \cdot (2^h - 1)2^{h-(i-1)c-1} > (2^h - 1)2^{h-((i-1)c-2)}$$

$$\iff |L_{ic+1}| > \frac{2^{h-(i-1)c-2}}{2^{h-(i-1)c-1}} = 2^{h(c-1)},$$

which contradicts our hypothesis. \qed

We now complete the proof of Proposition 1.

Proof. Let $v$ be the least common ancestor of the vertices which are incident to the edges in $S$. We now create a subtree $T'$ which is a sort of closure of $S$. $T'$ is obtained by adding edges to $S$ of any path between $v$ and some vertex incident to an edge $e \in S$. We also include the parent edge of $v$. We claim that $T'$ satisfies the hypothesis of Lemma 6. To see this, consider some level of $T'$ consisting of vertices $a_1, \ldots, a_p$. Let $E_i$ denote the set of edges which are either incident to vertex $a_i$ or lie in its subtree. Note that the $E_i$ are disjoint. Since each $a_i$ is either incident to an edge of $S$, or is the internal vertices of some path used to define the closure $T'$, it follows that $E_i \cap S \neq \emptyset$ for each $i$, and hence $p \leq \sum_{i=1}^p |E_i \cap S| \leq |S| \leq 2^{h(c-1)}$.

We now colour all the requests of $T$. We first invoke Lemma 6 to colour $R(T')$ using $c$ colours. We can partition $R(T) \setminus R(T')$ as $A \cup B$, where $B$ denotes the requests “below” $T'$ (their paths to the root of $T$ intersect $T'$) and $A$ denotes the remaining “above” requests. The set $B$ is $S$-routable by Lemma 2. Requests in the set $A$ do not even route on any edge of $S$. Hence, $A \cup B$ can be the $(c+1)^{st}$ colour class. \qed

5 Integrality Gap Upper Bound

In this section, we prove Theorem 5 namely that for instances $T_{FG}^h$ and $c > 0$, the integrality gap of both $P^k$ and $P_{\text{rank}}^k$ is $O(1/c)$.

Theorem 6. Let $\ell$ be the largest integer such that $n(\ell) \leq t$ (with $n(\ell)$ as defined in Section 3.2). The integrality gap for optimizing over $P^\ell$ (with profits defined in Section 3.2) for instances $T_{FG}^h$ is $O(h/\ell)$.

We saw in Section 3.2 that $\ell = \Theta(\log(n(\ell))/h)$ and $h = \Theta(\sqrt{\log k})$. For $c > 0$ and $t = k^c$, the theorem statement chooses $\ell = \Theta(\log(k^c)/h)$, so the integrality gap is $O(h/\ell) = O(1/c)$, proving Theorem 5.

We show a particular way to partition the requests of the tree into $O(h/\ell)$ sets, and then show that for each set the profit of any $x \in P^\ell$ which uses only the requests in that set is $O(1)$. Since the integral optimum for instances $T_{FG}^h$ is at most $2^{h/2}$, it follows that the integrality gap of $P^\ell$ is $O(h/\ell)$ on these instances. The proof relies on the self similar structure of the Friggstad-Gao instances, namely that every vertex except for the leaves and the root has exactly $2^{h-1}$ children and capacities and demands scale down by $2^h$ for each step away from the root.

For $v \neq r$ let $T_v^\ell$ be the subtree consisting of the first $\ell$ levels of the children of vertex $v$ along with the edge immediately above $v$. The edge immediately above $v$ has its upper endpoint outside of the subtree. We denote the edges of the subtree, vertices of the subtree, and requests with an endpoint inside the subtree by $E(T_v^\ell), V(T_v^\ell)$, and $R(T_v^\ell)$, respectively. For Friggstad-Gao instances, $|E(T_v^\ell)| = |V(T_v^\ell)| = |R(T_v^\ell)|$; we denote this size simply by $|T_v^\ell|$. Notice that we have $|T_v^\ell| \leq n(\ell)$ by self similarity, and this holds with equality unless $v$ is less than $\ell$ levels from the leaves. Since we assumed $n(\ell) \leq t$, we have $|T_v^\ell| \leq t$. For vectors $x \in \mathbb{R}^k$, we denote by $x_{T_v^\ell}$ the restriction of $x$ to those requests with an endpoint in $T_v^\ell$. 

We now define, for each \(0 \leq i < \lfloor h/\ell \rfloor\), a set of subtrees \(\mathcal{P}_i = \{T_v^\ell : v \in \text{level}_{i\ell + 1}\}\). Let \(x_{\mathcal{P}_i}\) denote the restriction of \(x\) to those requests with an endpoint in some \(T_v^\ell \in \mathcal{P}_i\). Observe that the union \(\mathcal{P} = \bigcup \mathcal{P}_i\) of these subtrees is a partition of \(T_{FG}^h \setminus \{r\}\) into edge and vertex disjoint subtrees. See Fig. 4 for a visual depiction of this. The following lemma bounds the profit obtainable using requests with an endpoint in some \(\mathcal{P}_i\).

**Lemma 7.** For any feasible vector \(x \in P^\ell\) we have \(w_{\mathcal{P}_i}^T x_{\mathcal{P}_i} \leq 2\) for all \(0 \leq i < \lfloor h/\ell \rfloor\).

**Proof.** Let \(T_v^\ell \in \mathcal{P}_i\). First we show that every feasible subset of \(R(T_v^\ell)\) has profit at most \(2^{-(h-1)i\ell+1}\). This follows by the self similarity of the instance; scaling all demands and capacities in \(T_v^\ell\) by \(2^{h\ell}\) and all profits by \(2^{(h-1)i\ell}\) produces a tree identical to \(T_v^\ell\); (recall \(v_1\) is the single child vertex of the root \(r\)). For instances \(T_{FG}^h\), every routable set has profit at most \(2^{(h-1)i\ell} [FG13]\), so if we only use requests in \(T_v^\ell\) the profit certainly must be less than 2. By scaling as necessary, it then follows that any feasible subset of \(R(T_v^\ell)\) has profit at most \(2^{-(h-1)i\ell+1}\), as desired.

Now, we show that to determine feasibility of a subset of \(R(T_v^\ell)\) it is sufficient to check only the capacity constraints of the edges \(E(T_v^\ell)\). If \(S\) is routable, then clearly no capacity constraints are violated, so assume conversely that \(S\) is not routable. By \[\text{Lemma 2}\] no edge which is outside of \(E(T_v^\ell)\) and is an ancestor (towards the root) of any edge in \(S\) has its capacity violated by routing all requests in \(T_v^\ell\). Furthermore, any other edge which is outside of \(E(T_v^\ell)\) is not routable by the requests in \(S\) and thus cannot be violated. Thus, in order for \(S\) to not be routable, the capacity of one of the edges in \(E(T_v^\ell)\) must be violated.

Since \(K_I(E(T_v^\ell))\) is an integer hull, any \(x \in K_I(E(T_v^\ell))\) can be written as a convex combination of integral vectors in \(K_I(E(T_v^\ell))\). We saw that to determine feasibility of a subset of \(R(T_v^\ell)\) it is sufficient to check the capacity constraints of edges in \(E(T_v^\ell)\). Thus, for \(x \in K_I(E(T_v^\ell))\) such that \(x \leq 1_{R(T_v^\ell)}\), we can write \(x\) as a convex combination of integral vectors \(1_S\) for routable sets \(S \subseteq R(T_v^\ell)\), which we know all have profit at most \(2^{-(h-1)i\ell+1}\). Given \(|T_v^\ell| \leq t\), any \(x \in P^\ell\) has \(x \in K_I(E(T_v^\ell))\), so \(w_{\mathcal{P}_i}^T x_{\mathcal{P}_i} \leq 2^{-(h-1)i\ell+1}\). Finally, \(|\mathcal{P}_i| = |\text{level}_{i\ell + 1}| = 2^{(h-1)i\ell}\), so we can conclude that \(w_{\mathcal{P}_i}^T x_{\mathcal{P}_i} \leq 2^{-(h-1)i\ell+1} \cdot 2^{(h-1)i\ell} = 2\). \(\square\)

**Proof (Theorem 6).** Let \(x \in P^\ell\). From \[\text{Lemma 7}\] we know that for each \(0 \leq i < \lfloor h/\ell \rfloor\) we have \(w_{\mathcal{P}_i}^T x_{\mathcal{P}_i} \leq 2\). Summing over all \(i\), we find that \(w^T x \leq 2\lfloor h/\ell \rfloor \leq 2h/\ell\). We know that the integral optimum is \(\Omega(1)\), so the integrality gap of \(P^\ell\) is \(O(h/\ell)\). Since the rank formulation is stronger than the natural LP formulation, the integrality gap of \(P^\ell_{\text{rank}}\) is \(O(h/\ell)\) as well. \(\square\)
6 Conclusion

It would be interesting to establish stronger links to existing hierarchies such as those given by Lasserre, Parillo, Lovász-Schrijver, Sherali-Adams, or Chvátal, or that induced by the split closure. In terms of achieving stronger approximations for ANF-Tree, we see two interesting directions. One is to consider a rank $t$ approximation $P^t$ based on intersecting a structured set of $t$-row cuts (as opposed to all possible $t$-row cuts, as we have done here). This may allow tractable formulations with larger values of $t$. A related idea is to consider the intersection of the integer hulls of sub-instances induced by keeping a subset of the requests (instead of keeping a subset of the edges). For example, to restrict to the set of requests which pass through at least one of some set of $t$ edges, as such instances are known to be easier to approximate [GMWZ17]. Lastly, the question of whether $P^*_{\text{rank}}$ has constant integrality gap for general ANF-Tree instances has so far eluded us; it remains a very interesting question.

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