Invariant differential operators for the Jacobi algebra $G_2$

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Abstract

In the present paper we construct explicitly the intertwining differential operators for the Jacobi algebra $G_2$. For the construction we use the singular vectors of the Verma modules over $G_2$ which we have constructed earlier. We construct the function spaces on which the operators act. We display two versions of the left (representation) action and the right action. The latter is inserted in the singular vectors to provide the intertwining differential operators.

1 Introduction

Consider a Lie group $G$, e.g., the Lorentz, Poincaré, conformal groups, and differential equations

$$\mathcal{I} f = j$$

which are $G$-invariant. These play a very important role in the description of physical symmetries - recall, e.g., the early examples of Dirac, Maxwell, d’Allember, equations and nowadays the latest applications of (super-)differential operators in conformal field theory, supergravity, string theory, see e.g. [1]. Naturally, it is important to construct systematically such invariant equations and operators.

To recall the notions, consider a Lie group $G$ and two representations $T, T'$ acting in the representation spaces $C, C'$, which may be Hilbert, Fréchet, etc. An invariant (or intertwining) operator $\mathcal{I}$ for these two representations is a continuous linear map

$$\mathcal{I} : C \rightarrow C'$$

such that

$$T'(g) \circ \mathcal{I} = \mathcal{I} \circ T(g), \quad \forall g \in G.$$  \hfill (1.3)

Then we say that the equation $\mathcal{I}(\mathcal{I})$ is a $G$-invariant equation. Note that $\ker \mathcal{I}, \text{im} \mathcal{I}$ are invariant subspaces of $C, C'$, resp.
If $G$ is semisimple then there exist canonical ways for the construction of the intertwining differential operators, cf., e.g., [2, 3]. In this method there is a correspondence between invariant differential operators and singular vectors of Verma modules over the (complexified) Lie algebra in consideration.

The procedure may be applied for more general classes of Lie groups. For instance, it was applied to the Schrödinger group [4, 5] in, e.g., [6, 7].

This is what we try to do in the present paper for the case of $G_2$.

2 Preliminaries

The procedure that we shall follow requires first that we find the singular vectors of the Verma modules over $G_2$. This task was fulfilled in [8]. Furthermore there are given all necessary details, and thus, we can present the preliminaries in a shorter fashion.

The Jacobi algebra is the semi-direct sum $G_2 := H_n \oplus sp(n, \mathbb{R})_C$ [9, 10]. The Heisenberg algebra $H_n$ is generated by the boson creation (respectively, annihilation) operators $a_i^+$ ($a_i^-$), $i, j = 1, \ldots, n$, which verify the canonical commutation relations

$$[a_i^-, a_j^+] = \delta_{ij}, \quad [a_i^-, a_j^-] = [a_i^+, a_j^+] = 0.$$  \hspace{1cm} (2.1)

$H_n$ is an ideal in $G_2$, i.e., $[H_n, G_n] = H_n$, determined by the commutation relations (following the notation of [11]):

$$[a_k^+, K_{ij}^+] = [a_k^-, K_{ij}^{-}] = 0, \quad [a_i^-, K_{ij}^+] = \frac{1}{2} \delta_{ik} a_j^+ + \frac{1}{2} \delta_{ij} a_k^+, \quad [K_{ij}^-, a_i^+] = \frac{1}{2} \delta_{ik} a_j^- + \frac{1}{2} \delta_{ij} a_k^-, \quad (2.2a)$$

$$[K_{ij}^0, K_{kl}^0] = [K_{ij}^0, K_{kl}^0] = 0, \quad 2 [K_{ij}^-, K_{kl}^0] = K_{ij}^0 \delta_{kl} + K_{kl}^0 \delta_{ij}, \quad (2.2a)$$

$$2 [K_{ij}^-, K_{kl}^0] = K_{ij}^0 \delta_{kl} + K_{kl}^0 \delta_{ij} + K_{ji}^0 \delta_{k,l} + K_{ij}^0 \delta_{k,l}. \quad (2.2b)$$

$$2 [K_{ij}^+, K_{kl}^0] = -K_{jk}^0 \delta_{il} - K_{jk}^0 \delta_{il}, \quad 2 [K_{ij}^0, K_{kl}^0] = K_{ij}^0 \delta_{kl} - K_{kl}^0 \delta_{ij}. \quad (2.2c)$$

$S^+_n$ are the generators of the $S_n \equiv sp(n, \mathbb{R})_C$ algebra:

$$[S_{ij}^+, S_{kl}^+] = [S_{ij}^-, S_{kl}^-] = 0, \quad 2 [S_{ij}^-, S_{kl}^0] = K_{ij}^0 \delta_{kl} + K_{kl}^0 \delta_{ij}, \quad (2.3a)$$

$$2 [S_{ij}^-, S_{kl}^0] = K_{ij}^0 \delta_{kl} + K_{kl}^0 \delta_{ij} + K_{ji}^0 \delta_{k,l} + K_{ij}^0 \delta_{k,l}. \quad (2.3b)$$

First, for simplicity, we introduce the following notations for the basis of $S_2$:

$$S^+ : \quad b_i^+ \equiv K_{ii}^+, \quad c^+ \equiv K_{12}^+, \quad d^+ \equiv K_{12}^0 \quad (2.4a)$$

$$S^- : \quad b_i^- \equiv K_{ii}^-, \quad c^- \equiv K_{12}^-, \quad d^- \equiv K_{21}^0 \quad (2.4b)$$

$$K : \quad h_i \equiv K_{ii}^0, \quad i = 1, 2. \quad (2.4c)$$

We need also the triangular decomposition of $G_2$

$$G_2^+ := \{ a_i^+, b_i^+ + c^+, d^+ \}, \quad i = 1, 2,$$

$$G_2^- := \{ a_i^-, b_i^- + c^-, d^- \}, \quad i = 1, 2,$$

$$K_2 := \{ h_i, \ 1 \}, \quad i = 1, 2. \quad (2.5)$$
For the explicit construction of the intertwining differential operators we need a parameter space. That would be some coset space of the Jacobi group $G$ as generated by $G_2$. Then we need its triangular decomposition $G = G^+ KG$ and Borel subgroup $B = KG$.

Now we can define the space of the right covariant functions:

$$C_\Lambda = \{ \mathcal{F} \in C^\infty(G) \mid \mathcal{F}(gkg^-) = e^{\Lambda(H)} \mathcal{F}(g) \}$$

where $g \in G$, $k = e^H \in K$, $g^- \in G^-$, $H \in K_2$, $\Lambda \in K^*$.

Correspondingly we define the right action of $G_2$ on $C_\Lambda$:

$$(\pi_R(X)\mathcal{F})(g) \overset{\text{def}}{=} \left. \frac{d}{dt} \mathcal{F}(g \exp(tX)) \right|_{t=0}, \quad X \in G_2, \; g \in G \quad (2.7)$$

and the left action of $G_2$

$$(\pi_L(X)\mathcal{F})(g) \overset{\text{def}}{=} \left. \frac{d}{dt} \mathcal{F}(\exp(-tX)g) \right|_{t=0}, \quad X \in G_2, \; g \in G \quad (2.8)$$

In the next section we present these construction in explicit detail.

### 3 Right and left actions on $G^+$

#### 3.1 Right action

For the elements $g^+$ of $G^+$ we write:

$$g^+ = \exp \left( x_1 a_1^\dagger + x_2 a_2^\dagger \right) \exp \left( y_1 b_1^+ + y_2 b_2^+ + z c^+ + wd^+ \right).$$

It is important that there are only three non-vanishing relations among the generators of $G_2^+$:

$$[b_2^\dagger, d^\dagger] = -c^\dagger, \quad [a_2^\dagger, d^\dagger] = -\frac{1}{2} a_1^\dagger, \quad [c^\dagger, d^\dagger] = -\frac{1}{2} b_1^\dagger.$$  \((3.2)\)

Using these relations, it is easy to compute the right action of $G_2^+$:

$$\pi_R(a_1^\dagger) = \partial_{x_1},$$

$$\pi_R(a_2^\dagger) = \partial_{x_2} + \frac{w}{2} \partial_{x_1},$$

$$\pi_R(b_1^\dagger) = \partial_{y_1},$$

$$\pi_R(b_2^\dagger) = \partial_{y_2} + \frac{w^2}{24} \partial_{y_1} + \frac{w}{2} \partial z,$$

$$\pi_R(c^\dagger) = \partial z + \frac{w}{4} \partial_{y_1},$$

$$\pi_R(d^\dagger) = \partial w - \frac{1}{4} \left( z + \frac{y_2 w}{6} \right) \partial_{y_1} - \frac{y_2}{2} \partial z.$$  \((3.3)\)
3.2 Left action

The left action is computed by using the Baker-Campbell-Hausdorff formula:

\[ \ln e^X e^Y = X + Y + \frac{1}{2} [X, Y] + \frac{1}{12} ((adX)^2(Y) + (adY)^2(X)) \]

\[ - \frac{1}{24} [Y, [X, [X, Y]]] - \frac{1}{720} ((adY)^4(X) + (adX)^4(Y)) + \cdots \]  

(3.4)

where \( \text{ad}X(Y) := [X, Y] \). We here present the final results and omit the computational details.

Left action of \( \mathcal{G}_2^+ \):

\[ \pi_L(a_1^+) = -\partial x_1, \]
\[ \pi_L(a_2^+) = -\partial x_2, \]
\[ \pi_L(b_1^+) = -\partial y_1, \]
\[ \pi_L(b_2^+) = -\partial y_2 - \frac{w^2}{24} \partial y_1 + \frac{w}{2} \partial z, \]
\[ \pi_L(c^+) = -\partial z + \frac{w}{4} \partial y_1, \]
\[ \pi_L(d^+) = -\partial w - \frac{x_2}{2} \partial x_1 - \frac{1}{4} \left( z - \frac{y_2 w}{6} \right) \partial y_1 - \frac{y_2}{2} \partial z. \]  

(3.5)

Left action of \( \mathcal{K}_2 \):

\[ \pi_L(h_1) = -\frac{x_1}{2} \partial x_1 - y_1 \partial y_1 - \frac{z}{2} \partial z - \frac{w}{2} \partial w - \Lambda(h_1), \]
\[ \pi_L(h_2) = -\frac{x_2}{2} \partial x_2 - y_2 \partial y_2 - \frac{z}{2} \partial z + \frac{w}{2} \partial w - \Lambda(h_2), \]
\[ \pi_L(1) = -\Lambda(1) \]  

(3.6)

where \( \Lambda(1) \) is the weight of the central element of the Heisenberg algebra: \([a_i^-, a_j^+] = \delta_{ij}1\).
Left action of $G_2^-$:

\[
\pi_L(a_1^-) = - \left( y_1 + \frac{wz}{4} + \frac{y_2w^2}{12} \right) \partial_{x_1} - \frac{1}{2} \left( z + \frac{y_2w}{2} \right) \partial_{x_2} - x_1 \Lambda(1),
\]

\[
\pi_L(a_2^-) = - \frac{1}{2} \left( z + \frac{y_2w}{2} \right) \partial_{x_1} - y_2 \partial_{x_2} - x_2 \Lambda(1),
\]

\[
\pi_L(b_1^-) = x_1 \pi_L(a_1^-) - \left( y_1^2 + \frac{w^2}{96} \left( z^2 + y_2wz + \frac{y_2w^2}{12} \right) \right) \partial_{y_1} - \frac{1}{4} \left( z + \frac{y_2w}{2} \right)^2 \partial_{y_2}
\]

\[
- \left( y_1z + \frac{w}{8} \left( z^2 + \frac{2y_2wz}{3} + \frac{y_2w^2}{4} \right) \right) \partial_z - w \left( y_1 - \frac{y_2w^2}{24} \right) \partial_w
\]

\[
+ \frac{x_1^2}{2} \Lambda(1) - 2 \left( y_1 - \frac{y_2w^2}{24} \right) \Lambda(h_1) - \frac{w}{2} \left( z + \frac{y_2w}{2} \right) \Lambda(h_2),
\]

\[
\pi_L(b_2^-) = x_2 \pi_L(a_1^-) - \frac{y_2wz}{12} \partial_{y_1} - y_2 \partial_{y_2}
\]

\[
\pi(c^-) = \frac{x_2^2}{2} \pi_L(a_1^-) + \frac{x_1^2}{2} \pi_L(a_2^-)
\]

\[
- \frac{1}{4} \left( y_1 \left( z - \frac{y_2w}{6} \right) + \frac{y_2w}{8} \left( wz + \frac{y_2w^2}{18} \right) \right) \partial_{y_1} - \frac{y_2}{2} \left( z + \frac{y_2w}{2} \right) \partial_{y_2}
\]

\[
- \frac{1}{2} \left( y_2 \left( y_1 + \frac{wz}{4} + \frac{5y_2w^2}{24} \right) + \frac{z^2}{2} \right) \partial_z - \left( y_1 + \frac{wz}{4} - \frac{y_2w^2}{6} \right) \partial_w
\]

\[
+ \frac{x_1x_2}{2} \Lambda(1) - \frac{1}{2} \left( z - \frac{y_2w}{2} \right) \Lambda(h_1) - \frac{1}{2} \left( z + \frac{3y_2w}{2} \right) \Lambda(h_2),
\]

\[
\pi_L(d^-) = \frac{x_1}{2} \partial_{x_2} + \frac{w}{4} \left( y_1 - \frac{wz}{12} \right) \partial_{y_1} - \frac{1}{2} \left( z + \frac{y_2w}{2} \right) \partial_{y_2}
\]

\[
- \left( y_1 + \frac{y_2w^2}{12} \right) \partial_z + \frac{w^2}{4} \partial_w + \frac{w}{2} \left( \Lambda(h_1) - \Lambda(h_2) \right).
\]

It has been verified by direct computation (with MAPLE) that the left action given above is compatible with the defining commutation relations of $G_2$.

4 Invariant differential operators: first version

First we give the list of singular vectors that were found in [8]. We denote the lowest weight vector of the Verma module by $|0\rangle$ and the lowest weight by $\Lambda_k = \Lambda(h_k)$. The parameters $p_k$ and $q_k$ take a positive integer and the weight of the singular vector is denoted by $\Lambda'$.

(i) $\Lambda_1 - \Lambda_2 = \frac{1}{2} (1 - p^4)$

$$\left| v_s^{\Lambda'_1} \right\rangle = (d^+)^{p^4} |0\rangle, \quad \Lambda' = \Lambda + p^4 (\delta_1 - \delta_2).$$

(ii) $\forall \Lambda_1, \Lambda_2 = \frac{3}{4} - \frac{p^2}{2}$

$$\left| v_s^{\Lambda'_1} \right\rangle = (b_2^+)^{p^2} |0\rangle, \quad \Lambda' = \Lambda + 2p^2 \delta_2.$$
(iii) $\Lambda_1 = \frac{5}{4} - \frac{1}{2} (p^3 - q^3)$, $\Lambda_2 = \frac{3}{4} - \frac{1}{2} q^3$, $(p^3 \neq q^3, p^3 \neq 2q^3)$

(a) $p^3 < q^3$

$$\left| v_\Lambda^\prime \right> = \sum_{k=0}^{p^3/2} \sum_{n=0}^{p^3-2k} c(k, n) \left| k, q^3 - k - n, n, p^3 - 2k - n \right>$$  \hspace{1cm} (4.3)

(b) $q^3 < p^3 < 2q^3$

$$\left| v_\Lambda^\prime \right> = \left( \sum_{k=0}^{p^3-q^3-k} \sum_{n=0}^{p^3/2} \sum_{k=p^3-q^3+1}^{p^3-2k} c(k, n) \left| k, q^3 - k - n, n, p^3 - 2k - n \right> \right)$$  \hspace{1cm} (4.4)

(c) $2q^3 < p^3$

$$\left| v_\Lambda^\prime \right> = \sum_{k=0}^{q^3-q^3-k} \sum_{n=0}^{q^3-k} c(k, n) \left| k, q^3 - k - n, n, p^3 - 2k - n \right>$$  \hspace{1cm} (4.5)

where $\Lambda'$ and $c(k, n)$ are common for (a) (b) (c) and given by

$$\Lambda' = \Lambda + p^3 \delta_1 + (2q^3 - p^3) \delta_2,$$  \hspace{1cm} (4.6)

$$c(k, n) = \frac{p^3! q^3!}{4^k k! n!(p^3 - 2k - n)!(q^3 - k - n)!}.$$  \hspace{1cm} (4.7)

(iv) $\Lambda_1 + \Lambda_2 = 2 - \frac{p^4}{2}$

$$\left| v_\Lambda^\prime \right> = \sum_{k=0}^{p^4/2} \sum_{n=0}^{p^4-2k} c(k, n) \left| k, p^4 - k - n, n, p^4 - 2k - n \right>, \quad \Lambda' = \Lambda + p^4 (\delta_1 + \delta_2),$$

$$c(k, n) = \frac{p^4!}{4^k k! n!(p^4 - 2k - n)!} \frac{\Gamma(2\Lambda_1 + p^4 - \frac{3}{2})}{\Gamma(2\Lambda_1 + p^4 - \frac{3}{2} - k - n)}. \hspace{1cm} (4.8)$$

(v) $\Lambda_1 = \frac{5}{4} - \frac{p^5}{2}$, $\forall \Lambda_2$

$$\left| v_\Lambda^\prime \right> = \sum_{k=0}^{p^5} \sum_{n=0}^{p^5-k} c(k, n) \left| k, p^5 - k - n, n, 2p^5 - 2k - n \right>, \quad \Lambda' = \Lambda + 2p^5 \delta_1, \hspace{1cm} (4.10)$$

$$c(k, n) = \frac{(-1)^n p^5!}{4^k k! n!(p^5 - k - n)!} \frac{\Gamma(2\Lambda_2 - p^5 - \frac{3}{2} + 2k + n)}{\Gamma(2\Lambda_2 - p^5 - \frac{3}{2})}. \hspace{1cm} (4.11)$$

Note that there is a change of basis w.r.t. [3], namely:

$$\left| k, \ell, n, m \right> := (\hat{b}_1^\dagger)^k (\hat{b}_2^\dagger)^\ell (\hat{c}^\dagger)^n (d^\dagger)^m |0\rangle$$  \hspace{1cm} (4.12)
and
\[ \hat{b}_k := b_k^+ - \frac{1}{2}(a_k^+)^2, \quad \hat{c} := c^+ - \frac{1}{2}a_1^+a_2^+. \] (4.13)

Then using the right action obtained in §3 we have using (4.13):

\[ \pi_R(\hat{b}_1^+) = \partial y_1 - \frac{1}{2}\partial x_1, \]
\[ \pi_R(\hat{b}_2^+) = \partial y_2 + \frac{w^2}{24}\partial y_1 + \frac{w}{2}\partial z - \frac{1}{2}\left(\partial x_2 + \frac{w}{2}\partial x_1\right)^2, \] (4.14)
\[ \pi_R(\hat{c}^+) = \partial z + \frac{w}{4}\partial y_1 - \frac{1}{2}\left(\partial x_1\partial x_2 + \frac{w}{2}\partial x_1\right). \]

Thus the invariant differential operators are given by substituting the above right action in the singular vectors given above:

(i) \( \Lambda_1 - \Lambda_2 = \frac{1}{2}(1 - p^1) \)

\[ \left(\partial w - \frac{1}{4}\left(z + \frac{y_2w}{6}\right)\partial y_1 - \frac{y_2}{2}\partial z\right)^{p^1} \] (4.15)

(ii) \( \forall \Lambda_1, \quad \Lambda_2 = \frac{3}{4} - \frac{p^2}{2} \)

\[ \left(\partial y_2 + \frac{w^2}{24}\partial y_1 + \frac{w}{2}\partial z - \frac{1}{2}\left(\partial x_2 + \frac{w}{2}\partial x_1\right)^2\right)^{p^2} \] (4.16)

(iii) \( \Lambda_1 = \frac{5}{4} - \frac{1}{2}(p^3 - q^3), \quad \Lambda_2 = \frac{3}{4} - \frac{1}{2}q^3, \quad (p^3 \neq q^3, p^3 \neq 2q^3) \)

(a) \( p^3 < q^3 \)

\[ \sum_{k=0}^{\lfloor p^3/2 \rfloor} \sum_{n=0}^{p^3 - 2k} c(k, n) \mathcal{P}(p^3, q^3, k, n) \] (4.17)

(b) \( q^3 < p^3 < 2q^3 \)

\[ \left(\sum_{k=0}^{p^3 - q^3} \sum_{n=0}^{q^3 - k} + \sum_{k=p^3 - q^3 + 1}^{\lfloor p^3/2 \rfloor} \sum_{n=0}^{p^3 - 2k} c(k, n) \mathcal{P}(p^3, q^3, k, n) \right) \] (4.18)

(c) \( 2q^3 < p^3 \)

\[ \sum_{k=0}^{q^3} \sum_{n=0}^{q^3 - k} c(k, n) \mathcal{P}(p^3, q^3, k, n) \] (4.19)
where $\mathcal{P}(p^3, q^3, k, n)$ and $c(k, n)$ are common for (a) (b) (c) and given by

\[
\mathcal{P}(p^3, q^3, k, n) = \left( \partial_y - \frac{1}{2} \partial_{x_1}^2 \right)^k \left( \partial_y + \frac{w^2}{24} \partial_y + \frac{w}{2} \partial_z - \frac{1}{2} \left( \partial_{x_2} + \frac{w}{2} \partial_{x_1} \right)^2 \right)^{q^3-k-n} \\
\times \left( \partial_z + \frac{w}{4} \partial_y - \frac{1}{2} \left( \partial_{x_1} \partial_{x_2} + \frac{w}{2} \partial_{x_1}^2 \right) \right)^{n} \\
\times \left( \partial_w - \frac{1}{4} \left( z + \frac{y_2 w}{6} \right) \partial_y - \frac{y_2}{2} \partial_z \right)^{p^3-2k-n},
\]

(4.20)

\[
c(k, n) = \frac{p^3 q^3!}{4^k k! n! (p^3 - 2k - n)! (q^3 - k - n)!}.
\]

(4.21)

(iv) $\Lambda_1 + \Lambda_2 = 2 - \frac{p^4}{2}$

\[
\sum_{k=0}^{\lfloor p^4/2 \rfloor} \sum_{n=0}^{p^4-2k} \frac{p^4!}{4^k k! n! (p^4 - 2k - n)! \Gamma(2\Lambda_1 + p^4 - \frac{3}{2})} \\
\times \left( \partial_y - \frac{1}{2} \partial_{x_1}^2 \right)^k \left( \partial_y + \frac{w^2}{24} \partial_y + \frac{w}{2} \partial_z - \frac{1}{2} \left( \partial_{x_2} + \frac{w}{2} \partial_{x_1} \right)^2 \right)^{p^4-k-n} \\
\times \left( \partial_z + \frac{w}{4} \partial_y - \frac{1}{2} \left( \partial_{x_1} \partial_{x_2} + \frac{w}{2} \partial_{x_1}^2 \right) \right)^{n} \\
\times \left( \partial_w - \frac{1}{4} \left( z + \frac{y_2 w}{6} \right) \partial_y - \frac{y_2}{2} \partial_z \right)^{p^4-2k-n}
\]

(4.22)

(v) $\Lambda_1 = \frac{5}{4} - \frac{p^5}{2}$, $\forall \Lambda_2$

\[
\sum_{k=0}^{p^5} \sum_{n=0}^{p^5-k} \frac{(-1)^n p^5!}{4^k k! n! (p^5 - k - n)! \Gamma(2\Lambda_2 - p^5 - \frac{3}{2} + 2k + n)} \\
\times \left( \partial_y - \frac{1}{2} \partial_{x_1}^2 \right)^k \left( \partial_y + \frac{w^2}{24} \partial_y + \frac{w}{2} \partial_z - \frac{1}{2} \left( \partial_{x_2} + \frac{w}{2} \partial_{x_1} \right)^2 \right)^{p^5-k-n} \\
\times \left( \partial_z + \frac{w}{4} \partial_y - \frac{1}{2} \left( \partial_{x_1} \partial_{x_2} + \frac{w}{2} \partial_{x_1}^2 \right) \right)^{n} \\
\times \left( \partial_w - \frac{1}{4} \left( z + \frac{y_2 w}{6} \right) \partial_y - \frac{y_2}{2} \partial_z \right)^{2p^5-2k-n}
\]

(4.23)
5 Final expressions for the intertwining differential operators

To simplify our results we make the following change of parameters:

\[(x_1, x_2, y_1, y_2, z, w) \rightarrow (\xi_1, \xi_2, \eta_1, \eta_2, \zeta, \omega),\] namely:

\[\begin{align*}
\xi_1 &= x_1 - \frac{w}{2} x_2, \\
\xi_2 &= x_2, \\
\eta_1 &= y_1 + \frac{w^2}{12} y_2 - \frac{w}{4} z, \\
\eta_2 &= y_2, \\
\zeta &= z - \frac{w}{2} y_2, \\
\omega &= w.
\end{align*}\] (5.1)

Inverse transform is given by

\[\begin{align*}
x_1 &= \xi_1 + \omega \xi_2, \\
x_2 &= \xi_2, \\
y_1 &= \eta_1 + \frac{\zeta \omega}{2} + \frac{\eta_2 \omega^2}{24}, \\
y_2 &= \eta_2, \\
z &= \zeta + \frac{\eta_2 \omega}{2}, \\
w &= \omega.
\end{align*}\] (5.2)

It follows that

\[\begin{align*}
\partial \xi_1 &= \partial x_1, \\
\partial \xi_2 &= \partial x_2 - \frac{\omega}{2} \partial \xi_1, \\
\partial \eta_1 &= \partial y_1, \\
\partial \eta_2 &= \partial y_2 + \frac{\omega^2}{12} \partial \eta_1 - \frac{\omega}{2} \partial \zeta, \\
\partial \zeta &= \partial \zeta - \frac{\omega}{4} \partial \eta_1, \\
\partial \omega &= \partial \omega - \frac{1}{4} \left( \zeta - \frac{\eta_2 \omega}{6} \right) \partial \eta_1 - \frac{\eta_2}{2} \partial \zeta - \frac{\xi_2}{2} \partial \xi_1.
\end{align*}\] (5.3)

Then, the right action becomes:

\[\begin{align*}
\pi_R(a_k^+) &= \partial \xi_k, \\
\pi_R(b_k^+) &= \partial \eta_k, \\
\pi_R(c^+) &= \partial \zeta, \\
\pi_R(d^+) &= \partial \omega - \frac{\zeta}{2} \partial \eta_1 - \eta_2 \partial \zeta - \frac{\xi_2}{2} \partial \xi_1.
\end{align*}\] (5.4)

The left action is also simplified and now reads as follows:

\[\begin{align*}
\pi_L(a_1^+) &= -\partial \xi_1, \\
\pi_L(a_2^+) &= -\partial \xi_2 + \frac{\omega}{2} \partial \xi_1, \\
\pi_L(b_1^+) &= -\partial \eta_1, \\
\pi_L(b_2^+) &= -\partial \eta_2 - \frac{\omega^2}{4} \partial \eta_1 + \omega \partial \zeta, \\
\pi_L(c^+) &= -\partial \zeta + \frac{\omega}{2} \partial \eta_1, \\
\pi_L(d^+) &= -\partial \omega,
\end{align*}\] (5.5)

\[\begin{align*}
\pi_L(h_1) &= -\frac{\xi_1}{2} \partial \xi_1 - \eta_1 \partial \eta_1 - \frac{\zeta}{2} \partial \zeta - \frac{\omega}{2} \partial \omega - \Lambda(h_1), \\
\pi_L(h_2) &= -\frac{\xi_2}{2} \partial \xi_2 - \eta_2 \partial \eta_2 - \frac{\zeta}{2} \partial \zeta + \frac{3}{2} \partial \omega - \Lambda(h_2), \\
\pi_L(1) &= -\Lambda(1).
\end{align*}\] (5.6)
\[ \pi_L(a_1^-) = -\left( \eta_1 + \frac{\xi \omega}{4} \right) \partial_{\xi_1} - \frac{1}{2}(\zeta + \eta_2 \omega) \partial_{\xi_2} - \left( \xi_1 + \frac{\xi_2 \omega}{2} \right) \Lambda(1), \]
\[ \pi_L(a_2^-) = -\frac{\zeta}{2} \partial_{\xi_1} - \eta_2 \partial_{\xi_2} - \xi_2 \Lambda(1), \]
\[ \pi_L(b_1^-) = \left( \xi_1 + \frac{\xi_2 \omega}{2} \right) \pi_L(a_1^-) + \frac{\xi_2 \omega}{4} \left( \eta_1 + \frac{\zeta \omega}{4} \right) \partial_{\xi_1} - \left( \eta_1^2 - \frac{\xi_2 \omega^2}{16} \right) \partial_{\eta_1} \]
\[ - \frac{1}{4}(\zeta + \eta_2 \omega)^2 \partial_{\eta_2} - \left( \eta_1 + \frac{\zeta \omega}{4} \right) (\zeta \partial_{\zeta} + \omega \partial_{\omega} - 2\Lambda(h_1)) \]
\[ + \frac{1}{2} \left( \xi_1 + \frac{\xi_2 \omega}{2} \right)^2 \Lambda(1) - \frac{\omega}{2}(\zeta + \eta_2 \omega) \Lambda(h_2), \]
\[ \pi_L(b_2^-) = \xi_2 \pi_L(a_2^-) + \frac{\xi_2 \zeta}{2} \partial_{\xi_1} + \frac{\xi_2 \omega}{4} \partial_{\eta_1} - \eta_2^2 \partial_{\eta_2} - \zeta \partial_{\zeta} + \frac{\xi_2^2 \omega}{2} \Lambda(1) - 2\eta_2 \Lambda(h_2), \]
\[ \pi_L(c^-) = \frac{\xi_2}{2} \pi_L(a_1^-) + \frac{\xi_2 \zeta}{2} \partial_{\xi_1} + \frac{\xi_2 \omega}{4} \partial_{\eta_1} - \left( \eta_1 + \frac{\zeta \omega}{2} \right) \left( \frac{\xi_2}{2} \partial_{\xi_1} - \partial_{\omega} \right) \]
\[ + \frac{\xi_2 \omega}{8} \partial_{\eta_1} - \frac{1}{2}(\zeta + \eta_2 \omega)(\eta_2 \partial_{\eta_2} + \Lambda(h_2)) - \frac{\xi_2^2}{4} \partial_{\zeta} \]
\[ + \frac{\xi_2}{2} \left( \xi_1 + \frac{\xi_2 \omega}{2} \right) \Lambda(1) - \frac{\zeta}{2} \Lambda(h_1), \]
\[ \pi_L(d^-) = -\frac{\omega \xi_1}{4} \partial_{\xi_1} - \frac{1}{2} \left( \xi_1 + \frac{\omega \xi_2}{2} \right) \partial_{\xi_2} + \frac{\omega \eta_1}{2} \partial_{\eta_1} - \frac{1}{2}(\zeta + \omega \eta_2) \partial_{\eta_2} \]
\[ - \eta_1 \partial_{\zeta} - \frac{\omega^2}{4} \partial_{\omega} + \frac{\omega}{2}(\Lambda(h_1) - \Lambda(h_2)). \] (5.7)

\[ \pi_L(b_k^-), \pi_L(c^-), \pi_L(d^-) \] have the following simpler expressions:
\[ \pi_L(b_1^-) = \xi_1 \pi_L(a_1^-) - \frac{\xi_1 \omega}{2} \pi_L(a_2^-) + \omega \pi_L(c^-) - \frac{\omega^2}{4} \pi_L(b_2^-) \]
\[ - \eta_2 \partial_{\eta_1} + \frac{\xi_2 \omega^2}{4} \partial_{\eta_2} - \eta_1 \zeta \partial_{\zeta} + \frac{\xi_1^2}{2} \Lambda(1) - 2\eta_1 \Lambda(h_1), \]
\[ \pi_L(b_2^-) = -\xi_2 \eta_2 \partial_{\xi_1} + \frac{\xi_2 \omega}{4} \partial_{\eta_1} - \eta_2^2 \partial_{\eta_2} - \zeta \partial_{\zeta} + \frac{\xi_2^2 \omega}{2} \Lambda(1) - 2\eta_2 \Lambda(h_2), \]
\[ \pi_L(c^-) = \frac{\xi_2}{2} \pi_L(a_1^-) + \frac{1}{2} \left( \xi_1 - \frac{\xi_2 \omega}{2} \right) \pi_L(a_2^-) + \frac{\omega}{2} \pi_L(b_2^-) \]
\[ + \frac{\xi_2 \eta_1}{2} \partial_{\xi_1} - \frac{\xi_2 \eta_2}{2} \partial_{\eta_2} - \frac{\xi_2 \omega}{4} \partial_{\zeta} - \eta_1 \partial_{\omega} + \frac{\xi_1 \xi_2}{2} \Lambda(1) - \frac{\zeta}{2} (\Lambda(h_1) + \Lambda(h_2)), \]
\[ \pi_L(d^-) = -\frac{\omega}{2} (\pi_L(h_1) - \pi_L(h_2)) - \frac{\xi_1}{2} \partial_{\xi_2} - \frac{\zeta}{2} \partial_{\eta_2} - \eta_1 \partial_{\zeta} - \frac{\omega^2}{4} \partial_{\omega}. \] (5.8)

Finally, we pass to the "hat" basis:
\[ \pi_R(b_k^+) = \partial_{\eta_1} - \frac{1}{2} \partial_{\xi_1}^2, \quad \pi_R(c^+) = \partial_{\zeta} - \frac{1}{2} \partial_{\xi_1} \partial_{\xi_2}. \] (5.9)

Then the final expressions for the invariant differential operators are:
(i) $\Lambda_1 - \Lambda_2 = \frac{1}{2}(1 - p^1)$
\[
\mathcal{D}_{(i)} = \left( \frac{\partial_\omega - \xi_2}{2} \partial_{\xi_1} - \frac{\xi}{2} \partial_{\eta_1} - \eta_2 \partial_\zeta \right)^{p^1} \tag{5.10}
\]

(ii) $\forall \Lambda_1, \Lambda_2 = \frac{3}{4} - \frac{p^2}{2}$
\[
\mathcal{D}_{(ii)} = \left( \partial_{\eta_2} - \frac{1}{2} \partial_{\xi_2} \right)^{p^2} \tag{5.11}
\]

(iii) $\Lambda_1 = \frac{5}{4} - \frac{1}{2}(p^3 - q^3), \Lambda_2 = \frac{3}{4} - \frac{1}{2}q^3$, $(p^3 \neq q^3, p^3 \neq 2q^3)$

(a) $p^3 < q^3$
\[
\mathcal{D}_{(iii,a)} = \sum_{k=0}^{p^3/2} \sum_{n=0}^{p^3-2k} \frac{p^3!q^3!}{4^k k! n!(p^3 - 2k - n)!(q^3 - k - n)!} \left( \partial_{\eta_1} - \frac{1}{2} \partial_{\xi_1} \right)^k \times \left( \partial_{\eta_2} - \frac{1}{2} \partial_{\xi_2} \right)^{q^3 - k - n} \left( \partial_\zeta - \frac{1}{2} \partial_{\xi_1} \partial_{\xi_2} \right)^n \times \left( \partial_\omega - \frac{\xi_2}{2} \partial_{\xi_1} - \frac{\xi}{2} \partial_{\eta_1} - \eta_2 \partial_\zeta \right)^{p^3 - 2k - n} \tag{5.12}
\]

(b) $q^3 < p^3 < 2q^3$
\[
\mathcal{D}_{(iii,b)} = \left( \sum_{k=0}^{p^3-q^3} \sum_{n=0}^{p^3-2k} \right) \frac{p^3!q^3!}{(p^3-q^3-k)! (q^3-k-n)!} \mathcal{P}(p^3, q^3, k, n) \tag{5.13}
\]

(c) $2q^3 < p^3$
\[
\mathcal{D}_{(iii,c)} = \sum_{k=0}^{q^3} \sum_{n=0}^{q^3-k} \mathcal{P}(p^3, q^3, k, n) \tag{5.14}
\]

where the summand $\mathcal{P}(p^3, q^3, k, n)$ for (b) (c) is same as (a).

(iv) $\Lambda_1 + \Lambda_2 = 2 - \frac{p^4}{4}$
\[
\mathcal{D}_{(iv)} = \sum_{k=0}^{p^4/2} \sum_{n=0}^{p^4-2k} \frac{p^4!}{4^k k! n!(p^4 - 2k - n)!} \frac{\Gamma(2\Lambda_1 + p^4 - \frac{3}{2})}{\Gamma(2\Lambda_1 + p^4 - \frac{3}{2} - k - n)} \times \left( \partial_{\eta_1} - \frac{1}{2} \partial_{\xi_1} \right)^k \left( \partial_{\eta_2} - \frac{1}{2} \partial_{\xi_2} \right)^{p^4-k-n} \left( \partial_\zeta - \frac{1}{2} \partial_{\xi_1} \partial_{\xi_2} \right)^n \times \left( \partial_\omega - \frac{\xi_2}{2} \partial_{\xi_1} - \frac{\xi}{2} \partial_{\eta_1} - \eta_2 \partial_\zeta \right)^{p^4-2k-n} \tag{5.15}
\]
\( \Lambda_1 = \frac{\delta}{4} - \frac{\nu}{2} \), \( \forall \Lambda_2 \)

\[
D_{(v)} = \sum_{k=0}^{p^5} \sum_{n=0}^{p^5 - k} \frac{(-1)^n p^5!}{4^k k! n! (p^5 - k - n)!} \frac{\Gamma(2\Lambda_2 - p^5 - \frac{3}{2} + 2k + n)}{\Gamma(2\Lambda_2 - p^5 - \frac{1}{2})} \\
\times \left( \partial_{\eta_1} - \frac{1}{2} \partial_{\xi_1} \right)^k \left( \partial_{\eta_2} - \frac{1}{2} \partial_{\xi_2} \right)^{p^5 - k - n} \left( \partial_{\zeta} - \frac{1}{2} \partial_{\xi_1} \partial_{\xi_2} \right)^n \\
\times \left( \partial_{\omega} - \frac{\xi_2}{2} \partial_{\xi_1} - \frac{\zeta}{2} \partial_{\eta_1} - \eta_2 \partial_{\zeta} \right)^{2p^5 - 2k - n} \tag{5.16}
\]

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