HARMONIC DEFORMATIONS OF HYPERBOLIC 3-MANIFOLDS

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ABSTRACT. This paper gives an exposition of the authors’ harmonic deformation theory for 3-dimensional hyperbolic cone-manifolds. We discuss topological applications to hyperbolic Dehn surgery as well as recent applications to Kleinian group theory. A central idea is that local rigidity results (for deformations fixing cone angles) can be turned into effective control on the deformations that do exist. This leads to precise analytic and geometric versions of the idea that hyperbolic structures with short geodesics are close to hyperbolic structures with cusps. The paper also outlines a new harmonic deformation theory which applies whenever there is a sufficiently large embedded tube around the singular locus, removing the previous restriction to cone angles at most $2\pi$.

1. Introduction

The local rigidity theorem of Weil [29] and Garland [15] for complete, finite volume hyperbolic manifolds states that there is no non-trivial deformation of such a structure through complete hyperbolic structures if the manifold has dimension at least 3. If the manifold is closed, the condition that the structures be complete is automatically satisfied. However, if the manifold is non-compact, there may be deformations through incomplete structures. This cannot happen in dimensions greater than 3 (Garland-Raghunathan [16]); but there are always non-trivial deformations in dimension 3 (Thurston [27]) in the non-compact case.

In [21] this rigidity theory is extended to a class of finite volume, orientable 3-dimensional hyperbolic cone-manifolds, i.e. hyperbolic structures on 3-manifolds with cone-like singularities along a knot or link. The main result is that such structures are locally rigid if the cone angles are fixed, under the extra hypothesis that all cone angles are at most $2\pi$. There is a smooth, incomplete structure on the complement of the singular locus; by completing the metric the singular cone-metric is recovered. The space of deformations of (generally incomplete) hyperbolic structures on this open manifold has non-zero dimension, so there will be deformations if the cone angles are allowed to vary. An application of the implicit function theorem shows that it is possible to deform the structure so that the metric completion is still a 3-dimensional hyperbolic cone-manifold, and it is always possible to deform the cone-manifold to make arbitrary (small) changes in the cone angles. In fact, the collection of cone angles locally parametrizes the set of cone-manifold structures.

A (smooth) finite volume hyperbolic 3-manifold with cusps is the interior of a compact 3-manifold with torus boundary components. Filling these in by attaching solid tori produces a closed manifold; there is an infinite number of topologically distinct ways to do this, parametrized by the isotopy classes of the curves on

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the boundary tori that bound disks in the solid tori. These curves are called the “surgery curves”. The manifold with cusps can be viewed as a cone-manifold structure with cone angles 0 on any of these closed manifolds. If it is possible to increase the cone angle from 0 to 2\(\pi\), this constructs a smooth hyperbolic structure on this closed manifold. This process is called hyperbolic Dehn surgery. Thurston proved that hyperbolic Dehn surgery fails for at most a finite number of choices of surgery curves on each boundary component.

The proof of local rigidity puts strong constraints on those deformations of hyperbolic cone-manifolds that do exist. It is possible to control the change in the geometric structure when the cone angles are deformed a fixed amount. Importantly, this control depends only on the geometry in a tubular neighborhood around the singular locus, not on the rest of the 3-manifold. In particular, it provides geometric and analytic control on the hyperbolic Dehn surgery process. This idea is developed in [22].

That paper provides a quantitative version of Thurston’s hyperbolic Dehn surgery theorem. Applications include the first universal bounds on the number of non-hyperbolic Dehn fillings on a cusped hyperbolic 3-manifold, and estimates on the changes in volume and core geodesic length during hyperbolic Dehn filling.

The local rigidity theory of [21] was generalized by Bromberg to include geometrically finite hyperbolic cone-manifolds. Recently, there have been some very imaginative and interesting applications of the deformation theory of geometrically finite hyperbolic cone-manifolds to well-known problems in Kleinian groups. In particular, the reader is referred to [9] in this volume for a description of some of these results and references to others.

Our purpose here is to provide a brief outline of the main ideas and results from [21] and [22] and how they are related to the Kleinian group applications. As noted above, the central idea is that rigidity results can be turned into effective control on the deformations that do exist. However, we wish to emphasize a particular consequence that provides the common theme between [22] and the Kleinian group applications in [9], [10], [5], [6] and [7]. As a corollary of the control provided by effective rigidity, it is possible to give precise analytic and geometric meaning to the familiar idea that hyperbolic structures with short geodesics are “close” to ones with cusps. Specifically, it can be shown that a structure with a sufficiently short geodesic can be deformed through hyperbolic cone-manifolds to a complete structure, viewed as having cone angle 0. Furthermore, the total change in the structure can be proved to be arbitrarily small for structures with arbitrarily short geodesics. Most importantly this control is independent of the manifolds involved, depending only on the lengths and cone angles.

There are varied reasons for wanting to find such a family of cone-manifolds. It is conjectured that any closed hyperbolic 3-manifold can be obtained by hyperbolic Dehn surgery on some singly cusped, finite volume hyperbolic 3-manifold. If this were true, it could have useful implications in 3-dimensional topology. In [22] it is proved that it is true for any such closed 3-manifold whose shortest geodesic has length at most 0.162. (See Theorem 5.6 in Section 5.)

In [9] Bromberg describes a construction that, remarkably, allows one to replace an incompressible, geometrically infinite end with a short geodesic by a geometrically finite one by gluing in a wedge that creates a cone angle of 4\(\pi\) along the short geodesic. Pushing the cone angle back to 2\(\pi\) provides an approximation of the
In general, a sequence of Kleinian groups with geodesics that are becoming arbitrarily short (or a single Kleinian group with a sequence of arbitrarily short geodesics) is very difficult to analyze. Things are often simpler when the lengths are actually equal to 0; i.e., when they are cusps. Thus, if structures with short geodesics can be uniformly compared with structures where they have become cusps, this can be quite useful. One example where this idea has been successfully employed is [7]. It seems likely that there will be others in the near future.

Note that the application to the Density Conjecture above involves cone angles between $4\pi$ and $2\pi$ whereas the theory in [21] and [8] requires cone angles to be at most $2\pi$. Thus, this application actually depends on a new version of the deformation theory ([23]) which applies to all cone angles, as long as there is a tube of a certain radius around the singular locus.

Because of its connection with these Kleinian group results, we use the current paper as an opportunity to outline the main points in this new theory. It is based on a boundary value problem which is used to construct infinitesimal deformations with the same essential properties as those utilized in [21] and [22]. Explaining those properties and how they are used occupies Sections 2 to 5. The discussion of Kleinian groups and the new deformation theory are both contained in Section 6, the final section of the paper.

The relaxation of the cone angle restriction has implications for hyperbolic Dehn surgery. Some of these are also described in the final section. (See Theorems 6.2 and 6.3.)

2. Deformations of hyperbolic structures

A standard method for analyzing families of structures or maps is to look at the infinitesimal theory where the determining equations simplify considerably. To this end, we first describe precisely what we mean by a 1-parameter family of hyperbolic structures on a manifold. Associated to the derivative of such a family are various analytic, algebraic, and geometric objects which play a central role in this theory. It is useful to be able to move freely among the interpretations provided by these objects and we attempt to explain the relationships between them.

The initial portion of this analysis is quite general, applying to hyperbolic structures in any dimension or, even more broadly, to structures modeled on a Lie group acting transitively and analytically on a manifold.

A hyperbolic structure on an $n$-manifold $X$ is determined by local charts modeled on $\mathbb{H}^n$ whose overlap maps are restrictions of global isometries of $\mathbb{H}^n$. These determine, via analytic continuation, a map $\Phi: \tilde{X} \to \mathbb{H}^n$ from the universal cover $\tilde{X}$ of $X$ to $\mathbb{H}^n$, called the developing map, which is determined uniquely up to post-multiplication by an element of $G = \text{isom}(\mathbb{H}^n)$. The developing map satisfies the equivariance property $\Phi(\gamma m) = \rho(\gamma)\Phi(m)$, for all $m \in \tilde{X}$, $\gamma \in \pi_1(X)$, where $\pi_1(X)$ acts on $\tilde{X}$ by covering transformations, and $\rho: \pi_1(X) \to G$ is the holonomy representation of the structure. The developing map also determines the hyperbolic metric on $\tilde{X}$ by pulling back the hyperbolic metric on $\mathbb{H}^n$. (See [28] and [26] for a complete discussion of these ideas.)
We say that two hyperbolic structures are equivalent if there is a diffeomorphism $f$ from $X$ to itself taking one structure to the other. We will use the term “hyperbolic structure” to mean such an equivalence class. A 1-parameter family, $X_t$, of hyperbolic structures defines a 1-parameter family of developing maps $\Phi_t: \tilde{X} \to \mathbb{H}^n$, where two families are equivalent under the relation $\Phi_t \equiv k_t \Phi_1 f_t$ where $k_t$ are isometries of $\mathbb{H}^n$ and $f_t$ are lifts of diffeomorphisms $f_t$ from $X$ to itself. We assume that $k_0$ and $f_0$ are the identity, and denote $\Phi_0$ as $\Phi$. All of the maps here are assumed to be smooth and to vary smoothly with respect to $t$.

The tangent vector to a smooth family of hyperbolic structures will be called an infinitesimal deformation. The derivative at $t = 0$ of a 1-parameter family of developing maps $\Phi_t : \tilde{X} \to \mathbb{H}^n$ defines a map $\dot{\Phi} : \tilde{X} \to T\mathbb{H}^n$. For any point $m \in \tilde{X}$, $\Phi_t(m)$ is a curve in $\mathbb{H}^n$ describing how the image of $m$ is moving under the developing maps; $\dot{\Phi}(m)$ is the initial tangent vector to the curve.

We will identify $\tilde{X}$ locally with $\mathbb{H}^n$ and $T\tilde{X}$ locally with $T\mathbb{H}^n$ via the initial developing map $\Phi$. Note that this identification is generally not a global diffeomorphism unless the hyperbolic structure is complete. However, it is a local diffeomorphism, providing identification of small open sets in $\tilde{X}$ with ones in $\mathbb{H}^n$.

In particular, each point $m \in \tilde{X}$ has a neighborhood $U$ where $\Psi_t = \Phi^{-1} \circ \Phi_t : U \to \tilde{X}$ is defined, and the derivative at $t = 0$ defines a vector field on $\tilde{X}$, $v = \dot{\Psi} : \tilde{X} \to T\tilde{X}$. This vector field determines the infinitesimal variation in developing maps since $\dot{\Phi} = d\Phi \circ v$, and also determines the infinitesimal variation in metric as follows. Let $g_t$ be the hyperbolic metric on $\tilde{X}$ obtained by pulling back the hyperbolic metric on $\mathbb{H}^n$ via $\Phi_t$ and put $g_0 = g$. Then $g_t = \Psi_t^* g$ and the infinitesimal variation in metrics $\dot{g} = \frac{d\Psi_t}{dt}|_{t=0}$ is the Lie derivative, $\mathcal{L}_v g$, of the initial metric $g$ along $v$.

Riemannian covariant differentiation of the vector field $v$ gives a $T\tilde{X}$ valued 1-form on $\tilde{X}$, $\nabla v : T\tilde{X} \to T\tilde{X}$, defined by $\nabla v(x) = \nabla_x v$ for $x \in T\tilde{X}$. We can decompose $\nabla v$ at each point into a symmetric part and skew-symmetric part. The symmetric part, $\tilde{\eta} = (\nabla v)_{\text{sym}}$, represents the infinitesimal change in metric, since

$$\tilde{\eta}(x, y) = \mathcal{L}_v g(x, y) = g(\nabla_x v, y) + g(x, \nabla_y v) = 2g(\tilde{\eta}(x), y)$$

for $x, y \in T\tilde{X}$. In particular, $\tilde{\eta}$ descends to a well-defined $T\tilde{X}$-valued 1-form $\eta$ on $X$. The skew-symmetric part $(\nabla v)_{\text{skew}}$ is the curl of the vector field $v$; its value at $m \in \tilde{X}$ describes the infinitesimal rotation about $m$ induced by $v$.

To connect this discussion of infinitesimal deformations with cohomology theory, we consider the Lie algebra $\mathfrak{g}$ of $G = \text{isom}(\mathbb{H}^n)$ as vector fields on $\mathbb{H}^n$ representing infinitesimal isometries of $\mathbb{H}^n$. Pulling back these vector fields via the initial developing map $\Phi$ gives locally defined infinitesimal isometries on $\tilde{X}$ and on $X$.

Let $E, E$ denote the vector bundles over $\tilde{X}, X$ respectively of (germs of) infinitesimal isometries. Then we can regard $E$ as the product bundle with total space $\tilde{X} \times \mathfrak{g}$, and $E$ is isomorphic to $(\tilde{X} \times \mathfrak{g})/\sim$ where $(m, \zeta) \sim (\gamma m, Ad\rho(\gamma) \cdot \zeta)$ with $\gamma \in \pi_1(X)$ acting on $\tilde{X}$ by covering transformations and on $\mathfrak{g}$ by the adjoint action of the holonomy $\rho(\gamma)$. At each point $p$ of $\tilde{X}$, the fiber of $E$ splits as a direct sum of infinitesimal pure translations and infinitesimal pure rotations about $p$; these can be identified with $T_p\tilde{X}$ and $so(n)$ respectively. The hyperbolic metric on $\tilde{X}$ induces a metric on $T_p\tilde{X}$ and on $so(n)$. A metric can then be defined on the fibers of $E$ in which the two factors are orthogonal; this descends to a metric on the fibers of $E$. 


Given a vector field \( v : \tilde{X} \to T\tilde{X} \), we can lift it to a section \( s : \tilde{X} \to \tilde{E} \) by choosing an “osculating” infinitesimal isometry \( s(m) \) which best approximates the vector field \( v \) at each point \( m \in \tilde{X} \). Thus \( s(m) \) is the unique infinitesimal isometry whose translational part and rotational part at \( m \) agree with the values of \( v \) and \( \text{curl} \ v \) at \( m \). (This is the “canonical lift” as defined in \([21]\).) In particular, if \( v \) is itself an infinitesimal isometry of \( \tilde{X} \) then \( s \) will be a constant section.

Using the equivariance property of the developing maps it follows that \( s \) satisfies an “automorphic” property: for any fixed \( \gamma \in \pi_1(X) \), the difference \( s(\gamma m) - Ad\rho(\gamma) s(m) \) is a constant infinitesimal isometry, given by the variation \( \hat{\rho}(\gamma) \) of holonomy isometries \( \rho(\gamma) \in G \) (see Prop 2.3(a) of \([21]\)). Here \( \hat{\rho} : \pi_1(X) \to \mathfrak{g} \) satisfies the cocycle condition \( \hat{\rho}(\gamma \gamma_2) = \hat{\rho}(\gamma_1) + Ad\rho(\gamma_1)\hat{\rho}(\gamma_2) \), so it represents a class in group cohomology \([\hat{\rho}] \in H^1(\pi_1(X); Ad\rho)\), describing the variation of holonomy representations \( \rho_t \).

Regarding \( s \) as a vector-valued function with values in the vector space \( \mathfrak{g} \), its differential \( \hat{\omega} = ds \) satisfies \( \hat{\omega}(\gamma m) = Ad\rho(\gamma)\hat{\omega}(m) \) so it descends to a closed 1-form \( \omega \) on \( X \) with values in the bundle \( E \). Hence it determines a de Rham cohomology class \([\omega] \in H^1(X; E)\). This agrees with the group cohomology class \([\hat{\rho}]\) under the de Rham isomorphism \( H^1(X; E) \cong H^1(\pi_1(X); Ad\rho) \). Also, we note that the translational part of \( \omega \) can be regarded as a \( TX \)-valued 1-form on \( X \). Its symmetric part is exactly the form \( \eta \) defined above (see Prop 2.3(b) of \([21]\)), describing the infinitesimal change in metric on \( X \).

On the other hand, a family of hyperbolic structures determines only an equivalence class of families of developing maps and we need to see how replacing one family by an equivalent family changes both the group cocycle and the de Rham cocycle. Recall that a family equivalent to \( \Phi_t \) is of the form \( k_t\Phi_t\hat{f}_t \) where \( k_t \) are isometries of \( \mathbb{H}^n \) and \( \hat{f}_t \) are lifts of diffeomorphisms \( f_t \) from \( X \) to itself. We assume that \( k_0 \) and \( \hat{f}_0 \) are the identity.

The \( k_t \) term changes the path \( \rho_t \) of holonomy representations by conjugating by \( k_t \). Infinitesimally, this changes the cocycle \( \hat{\rho} \) by a coboundary in the sense of group cohomology. Thus it leaves the class in \( H^1(\pi_1(X); Ad\rho) \) unchanged. The diffeomorphisms \( f_t \) amount to choosing a different map from \( X_0 \) to \( X_t \). But \( f_t \) is isotopic to \( f_0 = \text{identity} \), so the lifts \( \hat{f}_t \) don’t change the group cocycle at all. It follows that equivalent families of hyperbolic structures determine the same group cohomology class.

If, instead, we view the infinitesimal deformation as represented by the \( E \)-valued 1-form \( \omega \), we note that the infinitesimal effect of the isometries \( k_t \) is to add a constant to \( s : \tilde{X} \to \tilde{E} \). Thus, \( ds \), its projection \( \omega \), and the infinitesimal variation of metric are all unchanged. However, the infinitesimal effect of the \( f_t \) is to change the vector field on \( \tilde{X} \) by the lift of a globally defined vector field on \( X \). This changes \( \omega \) by the derivative of a globally defined section of \( E \). Hence, it doesn’t affect the de Rham cohomology class in \( H^1(X; E) \). The corresponding infinitesimal change of metric is altered by the Lie derivative of a globally defined vector field on \( X \).

### 3. Infinitesimal harmonic deformations

In the previous section, we saw how a family of hyperbolic structures leads, at the infinitesimal level, to both a group cohomology class and a de Rham cohomology class. Each of these objects has certain advantages and disadvantages. The group cohomology class is determined by its values on a finite number of group
generators and the equivalence relation, dividing out by coboundaries which represent infinitesimal conjugation by a Lie group element, is easy to understand. Local changes in the geometry of the hyperbolic manifolds are not encoded, but important global information like the infinitesimal change in the lengths of geodesics is easily derivable from the group cohomology class. However, the chosen generators of the fundamental group may not be related in any simple manner to the hyperbolic structure, making it unclear how the infinitesimal change in the holonomy representation affects the geometry of the hyperbolic structure. Furthermore, it is usually hard to compute even the dimension of $H^1(\pi_1(X); \text{Ad}\rho)$ by purely algebraic means and much more difficult to find explicit classes in this cohomology group.

The de Rham cohomology cocycle does contain information about the local changes in metric. The value of the corresponding group cocycle applied to an element $\gamma \in \pi_1(X)$ can be computed simply by integrating an $E$-valued 1-form representing the de Rham class around any loop in the homotopy class of $\gamma$ that element; this is the definition of the de Rham isomorphism map. However, it is generally quite difficult to find such a 1-form that is sufficiently explicit to carry out this computation. Furthermore, the fact that any de Rham representative can be altered, within the same cohomology class, by adding an exact $E$-valued 1-form, (which can be induced by any smooth vector field on $X$), means that the behavior on small open sets is virtually arbitrary, making it hard to extrapolate to information on the global change in the hyperbolic metric.

In differential topology, one method for dealing with the large indeterminacy within a real-valued cohomology class is to use Hodge theory. The existence and uniqueness of a closed and co-closed (harmonic) 1-form within a cohomology class for a closed Riemannian manifold is now a standard fact. Similar results are known for complete manifolds and for manifolds with boundary, where uniqueness requires certain asymptotic or boundary conditions on the forms. By putting a natural metric on the fibers of the bundle $E$, the same theory extends to the de Rham cohomology groups, $H^1(X; E)$, that arise in the deformation theory of hyperbolic structures. The fact that these forms are harmonic implies that they satisfy certain nice elliptic linear partial differential equations. In particular, for a harmonic representative $\omega \in H^1(X; E)$, the infinitesimal change in metric $\eta$, which appears as the symmetric portion of the translational part of $\omega$, satisfies equations of this type. As we will see in the next section, these are the key to the infinitesimal rigidity of hyperbolic structures.

For manifolds with hyperbolic metrics, the theory of harmonic maps provides a non-linear generalization of this Hodge theory, at least for closed manifolds. For non-compact manifolds or manifolds with boundary, the asymptotic or boundary conditions needed for this theory are more complicated than those needed for the Hodge theory. However, at least the relationship described below between the defining equations of the two theories continues to be valid in this general context.

It is known that, given a map $f : X \to X'$ between closed hyperbolic manifolds, there is a unique harmonic map homotopic to $f$. (In fact, this holds for negatively curved manifolds. See [14].) Specifically, if $X = X'$ and $f$ is homotopic to the identity, the identity map is this unique harmonic map. Associated to a 1-parameter family $X_t$ of hyperbolic structures on $X$ is a 1-parameter family of developing maps from the universal cover $\tilde{X}$ of $X$ to $\mathbb{H}^n$. Using these maps to pull back the metric on $\mathbb{H}^n$ defines a 1-parameter family of metrics on $\tilde{X}$, and dividing out by the group
of covering transformations determines a family of hyperbolic metrics \( g_t \) on \( X \). However, a hyperbolic structure only determines an equivalence class of developing maps. Because of this equivalence relation, the metrics, \( g_t \) are only determined, for each fixed \( t \), up to pull-back by a diffeomorphism of \( X \). For the smooth family of hyperbolic metrics \( g_t \) on \( X \), we consider the identity map as a map from \( X \), equipped with the metric \( g_0 \), to \( X \), equipped with the metric \( g_t \). For \( t = 0 \), the identity map is harmonic, but in general it won’t be harmonic. Choosing the unique harmonic map homotopic to the identity for each \( t \) and using it to pull back the metric \( g_t \) defines a new family of metrics beginning with \( g_0 \). (For small values of \( t \) the harmonic map will still be a diffeomorphism.) By uniqueness and the behavior of harmonic maps under composition with an isometry, the new family of metrics depends only on the family of hyperbolic structures. In this way, we can pick out a canonical family of metrics from the family of equivalence classes of metrics.

If we differentiate this “harmonic” family of metrics associated to a family of hyperbolic structures at \( t = 0 \), we obtain a symmetric 2-tensor which describes the infinitesimal change of metric at each point of \( X \). Using the underlying hyperbolic metric, a symmetric 2-tensor on \( X \) can be viewed as a symmetric \( TX \)-valued 1-form. This is precisely the form \( \eta \) described above which is the symmetric portion of the translational part of the Hodge representative \( \omega \in H^1(X; E) \), corresponding to this infinitesimal deformation of the hyperbolic structure.

Thus, the Hodge representative in the de Rham cohomology group corresponds to an infinitesimal harmonic map. The corresponding infinitesimal change of metric has the property that it is \( L^2 \)-orthogonal to the trivial variations of the initial metric given by the Lie derivative of compactly supported vector fields on \( X \).

We now specialize to the case of interest in this paper, 3-dimensional hyperbolic cone-manifolds. We recall some of the results and computations derived in [21]. Let \( M_t \) be a smooth family of hyperbolic cone-manifold structures on a 3-dimensional manifold \( M \) with cone angles \( \alpha_t \) along a link \( \Sigma \), where \( 0 \leq \alpha_t \leq 2\pi \). By the Hodge theorem proved in [21], the corresponding infinitesimal deformation at time \( t = 0 \) has a unique Hodge representative whose translational part is a \( TX \)-valued 1-form \( \eta \) on \( X = M - \Sigma \) satisfying

\[
\begin{align*}
D^*\eta &= 0, \\
D^*D\eta &= -\eta.
\end{align*}
\]

Here \( D : \Omega^1(X; TX) \to \Omega^2(X; TX) \) is the exterior covariant derivative, defined, in terms of the Riemannian connection from the hyperbolic metric on \( X \), by

\[
D\eta(v, w) = \nabla_v \eta(w) - \nabla_w \eta(v) - \eta([v, w])
\]

for all vectors fields \( v, w \) on \( X \), and \( D^* : \Omega^2(X; TX) \to \Omega^1(X; TX) \) is its formal adjoint. Further, \( \eta \) and \( *D\eta \) determine symmetric and traceless linear maps \( T_x X \to T_x X \) at each point \( x \in X \).

Inside an embedded tube \( U = U_R \) of radius \( R \) around the singular locus \( \Sigma \), \( \eta \) has a decomposition:

\[
\eta = \eta_m + \eta_l + \eta_c
\]

where \( \eta_m \), \( \eta_l \) are “standard” forms changing the holonomy of peripheral group elements, and \( \eta_c \) is a correction term with \( \eta_c, D\eta_c \) in \( L^2 \).

We think of \( \eta_m \) as an ideal model for the infinitesimal deformation in a tube around the singular locus; it is completely determined by the rate of change of cone
angle. Its effect on the complex length $L$ of any peripheral element satisfies
\[ \frac{dL}{d\alpha} = \frac{L}{\alpha}. \]
In particular, the (real) length $\ell$ of the core geodesic satisfies
\[ \frac{d\ell}{d\alpha} = \frac{\ell}{\alpha} \quad (4) \]
for this model deformation.

This model is then “corrected” by adding $\eta_l$ to get the actual change in complex length of the core geodesic and then by adding a further term $\eta_c$ that doesn’t change the holonomy of the peripheral elements at all, but is needed to extend the deformation in the tube $U$ over the rest of the manifold $X$.

One special feature of the 3-dimensional case is the complex structure on the Lie algebra $\mathfrak{g} \cong \mathfrak{sl}_2\mathbb{C}$ of infinitesimal isometries of $\mathbb{H}^3$. The infinitesimal rotations fixing a point $p \in \mathbb{H}^3$ can be identified with $\mathfrak{su}(2) \cong \mathfrak{so}(3)$, and the infinitesimal pure translations at $p$ correspond to $i \mathfrak{su}(2) \cong \mathfrak{t}_p \mathbb{H}^3$. Geometrically, if $t \in T_p \mathbb{H}^3$ represents an infinitesimal translation, then $i t$ represents an infinitesimal rotation with axis in the direction of $t$. Thus, on a hyperbolic 3-manifold $X$ we can identify the bundle $E$ of (germs of) infinitesimal isometries with the complexified tangent bundle $T X \otimes \mathbb{C}$.

In [21] it was shown that the corresponding harmonic 1-form $\omega$ with values in the infinitesimal isometries of $\mathbb{H}^3$ can be written in this complex notation as:
\[ \omega = \eta + i \ast D\eta. \quad (5) \]
There is decomposition of $\omega$ in the neighborhood $U$ analogous to that (3) of $\eta$ as
\[ \omega = \omega_m + \omega_l + \omega_c, \quad (6) \]
where only $\omega_m$ and $\omega_l$ change the peripheral holonomy and $\omega_c$ is in $L^2$.

The fact that the hyperbolic structure on $X = M - \Sigma$ is incomplete makes the existence and uniqueness of a Hodge representative substantially more subtle than the standard theory for complete hyperbolic structures (including structures on closed manifolds). Certain conditions on the behavior of the forms as they approach the singular locus are required. This makes the theory sensitive to the value of the cone angle at the singularity; in particular, this is where the condition that the cone angle be at most $2\pi$ arises. The fact that $\omega_c$ is in $L^2$ is a reflection of these asymptotic conditions. In the final section of this paper, we discuss a new version of this Hodge theory, involving boundary conditions on the boundary of a tube around the singular locus, that removes the cone angle condition, replacing it with a lower bound on the radius of the tube.

4. Effective Rigidity

In this section we explain how the equations satisfied by the harmonic representative of an infinitesimal deformation lead to local rigidity results. We then come to one of our primary themes, that the arguments leading to local rigidity can be made computationally effective. By this we mean that even when there does exist a non-trivial deformation of a hyperbolic structure, the same equations can be used to bound both the geometric and analytic effect of such a deformation. This philosophy carries over into many different contexts, but here we will continue to focus on finite volume 3-dimensional hyperbolic cone-manifolds.
The first step is to represent an infinitesimal deformation by a Hodge (harmonic) representative $\omega$ in the cohomology group $H^1(X; E)$, as discussed in the previous section. If $X$ is any hyperbolic 3-manifold, the symmetric real part of this representative is a 1-form $\eta$ with values in the tangent bundle of $X$, satisfying the Weitzenböck-type formula:

$$D^* D\eta + \eta = 0$$

where $D$ is the exterior covariant derivative on such forms and $D^*$ is its adjoint. First, suppose $X$ is closed. Taking the $L^2$ inner product of this formula with $\eta$ and integrating by parts gives the formula

$$||D\eta||_{X}^2 + ||\eta||_{X}^2 = 0.$$

(Here $||\eta||_{X}^2$ denotes the square of the $L^2$ norm of $\eta$ on $X$.) Thus $\eta = 0$ and the deformation is trivial. This is the proof of local rigidity for closed hyperbolic 3-manifolds, using the methods of Calabi [11], Weil [29] and Matsushima-Murakami [24].

When $X$ has boundary or is non-compact, there will be a Weitzenböck boundary term $b$:

$$||D\eta||_{X}^2 + ||\eta||_{X}^2 = b.$$  

(7)

If the boundary term is non-positive, the same conclusion holds: the deformation is trivial. When $X = M - \Sigma$, where $M$ is a hyperbolic cone-manifold with cone angles at most $2\pi$ along its singular set $\Sigma$, it was shown in [21] that, for a deformation which leaves the cone angle fixed, it is possible to find a representative as above for which the boundary term goes to zero on the boundary of tubes around the singular locus whose radii go to zero. Again, such an infinitesimal deformation must be trivial, proving local rigidity rel cone angles.

On the other hand, Thurston has shown ([27, Chap. 5]) that there are non-trivial deformations of the (incomplete) hyperbolic structures on $X = M - \Sigma$. By local rigidity rel cone angles such a deformation must change the cone angles, implying that it is always possible to alter the cone angles by a small amount. Using the implicit function theorem it is further possible to show that the variety of representations $\pi_1(X) \to PSL_2(\mathbb{C})$ is smooth near the holonomy representation of such a hyperbolic cone-manifold. This leads to a local parametrization of hyperbolic cone-manifolds by cone angles.

**Theorem 4.1 ([21]).** For a 3-dimensional hyperbolic cone-manifold with singularities along a link with cone angles $\leq 2\pi$, there are no deformations of the hyperbolic structure keeping the cone angles fixed. Furthermore, the nearby hyperbolic cone-manifold structures are parametrized by their cone angles.

The argument for local rigidity rel cone angles actually provides further information about the boundary term. To explain this, we need to give a more detailed description of some of the work in [21]. This will provide not only a fuller explanation of the proof that there are no deformations fixing the cone angles, but also additional information about the deformations that do occur.

Assume that $\eta$ represents a non-trivial infinitesimal deformation. Recall that, inside a tube around the singular locus, $\eta$ can be decomposed as $\eta = \eta_m + \eta_l + \eta_c$, where only $\eta_m$ changes the cone angle. Leaving the cone angle unchanged is equivalent to the vanishing of $\eta_m$. As we shall see below, the boundary term for $\eta_m$ by itself is positive. Roughly speaking, $\eta_m$ contributes positive quantities to
the boundary term, while everything else gives negative contributions. (There are also cross-terms which are easily handled.) The condition that the entire boundary term be positive not only implies that the \( \eta_m \) term must be non-zero (which is equivalent to local rigidity rel cone angles), but also puts strong restrictions on the \( \eta_l \) and \( \eta_r \) terms. This is the underlying philosophy for the estimates in this section.

In order to implement this idea, we need to derive a formula for the boundary term in (7) as an integral over the boundary of \( X \). For details we refer to [21].

Let \( U_r \) denote a tubular neighborhood of radius \( r \) about the singular locus of \( M \) and let \( X = M - U_r \); it will always be assumed that \( r \) is small enough so that \( U_r \) will be embedded. Let \( T_r \) denote the boundary torus of \( U_r \), oriented by the normal \( \frac{\partial}{\partial r} \), (which is the \textit{inward} normal for \( X \)). For any \( TX \)-valued 1-forms \( \alpha, \beta \) we define

\[
(8) \quad b_r(\alpha, \beta) = \int_{T_r} *D\alpha \wedge \beta.
\]

Note that in this integral, \( *D\alpha \wedge \beta \) denotes the real valued 2-form obtained using the wedge product of the form parts, and the geometrically defined inner product on the vector-valued parts of the \( TX \)-valued 1-forms \( *D\alpha \) and \( \beta \).

As above, we express the Hodge \( E \)-valued 1-form as \( \omega = \eta + i *D\eta \) where \( D^*D\eta + \eta = 0 \). Fix a radius \( R \), remove the tubular neighborhood \( U_R \), and denote \( M - U_R \) by \( X \). Then one computes that the Weitzenböck boundary term \( b \) in (7) equals \( b_R(\eta, \eta) \) (see Proposition 1.3 and p. 36 of [21]). This implies:

**Lemma 4.2.**

\[
(9) \quad b_R(\eta, \eta) = ||\eta||^2_X + ||D\eta||^2_X = ||\omega||^2_X.
\]

In particular, we see that \( b_R(\eta, \eta) \) is \textit{non-negative}. Writing \( \eta = \eta_0 + \eta_c \) where \( \eta_0 = \eta_m + \eta_l \), we analyze the contribution from each part. First, using the Fourier decomposition for \( \eta_c \) obtained in [21], it turns out that the cross-terms vanish so that the boundary term is simply the sum of two boundary terms:

\[
(10) \quad b_R(\eta, \eta) = b_R(\eta_0, \eta_0) + b_R(\eta_c, \eta_c).
\]

Next, we see that the contribution, \( b_R(\eta_c, \eta_c) \), from the part of the “correction term” that doesn’t affect the holonomy of the peripheral elements, is \textit{non-positive}. In fact,

**Proposition 4.3.**

\[
(11) \quad b_R(\eta_c, \eta_c) = -(||\eta_c||^2_{U_R} + ||D\eta_c||^2_{U_R}) = -||\omega_c||^2_{U_R}.
\]

We have assumed that \( \omega_c \) is harmonic in a neighborhood of \( U_R \) so the same argument applied above to \( \eta \) can be applied to \( \eta_c \) on this neighborhood. Consider a region \( N \) between tori at distances \( r, R \) from \( \Sigma \) with \( r < R \). As before, integration by parts over this region implies that the difference \( b_r(\eta_c, \eta_c) - b_R(\eta_c, \eta_c) \) equals \( ||\omega_c||^2_{U_r} \). Then the main step is to show that \( \lim_{r \to 0} b_r(\eta_c, \eta_c) = 0 \). This follows from the proof of rigidity rel cone angles (in section 3 of [21]), since \( \eta_c \) represents an infinitesimal deformation which doesn’t change the cone angle.

Combining (10) with (9) and (11), we obtain:

\[
(12) \quad b_R(\eta_0, \eta_0) = ||\omega||^2_{M - U_R} + ||\omega_c||^2_{U_R}.
\]

In particular, this shows that

\[
(13) \quad 0 \leq b_R(\eta_0, \eta_0),
\]
and that

\[(14) \quad ||\omega||_{M-U_R}^2 \leq b_R(\eta_0, \eta_0).\]

**Remark 4.4.** We emphasize that the only place in the derivation of (13) and (14) that we have used the analysis near the singular locus from [21] is in the proof of Proposition 4.3. Furthermore, all that is required from this Proposition is the fact that \(b_R(\eta_c, \eta_c)\) is non-positive. This, together with (9) and (10), implies both (13) and (14). In the final section of this paper, we describe another method for finding a Hodge representative for which \(b_R(\eta_c, \eta_c)\) is non-positive. This method requires a tube radius of at least a universal size, but no bound on the cone angle. Once this is established, all the results described here carry over immediately to the case where the tube radius condition is satisfied.

We will focus here on applications of the inequality (13), which is the primary use of this analysis in [22]. The work of Brock and Bromberg discussed in these proceedings ([6]) also requires the second inequality (14). This is discussed further in the final section.

As we show below, (13) implies that, in the decomposition \(\eta_0 = \eta_m + \eta_l\), the \(\eta_m\) term must be non-trivial for a non-trivial deformation. This is equivalent to local rigidity rel cone angles. The positivity result (13) can also be used to find upper bounds on \(b_R(\eta_0, \eta_0)\). On the face of it, this may seem somewhat surprising, but, as we explain below, the algebraic structure of the quadratic form \(b_R(\eta_0, \eta_0)\) makes it quite straightforward to derive such bounds.

The possible harmonic forms \(\eta_0 = \eta_m + \eta_l\) give a 3-dimensional real vector space \(W\) representing models for deformations of hyperbolic cone-manifold structures in a neighborhood of the boundary torus. Here \(\eta_m\) lies in a 1-dimensional subspace \(W_m\) containing deformations changing the cone angle, while \(\eta_l\) lies in the 2-dimensional subspace \(W_l\) consisting of deformations leaving the cone angle unchanged. In [21], we describe explicit \(TX\)-valued 1-forms giving bases for these subspaces.

Now consider the quadratic form \(Q(\eta_0) = b_R(\eta_0, \eta_0)\) on the vector space \(W\). One easily computes that in all cases, \(Q\) is positive definite on \(W_m\) and negative definite on \(W_l\), so \(Q\) has signature \(+ - -\). This gives the situation shown in the following figure.

![Diagram showing the cone angles and quadratic form Q](image-url)
The positivity condition (13) says that \( \eta_0 \) lies in the cone where \( Q \geq 0 \) for any deformation which extends over the manifold \( X \). Further, \( \eta_m \) must be non-zero if the deformation is non-trivial; so the cone angle must be changed. Thus (13) implies local rigidity rel cone angles.

As noted above, the local parametrization by cone angles (Theorem 4.1) follows from this, and a smooth family of cone-manifold structures \( M_t \) is completely determined by a choice of parametrization of the cone angles \( \alpha_t \). We are free to choose this parametrization as we wish. Then the term \( \eta_m \) is completely determined by the derivative of the cone angle.

Once \( \eta_m \) is fixed, the inequality \( Q(\eta_0) \geq 0 \) restricts \( \eta_0 = \eta_m + \eta_l \) to lie in an ellipse (as illustrated above). Since \( Q \) has a positive maximum on this compact set, this gives an explicit upper bound for \( Q(\eta_0) \) for any deformation.

It turns out to be useful to parametrize the cone-manifolds by the square of the cone angle \( \alpha \); i.e., we will let \( t = \alpha^2 \). With this choice of parametrization we obtain:

\[
(15) \quad b_R(\eta_m, \eta_m) = \frac{\text{area}(TR)}{16m^4} (\tanh R + \tanh^3 R),
\]

where \( m = \alpha \sinh(R) \) is the length of the meridian on the tube boundary \( TR \).

Essentially, any contribution to \( b_R(\eta_0, \eta_0) \) from \( \eta_l \) will be negative; the cross-terms only complicate matters slightly. Computation leads easily to the following upper bound:

\[
(16) \quad b_R(\eta_0, \eta_0) \leq \frac{\text{area}(TR)}{8m^4}.
\]

Combining (12) and (16), we see that the boundary formula (7) leads to the estimate:

**Theorem 4.5.**

\[
(17) \quad ||\omega||^2_{M - U_R} + ||\omega_c||^2_{U_R} \leq \frac{\text{area}(TR)}{8m^4},
\]

where \( m = \alpha \sinh(R) \) is the length of the meridian on the tube boundary \( TR \).

**Remark 4.6.** A crucial property of the inequalities (16) and (17) is their dependence only on the geometry of the boundary torus, not on the rest of the hyperbolic manifold. This is the reason that \textit{a priori} bounds, independent of the underlying manifold can be derived by these methods. Furthermore, it is possible to find geometric conditions, like a very short core geodesic, that force the upper bounds to be very small.

In particular, (17) provides an upper bound on the \( L^2 \) norm of \( \omega \) on the complement of the tubular neighborhood of the singular locus. Such a bound can be used to bound the infinitesimal change in geometric quantities, like lengths of geodesics, away from the singular locus. Arbitrarily short core geodesics typically lead to the conclusion that these infinitesimal changes are arbitrarily small, a useful fact when studying approximation by manifolds with short geodesics. This will be discussed further in the final section.

The principle that the contribution of \( \eta_l \) to \( b_R(\eta_0, \eta_0) \) is essentially negative (ignoring the cross-terms) while \( b_R(\eta_0, \eta_0) \) is positive also provides a bound on the size of \( \eta_l \). Although \( \eta_m \) does change the length of the singular locus, its effect is
fixed since $\eta_m$ is fixed by the parametrization. The only other term affecting the length of the singular locus is $\eta$. Explicit computation ultimately leads to the following estimate:

**Theorem 4.7** ([22]). Consider any smooth family of hyperbolic cone structures on $M$, all of whose cone angles are at most $2\pi$. For any component of the singular set, let $\ell$ denote its length and $\alpha$ its cone angle. Suppose there is an embedded tube of radius $R$ around that component. Then

$$\frac{d\ell}{d\alpha} = \frac{\ell}{\alpha}(1 + E),$$

where

$$-\frac{1}{\sinh^2(R)} \left( \frac{2\sinh^2(R) + 1}{2\sinh^2(R) + 3} \right) \leq E \leq \frac{1}{\sinh^2(R)}.$$

Note that the ‘error’ term $E$ represents the deviation from the standard model; compare (1).

This is the key estimate in [22]. The next section discusses some of the applications of this inequality to the theory of hyperbolic Dehn surgery.

### 5. A quantitative hyperbolic Dehn surgery theorem

We begin with a non-compact, finite volume hyperbolic 3-manifold $X$, which, for simplicity, we assume has a single cusp. In the general case the cusps are handled independently. The manifold $X$ is the interior of a compact manifold which has a single torus boundary, which is necessarily incompressible. By attaching a solid torus by a diffeomorphism along this boundary torus, one obtains a closed manifold, determined up to diffeomorphism by the isotopy class of the non-trivial simple closed curve $\gamma$ on the torus which bounds a disk in the solid torus. The resulting manifold is denoted by $X(\gamma)$ and $\gamma$ is referred to either as the meridian or the surgery curve. The process is called Dehn filling (or Dehn surgery).

The set of Dehn fillings of $X$ is thus parametrized by the set of simple closed curves on the boundary torus; after choosing a basis for $H_1(T^2, \mathbb{Z})$, these are parametrized by pairs $(p, q)$ of relatively prime integers. Thurston ([27]) proved that, for all but a finite number of choices of $\gamma$, $X(\gamma)$ has a complete, smooth hyperbolic structure. However, the proof is computationally ineffective. It gives no indication of how many non-hyperbolic fillings there are or which curves $\gamma$ they might correspond to. In particular, it left open the possibility that there is no upper bound to the number of non-hyperbolic fillings as one varies over all possible $X$.

One approach to putting a hyperbolic metric on $X(\gamma)$ is through families of hyperbolic cone-manifolds. The complete metric on the open manifold $X$ is deformed through incomplete metrics whose metric completions are hyperbolic cone-manifold structures on $X(\gamma)$, with the singular set equal to the core of the added solid torus. The complete structure can be considered as a cone-manifold with angle $0$. The cone angle is increased monotonically, and, if the angle of $2\pi$ is reached, it defines a smooth hyperbolic metric on $X(\gamma)$.

Thurston’s proof of his finiteness theorem actually shows that, for any non-trivial simple closed curve $\gamma$ on the boundary torus, it is possible to find such a cone-manifold structure on $X(\gamma)$ for sufficiently small (possibly depending on $X$) values of the cone angle. This also follows from Theorem 4.1, which further implies
that, for any angle at most $2\pi$, it is always possible to change the cone angle a small amount, either to increase it or to decrease it. Locally, this can be done in a unique way since the cone angles locally parametrize the set of cone-manifold structures on $X(\gamma)$. Although there are always small variations of the cone-manifold structure, the structures may degenerate in various ways as a family of angles reaches a limit. In order to find a smooth hyperbolic metric on $X(\gamma)$ it is necessary and sufficient to show that no degeneration occurs before the angle $2\pi$ is attained.

In [22] the concept of convergence of metric spaces in the Gromov-Hausdorff topology (see, e.g., [17], [18], or [13]) is utilized to rule out degeneration of the metric as long as there is a lower bound on the tube radius and an upper bound on the volume. The main issue is to show that the injectivity radius in the complement of the tube around the singular locus stays bounded below. One shows that, if the injectivity radius goes to zero, a new cusp develops. Analysis of Dehn filling on this new cusp leads to a contradiction of Thurston’s finiteness theorem. The argument is similar to a central step in the proof of the orbifold theorem (see, e.g., [13], [3], or [4]). One result proved in [22] is:

**Theorem 5.1.** Let $M_t$, $t \in [0, t_\infty)$, be a smooth path of closed hyperbolic cone-manifold structures on $(M, \Sigma)$ with cone angle $\alpha_t$ along the singular locus $\Sigma$. Suppose $\alpha_t \to \alpha \geq 0$ as $t \to t_\infty$, that the volumes of the $M_t$ are bounded above, and that there is a positive constant $R_0$ such that there is an embedded tube of radius at least $R_0$ around $\Sigma$ for all $t$. Then the path extends continuously to $t = t_\infty$ so that as $t \to t_\infty$, $M_t$ converges in the bilipschitz topology to a cone-manifold structure $M_\infty$ on $M$ with cone angles $\alpha$ along $\Sigma$.

As a result of this theorem, we can focus on controlling the tube radius. A general principle for smooth hyperbolic manifolds is that short geodesics have large embedded tubes around them. This follows from the Margulis lemma or, alternatively in dimension 3, from Jørgensen’s inequality. However, both of these results require that the holonomy representation be discrete which is almost never true for cone-manifolds. In fact, the statement that there is a universal lower bound to the tube radius around a short geodesic is easily seen to be false for cone-manifolds. To see this, consider a sequence of hyperbolic cone-manifolds whose diameters go to zero. Then, both the length of the singular locus and the tube radius go to zero. For example, $S^3$ with singular locus the figure-8 knot and cone angles approaching $\frac{2\pi}{3}$ from below behaves in this manner.

However, a more subtle statement is true. If, at the beginning of a family of hyperbolic cone-manifold structures, the tube radius is sufficiently large and the length of the core curve is sufficiently small, then as long as the core curve remains sufficiently small, the tube radius will be bounded from below. The precise statement is given below. It should be noted that it is actually the product of the cone angle and the core length that must be bounded from above.

**Theorem 5.2 ([22]).** Let $M_s$ be a smooth family of finite volume 3-dimensional hyperbolic cone-manifolds, with cone angles $\alpha_s$, $0 \leq s < 1$, where $\lim_{s \to 1} \alpha_s = \alpha_1$. Suppose the tube radius $R$ satisfies $R \geq 0.531$ for $s = 0$ and $\alpha_s \ell_s \leq 1.019$ holds for all $s$, where $\ell_s$ denotes the length of the singular geodesic. Then the tube radius satisfies $R \geq 0.531$ for all $s$.

The proof of this theorem involves estimates on “tube packing” in cone-manifolds. We look in a certain cover of the complement of the singular locus where the tube
around the singular geodesic lifts to many copies of a tube of the same radius around infinite geodesics which are lifts of the core geodesic. One of the lifts is chosen and then packing arguments are developed to show that other lifts project onto the chosen one in a manner that fills up at least a certain amount of area in the tube in the original cone-manifold. In this way, we derive a lower bound for the area of the boundary torus which depends only on the product $\alpha \ell$, assuming $R \geq 0.531$. This, in turn, further bounds the tube radius strictly away from 0.531 as long as $\alpha \ell$ is sufficiently small. Thus, if this product stays small the tube radius stays away from the value 0.531 so that the estimates continue to hold. The estimates derived actually prove that, as long as these bounds hold, then $R \to \infty$ as $\alpha \ell \to 0$. Furthermore, the rate at which the tube radius goes to infinity is bounded below.

This result shows that the tube radius can be bounded below by controlling the behavior of the holonomy of the peripheral elements. The cone angle is determined by the parametrization so it suffices to understand the longitudinal holonomy. In the previous section, we derived estimates on the derivative of the core length with respect to cone angle, where the bounds depend on the tube radius. On the other hand, from Theorem 5.2, the change in the tube radius can be controlled by the product of the cone angle and the core length. Putting these results together, we arrive at differential inequalities that provide strong control on the change in the geometry of the maximal tube around the singular geodesic, including the tube radius.

A horospherical torus which is a cross-section of the cusp for the complete structure on $X$ has an intrinsic flat structure (i.e., zero curvature metric). Any two such cross-sections differ only by scaling. Given a choice $\gamma$ of surgery curve, an important quantity associated to this flat structure is the normalized flat length of $\gamma$, which, by definition equals the geodesic length of $\gamma$ in the flat metric, scaled to have area 1. Clearly this is independent of the choice of cross-section. Its significance comes from the fact that its square (usually called the extremal length of $\gamma$) is the limiting value of the ratio $\frac{\ell}{\alpha}$ of the cone angle to the core length as $\alpha \to 0$ for hyperbolic cone-manifold structures on $X(\gamma)$. In particular, near the complete structure, even though $\ell$ and $\alpha$ individually approach zero, their ratio approaches a finite, non-zero value.

The estimate implies that, as long as the tube radius isn’t too small, $\frac{d\ell}{d\alpha}$ is approximately equal to the ratio $\frac{\ell}{\alpha}$. In the case of equality (when the error term $E$ in (18) equals zero), the ratio $\frac{\ell}{\alpha}$ will remain constant. A small error term implies that the ratio doesn’t change too much. If the initial value of the reciprocal, $\frac{1}{\frac{\ell}{\alpha}}$ is large, then $\frac{\ell}{\alpha}$ will be small and stay small as long as the tube radius doesn’t get too small. But this implies that the product of the cone angle and the core length will remain small. In turn, the packing argument then provides a lower bound to the tube radius.

Formally, this can be expressed as a differential inequality, bounding the change of core length with respect to the change in cone angle as a function of the core length and the cone angle. Solving this inequality, with initial conditions coming from the normalized length, gives a proof of the following theorem:

**Theorem 5.3** (22). Let $X$ be a complete, finite volume orientable hyperbolic 3-manifold with one cusp and let $T$ be a horospherical torus embedded as a cusp cross section. Fix $\gamma$, a simple closed curve on $T$. Let $X_\alpha(\gamma)$ denote the cone-manifold
structure on $X(\gamma)$ with cone angle $\alpha$ along the core, $\Sigma$, of the added solid torus, obtained by increasing the angle from the complete structure. If the normalized length of $\gamma$ on $T$ is at least $7.515$, then there is a positive lower bound to the tube radius around $\Sigma$ in $X_\alpha(\gamma)$ for all $\alpha$ satisfying $2\pi \geq \alpha \geq 0$.

Given $X$ and $T$ as in Theorem 5.3, choose any non-trivial simple closed curve $\gamma$ on $T$. There is a maximal sub-interval $J \subseteq [0, 2\pi]$ containing 0 such that there is a smooth family $M_\alpha$, where $\alpha \in J$, of hyperbolic cone-manifold structures on $X(\gamma)$ with cone angle $\alpha$. Thurston’s Dehn surgery theorem ([27]) implies that $J$ is non-empty and Theorem 4.1 implies that it is open. Theorem 5.1 implies that, with a lower bound on the tube radii and an upper bound on the volume, the path of $M_\alpha$’s can be extended continuously to the endpoint of $J$. Again, Theorem 4.1 implies that this extension can be made to be smooth. Hence, under these conditions $J$ will be closed. By Schläfli’s formula (see (19) below), the volumes decrease as the cone angles increase so they will clearly be bounded above. Theorem 5.3 provides initial conditions on $\gamma$ which guarantee that there will be a lower bound on the tube radii for all $\alpha \in J$. Thus, assuming Theorems 5.3 and 5.1, we have proved:

**Theorem 5.4.** Let $X$ be a complete, finite volume orientable hyperbolic 3-manifold with one cusp, and let $T$ be a horospherical torus which is embedded as a cross-section to the cusp of $X$. Let $\gamma$ be a simple closed curve on $T$ whose normalized length is at least $7.515$. Then the closed manifold $X(\gamma)$ obtained by Dehn filling along $\gamma$ is hyperbolic.

This result also gives a universal bound on the number of non-hyperbolic Dehn fillings on a cusped hyperbolic 3-manifold $X$, independent of $X$.

**Corollary 5.5.** Let $X$ be a complete, orientable hyperbolic 3-manifold with one cusp. Then at most 60 Dehn fillings on $X$ yield manifolds which admit no complete hyperbolic metric.

When there are multiple cusps the results are only slightly weaker. Theorem 5.4 holds without change. If there are $k$ cusps, the cone angles $\alpha_i$ and $\alpha$ are simply interpreted as $k$-tuples of angles. Having tube radius at least $R$ is interpreted as meaning that there are disjoint, embedded tubes of radius $R$ around all of components of the singular locus. The conclusion of Theorem 5.3 and hence of Theorem 5.4 holds when there are multiple cusps as long as the normalized lengths of all of the meridian curves are at least $7.515\sqrt{2} \approx 10.627$. At most 114 curves from each cusp need to be excluded. In fact, this can be refined to say that at most 60 curves need to be excluded from one cusp and at most 114 excluded from the remaining cusps. The rest of the Dehn filled manifolds are hyperbolic.

By Mostow rigidity the volume of a closed or cusped hyperbolic 3-manifold is a topological invariant. The set of volumes of hyperbolic 3-manifolds is well-ordered ([27]); the hyperbolic volume gives an important measure of the complexity of the manifold. It is therefore of interest to find the smallest volume hyperbolic manifold. This is conjectured to be the Weeks manifold which can be described as a surgery on the Whitehead link and has volume $\approx 0.9427$. On the other hand, the smallest volume of a hyperbolic 3-manifold with a single cusp is known ([12]) to equal $\approx 2.0299$, the volume of the figure eight knot complement.

Thus, one can attempt to estimate the volume of a closed hyperbolic 3-manifold by comparing it to the complete structure on the non-compact manifold obtained
by removing a simple closed geodesic from the closed manifold. It is conjectured
that these two hyperbolic structures are always connected by a smooth family of
cone-manifolds, with singular locus equal to the simple closed geodesic, with cone
angle $\alpha$ varying from $2\pi$ in the smooth structure on the closed manifold to 0 for the
complete structure on the complement of the geodesic. In such a family, Schl"afli’s
formula implies that the derivative of the volume $V$ satisfies the equation (see, e.g.
[20] or [13, Theorem 3.20]):

$$\frac{dV}{d\alpha} = -\frac{1}{2} \ell,$$

where $\ell$ denotes the length of the singular locus. Thus, controlling of the length
of the singular locus throughout the family of cone-manifolds would control the
change in the volume. In [22] it is shown that the derivative of the length of the
singular locus with respect to the cone angle is positive as long as the tube radius is
at least $\text{arctanh}(1/\sqrt{3})$ and much sharper statements are proved, using the packing
arguments, when the length of the singular locus is small.

It is also shown that, if the original simple closed geodesic is sufficiently short,
then such a family of cone-manifolds connecting the smooth structure to the com-
plete structure on the complement of the geodesic will always exist. To see this,
note that, for sufficiently short geodesics in a smooth structure, there will always be
a tube of radius greater than $\text{arctanh}(1/\sqrt{3})$. Thus the core length will decrease as
the cone angle decreases and, in particular, the product of the cone angle and the
core length will decrease. By Theorems 5.2 and 5.4 there can be no degeneration
and the complete structure will be reached. (It is not hard to show that the volume
is bounded above during the deformation.)

Combining these two ideas, we can bound the volume of a closed hyperbolic
3-manifold with a sufficiently short geodesic in terms of the associated cusped 3-
manifold. An example of an explicit estimate derived in this manner in [22] is:

**Theorem 5.6.** Let $M$ be a closed hyperbolic manifold whose shortest closed geodesic
$\tau$ has length at most 0.162. Then the hyperbolic structure on $M$ can be deformed
to a complete hyperbolic structure on $M - \tau$ by decreasing the cone angle along $\tau$
from $2\pi$ to 0. Furthermore, the volumes of these manifolds satisfy the inequality
$\text{Volume}(M) \geq \text{Volume}(M - \tau) - 0.329$. In particular, $\text{Volume}(M) \geq 1.701$ so that
it has larger volume than the closed hyperbolic manifold with the smallest known volume.

6. **Kleinian groups and boundary value theory**

In this section we give a brief description of generalizations and applications of
the deformation theory described in the previous sections.

The first way in which one could attempt to generalize the harmonic deformation
theory is to allow finitely many infinite volume, but geometrically finite, ends in
our hyperbolic cone-manifolds $X$. Without giving a formal definition, this structure
provides ends that, asymptotically, are like the ends of smooth geometrically finite
hyperbolic 3-manifolds. We further assume that there are no rank-1 cusps (so the
ends are like those Kleinian groups with compact convex cores). The term
“geometrically finite” will include this extra assumption throughout this section.
Each infinite volume end of such a geometrically finite cone-manifold determines a
conformal structure (in fact a complex projective structure) on a surface at infinity,
which can be used to (topologically) compactify the cone-manifold. We refer to these as *boundary conformal structures*.

When there is no singular set, then the quasi-conformal deformation theory developed by Ahlfors, Bers and others (see, e.g., [11], [2]) implies that such structures are parametrized by these conformal structures at infinity. In particular, they satisfy local rigidity relative to the boundary conformal structures; it is not possible locally to vary the hyperbolic structure without varying at least one of the conformal structures at infinity.

In [8] Bromberg extends the harmonic deformation theory outlined in the previous sections to such geometrically finite cone-manifolds, assuming that the cone angles are at most $2\pi$. In particular, it is proved there that there are no infinitesimal deformations of such structures that fix both the cone angles and the conformal structures at infinity, i.e., rigidity relative cone angles and boundary conformal structures. This generalizes the local theory both for smooth geometrically finite Kleinian groups and for finite volume cone-manifolds. However, it should be pointed out that, since the holonomy groups for these geometrically finite cone-manifolds are not usually discrete, the global quasi-conformal theory on the sphere at infinity which is the basis for the smooth (Ahlfors-Bers) theory can’t be used at all. So, new techniques are required. Similarly, since the deformation is only assumed to be “conformal at infinity”, not trivial at infinity, one cannot assume that there is a representative of this infinitesimal deformation for which the infinitesimal change of metric is asymptotically trivial. As a result, the asymptotic behavior of the $E$-valued forms representing the deformation must be analyzed at these infinite volume ends as well as near the singular locus. This makes the proof of the necessary Hodge theorem much more difficult. After the Hodge theorem is proved, the final step is to show that there is a harmonic (Hodge) representative for which the contribution to the Weitzenböck boundary term goes to zero near infinity. This requires some subtle analysis.

Once this is established, the analytic results from the finite volume cone-manifold theory go through without change. In particular, the inequalities (13) and (14) still apply. Once packing arguments, which imply non-degeneration results, are proved in this context, then the Dehn filling results, such as Theorem 5.4, can be generalized (with different numerical values). Similarly, there will be an analog, for smooth, geometrically finite hyperbolic manifolds, of Theorem 5.6. Recall that this theorem says that a closed, smooth hyperbolic manifold with a sufficiently short simple closed geodesic can be connected by a family of cone-manifolds to the complete structure on the manifold obtained by removing the geodesic.

Such packing and non-degeneration results are proved by Bromberg in [11]. However, his goal and the goal of subsequent papers by (various subsets of) Brock, Bromberg, Evans and Souto (see for example, [9], [5], [7]) is an analytically sharper version of this type of result. A goal in each of these papers is to approximate structures with very short (possibly singular) geodesics by ones with smaller cone angles, including ones with cusps. Not only is a path of cone-manifolds connecting the structures needed but this must be constructed so as to bound the distortion of the structure along the way.

As was remarked in Section 4, the results in [22] which we have discussed in the previous sections require only the inequality (13). However, the analysis in that section also led to the inequality (14) which bounds the $L^2$ norm of the harmonic
representative for the infinitesimal deformation. Further calculations there also provided an upper bound \( L_2 \) for this \( L^2 \) norm in terms of the geometry of the boundary torus. It is easy to check that, for any given non-zero cone angle, this upper bound will become arbitrarily small as the length of the singular locus goes to zero, assuming there is a lower bound to the tube radius.

This allows one to bound the \( L^2 \) norm of the harmonic representative \( \omega \) of an infinitesimal deformation at such a structure, where we are assuming that the deformation is infinitesimally conformal at infinity on the infinite volume ends. Such deformations are locally parametrized by cone angles. To be consistent with the previous sections, we use the parametrization \( t = \alpha^2 \), where \( \alpha \) is the cone angle. Then the following theorem holds:

**Theorem 6.1** ([10], [23]). Let \( X \) be a geometrically finite hyperbolic cone-manifold with no rank-1 cusps and with an embedded tube of radius at least \( \text{arctanh}(1/\sqrt{3}) \) around the singular locus \( \tau \). Suppose that \( \omega \) is the harmonic representative of an infinitesimal deformation of \( X \) as above. Then, for any fixed cone angle \( \alpha > 0 \) and any \( \epsilon > 0 \), there is a length \( \delta > 0 \), depending only on \( \alpha \) and \( \epsilon \), so that, if the length of \( \tau \) is less than \( \delta \), then \( ||\omega||_{L^2_{-U}} < \epsilon \), for some embedded tube \( U \) around \( \tau \).

Recall that the real part of \( \omega \) corresponds to the infinitesimal change in metric induced by the infinitesimal deformation. So, in particular, the above theorem gives an \( L^2 \) bound on the size of the infinitesimal change of metric. However, it is still necessary to bound the change in the hyperbolic structure in a more usable way. In [9] and [10] this was turned into a bound in the change of the projective structures at infinity. This is sufficient for the applications in those papers. For the applications in [5] and [7], an upper bound is needed on the bilipschitz constants of maps between structures along the path of cone-manifolds. For such a bound, it is necessary to turn the \( L^2 \) bounds on the infinitesimal change of metric into pointwise bounds. It is then also necessary to extend the bilipschitz maps into the tubes in a way that still has small bilipschitz constant. This is carried out in [5].

The work in [10] and [5] globalizes Theorem 6.1. It implies that, under the same assumptions, it is possible to find a path of cone-manifolds from the geometrically finite cone-manifold \( X \) to the complete structure on \( X \) with the geodesic removed, and that this can be done so that all the geometric structures along the path can be made arbitrarily close to each other. In [10] the distance between the structures is measured in terms of projective structures at infinity, whereas in [5] it is measured by the bilipschitz constant of maps. Since the core geodesic is removed in the cusped structure, the authors call these results “Drilling Theorems” (see [6]). These theorems provide very strong quantitative statements of the qualitative idea that hyperbolic structures with short geodesics are “close” to ones with cusps.

The careful reader will have noticed that Theorem 6.1 has no conditions on the size of the cone angles whereas the theory in [21] requires that the cone angles be at most \( 2\pi \). As stated, this theorem and hence the full Drilling Theorem depends on a harmonic deformation theory which has no conditions on the cone angle, but only requires the above lower bound on the tube radius. Such a theory is developed in [23].

Some uses of the Drilling Theorem (e.g., [7]) only involve going from a smooth structure (cone angle \( 2\pi \)) to a cusp, so only the analysis in [21] is needed. However, the proofs of the Density Conjecture in [9], [10], and [5], as described in these
proceedings ([6]), require a path of cone-manifolds beginning with cone angle $4\pi$ and ending at cone angle $2\pi$, so they depend on the new work in [23].

Below we give a brief description of the boundary value problem involved in this new version of the harmonic deformation theory, as well as some applications to finite volume hyperbolic cone-manifolds.

We will assume, for simplicity, that $X$ is a compact hyperbolic 3-manifold with a single torus as its boundary. Hyperbolic manifolds with multiple torus boundary components can be handled by using the same type of boundary conditions on each one. The theory extends to hyperbolic manifolds which also have infinite volume geometrically finite ends, whose conformal structure at infinity is assumed to be fixed, using the same techniques as in [8].

We further assume that the geometry near each boundary torus is modelled on the complement of an open tubular neighborhood of radius $R$ around the singular set of a hyperbolic cone-manifold. (A horospherical neighborhood of a cusp is included by allowing $R = \infty$.) In particular, we assume that each torus has an intrinsic flat metric with constant principal normal curvatures $\kappa$ and $\frac{1}{\kappa}$, where $\kappa \geq 1$. The normal curvatures and the tube radius, $R$, are related by $\coth R = \kappa$ so they determine each other. In fact, given such a boundary structure, it can be canonically filled in. In general, the filled-in structure has “Dehn surgery type singularities” (see [24, Chap. 4]), which includes cone singularities with arbitrary cone angle. We say that $X$ has tubular boundary. This structure is described in more detail in [22].

In [21] specific closed $E$-valued 1-forms, defined in a neighborhood of the singular locus, were exhibited which had the property that some complex linear combination of them induced every possible infinitesimal change in the holonomy representation of a boundary torus. As a result, by standard cohomology theory, for any infinitesimal deformation of the hyperbolic structure, it is possible to find a closed $E$-valued 1-form $\hat{\omega}$ on $X$ which equals such a complex linear combination of these standard forms in a neighborhood of the torus boundary. This combination of standard forms corresponds to the terms $\omega_m + \omega_l$ in equation (6) in Section 3.

The standard forms are harmonic so the $E$-valued 1-form $\hat{\omega}$ will be harmonic in a neighborhood of the boundary but not generally harmonic on all of $X$. Since it represents a cohomology class in $H^1(X; E)$, it will be closed as an $E$-valued 1-form, but it won’t generally be co-closed. If we denote by $d_E$ and $\delta_E$ the exterior derivative and its adjoint on $E$-valued forms, then this means that $d_E \hat{\omega} = 0$, but $\delta_E \hat{\omega} \neq 0$ in general. (Note that $E$ is a flat bundle so that $d_E$ is the coboundary operator for this cohomology theory.)

A representative for a cohomology class can be altered by a coboundary without changing its cohomology class. An $E$-valued 1-form is a coboundary precisely when it can be expressed as $d_E s$, where $s$ is an $E$-valued 0-form, i.e. a global section of $E$. Thus, finding a harmonic (closed and co-closed) representative cohomologous to $\hat{\omega}$ is equivalent to finding a section $s$ such that

\begin{equation}
\delta_E d_E s = -\delta_E \hat{\omega}.
\end{equation}

Then, $\omega = \hat{\omega} + d_E s$ satisfies $\delta_E \omega = 0$, $d_E \omega = 0$; it is a closed and co-closed (hence harmonic) representative in the same cohomology class as $\hat{\omega}$.

In [21] it was shown that in order to solve equation (20) for $E$-valued sections, it suffices to solve it for the “real part” of $E$, where we are interpreting $E$ as the complexified tangent bundle of $X$ as discussed at the end of Section 3. The real part of a section $s$ of $E$ is just a (real) section of the tangent bundle of $X$; i.e., a
vector field, which we denote by $v$. The real part of $\delta_E d_E s$ equals $(\nabla^* \nabla + 2) v$, where $\nabla$ denotes the (Riemannian) covariant derivative and $\nabla^*$ is its adjoint. The composition $\nabla^* \nabla$ is sometimes called the “rough Laplacian” or the “connection Laplacian”. We will denote it by $\triangle$.

To solve the equation $\delta_E d_E s = -\delta_E \hat{\omega}$, we take the real part of $-\delta_E \hat{\omega}$, considered as a vector field, and denote it by $\zeta$. We then solve the equation
\[(\triangle + 2) v = \zeta,\] (21)
for a vector field $v$ on $X$. As discussed in [21], this gives rise to a section $s$ of $E$ whose real part equals $v$ which is a solution to (20). We denote the correction term $d_E s$ by $\omega_c$. In a neighborhood of the boundary $\hat{\omega}$ equals a combination of standard forms, $\omega_m + \omega_l$. Thus, in a neighborhood of the boundary, we can decompose the harmonic representative $\omega$ as $\omega = \omega_m + \omega_l + \omega_c$, just as we did in (6) in Section 3.

Note that, since $\hat{\omega}$ was already harmonic in a neighborhood of the boundary, the correction term $\omega_c$ will also be harmonic on that neighborhood.

Because $\omega$ is harmonic, it will satisfy Weitzenböck formulae as described in Section 4. As will be outlined below will choose boundary conditions that will further guarantee that $\omega = \eta + i * D \eta$ where $\eta$ is a 1-form with values in the tangent bundle of $X$. It decomposes as $\eta = \eta_m + \eta_l + \eta_c$ in a neighborhood of the boundary and satisfies the equation $D^* D \eta + \eta = 0$ on all of $X$. As before, taking an $L^2$ inner product and integrating by parts leads to equation (7). The key to generalizing the harmonic deformation theory from the previous sections is finding boundary conditions that will guarantee that the contribution to the boundary term of (7) from the correction term $\eta_c$ will be non-positive. Once this is established, everything else goes through without change.

In order to control the behavior of $d_E s = \omega_c$ (hence of $\eta_c$), it is necessary to put restrictions on the domain of the operator $(\triangle + 2)$ in equation (21). On smooth vector fields with compact support the operator $(\triangle + 2)$ is self-adjoint. It is natural to look for boundary conditions for which self-adjointness still holds. It is possible to find boundary conditions which make this operator elliptic and self-adjoint with trivial kernel. Standard theory then implies that equation (21) is always uniquely solvable.

There are many choices for such boundary data. Standard examples include the conditions that either $v$ or its normal derivative be zero, analogous to Dirichlet and Neumann conditions for the Laplacian on real-valued functions. However, none of these standard choices of boundary data have the key property that the Weitzenböck boundary term for $\eta_c$ will necessarily be non-positive.

The main analytic result in [23] is the construction of a boundary value problem that has this key additional property. We give a very brief description of the boundary conditions involved. (In particular, we will avoid discussion of the precise function spaces involved.)

The first boundary condition on the vector fields allowed in the domain of the operator $(\triangle + 2)$ in equation (21) is that its (3-dimensional) divergence vanish on the boundary. This means that the corresponding infinitesimal change of metric is volume preserving at points on the boundary. The combination of standard forms, $\omega_m + \omega_l$, also induces infinitesimally volume preserving deformations. Once we prove the existence of a harmonic $E$-valued 1-form $\omega$ whose correction term comes from a vector field satisfying this boundary condition, we can conclude that $\omega$ is infinitesimally volume preserving at the boundary. If we denote by $\text{div}$ the function...
measuring the infinitesimal change of volume at a point, then this means that \( \text{div} \) vanishes at the boundary. However, for any harmonic \( E \)-valued 1-form \( \text{div} \) satisfies the equation:

\[
\triangle \text{div} = -4 \text{div},
\]

where \( \triangle \) here denotes the laplacian on functions given locally by the sum of the \textbf{negatives} of the second derivatives.

A standard integration by parts argument shows that any function satisfying such an equation and vanishing at the boundary must be identically zero. Thus we can conclude that the deformation induced by \( \omega \) is infinitesimally volume preserving at \textit{every} point in \( X \). This allows us to conclude that \( \omega \) can be written as \( \omega = \eta + i \ast D\eta \) where \( \eta \) satisfies \( D^*D\eta + \eta = 0 \). (See Proposition 2.6 in \cite{21}.) The computation of the Weitzenböck boundary term now proceeds as before.

The second boundary condition is more complicated to describe. Recall that the boundary of \( X \) has the same structure as the boundary of a tubular neighborhood of a (possibly singular) geodesic. In particular, it has a neighborhood which is foliated by tori which are equidistant from the boundary. In a sufficiently small neighborhood, these surfaces are all embedded and, on each of them, the nearest point projection to the boundary is a diffeomorphism. If we denote by \( u \) the (2-dimensional) tangential component of the vector field \( v \) at the boundary, we can use these projection maps to pull back \( u \) to these equidistant surfaces. We denote the resulting extension of \( u \) to the neighborhood of the boundary by \( \hat{u} \).

In dimension 3, the \textbf{curl} of a vector field is again a 3-dimensional vector field. The second boundary condition is that the (2-dimensional) tangential component of \( \text{curl} v \) agree with that of \( \text{curl} \hat{u} \) on the boundary. Note that, on the boundary, the normal component of \( \text{curl} v \) equals the curl of the 2-dimensional vector field \( u \) (this curl is a function) so it automatically agrees with that of \( \hat{u} \).

As a partial motivation for this condition, consider a vector field which generates an infinitesimal isometry in a neighborhood of the boundary which preserves the boundary as a set. Geometrically, it just translates the boundary and all of the nearby equidistant surfaces along themselves. Thus, this vector field is tangent to the boundary and to all the equidistant surfaces. It has the property that it equals the vector field \( \hat{u} \) defined above as the extension of its tangential boundary values. Thus, the above condition can be viewed as an attempt to mirror properties of infinitesimal isometries preserving the boundary.

To see why it might be natural to put conditions on the \textit{curl} of \( v \), rather than on \( v \) itself, consider the real-valued 1-form \( \tau \) dual (using the hyperbolic metric) to \( v \). Then, \( \delta \tau \) and \( *d\tau \) correspond, respectively, to the divergence and curl of \( v \), where \( d \) denotes exterior derivative, \( \delta \) its adjoint, and \( * \) is the Hodge star-operator. Our boundary conditions can be viewed as conditions on the exterior derivative and its adjoint applied to this dual 1-form.

However, the ultimate justification for these boundary conditions is that they lead to a Weitzenböck boundary term with the correct properties, as long as the tube radius is sufficiently large. A direct geometric proof of this fact is still lacking, as is an understanding of the geometric significance of the value of the required lower bound on the tube radius. Nonetheless, the fact that the contribution to the Weitzenböck boundary from the correction term \( \omega_c \) is always non-positive when it arises from a vector field satisfying these boundary conditions can be derived by
straightforward (though somewhat intricate) calculation. All the results from the previous harmonic theory follow immediately. For example, we can conclude:

**Theorem 6.2** ([23]). For a finite volume hyperbolic cone-manifold with singularities along a link with tube radius at least $\text{arctanh}(1/\sqrt{3}) \approx 0.65848$, there are no deformations of the hyperbolic structure keeping the cone angles fixed. Furthermore, the nearby hyperbolic cone-manifold structures are parametrized by their cone angles.

In the statement of this theorem, when the singular link has more than one component, having tube radius at least $R$ means that there are disjoint embedded tubes of radius $R$ around all the components.

Besides being able to extend local rigidity rel cone angles and parametrization by cone angles, the boundary value theory permits all of the estimates involved in the effective rigidity arguments to go through. In particular, inequalities (13), (14), and (18) continue to hold. The packing arguments require no restriction on cone angles so that the proofs of the results on hyperbolic Dehn surgery (e.g., Theorems 5.4 and 5.5) go through unchanged.

In order to give an efficient description of the conclusions of these arguments in this more general context we first extend some previous definitions. Recall that, if $X$ has a complete finite volume hyperbolic structure with one cusp and $T$ is an embedded horospherical torus, the normalized length of a simple closed curve $\gamma$ on $T$ is defined as the length of the geodesic isotopic to $\gamma$ in the flat metric on $T$, scaled to have area 1. A **weighted** simple closed curve is just a pair, $(\lambda, \gamma)$, where $\lambda$ is a positive real number. Its normalized length is then defined to be $\lambda$ times the normalized length of $\gamma$. If a basis is chosen for $H_1(T, \mathbb{Z})$, the set of isotopy classes of non-contractible simple closed curves on $T$ corresponds to pairs $(p, q)$ of relatively prime integers. Then a weighted simple closed curve $(\lambda, \gamma)$ can be identified with the point $(\lambda p, \lambda q) \in \mathbb{R}^2 \cong H_1(T, \mathbb{Z}) \otimes \mathbb{R} \cong H_1(T, \mathbb{R})$. It is easy to check that the notion of normalized length extends by continuity to any $(x, y) \in H_1(T, \mathbb{R})$.

The **hyperbolic Dehn surgery space** for $X$ (denoted $\mathcal{HDS}(X)$) is a subset of $H_1(T, \mathbb{R}) \cup \infty$ which serves as a parameter space for (generally incomplete) hyperbolic structures on $X$ (with certain restrictions on the structure near its end). In particular, if we view a weighted simple closed curve $(\lambda, \gamma)$ as an element of $H_1(T, \mathbb{R})$, then saying that it is in $\mathcal{HDS}(X)$ means that there is a hyperbolic cone-manifold structure on the manifold obtained from $X$ by doing Dehn filling with $\gamma$ as meridian which has the core curve as the singular locus with cone angle $\frac{2\pi}{\lambda}$. The point at infinity corresponds to the complete hyperbolic structure on $X$.

Thurston’s hyperbolic Dehn surgery theorem (see [27]) states that $\mathcal{HDS}(X)$ always contains an open neighborhood of $\infty$. This, in particular, implies that it contains all but a finite number of pairs $(p, q)$ of relatively prime integers, which implies that all but a finite number of the manifolds obtained by (topological) Dehn surgery are hyperbolic. However, since most of these pairs are clustered “near” infinity, the statement that it contains an open neighborhood of infinity is considerably stronger. Again, Thurston’s proof is not effective; it provides no information about the size of any region contained in hyperbolic Dehn surgery space. The following theorem, whose proof is analogous to that of Theorem 5.4, provides such information:
Theorem 6.3. Let $X$ be a complete, finite volume orientable hyperbolic 3-manifold with one cusp, and let $T$ be a horospherical torus which is embedded as a cross-section to the cusp of $X$. Let $(x, y) \in H_1(T, \mathbb{R})$ have normalized length at least 7.583. Then there is a hyperbolic structure on $X$ with Dehn surgery coefficient $(x, y)$. In particular, the hyperbolic Dehn surgery space for $X$ contains the complement of the ellipse around the origin determined by the condition that the normalized length of $(x, y)$ is less than 7.583. Furthermore, the volumes of hyperbolic structures in this region differ from that of $X$ by at most 0.306.

Remark: The homology group $H_1(T, \mathbb{R})$ can be naturally identified with the universal cover of $T$, so the flat metric on $T$, normalized to have area 1, induces a flat metric on $H_1(T, \mathbb{R})$. Then the ellipse in the above theorem becomes a metric disk of radius 7.583. From this point of view, the theorem provides a universal size region in $\text{HDS}(X)$ (the complement of a “round disk” of radius 7.583), which is independent of $X$. However, it is perhaps more interesting to note that, if $H_1(T, \mathbb{R})$ is more naturally identified with $H_1(T, \mathbb{Z}) \otimes \mathbb{R}$, then this region actually reflects the shape of $T$.

Finally we note that these techniques also provide good estimates on the change in geometry during hyperbolic Dehn filling as in Theorem 6.3. For example, Schlafli’s formula (19) together with control on the length of the singular locus (as in Theorem 4.7) leads to explicit upper and lower bounds for the decrease in volume, $\Delta V$. These bounds are independent of the cusped manifold $X$, and can be viewed as refinements of the asymptotic formula of Neumann and Zagier [25]:

$$\Delta V \sim \frac{\pi^2}{L(x, y)}$$

as $L(x, y) \to \infty$,

where $L(x, y)$ denotes the normalized length of the Dehn surgery coefficient $(x, y) \in H_1(T; \mathbb{R})$. The details will appear in [23].

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