Identities on Factorial Grothendieck Polynomials

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Abstract

Gustafson and Milne proved an identity on the Schur function indexed by a partition of the form \((\lambda_1 - n + k, \lambda_2 - n + k, \ldots, \lambda_k - n + k)\). On the other hand, Fehér, Némethi and Rimányi found an identity on the Schur function indexed by a partition of the form \((m - k, \ldots, m - k, \lambda_1, \ldots, \lambda_k)\). Fehér, Némethi and Rimányi gave a geometric explanation of their identity, and they raised the question of finding a combinatorial proof. In this paper, we establish a Gustafson-Milne type identity as well as a Fehér-Némethi-Rimányi type identity for factorial Grothendieck polynomials. Specializing a factorial Grothendieck polynomial to a Schur function, we obtain a combinatorial proof of the Fehér-Némethi-Rimányi identity.

1 Introduction

Throughout this paper, we let \(n\) be a positive integer. We shall write \([n] = \{1, 2, \ldots, n\}\) and use \(\binom{n}{k}\) to represent the set of \(k\)-subsets of \([n]\). For a \(k\)-subset \(S\) of \([n]\), we denote the elements of \(S\) by \(i_1 < i_2 < \cdots < i_k\). For a partition \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)\), Gustafson and Milne proved the following identity on Schur functions:

\[
s(\lambda_1 - n + k, \lambda_2 - n + k, \ldots, \lambda_k - n + k)(x_1, x_2, \ldots, x_n) = \sum_{S \in \binom{[n]}{k}} \frac{s_{\lambda}(x_S)}{\prod_{i \in S} \prod_{j \not\in S}(x_i - x_j)},
\]

where \(s_{\lambda}(x_S) = s_{\lambda}(x_{i_1}, x_{i_2}, \ldots, x_{i_k})\) and \(\overline{S} = [n] \setminus S\) is the complement of \(S\). In the case \(\lambda_k - n + k < 0\), the left-hand side of (1.1) is zero. Gustafson and Milne deduced (1.1) based on the determinantal formula of Schur functions. Chen and Louck gave an alternative approach by using an interpolation formula for symmetric functions.
Taking $\lambda = (n-1)$ and then replacing $x_i$ with $x_i^{-1}$ for $i \geq 1$, the Gustafson-Milne identity specializes to the Good’s identity

$$1 = \sum_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - \frac{x_i}{x_j}\right)^{-1},$$

which played a crucial role in the proof of the Dyson conjecture [9]. In the case $\lambda = (m)$, the Gustafson-Milne identity becomes an identity due to Louck [20, 21]

$$h_{m-n+1}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \frac{x_i^m}{\prod_{j=1}^{n} (x_i - x_j)},$$

where $h_k(x)$ is the complete homogeneous symmetric function. See [11] for various applications of the Louck’s identity. The Gustafson-Milne identity also appeared in the calculations of Thom polynomials [7].

On the other hand, in the study of equivariant cohomology classes of matrix matroid varieties, Fehér, Némethi and Rimányi [6, Theorem 7.5] found the following identity:

$$s_{(m-k, m-k, \ldots, m-k)_{n-k}}(x_1, x_2, \ldots, x_n) = \sum_{S \in \binom{[n]}{k}} s_{\lambda(S)}(x_S) \prod_{j \in S} \prod_{i \in S} (x_j - x_i).$$

Fehér, Némethi and Rimányi [6] gave a geometric explanation for the identity (1.4), and posed the question of finding a combinatorial proof.

In this paper, we establish a Gustafson-Milne type identity as well as a Fehé-Némethi-Rimányi type identity for factorial Grothendieck polynomials. The factorial Grothendieck polynomial $G_\lambda(x|y)$ is the double Grothendieck polynomial corresponding to a Grassmannian permutation. These polynomials can also be interpreted in terms of set-valued tableaux [3, 15, 16, 25, 26], or expressed as the quotient of determinants [12, 14, 24]. The lowest degree homogeneous component of $G_\lambda(x|y)$ is equal to the factorial Schur function $s_\lambda(x|y)$, which is the double Schubert polynomial of a Grassmannian permutation and has received extensive attention, see, for example, [1, 2, 5, 10, 17, 23, 27, 28]. Restricting a factorial Grothendieck polynomial to a Schur function, we are led to a combinatorial proof of the identity (1.4). This serves as an answer to the question posed by Fehér, Némethi and Rimányi.

Let us proceed with the tableau interpretation of factorial Grothendieck polynomials. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be an integer partition, that is, $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ are nonnegative integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0$. Write $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$. The Young diagram
of \( \lambda \) is a left-justified array of squares with \( \lambda_i \) squares in row \( i \). A square \( \alpha \) of \( \lambda \) in row \( i \) and column \( j \) is denoted \( \alpha = (i, j) \). If no confusion arises, we do not distinguish a partition and its Young diagram. A set-valued tableau \( T \) of shape \( \lambda \) is an assignment of finite sets of positive integers into the squares of \( \lambda \) such that the sets in each row (respectively, column) are weakly (respectively, strictly) increasing, where, for two finite sets \( A \) and \( B \) of positive integers, we write \( A \leq B \) if \( \max A \leq \min B \) and \( A < B \) if \( \max A < \min B \). The notion of set-valued tableaux was introduced by Buch \cite{bu15} in his study of the Littlewood-Richardson rule for stable Grothendieck polynomials. Let \( T(\alpha) \) denote the subset filled in a square \( \alpha \). We write \( |T| = \sum_{\alpha \in \lambda} |T(\alpha)| \) and write \( c(\alpha) = j - i \) for the content of \( \alpha = (i, j) \).

Let \( \mathcal{T}(\lambda, n) \) denote the set of set-valued tableaux \( T \) of shape \( \lambda \) such that each subset appearing in \( T \) is a subset of \( [n] \). For variables \( \beta, x, i \) and \( y, j \), we adopt the following notation as used by Fomin and Kirillov \cite{FK13}:

\[
\begin{align*}
    x_i + y_j &= x_i + y_j + \beta x_i y_j.
\end{align*}
\]

The factorial Grothendieck polynomial \( G_\lambda(x|y) \) is defined as

\[
G_\lambda(x|y) = \sum_{T \in \mathcal{T}(\lambda, n)} \beta^{|T| - |\lambda|} \prod_{\alpha \in T} \prod_{t \in T(\alpha)} (x_t \oplus y_{t+c(\alpha)}) \tag{1.5}.
\]

In the case \( \beta = 0 \), \( G_\lambda(x|y) \) becomes the factorial Schur function \( s_\lambda(x|y) \), while in the case \( \beta = 0 \) and \( y = 0 \), \( G_\lambda(x|y) \) specializes to the Schur function \( s_\lambda(x) \).

For example, there are three set-valued tableaux in \( \mathcal{T}(\lambda, n) \) for \( \lambda = (2, 1) \) and \( n = 2 \):

\[
\begin{array}{cc}
1 & 1 \\
2 & 2 \\
\end{array} \quad \begin{array}{cc}
1 & 2 \\
2 & 2 \\
\end{array} \quad \begin{array}{cc}
1 & 12 \\
2 & 2 \\
\end{array}
\]

By (1.5), we see that

\[
G_\lambda(x|y) = (x_1 \oplus y_1) \cdot (x_1 \oplus y_2) \cdot (x_2 \oplus y_1) + (x_1 \oplus y_1) \cdot (x_2 \oplus y_3) \cdot (x_2 \oplus y_1) + \beta \cdot (x_1 \oplus y_1) \cdot (x_1 \oplus y_3) \cdot (x_2 \oplus y_1) \tag{1.6}
\]

Factorial Grothendieck polynomials are double Grothendieck polynomials of Grassmannian permutations. The double Grothendieck polynomials were introduced by Lascoux and Schützenberger \cite{LS78} as polynomial representatives of the equivariant \( K \)-theory classes of structure sheaves of Schubert varieties in the flag manifold. Let \( S_p \) denote the symmetric group of permutations of \( [p] \). For a permutation \( w \in S_p \), the double Grothendieck polynomial \( G_w(x; y) \) is defined based on the isobaric divided difference operator \( \pi_i \), acting on \( \mathbb{Z}[\beta][x, y] \). For \( f \in \mathbb{Z}[\beta][x, y] \), let \( s_i f \) be obtained from \( f \) by interchanging \( x_i \) and \( x_{i+1} \). Then

\[
\pi_i f = \frac{(1 + \beta x_{i+1})f - (1 + \beta x_i)s_i f}{x_i - x_{i+1}}.
\]
The length of \( w \), denoted \( \ell(w) \), is equal to the number of pairs \((i, j)\) such that \( 1 \leq i < j \leq p \) and \( w_i > w_j \). For the longest permutation \( w_0 = p \cdots 21 \), set
\[
\mathfrak{G}_{w_0}(x; y) = \prod_{i+j \leq p} (x_i \oplus y_j).
\]
If \( w \neq w_0 \), then one can choose a simple transposition \( s_i \) such that \( \ell(ws_i) > \ell(w) \). Set
\[
\mathfrak{G}_w(x; y) = \pi_i \mathfrak{G}_{ws_i}(x; y).
\]

Note that the above defined double Grothendieck polynomials are called double \( \beta \)-polynomials [8], and reduce to the ordinary double Grothendieck polynomials in the case \( \beta = -1 \). In the case \( \beta = 0 \), \( \mathfrak{G}_w(x; y) \) equals the double Schubert polynomial \( S_w(x|y) \) [18,22].

A permutation \( w \in S_p \) is a Grassmannian permutation if there is at most one position, say \( n \), such that \( w_n > w_{n+1} \). To a Grassmannian permutation \( w \), one can associate a partition \( \lambda(w) = (\lambda_1, \ldots, \lambda_n) \) where \( \lambda_i = w_{n-i+1} - (n - i + 1) \). For a Grassmannian permutation \( w \) with \( \lambda(w) = \lambda \), it has been shown that \( \mathfrak{G}_w(x; y) = G_\lambda(x|y) \), see for example [3,15,16,25,26].

On the other hand, there have been determinantal formulas for factorial Grothendieck polynomials [12–14,24]. For the purpose of this paper, we need the following determinantal formula for \( G_\lambda(x|y) \) due to Ikeda and Naruse [14]:
\[
G_\lambda(x|y) = \frac{\det \left( [x_i|y]^{\lambda_j+n-j}(1+\beta x_i)^{j-1} \right)_{1 \leq i,j \leq n}}{\prod_{1 \leq i < j \leq n} (x_i - x_j)}, \tag{1.7}
\]
where
\[
[x_i|y]^j = (x_i \oplus y_1)(x_i \oplus y_2) \cdots (x_i \oplus y_j).
\]
For example, for \( \lambda = (2, 1) \) and \( n = 2 \), by (1.7), we have
\[
G_\lambda(x|y) = \frac{1}{x_1 - x_2} \begin{vmatrix} [x_1|y]^3 & [x_1|y](1+\beta x_1) \\ [x_2|y]^3 & [x_2|y](1+\beta x_2) \end{vmatrix},
\]
which is the same as (1.6) via a simple calculation.

2 The Gustafson-Milne type identity

In this section, we prove a Gustafson-Milne type identity for factorial Grothendieck polynomials which can be stated as follows.
**Theorem 2.1.** Let \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots) \) be two sets of variables. For a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \), we have

\[
G_{(\lambda_1-n+k, \lambda_2-n+k, \ldots, \lambda_k-n+k)}(x|y) = \sum_{S \in \binom{[n]}{k}} G_{(\lambda_1, \lambda_2, \ldots, \lambda_k)}(x_S|y) \prod_{j \in S} (1 + \beta x_j)^k \prod_{i \in S} \prod_{j \in S \setminus j} (x_i - x_j),
\]

(2.1)

where the left-hand side of (2.1) is zero in the case \( \lambda_k - n + k < 0 \).

Theorem 2.1 becomes an identity on factorial Schur functions in the case \( \beta = 0 \), and specializes to the Gustafson-Milne identity (1.1) in the case \( \beta = 0 \) and \( y = 0 \).

Taking \( \lambda = (n-1) \) and \( \lambda = (m) \) respectively in Theorem 2.1, we obtain two identities which contain the Good’s identity (1.2) and the Louck’s identity (1.3) as special cases.

**Corollary 2.2.** Let \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots) \) be two sets of variables. Then,

\[
1 = \sum_{i=1}^{n} [x_i|y]^{n-1} \prod_{j=1 \atop j \neq i}^{n} \frac{1 + \beta x_j}{x_i - x_j}
\]

(2.2)

and

\[
h_{m-n+1}(x|y) = \sum_{i=1}^{n} [x_i|y]^m \prod_{j=1 \atop j \neq i}^{n} \frac{1 + \beta x_j}{x_i - x_j}.
\]

The following lemma is key to the proof of Theorem 2.1 which can be viewed as a generalization of the Vandermonde determinant.

**Lemma 2.3.** For \( n \geq 1 \),

\[
\det \left( [x_i|y]^{n-c}(1 + \beta x_r)^{c-1} \right)_{1 \leq r, c \leq n} = \prod_{1 \leq i < j \leq n} (x_i - x_j),
\]

(2.3)

where, in the case \( n = 1 \), both sides of (2.3) are equal to one.

**Proof.** For a set \( S = \{i_1 < i_2 < \cdots < i_k\} \) of positive integers, let

\[
y_S = y_{i_1} y_{i_2} \cdots y_{i_k}
\]

and

\[(1 + \beta y)_S = (1 + \beta y_{i_1})(1 + \beta y_{i_2}) \cdots (1 + \beta y_{i_k}).\]
With the above notation, we define

\[ E_k^{(\beta)}(Y_n) = \sum_{S \in \binom{[n]}{k}} y_S (1 + \beta y)_S, \]

where \( \overline{S} = [n] \setminus S \). Here, \( E_k^{(\beta)}(Y_n) = 0 \) unless \( 0 \leq k \leq n \). Note that when \( \beta = 0 \), the polynomial \( E_k^{(\beta)}(Y_n) \) is the elementary symmetric function \( e_k(y_1, y_2, \ldots, y_n) \). It is easily checked that \( E_k^{(\beta)}(Y_n) \) satisfies the following recurrence relation:

\[ E_k^{(\beta)}(Y_n) = (1 + \beta y_n) E_k^{(\beta)}(Y_{n-1}) + y_n E_{k-1}^{(\beta)}(Y_{n-1}). \tag{2.4} \]

The entry on the left-hand side of (2.3) can be reformulated as

\[
[x_r | y]^{n-c} (1 + \beta x_r)^{c-1} = \left( \sum_{h=0}^{n-c} E_k^{(\beta)}(Y_{n-c}) x_r^{n-c-h} \right) \left( \sum_{i=0}^{c-1} \binom{c-1}{i} \beta^i x_r^i \right) \\
= \sum_{j=0}^{n-1} \left( \sum_{i=0}^{j} \binom{c-1}{i} E_{n-c-j+i}^{(\beta)}(Y_{n-c}) \beta^i \right) x_r^j \\
= \sum_{j=1}^{n} x_r^{n-j} \left( \sum_{i=0}^{n-j} \binom{c-1}{i} E_{i-c+j}^{(\beta)}(Y_{n-c}) \beta^i \right),
\]

which implies that the left-hand side of (2.3) can be written as a product involving a Vandermonde determinant:

\[
\text{det} \left( [x_r | y]^{n-c} (1 + \beta x_r)^{c-1} \right)_{1 \leq r, c \leq n} = \text{det}(x_r^{n-c})_{1 \leq r, c \leq n} \cdot \text{det} \left( \sum_{i=0}^{n-r} \binom{c-1}{i} E_{i-c+r}^{(\beta)}(Y_{n-c}) \beta^i \right)_{1 \leq r, c \leq n} \\
= \prod_{1 \leq i < j \leq n} (x_i - x_j) \cdot \text{det}(H(n)).
\]

It remains to evaluate that \( \text{det}(H(n)) = 1 \). Denote by \( \text{Col}_c \) the \( c \)-th column of \( H(n) \). For \( 1 \leq c \leq n-1 \), we apply the column transformation \( \text{Col}_c - y_{n-c} \text{Col}_{c+1} \), that is, \( \text{Col}_c \) is replaced by \( \text{Col}_c - y_{n-c} \text{Col}_{c+1} \). Let \( H'(n) \) be the resulting matrix after such column transformations. Then the entry of \( H'(n) \) in row \( 1 \leq r \leq n \) and column \( 1 \leq c \leq n-1 \) can be simplified as

\[
\sum_{i=0}^{n-r} \binom{c-1}{i} E_{i-c+r}^{(\beta)}(Y_{n-c}) \beta^i - y_{n-c} \sum_{i=0}^{n-r} \binom{c}{i} E_{i-c+r-1}^{(\beta)}(Y_{n-c-1}) \beta^i
\]
where the third equality follows from (2.4).

Notice that the first \( n - 1 \) entries in the \( n \)-th row of \( H'(n) \) are all zero, while that last entry in the \( n \)-th row of \( H'(n) \) is one. Therefore, by (2.5) and using the Laplace expansion along the \( n \)-th row of \( H'(n) \), we have

\[
\det(H(n)) = \det(H'(n)) = \det\left(\sum_{i=0}^{n-r-1} \binom{c-1}{i} E_{i-c+r}^{(\beta)}(Y_{n-c}) \beta^i \right)_{1 \leq r, c \leq n-1}
\]

With the initial value \( \det(H(1)) = 1 \) and using induction, we deduce that \( \det(H(n)) = 1 \) for \( n \geq 1 \). This completes the proof.

Now we are ready to present a proof of Theorem 2.1.

Proof of Theorem 2.1. Write \( \nu = (\lambda_1 - n + k, \lambda_2 - n + k, \ldots, \lambda_k - n + k) \). By (1.7),

\[
G_{\nu}(x|y) = \frac{\det \left( [x_i | y]^{\nu_i + n - j} (1 + \beta x_i)^{j-1} \right)_{1 \leq i, j \leq n}}{\prod_{1 \leq i < j \leq n} (x_i - x_j)},
\]

(2.6)

where \( \nu_j = 0 \) for \( j > k \). By the Laplace expansion of a determinant along the first \( k \) columns, the numerator on the right-hand side of (2.6) equals

\[
\det \left( [x_i | y]^{\nu_i + n - j} (1 + \beta x_i)^{j-1} \right)_{1 \leq i, j \leq n}
\]

(2.6)
By Lemma 2.3, we see that

\[ \sum_{S \subseteq [n]} (-1)^{\binom{k+1}{2} + \sum_{r \in S} r} \det([x_r | y]^{\lambda+\beta x_r c-1}) \prod_{i \in S} (1 + \beta x_r) \]  

Hence the denominator on the right-hand side of (2.6) can be rewritten as

\[ \sum_{S \subseteq [n]} (-1)^{\binom{k+1}{2} + \sum_{r \in S} r} \det([x_r | y]^{\lambda+\beta x_r c-1}) \prod_{i \in S} (1 + \beta x_r) \]  

and so (2.7) can be expressed as

\[ \det([x_r | y]^{\lambda+\beta x_r c-1}) \prod_{i \in S} (1 + \beta x_r) \]  

By Lemma 2.3 we see that

\[ \det([x_r | y]^{\lambda+\beta x_r c-1}) \prod_{i \in S} (1 + \beta x_r) = \prod_{i,j \in S} (x_i - x_j) \]  

and so (2.7) can be expressed as

\[ \det([x_r | y]^{\lambda+\beta x_r c-1}) \prod_{i \in S} (1 + \beta x_r) \]  

For any \( k \)-subset \( S \subseteq [n] \), notice that

\[ \prod_{i \in S} (x_i - x_j) = \prod_{i < j} (x_i - x_j) \prod_{i > j} (x_i - x_j) \]

\[ = (-1)^{\binom{k+1}{2} + \sum_{r \in S} r} \prod_{i < j} (x_i - x_j) \prod_{i > j} (x_i - x_j) \]

Hence the denominator on the right-hand side of (2.6) can be rewritten as

\[ \prod_{1 \leq i < j \leq n} (x_i - x_j) = \prod_{i < j} (x_i - x_j) \prod_{i \in S} (x_i - x_j) \prod_{i \geq j} (x_i - x_j) \prod_{j \in S} (x_j - x_i) \]

\[ = (-1)^{\binom{k+1}{2} + \sum_{r \in S} r} \prod_{i < j} (x_i - x_j) \prod_{i > j} (x_i - x_j) \prod_{j \in S} (x_j - x_i). \]
Putting (2.8) and (2.9) into (2.6), we deduce that

\[ G(\lambda_1-n+k,\lambda_2-n+k,\ldots,\lambda_k-n+k)(x|y) = \sum_{S \in \binom{[n]}{k}} \det \left( \left[ x_i \vline y_j \right]^{\lambda_c+k-c(1+\beta x_r)^{c-1}} \right)_{r \in S} \prod_{1 \leq i \leq k} (1+\beta x_j)^k \prod_{i \in S} (x_i - x_j) \prod_{i \leq j} (x_i - x_j) \prod_{i \in S} \prod_{j \in S} (x_j - x_i). \]  

By (1.7),

\[ G(\lambda_1,\lambda_2,\ldots,\lambda_k)(x|y) = \frac{\det \left( \left[ x_i \vline y_j \right]^{\lambda_c+k-c(1+\beta x_r)^{c-1}} \right)_{r \in S}}{\prod_{1 \leq i \leq k} (x_i - x_j) \prod_{i \leq j} (x_i - x_j)} \prod_{i \in S} \prod_{j \in S} (x_j - x_i), \]

which together with (2.10) yields (2.1). This completes the proof.

3 The Fehér-Némethi-Rimányi type identity

In this section, we provide a Fehér-Némethi-Rimányi type identity on factorial Grothendieck polynomials. The proof also relies on (1.7) and Lemma 2.3. Restricting to Schur functions, we obtain a combinatorial proof of the identity (1.4) given by Fehér, Némethi and Rimányi.

**Theorem 3.1.** Let \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots) \) be two sets of variables. Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) be a partition such that \( \lambda_1 \leq m - k \), and let

\[ \mu = (m-k, m-k, \ldots, m-k, \lambda_1, \lambda_2, \ldots, \lambda_k). \]

Then we have

\[ G_\mu(x|y) = \sum_{S \in \binom{[n]}{k}} G_\lambda(x_S|y) \frac{\prod_{i \in S} (1+\beta x_i)^{n-k} \prod_{j \in S} [x_j|y]^m}{\prod_{i \in S} \prod_{j \in S} (x_j - x_i)}. \]  

(3.1)

Note that in the case \( \beta = 0 \) and \( y_i = 0 \), Theorem 3.1 specializes to (1.4).

**Proof of Theorem 3.1.** Again, by (1.7), we have

\[ G_\mu(x|y) = \frac{\det \left( \left[ x_i \vline y_j \right]^{\mu_j+n-j(1+\beta x_r)^j} \right)_{1 \leq i \leq j \leq n}}{\prod_{1 \leq i < j \leq n} (x_i - x_j)}. \]  

(3.2)
Using the Laplace expansion by the last \( k \) columns, the numerator in (3.2) can be written as

\[
\sum_{S \subseteq \binom{[n]}{k}} (-1)^{nk - \binom{k}{2} + \sum_{r \in S} r} \det \left( [x_r | y]^{\lambda_r + k - c} (1 + \beta x_r)^{n - k + c - 1} \right)_{r \in S : 1 \leq r \leq k} \cdot \det \left( [x_r | y]^{n - k - m - c} (1 + \beta x_r)^{c - 1} \right)_{r \in \overline{S} : 1 \leq r \leq n - k} 
\]

\[
= \sum_{S \subseteq \binom{[n]}{k}} (-1)^{nk - \binom{k}{2} + \sum_{r \in S} r} \det \left( [x_r | y]^{\lambda_r + k - c} (1 + \beta x_r)^{n - k} \right)_{r \in S : 1 \leq r \leq k} \prod_{i \in S} (1 + \beta x_i)^{n - k} 
\]

Let us first evaluate the last factor in (3.3). It is easy to see that

\[
\det \left( [x_r | y]^{n - k - m - c} (1 + \beta x_r)^{c - 1} \right)_{r \in \overline{S} : 1 \leq r \leq n - k} = \prod_{j \in \overline{S}} [x_j | y]^m \cdot \det \left( [x_r | y]^{n - k - c} (1 + \beta x_r)^{c - 1} \right)_{r \in \overline{S} : 1 \leq r \leq n - k},
\]

where, for \( p \geq 0 \) and \( q \geq 0 \),

\[
[x_i | y]^q_p = (x_i \oplus y_{p+1}) \cdots (x_i \oplus y_{p+q}).
\]

Note that

\[
\det \left( [x_r | y]^{n - k - c} (1 + \beta x_r)^{c - 1} \right)_{r \in \overline{S} : 1 \leq r \leq n - k}
\]

is obtained from

\[
\det \left( [x_r | y]^{n - k - c} (1 + \beta x_r)^{c - 1} \right)_{r \in \overline{S} : 1 \leq r \leq n - k} = \prod_{i \leq j \leq \overline{S}} (x_i - x_j),
\]

by replacing \( y_i \) with \( y_{i+m} \) for \( i \geq 1 \). By Lemma 2.3, (3.4) is equal to

\[
\prod_{i \leq j \leq \overline{S}} (x_i - x_j),
\]

which is independent of the variables \( y_i \), and so we have

\[
\det \left( [x_r | y]^{n - k - c} (1 + \beta x_r)^{c - 1} \right)_{r \in \overline{S} : 1 \leq r \leq n - k} = \prod_{i \leq j \leq \overline{S}} (x_i - x_j).
\]

Hence we obtain that

\[
\det \left( [x_r | y]^{n - k + m - c} (1 + \beta x_r)^{c - 1} \right)_{r \in \overline{S} : 1 \leq r \leq n - k} = \prod_{j \in \overline{S}} [x_j | y]^m \cdot \prod_{i \leq j \leq \overline{S}} (x_i - x_j).
\]
As to the denominator in (3.2), by (2.9), it follows that
\[
\prod_{1 \leq i < j \leq n} (x_i - x_j) = \sum_{r \in S} \prod_{i,j \in S} (x_i - x_j) \prod_{i,j \in S} (x_i - x_j) \prod_{i \in S} (x_i - x_j)
\]
\[
= \left(-1\right)^{k(n-k)-(k+1)/2} \sum_{r \in S} \prod_{i,j \in S} (x_i - x_j) \prod_{i,j \in S} (x_i - x_j) \prod_{i \in S} (x_i - x_j)
\]
\[
= \left(-1\right)^{k(n-k)-(k+1)/2} \sum_{r \in S} \prod_{i,j \in S} (x_i - x_j) \prod_{i,j \in S} (x_i - x_j) \prod_{i \in S} (x_i - x_j).
\] (3.6)

Finally, notice that
\[
G_\lambda(x_S|y) = \det \left( [x_r | y]^{\lambda_i + k - c} (1 + \beta x_r)^{c-1} \right)_{r \in S} \prod_{1 \leq i < j \leq n} (x_i - x_j). \] (3.7)

Combining (3.2), (3.3), (3.5), (3.6) and (3.7), we arrive at (3.1), as desired.

Taking \( \lambda_1 = \lambda_2 = \ldots = \lambda_k = 0 \) and \( m = k \) in Theorem 3.1 we are led to another generalization of the Good’s identity.

**Corollary 3.2.** Let \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots) \) be two sets of variables. Then, for \( 0 \leq k \leq n \), we have
\[
1 = \sum_{S \in \binom{[n]}{k}} \frac{\prod_{i \in S} (1 + \beta x_i)^{n-k} \prod_{j \in S} [x_j | y]^k}{\prod_{i \in S} \prod_{j \in S} (x_j - x_i)}.
\]

Note that Corollary 3.2 specializes to identity (2.2) in the case \( k = n - 1 \).

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