Systematic construction of non-autonomous Hamiltonian equations of Painleve-type. III. Quantization

Maciej Blaszak and Krzysztof Marciniak

The self-archived postprint version of this journal article is available at Linköping University Institutional Repository (DiVA):
http://urn.kb.se/resolve?urn=urn:nbn:se:liu:diva-187439

N.B.: When citing this work, cite the original publication.
Blaszak, M., Marciniak, K., (2022), Systematic construction of non-autonomous Hamiltonian equations of Painleve-type. III. Quantization, Studies in applied mathematics (Cambridge). https://doi.org/10.1111/sapm.12514

Original publication available at:
https://doi.org/10.1111/sapm.12514

Copyright: Wiley
https://www.wiley.com/en-gb
Systematic construction of non-autonomous Hamiltonian equations of Painlevé-type. III. Quantization

Maciej Błaszak
Faculty of Physics, Department of Mathematical Physics and Computer Modelling, A. Mickiewicz University, 61-614 Poznań, Poland
blaszakm@amu.edu.pl

Krzysztof Marciniak
Department of Science and Technology
Campus Norrköping, Linköping University
601-74 Norrköping, Sweden
krzma@itn.liu.se

May 10, 2022

Abstract

This is the third article in our series of articles exploring connections between dynamical systems of Stäckel-type and of Painlevé-type. In this article we present a method of deforming of minimally quantized quasi-Stäckel Hamiltonians, considered in Part I to self-adjoint operators satisfying the quantum Frobenius condition, thus guaranteeing that the corresponding Schrödinger equations possess common, multi-time solutions. As in the classical case, we obtain here both magnetic and non-magnetic families of systems. We also show the existence of multitime-dependent quantum canonical maps between both classes of quantum systems.

1 Introduction

This is the third article in the suit of articles investigating relations between Painlevé-type systems and Stäckel-type systems. In the first paper (Part I, i.e. [9], see also [8]) we have constructed, starting from appropriate Stäckel-type systems, multi-parameter families of Frobenius integrable non-autonomous Hamiltonian systems with arbitrary number of degrees of freedom. In the second article (Part II, i.e. [10]) we have constructed the isomonodromic Lax representations for these systems, proving that Frobenius integrable systems constructed in Part I are indeed Painlevé-type systems. Each of such families was written in two different representations, called an ordinary one and a magnetic one, respectively, and they were connected by a multi-time canonical transformation [14]. In this paper we discuss the minimal quantization [6],[7] of Painlevé-type systems considered in Part I and Part II. We prove that this quantization turns the Hamiltonians $\tilde{H}_r$ of the Painlevé-type systems into their quantum counterparts, that is the self-adjoint operators $\hat{\tilde{H}}_r$, acting in an appropriate Hilbert space $\mathcal{H}$ and satisfying the quantum Frobenius condition

$$i\hbar \frac{\partial \hat{\tilde{H}}_r}{\partial t_s} - i\hbar \frac{\partial \hat{\tilde{H}}_s}{\partial t_r} + [\hat{\tilde{H}}_r, \hat{\tilde{H}}_s] = 0, \quad r,s = 1,\ldots,n. \quad (1.1)$$

We also prove that both types of quantum Painlevé-type systems are related by a quantum canonical transformation which resembles the classical result.

The paper is organized as follows. Section 2 is devoted to a concise presentation (with references to literature) of both classical and quantum Stäckel systems, both with ordinary and with magnetic potentials, as well as canonical transformations between them. In Section 3 we remind the construction of classical Painlevé-type systems from appropriate deformations of Stäckel-type systems. Section 4 contains
the first of two main results of this paper, namely the systematic method of minimal quantization of all the classical systems from Section 3 in such a way that they satisfy the quantum Frobenius condition (1.1). In Section 5 we present the second result of this article that is the multi-time quantum canonical transformations between quantum Painlevé-type systems with magnetic potentials and the corresponding quantum Painlevé-type systems with ordinary potentials.

2 Classical and quantum Stäckel systems with ordinary and magnetic potentials

Consider the following algebraic curve in the $(x, y)$-plane

\[ \sum_{\alpha \in I_\alpha} c_\alpha x^{\alpha} + \sum_{\gamma \in I_\gamma} d_\gamma x^{\gamma} y + \sum_{r=1}^{n} h_r x^{n-r} = \frac{1}{2} x^m y^2 \]  

(2.1)

where $m \in \{0, \ldots, n + 1\}$, $I_\alpha$ and $I_\gamma$ are finite subsets of $\mathbb{Z}$ and where $c_\alpha$ and $d_\gamma$ are real constants. Taking $n$ copies of (2.1) at points $(x, y) = (\lambda_i, \mu_i), i = 1, \ldots, n,$ we obtain a system of $n$ linear equations (separation relations) for $h_r$. Solving this system yields $n$ functions (Hamiltonians)

\[ h_r = \frac{1}{2} \mu^T A_r \mu + \sum_{\gamma \in I_\gamma} d_\gamma \mu^T \Gamma_\gamma + \sum_{\alpha \in I_\alpha} c_\alpha \nu_\alpha, \quad r = 1, \ldots, n \]  

(2.2)

on a $2n$-dimensional manifold (phase space) $M$ parametrized by the coordinates $(\lambda, \mu) = (\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)$, where

\[ E_r = \frac{1}{2} \mu^T A_r \mu \equiv \frac{1}{2} \mu^T K_r G \mu, \quad r = 1, \ldots, n, \]

$G$ is a contravariant metric tensor on the configurational space $Q$ (such that $M = T^*Q$), $K_r (K_1 = \text{Id})$ are $(1, 1)$-Killing tensors of $G$ (for any $m$ even though $G$ depends on $m$) [4], $P_\gamma$ are basic separable vector potentials and $\nu_\alpha$ are basic separable scalar potentials (see Part I). By construction, all the Hamiltonian functions $h_r$ are in involution

\[ \{ h_r, h_s \} = \pi (dh_r, dh_s) = 0, \quad r, s = 1, \ldots, n \]  

(2.3)

with respect to the Poisson bracket $\pi = \sum_{i=1}^{n} \frac{\delta}{\delta \lambda_i} \wedge \frac{\delta}{\delta \mu_i}$ on $M$. By construction, they also separate in coordinates $(\lambda, \mu)$. The Hamiltonians (2.2) are known in literature as (classical) Stäckel Hamiltonians, while $E_r$ are their geodesic parts.

In the separation coordinates $\lambda_i, i = 1, \ldots, n$, the geometric objects $A_r, G, K_r, P_\gamma$ and $\nu_\alpha$ are explicitly given by

\[ (A_r)^{ij} = -\frac{\partial \rho_r}{\partial \lambda_i} \chi_{m}^{n} \frac{\lambda_{i}^{m}}{\lambda_{j}} \delta^{ij}, \quad G^{ij} = \frac{\lambda_{i}^{n}}{\Delta_{j}} \delta^{ij}, \quad (K_r)^{ij} = -\frac{\partial \rho_r}{\partial \lambda_i} \delta^{ij}, \]  

(2.4)

\[ (P_\gamma)^{ij} = -\frac{\partial \rho_{\gamma}}{\partial \lambda_j} \frac{\lambda_{i}^{\gamma}}{\Delta_{j}}, \quad (K_r)^{ij} G^{jj} \lambda_{j}^{\gamma} = -A_{r}^{ij} \lambda_{i}^{\gamma} = -\frac{\partial \rho_{\gamma}}{\partial \lambda_j} \frac{\lambda_{i}^{\gamma}}{\Delta_{j}} \]  

(no summation in the above formulas unless explicitly stated) where $\Delta_j = \prod_{k \neq j} (\lambda_j - \lambda_k)$ and $\rho_r (\lambda) = (-1)^{r} s_r (\lambda)$ where $s_r (\lambda)$ are elementary symmetric polynomials. Note that the coordinates $\lambda$ are thus orthogonal coordinates for the metric $G$.

In what follows we will also work in the so called canonical Viète coordinates $(q, p)$ on $M$, related with the separation coordinates $(\lambda, \mu)$ through the point transformation

\[ q_i = \rho_i (\lambda), \quad p_i = -\sum_{k=1}^{n} \frac{\lambda_{i}^{n-k}}{\lambda_{k}} \mu_{k}, \quad i = 1, \ldots, n. \]  

(2.5)
Due to (2.4) the metric tensor $G$ for arbitrary $m$ is constructed by

$$G = L^m G_0,$$

(2.6)

where $L$ is the so called special conformal Killing tensor on $Q$ [1] and where $G_0$ is the metric tensor for $m = 0$ (thus $G_0^{ij} = \frac{1}{4} \delta^{ij}$ and $L_j = \lambda_i \delta^{ij}$). In Viète coordinates $L$ and $G_0$ have the form

$$L = \begin{pmatrix}
-q_1 & 1 & 0 & 0 \\
\vdots & 0 & \ddots & 0 \\
\vdots & 0 & 0 & 1 \\
-q_n & 0 & 0 & 0 \\
\end{pmatrix}, \quad G_0 = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & q_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 1 & q_1 & \cdots & q_{n-2} \\
1 & q_1 & \cdots & q_{n-2} & q_{n-1} \\
\end{pmatrix}. \quad (2.7)
$$

Further, the $(1,1)$-Killing tensors $K_r$ on $Q$ for every $G$ given by (2.6), can be effectively computed through [1]

$$K_1 = \text{Id}, \quad K_r = \sum_{k=0}^{r-1} q_k L^{r-1-k}, \quad r = 2, \ldots, n.$$

The scalar potentials $V_r^{(\alpha)}$ can be explicitly constructed by a recursion formula [5],[9]. Finally, the basic separable vector potentials $P_r^{(\gamma)}$ in Viète coordinates have the form

$$\left(P_r^{(\gamma)}\right)^j = - \sum_{s=0}^{r-1} q_s V_j^{(r+s-1)}.$$

(2.8)

More information about the structure of the above geometric objects is given in Part I and Part II.

Let us now turn to the issue of quantization of the classical Stäckel Hamiltonians (2.2). A natural way to quantize these Hamiltonians is by the procedure of minimal quantization [6],[7], that we shortly remind here.

**Definition 1.** Given a Poisson manifold $\mathcal{M} = T^*Q$ and a metric $g$ on $Q$, the minimal quantization of the quadratic in momenta function $H = \frac{1}{2} C^{kj} \dot{p}_k \dot{p}_j$ on $\mathcal{M}$, where $C^{kj}$ is a symmetric $(2,0)$-tensor on $Q$, is the second-order linear self-adjoint operator

$$\hat{H} = -\frac{1}{2} \hbar^2 \nabla_k C^{kj} \nabla_j = -\frac{1}{2} \hbar^2 |g|^{-\frac{1}{2}} \partial_k |g|^{\frac{1}{2}} C^{kj} \partial_j \quad (2.9)$$

acting in the Hilbert space $\mathcal{H} = L^2(Q, |g|^{1/2} \, dq)$, where $\nabla_j$ are covariant derivatives for the Levi-Civita connection of $g$ while $q$ are (any) variables on $Q$ and $|g| = \det g$. Likewise, the minimal quantization of the linear in momenta function $W = X^j \dot{p}_j$, where $X^j$ are components of the vector field $X = X^j \frac{\partial}{\partial q_j}$ on $Q$, is the first-order linear operator

$$\hat{W} = -\frac{1}{2} i \hbar \left( \nabla_j X^j + X^j \nabla_j \right) = -\frac{1}{2} i \hbar \left( |g|^{-\frac{1}{2}} \partial_j |g|^{\frac{1}{2}} X^j + X^j \partial_j \right), \quad (2.10)$$

acting in $\mathcal{H} = L^2(Q, |g|^{1/2} \, dq)$. Finally, the minimal quantization of a function $f$ on $Q$ is the operator $\hat{f}$ of pointwise multiplication by $f$ so that $\hat{f} = f$.

The second equalities in (2.9) and (2.10) follow by a direct calculation; note that the right hand sides of these expressions are still covariant, i.e. they look the same in all coordinate systems on $Q$. In general, the symbol $\circ$ denotes throughout the article the $\mathbb{R}$-linear operation of minimal quantization. Thus, the minimal quantization of Hamiltonians (2.2) is given by the following set of $n$ self-adjoint operators

$$\hat{h}_r = -\frac{1}{2} \hbar^2 \nabla_k A_r^{kj} \nabla_j - \frac{1}{2} i \hbar \sum_{\gamma \in I_r} d_\gamma \left( \nabla_j \left( P_r^{(\gamma)}\right)^j + (P_r^{(\gamma)})^j \nabla_j \right) + \sum_{\alpha \in I_a} c_{\alpha} V_r^{(\alpha)}, \quad r = 1, \ldots n \quad (2.11)$$
acting in the Hilbert space $\mathcal{H} = L^2(Q, |y|^{1/2} \, dq)$. One can show [2, 3, 7] that in the separation coordinates $\lambda$ the operators (2.11) take the form

$$
\hat{h}_r = -\frac{1}{2} \hbar^2 \sum_{j=1}^{n} \lambda_{ij} \left( \partial_j^2 - \Gamma_j \partial_j \right) - i \hbar \sum_{\gamma \in I_r} d_\gamma \left( \hat{P}_r^{(\alpha)} \right)^j \left( \partial_j - \frac{1}{2} \Gamma_j + \frac{1}{2} (\gamma-m) \lambda_j^{-1} \right) + \sum_{\alpha \in I_a} c_\alpha \hat{V}_r^{(\alpha)} , \quad r = 1, \ldots, n, \quad \partial_j = \frac{\partial}{\partial x_j}
$$

(2.12)

where $\Gamma_j$ are so called metrically contracted Christoffel symbols of $g$, expressed in the orthogonal coordinates $\lambda$, by

$$
\Gamma_j = \frac{1}{2} \partial_j \ln \frac{\prod_{k \neq j} G_{kk}}{G_{jj}} = -\frac{1}{2} m \lambda_j^{-1}.
$$

Thus, $\Gamma_j$ satisfy the Robertson condition

$$
\partial_k \Gamma_j = 0, \quad k \neq j
$$

(2.13)

and so the corresponding eigenvalue problems for $\hat{h}_r$

$$
\hat{h}_r \Psi = \varepsilon_r \Psi , \quad r = 1, \ldots, n
$$

are multiplicatively separable. It means that they have for each choice of eigenvalue $\varepsilon_r$ of $\hat{h}_r$ the common multiplicatively separable eigenfunction $\Psi(\lambda_1, \ldots, \lambda_n) = \prod_{i=1}^{n} \psi(\lambda_i)$ with $\psi$ satisfying the following ODE (quantum separation relation)

$$
(\varepsilon_1 \lambda_1^{n-1} + \varepsilon_2 \lambda_2^{n-1} + \cdots + \varepsilon_n) \psi(\lambda) = -\frac{1}{2} \hbar^2 \lambda^m \left( \frac{d^2 \psi(\lambda)}{d\lambda^2} + \frac{1}{2} m \lambda^{-1} \frac{d \psi(\lambda)}{d\lambda} \right) + i \hbar \sum_{\gamma \in I_r} \gamma_j \frac{d \psi(\lambda)}{d\lambda} + \frac{1}{4} (\gamma - 2m) \lambda^{-1} \psi(\lambda) + \sum_{\alpha \in I_a} c_\alpha \lambda^\alpha \psi(\lambda).
$$

The Robertson condition (2.13) implies that $\Gamma_j$ satisfy the pre-Robertson condition [2]

$$
\partial_k (\partial_j \Gamma_j - \frac{1}{2} \Gamma_j^2) = 0, \quad k \neq j
$$

and thus all the Hamiltonian operators (2.11) commute

$$
[\hat{h}_r, \hat{h}_s] = 0, \quad r, s = 1, \ldots, n.
$$

(2.15)

Observe that one can eliminate all the linear in $y$ terms in the algebraic curve (2.1) through the map

$$
x = x', \quad y = y' + \sum_{\gamma \in I_r} d_\gamma x^{\gamma-m}, \quad j = 1, \ldots, n
$$

(2.16)

which transforms this curve (2.1) to the algebraic curve without terms linear in $y$ (below we omit the prime at $x$ and $y$)

$$
\sum_{\alpha \in I_a} c_\alpha x^\alpha + \frac{1}{2} \sum_{\gamma, \gamma' \in I_r} d_\gamma d_{\gamma'} x^{\gamma+ \gamma'-m} + \sum_{r=1}^{n} \hat{h}_r x^{n-r} = \frac{1}{2} x^m y^2, \quad m, \alpha, \gamma \in \mathbb{Z}
$$

which leads to the Hamiltonians (2.2) in the new form (in the new variables) without the vector potential terms:

$$
\hat{h}_r = \frac{1}{2} \hbar^2 A_r(\lambda) \mu + \sum_{\alpha \in I_a} c_\alpha x^\alpha + \frac{1}{2} \sum_{\gamma, \gamma' \in I_r} d_\gamma d_{\gamma'} V_r^{(\gamma+ \gamma'-m)}, \quad r = 1, \ldots, n.
$$

(2.17)

The transformation (2.16) induces the following canonical transformation on $\mathcal{M}$:

$$
\lambda_j = \lambda_j', \quad \mu_j = \mu_j' + \sum_{\gamma \in I_r} d_\gamma \lambda_j^{\gamma-m}, \quad j = 1, \ldots, n
$$
which transforms the Hamiltonians (2.2) to the form (2.17).

The minimal quantization of (2.17) yields the operators (given here in \( \lambda \) variables)

\[
\hat{h}_r = -\frac{1}{\hbar^2} \sum_{j=1}^{n} A^{ij}_r (\partial_j^2 - \Gamma_j \partial_j) + \sum_{\alpha \in I_\alpha} c_\alpha V^{(\alpha)}_r + \frac{1}{2} \sum_{\gamma, \gamma' \in I_\gamma} d_{\gamma \gamma'} V_{r}^{(\gamma + \gamma' - m)}, \quad r = 1, \ldots, n, \quad \partial_j = \frac{\partial}{\partial \lambda_j}
\]  

(2.18)

acting in the same Hilbert space \( \mathcal{H} = L^2(Q, |g|^{1/2} \, dq) \) as the operators (2.12). Both sets of operators, (2.12) and (2.18), are related by the quantum canonical transformation

\[
\hat{h}_r = U \hat{h}_r U^\dagger, \quad U = U(\lambda) = e^{F(\lambda)}, \quad F(\lambda) = \left( -\frac{i}{\hbar} \sum_{\gamma \in I_\gamma} \frac{d_\gamma}{\gamma - m + 1} \sum_{j=1}^{n} \lambda_j^{\gamma - m + 1} \right). \]  

(2.19)

Note that the transformation (2.19) is covariant i.e. is valid in any coordinate system \( q \) on \( Q \). Note also that there is no singularity in (2.19) as in case \( \gamma = m - 1 \) the corresponding part of the unitary operation \( U \hat{h}_r U^\dagger \) is formally correctly defined.

### 3 Classical Painlevé-type systems

In Part I we constructed Frobenius integrable non-autonomous Hamiltonian systems with ordinary potentials, generated by the algebraic curve

\[
\sum_{\alpha = -m}^{2n - m + 2} c_\alpha(t) x^\alpha + \sum_{r=1}^{n} \hbar_r x^{n-r} = \frac{1}{2} x^m y^2, \quad m \in \{0, \ldots, n + 1\},
\]  

(3.1)

so that it has the form (2.1) with \( I_\alpha = \{-m, \ldots, 2n - m + 2\} \), \( I_\gamma = \emptyset \) but with \( c_\alpha(t) = c_\alpha(t_1, \ldots, t_n) \) no longer constant but some, for now undefined, smooth functions of all times \( t_r \). Solving the corresponding separation relations yields \( n \) Hamiltonians

\[
h_r = \frac{1}{2} \mu^T A_r \mu + \sum_{\alpha = -m}^{2n - m + 2} c_\alpha(t) V^{(\alpha)}_r, \quad r = 1, \ldots, n,
\]  

(3.2)

on the phase space \( \mathcal{M} \), with the \((2,0)\)-tensors \( A_r \) and the basic separable potentials given by (2.4). Following the method developed in [8] and in Part I, we now perturb the Hamiltonians \( h_r \) to the Hamiltonians

\[
h^A_1 = h_1, \quad h^A_r = h_r + W_r, \quad r = 2, \ldots, n,
\]  

(3.3)

where \( W_r \) are linear in momenta terms (quasi-Stäckel terms) given in \((q,p)\) coordinates by

\[
W_r = \sum_{j=1}^{n} J^j_r p_j
\]

where \( J_r, r = 2, \ldots, n, \) are \( n - 1 \) vector fields on \( Q \) given by

\[
J_r = \sum_{k=n+2-m-r}^{2n+2-m-r} (n + 1 - m - k) q_{m+r-n-2+k} \partial_k, \quad r \in I^m_1, \quad J_r = -\sum_{k=n+2-m}^{2n+2-m-r} (n + 1 - m - k) q_{m+r-n-2+k} \partial_k, \quad r \in I^m_2
\]  

(3.4)

(3.5)

(note that they do depend on \( m, \) as \( G \) does) that are Killing vector fields for the metric \( G \). Here and in what follows \( \partial_i = \partial/\partial q_i \). The index sets \( I^m_1 \) and \( I^m_2 \) are defined as follows:

\[
I^m_1 = \{2, \ldots, n - m + 1\}, \quad I^m_2 = \{n - m + 2, \ldots, n\}, \quad m = 0, \ldots, n + 1.
\]

(3.6)
with the degenerations $I^n_2 = \emptyset$ for $m = 0, 1$ while $I^n_1 = \emptyset$ for $m = n, n + 1$. Here and throughout the paper we use the notation $q_0 = 1$ and $q_r = 0$ for $r < 0$ and for $r > n$. We also set $W_1 = 0$.

This particular choice of Killing vector fields $J_r$ from the whole algebra of Killing vector fields for $G$ is motivated by the following important observation. It can be shown that the functions $E_r = E_r + W_r$ (called the geodesic quasi-Stäckel Hamiltonians) span the Lie algebra $\mathfrak{g} = \text{span}(E_r, \ r = 1, \ldots, n)$ with the commutation relations

$$\{E_i, E_r\} = 0, \quad r = 2, \ldots, n,$$

and

$$\{E_r, E_s\} = \begin{cases} 0 & \text{for } r \in I^n_1 \text{ and } s \in I^n_2, \\ (s - r)E_{r+s-(n-m+2)} & \text{for } r, s \in I^n_1, \\ -(s - r)E_{r+s-(n-m+2)} & \text{for } r, s \in I^n_2, \end{cases} \quad (3.7)$$

where $E_i = 0$ as soon as $i \leq 0$ or $i > n$. The algebra $\mathfrak{g}$ has an Abelian subalgebra

$$\mathfrak{a} = \text{span}(E_1, \ldots, E_{\kappa_1}, E_{n-\kappa_2+1}, \ldots, E_n) \quad (3.8)$$

where

$$\kappa_1 = \left[\frac{n + 3 - m}{2}\right], \quad \kappa_2 = \left[\frac{m}{2}\right]. \quad (3.9)$$

Finally, let us construct new Hamiltonians $H^A_r$ such that for $r \in \{1\} \cup I^n_1$

$$H^A_r = h^A_r, \quad \text{for } r = 1, \ldots, \kappa_1,$$

$$H^A_r = \sum_{j=1}^r \zeta_{r,j}(t_1, \ldots, t_{r-1}) h^A_j, \quad \zeta_{r,r} = 1, \quad \text{for } r = \kappa_1 + 1, \ldots, n - m + 1 \quad (3.10)$$

and for $r \in I^n_2$

$$H^A_r = \sum_{j=0}^{n-r} \zeta_{r,r+j}(t_{r+1}, \ldots, t_n) h^A_{r+j}, \quad \zeta_{r,r} = 1, \quad \text{for } r = n - m + 2, \ldots, n - \kappa_2,$$

$$H^A_r = h^A_r, \quad \text{for } r = n - \kappa_2 + 1, \ldots, n, \quad (3.11)$$

where $\zeta_{r,j}(t)$ are some functions of appropriate evolution parameters. The Hamiltonians $H^A_r$ define $n$ non-autonomous Hamiltonian systems on $\mathcal{M}$

$$\xi_{t_r} = Y_r(\xi, t) = \pi dH^A_r(\xi, t), \quad r = 1, \ldots, n \quad (3.12)$$

and according to results obtained in [8] and in Part I one can always obtain (by solving an appropriate compatible and overdetermined system of PDE's) the explicit form of functions $c_\alpha(t_1, \ldots, t_n)$, $\zeta_{r,j}(t_1, \ldots, t_{r-1})$ and $\zeta_{r,r+j}(t_{r+1}, \ldots, t_n)$ such that the Hamiltonians $H^A_r$ in (3.10) and (3.11) satisfy the (classical) Frobenius integrability condition

$$\frac{\partial H_r}{\partial t_s} - \frac{\partial H_s}{\partial t_r} + \{H_r, H_s\} = f_{rs}(t_1, \ldots, t_n), \quad r, s = 1, \ldots, n \quad (3.13)$$

with $H_r = H^A_r$. It means that $n$ Hamiltonian systems (3.12) have (at least for small intervals of times $t_i$) a unique common multi-time solution $\xi = \xi(t_1, \ldots, t_n, \xi_0)$ for any initial condition $\xi_0$ [12, 16]. In consequence, the non-autonomous Hamiltonian vector fields $Y_r$ on $\mathcal{M}$ satisfy the Frobenius condition

$$\frac{\partial Y_r}{\partial t_s} - \frac{\partial Y_s}{\partial t_r} - [Y_r, Y_s] = 0 \text{ for all } r, s = 1, \ldots, n \quad (3.14)$$

(see Part I for the details of this construction; the second minus sign in (3.14) is due to the convention used in (2.3)).

In Part I we also constructed Frobenius integrable non-autonomous Hamiltonian systems with vector potentials, generated by the following algebraic curve

6
\[
\sum_{\gamma=0}^{n+1} d_{\gamma}(t)x^\gamma y + \sum_{r=1}^{n} x^{n-r} h_r = \frac{1}{2} x^m y^2, \quad (3.15)
\]
i.e. the curve (2.1) with \( I_\alpha = 0, I_\gamma = \{0, \ldots, n+1\} \) and with \( d_{\gamma}(t) = d_{\gamma}(t_1, \ldots, t_n) \) no longer constant but, for the moment arbitrary, functions of all times \( t_i \). Solving the corresponding separation relations with respect to \( h_r \) yields \( n \) Hamiltonians
\[
h_r = \frac{1}{2} \mu^T A_r \mu + \sum_{\gamma=0}^{n+1} d_{\gamma}(t) \mu^T P_r(\gamma), \quad r = 1, \ldots, n \quad (3.16)
\]
on \( \mathcal{M} = T^*Q \), with the contravariant tensors \( A_r \) and with the vector potentials \( P_r(\gamma) \) given by (2.4) that enter the Hamiltonians (3.16) through the (linear in momenta) magnetic terms
\[
M_r(\gamma) = \mu^T P_r(\gamma). \quad (3.17)
\]
Deforming the Hamiltonians (3.16) to the Hamiltonians \( h^B_r = h_r + W_r \) (cf. (3.3)) using the same quasi-Stäckel terms \( W_r = J_f^p \) with the same vector fields \( J_f \) given by (3.4) and (3.5) we were able to construct, through an appropriate choice of the functions \( \varphi_r \) in (3.10), (3.11), as well as an appropriate choice of \( d_{\gamma}(t) \) in (3.16), the set of \( n \) non-autonomous Hamiltonians \( H^B_r(t), r = 1, \ldots, n \), satisfying the Frobenius integrability condition (3.13).

In Part II we have constructed the isomonodromic Lax representation for the non-homogeneous systems \( \xi_{tr} = \pi dH^A_r(\xi, t) \) and \( \xi_{tr} = \pi dH^B_r(\xi, t) \) thus proving that the are indeed of Painlevé-type.

4 Quantization of Painlevé-type systems

As we explained in Introduction, the aim of this paper is to show that the deformation procedures, developed in [8] and in Part I, and shortly presented in the previous section, have its quantum counterpart. In this section we prove that the minimally quantized Painlevé-type systems, both with ordinary and with magnetic potentials, satisfy the quantum Frobenius condition (4.2).

4.1 Quantum Frobenius condition

Consider the set of Schrödinger equations
\[
i\hbar \frac{\partial \Psi}{\partial t_r} = \hat{H}_r \Psi, \quad r = 1, \ldots, n \quad (4.1)
\]
where \( \hat{H}_r \) is a set of \( n \) linear operators acting in a Hilbert space \( \mathcal{H} \). A necessary condition for the existence of a common multi-time solution \( \Psi(t_1, \ldots, t_n) \) of the system (4.1) has the form
\[
\frac{\partial^2 \Psi}{\partial t_r \partial t_s} = \frac{\partial^2 \Psi}{\partial t_s \partial t_r} \quad \text{for any } r, s = 1, \ldots, n.
\]
Inserting it into (4.1) leads to the following necessary condition for the existence of common solutions of (4.1)
\[
i\hbar \frac{\partial \hat{H}_r}{\partial t_s} - i\hbar \frac{\partial \hat{H}_s}{\partial t_r} + [\hat{H}_r, \hat{H}_s] = 0, \quad r, s = 1, \ldots, n. \quad (4.2)
\]
We will refer to the condition (4.2) as the quantum Frobenius condition. We stress that while the right hand sides \( f_{rs} \) of the classical Frobenius condition (3.13) may in general depend on all the times \( t_i \), the right hand side of (4.2) must be zero. This however does not restrict our method, described below, as it is always possible in the classical regime to choose the functions \( c_\alpha \) so that \( f_{rs}(t) = 0 \) (while in the magnetic case \( f_{rs} \) are always zero).
4.2 Quantization of systems with ordinary potentials

We start with the case of ordinary potentials. According to Definition 1, the minimal quantization \( \hat{H}_r^A \) of Hamiltonians \( H_r^A \) is obtained by replacing the Hamiltonians \( h_r^A \) in (3.10) and (3.11) by the self-adjoint operators

\[
\hat{h}_r^A = \hat{E}_r + \hat{W}_r + \sum_{\alpha = -m}^{2n-m+2} c_\alpha (t_1, \ldots, t_n) V^{(\alpha)}_r, \quad r = 1, \ldots, n, \tag{4.3}
\]

where, due to (2.9) and (2.10)

\[
\begin{align*}
\hat{E}_r &= -\frac{1}{2} \hbar^2 \nabla_k A_{kj} \nabla_j = -\frac{1}{2} \hbar^2 |g|^{-1/2} \partial_k |g|^{1/2} A_{kj} \partial_j \\
\hat{W}_r &= -\frac{1}{2} i\hbar (\nabla_j J_j^r + J_j^r \nabla_j) = -\frac{1}{2} i\hbar \left( |g|^{-1/2} \partial_j |g|^{1/2} J_j^r + J_j^r \partial_j \right)
\end{align*}
\tag{4.4, 4.5}
\]

are operators acting in the Hilbert space \( \mathcal{H} = L^2(Q, \Gamma g)^{1/2} dq \).

**Theorem 2** For any fixed \( m \in \{0, \ldots, n+1\} \), the set of \( n \) operators \( \hat{H}_r^A \), \( r = 1, \ldots, n \), given by (3.10), (3.11) with \( h_r^A \) replaced by \( \hat{h}_r^A \) given by (4.3), satisfies the quantum Frobenius condition (4.2).

We will prove this theorem by showing that the proof of the classical version of this theorem, as it is presented in Part I, survives in the quantum regime. By results in [8] and in Part I, also mentioned in Section 3, the Hamiltonians \( \hat{H}_r^A \) will satisfy the quantum Frobenius condition (4.2) as soon as any pair of operators \( \hat{E}_r + V_r^{(\alpha)} = \hat{E}_r + \hat{W}_r + V_r^{(\alpha)} \) and \( \hat{E}_s + V_s^{(\alpha)} = \hat{E}_s + \hat{W}_s + V_s^{(\alpha)} \) will, for any fixed \( m \in \{0, \ldots, n+1\} \), satisfy the same, up to the factor \( \hbar \), commutation relations as their classical counterparts \( \mathcal{E}_r + V_r^{(\alpha)} = \mathcal{E}_r + W_r + V_r^{(\alpha)} \) and \( \mathcal{E}_s + V_s^{(\alpha)} = \mathcal{E}_s + W_s + V_s^{(\alpha)} \). That is, we have to prove that

\[
\hbar \left\{ E_r + W_r + V_r^{(\alpha)}, E_s + W_s + V_s^{(\alpha)} \right\} = \left[ \hat{E}_r + \hat{W}_r + V_r^{(\alpha)}, \hat{E}_s + \hat{W}_s + V_s^{(\alpha)} \right]
\tag{4.6}
\]

for all \( r, s = 1, \ldots, n \), for all \( m \in \{0, \ldots, n+1\} \) and for all \( \alpha \in \{-m, \ldots, 2n-m+2\} \), as the relation (4.6) means that we can follow the procedure from Section 3 also in the quantum case and this procedure will yield the same PDE’s for functions \( c_\alpha (t_1, \ldots, t_n) \), \( \zeta_r (t_1, \ldots, t_r-1) \) and \( \zeta_{r+j} (t_{r+1}, \ldots, t_n) \) as in the corresponding classical procedure. We will perform all calculations in Viète coordinates \((q,p)\).

Before we proceed, let us rewrite (4.4) and (4.5) with the operators \( \partial_j \) standing maximally to the right. Note that due to (2.7) we have \( \det L = (-1)^n q_n \) and \( \det G_0 = (-1)^{[n/2]} \) (where \([\cdot]\) denotes the integer part) so that, due to (2.6)

\[
|g| = \varepsilon q_n^{-m} \text{ with } \varepsilon = (-1)^{nm+[n/2]}. \tag{4.7}
\]

Thus, \( \partial_j |g| = -\varepsilon q_n^{-m-1} \delta_{j,n} \) and a direct calculation yields that

\[
\hat{E}_r = -\frac{1}{2} \hbar^2 \left( A^k_r \partial_k \partial_j + (A^k_r)_{q_k} \partial_j - \frac{1}{2} \varepsilon^2 m q_n^{-1} A^k_r \partial_j \right)
\]

while

\[
\hat{W}_r = -i\hbar \left( J_j^r \partial_j + \frac{1}{2} |g|^{-1/2} \left( \partial_j |g|^{1/2} J_j^r \right) \right).
\]

A straightforward calculation shows that the zero-order term in \( \hat{W}_r \) is equal to 0 for all \( r \). Indeed, due to (4.7) we have

\[
|g|^{-1/2} \left( \partial_j |g|^{1/2} J_j^r \right) = -\frac{1}{2} m q_n^{-1} J_r^m + (\partial_j J_j^r).
\]

Assume first that \( r \in I_n^m \). Then, due to (3.4), \( J_r^m = 0 \) as soon as \( m > 0 \) and thus \( m q_n^{-1} J_r^m \) is always (for all \( m \)) zero. Further, again due to (3.4)

\[
(\partial_j J_j^r) = (n - m - j + 1) \delta_{r,n-m+2} = 0 \text{ (no summation)}
\]

8
due to definition of $I^m_1$. A similar, albeit a little bit more tedious, calculation shows that the same result holds for any $r \in I^m_2$. In consequence the Hamiltonian operators (4.4) and (4.5) can be written as

$$\hat{E}_r = -\frac{1}{2}\hbar^2 \left( A^{kj}_{r}\partial_k \partial_j + (A^{kj}_{r})_{qs} \partial_j - \frac{1}{2} m q^{-1}_{rs} A^{nij}_{r} \partial_j \right)$$

$$\hat{W}_r = -i\hbar J^j_r \partial_j = -i\hbar J_r.$$

We are now in position to prove the condition (4.6). Obviously, for any $r, s \in \{1, \ldots, n\}$

$$\left\{ E_r + W_r + V^{(\alpha)}_r, E_s + W_s + V^{(\alpha)}_s \right\} = \{ \mathcal{E}_r, \mathcal{E}_s \} + \left\{ E_r, V^{(\alpha)}_r \right\} + \left\{ V^{(\alpha)}_r, E_s \right\} + \left\{ W_r, V^{(\alpha)}_s \right\} + \left\{ V^{(\alpha)}_r, W_s \right\} + \left\{ V^{(\alpha)}_r, V^{(\alpha)}_s \right\}.$$

The first term $\{ \mathcal{E}_r, \mathcal{E}_s \}$ is given by (3.7). The last term $\left\{ V^{(\alpha)}_r, V^{(\alpha)}_s \right\}$ is obviously 0. Moreover, Stäckel Hamiltonians themselves (i.e. without the quasi-Stäckel term $W_r$), both geodesic and with potentials, commute with each other, so that $\{ E_r, E_s \} = 0$ and $\{ E_r + V^{(\alpha)}_r, E_s + V^{(\alpha)}_s \} = 0$, which yields

$$\{ E_r, V^{(\alpha)}_s \} + \{ V^{(\alpha)}_r, E_s \} = 0.$$

Thus, the left hand side of (4.6) becomes

$$i\hbar \{ \mathcal{E}_r, \mathcal{E}_s \} + i\hbar \left( \{ W_r, V^{(\alpha)}_s \} + \{ V^{(\alpha)}_r, W_s \} \right).$$

On the other hand, the right hand side of (4.6) is

$$[\hat{\mathcal{E}}_r, \hat{\mathcal{E}}_s] + [\hat{E}_r, V^{(\alpha)}_s] + [V^{(\alpha)}_r, \hat{E}_s] + [\hat{W}_r, V^{(\alpha)}_s] + [V^{(\alpha)}_r, \hat{W}_s] + [V^{(\alpha)}_r, V^{(\alpha)}_s]$$

(also here the last term is 0) where the operators $\hat{\mathcal{E}}_r$ denote minimal quantization of respective geodesic quasi-Stäckel Hamiltonians $\mathcal{E}_r$

$$\hat{\mathcal{E}}_r = \hat{E}_r + \hat{W}_r, \ r = 1, \ldots, n. \quad (4.10)$$

We moreover know that the quantum Stäckel Hamiltonians themselves (i.e. without the quasi-Stäckel term $\hat{W}_r$), both geodesic and with potentials, commute with each other, due to (2.15), i.e. $[\hat{\mathcal{E}}_r, \hat{\mathcal{E}}_s] = 0$ and $[\hat{E}_r + V^{(\alpha)}_r, \hat{E}_s + V^{(\alpha)}_s] = 0$. This also yields

$$[\hat{\mathcal{E}}_r, V^{(\alpha)}_s] + [V^{(\alpha)}_r, \hat{\mathcal{E}}_s] = 0$$

and so the right hand side of (4.6) becomes

$$[\hat{\mathcal{E}}_r, \hat{\mathcal{E}}_s] + [\hat{W}_r, V^{(\alpha)}_s] + [V^{(\alpha)}_r, \hat{W}_s].$$

Further

$$\{ W_r, V^{(\alpha)}_s \} + \{ V^{(\alpha)}_r, W_s \} = -J^j_r (V^{(\alpha)}_s)_{qj} + J^j_s (V^{(\alpha)}_r)_{qj},$$

is thus an expression on $Q$ not on $\mathcal{M}$ i.e. a function on $Q$, while on the quantum level

$$[\hat{W}_r, V^{(\alpha)}_s] + [V^{(\alpha)}_r, \hat{W}_s] = -i\hbar \left( [J^j_r (V^{(\alpha)}_s)_{qj} + [V^{(\alpha)}_r, J^j_s (V^{(\alpha)}_r)_{qj}] \right)$$

$$= -i\hbar \left( J^j_r (V^{(\alpha)}_s)_{qj} - J^j_s (V^{(\alpha)}_r)_{qj} \right)$$

$$= i\hbar \left( \{ W_r, V^{(\alpha)}_s \} + \{ V^{(\alpha)}_r, W_s \} \right)$$

9
and thus (see Definition 1)
\[ i\hbar \left( \{ W_r, V^{(r)}_s \} + \{ V^{(r)}_s, W_r \} \right) = \left[ \hat{W}_r, V^{(r)}_s \right] + [V^{(r)}_s, \hat{W}_r]. \]

It remains to show that \( i\hbar \{ \mathcal{E}_r, \mathcal{E}_s \} = \left[ \hat{\mathcal{E}}_r, \hat{\mathcal{E}}_s \right] \) that actually shows that the operators \( \hat{\mathcal{E}}_r \) constitute a Lie algebra with the same, up to the factor \( i\hbar \), structure constants as the algebra (3.7) generated by their classical counterparts \( \mathcal{E}_r = E_r + W_r \).

**Theorem 3** The operators \( \hat{\mathcal{E}}_r \) in (4.10) satisfy the commutation relations
\[ [\hat{\mathcal{E}}_r, \hat{\mathcal{E}}_s] = 0, \quad r = 2, \ldots, n, \]
\[ [\hat{\mathcal{E}}_r, \hat{\mathcal{E}}_s] = \begin{cases} 
0, & \text{for } r \in I^m_1 \text{ and } s \in I^m_2, \\
-ih(s-r)\hat{\mathcal{E}}_{r+s-(n-m+2)}, & \text{for } r, s \in I^m_1, \\
\text{for } r, s \in I^m_2. 
\end{cases} \]

(since \( [\hat{\mathcal{E}}_r, \hat{\mathcal{E}}_s] = i\hbar \{ \mathcal{E}_r, \mathcal{E}_s \} \)).

Let us see why this theorem is true. Naturally
\[ \{ \mathcal{E}_r, \mathcal{E}_s \} = \{ E_r, E_s \} + \{ E_r, W_s \} + \{ W_r, E_s \} + \{ W_r, W_s \}. \]  
(4.12)
(with \( \{ E_r, E_s \} = 0 \) while
\[ [\hat{\mathcal{E}}_r, \hat{\mathcal{E}}_s] = \left[ \hat{E}_r + \hat{W}_r, \hat{E}_s + \hat{W}_s \right] = \left[ \hat{E}_r, \hat{E}_s \right] + \left[ \hat{E}_r, \hat{W}_s \right] + \left[ \hat{W}_r, \hat{E}_s \right] + \left[ \hat{W}_r, \hat{W}_s \right] \]  
(4.13)
and \( [\hat{E}_r, \hat{E}_s] = 0 \). In order to calculate the remaining terms in (4.13) we will use the link between the canonical Poisson bracket on \( \mathcal{M} = T^*Q \) and the Schouten bracket between symmetric contravariant tensors on \( Q \). For a pair of functions on \( \mathcal{M} \)
\[ F_K = \frac{1}{k!} K^{i_1 \ldots i_k}(q)p_{i_1} \ldots p_{i_k}, \quad F_R = \frac{1}{r!} R^{i_1 \ldots i_r}(q)p_{i_1} \ldots p_{i_r} \]
where \( K \) and \( R \) are two symmetric tensors on \( Q \), of type \( (k,0) \) and \( (r,0) \) respectively, the following relation holds [11]
\[ \{ F_K, F_R \} = -[K, R]S^{i_1 \ldots i_k + r - 1} p_{i_1} \ldots p_{i_k + r - 1} \]  
(4.14)
(the minus sign in (4.14) is due to the convention used in (2.3)) where
\[ [K, R]S^{i_1 \ldots i_k + r - 1} = \frac{1}{k!r!} \left[ kK^{i_1 \ldots i_k} \partial_{i_1} \partial_{i_2} \ldots \partial_{i_k} R^{i_1 \ldots i_k + r - 1} - r R^{(i_1 \ldots i_k + r - 1)} \partial_{i_1} \partial_{i_2} \ldots \partial_{i_k} K^{i_1 \ldots i_k + r - 1} \right] \]  
(4.15)
is a \( (k + r - 1,0) \)-type symmetric tensor on \( Q \), called the Schouten bracket of \( K \) and \( R \); the symbol \( (\ldots) \) on the right hand side of (4.15) denotes symmetrization over indices. Thus
\[ \{ W_r, W_s \} = \{ J_s, J_r \} \quad p_j = \left( J_s^{(r)}(J_r^{(s)})_{q_k} - J_s^{(r)}(J_r^{(s)})_{q_k} \right) p_j \]
(where \( (J_r^{(s)})_{q_k} \) denotes \( \frac{\partial}{\partial q_k} J_r^{(s)} \)) and by comparing (4.12) and (3.7) we obtain
\[ J_s^{(r)}(J_r^{(s)})_{q_k} - J_r^{(s)}(J_s^{(r)})_{q_k} = \begin{cases} 
0, & \text{for } r \in I^m_1 \text{ and } s \in I^m_2, \\
(s-r)J_r^{(s)}(J_s^{(r)})_{q_k} - (s-r)J_s^{(r)}(J_r^{(s)})_{q_k}, & \text{for } r, s \in I^m_1, \\
-(s-r)J_r^{(s)}(J_s^{(r)})_{q_k} - (s-r)J_s^{(r)}(J_r^{(s)})_{q_k}, & \text{for } r, s \in I^m_2. 
\end{cases} \]  
(4.16)
Moreover, again due to (4.14),
\[ \{ W_r, E_s \} + \{ E_r, W_s \} = \left( [A_s, J_r]^{(s)} + [J_s, A_r]^{(s)} \right) p_k p_j, \]
so, by comparing the appropriate terms in (4.12) and in (3.7) and due to (4.15), the above formula reads

\[ A_s^k(J_r^q)_q + A_s^j(J_r^q)_q - J_r^k(A_s^j)_q + J_r^j(A_s^k)_q = A_r^k(J_s^q)_q - A_r^j(J_s^q)_q \]

\[ (4.17) \]

Further, differentiating (4.17) with respect to \( q_k \) and summation over \( k \) yields

\[ (A_s^k)_q(J_r^q)_q - (A_r^q)_q(J_s^q)_q - J_r^k(A_s^j)_q + J_r^j(A_s^k)_q = \]

\[ (4.18) \]

due to fact that all components of all \( J_r \) are linear in \( q \). Using the above classical relations we can now calculate the remaining terms in (4.13).

**Lemma 1**

The commutator of differential operators \( \hat{W}_r \) and \( \hat{W}_s \) is

\[ [\hat{W}_r, \hat{W}_s] = \begin{cases} 0, & \text{for } r \in I_1^m \text{ and } s \in I_2^m, \\ i\hbar(s - r)\hat{W}_{r+s-(n-m+2)}, & \text{for } r, s \in I_1^m, \\ -i\hbar(s - r)\hat{W}_{r+s-(n-m+2)}, & \text{for } r, s \in I_2^m. \end{cases} \]

\[ (4.19) \]

(so that \( [\hat{W}_r, \hat{W}_s] = i\hbar\{\hat{W}_r, \hat{W}_s\} \)).

The proof is by direct computation:

\[ [\hat{W}_r, \hat{W}_s] = -\hbar^2[J_r, J_s] = -\hbar^2 (J_r^k(J_s^q)_q - J_s^k(J_r^q)_q) \partial_j \]

\[ (4.16) \]

\[ = \begin{cases} 0, & \text{for } r \in I_1^m \text{ and } s \in I_2^m, \\ i\hbar^2(s - r)J_r^1(J_s^q)_q, & \text{for } r, s \in I_1^m, \\ -i\hbar^2(s - r)J_s^1(J_r^q)_q, & \text{for } r, s \in I_2^m, \end{cases} \]

which yields immediately (4.19) as \( \hat{W}_r = -i\hbar \partial_r \).

**Lemma 2**

The mixed term in (4.13) is given by

\[ [\hat{W}_r, \hat{E}_s] + [\hat{E}_r, \hat{W}_s] = \begin{cases} 0, & \text{for } r \in I_1^m \text{ and } s \in I_2^m, \\ i\hbar(s - r)\hat{E}_{r+s-(n-m+2)}, & \text{for } r, s \in I_1^m, \\ -i\hbar(s - r)\hat{E}_{r+s-(n-m+2)}, & \text{for } r, s \in I_2^m. \end{cases} \]

\[ (4.20) \]

**Proof.** Assume \( r, s \in I_1^m \) (for \( r, s \in I_2^m \) the proof is analogous). Using (4.8) we obtain

\[ [\hat{W}_r, \hat{E}_s] + [\hat{E}_r, \hat{W}_s] = \frac{1}{2} i\hbar^3 \{ [J_s^i \partial_i, A_s^{ij} \partial_j] + [A_r^{ij} \partial_i \partial_j, J_r^j \partial_i] \\ + [J_r^j \partial_i, (A_s^{ij})_q \partial_j] + [(A_r^{ij})_{q_k} \partial_i, J_r^j \partial_i] \\ - \frac{1}{2} m([J_s^i \partial_i, q^{-1}_n A_r^{nq} \partial_j] + [q^{-1}_n A_r^{nq} \partial_j, J_s^i \partial_i]) \}

\[ (4.21) \]

Since \( (J_r^j)_q\partial_k = 0 \), the first two terms above read

\[ [J_r^j \partial_i, A_s^{ij} \partial_j \partial_k] + [A_r^{ij} \partial_i \partial_j, J_r^j \partial_i] \]

\[ (4.15) \]

\[ \overset{(4.17)}{=} -(s - r)A_r^{ij} \partial_r + (s - r)A_r^{ij} \partial_r, \]

\[ (4.17) \]
while the next two terms become
\[
[J_i^j \partial_i, (A^{kj}_s)_{q_k} \partial_j] + [(A^{kj}_r)_{q_k} \partial_j, J_i^j \partial_i]
\]
\[
= (J_i^j(A^{kj}_s))_{q_k} - (A^{kj}_s)_{q_k} (J_i^j)_{q_k} + (A^{kj}_r)_{q_k} (J_i^j)_{q_k} - J_i^j (A^{kj}_r)_{q_k} \partial_j
\]
\[
= -(s-r)(J_i^j)^{k+s-(n-m+2)} q_k \partial_j.
\]

The last two terms in (4.21) are
\[
-\frac{1}{2} m \left( [J_i^j \partial_i, q_n^{-1} A^{nj}_s \partial_j] + [q_n^{-1} A^{nj}_s \partial_j, J_i^j \partial_i] \right)
\]
\[
= -\frac{1}{2} m q_n^{-1} (J_i^j(A^{nj}_s))_{q_k} - A^{nj}_s(J_i^j)_{q_k} + A^{nj}_r(J_i^j)_{q_k} - J_i^j (A^{nj}_r)_{q_k} \partial_j
\]
\[
= \frac{1}{2} m q_n^{-1} J_i^j (A^{nj}_r)_{q_k} - J_i^j (A^{nj}_r)_{q_k} \partial_j
\]
\[
\equiv \frac{1}{2} m \left( (s-r) q_n^{-1} A^{nj}_r \partial_j \right).
\]

Gathering all the terms in (4.21) as calculated above and comparing the result with (4.8) we receive the relation
\[
J_n^r = 0, \quad r \neq n - m + 2 \quad \text{and} \quad J_{n-m+2}^n = (m-1) q_n
\]

we can formally write
\[
\frac{1}{q_n} J_n^r A^{nj}_s = (J_n^r)_{q_n} A^{nj}_s = (J_n^r)_{q_i} A^{ij}_s
\]

and thus
\[
-\frac{1}{2} m \left( [J_i^j \partial_i, q_n^{-1} A^{nj}_s \partial_j] + [q_n^{-1} A^{nj}_s \partial_j, J_i^j \partial_i] \right)
\]
\[
= -\frac{1}{2} m q_n^{-1} (J_i^j(A^{nj}_s))_{q_k} - A^{nj}_s(J_i^j)_{q_k} + A^{nj}_r(J_i^j)_{q_k} - J_i^j (A^{nj}_r)_{q_k} \partial_j
\]
\[
= \frac{1}{2} m \left( (s-r) q_n^{-1} A^{nj}_r \partial_j \right).
\]

Lemmas 1 and 2 immediately imply the thesis of Theorem 3 which in turn implies that the relation (4.6) is true and therefore Theorem 2 is proved.

The first example (Example 1 below) is rather detailed, to illustrate various aspects of the theory presented above.

**Example 1** (Non-autonomous quantum Hénon-Heiles system. Continuation of Example 3 from Part I) Choose \( n = 2 \) and \( m = 1 \). Then \( I_1 = \{2\} \) while \( I_2 = \emptyset \) and the only Killing vector field (3.4) is \( J_2 = \partial_1 = \frac{\partial}{\partial q_1} \) (and there are no fields (3.5)). The curve (3.1) becomes
\[
\sum_{\alpha=-1}^{5} c_\alpha(t) x^\alpha + h_1 x + h_2 = \frac{1}{2} xy^2
\]

and solving the corresponding separation relations yields the Hamiltonians \( h_r \) that in Viète coordinates \( (q,p) \) attain the form
\[
h_r = \frac{1}{2} p^T A_r p + \sum_{\alpha=-1}^{5} c_\alpha(t) V_r^{(\alpha)}, \quad r = 1, 2
\]

with \( A_r = K_r G \) of the form
\[
A_1 = \begin{pmatrix}
1 & 0 \\
0 & -q_2
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
0 & -q_2 \\
-q_2 & -q_1 q_2
\end{pmatrix}
\]
\[
(4.22)
\]

and where \( V^{(0)} = (0, -1)^T \) and \( V^{(1)} = (-1, 0) \) are trivial potentials and where
\[
V^{(-1)} = (q_2^{-1}, q_1 q_2^{-1})^T, \quad V^{(2)} = (q_1, q_2)^T, \quad V^{(3)} = (q_2 - q_1^2, -q_1 q_2)^T, \quad V^{(4)} = (q_3^2 - 2q_1 q_2, q_1 q_2 - q_2^2)^T
\]
\[
V^{(5)} = (-q_1^4 + 3q_1^2 q_2 - q_2^2, -q_1 q_2 + 2q_1 q_2)^T
\]
Deforming \( h_1 \) and \( h_2 \) respectively by the functions \( W_1 = 0 \) and \( W_2 = p_1 \), yields the Hamiltonians \( h_1^A = h_1 \) and \( h_2^A = h_2 + p_1 \). Further, due to the fact that \( \kappa_1 = 2 = n \) we have that \( H_r^A = h_r^A \) for \( r = 1, 2 \) (see (3.9) and (3.10)). We can now determine the functions \( c_\alpha(t) \) by demanding that \( H_r^A \) satisfy the Frobenius condition (3.13) with \( f_{12} \equiv 0 \). That leads to an overdetermined, but solvable, system of PDE’s for \( c_\alpha \) (note that the functions \( c_\alpha \) do not depend on the number \( r \) of the Hamiltonian) with a particular solution

\[
c_{-1} = -\frac{1}{4}a \text{ arbitrary}, \quad c_0 = \frac{1}{2}t_1^2 + 3t_1t_2, \quad c_1 = 0, \quad c_2 = -(t_1 + 3t_2^2), \quad c_3 = -3t_2, \quad c_4 = -1, \quad c_5 = 0.
\]

Another particular solution, which differs from the one above only in the non-dynamical (in the sense that they do not influence the Hamiltonian flows of \( h_1^A \)) functions \( c_0 \) and \( c_1 \) is

\[
c_{-1} = -\frac{1}{4}a \text{ arbitrary}, \quad c_0 = \frac{1}{2}t_1^2, \quad c_1 = -t_3^2, \quad c_2 = -(t_1 + 3t_2^2), \quad c_3 = -3t_2, \quad c_4 = -1, \quad c_5 = 0.
\]

In consequence, the sought Hamiltonians \( H_r^A \) become

\[
\begin{align*}
H_1^A &= \frac{1}{2} p_1^2 - \frac{1}{2} q_2 p_2^2 - q_1^3 + 2q_1 q_2 + 3t_2(q_1^2 - q_2) - (t_1 + 3t_2^2)q_1 - \frac{1}{4} a q_1^2 - c_1, \\
H_2^A &= -q_2 p_2 p_1 - \frac{1}{2} q_2 p_2^2 + p_1 + q_2 - q_1^2 q_2 + 3t_2 q_1 q_2 - (t_1 + 3t_2^2)q_2 - \frac{1}{4} a q_1 q_2 - c_0.
\end{align*}
\]

and one can verify, by direct computation, that they satisfy the Frobenius condition (3.13) with \( f_{rs} = 0 \). In the flat orthogonal coordinates \((x_1, x_2, y_1, y_2) / f_s\) (the reader should not confuse these coordinates with the variables \( x, y \) used in the considered algebraic curves)

\[
q_1 = -x_1, \quad q_2 = -\frac{1}{4} x_2^2, \quad p_1 = -y_1, \quad p_2 = -\frac{2 y_2}{x_2},
\]

the Hamiltonians \( H_1^A \) take the form

\[
\begin{align*}
H_1^A &= \frac{1}{2} y_1^2 + \frac{1}{2} y_2^2 + x_1^3 + \frac{1}{2} x_1 x_2^2 + \alpha x_2^{-2} + 3t_2(x_1^2 + \frac{1}{4} x_2^2) + (t_1 + 3t_2^2)x_1 - c_1 \\
&\equiv h_1^{HH} + 3t_2(x_1^2 + \frac{1}{4} x_2^2) + (t_1 + 3t_2^2)x_1 - c_1, \\
H_2^A &= \frac{1}{2} x_2 y_1 y_2 - \frac{1}{2} x_1 y_2^2 - y_1 + \frac{1}{16} x_1^4 + \frac{1}{4} x_1^2 x_2^2 - \alpha x_1 x_2^{-2} + \frac{1}{4} 3t_2 x_1 x_2^2 + \frac{1}{4} (t_1 + 3t_2^2)x_2^2 - c_0 \\
&\equiv h_2^{HH} + \frac{1}{4} 3t_2 x_1 x_2^2 + \frac{1}{4} (t_1 + 3t_2^2)x_2^2 - c_0.
\end{align*}
\]

(so that \( G = I \) and the coordinates are actually Euclidean) and constitute a non-autonomous deformation of the integrable case of the extended Hénon-Heiles system \( h_r^{HH} \). Moreover, the flow generated by \( h_r^{HH} \) is exactly the stationary flow of the 5th-order KdV [13]. Let us now perform the minimal quantization of the obtained \( H_1^A \) in the flat coordinates. The \((2,0)\)-tensors \( A_r \) (4.22) have the form

\[
A_1 = \text{Id}, \quad A_2 = \begin{pmatrix}
0 & \frac{1}{2} x_2 \\
\frac{1}{2} x_2 & -x_1
\end{pmatrix}
\]

and the minimal quantization \( \hat{H}_r \) of \( H_r \) can be calculated using (4.4) and (4.5). The result is

\[
\begin{align*}
\hat{H}_1^A &= -\frac{1}{2} h^2 (\partial_1^2 + \partial_2^2) + x_1^3 + \frac{1}{2} x_1 x_2^2 + \alpha x_2^{-2} + 3t_2(x_1^2 + \frac{1}{4} x_2^2) + (t_1 + 3t_2^2)x_1 - c_1, \\
\hat{H}_2^A &= -\frac{1}{2} h^2 \left( x_2 \partial_1 \partial_2 - x_1 \partial_2^2 + \frac{1}{2} \partial_1 \right) + i h \partial_1 + \frac{1}{16} x_1^4 + \frac{1}{4} x_1^2 x_2^2 - \alpha x_1 x_2^{-2} + \frac{1}{4} 3t_2 x_1 x_2^2 + \frac{1}{4} (t_1 + 3t_2^2)x_2^2 - c_0,
\end{align*}
\]

where \( \partial_i = \frac{\partial}{\partial x_i} \) now, and one can show by a direct computation that \( \hat{H}_r^A \) do indeed satisfy the quantum Frobenius condition (4.2).

13
Example 2  (Example 2 from Part I continued) Let us now choose $n = 3$ and $m = 1$. This time we will perform calculations in Viète coordinates $(p, q)$. We have $I^m_1 = \{2, 3\}$ and $I^m_2 = \emptyset$, while the vector fields $J_r$ are

$$J_2 = \partial_2, \quad J_3 = 2\partial_1 + q_1\partial_2$$

so that $W_1 = 0, W_2 = p_2$ while $W_3 = 2p_1 + q_1p_2$. The curve (3.1) becomes

$$\sum_{\alpha = -1}^7 c_\alpha(t)x^\alpha + h_1x^2 + h_2x + h_3 = \frac{1}{2}xy^2$$

and the Hamiltonians $h_r$ in become

$$h_r = \frac{1}{2}\mu^T A_r \mu + \sum_{\alpha = -1}^7 c_\alpha(t)V_r^{(\alpha)}, \quad r = 1, \ldots, 3,$$

with $A_r$ in Viète coordinates given by (see (2.6) and (2.7)):

$$A_1 = G = \begin{pmatrix} 0 & 1 & 0 \\ 1 & q_1 & 0 \\ 0 & 0 & -q_3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & q_1 & 0 \\ q_1 & q_1^2 - q_2 & -q_3 \\ 0 & -q_3 & -q_1q_3 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & -q_3 \\ 0 & -q_3 & -q_1q_3 \\ -q_3 & -q_1q_3 & -q_2q_3 \end{pmatrix}$$

and where the scalar potentials $V_r^{(\alpha)}$ are polynomials in $q$ that can be calculated from the recursion formula in Part I. Further, $h_r^A = h_r + W_r$ and, since $\kappa_1 = 2 < n$ and due to (3.10) and to results in Part I

$$H^A_1 = h_1^A, \quad H^A_2 = h_2^A, \quad H^A_3 = h_3^A + t_2h_1^A. \quad (4.23)$$

The functions $c_\alpha$ can now be determined from the Frobenius condition (3.13) (with $f_{rs}^A = 0$). It leads to an overdetermined but solvable system of PDE’s. A particular solution of this system is

$$c_{-1} = c_0 = 0, \quad c_1 = a(t_3^2 - 2t_2t_3 - 4t_1)t_3, \quad c_2 = 4a(t_3^2 - t_1), \quad (4.24)$$

$$c_3 = 2a(3t_2^2 + t_2), \quad c_4 = 4at_3, \quad c_5 = c_6 = c_7 = 0, \quad a \in \mathbb{R}.$$ 

Let us now perform the minimal quantization of the Hamiltonians $H_r$ (4.23) with $c_\alpha$ given by (4.24). Of course

$$\hat{H}_r^A = \hat{h}_r^A, \quad \hat{H}_2^A = \hat{h}_2^A, \quad \hat{H}_3^A = \hat{h}_3^A + t_2\hat{h}_1^A. \quad (4.25)$$

where $\hat{h}_r^A$ can be computed explicitly from (4.8). The final result is

$$\hat{h}_r^A = \hat{E}_r + i\hbar J_r + a(t_3^2 - t_2t_3 - 4t_1)t_3V_r^{(1)} + 4a(t_3^2 - t_1)V_r^{(2)} + 2a(3t_2^2 + t_2)V_r^{(3)} + 4at_3V_r^{(4)}, \quad r = 1, 2, 3$$

where

$$\hat{E}_1 = -\frac{1}{2}\hbar^2 \left( 2\partial_1\partial_2 + q_1\partial_2^2 - q_3\partial_3^2 - \frac{1}{2}\partial_3 \right)$$

$$\hat{E}_2 = -\frac{1}{2}\hbar^2 \left( \partial_1^2 + 2q_1\partial_1\partial_2 + (q_1^2 - q_2)\partial_2^2 - 2q_3\partial_2\partial_3 - q_1q_3\partial_3^2 - \frac{1}{2}\partial_2 - \frac{1}{2}q_1\partial_3 \right)$$

$$\hat{E}_3 = -\frac{1}{2}\hbar^2 \left( -2q_3\partial_1\partial_3 - q_3\partial_2^2 - 2q_1q_3\partial_2\partial_3 - q_2q_3\partial_3^2 - \frac{1}{2}\partial_1 - \frac{1}{2}q_1\partial_2 - \frac{1}{2}q_2\partial_3 \right)$$

and the resulting quantum operators $\hat{H}_r^A$ in (4.25) do satisfy the quantum Frobenius condition (4.2).

4.3 Quantization of systems with magnetic terms

We proceed now with the case of magnetic potentials. According to Definition 1, the minimal quantization $\hat{H}_r^B$ of Hamiltonians $H_r^B$ is in the magnetic case obtained by replacing the Hamiltonians $h_r^A$ in (3.10), (3.11) (with $h_r$ given now by the magnetic Hamiltonians (3.16)) by the self-adjoint operators

$$\hat{h}_r^B = \hat{E}_r + \hat{W}_r + \sum_{\gamma = 0}^{n+1} d_r(t)\hat{M}_r^{(\gamma)}, \quad r = 1, \ldots, n, \quad (4.26)$$
where  and  are given, as before by (4.8), while the quantized magnetic terms  are given by

\[
\hat{M}_{r}^{(\gamma)} = -\frac{1}{2} i \hbar \left[ \nabla_j (P_r^{(\gamma)})^j + (P_r^{(\gamma)})^j \nabla_j \right] = -\frac{1}{2} i \hbar \left[ |g|^{-\frac{1}{2}} \partial_j |g|^{-\frac{1}{2}} (P_r^{(\gamma)})^j + (P_r^{(\gamma)})^j \partial_j \right]
\]

\[
\hat{M}_{s}^{(\gamma)} = -\frac{1}{2} i \hbar (\gamma - \frac{1}{2}) V_{(\gamma-1)} - i \hbar (P_t^{(\gamma)})^j \partial_j, \quad \partial_j = \frac{\partial}{\partial q_j}
\]

where the explicit form of the first-order operator  in (4.27) is obtained by straightforward calculations. Note that the potentials  in (4.27) are nontrivial only for  and  are trivial. Note also that the minimal quantization  of the classical magnetic term (3.17) depends on  even though  itself does not depend on  in the non-magnetic case, we have

**Theorem 4** For any fixed , the set of operators  satisfies the quantum Frobenius condition (4.2). By the same reasons as in the non-magnetic case, the Hamiltonians  will satisfy the quantum Frobenius condition (4.2) as soon as any pair of operators  and  will satisfy the same, up to the factor , commutation relations as their classical counterparts  and  and  and  (again, we perform all calculations in V"iete coordinates). Thus, we have to show that for any  and any  and any  and any  such that

\[
i \hbar \left\{ E_r + W_r + M_r^{(\gamma)}, E_s + W_s + M_s^{(\gamma)} \right\} = \left[ \hat{E}_r + \hat{W}_r + \hat{M}_r^{(\gamma)}, \hat{E}_s + \hat{W}_s + \hat{M}_s^{(\gamma)} \right]
\]

as the relation (4.28) means that we can perform the appropriate deformation procedure described in Part I (and shortly revisited in Section 3) also in the quantum magnetic case and that this procedure will yield the same PDE's for functions , \( t_1, \ldots, t_n \), \( \zeta_{r,j}(t_1, \ldots, t_{r-1}) \) and \( \zeta_{r,r+j}(t_{r+1}, \ldots, t_n) \) as the corresponding classical procedure.

For any  and

\[
\left\{ E_r + W_r + M_r^{(\gamma)}, E_s + W_s + M_s^{(\gamma)} \right\} = \left\{ E_r, E_s \right\} + \left\{ E_r, M_s^{(\gamma)} \right\} + \left\{ M_r^{(\gamma)}, E_s \right\} + \left\{ W_r, M_s^{(\gamma)} \right\} + \left\{ M_r^{(\gamma)}, W_s \right\} + \left\{ M_r^{(\gamma)}, M_s^{(\gamma)} \right\}.
\]

Moreover, the magnetic St"ackel Hamiltonians themselves (i.e. without the quasi-St"ackel term  ) both geodesic and with magnetic terms, commute with each other, so that  and \( \left\{ E_r + M_r^{(\gamma)}, E_s + M_s^{(\gamma)} \right\} = 0 \), which yields

\[
\left\{ E_r, M_s^{(\gamma)} \right\} + \left\{ M_r^{(\gamma)}, E_s \right\} + \left\{ M_r^{(\gamma)}, M_s^{(\gamma)} \right\} = 0
\]

and thus the left hand side of (4.28) is \( \left\{ E_r, E_s \right\} \), is

\[
i \hbar \left\{ E_r, E_s \right\} + i \hbar \left\{ W_r, M_s^{(\gamma)} \right\} + \left\{ M_r^{(\gamma)}, W_s \right\}.
\]

Moreover, due to (2.15),

\[
\left[ \hat{E}_r, \hat{E}_s \right] = 0 \quad \text{and} \quad \left[ \hat{E}_r + \hat{M}_r^{(\gamma)}, \hat{E}_s + \hat{M}_s^{(\gamma)} \right] = 0
\]

and thus

\[
\left[ \hat{E}_r, \hat{M}_s^{(\gamma)} \right] + \left[ \hat{M}_r^{(\gamma)}, \hat{E}_s \right] + \left[ \hat{M}_r^{(\gamma)}, \hat{M}_s^{(\gamma)} \right] = 0
\]

so that the right hand side of (4.28) is actually

\[
\left[ \hat{E}_r, \hat{E}_s \right] + \left[ \hat{E}_r, \hat{M}_s^{(\gamma)} \right] + \left[ \hat{M}_r^{(\gamma)}, \hat{E}_s \right]
\]
By Theorem 3, \( i\hbar \{\hat{E}_r, \hat{E}_s\} = \{\hat{E}_r, \hat{E}_s\} \) so in order to prove (4.28) and thus Theorem 4 it remains to show that the following relation is valid:

\[
i\hbar \left( \left\{ W_r, M_s^{(\gamma)} \right\} + \left\{ M_r^{(\gamma)}, W_s \right\} \right) = \left[ \hat{W}_r, \hat{M}_s^{(\gamma)} \right] + \left[ \hat{M}_r^{(\gamma)}, \hat{W}_s \right]. \tag{4.30}
\]

A direct calculation shows that the right hand side of (4.30) is

\[
-H^2 \left( [J_r, P_s^{(\gamma)}] + [P_r^{(\gamma)}, J_s] + aJ_r \left( V_s^{(\gamma-1)} \right) - aJ_s \left( V_r^{(\gamma-1)} \right) \right) \tag{4.31}
\]

with a denoting (in this proof) \( \frac{1}{2} (\gamma - \frac{1}{2} m) \), so that it is a sum of a vector field and a function on \( \mathcal{M} \). Further

\[
\left\{ W_r, M_s^{(\gamma)} \right\} + \left\{ M_r^{(\gamma)}, W_s \right\} = \Theta \partial_{\gamma} \tag{4.32}
\]

for some vector field \( \Xi = \Xi(\partial_{\gamma}) \) on \( \mathcal{M} \). Thus, due to (2.10), the left hand side of (4.30) is

\[
h^2 \left( \Theta + \frac{1}{2} (\partial_j \Theta^j) + \frac{1}{4} |g|^{-1} (\partial_j |g|) \Theta \right) \tag{4.33}
\]

and as such is also a sum of a vector field and a function on \( \mathcal{M} \). The vector field parts of (4.31) and (4.32) are equal due to (4.15) so it remains to prove that

\[
\frac{1}{2} (\partial_j \Theta^j) + \frac{1}{4} |g|^{-1} (\partial_j |g|) \Theta = aJ_r \left( V_s^{(\gamma-1)} \right) - aJ_s \left( V_r^{(\gamma-1)} \right) \tag{4.33}
\]

A direct calculation shows that both sides of (4.33) are zero for all \( \gamma \) except \( \gamma = 0 \) and \( \gamma = n + 1 \). For \( \gamma = n + 1 \) both sides of (4.33) for a given choice of indices \((r, s)\) are

\[
\begin{cases}
0, & \text{for } r \in I_1^n \text{ and } s \in I_2^n, \\
a(r-s)q_{r+s-(n-m+2)}, & \text{for } r, s \in I_1^n, \\
-a(r-s)q_{r+s-(n-m+2)}, & \text{for } r, s \in I_2^n.
\end{cases}
\]

while for \( \gamma = 0 \) both sides of (4.33) for a given choice of indices \((r, s)\) are

\[
\frac{a(m-1)}{q_n} (q_{r-1}\delta_{s,n-m+2} - q_{s-1}\delta_{r,n-m+2}) + \begin{cases}
0, & \text{for } r \in I_1^n \text{ and } s \in I_2^n, \\
a(r-s)q_{r+s-(n-m+2)}, & \text{for } r, s \in I_1^n, \\
-a(r-s)q_{r+s-(n-m+2)}, & \text{for } r, s \in I_2^n.
\end{cases}
\]

Therefore, (4.30) is valid. This also concludes the proof of Theorem 4.

**Example 3** Consider the case \( n = 3 \) and \( m = 3 \) (see subsection 7.2 in Part I), which means that \( I_1^n = \emptyset \) while \( I_2^n = \{2, 3\} \). Further, \( \kappa_1 = \kappa_2 = 1 \) and thus the vector fields \( J_r \) are

\[
J_2 = q_2 \partial_2 + 2q_3 \partial_3, \quad J_3 = q_3 \partial_2
\]

so that \( W_1 = 0, \ W_2 = p_2q_2 + 2q_3p_3 \) while \( W_3 = q_3p_2 \). The curve (3.15) becomes

\[
\sum_{\gamma=0}^{4} d_{\gamma}(t)x^{\gamma}y + x^{2}h_1 + xh_2 + h_3 = \frac{1}{2}x^{3}y^{2}
\]

leading to Hamiltonians \( h_r \) (3.16) that in Viète coordinates attain the form

\[
h_r = \frac{1}{2} p_r^TA_r p + \sum_{\gamma=0}^{4} d_{\gamma}(t)p_r^TP_r^{(\gamma)} \quad r = 1, \ldots, 3,
\]

with \( A_r \) given by (see (2.6) and (2.7)):

\[
A_1 = G = \begin{pmatrix}
-q_1 & -q_2 & -q_3 \\
-q_2 & -q_3 & 0 \\
-q_3 & 0 & 0
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
-q_2 & -q_3 & 0 \\
-q_3 & -q_1q_3 + q_2^2 & q_2q_3 \\
0 & q_2q_3 & q_3^2
\end{pmatrix}, \quad A_3 = \begin{pmatrix}
-q_3 & 0 & 0 \\
0 & q_2q_3 & q_3^2 \\
0 & q_3 & 0
\end{pmatrix}.
\]
Further, the vector potentials $P_r^{(7)}$ are given by (2.8). Explicitly
\[
P_1^{(0)} = (0, 0, 1)^T, \quad P_2^{(0)} = (0, 1, q_1)^T, \quad P_3^{(0)} = (1, q_1, q_2)^T, \\
P_1^{(1)} = (0, 1, 0)^T, \quad P_2^{(1)} = (1, q_1, 0)^T, \quad P_3^{(1)} = (0, 0, -q_3)^T, \\
P_1^{(2)} = (1, 0, 0)^T, \quad P_2^{(2)} = (0, -q_2, -q_3)^T, \quad P_3^{(2)} = (0, -q_3, 0)^T, \\
P_1^{(3)} = (-q_1, -q_2, -q_3)^T, \quad P_2^{(3)} = (-q_2, -q_3, 0)^T, \quad P_3^{(3)} = (-q_3, 0, 0)^T, \\
P_1^{(4)} = (q_1^2 - q_2, q_1q_2 - q_3, q_1q_3)^T, \quad P_2^{(4)} = (q_1q_2 - q_3, q_2^2, q_2q_3)^T, \quad P_3^{(4)} = (q_1q_3, q_2q_3, q_3^2)^T. \tag{4.34}
\]

As usual, $h_r^B = h_r + W_r$ and, since $\kappa_1 = \kappa_2 < n$ and due to (3.10), (3.11) and results in Part I,
\[
H_1^B = h_1^B, \quad H_2^B = h_2^B + t_3h_3^B, \quad H_3^B = h_3^B. \tag{4.35}
\]

The functions $d_\alpha$ can now be determined from the Frobenius condition (3.13) (with $f_{rs} = 0$). It leads to an overdetermined but soluble system of PDE’s. The general solution of this system is
\[
d_0 = b_0 \exp(2t_2), \quad d_1 = b_0t_3 \exp(2t_2) + b_1 \exp(t_2), \quad d_2 = b_2, \quad d_3 = (b_3 + b_4t_1), \quad d_4 = b_4,
\]
(parametrized by the arbitrary constants $b_r$). Let us now minimally quantize the Hamiltonians $H_r$ in (4.35). Naturally
\[
\hat{H}_1^B = \hat{h}_1^B, \quad \hat{H}_2^B = \hat{h}_2^B + t_3\hat{h}_3^B, \quad \hat{H}_3^B = \hat{h}_3^B
\]
where $\hat{h}_r^B$ are given by (4.26) where $\hat{E}_r$ and $\hat{W}_r$ can computed using (4.8) and where $\hat{M}_r^{(7)}$ can be computed by (4.27). The result is
\[
\hat{h}_r^B = \hat{E}_r - i\hbar J_r + b_0 \exp(2t_2)\hat{M}_r^{(0)} + [b_0t_3 \exp(2t_2) + b_1 \exp(t_2)]\hat{M}_r^{(1)} + b_2\hat{M}_r^{(2)} + (b_3 + b_4t_1)\hat{M}_r^{(3)} + b_4\hat{M}_r^{(4)};
\]
where
\[
\hat{E}_1 = -\frac{1}{2} \hbar^2 \left( -q_1 \partial_1^2 - q_3 \partial_2^2 - 2q_2 \partial_1 \partial_2 - 2q_3 \partial_3 \partial_1 - \frac{3}{2} \partial_1 \right), \\
\hat{E}_2 = -\frac{1}{2} \hbar^2 \left( -q_2 \partial_1^2 + (-q_1q_3 + q_3^2) \partial_2^2 + q_2q_3 \partial_3^2 - 2q_3 \partial_1 \partial_2 + 2q_2q_3 \partial_2 \partial_3 + \frac{3}{2} q_2 \partial_2 + \frac{3}{2} q_3 \partial_3 \right), \\
\hat{E}_3 = -\frac{1}{2} \hbar^2 \left( -q_3 \partial_1^2 + 2q_2q_3 \partial_2^2 - 2q_2q_3 \partial_2 \partial_3 + \frac{3}{2} q_2 \partial_2 \right)
\]
and where
\[
\hat{M}_r^{(0)} = \frac{3}{4} \hbar^3 q_r - i\hbar(P_r^{(0)})^j \partial_j, \quad \hat{M}_r^{(1)} = -\frac{1}{4} i\hbar \delta_r \cdot \partial - i\hbar(P_r^{(1)})^j \partial_j, \\
\hat{M}_r^{(2)} = \frac{1}{4} i\hbar \delta_r \cdot \partial - i\hbar(P_r^{(2)})^j \partial_j, \quad \hat{M}_r^{(3)} = \frac{3}{4} i\hbar \delta_r \cdot \partial - i\hbar(P_r^{(3)})^j \partial_j, \\
\hat{M}_r^{(4)} = -\frac{5}{4} \hbar q_r - i\hbar(P_r^{(4)})^j \partial_j
\]
with $r = 1, 2, 3$ and with $P_r^{(7)}$ given by (4.34). It can be demonstrated by a direct computation that $\hat{H}_r^B$ do satisfy the quantum Frobenius condition (4.2).

5 Quantum canonical transformations between magnetic and non-magnetic quantum Painlevé systems

In this chapter we prove that the magnetic quantum Painlevé operator $\hat{H}_r^B$ can be transformed, by a multitime-dependent quantum canonical transformations (see [15]), to a corresponding non-magnetic
quantum Painlevé operator $\hat{H}_r^A$. Since $\hat{H}_r^A$ contains $n + 3$ parameters while $\hat{H}_r^B$ contains only $n + 2$ parameters, we have first to extend the quantum magnetic system $\hat{h}_r^B$ by one parameter to the system

$$\hat{h}_r^B = \hat{E}_r + \hat{W}_r + \sum_{\gamma=0}^{n+1} d_\gamma(t)\hat{M}_r^{(\gamma)} + \hat{b}_e(t)\hat{V}_r^{(n)} - \hat{b}_{e_{n-r}}(t), \quad r = 1, \ldots, n$$

(5.1)

(as we also did in the classical case, see Section 8 of Part I) where from the Frobenius condition it follows that $e_n(t_1, \ldots, t_n) = 1$ for $m = 0, \ldots, n$ and $e_n(t_1, \ldots, t_n) = \exp(t_1)$ for $m = n + 1$ and where $e_{n-r}$ are chosen so that $\hat{h}_r^B$ in (5.1) satisfy the quantum Frobenius condition (4.2). Each $\hat{H}_r^B$ is then obtained by deforming of $\hat{h}_r^B$ in (5.1) through an appropriate formula in (3.10) or (3.11).

**Theorem 5** The multi-time dependent quantum canonical transformation

$$\hat{H}_r^A = U^* \hat{H}_r^B U^+ + \frac{i\hbar}{2} \frac{\partial U^+}{\partial r}, \quad U = U(\lambda, t) = e^{F(\lambda, t)},$$

(5.2)

transforms the magnetic quantum Painlevé Hamiltonian operators $\hat{H}_r^B$ into the corresponding non-magnetic quantum Painlevé Hamiltonian operators $\hat{H}_r^A$, provided that the functions $\alpha_{r,j}$ and $d_\gamma$ satisfy the same set of first order linear PDE’s as in the classical case (see Part I, Theorem 4).

This theorem is the quantum counterpart of Theorem 4 in Part I. Note that (5.2) is in fact covariant, just as (2.19) is.

**Proof.** Let us demand that the transformation (5.2) maps the operator $\hat{H}_r^B$, onto the corresponding operator $\hat{H}_r^A$ with some functions $c_\alpha(t)$. Consider first the Hamiltonians $\hat{h}_r^B$ for which $\hat{H}_r^B = \hat{h}_r^B$ i.e. when $r \in \{1, \ldots, \kappa_1\} \cup \{n - \kappa_2 + 1, \ldots, n\}$. In such cases, the relation (5.2) is satisfied provided that (cf. Part I)

$$\sum_{\alpha = -m}^{2n-m+2} c_\alpha(t)V_r^{(\alpha)} = \sum_{\gamma, \gamma' = 0}^{n+1} d_\gamma(t)d_{\gamma'}(t)V_r^{(\gamma+\gamma'-m)} + \hat{b}_e(t)\hat{V}_r^{(n)} - \hat{b}_{e_{n-r}}(t) + S_r(t, \lambda) + \frac{\partial F(\lambda, t)}{\partial r}, \quad r = 1, \ldots, n,$$

(5.3)

where functions $S_r$ are given by

$$S_r = U\hat{W}_r U^+ - \hat{W}_r = \sum_{\gamma = 0}^{n+1} \sum_{j=1}^{n} d_\gamma(t)\hat{M}_r^{(\gamma)}\lambda_j^{-m}$$

(5.4)

The condition (5.3) is satisfied if and only if

$$\sum_{\alpha = -m}^{2n-m+2} c_\alpha(t)V_r^{(\alpha)} = \sum_{\gamma, \gamma' = 0}^{n+1} d_\gamma(t)d_{\gamma'}(t)V_r^{(\gamma+\gamma'-m)} + \hat{b}_e(t)\hat{V}_r^{(n)} - \hat{b}_{e_{n-r}}(t)$$

(5.5)

and

$$S_r(t, \lambda) + \frac{\partial F(\lambda, t)}{\partial r} = 0$$

(5.6)

separately. The condition (5.5) defines a map between the functions $e_r(t), d_\gamma(t)$ and $c_\alpha(t)$ which reconstructs the classical result. Since $S_r$ in (5.4) is equal to its classical counterpart (see Part I, Appendix B) and since the partial derivatives $\frac{\partial F(\lambda, t)}{\partial r}$ are also equal in the classical and in the quantum case, the condition (5.6) reconstructs exactly the system of PDE's for functions $d_\gamma(t), \alpha_{r,j}(t)$ from Theorem 4 in Part I. Further, in case that $r \in \{\kappa_1 + 1, \ldots, n - \kappa_2\}$ the operator $\hat{H}_r^B$ is the same linear combination of appropriate operators $\hat{h}_r^B$ as the corresponding classical counterparts. It means that the proof of the above theorem reduces to the proof of its classical counterpart (Theorem 4 Part I).
Example 4 Let us choose $n = 3$ and $m = 1$ and let us consider the system (7.2) in Part I with $b_1 = b_2 = b_1 = b_0 = 0$ and $b_3 = b$ an arbitrary parameter (i.e. the system from Example 4 from Part I). Its minimal quantization has the form (5.1) with $\omega = 0$ and explicitly reads
\[
\hat{h}_r^B = \hat{E}_r - i\hbar J_r + b(t_2 + t_3^2)\hat{M}_r^{(1)} + 2bt_3\hat{M}_r^{(2)} + b\hat{M}_r^{(3)}, \quad r = 1, 2, 3
\]
where $\hat{E}_r$ and $J_r$ are exactly as in Example 3 above, and where $\hat{M}_r^{(\gamma)} = -\frac{i}{2}\hbar(\gamma - \frac{1}{2})V_r^{(\gamma - 1)} - i\hbar(\gamma^{(\gamma)})^{\gamma}\partial_{\gamma}$ with $\gamma^{(\gamma)}$ given by (4.34). Then the operators $\hat{H}_r^B = \hat{h}_r^B$, $\hat{H}_2^B = \hat{h}_2^B$ and $\hat{H}_3^B = \hat{h}_3^B + t_2\hat{h}_1^B$ satisfy the quantum Frobenius condition (4.2). The unitary operator $U$ in (5.2) is explicitly given by $U = U(\lambda, t) = e^{F(\lambda, t)}$ with
\[
F(\lambda, t) = -\frac{i}{\hbar} \left[ b(t_2 + t_3^2)\sum_{j=1}^n \lambda_j + bt_3\sum_{j=1}^n \lambda_j^2 + \frac{1}{3}b\sum_{j=1}^n \lambda_j^3 \right] = \\
\frac{i}{\hbar} \left[ b(t_2 + t_3^2)q_1 + bt_3(2q_2 - q_1^2) + \frac{1}{3}b(q_3 - 3q_1q_2 + 3q_3) \right]
\]
and the quantum canonical transformation (5.2) yields the quantum Hamiltonians $\hat{H}_r^A$ of the form $\hat{H}_1^A = \hat{h}_1^A$, $\hat{H}_2^A = \hat{h}_2^A$ and $\hat{H}_3^A = \hat{h}_3^A + t_2\hat{h}_1^A$ with
\[
\hat{h}_r^A = \hat{E}_r - i\hbar J_r + \sum_{\alpha=0}^5 c_\alpha(t_1, \ldots, t_n) V_r^{(\alpha)}
\]
with
\[
c_5 = \frac{1}{2} b^2, \quad c_4 = 2b^2t_3, \quad c_3 = b^2(3t_3^2 + t_2), \quad c_2 = 2b^2(t_2 + t_3^2)t_3, \quad c_1 = \frac{1}{2} b^2(t_2^2 + t_3^2 + 2t_2t_3) + 2bt_3, \quad c_0 = 2b(t_2 + t_3^2)t_3 + 2b^2t_2t_3(t_2 + t_3^2).
\]
This is exactly the non-magnetic system from Example 2 in Part I provided that we put $a_5 = \frac{1}{2} b^2$ (note however that the form of non-dynamical parts is different above and in the mentioned Example). One can show by a direct computation that $\hat{H}_r^A$ do indeed satisfy the quantum Frobenius condition.

6 Conclusions

In the series of articles (Part I-III) we have constructed multi-parameter and multi-dimensional hierarchies of Painlevé-type systems, both classical and quantum, from the corresponding classical respectively quantum Stäckel-type systems. In particular, they contain the famous one-degree of freedom Painlevé equations $P_I - P_{IV}$. Each hierarchy was presented in the ordinary as well as in the magnetic regime. We also constructed the multi-time canonical maps between both regimes (on the classical level and on the quantum level). Also, the proposed Painlevé hierarchies $P_I - P_{IV}$, presented in Part II, can be written in the quantum version.

One of the directions of future research is to identify the obtained hierarchies with the known hierarchies constructed by other methods, such as appropriate reductions of soliton hierarchies. We stress that our method is much more complete than the existing methods as in our construction, for a given number $n$ of degrees of freedom, we obtain $n$ different Painlevé-type systems that mutually satisfy the Frobenius integrability condition, while the existing methods often produce only one system for a given $n$. Another, related, direction of future research is to find a systematic way of relating the obtained hierarchies of Painlevé-type systems with non-autonomous and non-homogeneous soliton-type hierarchies.

Acknowledgement 6 MB wished to express his gratitude to Department of Science and Technology of Linköping University, Sweden, for their hospitality during his visits.
References

[1] S. Benenti, *Intrinsic characterization of the variable separation in the Hamilton-Jacobi equation*, J. Math. Phys. **38** (1997), no. 12, 6578–6602.

[2] S. Benenti, C. Chanu, G. Rastelli, *Remarks on the connection between the additive separation of the Hamilton-Jacobi equation and the multiplicative separation of the Schrödinger equation. I. The completeness and Robertson conditions*, J. Math. Phys. **43** (11) (2002) 5183–5222.

[3] S. Benenti, C. Chanu, G. Rastelli, *Remarks on the connection between the additive separation of the Hamilton-Jacobi equation and the multiplicative separation of the Schrödinger equation. II. First integrals and symmetry operators*, J. Math. Phys. **43** (11) (2002) 5223–5253.

[4] M. Błaszak, A. Sergyeyev, *Natural coordinates for a class of Benenti systems*, Phys. Lett. A **365** (2007), no. 1–2, 28–33.

[5] M. Błaszak, A. Sergyeyev, *Generalized Stäckel systems*, Phys. Lett. A **375** (2011), no. 27, 2617–2623.

[6] M. Błaszak, K. Marciniak, Z. Domański, *Separable quantizations of Stäckel systems*, Ann. Phys. **371** (2016) 460–477.

[7] M. Błaszak, *Quantum versus Classical Mechanics and Integrability Problems*, Springer Nature, Switzerland AG, 2019.

[8] M. Błaszak, K. Marciniak, A. Sergyeyev, *Deforming Lie algebras to Frobenius integrable non-autonomous Hamiltonian systems*, Rep. Math. Phys. **87** (2021) 249-263.

[9] M. Błaszak, K. Marciniak, Z. Domański, *Systematic construction of non-autonomous Hamiltonian equations of Painlevé-type. I. Frobenius integrability*, Stud. Appl. Math. **148** (2022) pp. 1208–1250.

[10] M. Błaszak, Z. Domański, K. Marciniak, *Systematic construction of non-autonomous Hamiltonian equations of Painlevé-type. II. Isomonodromic Lax representation*, Stud. Appl. Math. (2022) online.

[11] P. Dolan, A. Kladouchou, C. Card, *On the significance of Killing tensors*, Gen. Rel. Grav. **21** (1989) 427-437.

[12] M. Fecko, *Differential geometry and Lie groups for physicists*, Cambridge University Press, New York, 2006.

[13] Fordy A., *The Hénon-Heiles system revisited*, Physica D **52** (1991) 204-210

[14] K. Iwasaki, H. Kimura, S. Shimomura, M. Yoshida, *From Gauss to Painlevé. A Modern Theory of Special Functions*, Vieweg & Sohn Verlagsgesellschaft mbH, Braunschweig, 1991.

[15] J-H. Kim, H-W. Lee, *Canonical transformations and the Hamilton-Jacobi theory in quantum mechanics*, Can. J. Phys. **77** (1999) 411–425

[16] A.T. Lundell, *A short proof of the Frobenius theorem*, Proc. Amer. Math. Soc. **116** (1992), no. 4, 1131–1133.

[17] K. Marciniak, M. Błaszak, *Non-Homogeneous Hydrodynamic Systems and Quasi-Stäckel Hamiltonians*, SIGMA **13** (2017), art. 077, 15 pages.