MEASURES AND PROCESSES ON SEMILATTICES

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Abstract. Every vector-valued, semi-modular set function on a semi-lattice of sets extends uniquely to an additive set function on the generated ring. Based on this theorem we outline a theory of stochastic processes when the underlying index set is a semi-lattice and obtain a characterization of processes as vector measures and a form of the Doob-Meyer decomposition.

1. Introduction and Notation

The need that often arises in stochastic analysis to examine the properties of processes along the hitting times of some family of subsets of $\mathbb{R}$, spurs interest for a theory of stochastic processes with an index set more general than the time domain. The history of such a theory is, of course, a long one, but most authors incline to credit Cairoli and Walsh [4] for having given start to it. A rather complete list of contributions, which is beyond scope here, may be found in the bibliographic references of the book of Ivanoff and Merzbach [15].

In the present paper we work with an index set $T$ which has the only properties of being partially ordered and closed with respect to one lattice operation, either $\vee$ or $\wedge$: a semi-lattice. The explicit aim is proving some of the most useful results of the classical theory of processes, such as the decomposition of Doob and Meyer, and see how much of that theory carries over to the more general setting adopted here. This attempt, that one should confront with a preceding paper [5], is made while trying to resist the temptation of restoring classical properties via additional ad hoc assumptions. Thus, e.g., $T$ will have no topological structure at all.

The problem considered here leads quite naturally to look for a convenient extension of the given index set. Proving the existence of a desirable extension has a direct translation into the somehow more general problem of extending set functions which are originally defined on a class of sets that is closed under only one set operation, either $\cap$ or $\cup$, a semi-lattice of sets. The existence of a measure theoretic counterpart to whatever a theory of stochastic processes comes, of course, at no surprise given the many crucial papers focusing exactly on this link, among which the important one of Doléans-Dade [7] and even the original contribution [14] of Štôka. A main source of inspiration for this paper has however been the unpublished work of Norberg [19].

The present paper deals in section 2 with the measure theoretic problem described above. We provide an exact characterization which is then easily extended to product spaces. This last result

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translates immediately, in section 5, into the statement that stochastic processes indexed by semi-lattices may be regarded as vector-valued, additive set functions. Eventually, in sections 5.1 and 5.2 we prove some classical results such as the Riesz decomposition for quasi martingales and a version of the decomposition of Doob and Meyer. To obtain the latter decomposition, we exploit a fairly simple, weak compactness condition, the class $DM$ property, which is shown to be nothing but an extension of the more classical class $D$ property originally introduced by Meyer [18]. It is shown, however, that the predictability property of the intervening compensator is in the present context more troublesome from the point of view of interpretation although quite easy mathematically. In particular, there is a multiplicity of possible definitions of predictability for the compensator, although, given an arbitrary choice among these, the decomposition is unique.

As in [5] we find convenient the following notation. When $S$ is a set, $2^S$ and $1_S$ indicate the collection of all of its subsets and its indicator function. If $\Sigma \subset 2^S$, then $\mathcal{S}(\Sigma)$ designates the collection of $\Sigma$ simple functions on $S$, $\mathcal{B}(\Sigma)$ its closure with respect to the topology of uniform convergence and $ba(\Sigma)$ (resp. $ca(\Sigma)$) the family of bounded, finitely (resp. countably) additive set functions on $\Sigma$. In general we omit to refer to $\Sigma$ whenever $\Sigma = 2^S$. Although, given an arbitrary choice among these, the decomposition is unique.

In particular, there is a multiplicity of possible definitions of predictability for the compensator, which is shown to be nothing but an extension of the more classical class $D$ property, originally introduced by Meyer [18]. It is shown, however, that the predictability property of the intervening compensator is in the present context more troublesome from the point of view of interpretation although quite easy mathematically. In particular, there is a multiplicity of possible definitions of predictability for the compensator, although, given an arbitrary choice among these, the decomposition is unique.

As an illustration take $\{A_1, \ldots, A_N\} \subset X$, $b \leq [N]$ and $A_b = (\bigcap_{n \in \varnothing} A_n^c) \cap (\bigcap_{k \in \nu \cap [N]} A_k)$ (with $\bigcap_{n \in \varnothing} A_n = X$). The equality $1_{\bigcap_{n \in \varnothing} A_n} = \sum_{k \geq a} 1_{A_k}$, [2] and [3] imply

$$\nu(\varnothing) 1_{A_b} = - \sum_{\varnothing \leq b \leq [N]} \nu(b) 1_{\bigcap_{n \in \varnothing} A_n} = \sum_{\varnothing \leq b \leq [N]} \nu(b) 1_{\bigcup_{n \in \varnothing} A_n} = \sum_{\varnothing \leq b \leq [N]} \nu(b) 1_{\bigcup_{n \in \varnothing} A_n}$$

and likewise $1_{\bigcup_{n \in [N]} A_n} = \sum_{\varnothing \leq b \leq [N]} \nu(b) 1_{\bigcap_{n \in \varnothing} A_n}$.
2. semi-modular Set Functions

In this section $X$ will be an arbitrary set and $Y$ a real vector space with neutral element denoted by $\emptyset$. $\mathcal{A} \subset 2^X$ is said to be a lattice of sets if it is non empty and closed with respect to $\cap$ and $\cup$. Several authors, including Bachman and Sultan [1] and Bhaskara Rao and Bhaskara Rao [2], have studied set functions on lattices of sets and the possibility of extending them to a larger class, typically the generated algebra or ring. This possibility hinges on the two properties introduced next.

We speak of $\Phi : \mathcal{A} \rightarrow Y$ as a modular set function whenever $\mathcal{A}$ is a lattice of sets and

$$\Phi(A \cup B) + \Phi(A \cap B) = \Phi(A) + \Phi(B) \quad A, B \in \mathcal{A}$$

(5)

Simple induction shows that (5) is equivalent to either one of the following two properties:

$$\Phi\left(\bigcup_{n \in [N]} A_n\right) = \sum_{\varnothing < b \leq [N]} \nu(b) \Phi\left(\bigcap_{n \in b} A_n\right) \quad A_1, \ldots, A_N \in \mathcal{A}$$

(6a)

$$\Phi\left(\bigcap_{n \in [N]} A_n\right) = \sum_{\varnothing < b \leq [N]} \nu(b) \Phi\left(\bigcup_{n \in b} A_n\right) \quad A_1, \ldots, A_N \in \mathcal{A}$$

(6b)

The property $\Phi(\emptyset) = \emptyset$ whenever $\emptyset \in \mathcal{A}$ does not follow from (5) and we then say that $\Phi : \mathcal{A} \rightarrow Y$ is strongly additive, in symbols $\Phi \in sa(\mathcal{A}, Y)$, if it is modular and satisfies

$$\text{either} \quad \emptyset \notin \mathcal{A} \quad \text{or} \quad \Phi(\emptyset) = \emptyset$$

(7)

In other words, $\Phi \in sa(\mathcal{A}, Y)$ if and only if it admits an extension $\Phi \in sa(\mathcal{A} \cup \{\emptyset\}, Y)$. It is also easy to see that each $\Phi \in sa(\mathcal{A}, Y)$ is additive and that, if $\mathcal{A}$ is a ring, then the converse is also true.

The case of modular or strongly additive set functions on lattices of sets was studied in depth by Pettis [20]. We are interested in defining similar properties when the underlying family $\mathcal{A}_0 \subset 2^X$ is just a semi-lattice, i.e. a collection of sets closed with respect to just one set operation, either $\cap$ or $\cup$. Important examples of semilattices are, of course, ideals and filters; a $\cap$-lattice is also known in probability as a $\pi$-system. $\Phi : \mathcal{A}_0 \rightarrow Y$ is then said to be semi modular if either (i) $\mathcal{A}_0$ is a $\cap$-lattice and (6a) holds for every finite collection $A_1, \ldots, A_N \in \mathcal{A}_0$ such that $\bigcup_{n=1}^N A_n \in \mathcal{A}_0$ or (ii) $\mathcal{A}_0$ is a $\cup$-lattice and (6b) holds for every finite collection $A_1, \ldots, A_N \in \mathcal{A}_0$ such that $\bigcap_{n=1}^N A_n \in \mathcal{A}_0$.

There exists, of course, a perfect symmetry between these two properties.

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1Some authors, e.g. Halmos [11 p. 25] and Bogachev [3 p. 75], include the property $\emptyset \in \mathcal{A}$ in the definition of a lattice.

2Pettis [20] calls this property 2-additivity.

3Chung [6 p. 329] credits Poincaré for being the first to obtain (6a).

4A $\cap$-lattice containing the empty set was called intersectional class by Groemer [10 p. 397]
Lemma 1. Let $\mathcal{A}_0 \subset 2^X$ and $y \in Y$. Then, (i) $\mathcal{A}_0$ is a semi-lattice if and only if so is $\mathcal{A}_0^c = \{ A^c : A \in \mathcal{A}_0 \}$; (ii) $\Phi : \mathcal{A}_0 \to Y$ is semi-modular if and only if so is its conjugate and its $y$ translation, i.e. the set functions $\Phi^c : \mathcal{A}^c \to Y$ and $\Phi_y : \mathcal{A} \to Y$ defined implicitly via

\begin{equation}
\Phi^c(A^c) = -\Phi(A) \quad \Phi_y(A) = \Phi(A) + y \quad A \in \mathcal{A}_0
\end{equation}

Proof. The first claim is obvious. Assume that $\mathcal{A}_0$ is a $\cap$-lattice and $\Phi : \mathcal{A}_0 \to Y$ is semi-modular. If $A_1, \ldots, A_N, \bigcup_{n=1}^N A_n \in \mathcal{A}_0$, then, we conclude

\[\Phi^c\left(\bigcap_{n=1}^N A_n^c\right) = \Phi\left(\bigcup_{n=1}^N A_n\right) = \sum_{\varnothing < b \leq [N]} \nu(b)\Phi\left(\bigcap_{n \in b} A_n\right) = \sum_{\varnothing < b \leq [N]} \nu(b)\Phi_y\left(\bigcup_{n \in b} A_n^c\right)\]

The fact that $\Phi_y$ is semi-modular follows from (2). The proof for the case in which $\mathcal{A}_0$ is a $\cup$-lattice is identical. \hfill \Box

Lemma 1 allows to restrict our proofs to the case of $\cap$-lattices.

Set functions on semilattices, even if not semi-modular, satisfy some minimal properties.

Lemma 2. Let $\mathcal{A}_0$ be a $\cap$-lattice, $A_1, \ldots, A_N \in \mathcal{A}_0$ and $\Phi : \mathcal{A}_0 \to Y$. The following is true:

\begin{equation}
\sum_{\varnothing < b \leq [N]} \nu(b)\Phi\left(\bigcap_{n \in b} A_n\right) = \Phi(A_N) + \sum_{\varnothing < b \leq [N-1]} \nu(b)\left[\Phi\left(\bigcap_{n \in b} A_n\right) - \Phi\left(A_N \cap \bigcap_{n \in b} A_n\right)\right],
\end{equation}

\begin{equation}
A_N \subset A_{N-1} \implies \sum_{\varnothing < b \leq [N-1]} \nu(b)\Phi\left(\bigcap_{n \in b} A_n\right) = \sum_{\varnothing < b \leq [N-1]} \nu(b)\Phi\left(\bigcap_{n \in b} A_n\right)
\end{equation}

(9) and (10) remain true after exchanging $\cap$ with $\cup$ and reversing inclusions.

Proof. (9) is a consequence of

\[\sum_{\varnothing < b \leq [N]} \nu(b)\Phi\left(\bigcap_{n \in b} A_n\right) = \sum_{\varnothing < b \leq [N-1]} \nu(b)\Phi\left(\bigcap_{n \in b} A_n\right) + \sum_{\{N\} \leq b \leq [N]} \nu(b)\Phi\left(\bigcap_{n \in b} A_n\right)\]

\[= \sum_{\varnothing < b \leq [N-1]} \nu(b)\Phi\left(\bigcap_{n \in b} A_n\right) - \sum_{\varnothing \leq b \leq [N-1]} \nu(b)\Phi\left(A_N \cap \bigcap_{n \in b} A_n\right)\]

and implies that $\sum_{\varnothing < b \leq [N-1]} \nu(b)\Phi\left(A_N \cap \bigcap_{n \in b} A_n\right)$ may equivalently be written as

\[\Phi(A_N \cap A_{N-1}) + \sum_{\varnothing < b \leq [N-2]} \nu(b)\left[\Phi\left(A_N \cap \bigcap_{n \in b} A_n\right) - \Phi\left(A_N \cap A_{N-1} \cap \bigcap_{n \in b} A_n\right)\right]
\]

or, if $A_N \subset A_{N-1}$, as $\Phi(A_N)$. (10) is then immediate. The last claim is clear. \hfill \Box

For some semilattices all set functions $\Phi : \mathcal{A}_0 \to Y$ are semi-modular.
Corollary 1. Assume that $\mathcal{A}_0$ is a $\cap$-lattice with the following property:

\begin{equation}
A_1, \ldots, A_N, \bigcup_{n=1}^N A_n \in \mathcal{A}_0 \text{ implies } \bigcup_{n=1}^N A_n = A_{n_0} \text{ for some } 1 \leq n_0 \leq N
\end{equation}

Then, every $\Phi : \mathcal{A}_0 \to Y$ is semi-modular.

Proof. This is a straightforward consequence of (10).

Property (11) appears as an assumption in [8] A1, p. 502 and [15] Assumption 1.1.5, p. 13 but it is in fact a natural condition in order lattices, see section 3.

We generalize the notion of a strongly additive set function given above to that of a semi-additive set function by saying that $\Phi : \mathcal{A}_0 \to \mathbb{R}$, in symbols $\Phi \in sa_{\mathcal{A}_0}(\mathcal{A}_0, Y)$, if it may be extended as a semi-modular set function to the semi-lattice $\mathcal{A}_0 \cup \{\emptyset\}$ with $\Phi(\emptyset) = 0$. Necessary and sufficient conditions for the existence of such an extension depend on the structure of $\mathcal{A}_0$. If $\mathcal{A}_0$ is $\cap$ closed $\Phi$ is semi-additive if and only if it satisfies (7); if $\mathcal{A}_0$ is $\cup$ closed, semi-additivity is equivalent to

\begin{equation}
\sum_{\emptyset \neq b \subseteq [N]} \nu(b) \Phi \left( \bigcup_{n \in b} A_n \right) = 0 \quad A_1, \ldots, A_N \in \mathcal{A}_0, \bigcap_{n=1}^N A_n = \emptyset
\end{equation}

In either case a semi-modular set function on a semi-lattice is semi-additive whenever $\bigcap_{n=1}^N A_n \neq \emptyset$ for all $A_1, \ldots, A_N \in \mathcal{A}_0$, again a property assumed in [15] Definition 1.1.1, p. 10] or [23] A3, p. 63.

We end this section with some examples of semi-modular set functions.

Example 1. Let $\mathcal{A}_0$ the collection of all the finite dimensional subspaces of a vector space $X$ and define $\Phi : \mathcal{A}_0 \to \mathbb{R}$ by letting $\Phi(A) = \dim(A)$. Then, $\mathcal{A}_0$ is a $\cap$-lattice and, by Grasmann formula, $\Phi$ is $\cap$-modular.

Example 2. Let $\mathcal{A}_0$ be a $\cup$-lattice of subsets of some set $X$, $Y$ the collection of real-valued functions on $X$ and define $\Phi : \mathcal{A}_0 \cup \{\emptyset\} \to Y$ by letting $\Phi(A) = 1_A$. Then, (11) implies that $\Phi$ is semi-modular. The same is true of $\delta_x$, the Dirac measure sitting at $x$, for all $x \in X$.

Example 3. Let $\mathcal{A}_0$ be the collection of all compact, finite-dimensional polytopes of some vector space $X$, a $\cap$-lattice. More explicitly, each $A \in \mathcal{A}_0$ may be written in the form $A = \text{co}\{x_1, \ldots, x_k\}$ for a minimal (and therefore unique) set $\{x_1, \ldots, x_k\}$ of its vertices so that $\dim(A) = k$. Equivalentl, we have the hyperplane representation $A = \{x \in X : Tx \geq b\}$ where $T$ is a linear map from $X$ to $\mathbb{R}^d$ and $b \in \mathbb{R}^d$ for some $d > 0$. Let $\Phi : \mathcal{A}_0 \to \mathbb{R}$ be given. If $A_1, \ldots, A_N \in \mathcal{A}_0$ and $A = \bigcup_{n=1}^N A_n \in \mathcal{A}_0$ then $\dim(A(N)) < \dim(A)$ implies $A_N \subseteq \bigcup_{n=1}^{N-1} A_n$. By compactness there would otherwise exist an open set $O \subset X$ such that $A_N \cap O \neq \emptyset = O \cap \bigcup_{n=1}^{N-1} A_n$. But then $A_N \cap O$ and $A \cap O$ coincide despite having different dimension. Choose $d$ to be the minimal linear dimension for which there is a collection $\{A_1, \ldots, A_N\}$ in $\mathcal{A}_0$ with $A = \bigcup_{n=1}^N A_n \in \mathcal{A}_0$, $\dim(A) = d$ but for which (ba) fails; choose $N$ to be the minimal number of sets for which this may occur. Of course
\(d > 0\) and \(N > 1\). Since \(N\) is minimal, Lemma 2 implies that all sets \(A_n\) must have the same dimension, \(d\). Let \(\mathcal{H}\) be the collection of all hyperplanes in \(X\) and, for each \(H \in \mathcal{H}\), let \(H^+, H^-\) be the corresponding half-spaces. Groemer [10] considered the following property:

\[
\Phi(A) = \Phi(A \cap H^+) + \Phi(A \cap H^-) - \Phi(A \cap H) \quad A \in \mathcal{A}_0, \ H \in \mathcal{H}
\]

Assume (13), choose \(H \in \mathcal{H}\) so that \(H^+\) contains \(A_1\) but not \(A\) and let \(A^+_n = A_n \cap H^+\), \(A^-_n = A_n \cap H^-\) and \(A_H^n = A_n \cap H\). By assumption, \(\dim(A^H) < d\) and \(\dim(A^-_1) = \dim(A^H_1) < \dim(A^-)\) so that \(A^- = \bigcup_{n=2}^N A^-_n\). Given that \(d\) and \(N\) were chosen to be minimal, \(\{A_1, \ldots, A_N\}\) fails to satisfy (6a) if and only if so does \(\{A_1^+, \ldots, A_N^+\}\). Repeating this reasoning for all \(H \in \mathcal{H}\) with \(A_1 \subset H^+\) leads to the conclusion that the collection \(\{A_1, A_2 \cap A_1, \ldots, A_1 \cap A_N\}\) does not meet (6a), a contradiction of Lemma 2. Thus,

**Lemma 3** (Groemer). If \(X, \mathcal{A}_0\) and \(\Phi\) are as above and \(\Phi\) satisfies (13), then, \(\Phi\) is semi-modular.

Klain and Rota [16] show that all set functions assigning to a compact polytope some polynomial of the length of its faces actually satisfy (13).

### 3. Additive Extensions

The main result of this section is a necessary and sufficient condition for extending semi-modular, vector-valued set functions from a semi-lattice of sets to the generated ring. The measure extension problem is, of course, a time honored one and, among the many contributions, a major one has been that of Horn and Tarski [12, theorem 1.21, p. 477] based on the notion of partial measure on which we shall briefly comment later on. A general treatment of this topic following these same guidelines is contained in [2, chapter 3]; see also [17].

Given their repeated use, we state hereafter some well known facts.

**Lemma 4.** Let \(\mathcal{A}_0 \subset 2^X\) be a \(\cup\) (resp. \(\cap\)) lattice and \(\mathcal{A}_1\) and \(\mathcal{A}\) the lattice and the ring generated by \(\mathcal{A}_0 \cup \{\emptyset\}\), respectively. \(\mathcal{A}_1\) consists of finite (possibly empty) intersections (resp. unions) of sets from \(\mathcal{A}_0\) while each \(A \in \mathcal{A}\) is the disjoint union \(\bigcup_{n=1}^N B_n \setminus C_n\) where \(B_n, C_n \in \mathcal{A}_1\) and \(C_n \subset B_n\) for \(n = 1, \ldots, N\). Moreover, \(\mathcal{I}(\mathcal{A}_0) = \mathcal{I}(\mathcal{A})\).

**Proof.** The representation of the members of \(\mathcal{A}_1\) is an easy fact; for \(\mathcal{A}\) see [11, p. 26] or [2, theorem 1.1.9, p. 4]. From (1) one easily establishes \(\mathcal{I}(\mathcal{A}_1) = \mathcal{I}(\mathcal{A}_0)\). Given the representation \(\bigcup_{n=1}^N B_n \cap C_n\) of each \(A \in \mathcal{A}\) as in the claim, we obtain \(1_A = \sum_{n=1}^N (1_{B_n} - 1_{C_n})\) so that \(\mathcal{I}(\mathcal{A}) = \mathcal{I}(\mathcal{A}_1)\).

The following is the main result of this section.

**Theorem 1.** Let \(\mathcal{A}_0 \subset 2^X\) be a semi-lattice and \(\mathcal{A}_1\) the generated lattice. \(\Phi_0 : \mathcal{A}_0 \to Y\) is semi-modular if and only if there exists a necessarily unique, modular set function \(\Phi_1 : \mathcal{A}_1 \to Y\) such that \(\Phi_1|_{\mathcal{A}_0} = \Phi_0\).
Proof. By Lemma 1 we just consider the case of a \( \cap \)-lattice. If an extension such as \( \Phi_1 \) in the statement exists it must take the form

\[
(14) \quad \Phi_1 \left( \bigcup_{n=1}^{N} A_n \right) = \sum_{\varnothing < b \leq [N]} \nu(b) \Phi_0 \left( \bigcap_{n \in b} A_n \right) \quad A_1, \ldots, A_N \in \mathcal{A}_0
\]

and is thus necessarily unique. Denote by \( \lambda(\{A_1, \ldots, A_N\}) \) the right hand side of (14). It is clear that \( \lambda \) is invariant with respect to a permutation of the indexes. Moreover, if \( A_n, B_k \in \mathcal{A}_0 \) for \( n = 1, \ldots, N \); \( k = 1, \ldots, K \) and \( \bigcup_{n=1}^{N} A_n = \bigcup_{k=1}^{K} B_k \) we know from (10) that \( \lambda(\{A_1, \ldots, A_N\}) = \lambda(\{A_1, \ldots, A_N, B_1, \ldots, B_K\}) = \lambda(\{B_1, \ldots, B_K\}) \).

Thus the right hand side of (14) indeed defines an extension \( \Phi_1 : \mathcal{A}_1 \rightarrow Y \) of \( \Phi_0 \). To prove that this is modular let \( A = \bigcup_{n=1}^{N} A_n \) and \( B = \bigcup_{k=1}^{K} B_k \) be elements of \( \mathcal{A}_1 \), with \( A_n, B_k \in \mathcal{A}_0 \) for \( n = 1, \ldots, N \) and \( k = 1, \ldots, K \). Define also the collection \( \{C_m : m = 1, \ldots, N + K\} \) with \( C_m = A_m \) for \( m \leq N \) or \( C_m = B_{m-N} \) when \( N < m \leq N + K \).

Then, since \( \nu(a \cup b) = \nu(a)\nu(b) \) when \( a \cap b = \varnothing \),

\[
\Phi_1(A \cup B) = \Phi_1 \left( \bigcup_{m=1}^{N+K} C_m \right)
= \sum_{\varnothing < b \leq [N+K]} \nu(b) \Phi_1 \left( \bigcap_{n \in b} C_n \right)
= \sum_{\varnothing < b \leq [N]} \nu(b) \Phi_1 \left( \bigcap_{n \in b} A_n \right) + \sum_{\varnothing < a \leq [K]} \nu(a) \Phi_1 \left( \bigcap_{k \in a} B_k \right)
+ \sum_{\varnothing < b \leq [N], \varnothing < a \leq [K]} \nu(a)\nu(b) \Phi_1 \left( \bigcap_{k \in a} B_k \cap \bigcap_{n \in b} A_n \right)
= \Phi_1(A) + \Phi_1(B) - \sum_{\varnothing < b \leq [N]} \nu(b) \Phi_1 \left( \bigcap_{n \in b} (B \cap A_n) \right)
= \Phi_1(A) + \Phi_1(B) - \Phi_1(A \cap B)
\]

Given that \( \mathcal{A}_1 \) is a lattice, this is equivalent to the statement that \( \Phi_1 \) is semi-modular. \( \square \)

The following is a minor generalization of a classical result of Pettis [20] theorem 1.2, p. 188].

Theorem 2 (Pettis). Let \( \mathcal{A}_0 \) be a semi-lattice of subsets of \( X \), \( \mathcal{A} \) the generated ring and let \( \Phi_0 : \mathcal{A}_0 \rightarrow Y \). \( \Phi_0 \) is semi-modular if and only if there exists a necessarily unique \( \Phi \in \text{sa}(\mathcal{A}, Y) \) satisfying

\[
(15) \quad \Phi(A \setminus B) = \Phi_0(A) - \Phi_0(B) \quad A, B \in \mathcal{A}_0, \ B \subset A
\]

Moreover, (i) \( \Phi \) extends \( \Phi_0 \) if and only if \( \Phi_0 \in \text{sa}_0(\mathcal{A}_0, Y) \) and (ii) if \( \mathcal{A} \) is any algebra containing \( \mathcal{A}_0 \) then \( \Phi_0 \in \text{sa}(\mathcal{A}_0, Y) \) has an additive extension to \( \mathcal{A} \) and this will be unique up to the choice of \( \Phi(X) \) if \( \mathcal{A} \) is the algebra generated by \( \mathcal{A}_0 \).
Proof. If \( \Phi_0 \) is semi-modular let \( \Phi_1 \) be its modular extension to the generated lattice \( \mathcal{A}_1 \). Let \( \mathcal{A}_2 = \{ A \setminus B : A, B \in \mathcal{A}_1, B \subset A \} \), a \( \cap \)-lattice. If \( A \setminus B, C \setminus D \in \mathcal{A}_2 \) and \( A \setminus B = C \setminus D \) then, since \( \Phi_1 \) is modular, \( \Phi_1(A \cup D) = \Phi_1(A) + \Phi_1(D) - \Phi_1(A \cap D) \) so that the equalities \( A \cup D = C \cup B \) and \( A \cap D = C \cap B \) imply \( \Phi_1(A) - \Phi_1(B) = \Phi_1(C) - \Phi_1(D) \). We are then free to define \( \Phi_2 : \mathcal{A}_2 \to \mathcal{Y} \) by letting
\[
\Phi_2(A \setminus B) = \Phi_1(A) - \Phi_1(B) \quad A \setminus B \in \mathcal{A}_1.
\]
Clearly \( \emptyset \in \mathcal{A}_2 \) and \( \Phi_2(\emptyset) = 0 \). By Lemma 1, \( \mathcal{A}_2 \) is the lattice generated by \( \mathcal{A}_2 \). Suppose \( A_n \setminus B_n, A \setminus B \in \mathcal{A}_2 \) for \( n = 1, \ldots, N \) and \( \bigcup_{n=1}^N A_n \setminus B_n = A \setminus B \). Then,
\[
A \setminus B = \bigcup_{n=1}^N \bar{A}_n \setminus \bigcap_{n=1}^N \bar{B}_n \quad \text{where} \quad \bar{A}_n = B \cup (A \cap A_n), \quad \bar{B}_n = B \cup (A \cap B_n).
\]
Thus, \( \bar{A}_n \setminus \bar{B}_n = A_n \setminus B_n \) and therefore
\[
\Phi_2(A \setminus B) = \Phi_1 \left( \bigcup_{n=1}^N \bar{A}_n \big) - \Phi_1 \left( \bigcup_{n=1}^N \bar{B}_n \big) \right) = \sum_{\emptyset < b \leq [N]} \nu(b) \left( \Phi_1 \left( \bigcap_{n \in b} \bar{A}_n \right) - \Phi_1 \left( \bigcup_{n \in b} \bar{B}_n \right) \right) = \sum_{\emptyset < b \leq [N]} \nu(b) \Phi_1 \left( \bigcap_{n \in b} (A_n \setminus B_n) \right)
\]
Thus, \( \Phi_2 \in sa_{0}(\mathcal{A}_2, \mathcal{Y}) \) and admits, by Theorem 11, an extension \( \Phi \in sa(\mathcal{A}, \mathcal{Y}) \) satisfying (15). Suppose that the same is true of \( \Psi \) and fix \( A, B \in \mathcal{A}_0 \). If \( \mathcal{A}_0 \) is a \( \cap \)-lattice, write \( \Psi(A) - \Psi(A \cap B) = \Phi(A) - \Phi(A \cap B) \) (otherwise \( \Psi(A \cup B) - \Psi(B) = \Phi(A \cup B) - \Phi(B) \)). Given that both \( \Psi \) and \( \Phi \) are modular on \( \mathcal{A}_1 \), this equality is easily extended first to all \( B \in \mathcal{A}_1 \) for fixed \( A \in \mathcal{A}_0 \) and then to all \( A, B \in \mathcal{A}_1 \). But then \( \Psi|_{\mathcal{A}_2} = \Phi|_{\mathcal{A}_2} \). Uniqueness then follows from Theorem 11. Claims (i) and (ii) are obvious.

A first implication of Theorems 11 and 2 is that if \( \Phi_0 : \mathcal{A}_0 \to \mathcal{Y} \) is a semi-modular set function then there always exist a semi-additive set function obtained from it by translation (see [13]).

Corollary 2. Every semi-modular \( \Phi_0 : \mathcal{A} \to \mathcal{Y} \) admits a unique semi-additive translation.

Proof. Let \( \Phi_1 \) be the modular extension of \( \Phi_0 \) to the generated lattice and write \( y = -\Phi_1(\emptyset) \), if \( \emptyset \in \mathcal{A}_1 \), or \( y = 0 \) otherwise and denote, as in [13], translations by subscripts. Then \( \Phi_{1,y} \in sa(\mathcal{A}_1, \mathcal{Y}) \) by construction while \( \Phi_{0,y} \in sa_{0}(\mathcal{A}_0, \mathcal{Y}) \) because of \( \Phi_{0,y} = \Phi_{1,y}|_{\mathcal{A}_0} \). If \( \Phi_{0,z} \) were another semiadditive translation of \( \Phi_0 \) then, by Theorem 2, \( \Phi_{0,y} \) and \( \Phi_{0,z} \) would have the same additive extension to \( \mathcal{A}_1 \). Thus, if \( A \in \mathcal{A}_0 \) one has \( 0 = \Phi_{0,y}(A) - \Phi_{0,z}(A) = y - z \).

Theorem 2 also implies a finitely additive version of Dynkin’s lemma:
Corollary 3. Two finitely additive probabilities agree on a semi-lattice if and only if they agree on the generated algebra.

The existence of an additive extension to the generated ring, established in Theorem 2 for semi-additive set functions on a semi-lattice, allows to associate this class of set functions with the family of linear functionals on the vector space of simple functions. It is easy to see that this relationship is isomorphic and in principle may be exploited to obtain even more general measure extensions. This approach was inaugurated by Horn and Tarski [12, Definition 1.6, p. 471] who first introduced the notion of partial measure, later generalized to that of real partial charge in [2] Definition 3.2.1, p. 64]. A real-valued set function $\lambda$ on some, arbitrary collection $B \subset 2^X$ is a real partial charge if $\sum_{n=1}^{N} 1_{C_n} = \sum_{k=1}^{K} 1_{D_k}$ with $C_n, D_k \in B$ implies $\sum_{n=1}^{N} \lambda(C_n) = \sum_{k=1}^{K} \lambda(D_k)$ and, if so, it may be extended to a real-valued, additive set function on any algebra containing $B$, [2] Theorem 3.2.5, p. 65]. In other words real partial charges may be extended with no restriction on the initial collection of sets, while we had to assume that $A_0$ is a semi-lattice. However, this condition belongs more to analysis than to measure theory and in several cases it is rather difficult to apply. It turns however very useful to establish further properties such as positivity or boundedness at least when $Y = \mathbb{R}$.

Lemma 5. Let $A_0$ be a semi-lattice, $\Phi_0 \in sa(A_0)$ and $\Phi \in sa(A)$ the unique extension of $\Phi_0$ to the ring $A$ generated by $A_0$. (i) $\Phi \geq 0$ if and only if $\sup\{\Phi_0(b) : 0 \geq b \in I(A_0)\} \leq 0$, (ii) $\Phi \in ba(A)$ if and only if $\sup\{|\Phi_0(b)| : 1 \geq |b| \in I(A_0)\} < \infty$, (iii) $\Phi \in ba(A)_+$ if and only if $\sup\{\Phi_0(b) : 1 \geq b \in I(A_0)\} < \infty$.

Proof. By Lemma 6 one may write (i)–(iii) with $\Phi$ and $A$ in place of $\Phi_0$ and $A_0$, respectively. An additive set function on a ring is positive or bounded if and only if its integral over simple functions is so, see e.g. [2] theorems 3.1.9, p. 63 and 3.2.5 p. 65].

Of some interest is also a version of Theorem 4.2 for product spaces. If $\Phi : A \times B \to Y$, $A \in A$ and $B \in B$ then we define $\Phi^A : B \to Y$ and $\Phi^B : A \to Y$ by letting

$$\Phi^A(B) = \Phi^B(A) = \Phi(A \times B) \quad A \in A, \quad B \in B$$

$\Phi$ is said to be separately in a given class if $\Phi^B$ and $\Phi^A$ belong to that class for each $A \in A$ and $B \in B$.

Lemma 6. Let $A_0$ and $B_0$ be semi-lattices of sets generating the lattices $A_1$ and $B_1$ and the rings $A$ and $B$, respectively. Let also $\Phi_0 : A_0 \times B_0 \to Y$. Then:

(i) $\Phi_0$ is separately semi-modular if and only if there exists a unique separately modular set function $\Phi_1 : A_1 \times B_1 \to Y$ such that $\Phi_1|\mathcal{A}_0 \times \mathcal{B}_0 = \Phi_0$;

(ii) $\Phi_0$ is separately semi-additive if and only if there exists a unique separately additive set function $\Phi : A \times B \to Y$ such that $\Phi|\mathcal{A}_0 \times \mathcal{B}_0 = \Phi_0$.

Proof. We prove the claim just for the case in which $A_0$ is a $\cup$-lattice and $B$ a $\cap$-lattice but it will be clear that the argument remains unchanged for any other choice. For each $B \in B$ extend
we obtain a unique extension \( \Phi \) to \( A \). Define the set function \( \Phi : \mathcal{A} \times \mathcal{B} \rightarrow Y \) obtained from this by letting \( B \) vary across \( \mathcal{B} \). We claim that \( \Phi \) is separately semi-modular. In fact, let \( B, B_1, \ldots, B_N \in \mathcal{B}_0 \) be such that \( B = \bigcup_{n=1}^{N} B_n \) and fix \( A \in \mathcal{A}_1 \). By Lemma [1] there exist \( A_1, \ldots, A_M \in \mathcal{A}_0 \) such that \( A = \bigcap_{m=1}^{M} A_n \). Then,

\[
\Phi^B (A \times B) = \sum_{\varnothing < b \leq [M]} \nu(b) \Phi_0 \left( \bigcup_{m \in b} A_m \times B \right)
\]

\[
= \sum_{\varnothing < a \leq [N]} \nu(a) \sum_{\varnothing < b \leq [M]} \nu(b) \Phi_0 \left( \bigcup_{m \in b} A_m \times \bigcap_{n \in a} B_n \right)
\]

\[
= \sum_{\varnothing < a \leq [N]} \nu(a) \Phi_0 \left( A \times \bigcap_{n \in a} B_n \right)
\]

\[
= \sum_{\varnothing < a \leq [N]} \nu(a) \Phi_A \left( A \times \bigcap_{n \in a} B_n \right)
\]

so that \( \Phi \) is semi-modular in its second coordinate while modular in the first one, by construction; moreover it is unique with these properties. A further application of this same argument proves the existence claim in \((i)\). If \( \Psi : \mathcal{A} \times \mathcal{B}_1 \rightarrow Y \) has the same properties of \( \Phi \) in \((i)\), then necessarily \( \Psi \big|_{\mathcal{A} \times \mathcal{B}} = \Phi \big|_{\mathcal{A} \times \mathcal{B}} \) and, by semi-modularity in the second coordinate, \( \Psi = \Phi \). The same argument applies in the proof of the existence in \((ii)\). For fixed \( B \in \mathcal{B}_1 \), consider the set function on \( \mathcal{A} \) associated with \( \Phi^B \) via (15) and obtain from this a set function \( \Phi^{\ast} : \mathcal{A} \times \mathcal{B}_1 \rightarrow Y \). As above, the linear nature of the relationship linking \( \Phi^{\ast} \) to \( \Phi \) implies that \( \Phi^A : \mathcal{B}_1 \rightarrow Y \) is modular. A further application of this same procedure concludes the proof of existence in \((ii)\). As above, uniqueness follows from Theorem [2] applied coordinatewise.

□

**Theorem 3.** Let \( \mathcal{A}_0, \mathcal{B}_0 \) be semi-lattices and \( \mathcal{C} \) the ring generated by \( \mathcal{A}_0 \times \mathcal{B}_0 \). \( \Phi_0 : \mathcal{A}_0 \times \mathcal{B}_0 \rightarrow Y \) is separately semi-additive if and only if there exists \( \Phi \in \text{sa}(\mathcal{C}, Y) \) such that \( \Phi \big|_{\mathcal{A}_0 \times \mathcal{B}_0} = \Phi_0 \).

**Proof.** Define the set function \( \Phi^\ast : (\mathcal{A}_0 \cup \varnothing) \times (\mathcal{B}_0 \cup \varnothing) \rightarrow Y \) by letting \( \Phi^\ast (A \times B) = \emptyset \) if either \( A = \varnothing \) or \( B = \varnothing \) and \( \Phi^\ast (A \times B) = \Phi (A \times B) \) if \( A \times B \in \mathcal{A}_0 \times \mathcal{B}_0 \). Given that \( \Phi \) is semi-additive on both coordinates, this is easily seen to be the unique separately semi-additive extension of \( \Phi \) to \( (\mathcal{A}_0 \cup \varnothing) \times (\mathcal{B}_0 \cup \varnothing) \). To avoid additional notation we simply assume \( \varnothing \in \mathcal{A}_0 \) and \( \varnothing \in \mathcal{B}_0 \) and \( \Phi_0(\varnothing) = 0 \). Let \( \mathcal{A} \) and \( \mathcal{B} \) be the rings generated by \( \mathcal{A}_0 \) and \( \mathcal{B}_0 \) respectively. By Lemma [6] we obtain a unique extension \( \Phi \) to \( \mathcal{A} \times \mathcal{B} \) that is separately additive. Let \( \mathcal{C}_0 \) be the collection of all finite unions \( \bigcup_{n=1}^{N} A_n \) from \( \mathcal{A} \times \mathcal{B} \) where the sets \( A_1, \ldots, A_N \) are pairwise disjoint. Let \( C, \tilde{C} \in \mathcal{C}_0 \) be given, with \( C = \bigcup_{j=1}^{J} A_j \times B_j \) and \( \tilde{C} = \bigcup_{k=1}^{K} \tilde{A}_k \times \tilde{B}_k \). Writing

\[
(17) \quad C \cup \tilde{C} = \bigcup_{j,k} \left( (A_j \cap \tilde{A}_k) \times (B_j \cup \tilde{B}_k) \right) \cup \bigcup_{j=1}^{J} \left( \left( A_j \setminus \bigcup_{k=1}^{K} \tilde{A}_k \right) \times B_j \right) \cup \bigcup_{k=1}^{K} \left( \left( \tilde{A}_k \setminus \bigcup_{j=1}^{J} A_j \right) \times \tilde{B}_k \right)
\]

shows that \( \mathcal{C}_0 \) is a \( \bigcup \)-lattice. If \( C = \tilde{C} \), then \( A_j \cap \tilde{A}_k \neq \varnothing \) implies \( B_j = \tilde{B}_k \) so that, \( \sum_{j=1}^{J} \Phi(A_j \times B_j) = \sum_{j=1}^{J} \sum_{k=1}^{K} \Phi((A_j \cap \tilde{A}_k) \times B_j) = \sum_{k=1}^{K} \sum_{j=1}^{J} \Phi((A_j \cap \tilde{A}_k) \times \tilde{B}_k) = \sum_{k=1}^{K} \Phi(\tilde{A}_k \times \tilde{B}_k) \).
Φ* : ℂ₀ → Y may thus be unambiguously defined by letting

\[ (18) \quad \Phi^*(C) = \sum_{j=1}^{J} \Phi(A_j \times B_j) \quad C = \bigcup_{j=1}^{J} A_j \times B_j \in ℂ₀ \]

To show that Φ* ∈ sa₀(ℂ₀, Y), fix C₁, . . . , Cᵢ, \( \bigcap_{j=1}^{J} C_j \in ℂ₀ \) with \( C_j = \bigcup_{k=1}^{K_j} A^j_k \times B^j_k \). Given that \( \mathcal{A} \) is a ring one may form a collection \( A^1, \ldots, A^N \in \mathcal{A} \) such that \( A^j_k \subseteq A^j_{k_n} \) for all \( j, k \). If \( 1 \leq n \leq N \) and \( 1 \leq j \leq J \) there is then at most one integer \( k_n \) such that \( A_n \subseteq A^j_{k_n} \), otherwise let \( A^j_{k_n} = \emptyset \). Then,

\[
\sum_{\emptyset < b \leq [J]} \nu(b)\Phi^* \left( \bigcup_{j \in b} C_j \right) = \sum_{\emptyset < b \leq [J]} \nu(b)\Phi^* \left( \bigcup_{n=1}^{N} \bigcup_{j \in b} A_n \times B^j_{k_n} \right)
\]

\[
= \sum_{\emptyset < b \leq [J]} \nu(b)\Phi^* \left( \bigcup_{n=1}^{N} A_n \times \left( \bigcup_{j \in b} B^j_{k_n} \right) \right)
\]

\[
= \sum_{n=1}^{N} \sum_{\emptyset < b \leq [J]} \nu(b)\Phi \left( A_n \times \left( \bigcup_{j \in b} B^j_{k_n} \right) \right)
\]

\[
= N \Phi \left( \bigcup_{n=1}^{N} A_n \times \left( \bigcap_{j=1}^{J} B^j_{k_n} \right) \right)
\]

\[
= \Phi^* \left( \bigcap_{j=1}^{J} C_j \right)
\]

i.e. Φ* ∈ sa₀(ℂ₀, Y). The claim thus follows from Theorem 2.

One may remark that Theorem 3 is somehow more general than Theorem 1 since the given initial collection, \( \mathcal{A}_0 \times \mathcal{B}_0 \), is not even a semi-lattice nor is the set function \( \Phi_0 \) semi-modular. Of course, Theorem 3 may be proved for any arbitrary but finite number of coordinates.

### 4. ORDER SEMILATTICES

Viewing the semi-lattice property considered above as a characteristic of the partial order induced by set inclusion induces to extend this notion to partially ordered sets. A set X endowed with a partial order \( \geq \) is a \( \land \)-lattice (resp. \( \lor \)-lattice) where each subset \( \{x, y\} \) of X admits an infimum \( x \land y \) (resp. suprema \( x \lor y \)) in X. This section proves a general version of a result originally due to Norberg [19] by which vector-valued functions on X may be associated to semiadditive set functions on special families of subsets of X. This fact may be regarded as a further example of this family of set functions, adding to the ones of section 2. But, more importantly, it shows, similarly
to the Stiltjes construction, how to associate additive set functions to point functions, even if of unbounded variation.

As usual, $\geq$ decomposes into an asymmetric part, $>$, and a symmetric part $\sim$, defined implicitly by writing $x > x'$ whenever $x \geq x'$ but not $x' \geq x$ and $x \sim x'$ whenever $x \geq x'$ and $x' \geq x$. On the basis of these orderings, define the order intervals

\begin{align}
(19a) & \quad [\infty, x) = \{ x' \in X : x' \leq x \} \\
(19b) & \quad (\infty, x] = \{ x' \in X : x' < x \}
\end{align}

A special property is the following:

\begin{equation}
|x, \infty[ \subset ]x', \infty[ \quad \text{implies} \quad x \geq x'
\end{equation}

Indeed property (20) is rather natural and is clearly true if $X$ is a space of real-valued functions ordered pointwise. Another example requires some form of continuity.

**Example 4.** Let $\geq$ be upper semicontinuous in the sense that $x$ is a point of closure of $]x, \infty[$ and that $]x, \infty[$ is closed. Then (20) holds. If fact, if $]x, \infty[ \subset ]x', \infty[ \quad \text{then} \quad x \in ]x, \infty[ \subset ]x', \infty[ \subset ]x', \infty[$.

From our point of view, (20) has an important implication.

**Lemma 7.** Let $X$ be a $\land$-lattice possessing property (20) and let $x_1, \ldots, x_n, y_1, \ldots, y_K \in X$. Then $\bigcup_{n=1}^{N} \bigcap_{k=1}^{K} [x_n, \infty[ \subset \bigcap_{k=1}^{K} [y_k, \infty[, \infty[ \quad \text{implies} \quad \bigcap_{n=1}^{N} \bigcup_{k=1}^{K} x_n \geq \bigcap_{k=1}^{K} y_k.

**Proof.** $\bigcup_{n=1}^{N} \bigcap_{k=1}^{K} [x_n, \infty[ \subset \bigcap_{k=1}^{K} [y_k, \infty[, \infty[ \quad \text{and} \quad (20) \quad \text{imply} \quad x_n \geq \bigcap_{k=1}^{K} y_k \quad \text{for} \quad n = 1, \ldots, N \quad \text{or}, \quad \text{equivalently,} \quad \bigcap_{n=1}^{N} x_n \geq \bigcap_{k=1}^{K} y_k \quad \text{which, under} \quad (20), \quad \text{is the same as} \quad \bigcap_{n=1}^{N} \bigcup_{k=1}^{K} [x_n, \infty[ \subset \bigcap_{k=1}^{K} [y_k, \infty[.$

Define also the following families of subsets of $X$

\begin{equation}
(21) \quad \mathcal{A}_0 = \{ [\infty, x] : x \in X \} \quad \mathcal{B}_0 = \left\{ \bigcup_{n=1}^{N} [x_n, \infty[ : N \in \mathbb{N}, \ x_1, \ldots, x_N \in X \right\}
\end{equation}

If $Y$ is a real vector space, then so is $\mathcal{F}(X, Y) = \{ F \in Y^X : x \sim x' \implies F(x) = F(x') \}$.

**Corollary 4** (Norberg). Let $X$ be a $\land$-lattice. Then $\mathcal{F}(X, Y)$ is linearly isomorphic to $sa_0(\mathcal{A}_0, Y)$ via the identity

\begin{equation}
(22) \quad \Phi( [\infty, x]) = F(x) \quad x \in X
\end{equation}

**Proof.** Observe that $\mathcal{A}_0$ is a $\cap$-lattice and $\emptyset \notin \mathcal{A}_0$. Given that $F \in \mathcal{F}(X, Y)$ and that $[\infty, x] = [\infty, x']$ if and only if $x \sim x'$, writing the right hand side of (22) as $\Phi([\infty, x])$ implicitly defines a set function $\Phi : \mathcal{A}_0 \to Y$. Assume that $\{ x_1, \ldots, x_N \} \subset X$ is such that $\bigcup_{n=1}^{N} [\infty, x_n] \in \mathcal{A}_0 \ i.e. \ such \ that \ \bigcup_{n=1}^{N} [\infty, x_n] = [\infty, x]$ for some $x \in X_0$. But then, upon relabeling, $x_1, \ldots, x_N \leq x_0 \leq x_1$ so that $\bigcup_{n=1}^{N} [\infty, x_n] = [\infty, x]$. Thus, $\Phi \in sa_0(\mathcal{A}_0, Y)$, by Lemma 2. \qed
Corollary 1 may be stated for \( \vee \)-lattices after replacing \([-\infty, x]\) with \([x, \infty[\) in the definition of \( \mathcal{A}_0 \). The proof is virtually unchanged.

This result was first stated by Norberg in a noteworthy but unpublished paper, [19, pp. 6-9]. His approach was later revived by the literature on set-indexed stochastic processes, although in a rather troublesome way (see section 5).

A strictly related but more difficult question is whether it is possible to associate \( F \in \mathfrak{F}(X, Y) \) with a set function on the family of strict order intervals \((19b)\). The novelty here is that this class of intervals is not itself a semi-lattice and the additional condition \((20)\) is therefore required.

**Theorem 4.** Let \( X \) be a \( \wedge \)-lattice and \((20)\) hold. Then \( \mathfrak{F}(X, Y) \) is linearly isomorphic to \( sa_0(\mathcal{B}_0, Y) \) via the identity

\[
\Psi \left( \bigcup_{n=1}^{N} [x_n, \infty[ \right) = F \left( \bigwedge_{n=1}^{N} x_n \right) + y_F \quad x_1, \ldots, x_N \in X
\]

where \( y_F \in Y \) is unique (given \( F \)).

**Proof.** \( \mathcal{B}_0 \) is clearly a \( \cup \)-lattice. By Lemma 7, the quantity \( F \left( \bigwedge_{n=1}^{N} x_n \right) \) induces a well defined set function \( \Psi_0 \) on \( \mathcal{B}_0 \) with the property that \( \Psi_0(\bigcup_{n=1}^{N} x_n, \infty[ ) = \Psi_0( \bigwedge_{n=1}^{N} x_n, \infty[ ) \). In order to prove that \( \Psi_0 \) is semi-modular, assume that \( x(n, m), y_k \in X \) for \( m = 1, \ldots, M_n \), \( n = 1, \ldots, N - 1 \) and \( k = 1, \ldots, K \) are such that

\[
\bigcap_{n=1}^{N-1} \bigcup_{m=1}^{M_n} [x(n, m), \infty[ = \bigcup_{k=1}^{K} [y_k, \infty[)
\]

Let for simplicity \( x_n = \bigwedge_{m} x(n, m) \), \( y = \bigcup_{k} y_k \), \( A_n = [x_n, \infty[ \) for \( n = 1, \ldots, N-1 \) and \( A_N = [y, \infty[ \). It is easily deduced from \((24)\) that \( A_N \subset \bigcap_{n=1}^{N-1} A_n \) so that \( \Psi_0(\bigcup_{n \leq b} A_n) = \Psi_0(A_N \cup \bigcup_{n \leq b} A_n) \) for all \( \emptyset < b \leq [N-1] \). But then,

\[
\sum_{\emptyset < n \leq [N-1]} \nu(b) \Psi_0 \left( \bigcup_{n \leq b} A_n \right) = \sum_{\emptyset < n \leq [N]} \nu(b) \Psi_0 \left( \bigcup_{n \leq b} A_n \right) \quad \text{by (10)}
\]

\[
= \Psi_0(A_N) \quad \text{by (9)}
\]

\[
= \Psi_0 \left( \bigcup_{k=1}^{K} [y_k, \infty[ \right) \quad \text{by (23)}
\]

\[
= \Psi_0 \left( \bigcap_{n=1}^{N-1} \bigcup_{m=1}^{M_n} [x(n, m), \infty[ \right) \quad \text{by (24)}
\]

The claim then follows from Corollary 2.

Given its importance, in the following sections when dealing with a \( \wedge \)-lattice we will always assume that property \((20)\) is satisfied.
5. Stochastic Processes Indexed by a semi-lattice

In this section we provide applications of the preceding measure-theoretic results to stochastic process when the index set \( T \) is a semi-lattice. The main advantage of this approach, as compared e.g. to the literature on set-indexed processes, lies in the limited number of assumptions required. We only need to fix a set \( \Omega \), a semi-lattice \( T \) and a collection \( (F_t : t \in T) \) of \( \sigma \)-algebras of subsets of \( \Omega \) such that \( s, t \in T \) and \( s \leq t \) imply \( F_s \subset F_t \); we also posit the existence of a (countably additive) probability \( P \) on \( \mathcal{F} = \bigvee_{t \in T} F_t \). We write \( L^1 \) for \( L^1(\Omega, \mathcal{F}, P) \) and \( \| \cdot \| \) for \( \| \cdot \|_{L^1} \). \( \mathcal{P} \) and \( \mathcal{O} \) will denote the rings generated by the collections

\[
\{ t, \infty : F \in \mathcal{F}, t \in T \} \quad \text{and} \quad \{ F \times [t, \infty] : F \in \mathcal{F}, t \in T \}
\]

respectively (see (19) for notation). We shall also refer to the collections

\[
(25) \quad \mathcal{P}_0^p = \left\{ \bigcup_{n=1}^N [t_n, \infty] : N \in \mathbb{N}, t_1, \ldots, t_N \in T \right\} \quad \text{and} \quad \mathcal{P}_0^o = \{ [t, \infty] : t \in T \}
\]

and to the generated rings, \( \mathcal{P}_0^p \) and \( \mathcal{P}_0^o \) respectively.

A similar setting was considered in [5], where the index set is a family of subsets of \( \Omega \times I \) with \( I \) a partially ordered set, and can thus be interpreted as a collection of stochastic intervals. The results in [5] can then be regarded as describing the properties of processes indexed by stopping times rather than deterministic times, as is the case here. If the underlying index set is not linearly ordered, this difference is rather deep as illustrated, e.g., by the failure of the optional sampling theorem, see [13].

We shall occasionally exemplify our results on the basis of the following setting:

**Example 5 (Experiment design).** Let \( (X_k(n) : n \in \mathbb{R}_+) \) describe the random outcome of some experiment in a given location \( k \), with \( X_k(0) = 0 \). Designing an experiment is a matter of choosing which locations will participate and at what time (time 0 corresponds to non participation). An experiment design may thus be viewed as a sequence \( \langle t(k) \rangle_{k \in \mathbb{N}} \) that assigns at each location \( k \) the exact time \( t(k) \) at which the experiment will have to take place. Given that only a limited number of locations may be involved, the set \( T \) of all designs consists of sequences with only a finite number of non null terms. \( T \) is endowed with pointwise order so that if \( t, u \in T \) and \( u \geq t \) then \( u \) involves a larger number of locations, each being active at a later moment in time than \( t \). This partial order reflects the view that the larger the number of units involved and the later the moment of the experiment the more informative its outcome will be. A special case occurs when the experiment may only involve one single location so that the corresponding index set \( T_0 \subset T \) consists of sequences with only one non null element. It is clear that \( T_0 \) is isomorphic to \( \mathbb{R}^2 \) as each sequence \( t \in T_0 \) such that \( t(n) = 0 \) whenever \( n \neq k \) may actually be represented as the pair \( (k, t(k)) \). The aggregate outcome of the experiment will be some statistics of the sample \( (X_k(t(k)) : k \in \mathbb{N}) \), e.g. the sample mean or the maximum

\[
(26) \quad X^m_t = \frac{1}{\# \{ k \in \mathbb{N} : t(k) > 0 \}} \sum_k X_k(t(k)) \quad X^*_t = \max_k X_k(t(k))
\]
A natural development would be to consider the possibility of random experimental schedules, that is of functions \( \kappa : S \to T \) where \((S, \Sigma)\) is some given measurable space. \( \kappa \) would then play the role of a generalized stopping time indicating the random time and the random place where the experiment will take place.

5.1. Stochastic Processes and Induced Set Functions. We denote the class of generalized stochastic processes by

\[
\Xi = \left\{ Y \in \times_{t \in T} L^0 : P(Y_s = Y_t) = 1 \text{ whenever } s \sim t \right\}
\]

and say that \( Y \in \Xi \) is locally integrable if \( Y_t \in L^1 \) for all \( t \in T \) or integrable if \( \sup_{t \in T} P(|Y_t|) < \infty \).

A stochastic process is an element \( X \in \Xi \) which is adapted, i.e. such that \( X_t \) is \( \mathcal{F}_t \) measurable for all \( t \in T \). Given the possibility of adding a point at infinity, we assume with no loss of generality that \( T \) admits a largest element \( t_\infty \) and write \( X_{\infty} = X_{t_\infty} \) or, if \( t_\infty \) is a point added at infinity, \( X_{\infty} = \hat{X} \). Looking just at ordered pairs \( s, t \in T, \ s \leq t \), we can define in the present setting most of the classical properties and say, e.g., that \( X \) is a supermartingale (resp. an increasing process) if \( X_s \geq P(X_t | \mathcal{F}_s) \) (resp. \( P(A_s \leq X_t) = 1 \)) for all such pairs. Given the extension procedure to be described shortly, it should be stressed that these properties depend on the filtration considered and on the index set adopted.

The next result associates any \( X \in \Xi \) with a set function on \( \hat{\mathcal{P}} \).

**Theorem 5.** Let \( T \) be a semi-lattice. \( \Xi \) is linearly isomorphic to either \( sa(\hat{\mathcal{P}}, L^0) \) or \( sa(\bar{\mathcal{P}}, L^0) \) via the following identities holding for all \( t_1, \ldots, t_N \in T \) and \( F \in \mathcal{P} \):

\[
\begin{align*}
\Phi_X^p & \left( F \times \bigcup_{n=1}^N [t_n, \infty[ \right) = 1_F \left( X_{\infty} - X_{\bigcup_{n=1}^N t_n} \right) \quad (28a) \\
\Phi_X^o & \left( F \times \bigcap_{n=1}^N [t_n, \infty[ \right) = 1_F \left( X_{\infty} - X_{\bigcap_{n=1}^N t_n} \right) \quad (28b)
\end{align*}
\]

**Proof.** Both \( \mathcal{P}_0^p \) and \( \mathcal{P}_0^o \) are semilattices under the corresponding assumption on \( T \), as \( \bigcap_{n=1}^N [t_n, \infty[ = \bigvee_{n=1}^N t_n, \infty[ \). Moreover, \( \hat{\mathcal{P}} \subset \mathcal{P} \times \mathcal{P}^p \) and \( \bar{\mathcal{P}} \subset \mathcal{P} \times \mathcal{P}^o \). In order to apply Theorem 3 we need to show that the right hand sides of (28) describe well defined set functions on \( \mathcal{P} \times \mathcal{P}_0^p \) and \( \mathcal{P} \times \mathcal{P}_0^o \) respectively. In fact semi-additivity is clear, in view of Theorem 4. Observe then that \( F \times [s, \infty[ = G \times [t, \infty[ \) implies \( F = G \) and either \( F = \emptyset \) or \( s \sim t \): thus \( 1_F (X_{\infty} - X_s) = 1_G (X_{\infty} - X_t) \).

On the other hand, \( F \times \bigcup_{n=1}^N [t_n, \infty[ = \emptyset \) only if either \( F = \emptyset \) or \( \bigcup_{n=1}^N [t_n, \infty[ = [t_{\infty}, \infty[ \) from which, given (20), follows that \( \bigwedge_{n=1}^N t_n = t_{\infty} \). In either case, \( 1_F (X_{\infty} - X_{\bigwedge_{n=1}^N t_n}) = \emptyset \). Moreover, \( F \times \bigcup_{n=1}^N [s_n, \infty[ = G \times \bigcup_{k=1}^K [t_k, \infty[ \) implies \( F = G \) and \( \bigcup_{n=1}^N [s_n, \infty[ = \bigcup_{k=1}^K [t_k, \infty[ \) and, as shown in the proof of Theorem 4 \( \bigwedge_{n=1}^N s_n \sim \bigwedge_{k=1}^K t_k \) so that indeed the right hand side of (28a) defines a set function on \( \mathcal{P} \times \mathcal{P}_0^p \).

\[\text{\footnote{The existence of a largest element } t_{\infty} \text{ implies } \emptyset \in \mathcal{P}_0^p \text{; while } \emptyset \notin \mathcal{P}_0^o.}\]
The relationship between processes and measures established in Theorem 5 is a first step of well-known importance in the construction of the stochastic integral proposed by Šiota as well as in the Doléans-Dade approach to the Doob Meyer decomposition. Our choice to define such measure on \( \hat{\mathcal{P}} \) (resp. \( \hat{\varnothing} \)) is somehow unconventional as it disregards the issue of adaptness but is motivated by the need to exploit the semi-lattice property.

Let us assume for the rest of the paper that \( T \) is a \( \wedge \)-lattice (the \( \vee \) case may be treated similarly). The first use of Theorem 5 is to obtain an extension of the original index set to the \( T^p \). If we identify \( t \in T \) with \( t, \infty \subset T \) then \( T^p \) is easily seen to be an extension of \( T \); moreover, \( T^p \) inherits the order from \( T \) if we let \( \tau \leq \upsilon \) whenever \( \upsilon \subset \tau \): \( \varnothing \) is thus the unique maximal element. Likewise, one may fix a filtration \( (\mathcal{F}_\tau : \tau \in T^p) \) with \( \mathcal{F}_t \subset \mathcal{F}_\tau \) whenever \( \tau \subset t, \infty \) and \( \mathcal{F}_\varnothing = \mathcal{F} \).

For example, for \( \sigma \in T^p_0, \tau \in T^p_1 \) and \( \upsilon \in T^p \) one may set

\[
\mathcal{F}_0^\sigma = \bigvee_{\{\sigma = \bigcup_{n=1}^N \{t_n, \infty\} \}} \bigcap_{n=1}^N \mathcal{F}_{t_n}, \quad \mathcal{F}_1^\tau = \bigcap_{\{\sigma \in T^p_0 : \tau \subset \sigma\}} \mathcal{F}_0^\sigma \quad \text{and} \quad \mathcal{F}_\upsilon = \bigcap_{\{\tau_1, \tau_2 \in T^p_1 : \tau_1 \cap \tau_2 \subset \upsilon\}} \mathcal{F}_{\tau_2}.
\]

In analogy with (27), we designate the class of generalized stochastic processes on this new index set by \( \Xi \). More importantly, we define the ring of predictable sets

\[
\mathcal{P} = \{ F \times \tau : F \in \mathcal{F}_\tau, \tau \in T^p \}
\]

and the restriction \( \Phi^p_X = \Phi^p_X|\mathcal{P} \). Our definition of \( \mathcal{P} \) compares somehow to that proposed by Saada and Slonowsky [23, p. 70].

The notion of predictability implicit in (30) is delicate as several alternative definitions are in fact possible. An example would be to let \( T^p \) be defined by the sets of the form \( ]-\infty, t[ \). In this case the intuitive meaning of prior to, naturally associated with the concept of prediction, would in fact coincide with the notion of no later than. The nice fact is that Theorem 5 above would carry through unchanged to this case as to all other definitions that allow to view each \( X \in \Xi \) as defining a set function on a semi-lattice of subsets of \( \Omega \times T \). Observe that in the classical case, \( T = \mathbb{R}_+ \), predictability may be equivalently defined in terms of order (i.e. starting from left-open stochastic intervals) or of topology (i.e. starting with the class of càglàd processes). Another possible line of approach, taken, e.g., by the literature on set-indexed processes, suggests to start with a convenient set of topological assumptions on \( T \). The difference with this way of thinking of predictability is indeed rather deep when \( T \) is just partially ordered and makes the comparison with the definition given here hardly possible.

Extending the process \( X \) to this enriched setting may now be done quite simply.

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\(^6\) In [23], however, in place of \( T^p \) appears the (semi) ring of all subsets of the indexing class. The authors show its utility in the construction of the stochastic integral.

\(^7\) See for example [15] where the indexing family consists of compact subsets of a \( \sigma \)-compact space and is in fact closed under arbitrary intersections. More or less the same setting appears in all contributions to this literature.
Corollary 5. For each $X \in \Xi$ there exists a unique process $\check{X} = (\check{X}_\tau : \tau \in \mathcal{T}^p) \in \widehat{\Xi}$ such that

$$\check{X}_\tau = X_{\bigwedge_{n=1}^N t_n} \quad t_1, \ldots, t_N \in T, \quad \tau = \bigcup_{n=1}^N ]t_n, \infty[$$

and that

$$\check{X}_{\bigvee_{n \in b} \wedge_{n \in b} t_n} = \sum_{\varnothing < b \leq \{N\}} \nu(b) \check{X}_{\bigwedge_{n \in b} \wedge_{n \in b} t_n} \quad t_1, \ldots, t_N \in \mathcal{T}^p$$

Moreover, $\check{X}$ is adapted to the filtration defined in (29).

Proof. It is enough to define:

$$\check{X}_\tau = X_\infty - \Phi^p_X (\Omega \times \tau) \quad \tau \in \mathcal{T}^p$$

Then (31) follows from (28) while (32) is an immediate consequence of the fact that $\Phi^p_X$ is additive. If $\check{Z} \in \widehat{\Xi}$ were another such extension, then, by (32), the set function on $\mathcal{F}$ defined implicitly by letting $\lambda(F \times \tau) = 1_{F}(X_\infty - Z_\tau)$ for all $F \in \mathcal{F}$ and $\tau \in \mathcal{T}^p$ would be additive. Moreover, $\lambda$ coincides with $\Phi^p_X$ in restriction to the semi-lattice $\mathcal{F} \times \mathcal{F}_0^p$. But we then conclude from Theorem 1 that necessarily $\lambda = \Phi^p_X$ and thus $\check{Y}_\tau = \check{X}_\tau$ for all $\tau \in \mathcal{T}^p$, by (33). That $\check{X}$ is adapted to $(\mathcal{F}_\tau : \tau \in \mathcal{T}^p)$ follows from Lemma 4 and (29).

The claim that $X$ extends additively to $(\mathcal{F}_\tau : \tau \in \mathcal{T}^p)$, proved in Corollary 5, had already appeared, in a slightly different context, in [8] and in several papers that followed that work, but unfortunately the argument presented was erroneous. This property has then later been taken as a key assumption in this literature (see [15] for details and [5] for remarks) while it follows here quite naturally from Theorem 1.

Given the better properties of rings versus semilattices, it is quite natural to prefer working with its extension rather than with the original process. However, one should be warned that the former may not possess some of the original mathematical properties, such as the supermartingale property.

To conclude on the role of $\Phi^p_X$, let us mention that the construction of the stochastic integral consists mainly of the extension of the map $\int h \, dX = \Phi^p_X (h) : \mathcal{F} \to L^0$ to more general spaces. Also relevant, when $X$ is locally integrable, is the possibility to define the Doléans-Dade measure associated with $X$, namely

$$\phi^p_X = P \otimes \Phi^p_X$$

We develop in the following subsections some implications of the preceding results aiming in particular at replicating some classical decompositions. All processes encountered will be element of $\widehat{\Xi}$.

---

The literature on set-indexed processes, started from Cairoli and Walsh [4, Definition p. 115], makes use of the terms weak and strong to distinguish between properties possessed by $(X_t : t \in T)$ rather than by its extension $(\check{X}_\tau : \tau \in \mathcal{T})$. 

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8The literature on set-indexed processes, started from Cairoli and Walsh [4, Definition p. 115], makes use of the terms weak and strong to distinguish between properties possessed by $(X_t : t \in T)$ rather than by its extension $(\check{X}_\tau : \tau \in \mathcal{T})$. 

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5.2. Quasi Martingales. A major role is played by the family $\mathcal{D}$ of all finite, disjoint collections $d$ of elements of $\mathcal{P}$ of the form $\tau \cap v^c$ with $v \subset \tau$. Since $\mathcal{P}$ is a ring, the refinement relation makes $\mathcal{D}$ into a directed set.

If $X$ is a locally integrable stochastic process one can define

$$ V^d(X) = \sum_{\tau \cap v^c \in d} |P(X_v|\mathcal{F}_{\tau}) - X_{\tau}|, \quad S^d(X) = \sum_{\tau \cap v^c \in d} P(X_v|\mathcal{F}_{\tau}) - X_{\tau} \quad d \in \mathcal{D} $$

and $\mathcal{S}(X) = \{S^d(X) : d \in \mathcal{D}\}$. We refer to $X$ as a quasi martingale, in symbols $X \in \mathcal{Q}$, whenever $\|X\|_\mathcal{Q} := \sup_{d \in \mathcal{D}} \|V^d(X)\| < \infty$. Differently from the classical case, supermartingales, submartingales or even increasing processes need not be quasi martingales. We provide an example.

**Example 6.** Resume the setting of Example 5 and let $(A_k)_{k \in \mathbb{N}}$ be a sequence of one dimensional increasing processes (thus $A_k(0) = 0$) bounded by 1 describing the local experiment. Let also $A^*$ be the sample maximum (26) of all local experiments, itself an increasing process bounded by 1. Consider the segment $]a, a + 1[ \subset \mathbb{R}_+$ and a subset $K \subset \mathbb{N}$ and define $\tau_K(a) \in T$ by letting $\tau_K(n) = 0$ if $n \notin K$ and $\tau_K(n) = a$ otherwise. We can then define $\tau_K([a, a + 1]) = \{t \in T : \tau_K(a) < t \leq \tau_K(a)\} \in \mathcal{P}$.

It is clear that $K \cap K' = \emptyset$ implies $\tau_K([a, a + 1]) \cap \tau_{K'}([a, a + 1]) = \emptyset$ so that we can easily form a disjoint countable collection $\{\tau_K_i([a, a + 1]) : i \in \mathbb{N}\}$. But then

$$ \|A^*\|_\mathcal{Q} \geq P \sum_i \left| \max_{k \in K_i} P(A_k(a + 1)|\mathcal{F}_{\tau_K_i(a)}) - A^*(\tau_K_i(a)) \right| $$

If, for example, $A_k(a + 1) > \eta + A_k(a)$ for all $k \in \mathbb{N}$ then clearly $A$ is not a quasi martingale. This conclusion applies also to the special case in which the index set is $T_0$ and is therefore isomorphic to $\mathbb{R}^2$.

Interest for quasi martingales originates from their role as integrators, following from the isometry

$$ \|\Phi^p_X\| = \sup_{\{h \in \mathcal{S}(\mathcal{P}) : \|h\|_\mathcal{Q} \leq 1\}} \|\Phi^p_X(h)\| = \|X\|_\mathcal{Q} $$

Then, $X \in \mathcal{Q}$ if and only if $\Phi^p_X : \mathcal{S}(\mathcal{P}) \rightarrow L^1$ is continuous when $\mathcal{S}(\mathcal{P})$ is endowed with the supremum norm. This provides a natural extension of the stochastic integral to the closure of $\mathcal{S}(\mathcal{P})$ in $\mathcal{B}$. Quasi martingales possess other, interesting properties.

If $\{\tau_\alpha \cap v^c_\alpha : \alpha \in A\}$ is a disjoint collection with $\tau_\alpha, v_\alpha \in \mathcal{P}$ and $v_\alpha \geq \tau_\alpha$, then,

$$ \sum_{\alpha \in A} \|P(X_{v_\alpha}|\mathcal{F}_{\tau_\alpha}) - X_{\tau_\alpha}\| \leq \|X\|_\mathcal{Q} $$

This inequality, completeness of the space $L^1$ and contractiveness of conditional expectation determine several interesting convergence results, as in the original work of Rao [21]. If, for example,
one chooses the version \( P(X_{\nu_0} | \mathcal{F}_{\tau_0}) = X_{\tau_0} \) whenever \( \| P(X_{\nu_0} | \mathcal{F}_{\tau_0}) - X_{\tau_0} \| = 0 \) then (36) implies that the infinite sums

(37) \[ S^A(X) = \sum_{\alpha \in A} P(X_{\nu_n} | \mathcal{F}_{\tau_0}) - X_{\tau_0} \quad \text{and} \quad V^A(X) = \sum_{\alpha \in A} |P(X_{\nu_n} | \mathcal{F}_{\tau_0}) - X_{\tau_0}| \]

converge \( P \) a.s. and in \( L^1 \). This conclusion applies, e.g., to monotone sequences \( \langle \tau_n \rangle \subseteq \mathbb{N} \) by setting \( \nu_n = \tau_{n-1} \lor \nu_{n+1} \) and \( A = \mathbb{N} \). Then, writing \( \mathcal{G}_{\tau_k} = \bigcap_{j \geq k} \mathcal{F}_{\tau_j} \), one concludes that the process \( (Y_{\tau_k} : k \in \mathbb{N}) \) with

(38)
\[
Y_{\tau_k} = P(X_{\nu_k} | \mathcal{G}_{\tau_k}) + \sum_{n \geq j} P(X_{\nu_n} - X_{\tau_n} | \mathcal{G}_{\tau_k}) \quad j \geq k
\]

is integrable with \( \| Y_{\tau_k} \| \leq \| X_{\nu_k} \| + \| X \|_Q \leq 2 \| X \|_Q \) and that

(39)
\[
\limsup_{p \searrow k} |P(X_{\nu_k+p} | \mathcal{G}_{\tau_k}) - Y_{\tau_k}| \leq \lim_{p} \sum_{j \geq p} |P(X_{\nu_j} | \mathcal{F}_{\tau_j}) - X_{\tau_j}| = 0 \quad P \text{ a.s. and in } L^1
\]

More interestingly,

**Lemma 8.** Let \( \langle \tau_n \rangle_{n \in \mathbb{N}} \) be an increasing sequence in \( \mathcal{T}^p \) and \( X \in Q \). Then, (i) \( Y_{\tau_k} = \lim_n P(X_{\nu_n} | \mathcal{F}_{\tau_k}) \) exists \( P \) a.s. and in \( L^1 \) and (ii) \( \lim_n \| Y_{\tau_k} - X_{\tau_n} \| = 0 \). If \( X \) is uniformly integrable then \( \langle X_{\tau_n} \rangle_{n \in \mathbb{N}} \) admits an \( L^1 \) limit which only depends on \( \bigvee_n \tau_n \).

**Proof.** If \( \langle \tau_n \rangle_{n \in \mathbb{N}} \) is increasing, \( \mathcal{G}_{\tau_k} = \mathcal{F}_{\tau_k} \). (i) is just (39); (ii) easily follows from (38). By (ii) and the fact that \( (Y_{\tau_k} : k \in \mathbb{N}) \) is a martingale, \( (X_{\tau_k} : k \in \mathbb{N}) \) converges in \( L^1 \) if and only if it is uniformly integrable. Assume that \( \langle \tau_n \rangle_{n \in \mathbb{N}} \) and \( \langle \bar{\tau}_n \rangle_{n \in \mathbb{N}} \) are increasing and that \( \bigwedge_n \bar{\tau}_n = \bigwedge_n \tau_n \).

Since each \( \tau_n \in \mathcal{T} \) is the intersection of finitely many intervals of the form \( ]t_n^k, \infty[ \) with \( t_n^k \in T \), then, for each pair \( n, k \) there must be a pair \( m, l \) such that \( t_n^k \leq \bar{t}_n^l \), i.e. for each \( n \in \mathbb{N} \) there is \( m \) such that \( \tau_n \leq \bar{\tau}_m \). In yet other words, moving to subsequences if necessary and relabeling terms, we may assume with no loss of generality that \( \tau_n \leq \bar{\tau}_n \leq \tau_{n+1} \). But then, considering the sequence \( \langle X_{\bar{\tau}_n} \rangle_{n \in \mathbb{N}} \) obtained by letting \( \bar{\tau}_{2j-1} = \tau_j \) and \( \bar{\tau}_{2j} = \bar{\tau}_j \) and in view of (39) we conclude that necessarily \( \lim_n X_{\tau_n} = \lim_n X_{\bar{\tau}_n} \).

The classical Riesz decomposition (see [2] theorem 1.1, p. 80) easily follows.

**Corollary 6.** \( X \in Q \) is uniformly integrable if and only if it decomposes uniquely into the sum

(40)
\[
X_{\tau} = P(M | \mathcal{F}_{\tau}) + Z_{\tau} \quad \tau \in \mathcal{T}^p
\]

where \( M \in L^1 \) and \( Z \in Q \) is uniformly integrable with \( \lim_{\tau \in \mathcal{T}} \| X_{\tau} - M \| = \lim_{\tau \in \mathcal{T}} \| Z_{\tau} \| = 0 \). Moreover \( Z \) in (40) is in addition a positive supermartingale (i.e. a quasi potential) if and only if \( X \) is a supermartingale.

**Proof.** Uniqueness is clear: if \( M' \) and \( Z' \) were another pair of processes satisfying (40), then \( |Z_{\tau} - Z_{\tau}'| = |M_{\tau} - M_{\tau}'| \leq |M_{\nu} - M_{\nu}'| = |Z_{\nu} - Z_{\nu}'| \) so that \( \| Z_{\tau} - Z_{\tau}' \| = \| M_{\tau} - M_{\tau}' \| = 0 \). To prove existence, define the monotone set function \( \lambda : \mathcal{T} \rightarrow [0, \| X_Q \|] \) implicitly by letting

\[
\lambda(\tau) = \sup_{d \in \mathcal{G}_{\tau}} P(V^d(X)) \quad \tau \in \mathcal{T}^p
\]
and construct an increasing sequence \( \langle \tau_n \rangle_{n \in \mathbb{N}} \) in \( \mathcal{T}^p \) such that \( \lambda(\tau_n) \leq \inf_{\tau \in \mathcal{F}} \lambda(\tau) + 2^{-n} \). Let \( M \) designate the \( L^1 \) limit of \( \langle X_{\tau_n} \rangle_{n \in \mathbb{N}} \), which exists by Lemma \([6]\) and put \( Z_\tau = X_\tau - P(M|\mathcal{F}_\tau) \). If \( \tau, \tau_n \in \mathcal{T}^p, \tau \geq \tau_n, d_\tau \in \mathcal{D}_\tau \) and \( d_n = \{ \tau_n \cap \tau^c \} \cup d_\tau \), then \( d_n \in \mathcal{D}_{\tau_n} \) and \( |X_\tau - X_{\tau_n}| = V^{d_n}(X) - V^{d_\tau}(X) \). It follows that \( \|X_\tau - X_{\tau_n}\| \leq \lambda(\tau_n) - \lambda(\tau) \leq 2^{-n} \) and, consequently, that \( \|Z_\tau\| \leq \|X_\tau - X_{\tau_n}\| + |X_{\tau_n} - M| \leq 2^{-n} + \|X_{\tau_n} - M\| \). \( Z \in \mathcal{Q} \) and is uniformly integrable since both \( X \) and \( M \) belong to this same class. If \( X \) is a supermartingale, then so is \( Z \) and for each \( F \in \mathcal{T}_\tau \), \( P(Z_\tau 1_F) \geq \lim_v P(Z_\tau 1_F) = 0 \) so that \( Z \) is positive and therefore a quasi potential. The converse is clear. \( \square \)

5.3. **Doob Meyer Decompositions.** We now consider general decompositions of locally integrable processes. In doing so we exploit techniques similar to those developed in \([5]\) but with the noteworthy difference that the processes considered here do not necessarily generate bounded, positive set functions as was for the case of class \( D_0 \) supermartingales treated in the mentioned reference.

For each \( d \in \mathcal{D} \) define the operator \( \mathcal{P}_d^d : L^\infty(P) \to \mathcal{B}(\mathcal{P}) \) as
\[
(41) \quad \mathcal{P}_d^d(b) = \sum_{\tau \cap \tau^c \in d} P(b|\mathcal{F}_\tau) 1_{\tau \cap \tau^c} \quad b \in L^\infty(P)
\]
Fix \( F \in \mathcal{F} \) and \( \tau \in \mathcal{F} \). If \( d \geq \{ \tau^c \} \) and \( \tau \cap \tau^c \in d \) then \( \tilde{\tau} \leq \bar{\tau} \leq \tau \) and thus \( \mathcal{P}_d^d(F) 1_{\tau^c} = \mathcal{P}_d^d(P(F|\mathcal{F}_\tau)) 1_{\tau^c} \). Therefore
\[
(42) \quad P(X_\tau 1_F) = \phi_X^d(1_\tau P(F|\mathcal{F}_\tau))
\]
\[
= \phi_X^d(1_\tau \mathcal{P}_d^d(P(F|\mathcal{F}_\tau)))
\]
\[
= \phi_X^d(\mathcal{P}_d^d(P(F|\mathcal{F}_\tau))) - \phi_X^d(1_{\tau^c} \mathcal{P}_d^d(P(F|\mathcal{F}_\tau)))
\]
\[
= \phi_X^d(\mathcal{P}_d^d(P(F|\mathcal{F}_\tau))) - \phi_X^d(1_{\tau^c} \mathcal{P}_d^d(F))
\]
\[
= \mu_X^d(P(F|\mathcal{F}_\tau)) - \alpha_{X,\tau}^d(F)
\]
The last line of \((41)\) defines \( \mu_X^d, \alpha_{X,\tau}^d \in \text{ca}(\Omega, \mathcal{F}, P) \) implicitly as
\[
(43) \quad \mu_X^d(F) = \phi_X^d(\mathcal{P}_d^d(F)) = P(S^d(X) 1_F) \quad \alpha_{X,\tau}^d(F) = \phi_X^d(1_{\tau^c} \mathcal{P}_d^d(F)) \quad \tau \in \mathcal{T}, \quad F \in \mathcal{F}
\]
**Definition 1.** A stochastic process is said to possess the strong Doob Meyer property, or simply to be of class \( DM_s \), if the limit \( \mu_X(F) = \lim_d \mu_X^d(F) \) exists for all \( F \in \mathcal{F} \) and if \( \mu_X \in \text{ca}(\mathcal{F}) \).

Given that \( \mu_{X+Y}^d = \mu_X^d + \mu_Y^d \) it is clear that if \( X \) and \( Y \) are of class \( DM_s \) then so is \( X + Y \). The following theorem motivates our terminology.

**Theorem 6.** A stochastic process \( X \) is of class \( DM_s \) if and only if it admits the decomposition
\[
(44) \quad X_\tau = P(M|\mathcal{F}_\tau) - A_\tau \quad \tau \in \mathcal{T}, \quad P \text{ a.s.}
\]
where \( M \in L^1 \) and \( A \) is a stochastic process of class \( DM_s \) such that
\[
(45) \quad P(A_\tau 1_F) = \lim_d P \int_{\tau^c} \mathcal{P}_d^d(F) dA \quad \tau \in \mathcal{T}, \quad F \in \mathcal{F}
\]
Decomposition (44), if it exists, is unique. If $X$ is a supermartingale then $A$ is increasing.

Proof. It follows from (42) that if the net $\{\mu^d_X\}_{d \in \mathcal{D}}$ converges set wise to some $\mu_X \in \mathcal{F}(\mathcal{F})$ then so does the net $\{\alpha^d_{X,\tau}\}_{d \in \mathcal{D}}$ for all $\tau \in \mathcal{T}$ and $\alpha_{X,\tau}$ be the corresponding limit and let $M, A_\tau \in L^1$ be versions of the Radon Nikodym derivatives of $\mu_X$ and $\alpha_{X,\tau}$, respectively. (44) soon follows. Moreover, if $\tau \in \mathcal{T}$ and $F \in \mathcal{F}$ then

$$P(A_\tau 1_F) = \lim_d P \int_{\tau^c} \mathcal{P}^d_F(F) dX$$

$$= \lim_d P \int_{\tau^c} \mathcal{P}^d_F dA$$

$$= \lim_d P \int_{\tau^c} \mathcal{P}^d_F(\mathcal{P}(F|\mathcal{F}_\tau)) dA$$

$$= P(A_\tau \mathcal{P}(F|\mathcal{F}_\tau))$$

proving (45) and that there is a version of $A_\tau$ which is $\mathcal{F}_\tau$ measurable and thus $A$ to be a stochastic process. $A$ is of class $DM_\tau$ since $M$ is so: if $d \in \mathcal{D}$ then $S^d(M) = 0$. Conversely, if (44) holds and $A$ is of class $DM_\tau$ then so necessarily is $X$, as was just shown. If $P(N|\mathcal{F}_\tau) - B_\tau$ were another decomposition such as (44), then $A - B$ would be a uniformly integrable martingale meeting (45) and thus such that $P((A_\tau - B_\tau) 1_F) = 0$ for all $F \in \mathcal{F}$ and $\tau \in \mathcal{T}$. Eventually, if $X$ is a supermartingale $F \in \mathcal{F}$ and $\tau \leq \upsilon$ then $\alpha^d_{X,\upsilon}(F) - \alpha^d_{X,\tau}(F) = \phi^p_X(1_{\tau \leq \upsilon}, \mathcal{P}^d_F) \geq 0$ as $\phi^p_X$ is positive.

Differently from Meyer [18] and similarly to the original result of Doob, decomposition (44) does not require the supermartingale property. Property (45) of the intervening compensator $A$ may be regarded as a version of the classical definition of a natural process. $A$ may however not be predictable as the relationship between these two properties is unclear outside of the classical setting.

The main drawback of the strong Doob Meyer property defined above, however, is its measure theoretic nature and, consequently, the intrinsic difficulty to translate it into a corresponding property on the process $X$. Thus, establishing sufficient conditions for the existence of the limit $\mu_X$ seems to be an open problem. Observe, however, that the collection of subsets of $\mathcal{D}$ of the form $\{d' \in \mathcal{D}: d' \geq d\}$ for $d \in \mathcal{D}$ is a filter base. Fix some ultrafilter $\mathfrak{u}$ containing it [9] lemma I.7.11, p. 30 and assume that the set $\mathcal{S}(X)$ is bounded in $L^1$. Then for each $F \in \mathcal{F}$, the collection of sets of the form $\{\mu^d_X(F): d \in U\}$ for all $U \in \mathfrak{u}$ is itself an ultrafilter in an interval and thus converges, [9] lemma I.7.12, p. 30: write $\mu^d_X(F) = \mathfrak{u} - \lim \mu^d_X(F)$. This way of writing simply amounts to saying that for each $\varepsilon$, the set $\{d \in \mathcal{D}: |\mu^d_X(F) - \mu^d_X(F)| < \varepsilon\}$ is in $\mathfrak{u}$.

The following property is the closest relative, in our framework, to the property $D$ considered by Meyer [18]. It shares with that original definition the honest judgment expressed by Meyer himself who considered it "(...) not easy to handle" (p. 193) and who admitted that "(...) no satisfactory condition is known, implying that sufficiently general classes of right continuous supermartingales are contained in the class (D)" (p. 195).
Definition 2. A stochastic process $X$ is said to possess the Doob Meyer property, or to be of class $DM$, if $S(X)$ is a relatively weakly compact subset of $L^1$. 

The preceding definition may be cast in terms of stopping times. If $\tau_n \in \mathcal{F}_n$, $F_n \in \mathcal{F}_n$ for $n = 1, \ldots, N$ and $F_n \cap F_m = \emptyset$ then it is natural to define the simple stopping time $\sigma$ and the value $X_\sigma$ of $X$ at $\sigma$ as

$$\sigma = \sum_{n=1}^{N} \tau_n 1_{F_n} \quad \text{and} \quad X_\sigma = \sum_{n=1}^{N} X_{\tau_n} 1_{F_n} + X_\emptyset 1_{\cap_n F_n}$$

Let $\Sigma$ be the collection of all simple stopping times and $D(X) = \{X_\sigma : \sigma \in \Sigma\}$. The following Lemma provides evidence of the strict relationship between Definition 2 and the class $D$ property (at least as long as supermartingales are concerned). We consider a slightly weaker version of this property, defined as follows

Definition 3. A stochastic process $X$ is said to be of class $D$ if $D(X)$ is a relatively weakly compact subset of $L^1$.

We start noting that if $\{Y_n : n = 0, \ldots, N\} \subset L^1_+$, $Y_0 = 0$ and $Z_n = \sum_{k=1}^{n} Y_k$, then

$$Z_N 1_{\{Z_N > 2k\}} = (Z_N - 2k)^+ + 2k 1_{\{Z_N > 2k\}} \leq (Z_N - 2k)^+ + 2(Z_N - k)^+ \leq 3(Z_N - k)^+$$

$$= 3 \sum_{n=1}^{N} [(Z_n - k)^+ - (Z_{n-1} - k)^+] \leq 3 \sum_{n=1}^{N} Y_n 1_{\{Z_n > k\}}$$

Lemma 9. Let $\mathcal{F}$ be linearly ordered. A submartingale is of class $DM$ if and only if it is of class $D$.

Proof. Of course, if $X$ is a submartingale, then $S^d(X) \geq 0$. Assume that $X$ is of class $D$. By the linearity of the order on $\mathcal{F}$, we may assume that each $d \in \mathcal{D}$ is of the form $d = \{\tau_n \cap \tau_{n+1} : n = 1, \ldots, N\}$ with $\tau_n < \tau_{n+1}$. Let $F_n = \{\sum_{j=1}^{n} P(X_{\tau_{j+1}} \mid \mathcal{F}_{\tau_j}) - X_{\tau_j} > k \geq \sum_{j=1}^{n-1} P(X_{\tau_{j+1}} \mid \mathcal{F}_{\tau_j}) - X_{\tau_j}\}$ and $\sigma_k = \sum_{n=1}^{N} \tau_n 1_{F_n}$. (47) implies

$$P(S^d(X) 1_{\{S^d(X) > 2k\}}) \leq 3P((X_{\tau_{N+1}} - X_{\sigma_k}) 1_{\{S^d(X) > k\}}) \leq 6 \sup_{\sigma \in \Sigma} P(|X_\sigma| 1_{\{S^d(X) > k\}})$$

Choose $k = 0$ in (48): $S(X)$ is norm bounded in $L^1$ and therefore $\lim_{k \to \infty} \sup_{d \in \mathcal{D}} P(S^d(X) > k) = 0$. Since $D(X)$ is uniformly integrable, then so is $S(X)$. Conversely, assume that $X$ is of class $DM$ and thus, being a supermartingale, a quasi martingale. Let $\sigma = \sum_{n=1}^{N} \tau_n 1_{F_n}$ with $\tau_1 < \tau_2 < \ldots < \tau_{N-1} < \emptyset$ and $F_N = \bigcap_{n=1}^{N-1} F_n^c$. Set $F_0 = \emptyset$, $F_n = \bigcup_{j=1}^{n} F_j$ and $d = \{\tau_n \cap \tau_{n+1} : n = 1, \ldots, N - 1\}$. 


Choose $\tau_N \in \mathcal{T}^p_{N-1}$ arbitrarily. Then, since $\bar{F}_n \cap \{X_\sigma > k\} \in \mathcal{F}_{\tau_N}$ we have

$$X_\sigma 1_{\{X_\sigma > k\}} = X_\emptyset 1_{\bar{F}_n} 1_{\{X_\sigma > k\}} + X_{\tau_N} 1_{\bar{F}_n} 1_{\{X_\sigma > k\}} - \sum_{n=1}^{N-1} (X_{\tau_n+1} - X_{\tau_n}) 1_{\{X_\sigma > k\}} 1_{\bar{F}_n}$$

and likewise

$$-X_\sigma 1_{\{X_\sigma < -k\}} = -X_\emptyset 1_{\bar{F}_n} 1_{\{X_\sigma < -k\}} - X_{\tau_N} 1_{\bar{F}_n} 1_{\{X_\sigma < -k\}} + \sum_{n=1}^{N-1} (X_{\tau_n+1} - X_{\tau_n}) 1_{\{X_\sigma < -k\}} 1_{\bar{F}_n}$$

In other words, $P(|X_\sigma| 1_{\{X_\sigma > k\}}) \leq P(1_{\{X_\sigma > k\}}(|X_\sigma| + S^d(X) + |X_{\tau_N}|))$ where $d = \{\tau_n \cap c_{n+1} : n = 1, \ldots, N - 1\} \in \mathcal{D}$. Since $\tau_N$ may be chosen arbitrarily large, we conclude from Corollary [6]

$$P(|X_\sigma| 1_{\{X_\sigma > k\}}) \leq P(1_{\{X_\sigma > k\}}(|X_\sigma| + |M|)) + \sup_{d \in \mathcal{D}} P(1_{\{X_\sigma > k\}} S^d(X))$$

Since $X_\emptyset, M \in L^1$ and $S(X)$ is bounded in $L^1$, we deduce that $D(X)$ is also bounded in $L^1$ and uniformly integrable too.

Not only does the above property provide a natural extension of the class $D$ property, at least as far as submartingales or supermartingales are concerned (the claim may in fact fail outside of this class); it is also the more convenient one in order to prove the existence of Doob Meyer decompositions. In fact,

**Theorem 7.** A stochastic process $X$ is of class $DM$ if and only if for each ultrafilter $\mathcal{U}$ on $\mathcal{D}$ there is a decomposition

$$X_\tau = P(M^\mathcal{U}|\mathcal{F}_\tau) - A_\tau^\mathcal{U} \quad \tau \in \mathcal{T}_p, \ P \text{ a.s.}$$

where $M^\mathcal{U} \in L^1$ and $A_\mathcal{U}$ is a process of class $DM$ such that $A_\emptyset^\mathcal{U} = M^\mathcal{U}$ and

$$P(A_\tau^\mathcal{U} 1_F) = u - \lim P \int_{\tau \in \mathcal{T}_p} \mathcal{D}_p(F) dA_\tau^\mathcal{U} \quad \tau \in \mathcal{T}_p, \ F \in \mathcal{F}$$

The decomposition (51) is unique. If $X$ is a supermartingale and $\tau \leq \nu \notin \mathcal{D}$ then $A_\tau^\mathcal{U} \leq A_\nu^\mathcal{V} \ P \text{ a.s.}$.

**Proof.** The proof is essentially the same as that of Theorem [6]. In fact we get from (42) the decomposition $P(X_\tau 1_F) = \mu_\mathcal{X}(P(F|\mathcal{F}_\tau)) - \alpha_{X,\tau}(F)$ for all $F \in \mathcal{F}$ and $\tau \in \mathcal{T}_p$, with $\alpha_{X,\tau} = u - \lim \alpha_{X,\tau}^d$. Looking at the collection $\{\mu_\mathcal{X}(F) : d \in \mathcal{D}\}$ as a function mapping $\mathcal{D}$ into $\mathbb{R}$, it is easy to conclude that ultrafilter limits are additive and monotone. This proves that indeed $\mu_\mathcal{X}$ is an additive set function. Choosing $k$ large enough so that $\sup_{d \in \mathcal{D}} P(|S^d(X)| 1_{\{S^d(X) > k\}}) < \epsilon$, we conclude by monotonicity that if $h \in \mathcal{F}(\mathcal{F})$ with $|h| \leq 1$, then $|\mu_\mathcal{X}(h)| \leq \epsilon + kP(|h|)$. In other words $\mu_\mathcal{X} \ll P$. Then, $\alpha_{X,\tau} \ll P$ by (42). Decomposition (51), property (51), adaptness of $A$ and uniqueness then easily follow. Conversely, if (51) holds with $A$ being of class $DM$, then the same must be true of $X$ since $S^d(X) = -S^d(A^\mathcal{U})$ for all $d \in \mathcal{D}$.

The main insight provided by Theorem 7 seems to be that, in the context considered here, there exists a multiplicity of suitable compensating schemes, each involving a different notion of naturality the choice among which appears as rather arbitrary.
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