Abstract
We present an orthogonal matrix outer product decomposition for the fourth-order conjugate partial-symmetric (CPS) tensor and show that the greedy successive rank-one approximation (SROA) algorithm can recover this decomposition exactly. Based on this matrix decomposition, the CP rank of CPS tensor can be bounded by the matrix rank, which can be applied to low-rank tensor completion. Additionally, we give the rank-one equivalence property for the CPS tensor based on the SVD of matrix, which can be applied to the rank-one approximation for CPS tensors.

Keywords Conjugate partial-symmetric tensor · Approximation algorithm · Rank-one equivalence property · Convex relaxation

Mathematics Subject Classification 15A69 · 15B57 · 90C26 · 41A50

1 Introduction
In this paper, we set out to study the approximation of the fourth-order tensors with conjugate partial-symmetricity, based on the so-called matrix outer product decomp-
position. Tensor decomposition and approximation have significant applications in computer vision, data mining, statistical estimation and so on, see for instance [1–3]. It is well-known that most tensor problems are NP-hard [4]. Extensive research focuses on tensors with special structures, exploring the nature like symmetricity, to help design tensor approximation algorithms for practical applications [5–7]. Recently, Jiang et al. [8] proposed the conjugate partially symmetric (CPS) tensor to characterize the functions in multivariate complex variables which always take real values. Ni et al. [9, 10] then generalized the concept of CPS tensors to Hermitian tensors and studied their decompositions. CPS tensors are encountered in various real applications such as blind identification [2], radar waveform design [11], beampattern optimization [12] and the power system state estimation [13], in forms of complex polynomial optimization problems.

CPS tensors, as a generalization of Hermitian matrices, inherit many nice properties. Fu et al. [14] studied the decomposition and rank-one approximation properties of the CPS tensor. Furthermore, the CPS tensor can be unfolded into a Hermitian matrix with careful matricization, which lead to a decomposition that inherits the nice properties of matrix factorization, the matrix outer product decomposition. In fact, such a decomposition or the idea to study tensor decomposition based on the matricization of tensors has been studied extensively. Lieven [15] proposed an algorithm to obtain the canonical components of tensors from a simultaneous matrix diagonalization. Jiang et al. [16] defined a new tensor rank, $M$-rank, based on the special matricization of tensors and discussed the connections between it and the CP rank, which help to design a new approach for the low-rank tensor completion and tensor robust PCA. On the other hand, many multi-dimensional data from real practice are fourth-order tensors, corresponding to the fourth-order complex polynomial functions, including the quartic minimization from radar waveform design [11], quadratic eigenvalue problem [17] and so on. Motivated by Lieven et al. [2], we focus on the orthogonal matrix outer product decomposition model for CPS tensors, which further explore the orthogonality of these matrices.

It is well-known that the best rank-$r$ ($r > 1$) approximation of a general tensor may not exist, and even if it admits a solution, it is NP-hard to solve [18]. It is very different from the matrix case, which has the well-known Eckart–Young theorem, thanks to the orthogonality of the matrix decomposition. The greedy successive rank-one approximation (SROA) algorithm is still used to compute the rank-$r$ ($r > 1$) approximation of tensor [19]. However, the theoretical guarantee for obtaining the best rank-$r$ approximation is less developed. Some progress has been made based on the orthogonality assumption for different tensor decompositions. For example, Zhang et al. [20] first proved that the successive algorithm exactly recovers the symmetric and orthogonal decomposition of the underlying real symmetrically and orthogonally decomposable tensors. Fu et al. [21] showed that SROA algorithm can exactly recover unitarily decomposable CPS tensors. We wonder if there is a theoretical guarantee for the SROA algorithm based on the orthogonality of the matrix decomposition of CPS tensors.

Among the low-rank approximation of tensors, the best rank-one approximation is of particular interest. Various methods have been proposed to find local or approximation solutions [22–24]. Recently, Jiang et al. [3] proposed convex relaxations for
solving a tensor optimization problem closely related to the best rank-one approximation problem for symmetric tensors and showed the possibility to find the global optimum. They proved an equivalence property between a rank-one symmetric tensor and its unfolding matrix. Yang et al. [25] studied the rank-one equivalence property for general real tensors. Based on these rank-one equivalence properties, the above-mentioned tensor optimization problem can be casted into a matrix optimization problem, which alleviates the difficulty of solving the tensor problem. Fu et al. [14] further studied the rank-one equivalence property in terms of general CPS tensors. In line with this idea, we prove the rank-one equivalence property for the fourth-order CPS tensor particularly and transform the best rank-one tensor approximation problem into a matrix optimization problem.

The remainder of this paper is organized as follows. In Sect. 2, we give some notations and definitions. The outer product approximation model based on matrix is proposed and the successive rank-one approximation (SMROA) algorithm is given to solve it in Sect. 3. We show that the SMROA algorithm can exactly recover the matrix outer product decomposition or approximation of the CPS tensor in Sect. 4. Section 5 discusses applications of our model simply. In Sect. 6, we present the rank-one equivalence property of fourth-order CPS tensor, and based on it an application is proposed. Numerical examples are in Sect. 7.

2 Preliminary

All tensors in this paper are fourth-order. For any complex number $z = a + ib \in \mathbb{C}$, $ar{z} = a - ib$ denotes the conjugate of $z$. “$\circ$” denotes the outer product of matrices, namely $A = X \circ Y$ means that

$$ A_{ijkl} = X_{ij} Y_{kl}. $$

$S^n$ denotes the set of $n$ by $n$ symmetric matrices, the entries of these matrices can be complex or only real according to the context, without causing ambiguity. The inner product between $A, B \in \mathbb{C}^{n^4}$ is defined as

$$ \langle A, B \rangle = \sum_{i,j,k,l=1}^{n} A_{ijkl} B_{ijkl}. $$

**Definition 1** A fourth-order tensor $A$ is called symmetric if $A$ is invariant under all permutations of its indices, i.e.,

$$ A_{ijkl} = A_{\pi(ijkl)}, \quad i, j, k, l = 1, \cdots, n. $$

**Definition 2** [9] A fourth-order complex tensor $A \in \mathbb{C}^{n_1 \times n_2 \times n_1 \times n_2}$ is called a Hermitian tensor if

$$ A_{i_1i_2j_1j_2} = A_{j_1j_2i_1i_2}. $$
Jiang et al. [8] introduced the concept of conjugate partial-symmetric tensors as follows.

**Definition 3** A fourth-order complex tensor \( A \in \mathbb{C}^{n^4} \) is called conjugate partial-symmetric (CPS) if
\[
A_{ijkl} = \overline{A}_{klij}, \quad i, j, k, l = 1, \ldots, n.
\]

**Definition 4** A fourth-order tensor \( A \in \mathbb{R}^{n^4} \) is called partial-symmetric if
\[
A_{ijkl} = A_{π(ik)π(jl)} = A_{π(kl)π(ij)}, \quad i, j, k, l = 1, \ldots, n.
\]

**Example 1** [2] In the blind source separation problem, the cumulant tensor is computed as
\[
C = \sum_{r=1}^{R} k_r a_r \circ \overline{a}_r \circ \overline{a}_r \circ a_r.
\]

By a permutation of the indices, it is in fact a conjugate partial-symmetric tensor.

**Definition 5** The square unfolding form \( M(A) \) of a fourth-order tensor \( A \in \mathbb{C}^{n^4} \) is defined as
\[
M(A)_{(j-1)n+s, (l-1)n+s} = A_{ijkl},
\]

where \( (U_i)_{st} = (u_i)_{(t-1)n+s}, \quad (V_i)_{st} = (v_i)_{(t-1)n+s}, \) for \( i = 1, 2, \ldots, n^2 \).

**3 Matrix Outer Product Approximation Model**

Jiang et al. [16] introduced the new notions of M-decomposition for an even-order tensor \( A \), which is exactly the rank-one decomposition of \( M(A) \), followed by the notion of tensor M-rank.

For each \( A \in \mathbb{R}^{n^4} \), let \( M(A) = UΣV^T = \sum_{i=1}^{n^2} \sigma_i u_i \circ v_i \) be the SVD of \( M(A) \), then \( A \) has the following decomposition form
\[
A = \sum_{i=1}^{n^2} \sigma_i U_i \circ V_i,
\]

where \( (U_i)_{st} = (u_i)_{(t-1)n+s}, \quad (V_i)_{st} = (v_i)_{(t-1)n+s}, \) for \( i = 1, 2, \ldots, n^2 \).

We are particularly interested in the tensor with some symmetric properties. And analogous to Lieven et al. [2], we prove that the CPS tensor has a decomposition based on matrix as follows.

**Theorem 1** If \( A \in \mathbb{C}^{n^4} \) is a conjugate partial-symmetric tensor, then it can be decomposed as
\[
A = \sum_{i=1}^{r} \lambda_i E_i \circ \overline{E}_i,
\]
where \( \lambda_i \in \mathbb{R} \), \( E_i \in \mathbb{C}^{n^2 \times n^2} \) are symmetric matrices and \( \langle E_i, E_j \rangle = \delta_{ij} \), for \( i, j = 1, 2, \ldots, r \). And the decomposition is unique when \( \lambda_i \) are different from each other.

**Proof** Since \( \mathcal{A} \) is conjugate partial-symmetric, then the unfold matrix \( M(\mathcal{A}) \) is Hermitian, and can be decomposed as

\[
M(\mathcal{A}) = \sum_{i=1}^{r} \lambda_i e_i e_i^*,
\]

where \( \lambda_i \in \mathbb{R}, e_i \in \mathbb{C}^{n^2}, i = 1, \ldots, r \) are mutually orthogonal. Folding \( e_i \) into matrix \( E_i \) via \( (E_i)_{ij} = (e_i)_{(j-1)\times n+i} \), thus \( E_i, i = 1, \ldots, p \) are mutually orthogonal, that is, \( \langle E_i, E_j \rangle = \delta_{ij} \). In this case, we have \( \mathcal{A} = \sum_{i=1}^{r} \lambda_i E_i \circ \bar{E}_i \).

From the eigen-decomposition of \( M(\mathcal{A}) \), we have \( M(\mathcal{A})e_{\tau} = \lambda_{\tau} e_{\tau} \), for \( \tau = 1, \ldots, r \), i.e., \( \sum_{k,l} a_{ijkl}(e_{\tau})_{(l-1)\times n+k} = \lambda_{\tau} (e_{\tau})_{(j-1)\times n+i} \), for any \( i, j = 1, \ldots, n \). Since \( a_{ijkl} = a_{jikl} \), for all \( i, j = 1, \ldots, n \), then \( (e_{\tau})_{(j-1)\times n+i} = (e_{\tau})_{(i-1)\times n+j} \), thus \( E_{\tau} \) is symmetric. The uniqueness of the decomposition follows the property of eigen-decomposition of Hermitian matrix naturally.

**Remark 1** Jiang et al. [8] gave the decomposition theorem for CPS tensor like Theorem 3. However, they established this theorem in the view of polynomial decomposition and did not explore the mutually orthogonality of matrices in the decomposition model.

**Definition 6** \( \mathcal{A} \in \mathbb{C}^{n^4} \) is a CPS tensor,

\[
\text{rank}_M(\mathcal{A}) = \min \left\{ r \mid \mathcal{A} = \sum_{i=1}^{r} \lambda_i E_i \circ \bar{E}_i, \lambda_i \in \mathbb{R}, E_i \in S^{n^2} \right\}.
\]

The \( \text{rank}_M(\mathcal{A}) \) is actually the strongly symmetric M-rank \( \text{rank}_{ssm}(\mathcal{A}) \) defined by Jiang [16]. For symmetric tensor \( \mathcal{A} \), they also proved the equivalence between \( \text{rank}_{ssm}(\mathcal{A}) \) and

\[
\text{rank}_{ssm}(\mathcal{A}) = \min \left\{ r \mid \mathcal{A} = \sum_{i=1}^{r} \lambda_i E_i \circ E_i, \lambda_i \in \mathbb{R}, E_i \in \mathbb{C}^{n^2} \right\}.
\]

This is also true for CPS fourth-order tensor.

**Theorem 2** Let \( \mathcal{A} \in \mathbb{C}^{n^4} \) be a CPS tensor, then \( \text{rank}_M(\mathcal{A}) = \text{rank}_{ssm}(\mathcal{A}) \).

**Proof** It is obvious that \( \text{rank}_M(\mathcal{A}) \geq \text{rank}_{ssm}(\mathcal{A}) \). On the other hand, if \( \text{rank}_{ssm}(\mathcal{A}) = r \), we have \( \text{rank}(M(\mathcal{A})) \leq r \). Since \( \text{rank}(M(\mathcal{A})) = \text{rank}_M(\mathcal{A}) \), we obtain the desired conclusion.
Corollary 3 Let $A \in \mathbb{R}^{n^4}$ be a partial-symmetric tensor, then one has

$$A = \sum_{i=1}^{r} \lambda_i E_i \circ E_i,$$  \hspace{1cm} (3)

where $\lambda_i \in \mathbb{R}$, $E_i$ are symmetric matrices and $[E_i, E_j] = \delta_{ij}$, for $i, j = 1, 2, \ldots, r$. 

$\text{rank}_M(A) \leq \frac{n(n+1)^2}{2}$. 

Proof The first part is obvious according to Theorem 1. Since all matrices belonging to $S^n$ form a $\frac{n(n+1)}{2}$-dimensional vector space, we have $\text{rank}_M(A) \leq \frac{n(n+1)^2}{2}$.

Fu et al. gave a rank-one decomposition of vector form for the CPS tensor based on Theorem 1 as follows.

Theorem 4 [14, Theorem 3.2] $A \in \mathbb{C}^{n^4}$ is CPS if and only if it has the following partial-symmetric decomposition

$$A = \sum_i \lambda_i \bar{a}_i \circ \bar{a}_i \circ a_i \circ a_i,$$

where $\lambda_i \in \mathbb{R}$ and $a_i \in \mathbb{C}^n$, that is, a CPS tensor can be decomposed as the sum of rank-one CPS tensors.

However, when we restricted the decomposition on real domain, the decomposition does not seem to hold, since $\sum_i \lambda_i a_i \circ a_i \circ a_i \circ a_i$, where $\lambda_i \in \mathbb{R}$, $a_i \in \mathbb{R}^n$, can only represent symmetric tensor. Thus, an extended rank-one approximation model for the partial-symmetric tensor can be proposed based on Corollary 3.

Corollary 5 Let $A \in \mathbb{R}^{n^4}$ be a partial-symmetric tensor, then it can be decomposed as the sum of simple low rank partial-symmetric tensor,

$$A = \sum_i \lambda_i (p_i \circ p_i \circ q_i \circ q_i + q_i \circ q_i \circ p_i \circ p_i).$$  \hspace{1cm} (4)

Proof From Corollary 3, partial-symmetric tensor $A = \sum_{i=1}^{r} \lambda_i E_i \circ E_i$, where $E_i$ are symmetric. So it can be decomposed as $\sum_{j=1}^{r_i} \beta_i^j u_i^j(\beta_i^j)^\top$, thus

$$A = \sum_{i=1}^{r} \lambda_i \left( \sum_{j=1}^{r_i} \beta_i^j u_i^j(\beta_i^j)^\top \right) \circ \left( \sum_{k=1}^{r_i} \beta_i^k u_i^k(\beta_i^k)^\top \right).$$

$$= \sum_{i=1}^{r} \lambda_i \left( \sum_{j=1}^{r_i} \sum_{k=1}^{r_i} \beta_i^j \beta_i^k (u_i^j \circ u_i^j \circ u_i^k \circ u_i^k + u_i^k \circ u_i^k \circ u_i^j \circ u_i^j) \right).$$

The desired decomposition form follows.
**Remark 2** From the proof of Corollary 5, we can see that if \( p_i \neq q_i \), \( p_i^\top q_i = 0 \). Whether this decomposition form is the compactest will be one of our future work.

We can discuss the case of skew partial-symmetric tensor in parallel.

**Theorem 6** We call \( A \in \mathbb{R}^{n^4} \) skew partial-symmetric tensor if

\[
A_{ijkl} = A_{\pi(i)\pi(j)\pi(k)\pi(l)} = -A_{\pi(k)\pi(l)\pi(i)\pi(j)}, \quad i, j, k, l = 1, 2, \ldots, n.
\]

Then, one has

\[
A = \sum_i \lambda_i (U_i \circ V_i - V_i \circ U_i)
\]

and

\[
A = \sum_i \lambda_i (p_i \circ p_i \circ q_i \circ q_i - q_i \circ q_i \circ p_i \circ p_i).
\]

**Proof** \( M(A) \) is skew-symmetric according to the definition of the skew partial-symmetric tensor. Then, \( M(A) = \sum_i \lambda_i (u_i v_i^\top - v_i u_i^\top) \). The rest of the proof is similar to that for partial-symmetric tensor, here we omit it.

Based on Theorem 1, we propose a matrix outer product approximation model for the CPS tensor as follows:

\[
\min_{\lambda_i \in \mathbb{R}, \ X_i \in S^n} \| A - \sum_{i=1}^r \lambda_i X_i \circ \bar{X}_i \|_F^2 \quad \text{subject to} \quad \langle X_i, \ X_j \rangle = \delta_{ij}.
\]

(5)

The main optimization problem in Algorithm 1 could be expressed as

\[
(\lambda_*, X_*) \in \arg \min_{\|X\|_F = 1, X \in S^n, \lambda \in \mathbb{R}} \| A - \lambda X \circ \bar{X} \|_F^2.
\]

(6)

**Algorithm 1** Successive Matrix Outer Product Rank-One Approximation (SMROA) Algorithm

Given a CPS tensor \( A \in \mathbb{C}^{n^4} \). Initialize \( A_0 = A \).

for \( j = 1 \) to \( r \) do

\[
(\lambda_j, X_j) \in \arg \min_{\|X\|_F = 1, X \in S^n, \lambda \in \mathbb{R}} \| A_{j-1} - \lambda X \circ \bar{X} \|_F.
\]

\[
A_j = A_{j-1} - \lambda_j X_j \circ \bar{X}_j.
\]

end for

Return \( \{\lambda_j, X_j\}_{j=1}^r \)
The objective function of (6) can be rewritten as

\[ \| A - \lambda X \circ \bar{X} \|_F^2 = \| A \|_F^2 + \lambda^2 - 2\lambda \langle A, X \circ \bar{X} \rangle, \]

from which we can derive that problem (6) is equivalent to

\[ X^* \in \arg\max_{\|X\|_F=1, X \in S^n} | \langle A, X \circ \bar{X} \rangle |, \tag{7} \]

and \( \lambda^* = \langle A, X^* \circ \bar{X}^* \rangle \). We can solve (7) by transforming it into matrix eigenproblem as follows:

\[ x^* \in \arg\max_{\|x\|_2=1, x \in \mathbb{C}^n} | x^* M(A)x |. \tag{8} \]

**Remark 3** Zhang et al. [6] proved that if \( A \in \mathbb{R}^{n^4} \) is symmetric,

\[ \min_{\|x_i\|_2=1} \| A - \lambda x_1 \circ x_2 \circ x_3 \circ x_4 \|_F = \min_{\|x\|_2=1} \| A - \lambda x \circ x \circ x \circ x \|_F; \]

if \( A \) is symmetric about the first two and the last two mode, respectively,

\[ \min_{\|x_i\|_2=1} \| A - \lambda x_1 \circ x_2 \circ x_3 \circ x_4 \|_F = \min_{\|x\|_2=1, \|y\|_2=1} \| A - \lambda x \circ x \circ y \circ y \|_F. \]

It is obvious that for partial-symmetric tensor, we also have

\[ \min_{\|X_i\|_F=1, X_i \in \mathbb{R}^{n^2}, \lambda \in \mathbb{R}} \| A - \lambda X_1 \circ X_2 \|_F = \min_{\|X\|_F=1, X \in S^n, \lambda \in \mathbb{R}} \| A - \lambda X \circ X \|_F. \]

**Remark 4** It is well-known that (6) is equivalent to the nearset Kronecker product problem [26] as follows:

\[ (\lambda^*, X^*) \in \arg\min_{\|X\|_F=1, X \in S^n, \lambda \in \mathbb{R}} \| A - \lambda X \otimes \bar{X} \|_F^2, \]

where \( A_{(i-1)n+k, (j-1)n+l} = A_{ijkl} \), "\( \otimes \)" denotes the kronecker product of matrices.

### 4 Exact Recovery for CPS Tensors

In this section, we give the theoretical analysis of exact recovery for CPS tensors by the SMROA algorithm.
Theorem 7 Let $A$ be a CPS tensor with $\text{rank}_M(A) = r$, that is,

$$A = \sum_{i=1}^{r} \lambda_i E_i \circ \tilde{E}_i.$$ 

If $\lambda_i$ are different from each other, then the SMROA algorithm will obtain the exact decomposition of $A$ after $r$ iterations.

We first claim the following lemma before proving the above theorem.

Lemma 8 Let $A$ be a CPS tensor with $\text{rank}_M(A) = r$, that is,

$$A = \sum_{i=1}^{r} \lambda_i E_i \circ \tilde{E}_i.$$ 

$\lambda_i$ are different from each other. Suppose

$$\hat{X}_1 \in \arg\max_{X \in S^n, \|X\|_F = 1} |\langle A, X \circ \tilde{X} \rangle|, \quad \hat{\lambda}_1 = \langle A, X \circ \tilde{X} \rangle.$$ 

Then, there exists $j \in \{1, 2, \cdots, r\}$ such that

$$\hat{\lambda}_1 = \lambda_j, \quad \hat{X}_1 = E_j.$$ 

Proof According to Theorem 1, $E_i$ is mutually orthogonal, thus $\{E_1, \cdots, E_r\}$ is a subset of an orthonormal basis $\{E_1, \cdots, E_{\frac{n(n+1)}{2}}\}$ of $S^n$ and $0 \neq \lambda_i \in \mathbb{R}$. Let $\hat{X}_1 = \sum_{i=1}^{r} x_i E_i$, where $x_i = \langle \hat{X}_1, E_i \rangle$ for $i = 1, 2, \cdots, \frac{n(n+1)}{2}$. Since $\|\hat{X}_1\|_F = 1$, we have $\sum_{i=1}^{r} |x_i|^2 = 1$. Reorder the indices such that

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_r|.$$ 

Then, we obtain

$$|\langle A, \hat{X}_1 \circ \tilde{\hat{X}} \rangle| = \sum_{i=1}^{r} |\lambda_i| |x_i|^2 \leq |\lambda_1|.$$ 

On the other hand, the optimality leads to

$$|\langle A, \hat{X}_1 \circ \tilde{\hat{X}} \rangle| \geq |\langle A, \xi_1 \circ \tilde{E}_1 \rangle| = |\lambda_1|.$$ 

Hence,

$$|\lambda_1| \leq |\langle A, \hat{X}_1 \circ \tilde{\hat{X}} \rangle| = |\hat{\lambda}_1| \leq |\lambda_1|.$$
So, $|\hat{\lambda}_1| = |\lambda_1|$, $|x_1| = 1$. Therefore, $\hat{X}_1 = e^{i\theta}E_1$, for any $\theta \in [0, 2\pi]$, and

$$\hat{\lambda}_1 = \left\langle A, \hat{X}_1 \circ \hat{X}_1 \right\rangle = \left\langle A, E_1 \circ \hat{E}_1 \right\rangle = \lambda_1.$$ 

Then, let $x_1 = 1$, we have $\hat{X}_1 = E_1$.

Now, we prove Theorem 7.

**Proof** By Lemma 8, there exists $j \in \{1, 2, \cdots, r\}$ such that $\hat{X}_1 = E_j$, $\hat{\lambda}_1 = \lambda_j$. Let

$$A_1 = A - \hat{\lambda}_1 \hat{X}_1 \circ \hat{X}_1 = \sum_{i \neq j} \lambda_i E_i \circ \hat{E}_i,$$

and

$$\hat{X}_2 \in \arg\max_{X \in S^n, \|X\|_F = 1} \left| \left\langle A_1, X \circ \bar{X} \right\rangle \right|, \quad \hat{\lambda}_2 = \left\langle A, \hat{X}_2 \circ \bar{X}_2 \right\rangle.$$ 

By the similar proof of Lemma 8, we know that there exists a $k \in \{1, 2, \cdots, n\} \setminus \{j\}$ such that $\hat{\lambda}_2 = \lambda_k$, $\hat{X}_2 = E_k$. Repeatedly, we can induce a permutation $\pi$ on $\{1, 2, \cdots, r\}$ such that

$$\hat{\lambda}_j = \lambda_{\pi(j)}, \quad \hat{X}_j = E_{\pi(j)}, \quad j = 1, 2, \cdots, r.$$ 

**Corollary 9** Let

$$A = \sum_{i = 1}^{r} \lambda_i E_i \circ \bar{E}_i,$$

where $\{E_1, \cdots, E_r\}$ is a subset of an orthonormal basis $\{E_1, \cdots, E_{n(n+1)/2}\}$ of $S^n$. $0 \neq \lambda_i \in \mathbb{R}$ are different from each other, and are ordered as

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_r|.$$ 

Suppose $\{(\hat{\lambda}_i, \hat{X}_i)\}_{i=1}^{n}$ is the output of the SMROA algorithm for input $A$. Then, $\hat{\lambda}_i = \lambda_i$, $\hat{X}_i = E_i$, for $i = 1, 2, \cdots, r$. In particular, if $\lambda_i > 0$, we have $\hat{\lambda}_1 > \hat{\lambda}_2 > \cdots > \hat{\lambda}_r$; if $\lambda_i < 0$, we have $\hat{\lambda}_1 < \hat{\lambda}_2 < \cdots < \hat{\lambda}_r$.

This proposition directly follows from the proof of Lemma 8.

**Remark 5** According to the proof in Lemma 8, if $A$ is a partial-symmetric tensor with entries whose imaginary part is not zero, the SMROA algorithm may fail. This is because that if $\hat{X}_1 = e^{i\theta}X_1$, $\hat{X}_1 \circ \hat{X}_1 \neq X_1 \circ X_1$. 

$\square$ Springer
5 Applications of Matrix Outer Product Model

5.1 Low-CP-Rank Tensor Completion

Definition 7 [1] The CP rank of the tensor $A \in \mathbb{C}^{n^4}$, denoted by $\text{rank}_{CP}(A)$, is the smallest number $r$ such that $A$ can be factorized as the following decomposition:

$$ A = \sum_{i=1}^{r} a_i^1 \circ a_i^2 \circ a_i^3 \circ a_i^4, $$

where $a_i^k \in \mathbb{C}^n$ for $k = 1, 2, 3, 4$ and $i = 1, 2, \ldots, r$.

The following theorem shows that the CP rank of the CPS tensor $A$ can be bounded by $\text{rank}_{M}(A)$.

Theorem 10 For CPS tensor $A \in \mathbb{C}^{n^4}$, $A = \sum_{i=1}^{\text{rank}_{M}(A)} \lambda_i E_i \circ \bar{E}_i$, it holds that

$$ \text{rank}_{M}(A) \leq \text{rank}_{CP}(A) \leq r^2 \text{rank}_{M}(A), $$

where $r = \max_{i} \{\text{rank}(E_i)\}$.

Proof Suppose $A$ can be decomposed as $A = \sum_{i=1}^{R} a_i^1 \circ a_i^2 \circ a_i^3 \circ a_i^4$, where $R = \text{rank}_{CP}(A)$. Then, $M(A)$ can be decomposed as $M(A) = \sum_{i=1}^{R} (a_i^1 \otimes a_i^4) \otimes (a_i^2 \otimes a_i^3)$. Thus, we obtain that $\text{rank}_{M}(A) \leq \text{rank}_{CP}(A)$. On the other hand, since $A = \sum_{i=1}^{\text{rank}_{M}(A)} \lambda_i E_i \circ \bar{E}_i$ and $r = \max_{i} \{\text{rank}(E_i)\}$, we have $A = \sum_{i=1}^{\text{rank}_{M}(A)} \lambda_i (\sum_{k_i=1}^{r_i} \bar{E}_i \circ E_i) \circ (\sum_{k_i=1}^{r_i} \bar{E}_i \circ E_i)$, where $r_i \leq r$ for $i = 1, 2, \ldots, \text{rank}_{M}(A)$. Thus, $\text{rank}_{CP}(A) \leq r^2 \text{rank}_{M}(A)$.

Data in real word usually can be modeled as low-CP-rank tensors, such as the colored video data with static background [16]. So the completion problem for this kind of data can be formulated as

$$ \min \ \text{rank}_{CP}(\mathcal{X}) \quad \text{s.t.} \quad P_{\Omega}(\mathcal{X}) = P_{\Omega}(A). \quad (10) $$

(10) is intractable to deal with directly, since the CP rank of a tensor is generally hard to estimate. Here, we follow the idea of Jiang et al. [16] to cope with the completion problem of partial-symmetric tensor. According to Theorem 10, we may approximate it by

$$ \min \ \text{rank}_{M}(\mathcal{X}) \quad \text{s.t.} \quad P_{\Omega}(\mathcal{X}) = P_{\Omega}(A). \quad (11) $$
The following example gives an intuitive explanation for the rationality of the approximation.

**Example 2** Suppose \( \mathcal{A} \in \mathbb{R}^{10^4} \), \( \mathcal{A} = \sum_{i=1}^{4} \lambda_i (x_i \odot x_i \odot y_i \odot y_i + y_i \odot y_i \odot x_i \odot x_i) \) is partial-symmetric. Then, \( \text{rank}_M(\mathcal{A}) \leq \text{rank}_{\mathcal{CP}}(\mathcal{A}) \leq 8 \), while \( M(\mathcal{A}) \in \mathbb{R}^{100 \times 100} \) is a low-rank matrix.

(11) is relaxed as a low-rank approximation problem with nuclear norm regular term,

\[
\min \mu \|X\|_* + \frac{1}{2} \|P_{\Omega}(X) - P_{\Omega}(\mathcal{A})\|_F^2
\]  

s.t. \( X = M(X) \).

\( X \) in above problems is required to be partial-symmetric. And the sample set \( \Omega \) is partial-symmetric.

We can apply the fixed point continuation (FPC) algorithm [27] to solve (12). 

\( \mu_{k+1} = \eta \mu_k \), where \( 0 < \eta < 1 \). The convergence of this algorithm is guaranteed [27]. Since the iteration in Algorithm 2 does not change the symmetric property of \( X_k \) when \( \Omega \) and \( \mathcal{A} \) are partial-symmetric, the solution \( X_* \) is still partial-symmetric.

---

**Algorithm 2** FPC for (12)

Given a PS tensor \( \mathcal{A} \in \mathbb{R}^{n^4} \). Initialize \( \lambda_0 = 0 \). Given parameter \( \tau, \mu_1, \mu_L \)

for \( \mu = \mu_1, \cdots, \mu_L \) do

for \( k = 1, 2, \cdots \) do

\( Y_k = X_{k-1} - \tau P_{\Omega}(X_{k-1} - A) \),

\( M(X)_k = D_{\tau \mu}(M(Y)_k) \).

end for

end for

Return \( \lambda_k \).

---

### 5.2 Low-Rank Matrix Outer Product Approximation

Parallel to the sparse rank-one approximation problem, we can also discuss the low-rank matrix outer product “rank-one” approximation as follows based on the matrix decomposition model proposed in last section.

\[
\min \|A - \alpha X \circ X\|_F^2 + \lambda \|X\|_* \\
\text{s.t. } \alpha \in \mathbb{R}, \quad X \in S^n, \quad \|X\|_F = 1,
\]  

where \( \mathcal{A} \in \mathbb{R}^{n^4} \) is partial-symmetric.
We modify the proximal linearized minimization algorithm (PLMA) proposed by Bolte et al. [28] to solve problem (13). The iterative scheme is

\[
\hat{X}_{k+1} \in \arg\min_{X} \left\{ f(\alpha_k, X_k) + \langle X - X_k, \nabla_X f(\alpha_k, X_k) \rangle + \frac{t_k}{2} \| X - X_k \|^2_F + \lambda \| X \|_* \right\},
\]

\[
X_{k+1} = \frac{\hat{X}_{k+1}}{\| \hat{X}_{k+1} \|_F},
\]

\[
\alpha_{k+1} = \langle A, X_{k+1} \circ X_{k+1} \rangle.
\]

(14)

where \( t_k > 0, f(\alpha, X) = \| A - \alpha X \circ X \|^2_F \) and

\[
\nabla_X f(\alpha, X) = 4\alpha^2 \| X \|^2_F X - 4\alpha A X.
\]

To solve (14), there exists simple singular value thresholding operator for the nuclear norm, see [29].

**Lemma 11** [29, Theorem 2.1] Let \( X \in \mathbb{R}^{n_1 \times n_2} \) be an arbitrary matrix and \( U \Sigma V^\top \) be its SVD. It is known that

\[
\partial \| X \|_* = \{ UV^\top + W : W \in \mathbb{R}^{n_1 \times n_2}, U^\top W = 0, WV = 0, \| W \|_2 \leq 1 \}.
\]

\[
D_\tau(X) = \arg\min_Y \frac{1}{2} \| Y - X \|^2_F + \tau \| Y \|_* = UD_\tau(\Sigma)V^\top = U_0(\Sigma_0 - \tau I)V_0^\top,
\]

(16)

where \( D_\tau(\Sigma) = \text{diag}((\sigma_i - \tau)_+) \), \( U_0, V_0 \) are the singular vectors associated with singular values greater than \( \tau \).

Thus, we can compute the analytical solution of \( \hat{X}_{k+1} \) in (14), that is,

\[
\hat{X}_{k+1} = D_\frac{1}{t} \left( X_k - \frac{1}{t} \nabla_X f(\alpha_k, X_k) \right).
\]

(17)

**Lemma 12** The \( \alpha \)-sublevel set of the objective function of (13), \( \{ (\alpha, X) \in \mathbb{R} \times S^n : f(\alpha, X) + \lambda \| X \|_* \leq \alpha, \| X \|_F = 1 \} \), is bounded.

It is obvious since \( f(\alpha, X) + \lambda \| X \|_* \geq \| A - \alpha X \|_F \| X \|_F \) \( \rightarrow +\infty \) when \( | \alpha | \rightarrow +\infty \).

For the iterative scheme, we have the following sufficient descent property.

**Lemma 13** Let \( f \) be continuously differentiable over \( X \) and its gradient \( \nabla_X f \) be \( L_f \)-Lipschitz continuous locally. Then for any \( t_k > L_f \), it holds that

\[
f(\alpha_{k+1}, X_{k+1}) + \lambda \| X_{k+1} \|_* \leq f(\alpha_k, X_k) + \lambda \| X_k \|_* - \frac{t_k - L_f}{2} \| X_k - X_{k+1} \|^2.
\]
Proof Since $\nabla f$ is $L_f$-Lipschitz continuous locally, we have

$$f(\alpha_k, \hat{X}_{k+1}) \leq f(\alpha_k, X_k) + \left( \hat{X}_{k+1} - X_k, \nabla_X f(\alpha_k, X_k) \right) + \frac{L_f}{2} \| \hat{X}_{k+1} - X_k \|_F^2.$$ 

(18)

According to (14), we also obtain that

$$\left( \hat{X}_{k+1} - X_k, \nabla_X f(\alpha_k, X_k) \right) + \frac{t_k}{2} \| \hat{X}_{k+1} - X_k \|_F^2 + \lambda \| \hat{X}_{k+1} \|_* \leq \lambda \| X_k \|_*.$$ 

(19)

Add (18) and (19), we have

$$f(\alpha_k, \hat{X}_{k+1}) + \lambda \| \hat{X}_{k+1} \|_* \leq f(\alpha_k, X_k) + \lambda \| X_k \|_* - \frac{t_k - L_f}{2} \| X_k - \hat{X}_{k+1} \|^2.$$ 

Since $\alpha_{k+1} = \langle A, X_{k+1} \circ X_{k+1} \rangle$ minimizes

$$f(\alpha, X_{k+1}) + \lambda \| X_{k+1} \|_*,$$

we obtain that

$$f(\alpha_{k+1}, X_{k+1}) + \lambda \| X_{k+1} \|_* \leq f(\alpha_k, \hat{X}_{k+1}) + \lambda \| \hat{X}_{k+1} \|_*.$$ 

The desired inequality then follows.

Theorem 14 For the sequence $\{X_k\}$ generated by (17), its any cluster $X_*$ is a stationary point of (13).

The proof is similar to that in [30, 31], we just omit it.

6 The Equivalence Property of CPS Tensors

In this section, we prove that a fourth-order CPS tensor, denoted as $T \in \mathbb{C}^n_4$, is rank-one if and only if a specific matrix unfolding of $T$ is rank-one. We first prove the following lemma.

Lemma 15 If $T \in \mathbb{C}^n_4$ is rank-one, then there exist $\lambda \in \mathbb{R}$ and $x \in \mathbb{C}^n$ such that $T = \lambda x \circ x \circ \bar{x} \circ \bar{x}$.

Proof Since $T$ is rank-one, we have

$$T = x \circ y \circ w \circ z, \quad x, y, w, z \in \mathbb{C}^n.$$ 

According to the conjugate symmetry of $T$, we obtain that $T_{[1,2;3,4]} = \text{vec}(x \circ y) \circ \text{vec}(w \circ z)$ is a Hermitian matrix, thus, there exists $\lambda \in \mathbb{R}$ such that $w \circ z = \lambda \bar{x} \circ \bar{y}$. 

Springer
This further implies that there are $\alpha, \beta \in \mathbb{C}$ such that $w = \alpha \bar{x}, z = \beta \bar{y}$ and $\alpha \beta = \lambda$. Therefore, we have

$$T = x \circ y \circ w \circ z = \lambda x \circ y \circ \bar{x} \circ \bar{y}.$$ 

On the other hand, $T$ is symmetric about the first and the second indexes, so $x = y$ and

$$T = \lambda x \circ x \circ \bar{x} \circ \bar{x}.$$ 

Then, we prove that $T \in \mathbb{C}^{n^4}_{\text{ps}}$ is rank-one if and only if $T[3,2;1,4]$ is a rank-one matrix. In fact, the following lemma can be deduced from [14, Theorem 4.14]. Nonetheless, we provide another proof for the fourth-order CPS tensor particularly, which is easier to understand.

**Lemma 16** Let $T \in \mathbb{C}^{n^4}_{\text{ps}}$, if the unfolding matrix $T[3,2;1,4]$ is rank-one, then $T$ is a rank-one tensor.

**Proof** According to Lemma 15, $T$ can be represented as

$$T = \sum_{i=1}^{r} \lambda_i a_i \circ a_i \circ \bar{a}_i \circ \bar{a}_i, \quad \lambda_i \in \mathbb{R}, a_i \in \mathbb{C}^n.$$ 

Then,

$$T[3,2;1,4] = \sum_{i=1}^{r} \lambda_i \bar{a}_i \circ a_i \circ \bar{a}_i, \quad T[3,2;1,4] = \sum_{i=1}^{r} \lambda_i \text{vec}(\bar{a}_i \circ a_i) \circ \text{vec}(a_i \circ \bar{a}_i).$$ 

The later is a rank-one Hermitian matrix, so there exists $Y \in \mathbb{C}^{n \times n}$ such that

$$T[3,2;1,4] = \text{vec}(Y) \circ \text{vec}(\bar{Y}).$$ 

Let $Y = \sum_{i=1}^{R} \sigma_i x_i \circ y_i$ be the SVD of $Y$, then

$$T[3,2;1,4] = \sum_{i=1}^{R} \sum_{j=1}^{R} \sigma_i \sigma_j x_i \circ y_i \circ \bar{x}_j \circ \bar{y}_j.$$ 

Since the second and third indexes of $T[3,2,1,4]$ are symmetric, we have

$$T[3,2,1,4] = T[3,1,2,4] = \sum_{i=1}^{R} \sum_{j=1}^{R} \sigma_i \sigma_j x_i \circ \bar{x}_j \circ y_i \circ \bar{y}_j.$$
This means that

\[ T_{[3,2;1,4]} = \sum_{i=1}^{R} \sum_{j=1}^{R} \sigma_i \sigma_j \text{vec}(x_i \circ \bar{x}_j) \circ \text{vec}(y_i \circ \bar{y}_j). \]

It is easy to verify that \( \{ \text{vec}(x_i \circ \bar{x}_j) \}_{i,j=1}^{R} \) and \( \{ \text{vec}(z_i \circ \bar{z}_j) \}_{i,j=1}^{R} \) are orthogonal basis, hence \( R = 1 \). Otherwise, we will have \( \text{rank}(T_{[3,2;1,4]}) = R^2 > 1 \). Thus,

\[ T_{[3,2,1,4]} = \sigma^2 x \circ \bar{x} \circ z \circ \bar{z}, \]

that is, \( \text{rank}(T_{[3,2,1,4]}) = \text{rank}(T) = \text{rank}_{\text{cps}}(T) = 1 \).

**Remark 6** It is pointed that a fourth-order real tensor \( \mathcal{X} \) is rank-one if and only if \( \mathcal{X}_{[1,2;3,4]} \) and \( \mathcal{X}_{[3,2;1,4]} \) are rank-one matrix [25, Lemma 3.1]. Lemma 16 reinforces this result for the CPS tensor. However, only \( T_{[1,2;3,4]} \) being rank-one cannot guarantee a partial-symmetric tensor \( T \) being rank-one. For example, the following \( T \in \mathbb{R}^{2 \times 2 \times 2 \times 2} \) is a partial-symmetric tensor,

\[ T = e_1 \circ e_1 \circ e_1 \circ e_1 + e_1 \circ e_1 \circ e_2 \circ e_2 + e_2 \circ e_2 \circ e_1 \circ e_1 + e_2 \circ e_2 \circ e_2 \circ e_2. \]

According to \( T_{[1,3;2,4]} = I \in \mathbb{R}^{4 \times 4} \), we have \( \text{rank}(T) > 1 \). However,

\[ T_{[1,2;3,4]} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \]

is a rank-one matrix.

### 6.1 Applications to the Best Rank-One Approximation Problem

This section concentrates on the best rank-one approximation problem,

\[
\min \| T - \lambda x \circ x \circ \bar{x} \circ \bar{x} \|_F^2 \quad \text{s.t.} \quad \lambda \in \mathbb{R}, x \in \mathbb{C}^n, \|x\| = 1, \tag{20}
\]

or its equivalent formulation

\[
\max | \langle T, x \circ x \circ \bar{x} \circ \bar{x} \rangle | \quad \text{s.t.} \quad x \in \mathbb{C}^n, \|x\| = 1, \tag{21}
\]

where \( T \in \mathbb{C}^{n^4}_{\text{ps}} \) is the conjugate partial-symmetric tensor. Solving (21) is further equal to solve two minimization problems, \( \min \langle T, x \circ x \circ \bar{x} \circ \bar{x} \rangle \) and \( \min \langle -T, x \circ x \circ \bar{x} \circ \bar{x} \rangle \) over \( \{ x \in \mathbb{C}^n, \|x\| = 1 \} \).

Without loss of generality, we only consider the following problem:

\[
\min \langle -T, x \circ x \circ \bar{x} \circ \bar{x} \rangle \quad \text{s.t.} \quad x \in \mathbb{C}^n, \|x\| = 1. \tag{22}
\]
6.1.1 Convex Relaxations of (22)

Let $\mathcal{X} := x \circ x \circ \tilde{x} \circ \tilde{x}$, the (22) can be reformulated equivalently as

$$\min -\langle T, \mathcal{X} \rangle \quad \text{s.t.} \quad \mathcal{X} \in \mathbb{C}^{n^4}_{ps}, \, \text{rank}(\mathcal{X}) = 1, \|\mathcal{X}\|_F = 1. \quad (23)$$

According to (16), the rank-one tensor constraint $\text{rank}(A) = 1$ can be replaced by a rank-one matrix constraint, i.e., (23) can be equivalently reformulated into

$$\min \langle -T, \mathcal{X} \rangle \quad \text{s.t.} \quad \mathcal{X} \in \mathbb{C}^{n^4}_{ps}, \, \text{rank}(\mathcal{X}[3,2;1,4]) = 1, \|\mathcal{X}\|_F = 1. \quad (24)$$

In recent years, it is popular to replace the rank function by the matrix nuclear norm. After relaxation, one expects to obtain a low rank solution via solving the nuclear norm based problems. The nuclear norm $\| \cdot \|_*$ is defined as the sum of singular values of a matrix [32]. Following this line, (24) can be relaxed to the following nuclear norm regularized convex optimization problem

$$\min \langle -T, \mathcal{X} \rangle + \rho \|\mathcal{X}[3,2;1,4]\|_* \quad \text{s.t.} \quad \mathcal{X} \in \mathbb{C}^{n^4}_{ps}, \|\mathcal{X}\|_F \leq 1, \quad (25)$$

where $\rho > 0$ is a regularization parameter. Analogously, we can also employ the nuclear norms as constraints, which results in the following problem:

$$\min \langle -T, \mathcal{X} \rangle \quad \text{s.t.} \quad \mathcal{X} \in \mathbb{C}^{n^4}_{ps}, \|\mathcal{X}[3,2;1,4]\|_* \leq 1. \quad (26)$$

Determining whether a solution to (25) or (26) is a global minimizer of (24). [25] proved that for $X \in \mathbb{R}^{M \times N}$, if $\|X\|_F = 1$ and $\|X\|_* = 1$, then $\text{rank}(X) = 1$. Based on this observation, we discuss in which case, the optimizer of (25) is the global minimizer of the original problem (24). Denote $\hat{\rho}$ as the optimal value of (25), the following corollary is presented.

Corollary 17 If $\hat{\mathcal{X}} \neq 0$ is an optimal value to (25), $\hat{\rho} \neq 0$, and $\mathcal{X}^*$ is a global minimizer of (24), then

1. $\|\hat{\mathcal{X}}\|_F = 1$,
2. $\|\hat{\mathcal{X}}[3,2;1,4]\|_* \geq 1$,
3. if $\|\hat{\mathcal{X}}[3,2;1,4]\|_* = 1$, then $\text{rank}_{CP}(\hat{\mathcal{X}}) = 1$,
4. $\lambda^* \leq \tilde{\lambda}$, where $\lambda^* = \langle T, \mathcal{X}^* \rangle$, $\tilde{\lambda} = \langle T, \hat{\mathcal{X}} \rangle$.

Proof

1. Since $\mathcal{X} = 0$ is a feasible solution to (25), it holds that $\hat{\rho} < 0$. Suppose $\|\hat{\mathcal{X}}\|_F < 1$, then $\hat{\mathcal{X}}/\|\hat{\mathcal{X}}\|_F$ is also a feasible solution and the objective value evaluated at it is $\hat{\rho}/\|\hat{\mathcal{X}}\|_F < \hat{\rho}$, which gives a contradiction to the optimality of $\hat{\mathcal{X}}$.
2. Denote $\hat{\sigma}_i, i = 1, \cdots, r$ as the singular values of $\hat{\mathcal{X}}[3,2;1,4]$, then according to $\|\hat{\mathcal{X}}\|_F = \sum_{i=1}^r \hat{\sigma}_i^2$ and $\|\hat{\mathcal{X}}[3,2;1,4]\|_* = \sum_{i=1}^r \hat{\sigma}_i$, it can be seen that when $\|\hat{\mathcal{X}}\|_F = 1, \|\hat{\mathcal{X}}[3,2;1,4]\|_* \geq \|\hat{\mathcal{X}}\|_F^2 = 1$.
3. It follows from (16) immediately.
4. Since $\mathcal{X}^*_{[3,2;1,4]}$ is rank-one and $\|\mathcal{X}^*\|_F = 1$, we have $\|\mathcal{X}^*_{[3,2;1,4]}\|_* = 1$ and $-\lambda^* + \rho \geq -\hat{\lambda} + \rho \|\hat{\mathcal{X}}_{[3,2;1,4]}\|_*$. Combined with $\|\hat{\mathcal{X}}_{[3,2;1,4]}\|_* \geq 1$, it is obtained that $\lambda^* \leq \hat{\lambda}$.

Corollary 17 implies that it is possible to identify whether an optimizer of (25) is an optimizer of (24) by computing the sum of balanced nuclear norms. The following theorem summarizes this result.

**Theorem 18** Assume that $\mathcal{T} \neq 0$ and $\hat{\mathcal{X}}$ is a global minimizer of (25). Then, $\hat{\mathcal{X}}$ is a global optimizer of (24) if and only if $\hat{\mathcal{X}} \neq 0$, $\|\hat{\mathcal{X}}_{[3,2;1,4]}\|_* = 1$ and $\hat{\rho} \neq 0$.

Next, we study (26). The following observations also characterize the conditions for the optimizer of (26) being the global optimizer of the original problem.

**Corollary 19** Assume $\mathcal{T} \neq 0$. If $\hat{\mathcal{X}}$ is an optimal solution of (26), then

1. $\|\hat{\mathcal{X}}_{[3,2;1,4]}\|_* = 1$,
2. $\|\hat{\mathcal{X}}\|_F \leq 1$,
3. if $\|\hat{\mathcal{X}}\|_F = 1$, then $\text{rank}_{\text{CP}}(\hat{\mathcal{X}}) = 1$.

**Proof** 1. Since $\mathcal{T} / \|\mathcal{T}_{[3,2;1,4]}\|_*$ is a feasible solution and the corresponding objective value is negative, so the optimal value of (26) must be negative. Suppose $\|\mathcal{X}_{[3,2;1,4]}\|_* < 1$, then $\mathcal{X} / \|\mathcal{X}_{[3,2;1,4]}\|_*$ is a feasible solution, whose objective value is smaller than the optimal value. It gives a contradiction.

2. When $\|\mathcal{X}_{[3,2;1,4]}\|_* \leq 1$, $\|\mathcal{X}\|_F^2 \leq \|\mathcal{X}_{[3,2;1,4]}\|_* \leq 1$.

3. It follows from Lemma 16.

Based on Corollary 19, we have the following theorem.

**Theorem 20** Assume $\mathcal{T} \neq 0$, then the optimizer $\hat{\mathcal{X}}$ of (26) is also a global optimizer of (24) if and only if $\|\hat{\mathcal{X}}\|_F = 1$.

**Implementation of (25) via ADMM.**

By introducing a variable $\mathcal{Y}$, problem (25) can be formulated as

$$
\text{min} \langle -\mathcal{T}, \mathcal{X} \rangle + \rho \|\mathcal{Y}_{[3,2;1,4]}\|_* \quad \text{s.t.} \quad \mathcal{X} \in \mathbb{C}^n_{\text{ps}}, \|\mathcal{X}\|_F \leq 1, \mathcal{Y} = \mathcal{X}.
$$

(27)

Denote the Lagrangian function of (27) as

$$
\mathcal{L}(\mathcal{X}, \mathcal{Y}, \Lambda) = \langle -\mathcal{T}, \mathcal{X} \rangle + \rho \|\mathcal{Y}_{[3,2;1,4]}\|_* + \langle \Lambda, \mathcal{Y} - \mathcal{X} \rangle + \frac{\tau}{2} \|\mathcal{Y} - \mathcal{X}\|_F^2.
$$

The ADMM iteration steps for (27) are given as follows:

$$
\mathcal{X}^{k+1} = \arg\min_{\|\mathcal{X}\|_F \leq 1, \mathcal{X} \in \mathbb{C}^n_{\text{ps}}} \mathcal{L}(\mathcal{X}, \mathcal{Y}^k, \Lambda^k) = \frac{P_{\text{cps}}[\frac{1}{\tau}(\tau \mathcal{Y}^k + \mathcal{T} + \Lambda^k)]}{\|P_{\text{cps}}[\frac{1}{\tau}(\tau \mathcal{Y}^k + \mathcal{T} + \Lambda^k)]\|_F}
$$

(28)

$$
\mathcal{Y}^{k+1} = \arg\min \mathcal{L}(\mathcal{X}^{k+1}, \mathcal{Y}, \Lambda^k);
$$

(29)
\[ \Lambda^{k+1} = \Lambda^k + \tau (\mathcal{Y}^{k+1} - \mathcal{X}^{k+1}). \]  

(30)

To compute the subproblem of \( \mathcal{Y} \), we first compute

\[ \mathcal{Y}^{k+1}_{[3,2;1,4]} = D_x \left[ \left( \mathcal{X} - \frac{1}{\tau} \Lambda^k \right)_{[3,2;1,4]} \right], \]

then fold the matrix \( \mathcal{Y}^{k+1}_{[3,2;1,4]} \) back into \( \mathcal{Y}^{k+1} \). \( P_{cps} (\cdot) \) denotes the projection onto the set \( C_{n^4}^{ps} \), which is given by the following lemma.

**Lemma 21** Let \( \mathcal{Y} \in C_{n^4}^{n \times n \times n \times n} \),

\[ P_{cps} (\mathcal{Y}) = \frac{1}{8} (\mathcal{Y} + \mathcal{Y}_{[1,2,4,3]} + \mathcal{Y}_{[2,1,3,4]} + \mathcal{Y}_{[2,1,4,3]} + \mathcal{Y}_{[3,4,1,2]} + \mathcal{Y}_{[4,3,1,2]} + \mathcal{Y}_{[3,4,2,1]} + \mathcal{Y}_{[4,3,2,1]}). \]

**Proof** It is easy to verify that for any \( \mathcal{Z} \in C_{n^4}^{n \times n \times n \times n} \),

\[ \langle \mathcal{Y}, \mathcal{Z} \rangle = \langle \mathcal{Z}, \mathcal{Y} \rangle = \langle P_{cps} (\mathcal{Y}), \mathcal{Z} \rangle. \]

Thus for any \( \mathcal{Z} \in C_{n^4}^{n^4} \), we have

\[ \| \mathcal{Y} - \mathcal{Z} \|^2_F = \| \mathcal{Y} - P_{cps} (\mathcal{Y}) + P_{cps} (\mathcal{Y}) - \mathcal{Z} \|^2_F \]

\[ = \| \mathcal{Y} - P_{cps} (\mathcal{Y}) \|^2_F + \| P_{cps} (\mathcal{Y}) - \mathcal{Z} \|^2_F \]

\[ \geq \| \mathcal{Y} - P_{cps} (\mathcal{Y}) \|^2_F. \]

That is, \( P_{cps} (\mathcal{Y}) = \arg \min_{\mathcal{Z} \in C_{n^4}^{n^4}} \| \mathcal{Z} - \mathcal{Y} \|^2_F. \)

**6.1.2 A Nonconvex Relaxation of (22)**

In this section, we consider a nonconvex relaxation of (22). By introducing an auxiliary variable \( \mathcal{Y} \), (24) can be reformulated as

\[ \min \frac{1}{2} \| T - \mathcal{X} \|^2_F \quad \text{s.t.} \quad \mathcal{X} = \mathcal{Y}, \mathcal{X} \in C_{n^4}^{n^4}, \text{rank}(\mathcal{X}_{[1,2;3,4]}) = 1, \text{rank}(\mathcal{Y}_{[3,2;1,4]}) = 1. \]  

(31)

By relaxing the constraint \( \mathcal{X} = \mathcal{Y} \) and imposing it on the objective function, we obtain the following nonconvex relaxation of (31)

\[ \min F (\mathcal{X}, \mathcal{Y}) := \frac{1}{2} \| T - \mathcal{Y} \|^2_F + \frac{\rho}{2} \| \mathcal{X} - \mathcal{Y} \|^2_F \]

s.t. \( \mathcal{X} \in C_{n^4}^{n^4}, \text{rank}(\mathcal{X}_{[1,2;3,4]}) = 1, \text{rank}(\mathcal{Y}_{[3,2;1,4]}) = 1, \)
where \( \rho > 0 \) is a regularization parameter. It is possible to employ the alternating minimization scheme to solve (32), namely optimize \( X \) and \( Y \) alternatively as

\[
Y^{k+1} = \arg \min_{\text{rank}(Y_{3,2,1,4})=1} F(X^k, Y)
\]

(33)

\[
X^{k+1} = \arg \min_{X \in \mathbb{C}^{n,4,ps}, \text{rank}(X_{1,2,3,4})=1} F(X, Y^{k+1}).
\]

(34)

In the above, the subproblem about \( Y \) is a simple matrix rank-one approximation problem, which can be solved by SVD efficiently. It can be simplified as

\[
Y^{k+1} = \arg \min_{\text{rank}(Y_{3,2,1,4})=1} \| \frac{2}{1+\rho} \left( \frac{1}{2} T + \rho X^k \right) - Y \|_F^2.
\]

(35)

Assume that \( X^k \) is a CPS tensor, then \( Y^{k+1}_{3,2,1,4} \) can be expressed as \( Y^{k+1}_{3,2,1,4} = \hat{\lambda} E \circ \bar{E} \), where \( E \) is a Hermitian matrix according to [2, Theorem 1]. We further denote \( E = \sum_{i=1}^{n} \sigma_i a_i \circ \bar{a}_i \) as the SVD of \( E \). Then,

\[
Y = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda \sigma_i \sigma_j \tilde{a}_i \circ \bar{a}_j \circ a_i \circ a_j.
\]

Thus, \( X^{k+1} = \lambda \sigma_1 \sigma_1 \tilde{a}_1 \circ \bar{a}_1 \circ a \circ a \) is always a rank-one CPS tensor.

7 Numerical Results

In this section, we present some numerical experiments to corroborate the results we obtain in previous sections. We conduct all the following algorithms in MATLAB (Realease 2016b) and performed them on a Lenovo laptop with an Intel(R) Core(TM) Processor with access to 8GB of RAM.

7.1 Recover CPS Tensor with SMROA Algorithm

For the real partial-symmetric tensor, we first construct it as \( A = \sum_{i=1}^{n} \lambda_i E_i \circ E_i \), where \( E_i \) is \( n \times n \) (\( n = 3, 4, \ldots, 7 \)) mutually orthogonal and \( \lambda_i \) is generated randomly and is different from each other. We find that the output satisfies that \( | \hat{\lambda}_1 | \geq | \hat{\lambda}_2 | \geq \cdots \geq | \hat{\lambda}_r | \) and \( X_i = \pm E_{\pi(i)} \). An example is given as follows.

**Example 3** For \( A = 0.054 7 E_1 \circ E_1 - 0.104 8 E_2 \circ E_2 - 0.220 9 E_3 \circ E_3, \hat{\lambda}_1 = -0.220 9 \), \( \hat{X}_1 = -E_3 \), \( \hat{\lambda}_2 = -0.104 8 \), \( \hat{X}_1 = -E_2 \), \( \hat{\lambda}_3 = 0.054 7 \), \( \hat{X}_1 = -E_3 \).

Then, we randomly generated 100 partial-symmetric tensors, the numerical results show that SMROA algorithm recover the matrix outer product decomposition successfully. The output \( \{ \hat{\lambda}_i, \hat{X}_i \}_{i=1}^{r} \) satisfy that \( | \hat{\lambda}_1 | \geq | \hat{\lambda}_2 | \geq \cdots \geq | \hat{\lambda}_r | \). \( \hat{X}_i \) are mutually orthogonal.
For the conjugate partial-symmetric tensor, we also design similar experiments as the real case. And the results are still consistent with our theories. The following is an example.

Example 4 $\mathcal{A} = 20.677 7 E_1 \odot \bar{E}_1 + 16.191 0 E_2 \odot \bar{E}_2 + 7.610 4 E_3 \odot \bar{E}_3 - 6.727 4 E_4 \odot \bar{E}_4 - 4.792 0 E_5 \odot \bar{E}_5 + 2.781 1 E_6 \odot \bar{E}_6$. $E_i$ are complex symmetric matrices and mutually orthogonal. Then, the SMROA algorithm generates $\lambda_i$ exactly the same as $\lambda_i$ in order.

7.2 Low-Rank Tensor Completion with Nuclear Norm

Since we are not quite sure what kinds of real data will be partial-symmetric tensor, we use the synthetic data as examples for low-CP-rank partial-symmetric tensor completion problem.

We generate randomly the PS tensor as

$$\mathcal{A} = \sum_{i=1}^{r} \lambda_i (x_i \odot x_i \odot y_i \odot y_i + y_i \odot y_i \odot x_i \odot x_i),$$

then $\text{rank}_{CP}(\mathcal{A}) \leq 2r$. We then recover it by (12) and report the average of the relative errors, which is defined as

$$\text{Err} := \frac{\|\mathcal{X}^* - \mathcal{A}\|_F}{\|\mathcal{A}\|_F},$$

where $\mathcal{X}^*$ is the optimizer of (12). The stop criteria for the inner iteration is $\frac{\|X_{k+1} - X_k\|_F}{\|X_k\|_F} < 10^{-10}$. Part of our numerical results are given in Table 1. The err denotes the average relative errors of 20 instances for each $(n, r, p)$ pair, where $n$ is the dimension of $\mathcal{A}$, $r$ is the same as (36) and $p$ is the sample ratio. The number in the rank$_M$ column with star superscript is the maximum rank$_M(\mathcal{X}^*)$ of the solution in the numerical examples.

From Table 1, we find that most of our examples satisfy that $\text{rank}_{CP}(\mathcal{A}) = \text{rank}_M(\mathcal{A}) = 2r$ except the cases for $(n = 10, r = 2, p = 0.5)$.

| $p$ | err | rank$_M$ | err | rank$_M$ | err | rank$_M$ |
|-----|-----|----------|-----|----------|-----|----------|
|     |     |          |     |          |     |          |
|     |     |          |     |          |     |          |
|     |     |          |     |          |     |          |
|     |     |          |     |          |     |          |
|     |     |          |     |          |     |          |
| $n = 10$ | | | | | | |
| 0.8 | $1.130 5 \times 10^{-8}$ | 2 | $7.401 6 \times 10^{-9}$ | 4 | $2.851 7 \times 10^{-9}$ | 6 |
| 0.5 | $3.489 4 \times 10^{-8}$ | 2 | $1.041 1 \times 10^{-8}$ | 6* | $1.088 3 \times 10^{-8}$ | 6 |
| $n = 15$ | | | | | | |
| 0.8 | $1.054 8 \times 10^{-8}$ | 2 | $3.097 0 \times 10^{-9}$ | 4 | $1.652 6 \times 10^{-9}$ | 6 |
| 0.5 | $5.518 3 \times 10^{-9}$ | 2 | $3.241 9 \times 10^{-9}$ | 4 | $2.930 0 \times 10^{-9}$ | 6 |
7.3 Rank-One Approximation for CPS Tensor

**Example 5** The CPS tensor $T$ in this experiment are generated as $T = \sum_{i=1}^{5} a_i \circ a_i \circ \bar{a}_i \circ \bar{a}_i$, where every $a_i$ is a randomly generated complex vector. The dimensions are 5. We generated 50 instances. We implement the ADMM for the convex relaxation (25) and the ALM for the nonconvex relaxation (32). We first apply ALM for (32) within five iterations, denote the output as $X$, and obtain $\bar{X} = X / \|X\|_F$. Let $\rho = \langle T, \bar{X} \rangle$ in (25). Then, we apply ADMM for (25). We observe that it finds a rank-one CPS tensor successfully, that is, the global optimum of (22), at average 21.740 0 steps.

**Example 6** Ambiguity function shaping for radar waveform. In this example, we conduct the numerical experiments for the radar wave form design problem, which is presented as a complex quartic minimization problem:

$$\min \phi(s) - \rho_0 \mid s^H s_0 \mid^2 \|s\|^2 \quad \text{s.t.} \|s\| = 1,$$

where

$$\phi(s) = \sum_{r=0}^{n-1} \sum_{j=1}^{m} \rho(r,k) \mid s^H J^r (s \odot p(x_j)) \mid^2,$$

\(\rho_0 > 0\) is a penalty parameter and \(s_0\) is a known code sharing some nice properties. \(\rho(r,k) = \sum_{k=1}^{n_0} \delta_{r,r_k} 1_{\Delta_k}(j) \frac{\sigma_k}{|\Delta_k|}\) with \(\delta_{r,r_k}\) being the Kronecker delta and \(1_{\Delta_k}(j)\) being an indicator function for the index set \(\Delta_k\) of discrete frequencies. \(J^r\) is the shifted matrix for \(r \in \{0, 1, \ldots, n - 1\}\), \(\odot\) denotes the Hadamard product, \(p(v) = (1, e^{i2\phi_1}, \ldots, e^{i2(n-1)\phi_n})^\top\). Please refer to [8, 11, 14] for more details. Since the objective function of (37) is a real-valued quartic conjugate form, there is a corresponding fourth-order CPS tensor. Thus, (37) can be formulated as (22), which can be solved by applying ADMM to its convex relaxation (25).

We construct the model using the data in [11] and let \(\rho_0 = 30\) in (37) as [14] to conduct a comparison. We randomly generate \(s_0\) for 100 instances and record the success rate of obtaining the rank-one optimum. We also compare our method with the nuclear norm penalty method (denoted as Funuclear) and the SDP relaxation method (denoted as SDP) proposed by Fu et al. [14]; see Table 2. They all find the rank-one optimum successfully and our method denoted as convex relaxation (25) seems to be much more efficient.

| \(n\) | Funuclear | SDP | Convex relaxation (25) |
|-------|-----------|-----|------------------------|
|       | rank-one/% | CPU | rank-one/% | CPU | rank-one/% | CPU |
| 5     | 100       | 0.346 | 100 | 0.688 | 100 | 0.015 |
| 10    | 100       | 3.765 | 100 | 9.991 | 100 | 0.160 |

Table 2 Results for radar waveform problems
8 Conclusions

In this paper, the matrix outer product decomposition for fourth-order CPS tensors was studied. It was shown that the SROA algorithm can be used to recovery the matrix outer product decomposition exactly. Then, we applied this decomposition for the low-CP-rank tensor completion. We also studied the rank-one equivalence property of fourth-order CPS tensors, which build the relationship between rank-one CPS tensor and a certain unfolded matrix of the CPS tensor. Based on this property, different relaxation approaches were proposed to solving the best rank-one approximation problem for the CPS tensor. Numerical experiments demonstrated the usefulness of the matrix outer product decomposition model.

Author Contributions Amina Sabir, Peng-Fei Huang and Qing-Zhi Yang designed the algorithms, performed the numerical experiments, drafted the manuscript, read and approved the final manuscript.

Conflict of interest The authors declare that they have no conflict of interest.

References

[1] Kolda, T.G., Bader, B.W.: Tensor decompositions and applications. SIAM Rev. 51(3), 455–500 (2009)
[2] De Lathauwer, L., Castaing, J., Cardoso, J.-F.: Fourth-order cumulant-based blind identification of underdetermined mixtures. IEEE Trans. Signal Process. 55(6), 2965–2973 (2007)
[3] Jiang, B., Ma, S., Zhang, S.: Tensor principal component analysis via convex optimization. Math. Program. 150(2), 423–457 (2015)
[4] Hillar, C.J., Lim, L.-H.: Most tensor problems are np-hard. J. ACM 60(6), 1–39 (2013)
[5] Kofidis, E., Regalia, P.A.: On the best rank-1 approximation of higher-order supersymmetric tensors. SIAM J. Matrix Anal. Appl. 23(3), 863–884 (2002)
[6] Zhang, X., Ling, C., Qi, L.: The best rank-1 approximation of a symmetric tensor and related spherical optimization problems. SIAM J. Matrix Anal. Appl. 33(3), 806–821 (2012)
[7] Wang, Y., Qi, L., Zhang, X.: A practical method for computing the largest m-eigenvalue of a fourth-order partially symmetric tensor. Numer. Linear Algebra Appl. 16(7), 589–601 (2009)
[8] Jiang, B., Li, Z., Zhang, S.: Characterizing real-valued multivariate complex polynomials and their symmetric tensor representations. SIAM J. Matrix Anal. Appl. 37(1), 381–408 (2016)
[9] Ni, G.: Hermitian tensors. arXiv:1902.02640v2 (2019)
[10] Nie, J., Yang, Z.: Hermitian tensor decompositions. SIAM J. Matrix Anal. Appl. 41(3), 1115–1144 (2020)
[11] Aubry, A., De Maio, A., Jiang, B., Zhang, S.: Ambiguity function shaping for cognitive radar via complex quartic optimization. IEEE Trans. Signal Process. 61(22), 5603–5619 (2013)
[12] Aittomaki, T., Koivunen, V.: Beampattern optimization by minimization of quartic polynomial. In: 2009 IEEE/SP 15th Workshop on Statistical Signal Processing. pp. 437–440. IEEE (2009)
[13] Madani, R., Lavaei, J., Baldick, R.: Convexification of power flow equations in the presence of noisy measurements. IEEE Trans. Autom. Control 64(8), 3101–3116 (2019)
[14] Fu, T., Jiang, B., Li, Z.: On decompositions and approximations of conjugate partial-symmetric complex tensors. arXiv preprint arXiv:1802.09013 (2018)
[15] Lathauwer, L.D., De Lathauwer, L.: A link between the canonical decomposition in multilinear algebra and simultaneous matrix diagonalization. SIAM J. Matrix Anal. Appl. 28(3), 642–666 (2006)
[16] Jiang, B., Ma, S., Zhang, S.: Low-m-rank tensor completion and robust tensor pca. IEEE J. Selected Topics Signal Process. 12(6), 1390–1404 (2018)
[17] Tisseur, F., Meerbergen, K.: The quadratic eigenvalue problem. Siam rev. Siam Rev. 43(2) (2001)
[18] De Silva, V., Lim, L.-H.: Tensor rank and the ill-posedness of the best low-rank approximation problem. SIAM J. Matrix Anal. Appl. 30(3), 1084–1127 (2008)
[19] Wang, Y., Qi, L.: On the successive supersymmetric rank-1 decomposition of higher-order supersymmetric tensors. Numer. Linear Algebra Appl. 14(6), 503–519 (2010)
[20] Zhang, T., Golub, G.H.: Rank-one approximation to high order tensors. SIAM J. Matrix Anal. Appl. 23(2), 534–550 (2001)

[21] Fu, T.R., Fan, J.Y.: Successive partial-symmetric rank-one algorithms for almost unitarily decomposable conjugate partial-symmetric tensors. J. Oper. Res. Soc. China 7(1), 147–167 (2018)

[22] Chen, B., He, S., Li, Z., Zhang, S.: Maximum block improvement and polynomial optimization. SIAM J. Optim. (2012). https://doi.org/10.1137/110834524

[23] Zhang, T., Golub, G.H.: Rank-one approximation to high order tensors. SIAM J. Matrix Anal. Appl. 23(2), 534–550 (2001)

[24] Kolda, T., Mayo, J.: Shifted power method for computing tensor eigenpairs. SIAM J. Matrix Anal. Appl. 32(4), 1095–1124 (2010)

[25] Yuning, Yang, Yunlong, Feng, Xiaolin, Huang, Johan, A.K.: Suykens: Rank-1 tensor properties with applications to a class of tensor optimization problems. SIAM J. Optim. 26(1), 171–196 (2016)

[26] Golub, G.H., Van Loan, C.F.: Matrix computations, 4th edn. Johns Hopkins (2013)

[27] Ma, S., Goldfarb, D., Chen, L.: Fixed point and bregman iterative methods for matrix rank minimization. Math. Program. 128(1–2), 321–353 (2011)

[28] Bolte, J., Sabach, S., Teboulle, M.: Proximal alternating linearized minimization for nonconvex and nonsmooth problems. Math. Program. 146(1–2), 459–494 (2014)

[29] Cai, J.F., Candès, E.J., Shen, Z.: A singular value thresholding algorithm for matrix completion. SIAM J. Optim. 20(4), 1956–1982 (2008)

[30] Wang, Y., Dong, M., Xu, Y.: A sparse rank-1 approximation algorithm for high-order tensors. Appl. Math. Lett. 102, 106140 (2020)

[31] Wang, X., Navasca, C.: Low rank approximation of tensors via sparse optimization. Numer. Linear Algebra Appl. 25(2), (2017). https://doi.org/10.1002/nla.2136

[32] Recht, B., Fazel, M., Parrilo, P.A.: Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization. SIAM Rev. 52(3), 471–501 (2010)