Phase Space Distribution of Riemann Zeros

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ABSTRACT: We present the partition function of a most generic $U(N)$ single plaquette model in terms of representations of unitary group. Extremising the partition function in large N limit we obtain a relation between eigenvalues of unitary matrices and number of boxes in the most dominant Young tableaux distribution. Since, eigenvalues of unitary matrices behave like coordinates of free fermions whereas, number of boxes in a row is like conjugate momenta of the same, a relation between them allows us to provide a phase space distribution for different phases of the unitary model under consideration. This proves a universal feature that all the phases of a generic unitary matrix model can be described in terms of topology of free fermi phase space distribution. Finally, using this result and analytic properties of resolvent that satisfy Dyson-Schwinger equation, we present a phase space distribution of unfolded zeros of Riemann zeta function.

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1. Introduction and summary

Hilbert and Pólya speculated that there is spectral origin for non-trivial (also called unfolded) zeros of Riemann zeta function. This means, if Riemann hypothesis is true i.e. non-trivial zeros of Riemann zeta function $\zeta(s)$ lie on $\Re[s] = 1/2$ line, (which mathematically means $\zeta(1/2 + i t_i) = 0$ for $t_i \in \mathbb{R}$), then $t_i$ corresponds to eigenvalues of an unbounded self-adjoint operator.

In 1972, H. Montgomery conjectured that pair correlation function between unfolded zeroes of zeta function to be same as the pair correlation function of random matrices taken from a Gaussian Unitary Ensemble [1]. In fact there exists a strong similarity between eigenvalues and unfolded zeros. Namely, they show repulsion at short range. Unfolded zeros repel each other quadratically which is similar to repulsion between eigenvalues of a random matrix, coming from the Harr measure. Montgomery’s conjecture led to numerous study on the analogy between the distribution of the zeroes of zeta function and random matrix theory. Berry and Keating showed that there exists a similarity between the formula for counting function of heights $t_i$ of unfolded zeroes of zeta function and energy eigenvalues $\tilde{E}_n$ of some unknown hermitian operator (in spirit of the Hilbert Polya conjecture) [2]. In [3] J. Keating¹ compare the value distribution of the logarithm of the zeta function on the critical line to the expectation value of the characteristic polynomial of a random unitary matrix under the Harr measure. Thus the similarity between the the eigen values of a unitary matrix (or Hermitian matrix) and the non trivial zeroes of the zeta function has long been known and studied.

One of the goals of this paper is to construct a phase space distribution for unfolded zeros of Riemann zeta function. It is well known that eigenvalues of unitary matrices behave

¹See also [4].
like positions of free fermions whereas representations of $U(N)$, at the same time, have an interpretation in the language of free fermions with number of boxes of Young tableaux being like momentum [5]. Thus, a relation between eigenvalues and number of boxes of a Young diagram provides a phase space picture of different phases of a unitary matrix model. Different phases are distinguished by different topologies of droplets [6, 7]. In this paper we first attempt to construct a unitary matrix model whose eigenvalue distribution for a particular phase captures information of distribution of unfolded zeros of Riemann zeta function. Then, we find the dominant representation(s) (i.e. number of boxes in different rows of a diagram) corresponding to that particular phase of the model. We observe that there exists a natural relation between eigenvalues and number of boxes. This enables us to draw a phase space distribution of unfolded zeros of Riemann zeta function.

Logical flow of our paper goes as follows. We start with the following partition function for one plaquette model

$$Z = \int [DU] \exp \left[ \sum_{n=1}^{\infty} \frac{\beta_n}{n} (\text{Tr} \ U^n + \text{Tr} \ U^*n) \right], \quad (1.1)$$

where $U$ is a $N \times N$ unitary matrix and $[DU]$ is corresponding Haar measure. The model shows a particular distribution $\rho(\{\theta_a\})$ of its eigenvalues depending on various parameters $\beta_n$ of the theory.

The first goal of this paper is to write down a unitary one plaquette model such that eigenvalue distribution function captures information of unfolded zeros of Riemann zeta function. In particular, we find a no-gap solution (eigenvalues are distributed between 0 and $2\pi$) where $\rho(\theta)$ has logarithmic divergences (spike like solution) at the positions of unfolded zeros of Riemann zeta function. Note that we are not mapping the eigenvalue distribution with the distribution of unfolded zeros, rather we are mapping divergences of eigenvalue distribution with the zeros of Riemann zeta function.

The second goal and the most important observation of this paper is following. The integral over unitary matrices in equation (1.1) can as well be done in a different way. Expanding the exponential in terms of character $\chi$ of conjugacy classes of a symmetric group and using the orthogonality relation between the characters of $U(N)$ in different representations, it is possible to write down the partition function in terms of a sum over representations of unitary group of rank $N$ [6]. Since representations of a unitary group can be described by Young diagrams, this is, therefore, equivalent to finding the Young
tableaux distribution which dominates the partition function in the large $N$ limit. It was suggested in [6] that in the large $N$ limit unitary matrix models admit a phase space description. This follows from the fact that eigenvalues of the holonomy matrix behave like position of free fermions whereas, the number of boxes in the Young tableaux are like conjugate momentum. Which means eigenvalue distribution is like position distribution and arrangement of boxes in the Young diagram is like momentum distribution for $N$ non-interacting fermions. [6] found an identification between these two descriptions which essentially led to free fermion picture.

This free fermi picture is very intriguing. One can ask why there should be a free fermion picture for large $N$ saddle points of an interacting theory. Does there exist an underlying many particle non-interacting quantum description for each large $N$ saddle point? Although we do not have clear understanding of these questions but before we try to attempt these, a natural question arises at this point if this free fermi phase space description is universal, i.e., if all large $N$ saddle points have an underlying free fermi picture.

The observed relation in [6] between eigenvalues and number of boxes in a Young tableaux was some what accidental. There was no natural way to arrive this relation. Therefore, universality of free fermi picture i.e. existence of a free fermi phase space distribution for any class of solution, remains under question. In this paper we explicitly show that relation between $h$ and $\theta$ follows from extremization condition in the large $N$ limit. This proves that a free fermi description is always possible for any class of solution or phase of a unitary matrix model. Although, we have proved this for one plaquette model, but it is easy to extend our proof to include the models considered in [21]. This is the most important result of this paper.

Finding out the most dominant Young representation in large $N$ limit is a technically challenging problem in general. This is because we do not have an exact formula for character of permutation groups in presence of arbitrary number of cycles. The most useful formula of character is given by Frobenius. However, this formula involves $N$ number of auxiliary variables. In presence of one cycles only, one can show that it is possible to get rid of these auxiliary variables and find an expression for character in terms of number of boxes in a diagram. [6] used this expression for character (because the model they considered in that paper corresponds to existence of one cycle only) and found the most dominant representations for different classes of solutions. But for a generic unitary matrix model like (1.1) the character involves arbitrary number of cycles, then it is technically difficult to extremize the partition function and obtain a Young tableaux
density. In this paper we show that although it is difficult to find the most dominant Young tableaux density for a generic unitary matrix model, but one can definitely find a relation between number of boxes $h_i$ in a representation and eigenvalues $\theta_i$ of a unitary matrix.

Proving the existence of free fermi description for any generic class of solution of a unitary matrix model, we proceed to our third and final goal, i.e. to draw a phase space distribution for unfolded zeros of Riemann zeta function. It turns out that the phase space has a shape of a broom (in two dimensions of course). There are infinite number of sticks on the right. Density of these sticks increases as one moves towards $\pi = 0$. These sticks corresponds to positions of unfolded zeros.

It has been observed in \cite{2} that if $t_n$ (height of critical line) is analogous to eigenvalue of some Hamiltonian, then the corresponding conjugate variable is time period of some primitive orbits. This time period is proportional to $\ln p$ where $p$ is prime number. In our analysis we see that eigenvalue distribution is related to distribution of unfolded zeros, therefore we expect that the Young distribution function captures some information about prime number distribution. Although our understanding regarding this is not yet complete, but in this paper we present some clue or hint about this.

Organization of this paper is following.

- In section 2 we discuss the properties of resolvent in the context of a most generic single plaquette model.

- Important properties of Riemann zeta function and useful relations are discussed in section 3

- In section 4 we present the unitary plaquette model in details and show how eigenvalue distribution function captures information of unfolded zeros.

- In section 5 we prove the universal feature that all the phases of a generic unitary matrix model can be described in terms of topology of free fermi phase space distribution.

- Phase space distribution of unfolded zeros has been plotted in section 6. Also the relation between Young distribution and prime number distribution has been studied in this section.
Finally, in the last section (section 7) we discuss some open problems related to this work.

2. Single plaquette model and properties of resolvent

Unitary matrix model of the following form has great importance in different branches of physics. Different 2d gauge theories, supersymmetric gauge theories in various dimensions can be cast in terms of unitary matrix models. These models also contain a rich phase structure. In this paper we consider a particular class of unitary matrix model, called single plaquette model. However, our results are also valid for a class of model considered in [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18] up to some re-definitions of parameters.

We consider the following partition function over $N \times N$ unitary matrices $U$

$$Z = \int \mathcal{D}U \exp \left( N \text{Tr}[f(U)] + N \text{Tr}[f(U^\dagger)] \right).$$  \hspace{2cm} (2.1)

Moreover let us assume that the function $f(z)$ has a convergent Taylor series expansion

$$f(U) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} U^n = \sum_{n=0}^{\infty} \frac{\beta_n}{n} U^n, \quad \beta_n = \frac{f^n(0)}{(n-1)!}. \hspace{2cm} (2.2)$$

Hence, the partition function can be written as,

$$Z = \int \mathcal{D}U \exp \left( N \sum_{n=0}^{\infty} \frac{\beta_n}{n} (\text{Tr}[U^n] + \text{Tr}[U^{\dagger n}]) \right).$$ \hspace{2cm} (2.3)

This is called a single plaquette model. $\beta_n$'s are parameters of this theory. They can depend on temperature of the system (for example [8, 19, 20]). Hence, as temperature changes, $\beta_n$'s change and the system undergoes phase transition form one phase to another phase.

2.1 Eigenvalue analysis

An eigenvalue analysis of this model has been discussed in [21]. Since integrand and Haar measure $\mathcal{D}U$ are invariant under unitary transformation, one can go to a diagonal
basis where $U$ has eigenvalues $\theta_i$, $i = 1, \cdots, N$ and write the partition function as,

$$Z = \int \prod_{i=1}^{N} d\theta_i e^{S[\theta_i]}$$

(2.4)

where,

$$S[\theta_i] = N \sum_{n=1}^{\infty} \sum_{i=1}^{N} \frac{2\beta_n}{n} \cos n\theta_i + \frac{1}{2} \sum_{i \neq j} \ln \left| 4 \sin^2 \left( \frac{\theta_i - \theta_j}{2} \right) \right|.$$  

(2.5)

In the large $N$ limit, one can define continuous variables

$$x = \frac{i}{N} \in [0, 1], \quad \theta(x) = \theta_i$$

(2.6)

and the partition function is given by,

$$Z = \int [D\theta] e^{-N^2 S[\theta]}, \quad \text{where}$$

$$S[\theta] = \sum_{n=1}^{\infty} \int_{0}^{1} dx \frac{2\beta_n}{n} \cos n\theta(x) + \frac{1}{2} \int_{0}^{1} dx \int_{0}^{1} dy \ln \left| 4 \sin^2 \left( \frac{\theta(x) - \theta(y)}{2} \right) \right|.$$  

(2.7)

In the large $N$ limit, the dominant contribution comes from extrema of $S[\theta]$ and that is determined by saddle point equation,

$$\int_{-\pi}^{\pi} d\theta' \rho(\theta') \cot \left( \frac{\theta - \theta'}{2} \right) = V'(\theta),$$

(2.8)

where,

$$\rho(\theta) = \frac{\partial x}{\partial \theta} \quad \text{and} \quad V'(\theta) = \sum_{n=1}^{\infty} 2\beta_n \sin n\theta.$$  

(2.9)

Thus for a given set of $\beta_n$’s one has to solve saddle point equation (2.8) to find $\rho(\theta)$.

### 2.2 The resolvent and its properties

It is, in general, difficult to solve an integral equation to find eigenvalue distribution. In this section we discuss an equivalent approach to study phases structure of unitary matrix models from analytic properties of resolvent.

The resolvent is defined by

$$R(z) = N^{-1} \langle Tr[(1 - zU)^{-1}] \rangle.$$  

(2.10)
Expanding the right hand side one can write,

\[ R(z) = 1 + \frac{1}{N} \left( z \langle \text{Tr} U \rangle + z^2 \langle \text{Tr} U^2 \rangle + z^3 \langle \text{Tr} U^3 \rangle + \cdots \right). \tag{2.11} \]

Since \( N^{-1} \langle \text{Tr} U^k \rangle \) lies between \(-1\) and \(+1\) the right hand side of eqn. (2.11) is convergent for \(|z| < 1\). Therefore, \( R(z) \) is an analytic and holomorphic function of \( z \) in the interior of unit disk.

Analyticity of resolvent outside the unit disk can also be shown easily. Expanding the function around \( z = \infty \) we find,

\[ R(z) = -\frac{1}{N} \left( \frac{1}{z} \langle \text{Tr} U \rangle + \frac{1}{z^2} \langle \text{Tr} U^2 \rangle + \frac{1}{z^3} \langle \text{Tr} U^3 \rangle + \cdots \right). \tag{2.12} \]

The series converges for \(|z| > 1\). From equation (2.11) and (2.12) we find that the resolvent satisfies,

\[ R(z) + R(1/z) = 1. \tag{2.13} \]

The advantage of this approach is that in large \( N \) limit resolvent satisfies an algebraic equation (quadratic) rather than an integral equation and hence easy to solve. The equation can be obtained from the fact that Haar measure is invariant under a generalized unitary transformation \( U \rightarrow U e^{tX} \), where \( X \) is a skew symmetric matrix and \( t \) is a real parameter. Partition function (2.1) being invariant under such transformation yields the following identity, which is known as Dyson-Schwinger equation

\[ \frac{d}{dt} \int D U \ N^{-1} \text{Tr} [(1 - z U e^{tX})] \exp(N \text{Tr} [(f(U e^{tX}))] + \text{Tr}[f(U^t e^{-tX})]) = 0. \tag{2.14} \]

A solution to this equation is given by,

\[ R(z) = \frac{1}{2} \left[ 1 + (f'(z)z - f'(1/z)/z) + \sqrt{F(z)} \right] \quad |z| < 1 \]

\[ R(z) = \frac{1}{2} \left[ 1 + (f'(z)z - f'(1/z)/z) - \sqrt{F(z)} \right] \quad |z| > 1 \tag{2.15} \]

where, \( F(z) \) is invariant under \( z \rightarrow 1/z \).

Expanding \( f(z) \) in Taylor series we find the solution is given by,

\[ R(z) = \frac{1}{2} \left[ 1 + \sum_{n=1}^{m} \beta_n (z^n - \frac{1}{z^n}) + \sqrt{F(z)} \right] \quad |z| < 1 \]

\[ R(z) = \frac{1}{2} \left[ 1 + \sum_{n=1}^{m} \beta_n (z^n - \frac{1}{z^n}) - \sqrt{F(z)} \right] \quad |z| > 1 \tag{2.16} \]
and $F(z)$ has the following form

$$
F(z) = \left[ 1 + \sum_{i=1}^{m} \beta_i \left( \frac{1}{z^i} + z^i \right) \right]^2 - 4 \left[ \sum_{i=1}^{m} \frac{\beta_i}{z^i} \right] \left[ \sum_{i=1}^{m} \beta_i z^i \right] + 4\beta_1 R'(0) \\
+ 4\beta_2 \left[ \frac{R'(0)}{z} + \frac{R''(0)}{2} + R'(0)z \right] + 4\beta_3 \left[ \frac{R'(0)}{z^2} + \frac{R''(0)}{2z} + \frac{R'''(0)}{3!} + \frac{R''(0)z}{2} + R'(0)z^2 \right] + \cdots
$$

Analyticity properties of $R(z)$ inside a unit circle depends on the form of $F(z)$ which determines phase structure of this model. A detailed discussion can be found in [7].

### 2.3 Eigenvalue distribution function from resolvent

The eigenvalue density can be obtained from real part of the resolvent [23],

$$
\rho(\theta) = 2\Re [R(e^{i\theta})] - 1
$$

which can also be written as,

$$
\rho(\theta) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \left[ R((1 + \epsilon) e^{i\theta}) - R((1 - \epsilon) e^{i\theta}) \right].
$$

On the other hand, imaginary part of $R(z)$ gives the derivative of potential $V'(\theta)$ which appears on the r.h.s. of saddle point equation (2.8)

$$
V'(\theta) = 2\Im [R(e^{i\theta})].
$$

### 3. Riemann zeta function and its properties

In $18^{th}$ century famous mathematician Leonhard Euler defined a function denoted by $\zeta$,

$$
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}
$$

for positive integer values of $s > 1$. Later, famous Russian mathematician Chebyshev extended the definition for any real values of $s > 1$.

In 1859 Bernhard Riemann extended zeta function defined by Euler to complex variable and showed that this function can be analytically continued to whole complex $s$ plane except the point $s = 1$. Thus the function defined in equation (3.1) as a function of complex variable $s$ is know as Riemann zeta function (or Euler-Riemann zeta function) and is a meromorphic function of complex variable $s$. See [24, 25] for detail discussion on Riemann zeta function.
3.1 Poles and zeros of zeta function and Riemann Hypothesis

Riemann zeta function has a simple pole at $s = 1$. It has infinite number of zeros at $s = -2n$ for positive integer $n$. These zeros are called *trivial zeros* since their distribution is known. The function also has infinitely many zeros, known as *non-trivial zeros* (or unfolded zeros), in the critical strip $0 < \Re[z] < 1$. Distribution of these non-trivial zeros are not known completely.

Riemann Hypothesis states that all on-trivial zeros lie on the critical line $\Re[z] = 1/2$. This means,

$$\zeta \left( \frac{1}{2} + it \right) = 0$$

has non-trivial solution for $t \in \mathbb{R}$. Riemann Hypothesis is not yet proved and is known to be true for at least 40% of non-trivial zeros.

3.2 Few important properties of zeta function

3.2.1 Functional equation and symmetric zeta function

Riemann zeta function satisfies the following functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1-s) \zeta(1-s). \quad (3.2)$$

Here $\Gamma(s)$ is gamma function. This equation relates value of zeta function at two different points in complex plane. Due to presence of sin function on the right hand side, it is easy to check that zeta function vanishes at $s = -2|n|$ for any integer. For $s = +2|n|$, the Gamma function has simple poles at non-positive integers and hence the product $\sin(\pi s/2)\Gamma(1-s)$ on the right is regular and non-zero.

Riemann also defined a symmetric version of zeta function, denoted by, $\xi(s)$,

$$\xi(s) = \frac{1}{2} \pi^{-s/2} s(s-1) \Gamma \left( \frac{s}{2} \right) \zeta(s). \quad (3.3)$$

Zeta function has zeros at $s = -2|n|$ whereas $\Gamma$ function has poles at those points, hence $\Gamma \left( \frac{s}{2} \right) \zeta(s)$ is regular at $s = -2|n|$. Since $\zeta(s)$ has a pole order one at $s = 1$ with residue 1, $\xi(s)$ has a normalization $\ln \xi(1) = -\ln 2$. Therefore, the symmetric version of zeta function $\xi(s)$ is analytic everywhere in complex $s$ plane. From functional equation (3.2) one can also show that, symmetric zeta function $\xi(s)$ satisfies

$$\xi(s) = \xi(1-s). \quad (3.4)$$
In our paper we shall mostly consider this symmetric \( \zeta \) function.

### 3.2.2 Product formula

There is a strong connection between \( \zeta \) function and prime numbers. Euler proved that there exists a product representation of \( \zeta \) function given by,

\[
\zeta(s) = \prod_{p=\text{prime}}^{\infty} \frac{1}{1 - p^{-s}}.
\]

The symmetric \( \zeta \) function also satisfies following product representation

\[
\xi(s) = \frac{1}{2} \prod_{i=1}^{\infty} \left( 1 - \frac{s}{\rho_i} \right)
\]

where \( \rho_i \) are non trivial zeroes of \( \zeta \) function.

### 3.2.3 Li criteria

In 1997, Xian-Jin Li proved that the Riemann hypothesis for \( \zeta \) function is equivalent to non-negativity of a sequence of real numbers \([26]^3\). Li introduced following sequence of numbers

\[
(n - 1)! \lambda_n = \frac{d^n}{ds^n} \left[ s^{n-1} \ln \xi(s) \right]_{s=1}
\]

and showed that a necessary and sufficient condition for nontrivial zeros of Riemann \( \zeta \) function to lie on critical line is that \( \lambda_n \) is non-negative for every positive integer \( n \).

Using product representation (3.6) of symmetric \( \zeta \) function one can show that the Li numbers can be written as,

\[
\lambda_n = \sum_i \left[ 1 - \left( 1 - \frac{1}{\rho_i} \right)^n \right].
\]

### 4. The matrix model

Having discussed relevant properties of plaquette model, resolvent and \( \zeta \) function, we are now in a position to construct a unitary matrix model whose eigenvalue distribution captures information about zeros of Riemann \( \zeta \) function.

\(^3\)See also [27]
4.1 From $s$–plane to $z$–plane

Since eigenvalues of unitary matrices are distributed on a unit circle in complex $z$ plane, we need to first map the conjectured zeros (on critical line in $s$ plane) of zeta function on a unit circle about the origin in $z$ plane. We use the following conformal map

$$s = \frac{1}{1 - z}.$$ (4.1)

All unfolded zeros on critical line (assuming Riemann hypothesis) in $s$–plane corresponds to $s = 1/2 + it_i$ where $t_i \in \mathbb{R}$. In $z$–plane these zeros correspond to $z = e^{i\theta_i}$. Hence, using equation (4.1) we find,

$$t_i = \frac{1}{2} \cot \frac{\theta_i}{2}.$$ (4.2)

The mapping is explained in figure 1. The critical line $\Re[s] = 1/2$ in $s$ plane is mapped to a unit circle about origin in $z$ plane (dashed line in figure 1). The region $\Re[s] > 1/2$ in $s$ plane is mapped inside the unit circle in $z$ plane (shaded region) and region $\Re[s] < 1/2$ is mapped outside the unit circle. $s = 1$ point is mapped to $z = 0$ point, $s = 1/2$ is mapped to $z = -1$.

Since symmetric zeta function (3.3) is analytic in $s$ plane, it is therefore analytic inside and outside the unit circle in $z$ plane with unfolded zeros located on $|z| = 1$. In $s$–plane zeros of Riemann zeta function are symmetrically distributed about $\Im[s] = 0$, this implies in $z$–plane zeros are symmetrically distributed on unit circle about $\Im[z] = 0$. It is observed that distribution of zeros of zeta function becomes denser as $t_i$ becomes larger. Therefore, in $z$–plane density of distribution increases as one goes to $\theta \rightarrow 0$.

Functional equation (3.4) for $\xi$ in $z$–plane is given by,

$$\xi \left( \frac{1}{1 - z} \right) = \xi \left( \frac{-z}{1 - z} \right).$$ (4.3)

4.2 The resolvent

As discussed in section 2, resolvent contains all the information about matrix model and eigenvalue distribution of its different phases. Eigenvalue distribution is given by real part of resolvent defined by equation (2.10), whereas imaginary part of resolvent contains information about potential.
Given a unitary matrix model, one needs to solve the Dyson-Schwinger equation to find resolvent with correct analytic structure. Eigenvalue distribution can be obtained from real part of the resolvent. Here we follow an inverse route. From the analytic properties, we construct a resolvent which captures information of unfolded Riemann zeros. Then, from resolvent we write down a consistent unitary matrix model such that the resolvent satisfies corresponding Dyson-Schwinger equation.

Following the properties of resolvent given in section 2.2, we take following form of the resolvent

\[ R_<(z) = \frac{1}{2} - \alpha \ln \xi \left( \frac{1}{1 - z} \right) \quad |z| < 1 \]
\[ R_>(z) = \frac{1}{2} + \alpha \ln \xi \left( \frac{1}{1 - z} \right) \quad |z| > 1. \]  

We fix value of \( \alpha \) from boundary condition \( R(0) = 1 \), which implies

\[ \alpha = -\frac{1}{2 \ln \xi(1)} = \frac{1}{2 \ln 2}. \]  

The above form of \( R(z) \) satisfies all the properties. First of all \( R_<(z) \) is analytic inside unit circle (assuming Riemann hypothesis). Hence there exists a Taylor expansion around
\( z = 0 \) inside unit circle. Symmetric properties of \( \xi(z) \) ensures condition (2.13),
\[
R_<(z) + R_>(1/z) = 1 + \alpha \ln \left[ \frac{\xi \left( \frac{1}{1-z} \right)}{\xi \left( \frac{1}{z-1} \right)} \right] = 1. \quad (4.6)
\]

Since \( \xi \left( \frac{1}{1-z} \right) \) has nontrivial zeros on unit circle in \( z \)-plane, \( \ln \xi \left( \frac{1}{1-z} \right) \) has logarithm branch points on unit circles.

Since \( \xi(z) \) is analytic in \( z \) plane, there are different possibilities to choose a resolvent which satisfies required properties. In the next section we see that this particular choice of resolvent gives rise to no-gap solution i.e. eigenvalue distribution has no gap. Also, corresponding unitary model turns out to be simple. This particular choice also ensures that corresponding Young distribution has a direct connection with prime counting function.

### 4.3 The eigenvalue density

The eigenvalue distribution according to equation (2.17) is given by,
\[
\rho(\theta) = -2\alpha \text{Re} \left[ \ln \xi \left( \frac{1}{1-z} \right) \bigg|_{z=e^{i\theta}} \right]. \quad (4.7)
\]

Our choice of resolvent indicates that unfolded zeros Riemann zeta function will play special role in \( \rho(\theta) \).

Using product representation (equation 3.6) of symmetric zeta function and denoting the positions of unfolded zeros by \( \theta_i \), we write a complete expression for eigenvalue distribution as (see appendix A for detailed calculation),
\[
\rho(\theta) = 1 + 2\alpha \sum_i \ln \left( \frac{\sin \left( \frac{\theta}{2} \right)}{2} \right) - \alpha \sum_i \ln \left[ \sin^2 \left( \frac{\theta - \theta_i}{2} \right) \right]. \quad (4.8)
\]

Since \( |\theta_i| < \pi \) for all \( i \), we see that from the last term \( \rho(\theta) \) diverges logarithmically at \( \theta_i \)'s. Therefore, this term in \( \rho(\theta) \) actually carries information about distribution of Riemann zeros. The second term also diverges at \( \theta = 0 \), but this is usual divergence as \( t \to \pm \infty \) in complex \( s \) plane. Although, we see logarithmic divergences in \( \rho(\theta) \), however, it is easy to check that the eigenvalue density satisfies normalization condition
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \rho(\theta) d\theta = 1. \quad (4.9)
\]
In figure 2 we plot $\rho(\theta)$ as a function of $\theta$. Sharp peaks correspond to logarithmic divergences at Riemann zeros. Since Riemann zeros are densely packed around $\theta = 0$, therefore most of the peaks appear near $\theta = 0$.

Eigenvalue distribution given by equation (4.8) corresponds to a no-gap solution because, the distribution does not have any gap between $-\pi < \theta \leq \pi$.

4.4 The plaquette model

In this section we construct a single plaquette model whose eigenvalue distribution exactly matches with equation (4.8).

A single plaquette model action is given by,

$$S(U) = N \sum_{n=1}^{\infty} \frac{\beta_n}{n} (Tr[U^n] + Tr[U^*n]) \quad (4.10)$$

where $\beta_n$’s are constant. Eigenvalue distribution in different phases of this model depends on values of $\beta_n$’s.

Since resolvent $R(z)$ is analytic inside unit circle one can write down a Taylor expansion of $R(z)$ about $z = 0$ for $|z| < 1$

$$R(z) = 1 + \sum_{n=1}^{\infty} \alpha_n z^n, \quad \text{for} \quad |z| < 1 \quad (4.11)$$
with \( \alpha_n = R^{(n)}(0)/n! \) and \( R(0) = 1 \). Plugging this expression in Dyson-Schwinger equation and demanding that the equation is satisfied inside unit circle one can find \( \alpha_n \) in terms of \( \beta_n \)'s for different phases. However, for no-gap phase there exists a trivial solution to Dyson-Schwinger equation : \( \alpha_n = \beta_n \).

Since, we are trying to construct a plaquette model with a no-gap phase, we have

\[
\alpha_n = \beta_n = \frac{1}{n!} R^{(n)}(0) = \frac{1}{2\pi i} \oint \frac{R(z)}{z^{n+1}} dz, \quad (4.12)
\]

where the contour is taken around \( z = 0 \).

Recall definition of Li numbers given by equation (3.7). These numbers can also be written in integral form using Cauchy theorem,

\[
(n-1)! \lambda_n = \frac{n!}{2\pi i} \oint_c \frac{ds}{(s-1)^{n+1}} \ln \xi(s), \quad (4.13)
\]

where the contour \( c \) is taken counter clockwise around \( s = 1 \). In \( z \)-plane the Li numbers are given by,

\[
\lambda_n \frac{n}{n!} = \frac{1}{2\pi i} \oint_c \frac{dz}{z^{n+1}} \ln \xi \left( \frac{1}{1-z} \right), \quad (4.14)
\]

Using the explicit form of \( R(z) \) inside unit circle (equation 4.4), one can write

\[
\frac{\alpha_n}{n} = -\frac{1}{2\pi i} \oint \frac{R(z)}{z^{n+1}} dz \quad (4.15)
\]

Hence, if we take

\[
\beta_n = -\frac{\lambda_n}{n}, \quad \text{for all } n \quad (4.16)
\]

then the above series (4.11) can be re-summed and written as

\[
R_<(z) = \frac{1}{2} - \alpha \ln \xi \left( \frac{1}{1-z} \right) \quad |z| < 1. \quad (4.17)
\]

Thus we get a plaquette model whose eigenvalue distribution for a particular phase is given by equation (4.8) and captures information about distribution of non-trivial zeros of Riemann zeta function. The model is given by,

\[
S(U) = -\alpha \sum_{n=1}^{\infty} \frac{\lambda_n}{n^2} (Tr[U^n] + Tr[U^\dagger n]). \quad (4.18)
\]
Since real part of resolvent contributes to eigenvalue distribution, one can compute from equation (4.11) that eigenvalue distribution is given by,

$$\rho(\theta) = \frac{1}{2\pi} \left( 1 + 2\alpha \sum_{n=1}^{\infty} \frac{\lambda_n}{n} \cos n\theta \right). \tag{4.19}$$

Upon substituting values of $\lambda_n$ given in equation (3.8)

$$\lambda_n = \sum_i \left[ 1 - \left( 1 - \frac{1}{\rho_i} \right)^n \right], \tag{4.20}$$

one can show that eigenvalue density can be written as (4.8).

### 4.5 The potential

We have constructed the matrix model from real part of the resolvent. As a consistency check we shall construct the potential from imaginary part of resolvent and compare with equation (4.18).

Derivative of the potential is given by imaginary part of the resolvent. Since $\xi$ function has a product representation given by equation (3.6), therefore $\Im[\ln \xi(s)]$ picks up a $i\pi$ factor everytime we cross a non-trivial zero $\rho_i$. We can write,

$$\ln \xi(s) = \sum_i \ln \left( 1 - \frac{s}{\rho_i} \right) = \sum_i \ln (\rho_i - s) - \sum_i \ln \rho_i, \tag{4.21}$$

where $i$ runs over all non-trivial zeros in upper and lower half plane. Denoting $\rho_i = 1/2 + i r_i$ and $s = 1/2 + i t$ for $r_i, \ t \in \mathbb{R}$ we can write

$$\ln \xi(s) = \Sigma_i^U \ln(r_i - t) + \Sigma_i^L \ln(t - r_i) - \Sigma_i \ln \rho_i \tag{4.22}$$

where $\Sigma_i^{U/L}$ implies sum over non-trivial zeros in upper/lower half plane.

Since non-trivial zeros are distributed symmetrically about the real axis, one can show that $\sum_i \ln \rho_i$ is a real number. Only the first two terms contribute to imaginary part of $\ln \xi(s)$. Every time one crosses a non-trivial zero in upper half plane $r_i - t$ becomes negative and the first logarithm gives an extra $i\pi$ term, i.e.

$$\ln(r_i - t) \rightarrow \ln(t - r_i) + i\pi \quad \text{for} \quad t > r_i. \tag{4.23}$$
Note that, in lower half plane all \( r_i \)'s are negative and hence second term in equation (4.22) is always positive for \( t > 0 \). Therefore imaginary part of \( \ln(\xi(s)) \) is a step function

\[
\Im \left[ \ln(s) \right] = \Sigma_i^U \Theta(t - r_i) \quad \text{for} \quad t > 0 \tag{4.24}
\]

The story is same for \( t < 0 \) also. This time second logarithm gives an imaginary part as one cross a non-trivial zero in lower half plane. However, for lower zeros we shall get an extra phase factor, i.e., when we cross \( i^{th} \) zero

\[
\ln(t - r_i) = \ln(|t| - |r_i|) - i\pi. \tag{4.25}
\]

Therefore, the final result is

\[
\Im \left[ \ln(s) \right] = \Sigma_i^U \Theta(t - r_i) \quad \text{for} \quad t > 0
\]
\[
= -\Sigma_i^L \Theta(r_i - t) \quad \text{for} \quad t < 0 \tag{4.26}
\]

Hence, in \( z \)-plane imaginary part of \( R(z = e^{i\theta}) \) can be written as,

\[
\Im \left[ \ln \xi \left( \frac{1}{1 - e^{i\theta}} \right) \right] = -\sum_{\text{zeros}} \Theta(\theta - \theta_i), \quad -\pi \leq \theta \leq 0
\]
\[
= \sum_{\text{zeros}} \Theta(\theta_i - \theta), \quad 0 < \theta \leq \pi. \tag{4.27}
\]

Thus derivative of the potential is biven by (see figure 3)

\[
V'(\theta) = -\alpha \sum_{\text{zeros}} \Theta(\theta - \theta_i), \quad -\pi \leq \theta \leq 0
\]
\[
= \alpha \sum_{\text{zeros}} \Theta(\theta_i - \theta), \quad 0 < \theta \leq \pi. \tag{4.28}
\]

From equation (4.11), imaginary part of resolvent can also be written as

\[
\Im \left[ \ln \xi \left( \frac{1}{1 - e^{i\theta}} \right) \right] = -\alpha \sum_n \frac{\lambda_n}{n} \sin n\theta. \tag{4.29}
\]

Again, plugging values of \( \lambda_n \) (equation 4.8) one can show that imaginary part of resolvent matches with equation (4.27).
Figure 3: Derivative of potential. We see that the derivative of potential has discrete jumps at Riemann zeros.

4.6 An alternate proof of Li theorem

One can prove Li theorem, i.e., positivity of Li numbers $\lambda_n$ from analytic properties of resolvent which satisfies Dyson-Schwinger equation of an underlying unitary matrix model.

The resolvent on unit circle is given by,

$$R(z = e^{i\theta}) = 1 + \sum_n \beta_n z^n$$

$$= 1 - \alpha \sum_n \frac{\lambda_n}{n} \cos n\theta - i\alpha \sum_n \frac{\lambda_n}{n} \sin n\theta. \quad (4.30)$$

Therefore Li numbers $\lambda_n$’s can be obtained either from real or imaginary part of resolvent. Writing, Li numbers in terms of imaginary part we find,

$$-\alpha \frac{\lambda_n}{n} = \int_{-\pi}^{\pi} \Im [R(z = e^{i\theta})] \sin n\theta d\theta. \quad (4.31)$$

Using equation (4.27) and doing integration by parts one can show that,

$$\lambda_n = \sum_i (1 - \cos n\theta_i) \geq 0. \quad (4.32)$$

This proves Li’s theorem. This proof is based on the fact that there exists a unitary model, whose eigenvalue density is given by equation (4.8) and corresponding resolvent satisfies Dyson-Schwinger equation which implies $\beta_n = -\alpha \lambda_n/n$. 
5. Generalised phasespace distribution of unitary matrix model

Possibility of free fermi phase space description of large N saddle points of a unitary matrix model was first observed in [5]. This was possible because partition can also be written as sum over representation of underlying $U(N)$ group. Since different representations can be described in terms of Young diagram, it was shown in [28] that in large $N$ limit partition function is dominated by class of Young diagrams. It turns out that, there exists a relation between the most dominant representation of $U(N)$ group and eigenvalue distribution which enables one to define a free fermi phase space distribution considering eigenvalues $\theta_i$ of $U(N)$ as coordinates of fermions and number of boxes $h_i$ in $i^{th}$ row of a Young diagram as conjugate momentum. This alternate possibility of writing partition function enables one to give a topological or geometrical description of large $N$ saddle points in terms of phase space distribution.

The difficulty in understanding universal nature of free fermi description lies in finding an exact expression of character of permutation group in presence of arbitrary number of cycles. In [6] simpler models were considered, where one need to calculate character in presence of only one cycles which is equal to dimension of the corresponding representation and thus a free fermi description arises. For a generic matrix model, there exists no compact expression for character in presence of arbitrary number of cycles. The closest and most useful formula for character is given by Frobenius and as we shall see, this formula gives an expression for character for arbitrary cycles up to integration over some auxiliary variables.

In this paper we show that although it is difficult to find the most dominant Young tableaux density for a generic unitary matrix model, but one can definitely find a relation between number of boxes $h_i$ in a representation and eigenvalues $\theta_i$ of a unitary matrix. This relation defines the boundary of free fermi phase space with constant density. In the next section we see that this relation allows us to provide a phase space distribution of unfolded Riemann zeros\footnote{In a follow up paper we shall present a complete construction of Young tableaux distribution for a generic single placquette model and show that universality of free fermi description for any phases of solution. However, finding the boundary relation is enough for our current purpose.}.

Partition function corresponding to single placquette model (4.10) is given by,

$$Z = \int [DU] \exp \left[ N \sum_{n=1}^{\infty} \frac{\beta_n}{n} (\text{Tr} U^n + \text{Tr} U^\dagger n) \right].$$

(5.1)
Expanding the exponential one can write partition function as,

\[ Z = \int [DU] \sum_{\vec{k}} \varepsilon(\vec{\beta}, \vec{k}) \frac{z_{\vec{k}}}{z_{\vec{k}}} \sum_{\vec{l}} \varepsilon(\vec{\beta}, \vec{l}) \frac{z_{\vec{l}}}{z_{\vec{l}}} \Upsilon_{\vec{k}}(U) \Upsilon_{\vec{l}}(U^\dagger) \]  \hspace{1cm} (5.2)

where,

\[ \varepsilon(\vec{\beta}, \vec{k}) = \prod_{n=1}^{\infty} N^{k_n} \beta_n^{k_n}, \quad z_{\vec{k}} = \prod_{n=1}^{\infty} k_n! n^{k_n} \quad \text{and} \quad \Upsilon_{\vec{k}}(U) = \prod_{n}(\text{Tr}U^n)^{k_n}. \]  \hspace{1cm} (5.3)

Here, \( n \) runs over positive integers and \( \vec{k} = (k_1, k_2, \ldots) \) (similarly \( \vec{l} = (l_1, l_2, \ldots) \)), \( k_n \) or \( l_n \)'s can be 0 or any positive integer.

\( \Upsilon_{\vec{k}}(U) \) can be rewritten in terms of the characters of the conjugacy class of the permutation group \( S_K \) as follows

\[ \Upsilon_{\vec{k}}(U) = \sum_{R} \chi_R(C(\vec{k})) \text{Tr}_R[U]. \]  \hspace{1cm} (5.4)

Here \( \chi_R(C(\vec{k})) \) is character of conjugacy class \( C(\vec{k}) \) of permutation group \( S_K \), \( K = \sum \alpha_k k_\alpha \) and \( R \) denotes representation of \( U(N) \). Finally, using following orthogonality relation between characters of representations of \( U(N) \)

\[ \int DU \text{Tr}_R[U]\text{Tr}_{R'}[U^\dagger] = \delta_{RR'} \]  \hspace{1cm} (5.5)

we obtain the following form for the partition function

\[ Z = \sum_{R} \sum_{\vec{k}} \frac{\varepsilon(\vec{\beta}, \vec{k})}{z_{\vec{k}}} \sum_{\vec{l}} \frac{\varepsilon(\vec{\beta}, \vec{l})}{z_{\vec{l}}} \chi_R(C(\vec{k})) \chi_R(C(\vec{l})). \]  \hspace{1cm} (5.6)

Any representation of an unitary group \( U(N) \) can be labeled by a Young diagram with maximum \( N \) number of rows and arbitrary numbers of boxes in each row up to a constraint that number of boxes in a particular row can not be greater than number of boxes in a row before . Therefore, sum of representations of \( U(N) \) can be cast as sum of different Young diagrams. If \( K \) is total number of boxes in a Young diagram with \( \lambda_i \) number of boxes in \( i^{th} \) row and \( \sum_i \lambda_i = K \), then sum over representations can be decomposed as

\[ \sum_{R} \to \sum_{K=1}^{\infty} \sum_{\{\lambda_i\}} \delta \left( \sum_{i=1}^{N} \lambda_i - K \right). \]  \hspace{1cm} (5.7)
The partition function, therefore, can be written as,

\[
Z = \sum_{\vec{\lambda}} \sum_{\vec{k},\vec{l}} \frac{\varepsilon(\vec{\beta}, \vec{k}) \varepsilon(\vec{\beta}, \vec{l})}{z_{\vec{k}} z_{\vec{l}}} \chi_{\vec{\lambda}}(C(\vec{k})) \chi_{\vec{\lambda}}(C(\vec{l})) \delta(\Sigma_n nk_n - \Sigma_i \lambda_i) \delta(\Sigma_n nl_n - \Sigma_i \lambda_i).
\]

(5.8)

Note that total number of boxes is same as order of permutation group \(S_K\). The characters of conjugacy class are determined recursively by Frobenius formula \([29, 30, 31]\). Explicit expressions for the most general case are not simple. We shall discuss about that in the next subsection.

In large \(N\) limit, we introduce continuous variables

\[
h(x) = \frac{h_i}{N}, \quad \text{where} \quad x = \frac{i}{N}, \quad \text{with} \quad x \in [0, 1]
\]

(5.9)

where \(h_i\) is given by,

\[
h_i = \lambda_i + N - i \quad \forall \quad i = 1, \cdots, N
\]

(5.10)

with

\[
h_1 > h_2 > \cdots > h_N \geq 0.
\]

(5.11)

Therefore, in large \(N\) limit partition function can be written in the following form

\[
Z = \int [Dh(x)] \prod_n \int dk'_n dl'_n \exp \left[ -N^2 S_{\text{eff}}[h(x), \vec{k}'_n, \vec{l}'_n] \right],
\]

(5.12)

where \(S_{\text{eff}}\) is some effective action constructed out of \(h(x)\) and other parameters. Dominant contribution to partition function comes from those representations which maximise effective action \(S_{\text{eff}}\) and those representations are characterized by a quantity called Young tableaux density defined as,

\[
u(h) = -\frac{\partial x}{\partial h}.
\]

(5.13)

Since \(h(x)\) is a monotonically decreasing function of \(x\), Young density \(\nu(h) > 0 \ \forall \ x \in [0, 1]\).

For a simple model, for example, \(\beta_1 \neq 0\) and other \(\beta_n = 0\) (for \(n \geq 2\)) it is possible to find \(\nu(h)\) corresponding to different phases \([6, 7]\). In this simple case one needs to calculate character of permutation group in presence of one cycles only. Character in presence of non-zero one cycles is given by dimension of the corresponding representation. However, in presence of other non-zero cycles it is hard to compute character of permutation group.
In fact there exists no exact expression for that. Therefore, it is difficult to find $u(h)$ by extremising the effective action for a generic unitary matrix model.

In that simple model, [6, 7] observed that there exists a beautiful identification between Young tableaux distribution and eigenvalue distribution,

$$h \leftrightarrow \rho(\theta), \quad \text{and} \quad u(h) \leftrightarrow \theta. \quad (5.14)$$

This identification allows one to write eigenvalue distribution and Young tableaux distribution in terms of a single constant phase space distribution function $\omega(h, \theta)$ such that,

$$\rho(\theta) = \int_{0}^{\infty} \omega(h, \theta) dh, \quad u(u) = \int_{-\pi}^{\pi} \omega(h, \theta) d\theta \quad (5.15)$$

where $\omega(h, \theta)$ is a distribution in a two dimensional phase space spanned by $h$ and $\theta$ and obeys,

$$\omega(h, \theta) = \frac{1}{2\pi}; \quad (h, \theta) \in R$$

$$= 0; \quad \text{otherwise.} \quad (5.16)$$

It is actually the shape (or boundary) of the region $R$ which contains all the information about two distributions for a given phase of the model. For other phases of solution (for example, one-gap or two-gap solution), a similar identification holds between eigenvalue description and Young tableaux description and hence phase space distribution.

The identification, observed in [6, 7], between two distribution was somewhat accidental. There was no clear understanding behind this. Specially, it was not known if such identification holds for a generic model like (5.1). In this section we show that, although it is difficult to find $u(h)$ for a generic model but identification between $h$ and $\rho(\theta)$ holds for this most generic case also.

5.1 Character of permutation group

We discuss in detail Frobenius character formula [29] for $\chi_{\vec{\lambda}}(C(\vec{k}))$ of permutation group. Let $C(\vec{k})$ denotes the conjugacy class of $S_K$ which is determined by a collection of sequence of numbers

$$\vec{k} = (k_1, k_2, \cdots), \quad \text{with,} \quad \sum_n nk_n = K. \quad (5.17)$$
\(C(\vec{k})\) consists of permutations having \(k_1\) number of 1-cycles, \(k_2\) number of 2-cycles and so on. We introduce a set of independent variables, \(x_1, \cdots, x_N\), and for a given Young diagram \((\vec{\lambda})\) we have \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0\). Then we define a power series and Vandermonde determinant as follows.

\[
P_j(x) = \sum_{i=1}^{N} x_i^j, \quad \Delta(x) = \prod_{i<j}(x_i - x_j).
\] (5.18)

For a set of non-negative integers, \((n_1, \cdots, n_N)\), we define

\[
[f(x)]_{(n_1, \cdots, n_N)} = \text{coefficient of } x_1^{n_1} \cdots x_N^{n_N} \text{ in } f(x).
\] (5.19)

The character corresponding to a conjugacy class \(C(\vec{k})\) and a representation characterized by \(\vec{\lambda}\) is given by famous Frobenius formula

\[
\chi_{\vec{h}}(C(\vec{k})) = \left[\Delta(x) \prod_j (P_j(x))^{k_j}\right]_{(h_1, \cdots, h_N)}.
\] (5.20)

where \(h_i\)'s are give by equation (5.10). This is the most generic formula for character. Using this formula one can in principle write down partition exactly.

In presence of only once cycles the character is nothing but dimension of the representation \(R\). In presence of other cycles, it is difficult to extract an exact expression for character.

5.2 Saddle point equations from character

Writing partition function (equation (5.8)) in terms of representations of \(U(N)\) group apparently a different approach in comparison with writing partition function in terms of eigenvalues of \(U(N)\) matrices (equation 2.4). Large \(N\) value of partition function depends of large \(N\) behavior of characters. However, it turns out, with our surprise, that large \(N\) behavior of character is controlled by saddle point equation of eigenvalue distribution. Not only that, upon integrating over representations one can show that partition function (5.8), in fact, boils down to equation (2.7).

From equation (5.8) it is obvious that for a given set of \(\beta_n\)'s, large \(N\) behavior of partition function is controlled by the same of character. Writing down the Frobenius formula explicitly we get

\[
\chi_{\vec{h}}(C(\vec{k})) = \left[\left(\sum_{i=1}^{N} x_i\right)^{k_1} \left(\sum_{i=1}^{N} x_i^2\right)^{k_2} \cdots \prod_{i<j}(x_i - x_j)\right]_{(h_1, \cdots, h_N)}.
\] (5.21)
Introducing \( N \) complex variables \((z_1, z_2, \cdots, z_N)\), character can be written as,

\[
\chi_{\vec{h}}(C_{\vec{k}}) = \prod_i \oint \frac{1}{2\pi i} \frac{dz_i}{z_i^{h_i+1}} \left[ \left( \sum_{i=1}^{N} z_i \right)^{k_1} \left( \sum_{i=1}^{N} z_i^2 \right)^{k_2} \cdots \prod_{i<j} (z_i - z_j) \right]. \quad (5.22)
\]

The contour is taken around origin. Note that, in the above expressions for character (equation 5.22 or 5.22) variables \( x_i \)'s or \( z_i \)'s are auxiliary variables. They do not carry any physical meaning as of now.

Integrand in the above expressions can be exponentiated and written as,

\[
\chi_{\vec{h}}(C_{\vec{k}}) = \prod_i \oint \frac{1}{2\pi i} \frac{dz_i}{z_i} \exp \left[ \sum_{n=1}^{\infty} k_n \ln \left( \sum_{i=1}^{N} z_i^n \right) + \frac{1}{2} \sum_{i \neq j} \ln |z_i - z_j| - \sum_{i=1}^{N} h_i \ln z_i \right] \quad (5.23)
\]

In large \( N \) limit we define following variables

\[
\frac{z_i}{N} = z(x), \quad \frac{h_i}{N} = h(x), \quad x = \frac{i}{N} \quad k_n = N^2 k_n. \quad (5.24)
\]

In large \( N \) limit summation over \( i \) is replaced by an integral over \( x \),

\[
\sum_{i=1}^{\infty} \rightarrow N \int_0^1 dx. \quad (5.25)
\]

Therefore, in large \( N \) limit character is given by,

\[
\chi[h(x), C(\vec{k}')] = \left( \frac{1}{2\pi i} \right)^N \oint \frac{[Dz(x)]}{z(x)} \exp[-N^2 S_{\chi}(h[x], \vec{k}')] \quad (5.26)
\]

where action \( S_{\chi} \) is given by\(^5\)

\[
-S_{\chi}(h[x], \vec{k}') = \sum_n k_n' \left[ (n + 1) \ln N + \ln \left( \int_0^1 dx z^n(x) \right) \right] + \frac{1}{2} \int_0^1 dx \int_0^1 dy \ln |z(x) - z(y)| - K' \ln N - \int_0^1 dx h(x) \ln z(x) \quad (5.27)
\]

where, \( N^2 K' = K = \) total number of boxes in a representation.

In large \( N \) limit, dominant contribution to character comes from a particular distribution of \( z(x) \), which extremises the function \( S_{\chi} \). Extremisation condition is given by,

\[
\sum_n \frac{n k'_n}{Z_n} z^n(x) + \int_0^1 dy \frac{z(x)}{z(x) - z(y)} - h(x) = 0 \quad (5.28)
\]

\(^5\)Note that, this is not a regular action.
where,
\[ Z_n = \int_0^1 dx z^n(x). \]  \hfill (5.29)
Introducing,
\[ \rho(z) = \frac{\partial x}{\partial z} \]  \hfill (5.30)
above equation can be written as,
\[ \sum_n \frac{n k'_n}{Z_n} z^n + \int dz' z \rho(z') - h(z) = 0. \]  \hfill (5.31)
Since original contour in equation (5.22) was about the origin, therefore we take \( z = e^{i\theta} \) and above equation can be written as,
\[ \sum_n \frac{n k'_n}{Z_n} e^{i\theta} + \int_{-\pi}^{\pi} d\theta' \rho(\theta') \frac{e^{i\theta}}{e^{i\theta} - e^{i\theta'}} - h(\theta) = 0 \]  \hfill (5.32)
where,
\[ \rho(\theta) = \frac{\partial x}{\partial \theta} = i e^{i\theta} \frac{\partial x}{\partial z} = i e^{i\theta} \rho(z). \]  \hfill (5.33)
Breaking this equation into real and imaginary part we find that the imaginary part is given by,
\[ \sum_n \frac{n k'_n}{Z_n} \sin n\theta - \frac{1}{2} \int_{-\pi}^{\pi} d\theta' \rho(\theta') \cot \left( \frac{\theta - \theta'}{2} \right) = 0. \]  \hfill (5.34)
This equation is exactly same as eigenvalue equation (2.8) once we identify, \( n k'_n/Z_n = \beta_n \).
In the next subsection we shall see that the above identification is correct and auxiliary variables which appear in Frobenius formula are indeed eigenvalues of unitary matrices under consideration. These auxiliary variables satisfy not only the same equation as eigenvalues, partition function written in terms of these variables exactly matches with partition function written in terms of eigenvalues. Hence, in large \( N \) limit values of \( k'_n \) (number of cycles) are fixed in terms of parameters of the model i.e. \( \vec{\beta} \) and density \( \rho(\theta) \) defined above is same as the density of eigenvalues.

The real part of equation (5.32) is an algebraic equation, which relates number of boxes in a particular Young diagram with eigenvalues of unitary matrices,
\[ h(\theta) = \frac{1}{2} + \sum_n \frac{n k'_n}{Z_n} \cos n\theta. \]  \hfill (5.35)
In terms of $\theta$ and $\rho(\theta)$ action $S_x$ (5.27) is given by,

$$-S_x(h[x], k') = \sum_n k'_n \left[ (n + 1) \ln N + \ln \left( \int_{-\pi}^{\pi} d\theta \rho(\theta) \cos n\theta \right) \right] + \frac{1}{2} \int_{-\pi}^{\pi} d\theta \rho(\theta) \int_{-\pi}^{\pi} d\theta' \rho(\theta') \ln |e^{i\theta} - e^{i\theta'}| - K' \ln N - i \int_{-\pi}^{\pi} d\theta \rho(\theta) h(\theta) \theta \quad (5.36)$$

Since $\rho(\theta)$ is even function of $\theta$, the last terms implies that there exists a redundancy in $h(\theta)$. For any even function $f(\theta)$

$$S_x \left[ h(\theta), k \right] = S_x \left[ h(\theta) + f(\theta), k \right] .$$

This means that different Young distributions which are related by this transformation correspond to the same eigenvalue distribution $\rho(\theta)$. Hence, the most generic relation between $h(\theta)$ and $\theta$ is given by,

$$h(\theta) = \frac{1}{2} + \sum_n \frac{n k'_n}{Z_n} \cos n\theta + f(\theta) \quad (5.37)$$

in large $N$ limit. This redundancy plays an important role to give a phase space description of different phases of a unitary matrix model. We name this equation "boundary equation" as it defines the boundary of phase space. We shall get back to this point in section 5.4. Before that we explicitly show auxiliary variables $\theta_i$'s are indeed eigenvalues of unitary matrices. They satisfy not only the same saddle point equation but partition function written in terms of these auxiliary variables is same as the partition function written in terms of eigenvalues.

5.3 The partition function at large N

Using the expression for character given in equation (5.26) partition function (5.12) can be written as,

$$\mathcal{Z} = N \prod_n \int dk'_n d\ell'_n \int Dh(x) \oint Dz(x) \oint Dw(x) \int_{\infty}^{\infty} dt \int_{\infty}^{\infty} ds \exp \left( -N^2 S_{\text{total}} \left[ h, z, w, k', \ell', t, s \right] \right) \quad (5.38)$$
where,

\[-S_{\text{total}}[h, z, w, k', l', t, s] = \sum_n \left[ k'_n \left( 1 + \ln \left( \frac{\beta_n Z_n}{nk'_n} \right) \right) + l'_n \left( 1 + \ln \left( \frac{\beta_n W_n}{nl'_n} \right) \right) \right] + \frac{1}{2} \int_0^1 dx \int_0^1 dy \ln |z(x) - z(y)| + \ln |w(x) - w(y)| - \int_0^1 dxh(x) \ln z(x)w(x) + it \left( \sum_n nk'_n - K' \right) + is \left( \sum_n nl'_n - K' \right).\] (5.39)

Varying effective action with respect to \(h(x)\) we find,

\[z(x)w(x) = e^{-i(t+s)} = \text{constant (independent of } x)\]. (5.40)

Variation with respect to \(k'_n\) and \(l'_n\) provides,

\[\frac{\beta_n Z_n}{nk'_n} = e^{-itn}, \quad \text{and} \quad \frac{\beta_n Z_n}{nl'_n} = e^{-isn}.\] (5.41)

Since, in equation (5.38) \(z(x)\) and \(w(x)\) vary over some contour around zero, we take these contours to be unit circle about the origin and hence one set of solutions to above equations are,

\[z(x) = e^{i\theta(x)}, \quad w(x) = e^{-i\theta(x)}, \quad \text{and} \quad t = s = 0.\] (5.42)

Therefore,

\[k'_n = l'_n = \frac{\beta_n \rho_n}{n}, \quad \text{where} \quad Z_n = \int d\theta \rho(\theta) \cos n\theta \equiv \rho_n.\] (5.43)

Evaluating the partition function (5.38) on these equations we find,

\[Z = N \int [D\theta] \exp \left[ N^2 \sum_{n=1}^{\infty} 2 \frac{\beta_n}{n} \rho_n + N^2 \frac{1}{2} \int d\theta \rho(\theta) \int d\theta' \rho(\theta') \ln \left| 4 \sin^2 \frac{\theta - \theta'}{2} \right| \right].\] (5.44)

This partition function is exactly same as the partition function of one plaquette model (equation 2.7) written as an integral over eigenvalues of unitary matrices. Hence, we identify that the auxiliary variables we used to write the character of symmetric group as eigenvalues of corresponding unitary matrix model involved.

### 5.4 Redundancy in boundary equation and phase space distribution

It is evident either from equation (5.27) or (5.44) that in large \(N\) limit there is a redundancy in Young tableaux distribution. However, there is further restriction on this
redundancy, which comes from the fact that redundancy can not change the topology of phase space distribution given by a constant density. In fact, this restriction on redundancy actually fixes phase space distribution. Let us consider tow cases.

**No-gap phase:** For no-gap, phase space distribution is given by a region around the origin, i.e. for a given $\theta$ $h$ has two solutions 0 and $h_{\pm}(\theta) = 2\pi \rho(\theta)$. From equation (5.37) we get,

$$h(h - S(\theta)) = f^2(\theta) - \frac{1}{4} S^2(\theta),$$

where,

$$S(\theta) = 1 + 2 \sum_n \frac{n k'_n}{Z_n} \cos n\theta.$$  

Therefore, $h$ has solution 0 and $\rho(\theta)$ provided

$$S(\theta) = 2\pi \rho(\theta) \quad \text{and} \quad f(\theta) = \frac{1}{2} S(\theta) = \pi \rho(\theta).$$

**Multi-gap phase:** Similarly for multi-gap solution, topology of phase space distribution implies that there exists two non-zero solution for $h$, denoted by $h_+$ and $h_-$ and

$$h_+(\theta) - h_-(\theta) = 2\pi \rho(\theta).$$

Again, from equation (5.37) we get,

$$h^2 - S(\theta)h + \frac{1}{4} S^2(\theta) - f^2(\theta) = 0$$

which has solution

$$h_\pm(\theta) = \frac{1}{2} S(\theta) \pm f(\theta).$$

Hence, equation (5.48) implies,

$$f(\theta) = \pi \rho(\theta).$$

One can, in fact, construct Young distribution for each phase considering the identification $\theta = \pi u(h)$ from this boundary relation. However, the complete construction is not necessary in our current analysis and hence we shall present that in details in a follow up paper.
5.5 A generalised phasespace distribution

It had been observed in [6, 7] that for a particular class of unitary model (namely \((a, b)\) model), eigenvalue distribution and Young tableaux distribution can be obtained from a single distribution function. This function, which takes values either zero or one in the entire two dimensional plane \((h, \theta)\) plane), was identified with the phase space distribution of free fermions in one dimension. In this section we show that it is indeed possible to get all the information about eigenvalue and Young tableaux distribution of different phases of a unitary matrix model of type \((5.1)\) from a single phase space distribution.

We define a complex phasespace distribution \(\Omega(h, \theta)\) in \((h, \theta)\) plane such that

\[
\int_{0}^{\infty} \Omega(h, \theta) dh = 2R(e^{i\theta}) - 1.
\]  

(5.52)

Being complex one can break the function \(\Omega(h, \theta)\) into real and imaginary part

\[
\Omega(h, \theta) = \omega_R(h, \theta) + i\omega_I(h, \theta).
\]  

(5.53)

When integrated over \(h\), real part gives eigenvalue distribution where as imaginary part gives derivative of potential.

\[
\int_{0}^{\infty} dh \ \omega_R(h, \theta) = \rho(\theta), \ \int_{0}^{\infty} dh \ \omega_I(h, \theta) = V'(\theta).
\]  

(5.54)

Since, \(\rho(\theta)\) and \(V'(\theta)\) are even and odd function of \(\theta\) respectively, the real and imaginary part of \(\Omega(h, \theta)\) are also real and imaginary respectively. Hence, when we integrate \(\Omega(h, \theta)\) over \(\theta\) only the real part contributes and gives Young tableaux distribution function

\[
u(h) = \int_{-\pi}^{\pi} \Omega(h, \theta) d\theta.
\]  

(5.55)

As mentioned in the beginning of this section, free fermi phases spaces are two dimensional droplets in \((h, \theta)\) plane with constant density inside each droplets. Therefore, real part of \(\Omega(h, \theta)\) is given by,

\[
\omega_R(h, \theta) = \frac{1}{2\pi} \Theta[(h - \underline{h}(\theta))(\underline{h}(\theta) - h)]
\]  

(5.56)

where \(\underline{h}(\theta)\) is given by equation \((5.50)\).
Imaginary part of $\Omega(h, \theta)$ can also be written in terms of constant density droplets in $(h, \theta)$ plane

$$\omega_\Im(h, \theta) = \Theta(h - \sum_n \beta_n \sin n\theta), \quad 0 \leq \theta \leq \pi$$

$$= -\Theta(h + \sum_n \beta_n \sin n\theta), \quad -\pi \leq \theta \leq 0. \quad (5.57)$$

Note that the shape or topology of imaginary part of $\Omega(h, \theta)$ is fixed for a given model. This is because the imaginary part contains information about derivative of potential, which is fixed for a particular model. Whereas, topology of real part of $\Omega(h, \theta)$ changes as we go from one phase to another phase.

6. Phase space distribution of Riemann zeros

6.1 Multiple Young distribution corresponding to Riemann zeros

In this section we shall find Young distribution for no-gap solution which captures information about distribution of unfolded zeros of Riemann zeta function, discussed in section 4. Since eigenvalue distribution, given by equation (4.8), has no gap and hence belongs to no-gap phase. Therefore, boundary relation for this phase is given by,

$$h = \frac{1}{2} + \sum_n \beta_n \cos n\theta + \pi \rho(\theta)$$

$$= \frac{1}{2} - \alpha \sum_n \lambda_n \cos n\theta + \pi \rho(\theta)$$

$$= 2\pi \rho(\theta). \quad (6.1)$$

From the plot of $\rho(\theta)$ (fig. 2) it is clear that the relation is not one two one. For a given $h$ there are multiple possible values of $\theta$. Since $\theta$ is identified with density of Young distribution $\pi u(h)$, this implies there exists many possible Young representations which correspond to the same eigenvalue distribution. Each of these representations are defined such that $h$ varies monotonically from a lowest value to a highest value. For example, consider a particular sector of the plot 2 as shown in figure 4. Representation $R_1$ is defined as,

$$u_{R_1}(h) = \alpha_1, \quad h \in [0, h_{L1}]$$

$$= u(h), \quad h \in [h_{L1}, \infty]. \quad (6.2)$$
Similarly representation $R_2$ is defined as,

$$u_{R_2}(h) = \alpha_2, \quad h \in [0, h_{L1}]$$
$$= u(h), \quad h \in [h_{L1}, \infty]$$

(6.3)

and so on. The constants $\alpha_1$, $\alpha_2$ etc. are fixed from the normalisation condition

$$\int_0^\infty u(h) \, dh = 1$$

(6.4)

for each representation.

Thus we see that, for a given eigenvalue distribution there exists multiple representations of $U(N)$ group which extremises the character and the partition function.

6.2 Phase space distribution of Riemann zeros

Since boundary of phase space for no-gap solution is given by $h = \rho(\theta)$ one can draw a phase space distribution in $(h, \theta)$ plane. Since $h > 0$ we take $h$ as radial variable and $\theta$ periodic. The distribution is plotted in fig. (5).

We see that the phase space looks like broom with sticks at different positions of Riemann zeros.

6.3 Young distributions and prime counting function

Resolvent $R(z)$ is written in terms of generalised distribution

$$R(e^{i\theta}) = \frac{1}{2} + \int_0^\infty \Omega(h, \theta) \, dh.$$  

(6.5)
To understand a relation between Young distribution and distribution between prime numbers we introduce a function $J(x)$, as defined in [25], which captures information about distribution of prime numbers.

$$J(x) = \text{Li}(x) - \sum_{i} \text{Li}(x^{\rho_i}) - \ln 2 + \int_{x}^{\infty} \frac{dt}{t(t^2 - 1)} \quad x > 1$$

where, $\text{Li}(x)$ is logarithmic integral function defined as,

$$\text{Li}(x) = \int_{0}^{x} \frac{dt}{\ln t}$$

and $\rho_i$'s are zeros of Riemann zeta function. Function $J(x)$ is a monotonically increasing function and increases by a jump of $1/n$ at each $p^n$ where $p$ is a prime number.

Riemann showed that zeta function $\zeta(s)$ and prime counting function $J(x)$ are related to each other by,

$$\frac{\ln \zeta(s)}{s} = \int_{0}^{\infty} J(x)x^{-s-1}dx \quad \Re[s] > 1$$

$$J(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \ln \zeta(s)x^{s} \frac{ds}{s} \quad a > 1.$$  \hspace{1cm} (6.8)

Using this relation one can write,

$$\ln \xi \left(\frac{1}{2} + \frac{i}{2} \cot \frac{\theta}{2}\right) = \int_{0}^{\infty} x^{-\frac{1}{2} - \frac{i}{2} \cot \frac{\theta}{2}} \frac{d\tilde{J}}{dx} dx$$

\hspace{1cm} (6.9)
where $\tilde{J}(x)$ is fluctuating part of $J(x)$ and explicitly given in terms of zeroes of zeta function \[25\]
\[
d\tilde{J}(x) = -\frac{1}{\ln x} \sum_{\rho_i} x^{\rho_i-1} dx.
\]

Hence, from equation (6.9) we find that,
\[
-2\alpha \ln \xi \left(\frac{1}{2} + \frac{i}{2} \cot \frac{\theta}{2}\right) = 2R(e^{i\theta}) - 1
\]
\[
= -2\alpha \int_0^\infty [x(\tilde{J})]^{-\frac{1}{2}-\frac{i}{2} \cot \frac{\theta}{2}} d\tilde{J}
\]
\[
= -2\alpha \int_0^\infty \Omega(\tilde{J}, \theta) d\tilde{J}, \quad \text{where} \quad \Omega(\tilde{J}, \theta) = [x(\tilde{J})]^{-\frac{1}{2}-\frac{i}{2} \cot \frac{\theta}{2}}.
\]

where the limit in the integral is kept over $x$ for simplicity. Comparing this result with equation (6.5 and 5.52) we see a surprising similarity between the variables $\tilde{J}$ and $h$. Eigenvalue density can be obtained from a distribution function $\tilde{\Omega}(\tilde{J}, \theta)$ defined on $(\tilde{J}, \theta)$ plane and integrating over $\tilde{J}$. The difference is, in $(h, \theta)$ plane it the boundary of droplets which determines the eigenvalue density whereas in $(\tilde{J}, \theta)$ plane distribution function $\tilde{\Omega}(\tilde{J}, \theta)$ determines $\rho(\theta)$.

In $(\tilde{J}, \theta)$ plane one can also integrate over $\theta$ and find,
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \tilde{\Omega}(\tilde{J}, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta [x(\tilde{J})]^{-\frac{1}{2}-\frac{i}{2} \cot \frac{\theta}{2}}
\]
\[
= [x(\tilde{J})]^{-1}.
\]

Since $\tilde{J}(x)$ is fluctuation part of $J(x)$, mapping between $x$ and $\tilde{J}$ is not one to one. For a given values of $\tilde{J}$ there are infinite solution for $x$. Therefore, we can split $x$ into domains where the map $\tilde{J} \rightarrow x$ is one to one which is analogous to what happened for the case of Young Tableaux distribution function from the boundary equation. Also, since $1 < x < \infty$ we have $1 > [x(\tilde{J})]^{-1} > 0$. Thus we see an interesting similarity between Young distribution for Riemann zeta function and distribution of prime numbers
\[
h \leftrightarrow \tilde{J}, \quad u(h) \leftrightarrow [x(\tilde{J})]^{-1}.
\]

It is interesting to see that for this particular choice of the phase space density we have $\tilde{J}$ playing the role of $h$ and $x^{-1} \sim \theta$. This suggest that there exists a mapping between these variables which bear resemblance to canonical transformations in phase space descriptions. Notably since $\tilde{J}$ is related to the Fourier transform of $\ln \xi$ and Fourier transform preserves
the symplectic structure of Hamiltonian mechanics, this further provides evidence of an underlying phase space description of the nontrivial zeroes of the zeta function and the prime numbers. We intend to look into this problem in our future work.

7. Discussion

In this paper we find a universal feature that all the phases of a generic unitary matrix model can be described in terms of topology of free fermi phase space distribution. This is therefore, the right time to get back to the original question which we raised in the introduction: “Does there exist an underlying many particle non-interacting quantum description for each large N saddle point?” A series of papers [32, 33, 34, 35] by Dhar, Mandal and Wadia will be useful to understand this question in details. It would be interesting to see if one can formulate a quantum theory from classical (large N) phase space description. The difference between free fermi droplets obtained for different phases of unitary matrix model considered in this paper and those considered in the above mentioned papers is the boundary relation is not quadratic in \( q \) (position) and \( p \) (momentum), rather it is trigonometric [6, 7] in general, which boils down to quadratic relation for small values of \( \theta \) (position). Another important difference is, in our case the free fermions are living on group manifold which is compact, \( i.e. \) position coordinate \( \theta \) is ranges between \([0, 2\pi]\) and momentum \( h \) is always positive.

The identification \( \theta = \pi u(h) \) for \( \theta > 0 \) is not proved in this article. Although this is the case we realize that the above has correct bound on \( u(h) \) and also is always positive. We intend to provide more rigourous arguments regarding this in our future work but here we present some heurisetic arguments in appendix C which suggests the the identification is correct.

The second interesting result we present in this paper is a phase space distribution of unfolded zeros of Riemann zeta function. We also found a similarity (or correspondence) between Young distribution function for Riemann zeros and prime counting function. This observation gives a strong hint that unfolded zeros of Riemann zeta function and prime number behave like conjugate pairs, as also discussed in [2]. The eigenvalue distribution considered in this paper has divergences at Riemann zeros, in stead one could consider a matrix model whose eigenvalue density is exactly peaked at the unfolded zeros \( i.e. \rho(\theta) \)
is equal to derivative of zero counting function, i.e.
\[ \rho(\theta) \sim \frac{d}{d\theta} \left[ \sum_{i=\text{zeros}} \Theta(\theta - \theta_i) \right] \sim \sum_{i=\text{zeros}} \delta(\theta - \theta_i) \]
and check the relation between Young tableaux distribution and prime counting function. We hope to present the result in near future.

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**A. Calculation of \( \rho(\theta) \)**

\[
\rho(\theta) = -2\alpha \Re \left[ \ln \xi \left( \frac{1}{1 - e^{i\theta}} \right) \right] = -\alpha \ln \left[ \frac{\xi \left( \frac{1}{1 - e^{i\theta}} \right)}{\xi \left( \frac{1}{1 - e^{-i\theta}} \right)} \right] = 2\alpha \ln 2 - \alpha \sum_i \ln \left[ \frac{\rho_i(1 - e^{i\theta})}{\rho_i(1 - e^{-i\theta})} \right] = 2\alpha \ln 2 - \alpha \sum_i \ln \left[ \frac{1 - e^{i(\theta + \theta_i)}}{1 - e^{-i\theta}} \right] \quad (A.1)
\]

Replacing \( \theta_i \) by \( -\theta_i \) and simplifying we obtain:

\[
\rho(\theta) = 1 - \alpha \sum_i \left[ e^{i\left(\theta - \frac{\theta_i}{2}\right)} \left( e^{-i\left(\frac{\theta - \theta_i}{2}\right)} - e^{i\left(\frac{\theta - \theta_i}{2}\right)} \right) \right] \left[ e^{-i\left(\frac{\frac{\theta - \theta_i}{2}}{2}\right)} \left( e^{-i\left(\frac{\theta - \theta_i}{2}\right)} - e^{i\left(\frac{\theta - \theta_i}{2}\right)} \right) \right] = 2\alpha \ln 2 - \alpha \sum_i \ln \left[ \frac{\sin^2 \frac{\theta - \theta_i}{2}}{\sin^2 \frac{\theta}{2}} \right] \quad (A.2)
\]

putting the value of \( \alpha = \frac{1}{2m^2} \) we obtain the result.
B. Normalisation condition for $\rho(\theta)$

The following integral can also be checked in Mathematica:

\[
\int_{-\pi}^{\pi} \rho(\theta) d\theta = \int_{-\pi}^{\pi} \left[ 1 + 2\alpha \sum_i \ln \sin \left( \frac{\theta}{2} \right) - 2\alpha \sum_i \ln \sin \left( \frac{\theta - \theta_i}{2} \right) \right] d\theta \\
= 2\pi + 2\alpha \sum_i (i\pi^2 - 2\pi \ln 2) - 2\alpha \sum_i (i\pi\theta_i + i\pi^2 - 2\pi \ln 2) \\
= 2\pi - 2\pi i\alpha \sum_i \theta_i \\
= 2\pi.
\]

Where we have used the fact that the $\theta_i$ appear symmetrically about the real line and hence for every $\theta_i$ there is a $\theta_i$ and thus their sum is 0. Equivalently this can be checked using the normalization of $R(z)$:

\[
\frac{1}{2\pi i} \oint \frac{dz}{z} R(z) = 1. \quad \text{(B.1)}
\]

C. Identification $\theta = \pi u(h)$

Since the auxiliary variables in the Frobenius character formula were identically mapped to the eigen values of the Unitary matrix models we have the condition:

\[
\int dx \theta(x) h(x) = 0 \quad \text{(C.1)}
\]

This means $h(x)$ is always an even function of $\theta(x)$. Now we start with the action defined in 5.36 and add an additional term

\[
\int dx \mu(x) \frac{\partial x}{\partial h(x)} h(x) \quad \text{(C.2)}
\]

where $\mu$ can be understood as a Lagrange multiplier which enforces the constraint $u(h) > 0$. Collecting the $h(x)$ dependent terms only, we have the following:

\[
\int dx \theta(x) h(x) + \int dx \mu(x) \frac{\partial x}{\partial h(x)} \theta(x) \quad \text{(C.3)}
\]
Since $h(x)$ is always positive we can write $h = \gamma^*\gamma$ following [21]. Variation with respect to the phase part of $\gamma^*$ yields the equation of motion:

$$\theta(x) + \mu(x) \frac{\partial x}{\partial h(x)} = 0$$

$$- \frac{\partial x}{\partial h(x)} = \frac{1}{\mu(x)} \theta(x)$$

(C.4)

Inserting this in C.2 we always have the value of the integral 0 as $h(x)$ has to be an even function of $\theta(x)$. Now for positivity of $u(h) = -\partial x/\partial h$ we must have $1/\mu(x)$ to be an odd function of $\theta(x)$ and for $u(h) < 1$ we must have $|\mu| > \pi$. For the particular choice $\mu = \pi \text{sign}(\theta)$ we have $u(h) = \frac{1}{\pi} \theta$, which is the identification obtained. Thus the multiplier $\mu$ shows us that the identification $\pi u(h) = \theta$ satisfies all the constraints needed.

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