Circle actions and geometric quantisation

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Abstract

The aim of this paper is to present unifying proofs for results in geometric quantisation by exploring the existence of symplectic circle actions. It provides an extension of Rawnsley’s results on the Kostant complex, and gives another proof for Śniatycki’s and Hamilton’s theorems; as well, a partial result for the focus-focus contribution to geometric quantisation.

Contents

1 Introduction 2
  1.1 Acknowledgements ........................................ 3

2 Geometric quantisation 3
  2.1 Prequantisation ............................................. 3
  2.2 Geometric quantisation à la Kostant ................... 5

3 Resolution approach 6

4 Circle actions and homotopy operators 9

5 The Bohr-Sommerfeld condition 14

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1 Introduction

Geometric quantisation tries to associate a Hilbert space to a symplectic manifold via a complex line bundle. Although it is possible to describe the canonical quantisation using this language, most of the difficulties arise when one tries to mimic this procedure for symplectic manifolds which are not naturally cotangent bundles. Those appear in the context of reduction and are far from being artificial mathematical models.

The first difficulty is to isolate in a global way position and momentum, in order to define wave functions as sections of a complex line bundle over the symplectic manifold. The second issue, that will not be address here, is how to define a Hilbert structure; however, all examples treated in this paper have a natural one.

As suggested by Kostant, wave functions will be associated to elements of higher cohomology groups and the quantum phase space will be built from these groups. Although the honest-to-goodness quantum phase space will not be constructed.

At least two approaches can be used to compute the cohomology groups: Čech and de Rham. The results of Hamilton [3] and Hamilton and Miranda [4] are based on a Čech approach. Here a de Rham approach, as in [8, 7], is used: by finding a resolution for the sheaf, as suggested in [4].

This paper follows closely Rawnsley’s ideas [7] and explores the existence of circle actions to provide an alternative proof for the theorems of Śniatycki
and Hamilton [3]. The tools developed here highlight and unravel the role played by symplectic circle actions in known results in geometric quantisation. Not only that, this approach casts some light on a conjecture about the contributions coming from focus-focus type of singularities.

Throughout this paper and otherwise stated, all the objects considered will be $C^\infty$; manifolds are real, Hausdorff, paracompact and connected; $C^\infty(V)$ denotes the set of complex valued functions over $V$; and the units are such that $\hbar = 1$.

1.1 Acknowledgements

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2 Geometric quantisation

2.1 Prequantisation

This subsection deals with some concepts needed to define wave functions. The first attempt was to see them as sections of a complex line bundle over the symplectic manifold, the so-called prequantum line bundle. The other notion described here, the polarisation, is a way to define a global distinction between momentum and position.

**Definition 2.1.** A symplectic manifold $(M, \omega)$ such that $[\omega]$ is integral is called prequantisable. A prequantum line bundle of $(M, \omega)$ is a hermitian line bundle over $M$ with connection, compatible with the hermitian structure, $(L, \nabla_\omega)$ that satisfies $\text{curv}(\nabla_\omega) = -i\omega$.

**Example 2.1.** Any exact symplectic manifold satisfies $[\omega] = 0$, in particular: cotangent bundles with the canonical symplectic structure. In that case the trivial line bundle is an example of a prequantum line bundle.

The following theorem provides a relation between the above definitions:

**Theorem 2.1** (Kostant). A symplectic manifold $(M, \omega)$ admits a prequantum line bundle $(L, \nabla_\omega)$ if and only if it is prequantisable.

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$^1$This result is also attributed to André Weil, Introduction à l’étude des variétés kählériennes (1958).
Lemma 2.1. The potential 1-forms of $\nabla^\omega$ for each unitary section defined in a subset of $M$ are cohomologous. Conversely, if there is an unitary section over a subset of $M$, any 1-form which is cohomologous to the potential 1-form of $\nabla^\omega$ is the potential 1-form of an unitary section. Moreover, any subset of $M$ where $\omega = d\theta$ has an unitary section such that $\theta$ is its potential 1-form.

Proof: Let $s$ and $r$ be two unitary sections defined over $N \subset M$, and $\Theta$ and $\vartheta$ the associated potential 1-forms. If $s = e^{i\theta}r$ for some real valued $f \in C^\infty(N)$, then

$$-i\Theta \otimes s = \nabla^\omega s = \nabla^\omega (e^{i\theta}r) = [de^{i\theta} - ie^{i\theta}\vartheta] \otimes r = -i[-df + \vartheta] \otimes e^{i\theta}r \Rightarrow \vartheta - \Theta = df.$$  

Conversely, by the same computation, if $s$ has $\Theta = \vartheta - df$ as potential 1-form, then $r = e^{-i\theta}s$ is a unitary section having $\vartheta$ as potential 1-form.

For $\omega = d\theta$ over $N \subset M$, let $\mathcal{A} = \{A_j\}_{j \in I}$ be a contractible open cover of $N$ such that each $A_j$ is a local trivialisation of $L$ with unitary section $s_j$ (this can always be obtained, e.g. using a convenient cover made of balls with respect to a riemannian metric). Each unitary section $s_j$ has $\Theta_j$ as a potential 1-form and since $\text{curv} (\nabla^\omega)|_{A_j} = -id\theta|_{A_j} = -id\Theta_j \Rightarrow$ there exists real valued functions $f_j \in C^\infty(A_j)$ such that $\theta|_{A_j} = \Theta_j - df_j$. By the above argument, the unitary sections $r_j = e^{-i\theta} s_j$ have $\theta|_{A_j}$ as potential 1-forms.

Any two sections $r_j$ and $r_k$ such that $A_j \cap A_k \neq \emptyset$ share the same potential 1-form, and because of that, they differ by a nonzero constant function, $r_j = c_{jk}r_k$ at $A_j \cap A_k$. Trivially, $c_{jk}$ can be extended to the same constant over $A_k$, and $c_{jk}r_k$ is a section defined over $A_k$ such that its restriction to $A_j \cap A_k$ is exactly $r_j$, and it still has $\theta|_{A_k}$ as potential 1-form. Hence, they can be glued together, using the gluing condition of sheaves, to a unitary section $r$ defined over $N$ and having $\theta$ as potential 1-form. $\blacksquare$

A real polarisation $\mathcal{P}$ is an integrable subbundle of $TM$ whose leaves are lagrangian submanifolds. But due to the example below, another definition is considered.

For an integrable system $F : M^{2n} \to \mathbb{R}^n$ on a symplectic manifold, the Liouville integrability condition implies that the distribution of the hamiltonian vector fields of the components of the moment map generates a lagrangian foliation (possible) with singularities. This is an example of a generalised real polarisation, i.e.: an integrable distribution on $TM$ whose leaves are lagrangian submanifolds, except for some singular leaves.
**Definition 2.2.** A real polarisation $\mathcal{P}$ is an integrable (in the Sussmann’s sense) distribution of $TM$ whose leaves are generically lagrangian. The complexification of $\mathcal{P}$ is denoted by $\hat{\mathcal{P}}$ and will be called polarisation.

From now on $(L, \nabla^\omega)$ will be a prequantum line bundle and $\hat{\mathcal{P}}$ the complexification of a real polarisation of $(M, \omega)$.

### 2.2 Geometric quantisation à la Kostant

The original idea of geometric quantisation is to associate a Hilbert space to a symplectic manifold via a prequantum line bundle and a polarisation. Usually this is done using flat global sections of the line bundle. In case these global sections do not exist, one can define geometric quantisation via higher cohomology groups by considering cohomology with coefficients in the sheaf of flat sections.

The existence of global flat sections is a nontrivial matter. Actually Rawnsley [7] (and also proposition 4.3 in this paper) showed that the existence of a $S^1$-action may be an obstruction for the existence of nonzero global flat sections.

In order to use flat sections as analogue for wave functions one is forced to work with delta functions with support over Bohr-Sommerfeld leaves, or deal with sheaves and higher order cohomology groups. Both approaches can be found in the literature, but here only the sheaf approach is treated: as suggested by Kostant.

**Definition 2.3.** Let $\mathcal{J}$ denotes the space of local sections $s$ of a prequantum line bundle $L$ such that $\nabla_X^\omega s = 0$ for all vector fields $X$ of a polarisation $\mathcal{P}$. The space $\mathcal{J}$ has the structure of a sheaf and it is called the sheaf of flat sections.

Considering the triplet prequantisable symplectic manifold $(M, \omega)$, prequantum line bundle $(L, \nabla^\omega)$, and polarisation $\mathcal{P}$:

**Definition 2.4.** The quantisation of $(M, \omega, L, \nabla^\omega, \mathcal{P})$ is given by

$$Q(M) = \bigoplus_{k \geq 0} \check{H}^k(M; \mathcal{J}) ,$$

where $\check{H}^k(M; \mathcal{J})$ are Čech cohomology groups with values in the sheaf $\mathcal{J}$. In this case, one implicitly assumes the extra structures and calls $M$ a quantisable manifold.

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2Rawnsley cites works of Simms, Śniatycki and Keller in [7].
Remark 2.1. Even though $Q(M)$ is just a vector space and a priori has no Hilbert structure, it will be called quantisation. The true quantisation of the triplet $(M, \omega, L, \nabla^\omega, P)$ shall be the completion of the vector space $Q(M)$, after a Hilbert structure is given, together with a Lie algebra homomorphism (possibly defined over a small set) between the Poisson algebra of $C^\infty(M)$ and operators on the Hilbert space. In spite of the problems that may exist in order to define geometric quantisation using $Q(M)$, the first step is to compute this vector space.

Remark 2.2. Flat sections behave in a different fashion for the Kähler case. This paper does not deal with this case, however much can be found in the literature (e.g.: [2, 3] and references therein).

3 Resolution approach

Following Rawnsley [7], given a prequantisable symplectic manifold with polarisation, it is possible to construct a fine resolution for the sheaf of flat sections. The propositions contained in this section are extensions of the results in [7]: it is mainly an opportunity to fix notation, the replacement of a subbundle of $TM$ by an integral distribution offers no obstruction and, therefore, proofs are omitted.

The restriction of the connection $\nabla^\omega$ to the polarisation induces a linear operator
\[ \nabla : \Gamma(L) \to \Gamma(P^* \otimes_{C^\infty(M)} \Gamma(L)) \] (4)
satisfying (by definition) the following property:
\[ \nabla(fs) = d^P f \otimes s + f \nabla s, \] (5)
for $f \in C^\infty(M)$ and $s \in \Gamma(L)$, where $d^P$ is the restriction of the exterior derivative to the distribution directions.

Remark 3.1. Although $P$ is not a subbundle of $TM^C$ when it is singular, for the sake of simplicity, the same notation for vector bundles will be used for it: $\Gamma(P) = \langle X_1, \ldots, X_n \rangle_{C^\infty(M)}$, where $X_1, \ldots, X_n \in \Gamma(TM)$ generates the real distribution $P$, and $\Gamma(\wedge^k P^*)$ is the space of smooth alternating $C^\infty(M)$-multilinear maps from the Cartesian product of $k$ copies of $\Gamma(P)$ to $C^\infty(M)$.

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As previously mentioned, for the concrete cases presented in this paper $Q(M)$ does admit Hilbert structures.
Definition 3.1. The space of line bundle valued polarised forms is $S_P^\bullet(L) = \bigoplus_{k \geq 0} S^k_P(L)$, where $S^k_P(L) = \Gamma(\wedge^k P^*) \otimes_{\mathcal{C}^\infty(M)} \Gamma(L)$.

Wherefore, $\nabla : S^0_P(L) \to S^1_P(L)$ and $S_P^\bullet(L)$ has a module structure, which enables an extension of $\nabla$ to a derivation of degree 1 on the space of line bundle valued polarised forms, as follows:

Definition 3.2. Denoting $\Gamma(\wedge^k P^*)$ by $\Omega^k_P(M)$, $\Omega_P^\bullet(M) = \bigoplus_{k \geq 0} \Omega^k_P(M)$ is the space of polarised forms.

If $\alpha \in \Omega^k_P(M)$ and $\beta = \beta \otimes s \in S^1_P(L)$,

$$\alpha \wedge \beta = \alpha \wedge (\beta \otimes s) = (\alpha \wedge \beta) \otimes s$$

(6)
defines a left multiplication of the ring $\Omega_P^\bullet(M)$ on $S_P^\bullet(L)$.

Definition 3.3. The derivation on $S_P^\bullet(L)$ is given by the degree +1 map $d^- : S_P^\bullet(L) \to S_P^\bullet(L)$,

$$d^- \beta = d^- (\beta \otimes s) = d_P \beta \otimes s + (-1)^k \beta \wedge \nabla s.$$  

(7)

Proposition 3.1. If $\alpha \in \Omega^k_P(M)$ and $\beta \in S^1_P(L)$, then

$$d^- (\alpha \wedge \beta) = d_P \alpha \wedge \beta + (-1)^k \alpha \wedge d^- \beta,$$

(8)
and

$$d^- \circ d^- \beta = curv(\nabla^\omega)|_P \wedge \beta.$$  

(9)

Since $\omega = i \ curv(\nabla^\omega)$ vanishes along $P$, $d^- \circ d^- = 0$.

Corollary 3.1. $d^-$ is a coboundary operator.

Remark 3.2. The only property of $L$ being used in this paper is the existence of flat connections along $P$; any complex line bundle would do, not only a prequantum one. In particular: the tensor product between a prequantum line bundle and a bundle of half forms normal to $P$ — the results here work if metaplectic correction is included.

If $S^k_P(L)$ denotes the associated sheaf of $S^k_P(L)$, one can extend $d^-$ to a homomorphism of sheaves, $d^- : S^k_P(L) \to S^{k+1}_P(L)$. $S^0_P(L) \cong S$, the sheaf of sections of the line bundle $L$, and $J$ is isomorphic to the kernel of $d^- : S \to S^1_P(L)$. Because $d^- \circ d^- = 0$, one is able to build a sequence.
Definition 3.4. The Kostant complex is

\[ 0 \rightarrow \mathcal{J} \hookrightarrow \mathcal{S} \xrightarrow{\nabla} S^1_P(L) \xrightarrow{d\nabla} \cdots \xrightarrow{d\nabla} S^n_P(L) \xrightarrow{d\nabla} 0. \] (10)

The sheaves $S^k_P(L)$ are fine; $\Gamma(L)$ and $\Omega^k_P(M)$ are free modules over the ring of functions of $M$, and by that, they admit partition of unity. Hence, via a Poincaré lemma, the abstract de Rham theorem offers a proof for the following:

Theorem 3.1. The Kostant complex is a fine resolution for $\mathcal{J}$. Therefore, each of its cohomology groups, $H^k(S^\bullet_P(L))$, are isomorphic to $\tilde{H}^k(M; \mathcal{J})$.

Remark 3.3. There are particular situations in which a Poincaré lemma is available, and only in these cases theorem holds. This is true when $\mathcal{P}$ is a subbundle of $TM$, and it can be extended to a more general setting, as it is announced in [9]. Using symplectic circle actions, this paper provides a proof when $\mathcal{P}$ is given by a locally toric singular lagrangian fibration (subsection 6.7) or an almost toric fibration in dimension 4 (subsection 6.8).

As expected, the notions of interior product and Lie derivative are available for $S^\bullet_P(L)$. The Lie derivative can be seen as a derivation along a flow, but for that, a nontrivial notion of pullback is needed.

Definition 3.5. The contraction between line bundle valued polarised forms and sections of $P$ is given by a map $i : \Gamma(P) \times S^\bullet_P(L) \rightarrow S^\bullet_P(L)$, which is a degree -1 map on $S^\bullet_P(L)$, i.e.: for each $X \in \Gamma(P)$ and $\beta = \beta \otimes s \in S^1_P(L)$ it holds that

\[ i_X(\nabla s) = \nabla_X s , \] (11)

\[ i_X\beta = i_X(\beta \otimes s) = (i_X\beta) \otimes s . \] (12)

Proposition 3.2. If $X \in \Gamma(P)$, $\alpha \in \Omega^k_P(M)$ and $\beta \in S^1_P(L)$, then $i_X \circ i_X = 0$ and

\[ i_X(\alpha \wedge \beta) = (i_X\alpha) \wedge \beta + (-1)^k \alpha \wedge i_X\beta . \] (13)

Using Cartan’s magic formula it is possible to define a Lie derivative on $S^\bullet_P(L)$:

Definition 3.6. The Lie derivative $\mathcal{L}^\nabla : \Gamma(P) \times S^\bullet_P(L) \rightarrow S^\bullet_P(L)$ is defined by:

\[ \mathcal{L}^\nabla_X(\beta) = i_X \circ d\nabla \beta + d\nabla \circ i_X\beta . \] (14)

Proposition 3.3. The Lie derivative $\mathcal{L}^\nabla$ commutes with the derivation $d\nabla$.

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4Both Śniatycki and Rawnsley attribute this to Kostant, a proof is provided in [7].
For $\Omega^k_P(M)$ the pullback still makes sense if one restricts to diffeomorphisms that preserve the polarisation $P$, but problems arise when one twists it with $\Gamma(L)$. A way to compare elements of $L$ is by parallel transport, which in general is path dependent. When it does not, the pullback on $S_P^\bullet(L)$ is well-defined (for diffeomorphisms that preserve the polarisation). When it depends on the path, it is possible to make sense of pullbacks over paths.

Now, if $X \in \Gamma(P)$ its flow $\phi_t$ already encodes both a curve and a diffeomorphism:

**Definition 3.7.** The pullback $\phi_t^* \alpha$ of $\alpha = \alpha \otimes s \in S^k_P(L)$ is defined by

$$\phi_t^* \alpha = (\phi_t^* \alpha) \otimes \Pi_{\phi_t}(s \circ \phi_t) ;$$

(15)

where, by the bundle automorphism property of the parallel transport, $\Pi_{\phi_t(p)}$ denotes the parallel transport between $p$ and $\phi_t(p)$ through the integral curve of the flow.

The following proposition justifies, somehow, the choices made for the interior product and pullback.

**Proposition 3.4.** The Lie derivative $\mathcal{L}^\nabla_X$ can be characterised by

$$\mathcal{L}^\nabla_X(\alpha) = \frac{d}{dt} \phi_t^* \alpha \bigg|_{t=0} .$$

(16)

**Corollary 3.2.** The Lie derivative $\mathcal{L}^\nabla_X$ commutes with the pullback $\phi_t^*$.

**Proposition 3.5.** If $X \in \Gamma(P)$ with flow $\phi_t$, $\alpha \in \Omega^k_P(M)$ and $\beta \in S^l_P(L)$, then

$$\phi_t^*(\alpha \wedge \beta) = (\phi_t^* \alpha) \wedge \phi_t^*(\beta) ,$$

(17)

and the pullback $\phi_t^*$ commutes with $d^\nabla$.

### 4 Circle actions and homotopy operators

This section explains the construction of an almost homotopy operator for the Kostant complex when one has a symplectic $S^1$-action, and how this implies the vanishing of the stalks of points with nontrivial holonomy. Most results of this section were previously provided in [7] with slightly less general hypothesis; some proofs automatically hold (propositions 4.1, 4.2 and 4.3), but one (lemma 4.2) had to be adapted.

Let $X \in \Gamma(P)$ be a generator of a symplectic $S^1$-action. If $\phi_t$ stands for the flow of $X$ at time $t$, it is possible to define an induced action on $S^k_P(L)$
by \( \phi_t^* \). Denoting by \( \operatorname{per(}\gamma) \) the period\(^3\) of the closed orbit \( \gamma \) (eventually constant in the case of a fixed point) of \( X \), it holds that:

\[
\phi_{\operatorname{per(}\gamma)}^* (\alpha \otimes s) = \phi_{\operatorname{per(}\gamma)}^* \alpha \otimes \Pi_{\phi_{\operatorname{per(}\gamma)}}^{-1} (s \circ \phi_{\operatorname{per(}\gamma)}) = \alpha \otimes [Q(\gamma)^{-1}] = Q(\gamma)^{-1} \alpha ,
\]

\( Q(\gamma) \) is the holonomy of \( \gamma \), and

\[
[Q(\gamma)^{-1} - 1] \alpha = \phi_{\operatorname{per(}\gamma)}^* \alpha - \phi_0^* \alpha = \int_0^{\operatorname{per(}\gamma)} \frac{d}{dt}(\phi_t^* \alpha) \, dt
\]

\[
= \int_0^{\operatorname{per(}\gamma)} \frac{d}{ds} \phi_{t+s}^* \alpha \bigg|_{s=0} \, dt = \int_0^{\operatorname{per(}\gamma)} \frac{d}{ds} \phi_s^* (\phi_t^* \alpha) \bigg|_{s=0} \, dt
\]

\[
= \int_0^{\operatorname{per(}\gamma)} \mathcal{L}_X^\nabla (\phi_t^* \alpha) \, dt = \int_0^{\operatorname{per(}\gamma)} [i_X \circ d^\nabla + d^\nabla \circ i_X] (\phi_t^* \alpha) \, dt
\]

\[
= i_X \left( \int_0^{\operatorname{per(}\gamma)} d^\nabla (\phi_t^* \alpha) \, dt \right) + d^\nabla \circ i_X \left( \int_0^{\operatorname{per(}\gamma)} \phi_t^* \alpha \, dt \right).
\]

Using that the pullback commutes with the derivative (proposition \([3.5]\)) one gets from the last equation

\[
[Q(\gamma)^{-1} - 1] \alpha = i_X \left( \int_0^{\operatorname{per(}\gamma)} \phi_t^* (d^\nabla \alpha) \, dt \right) + d^\nabla \circ i_X \left( \int_0^{\operatorname{per(}\gamma)} \phi_t^* \alpha \, dt \right),
\]

which resembles the equation satisfied by a homotopy operator.

**Proposition 4.1.** The expression \( J_X(\alpha) = i_X \left( \int_0^{\operatorname{per(}\gamma)} \phi_t^* \alpha \, dt \right) \) defines a degree \(-1\) derivation on \( S_{p^*} L \).

**Proof:** Propositions \([3.2]\) and \([3.5]\) imply that \( J_X \) is a derivation, and the degree comes from the fact that \( i_X \) has degree \(-1\). \( \blacksquare \)

The equation \([18]\) implies that \( J_X \) satisfies

\[
[Q(\gamma)^{-1} - 1] \alpha = J_X (d^\nabla \alpha) + d^\nabla J_X (\alpha),
\]

for any \( \alpha \in S_p^k(L) \) if \( k \geq 1 \), whilst for \( k = 0 \) it becomes

\[
[Q(\gamma)^{-1} - 1] \alpha = J_X (d^\nabla \alpha),
\]

since \( S_{p^{-1}}^1(L) \) is empty and \( J_X \) has degree \(-1\).

\(^3\)Indeed, \( \operatorname{per(}\gamma) \in C^\infty(M) \): for each \( p \in M \) the function \( \operatorname{per(}\gamma) \) gives the period of the orbit of \( X \) passing through \( p \).
Proposition 4.2. \(d \nabla [Q(\gamma)^{-1} - 1] \alpha = [Q(\gamma)^{-1} - 1] d \nabla \alpha\) for any \(\alpha \in S_L^k(L)\), hence \(Q(\gamma)\) is constant along \(P\).

Proof: It is a direct consequence of equation 19:

\[
d \nabla [Q(\gamma)^{-1} - 1] \alpha = d \nabla [J_X(d \nabla \alpha) + d \nabla J_X(\alpha)] = d \nabla J_X(d \nabla \alpha),
\]

(21)

\[
[Q(\gamma)^{-1} - 1] d \nabla \alpha = J_X(d \nabla \circ d \nabla \alpha) + d \nabla J_X(d \nabla \alpha) = d \nabla J_X(d \nabla \alpha).
\]

(22)

\[\blacksquare\]

Lemma 4.1. Let \(X\) be the generator of a symplectic \(S^1\)-action with orbits \(\gamma\), then its holonomy can be computed by

\[
Q(\gamma) = e^{i \text{per}(\gamma) \theta(X)};
\]

(23)

where \(\theta\) is a particular invariant potential 1-form for \(\omega\) in a neighbourhood of \(\gamma\).

Proof: Weinstein’s theorem for isotropic embeddings \[11\] asserts that in a neighbourhood \(N\) of an orbit \(\gamma\) the symplectic form is exact, \(\omega = d \theta\) —the potential 1-form can be chosen invariant by averaging it with respect to the flow of \(X\). Let \(s \in \Gamma(L|_N)\) be the unitary section given by lemma 2.1 which has \(\theta\) as the potential 1-form for \(\nabla\).

The parallel transport of a section \(r = f s \in \Gamma(L|_N)\) is given by

\[
r \circ \gamma(t) = e^{i \int_0^t \theta(\gamma(t')) dt'} r \circ \gamma(0).
\]

(24)

Indeed, \(\nabla_Xr = X(f)s - if \theta(X)s\) and \(X\gamma(t) = \dot{\gamma}(t)\), thus \(X(f)|_{\gamma(t)} = \frac{df}{dt} \circ \gamma(t)\),

\[
\nabla_{\dot{\gamma}(t)} r = \frac{d}{dt} f \circ \gamma(t) s \circ \gamma(t) - if \circ \gamma(t) \theta(\gamma(t)) s \circ \gamma(t)
\]

(25)

and the parallel transport equation becomes

\[
\begin{cases}
\frac{d}{dt} f \circ \gamma(t) = if \circ \gamma(t) \theta(\gamma(t)) \\
f \circ \gamma(0) = [s \circ \gamma(0)]^{-1} r \circ \gamma(0)
\end{cases}.
\]

(26)

Cartan’s magic formula and the invariance of \(\theta\) give:

\[
0 = \mathcal{L}_X(\theta) = i_X d \theta + d(i_X \theta) \Rightarrow i_X \omega = -d \theta(X),
\]

(27)

wherefore, near \(\gamma\), the action is hamiltonian, and \(\theta(X)\) is its hamiltonian function. In particular, since \(\gamma\) is an integral curve of the hamiltonian flow, \(\theta(\dot{\gamma}(t))\) is constant;

\[
r \circ \gamma(t) = e^{i \theta(X)} r \circ \gamma(0),
\]

(28)

proving the desired result when \(t = \text{per}(\gamma)\).

\[\blacksquare\]
Proposition 4.3. Supposing that \((M, \omega)\) admits a symplectic \(S^1\)-action preserving \(P\), flat sections of \(L\) vanish if the holonomy of the orbits of the circle action is nontrivial over a dense set.

Proof: Let \(s \in \Gamma(L)\) be a flat section, \(\nabla s = 0\). By (20) \([Q(\gamma)^{-1} - 1]s = 0\) and on the dense set where \(Q(\gamma) \neq 1\) the flat section vanishes. Consequently, there are no nonzero flat sections. ■

Lemma 4.2. Under the hypothesis that \(\{Q(\gamma) \neq 1\}\) is dense, a form \(\alpha \in S^k_P(L)\) vanishes where \(Q(\gamma) = 1\) if and only if there exists a \(\beta \in S^k_P(L)\) such that \(\alpha = [Q(\gamma)^{-1} - 1]\beta\).

Proof: If \(\alpha = [Q(\gamma)^{-1} - 1]\beta\) it is obvious that \(\alpha\) vanishes where \(Q(\gamma) = 1\). If the converse holds for functions on \(M\), in any trivialising neighbourhood \(A\) with unitary section \(s\) and coordinates \((z_1, \ldots, z_{2n})\) the form \(\alpha\) can be expressed by

\[
\alpha = \sum_{j_1, \ldots, j_k = 1}^{2n} \alpha_{j_1, \ldots, j_k}(z_1, \ldots, z_{2n}) dz_{j_1} \wedge \cdots \wedge dz_{j_k} \otimes s.
\]

Moreover, \(\alpha = 0\) at \(\{Q(\gamma) = 1\}\) if and only if all the functions \(\alpha_{j_1, \ldots, j_k}\) vanish on \(A \cap \{Q(\gamma) = 1\}\). Hence, there exists functions \(\beta_{j_1, \ldots, j_k}\) such that \(\alpha_{j_1, \ldots, j_k} = [Q(\gamma)^{-1} - 1]\beta_{j_1, \ldots, j_k}\). \(M\) can be covered by trivialising neighbourhoods and the local functions \(\beta_{j_1, \ldots, j_k}\) piece together to give the desired \(\beta \in S^k_P(L)\).

Therefore, given \(f \in C^\infty(A)\) satisfying \(f|_{A \cap \{Q(\gamma) = 1\}} = 0\) one must construct a \(g \in C^\infty(A)\) such that \(f = [Q(\gamma)^{-1} - 1]g\).

For points where 1 is a regular value of \(Q(\gamma)\), theorem 4 in [7] proves that this expression holds for functions. On the other hand, lemma 11 implies that critical points of \(Q(\gamma)\) are fixed points of the \(S^1\)-action, and that locally \(Q(\gamma) = e^{2\pi i h}\) for some function \(h\).

By shrinking \(A\), and possibly changing \(h\) by a constant, one can assume that only 0, and no other integer, satisfies \(A \cap h^{-1}(\{0\}) \neq \emptyset\). With the aid of any riemannian metric on \(A\), the gradient of \(h\) and the flow \(\varphi_t\) of \(-hZ\), the vector field \(Z = \text{grad}(h)\) times minus \(h\), are used to define a function \(g \in C^\infty(A)\):

\[
g = \int_0^\infty Z(f \circ \varphi_t) \, dt \quad \frac{2\pi i}{\int_0^1 e^{-2\pi ith} \, dt}.
\]

Indeed, for \(h = 0\)

\[
\int_0^1 e^{-2\pi ith} \, dt = 1.
\]
and for $h \neq 0$

$$\int_0^1 e^{-2\pi ith} \, dt = \frac{Q(\gamma)^{-1} - 1}{-2\pi ih}. \quad (32)$$

Thus the denominator in expression $30$ never vanishes, whilst

$$g = \frac{\int_0^\infty Z(f \circ \varphi_t) \, dt}{2\pi i [Q(\gamma)^{-1} - 1]} = \frac{-\int_0^\infty hZ(f \circ \varphi_t) \, dt}{Q(\gamma)^{-1} - 1}$$

$$= \frac{\int_0^\infty \frac{d}{dt} f \circ \varphi_t \, dt}{Q(\gamma)^{-1} - 1} = \frac{f - \lim_{t \to \infty} f \circ \varphi_t}{Q(\gamma)^{-1} - 1}. \quad (33)$$

Since the limit $\lim_{t \to \infty} \varphi_t(p)$ is a fixed point for any point $p \in A$ and $f|_{A \cap \{Q(\gamma) = 1\}} = 0$, on $A \cap \{Q(\gamma) \neq 1\}$ it holds that $f = [Q(\gamma)^{-1} - 1]g$. By continuity of $f$, $g$ and density of $\{Q(\gamma) \neq 1\}$ this must be true over all $A$.

**Proposition 4.4.** Let $\alpha \in S_k^p(L)$ be closed, $d^\nabla \alpha = 0$, and $k \neq 0$.

- The form $\alpha$ is exact everywhere $Q(\gamma) \neq 1$. It is also globally exact if $\{Q(\gamma) \neq 1\}$ is dense and $J_X(\alpha) = 0$ where $Q(\gamma) = 1$.

- When $\{Q(\gamma) = 1\}$ is a (not necessarily connected) submanifold, if $\alpha$ is exact on $M$, then $J_X(\alpha)|_{\{Q(\gamma) = 1\}}$ is exact; the converse holds if $\{Q(\gamma) \neq 1\}$ is dense.

**Proof:** At points satisfying $Q(\gamma) \neq 1$ a $(k - 1)$-form $\beta$ is well defined by

$$\beta = \frac{J_X(\alpha)}{Q(\gamma)^{-1} - 1}. \quad (34)$$

Proposition 4.2 and equation 19, together with the hypothesis of $\alpha$ being closed, imply that $d^\nabla \beta = \alpha$. In other words, $J_X/[Q(\gamma)^{-1} - 1]$ is a homotopy operator where $Q(\gamma) \neq 1$.

For $J_X(\alpha) = 0$ at $\{Q(\gamma) = 1\}$, lemma 4.2 gives a $\sigma \in S_{k-1}^p(L)$ such that $J_X(\alpha) = [Q(\gamma)^{-1} - 1]\sigma$; therefore, $\beta$ is again well defined by the expression $34$.

Assuming that $\{Q(\gamma) = 1\}$ is a submanifold, one consequence of proposition 4.2 (as it was observed in [17]) is that the polarisation is tangent to it, and all definitions make sense with $M$ replaced by $\{Q(\gamma) = 1\}$.

If $\alpha = d^\nabla \beta$, by applying equation 19

$$J_X(\alpha) = J_X \circ d^\nabla \beta = [Q(\gamma)^{-1} - 1]d^\nabla \beta - d^\nabla \circ J_X(\beta); \quad (35)$$
and \( J_X(\alpha)_{\{Q(\gamma)=1\}} \) is exact.

Conversely, if \( J_X(\alpha)_{\{Q(\gamma)=1\}} = d\nabla \big|_{\{Q(\gamma)=1\}} \zeta \), taking an extension \( \eta \in S^k_p(L) \) of \( \zeta \), the formula \( (J_X(\alpha) - d\nabla \eta)_{\{Q(\gamma)=1\}} = 0 \) holds and lemma 4.2—the density of \( \{Q(\gamma) \neq 1\} \) is assumed—provides a \( \beta \in S^{k-1}_p(L) \) such that \( J_X(\alpha) - d\nabla \eta = [Q(\gamma)^{-1} - 1]\beta \). Proposition 4.2 implies that

\[
d\nabla \circ J_X(\alpha) = d\nabla ([Q(\gamma)^{-1} - 1]\beta) = [Q(\gamma)^{-1} - 1]d\nabla \beta,
\]

but equation 19 reads

\[
d\nabla \circ J_X(\alpha) = [Q(\gamma)^{-1} - 1]\alpha,
\]

thus \( d\nabla \beta = \alpha \) holds where \( Q(\gamma) \neq 1 \). Since \( d\nabla \beta \) is everywhere defined, if \( \{Q(\gamma) \neq 1\} \) is a dense set, \( \alpha \) must be exact.

This proposition is a key tool to prove that the Kostant complex is a fine resolution, theorem 3.1, when the (singular) polarisation comes from an almost or locally toric structure.

5 \textbf{The Bohr-Sommerfeld condition}

The following definition plays a very important role in the computation of the cohomology groups appearing in geometric quantisation:

\textbf{Definition 5.1.} A leaf \( \ell \) of \( P \) is a Bohr-Sommerfeld leaf\(^6\) if there exists a nonzero section \( s : \ell \to L \) such that \( \nabla_X s = 0 \) for any vector field \( X \) of \( P \) restricted to \( \ell \).

\textbf{Example 5.1.} The cotangent bundle of the circle with the canonical symplectic structure: \( M = \mathbb{R} \times S^1 \) and \( \omega = dx \wedge dy \) in cylindrical coordinates \((x, y)\). \( L \) the trivial bundle with connection 1-form \( \Theta = xdy \) with respect to \( e^{ix} \) and \( \mathcal{P} = \langle \frac{d}{dy} \rangle \). Flat sections satisfy \( \nabla_X s = ds(X) - i\Theta(X)s = 0 \). Hence \( s(x, y) = f(x)e^{ixy} \), for some function \( f \), and Bohr-Sommerfeld leaves are given by the condition \( x \in \mathbb{Z} \).

\textbf{Proposition 5.1.} A leaf \( \ell \) of \( P \) is a Bohr-Sommerfeld leaf if and only if \( Q(\gamma) = 1 \) for any loop on a connected component of \( \ell \).

\(^6\text{Fibres which are composed of an union of Bohr-Sommerfeld leaves are called Bohr-Sommerfeld fibres.}\)
Proof: In a Bohr-Sommerfeld leaf $\ell$ the nonzero section $s$ can be used to define a potential 1-form $\Theta$ of the connection on the whole leaf. The potential 1-form vanishes on $\ell$, since $0 = \nabla_X s = i\Theta(X)s$. Thus if $\gamma$ is a loop on $\ell$, by equation (24), $Q(\gamma) = e^{i\int\Theta} = 1$.

Now supposing that $Q(\gamma) = 1$ for any loop on a connected component of a leaf $\ell$ of $P$, for any point $p \in \ell$ and a nonzero $s_p \in L_p$ (the fibre of $L$ over $p$) it is possible to define a nonzero section $s$ over $\ell$ by parallel transport, i.e.: $s(q) = \Pi_{\gamma_1}(s_p)$, where $\gamma_1$ is any curve connecting $p$ and $q \in \ell$. The section is well-defined because if $\gamma_2$ is another curve connecting $p$ and $q$, and $\gamma$ the loop formed by composing $\gamma_2$ and $\gamma_1^{-1}$,

$$s(q) = Q(\gamma)s(q) = \Pi_{\gamma}(s(q)) = \Pi_{\gamma_2} \circ [\Pi_{\gamma_1}]^{-1}(s(q))$$
$$= \Pi_{\gamma_2} \circ [\Pi_{\gamma_1}]^{-1} \circ \Pi_{\gamma_1}(s_p) = \Pi_{\gamma_2}(s_p).$$

The parallel transport respects the hermitian product, and this guarantees that the section defined in this way is nonzero.

There is a stronger characterisation for the Bohr-Sommerfeld leaves in the case of integrable systems.

**Theorem 5.1.** Under the assumption that the zero fibre is Bohr-Sommerfeld, the image of Bohr-Sommerfeld fibres by a nondegenerate moment map is contained in $\mathbb{R}^{n-k} \times \mathbb{Z}^k$; $k$ being the number of linearly independent hamiltonian $S^1$-actions generated by the moment map. Furthermore, the image of singular fibres is contained in a discrete set.

**Proof:** The Liouville tori case was proved by Guillemin and Sternberg in [2] —their theorem holds for lagrangian fibrations with compact connected fibres over simply connected basis. Lemma 4.1 and proposition 5.1 imply that over a Bohr-Sommerfeld fibre each component of the moment map generating a $S^1$-action takes an integer value, depending only on the fibre. It is consequence of nondegeneracy that the image of singular fibres is contained in a discrete set, the Morse-Bott condition insures it.

**Example 5.2.** For toric manifolds the Bohr-Sommerfeld fibres are the inverse image by the moment map of integer lattice points in the polytope, with regular ones inside the polytope.

---

7A moment map is nondegenerate when all of its critical points are nondegenerate in the Morse-Bott sense. The reader is referred to [12, 6] for precise definition and properties.
6 Applications I: local and semilocal computations

6.1 The cylinder: polarisation by circles

Recalling example 5.1: \((M = \mathbb{R} \times S^1, \omega = dx \wedge dy)\), \(L\) the trivial bundle with connection 1-form \(\Theta = xdy\) (with respect to the unitary section \(e^{ix}\)), and \(\mathcal{P} = (\frac{\partial}{\partial y})\).

The Hamiltonian vector field \(X = \frac{\partial}{\partial y}\) generates a \(S^1\)-action, and the holonomy of its orbits is given by \(Q(\gamma) = e^{i2\pi x}\) (lemma 4.2 for \(S^1 \cong \mathbb{R}/2\pi \mathbb{Z}\)). Since Bohr-Sommerfeld leaves satisfy \(x \in \mathbb{Z}\), the holonomy is nontrivial in a dense set of \(M\) and proposition 4.3 holds. Hence, applying theorem 3.1, one gets \(\check{H}^0(M; J) = \{0\}\).

Moreover, proposition 4.4 and theorem 3.1 can be applied implying \(\check{H}^l(V; J|_V) = \{0\}\), \(l \geq 1\), for each neighbourhood \(V = (a, b) \times S^1\) that does not contain a Bohr-Sommerfeld leaf.

Let \(\ell_k\) be the inverse image by the height function of the point \(x = k \in \mathbb{Z}\). Wherefore, \(\ell_k \cong S^1\) is a Bohr-Sommerfeld leaf and \(\{Q(\gamma) = 1\} = \bigcup_{k \in \mathbb{Z}} \ell_k\).

It is possible\(^8\) to define a linear map \(\Psi : S^1_p(L) \to \bigoplus_{k \in \mathbb{Z}} \Gamma(L|_{\ell_k})\) by:

\[
\Psi(\alpha) = \oplus_{k \in \mathbb{Z}} J_X(\alpha)\big|_{\ell_k}.
\]

Because the dimension of \(M\) is 2, \(S^1_p(L) = \{0\}\) for \(l \geq 2\); and for any \(\alpha \in S^1_p(L)\), equation 19 reads

\[
\nabla \circ J_X(\alpha) = [Q(\gamma)^{-1} - 1]\alpha \implies \nabla_X\Psi(\alpha) = 0.
\]

Thus the image of \(\Psi\) is contained in the set of flat sections over Bohr-Sommerfeld leaves.

Conversely, given \(\bigoplus_{k \in \mathbb{Z}} s_k \in \bigoplus_{k \in \mathbb{Z}} \Gamma(L|_{\ell_k})\), where \(s_k\) are flat sections \(\nabla_X s_k = 0\), there exists \(s \in \Gamma(L)\) such that \(s|_{\ell_k} = s_k\) for each \(k \in \mathbb{Z}\), due the closedness of \(\bigcup_{k \in \mathbb{Z}} \ell_k\). Lemma 4.2 implies the existence of an \(\alpha \in S^1_p(L)\) satisfying

\[
\nabla s = [Q(\gamma)^{-1} - 1]\alpha \implies [Q(\gamma)^{-1} - 1]J_X(\alpha) = J_X(\nabla s) = [Q(\gamma)^{-1} - 1]s.
\]

\(^8\)This construction is due to Rawnsley [7].
By density and continuity $J_X(\alpha) = s$, hence, the image of $\Psi$ is the set of flat sections over Bohr-Sommerfeld leaves.

Proposition 4.4 asserts that $\ker \Psi = \nabla(\Gamma(L))$, and the first isomorphism theorem
\[
\begin{array}{rcl}
S^1_p(L) & \longrightarrow & \Psi(S^1_p(L)) \\
\downarrow & & \searrow \Psi \\
S^1_p(L)/\ker \Psi & \cong & \Psi(S^1_p(L))
\end{array}
\] 
implies that $\hat{H}^1(M; \mathcal{J}) \cong \bigoplus_{k \in \mathbb{Z}} \mathbb{C}$; the ring of flat sections over $\ell_k$ is isomorphic to $\mathbb{C}$ (see example 5.1).

**Proposition 6.1.** The quantisation of a cylinder polarised by circles is $\mathbb{C}^{b_s}$, where $b_s$ is the number of Bohr-Sommerfeld leaves.

### 6.2 The complex plane: polarisation by circles

Let $(M = \mathbb{C}, \omega = dx \wedge dy)$ and $F : M \rightarrow \mathbb{R}$ be a nondegenerate integrable system of elliptic type, i.e.: $F(x, y) = x^2 + y^2$. For this case, the real polarisation is $\mathcal{P} = (-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y})$ and the hamiltonian vector field $X = -2y \frac{\partial}{\partial x} + 2x \frac{\partial}{\partial y}$ is the generator of a $S^1$-action; its periodic flow is given by $\phi_t(x, y) = (x \cos 2t - y \sin 2t, x \sin 2t + y \cos 2t)$.

As in the previous cases, $(M, \omega)$ is an exact symplectic manifold and the trivial line bundle is a prequantum line bundle for it: $L = \mathbb{C} \times \mathbb{C}$ with connection 1-form $\Theta = \frac{1}{2}(xdy - ydx)$ with respect to the unitary section $e^{i(x^2 + y^2)}$.

The holonomy of the orbits of $X$ is $Q(\gamma) = e^{i2\pi(x^2 + y^2)}$, lemma 4.2. Therefore, it is clear that $\{Q(\gamma) \neq 1\}$ is a dense set and proposition 4.3 can be applied to prove that $H^0(S^*_p(L)) = 0$.

The set $\{Q(\gamma) = 1\}$ is the union of the origin and concentric circles of integer radius, and since the origin is a fixed point, the operator $J_X$ is the null operator when restricted to the origin. Hence proposition 4.4, applied for each contractible neighbourhood of the origin that does not contain any other Bohr-Sommerfeld leaf, implies that elliptic singularities give no contribution to quantisation —this was first proved in [3] using different techniques. In other words: theorem 3.1 holds in this particular setting.

**Proposition 6.2.** The quantisation of an open disk polarised by circles is $\mathbb{C}^{b_s}$, where $b_s$ is the number of nonsingular Bohr-Sommerfeld leaves.

**Proof:** $M$ can be divided up\footnote{The complex plane will be considered; for an arbitrary open disk the same argument works.} into an open disk $V$ of radius $b < 1$ centred...
at the origin, and an annulus \( W \) centred at the origin with small radius \( a \in (0, b) \) and an infinite big radius: \( M = V \cup W \) and \( V \cap W \) is an annulus with small radius \( a \) and big radius \( b \). It was computed that \( \check{H}^k(V; \mathcal{F}|_{V \cap W}) = \{0\} \) for all \( k \), and since \( V \cap W \cong (a, b) \times S^1 \) (polarised by circles) it holds, by proposition \ref{6.1}, \( \check{H}^k(V \cap W; \mathcal{F}|_{V \cap W}) = \{0\} \) for all \( k \) as well. Thus the Mayer-Vietoris argument works and \( \check{H}^k(M; \mathcal{F}) \cong \check{H}^k(W; \mathcal{F}|_{W}) \). It happens that \( W \cong (a, \infty) \times S^1 \) (polarised by circles), and proposition \ref{6.1} concludes the proof. \(\square\)

6.3 Symplectic vector spaces: linear polarisation

All symplectic vector spaces polarised by lagrangian hyperplanes are equivalent to this particular case: \((M = \mathbb{C}^n, \omega = \sum_{j=1}^{n} dx_j \wedge dy_j)\) and \( \mathcal{P} = \langle \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} \rangle \).

Since \((M, \omega)\) is an exact symplectic manifold the trivial line bundle is a prequantum line bundle for it: \( L = \mathbb{C} \times \mathbb{C}^n \) with connection 1-form \( \Theta = \sum_{j=1}^{n} x_j dy_j \) with respect to the unitary section \( \exp \left( i \sum_{j=1}^{n} x_j y_j \right) \).

The solutions of the flat equation, \( \nabla s = 0 \), are complex valued functions of the type \( s(x_1, \ldots, x_n, y_1, \ldots, y_n) = h(x_1, \ldots, x_n) \exp \left( i \sum_{j=1}^{n} x_j y_j \right) \).

Therefore, \( \check{H}^0(M; \mathcal{F}) = \{s \in \Gamma(L) \; ; \; \nabla s = 0\} \cong C^\infty(\mathbb{R}^n) \).

Using the unitary flat section \( r = \exp \left( i \sum_{j=1}^{n} x_j y_j \right) \) as basis, if \( \alpha \otimes r \in S^k_p(L) \) is closed, wherefore:

\[ 0 = d^\nabla (\alpha \otimes r) = d_p \alpha \otimes r + (-1)^k \alpha \wedge \nabla r = d_p \alpha \otimes r \, . \quad (43) \]

Thus Poincaré lemma for regular foliations \cite{7} imply that \( H^k(S^\bullet_p(L)) = \{0\} \) for \( k \geq 1 \), and theorem \ref{3.1} asserts that:

**Proposition 6.3.** The quantisation of the cotangent bundle of \( \mathbb{R}^n \) with linear polarisation is \( C^\infty(\mathbb{R}^n) \).

This is by no means an example where the techniques developed in this paper are used; howbeit, this result is needed below.

6.4 Direct product type with a regular component

The following quantisation problem will be considered now: \( N = (-1, 1) \times S^1 \) endowed with the same structures (symplectic form, polarisation and
prequantum line bundle) of the model in subsection 6.1 and \((M, \omega)\) a pre-
quantiisable symplectic manifold with real polarisation \(P\) and prequantum
line bundle \((L, \nabla_\omega)\). The product \(N \times M\) admits \((\frac{\partial}{\partial y}) \oplus P\) as a real polarisation
for the symplectic form \(\text{d}x \wedge \text{d}y + \omega\) (its complexification will be denoted
by \(\mathcal{P}\)), and also a prequantum line bundle \((\mathcal{L}, \nabla_{\text{d}x \wedge \text{d}y + \omega})\).

The vector field \(\frac{\partial}{\partial y}\) generates a hamiltonian \(S^1\)-action with nontrivial
holonomy over a dense set; the holonomy of its orbits is given by \(Q(\gamma) = e^{2\pi i\gamma}\) (lemma 4.1). Wherefore, proposition 4.3 can be used to show that
\(H^0(S^*\mathcal{P}(\mathcal{L})) = \{0\}\).

For the other groups one has:

**Theorem 6.1.** The map \(\Psi : H^k(S^*\mathcal{P}(\mathcal{L})) \rightarrow H^{k-1}(S_P^\bullet(L))\) defined by
\(\Psi([\alpha]) = \left[ J_{\frac{\partial}{\partial y}}(\bar{\alpha}) \right]_{\{Q(\gamma) = 1\}}\) is an isomorphism.

**Proof:** Let \(\bar{\alpha} = \bar{\alpha} \otimes \bar{s} \in \mathcal{S}^k_P(\mathcal{L}); \bar{\alpha} = \text{d}y \wedge \bar{\beta} + \bar{\sigma},\) where \(\bar{\beta} = \frac{\partial}{\partial y} \bar{\alpha}\) and
\(\bar{\sigma} = \bar{\alpha} - \text{d}y \wedge \bar{\beta};\) thus \(\frac{\partial}{\partial y} \bar{\beta} = \frac{\partial}{\partial y} \bar{\sigma} = 0.\)

\[
J_{\frac{\partial}{\partial y}}(\bar{\alpha}) = \int_0^{2\pi} \phi_t^* \bar{\beta} e^{-itx} \text{d}t \otimes \bar{s} \Rightarrow J_{\frac{\partial}{\partial y}}(\bar{\alpha}) |_{\{Q(\gamma) = 1\}} = \eta \otimes \bar{s}|_{x = 0},
\]

where \(\eta = \int_0^{2\pi} \phi_t^* \bar{\beta} e^{-itx} \text{d}t |_{\{x = 0\}}.\) The flow of \(\frac{\partial}{\partial y}\) preserves \(\eta,\) and therefore, it belongs to \(\Omega_P^{k-1}(M).\)

For closed \(\bar{\alpha}\) it holds that

\[
d^\nabla \circ J_{\frac{\partial}{\partial y}}(\bar{\alpha}) = [Q(\gamma)^{-1} - 1] \bar{\alpha} \Rightarrow d^\nabla \circ J_{\frac{\partial}{\partial y}}(\bar{\alpha}) |_{\{Q(\gamma) = 1\}} = 0,
\]
so

\[
d_P \eta \otimes \bar{s}|_{x = 0} + (-1)^{k-1} \eta \wedge \nabla \bar{s}|_{\{Q(\gamma) = 1\}} = 0
\]
and contraction with \(\frac{\partial}{\partial y}\) gives \(\nabla \frac{\partial}{\partial y} \bar{s}|_{x = 0} = 0.\)

Therefore, for each point \(p \in M, \bar{s}|_{x = 0}\) is uniquely determined by its value
at \((0, 0, p) \in N \times M\) by parallel transport along integral curves of \(\frac{\partial}{\partial y}\). This
means that \(\bar{s}|_{x = 0}\) identifies itself as a section of \(L\): the restriction of \(\mathcal{L}\) to
\(\{(0, 0)\} \times M\) is a line bundle over \(M\) with a connection such that its curvature
is equal to \(\omega\), and consequently, it must be isomorphic to \(L\).

To summarise it, after some identifications, \(J_{\frac{\partial}{\partial y}}(\cdot) |_{\{Q(\gamma) = 1\}}\) maps closed
\(k\)-forms of \(S^*\mathcal{P}(\mathcal{L})\) to closed \((k - 1)\)-forms of \(S_P^\bullet(L),\) and proposition 4.3
proves that \(\Psi\) is injective — the set \(\{Q(\gamma) = 1\}\) is equal to \(\{0\} \times S^1 \times M.\)
Now, given \( r \in \Gamma(L) \), let \( \bar{r} \in \Gamma(L) \) be an extension of the following section defined on \( \{ x = 0 \} \): after identifying \( \mathcal{L} \mid \{(0,0)\} \times M \) with \( L \), for each point \( p \in M \), the parallel transport of \( r(p) \) by the integral curve of \( \frac{\partial}{\partial y} \) passing through \( p \) defines a section of \( \mathcal{L} \) over the set \( \{ x = 0 \} \).

By inclusion \( \Omega_{k-1}^P(M) \subset \Omega_{k-1}^P(N \times M) \), and for any \( [\zeta \otimes r] \in H^{k-1}(S_P^\bullet(L)) \), the expression \( \bar{\zeta} = \zeta \otimes \bar{r} \) defines an element in \( S_k^P(\mathcal{L}) \).

Via parallel transport (which commutes with the derivation), the form \( d\bar{\mathcal{V}}\bar{\zeta}\mid_{\{Q(\gamma)=1\}} \) is completely determined by the zero form, \( d\bar{\mathcal{V}}(\zeta \otimes r) \) and lemma 5.2 provides an \( \bar{\alpha} \in S_k^P(\mathcal{L}) \) such that \( d\bar{\mathcal{V}}\bar{\zeta} = [Q(\gamma)^{-1} - 1]\bar{\alpha} \). Consequently,

\[
0 = d\bar{\mathcal{V}} \circ d\bar{\mathcal{V}}\bar{\zeta} = [Q(\gamma)^{-1} - 1]d\bar{\mathcal{V}}\bar{\alpha},
\]

implying that \( \bar{\alpha} \) is closed over the dense set \( \{ x \neq 0 \} \), and by continuity, \( \bar{\alpha} \) is closed.

As consequence of \( \frac{\partial}{\partial y} \zeta \) being zero, \( J_{\frac{\partial}{\partial y}}(\bar{\zeta}) = 0 \) and equation 19 reads

\[
[Q(\gamma)^{-1} - 1]\bar{\zeta} = J_{\frac{\partial}{\partial y}} \circ d\bar{\mathcal{V}}\bar{\zeta} = J_{\frac{\partial}{\partial y}} ([Q(\gamma)^{-1} - 1]\bar{\alpha}) = [Q(\gamma)^{-1} - 1]J_{\frac{\partial}{\partial y}}(\bar{\alpha}),
\]

which implies \( J_{\frac{\partial}{\partial y}}(\bar{\alpha}) = \bar{\zeta} \) where \( x \neq 0 \); and again by density and continuity, it must hold true everywhere. This proves that \( \Psi \) is onto.

The theorem still holds if \( N \) is replaced by \( (a, b) \times S^1 \) with \( (a, b) \cap \mathbb{Z} = \{ k \} \). For \( (a, b) \cap \mathbb{Z} = \emptyset \), propositions 4.3 and 4.4 assert that all cohomology groups \( H^i(S_P^\bullet(\mathcal{L})) \) vanish; the quantisation of the product is trivial when there is no Bohr-Sommerfeld leaf.

By a Mayer-Vietoris argument\(^{10}\) similar to the one of subsection 6.2 one can compute the product quantisation for \( (a, b) \cap \mathbb{Z} = \{ k_1, \ldots, k_b \} \): the cover \( \mathcal{A} = \{ A_j \}_{j \in \{1, \ldots, b\}} \), where \( A_1 = (a, k_1 + 3/4) \times M \), \( A_{k_0} = (k_0 - 3/4, b) \times M \) and \( A_j = (k_j - 3/4, k_j + 3/4) \times M \) suffices (supposing \( k_1 \leq k_2 \leq \cdots \leq k_b \)).

**Corollary 6.1.** Assuming that the Kostant complex is a fine resolution for the sheaf of flat sections of \( L \), the quantisation of the product between a cylinder polarised by circles and an arbitrary quantisable manifold \( M \) is a direct sum of \( b_s \) copies of \( \mathcal{Q}(M) \); where \( b_s \) is the number of Bohr-Sommerfeld leaves with respect to the quantisation of the cylinder.

### 6.5 Direct product type with an elliptic component

The quantisation problem to be considered here is: \( N = \{(x, y) \in \mathbb{C} : x^2 + y^2 < 1 \} \) endowed with the same structures (symplectic form, polarisation

\(^{10}\)One might also use a global argument as in subsection 6.1 instead.
and prequantum line bundle) of the model in subsection 6.2, and \((M, \omega)\) a prequantisable symplectic manifold with real polarisation \(P\) and prequantum line bundle \((L, \nabla^\omega)\). The product \(N \times M\) admits \((-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}) \oplus P\) as a real polarisation for the symplectic form \(dx \wedge dy + \omega\) (the complexification of it will be denoted by \(\mathcal{P}\)), and also a prequantum line bundle \((\mathcal{L}, \nabla^{dx \wedge dy + \omega})\).

The group \(H^0(S\mathcal{P}(\mathcal{L}))\) is trivial because \(X = -2y \frac{\partial}{\partial x} + 2x \frac{\partial}{\partial y}\) generates a hamiltonian \(S^1\)-action with nontrivial holonomy over a dense set; lemma 4.1 gives \(Q(\gamma) = e^{i2\pi(x^2+y^2)}\) for the holonomy of its orbits, wherefore, proposition 4.3 holds. Whilst for higher order groups, one needs to note that the set \(\{Q(\gamma) = 1\}\) is equal to \(\{(0,0)\} \times M\), and that \((0,0,p)\) are fixed points for any \(p \in M\); thus the operator \(J_X\) is the null operator when restricted to \(\{(0,0)\} \times M\). Therefore, by applying proposition 4.4, \(H^k(S\mathcal{P}(\mathcal{L})) = \{0\}\) for \(k \geq 1\).

By a Mayer-Vietoris argument similar to the ones used in subsections 6.2 and 6.4:

**Proposition 6.4.** Assuming that the Kostant complex is a fine resolution for the sheaf of flat sections of \(L\), the quantisation of the product between an open disk polarised by circles and an arbitrary quantisable manifold \(M\) is a direct sum of \(b_s\) copies of \(Q(M)\); where \(b_s\) is the number of nonsingular Bohr-Sommerfeld leaves with respect to the quantisation of the open disk.

### 6.6 Neighbourhood of a Liouville fibre

The Liouville theorem for integrable systems provides a symplectic normal form for a neighbourhood of a regular fibre. What follows is the computation of the quantisation of that model.

Let \(M = \mathbb{R}^n \times (\mathbb{R}^{n-k} \times T^k)\) and \(0 \leq k \leq n\), where \(T^k \cong \mathbb{R}^k / 2\pi \mathbb{Z}^k\), with coordinates \((x_1, \ldots, x_n, y_1, \ldots, y_{n-k}, \ldots, y_n)\) and symplectic form \(\omega = \sum_{j=1}^{n} dx_j \wedge dy_j\). It admits as a real polarisation \(P = \left( \frac{\partial}{\partial y_n}, \ldots, \frac{\partial}{\partial y_1}\right)\), and since \((M, \omega)\) is an exact symplectic manifold, it also admits as a prequantum line bundle \(L = \mathbb{C} \times M\) with connection 1-form \(\Theta = \sum_{j=1}^{n} x_j dy_j\) with respect to the unitary section \(\exp \left( i \sum_{j=1}^{n} x_j \right)\).

The next proposition computes the contributions to geometric quantisation for each trivialising neighbourhood of a lagrangian fibre bundle.
Proposition 6.5. $H^{k+1}(Sp^\bullet(L)) = \{0\}$ for all $l \neq 0$ and

$$H^k(Sp^\bullet(L)) \cong \begin{cases} \bigoplus_{m \in \mathbb{Z}^k} C^\infty(\mathbb{R}^{n-k}) & \text{if } k \neq n \\ \bigoplus_{m \in \mathbb{Z}^k} C & \text{if } k = n \end{cases}. \quad (49)$$

Proof: Supposing $k \neq n$, when $M$ is written as $(\mathbb{R} \times S^1)^k \times \mathbb{C}^{n-k}$ it becomes clear that the use of theorem 6.1 (more precisely, corollary 6.1) $k$ times reduces the problem of computing the quantisation of $M$ to the computation of the quantisation of $\mathbb{C}^{n-k}$; which by proposition 6.3 is just $C^\infty(\mathbb{R}^{n-k})$. If $k = n$, one just need to apply theorem 6.1 $n-1$ times, and then proposition 6.1 to conclude. $\blacksquare$

6.7 Neighbourhood of an elliptic fibre

Toric, or locally toric, manifolds also have a normal form for a neighbourhood of its fibres, even if they are singular; Zung [12] attributes this normal form to Dufour and Molino and Eliasson. The model is described below, as well its quantisation.

For $0 \leq k \leq n$, let $(M = \mathbb{R}^{n+k} \times T^{n-k}, \omega = \sum_{j=1}^n dx_j \wedge dy_j)$, with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_k, \ldots, y_n)$, and $F : M \to \mathbb{R}^n$ be a nondegenerate integrable system of elliptic type, i.e.:

$$F(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1^2 + y_1^2, \ldots, x_k^2 + y_k^2, x_{k+1}, \ldots, x_n). \quad (50)$$

The real polarisation in this case is $P = \langle -y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1}, \ldots, -y_k \frac{\partial}{\partial x_k} + x_k \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_{k+1}}, \ldots, \frac{\partial}{\partial y_n} \rangle$, and since $(M, \omega)$ is an exact symplectic manifold, it also admits as a prequantum line bundle $L = \mathbb{C} \times M$ with connection 1-form

$$\Theta = \frac{i}{2} \sum_{j=1}^k (x_j dy_j - y_j dx_j) + \sum_{j=k+1}^n x_j dy_j \text{ with respect to the unitary section}$$

$$\exp \left[ i \sum_{j=1}^k (x_j^2 + y_j^2) + i \sum_{j=k+1}^n x_j \right].$$

Proposition 6.6. $\mathcal{Q}(M) \cong \mathbb{C}^{b_s}$, where $b_s$ is the number of nonsingular Bohr-Sommerfeld fibres.

Proof: One can first use proposition 6.4 $k$ times, then corollary 6.1 $n-k-1$ times, and finally proposition 6.1. $\blacksquare$
It is important to notice that when applying proposition 6.4 and corollary 6.1, one is only computing the cohomology groups of the underlying Kostant complex — theorem 3.1 is not being assumed. In fact, if it was considered \( x_1^2 + y_1^2, \ldots, x_k^2 + y_k^2 < 1 \), the previous proof would give that all cohomology groups vanish when \( k \neq 0 \). This proves that the Kostant complex computes geometric quantisation when the polarisation is given by a locally toric singular lagrangian fibration, theorem 3.1.

6.8 Focus-focus contribution to geometric quantisation

Let \( F = (f_1, f_2) : M^4 \to \mathbb{R}^2 \) be an integrable system for a prequantisable \((M^4, \omega)\), with a nondegenerate focus-focus singular fibre \( \ell_{ff} \) which is Bohr-Sommerfeld. In [13] it is demonstrated the existence of a neighbourhood of \( \ell_{ff} \) over which \( f_2 \) generates, via its hamiltonian vector field flow, a hamiltonian \( S^1 \)-action.

In a small enough neighbourhood \( W \) of a singular point of a focus-focus Bohr-Sommerfeld fibre \( J_X \) is the null operator over the points where \( \{ Q(\gamma) = 1 \} \). Indeed, near a singular point of \( \ell_{ff} \) the symplectic local model is given by[11]. \( W \cong \mathbb{C}^2 \) with coordinates \((x_1, x_2, y_1, y_2)\), \( L|_W \cong \mathbb{C} \times \mathbb{C}^2 \) with connection 1-form
\[
\Theta = \frac{1}{2}(x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2) \tag{51}
\]
with respect to the unitary section \( e^{i(x_1 y_2 - x_2 y_1)} \). The integrable system takes the form
\[
F(x_1, x_2, y_1, y_2) = (x_1 y_1 + x_2 y_2, x_1 y_2 - x_2 y_1) \tag{52}
\]
and therefore the polarisation is generated by
\[
X_1 = -x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} \tag{53}
\]
and
\[
X_2 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2}. \tag{54}
\]

The hamiltonian vector field \( X_2 \) is the generator of the \( S^1 \)-action. Its periodic flow is given by:
\[
\phi_t(x_1, x_2, y_1, y_2) = (x_1 \cos t + x_2 \sin t, x_2 \cos t - x_1 \sin t, y_1 \cos t + y_2 \sin t, y_2 \cos t - y_1 \sin t). \tag{55}
\]

\[\textsuperscript{11}\text{Zung [12, 13] cites Eliasson, Lerman and Umanskiy and Vey, but this particular model appears in Eliasson’s thesis, Normal forms for Hamiltonian systems with Poisson commuting integrals (1984).}\]
By lemma 4.1, the holonomy of its orbits is

\[ Q(\gamma) = e^{i2\pi(x_1y_2 - x_2y_1)} . \]  

(56)

Now given any \( \alpha \in S^1 P|w \) (L|W), using the unitary section \( e^{ix_1y_2 - x_2y_1} \), it can be written as

\[ \alpha = \alpha \otimes e^{i(x_1y_2 - x_2y_1)} = [\alpha_1 dx_1 + \alpha_2 dx_2 + \alpha_3 dy_1 + \alpha_4 dy_2] \otimes e^{i(x_1y_2 - x_2y_1)} \] and (57)

\[ \iota_{X_2} \phi_t^* \alpha(x_1, x_2, y_1, y_2) = \alpha \phi_t(x_1, x_2, y_1, y_2)(-x_1 \sin t + x_2 \cos t) + \alpha_2 \phi_t(x_1, x_2, y_1, y_2)(-x_2 \sin t - x_1 \cos t) + \alpha_3 \phi_t(x_1, x_2, y_1, y_2)(-y_1 \sin t + y_2 \cos t) + \alpha_4 \phi_t(x_1, x_2, y_1, y_2)(-y_2 \sin t - y_1 \cos t) . \]  

(58)

Therefore, using the unitary section \( e^{i(x_1y_2 - x_2y_1)} \), the expression \( J_X(\alpha) \) at a point \( (x_1, x_2, y_1, y_2) \) near the singular point is:

\[ J_X(\alpha) = \left[ \int_0^{2\pi} (x_2 \alpha_1 - x_1 \alpha_2 + y_2 \alpha_3 - y_1 \alpha_4) \circ \phi_t \ e^{-it(x_1y_2 - x_2y_1)} \cos t \ dt \right. 

\[ \left. - \int_0^{2\pi} (x_1 \alpha_1 + x_2 \alpha_2 + y_1 \alpha_3 + y_2 \alpha_4) \circ \phi_t \ e^{-it(x_1y_2 - x_2y_1)} \sin t \ dt \right] e^{i(x_1y_2 - x_2y_1)} . \]

The following upper bound proves that this is zero over the set \( \{(x_1, x_2, y_1, y_2) \in \mathbb{C}^2; x_1y_2 - x_2y_1 = 0\} \) (the points where \( Q(\gamma) = 1 \):

\[ |J_X(\alpha)| \leq \max_{t \in [0, 2\pi]} (x_2 \alpha_1 - x_1 \alpha_2 + y_2 \alpha_3 - y_1 \alpha_4) \circ \phi_t \left| \int_0^{2\pi} e^{-it(x_1y_2 - x_2y_1)} \cos t \ dt \right| 

\[ + \max_{t \in [0, 2\pi]} (x_1 \alpha_1 + x_2 \alpha_2 + y_1 \alpha_3 + y_2 \alpha_4) \circ \phi_t \left| \int_0^{2\pi} e^{-it(x_1y_2 - x_2y_1)} \sin t \ dt \right| 

\[ = \max_{t \in [0, 2\pi]} (x_2 \alpha_1 - x_1 \alpha_2 + y_2 \alpha_3 - y_1 \alpha_4) \circ \phi_t \left| \frac{(x_1y_2 - x_2y_1)(e^{-i2\pi(x_1y_2 - x_2y_1)} - 1)}{(x_1y_2 - x_2y_1)^2 - 1} \right| 

\[ + \max_{t \in [0, 2\pi]} (x_1 \alpha_1 + x_2 \alpha_2 + y_1 \alpha_3 + y_2 \alpha_4) \circ \phi_t \left| \frac{e^{-i2\pi(x_1y_2 - x_2y_1)} - 1}{(x_1y_2 - x_2y_1)^2 - 1} . \]  

(59)

This can be interpreted as a constructive proof of the Poincaré lemma needed for the proof of theorem 3.1 when the real distribution of a 4-dimensional manifold has focus-focus singularities.

**Proposition 6.7.** In the neighbourhood of \( \ell_{ff} \) over which a hamiltonian \( S^1 \)-action is defined there exists a neighbourhood \( V \) containing only \( \ell_{ff} \) as a Bohr-Sommerfeld fibre such that \( \hat{H}^0(V; \mathcal{J}|_V) = \{0\} \).
Proof: Lemma 4.1 guarantees that the holonomy of the orbits of the hamiltonian $S^1$-action is nontrivial over a dense set in $V$, hence proposition 4.3 asserts that there are no nonzero flat sections on $V$. □

This partially answers a conjecture from Hamilton and Miranda about how focus-focus singularities behave under geometric quantisation: they conjectured that focus-focus singularities do not contribute to geometric quantisation.

Believing that the conjecture is true one could try to use proposition 4.4 to prove it for the neighbourhood $V$. The first obstacle is that $\{Q(\gamma) = 1\}$ is not a submanifold, and one needs to prove that $J_X$ is the null operator over the points where $\{Q(\gamma) = 1\}$. Another approach would be to prove only the exactness of $J_X$ and check out convergence over the singular points of $\{Q(\gamma) = 1\}$. The second obstacle is that symplectically the neighbourhood $V$, as a singular fibre bundle, can be tangled, and a complete answer for the quantisation of $V$ might depend on it.

7 Applications II: global computations

7.1 Fibre bundles

In [8] Šniatycki studies the case when the polarisation is a lagrangian fibration. He uses a resolution of the sheaf and proves the vanishing of the groups $\tilde{H}^l(M; J)$, for $l \neq k$; $k$ being the rank of the fundamental group of a fibre.

Theorem 7.1 (Šniatycki). If the leaf space $N$ is a manifold and the natural projection $F : M \to N$ is a lagrangian fibration, then $Q(M) = \tilde{H}^k(M; J)$, and $\tilde{H}^k(M; J) \cong \tilde{H}^0(\ell_{BS}; J|_{\ell_{BS}})$, where $\ell_{BS} \subset M$ is the union of all Bohr-Sommerfeld fibres.

A slightly different proof of his theorem is given here when $k \neq 0$. When $k = 0$ there is no symplectic circle action and the techniques presented in this paper are of no use, wherefore, apart from the presentation, the proof is the same as the original one and is omitted.

Any atlas of the leaf space satisfies that the projection $F : M \to N$ on each open set $V$ of the atlas is a moment map; $F : F^{-1}(V) \to \mathbb{R}^n$ (assuming $\dim(M) = 2n$) is the composition of the restriction of $F$ to $F^{-1}(V)$ with the coordinate system over $V$.

The open sets $F^{-1}(V)$ are just the model in proposition 6.3 with a fixed number of Bohr-Sommerfeld fibres, thus the quantisation of it is just a sum.
of copies of $\mathbb{C}$, or $C^\infty(\mathbb{R}^{n-k})$, depending on the value of $k$, for each Bohr-Sommerfeld fibre.

Assuming that $k \neq 0$, so that theorem 5.1 can be used, the atlas can —and it will— be chosen in such a way that no Bohr-Sommerfeld fibre is contained in more than one of the open sets $\mathcal{F}^{-1}(V)$. In particular, if $V$ and $W$ are two open sets of the atlas such that $V \cap W \neq \emptyset$, then $\mathcal{F}^{-1}(V) \cap \mathcal{F}^{-1}(W)$ has no Bohr-Sommerfeld fibre. Proposition 5.1 implies that one of the periodic hamiltonian vector fields of the components of $F$ has orbits with nontrivial holonomy over $\mathcal{F}^{-1}(V) \cap \mathcal{F}^{-1}(W)$, thus by proposition 4.4, its quantisation is just the trivial vector space $\{0\}$. This means that a Mayer-Vietoris argument works for the cover $\{\mathcal{F}^{-1}(V)\}$ of $M$, and this finishes the proof for $k \neq 0$.

Remark 7.1. Śniatycki works with the prequantum line bundle twisted by a bundle of half forms normal to the polarisation, and here the result is presented for the nontwisted prequantum line bundle. As it was mentioned before, the techniques used here apply to any complex line bundle admitting a flat connection along the polarisation; the only difference being that the Bohr-Sommerfeld fibres may not be the same.

### 7.2 Locally toric manifolds

Hamilton [3] has shown, via Čech approach, that Śniatycki’s theorem holds for locally toric manifolds and that the elliptic singularities give no contribution to the quantisation:

**Theorem 7.2** (Hamilton). For $M$ a $2n$-dimensional compact symplectic manifold equipped with a locally toric singular lagrangian fibration:

$$
\mathcal{Q}(M) = \tilde{H}^n(M; \mathcal{I}) \cong \bigoplus_{p \in BS_r} \mathbb{C},
$$

(60)

$BS_r$ being the set of the regular Bohr-Sommerfeld fibres.

**Remark 7.2.** Regarding metaplectic correction, contrary to Śniatycki’s, Hamilton’s result does not include a twisted prequantum line bundle. Using the framework described in this paper, it is straightforward to twist the prequantum line bundle by a bundle of half forms normal to the polarisation and achieve the same result; only noticing that the Bohr-Sommerfeld fibres may not be the same.

The previous reasoning used for the fibre bundle case works in this singular setting. This provides a proof for Hamilton’s theorem via a de Rham approach.
A locally toric singular lagrangian fibration on a symplectic manifold \((M, \omega)\) is a surjective map \(F : M \to N\), where \(N\) is a topological space such that for every point in \(N\) there exist an open neighbourhood \(V\) and a homeomorphism \(\chi : V \to U \subset \{ z \in \mathbb{R}^k ; z \geq 0 \} \times \mathbb{R}^{n-k}\) satisfying that \((F^{-1}(V), \omega|_{F^{-1}(V)}, \chi \circ F|_{F^{-1}(V)})\) is an integrable system symplectomorphic to an open subset of the model of proposition 6.6.

Hence, by definition, the open sets \(F^{-1}(V)\) are just the model in proposition 6.6 with a fixed number of Bohr-Sommerfeld fibres, thus the quantisation of it is just a sum of copies of \(C\), or \(\{0\}\), depending on the fibre dimension, for each Bohr-Sommerfeld fibre.

Choosing an open cover for \(N\) in such a way that no Bohr-Sommerfeld fibre is contained in more than one of the open sets \(F^{-1}(V)\) (theorem 5.1 allows one to make this choice), if \(V\) and \(W\) are two open sets of the atlas such that \(V \cap W \neq \emptyset\), then \(F^{-1}(V) \cap F^{-1}(W)\) has no Bohr-Sommerfeld fibre. Proposition 5.1 implies that one of the periodic hamiltonian vector fields of the components of the integrable system has orbits with nontrivial holonomy over \(F^{-1}(V) \cap F^{-1}(W)\), wherefore, by proposition 4.4, its quantisation is just the trivial vector space \(\{0\}\). This means that a Mayer-Vietoris argument works for the cover \(\{F^{-1}(V)\}\) of \(M\), and this finishes the proof.

### 7.3 Almost toric manifolds

As it was seen from the quantisation of lagrangian fibrations and locally toric manifolds, quantisation of neighbourhoods of Bohr-Sommerfeld fibres computes the quantisation of the whole manifold. Consequently, if one knows how to compute, in dimension 4, the first and second cohomology groups for a neighbourhood of a focus-focus fibre, one is able to compute the quantisation for the almost toric case using the factorisation tools (corollary 6.1 and proposition 6.4).

For example, if \(M\) is a 4-dimensional compact almost toric manifold, \(BS_r\) and \(BS_{ff}\) are the image of the regular, respectively focus-focus, Bohr-Sommerfeld fibres by the moment map, and \(V\) neighbourhoods of focus-focus fibres admitting a hamiltonian \(S^1\)-action:

\[
Q(M) \cong \left( \bigoplus_{p \in BS_r} C \right) \oplus \left( \bigoplus_{p \in BS_{ff}} \check{H}^1(V; J|_V) \oplus \check{H}^2(V; J|_V) \right). \tag{61}
\]

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