INTEGRABLE FRACTIONAL MEAN FUNCTIONS ON SPACES OF HOMOGENEOUS TYPE.

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ABSTRACT. The class of Banach spaces $(L^q, L^p)_\alpha(X, d, \mu)$, $1 \leq q \leq \alpha \leq p \leq \infty$, introduced in [10] in connection with the study of the continuity of the fractional maximal operator of Hardy-Littlewood and of the Fourier transformation in the case $X = \mathbb{R}^n$ and $\mu$ is the Lebesgue measure, was generalized in [7] to the setting of homogeneous groups. We generalize it here to spaces of homogeneous type and we prove that the results obtained in [7] such as relations between these spaces and Lebesgue spaces, weak Lebesgue and Morrey spaces, remain true.

1. Introduction

In [23], Muckenhoupt raised the problem of characterizing weight functions $u$ and $v$ for which the inequality

\begin{equation}
\int_{-\infty}^{+\infty} |\hat{f}(x)|^p u(x)dx \leq C \int_{-\infty}^{+\infty} |f(x)|^p v(x)dx
\end{equation}

holds for every $f$ in the Lebesgue space $L^p(\mathbb{R})$.

Aguilera and Harboure showed in [1] that, in the case $v = 1$ and $1 < p < 2$, a necessary condition for (1) is

\begin{equation}
\left[ \sum_{k=-\infty}^{k=+\infty} \left( \int_{r^k}^{r^{(k+1)}} u(x) \right)^{b} \right]^{\frac{1}{b}} \leq C r^{p-1}, \quad r > 0
\end{equation}

where $b = \frac{2}{2-p}$.

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Let us assume that \( n \) is a positive integer and \( 1 \leq q \leq \alpha \leq p \leq \infty \). For any Lebesgue-measurable function \( f \) on \( \mathbb{R}^n \), we set

\[
\|f\|_{q,p,\alpha} = \begin{cases} 
\sup_{r>0} r^n \left( \frac{1}{p} \right) \left[ \sum_{k \in \mathbb{Z}^n} \left( \| f \chi_{I^r_k} \|_q \right)^p \right]^{\frac{1}{p}} & \text{if } p < \infty \\
\sup_{r>0} r^n \left( \frac{1}{p} \right) \sup_{k \in \mathbb{Z}^n} \| f \chi_{I^r_k} \|_q & \text{if } p = \infty
\end{cases}
\]

Here \( I^r_k = \prod_{j=1}^n \left( k_j r, (k_j + 1) r \right) \), \( k = (k_1, \ldots, k_n) \in \mathbb{Z}^n \), \( r > 0 \) and \( \| \cdot \|_q \) denotes the usual norm on the Lebesgue space \( L^q(\mathbb{R}^n) \). We denote by \( L_0(\mathbb{R}^n) \) the complex vector space of equivalent classes (modulo equality Lebesgue almost everywhere) of Lebesgue measurable complex-valued functions on \( \mathbb{R}^n \). It is clear that \( \| \cdot \|_{q,p,\alpha} \) may be looked at as a map of \( L_0(\mathbb{R}^n) \) into \([0, \infty)\). We define

\[
(L^q, L^p)^\alpha(\mathbb{R}^n) = \left\{ f \in L_0(\mathbb{R}^n) : \|f\|_{q,p,\alpha} < \infty \right\}.
\]

Fofana has proved in \[10\] that \(((L^q, L^p)^\alpha(\mathbb{R}^n), \| \cdot \|_{q,p,\alpha})\) is a complex Banach space and that the Lebesgue spaces \( L^q(\mathbb{R}^n) \), the Morrey spaces \( M^{(1-\frac{1}{\alpha})}_q(\mathbb{R}^n) \) and the Lorenz spaces \( L^{\alpha,\infty}(\mathbb{R}^n) \) (in the case \( q < \alpha < p \)) are its sub-spaces.

Note that condition \((2)\) can be written as \( u \in \left( L^1, L^b \right)^{\frac{1}{2-p}} \), with \( b = \frac{2}{2-p} \).

Further results on Fourier transform may be expressed in the setting of \((L^q, L^p)^\alpha(\mathbb{R}^n)\) and related spaces of Radon measures (see \[11, 20\]). These spaces are also related to \( L^q - L^p \) multiplier problems (see \[19, 25\]) and well-suited to establish norm inequalities for fractional maximal functions \[12\].

It is clear that \((L^q, \ell^p)^\alpha(\mathbb{R}^n)\) is a subspace of the so-called amalgam space of Wiener \((L^q, \ell^p)(\mathbb{R}^n)\), defined by

\[
(L^q, \ell^p)^\alpha(\mathbb{R}^n) = \left\{ f \in L_0(\mathbb{R}^n) : 1 \|f\|_{q,p} < \infty \right\}
\]

where for \( r > 0 \)

\[
r \|f\|_{q,p} = \begin{cases} 
\left( \sum_{n \in \mathbb{Z}^n} \left( \| f \chi_{I^r_n} \|_q \right)^p \right)^{\frac{1}{p}} & \text{if } p < \infty \\
\sup_{k \in \mathbb{Z}^n} \| f \chi_{I^r_k} \|_q & \text{if } p = \infty
\end{cases}
\]

These amalgam spaces have been used by Wiener (see \[31\]) in connection with Tauberian theorems. Long after, Holland undertook their systematic study (see \[18\]). Since then, they have been extensively studied (see the survey paper \[15\] and the references therein) and generalized to locally compact groups (see \[27, 3, 2\]). They may be looked at as spaces of functions that behave locally as elements of \( L^q(\mathbb{R}^n) \) and globally as belonging to \( L^p(\mathbb{R}^n) \). Taking this into account, Feichtinger has introduced Banach spaces whose elements belong locally to some Banach space, and globally to another (see \[6\]).
Replacing \( \mathbb{R}^n \) by a homogeneous group \( G \), the authors have defined and studied \((L^q, L^p) \alpha (G)\) spaces (see [7]). They proved that results obtained in [10] remain valid for \((L^q, L^p) \alpha (G)\).

In the present paper, we extend the definition of these spaces to spaces of homogeneous type. In this setting, we obtain interesting links between \((L^q, L^p) \alpha (X)\) and classical Banach function spaces.

These spaces are well suited for studying norm inequalities on fractional maximal operators. Actually, in [8] we established some continuity properties for these operators from \((L^q, L^p) \alpha (X)\) to weak-Lebesgue spaces, which extended in this context analogous results known in the Euclidean case (see [12], [24]).

The paper is organized as follows. Section 2 contains definitions and the main results, whose proofs are given in Section 4. Section 3 is devoted to auxiliary results.

Throughout the paper, \( C \) denotes positive constants that are independent of the main parameters involved, with values which may differ from line to line. Constants with subscripts, such as \( C_1 \), do not change in different occurrences.

2. Definitions-Results

A space of homogeneous type \((X, d, \mu)\) is a quasi metric space \((X, d)\) endowed with a non negative Borel measure \( \mu \) satisfying the doubling condition

\[
0 < \mu \left( B(x,2r) \right) \leq C\mu \left( B(x,r) \right) < \infty, \quad x \in X \text{ and } r > 0,
\]

where \( B(x,r) = \{ y \in X : d(x,y) < r \} \) is the ball with center \( x \) and radius \( r > 0 \). If \( C^{\prime}_\mu \) is the smallest constant \( C \) for which (7) holds, then \( D_\mu = \log_2 C^{\prime}_\mu \) is called the doubling order of \( \mu \). It is known (see [27]) that for all balls \( B_2 \subset B_1 \)

\[
\frac{\mu \left( B_1 \right)}{\mu \left( B_2 \right)} \leq C_\mu \left( \frac{r \left( B_1 \right)}{r \left( B_2 \right)} \right)^{D_\mu},
\]

where \( r(B) \) denote the radius of the ball \( B \) and \( C_\mu = C^{\prime}_\mu (2\kappa)^{D_\mu}, \kappa \geq 1 \) being a constant such that

\[
d(x,y) \leq \kappa \left( d(x,z) + d(z,y) \right), \quad x, y, z \in X.
\]

Two quasi metrics \( d \) and \( \delta \) on \( X \) are said to be equivalent if there exists constants \( C_1 > 0 \) and \( C_2 > 0 \) such that

\[
C_1 d(x,y) \leq \delta(x,y) \leq C_2 d(x,y), \quad x, y \in X.
\]

Observe that topologies defined by equivalent quasi metrics on \( X \) are equivalent. It is shown in [21] that on any space of homogeneous type \((X, d, \mu)\), there is a quasi metric \( \delta \) equivalent to \( d \) for which balls are open sets.

In the sequel we assume that \( X = (X, d, \mu) \) is a fixed space of homogeneous type and:

- all balls \( B(x,r) = \{ y \in X : d(x,y) < r \} \) are open subsets of \( X \) endowed with the \( d \)-topology,
- \( (X, d) \) is separable,
\[ \mu(X) = \infty, \]
\[ B_{(x,R)} \setminus B_{(x,r)} \neq \emptyset, \ 0 < r < R < \infty \text{ and } x \in X. \]

As proved in [30], the last assumption implies that there exist two constants \( \tilde{C}_\mu > 0 \) and \( \delta_\mu > 0 \) such that for all balls \( B_2 \subset B_1 \) of \( X \)

\[ \mu(B_1) \mu(B_2) \geq \tilde{C}_\mu \left( \frac{r(B_1)}{r(B_2)} \right)^{\delta_\mu}. \] (10)

For \( 1 \leq p \leq \infty, \| \cdot \|_p \) denotes the usual norm on the Lebesgue space \( L^p(X) \).

For any \( \mu \)-measurable function \( f \) on \( X \), we set:

- \( \lambda_f(\alpha) = \mu(\{ x \in X : |f(x)| > \alpha \}), \ \alpha > 0, \)
- \( f_*(t) = \inf \{ \alpha > 0 : \lambda_f(\alpha) \leq t \}, \ t > 0, \)
- \( f^*(t) = \frac{1}{t} \int_0^t f_*(u) du, \ t > 0, \)
- \( \| f \|_{p,q} = \left\{ \begin{array}{ll} \left[ \frac{p}{q} \int_0^\infty \left( t^\frac{1}{p} f^*(t) \right)^q \frac{dt}{t} \right]^\frac{1}{q} & \text{if } 1 \leq p, q < \infty \\ \sup_{t > 0} t^\frac{1}{p} f^*(t) & \text{if } 1 \leq p \leq \infty \text{ and } q = \infty \end{array} \right. \)
- \( \lambda_f(\alpha) = \mu(\{ x \in X : |f(x)| > \alpha \}), \ \alpha > 0, \)
- \( f_*(t) = \inf \{ \alpha > 0 : \lambda_f(\alpha) \leq t \}, \ t > 0, \)
- \( f^*(t) = \frac{1}{t} \int_0^t f_*(u) du, \ t > 0, \)
- \( \| f \|_{p,q} = \left\{ \begin{array}{ll} \left[ \frac{p}{q} \int_0^\infty \left( t^\frac{1}{p} f^*(t) \right)^q \frac{dt}{t} \right]^\frac{1}{q} & \text{if } 1 \leq p, q < \infty \\ \sup_{t > 0} t^\frac{1}{p} f^*(t) & \text{if } 1 \leq p \leq \infty \text{ and } q = \infty \end{array} \right. \)

Let \( L_0(X) \) be the complex vector space of equivalent classes (modulo equality \( \mu \)-almost everywhere) of \( \mu \)-measurable complex-valued functions on \( X \). Then \( \| \cdot \|_{p,q} \) is a map from \( L_0(X) \) into \([0, \infty)\). It is known (see [28]) that:

- for \( 1 < p, q \leq \infty, \) the space \( L^{p,q}(X) = \left\{ f \in L_0(X) : \| f \|_{p,q} < \infty \right\} \) endowed with \( f \mapsto \| f \|_{p,q} \), is a complex Banach space (called Lorentz space),
- \( f \mapsto \| f \|_{p,q}^* = \left\{ \begin{array}{ll} \left[ \frac{p}{q} \int_0^\infty \left( t^\frac{1}{p} f^*(t) \right)^q \frac{dt}{t} \right]^\frac{1}{q} & \text{if } 1 \leq p, q < \infty \\ \sup_{t > 0} t^\frac{1}{p} f^*(t) & \text{if } 1 \leq p \leq \infty \text{ and } q = \infty \end{array} \right. \)
- \( \sup_{t > 0} t^\frac{1}{p} f^*(t) = \sup_{\alpha > 0} \lambda_f(\alpha)^\frac{1}{p}. \)

In the sequel we assume that \( 1 \leq q \leq \alpha \leq p \leq \infty. \)

**Notation 2.1.** For any \( \mu \)-measurable function \( f \) on \( X \) and any \( r > 0 \), we put

\[ r \| f \|_{q,p,\alpha} = \left\{ \begin{array}{ll} \left[ \int_X \left( \mu(B(y,r))^{\frac{1}{q} - \frac{1}{p} - \frac{1}{2}} \left\| f \chi_{B(y,r)} \right\|_q \right)^p d\mu(y) \right]^\frac{1}{p} & \text{if } p < \infty \\ \sup_{y \in X} \mu(B(y,r))^{\alpha - \frac{1}{2}} \left\| f \chi_{B(y,r)} \right\|_q & \text{if } p = \infty \end{array} \right., \] (11)

where \( \chi_{B(y,r)} \) denotes the characteristic function of \( B(y,r) \), and we use the convention \( \frac{1}{q} = 0 \) if \( q = \infty \).

**Theorem 2.2.** For any \( \mu \)-measurable function \( f \) on \( X \) and \( r > 0 \), we have \( r \| f \|_{q,p,\alpha} = 0 \) if and only if \( f = 0 \) \( \mu \)-almost everywhere.

By the previous result we may (and shall) look at \( r \| f \|_{q,p,\alpha} \) as a map from \( L_0(X) \) into \([0, \infty] \).
Notation 2.3. For $r > 0$, we define

$$(L^q, L^p)^\alpha_r(X) = \left\{ f \in L_0(X) : r \| f \|_{q,p,\alpha} < \infty \right\}.$$  

Theorem 2.4. For any positive real number $r$, $((L^q, L^p)^\alpha_r(X), r \| \cdot \|_{q,p,\alpha})$ is a complex Banach space.

As in the Euclidean case we have the following results.

Theorem 2.5. Let $r$ be a positive real number, $1 \leq q_1 < q_2 \leq \alpha$ and $\alpha \leq p_1 < p_2 < \infty$. Then

\begin{align*}
(12) & \quad r \| \cdot \|_{q_1,p,\alpha} \leq r \| \cdot \|_{q_2,p,\alpha} \\
(13) & \quad r \| \cdot \|_{q,\infty,\alpha} \leq C r \| \cdot \|_{q_2,p,\alpha} \leq C' r \| \cdot \|_{q_1,p,\alpha},
\end{align*}

where $C > 0$ and $C' > 0$ are constants independent of $r$.

Theorem 2.6. There is a constant $C > 0$ such that

\begin{equation}
(14) \quad r \| \cdot \|_{q,p,\alpha} \leq C r \| \cdot \|_{\alpha}
\end{equation}

for any real number $r > 0$.

$(L^q, L^p)^\alpha_r(X)$ is actually a generalization of the Wiener amalgam space $(L^q, \ell^p)(\mathbb{R}^n)$. This appears clearly when we compare $r \| \cdot \|_{q,p}$ as define in (6), to the norm $\| \cdot \|^{d_{mr}}_{q,p,\alpha}$ which is equivalent to $r \| \cdot \|_{q,p,\alpha}$ (see Proposition 4.1). Now we define a subspace of $(L^q, L^p)^\alpha_r(X)$ which generalizes $(L^q, \ell^p)^\alpha(\mathbb{R}^n)$.

Definition 2.7. We set

$$(L^q, L^p)^\alpha(X) = \left\{ f \in L_0(X) : \| f \|_{q,p,\alpha} < \infty \right\},$$  

where $\| f \|_{q,p,\alpha} = \sup_{r>0} r \| f \|_{q,p,\alpha}$.

From Definition 2.7, Theorem 2.4, Theorem 2.6 and Theorem 2.5 the following results are straightforward.

Theorem 2.8. a) $((L^q, L^p)^\alpha(X), \| \cdot \|_{q,p,\alpha})$ is a complex Banach space and there exists a constant $C > 0$ such that

\begin{equation}
(15) \quad \| \cdot \|_{q,p,\alpha} \leq C \| \cdot \|_{\alpha}.
\end{equation}

b) Assume that $1 \leq q_1 < q_2 \leq \alpha$ and $\alpha \leq p_1 < p_2 < \infty$. Then

$$\| \cdot \|_{q_1,p,\alpha} \leq C \| \cdot \|_{q_2,p,\alpha}$$

and

$$\| \cdot \|_{q,p,\alpha} \leq C \| \cdot \|_{q_1,p,\alpha},$$

for some constant $C > 0$. 

The continuous embedding of $L^\alpha(X)$ into $(L^q, L^p)^\alpha(X)$ expressed by Inequality (15), may be an equivalence in some cases.

**Theorem 2.9.** There is a constant $C > 0$ such that $\|\cdot\|_\alpha \leq C \|\cdot\|_{q,p,\alpha}$ whenever $q = \alpha$ or $\alpha = p$.

In the case $q < \alpha < p$, the space $(L^q, L^p)^\alpha(X)$ contains properly $L^\alpha(X)$ as it appears in the following theorem.

**Theorem 2.10.** Assume that $1 \leq q < \alpha < p \leq \infty$. Then there is a constant $C$ such that

$$\|\cdot\|_{q,p,\alpha} \leq C \|\cdot\|_{\alpha,p}.$$ 

The previous result may be strengthened in some cases.

**Theorem 2.11.** Assume that $1 \leq q < \alpha < p$ and that there exists a non decreasing function $\varphi$ on $[0, \infty)$ and two constants $0 < a \leq b < \infty$ such that

$$(16) \quad a \varphi(r) \leq \mu(B(x,r)) \leq b \varphi(r), \ x \in X, r > 0.$$ 

Then there is $C > 0$ such that

$$\|\cdot\|_{q,p,\alpha} \leq C \|\cdot\|_{\alpha,\infty}.$$ 

From the doubling condition (8) and the reverse doubling condition (10), we obtain that the function $\varphi$ appearing in hypothesis (16) satisfies

$$(17) \quad a_0 r^{D_\mu} \leq \varphi(r) \leq b_0 r^{D_\mu}, \ r \leq 1,$$

$$(18) \quad a_1 r^{D_\mu} \leq \varphi(r) \leq b_1 r^{D_\mu}, \ 1 \leq r,$$

where $a_0, b_0, a_1$ and $b_1$ are positive constants.

Notice that Hypothesis (16) is fulfilled for example in the case where $X$ is an Ahlfors $n$ regular metric space, i.e., there is a positive integer $n$ and a constant $C > 0$ such that

$$C^{-1} r^n \leq \mu(B(x,r)) \leq C r^n, \ x \in X, \ r > 0,$$

and also in the case where $X$ is a Lie group with polynomial growth equipped with a left Haar measure $\mu$ and the Carnot-Carathéodory metric $d$ associated with a Hörmander system of left invariant vector fields (see [17], [22] and [29]).

The next result shows that the inclusion of $L^{\alpha,\infty}(X)$ into $(L^q, L^p)^\alpha(X)$ is proper.

**Theorem 2.12.** Under the hypothesis of Theorem 2.11 we have $(L^q, L^p)^\alpha(X) \setminus L^{\alpha,\infty}(X) \neq \emptyset$.
3. Auxiliary results

In order to establish various inclusions between the function spaces we study, we need the following ”dyadic cube decomposition” of $X$, proved in [27].

**Lemma 3.1.** There is $\rho > 1$, depending only on $\kappa$ in (9) (we may take $\rho = 8\kappa^5$), such that, given any integer $m$, there exists a family $\{(x_j^k, E_j^k) : k \in \mathbb{Z}, k \geq m, 1 \leq j < N_k\}$ where $x_j^k$ are points of $X$ and $E_j^k$ subsets of $X$ satisfying:

(i) $N_k \in \mathbb{N}^* \cup \{\infty\}, k \geq m$,
(ii) $B_{(x_j^k, \rho^k)} \subset E_j^k \subset B_{(x_j^k, \rho^{k+1})}$, $k \geq m$, $1 \leq j < N_k$,
(iii) $X = \bigcup_{j=1}^{N_k} E_j^k$, and $E_j^k \cap E_j^k = \emptyset$ if $i \neq j$, $k \geq m$,
(iv) $E_j^k \subset E_j^l$ or $E_j^k \cap E_i^l = \emptyset$, $\ell > k \geq m$, $1 \leq j < N_k, 1 \leq i < N_{\ell}$.

The $E_j^k$ are referred to as dyadic cubes of generation $k$.

**Notation 3.2.** Given an integer $k \geq m$ and $r > 0$, we set

(i) $T_r^k(x) = \{i : 1 \leq i < N_k$ and $E_j^k \cap B_{(x_r)} \neq \emptyset\}$, $x \in X$,
(ii) $S_r^k(j) = \{i : 1 \leq i < N_k$ and $E_i^k \cap B_{(y_r)} \neq \emptyset$ for some $y \in E_j^k\}$, $1 \leq j < N_k$.

Remark that $i \in S_r^k(j)$ if and only if $j \in S_r^k(i)$. Inequality (8) provides us with the following useful estimates on the cardinals $\#(S_r^k(x))$ and $\#(T_r^k(x))$ of the sets $S_r^k(x)$ and $T_r^k(x)$ respectively.

**Lemma 3.3.** Given integers $k \geq m$, $1 \leq j < N_k$ and $r > 0$, we have

\[\mu(B_{(y_r)}) \leq \mathcal{N}_1(k, r) \mu(E_j^k), \ y \in E_j^k,\]

(19)

\[\mu(E_j^k) \leq \mathcal{N}_2(k, r) \mu(E_j^k) \text{ and } \mu(E_j^k) \leq \mathcal{N}_2(k, r) \mu(E_j^k), \ i \in S_r^k(j),\]

(20)

\[\#(T_r^k(x)) \leq \mathcal{N}_2(k, r), \ x \in X,\]

(21)

\[\#(S_r^k(x)) \leq \mathcal{N}_3(k, r), \ x \in X,\]

and

(22)

where $\mathcal{N}_1(k, r) = C_{\mu} \left[\kappa + \frac{\rho}{\rho^k}\right]^{D_{\mu}}$, $\mathcal{N}_2(k, r) = C_{\mu} \left[\kappa \left(2\kappa^2 + 1\right)\rho + \frac{\rho}{\rho^k}\right]^{D_{\mu}}$ and $\mathcal{N}_3(k, r) = C_{\mu} \left[\kappa \left(2\kappa^2 + 1\right)\rho + \frac{\rho}{\rho^k}\right]^{D_{\mu}}$.

**Proof.** (a) Inequalities (19) and (20) are obtained immediately from inequality (8), the following inclusions:

- $B_{(x_j^k, \rho^k)} \subset B_{(x_j^k, \kappa(\rho^{k+1}+r))}$ and $B_{(y_r)} \subset B_{(x_j^k, \kappa(\rho^{k+1}+r))}$, $y \in E_j^k$
- $E_j^k \subset B_{(y, \kappa(2\kappa^2+1+r))}$ and $B_{(x_j^k, \rho^k)} \subset B_{(y, \kappa(2\kappa^2+1+r))}$, $y \in E_j^k$ and $E_j^k \cap B_{(y, \rho^k)} \neq \emptyset$,

and the remark stated after Notation 3.2.
Lemma 3.1 (iii) asserts that the $E^k_i$ ($1 \leq i < N_k$) are pairwise disjoints. Furthermore we have the following inclusions.

- $B(x_i^+, \rho^k) \subset E^k_i \subset B(x_i, \kappa(2\kappa \rho^{k+1} + r))$, $x \in X$ and $i \in T^k_r(x)$,
- $E^k_i \subset B((x_i^+)^{(2\kappa^2+1)} \rho^{k+1} + r)$ and $B(x_i^+, \rho^k) \subset B((x_i^+)^{(2\kappa^2+1)} \rho^{k+1} + r)$, $i \in T^k_r(j)$.

Thus by Inequality (8), we obtain for $x \in X$

$$\#(T^k_r(x))C^{-1} \left[ \kappa(2\kappa \rho^k + \frac{r}{\rho^k}) \right]^{-D_\mu} \mu\left(B(x, \kappa(2\kappa \rho^{k+1} + r))\right) \leq \sum_{i \in T^k_r(x)} \mu\left(B(x_i^+, \rho^k)\right) \leq \mu\left(B(x, \kappa(2\kappa \rho^{k+1} + r))\right)$$

and similarly

$$\#(S^k_r(j))N^{-1} \mu\left(E^k_j\right) \leq \sum_{i \in S^k_r(j)} \mu\left(E^k_i\right) \leq \mu\left(B(x_i^+, \kappa((2\kappa^2+1) \rho^{k+1} + r))\right)$$

$$\leq C \left[ \kappa(2\kappa^2 + 1) \rho^k + \frac{r}{\rho^k} \right]^{-D_\mu} \mu\left(E^k_j\right).$$

Inequalities (21) and (22) follow.

\[\Box\]

**Lemma 3.4.** Assume that $1 \leq q, p \leq \infty$, with $p \neq \infty$, $0 \leq s$, $m$ and $k$ are integers satisfying $k \geq m$, $1 \leq j < N_k$ and $2\kappa \rho^{k+1} \leq r$. Then, for any $\mu$-measurable function $f$ on $X$, we have

$$\mu\left(E^k_j\right)^{-s} \left\| f \chi_{E^k_j} \right\|_q^p \leq \mathcal{N}_1(k, r)^{s+1} \int_{E^k_j} \mu\left(B(y, r)\right)^{-s-1} \left\| f \chi_{B(y, r)} \right\|_q^p d\mu(y)$$

where $\mathcal{N}_1(k, r)$ is as in Inequality (19).

**Proof.** Notice that

$$\inf_{E^k_j} \left\| f \chi_{B(y, r)} \right\|_q^p \leq \mu\left(E^k_j\right)^{-1} \int_{E^k_j} \left\| f \chi_{B(y, r)} \right\|_q^p d\mu(y)$$

with equality only when $\left\| f \chi_{B(y, r)} \right\|_q$ is a constant almost everywhere on $E^k_j$. Thus, there exists an element $y^k_j$ of $E^k_j$ such that

$$\left\| f \chi_{B(y^k_j, r)} \right\|_q^p \leq \mu\left(E^k_j\right)^{-1} \int_{E^k_j} \left\| f \chi_{B(y, r)} \right\|_q^p d\mu(y).$$

Since $E^k_j$ is included in $B(y, r)$ for every $y$ in $E^k_j$, we have

$$\mu\left(E^k_j\right)^{-s} \left\| f \chi_{E^k_j} \right\|_q^p \leq \mu\left(E^k_j\right)^{-s} \left\| f \chi_{B(y^k_j, r)} \right\|_q^p \leq \mu\left(E^k_j\right)^{-s-1} \int_{E^k_j} \left\| f \chi_{B(y, r)} \right\|_q^p d\mu(y).$$

The result follows from inequality (19). \[\Box\]
We shall use the following result which may be viewed as a generalization of the Young inequality in a space without group structure.

**Lemma 3.5.** Let $\beta, \gamma$ be elements of $[1, \infty]$ such that $\frac{1}{\gamma} = \frac{1}{\beta} + \frac{1}{t} - 1$ and $K(x, y)$ a positive kernel on $X$. There is a constant $C > 0$ such that

$$
\|Tg\|_{\gamma} \leq C \|\|K\|_{\beta}\|_{\infty} \|g\|_{t, \gamma}^{*}, \ g \in L_{0}(X),
$$

where

$$
Tg(y) = \int_{X} g(x)K(x, y)d\mu(x),
$$

and

$$
\left\|\left\|K\right\|_{\beta}\right\|_{\infty} = \max \left(\sup \operatorname{ess} \|K(\cdot, y)\|_{\beta}; \sup \operatorname{ess} \|K(x, \cdot)\|_{\beta}\right).
$$

**Proof.** 1) Let $g \in L_{0}(X)$ and put $\tilde{g}(y) = \int_{X} |g(x)| K(x, y)d\mu(x)$. We claim that

$$
\|\tilde{g}\|_{t, \infty}^{*} \leq C \|\|K\|_{\beta}\|_{\infty} \|g\|_{t, \infty}^{*}.
$$

If $g \notin L^{t, \infty}(X)$ or $\|g\|_{t, \infty}^{*} = 0$, or $\|\|K\|_{\beta}\|_{\infty} \in \{0, \infty\}$ then the claim is trivially verified. So we assume that $0 < \|g\|_{t, \infty}^{*} < \infty$ and $0 < \|\|K\|_{\beta}\|_{\infty} < \infty$. Define

$$
g_{1}(x) = \begin{cases} 
    g(x) & \text{if } |g(x)| \leq M \\
    0 & \text{if not}
\end{cases}
$$

and $g_{2}(x) = g(x) - g_{1}(x), \ x \in X,$

where $M$ is a positive real number to be specified later. For $\alpha > 0$, we have

$$
\lambda_{\tilde{g}}(\alpha) \leq \lambda_{\tilde{g}_{1}} \left(\frac{\alpha}{2}\right) + \lambda_{\tilde{g}_{2}} \left(\frac{\alpha}{2}\right) \text{ since } \tilde{g} \leq \tilde{g}_{1} + \tilde{g}_{2}.
$$

a) We can estimate $\lambda_{\tilde{g}_{1}} \left(\frac{\alpha}{2}\right)$ as follows:

$$
\int_{X} |g_{1}(x)|^{\beta} d\mu(x) = \beta' \int_{0}^{\infty} s^{\beta' - 1} \lambda_{g_{1}}(s) ds \leq \beta' \int_{0}^{M} s^{\beta' - 1} \lambda_{g_{1}}(s) ds
$$

$$
\leq \beta' \left(\int_{0}^{M} s^{\beta' - 1} ds\right)^{t} \left\|\|g\|_{t, \infty}^{*}\right\|^{t}
= \frac{\beta'}{\beta' - t} M^{\beta' - t} \left\|\|g\|_{t, \infty}^{*}\right\|^{t}.
$$

So,

$$
|\tilde{g}_{1}(y)| = \int_{X} |g_{1}(x)| K(x, y)d\mu(x) \leq \left(\int_{X} |g_{1}(x)|^{\beta} d\mu(x)\right)^{\frac{1}{\beta'}} \left(\int_{X} K^{\beta}(x, y)d\mu(x)\right)^{\frac{1}{\beta'}}
\leq \left(\frac{\lambda_{\tilde{g}}(\alpha)}{2}\right)^{\frac{1}{\beta'}} M^{\frac{\beta'}{\beta' - t}} \left\|\|g\|_{t, \infty}^{*}\right\|^{\frac{t}{\beta'}} \|\|K\|_{\beta}\|_{\infty}.
$$

Let us choose

$$
M = \left(\frac{\alpha}{2}\right)^{\frac{t}{\beta'}} \left(\frac{1}{\gamma}\right)^{\frac{\gamma}{\beta'}} \left\|\|g\|_{t, \infty}^{*}\right\|^{\frac{\beta'}{\beta'}} \|\|K\|_{\beta}\|_{\infty}^{\frac{\beta}{\beta'}}.
$$
We have \( \| \tilde{g}_1 \|_{\infty} \leq \frac{\alpha}{2} \) and therefore \( \lambda_{\tilde{g}_1} \left( \frac{\alpha}{2} \right) = 0 \).

b) We also have the following estimate of \( \lambda_{\tilde{g}_2} \left( \frac{\alpha}{2} \right) \):

\[
\int_X |g_2(x)| \, d\mu(x) = \int_0^\infty \lambda_{g_2}(s) \, ds \leq \int_0^M \lambda_g(M) \, ds + \int_M^\infty \lambda_g(s) \, ds \\
\leq M^{1-t} \left( \| g \|_{t,\infty}^* \right)^t + \left( \int_M^\infty s^{-t} \, ds \right) \left( \| g \|_{t,\infty}^* \right)^t = \left( \frac{t}{t-1} \right) M^{1-t} \left( \| g \|_{t,\infty}^* \right)^t.
\]

Therefore,

\[
\lambda_{\tilde{g}_2} \left( \frac{\alpha}{2} \right) \leq \left( \frac{2}{\alpha} \right)^\beta \int_{\{u \in X : |\tilde{g}_2(u)| > \frac{\alpha}{2} \}} \left( \int_X |g_2(x)| \, K(x, y) \, d\mu(x) \right)^\beta \, d\mu(y) \\
\leq \left( \frac{2}{\alpha} \right)^\beta \left[ \int_X |g_2(x)| \left( \int_{\{u \in X : |\tilde{g}_2(u)| > \frac{\alpha}{2} \}} K^\beta(x, y) \, d\mu(y) \right)^\frac{1}{\beta} \, d\mu(x) \right]^\beta \\
\leq \left( \frac{2}{\alpha} \right)^\beta \| K \|_\beta \|_\infty \left[ \int_X |g_2(x)| \, d\mu(x) \right]^\beta \\
\leq \left( \frac{2}{\alpha} \right)^\beta \| K \|_\beta \|_\infty \left( \frac{t}{t-1} \right) M^{1-t} \left( \| g \|_{t,\infty}^* \right)^t \leq \left( C \alpha^{-1} \| K \|_\beta \|_\infty \| g \|_{t,\infty}^* \right)^\gamma,
\]

with \( C = (2)^\gamma \left( \frac{t}{t-1} \right)^\beta \left( \frac{t}{\gamma} \right)^{\gamma(t-1)} \).

From a) and b) we get

\[
\lambda_{\tilde{g}}(\alpha) \leq \left( C \alpha^{-1} \| K \|_\beta \|_\infty \| g \|_{t,\infty}^* \right)^\gamma.
\]

As this inequality is true for \( \alpha > 0 \), we have

\[
\| Tg \|_{\gamma,\infty}^* \leq C \| K \|_\beta \|_\infty \| g \|_{t,\infty}^*.
\]

2) Notice that \( T \) is a linear operator. Therefore, the result follows from 1) and Stein interpolation theorem (see [28]).

\[\square\]

4. PROOF OF THE MAIN RESULTS

Throughout this paragraph, for every \( r > 0 \), \( m_r \) denotes the unique integer which verifies

\[ (23) \quad \rho^{m_r+1} \leq \frac{r}{2\kappa} < \rho^{m_r+2}. \]

Notice that the constants in Lemma \ref{lem:N1} satisfy

\[ (24) \quad \mathcal{N}_1(m_r, \rho) \leq C \mu [\kappa \rho(1 + 2\kappa \rho)]^{D^\mu} = \mathcal{N}_1, \]

\[ (25) \quad \mathcal{N}_2(m_r, \rho) \leq C \mu [2\kappa^2 \rho(1 + \rho)]^{D^\mu} = \mathcal{N}_2, \]
and
\[(26) \quad \mathfrak{N}_3(m_r, r) \leq C_\mu \left[ \kappa \rho (2\kappa^2 + 2\kappa \rho + 1) \right]^{D_\mu} \mathfrak{N}_2 = \mathfrak{N}_3.\]

**Proof of Theorem 2.2.** Let \( f \) be a \( \mu \)-measurable function on \( X \) such that \( r \|f\|_{q,p,\alpha} = 0. \) Since balls in \( X \) have positive measure, \( \|\mu(B_{(r)}^{\frac{1}{q} - \frac{1}{p} - \frac{1}{q}}) \|_q = 0 \) implies that there exists a \( \mu \)-null subset \( E \) of \( X \) such that
\[ \|f\chi_{B_{(r)}}\|_q = 0 \text{ in } X \setminus E. \]
Similarly, for any \( y \) in \( X \setminus E \), there exists a \( \mu \)-null subset \( F_y \) of \( X \) out of which \( f\chi_{B_{(y,r)}} = 0 \). For \( 1 \leq j \leq N_{m_r} \), the intersection of \( X \setminus E \) and \( B_{(x^m_r, \rho \sum_{k>n})} \) is non-void. So we may pick in it an element \( y_j \). Since \( E_j^{m_r} \subset B_{(x^m_r, \rho \sum_{k>n})} \subset B_{(y_j, r)} \), we have \( X = \bigcup_{j=1}^{N_{m_r}} E_j^{m_r} = \bigcup_{j=1}^{N_{m_r}} B_{(y_j, r)} \). Setting \( F = \bigcup_{j=1}^{N_{m_r}} F_{y_j} \), we have \( f = 0 \) in \( X \setminus F \). The result follows from the fact that \( \mu(F) = 0. \)

**Proof of Theorem 2.4.** It is clear from Theorem 2.2 and the definition of \( r \|\cdot\|_{q,p,\alpha} \) that \((L^q, L^p)^\alpha_r(X)\) is a complex vector space and \( r \|\cdot\|_{q,p,\alpha} \) is a norm on it. All we need to prove is completeness.

Let \( (f_n)_{n>0} \) be a sequence of elements of \((L^q, L^p)^\alpha_r(X)\) such that \( \sum_{n>0} r \|f_n\|_{q,p,\alpha} < \infty. \)

Since \( \sum_{n>0} \left\| \mu(B_{(r)})^{\frac{1}{q} - \frac{1}{p} - \frac{1}{q}} \right\|_q = \sum_{n>0} r \|f_n\|_{q,p,\alpha} < \infty, \) there exists a \( \mu \)-null subset \( E \) of \( X \) out of which
\[ \sum_{n>0} \left\| f_n \chi_{B_{(y,r)}} \right\|_q < \infty. \]
Therefore, for any element \( y \) of \( X \setminus E \), there is a \( \mu \)-null subset \( F_y \) of \( X \) out of which
\[ \sum_{n>0} f_n \chi_{B_{(y,r)}} \] converges absolutely. Arguing as in the proof of Theorem 2.2, we shall obtain a \( \mu \)-null subset \( F \) of \( X \) such that \( \sum_{n>0} f_n \) converges absolutely on \( X \setminus F \). Define
\[ f(x) = \begin{cases} \sum_{n>0} f_n(x) & \text{if } x \in X \setminus F, \\ 0 & \text{otherwise}. \end{cases} \]
We have
\[ r \|f\|_{q,p,\alpha} \leq \sum_{n>0} r \|f_n\|_{q,p,\alpha} < \infty. \]
In addition, for any positive integer \( n \) and any element \( y \) of \( X \),
\[ \left\| f\chi_{B_{(y,r)}} - \sum_{k=1}^{n} f_k \chi_{B_{(y,r)}} \right\|_q \leq \sum_{k>n} \left\| f_k \chi_{B_{(y,r)}} \right\|_q. \]
Therefore
\[ r \left\| f - \sum_{k=1}^{n} f_k \right\|_{q,p,\alpha} \leq \sum_{k>n} r \| f_k \|_{q,p,\alpha}. \]

Thus \( \sum_{n>0} f_n \) converges to \( f \) in \( (L^q, L^p)_r^\alpha (X) \). \( \square \)

The norm \( r \| \cdot \|_{q,p,\alpha} \) is not easy to be used. The following proposition provides us with an equivalent norm.

**Proposition 4.1.** Let \( f \) be any \( \mu \)-measurable function on \( X \), and \( r > 0 \). Put
\[ \| f \|_{q,p,\alpha}^{dmr} = \left\{ \begin{array}{ll}
\left[ \sum_{j=1}^{N_{mr}} \left( \mu(E_j^{mr}) \frac{q}{p} \| f \|_{q,p,\alpha} \right)^\frac{1}{q} \right]^\frac{p}{q} & \text{if } p < \infty,
\sup_{1 \leq j < N_{mr}} \mu(E_j^{mr}) \frac{1}{q} \| f \|_{q,p,\alpha} & \text{if } p = \infty.
\end{array} \right. \]

Then, there are positive constants \( C_1 \) and \( C_2 \), not depending on \( f \) and \( r \), such that
\[ (27) \quad C_1 \ r \| f \|_{q,p,\alpha} \leq \| f \|_{q,p,\alpha}^{dmr} \leq C_2 \ r \| f \|_{q,p,\alpha}. \]

**Proof.** Let \( f \) be any \( \mu \)-measurable function on \( X \) and \( r > 0 \).

1st case. We suppose that \( p < \infty \).

a) We have
\[ r \left\| f \right\|_{q,p,\alpha}^p = \int_X \left\{ \mu(B(y,r))^{\frac{q}{p}} \frac{1}{p} \int_X \left( |f|^q \chi_{B(y,r)} \right) (x) d\mu(x) \right\}^{\frac{p}{q}} d\mu(y) \]
\[ = \sum_{j=1}^{N_{mr}} \int_{E_j^{mr}} \left\{ \sum_{i \in I_{mr}(y)} \mu(B(y,r))^{\frac{q}{p}} \frac{1}{p} \int_{E_i^{mr}} \left( |f|^q \chi_{B(y,r)} \right) (x) d\mu(x) \right\}^{\frac{p}{q}} d\mu(y) \]
\[ \leq \frac{N_{mr}^{\frac{1}{2}}}{\frac{1}{2}} \sum_{j=1}^{N_{mr}} \int_{E_j^{mr}} \mu(B(y,r))^{\frac{q}{p}} \frac{1}{p} \sum_{i \in I_{mr}(y)} \left[ \int_{E_i^{mr}} \left( |f|^q \chi_{B(y,r)} \right) (x) d\mu(x) \right]^{\frac{p}{q}} d\mu(y), \]

according to Inequalities (21) and (25). As \( 2\kappa \rho_{mr+1} \leq r \), we have \( E_i^{mr} \subset B(y,2\kappa r) \) for \( i \in I_{mr}(y) \) and therefore by Inequality (8),
\[ (28) \quad \mu(E_i^{mr}) \leq C_\mu(2\kappa)^{D_\mu} \mu(B(y,r)), \quad i \in I_{mr}(y). \]

Taking into account Inequalities (28), (20) and (25), we obtain
\[ r \left\| f \right\|_{q,p,\alpha}^p \leq C \sum_{j=1}^{N_{mr}} \sum_{i \in I_{mr}(y)} \mu(E_i^{mr})^{\frac{p}{q}} \| f \|_{E_i^{mr}}^p. \]

So by Inequalities (22) and (26) we get
\[ r \left\| f \right\|_{q,p,\alpha}^p \leq C N_3 \sum_{i=1}^{N_{mr}} \mu(E_i^{mr})^{\frac{p}{q}} \| f \|_{E_i^{mr}}^p \leq C N_3 \left( \| f \|_{q,p,\alpha}^{dmr} \right)^p. \]
b) Notice that if \( r \|f\|_{q,p,\alpha} = \infty \), then (27) follows trivially from the above inequality. Let us assume that \( r \|f\|_{q,p,\alpha} < \infty \). For \( 1 \leq j < N_{m_r} \), we have

\[
\mu \left( E_{j}^{m_{r}} \right) \left[ \frac{1}{p} - \frac{1}{q} \right] \left\| f \chi_{E_{j}^{m_{r}}} \right\|_{p}^{q} \leq \mathcal{N}_{1}^{\frac{1}{p} - \frac{1}{q} + 1} \int _{E_{j}^{m_{r}}} \mu \left( B(y,r) \right) \left[ \frac{1}{p} - \frac{1}{q} \right] \left\| f \chi_{B(y,r)} \right\|_{q}^{p} d\mu(y),
\]

according to Lemma 3.4. As the \( E_{j}^{m_{r}} (1 \leq j < N_{m_r}) \) are pairwise disjoints, this implies

\[
\|f\|_{d_{m_{r}},q,p,\alpha} \leq \mathcal{N}_{1}^{\frac{1}{p} - \frac{1}{q} + 1} r \|f\|_{q,p,\alpha}.
\]

\( 2^{\text{nd}} \text{ case.} \) We suppose that \( p = \infty \).

(a) We have

\[
r \|f\|_{q,\infty,\alpha} = \sup_{y \in X} \left[ \sum_{j \in T_{m_{r}}^{0}(y)} \mu \left( B(y,r) \right) \left[ \frac{1}{p} - \frac{1}{q} \right] \int _{E_{j}^{m_{r}}} \left\| f \chi_{B(y,r)} (x) \right\|_{q}^{p} d\mu(x) \right]^{\frac{1}{q}}
\]

\[
\leq \left[ C_{\mu} (2\kappa)^{D_{\mu}} \right]^{\frac{1}{p} - \frac{1}{q}} \sup_{y \in X} \sum_{j \in T_{m_{r}}^{0}(y)} \mu \left( E_{j}^{m_{r}} \right) \left[ \frac{1}{p} - \frac{1}{q} \right] \left\| f \chi_{E_{j}^{m_{r}}} \right\|_{q}
\]

\[
\leq \left[ C_{\mu} (2\kappa)^{D_{\mu}} \right]^{\frac{1}{p} - \frac{1}{q}} \mathcal{N}_{2} \|f\|_{d_{m_{r}},q,\infty,\alpha},
\]

according to Inequalities (28), (21) and (25).

(b) From Inequalities (19) and (24) we have

\[
\mu \left( E_{j}^{m_{r}} \right) \left[ \frac{1}{p} - \frac{1}{q} \right] \leq \mathcal{N}_{1} \mu \left( B(y,r) \right) \left[ \frac{1}{p} - \frac{1}{q} \right], \quad 1 \leq j < N_{m_r} \text{ and } y \in E_{j}^{m_{r}}
\]

and therefore

\[
\|f\|_{d_{m_{r}},q,\infty,\alpha} \leq \mathcal{N}_{1} \sup_{1 \leq j < N_{m_r}} \sup_{y \in E_{j}^{m_{r}}} \mu \left( B(y,r) \right) \left[ \frac{1}{p} - \frac{1}{q} \right] \left\| f \chi_{E_{j}^{m_{r}}} \right\|_{q}.
\]

As \( 2\kappa \rho_{m_{r}+1} \leq r \), we have

\[
E_{j}^{m_{r}} \subset B(y,r), \quad 1 \leq j < N_{m_r} \text{ and } y \in E_{j}^{m_{r}}.
\]

Thus,

\[
\|f\|_{d_{m_{r}},q,\infty,\alpha} \leq \mathcal{N}_{1} \sup_{y \in X} \mu \left( B(y,r) \right) \left[ \frac{1}{p} - \frac{1}{q} \right] \left\| f \chi_{B(y,r)} \right\|_{q} = \mathcal{N}_{1} r \|f\|_{q,\infty,\alpha}.
\]

\( 3^{\text{rd}} \text{ case.} \) For \( q = p = \infty \), it is clear that

\[
(29) \quad r \|f\|_{\infty,\infty,\infty} = \|f\|_{\infty} = \|f\|_{d_{m_{r}},\infty,\infty}.
\]

\( \square \)

**Proof of Theorem 2.5**

a) Inequality (12) is an immediate consequence of Hölder inequality.
b) Observe that as $0 < p_1 < p_2 < \infty$, we have for any sequence $(a_j)_{1 \leq j}$ of nonnegative numbers,

$$\sup_{1 \leq j} a_j \leq \left( \sum_{j=1}^{\infty} a_j^{p_2} \right)^{1/p_2} \leq \left( \sum_{j=1}^{\infty} a_j^{p_1} \right)^{1/p_1}$$

and therefore

$$\| \cdot \|_{q,\infty,\alpha} \leq \| \cdot \|_{q,p_2,\alpha} \leq \| \cdot \|_{q,p_1,\alpha}.$$  

Inequality (13) follows from these inequalities and Proposition 4.1. □

Proof of Theorem 2.6. Let $f$ be any $\mu$-measurable function on $X$.

1st case. We suppose that $p = \infty$.

By Hölder inequality we have

$$r \|f\|_{q,\infty,\alpha} \leq \sup_{y \in X} \left\| f \chi_{B(y,r)} \right\|_{\alpha} \leq \|f\|_{\alpha}.$$  

2nd case. We suppose that $p < \infty$. Then we have

$$\|f\|_{q,p,\alpha} \leq \left[ \sum_{j=1}^{N_m} \left( \mu(E_j^{m_{r}}) \right)^{\frac{1}{p}-\frac{1}{q}} \|f \chi_{E_j^{m_{r}}}\|_{q} \right]^{\frac{1}{p}} \leq \left( \sum_{j=1}^{N_m} \left( \|f \chi_{E_j^{m_{r}}}\|_{\alpha} \right)^{\frac{1}{p}} \right)^{\frac{1}{\alpha}} \leq \|f\|_{\alpha}$$

according to Hölder inequality, the fact that $0 < \alpha \leq p < \infty$ and the pairwise disjointness of the $E_j^{m_{r}}$ ($1 \leq j < N_m$). From this inequality and Proposition 4.1 we obtain (14).

□

Proof of Theorem 2.9.

1st case. We suppose that $q = \alpha = p$.

It is clear from Proposition 4.1 that there is a constant $C_2$, not depending on $f$, such that

$$\|f\|_{\alpha} = \|f\|_{q,p,\alpha} \leq C_2 \|f\|_{\alpha,\alpha,\alpha}, \quad r > 0$$

and therefore

$$\|f\|_{\alpha} \leq C_2 \|f\|_{\alpha,\alpha,\alpha}.$$  

2nd case. We suppose that $q = \alpha < p = \infty$.

For any element $y$ of $X$ formula (11) yields

$$\left\| f \chi_{B(y,r)} \right\|_{\alpha} \leq r \|f\|_{\alpha,\infty} \leq \|f\|_{\alpha,\infty}, \quad r > 0$$

and therefore

$$\|f\|_{\alpha} = \lim_{r \to \infty} \left\| f \chi_{B(y,r)} \right\|_{\alpha} \leq \|f\|_{\alpha,\infty}.$$
$3^{rd}$ case. We suppose that $q = \alpha < p < \infty$. For $y \in X$ and $r > 0$, we have

$$\|f \chi_{B(y,r)}\|_\alpha = \left(\sum_{j=1}^{N_mr} \int_{E_j^{mr}} |f(x)|^\alpha \chi_{B(y,r)}(x) d\mu(x)\right)^{\frac{1}{\alpha}}$$

$$= \left(\sum_{j \in T_r^{mr}(y)} \int_X \left| (f \chi_{E_j^{mr}})(x) \right|^\alpha \chi_{B(y,r)}(x) d\mu(x)\right)^{\frac{1}{\alpha}}$$

$$\leq \left(\sum_{j \in T_r^{mr}(y)} \|f \chi_{E_j^{mr}}\|_\alpha^\frac{1}{\alpha} \left(\sum_{j \in T_r^{mr}(y)} \|f \chi_{E_j^{mr}}\|_\alpha^p\right)^{\frac{1}{p}}\right)^{\frac{1}{\alpha}}$$

according to Inequalities (20) and (24). So by Proposition 4.1, we get

$$\|f \chi_{B(y,r)}\|_\alpha \leq \|f\|_{q,p,p} C_2 r \|f\|_{\alpha,\alpha,p}, \quad y \in X, \ r > 0$$

and therefore

$$\|f\|_\alpha \leq \|f\|_{q,p,p} C_2 r \|f\|_{\alpha,\alpha,p}.$$ 

$4^{th}$ case. We suppose that $q < \alpha = p$. We assume that $\|f\|_{q,p,p} < \infty$, since otherwise the result follows from Theorem 2.8. For $r > 0$, put

$$f_r(x) = \mu\left(B_{(y,r)}\right)^{-\frac{1}{\beta}} \left\|f \chi_{B(y,r)}\right\|_q.$$ 

On one hand, we have for $\mu$-almost every $x$ in $X$,

$$|f(x)| = \lim_{r \to 0} f_r(x) \leq \|f\|_{q,\infty,\infty}.$$ 

Consequently

$$\|f\|_\infty \leq \|f\|_{q,\infty,\infty}.$$ 

On the other hand,

$$\left[\int_X f_r^p(x) d\mu(x)\right]^{\frac{1}{p}} \leq C \|f\|_{q,p,p}.$$ 

So, according to Fatou’s lemma, $|f|^p$ is integrable and $\|f\|_p \leq C \|f\|_{q,p,p}$. 

$\square$

**Proof of Theorem 2.10.** Let $\frac{1}{\beta} = 1 - \frac{q}{\alpha} + \frac{q}{p}$, $f$ a $\mu$-measurable function on $X$ and $r > 0$. We have $1 < \beta$, $\frac{\alpha}{q} < \infty$ and $\frac{q}{p} = \frac{1}{\beta} + \frac{q}{\alpha} - 1$. Put

$$K(x, y) = \mu\left(B_{(x,r)}\right)^{-\frac{1}{\beta}} \chi_{B_{(y,r)}}(x), \quad x, y \in X$$

and

$$Tg(y) = \int_X g(x)K(x, y) d\mu(y), \quad g \in L_0(X).$$
If \( x \in B_{(y,r)} \) then \( B_{(y,r)} \subset B_{(x,2\kappa r)} \) and therefore \( \mu \left( B_{(y,r)} \right)^{-1} \leq C \mu \left( 2\kappa \right)^{D_\mu} \mu \left( B_{(x,r)} \right)^{-1} \).

Thus

\[
\left( \int_X |K(x,y)|^\beta \, d\mu(y) \right)^{\frac{1}{\beta}} = \left( \int_X \mu \left( B_{(y,r)} \right)^{-1} \chi_{B(x,r)}(y) \, d\mu(y) \right)^{\frac{1}{\beta}} \leq C \mu \left( 2\kappa \right)^{D_\mu},
\]

and

\[
\left( \int_X |K(x,y)|^\beta \, d\mu(x) \right)^{\frac{1}{\beta}} = \left( \int_X \mu \left( B_{(y,r)} \right)^{-1} \chi_{B(y,r)}(x) \, d\mu(x) \right)^{\frac{1}{\beta}} = 1.
\]

By Lemma 3.5, there is a constant \( C \) such that

\[
\|T(|f|^q)\|_q \leq C \|f\|_{\alpha,p}^\ast \|f\|_{\beta,q}^\ast.
\]

Furthermore

\[
r \|f\|_{q,p,\alpha} = \left[ \int_X (T(|f|^q)(y))^\frac{p}{q} \, d\mu(y) \right]^\frac{1}{p} = \left( \|T(|f|^q)\|_q \right)^\frac{1}{p}.
\]

Thus

\[
r \|f\|_{q,p,\alpha} \leq \left( C \|f\|_{\alpha,p}^\ast \|f\|_{\beta,q}^\ast \right)^\frac{1}{q} = C_{\alpha,p}^\frac{1}{p} \|f\|^\ast_{\alpha,p}.
\]

The result follows. \( \square \)

**Proof of Theorem 2.11.** Let \( f \) be any \( \mu \)-measurable function on \( X \). If \( f \) does not belong to \( L^\alpha,\infty(X) \) then \( \|f\|_{\alpha,\infty} = \infty \) and there is nothing to prove. So we assume that \( f \) is in \( L^\alpha,\infty(X) \) and put \( \|f\|_{\alpha,\infty}^\ast = A. \)

a) Let us fix \( r \) and \( \lambda \) in \( (0,\infty) \) and put

\[
E = \{ x \in X : |f(x)|^q > \beta \} \text{ with } \beta = \frac{\lambda}{4\varphi(\rho^{m_r+1})b}.
\]

For any integer \( 1 \leq j < N_{m_r} \) such that \( \|f_{E_{j}}\|_{q} > \lambda \), we have

\[
\lambda - \|f_{E_{j}\cap E_{j}}\|_{q} < \int_{E_{j}\cap E_{j}\setminus E} |f(x)|^q \, d\mu(x) \leq \beta \mu \left( E_{j}^{m_r} \setminus E \right) \leq \frac{\lambda}{4}.
\]

Therefore \( \frac{3\lambda}{4} < \|f_{E_{j}\cap E_{j}}\|_{q} \) and

\[
\# \left( \left\{ j : 1 \leq j < N_{m_r} \text{ and } \|f_{E_{j}}\|_{q} > \lambda \right\} \right) \leq \# \left( \left\{ j : 1 \leq j < N_{m_r} \text{ and } \|f_{E_{j}\cap E_{j}}\|_{q} > \frac{3\lambda}{4} \right\} \right).
\]

Thus

\[
\frac{3\lambda}{4} \# \left( \left\{ j : 1 \leq j < N_{m_r} \text{ and } \| f \chi_{E_j} \|_q^q > \lambda \right\} \right)
\]

\[
\leq \frac{3\lambda}{4} \# \left( \left\{ j : 1 \leq j < N_{m_r} \text{ and } \| f \chi_{E_j} \|_q^q > \frac{3\lambda}{4} \right\} \right)
\]

\[
\leq \sum_{j=1}^{N_{m_r}} \int_{E \cap E_{m_r}^j} |f(x)|^q d\mu(x) \leq \left( \frac{\alpha}{\alpha - q} \right) A^q \mu(E)^{1 - \frac{q}{\alpha}}
\]

according to Kolmogorov condition (see [16]). As

\[
\mu(E) = \lambda_f(\beta_1^\frac{1}{q}) \leq \left( \beta_1^{-\frac{1}{q}} A \right)^\frac{\alpha}{\alpha - q} = \left( \frac{4\varphi(\rho^{m_r+1})b}{\lambda} \right)^\frac{\alpha}{\alpha - q} A^\alpha,
\]

we obtain

\[
\frac{3\lambda}{4} \# \left( \left\{ j : 1 \leq j < N_{m_r} \text{ and } \| f \chi_{E_j} \|_q^q > \lambda \right\} \right) \leq \frac{\alpha}{\alpha - q} \left( \frac{4\varphi(\rho^{m_r+1})b}{\lambda} \right)^{\frac{\alpha}{\alpha - q} - 1} A^\alpha,
\]

that is

\[
\# \left( \left\{ j : 1 \leq j < N_{m_r} \text{ and } \| f \chi_{E_j} \|_q^q > \lambda \right\} \right) \leq C \varphi(\rho^{m_r+1})^{\frac{\alpha}{\alpha - q} - 1} \lambda^{-\frac{\alpha}{\alpha - q}} A^\alpha
\]

with \( C = \frac{4^\alpha a b^{\frac{\alpha}{\alpha - q} - 1}}{3(\alpha - q)} \).

b) Assume that \( p < \infty \). Suppose that \( 1 < s < \infty \) and \( r > 0 \) and put

\[
d_j = \left\| f \chi_{E_j} \right\|_q A^{-1} \left[ b \varphi(\rho^{m_r+1}) \right]^{\frac{1}{q} - \frac{1}{s}} \left( \frac{\alpha}{\alpha - q} \right)^{-\frac{1}{q}}, \quad 1 \leq j < N_{m_r}.
\]

From Kolmogorov condition, we obtain \( 0 \leq d_j \leq 1 \) for \( 1 \leq j < N_{m_r} \). In addition, for any number \( \lambda \), we have

\[
\# \left( \left\{ j : 1 \leq j < N_{m_r}, d_j > \lambda \right\} \right) \leq C \left[ \left( \frac{\alpha}{\alpha - q} \right)^{\frac{1}{q}} \lambda \right]^{-\alpha}
\]

according to Inequality (30). Thus, we have

\[
\sum_{j=1}^{N_{m_r}} d_j^p = \sum_{n=1}^{\infty} \left( \sum_{s^{-n-1} < d_k \leq s^{-n}} d_k^p \right) \leq \sum_{n=1}^{\infty} C \left[ \left( \frac{\alpha}{\alpha - q} \right)^{\frac{1}{q}} s^{-n-1} \right]^{-\alpha} s^{-np}
\]

\[
\leq C \left( \frac{\alpha}{\alpha - q} \right)^{\frac{\alpha}{q}} \sum_{n=1}^{\infty} s^{\alpha - (p - \alpha)n} = C \left( \frac{\alpha}{\alpha - q} \right)^{\frac{\alpha}{q}} s^{2\alpha - p} s^{p - \alpha - 1}.
\]
This implies that
\[
\|f\|_{q,p,\alpha}^{d_{nm}} \leq \left[ \sup_{1 \leq j < N_{mr}} \frac{\varphi(\rho_{mr+1})}{\mu(E_{j}^{mr})} \right]^{\frac{1}{q}} \cdot \left\{ \sum_{j=1}^{N_{mr}} \left[ \|f \chi_{E_{j}^{mr}}\|_{q} A^{-1} (b \varphi(\rho_{mr+1}))^{\frac{1}{q}} \left( \frac{\alpha}{\alpha - q} \right)^{\frac{1}{q}} \right]^{p} \right\}^{\frac{1}{p}}
\]
\[
\times Ab^{\frac{1}{q} - \frac{1}{\alpha}} \left( \frac{\alpha}{\alpha - q} \right)^{\frac{1}{q}}
\]
\[
\leq \left[ \sup_{1 \leq j < N_{mr}} \frac{\varphi(\rho_{mr+1})}{\mu(E_{j}^{mr})} \right]^{\frac{1}{q} - \frac{1}{\alpha}} \left( \frac{\alpha}{\alpha - q} \right)^{\frac{1}{q}(1-\frac{p}{\alpha})} \left( \frac{s^{2\alpha-p}}{s^{\alpha-q} - 1} \right)^{\frac{1}{p}} C_{q}^{\frac{1}{p}} A.
\]
As \(r > 0\) is arbitrary in \((0, \infty)\), we obtain
\[
\|f\|_{q,p,\alpha} \leq C \|f\|_{\alpha,\infty}^{*}
\]
with \(C\) a constant not depending on \(f\).
c) For any number \(r > 0\) and positive integer \(j < N_{mr}\), we have according to Kolmogorov condition
\[
\mu\left( E_{j}^{mr} \right)^{\frac{1}{\alpha} - \frac{1}{q}} \|f \chi_{E_{j}^{mr}}\|_{q} \leq \left( \frac{\alpha}{\alpha - q} \right)^{\frac{1}{q}} A.
\]
Thus
\[
\|f\|_{q,\infty,\alpha} \leq \left( \frac{\alpha}{\alpha - q} \right)^{\frac{1}{q}} \|f\|_{\alpha,\infty}^{*}.
\]
\[\square\]

Up to now we have used in our proofs the decomposition of \(X\) in dyadic cubes as given by Sawyer and Wheeden in [27]. The dyadic cubes \(E_{j}^{k} k \geq m, 1 \leq j < N_{k}\) have their size bounded below by \(\rho^{m}\) with \(\rho > 1\) and \(m\) a fixed integer. For the proof of the next theorem, we shall use the following decomposition given by Christ in [5].

**Lemma 4.2.** There exist a collection of open subsets \(\{Q_{k}^{\alpha} : k \in \mathbb{Z}, \alpha \in I_{k}\}\), and constants \(\rho \in (0, 1), c_{0} > 0, \eta > 0\) and \(c_{1}, c_{2} < \infty\) such that
\[
\begin{align*}
&\text{(i) } \mu\left( X \setminus \bigcup_{\alpha} Q_{k}^{\alpha} \right) = 0 \ \forall k, \\
&\text{(ii) } \text{if } \ell \geq k \text{ then either } Q_{k}^{\alpha} \subset Q_{\ell}^{\alpha} \text{ or } Q_{k}^{\alpha} \cap Q_{\ell}^{\alpha} = \emptyset, \\
&\text{(iii) } \text{for each } (k, \alpha) \text{ and each } \ell < k \text{ there is a unique } \beta \text{ such that } Q_{k}^{\alpha} \subset Q_{\beta}^{\ell}, \\
&\text{(iv) } \text{diameter}(Q_{k}^{\alpha}) \leq c_{1}\rho^{k}, \\
&\text{(v) } \text{each } Q_{k}^{\alpha} \text{ contains some ball } B_{(\rho_{\alpha}, c_{0}\rho^{k})}, \\
&\text{(vi) } \mu\left( \{x \in Q_{k}^{\alpha} : d(x, X \setminus Q_{k}^{\alpha}) \leq t\rho^{k} \} \right) \leq c_{2}t^{\eta}\mu(Q_{k}^{\alpha}) \ \forall k, \alpha, \forall t > 0.
\end{align*}
\]

**Proof of Theorem 2.12.** Throughout the proof, we shall use the notation of the above lemma.
A- (a) Let us consider an element \( \beta_1 \) of \( I_1 \) and put \( E_1 = Q_{\beta_1}^1 \). Then
\[
B_{(\varepsilon_{\beta_1}, \alpha \rho)} \subset Q_{\beta_1}^1 \subset B_{(\varepsilon_{\beta_1}, \alpha \rho)}.
\]
So by Inequalities (16), (17) and (18),
\[
\mu(E_1) = m \in \left[ a_0 c_0^\mu \rho^D \mu, b b_0 c_1^\mu \rho^D \mu \right].
\]
(b) Let \( \alpha_2 \in I_{-2^2-1} \) such that \( Q_{\beta_1}^1 \subset Q_{\alpha_2}^{-2^2-1} \).
Put
\[
F_1 = \emptyset, \quad F_2 = Q_{2^2-1}^{-2^2-1} \text{ and } J_2 = \left\{ j \in I_{-2^2-1} : d(Q_j^{-2^2-1}, F_2) > \epsilon_1 \rho^{-2^2-1} \right\}.
\]
For each \( j \in J_2 \), let \( \beta_j \in I_2 \) be so that \( d(z_j^{-2^2-1}, Q_{\beta_j}^2) < \rho^2 \). We have
\[
\mu(Q_{\beta_j}^2) \in \left[ a_0 c_0^\mu \rho^{2D \mu}, b b_0 c_1^\mu \rho^{2D \mu} \right], \quad j \in J_2.
\]
We can therefore choose a finite subset \( J_2 \) of \( J_2 \) such that
\[
\sum_{j \in J_2} \mu(Q_{\beta_j}^2) \in \left[ m, m + b b_0 c_1^\mu \right].
\]
Let us take
\[
E_2 = \bigcup_{j \in J_2} Q_{\beta_j}^2.
\]
(c) Let us consider for every \( j \in J_2 \) the element \( \alpha_j \) of \( I_{-2^3-1} \) such that \( Q_j^{-2^3-1} \subset Q_{\alpha_j}^{-2^3-1} \).
Put
\[
F_3 = \bigcup_{j \in J_3} Q_{\alpha_j}^{-2^3-1} \text{ and } J_3 = \left\{ j \in I_{-2^3-1} : d(Q_j^{-2^3-1}, F_3) > \epsilon_1 \rho^{-2^3-1} \right\}.
\]
For any \( j \in J_3 \), let \( \beta_j \in I_3 \) such that \( d(z_j^{-2^3-1}, Q_{\beta_j}^3) < \rho^3 \). We have
\[
\mu(Q_{\beta_j}^3) \in \left[ a_0 c_0^\mu \rho^{3D \mu}, b b_0 c_1^\mu \rho^{3D \mu} \right], \quad j \in J_3.
\]
Thus we can pick a finite subset \( J_3 \) in \( J_3 \) such that
\[
\sum_{j \in J_3} \mu(Q_{\beta_j}^3) \in \left[ m, m + b b_0 c_1^\mu \rho^{3D \mu} \right].
\]
Put \( E_3 = \bigcup_{j \in J_3} Q_{\beta_j}^3 \).
(d) By iteration we obtain two sequences \((E_n)_{n \geq 1}\) and \((F_n)_{n \geq 1}\) such that
- \( \mu(E_n) \in \left[ m, m + b b_0 c_1^\mu \rho^{nD \mu} \right] \) and \( E_n = \bigcup_{j \in J_n} Q_{\beta_j}^n \), where \( J_n \) is a finite subset of \( I_{-2^n-1} \),
- \( d(z_j^{-2^n-1}, Q_{\beta_j}^n) < \rho^n \) and \( d(Q_j^{-2^n-1}, F_n) > \epsilon_1 \rho^{-2^n-1} \).

B- We fix \( n \geq 1 \).
(a) Let \((x, r) \in X \times \mathbb{R}^*_+.\) Suppose that \(\ell, j \in J_n\) with \(B_{(x,r)} \cap Q^n_{\beta_j} \neq \emptyset \neq B_{(x,r)} \cap Q^n_{\beta_\ell} .\)

There exists \(x_1, x_2 \in Q^n_{\beta_j}\) and \(y_1, y_2 \in Q^n_{\beta_\ell}\) such that \(d(z_j^{-2^n-1}, x_1) < \rho^n, \)
\(x_2 \in B_{(x,r)}, d(z_\ell^{-2^n-1}, y_1) < \rho^n, y_2 \in B_{(x,r)} .\)

Therefore

\[
\rho^{-2^n-1} \leq d\left(z_j^{-2^n-1}, z_\ell^{-2^n-1}\right) \leq \kappa \left[d\left(z_j^{-2^n-1}, x_1\right) + d\left(x_1, z_\ell^{-2^n-1}\right)\right] \\
\leq \kappa \rho^n + 2\kappa^3 c_1 \rho^n + 2\kappa^4 r + \kappa^4 \left[d\left(y_2, y_1\right) + d\left(y_1, z_\ell^{-2^n-1}\right)\right] \\
< \left(\kappa + 2\kappa^3 c_1 + 2\kappa^5 c_1 + \kappa^4\right) \rho^n + 2\kappa^4 r .
\]

It follows that

\[
r > \frac{1}{2\kappa^4} \left[\rho^{-2^n-1} - \left(\kappa + 2\kappa^3 c_1 + 2\kappa^5 c_1\right) \rho^n\right] \\
= \frac{\rho^n}{2\kappa^4} \left[\rho^{-2^n-1-n} - \left(\kappa + 2\kappa^3 c_1 + 2\kappa^5 c_1\right)\right] .
\]

(b) In the sequel we assume that \(n\) is sufficiently great such that

\[
c_1 \rho^n < 1 \leq \frac{\rho^n}{2\kappa^4} \left[\rho^{-2^n-1-n} - \left(\kappa + 2\kappa^3 c_1 + 2\kappa^5 c_1 + \kappa^4\right)\right] = r_n .
\]

1st case. We suppose that \(0 < r \leq c_1 \rho^n .\)

Then every ball \(B_{(x,r)}\) meets at most one \(Q^n_{\beta_j} (j \in J_n) .\) Therefore,

\[
r \|\chi_{E_n}\|_{q,p,\alpha} = \left[\int_X \left(\mu\left(B_{(x,r)}\right)\right)^{\frac{1}{\alpha} - \frac{1}{p} - \frac{1}{q}} \|\chi_{E_n \cap B_{(x,r)}}\|_q^p \ d\mu(x)\right]^\frac{1}{p} \\
= \left[\sum_{j \in J_n} \int_{\left\{x \in X : B_{(x,r)} \cap Q^n_{\beta_j} \neq \emptyset\right\}} \left(\mu\left(B_{(x,r)}\right)\right)^{\frac{1}{\alpha} - \frac{1}{p} - \frac{1}{q}} \mu\left(E_n \cap B_{(x,r)}\right)^\frac{1}{p} \ d\mu(x)\right]^\frac{1}{p} \\
\leq \left[\sum_{j \in J_n} \mu\left\{x \in X : B_{(x,r)} \cap Q^n_{\beta_j} \neq \emptyset\right\} \sup_{B_{(x,r)} \cap Q^n_{\beta_j} \neq \emptyset} \mu\left(B_{(x,r)}\right)\right]^\frac{1}{p} \\
\leq \left[\sum_{j \in J_n} \mu\left(B_{(x,r)}\right)^{\frac{1}{p}} \left(\mathbf{b}\varphi\left(r\right)\right)^{\frac{1}{\alpha} - \frac{1}{p}}\right]^\frac{1}{p} \\
\leq \left(\mathbf{b}\varphi\left(r\right)\right)^{\frac{1}{p} - \frac{1}{p}} \left[\sum C_\mu\left(\frac{\kappa}{\c_0 \rho^n}\right) D_\mu \mu\left(Q^n_{\beta_j}\right)\right]^\frac{1}{p} \\
\leq C_\mu\left(\frac{2\kappa c_1}{c_0}\right) \frac{\mu\left(E_n\right)^\frac{1}{p}}{\left(\mathbf{b}\mathbf{b}_0 c_1 \rho^n\right)^{\frac{1}{\alpha} - \frac{1}{p}} \delta_n} .
\]
2nd case. We suppose that $c_1 \rho^n < r \leq r_n$.
Arguing as in the first case, we obtain

$$
 \|r \|X_{\alpha \eta} \|_{q,p,\alpha} \leq \left[ \sum_{j \in J_n} \mu \left( \{ x \in X : B(x,r) \cap Q_{\beta j}^n \neq \emptyset \} \right) \right] \left( \sup_{B(x,r) \cap Q_{\beta j}^n \neq \emptyset} \mu(B(x,r)) \right)^{\frac{\beta}{q} - \frac{\beta}{q} - 1} \mu \left( B(x,r) \cap Q_{\beta j}^n \right)^{\frac{\beta}{q}}
$$

$$
\leq \left\{ \sum_{j \in J_n} C_{\mu} \left( \frac{\kappa(r + c_1 \rho^n)}{r} \right)^D \mu \left( \varphi(r) (a \varphi(r))^{\frac{\beta}{q} - \frac{\beta}{q} - 1} \right) \right\} \left( \mu \left( B(x,r) \cap Q_{\beta j}^n \right) \right)^{\frac{\beta}{q}}.
$$

For the second inequality we have used the doubling condition of $\mu$, the relationship between $\mu$ and $\varphi$, the growth condition on $\varphi$ and the inclusion $Q_{\beta j}^n \subset B(z_{\beta j}^n, c_1 \rho^n)$. Thus

$$
\|r \|X_{\alpha \eta} \|_{q,p,\alpha} \leq C_{\mu} \left( \frac{\kappa(r + c_1 \rho^n)}{r} \right)^D \mu \left( B(x,r) \cap Q_{\beta j}^n \right)^{\frac{\beta}{q}} \mu \left( B(x,r) \cap Q_{\beta j}^n \right)^{\frac{\beta}{q}}
$$

$$
\leq C_{\mu} \left( \mu \left( E_n \right) \right)^{\frac{\beta}{q}} \left( \left( a \varphi(r) \right)^{\frac{\beta}{q} - \frac{\beta}{q} - 1} \right) \left( \mu \left( E_n \right) \right)^{\frac{\beta}{q}} \left( \left( a \varphi(r) \right)^{\frac{\beta}{q} - \frac{\beta}{q} - 1} \right).
$$

3rd case. We suppose $r > r_n$.

$$
\|r \|X_{\alpha \eta} \|_{q,p,\alpha} \leq \left[ \sum_{j \in J_n} \int \left\{ x \in X : B(x,r) \cap Q_{\beta j}^n \neq \emptyset \right\} \mu(B(x,r)) \mu \left( E_n \cap B(x,r) \right) d\mu(x) \right]^\frac{1}{p}
$$

$$
\leq \left[ \sum_{j \in J_n} \mu \left( E_n \right) \mu \left( B(z_{\beta j}^n, c_1 \rho^n) \right) \left( a \varphi(r) \right)^{\frac{\beta}{q} - \frac{\beta}{q} - 1} \right]^\frac{1}{p}
$$

$$
\leq \mu \left( E_n \right) \left( a \varphi(r) \right)^{\frac{\beta}{q} - \frac{\beta}{q} - 1} \left( \#(J_n) \right) C_{\mu} \left( \frac{\kappa}{\beta} \right)^{D} \varphi(r)
$$

But for all $j \in J_n$, $a_0 c_{\rho^n} \rho^n \mu \leq \mu \left( Q_{\beta j}^n \right) \leq b_0 c_{\rho^n} \rho^n \mu$. Thus

$$
\#(J_n) \leq \frac{m}{a_0 c_{\rho^n} \rho^n \mu},
$$

and

$$
\|r \|X_{\alpha \eta} \|_{q,p,\alpha} \leq C_{\mu} \left( \mu \left( E_n \right) \right)^{\frac{\beta}{q}} \left( \rho \left( \frac{\rho}{\rho} + \mu \left( \frac{\rho}{\rho} + \frac{1}{\rho} \right) \left( \rho^{n-1} - \left( \kappa + 2 \kappa^3 c_1 + 2 \kappa^5 c_1 \right) \right) \right)^{\frac{\beta}{q} - \frac{\beta}{q} - 1}
$$

$$
\leq C_{\mu} \left( \mu \left( E_n \right) \right)^{\frac{\beta}{q}} \left( \rho \left( \frac{\rho}{\rho} + \mu \left( \frac{\rho}{\rho} + \frac{1}{\rho} \right) \right) \left( \rho^{n-1} - \left( \kappa + 2 \kappa^3 c_1 + 2 \kappa^5 c_1 \right) \right) \right)^{\frac{\beta}{q} - \frac{\beta}{q} - 1}
$$

It follows that if we choose $n_0$ such that for all $n \geq n_0$

$$
\frac{\left( \rho \left( \frac{\rho}{\rho} + \mu \left( \frac{\rho}{\rho} + \frac{1}{\rho} \right) \right) \left( \rho^{n-1} - \left( \kappa + 2 \kappa^3 c_1 + 2 \kappa^5 c_1 \right) \right) \right)^{\frac{\beta}{q} - \frac{\beta}{q} - 1}}{\left( \rho^{n-1} - \left( \kappa + 2 \kappa^3 c_1 + 2 \kappa^5 c_1 \right) \right)^{\frac{\beta}{q} - \frac{\beta}{q} - 1}} < 1,
$$

then
then
\[ r \left\| \chi_{E_n} \right\|_{q,p,\alpha} \leq C \rho^{-n} D_{\alpha} \left( \frac{1}{2} - \frac{1}{q} + \delta \frac{1}{q} - \frac{1}{p} \right) \mu(E_n)^{\frac{1}{p}}, \quad r > 0. \]
That is
\[ \left\| \chi_{E_n} \right\|_{q,p,\alpha} \leq C \rho^{-n} D_{\alpha} \left( \frac{1}{2} - \frac{1}{q} + \delta \frac{1}{q} - \frac{1}{p} \right) \]
and therefore
\[ \left\| \chi_{U_n \geq n^q E_n} \right\|_{q,p,\alpha} < \infty \text{ and } \left\| \chi_{U_n \geq n^q E_n} \right\|_{\alpha,\infty}^* = \infty. \]
Thus, \( f = \chi_{U_n \geq n^q E_n} \) is in \( (L^q, L^p)^\alpha(X) \setminus L^{\alpha,\infty}(X) \). □

References

[1] N. E. Aguilera and E. O. Harboure, On the search for weighted norm inequalities for the Fourier transform. Pacific J. Math., 104 (1983), 1-14.
[2] J. P. Bertrandias et C. Dupuis, Transformation de Fourier sur les espaces \( \ell^p(L^p') \). Ann. Inst. Fourier, (1) 29 (1979), 189-206.
[3] R. C. Busby and H. A. Smith, Product-convolution operators and mixed-norm spaces. Trans. Amer. Math. Soc. 263 (1981), 309-341.
[4] R. Coifman, G. Weiss, Analyse Harmonique Non-commutative sur Certains Espaces Homogènes. Lecture Notes in Math, Vol. 242. Berlin: Springer, 1971
[5] M. Christ, A T(b) theorem with remarks on analytic capacity and the Cauchy integral. Colloq. Math 60 (1990), 601-628.
[6] H. G. Feichtinger, Wiener amalgams over Euclidean spaces and some of their applications. Function Spaces (Edwardsville, IL, 1990), Lecture Notes in Pure and Appl. Math. 136 Dekker New York (1992), 123-137.
[7] J. Feuto, I. Fofana and K. Koua, Espaces de fonctions à moyenne fractionnaire intégrable sur les groupes localement compacts. Afrika Mat. (3) 15 (2003), 73-91.
[8] J. Feuto, I. Fofana and K. Koua, Weighted norms inequalities for a maximal operator in some subspace of amalgams. to appear in Can. Math. Bull. (arxiv 09014197).
[9] I. Fofana, Étude d’une classe d’espaces de fonctions contenant les espaces de Lorentz. Afrika Mat. (3) 5 (1995), 53-76.
[10] I. Fofana, Continuité de l’intégrale fractionnaire et espace \( (L^q, \ell^p)^\alpha \). C.R.A.S Paris,(1) 308 (1989), 525-527.
[11] I. Fofana, Transformation de Fourier dans \( (L^q, \ell^p)^\alpha \) et \( M^p,\alpha \). Afrika Mat. (3) 5 (1995), 53-76.
[12] I. Fofana, Espaces \( (L^q, \ell^p)^\alpha \) et continuité de l’opérateur maximal fractionnaire de Hardy-Littlewood. Afrika Mat. (3) 12 (2001), 23-37.
[13] G. B Folland and E. M. Stein, Hardy spaces on Homogeneous groups, Mathematical Notes 28, Princeton University Press. Princeton, New Jersey 1982.
[14] J. J. Fournier, On the Hausdorff-Young theorem for amalgams. Monatsh. Math. 95 (1983), 117-135.
[15] J. J. F. Fournier and J. Stewart, Amalgams of \( L^p \) and \( L^q \). Bulletin of the AMS (13) 1 (1985), 1-21.
[16] J. Garcia-Cuerva and J. L. Rubio de Francia, Weighted Norm Inequalities and Related Topics. Amsterdam: North-Holland Publishing Co., 1985.
[17] Y. Han, D. Müller and D. Yang, A theory of Besov and Triebel-Lizorkin Spaces on Metric Measure Space Modeled on Carnot-Carathéodory Spaces. Abstr Appl Anal, 2008, Art. ID 893409, 252 pp.
[18] F. Holland, *Harmonic Analysis on amalgams of $L^p$ and $\ell^q$*, J. London Math. Soc. (2) 10 (1975), 295-305.
[19] A. Kpata, I. Fofana and K. Koua, *Necessary condition for measures which are $(L^q, L^p)$ multipliers*. to appear in An. Math. Blaise Pascal (16) n° 2 (2009).
[20] K. S. Lau, *Fractal Measure and Mean p-variations*. Journ. Funct. Anal. 108 (1992), 427-457.
[21] R. Marcíás and C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Adv. in Math. 33, (1979), 257-270.
[22] D. Mascrè, *Inégalités à poids pour l’opérateur de hardy-Littlewood-Sobolev dans les espaces métriques mesurés à deux demi-dimensions*. Coll. Math. (105) 1 (2006), 77-104.
[23] B. Muckenhoupt, *Weighted norm inequalities for classical operators*. Proc. Symp. Pure Math. 35 (1977), 69-83.
[24] B. Muckenhoupt and R. Wheeden, *Weighted norm inequalities for fractional integrals*. Trans. of the AMS, 192 (1974), 261-274.
[25] A. M. Oberlin, *Affine dimension: measuring the vestiges of curvature*. Michigan Math. J. 51, 1 (2003), 13-26.
[26] M. Rao and Z. Ren, *Theory of Orlicz spaces*. Dekker, New York, 1991.
[27] E. T. Sawyer and R. L. Wheeden, *Weighted inequalities for fractional integrals on euclidean and homogeneous spaces*. Amer. J. Math. 114 (1992), 813-874.
[28] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean spaces*. Princeton, New Jersey, Princeton University Press (1971).
[29] N. Th. Varopoulos, *Analysis on Lie groups*. J. Funct. Anal. 76 (1988), 346-410.
[30] R. L. Wheeden, *A characterization of some weighted norm inequalities for the fractional maximal function*. Studia Math. 107 (1993), 251-272.
[31] N. Wiener, *On the representation of functions by trigonometrical integrals*. Math. Z., 24 (1926), 575-616.

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