On Diophantine exponents in dimension 4

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1. Introduction.
We consider a vector \( \Theta = (\theta_1, ..., \theta_n) \), \( n \geq 2 \) and suppose that the numbers \( 1, \theta_1, ..., \theta_n \) are linearly independent over \( \mathbb{Z} \). Put

\[
\psi_\Theta(t) = \min_{q \in \mathbb{Z}^+, q \leq t} \max_{1 \leq j \leq n} ||q\theta_j||.
\]

We consider the ordinary Diophantine exponent \( \omega = \omega(\Theta) \) and the uniform Diophantine exponent \( \hat{\omega} = \hat{\omega}(\Theta) \) defined as

\[
\omega = \omega(\Theta) = \sup \left\{ \gamma : \liminf_{t \to +\infty} t^\gamma \psi_\Theta(t) < +\infty \right\},
\]

\[
\hat{\omega} = \hat{\omega}(\Theta) = \sup \left\{ \gamma : \limsup_{t \to +\infty} t^\gamma \psi_\Theta(t) < +\infty \right\}.
\]

It is clear that

\[
\frac{1}{n} \leq \hat{\omega} \leq \omega
\]

and

\[
\omega \geq \hat{\omega}.
\]  

(1)

In \[2\] V. Jarník proved that in the case \( n = 2 \) the trivial inequality \((1)\) may be improved to

\[
\omega \geq \frac{\hat{\omega}^2}{1 - \hat{\omega}}.
\]  

(2)

In \[3\] M. Laurent proved that the bound \((2)\) is optimal for the case \( n = 2 \). In \[5\] the author in the case \( n = 3 \) proved the inequality

\[
\omega \geq \frac{\hat{\omega}}{2} \left( \frac{\hat{\omega}}{1 - \hat{\omega}} + \sqrt{\left( \frac{\hat{\omega}}{1 - \hat{\omega}} \right)^2 + \frac{4\hat{\omega}}{1 - \hat{\omega}}} \right).
\]  

(3)

A different proof of the inequality \((3)\) was given by W. Schmidt and L. Summerer \[9\] by means of a new powerful method developed in \[7, 8\].

It turned out that the inequality \((3)\) is optimal. In \[9\] it was announced that D. Roy recently obtained such a result (see also author’s announcement from Section 3.3 from \[6\] concerning Theorem 14.) However no optimal inequality is known in dimensions \( n \geq 4 \). Probably the best known general inequality is due to W.M. Schmidt and L. Summerer. It is as follows. For an arbitrary \( n \geq 2 \) one has

\[
\omega \geq \frac{\hat{\omega}^2 + (n - 2)\hat{\omega}}{(n - 1)(1 - \hat{\omega})}.
\]

In the present paper we obtain some results concerning the case \( n = 4 \).

2. Index \( i(\Theta) \).
We consider a vector \( \Theta = (\theta_1, \theta_2, \theta_3, \theta_4) \) such that real numbers \( 1, \theta_1, \theta_2, \theta_3, \theta_4 \) are linearly independent over \( \mathbb{Z} \). Here we introduce the value \( i(\Theta) \) which is of importance for the
with a view to simplify the notation formulation of our results. We define it to be the index of the vector $\Theta$. We consider the sequence $z_\nu = (q_\nu, a_{1,\nu}, a_{2,\nu} a_{3,\nu}, a_{4,\nu}) \in \mathbb{Z}^5$, $\nu = 0, 1, 2, 3, 4, \ldots$ of all best approximations to the vector $\Theta$, so that

$$q_0 < q_1 < \ldots < q_\nu < q_{\nu+1} < \ldots, \quad \zeta_0 > \zeta_1 > \ldots > \zeta_\nu > \zeta_{\nu+1} > \ldots,$$

where

$$\zeta_\nu = \max_{1 \leq j \leq 4} ||q_\nu \theta_j||, \quad ||q_\nu \theta_j|| = |q_\nu \theta_j - a_{\nu,j}|$$

and

$$\zeta_\nu = \min_{x \in \mathbb{Z}} \max_{1 \leq j \leq 4} ||x \theta_j||.$$

Take $\alpha < \hat{\omega}(\Theta)$. Then

$$\zeta_\nu \leq q_{\nu+1}^{-\alpha} \quad (4)$$

for $\nu$ large enough.

It is a well known fact (see [4], Section 4.1) that for any $\nu_0$ all the vectors $z_\nu, \nu \geq \nu_0$ cannot lie in a common linear subspace $L \subset \mathbb{R}^5$ of dimension $\dim L \leq 4$. V. Jarník [2] proved that there exists infinitely many $\nu$ such that three vectors $z_{\nu-1}, z_\nu, z_{\nu+1}$ are linearly independent. One can easily deduce from these two facts the following

**Proposition 1.** There exist infinitely many sets of indices $(\nu = r_0, r_1, r_2, \ldots, r_n, r_{n+1} = k)_l$ (here $n = n_l$ depends on $l$) such that

(i) $\nu \to \infty$ as $l \to \infty$;

(ii) The triplets of vectors $(z_{r_i-1}, z_{r_i}, z_{r_i+1})$ are linearly independent for any $0 \leq i \leq n + 1$;

(iii) for any $i$ from the interval $1 \leq i \leq n$ each two consecutive couples of vectors

$$(z_{r_i}, z_{r_i+1}), (z_{r_i+1}, z_{r_i+2}), \ldots, (z_{r_{i+1}-1}, z_{r_{i+1}})$$

lie in the same two-dimensional linear subspace; we denote this subspace by $L_i$, so

$$L_i = \text{span} (z_{r_i}, z_{r_i+1}) = \text{span} (z_{r_{i+1}-1}, z_{r_{i+1}});$$

(iv) there exists a three-dimensional linear subspace $T_l$ such that for any $i$ from the interval $1 \leq i \leq n_l$ the triple $(z_{r_i-1}, z_{r_i}, z_{r_i+1})$ belongs to this subspace, so

$$T_l = \text{span} (z_{r_i-1}, z_{r_i}, z_{r_i+1}), \quad 1 \leq i \leq n_l;$$

(v) $z_{r_{i-1}} \not\in T_l$ and $z_{r_{i+1}} \not\in T_l$;

(vi) the collection $z_{\nu-1}, z_{r_{n-1}}, z_{k-1}, z_k, z_{k+1}$ consists of five linearly independent vectors.

It may happen that for the vector $\Theta$ there exists an integer $n$ and a sequence of indices $(\nu, r_1, r_2, \ldots, r_n, k)_l$ such that the conclusions (i) - (vi) of Proposition 1 hold for $n_l = n$. Then we define

$$i(\Theta) = \min \left\{ n : \text{there exists a sequence } (\nu, r_1, r_2, \ldots, r_n, k)_l \right. \quad (5)$$

such that the conditions (i) - (vi) hold for $n_l = n$.

If it is not so we define $i(\Theta) = \infty$.

For example in the case when there exist infinitely many $\nu$ such that every five vectors

$$(z_{\nu-1}, z_\nu, z_{\nu+1}, z_{\nu+2}, z_{\nu+3})$$

$^2$We have in mind that all the parameters are depend on $l$. However, sometimes we will not use the lower index $l$ with a view to simplify the notation.
are linearly independent, one has \( i(\Theta) = 1 \). In fact, certain lower bounds for \( \omega(\Theta) \) in terms of \( \hat{\omega}(\Theta) \) in this case were discussed in [6], Subsection 3.5, footnote 7. We believe that for simultaneous approximation these bounds are optimal in the case \( i(\Theta) = 1,2 \).

One can easily see that \( i(\Theta) \) can attain any positive integer value for a vector \( \Theta \) with the components \( \theta_1, \theta_2, \theta_3, \theta_1 \), linearly independent over \( \mathbb{Z} \) together with 1.

3. The main result.

We consider the polynomial

\[
f_1(x) = f_1^{[\omega]}(x) = x^3 - \frac{\omega}{1 - \omega} x^2 - \frac{\omega}{1 - \omega} x - \frac{\omega}{1 - \omega}.
\]

For every \( \hat{\omega} \in \left[\frac{1}{4}, 1\right) \) it has a unique real positive root \( G_1(\hat{\omega}) \). One can see that \( G_1 \left( \frac{1}{4} \right) = 1 \) and \( G_1(\hat{\omega}) \) increases to infinity as \( \hat{\omega} \) increases to 1.

Then we consider two polynomials

\[
f_{2,1}(x) = f_{2,1}^{[\omega]}(x) = x^4 - \frac{\omega}{1 - \omega} x^3 - \frac{\omega}{1 - \omega} x^2 + \left( \frac{\omega}{1 - \omega} \right)^2 x - \omega \quad \text{and}
\]

\[
f_{2,2}(x) = f_{2,2}^{[\omega]}(x) = x^4 - \frac{\omega}{1 - \omega} x^3 - \frac{\omega}{1 - \omega} x^2 + \frac{\omega}{1 - \omega} x - \omega \left( 1 - \omega \right)^2.
\]

One can see that for \( \hat{\omega} = \frac{1}{2} \) one has \( f_{2,1} = f_{2,2} \). For every \( \hat{\omega} \in \left[\frac{1}{4}, \frac{1}{2}\right] \) the polynomial \( f_{2,1} \) has the unique positive root \( G_{2,1}(\hat{\omega}) \). For every \( \hat{\omega} \in \left[\frac{1}{2}, 1\right) \) the polynomial \( f_{2,2} \) has the unique positive root \( G_{2,2}(\hat{\omega}) \). Put

\[
G_2(\hat{\omega}) = \max(G_{2,1}(\hat{\omega}), G_{2,2}(\hat{\omega})) = \begin{cases} G_{2,1}(\hat{\omega}) & \text{if } \frac{1}{4} \leq \hat{\omega} \leq \frac{1}{2}, \\ G_{2,2}(\hat{\omega}) & \text{if } \frac{1}{2} \leq \hat{\omega} < 1, \end{cases}
\]

It can be easily seen that \( G_2 \left( \frac{1}{4} \right) = 1 \) and \( G_2(\hat{\omega}) \) increases to infinity as \( \hat{\omega} \) increases to 1. It is clear that

\[
G_2(\hat{\omega}) < G_1(\hat{\omega}) \quad \text{for } \frac{1}{4} < \hat{\omega} < 1.
\] (6)

We also consider the polynomial

\[
f_3(x) = f_3^{[\omega]}(x) = x^5 - \frac{\omega}{1 - \omega} x^4 - \frac{\omega}{1 - \omega} x^3 + \frac{\omega^2}{(1 - \omega)^2} x^2 - \frac{\omega}{(1 - \omega)^2}.
\] (7)

Denote \( G_3(\hat{\omega}) \) a unique positive root of \( f_3(x) \). We can easily see that \( G_3 \left( \frac{1}{4} \right) = 1, G_3(\hat{\omega}) < G_2(\hat{\omega}) \) for \( \frac{1}{4} < \hat{\omega} < 1 \) and \( G_3(\hat{\omega}) \) increases to infinity as \( \hat{\omega} \) increases to 1.

Theorem 1.

1) Suppose that \( i(\Theta) = 1 \). Then

\[
\omega(\Theta) \geq \hat{\omega}(\Theta) G_1(\hat{\omega}(\Theta)).
\] (8)

2) Suppose that \( i(\Theta) = 2 \). Then

\[
\omega(\Theta) \geq \hat{\omega}(\Theta) G_2(\hat{\omega}(\Theta)).
\] (9)

3) Suppose that \( i(\Theta) = 3 \). Then

\[
\omega(\Theta) \geq \hat{\omega}(\Theta) G_3(\hat{\omega}(\Theta)).
\] (10)
It is not difficult to show that in the case \( i(\Theta) = 1 \) Theorem 1 gives the optimal lower bound. However we do not prove this fact in the present paper. The bound (9) of the case \( i(\Theta) = 2 \) should also be optimal. Moreover, we believe that there exists a constant \( \hat{\omega}_0 \) such that the bound (10) is optimal in the interval \( \frac{1}{4} \leq \hat{\omega} \leq \hat{\omega}_0 \).

In the sequel for the case \( i(\Theta) = 1 \) we give a complete proof of the statement, as well as for the case \( i(\Theta) = 2 \) and \( \hat{\omega} \leq \frac{1}{2} \). For the cases \( i(\Theta) = 2 \) and \( \hat{\omega} \geq \frac{1}{2} \) and \( i(\Theta) = 3 \).

**Remark 1.** In any cases Proposition 1 gives a system of equalities and inequalities. Some of these inequalities are essential for the case under the consideration, but some of them are not. We performed computer calculations to determine which inequalities are essential for each case. In such a way we found the systems (17,20,22) below. As these systems define simplicial cones in the considered spaces, this gives proofs of the main estimates.

**Remark 2.** We performed more extensive calculations in the case \( i(\Theta) = 3 \) and found out that sometimes for \( \hat{\omega} > \hat{\omega}_0 \) the polynomial \( f_3 \) from (7) does not give an optimal lower bound in (10). For some values of \( \hat{\omega} \) better bounds may be obtained due to the polynomials \( \hat{\omega} \)

\[
x^5 - x^4 \frac{\hat{\omega}}{1 - \omega} - x^3 \frac{\hat{\omega}}{1 - \omega} - \frac{(1 - x\hat{\omega})^2}{(1 - \omega)^3}, \quad x^5 - x^4 \frac{\hat{\omega}}{1 - \omega} - x^3 \frac{\hat{\omega}}{1 - \omega} - \frac{(1 - x\hat{\omega})(2 - x)}{(1 - \omega)^2}.
\]

However the calculations are too cumbersome and do not rely on any new ideas.

4. **Case** \( i(\Theta) = 1 \). Here one must note that if there are two indices \( j_1 \leq j_2 \) such that

\[
\text{span} \left( z_{j_1}, z_{j_1+1} \right) = \text{span} \left( z_{j_2-1}, z_{j_2} \right)
\]

then

\[
\zeta_{j_1} q_{j_1+1} \cong \Theta \zeta_{j_2-1} q_{j_2}.
\] (11)

This is a well known statement (see Lemma 1 from [5] or Theorem 2.13 from [1]). Now from (vi) we see that

\[1 \ll \Theta \zeta_{v-1} \zeta_{r_2-1} \zeta_{r_1-1} \zeta_{k-1} q_{k+1}.
\] (12)

We have

\[
\zeta_v q_{v+1} \cong \Theta \zeta_{r_1-1} q_{r_1}, \quad \zeta_{r_1-1} q_{r_1+1} \cong \Theta \zeta_{k-1} q_k.
\] (13)

Now we take two parameters \( u, v \in (0, 1) \) defined from the system of equalities

\[
\frac{\alpha}{(1 - \alpha)u} = -v + \frac{1}{1 - \alpha} = \frac{(1 - \alpha)u + \alpha}{(1 - \alpha)(1 - v)}.
\]

From (12) we have

\[
\zeta_{v-1} \zeta_v q_{v+1} \cdot \zeta_{r_1-1} \zeta_{r_1-1} q_{r_1+1} \cdot \zeta_{k-1} \zeta_k q_{k+1} \gg \Theta \left( \zeta_v q_{v+1} \right)^{1-u} \left( \zeta_{r_1-1} q_{r_1} \right)^u \left( \zeta_{k-1} q_{k+1} \right)^{1-v}.
\]

So at least one of the following three inequalities is valid:

\[
\zeta_{v-1} \zeta_v q_{v+1} \gg \Theta \left( \zeta_v q_{v+1} \right)^{1-u},
\] (14)

\[
\zeta_{r_1-1} \zeta_{r_1} q_{r_1+1} \gg \Theta \left( \zeta_{r_1-1} q_{r_1} \right)^u \left( \zeta_{r_1-1} q_{r_1+1} \right)^v,
\] (15)
\[ \zeta_{k-1} \zeta_k q_{k+1} \gg \theta (\zeta_{k-1} q_k)^{1-v}. \] (16)

We take \( \alpha < \hat{\omega}(\Theta) \) close to \( \hat{\omega} \). Then by (4) and the choice of parameters \( u, v \) we deduce the following.

Given any small positive \( \varepsilon \), if (14) holds then \( q_{\nu+1} \gg \theta q^G_{\nu+\varepsilon} \), if (15) holds then \( q_{r_1+1} \gg \theta q^G_{r_1+\varepsilon} \), if (16) holds then \( q_{k+1} \gg \theta q^G_{k+\varepsilon} \).

So in the case \( i(\Theta) = 1 \) everything is proven.

5. **Case** \( i(\Theta) = 2 \) and \( \hat{\omega} \leq \frac{1}{2} \).

Put

\[ \mathcal{X} = (\xi_{\nu-1}, \xi_{\nu}, \xi_{r_1-1}, \xi_{r_1}, \xi_{r_2-1}, \xi_{r_2}, \xi_{k-1}, \xi_{k}, X_1, X_{\nu+1}, X_{r_1}, X_{r_1+1}, X_{r_2}, X_{r_2+1}, X_k, X_{k+1}). \]

Then \( \mathcal{X} \) is a point in 16-dimensional space \( \mathcal{R} = \mathbb{R}^{16} \). We consider subspace

\[ \mathcal{R}' = \{ \mathcal{X} \in \mathcal{R} : X_\nu = 0 \}. \]

We are interested in the system in fifteen inequalities

\[
\begin{align*}
\xi_{\nu-1} + \xi_{r_2-1} + \xi_{k-1} + \xi_k + X_{k+1} & \geq 0, \\
\xi_{j-1} - \alpha X_j & \leq 0, \quad j = \nu, \nu + 1, r_1 + 1, r_2 + 1, k + 1, \\
X_{r_1+1} & \leq X_{r_2}, \quad X_{r_2+1} \leq X_k, \\
\xi_{r_1-1} & \geq \xi_{r_1} \geq \xi_{r_2-1}, \quad \xi_{r_2} \geq \xi_{k-1}, \\
X_{j+1} & \leq g X_j, \quad j = \nu, r_1, r_2, k.
\end{align*} \tag{17}
\]

and in the equation

\[ \xi_{\nu} + X_{\nu+1} = \xi_{r_1-1} + X_{r_1} \tag{18} \]

**Lemma 1.** Suppose that \( \alpha, g > 0 \). Suppose that there exists \( \mathcal{X} \in \mathcal{R}' \) such that its coordinates satisfy (17) and (18). Then

\[ g \geq G_{2,1}(\alpha). \tag{19} \]

**Proof.** As for the \( 15 \times 15 \) matrix

\[
\mathcal{G} = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & g & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

we have \( \det \mathcal{G} \neq 0 \), the set

\[ \mathcal{C} = \{ \mathcal{X} \in \mathcal{R}' : \text{coordinates of the point } \mathcal{X} \text{ satisfy (17)} \}. \]

5
is a simplicial cone in $\mathcal{R}'$. We may calculate the coordinates of the vertex of $\mathcal{C}$. If we substitute them into (18), we see that $g$ will be the root of the polynomial $f_{[2]}^{[a]}$. So we see that the condition

$$ \mathcal{C} \cap \{ \mathcal{X} : \xi_\nu + X_{\nu+1} = \xi_{r_1-1} + X_{r_1}\} \neq \emptyset $$

is equivalent to (23). $\square$

Now we prove (9). Suppose that $i(\Theta) = 2$ and $\hat{\omega} \leq \frac{1}{2}$. We take $\alpha < \hat{\omega}(\Theta)$ and take the 4-tiple $(\nu, r_1r_2, k) = (\nu_l, r_{l,1}r_{l,2}, k_l)$ with $l$ large enough. We may suppose that (4) holds for all the indices under consideration. Suppose that $q_{j+1} < q_j^d$ for $j \in \{\nu, r_1, r_2, k\}$. Put

$$ \xi_j = \log \xi_j \quad X_j = \log q_j, \quad j = 1, 2, 3, \ldots $$

From Proposition 1 we see that all the inequalities (17) are satisfied with the only one exception. Instead of the first inequality there will be the inequality

$$ \xi_{\nu-1} + \xi_{r_1-1} + \xi_{k-1} + \xi_k + x_{k+1} \geq \gamma(\Theta), $$

where $\gamma(\Theta)$ is bounded as $l \to \infty$. At the same time instead of (18) we have

$$ |\xi_\nu + X_{\nu+1} - (\xi_{r_1-1} + X_{r_1})| \leq \delta(\Theta), $$

where $\delta(\theta)$ is bounded as $l \to \infty$. We take into account that $|\log \xi_j|, |X_j| \to \infty$ for all indices under consideration, as $l \to \infty$. So from Lemma 1 we see that for any positive $\varepsilon$ for large $l$ either $X_{r_1+1} \geq (G_{2,1}(\hat{\omega}) - \varepsilon)X_\nu$, or $X_{r_1+1} \geq (G_{2,1}(\hat{\omega}) - \varepsilon)X_{r_1}$, or $X_{r_2+1} \geq (G_{2,1}(\hat{\omega}) - \varepsilon)X_{r_2}$, or $X_{k+1} \geq (G_{2,1}(\hat{\omega}) - \varepsilon)X_k$.

So there exist $j \in \{\nu, r_1, r_2, k\}$ such that $q_{j+1} \geq q_j^{G_{2,2}(\hat{\omega}) - \varepsilon}$ and everything follows.

6. Case $i(\Theta) = 2$ and $\hat{\omega} \geq \frac{1}{2}$.

Analogically, we consider the same vector $\mathcal{X}$ and a similar, but distinct system of inequalities

$$ \begin{align*}
\xi_{\nu-1} + \xi_{r_2-1} + \xi_{k-1} + \xi_k + X_{k+1} & \geq 0, \\
\xi_{r_1-1} + \xi_r + X_{r_1+1} & \geq 0, \\
\xi_j - \alpha X_j & \leq 0, \quad j = \nu, \nu + 1, r_1 + 1, r_2 + 1, k + 1 \\
X_{r_1+1} & \leq X_{r_2}, \quad X_{r_2+1} \leq X_k, \\
\xi_r & \geq \xi_{r_2-1}, \quad \xi_{r_2} \geq \xi_{k-1}, \\
X_{j+1} & \leq gX_j, \quad j = \nu, r_1, r_2, k.
\end{align*} $$

(20)

We deal with the same equation (18).

**Lemma 2.** Suppose that $\alpha, g > 0$. Suppose that there exists $\mathcal{X} \in \mathcal{R}'$ such that its coordinates satisfy (20) and (18). Then

$$ g \geq G_{2,2}(\alpha). $$

(21)

The proof is quite similar to the proof of Lemma 1. $\square$

Now we sketch a proof for the second part of (9). The argument from the previous case is valid if we take into account that from the Proposition 1 it follows that

$$ \xi_{r_1-1} + \xi_r + X_{r_1+1} \geq \gamma_1(\Theta) $$

where $\gamma_1(\Theta)$ is a constant. Now we may apply Lemma 1 similarly to the previous case and obtain the desired statement.

7. Case $i(\Theta) = 3$: a sketch.
Now we consider a vector
\[ \mathbf{X} = (\xi_{\nu-1}, \xi_{\nu}, \xi_{r_1-1}, \xi_{r_1}, \xi_{r_2-1}, \xi_{r_2}, \xi_{r_3-1}, \xi_{r_3}, \xi_{k-1}, \xi_{k}, X_{\nu}, X_{\nu+1}, X_{r_1}, X_{r_1+1}, X_{r_2}, X_{r_2+1}, X_{r_3}, X_{r_3+1}, X_{k}, X_{k+1}) \]
and a system of inequalities.

\[\begin{align*}
\xi_{\nu-1} + \xi_{r_3-1} + \xi_{k-1} + \xi_k + X_{k+1} &\geq 0, \\
\xi_{j-1} - \alpha X_j &\leq 0, \quad j = \nu, \nu + 1, r_2 + 1, r_3 + 1, k + 1 \\
X_{r_1+1} &\leq X_{r_2}, \quad X_{r_2+1} \leq X_{r_3}, X_{r_3+1} \leq X_k, \\
\xi_{r_1} &\geq \xi_{r_2-1} \geq \xi_{r_2} \xi_{r_3-1}, \quad \xi_{r_3} \geq \xi_{k-1}, \\
X_{j+1} &\leq g X_j, \quad j = \nu, r_1, r_2, r_3, k.
\end{align*}\]  
(22)

with the equation (18).

To obtain (10) we need Lemma 3.

**Lemma 3.** Suppose that \( \alpha, g > 0 \). Suppose that there exists \( \mathbf{X} \in \mathcal{R}' \) such that its coordinates satisfy (22) and (18). Then
\[ g \geq G_3(\alpha). \]  
(23)

To prove Lemma 3 one should consider a 19-dimensional cone in the subspace \( \{ X_{\nu} = 0 \} \) defined by (22).

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