Finite Codimensional Controllability, and Optimal Control Problems with Endpoint State Constraints

Xu Liu† Qi Lü‡ Xu Zhang§

Abstract

In this paper, motivated by the study of optimal control problems for infinite dimensional systems with endpoint state constraints, we introduce the notion of finite codimensional (exact/approximate) controllability. Some equivalent criteria on the finite codimensional controllability are presented. In particular, the finite codimensional exact controllability is reduced to deriving a Gårding type inequality for the adjoint system, which is new for many evolution equations. This inequality can be verified for some concrete problems (and hence applied to the corresponding optimal control problems), say the wave equations with both time and space dependent potentials. Moreover, under some mild assumptions, we show that the finite codimensional exact controllability of this sort of wave equations is equivalent to the classical geometric control condition.

Key Words. Finite codimensional controllability, finite codimensionality, optimal control, endpoint state constraint, Pontryagin type maximum principle.

AMS subject classifications. 93B05, 49J20, 93B07, 49K20, 35Q93.

1 Introduction

It is well known that control theory was founded by N. Wiener in 1948 ([44]). After that, this theory was greatly extended to various complicated setting and widely used in sciences and technologies. Particularly, after the seminal works [5, 6, 25, 37, 38], rapid development of mathematical control theory (for both deterministic and stochastic systems but this paper will focus only on deterministic ones) began in the 1960s (e.g., [4, 9, 10, 11, 13, 16, 18, 20, 22, 24, 31, 32, 33, 40, 41, 42, 45, 46, 47] and rich references cited therein). Usually, in terms of the so-called state-space technique, people describe the considered control system as a suitable state equation.

Roughly speaking, “control” means that one hopes to change the dynamics of the involved system, by means of a suitable way. In our opinion, there are two (most, in some sense) fundamental issues in control theory, i.e., feasibility and optimality, which we shall explain more below.

The first fundamental issue is feasibility, or in the terminology of control theory, controllability, which means that, one can find at least one way to achieve a goal. More precisely, for simplicity,
let us consider the following controlled system governed by a linear ordinary differential equation:

\[
\begin{aligned}
  y_t(t) &= Ay(t) + Bu(t), & t > 0, \\
  y(0) &= y_0.
\end{aligned}
\]  

(1.1)

In (1.1), \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m} \) \((n, m \in \mathbb{N})\), \( y(\cdot) \) is the state variable, \( u(\cdot) \) is the control variable, and \( \mathbb{R}^n \) and \( \mathbb{R}^m \) are the state space and control space, respectively. The system (1.1) is called exactly controllable (at time \( T > 0 \)) if for any initial state \( y_0 \in \mathbb{R}^n \) and any final state \( y_1 \in \mathbb{R}^n \), there is a control \( u(\cdot) \in L^2(0, T; \mathbb{R}^m) \) such that the solution \( y(\cdot) \) to (1.1) satisfies

\[
y(T) = y_1.
\]

(1.2)

The above definition of controllability can be easily extended to abstract evolution equations. In the general setting, it may happen that the requirement (1.2) has to be relaxed in one way or another. This leads to the approximate controllability, null controllability, partial controllability, and finite codimensional exact/approximate controllability (to be introduced in this paper), etc. Also, the above \( B \) can be unbounded for general controlled systems.

Clearly, the above controllability problem can be viewed as another equation problem, in which both \( y(\cdot) \) and \( u(\cdot) \) are unknowns. Namely, instead of viewing \( u(\cdot) \) as a control variable, we may simply regard it as another unknown variable. Nevertheless, the resulting equation problem is definitely ill-posed. Indeed, none of existence, uniqueness and continuous dependence of this equation problem is guaranteed. This is the main difficulty in the study of many controllability problems (both theoretically and numerically).

Controllability is strongly related to (or in some situation, even equivalent to) other important issues in control theory, say observability, stabilization and so on. One can find numerous literatures on these topics (see [1, 2, 4, 7, 8, 10, 11, 12, 20, 22, 23, 25, 27, 26, 28, 33, 40, 41, 42, 45, 46, 47] and rich references therein).

The second fundamental issue is optimality, or in the terminology of control theory, optimal control, which means that people are expected to find the best way, in some sense, to achieve their goal. As an example, we fix \( y_0, y_1 \in \mathbb{R}^n \). It is easy to see that, if there exists a control \( u(\cdot) \) such that the solution \( y(\cdot) \) to (1.1) satisfies (1.2), then very often one may find another control verifying the same conditions. Naturally, one hopes to find the “best” control fulfilling these conditions. To be more precisely, we fix a suitable function \( f^0(\cdot, \cdot, \cdot) : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \), and denote by \( U_{ad} \) the set of controls \( u(\cdot) \in L^2(0, T; \mathbb{R}^m) \) so that the solution \( y(\cdot) \) to (1.1) satisfies (1.2) and \( f^0(\cdot, y(\cdot), u(\cdot)) \in L^1(0, T) \). A typical optimal control problem for the system (1.1) is to find a \( \tilde{u}(\cdot) \in U_{ad} \), called an optimal control, which minimizes the following functional over \( U_{ad} \):

\[
J(u(\cdot)) = \int_0^T f^0(t, y(t), u(t))dt.
\]

The above formulation of optimal control problem can be easily extended to more general setting, in particular for the case that the controls take values in a more general set (instead of \( \mathbb{R}^m \)), say any metric space, which does not need to enjoy any linearity or convexity structure.

Optimal control problems are strongly related to the classical calculus of variations and optimization theory. Nevertheless, since the control set may be quite general, the classical variation technique cannot be applied to optimal control problems directly, especially in the case that the state space is infinite dimensional. Various optimal control problems are extensively studied in the literatures (e.g., [5, 6, 9, 13, 16, 18, 24, 30, 32, 37, 38, 43] and rich references cited therein).

Clearly, the study of controllability problems is a basis to investigate further optimal control problems. Indeed, the usual nonempty assumption on the set of feasible/admissible control-state
pairs (for optimal control problems) is actually a controllability condition. Nevertheless, in the previous literatures, it seems that the studies of controllability and optimal control problems are almost independent. Two typical exceptions that we know are the following:

1) In [23], some techniques from optimal control theory are employed to derive the observability estimate and null controllability for parabolic type equations.

2) In [43], some techniques developed in the study of controllability and observability problems are adopted to solve several time optimal control problems.

In our opinion, now it is the time to solve controllability and optimal control problems as a whole, at least in some sense and to some extend, though they are two different control issues. This is by no means an easy task. Actually, for many concrete problems, it is highly nontrivial to verify the above mentioned assumption that the set of feasible/admissible control-state pairs is nonempty.

The main purpose of this paper is to provide a new link between controllability and optimal control problems in infinite dimensions. Our work is motivated by the study of optimal control problems for abstract evolution equations with endpoint state constraints. In [17, 29, 30], in order to guarantee the nontriviality of Lagrange type multipliers in the corresponding Pontryagin type maximum principle, a finite codimensionality condition is introduced. However, it is usually very difficult to verify this condition directly except for some very special cases. Because of this, we shall reformulate this condition as a class of new controllability notion, i.e. finite codimensional exact controllability.

A key contribution in this work is to reduce further the above mentioned finite codimensional exact controllability to a suitable a priori estimate for the underlying adjoint system (see the estimate (3.13) for the equation (3.4)). We remark that, in some sense, the inequality (3.13) can be regarded as a Gårding type inequality, which concerns the lower bound of a bilinear form induced by a linear elliptic (pseudo-)differential operator. To see this, let us recall below the classical Gårding inequality (e.g., [21, Section 5 of Chapter X] for more details and more general results). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with a smooth boundary \( \partial \Omega \), and let \( L \) be a uniformly linear elliptic differential operator of order \( 2k \) (for some positive integer \( k \)) with smooth coefficients, i.e., there exists a constant \( s_0 > 0 \) such that

\[
\ell(x, \xi) \geq s_0 |\xi|^{2k}, \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^n,
\]

where \( \ell \) is the symbol of \( L \). Then there exist two constants \( C_0 > 0 \) and \( C_1 \geq 0 \) such that

\[
C_0 |v|_{H^k(\Omega)}^2 \leq \langle Lv, v \rangle_{H^{-k}(\Omega), H^k(\Omega)} + C_1 |v|_{L^2(\Omega)}^2, \quad \forall v \in H^k_0(\Omega).
\] (1.3)

Clearly, both (3.13) and (1.3) have an extra term, that is, \( |G\phi_T|_X \) and \( |v|_{L^2(\Omega)}^2 \), respectively. It is easy to observe that these two terms are accordingly compact with respect to the ones in the left hand sides of the corresponding estimates. Hence, we may call (3.13) a Gårding type inequality for the evolution equation (3.4). This inequality can be verified for some concrete problems, say the wave equations with both time and space dependent potentials (see Subsection 6.1). Though the later result (which seems not available in the previous literatures) might be known for some experts in the field of micro-local analysis, nobody knows how to use it. Interestingly, in this work we shall give its application in optimal control problems. Moreover, under some mild assumptions, we shall show that the finite codimensional exact controllability of this sort of wave equations is equivalent to the celebrated geometric control condition (introduced in the papers [3, 4]) for the classical wave equation.
In this work, in order to present the key idea in the simplest way, we shall not pursue the full technical generality. It deserves mentioning that the method and technique developed in this paper can be employed to handle many other problems. Especially, our finite codimensionality technique can be applied to solve some interesting problems in optimization, calculus of variations and stochastic control, and even gives new results for some finite dimensional optimal control problems under state constraints (see our forthcoming paper [34] for more details).

The rest of this paper is organized as follows. Section 2 is of preliminary nature, in which we present some notations, notions and simple results. In Section 3, some equivalent criteria for the finite codimensional exact controllability are given. Section 4 is devoted to a characterization of the finite codimensional approximate controllability. In Section 5, the finite codimensional controllability is applied to study some optimal control problems with state constraints. In Section 6, two examples are given. Finally, in Appendix, we prove a technical result used in this paper.

2 Notations, notions and some preliminary results

To begin with, we introduce some notations. Let $Y$ and $U$ be two reflexive Banach spaces. For a Banach space $Z$, denote by $\mathcal{L}(Z;Y)$ the set of all bounded linear operators from $Z$ to $Y$, and write it $\mathcal{L}(Y)$ for short when $Z = Y$. For any operator $P \in \mathcal{L}(Z;Y)$, write $P^*$ for its adjoint operator. Denote by $Y'$ the dual space of $Y$; by $D$ the closure of a subset $D$ of $Y$; by $\text{span}D$ the convex closed hull of $D$. For two subsets $D_1$ and $D_2$ of $Y$, set $D_1 - D_2 = \{ y \in Y \mid y = y_1 - y_2 \text{ for some } y_1 \in D_1 \text{ and } y_2 \in D_2 \}$ and $D_1 \setminus D_2 = \{ y \in Y \mid y \in D_1 \text{ and } y \notin D_2 \}$. Let $T > 0$, $p \in [1, \infty]$, $U_p = L^p(0,T;U)$ and $i$ be the usual imaginary unit.

Consider the following linear control system:

$$
\begin{cases}
    y(t) = Ay(t) + F(t)y(t) + B(t)u(t), & t \in (0,T], \\
    y(0) = y_0,
\end{cases}
$$

(2.1)

where $u(\cdot) \in U_p$ is the control variable and $y(\cdot)$ is the state variable, $y_0 \in Y$, $A : D(A) \subseteq Y \to Y$ generates a $C_0$-semigroup on $Y$, $F(\cdot) \in L^\infty(0,T;\mathcal{L}(Y))$, and $B(\cdot) \in L^\infty(0,T;\mathcal{L}(U;Y))$. Obviously, $A + F(\cdot)$ generates an evolution operator $S(\cdot,\cdot)$ on $Y$. For any $T > 0$, $y_0 \in Y$ and $u(\cdot) \in U_p$, (2.1) admits a mild solution $y(\cdot) = y(\cdot; y_0, u(\cdot)) \in C([0,T];Y)$, and

$$
y(t) = y(t; y_0, u(\cdot)) = S(t,0)y_0 + \int_0^t S(t,s)B(s)u(s)ds, \quad \forall \ t \in [0,T].
$$

Define the reachable set $\mathcal{R}(T; y_0)$ of (2.1) at time $T$ with the initial value $y_0$ as follows:

$$
\mathcal{R}(T; y_0) = \left\{ \ y(t; y_0, u(\cdot)) \in Y \mid y(\cdot) \text{ is the mild solution to (2.1) with some } u(\cdot) \in U_p \right\}.
$$

Next, let us recall the notions of finite codimensional subspace and finite codimensionality (e.g., [31]).

**Definition 2.1** A linear subspace $Y_0$ of $Y$ is called finite codimensional, if there exist an $m \in \mathbb{N}$ and linearly independent $y_1, y_2, \cdots, y_m \in Y \setminus Y_0$ such that $\text{span}\{Y_0, y_1, y_2, \cdots, y_m\} = Y$.

**Definition 2.2** A subset $D$ of $Y$ is called finite codimensional in $Y$, if

- $(H_1)$ There exists a $y_0 \in \overline{\text{co}}D$, such that $\text{span}\{D - y_0\}$ is a finite codimensional subspace of $Y$; and
- $(H_2)$ $\overline{\text{co}}(D - y_0)$ has at least an interior point in this subspace.
Now, we introduce two notions of finite codimensional controllability.

Definition 2.3 The system (2.1) is called finite codimensional exactly (resp., approximately) controllable at time $T$, if $\mathcal{R}(T;0)$ (resp., $\overline{\mathcal{R}}(T;0)$) is a finite dimensional subspace of $Y$.

Remark 2.1 Recall that (2.1) is exactly (resp., approximately) controllable at time $T$, if $\mathcal{R}(T;0) = Y$ (resp., $\overline{\mathcal{R}}(T;0) = Y$). Therefore, the finite codimensional exact (resp., approximate) controllability defined in Definition 2.3 is clearly weaker than the usual exact (resp., approximate) controllability for linear systems.

Remark 2.2 In general, the finite codimensional exact controllability cannot be reduced to the usual exact controllability problem. Indeed, this is possible only for the special case that $A + F(t)$ in (2.1) has an invariant subspace, which is finite codimensional in $Y$ and independent of $t \in [0,T]$.

As mentioned before, the notion of finite codimensional controllability is motivated by the study of some optimal control problems for infinite dimensional systems with endpoint state constraints. It is well known that Pontryagin’s maximum principle is one of the milestones in optimal control theory. As a necessary condition of optimal controls, for very general finite dimensional systems, Pontryagin type maximum principle was established in [38]. Nevertheless, surprisingly, it fails for infinite dimensional systems if there is no further assumption (see [15]). This leads to that for quite a long time, Pontryagin type maximum principle had been studied only for evolution equations without terminal state constraints. Until 1980s, by assuming the finite codimensionality of some subset in state spaces, Pontryagin type maximum principles on optimal control problems for infinite dimensional systems with endpoint constraints and general control domains were established in [17, 29, 30]. In the following, we present an optimal control problem with state constraints and recall how to use the finite codimensionality in deriving Pontryagin type maximum principle.

Consider the following evolution equation on $Y$:

$$y_t(t) = Ay(t) + f(t,y(t),u(t)), \quad t \in (0,T], \quad (2.2)$$

where $u(\cdot)$ is the control variable and $y(\cdot)$ is the state variable. Assume that $f : [0,T] \times Y \times U \to Y$ satisfies certain conditions (to be given later), such that for any $y(0) \in Y$ and $u(\cdot) \in \mathcal{U}_p$, (2.2) admits a mild solution $y(\cdot) = y(\cdot;y(0),u(\cdot)) \in C([0,T];Y)$. Also, let $\tilde{U}$ be a nonempty subset of $U$, and $S$ be a closed and convex subset of $Y \times Y$. Put

$$\mathcal{U}[0,T] = \left\{ u(\cdot) \in \mathcal{U}_p \mid u : (0,T) \to \tilde{U} \text{ is measurable} \right\},$$

$$\mathcal{U}_{ad} = \left\{ u(\cdot) \in \mathcal{U}[0,T] \mid \text{the mild solution } y(\cdot) \text{ to (2.2) satisfies } y(0),y(T) \in S \right\},$$

$$\mathcal{A}_{ad} = \left\{ (u(\cdot),y(\cdot)) \in \mathcal{U}_{ad} \times C([0,T];Y) : y \text{ is the mild solution to (2.2)} \right\}.$$

Write

$$J(u(\cdot),y(\cdot)) = \int_0^T f^0(t,y(t),u(t))dt,$$

where $f^0 : [0,T] \times Y \times U \to \mathbb{R}$ satisfies certain conditions (to be specified in the sequel), such that for any $u(\cdot) \in \mathcal{U}[0,T]$, $y(0) \in Y$ and the corresponding mild solution $y(\cdot)$ to (2.2), $f^0(\cdot,y(\cdot),u(\cdot)) \in L^1(0,T)$.

Consider the following optimal control problem for the system (2.2):

(P) Find a pair $(\overline{\mathcal{U}}(\cdot),\overline{\mathcal{Y}}(\cdot)) \in \mathcal{A}_{ad}$, such that $J(\overline{\mathcal{U}}(\cdot),\overline{\mathcal{Y}}(\cdot)) = \inf_{(u(\cdot),y(\cdot)) \in \mathcal{A}_{ad}} J(u(\cdot),y(\cdot)).$
Such a \((\bar{u}(\cdot), \bar{y}(\cdot))\) is called an optimal pair. As a necessary condition for optimal pairs, Pontryagin type maximum principle is stated as follows.

**Pontryagin type maximum principle**: Assume that \((\bar{u}(\cdot), \bar{y}(\cdot))\) is an optimal pair. Then there exists a pair \((\psi^0, \psi(\cdot)) \in \mathbb{R} \times C([0, T]; Y')\), such that

\[
\psi(t) = -A^* \psi(t) - f_y(t, \bar{y}(t), \bar{u}(t)) \psi(t) - \psi^0 f_y^0(t, \bar{y}(t), \bar{u}(t)), \quad \text{a.e. } t \in (0, T),
\]

\[
\langle \psi(0), y^0 - \bar{y}(0) \rangle_{Y', Y} = \langle \psi(T), y^1 - \bar{y}(T) \rangle_{Y', Y} \leq 0, \quad \forall (y^0, y^1) \in S,
\]

\[
H(t, \bar{y}(t), \bar{u}(t), \psi^0, \psi(t)) = \max_{u \in \bar{U}} H(t, \bar{y}(t), u, \psi^0, \psi(t)), \quad \text{a.e. } t \in (0, T),
\]

where \(A^*\) is the adjoint operator of \(A\), and

\[
H(t, y, u, \psi^0, \psi) = \psi^0 f^0(t, y, u) + \langle \psi, f(t, y, u) \rangle_{Y', Y},
\]

\[
\forall (t, y, u, \psi^0, \psi) \in [0, T] \times Y \times \bar{U} \times \mathbb{R} \times Y'.
\]

(2.3) is key in Pontryagin type maximum principle. Indeed, if it fails, then \(\psi^0 = 0\) and \(\psi(t) = 0\) for all \(t \in [0, T]\). Hence, (2.5) and (2.6) are trivial, since they are then simply “\(0 \leq 0\)” and “\(0 = 0\)”, respectively.

In order to ensure (2.3), the finite codimensionality of a suitable set was introduced. More precisely, consider the following system:

\[
\begin{cases}
\xi(t) = A \xi(t) + f_y(t, \bar{y}(t), \bar{u}(t)) \xi(t) + f(t, \bar{y}(t), u(t)) - f(t, \bar{y}(t), \bar{u}(t)), & t \in (0, T], \\
\xi(0) = 0,
\end{cases}
\]

(2.7)

and the homogenous equation:

\[
\begin{cases}
\eta(t) = A \eta(t) + f_y(t, \bar{y}(t), \bar{u}(t)) \eta(t), & t \in (0, T], \\
\eta(0) = y^0,
\end{cases}
\]

(2.8)

for \(y^0 \in Y\). Put

\[
\mathcal{R} = \left\{ \xi(T) \in Y \mid \xi(\cdot) \text{ is the mild solution to (2.7) with some } u(\cdot) \in \mathcal{U}[0, T] \right\}
\]

and

\[
\mathcal{Q} = \left\{ y^1 - \eta(T) \in Y \mid \eta(\cdot) \text{ is the mild solution to (2.8) and } (y^0, y^1) \in S \right\},
\]

and introduce the condition:

\[
(H) \quad \mathcal{R} - \mathcal{Q} \text{ is finite codimensional in } Y.
\]

It was proved in [31, Chapter 4] that, if the condition \(H\) holds, the optimal pair \((\bar{u}(\cdot), \bar{y}(\cdot))\) in the optimal control problem \((P)\) satisfies Pontryagin type maximum principle, i.e., (2.3)-(2.6) hold.

The proof is based on the following known result.

**Lemma 2.1** ([31, Lemma 3.6 on Page 142]) If \(M\) is finite codimensional in \(Y\), then for any \(\{f_j\}_{j=1}^{\infty} \subseteq Y'\) satisfying the following two conditions:

1. \(|f_j|_{Y'} \geq \delta\) for a positive constant \(\delta\) and \(f_j \to f\) weakly* in \(Y'\), as \(j \to \infty\); and
2. There exist positive constants \(\epsilon_j\), such that \(\lim_{j \to \infty} \epsilon_j = 0\) and \(\langle f_j, x \rangle_{Y', Y} \geq -\epsilon_j\), \(\forall x \in M\), it holds that \(f \neq 0\).


Lemma 2.1 means that, under some mild assumptions, the finite codimensionality on $M$ is sufficient to guarantee the weak limit point of a sequence to be nonzero in an infinite dimensional space. This is the reason why (2.3) holds in Pontryagin type maximum principle. On the other hand, this condition is also necessary, at least when $Y$ is a Hilbert space and $M$ is a linear closed subspace. In fact, we have the following result.

**Proposition 2.1** Suppose that $M$ is a linear closed subspace of a Hilbert space $Y$. Then $M$ is finite codimensional, if and only if for any $\{f_j\}_{j=1}^{\infty} \subseteq Y$ satisfying the conditions (1)-(2) in Lemma 2.1, it holds that $f \neq 0$.

**Proof.** By Lemma 2.1, we only need to prove the sufficiency. If $M$ is not finite codimensional, then there exists a subspace $Y_0 = \text{span}\{e_1,e_2,\cdots\}$ of $Y$, such that $M \oplus Y_0 = Y$. Also, $|e_j|_Y = 1$ for any $j \in \mathbb{N}$ and $\{e_j\}_{j=1}^{\infty}$ is pairwise orthogonal. Choose $f_j = e_j$, $\delta = 1$ and $\epsilon_j = 1/j$. Then $|f_j|_Y = 1$, $(f_j,x)_Y = 0 \geq -1/j$ for any $x \in M$ and $\lim_{j \to \infty} \epsilon_j = 0$. Notice that

$$e_j \to \hat{e}$$

weakly in $Y$ with $|\hat{e}|_Y \leq 1$.

Since $\sum_{j=1}^{\infty} (e_j,x)_Y < \infty$ for any $x \in Y$, $\lim_{j \to \infty} (e_j,x)_Y = 0$, which implies that $\hat{e} = 0$. This contradicts the assumptions on $\{f_j\}_{j=1}^{\infty}$ and therefore, $M$ is finite codimensional in $Y$. \hfill \square

**Remark 2.3** Proposition 2.1 indicates that the finite codimensionality seems closely related to the weak convergence method and existence of nontrivial solutions for partial differential equations. We shall study its applications in this respect in a future work.

Usually, unless $Q$ is finite codimensional in $Y$, it is quite difficult to verify the condition (H) directly, even for some simple linear systems. For example, if $S = \{y^0\} \times B_1$ for a given $y^0 \in Y$ and the unit ball $B_1$ of $Y$, then (H) holds trivially. But when $S = \{(y^0,y^1)\}$ with $y^0,y^1 \in Y$, it seems not easy to check this condition, since the set $\mathcal{R}$ is the reachable set of some system with control constraints. As we mentioned before, the motivation of this paper is to introduce a new method to verify the finite codimensionality condition appeared in optimal control problems. A little more precisely, first, the condition (H) is reduced to a finite codimensional exact controllability problem, as introduced in Definition 2.3. Then, by a duality argument, such a controllability problem is transformed into a suitable *a priori* estimate, called weak observability estimate (compared to the usual observability estimate), for its adjoint system, which is more easily verified or proved false, at least for some nontrivial examples (see Propositions 6.2 and 6.4).

### 3 Finite codimensional exact controllability

In this section, some equivalent results on finite codimensional exact controllability are established. First, consider the following linear control system:

$$
\begin{cases}
  y(t) = Ay(t) + F(t)y(t) + B(t)u(t), & t \in (0,T], \\
  y(0) = 0,
\end{cases}
$$

(3.1)

where $A$, $F(\cdot)$ and $B(\cdot)$ are the same as those in (2.1). Assume that

(A) \quad \overline{\mathcal{U}}$ is a nonempty bounded subset of $\mathcal{U}_p$ and $\overline{\mathcal{U}} \backslash \overline{\mathcal{U}}$ has at least an interior point.
Set
\[ M = \left\{ y(T) \in Y \mid y \text{ is the mild solution to (3.1) with some } u(\cdot) \in \overline{U} \right\}. \] (3.2)

Then it is easy to check that
\[
\begin{align*}
\text{span}M &= \left\{ y(T) \in Y \mid y \text{ is the mild solution to (3.1) with some } u(\cdot) \in \mathcal{U}_p \right\}, \\
\overline{\text{co}}M &= \left\{ y(T) \in Y \mid y \text{ is the mild solution to (3.1) with some } u(\cdot) \in \overline{\text{co}} \overline{U} \right\}.
\end{align*}
\] (3.3)

Also, we recall a known result on finite codimensional subspace.

**Lemma 3.1** ([31, Proposition 3.2 on Page 138]) Assume that \( Y_0 \) is a linear subspace of \( Y \). Then \( Y_0 \) is finite codimensional in \( Y \), if and only if there exist finitely many bounded linear functionals \( \{f_j\}_{j=1}^m \subseteq Y' \), such that \( Y_0 = \bigcap_{j=1}^m \ker \{f_j\} \).

The first result of this section is stated as follows.

**Theorem 3.1** Suppose that (A) holds. Then the following two assertions are equivalent:

1. The system (3.1) is finite codimensional exactly controllable in \( Y \).
2. The set \( M \) (in (3.2)) is finite codimensional in \( Y \).

**Proof.** Without loss of generality, we assume that 0 is an interior point of \( \overline{\text{co}} \overline{U} \). Otherwise, if \( u_0 \neq 0 \) is an interior point of \( \overline{\text{co}} \overline{U} \), it suffices to replace \( \overline{U} \) and \( M \), respectively, by \( \overline{U} - u_0 \) and \( M - y(T; u_0) \) with \( y(\cdot; u_0) \) being the mild solution to (3.1) associated to \( u = u_0 \). Hence, there is an \( r_0 > 0 \), such that \( \{u(\cdot) \in \mathcal{U}_p \mid |u|_{\mathcal{U}_p} \leq r_0 \} \subseteq \overline{\text{co}} \overline{U} \) and \( 0 \in M \).

First, we prove that (1) implies (2). For any \( n \in \mathbb{N} \), set
\[ N_n = \left\{ y(T) \in Y \mid y \text{ is the mild solution to (3.1) with some } u(\cdot) \in \mathcal{U}_p \text{ satisfying } |u|_{\mathcal{U}_p} \leq nr_0 \right\}. \]

Then \( N_1 \subseteq \overline{\text{co}}M \text{ and } \bigcup_{n \in \mathbb{N}} N_n = \mathcal{R}(T; 0) \). By (1) and (3.3), \( \mathcal{R}(T; 0) = \overline{\mathcal{R}(T; 0)} = \text{span}M \) is a finite codimensional subspace of \( Y \). Also, by the Baire category theorem, there exists an \( \tilde{n} \in \mathbb{N} \), such that \( N_{\tilde{n}} = \overline{N_{\tilde{n}}} \) has at least an interior point \( \tilde{y} \in \text{span}M \). Then \( \tilde{y} \) is an interior point of \( \overline{\text{co}}M \) in \( \text{span}M \). Hence, by Definition 2.2, \( M \) is finite codimensional in \( Y \).

On the other hand, we prove that (2) implies (1). Notice that \( \overline{\text{co}}M \subseteq \mathcal{R}(T; 0) \). By (H2) in Definition 2.2, \( \overline{\text{co}}M \) has at least an interior point in the subspace \( \text{span}M = \overline{\mathcal{R}(T; 0)} \). Hence, \( \mathcal{R}(T; 0) \) also has an interior point in \( \overline{\mathcal{R}(T; 0)} \). Since \( \mathcal{R}(T; 0) \) and \( \overline{\mathcal{R}(T; 0)} \) are two linear subspaces of \( Y \) and \( \mathcal{R}(T; 0) \) is dense in \( \overline{\mathcal{R}(T; 0)} \), it follows that \( \mathcal{R}(T; 0) = \overline{\mathcal{R}(T; 0)} \). Also, by (H1) in Definition 2.2, \( \mathcal{R}(T; 0) = \text{span}M \) is finite codimensional in \( Y \). Hence, (1) holds.

Next, by a duality technique, we prove that the finite codimensional exact controllability of (3.1) is equivalent to a suitable observability estimate for the following equation (or adjoint system):
\[
\begin{cases}
\phi_t(t) = -A^*\phi(t) - F(t)^*\phi(t), & t \in (0, T], \\
\phi(T) = \phi_T,
\end{cases}
\] (3.4)

where \( \phi_T \in Y' \). Set \( p' = p/(p - 1) \) for \( p \in (1, \infty) \), and \( p' = 1 \) for \( p = \infty \). In what follows, \( C \) is used to denote a generic positive constant, which may change from line to line in the sequel.

The second result of this section is as follows.
Theorem 3.2 The following two assertions are equivalent:

(1) The system (3.1) is finite codimensional exactly controllable in Y.

(2) There exists a finite codimensional subspace \( \hat{Y} \) of \( Y' \), such that any solution \( \phi \) to (3.4) satisfies

\[
|\phi_T|_{Y'} \leq C B(\cdot)^* \phi_{L^p(0,T;U')}, \quad \forall \phi_T \in \hat{Y}.
\]  (3.5)

Proof. First, we prove that (2) implies (1). The proof is divided into four parts.

Step 1. In this step, we prove that for the system (3.1), \( Y_1 \) of \( Y \) is finite dimensional:

\[
Y_1 = \{ x \in Y \mid \langle f, x \rangle_{Y', Y} = 0, \ \forall f \in \hat{Y} \},
\]

where \( \hat{Y} \) is the subspace given in (2).

Let the codimension of \( \hat{Y} \) be \( k_1 \). If \( Y_1 \) is an infinite dimensional space, then there is a linear subspace \( Y_1^0 \) of \( Y_1 \), whose dimension is \( k_1 + 1 \). Let \( \{ x_1, \ldots, x_{k_1+1} \} \subseteq Y \) be a basis of \( Y_1^0 \). By the Hahn-Banach theorem, one can find \( \{ f_1, \ldots, f_{k_1+1} \} \subseteq Y' \), such that for \( 1 \leq k, j \leq k_1 + 1 \),

\[
\langle f_k, x_j \rangle_{Y', Y} = \begin{cases} 1 & \text{for } k = j, \\ 0 & \text{for } k \neq j. \end{cases}
\]

It follows that \( \{ f_1, \ldots, f_{k_1+1} \} \) are linearly independent in \( Y' \). Hence, the dimension of the subspace \( \operatorname{span}\{ f_1, \ldots, f_{k_1+1} \} \) is \( k_1 + 1 \). Since for any \( j = 1, \ldots, k_1 + 1 \), \( x_j \in Y_1 \) and \( \langle f_j, x_j \rangle_{Y', Y} = 1 \neq 0 \), by the definition of \( Y_1 \), we get that \( f_j \notin \hat{Y} \) \( (j = 1, \ldots, k_1 + 1) \). This contradicts the fact that the codimension of \( \hat{Y} \) is \( k_1 \). Hence, \( Y_1 \) is finite dimensional and denote by \( k_2 \) its dimension.

Step 2. In this step, we prove that for any \( y_T \in Y \), there is a control \( u \in U_p \), such that the corresponding solution \( y(\cdot; u) \) to (3.1) satisfies

\[
y(T; u) - y_T \in Y_1. \quad (3.6)
\]

Write \( K = \left\{ B(\cdot)^* \phi \in L^p(0,T;U') \mid \phi \text{ is the mild solution to (3.4) with } \phi_T \in \hat{Y} \right\} \) and define a linear functional \( \ell \) on \( K \) as \( \ell(B(\cdot)^* \phi) = \langle \phi_T, y_T \rangle_{Y', Y} \). It follows from (3.5) that \( \ell \) is a bounded linear functional on \( K \). Moreover, \( |\ell|_{L(K;\mathbb{R})} \leq C |y_T|_Y \). Hence, by the Hahn-Banach theorem, \( \ell \) can be extended to be a bounded linear functional on \( L^p(0,T;U') \). This implies that there is a \( u \in U_p \), such that

\[
\langle \phi_T, y_T \rangle_{Y', Y} = \int_0^T \langle B(t)^* \phi, u(t) \rangle_{U', U} \, dt, \quad \forall \phi_T \in \hat{Y} \quad (3.7)
\]

and

\[
|u|_{U_p} \leq C |y_T|_Y. \quad (3.8)
\]

For this control \( u \in U_p \) and the corresponding solution \( y(\cdot; u) \) of (3.1), by (3.1) and (3.4), it is easy to show that

\[
\langle \phi_T, y(T; u) \rangle_{Y', Y} = \int_0^T \langle B(t)^* \phi, u(t) \rangle_{U', U} \, dt, \quad \forall \phi_T \in \hat{Y},
\]

which, together with (3.7), implies that

\[
\langle \phi_T, y(T; u) - y_T \rangle_{Y', Y} = 0, \quad \forall \phi_T \in \hat{Y}.
\]

This deduces (3.6).

Step 3. In this step, we prove that for the system (3.1), \( \mathcal{R}(T; 0) \) is a closed subspace of \( Y \).
Let $\mathbb{P}_Y$ be the projection operator from $Y$ to the subspace $Y_1$. Since $Y_1$ is finite dimensional, $\mathbb{P}_Y$ is well defined and a bounded linear operator on $Y$. Then for the identity operator $I$ on $Y$, $\mathcal{R}(T;0) = \mathbb{P}_Y \mathcal{R}(T;0) \oplus (I - \mathbb{P}_Y) \mathcal{R}(T;0)$. Since $\mathbb{P}_Y \mathcal{R}(T;0)$ is finite dimensional, it is closed. Furthermore, the linear subspace $(I - \mathbb{P}_Y) \mathcal{R}(T;0)$ is also closed in $Y$. Indeed, for any $\{y_j^T\}_{j=1}^\infty \subseteq (I - \mathbb{P}_Y) \mathcal{R}(T;0)$ satisfying that $\lim_{j \to \infty} y_j^T = \widetilde{y}_T \in Y$, similar to (3.8) and (3.6), there exists a sequence of controls $\{\tilde{u}_j\}_{j=1}^\infty \subseteq U_p$, such that

$$|\tilde{u}_j|_{U_p} \leq C|y_j^T|_Y, \quad \text{(3.9)}$$

and for the corresponding mild solution $\tilde{y}_j(\cdot) = y(\cdot; \tilde{u}_j)$ to (3.1), $\tilde{y}_j(T) - y_j^T \in Y_1$. Therefore, $(I - \mathbb{P}_Y)\tilde{y}_j(T) = y_j^T$. By (3.9), there exists a subsequence of $\{\tilde{u}_j\}_{j=1}^\infty$ (still denoted by itself) and $\tilde{u} \in U_p$, such that as $j \to \infty$,

$$\begin{align*}
\tilde{u}_j &\to \tilde{u} \quad \text{weakly in } U_p, \quad \text{for } p \in (1, \infty); \\
\tilde{u}_j &\to \tilde{u} \quad \text{weakly}^* \text{ in } U, \quad \text{for } p = \infty.
\end{align*}$$

Denote by $\tilde{y}$ the mild solution to (3.1) associated to $\tilde{u} \in U_p$. Then it is easy to show that as $j \to \infty$, $\tilde{y}_j(T)$ converges weakly to $\tilde{y}(T)$, and hence, $(I - \mathbb{P}_Y)\tilde{y}_j(T)$ converges weakly to $(I - \mathbb{P}_Y)\tilde{y}(T)$. Since $y_j^T$ converges strongly to $\tilde{y}_T$ in $Y$ and $(I - \mathbb{P}_Y)\tilde{y}_j(T) = y_j^T$, it holds that $(I - \mathbb{P}_Y)\tilde{y}(T) = \tilde{y}_T$. This implies that $\tilde{y}_T \in (I - \mathbb{P}_Y) \mathcal{R}(T;0)$. Therefore, $(I - \mathbb{P}_Y) \mathcal{R}(T;0)$ is a closed subspace of $Y$. So is $\mathcal{R}(T;0)$.

**Step 4.** In this step, we prove that the codimension of the closed subspace $\mathcal{R}(T;0)$ is not greater than the dimension $k_2$ of $Y_1$.

Otherwise, there exist linearly independent $x_1, \cdots, x_{k_2+1} \in Y$, such that for any $\tilde{x}_j \in \mathcal{R}(T;0)$ $(j = 1, \cdots, k_2 + 1)$,

$$\tilde{x}_1 - x_1, \cdots, \tilde{x}_{k_2+1} - x_{k_2+1} \text{ are linearly independent}. \quad \text{(3.10)}$$

By (3.8) and (3.6), there are controls $u_j \in U_p$ $(j = 1, \cdots, k_2 + 1)$, such that the corresponding mild solutions $y_j(\cdot) = y(\cdot; u_j)$ to (3.1) satisfies that $y_j(T) - x_j \in Y_1$, for $j = 1, \cdots, k_2 + 1$. Meanwhile, it follows from (3.10) that $y_1(T) - x_1, \cdots, y_{k_2+1}(T) - x_{k_2+1}$ are linearly independent. This contradicts the fact that the dimension of $Y_1$ is $k_2$. Therefore, the codimension of $\mathcal{R}(T;0)$ is finite and the assertion (1) holds.

Next, we prove that (1) implies (2). Assume the codimension of $\mathcal{R}(T;0)$ is $k_3$ for the system (3.1). Then there is a linear subspace $Y_2$ of $Y$, whose dimension is $k_3$, such that $Y = \text{span}\{Y_2, \mathcal{R}(T;0)\}$. Let $\{x_1, \cdots, x_{k_3}\}$ be a basis of $Y_2$. Then for any $x_j$ $(j = 1, \cdots, k_3)$, there exists an $x_j'' \in Y''$, such that $\langle x_j'', f \rangle_{Y'', Y'} = \langle f, x_j \rangle_{Y', Y'}$ for any $f \in Y'$. Set $\bar{Y} = \bigcap_{j=1}^{k_3} \ker\{x_j''\}$. Then by Lemma 3.1, $\bar{Y}$ is a finite codimensional subspace of $Y'$. Also, for any $y_1^T \in Y_2$ and $\phi_T \in \bar{Y}$,

$$\langle \phi_T, y_1^T \rangle_{Y', Y} = 0. \quad \text{(3.11)}$$

Now, we prove that (3.5) holds for the above subspace $\bar{Y}$ of $Y'$. If (3.5) fails, then there exists a sequence $\{\phi_j^T\}_{j=1}^\infty$ of $\bar{Y}$, such that the solution $\tilde{\phi}_j$ to (3.4) with the final datum $\tilde{\phi}(T) = \phi_j^T$ satisfies

$$|B(\cdot)\tilde{\phi}_j|_{L^p(0, T; U')} \leq \frac{1}{j^3} |\phi_j^T|_{Y'}, \quad \forall j \in \mathbb{N}.$$
Let $\bar{\phi}_T^j = \sqrt{T} \frac{\phi_T^j}{|\phi_T^j|_{Y'}}$. Then for the mild solution $\bar{\phi}_j$ to (3.4) with the final datum $\bar{\phi}_j(T) = \bar{\phi}_T^j$, it holds that
\[
|\bar{\phi}_j^j|_{Y'} = \sqrt{j} \quad \text{and} \quad |B(\cdot)^*\bar{\phi}_j|_{L^p(0,T;U')} \leq \frac{1}{\sqrt{j}}. \tag{3.12}
\]
By (3.1) and (3.4), for any $y_T^2 \in \mathcal{R}(T;0)$, one can find a control $v(\cdot) \in \mathcal{U}_p$, such that
\[
\langle \bar{\phi}_j^j, y_T^2 \rangle_{Y', Y} = \int_0^T \langle B(t)^*\bar{\phi}_j, v(t) \rangle_{U', U} dt.
\]
This, together with (3.11) and (3.12), implies that for any $y_T = y_T^1 + y_T^2 \in Y$, $\{\langle \bar{\phi}_j^j, y_T \rangle_{Y', Y}\}_{j=1}^\infty$ is uniformly bounded. Hence, $\{\bar{\phi}_j^j\}_{j=1}^\infty$ is uniformly bounded in $Y'$, but this contradicts (3.12). Hence, (3.5) holds for any $\phi_T \in \bar{Y}$.

In general, it is hard to find the finite codimensional subspace $\bar{Y}$ of $Y'$ in the assertion (2) of Theorem 3.2. Hence, we give another equivalent criterion for the finite codimensional exact controllability, where a priori estimate holds on the whole space $Y'$.

**Theorem 3.3** The following two assertions are equivalent:

1. There is a compact operator $G$ from $Y'$ to a Banach space $X$, such that any solution $\phi$ to (3.4) satisfies
\[
|\phi_T|_{Y'} \leq C(|B(\cdot)^*\phi|_{L^p(0,T;U')} + |G\phi_T|_X), \quad \forall \phi_T \in Y'. \tag{3.13}
\]
2. There is a finite codimensional subspace $\bar{Y}$ of $Y'$, such that any solution $\phi$ to (3.4) satisfies
\[
|\phi_T|_{Y'} \leq C|B(\cdot)^*\phi|_{L^p(0,T;U')}, \quad \forall \phi_T \in \bar{Y}. \tag{3.14}
\]

**Proof.** First, we prove that (1) implies (2). The proof is divided into three parts.

**Step 1.** In this step, we prove that the following subspace $\mathcal{E}$ of $Y'$ is finite dimensional:
\[
\mathcal{E} = \left\{ \phi_T \in Y' \left| \text{the corresponding mild solution } \phi \text{ to (3.4) satisfies that } B(\cdot)^*\phi = 0 \right. \right\}.
\]
Indeed, let $\{\phi_T^j\}_{j=1}^\infty \subseteq \mathcal{E}$ with $|\phi_T^j|_{Y'} = 1$ for every $j \in \mathbb{N}$. Then there exist a $\tilde{\phi}_T \in Y'$ and subsequence of $\{\phi_T^j\}_{j=1}^\infty$ (still denoted by itself), such that
\[
\phi_T^j \to \tilde{\phi}_T \quad \text{weakly* in } Y', \quad \text{as } j \to +\infty.
\]
Hence, $G\phi_T^j = G\tilde{\phi}_T$ in $X$. This, together with (3.13), implies that $\{\phi_T^j\}_{j=1}^\infty$ is strongly convergent in $Y'$ and therefore, $\mathcal{E}$ is a finite dimensional space.

**Step 2.** We find a suitable $\hat{\phi}_T \in \mathcal{E}$ with $\hat{\phi}_T \not= 0$, by assuming the following (3.15) fails.

Denote by $\mathbb{P}_\mathcal{E}$ the projection operator from $Y'$ to the subspace $\mathcal{E}$. Since $\mathcal{E}$ is finite dimensional, $\mathbb{P}_\mathcal{E}$ is well defined and a bounded linear operator. In the following, we prove that
\[
|\phi_T|_{Y'} \leq C|B(\cdot)^*\phi|_{L^p(0,T;U')}, \quad \forall \phi_T \in (\mathbb{I} - \mathbb{P}_\mathcal{E})Y', \tag{3.15}
\]
where $\mathbb{I}$ denotes the identity operator on $Y'$ and $\phi$ is the mild solution to (3.4) with the terminal value $\phi_T$. Otherwise, there exists a sequence $\{\phi_{T,j}\}_{j=1}^\infty$ of $(\mathbb{I} - \mathbb{P}_\mathcal{E})Y'$ with $|\phi_{T,j}|_{Y'} = 1$ for any $j \in \mathbb{N}$, such that the solution $\phi_j$ to (3.4) with the final datum $\phi_j(T) = \phi_{T,j}$ satisfies
\[
|B(\cdot)^*\phi_j|_{L^p(0,T;U')} < \frac{1}{j}. \tag{3.16}
\]
Then there exist a $\hat{\phi}_T \in Y'$ and subsequence of $\{\phi_{T,j}\}_{j=1}^\infty$ (still denoted by itself), such that
\[
\phi_{T,j} \to \hat{\phi}_T \quad \text{weakly* in } Y', \quad j \to +\infty.
\]
By (3.16), one has that $B(\cdot)^*\hat{\phi} = 0$, where $\hat{\phi}$ is the mild solution to (3.4) with the final datum $\hat{\phi}(T) = \hat{\phi}_T$. This implies that $\hat{\phi}_T \in \mathcal{E}$. Also,
\[
\lim_{j \to \infty} G\phi_{T,j} = G\hat{\phi}_T \quad \text{in } X. \tag{3.17}
\]
It follows from (3.13) that $|\phi_{T,j}|_{Y'} \leq C \left( |B(\cdot)^*\phi_j|_{L'(0,T;U')} + |G\phi_{T,j}|_X \right)$. Hence, by (3.16), for any $j > 2C$, one has that $|G\phi_{T,j}|_X \geq \frac{1}{2C}$, which, together with (3.17), indicates that $\hat{\phi}_T \neq 0$.

**Step 3.** In this step, we prove that $\hat{\phi}_T \in (I - P_E)Y'$.

By (3.13), for the above $\{\phi_{T,j}\}_{j=1}^\infty \subseteq (I - P_E)Y'$,
\[
|\phi_{T,n} - \phi_{T,m}|_{Y'} \leq C \left( |B(\cdot)^*\phi_n - B(\cdot)^*\phi_m|_{L'(0,T;U')} + |G\phi_{T,n} - G\phi_{T,m}|_X \right)
\leq C \left( \frac{1}{n} + \frac{1}{m} + |G\phi_{T,n} - G\phi_{T,m}|_X \right), \quad \text{for any } m, n \in \mathbb{N},
\]
which implies that $\{\phi_{T,j}\}_{j=1}^\infty$ is a Cauchy sequence in $Y'$. Since $(I - P_E)Y'$ is closed, $\hat{\phi}_T \in (I - P_E)Y'$. This contradicts the fact that $\hat{\phi}_T \in \mathcal{E}$ and $\hat{\phi}_T \neq 0$. Hence, (3.14) holds, provided that $\hat{\phi}_T$ belongs to the finite codimensional subspace $\tilde{Y} = (I - P_E)Y'$.

Next, we prove that (2) implies (1). Assume that there is a finite dimensional subspace $Y_3$ of $Y'$, such that $Y' = \text{span}\{\tilde{Y}, Y_3\}$. Denote by $P_{Y_3}$ and $P_{\tilde{Y}}$ the projections from $Y'$ to $Y_3$ and $\tilde{Y}$, respectively. Also, for any $\phi_T \in Y'$, denote by $F(\phi_T)$ the associated solution to (3.4). Then by (3.14), for any $\phi_T \in Y'$, it holds that
\[
|\phi_T|_{Y'} \leq |P_{Y_3}\phi_T|_{Y'} + |P_{\tilde{Y}}\phi_T|_{Y'} \leq |P_{Y_3}\phi_T|_{Y'} + C \left( |B(\cdot)^*F(\phi_T)|_{L'(0,T;U')} \right)
\leq C \left( |P_{Y_3}\phi_T|_{Y'} + |B(\cdot)^*F(\phi_T)|_{L'(0,T;U')} \right), \quad \text{for any } \phi_T \in Y'. \tag{3.18}
\]
Define a linear operator:
\[
G : Y' \to Y' \times L'(0,T;U'), \quad G(\phi_T) = (P_{Y_3}\phi_T, B(\cdot)^*F(P_{Y_3}\phi_T)), \quad \forall \phi_T \in Y'.
\]
Since $P_{Y_3}$ is compact, $G$ is also compact from $Y'$ to the Banach space $X = Y' \times L'(0,T;U')$ and therefore, (3.13) follows from (3.18).

## 4 Finite codimensional approximate controllability

This section is devoted to a characterization of the finite codimensional approximate controllability.

The main result of this section is stated as follows.

**Theorem 4.1** Suppose that (A) holds. Then the following three assertions are equivalent:

(1) The system (3.1) is finite codimensional approximately controllable in $Y$.

(2) $\text{span}M$ (in (3.3)) is a finite codimensional subspace of $Y$.

(3) There is a finite dimensional subspace $\tilde{Y}$ of $Y'$, such that for any solution $\phi$ to (3.4),
\[
B(\cdot)^*\phi = 0, \quad \text{if and only if} \quad \phi_T \in \tilde{Y}.
\]
Proof. First, by Definition 2.3 and (3.3), it is obvious that (1) and (2) are equivalent.

Next, we prove that (2) implies (3). Define a linear operator \( L : \mathcal{U}_p \to Y \) as \( \mathbb{L}(u(\cdot)) = y(T; u), \forall u(\cdot) \in \mathcal{U}_p, \) where \( y(\cdot; u) \) is the mild solution to (3.1) associated to \( u. \) Denote by \( \mathcal{R}(\mathbb{L}) \) the range of the operator \( L. \) Then \( L \) is a bounded linear operator and \( \text{span} M = \overline{\mathcal{R}(L)}. \) Therefore, (2) means that \( \overline{\mathcal{R}(L)} \) is a finite codimensional subspace of \( Y. \) Also, it is easy to show that the adjoint operator \( \mathbb{L}^* : Y' \to (\mathcal{U}_p)' \) of \( \mathbb{L} \) is \( \mathbb{L}^*(\phi_T) = B(\cdot)^* \phi, \) where \( \phi \) is the mild solution to (3.4) associated to \( \phi_T \in Y'. \) Since

\[
\ker(\mathbb{L}^*) = \left( \overline{\mathcal{R}(L)} \right)^\perp \triangleq \{ g \in Y' \mid \langle g, x \rangle_{Y', Y} = 0, \forall x \in \overline{\mathcal{R}(L)} \},
\]

it suffices to show that \( \left( \overline{\mathcal{R}(L)} \right)^\perp \) is finite dimensional.

Since \( \overline{\mathcal{R}(L)} \) is finite codimensional in \( Y, \) there exists a finite dimensional subspace \( Y_0 \) of \( Y, \) such that \( Y_0 \oplus \overline{\mathcal{R}(L)} = Y. \) Denote by \( m_0 \) and \( \mathbb{P}_{Y_0} \) the dimension of \( Y_0 \) and the projection operator from \( Y \) to \( Y_0, \) respectively. Since \( Y_0 \) is finite dimensional, \( \mathbb{P}_{Y_0} \) is well defined and a bounded linear operator. If \( \left( \overline{\mathcal{R}(L)} \right)^\perp \) is infinite dimensional in \( Y', \) there exist linearly independent \( f_1, f_2, \ldots, f_{m_0 + 1} \in \left( \overline{\mathcal{R}(L)} \right)^\perp. \) For the above given \( f_j \) \((j = 1, \ldots, m_0 + 1), \) define a bounded linear functional \( \tilde{f}_j \) on \( Y_0 \) as the limitation of \( f_j \) on \( Y_0. \) Since \( \tilde{f}_j \in Y'_0 \) \((j = 1, \ldots, m_0 + 1), \) the dimension of \( \text{span} \{ f_1, \ldots, f_{m_0 + 1} \} \) is not larger than \( m_0. \) Hence, there exists a nonzero vector \((a_1, \ldots, a_{m_0 + 1})^\top \in \mathbb{R}^{m_0+1}, \) such that \( \sum_{j = 1}^{m_0 + 1} a_j \tilde{f}_j(x) = 0, \forall x \in Y_0. \) Notice that for any \( x \in Y, \)

\[
\sum_{j = 1}^{m_0 + 1} a_j \tilde{f}_j(x) = \sum_{j = 1}^{m_0 + 1} a_j \tilde{f}_j(\mathbb{P}_{Y_0} x + (I - \mathbb{P}_{Y_0}) x) = \sum_{j = 1}^{m_0 + 1} a_j \tilde{f}_j(\mathbb{P}_{Y_0} x) = \sum_{j = 1}^{m_0 + 1} a_j \tilde{f}_j(\mathbb{P}_{Y_0} x) = 0.
\]

This implies that \( \sum_{j = 1}^{m_0 + 1} a_j f_j = 0 \) in \( Y'. \) This contradicts the linear independence of \( f_1, \ldots, f_{m_0 + 1}. \)

Finally, we prove that (3) implies (2). By (3), \( \ker(\mathbb{L}^*) = \left( \overline{\mathcal{R}(L)} \right)^\perp \) is finite dimensional in \( Y'. \) Assume that its dimension is \( m_1. \) If \( \overline{\mathcal{R}(L)} \) is not finite codimensional, then there exist \( x_j \notin \overline{\mathcal{R}(L)} \) \((j = 1, \ldots, m_1 + 1), \) which are linearly independent in \( Y. \) Let \( Y_1 = \text{span} \{ x_1, \ldots, x_{m_1 + 1} \} \oplus \overline{\mathcal{R}(L)}. \) Then \( Y_1 \) is closed and for any \( x \in Y_1, \) there exists a vector \((a_{1,x}, \ldots, a_{m_1+1,x})^\top \in \mathbb{R}^{m_1+1}, \) such that \( x = \sum_{j = 1}^{m_1 + 1} a_{j,x} x_j + \tilde{x} \) for some \( \tilde{x} \in \overline{\mathcal{R}(L)}. \) Define a functional \( g_j \) \((j = 1, \ldots, m_1 + 1) \) on \( Y_1 \) as follows:

\[
g_j(x) = a_{j,x}, \quad \forall x = \sum_{j = 1}^{m_1 + 1} a_{j,x} x_j + \tilde{x} \in Y_1.
\]

It is obvious that \( g_j \) is linear and bounded. By the Hahn-Banach theorem, \( g_j \) has an extension \( \tilde{g}_j \in Y', \) such that \( g_j(x) = \tilde{g}_j(x) \) for all \( x \in \overline{\mathcal{R}(L)}. \) Hence, \( \{ \tilde{g}_j \}_{j=1}^{m_1+1} \subseteq \left( \overline{\mathcal{R}(L)} \right)^\perp \) and \( \tilde{g}_1, \ldots, \tilde{g}_{m_1+1} \) are linearly independent in \( Y'. \) This contradicts the fact that the dimension of \( \left( \overline{\mathcal{R}(L)} \right)^\perp \) is \( m_1. \)

\[\square\]

Furthermore, suppose that the equation (3.4) satisfies the following forward uniqueness:

\[(U) \quad \text{For any solution } \phi \text{ to (3.4), if } \phi(0) = 0, \text{ then } \phi(\cdot) = 0 \text{ in } [0, T].\]

Then, it is easy to show the following equivalence result.
Corollary 4.1 Assume that (U) holds. Then the following two assertions are equivalent:

1. There is a finite dimensional subspace $\hat{Y}$ of $Y'$, such that for any solution $\phi$ to (3.4),
   $$B(\cdot)^*\phi = 0, \text{ if and only if } \phi_T \in \hat{Y}.$$

2. There is a finite dimensional subspace $\hat{Y}$ of $Y'$, such that for any solution $\phi$ to (3.4),
   $$B(\cdot)^*\phi = 0, \text{ if and only if } \phi(0) \in \hat{Y}.$$

5 Applications to optimal control problems with state constraints

In this section, as applications of results on the finite codimensional exact controllability in Section 3, we study the finite codimensionality (H) for the optimal control problem (P) with state constraints in Section 2.

In the following, suppose that $(\pi(\cdot), \psi(\cdot))$ is an optimal pair of the optimal control problem (P). We will study its necessary conditions in two different cases.

Case 1. The optimal control problem (P) without control constraints.

First, we give the following hypotheses:

(A) Let $f : [0, T] \times Y \times U \to Y$ and $f^0 : [0, T] \times Y \times U \to \mathbb{R}$ be strongly measurable with respect to $t$ in $(0, T)$, and continuously Fréchet differentiable with respect to $(y, u)$ in $Y \times U$ with $f(t, \cdot, \cdot), f_y(t, \cdot, \cdot), f_u(t, \cdot, \cdot), f^0(t, \cdot, \cdot), f_u^0(t, \cdot, \cdot)$ and $f_u^0(t, \cdot, \cdot)$ being continuous. Moreover, for any $u(\cdot) \in \mathcal{U}_2$ and $y(\cdot) \in C([0, T]; Y)$,
   $$f_y(\cdot, y(\cdot), u(\cdot)) \in L^1(0, T; \mathcal{L}(Y)), \quad f_u^0(\cdot, y(\cdot), u(\cdot)) \in L^1(0, T; Y'), \quad f^0(\cdot, y(\cdot), u(\cdot)) \in L^1(0, T),$$
   $$f_u(\cdot, y(\cdot), u(\cdot)) \in L^2(0, T; \mathcal{L}(U; Y)) \quad \text{and} \quad f_u^0(\cdot, y(\cdot), u(\cdot)) \in L^2(0, T; U').$$

(A1) $p = 2$ and $\overline{U} = U$.

Under the above assumptions, $\mathcal{U}[0, T] = \mathcal{U}_2 = L^2(0, T; U)$. For any $T > 0$, $y_0 \in Y$ and $u(\cdot) \in \mathcal{U}_2$, (2.2) admits a mild solution $y(\cdot) = y(\cdot; y_0, u) \in C([0, T]; Y)$ corresponding to the initial value $y_0$ and the control $u$. Also, we write $J(y_0, u(\cdot)) = J(u(\cdot), y(\cdot))$.

Consider the following linear control system:

$$\begin{cases}
\xi(t) = A\xi(t) + f_y(t, \pi(t), \pi(t))\xi(t) + f_u(t, \pi(t), \pi(t))v(t), \quad t \in (0, T], \\
\tilde{\xi}(0) = y^0,
\end{cases}$$

(5.1)

where $v(\cdot) \in \mathcal{U}_2$ and set

$$M_1 = \left\{ \tilde{\xi}(T) - y^1 \in Y \mid \tilde{\xi} \text{ is the solution to (5.1) with some } v(\cdot) \in \mathcal{U}_2 \text{ satisfying that } |v|_{\mathcal{U}_2} \leq 1 \right\},$$

and $(y^0, y^1) \in S$.

Further, let $\psi_1$ be the mild solution to the following equation:

$$\begin{cases}
\psi_{1,t}(t) = -A^*\psi_1(t) - \psi_1 f_y^0(t, \pi(t), \pi(t)) - f_y(t, \pi(t), \pi(t))^* \psi_1(t), \quad t \in (0, T], \\
\psi_1(T) = \psi_1^1,
\end{cases}$$

(5.2)

where $(\psi_1^0, \psi_1^1) \in \mathbb{R} \times Y'$. Then, similar to [31], using a convex variation technique, we can derive a necessary condition for the optimal pair $(\pi(\cdot), \psi(\cdot))$ as follows.
Proposition 5.1 Assume that (A_{11}) and (A_{12}) hold, and M_1 is finite codimensional in Y. Then there exists a pair \((\psi^0_1, \psi^1_1)\) \(\in \mathbb{R} \times Y'\), such that for the corresponding solution \(\psi_1\) to (5.2), \((\psi^0_1, \psi^1_1(\cdot)) \neq (0, 0)\) and
\[
f_u(t, \overline{y}(t), \overline{u}(t)) \psi_1(t) = \psi^0_1 f^0_u(t, \overline{y}(t), \overline{u}(t)) = 0, \quad \text{a.e. } t \in (0, T). \tag{5.3}
\]

Proof. The proof is divided into four steps.

Step 1. For any \(\varepsilon > 0\), \(y_0 \in Y\) and \(u(\cdot) \in U_2\), set \(\hat{J}(y_0, u(\cdot)) = J(y_0, u(\cdot)) - J(\overline{y}_0, \overline{u}(\cdot))\) and
\[
J_\varepsilon(y_0, u(\cdot)) = \left\{ \left[ ds(y_0, y(T; y_0, u)) \right]^2 + \left[ \hat{J}(y_0, u(\cdot)) + \varepsilon \right]^2 \right\}^{1/2},
\]
where \(\overline{y}_0 = \overline{y}(0)\) and \(ds(y_0, y_1) = \inf_{(y_0, y_1) \in S} \|y_0 - y_0\|^2 + |y_1 - y_1|^2\). Then \(J_\varepsilon(\cdot, \cdot)\) is continuous on \(Y \times U_2\) and
\[
J_\varepsilon(\overline{y}_0, \overline{u}(\cdot)) = \varepsilon \leq \inf_{(y_0, u(\cdot)) \in Y \times U_2} J_\varepsilon(y_0, u(\cdot)) + \varepsilon.
\]
By the Ekeland variational principle, there exists a pair \((y_0^\varepsilon, u_\varepsilon(\cdot)) \in Y \times U_2\), such that
\[
J_\varepsilon(y_0^\varepsilon, u_\varepsilon(\cdot)) \leq J_\varepsilon(\overline{y}_0, \overline{u}(\cdot)), \quad |y_0^\varepsilon - \overline{y}_0|_Y + |u_\varepsilon(\cdot) - \overline{u}(\cdot)|_{U_2} \leq \sqrt{\varepsilon},
\]
and
\[
-\sqrt{\varepsilon} |y_0^\varepsilon - y_0|_Y + |u_\varepsilon(\cdot) - v(\cdot)|_{U_2} \leq J_\varepsilon(y_0^\varepsilon, v(\cdot)) - J_\varepsilon(y_0^\varepsilon, u_\varepsilon(\cdot)), \quad \forall (y_0, v(\cdot)) \in Y \times U_2. \tag{5.4}
\]

Step 2. Set \(y_\varepsilon(\cdot) = y(\cdot; y_0^\varepsilon, u_\varepsilon)\) and for any \(\rho > 0\), \(\nu \in Y\) and \(v(\cdot) \in U_2\), write \(u^\varepsilon(\cdot) = u_\varepsilon(\cdot) + \rho v(\cdot), \quad y^\varepsilon(\cdot) = y(\cdot; y_0^\varepsilon, u^\varepsilon(\cdot)), \quad u^\varepsilon(\cdot) = y(\cdot; y_0^\varepsilon, u^\varepsilon(\cdot)), \quad v(\cdot) = y(\cdot; y_0^\varepsilon, u^\varepsilon(\cdot)), \quad v(\cdot) = y(\cdot; y_0^\varepsilon, u^\varepsilon(\cdot)), \quad v(\cdot) = y(\cdot; y_0^\varepsilon, u^\varepsilon(\cdot)) \)
\[
z_\varepsilon(\cdot) = \lim_{\rho \to 0} \frac{y^\varepsilon(\cdot) - y(\cdot)}{\rho} \quad \text{and} \quad z_0^\varepsilon = \lim_{\rho \to 0} \frac{\hat{J}(y_0^\varepsilon, u^\varepsilon(\cdot)) - \hat{J}(y_0^\varepsilon, u_\varepsilon(\cdot))}{\rho}.
\]
Then it is easy to check that \(z_\varepsilon(\cdot)\) and \(z_0^\varepsilon\), respectively, satisfy that
\[
\begin{cases}
z_\varepsilon(t) = A z_\varepsilon(t) + f_y(t, y_\varepsilon(t), u_\varepsilon(t)) z_\varepsilon(t) + f_u(t, \psi_\varepsilon(t), u_\varepsilon(t))) v(t), \quad t \in (0, T), \\
z_\varepsilon(0) = \nu, \\
t \in (0, T),
\end{cases}
\]
and
\[
z_0^\varepsilon = \int_0^T \left[ (f_y^\varepsilon(t, y_\varepsilon(t), u_\varepsilon(t)), z_\varepsilon(t))_{Y', Y} + (f_u^\varepsilon(t, y_\varepsilon(t), u_\varepsilon(t)), v(t))_{U', U} \right] dt.
\]
Furthermore, by the definition of \(J_\varepsilon(\cdot, \cdot)\), as \(\rho \to 0\),
\[
\frac{J_\varepsilon(y_0^\varepsilon, u^\varepsilon(\cdot)) - J_\varepsilon(y_0^\varepsilon, u_\varepsilon(\cdot))}{\rho} \rightarrow z_\varepsilon^0 \psi^0_1, \quad \psi^1_\varepsilon = \frac{1}{J_\varepsilon(y_0^\varepsilon, u_\varepsilon(\cdot))} \frac{ds(y_0^\varepsilon, y(\cdot; y_0^\varepsilon, u(\cdot)))}{J_\varepsilon(y_0^\varepsilon, u_\varepsilon(\cdot))}, \quad \psi^2_\varepsilon = \frac{ds(y_0^\varepsilon, y(T))}{J_\varepsilon(y_0^\varepsilon, u_\varepsilon(\cdot))} \frac{b_y}{J_\varepsilon(y_0^\varepsilon, u_\varepsilon(\cdot))},
\]
with \(|a^2|_{Y'} + |b^2|_{Y'} = 1, \quad |\psi^0_1|^2 + |\psi^1_\varepsilon|^2 + |\psi^2_\varepsilon|^2 = 1, \quad \varepsilon \to 0\), and
\[
\langle a_\varepsilon, y_0^\varepsilon - y_0 \rangle_{Y', Y} + \langle b_\varepsilon, y^1 - y_\varepsilon(\cdot) \rangle_{Y', Y} \leq 0, \quad \text{for any } (y^0, y^1) \in S. \tag{5.6}
\]
On the other hand, by (5.4),
\[ J_\epsilon(y_0^\epsilon, u^\epsilon_\nu(\cdot)) - J_\epsilon(y_0^\epsilon, u_\nu(\cdot)) \geq -\sqrt{\varepsilon}p(|\nu|_Y + |\nu(\cdot)|_{U_2}). \] (5.7)
(5.5) and (5.7) imply that for any \( \varepsilon > 0 \),
\[ -\sqrt{\varepsilon}[|\nu|_Y + |\nu(\cdot)|_{U_2}] \leq z^0_\nu \psi^0_1, \nu + \langle \psi^1_1, \nu \rangle_{Y', Y} + \langle \psi^2_1, z_\epsilon(T) \rangle_{Y', Y}. \] (5.8)

**Step 3.** Without loss of generality, we assume that as \( \varepsilon \to 0 \), \( \psi^1_1 \to \psi^1_1 \) weakly* in \( Y' \), \( \psi^2_1 \to \psi^2_1 \) weakly* in \( Y' \), \( \psi^1_1, \nu \to \psi^1_1, \nu \) in \( \mathbb{R} \), \( z_\epsilon^0 \to z^0 \) in \( \mathbb{R} \), and \( \sup_{t \in [0, T]} |z_\epsilon(t) - z(t)|_Y \to 0 \). Then by (5.8),
\[ z^0_\nu \psi^0_1 + \langle \psi^1_1, \nu \rangle_{Y', Y} + \langle \psi^2_1, z(T) \rangle_{Y', Y} \geq 0, \] (5.9)
where \( z^0_\nu \) and \( z(\cdot) \) satisfy, respectively, that
\[ z^0_\nu = \int_0^T \left[ \langle f^0_\nu(t, \overline{y}(t), \overline{u}(t)) \rangle_{Y', Y} \right] dt \]
and
\[ \left\{ \begin{array}{l}
  z(t) = A(t) + f_\nu(t, \overline{y}(t), \overline{u}(t))z(t) + f_\nu(t, \overline{y}(t), \overline{u}(t))v(t), \quad t \in (0, T], \\
  z(0) = \nu.
\end{array} \right. \] (5.10)

On the other hand, let \( \psi_1 \) be the solution to (5.2) associated to the above \( (\psi^0_1, \psi^2_1) \in \mathbb{R} \times Y' \). By (5.2), (5.9) and (5.10), choose \( \nu = 0 \) and it is easy to find that
\[ \int_0^T \left[ \langle \psi^0_1 f^0_\nu(t, \overline{y}(t), \overline{u}(t)), v(t) \rangle_{Y', Y} + \langle f_\nu(t, \overline{y}(t), \overline{u}(t)) \psi_1, v(t) \rangle_{Y', Y} \right] dt = 0, \forall \, v(\cdot) \in U_2. \]
This implies the necessary condition (5.3).

**Step 4.** The finite codimensionality of \( M_1 \) is given to guarantee that \( (\psi^0_1, \psi^1_1(\cdot)) \neq (0, 0) \). Indeed, if \( \psi^0_1 = 0 \), there exists a \( \delta_0 > 0 \), such that for sufficiently small \( \varepsilon > 0 \), \( \psi^1_1, \nu \) and \( \psi^2_1, \nu \) are \( \delta_0 \). Also, by (5.8) and (5.6), it follows that for any \( (\nu, \nu(\cdot)) \in Y \times U_2 \) with \( |\nu|_Y \leq 1 \) and \( |\nu(\cdot)|_{U_2} \leq 1 \),
\[ \langle \psi^1_1, \nu - y^0 + \overline{y}_0 \rangle_{Y', Y} + \langle \psi^2_1, z^0(T) \rangle_{Y', Y} \geq -\sqrt{\varepsilon} \left[ |\nu|_Y + |\nu(\cdot)|_{U_2} \right] \]
\[ - \langle \psi^2_1, y^1 - \overline{y}(T) \rangle_{Y', Y} + \langle \psi^2_1, z(T) - z_\epsilon(T) \rangle_{Y', Y} \]
\[ \Delta = -\delta_0 \to 0, \text{ as } \varepsilon \to 0. \]

Then, by Proposition 3.5 and Lemma 3.6 in [31], if \( M_1 \) is finite codimensional in \( Y \), \( (\psi^0_1, \psi^1_1, \psi^2_1) \neq (0, 0, 0) \). This implies that \( (\psi^0_1, \psi^1_1(\cdot)) \) is nontrivial, since \( \psi^1_1(0) = -\psi_1(0) \).

In the following, we study the finite codimensionality of the set \( M_1 \) under fixed endpoint constraints. Set \( S = \{ (y^0, y^1) \} \), where \( y^0, y^1 \in Y \) are arbitrarily given. Then the solution \( \xi \) to (5.1) can be rewritten as \( \xi = z_1 + \eta_1 \), where \( z_1 \) and \( \eta_1 \) solve, respectively, that
\[ \left\{ \begin{array}{l}
  z_{1, t}(t) = A z_1(t) + f_\nu(t, \overline{y}(t), \overline{u}(t))z_1(t) + f_\nu(t, \overline{y}(t), \overline{u}(t))v(t), \quad t \in (0, T], \\
  z_1(0) = 0,
\end{array} \right. \] (5.11)
and
\[
\begin{align*}
\eta_{1,t}(t) &= A\eta_1(t) + f_y(t, \bar{y}(t), \bar{w}(t))\eta_1(t), \quad t \in (0, T], \\
\eta_1(0) &= y^0.
\end{align*}
\]

Set
\[
M_2 = \left\{ z_1(T) \in Y \mid z_1 \text{ is the solution to (5.11) with some } v(\cdot) \in \mathcal{U}_2 \text{ satisfying } |v|_{\mathcal{U}_2} \leq 1 \right\}. 
\]

Then \( M_1 = M_2 + \{ \eta_1(T) - y^1 \} \) and therefore, \( M_1 \) is finite codimensional in \( Y \), if and only if \( M_2 \) is finite codimensional in \( Y \).

By Theorems 3.1-3.3, one has the following equivalent assertions on the finite codimensionality of \( M_2 \).

**Corollary 5.1** The following assertions are equivalent:

1. The system (5.11) is finite codimensional exactly controllable in \( Y \).
2. There is a finite codimensional subspace \( \tilde{Y} \subseteq Y' \), such that any solution \( \phi \) to the equation:
\[
\begin{align*}
\phi_t(t) &= -A^*\phi(t) - f_y(t, \bar{y}(t), \bar{w}(t))^\ast \phi(t), \quad t \in (0, T], \\
\phi(T) &= \phi_T,
\end{align*}
\]
satisfies that
\[
|\phi_T|_{Y'} \leq C|f_u(\cdot, \bar{y}(\cdot), \bar{w}(\cdot))^\ast \phi|_{L^2(0, T; U')}, \quad \forall \phi_T \in \tilde{Y}.
\]
3. There is a compact operator \( G \) from \( Y' \) to a Banach space \( X \), such that any solution \( \phi \) to (5.14) satisfies that
\[
|\phi_T|_{Y'} \leq C\left(|f_u(\cdot, \bar{y}(\cdot), \bar{w}(\cdot))^\ast \phi|_{L^2(0, T; U')} + |G\phi_T|_X\right), \quad \forall \phi_T \in Y'.
\]
4. The set \( M_2 \) (defined in (5.13)) is finite codimensional in \( Y \).

**Case 2.** The optimal control problem (P) with certain control constraint.

First, we give the following hypothesis:

(A21) Let \( f : [0, T] \times Y \times \bar{U} \rightarrow Y \) and \( f^0 : [0, T] \times Y \times \bar{U} \rightarrow \mathbb{R} \) satisfy that \( f \) and \( f^0 \) are strongly measurable with respect to \( t \) in \( (0,T) \), and continuously Fréchet differentiable with respect to \( y \) in \( Y \) with \( f(t, \cdot, \cdot), f_y(t, \cdot, \cdot), f^0(t, \cdot, \cdot) \) and \( f^0(t, \cdot, \cdot) \) being continuous, respectively. Moreover, there exists a positive constant \( L \), such that for any \( (t, y, u) \in [0, T] \times Y \times \bar{U} \),
\[
|f_y(t, y, u)|_{\mathcal{L}(Y')} + |f^0_y(t, y, u)|_{Y'} + |f(t, 0, u)|_Y + |f^0(t, 0, u)|_Y \leq L.
\]

(A22) For any \( (t, y, u) \in [0, T] \times Y \times \bar{U} \), \( f(t, y, u) = f_1(t, y) + B(t)u \) with \( B \in L^\infty(0, T; \mathcal{L}(U; Y)) \).

(A23) \( p = \infty \), \( \bar{U} \subseteq U \) is a bounded set and \( \overline{\mathbb{U}} \bar{U} \) has at least an interior point in \( U \).

Under the above assumptions, \( \mathcal{U}[0, T] = \left\{ u \in L^\infty(0, T; U) \mid u : (0, T) \rightarrow \bar{U} \text{ is measurable} \right\} \), and for any \( T > 0 \), \( y(0) \in Y \) and \( u(\cdot) \in \mathcal{U}[0, T], (2.2) \) admits a mild solution \( y(\cdot) \in C([0, T]; Y) \) and \( f^0(\cdot, y(\cdot), u(\cdot)) \in L^1(0, T) \). Also, the assumptions \((H_1)\) and \((H_2)\) on Page 130 in [31] hold.
Consider the following linear system:
\[
\begin{align*}
\tilde{\xi}(t) &= A\tilde{\xi}(t) + f_{1,y}(t,\bar{y}(t))\tilde{\xi}(t) + B(t)[v(t) - \bar{u}(t)], \quad t \in (0, T], \\
\tilde{\xi}(0) &= y^0,
\end{align*}
\]
where \(v \in \mathcal{U}[0, T]\) and set
\[
M_3 = \{\tilde{\xi}(T) - y^1 \in Y \mid \tilde{\xi} \text{ is the solution to (5.16) with some } v(\cdot) \in \mathcal{U}[0, T] \text{ and } (y^0, y^1) \in S\}.
\]
By Theorem 1.6 on Page 135 in [31], if \(M_3\) is finite codimensional in \(Y\), then the optimal pair \((\bar{v}(\cdot), \bar{y}(\cdot))\) for the optimal control problem (P) satisfies Pontryagin type maximum principle, that is, there exists a nontrivial pair \((\psi^0, \psi^1)\) and \(\tilde{\psi}\) such that
\[
\begin{align*}
\psi_{2,t}(t) &= -A^*\psi_2(t) - \psi_2^0 f_y^0(t, \bar{y}(t), \bar{u}(t)) - f_{1,y}(t, \bar{y}(t))^*\psi_2(t), \quad t \in (0, T], \\
\psi_2(T) &\in Y',
\end{align*}
\]
and \(H(t, \bar{y}(t), \bar{u}(t), \psi^0, \psi) = \max_{u \in \mathcal{U}} H(t, \bar{y}(t), u, \psi^0, \psi_2(t)), \text{ a.e. } t \in (0, T),\) where
\[
H(t, y, u, \psi^0, \psi) = \psi^0 f^0(t, y, u) + \langle \psi, f(t, y, u) \rangle_{Y'\times Y}, \forall (t, y, u, \psi^0, \psi) \in [0, T] \times Y \times \bar{\mathcal{U}} \times \mathbb{R} \times Y'.
\]
In the rest of this section, we study the finite codimensionality of \(M_3\) with fixed endpoint constraints. Set \(S = \{(y^0, y^1)\}\) with \(y^0, y^1 \in Y\) arbitrarily given. Then the solution \(\tilde{\xi}\) to (5.16) can be rewritten as \(\tilde{\xi} = z_2 + \eta_2\), where \(z_2\) and \(\eta_2\) satisfy, respectively, that
\[
\begin{align*}
\begin{cases}
z_{2,t}(t) &= A z_2(t) + f_{1,y}(t, \bar{y}(t)) z_2(t) + B(t) v(t), \quad t \in (0, T], \\
z_2(0) &= 0,
\end{cases}
\quad (5.17)
\]
and
\[
\begin{align*}
\begin{cases}
\eta_{2,t}(t) &= A \eta_2(t) + f_{1,y}(t, \bar{y}(t)) \eta_2(t) - B(t) \bar{u}(t), \quad t \in (0, T], \\
\eta_2(0) &= y^0.
\end{cases}
\end{align*}
\]
Set
\[
M_4 = \{z_2(T) \in Y \mid z_2 \text{ is the solution to (5.17) with some } v(\cdot) \in \mathcal{U}[0, T]\}.
\]
Then \(M_3 = M_4 + \{\eta_2(T) - y^1\}\) and therefore, \(M_3\) is finite codimensional in \(Y\), if and only if \(M_4\) is finite codimensional in \(Y\). Moreover, write
\[
\bar{\mathcal{U}} = \mathcal{U}[0, T] = \{u \in \mathcal{U}_\infty \mid u: (0, T) \to \bar{\mathcal{U}} \text{ is measurable}\}.
\]
By (A23), \(\bar{\mathcal{U}}\) is a bounded subset of \(\mathcal{U}_\infty\) and \(\overline{\mathcal{U}}\) has at least an interior point in \(\mathcal{U}_\infty\).

**Corollary 5.2** The following assertions are equivalent:

1. The system (5.17) is finite codimensional exactly controllable in \(Y\).
2. There is a finite codimensional subspace \(\bar{Y} \subseteq Y'\), such that any solution \(\phi\) to the equation
\[
\begin{align*}
\begin{cases}
\phi_{t}(t) &= -A^*\phi(t) - f_{1,y}(t, \bar{y}(t))^*\phi(t), \quad t \in (0, T], \\
\phi(T) &= \phi_T,
\end{cases}
\end{align*}
\]

satisfies that
\[ |\phi_T|_{Y'} \leq C|B(\cdot)^*\phi|_{L^1(0,T;U')}, \quad \forall \phi_T \in \tilde{Y}. \] \hspace{1cm} (5.20)

(3) There is a compact operator \( G \) from \( Y' \) to a Banach space \( X \), such that any solution \( \phi \) to (5.19) satisfies that
\[ |\phi_T|_{Y'} \leq C(|B(\cdot)^*\phi|_{L^1(0,T;U')} + |G\phi_T|_X), \quad \forall \phi_T \in Y'. \]

(4) The set \( M_4 \) (defined in (5.18)) is finite codimensional in \( Y \).

**Remark 5.1** Similar to the proofs of [48, Theorem 1.1] and Theorem 3.2 in this paper, one can get a weaker criterion than the estimate (5.20). Indeed, (5.20) is true, if and only if there is a finite codimensional subspace \( \tilde{Y} \subseteq Y' \) such that for any solution \( \phi \) to (5.19), it holds that
\[ |\phi_T|_{Y'} \leq C|B(\cdot)^*\phi|_{L^2(0,T;U')}, \quad \forall \phi_T \in \tilde{Y}. \]

6 Two examples

In this section, two examples on linear quadratic control (LQ for short) problems with fixed endpoint constraints for wave and heat equations are presented, respectively. By the finite codimensional exact controllability and its equivalent assertions introduced in this paper, the finite codimensionality of the sets appeared in these optimal control problems will be verified very easily.

6.1 Example 1. An LQ problem for wave equations

1) Formulation of problem

Recall that \( \Omega \subset \mathbb{R}^n \) is a bounded domain with a \( C^\infty \) boundary \( \partial \Omega \). Put \( Q = \Omega \times (0,T) \) and \( \Sigma = \partial \Omega \times (0,T) \). Assume that \( \omega \) is a nonempty open subset of \( \Omega \). Denote by \( \chi_\omega \) the characteristic function of \( \omega \). Consider the following wave equation:
\[
\begin{aligned}
&\left\{ \begin{array}{ll}
y_{tt} - \Delta y + a(x,t)y = \chi_\omega u & \text{in } Q, \\
y = 0 & \text{on } \Sigma, \\
y(0) = y_0, \ y_t(0) = y_1 & \text{in } \Omega,
\end{array} \right.
\end{aligned} \] \hspace{1cm} (6.1)

where \( u \in L^2(Q) \) is the control variable and \( (y, y_t) \) is the state variable, \( (y_0, y_1) \in H^1_0(\Omega) \times L^2(\Omega) \) is an initial value, and \( a(\cdot) \in L^\infty(Q) \). For a given \( (y^0, y^1) \in H^1_0(\Omega) \times L^2(\Omega) \), set
\[ U_{ad} = \left\{ u(\cdot) \in L^2(Q) \left| \text{ the solution } y \text{ to (6.1) satisfies that } (y(T), y_t(T)) = (y^0, y^1) \right. \right\} \]

and
\[ J(u(\cdot)) = \frac{1}{2} \int_Q \left[ q(x,t)|y(x,t)|^2 + r(x,t)|u(x,t)|^2 \right] dxdt, \]
where \( y(\cdot) \) is the solution to (6.1) associated to \( u(\cdot) \), and \( q, r \in L^\infty(Q) \) are given functions.

Assume that \( (\overline{u}(\cdot), \overline{y}(\cdot)) \) is an optimal pair of the optimal control problem:
\[ J(\overline{u}(\cdot)) = \inf_{u(\cdot) \in U_{ad}} J(u(\cdot)), \]

and set
\[ M_5 = \left\{ (y(T), y_t(T)) \in H^1_0(\Omega) \times L^2(\Omega) \left| y \text{ is the solution to (6.1) with } (y_0, y_1) = (0,0), \right. \right\}. \] \hspace{1cm} (6.2)
Consider the following equation:
\[
\begin{cases}
\psi_{tt} - \Delta \psi + a(x,t)\psi - \psi^0 q(x,t) \nabla(x,t) = 0 & \text{in } Q, \\
\psi = 0 & \text{on } \Sigma, \\
\psi(T) = \psi_1, \; \psi_t(T) = \psi_2 & \text{in } \Omega,
\end{cases}
\]  
(6.3)
where $\psi^0 \in \mathbb{R}$ and $(\psi_1, \psi_2) \in L^2(\Omega) \times H^{-1}(\Omega)$. It is easy to check that (A$_{11}$) and (A$_{12}$) in Proposition 5.1 hold. Hence, one has the following result.

**Proposition 6.1** If $M_5$ is finite codimensional in $H^1_0(\Omega) \times L^2(\Omega)$, then there exist $(\psi^0, \psi_1, \psi_2) \in \mathbb{R} \times L^2(\Omega) \times H^{-1}(\Omega)$, such that for the corresponding solution $\psi$ to (6.3), $(\psi^0, \psi(\cdot)) \neq (0, 0)$ and $\psi^0_x(x,t) \nabla(x,t) + \chi_\omega \psi(x,t) = 0, \; \text{a.e.} \; (x,t) \in Q.$

2) The geometric control condition and finite codimensionality

In this part, under some mild assumptions, we prove that the set $M_5$ (in Proposition 6.1) is finite codimensional in $H^1_0(\Omega) \times L^2(\Omega)$, if and only if the geometric control condition for $(\Omega, \omega, T)$ in (6.1) holds. To begin with, let us recall some related definitions on this condition (see [4] and also [14, 19, 26, 35] for more details and related results).

**Definition 6.1** (1) For a wave operator $W = \partial_4 - \Delta$, a null bicharacteristic $(t(\cdot), \hat{x}(\cdot), \tau(\cdot), p(\cdot)) : \mathbb{R} \to \mathbb{R}^{2n+1}$ is defined to be a solution to the system:
\[
\begin{align*}
t_s(s) &= 2\tau(s), \\
\hat{x}_s(s) &= -2p(s), \\
\tau_s(s) &= 0, \\
p_s(s) &= 0,
\end{align*}
\]  
(6.4)
with $t(0) = 0, \; \tau(0) = 1/2, \; \hat{x}(0) = \hat{x}_0 \in \mathbb{R}^n$ and $p(0) = p_0 \in \mathbb{R}^n$ satisfying that $|p_0| = 1/2$. $(t(\cdot), \hat{x}(\cdot), \tau(\cdot), p(\cdot))$ is called a ray of $W$.

(2) For any $T > 0$ and open set $\Omega$ of $\mathbb{R}^n$, $(t(\cdot), \hat{x}(\cdot), p(\cdot)) : [0, T] \to \overline{\Omega} \times \mathbb{R}^n$ with $\hat{x}(0), \hat{x}(T) \in \Omega$ is called a generalized ray of $W$ in $\overline{\Omega}$, if there exists a partition $0 = s_0 < s_1 < \cdots < s_m = T$ for any $m \in \mathbb{N}$, such that for any $j = 0, 1, \cdots, m - 1$, $(\hat{x}(s), p(s))|_{s_j \leq s \leq s_{j+1}} = (\hat{x}^j(s), p^j(s))$ satisfies (6.4), $\hat{x}(s_k) \in \partial \Omega \; (k = 1, 2, \cdots, m - 1)$, and the following law of geometric optics holds:
\[
p^{k+1}(s_k) = p^k(s_k) - 2\nu(\hat{x}^k(s_k)) \hat{x}^k(s_k),
\]  
where $\nu(x)$ denotes the unit outer normal vector on $x \in \partial \Omega$. $s_k$ is called the $k$-th reflected instant of this generalized ray. A generalized ray is denoted by $(t(\cdot), \hat{x}(\cdot), p(\cdot))|_{t \in [s_j, s_{j+1}]}^{m-1}$.

(3) $(\Omega, \omega, T)$ in (6.1) is called to satisfy the geometric control condition, if for any generalized ray $(t(s), \hat{x}(s), p(s))|_{s \in [s_j, s_{j+1}]}^{m-1}$ of $W$ in $\overline{\Omega}$, there are a $j \in \{0, 1, \cdots, m - 1 \}$ and $s_0 \in [s_j, s_{j+1}]$, such that $\hat{x}(s_0) \in \omega$.

**Remark 6.1** It is easy to show that a null bicharacteristics of $W$ is a straight line in $\mathbb{R}^{2(n+1)}$. Noting that $t(s) \equiv s$, in the rest of this paper, we simply denote by $(t(\cdot), \hat{x}(\cdot), p(\cdot))|_{t \in [0, T]}$ the generalized ray.

In order to prove that $M_5$ is finite codimensional in $H^1_0(\Omega) \times L^2(\Omega)$ (in Proposition 6.1), consider the following backward wave equation:
\[
\begin{cases}
\phi_{tt} - \Delta \phi + a(x, t)\phi = 0 & \text{in } Q, \\
\phi = 0 & \text{on } \Sigma, \\
\phi(T) = \phi_1, \; \phi_t(T) = \phi_2 & \text{in } \Omega,
\end{cases}
\]  
(6.5)
where $(\phi_1, \phi_2) \in L^2(\Omega) \times H^{-1}(\Omega)$. By Corollary 5.1, we have the following result.
Proposition 6.2 If \((\Omega, \omega, T)\) satisfies the geometric control condition, then \(M_5\) (defined in (6.2)) is finite codimensional in \(H^1_0(\Omega) \times L^2(\Omega)\).

Proof. First, denote by \(\phi\) the solution to (6.5) and set \(\phi = \xi_1 + \xi_2\), where \(\xi_j \ (j = 1, 2)\) satisfy
\[
\begin{cases}
\xi_{1,tt} - \Delta \xi_1 = 0 & \text{in } Q, \\
\xi_1 = 0 & \text{on } \Sigma, \\
\xi_1(T) = \phi_1, \ \xi_{1,t}(T) = \phi_2 & \text{in } \Omega,
\end{cases}
\]
and
\[
\begin{cases}
\xi_{2,tt} - \Delta \xi_2 + a(x, t)\phi = 0 & \text{in } Q, \\
\xi_2 = 0 & \text{on } \Sigma, \\
\xi_2(T) = 0, \ \xi_{2,t}(T) = 0 & \text{in } \Omega.
\end{cases}
\] (6.6)

Define operators \(G_1\), \(G_2\) and \(G_3\) as follows:
\[
G_1 : L^2(\Omega) \times H^{-1}(\Omega) \to L^2(Q), \quad G_1(\phi_1, \phi_2) = \phi, \quad \forall (\phi_1, \phi_2) \in L^2(\Omega) \times H^{-1}(\Omega),
\]
where \(\phi\) is the solution to (6.5) with the initial value \((\phi_1, \phi_2)\);
\[
G_2 : L^2(\Omega) \to \mathcal{L}(0, T; H_0^1(\Omega)) \cap W^{1, \infty}(0, T; L^2(\Omega)), \quad G_2(\phi) = \xi_2,
\]
where \(\xi_2\) is the solution to (6.6) associated to \(\phi \in L^2(\Omega)\); and
\[
G_3 : L^\infty(0, T; H_0^1(\Omega)) \cap W^{1, \infty}(0, T; L^2(\Omega)) \to L^2(\Omega), \quad G_3(\xi_2) = \xi_2.
\]

Let \(G = G_3 G_2 G_1\). Then \(G : L^2(\Omega) \times H^{-1}(\Omega) \to L^2(Q), \ \ G(\phi_1, \phi_2) = \xi_2\).

By the well-posedness results of wave equations, \(G\) is a compact operator. Since \((\Omega, \omega, T)\) fulfills the geometric control condition, by [3], we have that
\[
\|(\phi_1, \phi_2)\|^2_{L^2(\Omega) \times H^{-1}(\Omega)} \leq C \int_0^T \int_\omega \xi_2^2 dxdt, \quad \forall (\phi_1, \phi_2) \in L^2(\Omega) \times H^{-1}(\Omega).
\]

It follows that
\[
\|(\phi_1, \phi_2)\|^2_{L^2(\Omega) \times H^{-1}(\Omega)} \leq C \int_0^T \int_\omega (\phi^2 + \xi_2^2) dxdt \leq C \chi_\omega \phi^2_{L^2(\Omega)} + C G(\phi_1, \phi_2)^2_{L^2(Q)}.
\]

Take \(Y' = L^2(\Omega) \times H^{-1}(\Omega)\) and \(X = L^2(Q)\) in (5.15), and note that \(f(\cdot, \bar{y}(\cdot), \bar{\pi}(\cdot))\phi = \chi_\omega \phi\), by Corollary 5.1, \(M_5\) (defined in (6.2)) is finite codimensional in \(H^1_0(\Omega) \times L^2(\Omega)\).

Remark 6.2 By Theorem 3.1, the result in Proposition 6.2 implies that under the geometric control condition, the wave equation (6.1) is finite codimensional exactly controllable. Note that under the same condition, the exact controllability of this wave equation is still open. However, it is rather easy to show the weaker finite codimensional controllability, while this controllability is enough for us to study the optimal control problem with constraints.

Next, we prove that, under some mild assumptions, if the geometric control condition fails, the system (6.1) is not finite codimensional exactly controllable any more. Hence, the finite codimensionality of \(M_5\) fails in this case.
Proposition 6.3 Assume that there is a generalized ray \( \{(t, \hat{x}^j(t), \tilde{p}^j(t)) \mid t \in [s_j, s_{j+1}]\}_{j=0}^{m-1} \) of \( W \) in \( \mathbb{R} \), such that

1. It does not meet \( \omega \), i.e., \( x^j(t) \not\in \overline{\omega} \), for any \( j \in \{0,1,\cdots,m-1\} \) and \( t \in [s_j, s_{j+1}] \); and

2. It always meets \( \partial \Omega \) transversally, i.e., \( \nu(\hat{x}^k(s_k))^\top p^k(s_k) \neq 0 \), \( \forall k \in \{1,2,\cdots,m-1\} \), where \( s_k \) is the \( k \)-th reflected instant of this generalized ray.

Then the system (6.1) is not finite codimensional exactly controllable.

By Theorem 3.2, the finite codimensional exact controllability of the system (6.1) may be reduced to a suitable observability estimate (3.5) for solutions \( \phi \) to (6.5). Let \( \hat{\phi}(x,t) = \phi(x, T-t) \). Then it is easy to check that \( \hat{\phi}(\cdot) \) solves

\[
\begin{align*}
\hat{\phi}_{tt} - \Delta \hat{\phi} + \hat{a}(x,t)\hat{\phi} &= 0 \quad \text{in } Q, \\
\hat{\phi} &= 0 \quad \text{on } \Sigma, \\
\hat{\phi}(0) &= \phi_1, \quad \hat{\phi}_t(0) = \phi_2 \quad \text{in } \Omega,
\end{align*}
\]

where \( \hat{a}(x,t) = a(x, T-t) \) and \( (\phi_1, \phi_2) \in L^2(\Omega) \times H^{-1}(\Omega) \). Also, the observability estimate for solutions \( \phi \) to (6.5) is equivalent to the following one for solutions \( \hat{\phi} \) to (6.7):

\[
|((\phi_1, \phi_2))_{L^2(\Omega) \times H^{-1}(\Omega)}| \leq C|\hat{\phi}|_{L^2(0,T;L^2(\omega))}, \quad \forall (\phi_1, \phi_2) \in \hat{M},
\]

where \( \hat{M} \) is a finite codimensional subspace of \( L^2(\Omega) \times H^{-1}(\Omega) \).

As preliminaries to prove Proposition 6.3, some lemmas are given in order. First, consider the following wave equation:

\[
\begin{align*}
\varphi_{tt} - \Delta \varphi + \int_0^t \hat{a}(x,s)\varphi_s(x,s)ds &= 0 \quad \text{in } Q, \\
\varphi &= 0 \quad \text{on } \Sigma, \\
\varphi(0) &= \varphi_1, \quad \varphi_t(0) = \varphi_2 \quad \text{in } \Omega,
\end{align*}
\]

where \( (\varphi_1, \varphi_2) \in H^1_0(\Omega) \times L^2(\Omega) \). Then the observability estimate (6.8) for solutions \( \hat{\phi} \) to (6.7) implies a suitable estimate for solutions \( \varphi \) to (6.9) (Notice that conversely, it may be untrue).

Lemma 6.1 If any solution \( \hat{\phi} \) to the equation (6.7) satisfies (6.8), then there exists a finite codimensional subspace \( M_6 \) of \( H^1_0(\Omega) \times L^2(\Omega) \), such that solutions \( \varphi \) to (6.9) satisfy

\[
|((\varphi_1, \varphi_2))_{H^1_0(\Omega) \times L^2(\Omega)}| \leq C|\varphi_t|_{L^2(0,T;L^2(\omega))}, \quad \forall (\varphi_1, \varphi_2) \in M_6.
\]

Proof. For arbitrarily given \( (\varphi_1, \varphi_2) \in H^1_0(\Omega) \times L^2(\Omega) \), let \( \varphi \) be the corresponding solution to (6.9). Set \( \phi = \varphi_t \). Then \( \phi \) solves (6.7) with the initial value \( (\phi(0), \phi_t(0)) = (\varphi_2, \Delta \varphi_1) \in L^2(\Omega) \times H^{-1}(\Omega) \).

By the assumption, there exists a finite dimensional subspace \( M_6 \) of \( L^2(\Omega) \times H^{-1}(\Omega) \), such that \( L^2(\Omega) \times H^{-1}(\Omega) = M_6 \oplus \hat{M} \), and for any \( (\varphi_2, \Delta \varphi_1) \in \hat{M}, \) it holds that

\[
|((\varphi_2, \Delta \varphi_1))_{L^2(\Omega) \times H^{-1}(\Omega)}| \leq C|\varphi_t|_{L^2(0,T;L^2(\omega))}.
\]

Denote by \( \mathcal{A} \) be the Laplacian operator with homogeneous Dirichlet boundary condition. Set

\[
M_7 = \left\{ (\varphi_1, \varphi_2) \in H^1_0(\Omega) \times L^2(\Omega) \mid \varphi_2 = \hat{\phi}_1, \varphi_1 = (-\mathcal{A})^{-1}\hat{\phi}_2 \text{ for some } (\hat{\phi}_1, \hat{\phi}_2) \in \hat{M} \right\}.
\]
and

\[ M_8 = \left\{ (\varphi_1, \varphi_2) \in H_0^1(\Omega) \times L^2(\Omega) \mid \varphi_2 = \hat{\varphi}_1, \varphi_1 = (-A)^{-1} \hat{\varphi}_2 \text{ for some } (\hat{\varphi}_1, \hat{\varphi}_2) \in M_6 \right\} \]

Then by (6.11), it is clear that (6.10) holds. Also, \( M_6 \) is finite dimensional and \( H_0^1(\Omega) \times L^2(\Omega) = M_7 \oplus M_8 \). The proof is completed. \( \square \)

Further, a priori estimate for a compact operator is presented.

**Lemma 6.2** Assume that \( G \) is a compact operator from \( H_0^1(\Omega) \times L^2(\Omega) \) to itself. Then for any \( \delta > 0 \), there is a positive constant \( C \) such that

\[
|G(\varphi_1, \varphi_2)|_{H_0^1(\Omega) \times L^2(\Omega)} \leq \delta |(\varphi_1, \varphi_2)|_{H_0^1(\Omega) \times L^2(\Omega)} + C |(\varphi_1, \varphi_2)|_{L^2(\Omega) \times H^{-1}(\Omega)}, \quad \forall (\varphi_1, \varphi_2) \in H_0^1(\Omega) \times L^2(\Omega). \tag{6.12}
\]

**Proof.** Assume that (6.12) fails. Then there exist \( \delta_0 > 0 \) and a sequence \( \{(\varphi_{n,1}, \varphi_{n,2})\}_{n=1}^{\infty} \) in \( H_0^1(\Omega) \times L^2(\Omega) \), such that for any \( n \in \mathbb{N} \),

\[
\begin{cases}
|G(\varphi_{n,1}, \varphi_{n,2})|_{H_0^1(\Omega) \times L^2(\Omega)} = 1, \\
|G(\varphi_{n,1}, \varphi_{n,2})|_{H_0^1(\Omega) \times L^2(\Omega)} \leq \frac{1}{\delta_0}, \\
|G(\varphi_{n,1}, \varphi_{n,2})|_{L^2(\Omega) \times H^{-1}(\Omega)} \leq \frac{1}{n}.
\end{cases} \tag{6.13}
\]

Hence, there exist a subsequence of \( \{(\varphi_{n,1}, \varphi_{n,2})\}_{n=1}^{\infty} \) (still denoted by itself) and \( (\varphi_1, \varphi_2) \in H_0^1(\Omega) \times L^2(\Omega) \), such that

\[(\varphi_{n,1}, \varphi_{n,2}) \to (\varphi_1, \varphi_2) \text{ weakly in } H_0^1(\Omega) \times L^2(\Omega) \text{ as } n \to +\infty.\]

Since the embedding from \( H_0^1(\Omega) \times L^2(\Omega) \) to \( L^2(\Omega) \times H^{-1}(\Omega) \) is compact, then

\[
\lim_{n \to \infty} (\varphi_{n,1}, \varphi_{n,2}) = (\varphi_1, \varphi_2) \text{ weakly in } L^2(\Omega) \times H^{-1}(\Omega).
\]

This, together with the third inequality in (6.13), deduces that \( (\varphi_1, \varphi_2) = (0,0) \). Noting that \( G \) is compact, we obtain that

\[
\lim_{n \to \infty} G(\varphi_{n,1}, \varphi_{n,2}) = G(\varphi_1, \varphi_2) \text{ in } H_0^1(\Omega) \times L^2(\Omega),
\]

which implies that \( |G(\varphi_1, \varphi_2)|_{H_0^1(\Omega) \times L^2(\Omega)} = 1 \). It contradicts that \( (\varphi_1, \varphi_2) = (0,0) \). Hence, (6.12) holds. \( \square \)

Further, we construct a family of solutions to the equation (6.9).

**Lemma 6.3** Suppose that all assumptions in Proposition 6.3 hold and \( \hat{\alpha} \in L^\infty(0,T;W^{1,\infty}(\Omega)) \). Then there exist a family of solutions \( \{\varphi_\varepsilon\}_{\varepsilon > 0} \) to (6.9), and positive constants \( c_1 \) and \( c_2 \), independent of \( |\nabla \hat{\alpha}|_{L^\infty(\Omega)} \), such that for any \( 0 < \varepsilon < 1 \),

\[
\begin{cases}
|\varphi_{\varepsilon, t}(\cdot, 0)|_{L^2(\Omega)} \geq c_1, \\
|\varphi_{\varepsilon, t}(\cdot, 0)|_{L^2(\Omega)} \leq c_2, \\
|\varphi_{\varepsilon, t}(\cdot, 0)|_{H_0^1(\Omega)} \leq c_2, \\
|\varphi_\varepsilon|_{L^2(0,T;L^2(\omega))} = (|\hat{\alpha}|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + 1)O(\varepsilon^{1/2}), \\
|\varphi_\varepsilon(\cdot, 0)|_{L^2(\Omega)} + |\varphi_{\varepsilon, t}(\cdot, 0)|_{H^{-1}(\Omega)} = O(\varepsilon^{1/2}),
\end{cases} \tag{6.14}
\]

where \( O(\varepsilon^\alpha) \) denotes a function of order \( \varepsilon^\alpha \) for \( \alpha > 0 \).
The proof of Lemma 6.3 is similar to that of Theorem 7.1 in [19]. We shall only give its sketch in Appendix for completeness.

Now, we are in a position to prove Proposition 6.3.

**Proof of Proposition 6.3.** Assume that the system (6.1) is finite codimensional exactly controllable. Then by Theorem 3.2 and Lemma 6.1, there are a finite dimensional subspace $M_8$ and finite codimensional subspace $M_7$ of $H^1_0(\Omega) \times L^2(\Omega)$, such that $H^1_0(\Omega) \times L^2(\Omega) = M_8 \oplus M_7$, and for any $(\varphi_1, \varphi_2) \in M_7$, the corresponding solution $\varphi$ to the equation (6.9) satisfies (6.10). Denote by $P_{M_8}$ the projection from $H^1_0(\Omega) \times L^2(\Omega)$ to $M_8$. Then for any $(\varphi_1, \varphi_2) \in H^1_0(\Omega) \times L^2(\Omega)$, it holds that

$$
|\varphi_1, \varphi_2||^2_{H^1_0(\Omega) \times L^2(\Omega)}
= |\varphi_1, \varphi_2|^2 - P_{M_8}(\varphi_1, \varphi_2)|^2_{H^1_0(\Omega) \times L^2(\Omega)} + |P_{M_8}(\varphi_1, \varphi_2)|^2_{H^1_0(\Omega) \times L^2(\Omega)}
\leq C|\varphi_1|^2_{L^2(0,T;L^2(\omega))} + |P_{M_8}(\varphi_1, \varphi_2)|^2_{H^1_0(\Omega) \times L^2(\Omega)}
\leq C\rho^2|\varphi_1|^2_{L^2(0,T;L^2(\omega))} + C|\varphi_2|^2_{H^1_0(\Omega) \times L^2(\Omega)}
\leq C\left(\rho|\varphi_1|^2_{L^2(0,T;L^2(\omega))} + |P_{M_8}(\varphi_1, \varphi_2)|^2_{H^1_0(\Omega) \times L^2(\Omega)}\right),
$$

(6.15)

where $\varphi$ and $\bar{\varphi}$ are the solutions to the equation (6.9), respectively, with the initial values $(\varphi_1, \varphi_2) - P_{M_8}(\varphi_1, \varphi_2)$ and $P_{M_8}(\varphi_1, \varphi_2)$.

Since $P_{M_8}$ is compact, by Lemma 6.2, for any $\rho > 0$,

$$
|P_{M_8}(\varphi_1, \varphi_2)|^2_{H^1_0(\Omega) \times L^2(\Omega)} \leq \rho |(\varphi_1, \varphi_2)|^2_{H^1_0(\Omega) \times L^2(\Omega)} + C |(\varphi_1, \varphi_2)|^2_{L^2(\Omega) \times H^{-1}(\Omega)}.
$$

This, together with (6.15), implies that

$$
|(\varphi_1, \varphi_2)|^2_{H^1_0(\Omega) \times L^2(\Omega)} \leq C |\varphi_1|^2_{L^2(0,T;L^2(\omega))} + C |(\varphi_1, \varphi_2)|^2_{L^2(\Omega) \times H^{-1}(\Omega)}.
$$

(6.16)

For $\hat{a}(\cdot) \in L^\infty(Q)$ and any $\rho > 0$, there exists a function $a_\rho(\cdot) \in L^\infty(0,T;W^{1,\infty}(\Omega))$, such that $|a_\rho - \hat{a}|_{L^\infty(Q)} < \rho$ and $|a_\rho|_{L^\infty(Q)} \leq C$, where $C$ is independent of $\rho$. For any $\varepsilon > 0$, by Lemma 6.3, there exist $(\varphi^\varepsilon, 1, \varphi^\varepsilon, 2) \in H^1_0(\Omega) \times L^2(\Omega)$, such that the solution $\varphi^\varepsilon(\cdot)$ to (6.9) with $\hat{a}(\cdot) = a_\rho(\cdot)$ and $(\varphi_1, \varphi_2) = (\varphi^\varepsilon, 1, \varphi^\varepsilon, 2)$ satisfies the conclusions in (6.14). More precisely,

$$
\begin{cases}
| \varphi^\varepsilon_{x,t}(\cdot,0)|_{L^2(\Omega)} \geq c_1, \\
| \varphi^\varepsilon_{x,t}(\cdot,0)|_{L^2(\Omega)} \leq c_2, \\
| \varphi^\varepsilon_{x}(\cdot)|_{H^1_0(\Omega)} \leq (|a_\rho|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + 1)O(\varepsilon^{1/2}), \\
| \varphi^\varepsilon_{x}(\cdot)|_{L^2(\Omega)} + | \varphi^\varepsilon_{x,t}(\cdot,0)|_{H^{-1}(\Omega)} = O(\varepsilon^{1/2}),
\end{cases}
$$

where $c_1$ and $c_2$ are independent of $\rho$ and $\varepsilon$. Denote by $\psi^\varepsilon(\cdot)$ the solution to (6.9) with $\hat{a}(\cdot) \in L^\infty(Q)$ and $(\varphi_1, \varphi_2) = (\varphi^\varepsilon, 1, \varphi^\varepsilon, 2)$. Set $z^\varepsilon(\cdot) = \varphi^\varepsilon(\cdot) - \psi^\varepsilon(\cdot)$. Then $z^\varepsilon(\cdot)$ satisfies

$$
\begin{cases}
z^\varepsilon_{x,t} - \Delta z^\varepsilon + \int_0^t a_\rho(x,s)z^\varepsilon_{x,s}(x,s)ds = \int_0^t [\hat{a}(x,s) - a_\rho(x,s)]\psi^\varepsilon_{x,s}(x,s)ds \quad \text{in } Q, \\
z^\varepsilon = 0 \quad \text{on } \Sigma, \\
z^\varepsilon(0) = 0, \quad z^\varepsilon_{x,t}(0) = 0 \quad \text{in } \Omega,
\end{cases}
$$

and

$$
|z^\varepsilon|^2_{H^1(0,T;L^2(\omega))} \leq C|\psi^\varepsilon_{x,t}|_{L^2(Q)} \leq C|\psi^\varepsilon_{x,1}|_{H^1_0(\Omega)} + |\varphi^\varepsilon_{x,2}|_{L^2(\Omega)} \leq Cc_2\rho.
$$

24
Hence, 
\[ |\psi_\rho|^2_{H^1(0,T;L^2(\omega))} \leq Cc_2 \rho + C(|a_\rho|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + 1)\varepsilon^{1/2}. \]

By (6.16), it follows that 
\[ c_1 \leq Cc_2 \rho + C(|a_\rho|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + 1)\varepsilon^{1/2} + C\varepsilon^{1/2}. \]

Take \( \rho > 0 \) sufficiently small, such that \( Cc_2 \rho < c_1/2 \). Then, choose \( \varepsilon > 0 \) small enough, such that \( C(|a_\rho|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + 1)\varepsilon^{1/2} + C\varepsilon^{1/2} < c_1/2 \). This leads to a contradiction. Hence, the system (6.1) is not finite codimensional exactly controllable.

### 6.2 Example 2. An LQ problem for heat equations

Consider the following heat equation:
\[
\begin{cases}
  y_t - \Delta y = \chi_\omega u & \text{in } Q, \\
  y = 0 & \text{on } \Sigma, \\
  y(0) = y_0 & \text{in } \Omega,
\end{cases}
\]

where \( u \in L^2(Q) \) is the control variable, \( y \) is the state variable and \( y_0 \in L^2(\Omega) \) is an initial value.

Set 
\[ U_{ad} = \{ u(\cdot) \in L^2(Q) \mid \text{the solution } y \text{ of (6.17) satisfies that } y(T) = 0 \} \]

and 
\[ J(u(\cdot)) = \frac{1}{2} \int_Q \left[ q(x,t)|y(x,t)|^2 + r(x,t)|u(x,t)|^2 \right] dxdt, \]

where \( y(\cdot) \) is the solution to (6.17) associated to \( u(\cdot) \), and \( q, r \in L^\infty(Q) \) are given functions. Assume that \((\overline{u}(\cdot), \overline{y}(\cdot))\) is an optimal pair of the optimal control problem:
\[ J(\overline{u}(\cdot)) = \inf_{u(\cdot) \in U_{ad}} J(u(\cdot)). \]

Write
\[ M_9 = \left\{ y(T) \in L^2(\Omega) \mid y \text{ is the solution to (6.17) with } y_0 = 0 \text{ and } v \in L^2(Q) : |v|_{L^2(Q)} \leq 1 \right\}. \]

Then similar to the argument of Proposition 6.1, in order to guarantee (2.3) in Pontryagin type maximum principle to hold, it is required to check if the set \( M_9 \) is finite codimensional. By Corollary 5.1 and a contradiction argument, it is easy to check the following negative result on the finite codimensionality for the heat equation.

**Proposition 6.4** For any \( \Omega, \omega \) and \( T > 0 \), \( M_9 \) is not finite codimensional in \( L^2(\Omega) \).

**Proof.** By Corollary 5.1, it suffices to prove that (5.15) fails for the following heat equation:
\[
\begin{cases}
  \dot{\phi} + \Delta \phi = 0 & \text{in } Q, \\
  \phi = 0 & \text{on } \Sigma, \\
  \phi(T) = \phi_T & \text{in } \Omega,
\end{cases}
\]

with \( \phi_T \in L^2(\Omega) \). In (5.15), take \( U' = Y' = L^2(\Omega) \) and notice that \( f_u(\cdot, \overline{y}(\cdot), \overline{u}(\cdot))^* \phi = \chi_\omega \phi \).
Assume that \( \{ \phi^j \}_{j=1}^\infty \) are the solutions to (6.18) corresponding to terminal values \( \{ \phi_T^j \}_{j=1}^\infty \subseteq L^2(\Omega) \) with \( |\phi_T^j|_{L^2(\Omega)} = 1 \). Then there are a subsequence of \( \{ \phi_T^j \}_{j=1}^\infty \) (still denoted by itself) and \( \phi_T \in L^2(\Omega) \), such that as \( j \to \infty \),

\[
\phi_T^j \to \phi_T \quad \text{weakly in} \quad L^2(\Omega), \quad \phi^j \to \phi \quad \text{in} \quad L^2(Q) \quad \text{and} \quad G(\phi_T^j) \to G(\phi_T) \quad \text{in} \quad X, \quad (6.19)
\]

where \( \phi \) is the solution to (6.18) associated to \( \phi_T \) and \( G \) is a compact operator from \( L^2(\Omega) \) to a Banach space \( X \). If (5.15) is true, then for any \( j \in \mathbb{N} \),

\[
|\phi_T^j|_{L^2(\Omega)} \leq C(|\phi^j|_{L^2(Q)} + |G\phi_T^j|_X).
\]

By (6.19), this implies \( \lim_{j \to \infty} \phi_T^j = \phi_T \) in \( L^2(\Omega) \), which leads to a contradiction. Therefore, the set \( M_9 \) is not finite codimensional in \( L^2(\Omega) \).

Proposition 6.4 indicates that the finite codimensionality of \( M_9 \) fails for an LQ problem of heat equations with fixed endpoint constraints. Hence, the non-triviality of the Lagrange type multiplier cannot be established by the method introduced in this paper.

7 Appendix

This section is devoted to proving Lemma 6.3.

Proof of Lemma 6.3. We shall borrow some ideas from [19, 36, 39]. The proof is divided into four steps.

Step 1. We construct highly concentrated approximate solutions to the hyperbolic equation:

\[
\varphi_{tt} - \Delta \varphi + F(\varphi) = 0 \quad \text{in} \quad \mathbb{R}^n \times [-T, T], \quad (7.1)
\]

where \( F(\varphi) = \int_0^t \hat{a}(x, s)\varphi_s(x, s)ds \) and \( \hat{a} \in L^\infty(0, T; W^{1,\infty}(\mathbb{R}^n)). \)

Given a generalized ray \( (\hat{x}(\cdot), p(\cdot)) \) of the wave operator \( W \), for any \( \varepsilon \in (0, 1) \), construct a family of approximate solutions \( \phi_\varepsilon \) to the equation (7.1) as

\[
\phi_\varepsilon(x, t) = \varepsilon^{1-\frac{n}{4}} c(t) e^{i\psi(x, t)/\varepsilon} + \varepsilon^{2-\frac{n}{4}} \int_0^t A(s) e^{i\psi(x, s)/\varepsilon} ds, \quad (7.2)
\]

with

\[
\psi(x, t) = p^T(t) [x - \hat{x}(t)] + \frac{1}{2} [x - \hat{x}(t)]^T M(t) [x - \hat{x}(t)],
\]

where \( M(t) \) is a complex symmetric matrix with positive definite imaginary part. The construction of approximate solutions \( \phi_\varepsilon \) means an appropriate choice of \( c(\cdot), M(\cdot) \) and \( A(\cdot) \).

Notice that in the subsequent estimates, we are only concerned with the dependence of constants on \( |\nabla \hat{a}|_{L^\infty(Q)} \), rather than \( |\hat{a}|_{L^\infty(Q)} \).

By (7.2) and a direct computation, it is easy to check that

\[
\phi_{\varepsilon, tt} - \Delta \phi_\varepsilon + F(\phi_\varepsilon) = \varepsilon^{2-\frac{n}{4}} r_1 + \varepsilon^{1-\frac{n}{4}} r_2 + \varepsilon^{-\frac{n}{4}} r_3 + \varepsilon^{-1-\frac{n}{4}} r_4, \quad (7.3)
\]
where
\[
\begin{cases}
    r_1 = A(t)e^{i\psi(x,t)/\varepsilon} + \int_0^t \dot{a}(x,s)A(s)e^{i\psi(x,s)/\varepsilon} ds,
    \\
    r_2 = c(t)\dot{A}(t)e^{i\psi(x,t)/\varepsilon} + i\dot{\psi}_t(x,t)A(t)e^{i\psi(x,t)/\varepsilon}
    + \int_0^t \dot{a}(x,s)c(s)e^{i\psi(x,s)/\varepsilon} ds - i \int_0^t A(s)\Delta\psi(x,s)e^{i\psi(x,s)/\varepsilon} ds,
    \\
    r_3 = \dot{r} \left[ 2c(t)\psi_t(x,t) + c(t)(W\psi)(x,t) \right] e^{i\psi(x,t)/\varepsilon}
    + \int_0^t A(s)|\nabla\psi(x,s)|^2 e^{i\psi(x,s)/\varepsilon} ds + i \int_0^t \dot{a}(x,s)c(s)\psi_t(x,s)e^{i\psi(x,s)/\varepsilon} ds,
    \\
    r_4 = c(t)\left[ \nabla\psi(x,t) \cdot \nabla\psi(x,t) - \psi_t^2(x,t) \right] e^{i\psi(x,t)/\varepsilon}.
\end{cases}
\] (7.4)

By [19, 36], one first may choose an \( M(\cdot) \in C^2([-T,T]; \mathbb{R}^{n \times n}) \) with \( M(0) = M^0 \), such that
\[
\nabla\psi(x,t) \cdot \nabla\psi(x,t) - \psi_t^2(x,t) = O(|x - \hat{x}(t)|^3), \quad \text{as} \ x \to \hat{x}(t), \quad \forall \ t \in [-T,T].
\] (7.5)

This implies that
\[
|r_4(\cdot,t)|_{L^2(\mathbb{R}^n)} = O(\varepsilon^{\frac{3}{4} + \frac{1}{2}}), \quad \text{uniformly for a.e. } t \in (-T,T).
\] (7.6)

Also, one can choose a \( c(\cdot) \in C([-T,T]; \mathbb{R} \setminus \{0\}) \bigcap W^{2,\infty}((-T,T) \setminus \{0\}) \) with \( c(0) = c_0^1 \), such that
\[
2c(t)\dot{\psi}_t(x,t) + c(t)(W\psi)(x,t) = O(|x - \hat{x}(t)|), \quad \text{as} \ x \to \hat{x}(t).
\]

Meanwhile, one chooses \( A(s) = 2ic(s)\dot{a}(\hat{x}(s),s) \). Then
\[
A(t)|\nabla\psi(x,t)|^2 + ic(t)\dot{a}(x,t)\psi_t(x,t) = |\dot{a}|_{L^\infty(0,T;W^{1,\infty}(\mathbb{R}^n))} O(|x - \hat{x}(t)|) \quad \text{as} \ x \to \hat{x}(t),
\]
and
\[
|r_3(\cdot,t)|_{L^2(\mathbb{R}^n)} = (|\dot{a}|_{L^\infty(0,T;W^{1,\infty}(\mathbb{R}^n))} + 1)O(\varepsilon^{\frac{3}{4} + \frac{1}{2}}), \quad \text{uniformly for a.e. } t \in (-T,T).
\] (7.7)

Further, it is easy to check that
\[
|r_1(\cdot,t)|_{L^2(\mathbb{R}^n)} = |r_2(\cdot,t)|_{L^2(\mathbb{R}^n)} = O(\varepsilon^{\frac{1}{4}}), \quad \text{uniformly for a.e. } t \in (-T,T).
\] (7.8)

By [19, Lemma 3.4] and the definition of \( \phi_\varepsilon \), for any \( t \in [-T,T] \) and a positive constant \( c_2 \),
\[
|\phi_\varepsilon(\cdot,t)|_{L^2(\mathbb{R}^n)} = O(\varepsilon) \quad \text{and} \quad |\phi_{\varepsilon,t}(\cdot,0)|_{L^2(\mathbb{R}^n)}, |\phi_{\varepsilon}(\cdot,0)|_{H_0^1(\mathbb{R}^n)} \leq c_2.
\] (7.9)

Similar to arguments in [19], by (7.3)-(7.9), one can easily get the following results:
1) \( \{\phi_\varepsilon\}_{\varepsilon > 0} \) given in (7.2) is a sequence of approximate solutions to (7.1) in the sense that
\[
\text{esssup}_{t \in (-T,T)} |(W\phi_\varepsilon)(\cdot,t) + F(\phi_\varepsilon)(\cdot,t)|_{L^2(\mathbb{R}^n)} = (|\dot{a}|_{L^\infty(0,T;W^{1,\infty}(\mathbb{R}^n))} + 1)O(\varepsilon^{\frac{1}{4}}), \quad \text{as} \ \varepsilon \to 0.
\]

2) The initial energy of \( \phi_\varepsilon \) is bounded below, i.e., \( |\phi_{\varepsilon,t}(\cdot,0)|_{L^2(\mathbb{R}^n)} \geq c_1 \) for a positive constant \( c_1 \), independent of \( \varepsilon \), and \( |\phi_{\varepsilon}(\cdot,0)|_{L^2(\mathbb{R}^n)} = O(\varepsilon) \), as \( \varepsilon \to 0 \).

3) The energy of \( \phi_\varepsilon \) is polynomially small off the generalized ray \( (\hat{x}(\cdot),p(\cdot)) \):
\[
\text{esssup}_{t \in (-T,T)} \int_{\mathbb{R}^n \setminus B_{1/4}(\hat{x}(t))} (|\phi_{\varepsilon,t}(x,t)|^2 + |\phi_\varepsilon(x,t)|^2 + |\nabla\phi_\varepsilon(x,t)|^2) \ dx = O(\varepsilon^2), \quad \text{as} \ \varepsilon \to 0.
\]

27
Therefore, by the definition of $\phi$

\[ |\phi_{\varepsilon, t}(\cdot, 0)|_{H^{-1}(\mathbb{R}^n)} = O(\varepsilon^{1/2}), \quad \text{as } \varepsilon \to 0. \tag{7.10} \]

Indeed, by (7.9),

\[ |\nabla \phi_{\varepsilon}(\cdot, 0)|_{H^{-1}(\mathbb{R}^n)} \leq |\phi_{\varepsilon}(\cdot, 0)|_{L^2(\mathbb{R}^n)} = O(\varepsilon). \]

By (7.5), we get that

\[ \psi_1^2(x, 0) = \nabla \psi(x, 0) \cdot \nabla \psi(x, 0) + O(|x - \hat{x}(0)|^3), \quad \text{as } x \to \hat{x}(0). \]

Hence,

\[ |\psi_1(x, 0)|^2 - |\nabla \psi(x, 0)|^2 = O(|x - \hat{x}(0)|^3), \quad \text{as } x \to \hat{x}(0). \tag{7.11} \]

Therefore, by the definition of $\phi_{\varepsilon}$ and [19, Lemma 3.4],

\[
|\phi_{\varepsilon, t}(\cdot, 0)|_{H^{-1}(\mathbb{R}^n)} = \left|e^{-\frac{\varepsilon}{2} \left( C(t) + \frac{\varepsilon}{2} \right) e^{i \psi(\cdot, 0)/\varepsilon} + \varepsilon^{\frac{1}{4}} A(0) e^{i \psi(\cdot, 0)/\varepsilon}} \right|_{H^{-1}(\mathbb{R}^n)} \leq \left|e^{\epsilon - \frac{C(t)}{2} e^{i \psi(\cdot, 0)/\varepsilon}} \right|_{L^2(\mathbb{R}^n)} + \left|e^{\epsilon - \frac{C(t)}{2} e^{i \psi(\cdot, 0)/\varepsilon}} \right|_{H^{-1}(\mathbb{R}^n)} + \left|e^{\epsilon - \frac{C(t)}{2} e^{i \psi(\cdot, 0)/\varepsilon}} \nabla \psi(\cdot, 0) \right|_{L^2(\mathbb{R}^n)} + \left|\nabla \phi_{\varepsilon}(\cdot, 0) \right|_{H^{-1}(\mathbb{R}^n)} \leq O(\varepsilon) + \left|\varepsilon^{-\frac{C(t)}{2}} c(0) \psi_1(x, 0) e^{i \psi(\cdot, 0)/\varepsilon} \right|_{L^2(\mathbb{R}^n)} + \left|\nabla \psi_1(x, 0) \right|_{L^2(\mathbb{R}^n)}.
\]

Now, it is sufficient to estimate the last term in the above inequality. Set

\[ A_\varepsilon = \left\{ x \in \mathbb{R}^n \mid |\psi_1(x, 0)| + |\nabla \psi(x, 0)| \leq \varepsilon \right\}. \]

Then

\[
|\varepsilon^{-\frac{C(t)}{2}} c(0) \psi_1(x, 0)|_{L^2(\mathbb{R}^n)}^2 - |\nabla \psi(x, 0)|_{L^2(\mathbb{R}^n)}^2 \leq \frac{1}{4} \varepsilon^{-\frac{C(t)}{2}} c(0)^2 \int_{A_\varepsilon} (|\psi_1(x, 0)|^2 - |\nabla \psi(x, 0)|^2) e^{i \psi(\cdot, 0)/\varepsilon} dx + \frac{1}{4} \varepsilon^{-\frac{C(t)}{2}} c(0)^2 \int_{\mathbb{R}^n \setminus A_\varepsilon} (|\psi_1(x, 0)|^2 - |\nabla \psi(x, 0)|^2) e^{i \psi(\cdot, 0)/\varepsilon} dx \leq \frac{1}{4} \varepsilon^{-\frac{C(t)}{2}} c(0)^2 \int_{A_\varepsilon} e^{2i \psi(x, 0)/\varepsilon} |dx| \leq \frac{1}{4} \varepsilon^{-\frac{C(t)}{2}} c(0)^2 \int_{\mathbb{R}^n \setminus A_\varepsilon} e^{2i \psi(x, 0)/\varepsilon} |dx|.
\]

It is easy to show that

\[ \varepsilon^{-\frac{C(t)}{2}} c(0)^2 \int_{A_\varepsilon} e^{2i \psi(x, 0)/\varepsilon} |dx| \leq \varepsilon^{-\frac{C(t)}{2}} c(0)^2 \int_{\mathbb{R}^n} e^{2i \psi(x, 0)/\varepsilon} |dx| = O(\varepsilon^2). \tag{7.13} \]

Also, by (7.11), we find that

\[ (|\nabla \psi(x, 0)|^2 - |\psi_1(x, 0)|^2)^2 = O(|x - \hat{x}(0)|^6), \quad \text{as } x \to \hat{x}(0). \]

Hence, by [19, Lemma 3.4] again,

\[
\varepsilon^{-\frac{C(t)}{2}} c(0)^2 \int_{\mathbb{R}^n \setminus A_\varepsilon} (|\psi_1(x, 0)|^2 - |\nabla \psi(x, 0)|^2) e^{2i \psi(x, 0)/\varepsilon} |dx| 
\leq \varepsilon^{-\frac{C(t)}{2}} c(0)^2 \int_{\mathbb{R}^n} (|\psi_1(x, 0)|^2 - |\nabla \psi(x, 0)|^2) e^{2i \psi(x, 0)/\varepsilon} |dx| = \varepsilon^{-\frac{C(t)}{2}} O(\varepsilon^{3+\frac{C(t)}{2}}) = O(\varepsilon). \tag{7.14}
\]
By (7.12)-(7.14), we get the desired estimate (7.10).

**Step 2.** We construct highly concentrated approximate solutions to the hyperbolic equation in a bounded domain. Consider the following hyperbolic equation:

\[
\begin{aligned}
\varphi_{tt} - \Delta \varphi + F(\varphi) &= 0 \quad \text{in } Q, \\
\varphi &= 0 \quad \text{on } \Sigma.
\end{aligned}
\]  
(7.15)

Assume that \((\hat{x}^-(\cdot), p^-(\cdot))\) is a generalized ray of \(W\) starting from \(\hat{x}^-(0) \in \Omega\) and arriving \(x_0 = \hat{x}^-(t_0) \in \partial \Omega\). By Step 1, one can construct a family of approximate solutions \(\{\phi^-_\varepsilon\}_{\varepsilon > 0}\) to the first equation of (7.15). However, \(\phi^-_\varepsilon\) may not satisfy the homogeneous Dirichlet boundary condition on \(\Sigma\). To solve this problem, we superpose \(\phi^-_\varepsilon\) with another approximate solution \(\phi^+_\varepsilon\). The latter is constructed from a ray \((\hat{x}^+(\cdot), p^+(\cdot))\), which reflects \((\hat{x}^-(\cdot), p^-(\cdot))\) at the boundary \(\partial \Omega\).

The key point is to select an approximate solution \(\phi^+_\varepsilon\) concentrated in a small neighborhood of the reflected ray \((\hat{x}^+(\cdot), p^+(\cdot))\), such that \(\phi^-_\varepsilon + \phi^+_\varepsilon\) satisfies approximately the homogeneous Dirichlet boundary condition.

Choose \((\hat{x}^+(\cdot), p^+(\cdot))\), such that

\[
\begin{aligned}
\hat{x}^+_t(t) &= -2p^+_t(t), \quad p^+_t(t) = 0, \\
\hat{x}^+(t_0) &= x_0, \quad p^+(t_0) = p^-(t_0) - 2\nu^\top(x_0)p^-(t_0)\nu(x_0).
\end{aligned}
\]

Assume that \(p^-(\cdot)\) is transversal to the boundary \(\partial \Omega\) at the time \(t_0\), i.e., \([p^-(t_0)]^\top \nu(x_0) \neq 0\). Denote by \(t_1 > t_0\) the instant, when the reflected ray arrives at \(\partial \Omega\), i.e., \(\hat{x}^+(t_1) \in \partial \Omega\). For any \(T^* \in (t_0, t_1)\), choose a cut-off function \(\varphi^- \in C_0^\infty(\mathbb{R}^{n+1})\), which equals to 1 identically in a neighborhood of the set \(\{(t, \hat{x}^-(t)) \mid t \in [0, t_0]\}\) with \(\text{supp} \varphi^- \subseteq B_{(T^* - t_0)/4}\{(t, \hat{x}^-(t)) \mid t \in [0, t_0]\}\). Then by Step 1, we may construct approximate solutions to (7.1) as follows:

\[
\phi^-_\varepsilon(x, t) = \varepsilon^{1-n/4} \varphi^- (x, t)c^-(t)e^{i\psi^-(x,t)/\varepsilon},
\]

where

\[
\psi^-(x, t) = [p^-(t)]^\top [x - \hat{x}^-(t)] + \frac{1}{2} [x - \hat{x}^-(t)]^\top M^-(t)[x - \hat{x}^-(t)]
\]

and \(M^-(t)\) is a complex symmetric matrix with positive definite imaginary part.

Next, we construct another approximate solution to (7.1) as follows:

\[
\phi^+_\varepsilon(x, t) = \varepsilon^{1-n/4} \varphi^+ (x, t)c^+(t)e^{i\psi^+(x,t)/\varepsilon},
\]  
(7.16)

which is concentrated in a small neighborhood of the reflected ray \((\hat{x}^+\cdot, p^+\cdot)\), such that

\[
|\phi^-_\varepsilon + \phi^+_\varepsilon|_{H^1(\partial \Omega \times (0,T^*))} = \mathcal{O}(\varepsilon^{1/2}).
\]  
(7.17)

In (7.16), \(\varphi^+ \in C_0^\infty(\mathbb{R}^{n+1})\) is a cut-off function, which identically equals to 1 in a neighborhood of the set \(\{(t, \hat{x}^+(t)) \mid t \in [t_0, T_1]\}\), such that \(\text{supp} \varphi^+ \subseteq B_{\min\{t_0, T_1\}}(0, T^*/4)\{(t, \hat{x}^+(t)) \mid t \in [t_0, T_1]\}\), and

\[
\psi^+(t, x) = [p^+(t)]^\top [x - \hat{x}^+(t)] + \frac{1}{2} [x - \hat{x}^+(t)]^\top M^+(t)[x - \hat{x}^+(t)].
\]

Here \(M^+(\cdot)\) and \(c^+(\cdot)\) are determined in a similar way in Step 1. The only difference is that \(c^+(t_0) = -c^-(t_0)\). Also, \(M^+(t)\) is determined by its initial \(M^+(t_0)\) and the reflected ray \((\hat{x}^+, p^+)\). Similar to the choices in [19, 36], it is easy to check that (7.17) holds.
Step 3. We construct approximate solutions \( \{ \Phi_\varepsilon \}_{\varepsilon > 0} \) to (7.15), such that the energies of them are concentrated in a neighborhood of a generalized ray \( \{(t, \hat{x}^2(t), p^2(t)) \mid t \in [s_j, s_{j+1}]\}_{j=0}^{m-1} \). To this end, choose a cut-off function \( \varrho^1 \in C^\infty_0(\mathbb{R}^{n+1}) \), which identically equals to 1 in a neighborhood of \( \{(t, \hat{x}^1(t)) \mid t \in [0, s_1]\} \) with \( \text{supp} \varrho^1 \subseteq B_{(s_{2j-1} - s_{2j-2})/4}\{(t, \hat{x}^1(t)) \mid t \in [0, s_1]\} \). By Step 2, we can find a function \( \phi^1_\varepsilon(x, t) = \varepsilon^{1-n/4} \varrho^1(x, t)c_1(x) e^{i\psi(x,t)/\varepsilon} \), such that

\[
\text{esssup}_{t \in (0, s_1)} \left( |(W \phi^1_\varepsilon)(\cdot, t) + F(\phi^1_\varepsilon)(\cdot, t)|_{L^2(\Omega)} = (|\hat{a}|_{L\infty(0,T;W^{1,\infty}(\Omega))} + 1)O(\varepsilon^{1/2}) \right),
\]

\[
|\phi^1_\varepsilon(x,0)|_{L^2(\Omega)} \geq c_1, \quad |\phi^1_\varepsilon(x,0)|_{L^2(\Omega)} \leq c_2,
\]

\[
|\phi^1_\varepsilon(x,0)|_{L^2(\Omega)} + |\phi^1_\varepsilon(x,0)|_{H^{-1}(\Omega)} = O(\varepsilon^{1/2}),
\]

\[
|\phi^1_\varepsilon|_{H^1(0,T;L^2(\Omega))} + |\phi^1_\varepsilon|_{H^1(-T,0;L^2(\Omega))} = O(\varepsilon^{1/2}).
\]

Next, choose a cut-off function \( \varrho^2 \in C^\infty_0(\mathbb{R}^{n+1}) \), which identically equals to 1 in a neighborhood of \( \{(t, \hat{x}^2(t)) \mid t \in [s_1, s_2]\} \) with \( \text{supp} \varrho^2 \subseteq B_{\min(s_1, s_3 - s_2)}/4\{(t, \hat{x}^2(t)) \mid t \in [s_1, s_2]\} \). By Step 2, we can find a \( \phi^2_\varepsilon(x, t) = \varepsilon^{1-n/4} \varrho^2(x, t)c_2(t) e^{i\psi(x,t)/\varepsilon} \), such that for \( S \in (0, s_2) \),

\[
\text{esssup}_{t \in (s_1, s_2)} \left( |(W \phi^2_\varepsilon)(\cdot, t) + F(\phi^2_\varepsilon)(\cdot, t)|_{L^2(\Omega)} = (|\hat{a}|_{L\infty(0,T;W^{1,\infty}(\Omega))} + 1)O(\varepsilon^{1/2}) \right),
\]

\[
|\phi^2_\varepsilon(x,0)|_{L^2(\Omega)} \geq c_1, \quad |\phi^2_\varepsilon(x,0)|_{L^2(\Omega)} \leq c_2,
\]

\[
|\phi^2_\varepsilon(x,0)|_{L^2(\Omega)} + |\phi^2_\varepsilon(x,0)|_{H^{-1}(\Omega)} = O(\varepsilon^{1/2}),
\]

\[
|\phi^2_\varepsilon|_{H^1(0,T;L^2(\Omega))} + |\phi^2_\varepsilon|_{H^1(-T,0;L^2(\Omega))} = O(\varepsilon^{1/2}).
\]

Further, for \( j = 3, \ldots, m \), choose a cut-off function \( \varrho^j \in C^\infty_0(\mathbb{R}^{n+1}) \), which identically equals to 1 in a neighborhood of \( \{(t, \hat{x}^j(t)) \mid t \in [s_{j-1}, s_j]\} \) with \( \text{supp} \varrho^j \subseteq B_{\min(s_{j-1} - s_{j-2}, s_{j+1} - s_j)}/4\{(t, x^j(t)) \mid t \in [s_{j-1}, s_j]\} \). Similarly, we can find a \( \phi^j_\varepsilon(x, t) = \varepsilon^{1-n/4} \varrho^j(x, t)c_{j}(t) e^{i\psi(x,t)/\varepsilon} \), such that for \( S \in (0, s_j) \), if \( j = 3, \ldots, m - 1 \), and \( S \in (0, T) \), if \( j = m \).

Now, write \( \Phi_\varepsilon = \sum_{j=1}^{m} \phi^j_\varepsilon \). Then it is easy to show that

\[
\text{esssup}_{t \in (0, T)} \left( |(W \Phi_\varepsilon)(\cdot, t) + F(\Phi_\varepsilon)(\cdot, t)|_{L^2(\Omega)} = (|\hat{a}|_{L\infty(0,T;W^{1,\infty}(\Omega))} + 1)O(\varepsilon^{1/2}) \right),
\]

\[
|\Phi_\varepsilon|_{H^1(\Omega \times (0,T))} + |\Phi_\varepsilon|_{H^1(\Omega \times (-T,0))} = O(\varepsilon^{1/2}),
\]

\[
|\Phi_\varepsilon(x,0)|_{L^2(\Omega)} \geq c_1, \quad |\Phi_\varepsilon(x,0)|_{L^2(\Omega)} \leq c_2,
\]

\[
|\Phi_\varepsilon(x,0)|_{L^2(\Omega)} + |\Phi_\varepsilon(x,0)|_{H^{-1}(\Omega)} = O(\varepsilon^{1/2}),
\]

\[
|\Phi_\varepsilon|_{H^1(0,T;L^2(\Omega))} + |\Phi_\varepsilon|_{H^1(-T,0;L^2(\Omega))} = O(\varepsilon^{1/2}).
\]
Set $\Phi_\varepsilon(x,t) = \Phi_\varepsilon(x,t) + \Phi_\varepsilon(x,-t)$, for $(x,t) \in Q$. Then $\{\Phi_\varepsilon\}_{\varepsilon > 0}$ satisfies

\[
\begin{align*}
\text{ess sup}_{t \in (0,T)} \left| (W\Phi_\varepsilon)(\cdot, t) + F(\Phi_\varepsilon)(\cdot, t) \right|_{L^2(\Omega)} & = (|\hat{a}|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + 1)\mathcal{O}(\varepsilon^\frac{1}{2}), \\
|\Phi_\varepsilon|_{H^1(\partial\Omega \times (0,T))} & = \mathcal{O}(\varepsilon^\frac{1}{2}), \\
|\Phi_\varepsilon(\cdot, 0)|_{L^2(\Omega)} & \geq c_1, \quad |\Phi_\varepsilon, t(\cdot, 0)|_{L^2(\Omega)} |\Phi_\varepsilon(\cdot, 0)|_{H^1_0(\Omega)} \leq c_2, \\
|\Phi_\varepsilon(\cdot, 0)|_{L^2(\Omega)} + |\Phi_\varepsilon, t(\cdot, 0)|_{H^{-1}(\Omega)} & = \mathcal{O}(\varepsilon^\frac{1}{2}), \\
|\Phi_\varepsilon|_{H^1(0,T;L^2(\omega))} & = \mathcal{O}(\varepsilon^\frac{1}{2}).
\end{align*}
\]

(7.18)

**Step 4.** We construct a family of solutions $\{\phi_\varepsilon\}_{\varepsilon > 0}$ to (7.15). To this aim, let $\phi_\varepsilon = \Phi_\varepsilon + v_\varepsilon$, where $v_\varepsilon$ solves

\[
\begin{align*}
Wv_\varepsilon + F(v_\varepsilon) & = -W\Phi_\varepsilon - F(\Phi_\varepsilon) \quad \text{in } Q, \\
v_\varepsilon & = -\Phi_\varepsilon \quad \text{on } \Sigma, \\
v_\varepsilon(x,0) = 0, \quad v_{\varepsilon, t}(x,0) = 0 \quad \text{in } \Omega.
\end{align*}
\]

It is easy to see that

\[
\max_{t \in [0,T]} |(v_\varepsilon(\cdot, t), v_{\varepsilon, t}(\cdot, t))|_{H^1(\Omega) \times L^2(\Omega)} \leq C(|W\Phi_\varepsilon + F(\Phi_\varepsilon)|_{L^1(0,T;L^2(\Omega))} + |\Phi_\varepsilon|_{H^1(\partial\Omega \times (0,T)))}).
\]

This, together with the first two conclusions in (7.18), implies that

\[
\max_{t \in [0,T]} |(v_\varepsilon(\cdot, t), v_{\varepsilon, t}(\cdot, t))|_{H^1(\Omega) \times L^2(\Omega)} \leq C(|\hat{a}|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + 1)\varepsilon^\frac{1}{2}.
\]

(7.19)

By (7.18) and (7.19), we get that

\[
\begin{align*}
|\phi_{\varepsilon, t}(\cdot, 0)|_{L^2(\Omega)} & \geq c_1, \quad |\phi_{\varepsilon, t}(\cdot, 0)|_{L^2(\Omega)}, |\phi_\varepsilon(\cdot, 0)|_{H^1_0(\Omega)} \leq c_2, \\
|\phi_\varepsilon(\cdot, 0)|_{L^2(\Omega)} + |\phi_{\varepsilon, t}(\cdot, 0)|_{H^{-1}(\Omega)} & = \mathcal{O}(\varepsilon^\frac{1}{2}), \\
|\phi_\varepsilon|_{H^1(0,T;L^2(\omega))} & = (|\hat{a}|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + 1)\mathcal{O}(\varepsilon^\frac{1}{2}).
\end{align*}
\]

Hence, $\{\phi_\varepsilon\}_{\varepsilon > 0}$ are the desired family of solutions in Lemma 6.3. \square

**References**

[1] N. Anantharaman, M. Léautaud and F. Macià, *Wigner measures and observability for the Schrödinger equation on the disk*, Invent. Math., 206 (2016), 485–599.

[2] V. Barbu, T. Havârneanu, C. Popa and S. S. Sritharan, *Exact controllability for the magneto-hydrodynamic equations*, Comm. Pure Appl. Math., 56 (2003), 732–783.

[3] C. Bardos, G. Lebeau and J. Rauch, *Un exemple d’utilisation des notions de propagation pour le contrôle et la stabilisation de problèmes hyperboliques*, Rend. Sem. Mat. Univ. Politec. Torino, Special Issue (1988), 11–31.

[4] C. Bardos, G. Lebeau and J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim., 30 (1992), 1024–1065.

[5] R. Bellman, *Dynamic Programming*, Princeton Univ. Press, Princeton, New Jersey, 1957.
[6] R. Bellman, *Dynamic programming and modern control theory*, Proceedings of the International Congress of Mathematicians 1966, Vol. I, Moscow, U.S.S.R., 1968, 65–82.

[7] J. Bourgain, N. Burq and M. Zworski, *Control for Schrödinger operators on 2-tori: rough potentials*, J. Eur. Math. Soc., 15 (2013), 1597–1628.

[8] N. Burq and M. Zworski, *Geometric control in the presence of a black box*, J. Amer. Math. Soc., 17 (2004), 443–471.

[9] F. H. Clarke, *Nonsmooth analysis and optimization*, Proceedings of the International Congress of Mathematicians 1978, Vol. II, Helsinki, Finland, Academia Scientarum Fennica, Helsinki, 1980, 847–853.

[10] J.-M. Coron, *Control and Nonlinearity*, Mathematical Surveys and Monographs, 136. American Mathematical Society, Providence, RI, 2007.

[11] J.-M. Coron, *On the controllability of nonlinear partial differential equations*, Proceedings of the International Congress of Mathematicians 2010, Vol. I, Hyderabad, India, Hindustan Book Agency, New Delhi, 2010, 238–264.

[12] J.-M. Coron and P. Lissy, *Local null controllability of the three-dimensional Navier-Stokes system with a distributed control having two vanishing components*, Invent. Math., 198 (2014), 833–880.

[13] M. G. Crandall and P.-L. Lions, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., 277 (1983), 1–42.

[14] B. Dehman and S. Ervedoza, *Dependence of high-frequency waves with respect to potentials*, SIAM J. Control Optim., 52 (2014), 3722–3750.

[15] Ju. V. Egorov, *Necessary conditions for optimal control in Banach spaces*, Mat. Sb. (N. S.), 64 (1964), 79–101.

[16] I. Ekeland, *Problèmes variationnels non convexes*, Proceedings of the International Congress of Mathematicians 1978, Vol. II, Helsinki, Finland, Academia Scientarum Fennica, Helsinki, 1980, 855–858.

[17] H. O. Fattorini, *The maximum principle for nonlinear nonconvex systems in infinite-dimensional spaces*, Distributed Parameter Systems, Lect. Notes Control Inf. Sci., Vol. 75, Springer, Berlin, 1985, 162–178.

[18] H. Frankowska, *Optimal control under state constraints*, Proceedings of the International Congress of Mathematicians 2010, Vol. IV, Hyderabad, India, 2010, 2915–2943.

[19] X. Fu, J. Yong and X. Zhang, *Controllability and observability of a heat equation with hyperbolic memory kernel*, J. Differential Equations, 247 (2009), 2395–2439.

[20] A. V. Fursikov and O. Yu. Imanuvilov, *Controllability of Evolution Equations*, Lecture Notes Series 34, Seoul National University, Seoul, Korea, 1996.

[21] L. Hörmander, *Linear Partial Differential Operators*. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.
[22] O. Yu. Imanouilov, *Controllability of evolution equations of fluid dynamics*, Proceedings of the International Congress of Mathematicians 2006, Vol. III, Madrid, Spain, Eur. Math. Soc., Zürich, 2006, 1321–1338.

[23] O. Yu. Imanuvilov and M. Yamamoto, *Carleman inequalities for parabolic equations in Sobolev spaces of negative order and exact controllability for semilinear parabolic equations*, Publ. Res. Inst. Math. Sci., 39 (2003), 227–274.

[24] K. Itô and K. Kunisch, *Novel concepts for nonsmooth optimization and their impact on science and technology*, Proceedings of the International Congress of Mathematicians 2010, Vol. IV, Hyderabad, India, 2010, 3061–3090.

[25] R. E. Kalman, *On the general theory of control systems*, Proceedings of the First IFAC Congress 1960, Vol. 1, Moscow, Butterworth, London, 1961, 481–492.

[26] J. Le Rousseau, G. Lebeau, P. Terpolilli and E. Trélat, *Geometric control condition for the wave equation with a time-dependent observation domain*, Anal. PDE, 10 (2017), 983–1015.

[27] J. Le Rousseau and L. Robbiano, *Local and global Carleman estimates for parabolic operators with coefficients with jumps at interfaces*, Invent. Math., 183 (2011), 245–336.

[28] G. Lebeau, *Contrôle analytique. I. Estimations a priori*, Duke Math. J., 68 (1992), 1–30.

[29] X. Li and Y. Yao, *Maximum principle of distributed parameter systems with time lags*, Distributed Parameter Systems, Lect. Notes Control Inf. Sci., Vol. 75, Springer, Berlin, 1985, 410–427.

[30] X. Li and J. Yong, *Necessary conditions for optimal control of distributed parameter systems*, SIAM J. Control Optim., 29 (1991), 895–908.

[31] X. Li and J. Yong, *Optimal Control Theory for Infinite Dimensional Systems*, Systems & Control: Foundations & Applications, Birkhäuser, Boston, Inc., Boston, MA, 1995.

[32] J.-L. Lions, *Sur la théorie du contrôle*, Actes du Congrès International des Mathématiciens 1974, Vol. 1, Vancouver, Canada, Canadian Mathematical Congress, 1975, 139–154.

[33] J.-L. Lions, *Exact controllability, stabilization and perturbations for distributed systems*, SIAM Rev., 30 (1988), 1–68.

[34] X. Liu, Q. Lü and X. Zhang, *Work in progress*.

[35] Q. Lü, X. Zhang and E. Zuazua, *Null controllability for wave equations with memory*, J. Math. Pures Appl., 108 (2017), 500–531.

[36] F. Macià and E. Zuazua, *On the lack of observability for wave equations: A Gaussian beam approach*, Asymptot. Anal., 32 (2002), 1–26.

[37] L. S. Pontryagin, *Optimal processes of regulation* (in Russian), Proceedings of the International Congress of Mathematicians 1958, Edinburgh, UK, Cambridge Univ. Press, New York, 1960, 182–202.

[38] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mischenko, *The Mathematical Theory of Optimal Processes*, Interscience Publishers John Wiley & Sons, Inc., New York-London, 1962.
[39] J. Ralston, *Solutions of the wave equation with localized energy*, Comm. Pure Appl. Math., 22 (1969), 807–823.

[40] L. Robbiano, *Carleman estimates, results on control and stabilization for partial differential equations*, Proceedings of the International Congress of Mathematicians 2014, Vol. IV, Seoul, Korea, 2014, 897–919.

[41] D. L. Russell, *Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions*, SIAM Rev., 20 (1978), 639–739.

[42] H. J. Sussmann, *Analytic stratifications and control theory*, Proceedings of the International Congress of Mathematicians 1978, Vol. II, Helsinki, Finland, Academia Scientarum Fennica, Helsinki, 1980, 865–871.

[43] G. Wang, L. Wang, Y. Xu and Y. Zhang, *Time Optimal Control Of Evolution Equation*, Progress in Nonlinear Differential Equations and Their Applications: Subseries in Control, Vol. 92, Birkhäuser, Springer International Publishing AG, part of Springer Nature, 2018.

[44] N. Wiener, *Cybernetics or Control and Communication in the Animal and the Machine*, The MIT Press, Cambridge, Massachusetts, 1948.

[45] X. Zhang, *A unified controllability/observability theory for some stochastic and deterministic partial differential equations*, Proceedings of the International Congress of Mathematicians 2010, Vol. IV, Hyderabad, India, 2010, 3008–3034.

[46] E. Zuazua, *Control and numerical approximation of the wave and heat equations*, International Congress of Mathematicians, Vol. III, Madrid, Spain, Eur. Math. Soc., Zürich, 2006, 1389–1417.

[47] E. Zuazua, *Controllability and observability of partial differential equations: some results and open problems*, Handbook of Differential Equations: Evolutionary Differential Equations, Vol. 3, Elsevier Science, 2006, 527–621.

[48] E. Zuazua, *A remark on the observability of conservative linear systems*, Contemp. Math., 577 (2012), 47–59.