Numerical-Statistical and Analytical Study of Asymptotics for the Average Multiplication Particle Flow in a Random Medium

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Abstract—It is well known that, under rather general conditions, the particle flux density in a multiplying medium is asymptotically exponential in time $t$ with a parameter $\lambda$, i.e., with an exponent $\lambda t$. If the medium is random, then $\lambda$ is a random variable, and the time asymptotics of the average number of particles (over medium realizations) can be estimated in some approximation by averaging the exponent with respect to the distribution of $\lambda$. Assuming that this distribution is Gaussian, an asymptotic “superexponential” estimate for the average flux expressed by an exponential with the exponent $t\lambda + t^2\sigma^2/2$ can be obtained in this way. To verify this estimate in a numerical experiment, a procedure is developed for computing the probabilistic moments of $\lambda$ based on randomizations of Fourier approximations of special nonlinear functionals. The derived new formula is used to study the COVID-19 pandemic.

Keywords: statistical modeling, time asymptotics, random medium, particle flow, COVID-19

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1. PRELIMINARIES

We study the time asymptotics of the average flow of scattering and multiplying particles in a random medium. For this purpose, an averaging procedure is developed for corresponding analytical and numerical statistical estimates (i.e., Monte Carlo estimates) obtained for deterministic realizations of the medium.

It is well known that, under rather general conditions, the particle flux density $\Phi(t, r, v)$ in a multiplying medium occupying a domain $D$ is asymptotically exponential in time $t$ (see, e.g., [1]):

$$\Phi(t, r, v) \sim e^{\lambda t} \Phi(r, v), \quad t \to \infty.$$  

The time constant $\lambda$ is the leading eigenvalue of the corresponding homogeneous stationary kinetic equation

$$(v, \nabla \Phi) + (\sigma + \tau/v)\Phi = \sigma_s \int w_s(v' \to v, r) \Phi' dv' + \sigma_f \int w_f(v' \to v, r) \Phi' dv'. \quad (1.1)$$

Here, $\Phi \equiv \Phi(r, v)$ is the stationary flux density (characteristic function of Eq. (1.1)), $\Phi' \equiv \Phi(r, v')$; $\sigma \equiv \sigma(r, v)$ is the total cross section (attenuation coefficient); $\sigma = \sigma_s + \sigma_f + \sigma_c$, where $\sigma_s$ is the scattering cross section, $\sigma_f$ is the fission (multiplication) cross section ($w_s$ and $w_f$ are the corresponding phase functions), and $\sigma_c$ is the absorption cross section; $v$ is the number of particles leaving the multiplication point; $v = v\omega$ is the velocity vector, where $\omega$ is the unit direction vector and $v = |v|; \text{ and } r$ is a spatial point.

To construct and study Monte Carlo algorithms, as a mathematical model of the transport process corresponding to Eq. (1.1) (see, e.g., [2]), we use a terminating (with probability 1) homogeneous Markov chain of states that are the phase points of successive “collisions of a particle with substance elements,” i.e., points at which the particle velocity changes instantaneously. This Markov chain $x_0, x_1, ..., x_N$ is considered in the state space $X = R \times V \times T$ of coordinates, velocities, and time, i.e., $x_n = (r_n, v_n, t_n)$, where $r_n$
is the point of the \( n \)th collision, \( \mathbf{v}_n \) is the velocity immediately before the collision, and 
\( t_n = t_{n-1} + |\mathbf{r}_{n-1} - \mathbf{r}_n|/|\mathbf{v}_{n-1}| \) is the lifetime of the colliding particle. The considered chain is determined by 
the distribution density \( f(x) \) of the initial collision \( x_0 \) and by the transition density \( k(x',x) \) from the state \( x \) to \( x' \); moreover, it is assumed that 
\[
\int_k(x',x)dx = q(x') \leq 1 - \delta, \quad \delta > 0, \tag{1.2}
\]
i.e., the chain terminates with probability 1 and the average number of transitions is finite. Condition (1.2) 
is satisfied, for example, for a bounded system (see [2, 3]). A substochastic kernel \( k(x',x) \) is obtained from 
the transport process characterization (see [2, 3]), which also determines Eq. (1.1). The ratios \( \sigma_s(x)/\sigma(x) \), 
\( \sigma_f(x)/\sigma(x) \), and \( \sigma_a(x)/\sigma(x) \) are the probabilities of scattering, fission, and absorption immediately 
after a collision at the phase point \( x \), and the distribution density of the free path \( \ell \) from \( r \) to \( r' \) is 
\( p(\ell) = \sigma(r(\ell)) \exp(-\tau_{op}(\ell)) \), where \( \tau_{op}(\ell) \) is the optical path (see [2, 3]). It can be seen that the collision 
density \( \varphi(x) = \sum_{n=0}^{\infty} \varphi_n(x) \), where \( \varphi_n(x) \) is the distribution density of collisions of index \( n \), represents a 
Neumann series for the second-kind integral equation 
\[
\varphi = K \varphi + f, \quad f = \varphi_0,
\]
where \( K \) is an integral operator with kernel \( k(\cdot, \cdot) \). As was indicated, for example, in [2, 3], despite the 
generalized multipliers \( \delta(v'/|v'| - (r - r')/|r - r'|) \) and \( \delta(t' - t - |r - r'|/|v'|) \) present in the kernel \( k(x',x) \), the 
operator \( K \) can be treated as acting from \( L_1(\mathcal{X}) \) to \( L_1(\mathcal{X}) \), even more so that all functions in the problem 
are nonnegative. Under condition (1.2), we have \( \|K\|_1 < 1 \) and, hence, the spectral radius of the operator 
satisfies \( \rho(K) < 1 \).

Used usually in transport theory, the radiation intensity \( \Phi(x) \) (particle flux density) is connected with 
the collision density by the relation 
\[
\rho = \rho(K) < 1, \quad \text{where} \quad K_{\rho} \text{ is an operator with kernel } k^2(x',x)/p(x',x), \text{ and } f^2/f_0 \in L_1(\mathcal{X}), \text{ then } 
D_{\xi} < +\infty. \text{ The random variable } \xi \text{ is called the collision estimator for the functional } J_h.
\]

Note that the new velocity is usually modeled with probability \( \sigma_s(x')/\sigma(x') \) according to the indicatrix 
\( w_s(v' \rightarrow v; r') \) and with probability \( \sigma_f(x')/\sigma(x') \) according to \( w_f(v' \rightarrow v; r') \). To construct a weighted mod-
ification, the index of the modeling type (i.e., of the collision type) is added to the set of coordinates of 
the state space (see [2, 3]).

2. ESTIMATION OF \( \lambda \)

In this section, we consider two approaches for estimating the time constant \( \lambda \) of particle multiplication,
which determines the asymptotics \( C \exp(\lambda t) \) of the function \( \Phi \) in a deterministic medium (see, 
e.g., [1]). It is well known (see [1]) that, if \( \lambda/v \) (\( v \) is the particle velocity) is added to the absorption cross 
section, then the system becomes critical, i.e., \( \lambda \) is the eigenvalue of Eq. (1.1).
2.1. Differentiating Eq. (1.1) in operator form $k$ times with respect to $\tau$ yields

$$L\Phi^{(k)} + \left(\sigma + \frac{3}{v}\right)\Phi^{(k)} = S\Phi^{(k)} + S_f\Phi^{(k)} - \frac{k}{v} \Phi^{(k-1)}$$

(2.1)
or $R\Phi^{(k)} = -(k/v)\Phi^{(k-1)}$. Therefore,

$$\Phi^{(k)} = (-1)^{(k-1)} k! \left(R^{-1}\right)^{(k-1)} \Phi^{(i)}.$$  

In the case of standard spectral properties of the resolvent operator (see [1]), we have $k\Phi^{(k-1)}/\Phi^{(k)} \rightarrow \tau_i$ as $k \rightarrow \infty$, which yields the following limit form of Eq. (2.1):

$$L\Phi^* + \left(\sigma + \frac{3}{v}\right)\Phi^* = S\Phi^* + S_f\Phi^* - \frac{\tau_i}{v} \Phi^*.$$  

Thus,

$$\lambda = \tau + \tau_i \quad \text{and} \quad kJ^{(k-1)}/J^{(k)} \rightarrow \lambda - \tau.$$  

(2.2)

To estimate the derivatives of the linear functional $J$ with respect to $\tau$, we model a chain of collisions with parameters $\tau$ and $v$; moreover, $Q_n = \tilde{v}_n \exp(-\tau t_n)$, where $t_n$ is the total time required for the particle to move to $r_n$ and $\tilde{v}_n$ is a weight taking into account fissions for the simplest process modification without branching (see [2]). It follows that

$$\frac{\partial^k Q_n}{\partial \tau^k} = Q_n^{(k)} = \tilde{v}_n (-t_n)^k \exp(-\tau t_n).$$

Since $\tau/v$ is added to the absorption cross section, the quantity $\|K_\sigma\|$ modified by the substitutions $\sigma_f \mapsto \sigma_f + \sigma_v$ and $v \mapsto v \sigma_f/(\sigma_f + \sigma_v)$ can be made smaller than unity by choosing a sufficiently large $\tau$.

The unbiasedness of the corresponding estimators $\xi^{(k)}$ of $J^{(k)}$ and the finiteness of their variances for $\|K_\sigma\| < 1$ can easily be shown by applying the vector approach; the conditions that the variances are finite are less restrictive if branching is modeled (see [3]).

The considered approach was formulated in [4] and was elaborated in [5]. Its shortcoming is that it involves expensive computations of multiple derivatives $\xi^{(k)}$, whose variances grow rather significantly with increasing $k$ in real-world problems. Nevertheless, by additionally computing the derivatives of estimates of $\lambda$ and using the values of a piecewise constant random function $\sigma(\tau)$, rather accurate test estimates of $E\lambda$ and $D\lambda$ for a model spherically symmetric system were obtained in [5]; they will be used in Section 4 below.

Below, we formulated a more universal approach for estimating $\lambda$ based on Green’s function with respect to the time parameter $t$.

2.2. Let $\varphi_0(x; r_0, v_0)$ be the collision density (in $x$) produced by a single collision at the point $(r_0, v_0, 0)$, i.e., for $f(x) = \delta(r - r_0)\delta(v - v_0)\delta(t)$.

The functional

$$J(t) = \int_{R \times V} \varphi_0(r, v, t)h(r, v)d\varphi d\nu \quad \forall f \in L_4(X), \quad h \in L_\infty(R \times V),$$

can be represented in the form (see [5])

$$J(t) = \int_{R \times V} \int_{0}^{\infty} f(r_0, v_0, t - \tau)F(r_0, v_0, \tau)d\tau_0 d\nu \nu d \tau,$$

where

$$F(r_0, v_0, t) = \int_{R \times V} \varphi_0(r, v, t; r_0, v_0)h(r, v)d\tau d\nu.$$  

Assume that $f(r, v, -t) \equiv 0$ and, hence, $F(r, v, -t) \equiv 0$ for $t > 0$.

In what follows, the symbol $f^{(m)}_t$ denotes the $m$th derivative of the function $f$ with respect to $t$. 
Lemma 1 (see [2]). Suppose that the point \((\mathbf{r}_0, \mathbf{v}_0)\) is distributed at \(t_0 = 0\) with density \(f_0(\mathbf{r}, \mathbf{v})\), conditions (1.3) are satisfied,

\[
\left| f^{(m)}(\mathbf{r}, \mathbf{v}, t)/f_0(\mathbf{r}, \mathbf{v}) \right| < C_0 < +\infty, \quad \rho(K_\mu) < 1,
\]

and \(F(x) < C < +\infty\). Then \(J^{(m)}(t) = E_x^{(m)}\), where

\[
E_x^{(m)} = \sum_{k=0}^{N} Q_k h(\mathbf{r}_k, \mathbf{v}_k) f^{(m)}(\mathbf{r}_0, \mathbf{v}_0, t-t_k)/f_0(\mathbf{r}_0, \mathbf{v}_0), \quad Q_0 \equiv 1,
\]

where \(Q_0 = 1\) for \(m = 0, 1, \ldots, n\).

Note that, in [2], an estimate of type \(\xi\) (i.e., \(E_x^{(m)}\) for \(m = 0\)) is used to solve nonstationary optical sensing problems with actual radiation sources. In this paper, following [5], we use \(\xi\) and \(\xi'\) with an auxiliary density \(f(x)\) to estimate the parameter \(\lambda\) for the exponential time asymptotics of the particle flux.

It is well known, that under rather general conditions in the case of a source localized at the point \((\mathbf{r}_0, \mathbf{v}_0, 0)\), the following asymptotic relation holds as \(t \to \infty\) (see [1]):

\[
F(\mathbf{r}, \mathbf{v}, t) \sim C(\mathbf{r}, \mathbf{v}) e^{\lambda t}, \quad C(\mathbf{r}, \mathbf{v}) < C_0 < +\infty,
\]

where \(\lambda\) is the leading eigenvalue of Eq. (1.1). Specifically, these conditions hold for one-speed particle transport in a bounded medium with a source density rapidly decreasing with time. Relying on what was said above, we can construct an estimate of \(\lambda\).

Theorem 1. If

\[
\int f^{(n)}(\mathbf{r}, \mathbf{v}, t) e^{-\lambda t} dt < +\infty, \quad n = 0, 1,
\]

relation (2.4) holds, and the conditions of Lemma 1 are satisfied, then

\[
J(t) = \lambda \quad \text{as} \quad t \to +\infty.
\]

Proof. Direct integration combined with applying Lemma 1 and relations (2.4), (2.5) yields

\[
J(t) = C e^{\lambda t}[1 + \epsilon(t)], \quad J(t) = C_1 e^{\lambda t}[1 + \epsilon_1(t)],
\]

where \(\epsilon(t)\) and \(\epsilon_1(t)\) are the error terms. Integrating \(J(t)\) within the limits \((\tau_0, +\infty)\) as \(\tau_0 \to \infty\) for \(\lambda < 0\), we obtain

\[
J(t) = C e^{\lambda t}[1 + \epsilon(t)], \quad \text{i.e.,} \quad C_1 = \lambda C.
\]

In the case \(\lambda > 0\), the same result is obtained by introducing additional absorption with a coefficient \(\sigma^{(0)} \lambda / \nu\). This completes the proof of Theorem 1.

3. ESTIMATION OF THE AVERAGE PARTICLE FLUX IN A STOCHASTIC MEDIUM

3.1. To simplify the presentation, we consider one-speed particle transport. It is assumed that \(\sigma = \sigma(r)\) is a random field and the ratios \(\sigma_1/\sigma\) and \(\sigma_2/\sigma\) and the scattering and fission phase functions are fixed.

If \(h_\lambda(r) = I_D(r)/\sigma(r)\), where \(I_D\) is the indicator of the domain \(D\), then the functional \(J(t, \sigma)\) is the total particle flux in \(D\) at the given time \(t\). Assuming that the random variable \(\lambda(\sigma)\) is Gaussian and the passage to the limit \(J(t, \sigma) \to t \to +\infty, C(\sigma) e^{\lambda(\sigma)t}\) is uniform (in \(\sigma\)), we can estimate the asymptotics of the function \(EJ(t, \sigma) = J\) as \(t \to +\infty\):

\[
J(t) \equiv C e^{\lambda t/[\sigma_1/\sigma + \sigma_2/\sigma]}
\]

where \(\sigma_A = E\lambda(\sigma)\) and \(\sigma^2 = \text{D}\lambda(\sigma)\) are the expectation and variance of the corresponding variable. Additionally, we assume that the multipliers \(C(\sigma)\) and \(e^{\lambda(\sigma)t}\) in the asymptotics are weakly correlated; hence, \(C \approx EC(\sigma)\). The assumptions made are natural, for example, for a small random perturbation of a multi-
layered (multicomponent) medium under the condition $d \ll |\alpha|$. Their effectiveness has been checked by test computations (see Section 4 below). Using integral formula (2.3.15), no. 11, from [6], we obtain

$$J_r = C \exp \left( \frac{d^2}{2} t^2 + at \right).$$  

(3.1)

Therefore, we can assume that

$$\frac{d \ln J_r}{dt} = d^2 t + a.$$  

(3.2)

Note that formulas (3.1) and (3.2) can be used as a basis for numerical studies of particular problem instances. Determined by formula (3.1), the growth law for the average number of particles can be called “superexponential.” A more general and practically convenient definition of the superexponential property can be associated with an increase in the growth coefficient $n(t + \Delta t)/n(t)$ with increasing $t$.

In real-world problems, $-\infty < \lambda_1 < \lambda(\sigma) < \lambda_2 < +\infty$; therefore, the resulting asymptotic representation seems to approximate only in some interval $0 < T_1 \leq t \leq T_2 < +\infty$. Accordingly, an additional numerical study is required.

3.2. Practically important are the quantities $E \lambda(\sigma)$ and $D \lambda(\sigma)$. The conceptually simplest (direct) Monte Carlo technique for their estimation is to obtain fairly accurate estimates of $\lambda(\sigma = \sigma)$ for a sample $\sigma$ of sufficiently large size. However, this technique may be extremely expensive for realistic medium models and transport processes. For this reason, a randomized algorithm is constructed in [7] using the following relation ($\lambda(\sigma)$ is replaced by $\lambda_n$):

$$E[p' \sigma / \lambda(\sigma)] = \lambda_n = E \left[ \lambda_n \sum_{x=0}^{n} \left( J_{n+1} \left( J_0 - \tilde{J}_0 \right) \right) \right],$$

where $\tilde{J}_0$ is a preliminary statistical estimator of $EJ_0(\sigma)$, which can be made deterministic in the given case. The simplest (elementary) unbiased randomized estimator of $\lambda_n$ based on Lemma 1 is constructed by realizing $n + 1$ conditionally independent particle trajectories for a chosen value of $\sigma$: $\lambda_n = \tilde{\lambda}_n$, where

$$\tilde{\lambda}_n = \xi(\Omega_{n+1}, \sigma) \sum_{x=0}^{n} \left( J_{n+1} \left( J_0 - \tilde{J}_0 \right) \right) \xi(\sigma_{x+1}, \sigma) - \tilde{J}_0).$$

(3.3)

A similar decomposition is used to construct an estimator of $E\lambda(\sigma)$ (see [7]). It can be seen that $\tilde{\lambda}_0$ in (3.3) is a statistical estimator of $J'/J$. In this case, $J'$ and $J$ are simultaneously estimated using the double randomization method (see [3]). For definiteness, the result is stated as the following lemma.

**Lemma 2.** Under the conditions of Lemma 1,

$$\frac{J'}{J} = \frac{d \ln J}{dt} = \frac{E \xi(\Omega, \sigma)}{E \xi(\sigma)} = \frac{\tilde{J}'}{\tilde{J}},$$

(3.4)

where $\tilde{J}$ and $\tilde{J}'$ are the corresponding statistical estimators produced by the double randomization method.

For each realization of the medium, we can construct only one particle trajectory by using (2.3) for $m = 0; 1$. If the complexity of modeling the medium is substantially higher than the complexity of modeling the trajectory, then it is reasonable to construct $r$ conditionally independent trajectories for each medium realization. An optimal value of $r$ is estimated by analogy with the parameter of the splitting method (see [9]).

Note that the variance of $\tilde{J}'/\tilde{J}$ is estimated from above with the help of linearization of the fraction using the formula

$$D \left( \frac{J'}{J} \right) \leq \left( \sqrt{D J'} + \frac{|E J'|}{(E J')^3} \sqrt{D J} \right)^2.$$

(3.5)

3.3. For a medium with a random piecewise constant density $\rho$, an algorithm for estimating $E \lambda$ and $D \lambda$ based on the Taylor series expansions of $\lambda(\sigma)$ with respect to $\rho$, ..., $\rho_n$ at the point $(E \rho_1, ..., E \rho_m)$ is constructed in [5]. More specifically, truncated series up to the first and second orders inclusive are used
for the variance and the mean, respectively. Assuming that the random variables \( \{ p_i \} \) are independent, we have

\[
E \lambda = \lambda (E p_1, \ldots, E p_m) + \frac{1}{2} \sum_{i=1}^{m} \left( \frac{\partial^2 \lambda}{\partial p_i} \right)^2 D p_i,
\]

The variance estimate can be improved using second-order terms. In the example considered below, this improvement was found insignificant. Thus, estimate (3.6) is reduced to estimating the first and second derivatives of \( \lambda \) with respect to \( p_i \).

After replacing \( \lambda \) by an iterative resolvent approximation (see Subsection 2.1), such estimates are determined by the mixed derivatives

\[
\frac{\partial^j J^{(k-1)}_i}{\partial p^j_i}, \quad \frac{\partial^j J^{(k)}_i}{\partial p^j_i}.
\]

Quantities (3.7) can be computed as indicated in [5] by applying the Monte Carlo method with auxiliary weights corresponding to variations in the parameters \( \tau \) and \( \{ p_i \} \), \( i = 1, 2, \ldots, n \).

The results thus obtained can be independently verified using more expensive computations based on (3.3).

4. NUMERICAL RESULTS

4.1. To perform test computations, we considered one-speed particle transport in a spherically symmetric medium with a piecewise constant random density \( \rho = \rho(r) \) within a ball of radius \( R = 7.72043 \) with macroscopic sections \( \sigma^{(0)}_0, \rho \sigma^{(0)}_v, \) and \( \rho \sigma^{(0)}_f \), where

\[
\sigma^{(0)}_0 = 1, \quad \sigma^{(0)}_v = 0.97, \quad \sigma^{(0)}_f = 0.03, \quad v = 2.5, \quad v = 1.
\]

To construct a realization of the medium, the ball is divided into \( m \) spherical layers of identical volume. In each layer, the random variable \( \rho \equiv \rho_i \) is drawn independently and uniformly on the interval \([1 - \epsilon, 1 + \epsilon]\).

To construct an efficient Monte Carlo algorithm based on Lemma 2, into the formulated model, we introduced absorption with a constant nonrandom coefficient \( \sigma_i / v \), which led to the substitution \( \lambda \mapsto \lambda - \sigma_i / v \forall \sigma \). Note that this technique is universal and can significantly improve the efficiency of the weighted method, eliminating the necessity of branching modeled trajectories (see below).

By using the transport equation (see [2, 3]), we can make the substitution

\[
\sigma_f \mapsto \sigma_f + \sigma_v, \quad \nu \mapsto \nu \mathcal{S}/(\sigma_f + \sigma_v),
\]

In the numerical experiment, the transport process was modeled with constants \( \sigma^*_v = \sigma_v, \quad \sigma^*_f = \sigma_f + \sigma_v \), and \( \nu^* = 1 \). The auxiliary weights were determined by the formula \( Q_n = [\nu \sigma_f/(\sigma_f + \sigma_v)]^n \), where \( n_i \) is the number of fissions preceding the given collision (see [5]). We used the value \( \sigma_v = 0.059 \), for which \( \nu \mathcal{S}/(\sigma_f + \sigma_v) < 1 \) and, thus, \( \rho(K_\nu) < 1 \forall \sigma \) (see [2, 3]). The distribution density of the first collisions was specified as

\[
f(r, t) = 2 t \exp(-2t)f_0(r), \quad t > 0, \quad r = |r| < R,
\]

where \( f_0(r) = C \sin(\alpha(p)r)/r \) is the improved diffusion approximation of the spatial characteristic function for \( \sigma = 1 \) and \( \alpha(l) = 0.3739866 \) (see [8]). To estimate \( E \lambda \) and \( D \lambda \), we also set \( h(r, \nu) = h_0(r)/\mathcal{S}(r) \), where \( h_0(r) = \sin(\alpha(p)r)/r \) and (see [8])

\[
\alpha(p) = [\pi(\sigma_v + \nu \sigma_f)/R(\sigma_v + \nu \sigma_f) + 0.71044] \]

Moreover, \( J_x = (\Phi, h_0 f_0 / f_0) \), i.e., a functional of the particle flux was computed. The computations showed that these functional parameters of the algorithm significantly improve the convergence in (2.6) as compared with \( f_\Delta(r) \equiv \left( \frac{4}{3} \pi R^3 \right)^{-1} \) and even with the variant in which \( f(r, t) \) is defined by formula (4.1) and
It is easy to see that the conditions of Lemma 1 hold for the above-formulated characteristics of the computational model.

The corresponding estimates of $E\lambda$ and $\sqrt{D\lambda}$ are presented in the first two columns of Table 1. They were obtained using distributed computations at the Siberian Supercomputer Center of the Siberian Branch of the Russian Academy of Sciences. These estimates were determined on stabilizing two significant digits with increasing $t$ and $n$ taking into account the statistical error. The final results were obtained by averaging the estimates produced at the times $t = 17, 18, 19, 20$ for $n = 4$ and $m = 2$ and for $n = 2$ and $m = 6$. This corresponds to improved properties of the estimates as $m \to \infty$. The mean-square deviations are given as errors of these estimates.

For the problem under consideration, formulas (3.6) were also implemented in [5]. In view of, they have the form

$$y(t) = \frac{d \ln J_t}{dt}.$$  

By using point estimates of this function, the corresponding regression estimates $\hat{\alpha}$, $\hat{\beta}$ (for $15 \leq t_i \leq 20$) of the coefficients of the linear approximation $y(t) = \hat{\beta} t + \hat{\alpha}$ can be computed by applying the well-known formulas (see [10])

$$\hat{\alpha} = \frac{\sum_{i=1}^{n} y_i \left( \sum_{i=1}^{n} t_i^2 - t_i \sum_{j=1}^{n} t_j \right)}{n \sum_{i=1}^{n} t_i^2 - \left( \sum_{i=1}^{n} t_i \right)^2} = \sum_{i=1}^{n} A_i y_i,$$

$$\hat{\beta} = \frac{\sum_{i=1}^{n} y_i \left( nt_i - \sum_{j=1}^{n} t_j \right)}{n \sum_{i=1}^{n} t_i^2 - \left( \sum_{i=1}^{n} t_i \right)^2} = \sum_{i=1}^{n} B_i y_i.$$  

In the considered problem, the random variables $\{y_i\}$ are positively correlated, so

$$D\alpha = \sum_{i=1}^{n} A_i^2 D y_i < D\hat{\alpha} < \left( \sum_{i=1}^{n} |A_i| \sqrt{D y_i} \right)^2,$$

Table 1. Estimated $E\lambda$ and $\sqrt{D\lambda}$

| $m$ | Monte Carlo method, formula (3.3), $\varepsilon = 0.3$ | Formula (4.2) $\varepsilon = 0.3$ | Formula (4.2) $\varepsilon = 0.1$ |
|-----|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------|
|     | $E\lambda$ | $\sqrt{D\lambda}$ | $E\lambda$ | $\sqrt{D\lambda}$ | $E\lambda$ | $\sqrt{D\lambda}$ |
| 1   | $-0.00104 \pm 0.00001$ | $0.0143 \pm 0.0025$ | $-0.00102$ | $0.014$ | $0.00011$ | $0.00047$ |
| 6   | $0.000224 \pm 0.000003$ | $0.0065 \pm 0.0009$ | $0.00023$ | $0.0066$ | $0.000026$ | $0.00022$ |
An analysis of the resulting estimated variances with the incomplete dependence of $\alpha$ taken into account showed that the mean-square errors of $\bar{\alpha}$ and $\bar{\beta}$ can be estimated using the formulas

$$
\sigma(\bar{\alpha}) = 2\sqrt{D_{\alpha}} \quad \text{and} \quad \sigma(\bar{\beta}) = 2\sqrt{D_{\beta}}.
$$

The analytical and regression estimates for the coefficients of the linear approximation to the function $d \ln J_t/dt$ are given in Table 2. The required values of $a = E\lambda$ and $d^2 = D\lambda$ were determined using formulas (4.2) (see Table 1 and Table 7 in [3]).

Thus, we conclude that the analytical estimate for the exponent of the exponential time asymptotics of the particle flux is satisfactory in the interval $15 \leq t \leq 20$. For $\varepsilon = 0.1$, the statistical estimates of the coefficients agree better with their analytical values, which can be explained by the better Gaussian property of the distribution of $\lambda(\sigma)$ for this variant of the problem, according to the theory of small perturbations.

**Fig. 1.** Estimated logarithmic derivative for (a) $\varepsilon = 0.1$ and (b) $\varepsilon = 0.3$ and the regression approximation (straight line).
To conclude, we note that the results obtained above have a wider range of applications. They show that the average number of particles of an arbitrary nature (e.g., microorganisms) in the case of their distributed multiplication in a random medium can have a superexponential growth rate (see Section 3). This corresponds to an increasing growth coefficient of the numbers of particles at fixed equidistant times, i.e., to an increase in the ratio . However, if such an increase is observed experimentally, this may suggest a distributed nature of the multiplication “source.” Specifically, according to WHO statistics (see [11]), such behavior was exhibited by the COVID-19 pandemic developing around the world from March 9, 2020, to March 21, 2020. The corresponding number of cases (by days) is approximated within a 2% error by a formula of type (3.1), namely,
\[ n_i \approx 109577 \exp\{0.002965(i - 9)^2 + 0.037(i - 9)\}, \quad i = 9, 10, \ldots, 21. \] (4.3)

Individual numbers of cases are compared in Table 3.

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