Zel’ dovich states with very small mass and charge in nonlinear electrodynamics coupled to gravity

O. B. Zaslavskii

Astronomical Institute of Kharkov V.N. Karazin National University,
35 Sumskaya St., Kharkov, 61022, Ukraine

It is shown that in non-linear electrodynamics (in particular, Born-Infeld one) in the framework of general relativity there exist ”weakly singular” configurations such that (i) the proper mass $M$ is finite in spite of divergences of the energy density, (ii) the electric charge $q$ and Schwarzschild mass $m \sim q$ can be made as small as one likes, (iv) all field and energy distributions are concentrated in the core region. This region has an almost zero surface area but a finite longitudinal size $L = 2M$. Such configurations can be viewed as a new version of a classical analogue of an elementary particle.

PACS numbers: 04.70.Bw, 04.20.Dw, 04.20.Gz

I. INTRODUCTION

There is a long-standing problem of Coulomb divergences in classical electrodynamics. In a flat space-time, this drawback can be remedied in non-linear electrodynamics where corresponding localized soliton-like solutions replace point-like solutions of Maxwell theory [1]. These solutions represent extended objects which are free of divergences and can be considered as candidates to the role of classical analogues of an elementary particles. Meanwhile, in general relativity the situation changes. There exists a general theorem [2], [3], [4], [5] that states that a regular center is impossible, provided the nonlinear Lagrangian behaves like the Maxwell one in the limit of weak fields. This difficulty can be avoided by making Lagrangians inhomogeneous with non-Maxwellian behavior near the centre [6] that

*Electronic address: zaslav@ukr.net
looks, however, artificial. Although the nonzero global electric charge forbids the existence of globally regular solutions, some properties of electrically charged configurations deserve study.

First, it turns out that although globally regular solutions are impossible, one can sometimes achieve the finiteness of the proper mass in spite of the singular centre that represent so-called "weakly singular" solutions (see, e.g., textbook [11]). Second, it turns out that among such solutions there exists a particular class with rather unusual interesting properties: their Schwarzschild mass $m$ almost vanishes, the electric charge $q$ so does but the proper mass $M$ is still finite and non-zero. In this limit, all non-trivial field configuration is localized near the centre under the sphere of an almost vanishing radius but with finite longitudinal size. As a result, we obtain a classical analogue of an elementary particle which is in a sense combines features of a point-like and extended objects. For an external observer it reveals itself as almost "nothing" and, in this respect, is similar to a friedmon [7]. However, in contrast to it, there is no "other Universe" inside, all game is developed in the same space which is static everywhere.

In the present paper we describe the solutions with aforementioned properties. We do not pretend, of course, for comparison of the parameters of the obtained configurations with experimental data. Our goal is rather methodical: to demonstrate some unusual features that general relativity brings into non-linear electrodynamics and which have no analogues in flat space-times. Nonetheless, we would like to remind that, since, say, for an electron $m \ll q$ in geometrical units, this corresponds to a naked singularity that can be considered at least as an additional motivation for studying distributions which are singular (or smooth but become singular in some limits). The example of studies of such a kind in which a singular configuration was considered as a model of elementary particle within general relativity and linear electrodynamics, can be found in [8].

The configurations discussed in our paper are obtained on the basis of the energy distribution considered by Zel’dovich a long time ago [9] in a quite different context connected with relativistic astrophysics (the collapse of relativistic star of a small mass in general relativity). The similar distributions were also discussed in the context of black hole thermodynamics [10].
II. ZEL’DOVICH’S CONFIGURATIONS AND ULTIMATE GRAVITATION

MASS DEFECT

Let us consider the spherically-symmetric metric

$$ds^2 = -dt^2 \exp(2\gamma) + dr^2 \exp(2\alpha) + \exp(2\beta)d\omega^2, \quad d\omega^2 = d\theta^2 + \sin^2 \theta d\phi^2.$$  \hspace{1cm} (1)

It follows from the 00 component of the Einstein equations that

$$\exp(-2\alpha) = 1 - \frac{2m(r)}{r},$$  \hspace{1cm} (2)

$$m(r) = 4\pi \int_0^r d\bar{r}\bar{r}^2 \rho, \quad m(\infty) \equiv m,$$  \hspace{1cm} (3)

where $\rho$ is the energy density, $m$ has the meaning of the Schwarzschild mass. We assume that there is no horizon, so that $r > 2m(r)$, $\exp(-2\alpha) > 0$ for all $r > 0$.

The proper mass

$$M = 4\pi \int_0^\infty dr \exp(\alpha)\rho r^2.$$  \hspace{1cm} (4)

Now let us consider the situation in which for $r \leq r_0$ the energy density has the form

$$\rho = \frac{b}{8\pi r^2},$$  \hspace{1cm} (5)

where $b$ is a numerical coefficient, and for $r > r_0$ $\rho = 0$. Then, for $r \leq r_0$

$$\exp(-2\alpha) = 1 - b$$  \hspace{1cm} (6)

where we assume $b < 1$ to ensure $\exp(-2\alpha) > 0$. For $r \leq r_0$, the mass function reads

$$m(r) = \frac{br}{2}, \quad m = \frac{br_0}{2}.$$  \hspace{1cm} (7)

In doing so, the proper mass

$$M = \frac{br_0}{2\sqrt{1-b}}.$$  \hspace{1cm} (8)

It can be rewritten as

$$M = \frac{bL}{2}, \quad L = \frac{r_0}{\sqrt{1-b}}.$$  \hspace{1cm} (9)

where $L$ is the proper length that characterizes the longitudinal size of the system from the center to $r_0$. It is also worth noting a simple bound that follows from

$$M \leq \frac{L}{2}.$$  \hspace{1cm} (10)
In general, $r_0$ and $b$ are two independent parameters. However, if one arranges the special relationship between them, it is possible to obtain non-trivial realization of the gravitational mass defect. This was shown by Zel’dovich for the ultrarelativistic fermi gas of nucleons with the equation of state

$$\rho \sim n^{4/3}$$

where $n$ is the number density of nucleons. Their total number is equal to

$$N = 4\pi \int dr r^2 \exp(-\alpha)n.$$  \hspace{1cm} (12)

Then, it follows from (3), (4), (5) and (12) that

$$r_0 \sim N^{2/3}(1-b)^{1/3}, \quad m \sim N^{2/3}(1-b)^{1/3}, \quad M \sim N^{2/3}(1-b)^{-1/6}.$$  \hspace{1cm} (13)

Thus, taking the baryon number $N$ to be fixed we obtain that for $1-b \ll 1$ the quantities $r_0$ and $m \approx r_0^2$ can be made as small as one wishes.

Zel’dovich’s goal was to show that one can always rearrange the distribution of a fixed number of nucleons to achieve an almost vanishing gravitational mass $m$. Meanwhile, it remained unnoticed in [9] that for such a configuration the proper mass $M \to \infty$. This is interesting by itself since supplies us with the manifestation of the ultimate gravitational mass defect: the Schwarzschild mass vanishes but the proper mass diverges. Earlier, another example of such a situation was reported in [12] but it relied heavily on non-stationary geometries. Meanwhile, now the space-time is pure static, so one obtains one more type of the ultimate gravitational mass defect.

### III. MODIFICATION OF ZEL’DOVICH CONFIGURATION

Being interesting for astrophysical applications [13], the configurations suggested in [9] are not suitable for our purposes to find an analogue of classical elementary particle in nonlinear electrodynamics. The proper mass for configurations considered in [9] diverges as we saw above. In principle, this can be remedied if, instead of $N$ (which does not have sense at all for the problem under discussion), we just keep $M$ fixed by hand. However, this is insufficient since the density has somewhat different profile in all relevant electrodynamics theories (see
next section). This leads to modification of the original Zel’dovich’s distribution. Namely, let we have two typical regions: region I where \( r \to 0 \) (more exactly, \( r < r_0 \) where \( r_0 \) characterizes the scale on which density changes, \( r_0 \to 0 \) in final formulas) and region II where \( r \gtrsim r_0 \). In region I we suppose the validity of the asymptotic form of the type \( [5] \). However, we take into account in \( \rho \) not only the divergent term but the first small correction in expansion with respect to inverse powers of \( r \). More specifically, we suppose that there are no terms of \( r^{-1} \) (as it happens in physically relevant applications discussed below), so instead of \( [5] \) now

\[
\rho \approx \rho_1 - \rho_2, \quad \rho_1 = \frac{b}{8\pi r^2}, \quad \rho_2 = \frac{1}{8\pi \lambda}
\]

where \( \lambda > 0 \) is a constant. The role of \( r_0 \) is played by the parameter \( \lambda \), \( \rho_2 \ll \rho_1 \). We will see below that such a dependence arises, in particular, in nonlinear Born-Infeld electrodynamics.

In region II we suppose that \( \rho = \varepsilon f(r) \) where the parameter \( \varepsilon \to 0 \) and \( f \) is the bounded function. In particular, the role of \( \varepsilon \) can be played by an electric charge. Then, for \( \varepsilon \to 0 \) it is region I which gives the main contribution to the mass.

Then, for \( r \to 0 \)

\[
\exp(-2\alpha) \approx 1 - b + \frac{r^2}{3\lambda}.
\]

Now we want to examine under what conditions the proper energy \( M \) converges in the situation when \( r_0 \) and \( m \) can be made arbitrarily small. We assume that at infinity \( \rho \) falls off rapidly enough, so possible divergences are connected with the lower limit \( r = 0 \) of the integral \( [14] \) only. If \( b \neq 1 \), the integral trivially converges. However, when formally \( b \to 1 \), in general \( M \to \infty \). In doing so, the contribution from the constant term in \( [14] \) is finite and proportional to \( r_0 \). Let us evaluate the dominant contribution \( M_1 \) which stems from the first term in \( [14] \). It behaves like

\[
M_1 \approx \frac{\sqrt{3\lambda}}{2} [D + \frac{1}{2} \ln \frac{1}{\lambda(1 - b)}]
\]

where the exact value of the constant \( D \) is irrelevant for our purposes. If \( \lambda \) remains finite, \( M_1 \to \infty \) when \( b \) approaches the value \( b = 1 \). However, let us try to adjust two transitions \( \lambda \to 0 \) and \( b \to 1 \) in \( [16] \) in such a way that the quantity \( M_1 \) remain finite. It is indeed possible if

\[
1 - b \approx \frac{B}{\lambda} \exp(-\frac{4a}{\sqrt{3\lambda}})
\]
where $B$ and $a$ are constants. In other words, if $\lambda \to 0$ and $b \to 1$ along the curve (17) with an arbitrary value of constant $B$, the mass $M$ remains finite. More precisely, it follows from the substitution of (17) back into (16) that $M_1 = a + O(\sqrt{\lambda})$. The choice of the constant $B$ results in the corrections in $M_1$ absorbed by $D$ as it is clear from (16). However, in the limit $\lambda \to 0$ any such corrections are multiplied by the small factor proportional to $\sqrt{\lambda}$, so in this limit it does not affect the value of $M_1 \to a$. To be accurate, we must check that the quantity $M_2$ does not generate new divergences in this limit. Indeed, the core region I gives the contribution $M_2 \sim \frac{r^2}{\sqrt{\lambda}} \sim \sqrt{\lambda} \frac{r^2}{\lambda}$ represents a product of two small factors and is, obviously, finite and arbitrarily small. As a result, we obtain the configuration for which $M \to const \neq 0, \frac{m}{M}$ becomes arbitrarily small. Now we will see that such a situation can arise in nonlinear electrodynamics.

For all configurations under discussion the proper distance $L$ behaves in the way similar to $M$ since according to (4) both quantities differ from each other by the factor $\rho r^2$ which is finite according to (5). Moreover, for $r_0 \ll M, 1 - b \ll 1$ we obtain the relation $M = \frac{L}{2}$ that agrees with (9).

\section*{IV. NONLINEAR ELECTRODYNAMICS: BASIC FORMULAS}

In this section, we consider general basic formulas of nonlinear electrodynamics needed for what follows. In doing so, we follow the presentation of [11]. Consider the self-gravitating system with Lagrangian

\[ L = -\frac{R}{16\pi} - \Phi(I), \]

where $R$ is the Riemann curvature and we use the units with $G = c = 1$. Here, the field invariant

\[ I = -F_{\alpha\beta}F^{\alpha\beta} \]

where $F_{\alpha\beta}$ is the tensor of the electromagnetic field. Then, the stress-energy tensor takes the form

\[ T_{\mu}^{\nu} = diag(-\rho, p_r, p_\perp, p_\perp), \]

\[ (20) \]
where
\[ \rho = -p_r = 2I\Phi_I - \Phi, \quad \Phi_I = \frac{d\Phi}{dI}, \quad p_\perp = \Phi \] (21)

It follows from 00 and 11 Einstein equations that (up to the additive constant) that for our system
\[ \gamma = -\alpha. \] (22)

The Maxwell equations read
\[ \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\beta} \left( \sqrt{-g} F^{\alpha\beta} \Phi_I \right) = 0. \] (23)

We assume that the system is static and spherically-symmetric. Then, the 0-component of (23) together with (22) gives us that
\[ \Phi_I F^{0r} = \frac{q}{16\pi r^2} \] (24)

where \( q \) is an electric charge (for definiteness, in what follows we choose \( q > 0 \)). Now, \( I = -2F_{0r}F^{0r} = 2 (F^{0r})^2 > 0 \). With our sign convention, \( \Phi > 0 \). By the integration of (24) we obtain that
\[ \Phi = \frac{\sqrt{2}q}{16\pi} \int_0^I \frac{dI}{r^2\sqrt{I}} \] (25)

where we took into account that at \( r \to \infty \) \( I \to 0 \). Integrating by parts and taking into account (21), one obtains that
\[ \rho = \frac{q\sqrt{2}}{4\pi} \int_r^\infty \frac{d\tilde{r}\sqrt{\tilde{I}}}{\tilde{r}^3}. \] (26)

Thus, if \( I \leq \infty \) in the centre where \( r \to 0 \), the energy density has the universal behavior \( \rho \approx \frac{4}{r^2} + const \) where \( A \sim q \). This is just the behavior typical of the Zel’dovich gravitational configurations \([9]\) or their generalization \([13]\). However, in general, for \( q \neq 0 \), the Schwarzschild mass remains nonzero. Only in the situation when \( q \) becomes arbitrarily small and some other parameters are fine-tuned (see details below) one does indeed obtain the configuration with arbitrarily small \( m \) and finite \( M \).

V. BORN-INFELD ELECTRODYNAMICS

Let the system have the electrodynamic Lagrangian
$$\Phi_{BI} = \frac{1}{8\pi\lambda}(1 - \sqrt{1 - \lambda I}). \quad (27)$$

Then, applying formulas of the previous section, we obtain

$$I = \frac{2q^2}{r^4 + 2q^2\lambda}, \quad (28)$$

$$\rho = \frac{1}{8\pi\lambda r^2}(\sqrt{r^4 + 2\pi\lambda q^2} - r^2). \quad (29)$$

The value of the Schwarzschild mass

$$m \approx 0.84q. \quad (30)$$

Thus, for \( r \gg (2\pi\lambda q^2)^{1/4} \equiv r_0 \) we have the standard asymptotics

$$\rho \approx \frac{q^2}{8\pi r^4} \quad (31)$$

typical of the linear electrodynamics whereas for \( r \ll r_0 \) the density has the form \( [14] \) with

$$b = \sqrt{\frac{\lambda_{\min}}{\lambda}}, \lambda_{\min} \equiv q\sqrt{2}. \quad (32)$$

If \( \lambda \to \lambda_{\min}, r_0 \sim q \sim \lambda \) in accordance with general consideration in Sec. III. In general, in this limit the proper energy \( M \to \infty \). However, according to \( [17] \), it remains finite provided \( q \to 0 \) and the parameters are fine-tuned according to

$$\sqrt{\lambda} - 2^{1/4}\sqrt{q} \approx \frac{2^{-1/4}B}{3\sqrt{q}}\exp(-\frac{2^{7/4}M_1}{\sqrt{3q}}) \quad (33)$$

\( M_1 \) and \( B \) being finite fixed quantities.

It is worth noting that in this limit, for any intermediate value of \( r \) the density has the form \( [31] \) and, thus, tends to zero uniformly. In the core region \( r < r_0 \) it diverges but the size of the core \( r_0 \) shrinks in this limit. As a result, the Schwarzschild mass \( m \) becomes arbitrarily small but \( M \) remains finite nonzero quantity. This is true in spite of the fact that the invariant \( I(0) \sim q^{-1} \) diverges in this limit as well as the invariant \( \Phi(0) \).
VI. NONLINEAR SCHRÖDINGER LAGRANGIAN

Let now

\[ \Phi_{Sch} = -\frac{1}{8\pi\lambda} \ln(1 - \frac{\lambda I}{2}), \quad (34) \]

\[ \rho = \frac{1}{8\pi \lambda r^2} \left( \frac{4\lambda q^2}{f} + r^2 \ln(\frac{2r^2}{f}) \right), \quad I = \frac{8q^2}{f^2}, \quad f = \sqrt{r^4 + 4\lambda q^2 + r^2}. \quad (35) \]

Thus, at \( r \to 0 \) the energy density does indeed has the asymptotic form \( (14) \) with

\[ b = \frac{2q}{\sqrt{\lambda}} \equiv \sqrt{\frac{\lambda_{\text{min}}}{\lambda}}. \quad (36) \]

This example has some specific additional features as compared to the previous ones because of the logarithmic term that needs the modification of the described scheme but this difference is not crucial. Then,

\[ \exp(2\gamma) \approx 1 - b - r^2 \frac{\ln r^2}{3\lambda}, \quad m \approx 0.8q. \quad (37) \]

The potentially dangerous term in the proper mass has the form (we omit for simplicity numerical coefficients and give rough estimate)

\[ M \sim \sqrt{\lambda} |\ln |\ln x||, \quad x = \frac{1 - b}{|\ln(1 - b)|} \quad (38) \]

Thus, the finiteness of \( M \) is possible provided the parameters are adjusted in such a way that

\[ \frac{1 - b}{|\ln(1 - b)|} \sim \exp[-\exp(\frac{M}{\sqrt{\lambda}})]. \quad (39) \]

VII. DISCUSSION AND CONCLUSION

The configurations under discussion represent counterpart of those in [9] for a system with long-range forces. They share a rather unusual interesting property. Namely, when parameters of the system are fine-tuned in certain way, both the Schwarzschild mass \( m \) and charge \( q \sim m \) approach zero as closely as one likes but the proper mass \( M \) remains nonzero. It is worth stressing that one cannot simply put \( m = 0 \) or \( q = 0 \) (or both) from
the very beginning - this would have completely destroyed such configurations. In doing so, the imprint on space-time from the electric and gravitation fields disappears outside. All remaining fields are concentrated in the core region. In doing so, the core region has an almost zero surface area but with a longitudinal size $L = 2M$, so that a finite string-like object is obtained. Thus, although our particle-like configuration is localized within a sphere with an arbitrarily small size, it cannot be made point-like in agreement with discussion of other classical analogs of an elementary particle [14].

The configurations in question realize the strong gravitational mass defect since the ratio $\frac{m}{M}$ can be made as small as one wishes. It is also worth noting that, when the charge diminishes, the value of the field invariant in the centre $I(0) \sim q^{-1}$ grows unbound.

To some extent, the configurations under discussion resemble a semi-closed world [7] but, instead of the whole Universe inside a "particle", now we have a weak singularity in the same space which almost disjoins from the outer world.

Our consideration was pure classical. As is known, account for quantum effects places severe limitations on the relevance of Zel’dovich’s configurations in relativistic astrophysics [15], [16]. It would be of interest to carry out similar analysis for the systems considered in the present work. It is also of interest to extend the approach under discussion to the non-spherical (say, axially-symmetric) case when non-trivial static solutions with $m \to 0$ are known to exist [17]. A separate issue is the analysis of rotating configurations.

Also, other approaches to the problem of singularities of a point-like charge including quantum field theory effects can be of interest (see, e.g., the recent work [18]).

I thank K. A. Bronnikov for useful comments.

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