REGULARITY OF SOLUTIONS TO TIME FRACTIONAL DIFFUSION EQUATIONS

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Abstract. We derive some regularity estimates of the solution to a time fractional diffusion equation by using the Galerkin method. The regularity estimates partially unravel the singularity structure of the solution with respect to the time variable. We show that the regularity of the weak solution can be improved by subtracting some particular forms of singular functions.

1. Introduction. This paper considers the following time fractional diffusion problem:

\[
\begin{aligned}
\partial_t^\alpha (u - u_0) - \Delta u &= f \quad \text{in } \Omega \times (0, T), \\
u &= 0 \quad \text{on } \partial \Omega \times [0, T], \\
u &= u_0 \quad \text{on } \Omega \times \{0\},
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^d \) is a bounded domain with \( C^2 \) boundary, \( 0.5 < \alpha < 1 \), \( u_0 \in H_0^1(\Omega) \), and \( f \in L^2(\Omega) \) with \( \Omega := \Omega \times (0, T) \). Above \( \partial_t^\alpha : L^1(\Omega) \rightarrow D'(\Omega) \), the Riemann-Liouville fractional differential operator, is defined by \( \partial_t^\alpha := \partial_t I_{0+}^{1-\alpha} \), where \( \partial_t \) denotes the generalized differential operator with respect to the time variable \( t \), and \( I_{0+}^{1-\alpha} : L^1(\Omega) \rightarrow L^1(\Omega) \) is given by

\[
(I_{0+}^{1-\alpha}v)(x,t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} v(x,s) \, ds, \quad (x,t) \in Q_T,
\]

for all \( v \in L^1(\Omega) \), with \( \Gamma(\cdot) \) denoting the standard Gamma function.

The main goal of this paper is to investigate the regularity properties of the solution to problem (1.1). Let us first introduce some recent related works. In [2] Eidelman and Kochubei constructed and investigated fundamental solutions for Cauchy time-fractional diffusion problems. For the generalized time-fractional diffusion equation, Luchko [10, 11] established a maximum principle, and used this maximum principle to prove the uniqueness of the generalized solution; moreover, Luchko discussed the existence of the generalized solution of problem (1.1) with \( f = 0 \). Furthermore, Luchko [12] extended the maximum principle to the fractional diffusion equation, and discussed the properties of the solution to a one-dimensional time-fractional diffusion equation. Under the basic condition that \( f = 0 \) or \( u_0 = 0 \), Sakamoto and Yamamoto [18] discussed the uniqueness and regularity of the weak
solution to problem (1.1) for $0 < \alpha < 1$. Zacher [23] proposed a De Giorgi-Nash theorem for time fractional diffusion equations. Assuming the force function $f$ to be weighted Hölder continuous, Mu et al. [16] proved the unique existence of solutions to three types of time-fractional diffusion equations, and derived some new regularity estimates. For the abstract time-fractional evolution equations, Zhang and Liu [24] established sufficient conditions for the existence of mild solutions for fractional evolution differential equations by using a new fixed point theorem. Wang et al. [21] obtained the existence and uniqueness of mild solutions and classical solutions to abstract linear and semilinear fractional Cauchy problems with almost sectorial operators. For more works, we refer the reader to [13, 14, 15, 3, 1, 7, 6] and the references therein.

The motivation of paper is to provide regularity estimates for numerical analysis, and, to the best of our knowledge, for problem (1.1) there is no available regularity result for numerical analysis so far. Moreover, because of the nonlocal property of the operator $\partial_t^\alpha$, both the storage and computing costs for a numerical approximate of a time fractional diffusion problem are much more expensive than that of a corresponding standard diffusion problem. Therefore, designing high accuracy algorithms for time fractional diffusion problems is of great practical value [8, 25]. But whether a high accuracy algorithm works or not depends mainly on the regularity of the solution with respect to the time variable $t$. Unfortunately, it is well known that even $u_0$ and $f$ are regular enough, the solution to (1.1) has singularity in time. So it is natural to investigate the singularity structure of the solution in time, which not only is of theoretical value, but also can provide insight into developing efficient numerical algorithms.

In this paper, we employ the Galerkin method to investigate the regularity properties of the weak solution to problem (1.1). Compared to the work aforementioned, our regularity estimates are more applicable to numerical analysis. Furthermore, our regularity estimates demonstrate that, by subtracting some particular forms of singularity functions, we can improve the regularity of the solution with respect to the time variable $t$, which partially unravel the singularity structure of the solution in time.

The rest of this paper is organized as follows. In (2) we introduce some Sobolev spaces, the fractional integration and derivative operators, and some fundamental properties of these operators. In (3) we investigate the regularity properties of the solution of a fractional ordinary equation. Finally, in (4) we use the results developed in the previous sections to discuss the regularity of the solution to problem (1.1).

2. Preliminaries. We start by introducing some Sobolev spaces. Let $0 \leq \beta < \infty$. Define [20]

$$H^\beta(\mathbb{R}) := \left\{ v \in L^2(\mathbb{R}) \left| \left(1 + |\cdot|^2\right)^{\beta/2} \mathcal{F}v(\cdot) \in L^2(\mathbb{R}) \right. \right\},$$

and endow this space with the following norm:

$$\|v\|_{H^\beta(\mathbb{R})} := \|v\|_{L^2(\mathbb{R})} + |v|_{H^\beta(\mathbb{R})} \quad \text{for all } v \in H^\beta(\mathbb{R}),$$

where

$$|v|_{H^\beta(\mathbb{R})} := \left(\int_\mathbb{R} |\xi|^{2\beta} |\mathcal{F}v(\xi)|^2 \, d\xi\right)^{\frac{1}{2}}.$$
Above and throughout, $F: S'(\mathbb{R}) \to S'(\mathbb{R})$ denotes the well-known Fourier transform operator, where $S'(\mathbb{R})$, the dual space of $S(\mathbb{R})$, is the space of tempered distributions. For $-\infty \leq a < b \leq \infty$, define

$$H^\beta(a, b) := \{ v|_{(a,b)} \mid v \in H^\beta(\mathbb{R}) \},$$

and equip this space with the following norm:

$$\|v\|_{H^\beta(a, b)} := \inf_{\tilde{v} \in H^\beta(\mathbb{R})} \|\tilde{v}\|_{H^\beta(\mathbb{R})} \quad \text{for all } v \in H^\beta(a, b).$$

In addition, we use $H_0^\beta(a, b)$ to denote the closure of $\mathcal{D}(a, b)$ in $H^\beta(a, b)$, where $\mathcal{D}(a, b)$ denotes the set of $C^\infty$ functions with compact support in $(a, b)$.

**Remark 2.1.** It is well known that if $0 < \beta < 0.5$, then $H^\beta(a, b)$ coincides with $H_0^\beta(a, b)$. Also, there exists another equivalent definition of the space $H^\beta(0, T)$ for $0 < \beta < 1$:

$$H^\beta(0, T) := \left\{ v \in L^2(0, T) \left| \int_0^T \int_0^T \frac{|v(s) - v(t)|^2}{|s-t|^{1+2\beta}} \, ds \, dt < \infty \right. \right\}.$$

By this definition, a routine computation yields that if $0 < \beta < 0.25$, then $v \in H^\beta(0, T)$ with $v$ being given by

$$v(t) := t^{-\beta}, \quad 0 < t < 1.$$

This result will be used implicitly in the proof of (3.2).

Let $X$ be a separable Hilbert space with inner product $(\cdot, \cdot)_X$, and an orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$. For $-\infty < a < b < \infty$ and $0 \leq \beta < \infty$, define

$$H^\beta(a, b; X) := \left\{ v : (a, b) \to X \mid \sum_{k=0}^{\infty} \|(v, e_k)_X\|_{H^\beta(a,b)}^2 < \infty \right\},$$

and equip this space with the following norm:

$$\|v\|_{H^\beta(0,T;X)} := \left( \sum_{k=0}^{\infty} \|(v, e_k)_X\|_{H^\beta(0,T)}^2 \right)^{\frac{1}{2}} \quad \text{for all } v \in H^\beta(a, b; X).$$

It is easy to verify that $H^\beta(a, b; X)$ is a Banach space, and, in particular, we shall also use $L^2(a, b; X)$ to denote the space $H^0(a, b; X)$.

**Remark 2.2.** It is evident that the spaces $L^2(a, b; X)$ and $H^1(a, b; X)$ defined above coincide respectively with the corresponding standard $X$-valued Sobolev spaces [5], with the same norms. Using the $K$-method [20], we see that, for $0 < \beta < 1$, the space $H^\beta(a, b; X)$ coincides with the following interpolation space

$$\left(L^2(a, b; X), H^1(a, b; X)\right)_{\beta, 2},$$

with equivalent norms. Thus, the space $H^\beta(a, b; X)$, $0 \leq \beta \leq 1$, is independent of the choice of orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$ of $X$. The case of $\beta > 1$ is analogous.

Then, let us introduce the Riemann-Liouville fractional integration and derivative operators as follows [19, 17].
Definition 2.1. Let \( \beta > 0 \). Define \( I_+^\beta : V_+ \to V_+ \) by
\[
I_+^\beta v(x) := \frac{1}{\Gamma(\beta)} \int_{-\infty}^{x} (x-t)^{\beta-1} v(t) \, dt, \quad -\infty < x < \infty,
\]
for all \( v \in V_+ \), and define \( I_-^\beta : V_- \to V_- \) by
\[
I_-^\beta v(x) := \frac{1}{\Gamma(\beta)} \int_{x}^{\infty} (t-x)^{\beta-1} v(t) \, dt, \quad -\infty < x < \infty,
\]
for all \( v \in V_- \). Here \( \Gamma(\cdot) \) denotes the standard Gamma function, and
\[
V_+ := \{ v \in L_{loc}^1(\mathbb{R}) \mid \text{supp } v \subset [a, \infty) \text{ for some } a \in \mathbb{R} \},
\]
\[
V_- := \{ v \in L_{loc}^1(\mathbb{R}) \mid \text{supp } v \subset (-\infty, a] \text{ for some } a \in \mathbb{R} \},
\]
with \( \text{supp } v \) denoting the support of \( v \) and \( L_{loc}^1(\mathbb{R}) \) denoting the set of all locally integrable functions defined on \( \mathbb{R} \). In particular, for \( a \in \mathbb{R} \), we use \( I_{a+}^\beta \) to denote the restriction of \( I_+^\beta \) to
\[
\{ v \in V_+ \mid \text{supp } v \subset [a, \infty) \},
\]
and use \( I_{a-}^\beta \) to denote the restriction of \( I_-^\beta \) to
\[
\{ v \in V_- \mid \text{supp } v \subset (-\infty, a] \}.
\]

Definition 2.2. Let \( 0 < \beta < 1 \). Define \( D_+^\beta : V_+ \to D'(\mathbb{R}) \) and \( D_-^\beta : V_- \to D'(\mathbb{R}) \), respectively, by
\[
D_+^\beta := DI_+^{1-\beta}, \quad D_-^\beta := -DI_-^{1-\beta},
\]
where \( D : D'(\mathbb{R}) \to D'(\mathbb{R}) \) is the standard generalized differential operator. In particular, for \( a \in \mathbb{R} \), we use \( D_{a+}^\beta \) to denote the restriction of \( D_+^\beta \) to
\[
\{ v \in V_+ \mid \text{supp } v \subset [a, \infty) \},
\]
and use \( D_{a-}^\beta \) to denote the restriction of \( D_-^\beta \) to
\[
\{ v \in V_- \mid \text{supp } v \subset (-\infty, a] \}.
\]

Remark 2.3. Let \( 0 < \beta < 1 \) and \( v \in V_+ \). By the definition of \( I_+^{1-\beta} \), we have
\[
I_+^{1-\beta} v = h \ast v,
\]
where
\[
h(t) := \frac{1}{\Gamma(1-\beta)} \begin{cases} t^{-\beta}, & \text{if } t > 0, \\ 0, & \text{otherwise.} \end{cases}
\]
If the support of \( v \) is compact, then [22, Theorem 6, pp. 160-161] implies \( \mathcal{F}(h \ast v) = \mathcal{F}h \mathcal{F}v \), so that, using the fact that
\[
\mathcal{F}h(\xi) = (i\xi)^{\beta-1}, \quad -\infty < \xi < \infty,
\]
we obtain
\[
\mathcal{F}(I_+^{1-\beta} v)(\xi) = (i\xi)^{\beta-1} \mathcal{F}v(\xi),
\]
\[
\mathcal{F}(D_+^\beta v)(\xi) = (i\xi)^{\beta} \mathcal{F}v(\xi),
\]
for all \( -\infty < \xi < \infty \). This provides an approach to prove (2.3) below.

Remark 2.4. Note that these fractional integration and derivative operators introduced above act on functions defined on \( \mathbb{R} \). For the sake of rigorousness, we make the convention that, when applying one of these operators to a function \( v \) defined in some interval \( (a, b) \), we shall implicitly extend \( v \) to \( \mathbb{R} \setminus (a, b) \) by zero.
In the remainder of this section, we present some fundamental properties of these fractional integration and derivative operators.

**Lemma 2.1.** If $\beta, \gamma > 0$, then
\[
I_{a+}^{\beta+\gamma} = I_{a+}^\beta I_{a+}^\gamma, \quad I_{a-}^{\beta+\gamma} = I_{a-}^\beta I_{a-}^\gamma.
\]

**Lemma 2.2.** Let $-\infty < a < b < \infty$ and $\beta > 0$. If $u, v \in L^2(a, b)$, then
\[
\left( I_{a+}^\beta u, v \right)_{L^2(a, b)} = \left( u, I_{b-}^\beta v \right)_{L^2(a, b)}.
\]
If $v \in L^p(a, b)$ with $1 \leqslant p \leqslant \infty$, then
\[
\left\| I_{a+}^\beta v \right\|_{L^p(a, b)} \leqslant C \left\| v \right\|_{L^p(a, b)},
\]
\[
\left\| I_{b-}^\beta v \right\|_{L^p(a, b)} \leqslant C \left\| v \right\|_{L^p(a, b)},
\]
where $C$ is a positive constant that only depends on $a$, $b$, $\beta$ and $p$.

**Lemma 2.3.** Let $0 < \beta < 1$ and $v \in L^1(\mathbb{R})$ with compact support. Then
\[
\mathcal{F} I_{a+}^\beta v(\xi) = (i\xi)^{-\beta} \mathcal{F} v(\xi),
\]
\[
\mathcal{F} D_{a+}^\beta v(\xi) = (i\xi)^{\beta} \mathcal{F} v(\xi),
\]
\[
\mathcal{F} I_{a-}^\beta v(\xi) = (-i\xi)^{-\beta} \mathcal{F} v(\xi),
\]
\[
\mathcal{F} D_{a-}^\beta v(\xi) = (-i\xi)^{\beta} \mathcal{F} v(\xi),
\]
for all $\xi \in \mathbb{R}$.

The above three lemmas are contained in [19], and since Lemmas 2.1 and 2.2 are frequently used in this paper, we shall use them without notice for convenience.

**Remark 2.5.** Let $0 < \beta < 1$. If $v \in H^\beta(\mathbb{R})$ with compact support, then using (2.3) and the famous Fourier-Plancherel formula gives
\[
\left\| D_{a+}^\beta v \right\|_{L^2(\mathbb{R})} = \left| v \right|_{H^\beta(\mathbb{R})}.
\]
This is a remarkable and very useful property of fractional derivative operators. Also, when applying (2.3), we should be cautious: for $v \in H^\beta(\mathbb{R})$, in our setting $\mathcal{F}(D_{a+}^\beta v)$ may not make any sense since the support of $v$ is not necessarily compact.

Below we make the following conventions: by $x \lesssim y$ we mean that there exists a positive constant $C$ that only depends on $\alpha$, $T$ or $\Omega$, unless otherwise specified, such that $x \leqslant Cy$ (the value of $C$ may differ at each occurrence); by $x \sim y$ we mean that $x \lesssim y \lesssim x$.

**Lemma 2.4.** Let $0 < \beta < 1$. If $h \in L^2(0, T)$, then
\[
\left\| I_{0+}^\beta h \right\|_{H^\beta(0, T)} \lesssim \| h \|_{L^2(0, T)}, \quad (2.1)
\]
\[
\left\| I_{T-}^\beta h \right\|_{H^\beta(0, T)} \lesssim \| h \|_{L^2(0, T)}. \quad (2.2)
\]
If $h \in H^{1-\beta}_0(0, T)$ with $\beta \in (0, 1) \setminus \{0.5\}$, then
\[
\left\| I_{0+}^\beta h \right\|_{H^\beta(0, T)} \lesssim \| h \|_{H^{1-\beta}(0, T)}, \quad (2.3)
\]
and $I_{0+}^\beta h(0) = 0$. Here the implicit constants in the above three inequalities only depend on $\beta$ and $T$. 
Proof. Let us first prove (2.1). For $v \in L^2(0, T)$, a simple calculation yields
\[ \left\| I_{0+}^{\beta/2} v \right\|_{L^2(\mathbb{R})} \lesssim \| v \|_{L^2(0, T)}, \]
so that (2.3) and the Fourier-Plancherel formula imply
\[ \left\| I_{0+}^{\beta/2} v \right\|_{H^\beta(\mathbb{R})} \lesssim \| v \|_{L^2(0, T)}, \]
which indicates that
\[ \left\| I_{0+}^{\beta/2} v \right\|_{H^{\beta/2}(0, T)} \lesssim \| v \|_{L^2(0, T)} \quad \text{for all } v \in L^2(0, T). \tag{2.4} \]
For any $v \in H^1_0(0, T)$, since
\[ DJ_{0+} v = I_{0+}^2 D v \quad \text{in } (0, T), \]
by (2.4) we obtain
\[ \left\| I_{0+}^2 v \right\|_{H^{1+\beta}(0, T)} \lesssim \| v \|_{H^{\beta}(0, T)}. \]
Therefore, the standard properties of interpolation spaces implies that
\[ \left\| I_{0+}^2 v \right\|_{H^{\beta}(0, T)} \lesssim \| v \|_{H^{\beta}(0, T)} \quad \text{for all } v \in H^{\beta}(0, T). \tag{2.5} \]
By the fact that $I_{0+}^{\beta} h = I_{0+}^{\beta/2} I_{0+}^{\beta/2} h$, using (2.4) and (2.5) yields (2.1).

Then, since the proofs of (2.2) and (2.3) are similar to that of (2.1), we only prove $I_{0+}^{\beta} h(0) = 0$ under the condition that $h \in H^{1-\beta}_0(0, T)$ and $\beta \in (0, 1) \setminus \{0.5\}$. For $0 < \beta < 0.5$, since $H^{1-\beta}_0(0, T)$ is continuously embedded into $C[0, T]$, a simple calculation gives
\[ \left| I_{0+}^{\beta} h(t) \right| \lesssim t^{\beta} \| h \|_{H^{1-\beta}(0, T)} \],
which indicates $I_{0+}^{\beta} h(0) = 0$.

For $0.5 < \beta < 1$, from [9, Theorem 11.3] it follows
\[ \int_0^T s^{-2(1-\beta)} |h(s)|^2 \, ds \lesssim \| h \|_{H^{1-\beta}(0, T)}^2. \]
Therefore, using the Cauchy-Schwarz inequality yields that, for $0 < t < T$,
\[ \left| I_{0+}^{\beta} h(t) \right| \leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |h(s)| \, ds \]
\[ \leq \frac{t^{1-\beta}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |s^{\beta-1} h(s)| \, ds \]
\[ \lesssim \frac{1}{t^{1-\beta}} \| h \|_{H^{1-\beta}(0, T)}, \]
which also implies $I_{0+}^{\beta} h(0) = 0$. This lemma is thus proved. \hfill \Box

Lemma 2.5. If $v \in L^2(0, T)$, then $DJ_{0+}^2 v \in L^2(0, T)$ if and only if $v \in H^\frac{\beta}{2}(0, T)$, and $D_{T-}^2 v \in L^2(0, T)$ if and only if $v \in H^\frac{\beta}{2}(0, T)$. Furthermore, if $v \in H^\frac{\beta}{2}(0, T)$, then
\[ \left\| D_{0+}^2 v \right\|_{L^2(0, T)} \sim \| v \|_{H^\frac{\beta}{2}(0, T)} \sim \left\| D_{T-}^2 v \right\|_{L^2(0, T)} \],
\[ \left(D_{0+}^2 v, D_{T-}^2 v\right)_{L^2(0, T)} = \cos \left(\frac{\beta\pi}{2}\right) \| v \|_{H^\frac{\beta}{2}(0, T)}^2 \sim \| v \|_{H^\frac{\beta}{2}(0, T)}^2. \tag{2.7} \]
Proof. If $D^{\frac{\alpha}{2}}_{0^+} v \in L^2(0,T)$, then a simple calculation yields that

$$v(t) = ct^{\frac{\alpha}{2} - 1} + (I^{\frac{\alpha}{2}}_{0^+} D^{\frac{\alpha}{2}}_{0^+} v)(t), \quad 0 < t < T,$$

where $c$ is a constant. Since $v \in L^2(0,T)$, the constant $c$ is zero, and hence

$$v = I^{\frac{\alpha}{2}}_{0^+} D^{\frac{\alpha}{2}}_{0^+} v \quad \text{in } (0,T).$$

Therefore, (2.4) implies $v \in H^{\frac{\alpha}{2}}(0,T)$. Similarly, we can prove that $D^{\frac{\alpha}{2}}_{T^+} v \in L^2(0,T)$ implies $v \in H^{\frac{\alpha}{2}}(0,T)$.

Conversely, if $v \in H^{\frac{\alpha}{2}}(0,T)$, then extending $v$ to $\mathbb{R} \setminus (0,T)$ by zero gives

$$\|v\|_{H^{\frac{\alpha}{2}}(\mathbb{R})} \lesssim \|v\|_{H^{\frac{\alpha}{2}}(0,T)}.$$

From (2.3) and the Fourier-Plancherel formula it follows

$$\left\| D^{\frac{\alpha}{2}}_{0^+} v \right\|_{L^2(0,T)} \lesssim \left\| D^{\frac{\alpha}{2}}_{T^+} v \right\|_{L^2(\mathbb{R})} \lesssim \|v\|_{H^{\frac{\alpha}{2}}(0,T)},$$

$$\left\| D^{\frac{\alpha}{2}}_{T^+} v \right\|_{L^2(0,T)} \lesssim \left\| D^{\frac{\alpha}{2}}_{0^+} v \right\|_{L^2(\mathbb{R})} \lesssim \|v\|_{H^{\frac{\alpha}{2}}(0,T)}.$$

Finally, for the proofs of (2.6) and (2.7), we refer the reader to [4, Lemma 2.4 and Theorem 2.13]. This concludes the proof of the lemma. \qed

Remark 2.6. Observe that [4, Theorem 2.13] shows $H^{\frac{\alpha}{2}}(0,T) = J^{\frac{\alpha}{2}}_{L^0}(0,T)$, where $J^{\frac{\alpha}{2}}_{L^0}(0,T)$ denotes the closure of $D(0,T)$ with respect to the following norm:

$$\left\| \right\|_{J^{\frac{\alpha}{2}}_{L^0}(0,T)} := \left( \left\| v \right\|^2_{L^2(0,T)} + \left\| D^{\frac{\alpha}{2}}_{0^+} v \right\|^2_{L^2(0,T)} \right)^{\frac{1}{2}}, \quad \forall v \in D(0,T).$$

Evidently, this result does not lead to the conclusion that $v \in H^{\frac{\alpha}{2}}(0,T)$ if $v$, $D^{\frac{\alpha}{2}}_{0^+} v \in L^2(0,T)$.

Remark 2.7. By the definition of $H^{\frac{\alpha}{2}}(0,T;L^2(\Omega))$ and the above lemma, it is easy to verify that

$$\left\| \partial_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q_T)} \sim \|v\|_{H^{\frac{\alpha}{2}}(0,T;L^2(\Omega))},$$

for all $v \in H^{\frac{\alpha}{2}}(0,T;L^2(\Omega))$.

Lemma 2.6. If $v \in H^{\frac{\alpha}{2}}(0,T)$, then $D^{\alpha}_{0^+} v \in L^2(0,T)$ if and only if $v \in H^{\alpha}(0,T)$ with $v(0) = 0$. Moreover, if $D^{\alpha}_{0^+} v \in L^2(0,T)$, then

$$\|v\|_{H^{\alpha}(0,T)} \sim \left\| D^{\alpha}_{0^+} v \right\|_{L^2(0,T)}, \quad (2.8)$$

and

$$(D^{\alpha}_{0^+} v, \varphi)_{L^2(0,T)} = (D^{\frac{\alpha}{2}}_{0^+} v, D^{\frac{\alpha}{2}}_{T^+} \varphi)_{L^2(0,T)} \quad (2.9)$$

for all $\varphi \in H^{\frac{\alpha}{2}}(0,T)$.

Proof. Assuming $D^{\alpha}_{0^+} v \in L^2(0,T)$, let us first show that $v \in H^{\alpha}(0,T)$ with $v(0) = 0$, and

$$\|v\|_{H^{\alpha}(0,T)} \lesssim \left\| D^{\alpha}_{0^+} v \right\|_{L^2(0,T)} \quad (2.10)$$

A straightforward computing gives that

$$v(t) = ct^{\alpha - 1} + (I^{\alpha}_{0^+} D^{\alpha}_{0^+} v)(t), \quad 0 < t < T,$$
where $c$ is a constant. As $(2.4)$ implies $I_{0+}^\alpha D_0^{\alpha} v \in H^\alpha (0, T)$, the constant $c$ must be zero since $v \in H^{2\alpha} (0, T)$. Therefore, $v = I_{0+}^\alpha D_0^{\alpha} v$ in $(0, T)$, and hence using $(2.4)$ again yields $(2.10)$. Moreover, the Cauchy-Schwarz inequality implies

$$|v(t)| \leq \frac{1}{\Gamma(\alpha)} \sqrt{\frac{t^{2\alpha - 1}}{2\alpha - 1}} \|D_0^{\alpha} v\|_{L^2(0, t)}, \quad 0 < t < T,$$

so that $v(0) = \lim_{t \to 0^+} v(t) = 0$.

Next, assuming that $v \in H^\alpha (0, T)$ with $v(0) = 0$, let us prove

$$\|D_0^{\alpha} v\|_{L^2(0, T)} \lesssim \|v\|_{H^\alpha (0, T)}.$$  \hspace{1cm} (2.11)

Let $\tilde{v}$ be an $H^\alpha (\mathbb{R})$-extension of $v$ such that

$$\text{supp} \tilde{v} \subset [0, 2T] \quad \text{and} \quad \|\tilde{v}\|_{H^\alpha (\mathbb{R})} \sim \|v\|_{H^\alpha (0, T)}.$$

By $(2.3)$, the Fourier-Plancherel formula, and the fact that $D_0^{\alpha} v = D_0^\alpha \tilde{v}$ in $(0, T)$, we obtain

$$\|D_0^{\alpha} v\|_{L^2(0, T)} = \|D_0^\alpha \tilde{v}\|_{L^2(0, T)} \leq \|D_0^\alpha \tilde{v}\|_{L^2(\mathbb{R})} = \|\tilde{v}\|_{H^\alpha (\mathbb{R})} \lesssim \|v\|_{H^\alpha (0, T)},$$

which proves estimate $(2.11)$.

Now we have proved that $D_0^{\alpha} v \in L^2(0, T)$ if and only if $v \in H^\alpha (0, T)$ with $v(0) = 0$; moreover, combining $(2.10)$ and $(2.11)$ proves $(2.8)$. Therefore, it remains to prove $(2.9)$, and since $D(0, T)$ is dense in $H^{2\alpha} (0, T)$, by $(2.5)$ it suffices to show that $(2.9)$ holds for all $\varphi \in D(0, T)$. To this end, we argue as follows. Since $v \in H^{2\alpha} (0, T)$, $(2.4)$ implies that $I_{0+}^{1-\frac{2\alpha}{\alpha}} v \in H^1 (0, T)$ with $I_{0+}^{1-\frac{2\alpha}{\alpha}} v(0) = 0$. Also, an elementary computing yields that

$$I_{T-}^{1-\alpha} \varphi' = -I_{T-}^{1-\frac{2\alpha}{\alpha}} D^2 I_{T-}^{1-\frac{2\alpha}{\alpha}} \varphi \quad \text{for all} \quad \varphi \in D(0, T).$$

Consequently, using integration by parts gives that

$$\left( D_0^\alpha v, \varphi \right)_{L^2(0, T)} = -\left( I_{0+}^{1-\alpha} v, \varphi' \right)_{L^2(0, T)} = - \left( v, I_{T-}^{1-\alpha} \varphi' \right)_{L^2(0, T)}$$

$$= \left( v, I_{T-}^{1-\frac{2\alpha}{\alpha}} D^2 I_{T-}^{1-\frac{2\alpha}{\alpha}} \varphi \right)_{L^2(0, T)} = \left( I_{0+}^{1-\frac{2\alpha}{\alpha}} v, D^2 I_{T-}^{1-\frac{2\alpha}{\alpha}} \varphi \right)_{L^2(0, T)}$$

$$= - \left( D_0^{1-\frac{2\alpha}{\alpha}} v, D_T^{1-\frac{2\alpha}{\alpha}} \varphi \right)_{L^2(0, T)} = \left( D_0^{\frac{\alpha}{\alpha}} v, D_T^{\frac{\alpha}{\alpha}} \varphi \right)_{L^2(0, T)},$$

for all $\varphi \in D(0, T)$. This proves $(2.9)$ and thus concludes the proof of the lemma. \hfill \Box

**Remark 2.8.** Note that [8, Lemma 2.6] provides a proof of $(2.9)$ under the condition that $v \in H^1 (0, T)$ with $v(0) = 0$.

3. **Regularity of a fractional ordinary equation.** This section is devoted to investigating the regularity properties of the following problem: seek $y \in C[0, T]$ with $y(0) = y_0 \in \mathbb{R}$ such that

$$D_0^{\alpha} (y - y_0) + \lambda y = g \quad \text{in} \quad (0, T), \hspace{1cm} (3.1)$$

where $\lambda$ is a positive constant and $g \in L^2 (0, T)$. Throughout this section we assume that $1 \lesssim \lambda$, and, for the sake of rigorousness, we also understand $y_0$ by a function with support $[0, T]$.

The main results of this section are the following two theorems.
Theorem 3.1. Problem (3.1) has a unique solution \( y \in H^\alpha(0, T) \) with \( y(0) = y_0 \), and this solution satisfies that

\[
\|y\|_{H^2_x(0, T)} + \lambda^\frac{1}{2} \|y\|_{L^2(0, T)} \lesssim \lambda^{-\frac{1}{2}} \|g\|_{L^2(0, T)} + |y_0|, \tag{3.2}
\]

\[
\|y\|_{H^{1+\alpha}(0, T)} + \lambda^\frac{1}{2} \|y\|_{H^{\frac{2}{3}}_x(0, T)} \lesssim \|g\|_{L^2(0, T)} + \lambda^\frac{1}{2} |y_0|, \tag{3.3}
\]

\[
\lambda \|y\|_{H^1(0, T)} \lesssim \|g\|_{L^2(0, T)} + \lambda^\frac{1}{2} |y_0|. \tag{3.4}
\]

Moreover, if \( g \in H^{1-\alpha}(0, T) \) then

\[
\|y\|_{H^{1+\alpha}(0, T)} \lesssim \|g\|_{H^{1-\alpha}(0, T)} + \lambda |y_0|. \tag{3.5}
\]

Theorem 3.2. If \( g \in H^1(0, T) \), then the solution \( y \) to problem (3.1) satisfies the following estimates:

\[
\|y - S\|_{H^{1+\frac{2}{3}}(0, T)} + \lambda^\frac{1}{2} \|y\|_{H^1(0, T)} \lesssim \lambda^{-\frac{1}{2}} \|g\|_{H^{1+\alpha}(0, T)} + |y_0| + \lambda^\frac{1}{2} |g(0) - \lambda y_0|, \tag{3.6}
\]

\[
\|y - S\|_{H^{1+\alpha}(0, T)} + \lambda^\frac{1}{2} \|y - S\|_{H^{1+\frac{2}{3}}(0, T)} \lesssim \|g\|_{H^1(0, T)} + |y_0| + \lambda \|g(0) - \lambda y_0|, \tag{3.7}
\]

\[
\lambda \|y\|_{H^1(0, T)} \lesssim \|g\|_{H^1(0, T)} + \lambda^\frac{1}{2} |y_0| + \lambda \|g(0) - \lambda y_0|, \tag{3.8}
\]

where

\[
S(t) := \frac{g(0) - \lambda y_0}{\Gamma(1 + \alpha)} t^\alpha, \quad 0 < t < T.
\]

Moreover, if \( 0.75 < \alpha < 1 \) and \( g \in H^{2-\alpha}(0, T) \), then

\[
\|y - S\|_{H^2(0, T)} \lesssim \|g\|_{H^{2-\alpha}(0, T)} + |y_0| + \lambda \|g(0) - \lambda y_0|. \tag{3.9}
\]

Proof of (3.1). Let us first show that problem (3.1) has a unique solution \( y \in H^\alpha(0, T) \) with \( y(0) = y_0 \). By (2.5) and the famous Lax-Milgram theorem, there exists a unique \( \tilde{y} \in H^{\frac{2}{3}}(0, T) \) such that

\[
\left( D_{0+}^{\frac{2}{3}} \tilde{y}, D_T^{-\frac{2}{3}} z \right)_{L^2(0, T)} + \lambda \left( \tilde{y}, z \right)_{L^2(0, T)} = (g, z)_{L^2(0, T)} - \lambda (y_0, z)_{L^2(0, T)} \tag{3.10}
\]

for all \( z \in H^{\frac{2}{3}}(0, T) \). Since (2.4) implies \( I_{0+}^{1-\frac{2}{3}} \tilde{y}(0) = 0 \), using integration by parts gives that

\[
\left( D_{0+}^{\frac{2}{3}} \tilde{y}, D_{0+}^{\frac{2}{3}} z \right)_{L^2(0, T)} = - \left( D_{0+}^{1-\frac{2}{3}} \tilde{y}, D_{T-}^{1-\frac{2}{3}} \varphi \right)_{L^2(0, T)}
\]

\[
= \left( I_{0+}^{1-\frac{2}{3}} \tilde{y}, I_{T-}^{1-\frac{2}{3}} \varphi' \right)_{L^2(0, T)} = \left( \tilde{y}, I_{T-}^{2-\alpha} \varphi'' \right)_{L^2(0, T)}
\]

\[
= - \left( \tilde{y}, I_{T-}^{1-\alpha} \varphi' \right)_{L^2(0, T)} = - \left( I_{0+}^{1-\alpha} \tilde{y}, \varphi' \right)_{L^2(0, T)}
\]

\[
= \left( D_{0+}^{\alpha} \tilde{y}, \varphi \right)_{L^2(0, T)}
\]

for all \( \varphi \in \mathcal{D}(0, T) \), where \( \left< \cdot, \cdot \right> \) denotes the duality pairing between \( \mathcal{D}'(\mathbb{R}) \) and \( \mathcal{D}(\mathbb{R}) \). From (3.10) it follows that

\[
D_{0+}^{\alpha} \tilde{y} = g - \lambda (y_0 + \tilde{y}) \quad \text{in} \quad (0, T),
\]

and then, (2.6) implies that \( \tilde{y} \in H^\alpha(0, T) \) with \( \tilde{y}(0) = 0 \). Obviously, \( y := y_0 + \tilde{y} \) in \((0, T)\) is an \( H^\alpha(0, T) \) solution of (3.1), and, by Lemmas 2.6 and 2.5, it is clear that this \( H^\alpha(0, T) \) solution is unique.
Therefore, (3.3) follows from the following estimates:

\[ h \text{ Cauchy's inequality with } \varepsilon \]

so that

\[ (D_{0+}^\alpha (y - y_0), y - y_0)_{L^2(0, T)} + \lambda \| y \|_{L^2(0, T)}^2 = (g, y)_{L^2(0, T)} - (D_{0+}^\alpha (y - y_0), y_0)_{L^2(0, T)}. \]

From (2.6) it follows

\[ \left( D_{0+}^\alpha (y - y_0), D_{\alpha}^\varepsilon (y - y_0) \right)_{L^2(0, T)} + \lambda \| y \|_{L^2(0, T)}^2 = (g, y)_{L^2(0, T)} - (D_{0+}^\alpha (y - y_0), D_{\alpha}^\varepsilon y_0)_{L^2(0, T)}. \]

Therefore, using (2.5) and the Cauchy's inequality with \( \varepsilon \) gives

\[ \| y - y_0 \|_{H^{\varepsilon}(0, T)}^2 + \lambda \| y \|_{L^2(0, T)}^2 \lesssim \lambda^{-1} \| g \|_{L^2(0, T)}^2 + y_0^2, \]

which, together with the estimate

\[ \| y_0 \|_{H^{\varepsilon}(0, T)} \lesssim |y_0|, \]

yields (3.2).

Next, let us show (3.3). Multiplying both sides of (3.1) by \( D_{0+}^\alpha (y - y_0) \), and integrating in \((0, T)\) yield

\[ \| D_{0+}^\alpha (y - y_0) \|_{L^2(0, T)}^2 + \lambda \left( y, D_{0+}^\alpha (y - y_0) \right)_{L^2(0, T)} = (g, D_{0+}^\alpha (y - y_0))_{L^2(0, T)}, \]

so that

\[ \| D_{0+}^\alpha (y - y_0) \|_{L^2(0, T)}^2 + \lambda \left( y - y_0, D_{0+}^\alpha (y - y_0) \right)_{L^2(0, T)} = (g, D_{0+}^\alpha (y - y_0))_{L^2(0, T)} - \lambda \left( y_0, D_{0+}^\alpha (y - y_0) \right)_{L^2(0, T)}. \]

From Lemmas 2.5 and 2.6 it follows

\[ \| y - y_0 \|_{H^{\ast}(0, T)}^2 + \lambda \| y - y_0 \|_{H^{\varepsilon}(0, T)}^2 \lesssim \| g \|_{L^2(0, T)} \| y - y_0 \|_{H^{\ast}(0, T)} + \lambda \| y - y_0 \|_{H^{\varepsilon}(0, T)} |y_0|, \]

hence the Cauchy's inequality with \( \varepsilon \) implies

\[ \| y - y_0 \|_{H^{\ast}(0, T)}^2 + \lambda \| y - y_0 \|_{H^{\varepsilon}(0, T)}^2 \lesssim \| g \|_{L^2(0, T)}^2 + \lambda y_0^2. \]  \hspace{1cm} (3.11)

Therefore, (3.3) follows from the following estimates:

\[ \| y \|_{H^{\ast}(0, T)} \lesssim \| y - y_0 \|_{H^{\ast}(0, T)} + |y_0|, \]

\[ \| y \|_{H^{\varepsilon}(0, T)} \lesssim \| y - y_0 \|_{H^{\varepsilon}(0, T)} + |y_0|. \]

Now, let us show (3.4). Since (3.1) implies

\[ \lambda y = g - D_{0+}^\alpha (y - y_0) \text{ in } (0, T), \]

by (2.6) we obtain

\[ \lambda^2 \| y \|_{L^2(0, T)}^2 \lesssim \| g \|_{L^2(0, T)}^2 + \| y - y_0 \|_{H^{\ast}(0, T)}^2. \]

Then (3.4) is a direct conclusion of (3.11).

Finally, let us show (3.5). By (3.1) a simple computing gives

\[ y = y_0 + I_{0+}^\alpha (g - \lambda y) \text{ in } (0, T). \]
Since \( y \in H^\alpha(0, T) \subset H^{1-\alpha}(0, T) \), from (2.4) it follows \( y \in H^1(0, T) \), and so
\[
Dy = DI_{0+}^\alpha (g - ly) \quad \text{in } (0, T),
\]
which implies
\[
Dy + \lambda DI_{0+}^\alpha (y - y_0) = DI_{0+}^\alpha g - DI_{0+}^\alpha y_0 \quad \text{in } (0, T).
\]
Since \( y \in H^1(0, T) \) with \( y(0) = y_0 \) implies
\[
DI_{0+}^\alpha (y - y_0) = I_{0+}^\alpha Dy \quad \text{in } (0, T),
\]
it follows
\[
Dy + \lambda DI_{0+}^\alpha Dy = DI_{0+}^\alpha g - \lambda DI_{0+}^\alpha y_0 \quad \text{in } (0, T).
\]
Multiplying both sides of the above equation by \( D_y \), and integrating in \( (0, T) \), we obtain
\[
\|Dy\|^2_{L^2(0, T)} + \lambda \left( I_{0+}^\alpha Dy, Dy \right)_{L^2(0, T)} = \left( DI_{0+}^\alpha g, Dy \right)_{L^2(0, T)} - \lambda \left( DI_{0+}^\alpha y_0, Dy \right)_{L^2(0, T)},
\]
so that, using (2.4) and the Cauchy’s inequality with \( \epsilon \) gives
\[
\|Dy\|^2_{L^2(0, T)} + \lambda \left( I_{0+}^\alpha Dy, Dy \right)_{L^2(0, T)} \lesssim \|g\|^2_{H^{1-\alpha}(0, T)} + \lambda^2 y_0^2.
\]
Since (3.2) implies
\[
\|y\|_{L^2(0, T)} \lesssim \lambda^{-1} \|g\|_{L^2(0, T)} + \lambda^{-\frac{\alpha}{2}} |y_0|,
\]
to prove (3.5) it suffices to to show that
\[
\left( I_{0+}^\alpha Dy, Dy \right)_{L^2(0, T)} \geq 0.
\]
To this end, let us define
\[
v(t) := \begin{cases} Dy(t) & \text{if } 0 < t < T, \\ 0 & \text{otherwise}. \end{cases}
\]
Since \( \frac{\alpha}{2} < \frac{1}{2} \), it is easy to verify that \( I_{0+}^\frac{\alpha}{2} v, I_{T-}^\frac{\alpha}{2} v \in L^2(\mathbb{R}) \), and then, using (2.3) and the famous Parseval’s theorem yields
\[
\left( I_{0+}^\alpha v, I_{T-}^\alpha v \right)_{L^2(0, T)} = \left( I_{0+}^{\frac{\alpha}{2}} v, I_{T-}^{\frac{\alpha}{2}} v \right)_{L^2(\mathbb{R})} = \cos \left( \frac{\alpha \pi}{2} \right) \int_{\mathbb{R}} |w|^{-\alpha} |F v(\xi)|^2 \, dw \geq 0,
\]
which proves (3.12). This concludes the proof of (3.5) and thus the proof of (3.1). □

Proof of (3.2). By (3.1), there exists uniquely \( z \in H^\alpha(0, T) \) with \( z(0) = 0 \) such that
\[
D_{0+}^\alpha z + \lambda z = Dg - \lambda S^\alpha \quad \text{in } (0, T).
\]
Moreover,
\[
\|z\|_{H^{\frac{\alpha}{2}}(0, T)} + \lambda^{\frac{\alpha}{2}} \|z\|_{L^2(0, T)} \lesssim \lambda^{-\frac{\alpha}{2}} \|Dg\|_{L^2(0, T)} + \lambda^{\frac{\alpha}{2}} |g(0) - ly_0|,
\]
\[
\|z\|_{H^{\alpha}(0, T)} + \lambda^{\frac{\alpha}{2}} \|z\|_{H^{\frac{\alpha}{2}}(0, T)} \lesssim \|Dg\|_{L^2(0, T)} + \lambda |g(0) - ly_0|,
\]
\[
\lambda \|z\|_{L^2(0, T)} \lesssim \|Dg\|_{L^2(0, T)} + \lambda |g(0) - ly_0|,
\]
and if \( 0.75 < \alpha < 1 \) and \( g \in H^{2-\alpha}(0, T) \), then
\[
\|z\|_{H^1(0, T)} \lesssim \|Dg\|_{H^{1-\alpha}(0, T)} + \lambda |g(0) - ly_0|.
\]
since $Dg - \lambda S' \in H^{1-\alpha}(0, T)$. Set

$$y := y_0 + S + I_{0+}z \quad \text{in} \ (0, T).$$

(3.13)

If $y$ is the solution of problem (3.1), then, by (3.1) and the estimate

$$\|S\|_{H^1(0, T)} \lesssim |g(0) - \lambda y_0|,$$

some simple manipulation yields (3.6)-(3.9). Therefore, to complete the proof of this theorem, it remains to show that (3.13) is the solution of problem (3.1).

To do so, observe that the definition of $z$ implies

$$I_{0+}D_{0+}^\alpha z + \lambda I_{0+}z = g - g(0) - \lambda S \quad \text{in} \ (0, T).$$

Also, from the fact that $z \in H^\alpha(0, T)$ with $z(0) = 0$, it follows

$$I_{0+}D_{0+}^\alpha z = D_{0+}^\alpha I_{0+}z \quad \text{in} \ (0, T).$$

Consequently,

$$D_{0+}^\alpha(I_{0+}z + S) + \lambda(I_{0+}z + S + y_0) - D_{0+}^\alpha S + g(0) - \lambda y_0 = g \quad \text{in} \ (0, T).$$

Since a direct computing gives

$$-D_{0+}^\alpha S + g(0) - \lambda y_0 = 0 \quad \text{in} \ (0, T),$$

it is evident that (3.13) is the solution of (3.1) indeed. This completes the proof of (3.2).

4. **Main results.** In this section we shall employ the results developed in (3) to analyze the regularity properties of problem (1.1). Let us start by introducing some notation and conventions. For each $v \in L^2(Q_T)$, we can naturally regard it as an $L^2(\Omega)$-valued function with domain $(0, T)$, an element of $L^2(0, T; L^2(\Omega))$, and, for convenience, we also use $v$ to denote this $L^2(\Omega)$-valued function. It is well known that, there exists, in $H^1_0(\Omega)$, an orthonormal basis $\{\phi_k | k \in \mathbb{N}\}$ of $L^2(\Omega)$, and a nondecreasing sequence $\{\lambda_k > 0 | k \in \mathbb{N}\}$ such that

$$-\Delta \phi_k = \lambda_k \phi_k \quad \text{in} \ \Omega, \ \text{for all} \ k \in \mathbb{N}.$$

Meanwhile, $\{\lambda_k^{-1/2} \phi_k | k \in \mathbb{N}\}$ is an orthonormal basis of $H^1_0(\Omega)$ equipped with the inner product $(\nabla \cdot, \nabla \cdot)_{L^2(\Omega)}$. For each $k \in \mathbb{N}$, define $c_k \in C[0, T]$ with $c_k(0) := (u_0, \phi_k)_{L^2(\Omega)}$ by

$$-D_{0+}^\alpha (c_k - c_k(0)) + \lambda_k c_k = f_k \quad \text{in} \ (0, T),$$

where

$$f_k(t) := \int_\Omega f(x, t)\phi_k(x) \, dx, \quad 0 < t < T.$$

Finally, define

$$u(t) := \sum_{k=0}^{\infty} c_k(t)\phi_k, \quad 0 < t < T.$$

(4.1)

By Theorems 3.1 and 3.2, we readily obtain the following estimates on the above $u$. 

Theorem 4.1. Suppose that \( u_0 \in H^1_0(\Omega) \) and \( f \in L^2(0, T; L^2(\Omega)) \). Then \( u \) defined by (4.1) satisfies the regularity estimate
\[
\|u\|_{H^\alpha(0, T; L^2(\Omega))} + \|u\|_{H^{\frac{\alpha}{2}}(0, T; H^1_0(\Omega))} + \|u\|_{L^2(0, T; H^2(\Omega))} \leq \|f\|_{L^2(0, T; L^2(\Omega))} + \|u_0\|_{H^1_0(\Omega)}. \tag{4.2}
\]
Furthermore, the following regularity estimates also hold:

- If \( f \in H^{1-\alpha}(0, T; L^2(\Omega)) \) and \( u_0 \in H^1_0(\Omega) \cap H^2(\Omega) \), then
  \[
  \|u\|_{H^{1}(0, T; L^2(\Omega))} \leq \|f\|_{H^{1-\alpha}(0, T; L^2(\Omega))} + \|u_0\|_{H^2(\Omega)}. \tag{4.3}
  \]
- If \( f \in H^1(0, T; L^2(\Omega)) \) with \( f(0) + \Delta u_0 \in H^1_0(\Omega) \cap H^2(\Omega) \), then
  \[
  \|u - S\|_{H^{1+\alpha}(0, T; L^2(\Omega))} + \|u - S\|_{H^{1+\frac{\alpha}{2}}(0, T; H^1_0(\Omega))} + \|u\|_{H^1(0, T; H^2(\Omega))} \leq \|f\|_{H^{1}(0, T; L^2(\Omega))} + \|u_0\|_{H^1_0(\Omega)} + \|f(0) + \Delta u_0\|_{H^2(\Omega)}. \tag{4.4}
  \]
- If \( 0 < \alpha < 1, f \in H^{2-\alpha}(0, T; L^2(\Omega)) \) with \( f(0) + \Delta u_0 \in H^1_0(\Omega) \cap H^2(\Omega) \), then
  \[
  \|u - S\|_{H^2(0, T; L^2(\Omega))} \leq \|f\|_{H^{2-\alpha}(0, T; L^2(\Omega))} + \|f(0) + \Delta u_0\|_{H^2(\Omega)}. \tag{4.5}
  \]

Above,
\[
S(t) := \sum_{k=0}^{\infty} \frac{f_k(0) - \lambda_k c_k(0)}{\Gamma(1 + \alpha)} t^\alpha \phi_k, \quad 0 < t < T.
\]

Proof. Since by Theorems 3.1 and 3.2, the proofs of (4.2)-(4.5) are straightforward, below we only prove (4.2). To this end, note that \( f \in L^2(0, T; L^2(\Omega)) \) and \( u_0 \in H^1_0(\Omega) \) imply
\[
\sum_{k=0}^{\infty} \lambda_k c_k(0)^2 = \|u_0\|^2_{H^1_0(\Omega)},
\]
\[
\sum_{k=0}^{\infty} \|f_k\|^2_{L^2(0, T)} = \|f\|^2_{L^2(0, T; L^2(\Omega))}.
\]
From (3.1) it follows
\[
\sum_{k=0}^{\infty} \left( c_k \|u\|^2_{H^\alpha(0, T)} + \lambda_k \|c_k\|^2_{H^{\frac{\alpha}{2}}(0, T)} + \lambda_k^2 \|c_k\|^2_{L^2(0, T)} \right) \leq \|f\|^2_{L^2(0, T; L^2(\Omega))} + \|u_0\|^2_{H^1_0(\Omega)}. \tag{4.6}
\]
Obviously, the above estimate implies
\[
\|u\|_{H^\alpha(0, T; L^2(\Omega))} + \|u\|_{H^{\frac{\alpha}{2}}(0, T; H^1_0(\Omega))} \leq \|f\|_{L^2(0, T; L^2(\Omega))} + \|u_0\|_{H^1_0(\Omega)},
\]
and therefore, it remains to prove that
\[
\|u\|_{L^2(0, T; H^2(\Omega))} \leq \|f\|_{L^2(0, T; L^2(\Omega))} + \|u_0\|_{H^1_0(\Omega)}. \tag{4.7}
\]
To do so, using the standard estimate that \( \|v\|_{H^2(\Omega)} \lesssim \|\Delta v\|_{L^2(\Omega)} \) for all \( v \in H^1_0(\Omega) \) such that \( \Delta v \in L^2(\Omega) \), we obtain

\[
\|u\|_{L^2(0,T;H^2(\Omega))} = \left( \int_0^T \|u(t)\|_{H^2(\Omega)}^2 \, dt \right)^{\frac{1}{2}} \lesssim \left( \|\Delta u(t)\|_{L^2(\Omega)}^2 \, dt \right)^{\frac{1}{2}}
\]

\[
= \left( \int_0^T \sum_{k=0}^\infty \lambda_k^2 c_k(t)^2 \, dt \right)^{\frac{1}{2}} = \left( \sum_{k=0}^\infty \lambda_k^2 \|c_k\|_{L^2(0,T)}^2 \right)^{\frac{1}{2}},
\]

which, together with (4.6), yields (4.7). This proves the estimate (4.2), and concludes the proof of the theorem.

Finally, let us show in what sense \( u \), given by (4.1), is a solution to problem (1.1).

**Theorem 4.2.** Suppose that \( f \in L^2(0,T;L^2(\Omega)) \) and \( u_0 \in H^1_0(\Omega) \). Then \( u \), defined by (4.1), satisfies that \( u \in C([0,T];L^2(\Omega)) \) with \( u(0) = u_0 \) and that

\[
(\partial_t^\alpha (u - u_0), \varphi)_{L^2(\Omega)} + (\nabla u, \nabla \varphi)_{L^2(\Omega)} = (f, \varphi)_{L^2(\Omega)}, \tag{4.8}
\]

for all \( \varphi \in L^2(0,T;H^1_0(\Omega)) \).

**Proof.** By (4.1) we have \( u \in H^\alpha(0,T;L^2(\Omega)) \), and as \( H^\alpha(0,T) \) is continuously embedded into \( C^{\alpha-0.5}(0,T] \), modifying \( u \) on a set of measure zero yields

\[
\|u(t + h) - u(t)\|_{L^2(\Omega)}^2 = \sum_{k=0}^\infty |c_k(t + h) - c_k(t)|^2 \lesssim \sum_{k=0}^\infty |h|^{2\alpha-1} \|c_k\|_{H^\alpha(0,T)}^2 = |h|^{2\alpha-1} \|u\|_{H^\alpha(0,T;L^2(\Omega))}^2,
\]

for all \( 0 \leq t \leq T \) and \( h \) such that \( 0 \leq t + h \leq T \). This implies \( u \in C([0,T];L^2(\Omega)) \), and by the definition (4.1) of \( u \), it is obvious that \( u(0) = u_0 \).

Since \( u \in H^\alpha(0,T;L^2(\Omega)) \), (2.6) implies

\[
\sum_{k=0}^\infty D_{0+}^\alpha(c_k - c_k(0)) \phi_k \in L^2(\Omega) \cap L^2(0,T;H^1_0(\Omega)),
\]

and it is easy to verify that

\[
\partial_t^\alpha (u - u_0) = \sum_{k=0}^\infty D_{0+}^\alpha(c_k - c_k(0)) \phi_k \quad \text{in } L^2(\Omega) \cap L^2(0,T;H^1_0(\Omega)).
\]

Thus, the definitions of \( c_k \)'s indicate that

\[
(\partial_t^\alpha (u - u_0), \eta \phi_k)_{L^2(\Omega)} + (\nabla u, \nabla \phi_k \eta)_{L^2(\Omega)} = (f, \eta \phi_k)_{L^2(\Omega)}
\]

for all \( \eta \in D(0,T) \) and \( k \in \mathbb{N} \), and hence (4.8) follows from the density of \( \{\eta \phi_k \mid \eta \in D(0,T), \ k \in \mathbb{N}\} \)

in \( L^2(0,T;H^1_0(\Omega)) \). This proves the theorem.

**Remark 4.1.** Under the same condition as that in (4.2), suppose \( \tilde{u} \in H^\alpha(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega)) \) with \( \tilde{u}(0) = u_0 \) also satisfies that

\[
(\partial_t^\alpha (\tilde{u} - u_0), \varphi)_{L^2(\Omega)} + (\nabla \tilde{u}, \nabla \varphi)_{L^2(\Omega)} = (f, \varphi)_{L^2(\Omega)},
\]
for all \( \varphi \in L^2(0, T; H^1_0(\Omega)) \). Putting
\[
    w(t) := u(t) - \bar{u}(t), \quad 0 < t < T,
\]
we have
\[
    (\partial_t^\alpha w, w)_{L^2(Q_T)} + \|w\|_{L^2(0,T;H^1_0(\Omega))} = 0.
\]
Since \( w \in H^\alpha(0, T; L^2(\Omega)) \) with \( w(0) = 0 \), by (2.6) we obtain
\[
    (\partial_t^\alpha w, w)_{L^2(Q_T)} \sim \|w\|_{H^\alpha(0,T;L^2(\Omega))}^2,
\]
so that
\[
    \|w\|_{H^\alpha(0,T;L^2(\Omega))} = \|w\|_{L^2(0,T;H^1_0(\Omega))} = 0.
\]
This implies \( w = 0 \) and hence \( u = \bar{u} \). Therefore, if we call \( u \in H^\alpha(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)) \) a weak solution of (1.1) such that \( u(0) = u_0 \) and (4.8) holds for all \( \varphi \in L^2(0, T; H^1_0(\Omega)) \), then (1.1) has a unique weak solution given by (4.1).

REFERENCES

[1] O. P. Agarwal, Solution for a fractional diffusion-wave equation defined in a bounded domain, Nonlinear Dynamics, 29 (2002), 145–155.
[2] S. D. Eidelman and A. N. Kochubei, Cauchy problem for fractional diffusion equations, Journal of Differential Equations, 199 (2004), 211–255.
[3] A. M. A. El-Sayed, Fractional order diffusion-wave equation, Journal of Theoretical Physics, 35 (1996), 311–322.
[4] V. J. Ervin and J. P. Roop, Variational formulation for the stationary fractional advection dispersion equation, Numerical Methods for Partial Differential Equations, 22 (2006), 558–576.
[5] L. C. Evans, Partial Differential Equations, Graduate Studies in Mathematics, American Mathematical Society, 1998.
[6] Z. Fan, Existence and regularity of solutions for evolution equations with Riemann-Liouville fractional derivatives, Indagationes Mathematicae, 25 (2014), 516–524.
[7] V. D. Gejji and H. Jafari, Boundary value problems for fractional differential equations, The Australian Journal of Mathematical Analysis and Applications, 3 (2006), Art. 16, 8 pp.
[8] X. Li and C. Xu, A space-time spectral method for the time fractional diffusion equation, SIAM Journal on Numerical Analysis, 47 (2009), 2108–2131.
[9] J. L. Lions and E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, Vol. I. Springer-Verlag, Berlin Heidelberg, 1972.
[10] Y. Luchko, Maximum principle for the generalized time-fractional diffusion equation, Journal of Mathematical Analysis and Applications, 351 (2009), 218–223.
[11] Y. Luchko, Some uniqueness and existence results for the initial-boundary value problems for the generalized time-fractional diffusion equation, Computers & Mathematics with Applications, 59 (2010), 1766–1772.
[12] Y. Luchko, Initial-boundary-value problems for the one-dimensional time-fractional diffusion equation, Fractional Calculus and Applied Analysis, 15 (2012), 141–160.
[13] F. Mainardi, On the initial value problem for the fractional diffusion-wave equation, in: Waves and Stability in Continuous Media, World Scientific, Singapore, 23 (1994), 246–251.
[14] F. Mainardi, The time fractional diffusion-wave equation, Radiophysics and Quantum Electronics, 38 (1995), 13–24.
[15] F. Mainardi, The fundamental solutions for the fractional diffusion-wave equation, Applied Mathematics Letters, 9 (1996), 23–28.
[16] J. Mu, B. Ahmad, and S. Huang, Existence and regularity of solutions to time-fractional diffusion equations, Computers & Mathematics with Applications, 73 (2017), 985–996.
[17] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.
[18] K. Sakamoto and M. Yamamoto, Initial value/boundary value problems for fractional diffusion-wave equation and application to some inverse problems, Journal of Mathematical Analysis and Applications, 382 (2011), 426–447.
[19] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Yverdon, 1993.
[20] L. Tartar, *An Introduction to Sobolev Spaces and Interpolation Spaces*, Springer-Verlag Berlin Heidelberg, 2007.

[21] R. Wang, D. Chen and T. Xiao, Abstract fractional Cauchy problems with almost sectorial operators, *Journal of Differential Equations*, 252 (2012), 202–235.

[22] K. Yosida, *Functional Analysis*, sixth edition, Springer-Verlag, Berlin Heidelberg, 1980.

[23] R. Zacher, A De Giorgi-Nash type theorem for time fractional diffusion equations, *Mathematische Annalen*, 356 (2013), 99–146.

[24] Z. Zhang and B. Liu, Existence of mild solutions for fractional evolution equations, *Journal of Fractional Calculus and Applications*, 2 (2012), 1–10.

[25] M. Zheng, F. Liu, I. Turner and V. Anh, A novel high order space-time spectral method for the time fractional Fokker-Planck equation, *SIAM Journal on Scientific Computing*, 37 (2015), 701–724.

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