Energy constraints for evolving spherical and hyperbolic wormholes in $f(R, T)$ gravity

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Received: 30 November 2021 / Accepted: 6 June 2022
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Abstract The primary objective of this article is to study the energy condition bounds for spherical and hyperbolic wormholes in well-known $f(R, T)$ theory of gravity. For this purpose, we formulate the field equations for spherically and pseudospherically geometries using anisotropic matter and linear form of generic function $f(R, T)$. By imposing different conditions on radial and tangential pressures or by adopting some known choices for red shift and shape functions, we present the graphical analysis of energy conditions for both spherically and pseudospherically symmetric wormholes. It is seen that energy density for spherically symmetric wormhole is always positive for $\lambda > -4\pi$ and $\lambda < -8\pi$, while the energy conditions for radial pressure are negative at throat. Likewise, in case of pseudospherically symmetric wormhole, it is observed that energy density is always positive for negative $\lambda$, however conditions based on radial pressure may be positive or negative for the considered different cases.

1 Introduction

General relativity (GR) embodies the conceptual ingenuity and the very essence of scientific style in its each advancement. It provides considerable contributions at observational as well as experimental levels which range from millimeter to cosmological scales [1]. However, despite its excellence, the geometrical singularities challenge its potential and make it impotent in certain situations. It is argued that simple GR action cannot provide complete description of accelerated expansion eras of the universe and it is only possible by summoning new types of matter and energy components of the universe in its action [2]. To incorporate these adjustments, modified matter models or theories have been proposed in the literature. In this respect, modified theories have gained considerable attention of researchers after the revelation of accelerated expanding nature of universe in its current state. It is worth mentioning here that although several theories like torsion-based framework and its modified versions, scalar field theories, Gauss-Bonnet gravity have been presented [3–8] but the natural extensions of GR, namely $f(R)$ theory [9] and its extension $f(R, T)$ gravity [10] (with $R$ and $T$ as Ricci scalar and trace of stress energy tensor, respectively) have gained more attention due to their success and viability. Several works are available in the literature [11–16] which present discussions on various cosmic issues and proved as best candidates for the explanation of cosmic acceleration.

Geometrically, wormholes are tunnels type structures which join two different asymptotically flat distant universes or two different regions of the same universe. Such tube-like structures were firstly introduced by Einstein and Rosen [17], who investigated exact solutions of GR, and their proposed structures were named as Einstein-Rosen bridges. Later, Morris and Thorne (M-T) [18] explored the static spherically symmetric wormhole by adopting GR gravitational framework and proposed the traversable wormhole by describing its fundamental aspects, according to which human body can travel via wormhole tunnels. It is worthy to mention here that wormhole geometry is based on two functions, namely red shift and shape functions. It has been argued [18] that for the existence of a viable wormhole structure, these functions must satisfy some axioms, i.e., for $r_0 \leq r < \infty$, where $r_0$ is the throat radius, the shape function $b(r)$ must satisfy the condition $b(r_0) = r_0$ at $r = r_0$ and for $r > r_0$, the conditions $1 - \frac{b(r)}{r} > 0$ and $b'(r_0) < 1$ must hold. The wormhole spacetime geometry is asymptotically flat if it satisfies the condition, i.e., limit $\frac{b(r)}{r} \to 0$ as $|r| \to \infty$. Also, the red shift $\phi(r)$ must be finite throughout the spacetime. In literature, much work have already been done on this topic in both GR as well as modified gravity theories. In [19], authors have studied spherically symmetric traversable wormhole spacetime filled with single fluid involving anisotropic pressure. Cataldo and his collaborators [20] proposed the wormhole shape function defined by $b(r) = \mu + \nu(r)$, while Samanta et al. [21] discussed the exponential form of shape function, i.e., $b(r) = e^{c_1(r)}$ in GR framework. Furthermore, in another study [22], authors have used the power law type function given by $b(r) = (r_0)\pm 1(r)\mp 1$ by varying $\epsilon$. In [20]
23], the authors have studied the existence of spherically symmetric wormholes as well as pseudospherically symmetric spacetimes satisfying all necessary conditions for shape and red shift functions. They concluded that the energy density becomes negative and radial pressure remains positive for hyperbolic wormhole in GR. In a recent paper [24], Sharma and Ghosh proposed a generalized version of Ellis-Bronnikov wormholes which is embedded in a 4-dimensional wrapped background and discussed the validity of energy constraints in GR.

Many authors have explored the wormhole solutions in modified theory including Azizi [25] who studied wormhole geometries in the framework of $f(R, T)$ theory by considering a particular EoS for ordinary matter and concluded that modified stress-energy is responsible for the violation of the NEC. Sahoo with his collaborators [26] proposed the modeling of traversable wormholes (a wormhole in which any body can be passed safely) in the frame work of traceless $f(R, T)$ gravity theory. They discussed different features of energy condition bounds by considering hybrid form of shape function and a particular form of EoS parameter. Similarly, Dixit and his collaborators [27] explored the traversable wormhole by applying logarithmic shape function in $f(R, T)$ gravity and discussed the behavior of radial and tangential components of state along with anisotropy parameter in describing the universe geometry. Mishra et al. [28] presented the traversable wormhole models using different ansatz of shape functions and EoS parameter in the $f(R, T)$ gravitational framework. They also examined the suitability of these models by exploring all energy constraints. Furthermore, Sahoo and his collaborators [29] studied the wormhole solution for phantom case, i.e., $\omega < -1$ in $f(R, T)$ modified gravity. They adopted the EoS given by $p_r = \omega p_t$ to obtain the shape function which obeys all its axioms and also examined corresponding energy condition bounds. Similarly, Moraes and Sahoo [30] proposed the wormhole solution by applying power law form of shape function and analyzed all energy constraints in exponential $f(R, T)$ gravity. In the literature [31–45], wormholes have been discussed without non-exotic matter in modified theories. In this respect, Zubair et al. [46] proposed wormhole geometries by adopting simple linear as well as cubic forms of $f(R, T)$ function. They considered the non-commutative geometrical aspects of string theory in the context of $f(R, T)$ gravity. In another paper [47], the researchers have explored the idea of viable charged wormhole solutions in $f(R, T)$ gravity. They have assumed simple linear generic model given by $f(R, T) = R + 2T$ along with the ordinary matter as the total pressure of anisotropic fluid. Further, the existence of static wormhole model has been explored by utilizing different kind of shape functions in a study [48]. In another paper [49], authors have investigated the wormhole modeling by considering a specific general shape function in the quadratic $f(R, T)$ gravity. In a recent paper [50], Banerjee and his collaborator adopted different strategies to construct wormhole geometry using isotropic pressure in $f(R, T)$ theory and concluded that all energy constraints are valid for the proposed models.

Above literature motivated us to explore the spherically symmetric and pseudospherically symmetric wormhole solutions for M-T spacetime in $f(R, T)$ theory. Here, in order to calculate the $\phi(r)$ and $b(r)$ functions, we consider the matter with EoS parameter $p_r$ or $p_t$ to solve $f(R, T)$ field equations. Also, we shall present the validity of energy conditions for static spherically and pseudospherically symmetric wormhole spacetimes defined by the single perfect fluid, i.e., a source of matter with isotropic pressure in $f(R, T)$ gravity with wide range of $\lambda$. Up to our knowledge, till now, the only wormhole solution discussed in the literature is non-asymptotically flat wormhole with isotropic pressure as pointed out in Ref. [51–53]. In the present work, we shall discuss the asymptotically flat wormholes which may or may not satisfy all energy constraints with a wide range of $\lambda$ for both types of wormhole spacetimes. We have organized this paper in the following pattern. Section 2 provides the basic formulation of $f(R, T)$ field equations for static spherically symmetric spacetime and anisotropic matter. In its subsections, we shall discuss the energy conditions graphically for different cases: firstly, wormholes with zero tidal force and filled with perfect fluid, secondly, by taking the linear equation of state, and lastly, the analytic solution used by Tolman and static spheres filled with isotropic perfect fluid. Similarly, in Sect. 4, we express the $f(R, T)$ field equations for hyperbolic symmetric spacetime. In its subsection A, we discuss the pseudospherical symmetric wormhole and corresponding energy conditions using linear equation of state. The zero tidal hyperbolic wormholes with the corresponding energy conditions will be explained in subsection B. We choose non-vanishing red shift function and evaluate the more general form of shape function with isotropic pressure and explore the energy conditions for the obtained hyperbolic wormhole in subsection C. Finally, we summarize our discussion.

2 Field equations for spherically symmetric spacetime in $f(R, T)$ and wormholes construction

In this section, we shall present a brief review of $f(R, T)$ gravity and some necessary assumptions used for this work. The $f(R, T)$ theory of gravity is defined by the total action [10] given by

$$S = \frac{1}{16\pi} \int f(R,T)\sqrt{-g}d^4x + \int L_m\sqrt{-g}d^4x, \quad (1)$$

where $f(R, T)$ is an arbitrary function of Ricci scalar $R$ and the trace of energy-momentum tensor $T = g^{\mu\nu}T_{\mu\nu}$, and $L_m$ represents the Lagrangian density of ordinary matter. By varying the above action with respect to metric tensor $g_{ij}$, we have the following set of equations

$$8\pi T_{ij} - f_T(R, T)T_{ij} - f_T(R, T)\Theta_{ij} = f_R(R, T)R_{ij} - \frac{1}{2} f(R, T)g_{ij} + (g_{ij}\square - \nabla_i\nabla_j)f_R(R, T). \quad (2)$$
The above equation involves covariant derivative and d’Alembert operator denoted by $\nabla$ and $\square$, respectively, where $f_R(R, T)$ and $f_T(R, T)$ represent the function derivatives with respect to $R$ and $T$, respectively. Also, the term $\Theta_{ij}$ is defined by

$$
\Theta_{ij} = \frac{g^{\alpha\beta}\delta_{ij}}{\delta g^{\alpha\beta}} = -2T_{ij} + g_{ij}L_m - 2g^{\alpha\beta}\frac{\partial^2 L_m}{\partial g^{ij}\partial g^{\alpha\beta}},
$$

(3)

where $T_{ij} = diag(\rho, -p_r, -p_t, -p_t)$, with $\rho$ as the matter energy density, while $p_r$ and $p_t$ denote the radial and transverse components of pressure. Also, $L_m = -P$, where $P = \frac{[p_r - 2p_t]}{3}$ is the total pressure. As a result, Eq. (3) takes the form given by $\Theta_{ij} = -2T_{ij} - P g_{ij}$. For the sake of simplicity in calculations, let us consider the linear form of generic function $f(R, T)$ given by $f(R, T) = R + \lambda T$ and consequently, the field Eq. (2) can be re-written as

$$
R_{ij} - \frac{1}{2} R g_{ij} = (8\pi + \lambda)T_{ij} + \frac{\lambda}{2}(\rho - P)g_{ij},
$$

(4)

where $\lambda$ is an arbitrary constant.

In [18], Morris and Thorne introduced the wormhole geometry given by the following metric:

$$
ds^2 = e^{2\Phi(r)}dr^2 - \frac{dr^2}{1-b(r)/r} - r^2(d\theta^2 + \sin^2\theta d\Phi^2),
$$

(5)

where the functions $\Phi(r)$ and $b(r)$ refer to red shift and shape functions, respectively, and it is worthy to mention here that the essential characteristics of a viable wormhole geometry are encoded through these functions. Thus in order to construct the wormhole geometry, these functions must satisfy the constraints which are presented by Morris and Thorne in Ref. [18, 53]. In the present work, we assume $\phi(r)$ without horizons and hence it must be finite everywhere. Assuming single imperfect fluid and using Eqs. (4) and (5), we have

$$
\rho = \frac{1}{12r^2(4\pi + \lambda)(8\pi + \lambda)}\left[ b'(48\pi + \lambda(8 - r\phi^2) + \lambda((2r(2\phi' + r(\phi')^2 + r\phi''))) - b(3\phi' + 2r(\phi')^2 + 2r\phi'') \right],
$$

(6)

$$
p_r = \frac{1}{12r^3(4\pi + \lambda)(8\pi + \lambda)}\left[ r(b'\lambda(4 + r\phi') - 2r(-48\pi \phi' - 10\lambda\phi' + r(\phi')^2 + r\lambda\phi'')) - b(48\pi(1 + 2r\phi' + \lambda(12 + 21r\phi'' - 2r^2((\phi')^2 + \phi'')) \right],
$$

(7)

$$
p_t = \frac{1}{12r^3(4\pi + \lambda)(8\pi + \lambda)}\left[ b\left[ 24\pi(-1 + r\phi' + 2r^2(\phi')^2 + \phi'') + \lambda(-6 + 3r\phi' + 10r^2(\phi')^2 + \phi'') \right] + r\left[ - b\left[ 24\pi(1 + r\phi' + \lambda(2 + 5r\phi')) + 2r(24\pi(\phi' + r(\phi')^2 + r\phi'') + \lambda(4\phi' + 5r((\phi')^2 + \phi''))) \right] \right],
$$

(8)

where prime denotes the derivative with respect to radial coordinate. Also, from the conservation equation $T_{ij}^{\mu} = 0$, the hydrostatic equilibrium equation for the ordinary matter in the interior of wormhole can be defined as

$$
p_r' + \phi'(\rho + p_r) = \frac{\lambda}{(16\pi + 2\lambda)}\left( \rho' - \frac{p_r' + 2p_t'}{3} \right).
$$

(9)

It is obvious that for isotropic fluid, the condition $p_r = p_t$ must be imposed. Using this condition, we can obtain a differential equation connecting both redshift and shape functions given by

$$
\phi'' + \phi' - \frac{(b' + 2r - 3b)\phi'}{2r(r - b)} = \frac{b'r - 3b}{2r^2(r - b)}.
$$

(10)

The above differential equation (DE) can be solved for one of these two unknowns, for example, one can solve it for $b(r)$ by choosing a specific known form of function $\phi(r)$. Consequently, it can be written as

$$
b(r) = \left( \int \frac{2r(-\phi' + r\phi'^2 + r\phi'')e^{\int \frac{2\phi'^2 + \phi''}{2r + r\phi'} dr} - 1}{1 + r\phi'} dr + c \right) \times e^{\int \frac{2\phi'^2 + \phi''}{2r + r\phi'} dr}.
$$

(11)
where $c$ is an integration constant. It is interesting to point out that the $f(R, T)$ field equations has been reduced to two independent differential equations (DEs) given by Eqs. (9) and (13) which involve four unknowns, namely $\phi(r)$, $\rho(r)$, $b(r)$ and $p(r)$ under the assumption of isotropic fluid. Therefore, in order to obtain exact solutions, we have to consider some known forms of two functions and evaluate the others. In the upcoming subsections, we shall present the wormholes solutions by considering some interesting cases and also evaluate the energy constraint bounds in each case.

2.1 Energy conditions

Energy constraints play an important role in the framework of GR as well as alternative gravity theories for exploring the physical viability of proposed models. It is argued that the set of four energy constraints can be formulated by using the well-known Raychaudhuri’s equations and these bounds were firstly developed in GR [54] and then extended to modified gravity theories. These energy constraints are known as the null energy condition (NEC), the weak energy condition (WEC), the dominant energy condition (DEC) and the strong energy condition (SEC). The Raychaudhuri’s equations for congruence of geodesic with time-like and null-like energy constraints are known as the null energy condition (NEC), the weak energy condition (WEC), the dominant energy condition (DEC) and the strong energy condition (SEC). The Raychaudhuri’s equations for congruence of geodesic with time-like and null-like vectors are given by [55]

\[
\frac{d\theta}{d\tau} = -\frac{\theta^2}{3} - \sigma_{ij} \sigma^{ij} + \omega_{ij} \omega^{ij} - R_{ij} u^i u^j,
\]

\[
\frac{d\theta}{d\tau} = -\frac{\theta^2}{3} - \sigma_{ij} \sigma^{ij} + \omega_{ij} \omega^{ij} - R_{ij} k^i k^j,
\]

where $\sigma_{ij}$, $\omega_{ij}$ and $R_{ij}$ are the shear tensor, rotation and Ricci tensor, respectively, while $u^i$ represents the time-like vector and $u^i$ indicates the null-like geodesics in congruence. It will converge when $\theta < 0$ which further leads to $\frac{d\theta}{d\tau} < 0$. By ignoring second-order terms and integrating above equation, we obtain $\theta = -\tau R_{ij} u^i u^j$ and $\theta = -\tau R_{ij} k^i k^j$, where $\sigma_{ij} \sigma^{ij} \geq 0$ and $\omega = 0$. Using these conditions, one can define the following inequalities

\[ R_{ij} u^i u^j \geq 0, \quad R_{ij} k^i k^j \geq 0 \]

which can be re-written as the linear combination of energy momentum tensor and its trace by applying field equations as follows:

\[ (T_{ij} - \frac{Tg_{ij}}{2}) u^i u^j \geq 0, \quad T_{ij} k^i k^j \geq 0. \]

Now, by using above inequalities, we can define the energy constraints as WEC: $\rho = T_{ij} u^i u^j \geq 0$, DEC: $T^{0i} \geq |T^{ij}|$, i.e., in any orthonormal basis, the energy dominates as compared to any other components of $T^{ij}$, while NEC: $T_{ij} k^i k^j \geq 0$ and SEC: $(T_{ij} - \frac{Tg_{ij}}{2}) u^i u^j \geq 0$. The NEC and SEC come from Raychaudhuri equation which depends on the energy momentum tensor and also on alternative gravity theories, while the WEC and DEC only depend on the energy-momentum tensor. In upcoming sections, we shall discuss the energy condition bounds for wormhole models by using these constraints.

2.2 Wormhole with isotropic pressure using zero-tidal force condition in $f(R, T)$ gravity

In this section, we shall discuss the existence of wormhole geometries by taking two cases into account. In the first case, we consider the zero-tidal-force wormhole in which $\phi(r) = \phi_0$, where $\phi_0$ is a constant and consequently, Eq. (14) leads to the form of shape function given by $b(r_0) = r_0$, one can determine the value of constant $c$ and finally, the shape function takes the form as $b(r) = \frac{r_c}{r_r}$.

Consequently, the wormhole spacetime turns out to be

\[ ds^2 = dr^2 - \frac{dr^2}{1 - (r/r_0)^2} - r^2(d\theta^2 + \sin^2 \theta d\Phi^2). \]  

(15)

It is worthy to mention here that the above spacetime represents a metric with constant curvature. It is easy to compute the values of density and pressure and are given by $\rho = -\frac{2(8\pi+\lambda)}{r_0^2(4\pi+\lambda)(6\pi+\lambda)}$, $p = -\frac{4\pi}{r_0^2(4\pi+\lambda)(6\pi+\lambda)}$. Consequently, $\rho$ and $p$ are related by the expression $p = -\rho \frac{2\pi}{(8\pi+\lambda)}$. Also, $b'(r_0) = 3 \neq 1$ which implies that flaring out condition is not satisfied, therefore, no zero tidal force wormhole can be sustained in the present configuration.

In the second case, we consider the constant red shift function and a general form of shape function given by $b(r) = r_0^{\frac{\alpha}{r_0}}$, then M-T wormhole metric takes the form:

\[ ds^2 = dr^2 - \frac{dr^2}{1 - (r/r_0)^{\alpha}} - r^2(d\theta^2 + \sin^2 \theta d\phi^2), \]  

(16)

where $\alpha < 0$. In this case, Eqs. (9)–(12) lead to the following forms of energy density and pressures

\[ \rho = \frac{2(1+\alpha)(\frac{r_c}{r_0})^{\alpha}(6\pi+\lambda)}{3r^2(4\pi+\lambda)(6\pi+\lambda)}, \]  

\[ p = \frac{2(1+\alpha)(\frac{r_c}{r_0})^{\alpha}(6\pi+\lambda)}{3r^2(4\pi+\lambda)(6\pi+\lambda)}, \]  

\[ p = \frac{2(1+\alpha)(\frac{r_c}{r_0})^{\alpha}(6\pi+\lambda)}{3r^2(4\pi+\lambda)(6\pi+\lambda)}. \]
Fig. 1 Evolution of shape function $b(r)$ satisfying the conditions and energy density versus $r$. In the left panel, $b(r)$ (blue), $b'(r)$ (red), $b(r) - r$ (gray) and $\frac{b(r)}{r}$ (black). In the right panel, the plot indicates the behavior of energy density ($\rho$), where the chosen parameter are $r_0 = 1$ and $\alpha = -0.5$

$$p_r = \frac{(-12\pi + (-2 + \alpha)\lambda)(\frac{r}{r_0})^\alpha}{3r^2(4\pi + \lambda)(6\pi + \lambda)},$$

$$p_t = \frac{(-2\lambda + \alpha(12\pi + \lambda))(\frac{r}{r_0})^\alpha}{6r^2(4\pi + \lambda)(6\pi + \lambda)}.$$

For a viable wormhole geometry, the shape function shows increasing behavior and satisfies the conditions given by $b(r) - r < 0$, $b'(r) < 1$ as well as asymptotically flat condition ($\frac{b(r)}{r} \to 0$, $r \to \infty$). It is easy to check that all of these conditions are satisfied as shown in the left graph of Fig. 1. Next we explore the validity of energy conditions of the wormhole matter contents versus radial coordinate. Mathematically, these constraints are defined as WEC ($\rho \geq 0$), NEC ($\rho + p_i \geq 0$, $i = r, t$ represents the radial and tangential pressures), SEC ($\rho + p_r + 2p_t \geq 0$) and DEC ($\rho - p_r$, $\rho - p_t$). The right graph of Fig. 1 shows the graphical behavior of $\rho$ which represents that wormhole geometry satisfies the WEC for $\lambda < -8\pi$. From Fig. 2, it is observed that the expressions $\rho + p_i$, $i = r$ and $\rho + p_r + 2p_t$ attain positive values for $\lambda < -8\pi$, while $\rho + p_t$ exhibits negative behavior for $\lambda < -8\pi$. Thus, only NEC for radial pressure and SEC are satisfied for $\lambda < -8\pi$. Next the graphical behavior of $\rho - p_r$ and $\rho - p_t$ is provided in Fig. 3 which indicates that the DEC with tangential pressure is valid for $\lambda < -8\pi$, whereas DEC with radial pressure violates for $\lambda < -8\pi$.

Fig. 2 Evolution of NEC ($\rho + p_i$) and SEC ($\rho + p_r + 2p_t$) versus $r$, $i = r, t$. In the left panel, the yellow plot indicates the behavior of NEC ($\rho + p_r$) and gray plot shows behavior of NEC ($\rho + p_t$) while in the right panel, the plot indicates the behavior of SEC where the free parameters are fixed as $r_0 = 1$ and $\alpha = -0.5$.
construct a master differential equation for the shape function by using linear equation of state given by equation of state is defined as place, we consider the linear equation of state along with constant red shift function and evaluate the shape function. In this subsection, we will explore the wormhole construction by considering two interesting cases of EoS parameter. In the first

DEC (\( \rho - p_t \)) and properties of shape function given by Eq. (19) versus \( r \). In the left panel, the yellow plot indicates the behavior of DEC (\( \rho - p_t \)), while gray plot represents the behavior of DEC (\( \rho - p_t \)), where \( r_0 = 1 \) and \( \omega = -5 \). In the right panel, shape function (blue), \( b(r) \) (red), \( b(r) - r \) (gray) and \( \frac{\rho}{r^2} \) (black), where the chosen parameters are \( r_0 = 1 \), \( \lambda = 10 \) and \( \omega = -5 \)

2.3 Spherical wormholes with equation of state in \( f(R, T) \) gravity

In this subsection, we will explore the wormhole construction by considering two interesting cases of EoS parameter. In the first place, we consider the linear equation of state along with constant red shift function and evaluate the shape function. The linear equation of state is defined as

\[
p_r = \omega \rho. \tag{17}
\]

Using the above EoS along with Eqs. (9) and (11), we obtain the following differential equation:

\[
\frac{1}{r(4\pi + \lambda)(8\pi + \lambda)} (b(48\pi (1 + 2r\phi') - \lambda(-12 + 3r(-7 + \omega)\phi' + 2r^2(1 + \omega)(\phi'' + \phi)))
\]

\[
+ r(b' (48\pi \omega - \lambda(4 - 8\omega + r\phi' + r\omega \phi')) + 2r(-48\pi \phi' + \lambda(2(-5 + \omega)\phi' + r(1 + \omega)(\phi')^2 + r(1 + \omega)\phi^2))) = 0. \tag{18}
\]

For zero tidal force wormhole, i.e., \( \phi(r) = 0 \), the above differential equation leads to the following form of shape function:

\[
b(r) = C r^{-\frac{3(4\pi + \lambda)}{5 + \omega}}, \tag{19}
\]

where \( C \) is an integration constant and consequently, metric describing the wormhole geometry, energy density and tangential pressure take the form given by

\[
\frac{ds^2}{d\xi^2} = \frac{dr^2}{(r/\rho_0)^{(12\omega + 2\lambda)/(4\pi + \lambda)} - 1} - r^2 (d\theta^2 + \sin^2 \theta d\Phi^2),
\]

\[
\rho = \frac{2(6\pi + \lambda)(\frac{\xi}{\rho_0})^{-\frac{3(4\pi + \lambda + 2\lambda)}{12\omega + 2\lambda}}}{r_0^2 (8\pi + \lambda)(12\pi \omega - \lambda + 2\omega \lambda)},
\]

\[
p_t = \frac{(6\pi (3\omega + 2) - (1 - 3\omega)\lambda)(\frac{\xi}{\rho_0})^{-\frac{3(4\pi + \lambda + 2\lambda)}{12\omega + 2\lambda}}}{3r_0^2 (8\pi + \lambda)(12\pi \omega - \lambda + 2\omega \lambda)}.
\]

Now we will discuss the graphical behavior of \( b(r) \) defined by Eq. (19) and check if it satisfies all axioms for the existence of wormhole solution as discussed in the previous section. From the right plot of Fig. 3, it is clear that the obtained shape function exhibits physically viable behavior. Further, we plot the energy conditions of wormhole matter content versus radial coordinate. It is seen from the graphs of Fig. 4 that the energy density remains positive while the expressions \( \rho + p_t \) and \( \rho + p_t \) attain negative values for \( \lambda > -4\pi \) and thus, the NEC is not satisfied for this case. Also, from the graphs of Fig. 5, it is easy to check that expressions of DEC (\( \rho - p_i \), \( i = r, t \)) indicate positive behavior, while SEC takes negative values for \( \lambda > -4\pi \). Therefore, it can be concluded that DEC is satisfied, while SEC violates in this case.

In the second case, we suppose non-vanishing red shift function, i.e., \( \phi(r) \neq \text{constant} \). For wormhole construction, we shall construct a master differential equation for the shape function by using linear equation of state given by

\[
p_r = p_t = \omega \rho. \tag{20}
\]
Fig. 4 Evolution of energy density $\rho$ and NEC $(\rho + p_i, i = r, t)$ versus $r$. In the left panel, the plot indicates the behavior of $\rho$ and in right panel, the gray plot indicates the behavior of $\rho + p_r$, while orange plot indicates the behavior of $(\rho + p_t)$. Here we fixed $r_0 = 1$ and $\omega = -5$.

Fig. 5 Evolution of DEC $(\rho - p_i, i = r, t)$ and SEC $(\rho + p_r + 2p_t)$ versus $r$. In the left panel, the orange plot indicates the behavior of $\rho - p_r$, while gray plot indicates the behavior of $\rho - p_t$ and in right panel, the plot indicates the behavior of $\rho + p_r + 2p_t$

where $\omega$ is a constant. Thus from Eq. (13), it can be written as

$$\omega \rho' + \phi'(1 + \omega)\rho = \frac{\lambda(1 - \omega)\rho'}{16\pi + 2\lambda}$$

(21)

and consequently, the energy density $\rho$ takes the form as follows:

$$\rho(r) = ce^\frac{(16\omega \phi'(16\pi + 2\lambda)\phi(r))}{16(\omega + 3\lambda)}$$

(22)

where $c$ is an integration constant. It is interesting to mention here that if we consider vanishing red shift function, then we shall obtain $\rho$ as a constant and this case we have already discussed in the previous subsection. From Eqs. (9), (13) and (20), we obtain a differential equation for $b(r)$ as

$$(6 - 3r - r^2)(-16\pi + \lambda(-3 + \omega))^2b^2 + 2r^2( - (16\pi - \lambda(-3 + \omega))(16\pi(-1 + r + 2\omega) - \lambda(1 + r(-3 + \omega))n + 5\omega))b' + (-128\pi^2\omega(\omega - 1) + \lambda^2(1 - 2\omega - 3\omega^2) + 8\pi\lambda(1 - 2\omega + 5\omega^2))b^2 + r(256\pi^2\omega - 16\lambda\pi(-1 + \omega^2) + \lambda^2(3 - 10\omega + 3\omega^2))b'' + r(16\pi - \lambda(-3 + \omega))b((16\pi(-2 + r + 3\omega + 2r\omega) - \lambda(3 - 5\omega + 7r\omega)b')b' - 2(-3 + 2r)(16\pi + 3\lambda - \lambda\omega) - r(\lambda + 16\omega\pi - 3\omega\lambda)b'')) = 0.$$  

(23)

It is very difficult to find the analytic solution of above differential equation without assuming any condition. However, one may find the solution of this equation by taking some assumptions into account. Since first term of Eq. (24) vanishes when $\omega = \frac{16\pi + 3\lambda}{2\lambda}$, so by taking this condition, it is easy to obtain the solution: $b(r) = \Lambda$, $e^{2\phi(r)} = 1 - \frac{A}{r}$ and $\rho = p = 0$ which represents the Schwarzschild solution. It is worthy to mention here that Schwarzschild solution describes the non-traversable wormhole [53].
Fig. 6 Evolution of energy density versus \( r \) and NEC. The left plot indicates the behavior of energy density. In the right panel, the gray plot indicates the behavior of \( \rho + p_r \) and orange plot shows the behavior of \( \rho + p_t \) for \( r_0 = 1 \) and \( \beta = .6 \).

Fig. 7 Evolution of SEC (\( \rho + p_r + 2p_t \)) and DEC (\( \rho - p_l \)) versus \( r \). In the left panel, the plot indicates the behavior of SEC and in the right panel, the plot indicates the behavior of DEC, where \( \rho - p_r \) (gray) and \( \rho - p_t \) (orange) for \( r_0 = 1 \) and \( \beta = .6 \).

we consider the power law form of shape function given by \( b(r) = \frac{A}{r^n} \) which is a viable choice as it ensures the validity of M-T constraint given by \( \frac{2n}{r} \leq 1 \) for \( n > -1 \) and \( r \to \infty \). On substitution of this choice \( b(r) = \frac{A}{r^n} \) into Eq. (24), it is easy to check that this equation will be satisfied when the constraints \( n = -3 \) or \( \frac{3}{2} \) hold (for \( \omega = \frac{16\pi + 3\lambda}{\lambda} \) and \( \lambda = \pm 4\sqrt{2}\pi \)).

2.4 Spherical wormhole with \( e^{\phi(r)} = \left(\frac{r}{r_0}\right)\beta \) in \( f(R, T) \) gravity

In this subsection, we shall study the spherical wormhole solutions by imposing the most general form of red shift function given by \( e^{\phi(r)} = \left(\frac{r}{r_0}\right)^\beta \), where \( \beta \) is any arbitrary constant and try to explore the corresponding form of shape function. By inserting \( e^{\phi(r)} = \left(\frac{r}{r_0}\right)^\beta \) in Eq. (13), we obtain the shape function given by

\[
b(r) = \frac{\beta^2 - 2\beta}{\beta(\beta - 2) - 1} - cr^\frac{2\beta^2 - 5\beta - 3}{\beta^2 - \beta - 1},
\]

where \( c \) is an integration constant.

By considering the wormhole condition \( b(r_0) = r_0 \) in Eq. (24), we obtain the following expression for wormhole metric, \( \rho \) and \( p \):

\[
\text{d}s^2 = \left(\frac{r}{r_0}\right)^\beta \text{d}t^2 - \frac{\text{d}r^2}{1 - \beta^2 - 2\beta - 1 - c\left(\frac{r}{r_0}\right)^{2\beta^2 - 2\beta - 1}} - r^2 \text{d}\Omega^2,
\]

\[
\rho = \frac{1}{12r^2(4\pi + \lambda)(8\pi + \lambda)} \left[ \frac{6\beta(8\pi(-2 + \beta) + (-3 + \beta)\lambda)}{\beta(\beta - 2) - 1} + \frac{12\left(\frac{r}{r_0}\right)^{-2\beta^2 - 2\beta - 1}(1 + 2\beta)(4\pi(-3 + \beta) + (-2 + \beta)\lambda)}{(1 + \beta)(\beta(\beta - 2) - 1)} \right].
\]
In the following subsections, we shall discuss these cases separately and present the graphical analysis of obtained solutions. As we are interested in exploring the impact of matter terms (trace of energy-momentum tensor) involved in this theory, therefore we will fix the value of $\beta$ and plot $\rho$ in a wider range of $\lambda$. Moreover, we choose $r_0 = 1$ and $\beta = 0.6$ and analyze all energy conditions. It is easy to see that these constraints are valid for $\lambda > -4\pi$ as shown in the graphs of Figs. 6 and 7.

3 Field Equations for hyperbolic wormhole space time in $f(R, T)$ Gravity

In this section, we shall describe different cases of wormhole construction by taking hyperbolic spacetime into account. The spacetime representing wormhole geometry in hyperbolic coordinates is defined as

$$ds^2 = e^{2\Phi(r)}dr^2 - \frac{dr^2}{1 - b(r)/r} - r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

where $\Phi$ and $b(r)$ have the usual meaning, i.e., wormhole red shift and shape functions, respectively. From Eqs. (4) and (27), we obtain the following set of field equations:

$$\frac{b'}{r^2} = (8\pi + \lambda)\rho + \frac{\lambda}{2} \left( \rho - \frac{p_r + 2p_t}{3} \right),$$

$$\frac{2\phi'}{r} (1 - \frac{b}{r}) - \frac{b}{r^2} = (8\pi + \lambda)(-p_r) + \frac{\lambda}{2} \left( \rho - \frac{p_r + 2p_t}{3} \right),$$

$$\frac{1}{2r} \left[ \frac{1}{r} (\phi' b + b' - \frac{b}{r}) + 2(\phi'' + (\phi')^2) b - \phi'(2 - b') \right] = (8\pi + \lambda)(-p_t) + \frac{\lambda}{2} \left( \rho - \frac{p_r + 2p_t}{3} \right).$$

From the above equations, the explicit form of energy density, radial and tangential pressures denoted by symbols $\rho$, $p_r$, and $p_t$, respectively, can be obtained as follows:

$$\rho = \frac{1}{12r^3(4\pi + \lambda)(8\pi + \lambda)} \left[ (48(-2 + b')\pi r + \lambda(b(2 - r\phi' - 2r^2(\phi')^2 + \phi'')) + r(b' (8 - r\phi') + 2(-10 + r^2(\phi')^2 + \phi''))) \right]$$

$$p_r = \frac{1}{12r^3(4\pi + \lambda)(8\pi + \lambda)} \left[ b(48\pi(1 + 2r\phi') + \lambda(10 + 23r\phi' + 2r^2((\phi')^2 + \phi''))) - r(96\pi(1 + r\phi') + \lambda(-b' (4 + 12r\phi' + r^2((\phi')^2 + \phi'')))) \right]$$

$$p_t = \frac{1}{12r^3(4\pi + \lambda)(8\pi + \lambda)} \left[ -b \left[ 24\pi(-1 + r\phi' + 2r^2(\phi')^2 + \phi'')) + \lambda(-4 + 7r\phi' + 10r^2((\phi')^2 + \phi'')) \right] + r \left[ -b' (24\pi(1 + r\phi') + \lambda(2 + 5r\phi')) + 2(24\pi(\phi' + r(\phi')^2 + r\phi'') + \lambda(-2 + 6r\phi' + 5r^2((\phi')^2 + \phi'')))) \right] \right],$$

where the prime denotes the derivatives with respect to radial coordinate. It is worthy to mention here that the conservation equation for hyperbolic spherically symmetric spacetime is same as for symmetric metric and hence given by Eq. (12). It is easy to check that we have system of three equations with five unknown functions. To develop solutions representing hyperbolic wormholes, we have to fix two of these unknowns. In this work, we will solve these equations by considering the following three possibilities:

- EoS for radial or tangential pressure;
- Condition for isotropic pressure, i.e., $p_r = p_t$;
- Some known and interesting choices for $\phi(r)$ and $b(r)$.

In the following subsections, we shall discuss these cases separately and present the graphical analysis of obtained solutions.
Evolution of shape function given by Eq. (ref31) to check its consistency with the axioms. Here, shape function (blue dashes), \( b'(r) \) (red dashes), \( b(r) - r \) (gray dashes) and \( \frac{dr}{d\rho} \) (black), where the chosen parameters are \( \omega = -16, \lambda = -15 \) and \( r_0 = 1 \).

Evolution of energy density and NEC \( (\rho + p_r) \) versus \( r \). In the left panel, the plot indicates the behavior of energy density. In the right panel, the gray plot shows the behavior of NEC \( (\rho + p_r) \) and orange plot shows the behavior of NEC \( (\rho + p_t) \), where \( \omega = -16 \) and \( r_0 = 1 \).

3.1 Hyperbolic wormhole with equation of state in \( f(R, T) \) gravity

In this subsection, we shall try to explore the existence of pseudospherical wormhole solutions filled by the matter with anisotropic pressure by taking the following EoS for the radial pressure into account:

\[ p_r = \omega \rho. \tag{34} \]

From Eq. (32) and the above EoS parameter, it can be written as

\[
\frac{1}{r(4\pi + \lambda)(8\pi + \lambda)} \left[ -b(48\pi (1 + 2r\phi' + \lambda(-r\phi'(-23 + \omega) - 2(-5 + \omega) + 2r^2((\phi')^2 + \phi'')(1 + \omega))) \right. \\
\left. + r(48\pi (2 + 2r\phi' + (-2 + b')\omega) + \lambda(-b'(-4 - 8\omega + r\phi' (1 + \omega))) + 2(14 + 12r\phi' - 10\omega + r^2(\phi')^2 + \phi'')(1 + \omega))) \right] = 0.
\]

For zero tidal force wormhole, one needs to impose the condition \( \phi(r) = 0 \) and consequently, we obtain

\[ b(r) = 2r + A(2r(\lambda - 12\pi \omega - 2\lambda \omega))^{\frac{24\pi + 5\lambda - 5\lambda \omega}{24\pi + 5\lambda - 5\lambda \omega}}, \tag{35} \]

where \( A \) is an integration constant. For this shape function, the graphical illustration of different axioms which are necessary for a viable wormhole geometry is provided in Fig. 8.

Consequently, one may write wormhole metric, energy density and tangential pressure as follows:

\[
ds^2 = dt^2 - \frac{dr^2}{(\rho)^{\frac{1}{3}} - \frac{24\pi + 5\lambda - 5\lambda \omega}{24\pi + 5\lambda - 5\lambda \omega}} - r^2(d\theta^2 + \sinh^2 \theta d\phi^2), \tag{36} \\
\rho = -\frac{3r_0(\rho)^{\frac{24\pi + 5\lambda - 5\lambda \omega}{24\pi + 5\lambda - 5\lambda \omega}}}{r^3(24\pi \omega - 2\lambda + 4\lambda \omega)}. \tag{37} \]
Fig. 10 Evolution of SEC ($\rho + p_r + 2p_t$) and DEC ($\rho - p_r$) versus $r$. In the left panel, the plot indicates the behavior of SEC. In the right panel, the gray plot shows the behavior of DEC ($\rho - p_r$) and orange plot shows the behavior of DEC ($\rho - p_t$) for $\omega = -16$ and $r_0 = 1$.

Next we shall describe the energy conditions of wormhole matter content graphically with respect to radial coordinate. These graphs are provided in Figs. 9 and 10. The left panel of Fig. 9 provides the validity of WEC, i.e., $\rho \geq 0$ which indicates that energy density remains positive for $\lambda < 0$. From the right panel of same graph, it is easy to observe that NEC for both radial and tangential pressures remain invalid. Likewise, the left and right graphs of Fig. 10 show that SEC violates, while DEC remains valid for both radial and tangential pressures. Thus, it can be concluded that the WEC and DEC are valid in the present setup, whereas the SEC and NEC are incompatible for the choice $\lambda < 0$.

3.2 Zero tidal force hyperbolic wormholes along with isotropic fluid or a power law shape function in $f(R, T)$ gravity

In this section, we shall construct wormhole geometry by considering two cases. In the first case, we shall construct M-T hyperbolic wormhole by taking an interesting choice of shape function given by $b(r) = r_0 (\frac{r}{r_0})^\alpha$ along with zero red shift function, i.e., $\phi(r) = 0$. The graphical illustration of this shape function has already been discussed in the previous section which indicated that it is viable choice for $\alpha < 0$. It is worthy to mention here that in case of hyperbolic geometry, the behavior of this shape function is accurate and consistent with all axioms for $\alpha > 0$. Inserting the value of red shift as well as shape function in Eq. (34), we obtain the metric, $\rho$ and $p$ in the following form:

\begin{equation}
\rho = \frac{3r_0 (-1 + \omega)(\frac{r}{r_0})^{24\pi \omega - 2\lambda + 4\lambda \omega}}{2r^3(24\pi \omega - 2\lambda + 4\lambda \omega)}.
\end{equation}

(38)

Next we shall describe the energy conditions of wormhole matter content graphically with respect to radial coordinate. These graphs are provided in Figs. 9 and 10. The left panel of Fig. 9 provides the validity of WEC, i.e., $\rho \geq 0$ which indicates that energy density remains positive for $\lambda < 0$. From the right panel of same graph, it is easy to observe that NEC for both radial and tangential pressures remain invalid. Likewise, the left and right graphs of Fig. 10 show that SEC violates, while DEC remains valid for both radial and tangential pressures. Thus, it can be concluded that the WEC and DEC are valid in the present setup, whereas the SEC and NEC are incompatible for the choice $\lambda < 0$.
Fig. 12 Evolution of SEC \((\rho + p_r + 2p_t)\) and DEC \((\rho - p_t)\) versus \(r\). In the left panel, the plot shows the behavior of SEC. In the right panel, gray plot represents the behavior of DEC \((\rho - p_r)\) and yellow plot represents the behavior of DEC \((\rho - p_t)\) with \(\alpha = 0.5\) and \(r_0 = 1\).

\[
\rho = \frac{\pi (-48r + 24\alpha r_0(\frac{L}{r_0})^2) + (-10r + (1 + 4\alpha)r_0(\frac{L}{r_0})^2)\lambda}{6r^3(4\pi + \lambda)(8\pi + \lambda)},
\]

\[(40)\]

\[
p_t = \frac{\pi (-24\pi - 4\lambda)r_0((\frac{L}{r_0})^2) + r(-4\lambda + (24\pi + 2\lambda)\alpha(\frac{L}{r_0})^2)}{12r^3(4\pi + \lambda)(8\pi + \lambda)}.
\]

\[(41)\]

For the obtained solutions, we plot the models of all energy conditions versus radial coordinates as shown in Figs. 11 and 12. From these graphs, it can be seen that all energy conditions are valid for \(\lambda < -30\) except the DEC \((\rho - p_r)\) for radial pressure.

In the second case, we will construct new pseudospherical wormholes with isotropic pressure by imposing the condition given by \(p_r = p_t\). For hyperbolic spacetime, using Eq. (32), we obtain the following differential equation:

\[
\phi'' + \phi'^2 + \frac{(b' r - 3r + 5b)\phi'}{2r(b - r)} = -\frac{b' r - b + 4r}{2r^2(b - r)}
\]

\[(42)\]

which must be satisfied by the shape and red shift functions. This differential equation can be re-arranged and the shape function can be determined as

\[
b(r) = \left(\int \frac{2(2 + 3r\phi' + r^2(\phi'^2 + \phi^{''})e^{-\int \frac{1 + 2r^2 + 5\lambda r^2}{r(1 + r\phi)\phi} dr}}{1 + r\phi} dr + B\right) \times e^{-\int \frac{1 + 2r^2 + 5\lambda r^2}{r(1 + r\phi)\phi} dr},
\]

\[(43)\]

where \(B\) is an integration constant. Now we have two set of differential equations with four unknowns, namely shape function, red shift function, density and pressure. For solving these equations, one should use some valid assumptions. In the present work, we pick an interesting choice of red shift function given by \(\phi(r) = \phi_0\) = constant (zero tidal force). By inserting this condition in Eq. (43), we obtain a spacetime with constant curvature defined by \(b(r) = 2r + \frac{B}{r}\). In this case, the wormhole spacetime, energy density and isotropic pressure can be written as

\[
ds^2 = dt^2 - \frac{dr^2}{(\frac{L}{r_0})^2} - r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]

\[
\rho = \frac{\alpha (r_0)^2}{2r^4(4\pi + \lambda)},
\]

\[
p_r = p_t = -\frac{(r_0)^2}{2r^4(4\pi + \lambda)}.
\]

Also, by evaluating the flaring out condition at throat, we obtain \(b'(r_0) = 2 \neq 1\) which shows that no zero tidal force hyperbolic wormhole exists in \(f(R, T)\) gravity.
3.3 Hyperbolic wormhole with non-vanishing redshift function and isotropic pressure in $f(R, T)$

Here we shall consider an interesting power law form of red shift function given by $e^{\phi(r)} = (\frac{r}{r_0})^\beta$ with $\beta$ as some arbitrary constant. By inserting the value of $e^{\phi(r)}$ in Eq. (42), we obtain the following form of shape function given by

$$b(r) = \frac{\beta^2 - 2\beta - 2}{\beta(\beta - 2) - 1} r - c r \frac{2^{\beta - 2\beta - 3}}{r^{1+\beta}},$$

(44)

where $c$ is a constant of integration. The graphical behavior of this shape function is shown in Fig. 13 which indicates that all axioms, essential for a valid shape function, are satisfied.

By imposing, the wormhole condition $b(r_0) = r_0$, Eq. (44) leads to the following expression for wormhole metric, $\rho$ and $p$:

$$\rho = \frac{1}{6r^3(1 + \beta)(-1 - 2\beta + \beta^2)(8\pi + \lambda)(4\pi + \lambda)} \left[ (24\pi r\beta^2(2\beta + \beta^2) + \left( \frac{r}{r_0} \right)^{(3+5\beta-2\beta^2)/(1+\beta)} r_0(-3 - 5\beta + 2\beta^2)) 
+ (r\beta(9 + 5\beta - 4\beta^2) + \left( \frac{r}{r_0} \right)^{(3+5\beta-2\beta^2)/(1+\beta)} r_0(-13 - 21\beta + 10\beta^2)\lambda) \right].$$

$$p_r = \frac{1}{6r^3(1 + \beta)(-1 - 2\beta + \beta^2)(8\pi + \lambda)(4\pi + \lambda)} \left[ (24\pi(1 + \beta)(r\beta^2) + \left( \frac{r}{r_0} \right)^{(3+5\beta-2\beta^2)/(1+\beta)} r_0(1 + 2\beta)) 
+ (r\beta(3 + 7\beta + 4\beta^2) + \left( \frac{r}{r_0} \right)^{(3+5\beta-2\beta^2)/(1+\beta)} r_0(1 + 9\beta + 14\beta^2)\lambda) \right].$$

By the re-scaling $\xi^2 = (\beta(\beta - 2) - 1)r^2$ of the above wormhole metric, it can be written as

$$ds^2 = \left( \frac{\xi}{\xi_0} \right)^{2\beta} dr^2 - \frac{d\xi^2}{\left( \frac{\xi}{\xi_0} \right)^{(\beta-2)/(1+\beta)} - 1} - \frac{\xi^2}{(\beta(\beta - 2) - 1)} (d\theta^2 + \sin^2 \theta d\phi^2).$$

Since our main objective is to explore the impact of matter terms in the considered gravitational theory, therefore we fix the value of $\beta$ and plot energy density by varying the value of $\lambda$. Moreover, we choose $r_0 = 1$ and $\beta = 3.5$. We analyze the energy conditions graphically for the obtained model. From the right graph of Fig. 13, it is seen that the energy density remains positive versus radial coordinate. From the graphs of Fig. 14, it can be observed that the NEC ($\rho + p_r \geq 0$) is valid for $\lambda < -35$ while for tangential pressure, it is invalid. From the same Figure, it is seen that the validity of DEC ($\rho - p_i \geq 0$) is guaranteed for tangential pressure, whereas SEC violates for $\lambda < -35$ as shown in Fig. 15.)
Fig. 14 Evolution of NEC ($\rho + p_r$) and DEC ($\rho - p_t$) versus $r$. In the left panel, $\rho + p_r$ (yellow plot) and $\rho + p_t$ (gray plot) while, in the right panel, $\rho - p_r$ (yellow plot) and $\rho - p_t$ (gray plot) for $r_0 = .1$ and $\beta = 3.5$.

Fig. 15 Evolution of SEC ($\rho + p_r + 2p_t$) versus radial coordinate.

4 Conclusion and final remarks

The construction of viable wormhole geometries in modified gravity theories has always been a fascinating topic for researchers. In the present work, we propose interesting exact solutions for both classes of static spherically symmetric and pseudospherically symmetric spacetimes representing wormholes in the well-famed $f(R, T)$ gravitational framework. The considered gravitational theory is an interesting choice in this respect as it allows the interaction of curvature and matter. For the sake of simplicity, we have considered linear model of $f(R, T)$ function representing minimal interaction of matter and curvature. Our primary objective is to check what ranges of coupling parameter $\lambda$, present in $f(R, T)$ function, can allow the existence of wormholes as well as the validity of energy condition bounds. In this work, we have obtained some specific solutions either by considering EoS or by assuming some interesting choices of red shift or shape functions. The obtained solutions are then examined graphically and the validity of energy condition bounds have been explored in each case. It is seen that for all obtained spherically symmetric wormholes, we have positive energy density at the throat and radial pressure is negative for $\lambda > -4\pi$ and $\lambda < -8\pi$ satisfying all axioms for shape function. For pseudospherically symmetric wormhole, energy density is positive for all conditions and radial pressure may be positive or negative for different ranges of $\lambda$.

For zero tidal force wormhole one needs to assume $\phi(r) = \text{constant}$, and we have discussed two possibilities. Firstly, we obtain shape function using the main condition for perfect fluid, i.e., $p_r = p_t$. Secondly, we have imposed a restricted choice of shape function for both spherically and hyperbolic symmetric metrics. In both cases, energy density remains positive and radial pressure is negative for $\lambda > -4\pi$, but for pseudospherically symmetric wormhole, the energy density is also positive and radial pressure is negative for $\lambda > -4\pi$ in first possibility. But for the restricted choice of shape function, the energy density and pressure are positive for $\lambda \leq -10\pi$.
Table 1 In this table, we have listed all obtained wormhole solutions. It indicates that which of the energy conditions, i.e., WEC, NEC, DEC and SEC are valid. It is worthy to mention here that for wormhole with isotropic pressure, the SEC becomes $\rho + 3p$. Also, here $d\Omega_2^2 = d\theta^2 + r^2 \sin^2 \theta d\Phi^2$ and $d\Omega_{pr}^2 = d\theta^2 + r^2 \sin^2 \theta d\Phi^2$.

| Metric | Constraints | WEC ($\rho \geq 0$) | NEC ($\rho + pr \geq 0$) | DEC ($\rho - pr \geq 0$) | SEC ($\rho + pr + 2pr \geq 0$) |
|--------|-------------|---------------------|-------------------------|------------------------|-------------------------------|
| $ds^2 = dr^2 - \frac{dr^2}{1 - \left(\frac{r}{r_0}\right)^{w-1}} - r^2 d\Omega^2$ | $\phi(r) = \text{constant}$, $b(r) = r_0 \left(\frac{r}{r_0}\right)^{\alpha}$, $\lambda < -8\pi$ | Yes | No | Yes | Yes |
| $ds^2 = dr^2 - \frac{dr^2}{1 - \left(\frac{r}{r_0}\right)^{w-1}} - r^2 d\Omega^2$ | $\phi(r) = 0$, $b(r) = r_0 \left(\frac{r}{r_0}\right)^{\frac{-3\lambda_2}{2}}$, $\lambda \geq 12\pi + 2\lambda$ | Yes | No | Yes | No |
| $-r^2 d\Omega^2$, $\lambda_1 = 12\pi + 2\lambda$, $\lambda_2 = 12\pi \omega - \lambda + 2\omega \lambda$, $pr = \omega \rho$, $\lambda > -4\pi$ | $b(r) = r_0 \left(\frac{r}{r_0}\right)^{\frac{\alpha}{2}}$, $\lambda_3 = 4\pi + \lambda$, $\lambda > -4\pi$ | No | Yes | Yes | Yes |
| $ds^2 = (r \frac{r}{r_0})^2 dr^2 - \frac{\beta_1 dr^2}{\left(\frac{r}{r_0}\right)^{2\beta}} - r^2 d\Omega^2$ | $\phi(r) = \ln \left(\frac{r}{r_0}\right)^{2\beta}$, $b(r) = \frac{(\beta^2 - 2\beta) \nu}{pr}$, $\lambda > -4\pi$ | Yes | Yes | Yes | No |
| $-r^2 d\Omega^2$, $\beta_1 = \beta^2 - 2\beta - 1$ | $b(r) = \frac{(\beta^2 - 2\beta) \nu}{pr}$, $\lambda > -4\pi$ | Yes | Yes | Yes | No |
| $ds^2 = dr^2 - \frac{dr^2}{1 - \left(\frac{r}{r_0}\right)^{w-1}} - r^2 d\Omega^2$ | $\phi(r) = 0$, $b(r) = r_0 \left(\frac{r}{r_0}\right)^{\frac{3\lambda_3}{2}}$, $\lambda_2 = -24\pi (\omega - 1) + \lambda (7 - 5\omega)$ | Yes | No | Yes | No |
| $\nu = 2\pi + \lambda (-5 + \omega)$, $pr = \omega \rho$, $\lambda < 0$ | $b(r) = 2r - r_0 \left(\frac{r}{r_0}\right)^{\frac{\Omega_1}{2}}$, $\lambda_3 = 2\pi + \lambda (7 - 5\omega)$ | No | Yes | Yes | Yes |
| $ds^2 = dr^2 - \frac{dr^2}{1 - \left(\frac{r}{r_0}\right)^{w-1}} - r^2 d\Omega^2$ | $\phi(r) = 0$, $b(r) = r_0 \left(\frac{r}{r_0}\right)^{\frac{\alpha}{2}}$, $\lambda \leq -10\pi$ | Yes | Yes | No | Yes |
| $r^2 d\Omega_{pr}^2$, $\xi_1 = 24\pi \omega - 2\lambda + 4\omega \lambda$, $\xi_2 = -24\pi (\omega - 1) + \lambda (7 - 5\omega)$ | $b(r) = 2r - r_0 \left(\frac{r}{r_0}\right)^{\frac{\Omega_1}{2}}$, $\lambda \leq -10\pi$ | No | Yes | Yes | Yes |
| $pr = \omega \rho$, $\lambda < 0$ | $b(r) = \frac{(\beta^2 - 2\beta - 2) \nu}{pr}$, $\lambda \leq -12\pi$ | No | Yes | No | No |

We have also discussed a possibility with isotropic fluid using the linear EoS. We have obtained the shape function by imposing zero-tidal force condition for red shift function in both symmetric and pseudospherically symmetric wormhole cases by taking linear EoS into account. We have observed that energy density is positive and radial pressure is negative for $\lambda > -4\pi$ for spherically symmetric wormhole, whereas for hyperbolic wormhole energy density remains positive and pressure turns out as negative for $\lambda < 0$. Lastly, we have imposed a restricted choice of red shift function to evaluate the most general form of $b(r)$ using isotropic pressure condition. For this condition, in case of symmetric wormhole, same results have been obtained for energy density and radial pressure for $\lambda > -4\pi$. In case of hyperbolic wormhole, both energy density and radial pressure are positive for $\lambda \leq -12\pi$.

At the end, we have applied the energy conditions to material content of symmetric and pseudospherically symmetric wormhole solutions for all restricted choices of red shift and shape functions as well as linear equation of state with isotropic pressure. One can easily observe from all figures that either NEC, DEC and SEC are violated or valid for different ranges of $\lambda$, while WEC is always valid for both symmetric and pseudospherically symmetric wormholes. A brief comparison of both classes of wormholes is listed in Table 1. In this table, we indicate that which energy condition is satisfied or remained invalid corresponding to specified wormhole geometry, which can be explained as follows:
1. In case of spherical wormhole,

- When \( \phi(r) \) is constant and \( b(r) = r_0 \left( \frac{r}{r_0} \right)^\beta \), the WEC, SEC, NEC with transverse pressure and DEC with radial pressure are valid for \( \lambda < -8\pi \), while the NEC with radial pressure and DEC with transverse pressure are not satisfied for \( \lambda < -8\pi \).

- For \( \phi(r) = 0 \) and \( b(r) = r_0 \left( \frac{r}{r_0} \right)^{-\Delta \omega_1(\Delta \omega_1+1)} \), the WEC and DEC are satisfied for \( \lambda > -4\pi \), while the NEC and SEC are not satisfied for \( \lambda > -4\pi \).

- For \( \phi(r) = \ln(\frac{r}{r_0})^{2\beta} \) and \( b(r) = \frac{3}{2} \left[ \frac{r^2 - 2\beta - 2\pi}{r^2 - 2\beta - 1} \right] r - \frac{r_0}{\Phi_1} \left( \frac{r}{r_0} \right)^{-\Delta \omega_1(\Delta \omega_1+1)-\Delta \omega_1} \), all energy constraints are valid except SEC for \( \lambda > -4\pi \).

2. In case of hyperbolic wormhole,

- When \( \phi(r) = 0 \) and \( b(r) = 2r - r_0 \left( \frac{r}{r_0} \right)^{2\pi \beta \omega_1(\Delta \omega_1-1)} \), the WEC and DEC are valid for \( \lambda < 0 \), while the NEC and SEC are not valid for \( \lambda < 0 \).

- For \( \phi(r) = 0 \) and \( b(r) = r_0 \left( \frac{r}{r_0} \right)^{2\pi \beta \omega_1(\Delta \omega_1-1)} \), the WEC, NEC and SEC are valid, while the DEC is violated for \( \lambda \leq -10\pi \).

- For \( \phi(r) = \ln(\frac{r}{r_0})^{2\beta} \) and \( b(r) = \frac{3}{2} \left[ \frac{r^2 - 2\pi \omega_1(\Delta \omega_1+1)-\Delta \omega_1}{r^2 - 2\pi \omega_1(\Delta \omega_1+1)-1} \right] r - \frac{r_0}{\Phi_1} \left( \frac{r}{r_0} \right)^{-\Delta \omega_1(\Delta \omega_1+1)-\Delta \omega_1} \), the WEC, the NEC with radial pressure and the DEC with transverse pressure are satisfied for \( \lambda \leq -12\pi \), while the SEC and NEC, and DEC with other components of pressure are not satisfied for \( \lambda \leq -12\pi \).

Many authors have studied the topic of wormholes existence in the framework of \( f(R, T) \) theory. In this context, Sahoo and his collaborators [29] studied the spherically symmetric wormhole solution for phantom case, i.e., \( \omega < -1 \) in \( f(R, T) \) modified gravity. They adopted the EoS given by \( p_r = \omega p_t \) to obtain the form of shape function which can satisfy all the necessary axioms for the existence of wormhole geometry and also examined corresponding energy condition bounds. Similarly, Moraes and Sahoo [30] proposed the wormhole solution by applying the power law form of shape function and analyzed all energy constraints in exponential \( f(R, T) \) gravity. In another study, Zubair et al. [46] proposed wormhole geometries by adopting simple linear as well as cubic forms of \( f(R, T) \) function and further they considered the non-commutative geometrical aspects of string theory in the context of \( f(R, T) \) gravity. In a paper [47], the researchers have explored the idea of viable charged wormhole solutions in \( f(R, T) \) gravity. They have assumed simple linear generic model given by \( f(R, T) = R + 2T \) along with the ordinary matter as the total pressure of anisotropic fluid. Further, the existence of static wormhole model have been explored by some researchers where they utilized different kind of shape functions [48].

In another paper [49], authors have investigated the wormhole modeling by considering a specific general shape function in the quadratic \( f(R, T) \) gravity. In a recent papers [50], Banerjee and his collaborator adopted different strategies to construct wormhole geometry using isotropic pressure in \( f(R, T) \) theory and concluded that all energy constraints are valid for the proposed models, while Bhar and his collaborators [56], proposed the M-T wormhole solution which admits conformal motion in \( f(R, T) \) gravity. They employed the phantom energy EoS, i.e., \( p_r = \omega p_t \), \( \omega < -1 \) to constraint their model and showed that wormhole solution exist for both positive and negative values of coupling constant \( \lambda \).

In majority of the previously listed works, authors have considered a specific form of shape function and analyze the possibility of static and spherically symmetric wormhole existence satisfying energy condition bounds. In the present work, we have explored both the spherically symmetric and pseudospherically symmetric wormhole solutions for M-T spacetime in \( f(R, T) \) theory. The present work can be considered as an extension of the papers [20, 23], where authors have discussed similar cases of wormhole existence by using spherically and pseudospherically symmetric spacetimes in the framework of GR. In this work, we have not only checked the validity of energy bounds by taking some type of shape functions into account but also explored the possible forms of \( \phi(r) \) and \( b(r) \) functions by taking the matter with EoS parameter for \( p_r \) or \( p_t \) through \( f(R, T) \) field equations. Further, we have presented the validity of energy conditions bounds for both spherical and hyperbolic wormholes in the presence of single perfect fluid, i.e., a source of matter with isotropic pressure with a wider range of parameter \( \lambda \). Up to the best of our knowledge, the only wormhole solution discussed in literature are non-asymptotically flat wormhole with isotropic pressure as pointed out in the References [51–53]. In the present work, we have discussed the asymptotically flat wormholes which may or may not satisfy all energy constraints with a wider range of \( \lambda \) for both types of wormhole spacetimes.

Acknowledgements  M. Zubair thank the Higher Education Commission, Islamabad, Pakistan, for its financial support under the NRPU Project with Grant Number 5329/Federal/NRPU/R & D/HEC/2016.

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