Circle actions on 6-dimensional oriented manifolds with 4 fixed points

Donghoon Jang

Abstract. In this paper, we classify the fixed point data (weights and signs at the fixed points), of a circle action on a 6-dimensional compact connected oriented manifold with 4 fixed points. We prove that it agrees with that of an equivariant connected sum along free orbits of rotations on two 6-spheres, or that of a linear action on \( \mathbb{C}P^3 \). The former case includes that of Petrie’s exotic action on \( \mathbb{C}P^3 \).

Mathematics Subject Classification. 58C30.

Contents

1. Introduction 1
2. Background and preliminaries 4
3. Special multigraph 7
4. Dimension 6 and 4 fixed points: possible multigraphs 10
5. Proof of the main theorem 11
6. Comparison with results on different types of manifolds 19
7. Converting the fixed point data into the empty collection 20
References 22

1. Introduction

Let the circle group \( S^1 \) act on a compact connected oriented manifold \( M \) with a non-empty finite fixed point set. Let \( p \) be an isolated fixed point. The tangent space \( T_p M \) at \( p \) decomposes into \( n \) irreducible \( S^1 \)-representations \( T_p M = \bigoplus_{i=1}^{n} L_i \), where \( \dim M = 2n \). In particular, the dimension of \( M \) is even. On each \( L_i \), the circle group acts as multiplication by \( g^{w_{pi}} \) for all \( g \in S^1 \), for some non-zero integer \( w_{pi} \). We may choose an orientation of \( L_i \) so that \( w_{pi} \) is positive for all \( i \). The positive integers \( w_{p1}, \ldots, w_{pn} \) are called

Donghoon Jang was supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIT) (2021R1C1C1004158).
the weights at \( p \). Let \( \epsilon(p) = +1 \) if the orientation on \( M \) and the orientation on the representation space \( \oplus_{i=1}^n L_i \) agree, and \( \epsilon(p) = -1 \) otherwise, and call it the sign of \( p \). Define the fixed point data of \( p \) to be an ordered pair \( (\epsilon(p), \{w_{p1}, \ldots, w_{pn}\}) \) where \( \{w_{p1}, \ldots, w_{pn}\} \) is a multiset, and simplify it as \( \{\epsilon(p), w_{p1}, \ldots, w_{pn}\} \) by writing the sign first and the weights next. By the fixed point data of \( M \) we mean a collection \( \bigcup_{p \in M^S} \{\epsilon(p), w_{p1}, \ldots, w_{pn}\} \) of the fixed point datum of the fixed points of \( M \).

If the action on \( M \) has exactly one fixed point, say \( p \), then \( M \) must be the point itself, that is, \( M = \{p\} \); in particular \( \dim M = 0 \). If the action on \( M \) has exactly two fixed points, the fixed point data of \( M \) agrees with that of a rotation of \( S^{2n} \) (Theorem 2.8); in particular such an \( M \) exists in any even dimension. If there is an odd number of fixed points, the dimension of the manifold is a multiple of 4 (see Corollary 2.4). Examples of circle actions on compact connected oriented manifolds with 3 fixed points exist in dimensions 4, 8, and 16, and they are respectively circle actions on the complex projective space \( \mathbb{CP}^2 \), the quaternionic projective space \( \mathbb{HP}^2 \), and the octonionic projective space \( \mathbb{OP}^2 \).

Suppose that there exists a 2\( n \)-dimensional \( M \) with \( k \) fixed points, where \( n > 1 \). By taking an equivariant connected sum along free orbits of \( M \) and another 2\( n \)-dimensional compact connected oriented \( S^1 \)-manifold that has no fixed points (for instance, take the product of a rotation of \( S^1 \) and any (2\( n \)-1)-dimensional compact connected oriented manifold), we can construct infinitely many 2\( n \)-dimensional compact connected oriented \( S^3 \)-manifolds with \( k \) fixed points. In dimension 2, the 2-sphere is the only compact connected oriented \( S^1 \)-manifold with non-empty finite fixed point set.

While we can construct infinitely many compact connected oriented \( S^3 \)-manifolds in prescribed dimension and with a prescribed number of fixed points if there exists one, the number of types of fixed point data could be finite. This is true if \( k \leq 2 \) as noted above, and if \( \dim M \leq 6 \), see [8] and [11].

Circle actions with 4 fixed points have been studied on different types of 6-dimensional compact manifolds. Ahara studied almost complex manifolds with Todd genus 1 and \( \int_M c_1^3 \neq 0 \) [1], Tolman studied Hamiltonian actions on symplectic manifolds [19], and the author studied almost complex manifolds [9].

In this paper, we prove that if a 6-dimensional oriented \( M \) has 4 fixed points, only two types of fixed point data occur; one type has the same fixed point data as an equivariant connected sum along free orbits of rotations of two \( S^6 \)s, and the other type has the same fixed point data as a linear action on \( \mathbb{CP}^3 \). The former fixed point data is the same as the fixed point data of a disjoint union of rotations of two \( S^6 \)s, which is disconnected.

**Theorem 1.1.** Let the circle act on a 6-dimensional compact connected oriented manifold \( M \) with 4 fixed points. Then one of the following holds for the fixed point data of \( M \).

1. \( \{+, a, b, c\}, \{-, a, b, c\}, \{+, d, e, f\}, \{-, d, e, f\} \) for some positive integers \( a, b, c, d, e, \) and \( f \).
(2) $\{+, a, a+b, a+b+c\}, \{-, a, b+c\}, \{+, b, a+b\}, \{-, c, b+a+b+c\}$.

for some positive integers $a$, $b$, and $c$.

Note that in Theorem 1.1, Case (2) with $a = c$ belongs to Case (1). In Case (1) of Theorem 1.1, the fixed point data is the same as the equivariant sum (or a disjoint union) of rotations on two 6-spheres; see Example 2.9. In Case (2) of Theorem 1.1, the fixed point data is the same as a standard linear action of $S^1$ on the complex projective space $\mathbb{C}P^3$; see Example 2.10. On the other hand, in Example 2.11, we show that blowing up at a fixed point, of a rotation on $S^6$ with 2 fixed points also has the fixed point data of Case (2) of Theorem 1.1.

We note that Case (1) of Theorem 1.1 includes the fixed point data of the exotic $S^1$-action on $\mathbb{C}P^3$ constructed by Petrie [18]. The strong Petrie conjecture asserts that if a compact oriented manifold $M$ is homotopy equivalent to a complex projective space $\mathbb{C}P^n$ and admits a non-trivial $S^1$-action, then the total Pontryagin class of $M$ agrees with that of $\mathbb{C}P^n$ [18]. The weak Petrie conjecture adds an assumption that the fixed point set is discrete. If $\dim M = 6$, $M$ is a homotopy $\mathbb{C}P^3$, and the fixed point set is discrete, then $M$ has 4 fixed points. For the proof of the weak Petrie conjecture in dimension 6, see [4] and [15].

Petrie’s example is exotic in a sense that the weights at the 4 fixed points are distinct from those of linear action. For the exotic action, the (complex) weights at the fixed points are

$$\{7, 2, 3\}, \{-7, 2, 3\}, \{5, 2, 3\}, \{-5, 2, 3\},$$

respectively. The fixed point data of this action on $\mathbb{C}P^3$ as an oriented manifold is

$$\{+, 7, 2, 3\}, \{-, 7, 2, 3\}, \{+, 5, 2, 3\}, \{-, 5, 2, 3\}.$$

Therefore, this fixed point data belongs to Case (1) of Theorem 1.1.

We note that the two types of fixed point data of Theorem 1.1 include those of both the exotic action (as Case (1)) and the standard action (as Case (2)) on $\mathbb{C}P^3$, without assumption that a given manifold is homotopy equivalent to $\mathbb{C}P^3$.

The paper is organized as follows. In Sect. 2, we provide background and preliminaries. In Sect. 3, we discuss how to associate a multigraph to an oriented $S^1$-manifold $M$ with a discrete fixed point set. The vertex set of the multigraph is the fixed point set, and roughly speaking, if two fixed points $p$ and $q$ have weights $w$, we draw an edge between $p$ and $q$ with label $w$. The multigraph we assign is slightly different from previous work, but both encode the fixed point data while the former simplifies the proof of Theorem 1.1. In Sect. 4, we show that if $\dim M = 6$ and $|M^{S^1}| = 4$, there is a few number of possible multigraphs that encode the fixed point data of $M$. In Sect. 5, for each possible multigraph, we determine the fixed point data of $M$, proving Theorem 1.1. In Sect. 6, we compare our results for oriented manifolds with results for different types of manifolds mentioned above. In Sect. 7, we discuss how we convert the fixed point data of $M$ into the empty collection.
The author would like to thank the referee for a careful reading of this paper and many helpful suggestions.

2. Background and preliminaries

In this section, we review background and properties of a circle action on a compact oriented manifold with a discrete fixed point set.

For an action of a group \( G \) on a manifold \( M \), denote by \( M^G \) the fixed point set. That is,
\[
M^G = \{ m \in M \mid g \cdot m = m, \forall g \in G \}.
\]

Let the circle group act on a compact oriented manifold \( M \). The **equivariant cohomology** of \( M \) is
\[
H^*_S^1(M) := H^*(M \times S^1 S^\infty).
\]

The projection map onto the second factor \( \pi : M \times S^1 S^\infty \to \mathbb{C}P^\infty \) gives rise to a push-forward map
\[
\int_M := \pi_* : H^1_{s^1}(M) \to H^{i-dim M}(\mathbb{C}P^\infty) \text{ for all } i.
\]

The Atiyah–Bott–Berline–Vergne localization theorem allows us to compute the push-forward map in terms of the fixed point data.

**Theorem 2.1.** (Atiyah–Bott–Berline–Vergne localization theorem) \[2\] Let the circle act on a compact oriented manifold \( M \). Given \( \alpha \in H^*_S^1(M; \mathbb{Q}) \),
\[
\int_M \alpha = \sum_{F \subset M} \int_F \alpha \big|_{\epsilon_{s^1}(N_f)},
\]
where the sum is taken over all fixed components, and \( \epsilon_{s^1}(N_f) \) is the equivariant Euler class of the normal bundle to \( F \).

For a compact oriented manifold \( M \), the L-genus is the genus belonging to the power series \( \sqrt{z \sinh \sqrt{z}} \). The Atiyah–Singer index theorem proves that the L-genus of \( M \) is equal to the index of the signature operator on \( M \) \[3\]. The equivariant index of the signature operator on a compact oriented manifold \( M \) equipped with a circle action is independent of the choice of an element of \( S^1 \) and is equal to the signature of \( M \). As a result, the following formula holds.

**Theorem 2.2.** (Atiyah–Singer index theorem) \[3\] Let the circle act on a 2n-dimensional compact oriented manifold \( M \) with a discrete fixed point set. Then the signature of \( M \) is
\[
\text{sign}(M) = \sum_{p \in M^{S^1}} \epsilon(p) \cdot \prod_{i=1}^{n} \frac{1 + t w_{pi}}{1 - tw_{pi}},
\]
for all indeterminates \( t \), and is a constant.
Consider a circle action on a compact-oriented manifold with a discrete fixed point set. For each positive integer \( w \), the number of times the weight \( w \) occurs over all the fixed points, counted with multiplicity at each fixed point, is even.

**Lemma 2.3.** \([8, 15]\) Let the circle act on a \( 2n \)-dimensional compact oriented manifold \( M \) with a discrete fixed point set. For any positive integer \( w \),

\[
| \{ w_{pi} : w_{pi} = w, 1 \leq i \leq n, p \in M^{S^1} \} | \equiv 0 \mod 2.
\]

In particular, if \( \dim M = 2n \) and there are \( k \) fixed points, the total number \( nk \) of weights over all fixed points must be even. Thus the following statement holds.

**Corollary 2.4.** Let the circle act on a compact connected oriented manifold. If the number of fixed points is odd, then the dimension of the manifold is divisible by 4.

Consider a circle action on an oriented manifold \( M \). For a positive integer \( w \), the group \( \mathbb{Z}_w \) acts on \( M \) as a subgroup of \( S^1 \). H. Herrera and R. Herrera proved the orientability of the set \( M^{\mathbb{Z}_w} \) of points in \( M \) that are fixed by the \( \mathbb{Z}_w \)-action.

**Lemma 2.5.** \([5]\) Let the circle act on a \( 2n \)-dimensional oriented manifold \( M \). Consider \( \mathbb{Z}_w \subset S^1 \) and its corresponding action on \( M \). If \( w \) is odd then the fixed point set \( M^{\mathbb{Z}_w} \) is orientable. If \( w \) is even and a connected component \( F \) of \( M^{\mathbb{Z}_w} \) contains a fixed point of the \( S^1 \)-action, then \( F \) is orientable.

To prove Theorem 1.1, we introduce some terminologies. Let \( S^1 \) act on a \( 2n \)-dimensional compact oriented manifold \( M \) with a discrete fixed point set. Let \( w \) be a positive integer. Consider a connected component \( F \) of \( M^{\mathbb{Z}_w} \) such that \( F \cap M^{S^1} \neq \emptyset \). Then \( F \) is orientable by Lemma 2.5; choose an orientation of \( F \). Then the normal bundle \( NF \) of \( F \) is orientable. Take an orientation on \( NF \) so that the orientation of \( T_qF \oplus N_qF \) agrees with the orientation of \( T_qM \) for all \( q \in F \). If \( p \in F \cap M^{S^1} \), this gives orientations of \( T_pF \) and \( N_pF \), respectively. In the decomposition of the tangent space at \( p \) into \( n \) irreducibles \( T_pM = \bigoplus_{i=1}^n L_i \), for each \( i \) we choose an orientation of \( L_i \) so that \( S^1 \) acts on \( L_i \) with a positive weight \( w_{pi} > 0, 1 \leq i \leq n \). Rearrange \( L_i \)'s so that \( T_pF = L_1 \oplus \cdots \oplus L_m \) and \( N_pF = L_{m+1} \oplus \cdots \oplus L_n \).

**Definition 2.6.**

1. \( \epsilon_F(p) = +1 \) if the orientation on \( T_pF \) and the orientation on \( L_1 \oplus \cdots \oplus L_m \) (in which all \( w_{pi} \) are positive) agree, and \( \epsilon_F(p) = -1 \) otherwise.
2. \( \epsilon_N(p) = +1 \) if the orientation on \( N_pF \) and the orientation on \( L_{m+1} \oplus \cdots \oplus L_n \) (in which all \( w_{pi} \) are positive) agree, and \( \epsilon_N(p) = -1 \) otherwise.
3. \( \epsilon_M(p) = +1 \) if the orientation on \( T_pM \) and the orientation on \( L_1 \oplus \cdots \oplus L_n \) (in which all \( w_{pi} \) are positive) agree, and \( \epsilon_M(p) = -1 \) otherwise.

By the definitions, \( \epsilon(p) = \epsilon_M(p) \) and \( \epsilon(p) = \epsilon_F(p) \cdot \epsilon_N(p) \). The following lemma can be proved easily using Theorem 2.2.
Lemma 2.7. [8] Let the circle act on a compact oriented manifold $M$ with a discrete fixed point set. Suppose that every weight at any fixed point is equal to $w$ for some positive integer $w$. Then the number of fixed points $p$ with $\epsilon(p) = +1$ and the number of fixed points $p$ with $\epsilon(p) = -1$ are equal; in particular, there is an even number of fixed points. Moreover, $\text{sign}(M) = 0$.

We review known classification results on the fixed point data of an oriented $S^1$-manifold. If there are 2 fixed points, the fixed point data is the same as that of a rotation on an even-dimensional sphere.

Theorem 2.8. [13] Let the circle act on a compact connected oriented manifold with two fixed points $p$ and $q$. Then the weights at $p$ and $q$ agree up to order and $\epsilon(p) = -\epsilon(q)$.

In Theorem 1.1, two types of fixed point data occur for a circle action on a 6-dimensional compact oriented manifold with 4 fixed points. For each type, such a manifold exists.

Example 2.9. Let the circle act on $S^6$ by
\[ g \cdot (z_1, z_2, z_3, x) = (g^a z_1, g^b z_2, g^c z_3, x) \]
for all $g \in S^1 \subset \mathbb{C}$, where $S^6 = \{(z_1, z_2, z_3, x) \in \mathbb{C}^3 \times \mathbb{R} \mid \sum_{i=1}^3 |z_i|^2 + x^2 = 1\} \subset \mathbb{C}^3 \times \mathbb{R}$ and $a$, $b$, and $c$ are positive integers. The action has two fixed points $p_1 = (0, 0, 0, 1)$ and $p_2 = (0, 0, 0, -1)$, whose fixed point data are $\{+, a, b, c\}$ and $\{-, a, b, c\}$, respectively. Consider another $S^6$ with action
\[ g \cdot (z_1, z_2, z_3, x) = (g^d z_1, g^e z_2, g^f z_3, x). \]
For each $S^6$, there exists a free orbit, and the free orbit has a tubular neighborhood $S^1 \times D^5$. Thus, we can remove the interiors of the tubular neighborhoods of the two $S^6$’s and glue them along $S^1 \times S^4 \times [0, 1]$, to construct a 6-dimensional compact connected oriented $S^1$-manifold with 4 fixed points, which has the fixed point data of Case (1) of Theorem 1.1.

A standard linear action on $\mathbb{CP}^3$ has the fixed point data of Case (2) of Theorem 1.1.

Example 2.10. Let the circle act on $\mathbb{CP}^3$ by
\[ g \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : g^a z_1 : g^{a+b} z_2 : g^{a+b+c} z_3], \]
for all $g \in S^1 \subset \mathbb{C}$, for some positive integers $a$, $b$, and $c$. The action has 4 fixed points $[1 : 0 : 0 : 0]$, $[0 : 1 : 0 : 0]$, $[0 : 0 : 1 : 0]$, and $[0 : 0 : 0 : 1]$. As complex representations where the sign of every weight is well-defined, the weights at the fixed points are $\{a, a+b, a+b+c\}$, $\{-a, b, b+c\}$, $\{-a-b, -b, c\}$, and $\{-a-b-c, -b-c, -c\}$, respectively. As real representations, the fixed point data at each fixed point is $\{+, a+a+b, a+b+c\}$, $\{-, a, b, b+c\}$, $\{+, b, c+a+b\}$, and $\{-, c+b+c, a+b+c\}$, respectively. This fixed point data is the same as Case (2) in Theorem 1.1.

On the other hand, as explained in the introduction, blowing up at a fixed point, of a rotation on $S^6$ with two fixed points also has the fixed point data of Case (2) in Theorem 1.1.
Let $a$, $b$, and $c$ be mutually distinct non-zero integers. Let $S^1$ act on $\mathbb{C}^3$ by
\[ g \cdot (z_1, z_2, z_3) = (g^a z_1, g^b z_2, g^c z_3) \]
for all $g \in S^1$. The origin is fixed by this action, and the (complex) weights at the origin are $\{a, b, c\}$. Now we blow up the origin. Blowing up the origin replaces the origin with all complex straight lines through it. The blown-up space $\widetilde{\mathbb{C}^3}$ can be described by
\[ \widetilde{\mathbb{C}^3} = \{(z, \ell) \mid z \in \ell \} \subset \mathbb{C}^3 \times \mathbb{C}P^2. \]
The $S^1$-action on $\mathbb{C}^3$ naturally extends to act on $\widetilde{\mathbb{C}^3}$ by
\[ g \cdot ((z_1, z_2, z_3), [w_0 : w_1 : w_2]) = ((g^a z_1, g^b z_2, g^c z_3), [g^a w_0 : g^b w_1 : g^c w_2]) \]
for all $((z_1, z_2, z_3), [w_0 : w_1 : w_2]) \in \widetilde{\mathbb{C}^3}$. The action on $\widetilde{\mathbb{C}^3}$ has three fixed points $p_1 = ((0, 0, 0), [1 : 0 : 0]), p_2 = ((0, 0, 0), [0 : 1 : 0]),$ and $p_3 = ((0, 0, 0), [0 : 0 : 1]),$ that have (complex) weights $\{a, b-a, c-a\}, \{b, a-b, c-b\},$ and $\{c, a-c, b-c\}$, respectively.

**Example 2.11.** Let $S^1$ act on $S^6$ by $g \cdot (z_1, z_2, z_3, x) = (g^c z_1, g^b z_2, g^{a+b} z_3, x)$ for some positive integers $a$, $b$, and $c$. The fixed point datum at $p_1 = (0, 0, 0, 1)$ and $p_2 = (0, 0, 0, -1)$ are $\{+, c, b, a+b\}$ and $\{-, c, b, a+b\}$, respectively. Next, identify a neighborhood of $p_2$ with $\mathbb{C}^3$ so that the $S^1$-action near $p_2$ is identified with $g \cdot (z_1, z_2, z_3) = (g^{-c} z_1, g^b z_2, g^{a+b} z_3).$ Under this identification, we blow up the origin (the fixed point $p_2$) in $\mathbb{C}^3$ (a neighborhood of $p_2$) equivariantly to replace the origin by the set of complex straight lines through it; doing it equivariantly yields instead of one fixed point the origin (that has complex $S^1$-weights $\{-c, b, a+b\}$), three fixed points $q_1$, $q_2$, $q_3$, whose weights as complex $S^1$-representations are $\{-c, b+c, a+b+c\}, \{-b-c, a+b\},$ and $\{-a-b-c, -a, a+b\}$, respectively. The blown up manifold $S^6$ is then equipped with a circle action having 4 fixed points $p_1$, $q_1$, $q_2$, and $q_3$ that have the fixed point data as real $S^1$-representations $\{+, b, a+b\}, \{-c, b+c, a+b+c\}, \{-a, b, a+b+c\},$ and $\{+, a, a+b, a+b+c\}$. This fixed point data also agrees with Case (2) in Theorem 1.1.

3. Special multigraph

It has been already known that for a compact oriented $S^1$-manifold with a discrete fixed point set, we can associate a labeled multigraph, where the vertex set is the fixed point set and edges are drawn in terms of weights [8], [15]. To simplify the proof of our main result, we consider a special type of multigraph by requiring an additional property.

**Definition 3.1.** A **multigraph** $\Gamma$ is an ordered pair $\Gamma = (V, E)$ where $V$ is a set of vertices and $E$ is a multiset of unordered pairs of vertices, called edges. A multigraph is called **signed** if every vertex is assigned a number $+1$ or $-1$. A multigraph is called **labeled**, if every edge $e$ is labeled by a positive integer $w(e)$, called the **label**, or the **weight** of the edge. Alternatively, a multigraph is called labeled if there is a map from $E$ to the set of positive integers. Let $\Gamma$ be
a labeled multigraph. The weights at a vertex \( v \) is a multiset that consists of labels (weights) \( w(e) \) for each edge \( e \) at \( v \). A multigraph \( \Gamma \) is called \( n \)-regular, if every vertex has \( n \)-edges.

**Definition 3.2.** Let the circle act on a compact oriented manifold \( M \) with a discrete fixed point set. We say that a signed labeled multigraph \( \Gamma \) describes (the fixed point data of) \( M \), if the following hold.

1. The vertex set of \( \Gamma \) is the fixed point set \( M^{S^1} \).
2. For every fixed point \( p \), the weights (labels) of the edges at the vertex \( p \) are the weights at the fixed point \( p \).
3. For every fixed point \( p \), the sign of the vertex \( p \) is equal to the sign of the fixed point \( p \).

By definition, if a multigraph \( \Gamma \) describes \( M \), then \( \Gamma \) is \( n \)-regular, where \( \dim M = 2n \). To associate a special multigraph, we need the following lemma.

**Lemma 3.3.** Let the circle act on a \( 2n \)-dimensional compact oriented manifold \( M \) with a non-empty discrete fixed point set. Let \( W_+ = \{ w_{pi} | p \in M^{S^1}, \epsilon(p) = +1, 1 \leq i \leq n \} \) be the collection (multiset) of all the weights at the fixed points with sign \( +1 \) and \( W_- = \{ w_{pi} | p \in M^{S^1}, \epsilon(p) = -1, 1 \leq i \leq n \} \) the collection of all the weights at the fixed points with sign \( -1 \). Rearrange elements in \( W_+ = \{ a_1, a_2, \ldots, a_{i_0} \} \) and \( W_- = \{ b_1, b_2, \ldots, b_{i_0} \} \) so that \( a_1 \leq a_2 \leq \cdots \leq a_{i_0} \) and \( b_1 \leq b_2 \leq \cdots \leq b_{i_0} \). Then \( a_1 = b_1 \) and \( a_2 = b_2 \). Moreover, \( a_i = a_2 \) if and only if \( b_i = a_2 \).

**Proof.** By Theorem 2.2, the signature of \( M \) is

\[
\text{sign}(M) = \sum_{p \in M^{S^1}} \epsilon(p) \prod_{i=1}^{n} \frac{1 + t^{w_{pi}}}{1 - t^{w_{pi}}} = \sum_{p \in M^{S^1}} \epsilon(p) \prod_{i=1}^{n} \left[ (1 + t^{w_{pi}}) \left( \sum_{j=0}^{\infty} t^{j w_{pi}} \right) \right] = \sum_{p \in M^{S^1}} \epsilon(p) \prod_{i=1}^{n} \left( 1 + 2 \sum_{j=1}^{\infty} t^{j w_{pi}} \right).
\]

The signature of \( M \) is independent of the indeterminate \( t \) and is a constant. Therefore, comparing the term \( 2t^{a_1} \) that has the smallest positive exponent and positive coefficient and the term \( -2t^{b_1} \) that has the smallest positive exponent and negative coefficient, we must have \( a_1 = b_1 \). Cancel out all the terms \( 2t^{a_i} \) and \( -2t^{b_i} \) for any \( j > 0 \) in the equation above. Next, in what is left, we compare the terms whose exponent is equal to \( a_2 \). Let \( i_0 = \max\{ i \mid a_i = a_2 \} \). Then the fixed points \( p \) with \( \epsilon(p) = +1 \) contribute the term \( 2(i_0 - 1)t^{a_2} \), since each \( i \) contributes the term \( 2t^{a_2} \) for \( 2 \leq i \leq i_0 \). Since this term \( 2(i_0 - 1)t^{a_2} \) must be cancelled out, it follows that \( b_2 = b_3 = \cdots = b_{i_0} < b_{i_0+1} \), and (in particular) \( a_2 = b_2 \). \( \square \)
Remark 3.4. In Lemma 3.3, $a_3 \in W_+$ need not equal to $b_3 \in W_-$, since the next smallest exponent in the equation above can be $a_1 + a_2$. The simplest example is a linear action on $\mathbb{C}P^2$. Let $S^1$ act on $\mathbb{C}P^2$ by

$$g \cdot [z_0 : z_1 : z_2] = [z_0 : g^a z_1 : g^{a+b} z_2]$$

for all $g \in S^1 \subset \mathbb{C}$, for some positive integers $a$ and $b$. The action has 3 fixed points $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$, which have the weights $\{a, a + b\}$, $\{-a, b\}$, and $\{-b, -a - b\}$ as complex $S^1$-representations, and hence fixed point data $\{+, a, a + b\}$, $\{-, a, b\}$, and $\{+, b, a + b\}$, respectively.

In the notations as in Lemma 3.3, $W_+ = \{a, b, a + b, a + b\}$ and $W_- = \{a, b\}$.

For a multigraph, by a self-loop we mean an edge whose vertices coincide.

Lemma 3.5. Let the circle act on a compact oriented manifold $M$ with a discrete fixed point set. Then there exists a signed labeled multigraph $\Gamma$ describing $M$ that satisfies the following properties:

1. If the label of an edge is equal to $a_1$ or $a_2$ where $a_1$ and $a_2$ are as in Lemma 3.3, its vertices have different signs. In particular, $\Gamma$ has at least two edges whose vertices have different signs.
2. If the label $w$ of an edge is strictly bigger than $a_2$, the vertices of the edge lie in the same connected component of $M^{z_w}$.
3. The multigraph $\Gamma$ has no self-loops.

Proof. First, assign a vertex to each fixed point $p \in M^{S^1}$. To every vertex, assign the sign of the corresponding fixed point.

Let $W_+ = \{a_1, a_2, \ldots, a_i\}$ and $W_- = \{b_1, b_2, \ldots, b_i\}$ be as in Lemma 3.3. Suppose that a fixed point $p_1$ with $\epsilon(p_1) = 1$ has weight $a_1$. By Lemma 3.3, $b_1 = a_1$, i.e., there exists a fixed point $q_1$ with $\epsilon(q_1) = -1$ that has weight $a_1$. We draw an edge between $p_1$ and $q_1$, giving label $a_1$. Next, let $i_0 = \max \{i \mid a_i = a_2\}$. By Lemma 3.3, $b_2 = b_3 = \cdots = b_{i_0} < b_{i_0+1}$. Therefore, for each $j$ such that $2 \leq j \leq i_0$, if a fixed point $p$ with $\epsilon(p) = +1$ has weight $a_j (= a_2)$ and a fixed point $q$ with $\epsilon(q) = -1$ has weight $b_j (= a_j)$, then draw an edge between $p$ and $q$ giving label $a_j$, which is equal to $a_2$. For each fixed point, one weight is used to draw only one edge. This proves Claim (1).

Let $w > a_2$ be an integer. Suppose that a fixed point $p_0$ has weight $w$. Consider a connected component $F$ of $M^{z_w}$ that contains $p_0$, which is orientable by Lemma 2.5. Fix an orientation of $F$. The circle action on $M$ restricts to act on $F$, and the fixed point set of this $S^1$-action on $F$ is equal to $F^{S^1} = F \cap M^{S^1}$, which is discrete. If $q \in F^{S^1}$, every weight in $T_q F$ is a multiple of $w$. As for the circle action on $M$, for the $S^1$-action on $F$, let

$$W_{+,F} = \{w_{p_i} \mid p \in F^{S^1}, \epsilon_F(p) = +1, w_{p_i} \text{ is a weight in } T_p F\} = \{c_1, c_2, \ldots, c_{i_1}\},$$

$$W_{-,F} = \{w_{p_i} \mid p \in F^{S^1}, \epsilon_F(p) = -1, w_{p_i} \text{ is a weight in } T_p F\} = \{d_1, d_2, \ldots, d_{i_1}\},$$
be multisets, where \( c_1 \leq c_2 \leq \cdots \leq c_{i_3} \) and \( d_1 \leq d_2 \leq \cdots \leq d_{i_4} \). Let \( i_5 = \max\{j \mid c_j = c_1\} \). Applying Lemma 3.3 to the \( S^1 \)-action on \( F \), we have \( d_1 = \cdots = d_{i_5} < d_{i_5+1} \). Therefore, for each \( 1 \leq j \leq i_5 \), if a fixed point \( p \in F^{S^1} \) with \( \epsilon_F(p) = +1 \) has weight \( c_j \) and a fixed point \( q \in F^{S^1} \) with \( \epsilon_F(q) = -1 \) has weight \( d_j \), then draw an edge between \( p \) and \( q \) giving label \( c_j \).

Repeat the above for every positive integer \( w \) such that \( w > a_2 \) and for every connected component of \( M_{2w} \). This proves Claim (2). The resulting multigraph describes \( M \) and the whole procedure does not cause any self-loops. This proves Claim (3). \( \square \)

For a compact oriented \( S^1 \)-manifold with a discrete fixed point set, let \( \Gamma_1 \) and \( \Gamma_2 \) be two signed labeled multigraphs describing \( M \), where \( \Gamma_1 \) is arbitrary but \( \Gamma_2 \) satisfies the properties in Lemma 3.5. Since both of them describe \( M \), they encode the same fixed point data; the same sign and the same multiset of the weights at every fixed point \( p \in M^{S^1} \). On the other hand, only considering multigraphs that satisfy the properties in Lemma 3.5 reduces the proof of Theorem 1.1 as we shall see in the next section.

4. Dimension 6 and 4 fixed points: possible multigraphs

In this section, we show what kind of signed labeled multigraph can occur as a multigraph describing a 6-dimensional compact oriented \( S^1 \)-manifold having 4 fixed points.

Lemma 4.1. Let the circle act on a 6-dimensional compact connected oriented manifold \( M \) with 4 fixed points. Then one of the figures in Fig. 1 occurs as a signed labeled multigraph describing \( M \) that satisfies the properties in Lemma 3.5. In particular, one of the following occurs as the fixed point data of \( M \) for some positive integers \( a \), \( b \), \( c \), \( d \), \( e \), and \( f \).

- (A) \( \{+a, b, c\}, \{+d, e, f\}, \{-a, b, c\}, \{-d, e, f\} \) (Fig. 1A).
- (B) \( \{+a, b, c\}, \{+a, d, e\}, \{-b, c, f\}, \{-d, e, f\} \) (Fig. 1B).
- (C) \( \{+a, b, c\}, \{+d, e, f\}, \{-a, b, d\}, \{-c, e, f\} \) (Fig. 1C).
- (D) \( \{+a, b, c\}, \{+a, b, d\}, \{-c, e, f\}, \{-d, e, f\} \) (Fig. 1D).
- (E) \( \{+a, b, c\}, \{+a, d, e\}, \{-b, d, f\}, \{-c, e, f\} \) (Fig. 1E).

Proof. Since \( \dim M \equiv 2 \mod 4 \), the signature of \( M \) is zero, see [3, p. 582]. Taking \( t = 0 \) in Theorem 2.2,

\[
0 = \text{sign}(M) = \sum_{p \in M^{S^1}} \epsilon(p).
\]

This implies that there are two fixed points \( p_1 \) and \( p_2 \) with sign +1 and two fixed points \( p_3 \) and \( p_4 \) with sign −1. By Lemma 3.5, there exists a signed labeled multigraph \( \Gamma \) describing \( M \) that satisfies the following properties:

1. If the label of an edge is equal to \( a_1 \) or \( a_2 \) where \( a_1 \) and \( a_2 \) are as in Lemma 3.3, its vertices have different signs. In particular, \( \Gamma \) has at least two edges whose vertices have different signs.
(2) If the label $w$ of an edge is strictly bigger than $a_2$, the vertices of the edge lie in the same connected component of $M^{Z_w}$.

(3) The multigraph $\Gamma$ has no self-loops.

For $i \in \{2, 3, 4\}$, let $m_i$ be the number of edges between $p_1$ and $p_i$. By permuting $p_1$ and $p_2$, and by permuting $p_3$ and $p_4$ if necessary, we may assume that $m_3 \geq 1$ (by Property (1) above) and $m_3 \geq m_4$ (by permuting $p_3$ and $p_4$ if necessary). Since $m_i \geq 0$ for any $i$ and $m_2 + m_3 + m_4 = 3$, we have the following cases for $(m_2, m_3, m_4)$: $(0, 3, 0)$, $(1, 2, 0)$, $(0, 2, 1)$, $(2, 1, 0)$, and $(1, 1, 1)$ (in the order that $m_3$ is non-increasing). Each case in order corresponds to a multigraph in Fig. 1, when we complete the multigraph that satisfies the properties in Lemma 3.5.

\[\square\]

5. Proof of the main theorem

In this section, we prove our main result, Theorem 1.1, by determining the fixed point data for each multigraph in Fig. 1. Since the fixed point data of Fig. 1A is precisely Case (1) of Theorem 1.1, there is nothing to prove. Figure 1B–D are easy to deal with; the fixed point data in any of these figures falls into Case (1) of Theorem 1.1. Figure 1E (Case E) is the most difficult case and requires complicated arguments. The fixed point data from Fig. 1E either belongs to Case (1) or is precisely Case (2) of Theorem 1.1, depending on the relationship between the weights at the fixed points. For Fig. 1E, when
the fixed point data is the same as in Case (2) of Theorem 1.1, the multigraph
Fig. 1E describes a standard linear action on \( \mathbb{CP}^3 \).

**Lemma 5.1.** Let the circle act on a 6-dimensional compact connected oriented manifold \( M \) with 4 fixed points. Suppose that Fig. 1B describes \( M \) and satisfies the properties in Lemma 3.5. Then \( a = f \); the fixed point data of \( M \) belongs to Case (1) of Theorem 1.1.

**Proof.** The fixed point data of \( M \) is

\[ \{+, a, b, c\}, \{+, a, d, e\}, \{-, b, c, f\}, \{-, d, e, f\} \]

For a dimensional reason, the push-forward of the equivariant cohomology class 1 in equivariant cohomology vanishes;
\[ \int_M 1 = 0 \] for all \( i \). Therefore, by Theorem 2.1,
\[ 0 = \int_M 1 = \sum_{p \in M^{S^1}} \epsilon(p) \frac{1}{\prod_{i=1}^3 w_{pi}} = \frac{1}{abc} + \frac{1}{ade} - \frac{bcf}{def} \]

Thus \( a = f \) and hence the fixed point data of \( M \) belongs to Case (1) of Theorem 1.1. \( \square \)

**Lemma 5.2.** Let the circle act on a 6-dimensional compact connected oriented manifold \( M \) with 4 fixed points. Suppose that Fig. 1C describes \( M \) and satisfies the properties in Lemma 3.5. Then \( c = d \) or \( \{a, b\} = \{e, f\} \); the fixed point data of \( M \) belongs to Case (1) of Theorem 1.1.

**Proof.** The fixed point data of \( M \) is

\[ \{+, a, b, c\}, \{+, d, e, f\}, \{-, a, b, d\}, \{-, c, e, f\} \]

Since \( \dim M \neq 0 \) mod 4, the signature of \( M \) is equal to 0. By Theorem 2.2,
\[ 0 = \text{sign}(M) = \sum_{p \in M^{S^1}} \epsilon(p) \prod_{i=1}^3 \frac{1 + t^{w_{pi}}}{1 - t^{w_{pi}}} = \sum_{p \in M^{S^1}} \epsilon(p) \prod_{i=1}^3 \left(1 + 2 \sum_{j=1}^{\infty} t^{j w_{pi}}\right) \]

\[ \times \left(1 + 2 \sum_{j=1}^{\infty} t^{j c}\right) + \left(1 + 2 \sum_{j=1}^{\infty} t^{j d}\right) \left(1 + 2 \sum_{j=1}^{\infty} t^{j e}\right) \left(1 + 2 \sum_{j=1}^{\infty} t^{j f}\right) \]

\[ - \left(1 + 2 \sum_{j=1}^{\infty} t^{j a}\right) \left(1 + 2 \sum_{j=1}^{\infty} t^{j b}\right) \left(1 + 2 \sum_{j=1}^{\infty} t^{j d}\right) \]

\[ - \left(1 + 2 \sum_{j=1}^{\infty} t^{j c}\right) \left(1 + 2 \sum_{j=1}^{\infty} t^{j e}\right) \left(1 + 2 \sum_{j=1}^{\infty} t^{j f}\right) \]
\[
\begin{bmatrix}
\left(1 + 2 \sum_{j=1}^{\infty} t^{jc}\right) - \left(1 + 2 \sum_{j=1}^{\infty} t^{jd}\right) \\
- \left(1 + 2 \sum_{j=1}^{\infty} t^{je}\right) \left(1 + 2 \sum_{j=1}^{\infty} t^{jf}\right)
\end{bmatrix}
\]

= \left[\left(1 + 2 \sum_{j=1}^{\infty} t^{jc}\right) - \left(1 + 2 \sum_{j=1}^{\infty} t^{jd}\right)\right]
\left[\left(1 + 2 \sum_{j=1}^{\infty} t^{ja}\right) \left(1 + 2 \sum_{j=1}^{\infty} t^{jb}\right)\right]
- \left(1 + 2 \sum_{j=1}^{\infty} t^{je}\right) \left(1 + 2 \sum_{j=1}^{\infty} t^{jf}\right).

It follows that \(c = d\) or \(\{a, b\} = \{e, f\}\). In either case, the fixed point data of
\(M\) belongs to Case (1) of Theorem 1.1. \(\Box\)

**Lemma 5.3.** Let the circle act on a 6-dimensional compact connected oriented manifold \(M\) with 4 fixed points. Suppose that Fig. 1D describes \(M\) and satisfies the properties in Lemma 3.5. Then \(\{a, b\} = \{e, f\}\); the fixed point data of \(M\) belongs to Case (1) of Theorem 1.1.

**Proof.** The proof is similar to Lemma 5.2. The fixed point data of \(M\) is
\(\{+, a, b, c\}, \{+, a, b, d\}, \{-, c, e, f\}, \{-, d, e, f\}\).

By Theorem 2.2, we deduce that
\[
0 = \text{sign}(M) = \left[\left(1 + 2 \sum_{j=1}^{\infty} t^{jc}\right) + \left(1 + 2 \sum_{j=1}^{\infty} t^{jd}\right)\right]
\left[\left(1 + 2 \sum_{j=1}^{\infty} t^{ja}\right) \left(1 + 2 \sum_{j=1}^{\infty} t^{jb}\right)\right]
- \left(1 + 2 \sum_{j=1}^{\infty} t^{je}\right) \left(1 + 2 \sum_{j=1}^{\infty} t^{jf}\right).
\]

It follows that \(\{a, b\} = \{e, f\}\). The fixed point data of \(M\) then belongs to Case (1) of Theorem 1.1. \(\Box\)

We are left with Fig. 1E in Lemma 4.1. However, Fig. 1E requires many subcases to consider and more techniques.

**Lemma 5.4.** Let the circle act on a 6-dimensional compact connected oriented manifold \(M\) with 4 fixed points. Suppose that Fig. 1E describes \(M\) and satisfies the properties in Lemma 3.5. Assume that \(b\) is the largest weight and \(\dim F = 4\), where \(F\) is a connected component of \(M^{2b}\) that contains \(p_1\) and \(p_3\). Then the fixed point data of \(M\) belongs to Case (1) of Theorem 1.1.

**Proof.** Suppose that \(b\) is the largest weight. Since the multigraph Fig. 1E has an edge whose vertices have the same sign, Properties 1 and 2 of Lemma 3.5 implies that the largest weight is strictly bigger than \(a_2 \in W_+\) where \(a_2\) and \(W_+\) are as in Lemma 3.3; in particular \(b\) is strictly bigger than \(a_2\).

Since \(\dim F = 4\), exactly 2 weights at \(p_1\) are divisible by \((a\) and hence equal to) \(b\). That is, either \(a = b\) or \(c = b\). Suppose that \(a = b\). Since \(p_1\) and \(p_2\) are connected by the edge with label \(a\) (\(= b\)), by Property 2 of Lemma 3.5, \(p_2\) also lies in \(F\). The circle action on \(M\) restricts to a circle action on \(F\). By Lemma 2.5, \(F\) is orientable; fix an orientation of \(F\). If \(p \in F^{S^1}\), every weight in \(T_p F\) is a multiple of \((a\) and hence equal to) \(b\). Applying Lemma 2.7 to the \(S^1\)-action on \(F\), the \(S^1\)-action on \(F\) has an even number of fixed points. Since
that each $p_i$ has exactly two weights that are equal to $b$, and hence it follows that $e = b$ and $f = b$. Then the fixed point data of $M$ is

\[ \{+, b, b, c\}, \{+, b, b, d\}, \{-, b, b, d\}, \{-, b, b, c\}. \]

This belongs to Case (1) of Theorem 1.1.

Next, suppose that $c = b$. By an analogous argument as above, $c = b$ implies $p_4 \in F$ by Property 2 of Lemma 3.5, and applying Lemma 2.7 to the restriction of the circle action on $M$ to $F$, it follows that $p_i \in F$ for all $i$, and hence $d = e = b$. Since both $a$ and $f$ connect two fixed points with the same signs, $a$ and $f$ are also bigger than $a_2$, which leads to a contradiction. Therefore, $c \neq b$. \qed

**Lemma 5.5.** Let the circle act on a 6-dimensional compact connected oriented manifold $M$ with 4 fixed points. Suppose that Fig. 1E describes $M$ and satisfies the properties in Lemma 3.5. Assume that $a$ is the largest weight and $\dim F = 4$, where $F$ is a connected component of $M^{Z_a}$ that contains $p_1$ and $p_2$. Then the fixed point data of $M$ belongs to Case (1) of Theorem 1.1.

**Proof.** The proof is analogous to that of Lemma 5.4. Since the multigraph Fig. 1E has an edge whose vertices have the same sign, Properties 1 and 2 of Lemma 3.5 imply that the largest weight is strictly bigger than $a_2 \in W_+$; in particular $a$ is strictly bigger than $a_2$. Since $\dim F = 4$, exactly 2 weights at $p_1$ must be divisible by (and hence equal to) $a$. That is, either $b = a$ or $c = a$. If $b = a$, this case is the same as the case in Lemma 5.4 where $b$ is the largest weight and $a = b$. As in the proof of Lemma 5.4, the fixed point data of $M$ in this case belongs to Case (1) of Theorem 1.1.

Next, assume $c = a$. Since $p_1$ and $p_4$ are connected by the edge with label $c (= a)$, by Property 2 of Lemma 3.5, $p_4$ also lies in $F$. The circle action on $M$ restricts to a circle action on $F$. By Lemma 2.5, $F$ is orientable; fix an orientation of $F$. If $p \in F^{S^1}$, every weight in $T_p F$ is a multiple of (and hence equal to) $a$. Applying Lemma 2.7 to the $S^1$-action on $F$, the $S^1$-action on $F$ has an even number of fixed points. Since $p_1, p_2, p_4 \in F^{S^1}$ and $F^{S^1} \subset M^{S^1}$, this implies that $p_3 \in F^{S^1}$. This means that each $p_i$ has exactly two weights that are equal to $a$, and hence it follows that $d = f = a$. The fixed point data of $M$ is then $\{+, a, a, b\}, \{+, a, a, c\}, \{-, a, a, b\}, \{-, a, a, c\}$. This belongs to Case (1) of Theorem 1.1. \qed

**Lemma 5.6.** Let the circle act on a 6-dimensional compact connected oriented manifold $M$ with 4 fixed points. Suppose that Fig. 1E describes $M$ and satisfies the properties in Lemma 3.5. Assume that $b$ is the largest weight and $\dim F = 2$, where $F$ is a connected component of $M^{Z_b}$ that contains $p_1$ and $p_3$. Then one of the following holds for the fixed point data of $M$:

1. $\{+, a, b, c\}, \{-, a, b, c\}, \{+, d, e, f\}, \{-, d, e, f\}$ for some positive integers $a, b, c, d, e,$ and $f$.
2. $\{+, a + b, a + b + c\}, \{-, a, b + c\}, \{+, b, c, a + b\}, \{-, c, b + c, a + b + c\}$ for some positive integers $a, b,$ and $d$. 


Proof. Since there is an edge whose vertices have the same sign, Properties 1 and 2 of Lemma 3.5 imply that the largest weight is strictly bigger than $a_2 \in W_+$; in particular $b$ is strictly bigger than 1. Since $\dim F = 2$, the other weights $a$ and $c$ at $p_1$ are smaller than $b$. Similarly, the other weights $d$ and $f$ at $p_3$ are smaller than $b$.

The circle action on $M$ restricts to act on $F$. We apply Lemma 2.7 to the $S^1$-action on $F$. By Lemma 2.7 we have $\epsilon_F(p_1) = -\epsilon_F(p_3)$; we may choose an orientation of $F$ so that $\epsilon_F(p_1) = +1$ and $\epsilon_F(p_3) = -1$. Because $\epsilon(p_1) = +1$ and $\epsilon(p_3) = -1$, this implies that $\epsilon_N(p_1) = +1$ and $\epsilon_N(p_3) = +1$. Then the normal spaces of $F$ at $p_1$ and at $p_3$ have two weights, as complex $S^1$-representations, with the same signs. Without loss of generality we may assume that $N_{p_1}F$ has weights $\{a, c\}$ and $N_{p_3}F$ has weights $\{d, f\}$ or $\{-d, -f\}$. Since the weights at $p_1$ and at $p_3$ on the normal bundle of $F$ are equal as $\mathbb{Z}_b$-representations, we have

$$\{a, c\} \equiv \{d, f\} \mod b, \text{ or } \{a, c\} \equiv \{-d, -f\} \mod b.$$ 

Thus we have the following possibilities:

(i) $a \equiv d \mod b$ and $c \equiv f \mod b$.
(ii) $a \equiv f \mod b$ and $c \equiv d \mod b$.
(iii) $a \equiv -d \mod b$ and $c \equiv -f \mod b$.
(iv) $a \equiv -f \mod b$ and $c \equiv -d \mod b$.

Assume that Case (i) holds. Since $a, c, d, f < b$, this implies that $a = d$ and $c = f$. Because the weights $a$ and $f$ connect points with the same signs, $a > a_2$ and $f > a_2$, where $a_2$ is as in Lemma 3.3. Then only $e$ is left to be possibly $a_1$ or $a_2$, which is a contradiction. Hence Case (i) cannot occur.

Assume that Case (ii) holds. Then $a = f$ and $c = d$. The fixed point data of $M$ is

$$\{+, a, b, c\}, \{+, a, c, e\}, \{-, b, c, a\}, \{-, c, e, a\}.$$ 

This belongs to Case (1) of the lemma.

Assume that Case (iii) holds. Then $a + d = b$ and $c + f = b$, i.e., $d = b - a$ and $f = b - c$. With these, by Theorem 2.1,

$$0 = \int_M 1 = \sum_{p \in M_{S^1}} \frac{1}{\prod_{i=1}^3 w_{pi}} = \frac{1}{abc} + \frac{1}{a(b-a)e} - \frac{1}{b(b-a)(b-c)} - \frac{1}{ce(b-c)}.$$ 

Clearing the denominators and simplifying, we have $(b-a-c)(e+c-a) = 0$, that is, $b = a + c$ or $e = a - c$.

Suppose that $b = a + c$. Since $a + d = b$ and $c + f = b$, this implies that $c = d$ and $a = f$. This corresponds to Case (ii).

Suppose that $e = a - c$. Then the fixed point data of $M$ is

$$\{+, a, b, c\}, \{+, a, b - a, a - c\}, \{-, b, b - a, b - c\}, \{-, c, a - c, b - c\}.$$ 

Let $A = c$, $B = a - c$, $C = b - a$. We can rewrite the fixed point data of $M$;
\{+, A + B, A + B + C, A\}, \{+, A + B, C, B\}, \{−, A + B + C, C, B + C\},
\{−, A, B, B + C\}.

Changing capital letters to small letters, this is precisely Case (2) of the lemma.

Assume that Case (iv) holds. Then \(a + f = b\) and \(c + d = b\). Let \(a_1\) and \(a_2\) be as in Lemma 3.3. By Property 2 of Lemma 3.5, we have \(a_2 < a\) and \(a_2 < f\), since the edges of the labels \(a\) and \(f\) have vertices with the same sign. Next, because \(b, c, d,\) and \(e\) are the only labels whose edges have vertices with different signs and because \(b\) is the largest weight, by Property 1 of Lemma 3.5, we must have \(\{a_1, a_2\} \subset \{c, d, e\}\). We cannot have \(\{a_1, a_2\} = \{c, d\}\), since then we have \(b = c + d = a_1 + a_2 < a + f = b\). It follows that either \(\{c, e\} = \{a_1, a_2\}\) or \(\{d, e\} = \{a_1, a_2\}\). By reversing the orientation of \(M\), which amounts to changing the sign of each fixed point (in other words, by the vertical symmetry of the multigraph Fig. 1E), we may assume without loss of generality that \(\{c, e\} = \{a_1, a_2\}\). Since \(\{c, e\} = \{a_1, a_2\}\), \(a_2 < a\), \(a_2 < f\), \(a + f = b\), and \(c + d = b\), we have

\[
\max\{c, e\} < \min\{a, f\} \leq \frac{b}{2} \leq \max\{a, f\} < d < b. \tag{5.1}
\]

Since \(d > a_2\) and Fig. 1E satisfies the properties in Lemma 3.5, \(p_2\) and \(p_3\) lie in the same connected component \(F_d\) of \(M^{2a}\). As we did for \(F\) above, we apply Lemma 2.7 to the \(S^1\)-action on \(F_d\); we may assume that the weights as complex \(S^1\)-representations on the normal space of \(F_d\) at \(p_2\) are \(\{a, e\}\) and at \(p_3\) are \(\{b, f\}\) or \(\{-b, -f\}\). Since they are equal as \(Z_d\)-representations, we have the following possibilities:

1. \(a \equiv b \mod d\) and \(e \equiv f \mod d\).
2. \(a \equiv f \mod d\) and \(e \equiv b \mod d\).
3. \(a \equiv -b \mod d\) and \(e \equiv -f \mod d\).
4. \(a \equiv -f \mod d\) and \(e \equiv -b \mod d\).

Suppose that \(a \equiv b \mod d\) and \(e \equiv f \mod d\). Since \(a < b = c + b - c < a + b - c = a + d\), we cannot have \(a \equiv b \mod d\).

Suppose that \(a \equiv f \mod d\) and \(e \equiv b \mod d\). It follows that \(a = f\) and \(e + d = b\). Since \(f = b - a\) and \(d = b - c\), this means that \(b = 2a\) and \(c = e\).

The fixed point data of \(M\) is then

\{+, a, 2a, c\}, \{−, a, 2a − c, a\}, \{−, 2a, 2a − c, a\}, \{−, c, a\}.

By Theorem 2.1,

\[
0 = \int_M 1 = \sum_{p \in M^{S^1}} \prod_{1}^{2} \frac{1}{w_{pi}} = \frac{1}{2a^2c} + \frac{1}{ac(2a - c)} - \frac{1}{2a^2(2a - c)} - \frac{1}{ac^2}.
\]

This implies that \(a = c\) or \(2a = c\), but either case is not possible.

Suppose that \(a \equiv -b \mod d\) and \(e \equiv -f \mod d\). Since \(e, f < d, e \equiv -f \mod d\) implies that \(e + f = d\). Since \(f = b - a\), this means that \(e + b = a + d\).

Next, \(a \equiv -b \mod d\) implies that \(a + b = 2d\) because \(a < d < b < 2d\). Hence, \(b = 2d - a, c = b - d = d - a, e = a + d - b = 2a - d,\) and \(f = b - a = 2d - 2a\).
The fixed point data of $M$ is then
\[ \{+, a, 2d - a, d - a\}, \{+, a, d, 2a - d\}, \{-, 2d - a, d, 2d - 2a\}, \{-, d - a, 2a - d, 2d - 2a\}. \]

By Theorem 2.1,
\[ 0 = \int_M 1 = \sum_{p \in M S^1} \frac{1}{\prod_{i=1}^3 w_{pi}} = \frac{1}{a(2d - a)(d - a)} + \frac{1}{ad(2a - d)} - \frac{1}{(2d - a)d(2d - 2a)} - \frac{1}{(d - a)(2a - d)(2d - 2a)}. \]

We simplify this to $(2d - a)(d - 2a) = 0$, but this cannot hold because $2d - a = b > 0$ and $2a - d = e > 0$.

Suppose that $a \equiv -f \mod d$ and $e \equiv -b \mod d$. In this case, $a \equiv -f \mod d$ cannot hold, since $-f = a - b$ and $\frac{b}{2} < d < b$. $\square$

**Lemma 5.7.** Let the circle act on a 6-dimensional compact connected oriented manifold $M$ with 4 fixed points. Suppose that Fig. 1E describes $M$ and satisfies the properties in Lemma 3.5. Assume that $a$ is the largest weight and $\dim F = 2$, where $F$ is a connected component of $M^{2a}$ that contains $p_1$ and $p_2$. Then the fixed point data of $M$ belongs to Case (1) of Theorem 1.1.

**Proof.** The proof is similar to that of Lemma 5.6. Suppose that $a$ is the largest weight. By Property 2 of Lemma 3.5, since its edge has vertices of the same sign, the largest weight is strictly bigger than $a_2 \in W_+$ where $a_2$ and $W_+$ are as in Lemma 3.3; in particular $a$ is strictly bigger than 1.

Since $\dim F = 2$, the other weights $b$ and $c$ at $p_1$ are smaller than $a$, and the other weights $d$ and $e$ at $p_2$ are smaller than $a$. Applying Lemma 2.7 to $F$, the signs of $p_1$ and $p_2$ in $F$ are different; orient $F$ so that $e_F(p_1) = +1$. Then as complex $S^1$-representations the weights in the normal space of $F$ at $p_1$ ($p_2$) have the same (different) signs. We may assume that the normal space at $p_1$ has weights $\{b, c\}$ and the normal space at $p_2$ has weights $\{-d, e\}$ or $\{d, -e\}$. Since the weights at $p_1$ and at $p_2$ on the normal bundle of $F$ are equal as $\mathbb{Z}_a$-representations, we have the following possibilities:

(i) $b \equiv e \mod a$ and $c \equiv -d \mod a$.
(ii) $b \equiv -d \mod a$ and $c \equiv e \mod a$.
(iii) $b \equiv d \mod a$ and $c \equiv -e \mod a$.
(iv) $b \equiv -e \mod a$ and $c \equiv d \mod a$.

Up to changing the roles of $p_3$ and $p_4$, it suffices to do Cases (i) and (ii).

Suppose that $b \equiv e \mod a$ and $c \equiv -d \mod a$. This implies that $b = e$ and $c + d = a$. The fixed point data of $M$ is then
\[ \{+, c + d, b, c\}, \{+, c + d, d, b\}, \{-, b, d, f\}, \{-, c, b, f\}. \]

By Theorem 2.1,
\[ 0 = \int_M 1 = \sum_{p \in M S^1} \frac{1}{\prod_{i=3} w_{pi}} = \frac{1}{(c + d)bc} + \frac{1}{(c + d)db} - \frac{1}{bdf} - \frac{1}{cbf}. \]

Therefore, $f = c + d$. This belongs to Case (1) of Theorem 1.1.
Suppose that \( b \equiv -d \mod a \) and \( c \equiv e \mod a \). This implies that \( b + d = a \) and \( c = e \). The fixed point data of \( M \) is then
\[
\{+, b + d, b, c\}, \{+, b + d, d, c\}, \{-, b, d, f\}, \{-, c, c, f\}.
\]
By Theorem 2.2, we deduce that
\[
0 = \text{sign}(M) = \left( 1 + 2 \sum_{j=1}^{\infty} t^{j(b+d)} \right) \left( 1 + 2 \sum_{j=1}^{\infty} t^{jc} \right) \left[ \left( 1 + 2 \sum_{j=1}^{\infty} t^{jb} \right) + \left( 1 + 2 \sum_{j=1}^{\infty} t^{jd} \right) \right] - \left( 1 + 2 \sum_{j=1}^{\infty} t^{j} \right)^2 \left( 1 + 2 \sum_{j=1}^{\infty} t^{jb} \right) \left( 1 + 2 \sum_{j=1}^{\infty} t^{jd} \right) + \left( 1 + 2 \sum_{j=1}^{\infty} t^{jc} \right)^2.
\]
That is, we must have
\[
\left( 1 + 2 \sum_{j=1}^{\infty} t^{j(b+d)} \right) \left( 1 + 2 \sum_{j=1}^{\infty} t^{jc} \right) \left[ \left( 1 + 2 \sum_{j=1}^{\infty} t^{jb} \right) + \left( 1 + 2 \sum_{j=1}^{\infty} t^{jd} \right) \right] = \left( 1 + 2 \sum_{j=1}^{\infty} t^{j} \right)^2 \left( 1 + 2 \sum_{j=1}^{\infty} t^{jb} \right) \left( 1 + 2 \sum_{j=1}^{\infty} t^{jd} \right) + \left( 1 + 2 \sum_{j=1}^{\infty} t^{jc} \right)^2.
\]
This implies that either \( b = c \) and \( f = b + d \), or \( d = c \) and \( f = b + d \). In either case, this belongs to Case (1) of Theorem 1.1. □

With all of the above, we are ready to prove our main result, Theorem 1.1.

**Proof of Theorem 1.1.** By Lemma 4.1, one of the figures in Fig. 1 occurs as a signed labeled multigraph describing \( M \) that satisfies the properties in Lemma 3.5. If Fig. 1A describes \( M \), the fixed point data is precisely Case (1) of Theorem 1.1. By Lemmas 5.1, 5.2, and 5.3, if any of Fig. 1B–D describes \( M \), then the fixed point data of \( M \) belongs to Case (1) of Theorem 1.1.

Suppose that Fig. 1E describes \( M \). By permuting \( p_1 \) and \( p_2 \), by permuting \( p_3 \) and \( p_4 \), and by changing the sign of every fixed point if necessary, without loss of generality, we may assume that exactly one of the following cases holds for the largest weight:

1. \( b \) is the largest weight.
2. \( a \) is the largest weight.

Since the multigraph Fig. 1E has an edge whose vertices have the same sign, Properties 1 and 2 of Lemma 3.5 imply that the largest weight is strictly bigger than \( a_2 \in W_+ \) where \( a_2 \) and \( W_+ \) are as in Lemma 3.3.

Suppose that \( b \) is the largest weight. By Property 2 of Lemma 3.5, \( p_1 \) and \( p_3 \) lie in the same connected component \( F \) of \( M^2b \). Since \( b > a_2 \) is the largest weight, \( \dim F = 2 \) or \( \dim F = 4 \). If \( \dim F = 4 \), by Lemma 5.4, the fixed point data of \( M \) belongs to Case (1) of Theorem 1.1. Assume that \( \dim F = 2 \). Then by Lemma 5.6, one of the following holds for the fixed point data of \( M \):
(i) \{+, a, b, c\}, \{-a, b, c - a\}, \{-b, a - b, c - b\}, \{-c, a - c, b - c\}. Here, \(a, b,\)
and \(c\) are mutually distinct.

(ii) \{+, a + b, a + b + c\}, \{-a, b, a + b + c\}, \{+, b, a + b\}, \{-c, b + c, a + b + c\}.
for some positive integers \(a, b,\) and \(c\).

The fixed point data of Case (i) is Case (1) of Theorem 1.1, and the fixed point
data of Case (ii) is Case (2) of Theorem 1.1.

Suppose that \(a\) is the largest weight. By Property 2 of Lemma 3.5, \(p_1\) and \(p_2\) lie in the same connected component \(F\) of \(M^{2n}\). Since \(a > a_2\) is the largest
weight, \(\dim F = 2\) or \(\dim F = 4\). If \(\dim F = 4\), by Lemma 5.5, the fixed point
data of \(M\) belongs to Case (1) of Theorem 1.1. If \(\dim F = 2\), by Lemma 5.7, the fixed point data of \(M\) belongs to Case (1) of Theorem 1.1. □

6. Comparison with results on different types of manifolds

In this section, we compare circle actions on different types of 6-dimensional
manifolds that have 4 fixed points.

We consider almost complex manifolds. Throughout this section, sup-
pose that a circle action on an almost complex manifold \((M, J)\) preserves the
almost complex structure \(J\). Then the sign of every weight at a fixed point is
well-defined. While for oriented manifolds on any even dimension there exists a circle action with 2 fixed points (a rotation of \(S^{2n}\)), for almost complex
manifolds if the circle acts on a compact almost complex manifold with 2
fixed points, then the dimension of the manifold must be either 2 or 6; see
[7,12,13,16,17]. In addition, an almost complex manifold with 3 fixed points
only exists in dimension 4, see [7]; also see [6].

We compare circle actions on oriented manifolds and almost complex
manifolds, in dimension 6 and when there are 4 fixed points.

Theorem 6.1. Let the circle act on a 6-dimensional compact connected
almost complex manifold \(M\) with 4 fixed points. Then there exist positive
integers \(a, b, c,\) and \(d\) so that one of the following holds for the weights at
the fixed points as complex \(S^1\)-representations:

1. \(\{a, b, c\}, \{-a, b - a, c - a\}, \{-b, a - b, c - b\}, \{-c, a - c, b - c\}\). Here, \(a, b,\)
and \(c\) are mutually distinct.

2. \(\{a, a + b, a + 2b\}, \{-a, b, a + 2b\}, \{-a - 2b, -b, a\}, \{-a - 2b, -a - b, -a\}\).

3. \(\{1, 2, 3\}, \{-1, 1, a\}, \{-1, -a, 1\}, \{-1, -2, -3\}\).

4. \(\{-a - b, a, b\}, \{-c - d, c, d\}, \{-a, -b, a + b\}, \{-c, -d, c + d\}\).

5. \(\{-3a - b, a, b\}, \{-2a - b, 3a + b, 3a + 2b\}, \{-a, -a - b, 2a + b\}, \{-b, -3a -
2b, a + b\}\) up to reversing the circle action if necessary.

6. \(\{-a - b, 2a + b, b\}, \{-2a - b, a, b\}, \{-b, -2a - b, a + b\}, \{-a, -b, 2a + b\}\).

For a circle action on a compact almost complex manifold, let \(p\) be an
isolated fixed point. Suppose that \(p\) has weights \(\{w_{p1}, \ldots, w_{pn}\}\) as complex
\(S^1\)-representations, for some non-zero integers \(w_{pi}\)’s. If \(p\) has exactly \(n_p\) negative weights, the sign of \(p\) is \((-1)^{n_p}\), and hence the fixed point data of \(p\) as
real \(S^1\)-representations is \(\{(-1)^{n_p}, |w_{p1}|, \ldots, |w_{pn}|\}\). With this understand-
ing, we see that Case (1) and Case (5) of Theorem 6.1 belong to Case (2) of
Theorem 1.1, and all the other cases of Theorem 6.1 belong to Case (1) of Theorem 1.1.

Note that in Cases (1), (2), (4), and (5) in Theorem 6.1, there exists a manifold with the fixed point data. Case (3) with $a = 2$ or $3$ belongs to Case (1) in Theorem 6.1. In Case (3), if $a = 4$ or $5$, there are examples; see [1], [14]. Case (6) of Theorem 6.1 has a possibility that this might be realized as a blow-up $S^2$ in $S^6$ equipped with a rotation having 2 fixed points; see [9] for discussion on this.

If furthermore, $M$ admits a symplectic structure and the circle action is Hamiltonian, only the fixed point data of Cases (1)--(3) in Theorem 6.1 can occur, and in Case (3) only possible values for $a$ are 2, 3, 4, and 5; see [19]. Therefore, in this case for each fixed point data there exists a manifold.

A natural question is what is a possible fixed point data for an 8-dimensional compact oriented $S^1$-manifold with 4 fixed points. The examples of such a manifold are $S^2 \times S^6$, $S^4 \times S^4$, and an equivariant sum along free orbits of rotations of two $S^8$s. The fixed point data of the circle actions on $S^2 \times S^6$ and on $S^4 \times S^4$ belong to the fixed point data of the equivariant sum of rotations of two $S^8$s. On the other hand, if the circle acts on an 8-dimensional almost complex manifold with 4 fixed points, all the Chern numbers and the Hirzebruch $\chi_y$-genus agree with those of $S^2 \times S^6$ [10]. For an 8-dimensional oriented $S^1$-manifold with 4 fixed points, one may ask if the fixed point data is the same as that of an equivariant sum along free orbits of rotations of two $S^8$s.

7. Converting the fixed point data into the empty collection

In [11], the author showed that if the circle acts on a 6-dimensional compact oriented manifold $M$ with isolated fixed points, there is a systematic way of converting the fixed-point data of $M$ into the empty collection by applying a combination of a number of types of operations on it. This is proved by showing that we can successively take equivariant connected sums at fixed points of $M$ with itself, $M$ with $\mathbb{CP}^3$, or $M$ with 6-dimensional analogue of the Hirzebruch surfaces (and these with opposite orientations), to construct a fixed point free $S^1$-action on a compact connected oriented 6-manifold. In this section, we show how this result applies to the case that $M$ has 4 fixed points.

Consider a circle action on a compact oriented manifold $M$ with a non-empty finite fixed point set. Let $p$ be an (isolated) fixed point. In the introduction, in the decomposition of the tangent space $T_p M = \bigoplus_{i=1}^n L_i$ to $M$ at $p$, we chose an orientation of each $L_i$ so that $S^1$ acts on $L_i$ with positive weight $w_{pi}$. In this section, we allow any orientation of $L_i$: the circle acts on $L_i$ with weight $w_{pi}$, which is now a (non-zero) integer. Let $\epsilon(p) = +1$ if the orientation on $M$ agrees with the orientation on $\bigoplus_{i=1}^n L_i$, and let $\epsilon(p) = -1$ otherwise.
For an ordered pair \((a, \{a_1, \ldots, a_n\})\) where \(a \in \{-1, +1\}\) and \(\{a_1, \ldots, a_n\}\) is a multiset of non-zero elements of \(\mathbb{Z}\), we define an equivalence relation as follows:

- \((a, \{a_1, \ldots, a_i, \ldots, a_n\})\) is equivalent to \((-a, \{a_1, \ldots, -a_i, \ldots, a_n\})\).

With the equivalence relation, we define the **fixed point data** of \(p\) to be the equivalence class \([\epsilon(p), w_{p1}, \ldots, w_{pn}]\) of the ordered pair \((\epsilon(p), \{w_{p1}, \ldots, w_{pn}\})\). If all \(w_{pi}\) are positive, this agrees with the fixed point data of \(p\) in the introduction. The **fixed point data** of \(M\) is a collection of the fixed point data of the fixed points of \(M\).

**Theorem 7.1.** \cite{11} Let the circle group \(S^1\) act on a 6-dimensional compact oriented manifold \(M\) with isolated fixed points. To the fixed point data \(\Sigma_M\) of \(M\), we can apply a combination of the following operations to convert \(\Sigma_M\) to the empty collection.

1. Remove \([+, a, b, c]\) and \([-a, b, c]\) together.
2. Remove \([\pm, a, b, c]\) and \([\mp, c-a, c-b, c]\), and add \([\pm, a-b-a, c-a]\) and \([\mp, b, b-a, c-b]\), where \(0 < a < b < c\).
3. Remove \([\pm, a, b, c]\) and \([\pm, a-c-b, c]\), and add \([\pm, c-b, c-a, a]\), \([\pm, c-a, c-b, a]\), and \([\mp, c-a, a-b, a]\), where \(0 < a < b < c\) and \(a \neq b\).
4. Remove \([\pm, a, b, c]\) and \([\pm, c-a, c-a, a]\), and add \([\pm, c-a, c-2a, a]\), \([\pm, c-a, a, a]\), \([\mp, c-2a, a, a]\), where \(0 < a < c\).
5. Remove \([\pm, a, c, a]\) and \([\mp, c-a, c-a, a]\), and add \([\pm, a-c-a, c-2a, a]\), \([\pm, a-c-a, a, a]\), \([\mp, c-2a, a, a]\), \([\pm, a-c-a, c-a, a]\), \([\mp, c-2a, c-a, a]\), where \(0 < a < c\).

There is a definite procedure that this ends in a finite number of steps.

We check Theorem 7.1 for the case of 4 fixed points. Let the circle act on a 6-dimensional compact oriented manifold \(M\) with 4 fixed points. Suppose that Case (1) of Theorem 1.1 holds for the fixed point data of \(M\). That is, the fixed point data of \(M\) is

\([+, a, b, c], [-a, b, c], [+d, e, f], [-d, e, f]\)

for some positive integers \(a, b, c, d, e,\) and \(f\). We apply Operation (1) of Theorem 7.1 twice to convert the fixed point data of \(M\) to the empty collection, first by removing \([+, a, b, c]\) and \([-a, b, c]\) and second by removing \([+, d, e, f]\) and \([-d, e, f]\).

Next, suppose that Case (2) of Theorem 1.1 holds for the fixed point data of \(M\); the fixed point data of \(M\) is

\([+, a, a+b, a+b+c], [-a, b, b+c], [+b, c, a+b], [-c, b+c, a+b+c]\).

for some positive integers \(a, b,\) and \(c\). We apply Operation (2) of Theorem 7.1 to remove \([+, a+a+b, a+b+c]\) and \([-c, b+c, a+b+c]\), and add \([+, a, b+c]\) and \([-a+b, b, c]\) to have a collection

\([-a, b, b+c], [+b, c, a+b], [+a, b, b+c], [-a+b, b, c]\).
Next, we apply Operation (1) of Theorem 7.1 twice to the above collection, first to remove $[-, a, b, b+c]$ and $[+, a, b, b+c]$ and second to remove $[+, b, c, a+b]$ and $[-, a+b, b, c]$, to reach the empty collection.

**Data Availability** This paper has no associated data.

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

**References**

[1] Ahara, K.: 6-dimensional almost complex $S^1$-manifolds with $\chi(M) = 4$. J. Fac. Sci. Univ. Tokyo Sect. IA Math. 38(1), 47–72 (1991)

[2] Atiyah, M., Bott, R.: The moment map and equivariant cohomology. Topology 23, 1–28 (1984)

[3] Atiyah, M., Singer, I.: The index of elliptic operators: III. Ann. Math. 87, 546–604 (1968)

[4] Dejter, I.J.: Smooth $S^1$-manifolds in the homotopy type of $\mathbb{CP}^3$. Mich. Math. J. 23(1), 83–95 (1976)

[5] Herrera, H., Herrera, R.: $\hat{A}$-genus on non-spin manifolds with $S^1$ actions and the classification of positive quaternion-Kähler 12-manifolds. J. Differ. Geom. 61, 341–364 (2002)

[6] Jang, D.: Symplectic periodic flows with exactly three equilibrium points. Ergod. Theory Dyn. Syst. 34, 1930–1963 (2014)

[7] Jang, D.: Circle actions on almost complex manifolds with isolated fixed points. J. Geom. Phys. 119, 187–192 (2017)

[8] Jang, D.: Circle actions on oriented manifolds with discrete fixed point sets and classification in dimension 4. J. Geom. Phys. 133, 181–194 (2018)

[9] Jang, D.: Circle actions on almost complex manifolds with 4 fixed points. Math. Z. 294, 287–319 (2020)

[10] Jang, D.: Circle actions on 8-dimensional almost complex manifolds with 4 fixed points. J. Fixed Point Theory A 22, 95 (2020)

[11] Jang, D.: Graphs for torus actions on oriented manifolds with isolated fixed points and classification in dimension 6 (2022). arXiv:2202.10190

[12] Kosniowski, C.: Holomorphic vector fields with simple isolated zeros. Math. Ann. 208, 171–173 (1974)

[13] Kosniowski, C.: Fixed points and group actions. Lect. Notes Math. 1051, 603–609 (1984)

[14] McDuff, D.: Some 6-dimensional Hamiltonian $S^1$-manifolds. J. Topol. 2(3), 589–623 (2009)
[15] Musin, O.: Actions of a circle on homotopy complex projective spaces. Mat. Zametki 28(1), 139–152 (1980) (Russian) [Math. Notes 28(1), 533–540 (1980) (English translation)]

[16] Musin, O.: Circle actions with two fixed points. Math. Notes 100, 636–638 (2016)

[17] Pelayo, A., Tolman, S.: Fixed points of symplectic periodic flows. Ergod. Theory Dyn. Syst. 31, 1237–1247 (2011)

[18] Petrie, T.: Smooth $S^1$-actions on homotopy complex projective spaces and related topics. Bull. Am. Math. Soc. 78, 105–153 (1972)

[19] Tolman, S.: On a symplectic generalization of Petrie’s conjecture. Trans. Am. Math. Soc. 362(8), 3963–3996 (2010)

Donghoon Jang
Department of Mathematics
Pusan National University
Pusan
Korea
e-mail: donghoonjang@pusan.ac.kr

Accepted: June 18, 2023.