Nefness of the direct images of pluricanonical bundles

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Abstract

Given a fibration \( f \) between two projective manifolds \( X \) and \( Y \), we provide a sufficient condition such that the direct images \( f_* (K_X \otimes L \otimes I(f, \|L\|)) \) is nef, where \( L \) is a holomorphic line bundle with non-negative relative Iitaka dimension and \( I(f, \|L\|) \) is the relative asymptotic multiplier ideal sheaf.

1. Introduction

Assume that \( f : X \to Y \) is a fibration i.e. a surjective morphism with connected fibres between two projective manifolds \( X \) and \( Y \), and \( K_{X/Y}^m \) is the \( m \)-th tensor power of the relative pluricanonical bundle on \( X \). The positivity of the associated direct image \( f_* (K_{X/Y}^m) \) is of significant importance for understanding the geometry of this fibration. Fruitful results have been published on this subject, for example \([\text{Ber08}, \text{Ber09}, \text{Hor10}, \text{Kaw81}, \text{Kaw82}, \text{Kr86a}, \text{Kr86b}, \text{Kr87}, \text{Vie82b}, \text{Vie83}]\).

It turns out that the positivity of \( f_* (K_{X/Y}^m) \) is deeply influenced by \( K_{X/Y} \) when \( m > 1 \).

In this paper, we focus upon the situation that \( K_{X/Y} \) has non-negative relative Iitaka dimension \( \kappa(K_{X/Y}, f) \) (see Sect.2.1); our main theorem is as follows:

**Theorem 1.1.** Let \( f : X \to Y \) be a smooth fibration between projective manifolds \( X \) and \( Y \). Fix \( p \gg 0 \) and divisible enough that computes \( I(f, \|L\|) \). For any non-negative integer \( k \), if \( I(kp\|K_{X/Y}\|) = \mathcal{O}_X \),

\[
f_* (K_{X/Y}^m)
\]

is nef for any non-negative integer \( m \leq k + 1 \).

Here, and throughout the rest of this paper, \( \mathcal{I} (\|K_{X/Y}\|) \) (resp. \( \mathcal{I} (f, \|K_{X/Y}\|) \)) refers to the (resp. relative) asymptotic multiplier ideal sheaf (see Sect.2.3). Instead of Theorem 1.1, we would like to arrange all the things for the following refined version that is formulated for an arbitrary line bundle \( L \) on \( X \).

**Theorem 1.2.** Let \( f : X \to Y \) be a smooth fibration between projective manifolds \( X \) and \( Y \). Let \( L \) be a holomorphic line bundle on \( X \) with \( \kappa(L, f) \geq 0 \) and \( \kappa(K_{X/Y} \otimes L, f) \geq 0 \). Fix \( p \gg 0 \) and divisible enough that computes \( \mathcal{I} (f, \|L\|) \). Assume that the following conditions hold:

1. there exists a (singular) metric \( \tau \) on \( -K_{X/Y} \otimes L^p \) such that \( i^* \Theta_{-K_{X/Y} \otimes L^p, \tau} \geq 0 \) and \( \mathcal{I} (\tau) = \mathcal{O}_X \); and
2. let \( \varphi \) and \( \psi \) be the metrics on \( L \) and \( K_{X/Y} \otimes L \) associated to (see Sect.2.3)

\[
\mathcal{I} (f, \|L\|) \text{ and } \mathcal{I} (f, \|K_{X/Y} \otimes L\|)
\]

respectively, then \( \varphi \) is less singular than \( \psi \) (see [Dem12]), i.e. \( \varphi \preceq \psi \).

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Then
\[ \mathcal{E} := \pi_*(K_{X/Y} \otimes \mathcal{L} \otimes \mathcal{I}(f, \|\mathcal{L}\|)) \]
is nef.

For a line bundle \( H, H^k \) refers to its \( k \)-th tensor power with the convention that \( H^0 = \mathcal{O}_X \) and \( H^k = (H^*)^{-k} \) for \( k < 0 \). \(-K_{X/Y}\) is the relative anti-canonical bundle. Remember in [Wu21], Theorem 6.1 we show that \( \mathcal{E} \) is always torsion-free, hence the conclusion of Theorem 1.2 is interpreted as that \( \mathcal{E} \) is nef as a torsion-free coherent sheaf (see Sect.2.2).

The requirements (1), (2) are verified in other situations (see Corollaries 4.1 and 4.2) apart from Theorem 1.1.

We use the same strategy as in [Ko86a, Ko86b, Vie82b, Vie83] to prove Theorem 1.2. The ideal is expanded as follows: first we should prove an injectivity theorem.

**Theorem 1.3.** Let \( f : X \to Y \) be a fibration between projective manifolds \( X \) and \( Y \). Let \( L \) be a line bundle on \( X \) with \( \kappa(L, f) \geq 0 \), and let \( A' \) be the pullback of a sufficient ample line bundle \( A \) on \( Y \). Fix \( p \gg 0 \) and divisible enough that computes \( \mathcal{I}(f, \|\mathcal{L}\|) \). Assume that there exists a (singular) metric \( \tau \) on \(-K_{X/Y} \otimes \mathcal{L}^p\) such that \( i\Theta_{-K_{X/Y} \otimes \mathcal{L}^p, \tau} \geq 0 \) and \( \mathcal{I}(\tau) = \mathcal{O}_X \).

Then for a (non-zero) section \( s \) of \( A' \), the multiplication map induced by the tensor product with \( s \)
\[ \Phi : H^q(X, K_X \otimes L \otimes A' \otimes \mathcal{I}(f, \|\mathcal{L}\|)) \to H^q(X, K_X \otimes L \otimes (A')^2 \otimes \mathcal{I}(f, \|\mathcal{L}\|)) \]
is well-defined and injective for any \( q \geq 0 \).

Combine with Theorem 1.3 and the fact that \( \mathcal{E} \) is torsion-free, we obtain the following Kollár-type vanishing theorem.

**Theorem 1.4.** Let \( f : X \to Y \) be a fibration between projective manifolds \( X \) and \( Y \). Let \( L \) be a holomorphic line bundle on \( X \) with \( \kappa(L, f) \geq 0 \). Fix \( p \gg 0 \) and divisible enough that computes \( \mathcal{I}(f, \|\mathcal{L}\|) \). Assume that there exists a (singular) metric \( \tau \) on \(-K_{X/Y} \otimes \mathcal{L}^p\) such that \( i\Theta_{-K_{X/Y} \otimes \mathcal{L}^p, \tau} \geq 0 \) and \( \mathcal{I}(\tau) = \mathcal{O}_X \). If \( A \) is a sufficient ample line bundle (but independent of \( f \) and \( L \)) on \( Y \), then for any \( i > 0 \) and \( q \geq 0 \),
\[ H^i(Y, R^qf_*(K_X \otimes \mathcal{L} \otimes \mathcal{I}(f, \|\mathcal{L}\|)) \otimes A) = 0. \]

Here we emphasize that the choice of \( A \) in both of the theorems above is independent of \( f \) and \( L \). In particular, Theorem 1.3 is not an easy consequence of asymptotic Serre vanishing theorem [Har77].

A direct consequence of Theorem 1.4 is

**Corollary 1.1.** Under the same assumptions as in Theorem 1.4, if \( A \) is moreover globally generated and \( A' \) is a nef line bundle on \( Y \), then the sheaf
\[ R^qf_*(K_X \otimes \mathcal{L} \otimes \mathcal{I}(f, \|\mathcal{L}\|)) \otimes A^m \otimes A' \]
is globally generated for any \( q \geq 0 \) and \( m \geq \dim Y + 1 \).

Then we prove that \( \mathcal{E} \) is weakly positive in the sense of Viehweg (see Sect 2.2) via the fibre product method of Viehweg [Vie82b, Vie83], and Theorem 1.2 follows. We leave the details in the text.

This paper is organised as follows. We first recall some background materials, including the definition of the positivity concerning a torsion-free coherent sheaf, the asymptotic multiplier
ideal sheaf and so on. Then, we proceed to prove Theorems 1.3 and 1.4 in Sect. 3. The proof of Theorems 1.1 and 1.2 are presented in Sect. 4.

2. Preliminary

In this section we will introduce some basic materials. For clarity and for convenience of later reference, it will be done in the following setting: \( f : X \rightarrow Y \) is a fibration between two projective manifolds, and \( L \) is a holomorphic line bundle on \( X \) with \( \kappa(L, f) \geq 0 \). Here \( \kappa(L, f) \) is the relative Iitaka dimension that is explained immediately.

2.1 Relative Iitaka dimension

This part is borrowed from [FEM13].

Let \( l \) be the dimension of a general fibre \( F \) of \( f \). We have

**Proposition 2.1.** For every coherent sheaf \( G \) on \( X \), there is \( C > 0 \) (independent of \( L \)) such that

\[
\text{rank}(f_*(G \otimes L^k)) \leq Ck^l \text{ for all } k \gg 0.
\]

**Proof.** Let us write \( L = A \otimes B^{-1} \), with \( A \) and \( B \) are very ample line bundles. For every \( k \), if we choose \( E \) general in the complete linear system \( |B^k| \), then a local defining function of \( E \) is a non-zero divisor on \( G \), in which case we have an inclusion

\[
H^0(F, G \otimes L^k) \hookrightarrow H^0(F, G \otimes A^k).
\]

Since \( A \) is very ample, we know that there is a polynomial \( P \in \mathbb{Q}[t] \) [Laz04a] with \( \deg(P) \leq l \) such that \( h^0(F, G \otimes A^k) = P(k) \) for \( k \gg 0 \). Therefore \( h^0(F, G \otimes L^k) \leq P(k) \leq Ck^l \) for a suitable \( C > 0 \) and all \( k \gg 0 \).

**Definition 2.1.** The relative Iitaka dimension \( \kappa(L, f) \) of \( L \) is the biggest integer \( M \) such that there is \( C > 0 \) satisfying

\[
\text{rank}(f_*L^k) \geq Ck^M \text{ for all } k \gg 0
\]

with the convention that \( \kappa(L, f) = -\infty \) if \( \text{rank } f_*L^k = 0 \).

Note that \( \kappa(L, f) \) takes value in \( \{-\infty, 0, 1, \ldots, l\} \) by Proposition 2.1. In particular, if \( \kappa(L, f) = l \), we say that \( L \) is \( f \)-big.

2.2 Positivity

Let \( E \) be a holomorphic vector bundle of rank \( r \) over \( X \). By \( \mathbb{P}(E^*) \), we denote the projectivised bundle of \( E^* \) and by \( \mathcal{O}_E(1) := \mathcal{O}_{\mathbb{P}(E^*)}(1) \) we denote the tautological line bundle. Let

\[
\pi : \mathbb{P}(E^*) \rightarrow X
\]

be the canonical projection.

We now collect the definitions of positivity from [DPS94, DPS01, Pau16, PT14, Vie82b, Vie83] as follows.

**Definition 2.2.** (i) \( E \) is weakly positive in the sense of Viehweg if, on some Zariski open subset \( U \subset X \), for any integer \( a > 0 \), there exists an integer \( b > 0 \) such that \( S^{ab}(E) \otimes bA \) is generated by global sections on \( U \). Here, \( A \) is an auxiliary ample line bundle over \( X \) and \( S^{ab}(E) \) refers to the \( ab \)-th symmetric product of \( E \).
(ii) $E$ is pseudo-effective if $\mathcal{O}_E(1)$ is pseudo-effective and the image of the non-nef locus $\text{NNef}(\mathcal{O}_E(1))$

(i.e., the union of all curves $C$ on $\mathbb{P}(E^*)$ with $\mathcal{O}_E(1) \cdot C < 0$) under $\pi$ is a proper subset of $X$.

(iii) $E$ is nef if $\mathcal{O}_E(1)$ is nef.

(iv) $E$ is almost nef if there exists a countable family of proper subvarieties $Z_i$ of $X$ such that $E|_C$ is nef for any curve $C \not\subset \bigcup_i Z_i$.

(v) $E$ is ample if $\mathcal{O}_E(1)$ is ample.

Remark 2.1. The relationships among these notions are summarised below.

\[
\begin{array}{cccc}
\text{nef} & \downarrow & \cdots & \text{pseudo-effective} \\
& \downarrow & \cdots & \downarrow \\
& \rightarrow & \cdots & \rightarrow \\
\downarrow & \cdots & \downarrow & \rightarrow \\
\text{weakly positive} & \cdots & \text{almost nef} & \rightarrow \\
\end{array}
\]

These notions extend to a torsion-free coherent sheaf. Assume that $\mathcal{E}$ is a torsion-free coherent sheaf; then it is locally free outside of a 2-codimensional subvariety $Z$. We say that $\mathcal{E}$ is nef (resp. pseudo-effective, almost nef,...) if $\mathcal{E}|_X \setminus Z$ has the corresponding property. The reader can refer to \cite{Pau16, PT14} for the more details.

2.3 The asymptotic multiplier ideal sheaf

This part is mostly collected from \cite{Laz04b}.

Recall that for an arbitrary ideal sheaf $a \subset \mathcal{O}_X$, the associated multiplier ideal sheaf is defined as follows: let $\mu : \tilde{X} \to X$ be a smooth modification such that $\mu^*a = \mathcal{O}_{\tilde{X}}(-E)$, where $E$ has simple normal crossing support. Then given a positive real number $c > 0$ the multiplier ideal sheaf is defined as

$$\mathcal{I}(c \cdot a) := \mu_* \mathcal{O}_{\tilde{X}}(K_{\tilde{X}/X} - \lfloor cE \rfloor).$$

Here $\lfloor E \rfloor$ means the round-down.

Now assume that $\kappa(L) \geq 0$. Fix a positive real number $c > 0$. For $k > 0$ consider the complete linear series $|L^k|$, and form the multiplier ideal sheaf

$$\mathcal{I}(\frac{c}{k}|L^k|) \subseteq \mathcal{O}_X,$$

where $\mathcal{I}(\frac{c}{k}|L^k|) := \mathcal{I}(\frac{c}{k} \cdot a_k)$ with $a_k$ being the base-ideal of $|L^k|$. It is not hard to verify that for every integer $p \geq 1$ one has the inclusion

$$\mathcal{I}(\frac{c}{k}|L^k|) \subseteq \mathcal{I}(\frac{c}{pk}|L^{pk}|).$$

Therefore the family of ideals

$$\{\mathcal{I}(\frac{c}{k}|L^k|)\}_{k \geq 0}$$

has a unique maximal element from the ascending chain condition on ideals.

Definition 2.3. The asymptotic multiplier ideal sheaf associated to $c$ and $|L|$, $\mathcal{I}(c\|L\|)$
is defined to be the unique maximal member among the family of ideals $\{\mathcal{I}(\frac{c}{k}|L^k|)\}$.

By definition, $\mathcal{I}(c||L||) = \mathcal{I}(\frac{c}{k}|L^k|)$ for some $k$. Let $u_1, ..., u_m$ be a basis of $H^0(X, L^k)$, then the base-ideal of $|L^k|$ is just $\mathcal{I}(u_1, ..., u_m)$. Let $\varphi = \log(|u_1|^2 + \cdots + |u_m|^2)$, which is a singular metric on $L$. We verify that

$$\mathcal{I}(\frac{c}{k}|L^k|) = \mathcal{I}(\frac{c}{k}\varphi).$$

Indeed, let $\mu : \tilde{X} \to X$ be the smooth modification such that $\mu^*\mathcal{I}(u_1, ..., u_m) = \mathcal{O}_{\tilde{X}}(-E)$, where $E$ has simple normal crossing support. Then it is computed in [Dem12] that

$$\mathcal{I}(\frac{c}{k}\varphi) = \mu_*\mathcal{O}_{\tilde{X}}(K_{\tilde{X}/X} - [\frac{c}{k}E]),$$

which coincides with the definition of $\mathcal{I}(\frac{c}{k}|L^k|)$. In summary, we have

$$\mathcal{I}(c||L||) = \mathcal{I}(\frac{c}{k}\varphi),$$

and $\frac{1}{k}\varphi$ is called the singular metric on $L$ associated to $\mathcal{I}||L||$.

Next, we introduce the relative variant. Let $f : X \to Y$ be a surjective morphism between projective manifolds, and $L$ a line bundle on $X$ whose restriction to a general fibre of $f$ has non-negative Iitaka dimension. Then there is a naturally defined homomorphism

$$\rho : f^*f_*L \to L.$$

Let $\mu : \tilde{X} \to X$ be a smooth modification of $|L|$ with respect to $f$, having the property that the image of the induced homomorphism

$$\mu^*\rho : \mu^*f^*f_*L \to \mu^*L$$

is the subsheaf $\mu^*L \otimes \mathcal{O}_{\tilde{X}}(-E)$ of $\mu^*L$, $E$ being an effective divisor on $\tilde{X}$ such that $E + \operatorname{except}(\mu)$ has simple normal crossing support. Here $\operatorname{except}(\mu)$ is the exceptional divisor of $\mu$. Given $c > 0$ we define

$$\mathcal{I}(f, c||L||) = \mu_*\mathcal{O}_{\tilde{X}}(K_{\tilde{X}/X} - [cE]).$$

Similarly, $\{\mathcal{I}(f, \frac{c}{k}|L^k|)\}_{(k \geq 0)}$ has a unique maximal element.

**Definition 2.4.** The relative asymptotic multiplier ideal sheaf associated to $f$, $c$ and $|L|$, $\mathcal{I}(f, c||L||)$ is defined to be the unique maximal member among the family of ideals $\{\mathcal{I}(f, \frac{c}{k}|L^k|)\}$.

By definition, $\mathcal{I}(f, c||L||) = \mathcal{I}(f, \frac{c}{k}|L^k|)$ for some $k$. Let $\rho_k$ be the naturally defined homomorphism

$$\rho_k : f^*f_*L \to L^k.$$

Let $\mu : \tilde{X} \to X$ be the smooth modification of $|L^k|$ with respect to $f$ such that

$$\operatorname{Im}(\mu^*\rho_k) = \mu^*L^k \otimes \mathcal{O}_{\tilde{X}}(-E).$$

Consider $\mu_*\mathcal{O}_{\tilde{X}}(-E)$ which is an ideal sheaf on $X$. Pick a local coordinate ball $U$ of $Y$, and let $u_1, ..., u_m$ be the generators of $\mu_*\mathcal{O}_{\tilde{X}}(-E)$ on $f^{-1}(U)$. The existence of these generators is obvious concerning the fact that $\operatorname{Im}(\mu^*\rho_k) = \mu^*L^k \otimes \mathcal{O}_{\tilde{X}}(-E)$. Moreover they can be seen as the sections of $\Gamma(f^{-1}(U), L^k)$.
Now let \( \varphi_U = \log(|u_1|^2 + \cdots + |u_m|^2) \), which is a singular metric on \( L^k|_{f^{-1}(U)} \). It is then easy to verify that

\[
\mathcal{I}(\frac{c}{k} \varphi) = \mathcal{I}(f, \frac{c}{k} |L^k|) \text{ on } f^{-1}(U).
\]

Furthermore, if \( v_1, \ldots, v_m \) are alternative generators and \( \psi_U = \log(|v_1|^2 + \cdots + |v_m|^2) \), obviously we have \( \mathcal{I}(\frac{c}{k} \varphi U) = \mathcal{I}(\frac{c}{k} \psi_U) \). Hence all the \( \mathcal{I}(\frac{c}{k} \varphi) \) patch together to give a globally defined multiplier ideal sheaf \( \mathcal{I}(\frac{c}{k} \varphi) \) such that

\[
\mathcal{I}(\frac{c}{k} \varphi) = \mathcal{I}(f, \frac{c}{k} |L^k|) = \mathcal{I}(f, c||L||) \text{ on } X.
\]

One should be careful that \( \{ \frac{1}{k} \varphi_U \} \) won’t give a globally defined metric on \( L \) in general. Hence \( \frac{1}{k} \varphi \) is interpreted as the collection of functions \( \{ \frac{1}{k} \varphi_U \} \) by abusing the notation, which is called the collection of (local) singular metrics on \( L \) associated to \( \mathcal{I}(f, c||L||) \).

Now we collect some elementary properties from [Laz04b]. Recall that the relative base-ideal of \( |L| \) is by definition the image of the homomorphism

\[
f^* f_* L \otimes L^{-1} \rightarrow \mathcal{O}_X
\]
determined by \( \rho \).

**Proposition 2.2.** Let \( f : X \rightarrow Y \) be a surjective morphism between projective manifolds, and \( H_1, H_2 \) are line bundles on \( X \) with non-negative relative Iitaka dimension. \( k \) and \( m \) are arbitrary positive integers. Let \( L \) be a line bundle on \( X \) with non-negative Iitaka dimension.

1. Let \( a_{k,f} = a(f, |H^k_1|) \) be the base-ideal of \( |H^k_1| \) relative to \( f \). There exit an integer \( k_0 \) such that for every \( k \geq k_0 \), the canonical map \( \rho_k : f^* f_* H^k_1 \rightarrow H^k_1 \) factors through the inclusion \( H^k_1 \otimes \mathcal{I}(f, ||H^k_1||) \), i.e.

\[
a_{k,f} \subseteq \mathcal{I}(f, ||H^k_1||).
\]

Equivalently, the natural map

\[
f_* (H^k_1 \otimes \mathcal{I}(f, ||H^k_1||)) \rightarrow f_* (H^k_1)
\]

is an isomorphism.

2. \( a_{m,f} \cdot \mathcal{I}(f, ||H^k_2||) \subseteq \mathcal{I}(f, ||H^m_1 \otimes H^k_2||) \).

3. \( \mathcal{I}(f, ||H^k_1||) \supseteq \mathcal{I}(f, ||H^{k+1}_1||) \) for every \( k \).

4. \( \mathcal{I}(||L||) \subseteq \mathcal{I}(f, ||L||) \).

**Proof.** (i) is proved in [Laz04b], Proposition 11.2.15.

(ii) Fix \( p \gg 0 \) and divisible enough that computes all of the multiplier ideals \( \mathcal{I}(f, ||H^m_1||) \), \( \mathcal{I}(f, ||H^k_2||) \) and \( \mathcal{I}(f, ||H^m_1 \otimes H^k_2||) \). Let \( b_{k,f} \) be the base-ideal of \( |H^k_2| \) relative to \( f \), and let \( c_{m,k,f} \) be the base-ideal of \( |H^m_1 \otimes H^k_2| \) relative to \( f \). Let \( \mu : \tilde{X} \rightarrow X \) be the smooth modification of \( a_{m,f}, a_{pm,f}, b_{pk,f} \) and \( c_{pm, pk, f} \), such that

\[
\mu^* a_{m,f} = \mathcal{O}_{\tilde{X}}(-E), \mu^* a_{pm,f} = \mathcal{O}_{\tilde{X}}(-F), \mu^* b_{pk,f} = \mathcal{O}_{\tilde{X}}(-G) \text{ and } \mu^* c_{pm, pk, f} = \mathcal{O}_{\tilde{X}}(-H),
\]

where \( E = \sum a_i E_i, F = \sum b_i E_i, G = \sum c_i E_i \) and \( H = \sum d_i E_i \) have simple normal crossing support. Then for every \( i \),

\[
d_i \leq b_i + c_i \leq pa_i + c_i.
\]
and consequently
\[-a_i - \left\lfloor \frac{c_i}{p} \right\rfloor \leq -\left\lfloor \frac{d_i}{p} \right\rfloor.\]

Thus
\[a_{m,f} \cdot \mathcal{I}(f, \|H_k\|) \subseteq \mu_* O_X(-E + K_{X/X} - \left\lfloor \frac{1}{p}G \right\rfloor)\]
\[\subseteq \mu_* O_X(K_{X/X} - \left\lfloor \frac{1}{p}H \right\rfloor)\]
\[= \mathcal{I}(f, \|H_{m}^n \otimes H_k^2\|).\]

(iii) Fix \(p \gg 0\) and divisible enough that computes both of the multiplier ideals \(\mathcal{I}(f, k\|H_1\|)\) and \(\mathcal{I}(f, \|H_k\|)\). Then
\[\mathcal{I}(f, \|H_k\|) = \mathcal{I}(f, \|H_1\|) \supseteq \mathcal{I}(f, (k + 1)\|H_1\|) = \mathcal{I}(f, \|H_{k+1}^{k+1}\|).\]

(iv) Fix \(p \gg 0\) and divisible enough that computes both of the multiplier ideals \(\mathcal{I}(f, \|L\|)\) and \(\mathcal{I}(\|L\|)\). Let \(\{u_i\}\) be a basis of \(H^0(X, L^p)\). Then for a general fibre \(F\) of \(f\), \(u_i|_F\) is a section of \(H^0(F, L^p)\). Hence \(\{u_i\}\) forms an ideal that is contained in the relative base-ideal of \([L^p]\). Now the inclusion is obvious.

As a by-product of the formula
\[\mathcal{I}(f, \|H_k\|) = \mathcal{I}(f, k\|H_1\|)\]
in (iii), if \(\varphi = \{\varphi_U\}\) is the collection of metrics associated to \(\mathcal{I}(f, \|H_1\|)\), \(k\varphi = \{k\varphi_U\}\) is the collection of metrics associated to \(\mathcal{I}(f, \|H_k\|)\).

### 2.4 Fibration

Next, we recall the definition of a fibre product [Har77].

**Definition 2.5.** Let \(f : X \to Y\) be a fibration between two projective manifolds \(X\) and \(Y\). The fibre product, denoted by \((X \times_Y X, p_1^2, p_2^2)\), is a projective manifold coupled with two morphisms (we will also call the manifold \(X \times_Y X\) itself the fibre product if nothing is confused) that satisfies the following properties:

1. The diagram
\[
\begin{array}{ccc}
X \times_Y X & \stackrel{p_1^2}{\longrightarrow} & X \\
\downarrow & & \downarrow^f \\
X & \stackrel{f}{\longrightarrow} & Y
\end{array}
\]
commutes.
2. If there is another projective manifold $Z$ with morphisms $q_1, q_2$ such that the diagram
\[
\begin{array}{ccc}
Z & \xrightarrow{q_2} & X \\
\downarrow q_1 & \downarrow f & \downarrow f \\
X & \xrightarrow{f} & Y
\end{array}
\]
commutes, then there must exist a unique $g : Z \rightarrow X \times_Y X$ such that $p_1^2 \circ g = q_1, p_2^2 \circ g = q_2$.

We inductively define the $m$-fold fibre product, and denote it by $X \times_Y \cdots \times_Y X$. Then, the two projections are denoted by
\[
p_1^m : X \times_Y \cdots \times_Y X \rightarrow X
\]
and
\[
p_2^m : X \times_Y \cdots \times_Y X \rightarrow X \times_Y \cdots \times_Y X
\]
respectively.

The meaning of the fibre product is clear from the viewpoint of geometry. In particular, if $y$ is a regular value of $f$,
\[ (X \times_Y \cdots \times_Y X)_y = X_y \times \cdots \times X_y. \]

We collect the following two lemmas from [Har77] without proof for the later use.

**Lemma 2.1 (Projection formula).** If $f : X \rightarrow Y$ is a holomorphic morphism between two projective manifolds $X$ and $Y$, $\mathcal{F}$ is a coherent sheaf on $X$, and $\mathcal{E}$ is a locally free sheaf on $Y$, then there is a natural isomorphism
\[ f^!(\mathcal{F} \otimes \mathcal{E}) \cong f^!(\mathcal{F}) \otimes \mathcal{E}. \]

**Lemma 2.2 (Base change).** Assume that $f : X \rightarrow Y, v : X' \rightarrow X$ and $g : X' \rightarrow Y'$ are holomorphic morphisms between projective manifolds $X,X',Y$ and $Y'$. Let $\mathcal{F}$ be a coherent sheaf on $X$. $u : Y' \rightarrow Y$ is a smooth morphism, such that
\[
\begin{array}{ccc}
X' & \xrightarrow{u} & X \\
\downarrow g & \downarrow f & \downarrow f \\
Y' & \xrightarrow{u} & Y
\end{array}
\]
commutes. Then for all $q \geq 0$ there is a natural isomorphism
\[ u^* R^q f^!(\mathcal{F}) \cong R^q g^!(u^* \mathcal{F}). \]

### 3. The injectivity theorem and the vanishing theorem

#### 3.1 The injectivity theorem
Theorem [L3] is a direct consequence of the following variant of the Kollár-type injectivity theorem developed in [Eno93, Fuj12, GoM17, Ko86a, Ko86b, Mat14, Mat15a, Mat18].

**Theorem 3.1.** Let $(H, \varphi_H)$ and $(M, \varphi_M)$ be line bundles with (singular) metrics on a projective manifold $X$. Assume the following conditions:

1. $i\Theta_{H, \varphi_H} \geq 0$ and $i\Theta_{M, \varphi_M} \geq \gamma$ for some smooth real $(1,1)$-form $\gamma$ on $X$; and
2. $i\Theta_{H, \varphi_H} \geq \varepsilon i\Theta_{M, \varphi_M}$ for some positive number $\varepsilon$. 

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Then for a (non-zero) section $s$ of $M$ with $\sup_X |s|^2 e^{-\varphi M} < \infty$, the multiplication map induced by the tensor product with $s$

$$\Phi : H^q(X, K_X \otimes H \otimes \mathcal{I}(\varphi_H)) \to H^q(X, K_X \otimes H \otimes M \otimes \mathcal{I}(\varphi_H + \varphi_M))$$

is well-defined and injective for any $q \geq 0$.

**Proof.** The proof is nothing but repeating the argument in, say [GoM17], so we omit it here. \Box

Now we prove Theorem 1.3.

**Proof of Theorem 1.3.** We claim that for a general fibre $F$ of $f$, if $u$ is a section of $H^0(F, L^p)$, it extends to a global section $\tilde{u}$ of $H^0(X, L^p \otimes A')$.

In order to prove this claim, we first recall the Ohsawa–Takegoshi extension theorem as follows:

**Theorem 3.2** (Theorem 1, [Man93]). Let $X$ be a projective manifold, and let $Z \subset X$ be the zero set of a holomorphic section $s \in H^0(X, E)$ of a vector bundle $E \to X$; the subset $Z$ is assumed to be non-singular and of codimension $r = \text{rank}(E)$. Let $(G, \varphi)$ be a line bundle on $X$, endowed with a (singular) metric $\varphi$, such that

(a) $i\Theta_{G, \varphi} + i\partial \bar{\partial} \log |s|^2 \geq 0$ on $X$; 
(b) $i\Theta_{G, \varphi} + i\partial \bar{\partial} \log |s|^2 \geq \frac{1}{\alpha} \frac{<i\Theta_{G, \varphi}, s>}{|s|^2}$ for some $\alpha \geq 1$; and 
(c) $|s|^2 \leq e^{-\alpha}$ on $X$, and the restriction of $\varphi$ on $Z$ is well-defined.

Then every section $u \in H^0(Z, K_Z \otimes G \otimes \mathcal{I}(\varphi|_Z))$ admits an extension $\tilde{u}$ to $X$ such that

$$\int_X \frac{\tilde{u} \wedge \bar{\tilde{u}} e^{-\varphi}}{|s|^{2r}(-\log |s|)^2} \leq C_\alpha \int_Z \frac{|u|^2 e^{-\varphi}}{\Lambda^r(ds)^2},$$

provided the right hand side is finite.

Now the bundle that we are interested in can be decomposed as

$$L^p \otimes A' = K_X \otimes -K_{X/Y} \otimes L^p \otimes f^*(A \otimes -K_Y).$$

(The bundle $A'$ is chosen in a moment.) Our goal is to show that it is effective by extending the section $0 \neq u \in H^0(F, L^p \otimes \mathcal{I}(\tau))$. (Remember that $\mathcal{I}(\tau) = O_X$.) We choose now the bundle $A$ positive enough so that

(1) $H^0(Y, A) \neq 0$; and

(2) the point $y \in Y$ such that $f^{-1}(y) = F$ is the common zero set of the sections $s = (s_j)$ of an ample line bundle $B \to Y$ and $A \otimes -K_Y \geq B^2$, in the sense that the difference is an ample line bundle.

Obviously $A$ as well as $A' = f^*A$ is independent of $f$ and $L$. Property (2) gives a smooth metric $\varphi_{A \otimes -K_Y}$ with positive curvature. The bundle $-K_{X/Y} \otimes L^p \otimes f^*(A \otimes -K_Y)$ is endowed with the metric $\tau \otimes f^*\varphi_{A \otimes -K_Y}$; its curvature is semipositive on $X$, and the restriction to $F$ is well-defined. The section we want to extend is $s := u \otimes s_{A'}$, where $s_{A'}$ is the pullback of some nonzero section given by property (1) above. By property (2), the positivity conditions in Theorem 3.2 are satisfied with the bundle $G$ given by

$$G = -K_{X/Y} \otimes L^p \otimes f^*(A \otimes -K_Y).$$
Since
\[ i\Theta_{G,\tau} f^* \varphi_{A \otimes -K_Y} + i\partial \bar{\partial} \log |s|^2 \geq f^* \Theta_{A \otimes -K_Y \otimes B^{-1}}. \]

The right-hand side is semipositive, and it dominates the bundle \( B \); thus the requirements (a)-(c) are verified.

Now the integrability condition is obviously acceptable, and by Theorem 3.2 we can extend the section \( v \) over \( X \). The claim is proved.

We return to Theorem 1.3. Note \( p \) computes \( \mathcal{I}(f, \| L \|) \). Remember the discussion in Sect. 2.8 and keep the notations there, there exists a collection of singular metrics \( \varphi = \{ \varphi_U \} \) defined by the sections of \( \Gamma(f^{-1}(U), L^p) \), say \( \{ u_{i,U} \} \), with \( i\Theta_{L,\varphi_U} \geq 0 \) and

\[ \mathcal{I}(\varphi) = \mathcal{I}(f, \| L \|). \]

Let \( \{ s_j \} \) be the sections of \( A' \) that generates \( A' \). Due to the claim before, all of the sections \( \{ u_{i,U} \otimes s_j \} \) extend over \( X \) as the sections \( \{ v_{ij} \} \) of \( H^0(X, L^p \otimes A') \). Certainly they together define a (singular) metric \( \chi \) on \( L^p \otimes A' \) with positive curvature current. In particular, since \( \mathcal{I}(\{ u_{i,U} \}) = \mathcal{I}(\{ u_{i,U} \otimes s_j \}) = \mathcal{I}(\{ v_{ij} \}) \) on \( f^{-1}(U) \),

\[ \mathcal{I}(\frac{1}{p} \chi) = \mathcal{I}(\varphi) = \mathcal{I}(f, \| L \|) \text{ on } f^{-1}(U) \text{ hence everywhere}. \]

Now let \( \psi \) be a smooth metric on \( A' \) with semipositive curvature. Let

\[ (H, \varphi_H) = (L \otimes A', \frac{1}{p} \chi + \frac{p-1}{p} \psi) \text{ and } (M, \varphi_M) = (A', \psi). \]

The requirements (1) and (2) of Theorem 3.1 are easy to verified. We then obtain the desired injectivity result by Theorem 3.1.

\[ \square \]

3.2 The vanishing theorem

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let \( \varphi = \{ \varphi_U \} \) be the collection of metrics on \( L|_{f^{-1}(U)} \) such that

\[ \mathcal{I}(\varphi) = \mathcal{I}(f, \| L \|). \]

Let \( A \) be the ample line bundle on \( Y \) (independent of \( f \) and \( L \)) picked in Theorem 1.3.

By asymptotic Serre vanishing theorem [Har77], we can choose a positive integer \( m_0 \) such that for all \( m \geq m_0 \),

\[ H^i(Y, R^q f_* (K_X \otimes L \otimes \mathcal{I}(\varphi)) \otimes A^m) = 0 \]

for \( i > 0, q \geq 0 \). Fix an integer \( m \) such that \( m \geq m_0 \) and \( A^m \) is very ample.

We prove the theorem by induction on \( n = \dim Y \), the case \( n = 0 \) being trivial. Denote \( A' = f^* (A) \) and let \( H' \in |(A')^m| \) be the pull back of a general divisor \( H \in |A^m| \). It follows from Bertini’s theorem [Har77] that we can assume \( H \) is integral and \( H' \) is smooth (though possibly disconnected). Moreover, we can arrange the things that \( \mathcal{I}(\varphi|_{H'}) = \mathcal{I}(\varphi)|_{H'} \) by [FuM16], Theorem 1.10. Then we have a short exact sequence

\[ 0 \rightarrow K_X \otimes L \otimes A' \otimes \mathcal{I}(\varphi) \rightarrow K_X \otimes L \otimes (A')^{m+1} \otimes \mathcal{I}(\varphi) \rightarrow K_{H'} \otimes L \otimes A'|_{H'} \otimes \mathcal{I}(\varphi|_{H'}) \rightarrow 0 \]

(3.1)

which is induced by multiplication with a section defining \( H' \). We get from this short exact
sequence a long exact sequence
\[ 0 \to f_*(K_X \otimes L \otimes A' \otimes \mathcal{I}(\varphi)) \to f_*(K_X \otimes L \otimes (A')^{m+1} \otimes \mathcal{I}(\varphi)) \]
\[ \to f_*(K_{H'} \otimes L \otimes A'|_{H'} \otimes \mathcal{I}(\varphi|_{H'})) \to R^1 f_*(K_X \otimes L \otimes A' \otimes \mathcal{I}(\varphi)) \]
\[ \to R^1 f_*(K_X \otimes L \otimes (A')^{m+1} \otimes \mathcal{I}(\varphi)) \to \cdots \] (3.2)

By [Wu21], Theorem 6.1 all the higher direct images of \( K_X \otimes L \otimes A' \otimes \mathcal{I}(\varphi) \) are torsion-free. Also clearly the sheaves \( R^q f_*(K_{H'} \otimes L \otimes A'|_{H'} \otimes \mathcal{I}(\varphi|_{H'})) \) are torsion on \( H \). Hence the long exact sequence (3.2) can be split into a family of short exact sequences: for all \( q \geq 0 \),
\[ 0 \to R^q f_*(K_X \otimes L \otimes A' \otimes \mathcal{I}(\varphi)) \to R^q f_*(K_X \otimes L \otimes (A')^{m+1} \otimes \mathcal{I}(\varphi)) \]
\[ \to R^q f_*(K_{H'} \otimes L \otimes A'|_{H'} \otimes \mathcal{I}(\varphi|_{H'})) \to 0. \] (3.3)

On the other hand, applying the inductive hypothesis to each connected component of \( H' \), we have that for all \( i \geq 1 \)
\[ H^i(Y, R^q f_*(K_X \otimes L \otimes (A')^{m+1} \otimes \mathcal{I}(\varphi))) = 0. \]

Furthermore, by the choice of \( m \) we also have for all \( i \geq 1 \)
\[ H^i(Y, R^q f_*(K_X \otimes L \otimes A' \otimes \mathcal{I}(\varphi))) = 0. \] (3.4)

Now by taking the cohomology long exact sequence from the short exact sequence (3.3), we find for every \( i > 1 \)
\[ H^i(Y, R^q f_*(K_X \otimes L \otimes A' \otimes \mathcal{I}(\varphi))) = 0. \]

This proves the theorem for the cases where \( i > 1 \).

To prove the case where \( i = 1 \), we denote
\[ B_l := H^1(Y, R^q f_*(K_X \otimes L \otimes (A')^l \otimes \mathcal{I}(\varphi))). \]

By identity (3.4) for \( i = 1 \), we have \( B_{m+1} = 0 \). Hence we consider the following commutative diagram:
\[ \begin{array}{ccc}
B_1 & \xrightarrow{\phi} & H^{q+1}(X, K_X \otimes L \otimes A' \otimes \mathcal{I}(\varphi)) \\
\downarrow & & \downarrow \psi \\
B_{m+1} & \rightarrow & H^{q+1}(X, K_X \otimes L \otimes (A')^{m+1} \otimes \mathcal{I}(\varphi))
\end{array} \]

Here the horizontal maps are the canonical injective maps coming out of the Leray spectral sequence [Har77], and the vertical maps are induced by multiplication with sections defining \( H' \) and \( H \) respectively. By Theorem 1.3 the map \( \psi \circ \phi \) is also injective. So \( B_1 = 0 \) and we finish the proof of the theorem for the case where \( i = 1 \).

Using Theorem 1.3 we can prove the global generation of the higher direct images. We first review the definition and a basic result of the Castelnuovo–Mumford regularity [Mum66].

**Definition 3.1.** Let \( X \) be a projective manifold and \( L \) an ample and globally generated line bundle on \( X \). Given an integer \( m \), a coherent sheaf \( F \) on \( X \) is \( m \)-regular with respect to \( L \) if for all \( i \geq 1 \)
\[ H^i(X, F \otimes L^{m-i}) = 0. \]

**Theorem 3.3.** (Mumford, [Mum66]) Let \( X \) be a projective manifold and \( L \) an ample and globally generated line bundle on \( X \). If \( F \) is a coherent sheaf on \( X \) that is \( m \)-regular with respect to \( L \), then the sheaf \( F \otimes L^m \) is globally generated.
After this, we can prove Corollary 4.1.

Proof of Corollary 4.1. It follows from Theorem 1.4 that for every $i \geq 1$

$$H^i(Y, R^i f_*(K_X \otimes L \otimes \mathcal{I}(f, ||L||)) \otimes A^{m-i} \otimes A') = 0.$$ 

Hence the sheaf $R^i f_*(K_X \otimes L \otimes \mathcal{I}(f, ||L||)) \otimes A^m \otimes A'$ is 0-regular with respect to $A$. So it is globally generated by Theorem 3.3. \qed

4. The positivity of direct images

In this section, we shall prove Theorem 1.2 following Veilhag’s strategy in [Vie82b, Vie83]. Note $f$ is furthermore supposed to be smooth here. Recall that $\varphi = \{\varphi_U\}$ is the collection of metrics on $L|_{f^{-1}(U)}$ such that $\mathcal{I}(\varphi) = \mathcal{I}(f, ||L||)$. The following observation is needed:

**Lemma 4.1.** For any positive integer $m$, consider the $m$-fold fibre product

$$f_m : X_m = X \times_Y \cdots \times_Y X \to Y.$$ 

Using the same notation as Definition 2.3, we have

(i) $(f_m)_*(K_{X_m/Y} \otimes (p_1^m \otimes p_1^{m-1} p_2^m \otimes \cdots \otimes p_2^1 p_2^m)^*(L)) = f_*(K_{X/Y} \otimes L)^{\otimes m}$.

(ii) $\varphi_m = (p_1^m + p_1^{m-1} p_2^m + \cdots + p_2^1 p_2^m)^*\varphi$ is a collection of metrics on

$$L_m := (p_1^m \otimes p_1^{m-1} p_2^m \otimes \cdots \otimes p_2^1 p_2^m)^*(L)$$

such that $\mathcal{I}(\varphi_m) = \mathcal{I}(f_m, ||L_m||)$. In particular, if $p$ is an integer that computes $\mathcal{I}(f, ||L||)$, it also computes $\mathcal{I}(f_m, ||L_m||)$.

(iii) (Subadditivity) $\mathcal{I}(\varphi_m) \subset (p_1^m \otimes p_1^{m-1} p_2^m \otimes \cdots \otimes p_2^1 p_2^m)^*\mathcal{I}(\varphi)$.

**Proof.** (i) We simply prove it with $m = 3$, and the general case follows in the same way. The calculation involves nothing but the repeated use of Lemmas 2.1 and 2.2. We also need the following two facts in [Har77]:

1. $(g \circ f)_* = g_* f_*$ for arbitrary morphisms $f$ and $g$;
2. $K_{X_m/Y} = (p_1^m)^* K_{X/Y} \otimes (p_2^m)^* K_{X_m-1/Y}$.

Then, we complete the proof by carefully chasing the diagram.

$$\begin{align*}
(f_3)_*(K_{X_3/Y} \otimes (p_1^3)^* L \otimes (p_2^3)^* L \otimes (p_2^2)^* L) \\
= (f_3)_*((p_1^3)_* (K_{X/Y} \otimes (p_2^3)^* L)) \\
= (f_3)_*((p_1^3)_* ((p_1^2)_* (K_{X/y} \otimes (p_2^2)^* L))) \\
= (f_3)_*((p_1^3)_* ((p_1^2)_* ((p_1^1)_* (K_{X/y} \otimes (p_2^1)^* L))) \\
= (f_3)_*((p_1^3)_* ((p_1^2)_* ((p_1^1)_* (K_{X/y} \otimes (p_2^1)^* L)))) \\
= (f_3)_*((p_1^3)_* ((p_1^2)_* ((p_1^1)_* (K_{X/y} \otimes (p_2^1)^* L)))) \\
= (f_3)_*((p_1^3)_* ((p_1^2)_* ((p_1^1)_* (K_{X/y} \otimes (p_2^1)^* L)))) \\
= (f_3)_*((p_1^3)_* ((p_1^2)_* ((p_1^1)_* (K_{X/y} \otimes (p_2^1)^* L)))) \\
= (f_3)_*((p_1^3)_* ((p_1^2)_* ((p_1^1)_* (K_{X/y} \otimes (p_2^1)^* L)))) \\
= (f_3)_*((p_1^3)_* ((p_1^2)_* ((p_1^1)_* (K_{X/y} \otimes (p_2^1)^* L)))) \\
= (f_3)_*((p_1^3)_* ((p_1^2)_* ((p_1^1)_* (K_{X/y} \otimes (p_2^1)^* L)))) \\
= (f_3)_*((p_1^3)_* ((p_1^2)_* ((p_1^1)_* (K_{X/y} \otimes (p_2^1)^* L)))) \\
= (f_3)_*((p_1^3)_* ((p_1^2)_* ((p_1^1)_* (K_{X/y} \otimes (p_2^1)^* L)))) \\
= (f_3)_*((p_1^3)_* ((p_1^2)_* ((p_1^1)_* (K_{X/y} \otimes (p_2^1)^* L)))) \\
= (f_3)_*((p_1^3)_* ((p_1^2)_* ((p_1^1)_* (K_{X/y} \otimes (p_2^1)^* L)))) \\
= (f_3)_*((p_1^3)_* ((p_1^2)_* ((p_1^1)_* (K_{X/y} \otimes (p_2^1)^* L)))) \\
= (f_3)_*((p_1^3)_* ((p_1^2)_* ((p_1^1)_* (K_{X/y} \otimes (p_2^1)^* L)))) \\
= (f_3)_*((p_1^3)_* ((p_1^2)_* ((p_1^1)_* (K_{X/y} \otimes (p_2^1)^* L)))) \\
= (f_3)_*((p_1^3)_* ((p_1^2)_* ((p_1^1)_* (K_{X/y} \otimes (p_2^1)^* L)))) \\
= (f_3)_*((p_1^3)_* ((p_1^2)_* ((p_1^1)_* (K_{X/y} \otimes (p_2^1)^* L)))) \quad \text{(??)}
\end{align*}$$

(ii) Similar computation with (i) also implies that $(f_m)_*(L_m) = f_*(L)^{\otimes m}$. Let $U$ be a local coordinate ball of $Y$. Then any section $u$ of $\Gamma(f_m^{-1}(U), L_m)$ decomposes as

$$u = u_1 \otimes u_2 \otimes \cdots \otimes u_m.$$
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where \( u_1, \ldots, u_m \) are sections of \( \Gamma(f^{-1}(U), L) \). By definition, we have
\[
\mathcal{I}(\varphi_m) = \mathcal{I}(f_m, \|L_m\|).
\]

(iii) It comes from the subadditivity of the multiplier ideal sheaves proved in [DEL00]. In fact, since
\[
\varphi_m = (p_1^m + p_1^{m-1}p_2^m + \cdots + p_2^1p_2^{m-1}p_2^m)\varphi,
\]
we have
\[
\mathcal{I}(\varphi_m) \subset \mathcal{I}((p_1^m)^*\varphi) \cap \mathcal{I}((p_1^{m-1}p_2^m)^*\varphi) \cdots \cap \mathcal{I}((p_2^1p_2^{m-1}p_2^m)^*\varphi)
\]
by the main result (Theorem 2.6) in [DEL00]. One more application of Theorem 2.6 in [DEL00] implies that
\[
\mathcal{I}((p_1^m)^*\varphi) = (p_1^m)^*\mathcal{I}(\varphi),
\]
\[
\mathcal{I}((p_1^{m-1}p_2^m)^*\varphi) = (p_1^{m-1}p_2^m)^*\mathcal{I}(\varphi),
\]
\[
\mathcal{I}((p_2^1p_2^{m-1}p_2^m)^*\varphi) = (p_2^1p_2^{m-1}p_2^m)^*\mathcal{I}(\varphi),
\]
\[
\cdots
\]
\[
\mathcal{I}((p_2^1p_2^2\cdots p_2^m)^*\varphi) = (p_2^1p_2^2\cdots p_2^m)^*\mathcal{I}(\varphi).
\]

Indeed, let \( y \in Y \). Take a local coordinate neighbourhood \( U \) of \( y \), we have
\[
X|_{f^{-1}(U)} = U \times X_y
\]
and
\[
X_m|_{f_m^{-1}(U)} = \underbrace{X_y \times \cdots \times X_y}_{m-1} \times (U \times X_y).
\]
Therefore, locally \( X_m \) can be regarded as the product of two manifolds:
\[
X_1 = \underbrace{X_y \times \cdots \times X_y}_{m-1} \quad \text{and} \quad X_2 = U \times X_y.
\]
Let \( \phi_1 = 1 \), which is a function on \( X_1 \). Let \( \phi_2 = \varphi_U \), which is a function on \( X_2 \). Apply the first statement of Theorem 2.6 in [DEL00] to \( (X_1, \phi_1) \) and \( (X_2, \phi_2) \), we obtain
\[
\mathcal{I}((p_1^m)^*\varphi) = (p_1^m)^*\mathcal{I}(\varphi)
\]
at \( y \) hence everywhere.

The other formulas are the same. Combined with (4.1) and (4.2), the proof is complete. \( \square \)

Before introducing the next lemma, we need to fix some notations. We denote the \( m \)-fold fibre product \( X \times_Y \cdots \times_Y X \) by
\[
f_m : X_m \to Y.
\]
Furthermore, let
\[
L_m := (p_1^m \otimes p_1^{m-1}p_2^m \otimes \cdots \otimes p_2^1p_2^{m-1}p_2^m)^*(L),
\]
and \( \varphi_m = \{\varphi_U, m\} \) be the collection of metrics induced by \( \varphi = \{\varphi_U\} \). Now, given \( m \) sections \( u_1, \ldots, u_m \) of
\[
\Gamma(U, f_*(K_{X/Y} \otimes L)),
\]
by Lemma 4.3 they together induce a section
\[
u^m := (p_1^m)^*u_1 \otimes (p_1^{m-1}p_2^m)^*u_2 \otimes \cdots \otimes (p_2^1p_2^{m-1}p_2^m)^*u_m
\]
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of
\[ \Gamma(U, (f_m)_*(K_{X_m/Y} \otimes L_m)) = \Gamma(f_m^{-1}(U), K_{X_m/Y} \otimes L_m). \]

The next lemma shows that we even have
\[ u^\otimes_m \in \Gamma(U, (f_m)_*(K_{X_m/Y} \otimes L_m \otimes \mathcal{I}(\varphi_m))). \]

**Lemma 4.2.** Keep the notations. Let \( \varphi = \{\varphi_U\} \) and \( \psi = \{\psi_U\} \) be the collection of metrics associated to \( \mathcal{I}(f, \|L\|) \) and \( \mathcal{I}(f, \|K_{X/Y} \otimes L\|) \) respectively. Assume that \( \varphi \) is less singular than \( \psi \), i.e.
\[ \varphi_U \preceq \psi_U \text{ for every } U. \]

Then
\[ \int_{f_m^{-1}(U)} |u^\otimes_m|^2 e^{-\varphi_{U,m}} \]
is finite. In other words,
\[ u^\otimes_m \in \Gamma(U, (f_m)_*(K_{X_m/Y} \otimes L_m \otimes \mathcal{I}(\varphi_m))). \]

**Proof.** Let \( y \in Y \) be an arbitrary point. Take a local coordinate neighbourhood \( U \) of \( y \), such that \( X|_{f^{-1}(U)} = U \times X_y \).

If we take the local coordinate on \( f^{-1}(U) \) to be
\[ ((y_1, \ldots, y_n), (x_1, \ldots, x_l)), \]
\( \varphi_U \) can be written on \( f^{-1}(U) \) as:
\[ \varphi_U = \varphi_U((y_1, \ldots, y_n), (x_1, \ldots, x_l)). \]

Moreover, we have
\[ f_m^{-1}(U) = U \times X_1 \times \cdots \times X_m. \]

Here, we add the superscript \( \{1, \ldots, m\} \) to differentiate the fibres. Then, the corresponding local coordinate on \( f_m^{-1}(U) \) will be
\[ ((y_1, \ldots, y_n), (x_1^1, \ldots, x_l^1), \ldots, (x_1^m, \ldots, x_l^m)), \]
and \( \varphi_{U,m} \) becomes
\[ \varphi_{U,m} = (p_1^m + p_1^{m-1}p_2^m + p_1^{m-2}p_2^{m-1}p_3^m + \ldots + p_1p_2^2 \ldots p_2^m)^* \varphi_U = \sum_j \varphi_U((y_1, \ldots, y_n), (x_1^j, \ldots, x_l^j)). \]

We claim that, for any section \( u^\otimes_m \) of
\[ \Gamma(U, (f_m)_*(K_{X_m/Y} \otimes L_m)) = \Gamma(f_m^{-1}(U), K_{X_m/Y} \otimes L_m) \]
defined above, the integral
\[ \int_U \int_{X_1 \times \cdots \times X_m} |u^\otimes_m|^2 e^{-\varphi_{U,m}} \]
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is finite. In fact, we have

\[
\int_U \int_{X^1_y \times \cdots \times X^m_y} |u_1^{\otimes m}|^2 e^{-\varphi_{U,m}} \\
= \int_U \int_{X^1_y \times \cdots \times X^m_y} |u_1^{\otimes m}|^2 e^{-\sum_j \varphi_U((y_1, \ldots, y_n),(x_1', \ldots, x_l'))} \\
= \prod_j \int_{X^1_y} |u_j|^2 e^{-\varphi_U((y_1, \ldots, y_n),(x_1', \ldots, x_l'))} \\
\leq \prod_j (\int_U |u_j|^2 e^{-\varphi_U((y_1, \ldots, y_n),(x_1', \ldots, x_l'))})^{1/m}. 
\]

This last inequality is due to Hölder’s inequality. Since

\[
\int_U (\int_{X^1_y} |u_j|^2 e^{-\varphi_U((y_1, \ldots, y_n),(x_1', \ldots, x_l'))})^{m} = \int_U (\int_{X^1_y} |u_j|^2 e^{-\varphi_U((y_1, \ldots, y_n),(x_1', \ldots, x_l'))})^{m} 
\]

for every \( j \), this integral is finite by the definition of \( \varphi_U \) (see Sect. 2.3).

In fact, since \( u_j \) is a section of \( \Gamma(f^{-1}(U), K_{X/Y} \otimes L) \), \( u_j \in \mathfrak{a}_f \subseteq \mathcal{I}(f, \|K_{X/Y} \otimes L\|) \) by Proposition 2.2 (i). Here \( \mathfrak{a}_f \) is the base-ideal of \( |K_{X/Y} \otimes L| \) relative to \( f \). As a result, \( |u_j|^2 e^{-\varphi_U((y_1, \ldots, y_n),(x_1', \ldots, x_l'))} \) will be bounded. On the other hand, \( \varphi_U \) is less singular than \( \psi_U \) by assumption. So \( |u_j|^2 e^{-\varphi_U((y_1, \ldots, y_n),(x_1', \ldots, x_l'))} \) is also bounded on \( f^{-1}(U) \). We have finished the proof of this claim.

Then, we conclude that

\[
\int_{f^{-1}(U)} |u_1^{\otimes m}|^2 e^{-\varphi_{U,m}} 
\]

is also finite. Indeed, let \( Z := \{ y \in Y; \varphi_U|_{X_y} \equiv -\infty \} \), which is a set of measure zero. We have

\[
\int_{f^{-1}(U) \setminus f^{-1}(Z)} |u_1^{\otimes m}|^2 e^{-\varphi_{U,m}} = \int_{(U \setminus Z) \times X^1_y \times \cdots \times X^m_y} |u_1^{\otimes m}|^2 e^{-\varphi_{U,m}} \\
\leq \int_U \int_{X^1_y \times \cdots \times X^m_y} |u_1^{\otimes m}|^2 e^{-\varphi_{U,m}}. 
\]

Since the right hand side of the inequality is finite, the inequality is actually an equality. Therefore we conclude that

\[
\int_{f^{-1}(U)} |u_1^{\otimes m}|^2 e^{-\varphi_{U,m}} := \int_{f^{-1}(U) \setminus f^{-1}(Z)} |u_1^{\otimes m}|^2 e^{-\varphi_{U,m}} < +\infty. 
\]

Now we turn to Theorem 1.2.

Proof of Theorem 1.2 Keep the notations. Consider the \( m \)-fold fibre product

\[
f_m : X_m = X \times_Y \cdots \times_Y X \to Y.
\]

If we denote

\[
L_m := (p_1^{m} \otimes p_1^{m-1} p_2^{m} \otimes \cdots \otimes p_2^{1} p_2^{2} \cdots p_2^{m})^*(L),
\]

by Lemma 4.1 we have

\[
(f_* (K_{X/Y} \otimes L))^{\otimes m} = (f_m)_*(K_{X_m/Y} \otimes L_m).
\]
Moreover, Lemma 4.1 also implies that
\[ \mathcal{I}(\varphi_m) \subset (p^m_1 \otimes p^{m-1}_2 \otimes p^{m-2}_2 \otimes \cdots \otimes p^1 p^2 \cdots p^m)^* \mathcal{I}(\varphi), \]
so we have
\[ (f_m)_*(K_{X_m/Y} \otimes L_m \otimes \mathcal{I}(\varphi_m)) \subset (f_m)_*(K_{X_m/Y} \otimes L_m \otimes (p^m_1 \otimes p^{m-1}_2 \otimes p^{m-2}_2 \otimes \cdots \otimes p^1 p^2 \cdots p^m)^* \mathcal{I}(\varphi)) \]
\[ = f_*(K_{X/Y} \otimes L \otimes \mathcal{I}(\varphi))^{\otimes m}. \]
On the other hand, Lemma 4.2 says that the opposite direction of this inclusion holds, too. Thus, we actually have
\[ \mathcal{E}^{\otimes m} := f_*(K_{X/Y} \otimes L \otimes \mathcal{I}(\varphi))^{\otimes m} = (f_m)_*(K_{X_m/Y} \otimes L_m \otimes \mathcal{I}(\varphi_m)) \]
for all positive integer \( m \).

Then, we fix a very ample line bundle \( A \) on \( Y \). Let \( A' = O_Y \) and let \( H = K_Y \otimes A^{\otimes (\dim Y + 1)} \). By way of motivation, imagine for the moment that one had a singular metric \( \chi_m \) on \(-K_{X_m/Y} \otimes L^p_m\) such that
\[ i \Theta - K_{X_m/Y} \otimes L^p_m \chi_m \geq 0 \text{ and } \mathcal{I}(\tau_m) = O_{X_m}. \]
Then applying Corollary 1.1 to the fibration \( f_m \) as well as the direct image
\[ (f_m)_*(K_{X_m/Y} \otimes L_m \otimes \mathcal{I}(\varphi_m)), \]
we deduce that the sheaf \( \mathcal{E}^{\otimes m} \otimes H \) is generated by its global sections. In particular, \( \mathcal{E} \) is weakly positive in the sense of Viehweg. While in reality the existence of \( \chi_m \) may be too much to hope for, a simple observation is that
\[ \tau_m := (p^m_1 + p^{m-1}_2 + p^{m-2}_2 + \cdots + p^1 p^2 \cdots p^m)^* \tau \]
is just as good. In fact, recall the proof of Theorem 1.3 (and the notations there), we only use the fact that \( \mathcal{I}(\tau|_F) = O_F \). Returning to the situation of this theorem, the general fibre of \( f_m : X_m \to Y \) is \( F_m := \underbrace{F \times \cdots \times F}_{m} \). Let \( q_i : F_m \to F \) be the \( i \)-th projection. Obviously
\[ \mathcal{I}(\tau_m|_{F_m}) = \mathcal{I}(\sum_i q_i^*(\tau|_F)) = O_{F_m}, \]
hence Corollary 1.1 still applies here.

Now the torsion-free coherent sheaf \( S^m \mathcal{E} \otimes H \), being a quotient of \( \mathcal{E}^{\otimes m} \otimes H \), is also globally generated. Consider \( \pi : \mathbb{P}(\mathcal{E}^*) \to Y \). Here \( \mathbb{P}(\mathcal{E}^*) \) refers to the projective space bundle [Har77] associated to a coherent sheaf. Note that we have the surjective morphism
\[ \pi^* \pi_* O_{\mathcal{E}}(m) \cong \pi^*(S^m(\mathcal{E})) \to O_{\mathcal{E}}(m), \]
and we thus deduce that \( O_{\mathcal{E}}(m) \otimes \pi^* H \) is globally generated (hence nef) for every \( m \geq 1 \). This implies that \( O_{\mathcal{E}}(1) \) is nef, that is, \( \mathcal{E} \) is nef. \( \square \)

In the end, we discuss several special cases of Theorem 1.2. Firstly, we prove Theorem 1.1.

**Proof of Theorem 1.1** \( m = 0 \) is trivial. \( m = 1 \) is furnished by Griffiths [Gri84] and Fujita–Kawamata [Kaw82]. So we assume \( m > 1 \) without loss of generality.

Note that if \( \varphi = \{ \varphi_U \} \) is the collection of metrics associated to
\[ \mathcal{I}(f, ||K_{X/Y}||), \]
then \( m\varphi = \{m\varphi_U\} \) is the collection of metrics associated to \( \mathcal{I}(f, \|K_X^m\|) \). Obviously we have
\[
(m - 1)\varphi \leq m\varphi.
\]

On the other hand, let \( \psi \) be the metric on \( K_{X/Y} \) such that \( \mathcal{I}(\psi) = \mathcal{I}(\|K_X\|) \) (see Sect 2.3), then \( i\Theta_{K_{X/Y}^p(m-1)} - (m-1)\psi \geq 0 \) and
\[
\mathcal{I}((p(m-1)-1)\psi) = \mathcal{I}((p(m-1)-1)\|K_X\|) = \mathcal{O}_X
\]
for \( m \leq k + 1 \).

Apply Theorem 1.2 with \( L = K_{X/Y}^m \), and observe that \( \mathcal{I}(f, (m-1)\|K_X\|) \) is also trivial by Proposition 2.2 (iv), we then obtain that \( f_*(K_{X/Y}^m) \) is nef for \( m \leq k + 1 \).

**Corollary 4.1.** Let \( f : X \to Y \) be a smooth fibration between projective manifolds \( X \) and \( Y \). Let \( L \) be a holomorphic line bundle on \( X \) with \( \kappa(K_{X/Y} \otimes L, f) \geq 0 \). Fix \( p \gg 0 \) and divisible enough that computes \( \mathcal{I}(f, \|L\|) \). Assume that \( -K_{X/Y} \) is semi-ample, and there exists a (singular) metric \( \tau \) on \( L^p \) such that \( i\Theta_{\tau} \geq 0 \) and \( \mathcal{I}(\tau) = \mathcal{O}_X \). Then
\[
f_*(K_{X/Y} \otimes L \otimes \mathcal{I}(f, \|L\|))
\]
is nef.

**Proof.** Fix \( q \gg 0 \) and divisible enough that computes both of \( \mathcal{I}(f, \|K_{X/Y} \otimes L\|) \) and \( \mathcal{I}(f, \|L\|) \). Furthermore, \( |-K_{X/Y}^q| \) is base-point free.

Let \( \{\alpha_i\} \) be a basis of \( H^0(X, (K_{X/Y} \otimes L)^q) \) and let \( \{\beta_j\} \) be a basis of \( H^0(X, -K_{X/Y}^q) \). Then all of \( \alpha_i \otimes \beta_j \) form a linear subspace \( \mathcal{O} \) of \( |L|^q \). In particular, the base-ideal \( a \) of \( \mathcal{O} \) is contained in the base-ideal \( b \) of \( |L|^q \). Note that \( a \) is also the base-ideal of \( |(K_{X/Y} \otimes L)^q| \) since \( |-K_{X/Y}^q| \) is base-point free.

Let \( \{\gamma_k\} \) be the generators of \( b \). Then
\[
\log(\sum_k |\gamma_k|^2) \geq \log(\sum_{i,j} |\alpha_i \otimes \beta_j|^2)
\]
due to the inclusion of the corresponding ideals. Therefore the second requirement of Theorem 1.2 is verified by definition. The first requirement is by assumption. Now the conclusion follows by applying Theorem 1.2.

**Corollary 4.2.** Let \( f : X \to Y \) be a smooth fibration between projective manifolds \( X \) and \( Y \) with \( \kappa(-K_{X/Y}, f) \geq 0 \). Assume that \( \mathcal{I}(\|K_X\|) = \mathcal{O}_X \) for every \( k \geq 0 \). Then
\[
f_*(-K_{X/Y}^m)
\]
is nef for any integer \( m \geq -1 \).

**Proof.** The case where \( m = -1 \) is due to Griffiths [Gri84] and Fujita–Kawamata [Kaw82], which is even valid without any extra assumption.

Now \( m \geq 0 \). Since \( \mathcal{I}(\|K_X\|) = \mathcal{O}_X \), we have \( \mathcal{I}(f, m\|K_X\|) = \mathcal{O}_X \) by Proposition 2.2 (iv). Recall that there exists a metric \( \psi \) on \( -K_{X/Y} \) such that \( i\Theta_{-K_{X/Y} \cdot \psi} \geq 0 \) and
\[
\mathcal{I}(m\psi) = \mathcal{I}(m\| - K_{X/Y} \|).
\]
So the requirement (1) of Theorem 1.2 is verified.

On the other hand, let \( \varphi = \{\varphi_U\} \) be the collection of metrics associated to
\[
\mathcal{I}(f, \| - K_{X/Y} \|).
\]
Then
\[ \mathcal{I}(k\varphi) = \mathcal{O}_X \]
for any \( k \). It implies that \( \varphi \) is actually smooth concerning the fact that \( \varphi \) has algebraic singularities [Dem12]. The second requirement now is obviously satisfied. This proves the conclusion for the cases when \( k \geq 0 \) by Theorem [1.2].

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