A novel approach for coverings transformation to define a robot workspace

L A Rybak1*, L Behera2, M A Averbukh1 and A V Sapryka1

1BSTU named after V.G. Shukhov, 46, Kostyukova street, Belgorod, 308012, Russia
2Indian Institute of Technology Kanpur, Kanpur- 208016, India

E-mail: rl_bgtu@intbel.ru

Abstract. The article discusses the application of optimization algorithms to solve the problem of determining the workspace of robots. The development of a numerical method for approximating the set of solutions of a system of nonlinear inequalities that describe the restrictions on the geometric parameters of a robot based on the concept of non-uniform coverings is considered. An approach is proposed that allows you to reduce the number of boxes and numbers describing each of the boxes. Reducing the number of boxes is achieved by combining the boundary boxes of the covering set and the area between them along one of the axes. The dimensions along the other axes are equal for all boxes, which allows them to be described by one number. The transition to the integer space is described. Thus, converting non-uniform covering sets to a partially ordered set of integers reduces computational complexity. An algorithm for converting boxes of a covering set is presented. The approach has been tested for a 3-RPS robot.

1. Introduction
Workspace is one of the most important characteristic of robots. The issues of structural synthesis, methods for studying the workspace, movement trajectory optimization of parallel robots are discussed in detail in [1-4]. The task of the workspace determination can be solved using various methods, in particular, deterministic ones. The actual problem for applying these methods is often significant computational complexity. One of the most effective deterministic methods is the non-uniform coverings method. The application of this method to determine the workspace is considered in works [5-7]. In [8], a comparison of two approaches is considered, one of which is based on the use of a system of inequalities to describe the structural constraints of the robot, and the other on the use of a system of equations.

One of the tools for implementing the method is iterative bisection. With each division, the box decreases by 2 times, respectively, the ratio of the sizes of the initial box and one of the boxes forming an approximation of the workspace can be estimated by the degree of division d, and the ratio itself is $2^d$. With an increase in the degree of division d, the number of boxes, the combination of which describes the workspace, increases. Due to the increase in computational complexity for processing an approximated workspace of higher accuracy, the problem arises of reducing it by converting the resulting set of boxes describing the workspace. As part of this work, an approach is proposed for converting a covering set obtained using the method of non-uniform coverings into a partially ordered set of integers. It includes two components: reducing the number of boxes and the transition from the
space of real to the space of integers. Application of the proposed approach and assessment of its
effectiveness is considered on the problem of determining the workspaces of the 3-RPS robot.

2. Materials and methods
Consider the set of boxes which described the workspace. Using the system of inequalities \( g_j(x) = 0, j \in 1, m \), the workspace is described by the union of two sets \( Q_E = Q_I \cup Q_J \), where \( Q_I \) is an internal approximation set that is included in the set of solutions to the system of inequalities, \( Q_J \) is the boundary set. For each \( P_i \) from \( Q_I \), the condition

\[
\max_{j=1..m} \max_{x \in P_i} g_j(x) \leq 0.
\]

(1)

For each \( P_i \) from \( Q_J \), the system of inequalities is fulfilled:

\[
\begin{cases}
\max_{j=1..m} \max_{x \in P_i} g_j(x) > 0 \\
\max_{j=1..m} \min_{x \in P_i} g_j(x) \leq 0.
\end{cases}
\]

(2)

We introduce the following notation: \( \partial Q_E \) is the boundary of the set \( Q_E \), \( Q_A = \partial Q_E \cup (\mathbb{R}^n \setminus Q_E) \) is the outer region for which the condition

\[
\max_{j=1..m} \min_{x \in P_i} g_j(x) > 0.
\]

(3)

The elements of the set \( Q_J \) are necessarily located between the elements of the sets \( Q_I \) and \( Q_A \). Therefore, it is possible to describe the workspace as a set of boxes using the following approach. We introduce the following notation for the boundaries of the box:

\[
x_i \leq x_i \leq \bar{x}_i, \ i \in 1, n.
\]

(4)

We denote two subsets in the set \( Q_J \) as \( Q_{J1} \) and \( Q_{J2} \). The subset \( Q_{J2} \) includes only those boxes at the boundary of which there is a point \( x = (x_1, \ldots, x_n) \) for which the system is satisfied:

\[
\begin{cases}
x \in \partial Q_E \\
x_1 = x_{\bar{i}} \\
x_i < x_i < \bar{x}_i, \ i \in 2, n.
\end{cases}
\]

(5)

The condition for the subset \( Q_{J2} \) is similar, but for it \( c_1 = \bar{x}_1 \).

It should be noted that the number \( m \) of boxes in the subsets \( Q_{J1} \) and \( Q_{J2} \) is equal to each box \( P_{k,J1}, k \in 1, m \) from \( Q_{J1} \) corresponds to the box \( P_{k,J2}, k \in 1, m \) from \( Q_{J2} \) with equal values \( x_i, i \in 2, n \) and \( \bar{x}_i, i \in 2, n \), while for all points \( x = (x_1, \ldots, x_n) \) for which the condition \( x_{i,J1} < x_i < x_{i,J2} \) is satisfied, \( x \in Q_E \) is true. \( Q_F \) is the set of boxes and \( P_{k,F} = \left\{ x: x_{i,J1,k} \leq x_i \leq x_{i,J2,k}, i \in 1, n \right\}, k \in 1, m \).

The n-dimensional box is described by \( 2n \) real numbers. The proposed approach to the conversion of boxes allows us to describe boxes with a smaller quantity of numbers, while integers. First, we consider this concept in the general case of converting the set of real numbers \( Y \) into the set of integers \( Z \) with the approximation accuracy \( \delta \) (Fig. 1). For each of the set points, its coordinates in the space of integers are calculated:

\[
x_i^{(j)} = \left\lceil \frac{x_i^{(j)}}{\delta} \right\rceil, \ x^{(j)} \in \mathbb{R}, x^{(j)} \in Z
\]

(6)
Moreover, the Hausdorff distance between the sets depends on $\delta$:

$$0 < h(Y, Z) \leq \sqrt{\sum_i |\delta_i|^2}$$  \hspace{1cm} (7)

The integers of one of the coordinates are likewise combined into intervals, for each of which the values of the remaining coordinates are equal. Let us consider the application of this concept to the transformation of the set of boxes $Q_F$ to $Q_Z$. For $Q_F$ and $Q_Z$ the following holds:

$$h_{max}(Q_F, Q_Z) = \sqrt{\sum_i |\delta_i|^2}$$  \hspace{1cm} (8)

The accuracy of determining the workspace allows us to estimate the Hausdorff distance between the sets $Q_Z$ and $Y$ (Fig. 2a). The maximum distance $h_{max}(Y, Q_Z)$ is defined as

$$h_{max}(Y, Q_Z) = h_{max}(Y, Q_F) + h_{max}(Q_F, Q_Z)$$  \hspace{1cm} (9)

In order to reduce the Hausdorff distance, we modify the formula (7):

$$x'_i = \left[ \frac{2^d x_i + k_i}{x_i^{(0)} - x_i^{(0)}} \right], x_i \in \mathbb{R}, x'_i \in \mathbb{Z}$$  \hspace{1cm} (10)

where $k_i$ - bias coefficients, which are determined by the formula:

$$k_i = \left( \frac{2^d x_i^{(k)}}{x_i^{(0)} - x_i^{(0)}} + 0.5 \right) - 0.5 \left( \frac{x_i^{(0)} - x_i^{(0)}}{2^d} \right) - x_i^{(k)}, i \in 1, n$$  \hspace{1cm} (11)

In this case, $h_{max}(Q_F, Q_Z) = 0$ (Fig. 2b).

Figure 2. The Hausdorff distance: a) between the sets $Y$ and $Q_Z$, b) between the sets $Y$ and $Q_Z$ taking into account the displacement coefficients.

We introduce the following variables:

$$\alpha_i = \left[ \frac{2^d x_i + k_i}{x_i^{(0)} - x_i^{(0)}} \right], \alpha_i = \left[ \frac{2^d x_i + k_i}{x_i^{(0)} - x_i^{(0)}} - 0.5 \right]$$  \hspace{1cm} (12)

It is worth noting that an additional coefficient of 0.5 is added to exclude rounding of the upper boundary value to the next integer.

We synthesize a box conversion algorithm for $n = 2$, taking into account formulas (10) - (12):
Algorithm 1.
Input: $Q_E$.

$N_1 = 0$

for $k = 1, 2, ..., N$:
    Compute $k_1, k_2, a_1, \bar{a}_1, a_2, \bar{a}_2$.
    for $a_1 = a_1, a_1 + 1, ..., \bar{a}_1$:
        if $(N_1 = 0)$ then:
            $x_1^{(1)r} = a_1, x_2^{(1)(1)r} = \bar{a}_2, x_2^{(1)(1)r} = a_2 N_1 = 1, N_1^{(1)} = 1$
        else:
            for $j_1 = 1, 2, ..., N_1$:
                if $(a_1 = x_1^{(1)r})$ then:
                    for $j_2 = 1, 2, ..., N_2^{(j_1)}$:
                        if $(\bar{a}_2 < x_2^{(j_1)(j_2)r})$ then
                            if $(j_2 = 0)$ then
                                if $(x_2^{(j_1)(j_2)r} = \bar{a}_2 + 1)$ then
                                    $x_2^{(j_1)(j_2)r} = a_2$
                                else
                                    $x_2^{(j_1)(j_2)r} = a_2$
                                end if
                            else
                                for $i = 0, 1, ..., N_2^{(j_1)}$:
                                    $x_2^{(j_1)(j_2)r} = a_2, x_2^{(j_1)(j_2)r} = \bar{a}_2$
                                end for
                                $N_2^{(j_1)} = N_2^{(j_1)} + 1$
                            end if
                        else if $(a_1 = x_1^{(1)r})$ and $(a_1 = \bar{a}_2 + 1)$ then
                            $x_2^{(j_1)(j_2-1)r} = a_2 - 1, x_2^{(j_1)(j_2)r} = \bar{a}_2 + 1$ then
                            $x_2^{(j_1)(j_2-1)r} = x_2^{(j_1)(j_2)r}$
                            for $i = j_2, 1, ..., N_2^{(j_1)} - 1$:
                                $x_2^{(j_1)(i)r} = x_2^{(j_1)(i+1)r}$
                            end for
                            $N_2^{(j_1)} = N_2^{(j_1)} - 1$
                        else if $(x_2^{(j_1)(j_2-1)r} = a_2 - 1)$ then
                            $x_2^{(j_1)(j_2-1)r} = a_2$
                        else if $(x_2^{(j_1)(j_2)r} = \bar{a}_2 + 1)$ then
                            $x_2^{(j_1)(j_2)r} = a_2$
                        else:
                            for $i = 0, 1, ..., N_2^{(j_1)} - j_2$:
As the dimension increases, we add external loops and similar to actions with the $x_1$ coordinate.

3. Results and discussion

Consider the application of the method of non-uniform coverings to determine the workspace of one of the robot modules - a planar 3-RPS mechanism (Fig. 5), which consists of three kinematic chains containing variable-length rods pivotally attached to a fixed base at the vertices of an equilateral triangle (Fig. 3) The other ends of the rods are pivotally mounted at the vertices of an equilateral triangle on a movable platform. The input coordinates are the rod lengths $(l_1, l_2, l_3)$, the output...
coordinates are the position of the geometric center of the moving platform in Cartesian coordinates \((x, y)\) associated with the center of the base of the mechanism, and its rotation angle \((\phi)\) relative to the axis perpendicular to the plane of the base. \(R\) and \(r\) are the radii of circles describing triangles and, respectively.

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{image}
\caption{Scheme of a planar 3-RPS mechanism.}
\end{figure}

Define the workspace of the 3-RPS mechanism. To do this, we introduce restrictions on the geometric parameters of the mechanism

\[
l_{\min} \leq l_i \leq l_{\max}
\]

where \(l_{\min}, l_{\max}\) are determined by the design parameters of the mechanism, \(l_i\) is the current length of the \(i\)-th rod,

If the points \(A_i\) and \(B_i\) are located at the vertices of equilateral triangles, then the change in the length of the rods is determined by the formulas

\[
l_1^2 = (x + 0.5r(\sin \varphi - \sqrt{3} \cos \varphi) + 0.5\sqrt{3}R)^2 + (y - 0.5r(\sqrt{3} \sin \varphi + \cos \varphi) + 0.5R)^2
\]

\[
l_2^2 = (x + 0.5r(\sin \varphi + \sqrt{3} \cos \varphi) - 0.5\sqrt{3}R)^2 + (y + 0.5r(\sqrt{3} \sin \varphi - \cos \varphi) + 0.5R)^2
\]

\[
l_3^2 = (x - r \sin \varphi)^2 + (y + r \cos \varphi - R)^2
\]

Algorithms for approximating the set of solutions of nonlinear inequalities were considered earlier in the authors’ work [7]. To speed up the calculations, multithreaded calculations using the OpenMP library are used. This is considered in more detail in [8].

We use the proposed approach to transforming the covering set. In fig. 4a shows a visualization of the workspace symmetrical about the Y axis. The left half is described by the sets \(Q_1\) and \(Q_1\), the right half is described by the set \(Q_F\). The simulation results for \(R = 400\ \text{mm}, r = 50\ \text{mm},\)

\[
l_{1,2,3} \in [200\ \text{mm}, 500\ \text{mm}], \varphi = 0^\circ
\]

are presented in Fig. 4b. The calculation time for approximation accuracy \(\delta = 0.01\ \text{mm}\), the grid dimension for calculating 8x8 functions using parallelization of computations into 8 flows on a personal computer was 6 seconds.

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{image}
\caption{The 3-RPS mechanism workspace: a) before and after the transformation of the covering set, b) fixed angle \(\varphi = 0^\circ\): the blue area is the internal approximation, the yellow is the boundary.}
\end{figure}
With a degree of division of $2^{12}$, the number of boxes for two-dimensional space decreased by 86.7% from 3010 to 400, for three-dimensional - by 80.5% from 6 991 738 to 1 363 683. At the same time, boxes of the covering set are described by fewer numbers. For three-dimensional space, the set of numbers describing the workspace decreased by 90.2% from 41 950 428 to 4 099 241.

4. Summary
The proposed approach to the transformation of sets to reduce the number of covering boxes describing the workspace has shown its effectiveness. The application of the approach allowed reducing the quantity of numbers describing the 3-RPS mechanism workspace by 90.2%. It is shown that the approach can effectively optimize the workspace of various dimensions. It can be applied to optimize the workspace covering of various robots. It will reduce the computational complexity of visualization and other calculations that use the workspace, including trajectory planning, the safe relative positioning of robots as part of a multi-robot system, determination of a joint workspace for relative manipulation devices, and many others.

5. References
[1] Kong H, Gosselin C M 2007 Type Synthesis of Parallel Mechanisms Springer 272
[2] Merlet J-P 2007 Parallel Robots Springer 269-285
[3] Aleshin A K, Glazunov V A, Rashoyan G V, Shai O 2016 Analysis of kinematic screws that determine the topology of singular zones of parallel-structure robots Journal of Machinery Manufacture and Reliability 45(4) 291-296
[4] Srinivasa Rao P, Mohan Rao N 2013 Position Analysis of Spatial 3-RPS Parallel Manipulator International Journal of Mechanical Engineering and Robotics Research 2(2) 80-90
[5] Zeng Q, Ehmann F K, Cao J 2016 Tri-pyramid Robot: stiffness modeling of a 3-DOF translational parallel Manipulator Robotica 34(2) 383-402
[6] Virabyan L G, Khalapyan S Y, Kuzmina V S 2018 Optimization of the positioning trajectory of planar parallel robot output link Bulletin of BSTU named after V.G. Shukhov 9 106–113
[7] Posypkin M 2019 Automated Robot’s Workspace Approximation Journal of Physics: Conference Series 1163(1) 012050
[8] Evtushenko Y, Posypkin M, Rybak L, Turkin A 2018 Approximating a solution set of nonlinear inequalities Journal of Global Optimization 71(1) 129-145
[9] Malyshev D, Posypkin M, Rybak L, Usov A 2019 Approaches to the Determination of the Working Area of Parallel Robots and the Analysis of Their Geometric Characteristics Engineering Transactions 67(3) 333-345
[10] Malyshev D I, Posypkin M A, Gorchakov A Yu, Ignatov A D 2019 Parallel algorithm for approximating the work space of a robot International Journal of Open Information Technologies 7(1) 1–7

Acknowledgments
The research was funded by RFBR within the research project No 18-57-45014 and Science Committee, the Ministry of Education and Science, Republic of Kazakhstan, project no. AP05133190.