Black strings from minimal geometric deformation in a variable tension brane-world

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Abstract
We study brane-world models with variable brane tension and compute corrections to the horizon of a black string along the extra dimension. The four-dimensional geometry of the black string on the brane is obtained by means of the minimal geometric deformation approach, and the bulk corrections are then encoded in additional terms involving the covariant derivatives of the variable brane tension. Our investigation shows that the variable brane tension strongly affects the shape and evolution of the black string horizon along the extra dimension, at least in a near-brane expansion. In particular, we apply our general analysis to a model motivated by the Eötvös branes, where the variable brane tension is related to the Friedmann–Robertson–Walker brane-world cosmology. We show that for some stages in the evolution of the universe, the black string warped horizon collapses to a point and the black string has correspondingly finite extent along the extra dimension. Furthermore, we show that in the minimal geometric deformation of a black hole on the variable

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tension brane, the black string has a throat along the extra dimension, whose area tends to zero as time goes to infinity.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Brane-world (BW) models [1] represent a distinct branch of contemporary high-energy physics, inspired and supported by the string theory. Indeed, these models are a straightforward five-dimensional phenomenological realization of the Hořava–Witten supergravity solutions [2], when the hidden brane is moved to infinity, and the moduli effects from the further compact extra dimensions may be neglected (for a review, see e.g. [3]). Beside several precursory papers about the BW universe [4], there is nowadays a great variety of alternative models which have been developed in order to address specific aspects. For example, some studies focus on developing the dimensional reduction stemming from a strongly curved extra dimension, others provide a realization of the AdS/CFT correspondence to the lowest order, and further incorporate the self-gravity of the brane, namely the brane tension $\sigma$.

Among all BW models, those containing a brane endowed with variable tension, that is $\sigma = \sigma(x^\mu)$, where $x^\mu$ are coordinates on our four-dimensional brane, are a noteworthy attempt to ascertain signatures coming from high-energy physics beyond the Standard Model [5]. In fact, since the temperature has changed dramatically along the cosmological evolution, a variable tension BW scenario is indeed a natural candidate to describe our physical universe. The fully covariant dynamics of variable tension branes was established in [6], and further explored in [7]. Moreover, a variable tension was also implemented in the BW model consisting of two branes [8], and the cosmological evolution was investigated in a particular model in which the brane tension has an exponential dependence on the scale factor [9].

In this work, we consider black strings in BW models with the variable brane tension, whose four-dimensional geometry is determined according to the principle of minimal geometric deformation (MGD) [10]. In particular, we shall focus on the shape and evolution of the black string time-dependent horizon, and show that, in this setup, the black string warped horizon is drastically affected by the variable tension, since additional bulk terms are induced by the variable brane tension. For this purpose, we shall employ a Taylor expansion in the extra dimension about a black hole metric on the brane, in such a way that the corrections to the area of the five-dimensional black string horizon are singled out. A complete numerical analysis will then be displayed for a model physically motivated by Eötvös law.

Approximative methods involving expansions of the metric have been extensively used in order to investigate black strings in the BW. For instance, previous constructions of black holes and black strings in vacuum plane wave spacetimes employed the so-called method of matched asymptotic expansions [11]. Another construction was given in [12], based on a derivative expansion method along the direction of the black string, which provided a solution up to second order in derivatives and which showed, in particular, that the black string must shrink to zero size at the horizon of some black brane. In [12], the usual black string metric thus appears as the leading order solution in a given expansion that contains corrections to the black string metric order by order in derivatives. Other approximative methods, including Taylor and Fourier metric expansions, have been employed to provide black string profiles [13, 14]. Such expansions in the context of general relativity (GR) and black strings are thus commonly
used, as, schematically, the degrees of freedom of GR are split into long and short wavelength components. Also, approximations usually concern the far-zone and near-zone metric for the black string, the two regimes being defined where the radial coordinate is respectively much larger or much smaller than twice the black hole mass [15].

Regarding the time evolution of black strings, some seminal results showed that their warped horizon cannot pinch in a finite horizon time [16, 17]. This means that a black hole cannot be the end-state of their Hawking decay and, instead, the existence of a stable string phase should serve as an end-state. The argument that shows the pinching takes an infinite time is based on assuming the increasing area theorem for an event horizon and applying it to an area element at the throat, namely, the inward collapsing region of the event horizon. In addition, numerical analyses also support the conclusion that the warped horizon cannot pinch in finite time [17].

Our approach here is much more general, in that it provides the bulk metric near the brane, using Gaussian normal coordinates in order to extend into the bulk a BW black hole metric obtained from the MGD of the Schwarzschild metric. Computing the bulk metric when the radial coordinate equals the horizon radius of the brane black hole, will thus provide a description of the black string warped horizon, just as a particular case. Let us stress that the bulk shape of the horizon has only been investigated in very particular cases [3, 18], and, recently, the Schwarzschild black string was studied in this context, e.g. for a brane with variable tension [19]. Some realistic generalizations regarding a post-Newtonian parameter on the Casadio–Fabbri–Mazzacurati black string [20, 21], and the black string in a Friedmann–Robertson–Walker BW [22] also represent interesting applications.

In the next section, we shall recall the general expansion of the bulk metric elements in the Gaussian normal coordinate perpendicular to the brane and, in section 3, we shall review the MGD approach to generate BW solutions starting from GR solutions. In section 4, we shall then apply the latter method in order to obtain all the terms that determine the expansion of the bulk metric up to forth order, starting from the Schwarzschild metric on a brane with variable tension. We shall finally consider the particular case of Eötvös fluid branes endowed with a tension derived from a de Sitter-like cosmological scale factor, and investigate the role that the variable brane tension plays in this context. The black string warped horizon along the extra dimension will be analysed, and we shall show that our construction based on the MGD induces a throat on the black string, and prevents the black string warped horizon to pinch in finite horizon time.

2. Bulk metric in Gaussian coordinates

In this section, the general formalism of [3] is briefly introduced and reviewed. Hereon, \( \{ \theta_\mu \} \), with \( \mu = 0, 1, 2, 3 \) (and \( \{ \theta_A \} \), with \( A = 0, 1, 2, 3, 5 \)) denotes a basis for the cotangent space \( T^*_x M \) at a point \( x \) on a 3-brane \( M \), embedded in the five-dimensional bulk. Furthermore, \( \{ \epsilon_A \} \) is its dual basis and \( \theta^A = dx^A \), when a coordinate chart is chosen. Let \( n = n_A \theta^A \) be a time-like covector orthogonal to \( T^*_x M \) and \( y \) the associated Gaussian coordinate, which parameterizes geodesics starting from the brane and moving into the bulk. In particular, \( n_A dx^A = dy \) on the hypersurface defined by \( y = 0 \). The five-dimensional coordinate vector field \( u = x^\mu \epsilon_\mu \) thus splits into components parallel and orthogonal to the brane, and can be written as \( u = x^\mu \epsilon_\mu + y \epsilon_5 \) or \( u^A = (x^\mu, y) \). The brane metric \( g_{\mu\nu} \) and the corresponding components of the bulk metric \( \tilde{g}_{\mu\nu} \) are in general related by \( \tilde{g}_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \) [3]. Since with our choice of \( n \) we have \( g_{55} = 1 \) and \( g_{\mu5} = 0 \), the five-dimensional bulk metric \( \tilde{g}_{AB} dx^A dx^B = g_{\mu\nu}(x^\mu, y) dx^\mu dx^\nu + dy^2 \), and one can effectively use \( A, B = 0, 1, 2, 3 \).
There is a well-known relation between the effective four-dimensional cosmological constant $\Lambda_4$ on the brane, the bulk cosmological constant $\Lambda_5$ and the brane tension $\sigma$, given by the fine-tuning relations [3]

$$\kappa_4^2 = \frac{1}{6} \sigma \kappa_5^2, \quad \Lambda_4 = \frac{\kappa_5^2}{2} \left( \Lambda_5 + \frac{1}{6} \kappa_5^2 \sigma^2 \right), \quad \Lambda_5 = \frac{\kappa_5}{2} \left( \Lambda_4 + \frac{1}{6} \kappa_5^2 \sigma^2 \right), \quad \Lambda_4 = \frac{3}{2} \kappa_5 \left( \Lambda_5 + \frac{1}{6} \kappa_5^2 \sigma^2 \right), \quad \Lambda_5 = \frac{3}{2} \kappa_5 \left( \Lambda_4 + \frac{1}{6} \kappa_5^2 \sigma^2 \right),$$

where $\kappa_5$ ($\kappa_4$) denotes the five-dimensional (four-dimensional) gravitational coupling. The extrinsic curvature of the brane at $y = 0$ is given by $K_{\mu\nu} = \frac{1}{2} \gamma^2 g_{\mu\nu}$ in Gaussian normal coordinates, and the junction conditions thus imply that

$$K_{\mu\nu} = -\frac{\kappa_5^2}{2} \left[ T_{\mu\nu} + \frac{1}{3} (\sigma - T) \gamma_{\mu\nu} \right].$$

(2)

The symmetric and trace-free components of the bulk Weyl tensor $C_{\mu \nu \rho \sigma}$ are respectively given by

$$\mathcal{E}_{\mu\nu} = C_{\mu \nu \rho \sigma} n^\rho n^\sigma$$

and $B_{\mu\nu\rho\sigma} = g^\rho_\sigma C_{\mu \nu \rho \sigma} n^\rho n^\sigma$. The effective four-dimensional field equations are complemented by a set of equations obtained from the five-dimensional Einstein and Bianchi equations in [3, 23, 24] for constant $\sigma$, and in [6] for a brane with variable tension, all approaches being completely consistent. In particular, the bulk metric near the brane can be expressed as the Taylor expansion in the Gaussian coordinate $y$ as

$$g_{\mu\nu} (x^\alpha, y) = \sum_p g^{(p)}_{\mu\nu} (x^\alpha) \frac{y^p}{p!}. \quad (3)$$

By denoting $g_{\mu\nu} = g_{\mu\nu}^{(0)} (x^\alpha)$, and $R_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} (x^\mu, 0)$ the components of the bulk Riemann tensor ($R_{\mu\nu}$ and $R$ are obviously the associated Ricci tensor and scalar curvature) likewise computed on the brane, one then finds the above expansion up to order $p = 4$ is given by

$$g_{\mu\nu} (x^\alpha, y) = g_{\mu\nu} - \frac{\kappa_5^2}{2} \left[ T_{\mu\nu} + \frac{1}{3} (\sigma - T) \gamma_{\mu\nu} \right] y + \left[ \frac{1}{2} \kappa_5^3 \left( T_{\mu\nu} T^\alpha_\nu + \frac{2}{3} (\sigma - T) T_{\mu\nu} \right) \right] y^2$$

$$- 2 \mathcal{E}_{\mu\nu} + \frac{1}{3} \left( \frac{1}{6} \kappa_5^4 (\sigma - T)^2 - \Lambda_4 \right) g_{\mu\nu} \frac{y^3}{3!} + \cdots$$

where $B = B_{\mu\nu}$, and $B^2 = B_{\mu\nu} B^{\mu\nu}$, for any rank-2 tensor $B$. This expansion provides the bulk metric near the brane, and was analysed in [3, 18] only up to the second order, which is not sufficient to determine the additional terms arising from the brane variable tension. Moreover, alternative approaches do not take into account the $\mathbb{Z}_2$ symmetry [25]. In this paper, we shall investigate the warped horizon [26] by exploring the metric component $g_{\mu\nu} (x^\alpha, y)$ in equation (4). Indeed, let us consider a general spherically symmetric metric in spherical coordinates,

$$d \tilde{s}^2 = -f_1(r) \, dr^2 + f_2(r) \, r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right).$$

(5)
that can be associated with a black hole on the brane. Note, we wrote the metric by making use of the usual four-dimensional areal radial coordinate $r$, which, in the BW, is related to the five-dimensional metric by $\sqrt{g_{\mu\nu}(x^\alpha, y)} = r$ \cite{18, 19, 22}. The black string solution will display a horizon, on the brane, with (areal) radius $R = \sqrt{g_{\mu\nu}(x^\alpha, 0)}$, where $R$ denotes the coordinate singularity usually obtained from the condition $g_{rr}^{-1}(r) = 0$. For example, the Schwarzschild metric has a horizon at the well-known value $R = \frac{2GM}{c^2} = 2M$ (in the usual units with $G = c = 1$). (By abuse of notation, $R$ will also denote any coordinate singularities derived by the MGD procedure from the Schwarzschild metric.) The areal radius of the black string warped horizon, along the extra dimension, is then determined solely by the same metric component, namely

$$R(y) = \sqrt{g_{\mu\nu}(x^n, y)}\bigr|_{r=R}, \quad (6)$$

and will therefore be derived from equation (34) in section 4.3. In fact, our analysis of the term $g_{\mu\nu}(x^n, y)$ (given by equation (4) for $\mu = v = \theta$) holds for any value of $r$, corresponding to the bulk metric near the brane, and including the horizon at $r = R$ for $y = 0$. We finally recall that the term ‘black string’ was introduced by referring to a bulk geometry corresponding precisely to the Schwarzschild case on the brane \cite{26}.

The additional terms coming from the variable brane tension shall be shown to play an essential role for the subsequent analysis of the black string behaviour along the extra dimension. In particular, terms in the expansion (4) involving derivatives of the variable brane tension at order $|y|^3$ are given by \cite{19}

$$g^{(3)\text{variable}}_{\mu\nu} = \frac{2}{3} \kappa_5^2 [\nabla_\nu \nabla_\mu \sigma - g_{\mu\nu} \Box \sigma], \quad (7)$$

and, at order $|y|^4$, by

$$g^{(4)\text{variable}}_{\mu\nu} = -\frac{k_5^2}{3} \Box (\Box \sigma) g_{\mu\nu} - \nabla_\nu (\nabla_\mu (\Box \sigma))$

$$+ \left( \frac{1}{3} \kappa_5^2 + 2K \right) 
\left[ (\Box \sigma) \mathcal{E}_{\mu\nu} - \nabla^a [(\nabla_\mu (\sigma) \mathcal{E}_{\nu}^a)] \right]$$

$$+ \frac{k_5^2}{3} [(\Box \sigma) R_{\mu\nu} - \nabla^a [(\nabla_\mu (\sigma) R_{\nu}] )]$$

$$- 2K \tau \beta [(\Box \sigma) R_{\mu(\tau\nu)} - \nabla^a [(\nabla_\mu (\sigma) R_{\tau\nu}^\beta)]$$

$$+ \frac{k_5^2}{3} [(\Box \sigma) (K_{\mu(\tau\nu)} K_{\nu}^\beta) - K^\beta K_{\mu\tau}] - \nabla^a [(\nabla_\mu (\sigma) (K_{\tau\nu} K_{\nu}^\beta) - KK_{\mu\tau})]]$$

$$+ 6[(\Box \sigma) K_{\mu(\tau\nu)} e_{\nu}^a - \kappa_5^2 \nabla^a [(\nabla_\mu (\sigma) e_{\nu}^a)]$$

$$+ (2K^2 - \frac{1}{3} \Lambda) \left[ (\Box \sigma) g_{\mu\nu} - \nabla_\nu (\nabla_\mu (\sigma) \right]$$

$$+ 2 \left( K + \frac{7}{3} \kappa_5^2 \right) \left[(\Box \sigma) KK_{\mu\nu} - \nabla^a [(\nabla_\mu (\sigma) KK_{\nu}^a)] \right]. \quad (8)$$

In the following, we shall just consider a time-dependent brane tension $\sigma = \sigma(t)$, as we shall not be concerned with anisotropic branes.

3. Minimal geometric deformation

Solving the four-dimensional effective Einstein equations is not a straightforward task and, already in the simple case of a spherically symmetric metric we shall consider here

$$ds^2 = e^\nu dt^2 - e^\kappa dr^2 - r^2 d\Omega^2, \quad (9)$$

6 This calculation can be found in equation (II.6) of \cite{27} in the context of BW models.
only a few ‘vacuum’ solutions are known analytically [21, 28–31]. When stellar systems are studied, the search for solutions becomes even more difficult, mainly due to the presence of nonlinear terms in matter fields that arise from high-energy corrections [3, 23]. Nonetheless, two exact analytical solutions were found [32] by means of the MGD [10]. This approach has allowed in particular to generate physically acceptable interior solutions for stellar systems [33], to solve the tidally charged exterior solution found in [31] in terms of the ADM mass, and to study (micro) black hole solutions [34, 35], as well as to elucidate the role of exterior Weyl stresses from bulk gravitons on compact stellar distributions [36].

Let us start by reviewing the bases of the MGD approach, i.e. the deformation undergone by the radial metric component of the interior spacetime associated with a self-gravitating stellar system of radius \( R \). Such an approach has been employed in different contexts [37]. This radial metric component is deformed by bulk effects in such a way that, when we demand to recover GR at low energies \((\sigma^{-1} \to 0)\), it must be written as

\[
e^{-\lambda} = \mu(r) + e^{-I} \int_0^r \frac{e^{I}}{r^2 + x^2} \left[ H(p, \rho, \nu) + \frac{k^2}{\sigma} (\rho^2 + 3 \rho p) \right] dx + \beta e^{-I}
\]

Geometric deformation

\[
\equiv \mu(r) + f(r), \tag{10}
\]

where

\[
\mu(r) = \begin{cases} 
1 - \frac{k^2}{r} \int_0^x x^2 \rho \, dx & \text{for } r \leq R, \\
1 - \frac{2M_0}{r} & \text{for } r > R,
\end{cases} \tag{11}
\]

contains the usual GR mass function \( m(r) \) for \( r < R \), and \( M_0 \) for \( r > R \), whereas the function \( H(p, \rho, \nu) \) encodes anisotropic effects due to bulk gravity on the pressure \( p \), matter density \( \rho \) and the metric function \( \nu \). The function \( \beta = \beta(\sigma) \) in equation (10) depends on the brane tension \( \sigma \) and on the mass \( M_0 \) of the self-gravitating system, and must be zero in the GR limit. For interior solutions, the condition \( \beta(\sigma) = 0 \) must be imposed to avoid singular solutions at the centre \( r = 0 \). However, for a vacuum solution, or more properly, in the region \( r > R \) where there is a Weyl fluid filling the spacetime surrounding the spherically symmetric stellar distribution, the function \( \beta \) is not necessarily zero, and there must be a geometric deformation associated with the Schwarzschild solution.

When a spherically symmetric GR vacuum solution is considered, the quantity \( H = 0 \) and the geometric deformation \( f \) in vacuum \((p = \rho = 0)\), hereafter denoted \( f = g^*(r) \), will be consequently minimal and given by

\[
g^*(r) = \beta e^{-I}. \tag{12}
\]

The radial metric component in equation (10) then becomes

\[
e^{-\lambda} = 1 - \frac{2M_0}{r} + \beta(\sigma) e^{-I}, \tag{13}
\]

where

\[
I = \int \left( \frac{\nu'' + \nu^2 + \frac{2\nu'}{r} + \frac{2}{r^2}}{\left( \frac{\nu}{r^2} + \frac{2}{r} \right)} \right) dr. \tag{14}
\]

We then consider the general matching conditions between the generic interior MGD metric

\[
ds^2 = e^{\nu(r)} \left[ \frac{dr^2}{1 - \frac{2m(r)}{r} + f^*(r)} - r^2 d\Omega^2 \right]. \tag{15}
\]
characterizing the star interior \( r < R \), where \( f^*(r) \) is given by equation (10) with \( H = 0 \), and the most general exterior solution containing a Weyl fluid with \( \mathcal{U}^+, \mathcal{P}^+ \), and \( p = \rho = 0 \) for \( r > R \), which, according to the expression in equation (13), can be written as

\[
d s^2 = e^{2\gamma(r)} d r^2 - \frac{d r^2}{1 - \frac{2M}{r} + g^*(r)} - r^2 d \Omega^2, \tag{16}
\]

where the mass \( M \) in equation (16) is in general a function of the brane tension \( \sigma \). Continuity of the first fundamental form at the star surface \( \Sigma \) of radius \( r = R \) when the metrics in equations (15) and (16) are considered, leads to

\[
v_R^- = v_R^+ \tag{17}
\]

\[
\frac{2M}{R} = \frac{2M_0}{R} + (g_R^* - f_R^*), \tag{18}
\]

where \( f_R^* \equiv f(r \to R^+) \) for any function. Continuity of the second fundamental form on \( \Sigma \) likewise gives [45]

\[
[G_{\mu
u}r^\nu]_\Sigma = 0, \tag{19}
\]

where \( r_\mu \) is a unit radial vector and \([f]_\Sigma \equiv f(r \to R^+) - f(r \to R^-) \). Using equation (19) and Einstein field equations, one finds \([T_{\mu\nu}r^\nu]_\Sigma = 0\), which in our case reads

\[
\left[ p + \frac{1}{\sigma} \left( \frac{\rho^2}{2} + \rho p + \frac{2}{k^4} \mathcal{U} \right) + 4 \frac{\mathcal{P}}{k^4} \sigma \right]_\Sigma = 0. \tag{20}
\]

Since we assume the distribution is only surrounded by a Weyl fluid described by the functions \( \mathcal{U}^+ \) and \( \mathcal{P}^+ \), \( p = \rho = 0 \) for \( r > R \) and this matching condition takes the final form

\[
p_R + \frac{1}{\sigma} \left( \frac{\rho_R^2}{2} + \rho_R p_R + \frac{2}{k^4} \mathcal{U}_R \right) + 4 \frac{\mathcal{P}_R}{k^4} \sigma = \frac{2 \mathcal{U}_R}{k^4} \sigma + \frac{4 \mathcal{P}_R}{k^4} \sigma, \tag{21}
\]

with \( p_R \equiv \mathcal{P}_R \) and \( \rho_R \equiv \rho_R \).

The limit \( \sigma^{-1} \to 0 \) in the second fundamental form in equation (21) leads to the well-known GR matching condition \( p_R = 0 \) at the star surface. The expressions given by equations (17), (18) and (21) are the necessary and sufficient conditions for the matching of the interior MGD metric to a spherically symmetric ‘vacuum’ filled by a BW Weyl fluid [38, 39].

4. Black strings and variable tension brane

In this section, we proceed to apply the above formalism to the case of a black string. We shall first derive the brane geometry from the MGD approach applied to the standard Schwarzschild metric, in order to obtain the relevant projections of the bulk Weyl tensor that enter the bulk metric (4). Subsequently, we shall study the particular case of a phenomenological Eötvös brane, with a variable tension that depends on the time exponentially.

4.1. Brane geometry of a black string

Let us now find the explicit MGD function \( g^*(r) \) produced by the Schwarzschild solution

\[
e^{\gamma_s} = e^{-\lambda_s} = 1 - \frac{2M}{r} \tag{22},
\]
where we recall $M$ is a function of the brane tension $\sigma$. Using equation (22) in (12), we obtain

$$g^*(r) = -\frac{2\beta(\sigma)}{r} \left( 1 - \frac{2M}{r} \right),$$

and the deformed exterior metric components read

$$e^\nu = 1 - \frac{2M}{r},$$

$$e^{-\lambda} = \left( 1 - \frac{2M}{r} \right) \left[ 1 - \frac{\beta(\sigma)}{r - \frac{3M}{2}} \right],$$

which match the vacuum solution found in [38] in the particular case when $\beta(\sigma) = -\frac{C_0}{\sigma}$, for $C_0$ a positive constant. Below we shall show a general expression for the function $\beta$, which depends on the interior structure of the self-gravitating system surrounded by the geometry (24). The Weyl fluid associated with the solution in equations (24), (25) is then described by the functions (see, e.g. [36])

$$\frac{1}{k^2} \mathcal{P}^+ = - \frac{1 - \frac{4M}{3r}}{9 \left( 1 - \frac{3M}{2r} \right)^2} \beta,$$

and

$$\frac{1}{k^2} \mathcal{U}^+ = \frac{M}{12 \left( 1 - \frac{3M}{2r} \right)^2} \frac{\beta}{r^2}.$$  

The function $\beta = \beta(\sigma)$ must now be specified by considering the deformed Schwarzschild solution (24) and (25) in the matching conditions (17), (18) and (21). The first fundamental form leads to the expressions given by equations (17) and (18), where equation (17) becomes

$$e^\nu = 1 - \frac{2M}{R^2},$$

whereas the second fundamental form in equation (21) gives

$$p_R + \frac{f_R^2}{k^2} \left( \frac{v_R}{R} + \frac{1}{R^2} \right) = -\frac{g^*_{RR}}{R^2}.$$  

Note that if $M$ were the GR mass $M_0$ of equation (22), one would have $g^*_{RR} = f_R^2$, which represents an unphysical condition, according to the expression in equation (28). Equations (18), (27) and (28) are the necessary and sufficient conditions for matching the two minimally deformed metrics given in equation (15) and in equations (24) and (25). The matching condition (28) shows that the exterior geometric deformation $g^*(r)$ at the star surface, i.e. $g^*_{RR}$, is always negative. Therefore, the deformed horizon radius $r_H = 2M$ will always be smaller than the Schwarzschild radius $r_H = 2M_0$, as can clearly be seen from equation (18). This general result clearly shows that five-dimensional effects weaken the strength of the gravitational field produced by the self-gravitating stellar system.

Finally, when the explicit geometric deformation (23) is considered in the matching condition (28), the function $\beta = \beta(\sigma)$ becomes

$$\beta(\sigma) = R^3 \left( \frac{1 - \frac{3M}{2r}}{1 - \frac{2M}{R}} \right) \left[ \left( \frac{v_R}{R} + \frac{1}{R^2} \right) \frac{f_R}{8\pi} + p_R \right],$$

showing thus that $\beta$ is always positive and (interior) model-dependent. For instance, we can find $\beta(\sigma)$ by considering the exact interior BW solution found in [32], where the geometric deformation is given by

$$f^*(r) = \frac{1}{\sigma} \frac{4C(r(r))}{49\pi} \left[ \frac{240 + 589Cr^2 - 25C^2r^4 - 41C^2r^6 - 3C^4r^8}{3(1 + Cr^2)^3} - \frac{80 \arctan(\sqrt{Cr})}{\sqrt{Cr}} \right].$$

(30)
with $C$ a constant given by $CR^2 = \frac{M}{2} - \frac{\sqrt{R}}{2} = \alpha$, and the functions $(\tau(r))^{-1} \equiv (1 + Cr^2)^{3}(1 + 3Cr^2)$ and $\nu' = \frac{8Cr}{1+C^2r^2}$. Now, by using the explicit form of $f(R)$ we obtain
\[ \beta = \frac{\alpha \tau(R)}{98\pi^2\sigma} \left[ \frac{240 + 589\alpha - 25\alpha^2 - 41\alpha^3 - 3\alpha^4}{3(1 + \alpha)^2} \right] - \frac{80\arctan(\sqrt{\alpha})}{\sqrt{\alpha}}, \]  
\[ (31) \]

or
\[ \beta(\sigma) \propto \frac{1}{2\sigma R} \left( \frac{2R - 3M_0}{R - 2M_0} \right) = \frac{C_0}{\sigma}. \]  
\[ (32) \]

Note that we are using $M_0$ and not $M$, because $M = M_0 + O(\sigma^{-1})$, so that the exact $\beta = \beta(\sigma)$ will be given by the expression above plus terms of order $\sigma^{-2}$. The geometric deformation is proportional to the parameter $\beta$, as is clearly shown in equation (12). When $\beta = 0$, we recover the Schwarzschild solution. As a function of the star interior, $\beta$ can be found by using the matching conditions, which lead to the expression in equation (32). The new parameter $C_0$ essentially depends on the star compactness $M_0/R$, and increases when $M_0/R$ increases. Consequently, $\beta$ and therefore the geometric deformation of the GR solution, are larger the more compact is the star. In the following, we shall in fact find it convenient to express the dependence of the geometric from the star compactness via $C_0$ in equation (32).

Now, as the area of the five-dimensional horizon is determined by $g_{\theta\theta}(x^\alpha, y) = g_{\theta\theta}(r, y)$ shown in equation (23).

Upon using equation (9), we readily find
\[ \mathcal{E}_{\theta\theta} = -R_{\theta\theta} = \frac{r}{2} e^{-\lambda}(\lambda' - \nu') + 1 - e^{-\lambda} = \frac{\beta(\sigma)}{2} \left( 1 - \frac{M}{r} \right)^2. \]  
\[ (33) \]

This is the component of the projection of the Weyl tensor on the brane which we need in order to determine the horizon area in the bulk, where the function $\beta = \beta(\sigma)$ can be specified once we choose a specific interior BW solution to evaluate the expression in equation (29) and a model of the brane tension.

### 4.2. Time-dependent horizon area

As the main object of interest to us here is the black string horizon along the extra dimension, we shall focus on the term $g_{\theta\theta}$ which yields the area of the horizon, and evaluate the corresponding expansion (4). Clearly, a time-dependent brane tension will modify the black string Schwarzschild background. Since the complete solution is extremely difficult to compute, we shall take a more effective approach, and study how the horizon area changes with the brane tension in the Taylor expansion (4). As we shall show, even in this approximate description, very interesting results can be obtained.

Upon inserting the relevant expressions (24), (25) and (33) derived from the MGD of the Schwarzschild metric, the Taylor expansion of the term $g_{\theta\theta}(x^\alpha, y) = g_{\theta\theta}(r, y)$ in equation (4) reads
\[
g_{\theta\theta}(r, y) = r^2 - \frac{r^2}{3} \kappa_5^2 \sigma |y| + \left( \frac{\kappa_5^2 \sigma^2}{36} - \frac{\Lambda_5}{6} \right) r^2 |y|^2 \]
\[ - \left( \frac{193}{216} \sigma^3 \kappa_5^2 + \frac{5}{18} \Lambda_5 \kappa_5^2 \sigma + \frac{\kappa_5^2 \beta(\sigma)}{12r} \left( 1 - \frac{M}{r} \right)^2 \right) \frac{r^2}{3!} |y|^3 \]
\[ + \left( \frac{\Lambda_5}{18} \left( \Lambda_5^2 - 3\beta(\sigma) \left( 1 - \frac{M}{r} \right)^2 - \frac{\sigma^2}{6} + \frac{7\sigma R^4}{324} \right) \right) \]
of the brane \[6–9\].

shall do in the analysis hereupon, the brane tension can be understood as an intrinsic property where, from now on, \(\sigma\).

assumed in the context of string theory and supersymmetric branes \[40, 41\]. Otherwise, as we hand, one can consider the brane tension as a scalar field in the Lagrangian, as is widely

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4.3. Black string with minimal geometric deformation in a brane variable tension

A variable tension on the brane can be established in general by two approaches. On the one hand, one can consider the brane tension as a scalar field in the Lagrangian, as is widely assumed in the context of string theory and supersymmetric branes \[40, 41\]. Otherwise, as we shall do in the analysis hereupon, the brane tension can be understood as an intrinsic property of the brane \[6–9\].

In the BW scenario, the functional form of the variable brane tension is an open issue. However, taking into account the huge variation of the universe temperature during its cosmological evolution, it is indeed plausible to implement the brane tension as a function of the cosmological time. Although it lacks a complete scenario, the phenomenologically interesting case of Eötvös fluid membranes \[42\] can be useful to extract physical results. In this context, the cosmological evolution of a perfect fluid imposes cosmological symmetries, and the brane tension \(\sigma\), along with the constants \(\kappa_4\) and \(\Lambda_4\), become scale-factor (or cosmological time) dependent. The phenomenological Eötvös law asserts that the fluid membrane tension depends on the temperature as

\[
\sigma = \chi (T_c - T),
\]

where \(\chi\) is a constant and \(T_c\) represents a critical temperature equal to the highest temperature for which the membrane exists. By imposing the continuity equation, in such a way that the temperature dependence of the brane tension is balanced by the energy exchange between the brane and the bulk, it implies that the brane is formed in a very hot early universe when \(\sigma \simeq 0\), and initially \(\kappa_4\) and \(\sigma\) as well are small, strengthening BW effects. If there are no stresses in the bulk, and without taking into account the cosmological constant, the bulk is isolated from the brane, with no exchange of energy–momentum \[3\]. Thus the thermodynamical expression \(dQ = dE + pdV = 0\) holds for the brane. Furthermore, for photons of the CMB in a volume \(V\), we can use \(E = E_p = \sigma T^4 V\) and \(p = \frac{\rho}{\gamma} = \frac{2}{3} \rho V\). It is then easy to verify that \(\frac{1}{V} \frac{dT}{dt} = -\frac{1}{3V} \frac{dV}{dt}\). By relating the volume to the Friedmann–Robertson–Walker scale factor,
one obtains \( T \propto \frac{1}{a(t)} \) \cite{7, 19}. By considering the similarity with Eötvös membranes, one then infers that \( \sigma = \sigma_0 (1 - \frac{a_0}{a}) \), where \( \sigma_0 \) is a scalar associated with the four-dimensional coupling constants \cite{7}, and \( a_0 \) is the minimum scale factor such that the brane does not exist (meaning that for smaller \( a \), the brane tension would be negative). In any case, in what follows, it suffices to consider

\[
\sigma(t) = 1 - \frac{1}{a(t)}. \tag{38}
\]

We can also note that a similar behaviour is obtained from SUSY in inflationary cosmology. In fact, as shown in \cite{19}, one obtains

\[
\Lambda_4 \propto \left(1 - \frac{1}{a(t)}\right)^2, \tag{39}
\]

which means that the cosmological constant takes negative values before it becomes positive, as the universe expands.

To be more specific, a de Sitter brane profile is taken into account by setting \( a(t) \propto e^{\gamma t} \), with positive \( \gamma \) \cite{43}, so that

\[
\sigma(t) = 1 - e^{-\gamma t}. \tag{40}
\]

The phenomenological viability of this model was analysed in \cite{19}. Further, we shall set \( \Lambda_5 = 1 = \kappa_5 \) from here on. As the brane tension has the lower bound \( \sigma \sim 4.39 \times 10^8 \text{ MeV}^4 \) \cite{3, 44}, we shall normalize it accordingly in the analysis below.

The first result we present is the plot of the area of slices of constant \( y \) of the black string horizon \( r_H = 2M \) for different constant values of the brane tension \( \sigma \) (see figure 1). It is worth noting that when the brane tension reaches the value \( \sigma \approx 0.92 \), the black string warped horizon changes profile: for \( \sigma \gtrsim 0.92 \) the horizon area is always positive, whereas for \( \sigma \lesssim 0.92 \), there is always a point of coordinate \( y_c \) along the extra dimension where the horizon meets the axis of axial symmetry \( g_{\theta \theta} (r = r_H, y_c) = 0 \). For instance, when \( \sigma = 0.05 \), one finds \( y_c \simeq 0.51 \) in figure 1.

Next, in figure 2 one can see how the black string warped horizon varies along the extra dimension as a function of \( \beta(\sigma) = C_0/\sigma \) from the MGD described in section 4.1. As the brane tension \( \sigma \) is assumed heretofore constant, we have rescaled the parameter \( C_0 \) accordingly.
It is now important to note that, as time increases, the additional terms in (7) and (8), generated by the variable brane tension, play an increasingly important role in the Taylor expansion (4) for the warped horizon. Notwithstanding, since $\sigma = \sigma(t)$ is given by equation (40), all terms in equation (34) are under control. Indeed, in the expansion (4) and the terms (7) and (8) due to the variable brane tension, the only time-dependent terms are $\sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^{-1},$ and $\sigma''$ and $\sigma'''$. Their asymptotic values for $t \to \infty$ are respectively 1 (for the first five terms) and 0 (for the last two terms). Therefore, the Taylor expansion is well behaved for all values of time.

That said, the metric element $g_{\theta\theta}(r = 2M, y)$ displayed in figure 4 for $y > 0$ is obtained from the Taylor expansion (34) specified to our case. In order to guarantee that the expansion including the terms up to order $(y^4/4!)$ in equation (34) is reliable, we only considered values of $0 < y$ much smaller than $|t/\Lambda|^{-1/2}$. Furthermore, based on the results of [19], one can prove that the Taylor expansion along the extra dimension holds for values of $y$ after which the Gregory–Laflamme perturbation [46] makes the instability significant to alter the oscillating black string warped horizon, in our case.

Of course, there is no time dependence in the cases shown in figures 1 and 2. The plot in figure 3 instead displays the black string warped horizon given by the bulk metric (34) on the horizon $r = 2M$, with the additional terms (35) and (36) due to the variable brane tension given by expression (40). One can also see from figure 4 that, for values $\gamma t \gtrsim 0.52$, the horizon area increases monotonically along the extra dimension. For $\gamma t \lesssim 0.52$, there exists a point $y_c$ along the extra dimension beyond which the black string ceases to exist. We shall further discuss this point below.

In figure 5, the variable brane tension is considered for $\gamma t = 0.25$. The coordinate $y$ along the extra dimension beyond which the black string ceases to exist is $y_c \simeq 2.1$. Note that for $y > y_c$, the metric element $g_{\theta\theta}(t, r = 2M, y > y_c) < 0$, and such negative values are also displayed for clarity but have no physical meaning. Moreover, the four-dimensional Kretschmann scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ diverges for $r = 0$. This can be seen from the Gauss equation which relates the four- and five-dimensional Riemann curvature tensors according to $^{(5)}R_{\nu\rho\sigma} = ^{(4)}R_{\nu\rho\sigma} - K_{\nu\rho}K_{\sigma} + K_{\nu}^\mu K_{\rho\sigma}$. Now, equation (2) in this specific case
reads \( K_{\mu\nu} = -\frac{\kappa^2}{3} \sigma g_{\mu\nu} \) and, by inserting it in the Gauss equation for the five-dimensional Kretschmann scalar \((5)K_0 = (5)R_{\mu\nu\rho\sigma} (5)R^{\mu\nu\rho\sigma}\), one can see that terms involving the extrinsic curvature do not cancel the divergence provided by the four-dimensional Kretschmann scalar in \((5)K_0\). For such cases, the Kretschmann scalar diverges at \(r \to 0\), and also \(K_0\) diverges at \(y = y_c\), when \(r = 0\), characterizing indeed a singularity at \(y = y_c\).

The plots in figure 6 display the black string warped horizon for increasing values of the time variable. As the time passes, the horizon pinches at a critical point along the extra dimension. In the context of the MGD of a black hole on the variable tension brane, the black string presents a throat along the extra dimension, with area that tends to zero as time runs to infinity. As it is going to be discussed in the next section, our results completely support the ones obtained in [16, 17], in the context of the MGD. The area theorem prevents indeed the throat from shrinking to zero size in the finite horizon time.
Figure 5. Warped horizon along the extra dimension, for $t = 0.3$. Here $C_0 = 0.1$ and $\gamma t = 0.25$, for a variable brane tension.

Figure 6. Warped horizon along the extra dimension for $\gamma t = 0.5$, $\gamma t = 1.25$ and $\gamma t = 3$ (left to right). Here $C_0 = 1$.

5. Concluding remarks

It is well-known that BW models lead to cosmological evolutions whose background dynamics is completely understood, and can further reproduce general relativistic results with suitable restrictions on the BW parameters. In this paper, we focused our attention on branes whose variable tension only depends on the time, and obtained the Taylor expansion of the metric in the Gaussian coordinate along the extra dimension.

This Taylor expansion makes it possible to write the bulk metric in terms of the brane metric, and in a form that shows a time-dependence already in terms of the second order. It is however only including terms of (at least) third order that one can display the effects of a variable tension on the shape of the horizon in the bulk, since it is only from such terms that the covariant derivatives of the variable tension appear in the metric expansion along the extra dimension, and lead to sizeable corrections. Furthermore, as in some cosmological epochs the value of the tension could have been very small, and can thus be largely modified, it is very important to consider such terms, which contribute to a realistic description of black strings.
The additional terms in the expansion we investigated indeed alter the black string warped horizon, when the brane tension varies in a BW model based upon the Eötvös law.

We also showed that the MGD of a Schwarzschild black hole on the variable tension brane corresponds to a black string with a throat along the extra dimension, whose area tends to zero as time runs to infinity.

One could finally note that equation (39) implies that, for small values of the scale factor, the effective four-dimensional cosmological constant $\Lambda_4 < 0$, and it contributes like an attracting cosmic component. As the scale factor increases, one reaches $\Lambda_4 > 0$, which generates a dark energy-type repulsion. In addition, BW models with the varying brane tension might allow for energy exchange and other types of evolution that concretely lead to interesting new physics, supporting and generalizing some results in the literature, as those in [47–49] for instance. In particular, BW models can replace the dark matter with geometric effects, and we plan to investigate BW stars and the gravitational collapse on the brane [18, 32, 33, 38, 50, 51] by means of generalizations of spherically symmetric BW solutions to the case with the variable brane tension.

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References

[1] Randall L and Sundrum R 1999 A large mass hierarchy from a small extra dimension Phys. Rev. Lett. 83 3370 (arXiv:hep-ph/9905221)

Randall L and Sundrum R 1999 An alternative to compactification Phys. Rev. Lett. 83 4690 (arXiv:hep-th/9906064)

[2] Horava, P, and Witten E 1996 Heterotic and type I string dynamics from eleven-dimensions Nucl. Phys. B 460 306

[3] Maartens R and Koyama K 2010 Brane-world gravity Living Rev. Rel. 13 5 (arXiv:1004.3962 [hep-th])

[4] Rubakov V A and Shaposhnikov M E 1983 Do we live inside a domain wall? Phys. Lett. B 125 136

Visser M 1985 An exotic class of Kaluza–Klein models Phys. Lett. B 159 22 (arXiv:hep-th/9910093)

[5] Vacaru S I 2006 Clifford and Riemann Finsler Structures in Geometric Mechanics and Gravity ed P Stavrinos, E Gaburro and D Gonta (Bucharest: Geometry Balkan Press) (selected works by S Vacaru) chapter 7

[6] Gergely L A 2008 Friedmann branes with variable tension Phys. Rev. D 78 084006 (arXiv:0806.3857 [gr-qc])

[7] Gergely L A 2009 Eötvös branes Phys. Rev. D 79 086007 (arXiv:0806.4006 [gr-qc])

[8] Abdalla M C B, Hoff da Silva J M and da Rocha R 2009 Notes on the two-brane model with variable tension Phys. Rev. D 80 046003 (arXiv:1101.4214 [gr-qc])

[9] Wong K C, Cheng K S and Harko T 2010 Inflation and late time acceleration in braneworld cosmological models with varying brane tension Eur. Phys. J. C 68 241 (arXiv:1005.3101 [gr-qc])

[10] Ovalle J 2010 Braneworld stars: anisotropy minimally projected onto the brane Gravitation and Astrophysics ed J Luo (Singapore: World Scientific) pp 173–82

[11] Le Witt J and Ross S F 2010 Black holes and black strings in plane waves J. High Energy Phys. JHEP01(2010)101 (arXiv:0910.4332 [hep-th])

[12] Haddad N 2012 Black strings ending on horizons Class. Quantum Grav. 29 245001 (arXiv:1207.2305 [hep-th])
[13] Emparan R, Harmark T, Niarchos V and Obers N A 2009 Worldvolume effective theory for higher-dimensional black holes (blackfolds) Phys. Rev. Lett. 102 191301 (arXiv:0902.0427 [hep-th])
[14] Bhattacharyya S, Hubeny V E, Minwalla S and Rangamani M 2008 Nonlinear fluid dynamics from gravity J. High Energy Phys. JHEP02(2008)045 (arXiv:0712.2456 [hep-th])
[15] Emparan R, Harmark T, Niarchos V and Obers N A 2010 Essentials of blackfold dynamics J. High Energy Phys. JHEP03(2010)063 (arXiv:0910.1601 [hep-th])
[16] Horowitz G T and Maeda K 2001 Fate of the black string instability Phys. Rev. Lett. 87 131301 (arXiv:hep-th/0105111)
[17] Koli B 2006 The phase transition between caged black holes and black strings: a review Phys. Rep. 422 119 (arXiv:hep-th/0411240)
[18] Casadio R and Germani C 2005 Gravitational collapse and black hole evolution: do holographic black holes eventually ‘anti-evaporate’? Prog. Theor. Phys. 114 23 (arXiv:hep-th/0407191)
[19] da Rocha R and Hoff da Silva J M 2012 Black string corrections in variable tension braneworld scenarios Phys. Rev. D 85 046009 (arXiv:1202.1256 [gr-qc])
[20] Bazeia D, Hoff da Silva J M and da Rocha R 2013 Black holes in realistic branes: black string-like objects? Phys. Lett. B 721 306 (arXiv:1303.2243 [gr-qc])
[21] Casadio R, Fabbri A and Mazzacurati L 2002 New black holes in the brane world? Phys. Rev. D 65 084040 (arXiv:gr-qc/0111072)
[22] da Rocha R, Piloyan A, Kuerten A M and Coimbra-Araujo C H 2013 Casadio–Fabbri–Mazzacurati black strings and braneworld-induced quasars luminosity corrections Class. Quantum Grav. 30 045014 (arXiv:1301.4483 [gr-qc])
[23] Shiromizu T, Maeda K and Sasaki M 2000 The Einstein equations on the 3-brane world Phys. Rev. D 62 043523 (arXiv:gr-qc/9910076)
[24] Jennings D, Vernon I R, Davis A-C and van de Bruck C 2005 Bulk black holes radiating in non-z(2) brane-world spacetimes J. Cosmol. Astropart. Phys. JCAP04(2005)013 (arXiv:hep-th/0412281)
[25] Seahra S S, Clarkson C and Maartens R 2005 Detecting extra dimensions with gravity wave spectroscopy: the black string braneworld Phys. Rev. Lett. 94 121302 (arXiv:gr-qc/0408032)
[26] Casadio R and Harms B 2001 Black hole evaporation and compact extra dimensions Phys. Rev. D 64 024016 (arXiv:hep-th/0101154)
[27] Ovalle J, Linares F, Pascua A and Sotomayor R 2013 The role of exterior Weyl fluids on compact stellar structures in Randall–Sundrum gravity Class. Quantum Grav. 30 175019 (arXiv:1304.5995v2 [gr-qc])
[28] Ovalle J and Linares F 2013 Tolman IV solution in the Randall–Sundrum braneworld Phys. Rev. D 88 104026 (arXiv:1201.6145 [gr-qc])
[29] Casadio R, Ovalle J and Obers N A 2012 Non-uniform braneworld stars: an exact solution Int. J. Mod. Phys. D 21 1837 (arXiv:0809.3547 [gr-qc])
[30] Casadio R and Ovalle J 2012 Brane-world stars from minimal geometric deformation, and black holes arXiv:1212.0409 [gr-qc]
[31] Ovalle J 2008 Searching exact solutions for compact stars in braneworld: a conjecture Mod. Phys. Lett. A 23 3247 (arXiv:gr-qc/0703095)
[32] Ovalle J 2010 The Schwarzschild’s braneworld solution Mod. Phys. Lett. A 25 3323 (arXiv:1009.3674 [gr-qc])
[33] Casadio R and Ovalle J 2002 Brane-world stars and (microscopic) black holes Phys. Lett. B 715 251 (arXiv:1201.6145 [gr-qc])
[34] Casadio R and Ovalle J 2012 Brane-world stars from minimal geometric deformation, and black holes arXiv:1212.0409 [gr-qc]
[35] Ovalle J, Linares F, Pascau A and Sotomayor R 2013 The role of exterior Weyl fluids on compact stellar structures in Randall–Sundrum gravity Class. Quantum Grav. 30 175019 (arXiv:1304.5995v2 [gr-qc])
[36] Ovalle J and Linares F 2013 Tolman IV solution in the Randall–Sundrum braneworld Phys. Rev. D 88 104026 (arXiv:1201.6145 [gr-qc])
[37] Germani C and Maartens R 2001 Stars in the braneworld Phys. Rev. D 64 124010 (arXiv:hep-th/0107011)
[39] Gergely L A 2006 Brane-world cosmology with black strings Phys. Rev. D 74 024002 (arXiv:hep-th/0603244)
[40] Bergshoeff E and Townsend P K 1998 Super D-branes revisited Nucl. Phys. B 531 226 (arXiv:hep-th/9804011)
[41] Bergshoeff E, Kallosh R and Proeyen A Van 2000 Supersymmetry in singular spaces J. High Energy Phys. JHEP10(2000)03 (arXiv:hep-th/0007044)
[42] Eötvös R 1886 Ueber den Zusammenhang der Oberflächenspannung der Flüssigkeiten mit ihrem Molecularvolumen Ann. Phys., Lpz. 263 448
[43] Campos A and Sopuerta C F 2001 Evolution of cosmological models in the brane-world scenario Phys. Rev. D 63 104012 (arXiv:hep-th/0101060)
Campos A and Sopuerta C F 2001 Bulk effects in the cosmological dynamics of brane-world scenarios Phys. Rev. D 64 104011 (arXiv:hep-th/0105100)
[44] Garcia-Aspeitia M A 2013 Brane tension constrictions using astrophysical objects arXiv:1306.1283 [gr-qc]
[45] Israel W 1966 Singular hypersurfaces and thin shells in general relativity Nuovo Cimento B 44 1
Israel W 1966 Nuovo Cimento B 48 463
[46] Gregory R and Laflamme R 1993 Black strings and p-branes are unstable Phys. Rev. Lett. 70 2837 (arXiv:hep-th/9301052)
[47] Gergely L A 2007 Black holes and dark energy from gravitational collapse on the brane J. Cosmol. Astropart. Phys. JCAP02(2007)027 (arXiv:hep-th/0603254)
[48] Mak M K and Harko T 2004 Can the galactic rotation curves be explained in brane world models? Phys. Rev. D 70 024010 (arXiv:gr-qc/0404104)
[49] Gergely L A, Harko T, Dwornik M, Kupi G and Keresztes Z 2011 Galactic rotation curves in brane world models Mon. Not. R. Astron. Soc. 415 3275 (arXiv:1105.0159 [gr-qc])
[50] Bruni M, Germani C and Maartens R 2001 Gravitational collapse on the brane Phys. Rev. Lett. 87 231302 (arXiv:gr-qc/0108013)
[51] Gergely L A and Kepiro I 2007 Asymmetric Swiss-cheese brane-worlds J. Cosmol. Astropart. Phys. JCAP07(2007)007 (arXiv:hep-th/0608195)