The $H$-join of arbitrary families of graphs

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Abstract

The $H$-join of a family of graphs $\mathcal{G} = \{G_1, \ldots, G_p\}$, also called the generalized composition, $H[G_1, \ldots, G_p]$, where all graphs are undirected, simple and finite, is the graph obtained from the graph $H$ replacing each vertex $i$ of $H$ by $G_i$ and adding to the edges of all graphs in $\mathcal{G}$ the edges of the join $G_i \vee G_j$, for every edge $ij$ of $H$.

Some well known graph operations are particular cases of the $H$-join of a family of graphs $\mathcal{G}$ as it is the case of the lexicographic product (also called composition) of two graphs $H$ and $G$, $H[G]$, which coincides with the $H$-join of family of graphs $\mathcal{G}$ where all the graphs in $\mathcal{G}$ are isomorphic to a fixed graph $G$.

So far, the known expressions for the determination of the entire spectrum of the $H$-join in terms of the spectra of its components and an associated matrix are limited to families of regular graphs. In this paper, we extend such a determination to families of arbitrary graphs.

Keywords: $H$-join, lexicographic product, graph spectra.

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1 Introduction

Nearly five decades since the publication in 1974 of Allen Shweenk’s article [17], the determination of the spectrum of the generalized composition $H[G_1, \ldots, G_p]$ (recently designated $H$-join of $\mathcal{G} = \{G_1, \ldots, G_p\}$ [2]), in terms of the spectra of the graphs in $\mathcal{G}$ and an associated matrix, where all graphs are undirected, simple and finite, was limited to families of regular graphs. In this paper, the determination of this spectrum is extended to families of arbitrary graphs (which should be undirected, simple and finite).

The generalized composition $H[G_1, \ldots, G_p]$, introduced in [17] p. 167] was rediscovered in [2] under the designation of $H$-join of a family of graphs $\mathcal{G} = \{G_1, \ldots, G_p\}$, where $H$ is a graph of order $p$. In [17] Th. 7], assuming that $G_1, \ldots, G_p$ are all regular graphs and taking into account that $V(G_1) \cup \cdots \cup V(G_p)$ is an equitable partition $\pi$, the characteristic polynomial of $H[G_1, \ldots, G_p]$ is determined in terms of the characteristic polynomials of the graphs $G_1, \ldots, G_p$ and the matrix associated to $\pi$.

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Using a generalization of a Fiedler’s result [7 Lem. 2.2] obtained in [2 Th. 3], the spectrum of the $H$-join of a family of regular graphs (not necessarily connected) is determined in [2 Th. 5]. When the graphs of the family $\mathcal{G}$ are all isomorphic to a fixed graph $G$, the $H$-join of $\mathcal{G}$ is the same as the lexicographic product (also called the composition) of the graphs $H$ and $G$ which is denoted as $H[G]$ (or $H \circ G$). The lexicographic product of two graphs was introduced by Harary in [11] and Sabidussi in [10] (see also [12, 10]). From the definition, it is immediate that this graph operation is associative but not commutative.

In [1], as an application of the $H$-join spectral properties, the lexicographic powers of a graph $H$ were considered and their spectra determined, when $H$ is regular. The $k$-th lexicographic power of $H$, $H^k$, is the lexicographic product of $H$ by itself $k$ times (then $H^2 = H[H]$, $H^3 = H[H^2] = H^2[H]$, …). As an example, in [1], the spectrum of the 100-th lexicographic power of the Petersen graph, which has a gogool number (that is, $10^{100}$) of vertices, was determined. With these powers, $H^k$, in [3] the lexicographic polynomials were introduced and their spectra determined, for connected regular graphs $H$, in terms of the spectrum of $H$ and the coefficients of the polynomial.

Other particular $H$-joins appear in the literature under different designations, as it is the case of the mixed extension of a graph $H$ studied in [8], where special attention is given to the mixed extensions of $P_3$. The mixed extension of a graph $H$, with vertex set $V(H) = \{1, \ldots, p\}$, is the $H$-join of a family of graphs $\mathcal{G} = \{G_1, \ldots, G_p\}$, where each graph $G_i \in \mathcal{G}$ is a complete graph or its complement. From the $H$-join spectral properties, we may conclude that the mixed extensions of a graph $H$ of order $p$ has at most $p$ eigenvalues unequal to 0 and $-1$.

The remaining part of the paper is organized as follows. The focus of Section 2 is the preliminaries. Namely, the notation and basic definitions, the main spectral results of the $H$-join graph operation and the more relevant properties, in the context of this work, of the main characteristic polynomial and walk matrix of a graph. In section 3 the main result of this article, the determination of the spectrum of the $H$-join of a family of arbitrary graphs is deduced.

## 2 Preliminaries

### 2.1 Notation and basic definitions

Throughout the text we consider undirected, simple and finite graphs, which are just called graphs. The vertex set and the edge set of a graph $G$ is denoted by $V(G)$ and $E(G)$, respectively. The order of $G$ is the cardinality of its vertex set and when it is $n$ we consider that $V(G) = \{1, \ldots, n\}$. The eigenvalues of adjacency matrix of a graph $G$, $A(G)$, of order $n$ are also called the eigenvalues of $G$. For each distinct eigenvalue $\mu$ of $G$, $\mathcal{E}_G(\mu)$ denotes the eigenspace of $\mu$ whose dimension is equal to the algebraic multiplicity of $\mu$, $m(\mu)$. The spectrum of $G$ is denoted $\sigma(G) = \{\mu_1^{[m_1]}, \ldots, \mu_s^{[m_s]}, \mu_{s+1}^{[m_{s+1}]}, \ldots, \mu_t^{[m_t]}\}$, where $t \leq n$ and $\mu_i^{[m_i]}$ means that $m(\mu_i) = m_i$. When we say that $\mu$ is an eigenvalue of $G$ with zero multiplicity (that is, $m(\mu) = 0$) it means that $\mu \notin \sigma(G)$. The distinct eigenvalues of $G$ are indexed in such way that the eigenspaces $\mathcal{E}_G(\mu_i)$, for $1 \leq i \leq s$, are not orthogonal to $j_n$, the all-1 vector with $n$ entries. The eigenvalues $\mu_i$, with $1 \leq i \leq s$ are called main eigenvalues of $G$ and the remaining distinct eigenvalues non-main. The
concept of main (non-main) eigenvalue was introduced in [4] and further investigated in several publications. As it is well known, the largest eigenvalue of a connected graph is main and its remaining distinct eigenvalues are non-main [5]. A survey on main eigenvalues was published in [15].

2.2 The $H$-join operation

Now we recall the definition of the $H$-join of a family of graphs [2].

**Definition 2.1.** Consider a graph $H$ with vertex subset $V(H) = \{1, \ldots, p\}$ and a family of graphs $\mathcal{G} = \{G_1, \ldots, G_p\}$ such that $|V(G_1)| = n_1, \ldots, |V(G_p)| = n_p$. The $H$-join of $\mathcal{G}$ is the graph $G = \bigcup_{H} \mathcal{G}$ in which $V(G) = \bigcup_{j=1}^{p} V(G_j)$ and $E(G) = \left( \bigcup_{j=1}^{p} E(G_j) \right) \cup \left( \bigcup_{rs \in E(H)} E(G_r \lor G_s) \right)$, where $G_r \lor G_s$ denotes the join.

**Theorem 2.2.** Let $G$ be the $H$-join as in Definition 2.1, where $\mathcal{G}$ is a family of regular graphs such that $G_1$ is $d_1$-regular, $G_2$ is $d_2$-regular, \ldots and $G_p$ is $d_p$-regular. Then

$$\sigma(G) = \left( \bigcup_{j=1}^{p} (\sigma(G_j) \setminus \{d_j\}) \right) \cup \sigma(\tilde{C}),$$

(1)

where the matrix $\tilde{C}$ has order $p$ and is such that

$$\left( \tilde{C} \right)_{rs} = \begin{cases} d_r & \text{if } r = s, \\ \sqrt{n_r n_s} & \text{if } rs \in E(H), \\ 0 & \text{otherwise}, \end{cases}$$

(2)

and the set operations in (1) are done considering possible repetitions of elements of the multisets.

From the above theorem, if there is $G_i \in \mathcal{G}$ which is disconnected, with $q$ components, then its regularity $d_i$ appears $q$ times in the multiset $\sigma(G_i)$. Therefore, according to (1), remains as an eigenvalue of $G$ with multiplicity $q - 1$.

From now on, given a graph $H$, we consider the following notation:

$$\delta_{i,j}(H) = \begin{cases} 1 & \text{if } ij \in E(H), \\ 0 & \text{otherwise}. \end{cases}$$

Before the next result, it is worth observe the following. Considering a graph $G$, it is always possible to extend a basis of the eigensubspace associated to a main eigenvalue $\mu_j$, $\mathcal{E}_{G}(\mu_j) \cap j^\top$, to one of $\mathcal{E}_{G}(\mu_j)$ by adding an eigenvector $\hat{u}_{\mu_j}$ which is uniquely determined without considering its multiplication by a nonzero scalar. The eigenvector $\hat{u}_{\mu_j}$ is called the main eigenvector of $\mu_j$. The subspace with basis $\{\hat{u}_{\mu_1}, \ldots, \hat{u}_{\mu_s}\}$ is the main subspace of $G$ and is denoted as $\text{Main}(G)$. Note that for each main eigenvector $\hat{u}_{\mu_j}$ of the basis of $\text{Main}(G)$, $\hat{u}_{\mu_j}^\top j \neq 0$. 

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Lemma 2.3. Let $G$ be the $H$-join as in Definition 2.1 and $\mu_{i,j} \in \sigma(G)$. Then $\mu_{i,j} \in \sigma(G)$ with multiplicity

$$
\begin{cases}
    m(\mu_{i,j}) & \text{whether } \mu_{i,j} \text{ is a non-main eigenvalue of } G, \\
    m(\mu_{i,j}) - 1 & \text{whether } \mu_{i,j} \text{ is a main eigenvalue of } G.
\end{cases}
$$

Proof. Denoting $\delta_{i,j} = \delta_{i,j}(H)$, then $\delta_{i,j}J_{n_1}^i J_{n_2}^j$ is an $n_1 \times n_2$ matrix whose entries are 1 if $ij \in E(H)$ and 0 otherwise. Then the adjacency matrix of $G$ has the form

$$
A(G) = \left( \begin{array}{cccc}
A(G_1) & \delta_{1,2}J_{n_1}^i J_{n_2}^j & \cdots & \delta_{1,p-1}J_{n_1}^i J_{n_p}^j \\
\delta_{2,1}J_{n_2}^j J_{n_1}^i & A(G_2) & \cdots & \delta_{2,p-1}J_{n_2}^j J_{n_p}^j \\
& \vdots & \ddots & \vdots \\
\delta_{p-1,1}J_{n_{p-1}} J_{n_1}^i & \delta_{p,2}J_{n_p} J_{n_2}^j & \cdots & A(G_{p-1}) \\
\delta_{p,1}J_{n_p} J_{n_1}^i & \delta_{p,2}J_{n_p} J_{n_2}^j & \cdots & \delta_{p,1}J_{n_p} J_{n_p}^j \end{array} \right).$
$$

Let $u_{i,j}$ be an eigenvector of $A(G_i)$ associated to an eigenvalue $\mu_{i,j}$ whose sum of its components is zero (then, $\mu_{i,j}$ is non-main or it is main with multiplicity greater than one). Then,

$$
A(G) \begin{pmatrix}
0 \\
\vdots \\
0 \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
0 \\
\vdots \\
0 \\
0 \\
0
\end{pmatrix} + \begin{pmatrix}
\delta_{i,1}j_{n_1}^T (j_{n_1}^T u_{i,j}) j_{n_1} \\
\vdots \\
\delta_{i-1,1}j_{n_1}^T (j_{n_1}^T u_{i,j}) j_{n_1} \\
\delta_{i+1,1}j_{n_1}^T (j_{n_1}^T u_{i,j}) j_{n_1} \\
\vdots \\
\delta_{p,1}j_{n_1}^T (j_{n_1}^T u_{i,j}) j_{n_1}
\end{pmatrix}.
$$

(3)

It should be noted that when $\mu_{i,j}$ is main, there are $m(\mu_{i,j}) - 1$ linear independent eigenvectors belonging to $\mathcal{E}_G(\mu_{i,j}) \cap \mathcal{J}^T$.

2.3 The main characteristic polynomial and the walk matrix

If $G$ has $s$ distinct main eigenvalues $\mu_1, \ldots, \mu_s$, then the main characteristic polynomial of $G$ is the polynomial of degree $s$ [15]

$$
m_G(x) = \prod_{i=1}^s (x - \mu_i) = x^s - c_0 - c_1x - \cdots - c_{s-2}x^{s-2} - c_{s-1}x^{s-1}.
$$

(4)

Note that if $\mu$ is a main eigenvalue of $G$ so is its algebraic conjugate $\mu^*$. Therefore, the coefficients of $m_G(x)$ are integers as referred in [15] (see also [6]).

Let $G$ be a graph. From [15] Prop. 2.1] it is immediate that $m_G(A(G))j = 0$. Therefore,

$$
A^s(G)j = c_0j + c_1A(G)j + \cdots + c_{s-2}A^{s-2}(G)j + c_{s-1}A^{s-1}(G)j.
$$

(5)

Given a graph $G$ of order $n$, let us consider the $n \times k$ matrix [13] [14]

$$
W_{G:k} = \begin{pmatrix} j, A(G)j, A^2(G)j, \ldots, A^{k-1}(G)j \end{pmatrix}.
$$

The vector space spanned by the columns of $W_{G:k}$ is denoted by $ColSpW_{G:k}$.
\textbf{Theorem 2.4.} \cite{15} Let $G$ be a graph of order $n$ with $s$ distinct main eigenvalues. If $k \geq s$, then $W_{G;k}$ has rank $s$.

As an immediate consequence of Theorem 2.4 the number of distinct main eigenvalues is $s = \min \{ k : \{ j, A(G)j, A^2(G)j, \ldots, A^k(G)j \} \}$ is linearly dependent.

For a graph $G$ of order $n$ with $s$ distinct main eigenvalues, the $n \times s$ matrix $W_{G,s} = (j, A(G)j, A^2(G)j, \ldots, A^{s-1}(G)j)$ is referred to be the walk matrix of $G$ and is just denoted as $W_G$.

From \cite{9} we have the following corollary.

\textbf{Corollary 2.5.} The $s$-th column of $A(G)W_G$ is $A^s(G)j = W_G \begin{pmatrix} c_0 \\ \vdots \\ c_{s-2} \\ c_{s-1} \end{pmatrix}$, where $c_j$, for $j = 0, \ldots, s - 1$, are the coefficients of the main characteristic polynomial of $m_G(x)$, given in \cite{10}.

From this corollary we may conclude that the coefficients of the main characteristic polynomial of $G$ can be determined from its walk matrix $W_G$, solving the linear system $W_Gx = A^s(G)j$.

\textbf{Theorem 2.6.} \cite{15} Th. 2.4] Let $G$ be a graph with $s$ distinct main eigenvalues. Then the column space ColSp$W_G$ coincides with Main$(G)$.

Moreover Main$(G)$ and the vector space spanned by the vectors orthogonal to Main$(G)$, (Main$(G))^\perp$, are both $A$-invariant \cite{15} Th. 2.4.

From the above definitions, if $G$ is a $r$-regular graph of order $n$, since its largest eigenvalue, $r$, is the unique main eigenvalue, then $m_G(x) = x - r$ and $W_G = (j_n)$.

3 The spectrum of the $H$-join of a family of arbitrary graphs

Before the main result of this paper we need to define a special matrix $\tilde{W}$ which will be called the $H$-join associated matrix.

\textbf{Definition 3.1.} Let $G$ be the $H$-join as in Definition 2.1. The main eigenvalues of each $G_i \in \mathcal{G}$ are $\mu_{i,1}, \ldots, \mu_{i,s_i}$ and the corresponding main characteristic polynomial \cite{1} is $m_{G_i}(x) = x^s - c_{i,0} - c_{i,1}x - \cdots - c_{i,s_i-1}x^{s_i-1}$. For $1 \leq i \leq p$, let $W_{G_i}$ be the walk matrix of $G_i$ and consider the matrix

\[
\tilde{W} = \begin{pmatrix}
\delta_{i,1}j_n(W_{G_1}) & \cdots & \delta_{i,s_i-1}j_n(W_{G_{s-1}}) & 0 & 0 & 0 & \cdots & 0 & c_{i,0} \\
0 & \cdots & 0 & 1 & \cdots & 0 & c_{i,1} & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 1 & \cdots & 0 & c_{i,2} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & 0 & 1 & c_{i,s_i-1} & 0 & \cdots & 0 \\
\end{pmatrix}
\]
The $H$-join associated matrix is the $s \times s$ matrix, where $s = \sum_{i=1}^{p} s_i$,

$$\bar{W} = \begin{pmatrix} \bar{W}_1 \\ \bar{W}_2 \\ \vdots \\ \bar{W}_p \end{pmatrix}.$$

Observe that the submatrix in $\bar{W}_i$, $C(m_{G_i}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & c_{i,0} \\ 1 & 0 & \cdots & 0 & c_{i,1} \\ 0 & 1 & \cdots & 0 & c_{i,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c_{i,s_i-1} \end{pmatrix}$, is the Frobenius companion matrix of the main characteristic polynomial

$$m_{G_i}(x) = x^{s_i} - c_{i,0} - c_{i,1}x - \cdots - c_{i,s_i-1}x^{s_i-1},$$

whose roots (that is, eigenvalues of $C(m_{G_i})$) are the main eigenvalues of $G_i$.

Defining $M_i = \begin{pmatrix} J_n W_{G_i} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$, a $s_i \times s_i$ submatrix of the $s_i \times s$ matrix $\bar{W}_i$, then

$$\bar{W}_i = ( \delta_{i,1} M_1 \cdots \delta_{i,i-1} M_{i-1} \ C(m_{G_i}) \ \delta_{i,i+1} M_{i+1} \cdots \delta_{i,p} M_p ).$$

Using this notation,

$$\bar{W} = \begin{pmatrix} C(m_{G_1}) & \delta_{1,2} M_2 & \cdots & \delta_{1,p-1} M_{p-1} & \delta_{1,p} M_p \\ \delta_{2,1} M_1 & C(m_{G_2}) & \cdots & \delta_{2,p-1} M_{p-1} & \delta_{2,p} M_p \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta_{p,1} M_1 & \delta_{p,2} M_2 & \cdots & \delta_{p,p-1} M_{p-1} & C(m_{G_p}) \end{pmatrix}.$$

**Theorem 3.2.** Let $G$ be the $H$-join as in Definition 2.1 where $\mathcal{G}$ is a family of arbitrary graphs. If for each of the graphs $G_i$, with $1 \leq i \leq p$,

$$\sigma(G_i) = \{\mu_{i,1}^{(m_{i,1})}, \ldots, \mu_{i,t_i}^{(m_{i,t_i})}, \mu_{i,s_i}^{(m_{i,s_i})}, \ldots, \mu_{i,s_i}^{(m_{i,s_i+1})}, \ldots, \mu_{i,s_i}^{(m_{i,s_i+1})}\},$$

where $t_i \leq n_i$, $m_{i,j} = m(\mu_{i,j})$ and $\mu_{i,1}, \ldots, \mu_{i,s_i}$ are the main eigenvalues of $G_i$, then

$$\sigma(G) = \bigcup_{i=1}^{p} \{\mu_{i,1}^{[m_{i,1}-1]}, \ldots, \mu_{i,t_i}^{[m_{i,t_i}-1]}\} \cup \bigcup_{i=1}^{p} \{\mu_{i,s_i}^{[m_{i,s_i}-1]}, \ldots, \mu_{i,s_i}^{[m_{i,s_i}+1]}\} \cup \sigma(\bar{W}),$$

where the union of multisets is considered with possible repetitions.

**Proof.** From Lemma 2.4, it is immediate that

$$\bigcup_{i=1}^{p} \{\mu_{i,1}^{[m_{i,1}-1]}, \ldots, \mu_{i,t_i}^{[m_{i,t_i}-1]}\} \cup \bigcup_{i=1}^{p} \{\mu_{i,s_i}^{[m_{i,s_i}-1]}, \ldots, \mu_{i,s_i}^{[m_{i,s_i}+1]}\} \subseteq \sigma(G).$$
So it just remains to prove that $\sigma(\tilde{W}) \subseteq \sigma(G)$.

Let us define the vector $\hat{v} = \begin{pmatrix} \hat{v}_1 \\ \vdots \\ \hat{v}_p \end{pmatrix}$ such that

$$\hat{v}_i = \sum_{k=0}^{s_i-1} \alpha_{i,k} A^k(G_i) j_{n_i} = W_{G_i} \begin{pmatrix} \alpha_{i,0} \\ \alpha_{i,1} \\ \vdots \\ \alpha_{i,s_i-1} \end{pmatrix} = W_{G_i} \hat{\alpha}_i, \quad (6)$$

where $\hat{\alpha}_i = \begin{pmatrix} \alpha_{i,0} \\ \alpha_{i,1} \\ \vdots \\ \alpha_{i,s_i-1} \end{pmatrix}$, for $1 \leq i \leq p$. Then each $\hat{v}_i \in \text{Main}(G_i)$ and

$$A(G_i)\hat{v}_i = A(G_i)W_{G_i} \hat{\alpha}_i = \sum_{k=0}^{s_i-1} \alpha_{i,k} A^{k+1}(G_i) j_{n_i}, \quad \text{for } 1 \leq i \leq p. \quad (7)$$

Therefore,

$$A(G)\hat{v} = \begin{pmatrix} A(G_1) & \delta_{1,2} j_{n_1} j_{n_2}^T & \cdots & \delta_{1,p} j_{n_1} j_{n_p}^T \\ \delta_{2,1} j_{n_1} j_{n_2}^T & A(G_2) & \cdots & \delta_{2,p} j_{n_2} j_{n_p}^T \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{p,1} j_{n_p} j_{n_1}^T & \delta_{p,2} j_{n_p} j_{n_2}^T & \cdots & A(G_p) \end{pmatrix} \begin{pmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \vdots \\ \hat{v}_p \end{pmatrix}$$

$$= \begin{pmatrix} A(G_1)\hat{v}_1 + \left( \sum_{q \in [p]\setminus\{1\}} \delta_{1,q} j_{n_q}^T \hat{v}_q \right) j_{n_1} \\ A(G_2)\hat{v}_2 + \left( \sum_{q \in [p]\setminus\{2\}} \delta_{2,q} j_{n_q}^T \hat{v}_q \right) j_{n_2} \\ \vdots \\ A(G_p)\hat{v}_p + \left( \sum_{q \in [p]\setminus\{p\}} \delta_{p,q} j_{n_q}^T \hat{v}_q \right) j_{n_p} \end{pmatrix}, \quad (8)$$

$$= \begin{pmatrix} A(G_1)\hat{v}_1 + \left( \sum_{q \in [p]\setminus\{1\}} \delta_{1,q} j_{n_q}^T W_{G_q} \hat{\alpha}_q \right) j_{n_1} \\ A(G_2)\hat{v}_2 + \left( \sum_{q \in [p]\setminus\{2\}} \delta_{2,q} j_{n_q}^T W_{G_q} \hat{\alpha}_q \right) j_{n_2} \\ \vdots \\ A(G_p)\hat{v}_p + \left( \sum_{q \in [p]\setminus\{p\}} \delta_{p,q} j_{n_q}^T W_{G_q} \hat{\alpha}_q \right) j_{n_p} \end{pmatrix}, \quad (9)$$

where (9) is obtained applying (8) in (8). Defining

$$\beta_{i,0} = \sum_{q \in [p]\setminus\{i\}} \delta_{i,q} j_{n_q}^T \hat{v}_q = \sum_{q \in [p]\setminus\{i\}} \delta_{i,q} j_{n_q}^T W_{G_q} \hat{\alpha}_q, \quad \text{for } 1 \leq i \leq p,$$

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the $i$-th row of (10) can be written as

$$
\beta_{i,0}j_{n_i} + A(G_i)\hat{v}_i = \left(\sum_{k \in [p]\setminus\{i\}} \delta_{i,k} \hat{\beta}_k \hat{W}_{G_k} \hat{\alpha}_k\right) j_{n_i} + \sum_{k=0}^{s_i-1} \alpha_{i,k} A^{k+1}(G_i) j_{n_i}
$$

$$
\beta_{i,0}j_{n_i} + \sum_{k=1}^{s_i-1} \alpha_{i,k-1} A^{k}(G_i) j_{n_i} + \sum_{k=0}^{s_i-1} \alpha_{i,k} A^{k+1}(G_i) j_{n_i}
$$

(10)

$$
\beta_{i,0}j_{n_i} + \sum_{k=1}^{s_i-1} \alpha_{i,k-1} A^{k}(G_i) j_{n_i} + \sum_{k=0}^{s_i-1} \alpha_{i,k} A^{k+1}(G_i) j_{n_i} = \beta_{i,0}j_{n_i} + \sum_{k=1}^{s_i-1} \alpha_{i,k-1} A^{k}(G_i) j_{n_i} + \alpha_{i,s_i-1} A^{s_i}(G_i) j_{n_i}
$$

(11)

$$
\begin{pmatrix}
\beta_{i,0} + \alpha_{i,s_i-1} \hat{c}_{i,0} \\
\alpha_{i,0} + \alpha_{i,s_i-1} \hat{c}_{i,1} \\
\vdots \\
\alpha_{i,2} + \alpha_{i,s_i-1} \hat{c}_{i,s_i-1}
\end{pmatrix}
\begin{pmatrix}
\hat{\alpha}_1 \\
\vdots \\
\hat{\alpha}_i \\
\vdots \\
\hat{\alpha}_p
\end{pmatrix}
$$

(12)

Observe that (11) is obtained applying Corollary 2.5 to (10).

Finally, if $A(G)\hat{v} = \rho \hat{v}$, then $\hat{\alpha}_1, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_p$ can be determined as follows.

$$
A(G)\hat{v} = \begin{pmatrix}
W_{G_1} & 0 & \cdots & 0 \\
0 & W_{G_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & W_{G_p}
\end{pmatrix}\begin{pmatrix}
\hat{\omega}_1 \\
\hat{\omega}_2 \\
\vdots \\
\hat{\omega}_p
\end{pmatrix} = \rho \begin{pmatrix}
\hat{\omega}_1 \\
\hat{\omega}_2 \\
\vdots \\
\hat{\omega}_p
\end{pmatrix}
$$

taking into account (6).
Then we obtain
\[
\begin{pmatrix}
W_{G_1} & 0 & \cdots & 0 \\
0 & W_{G_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & W_{G_p}
\end{pmatrix}
\begin{pmatrix}
\tilde{W}_1 \\
\tilde{W}_2 \\
\vdots \\
\tilde{W}_p
\end{pmatrix}
- \rho I_s
\begin{pmatrix}
\hat{\alpha}_1 \\
\hat{\alpha}_2 \\
\vdots \\
\hat{\alpha}_p
\end{pmatrix}
= 0. \quad (13)
\]

Since the columns of each matrix \( W_{G_i} \) are linear independent, the columns of the matrix 

\((\ast)\) are also linear independent and, consequently, \((13)\) is equivalent to \[
\begin{pmatrix}
\tilde{W}_1 \\
\tilde{W}_2 \\
\vdots \\
\tilde{W}_p
\end{pmatrix}
- \rho I_s
\begin{pmatrix}
\hat{\alpha}_1 \\
\hat{\alpha}_2 \\
\vdots \\
\hat{\alpha}_p
\end{pmatrix}
= 0,
\]

where \( \tilde{W} = \begin{pmatrix}
\tilde{W}_1 \\
\tilde{W}_2 \\
\vdots \\
\tilde{W}_p
\end{pmatrix}. \) Therefore, the eigenvalue \( \rho \) is a root of the characteristic polynomial of the matrix \( \tilde{W}. \)

**Example 3.3.** Consider the graph \( H = P_3 \), the path with three vertices, and the graphs \( K_{1,3}, K_2 \) and \( P_3 \) depicted in the Figure 1.

![Diagram of graphs](image_url)

Figure 1: The \( P_3 \)-join of the family of graphs \( G_1, G_2 \) and \( G_3. \)

The spectra of the graphs \( G_1, G_2 \) and \( G_3 \), depicted in Figure 1, are

\[
\sigma(K_{1,3}) = \{\sqrt{3}, -\sqrt{3}, 0^{[3]}\},
\]

\[
\sigma(K_2) = \{1, -1\},
\]

\[
\sigma(P_3) = \{\sqrt{2}, -\sqrt{2}, 0\},
\]

and their main characteristic polynomials are \( m_{G_1}(x) = x^2 - 3, m_{G_2}(x) = x - 1 \) and \( m_{G_3}(x) = x^2 - 2 \), respectively. Since

\[
\begin{align*}
\tilde{W}_1 &= \begin{pmatrix} 0 & c_{1,0} & \delta_{1,2} & \delta_{1,3} & 0 \\
1 & c_{1,1} & 0 & 0 & 0 \\
\end{pmatrix}, \\
\tilde{W}_2 &= \begin{pmatrix} \delta_{2,1} & \delta_{2,1} & c_{2,0} & \delta_{2,3} & \delta_{2,4} \\
\end{pmatrix}, \\
\tilde{W}_3 &= \begin{pmatrix} \delta_{3,1} & \delta_{3,1} & \delta_{3,2} & c_{3,0} \\
0 & 0 & 0 & 1 & c_{3,1} \\
\end{pmatrix}.
\end{align*}
\]
it follows that
\[
\tilde{\mathbf{W}} = \begin{pmatrix}
\tilde{\mathbf{W}}_1 \\
\tilde{\mathbf{W}}_2 \\
\tilde{\mathbf{W}}_3
\end{pmatrix} = \begin{pmatrix}
0 & 3 & 2 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
4 & 6 & 1 & 3 & 4 \\
0 & 0 & 2 & 0 & 2 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

and thus the characteristic polynomial of \( \tilde{\mathbf{W}} \) is the polynomial
\[
p_{\tilde{\mathbf{W}}}(x) = -42 - 40x + 15x^2 + 19x^3 + x^4 - x^5.
\]

Therefore, applying Theorem 3.2, the characteristic polynomial of \( G \) is
\[
p_G(x) = x^3(x + 1)p_{\tilde{\mathbf{W}}}(x) = x^3(x + 1)(-42 - 40x + 15x^2 + 19x^3 + x^4 - x^5).
\]

When all graphs of the family \( \mathcal{G} \) are regular, that is, \( G_1 \) is \( d_1 \)-regular, \( G_2 \) is \( d_2 \)-regular, \ldots, \( G_p \) is \( d_p \)-regular, the walk matrices are \( W_{G_1} = (j_{n_1}) \), \( W_{G_2} = (j_{n_2}) \), \ldots, \( W_{G_p} = (j_{n_p}) \), respectively. Consequently, the main polynomials are \( m_{G_1}(x) = x - d_1 \), \( m_{G_2}(x) = x - d_2 \), \ldots, \( m_{G_p}(x) = x - d_p \). As direct consequence, for this particular case, the \( H \)-join associated matrix is
\[
\tilde{\mathbf{W}} = \begin{pmatrix}
d_1 & \delta_{1,2} j_{n_2}^T W_{G_2} & \cdots & \delta_{1,p} j_{n_p}^T W_{G_p} \\
\delta_{2,1} j_{n_1}^T W_{G_1} & d_2 & \cdots & \delta_{2,p} j_{n_p}^T W_{G_p} \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{p,1} j_{n_1}^T W_{G_1} & \delta_{p,2} j_{n_2}^T W_{G_2} & \cdots & d_p
\end{pmatrix} = \begin{pmatrix}
d_1 & \delta_{1,2} n_2 & \cdots & \delta_{1,p} n_p \\
\delta_{2,1} n_1 & d_2 & \cdots & \delta_{2,p} n_p \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{p,1} n_1 & \delta_{p,2} n_2 & \cdots & d_p
\end{pmatrix}.
\]

Therefore, it is immediate that when all the graphs of the family \( \mathcal{G} \) are regular, the matrix \( \tilde{\mathbf{W}} \) and the matrix \( \tilde{\mathbf{C}} \) in (2) are similar matrices. Note that \( \tilde{\mathbf{C}} = D \tilde{\mathbf{W}} D^{-1} \), where \( D = \text{diag}(\sqrt{n_1}, \sqrt{n_2}, \ldots, \sqrt{n_p}) \) and thus \( \tilde{\mathbf{W}} \) and \( \tilde{\mathbf{C}} \) are cospectral matrices as it should be.

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