GREEN POLYNOMIALS OF WEYL GROUPS, ELLIPTIC PAIRINGS, AND THE EXTENDED DIRAC INDEX

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Abstract. In this paper, we give a uniform construction of irreducible genuine characters of the Pin cover $\tilde{W}$ of a Weyl group $W$, and put them into the context of theory of Springer representations. In the process, we provide a direct connection between Springer theory, via Green polynomials, the irreducible representations of $W$, and an extended Dirac operator for graded Hecke algebras. We also introduce a $q$-elliptic pairing for $W$ with respect to the reflection representation $V$. These constructions are of independent interest. The $q$-elliptic pairing is a generalization of the elliptic pairing of $W$ introduced by Reeder, and it is also related to S. Kato’s notion of (graded) Kostka systems for the semidirect product $A_W = \mathbb{C}[W] \rtimes S(V)$.

1. Introduction

1.1. Graded affine Hecke algebras were defined by Lusztig [22] in his study of representations of reductive $p$-adic groups and Iwahori-Hecke algebras. A Dirac operator $D$ for graded affine Hecke algebras was defined in [3], and, by analogy with the setting of Dirac theory for $(\mathfrak{g}, K)$-modules of real reductive groups, the notion of Dirac cohomology was introduced. The Dirac cohomology and the Dirac index in the Hecke algebra setting were further studied in [10, 11]. The Dirac cohomology spaces are representations for a certain double cover (“pin cover”) $	ilde{W}$ of the Weyl group $W$. The irreducible representations of $	ilde{W}$ had been classified case by case in the work of Schur, Morris, Read and others, see for example [26, 28, 40]. Recently, it was noticed in [9] (again case by case) that there is a close relation between the representation theory of $	ilde{W}$ and the geometry of the nilpotent cone in semisimple Lie algebras $\mathfrak{g}$.

1.2. In this paper, we provide a direct link between:

(a) an (extended) Springer $W$-action on cohomology groups;
(b) the irreducible representations of $\tilde{W}$, and
(c) an (extended) Dirac index for tempered modules of graded Hecke algebras.

Here in (a) and (c), there is a canonical $\mathbb{Z}/2\mathbb{Z}$-action, induced from the conjugation of the longest Weyl group element $w_0$ on $W$. The $\mathbb{Z}/2\mathbb{Z}$-action looks irrelevant to the $\mathbb{Z}/2\mathbb{Z}$-covering $\tilde{W} \to W$. However, the remarkable thing is that one gets

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many more representation of $\tilde{W}$ with nontrivial extended Dirac index by using the
 twist given by such $\mathbb{Z}/2\mathbb{Z}$-action. One of the main result (see Theorem 1.2 below)
 shows that all the irreducible genuine $\tilde{W}$-representation (i.e., those do not factor
 through $W$) appears in this way. This leads to a new, uniform construction of the
 irreducible genuine $\tilde{W}$-characters.

1.3. To describe the role of this $\mathbb{Z}/2\mathbb{Z}$-action, let us first make a short digression
 on elliptic representation theory.

Elliptic representation theory, introduced by Arthur [1], studies the Grothendieck
 group of certain representations of a Lie-theoretic group modulo those induced
 from proper parabolic subgroups. The elliptic theory of representations of semisim-ple $p$-adic groups and Iwahori-Hecke algebras was further studied intensively, e.g.,
 Schneider-Stuhler [32], Bezrukavnikov [6], Reeder [29], Opdam-Solleveld [27]. In
 [29], Reeder also described the elliptic representations of $W$ in terms of Springer
 correspondence. He proved an orthogonality result of Springer representations and
 proved that it is exactly the quasidistinguished nilpotent elements (Definition 3.1)
 that appears in the elliptic representations of $W$ via Springer correspondence.

However, as remarked in [9], to study the representations of $\tilde{W}$, one needs the
 nilpotent elements with solvable connected centralizer. These elements form a
 subset $N_{\text{sol}}$ of the set $N$ of nilpotent elements in the semisimple Lie algebra $\mathfrak{g}$. In
 general, $N_{\text{sol}}$ is strictly larger than the set of quasidistinguished nilpotent elements.
 This is one of the obstacles when trying to relate the representation of $\tilde{W}$ to the
 Springer theory, or the elliptic representation theory.

1.4. In Proposition 3.3, we give a characterization of nilpotent elements with solv-
able connected centralizer. We show that an element $e \in N_{\text{sol}}$ if and if is $\delta$-
quasidistinguished, in the sense of Definition 3.2. Here $\delta$ is the diagram automor-
phism corresponding to $-w_0$.

This suggests that one needs to study the $\delta$-twisted version of elliptic represen-
tation theory of $W$. To do this in a systematic way, we introduce the $q$-elliptic
 pairing for $W$. It is the usual pairing on $W$-modules twisted by tensoring with a
 $q$-analogue of the reflection representation of $W$, see section 2.1).

For $q = 1$, one gets the elliptic pairing of $W$ introduced in [29], see also [27, 11,
 10]. The specialization of the pairing at $q = -1$ is the $\delta$-twisted version of elliptic
 representation of $W$ we mentioned above. It plays an important role in the study
 of representations of $\tilde{W}$.

The $q$-elliptic pairing is also related to Kato’s notion of (graded) Kostka systems
 [19] for the semidirect product $A_W = \mathbb{C}[W] \ltimes S(V)$ and the graded Euler-Poincaré
 pairing. The occurrence of the exterior powers of the reflection representation in the
 Lusztig-Shoji algorithm was also investigated in [35].

1.5. Now we describe explicitly the $q$-elliptic pairing on the Springer representa-
tions.

Let $G$ be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$ and Weyl group
 $W$. For every element $e \in N$, let $A(e) = Z_G(e)/Z_G(e)^0$ be the component group
 of its centralizer, $\widehat{A(e)}$ be the set of $A(e)$-representations of Springer type and $B_e$
 the Springer fiber over $e$. For $\phi \in \widehat{A(e)}_0$, we define

$$X_q(e, \phi) = \sum_{i \geq 0} q^{\dim(B_e) - i} \text{Hom}_{A(e)}[\phi, H^2(B_e) \otimes \text{sgn}].$$
This is a $q$-graded representations of $W$, defined using the Springer action [36]. By analyzing the Lusztig-Shoji algorithm [21, 33], we prove first:

**Theorem 1.1.** Let $e, e' \in \mathcal{N}$.

1. If $G \cdot e \neq G \cdot e'$, then $X_q(e, \phi)$ and $X_q(e', \phi')$ are orthogonal with respect to the $q$-elliptic pairing on $W$, for all $\phi \in A(e)_0$, $\phi' \in A(e')_0$.
2. The map $X_q(e, \phi) \to \phi$ is an isometry with respect to the $q$-elliptic pairing of $W$, and a certain $(q, M)$-elliptic pairing of $A(e)$, where $M$ is the $q$-graded $A(e)$-representation defined in (2.8.1). When $e \in \mathcal{N}^{\text{sol}}$, $M$ is the $q$-graded exterior representation of the natural representation of $A(e)$ on the space of complex characters of the central torus in $Z_G(e)^0$.

The case $q = 1$ was known before from [29], where it was obtained by different methods.

1.6. We explain next the main results concerning $\tilde{W}$-representations that we obtain via this approach.

Let $R(\tilde{W})$ be the (complexification of the) Grothendieck group of finite dimensional $\tilde{W}$-representations, and $R(\tilde{W})_{\text{gen}}$ the subspace spanned by the set $\text{Irr}_{\text{gen}} \tilde{W}$ of genuine irreducible $\tilde{W}$-representations. Let $V$ be the reflection representation of $W$. Let $C(V)$ be the Clifford algebra of $V$ with respect to a $W$-invariant inner product $(\cdot, \cdot)$, and let $S$ be the unique simple spin $C(V)$-module (when dim $V$ is even), respectively the sum of the two simple spin $C(V)$-modules (when dim $V$ is odd). For every $e \in \mathcal{N}$ and every $\phi \in A(e)_0$, set

$$\tilde{\Sigma}(e, \phi) = X_{-1}(e, \phi) \otimes S,$$

(1.6.1)

Let $\text{sgn}$ denote the pull-back to $\tilde{W}$ of the sign character of $W$. It is clear from the definition that $\tilde{\Sigma}(e, \phi)$ is a (virtual) character in $R(\tilde{W})_{\text{gen}}$, self dual under tensoring with $\text{sgn}$. The main results of section 6 may be summarized as follows.

**Theorem 1.2.**

1. Let $e \in \mathcal{N}$ and $\phi \in A(e)_0$. The character $\tilde{\Sigma}(e, \phi) \neq 0$ if and only if $e \in \mathcal{N}^{\text{sol}}$.

   In this case, $\tilde{\Sigma}(e, \phi)$ is the character of a genuine $\tilde{W}$-representation.

2. Every genuine irreducible $\tilde{W}$-character $\tilde{\sigma}$ occurs in a $\tilde{\Sigma}(e, \phi)$, $e \in \mathcal{N}^{\text{sol}}$.

   Moreover, the $G$-orbit of $e$ is uniquely determined by $\tilde{\sigma}$.

3. If $e \in \mathcal{N}^{\text{sol}}$, then for all $\phi, \phi' \in A(e)_0$,

   $$\langle \tilde{\Sigma}(e, \phi), \tilde{\Sigma}(e, \phi') \rangle_{\tilde{W}} = a_V \langle \phi, \phi' \rangle_{A(e)}^{-1}.$$

   Here $a_V = 1$, if dim $V$ is even, and $a_V = 2$, if dim $V$ is odd.

4. Suppose a genuine irreducible $\tilde{W}$-character $\tilde{\sigma}$ occurs in a $\tilde{\Sigma}(e, \phi)$, $e \in \mathcal{N}^{\text{sol}}$.

   Then the scalar $\tilde{\sigma}(\Omega_W)$ by which the central element $\Omega_W$ (Definition 5.7.3) acts in $\tilde{\sigma}$ equals $(h, h)$, where $h$ is a central element for a Lie triple of $e$.

1.7. Let $e \in \mathcal{N}^{\text{sol}}$ and $R(A(e))$ be the Grothendieck group of finite dimensional representations of $A(e)$. The $q$-elliptic pairing on $R(A(e))$ is degenerate when $q$ is specialized to $-1$. Let $\overline{\mathcal{N}}_{-1}(A(e))$ be the quotient of $R(A(e))$ by the radical of this form. Then the character $\tilde{\Sigma}(e, \phi)$ depends only on the image of $\phi$ in $\overline{\mathcal{N}}_{-1}(A(e))$. 
We calculate the structure of the spaces $R_{-1}(A(e))$, $e \in N^{\text{sol}}$, and we refine part (3) of Theorem 1.2 in Appendix A. For every $e \in N^{\text{sol}}$, we construct an orthogonal basis $\{[\chi_1], \ldots, [\chi_k]\}$ of $R_{-1}(A(e))$. We show that
\[
\tilde{\tau}(e, [\chi_j]) = \frac{1}{a_e} \tilde{\Sigma}(e, [\chi_j]), \quad \text{here } a_e \text{ is a certain power of } 2, \quad (1.7.1)
\]
is either an irreducible (sgn self dual) $\tilde{W}$-character or the sum of two (sgn dual to each other) irreducible $\tilde{W}$-characters, with two interesting exceptions: one family of nilpotent orbits in type $D_n$ and one orbit in $E_7$ (see Appendix A, particularly Remark A.7).

1.8. We comment on the relation with the results in [9]. In [9, Theorem 1.0.1], one constructed a surjection
\[
\Psi : \text{Irr}_{\text{gen}} \tilde{W} \twoheadrightarrow G \backslash N^{\text{sol}}
\]
with certain properties: given $e \in N^{\text{sol}}$, if $\tilde{\sigma} \in \Psi^{-1}(G \cdot e)$, then
(i) $\tilde{\sigma}(\Omega_{\tilde{W}}) = (h, h)$, where $h$ is a neutral element for $e$;
(ii) $\tilde{\sigma}$ occurs in the tensor product $\sigma \otimes S$ for a spin module $S$ and an irreducible Springer representation $\sigma$ corresponding to $e$.

The proofs in [9] are case-by-case for every simple $G$, using the combinatorics of Springer representations and of the tensor product decompositions, as well as the explicit character tables for $\tilde{W}$-representations as computed by Morris [26], Read [28], and Stembridge [40].

On the other hand, the proof of Theorem 1.2 is uniform and independent of the results of [9], and in particular, gives a uniform interpretation of [9, Theorem 1.0.1]. Moreover, together with other results in section 6 (e.g., Theorem 6.6, see also Appendix A), Theorem 1.2 gives a precise description of the fibers $\Psi^{-1}(G \cdot e)$ and it makes a natural connection between these fibers and the ($-1$)-elliptic space for the component group $A(e)$.

We should emphasize that our approach is also independent of the previous known classifications, e.g., [26, 28, 40]. Our proof uses Theorem 1.1 with $q = -1$, the extended Springer action of $W_{\#} = W \rtimes \langle \delta \rangle$ on the cohomology spaces of Springer fibers, and the relation with the extended Dirac index from section 5. Constructions of the Springer action of $W_{\#}$ are known from Baranovsky-Evens-Ginzburg [2] and Bezrukavnikov-Mirković [7]. In Appendix B, we also give our own explicit construction based on the Lusztig-Shoji algorithm.

1.9. We mention two applications of our results.

By comparing Theorem 1.2 and Appendix A with [9], the characters $\tilde{\tau}(e, [\phi])$ can be easily identified in terms of the previous known classifications. From this point of view, (1.6.1) can immediately be interpreted to give a character formula of $X_{-1}(e, \phi)$ on the set $W_{(1)}$-ell of ($-1$)-elliptic Weyl group elements. In this way, one obtains an extension of [11, Theorem 1.1]. See section 6.4 for details.

Theorem 1.2 can also be used to give a solution in terms of Kostka-type numbers to the problem of decomposing tensor products $\sigma \otimes S$, $\sigma \in W$. See (6.5.3) and Corollary 6.10.
1.10. We conclude the introduction by giving a brief summary of the structure of the paper.

In section 2, we recall certain elements of the Lusztig-Shoji algorithm, and prove Theorem 1.1.

In section 3, we study the nilpotent elements with solvable connected centralizer.

In section 4, we relate the \((-1)\)-elliptic pairing with a \(\delta\)-twisted elliptic pairing, and consider the corresponding spaces of virtual elliptic characters.

In section 5, we introduce the extended Dirac operator for the extended graded Hecke algebras \(H\#\), and define its index. Using Lusztig’s geometric realization of irreducible \(H\)-modules ([23, 24]), we compute the index of tempered modules in Theorem 5.11.

In section 6, we prove the results about \(\widetilde{W}\)-representations, in particular, Theorem 1.2.

In Appendix A, we compute explicitly the spaces \(R_{-1}(A(e))\) and the associated spin representations \(\widetilde{\tau}(e,\chi)\).

In Appendix B, we define an action of \(W\#\) on the cohomology groups \(H^j(B_e)\), extending the Springer action.

In Appendix C, we present a relation between \(q\)-elliptic pairings of \(W\) and the Kostka systems of [19].

2. The Lusztig-Shoji algorithm and the \(q\)-elliptic pairing

2.1. If \(\Gamma\) is a finite group, let \(R(\Gamma)\) denote the Grothendieck group of finite dimensional \(\mathbb{C}[\Gamma]\)-modules. Let \(\langle \ , \ \rangle_{\Gamma}\) be the character pairing of \(\Gamma\). If \(q\) is an indeterminate, set \(R_q(\Gamma) = R(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Z}[q]\) and extend \(\langle \ , \ \rangle_{\Gamma}\) \(\mathbb{Z}[q]\)-linearly to \(R_q(\Gamma)\).

Let \(U\) be a finite dimensional \(\mathbb{C}\)-representation of \(\Gamma\) and \(\bigwedge U\) the \(i\)-th exterior power of \(U\) viewed as a \(\Gamma\)-representation. Denote \(\bigwedge_{-q} U = \sum_{i \geq 0} (-q)^i \bigwedge^i U \in R_q(\Gamma)\).

Define the \(q\)-elliptic product in \(R_q(\Gamma)\) to be:

\[
\langle \chi, \chi' \rangle_q^{\Gamma} := \langle \chi \otimes \bigwedge_{-q} U, \chi' \rangle_{\Gamma} \in \mathbb{Z}[q].
\]

The case of interest for us will be when \(\Gamma\) is a Weyl group \(W\) acting on the reflection representation \(U = V\).

2.2. Let \(G\) be a connected semisimple algebraic group over \(\mathbb{C}\) with Lie algebra \(\mathfrak{g}\). Let \(N \subset \mathfrak{g}\) be the nilpotent cone. Let \(e \in N\) be given, and denote \(O_e\), the nilpotent orbit of \(e\). We set \(A(e) = Z_G(e)/Z_G(e)^0\). Denote \(B_e\) the variety of Borel subalgebras of \(\mathfrak{g}\) containing \(e\) and let \(d_e\) be its dimension. Springer [36] defined an action of \(W\) on the cohomology groups (with rational coefficients) \(H^j(B_e)\). This action commutes with the natural \(A(e)\)-action. Moreover, it is known that \(H^j(B_e) = 0\) unless \(j\) is even ([4, 13, 33]).

For any \(\phi \in A(e)\), set \(H^j(B_e)^\phi = \text{Hom}_{A(e)}[\phi, H^j(B_e)]\). This is a \(W\)-module since the action of \(W\) and the action of \(A(e)\) on \(H^j(B_e)\) commute. Set

\[
\widetilde{A}(e)_0 = \{ \phi \in \widetilde{A}(e) : H^{2d_e}(B_e)^\phi \neq 0 \},
\]

the set of \(A(e)\)-representations of Springer type. For every pair \((e, \phi), \phi \in \widetilde{A}(e)_0\), let \(\sigma(e, \phi) \in \widetilde{W}\) denote the irreducible Springer representation afforded by \(H^{2d_e}(B_e)^\phi\).
For later use, encode the Springer correspondence as the bijective map:
\[
\Psi : G\backslash \{(e, \phi) : e \in N, \phi \in \overline{A(e)_{0}}\} \to \overline{W}, \quad \Psi((e, \phi)) = \sigma(e, \phi).
\] (2.2.2)

Define
\[
X_q(e) = \sum_{i \geq 0} q^{d_i-1} H^{2i}(B_e) \otimes \text{sgn} \in R_q(W),
\]
\[
X_q(e, \phi) = \text{Hom}_{A(e)}[\phi, X_q(e)] \in R_q(W).
\] (2.2.3)

Thus \(\sigma(e, \phi)\) occurs in degree 0 in \(X_q(e, \phi)\).

We may identify \(X_q(0)\) with the graded representation of \(W\) on the space of coinvariants of \(W\) in \(S(V)\). Moreover with this interpretation of \(X_q(0)\), it is known that (see, e.g., [8, Prop. 11.1.1])
\[
X_q(0) \otimes \bigwedge^{-q} V = p(q)\text{triv}, \quad \text{where } p(q) = \prod q^{d_i};
\] (2.2.4)

here \(d_i\) are the fundamental degrees of \(W\).

Notice from the start that our normalization of \(X_q(e, \phi)\) is different than the usual one, in the sense that \(\sigma(e, \phi)\) has degree 0 in \(q\) rather than top degree.

Define \(R_q(W)^e\) to be the subspace of \(R_q(W)\) spanned by \(\{X_q(e, \phi) : \phi \in \overline{A(e)_{0}}\}\).

2.3. Let \(\Omega(q)\) be the symmetric matrix of fake degrees. More precisely, the \((e, \phi), (e', \phi')\) entry in \(\Omega(q)\) is the graded multiplicity of \(\sigma(e, \phi) \otimes \sigma(e', \phi')\) in \(X_q(0)\).

Fix a set \(\{e\}\) of representatives of \(G\)-orbits in \(N\) and for every such \(e\), a set \(\{\phi\}\) of representations in \(\overline{A(e)_{0}}\). Fix an order \(\leq\) on \(\{e\}\) such that if \(e \leq e'\) then \(d_e \leq d_{e'}\). Refine this to an order on \(\{(e, \phi)\}\) by fixing an ordering of \(\phi\)'s for a fixed \(e\).

The Lusztig-Shoji algorithm [21, 33] gives solutions \(K(q), \Lambda(q)\) to the matrix equation
\[
K(q)\Lambda(q)K(q)^t = \Omega(q),
\] (2.3.1)

where the matrices \(K(q), \Lambda(q)\) are square matrices with entries in \(Z[q]\) indexed by pairs \(((e, \phi), (e', \phi'))\) in the ordering chosen above, such that:

1. \(\Lambda(q)\) is a block diagonal matrix with blocks indexed by \(e\);
2. \(K(q)\) is an upper uni-triangular matrix. The \((e, \phi), (e', \phi')\) entry is given by the graded multiplicity of \(\sigma(e, \phi)\) in \(X_q(e', \phi')\).

Let \(M(q)\) be the symmetric matrix whose entries are \(\langle X_q(e, \phi), X_q(e', \phi')\rangle_q\). Using a linear algebra calculation, we relate first \(M(q)\) and \(\Lambda(q)\).

**Theorem 2.1.**
\[
\Lambda(q)M(q) = p(q)\text{id}.
\]

**Proof.** We compute \(\Lambda(q)\) from (2.3.1). We first calculate \(\Omega(q)^{-1}\). Using (2.2.4), we have \(X_q(0) \otimes \sigma \otimes \bigwedge^{-q} V = p(q)\sigma\), for every irreducible \(W\)-representation \(\sigma\), and then:
\[
X_q(0) \otimes \sigma \otimes \bigwedge^{-q} V = \sum_{\sigma'} \langle \sigma', X_q(0) \otimes \sigma \otimes \bigwedge^{-q} V\rangle_W \sigma'
\]
\[
= \sum_{\sigma'} \sum_{\sigma_1, \sigma_2} \langle \sigma_1, X_q(0) \otimes \sigma \rangle_W \langle \sigma_2, \bigwedge^{-q} V \rangle_W \langle \sigma', \sigma_1 \otimes \sigma_2 \rangle_W \sigma'
\]
\[
= \sum_{\sigma'} \sum_{\sigma_1, \sigma_2} \langle \sigma_1 \otimes \sigma^*, X_q(0) \rangle_W \langle \sigma_2, \bigwedge^{-q} V \rangle_W \langle \sigma', \sigma_1 \otimes \sigma_2 \rangle_W \sigma'.
\]
Thus
\[ \sum_{\sigma, \sigma'} \langle \sigma \otimes \sigma', X_q(0) \rangle_W \langle \sigma, \sigma' \rangle_W = \delta_{\sigma, \sigma'} p(q), \]
and therefore
\[ \Omega(q)^{-1}_{\sigma_1, \sigma_2} = p(q) \sum_{\sigma_3} \langle \sigma_3, -q \rangle_W \langle \sigma_2, \sigma_1 \otimes \sigma_3 \rangle_W. \tag{2.3.2} \]

(In the calculations above, \( \sigma' \) and \( \sigma_i \), \( i = 1, 2, 3 \), vary over \( \hat{W} \).)

Next, we compute \( p(q) K(q)^t \Omega(q)^{-1} K(q) \). The \((e, \phi), (e', \phi')\) entry of this matrix equals:
\[
p(q) \sum_{\sigma_1, \sigma_2} K(q)^t \Omega(q)^{-1}_{\sigma_1, \sigma_2} K(q)^{-1}_{\sigma_2, (\sigma_1, \sigma_2, (e', \phi')) \Omega(q)^{-1}_{(e'), \sigma_3} K(q)^{-1}_{(e'), \sigma_2} \Omega(q)^{-1}_{\sigma_3, (e', \phi')}
= p(q) \sum_{\sigma_1, \sigma_2} \langle \sigma_1, X_q(e, \phi) \rangle_W \langle \sigma_2, \sigma_1 \otimes \sigma_3 \rangle_W \langle \sigma_2, X_q(e', \phi') \rangle_W
= \sum_{\sigma_2} \sum_{\sigma_1, \sigma_3} \langle \sigma_1, X_q(e, \phi) \rangle_W \langle \sigma_3, -q \rangle_W \langle \sigma_2, \sigma_1 \otimes \sigma_3 \rangle_W \langle \sigma_2, X_q(e', \phi') \rangle_W
= \sum_{\sigma_2} \langle \sigma_2, X_q(e, \phi) \otimes -q \rangle_W \langle \sigma_2, X_q(e', \phi') \rangle_W
= \langle X_q(e, \phi) \otimes -q, X_q(e', \phi') \rangle_W = \langle X_q(e, \phi), X_q(e', \phi') \rangle_W^q.
\]
In other words, \( p(q) K(q)^t \Omega(q)^{-1} K(q) = M(q) \), and the conclusion follows from (2.3.1).

**Corollary 2.2.** If \( e \neq e' \), i.e., they are representatives of distinct \( G \)-orbits in \( N \), then
\[ \langle X_q(e, \phi), X_q(e', \phi') \rangle_W^q = 0, \]
for all \( \phi, \phi' \). Therefore, the subspaces \( R_q(W)^e \) and \( R_q(W)^{e'} \) are orthogonal with respect to \( \langle , \rangle_W^q \).

**Proof.** This is immediate from Theorem 2.1, since \( \Lambda(q) \) is block-diagonal.

**Example 2.3.** Suppose \( G = SL(3) \), so \( W = S_3 \) acting on a two dimensional reflection space \( V \). The nilpotent orbits are parameterized by partitions of 3, via the Jordan form, and so are the irreducible \( S_3 \)-representations, via Young diagrams. Since all component group representations of Springer type are trivial, we drop them from notation. We have: \( X_q(3) = (1^3), X_q(21) = (21) + q(1^3) \), and \( X_q(1^3) = (3) + q(21) + q^2(21) + q^3(1^3) \). Then we find
\[ M(q) = \text{diag}(1, 1 - q, (1 - q^2)(1 - q^3)). \]

**Example 2.4.** Suppose \( G = Sp(4) \). The nilpotent orbits are parameterized via the (analogue of) Jordan form by the partitions of 4: \((4), (22), (211)\) and \((1^4)\). The irreducible Weyl group representations are parameterized by bipartitions of 4, with \( 2 \times 0 \) denoting the trivial representation and \( 0 \times 11 \) the sign representation. We
have five Green polynomials: $X_q(4) = (0 \times 11), X_q(22), \text{triv} = (1 \times 1) + q(0 \times 11)$, $X_q(22), \text{sgn} = (11 \times 0), X_q(211)) = (0 \times 2) + q(1 \times 1) + q^2(0 \times 11)$, and

$$X_q(1^4) = (2 \times 0) + q(1 \times 1) + q^2(11 \times 0 + 0 \times 2) + q^3(1 \times 1) + q^4(0 \times 11).$$

Then we find

$$M(q) = \text{diag} \left( 1, \left( \begin{array}{cc} 1 & -q \\ -q & 1 \end{array} \right), 1 - q^2, (1 - q^2)(1 - q^4) \right).$$

2.4. To study $M(q)$ in more detail, we need to rely on the geometric interpretation of the matrix $A(q)$. Let $F$ be a finite field of size $q$. Regard $G$ as a group over $\mathbb{F}$ split over $\mathbb{F}$. Let $F : G \to G$ be the corresponding Frobenius map and $G^F = G(F)$ be the finite group of Lie type.

We assume furthermore that the characteristic of $F$ is sufficiently large. For our purposes, this could be taken to mean that the characteristic of $F$ is greater than $3(h - 1)$, where $h$ is the Coxeter number. Under this assumption, there is a bijection between nilpotent orbits in $g$ and unipotent classes in $G$. Moreover, the classification of nilpotent orbits is the same as in characteristic 0 and the type of the centralizers of nilpotent elements, as well as their component groups are the same as in characteristic 0. See [8, sections 5.5-5.11] for details.

Specialize $q$ from the previous section to $q$. Let $e \in N^F$ be given. Then $F$ acts trivially on $A(e)$ and there is a one-to-one correspondence between $G^F$-orbits in $O^F_e$ and conjugacy classes in $A(e)$.

The matrix $A(q)$ is block-diagonal, with one block of size $|A(e)_0|$ for each $e$. To define it precisely, we need more notation. For every conjugacy class $c$ of $A(e)$, denote by $O^F_e(c)$ the corresponding $G^F$-orbit in $O^F_e$. We fix an element $e_c \in O^F_e(c)$ for each conjugacy class $c$ of $A(e)$. For $c = \{1\}$, we may choose $e_c = e$. We choose an element $g_c \in G$ such that $g_c \cdot e = e_c$ and set $x_c = g_c^{-1}F(g_c) \in Z_G(e)$. Then the image $x_c$ of $x_c$ in $A(e)$ is contained in $c$.

For every $\phi \in A(e)$, define the $G^F$-class function $f^e_\phi : N^F \to \mathbb{Q}$, by

$$f^e_\phi(x) = \begin{cases} \text{tr} \phi(c), & \text{if } x \in O^F_e(c), \\ 0, & \text{if } x \notin O^F_e. \end{cases} \quad (2.4.1)$$

Define a bilinear form on $\mathbb{Q}$-valued functions on $N^F$, by

$$(f, f') = \sum_{x \in N^F} f(x)f'(x). \quad (2.4.2)$$

Define the matrix $\tilde{\Lambda}$ whose $(e, \phi), (e', \phi')$ entry is

$$(f^e_\phi, f^{e'}_{\phi'}) = \delta_{e,e'} \sum_c |O^F_e(c)| \text{tr} \phi(c) \text{tr} \phi'(c). \quad (2.4.3)$$

For every $(e, \phi)$, define the function $g^e_\phi : N^F \to \mathbb{Q}$, constant on $G^F$-orbits, by

$$g^e_\phi(x) = \frac{1}{|G^F||A(e)|} \begin{cases} \text{tr} \phi(c)||Z_G(e)||G^F|, & \text{if } x \in O^F_e(c), \\ 0, & \text{if } x \notin O^F_e. \end{cases} \quad (2.4.4)$$

It is immediate that

$$(f^e_\phi, g^{e'}_{\phi'}) = \delta_{e,e'} \delta_{\phi,\phi'}.$$

$$\quad(2.4.5)$$
in other words, \( \{ g_\mu^\circ \} \) is the basis dual to \( \{ f_\mu^\circ \} \). This means that the inverse matrix \( \widetilde{\Lambda}^{-1} \) has entries
\[
(g_\mu^\circ, g_{\nu}^\circ) = \frac{\delta_{\mu,\nu'}}{|A(e)|^2 |G^F|} \sum_c |Z_G(e_c)\phi| \text{tr} \phi(c) \text{tr} \phi'(c) |c|^2. \tag{2.4.6}
\]

We can regard \( \widetilde{\Lambda} \) as a matrix with entries which are polynomials in \( q \). To use this dependence, define \( \Lambda(q) \) as the matrix with polynomial entries in \( q \) such that \( \Lambda(q) = \widetilde{\Lambda} \). The relation between \( \Lambda(q) \) and \( \Lambda(q) \) in (2.3.1) is given by Lusztig [21, (24.2.7)] and Shoji [33, section 4]. Notice that both \( \Lambda(q) \) and \( \Lambda(q) \) are block-diagonal matrices with blocks indexed by the nilpotent representatives \( e \). Denote by \( \Lambda(q)_e \) and \( \Lambda(q)_e \), the blocks corresponding to \( e \). Since we need to account here for our normalization of the matrix \( K(q) \) in section 2.1, the relation is
\[
\Lambda(q)^{-1} = q^{\dim Z_G(e) - \dim G} A(q)_e. \tag{2.4.7}
\]

2.5. To compute \( M(q) \) further, in light of Theorem 2.1, we need to consider \( \Lambda(q)^{-1} = q^{\dim Z_G(e) - \dim G} (\Lambda^{-1})(q^{-1})_e \). Notice first that
\[
|G^F|(q^{-1}) = q^{-\dim G} \prod_i (1 - q^{-d_i}) = q^{-\dim G} p(q), \tag{2.5.1}
\]
which follows from the known formula for the order of finite groups of Lie type ([39, Theorem 25]). Using (2.4.6), we then see that the \((e, \phi), (e', \phi')\) entry in \( \Lambda(q)^{-1} \) equals
\[
\frac{1}{|A(e)|^2 p(q)} \sum_c \text{tr} \phi(c) \text{tr} \phi'(c) |c|^2 q^{\dim Z_G(e)} |Z_G(e_c)\phi| (q^{-1}). \tag{2.5.2}
\]

The map \( g \mapsto g \circ g_e \) gives an isomorphism from \( Z_G(e) \) to \( Z_G(e_c) \). Hence
\[
Z_G(e_c)^F = \{ g \in Z_G(e_c) ; F(g) = g \} \cong \{ g \in Z_G(e) ; F(g \circ g_e^{-1}) = g \circ g_e^{-1} \}
\]
\[
= \{ g \in Z_G(e) ; F(g) = g \} = Z_G(e)_F.
\]
Here \( F_c = \text{Ad}(x_c) \circ F \) is a Frobenius morphism on \( Z_G(e) \). It is easy to see that the image of \( Z_G(e)_F \) under the map \( Z_G(e) \rightarrow A(e) \) is \( Z_{A(e)}(\bar{x}_c) \). Hence we have the following short exact sequence
\[
1 \longrightarrow (Z^0_G(e))_F \longrightarrow Z_G(e)_F \longrightarrow Z_{A(e)}(\bar{x}_c) \longrightarrow 1. \tag{2.5.3}
\]

Let \( R_e \) be the unipotent radical of \( Z_G(e) \) and \( H_e = \frac{Z_G(e)}{R_e} \). Then \( H_e \) is a connected reductive group and
\[
H^F_e = (Z^0_G(e))_F \big/ R^F_e.
\]

So
\[
|Z_G(e_c)^F| = |Z_G(e)^F_c| = |Z_{A(e)}(\bar{x})_c| \cdot |(Z^0_G(e))_F^c| = |A(e)| \cdot (\frac{\dim R_e}{|c|}) |H^F_e| = q^{\dim R_e} |A(e)| |H^F_e|.
\]

Corollary 2.5. The \((e, \phi), (e', \phi')\) entry in the matrix \( M(q) \) is given by
\[
\delta_{e,e'} \frac{1}{|A(e)|} \sum_c \text{tr} \phi(c) \text{tr} \phi'(c) |c| \zeta_{e,e'}(q),
\]
with \( \zeta_{e,e}(q) = q^{\dim H_e} |H^F_e| (q^{-1}) \).
2.6. Now we follow the approach in [8, section 2.9].

We fix a conjugacy class $c$ of $A(e)$. Let $T$ be an $F_c$-stable maximal torus of $H_c$ contained in a $F_c$-stable Borel subgroup. Then we can define the $F_c$-action on the character group $X$ of $T$ and on $V_c = X_R$. We have that $F_c = qF_{c,0}$ on $V_c$ where $F_{c,0}$ is an automorphism of finite order.

The group $H_c$ is a product $H_c = H'Z^0$, where $H'$ is a semisimple group and $Z^0$ is the central torus. Then $T = SZ^0$, where $S = T \cap H'$ is a maximal torus of $H'$ and $S \cap Z^0$ is finite. We have the decomposition $V_c = V_1 \oplus V_2$, where $V_1 = (Z^0)_{Z^0}$ is the subspace spanned by the roots and $V_2 = S_{Z^0}$ is a complementary subspace. Both $V_1$ and $V_2$ are stable under the action of $F_c$ and $F_{c,0}$. Moreover, let $V_2$ be the vector space spanned by the characters of $Z^0$. Then $V_2 \cong V_Z$ as $F_c$-vector space and

$$|(Z^0)^{F_c}| = \det_{V_2}(q - F_{c,0}) = \det_{V_2}(q - F_{c,0}).$$

Let $W_c$ be the Weyl group of $H_c$. The $W_c$-invariants of the algebra of polynomial functions on $V_1$ is a polynomial ring and there exists homogeneous elements $I_1, \ldots, I_l$ of degree $d_1, \ldots, d_l$ such that

$$C[V_1] = C[I_1, \ldots, I_l]$$

and $F_{c,0}(I_l) = \epsilon_{c,l}I_l$, where $\epsilon_{c,l}$ is a root of unity. Moreover,

$$|H_c^{F_c}| = |Z^{F_c}|q^{N}P \cdot (q^{d_l} - \epsilon_{c,l}) = q^{N}\det_{V_2}(q - F_{c,0})P \cdot (q^{d_l} - \epsilon_{c,l}),$$

where $N$ is the number of positive roots.

We may reformulate the order of $H^{F_c}$ in the following way.

For any $d \in \mathbb{N}$, let $\mathcal{M}[d,c]$ be the complex vector space spanned by $I_i$ with $d_i = d$. Then $F_{c,0}$ acts on $\mathcal{M}[d,c]$ and

$$|H_c^{F_c}| = q^N\det_{V_2}(q - F_{c,0})P \cdot \det_{\mathcal{M}[d,c]}(q^{d} - F_{c,0}).$$

Notice that $N$ and $V_2$ are independent of the choice of $c$. Although $T$ and $I_i$ depends on the choice of $c$, the multisets $\{d_1, \ldots, d_l\}$ is the set of degrees of fundamental invariants for $H_c$ and thus for any given $d$, the dimension of the vector spaces $\mathcal{M}[d,c]$ is independent of the choice of $c$.

2.7. By (2.5.3), each coset of $A(e)$ contains a $F$-stable element. Moreover, if $g, g' \in Z_G(e)^F$ with $gZ_G(e)^0 = g'Z_G(e)^0$, then the actions of $\text{Ad}(g) \circ F$ and $\text{Ad}(g') \circ F$ on $V_Z$ are the same as $Z^0$ is the central torus of $H_c$. In other words, the map

$$Z_G(e)^F \to \text{End}(V_Z), \quad g \mapsto \text{Ad}(g) \circ F |_{V_Z}$$

factors through a map $A(e) \to \text{End}(V_Z)$. For any $x \in A(e)$, we denote the corresponding endomorphism on $V_Z$ by $F_x$. Then $F_x = qF_{x,0}$, where $F_{x,0}$ has finite order.

For any $g, g' \in Z_G(e)^F$, $(\text{Ad}(g) \circ F) \circ (\text{Ad}(g') \circ F) = \text{Ad}(gg') \circ F^2$. Notice that $F = F_1$ acts on $V_Z$ as $q \cdot \text{id}$. Thus the map $x \mapsto F_{x,0}$ gives a group homomorphism from $A(e) \to \text{GL}(V_Z)$.

**Proposition 2.6.** Let $d \in \mathbb{N}$. Then there exists a representation $\mathcal{M}[d]$ of $A(e)$ such that for any $x \in A(e)$, we have

$$\det_{\mathcal{M}[d]}(q - x) = \det_{\mathcal{M}[d,t]}(q - F_{x,0})$$

as a polynomial in $q$. Here $x_c$ is the conjugacy class of $x$. 
Proof. We follow the notations in [20].

We first consider the case where $G$ is a classical group.

If $G = \GL_n$, then $A(e) = 1$ and the statement is obvious.

If $G = \Sp_n$ and $e = \bigoplus_i J_i^+$ be a nilpotent element in $G$, where $J_i$ denotes a nilpotent Jordan block of length $i$. By [20, Theorem 3],

$$H_e = \prod_{i \text{ odd}} \Sp_{r_i} \times \prod_{i \text{ even}} \Or_{r_i}$$

and $A(e) = (S_2)^k$, where $k = \{ i; i \text{ even}, r_i > 0 \}$.

For any $d \in \mathbb{N}$, let $M[d]_{\text{odd}} = \bigoplus_{i \text{ odd}} M[d] S^i_{\text{Sp}}$, and $M[d]_{i} = M[d]^{\text{Or}_i}$ for $i$ even.

Here for any $H = \Sp_{r_i}$ or $\Or_{r_i}$, $M[d]_H$ is a complex vector space of dimension equal to $\dim M[d]_i$ for the group $H$, i.e., the number of the degree of the fundamental invariants for $H$ which equals $d$.

Let

$$M[d] = M[d]_{\text{odd}} \oplus \bigoplus_{i \text{ even}} M[d]_i.$$ 

We define the action of $A(e)$ on $M[d]$ as follows.

The action of $A(e)$ on $M[d]_{\text{odd}}$ is trivial. For any $i$ even with $r_i > 0$, the $i$-th copy of $S_2$ in $A(e)$ acts on $M[d]_{i}$ unless for $i' = i$, $d = \frac{q}{2}$. In the latter case, if $4 | r_i$, then $M[d]_{i}$ is 2-dimensional and the $i$-th copy of $S_2$ in $A(e)$ acts on $M[d]_{i}$ as permutation representation; if $4 \not| r_i$, then $M[d]_{i}$ is 1-dimensional and the $i$-th copy of $S_2$ in $A(e)$ acts on $M[d]_{i}$ as sign representation.

By [20, Theorem 2.12], this is the desired representation.

The case where $G = \OO_n$ can be proved in the same way.

Now we assume that $G$ is of exceptional type and the semisimple part of $H_e$ is nontrivial.

If $A(e) = S_2 = \{1, e\}$, then $F_\varepsilon^2 = \Ad(g) \circ F^2$ for some $g \in (Z_G(e)^0)^F$. By Lang’s theorem, there exists $h \in Z_G(e)^0$ such that $g = x F^2(x)^{-1}$. So $F_\varepsilon^2 = \Ad(x) \circ F^2 \circ \Ad(x)^{-1}$. Since $F^2$ acts on $M[d]_{i}$ as $q^2 \text{id}$, $F_\varepsilon^2$ acts on $M[d]_e$ as $q^2 \text{id}$. In particular, $F_{\varepsilon, 0}$ is an automorphism on $M[d]_e$ with $F_{\varepsilon, 0}^2 = \text{id}$. The statement holds in this case.

If $A(e) \neq \{1\}$ or $S_2$, we only have the following cases (see [20, Table 5.1 & 5.2]).

Class $D_4(a_1)$ in $E_6(q)$. Here $A(e) = S_3$, $H_e = D_4$, $M[2]$, $M[6]$ are one-dimensional trivial representations of $A(e)$ and $M[4]$ is the irreducible 2-dimensional representation of $A(e)$.

Class $D_4(a_1)$ in $E_6(q)$. Here $A(e) = S_3$, $H_e = A_1^3$ and $M[2]$ is the permutation representation of $A(e)$.

Class $E_7(a_5)$ in $E_8(q)$. Here $A(e) = E_7$, $H_e = A_1$ and $M[2]$ is one-dimensional trivial representation of $A(e)$.

Class $D_4(a_1)$ in $E_7(q)$. Here $A(e) = S_3$, $H_e = A_1^3$ and $M[2]$ is the permutation representation of $A(e)$. \hfill \square

2.8. We set

$$\mathcal{M} = \bigwedge_q^Z V \otimes \bigotimes_{d \in \mathbb{N}} \left( \bigwedge_q^Z \mathcal{M}[d] \right) \in R_q(A(e)) = R(A(e)) \otimes \mathbb{Z}[q]. \quad (2.8.1)$$

Recall that we defined $\mathcal{M}[d]$ such that $\mathcal{M}[d] \neq 0$ if and only if $d$ is the degree of a fundamental invariant of $H_e$. 
The action of $A(e)$ on $V_Z$ and $\mathcal{M}[d]$ extends in a unique way to an action on $\mathcal{M}$ and for any $x \in A(e)$,

$$\text{tr}_\mathcal{M}(x) = \text{tr}_{\Lambda_qV_Z}(x) \times \prod_d \text{tr}_{\Lambda_q\mathcal{M}[d]}(x)$$

$$= \det_{V_Z}(1 - qx) \times \prod_d \det_{\mathcal{M}[d]}(1 - q^d x)$$

$$= \det_{V_Z}(1 - qx) \times \prod_d \det_{\mathcal{M}[d]_{xe}}(1 - q^d F_{xe})$$

$$= q^\dim H_e - N \det_{V_Z}(q^{-1} - x) \times \prod_d \det_{\mathcal{M}[d]_{xe}}(q^{-d} - F_{xe,0})$$

$$= q^\dim H_e \mid_{He_{F,e}}^{|(q^{-1})}$$

We define the $(q, \mathcal{M})$-pairing in $R_q(A(e))$ to be

$$\langle \phi, \phi' \rangle^{q,\mathcal{M}}_{A(e)} := \langle \phi \otimes \mathcal{M}, \phi' \rangle_{A(e)} \in \mathbb{Z}[q].$$ (2.8.2)

We have proved, for every specialization $q = q$ large prime power, and thus for the indeterminate $q$ as well, the following statement.

**Theorem 2.7.** The map $X_q(e, \phi) \mapsto \phi$ induces an isometry with respect to the $q$-elliptic pairing in $R_q(W)^e$ and the $(q, \mathcal{M})$-pairing in $R_q(A(e))$. More precisely:

$$\langle X_q(e, \phi), X_q(e, \phi') \rangle^q_{W} = \langle \phi, \phi' \rangle_{A(e)}^{q,\mathcal{M}}.$$

**Remark 2.8.** The case we are mainly interested in is $e \in N^{\text{sol}}$. In this case, $\mathcal{M} = \Lambda^{-N} V_Z$, uniformly, and we will omit $\mathcal{M}$ in the $(q, \mathcal{M})$-pairing and simply write it as $\langle -, - \rangle_{A(e)}^{q}$.

### 3. Nilpotent Elements with Solvable Connected Centralizer

#### 3.1. Nilpotent Conjugacy Classes

In this section, we discuss certain nilpotent conjugacy classes that will play an essential role in our study of irreducible representations of $\tilde{W}$. We are back to the setting of a complex semisimple group $G$.

Let $e \in N$. A standard (Lie) triple of $e$ is a a triple $\{e, h, f\} \subset g$, such that $[h, e] = 2e$, $[h, f] = -2e$, and $[e, f] = h$. Every such triple corresponds to a Lie algebra homomorphism $\varphi : \mathfrak{sl}(2) \to g$. $\left(\begin{array}{cc}0 & 1 \\ 0 & 0\end{array}\right) \mapsto e$, $\left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right) \mapsto h$, $\left(\begin{array}{cc}0 & 0 \\ 1 & 0\end{array}\right) \mapsto f$.

By the Dynkin-Kostant classification, see for example [12, pages 35–36], the map $\{e, h, f\} \to e$ gives a one-to-one correspondence between the set of $G$-conjugacy classes of Lie triples and $G$-orbits of nilpotent elements in $g$. We refer to the element $h$ as a neutral element for $e$ and when we wish to emphasize the dependence of $h$ on $e$, we denote it by $h_e$.

**Definition 3.1.** An element $e \in N$ is called distinguished ([8]) if the centralizer $Z_G(e)$ does not contain any nonzero semisimple element.

An element $e \in N$ is called quasisimply if there exists a semisimple element $t \in Z_G(e)$ such that $Z_G(t)$ is semisimple and $e$ is a distinguished element in the Lie algebra of $Z_G(t)$.

Set $u = \exp(e)$. Then $e$ is quasisimply if and only if $u$ is quasisimply in the sense of [29], i.e., there exists a semisimple element $t \in G$ such that $u \in Z_G(t)$.
and \( Z_G(tu) \) does not contain any nontrivial torus. Indeed, one direction is obvious. For the converse, let \( g = ut \). By the uniqueness in the Jordan decomposition, any element that commutes with \( g \) also commutes with \( t \) and \( u \). Thus \( Z_G(g) = Z_{Z_G(t)}(u) \). Hence \( Z_G(g) \) doesn’t contain a nontrivial torus implies that \( Z_G(t) \) is semisimple and \( u \) is distinguished in \( Z_G(t) \). Hence \( e \) is distinguished in the Lie algebra of \( Z_G(t) \).

Recall that \( N^{\text{sol}} \) is the set of \( e \in N \) such that \( Z_G(e)^0 \) is a solvable group. It is clear that every distinguished nilpotent element \( e \) is also quasidistinguished and belongs to \( N^{\text{sol}} \). It is proved in [30, Lemma 7.1(1)] that if \( e \) is quasidistinguished, then necessarily \( e \in N^{\text{sol}} \). However, \( e \in N^{\text{sol}} \) does not imply that \( e \) is quasidistinguished.

3.2. Let \( \delta \) be the automorphism of \( G \) given by the action of \(-w_0 \) on the root datum, where \( w_0 \in W \) is the longest element. More precisely, \( \delta \) is the order two automorphism of the Dynkin diagram when \( G \) is of type \( A_n, D_{2n+1} \), or \( E_6 \), and \( \delta \) is trivial for other simple groups.

We set \( G_{\#} = G \rtimes \langle \delta \rangle \). An element \( g \in G_{\#} \) is called semisimple if \( \text{Ad}(g) \) is a semisimple endomorphism of \( g \). Following [38, section 9], we call an element \( g \in G_{\#} \) quasi-semisimple if there exists a Borel subgroup \( B \) of \( G \) and a maximal torus \( T \subset B \) such that \( gBg^{-1} = B \) and \( gTg^{-1} = T \). In this case \( Z_G(g) \) is a reductive group.

Moreover, by [38, Theorem 7.5], if \( g \in G_{\#} \) is semisimple, then \( g \) is quasi-semisimple.

**Definition 3.2.** An element \( e \in N \) is called \( \delta \)-quasidistinguished\(^1\) if there exists a semisimple element \( t\delta \in Z_G_{\#}(e) \) such that \( Z_G(t\delta) \) is semisimple and \( e \) is a distinguished element in the Lie algebra of \( Z_G(t\delta) \).

Suppose that \( t\delta \) is semisimple in \( G_{\#} \). The condition that \( Z_G(t\delta) \) be semisimple implies that \( t\delta \) is an isolated (torsion) element of \( G\delta \) in the terminology of [25, section 2] or [31, section 3.8]. For basic results about the isolated elements, see [25, section 2], particularly [25, Lemma 2.6]. The classification of isolated semisimple elements is known, and we recall it next, following [31, sections 3.8, 4.1-4.5]. Let \( t \subset \mathfrak{b} \) be \( \delta \)-stable Cartan and Borel subalgebras, respectively. If \( \Phi \) is the root system of \( \mathfrak{g} \) corresponding to \( t \), with positive roots given by \( \mathfrak{b} \). Call two roots \( \alpha, \beta \in \Phi \) \( \delta \)-equivalent if \( \alpha|_{\mathfrak{t}^\delta}, \beta|_{\mathfrak{t}^\delta} \) are proportional via a positive constant. If \( \alpha \) is a \( \delta \)-equivalence class in \( \Phi \), then \( a \) is a \( \delta \)-orbit in \( \Phi \), except in type \( A_{2n} \), when \( a \) could be of the form \( \{\alpha, \delta(\alpha), \alpha + \delta(\alpha)\} \).

Let

\[
\mathfrak{g}_a = \sum_{\alpha \in a} \mathfrak{g}_\alpha, \quad \gamma_a = \sum_{\alpha \in a} \alpha|_{\mathfrak{t}^\delta}, \quad \beta_a = \frac{1}{f_a} \gamma_a,
\]

where \( f_a = |a| \), unless \( a \) is the exception in type \( A_{2n} \), when \( f_a = 4 \).

With this notation, the root-space decomposition of \( \mathfrak{g}^\delta \), \( t = \exp(x) \), \( x \in \mathfrak{t}^\delta \) is ([31, Proposition 3.8]):

\[
\mathfrak{g}^\delta = \mathfrak{t}^\delta \oplus \sum_a \mathfrak{g}_a^\delta, \tag{3.2.1}
\]

where the sum of over the \( \delta \)-equivalence classes \( a \in \Phi/\delta \) such that \( \{\gamma_a, x\} \in \{-1, 0, 1\} \). Each \( \mathfrak{g}_a^\delta \) is one-dimensional, affording either a root \( \beta_a \) or \( 2\beta_a \), the latter case may only occur in the exceptional \( a \) in \( A_{2n} \).

\(^1\)It is not clear a priori that every quasidistinguished nilpotent element is \( \delta \)-quasidistinguished. However, this implication follows as a consequence of Proposition 3.3.
Proposition 3.3. A nilpotent element $e \in \mathcal{N}$ is $\delta$-quasidistinguished if and only if $e \in \mathcal{N}^{\text{sol}}$, i.e., the centralizer $Z_G(e)^0$ is solvable.

Proof. One can prove uniformly that “$e$ is $\delta$-quasidistinguished” implies “$e \in \mathcal{N}^{\text{sol}}$” analogously with the untwisted case [30, Lemma 7.1(1)], as follows. Suppose $e$ is $\delta$-quasidistinguished, and let $\delta \in G_\delta \subseteq G_\# \subseteq \mathfrak{g}_\#$ be a semisimple element as in Definition 3.2. Let $H_e$ be the (connected) reductive part of $Z_G(e)$ and let $\mathfrak{h}_e$ be the Lie algebra. Because $\delta \in G_\delta$ acts on $H_e$ by conjugation, one can consider $\text{Ad}(\delta)|_{\mathfrak{h}_e} : \mathfrak{h}_e \to \mathfrak{h}_e$ and let $\mathfrak{h}_e^{\delta}$ be the fixed points. The algebra $\mathfrak{h}_e^{\delta}$ is a reductive Lie algebra, since $t\delta$ is semisimple. However, $\mathfrak{h}_e^{\delta}$ does not contain any nonzero semisimple element. This means that $\mathfrak{h}_e^{\delta} = 0$. By [38, Corollary 10.12], $\mathfrak{h}_e$ has zero derived subalgebra, which implies $H_e$ is a torus, equivalently $Z_G(e)^0$ is solvable.

The proof of the converse direction is case by case.

In type $A$, we consider $\text{GL}(n)$ rather than $\text{SL}(n)$ for simplicity. The nilpotent orbits in $\mathcal{N}^{\text{sol}}$ are in one-to-one correspondence with partitions of $n$ into distinct parts via the Jordan canonical form. Let $\lambda$ be such a partition and break $\lambda$ into $\lambda_0$ containing all the even parts and $\lambda_1$ containing all the odd parts. Let $2m$ be the sum of parts in $\lambda_0$. Let $J_{2m} = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$, where $I_m$ is the identity matrix. Set $t_m = \text{diag}(J_{2m}, I_{n-2m})$. Then $t_m\delta$ is a semisimple element.

We consider the automorphism $\delta : \text{GL}(n) \to \text{GL}(n)$ given by $\delta(x) = (x^T)^{-1}$. Thus $Z_{\text{GL}(n)}(t_m\delta) = \text{Sp}(2m) \times O(n-2m)$. Let $e_{\lambda_0}$ be a distinguished nilpotent element in $\mathfrak{sp}(2m)$ parameterized by the even partition $\lambda_0$ and $e_{\lambda_1}$ a distinguished nilpotent element in $\mathfrak{o}(n-2m)$ parameterized by the odd partition $\lambda_1$. Then $e_{\lambda} = \begin{pmatrix} e_{\lambda_0} & 0 \\ 0 & e_{\lambda_1} \end{pmatrix}$ is a representative of the class in $\mathcal{N}^{\text{sol}}$ labeled by $\lambda$ and it is $\delta$-quasidistinguished by construction.

In $\text{Sp}(2n)$ (resp. $\text{SO}(2n+1)$ or $\text{SO}(4n)$), the automorphism $\delta$ is trivial, and one can see from the classification of nilpotent classes that the classes in $\mathcal{N}^{\text{sol}}$ are parameterized by partitions of $2n$ (resp. $2n+1$ or $4n$) where every part is even (resp. odd) and each part appears with multiplicity at most 2. It is easy to check that every such nilpotent class is quasidistinguished. For example, suppose $\lambda = (a_1, a_1, a_2, a_2, \ldots, a_k, a_k, a_{k+1}, \ldots, a_\ell)$ is a partition of $2n$, where $a_1 < a_2 < \cdots < a_k < a_{k+1} < \cdots < a_\ell$ are even numbers. Let $t_n \in \text{Sp}(2n)$ be a semisimple element whose centralizer is $\text{Sp}(2m) \times \text{Sp}(2n-2m)$, where $2m = \sum_{i=1}^{\ell} a_i$. We choose a distinguished nilpotent element $e_{1}$ in $\mathfrak{sp}(2m)$ corresponding to the partition $(a_1, a_2, \ldots, a_\ell)$ and a distinguished nilpotent element $e_{2}$ in $\mathfrak{sp}(2n-2m)$ corresponding to the partition $(a_1, a_2, \ldots, a_k)$. Then $e_{\lambda} = e_{1} \times e_{2}$ is a representative of the nilpotent class in $\mathfrak{sp}(2n)$ labeled by $\lambda$ and it is quasi-distinguished by construction.

If $G = \text{SO}(4n+2)$, $\delta$ corresponds to the automorphism of order 2 of the Dynkin diagram. Suppose the roots of the corresponding root system of type $D_{2n+1}$ are labeled $\{\alpha_1, \alpha_2, \ldots, \alpha_{2n+1}\}$ and that $\delta$ acts by interchanging $\alpha_{2n}$ and $\alpha_{2n+1}$ and fixes the other roots. The orbits in $\mathcal{N}^{\text{sol}}$ are parameterized by partitions of $4n+2$ into odd parts, each part of multiplicity at most 2. By [20, Lemma 2.9(iv)] and [15, Table 4.3.1], there exist $n+1$ classes of involutions in $G_\delta$ with representatives $t_{i-1}$ having centralizers of type $B_{n+1} \times B_{2n+1-i}$, where $1 \leq i \leq n+1$. Here $t_0 = 1$, and $t_i$, $1 \leq i \leq n$, is the order two element in the standard torus in $\text{SO}(4n+2)$.
corresponding to the root \( \alpha_i \), see for example [15, (4.4.4)] for the precise definition. In particular, \( t_{i-1} \) commutes with \( \delta \), so \( t_{i-1} \delta \) is semisimple, \( 1 \leq i \leq n + 1 \). The construction of \( \delta \)-quasi-distinguished orbits proceeds exactly as in the untwisted \( \text{Sp}(2n) \) example above.

When \( G \) is exceptional of type \( G_2 \), \( F_4 \), \( E_7 \), or \( E_8 \), the automorphism \( \delta \) is trivial, and one verifies the claim from the classification of nilpotent orbits and their centralizers. This is an easy direct calculation using the explicit classification of nilpotent classes. We give the results below, but skip the details, since the calculation is very similar to the twisted \( E_6 \) example which we’ll explain in detail.

If \( G \) is of type \( G_2 \) or \( F_4 \), the only nilpotent orbits in \( \mathcal{N}^{\text{sol}} \) are already distinguished, so there is nothing to check.

For groups of type \( E \), we use the following labeling of the Dynkin diagrams (this is not the Bourbaki notation):

\[
\begin{align*}
\alpha_1 & \quad \alpha_2 & \quad \alpha_3 & \quad \alpha_4 & \quad \alpha_5 & \quad \alpha_6 & \quad \alpha_7 & \quad \alpha_8. \\
& \text{(3.2.2)}
\end{align*}
\]

For types \( E_7 \) and \( E_8 \), denote \( t_i = \exp(\frac{1}{(\gamma, \alpha_i)\omega_i^\gamma}) \in T \), where \( \omega_i^\gamma \) is the fundamental coweight corresponding to the \( i \)-th simple root, and \( \gamma \) is the highest positive root.

If \( G \) is of type \( E_7 \), the non-distinguished nilpotent orbits in \( \mathcal{N}^{\text{sol}} \) are denoted in the Bala-Carter classification [8] by \( E_6(a_1) \) and \( A_1 + A_1 \). They come from the regular nilpotent orbits in \( Z_G(t_4) = A_7 \) and \( Z_G(t_4) = A_3 \times A_3 \times A_1 \), respectively.

If \( G \) is of type \( E_8 \), the non-distinguished nilpotent orbits in \( \mathcal{N}^{\text{sol}} \) are \( D_5 + A_2 \), \( D_7(a_1) \), \( D_7(a_2) \), and \( E_6(a_1) + A_1 \). They come from \( E_7(a_1) \) in \( Z_G(t_8) = E_7 \times A_1 \), \( E_7(a_3) \) in \( Z_G(t_8) = E_7 \times A_1 \), the regular nilpotent orbit in \( Z_G(t_6) = D_5 \times A_3 \), and the regular nilpotent orbit in \( Z_G(t_2) = A_7 \times A_1 \), respectively.

It remains to analyze the case \( G = E_6 \) and \( \delta \) coming from the automorphism of order 2 of the Dynkin diagram. There are seven nilpotent orbits in \( \mathcal{N}^{\text{sol}} \) labeled: \( E_6, E_6(a_1), E_6(a_1), D_5, D_5(a_1), A_4 + A_1 \), and \( D_4(a_1) \).

Suppose that \( t \delta \) is semisimple. We use (3.2.1) to realize \( \delta \)-quasi-distinguished nilpotent orbits. The explicit cases in \( E_6 \) are in [31, section 4.5]. For each \( x \in t^6 \) such that \( t = \exp(x) \) that appears (there are five cases), we compute the simple roots of the Lie algebra \( g^\delta \) as in (3.2.1). Then for each distinguished nilpotent element in \( g^\delta \) we match its Dynkin-Kostant diagram (in \( g^\delta \)) with a diagram in \( g \). This is done as follows: the Dynkin-Kostant diagram of \( e \in g^\delta \) gives the values of the simple roots \( \beta \) for \( g^\delta \) on the neutral element \( h_e \in t^6 \), and thus we can determine \( h_e \). Next, one makes \( h_e \) dominant with respect to the simple roots in \( g \) and computes the Dynkin-Kostant diagram in \( g \). The explicit results are below. We denote by \( \omega_i^\gamma \in t \) the fundamental coweight corresponding to \( \alpha_i \).

\[
(0) \quad x_0 = 0, \quad g^\delta = F_4, \quad \text{with simple roots:}
\]

\[
\alpha_4 \quad \alpha_3 \quad \frac{1}{2}(\alpha_2 + \alpha_5) \quad \frac{1}{2}(\alpha_1 + \alpha_6).
\]

The fixed point group \( F_4 \) has four distinguished nilpotent orbits: \( F_4, F_4(a_1), F_4(a_2), \) and \( F_4(a_3) \) which correspond in \( E_6 \) to: \( E_6, D_5, E_6(a_3) \), and \( D_4(a_1) \), respectively.
(1) \( x_1 = \frac{1}{4}(\omega^\vee_1 + \omega^\vee_6), \ g^{1,6} = B_3 \times A_1, \) with simple roots:
\[
\begin{array}{c}
\alpha_4 \rightarrow \alpha_3 \rightarrow \beta, \\
\end{array}
\]
where \( \beta = \frac{1}{4}[(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5) + (\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5)]. \)
There is only one distinguished nilpotent orbit in \( g^{1,6}, \) the regular one, which corresponds to \( D_5(a_1) \) in \( E_6. \)

(2) \( x_2 = \frac{1}{6}(\omega^\vee_2 + \omega^\vee_5), \ g^{2,5} = A_2 \times A_2, \) with simple roots:
\[
\begin{array}{c}
\alpha_4 \rightarrow \alpha_3 \rightarrow \frac{1}{2}(\alpha_1 + \alpha_6) \rightarrow \beta', \\
\end{array}
\]
where \( \beta' = \frac{1}{4}[(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5) + (\alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6)]. \)
There is only one distinguished nilpotent orbit in \( g^{2,5}, \) the regular one, which corresponds to \( D_4(a_1) \) in \( E_6. \)

(3) \( x_3 = \frac{1}{3}(\omega^\vee_3), \ g^{3} = A_3 \times A_1, \) with simple roots:
\[
\begin{array}{c}
\beta'' \rightarrow \frac{1}{2}(\alpha_2 + \alpha_5) \rightarrow \frac{1}{2}(\alpha_1 + \alpha_6) \rightarrow \alpha_4, \\
\end{array}
\]
where \( \beta'' = \frac{1}{4}[(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5) + (\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)]. \)
There is only one distinguished nilpotent orbit in \( g^3, \) the regular one, which corresponds to \( A_4 + A_1 \) in \( E_6. \)

(4) \( x_4 = \frac{1}{2}(\omega^\vee_4), \ g^{4} = C_4, \) with simple roots:
\[
\begin{array}{c}
\alpha_3 \rightarrow \frac{1}{2}(\alpha_2 + \alpha_5) \rightarrow \frac{1}{2}(\alpha_1 + \alpha_6) \rightarrow \beta''', \\
\end{array}
\]
where \( \beta''' = \frac{1}{4}[(\alpha_2 + \alpha_3 + \alpha_4) + (\alpha_3 + \alpha_4 + \alpha_5)]. \)
There are two distinguished nilpotent orbits in \( C_4, \) the regular orbit (8) and the subregular orbit (62), which correspond in \( E_6 \) to \( E_6(a_1) \) and \( E_6(a_3), \) respectively.

This finishes the proof for \( E_6 \) and therefore, the proof of the proposition. \( \square \)

Now we analyze the action of \( \delta \) on \( G \)-orbits of Lie triples in \( g. \)

**Lemma 3.4.** Let \( O \) be a nilpotent \( G \)-orbit in \( g. \) There exists a \( \delta \)-stable Levi subalgebra \( m \) and a Lie triple \( \{e, h, f\} \subset m, e \in O, \) such that \( e \) is \( \delta \)-quasidistinguished in \( m. \)

**Proof.** We prove the statement case by case. When \( \delta = 1, \) the claim is immediate from the Bala-Carter classification, in fact, \( m \) can be chosen so that \( e \) is distinguished in \( m. \)

Let \( G = GL(n). \) The nilpotent orbit \( O \) is given via Jordan form by a partition \( \lambda \) of \( n. \) Let \( I = \{1, 2, \ldots, n - 1\} \) be the indexing set for the simple roots. The automorphism \( \delta \) acts on \( I \) as \( \delta(i) = n - i. \) We construct a subset \( J \subset I, \) such that \( \delta(J) = J. \) Start by setting \( J = I. \) If \( \lambda \) has only distinct parts, by Proposition 3.3, \( O \) is \( \delta \)-quasidistinguished in \( g, \) so \( m = g. \) Suppose \( \lambda \) has equal parts. Write \( \lambda = \lambda' \cup \{r_1, r_1\} \cup \cdots \cup \{r_t, r_t\}, \) where \( \lambda' \) is the largest subset of \( \lambda \) having only distinct parts. Set \( k = \sum_{j=1}^{t} r_j. \) Remove from \( I \) the indices \( r_1, n - r_1, r_1 + r_2, n - (r_1 + r_2), \ldots, k, n - k, \) the resulting subset is \( J. \) By construction, \( \delta(J) = J. \) Then \( m \) is the Levi subalgebra corresponding to \( J, \) thus \( m = \sum_{j=1}^{t} (gl(r_j) \oplus gl(r_j)) \oplus gl(n - 2k). \) Let \( \{e, h, f\} \subset m \) be a Lie triple in \( m \) representing the regular nilpotent orbits on each factor \( gl(r_j) \) and the nilpotent orbit parameterized by \( \lambda' \) on \( gl(n - 2k). \) The latter is \( \delta \)-quasidistinguished by Proposition 3.3.
Let \( g = \mathfrak{so}(2n) \). Let \( I = \{1, 2, \ldots, n\} \) be the indexing set of simple roots, and suppose that the branch point of the Dynkin diagram is at \( n - 2 \), so that \( \delta(i) = i \) for all \( 1 \leq i \leq n - 2 \) and \( \delta(n) = n \). Suppose \( O \) is parameterized by a partition \( \lambda \) of \( 2n \). Then each even part in \( O \) appears with even multiplicity. Partition \( \lambda \) as \( \lambda = \lambda' \cup \{r_1, r_1\} \cup \cdots \cup \{r_k, r_k\} \), where \( \lambda' \) is the largest subset of \( \lambda \) such that \( \lambda' \) has only odd parts and each part occurs with multiplicity at most 2. As before, set \( k = \sum j \in I, \delta(j) = j \), is
\[ J = \{1, \ldots, r_1 - 1\} \cup \{r_1 + 1, \ldots, r_1 + r_2 - 1\} \cup \{k - r_2 + 1, \ldots, k - 1\} \cup \{k + 1, \ldots, n\}, \]
and the corresponding Levi subalgebra \( m = \oplus_{j=1}^{\ell} \mathfrak{g}(r_j) \oplus \mathfrak{so}(2n-2k) \). Let \( \{e, h, f\} \) be a Lie triple in \( m \) such that \( e \) represents the regular nilpotent orbits on the \( \mathfrak{g}(r_j) \) factors and the \( \delta \)-quasidistinguished orbit parameterized by \( \lambda' \) on the \( \mathfrak{so}(2n-2k) \) factor.

When \( g \) is of type \( E_6 \), we list the orbits with the corresponding \( m \) and \( e \in m \) in Table 1. The indexing set \( I = \{1, \ldots, 6\} \) corresponds to the Dynkin diagram (3.2.2). We do not include in the table the \( \delta \)-quasidistinguished orbits in \( E_6 \): \( E_6(a_1), E_6(a_3), D_5, D_6(a_1), A_4 + A_1 \), and \( D_4(a_1) \).

| \( O \) | \( m \) | \( \delta \)-quasidistinguished \( e \in m \) |
|---|---|---|
| \( A_1 \) | \{1, 2, 3, 5, 6\} | \{6\} |
| \( D_4 \) | \{2, 3, 4, 5\} | (7, 1) |
| \( A_4 \) | \{1, 2, 3, 5, 6\} | (5, 1) |
| \( A_3 + A_1 \) | \{1, 2, 3, 5, 6\} | (4, 2) |
| \( 2A_2 + A_1 \) | \{1, 2, 4, 5, 6\} | regular |
| \( A_2 \) | \{2, 3, 5\} | (4) |
| \( A_2 + 2A_1 \) | \{1, 3, 4, 6\} | regular |
| \( 2A_2 \) | \{1, 2, 5, 6\} | regular |
| \( A_2 + A_1 \) | \{1, 2, 3, 5, 6\} | (3, 2, 1) |
| \( A_2 \) | \{2, 3, 4, 5\} | (3, 3, 1, 1) |
| \( 3A_1 \) | \{2, 4, 5\} | regular |
| \( 2A_1 \) | \{2, 5\} | regular |
| \( A_1 \) | \{3\} | regular |
| 0 | 0 | 0 |

\[ \square \]

**Proposition 3.5.** Suppose that \( G \) is of adjoint type. Let \( O \) be a nilpotent \( G \)-orbit in \( g \). There exists a Lie triple \( \phi = \{e, h, f\} \), with \( e \in O \), \( h \in t^\delta \), and an element \( g \in G \) such that \( \delta(\phi) = \text{Ad}(g)\phi \), and \( \delta(g)g \in Z_G(\phi)^0 \).

**Proof.** Assume \( \delta \neq 1 \) (otherwise \( g = 1 \)). First, suppose \( e \) is \( \delta \)-quasidistinguished in \( g \). By definition, there exists \( t, t \in t^\delta \), an isolated semisimple element of \( G \delta \) such that \( e \in Z_p(t) \). We may choose the Lie triple so that \( \phi \subset Z_p(t) \). Thus \( \text{Ad}(t)\phi = \phi \), or equivalently \( \delta(\phi) = \text{Ad}(t^{-1})\phi \). So we may choose \( g = t^{-1} \in T^\delta \), and then \( \delta(g)g = t^{-2} \). We claim that \( t^{-2} \in Z_G(\phi)^0 \). Indeed, by the proof of Proposition 3.3, \( t^2 = 1 \) unless \( g \) is of type \( A_n \) or when \( g \) is of type \( E_6 \) and \( O \) is \( D_5(a_1) \) or \( A_4 + A_1 \). But in all of these cases, \( Z_G(\phi) \) is connected, so there is nothing to prove.
If $e$ is not $\delta$-quasidistinguished in $\mathfrak{g}$, by Lemma 3.4, there exists a $\delta$-stable Levi $\mathfrak{m}$ such that $\phi \subseteq \mathfrak{m}$ and $e$ is $\delta$-quasidistinguished in $\mathfrak{m}$. From the proof of Lemma 3.4, we see that whenever $\mathfrak{m}$ has two factors of the same type which are flipped by $\delta$, the two factors are of type $A_{r-1}$ (for some $r$) and the nilpotent elements on these factors are equal to a regular nilpotent element $e_{r-1}$. This means that $\delta$ fixes the pair $(e_{r-1}, e_{r-1})$, and therefore by the discussion in the case $\mathfrak{m} = \mathfrak{g}$, there exists $g \in \text{Ad}(\mathfrak{m})$ such that $\delta(g)g \in Z_M(\phi) = M \cap Z_G(\phi)$. □

4. THE TWISTED ELLIPTIC FORM

4.1. Let $e \in \{+1, -1\}$. Set

$$\mathcal{R}_e(W) = R(W)_{/\text{rad}}(\langle , \rangle)^e_W.$$

For any $e \in \mathcal{N}$, let $R_e(W)^e$ be the image of $R(W)^e$ in $R_e(W)$. By Corollary 2.2,

$$R_e(W) = \bigoplus e R_e(W)^e,$$

where $e$ runs over nilpotent conjugacy classes of $G$.

**Proposition 4.1.** Let $e \in \mathcal{N}$ be given.

1. $\mathcal{R}_1(W)^e \neq 0$ if and only if $e$ is quasidistinguished.
2. $\mathcal{R}_{-1}(W)^e \neq 0$ if and only if $e$ is $\delta$-quasidistinguished, or equivalently, $e \in \mathcal{N}^{\text{sol}}$.

**Remark 4.2.** Part (1) was first proved by Reeder [29] using representations of $p$-adic groups.

**Proof.** If $Z_G(e)^0$ is not solvable, then $Z_G(e)^F$ contains a subgroup the $F$-points of a rank one semisimple group, and in particular, $|Z_G(e)^F|$ (as a polynomial in $q$) is divisible by $(q^2 - 1)$. Therefore, $|Z_G(e)^F|/(\epsilon) = 0$ for $\epsilon \in \{+1, -1\}$. By Corollary 2.5, $\mathcal{R}_1(W)^e = \mathcal{R}_{-1}(W)^e = 0$.

Thus we may reduce to the case where $e \in \mathcal{N}^{\text{sol}}$. The statements then follow by an analysis similar to the proof of Proposition 2.6. □

4.2. In the rest of this section, we focus on the $(-1)$-elliptic form $\langle \cdot, \cdot \rangle_W^{-1}$. It is by definition

$$\langle \chi, \chi \rangle_W^{-1} = \langle \chi \wedge \bigwedge V, Y \rangle_W,$$

where $\bigwedge V = \bigoplus_{i \geq 0} \bigwedge^i V$. (4.2.1)

We will relate it to the $\delta$-twisted elliptic form, here $\delta$ is the automorphism of $(W, V)$ given by $-w_0$:

$$\delta(\xi) = -w_0(\xi), \quad \delta(w) = w_0w_0, \quad w \in W. \quad (4.2.2)$$

The $\delta$-twisted elliptic form is defined as follows. If $(\sigma, X)$ is a $W$-representation, let $(\sigma^\delta, X^\delta)$ be the $\delta$-twisted representation, i.e.,

$$\sigma^\delta(w)x = \sigma(\delta(w))x = \sigma(w_0w_0)x, \quad w \in W, \quad x \in X^\delta = X.$$

We choose the intertwiner $\phi : (\sigma^\delta, X^\delta) \to (\sigma, X)$ to be $\phi(x) = \sigma(w_0)x$. Clearly $\phi^2 = 1$. Define the $\delta$-twisted character of $\sigma$:

$$\text{tr}_\sigma^\delta(w) = \text{tr}(\sigma(w) \circ \phi^{-1}),$$

and using the explicit form of $\phi$, we find

$$\text{tr}_\sigma^\delta(w) = \text{tr}_\sigma(w_0), \quad w \in W. \quad (4.2.3)$$
**Definition 4.3.** Let $R^\delta(W)$ denote the $\mathbb{Z}$-span of $\delta$-twisted characters $\text{tr}_\delta^\sigma$, for all irreducible $W$-representations $\sigma$. Define the $\delta$-elliptic pairing in $R^\delta(W)$:

$$\langle \chi, \chi' \rangle_{W}^{\delta-\text{ell}} = \frac{1}{|W|} \sum_{w \in W} \chi(w)\chi'(w)\det_V(1 - w\delta). \quad (4.2.4)$$

**Lemma 4.4.** The map $\text{tr}_\delta^\sigma \to \text{tr}_{\sigma}$ induces a linear isometry between the spaces $(R^\delta(W), \langle \cdot, \cdot \rangle_{W}^{\delta-\text{ell}})$ and $(R(W), \langle \cdot, \cdot \rangle_{W}^{-1})$:

$$\langle \text{tr}_\delta^\sigma, \text{tr}_{\sigma}^{\delta} \rangle_{W}^{\delta-\text{ell}} = \langle \text{tr}_{\sigma}, \text{tr}_{\sigma} \rangle_{W}^{-1}. \quad (4.2.4)$$

**Proof.** This is a straightforward calculation:

$$\langle \text{tr}_\delta^\sigma, \text{tr}_{\sigma}^{\delta} \rangle_{W}^{\delta-\text{ell}} = \frac{1}{|W|} \sum_{w \in W} \text{tr}_\delta^\sigma(w)\text{tr}_{\sigma}^{\delta}(w)\det_V(1 - w\delta) = \frac{1}{|W|} \sum_{w \in W} \text{tr}_{\sigma}(ww)\text{tr}_{\sigma}(ww)\det_V(1 + ww) = \frac{1}{|W|} \sum_{w \in W} \text{tr}_{\sigma}(w)\text{tr}_{\sigma}(w)\det_V(1 + w)^{-1} = \langle \text{tr}_{\sigma}, \text{tr}_{\sigma} \rangle_{W}^{-1}. \quad \square$$

**4.3.** Set

$$\overline{R}^\delta(W) = R(W)_{\mathbb{C}}/\text{rad}(\cdot, \cdot)^{\delta-\text{ell}}.$$ 

Under the isometry in Lemma 4.4, $\overline{R}^{-1}(W)$ may be identified with $\overline{R}^\delta(W)$.

We give an explicit description of $\overline{R}^\delta(W)$ using the $\delta$-elliptic conjugacy classes.

**Definition 4.5.** The $\delta$-twisted conjugacy class of $w \in W$ is

$$C_w = \{w'w\delta(w')^{-1} : w' \in W\}.$$ 

A twisted class $C$ is called $\delta$-elliptic if $C \cap W_J = \emptyset$, for all $\delta$-stable proper parabolic subgroups $W_J$ of $W$. An element $w \in W$ is called $\delta$-elliptic if it belongs to a $\delta$-elliptic conjugacy class.

The following lemma is well-known, see for example [16].

**Lemma 4.6.** The following are equivalent for an element $w \in W$:

1. $w$ is $\delta$-elliptic;
2. $\det_V(1 - w\delta) \neq 0$;
3. $V^{w\delta} = 0$.

**4.4.** For every $\delta$-stable parabolic subgroup $W_J$ of $W$, let $\delta-\text{Ind}_{W_J}^W : R^\delta(W_J) \to R^\delta(W)$ be the induction functor: $\delta-\text{Ind}_{W_J}^W(\sigma) = \mathbb{C}[W]\delta \otimes_{\mathbb{C}[W_J]} \sigma$. Denote

$$R^\delta_{\text{ind}}(W) = \sum \delta-\text{Ind}_{W_J}^W(R^\delta(W_J)),$$

where the sum is over all $\delta$-stable proper parabolic subgroups $W_J$ of $W$.

The following proposition is a straightforward modification of [29, Proposition (2.2.2)], and we omit the proof.
Proposition 4.7. We keep the notations as above. Then
(1) $\text{rad}(\cdot, \cdot,W_{\delta-\text{ell}}) = R_{\text{rad}}^\delta(W)^C$.
(2) The dimension of $R^\delta(W)$ equals the number of $\delta$-elliptic conjugacy classes in $W$.

Remark 4.8. In light of Proposition 4.7, we record the number of $\delta$-elliptic conjugacy classes in the irreducible Weyl groups (where $\delta = -w_0$), see for example [29, section 3.1], [16, section 7] and [14, section 6].

1. $2A_{n-1}$: the number of partitions of $n$ into odd parts (this is the same as the number of partitions of $n$ into distinct parts);
2. $B_n$: the number of partitions of $n$;
3. $D_{2n}$: the number of partitions of $2n$ into an even number of parts;
4. $2D_{2n+1}$: the number of partitions of $2n+1$ into an odd number of parts;
5. $G_2$: 3; $F_4$: 9; $2E_6$: 9; $E_7$: 12; $E_8$: 30.

5. Extended Dirac operator

We retain the notation from the previous sections. Fix a $W$-invariant bilinear form $(\cdot, \cdot)$ on $V$. Then $W$ is a Weyl group in the Euclidean vector space $V$, $(\cdot, \cdot)$ with root system $\Phi \subset V^*$, positive roots $\Phi^+$, and simple roots $\Pi$. Let $r$ denote an indeterminate to be specialized later. Recall the automorphism $\delta$, $\delta^2 = 1$, of the root system given by $-w_0$.

Definition 5.1 (Lusztig,[22]). The graded affine Hecke algebra $H$ with equal parameters attached to $(\Phi, V, W)$ is the unique associative $\mathbb{C}[r]$-algebra with identity generated by $\{\xi \in V^*_C\}$ and $\{w \in W\}$ such that:

1. $H \cong \mathbb{C}[r] \otimes_C \mathbb{C}[W] \otimes_C S(V^*_C)$, as $(\mathbb{C}[r] \otimes_C \mathbb{C}[W], S(V^*_C))$-bimodules;
2. $\xi \cdot s_\alpha - s_\alpha \cdot s_\alpha(\xi) = 2r\xi(\alpha^\vee)$, $\alpha \in \Pi$, $\xi \in V^*_C$.

The center of $H$ is $\mathbb{C}[r] \otimes_C S(V^*_C)^W$, hence the central characters are parameterized by $W$-orbits in $\mathbb{C} \oplus V^*_C$.

Let $*$ denote the conjugate linear anti-automorphism of $H$ defined on generators via

$r^* = r$, $w^* = w^{-1}$, $\xi^* = w_0 \cdot \delta(\xi) \cdot w_0$, $w \in W$, $\xi \in V$. \hspace{1cm} (5.1.1)

For every $\xi \in V$, define

$\tilde{\xi} = \frac{1}{2}(\xi - \xi^*)$. \hspace{1cm} (5.1.2)

It is clear that $\tilde{\xi}^* = -\tilde{\xi}$. Moreover, it is known that $w \cdot \tilde{\xi} \cdot w^{-1} = \tilde{w}(\xi)$.

We consider the extended algebra $H_\# = (H, \delta)$, and extend the $*$-operation to $H_\#$ by setting $\delta^* = \delta$.

5.2. We recall the classification of simple $H$-modules from [23, 24], in particular [24, section 1]. We denote by $H^G_\#(\cdot)$ and $H^G_C(\cdot)$ the $G$-equivariant homology and cohomology as in the references.

If $x \in g$, denote $Z_{G \times C^\times}(x) = \{(g, \lambda) \in G \times C^\times : \text{Ad}(g)x = \lambda^2g\}$.

Let $e \in g$ be a nilpotent element, and $(s, r_0)$ a semisimple element in the Lie algebra of $Z_{G \times C^\times}(e)$. In particular, $[s, e] = 2r_0e$. Choose $A \subset Z_{G \times C^\times}(x)$ a torus containing $\exp(s, r_0)$. Let $\mathbb{C}_{s, r_0}$ denote the one dimensional $H^G_\#(\{pt\})$-module obtained by evaluation at $(s, r_0)$. (One may identify $H^G_\#(\{pt\})$ with the space of
polynomials $\mathbb{C}[a]$, where $a$ is the Lie algebra of $A$.) From ([23, 10.12], [24, 1.13])

$$E_{e,s,r_0} = H^*_\mathbb{C}(\mathcal{B}^*_e) = C_{s,r_0} \otimes H^*_\mathbb{C}(\mathcal{B}^*_e) = C_{s,r_0} \otimes H^*_\mathbb{C}(\mathcal{B}^*_e) = C_{s,r_0} \otimes H^*_\mathbb{C}(\mathcal{B}^*_e),$$

(5.2.1)

where $\mathcal{B}_e^*$ is the variety of Borel subalgebras of $\mathfrak{g}$ containing $e$ and $s$. The space $E_{e,s,r_0}$ carries an $\mathbb{H}$-action such that $r$ acts by $r_0$.

Let $Z(e,s) = Z_{G \times \mathbb{C}^\times}(e) \cap (Z_G(s) \times \mathbb{C}^\times)$. Denote by $A(e)$ and $A(e,s)$ the group of components of $Z_{G \times \mathbb{C}^\times}(e)$ and $Z(e,s)$, respectively. The natural map $A(e,s) \to A(e)$ is an injection, so $E_{e,s,r_0}$ carries an action of $A(e,s)$ obtained by restriction from the natural action of $A(e)$ on $H^*_\mathbb{C}(\mathcal{B}_e^*)$.

**Theorem 5.2** ([24, Theorem 1.15]). Let $r_0 \neq 0$.

1. Let $e$, $s$, $\psi$ be as above and $\psi \in \mathcal{A}(e,s)_0$. The $\mathbb{H}$-module $E_{e,s,r_0,\psi}$ has a unique maximal submodule. Let $\overline{E}_{e,s,r_0,\psi}$ be the irreducible quotient.

2. The map $(e,s,\psi) \to \overline{E}_{e,s,r_0,\psi}$ gives a one-to-one correspondence between the sets of $\mathbb{H}$-conjugacy classes of triples $(e,s,\psi)$ where $e \in \mathfrak{g}$ is nilpotent, $s \in \mathfrak{g}$ is semisimple with $[s,e] = 2r_0 e$, $\psi \in \mathcal{A}(s,e)_0$, and the set of simple $\mathbb{H}$-modules on which $r$ acts by $r_0$.

**5.3.** We recall next the classification and construction of irreducible tempered $\mathbb{H}$-modules following [24]. Suppose that $r_0 \in \mathbb{R}_{>0}$.

**Definition 5.3.** Denote $V^{+,*} = \{\xi \in V^* : \langle \xi(\alpha^\vee) \rangle > 0, \text{ for all } \alpha \in \Pi\}$ the set of dominant elements of $V^*$. Notice that $\delta(V^{+,*}) = V^{+,*}$. An irreducible $\mathbb{H}$-module $X$ is called tempered if every $S(V^*_e)$ weight $\nu \in V_C$ of $X$ satisfies

$$\langle \xi(\mathfrak{h}) \rangle \leq 0, \text{ for all } \xi \in V^{+,*}. $$

If all the inequalities are strict, then $X$ is called a discrete series module.

**Theorem 5.4** ([24, Theorem 1.21]). Let $e$, $s$, $\psi$ be as in Theorem 5.2 and $r_0 \in \mathbb{R}_{>0}$.

1. The simple module $\overline{E}_{e,s,r_0,\psi}$ is tempered if and only if there exists a Lie triple $\{e,h,f\}$ such that $[s,h] = 0$, $[s,f] = -2r_0 f$, and $\text{ad}(s-rh): \mathfrak{g} \to \mathfrak{g}$ has no real eigenvalues.

Moreover, in this case $\overline{E}_{e,s,r_0,\psi} = E_{e,s,r_0,\psi}$.

2. The module $\overline{E}_{e,s,r_0,\psi}(= E_{e,s,r_0,\psi})$ is a discrete series module if and only if $e$ is distinguished and $s = r_0 h$.

**5.4.** In light of Theorem 5.4, fix a Lie triple $\phi = \{e,h,f\}$, and consider the module $E_{e,h,1,\psi}$, with $\psi \in \mathcal{A}(e,h)_1$. This is a simple tempered $\mathbb{H}$-module (with real central character) on which $r$ acts by $r_0 = 1$. So we assume from now on that $r_0 = 1$ and drop it from the notations.

Assume that $\delta \neq \text{id}$. We extend the $\mathbb{H}$-module structure on $E_{e,h,\psi}$ to a $\mathbb{H}$-module structure. The construction is similar to section B. Assume, as we may, that $h \in \mathfrak{t}^\vee$. By [12, Lemma 3.7.3 and Remark 3.7.5 (ii)], $A(e) = A(h,e)$. Then the natural map

$$\{g \in G: \delta(\phi) = \text{Ad}(g)(\phi)\} \to \{g \in G: \delta(e) = \text{Ad}(g)(e)\}$$
induces a bijection on the connected components. Let \( g \in G \) with \( \delta(\phi) = \text{Ad}(g)(\phi) \) and that the image of \( g \) in \( \{ g \in G; \delta(e) = \text{Ad}(g)(e) \} \) is in the same connected component as the element \( g_1 \) in section B.4. Then \( \text{Ad}(g)^* \circ \delta^*: H_*(\mathcal{B}_e^h) \to H_*(\mathcal{B}_c^h) \). Since \( \delta(h) = h, \text{Ad}(g)^* \circ \delta^*: E_{c,h} \to E_{e,h} \). Similar to B.3.1, we have

\[
(\text{Ad}(g)^* \circ \delta^*)^2 = \text{id}. \tag{5.4.1}
\]

We have the following commuting diagram

\[
\begin{array}{ccc}
\mathbb{H} \times E_{e,h} & \xrightarrow{(\delta, \delta^*)} & \mathbb{H} \times E_{h(e),h} \xrightarrow{(\text{id}, \text{Ad}(g)^*)} \mathbb{H} \times E_{e,h} \\
\downarrow & & \downarrow \\
E_{e,h} & \xrightarrow{\delta^*} & E_{h(e),h} \xrightarrow{\text{Ad}(g)^*} E_{e,h}.
\end{array} \tag{5.4.2}
\]

Define the action of \( \delta \in \mathbb{H}_\# \) on \( E_{e,h} \) by \( \text{Ad}(g)^* \circ \delta^* \). By (5.4.1) and (5.4.2), this gives an action of \( \mathbb{H}_\# \) on \( E_{e,h} \). The map \( \text{Ad}(g)^* \circ \delta^*: H_*(\mathcal{B}_e^h) \to H_*(\mathcal{B}_c^h) \) induces an action on \( \tilde{A}(e,\hat{h})_0 \), which we denote by \( \delta \).

By [12, Lemma 3.7.3 and Remark 3.7.5 (ii)], \( A(e) = A(h, e) \). By [23, 10.13], there is an isomorphism of \( W \)-modules:

\[
E_{e,h,\psi} \cong X_1(e, \psi), \tag{5.5.1}
\]

where \( X_1(e, \psi) = H^*(\mathcal{B}_e) \psi \otimes \text{sgn} \) is the Springer \( W \)-representation from section 2. Since the \( W \)-structure of the module does not change under the \( \delta \)-twist, (5.5.3) implies that \( \delta(\psi) = \psi \).

Hence for any \( \psi \in \tilde{A}(e)_0 \), \( E_{e,h,\psi} \) is an \( \mathbb{H}_\# \)-module such that (5.5.3) is an isomorphism of \( W \)-modules, where the \( \delta \)-action on \( X_1(u, \psi) \) is as in section B.

5.5. Define the Clifford algebra \( C(V) \) of \( (V, \langle \ , \rangle) \) to be the real associative algebra with identity generated by \( \{ e \in V \} \) subject to the relations

\[
\xi \cdot \xi' + \xi' \cdot \xi = -2(\xi, \xi'), \quad \xi, \xi' \in V. \tag{5.5.1}
\]

The algebra \( C(V) \) is naturally \( \mathbb{Z}_{\geq 0} \)-filtered, where the \( n \)-th space \( C_n(V) \) in the filtration is the span of all elements of \( C(V) \) which are products of at most \( n \) elements of \( V \). The associated graded algebra is \( \bigwedge V \).

The Clifford algebra \( C(V) \) is also \( \mathbb{Z}/2\mathbb{Z} \)-graded \( C(V) = C(V)_{\text{even}} + C(V)_{\text{odd}} \) by the parity of the degree of homogeneous elements in the filtration just defined. Let \( \epsilon: C(V) \to C(V) \) be the involution which is \( +1 \) on \( C(V)_{\text{even}} \) and \( -1 \) on \( C(V)_{\text{odd}} \).

Let \( \xi: C(V) \to C(V) \) be the anti-automorphism defined by \( \xi(\xi') = -\xi' \) for all \( \xi \in V \).

Define the pin group

\[
\text{Pin}(V) = \{ g \in C(V)^* \mid g^* = g^{-1} \text{ and } \epsilon(g) \cdot \xi \cdot g^{-1} \in V, \text{ for all } \xi \in V \}. \tag{5.5.2}
\]

This is a central double extension of \( O(V) \) with the projection map \( p: \text{Pin}(V) \to O(V), p(g)(\xi) = \epsilon(g) \cdot \xi \cdot g^{-1}, g \in \text{Pin}(V), \xi \in V \). Since \( W \subset O(V) \), one considers

\[
\tilde{W} = p^{-1}(W) \subset \text{Pin}(V), \tag{5.5.3}
\]

a central double extension of \( W \).

When \( \text{dim} V \) is even, \( C(V) \) is a central simple algebra, and therefore it has a unique complex simple module \( S \) of dimension \( 2^{\text{dim} V/2} \). When \( \text{dim} V \) is odd, \( Z(C(V)) \) is two dimensional and \( C(V) = C(V)_{\text{even}} \otimes Z(C(V)) \). In this case, the unique simple module of \( C(V)_{\text{even}} \) can be extended in two inequivalent ways to
$C(V)$, $S^+$ and $S^-$. In what follows, we will refer to any one of $S$, $S^+$, $S^-$ as a spin module of $C(V)$. For convenience, we also set

$$S = \begin{cases} S, & \text{dim } V \text{ even,} \\ S^+ + S^-, & \text{dim } V \text{ odd.} \end{cases}$$  \hspace{1cm} (5.5.4)$$

One can restrict every spin module to $\text{Pin}(V)$ and furthermore to $\tilde{W}$. Since we assumed $V = 0$ (as the group $G$ is semisimple), $\tilde{W}$ generates $C(V)$, and therefore every spin module is an irreducible $\tilde{W}$-representation.

We have

$$S \otimes S = (\bigwedge V)^{a_V}$$  \hspace{1cm} (5.5.5)$$
as $W$-representations (or $O(V)$-representations), where $a_V = 1$ if $\text{dim } V$ is even, and $a_V = 2$ if $\text{dim } V$ is odd.

The group $\tilde{W}$ admits a Coxeter-like presentation. Let $z \in \text{Pin}(V)$ be the non-trivial element in $p^{-1}(1)$. Denote by $m(\alpha, \beta)$ the order in $W$ of $s_\alpha s_\beta$. Then:

$$\tilde{W} = \langle z, \tilde{s}_\alpha, \alpha \in I | z^2 = 1, (\tilde{s}_\alpha \tilde{s}_\beta)^{m(\alpha, \beta)} = z \rangle.$$  \hspace{1cm} (5.5.6)$$

5.6. Let $-1$ be the automorphism of $O(V)$ induced by $\xi \mapsto -\xi$ on $V$ and recall $W_\# = (W, \delta) = (W, -1) \subset O(V)$. Define $\tilde{W}_\# = p^{-1}(W_\#) \subset \text{Pin}(V)$.

We fix an orthonormal basis $\{\xi_1, \xi_2, \ldots, \xi_n\}$ of $V$ permuted by $\delta$, and set

$$\omega = \xi_1 \xi_2 \ldots \xi_n \in \text{Pin}(V).$$  \hspace{1cm} (5.6.1)$$

Lemma 5.5. The element $\omega$ satisfies the following properties:

1. $\omega^2 = z^{n(n+1)/2}$;
2. $p(\omega) = -1 \in O(V)$;
3. $\omega \xi = z^{n-1} \xi \omega$, $\xi \in V$.

Proof. Straightforward. \hfill \square

Therefore, $\omega$ is a central element in $C(V)$ if $\text{dim } V$ is odd, and $\omega$ is a central element of $C(V)_\text{even}$, when $\text{dim } V$ is even. When $\text{dim } V$ is even, denote by $S^\pm$ the two constituents in the restriction of spin module $S$ to $C(V)_\text{even}$. With this notation, $\omega$ acts by scalars on $S^\pm$ in both cases $\text{dim } V$ odd or even. Since the trace of $-1 \in O(V)$ on $\bigwedge V$ is zero, (5.5.5) implies that the trace of $\omega$ in $S^+ + S^-$ is zero as well. Therefore, the scalars by which $\omega$ acts on $S^+, S^-$ differ by a sign, i.e.,

$$\omega|_{S^+} = -\omega|_{S^-} = c \in \mathbb{C}.$$  \hspace{1cm} (5.6.2)$$

(We have $c^2 = (-1)^{n(n+1)/2}$, but we will not need to use this fact.)

For every $\tilde{w} \in \tilde{W}$, define

$$\text{sgn}(\tilde{w}) = \text{sgn}(p(\tilde{w})).$$  \hspace{1cm} (5.6.3)$$

We will use the formula ([3, Lemma 3.4])

$$\tilde{w} \cdot \xi \cdot \tilde{w}^{-1} = \text{sgn}(\tilde{w})w(\xi), \quad \tilde{w} \in \tilde{W}, \xi \in V.$$  \hspace{1cm} (5.6.4)$$

Fix once for all $\tilde{w}_0 \in p^{-1}(w_0)$ and set

$$\tilde{\delta} = \tilde{w}_0 \omega, \text{ so that } p(\tilde{\delta}) = \delta.$$  \hspace{1cm} (5.6.5)$$

It is easy to check that

$$\tilde{\delta} \xi = z^{n+\ell(w_0)} \delta(\xi) \tilde{\delta}, \quad \xi \in V.$$  \hspace{1cm} (5.6.5)$$
The group $\tilde{W}_\#$ is generated by $\tilde{W}$ and $\omega$. It is also generated by $\tilde{W}$ and $\tilde{\delta}$. Set

\[
\tilde{W}' = \begin{cases} 
\tilde{W} \cap C(V)_{\text{even}}, & \text{dim } V \text{ even}, \\
\tilde{W}, & \text{dim } V \text{ odd},
\end{cases}
\tag{5.6.6}
\]

and

\[
\tilde{W}'_\# = \begin{cases} 
(\tilde{W} \cap C(V)_{\text{even}}, \omega), & \text{dim } V \text{ even}, \\
\tilde{W}_\#, & \text{dim } V \text{ odd}.
\end{cases}
\tag{5.6.7}
\]

The discussion above shows that $S^\pm$ are $\tilde{W}'_\#$-representations.

**5.7.** Following [3, Definition 3.1], define the Dirac element

\[
D = \sum_{i=1}^{n} \tilde{\xi}_i \otimes \xi_i \in H \otimes C(V) \subset H_\# \otimes C(V),
\tag{5.7.1}
\]

where $\tilde{\xi}_i$ is defined in (5.1.2). Let $\rho: C[\tilde{W}_\#] \to \mathbb{H}_\# \otimes C(V)$ be the linear map extending $\tilde{w} \mapsto p(\tilde{w}) \otimes \tilde{w}$, $\tilde{w} \in \tilde{W}$ and $\tilde{\delta} \mapsto \tilde{\delta} \otimes \tilde{\delta}$. Notice that $\rho(\omega) = w_0\delta \otimes \omega \in \mathbb{H}_\# \otimes C(V)$.

The algebra $C(V)$ also has a $\ast$-operation defined by the transpose map $t$. On $\text{Pin}(V)$ this corresponds to the inversion operation. With respect to this operation, the spin modules $S, S^\pm$ admit positive definite invariant forms (i.e., they are unitary).

Define the Casimir element of $H$ ([3, Definition 2.3]):

\[
\Omega = \sum_{i=1}^{n} \xi_i^2.
\tag{5.7.2}
\]

This is an element in $S(V)^{W_\#}$, thus central in $\mathbb{H}_\#$. The central characters of irreducible $\mathbb{H}_\#$-modules are parameterized by $W_\#$-orbits in $V_C$. By [3, Lemma 2.5], if $(\pi, X)$ is an irreducible $\mathbb{H}_\#$-module with central character $W_\# \cdot \nu$, then $\pi(\Omega)$ acts on $X$ by the scalar $(\nu, \nu)$.

Let $\Phi^\vee \subset V$ be corresponding coroots and $\Phi^{\vee, +}$ the positive coroots, $\Phi^{\vee, -}$, the negative coroots. Define the Casimir element of $\tilde{W}$ ([3, section 3.4])

\[
\Omega_{\tilde{W}} = z \sum_{\alpha, \beta \in \Phi^+, \tilde{s}_\alpha \tilde{s}_\beta \in C[\tilde{W}]} |\alpha^\vee||\beta^\vee| \tilde{s}_\alpha \tilde{s}_\beta \in \mathbb{C}[\tilde{W}]^{\tilde{W}},
\tag{5.7.3}
\]

where $z, \tilde{s}_\alpha$ are as in (5.5.6).

**Proposition 5.6.** The element $D$ has the following properties:

1. $D^\ast = D$.
2. $\rho(\tilde{w})D = \text{sgn}(\tilde{w})D\rho(\tilde{w})$, for all $\tilde{w} \in \tilde{W}$;
3. $\rho(\omega)D = z^{\ell(w_0) + n}\text{sgn}(w_0)D\rho(\omega)$.
4. $D^2 = -\Omega \otimes 1 + r^2 \rho(\Omega_{\tilde{W}}) \in \mathbb{H} \otimes C(V)$.

**Proof.** (1) Straightforward.

(2) This is [3, Lemma 3.4].

(3) We have $\omega = \tilde{w}_0^{-1}\tilde{\delta}$. From (5.6.5), we see that $\rho(\tilde{\delta})D = z^{\ell(w_0) + n}D\rho(\tilde{\delta})$. The claim now follows from (2).

(4) This is [3, Theorem 3.5].
**Definition 5.7.** Let $X$ be an $\mathbb{H}_\#$-module, and $S$ be a spin $C(V)$-module (when $\dim V$ is odd, there are two choices). Since $\mathcal{D} \subset \mathbb{H}_\# \otimes C(V)$, it gives an operator on $X \otimes S$. We denote this operator by

$$D_\# : X \otimes S \to X \otimes S$$

and call it the Dirac operator (of $X$ and $S$). Define the extended Dirac cohomology of $X$ (with respect to $S$)

$$H^D_\#(X) = \ker(D_\#) / (\ker D_\# \cap \text{im} D_\#).$$

(5.7.4)

By Proposition 5.6(2),(3), $H^D_\#(X)$ is a $\tilde{\mathbb{W}}_\#$-representation. From Proposition 5.6(1), when $X$ is a $\ast$-unitary $\mathbb{H}_\#$-module, $H^D_\#(X) = \ker D_\#$.

**Definition 5.8.** Suppose dim $V$ is even. By restriction, $D_\#$ defines two operators $D_\#^\pm : X \otimes S^\pm \to X \otimes S^{\mp}$.

(5.7.5)

Suppose dim $V$ is odd. In this case, $S^+$ and $S^-$ are realized on the same vector space $U$ (coming from the unique simple module of $C(V)_{\text{even}}$). Then, as in [11, section 2.9], $D_\#$ : $X \otimes S^+ \to X \otimes S^-$ can be composed with the vector space identity map $S^+ \to S^-$ to yield $D_\#^+ : X \otimes S^+ \to X \otimes S^-$.

In both cases, set

$$H^{D^\pm}_\#(X) = \ker(D_\#^\pm) / (\ker D_\#^\pm \cap \text{im} D_\#^{\mp}).$$

(5.7.6)

**Definition 5.9.** The extended Dirac index is

$$I_\#(X) = H^{D^+}_\#(X) - H^{D^-}_\#(X).$$

By Proposition 5.6(2),(3), $H^{D^\pm}_\#(X)$ are $\tilde{W}_\#$-representations, and $I_\#(X)$ is a virtual $\tilde{W}_\#$-module.

Notice that

$$D_\#^\pm \circ D_\#^{\mp} = -\Omega \otimes 1 + r^2 \rho(\Omega \tilde{\omega}),$$

(5.7.7)

by Proposition 6.6(4), since $D_\#^\pm$ are given by the left action of $\mathcal{D}$.

**Proposition 5.10.** For every $\mathbb{H}_\#$-module $X$, we have $I_\#(X) = X \otimes (S^+ - S^-)$ as virtual $\tilde{W}_\#$-modules. In particular, for $\tilde{w} \in \tilde{W}_\#$, we have

$$\text{tr}(\tilde{w}, I_\#(X)) = \text{tr}(w, X) \text{tr}(\tilde{w}, S^+ - S^-),$$

$$\text{tr}(\tilde{w} \omega, I_\#(X)) = c \text{tr}(w w_0 \delta, X) \text{tr}(\tilde{w}, S),$$

(5.7.8)

where $S$ is as in (5.5.4), and $c$ is the scalar by which $\omega$ acts in $S^+$.

**Proof.** The first claim is proved identically with [10, Lemma 4.1]. The second claim is immediate from the first since $z$ acts by $c$ in $S^+$ and by $-c$ in $S^-$, and $\rho(z) = w_0 \delta \otimes z$. \qed

**5.8.** We are now in position to compute the extended Dirac index of the simple tempered $\mathbb{H}_\#$-modules $E_{e,h,\psi}$.

**Theorem 5.11.** Let $E_{e,h,\psi}$ be a simple tempered $\mathbb{H}_\#$-module as above.

(1) The $\delta$-twisted trace of $E_{e,h,\psi}$ on $W$ is given by:

$$\text{tr}(w w_0 \delta, E_{e,h,\psi}) = (-1)^d sgn(w_0) X_{-1}(e, \psi)(w), \text{ for all } w \in W.$$  

(5.8.1)
(2) The extended Dirac index of $E_{e,h,\psi}$ is given by the formula:
\[
\text{tr}(\tilde{w}, I_\#(E_{e,h,\psi})) = \text{tr}(\tilde{w}, X_1(e, \psi) \otimes (S^+ - S^-)),
\]
\[
\text{tr}(\tilde{w}, \nu) = c' \text{tr}(\tilde{w}, X_1(e, \psi) \otimes S),
\]
with $\tilde{w} \in \tilde{W}'$, $\omega$ as in (5.6.1), $c' = (-1)^{\delta \cdot \text{sgn}(w_0)}c$, where $c$ is the scalar from Proposition 5.10.

Proof. The above construction of the $\delta$-action on $E_{e,h,\psi}$ gives $\text{tr}(w\delta, E_{e,h,\psi}) = X_1(u, \psi)(w\delta)$. Then (1) follows from Corollary B.2.

For (2), Proposition 5.10 implies that $\text{tr}(\tilde{w}, I_\#(E_{e,h,\psi})) = \text{tr}(w, E_{e,h,\psi}) \text{tr}(\tilde{w}, S^+ - S^-)$ and $\text{tr}(\tilde{w}, I_\#(E_{e,h,\psi})) = c \text{tr}(w\omega_0\delta, E_{e,h,\psi}) \text{tr}(\tilde{w}, S)$. The formula now follows from part (1) and (5.4.3).

\[\Box\]

5.9. The analogue of Vogan’s conjecture from real reductive groups in the setting of the graded affine Hecke algebra was stated and proved in [3]. We need the following algebraic form.

**Theorem 5.12** ([3, Theorem 4.2]). For every $y \in Z(\mathbb{H})$, there exist a unique element $\zeta(y) \in Z(\mathbb{C}[\tilde{W}])$ and an element $a \in \mathbb{H} \otimes C(V)$ such that
\[
y \otimes 1 = \rho(\zeta(y)) + DA + aD,
\]
as elements of $\mathbb{H} \otimes C(V)$. Moreover, the map $\zeta : Z(\mathbb{H}) \to Z(\mathbb{C}[\tilde{W}])$ is an algebra homomorphism.

In fact, as one can see from the proof of [3, Theorem 4.2] (cf. [10, Theorem 3.2]), the element $a$ belongs to $\mathbb{H} \otimes C(V)_{\text{odd}}$. It is also noticed in [10, Corollary 3.3] that the image of the map $\zeta$ lies in $\mathbb{C}[\tilde{W}][\tilde{W}]$. Following [10, Definition 4.3] (compare also with [3, Definition 4.3]), if $\tilde{\sigma}$ is an irreducible $\tilde{W}'$-representation, one can attach canonically a homomorphism $\chi^{\tilde{\sigma}} : Z(\mathbb{H}) \to \mathbb{C}$ (i.e., a central character of $\mathbb{H}$-modules) by the requirement
\[
\chi^{\tilde{\sigma}}(y) = \tilde{\sigma}(\zeta(y)), \text{ for all } y \in Z(\mathbb{H}).
\]

Notice that if $\tilde{\sigma}$ is an irreducible $\tilde{W}$-representation, the central character $\chi^{\tilde{\sigma}}_1$ is the same for all irreducible $\tilde{W}'$-representations $\tilde{\sigma}_1$ that appear in the restriction of $\tilde{\sigma}$ to $\tilde{W}'$. Thus we can also denote $\chi^{\tilde{\sigma}}$ for an irreducible $\tilde{W}$-representation $\tilde{\sigma}$.

Let $W \cdot \nu$ denote the $W$-orbit in $V_\mathbb{C}$ corresponding to $\chi^{\tilde{\sigma}}$.

We following is a slight sharpening of [3, Theorem 4.4].

**Corollary 5.13.** Let $\epsilon \in \{+, -\}$. Suppose $(\pi, X)$ is an irreducible $\mathbb{H}_\#$-module with central character $W_\# \cdot \nu$ and that $\tilde{\sigma}$ is an irreducible $\tilde{W}'$-representation such that
\[
\text{Hom}_{\tilde{W}'}(\tilde{\sigma}, H_\#^{\tilde{\sigma}}(X)) \neq 0,
\]
where $H_\#^{\tilde{\sigma}}(X)$ is as in Definition 5.8. Then $W_\# \cdot \nu = W_\# \cdot \nu_{\tilde{\sigma}}$.

Proof. We show how the claim follows from Theorem 5.12. This is analogous with the proof of [3, Theorem 4.4], but we need some minor modification because we are considering $D^\pm$ (rather than operators $D$).

When dim $V$ is odd, $S^\epsilon$ and $S^{-\epsilon}$ are realized on the same vector space $U^\epsilon = U^{-\epsilon}$. When dim $V$ is even, $S^\epsilon$ and $S^{-\epsilon}$ are realized on the vector spaces $U^\epsilon$ and $U^{-\epsilon}$, respectively.
Let $\gamma$ denote Clifford multiplication by elements in $C(V)$ on the spin module $S^\epsilon$. When $\dim V$ is odd, $\gamma$ is a $C(V)$-action on $U^\epsilon$. When $\dim V$ is even, $\gamma(\xi)$, where $\xi \in C(V)_{\text{odd}}$ take $U^\epsilon$ to $U^{-\epsilon}$.

Let $y \in Z(\mathfrak{h})$ and $x \in X \otimes U^\epsilon$ be an element in the $\tilde{W}$-isotypic component of $\tilde{s}$ in $H^D_{\#}(X)$. Then $(\pi(y) \otimes \gamma(1))\tilde{x} = \chi_y(y)\tilde{x}$ and $(\pi \otimes \gamma^*)(\rho(\xi(y)))\tilde{x} = \tilde{s}(\rho(\xi(y)))\tilde{x} = \chi^\tilde{s}(y)\tilde{x}$. Together with (5.9.1), it follows that:

$$(\chi_y(y) - \chi^\tilde{s}(y))\tilde{x} = (\pi \otimes \gamma)(y \otimes 1 - \rho(\xi(y)))\tilde{x} = (\pi \otimes \gamma)(D_{\tilde{a}} + \tilde{a}D)\tilde{x},$$

$$(\pi \otimes \gamma)(D_{\tilde{a}})\tilde{x}.$$

By the discussion above $a \in H \otimes C(V)_{\text{odd}}$, and so when $\dim V$ is even, $ax \in X \otimes U^{-\epsilon}$. Therefore, regardless of the parity of $\dim V$, the right hand side is in $\text{im } D_{\#}^{-\epsilon}$, and it follows by the definition of $H^D_{\#}(X)$ that it must be zero. In conclusion, $\chi_y = \chi^\tilde{s}$.

6. Spin Weyl group representations

6.1. We denote by $\text{Irr}_{\text{gen}}(\tilde{W})$ the set of (isomorphism classes of) irreducible genuine representations of $\tilde{W}$, i.e., the irreducible representations of $\tilde{W}$ which do not factor through $W$. Let $R(\tilde{W})_{\text{gen}}$ be the subspace of $R(\tilde{W})$ spanned by $\text{Irr}_{\text{gen}}(\tilde{W})$. Denote

$$S_g : R(\tilde{W}) \rightarrow R(\tilde{W}), \quad S_g(\sigma) = \sigma \otimes \text{sgn}.$$ 

Let $R(\tilde{W})^{S_g}$ denote the (+1)-eigenspace of $S_g$. Notice that $S \otimes \text{sgn} = S$. Define the linear map

$$\iota : R(W) \rightarrow R(\tilde{W})^{S_g}, \quad \iota(\sigma) = \sigma \otimes S.$$ (6.1.1)

Recall that $a_V = 1$ if $\dim V$ is even, and $a_V = 2$ if $\dim V$ is odd.

**Proposition 6.1.** The map $\iota$ induces an injective linear map $\iota : \mathcal{R}_{-1}(W) \rightarrow R(\tilde{W})^{S_g}$ such that for $\sigma, \sigma' \in \mathcal{R}(W)$,

$$\langle \iota(\sigma), \iota(\sigma') \rangle_{\tilde{W}} = a_V \langle \sigma, \sigma' \rangle_{\tilde{W}}^{-1}.$$ 

Moreover, $\iota(X_{-1}(c, \phi)) \neq 0$ if and only if $c \in \mathcal{N}^\text{sol}$ and

$$\langle \iota(X_{-1}(c, \phi)), \iota(X_{-1}(c', \phi')) \rangle_{\tilde{W}} = \begin{cases} a_V \langle \phi, \phi' \rangle_{\mathcal{A}(c)}^{-1}, & \text{if } c = c' \in \mathcal{N}^\text{sol} \\ 0, & \text{otherwise.} \end{cases}$$ (6.1.2)

**Proof.** Since $S^* \cong S$, the first claim is immediate from (5.5.5) and the definition of $\langle \ , \ \rangle_{\tilde{W}}$. The second claim follows from Proposition 4.1(2).

By Theorem 2.7 in the case $q = -1$, the map $\phi : X(c, \phi)$ induces an isometric isomorphism $\mathcal{R}_{-1}(A(c))_0 \rightarrow \mathcal{R}_{-1}(W)^c$. Composing with $\iota$, this implies that $\iota(X_{-1}(c, \phi)) \in R(\tilde{W})_{\text{gen}}$, $c \in \mathcal{N}^\text{sol}$, is nonzero and that (6.1.2) holds. \(\square\)

We relate these facts with the extended Dirac index from section 5.

**Lemma 6.2.** If $w \in W$ is a (-1)-elliptic element, then $\text{sgn}(w) = 1$.

**Proof.** Suppose $w \in W$ is such that $\det_V(1 + w) \neq 0$. This means that $w$ acting on $V$ does not have the eigenvalue $-1$. Since $V$ is a real representation, the only real eigenvalue of $w$ is 1 and the complex eigenvalues come in pairs. Thus $\text{sgn}(w) = \det_V(w) = 1$. \(\square\)
Proposition 6.3. Let $E_{c,h,\phi}$ be a simple tempered $\mathbb{H}_{\mathfrak{g}}$-module as in section 5.4. The extended Dirac index $I_\#(E_{c,h,\phi}) \neq 0$ if and only if $c \in \mathcal{N}^{\text{sol}}$.

Proof. In one direction, suppose $c \in \mathcal{N}^{\text{sol}}$. By Proposition 6.1, $\iota(X_{-1}(e,\phi)) \neq 0$. By (5.5.5), $S$ is supported on $W_{(1)}$ and Lemma 6.2 implies that the support of $\iota(X_{-1}(e,\phi))$ is in $\tilde{W}'$. Therefore, (5.8.2) says that the restriction of $I_\#(E_{c,h,\phi})$ to $\tilde{W}^\prime \omega$ is nonzero.

For the converse, suppose that $I_\#(E_{c,h,\phi}) \neq 0$. Again by (5.8.2), there are two cases. If there exists $\tilde{w} \in \tilde{W}'$, then $\text{tr}(\tilde{w}, I_\#(E_{c,h,\phi})) \neq 0$, then $\text{tr}(\tilde{w}, X_1(e,\phi) \otimes (S^+ - S^-)) \neq 0$. By [11, Proposition 3.1], which is the analogue of Proposition 6.1 above, it follows that

$$\langle X_1(e,\phi), X_1(e,\phi) \rangle_{\mathcal{A}(\mathfrak{g})}^1 \neq 0.$$ 

Then Proposition 4.1(1) implies that $e$ is quasidistinguished and so $c \in \mathcal{N}^{\text{sol}}$. 

If, on the other case, $\text{tr}(\tilde{w}, I_\#(E_{c,h,\phi})) \neq 0$, then $\text{tr}(\tilde{w}, \iota(X_{-1}(e,\phi)) \neq 0$, and by Proposition 6.1, $c \in \mathcal{N}^{\text{sol}}$.

Corollary 6.4. Let $\{e, h, f\}$ be a Lie triple in $\mathfrak{g}$, $h \in \mathfrak{h}^\delta$, such that $e \in \mathcal{N}^{\text{sol}}$, and $\tilde{\sigma} \in \text{Irr}_{\mathfrak{h}} \tilde{W}$. Then $\tilde{W} \cdot \nu_\tilde{\sigma} = \tilde{W} \cdot h$, where $\nu_\tilde{\sigma}$ is the central character of $\tilde{W}$ defined by $\tilde{\sigma}$ in section 5.9. In particular, $\tilde{\sigma}(\Omega_{\tilde{W}}) = (h, h)$, where $\Omega_{\tilde{W}}$ is as in (5.7.3).

Proof. Let $\tilde{\sigma} \in \text{Irr}_{\mathfrak{h}} \tilde{W}$. There exists $\phi \in \mathcal{A}(e)_0$ such that $\text{Hom}_{\tilde{W}}[\tilde{\sigma}, \iota(X_{-1}(e,\phi))] \neq 0$. Let $\tilde{\sigma}_1$ be an irreducible $\tilde{W}^\prime$-representation such that $\text{Hom}_{\tilde{W}}[\tilde{\sigma}_1, \tilde{\sigma}] \neq 0$ and $\langle \tilde{\sigma}_1, \iota(X_{-1}(e,\phi)) \rangle_{\tilde{W}^\prime} \neq 0$.

Write $I_\#(E_{c,h,\phi}) = \sum_j a^j \tilde{\sigma}_j^\#$ for $\tilde{\sigma}_j^\#$ irreducible distinct $\tilde{W}^\prime$-representations, and $a^j \in \mathbb{Z}^+$. Suppose further that $\tilde{\sigma}_j^\# = \sum \tilde{\sigma}_i^\prime$ as $\tilde{W}^\prime$-representations, where $\tilde{\sigma}_i^\prime$ are irreducible $\tilde{W}^\prime$-representations. Since $\omega$ commutes with $\tilde{W}^\prime$, it acts by a nonzero scalar $u_i^\prime$ (in fact, a fourth root of 1) on $\tilde{\sigma}_i^\prime$. Thus

$$\text{tr}(\tilde{w}, I_\#(E_{c,h,\phi})) = \sum_j \sum_i a^j u_i^\prime \text{tr}(\tilde{w}, \tilde{\sigma}_i^\prime).$$

On the other hand, by Proposition 6.3, the left hand side equals (up to a nonzero scalar) $\text{tr}(\tilde{w}, \iota(X_{-1}(e,\phi))$, and so $\text{tr}(\tilde{w}, \tilde{\sigma}_j)$ appears in the linear combination. By the linear independence of irreducible $\tilde{W}^\prime$-characters, it follows that there exist $i, j$ such that $\tilde{\sigma}_i^\prime = \tilde{\sigma}_j$. In other words, there exists $j$ such that $\tilde{\sigma}_j^\#$ contains $\tilde{\sigma}_j$.

Since $\tilde{\sigma}_j^\#$ occurs in $I_\#(E_{c,h,\phi})$, it must occur in one of the spaces $H_{\mathfrak{g}}^{D^\pm}(E_{c,h,\phi})$ for a choice of sign $\pm$. This implies that $\text{Hom}_{\tilde{W}}[\tilde{\sigma}_1, H_{\mathfrak{g}}^{D^\pm}(E_{c,h,\phi})] \neq 0$. Corollary 5.13 says that $\tilde{W}_\# \cdot h = W_\# \cdot \nu_{\tilde{\sigma}_1} = W_\# \cdot \nu_{\tilde{\sigma}}$. 


Since $h$ is $\delta$-stable, the first claim of the corollary is proved. For the second claim, it is sufficient to notice that by definition $(\nu_{\tilde{\sigma}}, \nu_{\tilde{\sigma}}) = \tilde{\sigma}(\Omega_{\tilde{W}})$.

\section*{6.3. We state the main results of this section.}

**Theorem 6.5.** \( \text{lr}_{\text{gen}} \tilde{W} = \bigcup_{e \in G \setminus N^\text{sol}} \text{lr}_e \tilde{W}. \)

**Proof.** Let $\tilde{\sigma}$ be an irreducible genuine $\tilde{W}$-representation. Then $\tilde{\sigma} \otimes S$ is a $W$-representation. Since $\{X_{-1}(e, \phi) : e \in G \setminus N, \phi \in \tilde{A}(e)_{\tilde{0}}\}$ is a basis of $R(W)$, there exists $(e, \phi)$ such that $\langle \tilde{\sigma} \otimes S, X_{-1}(e, \phi) \rangle_W \neq 0$, or equivalently $\langle \tilde{\sigma}, \iota(X_{-1}(e, \phi)) \rangle_{\tilde{W}} \neq 0$. In particular, $\iota(X_{-1}(e, \phi)) \neq 0$, thus by Proposition 6.1, $e \in N^\text{sol}$. The disjointness follows from Corollary 6.4.

**Theorem 6.6.** Let $e \in N^\text{sol}$ and $\phi \in \tilde{A}(e)_{\tilde{0}}$ be given.

1. For every $\tilde{\sigma} \in \text{lr}_{e,[\phi]} \tilde{W}$, $m_{e,\phi}(\tilde{\sigma}) = \langle \sigma(e, \phi) \otimes S, \tilde{\sigma} \rangle_{\tilde{W}}$, and in particular, \( \iota(X_{-1}(e, \phi)) = X_{-1}(e, \phi) \otimes S \) is the character of a genuine representation of $\tilde{W}$.
2. If $(\phi, \phi)_{A(e)}^{-1} = 1$, then:
   - (a) $\text{lr}_{e,[\phi]} \tilde{W} = \{ \tilde{\sigma}(e, [\phi]) \}$, where $\tilde{\sigma}(e, [\phi])$ is irreducible sgn self dual, if $\dim V$ is even;
   - (b) $\text{lr}_{e,[\phi]} \tilde{W} = \{ \tilde{\sigma}(e, [\phi]^+), \tilde{\sigma}(e, [\phi]^-) \}$, where $\tilde{\sigma}(e, [\phi])^\pm$ are irreducible sgn dual representations, when $\dim V$ is odd.
3. Let $\phi \neq \phi'$ in $\tilde{A}(e)_{\tilde{0}}$. If $(\phi, \phi')_{A(e)}^{-1} = 0$, then $\text{lr}_{e,[\phi]} \tilde{W} \cap \text{lr}_{e,[\phi']} \tilde{W} = \emptyset$.

**Remark 6.7.** The assumptions in (2) and (3) of Theorem 6.6 are automatically satisfied when $e$ is distinguished.

**Proof.** Since the matrix of Green polynomials $K(q)$ is upper triangular with 1 on the diagonal, $K(q)^{-1}$ is again upper triangular with 1 on the diagonal and polynomials in $q$ above the diagonal. It is well-known that the $W$-types in $X_q(e, \phi)$, other than $\sigma(e, \phi)$ are all of the form $\sigma(e', \phi')$ with $e' \succ e$. Thus, in $R_q(W)$, we have:

\[
X_q(e, \phi) = \sigma(e, \phi) - \sum_{e' > e} K(q)_{e, e'}^{-1} X_q(e', \phi').
\]

(6.3.1)

Apply this identity when $q = -1$ and tensor with $S$:

\[
\iota(X_{-1}(e, \phi)) = \sigma(e, \phi) \otimes S - \sum_{e' > e} K(-1)_{e, e'}^{-1} \iota(X_{-1}(e', \phi')).
\]

(6.3.2)

an identity in $R(\tilde{W})_{\text{gen}}$. Suppose that $\tilde{\sigma}$ occurs in the LHS. By Corollary 6.4, $W \cdot \nu_{\tilde{\sigma}} = W \cdot h$, where $h$ is a neutral element for a Lie triple of $e$. In particular, $\tilde{\sigma}$ cannot occur in any $\iota(X_{-1}(e', \phi'))$ with $G \cdot e' \neq G \cdot e$, since $h$ determines $G \cdot e$. Thus $\tilde{\sigma}$ can only occur in $\sigma(e, \phi) \otimes S$. This proves (1).

For (2), suppose $(\phi, \phi)_{A(e)}^{-1} = 1$. (The fact that this is always the case when $e$ is distinguished is immediate.) By Proposition 6.1, $\langle \iota(X_{-1}(e, \phi)), \iota(X_{-1}(e, \phi)) \rangle_{\tilde{W}} = a_V$. Since $\iota(X_{-1}(e, \phi))$ is also sgn-dual, the only possibility when $\dim V$ is even is the one stated in (3).

If $\dim V$ is odd, then $\iota(X_{-1}(e, \phi)) = \tilde{\sigma}(e, [\phi])^+ + \tilde{\sigma}(e, [\phi])^-$. It remains to verify that $\tilde{\sigma}(e, [\phi])^- = \tilde{\sigma}(e, [\phi])^+ \otimes \text{sgn}$. Suppose this is not the case, so that $\tilde{\sigma}(e, [\phi])^-$
are \(\text{sgn}\) self-dual. By (6.3.2), they occur in \(\sigma(e, \phi) \otimes S\) with multiplicity 1. But \(S = S^+ + S^-\) and since \(S^+ = S^- \otimes \text{sgn}\), this is impossible.

Claim (3) follows similarly from Proposition 6.1.

\[6.4.\]
As mentioned in the introduction, Theorem 6.6 can be used to obtain character formulas for \(X_{-1}(e, \phi)\) and \(H^*(\mathcal{B}_e)^\phi\). Recall the notation from the introduction
\[\tilde{\Sigma}(e, \phi) = \iota(X_{-1}(e, \phi)) = X_{-1}(e, \phi) \otimes S.\]
For every \(w \in W_{(-1),\text{eli}}\), we have then
\[
\text{tr}(w, X_{-1}(e, \phi)) = \sum_{i=0}^{d_e} (-1)^{d_e-i} \text{tr}(w, H^{2i}(\mathcal{B}_e)^\phi) = \frac{\text{tr}(\tilde{w}, \Sigma(e, \phi))}{\text{tr}(\tilde{w}, S)},
\]
where \(\tilde{w}\) is a representative of the preimage of \(w\) in \(\tilde{W}\), and \(d_e = \dim \mathcal{B}_e\). In particular, when \(w = 1\), we find
\[
\sum_{i=0}^{d_e} (-1)^i \dim H^{2i}(\mathcal{B}_e)^\phi = (-1)^d_e \frac{\dim \tilde{\Sigma}(e, \phi)}{\dim S}.
\]
Using Corollary B.2, (6.4.1) can also be interpreted as a character formula of \(H^*(\mathcal{B}_e)^\phi\) on \(\delta\)-twisted elliptic conjugacy classes.

**Corollary 6.8.** Let \(w\) be a \(\delta\)-elliptic element of \(W\). Then
\[
\text{tr}(w, H^* (\mathcal{B}_e)^\phi) = (-1)^d_e \text{sgn}(w_0) \frac{\text{tr}(\tilde{w} \tilde{w}_0, \tilde{\Sigma}(e, \phi))}{\text{tr}(\tilde{w} \tilde{w}_0, S)},
\]
where \(\tilde{w}_0\) is a representative of the preimage of \(w_0\) in \(\tilde{W}\).

The calculations in Appendix A allow us to describe \(\tilde{\Sigma}(e, \phi)\) explicitly and by comparison to [9], to identify \(\tilde{\Sigma}(e, \phi)\) in terms of the known classifications of irreducible \(\tilde{W}\)-representations. Therefore, Corollary 6.8 can be effectively used as a character formula for \(H^*(\mathcal{B}_e)^\phi\) on \(\delta\)-elliptic classes.

**Example 6.9.** In \(\text{GL}(n)\), the class of the element \(e \in \mathcal{N}^\text{sol}\) is parameterized, via the Jordan canonical form, by a partition \(\lambda\) of \(n\) into distinct parts, and \(A(e) = \{1\}\). A partition \(\lambda\) of \(n\) is called even if \(\ell(\lambda) \equiv n \pmod{2}\), otherwise it is called odd, where \(\ell(\lambda)\) is the number of parts of \(\lambda\). Then
\[
\tilde{\Sigma}(e, \lambda) = a_\lambda \begin{cases} 
\tilde{\sigma}_\lambda, & \text{\(\lambda\) even}, \\
\tilde{\sigma}_\lambda^+ + \tilde{\sigma}_\lambda^-, & \text{\(\lambda\) odd}, 
\end{cases}
\]
where \(a_\lambda\) is as in Proposition A.3, and \(\sigma_\lambda^+, \sigma_\lambda^\pm\) are the irreducible \(\tilde{S}_n\)-representations constructed by Schur (cf. [40]). The dimension of \(\tilde{\sigma}_\lambda\), or of each of \(\tilde{\sigma}_\lambda^\pm, \lambda = (\lambda_1, \ldots, \lambda_r)\), is given by the formula
\[
2^{\frac{\lambda - \ell(\lambda)}{2}} g^\lambda, \quad \text{where } g^\lambda = \frac{n!}{\lambda_1! \cdots \lambda_r!} \prod_{i < j} \lambda_i - \lambda_j.
\]
Since \(\dim S = 2^{\left\lfloor \frac{n}{2} \right\rfloor}\), (6.4.2) becomes in this case:
\[
\sum_{i=0}^{d_e} (-1)^i \dim H^{2i}(\mathcal{B}_e) = (-1)^{d_e} g^\lambda.
\]
6.5. We end with an application to the decomposition of tensor products \( \sigma \otimes S \), \( \sigma \in \hat{W} \).

Let \( \sigma \) be an irreducible \( \hat{W} \)-representation. By Springer’s correspondence, write \( \sigma = \sigma(e, \phi) \) for \( e \in \mathcal{N}, \phi \in \hat{A}(e)_0 \). Since every \( \tilde{\sigma} \) occurs in a \( \Sigma(e', \phi') = \iota(X_{-1}(e', \phi')) \), with \( e' \in \mathcal{N}^{\text{nil}} \), we consider:

\[
(\sigma(e, \phi) \otimes S, \iota(X_{-1}(e', \phi')))_{\hat{W}}.
\]  

(6.5.1)

By (6.3.2),

\[
\sigma(e, \phi) \otimes S = \sum_{e'' \in \mathcal{E}} K(-1)_{(e, \phi), (e', \phi')} \iota(X_{-1}(e'', \phi'')),
\]

(6.5.2)

with \( K(-1)_{(e, \phi), (e', \phi')} = 1 \) if \( \phi'' = \phi \), and 0 if \( \phi'' \neq \phi \). Using Proposition 6.1, we find

\[
\langle \sigma(e, \phi) \otimes S, \iota(X_{-1}(e', \phi')) \rangle_{\hat{W}} = a_V \sum_{\phi'' \in \hat{A}(e)_0} K(-1)_{(e, \phi), (e', \phi')} (\phi', \phi'')_{\hat{A}(e)}^{-1}.
\]

(6.5.3)

In particular, this is zero unless \( e' \geq e \).

**Corollary 6.10.** Let \( e \in \mathcal{N} \) and \( \phi \in \hat{A}(e)_0 \). If \( \langle \sigma(e, \phi) \otimes S, \tilde{\sigma} \rangle_{\hat{W}} \neq 0 \), then \( \tilde{\sigma} \in \text{triv.}\hat{W} \) for some \( e'' \in \mathcal{N}^{\text{nil}} \) such that \( e'' \geq e \).

**Appendix A. Component Groups and Spin Representations**

In this appendix, we use a case by case analysis to determine the structure of the spaces \( \mathcal{R}_{-1}(A(e)) \), the image of the map \( \iota \), and the explicit decompositions of representations \( \iota(X_{-1}(e, \phi)) \). These calculations can also be used to relate our real-irreducible \( \hat{W} \)-representations with the known case by case classifications in [26, 28, 40], and also [9].

**Remark A.1.** The dimension of \( R(\hat{W})\text{gen}_g \) equals \(|\{ \sigma \in \hat{W}_\text{gen} : \sigma \cong \sigma \otimes \text{sgn} \}| + \frac{1}{2}|\{ \sigma \in \hat{W}_\text{gen} : \sigma \neq \sigma \otimes \text{sgn} \}| \). In particular, the classification of irreducible \( \hat{W} \)-representations [26, 28, 40] gives the following dimensions for \( R(\hat{W})\text{gen}_g \):

1. \( A_{n-1} \): the number of partitions of \( n \) into distinct parts;
2. \( B_n \): the number of partitions of \( n \);
3. \( D_n \): odd: \( \frac{1}{2}(|\lambda \vdash n : \lambda \neq \lambda'|) + |\lambda \vdash n : \lambda = \lambda'| \);
4. \( D_n \): even: \( \frac{1}{2}(|\lambda \vdash n : \lambda \neq \lambda'|) + 2|\lambda \vdash n : \lambda = \lambda'| \);
5. \( G_2 \): 3; \( F_4 \): 9; \( E_6 \): 13.; \( E_8 \): 30.

Comparing with the dimensions in Remark 4.8, we conclude that the map \( \iota : \mathcal{R}_{-1}(W) \to R(\hat{W})\text{gen}_g \) from Proposition 6.1 is an isomorphism for all irreducible \( W \), except when \( W = D_{2n} \) or \( E_7 \).

**A.1.** We first investigate type \( A \).

**Lemma A.2.** Suppose \( G = \text{PGL}(n) \) and \( e_\lambda \in \mathcal{N}^{\text{nil}} \) is a nilpotent element given in the Jordan form by the partition \( \lambda \) of \( n \) with distinct parts. Then \( A(e_\lambda) = \{1\} \) and

\[
(triv, triv)^{-1}_{A(e_\lambda)} = 2^{\ell(\lambda)} - 1.
\]

**Proof.** Straightforward.

\[ \square \]
Proposition A.3. For every distinct partition $\lambda$ of $n$, define

$$\tilde{\tau}_\lambda = \frac{1}{a_\lambda} X_{-1}(e_\lambda) \otimes S,$$

where $a_\lambda = 2^{\frac{a_\lambda}{2}}$, if both $n$ and $\lambda$ are even, and $a_\lambda = 2^{\lfloor \frac{a_\lambda}{2} \rfloor}$, otherwise. Then $\tilde{\tau}_\lambda$ is irreducible $\text{sgn}$ self dual, if $\lambda$ is an even partition, while $\tilde{\tau}_\lambda = \tilde{\tau}_\lambda^+ + \tilde{\tau}_\lambda^-$ if $\lambda$ is an odd partition, with $\tilde{\tau}_\lambda^\pm$ irreducible $\text{sgn}$ dual to each other.

Proof. The number of irreducible genuine $\hat{S}_n$-representations equals the number of conjugacy classes of $S_n$ that split when pulled back to $\hat{S}_n$. A well-known result (going back to Schur, see [40, Theorem 2.1]) says that this number equals the number of partitions of $n$ into odd parts plus the number of odd partitions of $n$ into distinct parts. Denote $\text{DP}(n)$ the set of distinct partitions of $n$ and for every $\lambda \in \text{DP}(n)$, set $b_\lambda$ equal to 1 or 2 if $\lambda$ is even or odd, respectively. Thus

$$|\langle \hat{S}_n \rangle_{\text{gen}}| = \sum_{\lambda \in \text{DP}(n)} b_\lambda. \quad (A.1.1)$$

By Proposition 6.1 and Lemma A.2, we see that for $\lambda \in \text{DP}(n)$,

$$\langle \iota(X_{-1}(e_\lambda)), \iota(X_{-1}(e_\lambda)) \rangle_{\hat{W}} = \begin{cases} a_\lambda^2, & \text{if } \lambda \text{ is even}, \\ 2a_\lambda^2, & \text{if } \lambda \text{ is odd}. \end{cases}$$

This means that $\iota(X_{-1}(e_\lambda))$ contains at least two distinct irreducible $\hat{S}_n$-representations when $\lambda \in \text{DP}(n)$ is odd. Since for $\lambda \neq \lambda'$, $\iota(X_{-1}(e_\lambda))$ and $\iota(X_{-1}(e_{\lambda'}))$ are orthogonal, the claim in the Proposition follows by comparison with (A.1.1). \qed

A.2. Next, we prove a criterion which will cover most of the remaining cases when $G$ is not type $A$ and $e \in \mathcal{N}^{\text{refl}}$, but $e$ is not distinguished.

Lemma A.4. Suppose $e \in \mathcal{N}^{\text{refl}}$ is such that $A(e) = (\mathbb{Z}/2\mathbb{Z})^k \times (\mathbb{Z}/2\mathbb{Z})^l$, $k \neq 0$, acts on the $k$-dimensional space $V_Z$ by the representation $\text{refl}_k \otimes \text{triv}_l$, where $\text{refl}_k$ is the reflection representation of $(\mathbb{Z}/2\mathbb{Z})^k$ and $\text{triv}_l$ is the trivial representation of $(\mathbb{Z}/2\mathbb{Z})^l$. Then $\dim \overline{R}_{-1}(A(e)) = 2^l$ and

$$\langle \phi, \phi \rangle_{A(e)}^{-1} = 1, \text{ for all } \phi \in \overline{A(e)}. \quad \text{Moreover, if } [\phi_1] \neq [\phi_2] \text{ in } \overline{R}_{-1}(A(e)), \text{ then}$$

$$\langle [\phi_1], [\phi_2] \rangle_{A(e)}^{-1} = 0.$$

Proof. Let $\text{sgn}^{(i)}$ denote the one dimensional $(\mathbb{Z}/2\mathbb{Z})^k$-representation, with $\text{sgn}$ on the $i$-th position and $\text{triv}$ everywhere else. Then $V_Z = \bigoplus_{i=1}^k \text{sgn}^{(i)}$ as an $(\mathbb{Z}/2\mathbb{Z})^k$-representation. Thus $\det_{V_Z}(1 + x) \neq 0$ if and only if $\text{proj}_{(\mathbb{Z}/2\mathbb{Z})^k} x = 1$, hence the $(-1)$-elliptic element in $A(e)$ are the subgroup $(\mathbb{Z}/2\mathbb{Z})^l$. It follows that $\dim \overline{R}_{-1}(A(e)) = 2^l$.

For the second claim, notice that since $\phi \otimes \phi = \text{triv}$, we have $\langle \phi, \phi \rangle_{A(e)}^{-1} = \langle \text{triv}, \text{triv} \rangle_{A(e)}^{-1}$. It is straightforward that $\langle \text{triv}, \text{triv} \rangle_{A(e)}^{-1} = (\text{triv}, \wedge V_Z)_{A(e)} = 1$.

For the last claim, let $[\phi_1] \neq [\phi_2]$ in $\overline{R}_{-1}(A(e))$. We can choose $\phi_i = \text{triv}_k \otimes \phi_i'$, $i = 1, 2$, where $\text{triv}_k$ is the trivial $(\mathbb{Z}/2\mathbb{Z})^k$-representation, and $\phi_1' \neq \phi_2'$ are one dimensional representations of $(\mathbb{Z}/2\mathbb{Z})^k$. Since $\phi_1' \otimes \phi_2' \neq \text{triv}$, it does not occur in $\wedge V_Z$, so $\langle [\phi_1], [\phi_2] \rangle_{A(e)}^{-1} = 0$. \qed
Lemma A.5. Let $G$ be simple and adjoint and $e \in \mathcal{N}^{sol}$, but not distinguished. Then $A(e)$ is as in Lemma A.4, except when:

1. $G$ is of type $D_n$, $n$ even, and $e = e_\lambda$ corresponds via the Jordan form to a partition $\lambda = (a_1, a_1, a_2, a_2, \ldots, a_k, a_k)$ of $2n$ where $a_i$ are distinct odd positive integers. In this case, $A(e_\lambda) = (\mathbb{Z}/2\mathbb{Z})^{k-1}$ acts on the $k$-dimensional space $V_Z$ by twice the reflection representation. Then $\overline{\mathcal{R}}_{-1}(A(e_\lambda))$ is one dimensional, and $\langle \text{triv, triv} \rangle_{A(e_\lambda)}^{-1} = 2$.

2. $G$ is of type $D_n$, $n$ odd, and $e = e_\lambda$ corresponds via the Jordan form to a partition $\lambda = (a_1, a_1, a_2, a_2, \ldots, a_k, a_k)$ of $2n$ where $a_i$ are distinct odd positive integers. In this case, $A(e_\lambda) = (\mathbb{Z}/2\mathbb{Z})^{k-1}$ acts on the $k$-dimensional space $V_Z = V^A_{Z(e)} \oplus V'_{Z}$, with $\dim V^A_{Z(e)} = 1$, by the reflection representation on $V'_Z$. Then $\overline{\mathcal{R}}_{-1}(A(e_\lambda))$ is one dimensional, and $\langle \text{triv, triv} \rangle_{A(e_\lambda)}^{-1} = 2$.

3. $G$ is of type $E_7$ and $e$ is of type $A_4 + A_1$. The component group $A(e) = \mathbb{Z}/2\mathbb{Z}$ acts on the two dimensional space $V_Z$ by twice the $\text{sgn}$ representation. Then $\overline{\mathcal{R}}_{-1}(A(e))$ is one dimensional, but $\langle \text{triv, triv} \rangle_{A(e)}^{-1} = 2$.

4. $G$ is of type $E_6$ and $e$ is of type $D_4(a_1)$. The component group $A(e) = S_3$ acts on the two dimensional space $V_Z$ by the reflection representation. Then $\overline{\mathcal{R}}_{-1}(A(e))$ is two dimensional, spanned by $[\text{triv}]$ and $[\text{refl}]$, and $\langle \text{triv, triv} \rangle_{A(e)}^{-1} = 1$, $\langle \text{refl, refl} \rangle_{A(e)}^{-1} = 3$, $\langle \text{triv, refl} \rangle_{A(e)}^{-1} = 1$. (A.2.1)

Proof. The proof is a direct calculation based on the classification of nilpotent orbits and their component groups. 

\[ \square \]

A.3. Suppose $W$ is of type $B_n$ and $G = \text{Sp}(2n)$. The nilpotent orbits $e \in \mathcal{N}^{sol}$ are in one to one correspondence with partitions $\mu$ of $2n$ such that $\mu$ has only even parts and the multiplicity of each part is at most 2. The distinguished nilpotent $e_\mu$ correspond to $\mu$ a partition with even distinct parts. Denote by $\text{DP}(2n)_{\text{even}}$ the set of distinct partitions with even parts of $2n$, and by $\text{qDP}(2n)_{\text{even}}$ the set of partitions with even parts of $2n$ where every part has multiplicity at most 2 and there is one part with multiplicity 2. By Remark A.1, the number of irreducible $\overline{W}$-representations, up to tensoring with \text{sgn}, equals $|P(n)|$, the number of partitions of $n$.

For every $\mu \in \text{DP}(2n)_{\text{even}} \cup \text{qDP}(2n)_{\text{even}}$ and $\phi \in \overline{A}(e_\mu)$, set

$$\overline{r}(e_\mu, \phi) = X_{-1}(e_\mu, \phi) \otimes S.$$  

Then, Lemma A.4 yields:

Proposition A.6. (1) If $n$ is even, $\overline{r}(e_\mu, \phi)$ is an irreducible $\text{sgn}$ self dual $\overline{W}$-representation.

(2) If $n$ is odd, $\overline{r}(e_\mu, \phi) = \overline{r}(e_\mu, \phi)^+ + \overline{r}(e_\mu, \phi)^-$, where $\overline{r}(e_\mu, \phi)^\pm$ are irreducible $\overline{W}$-representations $\text{sgn}$ dual to each other.

A.4. If $W$ (and $G$) is exceptional of type $G_2$, $F_4$, or $E_8$ for every $e \in \mathcal{N}$, either Theorem 6.6(2),(3) or Lemma A.4 applies. Since $\dim V$ is even, $a_V = 1$, and $\overline{r}(e, \phi) = \iota(X_{-1}(e, \phi))$ is an irreducible $\text{sgn}$ self dual $\overline{W}$-representation.

A.5. Let $W$ be of type $E_6$. There are seven orbits in $\mathcal{N}^{sol}$, three of which are distinguished. From Lemma A.5, it follows that when $u$ is of type $D_4(a_1)$, then $A(e) = S_3$ and

$$\overline{r}(D_4(a_1), \text{triv}) := X_{-1}(D_4(a_1), \text{triv}) \otimes S$$ (A.5.1)
is an irreducible $\text{sgn}$ self-dual representation of $\tilde{W}(E_6)$, while
\[
\tilde{\sigma}(D_{4}(a_1), \text{refl}) : = X^{-1}(D_{4}(a_1), \text{refl}) \otimes S
\]
\[= \tilde{\sigma}(D_{4}(a_1), \text{triv}) + \tilde{\sigma}(D_{4}(a_1), \text{refl})^+ + \tilde{\sigma}(D_{4}(a_1), \text{refl})^-, \tag{A.5.2}
\]
where $\tilde{\sigma}(D_{4}(a_1), \text{refl})^\pm$ are irreducible $\text{sgn}$ dual $\tilde{W}(E_6)$-representations. A basis of $\mathcal{R}_{-1}(A(e))$ consisting of orthogonal elements is $\{[\text{triv}], [\text{refl}] - [\text{triv}]\}$.

A.6. For type $E_7$, the interesting case is the nilpotent element $e$ of type $A_4 + A_1$. Then $A(e) = \mathbb{Z}/2\mathbb{Z}$, and $V_\mathbb{Z}$ is two dimensional. By Lemma A.5, $\langle \iota(X^{-1}(A_4 + A_1, \text{triv})), \iota(X^{-1}(A_4 + A_1, \text{triv})) \rangle_{\tilde{W}} = 4$ (since dim $V$ is odd). Using 6.6(2),(3) and Lemma A.4, the classes in $\mathcal{N}^{\text{sol}} \setminus \{A_4 + A_1\}$ account for 11 distinct irreducible $\tilde{W}$-representations (modulo $\otimes \text{sgn}$). This implies that $\iota(X^{-1}(A_4 + A_1, \text{triv}))$ is either two copies of a $\text{sgn}$ self-dual irreducible representation or a sum of two pairs of $\text{sgn}$ dual irreducible representations. The latter is in fact the correct one, and this can be seen either by invoking the fact that $\tilde{W}(E_7)$ does not have $\text{sgn}$ self-dual irreducible representations [26], or by refining the argument used to prove Theorem 6.6(2) as follows. If $\iota(X^{-1}(A_4 + A_1, \text{triv})) = 2\tilde{\sigma}$, where $\tilde{\sigma}$ is $\text{sgn}$ self dual, then the only possibility is that $\sigma(A_{4} + A_{1}, \text{triv}) \otimes S^\pm$ each contain $\tilde{\sigma}$ with multiplicity 1; moreover, there are no other $\tilde{W}$-representations $\tilde{\sigma}'$ such that $W \cdot \nu_{\tilde{\sigma}'} = W \cdot \nu_{\tilde{\sigma}'} = W \cdot h_{A_{4} + A_{1}}$. By Theorem 5.13, only $\tilde{\sigma}$ can occur in the Dirac cohomology spaces, and in particular, $X_{1}(A_{4} + A_{1}, \text{triv}) \otimes (S^+ - S^-) = \tilde{\sigma} - \tilde{\sigma} = 0$ (see [11, 10] for details about the Dirac index). Since $(S^+ - S^-) \otimes (S^+ - S^-)^* = 2\wedge^{-1} V$, it follows that $(X_{1}(A_{4} + A_{1}, \text{triv}), X_{1}(A_{4} + A_{1}, \text{triv}))_{\tilde{W}} = 0$. But this is a contradiction with Proposition 4.1(1), since $A_{4} + A_{1}$ is quasidistinguished in $E_7$.

A.7. Suppose $W$ is of type $D_n$ and $G = \text{PSO}(2n)$.

When $n$ is odd, all representatives of pairs $(e, \phi), e \in \mathcal{N}^{\text{sol}}$ are as in Theorem 6.6(2),(3) or as in Lemma A.4, except when $e = e_\mu$ corresponds to the partition $\mu = (a_1, a_1, a_2, a_2, \ldots, a_k, a_k)$ of $2n$, where $k$ is odd, $a_i$ are all distinct and odd, see Lemma A.5. In this case, since dim $V$ is odd, we have

$$\langle \iota(X^{-1}(e_\mu, \text{triv})), \iota(X^{-1}(e_\mu, \text{triv})) \rangle_{\tilde{W}} = 4.$$  

One may resolve the ambiguity in the same way as for $A_{4} + A_{1}$ in $E_7$ and find that
\[\iota(X^{-1}(e_\mu, \text{triv})) = 2\tilde{\sigma}(e_\mu, [\text{triv}]),\]
for some irreducible, $\text{sgn}$ self dual $\tilde{W}$-representation $\sigma(e_\mu, [\text{triv}])$. (In this case, $e_\mu$ is not quasidistinguished.)

When $n$ is even, all representatives of pairs $(e, \phi), e \in \mathcal{N}^{\text{sol}}$ are as in Theorem 6.6(2),(3) or as in Lemma A.4, except when $e = e_\mu$ corresponds to the partition $\mu = (a_1, a_1, a_2, a_2, \ldots, a_k, a_k)$ of $2n$, where $k$ is even, $a_i$ are all distinct and odd, see Lemma A.5. In this case, since dim $V$ is even, we have

$$\langle \iota(X^{-1}(e_\mu, \text{triv})), \iota(X^{-1}(e_\mu, \text{triv})) \rangle_{\tilde{W}} = 2,$$

so $\iota(X^{-1}(e_\mu, \text{triv})) = \tilde{\sigma}(e_\mu, [\text{triv}])_1 + \tilde{\sigma}(e_\mu, [\text{triv}])_2$. One can prove that $\tilde{\sigma}(e_\mu, [\text{triv}])_1, 2$ are $\text{sgn}$ self-dual by invoking an argument similar to that for $A_{4} + A_{1}$ in $E_7$, using the Dirac index in the even case ([11, 10]) and the fact that $e_\mu$ is quasidistinguished in $D_n$, $n$ even.

\textit{Erratum:} In [9, Theorem 3.8.1(2)], it is incorrectly stated that $(\tilde{\sigma})_{1, 2}$ are $\text{sgn}$ dual to each other.
Remark A.7. The nilpotent element \( u = A_4 + A_1 \) in \( E_7 \) can be realized as a regular nilpotent element in \( Z_G(t) = A_3 + A_3 + A_1 \), see the proof of Proposition 3.3. Similarly, the nilpotent element \( e_\mu \in D_n \) with \( \mu = (a_1, a_1, a_2, a_2, \ldots, a_k, a_k) \), \( a_i \) distinct and odd, can be realized as the pair \((e_\lambda, e_\lambda)\), \( \lambda = (a_1, a_2, \ldots, a_k) \), in \( Z_G(t) = D_{n/2} \times D_{n/2} \), when \( n \) is even, respectively \( Z_G(b) = B_{n+1} \times B_{n+1} \) when \( n \) is odd. Thus the automorphism \( \theta \) coming from the symmetry of the affine Dynkin diagram interchanges the two factors of \( e_\lambda \) and the exceptions appear to be related to this phenomenon.

Appendix B. An action of \( W_\# \) on \( H^*(B_e) \)

B.1. Let \( \delta \) be the automorphism on \( G \) and on \( W \) defined in section 3.2 and \( W_\# = W \rtimes (\delta) \). In this section, we construct a natural action of \( W_\# \) on \( H^*(B_e) \), which extends the action of \( W \) we discussed in section 2.2, such that the following theorem holds.

**Theorem B.1.** For every \( e \in \mathcal{N} \) and \( 0 \leq i \leq d_e \),

\[
\text{tr}(\delta w, H^{2i}(B_e)) = (-1)^i \text{sgn}(w_0) \text{tr}(w_{0w}, H^{2i}(B_e)).
\]

In particular, we have

**Corollary B.2.** For any \( w \in W \),

\[
X_e(\phi, \psi)(\delta w) = (-1)^{d_e} \text{sgn}(w_0) X_{-e}(\psi, \phi)(w_0w).
\]

B.2. We assume that \( \delta \neq \text{id} \).

We fix a \( \delta \)-stable Borel subalgebra \( b \) of \( g \). Let \( g_{\text{reg}} \) be the set of regular semisimple elements in \( g \) and \( b_{\text{reg}} = g_{\text{reg}} \cap b \). Define the action of \( B \) on \( G \times b \) by \( b \cdot (g, b') = (g^{-1}b, \text{Ad}(b)b') \). Let \( G \times b \) be the quotient scheme and \( G \times b_{\text{reg}} \) be the image of \( G \times b_{\text{reg}} \) in \( G \times B \). The Springer resolution (of \( g \)) is the map

\[
q : G \times b \rightarrow g, \quad (g, b) \mapsto \text{Ad}(g)b.
\]

Its restriction \( q_{\text{reg}} \) to \( G \times b_{\text{reg}} \) gives an unramified Galois covering of \( g_{\text{reg}} \) whose Galois group is \( W \).

We fix a prime number \( l \) invertible in \( F \). Let \( \Omega_{G \times b} \) be the trivial sheaf on \( G \times B \) and \( \Omega_{G \times b_{\text{reg}}} \) be the trivial sheaf on \( G \times b_{\text{reg}} \). Set \( \Psi = R\theta(\Omega_{G \times b})[\dim(G)] \) and \( \Psi_{\text{reg}} = R(\theta_{\text{reg}}(\Omega_{G \times b_{\text{reg}}})[\dim(G)]) \). Since \( q \) is a small map, \( \Psi \) is the intersection cohomology complex \( IC(g, \Psi_{\text{reg}}) \). Thus End(\( \Psi \)) = End(\( \Psi_{\text{reg}} \)) \cong \prod_{i} W_i \) and we have a natural action of \( W \) on \( \Psi \). Notice that the map \( q \) is in fact \( G_\# \)-equivariant. Hence the automorphism \( \delta \) on \( G \) induces an action \( \delta^* : \Psi \rightarrow \Psi \). We have that \( (\delta^*)^2 = 1 \) and \( (\delta)^*w = \delta(w)\delta^* : \Psi \rightarrow \Psi \).

B.3. Let \( e \in \mathcal{N} \). If \( g \in G \) such that \( \delta(e) = \text{Ad}(g)(e) \). Then \( \delta(g)g \in Z_G(e) \). We choose \( g \in G \) such that \( \delta(e) = \text{Ad}(g)(e) \) and the image of \( \delta(g)g \) in \( A(e) \) lies in the kernel of \( \phi \) for all \( \phi \in A(e)_0 \). By Proposition 3.5, such \( g \) always exists. Thus \( \text{Ad}(g)^* \circ \delta^* : \Psi_e \rightarrow \Psi_e \) satisfies

\[
(\text{Ad}(g)^* \circ \delta^*)^2 = (\text{Ad}(\delta(g)g))^* = \text{id}.
\]

For any \( w \in W \), we have the following commuting diagram.
\[ 
\begin{array}{c}
\Psi_e \xrightarrow{\delta^*} \Psi_{\text{Ad}(g)e} \xrightarrow{\text{Ad}(g)^*} \Psi_e \\
\downarrow \delta(w) \quad \quad \quad \downarrow \delta(w) \quad \quad \quad \quad \downarrow \delta(w)
\end{array}
\]

(B.3.2)

From (B.3.1) and (B.3.2), we see that the map

\[ w \mapsto w, \quad \delta \mapsto \text{Ad}(g)^* \circ \delta^* \]  

(B.3.3)

gives an action of \( W_\theta \) on \( \Psi_e \) and hence on the cohomology of \( \Psi_e \). By construction, this action commutes with the action of \( A(e) \) on \( \Psi_e \). In particular, for any \( i \geq 0 \) and \( \phi \in A(e)_0 \), we may regard \( H^2(B_e)^\phi \) as a \( W_\theta \)-module and thus \( X_q(e, \phi) \) as a virtual character of \( W_\theta \).

Notice that the \( W_\theta \)-module structure depends on the choice of \( g \). If we pick a different element \( g' \in G \) such that \( \delta(e) = \text{Ad}(g')(e) \) and \( \delta(g')g' \in Z_G(e)^0 \), then \( g'^{-1}g' \in Z_G(e) \) and thus the actions of \( \delta \) on \( H^2(B_e)^\phi \) (defined using \( g \) and \( g' \)) differ by \( \phi(g'^{-1}g') \).

**B.4.** Now we discuss the choice of \( g \) which makes the action of \( \delta \) on \( H^*(B_e) \) nice. We construct such an element \( g = g_1 \) without using Proposition 3.5. A similar action of \( A(e) \rtimes \langle \delta \rangle \) on \( H^*(B_e) \) is studied by Baranovsky-Evens-Ginzburg in [2, Appendix], and Bezrukavnikov-Mirković in [7, Appendix A].

As before we assume \( p \) and \( q \) are large. We assume furthermore that \( p \equiv 1 \mod 3 \) if \( G \) is of type \( E_6 \). Let \( F : G \to G \) be the split Frobenius map. For \( e \in N^F \), \( F(B_e) = B_e \). We say that \( e \) is split (with respect to \( F \)) if all the irreducible components of \( B_e \) are \( F \)-stable. By [33, Proposition 3.3] and [4, Section 3], each nilpotent orbit of \( g \) contains exactly one split \( G^F \)-orbit.

Let \( F' = F \delta = \delta F : G \to G \) be a twisted Frobenius map. The action of \( F' \) on \( W \) is conjugation by \( w_0 \). The following result is proved by Hotta and Springer for unitary groups in [17, Theorem 3.1], as a consequence of a specialization theorem, by Shoji for the other classical groups in [33, Theorem 4.18], and by Beynon and Spaltenstein for exceptional groups, in particular for \( E_6 \) in [4, Theorem 4.1], as a consequence of the Lusztig-Shoji algorithm.

**Theorem B.3.** Let \( O \) be a nilpotent orbit of \( g \). There exists a bijection \( \sigma \) from the set of \( G^F \)-orbits in \( O^{F^*} \) to the set of \( G^F \)-orbits in \( O^F \) such that

\[ \text{tr}((F')^* \circ w, H^2(B_e)) = (-1)^\text{sgn}(w_0) \text{tr}(F^* \circ w_0w, H^2(B_{\sigma(e)})). \]

We assume furthermore that \( \sigma(e) \) is split with respect to \( F \). Let \( g \in G \) such that \( \delta(e) = \text{Ad}(g)(e) \). Then \( e \in O^{F_0} \), where \( F_0 = \text{Ad}(g^{-1}) \circ F \) is again a split Frobenius morphism. There exists \( h \in G \) such that \( \text{Ad}(h)(e) \) is split with respect to \( F_0 \), i.e. \( F_0(\text{Ad}(h)(e)) = \text{Ad}(h)(e) \) and all the irreducible components of \( B_{\text{Ad}(h)(e)} \) are \( F_0 \)-stable. In other words, \( h^{-1}F_0(h) \in Z_G(e) \) and all the irreducible components of \( B_e \) are \( \text{Ad}(h^{-1}F_0(h)) \circ F_0 = \text{Ad}(h^{-1}) \circ F_0 \circ \text{Ad}(h) \)-stable. Set

\[ g_1 = g(h^{-1}F_0(h))^{-1}, \]  

(B.4.1)

and \( F_1 = \text{Ad}(g_1)^{-1} \circ F \). Then \( \delta(e) = \text{Ad}(g_1)(e) \) and \( F_1 \) is a split Frobenius morphism and \( e \) is split with respect to \( F_1 \).
By [4, section 5(C)], $F^*$ acts by $q^i$ on $H^{2i}(B_e)$ and $F^*_e$ acts by $q^i$ on $H^{2i}(B_e)$. Thus
\[
\text{tr}((F^*)^i \circ w, H^{2i}(B_e)) = \text{tr}(F^*_1 \circ (\text{Ad}(g_1)^* \circ \delta^*) \circ w, H^{2i}(B_e)) = q^i \text{tr}((\text{Ad}(g_1)^* \circ \delta^*) \circ w, H^{2i}(B_e))
\]
and
\[
\text{tr}(F^* \circ w_0 w, H^{2i}(B_{\sigma(e)})) = q^i \text{tr}(w_0 w, H^{2i}(B_{\sigma(e)})) = q^i \text{tr}(w_0 w, H^{2i}(B_e)).
\]

By Theorem B.3,
\[
\text{tr}((\text{Ad}(g_1)^* \circ \delta^*) \circ w, H^{2i}(B_e)) = (-1)^i \text{sgn}(w_0) \text{tr}(w_0 w, H^{2i}(B_e)). \quad \text{(BA.2)}
\]

In particular,
\[
\text{tr}((\text{Ad}(g_1)^* \circ \delta^*) \circ w_0, H^{2i}(B_e)) = (-1)^i \text{sgn}(w_0) \dim(H^{2i}(B_e)).
\]

Since $(\text{Ad}(g_1)^* \circ \delta^*)$ commutes with $w_0$ by (B.3.2), $(\text{Ad}(g_1)^* \circ \delta^* \circ w_0)^2 = \text{Ad}(\delta(g_1)g_1)^*$ acts on $H^{2i}(B_e)$ via the image of $\delta(g_1)g_1$ in $A(e)$, and so $\text{Ad}(g_1)^* \circ \delta^* \circ w_0$ acts on $H^{2i}(B_e)$ as an element of finite order. Hence it acts by the scalar $(-1)^i \text{sgn}(w_0)$.

So $(\text{Ad}(g_1)^* \circ \delta^* \circ w_0)^2 = \text{Ad}(\delta(g_1)g_1)^*$ acts on $H^{2d_4}(B_e)$ as the identity. Therefore the image of $\delta(g_1)g_1$ in $A(e)$ lies in the kernel of $\phi$ for all $\phi \in \overline{A(e)}_0$.

**Remark B.4.** In the rest of the paper, unless otherwise stated, we regard $H^{2i}(B_e)^\phi$ as a $W_{\phi}$-module via (B.3.3) with respect to the element $g_1$. As we discussed in section B.3, $g_1$ is uniquely determined by (B.4.2) up to right multiplication by $Z_G(e)^0$.

**B.5.** We proved Theorem B.1 over a finite field. In order to pass from (large) characteristic $p$ to characteristic 0, first note that the representation of $W$ on $H^{2i}(B_e)$ is independent of the characteristic [37, section 3]. Now we choose $q$ as in the proof of Proposition 3.5. Then the action of $\text{Ad}(g)^* \circ \delta^*$ is again independent of the characteristic. As explained in section B.3, there exists $z \in A(e)$ such that for any $i$ and $\phi$, $\text{Ad}(g)^* \circ \delta^* \circ w_0$ acts on $H^{2i}(B_e)^\phi$ as $(-1)^i \text{sgn}(w_0)\phi(z)$. In fact, in characteristic $p$, $z$ is the image of $g_1^{-1}g$ in $A(e)$. The component group $A(e)$ is independent of the choice of characteristic. In characteristic 0, set $g_1 = g_1^{-1} \in Z_G(e)$, for a representative $z_1 \in Z_G(e)$ of $z$. Then $\text{Ad}(g_1)^* \circ \delta^* \circ w_0$ acts on $H^{2i}(B_e)$ as $(-1)^i \text{sgn}(w_0)$, and the same argument as at the end of section B.4 shows that the image of $\delta(g_1)g_1$ in $A(e)$ lies in the kernel of $\phi$ for all $\phi \in \overline{A(e)}_0$.

**Remark B.4.** In the rest of the paper, unless otherwise stated, we regard $H^{2i}(B_e)^\phi$ as a $W_{\phi}$-module via (B.3.3) with respect to the element $g_1$. As we discussed in section B.3, $g_1$ is uniquely determined by (B.4.2) up to right multiplication by $Z_G(e)^0$.

**B.6.** Suppose that $\delta = 1$. We set $g_1 = 1$.

When $g$ is a classical Lie algebra $\mathfrak{sp}(2n)$, $\mathfrak{so}(2n+1)$, or $\mathfrak{so}(4n)$, Theorem B.3 is proved in [33, Theorem 4.18] and therefore (B.4.2) holds.

When $g$ is exceptional of type $G_2, F_4, E_6, E_7, E_8$, we do not know an explicit reference for Theorem B.3. However, the argument in [33, Theorem 4.18] (see also [4]) can be applied in these cases as well, as soon as we construct the correct matching $\sigma$ so that the analogues of [33, Proposition 1.12 and Lemma 4.20(ii)] hold. This is done as follows.

Let $e \in N^F$. For every $\phi \in \overline{A(e)}_0$, let $\text{hdg}(\sigma(e, \phi))$ denote the lowest harmonic degree of $\sigma(e, \phi)$, and set $d_\phi = (-1)^{\text{hdg}(\sigma(e, \phi))}$. 
Lemma B.5 (compare with [33, Proposition 1.12]). There exists a conjugacy class $c_0$ of $A(e)$ such that $\phi(c_0) = d_\phi d_1$, for all $\phi \in \hat{A}(e)$.

Proof. When $\mathfrak{g}$ is a simple exceptional Lie algebra not of type $F$, Definitions (C.1.2) and (C.1.3) extend to actions of $C_\infty$ and $C_\infty$ on $X$. By inspection of the tables in [8, pages 429-432], we see that all $\sigma(e, \phi)$ have the same parity of the lowest harmonic degrees. Thus, we may choose $c_0 = 1$ in these cases.

Define the map $\sigma : G^F \cdot e \to G^F \cdot e$, by $O^F(e) \mapsto O^F(e)$. This makes sense because in the cases when $A(e)$ is not abelian, we chose $c_0 = 1$. By inspection of tables in [20, Chapter 22], we see that the analogue of [33, Lemma 4.20(ii)] holds:

$$|Z_{G^F}(e_c)| (-q) = |Z_{G^F}(e_{c_0})|(q).$$

(B.6.1)

An alternative argument, again case by case, is as follows. The graded $W$-representations $H^*(e)^w$ are explicitly computed in all the exceptional cases: for $G_2, F_4$ in [34], and for type $E$ in [5]. One can check that if an irreducible $W$-representation $\mu$ occurs in $H^{2i}(E_\infty)$ then $(-1)^{hdeg(\mu)} = \text{sgn}(w_0)(-1)^i$. See also [5, page 19, Remark (a)]. The lowest harmonic degrees can be read from [8]. Since $w_0$ is central, it acts on every irreducible $\mu$ by $(-1)^{hdeg(\mu)}$, and (B.4.2) follows.

Appendix C. Relation with Kostka systems

We explain a relation between our approach and the results of Kato [19] about Kostka systems in the category of $A_W$-modules, where $A_W = \mathbb{C}[W] \ltimes S(V_C)$. The relevant homological properties of the category of $A_W$-modules are presented in [19, section 2].

C.1. Retain the notation from the previous section. Thus $W$ is a finite Weyl group acting on the real reflection representation $V$. Define

$$A_W = \mathbb{C}[W] \ltimes S(V_C);$$

(C.1.1)

in the language of section 5, this is the same as the graded affine Hecke algebra with zero parameters.

Let $X, Y$ be $A_W$-modules. Then one may define a structure of $A_W$-modules on $X \otimes_C Y$ and $\text{Hom}_C[X, Y]$ as follows:

1. $X \otimes_C Y$

$$w \cdot (x \otimes y) = w \cdot x \otimes w \cdot y, \quad w \in W,$$

$$\xi \cdot (x \otimes y) = \xi \cdot x \otimes y + x \otimes \xi \cdot y, \quad \xi \in V_C;$$

(C.1.2)

2. $\text{Hom}_C[X, Y]$

$$(w \cdot \phi)(x) = w \cdot \phi(w^{-1} x), \quad w \in W,$$

$$\xi \cdot (\phi(x)) = \xi \cdot \phi(x) - \phi(\xi \cdot x), \quad \xi \in V_C,$$

(C.1.3)

for all $\phi \in \text{Hom}_C[X, Y]$.

Lemma C.1. (1) Definitions (C.1.2) and (C.1.3) extend to actions of $A_W$.

(2) If $C$ is the trivial $A_W$-module (i.e., acts by $0$), then $X \otimes_C C \cong X$, as $A_W$-modules.

(3) $\text{Hom}_C[X, Y] \cong X^* \otimes_C Y$.

Proof. Straightforward.
C.2. Since $A_W \otimes_W M \cong S(V) \otimes_{C} M$, for every $W$-module $M$, the usual Koszul complex of vector spaces admits an interpretation as a projective resolution of the trivial $A_W$-module. More precisely, let
\[
\epsilon : A_W \otimes_{C} C \to C, \quad \epsilon(a \otimes \lambda) = a \cdot \lambda, \tag{C.2.1}
\]
i.e., the $A_W$-action in the trivial module, and
\[
\partial_{n-1} : A_W \otimes W \wedge^n V \to A_W \otimes W \wedge^{n-1} V,
\]
a $\otimes (\xi_1 \wedge \cdots \wedge \xi_n) \mapsto \sum_{\ell=1}^{n} (-1)^{\ell+1} a_{\xi_{\ell}} \otimes (\xi_1 \wedge \cdots \wedge \hat{\xi}_{\ell} \wedge \cdots \wedge \xi_n). \tag{C.2.2}
\]
It is easy to check that $\partial_{n-1}$ is a well-defined $A_W$-module homomorphism. Therefore
\[
0 \leftarrow C \leftarrow A_W \otimes W C \stackrel{\partial_0}{\leftarrow} A_W \otimes W V \stackrel{\partial_1}{\leftarrow} A_W \otimes W \wedge^2 V \leftarrow \ldots \tag{C.2.3}
\]
is a projective resolution of the trivial module in the category of $A_W$-modules. Since tensor products exist in this category, one may tensor this complex by $- \otimes_{C} X$ to obtain a projective resolution for every finite dimensional $A_W$-module $X$:
\[
0 \leftarrow X \leftarrow A_W \otimes W X \leftarrow \ldots \tag{C.2.4}
\]
where we used the isomorphism $C \otimes_{C} X \cong X$, and therefore the morphism $\epsilon : A_W \otimes W X \to X$ becomes the action of $A_W$ on $X$.

C.3. We regard $A_W$ as a graded algebra by assigning to $w$ degree 0 and $\xi \in V_C$ degree 1. This differs from the convention in [19], where the elements of $V_C$ have degree 2, but it will be consistent with the results in section 2. We consider the category of graded $A_W$-modules: $A_W$-gmod. If $X$ is a $\mathbb{Z}$-graded vector space, $X = \oplus_j X(j)$, let
\[
gdim X = \sum_j q^j \dim X(j)
\]
denote the graded dimension of $X$.

**Definition C.2** ([19]). If $X, Y$ are finite dimensional graded $A_W$-modules, define the graded Euler-Poincaré pairing
\[
\langle X, Y \rangle^{\text{EP}}_{A_W} := \sum_{i \geq 0} (-1)^i gdim \operatorname{Ext}^i_{A_W} (X, Y) \in \mathbb{Z}[q]. \tag{C.3.1}
\]

It is clear that the maps $\partial_n$ in the Koszul complex are graded maps of degree 0, and this makes (C.2.3) and (C.2.4) graded complexes.

Define the graded $W$-character of $X \in A_W$-gmod:
\[
gch_W X = \sum_{\sigma \in W} \sum_{j \in \mathbb{Z}} q^j \dim \operatorname{Hom}_W [\sigma, X(j)].
\]

**Proposition C.3.** If $X, Y$ are finite dimensional graded $A_W$-modules, then
\[
\langle X, Y \rangle^{\text{EP}}_{A_W} = \langle \text{gch}_W X, \text{gch}_W Y \rangle_{W}^q .
\]
Proof. The proof is a simple application of the (graded) Euler-Poincaré principle using the resolution (C.2.4) and it is an immediate analogue of the proof for the group algebra of the affine Weyl in the non-graded setting [27, Theorem 3.2]:

\[
\langle X, Y \rangle_{A_W}^{EP} = \sum_{i \geq 0} (-1)^i \text{gdim} \text{Ext}_{A_W}^i(X, Y) = \sum_{i \geq 0} (-1)^i \text{gdim} H^i(\text{Hom}_{A_W}(A_W \otimes_W X, Y)) \\
= \sum_{i \geq 0} (-1)^i \text{gdim} \text{Hom}_{A_W}(A_W \otimes_W \Lambda^i V \otimes X, Y), \quad \text{by Euler-Poincaré principle} \\
= \sum_{i \geq 0} (-1)^i \text{gdim} \text{Hom}_{W}(\text{gch}_W(X \otimes \Lambda^i V), \text{gch}_W Y), \quad \text{by Frobenius reciprocity} \\
= \langle X, Y \rangle_W^q, \quad \text{since } \text{gch}_W \Lambda^i V = q^i \Lambda^i V. 
\]

Example C.4. In [18], the Springer \( W \)-action on \( H^*(B_v) \) is upgraded to an action of the affine Weyl group. As a consequence, one can define a graded \( A_W \)-module structure \( \mathcal{X}_q(e, \phi) \) on \( H^*(B_v) \) [19], such that \( \text{gch}_W \mathcal{X}_q(e, \phi) = X_q(e, \phi) \). These are particular examples of Kostka systems [19, Definition A]. Thus:

\[
\langle \mathcal{X}_q(e, \phi), \mathcal{X}_q(e', \phi') \rangle_{A_W}^{EP} = \langle \text{gch}_W X_q(e, \phi), \text{gch}_W X_q(e', \phi') \rangle_W^q.
\]

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