RAN-REURINGS THEOREMS
IN ORDERED METRIC SPACES

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Abstract. The Ran-Reurings fixed point theorem [Proc. Amer. Math. Soc., 132 (2004), 1435-1443] is but a particular case of Maia’s [Rend. Sem. Mat. Univ. Padova, 40 (1968), 139-143]. A functional version of this last result is then provided, in a convergence-metric setting.

1. Introduction

Let $X$ be a nonempty set. Take a metric $d(\cdot, \cdot)$ over it; as well as a self-map $T : X \to X$. We say that $x \in X$ is a Picard point (modulo $(d, T)$) if i) $(T^n x; n \geq 0)$ (=the orbit of $x$) is $d$-convergent, ii) $z := \lim_n T^n x$ is in $\text{Fix}(T)$ (i.e., $z = Tz$). If this happens for each $x \in X$ and iii) $\text{Fix}(T)$ is a singleton, then $T$ is referred to as a Picard operator (modulo $d$); cf. Rus [23, Ch 2, Sect 2.2]. For example, such a property holds whenever $d$ is complete and $T$ is $d$-contractive; cf. (b04). A structural extension of this fact – when an order $(\leq)$ on $X$ is being added – was obtained in 2004 by Ran and Reurings [21]. For each $x, y \in X$, denote

(a01) $x <> y$ iff either $x \leq y$ or $y \leq x$ (i.e.: $x$ and $y$ are comparable).

This relation is reflexive and symmetric; but not in general transitive. Call the self-map $T$, $(d, \leq; \alpha)$-contractive (for $\alpha > 0$), if

(a02) $d(Tx, Ty) \leq \alpha d(x, y), \forall x, y \in X, x \leq y$.

If this holds for some $\alpha \in ]0, 1[$, we say that $T$ is $(d, \leq)$-contractive.

Theorem 1. Let $d$ be complete and $T$ be $d$-continuous. In addition, assume that $T$ is $(d, \leq)$-contractive and

(a03) $X(T, <>) := \{x \in X; x <> Tx\}$ is nonempty

(a04) $T$ is monotone (increasing or decreasing)

(a05) for each $x, y \in X$, $\{x, y\}$ has lower and upper bounds.

Then, $T$ is a Picard operator (modulo $d$).

According to many authors (cf. [1], [4], [8], [18], [19] and the references therein), this result is credited to be the first extension of the classical 1922 Banach’s contraction mapping principle [2] to the realm of (partially) ordered metric spaces. Unfortunately, the assertion is not true: some early statements of this type have been obtained two decades ago by Turinici [27], in the context of quasi-ordered metric spaces. (We refer to Section 5 below for details).

Now, the Ran-Reurings fixed point result found some useful applications to matrix and differential/integral equations. So, it cannot be surprising that, soon after, many extensions of Theorem [1] were provided; see the quoted papers for details. It

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is therefore natural to discuss the position of Theorem 1 within the classification scheme proposed by Rhoades [22]. The conclusion to be derived reads (cf. Section 2): the Ran-Reurings theorem is but a particular case of the 1968 fixed point statement in Maia [15, Theorem 1]. Further, in Section 3, some extensions are given for this last result, in the context of quasi-ordered convergence almost metric spaces. Some trivial quasi-order variants of these are then discussed in Section 4; note that, as a consequence of this, one gets the related contributions in the area due Kasahara [12] and Jachymski [10], as well as the order type statement in O’Regan and Petruşel [19]. Some other aspects will be delineated elsewhere.

2. Main result

Let $(X, d; \leq)$ be an ordered metric space; and $T : X \to X$, a self-map of $X$. Given $x, y \in X$, any subset $\{z_1, \ldots, z_k\}$ (for $k \geq 2$) in $X$ with $z_1 = x$, $z_k = y$, and $[z_i \not< z_{i+1}, i \in \{1, \ldots, k - 1\}]$ will be referred to as a $\not<$ chain between $x$ and $y$; the class of all these will be denoted as $C(x, y; \not<)$. Let $\sim$ stand for the relation over $X$ attached to $\not<$ as

$$(b_01) \quad x \sim y \text{ iff } C(x, y; \not<) \text{ is nonempty.}$$

Clearly, $(\sim)$ is reflexive and symmetric; because so is $\not<$. Moreover, $(\sim)$ is transitive; hence, it is an equivalence over $X$.

The following variant of Theorem 1 is our starting point.

**Theorem 2.** Let $d$ be complete and $T$ be $d$-continuous. In addition, assume that $T$ is $(d, \leq)$-contractive and

$$(b_02) \quad T \text{ is } \not< \text{-increasing } [x \not< y \text{ implies } Tz \not< Ty]$$

$$(b_03) \quad (\sim) = X \times X \ [C(x, y; \not<) \text{ is nonempty, for each } x, y \in X].$$

Then, $T$ is a Picard operator (modulo $d$).

This result includes Theorem 1 because (a04) $\implies$ (b02), (a05) $\implies$ (b03). For, given $x, y \in X$, there exist, by (a05), some $u, v \in X$ with $u \leq x \leq v$, $u \leq y \leq v$. This yields $x \not< u$, $u \not< y$; wherefrom, $x \sim y$. In addition, it tells us that the regularity condition (a03) is superfluous.

The remarkable fact to be noted is that Theorem 2 (hence the Ran-Reurings statement as well) is deductible from the Maia’s fixed point statement [15, Theorem 1]. Let $e(., .)$ be another metric over $X$. Call $T : X \to X$, $(e; \alpha)$-contractive (for $\alpha > 0$) when

$$(b_04) \quad e(Tx, Ty) \leq \alpha e(x, y), \forall x, y \in X;$$

if this holds for some $\alpha \in [0, 1[$, the resulting convention will read as: $T$ is $e$-contractive. Further, let us say that $d$ is subordinated to $e$ when $d(x, y) \leq e(x, y)$, $\forall x, y \in X$. The announced Maia’s result is:

**Theorem 3.** Let $d$ be complete and $T$ be $d$-continuous. In addition, assume that $T$ is $e$-contractive and $d$ is subordinated to $e$. Then, $T$ is a Picard operator (modulo $d$).

In particular, when $d = e$, Theorem 3 is just the Banach contraction principle [2]. However, its potential is much more spectacular; as certified by

**Proposition 1.** Under these conventions, we have Theorem $\exists \implies$ Theorem $\exists$; hence (by the above) Maia’s fixed point result implies Ran-Reurings’.
Proof. Let $\alpha \in [0, 1]$ be the number in (a02); and fix $\lambda$ in $[1, 1/\alpha]$. We claim that
\[ e(x, y) := \sum_{n \geq 0} \lambda^n d(T^n x, T^n y) < \infty, \text{ for all } x, y \in X. \] (2.1)
In fact, there exists from (b03), a ($\langle z_0, \ldots, z_k \rangle$) (for $k \geq 2$) in $X$ with $z_1 = x$, $z_k = y$. By (b02), $T^n z_i \to z_{i+1}$, $\forall n$, $\forall i \in \{1, \ldots, k-1\}$; hence, via (a02),
\[ d(T^n z_i, T^n z_{i+1}) \leq \alpha^n d(z_i, z_{i+1}), \forall n; \]
wherefrom (by the choice of $\lambda$)
\[ \sum_{n \geq 0} \lambda^n d(T^n x, T^n y) \leq \sum_{n \geq 0} (\lambda \alpha)^n \sum_{i=1}^{k-1} d(z_i, z_{i+1}) < \infty; \]
hence the claim. The obtained map $e : X \times X \to R_+$ is reflexive [\$e(x, x) = 0, \forall x \in X\$], symmetric [\$e(y, y) = e(x, y), \forall x, y \in X\$] and triangular [\$e(x, z) \leq e(x, y) + e(y, z), \forall x, y, z \in X\$]. Moreover, in view of
\[ e(x, y) = d(x, y) + \lambda e(Tx, Ty) \geq \lambda e(Tx, Ty), \forall x, y \in X, \]
\[ d \text{ is subordinated to } e. \]
Note that $e$ is sufficient in such a case $\{e(x, y) = 0 \implies x = y\}$; hence, it is a (standard) metric on $X$. On the other hand, the same relation tells us that $T$ is $(e, \mu)$-contractive for $\mu = 1/\lambda \in [\alpha, 1]$; hence, (by definition), $e$-contractive. This, along with the remaining conditions of Theorem 2, shows that Theorem 3 applies to these data; wherefrom, all is clear.

3. Extensions of Maia’s result

From these developments, it follows that Maia’s result \cite{15} Theorem 1] is an outstanding tool in the area; so, the question of enlarging it is of interest. A positive answer to this, in a convergence-metric setting, will be described below.

Let $X$ be a nonempty set. Denote by $S(X)$, the class of all sequences $(x_n)$ in $X$. By a (sequential) convergence structure on $X$ we mean, as in Kasahara \cite{12}, any part $C$ of $S(X) \times X$ with the properties
\begin{itemize}
  \item[(c01)] $x_n = x, \forall n \in N \implies ((x_n); x) \in C$
  \item[(c02)] $((x_n); x) \in C \implies ((y_n); x) \in C$, for each subsequence $(y_n)$ of $(x_n)$.
\end{itemize}
In this case, $(x_n); x) \in C$ writes $x_n \xrightarrow{C} x$; and reads: $x$ is the $C$-limit of $(x_n)$.
The set of all such $x$ is denoted $\lim_{n} x_n$; when it is nonempty, we say that $(x_n)$ is $C$-convergent; and the class of all these will be denoted $S_c(X)$. Assume that we fixed such an object, with
\begin{itemize}
  \item[(c03)] $C$=separated: $\lim_n x_n$ is a singleton, for each $(x_n)$ in $S_c(X)$;
\end{itemize}
as usually, we shall write $\lim_n x_n = \{z\}$ as $\lim_n x_n = z$. (Note that, in the Fréchet terminology \cite{6}, this condition is automatically fulfilled, by the specific way of introducing the ambient convergence; see, for instance, Petrusel and Rus \cite{20}). Let ($\leq$) be a quasi-order (i.e.: reflexive and transitive relation) over $X$; and take a self-map $T$ of $X$. The basic conditions to be imposed are
\begin{itemize}
  \item[(c04)] $X(T, \leq) := \{x \in X; x \leq Tx\}$ is nonempty
  \item[(c05)] $T$ is $\leq$-increasing ($x \leq y \implies Tx \leq Ty$). 
\end{itemize}
We say that $x \in X(T, \leq)$ is a Picard point (modulo $\langle C, \leq, T \rangle$) if i) $(T^n x; n \geq 0)$ is $C$-convergent, ii) $z := \lim_{n \to \infty} T^n x$ is in $\text{Fix}(T)$ and $T^n x \leq z, \forall n$. If this happens for each $x \in X(T, \leq)$ and iii) $\text{Fix}(T)$ is $(\leq)$-singleton $\{z, w \in \text{Fix}(T), z \leq w \implies z = w\}$, then $T$ is called a Picard operator (modulo $\langle C, \leq \rangle$). Note that, in this case, each $x^* \in \text{Fix}(T)$ fulfills
\begin{equation}
\forall u \in X(T, \leq) : \text{ } x^* \leq u \implies u \leq x^*; \tag{3.1}
\end{equation}
i.e.: $x^*$ is $(\leq)$-maximal in $X(T, \leq)$. In fact, assume that $x^* \leq u \in X(T, \leq)$. By i) and ii), $(T^n u; n \geq 0)$ $C$-converges to some $u^* \in \text{Fix}(T)$ with $T^n u \leq u^*, \forall n$; hence, $x^* \leq u \leq u^*$. Combining with iii) gives $x^* = u^*$; wherefrom $u \leq x^*$.

Concerning the sufficient conditions for such a property, an early statement of this type was established by Turinici [27]; cf. Section 5. Here, we propose a different approach, founded on ascending orbital concepts (in short: ao-concepts) and almost metrics. Some conventions are in order. Call the sequence $(x, \phi)$, $\phi$-ascending $\phi$-orbital concepts (in short: ao-concepts) $= z$ for each $i \in \mathbb{N}$, i.e.:
\begin{equation}
\forall \psi \in F, \phi \in F, \text{ if } \phi \leq \psi \text{ then } \phi = \psi; \tag{0) = 0 \text{ and } [16]; \text{ but, this is not essential for us]. A basic property of such functions (used in the sequel) is}
\end{equation}
\begin{equation}
(\forall \gamma > 0), (\exists \beta > 0), (\forall t) : 0 \leq t < \gamma + \beta \implies \phi(t) \leq \gamma. \tag{3.2}
\end{equation}
For completeness, we supply a proof of this, due to Jachymski [11]. Assume that the underlying property fails; i.e. (for some $\gamma > 0$):
\begin{equation}
\forall \beta > 0, \exists t \in [0, \gamma + \beta], \text{ such that } \phi(t) > \gamma \text{ (hence, } \gamma < t < \gamma + \beta).\tag{c06}\end{equation}
As $\phi \in F_1(R_+)$, this yields $\phi(t) > \gamma, \forall t > \gamma$. By induction, we get (for some $t > \gamma$) $\phi^n(t) > \gamma, \forall n$; so (passing to limit as $n \to \infty$) $0 \geq \gamma$, contradiction.

Denote, for $x, y \in X$: $H(x, y) = \max\{e(x, T x), e(y, T y)\}$, $L(x, y) = \frac{1}{2}[e(x, T y) + e(T x, y)]$, $M(x, y) = \max\{e(x, y), H(x, y), L(x, y)\}$. Clearly,
\begin{equation}
M(x, T x) = \max\{e(x, T x), e(T x, T^2 x)\}, \forall x \in X. \tag{3.3}
\end{equation}
Call the self-map $T$, $(\epsilon, M; \leq; \phi)$-contractive (for $\phi \in F(R_+)$), if
\begin{equation}
\epsilon(T x, T y) \leq \phi(M(x, y)), \forall x, y \in X, x \leq y, \tag{c06}\end{equation}
when this holds for at least one comparison function $\phi$, the resulting convention reads: $T$ is extended $(\epsilon, M; \leq)$-contractive.
Theorem 4. Suppose that \((c04)\) and \((c05)\), \(T\) is extended \((e, M; \leq)\)-contractive and \((ao, C)\)-continuous, \((e, C)\) is ao-complete, and \((\leq)\) is \((ao, C)\)-self-closed. Then, \(T\) is a Picard operator (modulo \((C, \leq)\)).

Proof. Let \(x^*, u^* \in \text{Fix}(T)\) be such that \(x^* \leq u^*\). By the contractive condition, \(e(x^*, u^*) = 0\); wherefrom, \(x^* = u^*\); and so, \(\text{Fix}(T)\) is \((\leq)\)-singleton. It remains to show that each \(x = x_0 \in X(T, \leq)\) is a Picard point (modulo \((C, \leq, T)\)). Put \(x_n = T^nx, n \geq 0\); and let \(\varphi \in \mathcal{F}(R_+)\) be the comparison function given by the extended \((e, M; \leq)\)-contractivity of \(T\).

\section{I} By the contractive condition and \((\cdot)\),

\[ e(x_{n+1}, x_{n+2}) \leq \varphi(M(x_n, x_{n+1})) = \varphi[\max\{e(x_n, x_{n+1}), e(x_{n+1}, x_{n+2})\}], \forall n. \]

If (for some \(n\)) the maximum in the right hand side is \(e(x_{n+1}, x_{n+2})\), then (via \(\varphi \in \mathcal{F}_1(R_+)\)) \(e(x_{n+1}, x_{n+2}) = 0\); so that (as \(e=\)sufficient) \(x_{n+1} \in \text{Fix}(T)\); and we are done. Suppose that this alternative fails: \(e(x_{n+1}, x_{n+2}) \leq \varphi(e(x_n, x_{n+1})), \forall n\). This yields (by an ordinary induction) \(e(x_n, x_{n+1}) \leq \varphi^n(e(x_0, x_1)), \forall n\); wherefrom \(e(x_n, x_{n+1}) \to 0\) as \(n \to \infty\).

\section{II} We claim that \((x_n; n \geq 0)\) is \(e\)-Cauchy in \(X\). Denote, for simplicity, \(E(k, n) = e(x_k, x_{k+n}), k, n \geq 0\). Let \(\gamma > 0\) be arbitrary fixed; and \(\beta > 0\) be the number appearing in \((3.2)\); without loss, one may assume that \(\beta < \gamma\). By the preceding step, there exists a rank \(m = m(\beta)\) such that

\[ k \geq m \text{ implies } E(k, 1) < \beta/2 < \beta < \gamma. \quad (3.4) \]

The desired property follows from the inductive type relation

\[ \forall n \geq 0: [E(k, n) < \gamma + \beta/2, \text{ for each } k \geq m]. \quad (3.5) \]

The case \(n = 0\) is trivial; while the case \(n = 1\) is clear, via \((3.4)\). Assume that \((3.5)\) is true, for all \(n \in \{1, \ldots, p\}\) (where \(p \geq 1\)); we want to establish that it holds as well for \(n = p + 1\). So, let \(k \geq m\) be arbitrary fixed. By the induction hypothesis and \((3.4)\), \(e(x_k, x_{k+p}) = E(k, p) < \gamma + \beta/2\) and \(H(x_k, x_{k+p}) = \max\{E(k, 1), E(k, p, 1)\} < \beta/2\). Moreover, the same premises give (by the triangular property)

\[ L(x_k, x_{k+p}) = (1/2)[E(k, p + 1) + E(k + 1, p - 1)] \leq (1/2)[E(k, p) + E(k + 1, p) + E(k + 1, p - 1)] < \gamma + \beta; \]

wherefrom \(M(x_k, x_{k+p}) < \gamma + \beta\); so, by the contractive condition and \((3.2)\),

\[ E(k + 1, p) = e(x_{k+1}, x_{k+p+1}) = e(Tx_k, Tx_{k+p}) \leq \varphi(M(x_k, x_{k+p})) \leq \gamma; \]

which “improves” the previous evaluation \((3.5)\) of our quantity. This, along with \((3.4)\) and the triangular property, gives \(E(k, p + 1) = e(x_k, x_{k+p+1}) < \gamma + \beta/2\).

\section{III} As \((e, C)\) is ao-complete, \((3.5)\) tells us that \(x_n \xrightarrow{C} x^*\) for some \(x^* \in X\). Moreover, as \((\leq)\) is \((ao, C)\)-self-closed, we have \(x_n \leq x^*, \forall n\); hence, in particular, \(x \leq x^*\). Combining with the \((ao, C)\)-continuity of \(T\), yields \(x_{n+1} = Tx_n \xrightarrow{C} Tx^*\); wherefrom (as \(C\) is separated), \(x^* \in \text{Fix}(T)\). \hfill \(\square\)

Now letting \(d\) be a metric on \(X\), the associated convergence \(C := (d)\) is separated; moreover, the ao-complete property of \((e, C)\) is holding whenever \(d\) is complete and subordinated to \(e\). Clearly, this last property is trivially assured if \(d = e\); when Theorem \([11]\) is comparable with the main result in Agarwal, El-Gebeily and O’Regan \([11]\). In fact, a little modification of the working hypotheses allows us getting the whole conclusion of the quoted statement; we do not give details.
4. Particular aspects

Let $X$ be a nonempty set; and $T : X \to X$ be a self-map of $X$. Further, take a separated (sequential) convergence structure $\mathcal{C}$ on $X$.

(A) Let $e(.,.)$ be an almost metric over $X$. A basic particular case of the previous developments corresponds to $(\leq) = X \times X$ (=the trivial quasi-order on $X$). Then, $(c04)+(c05)$ are holding; and the resulting Picard concept becomes a Picard property (modulo $\mathcal{C}$) of $T$, which writes: i) $\text{Fix}(T)$ is a singleton, $\{x^{*}\}$, if this holds for at least one comparison function $\varphi$, the resulting convention reads: $T$ is extended $(e,M)$-contractive. Putting these together, one gets the following version of Theorem 4.

**Corollary 1.** Suppose that $T$ is extended $e$-contractive and $(o,C)$-continuous, and $(e,C)$ is $o$-complete. Then, $T$ is a Picard operator (modulo $\mathcal{C}$).

The obtained statement includes Kasahara’s fixed point principle [12], when $e$ is a metric on $X$. On the other hand, if $d$ is a metric on $X$ and $\mathcal{C} := (\frac{d}{\alpha})$, the $o$-complete property of $(e,C)$ is assured when $d$ is complete and subordinated to $e$. This, under a linear choice of the comparison function $(\varphi(t) = \alpha t, t \in R_{+}$, for $0 < \alpha < 1$), tells us that Corollary 1 includes Theorem 6. Finally, when $d = e$, Corollary 1 reduces to Jachymski’s result [10].

(B) An interesting version of Corollary 1 was provided in the 2008 paper by O’Regan and Petruˇ sel [19, Theorem 3.3]. Let $(X,T,\mathcal{C})$ be endowed with their precise general meaning; and $d(.,.)$ be a (standard) metric on $X$. As before, we are interested to give sufficient conditions under which $T$ be a Picard operator (modulo $\mathcal{C}$). Take an order $(\leq)$ on $X$; and put $X_{(\leq)} = (\leq) \cup (\geq)$, where $(\geq)$ stands for the dual order. This subset is just the graph of the relation $\langle \rangle$ over $X$ introduced as in (a01); so, it may be identified with the underlying relation. As a consequence, $X_{(\leq)}$ is reflexive $[(x,x) \in X_{(\leq)}$, for each $x \in X]$ and symmetric $[(x,y) \in X_{(\leq)}$ iff $(y,x) \in X_{(\leq)}]$; but not in general transitive, as simple examples show. Further, let us say that $(d,\mathcal{C})$ is $o$-complete if $[(\text{for each sequence}) d\text{-Cauchy} \implies \mathcal{C}\text{-convergent}].$ Finally, call $T$, $(d,\leq;\varphi)$-contractive (for $\varphi \in \mathcal{F}(R_{+})$), if

\[ (d03) \quad d(Tx, Ty) \leq \varphi(d(x,y)), \forall x, y \in X, x \leq y; \]

when this holds for at least one comparison function $\varphi$, the resulting convention reads: $T$ is $(d,\leq)$-contractive.

**Corollary 2.** Assume that $(a03)+(b02)$ hold, $T$ is $(d,\leq)$-contractive and $(o,C)$-continuous, $(d,\mathcal{C})$ is $o$-complete, and

\[ (d03) \quad (x,y), (y,z) \in X_{(\leq)} \implies (x,z) \in X_{(\leq)} \quad (i.e.: X_{(\leq)} \text{ is transitive}) \]
\[ (d04) \quad (x,y) \notin X_{(\leq)} \implies \exists c = c(x,y) \in X: (x,c), (y,c) \in X_{(\leq)}. \]

Then, $T$ is a Picard map (modulo $\mathcal{C}$).
Proof. We claim that Corollary 1 is applicable to such data. This will follow from Theorem 5. In fact, let 
\[ (x, y, c) \in X(\leq) \] such that \((e_01)\) holds. Then, 
\[ T \in F \] is a Picard operator (modulo \((a_03)+(b_02)\) are superfluous.

In the following, a summary of the 1986 results in Turinici [27] is being sketched, for completeness reasons.

(A) Let \((X, d)\) be a complete metric space and \(T\) be a self-map of \(X\). Assume that for each \(x \in X\) there exists a \(n(x) \in N_0 := N \setminus \{0\}\) such that \(T^{n(x)}\) is (metrically) contractive at \(x\); then, we may ask of under which additional conditions is \(T\) endowed with a Picard property (cf. Section 1). A first answer to this question was given, in the continuous case, by Sehgal [24] through a specific iterative procedure; a reformulation of it for discontinuous maps was performed in Guseman’s paper [7]. During the last decade, some technical extensions — involving the contractive condition — of these results were obtained by Ciric [3], Khazanchi [13], Iseki [9], Matkowski [16], and Singh [25]. The most general statement of this kind, obtained by Matkowski [16], reads as follows. For each \(m \in N_0\), let \(F(R^m_+)\) stand for the class of all functions \(f : R^m_+ \to R_+\); and \(F_i(R^m_+)\) the subclass of all \(f \in F(R^m_+)\), increasing in each variable. The iterative contraction property below is considered:

\[ (e_01) \exists f \in F_i(R^m_+) \text{ such that: } \forall x \in X, \exists n(x) \in N_0 \text{ with } d(T^{n(x)}x, T^{n(x)}y) \leq f(d(x, T^{n(x)}x), d(x, y), d(T^{n(x)}x, y), d(T^{n(x)}y, y)), \forall y \in X. \]

Given \(f \in F_i(R^m_+)\) like before, denote \(g(t) = f(t, t, t, 2t, 2t)\), \(t \geq 0\); clearly, it is an element of \(F_i(R_+)\). We shall say that \(f\) is normal provided

\[ (e_02) \quad g \in F_1(R_+) \text{ and } |t - g(t) \to \infty \text{ as } t \to \infty \]

\[ (e_03) \lim_{n} g^n(t) = 0, \text{ for each } t > 0. \]

(As already remarked, \((e_03)\) implies the first part of \((e_02)\), under the properties of \(g\); we do not give details).

Theorem 5. Suppose that there exists a normal function \(f \in F_i(R^m_+)\) in such a way that \((e_01)\) holds. Then, \(T\) is a Picard operator (modulo \(d\)).

A direct examination of the above conditions shows that, by virtue of

\[ d(T^{n(x)}x, y) \leq d(x, T^{n(x)}x) + d(x, y), \quad x, y \in X, \]

\[ d(T^{n(x)}y, y) \leq d(x, T^{n(x)}y) + d(x, y), \quad x, y \in X, \]

a slight extension of Theorem 3 might be reached if one replaces \((e_01)\) by

\[ (e_04) \quad d(T^{n(x)}x, T^{n(x)}y) \leq F(d(x, T^{n(x)}x), d(x, y), d(x, T^{n(x)}y)), \quad y \in X, \]

where \(F : R^m_+ \to R_+\) is defined as

\[ F(\xi, \eta, \zeta) = f(\xi, \eta, \zeta + \eta, \zeta + \eta), \quad \xi, \eta, \zeta \in R_+. \]
A natural question to be solved is that of determining what happens when the right-hand side of (e04) depends on the (abstract) variable \( x \in X \) and the (real) variables \( (d(x, T^i x); 1 \leq i \leq n(x)), (d(x, T_j y); 0 \leq j \leq n(x)) \); or, in other words, when the function \( F = F(x) \) acts from \( R^{2n(x)+1}_+ \) to \( R_+ \). At the same time, observe that, from a "relational" viewpoint, the result we just recorded may be deemed as being expressed modulo the trivial quasi-ordering on \( X \); so that, a formulation of it in terms of genuine quasi-orderings would be of interest. It is precisely our main aim to get a generalization – under the above lines – of Theorem 5.

(B) Let \((X, d)\) be a metric space and \( \leq \) be a quasi-ordering (i.e.: reflexive and transitive relation) over \( X \). A sequence \((x_n; n \in N)\) in \( X \) will be said to be increasing when \( x_i \leq x_j \) for \( i \leq j \). Take the self-map \( T \) of \( X \) according to

\[
\begin{align*}
(e05) \quad & Y := \{ x \in X; x \leq T x \} \text{ is not empty} \\
(e06) \quad & T \text{ is increasing} (x \leq y \implies T x \leq T y).
\end{align*}
\]

In addition, the specific condition will be accepted:

\[
\begin{align*}
(e07) \quad & \text{for each } x \text{ in } Y \text{ there exist } n(x) \in N_0, f(x) \in \mathcal{F}_i(R^{2n(x)+1}_+), \text{ with} \\
& d(T^n x, T^n y) \leq f(x)(d(x, T x), ..., d(x, T^n x); d(x, y), ..., d(x, T^n y)), \\
& \text{for all } y \in Y \text{ with } x \leq y.
\end{align*}
\]

For the arbitrary fixed \( x \in Y \), let \( g(x) \) indicate the element of \( \mathcal{F}_i(R_+) \), given as \( g(x)(t) = f(x)(t, ..., t, t, ..., t), t \geq 0 \). We shall say that the family (of (e07)) \(((n(x), f(x))\); \( x \in Y \)) is iterative \( T \)-normal provided, for each \( x_0 \in Y \),

\[
\begin{align*}
(e08) \quad & g(x_0) \in \mathcal{F}_1(R_+) \text{ and } t - g(x_0)(t) \to \infty \text{ as } t \to \infty, \\
(e09) \quad & \lim_k g(x_k) \circ ... \circ g(x_0)(t) = 0, t > 0, \text{ where } [n_0 = n(x_0), x_1 = T^{n_0} x_0] \text{ and,} \\
& \text{inductively, } [n_i = n(x_i), x_{i+1} = T^{n_i} x_i], i \geq 1.
\end{align*}
\]

The following auxiliary fact will be useful.

**Proposition 2.** Let (e05)-(e07) hold; and the family \(((n(x), f(x)); x \in Y)\) [attached to (e07)] be iterative \( T \)-normal. Then, the following conclusions hold

- (i) for each \( x \in Y \), \((T^m x; m \in N)\) is increasing Cauchy (in \( X \))
- (ii) \( d(T^m x, T^m y) \to 0 \text{ as } m \to \infty, \text{ for all } y \in Y, x \leq y. \)

**Proof.** Let \( x \in Y \) be given. We firstly claim that

\[
d(x, T^m x) \leq t, m \in N, \text{ for some } t = t(x) > 0. \quad (5.1)
\]

Indeed, it follows by (e08) that, given \( \alpha > 0 \), there exists \( \beta = \beta(\alpha, x) > \alpha \) with

\[
t \leq \alpha + g(x)(t) \text{ implies } t \leq \beta. \quad (5.2)
\]

Put \( \alpha = \max\{d(x, T x), ..., d(x, T^n x)\} \). We claim that (5.1) holds with \( t = \beta \). In fact, suppose that the considered assertion would be false; and let \( m \) denote the infimum of those ranks for which the reverse of (5.1) takes place. Clearly, \( m > n(x), d(x, T^k x) \leq \beta, k \in \{1, ..., m - 1\}, \text{ and } d(x, T^m x) > \beta \); so that, by (e07),

\[
\begin{align*}
d(x, T^m x) & \leq d(x, T^n x) + d(T^n x, T^m x) \\
& \leq \alpha + f(x)(d(x, T x), ..., d(x, T^n x); d(x, T^{m-n(x)} x), ..., d(x, T^m x)) \\
& \leq \alpha + f(x)(\alpha, ..., \alpha, \beta, ..., \beta, d(x, T^m x)) \leq \alpha + g(x)(d(x, T^m x))
\end{align*}
\]

contradicting (5.2) and proving our assertion. In this case, letting \( x = x_0 \in Y \), put \( n_0 = n(x_0), m_0 = n_0, x_1 = T^{m_0} x_0 = T^{m_0} x_0 \text{ and,} \)

inductively,

\[
i = n(x), m_i = n_0 + \ldots + n_i, x_{i+1} = T^{n_i} x_i = T^{m_i} x_0, i \geq 1.
\]
By (5.1), $d(x_0, T^m x_0) \leq t_0$, $m \in N$, for some $t_0 > 0$; so combining with (e07):

\[
d(x_1, T^m x_1) = d(T^m x_0, T^m x_0) \leq f(x_0)(d(x_0, T x_0), \ldots, d(x_0, T^n x_0);
\]

or equivalently, $d(T^m x_0, T^m x_0) \leq g(x_0)(t_0), m \in N$;

or equivalently, $d(T^m x_0, T^m x_0) \leq g(x_1)(t_0), m \geq m_0$. Again via (e07),

\[
d(x_2, T^m x_2) = d(T^n x_1, T^n x_1) \leq f(x_1)(d(x_1, T x_1), \ldots, d(x_1, T^n x_1);
\]

or equivalently: $d(T^m x_0, T^m x_0) \leq g(x_1) \circ g(x_0)(t_0), m \geq m_1$; and so on. By a finite induction procedure one gets $d(x_k+1, T^m x_{k+1}) \leq g(x_k) \cdots g(x_0)(t_0), m, k \in N;$

or equivalently (for each $k \in N$)

\[
d(T^m x_0, T^m x_0) \leq g(x_k) \circ \cdots g(x_0)(t_0), m \geq m_k;
\]

wherefrom, taking (e09) into account, $(T^n x_0; n \in N)$ is an increasing Cauchy sequence. Finally, given $y_0 \in Y$ with $x_0 \leq y_0$, put $y_1 = T^n y_0$ and, inductively,

\[
y_{i+1} = T^n y_i = T^{m_i} y_0, i \geq 1.
\]

Again by (5.1),

\[
d(x_0, T^n x_0), d(x_0, T^n y_0) \leq t_0, m \in N, \text{ for some } t_0 > 0.
\]

This fact, combined with (e07), leads us, by the same procedure as before, at

\[
d(x_{k+1}, T^n x_{k+1}), d(x_{k+1}, T^n y_{k+1}) \leq g(x_k) \circ \cdots g(x_0)(t_0), m, k \in N;
\]

or equivalently (for each $k \in N$)

\[
d(T^n x_0, T^n y_0) \leq g(x_k) \circ \cdots g(x_0)(t_0), m \geq m_k;
\]

proving the desired conclusion and completing the argument. \hfill \Box

(C) Let $X, d$ and $\leq$ be endowed with their previous meaning. Given the sequence $(x_n; n \in N)$ in $X$ and the point $x \in X$, define $x_n \uparrow x$ as: $(x_n; n \in N)$ is increasing and convergent to $x$. Term the triplet $(X, d; \leq)$, quasi-order complete, provided each increasing Cauchy sequence converges. Note that any complete metric space is quasi-order complete; but the converse is not in general valid. Further, given the self-map $T$ of $X$, call it continuous at the left when $x_n \uparrow x$ and $x_n \leq x, n \in N$, imply $T x_n \rightarrow T x$. Also, the ambient quasi-ordering $\leq$ will be said to be self-closed when $x \leq y_n, n \in N$ and $y_n \uparrow y$ imply $x \leq y$; note that any semi-closed quasi-ordering in Nachbin’s sense [17, Appendix] is necessarily self-closed. The first main result of the present note is

**Theorem 6.** Let the conditions of Proposition [2] be fulfilled; and (in addition) $(X, d; \leq)$ is quasi-order complete, $\leq$ is self-closed, and $T$ is continuous at the left. Then, the following conclusions will be valid

\begin{enumerate}
  \item[iii)] $Z := \{x \in X; x = T x\}$ is not empty
  \item[iv)] for every $x \in Y$, $(T^n x; n \in N)$ converges to an element of $Z$
  \item[v)] if $x, y \in Y$ are comparable, $(T^n x; n \in N)$ and $(T^n y; n \in N)$ have the same limit (in $Z$).
\end{enumerate}

**Proof.** By Proposition [2] and the quasi-order completeness of $(X, d; \leq)$, it follows that, for the arbitrary fixed $x \in Y$, $T^n x \uparrow z$ for some $x \in X$. As $\leq$ is self-closed, $T^n x \leq z, n \in N$; so that, combining with the left continuity of $T$ one gets $T^n x \uparrow T z$; hence $z = T z$. The proof is thereby complete. \hfill \Box
Now, it is natural to ask of what happens when \( T \) is no longer continuous at the left. Some conventions are in order. Call \( \leq \), anti self-closed when \( y_n \leq x, n \in N \), and \( y_n \uparrow y \) imply \( y \leq z \); observe at this moment that a sufficient condition for \( \leq \) to be anti self-closed is that \( \geq \) (its dual) be semi-closed. Further, call \( \leq \), interval closed when it is both self-closed and anti self-closed. Our second main result is

**Theorem 7.** Let the conditions of Proposition 3 be fulfilled; and (in addition) \( (X, d; \leq) \) is quasi-order complete and \( \leq \) is an interval closed ordering. Then, conclusions iii)- vi) of Theorem 2 continue to hold; and, moreover,

vi) for each \( x \in Y \) the element \( z = \lim_n T^n x \) in \( Z \) has the properties (a) \( x \leq z \), (b) \( z \leq y \in Y \) implies \( z = y \).

**Proof.** Let \( x \in Y \) be arbitrary fixed. By Proposition 2 \( T^n x \uparrow z \), for some \( z \in X \). Hence (as \( \leq \) is self-closed), \( x \leq T^n x \leq z, n \in N \). It immediately follows that \( T^n x \leq Tz, n \in N \); so (by the anti self-closedness of \( \leq \)), \( z \in Y \). Now, \( x \leq z \in Y \) gives, again by Proposition 2 \( T^n z \uparrow z \) (hence \( Tz \leq T^n z \leq z, n \in N_0 \)) and therefore (as \( \leq \) is ordering) \( z \in Z \). The remaining part is evident. \( \square \)

It remains now to discuss the alternative:

\( (T \text{ is not continuous at the left}) \) and \( (\leq \text{ is not an interval closed ordering}) \).

To this end, assume that, for any \( x \in Y \), the function \( f(x) \in F_i(R^{2n(x)+1}_+ x) \) given by (e07) fulfills

\[
(e10) \text{ for each } (\alpha_1, ..., \alpha_{n(x)}) \in R^n_+ \text{ with } \alpha_{n(x)} > 0 \text{ there exists } \beta > 0 \text{ with } \\
\beta + f(x)(\alpha_1, ..., \alpha_{n(x)}; \beta, ..., \beta) < \alpha_{n(x)}
\]

\[
(e11) \text{ for each } (\alpha_1, ..., \alpha_{n(x)}) \in R^n_+ \text{ with } \alpha_1 > 0, \alpha_{n(x)} = 0, \text{ we have } \\
f(x)(\alpha_1, ..., \alpha_{n(x)}; \alpha_1, ..., \alpha_{n(x)}, \alpha_1) < \alpha_1.
\]

Now, as a completion of the above results, we have

**Theorem 8.** Let the conditions of Proposition 2 be fulfilled; and (in addition) \( (X, d; \leq) \) is quasi-order complete, (e10)+(e11) hold, and \( \leq \) is an interval closed quasi-ordering. Then, conclusions iii)- vi) of Theorem 7 still remain valid.

**Proof.** Let \( x \in Y \) be arbitrary fixed. By the above reasoning, \( T^n x \uparrow z \), for some \( z \in Y \); with, in addition (cf. Proposition 2): \( [x \leq T^n x \leq z, n \in N] \) and \( [T^n x \uparrow z] \). Assume that \( z \neq T^n(z); \text{ and let } \beta > 0 \text{ be the number attached (via (e10)) to } \alpha_1 := d(z, Tz), ..., \alpha_{n(x)} := d(z, T^n(z)). \text{ By the convergence property above, there exists } k(\beta) \in N \text{ such that } d(z, T^k z) \leq \beta, \forall k \geq k(\beta); \text{ and this gives for all ranks } m \geq k(\beta) + n(z), \\
\]

\[
d(z, T^{(z)} z) \leq d(z, T^m z) + d(T^{(z)} z, T^m z) \leq d(z, T^m z) + \\
f(z)(d(z, Tz), ..., d(z, T^n(z)); d(z, T^{m-n(z)} z), ..., d(z, T^m z)) \leq \\
\beta + f(z)(d(z, Tz), ..., d(z, T^n(z)) \leq \beta, ..., \beta) < d(z, T^n(z));
\]

contradiction; hence \( z = T^n(z) \). Moreover,

\[
d(z, Tz) = d(T^n(z) z, T^n(z) Tz) \leq \\
f(z)(d(z, Tz), ..., d(z, T^n(z)); d(z, Tz), ..., d(z, T^n(z)), d(z, T^n(z) Tz)) = \\
f(z)(d(z, Tz), ..., d(z, T^{n-1}(z)), 0; d(z, Tz), ..., d(z, T^{n-1}(z)), 0, d(z, Tz));
\]

wherefrom, if \( z \neq Tz \), (e11) will be contradicted. Hence the conclusion. \( \square \)
Some remarks are in order. Theorem 6 may be viewed as a quasi-order extension of Sehgal’s result we just quoted (cf. also Dugundji and Granas [5, Ch 1, Sect 3]) while Theorem 7 is a quasi-order “functional” version of Matkowski’s contribution (Theorem 5). At the same time, Theorem 8 - although formulated as a fixed point result - may be deemed in fact as a maximality principle in \((Y, \leq)\); so, it is comparable under this perspective with a related author’s one [20] obtained by means of a “compactness” procedure like in Krasnoselskii and Sobolev [14].

(D) Note added in 2011

From these developments, the following statement is deductible. Let the quasi-ordered metric space \((X, d, \leq)\) the self-map \(T\) of \(X\) be taken as in (e05)+(e06). In addition, the specific condition will be accepted:

\[(e12) \text{there exists } f \in F(R_+) \text{ such that: for each } x \in Y \text{ there exists } n(x) \in N_0 \text{ with } d(T^n(x), T^n(y)) \leq f(d(x, y)), \text{for all } y \in Y \text{ with } x \leq y.\]

Note that, in such a case, the iterative normality of \((n(x); f); x \in Y\) is characterized by (e02)+(e03), with \(f\) in place of \(g\) and referred to as: \(f\) is normal (see above). From Theorem 6 we then get, formally

**Theorem 9.** In addition to (e05)+(e06), assume that the function \(f\) (appearing in (e12)) is normal, \((X, d, \leq)\) is quasi-order complete, \(\leq\) is self-closed, and \(T\) is continuous at the left. Then, conclusions iii)–v) of Theorem 6 are retainable.

In particular, any linear comparison function \(f\) (in the sense: \(f(t) = \alpha t, t \in R_+, \text{ for } 0 < \alpha < 1\)) is normal. Then, Theorem 9 includes the essential conclusions of the Ran-Reurings result (Theorem 1). [In fact, under appropriate conditions, it may give us all conclusions in the quoted statement; we do not give details]. Note that Theorem 9 is not yet covered by the existing fixed point statements in the realm of quasi-ordered metric spaces. Further aspects will be delineated elsewhere.

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