Field Theory on $q = -1$ Quantum Plane

Andrzej Sitarz

Department of Theoretical Physics
Institute of Physics, Jagiellonian University
Reymonta 4, 30-059 Kraków, Poland
e-mail: sitarz@if.uj.edu.pl

Abstract

We build the $q = -1$ deformation of plane on a product of two copies of algebras of functions on the plane. This algebra contains a subalgebra of functions on the plane. We present general scheme (which could be used as well to construct quaternion from pairs of complex numbers) and we use it to derive differential structures, metric and discuss sample field theoretical models.

TPJU 4/95
February 1995

*Partially supported by KBN grant 2P 302 103 06
1 Introduction

The algebraic concept of the two-dimensional plane, conceived as the $C^*$ algebra of continuous functions it, can be easily generalised to the noncommutative case. The $q$-deformation of such algebra, called Manin plane, is the simplest example (see [1] for basic introduction, also [5] for general concepts of noncommutative geometry). Let us remind here its definition, we construct this algebra from the self-adjoint generators $X$ and $Y$ and the unit 1, satisfying the following commutation relations:

$$XY = qYX,$$

(1)

where $q$ is a non-zero complex number. Since we can choose the generators to be selfadjoint we have $qq^* = 1$.

In this paper we should restrict ourselves to one particular case when $q = -1$, and we shall call the algebra in this situation as corresponding to 'noncommutative plane' [2].

The quantum planes in general have been the topic of several papers [4, 3], focusing especially on the differential structures. What we shall discuss here is a special formulation of $q = -1$ quantum plane algebra, which is built on the product of algebras of functions on the standard plane. In this respect our approach differs from [2], moreover, we do not discuss the corresponding $q = -1$ quantum linear group. We shall describe the general construction, demonstrating that it could be applied also to some other objects. We shall give simple examples of this scheme (obtaining in this way quaternions from complex numbers, for instance).

The main body of this paper is devoted, however, to the application of this method to $q = -1$ deformation of the plane and simple models of field theory in this setup. We discuss the differential calculus, metric and construct sample actions for scalar field theory and gauge theory.

2 General concepts

In this section we shall introduce a general structure of an algebra built on the product of two copies of an arbitrary algebra. We shall see that this structure could be used to constuct the algebra of quaternions as well as the $q = -1$ deformation of the algebra of functions on the plane.

Let $\mathcal{A}$ be an algebra and $\hat{\,} : \mathcal{A} \to \mathcal{A}$ be an algebra automorphism such that $\hat{\,} \circ \hat{\,} = \text{id}.$
Then the following is true:

**Observation 1.**

*If* \( \xi \) *is an element of the centre of* \( \mathcal{A} \), *which is* \(^\wedge\) *invariant, \( \hat{\xi} = \xi \), then* \( \mathcal{A} \times \mathcal{A} \) *with pointwise addition and the following multiplication rule:

\[
(a, b) \wedge (A, B) = (aA + \xi b\hat{B}, b\hat{A} + aB) \tag{2}
\]

*is an algebra.*

As linearity is quite obvious, it remains to check the associativity, the proof of which is technical. We shall call this algebra \( \text{httA} \).

Additionally, if \( \mathcal{A} \) has a \( \star \)-structure and \( \xi^\star = \xi \) we can choose between the following star structures on \( \hat{\mathcal{A}} \):

\[
(a, b)^\star = (a^\star, \hat{b}^\star), \tag{3}
\]

or:

\[
(a, b)^\star = (a^\star, -\hat{b}^\star). \tag{4}
\]

**Example.**

*Let* \( \mathcal{A} \) *be just the field of complex numbers. Then if* \(^\wedge\) *is just the complex conjugation (as* \( \mathbb{C} \) *is commutative then there is no problem with it being an algebra automorphism) and* \( q = -1 \), *then* \( \hat{\mathcal{A}} \) *with the second \( \star \)-structure (4) is the algebra of quaternions.*

**Observation 2.**

\( \mathcal{A} \) *is included in* \( \hat{\mathcal{A}} \):

\[
\mathcal{A} \ni a \mapsto (a, 0) \in \hat{\mathcal{A}}, \tag{5}
\]

To end this section let us present two examples of construction based on the algebras of functions on discrete spaces. In the first case we take as \( \mathcal{A} \) the algebra of complex valued functions on \( \mathbb{Z}_2 \), and \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) in the second one. The algebra of functions on \( \mathbb{Z}_2 \) has one generator \( a \) such that \( a^2 = 1, a^\star = a \). Let us take \( \hat{a} = -a \) (this operation corresponds to underlying symmetry of
changing the points), then we must choose $\xi$ to be real (without loss of
generality it could be set to 1) and we get the algebra $\widehat{A}$ generated by
$a = (a, 0)$ and $b = (0, 1)$ with the relations:

$$a^2 = 1, \quad b^2 = 1 \quad ab + ba = 0.$$  \hspace{1cm} (6)

Now, let us turn to $\mathbb{Z}_2 \times \mathbb{Z}_2$, functions on this space are generated by com-
muting $a$ and $b$, $a^2 = b^2 = 1$. Let us take $\tilde{a} = a$, $\tilde{b} = -b$ and $\xi = a$. Then
the algebra $\widehat{A}$ is generated by $a = (a, 0)$ and $B = (0, 1)$ with the following
relations:

$$b^2 = 1, \quad A^2 = a, \quad Ab + bA = 0.$$  \hspace{1cm} (7)

Next we shall apply this procedure to obtain $q = -1$ Manin plane.

3 Anticommutative plane

Let us take $\mathcal{A}$ to be an algebra of polynomials on $\mathbb{R}^2$ and let us denote its
generators by $x$ and $y$. We choose $\tilde{\cdot}$ to be the automorphism of mirror
symmetry: $\tilde{x} = x, \tilde{y} = -y$ and, finally, $\xi = x$. Then the algebra $\widehat{\mathcal{A}}$ has two
generators: $Y = (y, 0)$ and $X = (0, 1)$ with the following relations:

$$XY + YX = 0.$$  \hspace{1cm} (8)

Of course, we have $X^2 = x$. The relation (8) is just the deformation relation
of the Manin plane for $q = -1$. What have we achieved here is that now we
can work with the presentation of such algebra as an algebra built upon the
cartesian product of algebras of functions on the regular plane.

The choice of the $\star$-structure corresponds now to the choice of $X$ to be a
hermitian or antihermitian operator (of, course in either case $X^2 = x$ would
be hermitian.

3.1 Differential calculus

In this section we shall construct a differential calculus on our anticommu-
tative plane. From many possibilities we shall choose the one, which has
the following properties. First, the bimodule of one-forms must be a free
module generated by $dX$ and $dY$, moreover, we require that the calculus,
when restricted to $\mathcal{A} \subset \widehat{\mathcal{A}}$, should be just a standard differential calculus on
the plane.
This gives us immediately:

\[ Y \, dY = dY \, Y \quad x \, dY = dY \, x \quad (9) \]

\[ x \, dx = dx \, x \quad Y \, dx = dx \, Y \quad (10) \]

where \( dx = d(X^2) = X \, dX + dX \, X \).

If we look for calculi, which are defined through relations of the type \( x^k dx^l = C_{ij}^{kl} dx^i x^j \) and assume (9-10) we obtain that there are only three possibilities of differential structures:

(A) \[ X \, dX = dX \, X, \]
\[ Y \, dX = -dX \, Y, \]
\[ X \, dY = -dY \, X \]

(B) \[ X \, dX = -dX \, X + wdY \, Y, \]
\[ Y \, dX = -dX \, Y, \]
\[ X \, dY = -dY \, X, \]

(C) \[ X \, dX = dX \, X + wdY \, Y, \]
\[ X \, dY = dY \, X, \]
\[ Y \, dX = -dX \, Y - 2dY \, X. \]

The first one (A) is the standard one, as it is built from two commutative calculi (one with \( X \, dX = dX \, X \), the other one with \( Y \, dY = dY \, Y \)). This is also the only calculus out these three that has Ker \( d = \mathbb{C} \). Indeed, let us observe that for the other calculi from the relations (B-C) we immediately get:

\[ d(X^2 - \frac{w}{2}Y^2) = 0. \quad (11) \]

In either of the presented versions of differential calculi we could have the higher order calculus:

\[ dX \wedge dX = 0, \quad dY \wedge dY = 0, \quad dX \wedge dY = dY \wedge dX, \quad (12) \]

however, only in the first one (A) the condition \( dX \wedge dX = 0 \) is necessary. In our further considerations we shall use only the first calculus, (however, we shall briefly discuss the existence of the metric structures on all of them), mainly due to the fact that it is the only one with Ker \( d = \mathbb{C} \), as we would normally assume for physical theories.
3.2 Metric

We use here the definition of metric as proposed and discussed in the general case of quantum plane [3]. Let us denote $g_{ab}$ the value of metric $g(da, db)$, $a, b$ being $X$ and $Y$.

It appears that only for the first two calculi (A,B) we can have a nonvanishing metric, for the third one, the bimodule properties of the metric enforce that it should vanish.

We find that for the calculi (A) and (B) $g_{XX}$ and $g_{YY}$ must be in the centre of $\hat{A}$ whereas $g_{xy}$ and $g_{yx}$ must be of the form $xyf$, where $f$ is in the centre of the algebra. Additionally, for (B) we must require that $g_{XY} = -g_{YX}$ and $wg_{yy} = 2X^2f$.

4 Scalar Field Theory

To construct a simple scalar field theory on $q = -1$ quantum plane, with the field $\Phi \in \hat{A}$ we need to choose the differential calculus as well as the $\star$ operation. As we have already dealt with the first problem, having chosen the calculus (A), we shall show the result for each $\star$ structure, using the notation: $(a, b)^\star = (a^\star, \pm \hat{b}^\star)$.

First, for $\Phi$ presented as $(\psi, \phi)$ according to the general scheme (see Observation 1.) and the calculus (A) we get:

$$d(\psi, \phi) = dX(\hat{\phi} + 2x\partial_x\hat{\phi}, 2\partial_x\hat{\psi}) + dY(\partial_y\psi, \partial_y\phi),$$

Let us mention here that, formally, in the above equations, $d(X\frac{1}{\sqrt{x}}) = 0$, however, this does not pose a problem as $\frac{1}{\sqrt{x}}$ does not belong to the algebra of functions, which we are considering.

Having calculated $d\Phi$ we can for any metric $g$ calculate $g(d\Phi^\star, \Phi)$, which is the kinetic term of the Lagrange function of any field theory. We shall do it for the simplest admissible metric $g_{XX} = g_{YY} = 1$ and $g_{XY} = g_{YX} = 0$, we use also notation $2\partial_X f = f + 2\partial_x f$. Then for (13) one obtains:

$$g(d\Phi^\star, d\Phi) = 4(\partial_X\hat{\phi}^\star, \pm \partial_x\psi^\star)(\partial_X\hat{\phi}, \partial_x\hat{\psi})$$

$$+ (\partial_y\psi^\star, \pm \partial_y\hat{\phi}^\star)(\partial_y\psi, \partial_y\phi),$$

which leads to the following final expression:

$$g(d\Phi^\star, d\Phi) = 4|\partial_X\hat{\phi}|^2 \pm 4x|\partial_x\psi|^2 + |\partial_y\psi|^2 \pm x|\partial\hat{\phi}|^2$$

$$+ \left(4(\partial_X\hat{\phi}^\star\partial_x\hat{\psi} \pm \partial_x\psi^\star\partial_X\hat{\phi}) + (\partial_y\psi^\star\partial_y\phi \pm \partial_y\hat{\phi}^\star\partial_y\hat{\psi})\right) X.$$

(15)
For fields $\Phi$, which belong to the subalgebra of functions on the regular plane $\Phi_r = (\psi, 0)$ the above expression reduces to:

$$g(d\Phi_r^*, d\Phi_r) = \pm 4x|\partial_x \psi|^2 + |\partial_y \psi|^2,$$

so it is the standard kinetic term on a plane with a nonconstant metric.

For fields, which are of the form $\Phi_n = (0, \phi)$, the kinetic term (15) becomes:

$$g(d\Phi_n^*, d\Phi_n) = 4x^2|\partial_x \hat{\phi}|^2 + x|\partial_y \hat{\phi}|^2 + |\hat{\phi}|^2 + 2x(\hat{\phi}^\star \partial_x \hat{\phi} + \hat{\phi} \partial_x \hat{\phi}^\star).$$

### 4.1 Integration and Action

The concept of the integration (or trace) on the algebra is one of crucial elements, which are required to build a satisfactory model. Though it is rather unclear in general whether such operation could be introduced for quantum spaces, even if it could be a number-valued operation, in this particular case of $q = -1$ deformation one formulate the problem without difficulties. We propose the trace to be a $\mathbb{C}$-valued, symmetric operator on our algebra $\hat{A}$, which is $\ast$-covariant, i.e. $\int(ab) = \int(ba)$ and $\int a^\star = (\int a)^\star$. Suppose we have a trace $\int_0$ on the algebra $\hat{A}$ (remember that $\hat{A}$ is built on the product of two copies of $A$), which is $\hat{\ast}$-invariant. Then the following is true:

**Observation 3.**

$$\hat{A} \ni (\psi, \phi) \rightarrow \int_0 \psi$$

is a trace on the algebra $\hat{A}$.

For the algebra of quaternions, as discussed at the beginning of our paper, this gives the natural answer: the $\ast$-invariant trace on the algebra of complex numbers is the real part: $\int_0 z = \text{Re} z$, consequently, a trace of the quaternion $(a, b)$ is just the real part of $a$.

Coming back to the algebra of $q = -1$ quantum plane we can now write the proposition for the simplest action of scalar field theory as the trace of (17):

$$S = \int_0 \left( 4x^2|\partial_x \hat{\phi}|^2 + x(\hat{\phi}^\star \partial_x \hat{\phi} + \hat{\phi} \partial_x \hat{\phi}^\star) \right).$$

where $\int_0$ is any integration on the plane, which is $\hat{\ast}$-invariant. As we can see, we have this action could be interpreted as a standard action (though in a non-trivial metric) for a doublet of scalar fields on the plane, with the usual kinetic terms as well as a mass term for one field.
5 Gauge Theory

In this section we shall briefly discuss the construction of a simple gauge theory on \( q = -1 \) noncommutative plane. We shall use the concepts of Connes’ approach \([5]\) to gauge theories in noncommutative geometry, choosing the gauge group as the unitary group of the algebra, being aware that this scheme may not be appropriate for arbitrary quantum deformation.

The unitary group \( U(\hat{A}) \) consist of all elements \((u, v)\) such that:

\[
u u^* \pm xv^* = 1, \quad v\hat{u}^* \pm u\hat{v}^* = 0.
\] (19)

Of course, the group all unitary functions on the plane is a subgroup of \( U(\hat{A}) \).

The connection one-form \( A \) must be antihermitian \( A = -A^* \), so that the curvature \( F = dA + A \wedge A \) is hermitian. The two-form \( F \) has, due to the rules of calculus (12), has only one component \( dX \wedge dyF \).

If \( A = dX(\psi_X, \phi_X) + dy(\psi_y, \phi_y) \) the relation \( A^* = -A \) gives us:

\[
\begin{align*}
\psi_X &= -\hat{\psi}_X^* \quad \phi_X = \mp \hat{\phi}_X^* \\
\psi_y &= -\hat{\psi}_y^* \quad \phi_y = \pm \hat{\phi}_y^*
\end{align*}
\] (20, 21)

and \( F \) becomes:

\[
F = dX \wedge dY \left( -\partial_y \psi_X \pm \phi_y^* \pm 2x\partial_x \phi_y^* \psi_X (\hat{\psi}_y + \hat{\psi}_y) + x\phi_y^* (\hat{\phi}_X - \phi_X) \right) \\
+ dX \wedge dY \left( -\partial_y \phi_X + 2\partial_x \psi_y^* + \psi_X \phi_y \mp \hat{\psi}_y^* \phi_y^* \right) X
\] (22)

The expression for \( F_{xy}F_{xy} \) and its integral (which would be the Yang-Mills action) is so complicated that it is rather unreadable and we shall not discuss it in detail. Let us observe, however, that it would certainly contain a potential term for fields \( \phi \) and \( \psi \), which is a feature that distinguishes such theory from its analogue on the commutative plane. The potential term \( V \) would be:

\[
V \sim x \left( \psi_X \phi_y \mp \psi_X^* \phi_y^* \right)^2.
\] (23)

To end this section we shall briefly present the calculation of \( F \) in situation, where we restrict ourselves to such connections that arise from the usual \( U(1) \) gauge group, which have the following form:

\[
A_r = dX(0, \phi_X) + dy(\psi_y, 0)
\] (24)
where (20,21) for $\phi_X$ and $\psi_y$ still hold. The curvature now reads:

$$F = dX \wedge dy \left(-\partial_y \phi_X + 2\partial_x \psi^*_y\right)$$

and its square gives (up to the factor 2) the usual action for the two-dimensional abelian Yang-Mills theory.

6 Conclusions

As we have demonstrated one could construct the $q = -1$ deformation in such a way that it is just an extension of the algebra of functions on the standard plane in the same way as quaternions are extension of complex numbers. It is unclear, whether such property holds only for $q = -1$ or could be generalised for other $q$ (roots of unity in particular).

Another interesting problem is the application the scheme to some other algebras, which could lead to the generalisation of the concept of $q = -1$ deformations for other objects (including discrete spaces).

Discussing the examples of field theory models on the anticomutative plane, we have found that they give some effective actions for usual (commutative) fields on the plane in a nontrivial metric background. The interesting features are the mass terms appearing automatically for scalar field action and the potential term in the gauge theory. Though the physical significance of such models might not be great, they remain interesting as a testing ground for implications and role of $q$-deformations in theoretical physics.

References

[1] Yu.Manin, *Topics in Noncommutative Geometry*, Princeton University Press, (1991)

[2] S.Zakrzewski, *Matrix Pseudogroups Associated with Anti-Commutative Plane*, Lett.Math.Phys. 21 309-321, (1991)

[3] C.Chryssomalakos, P.Schupp, B.Zumino *Induced Extended Calculus On The Quantum Plane*, LBL-35034 (1994)

[4] J.Wess, B.Zumino *Covariant Differential Calculus on the Quantum Hyperplane*, Nucl.Phys.B. (Proc.Suppl.) 18B 302-312 (1990)
[5] A. Connes, *Noncommutative Geometry*, Academic Press, to be published.

[6] A. Sitarz *Metric on Quantum Spaces*, Lett. Math. Phys. **31** 35-39 (1994)