Coherent distributions for the rigid rotator

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This work presents Wigner-type quasiprobability distributions for the rigid rotator which become, in the limit $\hbar \to 0$, coherent solutions of the classical Liouville equation. The results are consistent with the usual quantization of the intrinsic angular momentum, but for the expectation value of the Hamiltonian, a finite ”zero point” energy term is obtained.
1 Introduction

The models of rigid rotation concern finite many-particle systems having a well-defined instantaneous intrinsic frame. The space $Q$ of static configurations is in this case the group $SO(3, \mathbb{R})$ while the low-energy dynamics can be described as a hamiltonian flow on the phase-space $M = T^*SO(3, \mathbb{R})$ [1].

For the atomic nuclei such an intrinsic frame is defined by a deformed mean-field. The residual interactions between protons and neutrons can also be taken into account by models of two coupled rotators, used mainly to study the low-lying isovector magnetic excitations [2, 3]. Similar considerations for the isospin degree of freedom have been presented in [4, 5].

Distributions on $T^*SO(3, \mathbb{R})$ appear in the treatment of statistical ensembles of microscopic rotators. Such ensembles are particularly interesting because unlike the space $Q = \mathbb{R}^3$ of the translation coordinates, the spaces $SO(3, \mathbb{R})$ of the intrinsic rotations are physically distinct and of finite volume ($v_R = 8\pi^2$). For the polyatomic molecules the moments of inertia are large, and although the statistical weight may include the nuclear spin degeneracy [6], the thermal equilibrium is described by the classical Boltzmann distribution. At low temperatures the partition function is calculated using the spectrum of the quantum Hamiltonian, which is well-known up to a controversial ”zero-point” energy term. Proposed during the early days of the quantum theory [7] to explain the specific heats of diatomic gases, finite ground state-energy terms have been retrieved in the more recent years for the rigid rotator from geometrical considerations [8, 9, 10], and for the simple rotator by using reduced Wigner quasiprobability distributions [11]. Though, by adapting the known Wigner transform [12] from $\mathbb{R}^3$ to $SO(3, \mathbb{R})$, an additional ”quantum potential” term was obtained [13].

In this work Wigner-type distributions for the rigid rotator will be introduced along the lines of [14, 15, 16], by discretizing the Fourier transform in momentum of the ”action waves” on $T^*SO(3, \mathbb{R})$. The classical dynamics of the rigid rotator is presented in Section 2, both as a Hamiltonian system on $T^*SO(3, \mathbb{R})$ and as geodesic motion on $SO(3, \mathbb{R})$. The coherent solutions of the classical Liouville equation provided by the ”action waves” of the Hamilton-Jacobi theory are presented in Section 3. It is shown that by a suitable discretization of their Fourier transform in angular momentum it is possible to obtain quasiprobability distributions on $T^*SO(3, \mathbb{R})$ which are consistent with the usual quantum treatment. However, for the expectation value of the Hamiltonian, a finite ”zero point” energy is obtained.
Conclusions are summarized in Section 4.

2 The Euler equations

The group $SO(3, \mathbb{R})$ consists of the $3 \times 3$ real, orthogonal, unimodular matrices,

$$SO(3, \mathbb{R}) = \{ \mathcal{R} : \mathcal{R}^T \mathcal{R} = I, \det \mathcal{R} = 1 \} .$$

(1)

This group is a compact manifold, such that every rotation $\mathcal{R}$ can be specified by the versor $\mathbf{n}$ of the rotation axis and the angle $\gamma$ of rotation around this axis, which means by the vector $\gamma \mathbf{n} \in \mathbb{R}^3$, $|\mathbf{n}| = 1$, $\gamma \in [0, \pi]$.

Usually, $\mathcal{R}$ is specified by the Euler angles $\varphi, \theta, \psi$. These parameters are continuous variables, and $SO(3, \mathbb{R})$ is not the limit of a sequence of discrete subgroups. The largest discrete subgroup which covers uniformly $SO(3, \mathbb{R})$ is the icosahedron group, with 60 elements [17].

If $SO(3, \mathbb{R})$ acts on $\mathbb{R}^3$ by (covariant) rotations $e'_k = \sum_i \mathcal{R}^e_{ki} e_i$, of the coordinate axes, the position vector $\mathbf{r} = \sum_k q_k e_k$ of a fixed point in space remains invariant, and its coordinates are subject to the (contravariant) action $q'_k = \sum_i \mathcal{R}^q_{ki} q_i$, where $\mathcal{R}^q = (\mathcal{R}^e)^T$. Using the Euler parametrization these matrices take the form

$$\mathcal{R}^e = e^{\varphi \xi_3} e^{\theta \xi_1} e^{\psi \xi_3} , \quad \mathcal{R}^q = e^{\psi \ell_3} e^{\theta \ell_1} e^{\varphi \ell_3} ,$$

(2)

where $\xi_i = -\ell_i$, $i = 1, 2, 3$, are 3 independent, antisymmetric, $3 \times 3$ matrices, with elements $\left( \xi_i \right)_{jk} = -\delta_{ijk}$ ($\varepsilon_{ijk}$ is the Levi-Civita symbol) and commutation relations

$$[\xi_i, \xi_j] = \varepsilon_{ijk} \xi_k , \quad [\ell_i, \ell_j] = -\varepsilon_{ijk} \ell_k .$$

(3)

To describe the rigid rotator, the “laboratory” frame versors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are supposed to be fixed, and a rotated frame $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ is intrinsically attached to each orientation in $\mathbb{R}^3$ of the rotator, such that its configuration space can be identified with $SO(3, \mathbb{R})$. If $(\varphi, \theta, \psi)$ depend on time, the derivative

$$\dot{\mathcal{R}}^e = \sum_i \omega_i \xi_i \mathcal{R}^e = \mathcal{R}^e \sum_i \omega'_i \xi_i$$

(4)

defines the components of the angular velocity $\omega_i$, $\omega'_i$, $(\omega'_k = \sum_i \mathcal{R}^q_{ki} \omega_i)$ in the laboratory, respectively in the intrinsic frame,

$$\omega'_1 = \dot{\theta} \cos \psi + \dot{\varphi} \sin \theta \sin \psi$$
\[
\omega'_{2} = -\dot{\theta} \sin \psi + \dot{\varphi} \sin \theta \cos \psi \\
\omega'_{3} = \dot{\psi} + \dot{\varphi} \cos \theta
\]

Similarly to (4) it is convenient to define the generators of the translations to the left \((Y_k)\) and to the right \((Z_k)\) by

\[
Y_k \mathcal{R}^e = \xi_k \mathcal{R}^e \quad Z_k \mathcal{R}^e = \mathcal{R}^e \xi_k .
\]  

(5)

Explicitly,

\[
Y_1 = \cos \varphi \partial_\theta + \frac{\sin \varphi}{\sin \theta} (\partial_\psi - \cos \theta \partial_\varphi) , \quad Y_2 = \sin \varphi \partial_\theta - \frac{\cos \varphi}{\sin \theta} (\partial_\psi - \cos \theta \partial_\varphi) ,
\]

\[
Y_3 = \partial_\varphi , \quad Z_1 = \cos \psi \partial_\theta + \sin \psi \frac{\sin \theta}{\sin \theta} (\partial_\psi - \cos \theta \partial_\varphi) ,
\]

\[
Z_2 = -\sin \psi \partial_\theta + \frac{\cos \psi}{\sin \theta} (\partial_\varphi - \cos \theta \partial_\psi) , \quad Z_3 = \partial_\psi .
\]

Thus, \(Z_k(\varphi, \theta, \psi) = -(-1)^k Y_k(\psi, \theta, \varphi)\) and

\[
[Y_i, Y_j] = -\epsilon_{ijk} Y_k , \quad [Z_i, Z_j] = \epsilon_{ijk} Z_k , \quad [Y_i, Z_j] = 0 ,
\]

such that the components \(x_k\) of a left-invariant vector field \(X \in TSO(3, \mathbb{R})\), \(([X, Y_k] = 0)\), \(X = \sum_k x_k Z_k\), are interpreted as "intrinsic".

The basis \(Z_k\) of left-invariant vector fields on \(TSO(3, \mathbb{R})\) can be related to a local basis \(\zeta_k\) of one-forms on \(T^*SO(3, \mathbb{R})\),

\[
\zeta_1 = \sin \theta \sin \psi d\varphi + \cos \psi d\theta
\]

\[
\zeta_2 = \sin \theta \cos \psi d\varphi - \sin \psi d\theta
\]

\[
\zeta_3 = \cos \theta d\varphi + d\psi
\]

such that \(\zeta_i(Z_j) \equiv \langle \zeta_i, Z_j \rangle = \delta_{ij}\). It is easy to check that

\[
d\zeta_1 = -\zeta_2 \wedge \zeta_3 , \quad d\zeta_2 = -\zeta_3 \wedge \zeta_1 , \quad d\zeta_3 = -\zeta_1 \wedge \zeta_2
\]

and \(\zeta_1 \wedge \zeta_2 \wedge \zeta_3 = -\sin \theta d\varphi \wedge d\theta \wedge d\psi\). Thus, if the tangent to the trajectory \(\mathcal{R}^e(t)\) is expressed in the form \(X_\tau = \varphi \partial_\varphi + \theta \partial_\theta + \psi \partial_\psi\), then \(\zeta_i(X_\tau) = \omega'_i\) are the intrinsic components of the angular velocity.

The 1-forms \(\zeta_i\) can be used to define a symmetric 2-form on \(TSO(3, \mathbb{R})\),

\[
B(X, Y) = \sum_k I_k \zeta_k(X) \zeta_k(Y) , \quad X, Y \in TSO(3, \mathbb{R}) .
\]  

(6)
If $I_k > 0$ are the intrinsic moments of inertia (constants), then the kinetic energy of the rigid rotator can be written in the form $T = B(X, X) / 2$. By the Legendre transform $L_1$, the 2-form $B$ provides a map $L : TSO(3, \mathbb{R}) \to T^*SO(3, \mathbb{R})$, $X \to L_X$, $L_X(Y) \equiv B(X, Y)$. In particular $L_{X_r} = \sum_k \rho_k \zeta_k$, with $\rho_k = I_k \omega_k'$, such that the kinetic energy $T$ also defines the classical Hamilton function $H = \sum_k \rho_k^2 / 2I_k$.

Let $\Theta = \sum_k \rho_k \zeta_k$ be the canonical 1-form on $M = T^*SO(3, \mathbb{R})$, and $\Omega = -d\Theta$ the symplectic form. The dynamics of the rigid rotator can be described either as a hamiltonian flow on $(M, \Omega)$, induced by $H$, or as geodesic motion on $SO(3, \mathbb{R})$ for the metric $B$.

### 2.1 The Hamiltonian approach

In classical mechanics the observables are represented within the set $\mathcal{F}(M)$ of the smooth real functions on the symplectic manifold $(M, \Omega)$. Let $X_f$ be the vector field provided by $i_{X_f} \Omega = df$, where $i_{X_f} \Omega$ denotes the inner product between $X_f$ and $\Omega$. The set $\mathcal{F}(M)$ becomes a Lie algebra with respect to the Poisson bracket $\{*, *\}$,

$$\{f, g\} = \langle df, X_g \rangle = \omega(X_f, X_g) = -L_{X_f} g, \quad f, g \in \mathcal{F}(M),$$

in which $L_X$ denotes the Lie derivative with respect to $X$. In the case $M = T^*SO(3, \mathbb{R})$, $\Omega = -d\Theta$, we get

$$\{f, g\} = \sum_k (Z_k f \partial_{\rho_k} g - Z_k g \partial_{\rho_k} f) - \bar{\rho} \cdot (\bar{\partial}_f \times \bar{\partial}_g),$$

where $\partial_{\rho_k} \equiv \partial / \partial \rho_k$, $\bar{\rho} \equiv (\rho_1, \rho_2, \rho_3)$ and $f, g \in \mathcal{F}(M)$ are functions $f(\mathcal{R}, \bar{\rho})$, of $\mathcal{R} \equiv \mathcal{R}^3 \in SO(3, \mathbb{R})$ and $\bar{\rho} \in \mathbb{R}^3$.

Considering the vector fields $X$ on $M$ of the form $X = \sum k \omega_k Z_k + \dot{\rho}_k \partial_{\rho_k}$, the condition $i_{X_t} \Omega = dH$ reduces to the Euler equations

$$\omega_k' = \rho_k / I_k, \quad \dot{\rho} = \bar{\rho} \times \bar{\omega}'. \quad (7)$$

### 2.2 The geodesic approach

In the geodesic approach we take $B$ as a left-invariant metric on $TSO(3, \mathbb{R})$, $L_Y B = 0$. A curve $\mathcal{R}(t)$ on $SO(3, \mathbb{R})$ is a geodesic with respect to $B$ if its tangent field $X_r$ is "autoparallel", in the sense that $\nabla_{X_r} X_r = 0$, in the sense that $\nabla_{X_r} X_r = 0$,
where $\nabla_X$ is the covariant derivative with respect to $X$ induced by the metric $B$. This derivative is specified by the condition $L_X B(X_1, X_2) = B(\nabla_X X_1, X_2) + B(X_1, \nabla_X X_2)$, used to calculate the Christoffel symbols $\Gamma$. Because $L_Z \zeta_i \zeta_j (Z_k) = \epsilon_{ijk}$, in the basis $\{Z_k, k = 1, 2, 3\}$ of the left-invariant fields we get $\nabla_Z Z_j Z_k = \sum_i \Gamma^i_{jk} Z_i$ with $\Gamma^i_{jk} = -(I_j - I_k) \epsilon_{ijk} / I_i$. Taking $X\tau$ of the form $X\tau = \sum_k \omega'_k Z_k$, the condition $\nabla_X X\tau = 0$ yields $\dot{\omega}'_i = -\sum_{jk} \Gamma^i_{jk} \omega'_j \omega'_k$, which is the same as (7). For a spherical rotator $I_1 = I_2 = I_3 \equiv I$, $\Gamma = 0$, $T = I \dot{\theta}^2 + \dot{\phi}^2 + 2 \cos \theta \dot{\phi} \dot{\psi}$, which is the same as (8).

3 The Liouville equation

Let $(M_\mu, \Omega_\mu)$ be the classical phase-space of an elementary rotator $\mu$ (e.g. molecule), and $(M_N, \Omega_N)$ the phase-space of the ensemble (gas) consisting of $N$ identical elementary rotators [18],

$$M_N = M^1_\mu \times M^2_\mu \times \ldots M^N_\mu \ , \ \Omega_N = \sum_{k=1}^N \Omega^k_\mu \ .$$

For a statistical description of the ensemble we presume that on each manifold $M_\mu$ can be defined a partition in $K$ infinitesimal cells, $\{b_j ; j = 1, K\}$, of volume $\delta v^j = \int_{b_j} d\mathbf{v} \rho$, $\rho \equiv \sin \theta d\phi d\theta d\psi$, $d^3 \rho \equiv d\rho_1 d\rho_2 d\rho_3$.

This partition induces a partition of $M_N$ in $n_B = K^N$ cells $B_j$ of volume $\delta V^j_N$, $j = 1, n_B$. Denoting by $w_j$ the probability of finding the representative point $m \in M_N$, for the state of the ensemble at the time $t$, localized in $B_j$, the ratio $F_j = w_j / \delta V^j_N$ defines the distribution function $F$ of the probability density. This is symmetric at the permutation of the rotator indices and normalized by

$$\int_{M_N} dV_N F = 1 \ , \ \int_{M_N} dV_N = \Pi_{\mu=1}^N dv_{R\mu} d^3 \rho_\mu \ .$$

1The permutations of the rotators yield $N!$ possible representative points for the same physical state of the ensemble.
Because $dV_Rd^3\rho = |\Omega^3|$, the volume element $dV_N$ is invariant to the hamiltonian flow on $M_N$, and $F$ evolves according to the continuity (Liouville) equation
\[
\partial_t F = \{H_N, F\}_N , \tag{11}
\]
where $\{ , \}_N$ is the Poisson bracket on $M_N$ and $H_N$ is the Hamiltonian of the ensemble. The expectation value of an observable $A \in \mathcal{F}(M_N)$ is
\[
\langle A \rangle_F = \int_{M_N} dV_N \ F \ A ,
\]
and
\[
\frac{d}{dt} \langle A \rangle_F = \langle \{ A, H_N \}_N \rangle . \tag{12}
\]
For an ensemble of non-interacting rotators a particular solution of (11) is $F(m, t) = \Pi_{\mu=1}^N f(R_\mu, \vec{\rho}_\mu, t)$, where $f \in \mathcal{F}(M_\mu)$ is a solution of the Liouville equation on $T^*SO(3, \mathbb{R})$,
\[
\partial_t f = \{H, f\} ,
\]
with $H = \sum_k \rho_k^2 / 2 I_k$, namely
\[
\partial_t f + \sum_k \rho_k Z_k f + \sum_{ijk} \epsilon_{ijk} \rho_i \rho_j \partial_{\rho_k} f = 0 . \tag{13}
\]
Let $\tilde{f}$ be the Fourier transform of $f$ with respect to the intrinsic angular momentum,
\[
\tilde{f}(R, r, t) \equiv \int d^3 \rho \ e^{i r \cdot \vec{\rho}} \ f(R, \vec{\rho}, t) . \tag{14}
\]
Thus, if $f(R, \vec{\rho}, t)$ is a solution of (13), then $\tilde{f}(R, r, t)$ will satisfy
\[
\partial_t \tilde{f} - i \sum_i \frac{1}{I_i} Z_i \partial_{r_i} \tilde{f} + i \sum_{ijk} \epsilon_{ijk} \frac{r_i}{I_k} \partial_{r_j} \partial_{r_k} \tilde{f} = 0 . \tag{15}
\]
Particular solutions of this equation are related to local Lagrangian submanifolds $\Lambda \subset T^*SO(3, \mathbb{R})$, such that $\Theta|_\Lambda = dS$, where $S(R, t)$ is the generating function of the Hamilton-Jacobi theory. These solutions are of the form
\[
\tilde{f}_0(R, r, t) = n(R, t) e^{ir \cdot Z S} , \tag{16}
\]
where the density $n \geq 0$ and $S$ are real functions on $SO(3, \mathbb{R})$ which satisfy the continuity and, respectively, the Hamilton-Jacobi equations
\[
\partial_t n + \sum_k Z_k (n \frac{Z_k S}{I_k}) = 0 , \quad Z_k [\partial_t S + \sum_k (\frac{Z_k S}{2 I_k})^2] = 0 . \tag{17}
\]
The inverse of (14),
\[ f(R, \vec{\rho}, t) = \frac{1}{(2\pi)^3} \int d^3r \ e^{-ir\vec{\rho}} \tilde{f}(R, r, t) \] (18)
takes in this case the form of the classical "action distributions",
\[ f_0(R, \vec{\rho}, t) = n\delta(\vec{\rho} - ZS) \] . (19)
These are coherent solutions of (13) in the sense that during time evolution remain the same functionals of $n$ and $S$.

The quantum (Wigner-type) distributions are related to a peculiar form of $\tilde{f}(R, r, t)$, in which the variable $r$ (Fourier dual to $\vec{\rho}$) enters as a parameter for translations on the configuration space. To obtain this functional we note that in (16) $Z$ is the generator of the translations to the right, and therefore
\[ r \cdot ZS(R) = \lim_{\sigma \to 0} \frac{1}{\sigma} [S(R e^{\sigma r} \vec{\xi}/2) - S(R e^{-\sigma r} \vec{\xi}/2)] \] , (20)
where $\sigma$ is a real parameter, $\sigma r \cdot \vec{\xi}$ is an element of the Lie algebra $so(3, \mathbb{R})$, and $\gamma = \sigma |r|$ has the significance of a rotation angle. On the Cartan subalgebra of any semisimple Lie algebra we can introduce a lattice structure, dual to the lattice of the weights, while discrete rotation angles are associated with a complete orthonormal set of angle states [19, 20]. Thus, near $r = 0$ the derivative (20) may retain the nonlocal, finite differences expression, in which $\sigma$ is a small, but finite constant. Considering $\sigma = \hbar$, (16) takes the "quantum" form
\[ \tilde{f}_\Psi(R, r, t) = \Psi(R e^{\hbar r} \vec{\xi}/2) \Psi^*(R e^{-\hbar r} \vec{\xi}/2) \] , (21)
where $\Psi = \sqrt{n} e^{iS/\hbar}$ is the complex wave function. This expression, in principle, will provide by (18) a phase-space distribution $f_\Psi$, but due to the infinite integration domain over $r$, such a functional will not have the properties of the Wigner quasiprobability distributions. Therefore, we may consider instead the functional
\[ f_W(R, \vec{\rho}) = \frac{1}{(2\pi)^3} \int_{r \leq \pi/\hbar} d^3r j_0^2(\frac{\hbar r}{2}) \ e^{-ir\vec{\rho}} \tilde{f}_\Psi(R, r, t) \] , (22)
where $r = |r|$ and $j_0(x) = x^{-1} \sin x$. By introducing the new variable $\tilde{\gamma} = \hbar r$, (22) can also be written in the form
\[ f_W(R, \vec{\rho}) = \frac{1}{(2\pi \hbar)^3} \int_{\gamma \leq \pi} d^3\gamma j_0^2(\frac{\gamma}{2}) \ e^{-i\gamma^2/\hbar} \Psi(R e^{\gamma} \vec{\xi}/2) \Psi^*(R e^{-\gamma} \vec{\xi}/2) \] . (23)
The variable \( \vec{\gamma} = (\gamma_1, \gamma_2, \gamma_3) \) is related to the parametrization of the rotation matrices by the exponential map \( R = \exp(\gamma_1 \xi_1 + \gamma_2 \xi_2 + \gamma_3 \xi_3) \), and the factor \( j_0^2(\gamma/2), \gamma = |\vec{\gamma}| \), was introduced such that \( j_0^2(\gamma/2)d\gamma_1d\gamma_2d\gamma_3 \) becomes in the Euler parametrization the volume element \( dv_R = \sin \theta d\theta d\phi d\psi \). In this case, if \( f_{W_1} \) and \( f_{W_2} \) are given by \( \Psi_1 \), respectively \( \Psi_2 \), then \(< f_{W_1} | f_{W_2} > = |< \Psi_1 | \Psi_2 >|^2/(2\pi \hbar)^3 \).

Because \( \lim_{x \to 0} j_0(x) = 1 \) we note from (22) that \( \lim_{\hbar \to 0} f_W = f_0 \). However, in general \( f_W \) is not an exact solution of the Liouville equation (13).

The expectation values of the angular momentum operators,

\[
<\rho_k >_{t_w} = \int d^3\rho \ dv_R \ \rho_k \ f_W = \frac{1}{(2\pi \hbar)^3} \int d^3\rho \ dv_R \int_{\gamma \leq \pi} d^3j_0^2(\gamma/2) \ (i\hbar \partial_{\gamma_k} e^{-i\vec{\gamma} \cdot \vec{\rho}/\hbar}) \Psi(Re^{\vec{\gamma} \cdot \vec{\xi}/2}) \Psi^*(Re^{-\vec{\gamma} \cdot \vec{\xi}/2})
\]

can be estimated considering that the integral over \( d^3\rho \) yields a delta function \( \delta(\vec{\gamma}) \), the first derivative of \( j_0^2(\gamma/2) \) vanishes at \( \gamma = 0 \), and

\[
\partial_{\gamma_k} \Psi(Re^{\vec{\gamma} \cdot \vec{\xi}/2}) |_{\gamma = 0} = \frac{1}{2} \hat{Z} \Psi(R)
\]

such that

\[
<\rho_k >_{t_w} = \frac{1}{2} \int dv_R [(\hat{L}'_k \Psi) \Psi^* + \Psi(\hat{L}'_k \Psi)^*]
\]

where \( \hat{L}'_k = -i\hbar \hat{Z}_k \) is the intrinsic angular momentum operator. This operator is hermitian with respect to the volume element \( v_R \), and therefore

\[
<\rho_k >_{t_w} = \langle \Psi | \hat{L}'_k | \Psi \rangle.
\]

The calculation of \( <\rho_k^2 >_{t_w} \) involves second derivatives with respect to \( \gamma_k \) and proceeds similarly, excepting for the second derivative \( \partial^2_{\gamma_k} j_0^2(\gamma/2) \) at \( \gamma = 0 \), which is \(-1/6\). Thus, the operator associated to \( \rho_k^2 \) is \( (\hat{L}'_k)^2 + \hbar^2/6 \), and the Hamiltonian operator

\[
\hat{H} = \sum_k (\frac{\hat{L}'_k^2}{2I_k} + \frac{\hbar^2}{12I_k})
\]

contains beside the usual part a zero-point energy term \( \sum_k \hbar^2/12I_k \).

\( \gamma \) provides the character \( \chi_1(R) = Tr(R) = 2 \cos \gamma + 1. \)
4 Concluding remarks

The Wigner transform defines quasiprobability distributions on the phase-space $M = T^*\mathbb{R}^3$ which evolve as coherent solutions of the classical Liouville equation for ensembles of particles subject to uniform or elastic force fields. It also relates classical and quantum expectation values for many observables and provides a basis for the statistical interpretation of the scalar product between quantum wave functions. However, to find the geometrical structure underlying these properties, it is necessary to go beyond $T^*\mathbb{R}^3$.

The case $M = T^*SO(3, \mathbb{R})$ corresponds to the rigid rotator, and the proposed extension is $f_W$ of (23). This functional reduces to the ”action distribution” $f_0$ of (19) in the limit $\hbar \to 0$ and ensures the usual quantization of the intrinsic angular momentum. Though, the classical expectation value of the intrinsic Hamiltonian contains beside the usual ”quantum part”, a zero-point energy term. This term is positive definite and reflects the compact, rather than the non-abelian structure of $SO(3, \mathbb{R})$.

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