Semiparametric Optimal Estimation With Nonignorable Nonresponse Data

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Abstract: When the response mechanism is believed to be not missing at random (NMAR), a valid analysis requires stronger assumptions on the response mechanism than standard statistical methods would otherwise require. Semiparametric estimators have been developed under the model assumptions on the response mechanism. In this paper, a new statistical test is proposed to guarantee model identifiability without using instrumental variable assumption. Furthermore, we develop optimal semiparametric estimation for parameters such as the population mean. Specifically, we propose two semiparametric optimal estimators that do not require any model assumptions other than the response mechanism. Asymptotic properties of the proposed estimators are discussed. An extensive simulation study is presented to compare with some existing methods. We present an application of our method using Korean Labor and Income Panel Survey data.

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1. Introduction

Handling missing data often requires some assumptions about the response mechanism. If the study variable does not affect the probability of the response, the response mechanism is called missing at random (MAR) [33]. If, on the other hand, the response probability of a study variable depends on that variable directly, the response mechanism is called not missing at random (NMAR) [21]. Under NMAR, the response probability cannot be verified using the observed study variables only, therefore, additional assumptions about the study variable are often required.

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Let $r$ be the response indicator of the study variable $y$ with auxiliary variable $x$, where $r$ takes 1 if $y$ is observed, and takes 0 otherwise. In this paper, we consider a situation where the study variable $y$ is subject to missingness. Ignorable nonresponse or MAR can be understood as the conditional independence of $r$ and $y$ given $x$, namely $r \perp y \mid x$, which is usually untestable. Greenlees et al. [11] and Diggle & Kenward [6] proposed a fully parametric approach to analyze nonignorable nonresponse data; their method requires two parametric models: (i) an outcome model, $[y \mid x]$; and (ii) a response model $[r \mid x, y]$. In practice, it is difficult to verify models (i) and (ii), because some of $Y$ are not observed. For the fully parametric approach, model identification and model misspecification can be a problem, and sensitivity analysis becomes necessary [34, 32, 42, 41]. Sverchkov [38] and Riddles et al. [28] proposed a fully parametric approach that uses different model specifications based on (i) $[y \mid x, r = 1]$, and (ii) $[r \mid x, y]$. Their approach is attractive because one can verify a model for $[y \mid x, r = 1]$ from the observed responses; however, because it is a fully parametric approach, it is still subject to model misspecification error.

Recently, several semiparametric approaches have been proposed for nonignorable nonresponses. Ma et al. [23] studied identification and parameter estimation for binary study variables. Tang et al. [39] also considered model identification using an instrumental variable and proposed a maximum pseudo likelihood estimator that does not require model specification of the response mechanism. D’Haultfoeuille [5] also used the same instrumental variable assumption and considered a regression analysis using the nonparametric propensity score model. Zhao & Shao [45] extended the method of Tang et al. [39] and relaxed the condition on the instrumental variable, which is called nonresponse instrumental variable [43]. Fitzmaurice et al. [9] and Skrondal & Rabe-Hesketh [37] proposed protective estimators that do not require a model for the response mechanism, but the application of this approach is limited to situations in which $Y$ is binary. In the meantime, Kim & Yu [19] proposed a semiparametric method for estimating $E(Y)$ using a semiparametric response model, but a validation sample is required in order to estimate the parameters in the response mechanism. Tang et al. [40] used the method of empirical likelihood to extend the method of Kim & Yu [19] to estimate more general parameters. In Zhao et al. [46], the method of Qin et al. [27] was used to construct a $n^{1/2}$-consistent estimator without a validation sample. Morikawa et al. [26] used the kernel regression estimator to remove the parametric model assumption on model (i) $[y \mid x, r = 1]$. Chang & Kott [3] and Wang et al. [43] considered a generalized method of moments (GMM) estimator that uses the response model assumption only, but their method is generally lacking in efficiency. Recently, Shao & Wang [36] proposed a semiparametric inverse propensity weighting method using the nonresponse instrumental variable (NIV) assumption of Wang et al. [43]. However, the above papers do not address efficiency of their semiparametric estimation methods. Furthermore, the NIV assumption is difficult to verify from the sample. Developing an optimal semiparametric estimator and a test procedure for model identification under NMAR are important research topics in missing data analysis.

In this paper we use a parametric model for $[r \mid x, y]$ and a fully nonpara-
metric model for \([y \mid x, r = 1]\) to form a semiparametric model and develop a nonparametric test procedure for model identification of the semiparametric model. After that, we construct optimal estimators for parameters both related to the response mechanism and for the parameter of interest such as population mean. Efficiency under this setup has already been discussed by Rotnitzky & Robins [31] and Robins et al. [30]. However, their estimator requires many working models to achieve the semiparametric efficiency bound. Misspecification of the working models may lead to loss of efficiency. See the simulation study in §6 and real data analysis in §7 for comparison with the method of Rotnitzky & Robins [31].

Therefore, we consider an alternative approach and propose two semiparametric estimators that attain the semiparametric lower bound [2] (1) with a working model assumption or (2) without requiring working model assumptions. The first estimator is an adaptive estimator using a working model for \([y \mid x, r = 1]\). If the working model is correct, the first estimator attains the lower bound. The second one is based on the nonparametric regression model which does not require any additional assumptions, but it still attains the lower bound. All technical details are given in Appendix B.

2. Basic setup

Let \((z_i, r_i), i = 1, \ldots, n\) be \(n\) realizations from a joint distribution \([z, r]\), where 
\(z = (x^T, y)^T, x\) is a \(d\)-dimensional covariate vector, \(y\) is a response variable, and \(r\) is a response indicator of \(y\), i.e., it takes 1 if \(y\) is observed, and takes 0 otherwise. Also, let \(G_r(z)\) be the observed data when the response indicator is \(r\), i.e., \(G_1(z) = z\) and \(G_0(z) = x\). Suppose that the response model is \(\pi(z; \phi)\) with a \(q\)-dimensional parameter \(\phi \in \Phi\). Let \(\theta \in \Theta\) be a parameter satisfying \(E\{U(Z; \theta)\} = 0\), where \(U\) is a known function of \(z\). For example, if we are interested in \(E(Y)\), then \(U(z; \theta) = y - \theta\), and in regression coefficients \(E(Y \mid x) = \mu(x; \theta)\), then \(U(z; \theta) = a(x)\{y - \mu(x; \theta)\}\), where \(a(\cdot)\) is any linearly independent function of \(x\) having same dimension as \(\theta\). In this paper, we consider semiparametric estimation of \((\phi, \theta)\) from partial observations. In particular, we propose the efficient estimator among the regular asymptotically linear estimators \([2, 41]\) without relying on the correctness of \(U\) function and propose two adaptive estimators.

For model identification for a response model, Miao et al. [24] gives a sufficient condition when the outcome models are normal or normal mixture. However, the normality assumption cannot be checked directly from observed data. In the meantime, Wang et al. [43] developed a theory for identification by assuming that there exists a NIV \(x^2\) in the covariate vector \(x = (x_1^T, x_2^T)^T\) such that \(x_2\) is independent of \(r\), given \(x_1\) and \(y\). When \(x\) is the single variable, \(x\) itself is the NIV. Although the existence of such a NIV is a sufficient condition, it is hard to verify it from the observed data. Therefore, both identification conditions are not testable with observed data. In §3, we propose an alternative condition for the model identification by assuming a restriction on \([y \mid x, r = 1]\), not only on the response mechanism, and develop a test procedure for model identification.
Classical approaches for analyzing nonignorable nonresponse data are based on correct specification for $[y \mid x]$ as well as the response mechanism [11]. This requirement can be challenging because the specification cannot be verified under nonignorable nonresponse [25]. Chang & Kott [3] proposed a semiparametric estimator for $\phi$ based on the following estimating equation:

$$\sum_{i=1}^{n} \Gamma(x_i, y_i, r_i; \phi) = \sum_{i=1}^{n} \left\{ 1 - \frac{r_i}{\pi(z_i; \phi)} \right\} g(x_i; \phi) = 0,$$

(2.1)

where $g = \{g_1(x), g_2(x), \ldots, g_q(x)\}^T$, which can be called calibration function, is a function of $x$ whose elements are linearly independent; $q$ is the dimension of $\phi$. Note that although this estimator satisfies consistency and asymptotic normality under certain regularity conditions, its efficiency is not guaranteed.

Recently, Riddles et al. [28] proposed an efficient estimator that uses a parametric model for $[y \mid x, r = 1]$. Using the mean score theorem [22], the maximum likelihood estimator can be obtained by solving

$$\sum_{i=1}^{n} \left[ r_is_1(z_i; \phi) + (1 - r_i)E_0\{s_0(Z; \phi) \mid x_i\} \right] = 0,$$

where $s_r(z; \phi)$ is the score function of $\phi$, that is,

$$s_r(z; \phi) = \left\{ r - \pi(z; \phi) \right\} \frac{\dot{\pi}(z; \phi)}{\pi(z; \phi)} \frac{1 - \pi(z; \phi)}{\pi(z; \phi)}.$$

(2.2)

$$\dot{\pi}(z; \phi) = \frac{\partial \pi(z; \phi)}{\partial \phi},$$

and $E_0(\cdot \mid x)$ is the conditional expectation conditional on $x$ and $r = 0$. To compute $E_0(\cdot \mid x)$, under Bayes’ formula, Riddles et al. [28] proposed using

$$\sum_{i=1}^{n} \left[ r_is_1(z_i; \phi) + (1 - r_i) \frac{E_1\{O(Z; \phi)s_0(Z; \phi) \mid x_i\}}{E_1\{O(Z; \phi) \mid x_i\}} \right] = 0,$$

(2.3)

where $O(z; \phi) = \{1 - \pi(z; \phi)\}/\pi(z; \phi)$, and $E_1(\cdot \mid x)$ is the conditional expectation on $y$ given $x$ and $r = 1$. The conditional expectation is computed by assuming a parametric model $f_1(y \mid x; \gamma) = f(y \mid x, r = 1; \gamma)$. This may increase the efficiency, however, misspecification of the $f_1$ model could cause the solution to be inconsistent. Morikawa et al. [26] proposed a semiparametric method using a nonparametric estimator of $f_1$, assuming that the semiparametric model is identified. We now give more rigorous treatments of the model identification of the semiparametric model.

3. Identification

We consider a new identification condition for estimation of the response model with observed data. Our idea is to define the target parameter $\phi_0$ as a unique solution to

$$E \{ \Gamma(Z, R; \phi) \mid X \} = 0 \quad \text{a.s.,}$$

(3.1)
where $\Gamma$ is defined in (2.1), though natural definition of the parameter might be through either (i) $E\{\Gamma(Z, R; \phi) \mid Z\} = 0$ or (ii) $E\{\Gamma(Z, R; \phi)\} = 0$. Note that providing a sufficient condition for the parameter defined in (ii) is the strongest (and in (i) is the weakest) since $E\{\Gamma(Z, R; \phi)\} = E[E\{\Gamma(Z, R; \phi) \mid Z\}]$ and $E\{\Gamma(Z, R; \phi) \mid X\} = E[E\{\Gamma(Z, R; \phi) \mid Z\} \mid X]$ hold. This implies a sufficient condition for the parameter defined in (3.1) does not necessarily guarantee the model identification of (ii), which is the probability limit of the estimating equation (2.1). However, it is rare in practice that the model (3.1) is identifiable, but the model (ii) is not. Also even if we face such a problem, it can be solved by constructing an objective function with the integrated regression function and additional minor conditions [see 7, Assumptions 1–3]. For the above reasons, we focus on providing a sufficient condition of the model identification for the parameter defined in (3.1).

3.1. Identification condition with $f_1$ model

Let $O(z; \phi) = 1/\pi(z; \phi) - 1$ be the odds function of the response model, $E_1(\cdot \mid x)$ be the operator for the true conditional expectation given $x$ and $r = 1$. A new identification condition for the semiparametric model is given in the following theorem.

**Theorem 3.1.** The identification condition for a parameter (3.1) holds under the following conditions.

(I1). $E_1\{O(Z; \phi) \mid x\}$ exists and is bounded almost surely;

(I2). The weight function $g$ in (2.1) satisfies $P(\inf_{\phi \in \Phi} |g(X; \phi)| > 0) > 0$, and elements of $g(x; \phi)$ are linearly independent functions with respect to $x$ for all $\phi$;

(I3). $E_1\{O(Z; \phi) \mid x\} = E_1\{O(Z; \phi') \mid x\}$ a.s. implies $\phi = \phi'$.

By the condition (I1), the response model is almost limited to the logistic regression models. For example, let $\Psi$ be the cumulative function of the standard normal distribution. Assume that $\pi(y) = \Psi(y)$ (probit model) and the density of $f_1$ is the standard normal distribution, then

$$E_1\{O(Y) \mid x\} = E_1\left\{\frac{1}{\pi(Y)} - 1\right\} = \int_{-\infty}^{\infty} \frac{\partial}{\partial y} \log \Psi(y) dy - 1,$$

and $E_1\{O(Y) \mid x\}$ does not exist even for this simple probit model. Nevertheless, Theorem 3.1 is practically useful because the performance with the probit and logistic model is very similar, and thus, misspecification of the response model is not a serious problem in practice (see §7 for the performance with misspecified response models). Condition (I2) is required to avoid $g$ becomes identically zero.

The key condition is (I3), which implies that we should check the identification of $E_1\{O(Z; \phi) \mid x\}$. Checking the identification of $E_1\{O(Z; \phi) \mid x\}$ is
where \( y \) is easy to check. For example, if \( \pi \) mechanism is specified as cumulant-generating function reduces to \( K \) respect to \( x \) where \( K \phi \) infinitely differentiable at \( \phi \) respect to \( \ell = 1 \). From which we can verify the model identification. For example, for model \( x \) linear with respect to \( \ell = 1 \), where the superscript stands for the \( \ell \)-th partial derivative with respect to \( \phi_y \). Thus, by Theorem 3.1, \( \phi \) is identifiable unless the mean structure \( \tau(x) \) is linear for all \( \phi_y \), i.e., the null hypothesis \( H_0: K_{\phi_y}(x) = c_1(\phi_y) + c_2(\phi_y)x \) holds, where \( c_1(\phi_y) \) and \( c_2(\phi_y) \) are functions of \( \phi_y \). If \( c_1 \) and \( c_2 \) can be infinitely differentiable at \( \phi_y = 0 \), we have \( K_{\phi_y}^{(\ell)}(x) = c_1^{(\ell)}(0) + c_2^{(\ell)}(0)x \) for all \( \ell = 1, 2, \ldots \), where the superscript stands for the \( \ell \)-th partial derivative with respect to \( \phi_y \). Because the cumulant-generating function is expanded as

\[
K_{\phi_y}(x) = \sum_{\ell=0}^{\infty} \frac{\phi_y^{(\ell)}}{\ell!} K_{\phi_y}^{(\ell)}(x),
\]

Thus, by Theorem 3.1, \( \phi \) is identifiable unless the mean structure \( \tau(x) \) is linear since there are three parameters with two equations. If \( \tau(x) \) is linear, we may use a transformation approach which is introduced in §3.3.

On the other hand, checking the model identifiability with a nonparametric \( f_1(y \mid x) \) model is still challenging, because there is no way to estimate the cumulative function \( K_{\phi_y}(x) \) nonparametrically for every \( \phi_y \) up to our knowledge. Therefore, we propose a test statistic to test a reasonable necessary condition for the identification condition.

### 3.2. Nonparametric test statistics

In view of (3.2), the model is unidentifiable when the cumulative function is linear with respect to \( x \) for all \( \phi_y \), i.e., the null hypothesis \( H_0: K_{\phi_y}(x) = c_1(\phi_y) + c_2(\phi_y)x \) holds, where \( c_1(\phi_y) \) and \( c_2(\phi_y) \) are functions of \( \phi_y \). If \( c_1 \) and \( c_2 \) can be infinitely differentiable at \( \phi_y = 0 \), we have \( K_{\phi_y}^{(\ell)}(x) = c_1^{(\ell)}(0) + c_2^{(\ell)}(0)x \) for all \( \ell = 1, 2, \ldots \), where the superscript stands for the \( \ell \)-th partial derivative with respect to \( \phi_y \). Because the cumulant-generating function is expanded as

\[
E_1\{O(Z; \phi) \mid x\} = \exp\{\phi_{x0} + \phi_{x1}x + K_{\phi_y}(x)\}, \tag{3.2}
\]
the linearity of the cumulant-generating function can be checked by that of $K_0^{(L)}(x)$ for all $L$. Based on this idea, we obtain an alternative null hypothesis $H_0^{(L)} : K_0^{(L)}(x) = c_1^{(L)}(0) + c_2^{(L)}(0)x$, $\ell = 1, \ldots, L$, for a positive integer $L$ or $L = \infty$. When $L = 1$, this corresponds to a goodness-of-fit test of a simple linear regression with a normal distribution in $f_1$. Although $L = 1$ is just a necessary condition for nonparametric models, in many cases, it is enough to guarantee the model identification.

Let a general data-generating process be $y = \mu(x) + \varepsilon(x)$, where $\mu(x)$ is the conditional expectation of $y$ given $x$, and $\varepsilon(x)$ is the conditional mean-zero error. Consider a class of error functions $\mathcal{E}$: for $\varepsilon \in \mathcal{E}$, $\varepsilon(x) = \sum_{j=0}^{\infty} \xi_j e_j(x)$, where $\xi_j$ $(j \geq 0)$ are mean-zero random variables which are independent of $x$, and $e_j$ $(j \geq 0)$ are any measurable functions of $x$ satisfying $E[\{\sum_{j=0}^{\infty} |\xi_j e_j(X)|\}^k] < \infty$ for any positive integer $k$, and $e_k \neq e_l$ for $k \neq l$. This class of error functions include many functions with mean-zero conditional expectation such as the infinite normal mixture distribution. Under this setup, we can show the following proposition.

**Proposition 3.1.** Suppose that $\varepsilon \in \mathcal{E}$, then $H^{(\infty)}$ implies $H^{(1)}$.

For the above reasons, we test a data-generation structure

$$y = \mu(x) + \varepsilon,$$  \hspace{1cm} (3.3)$$

where $\mu(x)$ is a linear function of $x$, and $\varepsilon$ is a mean-zero random variable and independent of $x$, and consider a test to check the goodness-of-fit of the linear model. It is desirable that the statistical test enjoys two properties: (i) dimension free for $x$; (ii) no parametric assumption on $\varepsilon$. The first property is practically useful because classical nonparametric tests such as Eubank & Hart [8] suffer from curse of dimensionality. The second property can avoid subjectivity imposing some parametric assumption on the error variable. Recently, some nonparametric methods to check a goodness-of-fit have been proposed with Hilbert-Schmidt independence criterion (HSIC) [12, 13, 16, 35] and mutual information [1]. In this paper, we utilize an idea of HSIC proposed by [12, 13]. With HSIC, Sen & Sen [35] and Hidalgo et al. [16] proposed a test statistics to check goodness-of-fit of a (parametric/nonparametric) model, which has the two desirable properties. Their idea is based on the fact that independence of $X$ and $\varepsilon$ implies correctness of the mean function $\mu(x)$ because $\varepsilon$ is independent of $x$. The HSIC can be used to check the independence.

Let $\mathcal{F}$ be a reproducing kernel Hilbert space (RKHS) on a domain $\mathcal{X}$ with a positive-definite function $k : \mathcal{X} \times \mathcal{X} \to \mathcal{R}$. The Hilbert space $\mathcal{F}$ has inner product $\langle \cdot, \cdot \rangle$ satisfying a property called reproducing property $\langle f, k(x, \cdot) \rangle = f(x)$ ($f \in \mathcal{F}, x \in \mathcal{X}$). The kernel mean on the RKHS is defined by $E[k(\cdot, X)] = \int k(\cdot, x)dP(x)$ where $P$ is the probability measure of a random variable $X$. Then, a kernel $k$ is called characteristic if the kernel mean determines the probability measure $P$ uniquely. For example, Fukumizu et al. [10] showed the gaussian kernel $k(x, \tilde{x}) = \exp(-\sigma^{-1}\|x - \tilde{x}\|)$ is characteristic, where $\sigma$ is a tuning parameter and median is often used as a heuristic estimate of $\sigma$. Next, define the HSIC. Let
\( G \) be another characteristic RKHS with kernel \( l \). Then, define HSIC of between two random variables \( X \) and \( Y \), \( M_{XY} \), by

\[
M_{XY} = E\{k(X, \tilde{X})l(Y, \tilde{Y})\} + E\{k(X, \tilde{X})\}E\{l(Y, \tilde{Y})\} - 2E \left[ E\{k(X, \tilde{X}) \mid X\}E\{l(Y, \tilde{Y}) \mid Y\} \right],
\]

where \((\tilde{X}, \tilde{Y})\) is independent copy of \((X, Y)\). Gretton et al. [12] shows that if the product kernel \( kl \) is characteristic, \( M_{XY} = 0 \) implies independence between \( X \) and \( Y \). By checking \( M_{X\varepsilon} = 0 \) in the model (3.3), goodness-of-fit of a mean function \( \mu(x) \) can be tested with observed data. The HSIC \( M_{XY} \) is estimated with a V-statistics based estimator, \( \hat{M}_{XY} = n^{-2}tr(KHLH) \), where \( K_{ij} = k(X_i, X_j) \), \( L_{ij} = l(Y_i, Y_j) \), \( H = I_n - n^{-1}1_n1_n^\top \), \( I_n \) is the \( n \times n \) identity matrix, and \( 1_n \) is the \( n \times 1 \) vector of ones. Unlike \( \hat{M}_{XY} \), it is hard to derive the asymptotic distribution of \( \hat{M}_{X\varepsilon} \) under the null hypothesis because \( \mu \) in (3.3) is replaced with an estimated mean function, and the limiting distribution becomes more complicated. However, the bootstrap method is applicable to estimate the distribution as follows [35]. In the algorithm, let \( x = (x_1, \ldots, x_{n1})^\top \) and \( y = (y_1, \ldots, y_{n1}) \) be observed covariate variables and response variables, and \( \mu(x_i; c) = c_1 + c_2x_i \), where \( c = (c_1, c_2)^\top \) be a linear model under the null hypothesis. To make the algorithm simple and clear, a vector is used instead of use of each element, that is, \( \mu(x; c) \) implies the vector \((\mu(x_1; c), \ldots, \mu(x_{n1}; c))^\top \).

1: **procedure** Bootstrap Method for \( \hat{M}_{X\varepsilon}^{(b)} (b = 1, \ldots, B) \)
2: \( \hat{c} \leftarrow \arg \min \sum_{i=1}^{n1} \{y_i - \mu(x_i; \hat{c})\}^2 \)
3: \( \hat{\varepsilon} \leftarrow y - \mu(x; \hat{c}) \)
4: **for** \( b = 1 \) to \( B \) **do**
5: \( x^{(b)} \leftarrow \text{bootstrap sample from observed data } x \)
6: \( \varepsilon^{(b)} \leftarrow \text{bootstrap sample from } \hat{\varepsilon} \)
7: \( y^{(b)} \leftarrow \mu(x^{(b)}; \hat{c}) + \varepsilon^{(b)} \)
8: \( \hat{c}^{(b)} \leftarrow \arg \min \sum_{i=1}^{n1} \{y_i^{(b)} - \mu(x_i^{(b)}; \hat{c})\}^2 \)
9: \( \varepsilon^{(b)} \leftarrow y^{(b)} - \mu(x^{(b)}; \hat{c}^{(b)}) \)
10: \( (K^{(b)})_{ij} \leftarrow k(x_i^{(b)}, x_j^{(b)}); (E^{(b)})_{ij} = k(\varepsilon_i^{(b)}, \varepsilon_j^{(b)}) \) for all \( i, j \)
11: \( \hat{M}_{X\varepsilon}^{(b)} = n_1^{-2}tr(K^{(b)}HE^{(b)}H) \)
12: **end for**

Then, we have the following asymptotic result.

**Proposition 3.2.** Let \( \hat{M}_{X\varepsilon}^{(b)} \) be the bootstrap test statistics for \( M_{X\varepsilon} \). Suppose that the kernels \( k \) and \( l \), which prescribe RKHS of the random variables \( X \) and \( Y \), and the mean function \( \mu(x) \) satisfies Condition 1, 2, and 5 in Sen & Sen [35]. Then, under the null hypothesis \( H_0^{(1)} \), the asymptotic distribution of \( n_1\hat{M}_{X\varepsilon}^{(1)} \) is the same as that of \( n_1M_{X\varepsilon} \).
3.3. Doubly-normalized exponential transformation

When the null hypothesis is not rejected, an instrumental variable is required to make the estimator consistent. However, selecting the instrumental variable is very difficult even if it exists. Because the problem comes from using the same covariate between the response model and mean function, we can make an identifiable model artificially by transforming covariate variable \( x \) in a response model to a nonlinear variable \( T(x) \) such as \( \exp(x) \) and \( x^2 \), at the sacrifice of consistency. Although there are many choices of such functions, it would be desirable that the transformation enjoys three properties: (i) “nonlinearity” can be adjusted through a tuning parameter \( a \) such that \( \lim_{a \to 0} T_a(x) = x \); (ii) the value \( a \) does not depend on range/scale of \( x \); (iii) range of \( T_a(x) \) is same as that of \( x \). The first condition is necessary to adjust “nonlinearity”: small \( a \)-value holds the original data structure, and large \( a \)-value breaks the structure, but provides stronger identification. For example, one may come up with a transformation \( T_a(x) = \log\{\exp(a + x)\} \). However, nonlinearity of such a transformation may heavily depend on both \( a \) and range/scale of \( x \) so that it is necessary to find an appropriate value \( a \) (which is close to 0) for every covariate or dataset, hence, the second condition is required. The third condition is requisite to retain the value of response probability to some extent. Considerably large (small) value of \( T_a(x) \) may damage the bounded condition \( \pi(T_a(x), y) > 0 \), which is often assumed in this field.

We propose a simple nonlinear transformation having three desirable properties called doubly-normalized exponential transformation (DNET). Let \( S_a(x) \) be a normalized exponential transformation \( S_a(x) = \{\text{Var}(aX)\}^{-1/2}\{\exp(ax) - E(\exp(aX))\} \). By letting \( a \to 0 \), we obtain

\[
\lim_{a \to 0} S_a(x) = \lim_{a \to 0} \frac{a^{-1}\{\exp(ax) - 1\} + a^{-1}\{1 - E(\exp(ax))\}}{\{\text{Var}(X)\}^{1/2}} = \frac{x - E(X)}{\{\text{Var}(X)\}^{1/2}}.
\]

This indicates that the normalized exponential transformation \( S_a \) after data normalization is an identity map as \( a \to 0 \), i.e., with \( Z : x \mapsto \{\text{Var}(X)\}^{-1/2}(x - E(X)) \), a map \( S_a \circ Z \) becomes identity as \( a \to 0 \). Finally, after some minor modification to satisfy the third condition above, we have our proposed transformation method:

1: **procedure** \textsc{Compute DNET}(\( a \))
2: \( z \leftarrow \{\text{var}(x)\}^{-1/2}(x - \text{mean}(x)) \)
3: \( s \leftarrow \{\text{var}(az/5)\}^{-1/2}\{\exp(az/5) - \text{mean}(\exp(az/5))\} \)
4: \( r_x \leftarrow \max(x) - \min(x); \ r_s \leftarrow \max(s) - \min(s) \)
5: \( T_a(x) \leftarrow \min(x) + \{s - \min(s)\} \times r_x/r_s \)

In the algorithm, each mean, var, max, and min is sample mean, variance, maximum and minimum value of \( x = (x_1,\ldots,x_n) \). Obtained \( T_a(x_i) \) \((i = 1,\ldots,n)\) is the proposed nonlinear transformation. The reason divided by 5 is just for
scale adjustment. We call the transformation with a-value 0.5 (weak), 1 (moderate), and 2 (strong) nonlinearity. In Figure 1, we illustrate the scatterplot of $T_a(x_i)$ versus $y_i$, for $a = 0$ (original), 0.5, 1, 2, where $(x_i, y_i)$ ($i = 1, \ldots, 500$) are independently generated from a bivariate normal distribution with both mean 0, variance 1, and correlation 0.5. It can be seen that the transformation enjoys the three desirable properties.

![Scatterplots of T_a(x) vs y for a = 0, 0.5, 1, 2](image)

Fig 1. Illustration of DNET: each top left, top right, bottom left, and bottom right shows the scatterplot of $T_a(x)$ v.s. $y$, for $a = 0$ (original), 0.5, 1, 2, respectively. The red curve is conditional mean function of $y$ given $T_a(x)$.

4. Efficiency Bound

In this section, we provide an optimal influence function for the true parameter $(\phi_T^T, \theta_0)^T$ that is the most efficient among all regular and asymptotically linear estimators, but that does not depend on the correctness of the $U$-function, i.e., we put a constraint that the nuisance tangent space of $\theta$ and $\phi$ are orthogonal. For example, Rotnitzky & Robins [31] derived the semiparametric efficiency
bound for regression parameters, which prescribe the first moment of the distribution of \([y \mid x]\). However, their adaptive estimators require many working models, and misspecification of either a regression model or a response model leads to a biased estimator, but in most cases, we do not expect the regression model is true and assume a simple function such as a linear regression model. In this section, we first provide the efficiency bound under the response model only, without relying on the information of \(U\)-function because the most difficult task in nonignorable nonresponse missing data analysis is to obtain a consistent estimator of the response model. Optimal estimators achieving this lower bound will be considered in the next section.

In the following discussion, we abbreviate the parameter value or random variable, for example, \(\pi(z; \phi_0) = \pi(z) = \pi(\phi_0)\), unless this would lead to ambiguity.

**Lemma 4.1.** Let \(S_{\text{eff}} = (S_1^T, S_2)^T\), where \(S_1 = S_1(R, G_R(Z))\) and \(S_2 = S_2(R, G_R(Z))\) be defined as

\[
S_1(R, G_R(Z); \phi) = \left\{1 - \frac{R}{\pi(Z; \phi)}\right\} g^*(X; \phi_0), \tag{4.1}
\]
\[
S_2(R, G_R(Z); \phi, \theta) = \frac{R}{\pi(Z; \phi)} U(Z; \theta) + \left\{1 - \frac{R}{\pi(Z; \phi)}\right\} U^*(X; \phi_0, \theta), \tag{4.2}
\]

\[g^*(x; \phi_0) = E^*\{s_0(Z; \phi_0) \mid x; \phi_0\}, \quad U^*(x; \phi_0, \theta) = E^*\{U(Z; \theta) \mid x; \phi_0\},\]

\[
E^*\{g(Z) \mid x; \phi_0\} = \frac{E\{O(Z; \phi_0)g(Z) \mid x\}}{E\{O(Z; \phi_0) \mid x\}} \tag{4.3}
\]

with \(O(z; \phi_0) = \{1 - \pi(z; \phi_0)\}/\pi(z; \phi_0)\). Then, the efficient influence function is \(\varphi_{\text{eff}} = H^{-1} S_{\text{eff}}\), where \(H = E(S_{\text{eff}}^2) = E\{\partial S_{\text{eff}}(\phi_0, \theta_0)/\partial(\phi^T, \theta)^T\}\) and \(B_{\text{eff}}^2 = BB^T\). Therefore, the semiparametric efficiency bound is given by \(E(S_{\text{eff}}^2)^{-1}\).

This lemma implies that if we can compute \(E^*(\cdot \mid x)\) then estimating functions (4.1) and (4.2) will provide an optimal estimator. The optimal estimator is the solution to

\[
\sum_{i=1}^n S_{\text{eff}, i}(\phi, \theta) = \sum_{i=1}^n \{S_1^T(r_i, G_{r_i}(z_i); \phi), S_2(r_i, G_{r_i}(z_i); \phi, \theta)\}^T = 0. \tag{4.4}
\]

The equation based on \(S_1(\phi)\) in (4.1) gives an optimal estimator for \(\phi\), say \(\hat{\phi}\). Then, by using \(\hat{\phi}\), \(S_2(\hat{\phi}, \theta)\) in (4.2) can provide an optimal estimator for \(\theta\). However, the expectation \(E^*(\cdot \mid x)\) and the parameter \(\phi_0\) are unknown and need to be estimated. Also, to compute the conditional expectation, we may need to correctly specify the distribution of \([y \mid x]\), which is subjective and unverifiable, as is stated in §1. In the next section, two adaptive estimators are proposed to work around the problem and to attain the lower bound derived in Lemma 4.1.

**Remark 4.1.** Equation (4.1) can be viewed as a special case of the estimator of Chang & Kott [3] defined in (2.1). Thus, the optimal \(g\) function in (2.1) for the
Chang & Kott [3] method is given by $g^*(x, \phi_0)$ in (4.1) although $\phi_0$ is unknown. One might think that the efficiency can be improved with a larger dimension of $g$ because the above two methods can handle over-identified models with $q > d + 1$. However, according to Lemma 4.1, there is no need to use more $g$ functions and it is enough to consider only $g^*(x, \phi_0)$ (i.e., $q = d + 1$) as the calibration function.

Remark 4.2. The optimal score function in (4.1) can be derived differently as follows. Consider the class of estimating equations in (2.1) indexed by $g$. For given $g$, the asymptotic variance of the solution $\hat{\phi}_g$ to (2.1) can be written as

$$V(\hat{\phi}_g) = \frac{1}{n} A_g^{-1} B_g A_g^{-1}$$

where

$$A_g = E[E\{O(Z; \phi_0) \cdot s_0(Z; \phi_0) \mid X\} g(X; \phi_0)^T]$$

$$B_g = E[E\{O(Z; \phi_0) \mid X\} g(X; \phi_0) g(X; \phi_0)^T].$$

Using Cauchy-Schwarz inequality, the asymptotic variance is minimized at $g^*(x; \phi_0) = E\{s_0(Z; \phi_0) \mid x; \phi_0\}$. Similarly, we can obtain the optimal estimating function in (4.2) by considering a class of estimating equations of the form

$$\sum_{i=1}^{n} \left[ \frac{r_i}{\pi(z_i; \phi)} U(z_i; \theta) + \left(1 - \frac{r_i}{\pi(z_i; \phi)}\right) h(z_i) \right] = 0, \quad (4.5)$$

indexed by $h$. The asymptotic variance of the solution to (4.5) is minimized at $h = E\{U(Z; \theta) \mid x; \phi_0\}$.

Remark 4.3. In our estimation steps, $\phi$ and $\theta$ are separately estimated. Thus, it follows from the identifiability of $\phi$ that $\theta$ is also identifiable. This is because, under assumptions (II)–(I3), $\phi$ is identifiable, thus, the identification problem of $\theta$ reduces to that of the probability limit of (4.4) or expectation of (4.2), i.e., $E\{U(Z; \theta)\}$.

5. Adaptive Estimators

We now propose two adaptive estimators for $(\phi_0, \theta_0)$: (i) with a parametric working model for $f_1(y \mid x)$; (ii) with a nonparametric estimator for $f_1(y \mid x)$, where $f_1(y \mid x) = f(y \mid x, r = 1).$ Although the optimality result in Lemma 4.1 has already been discussed by Rotnitzky & Robins [31], the adaptive estimators proposed here are different from those of Rotnitzky & Robins [31]. See Appendix C for some discussion of Rotnitzky & Robins [31] estimator.

To discuss the first proposed method, let $f_1^*(y \mid x)$ be known up to the parameter $\gamma \in \Gamma$, and let $\gamma$ be the maximizer of $\sum_{i=1}^{n} r_i \log f_1(y_i \mid x_i; \gamma)$. This can be easily implemented, and the model selection can be implemented by using information criteria such as the Akaike information criterion (AIC) and...
the Bayesian information criterion (BIC). By using the idea similar to that used to derive (2.3), we can show that, for any function \(g(z)\),

\[
E^* \{g(Z) \mid x; \phi_0, \gamma\} = \frac{E_1 \{\pi^{-1}(Z; \phi_0)O(Z; \phi_0)g(Z) \mid x; \gamma\}}{E_1 \{\pi^{-1}(Z; \phi_0)O(Z; \phi_0) \mid x; \gamma\}} \tag{5.1}
\]

where \(E_1(\cdot \mid x) = E(\cdot \mid x, r = 1)\). Thus, the expectation can be estimated by using \(f_1(y \mid x; \hat{\gamma})\) and \(\pi(z; \phi_0)\). However, since \(\phi_0\) is unknown, we propose an efficient estimating equation \(\sum_{i=1}^n S_{\text{eff}, i}(\phi, \theta, \hat{\gamma}) = 0\), where

\[
S_{\text{eff}, i}(\phi, \theta, \hat{\gamma}) = \{S_1^T(r_i, G_r(z_i); \phi, \hat{\gamma}), S_2(r_i, G_r(z_i); \phi, \theta, \hat{\gamma})\}^T, \tag{5.2}
\]

with

\[
S_1(r, G_r(z); \phi; \hat{\gamma}) = \left\{1 - \frac{r}{\pi(z; \phi)}\right\} E^* \{s_0(z; \phi) \mid x_i; \phi, \hat{\gamma}\},
\]

\[
S_2(r, G_r(z); \phi, \theta, \hat{\gamma}) = \frac{r}{\pi(z; \phi)} U(z; \theta) + \left\{1 - \frac{r_i}{\pi(z; \phi)}\right\} E^* \{U(z; \theta) \mid x_i; \phi, \hat{\gamma}\}.
\]

What if \(f_1(y \mid x)\) is misspecified? One might expect the solution to the estimating equation with (5.2) to be inconsistent as a result. Note that the estimator that uses the function on the right-hand side of (5.1) is consistent even when the assumed model for \(f_1(y \mid x)\) is misspecified. Also, if the model is correctly specified, the estimator attains the lower bound. This leads to Theorem 5.1.

**Theorem 5.1.** Let \((\hat{\phi}^T, \hat{\theta}^T)\) be the solution to \(\sum_{i=1}^n S_{\text{eff}, i}(\phi, \theta, \hat{\gamma}) = 0\) in (5.2). Under conditions (I1)–(I3) and (C1)–(C7) given in Appendix A and the identification conditions assumed in Theorem 3.1, \((\hat{\phi}^T, \hat{\theta}^T)\) satisfies consistency and asymptotic normality with variance

\[
E \left\{ \frac{\partial S_{\text{eff}}^*}{\partial (\hat{\phi}^T, \hat{\theta})} \right\}^{-1} E(S_{\text{eff}}^* \otimes S_{\text{eff}}^*) E \left\{ \frac{\partial S_{\text{eff}}^*}{\partial (\hat{\phi}^T, \hat{\theta})} \right\}^{-1},
\]

even if \(f_1(y \mid x; \hat{\gamma})\) is misspecified, where \(\gamma^*\) is the probability limit of \(\hat{\gamma}\), and \(S_{\text{eff}}^* = \{S_1(\phi_0, \gamma^*), S_2(\phi_0, \theta_0, \gamma^*), S_3(\phi_0, \gamma^*), S_4(\phi_0, \gamma^*), \ldots\}\) is defined in (5.2). In particular, the asymptotic variance of \(\hat{\theta}\) is given as

\[
V^* = \text{var} \left[ \tau_{U^{-1}} S_2(\phi_0, \theta_0, \gamma^*) - \kappa^* S_1(\phi_0, \gamma^*) \right], \tag{5.3}
\]

where \(\kappa^* = \kappa^*_1 \kappa^*_2^{-1}\), \(\kappa^*_1 = E \{U^*(\phi_0, \theta_0, \gamma^*) - U(\theta_0)\} \hat{\pi}(\phi_0)^T / \pi(\phi_0)\},\n\kappa^*_2 = E \{g^*(\phi_0, \gamma^*) \hat{\pi}(\phi_0)^T / \pi(\phi_0)\},\) and \(\tau_U = E \{\partial U(\theta_0) / \partial \theta\}.\) In addition, if the model is correctly specified, the estimator attains the semiparametric efficiency bound.

Note that Theorem 5.1 does not require that \(f_1\) be correctly specified. Unlike the estimator of Riddles et al. [28], the parametric model \(f_1\) is irrelevant to the consistency and asymptotic normality of the estimator here. Therefore, we call \(f_1\) a working model, as in Liang & Zeger [20]. Also, though equation (4.2)
Theorem 5.2. Let $\text{Var}(\hat{\theta})$ be the solution to $\sum_{i=1}^{n} \hat{S}_{\text{eff},i}(\phi, \theta) = 0$, where $\hat{S}_{\text{eff},i}(\phi, \theta)$ is defined in (4.4) with the estimated conditional expectation in (5.5). Under Conditions (II)–(I3), (C1)–(C4), and, (C8)–(C13) given in Appendix A, $(\hat{\phi}^*, \hat{\theta})^T$ satisfies consistency and asymptotic normality, and the estimator attains the semiparametric efficiency bound.

Remark 5.1. The second proposed estimator is robust because it does not require any model assumptions on $f_1$, but it would not work well when the dimension of $x$ is high, as is common in any nonparametric estimation.

Variance estimation is also a difficult problem in semiparametric estimation. When we consider a parametric working model $f_1(y | x)$, $\hat{V} = n^{-1} \sum_{i=1}^{n} \left[ \hat{r}_i^{-1} \{ S_2(r_i, G_{r_i}(z_i); \hat{\phi}, \hat{\theta}, \hat{\gamma}) - \hat{r} \hat{S}_1(r_i, G_{r_i}(z_i); \hat{\phi}, \hat{\gamma}) \} \right]^{\otimes 2} (5.6)$
converges to $V^*$ in probability as defined in (5.3), where $\hat{\tau}_U$ and $\hat{\kappa}$ are consistent estimators for $\tau_U$ and $\kappa^* = \kappa_1^*(\kappa_2^*)^{-1}$, respectively, for $\kappa_1^*$ and $\kappa_2^*$ as defined in Theorem 5.1. To estimate $\kappa_1^*$, we propose using the same method that we used to compute $\theta_0$, i.e., let $U(\phi_0, k_1, \gamma^*) = k_1 - (U^*(\gamma^*) - U)^r(\phi_0)^T / \pi(\phi_0)$ be our new $U$-function and let the solution to $E\{U(\phi_0, k_1, \gamma^*)\} = 0$ with respect to $k_1$ be our target parameter; solve the following equation:

$$\sum_{i=1}^n \left[ \frac{r_i}{\pi(z_i; \phi)} U(z_i; \hat{\phi}, k_1, \hat{\gamma}) + \left\{ 1 - \frac{r_i}{\pi(z_i; \phi)} \right\} E^* \{ U(Z; \hat{\phi}, k_1, \hat{\gamma}) \mid x_i; \hat{\gamma} \} \right] = 0.$$ 

This is the optimal estimator for $(\phi_0, \kappa_1^*)$ in terms of the asymptotic variance, because $U$ is a known function and Theorem 5.1 is applicable. The best estimator for $\kappa_2^*$ can be obtained in the same way. When we use the nonparametric method stated in Theorem 5.2 to estimate $\theta_0$, the variance can be also estimated by using the nonparametric method (5.4) and (5.5), instead of using the parametric model $f_1(y \mid x; \gamma)$ in (5.6).

6. Simulation Study

In order to evaluate the performance of our proposed estimators and to compare their efficiency with other methods in finite samples, we conduct a Monte Carlo simulation study with three scenarios. In each scenario, two covariates $X_1 \sim N(0, 1/\sqrt{3})$ and $X_2 \mid X_1 = x_1 \sim N(-x_1/3, 1/2^2)$ are used. For each scenario $s (= 1, 2, 3)$, the response mechanism is set to a Bernoulli distribution with parameter $\pi_x(x_1, x_2, y)$, where $\pi_x(x_1, x_2, y) = 1/[1 + \exp(\phi_{x0} + 0.5x_1 + 0.5x_2 + \phi_{x}y)]$, and the response outcome variables are generated from $Y \mid (x, r = 1) \sim N(\mu_x(x), 1/2^2)$, where $\mu_x(x) = a_0^s + 0.4x_1 + 0.4x_2 + a_1^s x_1x_2$. The coefficients of the nonlinear term, which is the degree of nonlinearity, are set to $a_1^s = 0$, $a_2^s = 0.3$, $a_3^s = 0.6$, and the other parameters are set, so that the expectation of the outcome variable is zero and the marginal response probability is $70\%$, to $\phi_0^s = -0.959$, $\phi_{x0}^s = -0.914$, $\phi_{x}^s = -0.904, \phi_{x0}^s = 0.75, \phi_{x}^s = 0.4, \phi_{x}^s = 0.3, \phi_0^s = -0.0563, a_0^s = 0.2$, and $\phi_0^s = 0.0775$. Note that the scenario 2 and 3 are identifiable without using any instrumental variable because of the nonlinear term $x_1x_2$ in $f_1$, on the other hand, Scenario 1 is unidentifiable and Scenario 2 is weakly identified than Scenario 3. We estimate $\theta = E(Y)$, thus $U(\theta; Z) = \theta - Y$, with two different Monte Carlo samples of size $n = 500$ and $n = 2000$ being independently generated 2,000 times.

In Scenario 1, however, it is still possible to make the response model identifiable at the risk of misspecification of the response mechanism by using DNET. In this article, we change the variable $x_1 \rightarrow T(x_1)$ and $x_2 \rightarrow T_0(x_2)$ ($a = 0.5, 1, 2$).

From each sample, we compute six estimators, as follows:

1. CK: The estimator of Chang & Kott [3]. We use the estimating equation
(2.1), setting \( g \) as \((1, x_1, x_2)\): \( \theta \) is estimated by solving

\[
\sum_{i=1}^{n} r_i(\theta - y_i)/\hat{\pi}_i = 0,
\]

where \( \hat{\pi} \) is the estimated response model.

[2] RR: The estimator of Rotnitzky & Robins [31]. This estimator is defined through four steps (i)–(iv) in Appendix C. In the first step, a consistent estimator is set to be the CK estimator, and in the second step, each of (C.1), (C.3)–(C.6) is modeled by at most third order polynomial function of \( x_1 \) and \( x_2 \).

[3] RKI: The estimator of Riddles et al. [28]. In all scenarios, we specify a parametric model on \( f_1 \) based on normal distribution with the correct mean structure \( \mu(x) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2. \)

[4] P: Our proposed estimator with parametric \( f_1 \) model. As for the working model for \( f_1 \), the same model specification as in the RKI method is used.

[5] NP: Our proposed estimator with nonparametric \( f_1 \) model. As for the kernel function and its bandwidth, Gaussian kernel, and a rule-of-thumb bandwidth \( h_j = n_1^{-1/5} \hat{\sigma}_x J (j = 1, 2) \) is used, where \( n_1 \) is the sample size of observed outcome variable and \( \hat{\sigma}_j \) is the square root of the sample variance of \( x_j \) for \( j = 1, 2 \).

[6] DNET\((a)\): Same method as P and NP with the nonlinearly transformed data \( T_a(x_1) (a = 0.5, 1, 2) \) for the variables in response models.

Suppose that the response model is correctly specified in our proposed methods, and except for our proposed methods, for identifiability, suppose that \( x_2 \) is specified as the instrumental variable, i.e., the response model is specified as

\[
\logit \{ \pi(x_1, y) \} = \phi_{x1} x_1 + \phi_y y.
\]

Before estimating the parameters, we first check the model identifiability of our proposed method. The right panel in Figure 2 shows the p-values of the statistical tests proposed in §3.2 under the three scenarios with different sample sizes. The p-values in scenario 1 spread around 1/2 because the null hypothesis is correct or the model is unidentifiable. On the other hand, as the nonlinearity increases, p-values are close to zero. In particular, when \( n = 2000 \), model identification can be judged with the probability almost 1 even for Scenario 2 which has a small degree of the nonlinearity 0.3.

The left panel in Figure 2 shows the Monte Carlo simulation results with sample size \( n = 500 \). The results with sample size \( n = 2000 \) are omitted because they are almost the same. In some Monte Carlo samples, we encounter some numerical problems and there is no solution because the estimate of the response model does not converge due to weak identifiability. The rates of datasets not having converging estimators are reported at the bottom right in Figure 2. The following is a summary of the simulation results:

[1] The CK method estimates the parameter stably, but it is biased due to the misspecification of the response model. The standard error of the CK
estimators is a little larger than RKI, P, NP, and DNET methods due to the lack of efficiency.

[2] In many cases, the RR estimators do not converge. This comes from the difficulty of finding a good starting value of \( \phi \) in the first step and of modeling the working models defined in Appendix C.

[3] Performance of RKI method is similar to that of CK, but it is less biased and has a smaller standard error because the \( f_1 \) model is correctly specified.
[4] When the model is identifiable, the proposed P method works well. However, when it is unidentifiable, it is hard even to get a convergent sequence of estimators, though that can be inferred by testing linearity of the mean function.

[5] Surprisingly, the NP method can estimate the parameter stably despite of the unidentifiability of the model, and the estimates are biased according to the degree of linearity of $μ_s(x)$.

[6] Proposed DNET works well for all the transformations $T_n(x)$. In Scenario 1, when the model is not identifiable, the rate obtaining a non-convergent estimator and bias increase as the nonlinearity increases, in the meantime, the standard error decreases.

7. Real data analysis

In this section, our proposed estimators are applied to the Korea Labor and Income Panel Survey (KLIPS) data, which have been analyzed multiple times [19, 43, 36]. The data contain $n = 2,506$ Korean wage earners; the response variable $y$ is total wage income ($10^6$ Korean Won) in year 2008. There are three fully observed covariates: $x_1$: total wage income in the previous year (2007); $x_2$: gender; $x_3$: age. While $x_1$ is a continuous variable, $x_2$ has two categories 1 and 2 for male and female, respectively, and $x_3$ has three categories 1-3: $x_3 < 35$, $35 \leq x_3 < 51$, and $x_3 \geq 51$. We also identified three data points as outliers and excluded them from further analysis.

Although the data are completely observed, we took the approach of Kim & Yu [19] and created 1000 incomplete datasets with the following eight response mechanisms: M1 (linear nonignorable without $(x_2, x_3)$): $\text{logit}(\pi) = 0.48 - 0.3x_1 - 0.5y$; M2 (linear nonignorable): $\text{logit}(\pi) = -0.85 - 0.2x_1 + 0.5x_2 + 0.2x_3 - 0.4y$; M3 (nonlinear nonignorable, quadratic in $x_1$ without $(x_2, x_3)$): $\text{logit}(\pi) = 0.33 - 0.5x_1 - 0.1x_1^2 - 0.3y$; M4 (nonlinear nonignorable, quadratic in $x_1$): $\text{logit}(\pi) = -0.89 - 0.4x_1 - 0.1x_1^2 + 0.5x_2 + 0.2x_3 - 0.4y$; M5 (nonlinear nonignorable, quadratic in $y$ without $(x_2, x_3)$): $\text{logit}(\pi) = 0.24 - 0.25x_1 - 0.25y - 0.1y^2$; M6 (nonlinear nonignorable, quadratic in $y$): $\text{logit}(\pi) = -0.93 - 0.2x_1 + 0.5x_2 + 0.2x_3 - 0.2y - 0.1y^2$; M7 (probit nonignorable) $\pi = \Phi(-0.55 + 0.3x_1 + 0.4y)$; M8 (jump nonignorable) $\pi = 0.5I(0.5x_2 + 0.2x_3 + y \leq 2.6) + 0.9(0.5x_2 + 0.2x_3 + y > 2.6)$, where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution, and $I(A)$ is the indicator function that takes 1(0) if event $A$ is true (false). Note that there are NIVs for models M2, M4, M6, and M8. For all data sets, the response rate is about 70%. We estimated $\theta = E(Y)$ as considered in the simulation. The “true” average income in 2008 is $\hat{\theta}_n = 1.846$ as calculated using the complete data. In order to estimate the parameters, we assumed a response mechanism $\text{logit}\{\pi(x, y; \phi)\} = \phi_{x0} + \phi_{x1}x_1 + \phi_{x2}x_2 + \phi_{x3}x_3 + \phi_{y}y$. Therefore M1 and M2 are correctly specified while M3-M8 are misspecified.

We specified unknown $f_1$ models as normal distribution $Y \mid (x_1, x_2 = i, x_3 = j, r = 1) \sim N(\mu_{i,j}(x_1), \sigma_{i,j}^2)$ ($i = 1, 2; j = 1, 2, 3$), where $\mu_{i,j}(x_1) = \gamma_{0i,j} + \gamma_{1i,j}x_1 + \gamma_{2i,j}x_1^2 + \gamma_{3i,j}x_1^3 + \gamma_{4i,j}x_1^4$; ($\gamma_{1i,j}, \gamma_{2i,j}, \gamma_{3i,j}, \gamma_{4i,j}$) is the regression pa-
rameter when \((x_2, x_3) = (i, j)\). We chose the best model by AIC among \(2^5 - 1\) models for each \((x_2, x_3)\)'s 2 x 3 pattern. Using Theorem 3.1, one can show that this model is identifiable as one of the 6 mean structures are nonlinear, or all the structures are linear but all of them are not the same. One simple sufficient condition is to check whether the conditional mean of \(y\) given \(x_1\) is linear with respect to \(x_1\). In the real data, the correlation between \(x_1\) and \(y\) is too high because wage income does not change considerably within one year; the mean structure is almost linear. However, the p-values of the test statistics are almost zero in all datasets with M1–M8, therefore, all the response models are identifiable without using any instrumental variable nor transformation. In Table 1, Bias, S.E. (standard error), and RMSE (root mean square error) with five methods, CK, RR, RKI, P, NP methods same as in §5, are reported. Following are summary of the results:

[1] The CK method estimates the parameter stably, but it is inefficient compared to our proposed methods.

[2] As in §5, the RR estimators do not converge in many datasets.

[3] RKI methods can obtain estimates stably, but it is severely biased due to the misspecification of \(f_1\) model, which is generally unknown in real data.

[4] The proposed P method works well, but for some datasets, we encounter some numerical problems due to the misspecification of the response model. As for such datasets, we may get a reasonable estimator by using DNET. Note that our method is effective for the probit response mechanism (M7), even though the use of the probit model makes it hard to identify the parameter as stated in §3.1.

[5] Performance of the proposed NP method is the best among the five methods considered. However, the results with dataset M5 and M6 implies the difficulty of obtaining the estimator with misspecified response models.

8. Discussion

We have presented a test statistic for model identification, semiparametric efficiency bound for \((\phi^T_0, \theta_0)^T\) under nonignorable nonresponse; proposed two types of adaptive semiparametric estimators that attain the semiparametric lower bound. Identification is a challenging problem in nonignorable nonresponse [24]; previous methods require nonignorable NIVs to guarantee model identification [43]. Our new identifiability condition is not on the response mechanism, but on the distribution of \([y \mid x, r = 1]\).

The proposed method is based on the correct specification of the response model. There may be various other models for the true response mechanism, and thus the appropriate information criteria for choosing the response mechanism will be a topic of future research. Instead of specifying a single response model, one can consider multiple response models, and obtain consistency when one of the specified response models is correct. This multiple robustness property has been investigated under the ignorable nonresponse setup [15, 4]. Extension of
Table 1
Bias, S.E. (standard error), and RMSE (root mean square error) of our proposed estimator, where the full sample estimate $\hat{\theta}_n = 1.846$ is set to the true value, for datasets M1–M8. NA rate is the rate of dataset failed to get a convergence estimator. All values are multiplied by 1,000 except for NA rate.

| Model | Methods | CK | RR | RKI | P | NP |
|-------|---------|----|----|-----|---|----|
| M1    | Bias    | 73 | 118| 737 | 22| 17 |
|       | S.E.    | 130| 388| 676 | 48| 31 |
|       | RMSE    | 149| 406| 1000| 52| 35 |
|       | NA rate(%) | 0.7 | 31.9 | 0 | 1.8 | 1.6 |
| M2    | Bias    | 66 | 90 | 599 | 23| 13 |
|       | S.E.    | 119| 217| 580 | 49| 31 |
|       | RMSE    | 136| 235| 834 | 54| 33 |
|       | NA rate(%) | 1.1 | 27.9 | 0 | 1.2 | 1.0 |
| M3    | Bias    | 205| 197| 879 | 11| -6 |
|       | S.E.    | 128| 694| 810 | 49| 31 |
|       | RMSE    | 241| 722| 1195| 50| 32 |
|       | NA rate(%) | 8.1 | 48.4 | 0 | 4.3 | 2.5 |
| M4    | Bias    | -103| 10 | 245 | 55| 29 |
|       | S.E.    | 100| 179| 295 | 51| 26 |
|       | RMSE    | 144| 179| 383 | 75| 39 |
|       | NA rate(%) | 15.8 | 40.9 | 0 | 1.1 | 0.3 |
| M5    | Bias    | -39 | 78 | 840 | 59| 0 |
|       | S.E.    | 71 | 288| 780 | 211| 56 |
|       | RMSE    | 81 | 298| 1146| 219| 56 |
|       | NA rate(%) | 2.9 | 24.2 | 0 | 1.9 | 32.7 |
| M6    | Bias    | -68 | 57 | 776 | 44| 13 |
|       | S.E.    | 87 | 550| 734 | 51| 49 |
|       | RMSE    | 111| 553| 1068| 68| 51 |
|       | NA rate(%) | 7.4 | 34.0 | 0 | 1.7 | 37.4 |
| M7    | Bias    | 158| 125| 1131| 15| 11 |
|       | S.E.    | 155| 472| 844 | 42| 34 |
|       | RMSE    | 221| 489| 1412| 45| 36 |
|       | NA rate(%) | 2.2 | 36.7 | 0 | 5.8 | 2.0 |
| M8    | Bias    | 175| 136| 689 | 27| 9 |
|       | S.E.    | 115| 475| 579 | 48| 33 |
|       | RMSE    | 209| 494| 900 | 55| 34 |
|       | NA rate(%) | 0.6 | 37.2 | 0 | 1.4 | 1.1 |
multiple robustness to the nonignorable nonresponse case will also be a topic of
our future research.

Appendix A: Regularity conditions

(C1). \( \Phi \) and \( \Theta \) are compact.
(C2). \( W_i = (X_i, Y_i, R_i) \) are independently and identically distributed.
(C3). \( \Gamma \) is compact, \( S_\gamma(\gamma) = \partial \log f_1(y \mid x; \gamma)/\partial \gamma \) is continuously differentiable at
\( \gamma \in \Gamma \) with probability one, there exists \( e(W) \) such that \( ||S_\gamma(\gamma)|| \leq e(W) \)
for all \( \gamma \in \Gamma \) and \( E\{e(W)\} < \infty \), \( E\{S_\gamma(\gamma)\} = 0 \) has a unique solution
\( \gamma^* \in \Gamma \), \( \partial S_\gamma(\gamma)/\partial \gamma^T \) is continuous at \( \gamma^* \) with probability one, and there
is a neighborhood \( \Gamma_{\gamma^*} \) of \( \gamma^* \) such that \( ||E\{\sup_{\gamma \in \Gamma_{\gamma^*}} \partial S_\gamma(\gamma)/\partial \gamma^T\}|| < \infty \).
(C4). Identifiability of \( \theta \) for complete data: there exists \( \theta_0 \in \Theta \) such that \( E\{U(Z; \theta_0)\} = 0 \).
(C5). \( \partial S_{\text{eff}}(\phi, \theta, \gamma)/\partial (\phi^T, \theta, \gamma^T) \) is continuous at \( (\phi_0, \theta_0, \gamma^*) \) with probability
one, and there is a neighborhood \( \Phi_{\gamma^*} \times \Theta_{\gamma^*} \times \Gamma_{\gamma^*} \) of \( (\phi_0, \theta_0, \gamma^*) \) such that
\[
||E\{\sup_{(\phi, \theta, \gamma) \in \Phi_{\gamma^*} \times \Theta_{\gamma^*} \times \Gamma_{\gamma^*}} \partial S_{\text{eff}}(\phi, \theta, \gamma)/\partial (\phi^T, \theta, \gamma^T)\}|| < \infty.
\]
(C6). \( S_{\text{eff}}(\phi, \theta, \gamma) \) is continuously differentiable at each \( (\phi, \theta, \gamma) \in \Phi \times \Theta \times \Gamma \) with
probability one, and there exists \( d_1(W) \) such that \( ||S_{\text{eff}}(\phi, \theta, \gamma)|| \leq d_1(W) \)
for all \( (\phi, \theta, \gamma) \in \Phi \times \Theta \times \Gamma \) and \( E\{d_1(W)\} < \infty \).
(C7). \( E\{\partial S_{\text{eff}}(\phi, \theta, \gamma)/\partial (\phi^T, \theta, \gamma^T)\} \) is nonsingular at \( (\phi_0, \theta_0, \gamma^*) \).
(C8). The conditions (C5)-(C7) hold at the true value \( \gamma^* = \gamma_0 \).
(C9). Let \( \mathcal{X} \) be the support of \( x \). Then, \( f_1(x) > 0 \) and \( E_1\{\pi(x, Y; \theta_0) \mid x\} > 0 \)
for all \( x \in \mathcal{X} \).
(C10). The kernel \( K(u) \) has bounded derivatives of order \( k \), satisfies \( \int K(u)du = 1 \), has zero moments of order \( \leq m - 1 \), and has a nonzero \( m \)-th order
moment.
(C11). For all \( y, \pi(\cdot, y; \theta_0), \pi(\cdot, y; \phi_0) \), and \( U(\cdot, y; \theta_0) \) are differentiable to order \( k \)
and are bounded on an open set containing \( \mathcal{X} \).
(C12). Let \( a_1(z) = 1, a_2(z) = s_0(z; \phi_0), \) and \( a_3(z) = U(z) \). Then, there exists \( v \geq 4 \) such that \( E_1\{[\pi^{-1}(Z; \phi_0)O(Z; \phi_0)a_1(Z)]^v\} \) and \( E_1\{[\pi^{-1}(Z; \phi_0)O(Z; \phi_0)a_1(Z)[^v \mid x]\}
\( f_1(x) \) are bounded for all \( x \in \mathcal{X} \).
(C13). As \( h \to 0 \), \( n^{1-2/v}h^d/\ln n \to \infty, n^{1/2}h^{d+2k}/\ln n \to \infty \), and \( n^{1/2}h^{2m} \to 0 \).

Appendix B: Proofs of the technical results

Proof of Theorem 3.1. Let \( f_1(y \mid x) \) be the true density function of \( [y \mid x, r = 1] \).
Here, the distribution of \( [y \mid x] \) can be represented through the observed outcome
density and the response model, because by using Bayes’ formula, we have
\[
f(y \mid x; \phi) = \frac{f_1(y \mid x)\pi^{-1}(x, y; \phi)}{\int f_1(y \mid x)\pi^{-1}(x, y; \phi)dy}. \quad (B.1)
\]
Suppose that $\phi_0$ is the true value of the response model so that the true distribution of $[y \mid x]$ is $f(y \mid x; \phi_0)$. Then, it follows from (B.1) that the probability limit of the estimating equation is

$$
E \{ \Gamma(Z, R; \phi) \mid x \} = g(x; \phi) \left\{ 1 - \frac{\pi(Z; \phi)}{\pi(Z; \phi_0)} \right\} f(y \mid x; \phi_0)dy
$$

$$
= g(x; \phi) \left\{ 1 - \int \frac{\pi(Z; \phi)^{-1}}{\pi(Z; \phi_0)^{-1}} f(y \mid x; \phi_0)dy \right\}.
$$

By using (I2) and (I3), the conditional expectation can not be vanished unless $\phi = \phi_0$. Therefore, the solution is unique.

Proof of Proposition 3.1. For any error function $\varepsilon \in E$, under the null hypothesis $H_0^{(*)}$, there exist $c_1^{(\ell)}, c_2^{(\ell)}$ ($\ell = 2, 3$) such that $E(\varepsilon^2 \mid x) = c_1^{(2)} + (c_2^{(3)})^\top x$ and $E(\varepsilon^3 \mid x) = c_1^{(3)} + (c_2^{(3)})^\top x$. On the other hand, it holds that

$$
\varepsilon^2 = \sum_{j=0}^{\infty} \varepsilon_j^2 e_j(x) + \sum_{j \neq k} \xi_j \xi_k e_j(x)e_k(x).
$$

It follows from $e_j \neq e_k$ ($j \neq k$) that there must exist a positive integer $j$ such that $e_j = \{E(\xi_j^2)\}^{-1/2} (c_1^{(2)} + (c_2^{(2)})^\top x)^{1/2}$ and $e_k \equiv 0$ for $k \neq j$, so that $\varepsilon = (c_1^{(2)} + (c_2^{(2)})^\top x)^{1/2} \xi_j$. Let such $j$ be 1 without loss of generality. In a similar way, it follows from the third moment condition of $\varepsilon$ that $\varepsilon = (c_1^{(3)} + (c_2^{(3)})^\top x)^{1/3} \xi_1$, which implies $c_2^{(2)} = c_2^{(3)} = 0$ and $c_1^{(3)} = (c_1^{(2)})^{3/2}$. By using the induction, under the null hypothesis $H_0^{(*)}$, it can be shown that $c_2^{(\ell)} \equiv 0$ for $\ell \geq 2$. As a result, $\varepsilon$ is a random variable which is independent of $x$. Therefore, $H_0^{(1)}$ or testing linearity of mean function is enough to check the model identification.

Next, we provide a proof of Lemma 4.1 and Theorem 5.1 and 5.2. In order to prove Lemma 4.1, we will assume $U(z) = y$ just for simplicity. We specify the joint distribution $z = (x^T, y)^T$ by $f(z; \eta)$, where $\eta$ is an infinite-dimensional nuisance parameter, and $\eta_0$ is the true value. By “full model” we refer to the class of models in which the data are completely observed, and by “obs model” we refer to those in which some $Y$ are missing; that is, a full model consists of functions $h(Z)$ and an obs model consists of $h(R, G_R(Z))$. Furthermore, for each full and obs model, denote the nuisance tangent space by $\Lambda^F$ and $\Lambda$, respectively, and its orthogonal complement by $\Lambda^{F\bot}$ and $\Lambda^\bot$, respectively. Let $S_\phi$ be the score function with respect to $\phi$. Consider a Hilbert space $H = \{h^{(q+1)\times 1} \mid E(h) = 0; \|h\| < \infty\}$ with inner product $\langle h_1, h_2 \rangle = E(h_1^T h_2)$, where the expectation is taken under the true model. See Bickel et al. [2] and Tsiatis [41] for more details.

When $U$ is comprised of other functions, the proof is almost the same.

At first, we introduce a proposition of Rotnitzky & Robins [31], which provides the efficient score for $(\phi, \theta)$, as follows. Let $B$ and $D$ be functions of $(R, G_R(Z))$, and let $B^*$ and $D^*$ be functions of $Z$. Also, let us define the following three linear operators: $g(B^*) = E(B^* \mid R, G_R(Z))$, $m(B^*) = E(g(B^*) \mid Z)$, and $u(B^*) = RB^*/\pi(Z)$. Then, the efficient score for $(\phi, \theta)$ can be derived by
Thus, it remains to check that for any \( g \) which can be proved easily.

Obviously, the right-hand side of (B.4) belongs to \( \Lambda \).

Proof of Lemma B.3.
The efficient score for Lemma B.1.

The projection onto \( \Lambda \) is written as follows:

\[
P\left[ m^{-1}(D^*) \mid \Lambda F^{\perp} \right] = (Q, S_{\text{eff}, g}), \tag{B.3}
\]

where \( Q = P\left[ m^{-1}[E\{g(S_{\phi}^F \mid L)\} \mid \Lambda F^{\perp}] \right] \), \( A_{2, \text{eff}} = (P[S_{\phi} \mid A_{2}]^T, 0)^T = (g(S_{\phi}^F) - g[m^{-1}[E\{g(S_{\phi}^F \mid L)\}])^T, 0)^T \), and \( S_{\text{eff}, g} \) is the efficient score function of \( \theta \) in the full model.

This Lemma implies that the efficient score can be represented by (B.2) with \( D_{\text{eff}}^* \) satisfying condition (B.3). Thus, in the nonignorable nonresponse case, \( \Lambda F^{\perp} \) needs to be calculated, and it can be done in a way similar to that shown in Section 4.5 of Tsiatis [41].

Lemma B.2. The nuisance tangent space \( \Lambda^F \) and its orthogonal complement \( \Lambda^{F^{\perp}} \) in the full model are written as follows:

\[
\Lambda^F = \{ h(Z) \in \mathcal{H} \text{ such that } E\{Yh(Z)\} = 0 \},
\]

\[
\Lambda^{F^{\perp}} = \{ k(Y - \theta_0) \text{, where } k \text{ is any } q + 1 \text{ dimensional vector} \}.
\]

Finally, we give an explicit formula to calculate the projection onto \( \Lambda_2 \).

Lemma B.3. For \( h(R, G_R(Z)) = Rh_1(Z) + (1 - R)h_2(X) \), it holds that

\[
\Pi(h \mid \Lambda_2) = \left\{ 1 - \frac{R}{\pi(Z)} \right\} \frac{E\{1 - \pi(Z)\}h_2(X) - E\{h_1(Z)\} \mid X}{E\{O(Z) \mid X\}}. \tag{B.4}
\]

Proof of Lemma B.3. Obviously, the right-hand side of (B.4) belongs to \( \Lambda_2 \). Thus, it remains to check that for any \( g \),

\[
\left\langle h - \left\{ 1 - \frac{R}{\pi(Z)} \right\} \frac{h_2(X) - E\{h_1(Z) \mid X\}}{E\{O(Z) \mid X\}}, \left\{ 1 - \frac{R}{\pi(Z)} \right\} g(X) \right\rangle = 0,
\]

which can be proved easily. \( \square \)

We now give a proof of Lemma 4.1.

Proof of Lemma 4.1. Note that \( S_{\text{eff}, \theta} = Y - \theta_0 \) by Lemma B.2, since there exists only one influence function, and it is the efficient one under the assumption that \( \theta \) does not require any assumptions on the distribution of \( Z \) [see 41, Chap. 5]. By the projection theorem, there exists a unique \( k = (k_1, k_2, \ldots)^T \) such that \( D_{\text{eff}}^* = k(Y - \theta_0) \).

Then, we calculate \( A_{2, \text{eff}} \). The score function of \( \phi \) is

\[
S_{\phi} = g(S_{\phi}^F) = Rs_1(Z; \phi) + (1 - R)s_0(X; \phi),
\]
where \( s_r(\phi) \) is defined in (3). It follows from Lemma B.3 with \( h_1(z) = s_1(\phi) \) and \( h_2(x) = s_0(x; \phi) \) in (B.4) that \( \Pi(S_0 | A_2) = -\{1 - R/\pi(Z)\}g^*(X) \). Thus, \( A_{2,\text{eff}} = [0, -\{1 - R/\pi(Z)\}g^*(X)] \). Again, by using Lemma B.3, it follows that \( \Pi[u(D_{\text{eff}}) | A_2] = -\{1 - R/\pi(Z)\}E^*(Y - \theta_0 | X) \), by which (B.2) becomes

\[
S_1 = k_2 \left[ \frac{R(Y - \theta_0)}{\pi(\phi_0)} + \left\{ 1 - \frac{R}{\pi(\phi_0)} \right\} E^*(Y - \theta_0 | X) \right] - \left\{ 1 - \frac{R}{\pi(\phi_0)} \right\} g^*(X)
\]

and

\[
S_2 = k_1 \left[ \frac{R(Y - \theta_0)}{\pi(\phi_0)} + \left\{ 1 - \frac{R}{\pi(\phi_0)} \right\} E^*(Y - \theta_0 | X) \right].
\]

This \( S_{\text{eff}} = (S_1, S_2^T) \) can be transformed into \( \hat{S}_{\text{eff}} = (\hat{S}_1, \hat{S}_2^T) = AS_{\text{eff}} \),

\[
\hat{S}_1 = \left\{ 1 - \frac{R}{\pi(\phi_0)} \right\} g^*(X), \\
\hat{S}_2 = \frac{R(Y - \theta_0)}{\pi(\phi_0)} + \left\{ 1 - \frac{R}{\pi(\phi_0)} \right\} E^*(Y - \theta_0 | X)
\]

with a nonsingular matrix \( A \),

\[
A = \begin{bmatrix} -I_q & -k_2^T/k_1 \\ 0^T & k_1^{-1} \end{bmatrix},
\]

where \( I_q \) is a \( q \)-dimensional identity matrix. The score function multiplied by a nonsingular constant matrix does not have an influence on the asymptotic distribution. Thus, we have the desired efficient score. \( \square \)

**Proof of Theorem 5.1.** Consistency and asymptotic normality are proved under the assumptions (C1)–(C8) by using the standard argument for GMM. Next, we give the explicit form of the asymptotic variance. Let \( \xi = (\phi, \theta)^T \). Recall that each \( \hat{\gamma} \) and \( \hat{\xi} \) is a solution to \( \sum_{i=1}^n \partial \log f_i(y_i | x_i; \gamma) / \partial \gamma = \sum_{i=1}^n S_{\gamma i}(\gamma) = 0 \) and \( \sum_{i=1}^n S_{\text{eff},i}(\hat{\gamma}, \hat{\xi}) = 0 \), respectively, where \( S_{\text{eff},i}(\gamma, \xi) \) is defined in (10). By using standard asymptotic theory,

\[
\begin{bmatrix} \hat{\gamma} - \gamma^* \\ \hat{\xi} - \xi_0 \end{bmatrix} = -\mathcal{I}^{-1} \mathcal{I} \sum_{i=1}^n \begin{bmatrix} S_{\gamma i}(\gamma^*) \\ S_{\text{eff},i}(\gamma^*, \xi_0) \end{bmatrix},
\]

where

\[
\mathcal{I} = E \begin{bmatrix} \partial S_{\gamma}(\gamma^*)/\gamma^T & \partial S_{\gamma}(\gamma^*)/\xi^T \\ \partial S_{\text{eff}}(\gamma^*, \xi_0)/\gamma^T & \partial S_{\text{eff}}(\gamma^*, \xi_0)/\xi^T \end{bmatrix}
\]

\[
= E \begin{bmatrix} \partial S_{\gamma}(\gamma^*)/\gamma^T & O \\ \partial S_{\text{eff}}(\gamma^*, \xi_0)/\gamma^T & \partial S_{\text{eff}}(\gamma^*, \xi_0)/\xi^T \end{bmatrix}.
\]

Let the \((i, j)\) block of \( \mathcal{I} \) be \( \mathcal{I}_{ij} \). Then,

\[
\mathcal{I}^{-1} = \begin{bmatrix} \mathcal{I}_{11}^{-1} & O \\ -\mathcal{I}_{21} \mathcal{I}_{11}^{-1} & \mathcal{I}_{22}^{-1} \end{bmatrix}.
\]
Here, it follows that $I_{21} = O$ because
\[ E \left\{ 1 - \frac{R}{\pi(\phi_0)} \right\} \frac{\partial g^*(\gamma^*, \xi_0)}{\partial \gamma} = O \]
and
\[ E \left\{ 1 - \frac{R}{\pi(\phi_0)} \right\} \frac{\partial U^*(\gamma^*, \xi_0)}{\partial \gamma} = 0^T. \]

Therefore, we have,
\[ \mathbb{I}^{-1} = \begin{bmatrix} I_{11}^{-1} & O \\ O & I_{22}^{-1} \end{bmatrix}. \]

By applying exactly the same arguments for $I_{22}^{-1}$ used for $\mathbb{I}^{-1}$, we got the asymptotic variance of $\hat{\theta}$ as given in (11).

**Proof of Theorem 5.2.** Consistency and asymptotic normality of our proposed estimator are similar to proving Lemma 4.1 of Morikawa et al. [26]. We herein show our estimator attains the semiparametric lower bound derived in Lemma 4.1. Let $f_1(x)$ be the conditional distribution of $[x \mid r = 1]$. From the same arguments that were used to prove Lemma A.1 in Morikawa et al. [26], it can be shown that the estimating equation in Theorem 5.2, $\hat{S}_{\text{eff}}(\phi, \theta) = \{\hat{S}_1(\phi)^T, \hat{S}_2(\phi, \theta)\}^T$ is expanded as
\[
\hat{S}_1(\phi) = n^{-1} \sum_{i=1}^{n} \left\{ 1 - \frac{r_i}{\pi(\phi; z_i)} \right\} g^*(\phi; x_i) + r_i G(z_i; \phi) + o_p(n^{-1/2})
\]
\[
\hat{S}_2(\phi, \theta) = n^{-1} \sum_{i=1}^{n} \left[ \frac{r_i}{\pi(\phi; z_i)} U(\theta; z_i) + \left\{ 1 - \frac{r_i}{\pi(\phi; z_i)} \right\} U^*(\theta; \phi; x_i) + r_i H(\theta, \phi; z_i) \right] + o_p(n^{-1/2}),
\]
where $G(\phi; z_i) = G_1(\phi; x_i) G_2(\phi; z_i)$, $H(\theta, \phi; z_i) = G_1(\phi; x_i) H_2(\theta, \phi; z_i)$, and
\[
G_1(\phi; x_i) = 1 - E\left\{ \frac{\pi(\phi; Z)}{\pi(\phi; Z) \mid x_i} \right\} x_i,
\]
\[
G_2(\phi; z_i) = \frac{E_1\{\pi^{-1}(\phi; Z) O(\phi; Z) \mid x_i\} P(R = 1 | x_i)}{E_1\{\pi^{-1}(\phi; Z) O(\phi; Z) \mid x_i\} P(R = 1 | x_i)}.
\]
\[
H_2(\theta, \phi; z_i) = \frac{E_1\{\pi^{-1}(\phi; Z) O(\phi; Z) \mid x_i\} P(R = 1 | x_i)}{E_1\{\pi^{-1}(\phi; Z) O(\phi; Z) \mid x_i\} P(R = 1 | x_i)}.
\]

Therefore, the asymptotic variance may increase due to the additional terms $rG(\phi)$ and $rH(\phi)$, but this solution also attains the lower bound. At first, we focus on the estimator for $\phi$. Once we get an unbiased estimating equation $\sum_{i=1}^{n} \varphi(z_i; \phi) = 0$, the asymptotic variance can be given as $\text{Var}\{\hat{\phi}(\phi_0)\}^{-1} \varphi(\phi_0)\}$, where $\varphi(\phi_0) = \partial \varphi(\phi_0)/\partial \phi^T$. Thus, for the proving purpose, it suffices to show that $G(\phi_0) = 0$ and $E(RG(\phi_0)) = O$. The former equation is trivial, so we only need to work on the latter equation, which can be written as $E(RG(\phi_0)) = E(rG_1(\phi_0) G_2(\phi_0)) + E(rG_2(\phi_0) G_1(\phi_0))$. The first term is zero from $G_1(\phi_0) =
0. Also, the second term is $E(RG_2(\phi_0)\tilde{G}_1(\phi_0)) = E\{E(RG_2(\phi_0) \mid X)\tilde{G}_1(\phi_0)\} = O$. Hence, the last equation holds by the definition of $g^*(\phi; x)$. Therefore, $rG(\phi)$ has no effect on the asymptotic variance and our estimator also attains the semiparametric efficiency bound. The same conclusion can be made when estimating $\theta$. 

\[ \Box \]

**Appendix C: Comparison with Rotnitzky and Robins (1997)’s estimator**

In Rotnitzky & Robins [31], the semiparametric efficiency bound for NMAR data was derived in more general settings in Proposition A1 and A2, and an adaptive estimator for regression coefficients was proposed. However, to attain the efficiency bound, the estimator requires many working models to be correctly specified, and it would be practically impossible to correctly specify all of the models. For example, for the case of nonignorable nonresponse, seven working models, equations (32) to (38) in Rotnitzky & Robins [31], have to be specified.

In particular, if $\theta = E(Y)$ is our parameter of interest, three working models are required:

\[ E_1\{\pi^{-1}(Z; \phi_0)O(Z; \phi_0) \mid x\} =: \nu_1(x; \zeta_1), \quad (C.1) \]
\[ E_1\{\pi^{-1}(Z; \phi_0)O(Z; \phi_0)s_0(Z; \phi_0) \mid x\} =: \nu_2(x; \zeta_2), \quad (C.2) \]
\[ E_1\{Y \pi^{-1}(Z; \phi_0)O(Z; \phi_0) \mid x\} =: \nu_3(x; \zeta_3). \quad (C.3) \]

Note that (C.2) is a multi-dimensional function. For example, in the same setup as §6, i.e., $\logit\{\pi(x, y; \phi)\} = \phi_{0x} + \phi_{1x}x_1 + \phi_{2y}y$, where $x = (x_1, x_2)$, (C.2) can be written as

\[ E_1\{O(Z; \phi_0) \mid x\} =: \nu_4(x; \zeta_4), \quad (C.4) \]
\[ E_1\{x_1O(Z; \phi_0) \mid x\} =: \nu_5(x; \zeta_5), \quad (C.5) \]
\[ E_1\{YO(Z; \phi_0) \mid x\} =: \nu_6(x; \zeta_6), \quad (C.6) \]

where $\nu_2(x; \zeta_2) = \{\nu_4(x; \zeta_4), \nu_5(x; \zeta_5), \nu_6(x; \zeta_6)\}^\top$.

Then an adaptive estimator of $\phi$ and $\theta$ can be obtained from the following four steps:

(i). Find a consistent estimator $\hat{\phi}$ of $\phi_0$ by e.g. Chang & Kott [3]'s method;

(ii). Estimate $\zeta_k (k = 1, 2, 3)$ in (C.1)-(C.3) by the least square method with the estimated $\hat{\phi}$;

(iii). Let $\hat{\phi}$ be a solution to

\[ \sum_{i=1}^{n} \left\{1 - \frac{r_i}{\pi(z_i; \hat{\phi})} \right\} \frac{\nu_2(x_i; \hat{\zeta_2})}{\nu_1(x_i; \hat{\zeta_1})} = 0. \]

(iv). Let $\hat{\theta}$ be the solution to

\[ \sum_{i=1}^{n} \left[ \frac{r_i(y_i - \hat{\theta})}{\pi(z_i; \hat{\phi})} + \left\{1 - \frac{r_i}{\pi(z_i; \hat{\phi})} \right\} \frac{\nu_3(x_i; \hat{\zeta_3})}{\nu_1(x_i; \hat{\zeta_1})} \right] = 0. \]
Therefore, their adaptive estimator is similar to the two-step estimator in GMM. However, as shown in section 6, it may be practically difficult to find a valid consistent estimator of $\phi$ for NMAR data. Also, giving reasonable parametric models for (C.1)-(C.3) are challenging because the left-hand side of them are non-linear functions.

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