THE FREE BANACH LATTICE GENERATED BY A BANACH SPACE

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Abstract. The free Banach lattice over a Banach space is introduced and analyzed. This generalizes the concept of free Banach lattice over a set of generators, and allows us to study the Nakano property and the density character of non-degenerate intervals on these spaces, answering some recent questions of B. de Pagter and A.W. Wickstead. Moreover, an example of a Banach lattice which is weakly compactly generated as a lattice but not as a Banach space is exhibited, thus answering a question of J. Diestel.

1. Introduction

The purpose of this paper is to introduce the free Banach lattice generated by a Banach space and investigate its properties. The free Banach lattice generated by a set \( A \) with no extra structure, which is denoted by \( \text{FBL}(A) \), has been recently introduced and analyzed by B. de Pagter and A.W. Wickstead in [6]. Namely, \( \text{FBL}(A) \) is a Banach lattice together with a bounded map \( u : A \to \text{FBL}(A) \) having the following universal property: for every Banach lattice \( Y \) and every bounded map \( v : A \to Y \) there is a unique lattice homomorphism \( S : \text{FBL}(A) \to Y \) such that \( S \circ u = v \) and \( \| S \| = \sup \{ \| v(a) \| : a \in A \} \). Our aim here is to provide an analogous construction replacing the set \( A \) by a Banach space \( E \), in a way that the resulting free Banach lattice behaves well with respect to the Banach space structure of \( E \).

In the absence of topology, the free vector lattice generated by a set \( A \) was previously considered in [4, 5] and can be characterized as certain sublattice of \( \mathbb{R}^A \). Constructing the free Banach lattice \( \text{FBL}(A) \) becomes tantamount to finding the largest possible lattice norm that the free vector lattice over \( A \) can carry. Among other things, the existence of such a norm is proved in [6]. However, one can provide an explicit form of the norm of \( \text{FBL}(A) \) (see Corollary 2.8).

Loosely speaking, the free Banach lattice \( \text{FBL}[E] \) generated by a Banach space \( E \) is a Banach lattice which contains a subspace linearly isometric to \( E \) in a way that its elements work as lattice-free generators. In other words, the subspace of generators has two properties: first, the generators carry no lattice relation among them (except for the linear and metric ones coming from \( E \)), and second, the sublattice spanned by these generators is dense in the free Banach lattice. To be more precise, if \( \phi_E \) stands for the canonical isometric embedding of \( E \) into \( \text{FBL}[E] \), then the universal property of \( \text{FBL}[E] \) reads as follows: for every Banach lattice \( X \) and every operator \( T : E \to X \) there exists a unique lattice homomorphism

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\( \hat{T} : FBL[E] \to X \) such that \( \| \hat{T} \| = \| T \| \) and \( \hat{T} \circ \phi_E = T \), i.e. the following diagram commutes:

\[
\begin{array}{ccc}
E & \xrightarrow{T} & X \\
\phi_E \downarrow & & \downarrow \\
FBL[E] & \xrightarrow{\hat{T}} & X
\end{array}
\]

Section 2 is devoted to constructing the free Banach lattice \( FBL[E] \) generated by a Banach space \( E \). The construction will be achieved by defining a norm on a sublattice of \( \mathbb{R}^{E^*} \). This explicit description of the norm in the free Banach lattice is a helpful tool to tackle some questions that were raised in [6]. It should come as no great surprise that as \( \ell_1(A) \) is the free Banach space over the set \( A \), then \( FBL(A) = FBL[\ell_1(A)] \) (see Corollary 2.8). In particular, the free Banach lattice generated by a Banach space can be thought of as a generalization of the free Banach lattices of the form \( FBL(A) \).

In Sections 3 and 4 we discuss further properties of the free Banach lattice generated by a Banach space. In [6, Theorem 8.3] the authors show that, given any infinite set \( A \), the smallest cardinal \( a \) such that every order interval in \( FBL(A) \) has density character at most \( a \) is \( |A| \) (the cardinality of \( A \)). They ask whether all non-degenerate order intervals in \( FBL(A) \) must have the same density character (that would necessarily be equal to \( |A| \)). We will see that this is indeed the case and, more generally, that for any Banach space \( E \), every non-degenerate order interval in \( FBL[E] \) has the same density character as \( E \) (Theorem 3.2). Another intriguing question raised in [6] is whether the norm of a free Banach lattice of the form \( FBL(A) \) must be Fatou, or even Nakano. We will show that this is indeed the case (Theorem 4.11), while this property is not shared by all free Banach lattices generated by a Banach space (Theorem 4.13).

Finally, in Section 5 we revisit a question that J. Diestel raised during the conference “Integration, Vector Measures and Related Topics IV” held in La Manga del Mar Menor, Spain, 2011. A Banach lattice \( X \) is said to be \( \text{lattice weakly compactly generated} \) (LWCG for short) if there is a weakly compact set \( L \subseteq X \) such that the sublattice generated by \( L \) is dense in \( X \). This is formally weaker than being weakly compactly generated (WCG for short) as a Banach space, which means that there is a weakly compact set \( K \subseteq X \) such that \( X = \text{span}(K) \).

**Problem 1.1** (Diestel). Is every LWCG Banach lattice WCG?

This and related questions have been recently investigated in [3], where Problem 1.1 is solved affirmatively for Banach lattices which are order continuous or have weakly sequentially continuous lattice operations. Here we will provide a negative answer to Diestel’s question by showing that the free Banach lattice \( FBL[\ell_2(\Gamma)] \) is LWCG but not WCG as long as \( \Gamma \) is uncountable (Corollary 5.5).

**Terminology.** We only consider linear spaces over the real field. Given a Banach space \( E \), its norm is denoted by \( \| \cdot \|_E \) or simply \( \| \cdot \| \) if no confusion arises. The closed unit ball and the unit sphere of \( E \) are denoted by \( B_E \) and \( S_E \), respectively. The linear subspace generated by a set \( S \subseteq E \) is denoted by \( \text{span}(S) \) and its closure is denoted by \( \overline{\text{span}}(S) \). The symbol \( E^* \) stands for the (topological) dual of \( E \). Given a Banach lattice \( X \), we write \( X_+ = \{ x \in X : x \geq 0 \} \). By an operator between Banach spaces we mean a linear continuous map.
2. A description of the free Banach lattice

Throughout this section $E$ is a Banach space. Our first aim is to show that the free Banach lattice generated by $E$ exists and provide an explicit description (Theorem 2.4).

We denote by $H[E]$ the linear subspace of $\mathbb{R}^{E^*}$ consisting of all positively homogeneous functions $f : E^* \to \mathbb{R}$. For any $f \in H[E]$ we define

$$\|f\|_{FBL[E]} := \sup \left\{ \sum_{k=1}^{n} |f(x_k^*)| : n \in \mathbb{N}, x_1^*, \ldots, x_n^* \in E^*, \sup_{x \in B_E} \sum_{k=1}^{n} |x_k^*(x)| \leq 1 \right\}.$$  

**Remark 2.1.** If $f \in H[E]$, then

$$\|f\|_\infty := \sup \left\{ |f(x^*)| : x^* \in B_{E^*} \right\} \leq \|f\|_{FBL[E]}.$$

It is routine to check that $H_0[E] := \{ f \in H[E] : \|f\|_{FBL[E]} < \infty \}$ is a Banach lattice when equipped with the norm $\|\cdot\|_{FBL[E]}$ and the pointwise lattice operations.

**Definition 2.2.** Given any $x \in E$, let $\delta_x \in H_0[E]$ be defined by

$$\delta_x(x^*) := x^*(x) \quad \text{for all } x^* \in E^*.$$  

We define $FBL[E]$ to be the closed sublattice of $H_0[E]$ generated by $\{\delta_x : x \in E\}$.

The following lemma is straightforward.

**Lemma 2.3.** The mapping $\phi_E : E \to FBL[E]$ given by $\phi_E(x) := \delta_x$ defines a linear isometry between $E$ and its image in $FBL[E]$.

**Theorem 2.4.** Let $X$ be a Banach lattice and $T : E \to X$ an operator. There is a unique lattice homomorphism $\tilde{T} : FBL[E] \to X$ extending $T$ (in the sense that $\tilde{T} \circ \phi_E = T$). Moreover, $\|\tilde{T}\| = \|T\|$.

**Proof.** Recall that the free vector lattice over the set $E$, denoted by $FVL(E)$, is the vector sublattice of $\mathbb{R}^{E^*}$ generated by the family $\{\eta_x : x \in E\}$, where

$$\eta_x(f) := f(x) \quad \text{for all } f \in \mathbb{R}^E.$$  

(cf. [6, Theorem 3.6]). Thus, the mapping $\phi_E : E \to FBL[E]$ of Lemma 2.3 induces a lattice homomorphism $\varphi : FVL(E) \to FBL[E]$ such that $\varphi(\eta_x) = \delta_x$ for all $x \in E$. In particular, $\varphi$ has dense range. The universal property of $FVL(E)$ can be used again to obtain a lattice homomorphism $\tilde{T} : FVL(E) \to X$ such that $\tilde{T}(\eta_x) = T(x)$ for all $x \in E$.

**Claim.** For every $f \in FVL(E)$ we have

$$\|\tilde{T}(f)\|_X \leq \|T\| \|\varphi(f)\|_{FBL[E]}.$$  

Once the claim is proved the proof of the theorem finishes as follows. Inequality (2.1) and the density of $\varphi(FVL(E))$ in $FBL[E]$ allow us to define an operator $\tilde{T} : FBL[E] \to X$ with $\|\tilde{T}\| \leq \|T\|$ such that $\tilde{T} \circ \varphi = \tilde{T}$. Since $\varphi$ and $\tilde{T}$ are lattice homomorphisms, so is $\tilde{T}$. Clearly, $\tilde{T} \circ \phi_E = T$. Moreover, we have $\|\tilde{T}\| = \|T\|$, because $\|T(x)\| = \|T(\delta_x)\| \leq \|\tilde{T}\| \|\delta_x\|_{FBL[E]} = \|\tilde{T}\| \|x\|$ for every $x \in E$. For the uniqueness of $\tilde{T}$, bear in mind that any lattice homomorphism from $FBL[E]$ to a Banach lattice is uniquely determined by its values in $\{\delta_x : x \in E\}$.

**Proof of Claim.** The case $T = 0$ is trivial, so we assume that $T \neq 0$. Fix $f \in FVL(E)$. Actually, $f$ belongs to the sublattice of $FVL(E)$ generated by
\{\eta_{x_1}, \ldots, \eta_{x_n}\} for some finite set \{x_1, \ldots, x_n\} \subseteq E. By [2] Ex. 8, p. 204, we can write
\[
f = \sum_{i=1}^{m} f_i - \sum_{j=1}^{p} g_j
\]
for some \(f_1, \ldots, f_m, g_1, \ldots, g_p\) in \(\text{span}(\{\eta_{x_1}, \ldots, \eta_{x_n}\})\). Write \(f_i = \sum_{t=1}^{n} \lambda_{i}^{t} \eta_{x_t}\) and \(g_j = \sum_{l=1}^{n} \mu_{j}^{l} \eta_{x_l}\) for some \(\lambda_{i}^{t}, \mu_{j}^{l} \in \mathbb{R}\). Since \(\hat{T}\) is a lattice homomorphism, we have
\[
\hat{T}(f) = \sum_{i=1}^{m} u_i - \sum_{j=1}^{p} v_j
\]
where \(u_i := \sum_{t=1}^{n} \lambda_{i}^{t} T(x_t)\) and \(v_j := \sum_{l=1}^{n} \mu_{j}^{l} T(x_l)\).

Since \(\hat{T}\) and \(\varphi\) are lattice homomorphisms, it suffices to check (2.1) in the particular case that \(f \geq 0\). In this case, (2.1) is equivalent to the fact that \(y^*(\hat{T}(f)) \leq ||T|| \varphi(f)\) for every \(y^* \in B_{X^*} \cap (X^*)_+\) (bear in mind that \(\hat{T}(f) \geq 0\)). Fix \(y^* \in B_{X^*} \cap (X^*)_+\). Take an arbitrary decomposition \(y^* = \sum_{k=1}^{m} y_{k}^*\) where \(y_{1}^{*}, \ldots, y_{m}^{*} \in (X^*)_+\). Let us define \(x_{k}^{*} := \|T\|^{-1} T^*(y_{k}^*) \in E^*\) for every \(k \in \{1, \ldots, m\}\), which satisfy
\[
\sup_{x \in B_E} \sup_{k=1}^{m} |x_{k}^*(x)| \leq \sup_{x \in B_E} \sup_{k=1}^{m} y_{k}^* \left(\frac{1}{\|T\|} |T(x)|\right) = \sup_{x \in B_E} y^* \left(\frac{1}{\|T\|} |T(x)|\right) \leq \|y^*\| \leq 1.
\]
Hence,
\[
\|\varphi(f)\|_{FBL[E]} \geq \sum_{k=1}^{m} \varphi(f)(x_{k}^{*}) = \sum_{k=1}^{m} \left(\frac{m}{\|T\|} \varphi(f_i)(x_{k}^{*}) - p \varphi(g_j)(x_{k}^{*})\right)
\]
\[
= \frac{1}{\|T\|} \sum_{k=1}^{m} \left(\frac{m}{\|T\|} \varphi(f_i)(u_i) - p \varphi(g_j)(v_j)\right)
\]
\[
\geq \frac{1}{\|T\|} \sum_{k=1}^{m} \left(y_{k}^*(u_k) - y_{k}^*(v_j)\right)
\]
\[
= \frac{1}{\|T\|} \left(\sum_{k=1}^{m} y_{k}^*(u_k) - y_{k}^*(v_j)\right).
\]
By the classical Riesz-Kantorovich formulas (cf. [2] Theorem 1.18) we have
\[
y^* \left(\sum_{k=1}^{m} u_k\right) = \sup \left\{ \sum_{k=1}^{m} y_{k}^*(u_k) : y_{k}^* \in (X^*)_+, \ y^* = \sum_{k=1}^{m} y_{k}^* \right\}.
\]
Hence, by taking supremum over all such decompositions of \(y^*\), the above inequalities yield
\[
y^* \left(\sum_{i=1}^{m} u_i - \sum_{j=1}^{p} v_j\right) \leq ||T|| \varphi(f)\|_{FBL[E]}.
\]
as desired. The proof is complete. \(\square\)

**Corollary 2.5.** If \(E\) is a Banach lattice, then it is the range of a lattice projection \(P : FBL[E] \to E\).

**Proof.** Apply Theorem 2.3 to the identity \(T : E \to E\) to get \(P := \hat{T}\). \(\square\)
Corollary 2.6. A functional \( \varphi \in FBL[E]^* \) is a lattice homomorphism if and only if there is \( x^* \in E^* \) such that \( \varphi(f) = f(x^*) \) for all \( f \in FBL[E] \).

Proof. Given any \( x^* \in E^* \), the evaluation functional

\[
\varphi_{x^*} : FBL[E] \to \mathbb{R}, \quad \varphi_{x^*}(f) := f(x^*),
\]

is obviously a lattice homomorphism. Conversely, suppose \( \varphi \in FBL[E]^* \) is a lattice homomorphism. Then \( x^* := \varphi \circ \phi_E \) belongs to \( E^* \). Since \( \varphi \) and \( \varphi_{x^*} \) are lattice homomorphisms such that \( \varphi \circ \phi_E = \varphi_{x^*} \circ \phi_E \), the uniqueness part of Theorem 2.4 yields that \( \varphi = \varphi_{x^*} \). \( \square \)

In the spirit of [6] Corollary 4.10], we have the following:

Corollary 2.7. Let \( F \subseteq E \) be a closed subspace which is complemented by a contractive projection. Then:

(i) \( FBL[F] \) is isometrically order isomorphic to a closed sublattice of \( FBL[E] \).

(ii) \( FBL[F]^* \) is isometrically order isomorphic to a \( \omega^* \)-closed band of \( FBL[E]^* \).

Proof. Let \( P : E \to F \) be a contractive projection. Consider

\[
T := \phi_F \circ P : E \to FBL[F]
\]

and let \( \hat{T} : FBL[E] \to FBL[F] \) be the unique lattice homomorphism extending \( T \), which satisfies \( \|\hat{T}\| = \|T\| = \|P\| = 1 \) (Theorem 2.4). Let \( i : F \to E \) be the canonical inclusion, consider

\[
S := \phi_E \circ i : F \to FBL[E]
\]

and let \( \hat{S} : FBL[F] \to FBL[E] \) be the unique lattice homomorphism extending \( S \), which also satisfies \( \|\hat{S}\| = \|S\| = \|i\| = 1 \) (Theorem 2.4). For every \( x \in F \) we have

\[
\hat{S}(\delta_x) = \phi_E(i(x)) = \delta_x \quad \text{and} \quad \hat{T}(\delta_x) = \phi_F(P(x)) = \phi_F(x) = \delta_x,
\]

so \( \hat{T} \circ \hat{S} \) is the identity on \( FBL[F] \). It follows that \( \hat{S} \) is an isometric embedding (which yields statement (i)) and that \( \hat{T} \) is a contractive projection onto \( FBL[F] \). Statement (ii) now follows from [6] Proposition 4.9]. \( \square \)

In the particular case \( E = \ell_1(A) \) the space \( FBL[E] \) turns out to be the free Banach lattice generated by the set \( A \) (in the sense of [1]), as we next show.

Corollary 2.8. Let \( A \) be a non-empty set. Then:

(i) For every \( f \in H[\ell_1(A)] \) we have

\[
\|f\|_{FBL(A)} = \sup \left\{ \sum_{k=1}^n |f(x_k^n)| : n \in \mathbb{N}, x_1^*, \ldots, x_n^* \in \ell_\infty(A), \sup_{a \in A} \sum_{k=1}^n |x_k^n(a)| \leq 1 \right\}.
\]

(ii) \( FBL[\ell_1(A)] \) is the closed sublattice of \( H_0[\ell_1(A)] \) generated by the family \( \{\delta_a : a \in A\} \), where \( \{e_a : a \in A\} \) is the unit vector basis of \( \ell_1(A) \).

(iii) The pair \( (FBL[\ell_1(A)], i) \) is the free Banach lattice generated by \( A \), where \( i : A \to FBL[\ell_1(A)] \) is the bounded map given by \( i(a) := \delta_a \) for all \( a \in A \).

Proof. (i) is elementary, while (ii) is an immediate consequence of Lemma 2.3 and the fact that \( \ell_1(A) = \text{span}(\{e_a : a \in A\}) \).

In order to check (iii), fix a Banach lattice \( Y \) and a bounded map \( \kappa : A \to Y \). Consider the operator \( T : \ell_1(A) \to Y \) satisfying \( T(e_a) = \kappa(a) \) for all \( a \in A \). By
Let $\phi_{\ell_1(A)} = T$ (hence $\hat{T} \circ i = \kappa$), with
\[
\|\hat{T}\| = \|T\| = \sup\{\|\kappa(a)\| : a \in A\}.
\]
Moreover, by (ii), any lattice homomorphism $S : FBL[\ell_1(A)] \to Y$ such that $S \circ i = \kappa$ must coincide with $\hat{T}$. \qed

The following examples show that $H[\ell_1] \supseteq H_0[\ell_1] \supseteq FBL[\ell_1]$.

**Example 2.9.** Let $f \in H[\ell_1]$ be defined by $f(x) := \sup \{ \frac{x(a)}{a} : a \in \mathbb{N} \}$ for all $x \in \ell_1$. Then $\|f\|_{FBL[\ell_1]} = \infty$. Indeed, let $(e_n^a)_{n \in \mathbb{N}}$ be the unit vector basis of $c_0$. For each $n \in \mathbb{N}$ we have $\sup_{a \in \mathbb{N}} \sum_{k=1}^{n} |e_k^a(a)| \leq 1$, so Corollary 2.8(i) yields
\[
\|f\|_{FBL[\ell_1]} \geq \sum_{k=1}^{n} f(e_k^a) = \sum_{k=1}^{n} \frac{1}{k}.
\]
By taking limits when $n \to \infty$ we get $\|f\|_{FBL[\ell_1]} = \infty$. \qed

**Example 2.10.** (A function in $H_0[\ell_1] \setminus FBL[\ell_1]$.) Define a positively homogeneous function $f : \ell_\infty \to \mathbb{R}$ by
\[
f(x) := \min \left( |x(1)|, \sup \left\{ \frac{|x(a)|}{a} : a \geq 2 \right\} \right) \quad \text{for all } x \in \ell_\infty.
\]
The fact that $0 \leq f(x) \leq |x(1)|$ for all $x \in \ell_\infty$ implies that $\|f\|_{FBL[\ell_1]} \leq 1$. We will prove that $f \not\in FBL[\ell_1]$ by showing that $\|f - g\|_{FBL[\ell_1]} \geq \frac{1}{4}$ for every $g \in H_0[\ell_1]$ which belongs to the sublattice generated by $\{\delta_{e_n} : a \in \mathbb{N}\}$ (bear in mind Corollary 2.8(ii)). To this end, note first that such $g$ belongs to the sublattice generated by $\{\delta_{e_1}, \ldots, \delta_{e_n}\}$ for some $n \in \mathbb{N}$. For each $k \in \{1, \ldots, n\}$, we define $x_k^a, y_k^a \in \ell_\infty$ by declaring $x_k^a(1) = y_k^a(1) := \frac{1}{n}$, $x_k^a(n+k) := 1$, and all other coordinates of $x_k^a$ and $y_k^a$ are zero. Clearly, we have $\sum_{k=1}^{n} |x_k^a(a)| \leq 1$ and $\sum_{k=1}^{n} |y_k^a(a)| \leq 1$ for all $a \in \mathbb{N}$, so Corollary 2.8(i) yields
\[
\|f - g\|_{FBL[\ell_1]} \geq \max \left( \sum_{k=1}^{n} |f(x_k^a) - g(x_k^a)|, \sum_{k=1}^{n} |f(y_k^a) - g(y_k^a)| \right).
\]
But $g(x_k^a) = g(y_k^a)$ for every $k \in \{1, \ldots, n\}$, because $g$ belongs to the sublattice generated by $\{\delta_{e_1}, \ldots, \delta_{e_n}\}$ and $x_k^a(a) = y_k^a(a)$ for every $a \in \{1, \ldots, n\}$. Hence
\[
\|f - g\|_{FBL[\ell_1]} \geq \frac{1}{2} \sum_{k=1}^{n} |f(x_k^a) - f(y_k^a)|.
\]
By the definition of $f$, we have $f(x_k^a) = \frac{1}{n+k}$ and $f(y_k^a) = 0$ for every $k \in \{1, \ldots, n\}$, so
\[
\|f - g\|_{FBL[\ell_1]} \geq \frac{1}{2} \sum_{k=1}^{n} \frac{1}{n+k} \geq \frac{1}{4},
\]
as we wanted to show. \qed

Let $\mathcal{P}(B_{E^*})$ denote the set of regular Borel probabilities on $(B_{E^*}, w^*)$. Then $\mathcal{P}(B_{E^*})$ is a convex $w^*$-compact subset of the dual of $C(B_{E^*}, w^*)$. Note that each $\mu \in \mathcal{P}(B_{E^*})$ induces a function $f_\mu : E^* \to \mathbb{R}_+$ by
\[
f_\mu(x^*) := \int_{B_{E^*}} |x^*(\cdot)| \, d\mu \quad \text{for all } x^* \in E^*.
\]
This provides a link between \( H_0[E]_+ \) and \( \Psi(B_{E^{***}}) \), as we next explain.

**Proposition 2.11.** If \( \mu \in \Psi(B_{E^{***}}) \), then \( f_\mu \in H_0[E]_+ \) and \( \| f_\mu \|_{FBL[E]} \leq 1 \).

**Proof.** Clearly, \( f_\mu \) is positively homogeneous. Given \( x_1^*, \ldots, x_n^* \in E^* \) we have

\[
\sum_{i=1}^n f_\mu(x_i^*) = \int_{B_{E^{***}}} \sum_{i=1}^n |x_i^*(\cdot)| \, d\mu \leq \sup_{x^{**} \in B_{E^{***}}} \sum_{i=1}^n |x_i^*(x^{**})| = \sup_{x \in BE} \sum_{i=1}^n |x_i^*(x)|,
\]

where the last equality holds because \( B_E \) is \( w^* \)-dense in \( B_{E^{***}} \). It follows that \( \| f_\mu \|_{FBL[E]} \leq 1 \). \( \square \)

**Proposition 2.12.** For every \( f \in H_0[E]_+ \) there is \( \mu \in \Psi(B_{E^{***}}) \) such that

\[
f(x^*) \leq \| f \|_{FBL[E]} f_\mu(x^*) \quad \text{for all } x^* \in E^*.
\]

**Proof.** Assume without loss of generality that \( \| f \|_{FBL[E]} = 1 \). Given any finite collection \( x_1^*, \ldots, x_n^* \in E^* \), let

\[
\varphi_{x_1^*, \ldots, x_n^*} : \Psi(B_{E^{***}}) \rightarrow \mathbb{R}, \quad \mu \mapsto \sum_{i=1}^n \left( f(x_i^*) - \int_{B_{E^{***}}} |x_i^*(\cdot)| \, d\mu \right).
\]

It is clear that the function \( \varphi_{x_1^*, \ldots, x_n^*} \) is convex and \( w^* \)-continuous. Moreover, since \( \| f \|_{FBL[E]} = 1 \) and \( B_E \) is \( w^* \)-dense in \( B_{E^{***}} \), we have

\[
\sum_{i=1}^n f(x_i^*) \leq \sup_{x \in BE} \sum_{i=1}^n |x_i^*(x)| = \sup_{x^{**} \in B_{E^{***}}} \sum_{i=1}^n |x_i^*(x^{**})|.
\]

The last supremum is attained at some \( x_0^{**} \in B_{E^{***}} \) because \( (B_{E^{***}}, w^*) \) is compact and the map \( \sum_{i=1}^n |x_i^*(\cdot)| \) is \( w^* \)-continuous. This means that the value of \( \varphi_{x_1^*, \ldots, x_n^*} \) at the probability measure concentrated on \( x_0^{**} \) is less than or equal to 0.

By using that \( f \) is positively homogeneous, it is easy to check that the collection of all functions of the form \( \varphi_{x_1^*, \ldots, x_n^*} \) is a convex cone of \( \mathbb{R}^{\Psi(B_{E^{***}})} \). From Ky Fan’s lemma (see e.g. [7, 9.10]) it follows that there is \( \mu \in \Psi(B_{E^{***}}) \) such that \( \varphi_{x_1^*, \ldots, x_n^*}(\mu) \leq 0 \) for every \( n \in \mathbb{N} \) and \( x_1^*, \ldots, x_n^* \in E^* \). In particular, this yields that for every \( x^* \in E^* \) we have \( \varphi_{x^*}(\mu) \leq 0 \), that is,

\[
f(x^*) \leq \int_{B_{E^{***}}} |x^*(\cdot)| \, d\mu = f_\mu(x^*).
\]

The proof is complete. \( \square \)

### 3. Density Character of Order Intervals

Recall that the density character of a topological space \( T \), denoted by \( \text{dens}(T) \), is the least cardinality of a dense subset. Given any Banach space \( E \), we have

\[
\text{dens}(E) = \text{dens}(FBL[E])
\]

since the sublattice generated by \( \delta_x : x \in E \) is dense in \( FBL[E] \). In [6, Question 12.5], the authors ask whether every non-degenerate order interval in \( FBL(A) \) (for an arbitrary non-empty set \( A \)) has the same density character. Theorem 2.14 below shows that this is indeed the case, even in the more general setting of \( FBL[E] \). The proof requires the following lemma, which might be known.
Lemma 3.1. Let $E$ be an infinite-dimensional Banach space with $\text{dens}(E) = \kappa$. Then there exist $\delta > 0$ and a linearly independent set $S \subseteq S_E$ with $|S| = \kappa$ such that $\|x - \lambda x'\| \geq \delta$ for every distinct $x, x' \in S$ and every $\lambda \in \mathbb{R}$.

Proof. Fix $x^* \in S_{E^*}$ and $0 < \varepsilon < 1$. Define $S_0 := \{x \in S_E : x^*(x) > \varepsilon\}$. We will first check that $S_0$ is infinite. To this end, note that $W := \{x \in E : \|x\| < 1, x^*(x) > \varepsilon\}$ is open and non-empty, hence there exist $x_0 \in W$ and $r > 0$ in such a way that $x_0 + rB_E \subseteq W$. Since $T := \{x \in x_0 + rB_E : x^*(x) = x^*(x_0)\}$ is infinite (because it is a closed ball of the affine hyperplane $x_0 + \text{ker}(x^*)$) and the map $h : T \rightarrow S_0$ given by $h(x) := \|x\|^{-1}x$ is one-to-one, we conclude that $S_0$ is infinite.

We next show that $\text{dens}(S_0) = \kappa$. Indeed, take any dense set $D \subseteq S_0$. Then $D$ is infinite and so it has the same cardinality as $\tilde{D} := \{\lambda x : x \in D, \lambda \in \mathbb{Q}, \lambda > 0\}$, which is dense in $G := \{\lambda x : x \in S_0, \lambda \in \mathbb{R}, \lambda > 0\} = \{x \in E : x^*(x) > \varepsilon\|x\|\}$.

Since $G$ is open and non-empty, we have $\kappa = \text{dens}(G) \leq |\tilde{D}| = |D|$. This proves that $\text{dens}(S_0) = \kappa$.

So, there exist $0 < \delta < \varepsilon$ and a set $S_1 \subseteq S_0$ with $|S_1| = \kappa$ such that $\|x - x'\| \geq 2\delta$ for every distinct $x, x' \in S_1$. Let $S \subseteq S_1$ be a maximal linearly independent subset. We shall check that $S$ satisfies the required properties. By maximality, we have $S_1 \subseteq \text{span}(S)$ and therefore

(i) $S$ is infinite (bear in mind that the closed unit ball of a finite-dimensional Banach space is compact);

(ii) $\text{dens}(\text{span}(S)) = \kappa$.

From (i) and (ii) it follows that $|S| = \kappa$. Now, fix $x, x' \in S$ with $x \neq x'$ and take any $\lambda \in \mathbb{R}$. We next show that $\|x - \lambda x'\| \geq \delta$ by considering several cases:

- If $|1 + \lambda| < \delta$, then $\|(x - \lambda x') - (x + x')\| < \delta$ and so
  $\|x - \lambda x'\| > \|x + x'\| - \delta \geq x^*(x + x') - \delta > 2\varepsilon - \delta > \delta$.

- If $|1 - \lambda| < \delta$, then
  $\|x - \lambda x'\| \geq \|x - x'\| - \|x' - \lambda x'\| > \|x - x'\| - \delta \geq \delta$.

- If $|\lambda| \leq 1 - \delta$, then
  $\|x - \lambda x'\| \geq \|x\| - |\lambda|\|x'\| = 1 - |\lambda| \geq \delta$.

- If $|\lambda| \geq 1 + \delta$, then
  $\|x - \lambda x'\| \geq |\lambda|\|x'\| - \|x\| = |\lambda| - 1 \geq \delta$.

The proof is finished. \qed

Theorem 3.2. Let $E$ be a Banach space. Then every non-degenerate order interval in $FBL[E]$ has the same density character as $E$.

Proof. If $E$ is separable, then so is $FBL[E]$ and, in particular, every order interval in $FBL[E]$ is separable.

Let us assume now that $E$ is non-separable with $\text{dens}(E) = \kappa$. Pick $f \leq g$ in $FBL[E]$ with $f \neq g$. Rescaling, we can suppose that $f$ and $g$ belong to $B_{FBL[E]}$. There is a separable closed subspace $F \subseteq E$ such that $f$ and $g$ belong to the closed sublattice of $FBL[E]$ generated by $\{\delta_x : x \in F\}$. Bearing in mind that
This shows that \( \|E\| \) and consider the evaluation functional \( \varphi_{x^*} \in \mathcal{B}_{FBL[E]^*} \) given by \( \varphi_{x^*}(h) := h(x^*) \) for all \( h \in FBL[E] \) (see Corollary 2.6). Set \( C := 6\|\pi\|/\delta + 1 \) and fix \( s \neq t \) in \( S \).

Claim. The inequality
\[
(3.1) \quad \|x\| + |\alpha| + |\beta| \leq C\|x + \alpha u_s + \beta u_t\|
\]
holds for every \( x \in F \) and every \( \alpha, \beta \in \mathbb{R} \). Indeed, by the choice of \( S \) we have
\[
\|\pi\|\|x + \alpha u_s + \beta u_t\| \geq |\alpha s + \beta t| \geq \delta \max\{|\alpha|,|\beta|\} \geq \delta \frac{|\alpha| + |\beta|}{2}.
\]

On the other hand, since \( \|u_s\| \leq 2 \) and \( \|u_t\| \leq 2 \), we have
\[
\|x + \alpha u_s + \beta u_t\| \geq \|x\| - |\alpha u_s + \beta u_t| \geq \|x\| - 2(|\alpha| + |\beta|).
\]
Therefore
\[
C\|x + \alpha u_s + \beta u_t\| = \frac{6\|\pi\|}{\delta}\|x + \alpha u_s + \beta u_t\| + \|x + \alpha u_s + \beta u_t\|
\geq \frac{3(\|x\| + |\alpha| + |\beta|)}{\delta} \|x\| - 2(|\alpha| + |\beta|) = \|x\| + |\alpha| + |\beta|,
\]
which finishes the proof of the claim.

Note that \( u_s \) and \( u_t \) are linearly independent vectors in \( E \setminus F \). Let
\[
\xi : \text{span}(F \cup \{u_s, u_t\}) \to \mathbb{R}
\]
be the linear functional given by
\[
\xi(x + \alpha u_s + \beta u_t) := \varphi_{x^*}(\delta_x + \alpha f + \beta g) = x^*(x) + \alpha f(x^*) + \beta g(x^*) \quad \text{for all } x \in F \text{ and } \alpha, \beta \in \mathbb{R}.
\]

Bearing in mind (5.1), we get
\[
\|\varphi_{x^*}(\delta_x + \alpha f + \beta g)\| \leq ||\delta_x + \alpha f + \beta g||_{FBL[E]} \leq \|x\| + |\alpha| + |\beta| \leq C\|x + \alpha u_s + \beta u_t\|
\]
for every \( x \in F \) and \( \alpha, \beta \in \mathbb{R} \). By the Hahn-Banach theorem, \( \xi \) can be extended to an element of \( E^* \), still denoted by \( \xi \), with \( \|\xi\| \leq C \).

Now, let \( \hat{\xi} \in FBL[E]^* \) be the lattice homomorphism satisfying \( \hat{\xi} \circ \phi_E = \xi \) and \( \|\hat{\xi}\| = \|\xi\| \leq C \) (Theorem 2.4). Note that
\[
\hat{\xi}(\delta_x) = \xi(x) = \varphi_{x^*}(\delta_x) = x^*(x) \quad \text{for every } x \in F.
\]
Since \( \hat{\xi} \) and \( \varphi_{x^*} \) are lattice homomorphisms and \( f \) and \( g \) belong to the closed sublattice generated by \( \{\delta_x : x \in F\} \), it follows that \( \hat{\xi}(f) = \varphi_{x^*}(f) = f(x^*) \) and \( \hat{\xi}(g) = \varphi_{x^*}(g) = g(x^*) \). In particular, we have
\[
C\|z_s - z_t\| \geq \|\hat{\xi}((\delta_{u_s} \lor f) \lor g - (\delta_{u_t} \lor f) \lor g) = g(x^*) - f(x^*)\|
\]
This shows that \( \|z_s - z_t\| \geq C^{-1}(g(x^*) - f(x^*)) \) for every \( s \neq t \) in \( S \). The proof is finished.
4. The Nakano property

The norm of a Banach lattice $X$ is said to have the Nakano property if for every upwards directed order bounded set $\mathcal{F} \subseteq X_+$ we have

$$\sup\{\|x\| : x \in \mathcal{F}\} = \inf\{\|y\| : y \in X \text{ is an upper bound of } \mathcal{F}\}.$$  

This property was introduced in [10] (see also [11]) and is stronger than the Fatou property, which simply states that whenever an upwards directed set $\mathcal{F} \subseteq X_+$ has a supremum $y_0 \in X$, then

$$\sup\{\|x\| : x \in \mathcal{F}\} = \|y_0\|.$$  

In [3] Question 12.1, it was asked whether the norm of $FBL(A)$ has the Nakano property. We will show in Theorem 4.11 that this is the case. In fact, a stronger property holds:

**Definition 4.1.** We say that the norm of a Banach lattice $X$ has the strong Nakano property if for every upwards directed norm bounded set $\mathcal{F} \subseteq X_+$ there exists an upper bound $y_0$ of $\mathcal{F}$ in $X$ such that

$$\sup\{\|x\| : x \in \mathcal{F}\} = \|y_0\|.$$  

The supremum norm of a $C(K)$ space ($K$ being a compact Hausdorff topological space) has the strong Nakano property, because we can take $y_0$ as the constant function equal to $\sup\{\|x\|_\infty : x \in \mathcal{F}\}$. In a sense, we shall see that the free Banach lattices $FBL(A) = FBL[\ell_1(A)]$ have an analogous structure, the role of the positive constant functions being played by the elements of the form $|\delta_a|$ for $a \in A$. Our proof of Theorem 4.11 requires some preliminary lemmas.

**Lemma 4.2.** Let $E$ be a Banach space and let $\mathcal{F} \subseteq H[E]_+$ be upwards directed and pointwise bounded. Define $g : E^* \to \mathbb{R}_+$ by $g(x^*) := \sup\{f(x^*) : f \in \mathcal{F}\}$ for all $x^* \in E^*$. Then $g \in H[E]_+$ and

$$\|g\|_{FBL[E]} = \sup\{\|f\|_{FBL[E]} : f \in \mathcal{F}\}.$$

**Proof.** Clearly, $g$ is positively homogeneous and $\|g\|_{FBL[E]} \geq \|f\|_{FBL[E]}$ for every $f \in \mathcal{F}$. To prove that $\|g\|_{FBL[E]} \leq \sup\{\|f\|_{FBL[E]} : f \in \mathcal{F}\} := \alpha$ we can assume that the supremum is finite. Fix $\varepsilon > 0$. Take any $x_1^*, \ldots, x_n^* \in E^*$ such that $\sum_{k=1}^n |x_k^*(x)| \leq 1$ for every $x \in B_E$. Since $\mathcal{F}$ is upwards directed, we can find $f \in \mathcal{F}$ such that $g(x_k^*) - \frac{\varepsilon}{n} \leq f(x_k^*)$ for all $k \in \{1, \ldots, n\}$, therefore

$$\sum_{k=1}^n g(x_k^*) \leq \varepsilon + \sum_{k=1}^n f(x_k^*) \leq \varepsilon + \alpha.$$  

It follows that $\|g\|_{FBL[E]} \leq \varepsilon + \alpha$. As $\varepsilon > 0$ is arbitrary, $\|g\|_{FBL[E]} \leq \alpha$.  

**Definition 4.3.** Let $E$ be a Banach space. We say that $f \in H_0[E]_+$ is maximal if

$$\{g \in H_0[E]_+ : g \geq f, \|g\|_{FBL[E]} = \|f\|_{FBL[E]}\} = \{f\}.$$  

**Lemma 4.4.** Let $E$ be a Banach space and $f \in H_0[E]_+$. Then there exists a maximal $\tilde{f} \in H_0[E]_+$ such that $f \leq \tilde{f}$ and $\|f\|_{FBL[E]} = \|	ilde{f}\|_{FBL[E]}$.

**Proof.** The set $\mathcal{G} := \{g \in H_0[E]_+ : g \geq f, \|g\|_{FBL[E]} = \|f\|_{FBL[E]}\}$ is pointwise bounded by Remark 2.4. Note that every upwards directed subset of $\mathcal{G}$ has an upper bound in $\mathcal{G}$ (by Lemma 4.2). Thus, Zorn’s lemma ensures the existence of an element of $\mathcal{G}$ which is maximal for the pointwise ordering.  

\[\square\]
Our next aim is to identify the maximal elements in $H_0[\ell_1(A)]_+$ for an arbitrary non-empty set $A$. We shall use without explicit mention the formula to compute the norm $\| \cdot \|_{FBL[\ell_1(A)]}$ given in Corollary 2.8(i).

**Lemma 4.5.** Let $A$ be a non-empty set and let $f \in H_0[\ell_1(A)]_+$ be maximal.

(i) $f(x^*) \leq f(y^*)$ whenever $x^*, y^* \in \ell_\infty(A)$ satisfy $|x^*| \leq |y^*|$.

(ii) $\sum_{k=1}^n f(x_k^*) \leq f(\sum_{k=1}^n x_k^*)$ for every $n \in \mathbb{N}$ and $x_1^*, \ldots, x_n^* \in \ell_\infty(A)_+$.

(iii) $\|f\|_{FBL[\ell_1(A)]} = \|f\|_\infty$.

**Proof.** For any $z^* \in \ell_\infty(A)$ we write $R(z^*) := \{\lambda z^* : \lambda > 0\} \subseteq \ell_\infty(A)$.

(i): By contradiction, suppose that $f(x^*) > f(y^*)$. Define $g : \ell_\infty(A) \to \mathbb{R}_+$ by

$$
\begin{align*}
g(\lambda y^*) &:= f(\lambda x^*) \quad \text{for all } \lambda > 0, \\
g(z^*) &:= f(z^*) \quad \text{for all } z^* \in \ell_\infty(A) \setminus R(y^*).
\end{align*}
$$

It is easy to check that $g$ is positively homogeneous. Since $f(x^*) > f(y^*)$, we have $f \leq g$ and $f \neq g$. Bearing in mind that $f$ is maximal, in order to get a contradiction it suffices to check that $\|g\|_{FBL[\ell_1(A)]} = \|f\|_{FBL[\ell_1(A)]}$. To this end, take any $x_1^*, \ldots, x_n^* \in \ell_\infty(A)$ such that $\sum_{k=1}^n x_k^*(a) \leq 1$ for all $a \in A$. Let $I$ be the set of those $k \in \{1, \ldots, n\}$ such that $x_k^* \notin R(y^*)$ and let $J := \{1, \ldots, n\} \setminus I$, so that for each $k \in J$ we have $x_k^* = \lambda_k y^*$ for some $\lambda_k > 0$. Note that

$$\sum_{k \in I} |x_k^*(a)| + \sum_{k \in J} |\lambda_k x^*(a)| = \sum_{k \in I} |x_k^*(a)| + |x^*(a)| \sum_{k \in J} \lambda_k \leq \sum_{k \in I} |x_k^*(a)| + |y^*(a)| \sum_{k \in J} \lambda_k = \sum_{k=1}^n |x_k^*(a)| \leq 1 \quad \text{for all } a \in A,$$

hence

$$\sum_{k=1}^n g(x_k^*) = \sum_{k \in I} f(x_k^*) + \sum_{k \in J} f(\lambda_k x^*) \leq \|f\|_{FBL[\ell_1(A)]}.$$

This shows that $\|g\|_{FBL[\ell_1(A)]} \leq \|f\|_{FBL[\ell_1(A)]}$, a contradiction.

(ii): Set $x^* := \sum_{k=1}^n x_k^*$. By contradiction, suppose that $f(x^*) < \sum_{k=1}^n f(x_k^*)$. Define $g : \ell_\infty(A) \to \mathbb{R}_+$ by

$$
\begin{align*}
g(\lambda x^*) &:= \lambda \sum_{k=1}^n f(x_k^*) \quad \text{for all } \lambda > 0, \\
g(z^*) &:= f(z^*) \quad \text{for all } z^* \in \ell_\infty(A) \setminus R(x^*).
\end{align*}
$$

Clearly, $g$ is positively homogeneous, $f \leq g$ and $f \neq g$. Again by the maximality of $f$, to get a contradiction it suffices to show that $\|g\|_{FBL[\ell_1(A)]} = \|f\|_{FBL[\ell_1(A)]}$. Take $y_1^*, \ldots, y_m^* \in \ell_\infty(A)$ such that $\sum_{j=1}^m |y_j^*(a)| \leq 1$ for all $a \in A$. Let $I$ denote the set of all $j \in \{1, \ldots, m\}$ for which $y_j^* \notin R(x^*)$ and let $J := \{1, \ldots, m\} \setminus I$, so that for each $j \in J$ we can write $y_j^* = \lambda_j x^*$ for some $\lambda_j > 0$. Set $\mu := \sum_{j \in J} \lambda_j$. Since

$$\sum_{j \in I} |y_j^*(a)| + \sum_{k=1}^n |\mu x_k^*(a)| = \sum_{j \in I} |y_j^*(a)| + |x^*(a)| \sum_{k=1}^n \mu_k \leq \sum_{j \in I} |y_j^*(a)| + \sum_{j \in J} \lambda_j x^*(a) = \sum_{j=1}^m |y_j^*(a)| \leq 1 \quad \text{for all } a \in A,$$
we obtain
\[
\sum_{j=1}^{m} g(y_j) = \sum_{j \in I} f(y_j) + \sum_{j \in J} \lambda_j \left( \sum_{k=1}^{n} f(x_k^+) \right)
\]
\[
= \sum_{j \in I} f(y_j) + \sum_{k=1}^{n} f(\mu x_k^+) \leq \|f\|_{\mathcal{F}BL[\ell_1(A)]}.
\]

It follows that \(\|g\|_{\mathcal{F}BL[\ell_1(A)]} \leq \|f\|_{\mathcal{F}BL[\ell_1(A)]}\), which is a contradiction.

(iii): By Remark 2.1 we have \(\|f\|_{\mathcal{F}BL[\ell_1(A)]} \geq \|f\|_{\infty}\). To prove the equality, take finitely many \(x_1^*, \ldots, x_n^* \in \ell_\infty(A)\) such that \(\sum_{k=1}^{n} |x_k^*(a)| \leq 1\) for every \(a \in A\). Then \(x^* := \sum_{k=1}^{n} x_k^* \in B_{\ell_\infty(A)}\) and
\[
\sum_{k=1}^{n} f(x_k^+) \leq \sum_{k=1}^{n} f(\|x_k^*\|) \leq f(x^*) \leq \|f\|_{\infty}.
\]
This shows that \(\|f\|_{\mathcal{F}BL[\ell_1(A)]} \leq \|f\|_{\infty}\) and finishes the proof.

\[
\text{Lemma 4.6. Let } A \text{ be a non-empty set and let } \phi : \ell_\infty(A) \to \mathbb{R} \text{ be a linear functional. Define } g_\phi : \ell_\infty(A) \to \mathbb{R}_+ \text{ by }
\]
\[
g_\phi(x^*) := |\phi(\|x^*\|)| \quad \text{for all } x^* \in \ell_\infty(A).
\]

Then \(g_\phi \in H[\ell_1(A)]_+\) and
\[
\|g_\phi\|_{\mathcal{F}BL[\ell_1(A)]} = \sup \{ |\phi(x^*)| : x^* \in B_{\ell_\infty(A)} \}.
\]

\[
\text{Proof. Clearly, } g_\phi \text{ is positively homogeneous. Take any } x_1^*, \ldots, x_n^* \in \ell_\infty(A) \text{ such that } \sum_{k=1}^{n} |x_k^*(a)| \leq 1 \text{ for all } a \in A. \text{ For each } k \in \{1, \ldots, n\}, \text{ let } \varepsilon_k \in \{-1, 1\} \text{ be the sign of } \phi(|x_k^*|). \text{ Then } \sum_{k=1}^{n} \varepsilon_k x_k^* \in B_{\ell_\infty(A)} \text{ and so }
\]
\[
\sum_{k=1}^{n} g_\phi(x_k^*) = \sum_{k=1}^{n} \varepsilon_k \phi(|x_k^*|) = \phi \left( \sum_{k=1}^{n} \varepsilon_k x_k^* \right) \leq \sup \{ |\phi(x^*)| : x^* \in B_{\ell_\infty(A)} \} := \alpha.
\]
This immediately shows that \(\|g_\phi\|_{\mathcal{F}BL[\ell_1(A)]} \leq \alpha\).

For the converse, pick \(x^* \in B_{\ell_\infty(A)}\) and write it as \(x^* = (x^*)^+ - (x^*)^-\), the difference of its positive and negative parts. Since \(|(x^*)^+(a)| + |(x^*)^-(a)| = |x^*(a)| \leq 1\) for all \(a \in A\), we have
\[
|\phi(x^*)| \leq |\phi((x^*)^+)| + |\phi((x^*)^-)| = g_\phi((x^*)^+) + g_\phi((x^*)^-) \leq \|g_\phi\|_{\mathcal{F}BL[\ell_1(A)]}.
\]
This proves that \(\alpha \leq \|g_\phi\|_{\mathcal{F}BL[\ell_1(A)]}\).

\[
\text{Lemma 4.7. Let } A \text{ be a non-empty set and let } f \in H_0[\ell_1(A)]_+ \text{ be maximal. Then there exists } \phi \in \ell_\infty(A)^* \text{ such that } f = g_\phi.
\]

\[
\text{Proof. The case } f = 0 \text{ being trivial, we can suppose without loss of generality that } \|f\|_{\mathcal{F}BL[\ell_1(A)]} = 1. \text{ The set }
\]
\[
C := \{ x^* \in \ell_\infty(A)_+ : f(x^*) > 1 \}
\]
is convex as a consequence of Lemma 4.5(ii). Let \(U\) be the open unit ball of \(\ell_\infty(A)\). Since \(\|f\|_{\infty} = \|f\|_{\mathcal{F}BL[\ell_1(A)]} = 1\) (Lemma 4.5(iii)), we have \(C \cap U = \emptyset\). As an application of the Hahn-Banach separation theorem (cf. [9] Proposition 2.13(ii)), there is \(\phi \in \ell_\infty(A)^*\) such that
\[
\phi(y^*) < \inf \{ \phi(x^*) : x^* \in C \} \quad \text{for all } y^* \in U.
\]
We can suppose that \( \| \phi \| = 1 \) and so \( \| g_\phi \|_{\text{FBL}[\ell_1(A)]} = 1 \) (Lemma 4.6).

We claim that \( f = g_\phi \). Indeed, since \( f \) is maximal, it suffices to prove that \( f(x^*) \leq g_\phi(x^*) \) for every \( x^* \in \ell_\infty(A) \) with \( f(x^*) > 0 \). Fix \( t > 1 \). By Lemma 4.5(i), we have \( f(|x^*|) \geq f(x^*) > 0 \) and so
\[
 f \left( \frac{t}{f(|x^*|)} |x^*| \right) = t > 1.
\]

Therefore, (4.1) yields
\[
 \phi \left( \frac{t}{f(|x^*|)} |x^*| \right) \geq \sup \{ \phi(y^*) : y^* \in U \} = \| \phi \| = 1.
\]

We conclude that \( t \phi(|x^*|) \geq f(|x^*|) \) for any \( t > 1 \), so \( g_\phi(|x^*|) = \phi(|x^*|) \geq f(x^*) \).

The proof is complete. \( \square \)

Given any non-empty set \( A \), it is well-known that every \( \phi \in \ell_\infty(A)^* \) can be written in a unique way as \( \phi = \phi_0 + \phi_1 \), where

- \( \phi_0 \in \ell_1(A) \) (identified as a subspace of \( \ell_\infty(A)^* \)),
- \( \phi_1 \in \ell_\infty(A)^* \) vanishes on all finitely supported elements of \( \ell_\infty(A) \).

Moreover, \( \| \phi \| = \| \phi_0 \| + \| \phi_1 \| \).

**Lemma 4.8.** Let \( A \) be a non-empty set and \( \phi \in \ell_1(A) \). Then \( g_\phi \in \text{FBL}[\ell_1(A)] \).

**Proof.** Let \( (e_a)_{a \in A} \) be the unit vector basis of \( \ell_1(A) \). The series \( \sum_{a \in A} \phi(a)|\delta_{e_a}| \) is summable in \( \text{FBL}[\ell_1(A)] \) because \( \phi \in \ell_1(A) \) and \( \| \delta_{e_a} \|_{\text{FBL}[\ell_1(A)]} = 1 \) for every \( a \in A \). Let \( h \in \text{FBL}[\ell_1(A)] \) be its sum. By Remark 2.1 we have
\[
 h(x^*) = \sum_{a \in A} \phi(a)|x^*(a)| \quad \text{for all } x^* \in \ell_\infty(A).
\]

Therefore, \( |h(x^*)| = g_\phi(x^*) \) for all \( x^* \in \ell_\infty(A) \) and so \( g_\phi \in \text{FBL}[\ell_1(A)] \). \( \square \)

**Lemma 4.9.** Let \( A \) be a non-empty set, \( \xi : B_{\ell_\infty(A)} \to \mathbb{R} \) a \( w^* \)-continuous function and \( \phi \in \ell_\infty(A)^* \). If \( \xi \leq g_\phi \) on \( B_{\ell_\infty(A)} \), then \( \xi \leq g_{\phi_0} \) on \( B_{\ell_\infty(A)} \) as well.

**Proof.** The \( w^* \)-topology on \( B_{\ell_\infty(A)} = [-1, 1]^A \) agrees with the pointwise topology. Since the map \( x^* \mapsto |x^*| \) is \( w^* \)-\( w^* \)-continuous when restricted to \( B_{\ell_\infty(A)} \) and \( \phi_0 \) is \( w^* \)-continuous, we have that \( g_{\phi_0} \) is \( w^* \)-continuous on \( B_{\ell_\infty(A)} \). On the other hand, if \( x^* \in B_{\ell_\infty(A)} \) is finitely supported, then \( \phi_1(|x^*|) = 0 \) and, therefore, we have \( \xi(x^*) \leq g_\phi(x^*) = |\phi_0(|x^*|)| = g_{\phi_0}(x^*) \). Since the finitely supported elements of \( B_{\ell_\infty(A)} \) are \( w^* \)-dense and the functions \( \xi \) and \( g_{\phi_0} \) are \( w^* \)-continuous on \( B_{\ell_\infty(A)} \), we conclude that \( \xi \leq g_{\phi_0} \) on \( B_{\ell_\infty(A)} \). \( \square \)

**Lemma 4.10.** Let \( E \) be a Banach space. Then for every \( f \in \text{FBL}[E] \) the restriction \( f|_{B_{w^*}} \) is \( w^* \)-continuous.

**Proof.** The set \( S \) consisting of all \( f \in \text{FBL}[E] \) for which \( f|_{B_{w^*}} \) is \( w^* \)-continuous is clearly a sublattice of \( \text{FBL}[E] \) containing \( \{ \delta_x : x \in E \} \). Moreover, \( S \) is closed in \( \text{FBL}[E] \) (by Remark 2.1), so \( \text{FBL}[E] = S \). \( \square \)

We arrive at the main result of this section:

**Theorem 4.11.** The norm of \( \text{FBL}[\ell_1(A)] \) has the strong Nakano property for any non-empty set \( A \).
Proof. Let $\mathcal{F} \subseteq FBL[\ell_1(A)]_+$ be an upwards directed family such that
\[
\sup\{\|f\|_{FBL[\ell_1(A)]} : f \in \mathcal{F}\} = 1.
\]
We are going to show that $\mathcal{F}$ has an upper bound of norm 1.

Note that $\mathcal{F}$ is pointwise bounded (Remark 2.1) and let $h : \ell_\infty(A) \to \mathbb{R}_+$ be defined as $h(x^*) := \sup\{f(x^*) : f \in \mathcal{F}\}$ for all $x^* \in \ell_\infty(A)$. Lemma 4.2 ensures that $h \in H_0[\ell_1(A)]_+$ and $\|h\|_{FBL[\ell_1(A)]} = 1$. Now let $g \in H_0[\ell_1(A)]_+$ be maximal such that $g \geq h$ and $\|g\|_{FBL[\ell_1(A)]} = 1$ (apply Lemma 4.4). Then $g = g_\phi$ for some $\phi \in \ell_\infty(A)^*$ with $\|\phi\| = 1$ (combine Lemmas 4.7 and 4.9).

Given any $f \in \mathcal{F} \subseteq FBL[\ell_1(A)]$, we have $f \leq g_\phi$ and the restriction $f|_{B_{\ell_\infty(A)}}$ is $w^*$-continuous (Lemma 4.10), hence Lemma 4.9 yields $f(x^*) \leq g_\phi(x^*)$ for every $x^* \in \ell_\infty(A)$ (bear in mind that both $f$ and $g_\phi$ are positively homogeneous). Since $g_{\phi_0} \in FBL[\ell_1(A)]$ (Lemma 4.8 and $1 = \|\phi_0\| \geq \|\phi_0\| = \|g_{\phi_0}\|_{FBL[\ell_1(A)]}$ (Lemma 4.8), it turns out that $g_{\phi_0}$ is the upper bound of $\mathcal{F}$ in $FBL[\ell_1(A)]$ that we were looking for. The proof is finished. \hfill \Box

It is natural to wonder whether the norm of $FBL[E]$ also has the (strong) Nakano property for an arbitrary Banach space $E$. We will next show that the norm of $FBL[L_1]$ is not even Fatou, where $L_1$ denotes the space $L_1([0,1],\mu)$ and $\mu$ is the Lebesgue measure.

The following auxiliary lemma belongs to the folklore and we include its proof for the sake of completeness.

**Lemma 4.12.** For each $n \in \mathbb{N}$, define $f_n : L_1 \to \mathbb{R}_+$ by
\[
f_n(h) := \sum_{j=1}^{2^n} \int_{I_{n,j}} h \, d\mu \quad \text{for every } h \in L_1,
\]
where $I_{n,j} := \left[ \frac{j-1}{2^n}, \frac{j}{2^n} \right]$ for all $j \in \{1,\ldots,2^n\}$. The following properties hold:

(i) $f_n(h) \leq f_{n+1}(h)$ for every $n \in \mathbb{N}$ and $h \in L_1$.
(ii) $\lim_{n \to \infty} f_n(h) = \int_0^1 |h| \, d\mu$ for every $h \in L_1$.

**Proof.** Part (i) is straightforward. To prove (ii), note first that for every $h \in L_1$ we have $f_n(h) \leq \int_0^1 |h| \, d\mu$ for all $n \in \mathbb{N}$, so the increasing sequence $(f_n(h))$ is bounded and converges to its supremum. Let us denote
\[
\phi(h) := \sup_{n \in \mathbb{N}} f_n(h) = \lim_{n \to \infty} f_n(h) \quad \text{for } h \in L_1.
\]
We want to show that $\phi(h) = \int_0^1 |h| \, d\mu$ for every $h \in L_1$.

Observe first that this equality is clear whenever $h$ is of the form
\[
\sum_{j=1}^{2^n} a_j \chi_{I_{n,j}}
\]
for some $n \in \mathbb{N}$ and some $a_1,\ldots,a_{2^n} \in \mathbb{R}$. To prove the equality for arbitrary $h \in L_1$ it is enough to show that $\phi : L_1 \to \mathbb{R}$ is $\|\cdot\|_1$-continuous (because simple functions as in (4.2) are dense in $L_1$). In fact, we will check that
\[
|\phi(h) - \phi(h')| \leq \|h - h'\|_1 \quad \text{for every } h,h' \in L_1.
\]
Indeed, given any \( n \in \mathbb{N} \), we have
\[
|f_n(h) - f_n(h')| = \left| \sum_{j=1}^{2^n} \left( \left| \int_{I_{n,j}} h \, d\mu \right| - \left| \int_{I_{n,j}} h' \, d\mu \right| \right) \right|
\leq \sum_{j=1}^{2^n} \left| \int_{I_{n,j}} h \, d\mu - \int_{I_{n,j}} h' \, d\mu \right|
\leq 2^n \int_{I_{n,j}} |h - h'| \, d\mu = ||h - h'||_1.
\]

As \( n \in \mathbb{N} \) is arbitrary, (4.3) holds and the proof is finished. \( \square \)

**Theorem 4.13.** The norm of \( FBL[L_1] \) fails the Fatou property.

**Proof.** We use the notation of Lemma 4.12. By considering the natural inclusion of \( L_\infty = L_1^* \) in \( L_1 \), each \( f_n \) can be seen as an element of \( H[L_1]^+ \). In fact, we have \( f_n \in FBL[L_1]^+ \) and \( ||f_n||_{FBL[L_1]} = 1 \), because
\[
f_n = \sum_{j=1}^{2^n} |\delta_{\chi_{I_{n,j}}}| \quad \text{and} \quad 1 = f_n(\chi_{[0,1]}) \leq ||f_n||_\infty \leq ||f_n||_{FBL[L_1]} \leq \sum_{j=1}^{2^n} ||\delta_{\chi_{I_{n,j}}}||_{FBL[L_1]} = \sum_{j=1}^{2^n} ||\chi_{I_{n,j}}||_1 = 1.
\]

Fix any \( g \in FBL[L_1]^+ \) with \( ||g||_{FBL[L_1]} > 1 \), and let \( \tilde{f}_n := g \wedge f_n \in FBL[L_1]^+ \) for all \( n \in \mathbb{N} \). The sequence \( (f_n) \) is increasing (by Lemma 4.12), bounded above by \( g \) and
\[
\sup_{n \in \mathbb{N}} ||\tilde{f}_n||_{FBL[L_1]} \leq \sup_{n \in \mathbb{N}} ||f_n||_{FBL[L_1]} = 1.
\]

We claim that \( \sup_{n \in \mathbb{N}} \tilde{f}_n = g \) in \( FBL[L_1] \). Indeed, fix \( h \in L_\infty \) and let
\[
K := ||g||_{FBL[L_1]}||h||_\infty + ||h||_1 + 1.
\]

Define
\[
h_j := h + Kr_j \in L_\infty \quad \text{for all} \quad j \in \mathbb{N}
\]
(\( r_j \) denotes the \( j \)-th Rademacher function) and observe that the sequence \( (h_j) \) is \( w^* \)-convergent to \( h \) (as \( (r_j) \) is \( w^* \)-null in \( L_\infty \)). Since \( g : L_\infty \to \mathbb{R}_+ \) is \( w^* \)-continuous on bounded sets (Lemma 4.10), we have \( g(h_j) \to g(h) \) as \( j \to \infty \), so in particular there is \( j_0 \in \mathbb{N} \) such that for every \( j \geq j_0 \) we have
\[
(4.4) \quad g(h_j) \leq g(h) + 1 \leq ||g||_{FBL[L_1]}||h||_\infty + 1 = K - ||h||_1 \leq \int_0^1 |h_j| \, d\mu.
\]
(the second inequality being a consequence of Remark 2.1).
Now take any \( \varphi \in \text{FBL}[L_1] \) satisfying \( \varphi \geq \tilde{f}_n \) for every \( n \in \mathbb{N} \). By Lemma 4.12 for every \( j \geq j_0 \) we have
\[
\varphi(h_j) \geq \sup_{n \in \mathbb{N}} \tilde{f}_n(h_j) = g(h_j) \land \int_0^1 |h_j| d\mu \overset{4.13}{=} g(h_j).
\]
Since \( \varphi \) and \( g \) are \( w^* \)-continuous on bounded sets (Lemma 4.10), it follows that \( \varphi(h) \geq g(h) \).

Therefore, we have \( \sup_{n \in \mathbb{N}} \tilde{f}_n = g \) in \( \text{FBL}[L_1] \). Since \( \sup_{n \in \mathbb{N}} \| \tilde{f}_n \|_{\text{FBL}[L_1]} \leq 1 \) and \( \| g \|_{\text{FBL}[L_1]} > 1 \), the Fatou property cannot hold. □

5. AN APPLICATION TO WEAKLY COMPACTLY GENERATED BANACH LATTICES

The purpose of this section is to give a negative answer to Diestel’s question mentioned in the introduction (Problem 1.1). Interestingly enough, that question can be equivalently rephrased by asking:

Problem 5.1. If \( E \) is a WCG Banach space, is \( \text{FBL}[E] \) WCG as well?

The equivalence of Problems 1.1 and 5.1 follows at once from the following remarks:

Remark 5.2. If \( E \) is a WCG Banach space, then \( \text{FBL}[E] \) is LWCG.

Proof. Let \( K \subseteq E \) be a weakly compact set such that \( E = \overline{\text{span}}(K) \). Then the sublattice generated by the weakly compact set \( \phi_E(K) = \{ \delta_x : x \in K \} \) is dense in \( \text{FBL}[E] \).

Remark 5.3. Let \( X \) be an LWCG Banach lattice. Then there exist a WCG Banach space \( E \) and an operator \( \hat{T} : \text{FBL}[E] \to X \) with dense range.

Proof. Let \( K \subseteq X \) be a weakly compact set such that the sublattice generated by \( K \) is dense in \( X \). Define \( E := \overline{\text{span}}(K) \subseteq X \). Then there is a lattice homomorphism \( \hat{T} : \text{FBL}[E] \to X \) such that \( \hat{T} \circ \phi_E \) is the identity on \( E \) (Theorem 2.4). The sublattice generated by \( E \) in \( X \) is contained in the range of \( \hat{T} \), hence the range of \( \hat{T} \) is dense in \( X \).

So we might ask whether \( \text{FBL}[\ell_p(\Gamma)] \) or \( \text{FBL}[\ell_p(\Gamma)] \) \( (1 < p < \infty) \) are WCG for uncountable \( \Gamma \). We will next see that \( \text{FBL}[\ell_p(\Gamma)] \) is not WCG whenever \( \Gamma \) is uncountable and \( 1 < p \leq 2 \), hence answering in the negative Diestel’s question.

Theorem 5.4. Let \( \Gamma \) be a non-empty set and \( 1 < p \leq 2 \). Then \( \text{FBL}[\ell_p(\Gamma)] \) contains a subspace isomorphic to \( \ell_1(\Gamma) \).

Proof. Let \( (e_\gamma)_{\gamma \in \Gamma} \) denote the unit vector basis of \( \ell_p(\Gamma) \). We will prove that the family \( \{ [\delta_{e_\gamma}] : \gamma \in \Gamma \} \subseteq S_{\text{FBL}[\ell_p(\Gamma)]} \) is equivalent to the unit vector basis of \( \ell_1(\Gamma) \).

Let \( i : \ell_p \to \ell_2 \) be the formal inclusion operator (so that \( \| i \| = 1 \) and \( j : \ell_2 \to L_1 \) be the isomorphic embedding satisfying \( \| j \| = 1 \) and \( j(e_n) = r_n \) for all \( n \in \mathbb{N} \), where \( (e_n) \) is the unit vector basis of \( \ell_2 \) and \( r_n \) denotes the \( n \)-th Rademacher function (see e.g. [1], Theorem 6.2.3]).

Given any finite set \( A = \{ \gamma_1, \ldots, \gamma_n \} \subseteq \Gamma \), let \( P_A : \ell_p(\Gamma) \to \ell_p \) be the operator defined by
\[
P_A((x_\gamma)_{\gamma \in \Gamma}) := (x_{\gamma_1}, x_{\gamma_2}, \ldots, x_{\gamma_n}, 0, 0, \ldots)
\]
(for \( A = \emptyset \) we define \( P_0 := 0 \)) and consider
\[
T_A := j \circ i \circ P_A : \ell_p(\Gamma) \to L_1.
\]
so that $\|T_A\| \leq 1$. Let $\hat{T}_A : \text{FBL}([\ell_p(\Gamma)]) \to L_A$ be the unique lattice homomorphism extending $T_A$, which satisfies $\|\hat{T}_A\| = \|T_A\|$ (Theorem 2.4). Define $\xi^*_A := (\hat{T}_A)^*(\chi_{[0,1]}) \in \text{FBL}([\ell_p(\Gamma)])^*$ and note that $\|\xi^*_A\| \leq 1$. Moreover, for each $\gamma \in \Gamma$ we have

$$\langle \xi^*_A, |\delta_{e_{\gamma}}| \rangle = \langle \chi_{[0,1]}, \hat{T}_A(|\delta_{e_{\gamma}}|) \rangle = \langle \chi_{[0,1]}, |T_A(e_{\gamma})| \rangle = \begin{cases} 1 & \text{if } \gamma \in A \\ 0 & \text{if } \gamma \notin A. \end{cases}$$

Finally, take any finite non-empty set $B \subseteq \Gamma$ and pick $a_{\gamma} \in \mathbb{R}$ for each $\gamma \in B$. Write $B_+ := \{\gamma \in B : a_{\gamma} > 0\} \text{ and } B_- := \{\gamma \in B : a_{\gamma} < 0\}$. From (5.1) it follows that

$$2 \left\| \sum_{\gamma \in B} a_{\gamma} |\delta_{e_{\gamma}}| \right\|_{\text{FBL}([\ell_p(\Gamma)])} \geq \xi^*_B \left( \sum_{\gamma \in B} a_{\gamma} |\delta_{e_{\gamma}}| \right) - \xi^*_B \left( \sum_{\gamma \in B} a_{\gamma} |\delta_{e_{\gamma}}| \right) = \sum_{\gamma \in B} |a_{\gamma}|.$$

This shows that $\{|\delta_{e_{\gamma}}| : \gamma \in \Gamma\}$ is equivalent to the unit vector basis of $\ell_1(\Gamma)$, hence $\text{span}(\{|\delta_{e_{\gamma}}| : \gamma \in \Gamma\})$ is isomorphic to $\ell_1(\Gamma)$.

**Corollary 5.5.** Let $\Gamma$ be an uncountable set and $1 < p \leq 2$. Then $\text{FBL}([\ell_p(\Gamma)])$ is LWCG but it is not isomorphic to a subspace of a WCG Banach space.

**Proof.** Bear in mind that $\ell_1(\Gamma)$ does not embed isomorphically into any WCG Banach space. This can be deduced, for instance, from the fact that subspaces of WCG Banach spaces are weakly Lindelöf (see e.g. [2] Theorem 14.31), while $\ell_1(\Gamma)$ is not weakly Lindelöf (see e.g. [3] Proposition 5.11).

We do not know whether the spaces $\text{FBL}([c_0(\Gamma)])$ or $\text{FBL}([\ell_p(\Gamma)])$ for $2 < p < \infty$ and uncountable $\Gamma$ are WCG. In fact, we do not know any non-separable Banach space $E$ for which $\text{FBL}[E]$ is WCG.

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