The structure of boundary parameter property satisfying sets

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Abstract

Precise definitions of singularities in General Relativity rely on a set of curves for their mathematical definition and physical interpretation. The abstract boundary allows the set of curves to be chosen, as long as a condition called the boundary parameter property (b.p.p.) is satisfied. It is, therefore, important to determine the answer to the question, “What happens to the abstract boundary classification when we enlarge or take the union/intersection/complement of b.p.p. sets?” Unfortunately the standard set operations do not respect the b.p.p. In this paper we prove that there exists a correspondence between b.p.p. sets and sets of inextendible curves, modulo two conditions to do with compactness and parametrisation. Using the correspondence we are able to give meaning to the question above. In a forthcoming paper we will use the results given here to answer the question above.

1 Introduction

Boundary constructions in General Relativity provide rigorous definitions of boundary points. These boundary points can be divided into classes such as, singularities, points at infinity and regular boundary points. To do this most boundary constructions use, implicitly or explicitly, certain sets of curves, usually with a particular type of parametrisation. For example the $g$-boundary, [1], relies on incomplete geodesics with affine parameter, the $b$-boundary, [2], on incomplete curves with generalised affine parameter and

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the $c$-boundary, \[3\], on endless causal curves\[4\]. Papers such as \[6\] reiterate the point that careful consideration of the set of curves is needed to get a correct definition of singularity. Indeed it can be argued that the topological problems of the $g$-, $b$- and $c$-boundaries relate to their corresponding sets of curves being too big since the sets of curves that are used necessarily include affinely parametrised timelike geodesics contained in compact regions (if such geodesics exist). This choice is part of the root cause of the non-Hausdorff and non-$T_0$ separation issues with these boundaries, see \[7, 8\] for reviews.

The abstract boundary, \[9\], does not use sets of curves in its construction but does need such a set for its classification. Once a choice has been made it is possible to provide a physically motivated classification of boundary and abstract boundary points (see \[9\] for the details) that complements the classification in terms of polynomials of the Riemann tensor, see \[10\]. Note that the classification of boundary points, not abstract boundary points, can be applied to any boundary construction. Thus the results given below can be applied to any boundary construction in which the set of curves can be altered.

That one has to choose a set of curves begs the question of which choice is the “right” one and what happens when the set of curves changes. To answer these questions we need to be able to compare the effect of different sets of curves on the classification. Thus we need to be say what the relationship between these sets of curves is, i.e. describe unions, intersections, etc...

The sets of curves used in the abstract boundary must satisfy what is called the boundary parameter property or b.p.p. This property ensures that the curves have mutually compatible behaviour which allows for a physical interpretation of the boundary points. The standard set operations do not respect the b.p.p. Hence it is not easy to compare b.p.p. sets of curves. We address this by proving that a b.p.p. set corresponds to set of inextendible curves that are not contained in a compact set and which satisfy a simple property regarding reparametrisation. By using the correspondence we are then able to extend the standard set operations to the set of all b.p.p. sets. The correspondence also gives an easy way to construct b.p.p. sets.

Throughout the paper we illustrate the discussion with various, mostly

\[1\]At least this is the case for the older formulations of the $c$-boundary. The more modern constructions, e.g. \[4, 5\], because of their reliance on the structure of various relations obfuscate this point. It maybe the case that the recent constructions still impose a particular choice of curves. Given the importance, however, of providing physical interpretations of boundary points and the necessity of using curves to do so even if no choice of curves is forced a choice will need to be made at some point for physical applications.
trivial, examples. In a forth coming paper we give a decidedly non-trivial application by answering the question, “What happens to the abstract boundary classification when we enlarge or take the union/intersection/complement of b.p.p. sets?”

The paper is divided into four sections. Section 2 introduces the relationship between b.p.p. sets and certain sets of inextendible curves. Section 3 presents two pairs of equivalence relations under which the correspondence of section 2 becomes a bijection. Section 4 gives an initial application by modifying the standard set operations to respect the b.p.p.

We now give a few of the results from [9] that are needed for this paper.

**Definition 1** (Manifold). We shall only consider manifolds, $M$, that are paracompact, Hausdorff, connected, $C^\infty$--manifolds, equipped with a Lorentzian metric $g$.

**Definition 2.** An embedding, $\phi : M \to M_\phi$, of $M$ is an envelopment if $M_\phi$ has the same dimension as $M$. Let $\Phi(M)$ be the set of all envelopments of $M$.

**Definition 3** (The abstract boundary). Let

\[ \text{Bound}(M) = \{(\phi, U) : \text{where } \phi \in \Phi \text{ and } U \subset \partial \phi(M) = \overline{\phi M} - \phi M\}. \]

Define an partial order $\triangleright$ on $\text{Bound}(M)$ by $(\phi, U) \triangleright (\psi, V)$ if and only if for every sequence $\{x_i\}$ in $M$, $\{\psi(x_i)\}$ has a limit point in $V$ implies that $\{\phi(x_i)\}$ has a limit point in $U$. We can construct an equivalence relation $\equiv$ on $\text{Bound}(M)$ by $(\phi, U) \equiv (\psi, V)$ if and only if $(\phi, U) \triangleright (\psi, V)$ and $(\psi, V) \triangleright (\phi, U)$. Denote the equivalence class of $(\phi, U)$ by $[(\phi, U)]$.

The abstract boundary is the set

\[ \mathcal{B}(M) = \left\{ [(\phi, A)] \in \frac{\text{Bound}(M)}{\equiv} : \exists (\psi, \{p\}) \in [(\phi, A)] \right\}. \]

It is the set of all equivalence classes of $\text{Bound}(M)$ under the equivalence relation $\equiv$ that contain an element $(\psi, \{p\})$ where $p \in \partial \psi(M)$. The elements of the abstract boundary are referred to as abstract boundary points. Abstract boundary points have some nice properties in common with normal points [11].

Following the lead of [9] we work with $C^0$ piecewise $C^1$ curves.

**Definition 4** (Curve). A parametrised $C^0$ piecewise $C^1$ curve (or curve) $\gamma$ in the manifold $M$ is a $C^0$ map $\gamma : [a, b) \to M$ where $[a, b) \subset \mathbb{R}$, $a < b \leq \infty$
with a finite subset $a = \tau_0, \tau_1, \ldots, \tau_m = b$ so that on each segment $[\tau_i, \tau_{i+1})$ the tangent vector $\gamma' : [\tau_i, \tau_{i+1}) \to T_{\gamma}M$ is everywhere non-zero. We shall say that $\gamma$ is bounded if $b < \infty$ otherwise $\gamma$ is unbounded.

A curve $\delta : [a', b') \to M$ is a subcurve of $\gamma$ if $a \leq a' < b' \leq b$ and $\delta = \gamma|_{[a', b')}$. That is a curve $\delta$ is a subcurve of $\gamma$ if $\delta$ is the restriction of $\gamma$ to some right-half open interval of $[a, b)$. We shall denote this by $\delta < \gamma$. If $a' = a$ and $b' < b$ we shall say that $\gamma$ is an extension of $\delta$.

A change of parameter is a monotone increasing surjective $C^1$ function, $s : [a, b) \to [a', b')$. A curve $\delta$ is obtained from the curve $\gamma$ if $\delta = \gamma \circ s$. Note that our definition of $<$ considers the parameterisation chosen for $\gamma$. That is if $\delta(t) < \gamma(t)$ then we know that $\delta(t) < \gamma(2t)$. Another way to say this is that we draw a distinction between reparameterisations of the same image of a curve.

**Definition 5** (Bounded parameter property). A set $C$ of parameterised curves is said to have the bounded parameter property (or b.p.p.) if the following properties are satisfied;

1. For all $p \in M$ there exits $\gamma : [a, b) \to M \in C$ so that $p \in \gamma([a, b))$.
2. If $\gamma \in C$ and $\delta < \gamma$ then $\delta \in C$.
3. For all $\gamma, \delta \in C$, if $\delta$ is obtained from $\gamma$ by a change of parameter then either both curves are bounded or both are unbounded.

**Example 6.** Let $S$ be the maximally extended Schwarzschild solution in given in Kruskal-Szekeres coordinates, $(t, x, \theta, \phi)$, as

$$ds^2 = \frac{16m^2}{r} e^{\frac{r}{2m}} (-dt^2 + dx^2) + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

$r$ is defined implicitly by

$$t^2 - x^2 = (r - 2m) e^{\frac{r}{2m}}.$$

Let $S_e = \{p \in S : v(p)^2 - u(p)^2 < 0, \ u(p) > 0\}$ and $S_i = \{p \in S : 0 < v(p)^2 - u(p)^2 < 1, \ v(p) > 0\}$ be the submanifolds representing the exterior and interior of the black hole, respectively.

Let $M \in \{S, S_e, S_i\}$. Let $C_{gt}(M)$, be the set of all timelike geodesics in $M$ with affine parameter. Let $C_{gap}(M)$ be the set of all curves with generalised affine parameter. If $M \in \{S_e, S_i\}$, then $M$ is globally hyperbolic and there exists $f_M : M \to \mathbb{R}$ a $C^\infty$ time function so that $0 \in f(M)$. For
each \( x \in f^{-1}(0) \), let \( \gamma_x : J \to \mathcal{M} \) be the curve so that \( \gamma_x(0) = x \) and \( \gamma'_x \) is the gradient of \( f \), where \( J = \{ t \in f(\mathcal{M}) : t \geq 0 \} \). Thus \( \gamma_x(t) \in f^{-1}(t) \). Let \( \mathcal{S}(f_M) \) be the set of all such curves.

For \( \mathcal{M} \in \{ \mathcal{S}, \mathcal{S}_e, \mathcal{S}_i \} \) both \( \mathcal{C}_{gt}(\mathcal{M}) \) and \( \mathcal{C}_{gap}(\mathcal{M}) \) satisfy the b.p.p. in \( \mathcal{M} \), when considered as a submanifold of \( \mathcal{S} \). For \( \mathcal{M} \in \{ \mathcal{S}_e, \mathcal{S}_i \} \) the sets \( \mathcal{C}_{gt}(\mathcal{M}), \mathcal{C}_{gap}(\mathcal{M}) \) and \( \mathcal{S}(f_M) \) fail condition 1 of definition 5 with respect to \( \mathcal{S} \). The set \( \mathcal{S}(f_M) \) also fails to satisfy condition 2 of definition 5 in \( \mathcal{M} \). Note that condition 3 of definition 5 holds vacuously on \( \mathcal{S}(f_M) \) as there are no subcurves in \( \mathcal{S}(f_M) \).

We can build a b.p.p. set of curves from \( \mathcal{S}(f_M) \). For each \( p \in \mathcal{M} \), so that there is no \( \gamma \in \mathcal{S} \) such that \( p \in \gamma(J) \), we choose \( \delta_p : [-\epsilon_p, \epsilon_p) \to \mathcal{M} \) to be a geodesic so that \( \delta_p(0) = p \) and \( \delta_p([-\epsilon_p, \epsilon_p)) \) is compact. Then the set

\[
\mathcal{C}(\mathcal{S}(f_M)) = \{ \mu < \gamma : \gamma \in \mathcal{S}(f_M) \} \cup \{ \mu < \delta_p : p \in \mathcal{M}, \quad \exists \gamma \in \mathcal{S}(f_M), \quad p \in \gamma(J) \}
\]

satisfies the b.p.p. with respect to \( \mathcal{S} \). We will show this to be the case below, see definition 10 and proposition 21.

Since every affinely parameterised geodesic is a generalised affinely parameterised curve we see that

\[
\mathcal{C}_{gt}(\mathcal{M}) \subset \mathcal{C}_{gap}(\mathcal{M}).
\]

If the parameter induced by \( f_M \) on the curves of \( \mathcal{S}(f_M) \) is affine and if the parameter chosen on each \( \delta_p \) is affine then

\[
\mathcal{C}(\mathcal{S}(f_M)) \subset \mathcal{C}_{gap}(\mathcal{M}).
\]

If, in addition, the integral curves of \( f_M \) are timelike geodesics and each \( \delta_p \) is a timelike geodesic then

\[
\mathcal{C}(\mathcal{S}(f_M)) \subset \mathcal{C}_{gt}(\mathcal{M}).
\]

These are examples where the standard set operations respect the b.p.p. In these cases our guiding question already has meaning and the induced changes on the classification can be explored.

Only the relative bounded or unboundedness of curves is important in definition 5. Indeed from definitions 28, 31 and 37 of [9] it is clear that we need not pay close attention to the domain of any curve. We only need know the curve’s image and whether it is bounded or unbounded. Thus the relation \(< \) is, in this sense, too fine for the abstract boundary. In most uses
of the abstract boundary classification the b.p.p. satisfying set is fixed and nuanced points about distinctions between the domains of curves and their images is unimportant. It is, however, an important point when considering relationships between b.p.p. satisfying sets, as we shall see below. Despite this problem we will persist in using $<$ in order to maintain a stronger connection with the existing abstract boundary literature.$^2$

The classification of abstract boundary points [9] is given by a classification of boundary points $\partial\phi(M)$, $\phi \in \Phi$ and then by showing which ‘parts’ of this classification are invariant under the equivalence relation $\equiv$. From [9], and as we will show in a following paper, the classification of boundary points is constructed from $\mathcal{C}$ by studying the relative bounded/unbounded-ness of curves which limit to the same boundary points. For the purposes of this paper, therefore we need only divide boundary points into four classes, approachable and approachable with bounded parameter and the two complements of these sets.

Definition 7 (Approachable and Unapproachable Points). Let $\phi \in \Phi(M)$ and $\mathcal{C}$ be a set of curves with the bounded parameter property. A boundary point $p \in \partial\phi(M)$ is approachable if there exists $\gamma \in \mathcal{C}$ so that $p$ is a limit point of the image of the curve $\phi \circ \gamma$.

We make the following definitions;

\[
\text{App}(\phi, \mathcal{C}) = \{ p \in \partial\phi(M) : p \text{ is approachable} \}
\]

\[
\text{Nonapp}(\phi, \mathcal{C}) = \{ p \in \partial\phi(M) : p \text{ is unapproachable} \} = \partial\phi(M) - \text{App}(\phi, \mathcal{C}).
\]

Definition 8. Let $\text{App}_{\text{Sing}}(\phi, \mathcal{C})$ and $\text{App}_{\text{Inf}}(\phi, \mathcal{C})$ be defined by,

\[
\text{App}_{\text{Sing}}(\phi, \mathcal{C}) = \{ p \in \text{App}(\phi, \mathcal{C}) : \exists \gamma \in \mathcal{C} \text{ so that } \phi(\gamma) \rightarrow \phi(p) \text{ and } \gamma \text{ has bounded parameter} \}
\]

\[
\text{App}_{\text{Inf}}(\phi, \mathcal{C}) = \{ p \in \text{App}(\phi, \mathcal{C}) : \forall \gamma \in \mathcal{C}, \phi(\gamma) \rightarrow \phi(p) \text{ implies } \gamma \text{ has unbounded parameter} \}.
\]

It is easy to see that $\text{App}_{\text{Sing}}(\phi, \mathcal{C}) = \text{App}(\phi, \mathcal{C}) - \text{App}_{\text{Inf}}(\phi, \mathcal{C})$ and $\text{Nonapp}(\phi, \mathcal{C}) = \partial\phi(M) - \text{App}_{\text{Sing}}(\phi, \mathcal{C}) \cup \text{App}_{\text{Inf}}(\phi, \mathcal{C})$.

$^2$A suggestion for a more appropriate relation, for the abstract boundary classification, is that given two curves $\gamma : [a, b) \rightarrow \mathcal{M}$ and $\delta : [a', b') \rightarrow \mathcal{M}$ we have $\gamma < \delta$ if and only if there exists $c \in (a, b)$ and $c' \in (a', b')$ and a change of parameter $s : [c, b) \rightarrow [c', b')$ so that $\gamma|_{[c, b)} \equiv \delta|_{[c', b')}$ and $b = \infty$ if and only if $b' = \infty$. We use this idea in definition 33.
The set \( \text{App}_{\text{Sing}}(\phi, C) \) is the set of approachable boundary points that can be approached with bounded parameter. If \( C \) is the set of all affinely parameterised geodesics then an element of \( \text{App}_{\text{Sing}}(\phi, C) \) is a point that can be reached in finite time, e.g. the singularity of Schwarzschild spacetime. We use the subscript ‘Sing’ as the set is closely related to the abstract boundary definition of a singularity \([9]\). Similarly, the set \( \text{App}_{\text{Inf}}(\phi, C) \) is the set of approachable boundary points that are always approached with unbounded parameter. Once again, if \( C \) is the set of all affinely parameterised geodesics then an element of \( \text{App}_{\text{Inf}}(\phi, C) \) is a point that cannot be approached in finite time, e.g. future timelike infinity in the Schwarzschild spacetime. Just as before we use the subscript ‘Inf’ as the set is closely related to abstract boundary points at infinity \([9]\).

**Example 9.** We continue from example \([6]\).

Suppose that \( f_M \) induces an affine parameter on each \( \gamma \in S(f_M) \). We may reparametrise each curve by \( s(t) = t^2 \) so that the bounded/unboundedness of each \( \gamma \) is preserved and so that each curve is no longer affinely parametrised. We denote this new set of curves by \( \hat{S}(f_M) \). Since the classification of the abstract boundary depends only on the relative bounded/unboundedness of curves the sets \( C(S(f_M)) \) and \( C(\hat{S}(f_M)) \) give the same information about the classification.

Since \( f_M \) induces an affine parameter the information that the set \( C(S(f_M)) \) provides for the classification is ‘contained’ in the set \( C_{\text{gap}}(M) \). That is for each \( \gamma \in C(S(f_M)) \), where \( \gamma \) is non-compact, \( \gamma \in C_{\text{gap}}(M) \). It is not necessarily the case, however, that

\[
C(S(f_M)) \subset C_{\text{gap}}(M)
\]

or that

\[
C(S(f_M)) \cup C_{\text{gap}}(M)
\]

has the b.p.p. This is due to the presence of the curves \( \delta_p \in C(S(f_M)) \). Worse than this, by construction, for all \( \gamma \in C(\hat{S}(f_M)) \), where \( \gamma \) is non-compact, \( \gamma \notin C_{\text{gap}}(M) \). Thus despite \( C(S(f_M)) \) and \( C(\hat{S}(f_M)) \) providing the same information for the classification and despite the existence of a strong relationship between \( C(S(f_M)) \) and \( C_{\text{gap}}(M) \) there is no relationship between \( C(\hat{S}(f_M)) \) and \( C_{\text{gap}}(M) \).
2 Inextendible curves and boundary parameter property satisfying sets

We shall show that every b.p.p. set of curves corresponds to a set of curves with non-compact closures which satisfy a condition to do with reparametrisation. The reparametrisation condition is related to condition 3 of definition 5. The correspondence demonstrates that the structure of b.p.p. sets of curves is very simple. In practical situations, when choosing a b.p.p. set of curves the correspondence shows that it is enough to choose a set of curves that are not contained in a compact set and so that no two curves have the same image on non-compact portions of their domains.

The idea is the same as that used to construct $C(S(f_M))$ from $S(f_M)$. Once you have some collection of curves with non-compact images (and that satisfy the additional condition), it is possible to fill in the “gaps” of the images of the curves and add in the additional subcurves so as to ensure that the set satisfies the b.p.p. Since the way in which the “gaps” are filled makes no difference to the abstract boundary classification this method ensures that all b.p.p. sets can be constructed up to one of two equivalence relations. The details of the equivalence relations are left to section 3. We start by showing how to construct a particular b.p.p. satisfying set of curves that will be very useful in what follows.

**Definition 10.** Given any $p \in M$ there exists a normal neighbourhood $V_p$ so that $p \in V_p$ and $V_p$ is compact. Choose any $v \in T_pM$ and let $\gamma_p : [-\epsilon, \epsilon) \to M$ be the unique geodesic lying in $V_p$ so that $\gamma_p(0) = p$ and $\gamma_p'(0) = v$.

Let $C_{\text{Norm}} = \{ \delta \mid \text{a curve : } \exists \gamma_p \text{ so that } \delta < \gamma_p \}$. Let $C_{\text{Norm}}(U) = \{ \delta \mid \text{a curve : } \exists p \in U \exists \gamma_p \text{ so that } \delta < \gamma_p \}$. That is $C_{\text{Norm}}(U)$ is the restriction of our construction to include only curves $\gamma_p$ for the points in some $U \subset M$.

**Proposition 11.** The set $C_{\text{Norm}}$ satisfies the b.p.p. In addition all curves $\delta \in C_{\text{Norm}}$ have bounded parameter.

**Proof.**

1. For all $p \in M$ the curve $\gamma_p \in C_{\text{Norm}}$ is such that $p \in \gamma_p$.

2. Let $\delta \in C_{\text{Norm}}$ and let $\beta < \delta$. Since $\delta \in C_{\text{Norm}}$ there exists $\gamma_p \in C$ so that $\delta < \gamma_p$. We know that $\beta < \delta < \gamma_p$ and therefore, by construction, $\beta \in C_{\text{Norm}}$.

3. Let $\delta : [a, b) \to M \in C_{\text{Norm}}$ then there must exist $\gamma_p : [-\epsilon, \epsilon) \to M \in C_{\text{Norm}}$ so that $\delta < \gamma_p$. By definition, this implies that $[a, b) \subset [-\epsilon, \epsilon)$ and therefore that $\delta$ has bounded parameter. Suppose that there exists $\mu : [p, q) \to M \in C_{\text{Norm}}$ so that $\delta$ and $\mu$ are obtained from each other.
by a change of parameter. Since $\delta, \mu \in \mathcal{C}_{\text{Norm}}$ we know that both must be bounded.

Therefore $\mathcal{C}_{\text{Norm}}$ satisfies the b.p.p. and all curves in $\mathcal{C}_{\text{Norm}}$ are bounded. \hfill \Box

Sets of curves with the b.p.p. such as above are not useful for the abstract boundary classification as they make no distinction between approachable and non-approachable boundary points for envelopments of $\mathcal{M}$.

**Proposition 12.** For all $\phi \in \Phi$, $\text{Nonapp}(\phi, \mathcal{C}_{\text{Norm}}) = \partial \phi(\mathcal{M})$.

**Proof.** By definition we know that $\text{Nonapp}(\phi, \mathcal{C}_{\text{Norm}}) \subset \partial \phi(\mathcal{M})$ so we need only show that $\partial \phi(\mathcal{M}) \subset \text{Nonapp}(\phi, \mathcal{C}_{\text{Norm}})$.

Let $p \in \partial \phi(\mathcal{M})$. If $p \notin \text{Nonapp}(\phi, \mathcal{C}_{\text{Norm}})$ then $p \in \text{App}(\phi, \mathcal{C}_{\text{Norm}})$. Thus there exists $\delta : [a, b) \to \mathcal{M} \in \mathcal{C}_{\text{Norm}}$ so that $p$ is a limit point of the curve $\phi \circ \delta$. We may choose $\{t_i\} \subset [a, b)$ so that $\phi \circ \delta(t_i) \to p$. By construction the sequence $\{\delta(t_i)\}$ cannot have any limit points in $\mathcal{M}$. Since $\delta \in \mathcal{C}_{\text{Norm}}$ there exists $\gamma_q : (-\epsilon, \epsilon) \to \mathcal{M}$, for some $q$ so that $\gamma_q([\epsilon, \epsilon]) \subset V_q$ and $\delta < \gamma_q$. We can conclude that $\{\delta(t_i)\} \subset \gamma_q([-\epsilon, \epsilon]) \subset V_q$ which is compact. Hence $\{\delta(t_i)\}$ must have a limit point. Since this is a contradiction we know that $p \in \text{Nonapp}(\phi, \mathcal{C}_{\text{Norm}})$ as required. \hfill \Box

Thus from the point of view of the abstract boundary $\mathcal{C}_{\text{Norm}}$ provides no information. Hence given a set of curves, with non-compact closures, $S$ we can construct a set which satisfies conditions 1 and 2 of definition 5 and which gives the same abstract boundary information as $S$ by taking the union of $\mathcal{C}_{\text{Norm}}(U)$, for some suitable $U$, and $\{\delta < \gamma : \gamma \in S\}$. We will use this latter, for the moment we prove a more useful result than proposition 12.

**Proposition 13.** Let $\gamma$ be a curve in $\mathcal{M}$ then $\overline{\gamma}$ is compact if and only if for all $\phi \in \Phi$, $\partial \phi(\mathcal{M}) \cap \overline{\phi \circ \gamma} = \emptyset$.

**Proof.** Let $\gamma : [a, b) \to \mathcal{M}$ be a curve. Suppose that $\overline{\gamma}$ is compact and let $p \in \partial \phi(\mathcal{M}) \cap \overline{\phi \circ \gamma}$, from some $\phi \in \Phi$. The same argument used in the proof of proposition 12 can be used to derive a contradiction. Hence $\partial \phi(\mathcal{M}) \cap \overline{\phi \circ \gamma} = \emptyset$.

Suppose that for all $\phi \in \Phi$, $\partial \phi(\mathcal{M}) \cap \overline{\phi(\gamma)} = \emptyset$ and that $\overline{\gamma([a, b])}$ is not compact. Then there exists a sequence $\{x_i\} \subset \overline{\gamma([a, b])}$ with no limit points in $\mathcal{M}$. By the Endpoint Theorem (see 3) there exists $\psi \in \Phi(\mathcal{M})$ and $x \in \partial \psi(\mathcal{M})$ so that $\{\psi(x_i)\} \to x$. Then $x \in \partial \psi(\mathcal{M}) \cap \overline{\psi \circ \gamma}$ which is a contradiction. Thus $\overline{\gamma([a, b])}$ is compact. \hfill \Box
Proposition 13 proves our claim that the abstract boundary is only interested in the curves with non-compact closure in a b.p.p. set. This is an important point as we use this to show that b.p.p. sets are, up to an equivalence relation, the same as sets of curves with non-compact closures that also satisfy an additional condition to do with reparameterisations.

**Example 14.** We continue from example 6. If the parameter induced by $f_M$ is affine, but if we parametrised any $\delta_p$ with a non-affine parameter then

$$C(S(f_M)) \not\subseteq C_{gap}(\mathcal{M}).$$

This is undesirable since, as we have seen, in proposition 13, the curves $\delta_p$ do not contribute any information to the classification of abstract boundary points. We would like to be able to say that

$$C(S(f_M)) \subseteq C_{gap}(\mathcal{M})$$

without regard to curves that have no effect on the classification. Moreover, we would like to be able to consider the effect on the classification of enlarging the set of curves considered, for example by taking

$$C(S(f_M)) \cup C_g(\mathcal{M}).$$

Since all the curves in the subset of inextendible non-compact curves in $C(S(f_M)) \cup C_g(\mathcal{M})$ have affine parameters we know that on this subset satisfies condition 3 of definition 5. This subset contains all the curves that effect the classification so, from this point of view, the union above should give a well-defined classification of the boundary. As we have seen, the parameterisation of the curves $\delta_p$, however, can prevent this.

As similar situation occurs if we consider a different collection of $\delta_p$’s during the construction of $C(S(f_M))$. We will denote this new set by $\tilde{C}(S(f_M))$. Just as above the sets $C(S(f_M))$ and $\tilde{C}(S(f_M))$ differ only on curves with compact closures. Hence the sets provide the same classification, yet the preservation of the b.p.p. on, say, the set union depends on details that have no relevance to the classification (in this case the relative parameterisation of the two sets of curves with compact closures used in the construction of $C(S(f_M))$ and $\tilde{C}(S(f_M))$).

**Definition 15.** Let $BPP(\mathcal{M})$ be the set of all sets of curves with the b.p.p. That is $BPP(\mathcal{M}) = \{C : C$ is a set of curves with the b.p.p.$\}$. 

10
Recalling the fineness of $<$ we note that $BPP(\mathcal{M})$ will contain multiple sets with curves that have the same images but different domains. For example given $\mathcal{C} \in BPP(\mathcal{M})$. We can define a new b.p.p. set by $\mathcal{C}' = \{ \gamma(t-1) : \gamma \in \mathcal{C} \}$. Since we have not changed the “boundedness” of any curve in $\mathcal{C}$, from the point of view of the abstract boundary the sets $\mathcal{C}$ and $\mathcal{C}'$ will produce the same classification. That is $App(\phi, \mathcal{C}) = App(\phi, \mathcal{C}')$ and $App_{\text{Sing}}(\phi, \mathcal{C}) = App_{\text{Sing}}(\phi, \mathcal{C}')$ for all $\phi \in \Phi$. Thus $BPP(\mathcal{M})$ will contain multiple copies of each b.p.p. set that from the view point of the abstract boundary should be considered the same. We deal with this issue in definitions 29 and 33 by defining two equivalence relations.

**Definition 16.** Let $\text{NonCom}(\mathcal{M})$ be the set of curves with non-compact closures. Let $\text{NonCom}_{\text{b.p.p.}}(\mathcal{M})$ be the set of subsets of $\text{NonCom}(\mathcal{M})$ such that for all $S \in \text{NonCom}_{\text{b.p.p.}}(\mathcal{M})$ and for all $\gamma : [a,b) \to \mathcal{M}, \delta : [p,q) \to \mathcal{M} \in S$, if there exists $c \in [a,b), r \in [p,q)$ and a change of parameter $s : [c,b) \to [r,q]$ so that $\gamma|_{[r,q]} \circ s = \delta|_{[c,b)}$ then either both $\gamma$ and $\delta$ are bounded or both are unbounded.

**Example 17.** We continue from example 6. By construction $S(f_\mathcal{M}), C_g (\mathcal{M}), C_{\text{gap}} (\mathcal{M}) \in \text{NonCom}_{\text{b.p.p.}}(\mathcal{M})$. In a manifold with inextendible geodesics with compact closure it will be the case that the set of all geodesics, with affine parameter, is not in $\text{NonCom}_{\text{b.p.p.}}(\mathcal{M})$. The Misner spacetime [12] gives a concrete example of this. The set $\mathcal{C}(S(f_\mathcal{M}))$ is not in $\text{NonCom}_{\text{b.p.p.}}(\mathcal{M})$ as it contains curves with compact closure, i.e. the $\delta_p$’s.

Note that, by Zorn’s lemma and the definition of $<$, for each $S \in \text{NonCom}_{\text{b.p.p.}}(\mathcal{M})$ there exists $\tilde{S} \in \text{NonCom}_{\text{b.p.p.}}(\mathcal{M})$ so that each $\gamma \in \tilde{S}$ is a maximal element of $S$ with respect to $<^3$. In particular $\tilde{S}$ now vacuously satisfies condition 2 of definition 5. There is no guarantee that condition 2 of definition 5 holds for a general element of $\text{NonCom}_{\text{b.p.p.}}(\mathcal{M})$. Using a similar Zornological argument it is possible to find elements of $\text{NonCom}_{\text{b.p.p.}}(\mathcal{M})$ that vacuously satisfy the the reparametrisation condition of the definition of $\text{NonCom}_{\text{b.p.p.}}(\mathcal{M})$.

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3This requires definition 4 to be altered to allow for curves defined on open domains, not just right half-open domains.

4One might wonder then why we didn’t define $\text{NonCom}_{\text{b.p.p.}}(\mathcal{M})$ without the reparametrisation condition. This is certainly possible. However, one would still need to impose the condition that no two curves of an element of $\text{NonCom}_{\text{b.p.p.}}(\mathcal{M})$ have the same image, excluding a compact segment. Thus the definition of $\text{NonCom}_{\text{b.p.p.}}(\mathcal{M})$ would be no simpler. In addition, the reparameterisation condition is easy to work with (one need only study the properties of the parameterisation that one wishes to use). Checking the image condition involves more work.
The technical condition in definition 16 is to prevent us from choosing subsets of $NonCom(M)$ which are inappropriate from the point of view of the abstract boundary. It is a translation of condition 3 of definition 5 to subsets of $NonCom(M)$. Hence the subscript b.p.p. Note that in order to accommodate our distinction between curves with the same image but different domains, given by the relation $<$, the issue of redundant sets of curves in $BPP(M)$ also applies to $NonCom_{b.p.p.}(M)$. We take, below, the same approach to this problem as mentioned after definition 15, see definitions 30 and 37.

The technical condition of definition 16 does not reduce the utility of the correspondences that we will show between $BPP(M)$ and $NonCom_{b.p.p.}(M)$. This is because the correspondence will almost always be used when constructing b.p.p. sets and as such we can always choose a subset $S$ of $NonCom(M)$ so that no two curves have the same image on non-compact portions of their domains. In this case $S$ vacuously satisfies the technical condition and hence $S \in NonCom_{b.p.p.}(M)$.

Before we embark on removing the issue produced by the fineness of $<$. We show how $BPP(M)$ and $NonCom_{b.p.p.}(M)$ are related.

**Proposition 18.** There exists a function $f : BPP(M) \rightarrow NonCom_{b.p.p.}(M)$, given by $f(C) = NonCom(M) \cap C$.

**Proof.** The function is certainly well defined and by definition of $C$ we know that $NonCom(M) \cap C \in NonCom_{b.p.p.}(M)$. □

**Example 19.** We continue from example 6.

By construction we have that $f(C(S(f_M)))) = \{\mu : \exists \gamma \in S(f_M) \mu < \gamma\}$.

The function $f$ takes a b.p.p. set and extracts the inextendible curves that are not contained in a compact set. That is it extracts the curves that are important with regards to the abstract boundary classification using $C$.

We now show how to construct a b.p.p. satisfying set from an element of $NonCom_{b.p.p.}(M)$.

**Definition 20.** Given a set of curves $S$ let $S_p = \{p \in M : \exists \gamma : [a,b) \rightarrow M \in S \text{ so that } p \in \gamma([a,b))\}$.

Thus $S_p$ is the set of all points in the manifold that are contained in the image of some curve in $S$.

---

5 Or chose a suitable parameter with properties that ensure that the reparameterisation condition of definition 16 holds, e.g. the generalised affine parameter
**Proposition 21.** Let $S \in \text{NonCom}_{b.p.p.}(\mathcal{M})$, then we can define a function $g : \text{NonCom}_{b.p.p.}(\mathcal{M}) \rightarrow \text{BPP}(\mathcal{M})$ by letting

$$g(S) = \{ \delta : \exists \gamma \in S \delta < \gamma \} \cup \mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p).$$

**Proof.** As before, it is clear that $g$ is well defined, so we must only check that $g(S)$ satisfies the b.p.p. We must check the three conditions of definition 5.

1. Let $p \in \mathcal{M}$. If $p \in S_p$ then there exists $\alpha : [a, b) \rightarrow \mathcal{M} \in S$ so that $p \in \alpha([a, b))$. Since $\alpha \leq \beta$ we know that $\alpha \in g(S)$. If $p \notin S_p$ then $p \in \mathcal{M} - S_p$ then by definition of $\mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)$ we know that there must exist some $\gamma_p : [-\epsilon, \epsilon) \rightarrow \mathcal{M} \in \mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)$ so that $p \in \gamma_p([-\epsilon, \epsilon))$, as required.

2. Let $\alpha \in g(S)$ and $\beta < \alpha$. We know that either $\alpha \in \{ \delta : \exists \gamma \in S \delta < \gamma \}$ or $\alpha \in \mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)$. If $\alpha \in \{ \delta : \exists \gamma \in S \delta < \gamma \}$, then by definition $\beta \in \{ \delta : \exists \gamma \in S \delta < \gamma \}$. Likewise if $\alpha \in \mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)$ then by the definition of $\mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)$ we know that $\beta \in \mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)$ as required.

3. Let $\alpha : [a, b) \rightarrow \mathcal{M}, \beta : [p, q) \rightarrow \mathcal{M} \in g(S)$ be such that $\beta$ and $\alpha$ are obtained from each other by a change of parameter $s : [a, b) \rightarrow [p, q)$, so that $\alpha \circ s = \beta$.

Suppose that $\beta, \alpha \in \mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)$ then from corollary 11 we know that both must be bounded.

Suppose that $\beta \in \{ \delta : \exists \gamma \in S \delta < \gamma \}$, $\alpha \in \mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)$. Since $\beta \in \{ \delta : \exists \gamma \in S \delta < \gamma \}$ there exists $\gamma_x : [p', q') \rightarrow \mathcal{M} \in S$ so that $\beta < \gamma_x$, that is $[p, q) \subset [p', q')$ and $\beta = \gamma_x|_{[p, q)}$. Since $\gamma_x \in S$ it must be the case that $\gamma_x([p', q'))$ is not compact. As $\alpha \in \mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)$ we know that $\alpha([a, b])$ is necessarily compact. Hence as $\alpha([a, b]) = \alpha \circ s([p, q)) = \beta([p, q))$ we know that $\beta([p, q))$ must be compact. Since $\beta([p, q)) \subset \gamma_x([p', q'))$ and $\gamma_x([p', q'))$ is not compact we can conclude that $[p, q)$ is a proper subset of $[p', q')$ and in particular that $q < q'$. Hence $q \in \mathbb{R}$ and therefore $\beta$ must be bounded.

Now suppose $\beta, \alpha \in \{ \delta : \exists \gamma \in S \delta < \gamma \}$. As $\alpha \in \{ \delta : \exists \gamma \in S \delta < \gamma \}$ there must exist $\gamma_y : [a', b') \rightarrow \mathcal{M} \in S$ so that $\gamma_y([a', b'))$ is not compact, $[a, b) \subset [a', b')$ and $\alpha = \gamma_y|_{[a, b)}$. Let $\gamma_x : [p', q') \rightarrow \mathcal{M}$ be as above.

If $q < q'$ then $\beta([p, q))$ is compact and $\beta([p, q)) = \alpha \circ s([p, q)) = \alpha([a, b])$ must also be compact. Therefore $b < b'$ and $\alpha$ is bounded.
Applying the same argument for $\alpha$ shows that $q < q'$ if and only if $b < b'$. Thus, in this case, both $\beta$ and $\alpha$ are bounded.

Thus we need only now consider the case when $q = q'$ and $b = b'$. We see that $\gamma_x|_{[p,q')} = \beta = \alpha \circ s = \gamma_y|_{[a,b')} \circ s$. Since $\gamma_x, \gamma_y \in S$ from the technical condition in definition 16 we know that $\gamma_x$ and $\gamma_y$ are either both bounded or both unbounded. Since $q = q'$ and $b = b'$ the same applies to $\alpha$ and $\beta$. That is either both are bounded or both are unbounded, as required.

Example 22. Continuing from example 6. We have that $g(S(f_M)) = C(S(f_M))$ for some appropriate choice of the $\delta_p$, given in definition 10.

We have the following lemmas which tell us how $f$ and $g$ relate and, in particular, that $g$ is injective when restricted to the range of $f$ and $f$ is injective when restricted to the range of $g$. Neither function is surjective, however.

Lemma 23. Let $S \in \text{NonCom}_{b.p.p.}(M)$ then $S \subset f \circ g(S)$ and if $\gamma : [a,b) \to \mathcal{M} \in f \circ g(S) - S$ then there exists $\delta : [p,q) \to \mathcal{M} \subset S$ so that $\gamma < \delta$ and $\gamma \neq \delta$. In particular, $q = b$ and $p \neq a$.

Proof. We first note that

\[
\begin{align*}
f \circ g(S) &= g(S) \cap \text{NonCom}(\mathcal{M}) \\
&= \left(\{\delta : \exists \gamma \in S \delta < \gamma\} \cup \mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)\right) \cap \text{NonCom}(\mathcal{M}) \\
&= \left(\{\delta : \exists \gamma \in S \delta < \gamma\} \cap \text{NonCom}(\mathcal{M})\right) \cup \left(\mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p) \cap \text{NonCom}(\mathcal{M})\right).
\end{align*}
\]

The intersection $\mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p) \cap \text{NonCom}(\mathcal{M})$ must be empty by definition 11 and proposition 12. Since $S \subset \{\delta : \exists \gamma \in S \delta < \gamma\} \cap \text{NonCom}(\mathcal{M})$ we know that $S \subset g(S) \cap \text{NonCom}(\mathcal{M})$. Let $\gamma : [a,b) \to \mathcal{M} \in f \circ g(S) - S$. Thus $\gamma \in \{\delta : \exists \gamma \in S \delta < \gamma\} \cap \text{NonCom}(\mathcal{M})$. Hence there exists $\delta : [p,q) \to \mathcal{M} \in S$ so that $\gamma < \delta$. By definition we know that $b \leq q$. If $b < q$ then $\gamma((a,b))$ must be compact and hence $\gamma \notin \text{NonCom}(\mathcal{M})$. Since this is a contradiction it must be the case that $b = q$. If $a = p$ we would have $\gamma = \delta$ and thus $\gamma \in S$. Since this is also a contradiction it must be the case that $p < a$. Thus $\gamma < \delta$ and $\gamma \neq \delta$ as required. 

\[\square\]
Example 24. Continuing from example 6 we have that \( f \circ g (S(f_M)) = \{ \mu < \gamma : \gamma \in S(f_M) \} \). Whereas \( S(f_M) \) vacuously satisfied condition 2 of definition 5 the set \( f \circ g (S(f_M)) \) satisfies the condition because of the definition of \( g \).

Proposition 25. Let \( C \in BPP(M) \) then \( f \circ g \circ f(C) = f(C) \).

Proof. We can calculate that,

\[
\begin{align*}
  f \circ g \circ f(C) &= f \left( \{ \delta : \exists \gamma \in f(C) \; \delta < \gamma \} \cup C_{\text{Norm}}(M - f(C)_p) \right) \\
  &= \left( \{ \delta : \exists \gamma \in f(C) \; \delta < \gamma \} \cup C_{\text{Norm}}(M - f(C)_p) \right) \cap \text{NonCom}(M) \\
  &= \{ \delta : \exists \gamma \in f(C) \; \delta < \gamma \} \cap \text{NonCom}(M).
\end{align*}
\]

If \( \gamma \in \{ \delta : \exists \gamma \in f(C) \; \delta < \gamma \} \) then there exists \( \delta \in f(C) \) so that \( \gamma < \delta \). Since \( \delta \in f(C) = C \cap \text{NonCom}(M) \) we know that \( \delta \in C \). By definition of the b.p.p. we then know that \( \gamma \in C \). Hence \( f \circ g \circ f(C) \subset f(C) \). From lemma 23 we know that \( f(C) \subset f \circ g \circ f(C) \) as required.

Proposition 26. Let \( S \in \text{NonCom}_{b.p.p.}(M) \) then \( g \circ f \circ g(S) = g(S) \).

Proof. We first show that \( (f \circ g(S))_p = S_p \). Let \( x \in f \circ g(S) \) then there exists \( \gamma : [a, b] \to M \in f \circ g(S) \) so that \( x \in \gamma([a, b]) \). By lemma 23 we know that there exists \( \delta : [p, b] \to M \in S \) so that \( \gamma < \delta \). Hence \( x \in \delta([p, q]) \) and \( x \in S_p \). Again, from lemma 23 we know that \( S \subset f \circ g(S) \) and therefore \( S_p \subset (f \circ g(S))_p \) as required.

We now show that

\[ \{ \delta : \exists \gamma \in S \; \delta < \gamma \} = \{ \delta : \exists \gamma \in f \circ g(S) \; \delta < \gamma \}. \]

From lemma 23 we know that \( S \subset f \circ g(S) \) so that

\[ \{ \delta : \exists \gamma \in S \; \delta < \gamma \} \subset \{ \delta : \exists \gamma \in f \circ g(S) \; \delta < \gamma \}. \]

Let \( \mu \in \{ \delta : \exists \gamma \in f \circ g(S) \; \delta < \gamma \} \) then there exists \( \gamma \in f \circ g(S) \) so that \( \mu < \delta \). Since \( f \circ g(S) = g(S) \cap \text{NonCom}(M) \) we know that \( \gamma \in \{ \delta : \exists \gamma \in S \; \delta < \gamma \} \cup C_{\text{Norm}}(M - S_p) \) and that \( \gamma \in \text{NonCom}(M) \). By definition 10 we know that \( \text{NonCom}(M) \cap C_{\text{Norm}}(M - S_p) = \emptyset \). Hence \( \gamma \in \{ \delta : \exists \gamma \in S \; \delta < \gamma \} \) as required.

Thus as

\[
g \circ f \circ g(S) = \{ \delta : \exists \gamma \in f \circ g(S) \; \delta < \gamma \} \cup C_{\text{Norm}}(M - (f \circ g(S))_p)
\]

\text{\footnote{It may be the case that } \delta = \gamma.}
and
\[ g(S) = \{ \delta : \exists \gamma \in S \, \delta < \gamma \} \cup \mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p) \]
we have our result.

These particular properties of \( f \) and \( g \) strongly suggest that there is too much differentiation between objects. That is, we should look to equivalence relationship to get ride of the surplus information.

3 Formalising the one-to-one correspondence

The results of section 2 provide enough structure to use \( \text{NonCom}_{\text{b.p.p.}}(\mathcal{M}) \) as a replacement for \( \text{BPP}(\mathcal{M}) \). We are left, however, with three problems. The first is the fineness of \(<\). This fineness results in multiple sets \( C, C' \in \text{BPP}(\mathcal{M}) \) and \( S, S' \in \text{NonCom}_{\text{b.p.p.}}(\mathcal{M}) \) so that the parametrisations of the curves of \( C \) and \( C' \), or of \( S \) and \( S' \), differ, for example, by some collection of linear transformations \( \{s_\gamma\} \). Thus either both \( \gamma \in C \) and \( \gamma \circ s_\gamma \in C' \) are bounded or unbounded, likewise for \( S \) and \( S' \). Since this implies that \( \text{App}(\phi, C) = \text{App}(\phi, C') \) and \( \text{App}_{\text{Sing}}(\phi, C) = \text{App}_{\text{Sing}}(\phi, C') \) for all \( \phi \in \Phi(\mathcal{M}) \) there is no difference between \( C \) and \( C' \) or \( g(S) \) and \( g(S') \) from the point of view of the abstract boundary. The sets \( \hat{S}(f_{\mathcal{M}}) \) and \( S(f_{\mathcal{M}}) \) of example 9 have this property.

The second is the method of filling the “gaps” used in the construction of \( g(S) \). There are many ways to do this. To each \( p \in \mathcal{M} \) we have associated a vector \( v \in T_p\mathcal{M} \) and selected the unique geodesic \( \gamma_p : [-\epsilon, \epsilon) \to \mathcal{M} \) contained in some normal neighbourhood \( V_p \) of \( p \) so that \( \gamma_p(0) = p \) and \( \gamma'_p(0) = v \). Replacing \( v \) by \( v' \in T_p\mathcal{M} \) will result in a different method of filling the “gaps”. The result, is a function \( \tilde{g} : \text{NonCom}_{\text{b.p.p.}}(\mathcal{M}) \to \text{BPP}(\mathcal{M}) \) that differs from \( g \), \( g(S) \neq \tilde{g}(S) \) for all \( S \in \text{NonCom}_{\text{b.p.p.}}(\mathcal{M}) \), and yet since \( S = S \) we have that \( \text{App}(\phi, g(S)) = \text{App}(\phi, \tilde{g}(S)) \) and \( \text{App}_{\text{Sing}}(\phi, g(S)) = \text{App}_{\text{Sing}}(\phi, \tilde{g}(S)) \) for all \( \phi \in \Phi(\mathcal{M}) \). Once again, from the point of view of the abstract boundary there is no difference between \( g(S) \) and \( \tilde{g}(S) \). The sets \( \tilde{C}(S(f_{\mathcal{M}})) \) and \( \hat{C}(S(f_{\mathcal{M}})) \) of example 14 have this property.

The third is the possible existence of classes of curves \( C, C' \in \text{BPP}(\mathcal{M}) \) so that the elements of \( f(C) \) and \( f(C') \) have no pairs of curves with the same image and yet \( \text{App}(\phi, C) = \text{App}(\phi, C') \) and \( \text{App}_{\text{Sing}}(\phi, C) = \text{App}_{\text{Sing}}(\phi, C') \) for all \( \phi \in \Phi(\mathcal{M}) \). This problem has a substantially different flavour to the two problems above which deal with the construction of b.p.p. sets.
An example of this type of behaviour is, predictably, given by the Misner spacetime [12].

**Example 27.** Let \( \mathcal{M} = \mathbb{R}^+ \times S^1 \) be the Misner spacetime, [12], given in the coordinates \( 0 < t < \infty \) and \( 0 \leq \psi \leq 2\pi \). With respect to these coordinates we have \( ds^2 = -t^{-1}dt^2 + t d\psi^2 \). Let \( \phi : \mathcal{M} \to \mathbb{R} \times S^1 \) be defined by \( \phi(t, \psi) = (t, \psi - \log(t)) \). This is one of the two maximal extensions of \( \mathcal{M} \). Letting \( \psi' = \psi - \log(t) \) the metric becomes \( ds^2 = 2d\psi' dt + t (d\psi')^2 \). With respect to this extension we are able to select two sets of affinely parametrised null geodesics lying in \( \mathcal{M} \) those run vertically and those that approach but never reach the waist \( t = 0 \).

Let \( C_v \) be the set of all affinely parametrised null geodesics so that \( \phi(C) \) is the set of vertical null geodesics in the extension given by \( \phi \) and \( t > 0 \). Likewise let \( C_w \) be the set of all affinely parametrised null geodesics so that \( \phi(D) \) is the set of null geodesics that approach but do not reach the waist. Thus, with respect to \((t, \psi)\), \( \gamma \in C_v \) is given by \( \gamma(t) = (-a_0 \tau + b_0, \log(-a_0 \tau + b_0) + b_1) \) for \( a_0 \in \mathbb{R}^+, b_0 \in \mathbb{R} \) and \( 0 \leq b_1 < 2\pi \) where \( \tau \in \left(-\infty, -\frac{b_0}{a_0}\right) \). Similarly \( \gamma \in C_w \) is given by \( \gamma(t) = (-a_0 \tau + b_0, -\log(-a_0 \tau + b_0) + b_1) \) for \( a_0 \in \mathbb{R}^+, b_0 \in \mathbb{R} \) and \( 0 \leq b_1 < 2\pi \) where \( \tau \in \left(-\infty, -\frac{b_0}{a_0}\right), [8] \).

Let \( \gamma \in C_v \) and \( \mu \in C_w \). The equation \( \gamma(\tau) = \mu(\tau') \) has the solution

\[
\tau_i = \frac{-1}{a_0} \left( e^{\left(\frac{b_i - b_0}{2} - i\pi\right)} - b_0 \right)
\]

\[
\tau'_i = \frac{-1}{a_0} (-a_0 \tau + b_0 - b'_0),
\]

for each \( i \in \mathbb{Z} \). As \( i \to \infty \), \( \tau_i \to \frac{b_0}{a_0} \) and \( \tau'_i \to \frac{b'_0}{a_0} \). The sequence \( \{(t_i, \psi_i) = \gamma(\tau_i) = \mu(\tau'_i)\} \) has, as \( i \to \infty \), the solution \((0, b_1)\) as expected. As \( \gamma \) and \( \mu \) are arbitrary this calculation implies that given \( \psi \in \Phi(\mathcal{M}) \), \( \gamma \in C_v \) has \( p \in \partial \psi(\mathcal{M}) \) as a limit point if and only if every curve in \( C_w \) has \( p \) as a limit point and that \( \mu \in C_w \) has \( p \in \partial \psi(\mathcal{M}) \) as an endpoint if and only if every curve in \( C_v \) has \( p \) as a limit point.

Let \( \phi' \in \Phi(\mathcal{M}) \) and consider \( p \in \partial \phi'(\mathcal{M}) \). Since every curve in \( C_v \) and \( C_w \) has bounded parameter we know that \( \text{App}(\psi, C_v) = \text{App}_{\text{Sing}}(\phi, C_v) \), \( \text{App}(\psi, C_w) = \text{App}_{\text{Sing}}(\phi, C_w) \), \( \text{Nonapp}(\psi, C_v) = \partial \psi(\mathcal{M}) - \text{App}_{\text{Sing}}(\phi, C_v) \) and \( \text{Nonapp}(\psi, C_w) = \partial \psi(\mathcal{M}) - \text{App}_{\text{Sing}}(\phi, C_w) \). From the calculation above we know that \( p \in \text{App}_{\text{Sing}}(\phi', C_v) \) if and only if \( p \in \text{App}_{\text{Sing}}(\phi', C_w) \). That is that \( \text{App}_{\text{Sing}}(\phi', C_v) = \text{App}_{\text{Sing}}(\phi, C_w) \). This implies that \( \text{Nonapp}(\psi, C_v) = \text{Nonapp}(\psi, C_w) \). This demonstrates that for any envelopment \( \phi \in \Phi, C_v \) and
\(\mathcal{C}_w\) provide the same information despite containing no curves with the same images.

To remove this redundant additional information we define two equivalence relations. The result, in both cases, is that \(f\) and \(g\) become bijective. Thus with respect to either equivalence relation the sets \(\text{BPP}(\mathcal{M})\) and \(\text{NonCom}_{\text{b.p.p.}}(\mathcal{M})\) are in one-to-one correspondence. The first equivalence relation addresses all three pieces of redundant information. While the second address only the fineness of \(<\) and the different ways to fill in the “gaps”. The reason for doing this is that it is a priori very hard to calculate when two different sets \(\mathcal{C}, \mathcal{D} \in \text{BPP}(\mathcal{M})\) are such that \(\text{App}(\phi, \mathcal{C}) = \text{App}(\phi, \mathcal{D})\) and \(\text{App}_{\text{Sing}}(\phi, \mathcal{C}) = \text{App}_{\text{Sing}}(\phi, \mathcal{D})\) for any \(\phi \in \Phi(\mathcal{M})\). Thus the first equivalence relation is in some sense the more mathematically motivated one, while the second equivalence relation is in some sense the more practically motivated one. There is a nice relationship between the two relations that fits with what we expect, see proposition 38.

3.1 The mathematically motivated equivalence relation

As the abstract boundary cares only for the relative boundedness/ unboundedness of curves and as this can be expressed via the sets \(\text{App}(\phi, \mathcal{C})\) and \(\text{App}_{\text{Sing}}(\phi, \mathcal{C})\) the obvious equivalence relation is to say that two b.p.p. satisfying sets of curves, \(\mathcal{C}, \mathcal{D} \in \text{BPP}(\mathcal{M})\), are equivalent if \(\text{App}(\phi, \mathcal{C}) = \text{App}(\phi, \mathcal{D})\) and \(\text{App}_{\text{Sing}}(\phi, \mathcal{C}) = \text{App}_{\text{Sing}}(\phi, \mathcal{D})\) for all \(\phi \in \Phi\). This will ensure that \(\mathcal{C}\) and \(\mathcal{D}\) give the same classification of boundary points and therefore the same classification of abstract boundary points.

We must also apply an equivalence relation to \(g\). For example taking \(\mathcal{C}\) and \(\mathcal{D}\) from example 27 we see that \(f(\mathcal{C})\) will be equated with \(f(\mathcal{D})\) but we know that \(g \circ f(\mathcal{C}) \neq g \circ f(\mathcal{D})\). The following lemma suggests how to fix this.

**Lemma 28.** Let \(\mathcal{C} \in \text{BPP}(\mathcal{M})\) and let \(\mathcal{D} = g \circ f(\mathcal{C})\). Then \(\text{App}(\phi, \mathcal{C}) = \text{App}(\phi, \mathcal{D})\) and \(\text{App}_{\text{Sing}}(\phi, \mathcal{C}) = \text{App}_{\text{Sing}}(\phi, \mathcal{D})\) for all \(\phi \in \Phi\).

**Proof.** From lemma 25 we know that \(f(\mathcal{D}) = f(\mathcal{C})\). Thus \(\mathcal{D} \cap \text{NonCom}(\mathcal{M}) = \mathcal{C} \cap \text{NonCom}(\mathcal{M})\). This implies that \(\text{App}(\phi, \mathcal{C}) = \text{App}(\phi, \mathcal{D})\) and \(\text{App}_{\text{Sing}}(\phi, \mathcal{C}) = \text{App}_{\text{Sing}}(\phi, \mathcal{D})\) for all \(\phi \in \Phi\), by proposition 13.

---

In the case of example 27 we benefited from two facts. First that given a curve in one set every curve of the other set intersected it in an infinite sequence. Second that every curve in either set had bounded parameter.
This suggests that we should define an equivalence relation on $\text{NonCom}_{\text{b.p.p.}}(\mathcal{M})$ as well. Hence we make the following definitions.

**Definition 29.** Let $\approx_1$ be the equivalence relation on $\text{BPP}(\mathcal{M})$ given by $\mathcal{C} \approx_1 \mathcal{D}$ if and only if for all $\phi \in \Phi$, $\text{App}(\phi, \mathcal{C}) = \text{App}(\phi, \mathcal{D})$ and $\text{App}_{\text{Sing}}(\phi, \mathcal{C}) = \text{App}_{\text{Sing}}(\phi, \mathcal{D})$. We denote the equivalence class of $\mathcal{C}$ by $[\mathcal{C}]_1$.

**Definition 30.** Let $\simeq_1$ be the equivalence relation on $\text{NonCom}(\mathcal{M})$ given by $S \simeq_1 P$ if and only if $g(S) \approx_1 g(P)$. We denote the equivalence class of $S$ by $[S]_1$.

Both equivalence relations are clearly well defined, so we can now show that the induced functions are bijective and mutually inverse.

**Theorem 31.** The induced functions $f_1 : \text{BPP}(\mathcal{M}) \approx_1 \rightarrow \text{NonCom}(\mathcal{M}) \simeq_1$ and $g_1 : \text{NonCom}(\mathcal{M}) \simeq_1 \rightarrow \text{BPP}(\mathcal{M}) \approx_1$ are bijective and mutually inverse.

**Proof.** We need to show that $f_1$ and $g_1$ are well-defined. Suppose that $\mathcal{C} \approx_1 \mathcal{D}$ we need to show that $f_1(\mathcal{C}) \approx_1 f_1(\mathcal{D})$. That is we need to show that $g \circ f(\mathcal{C}) \approx_1 g \circ f(\mathcal{D})$. From lemma 28, however, we know that $g \circ f(\mathcal{C}) \approx_1 g \circ f(\mathcal{D})$ as required. Likewise suppose that $S \simeq_1 P$ we need to show that $g \circ f(\mathcal{C}) \approx_1 S$. From definition 30 this implies that $f_1(\mathcal{C}) \approx_1 S$ or that $f_1(\mathcal{C}) = [S]_1$ as required. Therefore $f_1$ and $g_1$ are both bijective are mutually inverse.

**Example 32.** Under this equivalence relation we have that $[\mathcal{S}(f_\mathcal{M})]_1 = [\mathcal{S}(S(f_\mathcal{M}))]_1$, $[\mathcal{C}(S(f_\mathcal{M}))]_1 = [\mathcal{C}(S(f_\mathcal{M}))]_1$ and $[\mathcal{C}_{\text{v}}]_1 = [\mathcal{C}_{\text{w}}]_1$, where the sets are taken from examples 29, 14 and 27, respectively.

The equivalence classes $[\mathcal{C}_{\text{gt}}(S)]_1$ and $[\mathcal{C}_{\text{gap}}(S)]_1$ are not equal since no timelike geodesic has limit points at null infinity, while null geodesics have endpoints at null infinity.
3.2 The computationally motivated equivalence relation

If we are willing to forgo the insistence that b.p.p. satisfying sets $C$ and $D$ must be identified if $App_{\text{Sing}}(\phi, C) = App_{\text{Sing}}(\phi, D)$ and $App(\phi, C) = App(\phi, D)$. Then another pair of equivalence relations can be investigated.

Rather than identifying b.p.p. satisfying sets based on limit points we can identify them based on images of curves. The result is a pair of equivalence relations with a clearer connection to the set of curves without compact closure. This allows us to work with elements of $BPP(M)$ via elements of $\text{NonCom}_{b.p.p.}(M)$ more easily than when using the equivalence relations given above. We will still remove the unnecessary fineness of $<$. Since we would also like to remove the distinction between different ways of adding extendible curves to elements of $\text{NonCom}_{b.p.p.}(M)$, it makes sense to start with the following definition.

**Definition 33.** Define an equivalence relation $\simeq_2$ on the set $\text{NonCom}_{b.p.p.}(M)$ by $S \simeq_2 P$ if and only if

$$
\forall \gamma : [a, b) \to M \in S, \exists \delta : [p, q) \to M \in P, c, r \in \mathbb{R}
$$

so that $\gamma([c, b)) = \delta([r, q))$ and either both $\gamma, \delta$ are bounded or unbounded, and,

$$
\forall \delta : [p, q) \to M \in P, \exists \gamma : [a, b) \to M \in S, c, r \in \mathbb{R}
$$

so that $\gamma([c, b)) = \delta([r, q))$ and either both $\gamma, \delta$ are bounded or unbounded.

That is $S \simeq_2 P$ if and only if the images of all curves in $S$ is equal to the images of all curves in $P$ and that the boundedness and unboundedness of curves that have the same image, excluding some finite length portion of the domains of each curves, are the same.

We shall write the equivalence class of $S$ by $[S]_2$.

**Lemma 34.** The equivalence relation $\simeq_2$ on $\text{NonCom}_{b.p.p.}(M)$ is well defined.

**Proof.** Let $S \in \text{NonCom}_{b.p.p.}(M)$ then as $\gamma < \gamma$ for all $\gamma \in S$ we can see that $S \simeq_2 S$.

It is clear that the symmetry of $\simeq_2$ is satisfied by definition.

Suppose that $S, P, Q \in \text{NonCom}_{b.p.p.}(M)$ are such that $S \simeq_2 P$ and $P \simeq_2 Q$. Let $\gamma : [a, b) \to M \in S$ then there exists $\delta : [p, q) \to M \in P$ and $c, r \in \mathbb{R}$ so that $\gamma([c, b)) = \delta([r, q))$ and either both $\gamma$ and $\delta$ are bounded or unbounded. Likewise as $\delta \in P$ there exists $\mu : [u, v) \to M \in Q$ and $s, w \in \mathbb{R}$ so that $\delta([s, q)) = \mu([w, v))$ and either both are bounded or
unbounded. Without loss of generality assume that \( r < s \), then as \( \gamma([c, b]) = \delta([r, q]) \) there must exist \( d \in \mathbb{R} \) so that \( \gamma([d, b]) = \delta([s, q]) = \mu([w, v]) \). If \( \gamma \) is bounded then \( \delta \) must be bounded and therefore \( \mu \) must also be bounded. Likewise, if \( \gamma \) is unbounded then \( \mu \) must be unbounded. The reverse direction follows similarly.

Therefore \( \simeq_2 \) is well defined.

**Lemma 35.** Let \( S \in \text{NonCom}_b \cdot \text{p.p.}(\mathcal{M}) \) then \( S \simeq_2 f \circ g(S) \).

*Proof.* This follows from lemma 23. Q.E.D.

**Corollary 36.** Let \( S, P \in \text{NonCom}_b \cdot \text{p.p.}(\mathcal{M}) \) then \( S \simeq_2 P \) if and only if \( f \circ g(S) \simeq_2 f \circ g(P) \).

*Proof.* This follows directly from lemma 35. Q.E.D.

**Definition 37.** Define \( \mathcal{C}, \mathcal{D} \in \text{BPP}(\mathcal{M}) \) to be equivalent, denoted \( \mathcal{C} \simeq_2 \mathcal{D} \), if and only if \( f(\mathcal{C}) \simeq_2 f(\mathcal{D}) \). It is clear from the definition that this provides a well-defined equivalence relation. Denote the equivalence class of \( \mathcal{C} \) by \([\mathcal{C}]_2\).

In effect we are saying that two b.p.p. sets are equivalent if the subset of curves with non-compact closures are equivalent. This makes sense as we know from proposition 13 that only the curves with non-compact closures tell us about the abstract boundary. This set of these curves is precisely the set \( f(\mathcal{C}) \). The following two results describe the relation between \( \simeq_1 \) and \( \simeq_2 \).

**Proposition 38.** Let \( \mathcal{C}, \mathcal{D} \in \text{BPP}(\mathcal{M}) \) then \( \mathcal{C} \simeq_2 \mathcal{D} \) implies that \( \mathcal{C} \simeq_1 \mathcal{D} \).

*Proof.* Suppose that \( \mathcal{C} \simeq_2 \mathcal{D} \) and let \( p \in \text{App}(\phi, \mathcal{C}) \). Then there exists \( \gamma \in \mathcal{C} \) so that \( p \) is an accumulation point of \( \phi \circ \gamma \). Since \( p \in \partial \phi(\mathcal{M}) \) it must be the case that \( \gamma \in f(\mathcal{C}) \). Since \( \mathcal{C} \simeq_2 \mathcal{D} \) we know that \( f(\mathcal{C}) \simeq_2 f(\mathcal{D}) \). Hence there must exist \( \delta \in f(\mathcal{D}) \) so that the images of \( \gamma \) and \( \delta \) agree, except on some compact portion. Thus \( p \) must be an accumulation point of \( \phi \circ \delta \) and therefore \( p \in \text{App}(\phi, \mathcal{D}) \). Moreover, if \( \gamma \) is bounded then by definition 33 \( \delta \) must also be bounded. Therefore \( p \in \text{Appsing}(\phi, \mathcal{C}) \) implies that \( p \in \text{Appsing}(\phi, \mathcal{D}) \). Therefore \( \mathcal{C} \simeq_1 \mathcal{D} \) as required. Q.E.D.

Thus \( \simeq_2 \subset \simeq_1 \), so that if \( \mathcal{C} \simeq_2 \mathcal{D} \) then \( \text{App}(\phi, \mathcal{C}) = \text{App}(\phi, \mathcal{D}) \) and \( \text{Appsing}(\phi, \mathcal{C}) = \text{Appsing}(\phi, \mathcal{D}) \) for all \( \phi \in \Phi \). Example 27 shows that this inclusion is proper. We now show that the functions \( f_2 \) and \( g_2 \) induced by \( \simeq_2 \) and \( \simeq_2 \) are bijective and mutually inverse.
Proposition 39. The induced functions \( f_2 : \frac{\text{BPP}(M) \approx_2 \text{NonCom}_{\text{B.P.P.}}(M)}{\approx_2} \rightarrow \frac{\text{NonCom}_{\text{B.P.P.}}(M) \simeq_2 \text{BPP}(M)}{\approx_2} \) and \( g_2 : \frac{\text{NonCom}_{\text{B.P.P.}}(M) \simeq_2 \text{BPP}(M)}{\approx_2} \rightarrow \frac{\text{BPP}(M) \approx_2 \text{NonCom}_{\text{B.P.P.}}(M)}{\approx_2} \) are bijective and mutually inverse.

Proof. We first show that \( f_2 \) and \( g_2 \) are well defined. Let \( C \approx_2 D \), we must show that \( f_2(C) \simeq_2 f_2(D) \). This is true by definition and therefore \( f_2 \) is well-defined. Now let \( S \simeq_2 P \), from corollary 36 we know that \( f_2 \circ g_2(S) \simeq_2 f_2 \circ g_2(P) \).

By definition 37 this implies that \( g_2(S) \approx_2 g_2(P) \). Thus \( g_2 \) is well defined.

We now show that \( f_2 \circ g_2(S)_2 = [S]_2 \) and \( g_2 \circ f_2([C]_2) = [C]_2 \). Let \( [C]_2 \in \text{BPP}(M) \) and note that \( f_2 \circ g_2 \circ f_2(C) = f_2(C) \) by lemma 25. By definition 37 we know that \( g_2 \circ f_2(C) \approx_2 C \) or rather that \( g_2 \circ f_2([C]_2) = [C]_2 \). Let \( S \in \text{NonCom}_{\text{B.P.P.}}(M) \) then by lemma 35 we know \( S \simeq_2 f_2 \circ g_2(S) \). That is \( f_2 \circ g_2([S]_2) = [S]_2 \) as required. This is sufficient to prove that \( f_2 \) and \( g_2 \) are bijective and mutually inverse as required.

Thus using \( \approx_2 \) and \( \simeq_2 \), rather than \( \approx_1 \) and \( \simeq_1 \), we still get a one-to-one correspondence. It is, however, easier to work with \( \simeq_2 \) and therefore easier to exploit the structure of elements of \( \text{BPP}(M) \) outlined here.

Example 40. Except that \([C_v]_2 \neq [C_w]_2 \) all the other comments of example 32 remain true.

4 Set-like operations that respect the b.p.p.

As an application of these results, that will be useful in a forthcoming publication, we define set-like operations on \( \text{BPP}(M) \) that preserve the b.p.p. This will give meaning to the question, “What happens to the abstract boundary classification when we enlarge or take the union/intersection/complement of b.p.p. sets?” Since the normal set operations are easier to use on \( \text{NonCom}_{\text{B.P.P.}}(M) \), the basic idea is to exploit this by defining the new operations on \( \text{BPP}(M) \) by using \( f_2 \), \( g_2 \) and the equivalence relations.

An important and surprising point is that there will not always be a unique result of a set operation. For example if \( \gamma : [0, \infty) \rightarrow M \in C \) and \( \gamma \circ \text{arctan}(t) \in D \) and \( \gamma \in \text{NonCom}(M) \) then a choice needs to be made when defining \( C \cup D \). Should \( \gamma \) or \( \gamma \circ \text{arctan} \) be in \( C \cup D \)? We can’t have both as they are reparametrisations of each other and one is bounded while the other is unbounded. The following definition thus defines the set operations as maximal elements of certain subsets of \( \text{NonCom}_{\text{B.P.P.}}(M) \).

We choose to use the equivalence relations \( \approx_2 \) and \( \simeq_2 \) as these have a stronger connection to the elements of b.p.p. sets. The definition below,
using $\approx_1$ and $\simeq_1$ instead, gives set relations that describe how the abstract boundary classification induced by each b.p.p. sets are related.

**Definition 41.** Let $\mathcal{C}, \mathcal{D} \in BPP(\mathcal{M})$ we make the following definitions.

**Subset.** We say that $\mathcal{C}$ is a subset of $\mathcal{D}$, denoted $\mathcal{C} \subset_{b.p.p.} \mathcal{D}$, if and only if there exists $S \in [f(\mathcal{D})]_2$ so that $f(\mathcal{C}) \subset S$. If $\mathcal{C} \subset_{b.p.p.} \mathcal{D}$ and $\mathcal{D} \subset_{b.p.p.} \mathcal{C}$ then $[\mathcal{C}]_2 = [\mathcal{D}]_2$

**Union.** Let

$$\text{Union}(f(\mathcal{C}), f(\mathcal{D})) = \{ P \in \text{NonCom}_{b.p.p.}(\mathcal{M}) : \exists Q \in \text{NonCom}_{b.p.p.}(\mathcal{M}), [P]_2 = [Q]_2, Q \subset f(\mathcal{C}) \cup f(\mathcal{D}) \}$$

where $\subset$ and $\cup$ are the standard set operations. A union of $\mathcal{C}$ and $\mathcal{D}$ is $g(S)$, where $S$ is a maximal element of $\text{Union}(f(\mathcal{C}), f(\mathcal{D}))$. When $f(\mathcal{C}) \cup f(\mathcal{D}) \in \text{NonCom}_{b.p.p.}(\mathcal{M}), \{ [f(\mathcal{C}) \cup f(\mathcal{D})]_2 \} \subset \text{Union}(f(\mathcal{C}), f(\mathcal{D}))$ is the set of equivalence classes of the maximal elements of $\text{Union}(f(\mathcal{C}), f(\mathcal{D}))$. In this case we say that $g(S)$ is the union of the sets $\mathcal{C}$ and $\mathcal{D}$. We denote $[g(S)]_2$ by $\mathcal{C} \cup_{b.p.p.} \mathcal{D}$.

**Intersection.** Let

$$\text{Inter}(f(\mathcal{C}), f(\mathcal{D})) = \{ P \in \text{NonCom}_{b.p.p.}(\mathcal{M}) : \exists Q \in \text{NonCom}_{b.p.p.}(\mathcal{M}), [P]_2 = [Q]_2, Q \subset f(\mathcal{C}) \cap f(\mathcal{D}) \}$$

where $\subset$ and $\cap$ are the standard set operations. An intersection of the $\mathcal{C}$ and $\mathcal{D}$ is $g(S)$, where $S$ is a maximal element of the subset of $\text{Inter}(f(\mathcal{C}), f(\mathcal{D}))$. When $f(\mathcal{C}) \cap f(\mathcal{D}) \in \text{NonCom}_{b.p.p.}(\mathcal{M}), \{ [f(\mathcal{C}) \cap f(\mathcal{D})]_2 \} \subset \text{Inter}(f(\mathcal{C}), f(\mathcal{D}))$ is the set of equivalence classes of the maximal elements of $\text{Inter}(f(\mathcal{C}), f(\mathcal{D}))$. In this case we say that $g(S)$ is the intersection of the sets $\mathcal{C}$ and $\mathcal{D}$. We denote $[g(S)]_2$ by $\mathcal{C} \cap_{b.p.p.} \mathcal{D}$.

**Relative Complement.** Let

$$\text{Relat}(f(\mathcal{C}), f(\mathcal{D})) = \{ P \in \text{NonCom}_{b.p.p.}(\mathcal{M}) : \exists Q \in \text{NonCom}_{b.p.p.}(\mathcal{M}), [P]_2 = [Q]_2, Q \subset f(\mathcal{C}) \setminus f(\mathcal{D}) \}$$

where $\subset$ and $\setminus$ are the standard set operations.

For any $S, P \in \text{NonCom}_{b.p.p.}(\mathcal{M})$ it is simple to show that $S \setminus P \in \text{NonCom}_{b.p.p.}(\mathcal{M})$. Thus the set of equivalence classes of the maximal elements of $\text{Relat}(f(\mathcal{C}), f(\mathcal{D}))$ is $\{ [f(\mathcal{C}) \setminus f(\mathcal{D})]_2 \}$. Thus we define the relative complement to be $g(S)$ where, $S \in [f(\mathcal{C}) \setminus f(\mathcal{D})]_2$. We denote $[g(S)]_2$ by $\mathcal{C} \setminus_{b.p.p.} \mathcal{D}$. 

23
Where there is no risk of ambiguity we shall drop the subscript b.p.p.

**Example 42.** Using these definitions, and continuing from example 6, 9 and 14 we see that if \( f_M \) induces a generalised affine parameter then

\[
\mathcal{C}(S(f_M)) \subset b.p.p. \mathcal{C}_{\text{gap}}(\mathcal{M})
\]

regardless of the properties of the \( \delta_p \)'s. Moreover, if each \( \gamma \in S(f_M) \) is unbounded then

\[
\mathcal{C}(S(f_M)) \cap b.p.p. \mathcal{C}_{\text{gt}}(\mathcal{M})
\]

is well defined, as an equivalence class in \( BPP(\mathcal{M}) \). Since \( [\mathcal{C}(S(f_M))]_2 = [\mathcal{C}(\hat{S}(f_M))]_2 \) we also have that

\[
\mathcal{C}(\hat{S}(f_M)) \subset b.p.p. \mathcal{C}_{\text{gap}}(\mathcal{M})
\]

and that

\[
\mathcal{C}(\hat{S}(f_M)) \cap b.p.p. \mathcal{C}_{\text{gt}}(\mathcal{M})
\]

if each \( \gamma \in S(f_M) \). Thus we have resolved the problems highlighted in examples 9 and 14.

In the case that, say, \( f(C) \cup f(D) \) is not in \( \text{NonCom}_{b.p.p.}(\mathcal{M}) \) it is natural that a choice needs to be made, since we must include some curves and exclude others, as mentioned above. Thus each maximal subset gives us one of those possible choices. The majority of the properties of the usual set operations remain true for the operations defined in definition 41 at least once the equivalence relations are used.

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