Hamiltonian formulation of \( f(\text{Riemann}) \) theories of gravity

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We present a canonical formulation of gravity theories whose Lagrangian is an arbitrary function of the Riemann tensor. Our approach allows a unified treatment of various subcases and an easy identification of the degrees of freedom of the theory.

I. INTRODUCTION

Since H. Weyl introduced them in 1919\(^[1]\), theories of gravity whose action is nonlinear in the Riemann tensor (contrarily to Hilbert’s) have been part of the “landscape” of General Relativity and its various extensions, up to present-day string theories. In this paper we shall be interested in the action:

\[
S_g[g_{\mu\nu}] = \frac{1}{2} \int_M d^D x \sqrt{-g} f(R_{\mu\nu\rho\sigma}).
\]

where \( g_{\mu\nu}(x^\rho) \), \( D \) is the dimension of the spacetime \( M \), and \( f \) is an arbitrary function of the Riemann tensor \( R_{\mu\nu\rho\sigma} \).\(^1\)

The Euler–Lagrange equations of motion derived from metric variation of \( (1.1) \) are

\[
R^{(\lambda}_{\rho\sigma} \frac{\partial f}{\partial R^{(\lambda}_{\rho\sigma}} - 2 \nabla_\rho \nabla_\sigma \frac{\partial f}{\partial R^{(\rho}_{\rho\sigma}} - \frac{1}{2} f g^{\mu\nu} = T^{\mu\nu},
\]

where \( \nabla_\mu \) is the covariant derivative associated with \( g_{\mu\nu} \), \( T^{\mu\nu} \) is the energy-momentum tensor of matter, and \( f_{(\mu\nu)} \equiv (f^{\mu\nu} + f^{\nu\mu})/2 \). These equations are generically fourth-order in the derivatives of the metric. To convert them into a set of first-order differential equations, one must introduce as new variables some well-chosen functions of the metric and its derivatives. The identification of these extra degrees of freedom (besides those of Einstein’s gravity) is a prerequisite to introducing non-minimal coupling to matter. It is also an important step to study for example the well-posedness of the initial value problem and the number of independent Cauchy data, the global charges associated with the solutions, the stability of the theory and the positivity of energy as well as the junction conditions.

At the linear approximation around \( D = 4 \) flat spacetime and in the case \( f = \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} \) the identification of the extra dynamical degrees of freedom was made long ago, see \( [2] \). They are generally six, corresponding to a “massive spin-0” together with a “massive spin-2” field (a “ghost” of negative energy), and reduce to five if \( f \) is the square of the Weyl tensor (\( \beta = -3 \alpha \)). The action was then recast in first-order form and the constraints were analysed in \( [3] \) and \( [4] \) with the result, among others, that when \( f \) is the square of the Weyl tensor there is an additional, “first-class,” constraint which generates conformal transformations (see also \( [5,6] \)). The analysis of \( f = \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} + \gamma R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \) in \( D \) dimensions was performed in \( [7] \) with the result that the number of extra degrees of freedom is reduced if \( f \) is the square of the Weyl tensor (\( \alpha = 2, \beta = -(D-1), \gamma = (D-1)(D-2) \) or the Gauss–Bonnet combination (\( \alpha = 1, \beta = -4, \gamma = 1 \)). Finally, a canonical formulation of \( f(R) \) theories has been proposed in \( [8] \) (see also \( [9] \)) and the Lovelock case \( [10] \) was treated in \( [11] \).

Our aim here will be to unify these results and generalise them to the full action \( (1.1) \). To do so we shall follow the procedure of Arnowitt, Deser and Misner (ADM) \( [12] \), and present a canonical formulation of theories of gravity yielding the field equations \( (1.2) \). In contradistinction with previous approaches which consist in choosing as the extra variables either the extrinsic curvature of the ADM foliation \( [4] \) or its time derivative \( [2] \), we shall ascribe a leading role to the components \( \mathcal{R}_{ij}^{0,0} \) of the Riemann tensor (in an ADM coordinate system adapted to the foliation). This will allow us to write the canonical equations of motion in a compact form. The extra degrees of freedom will be encoded in the \( D(D-1)/2 \) components of some spatial tensor \( \Psi^{ij} \) and their number will depend on the number of components of \( \Omega_{ij} \) that can be extracted from the equation

\[
2 \Psi^{ij} + \frac{\partial f}{\partial \Omega_{ij}} = 0.
\]

\(^{1}\) Our conventions are: \( \mathcal{R}_{\mu\nu\rho\sigma} = (1/2) (\partial_{\mu\rho} g_{\nu\sigma} - \partial_{\nu\sigma} g_{\mu\rho}) + \cdots ; g \) is the determinant of \( g_{\mu\nu} \); the signature is \(( -- + \cdots )\); spacetime indices \(( \mu, \nu, \cdots ) \) run from 0 to \( D - 1 \); space indices \(( i, j, \cdots ) \) will run from 1 to \( D - 1 \).
The paper is organised as follows. In Section II we introduce a number of auxiliary fields in the action, perform its ADM decomposition and simplify it by using some of the constraints which simply determine some of the auxiliary fields algebraically. This leads us to our second-order form Lagrangian, see (2.18). In Section III we first obtain the Hamiltonian of the theory, see (3.4) and (3.5), and, second, write Hamilton’s equations of motion, see (3.9) and (3.10). Section IV presents the way to reduce the Hamiltonian using constraints and shows how a number of well-known subcases are recovered (General Relativity, \(f(R)\) and “Weyl\(^2\)” theories). Finally we make the link in an Appendix between our formalism and the commonly used Ostrogradsky one.

We shall present in this paper only the thread of the argument. The details of the calculations will be presented elsewhere [13].

II. CHOOSING THE ACTION

A. Introducing auxiliary fields

In order to turn the equations of motion (1.2) into a set of first-order differential equations we shall start our analysis, not with (1.1) but with the related action

\[
S[g_{\mu\nu}, \varrho_{\mu\nu\rho\sigma}, \varphi_{\mu\nu\rho\sigma}] = \frac{1}{2} \int_M d^Dx \sqrt{-g} \left[f(\varrho_{\mu\nu\rho\sigma}) + \varphi_{\mu\nu\rho\sigma} (R_{\mu\nu\rho\sigma} - \varrho_{\mu\nu\rho\sigma})\right]. \tag{2.1}
\]

The two auxiliary fields \(\varrho_{\mu\nu\rho\sigma}\) and \(\varphi_{\mu\nu\rho\sigma}\) have all the symmetries of \(R_{\mu\nu\rho\sigma}\) and are chosen to be independent of each other and of \(g_{\mu\nu}\). The reason for introducing them is that the second derivatives of the metric appear only linearly in (2.1) and will be eliminated by means of an integration by parts, see below. If matter does not couple to \(\varphi_{\mu\nu\rho\sigma}\) and \(\varrho_{\mu\nu\rho\sigma}\), variation of (2.1) yields a set of field equations which is equivalent to (1.2). Indeed:

\[
\delta S = \frac{1}{2} \int_M d^Dx \sqrt{-g} \left[\mathcal{E}^{\mu\nu} \delta g_{\mu\nu} + (R_{\mu\nu\rho\sigma} - \varrho_{\mu\nu\rho\sigma}) \delta \varphi_{\mu\nu\rho\sigma} + \left(\frac{\partial f}{\partial \varrho_{\mu\nu\rho\sigma}} - \varphi_{\mu\nu\rho\sigma}\right) \delta \varrho_{\mu\nu\rho\sigma}\right] + \int_M d^Dx \sqrt{-g} \nabla_\sigma \left(\nabla_\rho \varphi_{\mu\nu\rho\sigma}\right) \delta g_{\mu\nu} - \varphi_{\mu\nu\rho\sigma} \left(\nabla_\rho \delta g_{\mu\nu}\right), \tag{2.2}
\]

where

\[
\mathcal{E}^{\mu\nu} = -R^{(\alpha\beta\gamma\rho)} \varphi^{\alpha\beta\gamma\rho} - 2 \nabla_\alpha \nabla_\beta \varphi^{\alpha\beta\mu\nu} + 2 \varrho^{(\mu\nu)\alpha\beta\gamma} \frac{\partial f}{\partial \varrho^{\alpha\beta\gamma\delta}} - \frac{1}{2} \left[\varphi^{\alpha\beta\gamma\delta} (R_{\alpha\beta\gamma\delta} - \varrho_{\alpha\beta\gamma\delta}) + f(\varrho_{\alpha\beta\gamma\delta})\right] g^{\mu\nu}. \tag{2.3}
\]

Ignoring the divergence term for the time being, the field equations are

\[
\mathcal{E}^{\mu\nu} = T^{\mu\nu}, \tag{2.4}
\]

and, if the matter action does not depend on \(\varrho_{\mu\nu\rho\sigma}\) and \(\varphi_{\mu\nu\rho\sigma}\),

\[
\varrho_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}, \tag{2.5}
\]

together with

\[
\varphi_{\mu\nu\rho\sigma} = \frac{\partial f}{\partial \varrho_{\mu\nu\rho\sigma}}. \tag{2.6}
\]

Substituting these extra equations into (2.4), we recover the original fourth-order equation of motion (1.2).

A remark is in order here. As is well known [16], one can plug the constraint (2.6) back into the action (2.1) and consider it as a functional of \(g_{\mu\nu}\) and \(\varrho_{\mu\nu\rho\sigma}\) only. Indeed variation of this new action still yields the same equations of motion, the only difference being that (2.5) is replaced by

\[
\frac{\partial^2 f}{\partial \varrho_{\mu\nu\rho\sigma} \partial \varrho_{\alpha\beta\gamma\delta}} (R_{\alpha\beta\gamma\delta} - \varrho_{\alpha\beta\gamma\delta}) = 0. \tag{2.7}
\]

Whether this equation can be inverted to yield (2.5) or not imposes an analysis of the various subcases before any canonical treatment. For this reason we shall refrain from using (2.6) straightaway and shall stick to the action (2.1). This will allow us to treat the different subcases in a unified manner.

\[\text{For an example of non-minimal coupling to matter, see e.g. [14, 15].}\]
B. ADM decomposition

Suppose that \( \mathcal{M} \) can be foliated by a family of spacelike surfaces \( \Sigma_t \), defined by \( t = x^0 \). Let \( h_{ij} \equiv g_{ij}|_{x^0=t} \) with \( i, j \) running from 1 to \( D - 1 \) be the metric on \( \Sigma_t \), \( h \) its determinant, \( h^{ij} \) its inverse, and \( D_i \) be the associated covariant derivative. Introduce the future-pointing unit normal vector \( n \) to the surface \( \Sigma_t \), that is, to the basis vector fields \( \partial_4 \) with components \( \delta^0_i \); the components of \( n \) are \( n_0 = 0 \), \( n_0 = -1/\sqrt{-g^{00}} \), \( n^0 = \sqrt{-g^{00}} \), \( n^i = -g^{0i}/\sqrt{-g^{00}}. \)

Decompose then the timelike basis vector \( \partial_i \) (with components \( \delta^n_i \)) on \( n \) and the spatial vectors \( \partial_j \) : \( \delta^n_i = N n^i + \beta^i \delta^n_i \). \( N = 1/\sqrt{-g^{00}} \) and \( \beta^i = -g^{0i}/g^{00} \) are the “lapse” and “shift,” respectively; together with the induced metric \( h_{ij} \), they constitute the “ADM variables” [12]. In terms of these variables we have

\[
n_0 = -N, \quad n_i = 0, \quad n^0 = \frac{1}{N}, \quad n^i = -\frac{\beta^i}{N}.
\]

The components of the spacetime metric read

\[
\begin{align*}
g_{00} &= -N^2 + \beta_i \beta^i, \quad g_{0i} = \beta_i, \quad g_{ij} = h_{ij}, \\
g^{00} &= -\frac{1}{N^2}, \quad g^{0i} = \frac{\beta^i}{N^2}, \quad g^{ij} = h^{ij} - \frac{\beta^i \beta^j}{N^2}
\end{align*}
\]

and \( \sqrt{-g} = N\sqrt{h} \). Introduce finally the extrinsic curvature of \( \Sigma \):

\[
K_{ij} \equiv \nabla_i n_j = \frac{1}{2N} (\dot{h}_{ij} - D_i \beta_j - D_j \beta_i),
\]

where a dot denotes a time derivative: \( \dot{h}_{ij} = \partial_t h_{ij} \).

A standard calculation using (2.8), (2.9) and (2.10) (see e.g. [17, 18, 19] for a geometrical derivation) yields the Gauss, Codazzi and Ricci equations, that is, the components of the Riemann tensor in terms of the ADM variables:

\[
\begin{align*}
\mathcal{R}_{ijkl} &= K_{ik} K_{jl} - K_{il} K_{jk} + R_{ijkl}, \\
\mathcal{R}_{i j k n} &\equiv n^\nu \mathcal{R}_{ij k} = D_i K_{jk} - D_j K_{ik}, \\
\mathcal{R}_{i j m n} &\equiv n^\nu n^\sigma \mathcal{R}_{i j \nu \sigma} = -N^{-1} (K_{ij} - \mathcal{L}_\beta K_{ij}) + (K \cdot K)_{ij} + N^{-1} D_{ij} N,
\end{align*}
\]

where \( R_{ijkl} \) is the Riemann tensor of \( h_{ij} \), \( (A \cdot B)_{ij} \equiv A_{ik} B_{jk} \), \( D_{ij} \equiv D_i D_j \), and the short-hand notation \( \mathcal{L}_\beta \) denotes the Lie derivative with respect to the shift: \( \mathcal{L}_\beta K_{ij} \equiv \beta^k D_k K_{ij} + K_{ik} D_j \beta^k + K_{jk} D_i \beta^k \). We can thus perform the following decomposition:

\[
\varphi^{\mu \nu \rho \sigma} (\mathcal{R}_{\mu \nu \rho \sigma} - g_{\mu \nu} g_{\rho \sigma}) = \phi^{ijkl} (R_{ijkl} - \rho_{ijkl}) - 4 \phi^{ijk} (\mathcal{R}_{i j k n} - \rho_{ijk}) - 2 \Psi^{ij} (\mathcal{R}_{i j m n} - \Omega_{ij}),
\]

where we have introduced the following spatial tensors evaluated on \( \Sigma_t \):

\[
\begin{align*}
\rho_{ijkl} &\equiv \varphi_{ijkl}, \quad \rho_{ijk} \equiv n^\mu \varphi_{ij k \mu}, \quad \Omega_{ij} \equiv n^\mu n^\nu \varphi_{\mu j k \nu}, \\
\phi^{ijkl} &\equiv h^{im} h^{jn} h^{kp} h^{tq} \varphi_{mnpq}, \quad \phi^{ijk} \equiv h^{im} h^{jn} h^{kn} \varphi_{lmn}, \quad \Psi^{ij} \equiv -2 h^{ik} h^{jl} n^\mu n^\nu \varphi_{k \mu \nu}.
\end{align*}
\]

In order to pass to canonical formulation, one must remove second derivatives from the action. It is done by integrating the second derivative by parts to cast it into a total divergence. A side exercise shows how the time derivative of the extrinsic curvature is transformed into a time derivative of \( \Psi^{ij} \):

\[
N^{-1} \Psi^{ij} (\dot{K}_{ij} - \mathcal{L}_{\beta K_{ij}}) = \nabla_\mu (n^\mu K \cdot \Psi) - K (K \cdot \Psi) - N^{-1} K_{ij} (\dot{\Psi}^{ij} - \mathcal{L}_{\beta \Psi}^{ij}),
\]

where \( K \equiv h_{ij} K_{ij} \) and \( A \cdot B \equiv A_{ij} B^{ij} \).

Armed with these preliminaries we can decompose the action (2.1) as

\[
S[h_{ij}, \beta^i, \Psi^{ij}, \Omega_{ij}, \phi^{ijkl}, \phi^{ijk}, \rho_{ijkl}, \rho_{ijk}] = \int_{\mathcal{M}} d^Dx \left[ \mathcal{L} + \partial_0 (\sqrt{-g} n^\mu K \cdot \Psi) \right],
\]

---

3 We here work in an ADM coordinate system. See e.g. [17, 18, 19] for a more geometrical approach.

4 Here and in the following indices of \((D - 1)\)-dimensional spatial tensors are moved with the induced metric \( h_{ij} \).
with
\[ L = \sqrt{\mathcal{H}} N \left[ \frac{1}{2} f(\mathcal{H}_{\mu\nu\rho\sigma}) + \frac{1}{2} \phi^{ijkl} (R_{ijkl} - \rho_{ijkl}) - 2 \phi^{ijk} (R_{ijk\kappa} - \rho_{ijk}) - \Psi^{ij} (K K_{ij} + (K \cdot K)_{ij} + N^{-1} D_{ij} N - \Omega_{ij}) - N^{-1} K_{ij} (\dot{\Psi}^{ij} - L_\beta \Psi^{ij}) \right], \]

(2.16)

where \( R_{ijkl} \) and \( R_{ijk\kappa} \) are given in (2.11) and where it is understood that \( f(\mathcal{H}_{\mu\nu\rho\sigma}) \) is expressed in terms of \( \rho_{ijkl} \), \( \rho_{ijk} \) and \( \Omega_{ij} \) (see below for examples of such a decomposition).

As one can see from (2.16) and (2.11) the spatial tensors \( \phi^{ijkl} \) and \( \phi^{ijk} \) are not dynamical and their equations of motion \( (\partial L/\partial \phi^{ijkl} = 0, \partial L/\partial \phi^{ijk} = 0) \) are simple constraints:

\[ \rho_{ijkl} = K_{ik} K_{j\ell} - K_{i\ell} K_{jk} + R_{ijkl}, \quad \rho_{ijk} = D_i K_{jk} - D_j K_{ik}, \]

(2.17)

which can harmlessly be incorporated into \( L \), see \[10\]. As a consequence the Lagrangian density of our theory reduces to

\[ L^* = \sqrt{\mathcal{H}} N \left[ \frac{1}{2} f(\mathcal{H}_{\mu\nu\rho\sigma}) - \Psi^{ij} (K K_{ij} + (K \cdot K)_{ij} + N^{-1} D_{ij} N - \Omega_{ij}) - N^{-1} K_{ij} (\dot{\Psi}^{ij} - L_\beta \Psi^{ij}) \right], \]

(2.18)

where the spatial tensors \( \rho_{ijkl} \) and \( \rho_{ijk} \) in \( f(\mathcal{H}_{\mu\nu\rho\sigma}) \) are given by (2.17).

The divergence in (2.15) is canceled by adding to the action the generalisation of the York–Gibbons–Hawking term \[19, 21\] (see also \[22\]):

\[ \bar{S} = - \oint_{\partial M} d\Sigma_\mu n^\mu \Psi \cdot K, \]

(2.19)

where \( d\Sigma_\mu \) is the normal to the boundary \( \partial M \) being proportional to the volume element of \( \partial M \). Here \( d\Sigma_\mu \) is outward-pointing if spacelike and inward-pointing if timelike. The field equations (2.4), (2.5) and (2.6) then derive from a Dirichlet variational principle where the induced metric \( h_{ij} \) and \( \Psi^{ij} \) have to be held fixed on \( \partial M \).

Let us note that there can be constraints on \( \Psi^{ij} \) depending on the form of \( f \) (see Section \[11\] for general discussion about this constraint) and in some specific cases \( \Psi^{ij} \) is given as a function of \( K_{ij} \). Then the surface term \( (2.19) \) cannot be used to remove the second derivatives from the action. This is for example what happens in the Lovelock case \[13\]. For simplicity, we do not consider such exceptional cases in this paper and the divergence in \( (2.19) \) is hereafter discarded. More details about the surface term will be presented elsewhere \[13\].

III. CANONICAL FORMALISM

A. First-order action and Hamiltonian

Momenta conjugate to the dynamical variables \( h_{ij} \) and \( \Psi^{ij} \) are defined as, recalling the definition (2.10) of \( K_{ij} \),

\[
\begin{align*}
\rho^{ij} &\equiv \frac{\delta L^*}{\delta h_{ij}} = -\frac{\sqrt{\mathcal{H}}}{2} \left[ h^{ij} \Psi \cdot K + K \Psi^{ij} + 2 (\Psi \cdot K)^{(ij)} + N^{-1} (\dot{\Psi}^{ij} - L_\beta \Psi^{ij}) - 2 \frac{\partial f}{\partial \rho_{ijkl}} K_{kl} \\
&\quad + N^{-1} D_k \left( N \frac{\partial f}{\partial \rho_{k(ij)}} \right) \right], \\
\Pi_{ij} &\equiv \frac{\delta L^*}{\delta \Psi^{ij}} = -\sqrt{\mathcal{H}} K_{ij},
\end{align*}
\]

(3.1)

where it is understood that the derivatives \( \partial f/\partial \rho_{ijkl} \) and \( \partial f/\partial \rho_{ijk} \) have all the symmetries of \( \rho_{ijkl} \) and \( \rho_{ijk} \), respectively, and we have used the constraints (2.17) and discarded a total divergence in \( \rho^{ij} \).\footnote{When integrated over space this divergence yields the following contribution to the momentum: \((1/2) \oint dS_k (\partial f/\partial \rho_{k(ij)})\), where \( dS_k \) is the outward-pointing normal to the \((D-2)\)-dimensional sphere \( S \) being proportional to the volume element of \( S \). Such a term, as well as all spatial divergences, will be discarded in this paper but are important when studying, e.g., the energy of the system or junction conditions \[13\].} Inversion yields the velocities
in terms of the canonical variables (see (3.9) below for their explicit expression) and the first-order Lagrangian then is
\[ L^* = p \cdot \dot{h} + \Pi \cdot \dot{\Psi} - \mathcal{H}^* - \partial_i (\sqrt{h} V^i) \] (3.2)
with
\[ V^i = \Psi^{ij} \partial_j N - N D_j \Psi^j + 2 \frac{h^{ij}}{\sqrt{h}} \beta_j - 2 \frac{\Pi_{jk}}{\sqrt{h}} \Psi^{ij} \beta^k , \] (3.3)
where \( \mathcal{H}^* \) is the Hamiltonian density:
\[ \mathcal{H}^*[h_{ij}, p^{ij}, \Psi^{ij}, \Pi_{ij}, N, \beta^i, \Omega_{ij}] = N C + \beta^i C_i \] (3.4)
with
\[
\begin{align*}
C &\equiv \frac{1}{\sqrt{h}} ( (\Psi \cdot \Pi) + \Psi \cdot \Pi - 2 \rho \cdot \Pi ) + \sqrt{h} \left[ D_{ij} \Psi^{ij} - \left( \Psi \cdot \Omega + \frac{1}{2} f(\varrho_{\mu
u\rho\sigma}) \right) \right], \\
C_i &\equiv -2 \sqrt{h} D_j \left( \frac{p_j}{\sqrt{h}} \right) + \Pi_{jk} D_k \Psi^{jk} + 2 \sqrt{h} D_k \left( \frac{\Pi_{ij}}{\sqrt{h}} \Psi^{jk} \right),
\end{align*}
\] (3.5)
where \( \Pi = h^{ij} \Pi_{ij} \) and \( A \cdot B \cdot C = A^k B^j C^k \), and the spatial tensors \( \rho_{ijkl} \) and \( \rho_{ijk} \) in \( f(\varrho_{\mu
u\rho\sigma}) \) are now given by
\[ \rho_{ijkl} = \frac{\Pi_{ik} \Pi_{jl} - \Pi_{il} \Pi_{jk}}{h} + R_{ijkl} \, , \quad \rho_{ijk} = -D_i \left( \frac{\Pi_{jk}}{\sqrt{h}} \right) + D_k \left( \frac{\Pi_{ij}}{\sqrt{h}} \right) . \] (3.6)

**B. Hamilton’s equations and the algebra of constraints**

The equations of motion consist first in two sets of constraint equations, \( \delta \mathcal{H}^* / \delta N = 0 \) and \( \delta \mathcal{H}^* / \delta \beta^i = 0 \), which are
\[ C = 0 , \quad C_i = 0 . \] (3.7)
The equation of motion for \( \Omega_{ij} \), \( \delta \mathcal{H}^* / \delta \Omega_{ij} = 0 \), is also a constraint:
\[ 2 \Psi^{ij} + \frac{\partial f}{\partial \Omega_{ij}} = 0 . \] (3.8)
Depending on the function \( f(\varrho_{\mu
u\rho\sigma}) \), this equation may or may not be invertible to give all the components of \( \Omega_{ij} \) in terms of \( \Psi^{ij} \) as well as \( h_{ij} \), \( \Pi_{ij} \) and their spatial derivatives. As announced in the Introduction, the number of extra degrees of freedom beyond those of General Relativity will depend on the invertibility properties of \( (3.8) \). We shall come back to this issue at the end of this Section. A systematic way to reduce the action using \( (3.8) \) will be presented in Section VI on some specific examples.

As for the dynamical equations, the first set, \( \dot{h}_{ij} = \delta \mathcal{H}^* / \delta p^{ij} \) and \( \dot{\Psi}^{ij} = \delta \mathcal{H}^* / \delta \Pi_{ij} \), gives the velocities in terms of the canonical variables:
\[
\begin{align*}
\dot{h}_{ij} &= L_\beta h_{ij} - 2 N \frac{\Pi_{ij}}{\sqrt{h}} , \\
\dot{\Psi}^{ij} &= L_\beta \Psi^{ij} + \frac{N}{\sqrt{h}} \left( \Pi \Psi^{ij} + (\Psi \cdot \Pi) h^{ij} + 2 (\Psi \cdot \Pi) h^{(ij)} - 2 p^{ij} \right) - 2 N \frac{\Pi_{kl}}{\sqrt{h}} \frac{\partial f}{\partial \rho_{ikjl}} - D_k \left( N \frac{\partial f}{\partial \rho_{k(ij)}} \right) .
\end{align*}
\] (3.9)
The second set, \( \dot{p}^{ij} = -\delta \mathcal{H}^* / \delta h_{ij} \) and \( \dot{\Pi}_{ij} = -\delta \mathcal{H}^* / \delta \Psi^{ij} \) yields, using (3.7):
\[
\begin{align*}
\dot{p}^{ij} &= \sqrt{h} L_\beta \left( \frac{p^{ij}}{\sqrt{h}} \right) + \frac{N}{\sqrt{h}} \left[ (\Psi \cdot \Pi) \Pi^{ij} + \Psi_{kl} \Pi^{ik} \Pi^{jl} + h^{ij} ( (\Psi \cdot \Pi) \Pi + 2 \rho \cdot \Pi - 2 p \cdot \Pi ) \right] \\
&\quad + \sqrt{h} \left[ D_k (\Psi^{ij} \partial_k N - 2 \Psi^{(ij)k} \partial^k N) + h^{ij} ( N D_{kl} \Psi^{kl} - \Psi^{kl} D_k N ) \right] + \frac{\sqrt{h}}{2} \frac{\delta (N f)}{\delta h_{ij}} , \\
\dot{\Pi}_{ij} &= \sqrt{h} L_\beta \left( \frac{\Pi_{ij}}{\sqrt{h}} \right) - \frac{N}{\sqrt{h}} ( \Pi \Pi_{ij} + (\Pi \cdot \Pi)_{ij} ) + \sqrt{h} ( N \Omega_{ij} - D_{ij} N ) ,
\end{align*}
\] (3.10)

\[ ^6 \text{Notice that nonlinearity of } f \text{ in } \Omega_{ij} = \mathcal{K}_{mn} \text{ is essential for the invertibility of } (3.8) \text{ as well as for the appearance of fourth-order time-derivatives in (1.2).} \]
where the last term of the right-hand side of the first equation reads, using (3.8),
\[
\frac{\delta (N f)}{\delta h_{ij}} = -N R^{ijklm}_{ij} \frac{\partial f}{\partial \rho_{ijklm}} + 2 D_{kl} \left( N \frac{\partial f}{\partial \rho_{k(ij)}} \right) - 4 N \frac{\Pi^{ijkl}}{h} \frac{\partial f}{\partial \rho_{ijkl}} - 2 N h_{ij} \frac{\Pi_{km}}{h} \frac{\partial f}{\partial \rho_{klm}}
\]
up to surface integrals which we omit.

We have checked that, as they must, the two constraint equations (3.7) with (3.8) reproduce the (00) and the (0\(i\)) components of the vacuum Euler–Lagrange equation of motion (1.2) and that (3.9) and (3.10) are nothing but their (\(ij\)) components modulo constraints (the calculation is fairly involved and the details will be presented elsewhere [13]).

To compute the Poisson brackets of the secondary constraints (3.5), it is useful to define smeared quantities by
\[
H[\nu] \equiv \int_{\Sigma_t} d^{D-1}x C \nu, \quad M[\xi^i] \equiv \int_{\Sigma_t} d^{D-1}x C_i \xi^i,
\]
where \(\nu(x^k)\) and \(\xi^i(x^k)\) are test functions. This allows us to integrate by parts and to ignore boundary terms, as everywhere else in this paper. A calculation shows that the Poisson brackets of the smeared Hamiltonian and momentum constraints, modulo (3.8), read the same as in General Relativity:
\[
\{ H[\nu_1], H[\nu_2] \} = M[\nu_1 \partial^i \nu_2 - \nu_2 \partial^i \nu_1],
\]
\[
\{ M[\xi^i_1], M[\xi^i_2] \} = M[\xi^i_1 \partial_j \xi^i_2 - \xi^i_2 \partial_j \xi^i_1],
\]
\[
\{ M[\xi^i], H[\nu] \} = H[\mathcal{L}_\xi \nu].
\]

Here it may be worth making a few comments on the relation between the constraint (3.8) and the Hamiltonian and momentum constraints.

If (3.8) is invertible with respect to \(\Omega_{ij}\), we may eliminate \(\Omega_{ij}\) from the action completely, leaving no further constraints. Then the remaining constraints are the Hamiltonian and momentum constraints, which are of “first-class” [23, 24] representing the diffeomorphism invariance of the \(f(\text{Riemann})\) action.

On the other hand, if (3.8) is not completely invertible, some of the components give rise to non-trivial extra “primary” constraints on the dynamical variable \(\Psi^{ij}\). Then time derivatives of these extra constraints may give rise to “secondary” constraints [23]. After all the extra constraints, irrespective of primary or secondary, are spelled out, one can classify them into first-class and second-class. In most cases, these constraints will be of second-class, which may be inserted in the action to reduce the dynamical degrees of freedom [16]. Then, for consistency, the Hamiltonian and momentum constraints expressed in terms of the reduced phase space variables should also satisfy Eq. (3.13). As examples, this will be confirmed below in the case of Einstein gravity as well as of \(f(\text{R})\) gravity.

In some exceptional cases, these extra constraints may happen to be of first-class, that is, the action may acquire a larger gauge invariance: It is known that the “Weyl” action in \(D = 4\) has a conformal invariance and in that case the constraint algebra of the Poisson brackets is extended to incorporate the generator of conformal transformations [3].

Now that we have completed the presentation of the general formalism, let us turn to some “practical” applications.

IV. PHASE SPACE REDUCTION

In this Section, we show how the constraint (3.8) is utilised to find “reduced” Hamiltonians for various sub-classes of \(f(\text{Riemann})\) gravity. We follow the procedure advocated in [16]: Second-class constraints arising from (3.8) are inserted into the first-order action (3.2) to eliminate as many components of \(\Omega_{ij}\) and of dynamical variables as possible. If there remain any constraints on \(\Psi^{ij}\), irrespective of whether they give rise to further constraints on the other dynamical variables or not, it becomes necessary to redefine canonical momenta conjugate to the reduced sets of \(h_{ij}\) and \(\Psi^{ij}\). As we shall see below, they are relatively easily read off from the first-order action. Consequently the Hamiltonian in terms of the reduced set of variables is obtained.
A. \( f = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \)

This is a simple “generic” example where all the components of \( \Omega_{ij} \) can be extracted from \( (3.8) \), so that there are no extra constraint on the dynamical variables.

The first step is to decompose \( f(\theta_{\mu\nu\rho\sigma}) \) as
\[
f(\theta_{\mu\nu\rho\sigma}) = \theta_{\mu\nu\rho\sigma} \theta^{\mu\nu\rho\sigma} = 4 \Omega \cdot \Omega + \rho_{ijkl} \rho^{ijkl} - 4 \rho_{ijk} \rho^{ijk},
\]
where \( \rho_{ijkl} \) and \( \rho_{ijk} \) are given in \( (3.6) \). Thus the constraint \( (3.8) \) reads
\[
\Omega_{ij} = -\frac{\Psi_{ij}}{4}.
\]
This constraint is inserted into the first-order action \( (3.2) \) to eliminate \( \Omega_{ij} \). The Hamiltonian density is therefore given by \( (3.4) \) and \( (3.5) \), where
\[
\Psi \cdot \Omega + \frac{1}{2} f(\theta_{\mu\nu\rho\sigma}) = -\frac{\Psi \cdot \Psi}{8} + \frac{1}{2} \rho_{ijkl} \rho^{ijkl} - 4 \rho_{ijk} \rho^{ijk}.
\]
It depends on the usual set of variables of General Relativity, \( \{h_{ij}, p^{ij}, N, \beta^j\} \), plus \( D(D-1)/2 \) extra degrees of freedom, \( \{\Psi^{ij}, \Pi_{ij}\} \).

Let us mention that the form of the canonical equations of motion \( (3.9) \) and \( (3.10) \) is unchanged even if \( (3.8) \) is inserted into the Hamiltonian before taking the variations. This is because the Hamiltonian depends on \( \Omega_{ij} \) only through the combination
\[
\Psi \cdot \Omega + \frac{1}{2} f(\theta_{\mu\nu\rho\sigma}),
\]
so that the additional terms appearing in the equations of motion due to the constraint \( (3.8) \) are all proportional to the derivative of \( (4.4) \) with respect to \( \Omega_{ij} \), which vanish by virtue of \( (3.8) \) itself.

It is also worth mentioning here a decomposition of \( \Psi^{ij} \) in the “generic” case. The \( D(D-1)/2 \) components of \( \Psi^{ij} \) can be decomposed into two irreducible parts:
\[
\Phi \equiv \frac{\Psi}{D-1}, \quad \psi^{ij} \equiv \Upsilon \Psi^{ij},
\]
where \( \Psi \equiv h_{ij} \Psi^{ij} \) and the symbol \( \Upsilon \) denotes the traceless part. Noting that there is an arbitrariness in choosing canonical momenta conjugate to the variables \( \{h_{ij}, \Phi, \psi^{ij}\} \), one finds it most convenient to introduce
\[
\tilde{p}^{ij} \equiv p^{ij} - \frac{1}{D-1} (\Pi \Psi^{ij} + \Psi \Upsilon \Pi^{ij}), \quad \pi_{ij} \equiv \Upsilon \Pi_{ij}
\]
resulting in
\[
\mathcal{L}^* = \tilde{p} \cdot \dot{h} + \Pi \dot{\Phi} + \pi \cdot \dot{\psi} + \mathcal{H}^*,
\]
where \( \mathcal{H}^* \) is to be given in terms of the new variables. Now it is manifest that the “generic” theory contains a scalar degree of freedom \( \{\Phi, \Pi\} \) and traceless tensor degrees of freedom \( \{\psi^{ij}, \pi_{ij}\} \) having \( D(D+1)(D-2)/2 \) components on top of the canonical metric degrees of freedom \( \{h_{ij}, \tilde{p}^{ij}\} \).

B. \( f = R \)

We show here how the ADM Hamiltonian for General Relativity follows from our general formalism. The decomposition of \( f(\theta_{\mu\nu\rho\sigma}) \) is
\[
f = g = g^{\mu\rho} g^{\nu\sigma} \theta_{\mu\nu\rho\sigma} = -2 \Omega + \rho,
\]
where \( \Omega \equiv h_{ij} \Omega_{ij} \) and \( \rho \equiv h^{ik} h^{jl} \rho_{ijkl} \) so that the primary constraint \( (3.8) \) reads
\[
\Psi^{ij} = h^{ij}.
\]
Hence it “freezes” all the extra degrees of freedom of $\Psi^{ij}$. Though (4.9) does not give $\Omega_{ij}$, the terms containing $\Omega_{ij}$ in the Hamiltonian will all drop out due to the linear dependence of $f$ on $\Omega_{ij}$.

The Hamilton equations (3.9) for the velocities give a secondary constraint. It tells us that, since $\dot{\Psi}^{ij} = \dot{h}^{ij}$, the momentum conjugate to $\Psi^{ij}$ is also frozen:

$$\Pi_{ij} = 2p_{ij} - \frac{2p}{D}h_{ij}, \quad (4.10)$$

where $p \equiv h_{ij}p^{ij}$.

We now follow the procedure advocated in [16] and insert (4.9) into the first-order action (3.2), which becomes (ignoring the divergence)

$$\mathcal{L}^* = \tilde{p} \cdot \dot{h} - \mathcal{H}^*, \quad (4.11)$$

with

$$\tilde{p}^{ij} \equiv -p^{ij} + 2\tilde{p}^{ij} D h^{ij}, \quad (4.12)$$

This $\tilde{p}^{ij}$ plays the role of the new momentum conjugate to $h_{ij}$.

We now gather the results, to wit, all the dynamical variables are expressed in terms of $\{h_{ij}, \tilde{p}^{ij}\}$ as

$$\begin{align*}
\Psi^{ij} &= h^{ij}, \\
\Pi_{ij} &= -2\tilde{p}_{ij} + 2\tilde{p}^{ij} D h^{ij}, \\
p_{ij} &= -\tilde{p}_{ij} + 2\tilde{p}^{ij} D h^{ij},
\end{align*} \quad (4.13)$$

where $\tilde{p} \equiv h_{ij} \tilde{p}^{ij}$, and we plug them into the Hamiltonian (3.4) and (3.5) to obtain

$$\mathcal{H}^* = N C + \beta \mathcal{C}_i, \quad (4.14)$$

with

$$C = \frac{2}{\sqrt{h}} \left( \tilde{p} \cdot \tilde{p} - \frac{\tilde{p}^2}{D - 2} \right) - \frac{\sqrt{h}}{2} R, \quad \mathcal{C}_i = -2 \sqrt{h} D_j \left( \frac{\tilde{p}^j_i}{\sqrt{h}} \right). \quad (4.15)$$

This is nothing but the ADM Hamiltonian for General Relativity in $D$ dimensions. As for the equations of motion (3.9) and (3.10), they reduce to the ADM equations, see e.g. [18]. Moreover, the constraints (4.15) are first-class as is well known.

### C. $f = f(R)$

We show here how our general formalism yields the “Jordan-frame” Hamiltonian of $f(R)$ gravity. As above we have $\varrho = -2\Omega + \rho$ with $\Omega \equiv h^{ij} \Omega_{ij}$ and $\rho \equiv h^{ik} h^{jl} \rho_{ijkl}$, that is, using (3.6):

$$\varrho = -2\Omega + \frac{\Pi^2 - \Pi \cdot \Pi}{h} + R, \quad (4.16)$$

so that Eq. (3.8) reads

$$\Psi^{ij} = f'(\varrho) h^{ij} \iff \Phi = \frac{h \cdot \Psi}{D - 1} = f'(\varrho), \quad \tau \Psi^{ij} = 0, \quad (4.17)$$

where the symbol $\tau$ denotes the traceless part. This primary constraint tells us, first, that the trace $\Omega$ is known in terms of $\Phi$ and other variables, and that the traceless part of $\Psi^{ij}$ is constrained to be zero so that $\Psi^{ij}$ reduces to one scalar degree of freedom, $\Phi$. The traceless part of $\Omega_{ij}$ automatically disappears from the Hamiltonian since $\Psi^{ij} = \Phi^{ij}$ and $f(\varrho)$ depends only on the trace $\Omega$ from the beginning.

The traceless part of $\Pi_{ij}$ is also constrained through the Hamilton equations for the velocities (3.9). Indeed, the traceless part of the velocities now satisfies $\tau(\Psi^{ij}) = \tau(\dot{h}^{ij}) \Phi$ giving a secondary constraint

$$\tau \Pi_{ij} = \frac{2}{\Phi} \tau p_{ij}. \quad (4.18)$$
In order now to find a new momentum $\tilde{p}^{ij}$ conjugate to $h_{ij}$, we again follow [10] and insert (4.17) and (4.18) into the first-order action (3.2), which now reads (ignoring the divergence)

$$\mathcal{L}^* = \tilde{p} \cdot \dot{h} + \Pi \Phi - \mathcal{H}^*, \tag{4.19}$$

with

$$\tilde{p}^{ij} = -p^{ij} + \frac{h_{ij}}{D-1} (2 p - \Phi \Pi), \tag{4.20}$$

where $p \equiv h_{ij} p^{ij}$. Gathering the results:

$$\Psi^{ij} = \Phi h^{ij}, \quad \Pi_{ij} = \frac{1}{\Phi} \left[ -2 \tilde{p}_{ij} + \frac{h_{ij}}{D-1} (2 \tilde{p} + \Phi \Pi) \right], \quad p^{ij} = -\tilde{p}^{ij} + \frac{h_{ij}}{D-1} (2 \tilde{p} + \Phi \Pi), \tag{4.21}$$

where $\tilde{p} \equiv h_{ij} \tilde{p}^{ij}$, and using (4.16) and (4.17), we have

$$\Psi \cdot \Omega = \frac{1}{2} \Phi (R - \varrho) + \frac{D-2}{2(D-1)} \Phi \Pi^2 - \frac{2}{\Phi} \left[ \sqrt{h} \left( \Phi \varrho - f(\varrho) - \Phi R + 2 D_i D^i \Phi \right) \right], \tag{4.22}$$

where it is understood that $\varrho$ is known in terms of $\Phi$ via $f'(\varrho) = \Phi$. Therefore the Hamiltonian (3.4) becomes

$$\mathcal{H}^* = N \mathcal{C} + \beta^i \mathcal{C}_i$$

with

$$\begin{aligned}
\mathcal{C} &= \frac{2}{\sqrt{h}} \left[ \frac{\sqrt{h} \Phi}{\Phi} \cdot \tau \tilde{p} + \frac{D-2}{4(D-1)} \Phi \Pi^2 - \frac{\tilde{p} \Pi}{D-1} \right] + \sqrt{h} \left( \Phi \varrho - f(\varrho) - \Phi R + 2 D_i D^i \Phi \right), \\
\mathcal{C}_i &= -2 \sqrt{h} D_j \left( \frac{\tilde{p}^j}{\sqrt{h}} \right) + \Pi \partial_i \Phi.
\end{aligned} \tag{4.23}$$

The constraints (4.23) give the standard Jordan-frame Hamiltonian for $f(R)$ gravity, see [9]. Moreover, it can be checked that they are first-class.

One can also make a canonical transformation of the Hamiltonian into that of Einstein gravity with a minimally coupled scalar field. These metrics are related by a $D$-dimensional conformal transformation, see e.g. [25] and [9].

**D.** $f = \mathcal{C}_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$

We consider here the case when $f = \mathcal{C}_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$ where $C_{\mu\nu\rho\sigma}$ is the $D$-dimensional Weyl tensor. In terms of the Riemann and Ricci tensors, it is expressed as

$$f(R_{\mu\nu\rho\sigma}) = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - \frac{4}{D-2} R_{\mu\nu} R^{\mu\nu} + \frac{2}{(D-1)(D-2)} R^2. \tag{4.24}$$

This theory yields conformally invariant equations of motion in $D = 4$.

To decompose $f(\varrho_{\mu\nu\rho\sigma})$, we need

$$\begin{aligned}
\varrho_{\mu\nu\rho\sigma} \varrho_{\rho^\sigma\mu^\nu} &= 4 \Omega \cdot \Omega + \rho_{ijkl} \rho^{ijkl} - 4 \rho_{ijk} \rho^{ij}, \\
\varrho_{\mu\nu \varrho^{\mu\nu}} &= \Omega^2 + \Omega \cdot \Omega - 2 \rho \cdot \Omega + \rho - 2 \rho_i \rho^i, \\
\varrho^2 &= (-2 \Omega + \rho)^2,
\end{aligned} \tag{4.25}$$

where $\Omega \equiv h^{ij} \Omega_{ij}$, $\rho_{ij} \equiv h^{kl} \rho_{ijkl}$, $\rho \equiv h^{ik} h^{jl} \rho_{ijkl}$ and $\rho_i \equiv h^{jk} \rho_{ijk}$ with

$$\begin{aligned}
\rho_{ijkl} &= \frac{\Pi_{ik} \Pi_{jl} - \Pi_{il} \Pi_{jk}}{h} + R_{ijkl}, \\
\rho_{ij} &= \frac{\Pi \Pi_{ij} - (\Pi \cdot \Pi)_{ij}}{h} + R_{ij}, \\
\rho &= \frac{\Pi^2 - \Pi}{h} + R,
\end{aligned} \tag{4.26}$$

For comparison with the Riemann squared case, see [4.1] and (3.6). The constraint equation (3.8) now reads

$$\Psi^{ij} = -\frac{4}{D-2} \left[ (D-3) \tau \Omega^{ij} + \tau \rho^{ij} \right], \tag{4.27}$$
where again \( \mathbb{T} \) denotes the traceless part. Thus only the traceless part of \( \Omega_{ij} \) is determined. As a consequence, there appears an extra primary constraint \( \hbar \cdot \dot{\Psi} = 0 \), that is, the trace part of \( \Psi^{ij} \) disappears from the action. This implies the number of degrees of freedom in \( \Psi^{ij} \) will be reduced by one.

Now we have a relation between the velocities \( \dot{h}_{ij} + \hbar \cdot \dot{\Psi} = 0 \) as a consequence of the above primary constraint so that the Hamilton equations for the velocities \( (3.9) \) yield a secondary constraint,

\[
p - \frac{D}{2} \tau \Psi \cdot \tau \Pi = 0,
\]

where \( p \equiv h_{ij} p^{ij} \). In contradistinction with the previous example of \( f(R) \) gravity, this equation does not constrain \( \Pi \) but \( p \). For the moment, we decide not to insert this constraint into the action. The reason will be clarified below.

In order now to find the new momenta \( \tilde{p}^{ij} \) and \( \pi_{ij} \) conjugate to \( h_{ij} \) and \( \psi^{ij} \equiv \tau \Psi^{ij} \), respectively, we again follow [10] and write the first-order action \( (3.2) \) as (ignoring the divergence)

\[
\mathcal{L}^* = \tilde{p} \cdot \dot{h} + \pi \cdot \dot{\psi} - \mathcal{H}^*,
\]

with

\[
\tilde{p}^{ij} \equiv p^{ij} - \frac{\Pi}{D - 1} \psi^{ij}, \quad \pi_{ij} \equiv \tau \Pi_{ij}.
\]

Let us gather the results:

\[
\Psi^{ij} = \psi^{ij}, \quad \Pi_{ij} = \pi_{ij} + \frac{\Pi}{D - 1} h_{ij}, \quad p^{ij} = \tilde{p}^{ij} + \frac{\Pi}{D - 1} \psi^{ij},
\]

where we see that \( \Pi \) is not determined. Plugging these into the Hamiltonian \( (3.4) \) one finds

\[
\mathcal{H}^* = N C + \beta^i C_i + \frac{2}{D - 1} N \frac{\Pi}{\sqrt{h}} C_W,
\]

with

\[
\begin{align*}
C &= \frac{1}{\sqrt{h}} \left( \frac{D - 4}{D - 3} \psi \cdot \pi - 2 \tilde{p} \cdot \pi \right) + \sqrt{h} \left[ \frac{D - 2}{8(D - 3)} \psi \cdot \psi + D_{ij} \psi^{ij} + \frac{1}{D - 3} \psi \cdot \tau R \right] \\
&\quad - \frac{\sqrt{h}}{2} \left( \tau \rho_{ijkl} \psi^{ijkl} - 4 \tau \rho_{ijkl} \psi^{ijkl} \right), \\
C_i &= -2 \sqrt{h} D_j \left( \frac{\tilde{p}^{ij}}{\sqrt{h}} + \pi_{jk} D_i \psi^{jk} \right) + \frac{\pi_{ij} \psi^{jk}}{\sqrt{h}}, \\
C_W &= \frac{D}{2} \psi \cdot \pi - \tilde{p},
\end{align*}
\]

where \( \tilde{p} \equiv h_{ij} \tilde{p}^{ij} \). We note that the Hamiltonian is linear in \( \Pi \). In other words, \( \Pi \) acts as a Lagrange multiplier whose equation of motion \( C_W = 0 \), coincides with the secondary constraint \( (4.28) \).

Boulware [8] showed that, in \( D = 4 \), the Poisson bracket algebra of the constraints closes, that is, \( C_W \) as well as \( C \) and \( C_i \) are first-class constraints. Boulware also showed that \( C_W \) is the generator of conformal transformations under which the theory is invariant. Thus, to summarise, the Lagrangian for the Weyl squared theory in \( D = 4 \) keeps the original form \( (4.29) \),

\[
\mathcal{L}^* = \tilde{p} \cdot \dot{h} + \pi \cdot \dot{\psi} - \mathcal{H}^*,
\]

where \( \mathcal{H}^* \) is given by \( (4.32) \), but with all the constraints \( C, C_i \) and \( C_W \) being first-class.

In passing, we note that it may be more transparent to separate out the determinant of \( h_{ij} \) as an independent canonical variable. Namely, in addition to \( \tilde{p} \equiv h_{ij} \tilde{p}^{ij} \), introducing

\[
\eta \equiv \ln h^{1/(D - 1)}, \quad \tilde{h}_{ij} \equiv e^{-\eta} h_{ij}, \quad \tilde{p}^{ij} \equiv e^\eta \tau \tilde{p}^{ij},
\]

we express the Lagrangian as

\[
\mathcal{L}^* = \tilde{p} \cdot \dot{\tilde{h}} + \pi \cdot \dot{\psi} - \mathcal{H}^*,
\]
where $H^*$ is given by

$$H^* = NC + \beta^i C_i + WC_W.$$  \hfill (4.37)

Here we have replaced $\Pi$ by

$$\Pi \equiv 2 \frac{(D-1)}{N} \sqrt{h}^{-1} \Pi,$$

and $C$, $C_i$ and $C_W$ are to be expressed in terms of the new canonical variables $\{\hat{h}_{ij}, \hat{p}^{ij}\}$, $\{\eta, \tilde{p}\}$ and $\{\psi^{ij}, \pi_{ij}\}$.

In $D > 4$, however, the Poisson brackets of $C_W$ with $C$ and $C_i$ no longer close, giving rise to additional secondary constraints. This is because, although the $D$-dimensional Weyl tensor, $C_{\mu \nu \rho \sigma}$, is conformally invariant in any dimensions, the action is not conformally invariant any longer in dimensions other than $D = 4$. In this case, there may be more secondary constraints from the time derivatives of these secondary constraints. They will be all second-class constraints. Inserting them into the action will reduce the phase space considerably. Because of rather involved calculations, we have not checked how many second-class constraints would appear in the end. We plan to come back to this issue in a future publication \[13\].

V. CONCLUSION

We have presented in this paper a canonical formulation of $f(Riemann)$ theories in a form as compact as possible. It includes in a unifying manner well-known subcases and should prove useful to analyse the properties of various theories of “extended gravity,” such as the global charges associated with the solutions, their (in)stability and the (non-)positivity of energy or the junction conditions.

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APPENDIX A: THE OSTROGRADSKY HAMILTONIAN OF $f(Riemann)$ GRAVITY

For completeness we give here the canonical transformation which relates the variables used in this paper to the “Ostrogradsky” one, that is, the extrinsic curvature of the ADM foliation, see [6] for an Ostrogradsky treatment of quadratic theories.

The Ostrogradsky action written in terms of $\varsigma_{\mu \nu \rho \sigma} \equiv R_{\mu \nu \rho \sigma}|_{K_{ij} = Q_{ij}}$, where $Q_{ij}$ is independent of other fields, is

$$S = \int_M d^D x \sqrt{-g} \left[ \frac{1}{2} f(\varsigma_{\mu \nu \rho \sigma}) + 2 u^{ij} (K_{ij} - Q_{ij}) \right].$$  \hfill (A1)

The momenta canonically conjugate to $h_{ij}$ and $Q_{ij}$, respectively, are

$$k_{ij} \equiv \frac{\delta S}{\delta \dot{h}_{ij}} = \sqrt{h} u^{ij}, \quad P_{ij} \equiv \frac{\delta S}{\delta \dot{Q}_{ij}} = -\frac{\sqrt{h}}{2} \frac{\partial f}{\partial \Sigma_{ij}}.$$  \hfill (A2)

where $\Sigma_{ij} \equiv n^\mu n^\nu \varsigma_{\mu \nu \rho \sigma}$. We have the following relations with the variables of this paper:

$$\Pi_{ij} = -\sqrt{h} Q_{ij}, \quad \Psi^{ij} = \frac{P^{ij}}{\sqrt{h}}.$$  \hfill (A3)

They give

$$p \cdot \dot{h} + \Pi \cdot \dot{\Psi} = k \cdot \dot{h} + P \cdot \dot{Q} - \frac{d}{dt} (Q \cdot P)$$  \hfill (A4)
with
\[ k_{ij} = p_{ij} + \frac{Q \cdot P}{2} h_{ij}. \]  
(A5)

Plugging these relations into our Hamiltonian (3.4), we obtain the Ostrogradsky Hamiltonian \( \mathcal{H}^* = N C + \beta^i C_i \) with
\[
\begin{align*}
C &= Q \cdot Q \cdot P + 2 Q \cdot k - P \cdot \Sigma + \sqrt{\hbar} D_{ij} \left( \frac{P_{ij}}{\sqrt{\hbar}} \right) - \frac{\sqrt{\hbar}}{2} f(\varsigma_{\mu\nu\rho\sigma}), \\
C_i &= -2 \sqrt{\hbar} D_j \left( \frac{k_{ij}}{\sqrt{\hbar}} \right) + P^{jk} D_i Q_{jk} - 2 \sqrt{\hbar} D_k \left( \frac{Q_{ij} P^{jk}}{\sqrt{\hbar}} \right),
\end{align*}
\]
(A6)

where \( \varsigma_{\mu\nu\rho\sigma} \) is expressed in terms of \( h_{ij}, Q_{ij} \) and \( \Sigma_{ij} \), and where \( \Sigma_{ij} \) is constrained to satisfy
\[
\frac{\partial f}{\partial \Sigma_{ij}} = -2 \frac{P_{ij}}{\sqrt{\hbar}}.
\]
(A7)

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