Positivity and nonadditivity of quantum capacities using generalized erasure channels

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Abstract—We consider various forms of a process, which we call gluing, for combining two or more complementary quantum channel pairs \((B,C)\) to form a composite. One type of gluing combines a perfect channel with a second channel to produce a \textit{generalized erasure channel pair} \((B_2,C_2)\). We consider two cases in which the second channel is (i) an amplitude-damping, or (ii) a phase-damping qubit channel; (ii) is the \textit{dephrasure channel} of Leditzky et al. For both (i) and (ii), \((B_2,C_2)\) depends on the damping parameter \(0 \leq p \leq 1\) and a parameter \(0 \leq \lambda \leq 1\) that characterizes the gluing process. In both cases we study \(Q^{(1)}(B_2)\) and \(Q^{(1)}(C_2)\), where \(Q^{(1)}\) is the channel coherent information, and determine the regions in the \((p,\lambda)\) plane where each is zero or positive, confirming previous results for (ii). A somewhat surprising result for which we lack any intuitive explanation is that \(Q^{(1)}(C_2)\) is zero for \(\lambda \leq 1/2\) when \(p = 0\), but is strictly positive (though perhaps extremely small) for all values of \(\lambda > 0\) when \(p > 0\) by even the smallest amount. In addition we study the nonadditivity of \(Q^{(1)}(B_2)\) for two identical channels in parallel. It occurs in a well-defined region of the \((p,\lambda)\) plane in case (i). In case (ii) we extend previous results for the dephrasure channel without, however, identifying the full range of \((p,\lambda)\) values where nonadditivity occurs. Again, an intuitive explanation is lacking.

Index Terms—Quantum Information, Quantum channel capacity, Non-additivity

I. INTRODUCTION

Understanding the capacity of a noisy quantum channel to transmit information is a central and challenging problem in quantum information theory. In contrast to the case of a classical channel one can define several capacities for a quantum channel, among them the capacity to transmit classical information \([1],[2]\), the private capacity \([3]\), and—the subject of the present paper—the \textit{quantum capacity}, a measure of its ability to transmit quantum information. The asymptotic capacity \(C\) of a classical channel was shown by Shannon \([4]\) to be equal to \(C^{(1)}\), the \textit{channel mutual information} i.e., the mutual information between input and output, when maximized over probability distributions of the input. An analog of the mutual information for a quantum channel \(B\) is the \textit{entropy bias or coherent information} \(\Delta(B,\rho)\) \([5]\), the difference of the von Neumann entropies of the outputs of \(B\) and its \textit{complementary channel} \(\tilde{C}\) for a given input density operator \(\rho\). Maximizing this over \(\rho\) yields a non-negative real number, the \textit{channel coherent information} (sometimes also called the \textit{single-letter quantum capacity}) \(Q^{(1)}(B)\). (A quantum channel \(B\) of the kind considered here is always a member of a complementary pair of channels \((B,C)\), \(C\) the complement \(B\) and vice versa, generated by a single isometry as discussed in Sec. II.)

A significant difference between quantum and classical channels is that when two classical channels are placed in parallel, the channel mutual information \(C^{(1)}\) of the combination is simply the sum of the individual channel mutual informations, whereas in the quantum case when channel \(B\) is placed in parallel with \(B'\) one has only an inequality:

\[Q^{(1)}(B \otimes B') \geq Q^{(1)}(B) + Q^{(1)}(B').\] (1)

This inequality can be strict, i.e., \(Q^{(1)}\) can be \textit{nonadditive} \([6]\). Nonadditivity makes it difficult to calculate the asymptotic quantum capacity \(Q(B)\) of a channel \(B\), the limit as \(n \to \infty\) of \(Q^{(1)}(B^{\otimes n})/n\) \([3],[7],[8]\). The asymptotic capacity \(C\) of two classical channels placed in parallel is simply the sum of the capacities of the individual channels. By contrast, due to nonadditivity (1) the asymptotic capacity \(Q(B \otimes B')\) of two quantum channels \(B\) and \(B'\) used in parallel may be greater than \(Q(B)+Q(B')\) \([9],[10]\), which implies that the asymptotic \(Q\), unlike its classical counterpart \(C\), does not completely capture a channel’s ability to transmit quantum information. The mathematical or physical principles behind nonadditivity are at present not well understood. Simple examples of nonadditivity are hard to construct. One source of difficulty is finding the global maximum of the function \(\Delta(B,\rho)\) which in general is not a concave function of \(\rho\).

If both \(B\) and \(B'\) are channels that are either degradable or antidegradable (see comments at the end of Sec. II) it is known that (1) is an equality, and therefore \(Q(B) = Q^{(1)}(B)\), and \(Q(B \otimes B') = Q(B) + Q(B')\) \([11],[12]\). For an antidegradable channel \(Q^{(1)}(B) = Q(B) = 0\), and the same is true for entanglement-binding channels \([13]\). But apart from special cases such as these it is in general not easy to determine whether \(Q^{(1)}\) or \(Q\) is positive or zero \([14],[15]\).

For the case of two identical channels in parallel, \(B' = B\), a simple example of nonadditivity has recently been constructed by Leditzky et al. \([16]\) using what they call the \textit{dephrasure channel}. To show nonadditivity they first find \(Q^{(1)}(B)\), and then make a guess or \textit{ansatz} \(\hat{\rho}_2\) for a bipartite input density operator for which \(\Delta(B^{\otimes 2},\hat{\rho}_2)\), a lower bound for \(Q^{(1)}(B^{\otimes 2})\), is larger than \(2Q^{(1)}(B)\). The ansatz approach can be extended to \(n\) identical channels in parallel in an obvious way to look for cases where \(\Delta(B^{\otimes n},\hat{\rho}_n)\) (and hence \(Q^{(1)}(B^{\otimes n}))\) exceeds \(nQ^{(1)}(B)\). This approach has been successfully applied for \(n \geq 5\) to the qubit depolarizing channel \([6]\) where \(Q^{(1)}\) is
known, and to other qubit Pauli channels [17]–[19] where $Q^{(1)}$ is believed to be known.

Our exploration of some of these issues begins with a general procedure for combining several quantum channels to form a new channel through a process we call gluing. It differs from the familiar procedures of placing channels in parallel or series, and it puts together in a single overall structure concepts such as subchannels [20], direct sums [21], and convex sums of channels (see Sec. 2.2 in [22]). A particular type of gluing results in what we call a block diagonal channel pair, an instance of which is the much-studied and well-understood erasure channel [23] with erasure probability $0 ≤ θ ≤ 1$, whose complement is also an erasure channel. The erasure channel can be regarded as the result of gluing together two perfect channels as discussed in Sec. IV. When one of the perfect channel pairs is replaced by an arbitrary complementary channel pair $(B_1, C_1)$, the result is a generalized erasure channel pair $(B_g, C_g)$. The $B_g$ channel can be viewed as a concatenation of $B_1$ with an erasure channel, and $C_g$ as an “incomplete erasure” channel.

We study two cases of such generalized erasure channel pairs. In the first, $B_1$ is a qubit-to-qubit amplitude damping channel, as is its complement $C_1$. In the second, $B_1$ is a qubit-to-qubit phase-damping channel whose complement is a measure-and-prepare channel; here $B_g$ is the dephrasure channel. In both cases the qubit channel pair $(B_1, C_1)$ depends on a parameter $0 ≤ p ≤ 1$, and thus $(B_g, C_g)$ depends on two parameters, $p$ and the erasure probability $θ$. For all values of these parameters we compute $Q^{(1)}(B_g)$ and $Q^{(1)}(C_g)$ by performing a global optimization and find the $(p, θ)$ values for which they are positive. The dependence of $Q^{(1)}(C_g)$ on these parameters is rather surprising—see Fig. 5 and the accompanying discussion—and worth further study.

In both the amplitude and phase damping cases we find nonadditivity, a strict inequality in (1), when both $B$ and $B'$ are $B_g$. Our results in the amplitude damping case indicate that nonadditivity occurs over a well-defined region in the space $(p, θ)$ of parameters, as shown in Fig. 2. For the phase-damping case, where $B_g$ is the dephrasure channel, our numerical results confirm and also extend the region of nonadditivity identified in [16], but without finding its precise boundaries. In addition we have carried out a limited exploration of higher-order nonadditivity by using various ansatzes, but without finding anything very interesting.

The remainder of this paper is structured as follows. Section II contains preliminary definitions and notation: in particular our use of isometries to construct a channel pair, and the use of projective decompositions of the identity (PDIs) to identify orthogonal subspaces. Definitions of the entropy bias $Δ(B, ρ)$ and the channel coherent information $Q^{(1)}(B)$ of a channel $B$, and (anti)degradable channels are also found in this section. Various gluing procedures for combining two or more channels are discussed at some length in Sec. III. The particular procedure that yields a block diagonal channel pair (see eq. (23)) is employed in Sec. IV to define a generalized erasure channel pair. The amplitude-damping case is discussed in Sec. V-A, and the phase-damping (dephrasure) case in Sec. V-B. The surprising positivity of $Q^{(1)}$ in the “incomplete erasure” situation found in both cases is discussed in Sec. V-C. A concluding Sec. VI contains a summary of our results and an indication of some unsolved problems that deserve further study. It is followed by two appendices devoted to some technical issues and details.

II. PRELIMINARIES

A quantum channel (completely positive trace preserving map) can always be constructed using an isometry $J$

$J : H_a \mapsto H_b \otimes H_c; \quad J^† J = I_a,
(2)$
mapping the Hilbert space $H_a$ representing the channel’s input to a subspace of $H_b \otimes H_c$, where $H_b$ and $H_c$ represent the direct and complementary channel outputs. The isometry preserves inner products, and this is ensured by the condition $J^† J = I_a$, where $I_a$ is the identity operator on $H_a$. We assume that the dimensions $d_b$, $d_b$, and $d_c$ of the three Hilbert spaces are finite and satisfy $d_a ≤ d_b d_c$, but are otherwise unrestricted. The isometry results in a pair of quantum channels with superoperators

$B(A) = Tr_c(J AJ^†), \quad C(A) = Tr_b(J AJ^†), \quad
(3)$
that map $H_a$, the space of operators on $H_a$, to the operator spaces $H_b$ and $H_c$, respectively. Given the superoperator $B$, the corresponding isometry $J$, and thus the superoperator $C$, is uniquely determined up to a unitary acting on $H_c$. Likewise, $C$ determines $B$ up to a unitary on $H_b$. One refers to $C$ as the complement of $B$, or $B$ as the complement of $C$, and the two together as a complementary pair of channels.

We shall be concerned with orthogonal subspaces of $H_a, H_b$, and $H_c$, and it is convenient to represent the subspaces using a projective decomposition of the identity (PDI), a collection of mutually orthogonal projectors that sum to the identity. Thus a PDI $\lbrace P_j \rbrace$ of $H_a$ is a set of projectors such that

$P_j = P_j^† = P_j^2; \quad P_j P_j = δ_{ij} P_j, \quad \sum_j P_j = I_a.
(4)$

Using it one can define the $j$th subspace $H_{aj}$ of $H_a$, as

$H_{aj} = P_j H_a; \quad
(5)$
that is to say, the collection of all $|ψ⟩$ such that $P_j |ψ⟩ = |ψ⟩$, which means they are orthogonal to any $|φ⟩$ in $H_{ak}$ with $k ≠ j$. Thus $H_a$ is a direct sum,

$H_a = \bigoplus_j H_{aj}; \quad
(6)$
of these orthogonal subspaces. Similarly, a PDI $\lbrace Q_j \rbrace$ can be used to partition $H_b$ into subspaces $H_{bj} = Q_j H_b$, and $\lbrace R_j \rbrace$ to partition $H_c$ into $H_{cj} = R_j H_c$. The coherent information or entropy bias of a channel $B$ with complement $C$ for an input density operator $ρ$ in $H_a$ is

$Δ(B, ρ) = S(B(ρ)) - S(C(ρ)), \quad
(7)$

This assumes the complementary output space $H_c$ is as small as possible, which is to say it is the support of $C(I_c)$. If one allows $H_c$ to have higher dimension than the support, “unitary” must be replaced with “partial isometry”, see Sec. 5.2 in [24].
The channel $\mathcal{B}$ is said to be degradable and $\mathcal{C}$ antidegradable if there exists a quantum channel $\mathcal{D}$ such that $\mathcal{C} = \mathcal{D} \circ \mathcal{B}$, i.e., if the output of $\mathcal{B}$ is made the input of $\mathcal{D}$, the result is $\mathcal{C}$. Various properties of such channel pairs are discussed in [11], [25], and [26]. Of particular relevance in what follows is the fact that $Q(\mathcal{C}) = Q^{(1)}(\mathcal{C}) = 0$ for an antidegradable channel $\mathcal{C}$; and the entropy bias $\Delta(\mathcal{B}, \rho)$ of a degradable channel $\mathcal{B}$ is a concave function of $\rho$ [27], making it relatively easy to compute its maximum $Q^{(1)}(\mathcal{B}) = Q(\mathcal{B})$.

### III. Glued Isometries and Channels

There are various ways of combining quantum channels and their corresponding isometries. A concatenation of two channels in which the output of the first becomes the input of the second corresponds to the concatenation of the two isometries. When two channels or channel pairs are placed in parallel, the input space of the combination is the tensor product of the two input spaces, likewise the direct and complementary output spaces are the corresponding tensor products, and the isometry for the combination is the tensor product of the individual isometries. But in addition isometries can be combined in such a way that one or more of the input, direct output and complementary output spaces are subspaces of larger Hilbert spaces, a process which we refer to as gluing. The idea will become clear from the following examples.

Consider a collection of isometries

$$J_j : \mathcal{H}_{a_j} \rightarrow \mathcal{H}_{b_j} \otimes \mathcal{H}_{c_j}$$

(9)

in the notation of Sec. II, where the $\mathcal{H}_{a_j}$ are either distinct orthogonal subspaces of $\mathcal{H}_a$, or all equal to $\mathcal{H}_a$, and the same for the $\mathcal{H}_{b_j}$ and $\mathcal{H}_{c_j}$; see (5) and (6). We shall, in what follows, assume the convenient, but not absolutely necessary, condition:

$$J_j^\dagger J_k = 0 \text{ for } j \neq k.$$  (10)

Finally, the overall isometry $J : \mathcal{H}_a \rightarrow \mathcal{H}_b \otimes \mathcal{H}_c$, obtained from gluing the collection of isometries in (9), is given by a sum

$$J := \sum_j \nu_j J_j,$$  (11)

where the $\nu_j$ are positive numbers. The condition $J^\dagger J = I_a$ for $J$ to be an isometry is then

$$\sum_j \nu_j^2 J_j^\dagger J_j = I_a.$$  (12)

The isometry $J$ in (11) defines a pair of channels $\mathcal{B}$ and $\mathcal{C}$ through (3).

Perhaps the simplest example of gluing is when each $J_j$ is simply $J$ applied to the subspace $\mathcal{H}_{a_j} = P_j \mathcal{H}_a$, while $\mathcal{H}_{b_j} = \mathcal{H}_b$ and $\mathcal{H}_{c_j} = \mathcal{H}_c$, independent of $j$, and thus

$$J_j = JP_j, \; J_j^\dagger J_j = P_j J^\dagger J P_j = P_j I_a P_j = P_j,$$  (13)

where the projector $P_j$ is the identity operator on $\mathcal{H}_{a_j}$. Then with every $\nu_j = 1$ in (11) one has

$$J = \sum_j J_j = \sum_j JP_j = JJ.$$  (14)

The corresponding subchannels $\mathcal{B}_j$ and $\mathcal{C}_j$ are given by the expressions

$$B_j(A) = B(P_j A P_j), \; C_j(A) = C(P_j A P_j),$$  (15)

in terms of the superoperators $B$ and $C$ for the full channel and its complement. In general $B(A)$ will not equal $\sum_j B_j(A)$, because the latter maps all “off-diagonal” parts, $P_j A P_k$ for $j \neq k$, of the operator $A$ to zero; similarly, $C(A)$ will in general not be the sum of the $C_j(A)$.

Another example of gluing arises given a collection of isometries $J_j : \mathcal{H}_a \rightarrow \mathcal{H}_b \otimes \mathcal{H}_{c_j}$; that is, the direct channels have the same input and output spaces $\mathcal{H}_a$ and $\mathcal{H}_b$, whereas the complementary channels map to orthogonal subspaces $\mathcal{H}_{c_j} = R_j \mathcal{H}_c$ of $\mathcal{H}_c$. If we write (11) in the form

$$J = \sum_j \sqrt{p_j} J_j,$$  (16)

i.e., $\nu_j = \sqrt{p_j}$, where the $p_j > 0$ are any set of probabilities that sum to 1, then

$$J_j = R_j J_j / \sqrt{p_j},$$  (17)

and it is easily checked that (10) and (12) are satisfied. It is straightforward to show that the $\mathcal{B}$ channel resulting from this gluing of the $\mathcal{H}_{c_j}$ spaces is given by

$$B(A) = \sum_j p_j B_j(A),$$  (18)

thus a weighted sum or convex combination of the $B_j$ channels corresponding to the different $J_j$ isometries. Furthermore, any convex combination $\sum_j p_j B_j$ of channels with a common input space $\mathcal{H}_a$ and output space $\mathcal{H}_b$ can be constructed in this manner by gluing together the different $\mathcal{H}_{c_j}$ output spaces of the complementary channels. But in general there is no simple relationship between the complementary channel $C$ and the different $C_j$.

One can combine the two previous examples and glue both the input spaces $\mathcal{H}_{a_j}$ and the complementary output spaces $\mathcal{H}_{c_j}$ of isometries $J_j : \mathcal{H}_{a_j} \rightarrow \mathcal{H}_b \otimes \mathcal{H}_{c_j}$, which have a common direct output space $\mathcal{H}_b$. With $\nu_j = 1$ in (11) one has

$$J = \sum_j J_j, \; J_j = R_j JP_j.$$  (19)

Again (10) is obviously satisfied. A simple calculation shows that

$$B(A) = \sum_j B_j(A) = \sum_j B_j(A),$$  (20)
and that $B_j$ and $C_j$ satisfy (15). But now $B$ (as well as $B_j$ and $C_j$) maps “off diagonal” parts, $P_j A P_k$ for $j \neq k$ of the operator $A$ to zero. There is in general no connection between $C$ and the $C_j$, analogous to (20).

Of particular interest for what follows later is a block diagonal channel pair obtained from gluing both the direct and complementary outputs of the isometries $J_j : \mathcal{H}_a \mapsto \mathcal{H}_{b_j} \otimes \mathcal{H}_{c_j}$, with $\mathcal{H}_{b_j} = Q_j \mathcal{H}_b$ and $\mathcal{H}_{c_j} = R_j \mathcal{H}_c$. As in (16) one writes

$$J = \sum_j \sqrt{P_j} J_j, \quad J_j = (Q_j \otimes R_j) J_j / \sqrt{P_j}. \quad (21)$$

It is then straightforward to show that

$$Q_j B(A) Q_k = \delta_{jk} p_j B_j(A), \quad R_j C(A) R_k = \delta_{jk} p_j C_j(A),$$

and as a consequence,

$$B(A) = \bigoplus_j p_j B_j(A), \quad C(A) = \bigoplus_j p_j C_j(A). \quad (22)$$

Instead of $\bigoplus$, one could have used $\sum$ in (23) to indicate that both $B$ and $C$ are convex sums of $\{B_j\}$ and $\{C_j\}$ respectively; however, using $\bigoplus$ rather than $\sum$ serves to emphasize that the output spaces of the $\{B_j\}$ are mutually orthogonal, as are the output spaces of the $\{C_j\}$. Thus the outputs of each channel in the $(B, C)$ pair are in separate, mutually orthogonal blocks, whence our name “block diagonal” channel pair. In addition, from eq. (21) it follows that the $B$ and $C$ blocks are correlated: if in a particular run the output of the $B$ channel falls in a particular block $Q_j$ (as could be determined by a suitable measurement), the $C$ channel output will be in the corresponding block $R_j$. This means the entropy of the $B$ output is given by

$$S(B(\rho)) = \sum_j p_j S(B_j(\rho)) + h(p), \quad (24)$$

where $h(p) = -\sum_j p_j \log_2 p_j$ is the Shannon entropy of the probability distribution $\{p_j\}$ (in base 2). There is an analogous expression for the entropy of the $C$ output. Thus the output entropy in each case is the weighted sum of the output entropies of the individual channels plus a “classical” term $h(p)$. This classical term cancels when one computes the entropy bias, (7), which is given by

$$\Delta(B, \rho) = \sum_j p_j \Delta(B_j, \rho). \quad (25)$$

These considerations suggest a simple physical picture: the channel $B$ can be obtained by randomly applying with probability $p_i$ the channel $B_i$ to the input $\mathcal{H}_a$, with the output going to $\mathcal{H}_{b_i}$. Thus $B$ is a convex combination of the $B_i$ if one regards each of these as a map into the full operator space $\mathcal{H}_b$. A similar comment applies to $C$ as a convex combination of the $C_i$. The structure of these block diagonal channel pairs is exemplified by the specific examples in Sec. IV.

It is possible to glue the inputs and both the direct and complementary outputs in a construction called the direct sum of channels. Let the corresponding isometries be $J_j : \mathcal{H}_{a_j} \mapsto \mathcal{H}_{b_j} \otimes \mathcal{H}_{c_j}$, with $\mathcal{H}_{a_j} = P_j \mathcal{H}_a$, $\mathcal{H}_{b_j} = Q_j \mathcal{H}_b$, $\mathcal{H}_{c_j} = R_j \mathcal{H}_c$. Thus the corresponding channels are completely independent of each other, with distinct input and output spaces. With $J$ the sum of these isometries one has:

$$J = \sum_j J_j; \quad J_j = (Q_j \otimes R_j) J P_j. \quad (26)$$

A straightforward calculation shows that $J$ gives rise to channels

$$B(A) = \bigoplus_j B_j(P_j A P_j), \quad C(A) = \bigoplus_j C_j(P_j A P_j). \quad (27)$$

Once again there is a block-diagonal structure with correlated blocks, but the physical picture is a bit different from (23). The channel $B$ acts on a density operator $\rho$ by projecting it to the sub-space $\mathcal{H}_{a_j}$ with probability $\text{Tr}(P_j \rho)$ (thus the “off-diagonal” $P_j \rho P_k$ parts of $\rho$ for $j \neq k$ always map to zero), and then applying the channel $B_j$. An analogous interpretation holds for $C$. This direct sum construction of a channel $B$ has been studied in [21] where it was used in simplifying the nonadditivity conjecture of the Holevo capacity of a quantum channel. The gluing picture reveals that $C$, the complement of $B$, is also a direct sum, and the two channels $B$ and $C$ have correlated blocks.

If an isometry $J$ has been produced by gluing other isometries together in the manner indicated above, one can recover the constituents by a process of slicing $J$, in which projectors corresponding to the different PDIs are placed to the left and right of $J$. For any map $J$ from $\mathcal{H}_a$ to $\mathcal{H}_b \otimes \mathcal{H}_c$, not necessarily an isometry, one can define a collection of operators

$$K_{jk} := (Q_k \otimes R_i) J P_j. \quad (28)$$

using PDIs $\{P_j\}$, $\{Q_k\}$, and $\{R_i\}$, which need not have the same number of projectors, on $\mathcal{H}_a$, $\mathcal{H}_b$, and $\mathcal{H}_c$, respectively. It is obvious that $J$ is the sum of all of the $K_{jk}$, but even if $J$ is an isometry, the individual $K_{jk}$ will in general not be isometries or proportional to isometries; that is, $K_{jk} K_{ik}$ will not be proportional to $P_j$. Only for special choices of the isometry $J$ and the PDIs, as in the examples considered above, will the operators resulting from slicing be proportional to isometries.

IV. Generalized Erasure Channel Pair

An erasure channel pair is an example of a block diagonal channel with two blocks, where the isometries in (21) are given by

$$J_1(\psi) = |\psi\rangle_{b_1} \langle \psi|_{c_1}, \quad J_2(\psi) = |\psi\rangle_{b_2} \langle \psi|_{c_2}. \quad (29)$$

Here $\mathcal{H}_{b_1}$ and $\mathcal{H}_{c_2}$ are isomorphic to $\mathcal{H}_a$, whereas $\mathcal{H}_{c_1}$ and $\mathcal{H}_{b_2}$ are one-dimensional Hilbert spaces spanned by the normalized kets $|e\rangle_{c_1}$ and $|f\rangle_{b_2}$, respectively. By definition, $\mathcal{H}_{b_1}$ and $\mathcal{H}_{b_2}$ are orthogonal subspaces of $\mathcal{H}_b = \mathcal{H}_{b_1} \oplus \mathcal{H}_{b_2}$, and $\mathcal{H}_{c_1}$ and $\mathcal{H}_{c_2}$ are orthogonal subspaces of $\mathcal{H}_c = \mathcal{H}_{c_1} \oplus \mathcal{H}_{c_2}$. The isometry $J_1$ generates a perfect channel pair $(I, T)$, where the identity channel $I$ maps any operator $A$ in $\mathcal{H}_a$ to the same operator $A$ in $\mathcal{H}_{b_1}$, while the trace channel $T$ maps $A$ to $\text{Tr}(A)|e\rangle_{c_1}$. Here and later we use the abbreviation $[\psi] = |\psi\rangle \langle \psi|$ for the projector corresponding to a normalized ket $|\psi\rangle$. In the same way $J_2$ generates the perfect channel pair $(T, I)$ mapping $\mathcal{H}_a$ to $\mathcal{H}_{b_2}$ and $\mathcal{H}_{c_2}$, respectively. Gluing
these perfect channel pairs together using $p_1 = 1 - \lambda$ and $p_2 = \lambda$ in (21) results in the channel pair
\begin{align}
B_c(A) = E_1^\lambda(A) = (1 - \lambda)A \bigoplus \lambda \text{Tr}(A)[1], \\
C_c(A) = E_2^{1-\lambda}(A) = (1 - \lambda)\text{Tr}(A)[c] \bigoplus \lambda A.
\end{align}
(30)

The channel $E_1^\lambda$ is the erasure channel with erasure probability $\lambda$, and $E_2^{1-\lambda}$ is its complement. The subspaces on the left and right side of $\bigoplus$ can be interchanged; the order used in (30) and (31) reflects the correlations discussed following (23): if $A$ occurs in the output of the direct channel, $[c]$ will be present in the complementary output.

It is easily shown that the entropy bias, (7), of $B_c = E_1^\lambda$ takes the form
\[ \Delta(E^{\lambda}, \rho) = (1 - 2\lambda)S(\rho). \]
(32)

Its maximum for $\lambda \geq 1/2$ is zero, and $(1 - 2\lambda)\log_2 d_a$ for $\lambda \leq 1/2$, since the maximum value $\log_2 d_a$ of $S(\rho)$ is achieved when $\rho$ is proportional to the identity operator. If $E_1^\lambda$ for $\lambda \leq 1/2$ is followed by $E_2^\mu$ with $\mu = (1 - 2\lambda)/(1 - \lambda)$, the resulting channel $E_2^\mu \circ E_1^\lambda = E_1^{1-\lambda}$ is the complement of $E_1^\lambda$. Hence for $\lambda \leq 1/2$, $E_1^\lambda$ is degradable, while for $\lambda \geq 1/2$ it is antidegradable. Consequently $Q$ and $Q^{(1)}$ are identical for an erasure channel, and
\[ Q^{(1)}(B_c) = Q(B_c) = Q(E^{\lambda}) = \max \{1 - 2\lambda, 0\} \log_2 d_a, \]
\[ Q^{(1)}(C_c) = Q(C_c) = Q(E_2^{1-\lambda}) = \max \{2\lambda - 1, 0\} \log_2 d_a. \]
(33)

We define the generalized erasure channel pair as one in which $J_2$ is the same as in (29) and corresponds to a perfect channel, but $J_1$ is replaced by any isometry from $H_1$ to $H_{b_1} \otimes H_{c_1}$, where the dimensions of $H_{b_1}$ and $H_{c_1}$ are arbitrary (except that the product cannot be less than the dimension of $H_a$). The result is a channel pair
\begin{align}
B_g(A) = (1 - \lambda)B_1(A) \bigoplus \lambda \text{Tr}(A)[1], \\
C_g(A) = (1 - \lambda)C_1(A) \bigoplus \lambda A,
\end{align}
(34)

where $(B_1, C_1)$ is the channel pair generated by $J_1$. The form of $B_g$ in (34) means that it either erases its input with probability $\lambda$, or else sends it through $B_1$ into the output subspace $H_{b_1}$ of $H_b$. Similarly, $C_g$ with probability $\lambda$ sends its input unchanged to $H_{c_1}$, or else sends it through $C_1$ to $H_{c_1}$. When $C_1$ is the trace channel $T$ that completely erases its input, $C_g$ is an erasure channel $C_1$ with erasure probability $1 - \lambda$. But in general $C_1$ need not erase completely, so we call $C_g$ an “incomplete erasure” channel.

The $B_g$ channel can be obtained by concatenating $B_1$ either with a preceding erasure channel $E_1^\lambda$, defined in (30), or one that follows it, $E_2^\lambda$:
\[ B_g(A) = B_1 \circ E_1^\lambda(A) = E_2^\lambda \circ B_1(A). \]
(36)

Here the superoperator $E_2^\lambda$ is an erasure channel whose input space is identical to the output space of $B_1$, which need not have the same dimension as its input space. The operator $B_1$ is the same as $B_1$ except that when it is applied to $[c]$ in (30) $B_1$ maps $[c]$ to the corresponding $[e]$ in (34), and maps any “off-diagonal” ket $|e\rangle\langle\alpha|$ or $|\alpha\rangle\langle e|$, $|\alpha\rangle$ any element of $H_a$, to zero. Given these definitions it is straightforward to check the validity of (36). There is no analog of (36) for $C_g(A)$. Since the concatenation of a channel with an antidegradable channel always results in an antidegradable channel (see App. A for a simple proof), $B_g$ is antidegradable when either $B_1$ or $E_1^\lambda$ is antidegradable (the latter happens when $\lambda \geq 1/2$).

As a consequence of data-processing [5], a channel obtained by concatenating two channels has a smaller channel coherent information and hence smaller quantum capacity than either of the channels that are being concatenated [28], [29]. Thus, from (36) it follows that $Q^{(1)}(B_g) \leq \min\{Q^{(1)}(B_1), Q^{(1)}(E_1^\lambda)\}$ and $Q(B_g) \leq \min\{Q(B_1), Q(E_1^\lambda)\}$. The channel $C_g$ in (31) can be obtained by concatenating the output of $C_g$ with a channel that traces out operators on $H_{c_1}$ to a fixed pure state $[f]$ and does nothing to $H_{c_2}$, thus $Q^{(1)}(C_g) \leq \min\{Q^{(1)}(C_g), \log d_a\}$ and $Q(C_g) \leq \min\{Q(C_g), \log d_a\}$.

V. APPLICATIONS
A. Generalized Erasure using Qubit Amplitude Damping Channel

The isometry $J_1 : H_a \rightarrow H_{b_1} \otimes H_{c_1}$ defined by
\[ J_1 |0\rangle_a = |0\rangle_{b_1} |1\rangle_{c_1}, \]
\[ J_1 |1\rangle_a = \sqrt{1 - p}|1\rangle_{b_1} |1\rangle_{c_1} + \sqrt{p}|0\rangle_{b_1} |0\rangle_{c_1}, \]
(37)

with $0 \leq p \leq 1$, and $|0\rangle$ and $|1\rangle$ are the usual orthonormal basis kets for a qubit, defines a channel pair $(B_1, C_1)$ in which $B_1$ is an amplitude-damping channel with $p$ the probability that the input state $|1\rangle_a$ decays to the output state $|0\rangle_{b_1}$. Similarly, $C_1$ is an amplitude damping channel with decay rate $(1 - p)$ if one interchanges $|0\rangle_{c_1}$ and $|1\rangle_{c_1}$ in (37). The Bloch vector parametrization for a qubit density operator,
\[ \rho(r) = \frac{1}{2}(I + r \sigma) := \frac{1}{2}(I + xx \sigma_x + yy \sigma_y + zz \sigma_z), \]
(38)

where $I$ is the identity and $(\sigma_x, \sigma_y, \sigma_z)$ are the three Pauli matrices, provides a convenient way to represent $B_1$ and $C_1$ as maps carrying $r$ to Bloch vectors
\[ r_a = (\sqrt{1 - p} x, \sqrt{1 - p} y, (1 - p)z + p) \]
and
\[ r_c = (\sqrt{p} x, -\sqrt{p} y, p - pz - 1), \]
(39)
respectively. See Fig. 1(a) for a convenient way to visualize this channel pair.

Let $(B_g, C_g)$ be the generalized channel pair resulting from inserting $B_1$ and $C_1$ in (34) and (35). The entropy bias $\Delta(B_g, \rho(r))$ of $B_g$ at $\rho(r)$ is a real-valued function of $r = (x, y, z)$, whose maximum and minimum over $r$ with $|r| \leq 1$ gives $Q^{(1)}(B_g)$ and $Q^{(1)}(C_g)$, respectively; see (8). Finding these extrema is simplified by the fact that the rotational symmetry of $\Delta(B_g, \rho(r))$ about the $z$ axis—see Fig. 1(a)—means that for a fixed $z$ it is a function of $x^2 + y^2$, so one can set $y = 0$. In addition, with $y = 0$, $\Delta(B_g, \rho(r))$ for a fixed $x^2 + z^2$ is monotone increasing in $z$ for $p \leq 1/2$, and monotone decreasing for $p \geq 1/2$. Thus one can also set $x = 0$ and look for its maximum or minimum as a function of the single parameter $-1 \leq z \leq 1$.

The range of the two parameters $p$ and $\lambda$ for which $Q^{(1)}(B_g)$ is greater than 0 can be determined as follows. For
$$1/2 \leq p \leq 1, \quad B_1 \text{ is antidegradable} \quad [30], \text{ while for } 1/2 \leq \lambda \leq 1, \quad E^\lambda \text{ is antidegradable. Thus } B_g, \text{ the concatenation of these two channels (see (36) and the following discussion) is antidegradable, and } Q^{(1)}(B_g) = Q(B_g) = 0. \text{ Thus } Q^{(1)}(B_g) \text{ can only be positive when both } p \text{ and } \lambda \text{ are less than } 1/2. \text{ At } p = 0, \quad B_1 \text{ is a perfect channel and } B_g \text{ an erasure channel, so } Q^{(1)}(B_g) = 1 - 2\lambda \text{ for } \lambda \leq 1/2 \text{ (see (33) with } d_a = 2). \text{ For } \lambda = 0, \quad B_g \text{ is just the amplitude damping channel, which is degradable with a positive } Q^{(1)} \text{ for } 0 \leq p < 1/2 \text{ (see [30] and Sec.IV A in [20]).}
$$

For other values of $\lambda$ and $p$ between 0 and 1/2, the numerical maximization $^1$ of $\Delta(B_g, \rho(r))$ together with an asymptotic analysis as $z$ approaches 1 (App. B) shows that $Q^{(1)}(B_g)$ is positive for $\lambda$ in the interval

$$0 \leq \lambda < \lambda_0(p) = (1 - 2p)/(2 - 2p), \quad (40)$$

(see Fig. 2), is zero for $\lambda \geq \lambda_0(p)$, and as

$$\delta \lambda = \lambda_0(p) - \lambda \quad (41)$$

tends to zero has the asymptotic form

$$Q^{(1)}(B_g) \simeq a(p)\delta \lambda \exp\left[-b(p)/\delta \lambda\right], \quad (42)$$

where $a(p)$ and $b(p)$ are positive functions of $p$. The exponentially rapid decrease of $Q^{(1)}(B_g)$ due to $\delta \lambda$ in the denominator makes a direct numerical study difficult when $\delta \lambda$ is very small. For $p = 1/4$ and $5 \times 10^{-3} < \delta \lambda < 10^{-1}$ we find good agreement between our numerical values and (42).

For all strictly positive $p$ and $\lambda$, $Q^{(1)}(C_g)$ is positive, see Sec. V-C, so its complementary channel $B_g$ cannot be degradable, and hence $Q^{(1)}(B_g)$ might conceivably be nonadditive. A relevant measure of nonadditivity for $n$ copies of this channel placed in parallel is

$$\delta_n := Q^{(1)}(B_g^\otimes n)/n - Q^{(1)}(B_g), \quad (43)$$

One says that nonadditivity occurs at the $n$-letter level if $\delta_n > 0$. We have found numerical evidence for nonadditivity at the 2-letter level for $\lambda$ in the range

$$\lambda_1(p) < \lambda < \lambda_0(p), \quad \text{thus } 0 < \delta \lambda < \delta \lambda_1(p) := \lambda_0(p) - \lambda_1(p), \quad (44)$$

where $\lambda_1(p)$, determined numerically, is shown in Fig. 2. In particular for $\delta \lambda$ between 0 and $\delta \lambda_1(p)$, an input density operator for $B_g^\otimes 2$ of the form

$$\sigma = (1 - \epsilon)[00] + \epsilon|\phi\rangle\langle\phi| = (|01\rangle + |10\rangle)/\sqrt{2} \quad (45)$$

with $0 < \epsilon < 1$ chosen to maximize $\Delta(B_g^\otimes 2, \sigma)$ gives a larger maximum value of $\Delta(B_g^\otimes 2)$ than the product density operator

$$\tau = \rho_m \otimes \rho_m, \quad \rho_m = (1 - z)[0] + z[1], \quad (46)$$

with $0 < z < 1$ chosen to maximize $\Delta(B_g^\otimes 2, \rho(r))$ for a single channel. When $\delta \lambda$ is sufficiently small, the asymptotic behavior of $\delta_2$ (App. B) is of the form (42), but with different choices for $a(p)$ and $b(p)$. For larger $\delta \lambda$, see Fig. 3 for $p = 0.25$, it rises to a maximum and then falls to zero with a finite slope at $\delta \lambda = \delta \lambda_1(p)$. While we can be quite confident of nonadditivity in the region $\lambda_1(p) < \lambda < \lambda_0(p)$, that $\delta_2$ is actually zero outside this range is less certain, since an input density operator different from (45) could conceivably give a maximum value of $\Delta(B_g^\otimes 2)$ larger than $2Q^{(1)}(B_g)$, even though we have found no indication of this in our numerical studies.

If $\delta_2$ is positive it is easy to show that $\delta_n$ is positive for all $n > 2$, and cannot be much smaller than $\delta_2$. As direct numerical searches become exponentially more difficult with increasing $n$, it is customary to make a guess or ansatz $\rho_n$ for the input density operator, which may depend upon a small number of parameters, and maximize $\Delta(B_g^\otimes n, \rho_n)$ over these parameters, see (8), to obtain a lower bound for $Q^{(1)}(B_g^\otimes n)$. When $n$ is even the pair ansatz consists in dividing the $n$ channels into $n/2$ pairs and employing the optimizing density operator $\sigma$ defined above as the input for each pair; this yields a lower bound $\delta_2$ for $\delta_n$. When $n$ is odd use $\sigma$ for each of $(n - 1)/2$ pairs, and for the remaining channel the density

---

$^1$ Here and elsewhere in this work, numerical optimizations use standard methods from SciPy [31]
Fig. 2: For a given $p$, $Q^{(1)}(B_g)$ is zero for $\lambda \geq \lambda_0(p)$, and positive for $\lambda < \lambda_0(p)$. It is nonadditive at the 2-letter level for $\lambda_1(p) < \lambda < \lambda_0(p)$.

Fig. 3: A plot of $\delta_2 = Q^{(1)}(B_g^{\otimes 2})/2 - Q^{(1)}(B_g)$ versus $\delta\lambda = \lambda_0(p) - \lambda$ at $p = 0.25$ shows $\delta_2$ is positive for $0 < \delta\lambda < 0.0406$ and attains a maximum value $\simeq 5.27 \times 10^{-3}$.

operator that gives rise to $Q^{(1)}(B_g)$; the resulting lower bound is a bit less than $\delta_2$. In the literature [6], [16]–[18], [32] various other ansatzes have been proposed, including the $Z$-diagonal ansatz, a particular case of which is the repetition ansatz. Our numerical studies for $n = 3, 4$ and 5 using these and others motivated by the functional form of $\sigma$ have not found any that improve on the pair ansatz.

B. Generalized Erasure with Qubit Dephasing Channel

The isometry $J_1 : \mathcal{H}_{a_1} \to \mathcal{H}_{b_1} \otimes \mathcal{H}_{c_1}$, with each space a qubit (dimension 2), giving rise to the dephasing channel $B_1$ and its complement $C_1$ can be written in the form

$$J_1 |0\rangle_a = |0\rangle_{b_1} |\phi_0\rangle_{c_1}, \quad J_1 |1\rangle_a = |1\rangle_{b_1} |\phi_1\rangle_{c_1},$$  \hspace{1cm} (47)

where

$$|\phi_0\rangle_{c_1} = \sqrt{1-p}|+\rangle_{c_1} + \sqrt{p}|-\rangle_{c_1},$$

$$|\phi_1\rangle_{c_1} = \sqrt{1-p}|+\rangle_{c_1} - \sqrt{p}|-\rangle_{c_1};$$

$$|\pm\rangle := (|0\rangle \pm |1\rangle)/\sqrt{2},$$  \hspace{1cm} (48)

with $0 \leq p \leq 1$ the dephasing probability. Interchanging $p$ with $1-p$ in (48) is equivalent to applying the unitary $\sigma_z = |0\rangle - |1\rangle$ to both $\mathcal{H}_{b_1}$ and $\mathcal{H}_{c_1}$, so for our purposes we can limit $p$ to the range $0 \leq p \leq 1/2$. The superoperators for the dephasing channel $B_1$ and its complement $C_1$ are

$$B_1(A) = pZAZ^\dagger + (1-p)A,$$

$$C_1(A) = \langle 0|A|0\rangle [\phi_0] + \langle 1|A|1\rangle [\phi_1],$$  \hspace{1cm} (49)
where \( Z(0)_{a1} = |0\rangle_{b1} \) and \( Z(1)_{a1} = |1\rangle_{b1} \). One can think of \( C_1 \) as first measuring the input in the \( \{0, 1\} \) basis, and for measurement outcome \( i \) preparing the channel output \( |\phi_i\rangle \). Such a measure-and-prepare or entanglement breaking channel is antidegradable and thus has zero quantum capacity [33]. The channel \( B_1 \) and its complement \( C_1 \) map \( \rho(r) \) in (38) to qubit density operators with Bloch vectors

\[
\begin{align*}
\rho_b &= ((1 - 2p)x, (1 - 2p)y, z), \\
\rho_c &= (1 - 2p, 0, 2\sqrt{p(1 - p)}z),
\end{align*}
\]

(50) respectively (see Fig. 1(b)).

Inserting \( B_1 \) and \( C_1 \) defined in (49) in (34) and (35) yields the generalized erased channel pair \( (B_g, C_g) \). The channel \( B_g \) is the same as the dephasing channel studied in [16], where it was defined using the second equality in (36). These authors showed that the global maximum or minimum of \( \Delta(B_g, \rho(r)) \) occurs along \( r = (x, 0, z) \). They also found that for any fixed \( p \) between 0 and 1/2, as \( \lambda \) increases from 0 a local maximum of \( \Delta(B_g, \rho(r)) \) remains at \( x = z = 0 \) until \( \lambda \) reaches the value

\[
j(p) = \frac{1 - 2p - 2p(1 - p)\ln((1 - p)/p)}{2 - 4p - 2p(1 - p)\ln((1 - p)/p)},
\]

(51) at which point this maximum begins moving to positive \( z \) values, while \( x \) remains at 0. As \( \lambda \) increases further, the local maximum of \( \Delta(B_g, \rho(r)) \) goes to zero at \( \lambda \) equal to

\[
g(p) = \frac{(1 - 2p)^2}{1 + (1 - 2p)^2},
\]

(52) and remains zero for \( \lambda > g(p) \).

We strengthen these results by showing that for any \( p \) and any \( \lambda \) between 0 and 1/2 the global maximum \( Q^{(1)}(B_g) \) of \( \Delta(B_g, \rho(r)) \) occurs along \( r = (0, 0, z) \), where it agrees with the local maximum found in [16], while the global minimum, equal to \( -Q^{(1)}(C_g) \) (see (8)) occurs on the line \( r = (x, 0, 0) \). To show these results, we first use rotational symmetry, see Fig. 1. This symmetry ensures \( \Delta(B_g, \rho(r)) \) only depends on \( x^2 + y^2 \). The dependence guarantees a maximum or minimum of \( \Delta(B_g, \rho(r)) \) can always be obtained at \( y = 0 \). Setting \( y = 0 \) makes \( \Delta(B_g, \rho(r)) \) a function of two independent variables \( x \) and \( z \) where \( x^2 + z^2 \leq 1 \). To understand \( \Delta(B_g, \rho(r)) \) better, we switch to different independent variables \( u = \sqrt{x^2 + z^2} \) and \( v = z \), where \( -u \leq v \leq u \) and \( u \leq 1 \). In these variables \( \Delta(B_g, \rho(r)) \) is a convex function of \( v \) for any fixed \( u \); in addition, this convex function is symmetric about \( v = 0 \). Together the convexity and symmetry ensure that \( \Delta(B_g, \rho(r)) \) is maximum at \( v = \pm u \) and minimum at \( v = 0 \). Translating these minimum and maximum conditions in the \( u \) and \( v \) variable to the \( x \) and \( z \) variables shows that the global maximum of \( \Delta(B_g, \rho(r)) \) occurs along \( r = (0, 0, z) \), and its global minimum along \( r = (x, 0, 0) \).

The study in [16] used a repetition ansatz,

\[
\hat{\rho}_2(\eta) = \eta|00\rangle\langle 00| + (1 - \eta)|11\rangle\langle 11|,
\]

(53) with \( \eta \) chosen appropriately between zero and one to maximize \( \Delta(B_g^{\otimes 2}, \hat{\rho}_2(\eta)) \), and showed that \( Q^{(1)}(B_g) \) is non-additive at the two-letter level. Recently another study [32], used a neural network ansatz to obtain non-additivity of \( Q^{(1)}(B_g) \). In both studies, non-additivity is found for some values of \( p \) between 0 and 1/2, and some values of \( \lambda \) in the range \( j(p) < \lambda < g(p) \). We extend these results by showing \( Q^{(1)}(B_g) \) is non-additive at the two-letter level for all \( 0 < p < 1/2 \) at \( \lambda = j(p) \). This extension is obtained by using a different ansatz

\[
\rho(\zeta) = \{(1 + \zeta)(|00\rangle + |11\rangle) + (1 - \zeta)(|01\rangle + |10\rangle)\}/4,
\]

(54) and varying \( \zeta \) between \(-1 \) and \( +1 \) to maximize \( \Delta(B_g^{\otimes 2}, \rho(\zeta)) \). Inserting this maximum, \( \Delta^*(B_g^{\otimes 2}, \zeta_0) \), in (43) gives a lower bound \( \delta^*_2 \) for \( \delta_2 \). We find that along the curve \( \lambda = j(p) \), \( 0 < p < 1/2 \), \( \delta^*_2 \) is positive and goes to zero at the two end points (see Fig. 4). We also noticed that for a fixed \( p \), \( \delta^*_2 \) rapidly goes to zero as \( \lambda \) increases or decreases from \( j(p) \). It remains an open question whether using a different ansatz than (54), or by some other method, the range of \( p \) and \( \lambda \) values for which \( Q^{(1)}(B_g) \) is nonadditive can be extended beyond those discussed here and in previous studies [16], [32].

C. Incomplete Erasure Channel

As discussed following (35), \( C_g \) resembles an erasure channel \( C_r \), except that instead of completely erasing its input it sends it through a noisy channel \( C_1 \). This “incomplete erasure” leads to an interesting effect shown in Fig. 5 for the amplitude damping case of Sec. V-A. When \( p = 0 \), which means that \( C_1 \) is the completely noisy trace channel \( T \), both \( Q^{(1)}(C_g) \) and \( Q^{(1)}(C_g) \) are exactly zero for \( 0 \leq \lambda \leq 1/2 \). But as soon as \( p \) is positive by the smallest amount, \( Q^{(1)}(C_g) \) is positive over the entire range \( \lambda > 0 \). As \( p \) tends to 0, the analysis in App. B yields the asymptotic behavior

\[
Q^{(1)}(C_g) \approx \frac{\lambda}{\ln 2} \exp[(p + \ln p)(1 - \lambda)/\lambda]
\]

(55) for \( 0 < \lambda \leq 1/2 \); see the inset in Fig. 5. A very similar behavior is found when \( C_1 \) is the complement of the phase-damping channel \( B_1 \) discussed in Sec. V-B, for which the corresponding asymptotic expression is

\[
Q^{(1)}(C_g) \approx \frac{\lambda}{\ln 2} \exp\left\{\left[p + (1 - 2p)\ln p\right](1 - \lambda)/\lambda\right\}.
\]

(56)

In both cases, when \( p \) is small \( C_1 \) is not only noisy but antidegradable, so that its quantum capacity is exactly zero. Thus its ability to make \( Q^{(1)}(C_g) \) positive in the entire range \( 0 < \lambda \leq 1/2 \) comes as something of a surprise. These two examples might suggest that \( Q^{(1)}(C_g) \) is positive for all \( \lambda > 0 \) if \( C_1 \) is any channel that is not completely noisy. But this is not the case. If \( C_1 \) is an erasure channel with erasure probability \( \mu \), \( C_g \) is an erasure channel with erasure probability

\[
\epsilon = (1 - \lambda)\mu,
\]

(57) and thus has zero capacity for \( \epsilon \geq 1/2 \). This means \( Q^{(1)}(C_g) = 0 \) for

\[
0 \leq \lambda \leq 1 - 1/(2\mu),
\]

(58) which is a finite interval for any \( \mu \) greater than 1/2. One way to derive (57) is to note that if one defines the transmission probability of an erasure channel as 1 minus the erasure probability, \( B_1 \) has a transmission probability of \( \mu \). Since \( B_g \) is the concatenation (36) of an erasure channel with transmission
probability $\mu$ with one whose transmission probability is $1 - \lambda$, it is an erasure channel with transmission probability equal to the product $\mu(1 - \lambda)$. And this is the erasure probability of its complement $C_g$.

VI. Summary and Conclusions

Following a review in Sec. II of how an isometry gives rise to quantum channel pair $(B, C)$, the process of combining channels or channel pairs that we call gluing is presented in Sec. III. Combining channels by placing them in parallel or series is of course well known. However, the gluing procedure, aside from particular cases like convex combinations and direct sums of channels, has so far as we know not been discussed earlier in the literature, and might well have some interesting applications in addition to those discussed in this paper.

Our focus is on a type of gluing procedure that leads to what we call a block diagonal channel pair, see (21) and the discussion following it. What makes this procedure of combining channels particularly useful for the study of quantum channel capacities is that the entropy of the output of a block diagonal channel is a weighted sum of the entropies of the outputs of the individual channels in the combination, plus a “classical” term, (24). Thus the entropy bias or coherent information, the difference of the entropies of the outputs of a block diagonal channel pair $B$ and $C$ for a given input, is a similar weighted sum (with the “classical” term cancelling out), (25). In addition one has a simple physical picture: with
some probability the input to a block diagonal channel pair is sent into one of several different channel pairs.

The quantum erasure channel with erasure probability \( \lambda \), together with its complement, an erasure channel with erasure probability \( 1 - \lambda \), is an example of a block diagonal channel pair formed by gluing together two perfect channel pairs, as discussed in Sec. IV. Replacing one of the perfect channel pairs with an arbitrary pair \((B_1, C_1)\) results in a generalized erasure channel pair \((B_g, C_g)\). The channel \( B_g \) can be viewed as a concatenation of \( B_1 \) with a suitable erasure channel, while one can think of \( C_g \) as an erasure channel with incomplete erasure.

In Sec. V we have analyzed two cases of generalized erasure channel pairs constructed using a pair \((B_1, C_1)\), with both \( B_1 \) and \( C_1 \) qubit-to-qubit channels. In the first case, Sec. V-A, \( B_1 \) and \( C_1 \) are complementary amplitude damping channels. In the second case, Sec. V-B, \( B_1 \) is a phase-damping channel, with complement \( C_1 \) a measure-and-prepare channel. This second case has been studied in [16]; some of our results confirm and extend the ones published there. In both cases the qubit channel pair depends on a single parameter \( 0 \leq p \leq 1 \), which determines the amount of amplitude or phase damping of \( B_1 \). Hence the corresponding generalized erasure channel is characterized by two parameters: \( p \) and the erasure probability \( \lambda \). Both the amplitude and phase damping cases exhibit interesting, and to some extent unexpected, behavior.

The nonadditive behavior of \( Q^{(1)}(B_g) \) for two identical channels in parallel is analyzed in Sec. V-A for the phase-damping case, starting with a global optimization, assisted by an asymptotic analysis, to yield accurate values of \( Q^{(1)}(B_g) \). In the \((p, \lambda)\) plane \( Q^{(1)}(B_g) \) is zero for \( \lambda \geq \lambda_0(p) \), the upper curve in Fig. 2, and positive for \( 0 \leq \lambda < \lambda_0(p) \). A numerical search for a positive \( \delta_2 = Q^{(1)}(B_g^2)/2 - Q^{(1)}(B_g) \) suggests a plausible form \((45)\) for the bipartite input density operator, and using this one finds a well-defined region in the \((p, \lambda)\) plane, lying between the two curves in Fig. 2, in which \( \delta_2 \) is positive. Figure 3 shows \( \delta_2 \) as a function of \( \delta \lambda = \lambda_0(p) - \lambda \) for \( p = 1/4 \).

The nonadditivity of \( Q^{(1)}(B_g) \) for the phase-damping case (the dephrasure channel) was first studied in [16]. Our global optimization results confirm their \( Q^{(1)}(B_g) \) calculation, showing that it is zero for \( \lambda \geq g(p) \) and positive for \( \lambda < g(p) \), and we have extended the range of \((p, \lambda)\) values over which nonadditivity occurs, without determining its full extent. See the discussion following \((52)\) in Sec. V-B, and Fig. 4.

Even when nonadditivity is absent for two identical channels in parallel it could be present for three or more. One very preliminary result in Sec. V-A for the multiple channel case suggests an interesting possibility: There might be channels for which \( Q^{(1)} \) is nonadditive when two are placed in parallel, but thereafter no additional nonadditivity arises when a collection of such “double” channels are placed in parallel with one another.

The behavior of \( Q^{(1)} \) of the incomplete erasure channel \( C_g \), discussed in Sec. V-C, is also quite surprising. In both the amplitude and phase damping cases, when \( p = 0 \) the channel \( C_g \) becomes an erasure channel with erasure probability \( 1 - \lambda \), so that \( Q^{(1)}(C_g) = Q(C_g) = 0 \) for \( \lambda \leq 1/2 \). However, as soon as \( p \) is positive, \( C_g \) is greater than zero over the range \( 0 < \lambda \leq 1 \), see Fig. 5. It is surprising that “assisting” the erasure channel with a very noisy \( C_1 \), which itself has zero quantum capacity, gives rise to this effect. Not every noisy \( C_1 \) provides such a dramatic improvement, and it would be interesting to determine which channels do so. While the behavior of \( Q^{(1)}(C_g) \) in Fig. 5 emerges quite clearly from the mathematics, we lack an intuitive explanation.

The results reported here could be extended in various ways. Two real numbers, \( p \) and \( q \), are needed to parametrize the family of channel pairs \((B_1, C_1)\) where both are qubit-to-qubit channels. The amplitude- and phase-damping cases discussed above correspond to different choices of \( q \). It may be possible to extend the results in Sec. V to this larger family of channels; however the absence of certain symmetries that simplified the analysis in Sec. V might lead to complications.

In addition, nonadditivity can, and undoubtedly does, occur in certain cases when two nonidentical \( B_g \) channels, with unequal choices for the parameter pair \((\lambda, p)\), are placed in parallel. We have no idea what might arise from a study of these, but analyzing what happens when the parameters \((\lambda, p)\) for one channel are varied while those for the other are held fixed might in some situations turn out be simpler than studying identical channels in which the two sets of parameters are identical.

The main advantage of the generalized erasure approach for studying positivity and nonadditivity of \( Q^{(1)} \) lies in the fact that when two channel pairs with a very simple structure, in our case the \((B_1, C_1)\) pair and the perfect channel pair, are glued together, this can give rise to new and interesting behavior not present in either of the separate components. One suspects that there are other instances of this sort worth exploring, and one can hope that analyzing them will yield additional insights into the behavior of the quantum capacity of noisy quantum channels—a very challenging, but at the same time very important, problem in quantum information theory, something which needs to be better understood. We hope our results, limited as they are, will make some contribution to this end.

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APPENDIX A

APPENDIX. CONCATENATION AND ANTIDEGRADABLE CHANNELS

Figure 6 will be of assistance in understanding the following proof that the concatenation

\[ E = B_2 \circ B_1, \]  

(59)
of two channels placed in series is antidegradable if either $B_1$ or $B_2$ is antidegradable. Let

$$J_1 : \mathcal{H}_a \rightarrow \mathcal{H}_{b1} \otimes \mathcal{H}_{c1}, \quad J_2 : \mathcal{H}_{b1} \rightarrow \mathcal{H}_{b2} \otimes \mathcal{H}_{c2},$$

be the isometries that give rise to the channel pairs $(B_1, C_1)$ and $(B_2, C_2)$. Then $B$ and its complement $C$ are generated by the isometry $J : \mathcal{H}_a \rightarrow \mathcal{H}_b \otimes \mathcal{H}_c$, where

$$\mathcal{H}_b = \mathcal{H}_{b2}, \quad \mathcal{H}_c = \mathcal{H}_{c1} \otimes \mathcal{H}_{c2}, \quad J = (J_2 \otimes I_{c1}) \circ J_1,$$

with $I_{c1}$ the identity map on $\mathcal{H}_{c1}$. Thus the complement $C$ of $B$ maps $\mathcal{H}_a$ to the tensor product $\mathcal{H}_{c1} \otimes \mathcal{H}_{c2}$, and its partial traces over these outputs are:

$$\text{Tr}_{c1}[C(A)] = C_2 \circ B_1(A), \quad \text{Tr}_{c2}[C(A)] = C_1(A)$$

(62)

If $B_1$ is antidegradable, i.e., $C_1$ is degradable, there exists a degrading map $D_1 : \mathcal{H}_{c1} \rightarrow \mathcal{H}_{c1}$, indicated by a dashed curve in Fig. 6 (ignore $D_2$), such that for any $A$ in $\mathcal{H}_a$,

$$D_1 \circ C_1(A) = B_1(A)$$

(63)

Given $D_1$, one can define a degrading map $\mathcal{D}$ that maps $\mathcal{H}_c = \mathcal{H}_{c1} \otimes \mathcal{H}_{c2}$ to $\mathcal{H}_b = \mathcal{H}_{b2}$ by its action on the tensor product of operators $F_1 \in \mathcal{H}_{c1}$ and $F_2 \in \mathcal{H}_{c2}$,

$$\mathcal{D}(F_1 \otimes F_2) = \text{Tr}(F_2) \cdot B_2 \circ D_1(F_1).$$

(64)

Then use linearity to extend this to the entire operator space $\mathcal{H}_c$. Intuitively (Fig. 6) $\mathcal{D}$ “throws away” the $\mathcal{H}_{c2}$ output, while mapping the $\mathcal{H}_{c1}$ output to $\mathcal{H}_{b1}$. Since $C_1$ followed by $D_1$ is the same as $B_1$, $D \circ C$ is identical to $B = B_2 \circ B_1$. Hence $C$ is degradable and $B$ antidegradable.

If instead of $B_1$ we assume that $B_2$ is antidegradable, the appropriate degrading map $\mathcal{D} : \mathcal{H}_c \rightarrow \mathcal{H}_{b1}$ is obtained by “throwing away” the $\mathcal{H}_{c1}$ output of $C$ and applying the degrading map $D_2$, see Fig. 6 and ignore $D_1$, to the $\mathcal{H}_{c2}$ output, with the result

$$\mathcal{D}(C_1 \otimes C_2) = \text{Tr}(C_1) \cdot D_2(C_2).$$

(65)

So again $\mathcal{D} \circ C$ is identical to $B = B_2 \circ B_1$, which means $C$ is degradable and $B$ antidegradable.

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**APPENDIX B**

**APPENDIX. ASYMPTOTIC ESTIMATES OF $Q^{(1)}(B_g)$ AND $Q^{(1)}(C_g)$**

Some use was made in Sec. V of asymptotic expressions for the channel coherent information $Q^{(1)}$ in circumstances in which a straightforward numerical approach runs into difficulties because one is trying to find the maximum or minimum of a function

$$f(\epsilon) = \alpha \epsilon \ln(\epsilon) + \beta \epsilon,$$

(66)

where $\epsilon > 0$ is small, and $\alpha$ and $\beta$ are real numbers. If $\alpha$ is positive, $f(\epsilon)$ will be negative for sufficiently small $\epsilon$, and positive if $\alpha$ is negative. If $\alpha$ and $\beta$ are both positive, $f$ has a minimum at

$$\epsilon = \epsilon_m := \exp[-(1 + \beta/\alpha)],$$

(67)

where it takes the value

$$f(\epsilon_m) = -\alpha \epsilon_m = -\alpha \exp[-(1 + \beta/\alpha)].$$

(68)

If both $\alpha$ and $\beta$ are negative, $f$ has a maximum rather than a minimum at (67), and the maximum value is again given by (68).

A first application of these formulas is to the amplitude damping case, Sec. V-A, where for $r = (0, 0, z)$ in the expression for $\rho(r)$ in (38)

$$f(\epsilon) = \Delta(B_g, \rho(r)), \quad \epsilon = 1 - z$$

(69)

has the form (66) for small $\epsilon$, where as a function of $0 < \lambda < 1/2$ and $0 < p < 1/2$,

$$\alpha = \left[p(1 - \lambda) + \lambda - 1/2\right]/\ln 2$$

(70)

This is zero along the line $\lambda = \lambda_0(p)$, (40), and negative when $\delta\lambda = \lambda_0(p) - \lambda$ is positive. Hence $\Delta(B_g, \rho(r))$, and therefore its maximum $Q^{(1)}(B_g)$, is greater than zero for sufficiently small $\delta\lambda > 0$. This is consistent with numerical results that indicate that $Q^{(1)}(B_g)$ is zero for $\lambda \geq \lambda_0(p)$ and positive elsewhere.

One can work out the asymptotic form of $Q^{(1)}(B_g)$ for small positive $\delta\lambda$ using (68) and

$$\alpha = \alpha_1(p) \delta\lambda, \quad \beta = \beta_0(p) + \beta_1(p) \delta\lambda$$

(71)

where

$$\alpha_1(p) = -(1 - p)/\ln 2, \quad \beta_0(p) = \left[(p \ln p)/(1 - p) - \ln(1 - p)\right]/4 \ln 2, \quad \beta_1(p) = (1 - p)[2\beta_0(p) + 1 + 1/\ln 2],$$

(72)

Both $\alpha_1(p)$ and $\beta_0(p)$ are negative in the range of interest, $0 < p < 1/2$, so $\Delta(B_g, \rho(r))$ will have a maximum at

$$\epsilon_m \simeq K \exp[-\beta_0/(\alpha_1 \delta\lambda)], \quad K = \exp[-1 - (\beta_1/\alpha_1)],$$

(73)

and thus $Q^{(1)}(B_g)$, the maximum of $\Delta(B_g, \rho(r))$, has the asymptotic form

$$Q^{(1)}(B_g) \simeq -\alpha_1 \delta\lambda \epsilon_m \simeq a(p)\delta\lambda \exp[-b(p)/\delta\lambda],$$

(74)

for small $\delta\lambda$, where

$$a(p) := -\alpha_1 K, \quad b(p) := \beta_0/\alpha_1,$$

(75)
are positive functions whose \( p \) dependence is determined by that of \( \alpha \) and \( \beta \). The final factor in (74) is exponentially small due to the \( \delta \lambda \) in the denominator of the exponent. The approximation (74) is in reasonable agreement with direct numerical calculations for small \( \delta \lambda \) at \( p = 1/4 \).

In the same way one can find the asymptotic behavior, for \( 0 < \lambda < 1/2 \) and \( p \) very small, of \( Q^{(1)}(C_g) \), equal to minus the minimum of \( f(\epsilon) = \Delta(B_g, \rho(\mathbf{r})) \) in (69). In this case \( z \) is close to \(-1\) and

\[
\epsilon = 1 + z,
\]

is a small quantity. Now \( \alpha \) and \( \beta \) are given by

\[
\alpha = \lambda/\ln 2,
\]
\[
\beta = \beta_0(\lambda) + \beta_1(\lambda) \ln p + \beta_2(\lambda) p \ln p + \beta_3(\lambda) p + \cdots ,
\]

(77)

where

\[
\beta_0 = -\alpha(1 + \ln 2), \quad \beta_1(\lambda) = -\alpha(1 - \lambda)/\lambda, \quad \beta_2(\lambda) = 0, \quad \beta_3(\lambda) = \beta_1(\lambda).
\]

(78)

Inserting these in (68) one arrives at the asymptotic formula (55) for \( Q^{(1)}(C_g) \approx -f(\epsilon_m) \).

An asymptotic estimate for small \( \delta \lambda \) of the nonadditivity of \( Q^{(1)}(B_g) \) at the 2-letter level, see (43), can be carried out assuming that \( \epsilon \) in the input density operator \( \sigma \), (45), is small, and looking for the maximum of

\[
\tilde{f}(\epsilon) = \Delta(B_g^{\otimes 2}, \sigma(\epsilon)) = \tilde{\alpha} \epsilon \ln(\epsilon) + \tilde{\beta} \epsilon.
\]

(79)

It turns out that

\[
\tilde{\alpha} = 2\alpha, \quad \tilde{\beta} = \tilde{\beta}_0(p) + \tilde{\beta}_1(p) \delta \lambda + \tilde{\beta}_2(p) \delta \lambda^2,
\]

(80)

where \( \alpha \) is the single channel quantity defined in (71), and

\[
\tilde{\beta}_0(p) = (-2 \ln 2)p + (4 \ln 2)p^2 + p \ln p + (1 - p(1 + 2p)) \ln(1 + p) - 2(1 - p)^2 \ln(1 - p),
\]

\[
\tilde{\beta}_1(p) = 2(1 - 2p) + 2p(1 + \ln 2) + p \ln p - (1 - p)^2 \ln(1 - p) - p(1 + p) \ln(1 + p),
\]

\[
\tilde{\beta}_2(p) = [p \ln(4p) - (1 + p) \ln(1 + p)]/\ln 2.
\]

(81)

That \( \tilde{\alpha} = 2\alpha \) makes it convenient to consider the ratio

\[
R = \frac{\tilde{Q}^{(1)}(B_g^{\otimes 2})}{2Q^{(1)}(B_g)} = \frac{\tilde{\epsilon}_m}{\epsilon_m} = \exp(\beta\alpha(1 - \tilde{\beta}/2\beta)),
\]

(82)

and thus

\[
Q^{(1)}(B_g^{\otimes 2}) \simeq \tilde{a}(p)\delta \lambda \exp[-\tilde{b}(p)/\delta \lambda],
\]

(83)

with

\[
\tilde{a}(p) = -2\alpha \epsilon_1 \exp[-1 - \tilde{b}_1/2\alpha], \quad \tilde{b}(p) = \tilde{b}_0/2\alpha.
\]

(84)

Provided

\[
\tilde{b}_0/(2\tilde{b}_0) < 1
\]

(85)

a condition fulfilled for all \( 0 < p < 1/2 \), \( \tilde{b}(p) \) is less than \( b(p) \). \( R \) tends to \( +\infty \) as \( \delta \lambda \) goes to zero, so that as \( \delta \lambda \to 0 \),

\[
\delta_2 = \frac{Q^{(1)}(B_g^{\otimes 2})/2 - Q^{(1)}(B_g)}{2} \simeq \frac{\tilde{a}(p)\delta \lambda \exp[-\tilde{b}(p)/\delta \lambda]}{2},
\]

(86)

In the case of the phase-damping channel, Sec. V-B, similar asymptotic estimates are possible, where the small parameter is now

\[
\delta \lambda = g(p) - \lambda,
\]

(87)

where \( g(p) \) is defined in (52). For \( Q^{(1)}(B_g) \) the coefficients in (71) and (72) are given by

\[
\alpha_1(p) = -[1 + (1 - 2p)^2]/2 \ln 2,
\]
\[
\beta_0(p) = 2p(1-p) \ln(4p(1-p)) \quad \text{[(1 + (1 - 2p)^2)/2 \ln 2]},
\]
\[
\beta_1(p) = [1 + \ln 2 - 2p(1-p)(1 + \ln(2p(1-p)))]/\ln 2.
\]

(88)

These coefficients when inserted in (73) and (75) yield the asymptotic form (74).

Similarly, an asymptotic formula for \( Q^{(1)}(C_g) \) is obtained by employing in (77) the quantities

\[
\alpha = \lambda/\ln 2, \quad \beta_0 = -\alpha(1 + \ln 2), \quad \beta_1(\lambda) = -\alpha(1 - \lambda)/\lambda, \quad \beta_2(\lambda) = -2\beta_1(\lambda), \quad \beta_3(\lambda) = \beta_1(\lambda),
\]

(89)

resulting in the asymptotic form (56).

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