Accurate Charge-Dependent Nucleon-Nucleon Potential at Fourth Order of Chiral Perturbation Theory

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We present the first nucleon-nucleon potential at next-to-next-to-next-to-leading order (fourth order) of chiral perturbation theory. Charge-dependence is included up to next-to-leading order of the isospin-violation scheme. The accuracy for the reproduction of the $NN$ data below 290 MeV lab. energy is comparable to the one of phenomenological high-precision potentials. Since $NN$ potentials of order three and less are known to be deficient in quantitative terms, the present work shows that the fourth order is necessary and sufficient for a reliable $NN$ potential derived from chiral effective Lagrangians. The new potential provides a promising starting point for exact few-body calculations and microscopic nuclear structure theory (including chiral many-body forces derived on the same footing).

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The theory of nuclear forces has a long history. Based upon the Yukawa idea [1], first field-theoretic attempts [2,3] to derive the nucleon-nucleon ($NN$) interaction focused on pion-exchange, resulting in the $NN$ potentials by Gartenhaus [4] and by Signell and Marshak [5]. However, even qualitatively, these potentials barely agreed with empirical information on the nuclear force. So, these “pion theories” of the 1950s are generally judged as failures—for reasons we understand today: pion dynamics is constrained by chiral symmetry, a crucial point that was unknown in the 1950s.

Historically, the experimental discovery of heavy mesons [6] in the early 1960s saved the situation. The one-boson-exchange (OBE) model [7,8] emerged which is still the most economical and quantitative phenomenology for describing the nuclear force [9,10]. The weak point of this model, however, is the scalar-isoscalar “sigma” or “epsilon” boson, for which the empirical evidence remains controversial. Since this boson is associated with the correlated (or resonant) exchange of two pions, a vast theoretical effort that occupied more than a decade was launched to derive the $2\pi$-exchange contribution of the nuclear force, which creates the intermediate range attraction. For this, dispersion theory as well as field theory were invoked producing the Paris [11,12] and the Bonn [13,8] potentials.

The nuclear force problem appeared to be solved; however, with the discovery of quantum chromo-dynamics (QCD), all “meson theories” had to be relegated to models and the attempts to derive the nuclear force started all over again.

The problem with a derivation from QCD is that this theory is non-perturbative in the low-energy regime characteristic of nuclear physics, which makes direct solutions impossible. Therefore, during the first round of new attempts, QCD-inspired quark models [14] became popular. These models were able to reproduce qualitatively some of the gross features of the nuclear force. However, on a critical note, it has been pointed out that these quark-based approaches were nothing but another set of models and, thus, did not represent any fundamental progress. Equally well, one may then stay with the simpler and much more quantitative meson models.

A major breakthrough occurred when the concept of an effective field theory (EFT) was introduced and applied to low-energy QCD. As outlined by Weinberg in a seminal paper [15], one has to write down the most general Lagrangian consistent with the assumed symmetry principles, particularly the (broken) chiral symmetry of QCD. At low energy, the effective degrees of freedom are pions and nucleons rather than quarks and gluons; heavy mesons and nucleon resonances are “integrated out”. So, in a certain sense we are back to the 1950s, except that we are smarter by 40 years of experience: broken chiral symmetry is a crucial constraint that generates and controls the dynamics and establishes a clear connection with the underlying theory, QCD.

The chiral effective Lagrangian is given by an infinite series of terms with increasing number of derivatives and/or nucleon fields, with the dependence of each term on the pion field prescribed by the rules of broken chiral symmetry [16]. Applying this Lagrangian to $NN$ scattering generates an unlimited number of Feynman diagrams, which may suggest again an untractable problem. However, Weinberg showed [16] that a systematic expansion of the nuclear amplitude exists in terms of $(Q/\Lambda_\chi)^\nu$, where $Q$ denotes a momentum or pion mass, $\Lambda_\chi \approx 1$ GeV is the chiral symmetry breaking scale, and $\nu \geq 0$. For a given order $\nu$, the number of contributing terms is finite and calculable; these terms are uniquely defined and the prediction at each order is model-independent. By going to higher orders, the amplitude can be calculated to any

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desired accuracy. The scheme just outlined has become known as chiral perturbation theory (\(\chi PT\)).

Following the first initiative by Weinberg [16], pioneering work was performed by Ordóñez, Ray, and van Kolck [17,18] who constructed a \(NN\) potential in coordinate space based upon \(\chi PT\) at next-to-next-to-leading order (NNLO; \(\nu = 3\)). The results were encouraging and many researchers [19] became attracted to the new field. Kaiser, Brockmann, and Weise [20] presented the first model-independent prediction for the \(NN\) amplitudes of peripheral partial waves at NNLO.Epelbaum et al. [21] developed the first momentum-space \(NN\) potential at NNLO.

In the 1990s, unrelated, parallel research showed that, for conclusive few-body calculations and meaningful microscopic nuclear structure predictions, the input \(NN\) potential must be of the highest precision; i.e., it must reproduce the \(NN\) data below about 300 MeV lab. energy with a \(\chi^2/datum \approx 1\). The family of high-precision \(NN\) potentials [9,22,23,10] was developed which fulfills this requirement. Due to the outstanding accuracy of these \(NN\) potentials, it was possible to pin down cases of few-body scattering and of nuclear structure that clearly require three-nucleon forces (3NF) for their microscopic explanation. Famous examples are the \(A_g\) puzzle of \(N\)-\(d\) scattering [24] and the ground state of \(^{10}\text{B}\) [25].

One important advantage of \(\chi PT\) is that it makes specific predictions for many-body forces. For a given order of \(\chi PT\), both \(2N\) and \(3N\) forces are generated on the same footing. At next-to-leading order (NLO), all 3NF cancel [16,26]; however, at NNLO and higher orders, well-defined, nonvanishing 3NF terms occur. As discussed, since 3NF effects are in general very subtle, it is only possible to demonstrate their necessity and relevance when the 2NF is of high precision.

\(NN\) potentials based upon \(\chi PT\) at NNLO are poor in quantitative terms; they reproduce the \(NN\) data below 290 MeV lab. energy with a \(\chi^2/datum \) of more than 20 which is totally unacceptable. Clearly, there is a strong need for more precision, implying that going to higher order is necessary.

It is the purpose of this note to present the first \(NN\) potential that is based consistently on \(\chi PT\) at next-to-next-to-next-to-leading order (\(N^3\)LO; fourth order). We will show that, at this order, the accuracy is comparable to the one of the high-precision phenomenological potentials. Thus, the \(NN\) potential at \(N^3\)LO is the first to meet the requirements for a reliable input-potential for exact few-body and microscopic nuclear structure calculations (including chiral 3NF consistent with the chiral 2NF).

In \(\chi PT\), the \(NN\) amplitude is uniquely determined by two classes of contributions: contact terms and pion-exchange diagrams. At \(N^3\)LO, there are two contacts of order \(O(Q^0)\), seven of \(O(Q^2)\), and 15 of \(O(Q^4)\), resulting in a total of 24 contact terms, which generate 24 parameters that are crucial for the fit of the partial waves with orbital angular momentum \(L \leq 2\) [27].

Now, turning to the pion contributions: At leading order [LO, \(O(Q^0)\), \(\nu = 0\)], there is only the well-known static one-pion exchange (OPE). Two-pion exchange (TPE) starts at next-to-leading order (NLO, \(\nu = 2\)), and there are further TPE contributions in any higher order. While TPE at NNLO was known for a while [17,20,21], TPE at \(N^3\)LO has been calculated only recently by Kaiser [28]. All 2\(\pi\) exchange contributions up to \(N^3\)LO are summarized in a pedagogical and systematic fashion in Ref. [29] where the model-independent results for \(NN\) scattering in peripheral partial waves are also shown. We use the analytic expressions published in Ref. [29]. Finally, there is also three-pion exchange, which shows up for the first time at \(N^3\)LO (two loops). In Ref. [30], it was demon-

### Table I. Low-energy constants applied in the \(N^3\)LO \(NN\) potential (column ‘\(N^3\)N’). The \(c_i\) belong to the dimension-two \(\pi N\) Lagrangian and are in units of GeV\(^{-1}\), while the \(d_i\) are associated with the dimension-three Lagrangian and are in units of GeV\(^{-2}\). The column ‘\(\pi N\)’ shows values determined from \(\pi N\) data.

| \(c_i\) | \(\pi N\) |
|------|-------|
| \(c_1\) | \(-0.81\) |
| \(c_2\) | \(2.80\) |
| \(c_3\) | \(-3.20\) |
| \(c_4\) | \(5.40\) |
| \(d_1 + d_2\) | \(3.06\) |
| \(d_3\) | \(-3.27\) |
| \(d_5\) | \(0.45\) |
| \(d_{14} - d_{15}\) | \(-5.65\) |

\(a\)Ref. [31]. \(b\)Ref. [32]. \(c\)Ref. [33].

### Table II. \(\chi^2/datum\) for the reproduction of the 1999 \(np\) database [38] below 290 MeV by various \(np\) potentials.

| Bin (MeV) | \# of data | \(N^3LO^a\) | NNLO\(^b\) | NLO\(^c\) | AV18\(^d\) |
|-----------|------------|-------------|-------------|-------------|-------------|
| 0–100     | 1058       | 1.06        | 1.71        | 5.20        | 0.95        |
| 100–190   | 501        | 1.08        | 12.9        | 49.3        | 1.10        |
| 190–290   | 843        | 1.15        | 19.2        | 68.3        | 1.11        |
| 0–290     | 2402       | 1.10        | 10.1        | 36.2        | 1.04        |

\(a\)This work. \(b\)Ref. [35]. \(c\)Ref. [22].
FIG. 1. $np$ phase parameters below 300 MeV lab. energy for partial waves with $J \leq 2$. The solid line is the result at N$^3$LO. The dotted and dashed lines are the phase shifts at NLO and NNLO, respectively, as obtained by Epelbaum et al. [35]. The solid dots show the Nijmegen multi-energy $np$ phase shift analysis [36], and the open circles are the VPI single-energy $np$ analysis SM99 [37].

FIG. 1, continued.

$2n$ is chosen to be sufficiently large so that the regulator generates powers which are beyond the order ($\nu = 4$) at which our calculation is conducted; i.e., terms up to $Q^4$ are not affected.

The contact terms plus irreducible pion-exchange expressions at N$^3$LO, multiplied by the above regulator, define the $NN$ potential at N$^3$LO. This potential is applied in a Lippmann-Schwinger equation to obtain the $T$-matrix from which phase shifts and $NN$ observables are calculated. The corresponding homogenous equation determines the properties of the two-nucleon bound state (deuteron).

The peripheral partial waves of $NN$ scattering with $L \geq 3$ are exclusively determined by OPE and TPE because the N$^3$LO contacts contribute to $L \leq 2$ only. OPE and TPE at N$^3$LO depend on the axial-vector coupling constant, $g_A$ (we use $g_A = 1.29$), the pion decay constant, $f_\pi = 92.4$ MeV, and eight low-energy constants (LEC) that appear in the dimension-two and dimension-three $\pi N$ Lagrangians (cf. Ref. [29]). In the optimization process, we varied three of them, namely, $c_2$, $c_3$, and $c_4$. We found that the other LEC are not very effective in the $NN$ system and, therefore, we kept them at the values determined from $\pi N$ (cf. Table I). The most influential constant is $c_3$, which has to be chosen on the low side (slightly more than one standard deviation below its $\pi N$ determination) for an optimal fit of the $NN$ data. Our choice for $c_4$, which is substantially above the value determined in $\pi N$, is necessary to bring the $^3F_2$ phase shift down.

strated that the $3\pi$ contributions at this order are negligible, which is why we leave them out.

For an accurate fit of the low-energy $pp$ and $np$ data, charge-dependence is important. We include charge-dependence up to next-to-leading order of the isospin-violation scheme (NLO, in the notation of Ref. [31]). Thus, we include the pion mass difference in OPE and the Coulomb potential in $pp$ scattering, which takes care of the LO contributions. At order NLO we have pion mass difference in the NLO part of TPE, $\pi\gamma$ exchange [32], and two charge-dependent contact interactions of order $Q^0$ which make possible an accurate fit of the three different $^1S_0$ scattering lengths, $a_{pp}$, $a_{nn}$, and $a_{np}$.

Chiral perturbation theory is a low-momentum expansion. It is valid only for momenta $Q \ll \Lambda \chi \approx 1$ GeV. To enforce this, we multiply all expressions (contacts and irreducible pion exchanges) with a regulator function,

$$\exp \left[ - \left( \frac{p}{\Lambda} \right)^2 - \left( \frac{p'}{\Lambda} \right)^2 \right],$$

where $p$ and $p'$ denote, respectively, the magnitudes of the initial and final nucleon momenta in the center-of-mass frame. We use $\Lambda = 0.5$ GeV throughout. The exponent
The most important set of fit parameters are the ones associated with the 24 contact terms that rule the partial waves with $L \leq 2$. In addition, we have two charge-dependent contacts, which brings the number of contact parameters to 26. Since we treated three LEC as semi-free, the total number of parameters of the $N^3$LO potential is 29.

In the optimization procedure, we fit first phase shifts, and then we refine the fit by minimizing the $\chi^2$ obtained from a direct comparison with the data. The phase shifts at $N^3$LO for $np$ scattering below 300 MeV lab. energy are displayed in Fig. 1. The $\chi^2$/datum for the fit of the $np$ data below 290 MeV is shown in Table II, and the corresponding one for $pp$ is given in Table III. The $\chi^2$ tables demonstrate a dramatic improvement of the $NN$ interaction order by order. It is clearly revealed that, at NLO and NNLO, the reproduction of the $NN$ data is of unacceptably poor quality. However, at $N^3$LO, the quantitative character is comparable to the phenomenological high-precision Argonne $V_{18}$ potential [22].

In conclusion, we have developed the first $NN$ potential at fourth order of $\chi$PT [40]. This potential is as quantitative as some so-called high-precision phenomenological potentials. Due to its basis in $\chi$PT, the many-body forces associated with this two-body force are well-defined. Thus, we have a promising starting point for exact few-body calculations and microscopic nuclear structure theory.

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