Numerical solution of one-dimensional Sine–Gordon equation using Reproducing Kernel Hilbert Space Method  
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Abstract: In this paper, we propose a reproducing kernel Hilbert space method (RKHSM) for solving the sine–Gordon (SG) equation with initial and boundary conditions based on the reproducing kernel theory. Its exact solution is represented in the form of series in the reproducing kernel Hilbert space. Some numerical examples have been studied to demonstrate the accuracy of the present method. The results obtained from the method are compared with the exact solutions and the earlier works. Results of numerical examples show that the presented method is simple and effective.

Keywords: Reproducing kernel method, series solutions, sine–Gordon equation, reproducing kernel space.

1. Introduction

Since Russell’s [1-2] first observation for some special waves with characteristic properties on a canal in 1834 many scientists among whom Korteweg and de Vries [3], who derived the equation concerning the propagation of waves in one direction on the free surface of a shallow canal also known as the KdV equation, have investigated them more extensively. The name soliton was first used in Zabusky and Kruskal [4] in order to emphasize that a soliton is a localized entity which keeps its identity after interaction. Equations which also lead to solitary waves are the sine-Gordon, the cubic Schrödinger equation, etc. Because of the great importance of the study of physical phenomena theoretical solutions of nonlinear and especially of soliton type equations have been developed during the last years [5].

The nonlinear one-dimensional Sine–Gordon (SG) equation is a very important nonlinear hyperbolic partial differential equation PDE. It appears in differential geometry gained its significance because of the collisional behaviors of solitons that arise from these equations. It was basically considered in the nineteenth century in the course of surface of constant negative curvature. This equation attracted a lot of attention in the 1970s due to the presence of soliton solutions [6-7]. The sine-Gordon equation comes out in a number of physical applications [8–10] including applications in the chain of coupled pendulums and modelling the propagation of transverse electromagnetic (TEM) wave on a superconductor transmission system.

Consider the one-dimensional nonlinear sine-Gordon equation

\[
\frac{\partial^2 u}{\partial t^2}(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t) - \sin(u(x, t)), \quad a \leq x \leq b, \quad t \geq 0, \quad (1.1)
\]
with initial conditions
\[ u(x, 0) = f(x), \ a \leq x \leq b, \] (1.2)
\[ \frac{\partial u}{\partial t}(x, 0) = g(x), \ a \leq x \leq b, \] and the boundary conditions
\[ u(a, t) = h_1(t), \ u(b, t) = h_2(t), \ t \geq 0. \] (1.3)

Since the exact solution of the SG equation can only be obtained in special situations, many numerical schemes are constructed to solve the SG equation. The study of numerical solutions of the sine-Gordon equation have been investigated considerably in the last few years. For solving (1.1), for instance, high-order solution of one-dimensional sine-Gordon equation using compact finite difference and DIRKN methods [8]. The authors of [11] proposed a numerical method form solving (1.1) using collocation and radial basis functions. Also, the boundary integral equation approach is used in [12]. Bratsos has proposed a numerical scheme for solving one dimensional sine-Gordon equation and a third-order numerical scheme for the two-dimensional sine-Gordon equation in [13-14] respectively. Also, in [15], a numerical method using radial basis function for the solution of two-dimensional sine-Gordon equation is used. In addition, several authors recommended spectral methods and Fourier pseudospectral method for solving nonlinear wave equation using a discrete Fourier series and Chebyshev orthogonal polynomials [16-18]. Ma and Wu [19] presented a meshless scheme by using a multiquadric (MQ) quasi-interpolation named \( L_\varepsilon \) without solving a large-scale linear system of equations, but a polynomial \( p(x) \) was needed to improved the accuracy of the scheme. More recently, another numerical work has been investigated by Khaliq et al. [20].

One difficult point of numerically solving the sine-Gordon equation is how to solve the nonlinear system resulting by discretization. The Newton iteration method is often used as a basis for designing numerical schemes. In this case, the Jacobian matrices have to be established, inverted and possibly updated during the iteration. When high approximation accuracy is desired, it requires one to use a sufficiently small time-step and sufficiently fine grids. Thus it demands a large amount of computational effort [21].

In this paper, we solve Eqs. (1.1) and (1.3) by using Reproducing Kernel Method (or RKM). The nonlinear problem is solved easily and elegantly by using RKM. The technique has many advantages over the classical techniques. It also avoids discretization and provides an efficient numerical solution with high accuracy, minimal calculation, avoidance of physically unrealistic assumptions. In the next section, we will describe the procedure.

The theory of reproducing kernels [22] was used for the first time at the beginning of the 20th century by Zaremba in his work on boundary value problems.
for harmonic and biharmonic functions. Reproducing kernel theory has important applications in numerical analysis, differential equations, probability and statistics and so on [23-35]. Recently, using the RKM, some authors discussed fractional differential equation, nonlinear oscillator with discontinuity, singular nonlinear two-point periodic boundary value problems, integral equations and nonlinear partial differential equations and so on [23-35].

The paper is organized as follows. Section 2 is devoted to several reproducing kernel spaces. Solution representation in \( W(\Omega) \) and a linear operator have been presented in Section 3. Section 4 provides the main results, the exact and approximate solution of Eqs.(1.1) and (1.3) and an iterative method are developed for the kind of problems in the reproducing kernel space. We have proved that the approximate solution converges to the exact solution uniformly. Some numerical experiments are illustrated in Section 5. We provide some conclusions in the last section.

2. Preliminaries

2.1. Reproducing Kernel Spaces

In this section, we define some useful reproducing kernel spaces.

**Definition 2.1. (Reproducing kernel).** Let \( E \) be a nonempty abstract set. A function \( K : E \times E \to C \) is a reproducing kernel of the Hilbert space \( H \) if and only if

a) \( \forall t \in E, K(., t) \in H \),

b) \( \forall t \in E, \forall \varphi \in H, \langle \varphi(.), K(., t) \rangle = \varphi(t) \).

The last condition is called ”the reproducing property” the value of the function \( \varphi \) at the point \( t \) is reproduced by the inner product of \( \varphi \) with \( K(., t) \).

**Definition 2.2.**

\[
W^3_2[0,1] = \left\{ u(x) \mid u(x), u'(x), u''(x) \text{ are absolutely continuous real value functions in } [0,1], \right. \\
\left. u^{(3)}(x) \in L^2[0,1], x \in [0,1], u(0) = 0, u(1) = 0, \right. \\
\left. \int_0^1 u^{(3)}(x)g^{(3)}(x)dx, u(x), g(x) \in W^3_2[0,1], \right. \\
\right\},
\]

The inner product and the norm in \( W^3_2[0,1] \) are defined respectively by

\[
\langle u(x), g(x) \rangle_{W^3_2} = u(0)g(0)+u'(0)g'(0)+u''(1)g''(1)+\int_0^1 u^{(3)}(x)g^{(3)}(x)dx, \quad u(x), g(x) \in W^3_2[0,1],
\]

and

\[
\|u\|_{W^3_2} = \sqrt{\langle u, u \rangle_{W^3_2}}, \quad u \in W^3_2[0,1].
\]
The space $W^3_2[0, 1]$ is a reproducing kernel space, i.e., for each fixed $y \in [0, 1]$ and any $u(x) \in W^3_2[0, 1]$, there exists a function $R_y(x)$ such that

$$u(y) = \langle u(x), R_y(x) \rangle_{W^3_2}.$$ 

**Definition 2.3.**

$$W^3_2[0, T] = \left\{ u(t) \mid u(t), u'(t), u''(t) \text{ are absolutely continuous in } [0, T], \right. \\
\left. u^{(3)}(t) \in L^2[0, 1], \ t \in [0, T], \ u(0) = 0, \ u'(0) = 0. \right\}$$

The inner product and the norm in $W^3_2[0, T]$ are defined respectively by

$$\langle u(t), g(t) \rangle_{W^3_2} = \sum_{i=0}^2 u^{(i)}(0)g^{(i)}(0) + \int_0^1 u^{(3)}(t)g^{(3)}(t)dt, \ u(t), g(t) \in W^3_2[0, T],$$

and

$$\|u\|_{W^3_2} = \sqrt{\langle u, u \rangle_{W^3_2}}, \ u \in W^3_2[0, T].$$

The space $W^3_2[0, T]$ is a reproducing kernel space and its reproducing kernel function $r_s(t)$ is given by

$$r_s(t) = \begin{cases} 
\frac{1}{4}s^2t^2 + \frac{1}{12}s^2t^3 - \frac{1}{24}st^4 + \frac{1}{120}t^5, & t \leq s, \\
\frac{1}{7}s^2t^2 + \frac{1}{12}s^3t^2 - \frac{1}{24}ts^4 + \frac{1}{120}s^5, & t > s.
\end{cases}$$

**Definition 2.4.**

$$W^1_2[0, 1] = \left\{ u(x) \mid u(x) \text{ is absolutely continuous in } [0, 1], \right. \\
\left. u'(x) \in L^2[0, 1], \ x \in [0, 1]. \right\}$$

The inner product and the norm in $W^1_2[0, 1]$ are defined respectively by

$$\langle u(x), g(x) \rangle_{W^1_2} = u(0)g(0) + \int_0^1 u'(x)g'(x)dx, \ u(x), g(x) \in W^1_2[0, 1],$$

and

$$\|u\|_{W^1_2} = \sqrt{\langle u, u \rangle_{W^1_2}}, \ u \in W^1_2[0, 1].$$

The space $W^1_2[0, 1]$ is a reproducing kernel space and its reproducing kernel function $Q_y(x)$ is given by

$$Q_y(x) = \begin{cases} 
1 + x, & x \leq y, \\
1 + y, & x > y.
\end{cases}$$

**Definition 2.5.**
$$W^1_2[0,T] = \left\{ u(t) \mid u(t) \text{ is absolutely continuous in } [0,T], \quad u'(t) \in L^2[0,T], \ t \in [0,T] \right\},$$

The inner product and the norm in $W^1_2[0,T]$ are defined respectively by

$$\langle u(t), g(t) \rangle_{W^1_2} = u(0)g(0) + \int_0^T u'(t)g'(t)dt, \ u(t), g(t) \in W^1_2[0,T],$$

and

$$\|u\|_{W^1_2} = \sqrt{\langle u, u \rangle_{W^1_2}}, \ u \in W^1_2[0,T].$$

The space $W^1_2[0,T]$ is a reproducing kernel space and its reproducing kernel function $q_s(t)$ is given by

$$q_s(t) = \begin{cases} 1 + t, & t \leq s, \\ 1 + s, & t > s. \end{cases}$$

**Theorem 2.1.** The space $W^3_2[0,1]$ is a complete reproducing kernel space whose reproducing kernel function $R_y(x)$ is given as,

$$R_y(x) = \begin{cases} \sum_{i=1}^6 c_i(y) x^{i-1}, & x \leq y, \\ \sum_{i=1}^6 d_i(y) x^{i-1}, & x > y. \end{cases} \quad (2.1)$$

where

$$c_1(y) = 0, \quad c_2(y) = -\frac{1}{122} y^5 + \frac{5}{244} y^4 - \frac{127}{244} y^2 + \frac{31}{61} y,$$

$$c_3(y) = -\frac{1}{2928} y^5 + \frac{127}{5856} y^4 - \frac{1}{12} y^3 + \frac{1137}{1952} y^2 - \frac{127}{244} y,$$

$$c_4(y) = 0, \quad c_5(y) = \frac{1}{2938} y^5 - \frac{5}{5856} y^4 + \frac{127}{5856} y^2 - \frac{31}{1464} y,$$

$$c_6(y) = -\frac{1}{7320} y^5 + \frac{1}{2928} y^4 - \frac{127}{2928} y^2 - \frac{1}{122} y + \frac{1}{120},$$

$$d_1(y) = \frac{1}{120} y^5; \quad d_2(y) = -\frac{1}{122} y^5 - \frac{31}{1464} y^4 - \frac{127}{244} y^2 + \frac{31}{61} y,$$

$$d_3(y) = -\frac{1}{2928} y^5 + \frac{127}{5856} y^4 + \frac{1137}{1952} y^2 - \frac{127}{244} y,$$
By Definition 2.2 and integrating by parts two times, we obtain

\[ d_4(y) = -\frac{1}{12}y^2. \]

\[ d_5(y) = \frac{1}{2928}y^6 - \frac{5}{5856}y^4 + \frac{127}{5856}y^2 + \frac{5}{244}y, \]

\[ d_6(y) = \frac{1}{7320}y^5 + \frac{1}{2928}y^4 - \frac{1}{2928}y^2 - \frac{1}{122}y. \]

Proof: Let \( u \in W^3_2[0,1] \) and \( 0 \leq y \leq 1 \). Define \( R_y \) by (2.1). Note that

\[ R_y'(x) = \begin{cases} 
\sum_{i=1}^{5} ic_{i+1}(y) x^{i-1}, & x < y, \\
\sum_{i=1}^{5} id_{i+1}(y) x^{i-1}, & x > y,
\end{cases} \]

\[ R_y''(x) = \begin{cases} 
\sum_{i=1}^{4} i(i+1) c_{i+2}(y) x^{i-1}, & x < y, \\
\sum_{i=1}^{4} i(i+1) d_{i+2}(y) x^{i-1}, & x > y,
\end{cases} \]

\[ R_y^{(3)}(x) = \begin{cases} 
\sum_{i=1}^{3} i(i+1)(i+2) c_{i+3}(y) x^{i-1}, & x < y, \\
\sum_{i=1}^{3} i(i+1)(i+2) d_{i+3}(y) x^{i-1}, & x > y,
\end{cases} \]

\[ R_y^{(4)}(x) = \begin{cases} 
\sum_{i=1}^{2} i(i+1)(i+2)(i+3) c_{i+4}(y) x^{i-1}, & x < y, \\
\sum_{i=1}^{2} i(i+1)(i+2)(i+3) d_{i+4}(y) x^{i-1}, & x > y,
\end{cases} \]

and

\[ R_y^{(5)}(x) = \begin{cases} 
120c_6(y), & x < y, \\
120d_6(y), & x > y.
\end{cases} \]

By Definition 2.2 and integrating by parts two times, we obtain

\[ \langle u(x), R_y(x) \rangle_{W^2_2} = u(0)R_y'(0) + u'(0)R_y'(0) + u'(1)R_y'(1) + u''(1)R_y^{(3)}(1) 
\]

\[ + u(1)R_y^{(5)}(1) - u(0)R_y^{(5)}(0) - \int_0^1 u(x)R_y^{(6)}(x)dx \]

\[ = u'(0)(R_y'(0) + R_y^{(4)}(0)) + u'(1)(R_y'(1) - R_y^{(4)}(1)) 
\]

\[ + u''(1)R_y^{(3)}(1) - u''(0)R_y^{(3)}(0) 
\]

\[ + \int_0^y R_y^{(5)}(x)u(x)dx + \int_y^1 R_y^{(5)}(x)u(x)dx \]

\[ = u'(0)(c_2(y) + 24c_5(y)) - u''(0)(6c_3(y)) 
\]

\[ + u'(1)(d_2(y) + 2d_3(y) + 3d_4(y) - 20d_5(y) - 115d_6(y)) 
\]

\[ + u''(1)(6d_4(y) + 24d_5(y) + 60d_6(y)) 
\]

\[ + \int_0^y 120c_6(y)u(x)dx + \int_y^1 120d_6(y)u(x)dx \]

\[ = 120u(y)(\frac{1}{720}) = u(y). \]
Definition 2.6.
\[ W(\Omega) = \left\{ u(x, t) \mid \frac{\partial^8 u}{\partial x^8 \partial t} \text{ is completely continous in } \Omega = [0, 1] \times [0, T], \right\} \]
\[ \frac{\partial^6 u}{\partial x^6 \partial t} \in L^2(\Omega), \quad u(x, 0) = 0, \quad \frac{\partial u(x, 0)}{\partial t} = 0, \quad u(0, t) = 0, \quad u(1, t) = 0. \]
The inner product and the norm in \( W(\Omega) \) are defined respectively by
\[ \langle u(x, t), g(x, t) \rangle_w = \sum_{i=0}^{2} \int_0^T \left[ \frac{\partial^i}{\partial t^i} u(0, t) \frac{\partial^3}{\partial t^3} g(0, t) \right] dt \]
\[ + \sum_{j=0}^{2} \left\langle \frac{\partial}{\partial t} u(x, 0), \frac{\partial}{\partial t} g(x, 0) \right\rangle_{W^2_2} \]
\[ + \int_0^1 \int_0^T \left[ \frac{\partial^3}{\partial x^3} \frac{\partial^3}{\partial t^3} u(x, t) \frac{\partial^3}{\partial x^3} \frac{\partial^3}{\partial t^3} g(x, t) \right] dtdx, \]
and
\[ \| u \|_W = \sqrt{\langle u, u \rangle_W}, \quad u \in W(\Omega). \]

Theorem 2.2. \( W(\Omega) \) is a reproducing kernel space and its reproducing kernel function is
\[ K_{(y,s)}(x, t) = R_y(x) r_s(t), \]
such that for any \( u(x, t) \in W(\Omega), \)
\[ u(y, s) = \langle u(x, t), K_{(y,s)}(x, t) \rangle_W, \]
and
\[ K_{(y,s)}(x, t) = K_{(x,t)}(y, s). \]

Definition 2.7.
\[ \hat{W}(\Omega) = \left\{ u(x, t) \mid u(x, t) \text{ is completely continous in } \Omega = [0, 1] \times [0, T], \right\} \]
The inner product and the norm in \( \hat{W}(\Omega) \) are defined respectively by
\[ \langle u(x, t), g(x, t) \rangle_{\hat{W}} = \int_0^T \left[ \frac{\partial}{\partial t} u(0, t) \frac{\partial}{\partial t} g(0, t) \right] dt \]
\[ + \langle u(x, 0), g(x, 0) \rangle_{W^2_2} \]
\[ + \int_0^1 \int_0^T \left[ \frac{\partial}{\partial x} \frac{\partial}{\partial t} u(x, t) \frac{\partial}{\partial x} \frac{\partial}{\partial t} g(x, t) \right] dtdx. \]
and 
\[ \|u\|_{\tilde{W}} = \sqrt{\langle u, u \rangle_{\tilde{W}}}, \quad u \in \tilde{W}(\Omega). \]

\( \tilde{W}(\Omega) \) is a reproducing kernel space and its reproducing kernel function \( G_{(y,s)}(x,t) \) is
\[ G_{(y,s)}(x,t) = Q_{y}(x)q_{s}(t). \]

3. Solution representation in \( W(\Omega) \).

In this section, the solution of equation (1.1) is given in the reproducing kernel space \( W(\Omega) \). On defining the linear operator \( L : W(\Omega) \rightarrow \tilde{W}(\Omega) \) as
\[ Lv = \frac{\partial^2 v}{\partial t^2}(x,t) - \frac{\partial^2 v}{\partial x^2}(x,t). \]

After homogenizing the initial and boundary conditions model problem (1.1) changes the following problem:
\[
\begin{aligned}
Lv &= M(x, t, v(x,t)), \quad (x,t) \in [0,1] \times [0,T], \\
v(x,0) &= \frac{\partial v}{\partial t}(x,0) = v(0,t) = v(1,t) = 0.
\end{aligned}
\]

(3.1)

We replace \( v(x,t) \) with \( u(x, t) \) in (3.1), for simplicity.

**Lemma 3.1.** \( L \) is a bounded linear operator.

**Proof:**
\[
\|L u\|_{\tilde{W}}^2 = \int_0^T \left[ \frac{\partial}{\partial t} Lu(0,t) \right]^2 dt + \langle Lu(x,0), Lu(x,0) \rangle_{W_2^2} \\
+ \int_0^1 \int_0^T \left[ \frac{\partial}{\partial x} \frac{\partial}{\partial t} Lu(x,t) \right]^2 dxdt \\
= \int_0^T \left[ \frac{\partial}{\partial t} Lu(0,t) \right]^2 dt + |Lu(0,0)|^2 \\
+ \int_0^1 \left[ \frac{\partial}{\partial x} Lu(x,0) \right]^2 dx + \int_0^1 \int_0^T \left[ \frac{\partial}{\partial x} \frac{\partial}{\partial t} Lu(x,t) \right]^2 dxdt,
\]

since
\[
\begin{aligned}
u(x,t) &= \langle u(\xi, \eta), K_{(x,t)}(\xi, \eta) \rangle_{W}, \\
Lu(x,t) &= \langle u(\xi, \eta), LK_{(x,t)}(\xi, \eta) \rangle_{W},
\end{aligned}
\]

from the continuity of \( K_{(x,t)}(\xi, \eta) \), we have
\[ |Lu(x,t)| \leq \|u\|_{W} \|LK_{(x,t)}(\xi, \eta)\|_{W} \leq a_0 \|u\|_{W}. \]
similarly for \( i = 0,1 \)

\[
\frac{\partial^i}{\partial x^i} Lu(x,t) = \left\langle u(\xi, \eta), \frac{\partial^i}{\partial x^i} L K(\xi, \eta) \right\rangle_W ,
\]

\[
\frac{\partial}{\partial t} \frac{\partial^i}{\partial x^i} Lu(x,t) = \left\langle u(\xi, \eta), \frac{\partial}{\partial t} \frac{\partial^i}{\partial x^i} L K(\xi, \eta) \right\rangle_W ,
\]

and then

\[
\left| \frac{\partial^i}{\partial x^i} Lu(x,t) \right| \leq e_i \| u \|_W ,
\]

\[
\left| \frac{\partial}{\partial t} \frac{\partial^i}{\partial x^i} Lu(x,t) \right| \leq f_i \| u \|_W .
\]

Therefore

\[
\| Lu(x,t) \|_W^2 \leq \sum_{i=0}^{1} \left( e_i^2 + f_i^2 \right) \| u \|_W^2 \leq a^2 \| u \|_W^2 .
\]

Now, choose a countable dense subset \( \{ (x_1, t_1), (x_2, t_2), \ldots \} \) in \( \Omega = [0,1] \times [0,T] \) and define

\[
\varphi_i(x,t) = G(x_i, t_i) (x,t), \quad \Psi_i(x,t) = L^* \varphi_i(x,t),
\]

where \( L^* \) is the adjoint operator of \( L \). The orthonormal system \( \{ \Psi_i(x,t) \}_{i=1}^{\infty} \) of \( W(\Omega) \) can be derived from the process of Gram-Schmidt orthogonalization of \( \{ \Psi_i(x,t) \}_{i=1}^{\infty} \) as

\[
\tilde{\Psi}_i(x,t) = \sum_{k=1}^{i} \beta_{ik} \Psi_k(x,t).
\]

**Theorem 3.1.** Suppose that \( \{ (x_i, t_i) \}_{i=1}^{\infty} \) is dense in \( \Omega \); then \( \{ \Psi_i(x,t) \}_{i=1}^{\infty} \) is complete system in \( W(\Omega) \) and

\[
\Psi_i(x,t) = L(y,s) K(y,s) (x,t) \bigg|_{(y,s)=(x_i, t_i)}.
\]

**Proof:** We have

\[
\Psi_i(x,t) = (L^* \varphi_i) (x,t) = \left\langle (L^* \varphi_i) (y,s), K(x,t) (y,s) \right\rangle_W
\]

\[
= \left\langle \varphi_i (y,s), L(y,s) K(x,t) (y,s) \right\rangle_W
\]

\[
= L(y,s) K(x,t) (y,s) \bigg|_{(y,s)=(x_i, t_i)}
\]

\[
= L(y,s) K(y,s) (x,t) \bigg|_{(y,s)=(x_i, t_i)}.
\]

Clearly \( \Psi_i(x,t) \in W(\Omega) \). For each fixed \( u(x,t) \in W(\Omega) \), if

\[
\langle u(x,t), \Psi_i(x,t) \rangle_W = 0, \quad i = 1, 2, \ldots
\]



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then

\[ \langle u(x, t), (L^* \varphi_i) (x, t) \rangle_W = (Lu(x, t), \varphi_i(x, t))_{\hat{W}} = (Lu)(x_i, t_i) = 0, \quad i = 1, 2, \ldots. \]

Note that \( \{(x_i, t_i)\}_{i=1}^{\infty} \) is dense in \( \Omega \), hence, \( (Lu)(x, t) = 0 \). It follows that \( u = 0 \) from the existence of \( L^{-1} \). So the proof is complete. \( \square \)

**Theorem 3.2.** If \( \{(x_i, t_i)\}_{i=1}^{\infty} \) is dense in \( \Omega \), then the solution of (3.1) is

\[ u(x, t) = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \beta_{ik} M(x_k, t_k, u(x_k, t_k)) \hat{\Psi}_i(x, t). \]  

(3.2)

**Proof.** Since \( \{\Psi_i(x, t)\}_{i=1}^{\infty} \) is complete system in \( W(\Omega) \), we have

\[
\begin{align*}
  u(x, t) &= \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \beta_{ik} \langle u(x, t), \Psi_k(x, t) \rangle_W \hat{\Psi}_i(x, t) \\
  &= \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \beta_{ik} \langle Lu(x, t), \varphi_k(x, t) \rangle_{\hat{W}} \hat{\Psi}_i(x, t) \\
  &= \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \beta_{ik} \langle Lu(x, t), L^* \varphi_k(x, t) \rangle_{\hat{W}} \hat{\Psi}_i(x, t) \\
  &= \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \beta_{ik} \langle Lu(x, t), \varphi_k(x, t) \rangle_{\hat{W}} \hat{\Psi}_i(x, t) \\
  &= \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \beta_{ik} \langle Lu(x, t), G(x_k, t_k) \rangle_{\hat{W}} \hat{\Psi}_i(x, t) \\
  &= \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \beta_{ik} \langle Lu(x, t), \varphi_k(x, t) \rangle_{\hat{W}} \hat{\Psi}_i(x, t) \\
  &= \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \beta_{ik} \langle Lu(x, t), \hat{\Psi}_i(x, t) \rangle_W \\
  &= \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \beta_{ik} M(x_k, t_k, u(x_k, t_k)) \hat{\Psi}_i(x, t).
\end{align*}
\]

Now the approximate solution \( u_n(x, t) \) can be obtained from the \( n \)-term intercept of the exact solution \( u(x, t) \) and

\[ u_n(x, t) = \sum_{i=1}^{n} \sum_{k=1}^{\infty} \beta_{ik} M(x_k, t_k, u(x_k, t_k)) \hat{\Psi}_i(x, t). \]
Obviously
\[ \|u_n(x, t) - u(x, t)\| \to 0, \quad (n \to \infty). \]

4. The method implementation

If we write
\[ A_i = \sum_{k=1}^{i} \beta_{ik} M(x_k, t_k, u(x_k, t_k)), \]
then (3.2) can be written as
\[ u(x, t) = \sum_{i=1}^{\infty} A_i \hat{\Psi}_i(x, t). \]

Now let \((x_1, t_1) = 0\); then from the initial and boundary conditions of (3.1), \(u(x_1, t_1)\) is known. We put \(u_0(x_1, t_1) = u(x_1, t_1)\) and define the \(n-\) term approximation to \(u(x, t)\) by
\[ u_n(x, t) = \sum_{i=1}^{n} B_i \hat{\Psi}_i(x, t). \tag{3.3} \]
where
\[ B_i = \sum_{k=1}^{i} \beta_{ik} M(x_k, t_k, u_{k-1}(x_k, t_k)). \tag{3.4} \]

In the sequel, we verify that the approximate solution \(u_n(x, t)\) converges to the exact solution, uniformly.

**Theorem 4.1.** Suppose that \(\|u_n\|\) is a bounded in (3.3) and (3.1) has a unique solution. If \(\{(x_i, t_i)\}_{i=1}^{\infty}\) is dense in \(W(\Omega)\), then the \(n-\) term approximate solution \(u_n(x, t)\) can be derived from the above method converges to the analytical solution \(u(x, t)\) of (3.1) and
\[ u(x, t) = \sum_{i=1}^{\infty} B_i \hat{\Psi}_i(x, t), \]
where \(B_i\) is given by (3.4).

**Proof:** First, we prove the convergence of \(u_n(x, t)\). From (3.3), we infer that
\[ u_{n+1}(x, t) = u_n(x, t) + B_{n+1} \hat{\Psi}_{n+1}(x, t), \]
The orthonormality of \(\{\hat{\Psi}_i\}_{i=1}^{\infty}\) yields that
\[ \|u_{n+1}\|^2 = \|u_n\|^2 + B_{n+1}^2 = \sum_{i=1}^{n+1} B_i^2. \tag{3.5} \]
In terms of (3.5), it holds that $\|u_{n+1}\| > \|u_n\|$. Due to the condition that $\|u_n\|$ is bounded, $\|u_n\|$ is convergent and there exists a constant $c$ such that

$$\sum_{i=1}^{\infty} B_i^2 = c.$$  

This implies that

$$\{B_i\}_{i=1}^{\infty} \in l^2.$$  

If $m > n$, then

$$\|u_m - u_n\|^2 = \|u_m - u_{m-1} + u_{m-1} - u_{m-2} + \ldots + u_{n+1} - u_n\|^2 \leq \|u_m - u_{m-1}\|^2 + \|u_{m-1} - u_{m-2}\|^2 + \ldots + \|u_{n+1} - u_n\|^2.$$  

On account of

$$\|u_m - u_{m-1}\|^2 = B_m^2,$$

consequently,

$$\|u_m - u_n\|^2 = \sum_{l=n+1}^{m} B_l^2 \to 0, \text{ as } n \to \infty.$$  

The completeness of $W(\Omega)$ shows that $u_n \to \hat{u}$ as $n \to \infty$. Now, let we prove that $\hat{u}$ is the solution of (3.1). Taking limits in (3.3) we get

$$\hat{u}(x, t) = \sum_{i=1}^{\infty} B_i \hat{\Psi}_i(x, t).$$  

Note that

$$(L\hat{u})(x, t) = \sum_{i=1}^{\infty} B_i L\hat{\Psi}_i(x, t),$$  

and

$$(L\hat{u})(x_l, t_l) = \sum_{i=1}^{\infty} B_i L\hat{\Psi}_i(x_l, t_l)$$

$$= \sum_{i=1}^{\infty} B_i \left \langle L\hat{\Psi}_i(x_l, t_l), \varphi_l(x, t) \right \rangle_{W}$$

$$= \sum_{i=1}^{\infty} B_i \left \langle \hat{\Psi}_i(x_l, t_l), L^*\varphi_l(x, t) \right \rangle_{W}$$

$$= \sum_{i=1}^{\infty} B_i \left \langle \hat{\Psi}_i(x_l, t_l), \hat{\Psi}_l(x_l, t_l) \right \rangle_{W}.$$  

Therefore

$$\sum_{i=1}^{i} \beta_{il} (L\hat{u})(x_l, t_l) = \sum_{i=1}^{\infty} B_i \left \langle \hat{\Psi}_i(x_l, t_l), \sum_{l=1}^{i} \beta_{il} \Psi_l(x, t) \right \rangle_{W}$$

$$= \sum_{i=1}^{\infty} B_i \left \langle \hat{\Psi}_i(x_l, t_l), \hat{\Psi}_l(x_l, t_l) \right \rangle_{W} = B_l.$$  

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In view of (3.4), we have
\[ L\hat{u}(x_l, t_l) = M(x_l, t_l, u(x_l, t_l)) \]
Since \( \{(x_i, t_i)\}_{i=1}^{\infty} \) is dense in \( \Omega \), for each \((y, s) \in \Omega\), there exists a subsequence \( \{(x_{n_j}, t_{n_j})\}_{j=1}^{\infty} \) such that
\[ (x_{n_j}, t_{n_j}) \to (y, s), \quad (j \to \infty). \]
We know that
\[ L\hat{u}(x_{n_j}, t_{n_j}) = M(x_{n_j}, t_{n_j}, u(x_{n_j}, t_{n_j})). \]
Let \( j \to \infty \); by the continuity of \( f \), we have
\[ (L\hat{u})(y, s) = M(y, s, u(y, s)). \]
which indicates that \( \hat{u}(x, t) \) satisfies (3.1).

**Remark 4.1.** In a same manner, it can be proved that
\[ \left\| \frac{\partial u_n(x, t)}{\partial x} - \frac{\partial u(x, t)}{\partial x} \right\| \to 0, \quad \text{as } n \to \infty, \]
where
\[ \frac{\partial u(x, t)}{\partial x} = \sum_{i=1}^{\infty} B_i \frac{\partial \hat{\Psi}_i(x, t)}{\partial x}, \]
and
\[ \frac{\partial u_n(x, t)}{\partial x} = \sum_{i=1}^{n} B_i \frac{\partial \hat{\Psi}_i(x, t)}{\partial x}, \]
B\(_i\) is given by (3.4).

5. Numerical Results

**Example 5.1.** In this example, we consider SG equation (1.1) without nonlinear term \( \sin(u) \) in the region \([0, 1] \times [0, T]\). The initial conditions are given by
\[ u(x, 0) = \sin(\pi x), \quad \frac{\partial u}{\partial t}(x, 0) = 0, \]
with the boundary conditions
\[ u(0, t) = u(1, t) = 0. \]
The exact solution is given in [36] as
\[ u(x, t) = \frac{1}{2}(\sin(\pi(x + t) + \sin \pi(x - t)). \]

After homogenizing the initial and boundary conditions we obtain (5.1) as
\[
\left\{ \begin{array}{l}
\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = -\pi^2 \sin(\pi x), \\
\quad u(x,0) = \frac{\partial u}{\partial t}(x,0) = u(0,t) = u(1,t) = 0.
\end{array} \right.
(5.1)
\]

**Example 5.2.** In this example, we take notice of SG equation (1.1) in the region \([0,1] \times [0,T]\). The initial conditions are given by
\[
u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 4 \sec h(x),
\]
The exact solution is
\[
u(x,t) = \arctan(\sec h(x)t).
\]
After homogenizing the initial conditions we obtain (5.2) as
\[
\left\{ \begin{array}{l}
\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = -\sin(u(x,t) + 4t \sec h(x)) - \frac{4t}{\cosh^2(x)} + \frac{8t \sinh^2(x)}{\cosh^2(x)}, \\
\quad u(x,0) = \frac{\partial u}{\partial t}(x,0) = u(0,t) = u(1,t) = 0.
\end{array} \right.
(5.2)
\]

| x  | t  | Exact Solution | Approximate Solution | Absolute Error | Relative Error | Time |
|----|----|----------------|----------------------|----------------|---------------|------|
| 0.1| 0.1| 0.2938926262  | 0.2938930965         | 4.703 \times 10^{-4} | 1.6002443 \times 10^{-6} | 3.860 |
| 0.2| 0.2| 0.4755282582  | 0.475531577          | 3.0995 \times 10^{-6} | 6.518014327 \times 10^{-6} | 3.016 |
| 0.3| 0.3| 0.4755282582  | 0.4755183355         | 9.9227 \times 10^{-6} | 2.086668842 \times 10^{-6} | 2.984 |
| 0.4| 0.4| 0.2938926261  | 0.2939007109         | 8.0848 \times 10^{-6} | 2.750936663 \times 10^{-6} | 3.000 |
| 0.5| 0.5| 0.0           | 0.0000282140         | 2.82140 \times 10^{-8} | \infty          | 3.094 |
| 0.6| 0.6| -0.2938926264 | -0.2939063137        | 1.36873 \times 10^{-8} | 4.657245120 \times 10^{-6} | 3.031 |
| 0.7| 0.7| -0.4755282583 | -0.4755305759        | 2.3176 \times 10^{-8} | 4.8737377 \times 10^{-6} | 3.031 |
| 0.8| 0.8| -0.4755282581 | -0.4755277748        | 4.833 \times 10^{-8}  | 1.016343386 \times 10^{-6} | 2.953 |
| 0.9| 0.9| -0.2938926260 | -0.2938966580        | 4.0320 \times 10^{-8} | 1.371929624 \times 10^{-6} | 3.204 |
| 1.0| 1.0| 0.0           | -3.690702068 \times 10^{-7} | 3.690702068 \times 10^{-7} | \infty          | 3.578 |

Table 1. Numerical solutions for Example 5.1.
Table 2. Numerical solutions for example 5.1 for $t = 1$.  

| $x$  | Exact Solution | Approximate Solution | Absolute Error | Relative Error | Time      |
|------|----------------|----------------------|----------------|---------------|-----------|
| 0.1  | 0.0991756307   | 0.09917562           | 1.307 $\times 10^{-8}$ | 1.31786035 $\times 10^{-7}$ | 1.688     |
| 0.2  | 0.1936096360   | 0.193606475          | 3.5885 $\times 10^{-8}$ | 1.853471797 $\times 10^{-9}$ | 1.719     |
| 0.3  | 0.397471925    | 0.2794797547         | 2.5622 $\times 10^{-8}$ | 9.167832814 $\times 10^{-9}$ | 0.593     |
| 0.4  | 0.593825410    | 0.354382187          | 3.540 $\times 10^{-7}$  | 9.98905422 $\times 10^{-9}$ | 0.608     |
| 0.5  | 0.788597173    | 0.417357842          | 1.8753 $\times 10^{-6}$ | 4.93246287 $\times 10^{-7}$ | 0.577     |
| 0.6  | 0.985390032    | 0.468538681          | 1.222 $\times 10^{-6}$  | 2.60852916 $\times 10^{-7}$ | 0.577     |
| 0.7  | 1.182709794    | 0.508730850          | 1.294 $\times 10^{-6}$  | 2.54358404 $\times 10^{-7}$ | 0.593     |
| 0.8  | 1.380654126    | 0.539066210          | 7.974 $\times 10^{-6}$  | 1.47922679 $\times 10^{-6}$ | 0.608     |
| 0.9  | 1.578645949    | 0.560770310          | 5.7151 $\times 10^{-6}$ | 1.019162061 $\times 10^{-6}$ | 0.546     |
| 1.0  | 1.776810972    | 0.575005129          | 1.0536 $\times 10^{-5}$ | 1.83232812 $\times 10^{-6}$ | 0.561     |

Table 3. Comparison Absolute Error and Relative Error for Example 5.1.  

| $x$  | $t$  | Exact Solution | Approximate Solution | Absolute Error | Relative Error | Time CPU(s) |
|------|------|----------------|----------------------|----------------|---------------|-------------|
| 0.1  | 0.1  | 0.00           | 0.00                 | 0.00           | 0.00          | 0.00        |
| 0.2  | 0.2  | 0.00           | 0.00                 | 0.00           | 0.00          | 0.00        |
| 0.3  | 0.3  | 0.00           | 0.00                 | 0.00           | 0.00          | 0.00        |
| 0.4  | 0.4  | 0.00           | 0.00                 | 0.00           | 0.00          | 0.00        |
| 0.5  | 0.5  | 0.00           | 0.00                 | 0.00           | 0.00          | 0.00        |
| 0.6  | 0.6  | 0.00           | 0.00                 | 0.00           | 0.00          | 0.00        |
| 0.7  | 0.7  | 0.00           | 0.00                 | 0.00           | 0.00          | 0.00        |
| 0.8  | 0.8  | 0.00           | 0.00                 | 0.00           | 0.00          | 0.00        |
| 0.9  | 0.9  | 0.00           | 0.00                 | 0.00           | 0.00          | 0.00        |
| 1.0  | 1.0  | 0.00           | 0.00                 | 0.00           | 0.00          | 0.00        |

Table 4. Numerical solutions for Example 5.2.  

| $x$  | Exact Solution | Approximate Solution | Absolute Error | Relative Error | Time CPU(s) |
|------|----------------|----------------------|----------------|---------------|-------------|
| −0.80| −0.536810972   | −0.536810972         | 4 $\times 10^{-9}$  | 1.55756584 $\times 10^{-9}$ | 0.733       |
| −0.40| −0.298584334   | −0.298584334         | 0.0             | 0.00          | 0.686       |
| 0.00 | 0.00           | 0.00                 | 0.0             | 0.00          | 0.702       |
| 0.40 | 0.298584334    | 0.298584334          | 0.0             | 0.00          | 0.686       |
| 0.80 | 0.536810972    | 0.536810972          | 4 $\times 10^{-9}$  | 1.55756584 $\times 10^{-9}$ | 0.733       |

Table 5. Numerical solutions for example 5.2 for $t = 1$.  

| $x$  | Exact Solution | Approximate Solution | Absolute Error | Relative Error | Time CPU(s) |
|------|----------------|----------------------|----------------|---------------|-------------|
| −0.80| −0.536810972   | −0.536810972         | 4 $\times 10^{-9}$  | 1.55756584 $\times 10^{-9}$ | 0.733       |
| −0.40| −0.298584334   | −0.298584334         | 0.0             | 0.00          | 0.686       |
| 0.00 | 0.00           | 0.00                 | 0.0             | 0.00          | 0.702       |
| 0.40 | 0.298584334    | 0.298584334          | 0.0             | 0.00          | 0.686       |
| 0.80 | 0.536810972    | 0.536810972          | 4 $\times 10^{-9}$  | 1.55756584 $\times 10^{-9}$ | 0.733       |

Table 6. Comparison Absolute error and Relative error for example 5.2.  

6. Conclusion
In this study, linear and nonlinear SG Equations were solved by RKHSM. We described the method and used it in some test examples in order to show its applicability and validity in comparison with exact and other numerical solutions. The obtained results show that this approach can solve the problem effectively and need few computations. The results are satisfactory. The results that we obtained were compared with the results that were obtained by [37]. Numerical experiments on test examples show that our proposed schemes are of high accuracy, and support the theoretical results. According to these results, it is possible to apply RKHSM to linear and nonlinear differential equations with initial and boundary conditions. It has been shown that the obtained results are uniform convergent and the operator that was used is a bounded linear operator. Thus the studies that were compared with this method and this study show that RKHSM can be apply to high dimensional partial differential equations, integral equations and fractional differential equations without any transformation or discretization and good results can be obtained.

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