PROPAGATORS OF GENERALIZED SCHRÖDINGER EQUATIONS RELATED BY HIGHER-ORDER SUPERSYMMETRY

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Abstract

We construct explicit integral relations between propagators of generalized Schrödinger equations that are linked by higher-order supersymmetry. Our results complement and extend the findings obtained in [9] for the conventional Schrödinger equation.

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1 Introduction

This note is directly motivated by the recent work [9], where surprisingly simple expressions for propagators of supersymmetry-related Schrödinger equations were constructed. In general, the quantum-mechanical supersymmetry (SUSY) formalism connects two Schrödinger equations for (usually) different potentials, such that solutions of one equation are mapped onto solutions of the second equation [3]. The mapping used for relating the solutions to each other is called SUSY- or Darboux transformation. While the Darboux transformation has been around for more than a hundred years [4], its application to quantum-mechanical problems within the SUSY context is much younger, starting with its relation to the Infeld-Hull factorization [6]. Ever since then, the quantum-mechanical SUSY formalism has been extensively used to study solvability and spectral properties of Schrödinger equations, for an overview the reader may consult [3] [7] [15]. Besides the fact that Schrödinger equations related by the SUSY formalism allow for mapping their respective solutions onto each other, such equations are connected in more ways. In particular, their propagators and the traces of their Green’s functions are linked by relatively simple formulas, as was shown in [9] and [10] [13], respectively. It turns out that the link between the Green’s functions even persists under linear generalizations of the Schrödinger equation [12], such as the effective mass case or minimal coupling to a magnetic field. Motivated by this result, in the present note we study the question whether the propagator
relation found in [9] can be extended to generalized Schrödinger equations. While for first-order SUSY transformations a positive answer has been given [11], here we will study higher-order transformations. Section 2 gives a short review on the necessary preliminaries, while in section 3 we construct our propagator relations for generalized Schrödinger equations.

2 Preliminaries

In the following we briefly summarize basic facts about generalized Schrödinger equations, the SUSY formalism, propagators and Green’s functions.

The generalized Schrödinger equation. We consider the following generalized Sturm-Liouville problem on the real interval \((a, b)\), equipped with Dirichlet boundary conditions:

\[
  f(x) \psi''(x) + f'(x) \psi'(x) + \left[ E h(x) - V(x) \right] \psi(x) = 0, \quad x \in (a, b) \tag{1}
\]

\[
  \psi(a) = \psi(b) = 0. \tag{2}
\]

Here \(f, h, V\) are smooth, real functions, with \(f, h\) positive and bounded in \(a\) and \(b\). The constant \(E\) will be referred to as energy, and in solutions of (1), (2) that belong to the discrete spectrum, \(E\) stands for the spectral value. Any solution \(\psi\) of (1), (2) belonging to a value \(E\) from the discrete spectrum, is located in the weighted Hilbert space \(L^2_h(a, b)\) with weight function \(h\) [5]. The lowest value of the discrete spectrum will be called the ground state and denoted by \(E_0\). The interval \((a, b)\) can be unbounded, that is, \(a\) or \(b\) can represent minus infinity or infinity, respectively (however, if \(a\) and/or \(b\) are finite, then we require \(f, h, V\) to be continuous there). We see that the problem (1), (2) can be singular, which means that its spectrum can admit a continuous part. Equation (1) will be referred to as generalized Schrödinger equation, since its special cases are frequently encountered in Quantum Mechanics, such as the Schrödinger equation for effective mass or with a linearly energy-dependent potential. In the quantum-mechanical context, \(E\) denotes the energy associated with a solution \(\psi\), and \(V\) stands for the potential.

Generalized SUSY formalism. We will now summarize basic facts from [14], for details see the latter reference. The boundary-value problem (1), (2) can be linked to another problem of the same kind by means of the SUSY transformation method. Consider

\[
  f(x) \Psi''(x) + f'(x) \Psi'(x) + \left[ E h(x) - U(x) \right] \Psi(x) = 0, \quad x \in (a, b) \tag{3}
\]

\[
  \Psi(a) = \Psi(b) = 0, \tag{4}
\]

where the same settings imposed for (1), (2) apply. Clearly, a solution \(\Psi = \Psi(x)\) and the potential \(U = U(x)\) are in general different from their respective counterparts \(\psi\) and \(V\). Now, suppose that \(\psi\) and \(u_0, ..., u_{n-1}\) are solutions of the boundary-value problem (1), (2) and of equation (1) at real energies \(E\) and \(\lambda_j \leq E, j = 0, ..., n - 1\), respectively. Define the \(n\)-th order SUSY transformation of \(\psi\) as

\[
  D_{u_0, ..., u_{n-1}}^\psi x \psi(x) = \left[ f(x) \right]^{\frac{1}{2}} \frac{W(u_0, ..., u_{n-1}, \psi(x))}{W(u_0, ..., u_{n-1})} h(x), \tag{5}
\]

where \(W\) denotes the Wronskian of the functions in its argument and the upper index \(x\) of \(D\) denotes the variable which the derivatives in the Wronskians are applied to. The function
Suppose problem (1), (2) admits a complete set of eigenfunctions ($\psi$, as defined in (5)) solves the boundary-value problem (3), (4), if the potential $U$ is given in terms of its counterpart $V$ as follows:

\[ U = V - 2 \frac{d}{dx} \left( \log [W(u_0, ..., u_{n-1})] \right) + 2 \frac{d}{dx} \left( \frac{f}{h} \right) \frac{d}{dx} \left( \log [W(u_0, ..., u_{n-1})] \right) + \]

\[ + n h \left[ \frac{f f'}{2 f h} \frac{(f')^2}{2} - \frac{f' h'}{2 h^2} + \frac{3 f (h')^2}{2 h^2} - \frac{f h''}{h^2} \right] + \]

\[ + \frac{n^2}{2} \left[ \frac{(f')^2}{2 f} + \frac{f' h'}{h} - \frac{3 f (h')^2}{2 h^2} - f'' + \frac{f h''}{h} \right], \quad (6) \]

note that for the sake of brevity the arguments were left out. In the special case $n = 1$, the transformation (5) simplifies to

\[ D_{u_0}^x \psi(x) = \sqrt{\frac{f(x)}{h(x)}} \frac{W(u_0(x), \psi(x))}{u_0(x)} = \sqrt{\frac{f(x)}{h(x)}} \left[ -\frac{u_0'(x)}{u_0(x)} \psi(x) + \psi'(x) \right]. \quad (7) \]

It is well-known [1] [14] that the $n$-th order transformation (5) can always be written as an iteration (or chain) of $n$ first-order transformations (7), that is,

\[ D_{u_0, ..., u_{n-1}}^x = \prod_{j=0}^{n-1} D_{v_j}^x, \quad (8) \]

where each function $v_j$ solves the equation (11) that is obtained after the $j$-th first-order SUSY transformation. Note that (5) and (7) remain valid when multiplied by a constant, which can be used for normalization. Now, depending on the choice of the auxiliary solution $u$ in (7), the discrete spectrum of problem (3), (4) can be affected in three possible ways: if $\lambda = E_0$ and $u = \psi_0$, then $E_0$ is removed from the spectrum of (3), (4). The opposite case, creation of a new spectral value $\lambda < E_0$, happens if the auxiliary solution $u$ does not fulfill the boundary conditions (4). Finally, the spectra of both problems (1), (2) and (3), (4) are the same, if we pick $\lambda < E_0$ and an $u$ that fulfills only one of the boundary conditions (2).

**Propagator and Green’s function.** The propagator governs a quantum system’s time evolution. For a stationary Schrödinger equation, the propagator $K$ has the defining property

\[ \exp(-i \ E \ t) \ \psi(x) = \int_{(a,b)} K(x, y, t) \ \psi(y) \ dy. \quad (9) \]

Suppose problem (1), (2) admits a complete set of eigenfunctions ($\psi_n$, $n = 0, 1, 2, ..., M \in \mathbb{N}_0$, where $M$ can stand for infinity, and ($\phi_k$, $k \in \mathbb{R}$, belonging to the discrete and the continuous part of the spectrum, respectively. Then the propagator $K$ has the representation

\[ K(x, y, t) = h(y) \left[ \sum_{n=0}^{M} \psi_n(x) \ \exp(-i \ E_n \ t) \ \psi_n(y) + \int_{\mathbb{R}} \phi_k(x) \ \exp(-i \ k^2 \ t) \ \phi_k(y) \ dk \right], \quad (10) \]

where $E_n$ and $k^2$ stand for the spectral values belonging to the discrete and continuous spectrum, respectively. The Green’s function $G$ of the problem (1), (2) has two equivalent representations
, both of which we will use here. In order to state the first representation, let \( \psi_{0,l} \) and \( \psi_{0,r} \) be solutions of equation \((\text{I})\) that fulfill the following unilateral boundary conditions:

\[
\psi_{0,l}(a) = 0 \quad \psi_{0,r}(b) = 0.
\] (11)

The Wronskian \( W_{\psi_{0,l},\psi_{0,r}} \) of these functions is given by

\[
W_{\psi_{0,l},\psi_{0,r}}(x) = \frac{c_0}{f(x)},
\] (12)

where \( c_0 \) is a constant that depends on the explicit form of \( \psi_{0,l} \) and \( \psi_{0,r} \). Now we can give the first representation of the Green’s function \( G_0 \) for our boundary value problem \((\text{I}), (\text{II})\):

\[
G(x, y) = -\frac{1}{c_0} \left[ \psi_{0,l}(y) \psi_{0,r}(x) \theta(x - y) + \psi_{0,l}(x) \psi_{0,r}(y) \theta(y - x) \right],
\] (13)

where \( c_0 \) is the constant from \((\text{II})\) and \( \theta \) stands for the Heaviside distribution. The second representation of the Green’s function \( G \) can be obtained as follows, provided problem \((\text{I}), (\text{II})\) admits a complete set of solutions:

\[
G(x, y) = \sum_{n=0}^{M} \frac{\psi_n(x) \psi_n(y)}{E_n - E} + \int_{\mathbb{R}} \frac{\phi_k(x) \phi_k(y)}{k^2 - E} \, dk,
\] (14)

where the notation is the same as in \((\text{II})\). Note that the Green’s function is taken at energy \( E \).

**Propagators related by first-order SUSY.** In order to obtain a relation between the propagators of the two boundary-value problems \((\text{I}), (\text{II}) \) and \((\text{III}), (\text{IV})\), we take the propagator \( K_1 \) of the second problem and express it through quantities related to the first problem. For the sake of simplicity we assume for now that the two boundary-value problems have the same discrete spectrum and that both of them admit a complete set of solutions belonging to a discrete and a continuous part of the spectrum. Furthermore, we assume that the solutions of problem \((\text{I}), (\text{II})\) are real-valued functions. This is no restriction, as equation \((\text{I})\) involves only real functions. We then find the following relations \((\text{III})\) between the propagators \( K_1 \) and \( K_0 \) of our boundary-value problems \((\text{I}), (\text{II}), (\text{III}), (\text{IV})\), respectively:

\[
K_1(x, y, t) = h(y) \int_{(a,b)} K_0(x, z, t) G_0(z, y) \, dz
\]

\[
K_1(x, y, t) = h(y) \left\{ \int_{(a,b)} K_0(x, z, t) G_0(z, y) \, dz + \phi_{-1}(x) \exp(-i \lambda t) \phi_{-1}(y) \right\}
\]

\[
K_1(x, y, t) = h(y) \int_{E_0}^{E_0} \lim_{E \to E_0} \left[ G_0(z, y) - \frac{\psi_0(z) \psi_0(y)}{E_0 - E} \right] \, dz.
\] (15)

The first of these relations is valid, if both boundary-value problems admit the same discrete spectrum. The second relation applies, if \((\text{III}), (\text{IV})\) admits an additional discrete spectral value \( \lambda \)
with corresponding solution $\phi_{-1}$. Finally, the third relation holds, if the initial problem \(1\), \(2\) has one discrete spectral value more than its transformed counterpart. In this last case, if we choose the auxiliary function to be the ground state $\psi_0$ of our initial problem, we can simplify (15) as follows:

$$K_1(x, y, t) = \sqrt{\frac{h(y)}{f(y)}} \frac{1}{\psi_0(y)} D^x_{\psi_0} \int_{y,b} K_0(x, z, t) \psi_0(z) \, dz. \quad (16)$$

It is immediate to see that the above propagator relations simplify to their well-known conventional forms [9], if we set $f = h = 1$.

3 Propagators related by higher-order SUSY

From now on we will assume that a higher-order SUSY transformation \(5\) was applied to the initial boundary-value problem \(1\), \(2\), giving the associated problem \(3\), \(4\). We are looking for an explicit relation between the propagators of these two problems. To this end, we distinguish whether the SUSY transformation adds new values to the discrete spectrum, removes some from it or whether the spectra of both boundary-value problems stay the same.

3.1 Creation of spectral values

Let us first assume that our $N$-th order SUSY transformation creates $N$ new discrete spectral values in the corresponding transformed boundary-value problem. We denote these values and their corresponding solutions by $E_{-n}$ and $\Psi_{-n}$, $n = 1, \ldots, N$, respectively. According to (15), the propagator of problem \(3\), \(4\) is then given by

$$K_N(x, y, t) = h(y) \left[ \sum_{n=0}^{M} \Psi_n(x) \exp(-i E_n t) \Psi_n(y) + \int_{\mathbb{R}} \Phi_k(x) \exp(-i k^2 t) \Phi_k(y) \, dk + \sum_{n=1}^{N} \Psi_{-n}(x) \exp(-i E_{-n} t) \Psi_{-n}(y) \right]. \quad (17)$$

Next, we take into account that all functions $\Psi_n$, $n = 0, \ldots, M$, and $\Phi_k$, $k \in \mathbb{R}$, have been obtained from solutions $\psi_j$ by means of an $N$-th order SUSY transformation \(5\), using the auxiliary solutions $u_j$, $j = 0, \ldots, N - 1$:

$$K_N(x, y, t) = h(y) \left[ D^x_{u_0,\ldots,u_{N-1}} D^y_{u_0,\ldots,u_{N-1}} \left[ \sum_{n=0}^{M} L^2_{\phi} \psi_n(x) \exp(-i E_n t) \psi_n(y) + \int_{\mathbb{R}} L^2_{\phi} \exp(-i k^2 t) \phi_k(y) \, dk \right] + h(y) \sum_{n=1}^{N} \Psi_{-n}(x) \exp(-i E_{-n} t) \Psi_{-n}(y) \right], \quad (18)$$
where normalization constants $L_\psi$ and $L_\phi$ were introduced. Before we determine these constants, we make use of the defining property \((9)\), transforming \((18)\) into

\[
K_N(x, y, t) = h(y) \left[ \sum_{p=0}^{N-1} \prod_{q=0}^{p} \frac{1}{\lambda_q - \lambda_p} \int_{(a,b)} K_0(x, z, t) \psi_n(z) \, dz \right] + h(y) \sum_{n=1}^{N} \Psi_{-n}(x) \exp(-i E_{-n} t) \Psi_{-n}(y),
\]

(19)

Our normalization constants $L_\psi$ and $L_\phi$ must be chosen as follows \([2]\)

\[
L_\psi = \prod_{p=0}^{N-1} \sqrt{\frac{1}{E - \lambda_p}} \quad L_\phi = \prod_{p=0}^{N-1} \sqrt{\frac{1}{k^2 - \lambda_p}},
\]

substitution of which renders our propagator relation \((19)\) in the following form:

\[
K_N(x, y, t) = h(y) \left[ \sum_{p=0}^{N-1} \prod_{q=0}^{p} \frac{1}{\lambda_q - \lambda_p} \int_{(a,b)} K_0(x, z, t) \psi_n(z) \, dz \right] + h(y) \sum_{n=1}^{N} \Psi_{-n}(x) \exp(-i E_{-n} t) \Psi_{-n}(y),
\]

(20)

Next, we rewrite our normalization constants according to the decomposition

\[
\prod_{p=0}^{N-1} \frac{1}{E - \lambda_p} = \sum_{p=0}^{N-1} \prod_{q=0}^{p} \frac{1}{\lambda_q - \lambda_p} \frac{1}{E - \lambda_p}
\]

\[
\prod_{p=0}^{N-1} \frac{1}{k^2 - \lambda_p} = \sum_{p=0}^{N-1} \prod_{q=0}^{p} \frac{1}{\lambda_q - \lambda_p} \frac{1}{k^2 - \lambda_p}.
\]

We plug this into our propagator relation \((20)\) and obtain after regrouping terms

\[
K_N(x, y, t) = \sum_{p=0}^{N-1} \prod_{q=0}^{p} \frac{1}{\lambda_q - \lambda_p} \int_{(a,b)} K_0(x, z, t) \sum_{n=0}^{M} \frac{\psi_n(z) \psi_n(y)}{E - \lambda_p} \, dz + h(y) \sum_{n=1}^{N} \Psi_{-n}(x) \exp(-i E_{-n} t) \Psi_{-n}(y),
\]

(21)
In the final step we make use of the representation (14) of our Green’s function, which converts expression (21) to the form

\[ K_N(x, y, t) = h(y) D_x u_j, u_{j+1}, ..., u_{n-1} \]

\[ + h(y) \sum_{n=1}^{N} \phi_{-n}(x) \exp(-i E_n t) \phi_{-n}(y). \]

Expression (22) reduces correctly to the conventional result [9], if \( h = 1 \) is substituted.

### 3.2 Annihilation of spectral values

We will now assume that our SUSY transformation removes \( n \) values from the discrete spectrum. Speaking in terms of the factored transformation, each step will remove the corresponding ground state from the current system. In other words, after a SUSY transformation of order \( n \), the first \( n \) discrete spectral values of the initial boundary-value problem will have been removed from the discrete spectrum. This particular ordering of the spectral values and solutions to be deleted does not constitute a restriction [9], but facilitates notation and calculation. Before we turn to the propagator relations, it is necessary to set up some notation in order to obtain representations for the auxiliary solutions. In order to do so, we will consider our higher-order SUSY transformation (5) in its factored form (8), that is, an \( n \)-th order transformation is seen as the \( n \)-fold application of a first-order transformation. Each first-order transformation yields a new boundary-value problem of the type (3), (4), such that in total a sequence of \( n \) boundary-value problems is generated. We will assume that in each transformation step the lowest value is deleted from the discrete spectrum, such that after \( n \) iterations of our first-order transformation the lowest \( n \) spectral values are removed. The auxiliary solutions used in each iteration of our SUSY transformation will be named \( u_{j,k} \), where the first index denotes the number of the boundary-value problem associated with \( u_{j,k} \), starting from \( j = 0 \). The second index in \( u_{j,k} \) stands for the solution number associated with the \( k \)-th discrete spectral value. In particular, \( u_{n,n} \) is the ground state of the \( n \)-th boundary-value problem and the functions \( u_{n,j} \) for \( j = n + 1, ..., M \) represent solutions of the \( n \)-th boundary-value problem. Finally, the \( u_{n,j} \) for \( j = 0, ..., n - 1 \) stand for solutions of the equation associated with our boundary-value problem at energies that do not belong to the discrete spectrum, which implies that they do not fulfill the boundary conditions (1). For the sake of convenience, let us now introduce abbreviations for the SUSY transformation operators (5), using the notation for our auxiliary functions. For natural numbers \( n \) and \( j \leq n - 1 \) we define

\[ L_{n,j}^x = D_x^z u_{j,j+1}, ..., u_{j,n-1}. \]

This operator maps solutions of the \( j \)-th boundary-value problem onto solutions of the \( n \)-th boundary-value problem. Therefore, the operator (23) admits the contraction property

\[ L_{n,j}^x L_{j,k}^x = L_{n,k}^x. \]
In terms of these operators, our auxiliary solutions \( u_{n,j} \) for \( j = n, ..., M \) are built by the following rule:

\[
    u_{n,j}(x) = L_{n+1,j}^x u_{n-1,j}(x).
\]  

(25)

The remaining auxiliary solutions \( u_{n,j} \) for \( j = 0, ..., n - 1 \), which do not fulfill the boundary conditions, are constructed as follows:

\[
    u_{n,j}(x) = L_{n,0}^x \hat{u}_{0,j}(x)
\]

\[
    \hat{u}_{0,j}(x) = u_{0,j}(x) \int \frac{1}{f(x) u_{0,j}^2(x)} \, dx.
\]

Note that the functions \( u_{0,j} \) and \( \hat{u}_{0,j} \) solve the same equation and are linearly independent [8].

Now we are in position to construct closed-form expressions for our auxiliary solutions. To this end, we take the well-known expressions from the conventional case \( f = h = 1 \), which were obtained in [9], and map them to the present, generalized context. Let \( v_{n,j} \) stand for the conventional auxiliary functions, where the indices have the same meaning and vary exactly as in our generalized case described above, then we have from [9]

\[
    v_{n,j}(x) = (E_{n-1} - E_j) (E_{n-2} - E_j) ... (E_{j+1} - E_j) \frac{W_n(v_{0,0}, ..., v_{0,n-1})(x)}{W(v_{0,0}, ..., v_{0,n-1})(x)},
\]  

(26)

where the modified Wronskian \( W_n \) is obtained from \( W \) by removing the \((n+1)\)-th row and column from the underlying matrix. We will now rewrite the auxiliary solutions \( v_{n,j} \) and their Wronskians in (26) in terms of the present auxiliary functions \( u_{n,j} \). In order to do so, we make use of the following results taken from [14]:

\[
    v_{n,j}(x) = [f(x) h(x)]^{\frac{1}{2}} u_{n,j}(x)
\]

\[
    W_n(v_{0,0}, ..., v_{0,n-1})(x) = [f(x)]^{\frac{(n-1)^2}{4}} [h(x)]^{\frac{(n-1)(n-3)}{4}} W_n(u_{0,0}, ..., u_{0,n-1})(x)
\]

\[
    W(v_{0,0}, ..., v_{0,n-1})(x) = [f(x)]^{\frac{n^2}{4}} [h(x)]^{-\frac{n(n-2)}{4}} W(u_{0,0}, ..., u_{0,n-1})(x).
\]

We plug these equalities into (26), which then becomes

\[
    [f(x) h(x)]^{\frac{1}{2}} u_{n,j}(x) =
\]

\[
    = (E_{n-1} - E_j) (E_{n-2} - E_j) ... (E_{j+1} - E_j) \frac{1}{h(x)} \left[ \frac{h(x)}{f(x)} \right]^{\frac{n}{2} - \frac{1}{2}} \frac{W_n(u_{0,0}, ..., u_{0,n-1})(x)}{W(u_{0,0}, ..., u_{0,n-1})(x)}.
\]

Using the abbreviation

\[
    C_{n,j} = (E_{n-1} - E_j) (E_{n-2} - E_j) ... (E_{j+1} - E_j),
\]

renders our auxiliary solution \( u_{n,j} \) in the following form:

\[
    u_{n,j}(x) = C_{n,j} \frac{1}{h(x)} \left[ \frac{h(x)}{f(x)} \right]^{\frac{n}{2}} \frac{W_n(u_{0,0}, ..., u_{0,n-1})(x)}{W(u_{0,0}, ..., u_{0,n-1})(x)}.
\]  

(27)
We will now show that the propagator $K_n$ of the boundary-value [3], [4], which is obtained after an $n$-chain of SUSY transformations, can be given in the following form:

$$K_n(x, y, t) = \left[ \frac{h(y)}{f(y)} \right]^\frac{n}{2} (-1)^{n-1} L^x_{n,0} \sum_{j=0}^{n-1} \frac{W_j(y)}{W(y)} \int_{(y,b)} K_0(x, q, t) u_{0,j}(q) \, dq,$$  \hspace{1cm} (28)

where $K_0$ is the propagator of the initial boundary-value problem [11], [12]. In order to establish relation (28), we will use induction, proceeding similarly to how it was done in the conventional case [9]. The first-order case $K_1$ has already been established [11] and is given in [16]. Now assume this relation to hold for $K_n$, then our induction step starts at

$$K_{n+1}(x, y, t) = \sqrt{\frac{h(y)}{f(y)}} \frac{1}{u_{n,n}(y)} L^x_{n+1,n} \int_{(y,b)} K_n(x, z, t) u_{n,n}(z) \, dz. \hspace{1cm} (29)$$

Since we assume that (28) is true for $n$, we can substitute it into (29). After ordering terms, we get

$$K_{n+1}(x, y, t) =$$

$$= \sqrt{\frac{h(y)}{f(y)}} \frac{(-1)^{n-1}}{u_{n,n}(y)} L^x_{n+1,n} L^x_{n,0} \sum_{j=0}^{n-1} \int_{(y,b)} \left( \frac{h(z)}{f(z)} \right)^\frac{n}{2} \frac{W_j(z)}{W(z)} u_{n,n}(z) \int_{(z,b)} K_0(x, q, t) u_{0,j}(q) \, dq \, dz.$$

Next, we substitute the ratio of Wronskians $W_j/W$ by means of our representation (26), make use of the contraction (24) and arrive after some simplification at

$$K_{n+1}(x, y, t) =$$

$$= \sqrt{\frac{h(y)}{f(y)}} \frac{(-1)^{n-1}}{u_{n,n}(y)} L^x_{n+1,0} \sum_{j=0}^{n-1} (-1)^j C_{n,j} \int_{(y,b)} h(z) u_{n,n}(z) u_{n,j}(z) \int_{(z,b)} K_0(x, q, t) u_{0,j}(q) \, dq \, dz.$$

The area of integration forms a triangle $T$ in $z$-$q$-space, which can be seen as the upper half of the rectangle $[y, b] \times [y, b]$. Therefore, integration over $T$ can be replaced by integration over the rectangle minus integration over its lower triangle:

$$K_{n+1}(x, y, t) = \sqrt{\frac{h(y)}{f(y)}} \frac{(-1)^{n-1}}{u_{n,n}(y)} L^x_{n+1,0} \sum_{j=0}^{n-1} (-1)^j C_{n,j} \times$$

$$\times \left[ \int_{(y,b)} h(z) u_{n,n}(z) u_{n,j}(z) \, dz \int_{(y,b)} K_0(x, q, t) u_{0,j}(q) \, dq - \int_{(y,b)} K_0(x, q, t) u_{0,j}(q) \int_{(q,b)} h(z) u_{n,n}(z) u_{n,j}(z) \, dz \, dq \right]. \hspace{1cm} (30)$$
Before we process this expression further, let us evaluate the integral with respect to $z$:

$$\int_{(\xi,b)} h(z) \ u_{n,n}(z) \ u_{n,j}(z) \ dz = \frac{f(\xi) \ W(u_{n,n}, u_{n,j})(\xi)}{E_j - E_n} - \frac{f(b) \ W(u_{n,n}, u_{n,j})(b)}{E_j - E_n}. \quad (31)$$

This can be verified in a straightforward way by differentiating the right hand side and replacing the second derivatives by means of our equation (1). The second term on the right-hand side of (31) is a constant and will cancel out in subsequent calculations, whereas the first term will now be modified further. First we make use of the relation

$$L_{n+1,n}^\xi u_{n,j}(\xi) = \sqrt{\frac{f(\xi)}{h(\xi)}} \frac{W(u_{n,n}, u_{n,j})(\xi)}{u_{n,n}(\xi)},$$

replacing the Wronskian in (31):

$$\int_{(\xi,b)} h(z) \ u_{n,n}(z) \ u_{n,j}(z) \ dz = \frac{\sqrt{f(\xi) h(\xi)}}{E_j - E_n} \ u_{n,n}(\xi) \ L_{n+1,n}^\xi u_{n,j}(\xi) - \frac{f(b) \ W(u_{n,n}, u_{n,j})(b)}{E_j - E_n} \frac{\sqrt{f(\xi) h(\xi)}}{E_j - E_n} \ u_{n,n}(\xi) u_{n+1,j}(\xi) = \frac{\sqrt{f(\xi) h(\xi)}}{E_j - E_n} \ u_{n,n}(\xi) u_{n+1,j}(\xi) - \frac{f(b) \ W(u_{n,n}, u_{n,j})(b)}{E_j - E_n}. \quad (32)$$

We now replace the function $u_{n+1,j}$ by its representation (27) for $n + 1$ and simplify the result:

$$\int_{(\xi,b)} h(z) \ u_{n,n}(z) \ u_{n,j}(z) \ dz = \sqrt{\frac{h(\xi)}{f(\xi)}} \ W_j(u_0,0, ..., u_0,n)(\xi) - \frac{f(b) \ W(u_{n,n}, u_{n,j})(b)}{E_j - E_n}. \quad (33)$$

We can replace the integral (33) in our propagator relation (30), note that the integral appears twice there. After some simplification and regrouping terms, we arrive at the following expression:

$$K_{n+1}(x,y,t) = \left[ \frac{h(y)}{f(y)} \right]^{n+1} \left( -1 \right)^n L_{n+1,0}^x \sum_{j=0}^{n-1} (-1)^j \frac{W_j(y)}{W(y)} \int_{(y,b)} K_0(x,q,t) \ u_{0,j}(q) \ dq +$$

$$+ \sqrt{\frac{h(y)}{f(y)}} \left( -1 \right)^{n-1} L_{n+1,0}^x \sum_{j=0}^{n-1} (-1)^j \frac{u_{n,n}(q)}{u_{n,n}(y)} \ u_{n,j}(q) \ W_j(q) \ dq. \quad (34)$$

We will now simplify the second line of this expression for $K_{n+1}$. The last sum can be seen as an application of Laplace’s theorem:

$$\sum_{j=0}^{n-1} (-1)^j \ u_{0,j}(q) \ W_j(q) = (-1)^{n+1} u_{0,n}(q) \ W(u_0,0, ..., u_0,n-1)(q). \quad (35)$$
that is, the determinant $W(u_0, \ldots, u_{0,n-1})$ is expanded with respect to its first row. Next, we combine (35) with its two factors in front, expressing $u_{n,n}$ by means of (25) and (5):

$$u_{n,n}(q) = \left[ \frac{f(q)}{h(q)} \right]^{\frac{n}{2}} \frac{W(u_0, \ldots, u_{0,n})(q)}{W(u_0, \ldots, u_{0,n-1})(q)}$$  (36)

which leads to the following result:

$$\left[ \frac{h(y)}{f(y)} \right]^{\frac{n}{2}} \frac{u_{n,n}(q)}{W(q)} \sum_{j=0}^{n-1} (-1)^j W_0(q) W_j(q) = (-1)^{n+1} u_{0,n}(q).$$

After substitution of this expression into (34), we get the following form of our propagator relation:

$$K_{n+1}(x, y, t) = \left[ \frac{h(y)}{f(y)} \right]^{\frac{n+1}{2}} (-1)^n \frac{W_n(x, y, t)}{W_n(y, b)} \sum_{j=0}^{n-1} (-1)^j \frac{W_j(y)}{W(y)} \int_{(y,b)} K_0(x, q, t) u_{0,j}(q) dq +$$

$$+ \frac{1}{2} \left[ \frac{h(y)}{f(y)} \right] \frac{1}{W_n(y, b)} \int_{(y,b)} K_0(x, q, t) u_{0,n}(q) dq.$$

Using once more our representation (36) for $u_{n,n}$, we obtain

$$K_{n+1}(x, y, t) = \left[ \frac{h(y)}{f(y)} \right]^{\frac{n+1}{2}} (-1)^n \frac{W_n(x, y, t)}{W_n(y, b)} \sum_{j=0}^{n-1} (-1)^j \frac{W_j(y)}{W(y)} \int_{(y,b)} K_0(x, q, t) u_{0,j}(q) dq +$$

$$+ \left[ \frac{h(y)}{f(y)} \right]^{\frac{n+1}{2}} \frac{W(u_0, \ldots, u_{0,n-1})(y)}{W(u_0, \ldots, u_{0,n})(y)} \frac{W_n(x, y, t)}{W_n(y, b)} \int_{(y,b)} K_0(x, q, t) u_{0,n}(q) dq.$$

The second line of the right-hand side turns out to be the $n$-th term of the sum in the first line, such that we get the following final form of our propagator relation:

$$K_{n+1}(x, y, t) = \left[ \frac{h(y)}{f(y)} \right]^{\frac{n+1}{2}} (-1)^n \frac{W_n(x, y, t)}{W_n(y, b)} \sum_{j=0}^{n} (-1)^j \frac{W_j(y)}{W(y)} \int_{(y,b)} K_0(x, q, t) u_{0,j}(q) dq.$$

This expression coincides with (29) if $n$ is set to $n+1$, and the induction is complete.

### 3.3 Isospectrality

We will now consider the remaining case of a SUSY transformation that renders the discrete spectrum of the initial boundary-value problem (11), (2) and its final counterpart (3), (4) the same. In particular, we assume that in each transformation step of the chain the discrete spectrum is preserved. Consequently, the auxiliary functions $u_0, \ldots, u_{n-1}$ used in the SUSY transformation (5) fulfill only one of the boundary conditions (2). Adopting the settings from (9), from now on we will require our boundary-value problem to be defined on the whole real line, that is, the quantities $a, b$ in (2) stand for negative infinity and infinity, respectively. For
the sake of simplicity let us first assume that our auxiliary solutions \( u_0, \ldots, u_{n-1} \) only fulfill the first boundary condition, that is,

\[
\lim_{x \to -\infty} u_j(x) = 0. \tag{37}
\]

The general case of some \( u_j \) fulfilling the first boundary condition, and some fulfilling the second boundary condition will arise easily once we have established our findings for the setting \( \text{(37)} \). More precisely, we will now prove that the following relation between the propagators \( K_0 \) and \( K_n \) of our boundary-value problems \( (1), (2) \) and \( (3), (4) \) holds:

\[
K_n(x, y, t) = (-1)^n \left[ \frac{h(y)}{f(y)} \right]^x L^n_{0,0} \sum_{j=0}^{n-1} (-1)^j \frac{W_n(y)}{W(y)} \int_{-\infty}^{y} K_0(x, z, t) \, u_j(z) \, dz. \tag{38}
\]

The proof of this propagator relation will follow the same steps that were taken in \( [9] \). We must show that \( K_n \) satisfies the generalized time-dependent Schrödinger equation associated to the transformed boundary-value problem \( (30), (41) \), and that for \( t = 0 \) the propagator \( (38) \) becomes a delta. In order to prove this latter statement, let us evaluate \( (38) \) for \( t = 0 \). Introducing the Heaviside distribution \( \theta \), we have

\[
K_n(x, y, 0) = \]

\[
= (-1)^n \left[ \frac{h(y)}{f(y)} \right]^x L^n_{0,0} \sum_{j=0}^{n-1} (-1)^j \frac{W_n(y)}{W(y)} \int_{-\infty}^{y} K_0(x, z, 0) \, u_j(z) \, dz \]

\[
= (-1)^n \left[ \frac{h(y)}{f(y)} \right]^x L^n_{0,0} \sum_{j=0}^{n-1} (-1)^j \frac{W_n(y)}{W(y)} \theta(y-x) \, u_j(x) \]

\[
= (-1)^n \left[ \frac{h(y)}{f(y)} \right]^x \sum_{j=0}^{n-1} (-1)^j \frac{W_n(y)}{W(y)} L^n_{0,0} \left( \theta(y-x) \, u_j(x) \right) \]

\[
= (-1)^n \left[ \frac{h(y)}{f(y)} \right]^x \sum_{j=0}^{n-1} (-1)^j \frac{W_n(y)}{W(y)} \left( \frac{f(x)}{h(x)} \right)^x \frac{W(u_0, \ldots, u_{j-1}, \theta(y-x) \, u_j(x))}{W(u_0, \ldots, u_{j-1})} \tag{39}.
\]

Let us now rewrite the Wronskian that involves the Heaviside distribution. Note that the following argument is the same that was used in \( [9] \), we include it here for the sake of completeness. In order to rewrite the last Wronskian in \( (39) \), we first need to consider the derivatives:

\[
\frac{\partial^m}{\partial x^m} \theta(y-x) \, u_j(x) = \sum_{k=0}^{m-1} c_{km} \left[ \frac{\partial^{m-k}}{\partial x^{m-k}} \theta(y-x) \right] u_j^{(k)}(x) + \theta(y-x) \, u_j^{(m)}(x), \tag{40}
\]

where the \( c_{km} \) denote constants. If we use this expression to replace the derivatives in the last Wronskian of \( (39) \) and convert the sum in \( (40) \) into a sum of two Wronskians, then the last term vanishes, as \( W(u_0, \ldots, u_{n-1}, u_j) = 0 \) for \( j = 0, \ldots, n-1 \). We then arrive at

\[
W(u_0, \ldots, u_{j-1}, \theta(y-x) \, u_j)(x) =
\]

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= \det \begin{pmatrix}
  u_0(x) & \cdots & u_{n-1}(x) & 0 \\
  u'_0(x) & \cdots & u'_{n-1}(x) & -\delta(x - y) u_j(x) \\
  \vdots & \ddots & \vdots & \vdots \\
  u^{(n)}_0(x) & \cdots & u^{(n)}_{n-1}(x) & \sum_{k=0}^{n-1} c_{kn} \left( \frac{\partial^{m-k}}{\partial y^{m-k}} \theta(y - x) \right) u_j^{(k)}(x)
\end{pmatrix}. \quad (41)

Let us point out that each element of the last column is the sum given by the entry in the lower right corner, evaluated at \( n = 0, 1, 2, \ldots \). Evaluation of the determinant in (41) with respect to the last column gives

\[ W(u_0, \ldots, u_{j-1}, \theta(y - x)u_j)(x) = \]

\[ = (-1)^n \sum_{m=0}^{n} (-1)^m W_{nm}(x) \sum_{k=0}^{m-1} c_{km} \left[ \frac{\partial^{m-k}}{\partial x^{m-k}} \theta(y - x) \right] u_j^{(k)}(x), \quad (42) \]

where as usual the \( W_{nm} \) denote determinants of the minor matrices obtained by deleting the \( n \)-th row and the \( m \)-th column. We will now substitute this result into our propagator (39) and show that it gives a delta, which is equivalent to

\[ \int_{\mathbb{R}} K_n(x, y, 0) \phi(x) \, dx = \phi(y), \quad (43) \]

for all admissible test functions \( \phi \), recall that these functions are smooth and vanish at the infinities. After substitution of (42) into (39), we obtain

\[ \int_{\mathbb{R}} K_n(x, y, 0) \phi(x) \, dx = \left[ \frac{h(y)}{f(y)} \right]^{\frac{n}{2}} \sum_{j=0}^{n-1} \sum_{m=1}^{n} \sum_{k=0}^{m-1} (-1)^{j+m} c_{mk} \frac{W_j(y)}{W(x)} \times \]

\[ \times \int_{\mathbb{R}} \left[ \frac{\partial^{m-k}}{\partial x^{m-k}} \theta(y - x) \right] \left[ \frac{h(y)}{f(y)} \right]^{\frac{n}{2}} W_{nm}(x) \frac{f(x) u_j^{(k)}(x)}{W(x)} \, dx. \quad (44) \]

The derivatives applied to the Heaviside function can be removed by using \( \theta' = \delta \) in the distributional sense. Taking into account \( \delta(y - x) = \delta(x - y) \) and applying the definition of the delta function’s derivative, we obtain from (41)

\[ \int_{\mathbb{R}} K_n(x, y, 0) \phi(x) \, dx = \left[ \frac{h(y)}{f(y)} \right]^{\frac{n}{2}} \sum_{j=0}^{n-1} \sum_{m=1}^{n} \sum_{k=0}^{m-1} (-1)^{j+m} c_{mk} \frac{W_j(y)}{W(y)} \times \]

\[ \times \int_{\mathbb{R}} \left[ \frac{\partial^{m-k}}{\partial x^{m-k}} \delta(y - x) \right] \left[ \frac{h(y)}{f(y)} \right]^{\frac{n}{2}} W_{nm}(x) \frac{f(x) u_j^{(k)}(x)}{W(x)} \, dx \]

\[ = \left[ \frac{h(y)}{f(y)} \right]^{\frac{n}{2}} \sum_{j=0}^{n-1} \sum_{m=1}^{n} \sum_{k=0}^{m-1} (-1)^{j-k} c_{mk} \frac{W_j(y)}{W(y)} \times \]

\[ \times \int_{\mathbb{R}} \frac{\partial^{m-k}}{\partial y^{m-k}} \left\{ \left[ \frac{h(y)}{f(y)} \right]^{\frac{n}{2}} W_{nm}(y) \frac{f(y) u_j^{(k)}(y)}{W(y)} \right\} \, dy. \]

(45)
We now apply the Leibniz rule to the last term on the right-hand side and make use of the rule

\[ \sum_{j=0}^{n-1} (-1)^j \frac{W_j(y)}{W(y)} u_j^{(s)}(y) = \delta_{s,n-1}, \]

which turns (45) into

\[ \int_{\mathbb{R}} K_n(x, y, 0) \phi(x) \, dx = (-1)^n \frac{W_{nn}(y)}{W(y)} \sum_{k=0}^{n-1} c_{nk} (-1)^k \]

(46)

because \( W_{nn} = W \) and the sum in (46) equals \((-1)^n\), see e.g. [9]. Hence, we have shown that \( K_n(x, y, 0) = \delta(x - y) \). It remains to prove that the propagator \( K_n \) solves the time-dependent Schrödinger equation associated with our generalized boundary-value problem (3), (4), with respect to both spatial variables \( x \) and \( y \). This is obvious in the first case, as \( K_n \) is obtained from \( K_0 \) by application of \( L^x_{n,0} \), which maps solutions of our initial boundary-value problem (1), (2) onto its transformed counterpart (3), (4). Since \( K_0 \) solves the initial problem, it follows that \( K_n \) must solve the transformed problem. Regarding the second variable \( y \) we must substitute the explicit form of \( K_n \), as given in (38) into the time-dependent Schrödinger equation and show that it is fulfilled. This short calculation follows exactly the same steps as in [9], such that we omit to show it here. Let us now consider the case where the auxiliary solutions in our SUSY transformation fulfill the second boundary condition in (2), namely,

\[ \lim_{x \to \infty} u_j(x) = 0. \]

In this case, one uses the same argumentation as given above and arrives at the propagator relation

\[ K_n(x, y, t) = (-1)^n \left[ \frac{h(y)}{f(y)} \right]^2 \cdot \int_{\mathbb{R}} K_0(x, z, t) u_j(z) \, dz. \]

Finally, if the first \( M \) auxiliary solutions fulfill the first boundary condition, and the remaining \( N - M \) auxiliary solutions satisfy the second boundary condition, then our propagator relation will read

\[ K_n(x, y, t) = (-1)^n \left[ \frac{h(y)}{f(y)} \right]^2 \cdot L^x_{n,0} \sum_{j=0}^{M-1} (-1)^j \frac{W_j(y)}{W(y)} \int_{-\infty}^{y} K_0(x, z, t) u_j(z) \, dz + \]

\[ + (-1)^n \left[ \frac{h(y)}{f(y)} \right]^2 \cdot L^x_{n,0} \sum_{j=M}^{n-1} (-1)^j \frac{W_j(y)}{W(y)} \int_{y}^{\infty} K_0(x, z, t) u_j(z) \, dz. \]

As before, it is straightforward to see that our propagator relations reduce correctly to the conventional ones constructed in [9], if we set \( f = h = 1 \).
4 Concluding remarks

We have shown that the relations between propagators of SUSY-linked Schrödinger equations extend to the linearly generalized case. In particular, all possible SUSY scenarios (creation, annihilation of spectral values and isospectrality) have been verified and found to match corresponding findings in [9].

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