Research on optimize application of Buckingham Pi theorem to wind tunnel test and its aerodynamic simulation verification

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Abstract. In physical modeling, especially in the field of wind tunnel tests, dimensional analysis is essential, but the existing dimensional analysis methods are difficult to select the appropriate physical quantities. Therefore, based on the improvement of existing dimensional analysis methods for wind tunnel tests, a set of optimized dimensional analysis methods for wind tunnel tests based on Buckingham Pi theorem was derived in this paper, and then we put forward an aerodynamic simulation experiment based on Fluent to verify this method.

1. Introduction

If we want to get the relationship between multiple physical quantities, the direct experiment method has a great limitation, which can only get the relationship between a few physical quantities, and even do not know how to carry out the experiment. Take fixed-wing aircraft with a certain attitude in a certain aerodynamics simulation as an example, which is affected by the force of air R. It is affected by free flow velocity $V_∞$, free flow density $ρ_∞$, fluid viscosity coefficient $μ_∞$, object size $c$, and compressibility of fluid $a_∞$. It will take a lot of time and cost to get the specific relationship between R and these five quantities directly measured by wind tunnel experiment.

According to the research made by Chen Bing-cong[1], the similarity method is a theorem that can generalize the observed individual results to all similar phenomena by virtue of the similarity theorem. It is the theoretical basis of modern experimental simulation method, which was first introduced into China by Shen Ziqiu[2]. 1914 American physicist Edgar Buckingham use π first refer to dimensionless parameters which is obtained by the theorem, it is a dimensional analysis method, which derives the implicit function relation between physical quantities through the dimensional relation between physical quantities, and then obtains the specific proportional constant in the relation through experimental or theoretical analysis. The dimensional analysis before carrying out the experiment can not only save time, but also make the direction of the experiment clear and improve the accuracy.

Huang Hengdong is one of the earliest scholar who began to use this theorem to clearly points out the limitations of Buckingham Pi theorem in his article[3,4]. It can be understood from two perspectives: one is the limitations of mankind: the correctness of the theorem completely depends on
the selection of physical quantities to describe physical phenomena, and the selection of physical quantities completely depends on whether researchers have a clear understanding of the physical phenomena or are familiar with experimental data. The theorem itself can not choose physical quantities. Secondly, the result is limited. The implicit function relation obtained by this theorem is not a complete and specific physical equation, and the proportionality constant still needs to be obtained by experiment or theoretical analysis and the correctness of the relation is verified. In addition, an article[5] also points out the limitations of similarity theory in solving chemical engineering problems with chemical reactions.

Many scholars in different areas has been widely using Buckingham Pi theorem in their research: based on PI theorem, the relation model of roof pillar thickness to span ratio and roof pillar stability in caving and transfilling transition stope is put forward, and the relation model is applied to the mining practice of transition stope in Chengchao iron mine, and the value of thickness to span ratio suitable for this mine is obtained according to the mining conditions in Chengchao iron mine in an article[6]; The implicit function relation of siphon formation time is obtained by applying Buckingham Pi theorem to siphon outlet pipe in pumping station according to Feng Jiangang’s research[7]. An article[8] apply similarity theory to mechanical engineering; the relationship between the modal constants obtained by Buckingham Pi theorem is used as the criterion to verify the correctness of the simplified method in Zhang Chunli’s research[9]. Hu Fengfeng[10] use the similarity theory is applied to the speed control of aero engine. There are many other achievements, which will not be demonstrated here.

In this paper, the wind tunnel test analogy method based on Buckingham Pi theorem is derived, and the aerodynamic simulation is carried out to verify this method.

2. **Optimization application of Buckingham Pi theorem in wind tunnel test**

Aerodynamics studies the aerodynamic force and moment of a body. Take an airfoil for example, with a constant attack angle, the resultant aerodynamic force of a plane is determined by the following five physical quantities:

1. Free stream velocity, \(V_\infty (\text{m} \cdot \text{s}^{-1})\).
2. Free stream density, \(\rho_\infty (\text{kg} \cdot \text{m}^{-3})\).
3. Viscosity of the fluid, \(\mu_\infty (\text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-1})\). Forces and moments are affected by the shear stress \(\tau\), and the shear stress \(\tau\) is effect by the velocity gradient. Set the velocity of a point on a streamline as \(u\), distance normal to the streamline as \(y\), we have \(\tau = \mu \frac{\partial u}{\partial y}\), \(\mu\) is a function of the temperature of the fluid, it represent the influence of the viscosity, denoted as \(\mu_\infty\).
4. The size of the body \(c (\text{m})\), usually represented by a chosen reference length. For an airfoil, \(c\) is the chord length.
5. The compressibility of the fluid. The compressibility is related to the density variation of the flow field, and consequently, a change of density will cause a change of the resultant force. The compressibility is characterized by the speed of sound \(a_\infty\) in the fluid, according the book written by Anderson Jr, John David[12].

According to the discussion above, set the reluctant aerodynamic force as \(R (\text{kg} \cdot \text{m} \cdot \text{s}^{-2})\). Then there is an abstract function between \(R\) and the physical quantities:

\[
R = (V_\infty, \rho_\infty, \mu_\infty, c, a_\infty)
\]  

(1)

Based on (1), we can fix the object in wind tunnel at a constant attack angle, when other quantities are maintained except the one chosen to change, researchers record the data and analyze them, they might be able to get the exact equation of (1), but that would bring a huge amount of work, requiring prolonged wind tunnel measurement and enormous expense. On the other hand, if researchers use Buckingham Pi theorem to analyze dimensions before that, the workload will be considerably reduced. Some brief description is followed, the detailed proof is in Curtis’s research[11].
It is widely acknowledged that equations must be dimensionally homogeneous. All the mechanical quantities can be divided into mass(kilogram), length(meter), time(second) these three elementary quantities and quantities derived from them. Each physical quantity in mechanics is derived from at least one among the mass, length and time three elementary quantities. Thus dimensions of a mechanical quantity must comprise at least one from kg, m, s, three elementary dimensions. We can write the next equation.

\[ f(V_1, V_2, V_3, \ldots V_n) = 0 \]  

(2)

The proof is omitted. Set \( V_1, V_2, V_3 \) as 1kg, 1 m, 1s, put them at the right side of the equation, \( V_4 \) is derived from them and set \( V_4 \) as force \( F \). Put the \( V_4 \) at the right side, for force \( F \) the form is \( 1N=1kg*m*s^{-2} \). Dividing both sides of the equation by \( V_4 \) then get 1=1, dimensions are concealed, which implies that \( V_4/(V_1*V_2*V_3) \) is a dimensionless product.

Buckingham Pi theorem points out, if we want to find the relationship of \( n \) physical quantities, which comprise \( k \) basic dimensions and each quantity comprise at least one dimension, we can choose \( k \) quantities to multiply by each other’s power at the right side of the equation, making sure they “contain” all the basic dimensions, then we randomly choose a quantity from the \( n-k \) leftover quantities, after that we find the unique form of the chosen quantity and cancel the dimensions, ending up with a dimensionless product, denoted as \( \Pi \). Repeat the actions above, we can get \( n-k \) dimensionless result \( \Pi_1, \Pi_2, \ldots \Pi_{n-k} \).

Back to the previous discussion, there is \( C_6^3=20 \) ways to choose three quantities out of six. After selection, placing \( \rho_\infty, \ V_\infty, \ c \) at the right side of the equation gives us most meaningful dimensionless product. Now choose \( R \) to create \( \Pi_1 \):

\[ \Pi_1 = \rho_\infty^3 V_\infty^5 c R \]  

(3)

Substitute the corresponding units into the equation(3) and get following equation:

\[ \Pi_1 = (kg \cdot m^3 \cdot J^d \cdot m \cdot s^{-3} \cdot J^b \cdot m^e \cdot kg \cdot m \cdot s^{-2}) \]  

(4)

For three exponent, there’s three following equations to express them:

\[ \text{kg exponent: } d+1=0 \]  

(5)

\[ \text{m exponent: } -3d+b+e+1=0 \]  

(6)

\[ \text{s exponent: } -b-2=0 \]  

(7)

Solving the equations, then substitute \( d = -1, \ b = -2, \ e = -2 \) into Equation(3), we get:

\[ \Pi_1 = \frac{R}{\rho_\infty V_\infty^2 c} \]  

(8)

\( \Pi_1 \) multiplied by 2 is still dimensionless, then replace \( c^2 \) by wing area \( S \) and get following equation:

\[ \Pi_1 = \frac{R}{S \cdot \frac{1}{2} \rho_\infty V_\infty^2} \]  

(9)

\( \frac{1}{2} \rho_\infty V_\infty^2 \) is the dynamic pressure \( q_\infty \), \( \Pi_1 \) is the coefficient of the force \( R \). L denote the component of \( R \) perpendicular to free stream velocity while \( D \) denote the component of \( R \) parallel to free stream velocity. Since \( L \) and \( D \) are both components of \( R \), they have such coefficient as Lift coefficient \( C_L \) and drag coefficient \( C_D \). A certain force \( R \) bring us a certain moment \( M \). Thus, we also have moment coefficient \( C_M \).
\[ C_R = \frac{R}{q_\infty S} \]  
(10)

\[ C_L = \frac{L}{q_\infty S} \]  
(11)

\[ C_D = \frac{D}{q_\infty S} \]  
(12)

\[ C_M = \frac{M}{q_\infty S} \]  
(13)

Still placing \( \rho_\infty, V_\infty, c \) three quantities at the right side and adding \( \mu_\infty \) gives us \( \Pi_2 \):

\[ \Pi_2 = \rho_\infty^h V_\infty^i c^j \mu_\infty \]  
(14)

\[ \Pi_2 = (kg \cdot m^{-3})^h (m \cdot s^{-1})^i (m^2 \cdot s^{-2})^j \]  
(15)

For three exponent, there’s three following equations to express them:

\[ \text{kg exponent:} \quad h+i = 0 \]  
(16)

\[ \text{m exponent:} \quad -3h+i+j = 0 \]  
(17)

\[ \text{s exponent:} \quad -i-j = 0 \]  
(18)

Thus \( h = -1, j = -1, i = -1 \). Transfer equation (14) into equation (19):

\[ \Pi_2 = \frac{\mu_\infty}{\rho_\infty V_\infty c} \]  
(19)

The reciprocal of \( \Pi_2 \) is still dimensionless, which is the Reynolds number \( \text{Re} \) defined in Aerodynamics:

\[ \text{Re} = \frac{\rho_\infty V_\infty c}{\mu_\infty} \]  
(20)

In the same way, choose \( \alpha_\infty \) to calculate \( \Pi_3 \):

\[ \Pi_3 = \rho_\infty^k V_\infty^r c^s \alpha_\infty \]  
(21)

\[ \Pi_3 = (kg \cdot m^{-3})^k (m \cdot s^{-1})^r (m^2 \cdot s^{-2})^s \]  
(22)

For three exponent, there’s still three following equations to express them:

\[ \text{kg exponent:} \quad k = 0 \]  
(23)

\[ \text{m exponent:} \quad -3k+r+s = 0 \]  
(24)

\[ \text{s exponent:} \quad -r+s = 0 \]  
(25)

After calculating we get the result that \( k = 0, r = -1, s = 0 \), then formula (21) can be transferred as follow:

\[ \Pi_3 = \frac{\alpha_\infty}{V_\infty} \]  
(26)

The reciprocal of \( \Pi_3 \) is still dimensionless, which is the Mach number \( \text{M}_\infty \) defined in Aerodynamics:
For incompressible flow (\(M_\infty < 0.3\)), we could neglect thermodynamical issues, the Mach and Reynolds numbers are our dominant similarity parameters for our present considerations. The above analysis essentially solves the equations of the exponents to ensure the dimensions of the quantities are canceled. Hence we get the three most important parameters in incompressible flow: force coefficient, the Reynold number, and the Mach number, which has specific physical significance. The Reynolds number is physically a measure of the ratio of inertia forces to viscous forces in a flow. The Mach number represents the compressibility of the flow.

Since lift coefficient, drag coefficient, the Reynold number, and the Mach number are dimensionless, we have:

\[
f(C_L, Re, M_\infty, \alpha) = 0
\]  
\[
f(C_D, Re, M_\infty, \alpha) = 0
\]

(28) (29)

Where \(\alpha\) is the attack angle. At a given attack angle, with different pairs of the Reynolds and Mach number, values of \(C_L\), \(C_D\) are uniquely determined:

\[
C_L = f(Re, M_\infty)
\]  
\[
C_D = f(Re, M_\infty)
\]

(30) (31)

(30) (31) has their guiding significance in wind tunnel experiments, in which researchers usually are unable to adopt integral scale model of the test plane. For a given aircraft, the flow field in the wind tunnel must be geometrically similar to the flow field in the air. As long as we keep the Reynolds and Mach numbers are the same between experiments and real flying, the lift coefficient and drag coefficient measured are same in the error range.

\[
L = q_\infty S C_L
\]  
\[
D = q_\infty S C_D
\]

(32) (33)

By using Buckingham Pi theorem, the dimension analysis, we cut five quantities to two, we save not only time but also costs, controlling variables more pertinently. It reveals the idea of similarity parameter. This paper aims at presenting this process of thinking, so we uses an aircraft at a constant attack angle in the incompressible flow as the experiment subject.

3. Aerodynamic simulation verification

We use ANSYS Fluent to prove the Optimized application methods of Buckingham theorem. First, we design a simple aerodynamic shape, then prepare a integral scale model and a half-scale model, set both aircraft at the attack angle at 0 degrees. Each parameter’s value is different between both models, but after careful calculation, the Reynolds and Mach numbers are the same. Finally, we run the simulation. According to the Buckingham Pi theorem, the lift coefficient of the two models are identical, comparing the lift coefficient can prove the theorem.

Due to the abundant known aerodynamic data of NACA four-position airfoil, which is convenient for verification and examination, NACA4412 airfoil is adopted as the airfoil of the test aircraft. According to the research made by Chia-kan Chang[13], increasing the forward sweep Angle will increase the lift-drag ratio and reduce the lift coefficient. With the increase of forward sweep Angle, the variation range of lift coefficient and lift-drag ratio will increase. However, because the forward swept wing will reduce the lift coefficient and greatly increase the torque on the connection between the wing and the fuselage in the state of vertical take-off and landing, so the straight wing design is proposed in the design of wing planform. The airfoil of vertical tail adopts a symmetrical airfoil based on splines. Since at low speeds, the drag under the same vertical stern area is not affected by the sweep back angle of the leading edge and the wing taper ratio, the wing taper ratio set to 0.78 and the sweep
back angle of the leading edge set as 30° for the sake of appearance. In order to prevent the influence of wing downwash airflow on the horizontal tail, the t-shaped tail was adopted, and the anti-mounted NACA4412 airfoil was used as the flat tail to verify the aerodynamic characteristics of the aircraft.

For integral scale model, under the condition of the ideal gas, we set the temperature as 251K, gauge pressure $P_2$ as 42387.62Pa, air density $\rho_2=1.99472\text{kg/m}^3$ and free stream velocity $V_2$ as 62.452m/s. According to the integral scale model, the reference length is 0.4m, and reference area is 0.96m$^2$. The Enthalpy equals 257969.5J/kg; viscosity equals 1.7894e-05, and the ratio of Specific Heats equals 1.4.

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According to Anderson Jr, John David,[12], assume that both $\mu_\infty$ and $a_\infty$ are proportional to $T^{0.5}$. Set the temperature of the half-scale model is 300K:

The equal Mach numbers, \[ V_1 = \frac{V_2}{\sqrt{T_1}} \], get the value of $V_1=68.277\text{m/s}$.

The equal \[ \frac{\rho_1 V_1 c_1}{\sqrt{P_1}} = \frac{\rho_2 V_2 c_2}{\sqrt{P_2}} \], $c^2=2c_1$, get the value of $\rho_1=2\rho_2=2.355349\text{kg/m}^3$.

State equation of ideal gas, \[ \frac{P_2}{P_1} = \frac{\rho_2 T_2}{\rho_1 T_1} \], get the value of $P_1=101325.1\text{pa}$.
Run aerodynamic simulation in accordance with the above setting. For integral scale model, after about 321 iterations, the lift curve of integral scale model has been stable.
The lift coefficient obtained from the simulation is 0.526. Above are the figures of the pressure contour and velocity streamline.

For half scale model, it can be seen that after 6786 iterations, the lift coefficient curve is stable, get the lift coefficient as 0.531, shown in figure 12. The degree of deviation of lift coefficient between integral scale model and half-scale model is 0.94%, it proves that the method in this paper is accurate and effective.

4. Conclusion
Based on the buckingham Pi theorem, this paper optimizes the simulation method of wind tunnel test, calculates and deduces the three most important parameters in wind tunnel test, Mach number, Reynolds number and coefficient of force by using Buckingham Pi theorem, and proves the superiority of this theorem in wind tunnel test. The optimized method of using Buckingham Pi theorem to guide wind tunnel test is also developed to solve the problem of the difficulty in selecting the appropriate physical quantity, it can greatly facilitate the development of various types of military fixed-wing aircraft. Finally, the accuracy and validity of the theory are proved by the aerodynamic simulation. However, the aerodynamic simulation only involves the verification of the outflow field and the lift coefficient, which is slightly insufficient in the richness.

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