Trace Class Toeplitz Operators with Singular Symbols

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Received February 16, 2020; revised April 7, 2020; accepted June 24, 2020

To Armen Sergeev
on the occasion of his 70th birthday

Abstract—We provide sufficient conditions for Toeplitz operators with distributional symbols acting on the Bergman space on the unit disk to be trace class. The Berezin transform of distributions, introduced in the paper, yields a formula for the trace. Several instructive examples are also given.

DOI: 10.1134/S0081543820060140

1. INTRODUCTION

The classical Bergman space $A^2(D)$ is a closed subspace in $L_2(D,dA)$, where $dA(z) = \pi^{-1} \, dx \, dy$ is the normalized Lebesgue measure, which consists of functions analytic on the unit disk $D$. It is a reproducing kernel Hilbert space, whose reproducing kernel is given by $K_z(w) = (1 - \overline{z}w)^{-2}$. The orthogonal Bergman projection $P : L_2(D,dA) \rightarrow A^2(D)$ is then given by $(Pf)(z) = \langle f, K_z \rangle$.

A classical Toeplitz operator $T_a$, with symbol $a \in L_\infty(D)$, is defined as the compression of the operator of multiplication by $a$ onto the Bergman space; i.e., it acts on $A^2(D)$ as $T_a f = P(a f)$. This classical definition of Toeplitz operators has been extended to various more general situations, including weighted Bergman spaces, unit ball case and more general multidimensional domains of holomorphy, measure and distributional symbols, etc.

Toeplitz operators with strongly singular (distributional) symbols were probably first considered in [1, 6]. Recently, in [7, 8], a new approach to Toeplitz operators was proposed. The authors used the language of bounded sesquilinear forms, which allowed them not only to cover all previous generalizations in a unified form but also to extend the notion of Toeplitz operators to highly singular symbols (hyperfunctions).

Let us briefly recall the approach (for proofs and details see [7, 8]). Let $F[\cdot, \cdot]$ be a bounded sesquilinear form on a reproducing kernel Hilbert space $A$, with $K_z$ being its reproducing kernel. Then the Toeplitz operator $T_F$ defined by the form $F$ acts on $A$ as follows: $(T_F f)(z) = F[f, K_z]$. In particular, [8] treats Toeplitz operators defined by $k$-Carleson measures for derivatives. Recall in this connection that a measure $\mu$ is said to be $k$-Carleson if

$$\left| \int_D |f^{(k)}(z)|^2 \, d\mu(z) \right| \leq C \|f\|^2_{A^2(D)},$$

where $f^{(k)}$ is the $k$-th derivative of $f$.
where the constant $C$ does not depend on $f \in A^2(\mathbb{D})$. Boundedness and compactness conditions have been found for such operators.

In the present paper we begin the study of the Schatten class membership of Toeplitz operators defined by sesquilinear forms. To demonstrate the ideas and make our considerations more transparent (by avoiding unnecessary technicalities), we consider in this paper the simplest case, namely, the unweighted Bergman space $A^2(\mathbb{D})$, Toeplitz operators defined by derivatives of $k$-Carleson measures, and the trace class membership. More general cases will be considered elsewhere.

Recall, for completeness, that the trace and Schatten class membership of Toeplitz operators with more regular symbols was studied first in [3] and then in [2, 5, 9], etc.

2. SUFFICIENT TRACE CLASS CONDITION

Following the pattern in [7, 8], we now introduce sesquilinear forms involving derivatives of functions $f$ and $g$ and thus corresponding to derivatives of Carleson measures.

**Definition 2.1.** Let $\mu$ be a regular complex measure on $\mathbb{D}$ and let $\alpha$ and $\beta$ be two nonnegative integers. We denote by $F_{\alpha,\beta,\mu}$ the sesquilinear form

$$F_{\alpha,\beta,\mu}[f,g] = (-1)^{\alpha + \beta} \int_{\mathbb{D}} \overline{\partial^\alpha f(z)} \partial^\beta g(z) \, d\mu(z), \quad f, g \in A^2(\mathbb{D}). \tag{2.1}$$

In [8] conditions were found for the sesquilinear form (2.1) to be bounded or compact. These conditions are sufficient for any measure $\mu$, and they are also necessary for a positive measure and $\alpha = \beta$. Such conditions are expressed using the notion of $k$-Carleson measures. Our aim is to find a condition for the form (2.1) with a $k$-Carleson measure $\mu$ to generate a trace class operator. We denote by $T_{\alpha,\beta,\mu} := T_{F_{\alpha,\beta,\mu}}$, or just $T_F$ or $T_\Phi$ with $\Phi = \partial^\alpha \overline{\partial^\beta \mu}$, the Toeplitz operator defined by this form.

Note first that for a measure $\mu$ with compact support in $\mathbb{D}$ the Toeplitz operator $T_{\alpha,\beta,\mu}$ has singular numbers that decay exponentially; therefore, $T_{\alpha,\beta,\mu}$ belongs to all Schatten classes $S^p, p > 0$, for any $\alpha$ and $\beta$. We recall the reasoning in [8] in a somewhat modified form.

**Theorem 2.1.** Let $\mu$ be a measure with compact support in $\mathbb{D}$ and let $\alpha$ and $\beta$ be some nonnegative integers. Then for any $f, g \in A^2(\mathbb{D})$ the integral in (2.1) converges; moreover,

$$F[f,g] = F_{\alpha,\beta,\mu}[f,g] = (\partial^\alpha \overline{\partial^\beta \mu}, f \overline{g}), \tag{2.2}$$

where the derivatives are understood in the sense of distributions in $E'(\mathbb{D})$ and the parentheses mean the intrinsic pairing of $E'(\mathbb{D})$ and $E(\mathbb{D})$. The singular numbers $s_n(T_F)$ of the operator $T_F$ satisfy the estimate

$$s_n(T_F) \leq C \exp(-n\sigma), \tag{2.3}$$

where $\sigma > 0$ is some constant determined by the measure $\mu$ and integers $\alpha$ and $\beta$.

**Proof.** Due to Ky Fan’s inequality for singular numbers of compact operators, it is sufficient to establish (2.3) for a positive measure $\mu$. Consider a concentric closed disk $D \subset \mathbb{D}$ of radius $R < 1$ such that the support of $\mu$ lies strictly inside $D$; thus dist$(z, \partial D) > r > 0$ for all $z \in \text{supp} \mu$. The Cauchy integral formula implies that for any $z \in \text{supp} \mu$ and any $\alpha \in \mathbb{Z}_+$,

$$|\partial^\alpha f(z)|^2 \leq C_\alpha \int_D |f(\zeta)|^2 \, dA(\zeta) \tag{2.4}$$

where the constant $C$ does not depend on $f \in A^2(\mathbb{D})$. Boundedness and compactness conditions have been found for such operators.
for any function \( f \in \mathcal{A}^2(\mathbb{D}) \). Here the constant \( C_{\alpha} \) depends only on \( \alpha \), but does not depend on \( f \) and \( z \). By the Cauchy–Schwarz inequality,

\[
|F_{\alpha,\beta,\mu}[f,g]| \leq \left( \int_{\text{supp} \mu} |\partial^{\alpha} f(z)|^2 \, d\mu(z) \right)^{1/2} \left( \int_{\text{supp} \mu} |\partial^{\beta} g(z)|^2 \, d\mu(z) \right)^{1/2}
\]

for all \( f, g \in \mathcal{A}^2(\mathbb{D}) \), and then, due to (2.4), we obtain the estimate

\[
|F_{\alpha,\beta,\mu}[f,g]| \leq C_{\alpha} C_{\beta} \mu(\mathcal{D}) \|f\|_{L^2(\mathbb{D})} \|g\|_{L^2(\mathbb{D})}.
\]

This last relation means that the sesquilinear form \( F_{\alpha,\beta,\mu} \) is bounded not only in \( \mathcal{A}^2(\mathbb{D}) \) but in \( \mathcal{A}^2(\mathbb{D}) \) as well. In particular, for \( f = g \) and \( \alpha = \beta \), inequality (2.5) means that the operator \( Z_{\alpha}(\mathbb{D}) \) that acts by \( f \mapsto f^{(\alpha)} \) from \( \mathcal{A}^2(\mathbb{D}) \) to the weighted space \( L^2(\mathbb{D},\mu) \) is bounded. The operator \( Z_{\alpha}(\mathbb{D}) \) acting in the same way but from \( \mathcal{A}^2(\mathbb{D}) \) admits the factorization \( Z_{\alpha}(\mathbb{D}) = Z_{\alpha}(\mathbb{D}) I \), where \( I: \mathcal{A}^2(\mathbb{D}) \to \mathcal{A}^2(\mathbb{D}) \) is the restriction operator. Now we use the fact that the singular numbers of \( I \) decay exponentially (see, e.g., [4]). This implies the exponential decay for the singular numbers of the product \( Z_{\alpha}(\mathbb{D}) = Z_{\alpha}(\mathbb{D}) I \). Finally, the sesquilinear form \( F_{\alpha,\beta,\mu}[f,g] \) can be represented as \( \langle Z_{\alpha}(\mathbb{D}) f, Z_{\beta}(\mathbb{D}) g \rangle_{L^2(\mathbb{D},\mu)} \); therefore, the operator \( T_F \) equals the product \( T_F = Z_{\beta}(\mathbb{D})^* Z_{\alpha}(\mathbb{D}) \), and this proves the exponential decay of its singular numbers. \( \square \)

The above result demonstrates that the “quality” of the Toeplitz operator with distributional symbol, in the sense of its membership in the Schatten classes of compact operators, is determined by the behavior of the symbol near the boundary.

3. THE BEREZIN TRANSFORM AND THE TRACE CLASS

Recall that, given a bounded operator \( T \) on the Bergman space \( \mathcal{A}^2(\mathbb{D}) \), its Berezin transform is defined by \( \mathcal{T}(z) = \langle T \kappa_z, \kappa_z \rangle \), where \( \kappa_z(w) = (1 - |z|^2)/(1 - \overline{z}w) \) is the normalized reproducing kernel for \( \mathcal{A}^2(\mathbb{D}) \). The Möbius invariant measure on \( \mathbb{D} \) is denoted by \( d\lambda(z) = (1 - |z|^2)^{-1} \, dA(z) \), where, as previously, \( dA(z) = \pi^{-1} \, dx \, dy \).

We now recall the basic result by K. Zhu (see [9, Theorem 6.4, Corollary 6.5]).

**Theorem 3.1.** Let \( T \) be a nonnegative bounded operator in \( \mathcal{A}^2(\mathbb{D}) \). Then in the equality

\[
\text{tr} \, T = \int_{\mathbb{D}} \mathcal{T}(z) \, d\lambda(z)
\]

one side is finite if and only if the other is finite, and the equality holds in this case. The same assertion holds true if the nonnegativity condition is replaced by the requirement that \( T \) belongs to the trace class \( \mathcal{S}^1 \).

We will apply Theorem 3.1 to Toeplitz operators generated by derivatives of measures. We start with the symmetric nonnegative case.

**Theorem 3.2.** Let \( \mu \) be a regular nonnegative \( k \)-Carleson measure, \( k \geq 1 \). Consider the Toeplitz operator \( T \equiv T_{k,k,\mu} \) generated by the distribution \( \Phi = \partial^k \overline{\partial}^k \mu \). Then the operator \( T \) belongs to the trace class as soon as

\[
\int_{\mathbb{D}} (1 - |w|^2)^{-2 - 2k} \, d\mu(w) < \infty,
\]

and in this case

\[
\text{tr} \, T = \left( \Phi, \frac{1}{(1 - |w|^2)^2} \right);
\]

that is, the trace of the Toeplitz operator is equal to the result of the action of the distributional symbol on a certain universal function.
The inner integral in (3.8) is exactly the Berezin transform of the constant function 1. Since this function is harmonic, its Berezin transform coincides with itself. This gives us

\[ \text{tr} \, T = \left( \Phi, \frac{1}{(1 - |w|^2)^2} 1 \right) = \left( \Phi, \frac{1}{(1 - |w|^2)^2} \right). \quad \square \quad (3.9) \]

Now we are able to resolve the trace class problem for more general distributions \( \Phi \).

**Theorem 3.3.** Let the distributional symbol \( \Phi \) have the form \( \Phi = \partial^\alpha \overline{\partial}^\beta \mu \), where \( \mu \) is a \( k \)-Carleson measure, with \( 2k = \alpha + \beta \). Then, as soon as

\[ \int_D (1 - |w|^2)^{-2k-2} d|\mu|(w) < \infty, \quad (3.10) \]

the operator \( T_\Phi \) belongs to the trace class \( \mathcal{S}^1 \) and

\[ \text{tr} \, T = \left( \Phi, \frac{1}{(1 - |w|^2)^2} \right). \quad (3.11) \]
**Proof.** As usual, it suffices to consider the case of a nonnegative measure \( \mu \). We represent the sesquilinear form of the operator \( T_\Phi \) as

\[
T_\Phi(f, g) = (-1)^{\alpha+\beta} \int_D ((1 - |w|^2)^{\alpha-k} \partial^\alpha f(w))((1 - |w|^2)^{\beta-k} \partial^\beta g(w)) \, d\mu(w). \tag{3.12}
\]

We introduce two operators acting from the Bergman space \( A^2(\mathbb{D}) \) to the weighted \( L^2 \) space, \( L^2_\mu(\mathbb{D}) \), in the following way:

\[
H_1: f(w) \mapsto (1 - |w|^2)^{\alpha-k} \partial^\alpha f(w), \quad H_2: g(w) \mapsto (1 - |w|^2)^{\beta-k} \partial^\beta g(w). \tag{3.13}
\]

Using these operators, we can write the sesquilinear form \( T_\Phi(f, g) \) as

\[
T_\Phi(f, g) = \langle H_1 f, H_2 g \rangle_{L^2_\mu(\mathbb{D})} = \langle H_2^* H_1 f, g \rangle_{A^2(\mathbb{D})}. \tag{3.14}
\]

This means that the Toeplitz operator \( T_\Phi \) is factored as

\[
T_\Phi = H_2^* H_1. \tag{3.15}
\]

Therefore, we need to prove that both operators \( H_1 \) and \( H_2 \) belong to the Hilbert–Schmidt class \( \mathcal{S}^2 \). Consider, for example, \( H_1 \). We will show now that the nonnegative operator \( H_1^* H_1 \) (acting in the Bergman space \( A^2(\mathbb{D}) \)) belongs to the trace class \( \mathcal{S}^1 \). Indeed, the sesquilinear form of this operator equals

\[
\langle H_1^* H_1 f, g \rangle_{A^2(\mathbb{D})} = \langle H_1 f, H_1 g \rangle_{L^2_\mu} = \int_D (\partial^\alpha f \overline{\partial^\beta g})(1 - |w|^2)^{2\alpha-2k} \, d\mu. \tag{3.16}
\]

We have already established in Theorem 3.2 (applied to the measure \( (1 - |w|^2)^{2\alpha-2k} \, d\mu(w) \) instead of \( d\mu(w) \)) that condition (3.10) is sufficient for the operator defined by the sesquilinear form (3.16) to belong to \( \mathcal{S}^1 \). Therefore, the operator \( H_1 \) is a Hilbert–Schmidt one. The same reasoning applies to \( H_2 \). Finally this means that \( H_2^* H_1 \) is a trace class operator.

To prove (3.11), we need, similarly to the reasoning in Theorem 3.2, to show that it is possible to change the order of integration in the expression

\[
\text{tr} \, T = \int_D \langle \Phi, |\kappa(z)|^2 \rangle (1 - |z|^2)^2 \, dA(z) = \int_D \int_D d\mu(w) \partial^\alpha \overline{\partial^\beta} |1 - z \overline{w}|^{-4} \, dA(z). \tag{3.17}
\]

We represent (3.17) as

\[
\text{tr} \, T = (\alpha + 1)! (\beta + 1)! \int_D z^\beta \overline{z}^\alpha \int_D (1 - z \overline{w})^{-2-\beta} (1 - |w|^2)^{2\beta-2} \, d\mu(w) \, dA(z) \tag{3.18}
\]

and then transform it similarly to (3.12):

\[
\text{tr} \, T = (\alpha + 1)! (\beta + 1)! \int_D z^\beta \overline{z}^\alpha \int_D (1 - |w|^2)^{2\beta-2k} (1 - |w|^2)^{\alpha-k} (1 - |z|^2)^{-\alpha} \, d\mu(w) \, dA(z). \tag{3.19}
\]

The last integral is estimated by applying the Cauchy–Schwarz inequality,

\[
|\text{tr} \, T| \leq (\alpha + 1)! (\beta + 1)! \left\{ \int_D (1 - |w|^2)^{2\beta-2k} |1 - z \overline{w}|^{-4-2\beta} \, d\mu(w) \, dA(z) \right\}^{1/2} \times \left\{ \int_D (1 - |w|^2)^{2\alpha-2k} |1 - z \overline{w}|^{-4-2\alpha} \, d\mu(w) \, dA(z) \right\}^{1/2}, \tag{3.20}
\]
and both integrals in (3.20) converge under our conditions. Therefore, the double integral in (3.19) converges absolutely and we can apply the Fubini theorem to change the order of integration. After this, the proof of (3.11) goes in the same way as the proof of Theorem 3.2. □

**Remark 3.1.** We might have tried to prove Theorem 3.3 in a more straightforward way, actually mimicking the reasoning in the proof of Theorem 3.2, i.e., estimating the Berezin transform this, the proof of (3.11) goes in the same way as the proof of Theorem 3.2. Moreover, without the nonnegativity assumption, formula (3.1) is not justified either: one must first know that the operator is a trace class one. At the same time, having Theorem 3.3 proved, we already know that the operator is trace class and can therefore legally use formula (3.1). After this, the change in the integration order becomes legal.

**4. EXAMPLES**

We illustrate the above general results by several examples involving \( k \)-Carleson measures considered in [8]. In what follows we will constantly use the fact that, for \( \alpha, \beta \in \mathbb{Z}_+ \),

\[
\partial_w^\alpha \frac{1}{(1 - \overline{w}w)^2} = \frac{(\alpha + 1)!}{(1 - \overline{w}w)^{2+\alpha}} \quad \text{and} \quad \partial_w^\beta \frac{1}{(1 - z\overline{w})^2} = \frac{(\beta + 1)!}{(1 - z\overline{w})^{2+\beta}}.
\]

**Example 4.1.** We start with the Carleson measure \( (1 + |z|)^{2k} dA(z) \). Then, by [8, Proposition 6.2], the measure \( d\mu = (1 - |z|)^{2k}(1 + |z|)^{2k} dA(z) = (1 - |z|^2)^{2k} dA(z) \) is a \( k \)-Carleson measure. Now with \( \alpha + \beta \leq 2k \), consider

\[
F_{\alpha,\beta,\mu}[f, g] = (\partial_w^\alpha \overline{\partial_w^\beta} \mu, f g),
\]

the corresponding Toeplitz operator \( T_{\alpha,\beta,\mu} := T_{F_{\alpha,\beta,\mu}} \), and its Berezin transform

\[
\mathcal{T}_{\alpha,\beta,\mu}(z) = F_{\alpha,\beta,\mu}(\kappa_z, \kappa_z) = \int_{\mathcal{D}} \partial_w^\alpha \kappa_z(w) \partial_w^\beta \overline{\kappa_z}(w)(1 - |w|^2)^{2k} dA(w)
\]

\[
= \int_{\mathcal{D}} \partial_w^\alpha \frac{1 - |z|^2}{(1 - \overline{w}w)^2} \partial_w^\beta \frac{1 - |z|^2}{(1 - z\overline{w})^2}(1 - |w|^2)^{2k} dA(w)
\]

\[
= (1 - |z|^2)^2(\alpha + 1)! (\beta + 1)! \int_{\mathcal{D}} \frac{(1 - |w|^2)^{2k}}{(1 - \overline{w}w)^{2+\alpha}(1 - z\overline{w})^{2+\beta}} dA(w).
\]

For \( \alpha = \beta \), we have

\[
\mathcal{T}_{\alpha,\alpha,\mu}(z) = (1 - |z|^2)^2[(\alpha + 1)!]^2 |z|^{2\alpha} \int_{\mathcal{D}} \frac{(1 - |w|^2)^{2k}}{|1 - \overline{w}w|^{2(2+\alpha)}} dA(w)
\]

\[
= \frac{\alpha! (\alpha + 1)! |z|^{2\alpha}}{(1 - |z|^2)^{\alpha}} B_{\alpha}((1 - |w|^2)^{2k-\alpha})(z),
\]

where \( B_{\alpha} \) is the Berezin transform on the weighted Bergman space \( \mathcal{A}_{\alpha}^2(\mathcal{D}) \).
In particular, if \( \alpha \leq 2(k - 1) \leq 2k \):

\[
\text{tr } T_{\alpha, \beta, \mu} = \left( F_{\alpha, \beta, \mu}, \frac{1}{(1 - |w|^2)^2} \right) = \int_{D} \left[ \partial_w \partial_{\overline{w}} \frac{1}{(1 - w\overline{w})^2} \right] (1 - |w|^2)^{2k} dA(w). 
\]

In particular, for an integer or half-integer \( k \geq 2 \) and \( \alpha = \beta = 1 \), we have

\[
\text{tr } T_{1,1,\mu} = \int_{D} \left[ \partial_w \partial_{\overline{w}} \frac{1}{(1 - w\overline{w})^2} \right] (1 - |w|^2)^{2k} dA(w) = \int_{D} \frac{1 + 2|w|^2}{(1 - |w|^2)^{4-2k}} dA(w)
\]

\[
= \frac{1}{(k - 1)(2k - 3)} 
\]

**Example 4.2.** Given a point \( z_0 \in D \), consider the distribution equal to the derivative of the \( \delta \)-distribution at \( z_0 \), i.e., \( \partial^\alpha \overline{\partial}^\beta \delta_{z_0} \). The corresponding sesquilinear form is given by

\[
F_{\alpha, \beta, z_0}(f, g) = \left( \partial^\alpha \overline{\partial}^\beta \delta_{z_0}, f\overline{g} \right) = (-1)^{\alpha+\beta} \left( \delta_{z_0}, \partial^\alpha_w f\partial^\beta_{\overline{w}} \right) = f(\alpha)(z_0)\overline{g}(\beta)(z_0). 
\]

Now we introduce the corresponding Toeplitz operator \( T_{\alpha, \beta, z_0} := T_{F_{\alpha, \beta, z_0}} \) and calculate its Berezin transform

\[
\tilde{T}_{\alpha, \beta, z_0}(z) = \text{tr} T_{\alpha, \beta, z_0} \quad \text{with} \quad z \neq z_0 
\]

Therefore, by (3.3),

\[
\text{tr } T_{\alpha, \beta, z_0} = \left( \partial^\alpha \overline{\partial}^\beta \delta_{z_0}, \frac{1}{(1 - w\overline{w})^2} \right) = (-1)^{\alpha+\beta} \left[ \partial^\alpha_w \partial_{\overline{w}} \frac{1}{(1 - w\overline{w})^2} \right]_{w=z_0}. 
\]

Consider now several particular cases.

Let \( z_0 = 0 \). Then

\[
\tilde{T}_{\alpha, \beta, 0}(z) = (-1)^{\alpha+\beta}(1 - |z|^2)^2 (\alpha + 1)! (\beta + 1)! \overline{z}^\alpha z^\beta 
\]

and, by (3.1),

\[
\text{tr } T_{\alpha, \beta, 0} = (-1)^{\alpha+\beta}(\alpha + 1)! (\beta + 1)! \int_D \overline{z}^\alpha z^\beta dA(z) = \begin{cases} \alpha! (\alpha + 1)! & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases} 
\]

Let \( \alpha = \beta \). Then, by (3.1),

\[
\text{tr } T_{\alpha, \alpha, z_0} = [(\alpha + 1)!]^2 \int_D \frac{|z|^{2\alpha}}{|1 - z\overline{z_0}|^{2(2\alpha)}} dA(z) = [(\alpha + 1)!]^2 \left\| \frac{z^\alpha}{1 - z\overline{z_0}} \right\|^2_{A^2(D)}. 
\]

In particular, if \( \alpha = \beta = 1 \),

\[
\text{tr } T_{1,1, z_0} = 4 \left\| \frac{z}{(1 - z\overline{z_0})^3} \right\|^2_{A^2(D)},
\]
while, by (3.3),
\[
\text{tr} \mathbf{T}_{1,1,z_0} = \left( \delta_{z_0} \partial_w \frac{1}{(1 - w \bar{w})^2} \right) = 2 \frac{1 + 2 |z_0|^2}{(1 - |z_0|^2)^4},
\]
implying that
\[
\left\| \frac{z}{(1 - z \bar{z}_0)^3} \right\|_{A^2(D)} = \frac{\sqrt{1 + 2 |z_0|^2}}{\sqrt{2(1 - |z_0|^2)^2}}.
\]

Let finally $\beta = 0$. Then
\[
\tilde{T}_{\alpha,0,z_0}(z) = (-1)^\alpha (1 - |z|^2)^2 \frac{(\alpha + 1)! |z|^\alpha}{(1 - |z|^2)^{2+\alpha}} \frac{1}{(1 - z \bar{z}_0)^2},
\]
and, by (3.3),
\[
\text{tr} \mathbf{T}_{\alpha,0,z_0} = \left( \partial^\alpha \delta_{z_0}, \frac{1}{(1 - w \bar{w})^2} \right) = (-1)^\alpha \frac{(\alpha + 1)! |z|^\alpha}{(1 - |z|^2)^{2+\alpha}}.
\]

**Example 4.3.** Given $r_0 \in (0,1)$, consider the distribution $\Phi = \partial_r \delta_{r_0} \otimes (2\pi)^{-1} d\theta$, $z = re^{i\theta}$. This is the radial derivative of the measure concentrated on the circle of radius $r_0$. We calculate the trace of the operator $\mathbf{T}_\Phi$ using (3.3):
\[
\text{tr} \mathbf{T}_\Phi = \left( \partial_r \delta_{r_0} \otimes (2\pi)^{-1} d\theta, \frac{1}{(1 - r^2)^2} \right) = - \frac{4r_0}{(1 - r_0^2)^3}.
\]

**ACKNOWLEDGMENTS**

The first author is grateful to CINVESTAV for hospitality.

**FUNDING**

The work of the first author is supported by the Russian Science Foundation under grant 20-11-20032.

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