ON UTILITY MAXIMIZATION WITH DERIVATIVES UNDER MODEL
UNCERTAINTY

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Abstract. We consider the robust utility maximization using a static holding in derivatives and a dynamic holding in the stock. There is no fixed model for the price of the stock but we consider a set of probability measures (models) which are not necessarily dominated by a fixed probability measure. By assuming that the set of physical probability measures is convex and weakly compact, we obtain the duality result and the existence of an optimizer.

1. Set-up

We assume that in the market, there is a single risky asset at discrete times $t = 1, \ldots, T$. Let $S = (S_t)_{t=1}^T$ be the canonical process on the path space $\mathbb{R}_+^T$, i.e., for $(s_1, \ldots, s_T) \in \mathbb{R}_+^T$ we have that $S_t(s_1, \ldots, s_T) = s_t$. The random variable $S_t$ represents the price of the risky asset at time $t = i$. We denote the current spot price of the asset as $S_0 = s_0$. In addition, we assume that in the market there are a finite number of options $g_i : \mathbb{R}_+^T \to \mathbb{R}$, $i = 1, \ldots, N$, which can be bought or sold at time $t = 0$ at price $g_i^0$. We assume $g_i$ is continuous and $g_i^0 = 0$.

Let

$$\mathcal{M} := \{Q \text{ probability measure on } \mathbb{R}_+^T : S = (S_t)_{t=1}^T \text{ is a } Q \text{-martingale; for } i = 1, \ldots, N, \mathbb{E}_Q g_i = 0.\}$$

We make the standing assumption that $\mathcal{M} \neq \emptyset$.

Let us consider the semi-static trading strategies consisting of the sum of a static option portfolio and a dynamic strategy in the stock. We will denote by $\Delta$ the predictable process corresponding to the holdings on the stock. More precisely, the semi-static strategies generate payoffs of the form:

$$x + \sum_{i=1}^N h_i g_i(s_1, \ldots, s_n) + \sum_{j=1}^{T-1} \Delta_j(s_1, \ldots, s_j)(s_{j+1} - s_j) =: x + h \cdot g + (\Delta \cdot S)_T, \ s_1, \ldots, s_T \in \mathbb{R}_+,$$

where $x$ is the initial wealth, $h = (h_1, \ldots, h_N)$ and $\Delta = (\Delta_1, \ldots, \Delta_{T-1})$.

We will assume that $U$ is a function defined on $\mathbb{R}_+$ that is bounded, strictly increasing, strictly concave, continuously differentiable and satisfies the Inada conditions

$$U'(0) = \lim_{x \to 0} U'(x) = \infty,$$

$$U'(-\infty) = \lim_{x \to \infty} U'(x) = 0.$$
We also assume that $U$ has asymptotic elasticity strictly less than 1, i.e,
\[
AE(U) = \limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1.
\]

Let $\mathcal{P}$ be a set of probability measures on $\mathbb{R}_+^T$, which represents the possible beliefs for the market. We make the following assumptions on $\mathcal{P}$:

**Assumption $\mathbf{P}$:**

(1) $\mathcal{P}$ is convex and weakly compact.

(2) For any $\mathbb{P} \in \mathcal{P}$, there exists a $\mathbb{Q} \in \mathcal{M}$ that is equivalent to $\mathbb{P}$.

Note that the second condition is natural in the sense that every belief in the market model is reasonable concerning no arbitrage, e.g., see [1]. We consider the robust utility maximization problem
\[
\hat{u}(x) = \sup_{(\Delta, h)} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_\mathbb{P} \left[ U(x + (\Delta \cdot S)_T + h \cdot g) \right].
\]

2. **Main result**

Theorem 2.1 and Theorem 2.2 are the main results of this paper. We will first introduce some spaces and value functions concerning the duality.

Let
\[
V(y) = \sup_{x > 0} [U(x) - xy], \quad y > 0,
\]
and
\[
I := -V' = (U')^{-1}.
\]

For any $\mathbb{P} \in \mathcal{P}$, we define some spaces as follows, where the (in)equalities are in the sense of $\mathbb{P}$-a.s.

- $\mathcal{X}_\mathbb{P}(x, h) = \{X : X_0 = x, \ x + (\Delta \cdot S)_T + h \cdot g \geq 0, \text{ for some } \Delta\}$
- $\mathcal{Y}_\mathbb{P}(y) = \{Y \geq 0 : Y_0 = y, \ XY \text{ is a } \mathbb{P}\text{-super-martingale}, \forall X \in \mathcal{X}_\mathbb{P}(1, 0)\}$
- $\mathcal{Q}_\mathbb{P}(y) = \{Y \in \mathcal{Y}_\mathbb{P}(y) : \mathbb{E}_\mathbb{P} (Y_T (X_T + h \cdot g)) \leq xy, \forall X \in \mathcal{X}_\mathbb{P}(x, h)\}$
- $\mathcal{C}_\mathbb{P}(x, h) = \{c \in L^0_+(\mathbb{P}) : c \leq X_T + h \cdot g, \text{ for some } X \in \mathcal{X}_\mathbb{P}(x, h)\}$
- $\mathcal{C}_\mathbb{P}(x) = \bigcup_h \mathcal{C}_\mathbb{P}(x, h)$
- $\mathcal{D}_\mathbb{P}(y) = \{d \in L^0_+(\mathbb{P}) : d \leq Y_T, \text{ for some } Y \in \mathcal{Q}_\mathbb{P}(y)\}$

Denote
\[
C_\mathbb{P} = C_\mathbb{P}(1), \ D_\mathbb{P} = D_\mathbb{P}(1).
\]

It is easy to see that for $x > 0$, $C_\mathbb{P}(x) = xC_\mathbb{P}$, $D_\mathbb{P}(x) = xD_\mathbb{P}$. Define the value of the optimization problem under $\mathbb{P} \in \mathcal{P}$:
\[
u_\mathbb{P}(x) = \sup_{c \in \mathcal{C}_\mathbb{P}(x)} \mathbb{E}_\mathbb{P} U(c), \quad \nu_\mathbb{P}(y) = \inf_{d \in \mathcal{D}_\mathbb{P}(y)} \mathbb{E}_\mathbb{P} V(d).
\]

Then define
\[
u(x) = \inf_{\mathbb{P} \in \mathcal{P}} \nu_\mathbb{P}(x), \quad v(y) = \inf_{\mathbb{P} \in \mathcal{P}} \nu_\mathbb{P}(y).
\]

Below are the main results of this paper.
Theorem 2.1. Under Assumption $P$, we have

\begin{equation}
\label{eq:2.1}
\hat{u}(x) = \inf_{y > 0} (v(y) + xy), \quad v(y) = \sup_{x > 0} (u(x) - xy).
\end{equation}

Besides, the value function $u$ and $v$ are conjugate, i.e.,

\begin{equation}
\label{eq:2.2}
u(x) = \inf_{y > 0} (v(y) + xy), \quad v(y) = \sup_{x > 0} (u(x) - xy).
\end{equation}

Theorem 2.2. Let $x_0 > 0$. Under Assumption $P$, there exists a probability measure $\hat{P} \in \mathcal{P}$, an optimal strategy $\hat{X}_T = x_0 + (\hat{\Delta} \cdot S)_T + \hat{h} \cdot g \geq 0$, and $\hat{Y}_T \in \mathcal{Y}_P(\hat{y})$ with $\hat{y} = \hat{v}^*_P(x_0)$ such that

(i) $u(x_0) = u^*_P(x_0) = \mathbb{E}_P[U(\hat{X}_T)]$,
(ii) $v(\hat{y}) = u(x_0) - \hat{y}x_0$,
(iii) $v(\hat{y}) = \mathbb{E}_P[V(\hat{Y}_T)]$,
(iv) $\hat{X}_T = I(\hat{Y}_T)$ and $\hat{Y}_T = U'(\hat{X}_T)$, $\hat{P}$-a.s., and moreover $\mathbb{E}_P[\hat{X}_T \hat{Y}_T] = x_0\hat{y}$.

3. Proof of the main results

This section is devoted to the proof of the main results, Theorem 2.1 and Theorem 2.2.

Proof of Theorem 2.1. For $\mathbb{P} \in \mathcal{P}$ and any measurable function $f$ defined on $\mathbb{R}^J_+$, there exists a sequence of continuous functions $(f_n)_{n=1}^\infty$ converging to $f$ $\mathbb{P}$-a.s. (see e.g., Page 70 in [5]). By a truncation argument, $f_n$ can be chosen to be bounded. Therefore, we have

\[ \sup_{(\Delta, h)} \mathbb{E}_\mathbb{P}[U(x + (\Delta \cdot S)_T + h \cdot g)] = \sup_{(\Delta, h), \Delta \in \mathcal{C}_h} \mathbb{E}_\mathbb{P}[U(x + (\Delta \cdot S)_T + h \cdot g)], \]

where $\Delta \in \mathcal{C}_h$ means that each component $\Delta_j$ is a continuous bounded function on $\mathbb{R}^J_+$, $j = 1, \ldots, T-1$. Hence,

\begin{align}
\hat{u}(x) &= \sup_{(\Delta, h) \in \mathcal{P}} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_\mathbb{P}[U(x + (\Delta \cdot S)_T + h \cdot g)] \\
&\geq \sup_{(\Delta, h), \Delta \in \mathcal{C}_h} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_\mathbb{P}[U(x + (\Delta \cdot S)_T + h \cdot g)] \\
&= \inf_{\mathbb{P} \in \mathcal{P}} \sup_{(\Delta, h), \Delta \in \mathcal{C}_h} \mathbb{E}_\mathbb{P}[U(x + (\Delta \cdot S)_T + h \cdot g)] \\
&\geq \hat{u}(x),
\end{align}

where (3) follows from the minimax theorem. Hence (2) is proved. The rest of this theorem can be proved following the arguments in the proofs of Lemmas 7 and 8 in [4].

Theorem 3.1. Under Assumption $P$, for any given $\mathbb{P} \in \mathcal{P}$, the set $\mathcal{C}_\mathbb{P}$, $\mathcal{D}_\mathbb{P}$ defined in (1) satisfy the properties in Proposition 3.1 in [6].

Proof. It is obvious that (i) $\mathcal{C}_\mathbb{P}$ and $\mathcal{D}_\mathbb{P}$ are convex and solid, (ii) $\mathcal{C}_\mathbb{P}$ contains the constant function 1, and (iii) for any $c \in \mathcal{C}_\mathbb{P}, d \in \mathcal{D}_\mathbb{P}$, we have $\mathbb{E}_\mathbb{P}[cd] \leq 1$. We will finish the proof by showing the next four lemmas, where we use the notation $d\mathbb{Q}/d\mathbb{P}$ to denote both the Radon-Nikodym process and the Radon-Nikodym derivative on the whole space $\mathbb{R}^T_+$, whenever $\mathbb{Q} \sim \mathbb{P}$. 

\[ \mathbb{Q} \]
Lemma 3.2. $C_\mathbb{P}$ is bounded in $L^0(P)$.

Proof. By Assumption P(2), there exists $Q \in \mathcal{M}$ that is equivalent to $\mathbb{P}$. Then

$$\sup_{c \in C_\mathbb{P}} \mathbb{E}_\mathbb{P} \left[ \frac{dQ}{d\mathbb{P}} c \right] = \sup_{c \in C_\mathbb{P}} \mathbb{E}_Q[c] \leq 1.$$  

Hence,

$$\sup_{c \in C_\mathbb{P}} \mathbb{P}(c > K) = \sup_{c \in C_\mathbb{P}} \mathbb{P} \left( \frac{dQ}{d\mathbb{P}} c > \frac{dQ}{d\mathbb{P}} K \right) \leq \sup_{c \in C_\mathbb{P}} \left[ \mathbb{P} \left( \frac{dQ}{d\mathbb{P}} \leq \frac{1}{\sqrt{K}} \right) + \mathbb{P} \left( \frac{dQ}{d\mathbb{P}} c > \sqrt{K} \right) \right] \leq \mathbb{P} \left( \frac{dQ}{d\mathbb{P}} \leq \frac{1}{\sqrt{K}} \right) + \frac{1}{\sqrt{K}} \sup_{c \in C_\mathbb{P}} \mathbb{E}_\mathbb{P} \left[ \frac{dQ}{d\mathbb{P}} c \right]$$

$$\leq \mathbb{P} \left( \frac{dQ}{d\mathbb{P}} \leq \frac{1}{\sqrt{K}} \right) + \frac{1}{\sqrt{K}} \to 0, \quad K \to \infty.$$  

□

Lemma 3.3. For $c \in L^0_+(\mathbb{P})$, if $\mathbb{E}_\mathbb{P}[cd] \leq 1, \forall d \in \mathcal{D}_\mathbb{P}$, then $c \in C_\mathbb{P}$.

Proof. It can be shown that for any $Q \in \mathcal{M}(\mathbb{P}) := \{Q \in \mathcal{M} : Q \sim \mathbb{P}\}$, the process $\frac{dQ}{d\mathbb{P}}$ is in $\mathcal{G}_\mathbb{P}$. Then for $c \in L^0_+(\mathbb{P})$

$$\sup_{Q \in \mathcal{M}(\mathbb{P})} \mathbb{E}_Q[c] = \sup_{Q \in \mathcal{M}(\mathbb{P})} \mathbb{E}_\mathbb{P} \left[ \frac{dQ}{d\mathbb{P}} c \right] \leq 1.$$  

Applying the super-hedging Theorem on Page 6 in [1], we have that there exists a trading strategy $(\Delta, h)$, such that $1 + (\Delta \cdot S)_T + h \cdot g \geq c$, $\mathbb{P}$-a.s., and thus $c \in C_\mathbb{P}$. □

Lemma 3.4. For $d \in L^0_+(\mathbb{P})$, if $\mathbb{E}_\mathbb{P}[cd] \leq 1, \forall c \in C_\mathbb{P}$, then $d \in \mathcal{D}_\mathbb{P}$.

Proof. Let $d \in L^0_+(\mathbb{P})$ satisfying $\mathbb{E}_\mathbb{P}[cd] \leq 1, \forall c \in C_\mathbb{P}$. Then applying Proposition 3.1 in [6] (here the space $C_\mathbb{P}$ is larger than $\mathcal{C}$ defined in (3.1) in [6]), we have that there exists $\tilde{Y} \in \mathcal{Y}_\mathbb{P}(1)$, such that $0 \leq d \leq \tilde{Y}_T$. Define

$$Y_k = \begin{cases} \tilde{Y}_k, & k = 0, \ldots, T - 1, \\ d, & k = T. \end{cases}$$

Then it’s easy to show that $Y \in \mathcal{G}_\mathbb{P}$, and therefore $d \in \mathcal{D}_\mathbb{P}$ since $d = Y_T$. □

Lemma 3.5. $C_\mathbb{P}$ and $\mathcal{D}_\mathbb{P}$ are closed in the topology of convergence in measure.

Proof. Let $\{c_n\}_{n=1}^\infty \subset C_\mathbb{P}$ converge to some $c$ in probability with respect to $\mathbb{P}$. By passing to a subsequence, we may without loss of generality assume that $c_n \to c \geq 0$, $\mathbb{P}$-a.s. Then for any $d \in \mathcal{D}_\mathbb{P}$,

$$\mathbb{E}_\mathbb{P}[cd] \leq \liminf_{n \to \infty} \mathbb{E}_\mathbb{P}[c_n d] \leq 1,$$

by Fatou’s lemma. Then from Lemma 3.3 we know that $c \in C_\mathbb{P}$, which shows that $C_\mathbb{P}$ is closed in the topology of convergence in measure. Similarly, we can show that $\mathcal{D}_\mathbb{P}$ is closed. □

The proof of Theorem 3.1 is completed at this stage.
Proof of Theorem 2.2. We use Theorem 3.1 to show the second equalities in (i) and (iii), as well as (iv), by applying Theorems 3.1 and 3.2 in [6]. The rest of the proof is purely convex analytic and can be done exactly the same way as that in the proofs of Lemmas 9-12 in [4]. □

4. A EXAMPLE OF \( \mathcal{P} \)

We will give an example of \( \mathcal{P} \) satisfying Assumption \( \mathcal{P} \) in this section. We assume that there exists \( M > 0 \), such that
\[
\mathcal{M}_M := \{ Q \in \mathcal{M} : Q(||S||_\infty > M) = 0 \} \neq \emptyset.
\]

Remark 4.1. The assumption above is not restrictive. For example, if we are given a finite set of prices of European call options over a finite range of strike prices at each time period, and that the prices are consistent with an arbitrage-free model, then the model can be realized on a finite probability space, see [3] for details.

Fix \( \alpha \in (0,1) \) and \( \beta \in (1, \infty) \). Let
\[
\mathcal{P} := \left\{ \mathcal{P} : \mathcal{P} \sim \mathcal{Q} \text{ and } \alpha \leq \frac{d\mathcal{P}}{d\mathcal{Q}} \leq \beta, \text{ for some } \mathcal{Q} \in \mathcal{M}_M \right\}.
\]

Remark 4.2. From the financial point of view, the boundedness condition on the Radon-Nikodym derivative \( d\mathcal{P}/d\mathcal{Q} \) means that the physical measures, which represents the personal beliefs, should not be too far away from the martingale measures.

Theorem 4.1. \( \mathcal{P} \) defined in (4) satisfies Assumption \( \mathcal{P} \).

Proof. It is obvious that \( \mathcal{P} \) is convex, nonempty, and satisfies Assumption \( \mathcal{P} \)(2). Let \( (\mathcal{P})_{n=1}^\infty \subset \mathcal{P} \). Then there exists \( \mathcal{Q}_n \in \mathcal{M}_M \) that is equivalent to \( \mathcal{P}_n \) satisfying \( \alpha \leq d\mathcal{P}_n/d\mathcal{Q}_n \leq \beta \). Since \( (\mathcal{P}_n) \) and \( (\mathcal{Q}_n) \) are supported on \( [0, M]^T \), they are tight, and thus relatively weakly compact from Prokhorov’s theorem. By passing to subsequences, we may without loss of generality assume that there exist probability measures \( \mathcal{P} \) and \( \mathcal{Q} \) supported on \( [0, M]^T \), such that \( \mathcal{P}_n \xrightarrow{w} \mathcal{P} \) and \( \mathcal{Q}_n \xrightarrow{w} \mathcal{Q} \). Since probability measures have a compact support it can be shown using the monotone class theorem that \( \mathcal{Q} \in \mathcal{M}_M \). Let \( f \) be any nonnegative, bounded and continuous function. Then
\[
\alpha \mathbb{E}_{\mathcal{Q}_n}[f] \leq \mathbb{E}_{\mathcal{P}_n}[f] \leq \beta \mathbb{E}_{\mathcal{Q}_n}[f].
\]
Letting \( n \to \infty \), we have
\[
\alpha \mathbb{E}_{\mathcal{Q}}[f] \leq \mathbb{E}_{\mathcal{P}}[f] \leq \beta \mathbb{E}_{\mathcal{Q}}[f].
\]
Hence, \( \mathcal{P} \in \mathcal{P} \), which completes the proof. □

5. AN EXTENSION

Now instead of assuming that the market has a finite number of options, we can assume that our model is calibrated to a continuum of call options with payoffs \( (S_i - K)^+ \), \( K \in \mathbb{R}_+ \) at each time \( t = i \), and price
\[
\mathcal{C}(i, K) = \mathbb{E}_{\mathcal{Q}}[(S_i - K)^+].
\]
It is well-known that knowing the marginal $S_i$ is equivalent to knowing the prices $C(i, K)$ for all $K \geq 0$; see [2]. Hence, we can assume that the marginals of the stock price $S = (S_i)_{i=1}^T$ are given by $S_i \sim \mu_i$, where $\mu_1, \ldots, \mu_T$ are probability measures on $\mathbb{R}_+$. Let

$$\mathcal{M} := \{ Q \text{ probability measure on } \mathbb{R}_+^T : S = (S_i)_{i=1}^T \text{ is } Q - \text{martingale; for } i = 1, \ldots, T, S_i \text{ has marginal } \mu_i \text{ and mean } s_0 \}.$$ 

If we make the similar assumptions as before on the utility function and $\mathcal{P}$, and if we can generalize the super-hedging theorem in [1] to the case of infinitely many options (we only need the version with one probability measure here, which is much weaker result then the full generalization of [1]), we can get similar results correspondingly. Indeed, this can be seen from the proof of Lemma 3.3.

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