The NSVZ relation and the NSVZ scheme for $\mathcal{N} = 1$ non-Abelian supersymmetric theories, regularized by higher covariant derivatives

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Abstract. We discuss, how the exact NSVZ $\beta$-function appears in $\mathcal{N} = 1$ supersymmetric non-Abelian gauge theories, regularized by higher covariant derivatives. In particular, we demonstrate that the renormalization group functions defined in terms of the bare couplings satisfy the NSVZ relation in the case of using this regularization. This occurs, because the loop integrals giving the $\beta$-function are integrals of double total derivatives with respect to loop momenta. It is also shown that for the renormalization group functions standardly defined in terms of the renormalized couplings the NSVZ scheme can be obtained if the theory is regularized by higher covariant derivatives and only powers of $\ln \Lambda/\mu$ are included into the renormalization constants. These statements are confirmed by the explicit calculations in the three-loop approximation, where the scheme dependence is essential.

1. Introduction
The exact Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) $\beta$-function [1, 2, 3, 4] is an all-order expression relating the $\beta$-function of $\mathcal{N} = 1$ supersymmetric gauge theories to the anomalous dimension of the matter superfields,

$$\beta(\alpha, \lambda) = -\frac{\alpha^2 \left(3C_2 - T(R) + C(R)_{i,j} \gamma_j^{i}(\alpha, \lambda)/r\right)}{2\pi(1 - C_2 \alpha/2\pi)}.$$  \hspace{1cm} (1)

In this equation $\alpha$ and $\lambda$ denote the gauge and Yukawa couplings, respectively, $r$ is the dimension of a simple gauge group $G$, $\text{tr} \left( (T^A T^B) \right) \equiv T(R) \delta^{AB}$, $(T^A)_{i,k} (T^A)_{j,k} \equiv C(R)_{i,j}$, and $C_2 = T(\text{Adj})$. If the matter superfields are absent, then the NSVZ relation gives the all-order $\beta$-function of the $\mathcal{N} = 1$ supersymmetric Yang–Mills (SYM) theory, which appears to be the geometric series.

Three- and four-loop calculations made for $\mathcal{N} = 1$ supersymmetric theories with the help of dimensional reduction supplemented by the modified minimal subtractions demonstrate [5, 6, 7, 9] that the NSVZ relation in the DR-scheme is valid only for the one- and two-loop $\beta$-function. In these approximations the renormalization group functions (RGFs) entering the NSVZ equation are scheme independent. (The scheme-independent two-loop $\beta$-function is related to the scheme-independent one-loop anomalous dimension of the matter superfields.) However, it turned out that in the three- and four-loop approximations the NSVZ relation can be restored by a specially tuned finite renormalization of the gauge coupling constant. It should be
noted that the possibility of this tuning is highly nontrivial. This implies that the disagreement can be explained by the scheme-dependence of the NSVZ equation. Thus, the NSVZ relation takes place only in certain (NSVZ) subtraction schemes, which do not include the DR-scheme.

Here we will discuss how to derive the NSVZ relation and construct the NSVZ scheme in all loops with the help of the higher covariant derivative (HD) regularization proposed by A.A.Slavnov in [10, 11] in subsequently generalized to the supersymmetric case in [12, 13].

2. The NSVZ relation with the higher derivative regularization in $\mathcal{N} = 1$ SQED

First, let us briefly discuss the simplest case of the $\mathcal{N} = 1$ supersymmetric electrodynamics (SQED) with $N_f$ flavours, for which the exact NSVZ equation [14, 15] relates the $\beta$-function in a certain loop to the anomalous dimension in the previous loop.

With the HD regularization the NSVZ relation is valid in all orders [16, 17] for the (scheme-independent for a fixed regularization) RGFs defined in terms of the bare coupling constant

$$\beta(\alpha_0) \equiv \frac{d\alpha_0(\alpha, \Lambda/\mu)}{d\ln \Lambda}|_{\alpha = \text{const}}; \quad \gamma(\alpha_0) \equiv -\frac{d\ln Z(\alpha, \Lambda/\mu)}{d\ln \Lambda}|_{\alpha = \text{const}},$$  \hspace{1cm} (2)

where $\alpha_0$ and $\alpha$ are the bare and renormalized coupling constants, respectively. $\Lambda$ is the dimensionful regularization parameter playing the role of the ultraviolet cut-off, and $\mu$ denotes the renormalization point. For RGFs (2) the equation

$$\beta(\alpha_0) = \frac{\alpha_0^2 N_f}{\pi} \left( 1 - \gamma(\alpha_0) \right)$$  \hspace{1cm} (3)

is valid for an arbitrary renormalization prescription in the case of using the HD regularization. Note that in the case of using the regularization by dimensional reduction it is not so, see [18].

The NSVZ equation for RGFs defined in terms of the bare couplings appears in the perturbation theory because the loop integrals giving $\beta(\alpha_0)$ are integrals of total [19] and even double total derivatives [20] in the momentum space in the case of using the HD regularization. For example, in the two-loop approximation (see, e.g., [21])

$$\beta(\alpha_0) = N_f \sum_I c_I \int \frac{d^4q}{(2\pi)^4} \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q^\nu} \ln(q^2 + M_I^2) q^2 + 4\pi \int \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4}$$

$$\times c_1^2 \frac{\partial}{\partial q^\mu} \frac{1}{k^2 R_k} \frac{\partial}{\partial q^\nu} \frac{1}{q^2(k + q)^2} - \sum_I c_I \frac{1}{(q^2 + M_I^2)((k + q)^2 + M_I^2)} + O(\varepsilon_0^4).$$  \hspace{1cm} (4)

The double total derivatives cut matter lines producing contributions to the anomalous dimension, while by attaching two gauge lines in all possible ways we obtain contributions to the $\beta$-function [20], see Fig. 1 as an illustration.

**Figure 1.** Cutting internal lines in a vacuum supergraph we obtain superdiagrams contributing to the anomalous dimension. From the other side, attaching two external gauge lines in all possible way to the points denoted by disks, we obtain superdiagrams contributing to $\beta(\alpha_0)$. 

Note that RGFs (2) should be distinguished from RGFs standardly defined in terms of the renormalized coupling constant,
\[
\tilde{\beta}(\alpha) = \frac{d\alpha_0(\alpha, \Lambda/\mu)}{d \ln \mu} \bigg|_{\alpha_0=\text{const}}; \quad \tilde{\gamma}(\alpha) = \frac{d \ln Z(\alpha_0, \Lambda/\mu)}{d \ln \mu} \bigg|_{\alpha_0=\text{const}}.
\]

RGFs defined by these equations appear to be scheme-dependent and satisfy the NSVZ equation only for special (NSVZ) renormalization prescriptions. One of these prescriptions (so-called HD+MSL) is obtained with the HD regularization, when Minimal Subtractions of Logarithms are used for renormalization. In this case only powers of \(\ln \Lambda/\mu\) are included into the renormalization constants and all finite constants are set to 0. Then according to \([22, 23, 24]\) both definitions of RGFs give the same result up to the change of the argument,

\[
\beta(\alpha_0 \rightarrow \alpha) = \tilde{\beta}(\alpha); \quad \gamma(\alpha_0 \rightarrow \alpha) = \tilde{\gamma}(\alpha).
\]

Taking into account that RGFs (2) satisfy Eq. (3) in all orders, we see that the HD+MSL prescription gives the NSVZ scheme in all loops. However, this NSVZ scheme is not unique. Various NSVZ schemes are related by the finite renormalizations of the form \(\alpha' = \alpha(\Lambda/\mu)\), \(Z'(\alpha', \Lambda/\mu) = z(\alpha)Z(\alpha, \Lambda/\mu)\) which satisfy the equation \([25]\)

\[
\frac{1}{\alpha'(\alpha)} - \frac{1}{\alpha} - \frac{N_f}{\pi} \ln z(\alpha) = B,
\]

where a constant \(B\) appears due to an arbitrariness in choosing the renormalization point \(\mu\). According to \([26]\), another example of an all-loop NSVZ prescription is the on-shell scheme. This implies that both the HD+MSL and on-shell schemes belong to the class of the NSVZ schemes.

3. The NSVZ relation in the non-Abelian case

For non-Abelian \(\mathcal{N} = 1\) supersymmetric theories the NSVZ equation (1) relates the \(\beta\)-function to the matter superfield anomalous dimension in all previous loops due to the coupling constant dependent denominator. Moreover, it has not a qualitative interpretation similar to the Abelian case, because the cuts of internal lines can also give superdiagrams contributing to the anomalous dimensions of the Faddeev–Popov ghosts and of the quantum gauge superfields. However, it is possible to rewrite the NSVZ equation in a different (but equivalent) form. This can be done with the help of the non-renormalization theorem for the triple gauge-ghost vertices \([27]\). This theorem states that the three-point vertices with two ghost legs and a single leg of the quantum gauge superfield are finite in all loops. There are 4 such vertices corresponding to \(\bar{c}Vc, \bar{c}cVc, \bar{c}cVc^+,\) and \(\bar{c}cVc^+\) external lines. All of them have the same renormalization constant \(Z_{\alpha}^{-1/2}Z_cZ_V\). Therefore, their finiteness leads to the identity

\[
\frac{d}{d \ln \lambda}(Z_{\alpha}^{-1/2}Z_cZ_V) = 0,
\]

where the renormalization constants are defined by the equations \(Z_\alpha = \alpha/\alpha_0, V = Z_V Z_{\alpha}^{-1/2}V_R, \bar{c}c = Z_cZ_{\alpha}^{-1}\bar{c}Rc_R,\) and \(\phi_i = (\sqrt{Z_{\phi}})\phi(R)\). Therefore, it is possible to choose such a renormalization prescription that \(Z_{\alpha}^{-1/2}Z_cZ_V = 1\).

For RGFs defined in terms of the bare couplings (see Eq. (18) below) the NSVZ equation can be equivalently presented in the form

\[
\frac{\beta(\alpha_0, \lambda_0)}{\alpha_0^2} = -\frac{3C_2 - T(R) + C(R)\gamma_R(\gamma_\phi)\gamma(\alpha_0, \lambda_0)/r}{2\pi} + \frac{C_2}{2\pi} \frac{\beta(\alpha_0, \lambda_0)}{\alpha_0}.
\]

The \(\beta\)-function in the right hand side can be expressed in terms of the anomalous dimensions of the quantum gauge superfield \((\gamma_V)\) and of the Faddeev–Popov ghosts \((\gamma_c)\) with the help of the non-renormalization theorem for the triple gauge-ghost vertices,
\[
\beta(\alpha_0, \lambda_0) = -\alpha_0 \frac{d \ln Z_\alpha}{d \ln \Lambda} \bigg|_{\alpha, \lambda = \text{const}} = -2\alpha_0 \frac{d \ln (Z_c Z_V)}{d \ln \Lambda} \bigg|_{\alpha, \lambda = \text{const}} = 2\alpha_0 \left( \gamma_c(\alpha_0, \lambda_0) + \gamma_V(\alpha_0, \lambda_0) \right).
\]

Substituting this expression into the right hand side of Eq. (9) one obtains the equivalent form of the NSVZ relation

\[
\frac{\beta(\alpha_0, \lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi} \left( 3C_2 - T(R) - 2C_2 \gamma_c(\alpha_0, \lambda_0) - 2C_2 \gamma_V(\alpha_0, \lambda_0) + C(R) \epsilon(\gamma_\phi)_j^i(\alpha_0, \lambda_0)/r \right). 
\]

This equation relates the \(\beta\)-function in a certain loop to the anomalous dimensions of the quantum superfields in the previous loop and does not contain the coupling dependent denominator in the right hand side. Its graphical interpretation is similar to the Abelian case, because cuts of internal lines in vacuum supergraphs produce superdiagrams contributing to \(\gamma_V\), \(\gamma_c\), and \((\gamma_\phi)_j^i\). This allows suggesting that it is this equation that follows from the perturbative calculations. This guess was confirmed by an explicit calculation in the two-loop approximation, see [28].

4. Outline of the all-loop derivation

The new form of the NSVZ equation for RGFs defined in terms of the bare couplings (11) can be derived in all orders in the case of using the higher covariant derivative regularization. The proof is based on the method proposed in Refs. [16, 17]. Here we briefly outline its main steps.

1. Let us consider the two-point Green function of the background gauge superfield \(V\)

\[
\Gamma^{(2)}_V = -\frac{1}{8\pi} \text{tr} \int \frac{d^4p}{(2\pi)^4} d^4\theta V(-p, \theta) \partial^2 \Pi_{1/2} V(p, \theta) d^{-1}(\alpha_0, \lambda_0, \Lambda/p) 
\]

and make the formal replacement \(V^A \rightarrow \theta^a v^A\), where \(v^A\) is a function slowly decreasing at a scale \(R \rightarrow \infty\). This allows extracting the \(\beta\)-function defined in terms of the bare couplings,

\[
\frac{d\Delta \Gamma^{(2)}_V}{d \ln \Lambda} = \frac{\mathcal{V}_4}{2\pi d \ln \Lambda} \left( d^{-1}(\alpha_0, \lambda_0, \Lambda/p) - \alpha_0^{-1} \right) \bigg|_{\alpha, \lambda = \text{const}; \ p \rightarrow 0} = \frac{\mathcal{V}_4}{2\pi} \frac{\beta(\alpha_0, \lambda_0)}{\alpha_0^2},
\]

where

\[
\Delta \Gamma^{(2)}_V \equiv \Gamma^{(2)}_V - S^{(2)}_V \\
\mathcal{V}_4 \equiv \int d^4x (v^A)^2.
\]

2. Using the rules for calculating supergraphs and the identity [17]

\[
\theta^2 A B \theta^2 + 2(-1)^{P_a + P_\phi} \theta^a A \theta^2 B \theta_a - \theta^2 A \theta^2 B - A \theta^2 B \theta^2 = O(\theta),
\]

where \((-1)^{P_a}\) and \((-1)^{P_\phi}\) are the Grassmannian parities of \(A\) and \(B\) (which are constructed from supersymmetric and usual derivatives and superspace \(\delta\)-functions), the expression for the \(\beta\)-function can be reduced to a formally vanishing integral of double total derivatives [29]. This integral is similar to

\[
\int \frac{d^4q}{(2\pi)^4} \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q_{\mu}} \left( f\left(q^2\right)/q^2 \right) = \frac{1}{8\pi^4} \int \frac{dS}{S^3} \frac{f\left(q^2\right)/q^2}{q^3} = \frac{1}{4\pi^2} f(0),
\]

where \(f\left(q^2\right)\) is a non-singular function rapidly decreasing at infinity. This implies that \(\beta(\alpha_0, \lambda_0)\) beyond the one-loop approximation is given by the sum of singular contributions.
3. The sum of various singular contributions was found to be

$$\frac{\beta(\alpha_0, \lambda_0)}{\alpha_0^2} - \frac{\beta_{1\text{-loop}}(\alpha_0)}{\alpha_0^2} = \frac{1}{\pi} C_2 \gamma_V(\alpha_0, \lambda_0) + \frac{1}{\pi} C_2 \gamma_c(\alpha_0, \lambda_0) - \frac{1}{2\pi r} C(R)_i^j(\gamma_\phi)_j^i(\alpha_0, \lambda_0). \quad (17)$$

(The paper describing the details of this calculation is in preparation.) After substituting the one-loop contribution to the $\beta$-function $\beta_{1\text{-loop}}(\alpha_0) = -\alpha_0^2(3C_2 - T(R))/2\pi$ we obtain the new form of the NSVZ relation.

5. The NSVZ equation and various definitions of RGFs

The reasoning sketched above allows deriving the NSVZ equation for RGFs defined in terms of the bare couplings

$$\beta(\alpha_0, \lambda_0) \equiv \frac{d\alpha_0}{d\ln \Lambda}; \quad (\gamma_\phi)_i^j(\alpha_0, \lambda_0) \equiv -\frac{d\ln(Z_\phi)_i^j}{d\ln \Lambda};$$

$$\gamma_V(\alpha_0, \lambda_0) \equiv \frac{d\ln Z_V}{d\ln \Lambda}; \quad \gamma_c(\alpha_0, \lambda_0) \equiv -\frac{d\ln Z_c}{d\ln \Lambda}. \quad (18)$$

For a fixed regularization they do not depend on the renormalization prescription. Therefore, the NSVZ equation for these RGFs is valid in an arbitrary subtraction scheme in the case of using the higher covariant derivative regularization. However, RGFs defined in terms of the renormalized couplings

$$\tilde{\beta}(\alpha, \lambda) \equiv \frac{d\alpha}{d\ln \mu}; \quad (\tilde{\gamma}_\phi)_i^j(\alpha, \lambda) \equiv -\frac{d\ln(Z_\phi)_i^j}{d\ln \mu};$$

$$\tilde{\gamma}_V(\alpha, \lambda) \equiv \frac{d\ln Z_V}{d\ln \mu}; \quad \tilde{\gamma}_c(\alpha, \lambda) \equiv -\frac{d\ln Z_c}{d\ln \mu} \quad (19)$$

are scheme dependent and satisfy the NSVZ equation only in certain (NSVZ) schemes.

Similarly to the $\mathcal{N} = 1$ SQED case, one of the NSVZ schemes for non-Abelian $\mathcal{N} = 1$ supersymmetric theories is given by the HD+MSL prescription [27],

$$\text{HD+MSL} = \text{NSVZ}, \quad (20)$$

because in this case both definitions of RGFs give the same result,

$$\beta(\alpha_0 \rightarrow \alpha, \lambda_0 \rightarrow \lambda) = \tilde{\beta}(\alpha, \lambda); \quad (\gamma_\phi)_i^j(\alpha_0 \rightarrow \alpha, \lambda_0 \rightarrow \lambda) = (\tilde{\gamma}_\phi)_i^j(\alpha, \lambda);$$

$$\gamma_V(\alpha_0 \rightarrow \alpha, \lambda_0 \rightarrow \lambda) = \tilde{\gamma}_V(\alpha, \lambda); \quad \gamma_c(\alpha_0 \rightarrow \alpha, \lambda_0 \rightarrow \lambda) = \tilde{\gamma}_c(\alpha, \lambda). \quad (21)$$

6. Explicit perturbative three-loop calculation for terms containing Yukawa couplings

The above results have been verified by the explicit calculation of terms containing the Yukawa couplings in the three-loop $\beta$-function [30, 31]. They are generated by the supergraphs presented in Fig. 2 to which one should attach two external lines of the background gauge superfield in all possible ways. From the other side, cuts of the internal lines in these graphs produce various two-loop contributions to the anomalous dimensions of the quantum gauge superfield and of the matter superfields. (For the supergraphs under consideration superdiagrams contributing to the anomalous dimension of the Faddeev–Popov ghosts do not appear.)
Fig. 2. Supergraphs generating the three-loop contributions to the β-function containing the Yukawa couplings.

Expressions for all graphs obtained after the direct calculation of the two-point superdiagrams and summing the results turned out to be given by integrals of double total derivatives. This makes possible the analytic evaluation of one of the loop integrals, which has lead to the relations

\[
\Delta_A \left( \frac{\beta(\alpha_0, \lambda_0)}{\alpha_0^2} \right) = \frac{1}{\pi} C_2 \Delta_A \gamma_V(\alpha_0, \lambda_0) - \frac{1}{2\pi r} C(R)_j (\Delta_A \gamma_\phi)_j(\alpha_0, \lambda_0)
\]

for all considered graphs. For instance, the terms quartic in the Yukawa couplings (first found in [30]) are written as

\[
\Delta \beta(\alpha_0, \lambda_0) \frac{\alpha_0}{\alpha_0^2} = -\frac{2\pi}{r} C(R)_j \frac{d}{d\ln \Lambda} \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \chi^0_{i00j0j} \frac{\partial}{\partial q^\mu} \left( \frac{1}{k^2 F_k q^2 F_q (q + k)^2 F_{q+k}} \right) + \frac{4\pi}{r} C(R)_j \frac{d}{d\ln \Lambda} \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \left( \chi^0_{i00j0j} \chi^0_{000i0j} \frac{\partial}{\partial k^\mu} \frac{\partial}{\partial q^\mu} + 2\chi^0_{i00j0j} \chi^0_{000i0j} \right) \times \chi^0_{000i0j} \chi^0_{000i0j} \chi^0_{000i0j}
\]

For all supergraphs such integrals have been calculated for the higher derivative regulators \( F(x) = 1 + x^n \); \( R(x) = 1 + x^m \). The result for RGFs defined in terms of the bare couplings appear to be

\[
\frac{\beta(\alpha_0, \lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi} (2 C_2 - T(R)) - \frac{1}{2\pi r} C(R)_j \left( \frac{1}{4\pi^2} \chi^0_{i00j0j} \chi^0_{000i0j} \chi^0_{000i0j} C_2 + \frac{\alpha_0}{8\pi^3} \chi^0_{i00j0j} \chi^0_{000i0j} \chi^0_{000i0j} C_2 \right)
\]

\[
+ O(\alpha_0^2, \lambda_0^4) + \text{terms without the Yukawa couplings}
\]

(24)

\[
(\gamma_\phi)_j (\alpha_0, \lambda_0) = -\frac{\alpha_0}{\pi} C(R)_j + \frac{1}{4\pi^2} \chi^0_{i00j0j} \chi^0_{000i0j} - \frac{\alpha_0}{8\pi^3} \chi^0_{i00j0j} \chi^0_{000i0j} C(R)_j \left( 1 - \frac{1}{n} \right)
\]

\[
+ \frac{\alpha_0}{4\pi^3} \chi^0_{i00j0j} \chi^0_{000i0j} C(R)_j \left( 1 - \frac{1}{n} \right)
\]

\[
\gamma_V(\alpha_0, \lambda_0) = -\frac{\alpha_0}{\pi} (3 C_2 - T(R)) - \frac{\alpha_0}{16\pi^3} \chi^0_{i00j0j} \chi^0_{000i0j} C(R)_j + O(\alpha_0^2, \lambda_0^4)
\]

(25)

RGFs defined in terms of the renormalized couplings are

\[
\frac{\beta(\alpha, \lambda)}{\alpha^2} = -\frac{1}{2\pi} (2 C_2 - T(R)) - \frac{1}{2\pi r} C(R)_j \left( \frac{1}{4\pi^2} \chi^0_{i00j0j} \chi^0_{000i0j} + \frac{\alpha}{8\pi^3} \chi^0_{i00j0j} \chi^0_{000i0j} C_2 \right)
\]

\[
+ \frac{\alpha}{4\pi^3} \chi^0_{i00j0j} \chi^0_{000i0j} C(R)_j \left[ b_2 - g_11 - \frac{1}{2} \left( 1 - \frac{1}{n} \right) \right] + \frac{\alpha}{2\pi^3} \chi^0_{i00j0j} \chi^0_{000i0j} C(R)_j \left[ b_2 - g_11 + \frac{1}{2} \left( 1 + \frac{1}{n} \right) \right]
\]

(26)
\[-\frac{1}{8\pi^4}\lambda_{abc}^{ab}\lambda^c_{def}\left[b_2 - g_{12} + \frac{1}{2}\right] + \frac{1}{16\pi^4}\lambda^{a}_{bde}\lambda^{cde}\left[g_{12} - b_2\right] + O(\alpha^2\lambda^2, \alpha\lambda^4, \lambda^6)\]

+ terms without the Yukawa couplings;

\[
(\tilde{\gamma}_\phi)_{ij}^{(2)}(\alpha, \lambda) = -\frac{\alpha}{\pi}C(R)_{ij}^{(2)} + \frac{1}{4\pi^2}\lambda^a_{imn}\lambda^{jin}\left[1 + \frac{1}{\pi^2}\lambda^a_{imn}\lambda^{jin}\left(\ln \frac{\Lambda}{\mu} + b_1\right)\right] + O(\alpha, \alpha^2, \alpha\lambda^4, \lambda^6);
\]

\[
(\tilde{\gamma}_V)_{ij}^{(2)}(\alpha, \lambda) = -\frac{\alpha}{\pi}\left[3C_2 - T(R)\right] - \frac{1}{16\pi^4}\lambda^a_{imn}\lambda^{jin}\left[\ln \frac{\Lambda}{\mu} + b_2\right] + O(\alpha, \alpha^2, \alpha\lambda^4, \lambda^6);
\]

The finite constants \(b_1, b_2, g_{11}, \) and \(g_{12}\) entering these expressions are defined by equations

\[
\frac{1}{\alpha} - \frac{1}{\alpha_0} = -\frac{1}{2\pi}\left(3C_2 - T(R)\right)\left[\ln \frac{\Lambda}{\mu} + b_1\right] - \frac{1}{8\pi^2}\lambda^a_{imn}\lambda^{jin}\left[\ln \frac{\Lambda}{\mu} + b_2\right] + O(\alpha, \alpha^2, \alpha\lambda^4, \lambda^6);
\]

\[
(\tilde{Z}_\phi)_{ij}^{(2)} = \delta_{ij} + \frac{\alpha}{\pi}\left[\ln \frac{\Lambda}{\mu} + g_{11}\right]C(R)_{ij}^{(2)} - \frac{1}{4\pi^2}\lambda^a_{imn}\lambda^{jin}\left(\ln \frac{\Lambda}{\mu} + g_{12}\right) + O(\alpha, \alpha^2, \alpha\lambda^4, \lambda^6);
\]

and (partially) fix the subtraction scheme in the lowest approximation. The HD+MSL prescription evidently corresponds to \(g_{11} = 0, g_{12} = 0, b_1 = b_2 = 0.\)

Comparing Eqs. (27), (28), and (29) we see that for the considered terms in the considered approximation the new form of the NSVZ relation

\[
\frac{\tilde{\beta}(\alpha, \lambda)}{\alpha^2} = -\frac{1}{2\pi}\left[3C_2 - T(R) - 2C_2\tilde{\gamma}_\phi^{(2)}(\alpha, \lambda) - 2C_2\tilde{\gamma}_V^{(2)}(\alpha, \lambda) + C(R)_{ij}^{(2)}(\tilde{\gamma}_\phi)_{ij}^{(2)}(\alpha, \lambda)/r\right]
\]

is not valid for a general renormalization prescription. However, in the HD+MSL renormalization scheme (for which \(g_{12} - g_{11} = 0, b_2 - g_{11} = 0, b_2 - g_{12} = 0\) Eq. (31) is really valid. Moreover, it is easy to see that in this case both definition of RGFs give the same functions, see Eq. (21). It is possible to verify that the original form of the NSVZ relation is also valid in this case. Thus, for the considered class of superdiagrams HD + MSL = NSVZ.

7. Conclusion

The NSVZ equation naturally appears in the perturbation theory if the higher covariant derivative method is used for regularization. However, it is obtained in the new form, which relates the \(\beta\)-function to the anomalous dimensions of the quantum gauge superfield, of the Faddeev–Popov ghosts, and of the matter superfields. With the HD regularization the NSVZ relation is satisfied by RGFs defined in terms of the bare couplings independently of the prescription used for renormalization. This relation originates from the factorization of integrals giving the \(\beta\)-function into integrals of double total derivatives. If RGFs are defined in terms of the renormalized couplings, then the NSVZ scheme in all orders is given by the HD+MSL prescription, when the theory is regularized by higher covariant derivatives and the renormalization constants include only powers of \(\ln\Lambda/\mu\).

All these general statements concerning the NSVZ relation and the NSVZ scheme have been confirmed by some explicit calculations in the three-loop approximation, where the scheme dependence is essential.

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