Frames and fusion frames in quantum optics

Andrzej Jamiołkowski
Institute of Physics, Nicholas Copernicus University, 87–100 Toruń, Poland
E-mail: jam@fizyka.umk.pl

Abstract. The main purpose of this paper is to show that the notions of frames and fusion frames introduced in non-harmonic Fourier analysis are also very natural in discussion of some basic problems in theory of open quantum systems and, in particular, in quantum optics. Frames are collections of vectors in a Hilbert space which assure a natural representation of each vector in the space, but may have infinitely many different representations for any given vector. For a given quantum system represented in a Hilbert space $\mathcal{H}$, the question of minimal number of observables $Q_1, \ldots, Q_\eta$, whose expectation values at some instants $t_1, \ldots, t_\eta$ determine the statistical state of the system is discussed. We assume that the time evolution of the system in question is governed by a semigroup of linear transformations with generator $\mathbb{L}$.

1. Introduction

It is a fundamental observation concerning quantum physics that the concept of vector space plays a prominent role in the formulation of basic notions of quantum systems theory. The observation that pure states of all microscopic systems satisfy the superposition principle led to postulate that the states of any quantum system must be represented by elements of complex linear space. On the other hand, all quantum observables should be represented by self-adjoint operators acting on this space, and as such also constitute a real vector space. A beautiful discussion of this issues was given by R. Feynman in his lectures on physics [1].

One of the main concepts in the study of vector spaces is the concept of basis for a given space, which allows every vector to be uniquely represented as a linear combination of the basis elements. However, the assumption of linear independence of vectors which constitute a basis is very restrictive; in many situations it is impossible to find vectors which both fulfill the basis requirements and also satisfy natural external conditions demanded by particular applications. We will see examples of such situation in some problems discussed in this paper. In such cases we need to look for more flexible types of spanning collections of vectors or, in more general situations, of spanning collections of subspaces. Frames and fusion frames provide these alternatives. It is a very important fact, that they not only have great variety of applications in many disciplines, but also have a rich theory from a purely mathematical point of view.

One of the basic consequences of the standard assumptions of quantum mechanics is the observation that determination of an unknown state can be achieved by appropriate measurements only if we have at our disposal a set of identically prepared copies of the system in question. Moreover, to set up a successful approach to the above problem of state reconstruction, one has to identify a collection of observables, a quorum [2], such that their expectation values contain complete information about the state of the system under consideration. The problems of state determination have gained new relevance in recent years, following the realization
that quantum systems and their evolutions can perform tasks such as teleportation, secure communication or dense coding (c.f. e.g. [3]). In this new situation the development of methods to control and measure quantum states becomes an issue of practical interest. It is important to realize that if we identify the quorum of observables, then we also have a possibility to determine the expectation values of physical quantities (observables) for which no measuring apparatuses are available [4].

The idea of stroboscopic tomography for open quantum systems appeared for the first time in the beginning of 1980’s (although expressed in different terms [5, 6, 7]). In particular, using the concept of observability, in [6] and [7] the question of the minimal number of observables \( Q_1, \ldots, Q_n \) for which the quantum systems can be \((Q_1, \ldots, Q_n)\)-reconstructible was discussed.

On the other hand, theory of frames, which are collections of vectors that provide robust and usually non-unique representations of vectors, has been the subject of research in the last two decades and has been applied in these disciplines where redundancy played a vital and useful role. However, in some applications it is natural to model and describe considered systems by collections of families of subspaces, and to split a large frame system into a set of much smaller frame systems in these subspaces. This has led to the development of a suitable theory based on fusion frames (families of subspaces), which provides the framework to model these more complex applications [8, 9]. In particular, a sequence of the so-called Krylov subspaces which appear naturally in stroboscopic tomography [7] and are defined by

\[
K_k(\ell, Q) := \text{Span}_{\mathbb{R}} \left\{ Q, \ell Q, \ldots, \ell^{k-1} Q \right\},
\]

where \( Q \) is a fixed observable and \( \ell \) is a generator of time evolution of the system in question, constitutes a fusion frame in the Hilbert-Schmidt space \( B_h(\mathcal{H}) \) (cf. below Sect 3.) if

\[
\bigoplus_{i=1}^{r} K_{\mu}(\ell, Q_i) = B_h(\mathcal{H}).
\]

In the above equality \( \mu \) denotes the degree of the minimal polynomial of the superoperator \( \ell \) and \( Q_1, \ldots, Q_r \) represent fixed observables. The symbol \( \bigoplus \) denotes Minkowski sum of subspaces (2) (cf. e.g. [10]). We recall that for two subspaces \( K_1 \) and \( K_2 \) of the vector space \( \mathcal{H} \), by \( K_1 \bigoplus K_2 \) one understands the smallest subspace of \( \mathcal{H} \) which contains \( K_1 \) and \( K_2 \).

It is well known that the Krylov subspaces \( K_k(\ell, Q) \) for \( k = 1, 2, \ldots \) form a nested sequence of subspaces of increasing dimensions that eventually become invariant under \( \ell \). Hence for a given \( Q \), there exists an index \( \mu = \mu(Q) \), often called the grade of \( Q \) with respect to \( \ell \) for which

\[
K_1(\ell, Q) \subseteq \cdots \subseteq K_{\mu}(\ell, Q) = K_{\mu+1}(\ell, Q) = K_{\mu+2}(\ell, Q) \cdots.
\]

It is easy to see, that for a given operator \( Q \), \( \mu(Q) \) is equal to the degree of the minimal polynomial of \( \ell \) with respect of \( Q \). Clearly, \( \mu(Q) \leq \mu(\ell) \) where \( \mu(\ell) \) denotes the degree of the minimal polynomial of superoperator \( \ell \) (cf. e.g. [11]). Now, let us observe that even if observables \( Q_1, \ldots, Q_r \) are linearly independent, the Krylov subspaces \( K_k(\ell, Q_i) \) for \( i = 1, \ldots, r \) can have nonempty intersections. At the same time they can constitute a fusion frame for the space of all observables \( B_h(\mathcal{H}) \).

The organisation of the paper is as follows: In Section 2 we summarize some concepts and results of the theory of frames; Section 3 presents the main ideas of stroboscopic tomography. We conclude the paper in Section 4 by discussing some applications of the notions of frames and fusion frames to problems of quantum optics.

2. Frames and fusion frames
Frames were first introduced by Duffin and Schaeffer in 1952 as a natural notion that appeared during their research in nonharmonic Fourier analysis [12]. After more than three decades
Daubechies, Grossman and Meyer [13] in 1986 initiated the use of frame theory in description of signal processing. Today, frame theory plays an important role in dozens of applied areas, cf. e.g. [14, 15].

Let us consider a Hilbert space $\mathcal{H}$ ($\dim \mathcal{H} = d < \infty$) with scalar product $\langle \cdot, \cdot \rangle$ which is linear in the second argument. A collection of vectors $\mathcal{F} = \{f_i : i \in I\}, f_i \in \mathcal{H}$, is called a frame if there are two positive constants $\alpha, \beta > 0$ such that for every vector $x \in \mathcal{H}$

$$\alpha \| x \|^2 \leq \sum_{i \in I} |\langle f_i, x \rangle|^2 \leq \beta \| x \|^2. \quad (4)$$

One assumes that the number of vectors $f_i$ is greater or equal to $d$. The frame is tight when the constants $\alpha$ and $\beta$ are equal, $\alpha = \beta$. If $\alpha = \beta = 1$, then $\mathcal{F}$ is called a Parseval frame. The numbers $\langle f_i, x \rangle$ are called frame coefficients.

For a given frame $\mathcal{F}$ we can introduce the analysis $\Theta$ and synthesis $\Theta^\ast$ operators. They are defined by the equality

$$\Theta(x) = \sum_{i \in I} \langle f_i, x \rangle |e_i\rangle, \quad (5)$$

where $|e_i\rangle$ stands for standard basis in $\mathbb{C}^m$ (we will consider only finite dimensional frames, so that $I = \{1, \ldots, m\}$). Composing $\Theta$ with its adjoint operator $\Theta^\ast$, we obtain the frame operator

$$F : \mathcal{H} \rightarrow \mathcal{H}, \quad (6)$$

defined by

$$Fx := \Theta^\ast \Theta x = \sum_{i=1}^m \langle f_i, x \rangle |f_i\rangle. \quad (7)$$

It is not difficult to see that any collection of vectors $\{|f_i\rangle\}_{i=1}^m$ constitutes a frame for the vector space $\mathcal{N} := \text{span}\{|f_i\rangle\}_{i=1}^m$, $\mathcal{N} \subseteq \mathcal{H}$. On the other hand a family of elements $\{|f_i\rangle\}_{i=1}^m$ in $\mathcal{H}$ is a frame for $\mathcal{H}$ if and only if $\text{span}\{|f_i\rangle\}_{i=1}^m = \mathcal{H}$. This means that a frame might contain more elements than necessary for it to be a basis. In particular, if $\{|f_i\rangle\}_{i=1}^m$ is a frame for $\mathcal{H}$ and $\{|g_i\rangle\}_{i=1}^n$ is an arbitrary finite collection of elements in $\mathcal{H}$, then the set $\{|f_1\rangle, \ldots, |f_m\rangle, |g_1\rangle, \ldots, |g_n\rangle\}$ is also a frame for $\mathcal{H}$.

Generally speaking, frame theory is the study of how the frame operator $\Theta^\ast \Theta$ is well-conditioned. In particular, $\{|f_i\rangle\}_{i=1}^m$ is a frame for $\mathcal{H}$ if there exist frame bounds $\alpha, \beta$ such that

$$\alpha \mathbb{I} \leq \Theta^\ast \Theta \leq \beta \mathbb{I}, \quad (8)$$

and is a tight frame iff $\Theta^\ast \Theta = \alpha \mathbb{I}$. It is an obvious observation that $F = \Theta^\ast \Theta$ is a self-adjoint and invertible operator.

Fusion frame theory (theory of frames of subspaces) is an emerging mathematical theory that provides a natural framework for performing a distributed data processing in many fields [8, 9]. In particular, one can apply these ideas in quantum state tomography. The notion of fusion frame was introduced in [8] and further developed by [9]. A fusion frame in a Hilbert space $\mathcal{H} \cong \mathbb{C}^d$ is a finite collection of subspaces $\{W_i\}_{i=1}^m$ of $\mathcal{H}$, such that there exist constants $0 < \alpha < \beta < \infty$ satisfying for any $|\varphi\rangle \in \mathcal{H}$ the two inequalities

$$\alpha \| |\varphi\rangle \|^2 \leq \sum_{i=1}^m \| \text{Proj}_i |\varphi\rangle \|^2 \leq \beta \| |\varphi\rangle \|^2, \quad (9)$$
where $P_i$ denotes the orthogonal projection on $W_i$. In other words, a collection $\{W_i\}_{i=1}^m$ is a fusion frame if and only if
\[
\alpha \mathbb{I} \leq \sum_{i=1}^m P_i \leq \beta \mathbb{I}.
\]

The constants $\alpha$ and $\beta$ are called fusion frame bounds. An important class of fusion frames is the class of tight fusion frames, for which $\alpha = \beta$. This equality leads to the operator relation $\sum_{i=1}^m P_i = \mathbb{I}$.

A graphic representation of the various types of frames is given in Fig. 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{frames_types.png}
\caption{Types of frames; ENF:= Equal-norm frames, TF:= Tight frames, UNF:= Unit-norm frames, PTF:= Parseval tight frames, ONB:= Orthonormal bases}
\end{figure}

Now, let us recall that for a given operator $M : \mathcal{H} \to \mathcal{H}$ and a given fixed nonzero vector $\ket{x} \in \mathcal{H}$, one introduces the $k$th Krylov subspace of $\mathcal{H}$ by the equality
\[
K_k(M, x) := \text{span}\{\ket{x}, M\ket{x}, \ldots, M^{k-1}\ket{x}\}.
\]

The above definition can also be written as
\[
K_k(M, x) := \text{span}\{p(M)\ket{x} ; \deg(p) \leq k - 1\},
\]

where $p$ denotes an arbitrary polynomial and $\deg(p)$ is its degree. It is an obvious observation that the size of a Krylov subspace depends on both $M$ and $\ket{x}$. Note also that there exists such $k$ that $K_k(M, x) = K_{k+1}(M, x)$ and this $k$ is the degree of the minimal polynomial of $M$ with respect to $\ket{x}$. If by $\mu(\lambda, M)$ we denote the minimal polynomial of the operator $M$, then the minimal polynomial of $M$ with respect to any vector $\ket{x} \in \mathcal{H}$ divides $\mu(\lambda, M)$.

For a given operator $M : \mathcal{H} \to \mathcal{H}$ Krylov subspaces generated by a fixed set of vectors $\{\ket{x_1}, \ldots, \ket{x_r}\}$ constitute a fusion frame in $\mathcal{H}$ if the following equality is satisfied
\[
\bigoplus_{i=1}^r K_{\mu}(M, x_i) = \mathcal{H},
\]
where \( \boxplus \) denotes the so-called Minkowski sum of subsets of \( \mathcal{H} \) and \( \mu := \deg \mu(\lambda, M) \). It is important that we have to use the notion of Minkowski sum of subspaces instead of direct sum. This fact follows from the observation that even if the vectors \( |x_1\rangle, \ldots, |x_r\rangle \) are linearly independent, then the Krylov subspaces \( \mathcal{K}_\mu(M, x_i), \; i = 1, \ldots, r \) can partially overlap. It can be easily seen that every Krylov subspace \( \mathcal{K}_\mu(M, x) \) is essentially spanned on all vectors of the form \( M^k|x\rangle \), where \( k = 0, 1, \ldots \), and it is the smallest invariant subspace of the operator \( M \) containing \( |x\rangle \).

### 3. Stroboscopic tomography of open quantum systems

During the two decades – 1960s and 1970s – the theory of quantum isolated systems was generalized. The so-called theory of open quantum systems (systems interacting with their environments) was established, and the main sources of inspiration for it were the quantum optics and the theory of lasers. This led to the generalization of states (now density operators are considered as natural representation of quantum states), and to generalized description of time evolution of quantum states. At that time the concepts of so-called quantum master equations – which preserve positive semi-definiteness of density operators – and the idea of quantum communication channel were born, cf. e.g. [16, 17, 18, 19, 20, 21]. On the mathematical level, this approach initiated the study of semigroups of completely positive maps and their generators.

Now we summarize the main ideas and methods of description of open quantum systems.

The time evolution of a quantum system of finitely many degrees of freedom (a qudit), coupled with an infinite quantum system, usually called reservoir, can be described, under certain limiting conditions, by a one-parameter semigroup of maps (cf. e.g. [5, 17]). Let \( \mathcal{H} \) be the Hilbert space of the first system (\( \dim \mathcal{H} = d \)) and let

\[
\Phi(t) : \mathcal{B}_h(\mathcal{H}) \to \mathcal{B}_h(\mathcal{H}), \quad t \in \mathbb{R}_1^+, \tag{14}
\]

be a dynamical semigroup, where \( \mathcal{B}_h(\mathcal{H}) \) denotes the real vector space of all self-adjoint operators on \( \mathcal{H} \). If one introduces the scalar product of operators \( A, B \) by the formula \( (A, B) = \text{Tr}(A^*B) \), then \( \mathcal{B}_h(\mathcal{H}) \) can be considered as another inner product space, namely so-called Hilbert-Schmidt space with norm defined by \( \| \rho \|^2 := \text{Tr}(\rho^*\rho) \). States of the system are described by density operators \( \rho \in S(\mathcal{H}) \), where

\[
S(\mathcal{H}) := \{ \rho \in \mathcal{B}_h(\mathcal{H}); \; \rho \geq 0, \text{Tr} \rho = 1 \} . \tag{15}
\]

Usually one assumes that the family of linear superoperators \( \Phi(t) \) satisfies

(i) \( \Phi(t) \) is trace preserving, \( t \in \mathbb{R}_1^+ \),
(ii) \( \| \Phi(t)\rho \| \leq \| \rho \| \) for all \( \rho \in \mathcal{B}_h(\mathcal{H}) \),
(iii) \( \Phi(t_1) \circ \Phi(t_2) = \Phi(t_1 + t_2) \),

for all \( t_1, t_2 \) from \( \mathbb{R}_1^+ \), and if \( t \to 0 \), then \( \lim \Phi(t) = \mathbb{I} \). Since such defined \( \Phi(t) \) is a contraction, it follows from the Hille-Yosida theorem [17, 5] that there exists a linear superoperator \( \mathbb{K} : \mathcal{B}_h(\mathcal{H}) \to \mathcal{B}_h(\mathcal{H}) \) such that \( \Phi(t) = \exp(t\mathbb{K}) \) for all \( t \geq 0 \) and

\[
\frac{d\rho(t)}{dt} = \mathbb{K}\rho(t), \tag{16}
\]

where \( \rho(t) = \Phi(t)\rho(0) \). One should stress that above conditions for semigroup \( \Phi(t) \) imply preservation of positivity of density operators, \( \rho(0) \geq 0 \Rightarrow \rho(t) = \Phi(t)\rho(0) \geq 0 \) for all \( t \in \mathbb{R}_1^+ \).

Now, the above equation (usually called master equation) defines the assignment (trajectory of \( \rho(0) \))

\[
\mathbb{R}_1^+ \ni t \mapsto \rho(t) \in S(\mathcal{H}), \tag{17}
\]
provided that we know the initial state of the system $\rho(0) \in S(\mathcal{H})$. The fundamental question of stroboscopic tomography reads: What can we say about the trajectories (initial state $\rho(0)$) if the only information about the system in question is given by the mean values

$$E_i(t_j) = \text{Tr} (Q_i \rho(t_j)),$$

(18)
of, say $r$, linearly independent self-adjoint operators $Q_1, \ldots, Q_r$ at some time instants $t_1, \ldots, t_p$, where $r < d^2 - 1$ and $t_j \in [0, T]$ for $j = 1, \ldots, p$, $T > 0$. In other words, the problem of the stroboscopic tomography consists in the reconstruction of the initial state $\rho(0)$, or a current state $\rho(t)$, for any $t \in \mathbb{R}_+$, by known expectation values (18). To be more precise we introduce the following description. Suppose that we can prepare a quantum system repeatedly in the same initial state and we make a series of experiments such that we know the expectation values $E_Q(t) = \text{Tr} (Q \rho(t))$ for a fixed set of observables $Q_1, \ldots, Q_r$ at different time instants $t_1 < t_2 < \cdots < t_p$. The basic question is:

Can we find the expectation value of any other operator $Q \in \mathcal{B}_h(\mathcal{H})$, that is any other observable from $\mathcal{B}_h(\mathcal{H})$, knowing the set of measured outcomes of a given set $Q_1, \ldots, Q_r$ at $t_1, \ldots, t_p$, i.e.

$$
\begin{bmatrix}
E_1(t_1) & E_2(t_1) & \cdots & E_r(t_1) \\
E_1(t_2) & E_2(t_2) & \cdots & E_r(t_2) \\
\vdots & \vdots & \ddots & \vdots \\
E_1(t_p) & E_2(t_p) & \cdots & E_r(t_p)
\end{bmatrix}
\begin{bmatrix}
t_1 \\
t_2 \\
\vdots \\
t_p
\end{bmatrix}
$$

(19)

where $0 \leq t_1 < t_2 < \cdots < t_p \leq T$, for an interval $[0, T]$?

If the problem under consideration is static, then the state of a $d$-level open quantum system (a qudit) can be uniquely determined only if $r = d^2 - 1$ expectation values of linearly independent observables are at our disposal. However, if we assume that we know the dynamics of our system i.e. we know the generator $\mathbb{K}$ or $L := (\mathbb{K})^*$ (in Heisenberg picture) of the time evolution, then we can use the stroboscopic approach based on (19). In general, we use the term “state-tomography” to denote any kind of state-reconstruction method.

With reference to the terminology used in system theory, we introduce the following definition:

A $d$-level open quantum system $\mathbb{S}$ is said to be $(Q_1, \ldots, Q_r)$-reconstructible on the interval $[0, T]$, if for every two trajectories defined by the equation (16) there exists at least one instant $\hat{t} \in [0, T]$ and at least one operator $Q_k \in \{Q_1, \ldots, Q_r\}$ such that

$$\text{Tr} (Q_k \rho_1(\hat{t})) \neq \text{Tr} (Q_k \rho_2(\hat{t})).$$

(20)
The above definition is equivalent to the following statement [5, 11]. A $d$-level open quantum system $\mathbb{S}$ is $(Q_1, \ldots, Q_r)$-reconstructible on the interval $[0, T]$, if there exists at least one set of time instants $0 < t_1 < \cdots < t_p \leq T$ such that the state trajectory can be uniquely determined by the correspondence

$$[0, T] \ni t_j \mapsto E_i(t_j) = \text{Tr} (Q_i \rho(t_j)),$$

(21)

for $i = 1, \ldots, r$ and $j = 1, \ldots, p$.

Let us observe that in the above definition of reconstructibility we discuss the problem of verifying whether the accessible information about the system in question is sufficient to determine the state uniquely and we do not insist on determining its explicitly.

The positive dynamical semigroup $\{\Phi(t), t \in \mathbb{R}_+\}$ is determined by the generator

$$\mathbb{K} : \mathcal{B}_h(\mathcal{H}) \to \mathcal{B}_h(\mathcal{H})$$

(22)
(the Schrödinger picture) and it is related to the generator $L$ of the semigroup in Heisenberg picture by the duality relation
\[ \text{Tr}[Q(K\rho)] = \text{Tr}[(LQ)\rho]. \] (23)

For a given set of observables $Q_1, \ldots, Q_r$, the subspace spanned on the operators $Q_i, LQ_i, \ldots, (L)^{k-1}Q_i$, will be denoted by
\[ K_k(L, Q_i) := \text{Span}_\mathbb{R}\{Q_i, LQ_i, \ldots, (L)^{k-1}Q_i\}, \] (24)
as the Krylov subspace in the Hilbert-Schmidt space $B_{\mathcal{H}}$. If $k = \mu$, where $\mu$ is the degree of the minimal polynomial of the generator $L$, then the subspace $K_\mu(L, Q_i)$ is an invariant subspace of the superoperator $L$ with respect to $Q_i$. It can be easily seen that the subspace $K_\mu(L, Q_i)$ is essentially spanned on all operators of the form $(L)^kQ_i$, where $k = 0, 1, \ldots$. Furthermore, it is smallest invariant subspace of the superoperator $L$ containing $Q_i$ (i.e. the common part of all invariant subspaces of the operator $L$ containing $Q_i$).

One can now formulate the sufficient conditions for the reconstructibility of a $d$-level open quantum system (c.f. [5, 11]). Let $S$ be a $d$-level open quantum system with the evolution governed by an equation of the form $\dot{Q}(t) = LQ(t)$ (the Heisenberg picture), where $L$ is the generator of the dynamical semigroup $\Phi(t) = \exp(tL)$. Suppose that by performing measurements, the correspondence
\[ [0, T] \ni t_j \mapsto E_i(t_j) = \text{Tr}(\rho(0)Q_i(t_j)) \] (25)
can be established for fixed observables $Q_1, \ldots, Q_r$ at selected time instants $t_1, \ldots, t_p$. The system $S$ is $(Q_1, \ldots, Q_r)$-reconstructible if
\[ \bigoplus_{i=1}^r K_\mu(L, Q_i) = B_{\mathcal{H}}. \] (26)

The above condition has been obtained by using the polynomial representation of the semigroup $\Phi(t)$ [11]. Indeed, if $\mu(\lambda, L)$ denotes the minimal polynomial of the generator $L$ and $\mu = \text{deg} \mu(\lambda, L)$, then $\Phi(t) = \exp(tL)$ can be represented in the form
\[ \Phi(t) = \sum_{k=0}^{\mu-1} \alpha_k(t)L^k, \] (27)
where the functions $\alpha_k(t)$ for $k = 0, \ldots, \mu - 1$ are those solutions of the scalar linear differential equation with characteristic polynomial $\mu(\lambda, L)$ for which equalities
\[ \frac{d^k \alpha_k(t)}{dt^k} \bigg|_{t=0} = \delta_{lk}, \] (28)
are satisfied [22]. Since the functions $\alpha_k(t)$ are mutually independent, therefore for arbitrary $T > 0$ there exists at least one set of moments $t_1, \ldots, t_\mu$ ($\mu = \text{deg} \mu(\lambda, L)$) such that
\[ 0 \leq t_1 < t_2 < \cdots < t_\mu \leq T \] (29)
and $\det[\alpha_k(t_j)] \neq 0$. Taking into account these conditions one finds that the state $\rho(0)$ can be determined uniquely if operators of the form
\[ f_{kl} := (L)^kQ_l \] (30)
for \( l = 1, \ldots, r \) and \( k = 0, 1, \ldots \) span the space \( B_0(\mathcal{H}) \). In other words, we can say that \( \rho(0) \) can be determined if vectors (30) constitute a frame in Hilbert-Schmidt space \( B_0(\mathcal{H}) \) or, equivalently, if Krylov subspaces \( K_\mu(\mathbb{L}, Q_1) \) for \( l = 1, \ldots, r \) constitute a fusion frame in \( B_0(\mathcal{H}) \).

The question of an obvious physical interest is to find the minimal number of observables \( Q_1, \ldots, Q_\eta \) for which a \( d \)-level quantum system \( \mathcal{S} \) with a fixed generator \( \mathbb{L} \) can be \((Q_1, \ldots, Q_\eta)\)-reconstructible. It can be shown that for an \( d \)-level generator there always exists a set of observables \( Q_1, \ldots, Q_\eta \), where

\[
\eta := \max_{\lambda \in \sigma(\mathbb{L})} \{ \dim \ker(\lambda I - \mathbb{L}) \},
\]

such that the system is \((Q_1, \ldots, Q_\eta)\)-reconstructible \([11]\). Moreover, if we have another set of observables \( \tilde{Q}_1, \ldots, \tilde{Q}_\tilde{\eta} \) such that the system is \((\tilde{Q}_1, \ldots, \tilde{Q}_\tilde{\eta})\)-reconstructible, then \( \tilde{\eta} \geq \eta \). The number \( \eta \) defined by (31) is called the index of cyclicity of the quantum open system \( \mathcal{S} \) \([11]\). The symbol \( \sigma(\mathbb{L}) \) in (31) denotes the spectrum of the superoperator \( \mathbb{L} \).

In particular, if we consider an isolated quantum system characterized by Hamiltonian \( H_0 \), then the minimal number of observables \( Q_1, \ldots, Q_\eta \) for which the system is \((Q_1, \ldots, Q_\eta)\)-reconstructible is given by

\[
\eta = n_1^2 + n_2^2 + \cdots + n_m^2,
\]

where \( n_i = \dim \ker(\lambda_i I - H_0) \) for all \( \lambda_i \in \sigma(H_0), \) \( i = 1, \ldots, m \) (for details cf. \([5, 11]\)).

Now let us assume that the time evolution of a \( d \)-level quantum system \( \mathcal{S} \) is described by the generator \( \mathbb{L} \) given by

\[
\mathbb{L}\rho = \frac{1}{2} \left\{ [N\rho, N] + [N, \rho H] \right\} = -\frac{1}{2} [N, [N, \rho]],
\]

that is, we consider the so-called Gaussian semigroup. The symbol \( N \) in (33) denotes a self-adjoint operator with the spectrum

\[
\sigma(N) = \{ \lambda_1, \ldots, \lambda_m \}.
\]

In the sequel \( n_i \) stands for the multiplicity of the eigenvalue \( \lambda_i \) for \( i = 1, \ldots, m \). One can assume that the elements of the spectrum of \( N \) are numbered in such a way that the inequalities \( \lambda_1 < \lambda_2 < \cdots < \lambda_m \) are fulfilled. The following theorem holds:

**The index of cyclicity of the Gaussian semigroup with a generator \( \mathbb{L} \) given by (33) is expressed by the formula**

\[
\eta = \max \{ \kappa, \gamma_1, \ldots, \gamma_r \},
\]

**where** \( r = (m - 1)/2 \) **if** \( m \) **is odd** **or** \( r = (m - 2)/2 \) **if** \( m \) **is even**, **and**

\[
\kappa := n_1^2 + n_2^2 + \cdots + n_m^2,
\]

\[
\gamma_k := 2 \sum_{i=1}^{m-k} n_i n_{i+k}.
\]

In order to prove the above theorem and to determine the value of \( \eta \) for the generator \( \mathbb{L} \) defined by (33) we must find the number of nontrivial invariant factors of the operator \( \mathbb{L} \). Let us observe that if \( \sigma(N) = \{ \lambda_1, \ldots, \lambda_m \} \), then the spectrum of the operator \( \mathbb{L} \) is given by

\[
\sigma(\mathbb{L}) = \{ \nu_{ij} \in \mathbb{R} \mid \nu_{ij} = (\lambda_i - \lambda_j)^2, \ i, j = 1, \ldots, m \}.
\]
The above statement follows from the fact that the operator $\mathbb{L}$ can also be represented as
\[ \mathbb{L} = N^2 \otimes I + I \otimes N^2 - 2N \otimes N, \] (39)
where $I$ denotes the identity in the space $\mathcal{H}$. Since $N$ is self-adjoint therefore the algebraic multiplicity of $\lambda_i$, i.e. the multiplicity of $\lambda_i$ as the root of the characteristic polynomial of $N$, is equal to the geometric multiplicity of $\lambda_i$, $n_i = \dim \text{Ker}(\lambda_i I - N)$. Of course, we have $n_1 + \ldots + n_m = \dim \mathcal{H}$.

From (39) we can see that the multiplicities of the eigenvalues of the operator $\mathbb{L}$ are not determined uniquely by the multiplicities of $\lambda_i \in \sigma(N)$. But if we assume that $\lambda_1 < \ldots < \lambda_m$ and $\lambda_k = (k-1)c + \lambda_1$, where $k = 1, \ldots, m$, and $c = \text{const} > 0$, then the multiplicities of all eigenvalues of $\mathbb{L}$ are given by
\[ \gamma[i-j] = \dim \text{Ker} \left[ (\lambda_i - \lambda_j)^2 \mathbb{I} - \mathbb{L} \right] \] (40)
for $i \neq j$ and
\[ \dim \text{Ker} (\mathbb{L}) = n_1^2 + \ldots + n_m^2 = \kappa \] (41)
when $i = j$. Now, as we know, the minimal number of observables $Q_1, \ldots, Q_\eta$ for which the qudit $\mathcal{S}$ can be $(Q_1, \ldots, Q_\eta)$-reconstructible is given by (31), so in our case
\[ \eta = \max_{i,j=1,\ldots,m} \left\{ \dim \text{Ker} \left[ (\lambda_i - \lambda_j)^2 \mathbb{I} - \mathbb{L} \right] \right\}, \] (42)
where $\lambda_i \in \sigma(N)$. Using the above formulae and the inequality $\gamma_k < \kappa$ for $k > r$, where $r$ is given by $(m-1)/2$ if $m$ is odd and $(m-2)/2$ if $m$ is even, we can observe that also without the assumption $\lambda_k = (k-1)c + \lambda_1$ one obtains
\[ \eta = \max \{ \kappa, \gamma_1, \ldots, \gamma_r \}. \] (43)

This completes the proof.

Another natural question arises: what are the criteria governing the choice of time instants $t_1, \ldots, t_\mu$? The following theorem holds:

Let us assume that $0 \leq t_1 < t_2 < \ldots < t_\mu \leq T$. Suppose that the mutual distribution of time instants $t_1, \ldots, t_\mu$ is fixed, i.e. a set of nonnegative numbers $c_1 < \ldots < c_\mu$ is given and $t_j := c_j t$ for $j = 1, \ldots, \mu$, and $t \in \mathbb{R}_+$. Then for $T > 0$ the set
\[ \tau(T) := \left\{ (t_1, \ldots, t_\mu) : \ t_j = c_j t, \ 0 \leq t \leq \frac{T}{c_\mu} \right\} \]
contains almost all sequences of time instants $t_1, \ldots, t_\mu$, i.e. all of them except a finite number.

As one can check, the expectation values $E_i(t_j)$ and the operators $(\mathbb{L}^*)^k Q_i$ are related by the equality
\[ E_i(t_j) = \sum_{k=0}^{\mu-1} \alpha_k(c_j) \left( (\mathbb{L}^*)^k Q_i, \rho_0 \right), \] (44)
where we assume that $t_j = c_j t$ and the bracket $(\cdot, \cdot)$ denotes the Hilbert-Schmidt product in $\mathcal{B}_g(\mathcal{H})$. One can determine $\rho_0$ from (44) for all those values $t \in \mathbb{R}_+$ for which the determinant $\Omega(t)$ is different from zero, i.e.
\[ \Omega(t) := \det \left[ \alpha_k(c_j t) \right] \neq 0. \] (45)
One can prove that the range of the parameter $t \in \mathbb{R}_+$ for which $\Omega(t) = 0$ consists only of isolated points on the semiaxis $\mathbb{R}_+$, i.e., does not possess any accumulation points on $\mathbb{R}_+$. To this end let us note that since the functions $t \rightarrow \alpha_k(t)$ for $k = 0, 1, \ldots, \mu - 1$, are analytic on $\mathbb{R}$, the determinant $\Omega(t)$ defined by (45) is also an analytic function of $t \in \mathbb{R}$. If $\Omega(t)$ can be proved to be nonvanishing identically on $\mathbb{R}$, then, making use of its analyticity, we shall be in position to conclude that the values of $t$, for which $\Omega(t) = 0$, are isolated points on the axis $\mathbb{R}$.

It is easy to check that for $k = \mu(\mu - 1)/2$

$$\frac{d^k \Omega(t)}{dt^k}
|_{t=0} = \prod_{1 \leq j < i \leq \mu} (c_i - c_j).\quad (46)$$

According to the assumption $c_1 < c_2 < \ldots < c_\mu$, we have $\Omega^{(k)}(0) \neq 0$ if $k = \mu(\mu - 1)/2$. This means that the analytic function $t \rightarrow \Omega(t)$ does not vanish identically on $\mathbb{R}$ and the set of values of $t$ for which $\Omega(t) = 0$ cannot contain accumulation points. In other words, if we limit ourselves to an arbitrary finite interval $[0, T]$, then $\Omega(t)$ can vanish only on a finite number of points belonging to $[0, T]$. This completes the proof.

4. Frames in stroboscopic tomography

As we have seen the concepts of frames and fusion frames appear in stroboscopic tomography in natural way. The conclusion is based on the discussed above polynomial representations of semigroups which describe evolutions of open systems. The possibility to represent the semigroup $\Phi(t) = \exp(t L)$ in the form

$$\Phi(t) = \sum_{k=0}^{\mu-1} \alpha_k(t) L^k,\quad (47)$$

where $\mu$ stands for the degree of the minimal polynomial of the superoperator $L$ and $\alpha_k(t)$, $k = 0, \ldots, \mu - 1$, denote some functions of the eigenvalues of $L$ gives the equality (26) as a sufficient condition for stroboscopic tomography. On the other hand, this equality means that the Krylov subspaces $K_{\mu}(L, Q_i)$, $i = 1, \ldots, r$, constitute a fusion frame in the Hilbert-Schmidt space $B_h(\mathcal{H})$ of all observables. Moreover, this also means that the collection of vectors

$$f_{jk} := L^k Q_j,\quad (48)$$

for $j = 1, \ldots, r$ and $k = 0, 1, \ldots, \mu - 1$, constitute a frame in $B_h(\mathcal{H})$ and the system in question is $(Q_1, \ldots, Q_r)$-reconstructible. In this case every element $Q$ of the space $B_h(\mathcal{H})$ can be represented as

$$Q = \sum_{j,k} \langle F^{-1} f_{jk}, Q \rangle f_{jk} = \sum_{j,k} \langle f_{jk}, Q \rangle F^{-1} f_{jk},\quad (49)$$

where $F$ denotes the frame operator of the collection of vectors (48). One can say even more. If $Q \in B_h(\mathcal{H})$ also has another representation $Q = \sum_{j,k} c_{jk} f_{jk}$ for some scalar coefficients $c_{jk}$, $j = 1, \ldots, r$ and $k = 0, 1, \ldots, \mu - 1$, then

$$\sum_{j,k} |c_{jk}|^2 = \sum_{j,k} |\langle F^{-1} f_{jk}, Q \rangle|^2 + \sum_{j,k} |c_{jk} - \langle F^{-1} f_{jk}, Q \rangle|^2.\quad (50)$$

It is obvious that every frame in finite-dimensional space contains a subset that is a basis. As a conclusion we can say that if $\{f_{jk}\}$ is a frame but not a basis, then there exists a set of scalars $\{d_{jk}\}$ such that $\sum_{j,k} d_{jk} f_{jk} = 0$. Therefore, any fixed element $Q$ of $B_h(\mathcal{H})$ can also be represented as

$$Q = \sum_{j,k} \langle F^{-1} f_{jk}, Q \rangle + d_{jk} f_{jk}.\quad (51)$$
The above equality means that every $Q \in B_h(\mathcal{H})$ has many representations as superpositions of elements from the set (48). But according to equality (50) among all scalar coefficients $\{c_{jk}\}$ for which

$$Q = \sum_{j,k} c_{jk} f_{jk},$$

the sequence $\{ (F^{-1}f_{jk}, Q) \}$ has minimal norm. This is a general method in frame theory [23] and at the same time the main observation connected with the idea of stroboscopic tomography.

In conclusion, one can say that the Krylov subspaces $\mathcal{K}_\mu(L, Q_i)$ in the space $B_h(\mathcal{H})$ generated by the superoperator $L$ can be used in an effective way for procedures of stroboscopic tomography if they constitute appropriate fusion frames in this space. Some details of this approach will be given elsewhere [24].

Acknowledgments
The main ideas of this paper were presented during the conference organized by Professors Barbara and Tadeusz Lulek in Myczkowce 2009. It is a great pleasure for me to thank them for their hospitality and for the effort of all organizers of this meeting.

References
[1] Feynman R 1963 The Feynman Lectures on Physics Vol. 3 (Addison-Wesley)
[2] Band W and Park J L 1979 Am. J. Phys. 47 188
[3] Nielsen M A and Chuang I 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge Unv. Press.)
[4] Weigert S 2000 New Insight in Quantum Mechanics Eds. H.-D. Doebner et al. (Singapore: World Scientific)
[5] Jamiołkowski A 1982 On Observability of Classical and Quantum Stochastic Systems (Toruń; N. Copernicus Unv. Press) in Polish
[6] Jamiołkowski A 1983 Internat. J. Theoret. Phys. 22 369
[7] Jamiołkowski A 1986 Frequency Domain and State Space Methods for Linear Systems Eds. C I Byrnes and A. Lindquist (Amsterdam: Elsevier) p 347
[8] Casazza P G and Kutyniok G 2004 Frames of subspaces Contemp. Math. 345 87
[9] Casazza P G et al. 2008 Appl. Comput. Harmon. Anal. 25 114
[10] Farouki R. T. et al. 2001 Geometriae Dedicata 85 283
[11] Jamiołkowski A 2000 Rep. Math. Phys. 46 469
[12] Duffin R J and Schaeffer A C 1952 Transactions AMS 72 341
[13] Daubechies I, Grossman A and Meyer Y 1986 J. Math. Phys. 27 1271
[14] Kovačević J and Chebira A 2008 An Introduction to Frames (Boston-Delft: NOW)
[15] Heil Ch (Ed.) 2006 Harmonic Analysis and Applications (Boston: Birkhäuser)
[16] Kraus K 1971 Ann. Phys. 64 119
[17] Kossakowski A 1972 Rep.Math. Phys. 3 247
[18] Jamiołkowski A 1974 Rep.Math. Phys. 5 415
[19] Gorini V et al. 1976 J. Math. Phys. 17 149
[20] Lindblad G 1976 Comm. Math. Phys. 48 119
[21] Ohya M 1981 J. Math. Anal. Appl. 84 318
[22] Zadeh L A and Desoer C A 1968 Linear System Theory (New York: Mc Graw-Hill)
[23] Christensen O 2008 Frames and Bases (Boston: Birkhäuser)
[24] Jamiołkowski A, in preparation