Quantum Brownian oscillator for the stock market

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Abstract
We pursue the quantum-mechanical challenge to the efficient market hypothesis for the stock market by employing the quantum Brownian motion model. We utilize the quantum Caldeira-Leggett master equation as a possible phenomenological model for the stock-market-prices fluctuations while introducing the external harmonic field for the Brownian particle. Two quantum regimes are of particular interest: the exact regime as well as the approximate regime of the pure decoherence ("recoilless") limit of the Caldeira-Leggett equation. By calculating the standard deviation and the kurtosis for the particle’s position observable, we can detect deviations of the quantum-mechanical behavior from the classical counterpart, which bases the efficient market hypothesis. By varying the damping factor, temperature as well as the oscillator’s frequency, we are able to provide interpretation of different economic scenarios and possible situations that are not normally recognized by the efficient market hypothesis. Hence we recognize the quantum Brownian oscillator as a possibly useful model for the realistic behavior of stock prices.

Keywords: Econophysics, Stock market irrationality, Quantum Brownian motion, Harmonic oscillator, Fat-tail phenomena

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1. Introduction

Deviation from the Gaussian (so-called normal) distribution of returns appears as a universal empirical fact for different markets, ranging from the markets in developed countries, such as Germany [1] and US [2, 3], to markets in developing countries, like India [4] and China [5]. It manifests as, inter alia, the "fat tails" as well as the positive excess kurtosis for the probability distribution of returns. The distribution is expected to converge to the standard Gaussian behavior after sufficiently long time interval [6-8] thus...
suggesting the presence of some, hopefully universal, regularities governing the complex financial systems [9-11].

Appearance of both the fat tail deviation and the positive excess kurtosis exhibit non-Markovian behavior thus making obvious that the stock market does not satisfy the classical Brownian motion model, which is characteristic of the remarkable efficient market hypothesis (EMH) [12]. As a natural step forward appear the attempts to properly include certain quantum models [13-15] that led to a number of different approaches and models pursuing the quantum paradigm in the quantitative finance field [15-23]. As a rationale for the quantum mechanical approach it is often invoked the market irrationality—in sharp contrast to EMH. Behavioral economists plead for the important role of the agents irrationality in the realistic stock transactions so much that irrationality might substantially contribute to the (empirically well-known) persistent fluctuations of the stock price even when there is no information to impel the stock price. As a possible model of irrationality appears the quantum mechanical uncertainty that is therefore interpreted as the market uncertainty, that drives volatility [18,20,22] (and the references therein).

Different quantum models have been used, such as the particle in the potential well [19,20], the quantum damped oscillator suggested in [20], harmonic oscillator [21,23], quantum Brownian motion [22] etc. Quantum Brownian motion is particularly interesting bearing in mind it could be regarded a quantum-mechanical counterpart of the classical Brownian motion, which, in turn, backs the profound efficient market hypothesis [12]. Hence an elaborated quantum-mechanical challenge, notably Ref. [22], for the efficient market hypothesis that is worth pursuit.

In this paper we utilize the quantum Brownian motion as modelled by the well-known Caldeira-Leggett (CL) master equation [22,24,25]. We go beyond the existing models in that we introduce the external harmonic field for the Brownian particle, while, on the other hand, we pay special attention to the pure decoherence (the so-called "recoilless") limit of the CL master equation. The decoherence limit regards the off-diagonal terms, $\rho(x, x') = \langle x|\hat{\rho}|x'\rangle$, that often remain an open issue [22]. As distinct from the similar approaches, we regard the CL equation as a "phenomenological" master equation, meaning that we are not concerned with the underlying microscopic physical details that lead [24,25] to the equation. Rather we investigate usefulness of the CL equation for a harmonic oscillator in the context of the econophysics studies.

Generally, appearance of the external field is an attempt to model the external macroscopic influence on the stock market, such as the daily price limitation of the stock markets in China [21,22,26] or to distinguish between the "regular" and "irregular" returns in the United Kingdom’s Financial
Times-Stock Exchange (FTSE) All Share Index [19]. In this regard, we
consider our model of the harmonic oscillator Brownian particle as possibly
more realistic than the model of the free Brownian motion [22].

By following the standard wisdom, we calculate the standard deviation
and the kurtosis for both the exact and the recoilless limit of the quantum
harmonic Brownian particle and compare the obtained results with the clas-
sical counterpart. Comparison regards not only the damping rate $\gamma$ or the
bath’s temperature, but also the oscillator’s frequency $\omega$. Justification of
our model of the harmonic Brownian particle comes also from the similar
considerations [21,23] as well as from the analogous model of the (classical)
damped harmonic oscillator [27] that was considered as a possible physical
basis for the EMH [12]. To the extent that the classical harmonic Brownian
particle properly bases the EMH, we provide some evidences against market
efficiency as well as possibly a useful physical (quantum-mechanical) model
for the stock market.

In Section 2, we introduce and briefly discuss the model. In Section 3 we
provide the main results of this paper. In Section 4 we provide discussion
and conclusion for the obtained results by paying special attention to the
interpretation of our findings in the context of the possible economic scenarios
and situations.

2. The Caldeira-Leggett master equation for the harmonic oscil-
lator

The Caldeira-Leggett master equation for quantum state ("density ma-
trix"), $\dot{\rho}$, of one-dimensional system, in the Schrödinger picture, reads [24,25]:

$$\frac{d\dot{\rho}(t)}{dt} = -\frac{i}{\hbar} [\hat{H}, \dot{\rho}(t)] - \frac{\gamma}{\hbar} [\dot{x}, \{\dot{p}, \dot{\rho}(t)\}] - \frac{2m\gamma k_B T}{\hbar^2} [\dot{x}, [\dot{x}, \dot{\rho}_R(t)]]$$

(1)

In eq.(1), the only degree of freedom of the particle is the Descartes co-
ordinate $\hat{x}$, while $\dot{p}$ stands for its conjugate momentum; the commutation
relation, $[\hat{x}, \hat{p}] = i\hbar$. The system’s Hamiltonian $\hat{H}$ generates the
unitary dynamics described by the first commutator on the rhs of eq.(1),
while the second and the third terms model the quantum mechanical dissi-
aption and decoherence (sometimes also referred to as ”dephasing”), respec-
tively, both determined by the non-negative and time-independent damping
coefficient $\gamma$. By $m$ we denote the system’s mass, while $k_B$ and $T$ stand for
the Boltzmann constant and the thermal-bath’s temperature, respectively.
The system’s Hamiltonian (while neglecting the Lamb-shift term)

$$\hat{H} = \hat{T} + \hat{V} = \frac{\hat{p}^2}{2m} + V(\hat{x}),$$

(2)
where the external potential $\hat{V} = 0$ describes the free Brownian particle, while $\hat{V} = m\omega^2 \hat{x}^2 / 2$ regards the particle in the external harmonic potential with the frequency $\omega$ and the zero equilibrium position. The square brackets stand for the commutator, while the curly brackets for the anticommutator, $\{A, B\} = AB + BA$.

In this paper we take the equation (1) as given, stipulated, without resorting to the details of its microscopic origin [28]. This gives us a freedom to vary the values of the parameters $\gamma, m, T, \omega$—those variations are limited by the assumptions of large temperature and weak interaction in the microscopic derivation of the CL equation (1) [24,25].

For very large $T$ and/or very large mass $m$, the exact equation (1) can be approximated thus giving rise to the decoherence-limit (the so-called ”recoilless limit”) [25]. In this limit, the third—the decoherence—term dominates the system’s dynamics thus allowing for neglecting the second (the dissipation) term of eq.(1). Then the particle undergoes the environment-induced decoherence [25,29] without dissipation. Bearing in mind that (in the Heisenberg picture) the CL equation (1) has a well-defined classical counterpart in the form of the Langevin equation [24,25], it is particularly interesting to compare the decoherence-limit results with the known and exact classical expressions.

Prescription of the general model assumptions into the econophysics context is standard, e.g., [22]: the degree of freedom $\hat{x}$ regards the (logarithmic) price, the momentum $\hat{p}$ the trend of the price, while $m$ now stands for the stock inertia quantifying the market capitalization. The bath’s temperature $T$ quantifies the externally-induced fluctuations while the damping coefficient $\gamma$ quantifies the externally-induced damping strength [22]. Therefore the parameters variations may regard the different scenarios for the stock transactions with respect e.g. to the market capitalization ($m$) and the frequency of the external interventions ($\omega$).

The moments characterizing the distribution $\rho(x, x')$ are all of the form $tr(\hat{A}\hat{\rho}(t))$ (with the time-independent $\hat{A}$ in the Schrödinger picture). With the use of the identities, $tr(A[B,C]) = tr([A, B]C)$ and $tr(A\{B,C\}) = tr(\{A,B\}C)$, it easily follow the differential equations for the moments, in the following general form:

$$\frac{d(tr\hat{A}\hat{\rho}(t))}{dt} = -\frac{i}{\hbar} tr([\hat{A}, \hat{H}]\hat{\rho}) - \frac{\gamma}{\hbar} tr(\{[\hat{A}, \hat{x}], \hat{p}\}\rho) - \frac{2m\gamma k_B T}{\hbar^2} tr([\hat{x}, [\hat{x}, \hat{A}]]\hat{\rho}).$$

\[3\]

3. The results
With the use of eq.(3), in this section we provide the results for the \( \hat{x} \) standard deviation, as well as for the kurtosis for the quantum harmonic Brownian particle. Two sets of the results are provided: the exact quantum-mechanical expressions as well as the decoherence limit, which assumes neglecting the second term in eq.(3). Those results are (separately) compared to the exact classical-physics results for the harmonic Brownian motion. Along with the variations of the damping coefficient \( \gamma \) and the temperature \( T \), we also consider the variations due to the oscillator’s frequency \( \omega \), which is absent from the considerations regarding the free Brownian particle.

### 3.1 The standard deviations

With the use of eq.(3) it is straightforward but tedious to obtain the standard deviation \( \Delta \hat{x} \), which we overtake from equation (B.2) in Ref. [30] while exchanging the quantities for a rotator with the quantities for the translational motion:

\[
(\Delta \hat{x}(t))^2 = \frac{k_B T}{m \omega^2 \Omega^2} \left( \Omega^2 + e^{-2\gamma t}(\omega^2 - \gamma^2 \cosh(2\Omega t) - \gamma \Omega \sinh(2\Omega t)) \right) + \\
\frac{(\Delta \hat{p}(0))^2}{m^2 \Omega^2} e^{-2\gamma t} \sinh^2(\Omega t) + \frac{(\Delta \hat{x}(0))^2}{\Omega^2} e^{-2\gamma t} (-\omega^2 \cosh^2(\Omega t) + \\
\gamma^2 \cosh(2\Omega t) + \gamma \Omega \sinh(2\Omega t)) + \frac{e^{-2\gamma t} \sigma(0)}{2m \Omega^2} (2\gamma \sinh^2(\Omega t) + \Omega \sinh(2\Omega t) \sigma(0)).
\]

(4)

In eq.(4): \( \Omega^2 = \gamma^2 - \omega^2 \), while the quantum variance \( \sigma \equiv \langle \hat{x} \hat{p} - \hat{p} \hat{x} \rangle - 2 \langle \hat{x} \rangle \langle \hat{p} \rangle \).

Placing the classically allowed zero initial moments, \( \Delta \hat{x}(0) = 0, \Delta \hat{p}(0) = 0 \) and \( \sigma(0) = 0 \), while assuming without any loss of generality, \( \langle \hat{x}(0) \rangle = 0 = \langle \hat{p}(0) \rangle \), in eq.(4) remains the first term, i.e. the classical expression for \( \Delta x(t) \) for a classical Brownian harmonic oscillator [30] (and the references therein):

\[
(\Delta \hat{x}(t))^2 = \frac{k_B T}{m \omega^2 \Omega^2} \left( \Omega^2 + e^{-2\gamma t}(\omega^2 - \gamma^2 \cosh(2\Omega t) - \gamma \Omega \sinh(2\Omega t)) \right).
\]

(5)

For the decoherence-limit, neglecting the second term on the rhs of eq.(1), follow the corresponding expression for \( \Delta \hat{x}(t) \). To this end, we directly overtake the equation (C.2) in Ref.[30], with the quantities for the translational motion:

\[
(\Delta \hat{x}(t))^2 = (\Delta \hat{x}(0))^2 \cos^2 \omega t + \frac{(\Delta \hat{p}(0))^2}{m^2 \omega^2} \sin^2 \omega t + \frac{\sigma(0)}{2m \omega} \sin 2\omega t \\
+ \frac{2\gamma k_B T}{m \omega^2} t - \frac{\gamma k_B T}{m \omega^3} \sin 2\omega t.
\]

(6)
Of course, being a purely quantum-mechanical effect, the pure decoherence equation (6) does not have a distinguished classical counterpart.

Therefore we compare the exact (equation (4)) and the decoherence-limit (equation (6)) quantum expressions with the (exact) classical equation (5). Dependence of \((\Delta x(t))^2\) is separately investigated for every parameter, \(\gamma, T\) and \(\omega\).

Graphical results are presented in Figure 1 and in Figure 2 (the Log-Log plots), respectively, for the initial values \((\Delta \hat{x}(0))^2 = 10^{-7},\) \((\Delta \hat{p}(0))^2 = 10^7\) and \(\sigma(0) = 0.01,\) in accordance with the uncertainty relation \([\hat{x}, \hat{p}] = i\hbar\) (keeping \(\hbar = 1\)) and the (quantum-mechanical) Cauchy-Schwartz inequality \(\sigma \leq 2\Delta \hat{x}\Delta \hat{p},\) while a long time interval \(t = 10\) is chosen. For brevity, we place \(\Delta x^2\) instead of \((\Delta \hat{x})^2\). The choice of the parameter values and ranges as well as of the initial conditions is made in order to facilitate comparison of the obtained results with the results presented for the free particle in Ref. [22]—as explicitly emphasized in Figure 1 captions.

![Figure 1](image1.png)

Figure 1: Comparison of the exact quantum (solid lines) and the exact classical (dashed lines) expressions. (Left) \(\omega = 0.1, m = 0.1, \gamma = 10,\) the variable \(k_B T \in [10^{-7}, 10^7]\); (Middle) \(\omega = 10, m = 10, k_B T = 0.1,\) the variable \(\gamma \in [10^{-2}, 10^7]\); (Right) \(m = 10, \gamma = 1, k_B T = 0.1,\) the variable \(\omega \in [10^{-2}, 10^2].\)

![Figure 2](image2.png)

Figure 2: Comparison of the quantum decoherence-limit (solid lines) and the exact classical (dashed lines) expressions. The meaning of the plots and of the respective values of the parameters are the same as for Figure 1, except for the choice of the very large mass \(m = 1000.\)

### 3.2 Kurtosis
Bearing in mind Section 3.1, in this section we only consider the kurtosis \( \kappa = \langle \hat{x}^4 \rangle / (\langle \hat{x}^2 \rangle^2) \) for the exact quantum-mechanical expression eq.(4). In Appendix A, we provide the basis for the calculation of the kurtosis for both the free (\( \kappa_{\text{free}} \)) and the harmonic (\( \kappa_{\text{harmonic}} \)) Brownian particle. For both, the free and the harmonic Brownian particle, we obtain (Appendix A) the closed set of the differential equations without a need to use (or calculate) the moments of any higher order. Since we find that the analytical expressions are beyond a succinct presentation and are physically non-transparent, here we present the results for the chosen values of the parameters as well as of the initial values of the relevant moments.

Figure 3 provides a comparison of the results for \( \kappa_{\text{free}} \) (dashed line) and \( \kappa_{\text{harmonic}} \) (solid line) as the functions of time \( t \) for the choice of the parameters (underdamped regime): \( m = 20, \gamma = 0.001, k_B T = 0.38 \). In Chinese stock market there is a price limit rule: the rate of return in a trading day cannot be larger than \( \pm 10\% \) comparing with the previous day’s closing price, which applies to most stocks in China. To this end we choose the realistic value of the circular frequency \( \omega = 18 \cdot 10^{-3} \text{min} \). The time unit for Figure 3 is therefore chosen 1min.

Figure 3: Kurtosis for the underdamped regime (assuming \( \hbar = 1 \)) for the two time-intervals. The time scale for both plots is 1min. The solid line presents the results for the harmonic oscillator while the dotted line presents the results for the free particle. For \( t \geq 120 \) (Right), the harmonic oscillator exhibits small oscillations around the asymptotic limit of \( \kappa = 3 \). Approach to the limit for the free particle is slower and almost monotonic.

The results are rather sensitive to the choice of the initial values for the moments of interest (up to the fourth order) and the presented results are chosen so as to provide the ”best” fit of the harmonic-oscillator-model with the evidence data presented by the blue line in Fig.2c in Ref. [22]. The following initial values are used for the first and the second moments: \( \langle \hat{x} \rangle = 0 = \langle \hat{p} \rangle, \langle \hat{x}^2 \rangle = 1/2 = \langle \hat{p}^2 \rangle \) and \( \langle \hat{x}^4 \rangle = 50 \).
3.3 Comments

Closeness of the quantum and the classical dynamics, Figure 1, exhibits the approximately-classical dynamics for the oscillator. Deviation of the quantum from the classical Brownian dynamics is qualitatively a desired behavior that is supported by evidence for different stock markets [1-5].

For smaller values of the parameters \((\gamma, T, \omega)\), the exact quantum mechanical behaviors, Figure 1, exhibit deviations from the classical counterpart that are much larger than those observed for the free Brownian motion [22]. While for large \(T\) and \(\omega\) the behavior is virtually indistinguishable from the classical behavior, for larger values of the \(\gamma\) parameter, the deviation is small but observable and quantitatively comparable with the results presented in Figure 2b in Ref. [22].

On the other hand, the approximate quantum behavior presented by Figure 2 is nowhere similar with the classical counterpart. That is, the large-mass-limit (the decoherence-limit) reveals the behavior that is apart from the classically predicted one; analogous result for the Brownian rigid rotator can be found in Appendix C in Ref. [30]. Inevitably, the quantum decoherence process does not prove to be sufficient for the classical-like behavior of the quantum Brownian oscillator. As we observe due to the knowledge of the exact classical behavior, only some interplay of decoherence and dissipation provides the classically well-known Brownian effect in the context of the quantum mechanical dynamics. Hence we offer an answer to the question [22] of the relevance of the off-diagonal terms of the density matrix. Bearing in mind the dynamical disappearance of the off-diagonal elements, i.e. \(\rho(x, x') = \langle x | \hat{\rho} | x' \rangle \propto \exp(-\Gamma(x - x')^2)\), [the only effect in the decoherence-limit], while decoherence cancels out the quantum coherence, this is not sufficient for the classically observed Brownian effect, which requires also dissipation in the system. In other words, the dynamically disappearing the off-diagonal elements constitute a necessary but insufficient condition for the classical-like behavior of the quantum harmonic Brownian oscillator.

Figure 3(Right) clearly exhibits a faster approach of \(\kappa_{\text{harmonic}}\) to the asymptotic Gaussian value of \(\kappa = 3\), which is approximately attained for \(t = 150\) min. This approach is not as smooth as for the free Brownian particle and is qualitatively in accordance with the evidence as presented by the blue line on Fig.2c in Ref.[22]. Comparison of our results with the estimates of the evidence-found results of Fig.2c in Ref. [22] is presented in Table 1. From Table 1 we can see that, initially, the free Brownian particle gives a better fit with the evidence, while for \(t > 60\), the harmonic Brownian particle gives a better fit with the evidence. The values for the evidence-obtained kurtosis are obtained as the (approximate) estimates stemming from Fig.2c.
in Ref. [22]. Therefore, at least for the time window \( t \in (61, 100) \), we find the better qualitative as well as the quantitative match of the harmonic oscillator model with the evidence than for the case of the free particle.

Table 1: Comparison of the kurtosis with the evidence-estimated values.

|                | \( t=40 \) | \( t=60 \) | \( t=80 \) | \( t=100 \) |
|----------------|------------|------------|------------|------------|
| \( \kappa_{\text{evidence}} \) | 12         | 11         | 7          | 7          |
| \( \kappa_{\text{free}} \)    | 14.4       | 13.61      | 11.8       | 10         |
| \( \kappa_{\text{harmonic}} \)| 15.3       | 13.65      | 9.8        | 6.4        |

4. Discussion and Conclusion

We use the Caldeira-Leggett master equation as a "phenomenological" equation without resorting to the microscopic details or quantum-mechanical interpretations of the model. This allows us a departure from the standard high-temperature and weak-interaction assumptions [24,25,28].

Observation of a quantum-like effect does not prove or even suggest that the system of interest is of the quantum mechanical nature. As repeatedly emphasized, cf. e.g. [31], quantum-mechanical effects are consequences, i.e. logical implications of (necessary conditions for) the quantum mechanical formalism but not necessarily vice versa. That is, the possible match of the evidence with the quantum mechanical predictions does not imply quantum-mechanical nature of the observed system.

Quantum contributions, Figure 1, increase the volatility measured by the standard deviation. Furthermore, the quantum contributions may even additionally increase due to certain fast actions repeated in short time interval of time. Actually, fast actions which are not accounted for by the CL model, equation (1), typically lead to the increase of the standard deviation. A series of such repeated actions in a short time interval (for which, the quantum corrections must be accounted for) can lead to a quick, non-negligible increase in \( \Delta \hat{x} \) [30]. This makes the overall dynamics even less predictable.

From Figure 1 we can learn that for large temperature \( T \) and for large frequency \( \omega \) as well as for not-very-small and not-very-large the damping factor \( \gamma \), the dynamics is essentially classical. From figure 2, we can see that for very large mass \( m \), the dynamics is never similar to the classical counterpart. Hence one might expect that for sufficiently large \( T \) and \( \omega \) and for the "medium" \( \gamma \) and small mass \( m \), the quantum corrections can be essentially avoided. However, there is a caveat to this expectation. On the one hand, in the limit of \( T \to 0 \) and \( \gamma = 0 \), the system becomes "closed", i.e. unitary and
therefore deterministic even in the classical limit. On the other hand, in the
limit $m \to 0$, equations (4)-(6) give the totally uncontrollable system since
in this limit one obtains $\Delta x(t) \to \infty, \forall t$. Therefore the "obvious" choice of
very small $T, \gamma$ and $m$ should be taken with care.

Results presented by Figure 3 are very sensitive to the choice of the higher-
momenta initial values, which are used for both the free and the harmonic
Brownian particle models. From Figure 3 and from Table 1, we can see the
better qualitative (the wavy parts in Figure 3) and quantitative (Table 1) fit
of the harmonic-particle model with the evidence-obtained results than for
the case of the free Brownian particle. Therefore we may conclude that the
harmonic-oscillator model may be useful for description of the realistic stock
markets.

The results of Section 3 have the clear economics interpretations. While
the externally induced fluctuations and damping, quantified by the temper-
ature $T$ and the damping coefficient $\gamma$, respectively, are practically out of
control, the capitalization $m$ and the frequency $\omega$ can in principle be con-
trolled in the realistic situations. From Figure 2 we can learn that too large
capitalization drives the system relatively far from the classical model and
therefore in opposition with EMH [12]. Figure 1(Right) exhibits that small
frequency of the external influence drives the market dynamics further from
the classical counterpart. That is, volatility is considerably larger for smaller
frequencies while it practically disappears for the relatively large frequency
$\omega$. Therefore the "modest" capitalization and the not-too-frequent external
interventions may reduce the volatility and therefore the risk. In order to
make this observation more precise, let us assume that there is an option,
or other financial derivative, that can be used to manage the risk, with the
initially small volatility (standard deviation). Then a series of such actions
performed by a large number of agents in a short time interval may lead to
a sharp, non-negligible increase of the volatility and therefore to a sudden
break of the initial stability of the market. That is, numerous agents ac-
tions performed in a short time window, especially right after the opening
of the stock market, inevitably induce increase of the quantum contributions
to volatility thus possibly inducing a sharp break of the initial market sta-
ibility. This is a classically unknown scenario [30], which reminds us of the
essence of the Minsky’s financial instability hypothesis [32], that "periods of
calm can project a false sense of security and lure agents into taking a riskier
investment, preparing for a crisis" [33]; to this end see also [34].

Collecting the told above, we may say that avoiding the quantum contri-
butions in order to make the stock prices "more predictable" may be regarded
a kind of the optimization problem, rather than a straightforward procedure
with the more-or-less weakly dependent parameters.
Hence, globally, Section 3 clearly emphasizes: there may be additional uncertainty not predicted by the classical Brownian model that, while quantitatively approximately fitting to the evidence data (notably Figure 3(Right)), do not offer the simple recipes for avoiding the possible risks. Rather, some optimal strategies are required for the optimal choice of the parameter values. Formulation of such strategies is beyond the scope of this paper for at least two reasons. First, we need quantitatively the more elaborated data for comparison. Second, such strategies pose a challenge even for the idealized theoretical models. To this end, the research is ongoing and the results will be presented elsewhere. Nevertheless, certain lessons are out of question. E.g. by controlling the frequency of the external actions and the capitalization, the market dynamics may partially reveal the effective ”temperature” and ”damping” on the market thus providing possibly a deeper insight into the market dynamics and conditions. Finally, very quick and numerous agent actions in a short time window are expected to increase the volatility and therefore make the investments more risky.

We conclude with our expectation, that the progress in quantum-mechanical modeling of real behavior of stock prices may be regarded a kind of justification of the idea of the investor’s irrationality as recognized by the behavioral economists. That is, it may be viable to assume that the quantum econophysics studies provide arguments for irrationality of the agents. Nevertheless, this raises a far-reaching question: if [assumed] irrationality is typical for the economy business, how could it be absent from the other kinds of human endeavors? We believe that this way comes a new broad perspective open by the quantum econophysics studies for different humanistic and social sciences, including sociophysics [36-38]–this time with a more elaborated quantitative criteria.

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Appendix A

From eq.(3) follows the set of the differential equations for the moments of the fourth order (to simplify notation, we omit the ”hat” operator symbol) that can be presented in the matrix form:

\[
\frac{d}{dt}X(t) = \mathcal{M}X(t) + F(t),
\]

(7)

where the matrix \( \mathcal{M} \) reads:
and the (transposed) vector

$$X^T = (\langle \hat{x}^4 \rangle, \langle \hat{x}^3 \hat{p} + \hat{p} \hat{x}^3 \rangle, \langle \hat{x}^2 \hat{p}^2 + \hat{p}^2 \hat{x}^2 \rangle, \langle \hat{x} \hat{p}^3 + \hat{p}^3 \hat{x} \rangle, \langle \hat{p}^4 \rangle),$$

while the (transposed) nonhomegeneous part reads:

$$F^T(t) = (0, 3\hbar^2/m, -4\hbar^2 \gamma + 8m \gamma k_B T (\Delta \hat{x}(t))^2, -3\hbar^2 m \omega^2 + 12m \gamma k_B T \sigma_{xp}(t), 24m \gamma k_B T (\Delta \hat{p})^2).$$

Appearance of the second moments in eq.(10) makes the set of the differential equations eq.(7) closed—there are no moments of the order larger than four.

The general solution of eq.(7) can be written as

$$X(t) = e^{M t} X(0) + \int_0^t ds e^{M(t-s)} F(s).$$

The free Brownian particle is obtained by setting $\omega = 0$ in eq.(8) and repeating the same procedure as for the harmonic Brownian particle.

The analytical expressions for $\langle \hat{x}^4 \rangle$ are rather large and non-transparent, for both cases of the free and the harmonic particle. Therefore we only provide the results for the proper choices of the initial values for the moments and for the system parameters $(\gamma, k_B T, \omega)$ as described in Section 3.2 of the body text. The exact analytical expression for the second moment $\langle \hat{x}^2 \rangle$ is well-known for the free Brownian particle, see eq.(3.438) in Ref.[25]:

$$\left( \Delta \hat{x}(t) \right)^2 = (\Delta \hat{x}(0))^2 + \frac{1 - e^{-2\gamma t}}{2\gamma} \left( \frac{(\Delta \hat{p}(0))^2}{m^2} + \frac{1 - e^{-2\gamma t}}{2} \sigma_{xp}(0) + \frac{k_B T}{m \gamma^2} \left( \gamma t - (1 - e^{-2\gamma t}) + \frac{1 - e^{-4\gamma t}}{4} \right).$$

This completes the necessary data for computing the kurtosis for both, the free and the harmonic Brownian particle.

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