Tachyonic dispersion in coherent networks

Y D Chong$^{1,2}$ and M C Rechtsman$^3$

1 Centre for Disruptive Photonic Technologies, Nanyang Technological University, Singapore 637371, Singapore
2 Division of Physics and Applied Physics, School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore 637371, Singapore
3 Department of Physics, The Pennsylvania State University, University Park, PA 16802, USA

E-mail: yidong@ntu.edu.sg

Received 9 August 2015, revised 24 October 2015
Accepted for publication 24 September 2015
Published 17 November 2015

Abstract

We propose a technique to realize a tachyonic band structure in a coherent network, such as an array of coupled ring resonators. This is achieved by adding ‘PT symmetric’ spatially balanced gain and loss to each node of the network. In a square-lattice network, the quasi-energy bandstructure exhibits a tachyonic dispersion relation, centered at either the center or corner of the Brillouin zone. There is one tachyonic hyperboloid in each gap, unlike in PT-symmetric tight-binding honeycomb lattices where the hyperboloids occur in pairs. The dispersion relation can be probed by measuring the peaks in transmission across a finite network as the gain/loss parameter is varied.

Keywords: tachyons, network models, photonic bandstructures

Most wave theories, including but not limited to quantum mechanics and classical electromagnetism, are formulated using equations of motion with Hermitian Hamiltonians. In quantum mechanics, Hermiticity ensures the general conservation of total probability under time evolution; conversely, its violation describes amplification (gain) and/or loss. Thus, for instance, non-Hermitian Hamiltonians are used in effective theories of decaying quantum systems, in which the wavefunction can leak away into unmonitored degrees of freedom. In optical physics, gain and loss processes are even more ubiquitous, in the context of the emission and absorption of light, and the prescriptions for dealing with these processes (e.g., introducing complex frequency-domain dielectric permittivities) are similarly well known.

Several years ago, Bender and co-workers made the striking observation that in systems possessing parity-time (PT) symmetry, corresponding to spatially-balanced gain and loss, the Hamiltonian can have purely real eigenvalues (i.e. probability conserving eigenstates) despite being non-Hermitian [1, 2]. Subsequently, a series of works showed both theoretically and experimentally that this effect could be demonstrated in optical structures, using optical gain and loss [3–9]. In PT symmetric optical lattices [8, 9], the photonic band structure has quite unusual features: the band energies can be real in one region of the Brillouin zone, where the Bloch eigenstates are PT symmetric, and complex in another region where the PT symmetry is spontaneously broken.

For 2D lattices, Szameit et al showed that the PT symmetry-breaking phenomenon has a startling interpretation in terms of emergent ‘tachyons’: hypothetical superluminal particles which are not known to exist in nature [30]. A two-dimensional honeycomb lattice can be realized using an array of coupled optical waveguides. In the Hermitian case, the bandstructure is graphene-like, featuring a pair of linear band crossing points (‘Dirac points’) with band velocity $v_D$. When gain and loss are added to alternating sites of the honeycomb lattice, the Bloch Hamiltonian becomes non-Hermitian, and in the vicinity of each Dirac point it takes the form of a Dirac Hamiltonian with imaginary mass. The eigenstates are tachyons whose group velocities are larger than $v_D$. In fact, the group velocities become infinite along a ‘critical’ ring in $k$-space surrounding each Dirac point, corresponding to the PT symmetry breaking transition points of the Bloch Hamiltonian. The propagation of wavepackets faster than $v_D$ has been verified numerically [30]; however,
the notion of group velocity as the slope of the dispersion relation must be reevaluated in non-Hermitian systems. Using the Hellman–Feynman theorem, it has been shown that as the critical ring is reached in $k$-space, significant corrections to this definition of the group velocity arise [31, 32].

This paper describes an alternative way to realize a tachyon-like bandstructure, using a 2D network of coherent waves [12, 13] with non-unitary evolution. Unlike the tight-binding models commonly used in condensed-matter and optical physics, a ‘network model’ does not describe a lattice in terms of a Hamiltonian. Instead, it uses an evolution matrix to describe the propagation of waves through a network of directed links and nodes. As discussed below, such networks can be realized in a variety of ways, such as coupled optical resonator lattices [17–20], microwave networks [23, 24], and RF circuits [25]. Network models can produce bandstructures with various unusual features that are not found in static Hamiltonian models [15]; formally, they can be mapped to the class of ‘Floquet’ systems, described by Hamiltonians that vary periodically in time [11, 26–28]. As we shall see, introducing PT symmetric gain and loss to a square-lattice (not honeycomb) network yields a bandstructure with tachyonic Dirac dispersion relations. But unlike the previously studied tight-binding honeycomb lattice, where the tachyonic Dirac hyperboloids occur in pairs, this network bandstructure contains a single hyperboloid in each gap. Finally, we will show how the tachyonic dispersion relation’s critical $k$-vector can be determined, using transmission measurements across a finite network.

Consider the network model shown schematically in figure 1. It consists of links and nodes, where each link carries a one-directional wave described by a complex scalar amplitude; these are arranged in a 2D square lattice, with each cell containing four links arranged in a chiral loop [12]. Adjacent loops are coupled at the nodes, which are described by $2 \times 2$ scattering matrices. This model was first introduced for studying the transport properties of disordered quantum Hall systems [12]; it captures the essential features of a disordered 2D electron gas in strong magnetic fields, where the electronic orbits follow chiral ‘race-tracks’ along the equipotentials of a disordered potential landscape, and can tunnel to adjacent race-tracks at potential saddle-points [12]. There is now an extensive literature on the use of network models for studying electronic transport; see [13] for a survey.

Recently, researchers have implemented chiral networks in classical electromagnetic settings. One type of realization is an on-chip coupled resonator lattice [16, 18], of the sort proposed and experimentally studied by Hafezi et al. Optical ring resonators are arranged in a lattice, playing the role of the network’s chiral loops. Each pair of adjacent resonators is coupled by an auxiliary ring waveguide, which acts as a node. Due to local momentum conservation at the inter-waveguide interfaces, the optical modes of the lattice decouple into one set of modes where light propagates clockwise in the main rings, and another that is counter-clockwise; each set maps onto a network model. Such resonator lattices can exhibit topological edge states and fractal Hofstadter spectra [16, 18], as well as topological transitions and anomalous topological phases [15, 21, 22, 24]. A chiral network can also be realized using a microwave circuit [23]. The nodes of the network are implemented using directional couplers; auxiliary rings are not necessary, since the microwave components need not be strictly planar. The chirality of the network can be enforced using microwave isolators.

Regardless of the network model’s underlying implementation, its properties can be described theoretically in terms of evolution matrices. And for a disorder-free, spatially infinite network, the evolution matrix description gives rise to a bandstructure [14, 15]. Let us briefly review how this is done. Figure 1(b) shows a schematic of one cell of the network, which contains two couplers connecting adjacent loops along the $\hat{x}$ and $\hat{y}$ directions. These couplers can be described by scattering matrices $S_x$ and $S_y$. We denote the four input wave amplitudes into these couplers by $\{b_1, \ldots, b_4\}$, and the wave amplitudes on the other side of those links by $\{a_1, \ldots, a_4\}$. We assume each link has equal phase delay $\phi$, so that $b_n = e^{i\phi}a_n$. For the moment, we ignore gain and loss, so that $S_x$ and $S_y$ are unitary and $\phi$ is real. Using Bloch’s theorem, we can relate the input and output amplitudes as follows:

\[
S_x \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = a_3 e^{i k_x} \begin{bmatrix} a_1 e^{i k_x} \\ a_2 e^{i k_y} \end{bmatrix},
\]

\[
S_y \begin{bmatrix} b_3 \\ b_4 \end{bmatrix} = a_1 e^{i (k_y - k_x)} \begin{bmatrix} a_2 e^{i k_y} \\ a_3 e^{i k_x} \end{bmatrix},
\]

where $k = [k_x, k_y]$ is the Bloch wave-vector, with the lattice spacings normalized to 1. These can be combined into a single $4 \times 4$ eigenvalue equation:

\[
U(k) \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = e^{-i\phi} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix},
\]

where

\[
U(k) = \begin{bmatrix} 0 & W_x(k)S_x & 0 \\ W_y(k)S_y & 0 & \end{bmatrix},
\]
There are Dirac points at \( \theta_x = \theta_y = \pi/4 \). This describes a coupler which is symmetric under 180° rotations.

Figure 2 shows the spectrum for \( \theta_x = \theta_y = \pi/4 \). There are four bands, joined by Dirac points at

\[
\begin{align*}
\{ & k_x = k_y = 0, \\
& \phi \in [-\pi/4, 3\pi/4] \}
\end{align*}
\]

and

\[
\begin{align*}
\{ & k_x = k_y = \pi, \\
& \phi \in [-3\pi/4, -\pi/4] \}.
\end{align*}
\]

These Dirac points can be conveniently derived by taking the squared matrix [14]:

\[
U^2(k) = \begin{bmatrix}
U_f(k) & 0 \\
0 & U_f(k)
\end{bmatrix},
\]

\[
U_f = W_x S_x W_x S_y, \quad U_s = W_x S_y W_x S_y.
\]

For each \( k \), \( U(k) \) is unitary, and its eigenvalues are \( \exp(-i\phi) \), where the ‘quasi-energies’ \( \phi \) are the discrete values of the link delay for which modes can propagate with the given \( k \).

Because the bandstructure is defined by an evolution operator, \( U(k) \), rather than a Hamiltonian, it falls into the same class as the ‘Floquet’ bandstructures describing periodically driven lattices [11, 26–28]. Note that \( \phi \) is an angle variable, unlike the energy occurring in a conventional Hamiltonian eigenproblem.

The bandstructure depends on the choice of the \( 2 \times 2 \) unitary matrices \( S_x \) and \( S_y \), each parameterized by four Euler angles. It turns out that the bandstructure topology is determined only by one pair of Euler angles, denoted by \( \theta_{x,y} \) in [21], which parameterize the coupling strengths between adjacent loops. The gaps in the quasi-energy bandstructure close when \( \theta_x + \theta_y = \pi/2 \) [21]. For simplicity, we fix the other Euler angles so that

\[
S_{\mu} = \begin{bmatrix}
\cos \theta_{\mu} & i \sin \theta_{\mu} \\
i \sin \theta_{\mu} & \cos \theta_{\mu}
\end{bmatrix}, \text{ for } \mu \in \{x, y\}.
\]

This describes a coupler which is symmetric under 180° rotations (see figure 1(b)), and behaves the same when the order of the two inputs, and the two outputs, are simultaneously swapped.
every band has zero Chern number [15]. This happens because, in the critical bandstructure, each band had a Dirac point above and below, a situation that is possible because \( \phi \) is an angular variable and hence not bounded above or below. Hence, for \( \Delta \theta > 0 \) the network is in an ‘anomalous Floquet insulator’ phase, exhibiting topological edge states despite all bands having zero Chern numbers. Similar anomalous phases are also known to occur in periodically driven Floquet topological insulators [26–28].

We are now ready to consider a network containing gain and loss. In the honeycomb lattice, Szameit et al have previously shown that adding PT symmetric gain and loss to the alternate sublattices distorts the bandstructure’s Dirac cones into hyperboloids, corresponding to two species of emergent tachyons [30]. This is caused by the Bloch states near each Dirac point undergoing spontaneous PT symmetry breaking. In the PT symmetric region, the bands are real and have group velocity exceeding the Dirac velocity \( v_D \). The group velocity approaches infinity at the waists of the hyperboloids, which are the PT symmetry breaking points of the Bloch Hamiltonian.

In the network model, tachyonic behavior can arise by setting \( \Delta \theta \) to be imaginary. This can potentially be realized in the optical resonator domain by using ‘auxiliary rings’ that lie in between the principal rings, and are optically pumped and thus have gain (unpumped rings would naturally exhibit loss). A candidate platform would be that used in [19]. In the context of microwave networks, auxiliary directional couplers could be used in a similar way, in combination with amplifiers. According to equation (10), this gives the effective Dirac Hamiltonian an imaginary mass. For \( \Delta \theta = \gamma \), the coupling matrices become

\[
S_y = \begin{bmatrix} \alpha & -i \alpha^* \\ i \alpha^* & \alpha \end{bmatrix}, \quad \alpha = \frac{\cosh \gamma - i \sinh \gamma}{\sqrt{2}}.
\]  

(12)

This yields the bandstructure shown in figure 3. Each Dirac cone becomes a hyperboloid, corresponding to a tachyonic dispersion relation. Since there was originally only one Dirac cone per gap, the hyperboloids are unpaired, unlike in the PT symmetric honeycomb lattice [30]. The band quasi-energies are all real, except for the regions of \( k_x \) inside the waists of the hyperboloids. Using equation (10), we find the critical wave-numbers

\[
k_c(\gamma) = 2^{3/2} \gamma + O(\gamma^2).
\]  

(13)

The coupling matrix of equation (12) has the same \( 180^\circ \) rotational symmetry as the previously discussed unitary coupling matrix of equation (6). However, for \( \gamma = 0 \) it is manifestly non-unitary. This may be seen from the eigenvalues \( \sigma_\pm = \alpha \pm i \alpha^* \), whose magnitudes are \( |\sigma_\pm| = e^{2\gamma} \). The corresponding eigenvectors are \( |1; \pm 1\rangle \); one of these eigenvectors is amplified, and the other is damped by an equal and opposite amount. This is very similar to the behavior of scattering matrices derived from the wave equation in PT symmetric media [33–35]. Furthermore, the coupling matrix can be decomposed as

\[
S_{\mu} = S_0 \begin{bmatrix} \sigma_+ & 0 \\ 0 & \sigma_- \end{bmatrix}, \quad \text{where } S_0 \equiv \begin{bmatrix} 1 & 1 \\ -i \sqrt{2} & i \sqrt{2} \end{bmatrix}.
\]

(14)

Thus, such a coupler can be implemented by passing the inputs through a unitary 50:50 coupler described by \( S_0 \), applying balanced gain and loss to the results, and then remixing through a second \( S_0 \) coupler.

How might the tachyonic bandstructure be experimentally verified, whether in the context of optical ring resonators or microwave networks? One possibility is to construct a wave-packet and show that its group velocity can exceed the effective Dirac velocity, as discussed in [30]. For the network model, this approach could work if \( \phi \) is proportional to frequency, and the coupling parameters are approximately frequency-independent [22]. However, the natural quantities to study in a network model are the steady-state reflection and transmission for fixed \( \phi \). Here, we present an alternative experimental approach for probing the tachyonic bandstructure, based on measuring the transmission across a set of finite networks.

Let us first examine the bandstructure of a ‘strip’ of network, shown schematically in figure 4(a). The strip extends infinitely in the \( x \) direction, and has a width of \( N \) cells in the \( y \) direction. There are two useful choices of boundary conditions that we can impose on the edges of the strip. Firstly, we can impose periodic edges by making row \( N + 1 \) equivalent to row 1 (i.e. rolling the strip into the surface of cylinder). Secondly, we can impose ‘Dirichlet’ edges by terminating the network at
the edges of the strip, setting $b_1^t = a_1^t$ and $b_1^N = a_1^N$. (We could also introduce phase factors into these edge relations; but that generates additional non-topological edge states, which we are not interested in here.)

Figure 4 shows the band diagram of real-\(\phi\) versus complex-\(k_x\), for the semi-infinite strip. This band diagram is calculated from the eigenvalues of the transfer matrix across one \(x\) period of the strip [15]. The reason for plotting real \(\phi\) versus complex \(k_x\), rather than complex \(\phi\) versus real \(k_x\), is that we will be interested in the propagation of modes at a fixed real quasi-energy \(\phi\), chosen to correspond to one of the Dirac points. We first focus on figures 4(b) and (c), which shows the case of periodic edges. For each real \(\phi\), all the modes are either purely propagating (real \(k_x\), or purely evanescent (imaginary \(k_x\)). The evanescent modes are completely non-propagating (Re\([k_x]\) = 0), and are similar to the evanescent modes which occur within the band gaps of ordinary Hermitian systems. As for the propagating modes, there are specific branches of these modes which have tachyonic dispersion relations, and are highlighted in red and gold in the figure. As indicated in figure 4(c), these modes propagate with no amplification nor dissipation (Im\([k_x]\) = 0).

Figures 4(d) and (e) shows the band diagram for Dirichlet edges. In this case, the modes are no longer purely propagating or purely evanescent, but have complex \(k_x\). This happens because the edge conditions spoil the PT symmetry of the network. Nonetheless, the projected band diagram remains qualitatively similar to figures 4(b) and (c). In particular, there are tachyon modes which are weakly damped (small \(\text{Im}[k_x]\)) compared to the other modes.

A direct experimental signature for these tachyon modes can be found by measuring the transmission across a finite strip. Consider a strip of length \(M\) (in the \(x\) direction). Along column 1, we inject equal wave amplitudes \(a_n^0 = N^{-1/2}\) into each of the rightward links (see the schematic in figure 4(a)); then, \(M\) columns to the right, we calculate the total transmittance \(T = \sum_{n=1}^{N} |a_n^M|^2\). Physically, this corresponds to connecting the left and right edges of a finite network to uniform multi-mode waveguides, which act as scattering leads. The total transmittance is then measured as a function of the gain/loss parameter \(\gamma\), which is assumed to be externally tunable, e.g. by electrical or optical pumping.

The variation of the transmittance with \(\gamma\) is shown in figure 5(a). Here, we take strip length \(M = 10\) and width \(N = 10\), and set the network links to \(\phi = -0.25\pi\), corresponding to one of the Dirac points. The transmission is found to be peaked at certain values of \(\gamma\); the peak positions depend on the choice of periodic or Dirichlet edge conditions, as well as the strip size. Figure 5(b) plots the values of \(\gamma\) at the transmission peaks, versus the strip length \(M\). For periodic edges, the peaks can be fitted to

$$k_v(\gamma) = \frac{(m + 1/2)\pi}{M}, \quad m \in \mathbb{Z}^+,$$  

(15)
where $k_\gamma(\gamma)$ is the tachyonic critical wave-number given by equation (13). For Dirichlet edges, on the other hand, the transmission peaks can be fitted to
\[
k_T(\gamma) = \sqrt{\left(\frac{(m + 1/2)\pi}{M}\right)^2 + \left(\frac{\pi}{N}\right)^2}, \quad m \in \mathbb{Z}^+.
\] (16)

The accuracy of these fits can be seen in figure 5(b), by comparing the solid curves, which are produced from equations (15) and (16), to the circles, which correspond to the numerically obtained transmission peaks.

The relationship between the transmission peaks and the critical wave-number of the tachyon modes can be understood as follows. At the mid-gap quasi-energy $\phi = -0.25\pi$, only the tachyon modes are propagating; the other modes are evanescent, and thus incapable of forming standing-wave resonances. The transmission peaks occur when the strip length $M$ equals $(m + 1/2)/2$ tachyon mode wavelengths, which allows for the largest intensity at the output column (intensity anti-node) relative to the input column (intensity node). Although the tachyon modes have real wave-numbers, they do not overlap exactly with the input amplitudes, so the presence of gain in the network results in overall amplification, i.e., transmission peaks higher than unity. For periodic edges, equation (15) follows from taking the tachyon modes to be plane waves propagating parallel to the strip. For Dirichlet boundary conditions, the tachyon modes must undergo reflections from the strip edges, and taking the lowest-order wave-guide modes leads to equation (16). The dependence of this equation on both the width $N$ and length $M$ emphasizes the fact that the resonances arise from tachyon modes propagating in 2D, described by a PT-symmetric 2D Dirac equation.

In conclusion, we have shown how an isolated tachyonic dispersion can be realized in a photonic network, be it based on optical ring resonators or microwave transmission lines. A possible use of this tachyonic dispersion may be delay lines of possible use of this tachyonic dispersion may be delay lines of wide tunability with low loss.

Acknowledgments

We are grateful to W Hu, H Wang, and Y Jia for helpful discussions. This research was supported by the Singapore National Research Foundation under grant No. NRFF2012-02, and by the Singapore MOE Academic Research Fund Tier 3 grant MOE2011-T3-1-005.

References

[1] Bender C M and Boettcher S 1998 Phys. Rev. Lett. 80 5243

[2] Bender C M, Berry M V and Mandilari A 2002 J. Phys. A: Math. Gen. 35 L467

[3] El-Ganainy R, Makris K G, Christodoulides D N and Musslimani Z H 2007 Opt. Lett. 32 2632–4

[4] Makris K G, El-Ganainy R, Christodoulides D N and Musslimani Z H 2008 Phys. Rev. Lett. 100 103904

[5] Musslimani Z H, Makris K G, El-Ganainy R and Christodoulides D N 2008 Phys. Rev. Lett. 100 030402

[6] Guo A, Salamo G J, Duchesne D, Morandotti R, Volatier-Ravat M, Aimez V, Siviloglou G A and Christodoulides D N 2009 Phys. Rev. Lett. 103 093902

[7] Rüter C E, Makris K G, El-Ganainy R, Christodoulides D N, Segev M and Kip D 2010 Nat. Phys. 6 192

[8] Makris K G et al 2008 Phys. Rev. Lett. 100 103904

[9] Regensburger A, Bersch C, Giri M-A, Onishchukov G, Christodoulides D N and Peschel U 2012 Nature 488 167

[10] Plotnik Y et al 2014 Nat. Mat. 13 57

[11] Reichman M C, Zeuner J M, Plotnik Y, Lumer Y, Podolsky D, Dreisow F, Nolte S, Segev M and Szameit A 2013 Nature 496 196

[12] Chalker J T and Coddington P D 1988 J. Phys. C: Solid State Phys. 21 2665

[13] Kramer B, Ohtsukib T and Kettammanan S 2005 Phys. Rep. 417 211

[14] Ho C-M and Chalker J T 1996 Phys. Rev. B 54 8708

[15] Pasek M and Chong Y D 2014 Phys. Rev. B 89 075113

[16] Hafezi M, Demler E A, Lukin M D and Taylor J M 2011 Nat. Phys. 7 907

[17] Cooper M L, Gupta G, Schneider M A, Green W M J, Assefa S, Xia F, Vlasov Y A and Moekherjea S 2010 Opt. Express 18 26505

[18] Hafezi M, Mittal S, Fan J, Migdall A and Taylor J M 2013 Nat. Photonics 7 1001

[19] Hodaei H, Miri M-A, Heinrich M, Christodoulides D N and Khajavikhan M 2014 Science 21 975

[20] Feng L, Wong Z J, Ma R-M, Wang Y and Zhang X 2014 Science 21 972

[21] Liang G Q and Chong Y D 2013 Phys. Rev. Lett. 110 203904

[22] Liang G Q and Chong Y D 2014 Int. J. Mod. Phys. B 28 1441007

[23] Hu W, Pillay J C, Wu K, Pasek M, Shum P P and Chong Y D 2015 Phys. Rev. X 5 011012

[24] Gao F et al in preparation (arXiv:1504.0789)

[25] Jia N, Owens C, Sommer A, Schuster D and Simon J 2015 Phys. Rev. X 5 021031

[26] Lindner N H, Refael G and Galitski V 2011 Phys. Rev. Lett. 106 075113

[27] Haldane F D M 1988 Phys. Rev. Lett. 61 2015

[28] Szameit A, Rechtsman M C, Bahat-Treidel O and Segev M 2011 Phys. Rev. A 84 021806(R)

[29] Wang L J, Kuzmich A and Dogariu A 2000 Nature 406 277

[30] Schomerus H and Wiersig J 2014 Phys. Rev. A 90 053819

[31] Schomerus H 2010 Phys. Rev. Lett. 104 233601

[32] Chong Y D, Ge L and Stone A D 2011 Phys. Rev. Lett. 106 093902

[33] Ge L, Chong Y D and Stone A D 2012 Phys. Rev. A 85 023802