THE FUNDAMENTAL THEOREMS FOR CURVES AND SURFACES IN 3D HEISENBERG GROUP

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Abstract. We study the local equivalence problems of curves and surfaces in 3-dimensional Heisenberg group via Cartan’s method of moving frames and Lie groups, and find a complete set of invariants for curves and surfaces. For surfaces, in terms of these invariants and their suitable derivatives, we also give a Gaussian curvature formula of the metric induced from the adapted metric on $H^1$, and hence form a new formula for the Euler number of a closed surface.

1. Introduction

In 3-dimensional Euclidean space, it is well known that any unit-speed curve is completely determined by its curvature and torsion. This means that given any two function $k(s)$ and $\tau(s)$ with $k(s) > 0$ everywhere, then there exists a unit-speed curve whose curvature and torsion are $k$ and $\tau$, respectively. In addition, such a unit-speed curve is unique up to a Euclidean rigid motion. This is the fundamental theorem of curves. On the other hand, the fundamental theorem of surfaces says that, instead of the scalar-invariants, the first and second fundamental forms are the complete invariants for surfaces. In this paper we will show that there are the analogous fundamental theorems of curves and surfaces in 3-dimensional Heisenberg group $H^1$.

The Heisenberg group $H^1$ is the space $\mathbb{R}^3$ associated with the group multiplication

\[(x_1, y_1, z_1) \circ (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + y_1 x_2 - x_1 y_2).\]

It is a 3-dimensional Lie group. The space of all left invariant vector fields is spanned by the following three vector fields:

\[\hat{e}_1 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}, \quad \hat{e}_2 = \frac{\partial}{\partial y} - x \frac{\partial}{\partial z} \quad \text{and} \quad T = \frac{\partial}{\partial z}.\]

The standard contact bundle on $H^1$ is the subbundle $\xi_0$ of the tangent bundle $TH^1$ which is spanned by $e_1$ and $e_2$. It is also defined to be the kernel of the contact form

\[\theta_0 = dz + xdy - ydx.\]
The CR structure on $H^1$ is the endomorphism $J_0 : \xi_0 \to \xi_0$ defined by
\begin{equation}
J_0(\hat{e}_1) = \hat{e}_2 \quad \text{and} \quad J_0(\hat{e}_2) = -\hat{e}_1.
\end{equation}

We sometimes view the Heisenberg group $H^1$ as a pseudohermitian manifold when we consider it together with the standard pseudo-hermitian structure $(J_0, \theta_0)$. For the details about pseudo-hermitian structure, we refer the readers to [8], [9], or [11]. Let $P_{SH}(1)$ be the group of Heisenberg rigid motions, that is, the group of all pseudo-hermitian transformations on $H^1$. Recall that a pseudo-hermitian transformation on $H^1$ is a diffeomorphism on $H^1$ which preserves the standard pseudo-hermitian structure $(J_0, \theta_0)$. In Subsection 3.1, we give an explicit expression for a pseudo-hermitian transformation.

Let $\gamma : (a, b) \to H^1$ be a parametrized curve. For each $t \in (a, b)$, the velocity $\gamma'(t)$ of $\gamma(t)$ has the natural decomposition
\begin{equation}
\gamma'(t) = \gamma'_{\xi_0}(t) + \gamma'_T(t),
\end{equation}
where $\gamma'_{\xi_0}(t)$ and $\gamma'_T(t)$ are, respectively, the orthogonal projection of $\gamma'(t)$ on $\xi_0$ along $T$ and the orthogonal projection of $\gamma'(t)$ on $T$ along $\xi_0$.

**Definition 1.1.** A **horizontally regular curve** is a parametrized curve $\gamma(t)$ such that $\gamma'_{\xi_0}(t) \neq 0$ for each $t \in (a, b)$.

Proposition 4.1 shows that a horizontally regular curve can always be reparametrized by a parameter $s$ such that $|\gamma'_{\xi_0}(s)| = 1$ for every $s$. We call such a parameter $s$ the horizontal arc-length, which is unique up to a constant.

For a horizontally regular curve $\gamma(s)$ parametrized by the horizontal arc-length $s$, we define the $p$-curvature $k(s)$ and $T$-variation $\tau(s)$ as
\begin{align}
k(s) &= \left< \frac{dX(s)}{ds}, Y(s) \right> \\
\tau(s) &= \left< \gamma'(s), T \right>,
\end{align}
where $X(s) = \gamma'_{\xi_0}(s)$ and $Y(s) = J_0X(s)$. We have the following fundamental theorem for curves in $H^1$ which says that horizontally regular curves are completely prescribed by the $p$-curvature and $T$-variation as well.

**Theorem 1.2.** Let $\gamma_1(s)$ and $\gamma_2(s)$ be two horizontally regular curves parametrized by the horizontal arc-length. Suppose that they have the same $p$-curvature $k(s)$ and $T$-variation $\tau(s)$. Then there exists $g \in P_{SH}(1)$ such that
\begin{equation}
\gamma_2(s) = g \circ \gamma_1(s), \text{ for all } s.
\end{equation}
In addition, given smooth functions $k(s), \tau(s)$, there exists a horizontally regular curve $\gamma(s)$, parametrized by the horizontal arc-length, having $k(s)$ and $\tau(s)$ as its $p$-curvature and $T$-variation, respectively.

We say $\gamma(t)$ is a horizontal curve if $\gamma'(t) = \gamma''_{30}(t)$ for each $t \in (a, b)$. By the previous definition $\gamma(s)$ is horizontal if and only if the $T$-variation $\tau(s) = 0$, we have immediately the following corollary.

**Corollary 1.3.** Let $\gamma_1(s)$ and $\gamma_2(s)$ be two horizontal unit-speed curves in $H^1$ with the same $p$-curvature $k(s)$. Then there exists $g \in \text{PSH}(1)$ such that

$$\gamma_2(s) = g \circ \gamma_1(s), \text{ for all } s.$$  

In addition, given a smooth function $k(s)$, there exists a horizontal unit-speed curve $\gamma(s)$ having $k(s)$ as its $p$-curvature.

In Subsection 4.2, we compute the explicit formulae for the $p$-curvature and $T$-variation and get the following theorem.

**Theorem 1.4.** Let $\gamma(t) = (x(t), y(t), z(t)) \in H^1$ be a horizontally regular curve, not necessarily parametrized by horizontal arc-length. Then the $p$-curvature $k(t)$ and $T$-variation $\tau(t)$ are having the forms

$$k(t) = \frac{x' y'' - x'' y'}{((x')^2 + (y')^2)^{3/2}}(t)$$

$$\tau(t) = \frac{xy' - x'y' + z'}{((x')^2 + (y')^2)^{1/2}}(t).$$

As an application, we proceed to compute the $p$-curvature and $T$-variation of the geodesics of $H^1$ in Subsection 4.2 and obtain a characteristic description of the geodesics in $H^1$.

**Theorem 1.5.** The geodesics of $H^1$ are just those horizontally regular curves with vanishing $T$-variation and constant $p$-curvature, that is, $\tau = 0$ and $k = c$ for some constant $c \in \mathbb{R}$.

Observing the formula (1.9), which says that the $p$-curvature of $\gamma(t) = (x(t), y(t), z(t))$ is just the signed curvature of the plane curve $\alpha(t) = \pi \circ \gamma(t) = (x(t), y(t))$, where $\pi$ is the projection on $xy$ plane along the $z$-axis. On the other hand, it is well known that the signed curvature completely describes the plane curves, therefore we have immediately the following corollary:

**Corollary 1.6.** If two horizontally regular curves in $H^1$ differ by a Heisenberg rigid motion then their projections on $xy$-plane differ by
a Euclidean rigid motion. In particular, two horizontal curves in $H^1$ differ by a Heisenberg rigid motion if and only if their projections on xy-plane are congruent in the Euclidean plane.

For a surface $\Sigma \subset H^1$ which is embedded in $H^1$, we can also say something about fundamental theorem. First of all, we recall that a singular point $p \in \Sigma$ is a point such that, at $p$, the tangent plane $T_p\Sigma$ coincides with the contact plane $\xi_0(p)$. Therefore outside the singular set (the non-singular part of $\Sigma$), it is integrated to be a one-dimensional foliation for the intersection of $T\Sigma$ and $\xi_0$, which is called the characteristic foliation. Now we define the normal coordinates.

**Definition 1.7.** Let $F : U \to H^1$ be a parametrized surface with coordinates $(u, v)$ on $U \subset \mathbb{R}^2$. We say $F$ is normal if

(1) $F(U)$ is a surface without singular points;
(2) $F_u = \frac{\partial F}{\partial u}$ defines the characteristic foliation on $F(U)$;
(3) $|F_u| = 1$ for each point $(u, v) \in U$, where the norm is respect to the levi-metric on $H^1$.

We call $(u, v)$ a normal coordinates.

It is easy to see that every non-singular point $p \in \Sigma$, there exists a normal coordinates around $p$. For a normal parametrized surface $F : U \to H^1$, let $X = F_u$, $Y = J_0X$ and $T = \frac{\partial}{\partial z}$, we define functions $a, b, c, l$ and $m$ on $U$ by

\[
\begin{align*}
    a &= <F_v, X> \quad b = <F_v, Y> \quad c = <F_v, T> \\
    l &= <F_{uu}, Y> \quad m = <F_{uv}, Y>. 
\end{align*}
\]

They satisfy the integrability conditions

\[
\begin{align*}
    a_u &= bl, \quad b_u = -al + m, \quad c_u = 2b \\
    l_u - m_u &= 0.
\end{align*}
\]

The following theorem says that these functions are complete differential invariants for the map $F$. We call $a, b$ and $c$ the coefficients of the first kind of $F$, and $l, m$ the second kind.

**Theorem 1.8.** Let $U \subset \mathbb{R}^2$ be a simply connected open set. Suppose that $a, b, c, l$ and $m$ are functions on $U$ which satisfy the integrability condition (1.11). Then there exists a normal parametrized surface $F : U \to H^1$ having $a, b, c$ and $l, m$ as the coefficients of the first kind and the second kind, respectively. In addition, if $\tilde{F} : U \to H^1$ is another such a normal parametrized surface, then it differs from $F$ by a Heisenberg rigid motion, that is, there exists a motion $g \in PSH(1)$ such that $\tilde{F}(u, v) = g \circ F(u, v)$ for all $(u, v) \in U$. 

Note that, from (5.30), we see that \( l \), up to a sign, is independent of the choice of the normal coordinates, hence it is a differential invariant of the surface \( F(U) \). Actually \( l \) is the \( p \)-mean curvature. Therefore \( l = 0 \) means that \( F(U) \) is a \( p \)-minimal surface. Such a parametrization \( F : U \to H^1 \) is called a normal \( p \)-minimal parametrized surface. From the integrability condition (1.11), we see that the second kind of coefficient \( m \) is entirely determined by the first kind as

(1.12) \[ m = b_u. \]

The integrability conditions (1.11) hence become to be

(1.13) \[ a_u = 0, \quad b_{uu} = 0, \quad c_u = 2b, \]

and thus we obtain the following corollary of Theorem 1.8.

**Theorem 1.9.** Let \( U \subset \mathbb{R}^2 \) be a simply connected open set. Suppose that \( a, b \) and \( c \) are three functions on \( U \) which satisfy the integrability condition (1.13). Then there exists a normal \( p \)-minimal parametrized surface \( F : U \to H^1 \) having \( a, b \) and \( c \) as the first kind of coefficients of \( F \), and the second kind of coefficient is determined by \( b \) as (1.12). In addition, if \( \tilde{F} : U \to H^1 \) is another such a normal \( p \)-minimal parametrized surface, then it differs from \( F \) by a Heisenberg rigid motion, that is, there exists a motion \( g \in PSH(1) \) such that \( \tilde{F}(u, v) = g \circ F(u, v) \) for all \( (u, v) \in U \).

Besides the \( p \)-mean curvature \( l \), in Section 5, we also show that both \( \alpha = \frac{b}{c} \), up to a sign (which is called the \( p \)-variation), and the adapted metric \( g_{\theta_0} \) restricted to the surface are also invariants of the surface \( F(U) \). Actually \( \alpha \) is the function such that the vector field \( \alpha e_2 + T \) is tangent to the surface, where \( e_2 = J_0 e_1 \) and \( e_1 \) is a unit vector field tangent to the characteristic foliation. Let \( e_\Sigma \) be another unit vector field tangent to the surface which is defined by

\[ e_\Sigma = \frac{\alpha e_2 + T}{\sqrt{1 + \alpha^2}}. \]

We have that these three invariants satisfy the integrability condition:

(1.14) \[ (1 + \alpha^2)^{\frac{3}{2}}(e_\Sigma l) = (1 + \alpha^2)(e_1 e_1 \alpha) - \alpha(e_1 \alpha)^2 + 4\alpha(1 + \alpha^2)(e_1 \alpha) + \alpha(1 + \alpha^2)^2 K + \alpha l(1 + \alpha^2)^{\frac{3}{2}}(e_\Sigma \alpha) + \alpha(1 + \alpha^2)l^2, \]

where \( K \) is the Gaussian curvature with respect to \( g_{\theta_0}|\Sigma \).

The following theorem says that the Riemannian metric induced from the adapted metric together with the \( p \)-mean curvature \( l \) and
$p$-variation $\alpha$ is a complete system of invariants for a surface without singular point.

**Theorem 1.10** (The fundamental theorem for surfaces in $H^1$). Let $(\Sigma, g)$ be a Riemannian 2-manifold with Gaussian curvature $K$, and let $\alpha', l'$ be two real-valued functions on $\Sigma$. Assume that $K$, together with $\alpha'$ and $l'$, satisfy the integrability condition (1.14), with $\alpha, l$ replaced by $\alpha', l'$, respectively. Then for every point $x \in \Sigma$ there exists an open neighborhood $U$ containing $x$, and an embedding $f : U \to H^1$ such that $g = f^*(g_{\theta_0}), \alpha' = f^*\alpha$ and $l' = f^*l$, where $\alpha, l$ are the induced $p$-variation and $p$-mean curvature on $f(U)$. Moreover, $f$ is unique up to a Heisenberg rigid motion.

In the proof of Theorem 1.10, we also get

**Theorem 1.11.** Let $\Sigma \subset H^1$ be an oriented surface. Then the Gaussian curvature $K$ of the restricted metric $g_{\theta_0}|_\Sigma$ can be expressed by means of $l, \alpha$ and the derivatives of $\alpha$.

\begin{equation}
K = \frac{(e_1 \alpha)^2 + 2(1 + \alpha^2)(e_1 \alpha) + 4\alpha^2(1 + \alpha^2) - l(e_\Sigma \alpha)(1 + \alpha^2)^{\frac{3}{2}}}{(1 + \alpha^2)^2}.
\end{equation}

By the Gauss-Bonnet formula, we immediately have the following corollary.

**Theorem 1.12.** Let $\Sigma \subset H^1$ be a closed, oriented surface. Then we have

\begin{equation}
2\pi \chi(\Sigma) = \int_{\Sigma} \frac{(e_1 \alpha)^2 + 2(1 + \alpha^2)(e_1 \alpha) + 4\alpha^2(1 + \alpha^2) - l(e_\Sigma \alpha)(1 + \alpha^2)^{\frac{3}{2}}}{(1 + \alpha^2)^2} d\sigma
= \int_{\Sigma} \frac{(e_1 \alpha)^2 + 2(1 + \alpha^2)(e_1 \alpha) + 4\alpha^2(1 + \alpha^2) - l(e_\Sigma \alpha)(1 + \alpha^2)^{\frac{3}{2}}}{(1 + \alpha^2)^{\frac{3}{2}}} \omega^1 \wedge \theta_0,
\end{equation}

where $d\sigma$ is the area form with respect to the induced metric from the adapted metric $g_{\theta_0}$, and $\chi(\Sigma)$ is the Euler number of $\Sigma$.

Substituting the Gaussian curvature formula (1.15) into (1.14), we see that the integrability condition (1.14) is equivalent to the Gaussian equation (1.15) together with the following Codazzi-Like equation:

\begin{equation}
e_{\Sigma} l = \frac{e_1 e_1 \alpha + 6\alpha(e_1 \alpha) + 4\alpha^3 + \alpha l^2}{(1 + \alpha^2)^{\frac{3}{2}}}.
\end{equation}

**Remark 1.13.** There is also an integrability condition for a surface expressed as a graph of a function $u$, which is called a Codazzi-Like equation and shown up in [5].
We now give a brief outline of this paper. In section 2, we state the two propositions about uniqueness and existence of mappings of a smooth manifold into a Lie group $G$ which underlie the theory. In section 3, we obtain the representation of $PSH(1)$ which is the group of pseudohermitian transformations on $H^1$. Also we discuss how the matrix Lie group $PSH(1)$ interpret as the set of ”frames” on the homogeneous space $H^1 = PSH(1)/SO(2)$. Then from the (left-invariant) Maurer-Cartan form, we immediately get the moving frame formula. In section 4, we compute the Darboux derivative of a lift of a horizontally regular curve in $H^1$ and then to get the fundamental theorem for curves in $H^1$. Moreover, we compute the $p$-curvature and the $T$-variation of a horizontally regular curve and geodesics in $H^1$. In section 5, we compute the Darboux derivative of the lift of a normal parametrized surface. Then we get complete differential invariants for a normal parametrized surface. In section 6, let $\Sigma$ be an oriented surface and $f: \Sigma \to H^1$ be an embedding. We compute the Darboux derivative of the lifting of $f$ to get the fundamental theorem for surfaces in $H^1$. In this section, we also compute the Gaussian formula (1.15) and the integrability condition (1.14). Finally, in section 7, we give another proof for Theorem 1.2.

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2. Calculus on Lie group

Let $M$ be a connected smooth manifold, and let $G \subset GL(n, R)$ be a matrix Lie group with Lie algebra $\mathfrak{g}$ and the (left-invariant) Maurer-Cartan form $\omega$. In this section, we shall give, without proofs, two simple and essential local results concerning smooth maps from a manifold $M$ into a Lie group $G$. These two results play a fundamental role in whole of the paper. For the details, we refer the readers to [6],[7],[10] and [4]. The first of these is

**Theorem 2.1.** Given two maps $f, \tilde{f}: M \to G$, then $\tilde{f}^* \omega = f^* \omega$ if and only if $\tilde{f} = g \cdot f$ for some $g \in G$.

The Lie algebra one-form $f^* \omega$ is usually called the **Darboux derivative** of the map $f: M \to G$. The second one is a well-known existence theorem:
Theorem 2.2. Suppose that \( \phi \) is a \( g \)-valued one form on a simply connected manifold \( M \). Then there exists a map \( f : M \to G \) with \( f^*\omega = \phi \) if and only if \( d\phi = -\phi \wedge \phi \).

Moreover, the resulting map \( f \) is unique up to a group action.

The proof of Theorem 2.2 is strongly dependent on the Frobenius theorem.

3. The group of pseudohermitian transformations on \( H^1 \)

3.1. The pseudohermitian transformations on \( H^1 \). A pseudohermitian transformation on \( H^1 \) is a diffeomorphism \( \Phi \) on \( H^1 \) which preserves both the CR structure \( J_0 \) and the contact form \( \theta_0 \), that is, it satisfies

\[
\Phi^* J = J_0 \Phi^* \quad \text{on } \xi_0 \quad \text{and} \quad \Phi^* \theta = \theta.
\]

Let \( L_p \) be the left translation by \( p \) on the Heisenberg group \( H^1 \). It is easy to see that \( L_p \) is a pseudohermitian transformation. We give another pseudohermitian transformation \( \Phi_R : H^1 \to H^1 \) which is defined by

\[
\Phi_R \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},
\]

where \( R \in SO(2) \) is a \( 2 \times 2 \) orthogonal matrix.

Let \( PSH(1) \) be the group of pseudohermitian transformations on \( H^1 \). The following theorem specifies that the group \( PSH(1) \) consists exactly of all the transformations of the forms \( \Phi_{p,R} = L_p \circ \Phi_R \), that is, a transformation \( \Phi_R \) followed by a left translation \( L_p \). We have

\[
\Phi_{p,R} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax + by + p_1 \\ cx + dy + p_2 \\ (ap_2 - cp_1)x + (bp_2 - dp_1)y + z + p_3 \end{pmatrix},
\]

where \( p = (p_1, p_2, p_3)^t \in H^1 \) and \( R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SO(2) \).

Theorem 3.1. Let \( \phi : H^1 \to H^1 \) be a pseudohermitian transformation. Then \( \Phi = L_p \circ \Phi_R \) for some \( R \in SO(2) \) and \( p \in H^1 \).

Proof. Let \( \Phi : H^1 \to H^1 \) be a pseudohermitian transformation such that \( \Phi(0) = p \). Then the composition \( L_{p^{-1}} \circ \Phi \) is a transformation fixing the origin. Therefore, we reduce the proof of Theorem 3.1 to prove that any pseudohermitian transformation \( \Phi \) with \( \Phi(0) = 0 \) has the form \( \Phi = \Phi_R \) for some \( R \in SO(2) \). This is equivalent to prove the following Lemma:
**Lemma 3.2.** Let $\Phi$ be a pseudohermitian transformation on $H^1$ such that $\Phi(0) = 0$. Then, for any $p \in H^1$, the matrix representation of $\Phi_*(p)$ with respect to $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ is

$$
\Phi_*(p) = \begin{pmatrix}
\cos \alpha(p) & -\sin \alpha(p) & 0 \\
\sin \alpha(p) & \cos \alpha(p) & 0 \\
0 & 0 & 1
\end{pmatrix} (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})
$$

for some real constant $\alpha(p)$ which is independent of $p$. That is $\Phi_*$ is a constant matrix.

Now we prove Lemma 3.2. First we compute the matrix representation of $\Phi_*(p)$ with respect to $(\hat{e}_1, \hat{e}_2, T = \frac{\partial}{\partial z})$. Since, for $i = 1, 2$,

$$
\theta_0(\Phi_* \hat{e}_i) = (\Phi^* \theta_0)(\hat{e}_i) = \theta_0(\hat{e}_i) = 0,
$$

we see that $\xi_0$ is invariant under $\Phi_*$. Furthermore, let $h$ be the Levi metric on $\xi_0$ defined by $h(X, Y) = d\theta_0(X, J_0Y)$. We have

$$
\Phi^* h(X, Y) = h(\Phi_* X, \Phi_* Y) = d\theta_0(\Phi_* X, J_0 \Phi_* Y)
$$

$$
= d\theta_0(\Phi_* X, \Phi_* J_0 Y) = \Phi^* (d\theta_0)(X, J_0 Y) = d(\Phi^* \theta_0)(X, J_0 Y)
$$

$$
= d\theta_0(X, J_0 Y) = h(X, Y).
$$

That is $h(\Phi_* X, \Phi_* Y) = h(X, Y)$ for every $X, Y \in \xi_0 = \ker \theta_0$. Thus $\Phi_*$ is orthogonal on $\xi_0$. On the other hand,

$$
\theta_0(\Phi_* T) = \theta_0 \left( \Phi_* \frac{\partial}{\partial z} \right) = (\Phi^* \theta_0) \left( \frac{\partial}{\partial z} \right) = \theta_0 \left( \frac{\partial}{\partial z} \right) = 1,
$$

and, for all $X \in \xi_0$,

$$
d\theta_0(X, \Phi_* T) = d\theta_0(\Phi_* \Phi_*^{-1} X, \Phi_* T) = (\Phi^* d\theta_0)(\Phi_*^{-1} X, T)
$$

$$
= (d\Phi^* \theta_0)(\Phi_*^{-1} X, T) = d\theta_0(\Phi_*^{-1} X, T) = 0.
$$

By the uniqueness of the characteristic vector field, we have $\Phi_* T = T$. From the above argument, we conclude that

$$
\Phi_*(p) = \begin{pmatrix}
\cos \alpha(p) & -\sin \alpha(p) & 0 \\
\sin \alpha(p) & \cos \alpha(p) & 0 \\
0 & 0 & 1
\end{pmatrix} (\hat{e}_1, \hat{e}_2, \frac{\partial}{\partial z})
$$

for some real valued function $\alpha$ on $H^1$.

Next, let $\Phi = (\Phi^1, \Phi^2, \Phi^3)$, we would like to change the matrix representation of $\Phi_*(p)$ from $(\hat{e}_1, \hat{e}_2, \frac{\partial}{\partial z})$ to $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$. Let $p = (p_1, p_2, p_3)$,

$$
\hat{e}_1(p) = \frac{\partial}{\partial x} + p_2 \frac{\partial}{\partial z} \text{ and } \hat{e}_2(p) = \frac{\partial}{\partial y} - p_1 \frac{\partial}{\partial z}.
$$

Then
\[ \Phi_*(p) \left( \frac{\partial}{\partial x} \right) = \Phi_*(p) \left[ \hat{e}_1(p) - p_2 \frac{\partial}{\partial z} \right] = \Phi_*(p) \left[ \hat{e}_1(p) \right] - p_2 \frac{\partial}{\partial z} \]

\[ = \cos \alpha(p) \hat{e}_1(\Phi(p)) + \sin \alpha(p) \hat{e}_2(\Phi(p)) - p_2 \frac{\partial}{\partial z} \]

\[ = \cos \alpha(p) \frac{\partial}{\partial x} + \sin \alpha(p) \frac{\partial}{\partial y} \]

\[ + \left[ \cos \alpha(p) \Phi^2(p) - \sin \alpha(p) \Phi^1(p) - p_2 \right] \frac{\partial}{\partial z}, \]

and

\[ \Phi_*(p) \left( \frac{\partial}{\partial y} \right) = \Phi_*(p) \left[ \hat{e}_2(p) + p_1 \frac{\partial}{\partial z} \right] = \Phi_*(p) \left[ \hat{e}_2(p) \right] + p_1 \frac{\partial}{\partial z} \]

\[ = -\sin \alpha(p) \hat{e}_1(\Phi(p)) + \cos \alpha(p) \hat{e}_2(\Phi(p)) + p_1 \frac{\partial}{\partial z} \]

\[ = -\sin \alpha(p) \frac{\partial}{\partial x} + \cos \alpha(p) \frac{\partial}{\partial y} \]

\[ + \left[ -\sin \alpha(p) \Phi^2(p) - \cos \alpha(p) \Phi^1(p) + p_1 \right] \frac{\partial}{\partial z}. \]

Thus,

\[ \Phi_*(p) = \begin{pmatrix} \cos \alpha(p) & -\sin \alpha(p) & 0 \\ \sin \alpha(p) & \cos \alpha(p) & 0 \\ \Phi_2^3(p) & \Phi_2^3(p) & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}, \]

where

\[ \Phi_2^3(p) = \cos \alpha(p) \Phi^2(p) - \sin \alpha(p) \Phi^1(p) - p_2, \]

\[ \Phi_3^3(p) = -\sin \alpha(p) \Phi^2(p) - \cos \alpha(p) \Phi^1(p) + p_1. \]

Observing first that, from (3.5), \( \Phi_2^1 = \Phi_3^2 = 0 \), so both \( \Phi_1 \) and \( \Phi_2 \) are function depending only on \( x \) and \( y \), hence so is \( \alpha \). Secondly, since \( \Phi_{xy} = \Phi_{yx} = \Phi_{yx} = \Phi_{yx} \), we have, from (3.5),

\[ \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \alpha_x \\ \alpha_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \]

which implies that \( \alpha_x = \alpha_y = 0 \). Thus \( \alpha \) is a constant on \( H^1 \), say \( \alpha = \alpha_0 \). From (3.5) again and note that \( \Phi(0) = 0 \), we have that

\[ \Phi^1 = x \cos \alpha_0 - y \sin \alpha_0 \]

\[ \Phi^2 = x \sin \alpha_0 + y \cos \alpha_0, \]
which implies that $\Phi^3_x = \Phi^3_y = 0$. Thus

$$\Phi_x(p) = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{\varphi}{\varphi_0 \varphi_1 \varphi_2}.$$ 

This completes the proof. \[\square\]

3.2. Representation of $\text{PSH}(1)$. We can represent $\Phi_{p,R}$ and points of $H^1$, respectively, as

$$\Phi_{p,R} \leftrightarrow M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ p_1 & a & b & 0 \\ p_2 & c & d & 0 \\ p_3 & ap_2 - cp_1 & bp_2 - dp_1 & 1 \end{pmatrix},$$ 

and

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \leftrightarrow X = \begin{pmatrix} 1 \\ x \\ y \\ z \end{pmatrix}.$$ 

Then

$$MX = \begin{pmatrix} 1 \\ \Phi_{p,R} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{pmatrix}.$$ 

That is, $\text{PSH}(1)$ may be represented as a matrix group by writing

$$\text{PSH}(1) = \left\{ M \in \text{GL}(4, R) \bigg| M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ p_1 & a & b & 0 \\ p_2 & c & d & 0 \\ p_3 & ap_2 - cp_1 & bp_2 - dp_1 & 1 \end{pmatrix} \right\}.$$ 

Let $\text{psh}(1)$ be the Lie algebra of $\text{PSH}(1)$. Then it is easy to see that the element of $\text{psh}(1)$ is look as

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ x_1 & 0 & -x_1^2 & 0 \\ x_2 & x_1^2 & 0 & 0 \\ x_3 & x_3 & -x_1 & 0 \end{pmatrix}. $$
Therefore the Maurer-Cartan form of $PSH(1)$ is look like

$$\omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & -\omega^2 & 0 \\ \omega^2 & \omega^1 & 0 & 0 \\ \omega^3 & \omega^2 & -\omega^1 & 0 \end{pmatrix},$$

here $\omega_i^2$ and $\omega^j$, $j = 1,2,3$ are 1-forms on $PSH(1)$.

3.3. The oriented frames on $H^1$. An oriented frame on $H^1$ is a frame of the form

$$(p; X, Y, T),$$

where $p \in H^1$, $Y = J_0X$ and $X \in \xi_0(p)$ are unit vectors with respect to the standard Levi metric on $H^1$. We can also identify $PSH(1)$ with the space of all oriented frames on $H^1$ as following:

$$(3.14) \quad M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ p_1 & a & b & 0 \\ p_2 & c & d & 0 \\ p_3 & ap_2 - cp_1 & bp_2 - dp_1 & 1 \end{pmatrix} \leftrightarrow (p; X, Y, T),$$

where

$$X = a \frac{\partial}{\partial x} + c \frac{\partial}{\partial y} + (ap_2 - cp_1) \frac{\partial}{\partial t}$$

$$Y = b \frac{\partial}{\partial x} + d \frac{\partial}{\partial y} + (bp_2 - dp_1) \frac{\partial}{\partial t}$$

$p = (p_1, p_2, p_3)^t$.

Actually, we have that $X = a\dot{e}_1(p) + c\dot{e}_2(p)$ and $Y = b\dot{e}_1(p) + d\dot{e}_2(p)$, hence $M$ is the unique $4 \times 4$ matrix such that

$$(3.16) \quad (p; X, Y, T) = (0; \dot{e}_1, \dot{e}_2, T)M.$$

3.4. Moving frame formula. Since $PSH(1)$ is a matrix Lie group, the Maurer-Cartan form is to be $\omega = M^{-1} dM$ or $dM = M \omega$. Thus we immediately get that

$$(3.17) \quad (dp; dX, dY, dT) = (p; X, Y, T) \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & -\omega^2 & 0 \\ \omega^2 & \omega^1 & 0 & 0 \\ \omega^3 & \omega^2 & -\omega^1 & 0 \end{pmatrix},$$
that is, we have the following moving frame formula:

\[
\begin{align*}
    dp &= X\omega^1 + Y\omega^2 + T\omega^3 \\
dX &= Y\omega_1^2 + T\omega^2 \\
dY &= -X\omega_1^2 - T\omega^1 \\
dT &= 0.
\end{align*}
\] (3.18)

4. DIFFERENTIAL INVARIANTS OF HORIZONTALLY REGULAR CURVES IN $H^1$

**Proposition 4.1.** We can reparametrize a horizontally regular curve $\gamma(t)$ by a horizontal arc-length $s$

**Proof.** Define $s(t) = \int_0^t |\gamma'_{\xi_0}(u)| du$. Then any horizontal arc-length differs $s$ by a constant. By the fundamental theorem of calculus, we have \( \frac{ds}{dt} = |\gamma'_{\xi_0}(t)| \). So

\[
\frac{d\gamma}{ds} = \frac{d\gamma}{dt} \frac{dt}{ds} = \frac{\gamma'(t)}{|\gamma'_{\xi_0}(t)|},
\]

hence $\gamma'_{\xi_0}(s) = \frac{\gamma'_{\xi_0}(t)}{|\gamma'_{\xi_0}(t)|}$, that is $|\gamma'_{\xi_0}(s)| = 1$. \[\Box\]

**Definition 4.2.** A lift of a mapping $f : M \to G/H$ is defined to be a map $F : M \to G$ such that the following diagram commutes:

\[
\begin{array}{ccc}
    F & \downarrow & G \\
    M & \xrightarrow{f} & G/H
\end{array}
\]

where $G$ is a Lie group, $H$ is a closed Lie subgroup and $G/H$ is a homogeneous space. Given a lift $F$ of $f$, any other lift $\tilde{F} : M \to G$ must be of the form

\[
\tilde{F}(x) = F(x)g(x)
\]

for some map $g : M \to H$.

4.1. **The Proof of Theorem 1.2.** Let $\gamma(s)$ be a horizontally regular curve with horizontal arc-length as parameter. For each point of the curve uniquely determines an oriented frame of $H^1$ of the form

\[
(\gamma(s); X(s), Y(s), T),
\]

where $X(s) = \gamma'_{\xi_0}(s)$ and $Y(s) = J_0X(s)$. Define $\tilde{\gamma}(s)$ by

\[
\tilde{\gamma}(s) = (\gamma(s); X(s), Y(s), T).
\] (4.3)
Then $\tilde{\gamma}(s)$ is a lift of $\gamma(s)$ to $PSH(1)$, which is uniquely determined by $\gamma(s)$. Let $\omega$ be the Maurer-Cartan form of $PSH(1)$. We would like to compute the Darboux derivative $\tilde{\gamma}^* \omega$ of the curve $\tilde{\gamma}(s)$: 

First note that all pull back one-forms by $\tilde{\gamma}$ are multiples of $ds$. By (3.18), we have that 

$$d\tilde{\gamma}(s) = \tilde{\gamma}^* dp = Y(s)\tilde{\gamma}^* \omega^2 + T\tilde{\gamma}^* \omega^3. \tag{4.4}$$

On the other hand, 

$$d\tilde{\gamma}(s) = \gamma'_\xi(s) ds + \gamma'_T(s) ds = X(s)ds + \gamma'_T(s)ds. \tag{4.5}$$

Comparing (4.4) and (4.5), we get 

$$\tilde{\gamma}^* \omega^1 = ds, \quad \tilde{\gamma}^* \omega^2 = 0 \tag{4.6}$$

$$\tilde{\gamma}^* \omega^3 = \langle \gamma'(s), T \rangle ds = \tau(s)ds.$$

Again from (3.18), we have 

$$dX(s) = Y(s)\tilde{\gamma}^* \omega^1 + T\tilde{\gamma}^* \omega^2 = Y(s)\tilde{\gamma}^* \omega^1, \tag{4.7}$$

hence 

$$\tilde{\gamma}^* \omega^2 = \langle \frac{dX(s)}{ds}, Y(s) \rangle ds = k(s)ds. \tag{4.8}$$

Thus we have already obtained the Darboux derivative of $\tilde{\gamma}$: 

$$\tilde{\gamma}^* \omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -k(s) & 0 \\ 0 & k(s) & 0 & 0 \\ \tau(s) & 0 & -1 & 0 \end{pmatrix} ds. \tag{4.9}$$

Now suppose that $\gamma_1$ and $\gamma_2$ have the same $p$-curvature $k(s)$ and $T$-variation $\tau(s)$. Then, from (4.9), we get 

$$\tilde{\gamma}_1^* \omega = \tilde{\gamma}_2^* \omega.$$

Therefore, by Theorem 2.1 there exists $g \in PSH(1)$ such that $\tilde{\gamma}_2(s) = g \circ \tilde{\gamma}_1(s)$, hence $\gamma_2(s) = g \circ \gamma_1(s)$, for all $s$. This completes the uniqueness up to a group action. To finish the proof of Theorem 1.2 we show the existence. Given two functions $k(s)$ and $\tau(s)$ defined on an open interval $I$. Define a $psh(1)$-valued one-form $\varphi$ on $I$ by 

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -k(s) & 0 \\ 0 & k(s) & 0 & 0 \\ \tau(s) & 0 & -1 & 0 \end{pmatrix} ds.$$
Then it is easy to show that \( d\varphi + \varphi \wedge \varphi = 0 \). Thus, by Theorem 2.2, there exists a curve

\[
\tilde{\gamma}(s) = (\gamma(s), X(s), Y(s), T) \in PSH(1)
\]
such that \( \tilde{\gamma}^* \omega = \varphi \). This means, by moving frame formula (3.18),

\[
d\gamma(s) = X(s) ds + \tau(s) T ds \\
dX(s) = k(s) Y(s) ds \\
dY(s) = -k(s) X(s) ds - T ds,
\]

which implies that

\[
X(s) = \gamma'_{\xi_0}(s), \quad \text{and} \\
k(s) = \langle \frac{dX(s)}{ds}, Y(s) \rangle \\
\tau(s) = \langle \frac{d\gamma(s)}{ds}, T \rangle.
\]

This completes the proof of the existence.

### 4.2. The computation of the \( p \)-curvature and the \( T \)-variation.

In this subsection, we will compute the \( p \)-curvature and the \( T \)-variation of a horizontally regular curve, and thus give the proof of Theorem 1.4. After this, we also want to compute the \( p \)-curvature and the \( T \)-variation of the geodesics of \( H^1 \). Let \( \gamma(t) = (x(t), y(t), z(t)) \) be a horizontally regular curve. The horizontal arc-length \( s \) is defined by

\[
s(t) = \int_0^t |\gamma'_{\xi_0}(u)| du,
\]

where \( \gamma'_{\xi_0}(t) \) is the projection of \( \gamma'(t) \) on \( \xi_0 \) along \( T \) direction. Now

\[
\gamma'(t) = (x'(t), y'(t), z'(t)) = x'(t) \frac{\partial}{\partial x} + y'(t) \frac{\partial}{\partial y} + z'(t) \frac{\partial}{\partial z}
\]

\[
= x'(t)e_1 + y'(t)e_2 + (z'(t) + xy'(t) - yx'(t)) \frac{\partial}{\partial z},
\]

which shows that

\[
\gamma'_{\xi_0}(t) = x'(t)e_1 + y'(t)e_2; \\
\gamma_T(t) = (z'(t) + xy'(t) - yx'(t)) T,
\]

where note that \( \frac{\partial}{\partial z} = T \). Let \( \bar{\gamma}(s) \) be the reparametrization of \( \gamma(t) \) by the horizontal arc-length \( s \). Then we have that \( \gamma'(t) = \bar{\gamma}'(s) \frac{ds}{dt} \), hence,
comparing with (4.14),
\[
\bar{\gamma}_0'(s) = \frac{dt}{ds}(x'(t)e_1 + y'(t)e_2); \\
(4.15)
\]
\[
\bar{\gamma}_T'(s) = \frac{dt}{ds}\left( (z'(t) + xy'(t) - yx'(t))T \right).
\]
So the \(T\)-variation is
\[
\tau(s) = \langle \bar{\gamma}'(s), T \rangle = \langle \bar{\gamma}'_T(s), T \rangle
\]
\[
= \frac{dt}{ds}(z'(t) + xy'(t) - yx'(t)) \\
= \frac{xy' - x'y + z'}{((x')^2 + (y')^2)^{3/2}}(t).
\]

For the \(p\)-curvature, first note that \(X(s) = \frac{dt}{ds}(x'(t)e_1 + y'(t)e_2)\), hence \(Y(s) = J_0X(s) = \frac{dt}{ds}(x'(t)e_2 - y'(t)e_1)\). We compute
\[
\frac{dX(s)}{ds} = \left. \frac{dt}{ds} \right( \frac{dt}{ds}(x'(t), y'(t), x'y(t) - xy'(t)) \right) \\
= \left( x''(t) \left( \frac{dt}{ds} \right)^2 + x'(t) \frac{d^2t}{ds^2} \right) e_1 + \left( y''(t) \left( \frac{dt}{ds} \right)^2 + y'(t) \frac{d^2t}{ds^2} \right) e_2.
\]
So
\[
k(s) = \langle \frac{dX(s)}{ds}, Y(s) \rangle \\
= - \left( x''(t) \left( \frac{dt}{ds} \right)^2 + x'(t) \frac{d^2t}{ds^2} \right) y'(t) \frac{dt}{ds} + \left( y''(t) \left( \frac{dt}{ds} \right)^2 + y'(t) \frac{d^2t}{ds^2} \right) x'(t) \frac{dt}{ds} \\
= - \left( x''(t)y'(t) - x'(t)y''(t) \right) \left( \frac{dt}{ds} \right)^3 \\
= \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{3/2}}(t).
\]
This completes the proof of Theorem 1.4.
Now we make use of (4.18) and (4.16) to compute the \(p\)-curvature and \(T\)-variation of the geodesics in \(H^1\). Recall that the Hamiltonian
we have (4.19)
\[ \dot{x}^k (t) = h^{kj} (x(t)) \xi_j (t) \]
\[ \dot{\xi}_k (t) = -\frac{1}{2} \sum_{i,j=1}^3 \frac{\partial h^{ij} (x)}{\partial x^i} \xi_i \xi_j, \ k = 1, 2, 3, \]

where
\[ h^{ij} (x^1, x^2, x^3) = \begin{pmatrix} 1 & 0 & x^2 \\ 0 & 1 & -x^1 \\ x^2 & -x^1 & (x^1)^2 + (x^2)^2 \end{pmatrix}. \]

So the Hamiltonian system (4.19) can be expressed by
\begin{align*}
\dot{x}^1 (t) &= \xi_1 + x^2 \dot{\xi}_3 \\
\dot{x}^2 (t) &= \xi_2 - x^1 \dot{\xi}_3 \\
\dot{x}^3 (t) &= x^2 \xi_1 - x^1 \xi_2 + \xi_3 \left[ (x^1)^2 + (x^2)^2 \right] \\
\dot{\xi}_1 (t) &= \xi_2 \dot{\xi}_3 - x^1 \xi_3^2 \\
\dot{\xi}_2 (t) &= -\xi_1 \xi_3 - x^2 \xi_3^2 \\
\dot{\xi}_3 (t) &= 0.
\end{align*}

Since \( \dot{\xi}_3 (t) = 0 \), thus \( \xi_3 (t) = c_3 \) where \( c_3 \) is some constant. In the case \( c_3 = 0 \), we have that \( x (t) = (c_1 t + d_1, c_2 t + d_2, (c_1 d_2 - c_2 d_1) t + d_3) \), thus \( k (t) = 0 \) and \( \tau (t) = 0 \). Next, in the case \( c_3 > 0 \), we have
\begin{align*}
x (t) &= (x^1 (t), x^2 (t), x^3 (t)), \text{ where} \\
x^1 (t) &= a_1 \sin (2c_3 t) + a_2 \cos (2c_3 t) + d_1 \\
x^2 (t) &= -a_2 \sin (2c_3 t) + a_1 \cos (2c_3 t) + d_2 \\
x^3 (t) &= (a_2 d_1 + a_1 d_2) \sin (2c_3 t) + (a_2 d_2 - a_1 d_1) \cos (2c_3 t) + 2c_3 (a_1^2 + a_2^2) t + d_3,
\end{align*}

hence \( k (t) = -\frac{1}{[a_1^2 + a_2^2]^{\frac{1}{4}}} < 0 \) and \( \tau (t) = 0 \). Finally, in the case \( c_3 < 0 \), we have
\begin{align*}
x (t) &= (x^1 (t), x^2 (t), x^3 (t)), \text{ where} \\
x^1 (t) &= a_1 \sin (-2c_3 t) + a_2 \cos (-2c_3 t) + d_1 \\
x^2 (t) &= a_2 \sin (-2c_3 t) - a_1 \cos (-2c_3 t) + d_2 \\
x^3 (t) &= (a_1 d_1 + a_2 d_2) \sin (-2c_3 t) - (a_2 d_1 - a_1 d_2) \cos (-2c_3 t) + 2c_3 (a_1^2 + a_2^2) t + d_3,
\end{align*}

hence \( k (t) = \frac{1}{[a_1^2 + a_2^2]^{\frac{1}{4}}} > 0 \) and \( \tau (t) = 0 \).
The above computation shows that a horizontal curve is congruent to a geodeic if it has positive constant $p$-curvature. Conversely, it is easy to see that a symmetry action of a geodesic is still a geodesic. Therefore we complete the proof of Theorem 1.5.

Remark 4.3. Actually, the geodesics (4.21) for $c_3 > 0$ are the reverse of the geodesics (4.22) for $c_3 < 0$. That is, they run in the reverse direction of each other.

5. Differential invariants of parametrized surfaces in $H^1$

5.1. The proof of Theorem 1.8. First we show the uniqueness. Let $F : U \rightarrow H^1$ be a normal parametrized surface with $a, b, c, l$ and $m$ as the coefficients. That is,

\begin{align}
a &= \langle F_v, X \rangle & b &= \langle F_v, Y \rangle & c &= \langle F_v, T \rangle \\
l &= \langle F_{uu}, Y \rangle & m &= \langle F_{uv}, Y \rangle.
\end{align}

Defining the unique lift $\widetilde{F}$ of $F$ to $PSH(1)$ as

\begin{align}
\widetilde{F} &= \langle F, X, Y, T \rangle, \quad X = F_u, \quad JX = Y,
\end{align}

we would like to compute the Darboux derivative $\widetilde{F}^*\omega$ of $\widetilde{F}$: By the moving frame formula (3.18), we see that

\begin{align}
dF(u, v) &= X(\widetilde{F}^*\omega^1) + Y(\widetilde{F}^*\omega^2) + T(\widetilde{F}^*\omega^3) \\
&= F_u du + F_v dv.
\end{align}

This implies that

\begin{align}
F_u &= dF\left(\frac{\partial}{\partial u}\right) = X(\widetilde{F}^*\omega^1)\left(\frac{\partial}{\partial u}\right) + Y(\widetilde{F}^*\omega^2)\left(\frac{\partial}{\partial u}\right) + T(\widetilde{F}^*\omega^3)\left(\frac{\partial}{\partial u}\right); \\
F_v &= dF\left(\frac{\partial}{\partial v}\right) = X(\widetilde{F}^*\omega^1)\left(\frac{\partial}{\partial v}\right) + Y(\widetilde{F}^*\omega^2)\left(\frac{\partial}{\partial v}\right) + T(\widetilde{F}^*\omega^3)\left(\frac{\partial}{\partial v}\right),
\end{align}

hence, comparing the coefficients and note that $F_u = X$, we have

\begin{align}
(\widetilde{F}^*\omega^1)\left(\frac{\partial}{\partial u}\right) &= 1, & (\widetilde{F}^*\omega^2)\left(\frac{\partial}{\partial u}\right) &= (\widetilde{F}^*\omega^3)\left(\frac{\partial}{\partial u}\right) = 0,
\end{align}

and

\begin{align}
(\widetilde{F}^*\omega^1)\left(\frac{\partial}{\partial v}\right) &= \langle F_v, X \rangle = a \\
(\widetilde{F}^*\omega^2)\left(\frac{\partial}{\partial v}\right) &= \langle F_v, Y \rangle = b \\
(\widetilde{F}^*\omega^3)\left(\frac{\partial}{\partial v}\right) &= \langle F_v, T \rangle = c.
\end{align}
From (5.3) and (5.6), we get
\[
\tilde{F}^*\omega^1 = du + adv
\]
(5.7)
\[
\tilde{F}^*\omega^2 = bdv
\]
\[
\tilde{F}^*\omega^3 = cdv.
\]
On the other hand, again using the moving frame formula (3.18),
\[
dX(u, v) = Y(\tilde{F}^*\omega^2_1) + T(\tilde{F}^*\omega^2)
\]
(5.8)
\[
= (\tilde{F}^*\omega^2_1)(\frac{\partial}{\partial u})Y du + (\tilde{F}^*\omega^2)(\frac{\partial}{\partial v})Y dv + bT dv.
\]
Note again that \(X = F_u\), we have
\[
dX(u, v) = dF_u(u, v) = F_{uu}du + F_{uv}dv.
\]
Comparing the above two formulae, we obtain
\[
(\tilde{F}^*\omega^2_1)(\frac{\partial}{\partial u}) = \langle F_{uu}, Y \rangle = l
\]
(5.10)
\[
(\tilde{F}^*\omega^2_1)(\frac{\partial}{\partial v}) = \langle F_{uv}, Y \rangle = m
\]
\[
b = \langle F_{uv}, T \rangle
\]
\[
0 = \langle F_{uv}, X \rangle = \langle F_{uu}, X \rangle = \langle F_{uu}, T \rangle.
\]
In particular, combining (5.7) and (5.10), we get the Darboux derivative \(\tilde{F}^*\omega\) which is
\[
\tilde{F}^*\omega = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
bdv & ldu + mdv & 0 & 0 & 0 \\
\end{pmatrix}
\]
(5.11)
This completes the proof of the uniqueness. Now we prove the existence. Suppose \(a, b, c\) and \(m, l\) are functions defined on \(U\). Define a \(psh(1)\)-valued one form \(\phi\) by
\[
\phi = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
bdv & ldu + mdv & 0 & 0 & 0 \\
\end{pmatrix}
\]
(5.12)
Then we have
\[
d\phi = \begin{pmatrix}
0 & 0 & \frac{\partial m}{\partial v} - \frac{\partial m}{\partial u} & 0 & 0 \\
\frac{\partial a}{\partial u} - \frac{\partial a}{\partial v} + \frac{\partial m}{\partial u} & \frac{\partial m}{\partial u} & 0 & 0 & 0
\end{pmatrix} du \wedge dv
\]
(5.13)
\[ \phi \wedge \phi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -lb & 0 & 0 & 0 \\ al - m & 0 & 0 & 0 \\ -2b & -m + al & b & 0 \end{pmatrix} \, du \wedge dv. \]

Therefore we get that \( \phi \) satisfies the integrability condition \( d\phi = -\phi \wedge \phi \) if and only if \( a, b, c, l \) and \( m \) satisfy the integrability condition (1.11). Therefore, by Theorem 2.2, there exists a map \( \tilde{F}^*(u, v) = (F(u, v), X(u, v), Y(u, v), T) \) such that \( \tilde{F}^* \omega = \phi \). Thus, by the moving frame formula (3.18), we see that \( F : U \to H^1 \) is a map with \( a, b, c, l \) and \( m \) as its coefficients.

5.2. Invariants of surfaces. Let \( \Sigma \hookrightarrow H^1 \) be a surface such that each point of \( \Sigma \) is regular. For each point \( p \in \Sigma \), one can choose a parametrization \( F : U \to \Sigma \) with coordinates \( (u, v) \) such that

\[ F_u = \frac{\partial F}{\partial u} = X, \]

where \( X \) is an unit vector field defining the characteristic foliation around \( p \). We call \( F \) and \( (u, v) \) a normal parametrization and a normal coordinates around \( p \), respectively.

**Lemma 5.1.** The normal coordinates is determined up to a transformation of the form

\[ \tilde{u} = \pm u + g(v) \]
\[ \tilde{v} = h(v), \]

for some smooth functions \( g(v), h(v) \) such that \( \frac{\partial h}{\partial v} \neq 0 \).

**Proof.** Suppose that \( (\tilde{u}, \tilde{v}) \) is another normal coordinates around \( p \), i.e.,

\[ F_{\tilde{u}} = \tilde{X}, \]

where \( \tilde{X} = \pm X \). We have

\[ F_u = F_{\tilde{u}} \frac{\partial \tilde{u}}{\partial u} + F_{\tilde{v}} \frac{\partial \tilde{v}}{\partial u}, \]
\[ F_v = F_{\tilde{u}} \frac{\partial \tilde{u}}{\partial v} + F_{\tilde{v}} \frac{\partial \tilde{v}}{\partial v}. \]
Expand $F_v = \tilde{a} \tilde{X} + \tilde{b} \tilde{Y} + \tilde{c} \tilde{T}$. By the first identity of (5.18), we have

$$X = \tilde{X} \frac{\partial \tilde{u}}{\partial u} + \left( \tilde{a} \frac{\partial \tilde{v}}{\partial u} \tilde{X} + \tilde{b} \frac{\partial \tilde{v}}{\partial u} \tilde{Y} + \tilde{c} \frac{\partial \tilde{v}}{\partial u} \tilde{T} \right)$$

(5.19)

$$= \left( \frac{\partial \tilde{u}}{\partial u} + \tilde{a} \frac{\partial \tilde{v}}{\partial u} \right) \tilde{X} + \tilde{b} \frac{\partial \tilde{v}}{\partial u} \tilde{Y} + \tilde{c} \frac{\partial \tilde{v}}{\partial u} \tilde{T}.$$ Since $p$ is regular, we see that $\tilde{c} \neq 0$ around $p$, we conclude from the above formula

(5.20) $\frac{\partial \tilde{v}}{\partial u} = 0$, that is, $\tilde{v} = h(v)$.

for some function $h(v)$. In addition, comparing the coefficient of $X$, we have

(5.21) $\pm 1 = \frac{\partial \tilde{u}}{\partial u} + \tilde{a} \frac{\partial \tilde{v}}{\partial u} = \frac{\partial \tilde{u}}{\partial u}$,

hence $\tilde{u} = \pm u + g(v)$ for some function $g(v)$. Finally we compute

(5.22) $\det \left( \frac{\partial \tilde{u}}{\partial u} \frac{\partial \tilde{v}}{\partial u} \frac{\partial \tilde{v}}{\partial v} \right) = \det \left( \pm 1 \frac{\partial \tilde{u}}{\partial u} \frac{\partial \tilde{u}}{\partial v} \right) = \pm \frac{\partial h}{\partial v} \neq 0.$

This completes the proof.

Recall that by means of a normal parametrization $F$, we compute the Darboux derivative $\tilde{F}^* \omega$ as (5.11). One can define four one-forms on $\Sigma$ locally as follows:

(5.23) $I = \tilde{F}^* \omega^1 = du + adv$, $II = \tilde{F}^* \omega^2 = bdv$, $III = \tilde{F}^* \omega^3 = cdv$

$IV = \tilde{F}^* \omega^4_1 = ldu + mdv$,

where functions $a, b, c, m$ and $l$ are defined as (1.10). Let $(\tilde{u}, \tilde{v})$ be another normal coordinates around $p$, we have

**Proposition 5.2.**

$$\tilde{I} = \pm I, \quad \tilde{II} = \pm II, \quad \tilde{III} = III, \quad \text{and} \quad \tilde{IV} = IV.$$

**Proof.** From the definition of normal coordinates, we see that $F_{\tilde{u}} = \tilde{X} = \pm X$. By definition

(5.24) $\tilde{I} = d\tilde{u} + \tilde{a} d\tilde{v}$, $\tilde{II} = b d\tilde{v}$, $\tilde{III} = \tilde{c} d\tilde{v}$

$\tilde{IV} = l d\tilde{u} + m d\tilde{v}$,

where

(5.25) $\tilde{a} = < F_{\tilde{u}}, \tilde{X} >$, $\tilde{b} = < F_{\tilde{v}}, \tilde{Y} >$, $\tilde{c} = < F_{\tilde{v}}, T >$,

and

(5.26) $\tilde{t} = < F_{\tilde{u}u}, \tilde{Y} >$, $\tilde{m} = < F_{\tilde{v}v}, \tilde{Y} >$, $\tilde{Y} = J_0 \tilde{X} = \pm Y$. 

By lemma 5.1 there exists functions $g(v)$ and $h(v)$ such that
\begin{align}
\tilde{u} &= \pm u + g(v) \\
\tilde{v} &= h(v),
\end{align}
(5.27)

We compute the transformation laws of the coefficients of the fundamental forms:
\begin{align}
a &= \langle F_v, X \rangle = \langle F_{\tilde{u}} \frac{\partial \tilde{u}}{\partial v} + F_{\tilde{v}} \frac{\partial \tilde{v}}{\partial v}, X \rangle \\
&= \langle \pm X \frac{\partial g}{\partial v} + F_{\tilde{v}} \frac{\partial h}{\partial v}, X \rangle \\
&= \pm \left( \frac{\partial g}{\partial v} + \frac{\partial h}{\partial v} \tilde{a} \right).
\end{align}
(5.28)

Similarly, we have
\begin{align}
b &= \pm \frac{\partial h}{\partial v} \tilde{b}, \quad c = \frac{\partial h}{\partial v} \tilde{c}
\end{align}
(5.29)

On the other hand, note that $F_u = \pm F_{\tilde{u}}$, hence $F_{uu} = \pm (F_{\tilde{u}} \frac{\partial \tilde{u}}{\partial u} + F_{\tilde{v}} \frac{\partial \tilde{v}}{\partial u}) = F_{\tilde{uu}}$. Thus
\begin{align}
l &= \pm \tilde{l}.
\end{align}
(5.30)

Similarly we have
\begin{align}
m &= \frac{\partial g}{\partial v} \tilde{m} + \frac{\partial h}{\partial v} \tilde{m}.
\end{align}
(5.31)

From the transformation laws (5.28), (5.29), (5.30) and (5.31), it is easy to see that
\begin{align}
\tilde{I} = \pm I, \quad \tilde{II} = \pm II, \quad \tilde{III} = III, \quad \text{and} \quad \tilde{IV} = IV.
\end{align}

This finishes the proof of the proposition.

Define $\alpha = \frac{b}{c}$ and $\tilde{\alpha} = \frac{\tilde{b}}{\tilde{c}}$, then from (5.29), we see that $\alpha = \pm \tilde{\alpha}$. Actually, $\alpha$ is the function defined on the non-singular part of $\Sigma$ such that $\alpha e_2 + T \in T\Sigma$. Up to a sign, $\alpha$ is a function which is independent of the choice of the normal coordinates, hence an invariant of $\Sigma$ on the non-singular part. Similarly, from (5.30), so is for $l$, which actually is the $p$-mean curvature.

Remark 5.3. Note that if we restrict us to choose normal coordinates with respect to a fixed orientation of the characteristic foliation on the nonsingular part, we see, from the proof of Proposition 5.2, that $\alpha = \tilde{\alpha}$ and $l = \tilde{l}$. That is, the sign appearing is due to the different choice of orientation.
Besides the two invariants $\alpha$ and $l$, we now proceed to introduce another invariant of $\Sigma$, which is defined on all of $\Sigma$, not just on the non-singular part. Again, from Proposition 5.2, it is easy to see that

$$I \otimes I + II \otimes II + III \otimes III = \tilde{I} \otimes \tilde{I} + \tilde{II} \otimes \tilde{II} + \tilde{III} \otimes \tilde{III}. \tag{5.32}$$

Therefore the form $I \otimes I + II \otimes II + III \otimes III$ is again independent of the choice of a normal coordinates, hence also an invariant of $\Sigma$.

**Lemma 5.4.** Let $g_{\theta_0}$ be the adapted metric on $H^1$. Then we have

$$g_{\theta_0}|_{\Sigma} = I \otimes I + II \otimes II + III \otimes III, \tag{5.33}$$

on the non-singular part of $\Sigma$.

**Proof.** This lemma is a easy consequence of the first one of the moving frame formula (3.18).

In the following section, we will show that the form $IV = \tilde{F}^* \omega_1^2$ is completely determined by all $g_{\theta_0}, \alpha$ and $l$. We therefore obtain a complete set of invariants for surfaces on the non-singular part.

### 6. A complete set of invariants for surfaces in $H^1$

Let $\Sigma$ be an oriented surface and suppose $f : \Sigma \to H^1$ be an embedding. For the convenient of expression, we will not distinguish surfaces $\Sigma$ and $f(\Sigma)$. For each non-singular point $p \in \Sigma$, we specify an orthonormal frame by $(p; e_1, e_2, T)$, here $e_1$ is tangent to the characteristic foliation and $e_2 = J_0 e_1$. A Darboux frame is a moving frame which is smoothly defined on $\Sigma$, except those singular points, hence giving a lifting of $f$ to $PSH(1)$ which is defined by $F$. Now we would like to compute the Darboux derivative $F^* \omega$ of $F$. In the following, instead of $\tilde{F}^* \omega$, we still use

$$\omega = \begin{pmatrix}
0 & 0 & 0 & 0 \\
\omega^1 & 0 & -\omega_1^2 & 0 \\
\omega^2 & \omega_1^2 & 0 & 0 \\
\omega^3 & \omega^2 & -\omega^1 & 0
\end{pmatrix}, \tag{6.1}$$

to express the Darboux derivative. It satisfies the integrability condition $d\omega + \omega \wedge \omega = 0$, that is,

$$d\omega^1 = \omega_1^2 \wedge \omega^2$$

$$d\omega^2 = -\omega_1^2 \wedge \omega^1$$

$$d\omega^3 = 2 \omega^1 \wedge \omega^2$$

$$d\omega_1^2 = 0 \tag{6.2}$$
Let $g_{\theta_0} = h + \theta_0^2$ be the adapted metric. From Section 5, which we see that $\omega^2 = \alpha \omega^3$ on the nonsingular part of $\Sigma$, it is easy to see that

$$g_{\theta_0}|_{\Sigma} = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3 = \omega^1 \otimes \omega^1 + (1 + \alpha^2)\omega^3 \otimes \omega^3.$$ 

Define

$$\hat{\omega}^1 = \omega^1$$
$$\hat{\omega}^2 = \sqrt{1 + \alpha^2} \omega^3.$$ 

This is an orthonormal coframe of $g_{\theta_0}|_{\Sigma}$ and the dual frame is

$$\hat{e}_1 = e_1$$
$$\hat{e}_2 = e_\Sigma = \frac{\alpha e_2 + T}{\sqrt{1 + \alpha^2}}.$$ 

Let $\hat{\omega}_1^2$ be the Levi-Civita connection of $g_{\theta_0}|_{\Sigma}$ with respect to the frame $\hat{\omega}^1, \hat{\omega}^2$. By the fundamental theorem in Riemannian geometry, this connection is uniquely defined by

$$\begin{align*}
\omega^1 &= -\hat{\omega}_1^1 \wedge \hat{\omega}^2 \\
\omega^2 &= -\hat{\omega}_1^2 \wedge \hat{\omega}^1 \\
\omega_1^2 &= -\hat{\omega}_1^2.
\end{align*}$$

The following Proposition point out that $\omega_1^2$ is completely determined by the induced fundamental form $g_{\theta_0}|_{\Sigma}$ and the functions $\alpha$ and $l$.

**Proposition 6.1.** We have

$$\begin{align*}
\omega_1^2 &= \frac{\alpha}{\sqrt{1 + \alpha^2}} \hat{\omega}_1^1 + \frac{l}{1 + \alpha^2} \hat{\omega}_1^1 + \frac{e_1 \alpha}{(1 + \alpha^2)^2} \hat{\omega}^2 \\
&= l \hat{\omega}^1 + \frac{2\alpha^2 + (e_1 \alpha)}{\sqrt{1 + \alpha^2}} \hat{\omega}^2,
\end{align*}$$

(6.6)

$$\begin{align*}
\hat{\omega}_1^2 &= \frac{\alpha}{\sqrt{1 + \alpha^2}} \omega^1 + \frac{2\alpha}{1 + \alpha^2} \hat{\omega}^2 \\
&= \frac{la}{\sqrt{1 + \alpha^2}} \hat{\omega}^1 + \left(2\alpha + \frac{\alpha(e_1 \alpha)}{1 + \alpha^2}\right) \hat{\omega}^2.
\end{align*}$$
Proof. Note that $\omega^2 = \alpha \omega^3$. Then from the second identity of (6.3), we have

$$d\omega^2 = d\left( \frac{\alpha}{(1 + \alpha^2)^{\frac{3}{2}}} \right) \wedge \hat{\omega}^2 + \frac{\alpha}{(1 + \alpha^2)^{\frac{3}{2}}} d\hat{\omega}^2$$

$$= e_1 \left( \frac{\alpha}{(1 + \alpha^2)^{\frac{1}{2}}} \right) \hat{\omega}^1 \wedge \hat{\omega}^2 - \frac{\alpha}{(1 + \alpha^2)^{\frac{1}{2}}} \hat{\omega}^2 \wedge \hat{\omega}^1$$

$$= \hat{\omega}^1 \wedge \left( e_1 \left( \frac{\alpha}{(1 + \alpha^2)^{\frac{1}{2}}} \right) \hat{\omega}^2 + \frac{\alpha}{(1 + \alpha^2)^{\frac{1}{2}}} \hat{\omega}^1 \right),$$

where at the third equality above, we have used the second formula of the structure equation (6.5) in Riemannian geometry. On the other hand, from the Maurer-Cartan structure equation (6.2)

$$d\omega^2 = -\omega^2 \wedge \omega^1 = \hat{\omega}^1 \wedge \omega^2.$$  

Together the above two formulae and by Cartan lemma, we see that there exists a function $D$ such that

$$\omega^2 = e_1 \left( \frac{\alpha}{(1 + \alpha^2)^{\frac{1}{2}}} \right) \hat{\omega}^2 + \frac{\alpha}{(1 + \alpha^2)^{\frac{1}{2}}} \hat{\omega}^1 + D \hat{\omega}^1$$

(6.7)

$$= \frac{e_1 \alpha}{(1 + \alpha^2)^{\frac{1}{2}}} \hat{\omega}^2 + \frac{\alpha}{(1 + \alpha^2)^{\frac{1}{2}}} \hat{\omega}^1 + D \hat{\omega}^1.$$

Similarly, we compute

$$-\hat{\omega}^1 \wedge \hat{\omega}^2 = d\hat{\omega}^1 = d\omega^1$$

$$= \omega^2 \wedge \omega^2$$

$$= \frac{\alpha}{\sqrt{1 + \alpha^2}} \omega_1^2 \wedge \omega^2.$$  

(6.8)

Again, by Cartan lemma, there exists a function $A$ such that

$$-\hat{\omega}^2 = \frac{\alpha}{\sqrt{1 + \alpha^2}} \omega_1^2 + A \omega^2.$$  

(6.9)
Finally, we compute

\[-\hat{\omega}^2_1 \wedge \hat{\omega}^1 = d\hat{\omega}^2 = d\left( (1 + \alpha^2)^{\frac{1}{2}} \omega^3 \right) \]
\[= (1 + \alpha^2)^{\frac{1}{2}} d\omega^3 + d(1 + \alpha^2)^{\frac{1}{2}} \wedge \omega^3 \]
\[= 2\alpha(1 + \alpha^2)^{\frac{1}{2}} \hat{\omega}^1 \wedge \omega^3 + \frac{\alpha}{(1 + \alpha^2)^{\frac{3}{2}}} d\alpha \wedge \omega^3 \]
\[= \left( 2\alpha + \frac{\alpha(e_1 \alpha)}{1 + \alpha^2} \right) \omega^1 \wedge \hat{\omega}^2, \]

where we have used the third formula of (6.2) and \( \hat{\omega}^2 \wedge \omega^3 = 0 \). Therefore, there exists a function \( B \) such that

\[(6.11) \quad \hat{\omega}^2_1 = \left( 2\alpha + \frac{\alpha(e_1 \alpha)}{1 + \alpha^2} \right) \omega^2 + B\hat{\omega}^1. \]

By (6.7) and (6.9), we get

\[ D = \omega^2_1(e_1) - \frac{\alpha}{\sqrt{1 + \alpha^2}} \hat{\omega}^2_1(e_1) \]
\[= \frac{\omega^2_1(e_1)}{1 + \alpha^2} = \frac{l}{1 + \alpha^2}. \]

Similarly, by (6.7), (6.9) and (6.11), we obtain

\[ A = \frac{2\alpha}{1 + \alpha^2} \]
\[B = \frac{l\alpha}{\sqrt{1 + \alpha^2}}. \]

These complete the proof. \( \square \)

6.1. The proof of Theorem 1.11 Let \( K \) be the Gaussian curvature of the induced metric \( g_{\theta_0}|_\Sigma \), hence we have

\[ (6.13) \quad d\hat{\omega}^2_1 = K d\sigma, \]
where \( d\sigma \) is the area form \( \hat{\omega}^1 \wedge \hat{\omega}^2 \). Using Proposition \(6.1\) and \(6.2\) and \(6.5\), we compute

\[
(6.14)
\]

\[
d\hat{\omega}^2_i = d \left( \frac{\alpha}{\sqrt{1 + \alpha^2}} \hat{\omega}_1^2 + \frac{2\alpha}{1 + \alpha^2} \hat{\omega}^2 \right) \\
= d \left( \frac{\alpha}{\sqrt{1 + \alpha^2}} \right) \wedge \omega_1^2 + d \left( \frac{2\alpha}{1 + \alpha^2} \right) \wedge \hat{\omega}^2 + \frac{2\alpha}{1 + \alpha^2} d\hat{\omega}^2 \\
= \frac{d\alpha}{(1 + \alpha^2)^2} \wedge \omega_1^2 + \frac{2(1 - \alpha^2)d\alpha}{(1 + \alpha^2)^2} \wedge \hat{\omega}^2 - \frac{2\alpha}{1 + \alpha^2} \omega_1^2 \wedge \hat{\omega}^1 \\
= \frac{(e_1\alpha)^2 + 2(1 + \alpha^2)(e_1\alpha) + 4\alpha^2(1 + \alpha^2) - l(e_\Sigma)(1 + \alpha^2)}{(1 + \alpha^2)^2} \omega_1^1 \wedge \hat{\omega}^1.
\]

These completes the proof of Theorem \(1.11\).

6.2. The derivation of the integrability condition \((1.14)\). We compute

\[
(6.15)
\]

\[
0 = d\hat{\omega}^2_i \\
= d \left( \frac{\alpha}{\sqrt{1 + \alpha^2}} \hat{\omega}_1^2 + \frac{l}{1 + \alpha^2} \hat{\omega}^1 + \frac{e_1\alpha}{(1 + \alpha^2)^2} \hat{\omega}^2 \right) \\
= \left\{ -(1 + \alpha^2)^{\frac{3}{2}}(e_\Sigma l) + (1 + \alpha^2)(e_1e_1\alpha) - \alpha(e_1\alpha)^2 + 4\alpha(1 + \alpha^2)(e_1\alpha) \\
+ \alpha(1 + \alpha^2)^2 K + al(1 + \alpha^2)^{\frac{3}{2}}(e_\Sigma\alpha) + \alpha(1 + \alpha^2)l^2 \right\} \frac{\hat{\omega}_1^1 \wedge \hat{\omega}^2}{(1 + \alpha^2)^2}.
\]

Therefore the integrability condition \((1.14)\) is equivalent to \(d\omega_i^2 = 0\).

6.3. The proof of Theorem \(1.10\). First we show the existence. Define an \( psh(1) \)-valued one-form \( \phi \) on the non-singular part of \( \Sigma \) by

\[
(6.16)
\]

\[
\phi = \begin{pmatrix}
0 & 0 & 0 & 0 \\
\hat{\omega}^1 & 0 & -\omega_1^2 & 0 \\
\frac{\alpha}{\sqrt{1 + (\alpha')^2}} \hat{\omega}^2 & \omega_1^2 & 0 & 0 \\
\frac{1}{\sqrt{1 + (\alpha')^2}} \hat{\omega}^2 & \frac{\alpha'}{\sqrt{1 + (\alpha')^2}} \omega_1^2 & -\hat{\omega}^1 & 0
\end{pmatrix},
\]

where

\[
(6.17)
\]

\[
\omega_1^2 = \frac{\alpha'}{\sqrt{1 + (\alpha')^2}} \hat{\omega}_1^2 + \frac{l'}{1 + (\alpha')^2} \hat{\omega}^1 + \frac{e_1\alpha'}{(1 + (\alpha')^2)^2} \hat{\omega}^2.
\]
Then it is easy to check that \( \phi \) satisfies \( d\phi + \phi \wedge \phi = 0 \) if and only if the integrability condition (1.14) holds. Therefore, by Theorem 2.2, for each \( x \in \Sigma \) there exists an open set \( U \) containing \( x \) and an embedding \( f : U \to H^1 \) such that \( g = f^*(g_{\theta_0}) \), \( \alpha = f^*\alpha \) and \( l' = f^*l \). Next we show the uniqueness. By Proposition 6.1, we see that the Darboux derivative is completely determined by the induced metric \( g_{\theta_0}|_{\Sigma} \), the \( p \)-variation \( \alpha \) and the \( p \)-mean curvature \( l \). Therefore, by Theorem 2.1 the embedding into \( H^1 \) is unique up to a Heisenberg rigid motion.

7. Appendix

In this Appendix, we give another proof of Theorem 1.2.

7.1. The second proof of Theorem 1.2 For a horizontally regular curve \( \gamma(s) \) parametrized by horizontal arc-length \( s \), we define a moving frames \((X(s), Y(s), T(s))\) by

\[
(7.1) \quad X(s) = \gamma'(s), \quad Y(s) = JX(s), \quad \text{and} \quad T(s) = T.
\]

Then we have that

\[
X'(s) = k(s)Y(s)
\]

\[
Y'(s) = -k(s)X(s) - T
\]

\[
T'(s) = 0.
\]

Note also that

\[
(7.3) \quad \gamma'(s) = X(s) + \tau(s)T.
\]

Now, assume that two curves \( \gamma(s) \) and \( \bar{\gamma}(s) \) satisfy the conditions

\[
(7.4) \quad k(s) = \bar{k}(s) \quad \text{and} \quad \tau(s) = \bar{\tau}(s), \quad s \in I.
\]

After performing a Heisenberg rigid motion (i.e., a pseudohermitian transformation on \( H^1 \)), we can assume, without loss of generality, that

\[
(7.5) \quad \bar{\gamma}(s_0) = \gamma(s_0), \quad \bar{X}(s_0) = X(s_0), \quad \text{and} \quad \bar{Y}(s_0) = Y(s_0),
\]

for a fixed \( s_0 \in I \). Define \( A(s) = < X(s), \bar{X}(s) > + < Y(s), \bar{Y}(s) > \).

By using the moving frames formula (7.2), we have

\[
(7.6) \quad A'(s) = < X'(s), \bar{X}(s) > + < X(s), \bar{X}'(s) > + < Y'(s), \bar{Y}(s) > + < Y(s), \bar{Y}'(s) >
\]

\[
= k < Y(s), \bar{X}(s) > + \bar{k} < X(s), \bar{Y}(s) > + < -kX - T, \bar{Y}(s) > + < Y, -\bar{k}X - \bar{T} >
\]

\[
= 0.
\]

Since \( A(s_0) = 2 \), we get \( A(s) = 2 \), hence that \( X(s) = \bar{X}(s) \) and \( Y(s) = \bar{Y}(s) \) for each \( s \in I \). In particular, we have \( \gamma'(s) = \bar{\gamma}'(s) \).

Also note that \( \tau(s) = \bar{\tau}(s) \), by (7.3), we have \( \gamma'_{\bar{T}}(s) = \gamma'_{T}(s) \). We
therefore obtain that $\gamma'(s) = \bar{\gamma}'(s)$, which implies that $\gamma(s) = \bar{\gamma}(s) + C$ for some constant $C$. Since $\gamma(s_0) = \bar{\gamma}(s_0)$, we see that $C = 0$, that is, $\gamma(s) = \bar{\gamma}(s)$ for all $s \in I$.

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