The number of vertices of a tropical curve is bounded by its area

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We prove that the number of vertices of a tropical curve defined in an open subset of \( \mathbb{R}^n \) is bounded by the area of the curve. The approach is totally elementary yet tricky. Our proof uses some ideas from algebraic geometry by making analogies. The result can be interpreted as the fact that the moduli space of tropical curves with area bounded below a given value is of finite type.

1 Introduction and statement of the theorem

We begin with some heuristic motivations from symplectic geometry and algebraic geometry. Let \( X \) be a symplectic manifold equipped with a tame almost complex structure, \( A \) a positive real number. Then the moduli space of pseudoholomorphic curves embedded in \( X \) with area bounded by \( A \) is of “finite type”, in the sense that it can be parametrized by a finite number of parameters. In the case when \( X \) is a projective variety, the area of an embedded curve with respect to the Kähler form can be interpreted as intersection numbers with a hyperplane section, and the finiteness follows from the fact that the corresponding Hilbert schemes are of finite type \([2]\). This article tries to establish an analogous result in tropical geometry.

**Theorem 1.1.** Let \( A \) be a positive real number, \( U \) an open subset of \( \mathbb{R}^n \) and \( K \subset U \) a compact subset. Then there exists an integer \( N \), such that for any tropical curve \( G \) in \( U \) with area bounded by \( A \), the number of vertices of \( G \) inside \( K \) is bounded by \( N \).

We explain some of the terminology used above.

**Definition 1.2.** Let \( \mathbb{Z}/2\mathbb{Z} \) act on \( \mathbb{Z}^n \setminus \{0\} \) by multiplication by \(-1\), and denote the quotient by \( W \). For any \( w \in W \), we define its norm \( |w| = \sqrt{\sum (w^i)^2} \) for some representative \((w^1, \ldots, w^n) \in \mathbb{Z}^n \setminus \{0\} \). We do the same construction for \( \mathbb{Q}^n \setminus \{0\} \), and denote it by \( W_Q \).

**Definition 1.3.** A tropical curve \( G \) in an open subset \( U \subset \mathbb{R}^n \) is a finite one-dimensional polyhedral complex in \( U \) satisfying the following properties:

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(i) \( G \) is closed in \( U \) as a topological space. We call the 0-dimensional faces of \( G \) vertices, and the one-dimensional faces of \( G \) edges. The set of vertices is denoted by \( V(G) \); the set of edges is denoted by \( E(G) \). There are two kinds of edges: those edges which have both endpoints in \( U \) are called internal edges; while the rest are called unbounded edges.

(ii) Each vertex of \( G \) is at least 3-valent.

(iii) Each edge \( e \) is equipped with a weight \( w_e \in W \) parallel to the direction of \( e \) inside \( \mathbb{R}^n \). If \( w_e \) is \( k \) times a primitive integral vector, we call \( |k| \) the multiplicity of the edge \( e \).

(iv) We further require that the balancing condition holds, i.e. for any vertex \( v \) of \( G \), we have \( \sum_{e \ni v} \tilde{w}_e = 0 \), where the sum is taken over all edges containing \( v \) as an endpoint, and \( \tilde{w}_e \) is the representative of \( w_e \) that points outwards from \( v \).

Remark 1.4. The balancing condition in Definition 1.3(iv) is a necessary condition for a tropical curve \( G \) to be the amoeba of an analytic curve (see for example [3]). This condition is in fact homological in nature. It is proved and generalized to the global setting in [4] using vanishing cycles in analytic étale cohomology.

Definition 1.5. For any open subset \( V \subset U \), we denote by \( G|_V \) the restriction of \( G \) to \( V \).

Definition 1.6. For an edge \( e \) of \( G \), we define its area as

\[
\text{Area}(e) = |e| \cdot |w_e|,
\]

where \(|e| \) means the Euclidean length of the segment \( e \), and \(|w_e| \) is the norm of the weight \( w_e \). The area of a tropical curve \( G \) is by definition the sum of area over all its edges.

Example 1.7. Let \( e \) be an edge connecting the point \( 0 = (0, \ldots, 0) \) with the point \( x = (x_1, \ldots, x_n) \), and let \( \tilde{w}_e = (w^1, \ldots, w^n) \in \mathbb{Z}^n \setminus 0 \) be a representative of the weight \( w_e \). By definition, there exists \( \lambda \in \mathbb{R} \) such that \( x = \lambda \cdot \tilde{w}_e \). We have

\[
\text{Area}(e) = |\lambda| \cdot \sum_{i=1}^{n} (w^i)^2.
\]

Having introduced all the notions, we now explain the proof. From the point of view of physics, if we regard tropicalization as a classical limit from strings to particles, then the balancing condition is a conservation of momentum. The idea of the proof is to cover our tropical curve by a collection of paths (Section 3), thought of as paths of particles, and then try to bound the number of vertices on each path (Section 4). Let us start with two simple observations:
Observation 1: The balancing condition defined locally around each vertex has the following global consequence:

**Lemma 1.8.** Let $G$ be a tropical curve in an open set $U \subset \mathbb{R}^n$, and let $W$ be an open subset of $\mathbb{R}^n$ such that

(i) $\overline{W} \subset U$.

(ii) $W$ is a smooth manifold with corners.

(iii) $V(G) \cap \partial W = \emptyset$.

(iv) $G$ intersects $\partial W$ transversely.

For each edge $e$ of $G$ that intersects $\partial W$, we can pick a representative $\tilde{w}_e$ of its weight $w_e$ pointing from the inside of $W$ to the outside. Then we have

$$\sum_{e \cap \partial W \neq \emptyset} \tilde{w}_e = 0. \quad (1.1)$$

**Proof.** Let $v_1, \ldots, v_l$ be the vertices of $G$ inside $W$, $e_1, \ldots, e_m$ the edges of $G$ contained in $W$. Let $B_1, \ldots, B_l$ be open balls of radius $r > 0$ and with center $v_1, \ldots, v_l$. Let $C_1, \ldots, C_m$ be open cylinders of radius $r$ and with central axis $e_1, \ldots, e_m$. We choose $r$ small enough so that the closures of the balls and the cylinders do not intersect nearby edges and that all of them are contained in $W$. Let $B = \bigcup_{i=1}^l B_i$. We consider a chain of open sets in $\mathbb{R}^n$ verifying (i)-(iv):

$$B \subset B \cup C_1 \subset B \cup C_1 \cup C_2 \subset \cdots \subset B \cup C_1 \cup \cdots \cup C_m \subset W.$$  

The equation $(1.1)$ holds for $B$ by the definition of the balancing condition. Then we show by induction that $(1.1)$ holds for every open set in the chain above, and in particular holds for $W$. \\ 

Observation 2: By arguments of general topology, we can reduce Theorem 1.1 to the following particular situation:

**Theorem 1.9.** Let $A$ be a positive real number. Let $K$ be the $n$-simplex obtained as the convex hull of the $n + 1$ points $(0, \ldots, 0), (1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ inside $\mathbb{R}^n$, and $K^\circ$ the interior of $K$. Let $U$ be the interior of the convex hull of the $n + 1$ points $(-\delta, -\delta, \ldots, -\delta), (1 + 3\delta, -\delta, -\delta, \ldots, -\delta), (-\delta, 1 + 3\delta, -\delta, \ldots, -\delta), \ldots, (-\delta, -\delta, -\delta, 1 + 3\delta)$, where $\delta$ is a positive real number. Then there exists an integer $N$ such that for any tropical curve $G$ in $U$ with area bounded by $A$, the number of vertices of $G_{K^\circ}$ is bounded by $N$.

The proof of Theorem 1.9 consists of two parts. The first part (Sections 2-4) treats the case where we have a nice interpretation of the area as intersection numbers; the second part (Sections 5-6) explains how to reduce the general case to the case considered in the first part by some modification.

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2 Interpretation of the area as intersection numbers

Let $K$ be as in Theorem 1.3. The boundary $\partial K$ is a simplicial complex of dimension $n - 1$. We denote by $(\partial K)^{n-2}$ its skeleton of dimension $n - 2$. In this section, we study an even simpler situation, where $G$ is a tropical curve in $K^\circ$, and $G$ is saturated in the sense of the following definition.

**Definition 2.1.** A tropical curve $G$ in $K^\circ$ is said to be saturated if $G \cap (\partial K)^{n-2} = \emptyset$ and if $G$ intersects $\partial K \setminus (\partial K)^{n-2}$ perpendicularly, where $G$ denotes the closure of $G$ in $\mathbb{R}^n$.

**Remark 2.2.** The word “saturated” is used because in this case, the area is concentrated in $K$ in some sense, and reaches maximal.

For an intersection point between $G$ and $\partial K$, we define its multiplicity to be the multiplicity of the corresponding edge of $G$.

**Proposition 2.3.** The balancing condition implies that $G$ intersects each face of $\partial K$ by the same number of times (counting with multiplicity as defined above), which we denote by $d$.

**Proof.** We use Lemma 1.8, where we take $U$ to be $K^\circ$ and $W = \{ x \in K^\circ \mid \text{dist}(x, \partial K) > \epsilon \}$ for $\epsilon$ a positive number sufficiently small such that $(U \setminus W) \cap V(G) = \emptyset$. For $1 \leq i \leq n$, let $d_i$ be the number of intersections (counting with multiplicity) between $G$ and the face of $K$ defined by $x_i = 0$. Let $d$ be the number of intersections (counting with multiplicity) between $G$ and the face of $K$ defined by $x_1 + \cdots + x_n = 1$. Then equation (1.1) means that

$$d_1 e_1 + d_2 e_2 + \cdots + d_n e_n = d(e_1 + \cdots + e_n),$$

where we denote by $e_1, \ldots, e_n$ the vectors with coordinates $(1, 0, \ldots, 0)$, $\ldots$, $(0, \ldots, 0, 1)$ respectively. Therefore we obtain that $d_1 = d_2 = \cdots = d_n = d$. \qed

**Proposition 2.4.** $\text{Area}(G) = d$.

**Proof.** Let $K^1$ be the union of the $n$ segments connecting 0 and $e_i$, for $i \in \{1, \ldots, n\}$. We define a measure $\mu$ on $K^1$. We start with the zero measure on $K^1$. For each edge $e$ of $G$, we add to $\mu$ a measure $\mu_e$ defined as follows: Let $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in K$ be the two endpoints of $e$ and let $(w^1, \ldots, w^n) \in \mathbb{Z}^n \setminus \{0\}$ be a representative of the weight of $e$. We define the restriction of $\mu_e$ to the segment connecting 0 and $e_i$ to be $1_{[x_i, y_i]} \cdot |w^i| \cdot \nu$, where $1_{[x_i, y_i]}$ is the characteristic function of the segment $[x_i, y_i] \subset \mathbb{R}$, and $\nu$ denotes the one-dimensional Lebesgue measure. Then by Definition 1.6, the area of $G$ is the total mass of $\mu$. Let us calculate the measure $\mu$. 


Lemma 2.5. Let \( z^{(1)} , \ldots , z^{(l)} \) be the intersection points between \( G \) and the face of \( K \) defined by \( x_1 + \cdots + x_n = 1 \) with multiplicity \( m^{(1)} , \ldots , m^{(l)} \) respectively. We have \( m^{(1)} + \cdots + m^{(l)} = d \) by Proposition 2.3. Let \( \{ z_1^{(k)} , \ldots , z_l^{(k)} \} \) be the coordinates of \( z^{(k)} \) for \( k = 1, \ldots , l \). We fix \( i \in \{ 1, \ldots , n \} \) and assume that
\[
 z_1^{(i)} \leq \cdots \leq z_l^{(i)} .
\]
Let \( \mu_i \) denote the restriction of \( \mu \) to the segment connecting 0 and \( e_i \). We have
\[
 \mu_i = \sum_{k=1}^{l} m^{(k)} \cdot 1_{[0,z^{(k)}]} \quad \text{almost everywhere.}
\]

Proof. Let \( z_1^{(0)} = 0 , z_l^{(l+1)} = 1 \), and \( \zeta \in (0,1) \). Assume that there is no vertex of \( G \) with \( i \)th coordinate equal to \( \zeta \) and that \( z_1^{(j)} < \zeta < z_l^{(j+1)} \), for some \( j \in \{ 0, \ldots , l \} \). Denote \( \tilde{\zeta} = \zeta \cdot e_i \). Let us show that the density of \( \mu_i \) at the point \( \tilde{\zeta} \) is \( d - \sum_{k=1}^{j} m^{(k)} \), which we denote by \( d_{\zeta} \). Let \( H^0_{\zeta} \) be the half space \( \{ (x_1, \ldots , x_n) \in \mathbb{R}^n \mid x_i \leq \zeta \} \), \( W \) the interior of \( K \cap H^0_{\zeta} \). By Lemma 1.8, \( G \) has exactly \( d_{\zeta} \) intersection points with \( \partial H^0_{\zeta} \) counting with multiplicity (here multiplicity is defined to be the absolute value of the \( i \)th coordinate of the weight of the corresponding edge). So by construction, the tropical curve \( G \) contributes \( d_{\zeta} \) to the density of \( \mu_i \) at the point \( \zeta \in [0,e_i] \). \( \Box \)

We continue the proof of Proposition 2.4. We calculate the total mass of \( \mu \), denoted by \( m(\mu) \). We have
\[
 m(\mu) = \sum_{i=1}^{n} m(\mu_i) = \sum_{i=1}^{n} \sum_{k=1}^{l} m^{(k)} \cdot z_i^{(k)} = \sum_{k=1}^{l} m^{(k)} \sum_{i=1}^{n} z_i^{(k)} = \sum_{k=1}^{l} m^{(k)} = d.
\]

3 Paths and collection of paths

Let \( R \) be an \( n \)-dimensional polyhedron in \( \mathbb{R}^n \), \( V \) an open subset of \( \mathbb{R}^n \) containing \( R \). In this section, we fix a direction \( i \in \{ 1, \ldots , n \} \) and assume that \( R \) has an \((n-1)\)-dimensional face \( F \) contained in a hypersurface defined by \( x_i = c \), for some \( c \in \mathbb{R} \), and that \( R \) is contained in the half space \( x_i \geq c \). Morally, we can think of the \( i \)th direction as time, and the rest as space directions. Let \( H \) be a tropical curve in \( V \) such that there is an edge \( e_0 \) of \( H \) whose interior intersects the relative interior of \( F \) transversely.

Definition 3.1. A path \( P \) starting from \( e_0 \) with direction \( i \) is a chain of weighted segments \( s_0 , s_1 , \ldots , s_l \) such that
\begin{enumerate}
    \item \( s_0 = e_0 \cap R , s_{l_p} = e'_0 \cap R \) for some edge \( e'_0 \) of \( H \) such that exactly one endpoint of \( e'_0 \) does not belong to the interior \( R^0 \).
    \item \( s_1 , \ldots , s_{l_p-1} \) are edges of \( H \), and \( s_1 , \ldots , s_{l_p-1} \subset R^0 \).
    \item Every two consecutive segments in the chain share one endpoint.
\end{enumerate}
(iv) The projection to the $i^{th}$ coordinate $\mathbb{R}^n \to \mathbb{R}$ restricted to $P$ is injective.

(v) Each segment $s_j$ carries the weight $w'_e = w_e/|w_e| \in W_Q$, where $e$ is the edge of $H$ containing $s_j$.

**Definition 3.2.** A union $U$ of $m$ paths $P_1, \ldots, P_m$ is a sub-polyhedral complex of $H$, such that

(i) Set theoretically $U = \bigcup_{j=1}^{m} P_j$.

(ii) Each segment $s$ of $U$ carries the weight $w'_s = k \cdot w_e/|w_e| \in W_Q$, where $e$ is the edge of $H$ containing $s$, and $k = \# \{ j \mid P_j \text{ contains } s \}$.

**Lemma 3.3.** Let $m = |w'_{e_0}|$. Then there exists a collection of $m$ paths $P_1, \ldots, P_m$ starting from $e_0$ with direction $i$ such that each segment $s$ in the union $U = \bigcup_{j=1}^{m} P_j$ verifies the following property:

Let $e$ be the edge of $H$ containing $s$, and let $\tilde{w'_s}$ and $\tilde{w}_e$ be representatives of the weights $w'_s$ and $w_e$ respectively. By construction $\tilde{w'_s}$ and $\tilde{w}_e$ are parallel so there exists $q \in \mathbb{Q}$ such that $\tilde{w'_s} = q \tilde{w}_e$. The property is that $|q| \leq 1$.

**Proof.** We assign to each edge $e$ of our tropical curve $H$ an integer $c_i(e)$ called capacity (in the $i^{th}$ direction). Initially we set $c_i(e_0) = |w'_e|$. To construct the path $P_1$, we start with the segment $s_0 = e_0 \cap R$, and we decrease the capacity $c_i(e_0)$ by 1. Suppose we have constructed a chain of segments $s_0, s_1, \ldots, s_j$. Let $B$ be the endpoint of $s_j$ with larger $i^{th}$ coordinate. If $B \in \partial R$ we stop, otherwise we choose $e_{j+1}$ to be an edge of $H$ such that:

(i) $B$ is an endpoint of $e_{j+1}$.

(ii) For any point $x \in e_{j+1} \setminus B$, the $i^{th}$ coordinate of $x$ is larger than the $i^{th}$ coordinate of $B$.

(iii) The capacity $c_i(e_{j+1})$ is positive.

The existence of such $e_{j+1}$ is ensured by the balancing condition on $H$. After choosing $e_{j+1}$, we decrease the capacity $c_i(e_{j+1})$ by 1 and set $s_{j+1} = e_{j+1} \cap R$. We iterate this procedure until we stop, and we obtain the path $P_j$. We apply the same procedure $m$ times and obtain the collection of paths $P_1, \ldots, P_m$ as required in the lemma.

### 4 Tropical vertex bound and genus bound

Let $K$ be as in Theorem 1.9 and let $G$ be a saturated tropical curve in $K^\circ$ with area $d$ as in Section 2. In this section, we give a very coarse bound on the number of vertices of $G$ in terms of the area $d$ and the dimension $n$.

**Proposition 4.1.** $\#V(G) \leq 2(n - 1)^2d^2$. 


Proof. Let \((x_1, \ldots, x_n)\) be the standard coordinates on \(\mathbb{R}^n\). We fix a direction \(i \in \{1, \ldots, n\}\). Let \(z_1^{(1)}, \ldots, z_l^{(l)}\) be the intersection points between \(\Gamma\) and the face of \(K\) defined by \(x_i = 0\) with multiplicity \(m^{(1)}, \ldots, m^{(l)}\) respectively. By Proposition 2.4, we have \(\sum m^{(j)} = d\). Let \(e_{0i}^{(k)}\) be the edge of \(G\) corresponding to the intersection point \(z_i^{(k)}\). For each intersection point \(z_i^{(k)}\), by Lemma 3.3, we obtain a collection of \(m^{(k)}\) paths starting from \(e_{0i}^{(k)}\) with direction \(i\). So for \(k = 1, \ldots, l\), we obtain in total \(d\) paths, and we label them as \(P_{ik}\) for \(i = 1, \ldots, n\), \(k = 1, \ldots, d\). We claim that (see Lemma 4.2)

\[
\#V_0(P) \leq 2d(n-1).
\] (4.1)

Now we vary \(i\), and in the same way, we get \(nd\) paths \(P_{ik}\) for \(i = 1, \ldots, n\), \(k = 1, \ldots, d\). We claim that (see Lemma 4.3)

\[
\bigcup_{i=1}^{n-1} \bigcup_{k=1}^{d} V_0(P_{ik}) \supset V(G). \quad (4.2)
\]

Combining equations (4.1) and (4.2), we have proved our proposition.

Lemma 4.2. For a path \(P\) among the paths \(P_{ik}\) constructed in the proof above, we have the following bound

\[
\#V_0(P) \leq 2d(n-1).
\]

Proof. Let \(S_{P,j} = \sum_{Q \in V_0(P)} |w_e(Q)|\) for \(j \in \{1, \ldots, \widehat{i}, \ldots, n\} := \{1, \ldots, n\} \setminus \{i\}\), where \(e(Q)\) is the edge of \(G\) associated to the vertex \(Q\) as in the definition of \(V_0(P)\) in the proof of Proposition 4.1. Now we fix \(j\), and let

\[
E_{0,j}^{-}(P) = \left\{ e(Q) \mid Q \in V_0(P), \tilde{w}_e(Q) < 0 \right\},
\]

\[
E_{0,j}^{+}(P) = \left\{ e(Q) \mid Q \in V_0(P), \tilde{w}_e(Q) > 0 \right\},
\]

where \(\tilde{w}_e(Q)\) is the representative of \(w_e(Q)\) that points outwards from \(Q\). Let

\[
S_{P,j} = \sum_{e \in E_{0,j}^{-}(P)} -\tilde{w}_e.
\]
Let $p_i : \mathbb{R}^n \to \mathbb{R}$ be the projection to the $i^{th}$ coordinate, and $p_j$ the projection to the $j^{th}$ coordinate. By Definition 3.3, $p_i|_P$ is injective. Assume that the image of $p_i|_P$ is the closed interval $[0, z_i^P]$. Let

$$q_i = \begin{cases} (p_i|_P)^{-1}(0) & \text{for } x_i \in (-\infty, 0] \\ (p_i|_P)^{-1}(x_i) & \text{for } x_i \in [0, z_i^P] \\ (p_i|_P)^{-1}(z_i^P) & \text{for } x_i \in [z_i^P, \infty) \end{cases}$$

$$R_j = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_j \leq p_j(q_i(x_i)) - 1 \}.$$  

We choose $\epsilon$ to be a sufficiently small positive real number such that

(i) $R_j \supset \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_j \leq 0 \}$.

(ii) $\partial R_j \cap V(G) = \emptyset$.

(iii) $\partial R_j$ intersects $\overline{G}$ transversely.

(iv) $\forall e \in E_{0,j}^{-}(P), e \cap R_j^{-} = \emptyset$. 

Let

$$T_j = \partial(R_j \cap K) \setminus (\partial K \cap \{ x_j = 0 \}).$$

Then for any $y \in T_j \cap \overline{G}$, let $e(y)$ denote the edge of $G$ corresponding to the intersection point $y$. By Lemma 1.8, we have

$$\sum_{y \in T_j \cap \overline{G}} |w_{e(y)}^j| = d.$$ 

Therefore $S_{P,j} \leq d$, and similarly $S_{P,j}^+ \leq d$, so $S_{P,j} = S_{P,j}^+ = S_{P,j}^+ + S_{P,j}^+ \leq 2d$. Let $S_P = \sum_{1 \leq i \leq n, j \neq i} S_{P,j}$. We have $S_P \leq 2d(n-1)$. By the definition of the set $V_0(P)$, each vertex $Q \in V_0(P)$ contribute at least 1 to the quantity $S_P$ so we obtain that $\#V_0(P) \leq 2d(n-1)$.

**Lemma 4.3.** Let $P_{ik}, V_0$ be as in the proof of Proposition 4.1, we have

$$\bigcup_{i=1}^{n-1} \bigcup_{k=1}^{d} V_0(P_{ik}) \supset V(G).$$

**Proof.** By Lemma 1.8 and Lemma 3.3 we see that for any edge $e \subset G$, any $i \in \{1, \ldots, n\}$ such that $w_i^e \neq 0$, there exists $k \in \{1, \ldots, d\}$ such that the path $P_{ik}$ constructed in the proof of Proposition 4.1 contains $e$. Now for any vertex $v$ of $G$, since $v$ is at least 3-valent by definition, there exists an edge $e$ of $G$ containing $v$ such that $w_i^e \neq 0$ for some $i \in \{1, \ldots, n-1\}$. This means that there exists $k \in \{1, \ldots, d\}$ such that the path $P_{ik}$ contains $e$ by what we have just said. However it can happen that $v \notin V_0(P_{ik})$. In such cases, by the definition of the set $V_0(P_{ik})$, there exists another edge $e' \notin P_{ik}$ such that $w_{e'} = 0$
Indeed, once we know how to bound the genus of our tropical curve, we can expect a much better bound on the number of vertices based on the Castelnuovo bound on the genus of a smooth curve of given degree in the projective space $\mathbb{P}^n$ (see for example [1]).

Indeed, once we know how to bound the genus of any tropical curve $G$, we can bound the number of vertices immediately. For example, using cellular homology to calculate the Euler characteristic of $G$, we have

$$1 - \text{rank } H_1(G) = \# V(G) - \# \{\text{internal edges of } G\}. $$

Then it suffices to observe that the number of internal edges is bounded below by the hypothesis that each vertex is at least 3-valent.

**Conjecture.** The number of vertices of $G$ is bounded by $2\pi(d,n) + (n+1)d - 2$, where $\pi(d,n)$ is defined by

$$\pi(d,n) = \frac{m(m-1)}{2}(n-1) + m\epsilon,$$

where $m = \left\lceil \frac{d-1}{n-1} \right\rceil$ and $\epsilon = d - 1 - m(n-1)$.

This should be achieved when $d > 2n$ by a tropical analogue of Castelnuovo curves.

## 5 Bound on the weights by the area

**Proposition 5.1.** Fix $i \in \{1, \ldots, n\}$. Let $R$ be the convex hull of the $2^n$ points

$$\left\{ (\epsilon_1, \ldots, \epsilon_n) \in \mathbb{R}^n \mid \epsilon_j \in \{-1, +1\} \text{ for } j \in \{1, \ldots, \hat{i}, \ldots, n\}, \epsilon_i \in \{0, 1\} \right\}. $$

Let $V$ be an open set in $\mathbb{R}^n$ containing $R$. Let $H$ be a tropical curve in $V$ such that there is an edge $e_0$ of $H$ whose interior contains the point $0 = (0, \ldots, 0) \in \mathbb{R}^n$. Then we have

$$\text{Area}(H_{|R^i}) \geq |w_{e_0}^i|,$$

where $w_{e_0}^i$ denotes the $i^{th}$ component of the weight of the edge $e_0$.

**Proof.** Denote $m = |w_{e_0}^i|$. By Lemma 3.3, we obtain a collection of $m$ paths $P_1, \ldots, P_m$ starting from $e_0$ with direction $i$. Each path $P_k$ connects the origin $O$ with a point on the boundary $\partial R$, denoted by $z_k$. By Definition 3.1 (iv), the $i^{th}$ coordinate of $z_k$ is strictly positive. This implies in particular that the length of $P_k$ under the Euclidean metric is at least one, so we have $\text{Area}(P_k) \geq 1$.

By summing up contributions from all $P_k$, for $k = 1, \ldots, m$, we obtain that $\text{Area}(H_{|R^i}) \geq m$. \[\square\]

\[\text{Many thanks to Olivier Debarre for pointing out this reference to me.}\]
Corollary 5.2. Let $A, U, K, G, \delta$ be as in Theorem 1.9, and denote by $I$ the number of intersection points between $G$ and $\partial K$ (with no multiplicity concerned). Then $I \leq A/\delta$.

Proof. By Proposition 5.1, each intersection point contributes at least $\delta$ to the total area of $G$, whence the corollary.

Corollary 5.3. Let $A, U, K, G, \delta$ be as in Theorem 1.9. For any edge $e$ of $G|_{K^c}$, any $i \in \{1, \ldots, n\}$, we have $|w_i^e| \leq A/\delta$.

Proof. By Proposition 5.1, for any edge $e$ of $G|_{K^c}$, any $i \in \{1, \ldots, n\}$, the weight $w_i^e$ contributes at least $|w_i^e| \cdot \delta$ to the total area of $G$, whence the corollary.

6 The saturation trick

Finally we perform a trick to reduce the general case to the saturated case considered in Sections 2 and 4. Using the notations and assumptions as in Theorem 1.9, our aim is to construct from $G$ a saturated tropical curve $G'$ in $K^c$ (in the sense of Definition 2.1).

Let $\epsilon$ be a positive real number and put

$$\tilde{K} = \{ x \in K^c | \text{dist}(x, \partial K) > \epsilon \}.$$ 

We choose $\epsilon$ small enough such that $V(G) \cap (K^c \setminus \tilde{K}) = \emptyset$.

Lemma 6.1. For any $w \in \mathbb{Z}^n$, there exists non-negative integers $a_0, \ldots, a_n$ such that

$$w = \sum_{i=0}^{n} a_i e_i',$$

where we denote $e_i' = -e_i$ for $i = 1, \ldots, n$, and $e_0' = e_1 + \cdots + e_n$. Furthermore we require that $a_i$ is zero for at least one $i \in \{0, \ldots, n\}$. This determines $a_0, \ldots, a_n$ uniquely.

Initially we set $G' = G|_{K^c}$. Then for each edge $e$ of $G'$ such that the closure $\overline{e}$ intersects $\partial K$ non-perpendicularly, or $\overline{e} \cap (\partial K)^{n-2} \neq \emptyset$, we do the following modification to $G'$. Let $w_e$ be the weight of $e$ and choose the representative $\tilde{w}_e$ that points from $\tilde{K}$ to $K^c \setminus \tilde{K}$. Now put $\tilde{w}_e$ into the lemma above and we get $(n+1)$ non-negative integers $a_0, \ldots, a_n$. Let $P = e \cap \partial K, \hat{e} = (K^c \setminus \tilde{K}) \cap e$. We first delete $\hat{e}$ from $G'$. Now $P$ becomes an unbalanced vertex. Then we add to $G'$ the rays starting from $P$ with direction $e_i'$ and multiplicity $a_i$ for all $i \in \{0, \ldots, n\}$. This makes the vertex $P$ balanced again and we finish our modification concerning the edge $e$.

Lemma 6.2. Using notations in Theorem 1.9. By construction we have

(i) $G'$ is a saturated tropical curve in $K^c$.

(ii) $\#V(G|_{K^c}) \leq \#V(G')$. 

10
(iii) \( \text{Area}(G') \leq \text{Area}(G|_K) + n(A/\delta)^2 \leq A + n(A/\delta)^2. \)

Proof. (i) follows directly from the construction. (ii) is obvious since our modification may add new vertices to \( G|_K \) but never decreases the number of vertices. For (iii), each time we do a modification to an edge, we add at most \( n \) rays, each of which has area less than \( A/\delta \) (Corollary 5.3). Moreover by Corollary 5.2 there are at most \( A/\delta \) edges of \( G \) intersecting with \( \partial K \), so the total area of all the rays we added to \( G|_K \) is bounded by \( n(A/\delta)^2 \).

To conclude, combining the previous lemma with Proposition 4.1, we have proved Theorem 1.9 with \( 2(n - 1)^2(A + n(A/\delta)^2)^2 \) being the bound on the number of vertices. This bound is far from being optimal but would suffice for our purposes. By Observation 2, we have also proved Theorem 1.1.

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