SMOOTH FUNCTIONS AND PARTITIONS
OF UNITY ON CERTAIN BANACH SPACES

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INTRODUCTION

In an earlier paper [4], the author sketched a method, based on the use of “Ta-
lagrand operators”, for defining infinitely differentiable equivalent norms on the
spaces $C_0(L)$ for certain locally compact, scattered spaces $L$. A special case of this
result was that a $C^\infty$ renorming exists on $C_0(L)$ for every countable locally compact
$L$. Recently, Hájek [3] extended this result by showing that a real normed space $X$
admits a $C^\infty$ renorming whenever there is a countable subset of the unit ball of $X^*$
on which every element of $X$ attains its norm, that is to say, a countable boundary.
This suggested to the author that the locally compact topology on $L$ was perhaps
not essential in [4], and in the first part of the present paper we shall develop the
methods of that work in a way that does not require such a topology. We obtain
infinitely differentiable norms on certain (typically non-separable) Banach spaces
$X$ as well as on some certain injective tensor products $X \otimes E$.

In the second part of the paper we present a lemma about partitions of unity.
It is an open problem whether a non-separable Banach space with a $C^k$ norm (or,
more generally, a $C^k$ “bump function”) admits $C^k$ partitions of unity, though many
partial results in this direction are known. Our lemma enables us to show that the
answer is yes for classes of Banach spaces that admit proctional resolutions of
the identity. In particular, we show that the space $C_0[0, \Omega)$ admits $C^\infty$ partitions of
unity for every ordinal $\Omega$. Results from this paper are used in [5] to give examples
of Banach spaces admitting infinitely differentiable bump functions and partitions
of unity but no smooth norms.

Our notation and terminology are mostly standard and, wherever possible, we
have followed the conventions of [2]. Although that work contains everything that
the reader will need in order to understand the present paper, we recall for conve-
nience a few facts and definitions. It should be noted that we are concerned only
with real, as opposed to complex, Banach spaces. When we refer to a function on
a Banach space as being of class $C^k$, where $k$ is a positive integer, it is the stan-
dard (Fréchet) notion of smoothness that we are employing. Making a mild abuse
of language, we shall say that a norm $\| \cdot \|$ on a Banach space $X$ is of class $C^k$ if
the function $x \mapsto \| x \|$ is of that class on the set $X \setminus \{ 0 \}$. (Of course, no norm is
differentiable at 0.)

A bump function on a Banach space $X$ is a function $\phi : X \to \mathbb{R}$ with bounded,
non-empty support. On finite-dimensional spaces, $C^\infty$ bump functions are plentiful
(and fundamental to the theory of distributions). The existence of a $C^1$ bump
function on an infinite-dimensional Banach space $X$ is already a strong condition.
For a separable space $X$, it is equivalent to separability of the dual space $X^*$, and to the existence of an equivalent $C^1$ norm. More generally, the existence of a $C^1$ bump on $X$ implies that $X$ belongs to the important class of Asplund spaces. Whether every Asplund space admits a $C^1$ bump is an open problem, as is the relationship between existence of a $C^k$ bump and of a $C^k$ norm on a separable space once $k$ is greater than 1. On the other hand, the equivalences in the following proposition are very easy to establish.

**Proposition 1.** For a real Banach space $X$ and $k \in \mathbb{N} \cup \{\infty\}$, the following are equivalent:

1. $X$ admits a $C^k$-bump function;
2. there exists a real number $R > 1$ and a function $\psi : X \to \mathbb{R}$, of class $C^k$, such that $0 \leq \psi \leq 1$ when $\|x\| \leq 0$ and $\|x\| = 1$ when $\|x\| \geq R$;
3. there is a function $\theta : X \to \mathbb{R}$, of class $C^k$, such that $\theta(x) \to \infty$ as $\|x\| \to \infty$.

**Proof.** (1) $\implies$ (2):
Let $\phi$ be a $C^k$ bump function with $\phi(0) = 1$. There exist positive real numbers $\delta$ and $M$ such that $\phi(x) \geq \frac{2}{3}$ when $\|x\| \leq \delta$ and $\phi(x) = 0$ when $\|x\| \geq M$. Let $\pi : \mathbb{R} \to [0, 1]$ be a $C^\infty$ function with $\pi(t) = 0$ for $t \geq \frac{2}{3}$ and $\pi(t) = 1$ for $t \leq \frac{1}{3}$. Then the function $\psi$ defined by

$$
\psi(x) = \pi(\phi(\delta x))
$$

has the required properties, with $R = \delta^{-1}M$.

(2) $\implies$ (3):
Given $R$ and $\psi$ as in (2), we may define

$$
\theta(x) = \sum_{n=0}^{\infty} \psi(R^{-n}x),
$$

noting that on each ball $\{x \in X : \|x\| < N\}$ the sum has only finitely many nonzero terms.

(3) $\implies$ (1):
Given $\theta$, we define $\phi(x) = \pi(\theta(x) - \theta(0))$, where $\pi : \mathbb{R} \to \mathbb{R}$ is the function already used above. $\square$

Many of our results concern spaces that are subspaces of the space $\ell_\infty(L)$ of bounded real-valued functions on a set $L$. For elements $f$ of $\ell_\infty(L)$ we use “coordinate” notation, writing $(f_t)_{t \in L}$ and thinking of the $f_t$ as coordinates. A certain class of very nice functions, already well-established in the literature, will be of particular importance. We shall say that a function $\phi$, defined on a subset $D$ of $\ell_\infty(L)$, depends locally on finitely many coordinates if, for each $f^0$ in $D$, there exist an open neighbourhood $G$ of $f^0$ in $D$ and a finite subset $M$ of $L$ such that, for $f \in G$, the value of $\phi(f)$ depends only on $f_t$ ($t \in M$).

**Infinitely differentiable norms**

**Theorem 1.** Let $L$ be a set and let $U(L)$ be the subset of the direct sum $\ell_\infty(L) \oplus c_0(L)$ consisting of all pairs $(f,x)$ such that $\|f\|_\infty$ and $\|x\|_\infty$ are both strictly less
than $\|f\| + \frac{1}{2}\|x\|_\infty$. The space $\ell_\infty(L) \oplus c_0(L)$ admits an equivalent norm $\| \cdot \|$ with the following properties:

1. $\| \cdot \|$ is a lattice norm, in the sense that $\|(g,y)\| \leq \|(f,x)\|$ whenever $|g| \leq |f|$ and $|y| \leq |x|$;
2. $\| \cdot \|$ is infinitely differentiable on the open set $U(L)$;
3. locally on $U(L)$, $\|(f,x)\|$ depends on only finitely many non-zero coordinates; that is to say, for each $(f^0,x^0) \in U(L)$ there is a finite $N \subseteq L$ and an open neighbourhood $V$ of $(f^0,x^0)$ in $U(L)$, such that for $(f,x) \in V$ the norm $\|(f,x)\|$ is determined by the values of $f_t$ and $x_t$ with $t \in N$ and such that $f_t \neq 0, x_t \neq 0$ for all such $(f,x)$ and $t$.

We start the proof of Theorem 1 by fixing a strictly increasing $C^\infty$ function $\varpi : [0,2) \to [0,\infty)$ such that $\varpi(u) \to \infty$ as $u \uparrow 2$ and $\varpi(u) = 0$ for $u \leq 1$. The inverse function $\varpi^{-1}$ is $C^\infty$ and strictly increasing from $(0,\infty)$ onto $(1,2)$. We define $\theta : [0,\infty) \to [0,\infty)$ by

$$\theta(c) = \int_0^c \frac{dv}{\varpi^{-1}(v)},$$

and start by recording some facts about this function. The easy proofs are left to the reader.

**Lemma 1.**

1. The function $\theta$ is strictly increasing and strictly concave from $[0,\infty)$ onto $[0,\infty)$. It is of class $C^\infty$ on $(0,\infty)$, with $\theta'(c) = 1/\varpi^{-1}(c)$ ($c > 0$), and differentiable at 0 with $\theta'(0) = \lim_{c \downarrow 0} 1/\varpi^{-1}(c) = 1$.
2. The composite function $\theta \circ \varpi : [0,2) \to (0,\infty)$ is infinitely differentiable, with

$$ (\theta \circ \varpi)'(u) = \begin{cases} u^{-1}\varpi'(u) & (u \geq 0) \\ 0 & (u = 0) \end{cases} $$

3. We have $\frac{1}{2}c < c\theta'(c) < \theta(c) < c$ for all positive $c$.

The next lemma can be regarded as the finite-dimensional part of the proof of Theorem 1.

**Lemma 2.** Let $N$ be a finite set, let $\eta$ be a positive real number and and let $W$ be the subset of $\mathbb{R}^N \times \mathbb{R}^N$ consisting of all $(f,x)$ such that $\|f\| + \frac{1}{2}\|x\|_\infty > \max\{\|f\|, \|x\|_\infty\} + \eta$. Let the functions $F : \mathbb{R}^N \times \mathbb{R}^N \times [0,\infty)^N \to \mathbb{R}$, $G : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be defined by

$$ F(f,x,c) = \exp\left[ - \sum_{t \in N} c_t \sum_{s \in N} c_s f_t + \theta(c_t)|x_t| \right] $$

$$ G(f,x) = \sup_{c \in [0,\infty)^N} F(f,x,c). $$

If $(f,x) \in W$ then the supremum in the definition of $G(f,x)$ is attained at a unique $c$; this $c$ has the property that $c_t = 0$ whenever $|f_t| \leq \eta$ or $|x_t| \leq \eta$. The function $G$ is of class $C^\infty$ on $W$. 
Thus the infinite differentiability of \( f \) as \( H \) such that
\[
\frac{\partial H}{\partial H}\bigg|_{(f, x)} = 0
\]
where \( \nu > 0 \) and \( |f_s| + \theta'(c_s)|x_s| = \nu \)
\( (1a) \) or \( c_s = 0 \) and \( |f_s| + \theta'(0)|x_s| \leq \nu \)
\( (1b) \)
where \( \nu = \sum_{t \in N} \left[ c_t|f_t| + \theta(c_t)|x_t| \right] \). In case (1a) we have
\[
\nu = \sum_{t \in N} \left[ \arctan \left( \frac{|x_t|}{\nu - |f_t|} \right) \right]
\]
(2)
because it is the function inverse to \( 1/\theta' \). In fact, this equality holds in case 1b as well because then \( |f_t| + |x_t| = |f_t| + \theta'(0)|x_t| \leq \nu \), whence \( |x_t|/(\nu - |f_t|) \leq 1 \)
and \( \arctan(|x_t|/(\nu - |f_t|)) = 0 = c_t \). Thus \( \nu \) is a solution of
\[
\nu = \sum_{t \in N} \left[ \arctan \left( \frac{|x_t|}{\nu - |f_t|} \right) \right] + \theta' \circ \arctan \left( \frac{|x_t|}{\nu - |f_t|} \right) |x_t|
\]
(3)
Since the right hand side of this equation is a decreasing function of \( \nu \) it has only one solution. By equation (2), we now see that \( c_s \) (s \( \in \mathbb{N} \)) are uniquely determined too.

Because \( (f, x) \in W \), there is some \( s \) such that \( |f_s| + \frac{1}{2}|x_s| > \max\{|f|, |x|\} + \eta \); since \( \theta'(c_s) > \frac{1}{2} \) we have \( \nu \geq |f_s| + \frac{1}{2}|x_s| \) by (1a) and (1b). Thus \( \nu \geq |f_t| + \eta \) and \( \nu \geq |x_t| + \eta \) for all \( t \). So if either \( |x_t| \leq \eta \) or \( |f_t| \leq \eta \) it must be that (1b) holds, with \( c_t = 0 \).

We now move on to consider the behaviour of \( \nu = \nu(f, x) \) and of \( c_t = c_t(f, x) \) as \( (f, x) \) varies over \( W \). We consider the function \( H \) defined on the open set \( V = \{(f, x, \nu) : (f, x) \in W \text{ and } \nu > \max\{|f|, |x|\} + \eta \} \) of \( \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \) by
\[
H(f, x, \nu) = \nu - \sum_{t \in N} \left[ \arctan \left( \frac{|x_t|}{\nu - |f_t|} \right) \right] + \theta' \circ \arctan \left( \frac{|x_t|}{\nu - |f_t|} \right) |x_t|
\]
We have already noted that for each \( (f, x) \in W \) there is a unique \( \nu = \nu(f, x) \) such that \( H(f, x, \nu) = 0 \). It is also easy to verify that \( \frac{\partial H}{\partial \nu} \geq 1 \) everywhere on \( V \).

Thus the infinite differentiability of \( (f, x) \rightarrow \nu(f, x) \) will follow from the Implicit Function Theorem provided we can show that \( H \) is itself infinitely differentiable.

The absolute value signs appear to present a problem on a neighbourhood of a point where one of the variables \( f_t \) or \( x_t \) is zero. However, as soon as either \( |f_t| \) or \( |x_t| \) is
smaller than \(\eta\), the terms \(\frac{|x_t|}{\|f\|_1} |f_t|\) and \(\theta \circ \frac{|x_t|}{\|f\|_1} |x_t|\) vanish, showing that we do not have a problem at all.

Once we have shown that \(v\) varies in an infinitely differentiable fashion with \((f, x)\), it follows from (2) that the same is true for all the \(c_t\) and hence for \(G\). \(\Box\)

We now take up the proof of the theorem. We define a norm \(\| \cdot \|\) on \(\ell_\infty(L) \oplus c_0(L)\) by

\[
\|(f, x)\| = \sup \left\{ e^{-\sum_{t \in L} d_t} \sum_{t \in L} \left[ d_t |f_t| + \theta(d_t) |x_t| \right] : \begin{array}{l} d_t \geq 0 \text{ for all } t \\ d_t = 0 \text{ for all but finitely many } t \end{array} \right\},
\]

and claim that this has the properties we require. It is clear that \(\| \cdot \|\) is a lattice norm and that

\[
e^{-1} \max \left\{ \|f\|_\infty, \frac{1}{2} \|x\|_\infty \right\} \leq \|(f, x)\| \leq e^{-1} (\|f\|_\infty + \|x\|_\infty).
\]

For \((f, x) \in U(L)\) we define

\[
\xi(f, x) = \|f\| + \frac{1}{2} |x|_\infty - \max \{\|f\|_\infty, |x|_\infty\}
\]

\[
M(f, x) = \{ t \in L : |f(t)| + |x(t)| \geq \|f\| + \frac{1}{2} |x|_\infty \}
\]

\[
N(f, x) = \{ t \in L : |f(t)| + |x(t)| \geq \|f\| + \frac{1}{2} |x|_\infty - \frac{1}{2} \xi(f, x) \}
\]

and note that \(N(f, x)\) is a finite set, because \(x \in c_0(L)\) and \(N(f, x) \subseteq \{ t : |x_t| \geq \frac{1}{2} \xi(f, x) \}\). We shall show first that in the definition of \(\|(f, x)\|\) it is enough to take the supremum over families \(d = (d_t)\) such that \(d_t = 0\) for all \(t \notin M(f, x)\). Indeed, let \((f, x)\) be in \(U(L)\), and suppose that \(d = (d_t)\) is such that \(d_t > 0\) for some \(t_1 \notin M(f, x)\). Let \(t_0\) be chosen so that \(|f(t_0)| + \frac{1}{2} |x(t_0)| = \|f\| + \frac{1}{2} |x|_\infty\) and let \(d' = (d'_t)\) be defined by

\[
d'_t = \begin{cases} d_t & \text{if } t \notin \{ t_0, t_1 \} \\ 0 & \text{if } t = t_1 \\ d_{t_0} + d_{t_1} & \text{if } t = t_0. \end{cases}
\]

We note that \(\sum d'_t = \sum d_t\) and that \(\theta(d'_t) - \theta(d_{t_0}) > \frac{1}{2} d_{t_1}\), because \(\theta'\) is everywhere greater than \(\frac{1}{2}\).

\[
\sum_{t \in L} [d'_t |f_t| + \theta(d'_t) |x_t|] - \sum_{t \in L} [d_t |f_t| + \theta(d_t) |x_t|]
\]

\[
= d_{t_1} \left[ |f_{t_0}| - |f_{t_1}| \right] + \theta(\theta_{t_0}) - \theta(d_{t_0}) |x_{t_0}| - \theta(d_{t_1}) |x_{t_1}| \]

\[
\geq d_{t_1} \left[ |f_{t_0}| - \frac{1}{2} |x_{t_0}| - |f_{t_1}| - |x_{t_1}| \right]
\]

and this is strictly positive by our assumptions about \(t_0\) and \(t_1\). In this way, we may reduce to 0 all coordinates \(d_t\) with \(t \notin M(f, x)\) while increasing the value of \(\exp[-\sum d_t \left[ \sum d_t f_t + \theta(t) |x_t| \right] ]\).

We now set about finding the neighbourhoods \(V\) and finite sets \(N\) referred to in (3). Given \((f^0, x^0) \in U(L)\), we set \(N = N(f^0, x^0)\) and define \(V\) to be the open set

\[
V = \{ (f, x) : \|f - f^0\|_\infty, \|x - x^0\|_\infty < \frac{1}{2} \xi(f^0, x^0) \}.
\]
It is easy to check that if \((f, x) \in V\) then \(\xi(f, x) > 1/2 \xi(f^0, x^0)\) and \(M(f, x) \subseteq N\). By what we have already proved, this shows that on the open set \(V\) our norm depends only on coordinates in the finite set \(N\).

Moreover, for \((f, x) \in V\) we have

\[
\|(f, x)\| = \sup_{c \in [0, \infty)^N} F(c, (f_t)_{t \in N}, (x_t)_{t \in N})
\]

where \(F : [0, \infty)^N \times \mathbb{R}^N \times \mathbb{R}^N\) is the function

\[
F(d, f, x) = \exp(- \sum_{t \in N} d_t \sum_{i \in N} [d_i|f_i| + \theta(d_t)|x_t|]).
\]

We can thus apply Lemma 2 (with \(\eta = \frac{1}{2} \xi(f^0, x^0)\)) to conclude that \(\|\cdot\|\) is infinitely differentiable on \(V\). □

Remark. The property of depending locally on finitely many non-zero coordinates is useful in applications. For instance, if \(G\) is an open subset of a Banach space \(X\) and \(\phi : G \to \ell_\infty(L) \oplus c_0(L)\) is a continuous mapping, taking values in \(U(L)\), then \(x \mapsto \|\phi(x)\|\) is of class \(C^k\) provided only that each coordinate map \(x \mapsto \phi(x)_t\) \((t \in L)\) is of that class on the set where it is non-zero. This is particularly handy when the coordinate maps are themselves norms or seminorms on \(X\).

In the following corollary to Theorem 1 we use the above remark to deduce a renorming result about injective tensor products. We recall some facts about such products. If \(X\) and \(E\) are Banach spaces and \(\xi, \eta\) are elements of the dual spaces \(X^*, E^*\) respectively, then a linear form \(\xi \otimes \eta\) may be defined on the algebraic tensor product \(X \otimes E\) by

\[
(\xi \otimes \eta)(\sum_{j=1}^n x_j \otimes e_j) = \sum_{j} \langle \xi, x_j \rangle \langle \eta, e_j \rangle.
\]

The injective tensor product \(X \otimes_i E\) is defined to be the completion of \(X \otimes E\) for the norm defined by

\[
\|z\|_i = \max\{|\langle (\xi \otimes \eta), z \rangle| : \xi \in \text{ball } X^*, \ \eta \in \text{ball } E^*\},
\]

The elementary tensor forms \(\xi \otimes \eta\) extend by continuity to \(X \otimes E\) and \(\{\xi \otimes \eta : \xi \in \text{ball } X, \ \eta \in \text{ball } E\}\) is a weak* compact subset of ball \((X \otimes E)^*\) on which every element of \(X \otimes E\) attains its norm.

If \(Q : X_1 \to X_2\) and \(R : E_1 \to E_2\) are bounded linear operators then a bounded linear operator \(Q \otimes R : X_1 \otimes E_1 \to X_2 \otimes E_2\) is determined by \((Q \otimes R)(x \otimes e) = (Qx) \otimes (Re)\). A special case is the so-called “slice map” \(I_X \otimes \eta : X \otimes E \to X\) derived from an element \(\eta\) of \(E^*\) and given by \((I_X \otimes \eta)(x \otimes e) = \langle \eta, e \rangle x\). By our earlier remark about the attainment of norms on elementary tensor forms, we see that for any \(z \in X \otimes E\) there is some \(\eta \in \text{ball } E^*\) with \(\|z\|_i = \|\eta\|_e\) such that \(\langle (I_X \otimes \eta)(z), e \rangle = \|z\|_i\) for all \(e \in E\).

In the special case where \(X\) is a space \(\ell_\infty(L)\), then \(X \otimes E\) identifies isometrically with a subspace of the vector-valued function space \(\ell_\infty(L; E)\) (the elementary tensor \((x_t)_{t \in L} \otimes e\) corresponding to \(t \mapsto x_t e\)). The effect of a slice map on \(z \in \ell_\infty(L) \otimes_e E\) regarded as an element \((z_t)_{t \in L}\) of \(\ell_\infty(L; E)\) is simply

\[
(I_{\ell_\infty(L)} \otimes \eta)(z) = \langle (\eta, z_t)\rangle_{t \in L}.
\]
Corollary 1. Let $X$ be a Banach space and let $L$ be a set. Suppose that there exist a linear homeomorphic embedding $S : X \rightarrow \ell_\infty(L)$ and a linear operator $T : X \rightarrow c_0(L)$ with the property that $(S(x), T(x))$ is in the set $U(L)$ of Theorem 1 whenever $x$ is a non-zero element of $X$. Then $X$ admits an equivalent $C^\infty$ norm. Moreover, whenever the Banach space $E$ admits an equivalent $C^k$ norm so does the injective tensor product $X \otimes_c E$.

Proof. It is clear that, using the norm on $\ell_\infty(L) \oplus c_0(L)$ that we defined in Theorem 1, we may set

$$
\|x\| = \|(S(x), T(x))\|
$$

and obtain an infinitely differentiable norm on $X$.

Now let $E$ be a Banach space with a $C^k$ norm $\| \cdot \|_E$. For $f \in \ell_\infty(L; E)$ we define $Nf \in \ell_\infty(L)$ by

$$
(Nf)_t = \|f_t\|_E.
$$

The operators $S$ and $T$ induce $S \otimes I_E$ and $T \otimes I_E$, taking the injective tensor product $X \otimes_c E$ into $\ell_\infty(L) \otimes_c E$ and $c_0(L) \otimes_c E$ respectively. Identifying $c_0(L) \otimes_c E$ with $c_0(L; E)$ and $\ell_\infty(L) \otimes_c E$ with a subspace of $\ell_\infty(L; E)$, we may define a norm on $X \otimes_c E$ by

$$
\|z\| = \|(N((S \otimes I_E)(z)), (N((T \otimes I_E)(z)))\|.
$$

The coordinate maps $x \mapsto (N((S \otimes I_E)(x)))_t = \|(S \otimes I_E)(x)_t\|$ and $x \mapsto (N((T \otimes I_E)(x)))_t = \|(T \otimes I_E)(x)_t\|$ are of class $C^k$ except where they vanish. Thus, by the above remark, we shall be able to conclude that we have a $C^k$ norm on $X \otimes_c E$ provided we can show that $(N((S \otimes I_E)(z)), (N((T \otimes I_E)(z)))$ is in $U(L)$ whenever $z \in (X \otimes_c E) \setminus \{0\}$. This is not really difficult, being just a matter of disentangling tensor notation.

For such a $z$ we choose some $\eta \in \text{ball } E^*$ with $\|(I_{\ell_\infty(L)} \otimes \eta)((S \otimes I_E)(z))\|_\infty = \|(S \otimes I_E)(z)\|_\infty$. We then note that $(I_{\ell_\infty(L)} \otimes \eta) \circ (S \otimes I_E) = S \circ (I_X \otimes \eta)$. Our hypothesis about $S$ and $T$, applied to $x = (I_X \otimes \eta)(z)$ tells us that there is some $t \in L$ with

$$
|S((I_X \otimes \eta)(z))_t| + \frac{1}{2}|T((I_X \otimes \eta)(z))_t| > \|S((I_X \otimes \eta)(z))\|_\infty.
$$

Thus

$$
N((S \otimes I_E)(z))_t + \frac{1}{2}N((T \otimes I_E)(z))_t = \|(S \otimes I_E)(z)_t\|_E + \frac{1}{2}\|(T \otimes I_E)(z)_t\|_E
$$

$$
\geq |\langle \eta, (S \otimes I_E)(z)_t \rangle| + \frac{1}{2}|\langle \eta, (T \otimes I_E)(z)_t \rangle|
$$

$$
= |(I_{\ell_\infty(L)} \otimes \eta)(S \otimes I_E)(z)_t| + \frac{1}{2}|(I_{c_0(L)} \otimes \eta)(T \otimes I_E)(z)_t|
$$

$$
= |S((I_X \otimes \eta)(z))_t| + \frac{1}{2}|T((I_X \otimes \eta)(z))_t|
$$

$$
> \|S((I_X \otimes \eta)(z))\|_\infty = \|(S \otimes I_E)(z)\|_\infty
$$

A similar argument involving an $\eta$ chosen so that $\|(I_{c_0(L)} \otimes \eta)((T \otimes I_E)(z))\|_\infty$ is equal to $\|(T \otimes I_E)(z)\|_\infty$ enables us to finish the proof. □

As we may now note, the above corollary allows us to reprove Hájek’s theorem from [3], though not, of course, the more recent, and very strong, result of [1], according to which any norm on a Banach space with countable boundary may be approximated by analytic norms.
Corollary 2 [Hájek]. Let $X$ be a Banach space which admits a countable boundary. Then $X$ admits a $C^\infty$ renorming and $X \otimes_E E$ admits a $C^k$ renorming whenever $E$ does.

Proof. Let $\{\xi_n : n \in \mathbb{N}\}$ be a countable boundary for $X$ and define $S : X \to \ell_\infty(\mathbb{N})$ and $T : X \to c_0(\mathbb{N})$ by $(Sx)_n = \langle \xi_n, x \rangle$, $(Tx)_n = 2^{-n} \langle \xi_n, x \rangle$. It is easy to see that $(S(x), T(x)) \in U(L)$ when $x$ is a non-zero element of $X$, since for any $x$ there exists $n$ with $\langle \xi_n, x \rangle = \|x\|$.

Extending slightly the terminology of [4], we shall say that a bounded linear operator $T$ from a subspace $X$ of $\ell_\infty(L)$ into $c_0(L)$ is a Talagrand operator for $X$ if for every non-zero $x$ in $X$ there exists $t \in L$ with $|x_t| = \|x\|_\infty$ and $(Tx)_t \neq 0$. It is clear that Corollary 1 is applicable to any such space, taking $S$ to be the identity operator. In the particular case where $L$ is equipped with a locally compact topology and $X$ is the space $C_0(L)$ of continuous functions, vanishing at infinity, we retrieve the results of [4]. The classic example of a Talagrand operator is defined on the space $C_0([0, \Omega])$, where $\Omega$ is an ordinal, by

$$(Tf)_\alpha = f_\alpha - f_{\alpha + 1}.$$ 

This has the required property since for any non-zero $f \in C_0([0, \Omega])$ there is a maximal $\alpha$ with $|f_\alpha| = \|f\|_\infty$. A non-linear version of a Talagrand operator is used in [4] to give an example of a space admitting a $C^\infty$ bump function but no smooth norm. The author’s earlier $C^1$ version of this result appears as Theorem VII.6.1 of [2]. The relevant application of our theorem is the following.

Corollary 3. Let $X$ be a Banach space and let $L$ be a set. Suppose that there exist continuous mappings $S : X \to \ell_\infty(L)$, $T : X \to c_0(L)$ with the following properties:

1. For all $x \in X$ the pair $(Sx, Tx)$ is in $U(L) \cup \{0\}$;
2. The coordinates of $S$ and of $T$ are all $C^k$ functions on the sets where they are non-zero;
3. $\|Sx\|_{\infty} \to \infty$ as $\|x\| \to \infty$.

Then $X$ admits a $C^k$ bump function.

Proof. Let $\theta : [0, \infty) \to [0, \infty)$ be a $C^\infty$ function which vanishes on $[0, 1]$ and which tends to infinity with its argument. The formula

$$\phi(x) = \theta(\|S(x, Tx)\|)$$

defines a $C^k$ function on $X$ which tends to infinity with $\|x\|$.

The author does not know whether the results of Corollary 1 about injective tensor products extend to the non-linear set-up of Corollary 3. However, in the special case of spaces of continuous functions we have the following proposition.

Proposition 2. Let $L$ be a locally compact space such that there exists a function $T : C_0(L) \to c_0(L)$ satisfying

1. For all $f \in C_0(L)$ the pair $(f, Tf)$ is in $U(L) \cup \{0\}$;
2. Each coordinate of $T$ is a $C^k$ function depending locally on finitely many coordinates.
Let \( E \) be a Banach space admitting a \( C^k \) bump function. Then the space \( C_0(L; E) \) also admits such a function. In particular, if each of the spaces \( L_1, \ldots, L_n \) is homeomorphic to an ordinal then \( C_0(L_1 \times \cdots \times L_n; E) \) admits a \( C^k \) bump function.

**Proof.** Let \( \theta \) be a \( C^k \) function on \( E \) such that \( \theta(x) \to \infty \) as \( \|x\| \to \infty \). For \( f \in C_0(L; E) \) the composition \( \theta \circ f \) is in \( C_0(L) \) and the pair \((\theta \circ f, T(\theta \circ f))\) is in \( U(L) \cup \{0\} \). Moreover, for \( t \in L \), the coordinate maps \( f \mapsto (\theta \circ f)_t \) and \( f \mapsto T(\theta \circ f)_t \) are of class \( C^k \) on \( C_0(L; E) \). Let \( \rho : \mathbb{R} \to \mathbb{R} \) be a \( C^\infty \) function such that \( \rho(u) = 0 \) for \( u \leq 1 \) and \( \rho(u) \to \infty \) as \( u \to \infty \). It is easy to check that the function \( \phi : C_0(L; E) \to \mathbb{R} \) defined by

\[
\phi(f) = \rho\left( \| (\theta \circ f, T(\theta \circ f)) \| \right)
\]

is of class \( C^k \) and tends to infinity with the norm of its argument.

When \( L \) is an ordinal \( \Omega \) (identified with the set \([0, \Omega)\) of ordinals smaller than itself), an operator \( T \) of the type considered above does exist. Indeed, we may use the Talagrand operator \((Tf)_\alpha = f_\alpha - f_{\alpha + 1}\). Thus \( C_0([0, \Omega); E) \) admits a \( C^k \) bump function whenever \( E \) does. Since \( C_0(L_1 \times \cdots \times L_n; E) \) may be identified with \( C_0(L_1; C_0(L_2 \times \cdots \times L_n; E)) \), an easy induction argument finishes the proof. \( \square \)

### Smooth partitions of unity

We say that a subset \( H \) of a Banach space \( X \) admits **partitions of unity of class \( C^k \)** if, for every open covering \( \mathcal{V} \) of \( H \), there is a locally finite partition of unity on \( H \), subordinate to the covering \( \mathcal{V} \), and consisting of the restrictions to \( H \) of functions that are of class \( C^k \) on \( X \). Once again, the reader is referred to [2] for more details, for the connection between partitions of unity and approximation by smooth functions and for Torunczyk’s criterion: \( H \) admits \( C^k \) partitions of unity if and only if there is a \( \sigma \)-locally finite base for the topology of \( X \) consisting of \( C^k \)-cozero sets (that is to say, sets of the form \( \{x \in H : \phi(x) \neq 0\} \) with \( \phi \in C^k(X) \)). It is not known whether \( C^k \) partitions of unity necessarily exist on every space that admits a \( C^k \) bump function, though many partial results are known ([2, VIII.3]). The following theorem has hypotheses that are involved but of fairly wide applicability.

**Theorem 2.** Let \( X \) be a Banach space, let \( \Gamma \) be a set and let \( k \) be a positive integer or \( \infty \). Let \( T : X \to c_0(\Gamma) \) be a function such that each coordinate \( x \mapsto T(x)_\gamma \) is of class \( C^k \) on the set where it is non-zero. For each finite subset \( F \) of \( \Gamma \), let \( R_F : X \to X \) be of class \( C^k \) and assume that the following hold:

1. for each \( F \), the image \( R_F[X] \) admits \( C^k \) partitions of unity;
2. \( X \) admits a \( C^k \) bump function;
3. for each \( x \in X \) and each \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( \| x - R_F x \| < \epsilon \) if we set \( F = \{ \gamma \in \Gamma : |(T x)(\gamma)| \geq \delta \} \).

Then \( X \) admits \( C^k \) partitions of unity.

**Proof.** By Torunczyk’s Criterion, it is enough to show that there is a \( \sigma \)-locally finite base for the topology of \( X \), consisting of \( C^k \)-cozero-sets. By hypothesis, each \( R_F[X] \) admits a \( \sigma \)-locally finite base \( \mathcal{V}_F \) consisting of \( C^k \)-cozero-sets. Since \( X \) admits a \( C^k \)-bump function, there is a neighbourhood base of 0 in \( X \) consisting of \( C^k \)-cozero-sets, say \( U_n \) \((n \in \mathbb{N})\). We introduce the covering \( \mathcal{W} \) of \( c_0(\Gamma) \) consisting of \( W_0 = c_0(\Gamma) \), together with all sets

\[
W_{F,q,r} = \{ y \in c_0(\Gamma) : \min_{\gamma \in F} |y(\gamma)| > r \quad \text{and} \quad \sup_{\gamma \in \Gamma \setminus F} |y(\gamma)| < q \}
\]
with \( F \) a finite non-empty subset of \( \Gamma \), and \( q, r \) positive rational numbers with \( q < r \). We note that \( W \) is \( \sigma \)-locally finite, and that its members are \( \mathcal{C}^\infty \)-cozero-sets.

In \( X \) we consider the family of all sets of the form

\[
T^{-1}[W_{F,q,r}] \cap R_F^{-1}[V] \cap (R_F - I)^{-1}[U_m]
\]

with \( m \) a positive integer, \( F \) a finite subset of \( \Gamma \), \( q, r \) positive rationals with \( q < r \) and \( V \in \mathcal{V}_F \). It is easy to check that this family is a \( \sigma \)-locally finite family of \( \mathcal{C}^k \) cozero sets. We have to show that it forms a base for the topology of \( X \).

Let \( x \) be in \( X \) and let \( \epsilon > 0 \) be given. We fix \( m \) so that

\[
U_m \subseteq \frac{1}{3}\epsilon \text{ ball } X,
\]

and, using (3), choose \( \delta > 0 \) so that

\[
x - R_F(x) \in U_m
\]

when we set \( F = \{ \gamma \in \Gamma : |(Tx)(\gamma)| \geq \delta \} \). Because \( Tx \in c_0 \) there exist rationals \( q, r \) with \( 0 < q < r < \delta \) such that \( |(Tx)(\gamma)| < q \) whenever \( \gamma \in \Gamma \setminus F \). Thus \( x \) is in \( T^{-1}[W_{F,q,r}] \). Since \( \mathcal{V}_F \) is a base for the topology of \( R_F[X] \), there exists \( V \in \mathcal{V}_F \) such that

\[
R_F(x) \in V \subseteq R_F(x) + \frac{1}{3}\epsilon \text{ ball } X.
\]

It follows that \( x \) is in the set

\[
T^{-1}[W_{F,n}] \cap R_F^{-1}[V] \cap (R_F - I)^{-1}[U_m].
\]

If \( x' \) is any other member of this set, then we have

\[
\|R_F(x) - R_F(x')\| \leq \epsilon/3
\]

because \( R_F(x') \in V \), while

\[
\|R_F(x') - x'\| \leq \epsilon/3,
\]

because \( (R_F(x') - x') \in U_m \). Thus \( \|x - x'\| < \epsilon \), which is what we wanted to prove. \( \square \)

It should be noted that the mappings \( T \) and \( R_F \) in the theorem are not assumed to be linear: a non-linear \( T \) is used in [5] to give an example where \( \mathcal{C}^\infty \) partitions of unity exist on a space with no smooth norm. However, the theorem offers some improvements on existing results even when only linear operators are involved. A special case of the corollary that follows occurs when the \( R_\alpha \) form a “projectional resolution of the identity” on \( X \). It is thus a result that is more general, as well as a bit simpler to prove, than the implication (vi) \( \Rightarrow \) (v) in Theorem VII.3.2 of [2].

**Corollary 4.** Let \( X \) be a Banach space admitting a \( \mathcal{C}^k \) bump function. Let \( \Omega \) be an ordinal and let \( R_\alpha \) (\( \alpha < \Omega \)) be a family of \( \mathcal{C}^k \) functions from \( X \) to \( X \) having the property that, for every \( x \in X \), the function \( R_x : [0, \Omega] \to X \) defined by \( (Rx)_\alpha = \)
$R_\alpha x$ ($\alpha < \Omega$), $(Rx)_\Omega = x$ is continuous. If for each $\alpha$ the image of $R_\alpha$ admits $C^k$ partitions of unity then so does $X$.

Proof. Since $X$ admits a $C^k$ bump function there exists a function $\phi : X \to [0,1]$, of class $C^k$ and such that $\phi(x) = 0$ on some neighbourhood of $0$ in $X$, while $\phi(x) = 1$ whenever $\|x\| \geq 1$. We set $\Gamma = \Omega \times \mathbb{N}$ and define $T : X \to \ell_\infty(\Gamma)$ by

$$(Tx)(\alpha, n) = 2^{-n} \phi(2^n (R_{\alpha+1} x - R_\alpha x)).$$

By construction, there is some $\eta > 0$ such that $\phi(x) = 0$ whenever $\|x\| \leq \eta$. Given $x \in X$ and $\epsilon > 0$ we fix $m$ such that $2^{-m} \epsilon < \epsilon$ and then note that, because of the continuity of $\alpha \mapsto R_\alpha x$, the quantity $\|R_{\alpha+1} x - R_\alpha x\|$ can exceed $2^{-m} \eta$ only for $\alpha$ in some finite set $H$. We thus have $|(Tx)_\gamma| \leq \epsilon$ except when $\gamma \in H \times \{0, 1, 2, \ldots, m-1\}$. This shows that $T$ takes its values in $c_0(\Gamma)$.

To define the “reconstruction operators” $R_F$ we set $R_0 = R_0$ and $R_F = R_{\alpha(F)+1}$ where, for a finite non-empty subset $F$ of $\Gamma$, $\alpha(F) = \max\{\alpha : \exists n \text{ with } (\alpha, n) \in F\}$. We shall show that Condition (3) of Theorem 2 is satisfied. Given $x \in X$ and $\epsilon > 0$, it may be that $\|x - R_\alpha x\| < \epsilon$ for all $\alpha < \Omega$; in this case there is clearly no problem. Otherwise, by the continuity of $\alpha \mapsto R_\alpha x$ on $[0, \Omega]$, there is a maximal $\beta < \Omega$ with $\|x - R_\beta x\| \geq \epsilon$. Again by the continuity of $\alpha \mapsto R_\alpha x$, we know that there is some $\gamma > \beta$ such that $\|R_{\gamma+1} x - R_\beta x\|$ takes a strictly positive value, $\eta$ say. Now we fix $n$ such that $2^n \eta \geq 1$, noting that $(Tx)(\gamma, n) = 2^{-n}$, and set $\delta = 2^{-n}$. If $F$ is the set $\{(\alpha, n) \in \Omega \times \mathbb{N} : (Tx)(\alpha, n) \geq \delta\}$ then $(\gamma, n) \in F$ and so $\alpha(F) \geq \gamma > \beta$, whence $\|x - R_F x\| < \epsilon$, as required. □

Corollary 5. Let $\Omega$ be an ordinal and let $E$ be a Banach space admitting $C^k$ partitions of unity. Then the space $C([0, \Omega]; E)$ also admits $C^k$ partitions of unity.

Proof. Proceeding by transfinite induction, we may suppose that $C([0, \gamma]; E)$ admits $C^k$ partitions of unity whenever $\gamma$ is an ordinal smaller than $\Omega$. If we define $R_\gamma : C([0, \Omega]; E) \to C([0, \Omega]; E)$ by

$$(R_\gamma f)\beta = \begin{cases} f\beta & (\beta \leq \gamma) \\ f\gamma & (\beta > \gamma) \end{cases}$$

then the range of $R_\gamma$ is isomorphic to $C([0, \Omega]; E)$ and so admits $C^k$ partitions of unity. Moreover, the continuity hypothesis in the preceding corollary is certainly satisfied, so that the proof will be finished if we know that $C([0, \Omega]; E)$ admits a $C^k$ bump function. This is true by Proposition 2, since $C([0, \Omega]; E)$ is isomorphic to $X = C_0([0, \Omega]; E) \oplus E$. □

References

1. R. Deville, V. Fonf and P. Hájek, Analytic and polyhedral approximation of norms in separable Banach spaces, Preprint.
2. R. Deville, G. Godefroy and V. Zizler, Smoothness and renormings in Banach spaces, Longman, Harlow, 1993.
3. P. Hájek, Preprint.
4. R.G. Haydon, Normes infiniment différentiables sur certains espaces de Banach, C.R. Acad. Sci. Paris 315 (1992), 1175–1178.
5. R.G. Haydon, Trees in renorming theory, in preparation.
6. H. Toruńczyk, Smooth partitions of unity on some nonseparable Banach spaces, Studia Math. 46 (1973), 43–51.

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