Statistics of torus piecewise isometries

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Abstract

By now, we have learned quite well how to study hyperbolic (locally expanding/contracting or both) chaotic dynamical systems, thanks to a large extent to the development of the so called operator approach. Contrary to this almost nothing is known about piecewise isometries, except for a special case of one-dimensional interval exchange mappings. The last case is fundamentally different from the general situation in the presence of an invariant measure (Lebesgue measure), which helps a lot in the analysis. We start by showing that already the restriction of the rotation of the plane to a torus demonstrates a number of rather unexpected properties. Our main results describe sufficient conditions for the existence/absence of invariant measures of torus piecewise isometries. Technically these results are based on the approximation of the maps under study by weakly periodic ones.

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1 Introduction

Let us start with a simple example of an isometry in $\mathbb{R}^2$ defined by the map $\tilde{T}x := \mathcal{R}x + b$, where $x, b \in \mathbb{R}^2$ are vectors and $\mathcal{R}$ is a rotation matrix. Applying the same map to the 2-dimensional torus $X$ we get $Tx := \tilde{T}x \pmod{1}$, which is no longer a global isometry. Moreover, the map $T$ is not a bijection and has both lines of discontinuities and regions of points having no preimages. On the other hand, one can easily make a partition of $X$ into a finite number of regions, such that the restriction of $T$ to each region is an isometry. This is a basic model for our study. Some nontrivial invariant sets for such a map are demonstrated at Fig. 1. In general we are interested in asymptotic properties of the dynamics of piecewise isometric torus maps like the one shown in this figure.

Recall that an isometry is a map which preserves distances between all points in the phase space. Thus a piecewise isometry (PI) is a map $T$ from a space $X$ equipped with a special partition $\{X_i\}$ into itself such that the restriction of $T$ to each element of the partition preserves distances (i.e. is a true isometry). An example of a torus PI is demonstrated at Fig. 2.

Naturally, the list of available isometries sensitively depends on the choice of a metric. In particular, under the discrete metric each bijection turns out to be an isometry. We will

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\[ x \rightarrow Ax + b \mod 1 \]
\[ A = A(\varphi), \ \varphi = \frac{\pi}{e}, \ b = 0 \]

**Figure 1:** Invariant sets of the torus rotation by the angle \( \pi/e \)

Consider a natural family of translational invariant metrics \( \rho_p(x, y) := \left( \sum_i |x_i - y_i|^p \right)^{\frac{1}{p}}, \ p \geq 1 \), where \( x_i, y_i \) are the coordinates of the points \( x, y \in X \in \mathbb{R}^d \). The Euclidean metric corresponds to \( \rho_2(\cdot, \cdot) \), while the uniform metric to \( \rho_\infty(\cdot, \cdot) \).

**Figure 2:** A special partition for a torus isometry. One of the elements of the partition and its image are marked in yellow and green colors correspondingly.

The paper is organized as follows. In Section 2 we give a brief historical description of results related to piecewise isometries and motivations of the present research. Section 3 is dedicated to basic definitions the formulation of the main result – Theorem 1. In Section 4 we study a seemingly very simple example of a PI: torus restriction of a plain rotation. Examples demonstrating the possibility of the absence of invariant measures for PI will be described in Section 5. In Section 6 we introduce and study weakly periodic maps and semigroups. Finally in Section 7 we apply the approximation by weakly periodic maps to prove Theorem 1.

## 2 Historical remarks

We start this short review with the discussion of known results related to “bijective” PI \( T : X \rightarrow X \), when the union of the images under \( T \) of the elements of the special
partition \( \{X_i\} \) covers the entire phase space \( X \). Since in the multidimensional setting \( (d > 1) \) this situation is rather exotic, we will discuss only the one-dimensional case, known in the literature as interval exchange transformations (IET). Quotes marks in the word “bijective” emphasize a problem with the boundary points of the elements of the partition \( \Gamma := \cup_i \partial X_i \). In fact, the authors either assume that either \( TX_i \cap TX_j \in T(\Gamma) \) for \( i \neq j \) (may intersect only on boundary points), or instead of the partition consider the covering \( \{\text{Clos}X_i\} \). The former case leads to an endomorphism, while the latter to a multi-valued map, because the map is not uniquely defined at \( \Gamma \). This is especially unpleasant because under the action of a multi-valued map a total measure of the phase space may grow, which cannot happen for a conventional single-valued map. Fortunately, due the fact that for any version of the action on boundary points the Lebesgue measure is preserved under dynamics, while \( \Gamma \) is the set of zero Lebesgue measure, one does not care much about these peculiarities.

Analysis of IETs is a very popular field of research and the list of publications related to them is very long and still actively growing. Therefore we mention only some basic publications in a historical order and refer the reader to look for further reference therein. The first results and important constructions related to orientation preserving IET were obtained in V.I. Oseledeets (1966) [16], M. Keane (1975) [11], A. Katok (1980) [10], W. Veech (1982) [20], H. Masur (1982) [13], M. Viana (2006) [21], A.I. Bufetov (2006) [4]. Starting already from the work of W. Veech the main object of interest here was shifted from a pure dynamical point of view to connections with geometry, in particular with Teichmuller flows. Since the latter goes out of the scope of the present paper, we will not discuss this.

In the present paper we are interested mainly in measure-theoretic points of view to dynamics, which is relatively simple and well understood in the case of IETs.

The situation is becoming rather different if a PI is not “bijective” even in the above mentioned weak sense. This means that the images of the elements of the special partition might have arbitrary intersections. The main body of results in this direction are again about the one-dimensional orientation preserving case, known in the literature as interval translation maps (ITM). The most interesting among them in a historical order belongs to M. Boshernitzan & I. Kornfeld (1995) [10], J. Buzzi & P. Hubert (2004) [5], H. Bruin (2007,2012) [3], S. Marmi, P. Moussa, J-C. Yoccoz (2012) [14], D. Volk (2014) [22]. A further generalization when a general cover is used instead of a special partition (which inevitably leads to a multi-valued map) was considered in A. Skripchenko & S. Troubetzkoy (2015) [18]. In any case the problem with the boundary points already mentioned in the case of IET is becoming a serious obstacle here. Nevertheless from a topological point of view or in the discussion of attractors one can go around it in order to reduce the analysis of ITM to a better understood IET. A popular approach introduced in [2] for the case of the circle orientation preserving PI is to classify them into finite and infinite types with respect to the number of iterations under which the images of the phase space are stabilized. Namely, the finite type is characterized by the property \( T^{n+1}X = T^nX \) for some \( n < \infty \), and otherwise we refer to an infinite type. Indeed, for the finite type case in the one-dimensional orientation preserving case one obtains an ITM in a finite number of iterations.

A number of attempts to follow this idea in the multidimensional setting have been tried (see, e.g. [22, 19] and further references therein). Unfortunately, in distinction to the one-dimensional case, the multidimensional dynamics is much more complicated and cannot be reduced to some version of ITM, which has been demonstrated by A. Goetz in
The Fig. 2 shows that despite the fact that already after the first iteration the image of the entire phase space coincides with itself (i.e. $TX = X$), the intrinsic dynamics of points inside of the elements of the partition is extremely complicated.

Figure 3: Dynamics of a triangle piecewise isometry (from [9]).

From the most interesting for us measure-theoretic point of view there are no results at all for $d > 1$ and only a few in the one-dimensional setting, which we will discuss now. The main problem is that there is no obvious candidate for an invariant measure, while the images of a good enough reference measure in general might concentrate at boundary points, which complicates their analysis. In [5] it has been shown that in the absence of periodic points the number of ergodic invariant measures of an orientation preserving circle PI cannot be larger than the number of elements of the special partition. It is somewhat strange that having this answer the authors even did not consider a question whether at least one invariant measure is present. Observe that PI maps are necessarily discontinuous and the classical Krylov-Bogolyubov argument cannot be applied. The first step in this direction was done by B. Pires (2016) [17], where the existence was proven under some (uncheckable) technical assumptions. The first and the only real result about the existence of invariant measures of an orientation preserving circle PI was obtained by S. Kryzhevich (2020) [12] by means of rational approximations. We will discuss this approach in some detail in Section 3.

3 Setup

Let $X$ be a subset of the Euclidean space $\mathbb{R}^d$, $d \geq 1$ equipped with a certain metric $\rho(\cdot, \cdot)$ belonging to the class of $\rho_p(x, y) := \left( \sum_i |x_i - y_i|^p \right)^{\frac{1}{p}}$, $p \geq 1$, and let $\{X_i\}$ be a partition of $X$ into disjoint regions. By a region we mean a convex set with a nonempty interior. The union of boundary points of all regions we denote by $\Gamma$, which by definition is of zero $d$-dimensional Lebesgue measure.

Definition 1 A piecewise isometry (PI) is a map $T : X \rightarrow X$, satisfying the property

\footnote{Basically this is a version of the Krylov-Bogolyubov theorem, specialized for discontinuous maps under study.}
that its restriction to each region $X_i$ is an isometry. We will refer to $\{X_i\}$ as a special partition.

**Definition 2** A restriction $T|_{X_i}$ is said to be an extendable isometry if it can be extended to an isometry $T_i$ on the entire $X$ such that $T_i|_{X_i} \equiv T|_{X_i}$. Correspondingly a piecewise isometry $T$ is said to be extendable if its restriction each to each region $X_i$ is an extendable isometry.

As we will see the innocent looking assumption of extendability is in fact somewhat restricting. Collecting well known geometric facts (see e.g. [15, 1]) we get the following

**Proposition 1** An isometry from a region $Y \subset \mathbb{T}^d$ to $\mathbb{T}^d$ is extendable if and only if it can be represented as a finite superposition of basic torus isometries: translations and coordinate flips or exchanges.

The main result of our study is the following theorem about extendable torus piecewise isometries.

**Theorem 1** Let $T$ be an extendable piecewise isometry of the unit torus $X$ with a finite special partition $\{X_i\}$. Then there exists at least one probabilistic $T$-invariant measure $\mu_T$. Additionally, if the boundary set $\Gamma$ contains no periodic points, this measure is non-atomic.

We expect that the extendability assumption can be avoided or significantly weaken, but the techniques applied in the present paper uses this assumption heavily. In the one-dimensional orientation preserving case the claim of Theorem 1 was proven in [12], but in the general (even one-dimensional) setting this claim is new. Moreover, in [12] the non-atomic property of invariant measures has been proven without the assumption of the absence of periodic points at $\Gamma$. One of the key-points here is to demonstrate that a periodic trajectory of a PI cannot be isolated (all nearby trajectories have different itineraries). It was expected that even in the one-dimensional non orientation preserving case the situation will be the same. This turns out to be wrong, and a counterexample will be discussed in the last part of the proof of Proposition 4.

To demonstrate that the claim of Theorem 1 is far from being obvious we will construct an example an extendable piecewise isometry with a countable number of regions without any invariant measure (see Section 5).

A general scheme of the proof of Theorem 1 is as follows. First, we consider a family of torus isometries $\{g_\theta\}$ satisfying the property that any finite subcollection of such isometries generates a group $G(g_{\theta_1}, \ldots, g_{\theta_n})$ having a finite number of elements. We refer to this property as weak periodicity. In Section 6 we will analyze properties of weakly periodic maps. In the case of commuting generators the situation is rather trivial and an important point here is that we are able to study some important cases when at least some of the the generators $g_\theta$ do not commute.

On the 2-nd step we consider torus piecewise isometries $T$ such that the restrictions $T|_{X_i}$ can be extended to the torus isometries belonging to the above family. In Section 6 we study properties of such maps and prove for them the claim of Theorem 1.

Finally in Section 7 we approximate an arbitrary extendable torus piecewise isometry by means of weakly periodic ones considered in Section 6, which will allow to finish the proof of Theorem 1.

\[\text{See Section 6 for discussion.}\]
It is worth noting, that this result does not contradict to the possibility of having attractors admitting singular invariant measures. Various numerical results (see e.g. [9, 1]) give some evidence of the possibility of such behavior. The point is that the corresponding measures might not be represented as limits of measures of maps from Section 6 and hence cannot be controlled by our construction.

A similar approach was used in [12], where a specialized version of the above construction was applied for the case of orientation preserving circle piecewise rotations. In that case the author used the approximation scheme based on rational translations and special partitions with rational boundary points. The last fact is a crucial point not allowing to use this approximation in the multidimensional setting. Indeed, for $d > 1$ the boundaries of regions cannot consist of points with rational coordinates only.

4 Torus restriction of a plane isometry

In this section we consider actions of plane isometries restricted to the unit two-dimensional torus $X := \mathbb{T}^2$. Recall that a general plane isometry can be written as $x \to Rx + b$, where $x, b \in \mathbb{R}^2$ and $R$ is an orthogonal matrix. Restricting this map to the torus we get the torus map $Tx := Rx + b \pmod{1} : \mathbb{T}^2 \to \mathbb{T}^2$.

Example 1 Let $R = R(\varphi)$ be the rotation around the origin by the angle $\varphi \in [0, 2\pi)$ and let $b = 0$.

Having in mind that that each trajectory of the plane rotation belongs to a single circle around the origin, while a typical torus translation $x \to x + b \pmod{1}$ fills the torus densely, one naively expects something of the same sort for the map $T$. It turns out that if the rotation angle $\varphi \neq 0$ the situation is completely different (see Fig. 1).

The map $T$ defined above, considered as a map from the unit torus $X := \mathbb{T}^2$ into itself is continuous everywhere inside the open unit square $\tilde{X} := (0, 1)^2$ and the set of discontinuities is represented by the boundary of this set $\partial \tilde{X}$. On the other hand, the same map may be considered as a map from the unit square $\hat{X} := [0, 1]^2$ into itself. In the latter case the set of discontinuous is much more involved (see Fig. 4).

If $\det(I - R) \neq 0$ there is at least one fixed point of the map $Tx := Rx + b \pmod{1}$. Here and in the sequel by $I$ we denote the unit matrix. Indeed, making the change of variables $x = y + a$, we rewrite the equation for the fixed point of the map $T$ as $y + a = R(y + a) + b \pmod{1}$, which gives a solution $y := 0$, $a := (I - R)^{-1}b \pmod{1}$. Thus $z := (I - R)^{-1}b \pmod{1} \pmod{1}$ is the fixed point. If additionally, $b = 0$ (case of a pure rotation) the origin is always the fixed point even without the assumption about the absence of the degeneracy of the matrix $I - R$.

Therefore the main question is about the existence of nontrivial periodic trajectories. Our numerical observations allow to formulate the following

**Hypothesis.** Let $Tx := jR(\varphi)x + b \pmod{1}$, where $j = \pm 1$, $\varphi \in [0, 2\pi)$, $x, b \in \mathbb{R}^2$. Then $\forall j, \varphi, b \exists N = N(j, \varphi, b) \in \mathbb{Z}_+$ and $z = z(j, \varphi, b, N)$ such that $T^N z = z$. Moreover for a given triple $j, \varphi, b$ one may find arbitrary large $N$ satisfying the above property.

**An idea of the possible proof.** We consider only the case of the pure rotation (i.e. $j = 1$, $b = 0$) by the irrational angle (i.e. $\varphi = \gamma \pi$, $\gamma \notin \mathbb{Q}$), leaving the general case (where a number of cumbersome combinatorial type calculations are necessary) for the future.
Observe that $Tx = Rx \pmod{1} = Rx + \xi(x)$, where $\xi(x) \in \mathbb{Z}^2$. Therefore to find a periodic point of period 1 one needs to solve the equation:

$$z = Rz + \xi_1,$$

and for the period 2:

$$z = R(Rz + \xi_1) + \xi_2 = R^2z + R\xi_1 + \xi_2.$$

Continuing this, for a point of period $n$ we have

$$z = R^nz + \sum_{j=1}^{n} R^{n-j}\xi_j,$$

which is equivalent to

$$(I - R^n)z = \sum_{j=1}^{n} R^{n-j}\xi_j.$$

If the angle $\varphi$ is irrational, then the inverse matrix $(I - R^n)^{-1}$ is well defined. Thus we have an explicit solution

$$z := (I - R^n)^{-1} \sum_{k=1}^{n} R^{n-k}\xi_k.$$

Unfortunately this is not the end of the story, since in fact the integer vectors $\xi_i$ depend on $z$ and should be chosen in such a way that $R^kz + \sum_{j=1}^{n} R^{n-j}\xi_j$ belong to the unit square for each $1 \leq k \leq n$, otherwise further corrections need to be taken into account.

Due to the already mentioned problems, instead of the general statement formulated as a Hypothesis we give a proof of a partial result leaving the general case for future studies.
**Proposition 2** Let \( \varphi \notin [-\varphi_0, \varphi_0] \), where \( \tan \frac{\varphi_0}{2} = \frac{1}{2} \). Then the equation \( A(\varphi)x = x \) always has a solution.

**Proof.** Consider a point \( x \) lying on a vertical line with the 1st coordinate being equal to \( \frac{1}{2} \), i.e. \( x = (\frac{1}{2}, x_2) \) (see Fig.4). Then under the rotation around the origin by the angle \( \varphi := -2 \arctan(\frac{1}{2x_2}) \) we get a point \( \tilde{x} := (-\frac{1}{2}, x_2) \), which coincides with \( x = (\frac{1}{2}, x_2) \) modulo 1. The assumption \( \varphi \notin [-\varphi_0, \varphi_0] \) is explained by the choice of the point \( x \). \( \square \)

![Figure 5: Construction of the fixed point of the torus rotation \( x \to A(-\varphi)x \).](image)

**Proposition 3** Let \( z \) be a \( N \)-periodic point and let \( R := \inf_k \dist(T^k z, \partial X) > 0 \). Then \( S(r) := \bigcup_{i=0}^{N-1} B_r(z) \) is a forward invariant set \( \forall r \in (0, R) \).

**Proof.** Consider any point \( x \) close enough to \( z \). Under the action of the map \( T \) the \( \rho_2(\cdot, \cdot) \) distance between \( T^n x \) and \( T^n z \) is preserved. Therefore each circle centered at \( z \) with a small radius \( r \) is mapped into itself. The value of the radius \( r \) is bounded from above by the condition that the rotated circle does not intersect with the boundary of the unit square. If additionally \( \varphi/\pi \) is an irrational number, then the trajectory of the point \( x \) fills densely the corresponding circle, otherwise, it is periodic. \( \square \)

Observe that the number of ergodic invariant measures is always countable in distinction to the one-dimensional case, where it is known that in a “typical” case the number of ergodic measures is bounded from above by the number of elements of the special partition.

A substantially more general result about the dynamics of a general PI in a neighborhood of a given trajectory based on the Proposition 3 may be formulated as follows.

Let \( T \) be a PI of \( X \in \mathbb{R}^d \) with respect to a metric \( \rho(\cdot, \cdot) \) and a special partition \( \{X_i\} \).

**Proposition 4** Let a trajectory of a point \( z \in X \) is separated from the set \( \Gamma \) - the boundary of the special partition, i.e. \( \inf_{n \geq 0} \rho(T^nz, \Gamma) := R > 0 \). Then \( \forall x \in B_R(z) \) the itineraries of the trajectories started at the points \( x, z \) coincide, i.e. \( T^n x \in X_i \iff T^n z \in X_i \forall n \geq 0 \). Moreover, \( \forall r \in (0, R) \) the spheres \( S_r(T^nz) \) of radius \( r \) centered at \( T^nz \) are
mapped one to another under dynamics, i.e. \( TS_r(T^n z) = S_r(T^{n+1} z) \). If the separation property breaks down the trajectory stated at \( z \) may be completely isolated in the sense that \( \liminf_{n \to \infty} \rho(T^n z, T^n z) > 0 \) whenever \( x \neq z \).

**Proof.** The separation assumption means that the ball \( B_R(z) \) at each iteration of the map \( T \) falls entirely to a single element of the special partition, which implies the claim about the itineraries. Now using the same argument as in the proof of Proposition 3 we see that the sequence of balls \( B_R(T^n z) \) splits into the continuous on-parametric family of concentric spheres \( S_r(T^n z) \) mapped one to another under dynamics.

Observe that in the case of a periodic trajectory the separation property becomes especially simple and trivially checkable: \( \bigcup_{n \geq 0} T^n z \setminus \Gamma = \emptyset \).

To demonstrate the last claim we return to the pure rotation of the 2-dimensional torus around the origin by the angle \( \pi/4 \). Clearly, then point \( z = 0 \) is fixed under dynamics. Observe that a sufficiently small neighborhood \( O_z \) of this point on the torus consists of 4 triangular sectors at the corners of the unit square and that each of these sectors belong to at most 2 elements of the special partition. This allows to use a simple numerical simulations to check that \( T^n(X \setminus \{z\}) \cap O_z = \emptyset \ \forall n \gg 1 \), which finishes the proof.

Our numerical simulations indicate that the last claim remain valid for any rotation angle \( \varphi \neq 0 \) and any shift \( b \), but this is not proven yet. Instead we consider an even more unexpected situation when a periodic trajectory of an one-dimensional non orientation preserving PI is isolated.

**Example 2** Consider an \( PI : X \to X := [0,1) \) with the special partition into 4 intervals \( X_1 := [0, \frac{1}{4}), X_2 := [\frac{1}{4}, \frac{1}{2}), X_3 := [\frac{1}{2}, \frac{3}{4}), X_4 := (\frac{3}{4}, 1) \), where the map is defined as follows \( TX_1 := X_4, TX_2 := -X_3, TX_3 := X_1, TX_4 := X_4 \). The minus in \( TX_2 := -X_3 \) means the flip (orientation change) of the of the interval \( X_3 \), i.e. \( T_\frac{1}{4} = \frac{3}{4}, T_\frac{1}{2} = \frac{1}{4} \).

![Diagram](https://via.placeholder.com/150)

**Figure 6:** A non orientation preserving circle PI with an isolated periodic trajectory \( 1/2, 3/4 \).

Observe that in this example the trajectory started at the point \( \frac{1}{2} \) is 2-periodic, while all points from the surrounding intervals \((0, \frac{1}{4})\) and \((\frac{1}{4}, \frac{3}{4})\) in 2 iterations are mapped to the 4-th interval \((\frac{3}{4}, 1)\) and never return back. \( \square \)
5 Absence of invariant measures

In this section we construct an example of 2-dimensional torus piecewise isometry having no invariant measures. In distinction to other parts of this paper the special partition in the example will be countable.

Example 3 Let \( X := [0, 1]^2 \) and the special partition \( \{ X_n \} \) be defined as follows: \( X_n := (1 - 2^{-n+1}, 1 - 2^{-n}] \times [0, 1), \ n \geq 1 \), while \( Tx := (x_1 + 2^{-n-1}, x_2) \) if \( x \in X_n \).

![Diagram of a piecewise torus isometry without invariant measures. The element \( X_2 \) of the special partition and its image are marked in yellow and green colors correspondingly.]

Observe that after exactly two iterations each element of the partition \( \{ X_n \} \) will be mapped to the union of elements with larger indices. In this situation obviously any measure converges under the action of \( T \) to a measure supported by the unit segment \( \{0\} \times [0, 1) \). Since this segment is not forward invariant under the action of \( T \), this implies the absence of invariant measures.

Nevertheless, in the Example 3 we do not have a complete concentration of measure, namely the situations when the limit (but not invariant) measure is a \( \delta \)-measure at a single point. To achieve this goal instead of basically one-dimensional example above we consider a bit more involved \( d \)-dimensional map.

Example 4 Let \( X := [0, 1]^d \) and the special partition \( \{ X_n \} \) be defined as follows:

\[
d(1 - \frac{1}{n}) \leq \sum_{i=1}^{d} x_i < d(1 - \frac{1}{n+1}),
\]

while \( Tx := x + \frac{1}{2(n+1)} \bar{1} \) (mod 1) for \( x \in X_n \), where \( \bar{1} \) is the \( d \)-dimensional vector with unit coordinates.

In distinction to the previous example the image of \( X_n \) may intersect with elements \( X_k \) with smaller indices \( k < n \). Since the diameters of the elements of the partition decrease when their indices grow, after the implementation of the operation (mod 1) peripheral parts of the image may be located below the central part and intersect with the elements with lower indices. In fact, \( TX_n \cap X_1 \neq \emptyset \ \forall n \). Nevertheless the Hausdorff distance from the set \( T^kX_n \) to the point \( \bar{1} \) goes to 0 when \( k \to \infty \). Therefore under the action of \( T \) any measure weakly converges to \( \delta _1 \), while the point \( \bar{1} \equiv 0 \) is not forward invariant under \( T \), namely \( T\bar{1} = T0 = \frac{1}{4}\bar{1} \neq \bar{1} \equiv 0 \).
\[
X_n := \{ x : 2(1 - \frac{1}{n}) \leq x_1 + x_2 < 2(1 - \frac{1}{n+1}) \}\]

\[
X := \bigcup_{n \geq 1} X_n = [0,1) \times [0,1)
\]

\[
T x := x + \frac{1}{2n(n+1)} \bar{1} \pmod{1} \text{ if } x \in X_n
\]

\[
T^k X \overset{k \to \infty}{\to} \{1\} \equiv \{0\}
\]

Figure 8: Example of a piecewise torus isometry converging to \(\bar{1}\). The element \(X_2\) of the special partition and its image are marked in yellow and green colors correspondingly.

6 Weakly periodic semigroups of maps

The main aim of this Section is to introduce the class of weakly periodic torus piecewise isometries and to prove Theorem 1 for them.

Definition 3 A collection of maps \(\{ g_i \}_{i=1}^n\) from a set \(X\) into itself is said to be weakly periodic (notation WPer) if the number of elements in the semigroup generated by them is finite, i.e. \(\# G(g_1, \ldots, g_n) < \infty\).

The reason under this terminology is that a periodic homeomorphism \(T\) (i.e. \(T^n = Id\)) generates a semigroup with a finite number of elements, i.e. \(T \in \text{WPer}\).

The simplest (and very instructive) example of a weakly periodic semigroup of maps is a semigroup generated by a finite number of rational circle translations.

An important observation about this property gives the following result.

Proposition 5 Let the maps \(g_i : X \to X\) commute between themselves, then \(\sum_i \# G(g_i) < \infty\) implies \(\# G(g_1, \ldots, g_n) < \infty\). Otherwise there exists a pair of maps \(g_1, g_2\) such that \(\sum_{i=1}^2 \# G(g_i) = 4\), but \(\# G(g_1, g_2) = \infty\).

Proof. Each element \(T \in G(g_1, \ldots, g_n)\) may be represented in the form of a finite superposition of the generators \(g_i\), and due to the commutativity of \(g_i\) we get

\[
T x = g_1^{k_1} \circ \ldots \circ g_n^{k_n} x, \quad 0 \leq k_i = k_i(x) < \infty \quad \forall x \in X.
\]

By the assumption \(\exists N < \infty\) such that \(\# G(g_i) < N \forall i\). Therefore,

\[
\#(\bigcup_{T \in G} T x) \leq N^n < \infty \quad \forall x \in X,
\]

which implies the first claim.

The 2nd claim is not so obvious as one might expect. In particular, in the one-dimensional case it is wrong for orientation preserving maps. The latter property turns out to be crucial here. Let \(X := [0,1), \quad g_1(x) := -x \pmod{1}, \quad g_2(x) := -x + \alpha \pmod{1}, \quad \alpha \notin \mathbb{Q}\). Observe that \(\# G(g_i) = 2, \quad i = 1, 2, \) but \((g_1g_2)^n x = x - n\alpha \pmod{1}\), which implies \(\# G(g_1, g_2) = \infty\) since \(\alpha \notin \mathbb{Q}\). ☐

The definition of the weakly periodic maps implies the following simple technical result.
Proposition 6 Let \( \{g_i\}_{i=1}^n \in \text{WPer} \) and let a PI map \( T : X \to X \) with a finite special partition \( \{X_i\}_{i=1}^n \) such that \( T|_{X_i} \equiv g_i|_{X_i} \), \( \forall i \). Then \( \forall x \in X \) the trajectory \( \{T^n x\}_{n \geq 0} \) starting at \( x \) is eventually periodic.

Proof. The claim follows immediately from the observation that the total number of different points obtained by successive applications of the generators \( g_i \) is finite. \( \Box \)

Moreover, applying this idea for a PI torus map we get

Proposition 7 Let \( \{g_i\}_{i=1}^n \) be a collection of torus diffeomorphisms and let \( \#G(g_1, \ldots, g_n) < \infty \). Let \( T \) be a piecewise isometry of the unit torus \( X \) with the finite partition \( \{X_i\} \) and such that each \( T|_{X_i} \) can be extended to a torus map \( T_i \in G(g_1, \ldots, g_n) \). Then there exists at least one absolutely continuous invariant measure, being a restriction of the Lebesgue measure to a certain region.

Proof. Denote by \( \tilde{\Gamma} := \bigcup_{g \in G(g_1, \ldots, g_n)} g^{-1} \Gamma \) – the union of all pre-images under the action of all maps \( g \in G(g_1, \ldots, g_n) \) of the boundary \( \Gamma \) of the elements of the partition \( \{X_i\} \). By the weak periodicity assumption \( \tilde{\Gamma} \) divides the torus \( X \) into a finite number of regions \( \{Y_j\} \) of positive Lebesgue measure, such that points belonging to the same region \( \{Y_j\} \) have the same itineraries under the action of the map \( T \). Therefore each of the regions \( Y_i \) is either periodic under the action of \( T \), or is wandering. The latter case is out of interest for us, while a collection of periodically exchanging regions \( Y_{i_1}, \ldots, Y_{i_k} \) supports the Lebesgue measure on them being invariant with respect to the map \( T \). Moreover, all absolutely continuous \( T \)-invariant measures may be described this way. \( \Box \)

Despite that according to Proposition 5 in general the WPer property for individual maps does not imply this property for a collection of them, the situation becomes more optimistic when we restrict ourselves to torus isometries.

As we already noted in Proposition 1, any \( d \)-dimensional torus isometry may be represented by a superposition of basic isometries of the following 3 types:

(i) translation: \( x \to x + v \mod 1 \), \( v \in \mathbb{R}^d \)

(ii) coordinate flip: \( x_i \to -x_i \)

(iii) coordinate exchange: \( x_i \leftrightarrow x_j \)

The maps belonging to different types do not commute, but at least the maps from the first two types commute with the maps of the same type, however the maps of type (iii) do not commute even between themselves. Weak periodicity of the maps from the first type is equivalent to their periodicity, while the maps of the other two types are always 2-periodic.

To apply the machinery developed in Proposition 7 one needs to show that a PI generated by a finite collection of periodic basic isometries satisfy the weak periodicity property.

Theorem 2 Let \( T \) be a torus PI such that \( T|_{X_i} \) is a finite composition of periodic basic isometries \( \forall i \). Then the map \( T \in \text{WPer} \).
The proof of this intuitively obvious claim (to which we devote the remaining part of this Section) turns out to be surprisingly cumbersome. In particular, we are not able to perform the proof directly for the map $T$ itself and are doing this instead for a certain semigroup generated by symmetrized torus extensions of the local isometries.

**Definition 4** By a symmetrized semigroup $G^*(g_1, \ldots, g_n)$ generated by basic torus isometries $\{g_i\}$ we mean a semigroup generated by the maps $\{g_i\}$, where each translation map $x \to x + v \pmod 1$ is considered together with all permutations and sign changes of the coordinates of the vector $v$, similarly each coordinate flip or exchange is considered together with all other coordinate flips or exchanges respectively. In the same manner by $(\{g_i\})^*$ we mean a symmetrized collection of basic torus isometries $\{g_i\}$.

Fortunately to the finite dimension of the torus, the number of additional generators in the symmetrized semigroup is finite as well.\(^3\)

**Proposition 8** Let $\{g_i\}_{i=1}^n$ be a collection of periodic basic torus isometries. Then $\#G^*(g_1, g_2, \ldots, g_n) \leq \prod_{i=1}^n \#G^*(g_i) < \infty$.

**Proof.** We proceed by induction. The claim for $n = 1$ is trivial, except for its finiteness. If $g_i$ belongs to (i) or (ii) types, then the finiteness follows from the periodicity assumption together with the commutation property. If $g_i$ is of the 1st type, the last one gets a direct estimate $G^*(g_i) \leq d!$ since all possible coordinate permutations are taken into account.

Let us make the induction step from $n$ to $n + 1$. Observe that the application of the semigroup $G^*(g_1, g_2, \ldots, g_n)$ to a given point $x \in X$, i.e. the set $G^*(g_1, g_2, \ldots, g_n)x$ is invariant with respect to each of the maps $(\{g_i\}_{i=1}^n)^*$. The latter means that

$$g(G^*(g_1, g_2, \ldots, g_n)x) = G^*(g_1, g_2, \ldots, g_n)x \quad \forall g \in (\{g_i\}_{i=1}^n)^*.$$ 

We will show that for any periodic basic isometry $g_{n+1}$ the set $G^*(g_{n+1})G^*(g_1, g_2, \ldots, g_n)x$ is invariant with respect to each of the maps $(\{g_i\}_{i=1}^n)^*$, which implies the claim.

Denote $A := G^*(g_1, g_2, \ldots, g_n)x$, $B := G^*(g_{n+1})G^*(g_1, g_2, \ldots, g_n)x$ and consider all possibilities:

- $g_{n+1}$ is a translation. If $A$ is invariant with respect to some translation $g_i$, then the invariance of $B$ with respect to $g_i$ follows from the commutation property. The invariance of $A$ with respect to a coordinate flip is equivalent to the fact that the set $A$ is a uniform lattice on the torus. The latter property cannot be destroyed by a torus translation. If $A$ is invariant with respect to a coordinate exchange, then this set is characterized by the symmetry with respect to the corresponding unit cube diagonal. Unfortunately the translation $g_{n+1}$ may destroy this invariance for $B$. Indeed a symmetric pair $(a, b), (b, a)$ may by mapped to $(a + c, b), (b + c, a)$. This is exactly the point, where we need the symmetrization, because $G^*(g_{n+1})A$ clearly preserves the symmetry under question.

- $g_{n+1}$ is a coordinate flip. If $A$ was invariant with respect to some translation, $B := G^*(g_{n+1})A$ preserves this invariance due to the symmetrization. The invariance of $A$ with respect to a coordinate flip $g_i$ implies the same property for $B$, since coordinate flips commute. The invariance of $A$ with respect to a coordinate exchange is inherited by $B$ again helps to the symmetrization with respect to the coordinate flips.

\(^3\)It cannot be enlarged in more than $(d!)^3$ times.
• $g_{n+1}$ is a coordinate exchange $x_i \leftrightarrow x_j$. This situation can be analyzed in exactly the same manner and we skip it.

The induction step is proven. \(\square\)

Now we are ready to prove Theorem 2. Denote the torus extensions of the local isometries $T|_{X_i}$ by $g_i$ respectively and assume that the maps $g_i$ are finite superpositions of periodic basic torus isometries. Then the trajectory of length $n$ under the action of $T$ of any given point $x \in X$ is a subset of $G^*(g_1, \ldots, g_n)x$, which is uniformly finite on $x$ and $n$ by Proposition 8. This proves the weak periodicity of the map $T$. \(\square\)

7 Approximation

In what follows to control the rate of recurrence we will need the following abstract result

**Lemma 9** Let $(T, X, \mathcal{B}, \mu)$ be a measurable dynamical system, let $A \in \mathcal{B}$, $\mu(A) > 0$, and let $\tau_A(x)$ be the first return time for a point $x \in A$ to return to the set $A$ under the action of $T$. Then $\mu(\{x \in A : \tau_A(x) \leq 1/\mu(A)\}) > 0$.

Before to prove this result, observe that according to the famous Kac Lemma (see e.g. [7]) in the ergodic case the mathematical expectation of the first return time is exactly equal to $1/\mu(A)$. On the other hand, in our setting we cannot a priori assume ergodicity without which the claim of Lemma 9 is as best as possible.

**Proof.** By the Poincare Recurrence Theorem $\mu$-a.a. points return to $A$ under the action of $T$. Set $A_n := \{x \in A : n \leq \tau_A(x) < \infty\}$ — the set of points returning at least after $n \geq 0$ iterations. Clearly $\mu(A \setminus A_n) = 0 \implies \mu(A) = \mu(A_n)$.

Since $T^{-k}A_n \cap A = \emptyset$ $\forall k < n$, we get $T^{-i}A_n \cap T^{-j}A_n = \emptyset$ $\forall i \neq j < N$ (otherwise the points from the intersection above will not return to $A$). Hence

$$1 \geq \mu(\cup_{k=0}^{n-1}T^{-k}A_n) = \sum_{k=0}^{n-1} \mu(T^{-k}A_n) = n\mu(A)$$

and thus $n \leq 1/\mu(A)$. \(\square\)

Let $\{T^{(n)}\}$ be a sequence of piecewise torus isometries with the common finite special partition $\{X_i\}$, $\Gamma := \bigcup_i \partial X_i$, satisfying the assumptions of Proposition 7, and let $\{\mu_n\}$ be a sequence of their absolutely continuous invariant measures.

**Lemma 10** Let $z \notin \text{Per}(T)$. Then $\forall \varepsilon > 0 \exists \delta > 0 \limsup_{n \to \infty} \mu_n(B_\delta(z)) < \varepsilon$.

**Proof.** Assume from the contrary that there is $\varepsilon > 0$ and a pair of sequences $\{n_k\} \xrightarrow{k \to \infty} \infty$ and $\{\delta_k\} \xrightarrow{k \to \infty} 0$ such that $\mu_{n_k}(B_{\delta_k}(z)) \geq \varepsilon$. Then by Lemma 9

$$\exists N < 1/\varepsilon : T^{-N}B_{\delta_k}(z) \cap B_{\delta_k}(z) \neq \emptyset,$$
which implies that \( z \in \text{Per}(T) \). We came to the contradiction. \( \square \)

Since we deal with a compact phase space, the sequence of the above mentioned absolutely continuous invariant measures of the approximating weakly periodic maps has a limit point. Denote this limit point by \( \hat{\mu} \) and choose a sequence of the measures \( \{\mu_n\} \) converging to it in the weak topology.

**Lemma 11** Let \( \Gamma \cap \text{Per}(T) = \emptyset \). Then \( \hat{\mu}(\Gamma) = 0 \) and \( \forall k, \varphi \in C^0(X, \mathbb{R}) \) we have
\[
\lim_{n \to \infty} \mu_n(1_{\text{Clos}(X_k)} \cdot \varphi) = \hat{\mu}(1_{\text{Clos}(X_k)} \cdot \varphi).
\]

**Proof.** The first statement follows from Lemma 10. To prove the second statement without loss of generality we assume that \( |\varphi(x)| \leq 1 \). Using again Lemma 10 we get: \( \forall \varepsilon > 0 \exists \delta > 0 \) such that \( \mu_n(B_\delta(\partial X_k)) < \varepsilon \). Choose now a nonnegative function \( h \in C^0(X, [0, 1]) \) such that \( h|_{X_k \setminus B_\delta(\partial X_k)} \equiv 1 \) and \( h|_{X \setminus B_\delta(\partial X_k)} \equiv 0 \). Then \( \lim_{n \to \infty} \mu_n(h\varphi) = \hat{\mu}(h\varphi) \). by the definition of the weak convergence of measures.

Therefore \( \lim_{n \to \infty} |\mu_n(\varphi) - \hat{\mu}(\varphi)| \leq m(B_\varepsilon(\Gamma)) \), which implies the result since the value of \( \varepsilon > 0 \) can be taken arbitrarily small. \( \square \)

**Lemma 12** Let \( \Gamma \cap \text{Per}(T) = \emptyset \) and let \( \{T^{(n)}\} \) be a sequence of approximating weakly periodic maps, whose invariant measures \( \mu_n \xrightarrow{n \to \infty} \hat{\mu} \). Then
\[
\mu_n(\varphi \circ T^{(n)} - \varphi \circ T) \xrightarrow{n \to \infty} 0 \quad \forall \varphi \in C^0(X, \mathbb{R}).
\]

**Proof.** Given \( \varepsilon > 0 \) by Lemma 10 we can choose \( \delta > 0 \) such that \( \mu_n(B_\delta(\Gamma)) < \varepsilon \).\(^4\) From the approximation assumption \( \exists n_0 \) such that \( |\varphi \circ T^{(n)}(x) - \varphi \circ T(x)| < \varepsilon \) \( \forall n > n_0 \) and \( x \notin B_\delta(\Gamma) \). This implies the claim. \( \square \)

Now we are ready to finish the proof of Theorem 1.

First, observe that each map, satisfying the assumption of Theorem 1, can be approximated arbitrary well by weakly periodic ones. Indeed, by since \( T|_{X_k} \) is extendable it can be represented by a finite superposition basic isometries. If there are isometries of type (i) among them, they can be approximated arbitrary well by rational translations (which are periodic). Basic isometries of other types are periodic from the very beginning.

The second step is to check that “good” invariant measures of the weakly periodic approximations, existing by Theorem 2 and Proposition 7, converge to a \( T \)-invariant measure.

**Lemma 13** If \( \exists \gamma \in \Gamma \cap \text{Per}(T) \), then the measure uniformly distributed on the trajectory of \( \gamma \) is \( T \)-invariant. Otherwise, if \( \Gamma \cap \text{Per}(T) = \emptyset \) the measure \( \hat{\mu} \) constructed in Lemmas above is \( T \)-invariant.

**Proof.**
\[
|\hat{\mu}(\varphi - \varphi \circ T)| \leq |\hat{\mu}(\varphi - \varphi \circ T) - \mu_n(\varphi - \varphi \circ T)| + |\mu_n(\varphi - \varphi \circ T^{(n)})| + |\mu_n(\varphi \circ T^{(n)} - \varphi \circ T)|.
\]

\(^4\)Note that the convexity of the elements of the special partition is important at this point. Otherwise one needs to assume that the boundaries of those elements are smooth enough.
The first term in the rhs goes to 0 as \( n \to \infty \) by Lemma 11, the second term vanishes for each \( n \) since the measures \( \mu_n \) are \( T^n \)-invariant, while the third term goes to 0 by Lemma 12. Hence \( \tilde{\mu}(\varphi) = \tilde{\mu}(\varphi \circ T) \), which implies the claim. \( \square \)

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