Exact analytical solution of the influence of an external centrifugal field and the heat transfer on a confined gas between two plates in the unsteady state

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Abstract
This paper aims to study the non-stationary situation of the effectiveness of an external centrifugal force (ECF) and the heat transfer (HT) on the manner of neutral gas (NG). We solve a system of non-linear non-stationary partial differential equations, which represents an enormous task. Our model is examined to follow the manner of the macroscopic properties of the NG that bounded between two parallel horizontal rigid fixed plate plates (HRFPs). The moment method and the traveling wave method are utilized. The draw an analogy among the perturbed distribution function (DF), and the equilibrium DF with time is studied. The thermodynamic predictions are calculated. System internal energy change (IEC) is investigated. We applied the results for laboratory helium NG. We detected that in specific conditions, we could compensate for the decreasing of particles' number adjacent to the lower HRFP because of the centrifugation process, with other particles adjacent to the higher heated HRFP. We did that with the help of the reverse heat current from the heated upper HRFP, which gave us a considerable enhancement and development of gases isotope separation processes. Furthermore, we approved that our model is compatible with the second law of thermodynamics, the rule of Le Chatelier, and the H-Theorem of Boltzmann. Those investigations were done with a non-restricted range of the temperatures ratio factor, the centrifugal Mach number, and the Knudsen numbers. The significance of this study was due to its vast applications in numerous fields, such as in physics, engineering, biomedical, and various commercial and industrial applications.

Keywords
Fluid dynamics, distribution functions, kinetic theory, natural gas industry, partial differential equations, separation processes, exact solutions

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Introduction
Enhancement of isotope separation processes quality of an NG affected by an ECF is one of the critical aims in commercial and industrial applications, especially in the peaceful use of uranium enrichment technology in the generation of electricity and radiation therapy in medicine. In the previous paper, Wahid and El-Malky introduced a new pattern that investigated the effect of...
the ECF and the HT on a gas bounded by parallel two HRFPPs. They solved a steady ordinary differential equation system and applied the results for argon gas. While in the present paper, a non-linear non-stationary partial differential equations system is solved, and the results applied for helium gas. The non-stationary manner of the temperature and concentration are investigated with non-restricted values of $Kn$, $M$, and $h$. For that purpose, we used the moment and traveling wave methods. The thermodynamic properties and various types of IEC are calculated. A full historical introduction of the previous related studies and the details of the stationary model can be found in Wahid and El-Malky and the references therein. HT plus ECF, affected on an NG bounded between two rotating cylinders or its related approximate pattern, such as Pomraning’s pattern, is vital for the NG isotope separation process; see Figure 1. Pomraning’s pattern behaves as if an NG case may approximate an NG case between two parallel cylinders between two parallel HRFPPs. In this new situation, the boundaries go in the invert orientation with equal velocities. The ECF is simulated by influencing a mass force normal to the two parallel HRFPPs and symmetrical to the distance from the uniformity plane.

Many papers were dealing with the fantastic applications of the Bhatnagar–Gross–Krook (BGK)-pattern of the Boltzmann kinetic equation (BKE) and its usedness in various essential physical cases. Some of those critical cases are the HT between two HRFPPs, shock waves, oscillation flow, irreversible non-equilibrium thermodynamics, sound propagation in an NG, plasma fluid, and other many exciting applications.

We aim to discuss the influence of an ECF and an HT on the performance of the particles’ non-equilibrium DF of an NG bounded by two parallel HRFPPs. This examination is done in several $Kn$ in transition, hydrodynamic and rarefied gas regimes, and a non-restricted range of $M$ in durable, moderated, and weak external radial fields. For this approach, an analytical solution of the BGK-pattern of the BKE is presented with two-sided non-equilibrium DFs. The IEC of the system is studied. Finally, the results were employed for laboratory helium NG. The graphics are drawn to follow their manner. The obtained results were carefully discussed.

**Basic equations and analytical solution procedures**

We deal with the NG between two illimitable HRFPPs situated at $y = \pm L/2$. The two HRFPPs are saved at two fixed temperatures; see Figure 2. However, inconsistent temperatures as $T_i$ for the lower HRFPP and $T_{II}$ for the upper HRFPP with $T_i = \eta T_{II}, 0 < \eta \ll 1$, here, $\eta$ represent the ratio of temperature factor. With a frame of reference co-moving with the gas, the gas manner is investigated in the non-stationary situation under the assumption that the heat interchange will be done between particles and two HRFPPs. It is taking the form of total energy replacement. We treat that situation where there is an ECF ($F_c = \Omega^2 y$) columnar to the two HRFPPs takes the direction from the lower HRFPP to the upper one.

For a non-stationary motion, an ECF y-component $F_c$ can be written in the force term of the BGK pattern as

$$\frac{\partial \phi(y,t)}{\partial t} + \xi_y \frac{\partial \phi}{\partial y} + \frac{F_c}{m} \frac{\partial \phi}{\partial \xi_y} = \nu(\phi_0 - \phi)$$  \hspace{1cm} (1)
\[ \varphi_0 = \frac{n}{(2\pi RT)^2} \exp \left[ -\frac{\xi^2}{(2RT)} \right] \cdot \xi^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 \]  

(2)

Here the collision frequency \( \nu = \nu_L + \nu_C \) consisted of two parts \( \nu_L = \frac{nu_T}{4K_B^2} \) that represent collisions because of HT from heated HRFPP to the cold one plus a collision frequency \( \nu_C \) that exemplifies collisions because of the ECF columnar to the lower HRFPP that had the formula expression as in\(^{25,26}\). \( \nu_C = n_L e^{-A(1-y)} \), where \( n_L \) is the collision frequency at the lower HRFPP, and \( A = M^2m_w \) as \( M = \frac{\text{mass}}{\text{moment}} \) represents collisions of the cold NG.

The moment method\(^{27,28}\) is utilized to get the solution of the BKE. Applying Liu-Lees pattern of the DFs in the form\(^{9,23-27}\):

\[ \varphi = \begin{cases} 
\varphi_1 = \frac{n_1}{(2\pi RT_1)^2} \exp \left[ -\frac{\xi^2}{(2RT_1)} \right], & \text{for } \xi_3 > 0 \\
\varphi_2 = \frac{n_2}{(2\pi RT_2)^2} \exp \left[ -\frac{\xi^2}{(2RT_2)} \right], & \text{for } \xi_3 < 0 
\end{cases} \]  

(3)

Moments are obtained by multiplying the BGK type by a velocity function \( \Psi_i(\xi) \) and integrating over velocity. Multiplying equation (1) by \( \Psi_i = \Psi_i(\xi) \), and integrating w.r.t. \( \xi \), considering the two-stream DFs\(^{9,23-27}\) we get:

\[ \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_i \varphi_2 d\xi + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_i \varphi_1 d\xi = 0 \]

\[ \frac{\partial}{\partial y} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_i \varphi_2 d\xi + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_i \varphi_1 d\xi = 0 \]

\[ \nu \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_i(\phi_0) d\xi - \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_1 d\xi \right) \Psi_1 d\xi \right] \]

\[ \frac{\partial}{\partial t} \left[ n_1 T_1 + n_2 T_2 \right] + \frac{\partial}{\partial y} \left[ n_1 T_1^2 + n_2 T_2^2 \right] = 0 \]

(4)

\[ \frac{\partial}{\partial t} \left( n_1 T_1^2 - n_2 T_2^2 \right) + 5 \frac{\partial}{\partial y} \left( n_1 T_1^2 + n_2 T_2^2 \right) = 0 \]

(11)

\[ \frac{1}{K_n} \left( e^{-A(1-y)} + n_L \right) \left( n_1 T_1^2 - n_2 T_2^2 \right) \]

Using the state equation:

\[ P = n \quad T = \text{const}. \]

(13)

With the condition that the velocity \( V \) is situated at the origin.\(^{12-15}\) So, by applying equation (7), we obtain:

\[ V = 0 \Rightarrow \left( n_1 \sqrt{T_1} - n_2 \sqrt{T_2} \right) = 0 \]

(14)

Applying traveling wave method\(^{1,16,19}\) as:
\[ \zeta = m_1y - m_2t \]  

(15)

Such that all functions will be dependent on \( \zeta \). Here \( m_1 \) and \( m_2 \) are transformation constants. Using (15), we get the formulas:

\[
\begin{align*}
\frac{\partial}{\partial t} &= -m_2 \frac{\partial}{\partial \zeta} \left( \frac{\partial}{\partial y} + \frac{\partial}{\partial t} \right), \\
\frac{\partial}{\partial y} &= m_1 \frac{\partial}{\partial \zeta}, \\
\int f(y) \, d\zeta &= \int f(y) \left( \frac{\partial \zeta}{\partial y} \, dy + \frac{\partial \zeta}{\partial t} \, dt \right)
\end{align*}
\]  

(16)

Here \( \varepsilon > 0 \) is an integer number. Substitution equations (15) to (16) into equations (11) to (12), we get:

\[
\begin{align*}
- m_2 \frac{\partial}{\partial \zeta} \left( n_1 T_1 + n_2 T_2 \right) + m_1 \frac{\partial}{\partial \zeta} \left( n_1 T_1^2 - n_2 T_2^2 \right) \\
- (M^2 y) \left( n_1 T_1^2 - n_2 T_2^2 \right) = 0
\end{align*}
\]  

(17)

Integrating equation (18) w.r.t. \( \zeta \), considering equations (13) and (21), to get:

\[
\begin{align*}
- m_2 \left( n_1 T_1^2 - n_2 T_2^2 \right) + \frac{5}{4} m_1 \left[ \left( n_1 T_1^2 + n_2 T_2^2 \right) \right] \\
- \frac{5M^2}{6} \left( \frac{m_1^2}{2} - m_2 y \right) \left( n_1 T_1 + n_2 T_2 \right) \\
= \frac{1}{n_1 T_1 + n_2 T_2} \left( n_1 T_1^2 + n_2 T_2^2 \right)
\end{align*}
\]  

(18)

We will solve equations (13), (14), (17), and (18) to get \( T_1, T_2, n_1 \) and \( n_2 \) as follows:

Using equation (14), we have

\[ n_2 \sqrt{T_2} = n_1 \sqrt{T_1} \]  

(19)

Substituting from equations (13) and (19), into equation (17), taking into consideration equation (9), we obtain:

\[ m_1 \frac{\partial}{\partial \zeta} \left( n_1 T_1^2 - n_2 T_2^2 \right) = m_1 \frac{\partial}{\partial \zeta} (n_2 \sqrt{T_2} (T_1 - T_2)) = 0 \]  

(20)

Integrating equation (20), for \( \zeta \), we collect after factorization

\[
\begin{align*}
\left( n_1 T_1^2 - n_2 T_2^2 \right) &= \left( \frac{n_2 \sqrt{T_2} (\sqrt{T_1} + \sqrt{T_2})}{\sqrt{T_1} - \sqrt{T_2}} \right) = \frac{1}{2} \frac{\partial}{\partial \zeta} = C_2,
\end{align*}
\]  

(21)

as we put

\[ \vartheta_1 = n_2 \sqrt{T_2} (\sqrt{T_1} + \sqrt{T_2}) \, , \, \vartheta_2 = (\sqrt{T_1} - \sqrt{T_2}) \]  

(22)

Here \( C_2 \) is the constant of integration. Here \( \vartheta_1, \vartheta_2 \) are constants. That is because of the uniform pressure as \( P_{yy} = \frac{2m_1}{2} \sqrt{T_2} (\sqrt{T_1} + \sqrt{T_2}) = \frac{\partial}{\partial \zeta} \) is invariable, using equation (22); this reveals that \( \vartheta_2 \) it is unchanging as well.\(^{12-15}\)

Making the best use of equation (19), we can define a new function \( \beta(\zeta) \) as:

\[ \beta(\zeta) = n_2 \sqrt{T_2} = n_1 \sqrt{T_1}. \]  

(23)

From (22) and (23), we can obtain

\[
\begin{align*}
T_1 &= \frac{(\vartheta_1 + \vartheta_2 \beta)^2}{4\beta^2}, \quad T_2 = \frac{(\vartheta_1 - \vartheta_2 \beta)^2}{4\beta^2}, \\
n_1 &= \frac{2\beta}{(\vartheta_1 + \vartheta_2 \beta)}, \quad n_2 = \frac{2\beta}{(\vartheta_1 - \vartheta_2 \beta)}.
\end{align*}
\]  

(24)

Here \( \text{Erfi}[y] \) is the imaginary error function \( \frac{\text{erfi}(y)}{\sqrt{\pi}} \) as \( \text{erfi}[y] = \frac{2}{\sqrt{\pi}} \int_0^y e^{-t^2} \, dt \), and \( \vartheta_3 \) is constant of integration.

Using equations (21) to (23) into equation (25) yields the equation:

\[
\begin{align*}
0 &= \frac{5}{48} m_1 \vartheta_1 \left( -4M^2 y^2 + \frac{3m_2}{\beta^2} + 9\beta_1^2 \right) + \\
&+ 4m_2 \left( -10m_2 M^2 y T_1 \vartheta_1 - \frac{6\beta_1^2}{\beta} - 6\beta_2^2 \right),
\end{align*}
\]  

(26)

We gain three values of \( \beta \); we chose the value compatible with the boundary and initial conditions. The \( \vartheta_1, \vartheta_2, \) and \( \vartheta_3 \) are calculated using the boundary and initial conditions in the non-dimensional form:

\[
\begin{align*}
\frac{1}{2} \frac{\partial}{\partial \zeta} (n_1 (\zeta = -m_1) + n_2 (\zeta = -m_1)) = 1, \\
\frac{1}{2} \frac{\partial}{\partial \zeta} (n_1 (\zeta = -m_1) T_1 (\zeta = -m_1) + n_2 (\zeta = -m_1) T_2 (\zeta = -m_1)) = 1,
\end{align*}
\]  

(27)

\[
\begin{align*}
\frac{1}{2} \frac{\partial}{\partial \zeta} (n_1 (\zeta = -m_1) \sqrt{T_1 (\zeta = -m_1)} - n_2 (\zeta = -m_1) \sqrt{T_2 (\zeta = -m_1)}) = 1, \\
\sqrt{T_2 (\zeta = -m_1)} = 0
\end{align*}
\]  

(28)

\[
\begin{align*}
\frac{1}{2} \frac{\partial}{\partial \zeta} (n_1 (\zeta = -m_1) \sqrt{T_1 (\zeta = -m_1)} - n_2 (\zeta = -m_1) \sqrt{T_2 (\zeta = -m_1)}) = 1, \\
\sqrt{T_2 (\zeta = -m_1)} = 0
\end{align*}
\]  

(29)

\[
\begin{align*}
\frac{1}{2} \frac{\partial}{\partial \zeta} (n_1 (\zeta = -m_1) \sqrt{T_1 (\zeta = -m_1)} - n_2 (\zeta = -m_1) \sqrt{T_2 (\zeta = -m_1)}) = 1, \\
\sqrt{T_2 (\zeta = -m_1)} = 0
\end{align*}
\]  

(30)
Solving the algebraic system of equations (27) to (30), we obtain:

\[ n_1(\xi = - m_1) = \frac{2\sqrt{\eta}}{1 + \sqrt{\eta}}, \quad n_2(\xi = - m_1) = \frac{2}{1 + \sqrt{\eta}}, \]
\[ T_1(\xi = - m_1) = \frac{1 + \sqrt{\eta}}{\eta + \sqrt{\eta}}, \quad T_2(\xi = - m_1) = \frac{\eta(1 + \sqrt{\eta})}{\eta + \sqrt{\eta}} \]

(31)

Substituting equation (31) into equation (22), we get:

\[ \vartheta_1 = 2, \quad \vartheta_2 = \frac{1 - \sqrt{\eta}}{\sqrt{\eta}} \]  

(32)

Substituting equations (31) to (32) into equation (26), we obtain:

\[ \vartheta_3 = \frac{1}{6\lambda^{1/4}} \left( Kn(12\eta^{1/4}(1 + \eta) + 5I(M^2(\sqrt{\eta + \eta}) + 15I(1 + \eta^{1/2})))\eta^{1/4} \right) \]
\[ - 12(-1 + \sqrt{\eta})\nu \frac{12(-1 + \sqrt{\eta})\vartheta_1}{\sqrt{\eta}} \]
\[ \vartheta_3 = \frac{\vartheta_1}{\sqrt{\eta}} \]

(33)

where DawsonF[y] exemplifies the Dawson integral function F(y), as \(30\):

\[ F(y) = e^{-y^2} \int_0^y e^{x^2} dx = 0.5\sqrt{\pi}e^{-y^2}erfi(y). \]

Substituting the obtained variables into the two-sided Maxwellian DF, \( \varphi \) we obtain its value.

These DF \( \varphi \) of the NG particles help us to discuss their manner. The examination of the non-stationary DF performance is a unique advantage related to the BKE.

**Non-stationary Non-Equilibrium Thermodynamic properties**

The non-equilibrium thermodynamics state characteristics of the system are described in this section under the effectiveness of an applied ECF. Entropy per unit mass \( S \) is \(30\):

\[ S = - \int \varphi \ln \varphi d\xi = \frac{\pi^2}{8} \left( n_1 \left( 3 - 4 \ln \left( \frac{\rho_m}{\tau_m^2} \right) \right) + n_2 \left( 3 - 4 \ln \left( \frac{\rho_m}{\tau_m^2} \right) \right) \right) \]  

(34)

The \( y \)-component of entropy flux is:

\[ J_y = - \int \xi_y \varphi \ln \varphi d\xi = \frac{\pi}{2} \left( n_1\sqrt{T_1} \left( 3 - 4 \ln \left( \frac{\rho_m}{\tau_m^2} \right) \right) + n_2\sqrt{T_2} \left( 3 - 4 \ln \left( \frac{\rho_m}{\tau_m^2} \right) \right) \right) \]

(35)

Entropy production as \(30\):

\[ \sigma = \frac{dS}{dt} \text{ or } \sigma = \frac{\partial S}{\partial t} + \nabla \cdot \mathbf{J}. \]

(36)

The thermodynamic force (THF), consonant with the modification of concentration, temperature, and ECF, could be calculated, respectively, as in \(30,31\):

\[ X_1 = X_n = \frac{L}{\eta} \frac{\partial m}{\partial y}, \quad X_2 = \frac{L}{T} \frac{\partial T}{\partial y}, \quad \text{and} \quad X_3 = X_C = F_C \]

(37)

Here \( F_C = M^2\gamma \) is the ECF. Kinetic factors \( L_y \) were evaluated form the equation \(30\):

\[ \sigma_S = \sum_i \sum_j L_{ij}X_{ij} = (X_1 \times X_2 \times X_3) \]

\[ \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \]

(38)

Gibb’s equation for the IEC employed to the system is:

\[ dU = dU_S + dU_V + dU_F, \]
\[ dU_V = -PdV, \]
\[ dU_F = -M^2\gamma dy \]

exemplifies the IEC because of the temperature modifications, the concentration modifications, and the ECF modifications, respectively. Here \( P = nT \), is the pressure, and \( dV = \frac{dn}{\pi} \) is the variation in volume, as

\[ dn = \frac{\partial n}{\partial y} dy + \frac{\partial n}{\partial t} dt \]
\[ dS = \frac{\partial S}{\partial y} dy + \frac{\partial S}{\partial t} dt. \]

**Discussions and results**

With a frame of reference that is co-moving with the gas, NG manner under ECF effectiveness, plus the HT in the non-stationary situation, is examined. The ECF is introduced in the modified BGK-pattern of the BKE, taking into consideration the effect of ECF on the collision frequency \( \nu \) formula, where it had to be dependent on the distances’ variable.

In all evaluations and graphics, we consider the values for helium NG:

\[ c = 2.9979 \times 10^8 \text{m s}^{-1}; R = 8.3145 JK^{-1} \text{mol}^{-1}; \]
\[ \rho_s = mn_s = 7.344 \times 10^{-11} \text{kg m}^{-3} \]
\[ \eta = 0.8; M = 0.5; m_w = 4.002602; \]
\[ Kn = 1; \quad \eta = 0.8; M = 0.5; m_w = 4.002602; \]
\[ Kn = 1; \quad \eta = 0.8; M = 0.5; m_w = 4.002602; \]
We calculate the transformation constants to get $m_2 = -0.95$ and $m_1 = 0.85$. All estimates are applied for helium NG for:

I. A non-restricted range of $Kn$.
II. A non-restricted range of $M$.
III. A non-restricted range of $\eta$ between the two parallel HRFPPs. The helium NG figure captions are as follows:

All Figures satisfy the boundary conditions as:

$$n(-1, 0) = 1 \text{ and } T(-1, 0) = 1.$$ 

Graphics are drawn with fixed values of the dimensionless numbers that we clarify the physical meaning as:

- $Kn = 1$ in the transition regime, therefore the NG is rarefied.
- The $\eta = 0.8$ reveals that the temperature amount of the upper HRFPP is equal to 1.25 times the same amount of the lower HRFPP.
- The $M = 0.5$ reveals that it exemplifies the moderate intensity of the ECF.

Our new mathematical model has no restrictions on the magnitude of $(M, Kn, \text{ and } \eta)$. Therefore, we have a vast number of graphics. Then we try to reduce the discussion as possible as we can. Therefore, we will talk about general notices of the figure’s manners. Therefore, when we analyzed the performance of all-macroscopic and thermodynamics variable, we will mention some critical notes. Figure 3(a) to (f) shed light upon the fact that the decrement in $\phi_2$ and the increment in $\phi_1$ of the DFs $\varphi$ in the non-equilibrium state try to compensate each other. This behavior compatibles with the rule of Le Chatelier’s. Figure 3(b) shows that correspondence between $\varphi_0$ and $\varphi$ at $\eta = 1$. That is because, at equilibrium state, we must have $\varphi = \varphi_0$. That gives a qualitative similarity in the performance of the DF, as mentioned in Naris et al.32

Figure 4(a) and (b) illustrate that the NG particles move far from the lower HRFPP because of the ECF, normal to the lower HRFPP takes the direction from the lower to the upper one. Besides, it had a reverse direction that moves far from the upper HRFPP towards the lower one because of the HT current from the heated HRFPP to the colder one. After many efforts and trials, we found that in specific conditions, for example, for helium NG at $(Kn = 1, M = 0.5, \text{ and } \eta = 0.8)$, we can compensate for the decreasing of

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![Figure 3](image_url)

**Figure 3.** (a) $\Phi[\psi_1, \psi_2]$ and $\phi_0$ (Green) for $y = -0.8, M = 0.5$ and $Kn = 1$ as $t = 0.5$, (b) $\Phi[\psi_1, \psi_2]$ and $\phi_0$ (Green) for $y = -0.8, M = 0.5$ and $Kn = 0.05$ as $t = 0.5$, (c) $\Phi[\psi_1, \psi_2]$ for $y = -0.8, \eta = 0.8$ and $Kn = 1$ as $t = 0.5$, (d) $\Phi[\psi_1, \psi_2]$ for $y = -0.8, \eta = 0.8$ and $Kn = 0.05$ as $t = 0.5$, (e) $\Phi[\psi_1, \psi_2]$ for $y = -0.8, \eta = 0.8$ and $M = 0.5$ as $t = 0.5$, and (f) $\Phi[\psi_1, \psi_2]$ for $y = -0.8, \eta = 0.8$ and $M = 1.5$ as $t = 0.5$. 
particles adjacent to the lower HRFPP because of the centrifugation process with other particles because of the reverse heat current from the heated upper HRFPP. It gave a high enhancement to gases isotope separation processes.

Figure 4(c) clarified the manner of $S$ as it increases with time that behavior coincides with thermodynamics second law and H-Theorem of Boltzmann. This manner reveals that the system in our new pattern goes towards an equilibrium state with time, which confirms with Le Chatelier as the entropy had maximum value at balance.

Figure 4(d) is evident that $\sigma$ is a non-negative decreasing function with time. That behavior coincides
with the laws of thermodynamics. This manner of \( \sigma \) reveals that the system in our new pattern goes towards equilibrium state with time, which compatible with the rule of Le Chatelier, as the entropy production should have a minimum value at equilibrium state.

Figure 4(e) to (h) illuminate the various contributions in the system IEC as \( dU_s \), \( dU_V \), and \( dU_C \). The \( dU \) exemplifies the total IEC, respectively. \( dU_s \) and \( dU_V \) figures had a reverse manner that is in harmony with the style of \( n \) and \( T \) figures themselves, see Figure 4(e), (f), (a), and (b).

The concentration manner behaves in the reverse manner of the temperature performance, as we clarified before. Accordingly, the \( X_T \) will have a reverse direction comparing with \( X_n \), as shown in Figure 4(i) and (j).

We can prove that manner mathematically, as the pressure is constant so,

\[
\begin{align*}
nT &= C \Rightarrow \ln (n) + \ln (T) = \ln (C) \\
\frac{\partial \ln (n)}{\partial y} + \frac{\partial \ln (T)}{\partial y} &= 0 \Rightarrow \frac{1}{n} \frac{\partial n}{\partial y} = -\frac{1}{T} \frac{\partial T}{\partial y}
\end{align*}
\]

But \( X_T = \frac{L}{T} \frac{\partial T}{\partial y} \) & \( X_n = \frac{L}{n} \frac{\partial n}{\partial y} \Rightarrow X_T = -X_n \)

Figure 4(k) reveals that \( X_C \) depends on \( y \) only and does not depend on time. Our calculations illustrate that \( L_{TT} = L_{nn}, \ L_{TT} \approx 0, \ L_{nn} \approx 0, \ L_{CC} \approx 0 \) which are harmonious with kinetic coefficients constraints because of the thermodynamics laws.

The negative sign on some kinetic factors concerning cross-effects reveals that in these situations, there is a heat flux (HF) in the invert orientation to the flux due to the related THF (slope), which is compatible with the results gained by Tij and Santos. They clarified that the force induces heat transport over the force orientation, even in the non-attendance of slopes over that orientation. Besides, it compatible with the results gained by Aoki et al. and Xiao et al. obtain identical results, where they concluded that the HF over the positive orientation of ECF acceleration was enhanced. In another meaning, the ECF will participate the HF transport in the forcing orientation. We conclude that we get a negative sign for \( L_{Ts} = L_{12} \) and \( L_{st} = L_{21} \), which reveals a gas flow produced by the temperature slope. That comes from the lower temperature plate to the upper-temperature plate, known as a Soret effect (or thermal diffusion), that denotes a good qualitative agreement in a similarity in a general behavior as in.12–14,32,36

In Figure 5(a) to (e), and Figure 6(a) to (e), the temperature manner behaves in a reverse direction manner of the concentration performance done. That is because of the uniform pressure. That reveals that \( n \) is comparative to \( 1/T \). Therefore, they act in a reverse scenario,
which is happening for all values of the main factors \(Kn, M, \) and \(h\).

The numerical ratios between the various involvement of the IEC based on the derivatives of the extensive factors are evaluated via the Gibbs rule. Which clarified that they are ordered in absolute value as:

The varied range of \(M\) is from 0.1 to 1, which exemplifies the intensity of the ECF in weak and moderated values. For \(t = 0.5\), we had:

\[
d_{US} : d_{UV} : d_{UC} \approx 1 : 310^{-1} : 210^{-1},
\]

where \(d_{US}\) involvement becomes the mighty change.

In the situation where \(Kn = 1, t = 0.5, h = 0.8,\) and \(M\) varied from 1 to 1.5, which exemplifies the vigorous relativity intensity of the ECF and \(t = 0.5\), we found:

\[
d_{US} : d_{UV} : d_{UC} \approx 1 : 1.210^{-2} : 10,
\]

where \(d_{UC}\) participation became a powerful participator.

In the situation where \(Kn = 1, t = 0.5, M = 0.5,\) and \(\eta\) varied from 0.1 to 0.9, which exemplifies the variable intensity of the temperature variation between the two parallel HRFPPs in several situations, we found:

\[
d_{US} : d_{UV} : d_{UC} \approx 1 : 0.5 : 2.510^{-2},
\]

where the \(d_{US}\) participation becomes the principal participator.

In the situation where \(\eta = 0.8, t = 0.5, M = 0.5,\) \(Kn\) varied from 0.1 to 1, that is, in the transition regime, where NG is rarefied, we found:

\[
d_{US} : d_{UV} : d_{UC} \approx 1 : 410^{-1} : 2.510^{-2},
\]

where the \(d_{US}\) participations became the powerful participator.

In the situation of \(\eta = 0.8, t = 0.5, M = 0.5, Kn\) varied from 1 to 10, that is, in the transition regime, where the NG is highly rarefied.

\[
d_{US} : d_{UV} : d_{UC} \approx 1 : 3.7510^{-1} : 1.2510^{-1},
\]

where \(d_{US}\) participation becomes the main one.

In the situation of \(h = 0.8, t = 0.5, M = 0.5,\) \(Kn\) varied from 0.1 to 1, that is, in the transition regime, where NG is rarefied, we had:

\[
d_{US} : d_{UV} : d_{UC} \approx 1 : 310^{-1} : 210^{-1},
\]

where \(d_{US}\) involvement becomes the mighty change.

In the situation where \(Kn = 1, M = 0.5\) as \(t = 0.5,\) we had:

\[
d_{US} : d_{UV} : d_{UC} \approx 1 : 310^{-1} : 210^{-1},
\]

which is happening for all values of the main factors \(Kn, M, \) and \(\eta\).

The numerical ratios between the various involvement of the IEC based on the derivatives of the extensive factors are evaluated via the Gibbs rule. Which clarified that they are ordered in absolute value as:

The varied range of \(M\) is from 0.1 to 1, which exemplifies the intensity of the ECF in weak and moderated values. For \(t = 0.5\), we had:

\[
d_{US} : d_{UV} : d_{UC} \approx 1 : 310^{-1} : 210^{-1},
\]

where \(d_{US}\) involvement becomes the mighty change.

In the situation where \(Kn = 1, t = 0.5, h = 0.8,\) and \(M\) varied from 1 to 1.5, which exemplifies the vigorous relativity intensity of the ECF and \(t = 0.5\), we found:

\[
d_{US} : d_{UV} : d_{UC} \approx 1 : 1.210^{-2} : 10,
\]

where \(d_{UC}\) participation became a powerful participator.

In the situation where \(Kn = 1, t = 0.5, M = 0.5,\) and \(\eta\) varied from 0.1 to 0.9, which exemplifies the variable intensity of the temperature variation between the two parallel HRFPPs in several situations, we found:

\[
d_{US} : d_{UV} : d_{UC} \approx 1 : 0.5 : 2.510^{-2},
\]

where the \(d_{US}\) participation becomes the principal participator.

In the situation where \(\eta = 0.8, t = 0.5, M = 0.5,\) \(Kn\) varied from 0.1 to 1, that is, in the transition regime, where NG is rarefied, we found:

\[
d_{US} : d_{UV} : d_{UC} \approx 1 : 410^{-1} : 2.510^{-2},
\]

where the \(d_{US}\) participations became the powerful participator.

In the situation of \(\eta = 0.8, t = 0.5, M = 0.5, Kn\) varied from 1 to 10, that is, in the transition regime, where the NG is highly rarefied.

\[
d_{US} : d_{UV} : d_{UC} \approx 1 : 3.7510^{-1} : 1.2510^{-1},
\]

where \(d_{US}\) participation becomes the main one.

We indicated that as \(n\) behaves reversely via \(T\) performance, the \(X_T\) manner would have the reverse direction of the \(X_n\) manner.

We shed light upon the dependence of the ECF on \(y\) and \(M\) only and not on time. It changes linearly with \(y\) and non-linearly with \(M\).

**Conclusion**

ECF influences and HT, on the performance of an NG, bounded between two HRFPPs, is introduced in the non-stationary situation. For this new model, an exact non-stationary analytical calculation of the BGK pattern is shown. With a frame of reference that is co-moving with the gas, a non-linear non-stationary partial differential equations system is investigated. Therefore, we applied the moment and traveling wave factors methods to get the unsteady exact analytical solution. The gas temperature, concentration, and a comparison among the perturbed DFs and the equilibrium DFs are studied. We discussed the thermodynamic properties of the considered new model. Besides, we examined the various participants of the IEC. The results are applied for a helium NG. Graphics are performed to introduce their behavior, which is discussed.

Similarly, many agreements with relevant previous papers are clarified. After many efforts, we detected that in specific conditions, we could compensate for the decrease of particle numbers adjacent to the lower
HRFPP because of the centrifugation process with other particles. We did this with the help of the counter current of HT from the heated upper HRFPP, which obtains a high enhancement to gases isotope separation processes. Our new pattern and all calculations were examined. We proved that it was compatible with thermodynamics second law, the rule of Le Chatelier, and the H-Theorem of Boltzmann. That tested was performed with a non-restricted range of the factor of temperatures ratio, Knudsen numbers, and Mach number of centrifugal force.

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Data availability
The data used to support the findings of this study are included within the article.

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Appendix

Abbreviations

| Abbreviation | Description                                      |
|--------------|--------------------------------------------------|
| BGK          | Bhatnagar–Gross–Krook                            |
| BKE          | Boltzmann kinetic equation                       |
| DF           | distribution Function                             |
| ECF          | external centrifugal force                       |
| HRFPP        | horizontal rigid fixed plane plates               |
| HT           | heat transfer                                     |
| HF           | Heat Flux                                         |
| IEC          | internal energy change                            |
| NG           | neutral gas                                       |
| THF          | thermodynamic force                               |