Index of a singular point of a vector field or of a 1-form on an orbifold *

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Abstract

Indices of singular points of a vector field or of a 1-form on a smooth manifold are closely related with the Euler characteristic through the classical Poincaré–Hopf theorem. Generalized Euler characteristics (additive topological invariants of spaces with some additional structures) are sometimes related with corresponding analogues of indices of singular points. Earlier there was defined a notion of the universal Euler characteristic of an orbifold. It takes values in a ring $R$, as an abelian group freely generated by the generators, corresponding to the isomorphism classes of finite groups. Here we define the universal index of an isolated singular point of a vector field or of a 1-form on an orbifold as an element of the ring $R$. For this index, an analogue of the Poincaré–Hopf theorem holds.

1 Introduction

A classical invariant of a singular point of a vector field or of a 1-form on a smooth manifold is its index. The index is closely related with the Euler characteristic through the Poincaré–Hopf theorem: the sum of indices of the (isolated) singular points of a vector field or of a 1-form on a closed (compact, without boundary) manifold is equal to the Euler characteristic of the manifold. The notion of the index has generalizations to vector fields and 1-forms on singular (e.g., semianalytic) varieties which are related in a certain form with the Euler characteristic.

The Euler characteristic (defined as the alternating sum of the ranks of the cohomology groups with compact support) is the only, up to proportionality,
additive topological invariant of smooth manifolds: see, e. g., [7]. For manifolds or topological spaces with additional structures one has other additive topological invariants — generalized Euler characteristics. For example, for spaces with actions of a finite group $G$, there is defined its \textit{equivariant Euler characteristic} as an element of the Burnside ring $A(G)$ of the group $G$: [13].

The equivariant Euler characteristic is a universal additive topological invariant of manifolds with actions of the group $G$ or of equivariant (in the sense of [13]) $G$-CW-complexes: any additive topological invariant with values in a group $A$ is obtained from the equivariant Euler characteristic via a group homomorphism $A(G) \to A$. For manifolds or topological spaces with actions of the group $G$, there are also defined \textit{the orbifold Euler characteristic} $\chi^{\text{orb}}$ (see, e. g., [1], [9]) and its higher order analogues $\chi^{(k)}$ ([1], [2], [12]). They can be regarded as specifications of the equivariant Euler characteristic, i. e. they are given by group homomorphisms $\chi^{\text{orb}}$ and $\chi^{(k)}$ from the group $A(G)$ to $\mathbb{Z}$. One has notions of equivariant indices of singular points of $G$-invariant vector fields or 1-forms on $G$-manifolds or on singular $G$-spaces as elements of the Burnside ring related with the equivariant Euler characteristic through analogues of the Poincaré–Hopf theorem: [5].

A generalization of the notion of a manifold is the notion of a (real) orbifold (sometimes called a $V$-manifold): see, e. g., [10], [3, Appendix]. Locally an orbifold is the quotient of a vector space by a (linear) action of a finite group (generally speaking its own for each point). In the paper [11], there were defined notions of “the Euler characteristic of an orbifold as an orbifold” (the Euler–Satake characteristic) which is a rational number and of the index of a singular point of a vector field on an orbifold (also as a rational number) so that the sum if the indices of the (isolated) singular points of a vector field on a closed (compact, without boundary) orbifold is equal to its Euler–Satake characteristic. In [7], there was defined \textit{the universal Euler characteristic} (a universal additive topological invariant) of orbifolds. It takes values in a ring $\mathcal{R}$, as an abelian group freely generated by the elements, corresponding to the isomorphism classes of finite groups. The ring $\mathcal{R}$ is the ring of polynomials in the variables corresponding to the isomorphism classes of indecomposable finite groups.

Here we define \textit{the universal index} of an isolated singular point of a vector field or of a 1-form on an orbifold as an element of the ring $\mathcal{R}$. For this index, an analogue of the Poincaré–Hopf theorem holds. One can say that the construction unites the ideas from the papers [5] and [7]. For this index, one has an analogue of the Poincaré–Hopf theorem: the sum of the universal indices of the (isolated) singular points on a closed orbifold is equal to the universal Euler characteristic of the orbifold. The Satake index of a singular point of a vector field on an orbifold is a specification of the universal one: it
is obtained from the latter by the application of the homomorphism $\mathcal{R} \to \mathbb{Q}$ which sends the generator of $\mathcal{R}$ corresponding to the isomorphism class of a finite group $G$ to $\frac{1}{|G|}$, where $|G|$ is the number of elements of $G$.

2 The universal Euler characteristic of orbifolds

Initially a definition of an orbifold (called a $V$-manifold there) was given in [10]. In more convenient terms one can find it, e. g., in [3, Appendix], [2]. One can also define the notion of an orbifold with boundary: see [11], [3, Appendix].

The ring $\mathcal{R}$ of values of the universal Euler characteristic of orbifolds can be defined as the Grothendieck group of finite $G$-sets for (all) finite groups $G$ with an “induction relation”. More precisely this means the following. For a subgroup $H$ of a finite group $G$ one has an induction operation $\text{Ind}^G_H$ which converts $H$-spaces into $G$-spaces. For an $H$-space $X$ the space $\text{Ind}^G_H X$ is defined as the quotient of the Cartesian product $G \times X$ by the equivalence relation: $(g_1, x_1) \sim (g_2, x_2)$ if there exists $h \in H$ such that $g_2 = g_1 h$, $x_2 = h^{-1} x_1$. The action of the group $G$ on $\text{Ind}^G_H X$ is defined in the natural way.

The group $\mathcal{R}$ is generated by the classes $[(X, G)]$ of finite $G$-sets for all finite groups $G$ modulo the relations:

1) if $(X_1, G_1) \cong (X_2, G_2)$, i. e. if there exist a bijective map $F : X_1 \to X_2$ and a group isomorphism $\varphi : G_1 \to G_2$ such that $F(gx) = \varphi(g)F(x)$ for $x \in X_1$, $g \in G_1$, then $[(X_1, G_1)] = [(X_2, G_2)]$;

2) if $Y$ is a $G$-invariant subset of a (finite) $G$-set $X$, then $[(X, G)] = [(Y, G)] + [(X \setminus Y, G)]$;

3) if $H$ is a subgroup of a finite group $G$, then, for a finite $H$-set $X$, one has $[(X, H)] = [(\text{Ind}^G_H X, G)]$.

The multiplication in $\mathcal{R}$ is defined by the Cartesian product:

$$[(X, G)] \cdot [(X', G')] = [(X \times X', G \times G')]$$.

Let us denote by $\mathcal{G}$ the set of the isomorphism classes of finite groups. It is not difficult to see that $\mathcal{R}$ is the free $\mathbb{Z}$-module with the generators $T^G$ corresponding to the isomorphism classes $G \in \mathcal{G}$ of finite groups. The generator $T^G$ is represented by the one-point set with the (unique) action of a representative $G$ of the class $\mathcal{G}$. The Krull–Schmidt theorem implies that $\mathcal{R}$ is the ring of polynomials in the variables $T^G$ corresponding to the isomorphism classes of finite indecomposable groups.
For a point $x$ of an orbifold $Q$, let us denote by $G_x \in G$ the class of the isotropy group of the point $x$ (this group is defined up to isomorphism). For $G \in G$, let $Q(G) = \{ x \in Q : G_x = G \}$. One can see that the orbifold $Q(G)$ is a global quotient (by a free action of a representative $G$ of the class $G$). Moreover, its reduction is a usual ($C^\infty$-) manifold (with the action of the trivial group). The representation of $Q$ as the union of the subsets $Q(G)$ is its stratification. (We allow stratifications with non-connected strata.) This stratification is obviously a Whitney stratification. A homeomorphism of two orbifolds $Q_1$ and $Q_2$ is a homeomorphism of the corresponding topological spaces preserving the described stratification, i.e. mapping $Q_1(G)$ to $Q_2(G)$.

In order for an Euler characteristic of an orbifold (in particular, the universal one) to make sense, one needs that the orbifold possesses certain “finiteness properties”. For example, one can assume that it is the interior of a compact orbifold with boundary. For short such orbifolds are called tame. In the framework of this paper we shall assume that all the orbifolds under consideration are tame.

**Definition:** The universal Euler characteristic of a (tame) orbifold $Q$ is defined by the equation

$$
\chi^\text{un}(Q) = \sum_{G \in G} \chi(Q(G))T^G \in \mathcal{R}.
$$

Additive and multiplicative invariants of (tame) orbifolds are, in particular, the usual Euler characteristic $\chi$, the Euler-Satake characteristic $\chi^\text{ES}$, the orbifold Euler characteristic $\chi^\text{orb}$ and its higher order analogues $\chi^{(k)}$, $k > 1$. The universal Euler characteristic $\chi^\text{un}$ is an additive and multiplicative invariant of orbifolds. Moreover, in [7] it is shown that the universal Euler characteristic is a universal additive topological invariant of orbifolds in the sense that any additive topological invariant $I$ with values in a group $\mathcal{A}$ is represented in the form $I = \varphi \circ \chi^\text{un}$ for a unique group homomorphism $\varphi : \mathcal{R} \to \mathcal{A}$. If $\mathcal{A}$ is a ring and the invariant $I$ is multiplicative, then $\varphi$ is a ring homomorphism. For the listed invariants the corresponding homomorphisms (which we denote by the same symbols as the invariants themselves) are defined by the equations $\chi(T^G) = 1$, $\chi^\text{ES}(T^G) = \frac{1}{|G|}$, $\chi^\text{orb}(T^G) = \chi^\text{orb}(G/G, G)$, $\chi^{(k)}(T^G) = \chi^{(k)}(G/G, G)$ for a representative $G$ of the class $G$. Let us recall that, for a topological $G$-space $X$ its orbifold Euler characteristic and its higher order analogues are defined by the equations (see, e.g., [11], [12])

$$
\chi^\text{orb}(X, G) = \frac{1}{|G|} \sum_{(g_1, g_2) \in G^2 : \text{sgn}(g_2) = g_2g_1} \chi(X^{(g_1, g_2)}),
$$
\[ \chi^{(k)}(X, G) = \frac{1}{|G|} \sum_{(g_1, \ldots, g_{k+1}) \in G^{k+1}} \chi \left( X^{(g_1, \ldots, g_{k+1})} \right), \]

where \((\ldots)\) is the subgroup of \(G\) generated by the corresponding elements, \(X^H\) is the fixed points set of the subgroup \(H \subset G\). (If the group \(G\) is abelian, then \(\chi^{\text{orb}}(G/G, G) = |G|\), \(\chi^{(k)}(G/G, G) = |G|^k\). The usual Euler characteristic, the Euler–Satake characteristic and the orbifold Euler characteristic can be interpreted as higher order characteristics \(\chi^{(k)}\) with \(k = 0, -1\) and 1 respectively.)

For a finite group \(H\), there is defined the group (!) homomorphism \(r_H\) from the Burnside ring \(A(H)\) to the ring \(\mathcal{R}\) mapping the generator \([H\!/G]\) of the ring \(A(H)\) \((G\) is a subgroup of \(H)\) to \(T^{|G|}\) \(([G] \in \mathcal{G}\) is the isomorphism class of the group \(G\)). The homomorphism \(r_H\) is not, in general, injective. If the orbifold \(Q\) is a global quotient: \(Q = M/\!\!/H\) for a smooth \(H\)-manifold \(M\), then \(\chi^{\text{un}}(Q) = r_H \left( \chi^H(M) \right)\), where \(\chi^H(M) \in A(H)\) is the equivariant Euler characteristic of the \(H\)-manifold \(M\) in the sense of [13] (see also [6]).

3 The index of a singular point on an orbifold

An \((n\text{-dimensional})\) orbifold \(Q\) is locally (in a neighbourhood of a point \(p_0\)) isomorphic to the semialgebraic set \(\mathbb{R}^n/\!\!/H\) (in a neighbourhood of its point 0) for a linear action of a finite group \(H\) on the space \(\mathbb{R}^n\). One may assume that \(\mathbb{R}^n/\!\!/H\) is embedded into an ambient space \(\mathbb{R}^N\). The quotient space \(\mathbb{R}^n/\!\!/H\) (as the orbifold \(Q\) itself) has a natural stratification by the subsets \((\mathbb{R}^n/\!\!/H)^{G_0}, G_0 \in \mathcal{G}\). (The subset \((\mathbb{R}^n/\!\!/H)^{G_0}\) can be nonempty only if the class \(G_0\) contains a subgroup of \(H\).)

A vector field on the orbifold \(Q\) is defined locally as a (continuous) vector field on the quotient space \(\mathbb{R}^n/\!\!/H\) coordinated with the described stratification, i.e. the vector at a point \(x \in Q\) is tangent to the stratum \((\mathbb{R}^n/\!\!/H)^{G_0}\) containing the point \(x\). (A vector tangent to a zero-dimensional stratum is, of course, equal to zero.) A 1-form on the orbifold \(Q\) is defined locally as a 1-form on the quotient space \(\mathbb{R}^n/\!\!/H\). A germ of a vector field on \((\mathbb{R}^n/\!\!/H, 0)\) is called radial if, in a sufficiently small punctured neighbourhood of the point 0, it is transversal to the spheres \(S_{\varepsilon}^{N-1}\) centred at the origin in \(\mathbb{R}^N\) and is directed outside these spheres. An example of a radial vector field on \((\mathbb{R}^n/\!\!/H, 0)\) is the direct image of the vector field \(\sum_{i=1}^n x_i \frac{\partial}{\partial x_i}\). A germ of a 1-form on \((\mathbb{R}^n/\!\!/H, 0)\) is called radial if, for any analytic curve \(\gamma : (\mathbb{R}_{\geq 0}, 0) \rightarrow (\mathbb{R}^n/\!\!/H, 0)\), the 1-form is positive on the vector \(\gamma_* \frac{\partial}{\partial t}\) for sufficiently small positive \(t \in (\mathbb{R}_{\geq 0}, 0)\). An example of a radial 1-form on \((\mathbb{R}^n/\!\!/H, 0)\) is the restriction of the form \(\sum_{i=1}^N x_i dx_i\) on \(\mathbb{R}^N\).

For \(a \in \mathbb{R}^n\), let \(\mathbb{R}^n = (\mathbb{R}^n)^{H_a} \oplus (\mathbb{R}^n)^{H_a \perp}\) be a decomposition of the \(H_a\)-module \(\mathbb{R}^n\) into the direct sum of its part fixed respect to the subgroup \(H_a\).
and of its complement. This way a neighbourhood of the corresponding point \( x \) in the quotient space \( \mathbb{R}^n/H \) is isomorphic to the Cartesian product \((\mathbb{R}^n)^{H_a} \times ((\mathbb{R}^n)_{H_a \perp} / H_a)\).

We shall describe the corresponding constructions for vector fields. Changes for 1-forms are more or less obvious.

Assume that a vector field \( U \) on \((\mathbb{R}^n/H,0)\) does not tend to zero in a punctured neighbourhood of the origin. Let \( B_\varepsilon \) be the closed ball of a sufficiently small radius \( \varepsilon \) in the ambient space \( \mathbb{R}^N \) centred at the origin such that the vector field \( U \) is defined on \((\mathbb{R}^n/H) \cap B_\varepsilon \) and does not tend to zero at the points of \(((\mathbb{R}^n/H) \cap B_\varepsilon) \setminus \{0\}\). It is not difficult to show that there exists a vector field \( \tilde{U} \) on \((\mathbb{R}^n/H) \cap B_\varepsilon \) such that:

1) \( \tilde{U} \) coincides with \( U \) on a neighbourhood of \((\mathbb{R}^n/H) \cap S_\varepsilon \) in \((\mathbb{R}^n/H) \cap B_\varepsilon \), where \( S_\varepsilon = \partial B_\varepsilon \) is the boundary of the ball \( B_\varepsilon \);

2) in a neighbourhood of each singular point \( a \) of the vector field \( \tilde{U} \) (equal to the Cartesian product of a neighbourhood of the point \( a \) in \((\mathbb{R}^n)^{H_a} \) and of a neighbourhood of the origin in \((\mathbb{R}^n)_{H_a \perp} / H_a)\) the latter is equal to the direct sum \( U'_a \oplus U''_a \) of a vector field \( U'_a \) on \(((\mathbb{R}^n)^{H_a},0)\) with an isolated singular point \( a \) and of a radial vector field \( U''_a \) on \(((\mathbb{R}^n)_{H_a \perp} / H_a,0)\).

**Definition:** The universal index of an (isolated) singular point \( 0 \) of the vector field \( U \) on \((\mathbb{R}^n/H,0)\) is defined as

\[
\text{ind}_{\mathbb{R}^n/H,0}^\text{un} U = \sum_{a \in \text{Sing} \tilde{U}} \left( \text{ind}_{(\mathbb{R}^n)^{H_a},a} U'_a \right) \cdot T^a \in \mathcal{R}
\]

for the vector field \( \tilde{U} \) described above, where \( \text{ind}_{(\mathbb{R}^n)^{H_a},a} U'_a \) is the usual index of the singular point of the vector field \( U'_a \) on the manifold \((\mathbb{R}^n)^{H_a} \). (If the space \((\mathbb{R}^n)^{H_a}\) is zero-dimensional (this may take place only in the case \( a = 0, H_a = H \)), then the index \( \text{ind}_{(\mathbb{R}^n)^{H_a},a} U'_a \) is assumed to be equal to one.)

**Proposition 1** The universal index \( \text{ind}_{\mathbb{R}^n/H,0}^\text{un} U \) is well defined, i.e. the right hand side of Equation (2) does not depend on the choice of a vector field \( \tilde{U} \).

*Proof.* For the proof of the statement, let us express the universal index of the vector field \( U \) on \((\mathbb{R}^n/H,0)\) in terms of the radial indices of the restrictions of the vector field \( U \) to certain semialgebraic subsets in \( \mathbb{R}^n/H \). The fact that the radial index is well defined is proved in [4].

On the set \( \mathcal{G} \), let us consider the partial order \( \prec \), defined by: \( \mathcal{G}_1 \prec \mathcal{G}_2 \) if (and only if) a representative of the class \( \mathcal{G}_1 \) is isomorphic to a subgroup of a representative of the class \( \mathcal{G}_2 \). Let \( \zeta(\mathcal{G}_1,\mathcal{G}_2) \) be the zeta function of the
partially ordered set $\mathcal{G}$: $\zeta(\mathcal{G}_1, \mathcal{G}_2) = 1$ if $\mathcal{G}_1 \prec \mathcal{G}_2$ and $\zeta(\mathcal{G}_1, \mathcal{G}_2) = 0$ in the opposite case. Let $\mu(\mathcal{G}_1, \mathcal{G}_2)$ be the Möbius function of the partially ordered set $\mathcal{G}$, i.e. the function Möbius inverse to the zeta function:

$$\sum_{\mathcal{H} \prec \mathcal{G}} \zeta(\mathcal{H}, \mathcal{K}) \mu(\mathcal{K}, \mathcal{G}) = \delta_{\mathcal{H} = \mathcal{G}} ,$$

where $\delta_{\bullet}$ is the Kronecker delta (see, e.g., [8, Section 2.2]).

For $\mathcal{G} \in \mathcal{G}$, let us denote by $(\mathbb{R}^n/H)^{\mathcal{G}}$ the union of all the strata $(\mathbb{R}^n/H)^{\mathcal{G}'}$ with $\mathcal{G}' \prec \mathcal{G}$. The semialgebraic set $(\mathbb{R}^n/H)^{\mathcal{G}}$ is closed. (It is not, in general, an orbifold. It is the union of several (intersecting) orbifolds.) It is not difficult to see that the restriction of the vector field $\tilde{U}$ to $(\mathbb{R}^n/H)^{\mathcal{G}}$ is appropriate for the computation of the radial index of the restriction $U|_{(\mathbb{R}^n/H)^{\mathcal{G}}}$ of the vector field $U$ to $(\mathbb{R}^n/H)^{\mathcal{G}}$ according to the definition from [4]. Moreover,

$$\text{ind}^{\text{rad}}_{(\mathbb{R}^n/H)^{\mathcal{G}}}(U|_{(\mathbb{R}^n/H)^{\mathcal{G}}}) = \sum_{\mathcal{G}' \prec \mathcal{G}} \text{ind}_{\mathbb{R}^n/H} U'_{\mathcal{G}'} = \sum_{\mathcal{G}' \prec \mathcal{G}} \sum_{a \in (\mathbb{R}^n/H)^{\mathcal{G}'}} \mu(\mathcal{G}', \mathcal{G}) \text{ind}^{\text{rad}}_{(\mathbb{R}^n/H)^{\mathcal{G}'}} U'_{\mathcal{G}'} .$$

By the Möbius inversion theorem ([8, Theorem 2.2.1]) we have

$$\sum_{a \in (\mathbb{R}^n/H)^{\mathcal{G}}} \text{ind}_{\mathbb{R}^n/H} U'_{\mathcal{G}} = \sum_{\mathcal{G}'} \mu(\mathcal{G}', \mathcal{G}) \text{ind}^{\text{rad}}_{(\mathbb{R}^n/H)^{\mathcal{G}'}} U_{(\mathbb{R}^n/H)^{\mathcal{G}'}} .$$

Therefore

$$\text{ind}^{\text{un}}_{\mathbb{R}^n/H, 0} U = \sum_{\mathcal{G}} \left( \sum_{\mathcal{G}' \prec \mathcal{G}} \mu(\mathcal{G}', \mathcal{G}) \text{ind}^{\text{rad}}_{(\mathbb{R}^n/H)^{\mathcal{G}'}} U_{(\mathbb{R}^n/H)^{\mathcal{G}'}} \right) \cdot T_{\mathcal{G}} .$$

For the universal index one has an analogue of the Poincaré–Hopf theorem. Let $Q$ be a closed (compact, without boundary) orbifold and let $U$ be a vector field on $Q$ with isolated singular points (zeroes).

**Proposition 2** One has the equality

$$\sum_{a \in \text{Sing} U} \text{ind}^{\text{un}}_{Q, a} U = \chi^{\text{un}}(Q).$$

**Proof.** The orbifold $Q$ is the disjoint union $\bigcup_{\mathcal{G} \in \mathcal{G}} Q^{(\mathcal{G})}$. Let $Q^{\mathcal{G}} := \bigcup_{\mathcal{G}' \prec \mathcal{G}} Q^{(\mathcal{G}')}$. We have

$$\chi(Q^{\mathcal{G}}) = \sum_{\mathcal{G}' \prec \mathcal{G}} \chi(Q^{(\mathcal{G}')}),$$

where $\chi$ is the Euler characteristic.
which implies

\[ \chi(Q(\mathcal{G})) = \sum_{\mathcal{G}'} \mu(\mathcal{G}', \mathcal{G}) \chi(Q^{\mathcal{G}'}) . \]

Therefore

\[ \chi^\text{un}(Q) = \sum_{\mathcal{G} \in G} \left( \sum_{\mathcal{G}'} \mu(\mathcal{G}', \mathcal{G}) \chi(Q^{\mathcal{G}'}) \right)^{} T^\mathcal{G}. \tag{6} \]

According to the Poincaré–Hopf theorem for the (singular, locally algebraic) variety \(Q^{\mathcal{G}'}\) (see [5]) we have

\[ \chi(Q^{\mathcal{G}'}) = \sum_{a \in Q^{\mathcal{G}'}} \text{ind}^{\text{rad}}_{Q^{\mathcal{G}'}, a} U_{|Q^{\mathcal{G}'}}. \]

Therefore Equations (4) and (6) imply (5).

Remarks. 1. The Poincaré–Hopf theorem for the Satake index from [11] is the reduction of Proposition 2 by the action of the homomorphism \(\chi^{ES} : \mathcal{R} \to \mathbb{Q}\).

2. If \(Q\) is a compact orbifold with the boundary \(\partial Q\), then analogues of the Poincaré–Hopf theorem are the following statements: if a vector field \(U\) on \(Q\) with isolated singular points does not vanish on the boundary \(\partial Q\) and is directed outside of \(Q\) on it, then the sum of the universal indices of the singular points of \(U\) is equal to the universal Euler characteristic of \(Q\); if the vector field \(U\) on the boundary is directed inside the orbifold \(Q\), then the sum of the universal indices of the singular points of \(U\) is equal to the universal Euler characteristic of the interior of \(Q\). For a 1-form \(\omega\) on \(Q\), an analogue of the property of a vector field to be directed outside of (respectively inside) \(Q\) is the property that, at singular points of the restriction of the 1-form \(\omega\) to the boundary \(\partial Q\), it is positive (respectively negative) on the vectors directed outside of \(Q\).

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