Vacuum polarization in Schwarzschild space-time
by anomaly induced effective actions

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Abstract

The characteristic features of $\langle T_{\mu\nu} \rangle$ in the Boulware, Unruh and Hartle-Hawking states for a conformal massless scalar field propagating in the Schwarzschild space-time are obtained by means of effective actions deduced by the trace anomaly. The actions are made local by the introduction of auxiliary fields and boundary conditions are carefully imposed on them in order to select the different quantum states.

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1 Introduction

One of the most interesting aspects of quantum matter fields propagating in curved space-time [1] is the existence of a trace anomaly discovered by Capper and Duff [2] in 1974 and its deep connection with Hawking’s black hole evaporation [3], [4], [5]. At the classical level if the action describing the matter fields is conformally invariant the trace \( T \) of the corresponding stress energy tensor vanishes. However, when the fields are quantized as a result of the renormalization procedure the expectation value of the trace \( \langle T \rangle \) differs from zero when space-time dimension is even. We then have an “anomaly”. This anomaly depends on geometrical quantities only: it is expressed in terms of the Riemann tensor of the space-time and its contractions. Furthermore, being an ultraviolet effect, the anomaly is general in the sense that its value does not depend on the quantum state in which the expectation value is taken. In four dimensions (4D) we have

\[
\langle T^\alpha \rangle = -\frac{1}{(4\pi)^2}(aC^2 + bE + c\Box R) \quad (1.1)
\]

where \( C^2 \equiv C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} \) is the square of the Weyl tensor and \( E \) is an integrand of the Gauss-Bonnet topological term

\[
E \equiv R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2 \quad (1.2)
\]

and \( a, b, c \) are known numerical coefficients that depend on the spin of the matter fields considered [1]. In 2D we have the much simpler form

\[
\langle T^\alpha \rangle = \langle T \rangle = gR \quad (1.3)
\]

where as before \( g \) depends on the matter species.

The knowledge of the trace anomaly allows some information on the complete quantum stress tensor expectation values \( \langle T_{\mu\nu} \rangle \) to be extracted.

In 2D the conservation equations \( \nabla^a \langle T_a^b \rangle = 0 \) and the trace anomaly, eq. (1.3), determine almost completely the stress tensor. Parametrizing the 2D spacetime metric \( g_{ab}(x) \) as

\[
ds^2 = -e^{2\rho}dx^+dx^-
\]

in the conformal gauge \( \{x^+, x^-\} \) one finds [3]

\[
\langle x^\pm |T_{\pm\pm} |x^\pm \rangle = -\frac{1}{12\pi}(\partial_+ \rho \partial_- \rho - \partial_\pm^2 \rho) + t_\pm(x^\pm), \\
\langle x^\pm |T_{\pm-} |x^\pm \rangle = -\frac{1}{12\pi}\partial_+ \partial_- \rho \quad (1.5)
\]

where \( t_\pm(x^\pm) \) are arbitrary functions of their arguments which depend on the choice of quantum state in which the expectation values are taken. They represent conserved
radiation which, being traceless, cannot be determined by the trace anomaly alone. These functions are fixed by boundary conditions. The above expressions for \(\langle T_{ab} \rangle\) can also be obtained by rigorous canonical quantization and point splitting regularization of the divergences [7].

In 4D the situation is much more fuzzy since the four conservation equations and the trace anomaly eq. (1.1) are clearly insufficient to determine the ten components of \(\langle T_{\mu\nu} \rangle\). Things improve if the background space-time on which the quantum matter fields propagate enjoys some symmetries which reduce the number of independent components of \(\langle T_{\mu\nu} \rangle\) [5]. Unfortunately in 4D it appears impossible (at least for the moment) to find a nice formula analogous to eqs. (1.5) giving \(\langle T_{\mu\nu} \rangle\) as a function of a general metric \(g_{\alpha\beta}\). Such an expression would allow us the study of the evolution of the background geometry driven by the quantum fluctuation of the matter fields propagating on it. This is the so called backreaction, governed by the semiclassical Einstein equations

\[
G_{\mu\nu}(g_{\alpha\beta}) = 8\pi \langle T_{\mu\nu}(g_{\alpha\beta}) \rangle ,
\]

where \(G_{\mu\nu}\) is the Einstein’s tensor.

In principle \(\langle T_{\mu\nu}(g_{\alpha\beta}) \rangle\) can be derived from an effective action \(S_{\text{eff}}(g_{\alpha\beta})\) describing the quantum matter fields propagating on \(g_{\alpha\beta}\). Although \(S_{\text{eff}}(g_{\alpha\beta})\) is unknown, the trace anomaly \(\langle T \rangle\) can be used to reconstruct at least a part of it, i.e. the so called “anomaly induced effective action”. \(S_{\text{an}}^{\text{eff}}(g_{\alpha\beta})\) is then defined as following

\[
\frac{2}{\sqrt{-g}} g^{\mu\nu} \frac{\delta S_{\text{an}}^{\text{eff}}}{\delta g_{\mu\nu}} = \langle T \rangle .
\]

By functional integration of this equation one obtains the explicit form of \(S_{\text{an}}^{\text{eff}}\).

The complete effective action \(S_{\text{eff}}(g_{\alpha\beta})\) can then be written as

\[
S_{\text{eff}}(g_{\alpha\beta}) = S_{\text{an}}^{\text{eff}}(g_{\alpha\beta}) + S^W(g_{\alpha\beta}) ,
\]

where \(S^W\) is a conformal invariant functional which can be regarded as an integration constant of eq. (1.7). As expected, the exact form of \(S^W\) is not known.

Taking the functional derivation of \(S_{\text{an}}^{\text{eff}}(g_{\alpha\beta})\) with respect to \(g_{\alpha\beta}\) we define the “anomaly induced stress tensor” \(S_{\mu\nu}\) as follows

\[
\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{an}}^{\text{eff}}}{\delta g_{\mu\nu}} = \langle S_{\mu\nu} \rangle
\]

where of course \(\langle S \rangle = \langle T \rangle\). It is tempting, given our ignorance on \(T_{\mu\nu}\), to use \(\langle S_{\mu\nu} \rangle\) in the semiclassical Einstein equations (1.6) to get at least a feeling on the backreaction problem.

The basic question is: how close \(\langle S_{\mu\nu} \rangle\) is to \(\langle T_{\mu\nu} \rangle\)? In other words, can we ignore the Weyl invariant part \(S^W\) to get correctly at least the qualitative features of the backreaction? It is hard to find answers to the above questions since we do not know
in general $\langle T_{\mu\nu}(g_{\alpha\beta}) \rangle$. However, for some particular background, say $\bar{g}_{\alpha\beta}$, $\langle T_{\mu\nu}(\bar{g}_{\alpha\beta}) \rangle$ is known and hence a check of the above conjecture can be given by direct comparison of $\langle T_{\mu\nu}(\bar{g}_{\alpha\beta}) \rangle$ and the corresponding $\langle S_{\mu\nu}(\bar{g}_{\alpha\beta}) \rangle$. The background $g_{\alpha\beta}$ we have in mind is the Schwarzschild black hole geometry

$$ds^2 = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$ (1.10)

or

$$ds^2 = -(1 - 2M/r)du dv + r^2d\Omega^2,$$ (1.11)

where

$$u = t - r - 2M \ln |r/2M - 1|,$$

$$v = t + r + 2M \ln |r/2M - 1|$$ (1.12)

and $d\Omega^2$ is the metric for the unit two-sphere. This, because of its physical implication, most notably Hawking black hole evaporation, is one of the most studied space-times in quantum field theory in curved space.

Detailed analytical and numerical investigations, by means of modes sum and point-splitting regularization or by axiomatic principles, of $\langle T_{\mu\nu} \rangle$ in this background have been performed [8]. We are rather confident on these results on which, we stress, our understanding of all quantum black hole physics is based.

In the Schwarzschild space-time three different quantum states (“vacua”) are defined by expanding field operators in suitable basis:

i) The Boulware vacuum $|B\rangle$ [9]. This is obtained by choosing in and out modes to be positive frequency with respect to the Killing vector $\partial_t$ of the metric (1.10). This state most closely reproduces the familiar notion of Minkowski vacuum $|M\rangle$ asymptotically. One finds, as $r \to \infty$, $\langle B | T_{\mu\nu} | B \rangle \to 0$ sufficiently rapidly, i.e. $O(r^{-6})$. Unfortunately, the behaviour on the horizon $r = 2M$ is rather pathological, being $\langle B | T_{\mu\nu} | B \rangle$ divergent there in a free falling frame. This state is appropriate for the description of vacuum polarization around a static star whose radius is necessary bigger than the horizon.

ii) The Unruh vacuum $|U\rangle$ [10]. in modes are chosen as before to be positive frequency with respect to $\partial_t$. This ensures that asymptotically in the past $|U\rangle \sim |M\rangle$. On the other hand, out modes are taken to be positive frequency with respect to Kruskal’s $U = -4Me^{-u/4M}$, the affine parameter along the past horizon. This mimics the late time behaviour of modes coming out of a collapsing star as its surface approaches the horizon. By this choice $\langle U | T_{\mu\nu} | U \rangle$ is regular on the future (but not on the past) event horizon and most remarkably asymptotically in the future $\langle U | T_{\mu\nu} | U \rangle$ has the form of a flux of radiation at the Hawking temperature $T_H = \frac{1}{8\pi M}$. This state is the most appropriate to discuss evaporation of black holes formed by
gravitational collapse of matter. In that case the divergence on the past horizon is spurious.

iii) The Israel-Hartle-Hawking state $|H\rangle$. This state is obtained by choosing in modes to be positive frequency with respect to Kruskal’s $V = 4Me^{v/4M}$, the affine parameter on the future horizon, whereas outgoing modes are positive frequency with respect to $U$. By this choice asymptotically both in the future and the past $|H\rangle \neq |M\rangle$.

$\langle H|T_{\mu\nu}|H\rangle$ for $r \to \infty$ describes in fact a thermal bath of radiation at the Hawking temperature $T_H$. By construction this state is regular on both the future and the past horizon. $|H\rangle$ is used to describe a black hole in thermal equilibrium with a surrounding bath.

We summarize here, for later comparison, the analytic expression of the stress tensor for a conformally invariant scalar field in the three vacua defined above. In 2D Schwarzschild spacetime (neglect the angular part in eq. (1.11)) we have [12]

\begin{align*}
\langle B|T_{tt}|B\rangle &= \frac{1}{12\pi} \left[ -\frac{2M}{r^3} + \frac{7M^2}{2r^4} \right], \\
\langle B|T_{rr}|B\rangle &= -\frac{1}{48\pi} \frac{2M^2}{r^4} (1 - 2M/r)^{-2}, \\
\langle B|T_{rt}|B\rangle &= 0
\end{align*}

and

\begin{align*}
\langle U|T_{tt}|U\rangle &= \frac{1}{12\pi} \left[ -\frac{2M}{r^3} + \frac{7M^2}{2r^4} \right] + (768\pi M^2)^{-1}, \\
\langle U|T_{rr}|U\rangle &= -\frac{1}{48\pi} (1 - 2M/r)^{-2} \frac{2M^2}{r^4} + (768\pi M^2)^{-1}, \\
\langle U|T_{rt}|U\rangle &= -(1 - 2M/r)^{-1}(768\pi M^2)^{-1}
\end{align*}

and finally

\begin{align*}
\langle H|T_{tt}|H\rangle &= \frac{1}{12\pi} \left[ -\frac{2M}{r^3} + \frac{7M^2}{2r^4} \right] + (384\pi M^2)^{-1}, \\
\langle H|T_{rr}|H\rangle &= -\frac{1}{48\pi} \frac{2M^2}{r^4} (1 - 2M/r)^{-2} + (384\pi M^2)^{-1}, \\
\langle H|T_{rt}|H\rangle &= 0
\end{align*}

Note that $\langle B|T_{\alpha\dot{\alpha}}|B\rangle = \langle U|T_{\alpha\dot{\alpha}}|U\rangle = \langle H|T_{\alpha\dot{\alpha}}|H\rangle$ as a consequence of the state independence of the trace anomaly.

In 4D only approximate expressions of $\langle T_{\mu\nu}\rangle$ can be given [8]. Here we just report the leading behaviour at infinity and on the horizon. For the Boulware vacuum $|B\rangle$ in Schwarzschild coordinates $(t, r, \theta, \varphi)$

$$\langle B|T_{\mu\nu}|B\rangle \to O(r^{-6}), \quad r \to \infty,$$
\begin{align}
\langle B | T_{\nu}^\mu | B \rangle & \sim - \frac{1}{30} \frac{2^{12} \pi^2 M^4 f^2}{2^{12} \pi^2 M^4 f^2} \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1/3 & 0 & 0 \\
0 & 0 & 1/3 & 0 \\
0 & 0 & 0 & 1/3
\end{pmatrix}, \ r \to 2M, \\
\end{align}

where \( f \equiv 1 - 2M/r \).

For the Unruh vacuum \( |U\rangle \)

\begin{align}
\langle U | T_{\nu}^\mu | U \rangle & \to \frac{L}{4\pi r^2} \begin{pmatrix}
-1 & -1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \ r \to \infty, \\
\end{align}

describing an outgoing flux, and

\begin{align}
\langle U | T_{a}^{\ b} | U \rangle & \sim \frac{L}{4\pi (2M)^2} \begin{pmatrix}
1/f & -1 \\
1/f & -1/f
\end{pmatrix}, \ r \to 2M, \\
\end{align}

\( a, b = r, t \), which describes a negative energy flux of radiation going down the hole compensating the escaping flux at infinity. \( L \) is the luminosity. In geometric optics approximation \( L = \frac{2.197 \times 10^{-4}}{\pi M^2} \) [13]. Numerical estimates set \( L = \frac{2.337 \times 10^{-4}}{\pi M^2} \) [14].

Finally for the \( |H\rangle \) state

\begin{align}
\langle H | T_{\nu}^\mu | H \rangle & \to \frac{1}{30} \frac{2^{12} \pi^2 M^4}{2^{12} \pi^2 M^4} \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1/3 & 0 & 0 \\
0 & 0 & 1/3 & 0 \\
0 & 0 & 0 & 1/3
\end{pmatrix}, \ r \to \infty, \\
\end{align}

describing a thermal bath at the temperature \( T_H \). Furthermore \( \langle H | T_{\nu}^\mu | H \rangle \) is regular on the horizons.

In this paper, starting from the trace anomaly, we shall consider \( S^{\text{an}}_{\text{eff}} \) for a massless scalar field propagating in the Schwarzschild space-time. From this, by functional derivation, we obtain the corresponding stress tensor \( \langle S_{\mu\nu} \rangle \) which should then be compared with the expected results of \( \langle T_{\mu\nu} \rangle \) given by eqs. (1.13)-(1.20).

2 Polyakov’s action

As a warm up exercise we start with the 2D Schwarzschild space-time and a conformal massless scalar field propagating on it. The purpose of it is purely pedagogical. The results will not be new, although their nonstandard derivation is new and illustrates in a simple way our strategy, the computational techniques and the basic physical ingredients we shall use to deal the most interesting 4D case (see next section).

The classical action for the massless field \( f \) reads

\begin{align}
S = \int d^2 x \sqrt{-g^{(2)}} \partial_\mu f \partial^\mu f, \\
\end{align}

(2.1)
where \( g^{(2)} \) is the determinant of the 2D Schwarzschild metric (see eqs. (1.10) and (1.11) with angular part omitted). By canonical quantization and renormalization the corresponding \( \langle T_{ab} \rangle \) can be computed in the three vacua and the results are given in eqs. (1.13)-(1.15).

Our procedure starts with the expression of the trace anomaly, which in arbitrary 2D spacetime reads [4]

\[
\langle T \rangle = (24\pi)^{-1} R .
\]  

(2.2)

The anomaly induced effective action is obtained by functional integration of this anomaly (see eq. (1.7)). It is the well known Polyakov action [15]

\[
S_{\text{eff}}^{\text{an}} = S_{P} = -\frac{1}{96\pi} \int d^2x \sqrt{-g} R \frac{1}{\Box} R ,
\]  

(2.3)

where \( \Box \) is the d’Alembertian.

This action is nonlocal, but it can be made local either by choosing a conformal gauge or by introducing an auxiliary field \( \psi \) (and keeping the gauge arbitrary). We can write

\[
S_{P} = -\frac{1}{96\pi} \int d^2x \sqrt{-g} (-\psi \Box \psi + 2\psi R) .
\]  

(2.4)

The equation of motion for \( \psi \) is

\[
\Box \psi = R .
\]  

(2.5)

Once this is substituted back in eq. (2.4) we reobtain the nonlocal form of eq. (2.3).

The anomaly induced stress tensor \( \langle S_{ab} \rangle \) is defined as

\[
\frac{2}{\sqrt{-g}} \frac{\delta S_{P}}{\delta g^{ab}} \equiv \langle S_{ab} \rangle = -\frac{1}{48\pi} \left[ 2 \nabla_{a} \nabla_{b} \psi - \nabla_{a} \psi \nabla_{b} \psi - g_{ab} \left( 2R - \frac{1}{2}(\nabla \psi)^2 \right) \right] .
\]  

(2.6)

Our strategy is to solve eq. (2.3) for the auxiliary field \( \psi \) by assuming the 2D Schwarzschild spacetime as background, substitute the result in eqs. (2.6) to find \( \langle S_{ab} \rangle \) and compare with eqs. (1.13)-(1.15).

Before we start the program we have to solve a basic problem, how to implement state dependence in our \( \langle S_{ab} \rangle \). Being \( \langle T \rangle \) state independent, \( S_{\text{eff}}^{\text{an}} \) makes no apparent reference to a particular quantum state.

Note however that eq. (2.3) determines the solution for \( \psi \) modulo arbitrary solution of the homogeneous equation \( \Box \psi = 0 \). It is through the choice of the homogeneous solution that the state dependence will be encoded in our framework. The homogeneous solution will be chosen by imposing appropriate boundary conditions on \( \psi \) that reflect the physical features of the states \( |B\rangle, |U\rangle \) and \( |H\rangle \).

Since by eq. (2.3) we have

\[
\psi = \frac{1}{\Box} R ,
\]  

(2.7)

by this procedure we impose boundary conditions on the operator \( 1/\Box \) characterizing the particular quantum state we are working with.

For the 2D Schwarzschild spacetime eq. (2.5) becomes

\[
\partial_{a}[\partial^{a} \psi] = \frac{4M}{r^3} ,
\]  

(2.8)
whose general solution we can write as

$$\psi = at - \ln(1 - 2M/r) + A[r + 2M \ln(r - 2M)] + B$$

(2.9)

with

$$\frac{\partial \psi}{\partial r} = (1 - 2M/r)^{-1}(-\frac{2M}{r^2} + A) ,$$

(2.10)

where $a$, $A$ and $B$ are arbitrary constants which once fixed determine the solution of the homogeneous equation.

The choice of a linear dependence in $t$ is simple. Given the structure of $\langle S_{ab} \rangle$ in terms of $\psi$ (see eq. (2.6)), it is clear that any $t$ dependence different from the linear would imply $\partial_t \langle S_{ab} \rangle \neq 0$, which contradicts the exact result $\partial_t \langle T_{ab} \rangle = 0$. The presence of such a linear term allows, as we shall see, that $\langle S_{rt} \rangle \neq 0$ which is indeed needed in the Unruh state.

Inserting the solution for $\psi$ in eq. (2.6) we obtain

$$\langle S_{tt} \rangle = \frac{1}{12\pi} \left[ -\frac{2M}{r^3} + \frac{7M^2}{2r^4} \right] + (48\pi)^{-1}\frac{A^2 + a^2}{2},$$

$$\langle S_{rr} \rangle = -\frac{1}{48\pi}(1 - 2M/r)^{-2} \left(\frac{2M^2}{r^4} - \frac{A^2 + a^2}{2}\right),$$

$$\langle S_{rt} \rangle = (48\pi)^{-1}(1 - 2M/r)^{-1}Aa .$$

(2.11)

Now we have to fix the arbitrary constants according to our choice of quantum state.

i) Boulware vacuum $|B\rangle$.

This state by construction reduces as $r \to \infty$ to Minkowski vacuum and there $\langle B|T_{ab}|B\rangle$ should vanish. It is clear from the above equations that in order to fulfill the asymptotic requirement we have to impose $a = A = 0$ and we find

$$\langle B|S_{tt}|B\rangle = \frac{1}{12\pi} \left[ -\frac{2M}{r^3} + \frac{7M^2}{2r^4} \right],$$

$$\langle B|S_{rr}|B\rangle = -\frac{1}{48\pi}\frac{2M^2}{r^4}(1 - 2M/r)^{-2},$$

$$\langle B|S_{rt}|B\rangle = 0 .$$

(2.12)

We see the exact agreement of our $\langle B|S_{ab}|B\rangle$ with eqs. (I.13), namely

$$\langle B|S_{ab}|B\rangle = \langle B|T_{ab}|B\rangle .$$

(2.13)

With the above choice of integration constants the auxiliary field $\psi$ becomes

$$\psi = -\ln(1 - 2M/r) + B$$

(2.14)

which vanishes for $r \to \infty$ if we further set $B = 0$. Note the singularity of $\psi$ for $r = 2M$ which causes the divergence of $\langle B|S_{ab}|B\rangle$ on the horizon.
ii) Unruh state $|U\rangle$.
By assumption there is no incoming flux (i.e. $\langle U|S_{vv}|U\rangle = 0$ as $r \to \infty$) and $\langle U|S_{ab}|U\rangle$ has to be regular on the future horizon (i.e. $\langle U|S_{uu}|U\rangle \to 0$ at least as $(r - 2M)^2$, $\langle U|S_{vv}|U\rangle \sim \text{reg.}$ and $\langle U|S_{uv}|U\rangle \sim (r - 2M)$ for $r \to 2M$). The asymptotic condition

$$\langle U|S_{vv}|U\rangle = \frac{1}{4}\left(\langle U|S_{tt}|U\rangle + (1 - 2M/r)^2\langle U|S_{rr}|U\rangle + 2(1 - 2M/r)\langle U|S_{rt}|U\rangle\right) \to 0$$

(2.15)

requires $A = -a$. Furthermore $\langle U|S_{uu}|U\rangle$ vanishes like $(r - 2M)^2$ for $r \to 2M$ if $A = (4M)^{-1}$. Summarizing we have

$$\langle U|S_{tt}|U\rangle = \frac{1}{12\pi}\left[-\frac{2M}{r^3} + \frac{7M^2}{2r^4}\right] + (768\pi M^2)^{-1},$$

$$\langle U|S_{rr}|U\rangle = -(1 - 2M/r)^{-2}\left[\frac{M^2}{24\pi r^4} - (768\pi M^2)^{-1}\right],$$

$$\langle U|S_{rt}|U\rangle = -(1 - 2M/r)^{-1}(768\pi M^2)^{-1}.$$  (2.16)

Comparison with eqs. (1.14) reveals

$$\langle U|S_{ab}|U\rangle = \langle U|T_{ab}|U\rangle.$$  (2.17)

It is quite interesting to examine the behaviour of the auxiliary field $\psi$. From eq. (2.9) we see that the asymptotic requirement $A = -a$ yields $\psi \sim -Au$ as $r \to \infty$, whereas the regularity condition on the horizon $A = (4M)^{-1}$ reveals that $\psi \sim -v/4M + \text{const.}$ as $r \to 2M$. We have therefore the nice connection:

a) for $r \to \infty$ (we set $B = 0$) $\psi \sim -u/4M \iff \langle U|S_{ab}|U\rangle$ describes outgoing radiation;

b) for $r \to 2M$ $\psi \sim -v/4M + \text{const.} \iff \langle U|S_{ab}|U\rangle$ describes an ingoing flux of negative energy radiation; $\psi$ and $\partial_\alpha \psi, a, b = v, r$ regular on the future horizon $\iff \langle U|S_{ab}|U\rangle$ regular on the future horizon.

iii) Israel-Hartle-Hawking state $|H\rangle$.
This state corresponds to thermal equilibrium and thus we demand $\psi$ to be time independent, i.e. $a = 0$ which implies $\langle H|S_{rt}|H\rangle = 0$. Furthermore $|H\rangle$ is constructed in a way that makes the stress tensor regular both on the future and past horizon, which is achieved by requiring that as $r \to 2M$ $\langle H|S_{uu}|H\rangle = \langle H|S_{vv}|H\rangle$ vanish like $(r - 2M)^2$. Inspection of eqs. (2.11) reveals that we can fulfill the above requirement by demanding

$$\frac{1}{48\pi} \frac{A^2}{2} = \frac{1}{384\pi M^2},$$

i.e.

$$A = \frac{1}{2M}.$$  (2.19)
With the above choice $\psi$ is regular as $r \to 2M$, since by eq. (2.9) we have that for the state $|H\rangle$

$$\psi = \ln r + \frac{r}{2M} + B$$

(2.20)

which is indeed regular. \[1\] We can make $\psi$ vanishing on the horizon by choosing the arbitrary constant $B$ such that

$$B = -(\ln 2M + 1)$$

(2.21)

i.e. for the $|H\rangle$ state we have

$$\psi = \ln r + \frac{r}{2M} - (\ln 2M + 1)$$

(2.22)

Note that this expression could have been derived by looking at the static solution of the auxiliary field equation (2.8) of the form

$$\psi(r) = \int_{2M}^{r} dr_1 (1 - 2M/r_1)^{-1} \int_{2M}^{r_1} dz \frac{4M}{z^3}.$$  

(2.23)

Inserting now $a = 0$ and $A = 1/2M$ in eqs. (2.11) we find

$$\langle H|S_{tt}|H\rangle = (12\pi)^{-1} \left[ -\frac{2M}{r^3} + \frac{7M^2}{2r^4} \right] + (384\pi M^2)^{-1} ,$$

$$\langle H|S_{rr}|H\rangle = -(48\pi)^{-1} (1 - 2M/r)^2 \left[ \frac{2M}{r^4} - \frac{1}{8M^2} \right]$$

(2.24)

$$\langle H|S_{tr}|H\rangle = 0.$$  

Direct check with eqs. (1.15) reveals again the identity

$$\langle H|S_{ab}|H\rangle = \langle H|T_{ab}|H\rangle.$$

(2.25)

We have therefore shown in this simple 2D example the perfect agreement of our $\langle S_{ab}\rangle$ and $\langle T_{ab}\rangle$. This is not surprising at all; it is a general feature not at all limited to the 2D Schwarzschild background. It is well known that the auxiliary field equation can be integrated without making reference to any particular background. An arbitrary 2D spacetime

$$ds^2 = -g_{ab}dx^adx^b$$

(2.26)

can always be parametrized as follows (conformal gauge)

$$ds^2 = -e^{2\rho}dx^+dx^-.$$  

(2.27)

In that gauge the general solution of the equation for $\psi$ (eq. (2.3) ) reads

$$\psi = -2\rho + F(x^+) + G(x^-),$$

(2.28)

\[1\] The other solution of eq. (2.18), namely $A = -1/2M$, makes $\psi$ singular on the horizon.
where $F(x^+)$ and $G(x^-)$ are arbitrary functions of their arguments. Inserting this in eq. (2.6) we obtain $\langle S_{ab} \rangle$. This has exactly the same form as $\langle T_{ab} \rangle$ in eqs. (1.5) which, as already said, can be derived by canonical quantization.

The motivation for our nonstandard derivation of the stress tensor for a conformal scalar in 2D Schwarzschild space-time was purely pedagogical. It has shown explicitly how to implement state dependence in the framework of the anomaly induced effective action by imposing appropriate boundary conditions on the auxiliary field $\psi$ (or equivalently on $\langle S_{ab} \rangle$) which reflect the physical properties of the quantum state.

### 3 Four dimensions: Reigert’s action

Having set the basis of our formalism we are ready to attack the physically more interesting example: a conformal invariant scalar field $f$ in the 4D Schwarzschild black hole space-time. The classical action describing massless conformal invariant scalar field $f$ is

$$S = \int d^4x \sqrt{-g} \left( \partial^\mu f \partial_\mu f - \frac{1}{6} R f^2 \right) + S_{\text{vacuum}}, \quad (3.1)$$

where the $S_{\text{vacuum}}$ is necessary for the renormalizability of the theory (see [16] for the general introduction to the renormalization in curved space-time). Performing the path integration over the scalar field, one meets divergences and the renormalization is necessary. The trace anomaly results from the renormalization of the vacuum action in (3.1).

The vanishing of the classical trace has its quantum counterpart in the presence of a trace anomaly, composed by the terms coming from the vacuum counterterms [4]

$$\langle T^\alpha_{\alpha} \rangle \equiv \langle T \rangle = \frac{1}{(4\pi)^2} \left( aC^2 + bE + c\Box R \right). \quad (3.2)$$

For the free conformal invariant matter fields the three structures presented in (3.2) are sufficient, and the $\sqrt{-g}R^2$-type counterterm does not arise. That is why the corresponding term is absent in (3.2).

The anomaly induced effective action which yields the anomaly (3.2) has been independently derived by Reigert [18] and Fradkin and Tseytlin [19] (see also [16, 20, 21, 22] for further applications and references about the Reigert’s action). The

\begin{footnote}
More detailed consideration, including critical analysis of the current literature, have been reported separately in [17].
\end{footnote}

\begin{footnote}
Also we notice that that there is no expression, local or nonlocal, in the effective action, which could produce the term $\sqrt{-g}R^2$ in the trace anomaly [18].
\end{footnote}
expression derived in [18] has the form:

\[ S_{\text{eff}}^{an} = \frac{1}{(4\pi)^2} \int d^4x \int d^4y \left\{ \sqrt{-g} \left[ E - \frac{2}{3} \Box R \right] \right\}_x G(x, y) \]

\[ \left\{ \sqrt{-g} \left[ \frac{a}{4} C^2 + \frac{b}{8} (E - \frac{2}{3} \Box R) \right] \right\}_y - \frac{c + \frac{2}{3} b}{12(4\pi)^2} \int d^4x \sqrt{-g(x)} R^2 \],

(3.3)

where \( a = 1/120, b = -1/360, c = 1/180 \). \( G(x, y) \) is the inverse of the fourth order operator \( \Delta_4 \)

\[ \Delta_4 = \Box^2 - 2 R^{\mu\nu} \nabla_\mu \nabla_\nu + \frac{2}{3} R \Box - \frac{1}{3} (\nabla^\mu R) \nabla_\mu \],

i.e.

\[ (\sqrt{-g} \Delta_4)_x G(x, y) = \delta^4(x - y) \].

(3.5)

Note that \( \Delta_4 \) acting on a scalar field of zero conformal weight is the unique self-adjoint conformal invariant differential operator in 4D [23]. This also implies the conformal invariance of \( G(x, y) \). Let us rewrite the Reigert’s action in a more symmetric way

\[ S_{\text{eff}}^{an} = - \frac{c + \frac{2}{3} b}{12(4\pi)^2} \int d^4x \sqrt{-g} R^2 - \frac{1}{2(4\pi)^2} \int d^4x \sqrt{-g(x)} \int d^4y \sqrt{-g(y)} \]

\[ \frac{\sqrt{-b}}{2} \left[ (E - \frac{2}{3} \Box R) + \frac{a}{b} C^2 \right]_x G(x, y) \frac{\sqrt{-b}}{2} \left[ (E - \frac{2}{3} \Box R) + \frac{a}{b} C^2 \right]_y + \frac{1}{2(4\pi)^2} \int d^4x \sqrt{-g(x)} \int d^4y \sqrt{-g(y)} \left( \frac{a}{2 \sqrt{-b}} C^2 \right)_x G(x, y) \left( \frac{a}{2 \sqrt{-b}} C^2 \right)_y \]

(3.6)

This nonlocal action can be made local by the introduction of two auxiliary fields \( \phi \) and \( \psi \) [18, 24]

\[ S_{\text{eff}}^{an} = + \int d^4x \sqrt{-g} \left[ \frac{1}{2} \phi \Delta_4 \phi + \phi \left( \frac{\sqrt{-b}}{8\pi} (E - \frac{2}{3} \Box R) - \frac{a}{8\pi \sqrt{-b}} C^2 \right) \right] + \int d^4x \sqrt{-g} \left( - \frac{1}{2} \psi \Delta_4 \psi + \frac{a}{8\pi \sqrt{-b}} C^2 \psi \right) - \frac{c + \frac{2}{3} b}{12(4\pi)^2} \int d^4x \sqrt{-g} R^2 \].

(3.7)

Once the equations of motion for \( \phi \) and \( \psi \) are used, eq. (3.7) reduces to the nonlocal form (3.3). We remark that the path integration over the auxiliary fields would give an extra contribution to the anomaly (3.2), modifying the values of the constants \( a, b, c \) in (3.2). Here we consider the introduction of the auxiliary fields \( \psi, \phi \) as a method of working with the nonlocalities of the expression (3.6) and hence require the correspondence to hold only on classical level.

Note that the last nonlocal term in eq. (3.7) (or equivalently the second integrand in eq. (3.7) ) is conformally invariant and hence could be removed and included in the Weyl invariant part of the effective action \( S^W \) (see eq. (1.8) ). For later use we shall keep this term. However its coefficient \( a/(8\pi \sqrt{-b}) \) can be replaced by an arbitrary parameter, say \( l_1 \), since a change in this parameter can be reabsorbed by the Weyl
invariant part $S^W(g_{\alpha\beta})$ of $S_{\text{eff}}$ which is not determined by the trace anomaly. So we shall work with the following anomaly induced effective action

\[ S_{\text{eff}}^{\text{an}} = -\frac{k_3}{12} \int d^4x \sqrt{-g} R^2 + \int d^4x \sqrt{-g} \left[ \frac{1}{2} \phi \Delta_4 \phi + \phi \left( k_1 C^2 + k_2 (E - \frac{2}{3} \Box R) \right) \right] \]

\[ + \int d^4x \sqrt{-g} \left( -\frac{1}{2} \phi \Delta_4 \psi + l_1 C^2 \psi \right) , \quad (3.8) \]

where we set $k_1 = a/(8\pi \sqrt{-b})$, $k_2 = \sqrt{-b}/8\pi$ and $k_3 = (c + \frac{2k_2}{3})/(4\pi)^2$, bearing in mind the values of $a, b, c$ specific for a single conformal scalar.

After eliminating the auxiliary fields $\phi$ and $\psi$ eq. (3.3) does not reduce to Reigert’s nonlocal action of eq. (3.3) (unless $l_1 = a/(8\pi \sqrt{-b})$). The difference, as we just remarked, is a conformally invariant functional, hence of the form $S^W$, on which we have no control.

From eq. (3.8) the equations of motion for the auxiliary fields follow

\[ \frac{1}{\sqrt{-g}} \frac{\delta S_{\text{eff}}^{\text{an}}}{\delta \phi} = \Delta_4 \phi + k_1 C^2 + k_2 (E - \frac{2}{3} \Box R) = 0 , \quad (3.9) \]

\[ \frac{1}{\sqrt{-g}} \frac{\delta S_{\text{eff}}^{\text{an}}}{\delta \psi} = -\Delta_4 \psi + l_1 C^2 = 0 . \quad (3.10) \]

Introducing now the traceless tensor $K_{\mu\nu}(\phi)$

\[ K_{\mu\nu}(\phi) = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int d^4x \sqrt{-g} \left\{ \phi \Delta_4 \phi \right\} = -\frac{1}{2} g_{\mu\nu} \left( (\Box \phi)^2 - 2 R^{\alpha\beta} \phi_{\alpha} \phi_{\beta} + \frac{2}{3} (\nabla \phi)^2 R \right) \]

\[ \phi_{\mu}(\nabla_{\nu} \Box \phi) - \phi_{\nu}(\nabla_{\mu} \Box \phi) + 2 \phi_{\mu\nu} (\Box \phi) + 2 \phi^\lambda \phi_{\mu\nu\lambda} - \frac{2}{3} \phi^\lambda (\phi_{\lambda\mu\nu} + \phi_{\lambda\nu\mu}) + \frac{1}{3} g_{\mu\nu} \phi^\lambda \phi_{\lambda\tau} \]

\[ + \frac{1}{3} g_{\mu\nu} \phi^\lambda (\Box \phi_{\lambda}) - \frac{4}{3} \phi^\lambda \phi_{\lambda\nu} - 2 \phi_{\lambda} (R^\lambda_{\mu\nu\rho} + R^\lambda_{\nu\rho\mu} + \frac{2}{3} R \phi_{\mu} \phi_{\nu} + \frac{2}{3} R_{\mu\nu} \phi_{\lambda} \phi_{\lambda} \phi_{\lambda} \phi_{\lambda} \phi_{\lambda} = \phi_{\lambda\tau} \equiv \nabla_{\lambda} \nabla_{\tau} \phi, \text{ etc.} . \quad (3.11) \]

we can write the anomaly induced stress tensor $\langle S_{\mu\nu} \rangle$ as

\[ \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{eff}}^{\text{an}}}{\delta g^{\mu\nu}} = \langle S_{\mu\nu} \rangle = K_{\mu\nu}(\phi) - K_{\mu\nu}(\psi) + 8 \nabla^\lambda \nabla^\tau Z R_{\mu\nu\lambda\tau} - g_{\mu\nu} Z R_{\rho\sigma\alpha\beta} \]

\[ + 4 \mathcal{Z} R_{\mu\rho\lambda\tau} R_{\nu}^\rho \phi^\lambda \phi^\tau + \frac{4k_2}{3} \left[ (\nabla_{\mu} \nabla_{\nu} \Box \phi) - g_{\mu\nu} (\Box^2 \phi) \right] + ... \quad (3.12) \]

where we have defined

\[ Z \equiv (k_1 + k_2) \phi + l_1 \psi \]

and $\phi_{\lambda\tau} \equiv \nabla_{\lambda} \nabla_{\tau} \phi$, etc. . The dots in eq. (3.12) indicate terms containing either the Ricci tensor $R_{\mu\nu}$ or its contraction $R$. In our subsequent analysis the background space-time chosen will be Schwarzschild and therefore these terms, together with the sum of the two terms linear in $Z$ in eq. (3.12), vanish identically.

The equation of motion for $\phi$ in the Schwarzschild spacetime (see eq. (1.10)) reads

\[ \Box^2 \phi = \alpha \frac{M^2}{r^6} , \quad (3.14) \]
where $\alpha \equiv -48(k_1 + k_2)$ and $\Box_s$ is the Delambertian written in Schwarzschild coordinates. The solution for $\phi$ can be given in the following form

$$\phi(r, t) = d \cdot t + w(r),$$

where $w(r)$ is such that

$$\frac{dw}{dr} = \frac{B}{3}r + \frac{2}{3}MB - \frac{A}{6} - \frac{\alpha}{72M} + \left( \frac{4}{3}BM^2 + \frac{C}{2M} - AM - \frac{\alpha}{24} \right) \frac{1}{r - 2M} - \frac{C}{2M} \frac{1}{r}$$

and $\beta \equiv 48l_1$ and $(d, A, B, C) \rightarrow (d', A', B', C')$.

Substituting the solutions for the auxiliary fields in eqs. (3.12) we obtain the effective stress tensor for the Schwarzschild space-time in the form

$$\langle S_{\mu \nu} \rangle = \langle S_{\mu \nu}(\phi) \rangle + \langle S_{\mu \nu}(\psi) \rangle$$

separating the contribution of each individual field to $\langle S_{\mu \nu} \rangle$. Both $\phi$ and $\psi$ are related to the inverse of the fourth order differential operator $\Delta_4$, but they are independently defined, and therefore the boundary conditions have to be imposed on them individually. For later use we reproduce the limiting behaviour of $\partial_r \phi$ as

$$\partial_r \phi \sim \frac{E}{r - 2M} + 2M(\frac{A}{2M} - \frac{\alpha}{48M^2}) \ln(r - 2M) + \text{reg. terms},$$

where

$$E = -\frac{\alpha}{12} + \frac{4}{3}BM^2 + \frac{C}{2M} - 2M^2(\frac{A}{2M} - \frac{\alpha}{48M^2}) - \frac{2}{3}AM \ln 2M,$$

in the limit $r \to 2M$, and

$$\partial_r \phi \to \frac{B}{3}r + \frac{2}{3}MB - \frac{A}{2},$$

at infinity.

i) Boulware state $|B\rangle$. 

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For flat space and Minkowski vacuum $\phi = \text{const.}$ (which we can set to zero without any consequences). This implies $\langle M|S_{\mu\nu}|M \rangle = 0$ as it should be. Since as $r \to \infty$ $|B \rangle \to |M \rangle$, the correct asymptotic limit on $\phi$ is obtained in eqs. (3.13), (3.14) by setting all constants to zero. So for the state $|B \rangle$ we have

$$
\frac{dw}{dr} = -\frac{\alpha}{72M} - \frac{1}{24\alpha - 2M} + \ln r \left[ -\frac{\alpha M}{18r(2M)} + \frac{r^2}{48M^23(r-2M)} \right]
+ \ln(r - 2M) \left[ -\frac{\alpha}{48M^23r(r-2M)} \right]
$$

(3.21)

and an analogous expression for $\psi$ ($\alpha \to \beta$).

Starting from these expressions one can obtain $\langle B|S_{\mu\nu}|B \rangle$. Being the calculations rather long and boring, we report just the limiting behaviours

$$
\langle B|S_{\mu}^{\nu}|B \rangle \to \frac{1}{2} \frac{\alpha^2 - \beta^2}{(24)^2} \frac{1}{(2M)^4 f^2} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/3 \end{pmatrix}
$$

(3.22)

for $r \to 2M$ ($f = 1 - 2M/r$) and

$$
\langle B|S_{\mu}^{\nu}|B \rangle \to O\left( \frac{1}{r_0^6} \right)
$$

(3.23)

as $r \to \infty$.

The qualitative agreement of our $\langle B|S_{\mu\nu}|B \rangle$ and $\langle B|T_{\mu\nu}|B \rangle$ of eqs. (1.10), (1.17) is rather nice. We see the expected $r^{-6}$ fall off and the $1/f^2$ divergence on the horizon.

ii) Unruh state $|U \rangle$.

The Unruh state $|U \rangle$ agrees with Minkowski vacuum on past null infinity, i.e. no incoming radiation. This requires $\phi \sim u$ as $r \to \infty$, which implies $\langle U|S_{\nu\nu}|U \rangle \to 0$ for $r \to \infty$. This asymptotic behaviour of $\phi$ is achieved by requiring $B = 0$ and $d = A/2$. Regularity along the future horizon requires $\phi \sim v$ and $\partial_a \phi \sim \text{reg.}$, $a = v, r$ (see the discussion following eq. (2.17)). This is achieved by fixing $E = 2dM$ ($\phi \sim v$) and $A = \alpha/24M$ ($\partial_a \phi$ finite). That this is the correct choice it can be seen by direct evaluation of $\langle U|S_{\mu\nu}|U \rangle$.

Near the horizon to the order $f^{-2}$ we have

$$
\langle U|S_{\mu\nu}(\phi)|U \rangle \sim \frac{(E^2 - 4d^2M^2)}{32M^4 f^2} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/3 \end{pmatrix},
$$

(3.24)

which indeed vanishes for $E = 2dM$. Logarithmic divergence in the pressure $\langle T_{\theta\theta} \rangle$ are eliminated by $A = \alpha/24M$. Repeating the same arguments for $\psi$ gives $E' = 2d'M$, $A' = \beta/24M, B' = 0$. The following leading behaviour of $\langle S_{ab} \rangle$ emerges

$$
\langle U|S_{\alpha}^{\beta}|U \rangle \sim \frac{\alpha^2 - \beta^2}{2(48M^2)^2} \begin{pmatrix} 1/f & -1 \\ 1/f & -1/f \end{pmatrix}, \ r \to 2M
$$

(3.25)
which is indeed regular on the future horizon. By the above choice of the constants, we can find the asymptotic form of $\langle U|S_{\mu\nu}|U \rangle$

$$\langle U|S_{\mu\nu}|U \rangle \rightarrow \frac{\alpha^2 - \beta^2}{2r^2(24M)^2} \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ r \rightarrow \infty .$$  

(3.26)

Our results eqs. (3.25), (3.26) are in exact agreement with $\langle U|T_{\mu\nu}|U \rangle$ given by eqs. (1.18), (1.19) once the luminosity $L$ is identified with

$$\frac{L}{4\pi} = \frac{(\alpha^2 - \beta^2)}{2(24M)^2} .$$

(3.27)

iii) Israel-Hartle-Hawking state $|H \rangle$.

This state is an equilibrium state regular both on the future and the past horizons. For an equilibrium state $d = 0$ which implies no net fluxes ($\langle H|S_{rt}(\phi)|H \rangle = 0$). Inspection of eq. (3.18) reveals that $\phi$ and $\partial_r \phi$ are regular both on the future and the past horizons by imposing $E = 0$ and $A = \alpha/24M$. This leaves the parameter $B$ free. However, if the solution for $\phi$ is obtained by an integral formula like eq. (2.23), which sets the lower limit of integration to be $r = 2M$, we would obtain

$$A = \frac{\alpha}{24M} , \quad B = \frac{11\alpha}{288M^2} , \quad C = \frac{7\alpha M}{108} + \frac{\alpha M}{18} \ln 2M$$

(3.28)

which implies $E = 0$. By this choice, which we think characterizes the $|H \rangle$ state, we eventually arrive at the following expression

$$\langle H|S_{\mu\nu}|H \rangle \rightarrow -\frac{k_2\alpha}{96M^2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

(3.29)

in the limit $r \rightarrow 2M$ and

$$\langle H|S_{\mu\nu}|H \rangle \rightarrow \frac{7}{2} \left( \frac{11}{864M^2} \right)^2 (\alpha^2 - \beta^2) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/3 \end{pmatrix}$$

(3.30)

as $r \rightarrow \infty$. The asymptotic limit of eq. (3.30) describes indeed a thermal bath as expected (compare to eq. (1.20)). Note the appearance of the factor $(\alpha^2 - \beta^2)$ just as in eq. (3.26).

Putting some numbers, if we choose $l_1 = \frac{a}{8\sqrt{-b}}$ which reconduces our $S_{eff}$ to Reigert’s one, we have the disappointing result $(\alpha^2 - \beta^2) < 0$ that is physically meaningless. A similar situation was found in investigation of the anomaly induced effective action
representing 4D minimally coupled scalar fields spherically reduced to 2D at the classical level (see [26]). For $l_1 = 0$ (i.e. complete absorption of the conformally invariant part of our $S^{an}_{eff}$ in $S^W$) we find that the overall coefficient of eq. (3.30) is $\sim \frac{2.2 \times 10^{-4}}{\pi^2 M^4}$, which is much bigger than the correct one $\sim \frac{8 \times 10^{-6}}{\pi^2 M^4}$. Exact agreement of our result eq. (3.30) and eq. (1.20) requires a fine tuning of the parameter $\beta$, namely $\beta \sim \frac{6.2 \times 10^{-1}}{\pi}$. For this value of $\beta$ the luminosity of an evaporating black hole (see eq. (3.27)) turns out to be $L = \frac{4.9 \times 10^{-5}}{\pi M^2}$, which is roughly four times smaller than the value given in [13].

Despite many nice features, there are some disappointing aspects of our $\langle S_{\mu\nu} \rangle$. The overall coefficient of $\langle B|T_{\mu\nu}|B \rangle$ on the horizon is expected to be the same as the asymptotic limit of $\langle H|T_{\mu\nu}|H \rangle$ up to a minus sign (see eqs. (1.17) and (1.20)). This is not the case for $\langle S_{\mu\nu} \rangle$ (compare eq. (3.22) and eq. (3.30)). Unfortunately this is not all. If we go beyond the leading terms there is no agreement of our $\langle S_{\mu\nu} \rangle$ and $\langle T_{\mu\nu} \rangle$. Just to mention some negative aspects: $\langle U|S_{\theta\theta}|U \rangle \sim r^{-3}$ in the limit $r \to \infty$ while $\langle U|T_{\theta\theta}|U \rangle$ is expected to fall off as $r^{-4}$. Furthermore in the same limit the coefficients in $r^{-1}$ and $r^{-2}$ of our $\langle H|S_{\mu\nu}|H \rangle$ do not correspond to redshifted thermal radiation as we have for $\langle H|T_{\mu\nu}|H \rangle$. One can guess that this discrepancy may be removed by a careful modelling of the conformal part in (1.8) through the introduction of other conformal structures [28], or, hopefully, by using the results of some of the direct approximate calculations of the effective action (see, for example, [23]).

4 Conclusions

In a field theory the knowledge of the trace anomaly allows part of the effective action to be reconstructed; this is the so called anomaly induced effective action, i.e. $S^{an}_{eff}$. The basic features of this object are the nonlocality and the apparent absence of any reference to a particular quantum state. Auxiliary fields have been introduced to make $S^{an}_{eff}$ local and boundary conditions have been imposed on them to select the appropriate quantum states. This procedure has been positively tested in a simple 2D example where $S^{an}_{eff}$ is the well known Polyakov action. We found exact agreement of the stress tensor calculated from this effective action (i.e. $\langle S_{\mu\nu} \rangle$) and the one ($\langle T_{\mu\nu} \rangle$) resulting from standard canonical quantization.

The expertise gained from the analysis of this simple 2D model has allowed us to attack by similar methods the physically much more interesting example of 4D Schwarzschild black hole. We have been able to construct explicitly $\langle S_{\mu\nu} \rangle$ for the three quantum states relevant for the discussion of quantum matter fields in a black hole spacetime, namely Boulware state (vacuum polarization around a static star), Unruh state (evaporation of a black hole formed by gravitational collapse) and Israel-Hartle-Hawking state (equilibrium of a black hole and a thermal bath).
By appropriate choice of the arbitrary parameter $l_1$ of our model (or, equivalently, of $\beta$), we were able to show the qualitative agreement of our $\langle S_{\mu\nu} \rangle$ and the canonically computed $\langle T_{\mu\nu} \rangle$ at infinity and near the horizon. Unfortunately these nice features cannot be extended beyond leading terms. This is expected since the fundamental brick of our 4D investigation, namely $S_{\text{eff}}^{\text{an}}$, up to a Weyl invariant term coincides with Reigert’s action. It is well known [27] that this action is rather deficitary if considered as full effective action. Unlike Polyakov’s action Reigert’s one is unable even to correctly reproduce the three-point correlation function of the theory on the flat background. In view of this fact, the discrepancies found in our $\langle S_{\mu\nu} \rangle$ are not surprising at all.

What in our opinion is however remarkable is that, nevertheless, our $S_{\text{eff}}^{\text{an}}$ does indeed reproduce the expected behaviour of the stress tensor near the horizon and at infinity upon which our understanding of black hole evaporation is based.

In view of our investigation, a careful use (not a straightforward one) of $\langle S_{\mu\nu} \rangle$ in the semiclassical Einstein equations to get at least a feeling of the backreaction in an evaporating black hole can be made. We hope to come back to this point in a future work.

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