NON-DIVERGENCE PARABOLIC EQUATIONS OF SECOND ORDER WITH CRITICAL DRIFT IN LEBESGUE SPACES

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Abstract. We consider uniformly parabolic equations and inequalities of second order in the non-divergence form with drift
\[-u_t + Lu = -u_t + \sum_{ij} a_{ij} D_{ij} u + \sum_i b_i D_i u = 0 \quad (\geq 0, \leq 0)\]
in some domain \(Q \subset \mathbb{R}^{n+1}\). We prove growth theorems and the interior Harnack inequality as the main results. In this paper, we will only assume the drift \(b\) is in certain Lebesgue spaces which are critical under the parabolic scaling but not necessarily to be bounded. In the last section, some applications of the interior Harnack inequality are presented. In particular, we show there is a “universal” spectral gap for the associated elliptic operator. The counterpart for uniformly elliptic equations of second order in non-divergence form is shown in [S10].

1. Introduction

1.1. General Introduction. The qualitative properties of solutions to partial differential equations have been intensively studied for a long time. In this paper, we consider the qualitative properties of solutions to the uniformly parabolic equation in non-divergence form,
\[-u_t + Lu := -u_t + \sum_{ij} a_{ij} D_{ij} u + \sum_i b_i D_i u = 0\]
and the associated inequalities: \(-u_t + Lu \geq 0\) and \(-u_t + Lu \leq 0\). Throughout the paper, we use the notations \(D_i := \frac{\partial}{\partial x_i}, D_{ij} := \frac{\partial^2}{\partial x_i \partial x_j}\), and \(u_t := \frac{\partial u}{\partial t}\). We assume \(b = (b_1, \ldots, b_n)\) and \(a_{ij}\)'s are real measurable, \(a_{ij}\)'s also satisfy the uniform parabolicity condition
\[\forall \xi \in \mathbb{R}^n, \nu^{-1} |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(X) \xi_i \xi_j, \quad \sum_{i,j=1}^n a_{ij}^2 \leq \nu^2\]
with some constant \(\nu \geq 1\), \(\forall X = (x,t)\) in the domain of definition \(Q \subset \mathbb{R}^{n+1}\).

For the drift \(b\), we will only require it is in certain Lebesgue spaces which are critical under the parabolic scaling. To formulate our setting more precisely, we assume over the domain of definition \(Q\),
\[\|b\|_{L^p_x L^{\frac{n}{p}}_t} := \left( \int \left[ \int |b(x,t)|^q \, dx \right]^\frac{p}{q} \, dt \right)^\frac{1}{p} =: S(Q) < \infty,\]

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for some constants $p, q \geq 1$ such that

\begin{equation}
\frac{n}{p} + \frac{2}{q} = 1.
\end{equation}

By "critical", we mean that with the $L^p_x L^q_t$ norm, the drift is scaling invariant under the parabolic scaling: for $r > 0$,

\[ x \to r^{-1}x, \ t \to r^{-2}t. \]

Indeed, suppose $u$ satisfies

\[-u_t + \sum_{ij} a_{ij} D_{ij} u + \sum_i b_i D_i u = 0.\]

in a domain $Q \in \mathbb{R}^{n+1}$. Then for any constant $r > 0$, let

\[ \tilde{x} = r^{-1}x, \ \tilde{t} = r^{-2}t. \]

Then $u(\tilde{x}, \tilde{t}) = u(rx, r^2t)$ satisfies the equation

\[-\tilde{u}_t + \sum_{ij} \tilde{a}_{ij} D_{ij} \tilde{u} + \sum_i \tilde{b}_i D_i \tilde{u} = 0,\]

in $Q_r := \{(x,t), (rx, r^2t) \in Q\}$. Note that $\tilde{b} = rb$, so

\[ S(Q_r) = \|\tilde{b}\|_{L^p_x L^q_t} = \|b\|_{L^p_x L^q_t} = S(Q). \]

In general, regarding the scaling, intuitively, there is a competition between the transport term and the diffusion part. One might expect that for the supercritical scaling case, $\frac{n}{p} + \frac{2}{q} > 1$: the solutions of the equations have discontinuities [SYZ, GC]. For the critical situation we are considering here, we have Hölder continuous solutions, see Theorem 22. Finally, if the drift is subcritical with respect to the scaling, i.e. $\frac{n}{p} + \frac{2}{q} < 1$, we expect the solutions will be smooth.

We will concentrate on the interior Harnack inequality for parabolic equations in non-divergence form with critical drift. Given constants $p, q$ satisfying condition (1.4), let $Q$ be an open set in $\mathbb{R}^{n+1}$, we define

\[ W_{p,q}(Q) := C(Q) \cap W^{2,1}_{p,q}(Q) \]

where $f \in W^{2,1}_{p,q}(Q)$ means $f_t, D_i f, D_{ij} f \in (L^p_x L^q_t)_{loc}$.

With the assumptions above, the main result in this paper is then expressed by Theorem 1.

**Theorem 1** (Interior Harnack Inequality). *Given constants $p, q$ satisfying condition (1.4), suppose $u \in W_{p,q}$ and $-u_t + Lu = 0$ in $Q_{2r}(Y) := B_{2r}(y) \times (s-(2r)^2, s)$, $Y = (y,s) \in \mathbb{R}^{n+1}$ and $r > 0$. If $u \geq 0$, then*

\begin{equation}
\sup_{Q^{0}_{r}} u \leq N \inf_{Q^{1}_{r}} u,
\end{equation}

*where $N := N(n, \nu, p, q, S)$, $Q^{0}_{r}(Y) := B_{r}(y) \times (s-r^2, s)$, $Q^{1}_{r} := B_{r}(y) \times (s-3r^2, s-2r^2)$ and $S$ is from condition (1.3).*

**Remark.** We will see the most general form of Harnack principle in the later section on the applications of Harnack inequality.
Harnack inequalities have many important applications, not only in differential equations, but also in other areas, such as diffusion processes, geometry, etc. Unlike the classical maximum principle, the interior Harnack inequality is far from obvious. For elliptic and parabolic equations with measurable coefficients in the divergence form, it was proved by Moser in the papers [M61], [M64]. However, a similar result for non-divergence equations was obtained 15 years later after Moser’s papers by Krylov and Safonov [KS], [S80] in 1978-80. Their proofs relied on some improved versions of growth theorems from the book by Landis [LML]. These growth theorems control the behavior of (sub-, super-) solutions of second order elliptic and parabolic equations in terms of the Lebesgue measure of areas in which solutions are positive or negative. In [FS], Ferretti and Safonov used growth theorems as a common background for both divergence and non-divergence equations and used these three growth theorems to derive the interior Harnack inequality. Even in the one-dimensional case, the Harnack inequality fails for equations of a “joint” structure, which combine both divergence and non-divergence parts. One can find detailed discussion in [CS13].

At the beginning, the interior Harnack inequality was proved with bounded drift. Later on, this condition was relaxed to subcritical drift $b$. For the subcritical case, we can always rescale the problem. In small scale, the drift will work like a perturbation from the case without drift. But for the critical situation, our common tricks do not work. One can find a historical overview of this progress in [NU]. For non-divergence elliptic equations of second order, in [S10], Safonov shown the interior Harnack inequality for the scaling critical case $b \in L^n$. In this paper, we adapt Safonov’s idea to the parabolic setting. We will consider the case that the drift $b$ is in critical scaling Lebesgue spaces given by the conditions (1.3) and (1.4) above. In a later paper [GC], we will consider critical scaling Morrey spaces with different approaches. Similar results for both divergence form elliptic and parabolic equations are presented in [NU].

We will follow the unified approach to growth theorems and the interior Harnack inequality developed in [FS]. For this purpose, we need to prove three growth theorems and derive the interior Harnack inequality as a consequence for parabolic equations with critical drift formulated as above. We only present the case $b \in L^n_t L^\infty_x$. For other cases ($p > n, q < \infty$) the proofs are more or less identical to the situation we are considering here. We will see remarks about them later on. For certain points, other cases ($p > n, q < \infty$) are simpler than the endpoint case ($p = n, q = \infty$) we are discussing here. Namely, throughout the paper, we assume over the domain of definition $Q$,

\begin{equation}
\int \left( \sup_t |b(x,t)| \right)^n dx = S < \infty
\end{equation}

where $\sup$ means essential supremum. One point compared with the results for divergence form equations in [NU] is that our conclusions for $L^n_t L^\infty_x$ case do not depend on the modulus of continuity of the norm. For the sake of simplicity, we assume that all functions (coefficients and solutions) are smooth enough. It is easy to get rid of extra smoothness assumptions by means of standard approximation procedures, see Section 7. We should notice that it is important to have appropriate estimates for solutions with constants depending only on the prescribed quantities,
such as the dimension $n$, the parabolicity constant, etc., but not depending on “additional” smoothness.

This paper is organized as follows: In Section 1, we introduce our basic assumptions and notations. In Section 2, we formulate a weak version of the classical maximum principle, the Alexandrov-Bakelman-Pucci-Krylov-Tso estimate, and some consequences of it. In Sections 3, 4, and 5, we formulate and prove three growth theorems. In Section 6, we derive the interior Harnack inequality. In Section 7, we use approximation to show all results are valid without extra smoothness assumption. Finally, in Section 8, we discuss some applications of the interior Harnack inequality. In particular, we show there is a “universal” spectral gap between the principal eigenvalue and other eigenvalues for the elliptic operator $L$ with drift $b \in L^{\infty}$. In the appendix, we will prove the Alexandrov-Bakelman-Pucci-Krylov-Tso estimate we use in this paper.

1.2. Notations: In this paper, we use summation convention.

"$A := B$" or "$B =: A$" is the definition of $A$ by means of the expression $B$.

**Definition 2.** For any open set $Q \subset \mathbb{R}^{n+1}$, we define the space

$$W(Q) := C(Q) \cap W^{2,1}_{n,\infty}(Q),$$

where $f \in W^{2,1}_{n,\infty}(Q)$ means $f_t, D_i f, D_{ij} f \in (L^2_{x,t})_{\text{loc}}$.

$\mathbb{R}^n$ is the n-dimensional Euclidean space, $n \geq 1$, with points $x = (x_1, \ldots, x_n)$, where $x_i$'s are real numbers. Here the symbol $t$ stands for the transposition of vectors which indicates that vectors in $\mathbb{R}^n$ are treated as column vectors. For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, the scalar product $(x, y) := \sum x_i y_i$, the length of $x$ is $|x| := (x, x)^{\frac{1}{2}}$.

For a Borel set $\Gamma \subset \mathbb{R}^n$, $\bar{\Gamma} := \Gamma \cup \partial\Gamma$ is the closure of $\Gamma$, $|\Gamma|$ is the n-dimensional Lebesgue measure of $\Gamma$. Sometimes we use the same notation for the surface measure of a subset $\Gamma$ of a smooth surface $S$.

For real numbers $c$, we denote $c_+ := \max(c, 0)$, $c_- := \max(-c, 0)$.

In order to formulate our results, we need some standard definitions and notations for the setting of parabolic equations.

**Definition 3.** Let $Q$ be an open connected set in $\mathbb{R}^{n+1}$, $n \geq 1$. The parabolic boundary $\partial_p Q$ of $Q$ is the set of all points $X_0 = (x_0, t_0) \in \partial Q$, such that there exists a continuous function $x = x(t)$ on the interval $[t_0, t_0 + \delta)$ with values in $\mathbb{R}^n$, such that $x(t_0) = x_0$ and $(x(t), t) \in Q$ for all $t \in (t_0, t_0 + \delta)$. Here $x = x(t)$ and $\delta > 0$ depend on $X_0$. In particular, for cylinders $Q_\Omega = \Omega \times (0, T)$ with $\Omega \subset \mathbb{R}^n$, the parabolic boundary $\partial_{p} Q_\Omega := (\partial_{x} Q_\Omega) \cup (\partial_{t} Q_\Omega)$, where $\partial_{x} Q_\Omega := (\partial \Omega) \times (0, T)$, $\partial_{t} Q_\Omega := \Omega \times \{0\}$.

We will use the following notation for the “standard” parabolic cylinder. For $Y = (y, s)$ and $r > 0$, we define $Q_r(Y) := B_r(y) \times (s - r^2, s)$, where $B_r(y) := \{x \in \mathbb{R}^n : |x - y| < r\}$.

2. Preliminaries

In this section, we briefly discuss some well-known theorems and results which are crucial for us to carry out the discussion in the later parts of this paper. We use the notation $u \in W^{2,1}_{n,\infty}(Q)$ in the sense of Definition 2. Also we denote $S = \int [\sup_{t} |b(x, t)|^n] dx < \infty$.
Theorem 4 (Alexandrov-Bakelman-Pucci-Krylov-Tso estimate). Suppose \( u \in W(\Omega) \) and \( \Omega \subset Q_r \) and \(-u_t + Lu \geq f\). If \( \sup_{\partial_0 \Omega} u \leq 0 \), then

\[
\begin{align*}
\sup_{\Omega} u &\leq N(\nu, n, S) r \| f \|_{L^\infty_r}.
\end{align*}
\]

We will present the detailed proof of above theorem in Appendix A.

Remark 5. In [AIN], Nazarov shown the Alexandrov-Bakelman-Pucci-Krylov-Tso estimate holds for the drift \( b \in L^p x L^q_t \), i.e.,

\[
\| b \|_{L^p x L^q_t} = \left( \int \left[ \int |b(x, t)|^q \, dt \right]^{\frac{p}{q}} \, dx \right)^{\frac{1}{p}} < \infty,
\]

for

\[
\frac{n}{p} + \frac{1}{q} \leq 1, \quad p, q \geq 1.
\]

The proof was based on Krylov’s ideas and methods [NVK]. In this paper, we mainly focus the case \( p = n \) and \( q = \infty \). The approach in this paper is easily modified to show the general scaling invariant cases, i.e., for some constants \( p, q \geq 1 \) such that

\[
\frac{n}{p} + \frac{2}{q} = 1, \quad q < \infty.
\]

We will discuss this point again later on.

Theorem 6 (Maximal Principle). Let \( Q \) be a bounded open set in \( \mathbb{R}^{n+1} \), and let a function \( u \in C^{2,1}(\overline{Q} \setminus \partial_p Q) \cap C(\overline{Q}) \) satisfy the inequality \(-u_t + Lu \geq 0\) in \( Q \). Then

\[
\begin{align*}
\sup_{Q} u &= \sup_{\partial_p Q} u
\end{align*}
\]

As an easy consequence of the maximal principle and the Alexandrov-Bakelman-Pucci-Krylov-Tso estimate, we have the well-known comparison principle.

Theorem 7 (Comparison Principle). Let \( Q \) be a bounded domain in \( \mathbb{R}^{n+1} \), \( u, v \in W(\Omega) \cap C(\overline{Q}) \), \(-u_t + Lu \leq -v_t + Lv\) in \( Q \), and \( u \geq v \) on \( \partial_p Q \), then \( u \geq v \) on \( \overline{Q} \).

3. First Growth Theorem

Suppose \( R \) is the region in a cylinder where a subsolution \( u \) of our equation is positive. The first growth theorem, Theorem 12, basically tells us if the measure of \( R \) is small, then the maximal value of \( u \) over half of the cylinder is strictly less than the maximal value over the whole cylinder. In other words, it gives us some quantitative decay properties.

Before we start to prove the first growth theorem, we need to prove several intermediate results based on the comparison principle and the Alexandrov-Bakelman-Pucci-Krylov-Tso estimate. Let us first do some preliminary calculations in order to carry out some comparison arguments.

For fixed numbers \( \alpha > 0 \) and \( 0 < \epsilon < 1 \), in the cylinder \( Q = B_r(0) \times (-r^2, (\alpha - 1)r^2) \), we can define

\[
\begin{align*}
\psi_0 &= \frac{(1 - \epsilon^2)(t + r^2)}{\alpha} + \epsilon^2 r^2, \\
\psi_1 &= (\psi_0 - |x|^2)_+.
\end{align*}
\]
where $(\cdot)_+$ means positive part of the function. And we also define
\begin{equation}
\psi = \psi_1^2 \psi_0^{-q}
\end{equation}
for some number $q \geq 2$ to be determined later. First of all, we notice $\psi$ is $C^{2,1}$ in $\tilde{Q} := \{(x,t)| |x|^2 < \psi_0, -r^2 < t < (\alpha - 1)r^2\}$. It is clear that $-\psi_t + L\psi = 0$ if $\psi_0 \leq |x|^2$. Now if $\psi_0 > |x|^2$, by some computations, we obtain
\[-\psi_t + \sum a_{ij} D_{ij} \psi = \psi_0^{-q} \left[ 8a_{ij} x_i x_j - 4\psi_1 \text{trace}(a_{ij}) + \frac{(1-\epsilon^2)q}{\alpha \psi_0} \psi_1^2 - 2\frac{(1-\epsilon^2)}{\alpha} \psi_1 \right].\]
Set $F_1 = \frac{2}{\alpha} + 8n\nu^{-1}$, and $\xi = \frac{\psi_1}{\psi_0}$ then
\begin{equation}
-\psi_t + \sum a_{ij} D_{ij} \psi \geq \psi_0^{-q} \left[ (1-\epsilon^2)q \xi^2 - F_1 \xi + 8\lambda \right].
\end{equation}
Pick
\begin{equation}
q = 2 + \frac{\alpha}{32(1-\epsilon^2)},
\end{equation}
so that the quadratic form in (3.4) is non-negative. Then we get
\[-\psi_t + \sum a_{ij} D_{ij} \psi \geq 0, \ \forall (x,t) \in Q.\]
We also notice that
\[\psi(x, -r^2) \leq (er)^{-2q+4}, \ \forall |x| \leq r,\]
and
\begin{equation}
\psi(x, (\alpha - 1)r^2) \geq \frac{9}{16} r^{-2q+4}, \ \forall |x| \leq \frac{r}{2}.
\end{equation}
Finally, we notice that by the monotonicity of $\psi$ with respect to $t \in [-r^2, (\alpha - 1)r^2]$ for $x = 0$, we obtain
\begin{equation}
\psi(0, t) \geq \frac{9}{16} r^{-2q+4}, \ t \in [-r^2, (\alpha - 1)r^2].
\end{equation}
With the help of $\psi$ we just constructed in (3.3), in Lemma 8 we first show that when the drift $b$ is small enough, if we stay away from the lateral boundary of a standard cylinder. Then we have a lower bound for a positive supersolution $u$ ($-u_t + Lu \leq 0$) in a spacial region of the bottom of the cylinder, then we also have a lower bound for $u$ in the same region on the top of the cylinder. Intuitively, it basically tells us $u$ will not decay dramatically in the same spatial region if we stay away from the lateral boundary.

**Lemma 8.** Let $\alpha$ be positive constant and $-u_t + Lu \leq 0$ in $\Omega$. Suppose $Q := B_r(0) \times (-r^2, (\alpha - 1)r^2) \subset \Omega$ and $u > 0$ in $Q$. Then there are positive constants $s_1 := s_1(n, \nu), C_1 := C_1(n, \nu)$ and $k := k(n, \nu, \alpha)$ such that if
\begin{equation}
\tag{3.8}
 u \geq \ell \\
on B_{2\ell}(0) \times \{-r^2\} \text{ and } \left( \|b\|_{L^\infty_t L^n_x(Q)} \right)^n \leq s_1 \leq S,
\end{equation}
then
\begin{equation}
\tag{3.9}
 u \geq C_1 \left( \frac{1}{2} \right)^k \ell \text{ on } B_{2\ell}(0) \times \{(\alpha - 1)r^2\}.
\end{equation}
The above lemma is a special situation of Lemma 7.39 in [GL2] with some mollification.

Proof. We apply the results from the preliminary calculations with case \( \epsilon = \frac{1}{2} \).

Consider
\[
v = u - \ell \left( \frac{1}{2} r \right)^{2q - 4} \psi.
\]

Notice that with the \( q \) from the above calculation, we have
\[
- \psi_t + L\psi \geq b_1 D_1 \psi = -4 \psi_1 \psi_0 \left( b, x \right) \geq -4 |b| r \left( \frac{1}{2} \right)^{-2q} r^{2q - 2q}.
\]

Also it is clear \( \psi = 0 \) for \( |x| = r \). So we can conclude that \( v \geq 0 \) on \( \partial Q \) by the above calculation and \( u \geq \ell \) on the bottom. Finally, we apply Theorem 4 to \(-v\), we obtain
\[
v \geq -N(n, \nu, s, 1) \ell \left( \frac{1}{2} \right)^{-4} s_1^{\frac{1}{11}}
\]
in \( \bar{Q} \). In other words, we have
\[
u \geq \ell \left( \frac{1}{2} r \right)^{2q - 4} \psi - N(n, \nu, s, 1) \ell \left( \frac{1}{2} \right)^{-4} s_1^{\frac{1}{11}}.
\]

By the calculation above again, we conclude that
\[
u \geq \ell \left( \frac{1}{2} \right)^{-4} \left[ \frac{9}{16} \left( \frac{1}{2} \right)^{2q} - N(n, \nu, s_1) s_1^{\frac{1}{11}} \right].
\]

for \( |x| \leq \frac{r}{2} \). Pick \( s_1 := s_1(n, \nu) \) small enough \( (N(n, \nu, s_1) \) is decreasing when \( s_1 \) decays) to force the inequality,
\[
\left[ \frac{9}{16} \left( \frac{1}{2} \right)^{2q} - N(n, \nu, s_1) s_1^{\frac{1}{11}} \right] \geq \frac{1}{2} \left( \frac{1}{2} \right)^{2q}.
\]

We conclude
\[
u \geq C_1 (\frac{1}{2})^k \ell
\]
on \( B_2(0) \times \{ -r^2 \} \), with \( k = 2q - 4 \) and \( C_1 \) does not depend on \( u \). As a byproduct, we can also conclude that
\[
u(0, t) \geq C_1 (\frac{1}{2})^k \ell, \quad \forall t \in [-r^2, (\alpha - 1)r^2].
\]

Next, by iterating Lemma 9 and applying the pigeonhole principle, we show that under the same assumptions on \( u \) as above but without the assumption of the smallness of \( b \), if \( u \) has a lower bound on the bottom of a cylinder, then \( u \) still has a lower bound for later time at least in some small region in space.

Lemma 9. Let \( \alpha \) be a positive constant and \( -u_t + Lu \leq 0 \) in \( \Omega \). Suppose \( Q := B_r(0) \times (-r^2, (\alpha - 1)r^2) \subset \Omega \) and \( u > 0 \) in \( Q \). If \( u \) has a lower bound on the bottom of the cylinder,
\[
u \geq \ell
\]
on \( B_r(0) \times \{ -r^2 \} \), then it has a lower bound on \( B_{\frac{r}{2}}(0) \times \{ (\alpha - 1)r^2 \},
\]
\[
u \geq C(n, \nu, S, \alpha) \ell
\]
for some positive constant $C := C(n, \nu, S, \alpha)$ where $S := \left(\left\|b\right\|_{L^2_t L^\infty_x}\right)^n$. In particular

$$u(0, (\alpha - 1)r^2) \geq C(n, \nu, S, \alpha)\ell.$$

**Proof.** The result can be proved by iterating Lemma $\mathbb{8}$. We divide

$$[B_r(0) \times (-r^2, (\alpha - 1)r^2)] \setminus [B_{\frac{r}{2}}(0) \times (-r^2, (\alpha - 1)r^2)]$$

into $m$ pieces of cylindrical shells where $m = \left[\frac{1}{s_1}\right] + 1$, where $s_1$ satisfies the conditions in Lemma $\mathbb{S}$. If we denote $r_k = \frac{r}{2} + \frac{kr}{m}$ for $k = 1, \ldots, m$, each of those shells is of the form

$$V_k = \left(B_{r_k}(0) \setminus B_{r_{k-1}}(0)\right) \times (-r^2, (\alpha - 1)r^2)$$

where $k = 1, \ldots, m$. Then at least over one of these shells, say, $V_{k_0}$, such that $\left[\|b\|_{L^2_t L^\infty_x(V_{k_0})}\right] \leq s_1$, i.e. the norm of the drift is small over $V_{k_0}$. For any $|y_{k_0}| = \frac{r}{2} + \frac{kr}{m} - \frac{1}{4}r$, we can apply the above lemma iteratively to the cylinders

$$B_{\frac{r}{4m}}(y_{k_0}) \times \left(-r^2, -r^2 + \alpha \left(\frac{1}{4m}\right)^2\right), \ldots,$$

$$B_{\frac{r}{4m}}(y_{k_0}) \times \left(-r^2 + h\alpha \left(\frac{1}{4m}\right)^2, -r^2 + (h + 1)\alpha \left(\frac{1}{4m}\right)^2\right)$$

where $h \in \mathbb{N}$ such that $-r^2 + (h + 1)\left(\frac{1}{4m}\right)^2 = (\alpha - 1)r^2$. In other words, we apply Lemma $\mathbb{S}$ iteratively and put those small cylinders together vertically in one particular cylinder shell. We have $\forall |y_{k_0}| = \frac{r}{2} + \frac{kr}{m} - \frac{1}{4}r$ and $\forall t \in (-r^2, (\alpha - 1)r^2)$, $u(y_{k_0}, t) \geq C'(n, \nu, S, \alpha)\ell$ for some constant $C'$. Finally, with the maximal principle applied to $-u$ and the lower bound on $u$ on the bottom, we conclude that on $B_{\frac{r}{2}}(0) \times \{(\alpha - 1)r^2\}$, for some constant $C(n, \nu, S, \alpha)$

$$u \geq C(n, \nu, S, \alpha)\ell,$$

and

$$u(0, (\alpha - 1)r^2) \geq C(n, \nu, S, \alpha)\ell.$$

\[\square\]

Analogous results to Lemmas $\mathbb{S}$ $\mathbb{9}$ also hold for a slanted cylinder setting. For a fixed point $Y = (y, s) \in \mathbb{R}^{n+1}$ with $s > 0$, and $r > 0$, define the slanted cylinder

$$V_r = V_r(Y) := \left\{X = (x, t) \in \mathbb{R}^{n+1}; \left|x - \frac{t}{s}y\right| < r, 0 < t < s\right\}.$$

**Lemma 10.** Let a function $u \in C^{2,1}(V_r)$ satisfy $-u_t + Lu \leq 0$ in a slanted cylinder $V_r$, which is defined in $(3.23)$ with $Y = (y, s) \in \mathbb{R}^{n+1}$, $s > 0$, $r > 0$, such that

$$K^{-1}r|y| \leq s \leq Kr^2$$

where $K > 0$ is a constant. In addition, suppose $u \geq \ell$ on $D_r := B_r(0) \times \{0\}$. Then

$$u \geq C(n, \nu, S, K)\ell.$$
on $B_{1/2}(y) \times \{s\}$. In particular,

$$u(y, s) \geq C(n, \nu, S, K)\ell. \tag{3.26}$$

**Proof.** This can be proved using the result for the standard cylinder setting with a change of variables. First of all, we notice that $s/r^2 \leq K$ so $\alpha$ in above lemmas is bounded by $K$. We rescale $V_r(Y)$ to $V_1(Y)$ since all the quantities we are considering are scaling invariant. Next we make a change of variables.

We notice that with $k_i := y_i/s$ and $|k_i| \leq K$, then define $w_i = x_i - k_it$ and $z = t$. In this coordinate, the slanted cylinder is transformed to a standard cylinder. The equation with respect to the new coordinate is

$$-u_z + \sum_{ij} a_{ij} D_{w_i} w_j u + \sum_i (b_i + k_i) D_{w_i} u \leq 0. \tag{3.27}$$

Then we apply the stand cylinder result to the equation with respect to coordinate $(w, z)$. we have

$$u(y, s) \geq C(n, \nu, S, K)\ell. \tag{3.29}$$

□

Now the useful slanted cylinder lemma [FS] follows easily from Lemma 10. We can apply Lemma 10 to $1 - u$ after we multiply $u$ by a constant to reduce our problem to the case $1 = \sup_{V_r(Y)} u_+$. We have the following result:

**Lemma 11** (Slanted Cylinder Lemma). Let a function $u \in C^{2,1}(V_r)$ satisfy

$$-u_t + Lu \geq 0 \text{ in } V_r, \text{ which is defined in } (3.23) \text{ with } Y = (y, s) \in \mathbb{R}^{n+1}, $$

$$s > 0, \ r > 0, \text{ such that} \tag{3.28}$$

$$K^{-1}r|y| \leq s \leq Kr^2$$

where $K > 1$ is a constant. In addition, suppose $u \leq 0$ on $D_r := B_r(0) \times \{0\}$. Then

$$u(Y) \leq \beta_2 \sup_{V_r(Y)} u_+ \tag{3.29}$$

with a constant $\beta_2 = \beta_2(\nu, n, K, S) < 1$.

With the above comparison results, we can proceed to our proof of the first growth theorem. We will do the construction in the spirit of [S10] and use the structure of the parabolic maximal principle.

**Theorem 12** (First Growth Theorem). Let a function $u \in C^{2,1}(\overline{Q_r})$ where $r > 0$ and $Q_r = Q_r(Y)$ is a standard cylinder, in $\mathbb{R}^{n+1}$ containing $Y := (y, s)$. Suppose $-u_t + Lu \geq 0 \text{ in } Q_r$, then $\forall \beta_1 \in (0, 1)$, there exists $0 < \mu < 1$ such that if we know

$$|\{u > 0\} \cap Q_r(Y)| \leq \mu|Q_r(Y)|, \tag{3.30}$$

then

$$\mathcal{M}_2(Y) \leq \beta_1 \mathcal{M}_r(Y), \tag{3.31}$$

where

$$\mathcal{M}_r(Y) := \max_{Q_r(Y)} u_+.$$

In addition, we also notice that $\beta_1 \to 0^+$ as $\mu \to 0^+$. Theorem 12
Roughly, in order to prove the first growth theorem, one can use a elliptic type argument similar to the one in [110] to find a certain region where the drift is small. Then we just treat the small drift as a perturbation or an error term in the proof of the case without drift term. With the above comparison results. We can use a slanted cylinder to joint an arbitrary point in the standard cylinder and some portion of the region we found by the elliptic argument. Finally we apply Lemma [10] to the slanted cylinder to have some control of the value of $u$.

Remark 13. First of all, we make some reductions. In our problem, we want to show under some conditions, given $-u_t + Lu \geq 0$ in a cylinder $Q_r(Y)$, and some information about the set $\{ u \leq 0 \}$, we want to show that

$$M_2(Y) \leq \beta_1 M_r(Y).$$

Clearly, in order to derive the above estimate, we only need to consider positive part of $u$. We observe that to obtain the above estimate, it actually suffices to get

$$u(Y) \leq \beta_1 M_r(Y),$$

for some $\beta_1 \in (0, 1)$. Indeed, for an arbitrary point $Z \in Q_\beta(Y)$, we notice $Q_\beta(Z) \subset Q_r(Y)$, we can apply the above estimate (3.32) to $Q_\beta(Z)$ with $Y$ replaced by $Z$ and $r$ replaced by $\beta$ with some measure condition $\mu'$. With respect to the measure condition in the first growth theorem, we also observe that

$$|\{ u > 0 \} \cap Q_\beta(Z) | \leq |\{ u > 0 \} \cap Q_r(Y) | \leq \mu |Q_r| = 2^{n+2} \mu |Q_\beta(Z)|.$$

So we just need to take $\mu = 2^{-n-2} \mu'$ for the measure condition in the first growth theorem.

Remark 14. In the first growth theorem, we point out that when $\mu$ goes to $0$, $\beta_1$ goes to $0$. But actually, it suffices to show that if $\mu$ is small enough, $\beta_1 := \beta_1(n, \nu, S, \mu) < 1$. Indeed, we can apply the above estimate inductively to $Q_{2^{-k}}(Y)$. To illustrate the idea, without loss of generality, we can apply the above estimate to $Z_1 \in Q_{\beta_1}(Y)$ and $Q_{\beta_1}(Z_1)$, we have $u(Z_1) < \beta_1 < 1$, $\forall Z_1 \in Q_{\beta_1}(Y)$. Then we apply the above estimate again to $u/\beta_1$ in all points $Z_2 \in Q_{\beta_1}(Y)$ with $Q_{\beta_1}(Z_2)$ to obtain $u/\beta_1(Z_2) < \beta_1$, i.e., $\forall Z_2 \in Q_{\beta_1}(Y), u(Z_2) < \beta_1^2$ provided $\mu(2) \leq (2^{n+2})^{-2} \mu$ in the first growth lemma. We can do this process inductively. For any $\beta_0 \in (0, 1)$, we can find $m \in \mathbb{N}$, such that $\beta_1^m < \beta_0$, we choose $\mu^{(m)} \leq (2^{n+2})^{-m} \mu$. After we do the above process $m$ times, we conclude that $\forall Z_m \in Q_{\beta_0 m}(Y), u(Z_m) < \beta_1^m < \beta_0$. In particular, we infer that

$$u(Y) \leq \beta_0 M_r(Y).$$

Proof. Since every quantity is scaling invariant, we can assume $r = 1$. And we can multiply $u$ by a constant, so without loss of generality, we can also assume $M_1(Y) = 1$. Also we assume $u(Y) > 0$, otherwise the result is trivial.

Step 1: We show the first growth theorem holds for $\left( \| b \|_{L^q_t L^r_x(Q)} \right)^n \leq s_0 := s_0(\beta_1, n, \nu) < S$, where $s_0$ is small enough. Consider

$$v(X) = v(x, t) = u(x, t) + t - s - |x - y|^2$$

in $Q := \{ v > 0 \} \cap Q_1(Y)$. Clearly, $Q \neq \emptyset$ since $v(Y) = u(Y) > 0$ and $Y \in \partial Q_1(Y)$.

It is easy to see that $v \leq u$ in $Q$. By the measure condition, we obtain

$$|Q| \leq |\{ u > 0 \} \cap Q_1(Y) | \leq \mu |Q_1(Y) | \leq \mu.$$
Note that $v \leq 0$ on $\partial_y Q_1(Y)$, so $v = 0$ on $\partial_y Q$. Since $-u_t + Lu \geq 0$, we obtain
\begin{equation}
(-\partial_t + L)v \geq 0 - 1 - 2\text{trace}(a_{ij}) - 2|b| \geq -1 - 2nv^{-1} - 2|b|.
\end{equation}
By the Alexandrov-Bakelman-Pucci-Krylov-Tso estimate,
\begin{equation}
u(Y) \leq N(n, n, S)\left| -1 - 2nv^{-1} - 2|b| \right|_{L_q L^{\infty}(Q)} \leq N_1(n, n, S)\left( \mu^\frac{1}{p} + s_0^\frac{1}{q} \right).
\end{equation}
Fix a $\beta_1 \in (0, 1)$, let $s_1^\frac{1}{q} := \frac{\beta_1}{2N_1}$, then let $s_0 < s_1$ and $\mu < s_1^n$. Then we obtain
\begin{equation}
u(Y) < \beta_1.
\end{equation}

**STEP 2:** We follow the spirit in [S10] to find a region, such that the drift over it is small.

By the above discussion, we can choose $s_1$ from **STEP 1**. Now just for convenience, we translate $Y$ to $(0, 0)$. We divide $[B_1(0) \times (-1, 0)] \setminus [B_\frac{1}{2}(0) \times (-1, 0)]$ into $m$ pieces of cylindrical shells where $m = \lfloor \frac{s_1}{s} \rfloor + 1$. If we denote $r_k = \frac{1}{2} + \frac{k}{2m}$ for $k = 0, 1, \ldots, m$, each of those shells is of the form
\[V_k = (B_{r_k}(0) \setminus B_{r_{k-1}}(0)) \times (-1, 0),\]
where $k = 1, \ldots, m$. Then at least on one of these shells, say, $V_{k_0}$, such that $\|b\|_{L_q L^{\infty}(V_{k_0})} \leq s_1$, i.e. the norm of the drift is small over $V_{k_0}$ is small. For any $|y_{k_0}| \in \left[ \frac{1}{2} + \frac{k_0}{m} - \frac{3}{8} \frac{1}{m}, \frac{1}{2} + \frac{k_0}{m} - \frac{1}{8} \frac{1}{m} \right]$, and $\forall t_0 \in \left[ \frac{1}{1 + \frac{1}{8m}}, 0 \right]$, denote $Y_0 = (y_{k_0}, t_0)$. Then $Q_{\frac{1}{8m}}(Y_0) \subset Q_1(Y)$. Now we take $\mu' \leq \left( \frac{1}{8m} \right)^n \mu$ where $\mu$ is from the calculation in **STEP 1** to get
\begin{equation}
u(Y_0) < \beta_1.
\end{equation}

**STEP 3:** Now we can apply the preliminary comparison results to $v = 1 - u$. From **STEP 2**, we can find a cylinder shell, in which $u \leq \beta_1$. Then we can joint $Y$ and a $n$-dimensional ball in this region with a slanted cylinder. More precisely, from **STEP 2**, we can fix a point $Y^0$ such that $|y^0| = \frac{1}{2} + \frac{k_0}{m} - \frac{1}{8} \frac{1}{m}$ and a ball $B_{\frac{1}{8m}}(y^0)$.

Then we use a slanted cylinder of radius $\frac{1}{1 + \frac{1}{8m}}$ to joint $B_{\frac{1}{8m}}(y^0)$ and $Y$. Now apply Lemma [**11**] to $v = 1 - u$. We notice the quantity $K$ in Lemma [**11**] in this situation is bounded above, independent of $u$, and the upper bound only depends on $m$. Clearly, $v \geq 1 - \beta_1 =: \ell$ on the bottom, then Lemma [**11**] gives
\[v(Y) = 1 - u(Y) \geq C(n, v, S, K)(1 - \beta_1).
\]
So we can conclude that
\begin{equation}
u(Y) \leq 1 - C(n, v, S, K)(1 - \beta_1) =: \beta < 1.
\end{equation}
Finally, by remarks [**13**] and [**14**], we are done.  

**Remark 15.** For other scaling invariant drift cases, i.e.,
\[
\left( \int \left[ \int |b(x, t)|^q \, dt \right]^\frac{q}{p} \, dx \right)^\frac{1}{q} < \infty
\]
for some constants $p, q \geq 1$ such that
\[
\frac{n}{p} + \frac{2}{q} = 1, \quad q < \infty.
\]
The first growth theorem is easier to show. After rescaling and do translation again, we may still assume $r = 1$ and $Y = 0$. Similarly as above, we know when $\beta_1$ is fixed, the first growth theorem holds when the norm of the drift $b$ is small norm, say, $\|b\|_{L^p_t L^q} < s_1$. Similarly as step 2 above, we can find cylinder shell over which the drift is small, and over this region $u(Y^0) < \beta_1$. Next we divide $Q_1(0)$ evenly along $t$ direction to $m^2$ pieces. i.e. each shell is of the form $U_k = B_1(0) \times \left[ -\frac{k}{(8m)^2}, -\frac{k-1}{(8m)^2} \right]$, for $m$ large (say, here $m$ is larger than the number $m = \left\lfloor \frac{S}{\delta_1} \right\rfloor + 1$), we can find at least over one of $U_k$’s, say, $U_j$.

Now for $(y_{j_0}, t_{j_0})$, we can build a parabolic cylinder $Q_{\frac{1}{8}m} (y_{j_0}, t_{j_0})$. Over this cylinder, by step 1, we know $u(y_{j_0}, t_{j_0}) < \beta_1$.

The above estimate holds for $y_{j_0} \in B_1 - \frac{1}{8}m(0)$. Finally, we apply the maximal principle and step 2 above, we have $u(y, s) = u(Y) < \beta_1$.

4. Second Growth Theorem

The slanted cylinder lemma, i.e., Lemma 11 above plays a crucial role in this section to build a connection between different time slides. The second growth theorem helps us control the oscillation between different time slides. We follow the arguments in [FS].

**Theorem 16** (Second Growth Theorem). Let a function $u \in C^{2,1}(Q_r)$, where $Q_r := Q_r(Y)$, $Y := (y, s) \in \mathbb{R}^{n+1}$, $r > 0$, and let $-u_t + Lu \geq 0$ in $Q_r$. In addition, suppose $u \leq 0$ on $D_\rho := B_\rho(z) \times \{\tau\}$, where $B_\rho(z) \subset B_r(y)$ and

\[
s - r^2 \leq \tau \leq s - \frac{1}{4}r^2 - \rho^2.
\]

Then

\[
u(Y) \leq \beta_3 \sup_{Q_r(Y)} u_+
\]

where $\beta_3 := \beta_3(n, \nu, \rho/r, S) < 1$ is a constant.

**Proof.** After rescaling and translation in $\mathbb{R}^{n+1}$, we reduce our problem to $r = 1$, and $(z, \tau) = (0, 0) \in \mathbb{R}^{n+1}$. For an arbitrary point $Y' \in Q_{\frac{1}{2}}(Y)$, we can apply the slanted cylinder lemma to the slanted cylinder $V_\rho(Y') \subset Q_1(Y)$. Note that in this situation, the constant $K$ in slanted cylinder lemma only depends on $\rho$. Therefore, with the parameter $\beta_2$ from the slanted cylinder lemma, we have

\[
u(Y') \leq \beta_2 \sup_{V_\rho(Y')} u_+ \leq \beta_2 \sup_{Q_1(Y)} u_+.
\]

The above estimate holds for all $Y' \in Q_{\frac{1}{2}}(Y)$. Then, in particular, we obtain

\[
\sup_{Q_{\frac{1}{2}}(Y)} u_+ \leq \beta_2 \sup_{Q_1(Y)} u_+.
\]

\[\square\]
Lemma 17. Let a function $v \in C^{2,1}(\overline{Q_r})$ satisfy $v \geq 0$, $-\nu_t + Lv \leq 0$ in $Q_r := Q_r(Y), Y = (y, s) \in \mathbb{R}^{n+1}, r > 0$. For arbitrary disks $D_\rho := B_\rho(z) \times \{r\}$ and $D^0 := B_\rho(y) \times \{\sigma\}$, such that $B_\rho(z) \subset B_r(y)$ and

\begin{equation}
\inf_{D^0} v = \gamma \inf_{D_\rho} v
\end{equation}

where $\gamma = \gamma(n, \nu, h, S)$.

Proof. Without loss of generality, we may assume $m := \inf_{D_\rho} v > 0$, $r = 1$, $z = 0$, $\tau = 0$, $\sigma = s = h^2$. So $D_\rho = B_\rho(0) \times \{0\}$. We can apply an additional linear transformation along $t$-axis, we can also reduce the proof to the case $h = 1$. Now fix the integer $k$ such that $2^{-k-1} < \rho \leq 2^{-k}$, and for $j = 0, 1, \ldots$, and we define $y_j := y^* + 2^{-j}(y - y^*), B_j := B_\rho(j)$, where $y^* := \frac{1}{2}y, Y^j := (y^j, 4^{-j})$, $Q^j := Q_{2^{-j}}(Y^j), D^j := B_{2^{-j}}(y^j) \times \{4^{-j}\}$. By construction, $0 = y^* + \rho(y - y^*)$, so that

\begin{equation}
B_\rho(0), B_j \in \{B_\theta(y^* + \theta(y - y^*)): 0 \leq \theta \leq 1\}.
\end{equation}

Then by the assumption, $B_\rho(0) \subset B_1(y)$ it follows $|y| \leq 1 - \rho, |y - y^*| \leq 1$, and

\begin{equation}
B_{k+1} \subset B_\rho(0) \subset B_k \subset B_{k-1} \subset \ldots \subset B_1 \subset B^0 = B_1(y).
\end{equation}

Apply Theorem 16 to the function $u = 1 - \frac{1}{m}v$ in $Q^k$ with

\begin{equation}
r = 2^{-k}, \rho = 2^{-k-1}, Y = Y^k, z = 0, \tau = 0.
\end{equation}

Then we conclude that

\begin{equation}
\sup_{Q^k} u \leq \sup_{Q_{2^{-k-1}(y^k)}} u \leq \beta_3 \sup_{Q^k} u \leq \beta_3 = \beta_2(n, \nu, S, \frac{1}{2}) < 1,
\end{equation}

which is equivalent to

\begin{equation}
\inf_{D_\rho} v = m \leq (1 - \beta_3)^{-1} \inf_{D^k} v = 2^\gamma \inf_{D^k} v,
\end{equation}

where $\gamma := -\log_2(1 - \beta_3) > 0$. Similarly, if $k \geq 1$, we also have

\begin{equation}
\inf_{D^j} v \leq 2^\gamma \inf_{D^{j-1}} v,
\end{equation}

for $j = 1, 2, \ldots, k$. Finally we obtain

\begin{equation}
\inf_{D_\rho} v \leq 2^\gamma \inf_{D^k} v \leq 2^\gamma \inf_{D^{k-1}} v \leq \ldots \leq 2^{(k+1)^\gamma} \inf_{D^0} v \leq \left(\frac{2r}{\rho}\right)^\gamma \inf_{D^0} v.
\end{equation}

$\square$
5. Third Growth Theorem

The first growth theorem tells us if \( \mu \to 0^+ \) then \( \beta_1 \to 0^+ \). The third growth theorem tells us if we have a nice control of the measure of the set \( \{ u > 0 \} \) near the bottom, then we can have a more precise estimate. In other words, if we have the similar measure condition for

\[
Q^0 := Q_\frac{1}{2}(Y^0), Y^0 = \left(y, s - \frac{3}{4}r^2\right).
\]

Then if \( \mu < 1 \), then \( \beta_1 < 1 \). In order to carry out the third growth theorem, we need a covering lemma. We proceed as in [FS].

**Lemma 18.** Let a constant \( \mu_0 \in (0, 1) \) be a fixed number. For an arbitrary measurable set \( \Gamma \subset \mathbb{R}^{n+1} \) with finite Lebesgue measure \( |\Gamma| \), we introduce the family of cylinders

\[
\mathcal{A} := \{ Q = Q_r(Y) : |Q \cap \Gamma| \geq (1 - \mu_0) |Q| \}.
\]

Then the open set \( E := \bigcup_{Q \in \mathcal{A}} Q \) satisfies,

\[
|\Gamma \setminus E| = 0,
\]

and

\[
|E| \geq q_0 |\Gamma|, \quad q_0 := 1 + 3^{-n-1} \mu_0 > 1.
\]

**Proof.** From the fact that almost every point of \( \Gamma \) is a point of density, we have \( |\Gamma \setminus E| = 0 \). More precisely, suppose \( |\Gamma \setminus E| > 0 \), then we can choose a cylinder \( Q^* := B_\frac{1}{m}(y) \times (s - \frac{1}{m}, s) \), with some \( m \in \mathbb{N} \) such that

\[
|Q^* \cap (\Gamma \setminus E)| \geq (1 - \mu_0) |Q^*|.
\]

Notice that \( Q^* \) is a union of \( m \) disjoint parabolic cylinders

\[
Q_k^* := B_\frac{1}{m}(y) \times \left(s - \frac{k+1}{m^2}, s - \frac{k}{m^2}\right), \quad k = 0, 1, \ldots, m - 1,
\]

therefore the above inequality (5.4) must be true for some \( Q_k^* \) instead of \( Q^* \). Then \( Q_k^* \in \mathcal{A}, Q_k^* \subset E, Q_k^* \cap (\Gamma \setminus E) \) is empty and of course the above inequality cannot be true for \( Q_k^* \). Therefore we have \( |\Gamma \setminus E| = 0 \).

Now, for each \( Q = Q_r(Y) \in \mathcal{A} \) with \( |Q \cap \Gamma| \geq (1 - \mu_0) |Q| \), we continuously increase \( r \) such that we achieve the exact equality, i.e., \( |Q \cap \Gamma| = (1 - \mu_0) |Q| \). Therefore, we can write

\[
E := \cup_{Q \in \mathcal{A}_0} Q, \quad \mathcal{A}_0 := \{ Q = Q_r(Y) : |Q \cap \Gamma| = (1 - \mu_0) |Q| \}.
\]

Next, we follow the well-known argument in the classical Vitali covering lemma with parabolic cylinders instead of balls or cubes. We construct an at most countable sequence of cylinders \( Q^k, k = 1, 2, \ldots \) as follows: we denote

\[
R_1 := \sup \{ r : Q_r(Y) \in \mathcal{A}_0 \}.
\]

By an easy compactness argument gives us that this supremum is obtained for some cylinder \( Q_{R_1}(Y_1) \in \mathcal{A}_0 \). Define \( Q^1 := Q_{R_1}(Y_1) \). Now assume that \( Q^k := Q_{R_i}(Y_i) \) for \( i = 1, 2, \ldots, k \) have been selected for some \( k \geq 1 \), we set

\[
\mathcal{A}_{k+1} := \{ Q = Q_r(Y) \in \mathcal{A}_0 : Q \cap Q^i = \emptyset, i = 1, 2, \ldots, k \}.
\]

If \( \mathcal{A}_{k+1} \) is nonempty, then we denote

\[
R_{k+1} := \sup \{ r : Q_r(Y) \in \mathcal{A}_{k+1} \}.
\]
this supremum is obtained for \( Q^{k+1} := Q_{R_{k+1}}(Y_{k+1}) \in \mathcal{A}_{k+1} \).

In the case when all of the sets \( \mathcal{A}_0 \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \ldots \) are nonempty, we get a countable sequence of cylinder \( Q^i = Q_{R_i}(Y_i) \). If however \( \mathcal{A}_i \neq \emptyset \) for \( i = 1, 2, \ldots, k \) and \( \mathcal{A}_{k+1} = \emptyset \), then we have a finite sequence of cylinders \( Q^i, i = 1, 2, \ldots, k \). In the latter case we set, by definition, \( R_{k+1} = R_{k+1} = \cdots = 0 \).

Clearly, by our construction, the cylinders \( Q^i \) are pairwise disjoint, \( R_1 \geq R_2 \geq \cdots \), and \( R_i \to 0 \) as \( i \to \infty \).

Take an arbitrary cylinder \( Q_r(Y) \in \mathcal{A}_0 \). We have \( R_1 \geq R_2 \geq \cdots \geq R_k \geq r > R_{k+1} \) for some integer \( k \geq 1 \). Since \( r > R^{k+1} \), the cylinder \( Q_r(Y) \) does not belong to \( \mathcal{A}_{k+1} \) and therefore \( Q_r(Y) \cap Q^i \neq \emptyset \) for some \( i \leq k \). Since \( r < R_i \), this implies \( Q_r(Y) \subset \hat{Q}^i \), where

\[
\hat{Q} := B_{3\rho} \times (s - 2\rho^2, s + \rho^2)
\]

for

\[
Q = Q_{\rho}(z, s) = B_{\rho} \times (s - \rho^2, s).
\]

Therefore for arbitrary \( Q_r(Y) \in \mathcal{A}_0 \) is a subset of \( \hat{Q}^i \) for some \( i \geq 1 \). Then we have

\[
E \subset \bigcup_i \hat{Q}^i, \quad |E| \leq \sum_i |\hat{Q}^i| = 3^{n+1} \sum_i |Q^i|.
\]

On the other hand since \( Q^i \in \mathcal{A}_0 \) are pairwise disjoint,

\[
|E \setminus \Gamma| \geq \sum_i |(E \setminus \Gamma) \cap Q^i| = \mu_0 \sum_i |Q^i|.
\]

Then with the above two relations, we obtain

\[
\frac{|E|}{|\Gamma|} = 1 + \frac{|E \setminus \Gamma|}{|\Gamma|} \geq 1 + \frac{|E \setminus \Gamma|}{|E|} \geq 1 + \frac{\mu_0}{3^{n+1}} = q_0 > 1.
\]

Therefore

\[
|E| \geq q_0 |\Gamma|.
\]

\[ \square \]

Now we consider the change of measures after we shift cylinders with respect to certain rule, which will be helpful when we prove the third growth theorem. The lemma below seems trivial but we need to be cautious. The way a cylinder shifted depends on the size and the position of the cylinder, so some originally overlapped cylinders might be disjoint after we shift them, or vice versa.

**Lemma 19.** For a fixed constant \( K_1 > 1 \) and any standard cylinder \( Q = Q_r(Y) = B_{r}(y) \times (s-r^2, s) \), we denote \( \hat{Q} := B_{r}(y) \times (s+r^2, s+K_1r^2) \). Then for an arbitrary family of standard cylinders, the \( n+1 \) dimensional Lebesgue measures of the sets

\[
E := \bigcup_{Q \in \mathcal{A}} Q
\]

and

\[
\hat{E} := \bigcup_{Q \in \mathcal{A}} \hat{Q}
\]

have the following quantitative relation:

\[
|\hat{E}| \geq q_1 |E|, \quad q_1 := \frac{K_1 - 1}{K_1 + 1}.
\]


Proof. By Fubini’s theorem, we obtain

\[ |E| = \int |E_x| \, dx, \quad |\hat{E}| = \int |\hat{E}_x| \, dx \]

where we use the standard notation for \( x \in \mathbb{R}^n \),

\[ E_x := \{ t \in \mathbb{R} : (x,t) \in E \}, \quad \hat{E}_x := \{ t \in \mathbb{R} : (x,t) \in \hat{E} \}. \]

Now we see it suffices to show \( |\hat{E}_x| \geq q_1 |E_x|, \forall x \in \mathbb{R}^n \). Then everything is reduced to one-dimensional topology.

Now fix an \( x \) such that \( E_x \) is not empty. Then for this fixed \( x \) the open set \( \hat{E}_x \) is a union of disjoint open intervals \( \hat{I}_k, k = 1, 2, \ldots \). Here we use the basic fact from 1-D topology about the structure of open sets. If \( t \in E_x \), then \( (x,t) \in Q_r = B_r(y) \times (s-r^2, s) \) for some cylinder \( Q_r \in \mathcal{A} \), and \( (s + r^2, s + K_1 r^2) \subset \hat{I}_k \) for some \( k \). Then clearly, \( r^2 \leq r_k^2 := (K_1 - 1)^{-1} \hat{I}_k \). We can also choose \( s_k \) such that \( \hat{I}_k = (s_k + r_k^2, s_k + K_1 r_k^2) \) by our construction. Then, we observe that

\[ s_k + r_k^2 \leq s + r^2, \quad s_k - r_k^2 \leq s - r^2. \]

The first inequality is trivial, and the second one follows from the fact \( r_k \geq r \) by the construction. Therefore,

\[ t \in (s - r^2, s) \subset J_k := (s_k - r_k^2, s_k + K_1 r_k^2). \]

Hence we get

\[ E_x \subset \bigcup_k J_k, \]

which implies

\[ q_1 |E_x| \leq q_1 |\bigcup_k J_k| \leq q_1 \sum_k |J_k| = \sum_k |\hat{I}_k| = |\hat{E}_x|. \]

So we conclude that

\[ |\hat{E}| \geq q_1 |E|. \]

\[ \square \]

Theorem 20 (Third Growth Theorem). Let a function \( u \in C^{2,1}(\overline{Q_r}) \), where \( Q_r = Q_r(Y), Y = (y,s) \in \mathbb{R}^{n+1}, r > 0 \), and let \(-u_t + Lu \geq 0 \) in \( Q_r \). In addition, we assume

\[ \{u > 0\} \cap Q^0 \leq \mu |Q^0|, \quad \text{(5.8)} \]

where

\[ Q^0 := Q_{\frac{3}{4}r^2}(Y^0), \quad Y^0 = \left(y, s - \frac{3}{4}r^2 \right) \]

and \( \mu < 1 \) is a constant. Then we have

\[ \mathcal{M}_S(Y) \leq \beta \mathcal{M}_r(Y) \quad \text{(5.10)} \]

with a constant

\[ \beta := \beta(n, \nu, S, \mu) < 1. \]
Proof. Without loss of generality, we may rescale and translate our problem so that \( r = 2, Y = (y, s) = (0, 0) \). Now under this setting, \( Q^0 = B_1(0) \times (-4, -3) \) and \( |Q^0| = |B_1(0)| \), which only depends on \( n \).

Now consider \( \Gamma := \{ u \leq 0 \} \cap Q^0 \). Then from the above measure condition, we have

\[
|\Gamma| \geq (1 - \mu) |Q^0| = (1 - \mu) |B_1(0)| =: c_0 = c_0(\mu, n) > 0.
\]

From the first growth theorem, we know the constant \( \beta_1 = \beta_1(n, \nu, S, \mu) = 0^+ \) as \( \mu \to 0^+ \). So we can find a constant \( \mu_0 = \mu_0(n, \nu, S) \in (0, 1) \) such that the first growth theorem holds for a constant \( \beta_1 \leq 1/2 \). With this constant \( \mu_0 \) and \( \Gamma \), we perform the covering lemma, Lemma 18, to obtain a family of cylinders \( A \) defined by the formula in the above lemma. Then by the results from the above lemmas, if we denote \( E := \cup_{Q \in A} Q \), then we obtain

\[
|\Gamma \setminus E| = 0,
\]

and

\[
|E| \geq q_0 |\Gamma|, \quad q_0 := 1 + 3^{-n-1} \mu_0 > 1.
\]

Denote \( c_0 := 3^{-n-2} \mu_0 \), i.e., \( q_0 = 1 + 3c_0 \). Now choose constant \( K_1 > 0 \) such that

\[
q_1 := \frac{K_1 - 1}{K_1 + 1} = \frac{1 + 2c_0}{1 + 3c_0},
\]

i.e.,

\[
K_1 = K_1(n, \nu, S) = 5 + 2c_0^{-1}.
\]

By the above two lemmas, we conclude that

\[
|\hat{E}| \geq q_1 |E| \geq q_0 q_1 |\Gamma| = (1 + 2c_0) |\Gamma|.
\]

In order to have some estimate of the size of the cylinders in the family \( A \), we introduce another cylinder \( Q^1 \) such that

\[
Q^0 \subset Q^1, \quad |Q^1 \setminus Q^0| \leq c_0 c_0, \quad \text{dist} \ (Q^0, \partial Q^1) \geq c_1 = c_1(n, \nu, \mu).
\]

Then there are two possibilities about \( \hat{E} \) and \( Q^1 \). (a) \( \hat{E} \setminus Q^1 \neq \emptyset \) and (b) \( \hat{E} \setminus Q^1 = \emptyset \).

(a) If \( \hat{E} \setminus Q^1 \neq \emptyset \), this can only be true if there are some cylinders \( Q \in A \) which are large enough, i.e., there exists \( Q = Q_r(Y) \in A \) with \( r \geq r_0 = r_0(n, \nu, \mu) > 0 \). Notice that

\[
|\{ u > 0 \} \cap Q| \leq |Q \setminus \Gamma| \leq \mu_0 |Q|,
\]

for \( Q \in A \). By the first growth theorem and the choice of \( \mu_0 \), we have

\[
\sup_{Q_r(Y)} u \leq \frac{1}{2} \sup_{Q_r(Y)} u_+ \leq \frac{1}{2} M,
\]

for \( Q = Q_r(Y) \in A \) where

\[
M := \sup_{Q_r(0)} u_+.
\]

Therefore,

\[
u - \frac{1}{2} M \leq 0
\]
on $D := B_{2}^{s}(Y) \times \{s\}$ for $Q_{s}(Y) \in A$, $Y = (y, s)$. Now we apply the second growth theorem to $u - \frac{1}{2} M$ with $\rho = \frac{1}{2} r_{0}$. By the theorem, we obtain $u - \frac{1}{2} M \in M(\beta_{2}, 0, 2)$ with $\beta_{2} = \beta_{2}(n, \nu, S, \mu) < 1$. Then

$$\sup_{Q_{1}} u = \frac{1}{2} M + \sup_{Q_{1}} \left(u - \frac{1}{2} M\right)$$

$$\leq \frac{1}{2} M + \beta_{2} \sup_{Q_{2}} \left(u - \frac{1}{2} M\right) +$$

$$= \frac{1}{2} \left(1 + \beta_{2}\right) M,$$

and $u \in M(\beta_{0}, 0, 2)$ with $\beta_{0} = \frac{1}{2} (1 + \beta_{2}) < 1$.

(b) If $\hat{E} \setminus Q^{1} = \emptyset$, then $\hat{E} \subset Q^{1}$, and by the measure relations, if we set $\Gamma_{1} := \hat{E} \cap Q^{0}$, then $\Gamma_{1}$ satisfies

$$\sup_{Q_{1}} u = \frac{1}{2} M + \sup_{Q_{1}} \left(u - \frac{1}{2} M\right)$$

$$\leq \frac{1}{2} M + \beta_{2} \sup_{Q_{2}} \left(u - \frac{1}{2} M\right) +$$

$$= \frac{1}{2} \left(1 + \beta_{2}\right) M,$$

and $u \in M(\beta_{0}, 0, 2)$ with $\beta_{0} = \frac{1}{2} (1 + \beta_{2}) < 1$.

From the above argument in case (a), we know that $\forall Q \in A$, we have $u \leq \beta_{0} M$ on $\hat{Q}$ with $\beta_{0} = \beta_{0}(n, \nu, S, \mu) < 1$. Since the sets $\hat{Q}$ cover $\Gamma_{1}$, we know $u \leq \beta_{0} M$ on $\Gamma_{1}$. Therefore,

$$\left|\{u \leq \beta_{0} M\} \cap Q^{0}\right| \geq \left|\hat{E} \setminus Q^{1}\right| \geq (1 + 2 \epsilon_{0}) \left|\Gamma\right| - \epsilon_{0} c_{0} \geq (1 + \epsilon_{0}) \left|\Gamma\right|.$$ 

Now we have proved that either (a) $u \leq \beta_{0} M$ on $Q_{1}$, or in case (b) $u$ satisfies the measure relation $\left|\{u \leq \beta_{0} M\} \cap Q^{0}\right| \geq (1 + \epsilon_{0}) \left|\{u \leq 0\} \cap Q^{0}\right|$. Now we proceed recursively starting from

$$u_{0} := u, \ M_{0} := M,$$

and define

$$u_{k+1} := u_{k} - \beta_{0} M_{k}, \ M_{k+1} := \sup_{Q_{k}} u_{k+1}$$

for $k = 0, 1, 2, \ldots$ Then we can easily derive that $\forall k$

$$u_{k} = u - \left[1 - (1 - \beta_{0})^{k}\right] M, \ M_{k} = (1 - \beta_{0})^{k} M.$$ 

If case (b) holds for all $u_{k}$ with $k = 0, 1, 2, \ldots, m - 1$, then

$$\left|Q^{0}\right| \geq \left|\{u_{m} \leq 0\} \cap Q^{0}\right|$$

$$= \left|\{u_{m-1} \leq \beta_{0} M_{m-1}\} \cap Q^{0}\right|$$

$$\geq (1 + \epsilon_{0}) \left|\{u_{m-1} \leq 0\} \cap Q^{0}\right|$$

$$\geq \cdots$$

$$\geq (1 + \epsilon_{0})^{m} \left|\{u_{0} \leq 0\} \cap Q^{0}\right|$$

$$\geq (1 + \epsilon_{0})^{m} c_{0}.$$ 

If now, we take $m \in \mathbb{N}$ such that $(1 + \epsilon_{0})^{m} c_{0} > \left|Q^{0}\right|$, then (b) fails for some $u_{k}$ with $k \leq m - 1$. Therefore, we have

$$u_{m} \leq u_{k+1} \leq 0,$$

and

$$u \leq [1 - (1 - \beta_{0})^{m}] M.$$
on \(Q_1\). We can conclude that

\[ u \in M(\beta, 0, 2) \]

with

\[
\beta := 1 - (1 - \beta_0)^m < 1.
\]

**Corollary 21.** Let a function \(v \in C^{2,1}(\overline{Q_r})\) be such that \(v \geq 0\), and \(-v_t + L v \leq 0\) in \(Q_r\), and

\[
\{v \geq 1\} \cap Q^0 > (1 - \mu) |Q^0|.
\]

Then

\[
v \geq 1 - \beta > 0
\]
on \(Q_2\), where \(\beta = \beta(n, \nu, \mu, S) < 1\) for \(\mu < 1\).

The corollary basically tells us quantitatively that if \(v\) is large in a large region of a cylinder, then \(v\) is large in half of the cylinder. We can notice the above corollary is much more precise than the intermediate comparison results in section for the first growth theorem.

**Proof.** We can rescale our problem again, and assume \(r = 2\). Then the function \(u = 1 - v\) satisfies \(-u_t + Lu \geq 0\) in \(Q_2\), and

\[
\{u > 0\} \cap Q^0 = \{v < 1\} \cap Q^0 = |Q^0| - \{v \geq 1\} \cap Q^0 \leq |Q^0| - (1 - \mu) |Q^0| \leq \mu |Q^0|.
\]

Then we can apply the third growth theorem to \(u\), we obtain

\[
\sup_{Q_1} u \leq \beta \sup_{Q_2(X_0)} u_+ \leq \beta,
\]
and

\[
\inf_{Q_1} v = 1 - \sup_{Q_1} u \geq 1 - \beta.
\]

\[\square\]

6. **Interior Harnack Inequality**

With the first growth theorem, we can do the following useful argument, which is helpful for us to find a non-degenerate point to build a bridge between two regions we are interested in. Without loss of generality, we still assume \(r = 1\), for \(X \in Q_1(Y)\), we define

\[
d(X) := \sup \{\rho > 0 : Q_\rho(X) \subset Q_1(Y)\}.
\]

Roughly here \(d\) plays roles of weights with which we can make sure the point we are interested in is not degenerate, i.e., it is in the interior of the cylinder. For \(\gamma > 0\), we consider \(d^\gamma u(X)\) instead of \(u(X)\). \(d^\gamma u(X)\) is a continuous function in \(Q_1(Y)\). Clearly, \(d(Y) = 1\), we have

\[
u(Y) = d^\gamma u(Y) \leq M := \sup_{Q_1(Y)} d^\gamma u.
\]

By our construction, \(d^\gamma u\) vanishes on \(\partial_\rho Q_1\), so \(\exists X_0 \in Q_1(Y) \setminus \partial_\rho Q_1\) such that

\[
M = d^\gamma u(X_0).
\]

**Proof.** We can rescale our problem again, and assume \(r = 2\). Then the function \(u = 1 - v\) satisfies \(-u_t + Lu \geq 0\) in \(Q_2\), and

\[
\{u > 0\} \cap Q^0 = \{v < 1\} \cap Q^0 = |Q^0| - \{v \geq 1\} \cap Q^0 \leq |Q^0| - (1 - \mu) |Q^0| \leq \mu |Q^0|.
\]

Then we can apply the third growth theorem to \(u\), we obtain

\[
\sup_{Q_1} u \leq \beta \sup_{Q_2(X_0)} u_+ \leq \beta,
\]
and

\[
\inf_{Q_1} v = 1 - \sup_{Q_1} u \geq 1 - \beta.
\]

\[\square\]
Let \( r_0 := \frac{1}{2} d(X_0) \), we consider the intermediate region \( Q_{r_0}(X_0) \). In this region, we have

\[
\forall X \in Q_{r_0}(X_0), \ d(X) \geq r_0.
\]

Therefore, we can conclude that

\[
\text{(6.4)} \quad \sup_{Q_{r_0}(X_0)} u \leq r_0^{-\gamma} \sup_{Q_{r_0}(X_0)} d'u \leq r_0^{-\gamma} M \leq 2^\gamma u(X_0).
\]

Now, we define \( v = u - \frac{1}{2} u(X_0) \), then

\[
v(X_0) = \frac{1}{2} u(X_0) \geq 2^{-1-\gamma} \sup_{Q_{r_0}(X_0)} u > 2^{-1-\gamma} \sup_{Q_{r_0}(X_0)} v.
\]

From Theorem \[12\] we know \( \exists \mu(n, \nu, \gamma, S) \in (0, 1) \) such that the first growth theorem holds with \( \beta_1 = 2^{-1-\gamma} \). Now the above inequality tells us that \( v \) does not satisfy the measure condition in the first growth theorem. So

\[
\text{(6.5)} \quad \left| \{ v > 0 \} \cap Q_{r_0}(X_0) \right| = \left| \left\{ u > \frac{1}{2} u(X_0) \right\} \cap Q_{r_0}(X_0) \right| > \mu \left| Q_{r_0}(X_0) \right|.
\]

With the above preparation, we are ready to prove the interior Harnack inequality.

**Theorem (Interior Harnack Inequality).** Suppose \( u \in C^{2,1}(Q_{r_0}(Y)) \cap C(Q_{r_0}(Y)) \) and \(-u_t + Lu = 0\) in \( Q_{r_0}(Y) \), \( Y = (y, s) \in \mathbb{R}^{n+1} \) and \( r > 0 \). If \( u \geq 0 \), then

\[
\text{(6.6)} \quad \sup_{Q^0} u \leq N \inf_{Q^0} u,
\]

where \( N = N(n, \nu, S) \) and \( Q^0 = B_r(y) \times (s - 3r^2, s - 2r^2) \).

We will build a non-degenerate intermediate region to get a quantitative relation between two regions we are interested in with the help of three growth theorems.

**Proof.** After rescaling and translating as necessary, we can assume \( Y = 0 \) and \( r = 1 \). Now \( Q_1 = B_1(0) \times (-1, 0), Q^0 = B_1(0) \times (-3, -2) \). It is easy to see that if we define \( d(X) := \sup \{ \rho > 0 : Q\rho(X) \subset Q_2(0) \} \), then \( d(X) \geq 1 \) in \( Q^0 \). Hence, if we consider \( Q^1 := B_2(0) \times (-3, -2) \) we have

\[
\text{(6.7)} \quad \sup_{Q^1} u \leq M := \sup_{Q^0} d'u,
\]

where \( \gamma \) is chosen at the same as the \( \gamma \) in Lemma \[17\] with \( h = \frac{1}{2} \). From the above discussion, we can find \( \exists X_0 \in Q^1 \setminus \left[ \partial_p Q^1 \cap \partial_p Q_2 \right] \) such that

\[
\text{(6.8)} \quad d'u(X_0) = M.
\]

Similarly as above, we define

\[
\rho = \frac{4}{3} d(X_0) \in (0, \frac{1}{2}),
\]

and

\[
\text{(6.9)} \quad Q_0 = Q_\rho(X_0) \cap \left( u > \frac{1}{2} u(X_0) \right).
\]

By the above discussion, we conclude that

\[
\left| Q_0 \right| > \mu_1 \left| Q_\rho(X_0) \right|
\]
for some constant $\mu_1 = \mu_1(n, \nu, S, \gamma) > 0$. Now we apply Corollary \ref{corollary:1} with
\[
v = \frac{2}{u(X_0)}u, \quad Q_r = Q_{2\rho}(Y_0), \quad Y_0 = (x_0, t_0 + 3\rho^2), \quad Q^0 = Q_{\rho}(X_0), \quad 1 - \mu = \mu_1.
\]
Then we have
\[
\tag{6.11}
u \geq \beta u(X_0)
\]
on $Q_\rho(Y_0)$ with $\beta = \beta(n, \nu, S) > 0$. Next we apply Lemma \ref{lemma:17} with
\[
v = u, \quad r = 2, \quad D_\rho = B_\rho(x_0) \times \{t_0 + 2\rho^2\} \subset Q_{\rho}(Y_0),
\]
and
\[
D^0 = B_1(0) \times \{\tau\}, \quad \forall \tau \in (-1, 0).
\]
So we obtain
\[
\tag{6.12}\beta u(X_0) \leq \inf_{D_\rho} u \leq \left(\frac{4}{\rho}\right)^\gamma \inf_{Q_1(0)} u.
\]
Finally, with the help of the intermediate region, we conclude that
\[
\tag{6.13}\sup_{Q^0} u \leq M = d^\gamma u(X_0) = (4\rho)^\gamma u(X_0) \leq \beta^{-1}4^{2\gamma} \inf_{Q_1(0)} u.
\]
Taking $N = N(n, \nu, S) = \beta^{-1}4^{2\gamma}$ gives the desired result. \hfill \Box

It is well-known that it is easy to derive the Hölder continuity of solutions from the Harnack inequality by standard oscillation and iteration arguments.

**Theorem 22.** If $u \in W(Q_r)$ and is a solution of $-u_t + Lu = 0$ in $Q_r$, then $u$ is Hölder continuous in $Q^r_\frac{1}{2}$.

7. Approximation

In all the proofs from above sections, we always assume $u$ is $C^{2,1}$. In this section, we briefly show we can use an approximation argument to show that all results hold for $u \in W(Q_{2r}) = W_{n,2,1}^1(Q_{2r}) \cap C(Q_{2r})$, where $u \in W_{n,2,1}^1(Q_{2r})$ means $u_t, D_iu, D_{ij}u \in (L^2_{x}, L^\infty_{t})_{loc}$. Throughout, we assume
\[
\tag{7.1} u \geq 0, \quad -u_t + Lu = -u_t + \sum_{ij} a_{ij} D_{ij}u + \sum_i b_i D_iu = 0
\]
in $Q_{2r}$. We can approximate $a_{ij}, b_i$ and $u$ by smooth functions $a^\epsilon_{ij} \to a_{ij}, b^\epsilon_i \to b_i$ a.e. as $\epsilon \to 0^+$. And $u^\epsilon \to u$ in $W_{n,2,1}^1$ as $\epsilon \to 0^+$. Then
\[
\tag{7.2} f^\epsilon = -u^\epsilon_t + L^\epsilon u^\epsilon = -u^\epsilon_t + \sum_{ij} a^\epsilon_{ij} D_{ij}u^\epsilon + \sum_i b^\epsilon_i D_iu^\epsilon \to 0
\]
in $(L^2_{x}, L^\infty_{t})_{loc}(Q_{2r})$ as $\epsilon \to 0^+$. With the existence of solution to for equations with smooth coefficients, therefore we can write
\[
u^\epsilon = v^\epsilon + w^\epsilon,
\]
where
\[
-v^\epsilon_t + L^\epsilon v^\epsilon = 0
\]
in $Q_{2r}$ and
\[
u^\epsilon = u^\epsilon
\]
on $\partial_\rho Q_{2r}$:
\[
-w^\epsilon_t + L^\epsilon w^\epsilon = f^\epsilon,
\]
\[ w^\varepsilon = 0 \]
on \partial \bar{Q}_{2r}. By the Alexandrov-Bakelman-Pucci-Krylov-Tso estimate, we know \( w^\varepsilon \to 0 \) in \( L^\infty \) and \( w^\varepsilon \) satisfies the Harnack inequality. Finally, by an easy limiting argument, \( u \) also satisfies the Harnack inequality.

**Remark 23.** For other values for \( p, q \) for \( q < \infty \), the approximation argument for \( u \in W^{2,1}_{p,q}(Q_{2r}) \cap C(Q_{2r}) \) is similar but actually easier from standard results about \( L^p \) spaces.

### 8. Applications

This section, we show some applications of the interior Harnack inequality. We will just formulate some results based on the interior Harnack inequality. In particular the boundary Harnack inequality, the boundary backward Harnack inequality, and the Hölder continuity of quotients. The detailed proofs are provided \([FSY]\). And one can find more details on applications of the interior Harnack inequality in \([FSY, S08]\). We start with some additional basic notations in order to formulate our results.

For \( X = (x, t) \in \mathbb{R}^{n+1} \) and \( r > 0 \), a standard cylinder is defined as
\[
Q_r(X) = Q_r(x, t) = B_r(x) \times (t - r^2, t),
\]
where \( B_r(x) = \{ y \in \mathbb{R}^n, |y - x| < r \} \). For a constant \( \delta > 0, \Omega \subseteq \mathbb{R}^n \), \( Q_\Omega := \Omega \times (0, T) \), we define
\[
\Omega^\delta = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \delta \} = \{ x \in \Omega : \bar{B}_\delta(x) \subset \Omega \},
\]
\[
Q^\delta_\Omega = \Omega^\delta \times (\delta^2, T) = \{ X \in Q_\Omega : \bar{B}_\delta(X) \subset Q_\Omega \}.
\]

We assume \( \Omega \) to be a bounded Lipschitz domain in \( \mathbb{R}^n \). By a Lipschitz domain, we mean there are positive constants \( r_\Omega \) and \( m_\Omega \) such that \( \forall y \in \partial \Omega \), we can find an orthonormal frame centered at \( y \), in which we have
\[
\Omega \cap B_{r_\Omega}(y) = \{ x = (x', x_n) : x' \in \mathbb{R}^{n-1}, x_n > \phi(x'), |x| < r_\Omega \},
\]
and
\[
\| \nabla \phi \|_{L^\infty} \leq m_\Omega.
\]

Also in such coordinates, \( y \in \partial \Omega \) is represented as \( (0, 0) \) and \( (0, r) \in \Omega \) for all \( r \in (0, r_\Omega) \). For \( Q_\Omega = \Omega \times (0, T) \), \( Y = (y, s) = (0, 0, s) \in \partial_x Q_\Omega = \partial_x \Omega \times (0, T) \), and for \( r > 0 \), we denote
\[
Y_r = (0, r, s + r^2), \quad Y_{r'} = (0, r, s - 2r^2).
\]
For fixed positive constants \( r_0, R_0 \) and \( M_0 \), we assume our domain \( \Omega \) satisfies
\[
r_\Omega \geq r_0, \quad m_\Omega \leq M_0, \quad \text{diam}(\Omega) \leq R_0.
\]
In the most general form, we can state the interior Harnack principle as the following:

**Theorem 24.** (Harnack Principle). Suppose \( u \in W \) in the sense of Definition ?? and \( u \geq 0 \) satisfy \( -u_t + Lu = 0 \) in a bounded domain \( Q_\Omega = \Omega \times (0, T) \), constant \( \delta > 0 \) such that \( \Omega^\delta \) is connected set, \( X = (x, t) \), \( Y = (y, s) \in Q^\delta_\Omega \), and \( s - t \geq \delta^2 \). Then
\[
u(X) \leq Nu(Y),
\]
where the constant \( N = N(n, \nu, S, R_0, T, \delta) \).
Now we can state the theorem on a boundary Harnack inequality. This kind of inequality is also called Carleson type inequality.

**Theorem 25.** Let \( Y = (y, s) \in \partial \Omega \cap Q \) and \( 0 < r < \frac{1}{2} \min \{ r_0, \sqrt{T - s}, \sqrt{s} \} \) be fixed. Then for any non-negative solution \(-u_t + Lu = 0\) in \( Q \Omega = \Omega \times (0, T)\), which continuously vanishes on \( \Gamma = \partial_\Omega Q \cap Q_{2r}(Y) \), we have

\[
M_0 = \sup_{Q \Omega \cap Q_{2r}(Y)} d_0 u \leq Nr^\gamma u(Y),
\]

where

\[
d_0 = d_0(X) := \sup \{ \rho > 0 : Q_\rho(X) \subset Q_{2\rho}(Y) \},
\]

and \( \gamma \) and \( N \) are positive constants depending only on \( n, \nu, S \) and \( m_\Omega \). In particular

\[
\sup_{Q \Omega \cap Q_{2r}(Y)} u \leq Nu(Y).
\]

Again, one can find the detailed proof in [FSY]. With our growth theorems and interior Harnack inequality, the remaining steps in the proof are more or less independent of the specific structure of the equations.

We also state a elliptic-type Harnack inequality.

**Theorem 26.** Let \( u \) be a non-negative solution \(-u_t + Lu = 0\) in \( Q \Omega = \Omega \times (0, T)\) which continuously vanishes on \( \partial \Omega Q \Omega \), and let \( 0 < \delta \leq \frac{1}{2} \min \{ r_0, \sqrt{T} \} \). Then there exists a positive constant \( N = N(n, \nu, S, m_\Omega, r_0, R_0, T, \delta) \), such that

\[
\sup_{Q \Omega} u \leq N \inf_{Q \Omega} u.
\]

Next, the boundary backward Harnack inequality is formulated as follows:

**Theorem 27.** Let \( u \) be a non-negative solution \(-u_t + Lu = 0\) in \( Q \Omega = \Omega \times (0, T)\) which continuously vanishes on \( \partial \Omega Q \Omega \), and let \( \delta > 0 \) be a constant. Then there exists a positive constant \( N = N(n, \nu, S, M_\Omega, r_0, R_0, T, \delta) \), such that

\[
u(x, s) \leq Nu(x, t)
\]

where \( T > s \geq t \geq s - d^2 \geq \delta^2 > 0 \), \( d = \text{dist}(x, \partial \Omega) \).

Again interested readers can find details in [FSY].

Finally, we state a result related to the Hölder continuity of quotients.

**Theorem 28.** Let \( u \) and \( v \) be strictly positive solutions \(-u_t + Lu = 0\) in \( Q \Omega = \Omega \times (0, \infty)\), vanishing on \( Q_{2r}(Y_0) \cap \partial_\Omega Q \Omega \), where \( Y_0 = (y_0, s_0) \in \partial_\Omega Q \Omega = \partial \Omega \times (0, \infty) \) and \( s_0 \geq 4r^2 > 0 \). Then \( \frac{u}{v} \) is Hölder continuous in \( Q \Omega \cap Q_r(Y_0) \).

8.1. **A universal spectral gap for the elliptic problem.** Given the ellipticity condition and estimates on the coefficients, there is a universal gap in the spectrum of the operator \( L = \sum_{ij} a_{ij} D_{ij} + \sum b_i D_i \) between the principal eigenvalue and the rest of the eigenvalues. We first list two results about the Harnack principle for quotients of solutions which are helpful to show the desired spectral gap. As with the applications of interior Harnack inequality above, the proofs are more or less independent of the specific structure of the equations, so we will omit them. The detailed proofs are presented in [HPS1], [HPS2] and [FSY]. We will proceed as in [HPS1].

We consider the following problem for a linear parabolic equation.
Theorem 29. Let \( u_1 \) and \( u_2 \) be two real solutions of the above problem (8.12) and let \( u_1 > 0 \) in \( Q_\Omega := \Omega \times (s, \infty) \). Then
\[
\sup_{Q_\Omega} \frac{u_2}{u_1} = M(s) := \sup_{\Omega \times \{s\}} \frac{u_2}{u_1},
\]
and
\[
\inf_{Q_\Omega} \frac{u_2}{u_1} = m(s) := \inf_{\Omega \times \{s\}} \frac{u_2}{u_1}.
\]

Theorem 30. Let \( u_1 \) and \( u_2 \) be two positive solutions of the above problem (8.12) in the cylinder \( Q_\Omega := \Omega \times (s, \infty) \), and let \( M \) and \( m \) be defined as in the above corollary, then
\[
M(t) \leq N_1 m(t)
\]
for 
\[
t \geq s + 1
\]
with a constant \( N_1 \) depends on \( n, \nu, S, r_0, R_0 \) and \( M_0 \).

Recall the oscillation of a real function can be defined as
\[
osc_{\Omega} f := \sup_{x,y \in \Omega} |f(x) - f(y)| = \sup_{\Omega} f - \inf_{\Omega} f.
\]
For a complex function, we can also define the oscillation, it can be formulated as following:
\[
osc_{\Omega} f := \sup_{x,y \in \Omega} |f(x) - f(y)| = \sup_{0 \leq \phi \leq 2\pi} osc_{\Omega} \Re(e^{i\phi} f).
\]

Proposition 31. \( u_1 \) and \( u_2 \) be two real solutions of the above problem (8.12) in the cylinder \( Q_\Omega := \Omega \times (s, \infty) \), and \( u_1 > 0 \) in \( Q_\Omega \) but \( u \) is allowed to be complex valued. Then
\[
\omega(t) := osc_{\Omega \times \{t\}} \frac{u}{u_1} \leq \omega(t)
\]
for \( t \geq s \), and
\[
\omega(t) \leq \theta_0 \omega(s)
\]
for \( t \geq s + 1 \), where \( \theta_0 := 1 - N_1^{-1} \in (0, 1) \) where \( N_1 \geq 1 \) is from the above theorem [7].

Now we consider the operator \( L = \sum_{ij} a_{ij} D_{ij} + \sum b_i D_i \) with coefficients independent of time. We will show the above results will give us the existence of a gap in the spectrum of \( L \) that only depends on constants \( n, \nu, S, r_0, R_0 \) and \( M_0 \) but not on \( L \) itself. we will call this gap a universal gap. More precisely, we consider the following eigenvalue problem:
\[
-Lv = \lambda v
\]
in $\Omega$, and 

$$v = 0$$

on $\partial \Omega$. The principal eigenvalue $\lambda_1$ is defined as the eigenvalue with the smallest real part. It is well-known actually $\lambda_1$ is real, algebraically simple, and the associated eigenfunction $v_1$ can be chosen positive. No other eigenvalue has a positive eigenfunction and we also have $\Re(\lambda) > \lambda_1$ for any other eigenvalue $\lambda$. One can find details in [BHV].

**Theorem 32.** Let $\lambda_1$ be the principal eigenvalue of the above eigenvalue problem and let $\lambda$ be any other eigenvalue of it. Then

$$\Re(\lambda) - \lambda_1 \geq \gamma > 0,$$

where $\gamma$ is a constant only depending on constants $n, \nu, S, r_0, R_0$ and $M_0$.

**Proof.** First of all, we notice that if $v(x)$ is an eigenfunction of the eigenvalue problem associated to an eigenvalue $\lambda$, then $u(x, t) := e^{-\lambda t}v(x)$ is a solution to the parabolic problem with $f = 0$. Now when $\lambda = \lambda_1$, $v = v_1$, then the function $u_1(x, t) := e^{-\lambda_1 t}v_1(x)$ is a positive solution of the parabolic problem on $\Omega \times I$. For $\lambda \neq \lambda_1$, clearly, $v$ is not a constant multiple of $v_1$ so

$$\omega(t) := \operatorname{osc}_{\Omega \times I(t)} \frac{u}{u_1} = e^{(\lambda_1 - \Re(\lambda))t} \omega(0),$$

where $\omega(0) = \operatorname{osc}_{\Omega} \frac{u}{u_1} \neq 0$. From [HPS2], we also know $\omega(0) < \infty$. Now applying the result from the above Proposition 31 we conclude that

$$\omega(1) := e^{\lambda_1 - \Re(\lambda)} \omega(0) \leq \theta_0 \omega(0),$$

therefore

$$\Re(\lambda) - \lambda_1 \geq c_0 := -\ln \theta_0 > 0.$$  

We notice that $c_0$ only depends on the prescribed constants. \hfill \square

## 9. Appendix

In the section 2, we briefly discussed a version of Alexandrov-Bakelman-Pucci-Krylov-Tso estimate which plays an important role in this paper. In this appendix, we prove the version of Alexandrov-Bakelman-Pucci-Krylov-Tso estimate we used. For more general cases, one can find details in [AIN]. Again, we use the notations $u \in W^{2,1}_{n,\infty}(Q)$ which means $u_t, D_iu, D_{ij}u \in (L^p_{\infty}L^\infty_{\nu})_{t\nu}$. Also we assume $S := \int |\sup_{Q_{\Omega}} |b(x, t)|| dx < \infty$.

We will start with the associated version without drift. Consider

$$-u_t + \sum_{ij} a_{ij} D_{ij}u \geq f$$

where $f \in L^p_{t,\infty}L^\infty_{\nu}$.

In the following arguments, we will assume $u \in C^{2,1}$ instead of $u \in W^{2,1}_{n,\infty}$, but the results hold for $u \in W^{2,1}_{n,\infty}$ by standard approximation arguments as [KT].

**Lemma 33.** Let $u \in C^{2,1}(Q_{\Omega})$ and suppose $Q_{\Omega} = \Omega \times (0, T)$ with the diameter of $\Omega$ is $r$. Also assume $-u_t + \sum_{ij} a_{ij} D_{ij}u \geq f$ and $\sup_{\partial_{Q_{\Omega}}} u \leq 0$. Then

$$\sup_{Q_{\Omega}} u \leq N(\nu, r) \| f \|_{L^p_{t,\infty}L^\infty_{\nu}},$$

where $f_-$ denotes the negative part of $f$. 

Proof. First of all, we notice that it suffices to consider the positive part of the function \( u \). So without loss of generality, we might assume \( u = 0 \) on \( \partial_p Q_\Omega \). Following [KT] [LE1], we might also assume for some \((x_0, \tau) \in \Omega\) with \( 0 < \tau \leq T \) such that \( M = u(x_0, \tau) = \sup_{\Omega} u \). From the proof of Proposition 2.1 in [KT], we obtain the following estimate,

\[
(u(x_0, \tau))^{n+1} \leq C r^n \int_{A_u} \left| \det (D_{ij} u(x, t)) u_t(x, t) \right| \, dxdt
\]

where \( A_u = \{(x, t) \in \partial \Omega \times [0, \tau) : \exists \xi \in \mathbb{R}^n, \text{s.t. } u(y, s) \leq u(x, t) + \xi(y-x) \, \forall y \in \partial \Omega, \, 0 \leq s \leq \tau \} \) and \( C \) only depends on \( n \). Also by the discussion in [KT], for \((x, t) \in A_u\), \((D_{ij} u(x, t))\) is nonpositive and \( u_t(x, t) \geq 0 \). We have \( \sum_{ij} a_{ij} D_{ij} u \geq f \), i.e.,

\[
-\sum_{ij} a_{ij} D_{ij} u \leq f
\]

on \( A_u \). Therefore,

\[
|\det (D_{ij} u(x, t))| = -\det (D_{ij} u(x, t)) \leq C(n) \left(-\sum_{ij} a_{ij} D_{ij} u \right)^n \leq |f|^{n}
\]

for all \((x, t) \in A_u\). Hence

\[
u \leq \int_{A_u} \left| f \right|^n u_t(x, t) \, dxdt \leq C r^n \int_{A_u} \left( \sup_t |f|^{n} \right) u_t(x, t) \, dxdt.
\]

We project \( A_u \) onto \( \Omega \), and we denote the projected area in \( \Omega \) as \( P_u \). Let

\[
u(x) = \left\{ t \in [0, \tau[ : (x, t) \in A_u \right\}.
\]

Then one can write

\[
u \leq \int_{P_u} \left( \sup_t |f|^{n} \right) u_t(x, t) \, dxdt.
\]

Let \( y = x - h \xi \), where \( h \geq d(x) \) is a positive number so that \( y \) on the boundary of \( \Omega \). By our condition on the boundary, we have \( u(y, t) = 0 \). Now by the definition of \( A_u \),

\[
u \leq u(x, t) - h |\xi|.
\]

Hence if \((x, t) \in A_u \) with \( \xi \) in the definition \( A_u \), we obtain \( |\xi| \leq \frac{u(x, t)}{d(x)} \) where \( d(x) \) denotes the distance from \( x \) to \( \partial \Omega \).

In order to analyze the time integration, we must understand the topology of \( I_u(x) \). From above discussion, given the condition \( |\xi| \leq \frac{u(x, t)}{d(x)} \), suppose we pick \( t_{ij} \in I_u(x) \) with associated \( \xi_{ij} \). Then \( \{t_{ij}\} \) is bounded and \( \{||\xi_{ij}|\} \) is also bounded, so we can pick a subsequence \( \{t_{ij}\} \) with \( \xi_{ik} \) so that \( t_{ik} \to t_0 \) and \( \xi_{ik} \to \xi_0 \). From the definition of \( t_{ij} \) and \( \xi_{ij} \), we can conclude \( t_{0} \in I_u(x) \) since \( \xi_0 \) satisfies the condition in the definition of \( A_u \). So we can also conclude that \( I_u(x) \) is compact and is relatively closed to \([0, \tau[\) for all \( x \in P_u \).

By the basic 1-dimensional topology, we know we can write \([0, \tau[\backslash I_u(x)\) as a disjoint union of finite intervals \( I_j \), and each of them is one of the following four
forms: 

\[ [0, \alpha), (\beta, \gamma), (\delta, \tau] \]

with \( \alpha \leq \beta \leq \gamma \leq \tau \). Notice that by the definition of \( A_u \) and \( I(x) \),

\[
\int_{[0, \alpha)} u_t(x, t) \, dt = u(x, \alpha) - u(x, 0) \geq 0
\]

since \( \alpha \in I_u(x) \).

\[
\int_{(\beta, \gamma)} u_t(x, t) \, dt = u(x, \gamma) - u(x, \beta) \geq 0
\]

since \( \gamma \in I_u(x) \). For each \( x \in P \), only at most one of the intervals in the decomposition of \( [0, \tau] \triangle I(x) \) is of the form \( (\delta, \tau] \). So

\[
\int_{I_u(x)} u_t(x, t) \, dt \leq u(x, \tau) - u(x, \delta).
\]

Therefore,

\[
\int_{I_u(x)} u_t(x, t) \, dt \leq u(x_0, \tau),
\]

which implies

\[
u(t) \leq C \nu(t) \sup_{\Omega} u_0 \leq C r \| f \|_{L^\infty_{t}}.
\]

\( \Box \)

**Theorem 34.** (Alexandrov-Bakelman-Pucci-Krylov-Tso estimate) Let \( u \in C^{2,1}(Q_\Omega) \) and suppose \( Q_\Omega = \Omega \times (0, T) \) with the diameter of \( \Omega \) is \( r \). Also assume \(-u_t + Lu \geq f\) and \( \sup_{\partial \Omega} u \leq 0 \). Then

\[
\sup_{Q_\Omega} u \leq N(\nu, n, S) r \| f \|_{L^\infty_{t}}.
\]

**Proof.** Again as above we assume for some \( (x_0, \tau) \in \Omega \) with \( 0 < \tau \leq T \) such that

\[
M = u(x_0, \tau) = \sup_{\Omega} u > 0.
\]

Given \(-u_t + Lu \geq f\) with drift, we move the drift to the right hand side. Then by Cauchy-Schwarz inequality and Hölder’s inequality, we obtain for a fixed constant \( \mu \neq 0 \) to be determined later

\[
u_t - \sum_{ij} a_{ij} D_{ij} u \leq f_+ + \sum_i b_i D_i u
\]

\[
\leq (\bar{b}, \mu^{-1} f_+) \cdot (|\nabla u|, \mu)
\]

\[
\leq \left( \sup_{t} b_+ \right)^n + \left( \mu^{-1} \sup_{t} f_- \right)^n \left( |\nabla u|^n + \nu \mu^n \right)^{\frac{1}{n}} (1 + 1)^{\frac{n}{2^n}}
\]

We consider

\[
D = \{ (\xi, h) : |\xi| \leq M/r, r |\xi| < h < M \}
\]
following the proof of Proposition 2.1 in [KT]. We know if \( g \in C(\mathbb{R}^{n+1}) \) is nonnegative, we have

\[
\int_D g(\xi, h) \, d\xi dh \leq \int_{A_n} g(\nabla u, u_t) \left| \det (D_{ij} u(x, t) u_t(x, t)) \right| \, dx dt.
\]

Take

\[
g(\xi, h) = (|\xi|^n + \mu^n)^{-1}.
\]

Then the left hand side of (9.17) is

\[
\hat{D} g(\xi, h) \, d\xi dh = C \hat{M}/r (M - kr) k^{n-1} (k^n + \mu^n)^{-1} \, dk.
\]

For the right hand side of (9.17), by a similar argument as the above theorem, we can conclude

\[
\int_{A_n} |f_-|^n u_t(x, t) \, dx dt \leq C \int_{A_n} \left( \sup_t |f_-|^n \right) u_t(x, t) \, dx dt.
\]

So we need to calculate

\[
\int_0^{M/r} (M - kr) k^{n-1} (k^n + u^n)^{-1} \, dk.
\]

Notice that

\[
(kr) k^{n-1} (k^n + u^n)^{-1} \leq r,
\]

\[
\int_0^{M/r} M k^{n-1} (k^n + \mu^n)^{-1} \, dk = M \log \left( \left( \frac{M}{r \mu} \right)^n + 1 \right).
\]

So we can bound the left hand side of equation (9.21) from below,

\[
M \log \left( \left( \frac{M}{r \mu} \right)^n + 1 \right) - M \leq \int_0^{M/r} (M - kr) k^{n-1} (k^n + u^n)^{-1} \, dk.
\]

Therefore,

\[
M \log \left( \left( \frac{M}{r \mu} \right)^n + 1 \right) - M \leq C \int \left( \left| \sup_t b \right|^n + \mu^n \left( \sup_t f_- \right)^n \right) M \, dx.
\]

Since \( M > 0 \), and if we take \( \mu = \int (\sup_t f_-) \), one can conclude

\[
\log \left( \left( \frac{M}{r \| f_- \|_{L^\infty_x L^\infty_t}} \right)^n + 1 \right) \leq C \int \left( \left| \sup_t b \right|^n + C_1 \right) \, dx,
\]

where \( C \) and \( C_1 \) only depend on \( \nu \) and \( n \). Finally, we exponentiate both sides to obtain

\[
M \leq N \left( n, \nu, \int \left| \sup_t b \right|^n \right) r \| f_- \|_{L^\infty_x L^\infty_t}.
\]
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