A NEW PROOF FOR THE BORNOLOGICITY OF THE SPACE OF SLOWLY INCREASING FUNCTIONS

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Abstract. A. Grothendieck proved at the end of his thesis that the space $O_M$ of slowly increasing functions and the space $O'_C$ of rapidly decreasing distributions are bornological. Grothendieck’s proof relies on the isomorphy of these spaces to a sequence space and we present the first proof that does not utilize this fact by using homological methods and, in particular, the derived projective limit functor.

1. Introduction and notation

In [Sch66, p. 243] L. Schwartz introduced the space of multipliers of temperate distributions, i.e., the space of slowly increasing functions

$$O_M = \{ f \in C^\infty(\mathbb{R}^d); \forall \alpha \in \mathbb{N}_0^d \exists N \in \mathbb{N} : \langle x \rangle^{-N} \partial^\alpha f \in L^\infty \},$$

where $C^\infty(\mathbb{R}^d)$ is the space of complex valued, infinitely differentiable functions on $\mathbb{R}^d$, $\langle x \rangle = 1 + |x|^2$, $\partial^\alpha$ is the partial derivative, and $L^\infty$ is the Lebesgue space of bounded functions. The dual $O'_M$ of $O_M$ is the space of very rapidly decreasing distributions. Schwartz also introduced the space of convolutors of temperate distributions, i.e., the space

$$O'_C = \{ f \in C^\infty(\mathbb{R}^d); \exists N \forall \alpha \in \mathbb{N}_0^d : \langle x \rangle^{-N} \partial^\alpha f \in L^\infty \}$$

do of very slowly increasing functions. These spaces are related as in the diagram

$$O_C \subseteq O_M \quad \| \| \quad \| \|

O'_M \subseteq O'_C$$

where in both cases the Fourier transform can be taken as the isomorphism.

It is comparatively easy to see that the four spaces are nuclear and semi-reflexive, that $O_M$ and $O'_C$ are complete and that $O_C$ and $O'_M$ are (LF)-spaces and hence bornological. But the completeness of $O_C$ and $O'_M$ and the bornologicity of $O_M$ and $O'_C$ are not trivial (which was even asserted by Grothendieck, [Gro55, Chap. II, p. 130]). Since the dual of a bornological space is complete and the dual of a complete nuclear space is bornological, these two problems are equivalent (for the definitions of these topological properties and relations between them see [Itô87, Section 424]).

Grothendieck proved that $O_M$ is bornological by showing that it is isomorphic to a complemented subspace of the sequence space $s \otimes s$ [Gro55, Chap. II, Lemme 18, p. 132] and verified “directly” that the space $s \otimes s$ is bornological [Gro55, Chap.II, Prop. 15, p. 125, Cor. 2, p. 128]. We will find out more about this isomorphy in

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Section 2 and also give a homological proof of the bornologicity of \( s \hat{\otimes}_\pi s' \).

In [Kuc85], J. Kučera claimed to have presented a new (and simple) proof for the main properties of the space \( O_M \). That Kučera’s proof contains severe mistakes and that it is based on incorrect propositions is clarified in [Lar12], where also the lack of a proof of the bornologicity of \( O_M \), that does not use the isomorphy \( O_M \cong s \hat{\otimes}_\pi s' \), is pointed out. In Section 3 we will give such a proof.

2. Projective limits and the space \( s \hat{\otimes}_\pi s' \)

Since quotients (and, in particular, complemented subspaces) of bornological spaces are bornological, it was sufficient for Grothendieck to prove that \( O_M \) is isomorphic to a complemented subspace of \( s \hat{\otimes}_\pi s' \), where \( s \) is the space of rapidly decreasing sequences

\[
s = \{(x_j)_{j \in \mathbb{N}} \in \mathbb{C}^\mathbb{N}; \forall k : \sup_{j \in \mathbb{N}} j^k |x_j| < \infty \}
\]

and \( s' \) is its dual, the space of slowly increasing sequences

\[
s' = \{(x_j)_{j \in \mathbb{N}} \in \mathbb{C}^\mathbb{N}; \exists k : \sup_{j \in \mathbb{N}} j^{-k} |x_j| < \infty \}.
\]

By \( s \hat{\otimes}_\pi s' \) we denote the completed projective tensor product of these spaces. E.g., by [Bar12, Remark 1, p. 321], this space \( s \hat{\otimes}_\pi s' \) is canonically isomorphic to

\[
s \hat{\otimes}_\pi s' \cong \{x \in \mathbb{C}^\mathbb{N} \times \mathbb{N}; \forall n \exists N : \sup_{i,j} n^{-N} |x_{i,j}| < \infty \}.
\]

In [Val81], M. Valdivia proved that \( O_M \) is even isomorphic to \( s \hat{\otimes}_\pi s' \) itself which answered a question posed in [Gro55, Chap. II, p. 134]. C. Bargetz used this fact, the bornologicity of \( s \hat{\otimes}_\pi s' \), and methods of the theory of topological tensor products to obtain the isomorphy \( O_C \cong s \otimes_\pi s' \) [Bar12, Prop. 1, p. 318].

The descriptions of the spaces \( O_M \) and \( s \hat{\otimes}_\pi s' \) already indicate how they can be written as projective limits of LB-spaces (countable inductive limits of Banach spaces)

\[
(1) \quad O_M = \bigcap_{n \in \mathbb{N}} X_n = \bigcap_{n \in \mathbb{N}} \bigcup_{N \in \mathbb{N}} X_{n,N},
\]

\[
(2) \quad s \hat{\otimes}_\pi s' = \bigcap_{n \in \mathbb{N}} Y_n = \bigcup_{n \in \mathbb{N}} \bigcup_{N \in \mathbb{N}} Y_{n,N},
\]

where \( X_{n,N} \) and \( Y_{n,N} \) are the Banach spaces

\[
X_{n,N} = \{f \in \mathbb{C}^n(\mathbb{R}^d); \|f\|_{n,N} = \sup_{x \in \mathbb{R}^d, |\alpha| \leq n} (x)^{-N} |\partial^\alpha f(x)| < \infty \},
\]

\[
Y_{n,N} = \{x \in \mathbb{C}^N \times \mathbb{N}; \|x\|_{n,N} = \sup_{i,j} n^{-N} j^{-N} |x_{i,j}| < \infty \}.
\]

These representations as projective limits of LB-spaces are not only natural but also extremely useful since there are very good criteria for checking bornologicity. They are related to the derived projective limit functor \( \text{Proj}^1 \mathcal{F} \) (which can be defined as the cokernel of the map \( \prod X_n \to \prod X_n, (x_n)_n \mapsto (x_n - g^n_{n+1}(x_{n+1}))_n \) where \( g^n_m \) are the connecting maps of the projective spectrum \( \mathcal{F} \); in our cases, \( g^n_m \) are just inclusions). Indeed, an unpublished theorem of D. Vogt (his proof reproduced in [Wen03, Th. 3.3.4]) says that \( \text{Proj}^1 \mathcal{F} \) is bornological whenever \( \text{Proj}^1 \mathcal{F} = 0 \). Moreover, there is a variety of evaluable conditions ensuring \( \text{Proj}^1 \mathcal{F} = 0 \). We are going to apply the following results of Palamodov-Retakh [Pal71] and the second named author, respectively:
A spectrum $\mathcal{X}$ of LB-spaces satisfies $\text{Proj}^1 \mathcal{X} = 0$ if and only if there are Banach discs $D_n$ in $X_n$ with $g_m^n(D_m) \subseteq D_n$ and
\[
\forall n \in \mathbb{N} \exists m \geq n \forall k \geq m : g_m^n(X_m) \subseteq g_k^n(X_k) + D_n.
\]
The requirement $g_m^n(D_m) \subseteq D_n$ is sometimes very easy to fulfill but in many cases it is very inconvenient. It can be omitted if either all steps $X_n$ are LS-spaces (i.e., the inclusions $X_{n,N} \hookrightarrow X_{n,N+1}$ are compact) or if a slightly stronger condition of Palamodov-Retakh type is required. Denoting by $g_\infty^n : \text{Proj} \mathcal{X} \to X_n$ the obvious map we have:
\[
\text{A spectrum } \mathcal{X} \text{ of LB-spaces satisfies } \text{Proj}^1 \mathcal{X} = 0 \text{ if and only if, for every } n \in \mathbb{N}, \text{ there are a Banach discs } D_n \text{ in } X_n \text{ and } m \geq n \text{ with}
\]
\[
g_m^n(X_m) \subseteq g_\infty^n(\text{Proj} \mathcal{X}) + D_n.
\]
We refer to [Wen03] for the proofs of these characterizations and much more information about derived functors. Typically, the decompositions required in conditions of Retakh-Palamodov type are quite easy to produce in the case of spaces of sequences (or matrices) since one can write $x = x + (1 - \chi)x$ where $\chi$ is the indicator function of a suitably chosen set. We want to exemplify this by giving a very short proof for the bornologicity of $s\hat{\otimes}_c s'$ (which is similar to Vogt’s proof of $\text{Ext}^1(s,s) = 0$ [Vog84, Lemma 2.1, p. 359]).

**Proposition 1.** The space $s\hat{\otimes}_c s'$ is bornological.

**Proof.** We keep the notation $s\hat{\otimes}_c s' \cong \bigcap_{n \in \mathbb{N}} Y_n = \bigcap_{n \in \mathbb{N}} \bigcup_{N \in \mathbb{N}} Y_{n,N}$ from above and we will verify the Palamodov-Retakh condition for the unit balls $D_n$ of $Y_{n,0}$ which trivially satisfy $D_{n+1} \subseteq D_n$. For $n \in \mathbb{N}$ we take $m = n + 1$ and fix $x \in Y_n$ as well as $k \geq n + 1$. Since $x \in Y_{m,M}$ for some $M \in \mathbb{N}$ we have
\[
||x||_{m,M} = \sup_{i,j} i^m j^{-N} |x_{i,j}| = c < \infty.
\]
We set $y_{i,j} = x_{i,j}$ if $i < cj^M$ and $y_{i,j} = 0$ else, as well as $z = x - y$. For $i < cj^m$ we have $z_{i,j} = 0$ and for $i \geq cj^M$ we estimate
\[
i^m j^{M-M} |z_{i,j}| j^M/i \leq ||x||_{m,M}/c = 1
\]
which proves $z \in D_n$. It remains to show $y \in Y_{k,K}$ for $K$ sufficiently large. Indeed, for $K = M(k-m+1)$ we have $y_{i,j} = 0$ if $i < cj^M$ and if $i < cj^M$ we estimate
\[
i^k j^{-K} |y_{i,j}| = i^m j^{-M} |y_{i,j}| i^{k-m} j^{M-M-K} \leq ||x||_{m,M} c^{k-m} j^{(k-m)M+M-K} = c^{k-m+1}.
\]
This proves $\|y\|_{k,K} < \infty$, as required. \(\square\)

3. The new proof

Now we want to prove $\text{Proj}^1 \mathcal{X} = 0$ for the spectrum $\mathcal{X} = (X_n)_{n \in \mathbb{N}}$ in (1) in order to obtain that $\mathcal{O}_M$ is bornological. Splitting up a given function $f \in X_m$ as $f = \chi f + (1 - \chi)f$ with a cut-off function $\chi$ (as in the proof of Proposition 1) does not work in this case. But we will see how $f$ can be “split up” in the following proof of Grothendieck’s result.

**Proposition 2.** The space $\mathcal{O}_M$ is bornological.

**Proof.** To obtain $\text{Proj}^1 \mathcal{X} = 0$ we will show
\[
\forall n \exists m, N : X_m \subseteq \mathcal{O}_M + B_{n,N}.
\]
we can set

\[
K \geq 0, \int_{\mathbb{R}^d} K(t, x) dt = 1 \text{ for all } x \in \mathbb{R}^d, \quad \text{and}
\]

\[
\supp K(t, x) \subseteq \prod_{j=1}^d [x_j, x_j + \varepsilon(x)^{-1}] =: A_x \text{ for all } x \in \mathbb{R}^d
\]

where we will see later how \( \varepsilon \) and \( \mu \) have to be chosen in dependence on \( f \in X_m \).

We can obtain such a kernel by defining

\[
K(t, x) = \varepsilon^{-d}(x)^{-d} \varphi(\varepsilon^{-1}(x)^{\mu}(t - x))
\]

for a positive test function \( \varphi \in C^\infty(\mathbb{R}^d) \) with support in \([0, 1]^d\) and \( \int_{\mathbb{R}^d} \varphi(t) dt = 1 \) (the conditions above can be checked easily and \( K \in \mathcal{O}_M \) since every derivative of \( K \) can be estimated by a polynomial).

We start with the one-dimensional case \( d = 1 \) where we can take \( m = n + 1 \) and \( N = 0 \). So let \( f \in X_{n+1,M} \) for some \( M \in \mathbb{N} \). We want to find \( g \in \mathcal{O}_M \) such that \( f - g \in B_{n,0} \). At first we set

\[
g_n(x) = \int_{\mathbb{R}} f^{(n)}(t) K(t, x) dt
\]

and show that this is a good approximation to \( f^{(n)} \). Since for \( l \in \mathbb{N}_0 \)

\[
\left| g^{(l)}_n(x) \right| = \left| \int_{A_x} f^{(n)}(t) \frac{d^l}{dt^l} K(t, x) dt \right| \leq \int_{A_x} |P(t)| |Q(t, x)| dt \leq |R(x)|
\]

for some polynomials \( P, Q, R \), the function \( g_n \) is contained in \( \mathcal{O}_M \). Furthermore we can estimate in virtue of Taylor’s formula

\[
|f^{(n)}(t) - f^{(n)}(x)| \leq |t - x| |\xi(t, x)| M |f|_{n+1,M}
\]

with a point \( \xi(t, x) \) between \( t \) and \( x \). For \( \varepsilon \) small enough the inequality \( |\xi(t, x)| \leq 2 \langle x \rangle \) holds for every \( x \in \mathbb{R} \) and \( t \in A_x \). We obtain

\[
|g_n(x) - f^{(n)}(x)| = \left| \int_{\mathbb{R}} \left( f^{(n)}(t) - f^{(n)}(x) \right) K(t, x) dt \right| \\
\leq \int_{A_x} \left| f^{(n)}(t) - f^{(n)}(x) \right| K(t, x) dt \\
\leq \int_{A_x} |t - x| |\xi(t, x)| M |f|_{n+1,M} K(t, x) dt \\
\leq \varepsilon 2^M \langle x \rangle^{M-\mu} |f|_{n+1,M} \int_{A_x} K(t, x) dt \\
= \varepsilon 2^M \langle x \rangle^{M-\mu} |f|_{n+1,M}.
\]

Now if

\[
T : \mathcal{O}_M(\mathbb{R}) \to \mathcal{O}_M(\mathbb{R}), h \mapsto \left( x \mapsto \int_0^x h(t) dt \right),
\]

we can set

\[
g(x) = \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{j!} x^j + (T^n g_n)(x).
\]
Then \( g \in \mathcal{O}_M \) and since
\[
(T^n f^{(n)})(x) = f(x) - \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{j!} x^j,
\]
integrating (4) (the integral starting at 0) yields
\[
|g^{(l)}(x) - f^{(l)}(x)| \leq 1, \quad x \in \mathbb{R}^d, \quad l \leq n
\]
for \( \varepsilon \) small enough and \( \mu \) large enough. Hence \( g - f \in B_{n,0} \) and the proof is complete for the one-dimensional case.

Now we will prove the two-dimensional case \( d = 2 \). We set \( m = 2n + 1 \) and \( N = n - 1 \) in (3). So let \( f \in X_{2n+1,M} \) for some \( M \). With the help of a kernel \( K \in \mathcal{O}_M(\mathbb{R}^2 \times \mathbb{R}^2) \) like above, we set
\[
g_n(x) = \int_{\mathbb{R}^2} \partial^{(n,n)} f(t) K(t, x) \, dt
\]
in order to approximate \( \partial^{(n,n)} f \) by \( g_n \). Similar to the one-dimensional case we have
\[
|\partial^{(n,n)} f(t) - \partial^{(n,n)} f(x)| \leq c \cdot |t - x| |\xi(t, x)|^M \|f\|_{2n+1,M}
\]
and \( |\xi(t, x)| \leq 2 \varepsilon \) for \( t \in \mathcal{A}_x \) and \( \varepsilon \) small enough and thus
\[
|g_n(x) - \partial^{(n,n)} f(x)| \leq c \int_{\mathcal{A}_x} |t - x| |\xi(t, x)|^M \|f\|_{2n+1,M} K(t, x) \, dt
\]
(5)
\[
\leq \tilde{c} \varepsilon \langle x \rangle^{M-\mu} \|f\|_{2n+1,M}.
\]
Let us denote \( T_j \) the integral with respect to the \( j \)-th component (the integral starting at 0). Applying \( T_1 \circ T_2 \) \( n \)-times to \( \partial^{(n,n)} f(x) \) yields
\[
(T_1^n T_2^n f)(x) = f(x) + \sum_{\alpha \leq (n,n)} \partial^{\alpha} f(0,0) \frac{x^{\alpha}}{\alpha!} - \sum_{j=0}^{n-1} \partial^{(j,0)} f(0,x_2) \frac{x^j j!}{j!} - \sum_{j=0}^{n-1} \partial^{(0,j)} f(x_1,0) \frac{x^j j!}{j!}
\]
As in the one-dimensional case we can choose \( g_1, g_2, \ldots, g_{n-1}, \hat{g}_0, \ldots, \hat{g}_2 \in \mathcal{O}_M(\mathbb{R}) \) such that \( \|g^1_j - \partial^{(0,j)} f(\cdot, 0)\|_{n,0} \leq \varepsilon \) and \( \|g_j^2 - \partial^{(j,0)} f(\cdot, 0)\|_{n,0} \leq \varepsilon \). Defining
\[
g(x) = (T_1^n T_2^n) g_n(x) - \sum_{\alpha \leq (n,n)} \partial^{\alpha} f(0,0) \frac{x^{\alpha}}{\alpha!} + \sum_{j=0}^{n-1} g_j^2(x_2) \frac{x^j j!}{j!} + \sum_{j=0}^{n-1} g_j^1(x_1) \frac{x^j j!}{j!}
\]
and applying \( T_1^n T_2^n \) to (5) yields
\[
|g(x) - f(x)| \leq \varepsilon + \sum_{j=0}^{n-1} \left( |g^1_j(x_1) - \partial^{(0,j)} f(x_1,0)| \frac{|x_2|^j}{j!} + |g^2_j(x_2) - \partial^{(j,0)} f(0,x_2)| \frac{|x_1|^j}{j!} \right)
\]
for \( \mu \) large enough which implies
\[
|g(x) - f(x)| \leq \varepsilon + \varepsilon \sum_{j=0}^{n-1} \frac{|x_2|^j}{j!} + \varepsilon \sum_{j=0}^{n-1} \frac{|x_1|^j}{j!} \leq \varepsilon c \langle x \rangle^{n-1}
\]
for some \( c > 1 \). Since similar estimates also hold for \( |\partial^{\alpha} g(x) - \partial^{\alpha} f(x)|, |\alpha| \leq n \), we obtain \( g - f \in B_{n,n-1} \) and the proof is complete for \( d = 2 \).

The general case \( d \in \mathbb{N} \) is very similar. Inductively we want to show
\[
X_{dn+1} \subseteq \mathcal{O}_M + B_{n,(d-1)(n-1)}
\]
and start by approximating $\partial^{(n,\ldots,n)}f$ by $g_n(x) := \int_R \partial^{(n,\ldots,n)}f(t)K(t,x) dt$. Then we integrate the estimate of $g_n - \partial^{(n,\ldots,n)}f$ $n$-times with respect to each component. The integral $T_1^{} \cdots T_d^{} \partial^{(n,\ldots,n)}f$ contains $f$ as a summand and terms that are the product of a derivative of $f$ that only depends on less than $d$ components and a polynomial in less than $d$ components with exponents less than $n$. But we can estimate the functions that only depend on less than $d$ variables by the induction hypothesis and hence we can obtain $g \in O_M$ with $g - f \in B_{n,(d-1)(n-1)}$. □

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