Pattern Containment and Combinatorial Inequalities

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Abstract
We use a probabilistic method to produce some combinatorial inequalities by considering pattern containment in permutations and words.

If \( \sigma \in S_n \) and \( \tau \in S_m \), we say that \( \sigma \) contains \( \tau \), or \( \tau \) occurs in \( \sigma \), if \( \sigma \) has a subsequence order-isomorphic to \( \tau \). In this situation, \( \tau \) is called a pattern. Similarly, if \( \sigma \in [k]_n \) is a string of \( n \) letters over the alphabet \( [k] = \{1, \ldots, k\} \), and \( \tau \in [l]_m \) is a map from \( [m] \) onto \( [l] \) (i.e. \( \tau \) contains all letters from 1 to \( l \)), then we say that \( \sigma \) contains the pattern \( \tau \) if \( \sigma \) has a subsequence order-isomorphic to \( \tau \). An instance (or occurrence) of \( \tau \) in \( \sigma \) is a choice of \( m \) positions \( 1 \leq i_1 < \ldots < i_m \leq n \), such that the subsequence \( (\sigma(i_1), \ldots, \sigma(i_m)) \) is order-isomorphic to \( \tau = (\tau(1), \ldots, \tau(m)) \).

Most of the work on pattern containment concentrated on pattern avoidance, that is on characterizing and counting permutations that contain no occurrences of a given pattern or a set of patterns. Less attention has been given to counting the number of times a given pattern occurs in permutations of a given size, in particular, packing patterns into permutations (but see [1, 4], for example), and, to our knowledge, packing patterns into words (where repeated letters are allowed) has not yet been considered.

Here we consider pattern containment and use a simple probabilistic fact (the variance of a random variable is nonnegative) to produce nontrivial combinatorial inequalities.

1 Patterns in permutations
In this section, we consider permutation patterns contained in other permutations.
Theorem 1 Let $\tau$ be a permutation of $\{0, 1, \ldots, m\}$ and define
\[ [i,j]_m = \binom{i+j}{i} \binom{2m-i-j}{m-i}. \]
Then for any nonnegative integer $m$ and any $\tau$ as above,
\[ \sum_{i,j=0}^m [i,j]_m [\tau(i), \tau(j)]_m \geq \binom{2m+1}{m}^2. \tag{1} \]

Remark 1 Notice that $[i,j]_m$ is the number of northeast integer lattice paths from $(0,0)$ to $(m,m)$ through $(i,j)$. Hence the left-hand side is the number of pairs $(P,Q)$ of northeast integer lattice paths $P : (0,0) \rightarrow (i,j) \rightarrow (m,m)$ and $Q : (0,0) \rightarrow (\tau(i), \tau(j)) \rightarrow (m,m)$ over all $(i,j) \in \{0,m\}^2$.

Remark 2 The numbers $[i,j]_m$, $0 \leq i,j \leq m$ have been found to have other interesting properties as well. For example, Amdeberhan and Ekhad showed that
\[ \det([i,j]_m)_{0 \leq i,j \leq m} = \frac{(2m+1)!^{m+1}}{(2m+1)!}, \]
where $a! = 0! \cdot 1! \cdot 2! \cdots \cdot a!$.

It is a well-known result $[3]$ that for any nondecreasing subsequence $a_1 \geq \cdots \geq a_n$ and a permutation $\varphi \in S_n$, the sum $\sum_{i=1}^n a_i a_{\varphi(i)}$ attains its maximum when $\varphi = id_n = 12\ldots n$ and its minimum when $\varphi = n(n-1)\ldots 1$. Now if we arrange $(m+1)^2$ numbers $[i,j]_m$ ($0 \leq i,j \leq m$) in nondecreasing order, there is no permutation $\tau$ of $\{0, 1, \ldots, m\}$ which reverses that order (other than in the trivial case $m = 0$). For example, even when $m = 1$, we have
\[ [0,0]_2 = [1,1]_2 = 2 > 1 = [0,1]_2 = [1,0]_2, \]
reversing $(2, 2, 1, 1)$ gives $(1, 1, 2, 2)$ and the estimate of $[3]$ gives us the lower bound of $2 \cdot 1 + 2 \cdot 1 + 1 \cdot 2 + 1 \cdot 2 = 8$, our estimate yields the lower bound of $\binom{4}{1}^2 = 9$, while the left-hand side is actually equal to 10 for both $\tau = 01$ (the identity) and $\tau = 10$ (which transposes $[0,0]_2$ and $[1,1]_2$ as well as $[0,1]_2$ and $[1,0]_2$ in the above ordering).

The estimate in Theorem $[3]$ appears to be stronger than that of $[3]$. For example, the lower bounds for $m = 2, 3, 4, 5$ are 75, 792, 8660, 98876, respectively, according to $[3]$, while our lower bounds are 100, 1225, 15876, 213444, respectively. In fact, as the following proposition shows, the lower bound of $[3]$ can never be achieved in our case for $m > 0$.

Proposition 1 Arrange $(m+1)^2$ numbers $\{[i,j]_m \mid 0 \leq i,j \leq m\}$ into a nondecreasing order $a_1 \geq \cdots \geq a_{(m+1)^2}$. A permutation $\tau$ of $\{0, 1, \ldots, m\}$ induces an equivalence class of permutations $\varphi_\tau$ on the $a_i$’s (equivalence relation being a permutation of equal elements). Then for any $\tau$, reversal of the identity $(m+1)^2((m+1)^2-1)\cdots 21 \notin \varphi_\tau$. 

Proof. Suppose that there is a permutation $\tau$ which induces an order-reversing permutation of the $a_i$’s. Note that $[0, m]_m = [m, 0]_m = 1$ for any $m$, hence, $[\tau(0), \tau(m)]_m$ must have the greatest value among all $[\tau(i), \tau(j)]_m$. Note that

$$[i, j]_m = [j, i]_m = [m - i, m - j]_m = [m - j, m - i]_m$$

for any $i$ and $j$, so assume that $i \leq j$. Then it is a straightforward exercise to prove that

$$[i, j]_m > [i - 1, j]_m \text{ for } i > 0, \text{ and}$$

$$[i, j]_m > [i, j + 1]_m \text{ for } j < m,$$

so for $0 < i < j < m$,

$$[i, i]_m > [i, j]_m > [0, j]_m > [0, m]_m = 1.$$  

Similarly, we can assume that $i \leq \lfloor \frac{m}{2} \rfloor$ (since $[i, i]_m = [m - i, m - i]_m$), then it is just as easy to see that for any $i > 0$

$$[i, i]_m < [i - 1, i - 1]_m < \cdots < [0, 0]_m = \binom{2m}{m}.$$ 

Thus, for any $0 \leq i, j \leq m$,$$
1 = [0, m]_m = [m, 0]_m \leq [i, j]_m \leq [0, 0]_m = [m, m]_m = \binom{2m}{m},$$

and one of the two inequalities becomes an equality if and only if $i, j \in \{0, m\}$. Hence, for our permutation $\tau$, we must have $\tau(0) = \tau(m) = 0$ or $\tau(0) = \tau(m) = m$, neither of which is possible when $m \neq 0$. The resulting contradiction implies our proposition. \hfill \Box

Finally, before we begin with the proof of Theorem 1, let us note that a permutation of summands in (4) yields the following corollary.

**Corollary 1** For any two permutations $\tau_1, \tau_2$ of $\{0, 1, \ldots, m\}$ and any $m \in \mathbb{N}$,

$$\sum_{i,j=0}^{m} |\tau_1(i), \tau_1(j)|_m |\tau_2(i), \tau_2(j)|_m \geq \binom{2m+1}{m}^2.$$ 

Another immediate corollary is a consequence of the fact that

$$[i, j]_m = \binom{2m}{m} \frac{m}{i+j} \binom{m}{i+j} = \binom{2m}{m} \{i, j\}_m$$

where $\{i, j\}_m := \frac{m}{i+j} \binom{m}{i+j}$.

**Corollary 2** For any $m \in \mathbb{N}$ and any permutation $\tau$ of $\{0, 1, \ldots, m\}$,

$$\sum_{i,j=0}^{m} \{i, j\}_m |\tau(i), \tau(j)|_m \geq \frac{2m+1}{m+1} \binom{2m+1}{m+1} = \left(2 - \frac{1}{m+1}\right)^2.$$
Note that Corollary 2 no longer holds if we substitute 4 on the right side of this inequality.

**Proof of Theorem 1.** Consider $S_n$ as a sample space with uniform distribution. Let $\tau \in S_m$ (notation-wise, it is more convenient if, in the proof, $\tau$ is a permutation of $\{1,2,\ldots,m\}$), and let $X_\tau$ be a random variable such that $X_\tau(\sigma)$ is the number of occurrences of pattern $\tau$ in given permutation $\sigma \in S_n$. We will show that our inequality follows from the fact that

$$Var(X_\tau) = E(X_\tau^2) - E(X_\tau)^2 \geq 0$$

We start by finding $E(X_\tau)$. Pick an $m$-letter subset $S$ of $[n] = \{1,2,\ldots,n\}$ in $\binom{n}{m}$ ways. There is a unique permutation $\tau(S)$ of $S$ which is order-isomorphic to $\tau$. There are $m!$ equally likely permutations in which the elements of $S$ can occur in $\sigma$, but we need only 1 of them, namely, $\tau(S)$. Hence, $\tau(S)$ either occurs once or does not occur in a given permutation $\sigma$. Therefore, the probability that a random $\sigma$ contains $\tau(S)$ as a subsequence is $1/m!$. Let $Y_{\tau(S)}$ be a random variable such that $Y_{\tau(S)}(\sigma)$ is the number of occurrences of $\tau(S)$ in $\sigma$. Since

$$P(Y_{\tau(S)}(\sigma) = 1) = \frac{1}{m!} \quad \text{and} \quad P(Y_{\tau(S)}(\sigma) = 0) = 1 - \frac{1}{m!},$$

we have $E(Y_{\tau(S)}) = 1/m!$. But this is true for any $S \subseteq [n]$ such that $|S| = m$, and we have

$$X_\tau = \sum_{S \subseteq [n], |S|=m} Y_{\tau(S)}$$

hence,

$$E(X_\tau) = \sum_{S \subseteq [n], |S|=m} E(Y_{\tau(S)}) = \frac{1}{m!} \binom{n}{m}.$$ 

Next, we look at $E(X_\tau^2)$. We have

$$E(X_\tau^2) = E\left(\sum_{S \subseteq [n], |S|=m} Y_{\tau(S)}\right) = \sum_{S_1, S_2 \subseteq [n]} E\left(Y_{\tau(S_1)} Y_{\tau(S_2)}\right).$$

Of course, $Y_{\tau(S_1)} Y_{\tau(S_2)} = 1$ if and only if both $\tau(S_1)$ and $\tau(S_2)$ are subsequences of $\tau$, otherwise, $Y_{\tau(S_1)} Y_{\tau(S_2)} = 0$.

Let $S = S_1 \cup S_2$, and $|S_1 \cap S_2| = \ell$, so $|S| = |S_1 \cup S_2| = 2m - \ell$. We can pick a subset $S \subseteq [n]$ in $\binom{n}{2m-\ell}$ ways. Note that any such $S$ is order-isomorphic to $[2m-\ell] = \{1,2,\ldots,2m-\ell\}$. Therefore, the number of permutations $\rho(S)$ of $S$ such that $\rho(S) = \tau(S_1) \cup \tau(S_2)$ for some $S_1, S_2 \subseteq S$, $S_1 \cup S_2 = S$, is the same for any $S$ of cardinality $2m-\ell$ and depends only on $m$ and $\ell$.

Therefore, $E(X_\tau^2)$ is a linear combination of $\left\{ \binom{n}{2m-\ell} \mid 0 \leq \ell \leq m \right\}$ with coefficients which are rational functions of $m$ and $\ell$. The degrees in $n$ of both
$E(X_2^2)$ and $E(X_r^2)^2$ are $2m$, and the coefficient of $n^{2m}$ in $E(X_r^2)^2$ is $1/(m!)^4$. On the other hand, $S = S_1 \cup S_2$, $|S| = 2m$ and $|S_1| = |S_2| = m$ imply that $S_1 \cap S_2 = \emptyset$, so $Y_{\tau(S_1)}$ and $Y_{\tau(S_2)}$ are independent, and hence

$$P \left( Y_{\tau(S_1)}Y_{\tau(S_2)} = 1 \right) = P \left( Y_{\tau(S_1)} = 1 \right) P \left( Y_{\tau(S_2)} = 1 \right) = \left( \frac{1}{m!} \right)^2.$$ 

Since the number of ways to partition a set $S$ of size $2m$ into two subsets of size $m$ is $\binom{2m}{m}$, the coefficient of $\binom{n}{m}$ in $E(X_2^2)$ is $\frac{1}{(m!)^2} \binom{2m}{m}$, so $\text{deg}_n(\text{Var}(X_r)) \leq 2m - 1$, and hence, $\text{deg}_n(\text{Var}(X_r)) \geq 0$.

We have

$$[n^{2m-1}]E(X_r^2)^2 = [n^{2m-1}] \left( \frac{1}{m!} \binom{n}{m} \right)^2 = \frac{2}{(m!)^2} \cdot [n^m] \binom{n}{m} \cdot [n^{m-1}] \binom{n}{m} = \frac{2}{(m!)^2} \cdot \frac{1}{m!} \cdot \left( \frac{n}{m} \right) = -\frac{m(m-1)}{(m!)^4}.$$ 

Similarly, the coefficient of $n^{2m-1}$ in the $\binom{n}{2m}$-term of $E(X_2^2)$ is

$$-\frac{\binom{2m}{2}}{(2m)!} \frac{1}{(m!)^2} \binom{2m}{m} = -\frac{m(2m-1)}{(m!)^4},$$

so we only need to find the coefficient of the $\binom{n}{2m}$-term of $E(X_2^2)$.

As we noted before, all subsets $S \subseteq [n]$ of the same size (in our case, of size $2m-1$) are equivalent, so we may assume $S = [2m-1] = \{1, 2, \ldots, 2m-1\}$. We want to find the number of permutations $\rho$ of $S$ such that there exist subsets $S_1, S_2 \subseteq S$ of size $m$ for which we have $|S_1 \cap S_2| = 1$ (so $S_1 \cup S_2 = S$) and $\rho(S) = \tau(S_1) \cup \tau(S_2)$.

Suppose that we want to choose $S_1$ and $S_2$ as above, together with their positions in $S$, in such a way that the intersection element $e$ is in the $i$th position in $\tau(S_1)$ and the $j$th position in $\tau(S_2)$ (of course, $1 \leq i, j \leq m$). Then $e$ occupies position $(i-1) + (j-1) + 1 = i + j - 1$ in $S$. Hence, there are $\binom{i-1+j-1}{i-1}$ ways to choose the positions for elements of $\tau(S_1)$ and $\tau(S_2)$ on the left of $e$, and $\binom{n-i+m-j}{m-j}$ ways to choose the positions for elements of $\tau(S_1)$ and $\tau(S_2)$ on the right of $e$. On the other hand, both $\tau(S_1)$ and $\tau(S_2)$ are naturally order-isomorphic to $\tau$, hence, under that isomorphism $e$ maps to $\tau(i)$ as an element of...
$S_1$ and to $\tau(j)$ as an element of $S_2$. Since $e$ is the unique intersection element, it is easy to see that we must have $e = (\tau(i) - 1) + (\tau(j) - 1) + 1 = \tau(i) + \tau(j) - 1$ (exactly $\tau(i) - 1$ elements in $S_1$ and exactly $\tau(j) - 1$ elements in $S_2$, all distinct from those in $S_1$, must be less than $e$, the rest of the elements of $S$ must be greater than $e$). There are $(\tau(i)-1+\tau(j)-1)$ ways to choose the elements of $S_1$ and $S_2$ which are less than $e$, and $(m-\tau(i)+m-\tau(j))$ ways to choose the elements of $S_1$ and $S_2$ which are greater than $e$.

Thus, its positions in $\tau(S_1)$ and $\tau(S_2)$ uniquely determine the position and value of the intersection element $e$; there are $[i-1, j-1]_m$ ways to choose which other positions are occupied by $\tau(S_1)$ and which ones by $\tau(S_2)$; and, there are $[\tau(i)-1, \tau(j)-1]_m$ ways to choose which other values are in $\tau(S_1)$ and which ones are in $\tau(S_2)$.

Now that we have chosen both positions and values of elements of $S_1$ and $S_2$, we can produce a unique permutation $\rho(S)$ of $S$ which satisfies our conditions above. Simply fill the positions for $S_1$, resp. $S_2$, by elements of $\tau(S_1)$, resp. $\tau(S_2)$, in the order in which they occur.

Since the total number of permutations of $S$ is $(2m-1)!$, the coefficient of the $(2m-1)$-term of $E(X_r^2)$ is

\[
\sum_{i,j=1}^{m} \frac{(i-1+j-1)!}{(i-1)!} \frac{(m-i+m-j)!}{(m-j)!} \frac{(\tau(i)-1+\tau(j)-1)!}{\tau(i)-1} \frac{(m-\tau(i)+m-\tau(j))}{m-\tau(j)} = \sum_{i,j=1}^{m} \frac{[i-1, j-1]_m-1[\tau(i)-1, \tau(j)-1]_m-1}{(2m-1)!},
\]

the coefficient of $n^{2m-1}$ in $\text{Var}(X_r)$ is, by the previous equations,

\[
\sum_{i,j=1}^{m} \frac{[i-1, j-1]_m-1[\tau(i)-1, \tau(j)-1]_m-1}{((2m-1)!)^2} - \frac{m(2m-1)}{(m!)^4} + \frac{m(m-1)}{(m!)^4} = \sum_{i,j=1}^{m} \frac{[i-1, j-1]_m-1[\tau(i)-1, \tau(j)-1]_m-1}{((2m-1)!)^2} - \frac{1}{(m!(m-1)!)^2} \geq 0,
\]

so we finally get

\[
\sum_{i,j=1}^{m} [i-1, j-1]_m-1[\tau(i)-1, \tau(j)-1]_m-1 \geq \left(\frac{(2m-1)!(m!)}{(m!(m-1)!)}\right)^2 = \left(\frac{2m-1}{m-1}\right)^2,
\]

which is easily reducible to \[\Box\] by $m \leftarrow m + 1$, then $\bar{\tau}(i) \leftarrow \tau(i + 1) - 1$. \[\square\]

It seems, however, that a stronger form of our Theorem should be true, namely, the following

**Conjecture 1** The strict inequality holds in [1] for all $m > 0$.

This would imply that $\text{Var}(X_r)$ has order $2m-1$ in $n$, i.e. the standard deviation of $X_r$ is $1/2$ order smaller than its expected value.
Remark 3 Similarly, the leading coefficient of the covariance $\text{Cov}(X_{\tau_1}, X_{\tau_2})$ is

$$\sum_{i,j=1}^{m} [i-1, j-1]_{\tau_1[i] - 1, \tau_2[j] - 1}_{m-1} - \frac{1}{(m! (m-1)!)^2}$$

but $\sum_{i,j=1}^{m} [i-1, j-1]_{\tau_1[i] - 1, \tau_2[j] - 1}_{m-1}$ can be (and often is) less than $\binom{2m-1}{m-1}^2$.

As of now, we only have some results on the sign of covariance for small patterns. We hope to explore this topic further in subsequent papers.

Note that the reversal map, $\tau(i) \mapsto \tau(m + 1 - i)$, the complement map, $\tau(i) \mapsto m + 1 - \tau(i)$, preserve the variance and covariance (we also make a note for the next section that, for words $\tau \in [l]^m$, the reversal map is the same, while the complement is $\tau(i) \mapsto l + 1 - \tau(i)$).

Considering symmetry classes of pairs of patterns (i.e. equivalence classes with respect to reversal and complement), we see that there are 8 classes of pairs of 3-letter patterns: $\{123, 132\}$, $\{132, 132\}$, $\{123, 132\}$, $\{132, 213\}$, $\{132, 231\}$, $\{132, 312\}$, $\{123, 312\}$, $\{123, 321\}$ (listed in order of decreasing covariance). Of those, the first two pairs obviously have a positive covariance, and of the remaining six, only $\{123, 132\}$ has a positive covariance.

Finally, denote the left-hand side and right-hand side of equation (1) by $L(m, \tau)$ and $R(m, \tau)$, respectively, and let

$$M^*(m) = \max_{\tau \in S_m} (L(m, \tau) - R(m, \tau)),$$

$$M_*(m) = \min_{\tau \in S_m} (L(m, \tau) - R(m, \tau)).$$

It is not hard to see that $M^*(m) = L(m, id_m) - R(m, id_m) > 0$, where $id_m$ is the identity permutation of $\{0, 1, \ldots, m\}$ (use Chebyshev’s inequality, or dot product, or Cauchy-Schwarz inequality). It would be interesting to characterize the permutations $\tilde{\tau}_m$ such that $M_*(m) = L(m, \tilde{\tau}_m) - R(m, \tilde{\tau}_m)$. We also make the following conjecture.

Conjecture 2 $\exists \lim_{m \to \infty} M_*(m) = 0$.

2 Patterns in words

We now consider patterns contained in words, where repeated letters are allowed both in the pattern and the ambient string.

Theorem 2 Let $\tau$ be a map of $[0, m] = \{0, 1, \ldots, m\}$ onto $[0, l] = \{0, 1, \ldots, l\}$. Then for any nonnegative integers $0 \leq l \leq m$ and any $\tau$ as above,

$$\sum_{i,j=0}^{m} [i,j]_{\tau(i), \tau(j)\mid l} \geq \frac{(2m + 1)!(2l + 1)!}{(m!)^2 (l + 1)!^2}.$$  \hspace{1cm} (2)
Remark 4 Not e that Theorem \ref{thm:second} reduces to Theorem \ref{thm:first} when \( l = m \). Note also that, given \( 0 \leq l \leq m \), Theorem \ref{thm:second} applies to \((l + 1)!S(m + 1, l + 1)\) patterns \( \tau \), where \( S(m + 1, l + 1) \) is the Stirling number of the second kind.

Remark 5 As in Theorem \ref{thm:first}, the left-hand side of Theorem \ref{thm:second} is the number of \( \{0\} \) of northeast integer lattice paths \( P : (0, 0) \to (i, j) \to (m, m) \) and \( Q : (0, 0) \to (\tau(i), \tau(j)) \to (l, l) \) over all \((i, j) \in [0, m]^2\).

Proof of Theorem \ref{thm:second}. The proof follows the same outline as that of Theorem \ref{thm:first}, so we will use the same notation as well. Again, it will be convenient to assume in the proof that \( \tau \in [l]^m \) is map of \([m]\) onto \([l]\) (i.e. use \( \{1, \ldots, m\} \) instead of \( \{0, 1, \ldots, m\} \) and \( \{1, \ldots, l\} \) instead of \( \{0, 1, \ldots, l\} \)) and, similarly, that the ambient permutations \( \sigma \in [k]^n \). Note that for any subset \( S \subseteq [n] \) of positions, the probability that the subsequence of elements at positions in \( S \), i.e. \( \sigma(S) \), in a random word \( \sigma \in [k]^n \), is order-isomorphic to \( \tau \) is \( \binom{k}{l}/k^m \). This is because \( k^m \) is the total number of subsequences of \( m \) letters in \([k]\), \( \tau \) has exactly \( l \) distinct letters, and there are \( \binom{l}{l} \) ways to choose \( l \) distinct letters out of \( k \). Hence, as in Theorem \ref{thm:first}, we obtain

\[
E(X_\tau) = \frac{1}{k^m} \binom{k}{l} \binom{n}{m},
\]

which is a polynomial in \( n \) and \( k \). Therefore, the leading coefficient of \( E(X_\tau) \) as a polynomial in \( n \) is

\[
[n^m]E(X_\tau) = \frac{1}{k^m} \binom{k}{l} \frac{1}{m!},
\]

so the leading coefficient of \( E(X_\tau)^2 \) is

\[
[n^{2m}]E(X_\tau)^2 = \frac{1}{k^{2m}} \binom{k}{l}^2 \frac{1}{(m!)^2}.
\]

However, as in the proof of Theorem \ref{thm:second}, we have that \( E(X_\tau^2) \) is a linear combination of \( \binom{n}{2m} \), \( 0 \leq l \leq m \), with coefficients being polynomials in \( k \) and rational functions in \( l, m \). A similar analysis shows that the leading coefficient in \( n \) of \( E(X_\tau^2) \) is

\[
[n^{2m}]E(X_\tau^2) = \frac{1}{(2m)!} \binom{n}{2m} E(X_\tau^2) = \frac{1}{(2m)!} \binom{2m}{m} \binom{k}{l}^2 \frac{1}{k^{2m}} = [n^{2m}]E(X_\tau)^2,
\]

so \( \deg_n(\text{Var}(X_\tau)) \leq 2m - 1 \), and hence, \( [n^{2m-1}]\text{Var}(X_\tau) \geq 0 \).

As in the proof of Theorem \ref{thm:first}, we have that

\[
[n^{2m-1}]E(X_\tau)^2 = 2[n^{m-1}]E(X_\tau)[n^m]E(X_\tau) = \frac{m(m - 1)}{(m!)^2} \binom{k}{l}^2 \frac{1}{k^{2m}}.
\]
and the coefficient of $n^{2m-1}$ in the $\binom{n}{2m}$-term of $E(X_r^2)$ is

$$-\binom{2m}{2} \frac{1}{(2m)!} \binom{2m}{m} \binom{k}{l}^2 \frac{1}{k^{2m}} = -\frac{m(2m-1)}{(m!)^2} \binom{k}{l}^2 \frac{1}{k^{2m}}.$$ 

The remaining summand in $[n^{2m-1}]\text{Var}(X_r)$ is the coefficient of $n^{2m-1}$ in the $\binom{n}{2m-1}$-term of $E(X_r^2)$, i.e.

$$\frac{1}{(2m-1)!} \left[ \binom{n}{2m-1} \right] E(X_r^2)$$

$$-\frac{m(2m-1)}{(m!)^2} \binom{k}{l}^2 \frac{1}{k^{2m}} + \frac{m(m-1)}{(m!)^2} \binom{k}{l}^2 \frac{1}{k^{2m}} \geq 0,$$

which is equivalent to

$$\left[ \binom{n}{2m-1} \right] E(X_r^2) \geq \frac{(2m-1)!}{(m-1)!} \binom{k}{l}^2 \frac{1}{k^{2m}}.$$

As in the proof of Theorem 3, it is easy to see that $\binom{n}{2m-1}E(X_r^2)$ is equal to the probability that a sequence $\rho \in [k]^{2m-1}$ is a union of two subsequences order-isomorphic to $\tau$. Therefore, assume $[2m-1] = S_1 \cup S_2$, $\rho(S_1) \cong \tau \cong \rho(S_2)$. But then $S_1$ and $S_2$ have $m$ elements, so they intersect at a single element $e$.

Suppose that $e$ is at position $i$ in $S_1$ and at position $j$ in $S_2$. Then, as in the proof of Theorem 3, there are $\binom{i-1+j-1}{i-1} \binom{m-i+j-1}{m-i} = [i-1, j-1]_{m-1}$ ways to choose positions to the left and to the right of $e$ in $S_1$ and which ones are in $S_2$.

Suppose that $\rho$ contains $l + L$ distinct letters, then $0 \leq L \leq l - 1$. Because of the positions of $e$ in $S_1$ and $S_2$, we know that $e$ must map to $\tau(i)$ in $\rho(S_1)$ and to $\tau(j)$ in $\rho(S_2)$ under our order-isomorphism. Suppose that the value of $e$ in $\rho$ is $r$. Consider the $r - 1$ letters in $[l + L]$ which are less than $r$. Then

$$(r - 1) - (\tau(j) - 1) = r - \tau(j)$$

of those occur only in $S_1$,

$$(r - 1) - (\tau(i) - 1) = r - \tau(i)$$

occur only in $S_2$, and

$$(r - 1) - (r - \tau(i)) - (r - \tau(j)) = \tau(i) + \tau(j) - 1 - r$$

occur in both $\rho(S_1)$ and $\rho(S_2)$. Similarly, of the $l + L - r$ letters in $\rho$ which are greater than $r$,

$$(l + L - r) - (l - \tau(j)) = L - r + \tau(j)$$

occur only in $\rho(S_1)$,

$$(l + L - r) - (l - \tau(i)) = L - r + \tau(i)$$

occur only in $\rho(S_2)$. Similarly, of the $l + L - r$ letters in $\rho$ which are greater than $r$,
occur only in $\rho(S_2)$, and

$$(l + L - r) - (L - r + \tau(i)) - (L - r + \tau(j)) = l - L + r - \tau(i) - \tau(j)$$

occur in both $\rho(S_1)$ and $\rho(S_2)$.

Thus, the number of sequences $\rho \in [k]^{2m-1}$ which are a union of two subsequences order-isomorphic to $\tau$ is

$$f(\tau, k) = \sum_{L=0}^{l-1} \binom{k}{L} \sum_{r=0}^{l+L} \sum_{i,j=1}^{m} \sum_{i-1, j-1} \delta(\tau, L, r, i, j),$$

where

$$h(\tau, L, r, i, j) = \left( \frac{r - 1}{r - \tau(i), r - \tau(j), \tau(i) + \tau(j) - 1 - r} \right) \times \left( \frac{l + L - r}{l + L - r + \tau(i), \tau(i) + \tau(j), \tau(i) + \tau(j) - 1 - r} \right).$$

Hence, the probability that a sequence $\rho \in [k]^{2m-1}$ is a union of two subsequences order-isomorphic to $\tau$ is $f(\tau, k)/k^{2m-1}$, so we have

$$\left[ \sum_{2m - 1}^n \binom{n}{2m - 1} \right] E(X^2) = \frac{f(\tau, k)}{k^{2m-1}} \geq \left( \frac{(2m - 1)!}{(m - 1)!^2} \right)^2 \frac{1}{k^{2m}},$$

or, equivalently,

$$kf(\tau, k) \geq \left( \frac{(2m - 1)!}{(m - 1)!^2} \right)^2 \frac{1}{l^{2m}},$$

for all positive integers $k$ and all patterns $\tau \in [l]^m$. But both sides of this inequality are polynomials in $k$ of degree $2l$, hence the same inequality should hold for their leading coefficients. The leading coefficient on the right is

$$\frac{(2m - 1)! \cdot 1}{(m - 1)!^2 \cdot l^{2m}}.$$ 

On the left, $k^{2l}$ only occurs when $L = l - 1$. But then $\tau(i) + \tau(j) - 1 - r \geq 0$ and $l - L + r - \tau(i) - \tau(j) = r + 1 - \tau(i) - \tau(j) \geq 0$, so $r = \tau(i) + \tau(j) - 1$, and hence

$$h(\tau, L, r, i, j) = h(\tau, l - 1, \tau(i) + \tau(j) - 1, i, j) = \left( \frac{\tau(i) + \tau(j) - 2}{\tau(i) - 1} \right) \left( \frac{2l - \tau(i) - \tau(j)}{l - \tau(i)} \right) = \delta(\tau(i) - 1, \tau(j) - 1)_{l-1}.$$ 

Therefore,

$$[k^{2l}](kf(\tau, k)) = \frac{1}{(2l - 1)!} \sum_{i,j=1}^{m} \delta(\tau(i) - 1, \tau(j) - 1)_{l-1}.$$ 

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so

\[ \sum_{i,j=1}^{m} [i-1,j-1]_{m-1}[\tau(i)-1,\tau(j)-1]_{l-1} \geq \frac{(2m-1)!}{(m-1)!^2} \frac{(2l-1)!}{(l)!^2}. \]

Now, letting \( m \leftarrow m + 1, l \leftarrow l + 1 \), then \( \bar{\tau}(i) \leftarrow \tau(i+1) - 1 \), we obtain the inequality \([\mathcal{E}]\). \( \square \)

Note that, for \( l = 0 \) (which includes the case \( m = 0 \)), the inequality \([\mathcal{E}]\) becomes an equality. We conjecture, however, that the strict inequality holds if \( l > 0 \), i.e. if \( \tau \) is not a constant string.

As in the case of patterns in permutations, it would be interesting to characterize the patterns \( \tau \in \mathcal{L}^m \), where the difference between the two sides of \([\mathcal{E}]\) is minimal.

We also note that the covariance \( \text{Cov}(X_{\tau_1}, X_{\tau_2}) \) of patterns \( \tau_1, \tau_2 \in \mathcal{L}^m \) is positive (resp. negative) if

\[ \sum_{i,j=1}^{m} [i-1,j-1]_{m-1}[\tau_1(i)-1,\tau_2(j)-1]_{l-1} - \frac{(2m-1)!}{(m-1)!^2} \frac{(2l-1)!}{(l)!^2} \]

is positive (resp. negative). Hence, it would be interesting to characterize pairs of patterns \( \tau_1, \tau_2 \in \mathcal{L}^m \) based on the sign of the covariance \( \text{Cov}(X_{\tau_1}, X_{\tau_2}) \).

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