Abstract. In this paper, we show that if a quasi-Anosov diffeomorphism has the various types of shadowing property then it is Anosov.

1. Introduction

Let $M$ be a closed smooth Riemannian manifold and let $f : M \to M$ be a diffeomorphism. Denote by Diff($M$) the set of all diffeomorphisms of $M$ endowed with the $C^1$ topology.

Let $f \in \text{Diff}(M)$ and let $\Lambda$ be a closed $f$-invariant set. We say that $\Lambda$ is hyperbolic for $f$ if the tangent bundle $T_\Lambda M$ has a $Df$-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

\[ \|D_x f^n|_{E^s_x}\| \leq C\lambda^n \quad \text{and} \quad \|D_x f^{-n}|_{E^u_x}\| \leq C\lambda^n \]

for all $x \in \Lambda$ and $n \geq 0$. If $\Lambda = M$ then we say that $f$ is Anosov. We say that $f$ is quasi-Anosov if for every $0 \neq v \in TM$ then set $\{\|Df^n(v)\| : n \in \mathbb{Z}\}$ is unbounded.

Note that a quasi-Anosov diffeomorphism $f$ is not Anosov (see [3]). But if $\dim M = 2$ then a Anosov diffeomorphism is a quasi-Anosov diffeomorphism (see [3]). A point $p \in M$ is said to be periodic if there is $n > 0$ such that $f^n(p) = p$. Denote by $P(f)$ the set of all periodic points of $f$. A point $x \in M$ is said to be non-wandering if for any neighborhood $U$ of $x$ there is $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. Denote by $\Omega(f)$ the set of all non-wandering points of $f$. It is clear that $P(f) \subset \Omega(f)$. We
say that $f$ satisfies Axiom A if $\Omega(f) = \overline{P(f)}$ is hyperbolic. We say that $f$ is structurally stable if there is a neighborhood $\mathcal{U}(f) \subset \text{Diff}(M)$ such that for every $g \in \mathcal{U}(f)$, there is a homeomorphism $h : M \to M$ such that $f \circ h = h \circ g$. We define the stable set of $x$ as follows:\ \[ W^s(x) = \{ y \in M : d(f^n(x), f^n(y)) \to 0 \quad \text{as} \quad n \to \infty \} \]\ \[ W^u(x) = \{ y \in M : d(f^n(x), f^n(y)) \to 0 \quad \text{as} \quad n \to -\infty \} \]. We say that an Axiom A diffeomorphism $f$ satisfies the transversality condition if for any $x \in M$, $T_xM = T_xW^s(x) + T_xW^u(x)$. In [8], Mañé proved that a diffeomorphism $f$ is Anosov if and only if $f$ is quasi-Anosov and satisfies the transversality condition if and only if $f$ is quasi-Anosov and structurally stable.

For $\delta > 0$, a sequence of points $\{x_i\}_{i \in \mathbb{Z}}$ in $M$ is called a $\delta$-pseudo orbit of $f$ if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. We say that $f$ has the shadowing property if for every $\epsilon > 0$ there is $\delta > 0$ such that for any $\delta$-pseudo orbit $\{x_i\}_{i \in \mathbb{Z}}$, there is a point $y \in M$ such that $d(f^i(y), x_i) < \epsilon$ for all $i \in \mathbb{Z}$. Mañé [8] proved that a quasi-Anosov diffeomorphism $f$ if and only if an Axiom A diffeomorphisms satisfying $T_xW^s(x) \cap T_xW^u(x) = \{0_x\}$ for every $x \in M$.

Sakai [11] proved that every quasi-Anosov diffeomorphism with shadowing property is Anosov. From the results, we consider that if a quasi-Anosov diffeomorphism with the various shadowing properties (asymptotic average shadowing, average shadowing, ergodic shadowing) then it is Anosov.

The asymptotic average shadowing property introduced by Gu [5]. A sequence $\{x_i\}_{i \in \mathbb{Z}}$ is called an asymptotic average pseudo orbit of $f$ if

$$\lim_{n \to \infty} \frac{1}{2n} \sum_{i=-n}^{n-1} d(f(x_i), x_{i+1}) = 0.$$ 

A sequence $\{x_i\}_{i \in \mathbb{Z}}$ is said to be asymptotic average shadowed in average by the point $z$ in $M$ if

$$\lim_{n \to \infty} \frac{1}{2n} \sum_{i=-n}^{n-1} d(f^i(z), x_i) = 0.$$ 

We say that $f$ has the asymptotic average shadowing property if every asymptotic average pseudo orbit of $f$ can be asymptotic average shadowed in average by some point in $M$. The average shadowing property was introduced by Blank [1]. For $\delta > 0$, a sequence $\{x_i\}_{i \in \mathbb{Z}}$ of points in $M$ is called a $\delta$-average pseudo orbit of $f$ if there is $N(\delta) > 0$ such that
for all $n \geq N, k \in \mathbb{Z}$,
\[
\frac{1}{n} \sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta.
\]

We say that $f$ has the \textit{average shadowing property} if for any $\epsilon > 0$ there is a $\delta > 0$ such that every $\delta$-average pseudo orbit $\{x_i\}_{i \in \mathbb{Z}}$ is $\epsilon$-shadowed in average by some $z \in M$, that is,
\[
\limsup_{n \to \infty} \frac{1}{2n} \sum_{i=-n}^{n-1} d(f^i(z), x_i) < \epsilon.
\]

The notion of ergodic shadowing property for continuous onto maps over compact metric spaces was defined by Fakhari and Ghane in \cite{2}. For any $\delta > 0$, a sequence $\xi = \{x_i\}_{i \in \mathbb{Z}}$ is $\delta$-ergodic pseudo orbit of $f$ if for $Np^+_n(\xi, f, \delta) = \{i : d(f(x_i), x_{i+1}) \geq \delta\} \cap \{0, 1, \ldots, n-1\}$, and $Np^-_n(\xi, f, \delta) = \{i : d(f(x_i), x_{i+1}) \geq \delta\} \cap \{0, -1, \ldots, -n+1\},$
\[
\lim_{n \to \infty} \frac{\#Np^+_n(\xi, f, \delta)}{n} = 0 \text{ and } \lim_{n \to \infty} \frac{\#Np^-_n(\xi, f, \delta)}{n} = 0.
\]

We say that $f$ has the \textit{ergodic shadowing property} if for any $\epsilon > 0$, there is a $\delta > 0$ such that every $\delta$-ergodic pseudo orbit $\xi = \{x_i\}_{i \in \mathbb{Z}}$ of $f$ there is a point $z \in M$ such that for $Ns^+_n(\xi, f, z, \epsilon) = \{i : d(f^i(z), x_i) \geq \epsilon\} \cap \{0, 1, \ldots, n-1\}$, and $Ns^-_n(\xi, f, z, \epsilon) = \{i : d(f^i(z), x_i) \geq \epsilon\} \cap \{0, -1, \ldots, -n+1\},$
\[
\lim_{n \to \infty} \frac{\#Ns^+_n(\xi, f, z, \epsilon)}{N} = 0 \text{ and } \lim_{n \to \infty} \frac{\#Ns^-_n(\xi, f, z, \epsilon)}{N} = 0.
\]

Then we have the following which is a main theorem in this paper.

**Theorem 1.1.** Let $f \in \text{Diff}(M)$ be quasi-Anosov. If any of the following statements hold:
(a) $f$ has the asymptotic average shadowing property,
(b) $f$ has the average shadowing property,
(c) $f$ has the ergodic shadowing property,
then $f$ is Anosov.

2. Proof of Theorem 1.1

Let $M$ be as before and let $f \in \text{Diff}(M)$. For given $x, y \in M$, we write $x \leadsto y$ if for any $\delta > 0$, there is a finite $\delta$-pseudo orbit $\{x_i\}_{i=0}^{n}(n \geq 1)$ of $f$ such that $x_0 = x$ and $x_n = y$. For any $x, y \in \Lambda$, we write $x \leadsto \Lambda y$
if $x \sim y$ and $\{x_i\}_{i=0}^n \subset \Lambda(n \geq 1)$. We say that the set $\mathcal{C}(f)$ is chain transitive if for any $x, y \in \mathcal{C}(f)$, $x \sim_{\mathcal{C}(f)} y$. If $\mathcal{C}(f) = M$ then $f$ is said to be chain transitive.

We say that $f$ is robustly chain transitive if there are a $C^1$ neighborhood $\mathcal{U}(f)$ of $f$ and a neighborhood $U$ of $\mathcal{C}(f)$ such that for any $g \in \mathcal{U}(f)$, $\Lambda_g(U) = \bigcap_{n \in \mathbb{Z}} g^n(U)$ is chain transitive, where $\Lambda_g(U)$ is the continuation of $\mathcal{C}(f)$. Lee [7] proved that for any periodic points $p, q \in \mathcal{C}(f)$ if $\mathcal{C}(f)$ is robustly chain transitive and $\text{index}(p) = \text{index}(q)$ then it is hyperbolic, where $\text{index}(p) = \dim W^s(p)$.

Lee and Park [6] proved that $C^1$ generically, if a diffeomorphism $f$ has the asymptotic average, or average shadowing property and $W^s(p) \cap W^u(q) \neq \emptyset$ and $W^u(p) \cap W^s(q) \neq \emptyset$ then it is hyperbolic. For that, chain transitive diffeomorphisms and various types of shadowing properties are related to the hyperbolicity. The set $\{x \in M : x \sim x\}$ is called the chain recurrent set of $f$ and is denoted by $R(f)$. It is easy to see that the set is closed and $f(R(f)) = R(f)$. The relation $\sim$ induces an equivalence relation on $R(f)$ whose equivalence classes are called chain component of $f$ and is denoted by $C_f$. In general, the chain component is a closed and invariant set. Note that a chain component $C_f$ is a maximal chain transitive set.

**Lemma 2.1.** If $f$ is chain transitive then the chain recurrence set $R(f)$ is $M$.

*Proof.* Clearly, $R(f) \subset M$. Thus we show that $M \subset R(f)$. Note that a chain component $C_f$ is a maximal chain transitive. Since $f$ is chain transitive, we know that $M$ is contained in a chain component $C_f$. Since the chain component $C_f \subset R(f)$, we have $M \subset R(f)$. Thus if $f$ is chain transitive then $R(f) = M$. \hfill $\square$

**Lemma 2.2.** Let $f \in \text{Diff}(M)$ be $\Omega$-stable. If $f$ is chain transitive then it is Anosov.

*Proof.* Suppose that $f$ is $\Omega$-stable. Note that if $f$ is $\Omega$-stable then $f$ satisfies Axiom A without cycles (see [9]). Since $f$ satisfies Axiom A, we know that $\Omega(f) = \overline{P(f)}$ is hyperbolic. The result of Franks and Selgrade [4, Theorem A], that is, the chain recurrence set $R(f)$ is hyperbolic if and only if it is $\Omega$-stable. Since $f$ is chain transitive, by Lemma 2.1, $R(f) = M$. Since $f$ is $\Omega$-stable, $\Omega(f) = R(f) = M$ is hyperbolic. Thus $f$ is Anosov. \hfill $\square$

Gu [5, Theorem 3.1] proved that if a diffeomorphism $f$ has the asymptotic average shadowing property then it is chain transitive, Park and
Zhang [10, Theorem 3.4] proved that if a diffeomorphism $f$ has the average shadowing property then it is chain transitive. Fakhari and Ghane [2, Lemma 3.1] proved that if a diffeomorphism $f$ has the ergodic shadowing property then it is chain transitive. From the above results, we rewrite as the following.

**Lemma 2.3.** If $f$ has the asymptotic average, average, ergodic shadowing property then it is chain transitive.

**Proof of Theorem 1.1.** Let $f$ be a quasi-Anosov diffeomorphism. Suppose that a diffeomorphism $f$ has the asymptotic average, average, or ergodic shadowing property. By Lemma 2.3, it is chain transitive. Since $f$ satisfies Axiom A, by Lemma 2.2 $f$ is Anosov. □

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