Generalized second law of thermodynamics in $f(T, T_G)$ gravity

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Abstract We discuss the equilibrium picture of thermodynamic at the apparent horizon of FRW universe in $f(T, T_G)$ gravity, where $T$ represents the torsion invariant and $T_G$ is the teleparallel equivalent of the Gauss-Bonnet term. It is found that one can translate the Friedmann equations to the standard form of first law of thermodynamics. We discuss GSLT in the locality of assumption that temperature of matter inside the horizon is similar to that of apparent horizon. Furthermore, we consider particular models in this theory and generate constraints on the coupling parameters for the validity of GSLT. For this purpose we set the present day values of cosmic parameters and find the possible constraints on $f(T, T_G)$ models. We also choose the power law cosmology and found that GSLT can be met in accelerated cosmic expansion. We have also presented the cosmological reconstruction of some viable $f(T, T_G)$ models and discussed the cosmic evolution and validity of GSLT.

Keywords $f(T, T_G)$ theory · Dark energy · Thermodynamics

1 Introduction

The discovery of black hole (BH) thermodynamics suggest that there is a fundamental connection between relativistic gravity and thermodynamics laws, however people have been trying to find a significant way to develop such connection (Bekenstein 1973; Hawking 1975). BH acts as a thermodynamic system with temperature being related to surface gravity and entropy with horizon area (Bardeen et al. 1973). Jacobson (1995) unveiled the issue of relating BH thermodynamics to the Einstein gravity and derived Einstein field equations in local Rindler spacetime using the entropy $S = A/4G$ and Clausius relation $T dS = dQ$. Frolov and Kofman (2003) showed that for the flat quasi de-Sitter geometry of inflationary universe, Friedmann equations can result from $dE = T dS$ for slowly rolling scalar field. In (Padmanabhan 2002), Padmanabhan explored such connection in case of spherically symmetric BHs and showed that field equations can be stated in the form $dE + P dV = T dS$. This study is further extended to generic static spacetimes in Lanczos-Lovelock gravity and shown that the near-horizon field equations again represent a thermodynamic identity in all these models (Paranjape et al. 2006; Kothawala and Padmanabhan 2009).

Cai and Kim (2005) showed that Friedmann equations with any spatial curvature can be derived from the Clausius relation $T dS = dQ$. The relation between gravity and thermodynamics has also been tested in Einstein as well as Gauss-Bonnet and Lovelock gravities. Cai and Cao (2007a) showed that Friedmann equations in braneworld scenario can be cast to the form of first law of thermodynamics at the apparent horizon. This work is also extended in the framework of warped DGP braneworld (Cai and Cao 2007a) and Gauss-Bonnet Braneworld (Cai and Cao 2007b). Akbar and Cai (2007) found that formulation of thermodynamic laws in $f(R)$ and scalar tenser gravities is not trivial when compared to Einstein gravity and Clausius relation is to be modified. In this perspective, Eling et al. (2006) studied the thermodynamic laws in $f(R)$ gravity and remarked that non-equilibrium description of thermodyn-
ics needed, whereby the Clausius relation is modified to the form \( \delta Q = T (dS + d\mathbb{S}) \), where \( d\mathbb{S} \) is the additional entropy term. Cai and Cao (2007b) found that in scalar tensor theories thermodynamics associated with the apparent horizon of the FRW universe results in non-equilibrium description which modifies the standard Clausius relation. The thermodynamics properties have been discussed in various modified theories (Bamba and Geng 2009, 2010, 2011, 2013; Jamil et al. 2010; Karami and Abdolmaleki 2012; Sadjadi 2006; Sheykhi et al. 2007; Sheykhi 2013; Wu et al. 2008).

The development of cosmology and gravitation can be seen as one of the scientific triumphs of the twentieth century. In current situation modified theories of gravity have been appeared as significant tool to discuss various cosmic issues (Bamba et al. 2012a, 2012b; Nojiri and Odintsov 2007, 2011; De Felice and Tsujikawa 2010). The introduction of non-minimal coupling between matter and curvature in the context of modified theories has become a center of interest for the researchers (Allemandi et al. 2005; Nojiri and Odintsov 2004; Alvarenga et al. 2013; Bertolami et al. 2009; Harko et al. 2011; Haghani et al. 2013; Houndjo 2012; Odintsov and Saez-Gomez 2013; Sharif and Harko 2014a, 2014b). Another important and conceptually rich class consists of gravitational modifications involving torsion description of gravity. It is interesting to mention here that teleparallel equivalent of GR has been constructed by Einstein himself by including torsionless Levi-Civita connection instead of curvatureless Weitzenböck connection and the vielbein as the fundamental ingredient for the theory (Moller 1961; Pellegrini and Plebanski 1963). Harko et al. (2014) constructed a more general type of \( f(T) \) gravity by introducing a non-minimal interaction of torsion with matter in the Lagrangian density. We (Zubair and Waheed 2015) have discussed the validity of energy bounds for specific models and find the feasible constraints on the involved free parameters.

Kofinas and Saridakis (2014a, 2014b) and Kofinas et al. (2014) proposed a novel theory namely \( f(T_G) \) gravity and then its generalized form \( f(T, T_G) \) gravity and they also discussed its cosmological significance. In Kofinas and Saridakis (2014a, 2014b) and Kofinas et al. (2014), it has been argued that the Lorentz invariance has been lost because of choice of specific class of frames. This is not a deficit, it is a sort of analogue of gauge fixing in gauge theories. Thus, at this level of formulation of the theory, the question of Lorentz invariance has no meaning anymore. Also, the quantity \( T_G \) is a diffeomorphism invariant containing quartic scalars of the torsion (or contorsion) tensor. However, Lorentz invariance is lost since preferred autoparallel orthonormal frames have been chosen. In case of Einstein gravity, this is not a deficit, it is a sort of analogue of gauge fixing in gauge theories (Kofinas and Saridakis 2014a, 2014b; Kofinas et al. 2014).

Recently, we have discussed the energy condition bounds in this modified gravity and tested two well known models (Waheed and Zubair 2015) which are proposed in Kofinas and Saridakis (2014a, 2014b) and Kofinas et al. (2014). In this study, we are interested to explore laws of thermodynamics in \( f(T, T_G) \) gravity which is a more generic modified theory involving torsion and Gauss-Bonnet contributions (Kofinas and Saridakis 2014a, 2014b; Kofinas et al. 2014). In previous studies, we have explored the issue of equilibrium thermodynamics in \( f(R, T) \) (Sharif and Zubair 2012a, 2013a), \( f(R, L_m) \) (Sharif and Zubair 2013b) and \( f(R, T, R_\mu\nu T^{\mu\nu}) \) (Sharif and Zubair 2013c) theories of gravity. We find that equilibrium picture of thermodynamics in such theories needs more study to follow. However, in this paper we find that one can develop the equilibrium picture of thermodynamics in generic modified theory models which involve contribution from torsion scalar. The paper has the following format: In Sect. 2, we present the general formalism of field equation in \( f(T, T_G) \) gravity for FRW universe. In Sect. 3, the first law of thermodynamics (FLT) is established and we discuss the validity of GSLT for different \( f(T, T_G) \) models in Sect. 4. In Sect. 5, we have presented the cosmological analysis of \( f(T, T_G) \) models corresponding to different scale factors. In Sect. 6, we have included discussion on stability of the reconstructed models. Finally, Sect. 7 summarizes our findings.

2 \( f(T, T_G) \) gravity

In Kofinas and Saridakis (2014a, 2014b) and Kofinas et al. (2014), Kofinas and Saridakis proposed a modified theory involving both torsion scalar \( T \) and the teleparallel equivalent of Gauss-Bonnet term \( T_G \) as basic ingredient, defined by the following action (Kofinas and Saridakis 2014a, 2014b; Kofinas et al. 2014)

\[
S = \frac{1}{2\kappa^2} \int_M d^4x e f(T, T_G),
\]

(1)

where \( e = \det(e^a_\mu) = \sqrt{\text{det} g} \) and \( \kappa^2 = 8\pi G \). In some certain limits of the function \( f(T, T_G) \), other theories like GR, TTEGR, Einstein-Gauss-Bonnet theory etc. can be discussed.

We consider the flat FRW universe model with \( a(t) \) as expansion scalar given by

\[
ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2).
\]

(2)

The diagonal vielbein and the dual vielbein for this metric are

\[
e^a_\mu = \text{diag}(1, a(t), a(t), a(t)),
\]

\[
e^a_{\mu} = (1, a^{-1}(t), a^{-1}(t), a^{-1}(t)),
\]
while the corresponding determinant is given by $e = a(t)^3$.
The torsion scalar and Gauss-Bonnet equivalent term $T_G$ for this geometry are

$$T = 6H^2, \quad T_G = 24H^2(\ddot{H} + H^2),$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter. In this study, we consider the matter action $S_m = \int L_{\text{matter}} \sqrt{-g} dx^4$ corresponding to matter energy momentum tensor $\Theta_{\mu\nu}$, which is assumed as perfect fluid. Now the variation of action $S + S_m$ implies the following gravitational equations for FRW geometry

$$f - 12H^2 f_T - T_G f_{T_G} + 24H^3 \dot{f}_{T_G} = 2\kappa^2 \rho_m,$$

$$f - 4(\dot{H} + 3H^2) f_T - 4H \dot{f}_T - T_G f_{T_G} + \frac{2}{3H} T_G \dot{f}_{T_G} + 8H^2 \dot{f}_{T_G} = -2\kappa^2 p_m,$$

where $\rho_m$ and $p_m$ indicates the density and pressure of ordinary matter, $f f_T$, $f T_T$, etc. represent the second and higher-order derivatives with respect to $T$ and $T_G$ respectively. Moreover dot represents the time derivative and these derivatives are given by

$$\dot{f}_T = f_T \ddot{T} + f_{TT} \dot{T}_G, \quad \dot{f}_{T_G} = f_{TT} \dot{T} + f_{T_T} \ddot{T}_G,$$

$$\ddot{f}_{T_G} = f_T T_{\dot{T}} + 2 f_{T_T} \dot{T}_G + f_{T_T} \dddot{T_G} + f_{T_{TT}} \dot{T}_G + f_{T_G} \dddot{T}_G,$$

where the derivatives of torsion scalar $T$ and teleparallel equivalent to Gauss-Bonnet term $T_G$ can be set in terms of Hubble parameter $H$ as

$$\ddot{T} = 12H \dddot{H}, \quad \dot{T}_G = 24H^2(\ddot{H} + 2H \dot{H}) + 48H \dddot{H}(\dot{H} + H^2),$$

$$\dddot{T} = 12(\dddot{H}^2 + H \dddot{H}), \quad \dddot{T}_G = 48H^3 + 144H \dddot{H} \dddot{H} + 288\dot{H}^2 H^2 + 24H^2 \dddot{H} + 96H^3 \dot{H},$$

The dynamical equations (4) and (5) can be rewritten as

$$H^2 = \frac{\kappa^2}{3}(\rho_m + \rho_\phi),$$

$$\dot{H} = -\frac{\kappa^2}{2}(\rho_m + p_m + \rho_\phi + p_\phi),$$

where $\rho_\phi$ and $p_\phi$ are the density and pressure of dark energy, respectively given by

$$\rho_\phi = \frac{1}{2\kappa^2} \left[ 6H^2 - f + 12H^2 f_T + T_G f_{T_G} \right. \left. - 24H^3 \dot{f}_{T_G} \right].$$

For FRW spacetime, the energy density $\rho_\phi$ and pressure $p_\phi$ of torsion contributions satisfy the following relation

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = 0.$$

### 3 Thermodynamics in $f(T, T_G)$ gravity

Here, we discuss the first and second laws of thermodynamics at the apparent horizon of FRW universe in $f(T, T_G)$ gravity.

#### 3.1 First law of thermodynamics

The condition $h^\alpha\beta \partial_\alpha \tilde{r} \partial_\beta \tilde{r} = 0$, implies the radius of dynamical apparent horizon. For flat FRW geometry, radius $\tilde{r}_A$ is

$$\tilde{r}_A = \frac{1}{H}.$$  

Taking the time derivative of the above equation, it follows that

$$\frac{d\tilde{r}_A}{dt} = H \tilde{r}_A.$$  

Substituting above result in Eq. (9), we get

$$\frac{1}{2\pi \tilde{r}_A} \left( \frac{2\pi \tilde{r}_A d\tilde{r}_A}{G} \right) = 4\pi \tilde{r}_A^3 H \left[ \rho_m + p_m + \frac{1}{16\pi G} (-4\dot{H}(1 + f_T) - 4H \dot{f}_T + (16H \ddot{H} - 8H^3) \dot{f}_{T_G} + 8H^2 \dot{f}_{T_G}) \right] dt. \quad (14)$$

Multiplying Eq. (14) with $(1 - \frac{\tilde{r}_A}{2H \tilde{r}_A})$, it leads to

$$\frac{|\kappa_{sg}|}{2\pi} \int \left( \frac{A}{4G} \right) $$

$$= \left( 1 - \frac{\tilde{r}_A}{2H \tilde{r}_A} \right) \tilde{r}_A A H \left[ \rho_m + p_m + \frac{1}{16\pi G} (-4\dot{H}(1 + f_T) - 4H \dot{f}_T + (16H \ddot{H} - 8H^3) \dot{f}_{T_G} + 8H^2 \dot{f}_{T_G}) \right] dt. \quad (15)$$

where $A = 4\pi \tilde{r}_A^2$ is the area of apparent horizon, $\kappa_{sg} = \frac{1}{A}(1 - \frac{\tilde{r}_A}{2H \tilde{r}_A})$ is the surface gravity and $T = \frac{|\kappa_{sg}|}{2\pi}$ is identified as temperature of apparent horizon. Furthermore,
Eq. (15) involves Bekenstein-Hawking entropy relation (Bekenstein 1973; Bardeen et al. 1973; Hawking 1975) $S = A/4G$ defined in Einstein gravity. Consequently, Eq. (15) can be rewritten as

$$T dS = \tilde{r}_A \Delta H (\rho_m + p_m) dt - 2\pi \tilde{r}_A^2 (\rho_m + p_m) d\tilde{r}_A + \frac{(\tilde{r}_A \Delta H - 2\pi \tilde{r}_A^2 \tilde{\dot{r}}_A)}{16\pi G} (-4\dot{H} (1 + f_T) - 4H \dot{f}_T) + (16H \dot{H} - 8H^3) \dot{f}_T G + 8H^2 \dot{f}_G G dt.$$  \hspace{1cm} (16)

The matter energy density inside the apparent horizon is defined by the relation $E = V \rho_{tot}$ with $V = 4/3\pi \tilde{r}_A^3$ and for this theory it results in

$$dE = 4\pi \tilde{r}_A^2 (\rho_m + \rho_\phi) d\tilde{r}_A - 4\pi \tilde{r}_A^3 (\rho_m + \rho_\phi + p_m + p_\phi) H dt,$$  \hspace{1cm} (17)

where we have employed the standard continuity equation which holds in $f(T, T_G)$ gravity.

Inserting $dE$ in Eq. (16), we get

$$T dS = -dE + 2\pi \tilde{r}_A^2 (\rho_m + \rho_\phi - p_m - p_\phi) d\tilde{r}_A.$$  \hspace{1cm} (18)

Now introducing the total work density which is defined as (Izquierdo and Pavon 2006)

$$W = -\frac{1}{2} T^{(tot)\alpha\beta} h_{\alpha\beta} = \frac{1}{2} (\rho_{tot} - p_{tot}).$$  \hspace{1cm} (19)

Equation (18) takes the standard form of FLT

$$T dS = -dE + dW,$$  \hspace{1cm} (20)

which is FLT in $f(T, T_G)$ gravity identical to that in Einstein, Gauss-Bonnet and Lovelock gravities and usual FLT is satisfied by the respective field equations (Cai and Cao 2007a; Sheykhi et al. 2007).

4 GSLT via $f(T, T_G)$ models

Here, we explore the validity of GSLT in the framework of $f(T, T_G)$ gravity at the apparent horizon. According to GSLT, the sum of the horizon entropy and entropy of ordinary matter fluid components is not decreasing with time (Cai and Kim 2005). In literature, it is shown that GSLT can be met in the framework of modified theories of gravity (Bamba and Geng 2009, 2010, 2011, 2013; Jamil et al. 2010; Karami and Abdolmaleki 2012; Sadjadi 2006; Sharif and Zubair 2012a, 2013a, 2013b, 2013c; Sheykhi et al. 2007; Sheykhi 2013; Wu et al. 2008). It would be interesting to examine the GSLT in $f(T, T_G)$ modified theory. The Gibb’s equation which relates the entropy of matter and energy sources inside the horizon $S_{in}$ to the density and pressure in the horizon is defined as

$$T\text{ind} S_{in} = d(\rho_m V) + p_m dV.$$  \hspace{1cm} (21)

Equivalently, it can be expressed as

$$T\text{ind} S_{in} = 4\pi \tilde{r}_A^3 (\rho_m + p_m) (\dot{\tilde{r}}_A - H \tilde{\dot{r}}_A),$$  \hspace{1cm} (22)

where $\rho_m$ and $p_m$ can be evaluated of the form

$$\rho_m = \frac{1}{2} [f - 12H^2 f_T - T_G f_T + 24H^3 \dot{f}_T G],$$  \hspace{1cm} (23)

$$p_m = \frac{1}{2} [-f + 4(\dot{H} + 3H^2) f_T + 4H \dot{f}_T + T_G f_T G$$

$$- \frac{2}{3H} T_G \dot{f}_T G - 8H^2 \ddot{f}_G G].$$  \hspace{1cm} (24)

Substituting Eqs. (22) and (23) in Eq. (22), it follows

$$T\text{ind} S_{in} = 2\pi \tilde{r}_A^3 (\dot{\tilde{r}}_A - 4H \tilde{\dot{r}}_A) [\delta_t (4H f_T)$$

$$+ (8H^3 - 16H \dot{H}) \dot{f}_T G - 8H^2 \ddot{f}_T G].$$  \hspace{1cm} (25)

Using the horizon entropy relation, one can find

$$T\text{h} S_{\text{h}} = \frac{1}{2\tilde{r}_A H G} (2H \tilde{\dot{r}}_A \tilde{\dot{r}}_A - \tilde{r}_A^2).$$  \hspace{1cm} (26)

Now we discuss the validity of GSLT which requires $(\tilde{r}_h \dot{S}_h + \tilde{r}_i \dot{S}_i) \geq 0$. In this setting, we assume a relation between the temperature of matter and energy sources within the horizon and temperature of apparent horizon i.e., $T_h = bT_h$, where $0 < b < 1$. It is natural to assume a relation between the temperature of apparent horizon and entire contents within the horizon which results in thermal equilibrium for the choice of $b = 1$. Generally speaking, the horizon temperature varies from the temperature of all energy sources inside the horizon and this variation makes the spontaneous flow of energy between the horizon and fluid components so that thermal equilibrium is no longer preserved (Izquierdo and Pavon 2006). Here, we are discussing the equilibrium description of thermodynamics in $f(T, T_G)$ gravity, so that we limit our results to the case of thermal equilibrium $b = 1$ i.e., the horizon temperature is equal to that of fluid components inside the horizon.

After some manipulation Eqs. (25) and (26) can be summed to the following form

$$T\text{h} S_{\text{tot}} = \frac{-\dot{H}}{2G H^4} (2H^2 + \dot{H}) - \frac{2\pi}{H^4} (H^2 + \dot{H}) [\delta_t (4H f_T)$$

$$+ (8H^3 - 16H \dot{H}) \dot{f}_T G - 8H^2 \ddot{f}_T G].$$  \hspace{1cm} (27)

which is a condition to validate the GSLT in $f(T, T_G)$ gravity and it can be verified for different choices of Lagrangian.
To illustrate the validity of GSLT in \( f(T, T_G) \) gravity, we consider some generic \( f(T, T_G) \) models of the following form (Kofinas and Saridakis 2014a, 2014b; Kofinas et al. 2014):

1. \( f(T, T_G) = -T + \alpha_1 T^2 + \alpha_2 T_G \),
2. \( f(T, T_G) = -T + \beta_1 T^2 + \beta_2 T_G + \alpha_1 T^2 + \alpha_2 T \sqrt{|T_G|} \),
3. \( f(T, T_G) = -T + \beta_1 (T^2 + \beta_2 T_G) + \beta_3 (T^2 + \beta_4 T_G)^2 \),

where \( T_G \) contains the quartic torsion term, \( \alpha_i \)'s and \( \beta_i \)'s are dimensionless coupling parameters. These models have been proposed in Kofinas and Saridakis (2014a, 2014b) and Kofinas et al. (2014), where authors discussed the phase space analysis and expansion history from early-times inflation to late-times cosmic acceleration with no need of introducing cosmological constant. It is found that effective EoS parameter can represent different eras of the universe, namely, quintessence, phantom and quintom phase (crossing of phantom divide line).

- \( f(T, T_G) = -T + \alpha_1 T^2 + \alpha_2 T_G \)

Initially, we consider the model of the form \( f(T, T_G) = -T + \alpha_1 T^2 + \alpha_2 T_G \), where \( \alpha_i \)'s are constrained for the validity of GSLT.

For the above model, one can find GSLT of the following form

\[
T_h \dot{S}_{tot} = \frac{-\dot{H}}{2G H^4} \left(2 H^2 + 3 \dot{H} + \frac{2 \pi}{H^4} (H^2 + \dot{H}) \right) \\
\times \left[ 4 \dot{H} (T + \alpha_1 T^2 + \alpha_2 T_G) \right. \\
\left. + 4 H \left\{ \frac{\alpha_1}{2} (2 T^2 + \alpha_2 T_G) (T^2 + \alpha_2 T_G)^{-3/2} \right. \\
\left. + \frac{\alpha_1}{2} (2 T^2 + \alpha_2 T_G) (T^2 + \alpha_2 T_G)^{-3/2} \right. \\
\left. + (8 H^3 - 16 H \dot{H}) \left\{ -\frac{\alpha_1 \alpha_2}{4} (T^2 + \alpha_2 T_G)^{-3/2} \right. \\
\left. \times (2 T \dot{T} + \alpha_2 \dot{T}_G) \right) - 8 H^2 \right. \\
\left. \times \left\{ \frac{\alpha_1 \alpha_2}{4} (T^2 + \alpha_2 T_G)^{-3/2} \left\{ \frac{3}{2} (T^2 + \alpha_2 T_G)^{-1} \right. \\
\left. \times (2 T \dot{T} + \alpha_2 \dot{T}_G) + (2 \dot{T}^2 + 2 T \dot{T} + \alpha_2 \ddot{T}_G) \right) \right\} \right].
\]

(28)

Here, we define some cosmic parameters namely deceleration, jerk and snap parameters in terms of \( H \) as

\[
q = -\left(1 + \frac{\dot{H}}{H^2}\right), \quad r = 2q^2 + q - \frac{\dot{q}}{H}, \\
s = \frac{(r - 1)}{3(q - 1/2)}.
\]

Fig. 1 Evolution of GSLT for the model \( f(T, T_G) = -T + \alpha_1 \sqrt{T^2 + \alpha_2 T_G} \) in terms of coupling parameters \( \alpha_1 \) and \( \alpha_2 \).

so that the time derivatives of \( H \) can be expressed in terms of these parameters as

\[
\dot{H} = -H^2 (1 + q), \quad \ddot{H} = H^3 (j + 3q + 2), \\
\dddot{H} = H^4 (s - 4j - 3q(q + 4) - 6).
\]

Hence, one can represent \( T, T_G \) and their derivatives in terms of recent value of Hubble parameter \( H_0 \) and cosmic parameters of the following form

\[
T = 6 H_0^2, \quad T_G = -24 q H_0^4, \quad \dot{T} = 12 H_0^3 (1 + q), \\
\dot{T}_G = 48 H_0^4 (1 + q)^2 - 96 H_0^5 (1 + q) + 24 H_0^5 (j + 3q + 2), \\
\ddot{T} = 12 H_0^4 (1 + q)^2 + 12 H_0^4 (j + 3q + 2), \\
\dddot{T}_G = -48 H_0^5 (1 + q)^3 - 144 H_0^6 (1 + q) (j + 3q + 2) \\
+ 288 H_0^6 (1 + q)^2 + 24 H_0^6 (s - 4j - 3q(q + 4) - 6) \\
+ 96 H_0^6 (s + 3q + 2).
\]

Substituting relations (29) in (28), it implies the GSLT in terms of recent values of cosmic parameters. In this study, we set the present day values of Hubble, deceleration, jerk and snap parameters as \( H_0 = 73.8, \quad q_0 = -0.81 \pm 0.14, \quad j_0 = 2.16^{+0.81}_{-0.75} \) and \( s_0 = -0.22^{+0.21}_{-0.19} \) (Rapetti et al. 2007; Riess et al. 2011; Poplawski 2007). In Fig. 1, we show the evolution of GSLT for model 1 in terms of parameters \( \alpha_1 \) and \( \alpha_2 \). It can be seen that GSLT is satisfied for \( \alpha_i > 0 \).

Cosmic expansion history is thought to have experienced the decelerated phase and hence transition to accelerating epoch. Thus, power law solutions can play vital role to connect the matter dominated phase with accelerating paradigm. The existence of power law solutions in FRW setting is particularly relevant to intimate all possible cosmic evolutions. The scale factor for power law cosmology is
defined as

\[ a(t) = a_0 t^m, \]

where \( m \) is a positive real number. If \( 0 < m < 1 \), then the required power law solution is decelerating while for \( m > 1 \) it exhibits accelerating behavior. To be more explicit for the above constraint, we set the power law cosmology for accelerated cosmic expansion (\( m > 1 \)). For FRW universe, we show the evolution of GSLT in Fig. 2. It can be seen that validity of GSLT requires \( m > 2 \) with \( a_1 = 14 \) and \( a_2 = 0.02 \).

\[ f(T, TG) = -T + \frac{\sqrt{T^3} + \alpha_2 TG}{\sqrt{T^2}} \]

This model is modified version of previous model and involves higher order correction terms like \( T^2 \) and \( T \sqrt{|T_G|} \). We can represent the GSLT as

\[ T_h \dot{S}_{tot} = \frac{-\dot{H}}{2GH^4} \left( 2H^2 + \dot{H} \right) - \frac{2\pi}{H^4} \left( H^2 + \dot{H} \right) \]

\[ \times \left[ 4H \left( (T^2 + \beta_2 TG)^{-1/2} + 2\alpha_1 \right) \dot{T} + \frac{\alpha_1}{2} \left( (T_G)^{-1/2} \dot{T}_G \right) \right] \]

\[ + 4\dot{H} \left( -1 + \beta_1 \left( T^2 + \beta_2 TG \right)^{-1/2} + 2\alpha_1 \right) \dot{T} + \frac{\alpha_2}{2} \left( \sqrt{|T_G|} \right) \left( 8H^3 - 16H \dot{H} \right) \]

\[ \times \left\{ -\frac{\alpha_2 T}{4} \left( (T_G)^{-3/2} \right) \dot{T}_G \right\} \]

\[ - \frac{\alpha_2 T}{4} \left( (T_G)^{-3/2} \right) \dot{T}_G \right\} \]

\[ - \frac{3\beta_1 \beta_2 T}{2} \left( T^2 + \beta_2 TG \right)^{-5/2} \right\} \dot{T}^2 \]

\[ + \left( \frac{3}{2} \beta_1 \beta_2 T \left( T^2 + \beta_2 TG \right)^{-5/2} \right) \dot{T} \dot{T}_G \]

\[ + \left( \frac{3}{8} \beta_1 \beta_3 (T_G)^{-5/2} \right) \dot{T}_G \]

\[ + \frac{3\alpha_2 T}{8} \left( (T_G)^{-3/2} \right) \dot{T}_G \right\} \right]. \]  

(30)

We first analyze the evolution of GSLT in terms of present day values of cosmic parameters and show the respective behavior in Fig. 3. In left plot we present the validity of GSLT in terms of parameters \( \beta_i \), which can be met only if \( a_i < 0 \). Similarly, in right plot we fix \( \beta_i \) and GSLT is satisfied if \( a_i < 0 \). Furthermore, we consider the power law cosmology and present the validity of GSLT in terms of \( t \) and \( m \) as shown in Fig. 4.

\[ f(T, TG) = -T + \beta_1 \left( T^2 + \beta_2 TG \right) + \beta_3 \left( T^2 + \beta_4 TG \right)^2 \]

Here, \( f(T, TG) \) model involves fourth order torsion terms and second order contribution from \( T_G \). For this model, the derivatives of \( f \) are obtained as

\[ f_T = -1 + 2\beta_1 T + 4\beta_3 T \left( T^2 + \beta_4 TG \right), \]

\[ f_T = 2\beta_1 + 12\beta_3 T^2 + 4\beta_3 \beta_4 T \dot{T} + 4\beta_3 \beta_4 T \dot{T}_G, \]

\[ f_T = 4\beta_3 \beta_4 T \dot{T} + 2\beta_3 \beta_4^2 \dot{T}_G. \]

\[ f_T = 4\beta_3 \beta_4 T \dot{T} + 2\beta_3 \beta_4^2 \dot{T}_G. \]

Using the above expressions we find constraint for GSLT of the form

\[ T_h \dot{S}_{tot} = \frac{-\dot{H}}{2GH^4} \left( 2H^2 + \dot{H} \right) - \frac{2\pi}{H^4} \left( H^2 + \dot{H} \right) \]

\[ \times \left[ 4H \left( (2T^2 + 4\beta_3 \beta_4 T \dot{T} + 4\beta_3 \beta_4 T \dot{T}_G) \dot{T} + 4\beta_3 \beta_4 T \dot{T}_G \right) \right] \]

\[ + \left( \frac{3}{8} \beta_1 \beta_3 (T_G)^{-5/2} \right) \dot{T}_G \right\} \right]. \]  

(31)

One can see that constraint (31) depends only on the parameters \( \beta_1, \beta_3 \) and \( \beta_4 \). To specify the values of these parame-
Generalized second law of thermodynamics in \( f(T, T_G) \) gravity

Fig. 3 Evolution of GSLT for the model \( f(T, T_G) = -T + \beta_1 \sqrt{T^2 + \beta_2 T_G} + \alpha_1 T^2 + \alpha_2 T \sqrt{\sqrt{T_G}} \) versus the parameters \( \alpha_i \) \((i = 1, 2)\) and \( \beta_i \) \((i = 1, 2)\). The left plot corresponds to parameters \( \alpha_1 = -0.2 \) and \( \alpha_2 = -0.1 \) and right plot corresponds to \( \beta_1 = 0.2 \) and \( \beta_2 = 2 \).

Fig. 4 Evolution of GSLT for the model \( f(T, T_G) = -T + \beta_1 \sqrt{T^2 + \beta_2 T_G} + \alpha_1 T^2 + \alpha_2 T \sqrt{\sqrt{T_G}} \) versus \( m \) and \( t \) with \( \alpha_1 = -0.2, \alpha_2 = 0.1, \beta_1 = 0.2 \) and \( \beta_2 = 0.1 \).

5 Cosmological analysis and GSLT via various well-known scale factors

Here we will discuss the cosmological parameters (such as equation of state parameter (EoS), square speed of sound) as well as GSLT for reconstructed \( f(T, T_G) \) functions corresponding to three cosmic scale factors. The EoS parameter can be defined as follows

\[
  w_\beta = \frac{p_\beta}{\rho_\beta},
\]

where \( m > 0 \). This scale gives

\[
  H = \frac{m}{t}, \quad \dot{H} = -\frac{m}{t^2}, \quad T = \frac{6m^2}{t^2},
\]

\[
  T_G = \frac{24(-1 + m)m^3}{t^4}, \quad \dot{T_G} = -\frac{96(-1 + m)m^3}{t^5}.
\]

By inserting the above quantities in Eqs. (9) and (10), we get the function \( f(t) \) numerically then plot it along cosmic time as shown in Fig. 6 (upper left panel) for three different values \( m = 1.2 \) (red), \( m = 1.3 \) (green) and \( m = 1.4 \) (green).

The GSLT in this scenario can be obtained by using Eqs. (27) and (33). It can be observed from Fig. 6 (lower right panel) that The GSLT also remains valid for this reconstructed model.

The sign of \( v_s^2 \) is very important to see the stability of background evolution of the model. A positive value indicates a stable model whereas instability of a given perturbation corresponds to the negative value of \( v_s^2 \).

5.1 Power law scale factor

The power law scale factor can be written as follows

\[
  a(t) = a_0 t^m
\]

where \( m > 0 \). This scale gives

\[
  H = \frac{m}{t}, \quad \dot{H} = -\frac{m}{t^2}, \quad T = \frac{6m^2}{t^2},
\]

\[
  T_G = \frac{24(-1 + m)m^3}{t^4}, \quad \dot{T_G} = -\frac{96(-1 + m)m^3}{t^5}.
\]

By inserting the above quantities in Eqs. (9) and (10), we get the function \( f(t) \) numerically then plot it along cosmic time as shown in Fig. 6 (upper left panel) for three different values \( m = 1.2 \) (red), \( m = 1.3 \) (green) and \( m = 1.4 \) (green).

It can be observed from Figure that the reconstructed function shows increasing behavior with respect to cosmic time. We can also discuss the EoS parameter by using Eqs. (10), (32) and (33) for this reconstructed model. It can be observed that the EoS parameter (upper right panel) shows phantom like behavior throughout cosmic time. Moreover, the square speed of sound can be obtained by using Eqs. (10), (32) and (33). We can observe from lower left panel of Fig. 6 that the square speed of sound remains positive for all three values of \( m \) which exhibits the stability of the model. The GSLT in this scenario can be obtained by using Eqs. (27) and (33). It can be observed from Fig. 6 (lower right panel) that The GSLT also remains valid for this reconstructed model.
Fig. 5 Evolution of GSLT for the model \( f(T, T_G) = -T + \beta_1(T^2 + \beta_2 T_G) + \beta_3(T^2 + \beta_4 T_G)^2 \). In left plot we fix the \( \beta_1 = 2 \) and show the variation of \( \beta_3 \) and \( \beta_4 \). The right plot shows the evolution of GSLT in terms of parameter \( m \) and time \( t \) with \( \beta_1 = 0.1, \beta_3 = 1 \) and \( \beta_4 = 2 \).

Fig. 6 Plots of reconstructed \( f(t) \) (upper left panel), EoS parameter (upper right panel) and GSLT (lower panel) versus \( t \) for power law scale factor.

5.2 Intermediate scale factor

The intermediate form of scale factor can be defined as follows (Barrow et al. 2006)

\[
a(t) = \exp(At^\beta), \quad 0 < \beta < 1, \quad H(t) = \Lambda t^{\beta-1}. \tag{35}
\]

The scale factor is necessary to perform the analysis and therefore working with a hypothetical scale factor may not be consistent with the inflationary scenario. Hence we picked the intermediate scale factor which is also consistent with astrophysical observations (Barrow et al. 2006).

In the similar way, we reconstruct the model numerically and displayed it Fig. 7 (upper left panel) for three different
values $\beta = 2.2$ (red), $\beta = 2.4$ (green) and $\beta = 2.6$ (green), respectively. The reconstructed model exhibits the increasing behavior with the passage of time. The EoS parameter shows phantom behavior of the universe (upper right panel of Fig. 7). The squared speed of sound remains positive for all values of $\beta$ and sustain the stability of the model as shown in Fig. 7. For this reconstructed model, the GSLT remains positive and exhibits the validity.

5.3 Unification of matter dominated and accelerated phases

We consider that the Hubble rate $H$ is given by (Nojiri and Odintsov 2006)

$$H(t) = H_0 + \frac{H_1}{t}. \quad (36)$$

For $t \ll t_0$ (in the early Universe and $H(t) \sim \frac{H_1}{t}$) the universe was filled with perfect fluid with EOS parameter as $w = 1 + \frac{3}{\beta T}$. For $t \gg t_0$ (i.e., $H \rightarrow H_0$) and the universe seems to de-Sitter. So, this form of $H(t)$ provides transition from a matter dominated to the accelerating phase.

For this model, we plot reconstruct the model numerically as shown in Fig. 8 (upper left panel) for three different values $H_0 = 2.2$ (red), $H_0 = 2.3$ (green) and $H_0 = 2.4$ (green), respectively. The reconstructed model exhibits the decreasing behavior initially, then gets steep and afterwards it shows increasing behavior later time. The EoS parameter shows transition from quintessence region, then goes towards phantom region by evolving the $\Lambda$CDM limit of the universe (upper right panel of Fig. 8). The squared speed of sound remains positive and sustain the stability of the model as shown in Fig. 8. For this reconstructed model, the GSLT remains positive and exhibits the validity.

6 Stability analysis

Now, we use squared speed of sound for the stability analysis of $F(T, T_G)$ models. This parameter is given by

$$v_s^2 = \frac{\dot{p}_\theta}{\dot{\rho}_\theta}. \quad (37)$$

The sign of this parameter is very important in order to analyze the stability of model. This depicts the stable behavior for positive $v_s^2$ while its negativity expresses instability of the under consideration model.

- **For Power Law Scale Factor:**
  Figure 9 exhibits the behavior of $v_s^2$ versus $t$ for power law scale factor. The graph represents the stability of the model because $v_s^2 \geq 0$ for all time.
For Intermediate Scale Factor:
In this case, squared speed of sound shows increasing and positive behavior which exhibits the stability of the reconstructed model. The corresponding plot is given in Fig. 10.

For Unifying Scale Factor:
Taking into account the case of unifying model, we plot the squared speed of sound parameter versus \( t \) as shown in Fig. 11. The \( v_s^2 \) represents instability at the present as well as recent future. However, it shows stability of the reconstructed model at the later epoch.

7 Concluding remarks
In this paper, the thermodynamics properties have been discussed in \( f(T,T_G) \) theory, where \( T \) stands for torsion...
and $T_G$ represents the teleparallel equivalent of the Gauss-Bonnet term. We present the equilibrium picture of thermodynamics at the apparent horizon of FRW spacetime. We show that field equations can be cast to the form of FLT dynamics at the apparent horizon of FRW spacetime. We present the equilibrium picture of thermodynamics involving curvature matter coupling (Sharif and Zubair 2012a, 2013a, 2013b, 2013c). The results of this theory coincide with that in Einstein, Gauss-Bonnet, Lovelock and braneworld modified theories (Cai and Cao 2007a, 2007b; Sheykhi et al. 2007).

We also explore the validity of GSLT in the framework of $f(T, T_G)$ gravity. In this perspective, we consider three generic $f(T, T_G)$ models namely, $f(T, T_G) = -T + \alpha_1 \sqrt{T^2 + \omega_2 T_G} \ln(T)$, $f(T, T_G) = -T + \beta_1 \sqrt{T^2 + \omega_2 T_G}$, and $f(T, T_G) = -T + \beta_1 (T^2 + \omega_2 T_G) + \beta_3 (T^2 + \omega_4 T_G^2)$. We set the constraint for GSLT in terms of present day values of Hubble, deceleration, jerk and snap parameters. In Fig. 1, we show the evolution of GSLT for model 1 and it is found to be satisfied if $\alpha_1 > 0$. In this discussion we further consider the power law cosmology and found the constraints for accelerated cosmic expansion. Figure 2 shows that GSLT can be met for $m > 2$ with $\alpha_1 = 14$ and $\alpha_2 = 0.02$. For second model one requires coupling parameters $\alpha_i < 0$ with $\beta_i > 0$. Moreover in case of model 3, we fix $\beta_1 = 2$ and show the variation of $\beta_3$ and $\beta_4$ in left plot of Fig. 5. In right plot the evolution of GSLT for power law cosmology with fixed parameters $\beta_i$.

We have also explored the cosmological parameters as well as validity of GSLT for reconstructed $f(T, T_G)$ models corresponding to three different values of scale factors. We have observed that the reconstructed functions exhibit the increasing behavior for all scale factors. Also, EoS parameter exhibits the phantom behavior for power law as well as intermediate scale factors. However, it has shown transition from quintessence region, then goes towards phantom region by evolving the $\Lambda$CDM limit of the universe for unifying scale factor. The squared speed of sound has sustained the stability of the for all models. Also, the GSLT has remained positive and which exhibits the validity of it for all scale factors.

The authors (Bamba et al. 2012a, 2012b) have constructed $f(T)$ models by realizing inflation in the early universe, the $\Lambda$CDM model, little rip cosmology and pseudo-rip cosmology. They also explored that the disintegration of bound structures for little rip and pseudo-rip cosmologies occurs in the same way as in gravity with corresponding DE fluid. They also found that the second law of thermodynamics can be valid around the finite-time future singularities for the universe with the temperature inside the horizon being the same as that of the apparent horizon. In our case, we have reconstructed $f(T, T_G)$ models by realizing power law, intermediate and unifying scale factors. We have found that GSLT holds for all scale factors.

Moreover, the classification of the future singularities in the following way (Nojiri and Odintsov 2005):

- Type I: for $t \to t_s$, $a \to \infty$, $\rho, |p| \to \infty$, this singularity corresponds to Big Rip singularity.
- Type II: for $t \to t_s$, $a \to a_s$, $\rho \to \rho_s$, $|p| \to \infty$, this singularity corresponds to sudden future singularity.
- Type III: for $t \to t_s$, $a \to a_s$, $\rho, |p| \to \infty$.
- Type IV: for $t \to t_s$, $a \to \infty$, $\rho, |p| \to 0$.

Our scenario correspond to Type I singularity because the EoS parameter attains phantom-like universe which describes the Big Rip singularity in future.

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