Transformation formulas of a character analogue
of \( \log \theta_2 (z) \)

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Abstract

In this paper, transformation formulas for the function

\[
A_1 (z, s : \chi) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \chi(n) \chi(m) (-1)^{m-1} \frac{e^{2\pi i mnz/k}}{n^s-1}
\]

are obtained. Sums that appear in transformation formulas are generalizations of the Hardy–Berndt sums \( s_j(d, c), j = 1, 2, 5 \). As applications of these transformation formulas, reciprocity formulas for these sums are derived and several series relations are presented.

Keywords: Dedekind sums; Hardy-Berndt sums; Bernoulli and Euler polynomials.

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1 Introduction

Hardy sums or Berndt’s arithmetic sums are defined for \( c > 0 \) by

\[
S(d, c) = \sum_{n=1}^{c-1} (-1)^{n+1+\lfloor dn/c \rfloor}, \quad s_1(d, c) = \sum_{n=1}^{c-1} (-1)^{\lfloor dn/c \rfloor} \mathcal{B}_1 \left( \frac{n}{c} \right),
\]

\[
s_2(d, c) = \sum_{n=1}^{c-1} (-1)^n \mathcal{P}_1 \left( \frac{n}{c} \right) \mathcal{P}_1 \left( \frac{dn}{c} \right), \quad s_3(d, c) = \sum_{n=1}^{c-1} (-1)^n \mathcal{P}_1 \left( \frac{dn}{c} \right),
\]

\[
s_4(d, c) = \sum_{n=1}^{c-1} (-1)^{\lfloor dn/c \rfloor}, \quad s_5(d, c) = \sum_{n=1}^{c-1} (-1)^{n+\lfloor dn/c \rfloor} \mathcal{P}_1 \left( \frac{n}{c} \right),
\]

where \( \mathcal{P}_p (x) \) are the Bernoulli functions (see Section 2) and \( \lfloor x \rfloor \) denotes the greatest integer not exceeding \( x \). These sums arise in transformation formulas for the logarithms of the classical theta functions \([6, 17]\). In particular,
Hardy–Berndt sums $s_j(d, c)$ ($j = 1, 2, 5$) appear in transformation formulas of $\log \theta_2(z)$:

Let $Tz = (az + b)/(cz + d)$ where $a, b, c$ and $d$ are integers with $ad - bc = 1$ and $c > 0$. Berndt [6] proves that if $d$ is even, then

$$\log \theta_2(Tz) = \log \theta_4(z) + \frac{1}{2} \log(cz + d) - \frac{\pi i a}{4c} - \frac{\pi i}{4} + \frac{\pi i}{2} s_1(d, c), \quad (1.1)$$

if $c$ is even, then

$$\log \theta_2(Tz) = \log \theta_2(z) + \frac{1}{2} \log(cz + d) + \frac{\pi i a + d}{4c} - \frac{\pi i}{4} - \pi i s_2(d, c), \quad (1.2)$$

and Goldberg [17] shows that if $c$ and $d$ are odd, then

$$\log \theta_2(Tz) = \log \theta_3(z) + \frac{1}{2} \log(cz + d) - \frac{\pi i a}{4c} - \frac{\pi i}{4} + \frac{\pi i}{2} s_5(d, c). \quad (1.3)$$

Moreover, Goldberg [17] shows that these sums also arise in the theory of $r_m(n)$, the number of representations of $n$ as a sum of $m$ integral squares and in the study of the Fourier coefficients of the reciprocals of $\theta_j(z)$, $j = 2, 3, 4$. Analogous to Dedekind sums, these sums also satisfy reciprocity formulas: For coprime positive integers $d$ and $c$ we have [6,17]

$$s_1(d, c) - 2s_2(c, d) = \frac{1}{2} - \frac{1}{2} \left( \frac{1}{dc} + \frac{c}{d} \right), \quad \text{if } d \text{ is even}, \quad (1.4)$$

$$s_5(d, c) + s_5(c, d) = \frac{1}{2} - \frac{1}{2cd}, \quad \text{if } c \text{ and } d \text{ are odd.} \quad (1.5)$$

Various properties of Hardy–Berndt sums have been investigated ([2,6–8,17,22–24,26–32]) and several generalizations have been studied ([9–11,14,15,20,21,25]).

A character analogue of classical Dedekind sum, called as Dedekind character sum, appears in the transformation formula of a generalized Eisenstein series $G(z, s: \chi; r_1, r_2)$ (see (2.8) below) associated to a non-principle primitive character $\chi$ of modulus $k$ [3, p. 12]. This sum is defined by

$$s(d, c : \chi) = \sum_{n=1}^{ck} \chi(n) \mathfrak{B}_{1,\chi} \left( \frac{dn}{c} \right) \mathfrak{B}_1 \left( \frac{n}{ck} \right)$$

and possesses the reciprocity formula [3, Theorem 4]

$$s(c, d : \chi) + s(d, c : \overline{\chi}) = B_{1,\chi} B_{1,\overline{\chi}}.$$
and corresponding reciprocity formula is established [13].

Generalizations of Hardy–Berndt sums $S(d, c)$, $s_3(d, c)$ and $s_4(d, c)$, in the sense of $s_p (d, c : \chi)$, are presented in [10] by obtaining transformation formulas for the function

$$B(z, s : \chi) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \chi(m) \chi(2n+1) (2n+1)^{s-1} e^{\pi i m(2n+1)z/k},$$

(1.6)

which is a character extension of $\log \theta_4 (z)$.

Inspiring by [3, 10] and the fact

$$\log \left( \frac{\theta_2 (z)}{2 e^{\pi i z/4}} \right) = -\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^m n^{-1} e^{2\pi i n z}$$

we set the function $A_1 (z, s : \chi)$ to be

$$A_1 (z, s : \chi) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^m \chi(m) \chi(n) n^{s-1} e^{2\pi i n z/k},$$

for $\text{Im} (z) > 0$ and for all $s$.

In this paper, we derive transformation formulas for the function $A_1 (z, s : \chi)$. Sums appearing in transformation formulas are generalizations, involving characters and generalized Bernoulli and Euler functions, of Hardy–Berndt sums $s_1(d, c)$, $s_2(d, c)$ and $s_5(d, c)$. These new sums still obey reciprocity formulas.

2 Preliminaries

Throughout this paper $\chi$ denotes a non-principal primitive character of modulus $k$. The letter $p$ always denotes positive integer. We use the modular transformation $(az + b)/(cz + d)$ where $a$, $b$, $c$ and $d$ are integers with $ad - bc = 1$ and $c > 0$. The upper half-plane $\{ x + iy : y > 0 \}$ will be denoted by $\mathbb{H}$ and the upper quarter-plane $\{ x + iy : x > -d/c, y > 0 \}$ by $\mathbb{K}$. We use the notation $\{ x \}$ for the fractional part of $x$. Unless otherwise stated, we assume that the branch of the argument is defined by $-\pi \leq \text{arg} z < \pi$.

The Bernoulli polynomials $B_n(x)$ and the Euler polynomials $E_n(x)$ are defined by means of the generating functions

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi, \quad \text{and} \quad \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad |t| < \pi,$$

respectively (see [18]). $B_n(0) = B_n$ are the Bernoulli numbers with $B_0 = 1$, $B_1 = -1/2$ and $B_{2n-1} (1/2) = B_{2n+1} = 0$ for $n \geq 1$. For $0 \leq x < 1$ and $m \in \mathbb{Z}$, the Bernoulli functions $\mathfrak{B}_n (x)$ are defined by

$$\mathfrak{B}_n (x + m) = B_n (x) \quad \text{when} \ n \neq 1 \ \text{or} \ x \neq 0, \ \text{and} \ \mathfrak{B}_1 (m) = \mathfrak{B}_1 (0) = 0$$
and satisfy Raabe theorem for any \( x \)

\[
\sum_{j=0}^{r-1} B_n \left( x + \frac{j}{r} \right) = r^{1-n} B_n (rx). \tag{2.1}
\]

Also we have [12, Eq. (4.5)]

\[
r^{n-1} \sum_{j=0}^{r-1} (-1)^j B_n \left( \frac{x + j}{r} \right) = -\frac{n}{2} E_{n-1} (x) \tag{2.2}
\]

for even \( r \) and any \( x \). Here \( E_n (x) \) are the Euler functions defined by

\[
E_n (x) = E_n (x) \quad \text{and} \quad E_n (x + m) = (-1)^m E_n (x) \tag{2.3}
\]

for \( 0 \leq x < 1 \) and \( m \in \mathbb{Z} \). The generalized Bernoulli function \( B_{m, \chi} (x) \) are defined by Berndt [4]. We will often use the following property that can confer as a definition

\[
B_{m, \chi} (x) = k^{m-1} \sum_{j=0}^{k-1} \overline{\chi} (j) B_m \left( \frac{j + x}{k} \right), \quad m \geq 1, \tag{2.4}
\]

and satisfy

\[
B_{m, \chi} (x + nk) = B_{m, \chi} (x), \quad B_{m, \chi} (-x) = (-1)^m \chi (-1) B_{m, \chi} (x). \tag{2.5}
\]

For the convenience with the definition of \( B_{m, \chi} (x) \), let the character Euler function \( E_{m, \chi} (x) \) be defined by

\[
E_{m, \chi} (x) = k^m \sum_{j=0}^{k-1} (-1)^j \overline{\chi} (j) E_m \left( \frac{j + x}{k} \right), \quad m \geq 0 \tag{2.6}
\]

for odd \( k \), the modulus of \( \chi \). It is easily seen that

\[
E_{m, \chi} (x + nk) = (-1)^n E_{m, \chi} (x), \quad E_{m, \chi} (-x) = (-1)^{m-1} \chi (-1) E_{m, \chi} (x). \tag{2.7}
\]

The Gauss sum \( G (z, \chi) \) is defined by

\[
G (z, \chi) = \sum_{v=0}^{k-1} \chi (v) e^{2\pi ivz/k}.
\]

We put \( G (1, \chi) = G (\chi) \). If \( n \) is an integer, then [1, p. 168]

\[
G (n, \chi) = \overline{\chi} (n) G (\chi).
\]

Let \( r_1 \) and \( r_2 \) be arbitrary real numbers. For \( z \in \mathbb{H} \) and \( \text{Re} (s) > 2 \), Berndt [3] defines the function

\[
G (z, s : \chi ; r_1, r_2) = \sum_{m, n = -\infty}^{\infty} \frac{\chi(m) \overline{\chi}(n)}{(m + r_1) z + n + r_2), \tag{2.8}
\]
where the dash means that the possible pair \( m = -r_1, n = -r_2 \) is omitted from the summation. In accordance with the subject of this study we present Berndt’s formulas for \( r_1 = r_2 = 0 \). Set \( G(z, s : \chi) = G(z, s : \chi : 0, 0) \) and

\[
A(z, s : \chi) = \sum_{m=1}^{\infty} \chi(m) \sum_{n=1}^{\infty} \chi(n)n^{s-1}e^{2\pi inmz/k}, \quad z \in \mathbb{H} \text{ and } s \in \mathbb{C}.
\]

Then, it is shown that

\[
\Gamma(s) G(z, s : \chi) = G(\bar{\chi}) \left( -\frac{2\pi i}{k} \right)^s H(z, s : \chi)
\]

where \( H(z, s : \chi) = (1 + e^{\pi is}) A(z, s : \chi). \)

The following lemma is due to Lewittes \([19, \text{Lemma } 1]\).

**Lemma 2.1** Let \( A, B, C \) and \( D \) be real with \( A \neq 0 \) and \( C \neq 0 \). Then for \( z \in \mathbb{H} \),

\[
\arg((Az + B) / (Cz + D)) = \arg(Az + B) - \arg(Cz + D) + 2\pi l,
\]

where \( l \) is independent of \( z \) and \( l = \begin{cases} 1, & A \leq 0 \text{ and } AD - BC > 0, \\ 0, & \text{otherwise}. \end{cases} \)

We need the following Berndt’s transformation formulas (see \([16, \text{Theorem } 1] \) and \([25, \text{Theorem } 2] \) for generalizations).

**Theorem 2.2** \([3, \text{Theorem } 2] \) Let \( Tz = (az + b) / (cz + d) \). Suppose first that \( a \equiv d \equiv 0 \text{(mod } k) \). Then for \( z \in \mathbb{K} \) and \( s \in \mathbb{C} \),

\[
(z + d)^{s} \Gamma(s) G(Tz, s : \chi) = \bar{\chi}(b) \chi(c) \Gamma(s) G(z, s : \bar{\chi})
\]

\[
+ \bar{\chi}(b) \chi(c)e^{-\pi is} \sum_{j=1}^{c} \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{k-1} \chi(\mu c + j) \chi\left(\left[\frac{d\mu}{c}\right] + \nu\right) f(z, s, c, d),
\]

where

\[
f(z, s, c, d) = \int_C \frac{e^{-(\mu c + j)(cz + d)u/c \mu + 0} + \nu(\mu c + j)u}{e^{ku} - 1} \frac{u^{s-1}du}{u^{s-1}}.
\]

where \( C \) is a loop beginning at \( +\infty \), proceeding in the upper half-plane, encircling the origin in the positive direction so that \( u = 0 \) is the only zero of \((e^{-(cz + d)ku} - 1) / (e^{ku} - 1) \) lying “inside” the loop, and then returning to \( +\infty \) in the lower half-plane. Here we choose the branch of \( u^s \) with \( 0 < \arg u < 2\pi \).

Secondly, if \( b \equiv c \equiv 0 \text{(mod } k) \), we have for \( z \in \mathbb{K} \) and \( s \in \mathbb{C} \),

\[
(z + d)^{-s} \Gamma(s) G(Tz, s : \chi) = \bar{\chi}(a) \chi(d) \Gamma(s) G(z, s : \chi)
\]

\[
+ \bar{\chi}(a) \chi(d)e^{-\pi is} \sum_{j=1}^{c} \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{k-1} \chi(j) \chi\left(\left[\frac{d\mu}{c}\right] + d\mu - \nu\right) f(z, s, c, d).
\]
3 Transformation Formulas

In the sequel, unless otherwise stated, we assume that $k$ is odd.

From definition, $A_1 (z, s; \chi)$ can be written in terms of $A (z, s; \chi)$ as

$$A_1 (z, s; \chi) = 2 \chi (2) A (2z, s; \chi) - A (z, s; \chi).$$

Thus, transformation formulas can be achieved for the function $H_1 (z, s; \chi) = (1 + e^{\pi i s}) A_1 (z, s; \chi)$ with the help of Theorem 2.2. We have following transformation formulas according to $d$ or $c$ is even.

**Theorem 3.1** Let $Tz = (az + b) / (cz + d)$ and $d$ be even. If $a \equiv d \equiv 0 \pmod{k}$, then for $z \in \mathbb{K}$ and $s \in \mathbb{C}$

$$G (\bar{\chi}) (cz + d)^{-s} H_1 (Tz, s; \chi) = \bar{\chi} (b) \chi (c) G (\chi) 2^{1-s} \chi (2) B_1 (z, s; \chi)$$

$$+ \bar{\chi} (b) \chi (c) \left( -\frac{k}{2\pi i} \right)^s e^{-\pi i s} \sum_{j=1}^{c-1} \sum_{\mu=0}^{k-1} \frac{\chi (j)}{\sum_{\nu=0}^{e}} + \bar{\chi} (\mu + j)$$

$$\times \left\{ \frac{\chi (2)}{2^{s-1} \chi} \left( \left[ \frac{dj}{2c} \right] - \nu \right) f \left( \frac{z}{2}, s, \frac{d}{2} \right) - \chi \left( \left[ \frac{dj}{c} \right] - \nu \right) f \left( z, s, c, d \right) \right\},$$

(3.1)

where $B_1 (z, s; \chi) = (1 + e^{\pi i s}) B (z, s; \chi)$ is given by (1.6) and $f (z, s, c, d)$ is given by (2.9). If $b \equiv c \equiv 0 \pmod{k}$, then

$$G (\bar{\chi}) (cz + d)^{-s} H_1 (Tz, s; \chi) = \bar{\chi} (a) \chi (d) G (\bar{\chi}) 2^{1-s} \chi (2) B_1 (z, s; \chi)$$

$$+ \bar{\chi} (a) \chi (d) \left( -\frac{k}{2\pi i} \right)^s e^{-\pi i s} \sum_{j=1}^{c-1} \sum_{\mu=0}^{k-1} \frac{\chi (j)}{\sum_{\nu=0}^{e}}$$

$$\times \left\{ 2^{1-s} \chi \left( \left[ \frac{dj}{2c} \right] + \frac{d}{2} \mu - \nu \right) f \left( \frac{z}{2}, s, c, \frac{d}{2} \right) - \chi \left( \left[ \frac{dj}{c} \right] + d \mu - \nu \right) f \left( z, s, c, d \right) \right\}.$$

**Proof.** For even $d$, let $S_z = (2az + b) / (cz + d/2)$. Since $S (z/2) = 2T (z)$, one can write

$$H_1 (Tz, s; \chi) = 2 \chi (2) H (S (z/2), s; \chi) - H (Tz, s; \chi).$$

(3.2)

Thus, the desired result follows from (3.2) and Theorem 2.2 with [10, Eq. (3.1)]

$$2^{1-s} \chi (2) H \left( \frac{z}{2}, s; \chi \right) - H (z, s; \chi) = 2^{1-s} \chi (2) B_1 (z, s; \chi).$$

Theorem 3.2 Let $Tz = (az + b) / (cz + d)$ and let $c$ be even. If $a \equiv d \equiv 0 \pmod{k}$, then for $z \in \mathbb{K}$ and all $s \in \mathbb{C}$

$$G (\bar{\chi}) (cz + d)^{-s} H_1 (Tz, s; \chi) = \bar{\chi} (b) \chi (c) G (\chi) H_1 (z, s; \chi)$$

$$- \bar{\chi} (b) \chi (c) \left( -\frac{k}{2\pi i} \right)^s e^{-\pi i s} \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{e} \frac{\chi (j)}{\sum_{\nu=0}^{e}} + \bar{\chi} (\mu + j)$$

$$\times \left\{ \frac{\chi (2)}{2^{s-1} \chi} \left( \left[ \frac{dj}{2c} \right] - \nu \right) f \left( \frac{z}{2}, s, \frac{d}{2} \right) - \chi \left( \left[ \frac{dj}{c} \right] - \nu \right) f \left( z, s, c, d \right) \right\}.$$
\begin{align*}
&\times \left\{ \sum_{j=1}^c \bar{\chi} (\mu c + j) \chi \left( \left[ \frac{dj}{c} \right] - \nu \right) f (z, s, c, d) \\
&- \sum_{j=1}^{c/2} 2\bar{\chi} (2) \bar{\chi} \left( \left[ \frac{2dj}{c} \right] - \nu \right) f \left( 2z, s, \frac{c}{2}, d \right) \right\}.
\end{align*}

If \( b \equiv c \equiv 0 \pmod{k} \), then
\begin{align*}
(cz + d)^{-s} G (\bar{\chi}) H_1 (Tz, s : \chi) &= \bar{\chi} (a) \chi (d) G (\bar{\chi}) H_1 (z, s : \chi) \\
&- \bar{\chi} (a) \chi (d) \left( - \frac{k}{2\pi i} \right)^s e^{-\pi is} \sum_{\mu = 0}^{k-1} \sum_{\nu = 0}^{k-1} \\
&\times \sum_{j=1}^c \chi (j) \chi \left( \left[ \frac{dj}{c} \right] + d\mu - \nu \right) f (z, s, c, d) \\
&- \sum_{j=1}^{c/2} 2\chi (2) \chi (j) \chi \left( \left[ \frac{2dj}{c} \right] + d\mu - \nu \right) f \left( 2z, s, \frac{c}{2}, d \right) \right\}.
\end{align*}

**Proof.** For even \( c \), if we set \( Vz = (az + 2b) / \left( \frac{c}{2} z + d \right) \), then \( V (2z) = 2T (z) \) and
\begin{equation}
H_1 (Tz, s : \chi) = 2\chi (2) H (V (2z), s : \chi) - H (Tz, s : \chi). \tag{3.3}
\end{equation}
Using (3.3) and Theorem 2.2 completes the proof. \( \blacksquare \)

Theorem 3.1 and Theorem 3.2 can be simplified when \( s = 1 - p \) is an integer for \( p \geq 1 \). In this case, by the residue theorem, we have
\begin{equation}
f (z, 1 - p, c, d) = \frac{2\pi ik^{p-1}}{(p+1)!} \sum_{m=0}^{p+1} \binom{p+1}{m} \left( - (cz + d) \right)^{m-1} \\
\times B_{p+1-m} \left( \frac{\nu + \left\{ \frac{dj}{c} \right\}}{k} \right) B_m \left( \frac{\mu c + j}{ek} \right). \tag{3.4}
\end{equation}

The following is character extension of (1.1).

**Theorem 3.3** Let \( p \geq 1 \) be odd and \( d \) be even. If \( a \equiv d \equiv 0 \pmod{k} \), then for \( z \in \mathbb{H} \)
\begin{align*}
G (\bar{\chi}) (cz + d)^{p-1} H_1 (Tz, 1 - p : \chi) \\
= \bar{\chi} (b) \chi (c) \left( 2^p \chi (2) G (\chi) B_1 (z, 1 - p : \bar{\chi}) - \frac{\chi (-1) (2\pi i)^p}{2 (p!)} g_1 (c, d, z, p, \bar{\chi}) \right), \tag{3.5}
\end{align*}
where
\begin{equation}
g_1 (c, d, z, p, \chi)
\end{equation
\[
\sum_{m=1}^{p} \left( \frac{p}{m} \right) k^{m-p} (- (cz + d))^{m-1} \sum_{n=1}^{ck} \chi(n) \mathcal{E}_{p-m, \chi} \left( \frac{dn}{c} \right) \mathfrak{B}_m \left( \frac{n}{ck} \right). \tag{3.6}
\]

If \( b \equiv c \equiv 0 \) (mod \( k \)), then for \( z \in \mathbb{H} \)
\[
G(\bar{\chi})(cz + d)_{p-1} H_1(Tz, 1 - p : \chi)
= \bar{\chi}(a) \chi(d) \left( 2^p \bar{\chi}(2) G(\bar{\chi}) B_1(z, 1 - p : \chi) - \frac{\chi(-1) (2\pi i)^p}{2 (p!)} g_1(c, d, z, \chi) \right).
\]

**Proof.** Let us consider the case \( a \equiv d \equiv 0 \) (mod \( k \)). By aid of (3.4), equation (3.1) turns into
\[
G(\bar{\chi})(cz + d)_{p-1} H_1(Tz, 1 - p : \chi)
= \bar{\chi}(b) \chi(c) G(\bar{\chi}) (2^p \chi(2)) B_1(z, 1 - p : \bar{\chi})
+ \bar{\chi}(b) \chi(c) \frac{(2\pi i)^p}{(p+1)!} \sum_{m=1}^{p} \left( \frac{m+1}{m} \right) (- (cz + d))^{m-1} (T_1 - T_2),
\]
where
\[
T_1 = 2^{p+1-m} \chi(2) \sum_{j=1}^{c} \sum_{k=0}^{k-1} \sum_{\mu=0}^{k-1} \bar{\chi}(\mu c + j) \chi(\left\lfloor \frac{dj}{2c} \right\rfloor - \nu) \times B_{p+1-m} \left( \frac{\nu + (dj/2c)}{k} \right) B_{p+1-m} \left( \frac{\mu c + j}{c} \right), \tag{3.7}
\]
\[
T_2 = \sum_{j=1}^{c} \sum_{k=0}^{k-1} \sum_{\mu=0}^{k-1} \bar{\chi}(\mu c + j) \chi(\left\lfloor \frac{dj}{c} \right\rfloor - \nu) B_{p+1-m} \left( \frac{\nu + (dj/c)}{k} \right) B_{p+1-m} \left( \frac{\mu c + j}{c} \right)
\]
and we have used that the sum over \( \mu \) is zero for \( m = 0 \) and the sum over \( \nu \) is zero for \( m = p+1 \). We first note that the triple sum in (3.7) is invariant by replacing \( B_{p+1-m} \left( \frac{\nu + (dj/2c)}{k} \right) \) by \( \mathfrak{B}_{p+1-m} \left( \frac{\nu + (dj/2c)}{k} \right) \) since \( B_{p+1-m} \left( \frac{\nu + (dj/2c)}{k} \right) = \mathfrak{B}_{p+1-m} \left( \frac{\nu + (dj/2c)}{k} \right) \) for \( 0 < \frac{\nu + (dj/2c)}{k} < 1 \), and \( \chi(d/2) = 0 \) (\( d \equiv 0 \) (mod \( k \)) and \( k \) is odd) for \( \nu + (dj/2c) \leq 0 \). Similarly, one can write \( \mathfrak{B}_m \left( \frac{\nu + j}{c} \right) \) in place of \( B_m \left( \frac{\nu + j}{c} \right) \). After some manipulations, we see that
\[
T_1 = 2^{p+1-m} \chi(-2) k^{m-p} \sum_{n=1}^{ck} \bar{\chi}(n) \mathfrak{B}_{p+1-m, \bar{\chi}} \left( \frac{dn}{2c} \right) \mathfrak{B}_m \left( \frac{n}{ck} \right)
\]
and
\[
T_2 = \chi(-1) k^{m-p} \sum_{n=1}^{ck} \bar{\chi}(n) \mathfrak{B}_{p+1-m, \bar{\chi}} \left( \frac{dn}{c} \right) \mathfrak{B}_m \left( \frac{n}{ck} \right).
\]
So, we have
\[
G(\bar{\chi})(cz + d)_{p-1} H_1(Tz, 1 - p : \chi)
\]
\[= \bar{\chi} (b) \chi (c) G (\chi) 2^p \chi (2) B_1 (z, 1 - p : \bar{\chi})
+ \bar{\chi} (b) \chi (-c) \frac{(2\pi i)^p}{(p + 1)!} \sum_{m=1}^{p} \left( \frac{p + 1}{m} \right)^{k^m} \left( (cz + d)^{m-1} \right)
\times \sum_{n=1}^{c_k} \bar{\chi} (n) \mathfrak{B}_m \left( \frac{n}{c k} \right) \left( 2^{p+1-m} \chi (2) \mathfrak{B}_{p+1-m, \bar{\chi}} \left( \frac{dn}{2c} \right) - \mathfrak{B}_{p+1-m, \bar{\chi}} \left( \frac{dn}{c} \right) \right). \tag{3.8}\]

Now, consider the difference
\[T_3 = 2^{p+1-m} \chi (2) \mathfrak{B}_{p+1-m, \bar{\chi}} \left( \frac{dn}{2c} \right) - \mathfrak{B}_{p+1-m, \bar{\chi}} \left( \frac{dn}{c} \right). \]

Using the property \[10, \text{Eq. (3.13)}\]
\[r - 1 \sum_{j=0}^{r-1} B_{m, \chi} (x + j k) = \chi (r) r^{1-m} \mathfrak{B}_{m, \chi} (r x) \tag{3.9}\]
for \(r = 2\) and \(x = dn/2c\), and utilizing (2.2) we find that
\[T_3 = (2k)^{p-m} \sum_{\mu=0}^{k-1} \chi (2\mu) \left( \mathfrak{B}_{p+1-m} \left( \frac{\mu + dn/2c}{k} \right) - \mathfrak{B}_{p+1-m} \left( \frac{\mu + dn/2c}{k} + \frac{1}{2} \right) \right)
= - \frac{p + 1 - m}{2} k^{p-m} \sum_{\mu=0}^{k-1} \chi (2\mu) \mathcal{E}_m \left( \frac{2\mu + dn/c}{k} \right). \tag{3.10}\]

The sum in the last line can be evaluated as
\[\sum_{\mu=0}^{(k-1)/2} \chi (2\mu) \mathcal{E}_m \left( \frac{2\mu + x}{k} \right) + \sum_{\mu=(k+1)/2}^{k-1} \chi (2\mu) \mathcal{E}_m \left( \frac{2\mu + x}{k} \right)
= \sum_{\mu=0}^{k-1} \chi (2\mu) \mathcal{E}_m \left( \frac{2\mu + x}{k} \right) - \sum_{\mu=0}^{k-1} \chi (2\mu + 1) \mathcal{E}_m \left( \frac{2\mu + 1 + x}{k} \right)
= \sum_{\mu=0}^{k-1} (-1)^{\mu} \chi (\mu) \mathcal{E}_m \left( \frac{\mu + x}{k} \right). \tag{3.11}\]

Setting this in (3.10) and using (2.6) give
\[2^{p+1-m} \chi (2) \mathfrak{B}_{p+1-m, \bar{\chi}} \left( \frac{x}{2} \right) - \mathfrak{B}_{p+1-m, \bar{\chi}} (x) = - \frac{p + 1 - m}{2} \mathcal{E}_{p-m, \bar{\chi}} (x). \tag{3.12}\]

Therefore, we arrive at
\[G (\bar{\chi}) (cz + d)^{p-1} H_1 (T z, 1 - p : \chi)\]
\[ G(\bar{z}) = \sum_{n=1}^{ck} \bar{\chi}(n) E_{p-m,\bar{\chi}} \left( \frac{dn}{c} \right) \mathcal{B}_m \left( \frac{n}{ck} \right). \]

The result holds for \( z \in \mathbb{H} \) by analytic continuation.

The proof for \( b \equiv c \equiv 0 \pmod{k} \) is completely analogous. ■

Note that for odd \( d \) and \( c \), if we take \( Tz = T(z + k) = \frac{az + b + ak}{cz + d + ck} \) instead of \( Tz = \frac{az + b}{cz + d} \) in Theorem 3.3, then the function \( g_1(c, d + ck, z, p, \chi) \) turns into

\[ g_1(c, d + ck, z, p, \chi) = \sum_{m=1}^{p} \left( \frac{p}{m} \right) k^{m-p} (- (cz + d + ck))^{m-1} \]

\[ \times \sum_{n=1}^{ck} (-1)^n \chi(n) E_{p-m,\chi} \left( \frac{dn}{c} \right) \mathcal{B}_m \left( \frac{n}{ck} \right), \tag{3.13} \]

by (2.7). So, it is convenient to present the following theorem since it is observed a new sum.

**Theorem 3.4** Let \( p \geq 1 \) be odd and \( Rz = (az + b + ak) / (cz + d + ck) \) with \( d \) and \( c \) odd. If \( a \equiv d \equiv 0 \pmod{k} \), for \( z \in \mathbb{H} \),

\[ G(\bar{z}) (cz + d + ck)^{p-1} H_1(Rz, 1 - p : \chi) \]

\[ = \bar{\chi}(b) \chi(c) G(\chi) 2^p \chi(2) B_1(z, 1 - p : \bar{\chi}) - \frac{\bar{\chi}(b) \chi(-c) (2\pi i)^p}{2 p!} g_1(c, d + ck, z, p, \bar{\chi}), \tag{3.14} \]

where \( g_1(c, d + ck, z, p, \chi) \) is given by (3.13).

If \( b \equiv c \equiv 0 \pmod{k} \), for \( z \in \mathbb{H} \),

\[ G(\bar{z}) (cz + d + ck)^{p-1} H_1(Rz, 1 - p : \chi) \]

\[ = \bar{\chi}(a) \chi(d) G(\chi) 2^p \bar{\chi}(2) B_1(z, 1 - p : \bar{\chi}) - \frac{\bar{\chi}(a) \chi(-d) (2\pi i)^p}{2 p!} g_1(c, d + ck, z, p, \chi). \tag{3.15} \]

For \( s = 1 - p \), Theorem 3.2 turns into following, which is character analogous of (1.2).

**Theorem 3.5** Let \( p \geq 1 \) be odd and let \( c \) be even. If \( a \equiv d \equiv 0 \pmod{k} \), for \( z \in \mathbb{H} \),

\[ G(\bar{z}) (cz + d)^{p-1} H_1(Tz, 1 - p : \chi) \]

\[ = \bar{\chi}(b) \chi(c) G(\chi) H_1(z, 1 - p : \bar{\chi}) + \bar{\chi}(-b) \chi(c) \frac{(2\pi i)^p}{(p+1)!} g_2(c, d, z, p, \bar{\chi}), \tag{3.16} \]
where
\[
g_2(c, d, z, p, \chi) = \sum_{m=1}^{p} \binom{p+1}{m} (- (cz + d))^{m-1} k^{m-p} \\
\times \sum_{n=1}^{ck} (-1)^n \chi(n) \mathfrak{B}_{p+1-m,\chi} \left(\frac{dn}{c}\right) \mathfrak{B}_m \left(\frac{n}{ck}\right).
\]
(3.17)

If \( b \equiv c \equiv 0 \pmod{k} \), for \( z \in \mathbb{H} \),
\[
G(\bar{\chi}) (cz + d)^{p-1} H_1(Tz, 1-p: \chi) \\
= \bar{\chi}(a) \chi(d) G(\bar{\chi}) H_1(z, 1-p: \chi) + \bar{\chi}(a) \chi(-d) \frac{(2\pi i)^p}{(p+1)!} g_2(c, d, z, p, \chi).
\]
(3.18)

**Proof.** Similar to (3.8), it can be found that
\[
G(\bar{\chi})(cz + d)^{p-1} H_1(Tz, 1-p: \chi) \\
= \bar{\chi}(b) \chi(c) G(\chi) H_1(z, 1-p: \chi) \\
- \bar{\chi}(-b) \chi(c) \frac{(2\pi i)^p}{(p+1)!} \sum_{m=1}^{p} \binom{p+1}{m} (- (cz + d))^{m-1} k^{m-p} \\
\times \left( \sum_{n=1}^{ck} \bar{\chi}(n) \mathfrak{B}_{p+1-m,\chi} \left(\frac{dn}{c}\right) \mathfrak{B}_m \left(\frac{n}{ck}\right) \right) \\
- 2 \sum_{n=1}^{ck/2} \bar{\chi}(2n) \mathfrak{B}_{p+1-m,\bar{\chi}} \left(\frac{2dn}{c}\right) \mathfrak{B}_m \left(\frac{2n}{ck}\right).
\]

Then, (3.16) follows from
\[
2 \sum_{n=1}^{ck/2} \bar{\chi}(2n) \mathfrak{B}_{p+1-m,\bar{\chi}} \left(\frac{2dn}{c}\right) \mathfrak{B}_m \left(\frac{2n}{ck}\right) \\
- \sum_{n=1}^{ck} \bar{\chi}(n) \mathfrak{B}_{p+1-m,\bar{\chi}} \left(\frac{dn}{c}\right) \mathfrak{B}_m \left(\frac{n}{ck}\right) \\
= \sum_{n=1}^{ck} (-1)^n \chi(n) \mathfrak{B}_{p+1-m,\bar{\chi}} \left(\frac{dn}{c}\right) \mathfrak{B}_m \left(\frac{n}{ck}\right).
\]

The proof for \( b \equiv c \equiv 0 \pmod{k} \) is completely analogous. □

Note that for \( p = 1 \) transformation formulas (3.16) and (3.18) coincide with Meyer’s [25] second formulas in Theorems 10 and 11, respectively.

## 4 Reciprocity Theorems

In this section, we first give reciprocity formulas for the functions \( g_1(d, c + dk, z, p, \chi) \), \( g_1(d, c, z, p, \chi) \) and \( g_2(d, c, z, p, \chi) \). In particular, these formulas yield reciprocity formulas analogues to the reciprocity formulas (1.4) and (1.5).

We need the following theorem, offered by Can and Kurt [10].
Theorem 4.1 (see [10, Eqs. (3.4) and (3.20)]) Let \( p \geq 1 \) be odd integer and \( Tz = (az + b) / (cz + d) \). If \( b \) is even and \( a \equiv d \equiv 0 \pmod{k} \), then for \( z \in \mathbb{H} \)

\[
(cz + d)^{p-1} G(\bar{\chi}) B_1(Tz, 1 - p : \chi)
= \bar{\chi}(b) \chi(2c) G(\chi) B_1(z, 1 - p : \bar{\chi})
- \bar{\chi}(b) \chi(-2c) \frac{(2\pi i)^p}{(p+1)!} \sum_{m=1}^{p+1} \left( \frac{p+1}{m} \right) k^{m-p} (- (cz + d))^{m-1}
\times \frac{m}{2m} \sum_{n=1}^{ck} \chi(n) \mathfrak{B}_{p+1-m, \chi} \left( \frac{dn}{2c} \right) \mathcal{E}_{m-1} \left( \frac{n}{ck} \right).
\]

(4.1)

If \( a \) is even and \( a \equiv d \equiv 0 \pmod{k} \), then for \( z \in \mathbb{H} \)

\[
2^p \bar{\chi}(2) (cz + d)^{p-1} G(\bar{\chi}) B_1(Tz, 1 - p : \chi)
= \bar{\chi}(b) \chi(c) G(\chi) H_1(z, 1 - p : \bar{\chi})
+ \bar{\chi}(b) \chi(-c) \frac{(2\pi i)^p}{(p+1)!} \sum_{m=1}^{p+1} \left( \frac{p+1}{m} \right) (- (cz + d))^{m-1} k^{m-p}
\times \left( - \frac{m}{2} \sum_{n=1}^{ck} (-1)^n \chi(n) \mathfrak{B}_{p+1-m, \chi} \left( \frac{dn}{c} \right) \mathcal{E}_{m-1} \left( \frac{n}{ck} \right) \right).
\]

(4.2)

The function \( g_1(d, c + dk, z, p, \chi) \), given by (3.13), satisfies the following reciprocity formula:

Theorem 4.2 Let \( p \geq 1 \) be odd and \( d \) and \( c \) be coprime integers. If \( d \) or \( c \equiv 0 \pmod{k} \), then

\[
g_1(d, -c - dk, z, p, \chi) - \chi(-1)(z - k)^{p-1} g_1(c, d + ck, V_1(z), p, \bar{\chi})
= \bar{\chi}(4) \frac{p}{2kp^r} \sum_{m=0}^{p-1} \left( \frac{p-1}{m} \right) (z - k)^m \mathcal{E}_{p-1-m, \chi}(0) \mathcal{E}_{m, \chi}(0),
\]

where \( V_1(z) = \frac{-kz + k^2 - 1}{z - k} \).

Proof. For even \((c + d)\), consider the modular substitutions \( R(z) = \frac{az + b + ak}{cz + d + ck}, \) \( R^*(z) = \frac{bz - a - bk}{dz - c - dk} \) and \( V_1(z) = \frac{-kz + k^2 - 1}{z - k} \). Suppose \( a \equiv d \equiv 0 \pmod{k} \).

Replacing \( z \) by \( V_1(z) \) in (3.14) gives

\[
G(\bar{\chi}) \left( \frac{dz - c - dk}{z - k} \right)^{p-1} H_1(R^*(z), 1 - p : \chi)
= \bar{\chi}(b) \chi(c) 2^p \chi(2) G(\chi) B_1(V_1(z), 1 - p : \bar{\chi})
\]

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Replacing $R (z)$ by $R^* (z)$ in (3.15) yields

$$G (\tilde{\chi}) (dz - c - dk)^{p-1} H_1 (R^* (z), 1 - p : \chi)$$

$$= \tilde{\chi} (b) \chi (c) \left( 2^p \tilde{\chi} (-2) G (\tilde{\chi}) B_1 (z, 1 - p : \chi) - \frac{(2\pi i)^p}{2 (pl)} g_1 (d, -c - dk, z, p, \tilde{\chi}) \right).$$

(4.4)

Taking $a = -k, b = k^2 - 1, c = 1$ and $d = -k$ and writing $\tilde{\chi}$ in place of $\chi$ in (4.1) lead to

$$(z - k)^{p-1} 2^p \chi (2) G (\chi) B_1 (V_1 (z), 1 - p : \tilde{\chi})$$

$$= \chi \left( \frac{k^2 - 1}{2} \right) \tilde{\chi} (2) 2^p \chi (2) G (\tilde{\chi}) B_1 (z, 1 - p : \tilde{\chi})$$

$$- \chi \left( \frac{k^2 - 1}{2} \right) \chi (-1) 2^p \left( \frac{2\pi i}{p+1} \right)^p \sum_{m=1}^{p} \left( \frac{p+1}{m} \right) k^{m-p} (- (z - k))^{m-1}$$

$$\times \frac{m}{2^m} \sum_{n=1}^{k} \tilde{\chi} (n) \mathcal{B}_{p+1-m,\tilde{\chi}} \left( - \frac{kn}{2} \right) \mathcal{E}_{m-1} \left( \frac{n}{k} \right).$$

(4.5)

Thus, consider (4.4) and (4.5) with multiplying both sides of (4.3) by $(z - k)^{p-1}$ to obtain

$$g_1 (d, -c - dk, z, p, \tilde{\chi}) - \chi (-1) (z - k)^{p-1} g_1 (c, d + ck, V_1 (z), p, \tilde{\chi})$$

$$= \tilde{\chi} (-2) \frac{2^{p+1}}{p+1} \sum_{m=1}^{p} \left( \frac{p+1}{m} \right) k^{m-p} (- (z - k))^{m-1}$$

$$\times \frac{m}{2^m} \sum_{n=1}^{k} \tilde{\chi} (n) \mathcal{B}_{p+1-m,\tilde{\chi}} \left( - \frac{kn}{2} \right) \mathcal{E}_{m-1} \left( \frac{n}{k} \right).$$

(4.6)

Now, let us concern the sum over $n$ in (4.6). Using (3.9) for $r = 2$ yields

$$\sum_{n=1}^{k} \tilde{\chi} (n) \mathcal{B}_{p+1-m,\tilde{\chi}} \left( - \frac{kn}{2} \right) \mathcal{E}_{m-1} \left( \frac{n}{k} \right)$$

$$= \mathcal{B}_{p+1-m,\tilde{\chi}} (0) \sum_{n} \tilde{\chi} (2n) \mathcal{E}_{m-1} \left( \frac{2n}{k} \right) + \mathcal{B}_{p+1-m,\tilde{\chi}} \left( \frac{k}{2} \right) \sum_{n} \tilde{\chi} (2n - 1) \mathcal{E}_{m-1} \left( \frac{2n - 1}{k} \right)$$

$$= \left\{ \mathcal{B}_{p+1-m,\tilde{\chi}} (0) + \mathcal{B}_{p+1-m,\tilde{\chi}} \left( \frac{k}{2} \right) \right\} \sum_{n} \tilde{\chi} (2n) \mathcal{E}_{m-1} \left( \frac{2n}{k} \right)$$

$$- \mathcal{B}_{p+1-m,\tilde{\chi}} \left( \frac{k}{2} \right) \sum_{n=0}^{k-1} (-1)^n \tilde{\chi} (n) \mathcal{E}_{m-1} \left( \frac{n}{k} \right).$$
\[ = 2^{m-p} \chi (2) \mathfrak{B}_{p+1-m, \bar{\chi}} (0) \sum_{n=0}^{(k-1)/2} \bar{\chi} (2n) \mathcal{E}_{m-1} \left( \frac{2n}{k} \right) \]

\[ - \mathfrak{B}_{p+1-m, \bar{\chi}} \left( \frac{k}{2} \right) k^{1-m} \mathcal{E}_{m-1, \bar{\chi}} (0). \]

It follows from (2.3) that

\[
\sum_{\mu=0}^{k-1} \chi (2\mu) \mathcal{E}_{m-1} \left( \frac{2\mu}{k} \right) = \sum_{\mu=0}^{(k-1)/2} \chi (2\mu) \mathcal{E}_{m-1} \left( \frac{2\mu}{k} \right) + \sum_{\mu=0}^{(k-1)/2} \chi (2\mu) \mathcal{E}_{m-1} \left( \frac{2\mu}{k} \right)
\]

\[
= \sum_{\mu=0}^{(k-1)/2} \chi (2\mu) \mathcal{E}_{m-1} \left( \frac{2\mu}{k} \right) + \sum_{\mu=1}^{(k-1)/2} \chi (-2\mu) \mathcal{E}_{m-1} \left( \frac{-2\mu}{k} \right)
\]

\[
= (1 + (-1)^m \chi (-1)) \sum_{\mu=1}^{(k-1)/2} \chi (2\mu) \mathcal{E}_{m-1} \left( \frac{2\mu}{k} \right). \]

So, using (2.5) gives

\[
\mathfrak{B}_{p+1-m, \bar{\chi}} (0) \sum_{n=0}^{k-1} \bar{\chi} (2n) \mathcal{E}_{m-1} \left( \frac{2n}{k} \right) = \{ \mathfrak{B}_{p+1-m, \bar{\chi}} (0) + (-1)^m \chi (-1) \mathfrak{B}_{p+1-m, \bar{\chi}} (0) \} \sum_{n=1}^{(k-1)/2} \bar{\chi} (2n) \mathcal{E}_{m-1} \left( \frac{2n}{k} \right)
\]

\[
= 2 \mathfrak{B}_{p+1-m, \bar{\chi}} (0) \sum_{n=1}^{(k-1)/2} \bar{\chi} (2n) \mathcal{E}_{m-1} \left( \frac{2n}{k} \right). \]

Here, using (3.11) and taking \( x = k \) in (3.12) give rise to

\[
\sum_{n=1}^{k} \bar{\chi} (n) \mathfrak{B}_{p+1-m, \bar{\chi}} \left( \frac{-kn}{2} \right) \mathcal{E}_{m-1} \left( \frac{n}{k} \right)
\]

\[
= \frac{\bar{\chi} (2)}{2^{p+1-m}} \left\{ \mathfrak{B}_{p+1-m, \bar{\chi}} (0) - 2^{p+1-m} \chi (2) \mathfrak{B}_{p+1-m, \bar{\chi}} \left( \frac{k}{2} \right) \right\} k^{1-m} \mathcal{E}_{m-1, \bar{\chi}} (0)
\]

\[
= k^{1-m} 2^{m-p-2} \bar{\chi} (2) (p + 1 - m) \mathcal{E}_{p-m, \bar{\chi}} (k) \mathcal{E}_{m-1, \bar{\chi}} (0). \]  \hspace{1cm} (4.7)

Gathering (4.6), (4.7) and (2.7) completes the proof. \( \blacksquare \)

Theorem 4.2 can be simplified according to special values of \( z \). Firstly, let us consider the case \( z = \frac{c}{d} + k \). Then,

\[
g_1 \left( d, -c - dk, \frac{c}{d} + k, p, \chi \right) = \frac{p}{k^{p-1}} \sum_{n=1}^{dk} (-1)^n \chi (n) \mathcal{E}_{p-1, \chi} \left( \frac{-cn}{d} \right) \mathfrak{B}_1 \left( \frac{n}{dk} \right) \]  \hspace{1cm} (4.8)
and
\[
\left(\frac{c}{d}\right)^{p-1} g_1\left(c, d + ck, V_1\left(\frac{c}{d} + k\right), p, \tilde{\chi}\right) = \left(\frac{c}{kd}\right)^{p-1} p \sum_{n=1}^{ck} \left(-1\right)^n \tilde{\chi}(n) \mathcal{E}_{p-1, \tilde{\chi}}\left(\frac{dn}{c}\right) \mathfrak{B}_1\left(\frac{n}{ck}\right). \tag{4.9}
\]

Since \(E_0(x) = (-1)^{|x|} E_0(\{x\}) = (-1)^{|x|}\) and
\[
s_5(d, c) = \sum_{n=1}^{c} (-1)^{n+\lfloor dn/c \rfloor} \mathfrak{B}_1\left(\frac{n}{c}\right) = \sum_{n=1}^{c} (-1)^{n} E_0\left(\frac{dn}{c}\right) \mathfrak{B}_1\left(\frac{n}{c}\right),
\]
it is convenient to make the following definition.

**Definition 4.3** The character Hardy–Berndt sum \(s_{5,p}(d, c : \tilde{\chi})\) is defined for \(c > 0\) by
\[
s_{5,p}(d, c : \tilde{\chi}) = \sum_{n=1}^{ck} (-1)^n \tilde{\chi}(n) E_{p-1, \tilde{\chi}}\left(\frac{dn}{c}\right) \mathfrak{B}_1\left(\frac{n}{ck}\right).
\]

Observing that
\[
s_{5,p}(-c, d : \chi) = -\chi(-1) s_{5,p}(c, d : \chi),
\]
by (2.7), and using (4.8) and (4.9) in Theorem 4.2 we have proved the following reciprocity formula.

**Theorem 4.4** Let \(p \geq 1\) be odd and \(d\) and \(c\) be odd coprime integers. If \(c\) or \(d \equiv 0 \pmod{k}\), then
\[
\begin{align*}
    cdp s_{5,p}(c, d : \chi) + dc^p s_{5,p}(d, c : \tilde{\chi}) &= -\frac{\tilde{\chi}(-4)}{2} \sum_{m=0}^{p-1} \binom{m+1}{m} c^{m+1} d^{p-m} E_{p-1-m, \tilde{\chi}}(0) E_{m, \chi}(0).
\end{align*}
\]

In particular, we have the character analogue of (1.5) as
\[
s_5(c, d : \chi) + s_5(d, c : \tilde{\chi}) = -\frac{\tilde{\chi}(-4)}{2} E_{0, \tilde{\chi}}(0) E_{0, \chi}(0),
\]
where \(s_5(c, d : \chi) = s_{5,1}(c, d : \chi)\).

Now we let \(z = k\) in Theorem 4.2. Then
\[
g_1(d, c - dk, z, p, \chi)|_{z=k} = \sum_{m=1}^{p} \binom{p}{m} k^{m-p} c^{m-1} d^k \sum_{n=1}^{dk} (-1)^n \chi(n) E_{p-m, \chi}\left(\frac{-cn}{d}\right) \mathfrak{B}_m\left(\frac{n}{dk}\right)
\]

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and

\[(z - k)^{p-1} g_1 (c, d + ck, V_1 (z), p, \bar{\chi}) |_{z=k} \]

\[= \sum_{m=1}^{p} \binom{p}{m} k^{m-p} \left( - (c (-kz + k^2 - 1) + (d + ck) (z - k)) \right)^{m-1} (z - k)^{p-m} |_{z=k} \]

\[\times \sum_{n=1}^{ck} (-1)^n \bar{\chi} (n) \mathcal{E}_{p-m, \bar{\chi}} \left( \frac{dn}{c} \right) \mathcal{B}_m \left( \frac{n}{ck} \right) \]

\[= e^{n-1} \sum_{n=1}^{ck} (-1)^n \bar{\chi} (n) \mathcal{E}_{0, \bar{\chi}} \left( \frac{dn}{c} \right) \mathcal{B}_p \left( \frac{n}{ck} \right). \]

If we define

\[s_{5,p+1-m,m} (c, d : \chi) = \sum_{n=1}^{dk} (-1)^n \chi (n) \mathcal{E}_{p-m, \chi} \left( \frac{cn}{d} \right) \mathcal{B}_m \left( \frac{n}{dk} \right) \]

and use (2.7), then we see that

\[\sum_{m=1}^{p} \binom{p}{m} (-kc)^m s_{5,p+1-m,m} (c, d : \chi) \]

\[= - (kc)^{p-1} s_{5,1,p} (d, c : \bar{\chi}) - \bar{\chi} (-4) \frac{p}{2} \mathcal{E}_{p-1, \bar{\chi}} (0) \mathcal{E}_{0, \chi} (0). \]

Conditions \(d\) even and \(c\) even in Theorem 3.3 and Theorem 3.5 do not allow to present reciprocity theorems in the sense of Theorem 4.2 for the functions \(g_1(d, c, z, p, \chi)\) and \(g_2(d, c, z, p, \chi)\), respectively. However, the following relation is valid for these functions.

**Theorem 4.5** Let \(d\) be even. If \(d \equiv 0 \pmod{k}\), then

\[\frac{p + 1}{2} z^{p-1} g_1 \left( c, d, -\frac{1}{z}, p, \bar{\chi} \right) + g_2 \left( d, -c, z, p, \chi \right) \]

\[= - \frac{\chi (-1)}{k^{p-1}} \sum_{m=1}^{p} \binom{p + 1}{m} m \left( -z \right)^{m-1} \mathcal{B}_{p+1-m, \chi} (0) \mathcal{E}_{m-1, \bar{\chi}} (0), \quad (4.10) \]

where the functions \(g_1(d, c, z, p, \chi)\) and \(g_2(d, c, z, p, \chi)\) are given by (3.6) and (3.17), respectively.

**Proof.** For even \(d\), consider \(T(z) = (az + b) / (cz + d)\) and \(T^*(z) = (bz - a) / (dz - c) = T (-1/z)\) and \(a \equiv d \equiv 0 \pmod{k}\). Then, (4.10) follows by applying \(T^*(z)\) in (3.18) and replacing \(z\) by \(-1/z\) in (3.5), and then replacing \(T(z)\) by \(-1/z\) in (4.2).

To simplify Theorem 4.5 we first consider \(z = c/d\). Then,

\[g_1 \left( c, d, -\frac{d}{c}, p, \bar{\chi} \right) = \frac{p}{k^{p-1}} \sum_{n=1}^{ck} \bar{\chi} (n) \mathcal{E}_{p-1, \bar{\chi}} \left( \frac{dn}{c} \right) \mathcal{B}_1 \left( \frac{n}{ck} \right) \]

\[= \frac{p}{k^{p-1}} \sum_{n=1}^{ck} \bar{\chi} (n) \mathcal{E}_{p-1, \bar{\chi}} \left( \frac{dn}{c} \right) \mathcal{B}_1 \left( \frac{n}{ck} \right). \quad (4.11) \]
\[
g_2 \left( d, -c, \frac{c}{d}, p, \bar{\chi} \right) = \frac{p + 1}{kp - 1} \sum_{n=1}^{dk} (-1)^n \chi(n) \mathfrak{B}_{p,\bar{\chi}} \left( \frac{-cn}{d} \right) \mathfrak{B}_1 \left( \frac{n}{dk} \right). \quad (4.12)
\]

**Definition 4.6** The character Hardy–Berndt sums \(s_{1,p}(d,c,\chi)\) and \(s_{2,p}(d,c,\chi)\) are defined for \(c > 0\) by

\[
s_{1,p}(d,c) = \sum_{n=1}^{ck} \chi(n) \mathcal{E}_{-1,\chi} \left( \frac{dn}{c} \right) \mathfrak{B}_1 \left( \frac{n}{ck} \right),
\]

\[
s_{2,p}(d,c) = \sum_{n=1}^{ck} (-1)^n \chi(n) \mathfrak{B}_{p,\chi} \left( \frac{dn}{c} \right) \mathfrak{B}_1 \left( \frac{n}{ck} \right).
\]

Using (4.11) and (4.12) in Theorem 4.5 we have proved the following reciprocity formula for \(s_{1,p}(d,c,\chi)\) and \(s_{2,p}(d,c,\chi)\).

**Theorem 4.7** Let \(p \geq 1\) be odd, \((d,c) = 1\) and \(d\) be even. If \(d\) or \(c \equiv 0 \pmod{k}\), then

\[
pdc^{p}s_{1,p}(d,c) - \chi(-1) 2cdp s_{2,p}(c,d)
\]

\[
= \chi(-1) \sum_{m=1}^{p} (-1)^m \binom{p}{m-1} d^{p+1-m}s_{p+1-m,\chi}(0) \mathcal{E}_{m-1,\bar{\chi}}(0).
\]

In particular,

\[
s_1(d,c) - 2\chi(-1)s_2(c,d) = -\chi(-1) \mathfrak{B}_{1,\chi}(0) \mathcal{E}_{0,\bar{\chi}}(0).
\]

**Remark 4.8** The sum \(s_2(c,d)\) is first presented by Meyer [25, Definition 6] as \(s_1^*(c,d)\).
\[ s_{2,p+1-m,m} (d, c : \chi) = \sum_{n=1}^{ck} (-1)^n \chi(n) \mathfrak{B}_{p+1-m,\chi} \left( \frac{dn}{c} \right) \mathfrak{B}_m \left( \frac{n}{ck} \right). \]

Using (2.5), Theorem 4.5 reduces to
\[
\sum_{m=1}^{p} \binom{p+1}{m} (-ck)^{m-1} s_{2,p+1-m,m} (c, d : \bar{\chi})
= \frac{p+1}{2} \left( \chi(-1)(ck)^{p-1} s_{1,1,p} (d, c : \chi) + \mathfrak{B}_{p,\chi}(0) \mathcal{E}_0(\bar{\chi}(0)) \right).
\]

The following lemma shows that reciprocity formulas given by Theorems 4.4 and 4.7 are still valid for \( \gcd(d, c) = q \).

**Lemma 4.9** Let \( q \in \mathbb{N} \), \( p \geq 1 \), \( (d, c) = 1 \) and \( c > 0 \). If \( p \) is odd and \( d \) is even,
\[ s_{1,p} (qd, qc : \chi) = s_{1,p} (d, c : \chi), \]
if \( p \) is odd and \( c \) is even,
\[ s_{2,p} (qd, qc : \chi) = s_{2,p} (d, c : \chi), \]
if \( p \) is odd and \( (d+c) \) is even,
\[ s_{5,p} (qd, qc : \chi) = s_{5,p} (d, c : \chi). \]
Furthermore, \( s_{1,p} (d, c : \chi) = 0 \) if \( (d+p) \) is even, \( s_{2,p} (d, c : \chi) = 0 \) if \( (c+p) \) is even and \( s_{5,p} (d, c : \chi) = 0 \) if \( (d+c+p) \) is even.

**Proof.** Let \( p \) be odd and \( d \) be even. Then, setting \( \mu = n + mck \), \( 1 \leq n \leq ck \), \( 0 \leq m \leq q-1 \) and using (2.1) and (2.7) yield
\[
\begin{align*}
\sum_{\mu=1}^{qck} \chi(\mu) \mathcal{E}_{p-1,\mu} \left( \frac{d\mu}{c} \right) \mathfrak{B}_1 \left( \frac{\mu}{qck} \right) \\
= \sum_{n=1}^{ck} \chi(n) \mathcal{E}_{p-1,n} \left( \frac{dn}{c} \right) \sum_{m=0}^{q-1} (-1)^{dm} \mathfrak{B}_1 \left( \frac{n}{qck} + \frac{m}{q} \right) \\
= s_{1,p} (d, c : \chi).
\end{align*}
\]
On the other hand, using (2.7),
\[
\begin{align*}
\sum_{n=1}^{ck} \chi(n) \mathcal{E}_{p-1,n} \left( \frac{dn}{c} \right) \mathfrak{B}_1 \left( \frac{n}{ck} \right) \\
= \sum_{n=1}^{ck} \chi(-n) \mathcal{E}_{p-1,-n} \left( dk - \frac{dn}{c} \right) \mathfrak{B}_1 \left( 1 - \frac{n}{ck} \right) \\
= (-1)^{d+p+1} s_{1,p} (d, c : \chi)
\end{align*}
\]
which leads to \( s_{1,p} (d, c : \chi) = 0 \) for even \( d+p \).

Other statements can be shown in a similar way. \( \blacksquare \)
5 Some series relations

In this final section, we deal with (3.5) and (3.16) for special values of $Tz$ to present series relations, motivated by [5] (see also [16, 25]).

Summing over $m$ we see that

$$G(\bar{\chi}) A_1(z, 1 - p : \chi) = - \sum_{j=1}^{k-1} \bar{\chi}(j) \sum_{n=1}^\infty \frac{\chi(n)}{n^p (e^{-2\pi i (j+nx)/k} + 1)}$$

and

$$G(\chi) B(z, 1 - p : \bar{\chi}) = \sum_{j=1}^{k-1} \chi(j) \sum_{n=0}^\infty \frac{\bar{\chi}(2n+1)}{(2n+1)^p (e^{-\pi i (2n+(2n+1))z/k} - 1)}.$$ (5.2)

**Theorem 5.1** Let $p \geq 1$ be odd and $\alpha \beta = (\pi/k)^2$ with $\alpha, \beta > 0$. Then,

$$(-\beta)^{(p-1)/2} \sum_{j=1}^{k-1} \bar{\chi}(j) \sum_{n=1}^\infty \frac{\chi(n)}{n^p (e^{2\pi i - 2\pi ij/k} + 1)}$$

$$+ 2^p \alpha^{(p-1)/2} \sum_{j=1}^{k-1} \chi(j) \sum_{n=0}^\infty \frac{\bar{\chi}(2n+1)}{(2n+1)^p (e^{2\pi j - 2\pi ij/k} - 1)}$$

$$= 2^{p-2} \frac{k}{p!} \sum_{m=1}^p \binom{p}{m} (i)^{p+1-m} \mathcal{E}_{p-m,\bar{\chi}}(0) \mathcal{B}_{m,\chi}(0) \alpha^{p-m/2} \beta^{(p+m)/2}.$$ (5.2)

**Proof.** We put $a = d = 0$, $b = -1$ and $c = 1$ in (3.5) to obtain

$$z^{p-1} G(\bar{\chi}) A_1 \left( -\frac{1}{z}; 1 - p : \chi \right) = 2^p \chi(-2) G(\chi) B(z, 1 - p : \bar{\chi})$$

$$- \frac{k}{4} \frac{(2\pi i/k)^p}{p!} \sum_{m=1}^p \binom{p}{m} \mathcal{E}_{p-m,\bar{\chi}}(0) \mathcal{B}_{m,\chi}(0) (-z)^{m-1}.$$ (5.2)

Then, the proof follows by setting $z = \pi i/k\alpha$ and using (5.1), (5.2) and that $\alpha \beta = (\pi/k)^2$, and then multiplying both sides by $\alpha^{(p-1)/2}$. ■

**Corollary 5.2** Let $p \geq 1$ be odd and let $\chi$ be the primitive character of modulus 3 defined by

$$\chi(n) = \begin{cases} 1, & n \equiv 1 \pmod{3}, \\ -1, & n \equiv 2 \pmod{3}, \\ 0, & n \equiv 0 \pmod{3}. \end{cases}$$ (5.3)

Then,

$$\sum_{n=1}^\infty \frac{(-1)^n(\pi/3)\chi(n)}{n^p (2 \cosh (n\pi/3) - (-1)^n)}$$
we have

\[ \sum_{n=1}^{\infty} \frac{(-1)^n \chi(n)}{n (2 \cosh (n \pi/3) - (-1)^n)} = -\frac{\pi}{4 \sqrt{3}} \mathcal{E}_{0, \chi}(0) \mathfrak{B}_{1, \chi}(0) = -\frac{\pi}{6 \sqrt{3}} \]

and

\[ \sum_{n=1}^{\infty} \frac{\chi(n)}{n^3 (2 \cosh (n \pi/3) - (-1)^n)} = \frac{(\pi/3)^3}{8 \sqrt{3}} (\mathcal{E}_{0, \chi}(0) \mathfrak{B}_{3, \chi}(0) - 3 \mathcal{E}_{2, \chi}(0) \mathfrak{B}_{1, \chi}(0)) \]

**Proof.** Setting \( \alpha = \beta = \pi/3 \) in Theorem 5.1 we have

\[
\begin{align*}
(-1)^{(p-1)/2} &\sum_{n=1}^{\infty} \frac{\chi(n)}{n^p} \frac{1}{e^{2n\alpha - 2\pi n/3} + 1} - \frac{1}{e^{2n\alpha - 4\pi n/3} + 1} \\
+ 2^p &\sum_{n=0}^{\infty} \frac{\chi(2n+1)}{(2n+1)^p} \frac{1}{e^{(2n+1)\alpha - 2\pi n/3} - 1} - \frac{1}{e^{(2n+1)\alpha - 4\pi n/3} - 1} \\
= 2^{p-2} &\frac{3}{p!} \left( \frac{\pi}{3} \right)^p \sum_{m=1}^{p} \left( \frac{p}{m} \right) (i)^{p-m} \mathcal{E}_{p-m, \chi}(0) \mathfrak{B}_{m, \chi}(0) .
\end{align*}
\]

Some simplification gives

\[
\begin{align*}
\sum_{n=1}^{\infty} \frac{(-1)^{(p+1)/2} \chi(2n)}{(2n)^p (2 \cosh 2n\alpha - 1)} + \sum_{n=0}^{\infty} \frac{\chi(2n+1)}{(2n+1)^p (2 \cosh (2n+1) \alpha + 1)} \\
= \frac{1}{4 \sqrt{3}} &\frac{3}{p!} \left( \frac{\pi}{3} \right)^p \sum_{m=1}^{p} \left( \frac{p}{m} \right) (i)^{p-m} \mathcal{E}_{p-m, \chi}(0) \mathfrak{B}_{m, \chi}(0) ,
\end{align*}
\]

which is equivalent to (5.4).  

Observe that for \( \alpha = \beta = \pi/3 \) and real-valued primitive character \( \chi \), Theorem 5.1 can be composed as

\[
\begin{align*}
\chi(2) &\sum_{j=1}^{k-1} \chi(j) \sum_{n=1}^{\infty} \frac{(-1)^n \chi(n)}{n^p \left( e^{2n\alpha - 2\pi ij/k} + (-1)^n \right)} \\
= \frac{(-1)^{(p-1)/2}}{4} &\left( \frac{\pi}{k} \right)^p \frac{k}{p!} \sum_{m=1}^{p} \left( \frac{p}{m} \right) (i)^{p-m} \mathcal{E}_{p-m, \chi}(0) \mathfrak{B}_{m, \chi}(0) ,
\end{align*}
\]

where \( \delta = \begin{cases} (p-1)/2, & \text{if } \chi(-1) = 1, \\ (p+1)/2, & \text{if } \chi(-1) = -1. \end{cases} \)

**Theorem 5.3** Let \( \alpha \beta = (\pi/3)^2 \) with \( \alpha, \beta > 0 \) and let \( \chi \) be the primitive character of modulus 3 given by (5.3). Then

\[
\sum_{n=1}^{\infty} \frac{(-1)^n \chi(n)}{n (2 \cosh 2n\alpha - (-1)^n)} + \sum_{n=1}^{\infty} \frac{(-1)^n \chi(n)}{n (2 \cosh 2n\beta - (-1)^n)} = -\frac{\pi}{3 \sqrt{3}} .
\]

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In particular,
\[ \sum_{n=1}^{\infty} \frac{(-1)^n \chi(n)}{n(2 \cosh \frac{2n\pi}{3} - (-1)^n)} = -\frac{\pi}{6\sqrt{3}}. \]

**Proof.** We apply (3.16) with \( a = d = 3, b = 4 \) and \( c = 2 \). Setting \( 2z + 3 = \pi i/3 \alpha \) we have \( Tz = 3(1 - \alpha/\pi i)/2 \) and \( z = -3(1 + \beta/\pi i)/2 \), where \( \alpha \beta = (\pi/3)^2 \).

Straightforward calculation gives
\[
\sum_{n=1}^{\infty} \frac{\chi(n)}{n(2 \cosh (2n\alpha - n\pi i) - 1)} + \sum_{n=1}^{\infty} \frac{\chi(n)}{n(2 \cosh (2n\beta + n\pi i) - 1)} = -\frac{\pi}{\sqrt{3}} \sum_{j=1}^{5} (-1)^j \chi(j) \mathfrak{B}_{1,\chi} \left( \frac{3j}{2} \right) \mathfrak{B}_1 \left( \frac{j}{6} \right).
\]

Using (2.4) and the fact \( \mathfrak{B}_1 (x + 1) = \mathfrak{B}_1 (x) = x - 1/2 \) when \( 0 < x < 1 \), we find that
\[
\sum_{j=1}^{5} (-1)^j \chi(j) \mathfrak{B}_{1,\chi} \left( \frac{3j}{2} \right) \mathfrak{B}_1 \left( \frac{j}{6} \right) = \frac{1}{3}
\]
which completes the proof. ■

**References**

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Undergraduate Texts in Mathematics, Springer-Verlag, New York 1976.

[2] T. M. Apostol and T. H. Vu, Elementary proofs of Berndt’s reciprocity laws, *Pacific J. Math.* 98 (1982) 17–23.

[3] B. C. Berndt, Character transformation formulæ similar to those for the Dedekind Eta-function, *in ‘Analytic Number Theory’, Proc. Sym. Pure Math. XXIV*, Amer. Math. Soc., Providence, R. I., (1973) 9–30.

[4] B. C. Berndt, Character analogues of Poisson and Euler–Maclaurin summation formulæ with applications, *J. Number Theory* 7 (1975) 413–445.

[5] B. C. Berndt, Modular transformations and generalizations of several formulæ of Ramanujan, *Rocky Mt. J. Math.* 7(1977) 147–190.

[6] B. C. Berndt, Analytic Eisenstein series, theta functions and series relations in the spirit of Ramanujan, *J. Reine Angew. Math.* 303/304 (1978) 332–365.

[7] B. C. Berndt and L. A. Goldberg, Analytic properties of arithmetic sums arising in the theory of the classical theta functions, *Siam J. Math. Anal.* 15 (1) (1984) 143–150.
[8] M. Can, Some arithmetic on the Hardy sums $s_2(h,k)$ and $s_3(h,k)$, Acta Math. Sin. Engl. Ser. 20 (2) (2004) 193–200.

[9] M. Can, M. Cenkci and V. Kurt, Generalized Hardy–Berndt sums, Proc. Jangjeon Math. Soc. 9 (1) (2006) 19–38.

[10] M. Can and V. Kurt, Character analogues of certain Hardy–Berndt sums, Int. J. Number Theory 10 (2014) 737–762.

[11] M. Can and M. C. Dağlı, Character analogue of the Boole summation formula with applications, Turk. J. Math. (accepted).

[12] L. Carlitz, Eulerian numbers and polynomials, Math. Mag. 32 (5) (1959) 247–260.

[13] M. Cenkci, M. Can and V. Kurt, Degenerate and character Dedekind sums, J. Number Theory 124 (2007) 346–363.

[14] M. C. Dağlı and M. Can, A new generalization of Hardy–Berndt sums, Proc. Indian Acad. Sci. (Math. Sci.) 123 (2) (2013) 177–192.

[15] M. C. Dağlı and M. Can, On reciprocity formulas for Apostol’s Dedekind sums and their analogues, J. Integer Seq. 17 (2014) Article 14.5.4.

[16] M. C. Dağlı and M. Can, Periodic analogues of Dedekind sums and transformation formulas of Eisenstein series, Ramanujan J. doi: 10.1007/s11139-016-9808-y.

[17] L. A. Goldberg, Transformations of theta-functions and analogues of Dedekind sums, thesis, University of Illinois, Urbana, 1981.

[18] C. Jordan, Calculus of Finite Differences, Chelsea, New York 1965.

[19] J. Lewittes, Analytic continuation of the Eisenstein series, Trans. Amer. Math. Soc. 171 (1972) 469–490.

[20] H. Liu and W. Zhang, Generalized Cochrane sums and Cochrane–Hardy sums, J. Number Theory 122 (2) (2007) 415–428.

[21] H. Liu and J. Gao, Generalized Knopp identities for homogeneous Hardy sums and Cochrane-Hardy sums, Czech. Math. J. 62 (2012) 1147–1159.

[22] J. L. Meyer, Analogues of Dedekind sums, thesis, University of Illinois, Urbana, 1997.

[23] J. L. Meyer Properties of certain integer-valued analogues of Dedekind sums, Acta Arith. LXXXII (3) (1997) 229–242.

[24] J. L. Meyer, A reciprocity congruence for an analogue of the Dedekind sum and quadratic reciprocity, J. Théor. Nombres Bordeaux 12 (1) (2000) 93–101.
[25] J. L. Meyer, Character analogues of Dedekind sums and transformations of analytic Eisenstein series, *Pacific J. Math.* 194 (1) (2000) 137–164.

[26] W. Peng and T. Zhang, Some identities involving certain Hardy sum and Kloosterman sum, *J. Number Theory* 165 (2016) 355–362.

[27] M. R. Pettet and R. Sitaramachandrarao, Three-term relations for Hardy sums, *J. Number Theory* 25 (3) (1987) 328–339.

[28] Y. Simsek, Relations between theta functions, Hardy sums, Eisenstein and Lambert series in the transformation formulae of $\log \eta_{g,h}(z)$, *J. Number Theory* 99 (2003) 338–360.

[29] R. Sitaramachandrarao, Dedekind and Hardy sums, *Acta Arith.* XLIII (1987) 325–340.

[30] W. Wang and D. Han, An identity involving certain Hardy sums and Ramanujan’s sum, *Adv. Difference Equ.* (2013) 2013:261.

[31] Z. Xu and W. Zhang, The mean value of Hardy sums over short intervals, *Proc. R. Soc. Edinburgh* 137 (2007) 885–894.

[32] H. Zhang and W. Zhang, On the identity involving certain Hardy sums and Kloosterman sums, *J. Inequal. Appl.* (2014) 2014:52.