Symmetries between Untwisted and Twisted Strings
on Asymmetric Orbifolds

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Abstract

We study symmetries between untwisted and twisted strings on asymmetric orbifolds. We present a list of asymmetric orbifold models to possess intertwining currents which convert untwisted string states to twisted ones, and vice versa. We also present a list of heterotic strings on asymmetric orbifolds with supersymmetry between untwisted and twisted string states. Some of properties inherent in asymmetric orbifolds, which are not shared by symmetric orbifolds, are pointed out.

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1. Introduction

In the construction of realistic four-dimensional string models, various approaches have been proposed [1-8]. Among them, the orbifold compactification [1] is probably the most efficient method and the orbifold compactification of the heterotic string [9] is believed to provide a phenomenologically realistic string model. The heterotic string has asymmetric nature: The left-movers consist of a 26-dimensional bosonic string and the right-movers consist of a 10-dimensional superstring. This asymmetric nature of the heterotic string naturally leads to the idea of asymmetric orbifolds [10]. Although the search for realistic orbifold models has been continued by many authors [11-14], a more general and systematic investigation of asymmetric orbifolds should be done. In this paper we shall reveal some of properties inherent in asymmetric orbifolds, which are not shared by symmetric orbifolds.

Suppose that there exists an intertwining current operator which converts string states in an untwisted sector to string states in a twisted sector in an asymmetric orbifold model. This current operator will correspond to a state of the conformal weight (1,0) (or (0,1)) in the twisted sector and connect the ground state of the untwisted sector to the (1,0) (or (0,1)) twisted state. Therefore, the existence of a (1,0) (or (0,1)) twisted state implies a symmetry between the untwisted and twisted sectors. Since a total Hilbert space of strings on the orbifold is a direct sum of the untwisted and twisted Hilbert spaces, the existence of (1,0) (or (0,1)) physical twisted states implies that the symmetry of the total Hilbert space is larger than the symmetry of each (untwisted or twisted) Hilbert space *. It should be emphasized that this symmetry “enhancement” does not occur in the case of symmetric orbifolds because the left- and right- conformal weights, $h$ and $\bar{h}$, of a ground state of any twisted sectors are both positive (and equal) for symmetric orbifolds and hence no (1,0) (or (0,1)) state appears in any twisted sector.

Symmetry “enhancement” stated above will mean “enhancement” of gauge symmetries. In the case of superstring theories, another interesting type of symmetry “enhancement” might occur, i.e., supersymmetry “enhancement”. N=1 space-time supersymmetry might appear in a spectrum of a total Hilbert space through supersym-

* Some examples have been discussed in refs. [1,15,16] and in our previous papers [17,18].
metry “enhancement” even though there is no unbroken space-time supersymmetry in each untwisted or twisted Hilbert space. It would be of interest to investigate such a new class of four-dimensional orbifold models with N=1 space-time supersymmetry.

Another peculiarity of asymmetric orbifolds is related to the “torus-orbifold equivalence” [1,3,19-21]. An orbifold will be obtained by dividing a (D-dimensional) torus $T^D$ by the action of a discrete symmetry group $P$ of the torus. We may denote the orbifold by $T^D/P$. An orbifold model with $T^D/P$ can be equivalent to a torus model with $T'^D$ for asymmetric as well as symmetric orbifolds. However, $T^D$ can be equal to $T'^D$ only for asymmetric orbifolds but not symmetric ones. If the symmetry group $P$ includes an outer automorphism of the lattice defining the torus, then the orbifold model cannot be rewritten as a torus model for symmetric orbifolds. Our results, however, suggest a new class of the “torus-orbifold equivalence”, that is, some asymmetric orbifold models can be rewritten as torus models even in the case of outer automorphisms.

In the next section, we discuss general properties of asymmetric $\mathbb{Z}_N$-orbifold models, which do not depend on specific momentum lattices on which left- and right-moving momenta lie. In sect. 3, we investigate $\mathbb{Z}_N$-automorphisms of Lie algebra lattices. We will be concerned with $\mathbb{Z}_N$-automorphisms which have no fixed direction and give a classification of momentum lattices associated with Lie algebras and their $\mathbb{Z}_N$-automorphisms. In sect. 4, we briefly review the “torus-orbifold equivalence”, which may be used to determine full symmetries of asymmetric orbifold models. In sect. 5, we present a list of asymmetric $\mathbb{Z}_N$-orbifold models which possess (1,0) twisted states and show that these states correspond to twist-untwist intertwining currents which convert untwisted string states to twisted string states, and vice versa. In sect. 6, we study $E_8 \times E_8$ heterotic strings on asymmetric $\mathbb{Z}_N$-orbifolds and present four orbifold models with supersymmetry between untwisted and twisted string states. In sect. 7, various properties inherent in asymmetric orbifold models are summarized. In an appendix, shift vectors which are introduced in rewriting orbifold models into torus models are given.

### 2. Asymmetric $\mathbb{Z}_N$-orbifolds
An orbifold \[1\] will be obtained by dividing a torus by the action of a discrete symmetry group \( P \) of the torus. In the construction of an orbifold model, we start with a \( D \)-dimensional toroidally compactified closed bosonic string theory which is specified by a \((D + D)\)-dimensional lorentzian even self-dual lattice \( \Gamma^{D,D} [22] \). The left- and right-moving momentum \((p^I_L, p^I_R)\) \((I = 1, \ldots, D)\) lies on the lattice \( \Gamma^{D,D} \). Let \( g \) be a group element of \( P \). The \( g \) will, in general, act on the left-movers and the right-movers differently. If both the left- and right-moving string coordinates \( X^I_L \) and \( X^I_R \) obey twisted boundary conditions, there are no \((1,0)\) (or \((0,1)\)) states in the twisted sector because the conformal weight \((h, \bar{h})\) of the ground state in the twisted sector will be positive, i.e., \( h > 0 \) and \( \bar{h} > 0 \). For \((1,0)\) \((0,1)\) states to appear in a twisted sector, the right- (left-) moving string coordinate must obey the untwisted boundary condition. Hence we will restrict our considerations to the following class of the \( \mathbb{Z}_N \)-transformation:

\[
g : (X^I_L, X^I_R) \rightarrow (U^{IJ} X^J_L, X^I_R), \quad (I, J = 1, \ldots, D),
\]

(2.1)

where \( U \) is a rotation matrix which satisfies \( U^N = 1 \). The \( \mathbb{Z}_N \)-transformation must be an automorphism of the lattice \( \Gamma^{D,D} \), i.e.,

\[
(U^{IJ} p^I_L, p^I_R) \in \Gamma^{D,D} \quad \text{for all} \quad (p^I_L, p^I_R) \in \Gamma^{D,D}.
\]

(2.2)

For simplicity, we will assume that the rotation matrix \( U \) and all its powers \( U^\ell \) \((\ell = 1, \ldots, N - 1)\) do not have any fixed direction.

Let us consider the \( g^\ell \)-twisted sector in which strings close up to the \( g^\ell \)-action. The one loop partition function of the \( g^\ell \)-sector twisted by \( g^m \) is given by

\[
Z(g^\ell, g^m; \tau) = \text{Tr}[g^m e^{i2\pi \tau(L_0 - \frac{D}{24})} e^{-i2\pi \tau(\bar{L}_0 - \frac{D}{24})}],
\]

(2.3)

where the trace is taken over the Hilbert space of the \( g^\ell \)-sector and \( L_0 \) (\( \bar{L}_0 \)) is the zero mode of the left- (right-) moving Virasoro operators. Modular invariance of the one loop partition function will require

\[
Z(g^\ell, g^m; \tau + 1) = Z(g^\ell, g^{m+\ell}; \tau),
\]

(2.4)

\[
Z(g^\ell, g^m; -1/\tau) = Z(g^{-m}, g^\ell; \tau).
\]

(2.5)
Let $N_\ell$ be the minimum positive integer such that $(g^\ell)^{N_\ell} = 1$. Since $g^N = 1$ and hence $Z(g^\ell, g^m; \tau)$ has to be invariant under the modular transformation $\tau \rightarrow \tau + N_\ell$, the necessary condition for modular invariance is

$$N_\ell(L_0 - \bar{L}_0) = 0 \mod 1. \quad (2.6)$$

This is called the left-right level matching condition and it has been proved that this condition is also sufficient for modular invariance [10,23]. Let $\Gamma_0$ be the $g^\ell$-invariant sublattice of $\Gamma^{D,D}$. Then the left- and right-moving momentum in the $g^\ell$-sector lies on the lattice $\Gamma_0^*$ [10], which is the dual lattice of $\Gamma_0^*$. The degeneracy $d_\ell$ of the ground states in the $g^\ell$-sector is given by [10]

$$d_\ell = \frac{\sqrt{\det'(1 - U)}^-}{V_{\Gamma_0}}, \quad (2.7)$$

where $V_{\Gamma_0}$ is the volume of the unit cell of the lattice $\Gamma_0$ and the determinant is taken over the eigenvalues of $U^\ell$ not equal to one. The left-right level matching condition (2.6) can equivalently be rewritten as the following two conditions:

$$N_\ell(h_\ell - \bar{h}_\ell) = 0 \mod 1, \quad (2.8)$$

$$N_\ell((p^I_L)^2 - (p^I_R)^2) = 0 \mod 2 \quad \text{for all } (p^I_L, p^I_R) \in \Gamma_0^*, \quad (2.9)$$

where $h_\ell$ ($\bar{h}_\ell$) denotes the conformal weight of the ground state in the $g^\ell$-sector with respect to the left- (right-) movers. Since the $Z_N$-transformation is given by eq. (2.1) and $U^\ell$ ($\ell = 1, \ldots, N - 1$) is assumed to have no fixed direction, the two conditions (2.8) and (2.9) reduce to

$$N_\ell h_\ell = 0 \mod 1, \quad (2.10)$$

$$N_\ell(p^I_R)^2 = 0 \mod 2 \quad \text{for all } p^I_R \in \Gamma_0^*. \quad (2.11)$$

Let $\omega^a$ ($a = 1, \ldots, D$) be an eigenvalue of $U^\ell$ with $0 \leq s_a \leq N - 1$, where $\omega = e^{i2\pi/N}$. Then $h_\ell$ is given by

$$h_\ell = \frac{1}{4} \sum_{a=1}^D \frac{s_a}{N} \left(1 - \frac{s_a}{N}\right). \quad (2.12)$$

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* This is not true in general, as pointed out in refs. [24,25]. We will, however, restrict our considerations to the models in which the left- and right-moving momentum in the $g^\ell$-sector lies on the lattice $\Gamma_0^*$. 

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- 5 -
From eqs. (2.3) and (2.4), it turns out that the operator \( g^\ell \) in the \( g^\ell \)-sector is given by

\[
g^\ell = \exp[i 2\pi (L_0 - \bar{L}_0)].
\] (2.13)

If \( \ell \) is relatively prime to \( N \), \( g^\ell = 1 \) means \( g = 1 \). Thus any state of \( L_0 - \bar{L}_0 = 0 \mod 1 \), in particular, \((L_0, \bar{L}_0) = (1, 0)\) is \( \mathbb{Z}_N \)-invariant and hence physical. If \( \ell \) is not relatively prime to \( N \), \( g^\ell = 1 \) does not mean \( g = 1 \). Thus, to determine a physical spectrum of the \( g^\ell \)-sector, a detailed analysis of the one loop partition function is required.

Let \( \omega^{r_a} \ (a = 1, \ldots, D) \) be an eigenvalue of \( U \), where \( \omega = e^{i 2\pi / N} \). Since we have assumed that the rotation matrix \( U \) and all its powers \( U^\ell \ (\ell = 1, \ldots, N - 1) \) have no fixed direction, \( \omega^{\ell r_a} \) is not equal to one for \( \ell = 1, \ldots, N - 1 \). This implies that \( \omega^{r_a} \ (a = 1, \ldots, D) \) must be a primitive \( N \)th root of unity. Every primitive \( N \)th root of unity can be written as \( \omega^m \ (m = 1, \ldots, N - 1) \), where \( m \) is relatively prime to \( N \). The number of the primitive \( N \)th roots of unity is denoted by \( \varphi(N) \), which is called the Euler function. If we change the \((D + D)\)-dimensional basis vectors to the lattice basis of \( \Gamma^{D,D} \), then the \( \mathbb{Z}_N \)-automorphism \( g \) in eq. (2.1) is represented by an integer matrix. This means that the characteristic polynomial \( \det(\lambda 1 - U) \) must have integer coefficients. It turns out that \( \det(\lambda 1 - U) \) is given by a multiple of the cyclotomic polynomial \( \Phi_N(\lambda) \) [26], i.e.,

\[
\det(\lambda 1 - U) = [\Phi_N(\lambda)]^{D/\varphi(N)}.
\] (2.14)

In eq. (2.14), \( \Phi_N(\lambda) \) is the polynomial of the degree \( \varphi(N) \) and is defined by

\[
\Phi_N(\lambda) = \prod_{(m,N)=1}^{(m,N)=1} \prod_{m=1,\ldots,N-1} (\lambda - \omega^m),
\] (2.15)

where \((m, N)\) denotes the greatest common divisor of \( m \) and \( N \). Therefore, we have found that the eigenvalues of \( U \) consist of the primitive \( N \)th roots of unity and that each primitive \( N \)th root of unity appears \( D/\varphi(N) \) times. This implies that the dimension \( D \) must be a multiple of \( \varphi(N) \), i.e.,

\[
D = 0 \mod \varphi(N).
\] (2.16)

A list of \( \varphi(N) \) is given in table 1.
We can further put some constraints on the dimension $D$, irrespective of the specific lattice $\Gamma^{D,D}$. Since $U^\ell (\ell = 1, \ldots, N-1)$ is assumed to have no fixed direction, we have

\[
\det(\lambda 1 - U^\ell) = [\Phi_{N_\ell}(\lambda)]^{D/\varphi(N_\ell)},
\]

where $N_\ell$ is the minimum positive integer such that $(g^\ell)^{N_\ell} = 1$. It follows from the formula (2.12) that

\[
h_\ell = \frac{D}{\varphi(N_\ell)} \frac{1}{4} \sum_{(m,N_\ell)^\ell=1}^{m=1,\ldots,N_\ell-1} \frac{m}{N_\ell} \left(1 - \frac{m}{N_\ell}\right).
\]

Since modular invariance requires eq. (2.10), the dimension $D$ must satisfy

\[
D = 0 \mod \frac{\varphi(N_\ell)}{N_\ell h'_\ell} \quad \text{for } \ell = 1, \ldots, N - 1,
\]

where

\[
h'_\ell = \frac{1}{4} \sum_{m=1,\ldots,N_\ell-1} \frac{m}{N_\ell} \left(1 - \frac{m}{N_\ell}\right).
\]

For example, $D$ must be a multiple of 8 for $N = 2$ because $\varphi(2) = 1$ and $h'_1 = \frac{1}{16}$. In the third column of table 1, allowed values of $D$, which are consistent with the constraints (2.16) and (2.19), are listed up to $N = 30$.

Since we are interested in asymmetric orbifold models which possess $(1,0)$ states in the twisted sectors, $h_\ell (\ell = 1, \ldots, N-1)$ must be less than or equal to one for some $\ell$. This requirement severely restricts allowed values of $D$. Since $h_\ell$ is proportional to $D$, $h_\ell$ will exceed one for appropriately large $D$. Therefore, only lower dimensional orbifold models might possess $(1,0)$ states in twisted sectors. In the fourth column of table 1, a list of the dimensions $D$ in which $(1,0)$ states might appear in twisted sectors is given up to $N = 30$. In sect. 5, we will see all asymmetric orbifold models listed in the forth column of table 1, which possess $(1,0)$ states in twisted sectors, except for $N = 14, 21, 25$ and 26.

### 3. Automorphisms of $\Gamma^{D,D}$

In this section we will investigate automorphisms of the momentum lattice $\Gamma^{D,D}$. Since a complete classification of automorphisms of general lorentzian even self-dual
lattices is not known, we will restrict our considerations to lattices associated with Lie algebras.

We take the lattice $\Gamma^{D,D}$ of an asymmetric $\mathbb{Z}_N$-orbifold to be of the form:

$$\Gamma^{D,D} = \{ (p^I_L, p^I_R) \mid p^I_L, p^I_R \in \Lambda^*, p^I_L - p^I_R \in \Lambda \}, \quad (3.1)$$

where $\Lambda$ is a $D$-dimensional lattice and $\Lambda^*$ is the dual lattice of $\Lambda$. It turns out that $\Gamma^{D,D}$ is lorentzian even self-dual if $\Lambda$ is even integral. In the following, we will take $\Lambda$ in eq. (3.1) to be a root lattice of a simply-laced semi-simple Lie algebra $G$ [27]. Then the lattice $\Lambda$ is even integral if the squared length of the root vectors is normalized to $2m$ ($m = 1, 2, \ldots$). Since the $\mathbb{Z}_N$-transformation defined in eq. (2.1) must be an automorphism of the lattice (3.1), the rotation matrix $U$ must be an automorphism of $\Lambda$ as well as $\Lambda^*$. Furthermore, the condition (2.2) requires $U$ to satisfy

$$U^{IJ} p^J_L - p^J_R \in \Lambda \quad \text{for all } p^I_L \in \Lambda^*. \quad (3.2)$$

Let us first consider the case that the squared length of the root vectors in the root lattice $\Lambda$ is normalized to two. Then the dual lattice $\Lambda^*$ is equivalent to the weight lattice of the Lie algebra $G$. For simplicity, we assume that the rotation matrix $U$ and all its powers $U^\ell$ ($\ell = 1, \ldots, N - 1$) have no fixed direction. Let $\Phi$ be a root system of the semi-simple Lie algebra $G$ and let $\text{Aut}\Phi$ be the group of automorphisms of this root system. The group $\text{Aut}\Phi$ is a semi-direct product of two groups [28]:

$$\text{Aut}\Phi = W \rtimes \text{Aut}(\Phi, \Delta),$$

$$W \cap \text{Aut}(\Phi, \Delta) = \{1\}, \quad (3.3)$$

where $W$ is the Weyl group of $\Phi$, which is a normal subgroup of $\text{Aut}\Phi$, and $\text{Aut}(\Phi, \Delta)$ is defined as

$$\text{Aut}(\Phi, \Delta) = \{ \varphi \in \text{Aut}\Phi \mid \varphi(\Delta) = \Delta \}. \quad (3.4)$$

Here, $\Delta$ is a fixed basis of $\Phi$. $\text{Aut}(\Phi, \Delta)$ corresponds to the group of symmetries of the Dynkin diagram of $G$. The condition (3.2) tells us that the automorphism $U$ of $\Lambda$ must not change a conjugacy class of any vector in $\Lambda^*$. Since the squared length of the root vectors is normalized to two, any element of the Weyl group $W$ in eq. (3.3) does not change a conjugacy class of any vector in $\Lambda^*$. On the other hand, for any
nontrivial element of $\text{Aut}(\Phi, \Delta)$ in eq. (3.3), there always exists a vector in $\Lambda^*$ which is mapped to a different conjugacy class of $\Lambda^*$ by the automorphism. Consequently, the automorphism $U$ of $\Lambda$ must be an element of the Weyl group $W$ of the root system $\Phi$.

All automorphisms of the root systems of simple Lie algebras such that they and all their powers have no fixed directions have been discussed in ref. [29]. In the following, we will give the result concerning the Weyl group elements of the root system of simply-laced simple Lie algebras. In the case of $G = SU(n+1)$, there exists a Weyl group element of $SU(n+1)$ which has no fixed directions if and only if $n+1$ is prime. The order of this Weyl group element is $n+1$. In the case of $G = SO(2n)$, there exist Weyl group elements of $SO(2n)$ which have no fixed directions if and only if $n$ is even, i.e., $n = 2^\ell p$ ($\ell > 0$, $p$ = odd). Then the allowed orders are given by $2, 2^2, \ldots, 2^\ell$.

In the cases of $G = E_6, E_7$ and $E_8$, the orders of the Weyl group elements which have no fixed directions are

- $E_6 : 3, 9$
- $E_7 : 2$
- $E_8 : 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30$.

The explicit expressions of the Weyl group elements stated above are given in ref. [29].

Let us next consider the case that the squared length of the root vectors in $\Lambda$ is normalized to $2m$ ($m = 2, 3, \ldots$). Then we have $\Lambda = \sqrt{m} \Lambda_R$ and $\Lambda^* = \frac{1}{\sqrt{m}} \Lambda_W$ ($m = 2, 3, \ldots$), where $\Lambda_R$ denotes the root lattice of $G$ in which the squared length of the root vectors is normalized to two and $\Lambda_W = \Lambda_R^*$. Let $\mu^i$ ($i = 1, \ldots, \text{rank}G$) be a fundamental weight of $G$. Since $\Lambda_R$ is an even lattice, the condition (3.2) implies that

$$(\mu^i)^2 - \mu^i \cdot U \mu^i = 0 \mod m^2 \text{ for } i = 1, \ldots, \text{rank}G. \quad (3.5)$$

Since $U$ is a rotation matrix which has no fixed directions, the left hand side of eq. (3.5) is restricted to

$$0 < (\mu^i)^2 - \mu^i \cdot U \mu^i \leq 2(\mu^i)^2, \quad (3.6)$$

where the equality holds if and only if $U \mu^i = -\mu^i$. It is known that there always exists a fundamental weight $\mu^i$ such that $(\mu^i)^2 < 2$ except for $G = (E_8)^\ell$ ($\ell = 1, 2, \ldots$) [30]. For $G = (E_8)^\ell$ ($\ell = 1, 2, \ldots$), all fundamental weights have the squared length two because the root lattice of $G = (E_8)^\ell$ is even self-dual. Hence the conditions (3.5) are satisfied if and only if $m = 2$, $U = -1$ and $G = (E_8)^\ell$ ($\ell = 1, 2, \ldots$).
We have investigated the allowed automorphisms of the lattice (3.1) associated with Lie algebra lattices $\Lambda$ and have found that the orders $N$ of the allowed automorphisms are given by $N = \text{prime numbers}, 2^M \ (M = 1, 2, \ldots)$ and $6, 9, 10, 12, 15, 20, 24, 30$. Since we are interested in asymmetric orbifold models which possess $(1,0)$ states in twisted sectors, the allowed orders further reduce as follows: If the order $N$ is prime, the left conformal weight of the ground states of every $g^\ell$-twisted sector ($\ell = 1, \ldots, N - 1$) has the common value, i.e.,

$$h_\ell = \frac{1}{4} \sum_{i=1}^{N-1} \frac{i}{N} \left(1 - \frac{i}{N}\right) \frac{D}{N-1} = \frac{N+1}{24N} D. \quad (3.7)$$

It follows that the existence of $(1,0)$ states in some twisted sectors implies $D < 24$. Thereby, the allowed prime orders are $N = 2, 3, 5, 7, 11, 13, 17, 19, 23$. If the order $N$ is $2^M \ (M = 1, 2, \ldots)$, the left conformal weight of the ground states of the $g^\ell$-twisted sector ($\ell = 1, \ldots, N - 1$) is given by

$$h_\ell = \frac{1}{4} \sum_{j=1}^{N_\ell/2} \frac{2j - 1}{N_\ell} \left(1 - \frac{2j - 1}{N_\ell}\right) \frac{D}{N_\ell/2} = \frac{(N_\ell)^2 + 2}{24(N_\ell)^2} D. \quad (3.8)$$

It follows that the existence of $(1,0)$ states in some twisted sectors implies $D < 24$. Thereby, the allowed orders of the form $2^M \ (M = 1, 2, \ldots)$ are $N = 2, 4, 8, 16$.

### 4. Torus-Orbifold Equivalence

In the next section, we will present asymmetric $\mathbb{Z}_N$-orbifold models with intertwining currents which convert untwisted string states to twisted ones, and vice versa. To investigate symmetries of those orbifold models, we might construct the intertwining currents explicitly. Such currents correspond to twisted state emission vertex operators with the conformal weight $(1,0)$. It is not, however, easy to explicitly construct twisted state emission vertex operators [15,16,31-33]. To avoid this difficulty, we will use a trick, the “torus-orbifold equivalence” [1,3,19-21]: Any closed bosonic string theory compactified on a $\mathbb{Z}_N$-orbifold is equivalent to a closed bosonic string theory on a torus if the dimension of the orbifold is equal to rank of a gauge symmetry of strings in each of the untwisted and twisted sectors of the orbifold model, or equivalently, if the $\mathbb{Z}_N$-transformation is an inner automorphism of the lattice $\Gamma^{D,D}$. 

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In the next section, we may rewrite orbifold models into equivalent torus models using the “torus-orbifold equivalence” and investigate full symmetries of the torus models instead of the orbifold models themselves. For later convenience, we will give a brief review of the “torus-orbifold equivalence” in this section. See ref. [21] for details.

Let us start with a $D$-dimensional torus model associated with the root lattice $\Lambda_R(G)$ of a simply-laced Lie algebra $G$ ($D = \text{rank } G$). Suppose that an affine Kac-Moody algebra $\hat{G} \oplus \hat{G}$ is realized in the vertex operator representation à la Frenkel-Kac and Segal [34]:

$$P_I^L(z) \equiv i\partial_z X^I_L(z), \quad (4.1)$$
and

$$V_L(\alpha; z) \equiv \exp\{i\alpha \cdot X_L(z)\}:,$$

$$P_I^R(\bar{z}) \equiv i\partial_{\bar{z}} X^I_R(\bar{z}), \quad (4.2)$$
and

$$V_R(\alpha; \bar{z}) \equiv \exp\{i\alpha \cdot X_R(\bar{z})\}:,$$

where $\alpha$ is a root vector of $G$ and its squared length is normalized to two. A $\mathbb{Z}_N$-orbifold model is obtained by modding out of this torus model by a $\mathbb{Z}_N$-rotation which is an automorphism of the lattice defining the torus. Since every physical string state on the $\mathbb{Z}_N$-orbifold must be invariant under the $\mathbb{Z}_N$-transformation, the $\mathbb{Z}_N$-invariant subgroup $G_0$ of $G$ is the unbroken gauge symmetry in the untwisted sector.

Let us consider the $g^{\ell}$-sector (the untwisted sector for $\ell = 0$ and the twisted sectors for $\ell = 1, \ldots, N-1$). As eqs. (4.1) and (4.2), we will have the operators $P_I^L(z)$, $V_L(\alpha; z)$ and $P_I^R(\bar{z})$, $V_R(\alpha; \bar{z})$ which generate the untwisted (for $\ell = 0$) or twisted (for $\ell = 1, \ldots, N-1$) affine Kac-Moody algebra $\hat{G} \oplus \hat{G}$. If rank of $G_0$ is equal to $D$, we can always construct the $\mathbb{Z}_N$-invariant operators $P_I^L(z)$ and $P_I^R(\bar{z})$ ($I = 1, \ldots, D$) from suitable linear combinations of $P_I^L(z)$, $V_L(\alpha; z)$ and $P_I^R(\bar{z})$, $V_R(\alpha; \bar{z})$ such that

$$g(P_I^L(z), P_I^R(\bar{z})){g^{-1}} = (P_I^L(z), P_I^R(\bar{z})), \quad (4.3)$$
and

$$P_I^L(w)P_I^L(z) = \frac{\delta_{IJ}}{(w-z)^2} + \text{(regular terms)},$$

$$P_I^R(\bar{w})P_I^R(\bar{z}) = \frac{\delta_{IJ}}{(\bar{w}-\bar{z})^2} + \text{(regular terms)}, \quad (4.4)$$

where $g$ is the operator which generates the $\mathbb{Z}_N$-transformation in the $g^{\ell}$-sector. It follows from (4.4) that $P_I^L(z)$ and $P_I^R(\bar{z})$ can be expanded as

$$P_I^L(z) \equiv i\partial_z X_I^L(z) \equiv \sum_{n \in \mathbb{Z}} \alpha^I_{Ln} z^{-n-1},$$

and
\[ P^I_R(\bar{z}) \equiv i \partial_{\bar{z}} X^I_R(\bar{z}) \equiv \sum_{n \in \mathbb{Z}} \alpha^I_{Rn} \bar{z}^{-n-1}, \quad (4.5) \]

with
\[
[\alpha^I_{Lm}, \alpha^J_{Ln}] = m \delta^{IJ} \delta_{m+n,0},
\]
\[
[\alpha^I_{Rm}, \alpha^J_{Rn}] = m \delta^{IJ} \delta_{m+n,0}. \quad (4.6) \]

In this basis, a vertex operator in the \( g^\ell \)-sector will be given by
\[
V'(k_L, k_R; z) = \text{\textcircled{\textcircled{o}}} \exp\{ i k_L \cdot X'_L(z) + i k_R \cdot X'_R(\bar{z}) \} \text{\textcircled{\textcircled{o}}}, \quad \text{for } (k_L^I, k_R^I) \in \Gamma^{D,D}, \quad (4.7) \]

where \( \text{\textcircled{\textcircled{o}}} \text{\textcircled{\textcircled{o}}} \) denotes the normal ordering with respect to the new basis of the operators. Since \( P^I_L(z) \) and \( P^I_R(\bar{z}) \) are invariant under the \( \mathbb{Z}_N \)-transformation, the vertex operator (4.7) will transform as
\[
gV'(k_L, k_R; z) g^{-1} = e^{i 2\pi (k_L \cdot v_L - k_R \cdot v_R)} V'(k_L, k_R; z), \quad (4.8) \]

for some constant vector \((v^I_L, v^I_R)\). It follows that \( g \) will be given by
\[
g = \eta(\ell) \exp\{ i 2\pi (p^I_L \cdot v_L - p^I_R \cdot v_R) \}, \quad (4.9) \]

where \( p^I_L = \alpha^I_{L0}, p^I_R = \alpha^I_{R0} \) and \( \eta(\ell) \) is a constant phase with \( (\eta(\ell))^N = 1 \). Thus, the string coordinate in the new basis transforms as
\[
g(X^I_L(z), X^I_R(\bar{z})) g^{-1} = (X^I_L(z) + 2\pi v^I_L, X^I_R(\bar{z}) - 2\pi v^I_R). \quad (4.10)\]

This implies that the string coordinate \((X^I_L(z), X^I_R(\bar{z}))\) in the \( g^\ell \)-sector obeys the following boundary condition:
\[
(X^I_L(e^{2\pi i} z), X^I_R(e^{-2\pi i} \bar{z})) = (X^I_L(z) + 2\pi \ell v^I_L, X^I_R(\bar{z}) - 2\pi \ell v^I_R) + \text{(torus shift)}, \quad (4.11)\]

and hence that the eigenvalues of the momentum \((p^I_L, p^I_R)\) in the new basis are of the form
\[
(p^I_L, p^I_R) \in \Gamma^{D,D} + \ell (v^I_L, v^I_R). \quad (4.12) \]

In the new basis, \( g^\ell \) in the \( g^\ell \)-sector will be given by
\[
g^\ell = e^{i 2\pi (L'_0 - \bar{L}'_0)}, \quad (4.13) \]
where
\[ L'_0 = \sum_{I=1}^{D} \left\{ \frac{1}{2} (p'^I_L)^2 + \sum_{n=1}^{\infty} \alpha'^I_{L-n} \alpha'^I_{Ln} \right\}, \]
\[ \bar{L}'_0 = \sum_{I=1}^{D} \left\{ \frac{1}{2} (p'^I_R)^2 + \sum_{n=1}^{\infty} \alpha'^I_{R-n} \alpha'^I_{Rn} \right\}. \] (4.14)

Comparing eq. (4.13) with eq. (4.9), we find
\[ \eta(\ell) = \exp \left\{ -i\pi \ell ((v^I_L)^2 - (v^I_R)^2) \right\}. \] (4.15)

Since \( g^N = 1 \), \((v^I_L, v^I_R)\) must satisfy
\[ N((v^I_L)^2 - (v^I_R)^2) = 0 \mod 2, \]
\[ N(v^I_L, v^I_R) \in \Gamma^{D,D}. \] (4.16)

Every physical state must obey the condition \( g = 1 \) because it must be invariant under the \( \mathbb{Z}_N\)-transformation. Thus the allowed momentum eigenvalues \((p'^I_L, p'^I_R)\) of the physical states in the \( g^\ell\)-sector are restricted to
\[ (p'^I_L, p'^I_R) \in \Gamma^{D,D} + \ell(v^I_L, v^I_R) \text{ with } p'_L \cdot v_L - p'_R \cdot v_R - \frac{1}{2} \ell((v^I_L)^2 - (v^I_R)^2) = 0 \mod 1. \] (4.17)

The total physical Hilbert space \( \mathcal{H} \) of the \( \mathbb{Z}_N\)-orbifold model is a direct sum of a physical space \( \mathcal{H}_\ell \) in each sector:
\[ \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{(N-1)}. \] (4.18)

In the above consideration we have shown that \( \mathcal{H} \) is equivalent to
\[ \mathcal{H} = \{ \alpha'^I_{L-m} \cdots \alpha'^I_{R-n} \cdots |p'^I_L, p'^I_R > |m, \ldots, n, \ldots \in \mathbb{Z} > 0, (p'^I_L, p'^I_R) \in \Gamma'^{D,D} \}, \] (4.19)

where
\[ \Gamma'^{D,D} = \{ (p'^I_L, p'^I_R) \in \bigcup_{\ell=0}^{N-1} [\Gamma^{D,D} + \ell(v^I_L, v^I_R)] | p'_L \cdot v_L - p'_R \cdot v_R - \frac{1}{2} \ell((v^I_L)^2 - (v^I_R)^2) \in \mathbb{Z} \}. \] (4.20)

From the conditions (4.16), \( \Gamma'^{D,D} \) is lorentzial even self-dual lattice if \( \Gamma^{D,D} \) is. Therefore, the total physical Hilbert space of the \( \mathbb{Z}_N\)-orbifold model is equivalent to that of the torus model associated with the lattice \( \Gamma'^{D,D} \).
5. Asymmetric $Z_N$-Orbifold Models

with Twist-Untwist Intertwining Currents

In this section we present a list of asymmetric $Z_N$-orbifold models with (1,0) states in twisted sectors and show that these states correspond to twist-untwist intertwining currents which enlarge symmetries in each untwisted or twisted sector to larger symmetries of the total Hilbert space.

5.1. Asymmetric $Z_2$-orbifold models with twist-untwist intertwining currents

Let us first consider asymmetric $Z_2$-orbifolds. The lattice $\Lambda$ in eq. (3.1) is taken to be a root lattice of a simply-laced semi-simple Lie algebra $G$ having the $Z_2$-automorphism discussed in sect. 3. The squared length of the root vectors in the root lattice $\Lambda$ is normalized to two. The $Z_2$-transformation is defined by

$$(X_I^L, X_I^R) \rightarrow (-X_I^L, X_I^R), \quad (I = 1, \ldots, D).$$

(5.1)

In this case, the necessary and sufficient conditions for modular invariance are

$$D = 0 \mod 8,$$

(5.2)

$$2(p^I_L)^2 = 0 \mod 2 \quad \text{for all } p^I_R \in \Gamma_0^*,$$

(5.3)

where

$$\Gamma_0 = \{ p^I_R \mid (p^I_L = 0, p^I_R) \in \Gamma^{D,D} \}$$

$$= \{ p^I_R \mid p^I_R \in \Lambda \}. $$

(5.4)

The first condition (5.2) comes from eq. (2.10) because the conformal weight of the ground states in the twisted sector is given by $(D_{16},0)$. The condition (5.2) requires that the dimension $D$ of the root lattice $\Lambda$ of a Lie algebra $G$ must be a multiple of eight, that is, rank$G$ must be a multiple of eight. For our purpose, it is sufficient to consider models possessing (1,0) twisted states although some of these (1,0) twisted states might be unphysical. Thus it is sufficient to consider only the cases of $D = 8$ and 16 since there appear no (1,0) states in the twisted sector if $D > 16$. For $D = 16$ the ground states in the twisted sector have the conformal weight (1,0). For $D = 8$, the ground states in the twisted sector have the conformal weight $(\frac{1}{2},0)$ but the first
excited states have the conformal weight (1,0) because the left-moving oscillators are expanded in half-odd-integral modes in the twisted sector.

In sect. 3 we have obtained the simply-laced Lie algebras having the $\mathbb{Z}_2$-automorphism. Taking account of the condition (5.3), we conclude that $G$ must be products of $SO(8)$, $SO(16)$, $SO(24)$, $SO(32)$ and $E_8$ with rank $G = 8$ or 16. All the models we have to investigate are given in table 2.

In the following, we will mainly concentrate on the left-moving Hilbert space. The $G_0$ in table 2 denotes the $\mathbb{Z}_2$-invariant subalgebra of $G$, which is the symmetry in each (untwisted or twisted) left-moving physical Hilbert space. However, since there appear twisted states with the conformal weight (1,0), the symmetry $G_0$ will be “enlarged”: The physical (i.e., $\mathbb{Z}_2$-invariant) (1,0) states in the untwisted sector correspond to the adjoint representation of $G_0$. Each physical (1,0) state in the twisted sector corresponds to an intertwining current which converts untwisted states to twisted states, and vice versa. Thus the physical (1,0) states in the untwisted sector together with the physical (1,0) states in the twisted sector will form an adjoint representation of a larger group $G'$ than $G_0$, which is the full symmetry of the total physical Hilbert space.

To see what the full symmetry $G'$ is, we may rewrite each asymmetric $\mathbb{Z}_2$-orbifold model into an equivalent torus model using the “torus-orbifold equivalence”, as explained in sect. 4. The torus model which is equivalent to the asymmetric $\mathbb{Z}_2$-orbifold model with $\Gamma^{D,D}$ is specified by the following momentum lattice:

$$\Gamma'_{D,D} = \{ (p'_L, p'_R) \in \bigcup_{\ell=0}^1 [\Gamma^{D,D} + \ell(v^I_L, 0)] \mid p'_L \cdot v_L - \frac{1}{2} \ell(v^I_L)^2 \in \mathbb{Z} \}.$$  \hfill (5.5)

where the shift vector $v^I_L$ for each model is given in an appendix. It is easy to determine the full symmetry $G'$ from the lattice $\Gamma'_{D,D}$. The results are summarized in table 2. It is interesting to note that $G'$ is equal to $G$ for $D = 8$ although $G'$ is not necessarily the same as $G$. In fact, it is not difficult to show that the momentum lattice $\Gamma'^{D,D}$ is isomorphic to $\Gamma^{D,D}$ for each model with $G' = G$.

5.2. Asymmetric $\mathbb{Z}_N$-orbifold models with twist-untwist intertwining currents

Let us consider asymmetric $\mathbb{Z}_N$-orbifolds. The lattice $\Lambda$ in eq. (3.1) is taken to be a root lattice of a simply-laced semi-simple Lie algebra $G$ having a $\mathbb{Z}_N$-automorphism
discussed in sect. 3. The squared length of the root vectors in the root lattice $\Lambda$ is normalized to two.

Let $g$ be a group element of the cyclic group $\mathbb{Z}_N$. The action of $g$ on the string coordinate is defined by

$$g : (X^I_L, X^I_R) \rightarrow (U^{IJ} X^J_L, X^I_R), \quad (I, J = 1, \ldots, D), \quad (5.6)$$

where $U$ is a $D \times D$ rotation matrix which satisfies $U^N = 1$ and where $U$ and all its powers $U^\ell (\ell = 1, \ldots, N - 1)$ are assumed to have no fixed directions.

Let $N_\ell$ be the minimum positive integer such that $(g^\ell)^{N_\ell} = 1$. As discussed in sect. 2, $D$ must satisfy

$$D = 0 \mod \varphi(N), \quad (5.7)$$

and

$$D = 0 \mod \frac{\varphi(N_\ell)}{N_\ell h'_\ell} \quad \text{for } \ell = 1, \ldots, N - 1, \quad (5.8)$$

where

$$h'_\ell = \frac{1}{4} \sum_{(m,N_\ell) = 1}^{N_\ell} \frac{m}{N_\ell} \left(1 - \frac{m}{N_\ell}\right). \quad (5.9)$$

Modular invariance further requires

$$N_\ell (p^{I}_R)^2 = 0 \mod 2 \quad \text{for all } p^{I}_R \in \Gamma_0^*, \quad (5.10)$$

where

$$\Gamma_0 = \{ p^{I}_R \mid (p^{I}_L = 0, p^{I}_R) \in \Gamma^{D,D} \} = \{ p^{I}_R \mid p^{I}_R \in \Lambda \}. \quad (5.11)$$

For our purpose, it is sufficient to consider models possessing $(1,0)$ twisted states although some of these $(1,0)$ twisted states might be unphysical. The conditions (5.7) and (5.8) and the existence of $(1,0)$ states in twisted sectors restrict the allowed dimensions $D$ for a given order $N$. The results have been summarized in the fourth column of table 1 up to $N = 30$.

In sect. 3, we have classified the $\mathbb{Z}_N$-automorphism of the simply-laced Lie algebras and have shown that the allowed orders $N$ are restricted to $N = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 19, 20, 23, 24$ and 30. The conditions (5.7), (5.8) and (5.10) further restrict the possible orders $N$ and lattices $\Lambda$. 

– 16 –
All the models we have to investigate are given in tables 2−6. In the following, we will mainly concentrate on the left-moving Hilbert space. The \( G_0 \) in tables 2−6 denotes the \( \mathbb{Z}_N \)-invariant subalgebra of \( G \), which is the symmetry in each (untwisted or twisted) left-moving physical Hilbert space. However, since there appear twisted states with the conformal weight \((1,0)\), the symmetry \( G_0 \) will be “enlarged”: The physical (i.e., \( \mathbb{Z}_N \)-invariant) \((1,0)\) states in the untwisted sector correspond to the adjoint representation of \( G_0 \). Each physical \((1,0)\) state in the twisted sector corresponds to an intertwining current which converts untwisted states to twisted states, and vice versa. Thus, the physical \((1,0)\) states in the untwisted sector together with the physical \((1,0)\) states in the twisted sector will form an adjoint representation of a larger group \( G' \) than \( G_0 \), which is the full symmetry of the total physical Hilbert space.

To see what the full symmetry \( G' \) is, we may rewrite each asymmetric \( \mathbb{Z}_N \)-orbifold model into an equivalent torus model using the “torus-orbifold equivalence”, as explained in sect. 4. The torus model which is equivalent to the asymmetric \( \mathbb{Z}_N \)-orbifold model with \( \Gamma^{D,D} \) is specified by the following momentum lattice:

\[
\Gamma'^{D,D} = \left\{ (p_L^I, p_R^I) \in \bigcup_{\ell=0}^{N-1} [\Gamma^{D,D} + \ell(v^I_L, 0)] \mid p_L^I \cdot v_L - \frac{1}{2} \ell(v^I_L)^2 \in \mathbb{Z} \right\}, \tag{5.12}
\]

where the shift vector \( v^I_L \) must satisfy

\[
N v^I_L \in \Lambda, \tag{5.13}
\]

\[
\ell v^I_L \not\in \Lambda \ (\ell = 1, \ldots, N - 1). \tag{5.14}
\]

The shift vector \( v^I_L \) can be chosen to satisfy

\[
\frac{1}{2} (v^I_L)^2 = h_1, \tag{5.15}
\]

where \( h_1 \) is the left-conformal weight of the ground states in the \( g \)-twisted sector. For each model the shift vector \( v^I_L \) which satisfies the conditions (5.13), (5.14) and (5.15) are explicitly given in an appendix. It is easy to determine the full symmetry \( G' \) from the lattice \( \Gamma'^{D,D} \). The results are summarized in tables 2−6. It is interesting to note that in some cases \( G' \) is equal to \( G \) although \( G' \) is not necessarily the same as \( G \). In fact, it is not difficult to show that the momentum lattice \( \Gamma'^{D,D} \) is isomorphic to \( \Gamma^{D,D} \) for each model with \( G' = G \).
We should make a comment on the case of the asymmetric $\mathbb{Z}_{30}$-orbifold model with $G = (E_8)^3$. In this model, there would appear $(1,0)$ states in the $g^\ell$-twisted sector with $\ell = 2,3,4,5,8,9,14,16,21,22,25,26,27$ and $28$. However, all such states are not physical and are removed from the physical spectrum. This means that the lattice $\Gamma^{24,24}$ of the equivalent torus model to the asymmetric $\mathbb{Z}_{30}$-orbifold model contains no vectors with $(p^I_L)^2 = 2$ and $p^I_R = 0$. Since the root lattice of $(E_8)^3$ is even self-dual, $\Gamma^{24,24}$ can be written as

$$\Gamma^{24,24} = \Gamma^{24,0} \oplus \Gamma^{0,24}_{(E_8)^3},$$

where $\Gamma^{0,24}_{(E_8)^3}$ is the root lattice of $(E_8)^3$. Since $\Gamma^{24,0}$ must be even self-dual and contains no vectors of norm two, it must be the Leech lattice [35].

5.3. The other choices of the squared length of the root vectors

Let us consider asymmetric $\mathbb{Z}_N$-orbifold models, where the squared length of the root vectors in the root lattice $\Lambda$ of eq. (3.1) is normalized to $2m$ ($m = 2,3,\ldots$). The $\mathbb{Z}_N$-transformation is defined in eq. (5.6). In sect. 3 we have proved that such consistent asymmetric orbifold models are only the asymmetric $\mathbb{Z}_2$-orbifold models with $m = 2$, $U = -1$ and $G = (E_8)^\ell$ ($\ell = 1,2,\ldots$). Since we are interested in asymmetric orbifold models with $(1,0)$ twisted states, it is sufficient to consider only the cases of $D = 8$ and $16$, i.e., $\Lambda = \sqrt{2}\Lambda_R(G)$ with $G = E_8$ and $(E_8)^2$, where $\Lambda_R(G)$ denotes the root lattice which is spanned by the root vectors with the squared length two.

Since there is no momentum such that $(p^I_L)^2 = 2$ and $p^I_R = 0$ in the untwisted sector, there are no $(1,0)$ physical states in the untwisted sector. In the twisted sector, there appear 8 and 1 $(1,0)$ states for the models with $\Lambda = \sqrt{2}\Lambda_R(E_8)$ and $\sqrt{2}\Lambda_R(E_8)^2$, respectively and they are found to be physical. Therefore, we may conclude that the full symmetries of the total Hilbert spaces are $(U(1))^8$ and $U(1)$ for the models with $\Lambda = \sqrt{2}\Lambda_R(E_8)$ and $\sqrt{2}\Lambda_R(E_8)^2$, respectively although there is no symmetry within each (untwisted or twisted) physical Hilbert space.

It should be noticed that the partition function of the above asymmetric $\mathbb{Z}_2$-orbifold model with $\Lambda = \sqrt{2}\Lambda_R(E_8)$ can be shown to be identical to that of the torus model with which we have just started to construct this orbifold model [17]. Hence this orbifold model can probably be rewritten into the torus model although
6. Supersymmetry Enhancement

In this section we will consider $E_8 \times E_8$ heterotic strings on asymmetric $\mathbb{Z}_N$-orbifolds. The orbifold models are to be regarded only as illustrative of the new features of asymmetric orbifolds and not of a direct phenomenological relevance. In the light cone, right-movers consist of eight bosons $X^i_R (i = 1, \ldots, 8)$ and eight Neveu-Schwarz-Ramond (NSR) fermions $\lambda^i_R (i = 1, \ldots, 8)$. These complete a right-moving superstring. Left-movers consist of eight bosons $X^i_L (i = 1, \ldots, 8)$ and another sixteen bosons $\phi^I_L (I = 1, \ldots, 16)$.

A four-dimensional string can be constructed by compactifying the 22 left-moving ($X^a_L, \phi^I_L; a = 3, \ldots, 8, I = 1, \ldots, 16$) and 6 right-moving ($X^a_R; a = 3, \ldots, 8$) extra bosonic coordinates on a torus. Then momenta $(p^a_L, P^I_L; p^a_R)$ of the string coordinates $(X^a_L, \phi^I_L; X^a_R)$ lie on a $(22+6)$-dimensional lorentzian even self-dual lattice $\Gamma^{22,6}$, i.e.,

$$(p^a_L, P^I_L; p^a_R) \in \Gamma^{22,6} \quad (a = 3, \ldots, 8, I = 1, \ldots, 16).$$

(6.1)

The lattice $\Gamma^{22,6}$ is assumed to be of the form

$$\Gamma^{22,6} = \Gamma^{6,6} \oplus \Gamma^{16,0}_{E_8 \times E_8},$$

(6.2)

where $\Gamma^{16,0}_{E_8 \times E_8}$ is the root lattice of $E_8 \times E_8$ on which $P^I_L (I = 1, \ldots, 16)$ lies. The $(6+6)$-dimensional lattice $\Gamma^{6,6}$ has to be even self-dual and is taken to be the lattice defined in (3.1). The lattice $\Lambda$ in eq. (3.1) is a root lattice of a simply-laced semi-simple Lie algebra with rank $G = 6$. The squared length of the root vectors is normalized to two.

To construct an asymmetric $\mathbb{Z}_N$-orbifold, we will consider the following $\mathbb{Z}_N$-transformation:

$$g : X^a_L \to X^0_L,$$

$$X^a_R \to (UX^a_R)^0 \quad (a = 3, \ldots, 8),$$

(6.3)

where $U$ is a 6-dimensional rotation matrix with $U^N = 1$. The $\mathbb{Z}_N$-transformation must be an automorphism of the lattice $\Gamma^{6,6}$, i.e.,

$$(p^a_L, (Up^a_R)^0) \in \Gamma^{6,6} \quad \text{for all } (p^a_L, p^a_R) \in \Gamma^{6,6}.$$

(6.4)

* This orbifold model seems to give a counterexample of ref. [36].
The action of $g$ on the NSR fermions is the same as the action on $X_R^a$ to preserve world sheet supersymmetry [2]. The action of $g$ on the remaining fields is taken to be trivial.

As before, we will assume that $U$ and all its powers $U^\ell \ (\ell = 1, \ldots, N-1)$ have no fixed direction. Then it is not difficult to show that there are only four modular invariant asymmetric $\mathbb{Z}_N$-orbifold models: two $\mathbb{Z}_3$-orbifold models with $G = (SU(3))^3$ and $E_6$, one $\mathbb{Z}_7$-orbifold model with $G = SU(7)$ and one $\mathbb{Z}_9$-orbifold model with $G = E_6$. Let $\exp(i2\pi v^t)$ and $\exp(-i2\pi v^t) \ (t = 1, 2, 3)$ be the eigenvalues of $U$. The $v^t$ will be taken to be

$$v^t = \left( \frac{1}{3}, \frac{1}{3}, \frac{2}{3} \right) \text{ for the } \mathbb{Z}_3-\text{models},$$

$$= \left( \frac{1}{7}, \frac{2}{7}, \frac{3}{7} \right) \text{ for the } \mathbb{Z}_7-\text{model},$$

$$= \left( \frac{1}{9}, \frac{2}{9}, \frac{5}{9} \right) \text{ for the } \mathbb{Z}_9-\text{model.} \quad (6.5)$$

It should be noted that the $\mathbb{Z}_3$- and $\mathbb{Z}_7$-transformations leave one unbroken space-time supersymmetry while the $\mathbb{Z}_9$-transformation leaves no unbroken space-time supersymmetry [1]. That is, $N=1$ ($N=0$) space-time supersymmetry survives in each of untwisted and twisted sectors for the $\mathbb{Z}_3((SU(3))^3)$-, $\mathbb{Z}_3(E_6)$- and $\mathbb{Z}_7(SU(7))$- ($\mathbb{Z}_9(E_6)$-) orbifold models. In fact, for the $\mathbb{Z}_3((SU(3))^3)$-, $\mathbb{Z}_3(E_6)$- and $\mathbb{Z}_7(SU(7))$- ($\mathbb{Z}_9(E_6)$-) orbifold models an $N=1$ ($N=0$) supergravity multiplet coupled to an $N=1$ ($N=0$) super Yang-Mills multiplet with $E_8 \times E_8 \times G$ with $G = (SU(3))^3, E_6$ and $SU(7) \ (E_6)$ appears in the untwisted sector, respectively.

This is not the end of the story. A number of massless states appear in the twisted sectors. The degeneracy $d_\ell$ of the ground states in the $g^\ell$-sector ($\ell = 1, \ldots, N-1$) is given by

$$d_\ell = \begin{cases} 
1 & \ell = 1, 2 \text{ for the } \mathbb{Z}_3((SU(3))^3)\text{-model,} \\
3 & \ell = 1, 2 \text{ for the } \mathbb{Z}_3(E_6)\text{-model,} \\
1 & \ell = 1, \ldots, 6 \text{ for the } \mathbb{Z}_7(SU(7))\text{-model,} \\
\ell = 1, 2, 4, 5, 7, 8 \text{ for the } \mathbb{Z}_9(E_6)\text{-model.} 
\end{cases} \quad (6.6)$$

* The $N=0$ super multiplet means the $N=1$ super multiplet without the fermion contents.
For the $\mathbb{Z}_3((SU(3))^3)$-orbifold model, there appear one massless spin $\frac{3}{2}$ fermion, two $U(1)$ gauge bosons, one massless spin $\frac{1}{2}$ fermion and two massless scalars in the twisted sectors. The massless spin $\frac{1}{2}$ fermion and the two massless scalars belong to the adjoint representation of $E_8 \times E_8 \times (SU(3))^3$. Those massless fields in the twisted sectors together with the massless fields in the untwisted sector will form an $N=2$ supergravity multiplet coupled to an $N=2$ super Yang-Mills multiplet with $E_8 \times E_8 \times (SU(3))^3$.

For the $\mathbb{Z}_3(E_6)$- and $\mathbb{Z}_7(SU(7))$-orbifold models, there appear three massless spin $\frac{3}{2}$ fermions, six $U(1)$ gauge bosons, three massless spin $\frac{1}{2}$ fermions and six massless scalars in the twisted sectors. The three massless spin $\frac{1}{2}$ fermions and the six massless scalars belong to the adjoint representation of $E_8 \times E_8 \times G$ with $G = E_6$ or $SU(7)$. Thus those massless fields in the twisted sectors together with the massless fields in the untwisted sector will form an $N=4$ supergravity multiplet coupled to an $N=4$ super Yang-Mills multiplet with $E_8 \times E_8 \times E_6$ or $E_8 \times E_8 \times SU(7)$.

For the $\mathbb{Z}_9(E_6)$-orbifold model, there appear three massless spin $\frac{3}{2}$ fermions, six $U(1)$ gauge bosons, three massless spin $\frac{1}{2}$ fermions and six massless scalars in the $g^\ell$-twisted sectors with $\ell = 1, 2, 4, 5, 7$ and $8$. All these massless states are found to be physical. The three massless spin $\frac{1}{2}$ fermions and the six massless scalars belong to the adjoint representation of $E_8 \times E_8 \times E_6$. There would also appear three massless spin $\frac{3}{2}$ fermions, six $U(1)$ gauge bosons, three massless spin $\frac{1}{2}$ fermions and six massless scalars in the $g^\ell$-twisted sectors with $\ell = 3$ and $6$. However, these massless fields are not always physical. In fact, a detailed analysis of the one loop vacuum amplitude tells us that only one massless spin $\frac{3}{2}$ fermion and one massless spin $\frac{1}{2}$ fermion survive as physical states. Other states, in particular, all bosonic states are removed from the physical spectrum. Therefore, the massless fields in the twisted sectors together with the massless fields in the untwisted sector will form an $N=4$ supergravity multiplet coupled to an $N=4$ super Yang-Mills multiplet with $E_8 \times E_8 \times E_6$ although no supersymmetry survives in the untwisted sector.

We have investigated a restricted class of heterotic strings on asymmetric orbifolds and have found that $N=1$, 1, 1 and 0 space-time supersymmetry in each of the untwisted and twisted sectors is enlarged to $N=2$, 4, 4 and 4 space-time supersymmetry in the total Hilbert spaces for the $\mathbb{Z}_3((SU(3))^3)$-, $\mathbb{Z}_3(E_6)$-, $\mathbb{Z}_7(SU(7))$- and $\mathbb{Z}_9(E_6)$-orbifold models, respectively although this conclusion has not rigorously
been proved. It is worth while noting that space-time supersymmetry “enhancement” discussed above will also imply world-sheet supersymmetry “enhancement”[37].

7. Discussions

We have presented a list of asymmetric $\mathbb{Z}_N$-orbifold models to possess (1,0) twisted states. We have found that these twisted states play a role of intertwining currents which convert untwisted string states to twisted ones, and vice versa and that full symmetries of such orbifold models are larger than symmetries in each of the untwisted and twisted sectors *. As mentioned in the introduction, this symmetry “enhancement” is inherent in asymmetric orbifolds. We have seen that the conditions for the momentum lattices with $\mathbb{Z}_N$-automorphisms, modular invariance and the existence of (1,0) twisted states put severe restrictions on such orbifold models. It may be interesting to point out that for all the orbifold models which we encountered in this paper (1,0) twisted states can appear only for $D \leq 24$. The maximum dimension $D = 24$ is equal to the transverse dimension of the bosonic string theory in the light-cone gauge.

In tables 2–6, $G_0$ denotes the symmetries of the untwisted and twisted sectors and $G'$ denotes the full symmetries of the total Hilbert spaces. Each orbifold model listed in tables 2–6 has a symmetry $G' \times G$, where the symmetry $G'$ ($G$) comes from the left- (right-) moving modes. In general, the symmetry $G'$ of the left-movers is different from the symmetry $G$ of the right-movers due to asymmetric nature of the orbifolds. For some orbifold models, the symmetry $G'$ is, however, equal to $G$ and hence asymmetric nature disappears. Specifically, for $D \leq 10$, all the orbifold models we considered have this property.

All the orbifold models in tables 2–6 have been shown to be equivalently rewritten into the torus models because the $\mathbb{Z}_N$-transformations are inner automorphisms of the momentum lattices $\Gamma^{D,D}$. We may schematically write the “torus-orbifold equivalence” as $T^D/P \simeq T'^D$, where $T^D/P$ ($T'^D$) denotes a $D$-dimensional orbifold (torus) model. For symmetric orbifold models, there exist no intertwining currents and hence

* The asymmetric $\mathbb{Z}_{30}$-orbifold model with $(E_8)^3$ is an exception because all (1,0) twisted states are unphysical.
\( G' \) must be equal to \( G_0 \). This implies that \( T^D \neq T'^D \) for symmetric orbifolds. On the other hand, for asymmetric orbifold models, there exist examples of \( T^D/P \simeq T'^D \) with \( T^D = T'^D \). Specifically, for \( D \leq 10 \), all the orbifold models we considered have this property. It would be of interest to examine the above peculiarity for lower dimensional asymmetric orbifolds. More interesting observation is given in subsect. 5.3. for \( D = 8 \). Since rank of \( G_0 \) is less than rank of \( G \) (rank\( G_0 = 0 \) and rank\( G = 8 \)), the \( \mathbb{Z}_2 \)-transformation will correspond to an outer automorphism of the momentum lattice. Hence it seems that this orbifold model could not be rewritten into a torus model. However, we have shown that \( G' = G = (U(1))^8 \) and can prove that the one loop partition function of the orbifold model is identical to that of the torus model which we just started with to define this orbifold model. This result strongly suggests that the orbifold model is equivalent to the torus model even though the \( \mathbb{Z}_2 \)-transformation is an outer automorphism and that the “torus-orbifold equivalence” stated in sect. 4 should be replaced by the following statement: Any closed bosonic string theory compactified on an orbifold is equivalent to that on a torus if rank of the symmetry of the \emph{total} physical Hilbert space is equal to the dimension of the orbifold.

In sect. 6 we have considered the \( E_8 \times E_8 \) heterotic strings on the asymmetric \( \mathbb{Z}_N \)-orbifolds and found supersymmetry “enhancement”, instead of gauge symmetry “enhancement”. This mechanism will lead to a new class of four-dimensional string models with N=1 space-time supersymmetry: There is no unbroken supersymmetry in each of untwisted and twisted sectors but there exists a space-time supercharge which convert untwisted string states to twisted string states with opposite statistics, and vice versa. It would be of importance to study the new class of four-dimensional string models with N=1 space-time supersymmetry.

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Appendix

In this appendix, we will give the shift vectors $v^I_L$ which are introduced in rewriting the asymmetric $\mathbb{Z}_N$-orbifold models into the equivalent torus models. It will be sufficient to give the shift vectors only for the simple Lie algebra root lattices.

In the usual orthonormal basis, the root lattices $\Lambda_R$ of the simple Lie algebras are given by

\[
\Lambda_R(SU(n+1)) = \{ (m_1, m_2, \ldots, m_{n+1}) \mid m_i \in \mathbb{Z}, \sum_{i=1}^{n+1} m_i = 0 \},
\]

\[
\Lambda_R(SO(2n)) = \{ (m_1, m_2, \ldots, m_n) \mid m_i \in \mathbb{Z}, \sum_{i=1}^{n} m_i \in 2\mathbb{Z} \},
\]

\[
\Lambda_R(E_8) = \Lambda_R(SO(16)) \cup \{ \Lambda_R(SO(16)) + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \},
\]

\[
\Lambda_R(E_6) = \{ p^I \in \Lambda_R(E_8) \mid p \cdot (e_6 + e_8) = p \cdot (e_7 + e_8) = 0 \},
\]

where we have normalized the squared length of the root vectors to two.

The shift vectors $v^I_L$ for the asymmetric $\mathbb{Z}_N$-orbifolds which satisfy the conditions (5.13), (5.14) and (5.15) are given in the usual orthonormal basis as follows:

(a) Asymmetric $\mathbb{Z}_2$-orbifolds

\[
v^I_L(E_8) = \frac{1}{2}(2, 0, 0, 0, 0, 0, 0, 0),
\]

\[
v^I_L(SO(32)) = \frac{1}{2}(1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0),
\]

\[
v^I_L(SO(24)) = \frac{1}{2}(1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0),
\]

\[
v^I_L(SO(16)) = \frac{1}{2}(1, 1, 1, 0, 0, 0, 0),
\]

\[
v^I_L(SO(8)) = \frac{1}{2}(1, 1, 0, 0),
\]

(b) Asymmetric $\mathbb{Z}_3$-orbifolds

\[
v^I_L(E_8) = \frac{1}{3}(0, 2, 1, 0, -1, 1, 0, 1),
\]

\[
v^I_L(E_6) = \frac{1}{3}(0, 1, 2, 0, 1, 0, 0, 0),
\]

\[
v^I_L(SU(3)) = \frac{1}{3}(1, 0, -1),
\]

(c) Asymmetric $\mathbb{Z}_4$-orbifold

\[
v^I_L(E_8) = \frac{1}{4}(0, -1, 2, 1, 0, -1, 2, 1),
\]

\[
v^I_L(SO(32)) = \frac{1}{4}(0, -1, 2, 1, 0, -1, 2, 1, 0, -1, 2, 1, 0, -1, 2, 1),
\]

\[
v^I_L(SO(24)) = \frac{1}{4}(0, -1, 2, 1, 0, -1, 2, 1, 0, -1, 2, 1),
\]

\[
v^I_L(SO(16)) = \frac{1}{4}(0, -1, 2, 1, 0, -1, 2, 1),
\]

\[
v^I_L(SO(8)) = \frac{1}{4}(0, -1, 2, 1),
\]

(d) Asymmetric $\mathbb{Z}_5$-orbifolds

\[
v^I_L(E_8) = \frac{1}{5}(0, -1, 3, 2, 1, 0, -1, 2),
\]

\[
v^I_L(SU(5)) = \frac{1}{5}(2, 1, 0, -1, -2),
\]
(e) Asymmetric $\mathbb{Z}_6$-orbifold
\[ v_L^I(E_8) = \frac{1}{6}(0, -1, -2, 3, 2, 1, 0, 1), \]

(f) Asymmetric $\mathbb{Z}_7$-orbifolds
\[ v_L^I(SU(7)) = \frac{1}{7}(3, 2, 1, 0, -1, -2, -3), \]

(g) Asymmetric $\mathbb{Z}_9$-orbifold
\[ v_L^I(E_6) = \frac{1}{9}(0, -1, -2, -3, -4, -2, 2), \]

(h) Asymmetric $\mathbb{Z}_{10}$-orbifolds
\[ v_L^I(E_8) = \frac{1}{10}(0, -3, 4, 1, -2, 5, 2, 1), \]

(i) Asymmetric $\mathbb{Z}_{11}$-orbifolds
\[ v_L^I(SU(11)) = \frac{1}{11}(5, 4, 3, 2, 1, 0, -1, -2, -3, -4, -5), \]

(j) Asymmetric $\mathbb{Z}_{13}$-orbifolds
\[ v_L^I(SU(13)) = \frac{1}{13}(6, 5, 4, 3, 2, 1, 0, -1, -2, -3, -4, -5, -6), \]

(k) Asymmetric $\mathbb{Z}_{15}$-orbifolds
\[ v_L^I(E_8) = \frac{1}{15}(0, -1, -2, -3, -4, -5, -6, 7), \]

(l) Asymmetric $\mathbb{Z}_{17}$-orbifold
\[ v_L^I(SU(17)) = \frac{1}{17}(8, 7, 6, 5, 4, 3, 2, 1, 0, -1, -2, -3, -4, -5, -6, -7, -8), \]

(m) Asymmetric $\mathbb{Z}_{19}$-orbifold
\[ v_L^I(SU(19)) = \frac{1}{19}(9, 8, 7, 6, 5, 4, 3, 2, 1, 0, -1, -2, -3, -4, -5, -6, -7, -8, -9), \]

(n) Asymmetric $\mathbb{Z}_{20}$-orbifold
\[ v_L^I(E_8) = \frac{1}{20}(0, 1, 1, 2, 2, 3, 4, 15), \]

(o) Asymmetric $\mathbb{Z}_{23}$-orbifold
\[ v_L^I(SU(23)) = \frac{1}{23}(11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, -1, -2, -3, -4, -5, -6, -7, -8, -9, -10, -11), \]

(p) Asymmetric $\mathbb{Z}_{30}$-orbifold
\[ v_L^I(E_8) = \frac{1}{30}(0, -1, -2, -3, -4, -5, -6, -23). \]
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Table Captions

Table 1. \( \varphi(N) \) denotes the Euler function. In the third column allowed values of \( D \), which are consistent with the constraints (2.16) and (2.19), are given. In the fourth column the dimensions \( D \) in which (1,0) states might appear in twisted sectors are given.

Table 2. \( G_0 \) denotes the \( \mathbb{Z}_2 \)-invariant subgroup of \( G \), which is the symmetry in each sector and \( G' \) denotes the full symmetry of the total Hilbert space.

Table 3. \( G_0 \) denotes the \( \mathbb{Z}_3 \)-invariant subgroup of \( G \), which is the symmetry in each sector and \( G' \) denotes the full symmetry of the total Hilbert space.

Table 4. \( G_0 \) denotes the \( \mathbb{Z}_4 \)-invariant subgroup of \( G \), which is the symmetry in each sector and \( G' \) denotes the full symmetry of the total Hilbert space.

Table 5. \( G_0 \) denotes the \( \mathbb{Z}_5 \)-invariant subgroup of \( G \), which is the symmetry in each sector and \( G' \) denotes the full symmetry of the total Hilbert space.

Table 6. \( G_0 \) denotes the \( \mathbb{Z}_N \)-invariant subgroup of \( G \), which is the symmetry in each sector and \( G' \) denotes the full symmetry of the total Hilbert space.
Table 1.

| $\mathbb{Z}_N$ | $\varphi(N)$ | $D$     | $D$  |
|--------------|-------------|---------|------|
| $\mathbb{Z}_2$ | 1           | $8\mathbb{Z}$ | 8, 16 |
| $\mathbb{Z}_3$ | 2           | $6\mathbb{Z}$  | 6, 12, 18 |
| $\mathbb{Z}_4$ | 2           | $16\mathbb{Z}$ | 16   |
| $\mathbb{Z}_5$ | 4           | $4\mathbb{Z}$  | 4, 8, 12, 16, 20 |
| $\mathbb{Z}_6$ | 2           | $24\mathbb{Z}$ | 24   |
| $\mathbb{Z}_7$ | 6           | $6\mathbb{Z}$  | 6, 12, 18 |
| $\mathbb{Z}_8$ | 4           | $32\mathbb{Z}$ | non  |
| $\mathbb{Z}_9$ | 6           | $18\mathbb{Z}$ | 18   |
| $\mathbb{Z}_{10}$ | 4         | $8\mathbb{Z}$  | 8, 16, 24 |
| $\mathbb{Z}_{11}$ | 10         | $10\mathbb{Z}$ | 10, 20 |
| $\mathbb{Z}_{12}$ | 4           | $48\mathbb{Z}$ | non  |
| $\mathbb{Z}_{13}$ | 12          | $12\mathbb{Z}$ | 12   |
| $\mathbb{Z}_{14}$ | 6           | $24\mathbb{Z}$ | 24   |
| $\mathbb{Z}_{15}$ | 8           | $24\mathbb{Z}$ | 24   |
| $\mathbb{Z}_{16}$ | 8           | $64\mathbb{Z}$ | non  |
| $\mathbb{Z}_{17}$ | 16          | $16\mathbb{Z}$ | 16   |
| $\mathbb{Z}_{18}$ | 6           | $72\mathbb{Z}$ | non  |
| $\mathbb{Z}_{19}$ | 18          | $18\mathbb{Z}$ | 18   |
| $\mathbb{Z}_{20}$ | 8           | $16\mathbb{Z}$ | 16   |
| $\mathbb{Z}_{21}$ | 12          | $12\mathbb{Z}$ | 12, 24 |
| $\mathbb{Z}_{22}$ | 10          | $40\mathbb{Z}$ | non  |
| $\mathbb{Z}_{23}$ | 22          | $22\mathbb{Z}$ | 22   |
| $\mathbb{Z}_{24}$ | 8           | $96\mathbb{Z}$ | non  |
| $\mathbb{Z}_{25}$ | 20          | $20\mathbb{Z}$ | 20   |
| $\mathbb{Z}_{26}$ | 12          | $24\mathbb{Z}$ | 24   |
| $\mathbb{Z}_{27}$ | 18          | $54\mathbb{Z}$ | non  |
| $\mathbb{Z}_{28}$ | 12          | $48\mathbb{Z}$ | non  |
| $\mathbb{Z}_{29}$ | 28          | $28\mathbb{Z}$ | non  |
| $\mathbb{Z}_{30}$ | 8           | $24\mathbb{Z}$ | 24   |
Table 2.
Asymmetric $\mathbb{Z}_2$-orbifold models with (1,0) twisted states.

| $G$ | $G_0$ | $G'$ |
|-----|-------|------|
| $D = 8$ | | |
| $E_8$ | $SO(16)$ | $E_8$ |
| $SO(16)$ | $(SO(8))^2$ | $SO(16)$ |
| $(SO(8))^2$ | $(SU(2))^8$ | $(SO(8))^2$ |
| $D = 16$ | | |
| $(E_8)^2$ | $(SO(16))^2$ | $SO(32)$ |
| $SO(32)$ | $(SO(16))^2$ | $E_8 \times SO(16)$ |
| $SO(24) \times SO(8)$ | $(SO(12))^2 \times (SU(2))^4$ | $E_7 \times SO(12) \times (SU(2))^3$ |
| $E_8 \times SO(16)$ | $SO(16) \times (SO(8))^2$ | $SO(24) \times SO(8)$ |
| $(SO(16))^2$ | $(SO(8))^4$ | $SO(16) \times (SO(8))^2$ |
| $E_8 \times (SO(8))^2$ | $SO(16) \times (SU(2))^8$ | $SO(20) \times (SU(2))^6$ |
| $SO(16) \times (SO(8))^2$ | $(SO(8))^2 \times (SU(2))^8$ | $SO(12) \times SO(8) \times (SU(2))^6$ |
| $(SO(8))^4$ | $(SU(2))^{16}$ | $SO(8) \times (SU(2))^{12}$ |
Table 3.
Asymmetric $\mathbb{Z}_3$-orbifold models with (1,0) twisted states.

| $G$          | $G_0$            | $G'$            |
|--------------|------------------|-----------------|
| $D = 6$      |                  |                 |
| $E_6$        | $(SU(3))^3$      | $E_6$           |
| $(SU(3))^3$  |                  | $(SU(3))^3$     |
| $D = 12$     |                  |                 |
| $(E_6)^2$    | $(SU(3))^6$      | $(E_6)^2$       |
| $E_6 \times (SU(3))^3$ | $(SU(3))^3 \times (U(1))^6$ | $SU(6) \times SO(8) \times (U(1))^3$ |
| $(SU(3))^6$  |                  | $(SU(2))^6 \times (U(1))^6$ |
| $E_8 \times (SU(3))^2$ | $SU(9) \times (U(1))^4$ | $SO(20) \times (U(1))^2$ |
| $D = 18$     |                  |                 |
| $(E_6)^3$    | $(SU(3))^9$      | $E_6 \times (SU(3))^6$ |
| $(E_6)^2 \times (SU(3))^3$ | $(SU(3))^6 \times (U(1))^6$ | $SU(6) \times (SU(3))^4 \times (U(1))^5$ |
| $E_6 \times (SU(3))^6$ | $(SU(3))^3 \times (U(1))^{12}$ | $SU(4) \times (SU(3))^2 \times (U(1))^{11}$ |
| $(SU(3))^9$  |                  | $SU(2) \times (U(1))^{17}$ |
| $E_8 \times E_6 \times (SU(3))^2$ | $SU(9) \times (SU(3))^3 \times (U(1))^4$ | $SU(12) \times (SU(3))^2 \times (U(1))^3$ |
| $E_8 \times (SU(3))^5$ | $SU(9) \times (U(1))^{10}$ | $SU(10) \times (U(1))^9$ |
| $(E_8)^2 \times SU(3)$ | $(SU(9))^2 \times (U(1))^2$ | $SU(18) \times U(1)$ |
Table 4.
Asymmetric $\mathbb{Z}_4$-orbifold models with (1,0) twisted states.

| $G$                  | $G_0$                                           | $G'$                     |
|----------------------|-------------------------------------------------|--------------------------|
| $D = 16$             |                                                 |                          |
| $(E_8)^2$            | $(SO(10))^2 \times (SU(4))^2$                   | $(E_8)^2$                |
| $SO(32)$             | $SU(8) \times (SO(8))^2 \times U(1)$           | $SO(24) \times SO(8)$   |
| $SO(24) \times SO(8)$| $SU(6) \times (SU(4))^2 \times SU(2) \times (U(1))^4$ | $SU(12) \times SU(4) \times (U(1))^2$ |
| $E_8 \times SO(16)$  | $SO(10) \times (SU(4))^2 \times (SU(2))^4 \times U(1)$ | $(E_7)^2 \times (SU(2))^2$ |
| $(SO(16))^2$         | $(SU(4))^2 \times (SU(2))^8 \times (U(1))^2$    | $(SO(12))^2 \times (SU(2))^4$ |
| $E_8 \times (SO(8))^2$| $SO(10) \times SU(4) \times (SU(2))^2 \times (U(1))^6$ | $(E_6)^2 \times (U(1))^4$ |
| $SO(16) \times (SO(8))^2$ | $SU(4) \times (SU(2))^6 \times (U(1))^7$       | $(SU(6))^2 \times (SU(2))^2 \times (U(1))^4$ |
| $(SO(8))^4$          | $(SU(2))^4 \times (U(1))^{12}$                 | $(SU(3))^4 \times (U(1))^8$ |
Table 5.
Asymmetric $\mathbb{Z}_5$-orbifold models with (1,0) twisted states.

| $G$                     | $G_0$                      | $G'$                      |
|-------------------------|----------------------------|----------------------------|
| $D = 4$                 |                            |                            |
| $SU(5)$                 | $(U(1))^4$                 | $SU(5)$                    |
| $D = 8$                 | $(SU(5))^2$                | $E_8$                      |
| $(SU(5))^2$             | $(U(1))^8$                 | $(SU(5))^2$                |
| $D = 12$                | $(SU(5))^2 \times (U(1))^4$| $SO(22) \times U(1)$      |
| $E_8 \times SU(5)$     | $(U(1))^{12}$             | $(SU(4))^3 \times (U(1))^3$|
| $(SU(5))^3$             |                            |                            |
| $D = 16$                | $(SU(5))^4$                | $(E_8)^2$                  |
| $(SU(5))^2$             | $(SU(5))^2 \times (U(1))^8$| $(SO(12))^2 \times (U(1))^4$|
| $(SU(5))^4$             | $(U(1))^{16}$             | $(SU(2))^8 \times (U(1))^8$|
| $D = 20$                | $(SU(5))^4 \times (U(1))^4$| $(SU(10))^2 \times (U(1))^2$|
| $(E_8)^2 \times SU(5)$ | $(SU(5))^2 \times (U(1))^{12}$| $(SU(6))^2 \times (U(1))^{10}$|
| $(SU(5))^3$             | $(U(1))^{20}$             | $(SU(2))^2 \times (U(1))^{18}$|
| $(SU(5))^5$             |                            |                            |
Table 6.
Asymmetric $\mathbb{Z}_N$-orbifold models with (1,0) twisted states.

| $\mathbb{Z}_N$ | $G$ | $G_0$ | $G'$ |
|----------------|-----|-------|------|
| $\mathbb{Z}_6$ | $D = 24$ | $E_8^3$ | $(SU(5))^3 \times (SU(4))^3 \times (U(1))^3$ | $(SU(5))^6$ |
| $\mathbb{Z}_7$ | $D = 6$ | $SU(7)$ | $(U(1))^6$ | $SU(7)$ |
| $\mathbb{Z}_9$ | $D = 8$ | $E_8$ | $(SU(3))^2 \times (SU(2))^2 \times (U(1))^2$ | $E_8$ |
| $\mathbb{Z}_{10}$ | $D = 12$ | $SU(11)$ | $(U(1))^{12}$ | $(SU(6))^2 \times (U(1))^2$ |
| $\mathbb{Z}_{11}$ | $D = 18$ | $SU(13)$ | $(U(1))^{18}$ | $(SU(2))^9 \times (U(1))^9$ |
| $\mathbb{Z}_{13}$ | $D = 24$ | $E_8^3$ | $(SU(3))^6 \times (SU(2))^6 \times (U(1))^6$ | $(SU(3))^{12}$ |
| $\mathbb{Z}_{15}$ | $D = 20$ | $SU(11)^2$ | $(U(1))^{20}$ | $(SU(2))^{10} \times (U(1))^{10}$ |
| $\mathbb{Z}_{17}$ | $D = 12$ | $SU(13)$ | $(U(1))^{12}$ | $SU(12) \times U(1)$ |
| $\mathbb{Z}_{19}$ | $D = 24$ | $E_8^3$ | $(SU(2))^{12} \times (U(1))^{12}$ | $(SU(2))^{24}$ |
| $\mathbb{Z}_{20}$ | $D = 18$ | $SU(17)$ | $(U(1))^{16}$ | $(SU(8))^2 \times (U(1))^2$ |
| $\mathbb{Z}_{23}$ | $D = 18$ | $SU(19)$ | $(U(1))^{18}$ | $(SU(6))^3 \times (U(1))^3$ |
| $\mathbb{Z}_{23}$ | $D = 22$ | $SU(23)$ | $(U(1))^{22}$ | $(SU(2))^{11} \times (U(1))^{11}$ |
| $\mathbb{Z}_{30}$ | $D = 24$ | $E_8^3$ | $(U(1))^{24}$ | $(U(1))^{24}$ |