Abstract. The Calkin-Wilf tree is an infinite binary tree whose vertices are the positive rational numbers. Each number occurs in the tree exactly once and in the form $a/b$, where $a$ and $b$ are relatively prime positive integers. In this paper, certain subsemigroups of the modular group are used to construct similar trees in the set $D_0$ of positive complex numbers. Associated to each semigroup is a forest of trees that partitions $D_0$. The fundamental domain and the set of cusps of the semigroup are defined and computed.

1. Forests generated by left-right pairs of matrices

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, and $\mathbb{Q}$ denote, as usual, the sets of positive integers, nonnegative integers, integers, and rational numbers, respectively. The Calkin-Wilf tree is a rooted infinite binary tree whose vertices are the positive rational numbers. The root of the tree is 1, and the generation rule is

```
1 -> a
   `/\
  a+b  a+b
   `/\   `/\
   z  z+1  z+1
```

Writing $z = a/b$, we can redraw this as follows:

```
1  1
  `/\` /
  2 2
```

Every positive rational number occurs exactly once as a vertex in the Calkin-Wilf tree, and the geometry of the tree encodes beautiful arithmetical relations between rational numbers (Bates, Bunder, and Tognetti [1], Bates and Mansour [2], Calkin and Wilf [3], Chan [5], Dilcher and Stolarsky [4], Gibbons, Lester, and Bird [7], Han, Masuda, Singh, and Thiel [8], Mallows [10], Mansour and Shattuck [11], Nathanson [12, 14, 13], Reznick [15]).

Let

$$SL_2(\mathbb{N}_0) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{N}_0 \text{ and } ad - bc = 1 \right\}$$
be the semigroup of $2 \times 2$ matrices with nonnegative integral coordinates and determinant 1. Every matrix $T = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{N}_0)$ defines a function $z \mapsto T(z)$ by

$$T(z) = \frac{az + b}{cz + d}.$$ 

For example, if $L_1 = \left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right)$ and $R_1 = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)$ then $L_1(z) = \frac{z}{z + 1}$ and $R_1(z) = z + 1$.

Thus, the generation rule for the Calkin-Wilf tree can also be presented in the form

$$z \quad \rightarrow \\
L_1(z) \quad \rightarrow \\
R_1(z)$$

Denote the real and imaginary parts of a complex number $z$ by $\Re(z)$ and $\Im(z)$, respectively. Let $K$ be a non-real subfield of the complex numbers, that is, $K \subseteq \mathbb{C}$ and $K \not\subseteq \mathbb{R}$. In this paper we consider the set of “positive complex numbers”

$$\mathcal{D}_0 = \{ z = x + yi \in K : x > 0 \text{ and } y > 0 \}.$$

**Theorem 1.** If $z \in \mathcal{D}_0$ and $T \in SL_2(\mathbb{N}_0)$, then $T(z) \in \mathcal{D}_0$. Moreover, the function $(T, z) \mapsto T(z)$ from $SL_2(\mathbb{N}_0) \times \mathcal{D}_0$ into $\mathcal{D}_0$ defines a semigroup action of $SL_2(\mathbb{N}_0)$ on the set $\mathcal{D}_0$.

**Proof.** If $T = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{N}_0)$ and if $z = x + yi \in \mathcal{D}_0$, then $x > 0$, $y > 0$, and $ad - bc = 1$. A standard calculation gives

$$T(z) = \frac{az + b}{cz + d} = \frac{(az + b)(c\bar{z} + d)}{|cz + d|^2} = \frac{a|z|^2 + (ad + bc)x + bd}{|cz + d|^2} + \frac{yi}{|cz + d|^2}.$$ 

Because $a|z|^2 + (ad + bc)x + bd$, $|cz + d|^2$, and $y$ are positive real numbers, it follows that $T(z) \in \mathcal{D}_0$. Because $(T_1T_2)z = T_1(T_2z)$ for all $T_1, T_2 \in SL_2(\mathbb{N}_0)$ and $z \in \mathcal{D}_0$, the function $(T, z) \mapsto T(z)$ from $SL_2(\mathbb{N}_0) \times \mathcal{D}_0$ into $\mathcal{D}_0$ defines a semigroup action of $SL_2(\mathbb{N}_0)$ on the set $\mathcal{D}_0$. This completes the proof. □

**Lemma 1.** If $T \in SL_2(\mathbb{N}_0)$ and $T \neq I$, then the fractional linear transformation $z \mapsto T(z)$ has no fixed points in $\mathcal{D}_0$.

**Proof.** Let $T = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$, where $a, b, c, d \in \mathbb{N}_0$ and $ad - bc = 1$. If $z$ is a fixed point of $T$, then

$$\frac{az + b}{cz + d} = z.$$ 

Equivalently,

$$cz^2 + (d - a)z - b = 0.$$
If $c = 0$, then $a = d = 1$ and so $z = T(z) = z + b$, hence $b = 0$ and $T = I$, which is absurd. If $c \neq 0$, then
\[
z = \frac{a - d \pm \sqrt{(d - a)^2 + 4bc}}{2c} = \frac{a - d \pm \sqrt{(a + d)^2 - 4}}{2c}.
\]
If $z \in \mathcal{D}_0$, then $\Re(z) > 0$ and so $(a + d)^2 - 4 < 0$. Equivalently, $a + d = |a + d| < 2$. Because $a, b, c, d$ are nonnegative integers, it follows that $0 = ad = 1 + bc \geq 1$, which is also absurd. Therefore, $T$ has no fixed points in $\mathcal{D}_0$. \qed

Let $L$ and $R$ be non-identity matrices in $SL_2(\mathbb{N}_0)$. The ordered pair $(L, R)$ will be called a left-right pair if
\[
L(\mathcal{D}_0) \cap R(\mathcal{D}_0) = \emptyset.
\]
Equivalently, $(L, R)$ is a left-right pair if
\[
L(z_1) \neq R(z_2) \quad \text{for all } z_1, z_2 \in \mathcal{D}_0.
\]

**Lemma 2.** If $(L, R)$ is a left-right pair, then the subsemigroup $\langle L, R \rangle$ of $SL_2(\mathbb{N}_0)$ generated by $\{L, R\}$ is free.

**Proof.** The semigroup $\langle L, R \rangle$ is free if and only if the unique solution of the matrix equation
\[
T_1T_2 \cdots T_k = T'_1T'_2 \cdots T'_\ell
\]
with $k, \ell \in \mathbb{N}_0$ and $T_i, T'_j \in \{L, R\}$ for all $i \in \{1, 2, \ldots, k\}$ and $j \in \{1, 2, \ldots, \ell\}$ is the trivial solution $k = \ell$ and $T_i = T'_i$ for all $i \in \{1, 2, \ldots, k\}$. If there is a nontrivial solution, then there is a minimal solution, that is, a matrix identity (1) with $k \geq \ell$ and $k$ minimal.

If $\ell \geq 1$, then for every $z \in \mathcal{D}_0$ we have
\[
T_1T_2 \cdots T_k(z) = T'_1T'_2 \cdots T'_\ell(z)
\]
and so
\[
T_1(z_1) = T'_1(z_2)
\]
where
\[
z_1 = T_2 \cdots T_k(z) \in \mathcal{D}_0 \quad \text{and} \quad z_2 = T'_2 \cdots T'_\ell(z) \in \mathcal{D}_0.
\]
Because $(L, R)$ is a left-right pair, it follows that $T_1 = T'_1$, and so $T_2 \cdots T_k = T'_2 \cdots T'_\ell$, which contradicts the minimality of $k$.

If $\ell = 0$, then $k \geq 1$ and $T_1T_2 \cdots T_k = I$. Let $T_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{N}_0)$. Then
\[
\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = T_1^{-1} = T_2 \cdots T_k \in SL_2(\mathbb{N}_0)
\]
and so $b = c = 0$ and $a = d = 1$, that is, $T_1 = I \in \{L, R\}$, which is absurd. This completes the proof. \qed

**Lemma 3.** For positive integers $u$ and $v$, let
\[
(2) \quad L_u = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \quad \text{and} \quad R_v = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}.
\]
Then $(L_u, R_v)$ is a left-right pair.
Proof. If \( z = x + yi \in \mathcal{D}_0 \), then

\[
L_u(z) = \frac{z}{uz + 1} = \frac{u(x^2 + y^2) + x}{|uz + 1|^2} + \frac{yi}{|uz + 1|^2}.
\]

Because

\[
u(x^2 + y^2) + x < u^2(x^2 + y^2) + 2ux + 1 = (ux + 1)^2 + (uy)^2 = |uz + 1|^2
\]

it follows that

\[
\Re(L_u(z)) < 1.
\]

Similarly, \( R_v(z) = (x + v) + yi \) and so

\[
\Re(R_v(z)) = x + v > v \geq 1.
\]

Therefore,

\[
\Re(L_u(z_1)) < 1 < \Re(R_v(z_2))
\]

for all \( z_1, z_2 \in \mathcal{D}_0 \), and so \( L(\mathcal{D}_0) \cap R(\mathcal{D}_0) = \emptyset \). This completes the proof. \( \square \)

Lemmas \[\text{(2)}\] and \[\text{(3)}\] imply that the set \( \{L_u, R_v\} \) freely generates the semigroup \( \langle L_u, R_v \rangle \). Note that \( L_u = L_1^u \) and \( R_v = R_1^v \). It is a classical result that \( \{L_1, R_1\} \) freely generates \( SL_2(\mathbb{N}_0) \), and this also proves that \( \{L_u, R_v\} \) freely generates \( \langle L_u, R_v \rangle \) (cf. Nathanson \[\text{(1)}\]).

Let \( (L, R) \) be a pair of matrices in \( SL_2(\mathbb{N}_0) \). We consider the directed graph \( \mathcal{F}(L, R) \) whose vertex set is \( \mathcal{D}_0 \) and whose edge set is

\[
\{(z, L(z)) : z \in \mathcal{D}_0\} \cup \{(z, R(z)) : z \in \mathcal{D}_0\}.
\]

In this graph, every vertex has outdegree 2:

(3)

\[
\begin{align*}
L(z) & \to z & R(z)
\end{align*}
\]

We call \( L(z) \) the left child of \( z \) and \( R(z) \) the right child of \( z \), and we call \( z \) the parent of \( L(z) \) and of \( R(z) \). Because \( L \) and \( R \) are invertible matrices, if \( L(z_1) = L(z_2) \) or if \( R(z_1) = R(z_2) \) for some \( z_1, z_2 \in \mathcal{D}_0 \), then \( z_1 = z_2 \). If \( (L, R) \) is a right-left pair, then there do not exist \( z_1, z_2 \in \mathcal{D}_0 \) such that \( L(z_1) = R(z_2) = z \), and so the indegree of \( z \) is either 0 or 1. We call \( z \) an orphan if it has no parent, that is, if \( z \) has indegree 0.

For example, let \( (L_1, R_1) \) be the pair of matrices defined by \[\text{(2)}\] with \( u = v = 1 \). The first three generations of descendants of the complex number \( z \in \mathcal{D}_0 \) are:

\[
\begin{align*}
&z \quad &z + 1 \\
&\frac{z}{z + 1} \quad &\frac{z + 1}{z + 2} \\
&\frac{z^2 + 1}{z + 1} \quad &\frac{z^2 + 2}{z + 2} \quad &\frac{z^2 + 3}{z + 3}
\end{align*}
\]
If $z$ is a variable, then the vertices of this tree are the linear fractional transformations associated with the semigroup $SL_2(\mathbb{N}_0)$. Nathanson [12] described some remarkable arithmetical properties of this tree.

**Theorem 2.** For every left-right pair $(L, R)$ of matrices in $SL_2(\mathbb{N}_0)$, the directed graph $\mathcal{F}(L, R)$ is a forest of infinite binary trees.

**Proof.** We must prove that every connected component of the graph is a tree. If not, then some component of the graph contains an undirected cycle of length $n \geq 2$, that is, a sequence of $n$ vertices $z_0, z_1, \ldots, z_{n-1}, z_n$ such that

(i) $z_i \neq z_j$ for $0 \leq i < j \leq n-1$ and $z_n = z_0$;

(ii) for $i = 0, 1, \ldots, n-1$, either $(z_i, z_{i+1})$ is an edge or $(z_{i+1}, z_i)$ is an edge.

Suppose that $(z_0, z_1)$ is an edge. Then $z_0$ is the parent of $z_1$. Because $(L, R)$ is a left-right pair, every vertex has at most one parent. This implies that $z_1$ is the parent of $z_2$, and so $z_2$ is the parent of $z_3$. Continuing inductively, we conclude that $z_i$ is the parent of $z_{i+1}$ for $i = 0, 1, \ldots, n-1$. Thus, $(z_0, z_1, \ldots, z_n)$ is not only a cycle in the graph, but is a directed cycle. It follows that there is a sequence of matrices $T_0, T_1, \ldots, T_{n-1}$ such that $T_i \in \{L, R\}$ and $T_i z_i = z_{i+1}$ for all $i = 0, 1, \ldots, n-1$. Thus, $T = T_{n-1} \cdots T_1 T_0 \in SL_2(\mathbb{N}_0)$ and $T(z_0) = z_0$, that is, $z_0 \in D_0$ is a fixed point of $T$. Lemma [1] implies that $T = I$ and so

$T_0^{-1} = T_{n-1} \cdots T_1 \in SL_2(\mathbb{N}_0)$

which is impossible because (as observed in the proof of Lemma [2] the only invertible matrix in $SL_2(\mathbb{N}_0)$ is the identity matrix $I$. The same argument applies if $(z_1, z_0)$ is an edge. Thus, every component of the directed graph $\mathcal{F}(L, R)$ is a tree, and so $\mathcal{F}(L, R)$ is a forest. Because every vertex in $\mathcal{F}(L, R)$ has outdegree 2, it follows that every component of $\mathcal{F}(L, R)$ is an infinite binary tree.

Let $(L, R)$ be a left-right pair, and let $(L, R)$ be the subsemigroup of $SL_2(\mathbb{N}_0)$ generated by $\{L, R\}$. Every complex number $w \in D_0$ is the root of an infinite binary tree whose vertices are the complex numbers in $D_0$ constructed from the generation rule [3]. The orbit of $w$, denoted $\text{orbit}(w)$, is the set of vertices in this tree. Equivalently,

$\text{orbit}(w) = \{T(w) : T \in \langle L, R \rangle \}$.

**Lemma 4.** Let $(L, R)$ be a left-right pair, and consider the forest $\mathcal{F}(L, R)$. Let $w_1, w_2 \in D_0$. If $\text{orbit}(w_1) \cap \text{orbit}(w_2) \neq \emptyset$, then either $\text{orbit}(w_1) \subseteq \text{orbit}(w_2)$ or $\text{orbit}(w_2) \subseteq \text{orbit}(w_1)$.

**Proof.** If $\text{orbit}(w_1) \cap \text{orbit}(w_2) \neq \emptyset$, then there exist matrices $T, T' \in \langle L, R \rangle$ such that

$T(w_1) = T'(w_2)$.

There are sequences of matrices $(T_i^k)_{i=1}^k$ and $(T'_j^l)_{j=1}^l$ such that $T_i, T'_j \in \{L, R\}$ for $i = 1, \ldots, k$ and $j = 1, \ldots, \ell$, with $T = T_1 T_2 \cdots T_k$ and $T' = T'_1 T'_2 \cdots T'_\ell$. Therefore,

$T_1 (T_2 \cdots T_k(w_1)) = T(w_1) = T'(w_2) = T'_1 (T'_2 \cdots T'_\ell(w_2))$.

Because $(L, R)$ is a left-right pair, it follows that $T_1 = T'_1$ and so $T_2 \cdots T_k(w_1) = T'_2 \cdots T'_\ell(w_2)$. Suppose that $k > \ell$. Continuing inductively, we obtain $T_{k+1} \cdots T_k(w_1) = w_2$, and so $w_2 \in \text{orbit}(w_1)$. It follows that $\text{orbit}(w_2) \subseteq \text{orbit}(w_1)$. This completes the proof.

$\square$
Let \((L, R)\) be a left-right pair, and let \(F(L, R)\) be the associated forest whose vertices are the complex numbers in \(D_0\). Let \(w \in D_0\). Every element in \(\text{orbit}(w) \setminus \{w\}\) is a descendant of \(w\), and \(w\) is an ancestor of every element in \(\text{orbit}(w) \setminus \{w\}\).

An orphan is a complex number in \(D_0\) with no ancestors. A complex number in \(D_0\) is a descendant of an orphan if and only if it has only finitely many ancestors.

There is a one-to-one correspondence between the rooted infinite binary trees in the forest \(F(L, R)\) and the set of orphans. The set of orphans, denoted \(\Omega(L, R)\), is called the fundamental domain of the semigroup \(\langle L, R \rangle\).

An infinite path in the forest \(F(L, R)\) is a sequence \((w_n)_{n=1}^{\infty}\) of complex numbers in \(D_0\) such that \(w_{n+1} = L(w_n)\) or \(w_{n+1} = R(w_n)\) for all \(n \in \mathbb{N}\). A cusp of the semigroup \(\langle L, R \rangle\) is the limit of an infinite path in \(F(L, R)\). Thus, \(w^*\) is a cusp if \(w^* = \lim_{n \to \infty} w_n\), where \((w_n)_{n=1}^{\infty}\) is a path in \(D_0\). Of course, not every infinite path has a limit.

For every left-right pair \((L, R)\) of matrices in \(SL_2(\mathbb{N}_0)\), we have the following problems:

1. Compute the fundamental domain \(\Omega(L, R)\).
2. Determine if the forest \(F(L, R)\) contains infinite binary trees without roots, and describe them.
3. Determine the cusps of the semigroup \(\langle L, R \rangle\).

2. Example: Trees in the forest \(F(L_u, R_v)\)

Let \(u\) and \(v\) be positive integers, and let \((L_u, R_v)\) be the left-right pair of matrices defined by (2). The special case \(u = v = 1\) is a complex version of the Calkin-Wilf tree.

Let \(u > 0\). For every positive integer \(n\), we define the open half disk

\[
D_n = \left\{ x + yi \in D_0 : \left( x - \frac{1}{2nu} \right)^2 + y^2 < \left( \frac{1}{2nu} \right)^2 \right\}
\]

and \(D_n \setminus D_{n+1} = \left\{ x + yi \in D_0 : nu(x^2 + y^2) < x \right\}\).

The half disks and half crescents satisfy the relations

\[
D_0 \supset D_1 \supset D_2 \supset \cdots \supset D_n \supset D_{n+1} \supset \cdots,
\]

\[
\bigcap_{n=1}^{\infty} D_n = \emptyset,
\]

and, for \(0 \leq n < m\),

\[
(D_n \setminus D_{n+1}) \cap D_m = \emptyset.
\]

If \(w\) is in the half disk \(D_n\), then

\[
|w| \leq \left| w - \frac{1}{2nu} \right| + \frac{1}{2nu} < \frac{1}{nu}.
\]
Lemma 5. Let \( z \in D_0 \), and let \( u, v, \) and \( n \) be positive integers. Then \( R_{v}^{-n}(z) \in D_0 \) if and only if \( \Re(z) > z + nv \), and \( L_{u}^{-n}(z) \in D_0 \) if and only if \( z \in D_n \).

Proof. Let \( z = x + yi \in D_0 \). Because \( x > 0 \) and \( y > 0 \), we have

\[
R_{v}^{-n}(z) = z - nv = x - nv + yi \in D_0
\]

if and only if \( \Re(R_{v}^{-n}(z)) = x - nv > 0 \).

Let \( w = L_{u}^{-n}(z) \). Note that both \( y \) and

\[
|1 - nuz|^2 = (1 - nx)^2 + (ny)^2
\]

are positive real numbers. We have

\[
w = L_{u}^{-n}(z) = \frac{z}{1 - nuz} = \frac{z(1 - nuz)}{|1 - nuz|^2} = \frac{|z - nu|z|^2}{|1 - nuz|^2}
\]

\[
= \frac{x - nu(x^2 + y^2)}{|1 - nuz|^2} + \frac{yi}{|1 - nuz|^2} \in Q(i)
\]
and \( \Im(w) = y/|1 - nuz|^2 > 0 \). It follows that \( w \in D_0 \) if and only if \( \Re(w) > 0 \) if and only if

\[
\nu u(x^2 + y^2) < x.
\]

Completing the square, we see that \( w \in D_0 \) if and only if

\[
\left(x - \frac{1}{2nu}\right)^2 + y^2 < \left(\frac{1}{2nu}\right)^2.
\]

Thus, \( L_u^{-n}(z) \in D_0 \) if and only if \( z \in D_n \). This completes the proof. \( \square \)

**Theorem 3.** Let \( u \) be a positive integer. The linear fractional transformation

\[
L_u(z) = \frac{z}{uz + 1}
\]

maps \( D_n \setminus D_{n+1} \) onto \( D_{n+1} \setminus D_{n+2} \) for all integers \( n \geq 0 \).

**Proof.** Let \( n \geq 0 \). If \( z \in D_u \setminus D_{n+1} \) and \( w = L_u(z) \), then Lemma 5 implies that \( L_u^{-n-1}(w) = L_u^{-n}(z) \in D_0 \) and so \( w \in D_{n+1} \). If \( w \in D_{n+2} \), then \( L_u^{-n-1}(z) = L_u^{-n-2}(w) \in D_0 \) and so \( z \in D_{n+1} \), which is absurd. Therefore, \( L_u(z) \in D_{n+1} \setminus D_{n+2} \).
Conversely, let \( w \in D_{n+1} \setminus D_{n+2} \), and let \( z = L_u^{-1}(w) \). Lemma 5 implies that \( L_u^{-n}(z) = L_u^{-n-1}(w) \in D_0 \) and so \( z \in D_n \). If \( z \in D_{n+1} \), then \( L_u^{-n-2}(w) = L_u^{-n-1}(z) \in D_0 \) and so \( w \in D_{n+2} \), which is absurd. Therefore, \( z \in D_n \setminus D_{n+1} \), and \( L_u(z) = w \). It follows that the function

\[
L_u : D_n \setminus D_{n+1} \to D_{n+1} \setminus D_{n+2}
\]

is onto. This completes the proof. \( \square \)

**Theorem 4.** For all positive integers \( u \) and \( v \), the fundamental domain of the left-right pair \( (L_u, R_v) \) is

\[
\Omega(L_u, R_v) = \{ z \in D_0 : z \notin D_1 \text{ and } \Re(z) \leq v \}
= \{ x + yi \in D_0 : u(x^2 + y^2) \geq x \text{ and } x \leq v \}.
\]

**Proof.** Let \( z = x + yi \in D_0 \). Then \( R_v^{-1}(z) = (x - v) + yi \in D_0 \) if and only if \( x > v \). Similarly, \( L_u^{-1}(z) \in D_0 \) if and only if \( z \in D_1 \). Thus, the complex number \( z \) in \( D_0 \) has a parent if and only if either \( \Re(z) > v \) or \( z \in D_0 \). Equivalently, \( z \) is an orphan if and only if \( z \in D_0 \setminus D_1 \) and \( \Re(z) > v \). This completes the proof. \( \square \)

Bumby [3] and Thiel [17] have independently proved that every complex number in \( D_0 \) is descended from an orphan with respect to the left-right pair \( (L_u, R_v) \).

**Theorem 5.** For all positive integers \( v \) and \( u \), the set of cusps of the semigroup \( \langle L_u, R_v \rangle \) is \( \{0, \infty\} \).

**Proof.** Let \((w_n)_{n=1}^{\infty}\) be an infinite path in the forest \( F(L_u, R_v) \). If \( w_{n+1} = R_v(w_n) \) for all \( n \geq n_0 \), then

\[
w_n = R_v^{n-n_0}(w_{n_0}) = \Re(w_0) + (n - n_0)v + \Im(w_0)i
\]

and so \( \lim_{n \to \infty} w_n = \infty \).

If \( w_{n+1} = L_u(w_n) \) for all \( n \geq n_0 \), then

\[
w_n = L_u^{n-n_0}(w_{n_0}) = \frac{w_{n_0}}{(n - n_0)(uw_{n_0} + 1)}
\]

and so \( \lim_{n \to \infty} w_n = 0 \).

For all \( w \in D_0 \), we have \( L_u(w) \in D_1 \) and so

\[
\Re(L_u(w)) < \frac{1}{u} \leq 1.
\]

Similarly, for all \( w \in D_0 \), we have

\[
R_v(w) - w = v \geq 1.
\]

If \((w_n)_{n=1}^{\infty}\) is an infinite path in the forest \( F(L_u, R_v) \) such that \( w_{n+1} = L_u(w_n) \) for infinitely many \( n \) and \( w_{n+1} = R_v(w_n) \) for infinitely many \( n \), then \( w_{n+1} \in D_1 \) infinitely often and \( w_{n+1} - w_n = v \geq 1 \) infinitely often. It follows that \( \lim_{n \to \infty} w_n \) does not exist. Therefore, the set of cusps of the semigroup \( \langle L_u, R_v \rangle \) is \( \{0, 1\} \). This completes the proof. \( \square \)
3. Open problems

(1) Classify the left-right pairs in $SL_2(\mathbb{N}_0)$.

(2) Determine the left-right pairs $(L, R)$ whose associated forests contain infinite binary trees without roots.

(3) Let $u, v \in \mathbb{N}$ with $(u, v) \neq (1, 1)$. Find a algorithm to determine if a matrix belongs to the semigroup generated by $L_u$ and $R_v$.

(4) Is there an efficient algorithm to determine if two numbers in $D_0$ are in the same tree?

(5) Construct a class of freely generated subsemigroups of $SL_2(\mathbb{N})$ of rank $k \geq 3$, and describe their associated forests of $k$-regular trees of positive complex numbers.

References

[1] B. Bates, M. Bunder, and K. Tognetti, Linking the Calkin-Wilf and Stern-Brocot trees, European J. Combin. 31 (2010), no. 7, 1637–1661.

[2] B. Bates and T. Mansour, The q-Calkin-Wilf tree, J. Combin. Theory Ser. A 118 (2011), no. 3, 1143–1151.

[3] R. T. Bumby, personal communication, 2014.

[4] N. Calkin and H. S. Wilf, Recounting the rationals, Amer. Math. Monthly 107 (2000), no. 4, 360–363.

[5] S. H. Chan, Analogs of the Stern sequence, Integers 11 (2011), #A26, pp. 1–10.

[6] K. Dilcher and K. B. Stolarsky, A polynomial analogue to the Stern sequence, Int. J. Number Theory 3 (2007), no. 1, 85–103.

[7] J. Gibbons, D. Lester, and R. Bird, Functional pearl: Enumerating the rationals, Journal of Functional Programming 16 (2006), 281–291.

[8] S. Han, A. M. Masuda, S. Singh, and J. Thiel, The $(u,v)$-Calkin-Wilf forest, arXiv: 1411:1747, 2014.

[9] R. C. Lyndon and P. E. Schupp, Combinatorial Group Theory, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1977 edition.

[10] C. L. Mallows, A variation of the Stern-Brocot tree, J. Comb. 2 (2011), no. 4, 501–506.

[11] T. Mansour and M. Shattuck, Two further generalizations of the Calkin-Wilf tree, J. Comb. 2 (2011), no. 4, 507–524.

[12] M. B. Nathanson, A forest of linear fractional transformations, International J. Number Theory (2015), to appear, arXiv:1401.0012.

[13] ______, Free monoids and forests of rational numbers, Discrete Applied Math. (2015), to appear, arXiv:1406.2054.

[14] ______, Pairs of matrices in $GL_2(\mathbb{R}_{\geq 0})$ that freely generate, Amer. Math. Monthly (2015), to appear, arXiv:1406.1194.

[15] B. Reznick, Some binary partition functions, Analytic Number Theory (Allerton Park, IL, 1989), Progr. Math., vol. 85, Birkhäuser Boston, Boston, MA, 1990, pp. 451–477.

[16] I. N. Sanov, A property of a representation of a free group, Doklady Akad. Nauk SSSR (N. S.) 57 (1947), 657–659.

[17] J. Thiel, personal communication, 2014.

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