Stability of the $C^*$-algebra associated with the twisted CCR.

Daniil Proskurin and Yurii Samoilenko

Abstract

The universal enveloping $C^*$-algebra $\mathbf{A}_\mu$ of twisted canonical commutation relations is considered. It is shown that for any $\mu \in (-1, 1)$ the $C^*$-algebra $\mathbf{A}_\mu$ is isomorphic to the $C^*$-algebra $\mathbf{A}_0$ generated by partial isometries $t_i, t_i^*$, $i = 1, \ldots, d$ satisfying the relations

$$t_i^* t_j = \delta_{ij} (1 - \sum_{k<i} t_k t_k^*), \quad t_j t_i = 0, \quad i \neq j.$$

It is proved that Fock representation of $\mathbf{A}_\mu$ is faithful.

Mathematics Subject Classifications (2000): 46L55, 46L65, 81S05, 81T05. Key words: Fock representation, deformed commutation relations, universal bounded representation.

Introduction

Recently the interest to the *-algebras defined by generators and relations, their representations, particularly faithful representations, and the universal enveloping $C^*$-algebras has been growing because of their applications in mathematical physics, operator theory etc.

A lot of interesting classes of a *-algebras depending on the parameters are constructed as deformations of canonical commutation relations of quantum mechanics (CCR). A well-known examples of a such deformations are

• $q_{ij}$-CCR introduced by M. Bozejko and R. Speicher (see [3])

$$\mathbb{C}\langle a, a_i^* | a_i^* a_j = \delta_{ij} 1 + q_{ij} a_j a_i^*, \quad i, j = 1, \ldots, d, \quad q_{ji} = q_{ij} \in \mathbb{C}, \quad |q_{ij}| \leq 1 \rangle$$

and

• Twisted canonical commutation relations (TCCR) constructed by W. Pusz and S.L. Woronowicz (see [8]). The TCCR have the following form

$$a_i^* a_i = 1 + \mu^2 a_i^* a_i^* - (1 - \mu^2) \sum_{k < i} a_k a_k^*, \quad i = 1, \ldots, d$$

$$a_i^* a_j = \mu a_j a_i^*, \quad i \neq j, \quad a_j a_i = \mu a_i a_j, \quad i < j, \quad 0 < \mu < 1 \quad (1)$$
The universal $C^*$-algebra $A_{(q_{ij})}$ for $q_{ij}$-CCR, $|q_{ij}| < \sqrt{2} - 1$, was studied in [4]. Particularly it was shown that under above restrictions on the coefficients $A_{(q_{ij})}$ is isomorphic to the Cuntz-Toeplitz algebra generated by isometries $\{s_i, s^*_i, i = 1, \ldots, d\}$ satisfying relations $s^*_i s_j = 0, i \neq j$. This implies that Fock representation of $A_{(q_{ij})}$ is faithful. The conjecture that the same results are true for any choice of $|q_{ij}| < 1$ was discussed in [4] also. When $|q_{ij}| = 1, i \neq j$, the universal $C^*$-algebra is isomorphic to the extension of noncommutative higher-dimensional torus generated by isometries satisfying $s^*_i s_j = q_{ij} s_j s^*_i, i \neq j$ and the Fock representation is faithful also (see [9]).

In the present paper we consider the universal $C^*$-algebra $A_{\mu}$ corresponding to the TCCR. Recall that the irreducible representations of TCCR, including unbounded, were described in [8] and for any bounded representation $\pi$ of TCCR

$$\|\pi(a_i a^*_i)\| \leq \frac{1}{1 - \mu^2},$$

i.e. TCCR generate a *-bounded *-algebra (see, for example [6] and [7]). We show in the Sec. 1 that $A_{\mu} \simeq A_0$ for any $\mu \in (-1, 1)$. Note that $A_0$ is generated by partial isometries $\{s_i, s^*_i, i = 1, \ldots, d\}$ satisfying the relations

$$s^*_i s_j = \delta_{ij} (1 - \sum_{k<i} s_k s^*_k).$$

In the Sec. 2 we prove that Fock representation of $A_{\mu}$ is faithful. 

**Remark 1.** It follows from the main result of [3] that Fock representations of a *-algebras generated by $q_{ij}$-CCR, $|q_{ij}| < 1$, and TCCR are faithful. In the case when $|q_{ij}| = 1, i \neq j$, the kernel of Fock representation is generated as a *-ideal by the family $\{a_j a_i - q_{ij} a_i a_j\}$.

Finally, let us recall that by the universal $C^*$-algebra for a certain *-algebra $\mathcal{A}$ we mean the $C^*$-algebra $\mathcal{A}$ with the homomorphism $\psi: \mathcal{A} \to \mathcal{A}$ such that for any homomorphism $\varphi: \mathcal{A} \to B$, where $B$ is a $C^*$-algebra, there exists $\theta: \mathcal{A} \to B$ satisfying $\theta \psi = \varphi$. It can be obtained by the completion of $\mathcal{A}/J$ by the following $C^*$-seminorm on $\mathcal{A}$

$$\|a\| = \sup_\pi \|\pi(a)\|,$$

where sup is taken over all bounded representations of $\mathcal{A}$ and $J$ is the kernel of this seminorm. Obviously this process requires the condition $\sup_\pi \|\pi(a)\| < \infty$ for any $a \in \mathcal{A}$.

Through the paper we suppose that all $C^*$-algebras are realised by the Hilbert space operators. Particularly, it is correct to consider the polar decomposition of elements of a $C^*$-algebra. Obviously, we do not claim that in general the partial isometry from the polar decomposition lies in this $C^*$-algebra.
1 Stability of $\mu$-CCR.

Let us recall some properties of the $C^\ast$-algebra generated by one-dimensional $q$-CCR. Namely we need the following proposition (see [9]).

**Proposition 1.** Let $B$ be the unital $C^\ast$-algebra generated by the elements $a$, $a^*$ satisfying the relation

$$a^*a = 1 + qaa^*, \quad -1 < q < 1$$

and $a^* = S^*C$ is a polar decomposition. Then $S \in B$, $B = C^\ast(S, S^*)$ and

$$a = \left(\sum_{n=1}^{\infty} q^{n-1} S^n S^\ast n\right)^{1/2} S.$$

Let us show that any $C^\ast$-algebra generated by the operators satisfying (1) can be generated by some family of partial isometries.

**Proposition 2.** Let $A_\mu$ be the unital $C^\ast$-algebra generated by operators $a_i$, $a_i^\ast$, $i = 1, \ldots, d$, satisfying relations (1). Let $a_i^\ast = S_i^\ast C_i$ be the polar decomposition. Construct the following family of partial isometries inductively:

$$\hat{S}_1 := S_1, \quad \hat{S}_i = (1 - \sum_{j<i} \hat{S}_j \hat{S}_j^\ast) S_i.$$

Then for all $i = 1, \ldots, d$ we have $\hat{S}_i \in A_\mu$, $A_\mu = C^\ast(\hat{S}_i, \hat{S}_i^\ast, i = 1, \ldots, d)$ and the following relations hold

$$\hat{S}_i^\ast \hat{S}_j = \delta_{ij} (1 - \sum_{j<i} \hat{S}_j \hat{S}_j^\ast) \quad i, j = 1, \ldots, d \quad (2)$$

$$\hat{S}_j \hat{S}_i = 0, \quad j > i.$$

**Proof.** We use induction on the number of generators.

$d = 1$.

In this case we have $a_1^\ast a_1 = 1 + \mu^2 a_1 a_1^\ast$, $a_1^\ast = S_1^\ast C_1$ and as shown in Proposition [9] we have $S_1 \in C^\ast(a_1, a_1^\ast)$ and

$$a_1^\ast = \left(\sum_{n=1}^{\infty} \mu^{2(n-1)} S_1^n S_1^\ast n\right)^{1/2}$$

with $S_1^\ast S_1 = 1$.

$d - 1 \rightarrow d$.

Denote by $a_i^{(1)} := (1 - S_i S_i^\ast) a_i$, $i = 2, \ldots, d$. Note, that the relations (1) are equivalent to

$$C_i^2 S_i = S_i(1 + \mu^2 C_i^2 - (1 - \mu^2) \sum_{j<i} C_j^2)$$

$$C_i^2 S_j = \mu^2 S_j C_i^2, \quad j < i$$

$$C_i^2 S_j = S_j C_i^2, \quad j > i$$

$$C_i C_j = C_j C_i, \quad S_i^\ast S_j = S_j S_i^\ast, \quad S_i S_j = S_j S_i$$

3
Then it is easy to see that \((1 - S_i S_j^*)a_i = a_i(1 - S_i S_j^*), i = 2, \ldots, d\) and

\[
a_i^{* (1)} a_j^{(1)} = (1 - S_i S_j^*) a_i^* (1 - S_i S_j^*) a_j
= (1 - S_i S_j^*) a_i^* a_j (1 - S_i S_j^*)
= \mu (1 - S_i S_j^*) a_i^* (1 - S_i S_j^*)
= \mu (1 - S_i S_j^*) a_j (1 - S_i S_j^*) a_i
= \mu a_i^{(1)} a_j^{* (1)}, i \neq j
\]

Analogously \(a_i^{(1)} a_i^{(1)} = \mu a_i^{(1)} a_j^{(1)}, j > i > 1\). Multiplying the relation

\[
a_i^* a_i = 1 + \mu^2 a_i a_i^* - (1 - \mu^2) \sum_{k < i} a_k a_k^*
\]

by \(1 - S_i S_j^*\) we get

\[
a_i^{* (1)} a_i^{(1)} = (1 - S_i S_j^*) + \mu^2 a_i^{(1)} a_i^{* (1)} - (1 - \mu^2) \sum_{2 \leq k < i} a_k^{(1)} a_k^{* (1)}.
\]

Evidently, the element \(1 - S_i S_j^*\) is the unit of the \(C^*\)-algebra of operators \(C^* (1 - S_i S_j^*, a_i^{(1)}, a_i^{* (1)}, i = 2, \ldots, d)\). Using the assumption of induction we conclude that

\[
C^* (S_i, S_i^*, a_i^{(1)}, a_i^{* (1)}, i = 2, \ldots, d) = C^* (S_i, S_i^*, \hat{S}_i^{(1)}, \hat{S}_i^{* (1)}, i = 2, \ldots, d)
\]

and partial isometries \(\hat{S}_i^{(1)}, i = 2, \ldots, d\), satisfy the relations (B). Note that \(\hat{S}_i^{(1)} = \hat{S}_i, i = 2, \ldots, d\). Indeed, evidently if \(a_i^* = S_i C_i\) is a polar decomposition then \(a_i^{(1)} = (1 - S_i S_i^*) S_i^* (1 - S_i S_i^*) C_i, i = 2, \ldots, d\), is a polar decomposition too. I.e. \(S_i^{(1)} = (1 - S_i S_i^*) S_i\) and we have \(\hat{S}_i^{(1)} := \hat{S}_i^{(1)} = (1 - S_i S_i^*) S_i = \hat{S}_2\), further

\[
\hat{S}_i^{(1)} := (1 - S_i S_i^*) S_i - (1 - S_i S_i^*) S_i^* (1 - S_i S_i^*) S_i
= (1 - S_i S_i^*) - (1 - S_i S_i^*) S_i^* (1 - S_i S_i^*) S_i
= (1 - S_i S_i^*) S_i - (1 - S_i S_i^*) S_i^* (1 - S_i S_i^*) S_i = \hat{S}_i
\]

Obviously the conclusion above is obtained by the induction. Then \(\hat{S}_1 S_i = S_i^* (1 - S_i S_i^*) \hat{S}_i = 0\) and, analogously, \(\hat{S}_i \hat{S}_i = 0\, i = 2, \ldots, d\). It remains only to show that \(C^* (a_i, a_i^*, i = 1, \ldots, d) = C^* (\hat{S}_i, \hat{S}_i^*, i = 1, \ldots, d)\). It follows from the assumption of induction and the decomposition

\[
a_i = \sum_{n=0}^{\infty} \mu^n S_1^a (1) S_1^a
\]

4
Now we have to prove the converse statement, i.e. that any $C^*$-algebra generated by partial isometries satisfying (5) can be generated by the elements satisfying (4). Let us consider the unital $C^*$-algebra $A_0$ generated by the operators $t_i$, $t_i^*$, $i = 1, \ldots, d$, satisfying relations (5). Note that $t_i$, $i = 1, \ldots, d$, are partial isometries. Indeed we have

$$t_it_i^* = t_i(1 - \sum_{j<i} t_jt_j^*) = t_i.$$

For any $i = 1, \ldots, d$ define a family $\{a_i^{(j)}\}$ inductively:

$$a_i^{(1)} = (\sum_{n=1}^{\infty} \mu^n t_i^n t_i^*)^k t_i,$$ (3)

$$a_i^{(j)} = (\sum_{n=0}^{\infty} \mu^n t_i^n a_i^{(j-1)} t_i^n, j = 1, \ldots, i - 1.$$ (3)

We shall use the following evident decomposition also

$$a_i^{(j)} = \sum_{n_j, \ldots, n_{i-1} = 0}^{\infty} \mu^{n_j + n_{j+1} + \cdots + n_{i-1}} a_i^{(i)} t_i^{n_{i-1}} \cdots t_i^{n_{j}}$$

Denote $a_i^{(1)} := a_i$. Our goal is to show that $a_i$, $a_i^*$ satisfy the relations (4) and $S_i(a_1, \ldots, a_d) = t_i$, $i = 1, \ldots, d$. To do it we prove a few auxiliary lemmas.

**Lemma 1.** $(1 - t_1t_1^* - \cdots - t_jt_j^*)a_i^{(j)} = a_i^{(j+1)}$

**Proof.** In the following we denote $P_j := 1 - \sum_{l\leq j} t_jt_j^*$, $P_0 := 1$. It is easy to see that $P_j t_k = 0$, $k \leq j$, and $P_j t_k = t_k$, $k > j$.

Then

$$P_j t_i^{n_j} \cdots t_i^{n_{i-1}} = \begin{cases} t_i^{n_j+1} \cdots t_i^{n_{i-1}}, & n_j = 0, \forall n_l \neq 0, j + 1 \leq l \leq i - 1 \\ P_j, & n_l = 0, l = j, \ldots, i - 1 \end{cases}$$

Then

$$P_j a_i^{(j)} = \sum_{n_j, \ldots, n_{i-1} = 0}^{\infty} \mu^{n_j + n_{j+1} + \cdots + n_{i-1}} P_j t_j^{n_j} \cdots t_i^{n_{i-1}} a_i^{(i)} t_i^{n_{i-1}} \cdots t_j^{n_j}$$

$$= P_j a_i^{(i)} + \sum_{n_{j+1}, \ldots, n_{i-1} = 0, \sum_n n_l^2 \neq 0}^{\infty} \mu^{n_{j+1} + \cdots + n_{i-1}} t_j^{n_j+1} \cdots t_i^{n_{i-1}} a_i^{(i)} t_i^{n_{i-1}} \cdots t_j^{n_j+1}$$

$$= \sum_{n_j+1, \ldots, n_{i-1} = 0}^{\infty} \mu^{n_j+1 + \cdots + n_{i-1}} a_i^{(i)} t_i^{n_{i-1}} \cdots t_j^{n_j+1} = a_i^{(j+1)}$$

\[ \square \]
Where we have used that $P_j a^{(i)}_i = a^{(i)}_i$, $j < i$. Indeed

$$a^{(i)}_i = T_i t_i, \quad T_i^2 = \sum_{n=1}^{\infty} \mu^{2(n-1)} t^n_i t^*_i$$

and $t_k t^*_k T_i^2 = T_i^2 t_k t^*_k = 0$, $i \neq k$ implies $t^*_k T_i = 0$, $i \neq k$, hence $t^*_k a^{(i)}_i = 0$ and

$$P_j a^{(i)}_i = (1 - \sum_{k \leq j} t_k t^*_k)a^{(i)}_i = a^{(i)}_i, \quad j < i.$$  

□

**Corollary 1.** $P_k a^{(j+1)}_i = a^{(j+1)}_i$, $k \leq j$

*Proof.* We note only that $P_k P_j = P_j$, $k \leq j$.

□

**Lemma 2.** $t^*_k a^{(j+1)}_i = 0$, $a^{(j+1)}_i t_k = 0$, $t^*_k a^{*(j+1)}_i = 0$, $a^{*(j+1)}_i t_k = 0$, for any $k \leq j < i$.

*Proof.* As in the previous lemma we have $t^*_k a^{(j+1)}_i = t^*_k a^{(i)}_i = 0$ and $a^{(j+1)}_i t_k = a^{(i)}_i t_k = T_i t_i t_k = 0$ since $t, t_k = 0$, $i > k$. The other relations are adjoint to the proved above.

□

**Lemma 3.**

$$t^*_j t^*_m = \begin{cases} t^*_j t^*_{m-n}, n > m \\ P_j, n = m \\ t^*_m t^*_n, n < m \end{cases}$$

*Proof.* Induction on $n, m$ using the basic relations (2).

□

Now we are able to prove the following proposition.

**Proposition 3.** For any $i = 1, \ldots, d$ and $1 \leq j \leq i$ we have

$$a^{*(j)}_i a^{(j)}_i = P_{j-1} + \mu^2 a^{(j)}_i a^{*(j)}_i - (1 - \mu^2) \sum_{j \leq k < i} a^{(j)}_k a^{*(j)}_k$$

*Proof.* We use the induction on $j$ for a fixed $i = 1, \ldots, d$.

For $j = i$ we have

$$a^{*(i)}_i a^{(i)}_i = t^*_i \left( \sum_{n=1}^{\infty} \mu^{2(n-1)} t^n_i t^*_i \right) t_i$$

$$= t^*_i t_i + \mu^2 \sum_{n=1}^{\infty} \mu^{2(n-1)} t^n_i t^*_i$$

$$= P_{i-1} + \mu^2 a^{(i)}_i a^{*(i)}_i$$

□
Lemma 5. \( a_i^{*(j)} a_i^{(j)} = \sum_{n,m=0}^{\infty} \mu^{n+m} \gamma_j a_i^{*(j+1)} t_j^n a_i^{(j+1)} t_j^m = \sum_{n=0}^{\infty} \mu^{2n} t_j^n a_i^{*(j+1)} a_i^{(j+1)} t_j^n \)

\[
= \sum_{n=0}^{\infty} \mu^{2n} t_j^n (P_j + \mu^2 a_i^{*(j+1)} a_i^{(j+1)}) - (1 - \mu^2) \sum_{j+1 \leq k < i} a_k^{(j+1)} a_k^{(j)} t_j^n \\
= \sum_{n=0}^{\infty} \mu^{2n} t_j^n P_j t_j^n + \mu^2 a_i^{*(j)} a_i^{(j)} - (1 - \mu^2) \sum_{j+1 \leq k < i} a_k^{(j)} a_k^{(j)} \\
= P_j - (1 - \mu^2) \sum_{n=1}^{\infty} \mu^{2(n-1)} t_j^n a_i^{*(j)} a_i^{(j)} - (1 - \mu^2) \sum_{j+1 \leq k < i} a_k^{(j)} a_k^{(j)} \\
= P_j - (1 - \mu^2) \sum_{j \leq k < i} a_k^{(j)} a_k^{(j)}
\]

Particulary, for \( j = 1 \) we have

\[
\overline{a}^*_i a_i = 1 + \mu^2 \overline{a}^*_i \overline{a}_i - (1 - \mu^2) \sum_{k < i} a_k \overline{a}^*_k \]

It remains to show that \( \overline{a}^*_i \overline{a}_j = \mu \overline{a}^*_j \overline{a}_i \). Then \( \overline{a}^*_j \overline{a}_i = \mu \overline{a}^*_i \overline{a}_j, j > i \), hold automatically (see \( \boxdot \)).

Lemma 4. \( a_i^{*(k)} a_j^{(j)} = 0, j < k \leq i \).

Proof. We use induction again. For \( k = i \) one has

\[
a_i^{*(i)} a_j^{(j)} = t_i^* T_i T_j = 0
\]

since \( T_i^2 T_j = 0, i \neq j \), and \( T_i, T_j \geq 0 \).

\( k + 1 \rightarrow k \).

\[
a_i^{*(k)} a_j^{(j)} = \sum_{n=0}^{\infty} \mu^n t_k^n a_i^{*(k+1)} t_k^n a_j^{(j)} = a_i^{*(k+1)} a_j^{(j)} = 0
\]

since \( t_k^* a_j^{(j)} = 0, k > j \) (see Lemma \( \boxdot \)).

Lemma 5. \( a_i^{*(j)} a_j^{(j)} = a_j^{(j)} a_i^{*(j)} \), j < i.
Proof. Let us show that $a_i^{*\,(j)} T_j^2 = T_j^2 a_i^{*\,(j)}$ and $a_i^{*\,(j)} t_j = \mu t_j a_i^{*\,(j)}$.

\[
\begin{align*}
a_i^{*\,(j)} T_j^2 &= (\sum_{n=0}^{\infty} \mu^n t_j^n a_i^{*\,(j+1)} t_{j}^n)(\sum_{m=1}^{\infty} \mu^{2(m-1)} t_j^m t_{j}^n) \\
&= \sum_{n=0, m=1}^{\infty} \mu^{n+2(m-1)} t_j^n a_i^{*\,(j+1)} t_{j}^n t_j^m t_{j}^n \\
&= \sum_{n=1, m \leq n}^{\infty} \mu^{n+2(m-1)} t_j^n a_i^{*\,(j+1)} t_{j}^n .
\end{align*}
\]

where we have used Lemmas 2, 3. Analogously

\[
T_j^2 a_i^{*\,(j)} = \sum_{n=1, m \leq n}^{\infty} \mu^{n+2(m-1)} t_j^n a_i^{*\,(j+1)} t_{j}^n = a_i^{*\,(j)} T_j^2 .
\]

Finally

\[
\begin{align*}
a_i^{*\,(j)} t_j &= (\sum_{n=0}^{\infty} \mu^n t_j^n a_i^{*\,(j+1)} t_{j}^n) t_j \\
&= \mu t_j a_i^{*\,(j+1)} t_{j}^n + \sum_{n=2}^{\infty} \mu^n t_j^n a_i^{*\,(j+1)} t_{j}^{n-1} \\
&= \mu t_j a_i^{*\,(j+1)} t_{j}^n + \mu t_j \sum_{n=1}^{\infty} \mu^n t_j^n a_i^{*\,(j+1)} t_{j}^n \\
&= \mu t_j \sum_{n=0}^{\infty} \mu^n t_j^n a_i^{*\,(j+1)} t_{j}^n = \mu t_j a_i^{*\,(j)} .
\end{align*}
\]

Then

\[
a_i^{*\,(j)} a_j^{*\,(j)} = a_i^{*\,(j)} T_j t_j = \mu T_j t_j a_i^{*\,(j)} = \mu a_j^{*\,(j)} a_i^{*\,(j)} , \ i > j .
\]

\[\square\]

Lemma 6. $a_i^{*\,(k)} a_j^{*\,(k)} = \mu a_j^{*\,(k)} a_i^{*\,(k)} , 1 \leq k < j < i$

Proof. We use induction. The case $k = j$ is considered in the Lemma above. $k+1 \rightarrow k$. As in the Proposition 3 we have

\[
\begin{align*}
a_i^{*\,(k)} a_j^{(k)} &= \sum_{n=0}^{\infty} \mu^{2n} t_k^n a_i^{*\,(k+1)} a_j^{(k+1)} t_k^n \\
&= \mu \sum_{n=0}^{\infty} \mu^{2n} t_k^n a_j^{(k+1)} a_i^{*\,(k+1)} t_k^n = \mu a_j^{(k)} a_i^{*\,(k)} .
\end{align*}
\]

Particularly, for $k = 1$ we have $\tilde{a_i}^* \tilde{a_j} = \mu \tilde{a_j} \tilde{a_i}^* , i > j .
\]

\[\square\]
So, we have proved the following theorem.

**Theorem 1.** Let \( A_0 = C^*(t_i, t^*_i, i = 1, \ldots, d) \) where \( \{t_i, t^*_i, i = 1, \ldots, d\} \) satisfy relations (3), and the family \((\tilde{a}, \tilde{a}^*_i, i \geq 1)\) is constructed according to formulas (3). Then the relations (1) are satisfied and we have \( \hat{S}_i(\tilde{a}_1, \ldots, \tilde{a}_d) = t_i, \ i = 1, \ldots, d. \)

**Corollary 2.** For any \( \mu \in (-1, 1) \) the \( C^*\)-algebra \( A_\mu \) is isomorphic to \( A_0. \)

**Proof.** Using the universal property of \( A_0 \) we can define the surjective homomorphism \( \varphi: A_0 \rightarrow A_\mu \) by rule \( \varphi(t_i) = \hat{S}_i, \ i = 1, \ldots, d. \) Analogously, we have \( \psi: A_\mu \rightarrow A_0, \ \psi(a_i) = a_i^{(1)}, \ i = 1, \ldots, d. \) Obviously, \( \psi \varphi = \text{id} \) and \( \varphi \psi = \text{id}. \)

## 2 Fock representation.

Recall that Fock representation of TCCR is the irreducible representation determined by the cyclic vector \( \Omega \) such that \( a^*_i \Omega = 0, \ i = 1, \ldots, d. \)

Let us prove that Fock representation of \( A_\mu \) is faithful. Firstly note that Fock representation of \( A_0 \) corresponds to the Fock representation of \( A_\mu \) (it can be easily seen from the formulas connecting \( \{t_j\} \) and \( \{a_j\} \)).

In the following we need the description of classes of unitary equivalence of irreducible representations of \( A_0. \) As we have noted above, the irreducible representations of TCCR, including unbounded representations, were classified in [8]. However it is more convenient for us to present the representations of \( A_0 \) in some different form.

**Proposition 4.** Let \( \pi \) be an irreducible representation of \( A_0 \) acting on the Hilbert space \( \mathcal{H}, \) then for some \( j = 1, \ldots, d \) we have \( \mathcal{H} \cong \bigotimes_{k=1}^{i-1} l_2(\mathbb{N}) \) and

\[
\pi(t_i) = \bigotimes_{k=1}^{i-1} (1 - SS^*) \otimes S \otimes \bigotimes_{k=i+1}^{j} 1, \ i \leq j
\]

\[
\pi(t_{j+1}) = e^{i\varphi} \bigotimes_{k=1}^{j} (1 - SS^*), \ \varphi \in [0, 2\pi)
\]

\[
\pi(t_i) = 0, \ i > j + 1,
\]

where \( S \) is a unilateral shift on \( l_2(\mathbb{N}). \) The case \( j = d \) corresponds to the Fock representation.

**Proof.** It follows from (2) that \( \pi(t_1) \) is isometry. Hence, either \( \ker \pi(t_1^*) \neq \{0\} \) or \( \pi(t_1) \) is unitary. We shall use here \( t_i \) instead \( \pi(t_i). \)

Let \( \ker t_i^* = \mathcal{H}_1 \neq \{0\}. \) Then the relations (3) imply that \( \mathcal{H} = \bigoplus_{n=0}^{\infty} t_i^n \mathcal{H}_1 \) and

\[
t_i, t^*_i: \mathcal{H}_1 \rightarrow \mathcal{H}_1, \ t_i, t^*_i: t_i^n \mathcal{H}_1 \rightarrow \{0\}, \ n > 1, \ i > 1.
\]
If we identify \( t_1^n \mathcal{H}_1 \) with \( e_n \otimes \mathcal{H}_1, n \geq 0 \), then
\[
t_1 = S \otimes 1, \quad t_i = (1 - SS^*) \otimes t_i^{(1)}, \quad i > 1
\]
where \( Se_n = e_{n+1}, n \geq 0 \) and the family \( \{ t_i^{(1)} \}, i > 1 \) satisfy (2) on the space \( \mathcal{H}_1 \). Moreover, it is easy to show that the family \( \{ t_i, i = 1, \ldots, d \} \) is irreducible iff \( \{ t_i^{(1)}, i = 2, \ldots, d \} \) is irreducible.

If \( t_1 \) is unitary, then \( t_i t_1 = 0, i > 1, \) implies \( t_i = 0. \)

Using the previous proposition we can prove the following theorem.

**Theorem 2.** The Fock representation of \( A_0 \) is faithful.

**Proof.** Let \( C_F \) be the \( C^* \)-algebra generated by the operators of Fock representation and \( C_\pi \) be the \( C^* \)-algebra generated by some irreducible representation \( \pi \) of \( A_0 \). To prove the statement it is sufficient to construct a homomorphism
\[
\psi: C_F \to C_\pi
\]
such that \( \pi = \psi \pi_F \) (then \( \pi(\ker \pi_F) = \{0\} \) for any irreducible representation of \( A_0 \), i.e. \( \ker \pi_F = \{0\} \), where we denote by \( \pi_F \) the Fock representation).

To do it, we note that if \( \pi \) corresponds to some \( j = 1, \ldots, d - 1 \), then \( C_F \) and \( C_\pi \) are the \( C^* \)-subalgebras of the \( \bigotimes_{k=1}^d C^*(S,S^*) \) and \( \bigotimes_{k=1}^j C^*(S,S^*) \) respectively. Recall that \( C^*(S,S^*) \simeq \mathcal{T}(C(T)) \) is a nuclear \( C^* \)-algebra of the Toeplitz operators. Then we can define the homomorphism
\[
\psi: \bigotimes_{k=1}^d C^*(S,S^*) \to \bigotimes_{k=1}^j C^*(S,S^*),
\]
defined by
\[
\psi(\bigotimes_{k=1}^{i-1} 1 \otimes S \otimes_{k=i+1}^d 1) = \bigotimes_{k=1}^{i-1} 1 \otimes S \otimes_{k=i+1}^d 1, \quad i \leq j
\]
\[
\psi(\bigotimes_{k=1}^j 1 \otimes S \otimes_{k=i+1}^d 1) = e^{i \varphi} \otimes_{k=i+1}^d 1,
\]
\[
\psi(\bigotimes_{k=1}^{i-1} 1 \otimes S \otimes_{k=i+1}^d 1) = \bigotimes_{k=1}^d 1, \quad i > j
\]
It remains only to restrict \( \psi \) onto \( C_F \) and to note that \( \psi(C_F) = C_\pi. \)

**Acknowledgements.**

We express our gratitude to Vasyl Ostrovskyi and Stanislav Popovich for their critical remarks and helpful discussions.

This work was partially supported by the State Fund of Fundamental Researches of Ukraine, grant no. 01.07/071.
References

[1] K. Dykema and A.Nica. On the Fock representation of the q-commutation relations. *J. Reine Angew. Math.* **440** (1993), 201–212.

[2] Bożejko, M. and Speicher, R.: Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces. *Mat. Ann.* **300** (1994), 97–120.

[3] Jørgensen, P.E.T, Proskurin, D.P. and Samoilenko, Yu.S.: The kernel of Fock representation of Wick algebras with braided operator of coefficients. Accepted to publication in *Pacific. Journ. Math.*, [math-ph/0001011](http://arxiv.org/abs/math-ph/0001011).

[4] Jørgensen, P.E.T., Schmitt L.M., and Werner, R.F.: q-canonical commutation relations and stability of the Cuntz algebra. *Pacific Journ. Math.* **163**, no. 1 (1994), 131–151

[5] Jørgensen, P.E.T. Schmitt, L.M. and Werner, R.F.. Positive representations of general commutation relations allowing Wick ordering. *J. Funct. Anal.* **134** (1995), 33-99.

[6] Khelemski˘ı, A.Ya.: Banach algebras and poly-normed algebras: general theory, representations, homologies. *Nauka, Moscow* (1989), (Russian).

[7] Ostrovsky˘ı, V. and Samoilenko, Yu.: Introduction to the Theory of Representations of Finitely Presented *-Algebras. I. Representations by bounded operators. *The Gordon and Breach Publishing group, London* (1999).

[8] Pusz, W. and Woronowicz, S.L.: Twisted second quantization. *Reports Math. Phys.* **27** (1989), 251–263.

[9] Proskurin, D.: Stability of a special class of $q_{ij}$-CCR and extensions of higher-dimensional noncommutative tori, to be printed in *Lett. Math. Phys.*

Kyiv Taras Shevchenko University, cybernetics department, Volodymyrska, 64, Kyiv, 01033, Ukraine
Institute of Mathematics National Academy of Sci., Tereschenkivska, 3, Kyiv, 01601, Ukraine

e-mail:
prosk@imath.kiev.ua
Yurii_Sam@imath.kiev.ua