Acoustic radiation pressure in laterally unconfined plane wave beams

John H Cantrell ©
NASA Langley Research Center, Hampton, Virginia 23681, United States of America
E-mail: john.h.cantrell@gmail.com

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Abstract
The acoustic radiation pressure in laterally unconfined, plane wave beams in inviscid fluids is derived via the direct application of finite deformation theory for which an analytical accounting is made ab initio that the radiation pressure is established under static, laterally unconstrained conditions, while the acoustic wave that generates the radiation pressure propagates under dynamic (sinusoidal), laterally constrained conditions. The derivation reveals that the acoustic radiation pressure for laterally unconfined, plane waves along the propagation direction is equal to \( \langle 3/4 K \rangle \), where \( \langle K \rangle \) is the mean kinetic energy density of the wave, and zero in directions normal to the propagation direction. The results hold for both Lagrangian and Eulerian coordinates. The value \( \langle 3/4 K \rangle \) differs from the value \( \langle 2K \rangle \), traditionally used in the assessment of acoustic radiation pressure, obtained from the Langevin theory or from the momentum flux density in the Brillouin stress tensor. Errors in traditional derivations leading to the Brillouin stress tensor and the Langevin radiation pressure are pointed out and a long-standing misunderstanding of the relationship between Lagrangian and Eulerian quantities is corrected. The present theory predicts a power output from the transducer that is 4/3 times larger than that predicted from the Langevin theory. Tentative evidence for the validity of the present theory is provided from measurements previously reported in the literature, revealing the need for more accurate and precise measurements for experimental confirmation of the present theory.

1. Introduction

The radiation pressure generated by an acoustic wave is used in a variety of applications such as acoustic radiation force-based elasticity imaging [1–9], acoustophoretic drug delivery [10], acoustic tweezers [11–15], the characterization of atomic force microscope cantilevers [16, 17], and the calibration of ultrasonic transducers [18–24]. The search for the proper understanding of radiation pressure in acoustic fields has been controversial and elusive since the pioneering efforts of Lord Rayleigh [25], Brillouin [26, 27], Hertz and Mende [28], and Langevin [29, 30] in the early twentieth century and has continued to the present time [31–56]. As pointed out by Beyer [49], the controversy is fueled by confusion arising from differing definitions, faulty assumptions, and simplifying idealizations among other factors. Much of the confusion results from a widespread misunderstanding of Lagrangian and Eulerian coordinates and of the transformation between Lagrangian and Eulerian quantities.

It has long been known that radiation pressure in fluids is highly dependent on whether motion of fluid normal to the wave propagation direction is allowed—i.e., on whether the acoustic beam is laterally confined or laterally unconfined. The focus of the present work is to understand acoustic radiation pressure in a progressive, laterally unconfined, plane wave, acoustic beam via the direct application of finite deformation theory. The derivation shows that for both Lagrangian and Eulerian coordinates the radiation pressure for plane waves along the propagation direction is equal to \( \langle 3/4 K \rangle \), where \( \langle K \rangle \) is the mean kinetic energy density of the acoustic...
wave. This result differs considerably from the value \(2K\) for the radiation pressure obtained from the Langevin theory [29] or from the Brillouin stress tensor [26, 27]. In the directions normal to the propagation direction, the radiation pressure is zero for both Lagrangian and Eulerian coordinates.

The difference between the present assessment of radiation pressure and that obtained from the Langevin theory or from the Brillouin stress tensor is significant, since the assessment is used to link the radiation force on a target with the power generated by acoustic sources. As pointed out by Beissner [21], the ‘measured radiation force must be converted to the ultrasonic power value and this is carried out with the help of theory.’ It is generally assumed that for laterally unconfined, plane wave beams the relationship between the acoustic radiation pressure and the energy density for plane waves in the direction of wave propagation is that obtained from the Langevin theory [29] or from the Brillouin stress tensor [26, 27]. The present model predicts a transducer output power \(4/3\) times larger than that predicted from the Langevin theory or from the Brillouin stress tensor. This has considerable implications regarding safety issues for medical transducers, calibrated using radiation pressure.

Issenmann et al [52] point out that ‘despite the long-lasting theoretical controversies … the Langevin radiation pressure … has been the subject of many very few experimental studies.’ Indeed, absolute measurements obtained independently in the same experiment of the acoustic power generated by an acoustic source and the radiation force incident on a target to assess the radiation pressure-energy density relationship are relatively rare. If the transducer acoustic output power is calibrated from the measured radiation force and the Langevin theory is used as the radiation pressure-energy density relationship in the calibration (directly or indirectly via a secondary standard), then measurements taken with the transducer necessarily reflect (and, hence, by default ‘confirm’) the Langevin theory. Thus, radiation pressure measurements taken with ‘Langevin-calibrated’ transducers cannot in turn be used to validate the Langevin theory. Indeed, radiation pressure measurements taken with a transducer calibrated against a measured radiation force using any assumed radiation pressure-energy density relationship will necessarily reflect, and by default ‘confirm,’ the assumed pressure-energy density relationship. The correct radiation pressure-energy density relationship can only be validated from absolute, independent measurements of the radiation pressure and energy density in a single experiment under identical conditions. Such an experiment has been reported by Breazeale and Dunn [23] of the absolute pressure amplitude of a progressive, plane wave beam using a force balance, three different optical techniques, a thermoelectric probe, and a direct assessment of the transducer output utilizing the piezoelectric constant. Experiments by Haran et al [24], have also been reported employing Raman–Nath diffraction measurements of the intensity of a progressive, plane wave impinging on a calibrated force balance. The measurements of Breazeale and Dunn [23] and Haran et al [24] are shown to provide tentative evidence for the validity of the present model, but reveal that more accurate and precise measurements are necessary for model confirmation.

A critical analysis of Lagrangian and Eulerian coordinates and quantities is presented in section 2. The analysis is necessary to understand errors made in previous derivations of the acoustic radiation pressure and to provide the framework for a radical (though mathematically punctilious and physically realistic) departure from previous approaches to resolve the long-lasting confusion surrounding the subject. Various widespread misconceptions concerning Lagrangian and Eulerian quantities and the relationships between them are pointed out—in particular, those related to Lagrangian and Eulerian pressures. A derivation via the direct application of finite deformation theory of the acoustic radiation pressure for laterally unconfined, plane wave beams is given in section 3.1 where, in contrast to previous derivations, an analytical accounting is made \textit{ab initio} that the radiation pressure is established under static, laterally unconstrained conditions, while the acoustic wave that generates the radiation pressure propagates under dynamic (sinusoidal), laterally constrained conditions. Section 3.2 presents an assessment of the radiation pressure using the approach of Brillouin [26, 27], who employed the Boltzmann-Ehrenfest Principle of Adiabatic Invariance in his derivation. It is shown that the terms in the Brillouin stress tensor are incorrectly referred to Eulerian coordinates and do not apply to laterally unconfined beams. Section 4 addresses various derivations of the Langevin relation for the radiation pressure. It is shown that the derivations improperly assume that the pressure in question is the Eulerian pressure rather than, correctly, the thermodynamic pressure (thermodynamic tensions or second Piola-Kirchhoff stress). The present theory is shown in section 5 to predict a power output from the transducer that is \(4/3\) times larger than that predicted from the Langevin theory. Tentative experimental evidence for the validity of the present model is presented.

2. Elements of finite deformation theory

In contrast to previous derivations, the present derivation of the acoustic radiation pressure for laterally unconfined, plane wave beams analytically accounts \textit{ab initio} and \textit{a priori} that the radiation pressure is established under static, laterally unconstrained conditions, while the acoustic wave that generates the radiation pressure propagates under dynamic (sinusoidal), laterally constrained conditions. The present model is based on
the application of finite deformation theory to derive the relevant relationships between Lagrangian and Eulerian quantities. The derivation of the relationships via finite deformation theory is presented in some detail (a) to avoid the misinterpretation of mathematical operations and terms that has led to much of the confusion and controversy in the literature, (b) to provide the analytical underpinning critical to the derivation of the radiation pressure-energy density relationship, and (c) to allow the identification of errors made in previous derivations of the acoustic radiation pressure. The relationships between Lagrangian and Eulerian quantities are central to the theory of finite deformations, which was originally developed by Murnaghan [57], codified as a field theory by Truesdell and Toupin [58], Truesdell and Noll [59], and applied to acoustic wave propagation by Truesdell [60], Thurston [61], Thurston and Brugger [62], Thurston and Shapiro [63], and Wallace [64]. Finite deformation theory applies to any material of arbitrary crystalline symmetry including ideal fluids, which can be viewed as an isotropic material with zero shear modulus.

2.1. Lagrangian and Eulerian coordinates

Consider a material for which the initial (rest) configuration of particles comprising the material body is denoted by the set of position vectors \( \{ \mathbf{X} \} = \{ X_1, X_2, X_3 \} \) in a three-dimensional Cartesian reference frame having unit vectors \( \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \) along the coordinate axes. The \( \{ X_1, X_2, X_3 \} \) coordinates are known as Lagrangian or material (initial or rest) coordinates. Under an impressed stress the positions of the material particles will move from the initial (rest) set of vectors \( \{ \mathbf{X} \} \) to new positions described by the set of position vectors \( \{ \mathbf{x} \} = \{ x_1, x_2, x_3 \} \) in the same three-dimensional Cartesian reference frame. The \( \{ x_1, x_2, x_3 \} \) coordinates are known as Eulerian or spatial (present) coordinates in the Cartesian reference frame. It is assumed that \( \mathbf{x} \) and \( \mathbf{X} \) are functionally related as \( \mathbf{x} = \mathbf{x}(\mathbf{X}, t) \) where \( t \) is time. The present configuration of particles \( \{ \mathbf{x} \} \) is then related to the initial configuration \( \{ \mathbf{X} \} \) by means of an elastic deformation defined by the set of transformation (deformation) coefficients \( \alpha_{ij} = \partial x_i / \partial X_j \), where \( x_i \) and \( X_j \), respectively, are the Cartesian components of the vectors \( \mathbf{x} \) and \( \mathbf{X} \). The indices \( i \) and \( j \) take the values 1, 2, 3 representing the three mutually orthogonal Cartesian axes. An elemental length \( dx \) in the Lagrangian coordinates is transformed to an elemental length \( dx \) in the Eulerian coordinates as \( dx = \alpha_{ij} dx_i \). The Einstein convention of summation over repeated indices is used in the present work. The inverse deformation is described by the set of transformation coefficients \( \gamma_{ij} \) defined such that \( \gamma_{ij} \alpha_{jk} = \delta_{ik} \), where \( \delta_{ik} \) is the Kronecker delta. If the deformation is non-uniform (i.e., varies with spatial position), the deformation is considered to be local in \( \mathbf{x} \) and time \( t \).

The deformation is defined by following the motion of a given particle originally at rest in the Lagrangian position \( \mathbf{X} \), which during deformation is displaced to the Eulerian position \( \mathbf{x} \). The particle displacement \( \mathbf{u} \) is defined by \( \mathbf{u} = \mathbf{x} - \mathbf{X} \). The transformation coefficients \( \alpha_{ij} \) are related to the displacement gradients \( \partial u_i / \partial X_j \) as

\[
\alpha_{ij} = \delta_{ij} + u_{ij}.
\]  (1)

For finite deformations Murnaghan [57] pointed out that the Lagrangian strains \( \eta_{ij} \) defined as

\[
\eta_{ij} = \frac{1}{2} (\alpha_{ik} \alpha_{kj} - \delta_{ij}) = \frac{1}{2} (u_{ij} + u_{ji} + u_{ki} u_{kj})
\]  (2)

are rotationally invariant and provide an alternative to the displacement gradients \( u_{ij} \) as a strain measure. Equations (1) and (2) hold for any material system having arbitrary crystalline symmetry - solid or fluid.

2.2. Lagrangian and Eulerian quantities

A physical quantity \( q \) in the deformed state but referred to the Lagrangian (initial, rest, or un-deformed state) coordinates at time \( t \) is defined as the Lagrangian quantity \( q^L(\mathbf{X}, t) \). The same quantity referred to the Eulerian (present or deformed state) coordinates at the same time \( t \) is defined as the Eulerian quantity \( q^E(\mathbf{x}, t) \). Since \( q^L(\mathbf{X}, t) \) and \( q^E(\mathbf{x}, t) \) represent the same physical quantity \( q \) in the deformed state at the same position \( \mathbf{x} = \mathbf{x}(\mathbf{X}, t) = \mathbf{X} + \mathbf{u}(t) \) and same time \( t \) in Cartesian space, the relationship between the Lagrangian and Eulerian expressions of that quantity must necessarily be [61]

\[
q^L(\mathbf{X}, t) = q^E(\mathbf{x}, t)|_{\mathbf{x}=\mathbf{X}+\mathbf{u}(t)}
\]  (3)

or, inversely, as

\[
q^E(\mathbf{x}, t) = q^L(\mathbf{X}, t)|_{\mathbf{X}(t)=\mathbf{x}-\mathbf{u}(t)}.
\]  (4)

It is generally assumed in traditional derivations of the radiation pressure that in equation (3), for example, the Lagrangian quantity is \( q^L(\mathbf{X}, t) \) and that the relevant Eulerian quantity is not \( q^E(\mathbf{x}, t)|_{\mathbf{x}=-\mathbf{X}+\mathbf{u}(t)} \) but, rather, the first term \( q^E(\mathbf{x}, t)|_{\mathbf{x}=\mathbf{X}} \) in a power series expansion of \( q^E(\mathbf{x}, t)|_{\mathbf{x}=\mathbf{X}+\mathbf{u}(t)} \) in terms of the displacement \( \mathbf{u} \). This cannot be true, because equation (3) already states that the Lagrangian quantity \( q^L(\mathbf{X}, t) \) is equal to the Eulerian quantity \( q^E(\mathbf{x}, t)|_{\mathbf{x}=\mathbf{X}+\mathbf{u}(t)} \) at the same point in Cartesian space at all times \( t \) [61]. Indeed, the Lagrangian quantity \( q^L(\mathbf{X}, t) \) corresponds to the value of the quantity \( q^L \) in the deformed state at the Eulerian position \( \mathbf{x} \) at time \( t \) that
previously had the value \(q^t(X, t_0)\) in the un-deformed state at the initial (Lagrangian) position \(X\) at the initial time \(t_0\). Equation (3) states that both the Lagrangian quantity and the Eulerian quantity involve the same particle that initially is in the un-deformed position \(X\) at time \(t_0\) in Cartesian space but has moved at time \(t\) from the un-deformed position \(X\) to the deformed position \(x(t) = X + u(t)\) in Cartesian space [61].

A Eulerian quantity represents the value of a quantity associated with a particle in the present (deformed) position. Equation (4) states that the value of a Eulerian quantity at position \(x\) at time \(t\) corresponds to that of a particle whose present position at \(x\) originates from some un-deformed (Lagrangian) position \(X\), but at a different time \(t'\) the particle that appears at \(x\) is a different particle, originating from a different un-deformed (Lagrangian) position \(X'\). More importantly, both equations (3) and (4) state that the Lagrangian and Eulerian quantities corresponding to the same position in Cartesian space are exactly equal at all times \(t\). A more complete analysis of the relationship between Lagrangian and Eulerian quantities is given in [65], where it is shown that the correct transformation between Lagrangian and Eulerian quantities must be obtained via the transformation coefficients \(\alpha_{ij}\) - not the displacement \(u\). It is also important to note that the quantity \(q\) in equations (3) and (4) is assumed to be singly defined and is generally treated as a scalar (such as mass density being singly defined as a mass per unit volume). For tensor quantities, such as stress or pressure, the relationship between Lagrangian and Eulerian quantities can be more complicated, as shown in section 2.4.

2.3. Mass density in Lagrangian and Eulerian coordinates

A direct application of the transformation coefficients given in equation (1) for an initially un-deformed volume of material results in the well-known relationship [65]

\[
\rho_0 \rho = \det \alpha_{ij} = 1 + u_{11} + u_{22} + u_{33} + u_{11}u_{22} + u_{11}u_{33} + u_{22}u_{33} + u_{11}u_{22}u_{33}. \tag{5}
\]

where \(\rho_0\) is the mass density in the initial (un-deformed) state, \(\rho\) is the mass density in the deformed state, and \(J\) is the Jacobian of the transformation defined as the determinant of the transformation coefficients \(\alpha_{ij}\). It is important to recognize that \(\rho_0\) is the mass density in the un-deformed state for both the Lagrangian and Eulerian coordinates and that \(\rho\) is the mass density in the deformed state for both the Lagrangian and Eulerian coordinates. This is apparent from equation (4), which states that the mass density \(\rho_0 = \rho^t(X, t_0) = \rho^t(X, t_0)|_{x=x-}\) is the mass density in the initial (un-deformed) state at time \(t_0\), where \(u(t_0) = 0\), and that \(\rho = \rho^t(X, t) = \rho^t(X, t)|_{x=x-u}\) is the mass density in the deformed state at time \(t\), where \(u(t) \neq 0\). Similarly, from equation (3) \(\rho_0 = \rho^t(X, t_0) = \rho^t(X, t_0)|_{x=x+X}\), where \(u(t_0) = 0\), and \(\rho = \rho^t(X, t) = \rho^t(X, t)|_{x=x+X+u}\), where \(u(t) \neq 0\). Thus, both equations (3) and (4) state that the mass density \(\rho\) has the same value at the same point and time in Cartesian space whether referred to Lagrangian or Eulerian coordinates, since for either coordinates the mass density refers to the same state of deformation at a given point and time \(t\). Different values for the mass density in the two coordinates are obtained, when, as often occurs in the acoustics literature, the first term in a power series expansion of equations (3) or (4) in terms of the displacement \(u\) is assumed to represent the relevant conjugate density. In view of the equalities in equations (3) and (4), this assumption is clearly incorrect. Although equation (5) provides an expression of \(\rho\) as a function of the displacement gradients, which are referred to the Lagrangian coordinates, this does not mean that \(\rho\) in equation (5) now becomes exclusively the Lagrangian mass density, as often assumed. It is still the mass density in the deformed state for both Lagrangian and Eulerian coordinates in accordance with equations (3) and (4).

2.4. Stress in Lagrangian and Eulerian coordinates

Stress is defined in terms of the derivative of the internal energy per unit volume with respect to the relevant strain measure (a second rank tensor), which leads to the stress-strain relationships. Stress is thus a second rank tensor - not a scalar as is the mass density. The internal energy per unit mass \(U(X, S_m)\), of material depends on the relative positions of the particles comprising the material and the entropy per unit mass \(S_m\). This means that the internal energy per unit volume \(\phi = \rho_0 U(X, S_m)\), from which the stress is obtained by differentiation, can be expressed as a function of the displacement gradients \(u_{ij}\) or as a function of the Lagrangian strains \(\eta_{ij}\) as [64]

\[
\phi = \rho_0 U(X, \eta_{ij}, S_m) = \rho_0 U(X, 0, S_m) + C_{ij} \eta_{ij} + \frac{1}{2} C_{ijkl} \eta_{ij} \eta_{kl} + \frac{1}{3!} C_{ijklpq} \eta_{ij} \eta_{kl} \eta_{pq} + \cdots = \rho_0 U(X, u_{ij}, S_m) = \rho_0 U(X, 0, S_m) + A_{ij} u_{ij} + \frac{1}{2} A_{ijkl} u_{ij} u_{kl} + \frac{1}{3!} A_{ijklpq} u_{ij} u_{kl} u_{pq} + \cdots \tag{6}
\]

where \(A_{ij}, A_{ijkl}\), and \(C_{ijklpq}\), respectively, are the first, second, and third-order Huanc coefficients and \(C_{ij}, C_{ijkl}\) and \(C_{ijklpq}\) respectively, are the first, second, and third-order Brueger elastic constants [61, 64]. Substituting
equation (2) in equation (6) and comparing the coefficients of like powers of the displacement gradients yield the relations [64]

\[
A_{ij} = C_{ij} = T_{ij}(X) = (T_{ij})_0 = \sigma_{ij}(X) = (\sigma_{ij})_0
\]

(7)

\[
A_{ijkl} = T_{ijkl}(X) = (T_{ijkl})_0 + C_{ijkl}\delta_{ik}
\]

(8)

\[
A_{ijklpq} = C_{ijklpq}\delta_{ik} + C_{ijklq}\delta_{ip} + C_{ijklp}\delta_{iq} + C_{ijklq}\delta_{ipq}
\]

(9)

The first-order constants \( A_{ij} \) are the initial stresses at position \( x = X \) in the material and are denoted in various alternative ways in equation (7) that will become apparent below.

A stress is a force per unit area obtained by differentiating equation (6) with respect to the appropriate strain measure, \( \eta_{ij} \) or \( u_{ij} \). It is noted that while the strain is defined with respect to the initial state of the material (i.e., with respect to the Lagrangian coordinates), the force \( F_i \) is usually defined with respect to a unit area of deformed material (i.e., with respect to the Eulerian coordinates) [58, 61]. An exception is the thermodynamic tensions (second Piola-Kirchhoff stress) for which both the strain and the force are referred to the initial state [58, 61]. The stresses most relevant to acoustic wave propagation are the Eulerian (Cauchy) stresses and Lagrangian (first Piola-Kirchhoff) stresses. The Eulerian or Cauchy stress \( T_{ij} \) is the force per unit area referred to the present configuration. It is a force per unit area for which both the force and the area are referred to the deformed state \( x \) [58–64]. The Cauchy stresses, evaluated in the present (perturbed or deformed) configuration \( x \), are given in terms of the derivatives of the internal energy per unit volume with respect to the Lagrangian strains \( \eta_{ij} \) as [58–64]

\[
T_{ij} = J^{-1} \alpha_{ik} \alpha_{jl} \rho_0 \left( \frac{\partial U}{\partial \eta_{kl}} \right)_{x,s_n}.
\]

(10)

The Lagrangian or first Piola-Kirchhoff stress \( \sigma_{ij} \) is a stress for which the force is referred to the deformed state \( x \) but the area is referred to the initial state \( X \) of the material [58–64]. The Cauchy stresses \( T_{ij} \) are related to the first Piola-Kirchhoff stresses \( \sigma_{ij} \) as [58–65]

\[
\sigma_{ij} = J \gamma_{ij} T_{ij} = \rho_0 \alpha_{ik} \left( \frac{\partial U}{\partial \eta_{ik}} \right)_{x,s_n}.
\]

(11)

or, equivalently, as

\[
T_{ij} = \frac{1}{J} \alpha_{ik} \sigma_{jk}.
\]

(12)

Equations (11) and (12) reveal that the relationship between the Cauchy (Eulerian) and first Piola-Kirchhoff (Lagrangian) stresses is more complicated than that of the Lagrangian and Eulerian mass densities, given by equations (3) and (4). The complication results from the differing definitions of the Eulerian and Lagrangian stresses, in contrast to the single definition of the mass density as simply a mass per unit volume. Note that when evaluated at \( x = X \) (the initial or un-deformed state) equation (11) yields \( A_{ij} = T_{ij}(X) = \sigma_{ij}(X) \), the initial stress.

Brillouin [26, 27] preferred to use the Boussinesq stress tensor \( B_{ij} \), which is defined directly in terms of the derivatives of the internal energy per unit volume with respect to the displacement gradients \( u_{ij} \). The Boussinesq stress tensor is related to the first Piola-Kirchhoff stress tensor as [66]

\[
B_{ij} = \frac{\partial \varphi}{\partial u_{ij}} = \rho_0 \frac{\partial U}{\partial u_{ij}} = \rho_0 \frac{\partial \eta_{ik}}{\partial u_{ij}} \frac{\partial U}{\partial \eta_{kl}} = \rho_0 \alpha_{ik} \frac{\partial U}{\partial \eta_{kj}} = \sigma_{ij}.
\]

(13)

From equations (6) and (13)

\[
\sigma_{ij} = B_{ij} = A_{ij} + A_{ijkl} u_{ik} u_{jl} + \frac{1}{2} A_{ijklpq} u_{ik} u_{jl} u_{pq} + \cdots.
\]

(14)

It is extremely important to note that for purely longitudinal, plane wave propagation along the Cartesian direction \( e_1 \), the shear strains \( (\partial u_i / \partial X_j)_{e_1} = (u_{ij})_{e_1} = 0 \), \( T_{ij} \rightarrow T_{11} \), \( \sigma_{ij} \rightarrow \sigma_{11}, \alpha_{ij} \rightarrow \alpha_{11} = 1 + (\partial u_i / \partial X_1) = 1 + u_{11}, \alpha_{11} \rightarrow \alpha_{11} = 1 + (\partial u_i / \partial X_1) = 1 + u_{11}, \) and \((1/\alpha) = [1 + (\partial u_i / \partial X_1)]^{-1} = [1 + u_{11}]^{-1}\). Hence, \((1/\alpha) \alpha_{11} = 1\) and equation (12) simplifies without approximation to

\[
T_{11} = \sigma_{11}.
\]

(15)

Equation (15) states that for longitudinal, plane propagation the Lagrangian and Eulerian stresses are exactly equal. For fluids, the Cauchy stress component \( T_{11} \) is related to the Eulerian pressure \( p_1^E \) along \( e_1 \) as \( T_{11} = -p_1^E \), and the first Piola-Kirchhoff stress component \( \sigma_{11} \) is related to the Lagrangian pressure \( p_1^L \) along \( e_1 \) as \( \sigma_{11} = -p_1^L \) [57–64]. Equation (15) can thus be re-written as
\[ T_{i1} = -p_i^E = \sigma_{i1} = -p_i^L. \]  

Equation (16) shows, in contrast to previous derivations in the literature, that the Eulerian pressure \( p_i^E \) is exactly equal to the Lagrangian pressure \( p_i^L \) for longitudinal, plane wave propagation along \( e_1 \) in materials. The subscript ‘1’ in equation (16) denotes that the pressure corresponding to longitudinal, plane wave propagation along \( e_1 \) is the \( i = j = 1 \) component of the second rank tensors \( T_{ij} \) and \( \sigma_{ij} \). More generally, from equation (5) \( J^{-1} \) can be approximated as \([61, 64] J^{-1} \approx 1 - u_{\text{mean}}. \) Substituting \( J \) and \( \sigma_{ik} \) from equation (1) in equation (12) yields

\[ T_{ij} \approx \sigma_{ij} + (u_{ik} - \delta_{ik} u_{\text{mean}}) \sigma_{jk}. \]  

The equations-of-state for fluids are generally defined as functions of pressure in terms of the mass density \( \rho \). As shown in section 2.3, \( \rho \) refers to the mass density in the present state of deformation and has the same value whether referred to Lagrangian or Eulerian coordinates. The pressure, in contrast to the mass density, is generally different in Lagrangian and Eulerian coordinates, except for the case of purely longitudinal wave propagation. Since \( \rho \) refers to the mass density in the present state of deformation, the pressure in the equations-of-state for fluids is quite naturally referred to Eulerian coordinates. Thus, the equations-of-state for fluids, when expressed as functions of the mass density, are Eulerian equations. For liquids, the equation-of-state is given as an expansion of the Eulerian pressure \( p^E \) in terms of the mass density \( \rho \) as [34]

\[ p^E = p_0 + A \left( \frac{\rho - \rho_0}{\rho_0} \right) + \frac{1}{2} B \left( \frac{\rho - \rho_0}{\rho_0} \right)^2 + \cdots \]  

where \( p_0 \) is the initial hydrostatic pressure and \( A \) and \( B \) are the Fox-Wallace-Beyer coefficients. For plane wave propagation along \( e_1 \), where \( u_{11} = 0 \) and \( u_{23} = u_{33} = 0 \), substituting \( \rho = \rho_0(1 + u_{11})^{-1} \) from equation (5) in equation (18) leads to an expression of the pressure in terms of the strain measure \( u_{11} \) as

\[ p_{11}^E = p_{11}^L = p_0 - A u_{11} + \left( \frac{B}{2} + A \right) u_{11}^2 + \cdots \]  

where the relation \( p_{11}^E = p_{11}^L \) follows from equation (16) for plane waves.

The Brugger elastic coefficients are related to the Fox-Wallace-Beyer coefficients as [66]

\[ C_1 = C_2 = C_3 = -p_0 \]  

\[ C_{11} = C_{22} = C_{33} = A + p_0 \]  

\[ C_{12} = C_{21} = C_{33} = C_{31} = C_{32} = A - p_0 \]  

\[ C_{111} = C_{222} = C_{333} = -(B + 5A + 3p_0) \]  

\[ C_{112} = C_{211} = C_{311} = C_{312} = C_{321} = -(B + A - p_0) \]  

\[ C_{223} = C_{323} = C_{233} = C_{332} = C_{132} = C_{131} = C_{123} = A - B - 3p_0 \]  

where \( p_0 \) is the initial hydrostatic pressure and the Voigt contraction of indices \((11 \rightarrow 1, 22 \rightarrow 2, 33 \rightarrow 3, 23 \rightarrow 32 \rightarrow 4, 13 \rightarrow 31 \rightarrow 5, 12 \rightarrow 21 \rightarrow 6)\) has been used in equations (20)–(25) for the Brugger coefficients. The Huang elastic coefficients \( A_{ij} \) are assessed from equations (7)–(9), (20)–(25) in terms of the Fox-Wallace-Beyer coefficients as (using Voigt contraction of indices)

\[ A_1 = A_2 = A_3 = -p_0 \]  

\[ A_{11} = A_{22} = A_{33} = A \]  

\[ A_{12} = A_{21} = A_{31} = A_{33} = A_{32} = A - p_0 \]  

\[ A_{111} = A_{222} = A_{333} = -B - 2A \]  

\[ A_{112} = A_{211} = A_{311} = A_{312} = A_{321} = -B \]  

\[ A_{123} = A_{323} = A_{332} = A_{132} = A_{131} = A_{123} = A - B - p_0. \]  

The equation-of-state for ideal gases is given as \( p^E = p_0(\rho/\rho_0)^\gamma \), where \( \gamma \) is the ratio of specific heats. The relationships between the Huang coefficients and the corresponding elastic parameters for ideal gases are obtained by setting \( A = p_0 \gamma \) and \( B = p_0 \gamma (\gamma - 1) \) in equations (27)–(31).

### 2.5. Time-averaging of Lagrangian and Eulerian quantities

Since the acoustic radiation pressure is a time-averaged, steady-state property of the wave, it is useful to define the time-average of a continuous periodic function \( f(t) \) under steady-state conditions by the operation

\[ \langle f(t) \rangle = \lim_{t \to \infty} \frac{1}{t} \int_0^t f(t') dt' \]  

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where the angular bracket denotes time-averaging of the function enclosed in the bracket. It is often assumed that time-averaging a Lagrangian quantity yields values different from that of time-averaging the corresponding Eulerian quantity, since time-averaging the Lagrangian quantity \( \langle q^E(X, t) \rangle \) occurs while holding the Lagrangian coordinates fixed, but time-averaging the Eulerian quantity \( \langle q^E(x, t) \rangle \) occurs while holding the Eulerian coordinates constant. For fixed Lagrangian coordinates \( \mathbf{X} \), \( \langle q^E(x, t) \rangle = \langle q^E(x, t) \rangle_{\mathbf{X}(t)=\mathbf{X}(t)} = \langle q^E(X, t) \rangle \) where \( q^E(X, t) \) results from the fact that for sinusoidal waves \( \mathbf{u}(t) \) averages to zero and \( \mathbf{X}(t) \) averages to \( \mathbf{X} \). For fixed Eulerian coordinates \( \mathbf{x} \), \( \langle q^E(x, t) \rangle = \langle q^E(x, t) \rangle_{\mathbf{x}(t)=\mathbf{x}(t)} = \langle q^E(x, t) \rangle \), where \( q^E(x, t) \) results from the fact that for sinusoidal waves \( \mathbf{u}(t) \) averages to zero and \( \mathbf{X}(t) \) averages to \( \mathbf{X} \). When \( \mathbf{x} \) and \( \mathbf{X} \) correspond to the same point \( \mathbf{Y} \) in Cartesian space, then \( \mathbf{x} = \mathbf{X} = \mathbf{Y} \) and \( \langle q^E(Y, t) \rangle = \langle q^E(Y, t) \rangle \).

It is often assumed in the acoustics literature for fluids that the Eulerian coordinates correspond to surfaces fixed in Cartesian space and that the Lagrangian coordinates correspond to surfaces that oscillate in space under an impressed sinusoidal wave [28, 34, 35, 41, 48]. An oscillating material surface is defined by a set of \( n \) contiguous, particle displacements \( \mathbf{u}_n(t) (n = 1, 2, 3, \cdots) \) that vary sinusoidally in time \( t \). Relative to fixed Lagrangian coordinates \( \mathbf{X}_n \), the time-dependent particle displacements \( \mathbf{u}_n(t) \) are defined by \( \mathbf{u}_n(t) = \mathbf{x}_n(t) - \mathbf{X}_n (n = 1, 2, 3, \cdots) \). The Eulerian coordinates \( \mathbf{x}_n \) in this case are time-dependent. Relative to fixed Eulerian coordinates \( \mathbf{x}_n \), the particle displacements \( \mathbf{u}_n(t) \) are defined by \( \mathbf{u}_n(t) = \mathbf{x}_n - \mathbf{X}_n \), where it is the Lagrangian coordinates that are now dependent on time \( t \). Since it is the displacements that define the motion of the surface, it is apparent that the displacements can occur with respect to either fixed Lagrangian or fixed Eulerian coordinates - a consequence of the relativistic principle that for coordinate systems moving relative to each other it does not matter in regard to the relative displacement which system is regarded as moving and which is considered fixed.

It is noted that the quantity \( q \) in equations (3) and (4) is assumed to be a singly defined scalar quantity. The relationship between the time-averaged Cauchy (Eulerian) stress and the first Piola–Kirchhoff (Lagrangian) stress is more complicated, since stress is not a scalar but, rather, a second rank tensor, defined as a force per unit area for which the area is defined differently for the two stresses. The force in the definition of both stresses refers to Eulerian coordinates (present or deformed state) but the area in the first Piola–Kirchhoff stress refers to Lagrangian coordinates (initial or un-deformed state) and the area in the Cauchy stress refers to Eulerian coordinates. The relationship between the two stresses is thus governed by the transformation between the Lagrangian and Eulerian areas and the time-averaging must be assessed from the equation, obtained from finite deformation theory, linking the quantities [64]. For plane, longitudinal acoustic stresses such that \( \sigma_{11} = T_{11} \) at a given point in Cartesian space, \( \langle \sigma_{11} \rangle = \langle T_{11} \rangle \), exactly, resulting from the fact that the areas in the two stresses transform such that the areas are equal in magnitude. In other cases, equations (11), (12) or (17) must be used to assess the time-averaged relationship between Eulerian and Lagrangian stresses.

The time-averaged displacement gradient \( \langle \mathbf{u}_{ij} \rangle \) is the radiation-induced static strain. Since \( \sigma_{ij} \) is the force per unit area referred to the Lagrangian coordinates, \( \langle \sigma_{ij} \rangle \) is the Lagrangian radiation stress (also known as the first Piola–Kirchhoff radiation stress). Since \( T_{ij} \) is the force per unit area referred to the Eulerian coordinates, \( \langle T_{ij} \rangle \) is quite properly the Eulerian radiation stress (or Cauchy radiation stress). It is generally assumed in the acoustics literature that for plane wave propagation the radiation stress in Eulerian coordinates is not \( \langle T_{ij} \rangle \) but, rather, the momentum flux density \( \langle \rho \mathbf{v} \rangle \). It is shown in section 3.2 that this assumption is based on an incorrect interpretation, originally proposed by Brillouin [26, 27], of the terms in what is now known as the Brillouin stress tensor. It is shown that the term in the Brillouin stress tensor that Brillouin assumed to be the momentum flux density is actually the contribution to the fractional variation in the period of the acoustic wave resulting from a change in the sample length from slow, virtual variations in the strain parameter. Equally important, it is shown that the Brillouin stress tensor is not a Eulerian tensor at all, as generally assumed, but a Lagrangian tensor. The incorrect identification of the momentum flux density with the radiation stress for plane waves has greatly contributed to the considerable misunderstanding of acoustic radiation stress in the literature.

3. Acoustic radiation pressure in laterally unconstrained, plane waves

It has been known since the work of Hertz and Mende [28] that for fluids the radiation pressure in an acoustic beam is highly dependent on whether motion of fluid normal to the wave propagation direction is allowed—i.e., on whether the acoustic beam is laterally confined or laterally unconstrained. Brillouin [26, 27], Hertz and Mende [28], and Beyer [34, 49] assess the radiation pressure by assuming a longitudinal, plane wave acoustic beam of cylindrical cross-section incident on a target in laterally confined and laterally unconstrained volumes. Their derivation is questionable for several reasons given in [65], including an incorrect assessment of the relationship between Lagrangian and Eulerian quantities. Moreover, as pointed out by Beissner [50, 51], a beam of finite cross-section is three-dimensional and, thus, diffracted, which leads to additional issues in assessing the radiation pressure. The seminal papers [25–30, 40–42] on acoustic radiation pressure, however, assumed idealized, one-dimensional, plane wave propagation. Since these papers are responsible for much of the
confusion and misunderstanding surrounding acoustic radiation pressure, it is appropriate to focus on the derivation of the radiation pressure for plane wave propagation, beginning with a derivation based on the direct application of finite deformation theory to laterally unconfined beams.

3.1. Derivation of radiation pressure via direct application of finite deformation theory

In assessing the acoustic radiation pressure for laterally unconfined, plane waves propagating along \( e_1 \) in inviscid fluids, it is crucial to recognize that the time-averaged energy density \( \langle 2K \rangle \) (\( K \) = kinetic energy density) that drives the radiation pressure is produced by a sinusoidally oscillating plane wave of finite beam cross-section (usually cylindrical) propagating under laterally constrained conditions. That is, the dynamic (sinusoidal) plane wave propagation is defined such that \( u_{11} = 0, u_{22} = u_{33} = 0 \). In contrast, the radiation pressure itself is governed by static (time-averaged, steady-state) conditions associated with a laterally unconfined volume. This is quite unlike the case for laterally confined, plane wave beams where both the dynamical wave and the radiation (static) pressure are subject to the same lateral constraints, \( u_{11} = 0, u_{22} = u_{33} = 0 \).

Strictly, for laterally unconfined conditions a cylindrical acoustic beam of finite cross-section is not planar because of diffraction but becomes increasingly planar in an area around the center of the beam as the ratio of the acoustic wavelength to the beam radius \( r \) approaches zero. Further, as pointed out by Lee and Wang [48] the amplitude of the wave does not abruptly decrease to zero beyond the beam radius but does so smoothly in a manner approximated by the zeroth order Bessel function \( J_0(\alpha_c r) \), where \( \alpha_c \) is a constant corresponding to the reciprocal of some characteristic beam radius. For present purposes, there is no loss in generality for one-dimensional wave propagation to assume an idealized plane wave beam of cylindrical cross-section with a ‘top-hat’ amplitude profile. More importantly, as shown below, lateral unconfinement modifies the assessment of the elastic coefficients relevant to static conditions.

The radiation-induced static strain generated by the acoustic plane wave statically deforms the volume of material through which the wave propagates. For a laterally unconfined beam, the reaction of the statically deformed volume, however, is governed not by the dynamic, laterally constrained conditions associated with plane wave propagation but by the static (time-averaged, steady state) conditions governing a laterally unconstrained volume. To emphasize that the elastic properties associated with static, laterally unconstrained conditions are distinct from those of the laterally constrained conditions associated with dynamic, acoustic, plane wave propagation, the functions, parameters, and variables associated with static, laterally unconstrained conditions are designated by the superscript ‘\( S \)’. Thus, under static, laterally unconstrained conditions

\[
\begin{align*}
(u_{ij}) &\rightarrow (u_{ij}^S), (u_{ij} u_{kl}) &\rightarrow & (u_{ij}^S u_{kl}^S), \langle \sigma_{ij} \rangle &\rightarrow & \langle \sigma_{ij}^S \rangle, A_{ij} &\rightarrow & A_{ij}^S, etc.
\end{align*}
\]

For static, laterally unconstrained conditions in inviscid fluids, not only are the mean shear displacements gradients \( \langle u_{ij}^S \rangle_{ij} = 0 \) but also the mean dilatation \( \langle \Theta^S \rangle = \langle \Delta V^S / V_0^S \rangle = \langle u_{11}^S \rangle + \langle u_{22}^S \rangle + \langle u_{33}^S \rangle = 0 \), where \( \langle \Delta V^S \rangle \) is the mean change in the initial volume \( V_0^S \) [67]. For plane wave propagation along \( e_1 \), the symmetry in directions \( e_2 \) and \( e_3 \) and the null mean dilatation \( \langle \Theta^S \rangle = 0 \) require that \( \langle u_{22}^S \rangle = \langle u_{33}^S \rangle = -(1/2) \langle u_{11}^S \rangle \). Thus, unlike the case for laterally constrained conditions, where the strains \( \langle u_{11}^S \rangle, \langle u_{22}^S \rangle, \) and \( \langle u_{33}^S \rangle \) are independent, the relationship \( \langle u_{22}^S \rangle = \langle u_{33}^S \rangle = -(1/2) \langle u_{11}^S \rangle \) for static, laterally unconstrained conditions reduces the number of independent static strains to one, which is here chosen to be \( \langle u_{11}^S \rangle \), corresponding to the direction of wave propagation. The relationship \( \langle u_{22}^S \rangle = \langle u_{33}^S \rangle = -(1/2) \langle u_{11}^S \rangle \) implies that there is free flow of fluid in the directions \( e_2 \) and \( e_3 \) under static, laterally unconstrained conditions.

It is convenient to begin with the derivation of acoustic radiation pressure in Lagrangian coordinates. Performing the summation in the last equality in equation (6), time-averaging, and substituting within the time-average expressions the equations (26)–(31) for the Huang coefficients (in Voigt notation) and the relation \( u_{11}^S = u_{33}^S = -(1/2) u_{11}^S \) corresponding to laterally unconstrained conditions lead to an assessment of the mean internal energy density \( \phi^S \) as

\[
\langle \phi^S \rangle = \rho_0 U(X, 0, S) + A_{11}^S \langle u_{11}^S \rangle + \frac{1}{2} A_{11}^S \langle (u_{11}^S)^2 \rangle + \frac{1}{3!} A_{111}^S \langle (u_{11}^S)^3 \rangle + \cdots
\]  

(33)

where

\[
A_{11}^S = -p_0, \quad A_{111}^S = \frac{3}{2} p_0, \quad A_{111}^S = -\frac{3}{2} p_0.
\]

(34)

Note that the term \( A_{11} u_{11} \) in equation (6) is the contribution to the energy density from the hydrostatic pressure \( p_0 \) and thus not subject to the constraint \( u_{11}^S = u_{33}^S = -(1/2) u_{11}^S \), since the hydrostatic pressure is uniformly constant in all spatial directions. Upon time-averaging, \( A_{11} u_{11} \rightarrow A_{11}^S \langle u_{11}^S \rangle = A_{11}^S \langle (u_{11}^S)^0 \rangle + \langle u_{22}^S \rangle + \langle u_{33}^S \rangle \rangle \) where from equation (26) \( A_{11}^S = A_{22}^S = A_{33}^S = -p_0 \). The \( \langle \sigma_{11}^S \rangle \) component of the static (radiation) stress along \( e_1 \) is obtained from equation (33) as
\( (\sigma^S_{ij}) = \frac{\partial (\phi^S)}{\partial u^S_{ij}} = A_i^S + A_{11}^S (u_{ij}^S) + \frac{1}{2} A_{11}^S ((u_{ij}^S))^2 + \cdots. \) (35)

It is noted from equations (34) that the Fox-Wallace-Beyer coefficients \( A \) and \( B \) do not appear in equations (33) and (35)—only the initial pressure \( p_0 \) appears. This results from the fact that, because the shear modulus for inviscid fluids is zero, the fluid freely deforms under static loads.

The nonlinear wave equation for longitudinal, plane wave propagation along \( \mathbf{e}_1 \) is \([44, 61, 64, 66]\)

\[ \frac{\partial^2 u_{11}}{\partial t^2} = c_0^2 \left( 1 + \frac{A_{111}}{A_{11}} \frac{\partial u_{11}}{\partial X_1} \right) \frac{\partial^2 u_{11}}{\partial X_1^2}; \] (36)

where \( c_0 = (A_{11}/\rho_0)^{1/2} \) is the infinitesimal wave velocity. The relationship between the displacement gradient \( u_{11} = \partial u_{11}/\partial X_1 \) and the particle velocity (\( \partial u_{11}/\partial t \)) is given by the compatibility condition. The compatibility condition for wave propagation along \( \mathbf{e}_1 \) can be written most generally in the form

\[ \frac{\partial u_{11}}{\partial t} = f(u_{11}, u_{22}, u_{33}) \] (37)

where \( f \) is a function of the displacement gradients \( u_{11}, u_{22}, \) and \( u_{33} \).

For laterally confined beams in both solids and fluids the lateral strains \( u_{22} = u_{33} = 0 \). In such case \( f = f(u_{11}) \) is a function only of \( u_{11} \) for both dynamic and static conditions in laterally constrained beams. For laterally unconfined beams, \( f = f(u_{11}) \) continues to hold for dynamic conditions, but because the lateral strains under static conditions now vary as \( u_{11}^S = -(1/2) u_{11}^i \), it is necessary to retain the functional relationship \( f = f(u_{11}^i, u_{22}^i, u_{33}^i) \) for static conditions. Taking the time derivative of equation (37) yields

\[ \frac{\partial^2 u_{11}^S}{\partial t^2} = \frac{\partial f}{\partial u_{11}^S} \frac{\partial u_{11}^S}{\partial t} + \frac{\partial f}{\partial u_{22}^S} \frac{\partial u_{22}^S}{\partial t} + \frac{\partial f}{\partial u_{33}^S} \frac{\partial u_{33}^S}{\partial t} = 3f' \frac{\partial^2 u_{11}^i}{\partial t \partial X_1}. \] (38)

The last equality in equation (38) results by imposing the static, laterally unconfined conditions \( u_{11}^S = u_{11}^S = -(1/2) u_{11}^i \) and writing \( f' = df/du_{11}^i \).

Taking the derivative of equation (37) with respect to \( X_1 \) yields

\[ \frac{\partial^2 u_{11}^S}{\partial X_1^2} = f' \frac{\partial^2 u_{11}^i}{\partial X_1^2}, \] (39)

since for fluids the shear displacement gradients are zero. Substituting equation (39) in equation (38) leads to

\[ \frac{\partial^2 u_{11}^S}{\partial t^2} = 3(f')^2 \frac{\partial^2 u_{11}^i}{\partial X_1^2}. \] (40)

Comparing equation (40) to equation (36) for static, laterally unconfined conditions results in

\[ f' = \frac{\partial f}{\partial u_{11}^S} = \frac{1}{\sqrt{3}} c_0 \left( 1 + \frac{A_{111}^S}{A_{11}^S} u_{11}^i \right)^{1/2}. \] (41)

Integrating equation (41) with respect to \( u_{11} \) yields

\[ f = \frac{\partial u_{11}^S}{\partial t} = -\frac{2c_0}{3\sqrt{3}} \frac{A_{11}^S}{A_{11}^S} \left( 1 + \frac{A_{111}^S}{A_{11}^S} u_{11}^i \right)^{2/3} + \frac{2c_0}{3\sqrt{3}} A_{11}^S, \] (42)

where the last term in equation (42) is the constant of integration evaluated for the condition that when no wave is present both the displacement gradient \( u_{11}^i \) and particle velocity \( \partial u_{11}^i/\partial t \) are zero. Solving equation (42) for \( u_{11}^i \) in terms of \( \partial u_{11}^S/\partial t \) yields the compatibility relation for static, laterally unconfined conditions as

\[ \frac{\partial u_{11}^S}{\partial X_1} = u_{11}^S = \frac{\sqrt{3}}{c_0} \frac{\partial u_{11}^S}{\partial t} - \frac{3}{4c_0^2 A_{11}^S} \left( \frac{\partial u_{11}^S}{\partial t} \right)^2. \] (43)

When performing the time-average of terms involving \( u_{11} \) to establish the static relationships under laterally unconfined conditions in terms of \( \partial u_{11}^S/\partial t \) it is appropriate to employ the time-average of equation (43). Since for sinusoidal oscillations the displacement \( u_{11}^S \) is finite in amplitude for all times, substituting equation (43) in equation (32) yields that the time-average of the particle velocity (\( \partial u_{11}^S/\partial t \)) is zero. However, for sinusoidal displacements the time-average of \( (\partial u_{11}^S/\partial t)^2 \) = 0. This is shown directly, for example, by time-averaging \( (\sin \omega t)^2 \) to obtain \( ((\sin \omega t)^2) = \frac{1}{2} - \frac{1}{2} (\cos 2\omega t) = \frac{1}{2} \), i.e., time-averaging leaves only the static term \( \frac{1}{2} \). Thus, the distinction between a quantity and the time-average of that quantity is that time-averaging removes sinusoidal contributions to the quantity, leaving only the static contribution—if any. The static strain (time-averaged displacement gradient) in direction \( \mathbf{e}_1 \) under laterally unconfined
conditions is assessed by time-averaging equation (43) to obtain

\[
\langle u_{i1}^S \rangle = -\frac{3}{4(c_0^S)^2} A_{i11} \left( \frac{\partial u_{i1}^E}{\partial t} \right)^2
\]  

(44)

and

\[
\langle u_{i1}^E \rangle^2 = \frac{3}{(c_0^S)^2} \left( \frac{\partial u_{i1}^E}{\partial t} \right)^2
\]  

(45)

where \( c_0^S = (A_{i1}^S / \rho_0)^{1/2} \).

It is noted in equations (44) and (45) that the elastic constants for static, unconstrained conditions are \( A_{i1}^S \) and \( A_{i11}^S \), while the factor \( \left( \frac{\partial u^E}{\partial t} \right)^2 \) refers to the dynamic, plane wave oscillations as the sole energy source driving the static strain. Equations (44) and (45) thus establish the connection between the static strains and the mean (time-averaged) energy density of the driving dynamical wave. Substituting equations (44) and (45) in equation (35), noting from equation (16) that for plane waves the Lagrangian and Eulerian pressures are exactly equal such that \( \langle \sigma_{i1}^E \rangle = \langle T_{i11}^E \rangle = -\langle p_l^E \rangle = -\langle p_f^E \rangle \), and writing \( \rho_0 \left( \frac{\partial u^E}{\partial t} \right)^2 \) = \( \langle 2K \rangle \) lead to

\[
\langle p_f^E \rangle - \rho_0 = \langle p_l^E \rangle - p_0 = -\frac{3A_{i11}^S}{4A_{i1}^S} \langle 2K \rangle = \frac{3}{4} \langle 2K \rangle
\]  

(46)

where from equation (34) \( A_{i11}^S / A_{i1}^S = -1 \).

Equation (46) shows that the mean Eulerian and Lagrangian excess pressures for laterally unconstrained, plane waves are exactly equal with magnitude \( (3/4) \langle 2K \rangle \) along the direction of wave propagation. Equation (46) is quite different from the Langevin expression for laterally unconstrained plane waves, which posits that the mean Lagrangian pressure \( \langle p_l^E \rangle \) along \( e_1 \) is obtained from Langevin’s first relation \( \langle p_f^E \rangle = \langle K \rangle + \langle \phi \rangle + C \), where \( C = \text{constant} \). Assuming \( C = p_0 \) leads to Langevin’s result for the acoustic radiation pressure in laterally unconstrained, plane wave beams as

\[
\langle p_f^E \rangle - p_0 = \langle p_l^E \rangle \approx \langle 2K \rangle \approx \langle E \rangle,
\]  

(47)

where, as shown in section 3.2, for plane waves \( \langle 2K \rangle \) is approximately equal to the energy density \( \langle E \rangle \).

It is relevant to point out that for laterally constrained conditions, \( u_{i2} = u_{i3} = 0 \) and \( f = f(u_{i1}) \). Applying these conditions and following the derivation leading to equation (43) now results in the compatibility relation [44]

\[
\frac{\partial u_{i1}}{\partial X_i} = u_{i1} = -\frac{1}{c_0} \frac{\partial u_{i1}}{\partial t} - \frac{1}{4c_0^2} A_{i11} \left( \frac{\partial u_{i1}}{\partial t} \right)^2.
\]  

(48)

Equation (48) leads to the relations

\[
\langle \sigma_{i11}^E \rangle - \langle \sigma_{i11}^E \rangle_0 = \langle \sigma_{i11}^E \rangle_0 = -\frac{A_{i11}}{4A_{i1}} \langle 2K \rangle
\]  

(49)

and

\[
\langle u_{i1} \rangle = -\frac{1}{4c_0^2} A_{i11} \left( \frac{\partial u_{i1}}{\partial t} \right)^2.
\]  

(50)

Equation (50) has been experimentally confirmed along the three, independent, pure mode propagation directions in monocrystalline silicon [45, 55] and in isotropic vitreous silica [45, 56]. The experimental confirmation attests the validity of the derivation leading to equations (48)–(50) and lends support to the analogous derivation leading to equation (46). (It is noted that equation (49) with \( A_{i11}/A_{i1} = -1 \) is mistakenly used in reference [63] to assess the acoustic radiation pressure for laterally unconstrained, plane wave beams in fluids. The appropriate relationship for fluids is clearly given by equation (46).)

The static stresses \( \langle \sigma_{i11}^S \rangle \) and \( \langle \sigma_{i12}^S \rangle \), respectively, in the directions \( e_2 \) and \( e_3 \) are obtained from the mean internal energy density \( \langle \phi \rangle \), equation (33), as \( \langle \sigma_{i12}^S \rangle = \partial \langle \phi \rangle / \partial u_{i2}^S \) and \( \langle \sigma_{i33}^S \rangle = \partial \langle \phi \rangle / \partial u_{i3}^S \). However, because of the symmetry in the directions \( e_2 \) and \( e_3 \) and the dilatation relationship \( \langle \Theta^S \rangle = \langle \Delta V^S / V_0^S \rangle = \langle u_{i1}^S \rangle + \langle u_{i2}^S \rangle + \langle u_{i3}^S \rangle \) = 0 under static, laterally unconstrained conditions, the only independent strain relevant to the acoustic radiation pressure is \( u_{i1}^S \). There is no dependence of \( \langle \phi \rangle \) on \( u_{i2}^S \) and \( u_{i3}^S \), except for the term \( A_{i1}^S \langle u_{i1}^S \rangle \) corresponding to the hydrostatic pressure \( A_{i1}^S = A_{i1}^S = A_{i3}^S = -p_0 \). Thus, \( \langle \sigma_{i22}^S \rangle - \langle \sigma_{i33}^S \rangle = \partial \langle \phi \rangle / \partial u_{i3}^S - A_{i1}^S = 0 \) and \( \langle \sigma_{i33}^S \rangle - \langle \sigma_{i11}^S \rangle = \partial \langle \phi \rangle / \partial u_{i1}^S - A_{i3}^S = 0 \), which are exactly the conditions, together with the zero shear modulus, that result in the null dilatation \( \langle \Theta^S \rangle = 0 \) [67]. This means that the excess Lagrangian pressures normal to the plane wave propagation direction are obtained as
\[
(p_\ell^E) - p_0 = \langle p_\ell^E \rangle - p_0 = -\langle \sigma_{\ell \ell}^2 \rangle + A_\ell^2 = -\langle \sigma_{\ell \ell}^2 \rangle + A_\ell^2 = 0
\] (51)

This also means, from equation (17) and the null value under static conditions of the excess Lagrangian pressures in directions \(e_2\) and \(e_3\), that the excess Eulerian pressures normal to the wave propagation direction are given as

\[
\langle p_\ell^E \rangle - p_0 = \langle p_\ell^E \rangle - p_0 = 0.
\] (52)

Thus, for laterally unconfined, plane wave propagation the acoustic radiation pressure along the direction of propagation is \(\frac{\lambda}{2}(2K)\) and zero in directions normal to the propagation direction. This result holds for both Lagrangian and Eulerian coordinates for laterally unconfined, plane wave beams.

### 3.2. Acoustic radiation pressure and the Boltzmann-Ehrenfest Adiabatic Principle

Brillouin [26, 27] approached the problem of acoustic radiation stress (pressure) by applying the Boltzmann-Ehrenfest Principle of Adiabatic Invariance to longitudinal, plane wave propagation. His result, as shown below, differs considerably from equation (46). Since Brillouin’s theory has played such a pivotal role in assessing the radiation pressure for fluids, it is instructive to reconsider the Boltzmann-Ehrenfest approach in detail.

The Boltzmann-Ehrenfest (B-E) Adiabatic Principle [68, 69] states that if the constraints of a periodic system are allowed to vary sufficiently slowly, then the product of the mean (time-averaged, steady-state) kinetic energy \(\langle K^* \rangle\) and the period \(T\) of the system is an adiabatic invariant or constant of the motion such that the virtual variation \(\delta \langle K^* \rangle = 0\) or

\[
\delta \langle K^* \rangle = - \langle K^* \rangle \frac{\delta T}{T}.
\] (53)

According to the B-E Principle, a slow virtual variation \(\delta q^*\) in a constraint \(q^*\) (generalized displacement) of a conservative, oscillatory system leads to a change in the system configuration that results in a change \(\delta \langle E^* \rangle\) in the mean total energy \(\langle E^* \rangle\) of the system. The change in the mean total energy is quantified by the product of the generalized reaction force \(Q^*\) and virtual constraint variation \(\delta q^*\) such that \(\delta \langle E^* \rangle = Q^* \delta q^*\).

To understand Brillouin’s results, it is instructive to consider first the derivation in Lagrangian coordinates for laterally confined, longitudinal, plane wave propagation along \(e_1\). In Lagrangian coordinates, the virial theorem states that [70, 71]

\[
\langle K \rangle = \frac{1}{2} \left\langle \frac{\partial \phi}{\partial u_{11}} \right\rangle u_{11} = \frac{1}{2} \langle \sigma_{11} u_{11} \rangle
\] (54)

where for longitudinal plane waves, the potential energy density corresponding to the excess stress \([\sigma_{11} - \langle \sigma_{11} \rangle_0]\) is obtained from equation (6) by letting \(\phi - A_{11} u_{11} \to \phi',\) dropping the prime on \(\phi',\) and writing

\[
\phi = \rho_0 U(X, 0, S) + \frac{1}{2} A_{11} u_{11}^2 + \frac{1}{6} A_{111} u_{11}^3 + \cdots.
\]

The relationship between the mean kinetic energy density and the mean internal (potential) energy density for plane waves can be established by substituting equations (55) in (54) to obtain a power series expansion of equation (54), and then solving equation (55) for \(A_{11} u_{11}^2\) and iteratively substituting for \(u_{11}^2\) in the terms of the expanded equation (54) to obtain

\[
\langle K \rangle = \langle \phi \rangle + \frac{1}{6} \frac{A_{111}}{A_{11}} \langle \phi u_{11} \rangle + \cdots
\] (56)

where the constant term \(\rho_0 U(X, 0, S)\) has been dropped, since it makes no contribution to the kinetic energy. The mean total energy density \(\langle E \rangle\) for nonlinear plane waves is then

\[
\langle E \rangle = \langle K \rangle + \langle \phi \rangle = 2 \langle K \rangle - \frac{1}{6} \frac{A_{111}}{A_{11}} \langle \phi u_{11} \rangle + \cdots
\] (57)

where the last equality follows from equation (56). It is interesting to note from equation (57) that for nonlinear waves the total average energy density \(\langle E \rangle\) is not exactly equal to \(\langle 2K \rangle\).

According to the B-E Adiabatic Principle [70, 71], a slow virtual variation \(\delta q^*\) in a constraint \(q^*\) (generalized displacement) of a conservative, oscillatory system leads to a change in the system configuration that results in a change \(\delta \langle E^* \rangle\) in the mean (time-averaged, steady-state) total energy \(\langle E^* \rangle\) of the system. The change in the mean total energy is quantified by the product of the generalized reaction force \(Q^*\) and virtual constraint variation \(\delta q^*\) such that \(\delta \langle E^* \rangle = Q^* \delta q^*\). For longitudinal, acoustic plane wave propagation, the generalized reaction force in Lagrangian coordinates is the mean excess radiation stress \(\langle \sigma_{11} - \langle \sigma_{11} \rangle_0 \rangle = \langle \sigma_{11} \rangle - \langle \sigma_{11} \rangle_0\), the constraint (generalized displacement) is the displacement gradient \(u_{11}\), the mean kinetic energy \(\langle K^* \rangle\) corresponds to the mean kinetic energy density \(\langle K \rangle\), and the mean total energy \(\langle E^* \rangle\) corresponds to the mean total energy density \(\langle E \rangle\). Thus, for plane wave propagation the relation \(\delta \langle E^* \rangle = Q^* \delta q^*\) becomes
\[(\langle \sigma_{11} \rangle - \langle \sigma_{11} \rangle_0) \delta u_{11} = \delta \langle E \rangle = \delta \langle 2K \rangle - \frac{1}{6} \frac{A_{111}}{A_{11}} \delta \langle \phi u_{11} \rangle + \cdots \] (58)

where the last equality in equation (58) follows from equation (57).

Writing \(2\langle K^p \rangle \rightarrow 2\langle K \rangle\) in equation (53) and substituting in equation (58) lead, to first order in the nonlinearity, to the relation for the acoustic radiation stress

\[\langle \sigma_{11} \rangle - \langle \sigma_{11} \rangle_0 = -2\langle K \rangle \left( \frac{1}{T} \frac{\delta T}{\delta u_{11}} \right)_0 - \frac{1}{A_{11}} \frac{A_{111}}{4} \delta \langle \phi u_{11} \rangle \] (59)

where the subscripted '0' denotes evaluation at \(u_{11} = 0\). The factor \(\frac{\delta u_{11}}{\delta u_{11}}\) in equation (59) is evaluated as

\[
\frac{\delta \langle \phi u_{11} \rangle}{\delta u_{11}} = \langle \phi \rangle + \left\{ \frac{\partial \phi}{\partial u_{11}} \right\}_0 \approx \frac{3}{2} 2\langle K \rangle.
\] (60)

It is noted that Brillouin omitted in his derivation the nonlinear contribution corresponding to the last term in equation (59).

It is extremely important to note that the radiation stress given by equation (59) is the Lagrangian radiation stress. The fractional change in the oscillation period \(T^{-1} \delta T / \delta u_{11}\) with respect to the virtual variation \(\delta u_{11}\) can be easily assessed from the fractional change in the natural velocity \(W\). The natural velocity is the velocity defined as the ratio of the length of the sound path in the un-deformed state to the propagation time in the deformed state [61–63]. Since the path length in the un-deformed state is constant, only the propagation time in the deformed state plays a role in assessing the fractional variation in the system period when using the natural velocity for the assessment. The natural velocity is the velocity referred to the Lagrangian coordinates and is obtained from equation (14) as [61–63]

\[W^2 = \frac{1}{\rho_0} \frac{\partial \sigma_{11}}{\partial u_{11}} = \frac{1}{\rho_0} (A_{11} + A_{111} u_{11} + \cdots).\] (61)

The fractional change in the period \((T^{-1} \delta T / \delta u_{11})_0\) is assessed from the fractional change in the natural velocity as

\[
\left( \frac{1}{T} \frac{\delta T}{\delta u_{11}} \right)_0 = - \left( 1 \frac{\delta W}{W} \frac{\delta u_{11}}{\delta u_{11}} \right)_0 = - \frac{1}{2} A_{111} \frac{A_{111}}{A_{11}}.
\] (62)

The excess acoustic radiation stress in Lagrangian coordinates \(\langle \sigma_{11} \rangle - \langle \sigma_{11} \rangle_0\) is generally known as the Rayleigh radiation stress \(\langle \sigma_{11} \rangle_{\text{Rayleigh}}\) and is evaluated from equations (59), (60), and (62) as

\[\langle \sigma_{11} \rangle_{\text{Rayleigh}} = \langle \sigma_{11} \rangle - \langle \sigma_{11} \rangle_0 = \frac{1}{4} \frac{A_{111}}{A_{11}} \langle 2K \rangle.\] (63)

For laterally confined beams in liquids with initial (hydrostatic) pressure \(p_0\), equation (16) yields that the excess Eulerian pressure and the excess Lagrangian radiation pressure are equal along the wave propagation direction \(e_1\) and equations (27) and (29) yield \(A_{111} / A_{11} = -[(B/A) + 2]\). Thus, for laterally confined beams in liquids equation (63) gives

\[\langle p_1 \rangle_{\text{Rayleigh}} = \langle p_1 \rangle - p_0 = \langle p_1 \rangle - p_0 = \frac{1}{4} \left( \frac{B}{A} + 2 \right) \langle 2K \rangle.\] (64)

Equation (64) is in agreement with the results of Brillouin (except for the contribution, omitted by Brillouin, of the nonlinear factor corresponding to the last term in equation (59)). However, Brillouin incorrectly assumed that the equations he used, leading to equation (64), refer to Eulerian coordinates rather than to Lagrangian coordinates, as shown in the present derivation. It is crucial to understand how Brillouin came to such an assumption, since the assumption has led to a deep foundational misunderstanding of acoustic radiation stress and pressure. It is noted that the equality of the Lagrangian and Eulerian stresses (pressures), obtained from finite deformation theory and given by equation (16), for longitudinal, plane wave propagation was not known to Brillouin.

The equations that Brillouin interpreted as Eulerian equations can be obtained from a consideration of the relationship between the natural velocity \(W\) and the true velocity \(c\). The true velocity \(c\) is the velocity defined as the ratio of the length of the sound path in the deformed state to the propagation time in the deformed state. The true velocity is related to the natural velocity \(W\) as \(W = c / \ell_0\), where \(\ell_0\) is the length of material in the un-deformed state and \(\ell\) is the length of material in the deformed state [61–63, 66]. It is shown Appendix B of
[65] that the ratio \( R = \frac{\varepsilon}{\varepsilon} \) is obtained as \( R = \frac{\varepsilon}{\varepsilon} = \left[ \left( \varepsilon_i - 2\frac{\partial u_i}{\partial x_k} \right) N_j N_j \right]^{1/2} \), where \( N_j \) are the Cartesian components of the unit vector normal to the material surface in the un-deformed (initial) state. Thus, the true velocity and the natural velocity are related as

\[
W = \frac{\varepsilon}{\varepsilon} = \left[ \left( \varepsilon_i - 2\frac{\partial u_i}{\partial x_k} \right) N_j N_j \right]^{1/2} c. \quad (65)
\]

In terms of variations in the true velocity \( \frac{1}{\varepsilon} \), the variation of the system period \( T \) is obtained from equations (62) and (65) as

\[
\left( \frac{1}{\varepsilon} \frac{\delta T}{\varepsilon} \right)_0 = - \left( \frac{1}{\varepsilon} \frac{\delta W}{\varepsilon} \right)_0 = - \left( \frac{1}{\varepsilon} \frac{\delta c}{\varepsilon} \right)_0 = - \left( \frac{1}{\varepsilon} \frac{\delta R}{\varepsilon} \right)_0 = - \frac{1}{\varepsilon} \frac{\delta c}{\varepsilon} + 1 = - \frac{1}{2} \frac{A_{11}}{A_{11}} (2K). \quad (66)
\]

where \( R^{-1}(\delta R/\delta u_{11})_0 = -1. \) It is significant to point out that the terms in equation (66) are evaluated at \( u_{11} = 0 \), i.e., at the Lagrangian coordinates \( X \).

The Lagrangian radiation stress for laterally confined, longitudinal, plane waves along \( e_1 \) can be assessed in terms of the true velocity by substituting equation (66) in equation (59) to obtain

\[
\langle \sigma_{11} \rangle - \langle \sigma_{11} \rangle_0 = \langle 2K \rangle \left( \frac{1}{\varepsilon} \frac{\delta W}{\varepsilon} \right)_0 = \frac{1}{4} \frac{A_{11}}{A_{11}} \langle 2K \rangle = \langle 2K \rangle \left( \frac{1}{\varepsilon} \frac{\delta c}{\varepsilon} - 1 \right)_0 = \frac{1}{4} \frac{A_{11}}{A_{11}} \langle 2K \rangle. \quad (67)
\]

It is critically important to recognize that equation (67) is an equation in Lagrangian coordinates. Although the path length for the true velocity \( c \) refers to the deformed state, the term \( \frac{1}{\varepsilon} \frac{\delta c}{\varepsilon} \) in the last equality in equation (67) is evaluated at the un-deformed state, i.e., at the Lagrangian coordinate position \( X \), where \( u_{11} = 0 \). Brillouin [26, 27] identified the term \( \langle 2K \rangle = -\langle 2K \rangle \left( \frac{1}{\varepsilon} \frac{\delta c}{\varepsilon} \right)_0 \) in equation (67) as the momentum flux density \( \langle \rho \varepsilon_{11} \rangle \) and identified the term \( \langle 2K \rangle \left( \frac{1}{\varepsilon} \frac{\delta c}{\varepsilon} \right)_0 \) as the mean excess Cauchy (Eulerian) stress, which for fluids is generally called the ‘mean excess Eulerian pressure.’ Since \( c^2 \) is obtained for fluids from the Eulerian pressure \( p^E \) as \( c^2 = \partial p^E / \partial \rho \), this is, indeed, tempting to assume that \( \langle 2K \rangle \left( \frac{1}{\varepsilon} \frac{\delta c}{\varepsilon} \right)_0 \) corresponds to the mean excess radiation pressure in Eulerian coordinates, but the derivative in \( \langle 2K \rangle \left( \frac{1}{\varepsilon} \frac{\delta c}{\varepsilon} \right)_0 \) is not taken with respect to \( \rho \). Rather, it is taken with respect to \( u_{11} \) and evaluated at \( u_{11} = 0 \), i.e., at the Lagrangian coordinates \( X \). Moreover, taking the derivative of \( c \) with respect to \( u_{11} \) requires that \( c \) be expressed in terms of \( u_{11} \). Since \( u_{11} \) is defined with respect to the Lagrangian coordinates, the expression of \( c \) in terms of \( u_{11} \) allows \( c \) to be evaluated with respect to Lagrangian coordinates. This does not imply that \( c \) is the natural velocity; \( c \) is still the true velocity but simply expressed with respect to (referred to) the Lagrangian coordinates. This means that \( \langle 2K \rangle \left( \frac{1}{\varepsilon} \frac{\delta c}{\varepsilon} \right)_0 \) is also referred to the Lagrangian coordinates and confirms that equation (67) is a Lagrangian equation, contrary to Brillouin’s assumption.

Further, equation (67) shows that \( \langle 2K \rangle \left( \frac{1}{\varepsilon} \frac{\delta c}{\varepsilon} \right)_0 \) cannot be the mean excess Cauchy (Eulerian) stress \( \langle T_{11} \rangle = \langle T_{11} \rangle_0 \), since equation (16) already establishes that \( \langle T_{11} \rangle = \langle T_{11} \rangle_0 \). The term \( \langle 2K \rangle \left( \frac{1}{\varepsilon} \frac{\delta c}{\varepsilon} \right)_0 \) cannot be equal to both the mean excess Lagrangian stress and the mean excess Eulerian stress, since the term \( \langle 2K \rangle = -\langle 2K \rangle \left( \frac{1}{\varepsilon} \frac{\delta c}{\varepsilon} \right)_0 \) also appears in equation (67). Rather, \( \langle 2K \rangle \left( \frac{1}{\varepsilon} \frac{\delta c}{\varepsilon} \right)_0 \) is simply the contribution to the Lagrangian radiation stress along \( e_1 \), resulting from the change in the true sound velocity, when the variation in the system period is assessed using the true velocity. The radiation stress (pressure) in Eulerian coordinates is obtained from the time-averaged Lagrangian stress (pressure) via the transformation given by equations (12) or (17), which for longitudinal, plane wave propagation along \( e_1 \) for laterally confined beams in fluids leads to equation (64). The relationships in directions \( e_2 \) and \( e_3 \) are derived in [65].

It is seen from equations (66) and (67) that the term \( \langle 2K \rangle = -\langle 2K \rangle \left( \frac{1}{\varepsilon} \frac{\delta c}{\varepsilon} \right)_0 \), which Brillouin identified as the momentum flux density is simply the contribution to the Lagrangian radiation stress along \( e_1 \), resulting from the variation in the acoustic path length in response to the virtual variation in the strain \( u_{11} \), when the variation in the system period is assessed using the true velocity. The momentum flux density appears in acoustical equations in Eulerian coordinates as a consequence of the convective derivative in the Eulerian equations of motion. It does not appear in acoustical equations in Lagrangian coordinates, since the convective derivative does not appear in the Lagrangian equations of motion. Since equation (67) provides an assessment of the
acoustic radiation stress in Lagrangian coordinates, it follows that $\langle 2K \rangle = -\langle 2K \rangle \left( \frac{1}{R} \frac{8R}{8u_{ij}} \right)_0$ cannot be the momentum flux density.

Adding to the confusion generated by the incorrect identification of the $\langle 2K \rangle \left( \frac{1}{R} \frac{8R}{8u_{ij}} \right)_0$ and $\langle 2K \rangle \left( \frac{1}{R} \frac{8R}{8u_{ij}} \right)_0$ terms in equations (66) and (67), Brillouin failed to recognize that the term $\langle \sigma_{ij} \rangle - \langle \sigma_{ij} \rangle_0$ in equation (67) is actually the Lagrangian (first Piola-Kirchhoff) stress. Brillouin identified the term $\langle \sigma_{ij} \rangle - \langle \sigma_{ij} \rangle_0$ in equation (67) as the $(i = j = 1)$ component of an entirely different tensor, known today as the Brillouin stress tensor $S_{11}$, which he assumed to represent the radiation stress in Eulerian coordinates. Brillouin’s tensor is written more generally as $S_{ij} = \langle \sigma_{ij} \rangle - \langle \sigma_{ij} \rangle_0$, but, as shown in [65], care must be exercised in employing the B-E Principle in directions $e_1$ and $e_3$.

The improper identification of $S_{ij}$ as the radiation stress tensor in Eulerian coordinates has also occurred from a consideration of the wave equation written in ‘conservative’ form as [35, 44, 48]

$$\frac{\partial \langle T_{ij} \rangle - \rho v_i v_j}{\partial t} = \left( \frac{\partial \rho v_i}{\partial t} \right) = 0$$

where $T_{ij}$ is the Cauchy stress, $\rho v_i v_j$ is the momentum flux density, and the last equality results from equation (32) and the boundedness of $\rho v_i$ as $t \to \infty$. For longitudinal, plane wave propagation along $e_1$, integration of equation (68) yields $\langle T_{11} \rangle - \langle T_{11} \rangle_0 = \langle \rho v_1 v_1 \rangle = 0$ where the constant of integration is set equal to the initial Cauchy stress $\langle T_{11} \rangle_0$. The factor $\langle T_{11} \rangle - \langle T_{11} \rangle_0$ is identified as the ‘mean Eulerian excess stress’ and the linear combination of terms, $\langle T_{11} \rangle - \langle T_{11} \rangle_0 = \langle \rho v_1 v_1 \rangle$ is identified as the $S_{11}$ component of the Brillouin stress tensor.

According to equation (69), however, $\langle S_{11} \rangle$ is actuality $\langle \sigma_{11} \rangle = \langle \sigma_{11} \rangle - \langle \sigma_{11} \rangle_0$ and, according to equation (16), $\langle \sigma_{11} \rangle - \langle \sigma_{11} \rangle_0 = \langle T_{11} \rangle - \langle T_{11} \rangle_0$, where the initial stress $\langle \sigma_{11} \rangle_0 = \langle T_{11} \rangle_0$. Substituting these equalities in equation (69) leads to the result that the momentum flux density is zero along $e_1$—a result consistent with the Brillouin stress tensor being a Lagrangian tensor, since the momentum flux density does not appear in the equations of motion in Lagrangian coordinates. It is again concluded that the Brillouin stress tensor does not represent the radiation stress in Eulerian coordinates, as posited by Brillouin.

It is extremely important to recognize that the derivation of equations (63) and (64) from the B-E Principle is enabled by imposing lateral constraints on the acoustic beam. The lateral constraints meet the conditions under which the B-E Principle is strictly applicable, which requires a slow, virtual variation in the displacement gradient $u_{ij}$, serving as the constraint parameter (generalized displacement) directly affecting the system period. It is noted that for laterally confined beams the lateral constraints, $u_{22} = u_{33} = 0$, are imposed for both static and dynamic conditions. The derivation of the radiation pressure for laterally unconfined beams via the B-E Principle is more challenging because the condition of static lateral unconfinement, $\langle u_{ij}^2 \rangle = \langle u_{ij}^2 \rangle_0 = -(1/2) \langle u_{ij}^2 \rangle_0$, is different from the condition of lateral constraint, $u_{22} = u_{33} = 0$, imposed on the dynamic wave. A derivation of radiation pressure for laterally unconfined beams using the B-E Principle is not pursued here. More important to the historical and scientific development of the subject, Brillouin, as well as numerous researchers that followed, did not recognize the necessity for imposing conditions on static, laterally unconfined beams that are different from those of static, laterally confined beams, as done in section 3.1. Thus, the derivation of Brillouin [26, 27], although incorrectly labeled as Eulerian but fortuitously by virtue of equation (16) leads to the correct equation along $e_1$ for laterally confined beams, is not at all valid for laterally unconfined beams.

4. Langevin radiation pressure

Various attempts to assess the acoustic radiation pressure along the propagation direction for laterally unconfined, plane waves rely on establishing that the ‘mean excess Eulerian pressure’ $\langle p_{ij} \rangle \neq 0$, along the propagation direction $e_1$ is zero, leaving the radiation pressure to be equal to the momentum flux density, $\langle \rho v_i v_i \rangle = \langle E \rangle$, in the Brillouin stress tensor, equation (69). However, as shown in section 3.2, the term identified in the Brillouin stress tensor as the momentum flux density is not the momentum flux density at all, but rather the contribution to the change in the system period resulting from the variation in the acoustic path length in response to the virtual variation in the strain, when the variation in the system period is assessed using the true velocity. And the term identified by Brillouin as the ‘mean excess Eulerian pressure’ is simply the contribution to the change in system period resulting from the change in the true sound velocity, when the variation in the system period is assessed using the true velocity.
Brillouin largely ignored the ‘mean excess Eulerian pressure’ term in applications of the Brillouin stress tensor, assuming the term to be irrelevant in assessing the radiation pressure. Several attempts to justify Brillouin’s assumption have appeared in the literature. The approach to establishing a null ‘mean excess Eulerian pressure’ has been to utilize the relationship between pressure and enthalpy. The derivations of Lee and Wang [48], Beissner [50], Beissner and Makarov [51], and Hasegawa et al [47] are representative of such approaches. Consider the wave equation in Eulerian coordinates

$$\rho \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = - \frac{\partial p^E}{\partial x_i}.$$  \((70)\)

For irrotational sound waves \(u = \nabla \varphi^{SP}\), where \(\varphi^{SP}\) is the scalar potential. Equation \((70)\) can thus be re-written as

$$\nabla \left( \frac{\partial \varphi^{SP}}{\partial t} + \frac{1}{2} |\nabla \varphi^{SP}|^2 \right) = - \frac{\nabla p^E}{\rho}.$$  \((71)\)

It is generally assumed that equation \((71)\) can be solved using the thermodynamic relationship

\[dH = (dp^\text{Th}/\rho_0) + T dS_0,\]

where \(H = H(p^\text{Th}, S_0)\) is the enthalpy per unit mass, \(S_0\) is the entropy per unit mass, and \(p^\text{Th}\) is the thermodynamic pressure [58, 61, 64]. It is critically important to recognize that the thermodynamic pressure \(p^\text{Th}\) is not the Eulerian pressure \(p^E\). The thermodynamic pressure is obtained from the thermodynamic tensions (second Piola-Kirchhoff stress) \(\epsilon_{ij}^s = \rho_0 (\partial U / \partial x_i \partial x_j)\) via the relation \(p^\text{Th} \epsilon_{ij}^s = -\sigma_{ij}\) [58, 61, 64]. For longitudinal, plane wave propagation along direction \(e_1\), \(p_{T1}^E = p_{T1}^E = (1 + u_1) p_{T1}^\text{Th} [58, 61].\) It is generally overlooked that the pressure in the thermodynamic relationship for enthalpy is the thermodynamic pressure \(p^\text{Th}\) and it is incorrectly assumed in equation \((71)\) that for adiabatic motion \(\nabla p^E / \rho = \nabla H\). It is then assumed that the pressure \(p^E\) can be expanded in a power series in the enthalpy \(H\) [48] or that

\[H - H_0 = \int_{p_0}^{p^E} \rho^{-1} dp^E\]

can be expanded in terms of the pressure \(p^E [47, 50, 51]\). The power series expansion is a key operation in the derivations to obtain the relationship for propagation along \(e_1\) (see [47, 48, 50, 51] for details)

$$\langle p^E \rangle = \langle \phi \rangle - \langle K \rangle + C.$$  \((72)\)

where \(C\) is a constant and \(\langle \phi \rangle\) and \(\langle K \rangle\) are, respectively, the time-averaged potential and kinetic energies of the wave.

Equation \((72)\) is known as Langevin’s second relation [30, 50] and is generally regarded as the expression defining the mean Eulerian pressure in a wave. For plane, progressive waves it is traditionally assumed that \(\langle \phi \rangle = \langle K \rangle\). If \(C\) is assumed to be the initial pressure \(p_0\), equation \((72)\) predicts that the mean excess Eulerian pressure is zero. This means, in regard to the traditional (incorrect) interpretation of the Brillouin stress tensor, that the acoustic radiation pressure depends only on the momentum flux density \(\langle \rho u_1 u_1 \rangle = \langle E \rangle\) for laterally unconfined, plane wave propagation along \(e_1\).

The problem with the derivations leading to equation \((72)\) is that the Eulerian pressure \(p^E\) is not the thermodynamic pressure \(p^\text{Th}\). Substituting \(\nabla H\) for \(\nabla p^E / \rho\) in equation \((71)\) does not affect the terms on the left-hand side of the equation but it changes the right-hand side of the equation from a dependence on \(p^E\) to a dependence on \(p^\text{Th}\). By substituting \(p^\text{Th}\) for \(p^E\) in the relevant equations, the arguments of [47, 48, 50, 51] lead to the relation

$$\langle p_{T1}^\text{Th} \rangle = \langle \phi \rangle - \langle K \rangle + C.$$  \((73)\)

for propagation along direction \(e_1\), rather than to equation \((71)\). For longitudinal wave propagation along \(e_1\) [58, 61, 64],

$$\langle p_{T1}^\text{Th} \rangle = \langle (1 + u_1)^{-1} p_{T1}^E \rangle = \langle (1 + u_1)^{-1} p_{T1}^L \rangle$$

$$\approx \langle p_{T1}^E \rangle - \langle u_1 p_{T1}^E \rangle = \langle p_{T1}^L \rangle = \langle p_{T1}^L \rangle + 2 \langle K \rangle.$$  \((74)\)

The last equality in equation \((74)\) follows from the virial theorem, equation \((54)\), where \(\sigma_1 = -p_{T1}^L = -p_{T1}^E\).

Substituting equation \((74)\) in equation \((73)\) leads to

$$\langle p_{T1}^E \rangle = \langle p_{T1}^L \rangle = \langle \phi \rangle - 3 \langle K \rangle + C.$$  \((75)\)

Equation \((75)\) does not yield Langevin’s second relation, equation \((72)\).

It is concluded that Langevin’s second relation, equation \((72)\), is incorrect and does not provide an assessment of the mean Eulerian pressure, as traditionally assumed. Indeed, the results of equations \((16)\) and
already suggest that since \( \langle p_1^{E} \rangle = \langle p_1^{L} \rangle \) for plane wave propagation along \( \varepsilon_1 \), Langevin’s second relation [30], \( \langle p_1^{E} \rangle = \langle \phi \rangle - \langle K \rangle + C \), and Langevin’s first relation [29], \( \langle p_1^{L} \rangle = \langle \phi \rangle + \langle K \rangle + C \), cannot both be correct. Equally important, it is seen from equation (46) that Langevin’s first relation is also incorrect, as the equation does not account analytically a priori for the difference between the elastic properties under laterally unconfined, static conditions and that of the driving acoustic wave propagating under laterally confined, dynamic conditions.

5. Experimental evidence for the present theory

Issenmann et al [52] point out that ‘despite the long-lasting theoretical controversies … the Langevin radiation pressure … has been the subject of very few experimental studies.’ Indeed, few absolute measurements obtained independently in the same experiment of the acoustic power generated by an acoustic source and the radiation force incident on a target to assess the radiation pressure-energy density relationship have been reported. Typically, experimental assessments of the radiation pressure have either relied on the assumption of the Langevin relation a priori in evaluating the transducer power output [18–22] or have considered relative measurements without directly evaluating the transducer power output (cf. [8, 9, 38, 39]). Beissner [21] points out that if acoustic radiation pressure is used to calibrate acoustic sources the ‘measured radiation force must be converted to the ultrasonic power value and this is carried out with the help of theory.’ The measured radiation force on a target is generally assumed to result from the Langevin relation, equation (47), between the radiation pressure generated by the acoustic source and the energy density of the wave [18–22]. It is appropriate to consider the implications of this assumption.

Consider a planar transducer of active area \( S_A \) that emits an idealized plane wave. The average ultrasonic power \( \langle W_{pu} \rangle \) emitted by the transducer is related to the energy density \( \langle E \rangle \) of the plane wave as

\[
\langle W_{pu} \rangle = \langle E \rangle S_A c,
\]

where \( c \) is the sound velocity in the fluid. If Langevin’s theory, equation (47), is assumed to be correct, then along the direction of plane wave propagation, the average force \( \langle F \rangle \) generated over the area \( S_A \) normal to the propagation direction is

\[
\langle F \rangle = \langle E \rangle S_A
\]

and the average ultrasonic power is \( \langle W_{pu}^{\text{Langevin}} \rangle = \langle F \rangle c \).

The present theory, however, predicts from equation (46) that for laterally unconfined plane waves along \( \varepsilon_1 \)

\[
\langle p_1^{E} \rangle - p_0 = \langle p_1^{L} \rangle - p_0 = \frac{2}{7} \langle 2K \rangle \approx \frac{3}{7} \langle E \rangle.
\]

Along the direction of plane wave propagation, the average force \( \langle F \rangle \) generated over the area \( S_A \) normal to the propagation direction is

\[
\langle F \rangle = \langle p_1^{L} \rangle - p_0 S_A = \langle \langle p_1^{L} \rangle - p_0 \rangle S_A = \frac{3}{4} \langle E \rangle S_A.
\]

Thus, \( \langle E \rangle = \langle F \rangle / S_A \) and the average ultrasonic power \( \langle W_{pu} \rangle \) generated by the transducer is assessed from the present theory to be

\[
\langle W_{pu} \rangle = \langle E \rangle S_A c = \frac{4}{3} \langle F \rangle c = \frac{4}{3} \langle W_{pu}^{\text{Langevin}} \rangle.
\]

Equation (76) states that the power emitted by the transducer is \((4/3)\) times larger than the power \( \langle W_{pu}^{\text{Langevin}} \rangle = \langle F \rangle c \) predicted by the Langevin theory.

Alternatively, Langevin’s theory predicts from equation (47) that \( \langle p_1^{L} \rangle - p_0 = \langle 2K \rangle^{\text{Langevin}} \approx \langle E \rangle^{\text{Langevin}} \)

while the present theory predicts from equation (46) that \( \langle p_1^{L} \rangle - p_0 = \langle \langle p_1^{L} \rangle - p_0 \rangle \approx \langle 2K \rangle \approx \langle E \rangle \). Since the measured pressure \( \langle p_1^{L} \rangle - p_0 \) must be the same whether using the present theory or the Langevin theory, equating the left-hand sides of equations (46) and (47) yields in agreement with equation (76).

Independent measurements of the absolute ultrasonic pressure amplitude in water have been reported by Breazeale and Dunn [23] using different methods subject to the same experimental conditions. They report a direct assessment of the transducer output, calculated from the piezoelectric constant and the voltage applied to the transducer, and measurements, referred to a force balance, of three different optical techniques and a thermoelectric probe. They find that ‘the experimental results exhibit a total range of approximately \( \pm 10 \) percent about the mean and that this mean is approximately 27 percent below that calculated (from the piezoelectric constant) from the voltage applied to the transducer.’ If it is assumed that the direct piezoelectric assessment represents the correct transducer output, then the measurements of Breazeale and Dunn [23] imply that the acoustical measurements, referred to the radiation force balance using the Langevin theory, are consistently well below the correct transducer output. This is consistent with the prediction of the present theory that the correct energy density is 33 percent greater than the value calculated from the Langevin equation.

In a similar experiment, Haran et al [24] report independent, simultaneous assessments of the acoustic power generated from a 1 MHz acoustic transducer from acousto-optic Raman-Nath diffraction measurements and measurements from a force balance in water using the Langevin equation. Critical to their assessment from the Raman-Nath measurements is the value \( 1.46 \times 10^{-10} \text{ Pa}^{-1} \) assumed for the piezo-optic coefficient. There is considerable variation reported in the literature regarding the value of the piezo-optic coefficient in water, which is found to range from \( 1.32 \times 10^{-10} \text{ Pa}^{-1} \) to \( 1.51 \times 10^{-10} \text{ Pa}^{-1} \) [72–75]. Further, the active area of the transducer may be smaller than that of the stated physical size of the transducer. For a given measured value of the
Raman-Nath parameter, the uncertainty in the magnitude of the piezo-optic coefficient and active transducer radius, as well as contributions from acoustic diffraction and attenuation, can lead to large changes in the calculated value of the intensity from the Raman-Nath measurements. For example, using in [24] the value $1.32 \times 10^{-10}$ Pa$^{-1}$ for the piezo-optic coefficient and assuming that the active diameter of the transducer is 5 percent smaller than the stated physical diameter lead to an increase in the intensity assessed from the Raman-Nath measurements by 36 percent. This value is consistent with the prediction of the present theory that the correct energy density (and intensity) is 33 percent greater than the value calculated from the Langevin equation. It is concluded that although the measurements of Breazeale and Dunn [23] and Haran et al [24] provide tentative evidence for the validity of the present model, more accurate and precise measurements are needed for model confirmation.

6. Conclusion

Equation (46) shows that the mean excess Eulerian and Lagrangian pressures along the propagation direction for longitudinal waves in laterally unconfined, plane wave beams are exactly equal with magnitude $(3/4)(2K)$. This result is quite different from that derived from the Langevin expression for laterally unconfined, plane waves, which posits that the mean Lagrangian pressure is obtained as $\langle p_{1}^{Langevin} \rangle = \langle 2K \rangle \approx [E]$ [29]. A number of analytical efforts have been published [28, 29, 34, 35, 40–42, 47–51] in various attempts to validate Langevin’s result, equation (47). The derivations do not distinguish analytically a priori, as done in section 3.1, that the elastic properties under static, laterally unconstrained conditions associated with the radiation pressure are quite different from those of the driving (dynamical) acoustic wave propagating under laterally constrained conditions. Recognition of the difference occurs a posteriori in [28, 29, 34, 35, 40–42, 47–51], which entail various erroneous and, in some cases, somewhat contrived arguments involving fluid flow to establish Langevin’s result. Central to the arguments is the assumption that for laterally unconfined, plane wave propagation the Lagrangian and Eulerian radiation pressures are different, which, as shown in section 2.4, is not correct.

Other attempts to assess the acoustic radiation pressure along the propagation direction for laterally unconfined, plane waves are based on an application of the Brillouin stress tensor, which Brillouin assumed is an expression of the acoustic radiation stress in Eulerian coordinates [26, 27]. Brillouin, who employed the Boltzmann-Ehrenfest Principle of Adiabatic Invariance in his derivation, obtained that the Brillouin stress tensor is composed of two contributions – a ‘mean excess Eulerian stress’ contribution and a momentum flux density contribution. It is shown in section 3.2 that the Brillouin stress tensor is not an expression in Eulerian coordinates, but rather an expression in Lagrangian coordinates. The ‘mean excess Eulerian stress’ in the Brillouin stress tensor is not a Eulerian stress at all, but rather the contribution to the fractional change in the system period (in terms of the B-E Principle) associated with the change in the true sound velocity (defined as the ratio of the length of the sound path in the deformed state to the propagation time in the deformed state). The contribution that Brillouin attributed to the momentum flux density is actually the contribution to the fractional variation in the period of acoustic oscillations resulting from a change in the sample length from slow, virtual variations in the strain parameter (generalized displacement in terms of the B-E Principle).

Efforts to assess the acoustic radiation pressure along the propagation direction for laterally unconfined, plane waves from the Brillouin stress tensor rely on establishing that the ‘mean excess Eulerian pressure’ $\langle p_{1}^{E} \rangle - p_{0}$ along the propagation direction $\epsilon_{1}$ is zero, leaving the radiation pressure equal, incorrectly, to the momentum flux density $\langle \rho v_{1} v_{1} \rangle$ (where $v_{1} = \partial u_{1}/\partial t =$ particle velocity) in the Brillouin stress tensor [26, 27]. Indeed, Brillouin largely ignored the ‘mean excess Eulerian pressure’ term in applications of the Brillouin stress tensor. The approach to establishing $\langle p_{1}^{E} \rangle - p_{0} = 0$ has been to utilize the relationship between pressure and enthalpy to obtain ‘Langevin’s second relation’ [47, 48, 50, 51], $\langle p_{1}^{E} \rangle = \langle \phi \rangle - \langle K \rangle + C$, equation (72). For plane waves it is generally assumed in equation (72) that $\langle \phi \rangle = \langle K \rangle$ and $C = p_{0}$, which lead to the ‘mean excess Eulerian pressure’ $\langle p_{1}^{E} \rangle - p_{0} = 0$. The derivations in [47, 48, 50, 51] are representative of the thermodynamic approaches leading to $\langle p_{1}^{E} \rangle - p_{0} = 0$. It is shown in section 4 that the error in [47, 48, 50, 51] results from the incorrect assumption that the thermodynamic pressure associated with the enthalpy is the Eulerian pressure $p_{1}^{E}$ rather than the thermodynamic tensions (second Piola-Kirchhoff stress or pressure) $p_{1}^{Th}$ [58, 61, 64]. Using the second Piola-Kirchhoff pressure $p_{1}^{Th}$ in the derivations in [47, 48, 50, 51] and employing the relationship $\langle p_{1}^{Th} \rangle = \langle p_{1}^{E} \rangle + 2\langle K \rangle$, equation (74), between the second Piola-Kirchhoff radiation pressure $p_{1}^{Th}$ and the Eulerian radiation pressure $p_{1}^{E}$ lead to $\langle p_{1}^{E} \rangle = \langle \phi \rangle - 3\langle K \rangle + C$, equation (75), rather than Langevin’s second relation, $\langle p_{1}^{E} \rangle = \langle \phi \rangle - \langle K \rangle + C$, equation (72). It is concluded that Langevin’s second relation, equation (72), is incorrect and does not provide an assessment of the mean Eulerian pressure, as traditionally assumed. Indeed, the results of equations (16) and (46) already suggest that since $\langle p_{1}^{E} \rangle = \langle p_{1}^{Th} \rangle$ for plane wave propagation along $\epsilon_{1}$, Langevin’s second relation, equation (72), and Langevin’s first relation, $\langle p_{1}^{E} \rangle = \langle \phi \rangle + \langle K \rangle + C$, cannot both be correct.
The derivations of Langvin and Brillouin fail to recognize, as shown from finite deformation theory, that for laterally unconfined plane waves the Lagrangian and Eulerian radiation pressures are exactly equal and that the Brillouin stress tensor is a Lagrangian tensor - not a Eulerian tensor. Equation (46) establishes that the radiation pressure along the propagation direction is \( (3/4)(2K) \) and zero in the directions normal to the wave propagation direction. This result is consistent with the experimental studies of Herrey \(^1\), who shows that the radiation pressure in laterally unconfined fluids is anisotropic, and of Rooney \(^2\), who shows that the radiation pressure in such media is independent of the dynamic acoustic nonlinear parameter \( \beta = (-A_{111}/A_{11}) \) of the fluid - although equations (34) show that for laterally unconstrained beams the static nonlinearity parameter \( \beta_s = (-A_{111}/A_{11}) = 1 \) for all inviscid fluids.

The present model deviates from previous approaches in several, quite fundamental ways. The traditional derivations for laterally unconfined, plane waves in fluids do not account \textit{a priori} for the difference between the elastic properties under static, laterally unconfined conditions (giving rise to free fluid flow) and that of the driving acoustic wave propagating under laterally confined conditions (that do not permit free flow). Rather than accounting \textit{a priori} for the difference in elastic properties, a patchwork of \textit{a posteriori} assumptions, definitions, and arguments has been used in various attempts to quantify the radiation pressure for laterally unconfined, plane waves. The previous derivations are typically based on a number of misconceptions that have permeated the acoustics literature including (a) a widespread misunderstanding of Lagrangian and Eulerian quantities and of the transformation between them (addressed in detail in section 2), (b) the misinterpretation by Brillouin of terms leading to the Brillouin stress tensor, discussed in section 3.2, and (c) the assumption that the pressure defined by the enthalpy in deriving Langvin’s second relation is the Eulerian pressure rather than the thermodynamic pressure (second Piola-Kirchhoff pressure), as discussed in section 4. The present work corrects these misconceptions and provides a coherent, first principles examination of acoustic radiation pressure based on finite deformation theory. Equally important, as shown in section 5, a limited amount of experimental data have been reported that provide tentative evidence for the validity of the present theory. More precise and accurate measurements, however, are necessary for confirmation of the present theory.

The acoustic radiation pressure is used in a variety of applications \([1–24]\), many of which depend on a reliable assessment of the force on a target generated by an acoustic source. It is appreciated that the measurements for diffracted and focused beams are not described by simple plane wave propagation, but because of the difference between the present value of \((3/4)(2K)\) for the acoustic radiation pressure along the propagation direction for laterally unconfined, plane waves and the value \((2K) \approx \langle E \rangle\) from the Langvin theory or the momentum density \((\rho \nu \nu)\) from the Brillouin stress tensor, it would seem prudent to re-examine relevant applications in view of the present theoretical results. For example, a recent experimental study \([76]\) suggests that acoustic radiation pressure may play a role in pulmonary capillary hemorrhage (PCH) but ‘only a full understanding of PCH mechanisms will allow development of science-based safety assurance for pulmonary ultrasound.’ The larger acoustic power levels predicted in the present manuscript could contribute to acquiring ‘a full understanding.’

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ORCID iDs

John H Cantrell @ https://orcid.org/0000-0002-6785-695X

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