DUALITY AND TRIPLE STRUCTURES

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Dedicated to Alan Weinstein
on the occasion of his sixtieth birthday

Summary. We recall the basic theory of double vector bundles and the canonical pairing of their duals introduced by the author and by Konieczna and Urbanski. We then show that the relationship between a double vector bundle and its two duals can be understood simply in terms of an associated cotangent triple vector bundle structure. In particular we show that the dihedral group of the triangle acts on this triple via forms of the isomorphisms $R$ introduced by the author and by Ping Xu. We then consider the three duals of a general triple vector bundle and show that the corresponding group is neither the dihedral group of the square, nor the symmetry group on four symbols.

Double structures first appeared in Poisson geometry with Alan’s ground-breaking work on symplectic groupoids [Coste, Dazord, and Weinstein 1987, Weinstein 1987] and Poisson groupoids [Weinstein 1988]. The most fundamental example of a symplectic groupoid, the cotangent groupoid $T^*G$ of an arbitrary Lie groupoid $G$, introduced in [Coste et al. 1987], is a groupoid object in the category of vector bundles. An arbitrary Poisson Lie group can be integrated to a symplectic double groupoid [Lu and Weinstein 1989]. At a simpler level, a Poisson structure on a vector bundle is linear [Courant 1990] if and only if the associated anchor is a morphism of certain double vector bundles.

All these phenomena involve doubles in the categorical sense: taking $S$ to denote, for example, ‘vector bundle’ or ‘Lie groupoid’, a double $S$ is an $S$ object in the category of all $S$. (Groupoid objects in the category of vector bundles, named $\mathbb{VB}$–groupoids by Pradines 1988, may be regarded as double groupoids of a special type.) More generally, multiple $S$ structures are the $n$–fold extension of this notion of double.
The key link between Poisson geometry and double structures lies in properties of the Poisson anchor. If a Poisson manifold \( P \) is a vector bundle on base \( M \), then the Poisson structure is linear if and only if \( \pi^\# : T^*P \to TP \) is a morphism of double vector bundles. If \( P \) is instead a Lie groupoid on base \( M \), then the groupoid is a Poisson groupoid if and only if \( \pi^\# \) is a morphism of \( \mathcal{VB} \)–groupoids. Thus the Poisson anchor naturally appears as a map of double structures, and indeed many of the surprising basic features of Poisson and symplectic groupoids are not really so much consequences of Poisson or symplectic geometry, as consequences of the duality properties of the associated double structures. This point of view is developed further in Mackenzie 2004; in particular the theory of Poisson groupoids may be developed entirely in terms of groupoid theory and double structures of various kinds.

The present paper is concerned with the duality of double and higher multiple vector bundles. A double vector bundle has two duals which are themselves in duality and we show here that the various combinations of the two dualization operations gives rise to the dihedral (or symmetric) group of order six. We show in §5 and §6 that a double vector bundle and its two duals form the three lower faces of a triple vector bundle, the opposite vertex of which is the cotangent of the double space. This encapsulates and makes symmetric the relations between a double vector bundle and its duals, which can otherwise seem rather involved. One may think of three double vector bundles with a common vertex and appropriate pairings as constituting a two (sic) dimensional version of the familiar notion of pairing of vector bundles; we call this a cornering.

In §7 we consider the process of dualizing the structures in a triple vector bundle. This may appear to be a routine extension of the double case, but we show that the group of dualization operations here is not the dihedral group of the square, or the symmetric group on four symbols, but a group of order 72. This appears to demonstrate that the behaviour of duality for \( n \)–fold vector bundles may be a less routine extension of the double case than one might have expected. In the final §8 we formulate some general principles which we believe do hold for general multiple vector bundles.

The study of general double vector bundles was begun by Pradines 1974, though the case of the tangent double of an ordinary vector bundle had been used in connection theory since the late 1950s. More than a decade later, Pradines 1988 introduced the dualization process for \( \mathcal{VB} \)–groupoids; in the case of double vector bundles this is the duality construction presented here in §3 Theorem 3.2 is from Mackenzie 1999 and was also found by Konieczna and Urbaniski 1999. The idea of deriving the pairing (17) from the cotangent triple was noted in Mackenzie 2002.

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I am grateful to Moty Katzman for introducing me to the GAP software, which was very valuable in §7.

1 Double vector bundles

**Definition 1.1** A double vector bundle \((D; A, B; M)\) is a system of four vector bundle structures

\[
\begin{array}{c}
D \\ q^D_B \\
q^D_A \\
\downarrow \\
A \\ q_A \\
\rightarrow \\
B \\ q_B \\
\downarrow \\
M \\
\end{array}
\]

in which \(D\) has two vector bundle structures, on bases \(A\) and \(B\), which are themselves vector bundles on \(M\), such that each of the structure maps of each vector bundle structure on \(D\) (the bundle projection, addition, scalar multiplication and the zero section) is a morphism of vector bundles with respect to the other structure.

We refer to \(A\) and \(B\) as the side bundles of \(D\), and to \(M\) as the double base. In the two side bundles the addition, scalar multiplication and subtraction are denoted by the usual symbols +, juxtaposition, and -. We distinguish the two zero-sections, writing \(0^A: M \rightarrow A, m \mapsto 0^A_m\), and \(0^B: M \rightarrow B, m \mapsto 0^B_m\). We may denote an element \(d \in D\) by \((d; a, b; m)\) to indicate that \(a = q^D_A(d), b = q^D_B(d), m = q_B(q^D_B(d)) = q_A(q^D_A(d))\).

The notation \(q^D_A\) is clear; when the base of the bundle is the double base we write \(q_A\), for example, rather than \(q^D_A\).

In the vertical bundle structure on \(D\) with base \(A\) the vector bundle operations are denoted \(\hat{+}, \hat{}, \hat{-}\), with \(\hat{0}^A: A \rightarrow D, a \mapsto 0^A_a\), for the zero-section. Similarly, in the horizontal bundle structure on \(D\) with base \(B\) we write \(\hat{+}, \hat{}, \hat{-}\) and \(\hat{0}^B: B \rightarrow D, b \mapsto 0^B_b\). For \(m \in M\) the double zero \(\hat{0}^A_m = \hat{0}^B_m\) is denoted \(\circ_m\) or \(0^2_m\). The two structures on \(D\), namely \((D, q^D_B, B)\) and \((D, q^D_A, A)\), will occasionally be denoted \(\hat{D}_B\) and \(\hat{D}_A\), respectively.

In dealing with general double vector bundles such as \(\mathfrak{H}\), we thus usually label objects and operations in the two structures on \(D\) by the symbol for the base over which they take place. The words ‘horizontal’ and ‘vertical’ may be used as an alternative, but need to be referred to the arrangement in the diagram or the sequence in \((D; A, B; M)\). When considering examples in which \(A = B\), the words ‘horizontal’ and ‘vertical’ become necessary, and we use \(H\) and \(V\) as labels to distinguish the two structures on \(D\).
Although the concept of double vector bundle is symmetric, most examples are not; in the sequel it will be important to distinguish between $D$ and its flip in Figure 1(a), in which the arrangement of the two structures is reversed.

\[
\begin{array}{c}
\begin{array}{ccc}
D^{A_{0}} & \xrightarrow{q_{A}} & A \\
q_{B} & \downarrow & \\
B & \rightarrow & M
\end{array}
& & 
\begin{array}{ccc}
T(q) & \xrightarrow{T} & TM \\
\downarrow & & \downarrow \\
E & \rightarrow & M
\end{array}
\end{array}
\]

Fig. 1.

In such processes it is not the absolute arrangement which is significant, but the distinction between whichever arrangement is taken at the start, and its flip.

The condition that each addition in $D$ is a morphism with respect to the other is:

\[
(d_{1} + b_{2}) + (d_{3} + d_{4}) = (d_{1} + d_{3}) + (d_{2} + d_{4})
\]  

(2)

for quadruples $d_{1}, \ldots, d_{4} \in D$ such that $q_{B}^{D}(d_{1}) = q_{B}^{D}(d_{2})$, $q_{B}^{D}(d_{3}) = q_{B}^{D}(d_{4})$, $q_{A}^{D}(d_{1}) = q_{A}^{D}(d_{3})$, and $q_{A}^{D}(d_{2}) = q_{A}^{D}(d_{4})$. Next,

\[
t \cdot (d_{1} + b_{2}) = t \cdot d_{1} + t \cdot d_{2},
\]

(3)

for $t \in \mathbb{R}$ and $d_{1}, d_{2} \in D$ with $q_{B}^{D}(d_{1}) = q_{B}^{D}(d_{2})$; similarly

\[
t \cdot (d_{1} + d_{2}) = t \cdot d_{1} + t \cdot d_{2},
\]

(4)

for $t \in \mathbb{R}$ and $d_{1}, d_{2} \in D$ with $q_{A}^{D}(d_{1}) = q_{A}^{D}(d_{2})$. The two scalar multiplications are related by

\[
t \cdot (u \cdot b_{d}) = u \cdot (t \cdot d),
\]

(5)

where $t, u \in \mathbb{R}$ and $d \in D$.

Lastly, for compatible $a, a' \in A$ and compatible $b, b' \in B$, and $t \in \mathbb{R}$,

\[
\tilde{0}_{a+a'} = \tilde{0}_{a} + \tilde{0}_{a'}, \quad \tilde{0}_{ta} = t \cdot \tilde{0}_{a},
\]

(6)

and

\[
\tilde{0}_{b+b'} = \tilde{0}_{b} + \tilde{0}_{b'}, \quad \tilde{0}_{tb} = t \cdot \tilde{0}_{b}.
\]

(7)

Equations (2)–(7) are known as the interchange laws.

**Definition 1.2** A morphism of double vector bundles

\[
(\varphi; \varphi_{A}, \varphi_{B}; f): (D; A, B; M) \rightarrow (D'; A', B'; M')
\]
Example 1.3 For an ordinary vector bundle \((E, q, M)\), applying the tangent functor to each of the bundle operations yields the tangent prolongation vector bundle \((TE, T(q), TM)\). The zero section is \(T(0): TM \to TE\). We denote the addition by \(+\) and the scalar multiplication and subtraction by \(-\). Together with the standard structure \((TE, p_E, E)\), we have a double vector bundle \((TE; E, TM; M)\), shown in Figure 1(b). There is no preferred arrangement for the side bundles of \(TE\).

Example 1.4 Let \(A, B\) and \(C\) be any three vector bundles on the one base \(M\), and write \(D\) for the pullback manifold \(A \times_M B \times_M C\) over \(M\). Then \(D\) may be regarded as the direct sum \(q_A^1B \oplus q_A^1C\) over \(A\), and as the direct sum \(q_B^1A \oplus q_B^1C\) over \(B\), and with respect to these two structures, \(D\) is a double vector bundle with side bundles \(A\) and \(B\). We call this the trivial double vector bundle over \(A\) and \(B\) with core \(C\). It is tempting, but incorrect, to denote it by \(A \oplus B\).

Example 1.5 A double vector bundle \((D; A, B; M)\) may be pulled back over both of its side structures simultaneously. Suppose given vector bundles \((A', q_{A'}, M')\) and \((B', q_{B'}, M')\) and morphisms \(\varphi: A' \to A\) and \(\psi: B' \to B\), both over a map \(f: M' \to M\). Let \(D'\) denote the set of all \((a', d, b')\) such that \(\varphi(a') = q_A^D(d)\), \(\psi(b') = q_B^D(d)\) and \(q_A^D(a') = q_{B'}(b')\). Then, with the evident structures, \((D'; A', B'; M')\) is a double vector bundle and the projection \(D' \to D\) is a morphism over \(\varphi\), \(\psi\) and \(f\).

Further examples follow later in the paper.

2 The core and the core sequences

Until Example 2.2 consider a fixed double vector bundle \((D; A, B; M)\). Each of the bundle projections is a morphism with respect to the other structure and so has a kernel (in the ordinary sense); denote by \(C\) the intersection of the two kernels:

\[ C = \{ c \in D \mid \exists m \in M \text{ such that } q_B^D(c) = 0^B_m, \; q_A^D(c) = 0^A_m \}. \]

This is an embedded submanifold of \(D\). We will show that it has a well–defined vector bundle structure with base \(M\), projection \(q_C\) which is the restriction
of \( q_B \circ q_B^D = q_A \circ q_A^D \) and addition and scalar multiplication which are the restrictions of either of the operations on \( D \).

Note first that the two additions coincide on \( C \) since
\[
c + ^c = \left( c + _c \circ_m \right) + _b \left( \circ_m + ^c \right) = \left( c + ^c \right) = c + ^c,
\]
for \( c, c' \in C \) with \( q_C(c) = q_C(c') \), using (2). From this it follows that \( t \cdot _c = t \cdot _a c \) for integers \( t \), and consequently for rational \( t \), and thence for all real \( t \) by continuity.

It will often be helpful to distinguish between \( c \in C \), regarding \( C \) as a distinct vector bundle, and the image of \( c \) in \( D \), which we will denote by \( \overline{c} \). Given \( c, c' \in C \) with \( q_C(c) = q_C(c') \) there is a unique \( c + c' \in C \) with
\[
\overline{c + c'} = \overline{c} + \overline{c'} = \overline{c} + \overline{c'},
\]
and given \( t \in \mathbb{R} \) there is a unique \( t \cdot \overline{c} \in C \) such that
\[
\overline{t \cdot c} = t \cdot \overline{c} = t \cdot \overline{c}.
\]

It is now easy to prove that \((C, q_C, M)\) is a (smooth) vector bundle, which we call the core of \((D; A, B; M)\).

**Theorem 2.1** There is an exact sequence
\[
\begin{array}{c}
q_A C \xrightarrow{\tau_A} D_A \xrightarrow{(q_B^D)^\dagger} q_A B
\end{array}
\tag{8}
\]
of vector bundles over \( A \), and an exact sequence
\[
\begin{array}{c}
q_B C \xrightarrow{\tau_B} D_B \xrightarrow{(q_A^D)^\dagger} q_B A
\end{array}
\tag{9}
\]
of vector bundles over \( B \), where the injections are \( \tau_A: (a, c) \mapsto \overline{a} \cdot \overline{c} \) and \( \tau_B: (b, c) \mapsto \overline{b} + \overline{c} \), respectively, and \((q_B^D)^\dagger\) and \((q_A^D)^\dagger\) denote the maps induced by \( q_B^D \) and \( q_A^D \) into the pullback bundles.

**Proof.** Take \( a \in A_m, c \in C_m \) where \( m \in M \). Then both \( \overline{a} \cdot \overline{c} \) and \( \overline{c} \) project under \( q_B^D \) to \( 0_m \). So \( \overline{a} \cdot \overline{c} \) is defined and also projects under \( q_B^D \) to \( 0_m \). That \( \tau_A \) is linear over \( A \) follows from the interchange laws.

Suppose that \( d \in D \) has \( q_B^D(d) = 0_m \) for some \( m \in M \). Write \( a = q_A^D(d) \). Then \( d - \overline{a} \) is defined and \( q_B^D(d - \overline{a}) = 0_m \). On the other hand, \( \overline{a} \cdot (d - \overline{a}) = a - a = 0_m \). So \( d - \overline{a} \in C_m \). This establishes the exactness of (8). The proof of (9) is similar. \( \square \)

We refer to (8) as the core sequence of \( D \) over \( A \), and to (9) as the core sequence of \( D \) over \( B \).

If \( (\varphi; \varphi_A, \varphi_B; f): (D; A, B; M) \to (D'; A', B'; M') \) is a morphism of double vector bundles, then \( \varphi: D \to D' \) maps \( C \) into \( C' \), the core of \( D' \). It is clear that the restriction, \( \varphi_C: C \to C' \), is a morphism of the vector bundle structures on the cores, over \( f: M \to M' \).
Examples 2.2 For $E$ an ordinary vector bundle, consider the tangent double vector bundle $(TE; E, TM; M)$. The kernel of $T(q)$ consists of the vertical tangent vectors and the kernel of $p_E$ consists of the vectors tangent along the zero section; their intersection is naturally identified with $E$ itself. For clarity we distinguish $X \in E$ from the core element $\overline{X} \in TE$.

The injection map for $TE$ over $E$ is the map $\tau$ which sends $(X, Y) \in E_m \times E_m$ to the vector in $E_m$ which has tail at $X$ and is parallel to $Y$. In terms of the prolongation structure, $\tau(X, Y) = \overline{0}_X + Y$. The injection map over $TM$ is $\upsilon: (x, Y) \mapsto T(0)(x) + Y$.

For $\varphi: E \to E'$ a morphism of vector bundles over $f: M \to M'$, the morphism $T(\varphi)$ of the tangent double vector bundles induces $\varphi: E \to E'$ on the cores. In the case where $\varphi$ and $f$ are surjective submersions, the vertical subbundles form a double vector subbundle (in an obvious sense) $(T^\varphi E; E, T^f M; M)$ of $TE$, the core of which is the kernel (in the ordinary sense) of $\varphi$.

The trivial double vector bundle $A \times_M B \times_M C$ of [14] has core $C$.

3 Duals of double vector bundles

Throughout this section we consider a double vector bundle as in [11], with core bundle $C$. We will show that dualizing either structure on $D$ leads again to a double vector bundle; in the case of the dual of the structure over $A$ we denote this by

$$
\begin{array}{ccc}
D \uparrow A & \xrightarrow{q^A_C} & C^* \\
\downarrow q^A & & \downarrow q_{C^*} \\
A & \xrightarrow{q_A} & M,
\end{array}
$$

(10)

Here $C^*$ is the ordinary dual of $C$ as a vector bundle over $M$. We denote the dual of $D$ as a vector bundle over $A$ by $D \uparrow A$. (We will later modify this notation for cases in which $A$ and $B$ are identical.)

The vertical structure in [11] is the usual dual of the bundle structure on $D$ with base $A$, and $q_{C^*}: C^* \to M$ is the usual dual of $q_C: C \to M$. The additions and scalar multiplications in the side bundles of [11] will be denoted by the usual plain symbols as before. In the two structures on $D \uparrow A$ we write $\pm_A$, $\cdot_A$, $\overline{.}$ and $\pm^*_A$, $\cdot^*_A$, $\overline{.}$. The zero of $D \uparrow A$ above $a \in A$ is denoted $0_A^A$.

The unfamiliar projection $q^{A, A}_{C^*}: (D \uparrow A) \to C^*$ is defined by

$$
\langle q^{A, A}_{C^*}(\Phi), c \rangle = \langle \Phi, 0^A_a + \overline{c} \rangle
$$

(11)
where \( c \in C_m \), \( \Phi: (q_A^D)^{-1}(a) \to \mathbb{R} \) and \( a \in A_m \). The addition \( + \) in \( D \uparrow A \to C^* \) is defined by
\[
(\Phi + \Phi', d + d') = (\Phi, d) + (\Phi', d')
\] (12)
That this is well-defined depends strongly on the condition \( q_A^D \circ \Phi = q_A^D \circ \Psi \).

Similarly, define
\[
\langle t \cdot C^* \Phi, t \cdot B^* d \rangle = t \langle \Phi, d \rangle,
\]
for \( t \in \mathbb{R} \) and \( d \in D \) with \( q_A^D(d) = \tilde{\tau} \).
The zero above \( \kappa \in C_m^* \) is denoted \( \tilde{0} \) and is defined by
\[
\langle \tilde{0} \cdot C^* \kappa, \tilde{0} \cdot B^* b \rangle = \langle \kappa, c \rangle
\] (13)
where \( b \in B_m \), \( c \in C_m \).

The core element \( \psi \) corresponding to \( \psi \in B^*_m \) is
\[
\langle \psi, \tilde{0} \cdot C^* \kappa \rangle = \langle \psi, c \rangle.
\]

It is straightforward to verify that (10) is a double vector bundle, and that its core is \( B^* \). We call (10) the \textit{vertical dual} or \textit{dual over} \( A \) of (1).

As for any double vector bundle, there are exact sequences
\[
q_A^D B^* \xrightarrow{\sigma_A} D \uparrow A \quad (q_A^{D'})^! \xrightarrow{q_A^C} q_A^C^*,
\] (14)
of vector bundles over \( A \) and
\[
q_C^B B^* \xrightarrow{\sigma_{C^*}} D \uparrow A \quad (q_C^{\tau})^! \xrightarrow{q_C^A} q_C^A,
\] (15)
of vector bundles over \( C^* \). Here the two injections are given by
\[
\sigma_A(a, \psi) = \tilde{0}_A^D + \tilde{\psi}, \quad \sigma_{C^*}(\kappa, \psi) = \tilde{0}_A^D + \tilde{\psi},
\]
where \( a \in A, \psi \in B^*, \kappa \in C^* \). It is easily seen that
\[
\langle \sigma_A(a, \psi), d \rangle = \langle \psi, q_A^D(d) \rangle
\]
for \( d \in D \) and so \( \sigma_A \) is precisely the dual of \( (q_A^D)^! \). It is clear from the definition of \( q_A^D \) that \( (q_A^{D'})^! = \tau^*_\psi \). Thus (13) is precisely the dual of the core exact sequence (3).

For the sequence over \( C^* \) we have
\[
\langle \sigma_{C^*}(\kappa, \psi), \tilde{0}_A^D + \tilde{\psi} \rangle = \langle \kappa, c \rangle + \langle \psi, b \rangle
\]
for \( \kappa \in C_m^*, \psi \in B^*_m, x \in B_m, c \in C_m \).

The proof of the following result is straightforward. In Figure 2(a) and in similar figures in future, we omit arrows which are the identity.

\textbf{Proposition 3.1} Consider a morphism of double vector bundles, as in Figure 2(a), which preserves the horizontal side bundles, and which has core morphism \( \varphi_C: C \to C' \), where \( C' \) is the core of \( D' \). Dualizing \( \varphi \) as a morphism of vector bundles over \( A \), we obtain a morphism \( \varphi^! \) of double vector bundles over \( A \) and \( \varphi^* \), as in Figure 2(b), with core morphism \( \varphi^*_B \).
This completes the description of the vertical dual of (1). There is of course also a horizontal dual

\[ D \times A \rightarrow C^* \]

with core \( A^* \rightarrow M \), defined in an analogous way.

The following result is an entirely new phenomenon, arising from the double structures.

**Theorem 3.2** [Mackenzie 1999, Konieczna and Urbański 1999] There is a natural (up to sign) duality between the bundles \( D \times A \) and \( D \times B \) over \( C^* \) given by

\[ [\Phi, \Psi] = \langle \Phi, d \rangle_A - \langle \Psi, d \rangle_B \]

(17)

where \( \Phi \in D \uparrow A \), \( \Psi \in D \uparrow B \) have \( q_C^A(\Phi) = q_C^B(\Psi) \) and \( d \) is any element of \( D \) with \( q_A^D(d) = q_A^A(\Phi) \) and \( q_B^D(d) = q_B^B(\Psi) \).

Each of the pairings on the RHS of (17) is a canonical pairing of an ordinary vector bundle with its dual, the subscripts there indicating the base over which the pairing takes place.

**Proof.** Let \( \Phi \) and \( \Psi \) have the forms \((\Phi; a, \kappa; m)\) and \((\Psi; \kappa, b; m)\). Then \( d \) must have the form \((d; a, b; m)\). If \( d' \) also has the form \((d'; a, b; m)\) then there is a \( c \in C_m \) such that \( d = d' + \frac{1}{a} (0^A_a \uparrow_\beta) \), and so...
\[ \langle \Phi, d \rangle_A = \langle \Phi, d' \rangle_A + \langle \kappa, c \rangle \]

by (11). By the interchange law (2) we also have \( d = d' + \langle 0_B \rangle_A + \langle \tau \rangle_A \) and so

\[ \langle \Psi, d \rangle_B = \langle \Psi, d' \rangle_B + \langle \kappa, c \rangle. \]

Thus (17) is well defined. To check that it is bilinear is routine. It remains to prove that it is non-degenerate.

Suppose \( \Phi, \) given as above, is such that \( \langle \Phi, \Psi \rangle = 0 \) for all \( \Psi \in (q_0^B)^{-1}(\kappa). \) Take any \( \varphi \in A_m^* \) and consider \( \Psi = \tilde{\psi} \). Then, taking \( d = \tilde{0}_a \) we find \( \langle \Phi, d \rangle_A = 0 \) and \( \langle \Psi, d \rangle_B = \langle \varphi, a \rangle. \) Thus \( \langle \varphi, a \rangle = 0 \) for all \( \varphi \in A_m^* \) and so \( a = 0_A^A. \) It therefore follows from the horizontal exact sequence for \( D \uparrow A \) that

\[ \Phi = \tilde{0}_A + \tilde{\psi} \]

for some \( \psi \in B_m^* \). Now taking any \( c \in C_m \) and defining \( d = \tilde{0}_b + \tilde{\tau}, \) we find that

\[ \langle \Phi, d \rangle_A = \langle \kappa, c \rangle + \langle \psi, b \rangle \quad \text{and} \quad \langle \Psi, d \rangle_B = \langle \kappa, c \rangle. \]

So \( \langle \psi, b \rangle = 0 \) for all \( b \in B_m, \) since a suitable \( \Psi \) exists for any given \( b. \) It follows that \( \psi = 0 \in B_m^* \) and so \( \Phi \) is indeed the zero element over \( \kappa. \) Thus the pairing (17) is nondegenerate.

Note several special cases:

\[ \begin{align*}
\langle 0_A^A, 0_B^B \rangle &= 0, & \langle 0_A^A, \tilde{0}_b^B \rangle &= 0, \\
\langle \tilde{0}_a^A, \tilde{\psi} \rangle &= -\langle \varphi, a \rangle, & \langle \tilde{\psi}, \tilde{0}_b^B \rangle &= \langle \psi, b \rangle, & \langle \tilde{\psi}, \tilde{\tau} \rangle &= 0, \\
\langle \tilde{0}_a^A + \tilde{\psi}, \tilde{\psi} \rangle &= \langle \psi, b \rangle, & \langle \Phi, \tilde{0}_a^A + \tilde{\tau} \rangle &= -\langle \varphi, a \rangle,
\end{align*} \]

where \( b \in B, \) \( a \in A, \) \( \varphi \in A^*, \) \( \psi \in B^* \) and we have \( \langle \Psi; \kappa, b; m \rangle \in D \uparrow B \) and \( \langle \Phi; a, \kappa; m \rangle \in D \uparrow A. \)

Although we have proved that \( D \uparrow A \) and \( D \uparrow B \) are dual as vector bundles over \( C^* \), we have not yet considered the relationships between the other structures present. This is taken care of by the following result, the proof of which is straightforward.

**Proposition 3.3** [Mackenzie 1999] Let \( (D; A, B; M) \) and \( (E; A, W; M) \) be double vector bundles with a side bundle \( A \) in common, and with cores \( C \) and \( L \) respectively. Suppose given a nondegenerate pairing \( \langle , \rangle \) of \( D \) over \( A \) with \( E \) over \( A \), and two further nondegenerate pairings, both denoted \( \langle , \rangle \), of \( B \) with \( L \) and of \( C \) with \( W \), such that

(i). for all \( b \in B, \ell \in L, \) \( \langle \tilde{0}_b^B, \tilde{\ell} \rangle = \langle b, \ell \rangle; \)

(ii). for all \( c \in C, \) \( w \in W, \) \( \langle \tilde{\psi}, \tilde{0}_w^W \rangle = \langle c, w \rangle; \)

(iii). for all \( c \in C, \ell \in L, \tilde{\ell}, \tilde{\psi} \) such that \( \langle \tilde{0}_b^B(d_1), \tilde{0}_w^W \rangle = \langle \tilde{0}_b^B(d_2), \tilde{0}_w^W \rangle, \)

(iv). for all \( d_1, d_2 \in D, e_1, e_2 \in E \) such that \( q_0^D(d_1) = q_0^D(d_2), \)

\[ \langle q_E^E(e_1), q_E^E(e_2) \rangle = \langle q_E^E(e_1), q_E^E(e_2) \rangle, \]

we have

\[ \langle d_1 \uplus \tilde{d}_2, e_1 \uplus \tilde{e}_2 \rangle = \langle d_1, e_1 \rangle + \langle \tilde{d}_2, \tilde{e}_2 \rangle. \]
(v). for all \( d \in D, \ e \in E \) such that \( q_D^A(d) = q_E^C(e) \) and all \( t \in \mathbb{R} \), we have
\[
\langle t, d, t \rangle = t, d, e
\]
(In all the above conditions we assume the various elements of the side bundles lie in compatible fibres over \( M \).)

Then the map \( Z : D \to E \) defined by \( (Z(d), e)_A = d, e \) is an isomorphism of double vector bundles, with respect to \( id : A \to A \) and the isomorphisms \( B \to L^* \) and \( C \to W^* \) induced by the pairings in (i) and (ii).

A pairing \( \langle \ , \ \rangle \) satisfying the conditions of 3.3 is called a pairing of the double vector bundles.

Applying this result to the pairing (17) of \( D \times A \) and \( D \times B \), we find that the induced pairing of \( B \) with \( B^* \) is the standard one, but that of \( A^* \) with \( A \) is the negative of the standard pairing. Hence the signs in the following result are unavoidable.

**Corollary 3.4** The pairing (17) induces isomorphisms of double vector bundles
\[
Z_A : D \to D \times B \times C^*, \quad \langle Z_A(\Phi), \Psi \rangle_{C^*} = \langle \Phi, \Psi \rangle
\]
\[
Z_B : D \to D \times A^* \times B^* \times C^*, \quad \langle Z_B(\Phi), \Psi \rangle_{C^*} = \langle \Phi, \Psi \rangle
\]
with \( (Z_A)_{C^*} = Z_B \). Both isomorphisms induce the identity on the sides \( C^* \to C^* \).

\( Z_A \) is the identity on the cores \( B^* \to B^* \), and induces \( -id \) on the side bundles \( A \to A \).

\( Z_B \) is the identity on the side bundles \( B \to B \), and induces \( -id \) on the cores \( A^* \to A^* \).

**Example 3.5** Consider a trivial double vector bundle
\[
D = A \times_M B \times_M C.
\]
Let \( \Phi = (a, \psi, \kappa) \) be an element of \( D \times A = A \times_M B^* \times_M C^* \) and let \( \Psi = (\varphi, b, \kappa) \) be an element of \( D \times B \). Then taking any \( d = (a, b, c) \in D \), we find that
\[
\langle \Phi, \Psi \rangle = (\psi, b) - (\varphi, a).
\]
The associated maps are given by
\[
Z_A : A \times_M B^* \times M C^* \to A \times_M B^* \times_M C^*, \quad (a, \psi, \kappa) \mapsto (-a, \psi, \kappa);
\]
\[
Z_B : A^* \times_M B \times_M C^* \to A^* \times_M B \times M C^*, \quad (\varphi, b, \kappa) \mapsto (-\varphi, b, \kappa).
\]

The following result is essentially equivalent to Theorem 3.2, but deserves independent statement.

**Theorem 3.6** For any double vector bundle \( (D ; A, B ; M) \) there is a canonical isomorphism \( Q \) from \( D \) to the flip of \( (D \) \) which preserves the side bundles \( A \) and \( B \) and is \( -id \) on the cores \( C \).
Proof. Let $\Pi = Z_A \downarrow A$ be the dualization of $Z_A$ over $A$. Denote by $F: D \to D$ the map $d \mapsto \frac{\partial}{\partial t} d$, and define $Q = (F \circ \Pi)^{-1}$.

There are now three operations on double vector bundles: taking the vertical dual, denoted by $V$, taking the horizontal dual, denoted $H$, and the operation $VHV$ which by Theorem 3.6 combines the flip and reversal of the sign on the core; we denote this by $P$. We have $V^2 = H^2 = P^2 = I$, the identity operation and, by the same method as Theorem 3.6, $HVH = P$. The group generated by $V$, $H$ and $P$ therefore has elements

$I, \quad V, \quad HV, \quad VHV = P, \quad HVHV = HP = VH, \quad VHVHV = H,$

(21)

and is the dihedral group $\Delta_3$ of the triangle, or the symmetric group $S_3$.

4 The duals of $TE$

Consider the tangent prolongation double vector bundle (Figure 1(b)) of a vector bundle $(E, q, M)$.

First consider the horizontal dual. The canonical pairing of $E^*$ with $E$ prolongs to a pairing of $T(E^*) \to TM$ with $TM \to TM$. Suppose given $X \in T(E^*)$ and $\xi \in TE$ with $T(q)(X) = T(q_*)(\xi)$. Then $X = \frac{d}{dt} \varphi_t \bigg|_0 \in T(E^*)$ and $\xi = \frac{d}{dt} e_t \bigg|_0 \in TE$ where $e_t \in E$ and $\varphi_t \in E^*$ can be taken so that $q_*(\varphi_t) = q(e_t)$ for $t$ near zero. Now define the tangent pairing $\langle \langle \, , \, \rangle \rangle$ by

$\langle \langle X, \xi \rangle \rangle = \frac{d}{dt} \langle \varphi_t, e_t \rangle \bigg|_0$.

(22)

To show that this is non-degenerate it is sufficient to work locally. Suppose, therefore, that $E = M \times V$. Regard $\xi$ as $(x_0, v_0, w_0) \in T_{m_0}M \times V \times V$ and $X$ as $(x_0, \varphi_0, \psi_0) \in T_{m_0}M \times V^* \times V^*$. Then

$X = \frac{d}{dt} (m_t, \varphi_0 + t\psi_0) \bigg|_0, \quad \xi = \frac{d}{dt} (m_t, v_0 + tw_0) \bigg|_0,$

where $\frac{d}{dt} m_t \bigg|_0 = x = T(q)(X) = T(q_*)(\xi)$. So

$\langle \langle X, \xi \rangle \rangle = \frac{d}{dt} \langle \varphi_0 + t\psi_0, v_0 + tw_0 \rangle \bigg|_0$.

Expanding out the RHS, the constant term and the quadratic term vanish in the derivative, and we are left with

$\langle \langle X, \xi \rangle \rangle = \langle \psi_0, v_0 \rangle + \langle \varphi_0, w_0 \rangle$

from which it is clear that $\langle \langle \, , \, \rangle \rangle$ is non-degenerate. We now need to establish that this is a pairing of the double vector bundles.
Proposition 4.1 The tangent pairing $\langle \langle \cdot, \cdot \rangle \rangle$ of $T(E^*)$ with $TE$ over $TM$ satisfies the conditions of Proposition 3.3. In particular, for $m \in M$ and $\varphi, \varphi_1, \varphi_2 \in E^*_m$, $e, e_1, e_2 \in E_m$,

$$\langle \langle \varphi, \tau \rangle \rangle = 0, \quad \langle \langle 0_\varphi, 0_e \rangle \rangle = 0,$$

$$\langle \langle 0_\varphi, e \rangle \rangle = \langle \varphi, e \rangle, \quad \langle \langle \varphi, 0_e \rangle \rangle = \langle \varphi, e \rangle.$$

and

$$\langle \langle \tau^*_\varphi (\varphi_1, \varphi_2), \tau (e_1, e_2) \rangle \rangle = \langle \varphi_1, e_2 \rangle + \langle \varphi_2, e_1 \rangle,$$

where $\tau^*$ and $\tau$ are the injections in the core sequences of $T(E^*)$ and $T(E)$.

Proof. These are easily verified from the definition. For example, $\varphi = \frac{d}{dt}(t\varphi)|_0$ and $e = \frac{d}{dt}(te)|_0$ so

$$\langle \langle \varphi, e \rangle \rangle = \left. \frac{d}{dt} \langle \varphi, e \rangle \right|_0 = 0$$

whereas $0_\varphi = \left. \frac{d}{dt} \varphi \right|_0$ so

$$\langle \langle 0_\varphi, e \rangle \rangle = \left. \frac{d}{dt} \langle \varphi, e \rangle \right|_0 = \langle \varphi, e \rangle.$$

The bilinearity conditions are easily verified and the final equation follows.

Thus the pairing of the cores of $T(E^*)$ and $T(E)$ is the zero pairing, and so too is the pairing of the zero sections above $E^*$ and $E$. However the core of $T(E^*)$ and the zero section of $T(E)$ are paired under the standard pairing, and the same is true of the zero section of $T(E^*)$ and the core of $T(E)$.

It now follows that there is an isomorphism of double vector bundles from $T(E^*)$ to the dual $TE \times TM$ of $TE$ over $TM$. For convenience we denote this simply by $T^*E$ and call it the prolongation dual of $TE$. The next result follows from the general theory of §3.

Proposition 4.2 [Mackenzie and Xu 1994] The map $I: T(E^*) \to T^*(E)$ defined by

$$\langle I(\xi), \eta \rangle_{TM} = \langle [\xi, \eta] \rangle$$

where $\xi \in T(E^*)$, $\eta \in TE$, is an isomorphism of double vector bundles preserving the side bundles $E^*$ and $TM$ and the core bundles $E^*$.

When a name is needed we call $I$ the internalization map. In future we will almost always work with $T(E^*)$ and the tangent pairing, rather than with $T^*E$ and $I$.

Now consider the vertical dual of $TE$. Since the core of the double vector bundle $TE$ is $E$, dualizing the structure over $E$ leads to a double vector bundle of the form
We refer to this as the cotangent dual of $TE$. We will give a detailed description of the structures involved. Although this is a special case of the general construction, this example is so basic to the rest of the paper that it merits a specific treatment.

In (23) the vertical bundle is the standard cotangent bundle of $E$, and the notation $T_X^* (E)$ will always refer to the fibre with respect to $c_E$. In this bundle we use standard notation, and denote the zero element of $T_X^* (E)$ by $\tilde{0}_X$. We drop the subscripts $E$ from the maps when no confusion is likely.

The map $r: T^* E \to E^*$ takes the form

$$\langle r(\Phi), Y \rangle = \langle \Phi, \tau(X, Y) \rangle$$

(24)

where $\Phi \in T_X^* (E)$, $X \in E_m$ and $Y \in E_m$. Thus $r(\Phi) \in E^*_m$. For the addition over $E^*$ we have

$$\langle \Phi + \Psi, \xi + \eta \rangle = \langle \Phi, \xi \rangle + \langle \Psi, \eta \rangle,$$

where $\Phi \in T_X^* (E)$, $\Psi \in T_Y^* (E)$ with $r(\Phi) = r(\Psi) \in E^*_m$, and $\xi \in T_X (E)$, $\eta \in T_Y (E)$ with $T(q)(\xi) = T(q)(\eta)$. This defines $\Phi + \Psi \in T_{X+Y}^* (E)$. Similarly, we have

$$\langle t \cdot \Phi, t \cdot \xi \rangle = t(\Phi, \xi),$$

for $t \in \mathbb{R}$ and $\xi \in T_X (E)$. The zero element of $r^{-1}(\varphi)$, where $\varphi \in E^*_m$, is $\tilde{0}_x \in T^*_m (E)$ where

$$\langle \tilde{0}_x \varphi, T(0)(x) + X \rangle = \langle \varphi, X \rangle$$

for $x \in T_m (M), X \in E_m$.

Given $\omega \in T^*_m (M)$, the corresponding core element $\overline{\omega}$ is

$$\langle \overline{\omega}, T(0)(x) + X \rangle = \langle \omega, x \rangle,$$

for $x \in T_m (M), X \in E_m$. The injection over $E$,

$$q^! T^* M \to T^* E, \ (X, \omega) \mapsto \tilde{0}_X + \overline{\omega},$$

is precisely the dual of $T(q)^!$; that is to say, it is the map corresponding to the lifting of 1–forms from $M$ to $E$. Thus $\tilde{0}_X + \overline{\omega}$ is the pullback of $\omega \in T^*_m (M)$ to $E$ at the point $X \in E_m$.

The core exact sequence for $c$ is

$$q^! T^* M \longrightarrow T^* E \xrightarrow{r^! = r^*} q^! E^*, \tag{25}$$
and this is the dual of the core exact sequence for $TE$ and $p_E$. The other core exact sequence is

$$q^*_{1} T^* M \longrightarrow T^* E \longrightarrow q^*_{1} E,$$

(26)

where each bundle here is over $E^*$. The injection $q^*_{1} T^* M \to T^* E$ is $(\varphi, \omega) \mapsto \tilde{0}_c + \tilde{\omega}$ and $(\tilde{0}_c + \tilde{\omega}, T(0)(x) + \tilde{X}) = \langle \varphi, X \rangle + \langle \omega, x \rangle$.

Given $\omega \in T^* M$, the corresponding core element is $\tilde{\omega} = (\omega, 0, 0)$.

To summarize, the two dual double vector bundles of $D = TE$ are

$$D\downarrow E = T^* E \longrightarrow E^* \quad \quad D\downarrow TM = T^* E \longrightarrow TM$$

and the pairing

$$\langle \Phi, \xi \rangle_E = \langle \Phi, \xi \rangle - \langle \xi, \xi \rangle_{T^* M}$$

(27)

for suitable $\xi \in T^* E$. Composing the isomorphism $Z_E$ from (34) with the dual over $E^*$ of the internalization isomorphism $I$, we get an isomorphism of double vector bundles

$$(I \downarrow E^*) \circ Z_E : T^* E \to T^* (E^*)$$

denote this temporarily by $S^{-1}$. For $\Phi \in T^* E$ we have

\[
\langle S^{-1}(\Phi), X \rangle_{E^*} = \langle (I \downarrow E^*) \circ Z_E(\Phi), X \rangle_{E^*} = \langle Z_E(\Phi), I(X) \rangle_{E^*} = \langle \Phi, I(X) \rangle_E - \langle \xi, \xi \rangle_{T^* M}.
\]

Here we used the definition of $Z_E$, the definition (27), and the definition of $I$. It follows that for $\tilde{\Phi} \in T^* (E^*)$, writing $\Phi = S(\tilde{\Phi})$, we have

$$\langle \tilde{\Phi}, X \rangle_{E^*} = \langle S(\tilde{\Phi}), X \rangle_E - \langle \xi, \xi \rangle.$$  

Recall that $I$, and hence its dual, preserves both sides and the core, whereas $Z_E$ induces $-\text{id}$ on the sides $E$. We therefore define

$$R : T^* (E^* ) \to T^* (E), \quad R(\tilde{\Phi}) = S(-_{E^*} \tilde{\Phi}).$$

To summarize:

**Theorem 4.3** [Mackenzie and Xu 1994] The map $R$ just defined is an isomorphism of double vector bundles, preserving the side bundles $E$ and $E^*$, and inducing $-\text{id}$: $T^* M \to T^* M$ on the cores. Further, for all $\xi \in T E$, $X \in T(E^*)$, $\tilde{\Phi} \in T^* (E^*)$ such that $\xi$ and $X$ have the same projection into $TM$, ...
X and \( \mathfrak{X} \) have the same projection into \( E^* \), and \( \mathfrak{F} \) and \( \xi \) have the same projection into \( E \),

\[
\langle \mathfrak{X}, \xi \rangle = \langle R(\mathfrak{F}), \xi \rangle_E + \langle \mathfrak{F}, X \rangle_{E^*}
\]  

(28)

We call \( R \) the reversal isomorphism. It is proved in [Mackenzie and Xu 1994] that \( R \) is an antisymplectomorphism of the canonical symplectic structures.

5 Triple vector bundles

The definition of a triple vector bundle follows the same pattern as in the double case. There are a number of evident reformulations.

**Definition 5.1** A triple vector bundle is a manifold \( \mathcal{I} \) together with three vector bundle structures, over bases \( D_1, D_2, D_3 \), each of which is a double vector bundle with side bundles respectively \( E_2 \) and \( E_3 \), \( E_3 \) and \( E_1 \), \( E_1 \) and \( E_2 \), where \( E_1, E_2, E_3 \) are vector bundles over a shared base \( M \), such that each pair of vector bundle structures on \( \mathcal{I} \) forms a double vector bundle, the operations of which are vector bundle morphisms with respect to the third vector bundle structure.

We display a triple vector bundle in a diagram such as Figure 3(a). (We always read figures of this type with \( (\mathcal{I}; D_1, D_2; E_3) \) at the rear and \( (D_3; E_2, E_1; M) \) coming out of the page toward the reader.) The three structures of double vector bundle on \( \mathcal{I} \) are the upper double vector bundles, and \( D_1, D_2, D_3 \) are the lower double vector bundles. We refer to \( (D_1; E_2, E_3; M) \) as the floor of \( \mathcal{I} \) and to \( (\mathcal{I}; D_2, D_3; E_1) \) as the roof of \( \mathcal{I} \).

\[\begin{array}{ccc}
\mathcal{I} & \rightarrow & D_2 \\
\downarrow & & \downarrow \\
D_1 & \rightarrow & E_3 \\
\downarrow & & \downarrow \\
E_2 & \rightarrow & M \\
\end{array}\]

\[\begin{array}{ccc}
\mathcal{I} & \rightarrow & D_2 \\
\downarrow & & \downarrow \\
D_1 & \rightarrow & E_3 \\
\downarrow & & \downarrow \\
E_2 & \rightarrow & M \\
\end{array}\]

(a) (b)

**Fig. 3.**

We have found that, rather than assembling a notation capable of handling any calculation in a triple vector bundle without ambiguity, it is generally preferable to develop an ad hoc notation for each occasion. The great majority
of calculations use only certain parts of the structure, and in such cases a modification of the notation of §1 is often sufficient.

Each of the lower double vector bundles $D_i$ has a core, which is denoted $C_i$. The core of the upper double vector bundle $(III; D_3, D_2; E_1)$ is denoted $K_1$. Consider a core element $K_1 \in III$ where $k_1$ projects to $e_1 \in E_1$. Then the $d_2$ in Figure 4(b) is the zero over $e_1$ for $D_2 \to E_1$, and $d_3$ is the zero over $e_1$ for $D_3 \to E_1$. From the morphism condition we then have that $e_2$ and $e_3$ are zeros over $m$. So $d_1 = \overline{e}$ is a core element for some $c \in C_1$. This defines a map $K_1 \to C_1$. For $k, k' \in K_1$ over the same element of $C_1$, define

$$k + c_1 k' = \overline{k} + D_1 \overline{k'},$$

(29)

where each of the three bars refers to the roof double vector bundle. With scalar multiplication defined in a similar fashion, $K_1 \to C_1$ is a vector bundle, and a double vector bundle as shown in Figure 4(a).

The cores of the other upper double vector bundles are likewise denoted $K_2$ and $K_3$ and form double vector bundles as in Figure 4(b)(c). These three are the core double vector bundles. Although defined by restrictions of the operations in $III$, they are not substructures of $III$.

$$K_1 \to E_1 \quad K_2 \to E_2 \quad K_3 \to E_3$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$C_1 \to M \quad C_2 \to M \quad C_3 \to M$$

(a) \hspace{2cm} (b) \hspace{2cm} (c)

Fig. 4.

Denote the core of $(K_1; C_1, E_1; M)$ by $W$. In Figure 4(b), let $k_1 = \overline{w}$ where $w \in W$. Then $c_1$ is the zero of $C_1$ over $m$, and so $d_1 = \overline{c}_1$ is a double zero of $D_1$.

Next, since $e_2$ is a zero, $(d_3; e_2, e_1; m)$ must be of the form $d_3 = 0_{e_2} + e_2 \overline{e}_3$. But $e_1$ is zero, since $w$ is in the core, so $d_3$ is a core element. But it is known to be a zero over $e_1$, so must be a double zero. Similarly $d_2$ is a double zero.

This proves most of the next result, and the remainder is an easy verification.

**Proposition 5.2** Each of the core double vector bundles has as core the set $W$ of elements $w \in III$ for which the projection to each of $D_1, D_2, D_3$ is a double zero of the lower double vector bundle. Further, the vector bundle structures with base $M$ induced on $W$ by the various core double vector bundles coincide.

Given $w \in W$, the core elements in $III$ corresponding to $\overline{w}^1 \in K_1$, $\overline{w}^2 \in K_2$ and $\overline{w}^3 \in K_3$, coincide.
We call $W$ the ultracore of $\mathbb{I}$. 

**Example 5.3** For a double vector bundle $(D; A, B; M)$, the tangent prolongation triple vector bundle is as shown in Figure 5(a). Two of the three core double vector bundles are copies of $(D; A, B; M)$ and the third is $(TC; T'A, TM; M)$. The ultracore of $TD$ is $C$, the core of $D$. It is illuminating to verify (29) and 5.2 directly in this example.

![Diagram](attachment:figure5.png)

**Fig. 5.**

The key to understanding the relations between the duals of a double vector bundle, and the role of the dihedral group (21), lies in constructing a cotangent form of this example. Both the left and the rear faces of Figure 5(a) are tangent prolongation double vector bundles of ordinary vector bundles and so we can form the figure in Figure 5(b).

In Figure 5(a) we have added the two double vector bundle duals associated to $D$. In this diagram each of the four vertical sides, and of course the floor, is a double vector bundle. We need to prove that the roof is a double vector bundle and that Figure 5(a) is a triple vector bundle.

First consider the roof, shown in Figure 5(b). We use a short notation for the projections.

**Proposition 5.4** The structure in Figure 5(b) is a double vector bundle with core $T^*C$.

**Proof.** First we must prove that the projections form a commutative square. Take $f \in T^*_d D$, where $d$ has the form $(d; a, b; m)$. Then, for all $(d'; a, b'; m)$,

$$\langle r_A(f), d' \rangle = \langle f, \tilde{0}_{d'} + A d'^{-A} \rangle.$$  

Here $\tilde{0}_{d'}$ is the zero of $TD \to D$ above $d$ and the subscript on $+_{A}$ indicates that this is the tangent of the addition in $D \to A$. The superscript $A$ on the bar indicates that $D$ is here the core of $(TD; T'A, TM; A)$.

Writing $\varphi = r_A(f)$ we next have
Here $0^D_a$ is the zero of $D \to A$ over $a$ and $\overline{c}$ is the core element of $D$ corresponding to $c$. Writing out the corresponding formulas for the other side, we must prove that

$$\tilde{0}_d + A (0^D_a + B \overline{c}) = \tilde{0}_d + B (0^D_a + A \overline{c}).$$

Using (29), the LHS becomes

$$\tilde{0}_d + A (0^D_a + B \overline{c}) = \frac{d}{dt} A (t \cdot 0^D_a) |_{0} = \frac{d}{dt} 0^D_a |_{0}.$$ 

Now, using the interchange rule,

$$d + A (0^D_a + B \overline{c}) = (d + B 0^D_b) + A (0^D_a + B \overline{c})$$

$$= (d + A 0^D_a) + B (0^D_b + A \overline{c}) = d + B (0^D_b + A \overline{c}).$$

and from this (30) follows. The proof that $r_A$ preserves the addition and scalar multiplication proceeds in a similar way.

Next we show that the core is $T^*C$. Suppose that $f \in T^* D$ maps to zero under both $r_A$ and $r_B$. Then $d = \overline{c}$ is a core element and $f$ vanishes on elements $\xi \in T_D D$ which are vertical with respect to either $q_A^D$ or $q_B^D$. If $\xi$ is vertical with respect to $q_A^D$, then, in the notation of Figure 7(a), $X = \tilde{0}_m^A$, and it follows that $Z = \tilde{0}_m^A$ and so $Y$ is a core element. Likewise if $\xi$ is vertical with respect to $q_B^D$, then $Y = \tilde{0}_m^B$, and $X$ is a core element. Adding two such representative elements, it follows that $f$ vanishes on all $X$ as shown in Figure 7(b).

Now take $\xi \in T_D D$ as shown in Figure 7(a). Because $X \in T_0^D A$, it has the form $X = T(0^A)(Z) + \overline{c}$ for some $a \in A_m$; likewise $Y$ has the form $Y = T(0^B)(Z) + \overline{b}$ for some $b \in B_m$. Now define

$$\langle q_A \xi, c \rangle = \langle \xi, 0^D_a + B \overline{c} \rangle.$$
Then $\gamma = \xi - X$ has the form shown in Figure 8(a) and is an element of $T\mathcal{C}$.

It is now possible to extend a given $\omega \in T^*_c C$ to $\overline{\omega} \in T^*_D D$ by

$$\langle \overline{\omega}, \xi \rangle = \langle \omega, \xi - X \rangle$$

and $\overline{\omega}$ is annulled by both $r_A$ and $r_B$. This $\overline{\omega}$ is the core element of the double vector bundle in Figure 8(b).

Now that the commutativity of the projections has been established, verification that addition and scalar multiplication are preserved is straightforward.

The double vector bundle in Figure 8(b) is of a type not previously encountered.
Theorem 5.5 The structure in Figure 5(a) is a triple vector bundle.

The core double vector bundles are \((T^*A; A, A^*; M)\), \((T^*B; B, B^*; M)\) and \((T^*C; C, C^*; M)\), and the ultracore is \(T^*M\). Observe that the triple vector bundle \(T^*D\) has a much higher degree of symmetry than \(TD\).

Example 5.6 Consider seven vector bundles \(E_1, E_2, E_3, C_1, C_2, C_3, W\) over a shared base \(M\). Let \(D_1\) be the trivial double vector bundle with sides \(E_2\) and \(E_3\) and core \(C_1\), and likewise form \(D_2\) and \(D_3\). Similarly, let \(K_1\) be the trivial double vector bundle with sides \(C_1\) and \(E_1\) and core \(W\), and form \(K_2\) and \(K_3\) in the same way. Lastly, let \(I I I\) be the pullback of all seven vector bundles over \(M\). Then \(I I I\) can be considered as the trivial double vector bundle with side bundles \(D_1 \rightarrow E_3\) and \(D_2 \rightarrow E_3\) and core \(K_3 \rightarrow E_3\). Likewise, \(I I I\) can be considered the trivial double vector bundle over \(D_2\) and \(D_3\) with core \(K_1\) and over \(D_3\) and \(D_1\) with core \(K_2\). With these structures, \(I I I\) is a triple vector bundle, the \textit{trivial triple vector bundle} determined by the given seven vector bundles.

More refined versions of this construction exist. For example, suppose given four vector bundles \(E_1, E_2, E_3, W\) on \(M\) and three double vector bundles \((D_1; E_2, E_3; M)\), \((D_2; E_3, E_1; M)\), \((D_3; E_1, E_2; M)\). Then there is a triple vector bundle for which \(D_1\), \(D_2\), \(D_3\) are the lower double vector bundles and \(W\) is the ultracore, and for which each of the core double vector bundles is trivial.

6 Cornerings

Continue with a double vector bundle \(D\) as in the previous section. Since \(D\) is a vector bundle over \(A\), we have \(T^*D \cong T^*(D \downarrow A)\) by \(\Theta\) and similarly \(T^*D \cong T^*(D \uparrow B)\). Once it has been shown that these isomorphisms respect the triple structures, we can regard \(\Delta_3\) as acting on the cube \(T^*D\) by rotations about the axis from \(T^*D\) to \(M\).

Theorem 6.1 The map \(R^{-1} : T^*D \rightarrow T^*(D \downarrow A)\) arising from the vector bundle \(D \rightarrow A\) is an isomorphism of triple vector bundles over \(Z_B : D \uparrow B \rightarrow D \downarrow A \uparrow C^*\), the other maps on the side structures being identities.

Proof. The main work is to show that \(R^{-1}\) is a morphism of vector bundles over \(Z_B\). Take \(\Phi \in T^*_d D\) and denote \(R^{-1}(\Phi)\) by \(\tilde{\Phi}\). Let \(d\) have the form \((d; a, b, m)\) and let the projections of \(\tilde{\Phi}\) to \(D \uparrow A\), \(D \uparrow B\) and \(C^*\) be \(\chi\), \(\psi\) and \(\kappa\) respectively. Since \(R\) preserves \(D\) and \(D \uparrow A\), it follows that \(\tilde{\Psi}\) projects to \(d \in D\) and to \(\chi \in D \downarrow A\).

For \(\phi\) we have, from \(24\),

\[
(\phi, d_1)_B = (\Phi, \tilde{\Phi} + B \, \tilde{d_1})_D
\]

for any \(d_1\) of the form \((d_1; a_1, b; m)\). For \(\tilde{\Psi}\) and \(\Phi\) we have, by \(28\),
where $X \in T(D \downarrow A)$ has the form $(X; \chi, X; a)$ for some $X \in TA$, and $\xi \in TD$ then has the form $(\xi; d, X; a)$. Next, for $Z_B(\psi) \in D \downarrow A \downarrow C^*$, we have, for each $\varphi \in D \downarrow A$ of the form $(\varphi; a_2, \kappa; m)$,

$$\langle Z_B(\psi), \varphi \rangle = \langle \varphi, \psi \rangle_{C^*} = \langle \varphi, d_2 \rangle_A - \langle \psi, d_2 \rangle_B$$

for any $d_2 \in D$ of the form $(d_2; a_2, b; m)$. Lastly, for the same $\varphi$ we have

$$\langle r_{C^*}(f), \varphi \rangle = \langle f, \tilde{0}^{(D \downarrow A)} + C^* \varphi (D \downarrow A) \rangle$$

Here $\tilde{0}^{(D \downarrow A)}$ is the zero in $T(D \downarrow A)$ over $\chi$ and $\varphi (D \downarrow A)$ is the core element of $T(D \downarrow A)$ corresponding to $\varphi$. The addition is in the bundle $T(D \downarrow A) \to TC^*$.

We must prove that the RHSs of (33) and (34) are equal. We substitute

$$X = \tilde{0}^{(D \downarrow A)} + C^* \varphi (D \downarrow A) \quad \text{and} \quad \xi = \tilde{0}_A \# B \varphi (D \downarrow A)$$

into (32). Providing $a_1 = a_2$, the relevant projections match. Now applying (iv) of 3.3 to the double vector bundles $(T(D \downarrow A); TA, TC^*; TM)$ and $(TD; TA, TB; TM)$, we have

$$\langle X, \xi \rangle_{TA} = \langle \tilde{0}^{(D \downarrow A)} + C^* \varphi (D \downarrow A), \tilde{0}_A \# B \varphi (D \downarrow A) \rangle_{TA}$$

$$= \langle \tilde{0}^{(D \downarrow A)} \rangle_{TA} + \langle \varphi (D \downarrow A), \varphi (D \downarrow A) \rangle_{TA}.$$

In the first term, $\tilde{0}^{(D \downarrow A)}$ is tangent to the path constant at $\chi$, and $\tilde{0}_A$ is tangent to the path constant at $d$; therefore $\langle \tilde{0}^{(D \downarrow A)}, \tilde{0}_A \rangle_{TA}$ is tangent to the path constant at $(\chi, d)_A$, and is therefore zero. For the second term, $\varphi (D \downarrow A)$ is tangent to the path $t \varphi$ and $\varphi (D \downarrow A)$ is tangent to the path $t \varphi$, so $\langle \varphi (D \downarrow A), \varphi (D \downarrow A) \rangle_{TA}$ is tangent to the path $(t \varphi, t \varphi, d_1)_A$ and by (v) of 3.3 this is $t \langle \varphi, d_1 \rangle_A$. Altogether we have that $\langle X, \xi \rangle_{TA} = \langle \varphi, d_1 \rangle_A$. Using (31), we have that the RHS of (34) is

$$\langle \varphi, d_1 \rangle_A - \langle \psi, d_1 \rangle_B$$

and this is equal to the RHS of (33) by the proof of 3.2.

The rest of the proof is now straightforward.

For a single vector bundle $E \to M$, the pairing of $E^*$ with its dual $E^{**}$ can be identified in a straightforward way with the pairing of $E$ with its dual. For double vector bundles it is first necessary to ensure that pairings are chosen in a consistent way.

Consider a double vector bundle $(D; A, B; M)$ and assign signs to the two upper structures as in Figure 8(a) in order to show that we pair the duals according to (18). Now, referring to Figure 8(b), we assign signs in such a way
that each of \( A, B, C^* \) has one positive and one negative arrow approaching it, and each of \( D, D \times A, D \times B \) has one positive and one negative arrow departing from it; see Figure 9(b)(c). We therefore, for example, take the pairing of the duals of \( D \times A \) to be

\[
\langle D, D \times A \rangle_{C^*} = \langle D \times A, D \times A \rangle_{C^*} - \langle D, D \times A \rangle_A.
\]

**Proposition 6.2** For the isomorphism \( Z_B : D \times B \rightarrow (D \times A \times C^*)_B \),

\[
\langle \Phi, \Psi \rangle_{C^*} = \langle \Phi, Z_B(\Psi) \rangle_{C^*}, \quad (d, \Psi)_B = -d, Z_B(\Psi)_B
\]

for compatible \( \Phi \in D \times A, \Psi \in D \times B, d \in D \).

**Proof.** The first is the definition of \( Z_B \). The second follows from \( Z_B \times C^* = Z_A \).

If we insert these equations into (35), we get (17). Thus the signing on \( D \times A \) is compatible with that on \( D \).

For ordinary vector bundles \( E_1 \) and \( E_2 \) on the same base \( M \), one could take the view that a pairing of \( E_1 \) with \( E_2 \) is what enables one to construct a cotangent double vector bundle with sides \( E_1 \) and \( E_2 \). In a similar way, three double vector bundles with suitably overlapping sides can be completed to a cotangent triple vector bundle if and only if any two of them are the duals of the third.

**Definition 6.3** Consider three double vector bundles as in Figure 10(a), together with six pairings: \( \langle , \rangle_{E_1} \) of \( D_2 \) and \( D_3 \) over \( E_1 \), \( \langle , \rangle_{E_2} \) of \( D_3 \) and \( D_1 \) over \( E_2 \), \( \langle , \rangle_{E_3} \) of \( D_1 \) and \( D_2 \) over \( E_3 \), and \( \langle , \rangle_1 \) of \( C_1 \) and \( E_1 \) over \( M \), \( \langle , \rangle_2 \) of \( C_2 \) and \( E_2 \) over \( M \), \( \langle , \rangle_3 \) of \( C_3 \) and \( E_3 \) over \( M \), such that each pairing of \( D \) bundles is a pairing of double vector bundles (as defined in 3.3) with respect to the pairing of the relevant cores and sides. Then if

\[
\langle D_2, D_3 \rangle_{E_1} = \langle D_1, D_2 \rangle_{E_3} - \langle D_1, D_3 \rangle_{E_2}
\]

holds, we say that the system is a cornering of \( D_1 \) with \( D_2 \) and \( D_3 \).
Clearly, choosing any double vector bundle in a cornering, the other two double bundles may be identified with its duals, and the cornering may be identified with the lower sides of the cotangent triple vector bundle associated with the chosen double.

**Remark 6.4** For an ordinary vector bundle $E$ one may form the cotangent triple of $D = TE$. Now the canonical diffeomorphism between $T^*TE$ and $TT^*E$ [Abraham and Marsden 1985] is, since $E$ is a vector bundle, an isomorphism of double vector bundles, and so the triple $T^*TE$ is isomorphic to the tangent prolongation of $T^*E$, as shown in Figure 10(b). Now the pairing of the bundles over $E$ in the left face gives rise to the canonical 1–form on $T^*E$, and the pairing of the bundles in the roof gives rise to the canonical 1–form on $T^*E^*$.

### 7 Duals of triple vector bundles

Consider now a general triple vector bundle $\mathbf{III}$ as in Figure 3(a). Dualize $\mathbf{III}$ over the base $D_1$. Each of the upper double vector bundles of which $\mathbf{III} \to D_1$ is a side has a dual which is familiar from §4. Following the example of 5.5, we complete the cube as in Figure 11(a).

**Theorem 7.1** There is a triple vector bundle as shown in Figure 11(a) in which the four vertical sides are dual double vector bundles as just described.

We omit the details of this. As with the case $\mathbf{III} = TD$, five of the six faces are double vector bundles of known types and the main work is to show that the roof — which belongs to a new class of examples — is a double vector bundle, and calculate its core, which is $K_1 \frak I C_1$. The proof follows exactly the same outline as in 5.3 though steps involving derivatives must be replaced with forms of the interchange laws.

Notice that in Figure 11(a), two of the three upper double vector bundles are standard duals of the double vector bundles in the corresponding positions.
in Figure 3(a). Two of the lower double vector bundles are duals of core double vector bundles of III.

The core double vector bundles of \( \text{III} \uparrow D_1 \) are given in Figure 12. The ultracore is \( E_1^* \), the dual of the bundle which in the original was diagonally opposite \( \text{III} \) in the plane perpendicular to the axis of dualization.

\[
\begin{align*}
\text{K}_1 \uparrow C_1 & \rightarrow W^* \\
& \downarrow \\
C_1 & \rightarrow M
\end{align*}
\]

\[
\begin{align*}
D_2 \uparrow E_3 & \rightarrow C_2^* \\
& \downarrow \\
E_3 & \rightarrow M
\end{align*}
\]

\[
\begin{align*}
D_3 \uparrow E_2 & \rightarrow C_3^* \\
& \downarrow \\
E_2 & \rightarrow M
\end{align*}
\]

Fig. 12.

The relationship between the three duals of III is embodied in the cotangent quaternary vector bundle of III, as shown in Figure 13.

Denote dualization of III along the three axes by \( X \), \( Y \) and \( Z \). In terms of Figure 3(a), take \( Z \) to be dualization along the vertical axis, \( Y \) to be along \( D_3 \) and \( X \) to be along \( D_2 \). Compositions such as \( ZXZ \) are triple versions of the operation \( P \) studied in §3. Precisely, applying \( ZXZ \) to III in Figure 3(a) applies \( P \) to the rear face and to the front face; denote this by \( P_Y \). This operation may also be regarded as reflection of III in the plane through III, \( D_3 \), \( M \) and \( E_3 \); see Figure 11(b). Notice that each face has been flipped in the sense that it cannot be returned to its original position by a proper rotation of the cube. Further, the core double vector bundle which lies in the plane through III, \( D_3 \), \( M \) and \( E_3 \) is left fixed; the other two are flipped and interchanged.

With similar definitions of \( P_X \) and \( P_Z \) we have

\[
P_X = YZY = ZYZ, \quad P_Y = ZXZ = XZX, \quad P_Z = XYX = YXY, \quad (36)
\]
each of $P_X$, $P_Y$, $P_Z$ having order 2. Equivalently, each of $XY$, $YZ$, $ZX$ has order 3.

New in the triple case are the products $Q_Z = ZXYZ$, $Q_X = XYZX$ and $Q_Y = YZXZ$ and their inverses. It is easily found from (36) that $Q_X$, $Q_Y$ and $Q_Z$ have order 3 and that $Q_Z Q_Y Q_X = I$. Curiously, the equation $Q_X Q_Y Q_Z = I$ or, equivalently, $(XYZ)^4 = I$, is not a consequence of (36), but it may be verified directly by calculating the effect on $I I I$. We now have:

**Theorem 7.2** The group of operations on $I I I$ generated by $X$, $Y$ and $Z$ satisfies the relations $X^2 = I$, $Y^2 = I$, $Z^2 = I$, $(XYZ)^4 = I$, $(YXZ)^4 = I$, $(ZXY)^4 = I$, together with (36).

By a calculation with GAP, the group defined by these relations has order 72. Denote the group of operations generated by $X$, $Y$ and $Z$ by $\mathcal{V}B_3$. It is straightforward to find more than 36 distinct elements of $\mathcal{V}B_3$ and so it must have order 72. It thus cannot be, as one might have expected, a subgroup
of the full symmetry group of the hypercube, which has order 384 [Coxeter 1973].

This shows that the situation with double vector bundles, in which the operations generated by dualization can be identified with symmetries of the cotangent triple, does not extend in the analogous fashion to triple vector bundles and symmetries of the hypercube.

8 General principles

On the basis of the duality theory for duals and triples, we may formulate some likely principles for the duality of general multiple vector bundles. It may be that the proofs are mainly a matter of acquiring sufficient motivation and notation, but we cannot rule out the possibility that new phenomena arise with increasing dimension.

There are three groups associated with an $n$–fold vector bundle $N$. Firstly, the various operations generated by flips of the constituent double vector bundles generate an action of the symmetric group $S_n$. Secondly, there is an obvious sense in which individual $n$–fold vector bundles may have more symmetry than others, as we remarked in the case of $T^*D$ and $TD$ in §5.

Of most interest, however, is the group $\mathbb{VB}_n$ generated by the dualization operations. We have seen that $\mathbb{VB}_2$ is $S_3$ and that $\mathbb{VB}_3$ has order 72. Further, the subgroup of $\mathbb{VB}_3$ generated by $X Y X Z$, $Y Z Y X$ and $Z X Z Y$ has order 12 and is normal, with quotient isomorphic to $S_3$.

In the general case, the $n$ duals of an $n$–fold vector bundle $N$ and $N$ itself form the $(n+1)$ lower $n$–faces of an $(n+1)$–fold vector bundle, which may be completed to be the cotangent $(n+1)$–fold vector bundle of $N$, or of any of the duals of $N$.

The $(n+1)$ upper $n$–faces of $T^*N$ consist of the $n$ cotangent $n$–fold vector bundles of the upper $(n-1)$–faces of $N$, together with one $n$–fold vector bundle of a new type, which incorporates data from all of the structure of $N$. It is reasonable to conjecture that if $X_1, \ldots, X_n$ denote the dualization operations, each of order 2, then we have, for each $1 \leq k \leq n$ and each string $i_1, i_2, \ldots, i_k$ of $k$ distinct elements of $\{1, \ldots, n\}$,

$$(X_{i_1} X_{i_2} \cdots X_{i_k})^{k+1} = 1.$$ 

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