Nonlinear $q$-Stokes phenomena for $q$-Painlevé I

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Abstract
We consider the asymptotic behavior of solutions of the first $q$-difference Painlevé equation in the limits $|q| \to 1$ and $n \to \infty$. Using asymptotic power series, we describe four families of solutions that contain free parameters hidden beyond-all-orders. These asymptotic solutions exhibit Stokes phenomena, which is typically invisible to classical power series methods. In order to investigate such phenomena we apply exponential asymptotic techniques to obtain mathematical descriptions of the rapid switching behavior associated with Stokes curves. Through this analysis, we also determine the regions of the complex plane in which the asymptotic behavior is described by a power series expression, and find that the Stokes curves are described by curves known as $q$-spirals.

Keywords: discrete Painlevé equations, nonlinear discrete asymptotics, exponential asymptotics, Stokes phenomena, $q$ difference equations

(Some figures may appear in colour only in the online journal)
1. Introduction

In this paper we study the first $q$-difference Painlevé equation ($q$-PI)

$$\frac{w'}{w} = \frac{1}{w} - \frac{1}{3w^2},$$

(1)

where $w = w(x)$, $\bar{w} = w(qx)$, and $q \in \mathbb{C}$ such that $|q| \neq 0, 1$. In particular, we assume that $|q| > 1$ and consider (1) under the limits $|q| \to 1$ and $n \to \infty$ where $x = x_0 q^n$ for some initial $x_0$. Equation (1) is part of a class of integrable, second-order nonlinear difference equations known as the discrete Painlevé equations that tend to the ordinary Painlevé equations in the continuum limit. More generally, equation (1) is known as a $q$-difference equation since the evolution of the independent variable takes the form $x = x_0 q^n$ for some initial $x_0$.

Motivated by Boutroux’s study of the first Painlevé equation [7], the asymptotic behavior of (1) in the limit $|x| \to \infty$ has been considered in [30, 32]. Joshi [30] showed that there exists a true solution satisfying $w \to 0$ as $|x| \to \infty$, which is asymptotic to a divergent series. However [30], does not describe the Stokes switching behavior that is typically associated with (divergent) asymptotic series.

In this study we uncover the Stokes behavior present in the solutions of (1) using exponential asymptotic methods. We first introduce a parameter, $\epsilon$, such that the double limit $|q| \to 1$ and $n \to \infty$ is equivalent to $\epsilon \to 0$. Since we may parametrize $x$ by $x = x_0 q^n$, one possible scaling choice which captures the desired behavior is $n = s/\epsilon$ and $q = 1 + \epsilon$. Under this choice of scaling we have $x \sim x_0 e^s + O(\epsilon)$ as $\epsilon \to 0$. We note that the large $x$ limit may be obtained if $s$ is taken to be sufficiently large and positive.

Exponential asymptotic techniques for differential-difference equations were developed by King and Chapman [37] in order to study a nonlinear model of atomic lattices based on the works of [13, 50]. The authors of [33, 34] applied the Stokes smoothing technique described in [37] to the first and second discrete Painlevé equations and obtained asymptotic approximations which contain exponentially small contributions. The difference equations considered in [33, 34] are of additive type as the independent variable is of the form $z_n = \alpha n + \beta$.

Motivated by their work, we extend this to $q$-PI in order to study asymptotic solutions of $q$-difference equations which display Stokes phenomena.

We note that there are other exponential asymptotic approaches used to study difference equations [27, 49, 51]. In particular, Olde Daalhuis [49] considered a certain class of second-order linear difference equations, and applied Borel summation techniques in order to obtain asymptotic expansions with exponentially small error. The methods used in these studies involves applying Borel resummation to the divergent asymptotic series of the problem, which allows the optimally-truncated error to be computed directly. We note that Borel summation techniques may be applied whenever a series possess factorial-over-power divergence in the late-order terms and therefore could be applied as an alternative approach to that utilised in the present study.

1.1. Background

General solutions to the Painlevé equations are higher transcendental functions, which cannot be expressed exactly in terms of elementary functions and therefore much of the analytic information regarding their general solutions are very limited. In fact, Nishioka [47] proved that the general solutions of $q$-PI are not expressible in terms of solutions of first-order $q$-difference equations.
The discrete Painlevé equations have appeared in areas of physical interest such as in mathematical physics models describing two-dimensional quantum gravity [10, 22, 24, 53, 54, 61]. The basis of these models have origins in orthogonal polynomial theory, in which discrete Painlevé equations have been found to commonly appear [38, 40–42, 57–59]. Since the discrete Painlevé equations often arise in many nonlinear models of mathematical physics they are often regarded as defining new nonlinear special functions [15, 28].

Motivated by these applications, much research has gone into the asymptotic study of solutions to the (discrete) Painlevé equations. Previous asymptotics studies for the first discrete Painlevé equation have been conducted in [29, 33, 60] where the authors found solutions asymptotically free of poles in the large independent variable limit, which share features with the (tri)-tronquée solutions of the first Painlevé equation found by Boutroux [7]. Asymptotic solutions have also been found for the so called alternate discrete Painlevé I equation [16, 36].

Extending the work of [33], the authors of [34] also find solutions of the second discrete Painlevé equation (dPII), which are asymptotically free of poles in some domain of the complex plane containing the positive real axis. This was achieved by rescaling the problem such that the step size of the rescaled difference equation is small in the asymptotic limit. A similar study on dPII was also investigated by Shimomura [56]. Shimomura showed that an asymptotic solution of dPII which reduces to the tri-tronquée solution of PII can be found by choosing a scaling of dPII such that the rescaled equation tends to the second continuous Painlevé equation (PII) in the limit $n \to \infty$.

The isomonodromic deformation method has also been used to study the asymptotics for nonlinear difference equations [11, 23, 39, 61]. Using the nonlinear steepest descent method developed by Deift and Zhou [19], the authors of [62] find the asymptotic behavior of solutions of the fifth discrete Painlevé equation in terms of the solutions of the fifth continuous Painlevé equation.

However, there have been very few asymptotic studies on $q$-Painlevé equations. The asymptotic study of variants of the sixth $q$-Painlevé equation have been investigated by Mano [43] and Joshi and Roffelson [35] using the $q$-analogue of the isomonodromic deformation approach. The first $q$-Painlevé equation has also been investigated in [30, 32]. Such expansions are known to exhibit Stokes phenomena in the complex plane.

Stokes phenomena are generally well understood in the case of linear and nonlinear differential equations [9, 12, 14, 26, 52]. The study of Stokes phenomena have also been investigated for nonlinear difference equations [37], and have been extended to study discrete Painlevé equations [33, 34].

In the continuous theory, Borel summation methods are also used to describe Stokes behaviors by resumming divergent asymptotic series expansions. The $q$-analogues of these methods have also been developed for linear $q$-difference equations [20, 55, 63–65] in order to describe behaviors known as $q$-Stokes phenomena. These methods have been explicitly applied to certain classes of second-order linear $q$-difference equations by Morita [44–46] and Ohyama [48].

However, the $q$-Stokes phenomenon is unlike the classical Stokes phenomenon for differential equations. The notion of $q$-Stokes phenomenon and its differences to classical Stokes phenomenon are detailed in [20, 55]. To the best of our knowledge there have been no corresponding studies for nonlinear $q$-difference equations. The goal of this study is to extend the exponential asymptotic methods used in [33, 34, 37] to $q$-difference equations in order to describe Stokes behavior present in the asymptotic series expansions.
1.2. Exponential asymptotics and Stokes curves

Conventional asymptotic power series methods fail to capture the presence of exponentially small terms, and therefore these terms are often described as lying beyond-all-orders. In order to investigate such terms, exponential asymptotic methods are used. The underlying principle of these methods is that divergent asymptotic series may be truncated so that the divergent tail, also known as the remainder term, is exponentially small in the asymptotic limit [8]. This is known as an optimally-truncated asymptotic series. Thereafter, the problem can be rescaled in order to directly study the behavior of these exponentially small remainder terms. This idea was introduced by Berry [3–5], and Berry and Howls [6], who used these methods to determine the behavior of special functions such as the Airy function.

The basis of this study uses techniques of exponential asymptotics developed by Olde Daalhuis et al [50] for linear differential equations, extended by Chapman et al [13] for application to nonlinear differential equations, and further developed by King and Chapman [37] for nonlinear differential-difference equations. A brief outline of the key steps of the process will be provided here, however more detailed explanation of the methodology may be found in these studies.

In order to optimally truncate an asymptotic series, the general form of the coefficients of the asymptotic series is needed. However, in many cases this is an algebraically intractable problem. Dingle [21] used previous work by Darboux [17, 18] to show that, although it is often impossible to calculate asymptotic series terms in general, an asymptotic form may often be found for series terms of sufficiently large order. In [21], Dingle showed that if the leading term of an asymptotic power series (say, $a_0$) contains singular points in the complex plane, and subsequent terms (say, $a_m$ for $m \geq 1$) are obtained by repeated differentiation of earlier terms, the series terms can be written as a sum of contributions with a predictable factorial-over-power form. A general version of this form is given by

$$a_m \sim A \frac{\Gamma(km + \gamma)}{\chi^{km+\gamma}},$$

as $m \to \infty$ where $\Gamma$ is the gamma function defined in [1], $k$ is the number of differentiations required to obtain $a_m$ from $a_{m-1}$. The terms $A$ and $\chi$ are functions of the independent variable which do not depend on $m$, known as the prefactor and singulant respectively [13]. The singulant is subject to the condition that it vanishes at the singular points of the leading order behavior, which ensures that all subsequent terms are also singular at these points.

The terms takes this form due to repeated differentiation of the singular contributions in the leading term. This causes the strength of the singularity to increase at each order, eventually becoming the dominant effect in $a_m$ as $m \to \infty$. Chapman et al [13] noted this behavior in their investigations and utilize (2) as an ansatz for the late-order terms, which may then be used to optimally truncate the asymptotic expansion.

Following [50] we substitute the optimally-truncated series back into the governing equation and study the exponentially small remainder term. When investigating these terms we will discover two important curves known as Stokes and anti-Stokes curves [2]. Stokes curves are curves on which the switching exponential is maximally subdominant compared to the leading order behavior. As Stokes curves are crossed, the exponentially small behavior experiences a smooth, rapid change in value in the neighbourhood of the curve; this is known as Stokes switching. Anti-Stokes curves are curves along which the exponential term switches from being exponentially small to exponentially large (and vice versa). We will use these definitions to determine the locations of the Stokes and anti-Stokes curves in this study.
By studying the switching behavior of the exponentially small remainder term in the neighbourhood of Stokes curves, it is possible to obtain an expression for the remainder term. The behavior of the remainder associated with the late-order terms in (2) typically takes the form $S \exp(-\chi/\epsilon)$, where $S$ is a Stokes multiplier that is constant away from Stokes curves, but varies rapidly between constant values as Stokes curves are crossed. From this form, it can be shown that Stokes lines follow curves along which $\chi$ is real and positive, while anti-Stokes lines follow curves along which $\chi$ is imaginary. A more detailed discussion of the behavior of Stokes curves is given in [2].

1.3. Paper outline

In section 2, we find two classes of solution behavior of $q$-$P_1$, which we refer to as type A and type B solutions. We also determine the formal series expansions of these solutions, and provide the recurrence relations for the coefficients.

In section 3, we determine the form of the late-order terms for type A solutions and use this to determine the Stokes structure of these asymptotic series expansions. We then calculate the behaviors of the exponentially small contributions present in these solutions as Stokes curves are crossed. This is then used to determine the regions in which the asymptotic power series are accurate representations of the dominant asymptotic behavior.

In section 4, we consider type B solutions of $q$-$P_1$ following the analysis in section 3. In section 5 we establish a connection between types A and B solutions to the vanishing and non-vanishing asymptotic solutions of $q$-$P_1$ found by Joshi [30]. Finally, we discuss the results and conclusions of the paper in section 5. Appendices A–C contain detailed calculations needed in section 3.

2. Asymptotic series expansions

In this section, we expand the solution as a formal power series in the limit $\epsilon \to 0$, obtain the recurrence relation for the coefficients of the series and deduce the general expression of the late-order terms.

We first rewrite (1) as an additive difference equation by setting $x = x_0 q^n$, which gives

$$w_{n+1}w_{n-1} = \frac{1}{w_n} - \frac{1}{x_0 q^n w_n^n},$$

where $w_n = w(x_0 q^n)$. In our analysis, we will introduce a small parameter, $\epsilon$, by rescaling the variables appearing in (3). The choice of scalings we apply are given by

$$s = \epsilon n, \quad q = 1 + \epsilon, \quad w_n = W(s).$$

Under these scalings, equation (3) becomes

$$W(s + \epsilon)W(s)^2W(s - \epsilon) = W(s) - \frac{1}{x_0(1 + \epsilon)^{1/\epsilon}},$$

and consider the limit $\epsilon \to 0$. We also note that under the scalings given by (4), the independent variable, $x$, has behavior described by

$$x = (1 + \epsilon)^{1/\epsilon} \sim x_0 e^\epsilon + O(\epsilon),$$

as $\epsilon \to 0$. As $e^\epsilon$ is an entire function, we set $x_0 = 1$ for the remainder of this analysis. It can be shown that equation (5) is invariant under the mapping
as $\epsilon \to 0$, and where $\lambda^3 = 1$. We note that this corresponds to the rotational symmetry of (1) found in [30].

We expand the solution, $W(s)$, as an asymptotic power series in $\epsilon$ by writing

$$W(s) \sim \sum_{r=0}^{\infty} \epsilon^r W_r(s),$$

as $\epsilon \to 0$. Substituting (8) into (5) and matching terms of $O(\epsilon^r)$ we obtain the nonlinear recurrence relation

$$r \sum_{q=0}^{r} q \sum_{m=0}^{q} \frac{(-1)^{q-k} W_{m-k}^{(k)}}{k!} \sum_{j=0}^{q-m} \frac{W_{q-m-j}^{(j)}}{j!} \sum_{b=0}^{r-q} W_{r-q-b} W_b = W_r - e^{-s} P_r(-s)$$

for $r \geq 0$ and where the polynomials $P_n(s)$ are given by

$$P_n(s) = \sum_{r=0}^{n} (-1)^{n-k} s_1(k + n, k) \frac{s_{r-k}^r}{(r-k)! (k+n)!},$$

where $s_1(n, k)$ are the Stirling numbers of the first kind [1]. This recurrence relations allows us to calculate $W_r$ in terms of the previous coefficients. From (9) we find that the leading order behavior satisfies

$$W_0 = W_0 - e^{-s}.$$  

Equation (10) is invariant under $s \mapsto s + 2\pi i$ as the function $e^s$ is $2\pi i$-periodic and hence $W_0(s)$ is $2\pi i$-periodic. Furthermore, it will be shown in section 3.2 that the Stokes switching behavior of (8) depends on $W_0(s)$ to leading order. Hence we restrict our attention to the domain, $D_0$, described by

$$D_0 = \{ s \in \mathbb{C} | \text{Im}(s) \in (-\pi, \pi) \}.$$  

We also define the domain, $D_k$, which we call $k$th-adjacent domain, by

$$D_k = \{ s \in \mathbb{C} | \text{Im}(s) \in (-\pi + 2k\pi, \pi + 2k\pi) \}.$$  

As $W_0$ satisfies a quartic we therefore have four possible leading order behaviors as $\epsilon \to 0$. We first define the following

$$A = 4 \left( \frac{2}{3} \right)^{1/3} e^{-s}, \quad B = 9 + \sqrt{3} \sqrt{27 - 256e^{-3s}}, \quad C = 2^{1/3} 3^{2/3},$$

and

$$D = \frac{A}{B^{1/3}} + \frac{B^{1/3}}{C}. $$

Then the four solutions for $W_0$ are given by

$$W_{0,1} = -\frac{\sqrt{D}}{2} + \frac{i}{2} \sqrt{D + \frac{2}{\sqrt{D}}}, \quad W_{0,2} = -\frac{\sqrt{D}}{2} - \frac{i}{2} \sqrt{D + \frac{2}{\sqrt{D}}}. $$
and

\[ W_{0,3} = \frac{\sqrt{D}}{2} + \frac{1}{2} \sqrt{-D + \frac{2}{\sqrt{D}}} \], \quad W_{0,4} = \frac{\sqrt{D}}{2} - \frac{1}{2} \sqrt{-D + \frac{2}{\sqrt{D}}}. \]  \quad (16)

From (10) it follows that

\[ W_{0,4}(s) = -\sum_{j=1}^{3} W_{0,j}(s). \]  \quad (17)

Each of the leading order solutions, \( W_0 \), are singular at points for which the discriminant in (13) vanishes, i.e. the points

\[ s_0 = \frac{1}{3} \left( \log \left( \frac{256}{27} \right) + 2i\pi \right), \]  \quad (18)

where \( k \in \mathbb{Z} \). Let us denote the singularities in \( D_0 \) by

\[ s_{0,1} = \frac{1}{3} \left( \log \left( \frac{256}{27} \right) - 2i\pi \right), \quad s_{0,2} = \frac{1}{3} \left( \log \left( \frac{256}{27} \right) + 2i\pi \right), \quad s_{0,3} = \frac{1}{3} \log \left( \frac{256}{27} \right). \]  \quad (19)

Then it can be shown that the local behavior of \( W_{0,j}(s) \) near the singular points (18) is given by

\[ W_{0,j} \sim \left( \frac{1}{4} \right)^{1/3} e^{2i\pi/3} + \left( \frac{1}{8\sqrt{2}} \right)^{1/3} e^{2i\pi/3} \sqrt{s - s_{0,j}} + \mathcal{O}(s - s_{0,j}), \]  \quad (20)

as \( s \to s_{0,j} \) for \( j = 1, 2, 3 \). Equation (17) shows that \( W_{0,4}(s) \) is the sum of \( W_{0,j}(s) \) for \( j = 1, 2, 3 \), and is therefore singular at the points \( s_{0,1}, s_{0,2} \) and \( s_{0,3} \) in \( D_0 \).

Two types of leading order behaviors can be characterized by the number of points at which they are singular in \( D_0 \). In the subsequent analysis, we will refer to solutions with leading order behavior described by \( W_{0,j} \) for \( j = 1, 2, 3 \) as type A, while those with leading order behavior described \( W_{0,4} \) as type B.

In sections 3.3 and 4.1 we will show that the (anti-) Stokes curves emanate from these singularities. Consequently, the Stokes structure of type A solutions will be shown to emerge from a single singularity in \( D_0 \), while type B solutions will have (anti-) Stokes curves emanating from three singular points in \( D_0 \). In this sense, type B solutions will have the most complicated Stokes behavior due to possible interaction effects between the distinct singularities.

### 3. Type A exponential asymptotics

In this section we will investigate the Stokes phenomena exhibited in type A solutions with leading order behavior \( W_{0,3} \) as \( \epsilon \to 0 \). This is done by first determining the leading order behavior of the late-order terms, which will allow us to optimally truncate (8) and study the optimally-truncated error. The results for the remaining type A solutions may be obtained using the symmetry (7).

#### 3.1. Late-order terms

As discussed in section 1.2, the ansatz for the late-order terms is given by a factorial-over-power form since the determination of \( W_r \) in (9) involves repeated differentiation of the the previous terms. In particular, determining \( W_r \) requires differentiating \( W_{r-1} \) once, \( W_{r-2} \) twice,
and so on. We therefore apply a variant of (2) and assume that the coefficients of (8) with \( W_0 = W_{0,3} \) are described by

\[
W_s(r) \sim \frac{U_3(s) \Gamma(r + \gamma_1)}{\chi_3(s)^{r+\gamma_1}},
\]

(21)
as \( r \to \infty \), where \( \chi_3(s) \) is the singulant, \( U_3(s) \) is the prefactor and \( \gamma_1 \) a constant. We substitute (21) into (9) to obtain

\[
2W_{0,3}^3 W_r + W_{0,3}^3 \sum_{k=0}^{r} \frac{((-1)^k + 1)}{k!} (-\chi_3')^k W_r + 6W_{0,3}^2 W_1 W_{r-1}
\]

\[+ W_{0,3}^{3} \sum_{k=0}^{r} \frac{((-1)^k + 1)}{k!} \left( \binom{k}{1} (-\chi_3')^{k-1} \frac{U_3'}{U_3} + \binom{k}{2} (-\chi_3')^{k-2} (-\chi_3'') \right) W_{r-1}
\]

\[+ \sum_{k=0}^{r-1} \frac{((-1)^k + 1)}{k!} (-\chi_3')^k (3W_{0,3}^2 W_1 + W_{0,3}^2 W_{0,3}'') W_{r-1} + O(W_{r-2}) = W_r + \cdots ,
\]

(22)

where the remaining terms are negligible for the purposes of this demonstration as \( r \to \infty \). By matching terms of \( O(W_r) \) as \( r \to \infty \), we find that the leading order equation is given by

\[
2W_{0,3}^3 + W_{0,3}^{2} \sum_{k=0}^{r} \frac{((-1)^k + 1)}{k!} (-\chi_3')^k = 1,
\]

(23)
as \( r \to \infty \). Matching at the next subsequent order involves matching terms of \( O(W_{r-1}) \) in (22). Doing so gives

\[
W_{0,3}^{3} \sum_{k=0}^{r} \frac{((-1)^k + 1)}{k!} \left( \binom{k}{1} (-\chi_3')^{k-1} \frac{U_3'}{U_3} + \binom{k}{2} (-\chi_3')^{k-2} (-\chi_3'') \right) + 6W_{0,3}^2 W_1
\]

\[+ 3W_{0,3}^3 W_r \sum_{k=0}^{r-1} \frac{((-1)^k + 1)}{k!} (-\chi_3')^k + W_{0,3}^2 W_{0,3}' \sum_{k=0}^{r-1} \frac{((-1)^k - 1)}{k!} (-\chi_3')^k = 0
\]

(24)
as \( r \to \infty \). In order to determine the singulant, \( \chi_3(s) \), we consider (23). The leading order behavior of \( \chi_3(s) \) may be determined by replacing the upper limit of the sum in (23) by infinity. Taking the series to be infinite introduces error in the singulant behavior which is exponentially small in the limit \( r \to \infty \), which is negligible here [33, 34, 37]. Evaluating the sum appearing in (23) as an infinite series gives

\[
\cosh(-\chi_3') = \frac{1 - 2W_{0,3}^3}{2W_{0,3}}, \quad \chi(s_{0,3}) = 0.
\]

(25)
The solution of (25) is given by

\[
\chi_3(s; M) = \pm \int_{\sigma_1} \left( \cosh^{-1}(\sigma(t)) + 2iM\pi \right) dt, \quad \sigma(s) = \frac{1 - 2W_{0,3}(s)^3}{2W_{0,3}(s)^3}
\]

(26)

where \( M \in \mathbb{Z} \). Noting that there are two different expressions for the singulant, we name them \( \chi_3(s; M) \) and \( \chi_3^s(s; M) \) with the choice of the positive and negative signs respectively. In general, the behavior of \( W_r \) will be the sum of expressions (21), with each value of \( M \) and sign of the singulant [21]. However, this sum will be dominated by the two terms associated with \( M = 0 \) as this is the value for which \( |\chi| \) is smallest [13]. Thus, we consider the \( M = 0 \) case in the subsequent analysis. Hence we set
\[
\chi_3(s) = \int_{\sigma_3}^s \cosh^{-1}(\sigma(t)) \, dt, \quad \chi_3^-(s) = -\int_{\sigma_3}^s \cosh^{-1}(\sigma(t)) \, dt.
\]

In order to determine the prefactor associated with each singulant we solve (24). As before, we replace the upper limit of the sum appearing in (24) by infinity and use (26) in order to obtain

\[
-2W_{0,3}^2 \sinh(\chi^3) U_3'' U_3 - W_{0,3}^2 \chi_3'' \cosh(\chi_3^3) + 2W_{0,3}^2 W_0' \sinh(\chi_3) + 3 \frac{W_1}{W_{0,3}} = 0.
\]

(28)

It can be verified that

\[
F(s) = \frac{\Upsilon W_{0,3}}{\sqrt{\sinh(\chi_3^3)}},
\]

(29)

where \( \Upsilon \) is a constant of integration, solves the differential equation (28) without the 3\( W_1/W_{0,3} \) term. By setting \( U_3(s) = F(s) \phi(s) \) and substituting this into (28) we obtain the differential equation

\[
\phi' \phi = -\frac{W_1 \chi_3''}{W_{0,3}},
\]

(30)

where we have used (25) in order to express (30) in terms of \( \chi_3'' \). Integrating (30) then gives

\[
\phi(s) = \tilde{\Upsilon} \exp \left(-\int r W_1(t) \chi_3''(t) \sqrt{W_{0,3}(t)} \, dt\right),
\]

(31)

where \( \tilde{\Upsilon} \) is a constant of integration. Hence, if we let \( U_3 \) and \( U_3^- \) denote the prefactors associated with the singulants \( \chi_3 \) and \( \chi_3^- \), respectively, then the solutions of (28) are given by

\[
U_3(s) = \frac{\Lambda W_{0,3} e^{-G}}{\sqrt{\sinh(\chi_3^3)}}, \quad U_3^-(s) = \frac{\tilde{\Lambda} W_{0,3} e^G}{\sqrt{\sinh(\chi_3^3)}},
\]

(32)

where \( \Lambda, \tilde{\Lambda} \) are constants (which depend on \( \Upsilon \) and \( \tilde{\Upsilon} \)) and

\[
G(s) = \int_{s}^\infty \frac{W_1(t) \chi_3''(t)}{W_{0,3}(t)} \, dt.
\]

(33)

Substituting the (27) and (32) into (21) shows that

\[
W_r(s) \sim \frac{W_{0,3} \Gamma(r + \gamma_1)}{\sqrt{\sinh(\chi_3^3)}} \frac{\Lambda e^{-G} + \tilde{\Lambda} e^G}{(-1)^{r+\gamma_1}},
\]

(34)

as \( r \to \infty \).

To completely determine the form of the \( W_r \) we must also determine the value of \( \gamma_1 \). This requires matching the late-order expression given in (21) to the leading-order behavior in the neighbourhood of the singularity. This procedure is described in appendix B, and shows that \( \gamma_1 = -1/2 \) and \( \tilde{\Lambda} = -i\Lambda \).

From the results obtained in appendix B, we find that

\[
W_{2r}(s) \sim \frac{2W_{0,3} \Lambda \sinh(G) \Gamma(2r - 1/2)}{\sqrt{\sinh(\chi_3^3)}} \frac{1}{\chi_3^{2r-1/2}}, \quad W_{2r+1}(s) \sim \frac{-2W_{0,3} \Lambda \sinh(G) \Gamma(2r + 1/2)}{\sqrt{\sinh(\chi_3^3)}} \frac{1}{\chi_3^{2r+1/2}},
\]

(35)

as \( r \to \infty \). Hence, (35) reveals that the asymptotic series (8) may be expressed as the sum of two asymptotic series in even and odd powers of \( \epsilon \).
In the following analysis, we apply optimal truncation methods to the asymptotic series, (8) and show that the optimal truncation error is proportional to \( U_3 e^{-\chi_3/\epsilon} \).

### 3.2. Stokes smoothing

In order to determine the behavior of the exponentially small contributions in the neighbourhood of the Stokes curve we optimally truncate (8). We truncate the asymptotic series at the least even term by writing

\[
W(s) = \sum_{r=0}^{2N_{\text{opt}}-1} \epsilon^r W_r(s) + R_N(s),
\]

where \( N_{\text{opt}} \) is the optimal truncation point and \( R_N \) is the optimally-truncated error. As the analysis is technical, we will summarize the key results in this section with the details provided in appendix C.

In appendix C, we show that the optimal truncation point is given by

\[
N_{\text{opt}} \sim \left( \frac{|\chi_3|}{\epsilon} + \kappa \right) / 2,
\]

as \( \epsilon \to 0 \), and where \( \kappa \in [0, 1) \) is chosen such that \( N_{\text{opt}} \in \mathbb{Z} \).

The leading order behavior of the remainder term in the small \( \epsilon \) limit can be shown to take the form

\[
R_N \sim S_3 U_3 e^{-\chi_3/\epsilon},
\]

where \( S_3 \) is the Stokes switching parameter which varies rapidly between constant values in the neighbourhood of Stokes curves.

For algebraic convenience, we let \( T(s) \) represent the finite sum in (36) and use the shift notation denoted by \( F(s) = F(s + \epsilon), F(s - \epsilon) \). By substituting the truncated series, (36), into the governing equation (5) we obtain

\[
T^2 T R_N + 2T^2 T R_N + T^2 T R_N + \epsilon^{2N}(W_{2N} - 4W_{0.3} W_{2N}) + O(\epsilon^{2N+1}) = R_N,
\]

(38)

where the terms neglected are of order \( O(\epsilon^{2N+1}) \) and quadratic in \( R_N \). Terms of these sizes are negligible compared to the terms kept in (38) as \( \epsilon \to 0 \).

Recall that Stokes switching occurs across Stokes curves, which are characterized by \( \text{Im}(\chi_3) = 0 \) and \( \text{Re}(\chi_3) > 0 \). We define the value of the Stokes multiplier, \( S_3 \), to be

\[
S_3 = \begin{cases} S_3^+, & \text{in the regions where } \text{Im}(\chi_3) > 0 \text{ and } \text{Re}(\chi_3) > 0, \\ S_3^-, & \text{in the regions where } \text{Im}(\chi_3) < 0 \text{ and } \text{Re}(\chi_3) > 0. \end{cases}
\]

(39)

In appendix C we show that the Stokes multiplier, \( S_3 \), changes in value by

\[
\Delta S_3 = S_3^+ - S_3^- \sim i\sqrt{2\pi\epsilon} H(|\chi_3|) \int_{-\infty}^{\infty} e^{-x^2/2} dx = 2i\pi \sqrt{H(|\chi_3|)}.
\]

as Stokes curves are crossed and \( H \) is the function defined by \( H(\chi_3') = (1 - 4W_{0.3}^3)^{1/2}/\chi_3' \).

Let \( S_j \) denote the Stokes multiplier associated with \( \chi_j \) to be

\[
S_j = i\pi \sqrt{H(|\chi_j|)} \left( \text{erf} \left( \frac{\sqrt{|\chi_j|}}{2\epsilon} \right) + C_j \right).
\]

(40)

for \( j = 1, 2, 3 \) and \( C_j \) is an arbitrary constant. Then the asymptotic power series expansion of type A solutions of (5) with leading order behavior, \( W_{0.3} \), up to exponentially small corrections is given by
\[ W(s) \sim W_{0,3}(s) + \sum_{r=1}^{2N-1} e^{2r} W_r(s) + \mathcal{S}_3(s) U_3(s) e^{-\chi_j(s)/\epsilon}, \quad (41) \]

as \( \epsilon \to 0 \). By using the symmetry given by (7) we find that the other type A solutions are given by

\[ W(s) \sim W_{0,1}(s) + \lambda_1 \sum_{r=1}^{2N-1} e^{2r} W_{2r}(s) + \mathcal{S}_1(s) U_1(s) e^{-\chi_j(s)/\epsilon}, \quad (42) \]

\[ W(s) \sim W_{0,2}(s) + \lambda_2 \sum_{r=1}^{2N-1} e^{2r} W_{2r}(s) + \mathcal{S}_2(s) U_2(s) e^{-\chi_j(s)/\epsilon}, \quad (43) \]

as \( \epsilon \to 0 \), where \( \mathcal{S}_j \) is given by (40), \( \lambda_1 = e^{-2i\pi/3}, \lambda_2 = e^{2i\pi/3} \) and

\[ \chi_1(s) = \chi(s + 2i\pi/3), \quad U_1(s) = U_3(s + 2i\pi/3), \quad \chi_2(s) = \chi(s - 2i\pi/3), \quad U_2(s) = U_3(s - 2i\pi/3). \]

In particular, the values of \( \Lambda_j \) are

\[ \Lambda_1 = \lambda_1 \Lambda_3 = e^{-2i\pi/3} \Lambda_3, \quad \Lambda_2 = \lambda_2 \Lambda_3 = e^{-2i\pi/3} \Lambda_3, \]

where \( \Lambda_3 \) is given by (B.9). Furthermore, the complete asymptotic series expansions of type A solutions of (5) have Stokes multipliers which contain a free parameter in the form of \( C_j \). The free parameters, \( C_j \), and the validity of type A solutions will be discussed further in section 3.3.

We have successfully determined a family of asymptotic solutions of (5) which contains exponentially small errors. These exponentially small terms exhibit Stokes switching and therefore the expressions (41)–(43) describe the asymptotic behavior in certain regions of the complex plane. The regions of validity for type A solutions will be determined in section 3.3.

### 3.3. Stokes structure

In this section we determine the Stokes structure of type A solutions in both the complex \( s \) and \( x \)-planes. As demonstrated in section 3.2, we found that the exponential contributions present in the series expansions (41)–(43) are proportional to \( \exp(-\chi_j/\epsilon) \). These terms are exponentially small when \( \text{Re}(\chi_j) > 0 \) and exponentially large when \( \text{Re}(\chi_j) < 0 \). Hence, by considering (27) we may determine the behavior of these terms and the location of the (anti-) Stokes curves. Once the Stokes structure has been determined in the complex \( s \)-plane, we may reverse the scalings given by (4) in order to determine the Stokes structure in the complex \( x \)-plane.

We first illustrate and explain the Stokes structure and Stokes switching behavior of (41) in the domain \( D_0 \), described by (11). The upper and lower boundaries of \( D_0 \) are described by the curves \( \text{Im}(s) = \pm \pi \) and are denoted by the dot-dashed curves in figure 1(a). In this figure we also see that there are two Stokes curves (red curves) and three anti-Stokes curves (dashed blue curves) emanating from the singularity \( s_{0,3} \). The Stokes curves extending towards the upper boundary of \( D_0 \) switches the exponential contribution associated with \( \chi_3 \) as \( \text{Re}(\chi_3) > 0 \). This Stokes curve is denoted by \( \otimes \) in figure 1(b). While the Stokes curve extending towards the lower boundary of \( D_0 \) does not switch any exponential contributions as \( \text{Re}(\chi_3) < 0 \). Additionally, there is a branch cut (zig–zag curve) of \( \chi_3 \) located along the negative real \( s \) axis emanating from \( s_{0,3} \). Using this knowledge, we can determine the switching behavior as Stokes curves are crossed. Additionally, we observe in figure 1(a) that the (anti-) Stokes curves and the branch cut separate \( D_0 \) into six regions.
We now determine regions in $D_0$ in which the asymptotic behavior of (5) is described by the power series expansion (41), referred to as regions of validity. From figure 1(a) we observe that the exponential contribution associated with $\chi_3$ is exponentially small in the neighborhood of the upper Stokes curves since $\text{Re}(\chi_3) > 0$, and therefore the presence of $\exp(-\chi_3/\epsilon)$ does not affect the dominance of the leading order behavior in (41). Hence, the value of the Stokes multiplier, $S_3$, in the neighborhood of the upper Stokes curve may be freely specified, and will therefore contain a free parameter hidden beyond-all-orders. The values of $S_3$ in the regions of $D_0$ is illustrated in figure 2(a).

The remainder term associated with $\chi_3$ will exhibit Stokes switching and therefore the value of $S_3$ varies as it crosses a Stokes curve, say, from state 1 to state 2. Using the naming convention described by (39), we denote these states by $S^-_3$ and $S^+_3$ respectively. If we assume that the value of $S_3$ is nonzero on either side of the upper Stokes curve, then we conclude that the exponentially small contribution associated with $\chi_3$ is present in the regions bounded by the upper anti-Stokes curve, the anti-Stokes curve emanating from the singularity along the positive real $s$-axis and the curve Im$(s) = \pi$. Furthermore, the exponential contribution associated with $\chi_3$ is also exponentially small in the region bounded by the branch cut and the lower anti-Stokes curve. The regions of validity of the asymptotic solution described by (41) are illustrated in figure 2(b).

However, for special choices of the free parameter hidden beyond-all-orders, we can obtain asymptotic solutions with an extended range of validity in $D_0$. If we specify the value of $S^-_3$ to be equal to zero, then the exponential contribution associated with $\chi_3$ is no longer present in regions where it is normally exponentially large. In this case, the region of validity is extended by two additional adjacent sectorial regions in $D_0$ as illustrated in figure 3(b). We note that the case where $S^+_3 = 0$ is specified can also give type A solutions with an extended region of validity. However, this only extends the regions of validity of (41) by one additional sectorial region. In both cases the value of $S_3$ is specified, and therefore the asymptotic solution described by (41) is uniquely determined; we call these special type A asymptotic solutions.

Figure 4 illustrates the Stokes structure in the adjacent domains $D_1$ and $D_{-1}$ as described by (12). Due to the $2\pi i$-periodic nature of $W_0$, the Stokes structure is also $2\pi i$-periodic as shown in figure 4. Hence, we obtain an exponential contribution in each adjacent domain $D_k$. For integers $k \leq -1$, there are exponentially small contributions present in the adjacent domains $D_k$. The
presence of these exponentially small contributions do not affect the asymptotic behavior in the principal domain, \( D_0 \) and hence the corresponding Stokes multipliers may be freely specified. However, for integers \( k \geq 1 \) the exponential contributions originating from the adjacent domains \( D_k \) do affect the asymptotic behavior in \( D_0 \). In order for the asymptotic solution (41) to correctly describe the solution behavior in \( D_0 \), the value of the Stokes multipliers must be specified such that there are not present in \( D_0 \). The presence of the exponential contributions in the domains \( D_{-1}, D_0 \) and \( D_1 \) is illustrated in figure 4(b).

The corresponding analysis of the Stokes structure and switching behavior of the exponential contributions associated with the asymptotic solutions (42) and (43) may be obtained by using the symmetry (7). Consequently, the Stokes structure associated with \( \chi_1 \) and \( \chi_2 \) are vertical translations of the \( \chi_3 \)-Stokes structure by \( \pm 2i\pi \), respectively. The corresponding figures are contained in figure 5.

As the Stokes structure and switching behavior of the exponential contributions have been determined in the domain \( D_0 \), we may finally determine the Stokes structure in the original
In order to determine the Stokes structure in the $x$-plane we reverse the scalings transformations. In particular, from the scalings given by (6) we find that the leading order mapping from $s$ to $x$ is given by

$$s = \frac{\epsilon \log(x)}{\log(1 + \epsilon)} \sim \log(x) + O(\epsilon),$$

as $\epsilon \to 0$.

We illustrate the Stokes structures of type A solutions for the choice of $q = 1 + 0.2i$. Using (44), the singulants, $\chi_j(x)$, can be written as a function of $x$. We then compute the Stokes structure in the complex $x$-plane using MATLAB. The corresponding Stokes structure for the complex $x$-plane. In order to determine the Stokes structure in the $x$-plane we reverse the scaling transformations. In particular, from the scalings given by (6) we find that the leading order mapping from $s$ to $x$ is given by

$$s = \frac{\epsilon \log(x)}{\log(1 + \epsilon)} \sim \log(x) + O(\epsilon),$$

as $\epsilon \to 0$.

We illustrate the Stokes structures of type A solutions for the choice of $q = 1 + 0.2i$. Using (44), the singulants, $\chi_j(x)$, can be written as a function of $x$. We then compute the Stokes structure in the complex $x$-plane using MATLAB. The corresponding Stokes structure for the complex $x$-plane.
asymptotic solution described by (41) in the complex $x$-plane is illustrated in figure 6(a). In these figures, the (anti-) Stokes curves and branch cuts in the complex $x$-plane follow the convention illustrated in figure 1.

In particular, the complex $x$-plane contains a logarithmic branch cut (dot-dashed curve) along the negative real $x$ axis as a result of using the leading order term of the inverse transformation described by (44). In fact, the boundaries of $D_0$ are mapped to this branch cut in the complex $x$-plane. The inverse transformation maps the Stokes and anti-Stokes curves in the $s$-plane to $q$-spirals in the complex $x$-plane. From figure 6, we find the Stokes and anti-Stokes curve separate the complex $x$-plane into sectorial regions bounded by arcs of spirals.

4. Type B asymptotics

In this section we investigate type B solutions of (5). These are the asymptotic solutions of (5), which are described by $W_{0,4}$ to leading order as $\epsilon \to 0$. The analysis involved in the
subsequent sections is nearly identical to sections 2 and 3. Hence, we will omit the details and only provide the key results.

Type B solutions may be expanded as a power series in $\epsilon$ of the form

$$W(s) \sim W_{0,4}(s) + \sum_{r=1}^{\infty} \epsilon^r y_r(s),$$

(45)
as $\epsilon \to 0$. The terms $y_r$ (for $r = 1, 2, 3 \ldots$) may be determined by recursively solving (9). Following the analysis in section 3, the late-order terms behavior of $y_r$ is also described by the factorial-over-power form. We therefore apply the late-order terms ansatz:

$$y_r(s) \sim \frac{Y_j(s)\Gamma(r + \nu)}{\eta(s)^{r+\nu}},$$

as $r \to \infty$. Recall that the main difference between type A and type B solutions is that type B solutions are singular at three distinct points in $D_0$ rather than one. This feature will therefore be encoded in the calculation of the singulant function, $\eta$.

Since the leading order behavior $W_{0,4}$ is singular at $s_{0,1}, s_{0,2}$ and $s_{0,3}$, we obtain three singulant contributions, which we denote by $\eta_j(s)$. This feature can be deduced from equation (17), which shows that $W_{0,4}$ is expressible as the sum of $W_{0,1}, W_{0,2}$ and $W_{0,3}$. Following the analysis in section 3.1, $\eta_j(s)$ is given by

$$\eta_j(s) = \int_{s_{0,j}}^{s} \cosh^{-1}(\sigma(t))dt,$$

for $j = 1, 2, 3$ and where $\sigma$ is given in (26) with $W_{0,3}$ replaced by $W_{0,4}$. Hence we obtain three contributions for $\eta$. Using the results for the late-order terms found in section 3.1, we find that the late-order terms of (45) is given by

$$y_r(s) \sim \sum_{j=1}^{3} \frac{Y_j(s)\Gamma(r + \nu)}{\eta_j(s)^{r+\nu}},$$

as $r \to \infty$, and where $Y_j(s)$ are the prefactor terms associated with $\eta_j(s)$. Hence the asymptotic expansion of type B solutions of (5) is given by

$$W(s) \sim W_{0,4}(s) + \sum_{j=1}^{3} \sum_{r=1}^{\infty} \frac{\epsilon^r Y_j(s)\Gamma(r + \nu)}{\eta_j(s)^{r+\nu}},$$

as $\epsilon \to 0$. As there are three distinct singulant terms in (48) there will be three subdominant exponentials present (after optimal truncation) and hence type B solutions will also display Stokes behavior.

By applying the Stokes smoothing technique demonstrated in section 3.2 to (48), the expression which captures the Stokes behavior of the subdominant exponential correction terms is given by

$$W(s) \sim W_{0,4} + \sum_{j=1}^{3} \sum_{r=1}^{2N_{\text{opt}}-1} \frac{\epsilon^r Y_j(s)\Gamma(r + \nu)}{\eta_j(s)^{r+\nu}} + \sum_{j=1}^{3} \hat{S}_j(s) \hat{\phi}(s) Y_j(s)e^{-\eta_j(s)/\epsilon},$$

as $\epsilon \to 0$, and where $N_{\text{opt}}$ is the optimal truncation point. In particular, the type B prefactor terms satisfy equation (32) with $W_0$ replaced by $W_{0,4}$. Similarly, the Stokes multipliers $\hat{S}_j(s)$ satisfy (C.10) with $W_0$ and $\chi$ replaced by $W_{0,4}$ and $\eta$ respectively. In view of the formula (17), the leading order behavior of the type B solution is a composition of the leading order behaviors of type A solutions. Consequently, the Stokes behavior present in this solution will be more complicated as the Stokes curves emanate from more than one point as this allows the possibility of interaction effects. In order to determine the Stokes structure of type B solutions, we analyze the singulant (46).
The Stokes structure of type B asymptotic solutions in $D_0$ is illustrated in figure 7. In figure 7 we see that there are three Stokes and two anti-Stokes curves emanating from each of the singularities, $s_{0,j}$, for $j = 1, 2, 3$. The Stokes structure for type B solutions is more complicated as there are Stokes curves which cross into the branch cuts (zig–zag curves) of $\eta_j$ as illustrated in figure 7. As these Stokes curves continue onto another Riemann sheet of $\eta_j$, they may be subject to possible interaction effects from singularities originating in these Riemann sheets.

However, we observe from figure 7 that there are regions which do not contain the Stokes curves continuing into the different Riemann sheets of $\eta_j$. These regions are labelled as regions I–IV in figure 7. We first note that the Stokes curves emanating from the singularities $s_{0,1}$ and $s_{0,3}$ asymptote to infinity as $\text{Re}(s) \to \infty$ as illustrated in figure 7(a). In figure 7(a), we find that region I is the region bounded by the upper boundary of $D_0$, the upper anti-Stokes curve emanating from $s_{0,1}$ and the Stokes curve emanating from $s_{0,2}$, which is labelled by $\varnothing$; region II is the region bounded by the upper anti-Stokes curve emanating from $s_{0,1}$, and the Stokes curves labelled by $\varnothing$ and $\Theta$; region III is the region bounded by the lower anti-Stokes curve emanating from $s_{0,2}$, and the Stokes curves labelled by $\Theta$ and $\varnothing$; and finally region IV is the region bounded by the lower boundary of $D_0$, the lower anti-Stokes curve emanating from $s_{0,2}$ and the Stokes curve emanating from $s_{0,3}$, which is labelled by $\varnothing$.

Furthermore, $\text{Re}(\eta_j)$ is positive in each of these four regions and hence the exponential contributions associated with $\eta_j$ are exponentially small there. This is illustrated by the light grey shaded regions in figure 7(b). We therefore restrict our analysis to regions I–IV as these are the regions in which the dominant asymptotic behavior is described by (49). Consequently, the regions of validity of type B solutions are the regions bounded by the upper and lower anti-Stokes curves emanating from the singularities, $s_{0,1}$ and $s_{0,2}$, respectively, and the boundaries of $D_0$ containing the positive real $s$ axis. This is the union of regions I–IV and is illustrated in figure 7(b).

In order to determine Stokes behavior present in the asymptotic solution (49) we investigate the behavior of $\eta_j$ in regions I–IV. In figure 7, regions I–IV denote those of $D_0$ in which $\text{Re}(\eta_j) > 0$. Additionally, the imaginary parts of $\eta_j$ for $j = 1, 2, 3$ are all positive in region I,
whereas they are all negative in region IV. Furthermore, we have $\text{Im}(\eta_1) > 0$, $\text{Im}(\eta_2) < 0$, $\text{Im}(\eta_3) > 0$ in region II while $\text{Im}(\eta_1) > 0$, $\text{Im}(\eta_2) < 0$, $\text{Im}(\eta_3) < 0$ in region III.

Hence, the Stokes curve separating regions I and II switches on the exponential contribution associated with $\eta_2$, the Stokes curve separating regions II and III switches on the exponential contribution associated with $\eta_1$ and the Stokes curve separating regions III and IV switches on the exponential contribution associated with $\eta_3$. To denote the Stokes switching behavior of these subdominant exponentials, the Stokes curves are labelled by $\text{\textcopyright}$, $\text{\textcopyright}$ and $\text{\textcopyright}$.

In regions I–IV, the presence of the exponential contributions associated with $\eta_j$ are exponentially small since $\text{Re}(\eta_j) > 0$, and therefore do not affect the dominance of the leading order behavior in (49). Hence, the values of $\delta_j$ may be freely specified in these regions and therefore the asymptotic solution described by (49) contains free parameters hidden beyond-all-orders.

In section 3.3 we were able to obtain special asymptotic solutions by uniquely specifying the free parameters present in the asymptotic expansion. However, this cannot be done for type B solutions in the same way because of the possible interaction effects of singularities originating from the different Riemann sheets of $\eta_j$.

It may be possible to find special type B solutions by considering the behavior of the exponential contributions on the different Riemann sheets of $\eta_j$ and how they interact with those on the principal Riemann sheet. As this is beyond the scope of this study, we restrict our analysis to regions I–IV and therefore only obtain asymptotic solutions described by (49) which contain free parameters hidden beyond-all-orders.

Following the analysis in section 3.3 we obtain the Stokes structure in the original $x$-plane by applying the inverse transformation given by (44). As in section 3.3 we demonstrate the Stokes structure for the value of $q = 1 + 0.2i$. Using MATLAB, the Stokes structure of type B solutions is illustrated in figure 8.

Figure 8 shows the corresponding regions I–IV in the complex $x$-plane. In this figure, the Stokes and anti-Stokes curves are denoted by the solid red and dashed blue curves respectively. The branch cuts of $\eta_j$ are depicted as the zig–zag curves, which connect the singularities to the origin in the complex $x$-plane. Furthermore, the dot-dashed curve denotes the logarithmic branch cut defined by the reverse transformation (44) for the choice of $q = 1 + 0.2i$.

Recall that the inverse transformation maps the Stokes and anti-Stokes curves in the complex $s$-plane to $q$-spirals in the complex $x$-plane. In particular, we see in figure 8 that the Stokes curves of type B solutions extend to infinity in the complex $x$-plane. This is due to the fact that the Stokes curves in the complex $x$-plane extend to infinity as $\text{Re}(x) \to \infty$ as shown in figure 7(a). This was not case for the Stokes curves of type A solutions in section 3.3. Instead the Stokes curves of type A solutions emanate from the singularities and approach the logarithmic branch cuts and enter a different Riemann sheet of the inverse transformation; this is illustrated in figures 6(a), (d) and (g).

We have therefore determine the regions of validity for type B solutions of $q$-$P_1$, (5), in the complex $x$-plane. Type B solutions are described by the asymptotic power series expansion (49) as $\epsilon \to 0$, and contain free parameters hidden beyond-all-orders. Furthermore, we have also calculated the Stokes behavior present within these asymptotic solution, (49), which allowed us to determine regions in which this asymptotic description is valid.

5. Connection between type A and type B solutions and the nonzero and vanishing asymptotic solutions of $q$-Painlevé I

In this section we establish a connection between both types A and B solutions found in this study to the nonzero and vanishing asymptotic solutions of $q$-Painlevé I respectively. We recall
that the nonzero and vanishing asymptotic solutions were first found by Joshi in [30]. In particular, equation (1) admits solutions with the following behavior

\[ w(x) \sim w_n(x) = \omega^3 + O \left( \frac{1}{x} \right), \quad w(x) \sim w_v(x) = \frac{1}{x} + O \left( \frac{1}{x^3} \right), \]  

(50)

as \( |x| \to \infty \), where \( w_n \) and \( w_v \) are the nonzero and vanishing asymptotic solutions and \( \omega^3 = 1 \).

Recall that the four possible solutions for \( W_0(s) \) are denoted by \( W_{0j}(s) \), which are defined by (15) and (16). In our investigation, we applied the scalings given in (4), in which the variable \( x \) has the behavior described by (6) as \( \epsilon \to 0 \). The analysis for both types A and B solutions are valid in the limit \( \epsilon \to 0 \), which was shown to be equivalent to the double limit.
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and $n \to \infty$. Under the additional limit, $s \to +\infty$, we see that the behavior of $x$ in (6) approaches infinity. Therefore, the limits $\epsilon \to 0$ and $s \to +\infty$ are equivalent to the limit $|x| \to \infty$.

We now study the behavior of $W_{0,j}(s)$ under the additional limit $s \to +\infty$. By applying the limit $s \to +\infty$ to the terms appearing in (13) and (14) we find from equations (15) and (16) that

$$
\lim_{s \to +\infty} W_{0,1} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \lim_{s \to +\infty} W_{0,2} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}, \quad \lim_{s \to +\infty} W_{0,3} = 1, \quad \lim_{s \to +\infty} W_{0,4} = 0.
$$

(51)

The limiting behavior of type A solutions under the limit $s \to +\infty$ are therefore described by cube roots of unity. That is $W_{0,j} \sim \omega^j$ as $s \to +\infty$ for $j = 1, 2, 3$. However, we also find from (51) that type B solutions vanish in the limit $s \to +\infty$. We therefore find that type A solutions of (5) tend to the nonzero asymptotic behavior solutions, $w_{nv}$, found by Joshi [30] under the additional limit $s \to +\infty$.

In order to calculate the leading order behavior of $W_{0,4}(s)$ as $s \to +\infty$, we need to keep terms up to order $O(e^{-3s})$. Be carefully tracking such terms we find that the behavior of type B solutions are given by

$$
W_{0,4} \sim \frac{1}{2} \sqrt{1 - \frac{4e^{-s}}{3}} - \frac{1}{2} \sqrt{1 - \frac{4e^{-s}}{3} - \frac{2}{1 - 4e^{-s/3}}} \sim e^{-s} + O(e^{-3s}),
$$

(52)
as $s \to +\infty$. From (6) we find that (52) is equivalent to the behavior $1/|x|$ as $|x| \to \infty$ and hence type B solutions correspond to the vanishing asymptotic behavior, $w_v$, of (1).

5.1. Numerical computation for $q$-Painlevé I

In this section we give a numerical example of (3) with the parameter choice of $q = 1 + 0.2i$ and where $x = x_0 q^n$ with $x_0 = 1$. Given two initial conditions, $w_0$ and $w_1$, a sequence of
solutions of (3) may be obtained by repeated iteration. In general, only a certain choice of initial conditions will give a solution of (3) which tends to the asymptotic behavior of interest. We follow the numerical method demonstrated by [33], originally based on the works of [31] to find appropriate initial conditions which tend to type A solutions of (3).

Figure 9 illustrates a comparison between the numerical solution of (3) with $q = 1 + 0.2i$ and the initial conditions $w_0 = 0.846885522 + i0.798385416$ and $w_1 = -0.502881648 - i0.650433326$ and the nonzero asymptotic solution, $w_{nv}$ in (50) with $\omega = (-1 + i\sqrt{3})/2$. Figures 9(a) and (b) show that real and imaginary part of $w_n$ converges to $-1/2$ and $\sqrt{3}/2$ respectively for large $n$, which is precisely described by the leading order term of $w_{nv}$ in (50).

6. Conclusions

In this study, we extended the exponential asymptotic methods used in [33, 34] to compute and investigate Stokes behavior present in the asymptotic solutions of $q$-$P_1$ in the double limit $|q| \to 1$ and $n \to \infty$. In order to investigate the solution behavior of $q$-difference equations we rescaled the variables in the problem such that the double limit $|q| \to 1$ and $n \to \infty$ was equivalent to the limit $\epsilon \to 0$. We found two types of solutions for $q$-$P_1$, which we call type A and type B solutions. Type A solutions of $q$-$P_1$ are described asymptotically by either (41)–(43), and type B solutions are described by (49). The asymptotic descriptions obtained in this analysis are given as a sum of a truncated asymptotic power series and an exponentially subdominant correction term. We then determined the Stokes structure and used this information to deduce the regions of the complex plane in which these asymptotic solutions are valid.

In section 3 we first considered the asymptotic solutions of (5) described by type A solutions. Using exponential asymptotic methods, we determined the form of the subdominant exponential contribution present in the asymptotic solutions, which were found to be defined by one free Stokes switching parameter. From this behavior, we deduced the associated Stokes structure, illustrated in figures 1(a) and 5. By considering the Stokes switching behavior of these subdominant exponentials, we found that the dominant asymptotic behavior is described by either (41)–(43) in a region in the complex $s$-plane containing the positive real axis. Furthermore, we found that it is possible to select the Stokes parameters so that the exponential contribution is absent in the regions where it would normally be large. Consequently, the regions of validity for the special type A solutions are larger than the regions of validity for generic type A solutions as illustrated in figures 3 and 5(b), (d).

In section 4, we considered the equivalent analysis for type B solutions of (5). Compared to type A solutions, type B solutions are those which are singular at three points rather than one. Although exponential asymptotic methods may be used again to determine the form of the exponential small contributions present in type B solutions, we noted that the analysis is near identical except for the determination of the singulant of this problem, $\eta$. The main difference is due to the fact that type B solutions are singular at three points rather than one, and the remaining analysis was therefore identical as for type A solutions. Type B asymptotic solutions are given as a sum of a truncated asymptotic power series and three exponentially subdominant correction terms as described in (49). The Stokes structure for type B solutions, illustrated in figure 7, is significantly more complicated than the Stokes structure of type A solutions. In order to describe the Stokes switching behavior in the domain $\mathcal{D}_0$ we must understand how the asymptotic solution (49) interacts with solutions on different Riemann sheets. However, we restrict ourselves to regions I–IV as these regions are free of possible interaction.
effects with singularities originating from the different Riemann sheets. Furthermore, all three exponential contributions present in type B solutions are exponentially subdominant in regions I–IV and therefore represent the regions of validity of (49). Consequently, the asymptotic solutions described by (49) contain one free parameter defined by the Stokes multiplier.

By reversing the rescaling transformations we were then able to obtain the corresponding Stokes structure in the complex $x$-plane and found that the Stokes and anti-Stokes curves are described by $q$-spirals in the complex $x$-plane. As a result, the regions of validity are no longer described by traditional sectors bounded by rays but sectorial regions bounded by arcs of spirals. Consequently, this methodology is applicable to other $q$-Painlevé equations or more generally nonlinear $q$-difference equations in order to obtain asymptotic solutions which display Stokes phenomena.

In section 5, we demonstrated that types A and B solutions are related to the nonzero asymptotic and vanishing asymptotic solutions found by Joshi [30]. If the additional limit $s \to +\infty$ is also taken, then we find that type A solutions correspond to the nonzero asymptotic solutions of $q$-$P_1$ while type B solutions correspond to the quicksilver solutions of $q$-$P_1$ [30].

Finally, we note that the Riemann–Hilbert approach is typically used to investigate the solution behavior of the Painlevé equations; this approach typically utilizes the integrability of the Painlevé equations in order to obtain and make use of their associated Lax pairs. However, the solvability of the Riemann–Hilbert problem for $q$-difference Painlevé equations still remains unknown. The exponential asymptotic approach used here does not exploit the integrability of the equation, but rather computes the asymptotic behavior directly using the governing equation itself [13, 34, 37]. As discussed in the previous analysis of the first discrete Painlevé equation from [33], this implies that the direct exponential asymptotic method is applicable to both integrable and non-integrable nonlinear difference equations.

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Appendix A. Calculating the late-order terms near the singularity

In order to determine the value of $\gamma_1$ in (21) we calculate the local behavior of $\chi_3$ and $U_3$. Using (20) in equations (26) and (32) we can show that

$$\chi_3(s) \sim \frac{4i\sqrt{6\sqrt{2}}}{5}(s - s_{0,3})^{5/4}, \quad U_3(s) \sim \frac{\Lambda}{\sqrt{i\sqrt{6\sqrt{2}}(s - s_{0,3})^{1/8}}},$$  \(\text{(A.1)}\)

as $s \to s_{0,3}$. Furthermore, by calculating the expression for $W_1$ (using equation (9)) it can be shown that $W_1(s) = O \left((s - s_{0,3})^{-1/2}\right)$ and from (33) that

$$e^{\pm G(s)} = 1 \pm A_1(s - s_{0,3})^{1/4} + O \left((s - s_{0,3})^{1/2}\right),$$  \(\text{(A.2)}\)

as $s \to s_{0,3}$ for some constant $A_1$. By substituting (A.1) into the expression for $W_r$ as given by (34) we find that the local behavior of of the late-order terms near the singularity is given by
\( W_r(s) \sim \left( \frac{5}{4} \right)^{r+\gamma} \frac{\Gamma(r+\gamma_1)}{(i\sqrt{6/5})^{r+\gamma_1+1/2}} (s - s_{0,3})^{5(r+\gamma_1)/4+1/8} \left( \Lambda e^{-\gamma} + \tilde{\Lambda} e^{\gamma} \right), \) \hspace{1cm} (A.3)

as \( s \to s_{0,3} \). From (9), the dominant behavior of \( W_r \) is due to the term \( W_r/(4W_2^3 - 1) \). Hence, if \( W_r \to 2 \) has a singularity of strength \( \nu \), then \( W_r \) has one of strength \( \nu + 5/2 \). In particular, since we know that \( W_0 \) is singular at \( s_{0,3} \) of strength \(-1/2 \) then \( W_2 \) will also be singular at \( s_{0,3} \) but with strength \( 5r/2 - 1/2 \). Moreover, since \( W_1 \) is also singular at \( s_{0,3} \) of strength \( 1/2 \), then \( W_{2r+1} \) has one of strength \( 5r/2 + 1/2 \).

We first observe from (A.2) that
\[
\left( \Lambda e^{-\gamma} + \tilde{\Lambda} e^{\gamma} \right) \sim \begin{cases} 
\Lambda + i\tilde{\Lambda} + A_1(-\Lambda + i\tilde{\Lambda})(s - s_{0,3})^{1/2} + \mathcal{O}((s - s_{0,3})^{1/2}), & \text{for even } r,
\Lambda - i\tilde{\Lambda} + A_1(-\Lambda - i\tilde{\Lambda})(s - s_{0,3})^{1/2} + \mathcal{O}((s - s_{0,3})^{1/2}), & \text{for odd } r,
\end{cases} \hspace{1cm} (A.4)
\]
as \( s \to s_{0,3} \). Thus, in order for the singularity behavior of (A.3) to be consistent with that of \( W_2 \), we require that \( 5r/2 - 1/2 = 5(2r + \gamma_1)/4 + 1/8 \) under the condition \( \Lambda + i\tilde{\Lambda} \neq 0 \). Hence, we deduce that \( \gamma_1 = -1/2 \). Using the same argument, it can be shown that in order for \( W_{2r+1} \) to have the correct singular behavior in the limit \( s \to s_{0,3} \) we must impose the condition \( \Lambda - i\tilde{\Lambda} = 0 \), which gives
\[
\tilde{\Lambda} = -i\Lambda. \hspace{1cm} (A.5)
\]

Appendix B. Calculating the prefactor constants via the inner problem

The expression for the late-order terms given by (21) contains a constant \( \Lambda_3 \), which is yet to be determined. In order to determine the value of \( \Lambda_3 \) we perform an inner analysis of (5) near the singularity, \( s_{0,3} \) and determine the inner expansion of the inner solution. We then use the method of matched asymptotics to match the outer expansion to the inner expansion following Van Dyke’s matching principle [25].

In view of the leading order behavior given by (20), we study the inner solution by applying the scalings
\[
s = s_{0,3} + \epsilon^{4/5} \zeta, \quad W(s) = a_3 + \epsilon^{2/5} \psi_1(\zeta) + \epsilon^{3/5} \psi_2(\zeta), \hspace{1cm} (B.1)
\]
where \( \zeta \) is the inner variable, \( \psi_1 \) and \( \psi_2 \) are the first two terms of the inner solution, \( \psi \). In particular, we recall that
\[
s_{0,3} = \frac{1}{3} \log \left( \frac{256}{27} \right), \quad a_3 = \left( \frac{1}{4} \right)^{1/3}, \quad b_3 = \left( \frac{1}{8\sqrt{2}} \right)^{1/3}. \hspace{1cm} (B.2)
\]
Substituting (B.1) into (5) we obtain
\[
(a_3^4 - a_3 + e^{-s_{0,3}}) + \epsilon^{2/5}(4a_3^4 - 1)\psi_1 + \epsilon^{3/5}(4a_3^4 - 1)\psi_2 + \epsilon^{4/5}(6a_3^2\psi_1^2 - e^{-s_{0,3}}\zeta + a_3^2\psi_1') + \epsilon\left(\frac{s_{0,3}}{2}e^{-s_{0,3}} + 12a_3^2\psi_1\psi_2 + a_3^3\psi_2' \right) + \mathcal{O}(\epsilon^{6/5}) = 0, \hspace{1cm} (B.3)
\]
as \( \epsilon \to 0 \), and where the prime denotes derivatives with respect to \( \zeta \). Using the values of \( a_3, b_3 \) and \( s_{0,3} \) given in (B.2) we find that the coefficients of \( \epsilon^0, \epsilon^{2/5} \) and \( \epsilon^{3/5} \) are identically zero. Therefore, the leading order equation of the inner solution is given by
\[ \psi_2^2 - b_0^2 \zeta + a_3 \frac{d^2 \psi_1}{d\zeta^2} = 0, \quad (B.4) \]

as \( \epsilon \to 0 \). From equation (B.3) we see that the term \( \psi_2 \) does not appear in the leading order equation in the limit \( \epsilon \to \infty \). This reinforces the fact that the odd terms are indeed negligible in the limit \( \epsilon \to 0 \) as the term \( \psi_2 \) corresponds to the first odd coefficient term in the outer problem (far field expansion), (8).

To study the inner solution, we analyze (B.4) in the limit \( |\zeta| \to \infty \). Using the method of dominant balance, equation (B.4) has a solution described by \( \psi_1 \sim b_3 \sqrt{\zeta} \) as \( |\zeta| \to \infty \). For algebraic convenience, we rescale the inner solution by setting

\[ \psi_1(\zeta) \sim b_3 \sqrt{\zeta} \sum_{r=0}^{\infty} E_r \zeta^{-5r/2}. \quad (B.5) \]

with \( E_0 = 1 \). We then substitute (B.5) into (B.4) and match terms of \( O(\zeta) \) in the limit \( |\zeta| \to \infty \). Doing this, we obtain the following nonlinear recurrence relation

\[ E_r = -\frac{1}{2b_3^2} \left( \frac{a_3 b_3}{24} (5r - 4)(5r - 6)E_{r-1} + b_3^2 \sum_{k=1}^{r-1} E_{r-k} E_k \right), \quad (B.6) \]

for \( r \geq 1 \). We can express (B.5) in terms of the outer variables by reversing the scalings given in (B.1). Doing this, we find that the inner expansion of the outer solution is given by

\[ W(s) \sim a_3 + b_3 \sum_{r=0}^{\infty} \frac{e^{2r} E_r}{(s - s_{0,3})^{(5r-1)/2}}, \quad (B.7) \]

as \( \epsilon \to 0 \). Recall that the outer expansion is given by the expression (8) as \( \epsilon \to 0 \), and where the behavior of coefficients are given by (35). By matching the expansions (8) and (B.7) it follows that
\[
\Lambda_3 = \lim_{r \to \infty} b_3 E_r \sqrt[4]{\frac{i\sqrt{6\sqrt{2}}}{\Gamma(2r - 1/2)}} \left( \frac{4i\sqrt{6\sqrt{2}}}{5} \right)^{2r - \frac{1}{2}}.
\]  
(B.8)

We then compute the first 1000 \(E_r\) terms using the recurrence relation (B.6). Then, by using the formula (B.8) we find numerically that the approximate value of \(\Lambda_3\) is
\[
\Lambda_3 \approx -0.04364,
\]  
(B.9)
to four significant figures. The approximate value for \(\Lambda_3\) is shown in figure B1.

**Appendix C. Stokes smoothing**

To apply the exponential asymptotic method, we optimally-truncate the asymptotic series (8). One particular way to calculate the optimal truncation point is to consider where the terms in the asymptotic series is at its smallest [8]. This heuristic is equivalent to the finding \(N\) such that
\[
\epsilon^2 N + 2W_{2N}^2 - \epsilon \to 1,
\]  
in the limit \(N \to \infty\). By using the late-order form ansatz described by (21) we find that \(N \sim |\chi_3|/(2\epsilon)\) as \(\epsilon \to 0\) (which is equivalent to the limit \(N \to \infty\)). As this quantity may not necessarily be integer valued, we therefore choose \(\kappa \in [0, 1)\) such that
\[
N_{\text{opt}} \sim \frac{1}{2} \left( \frac{|\chi_3|}{\epsilon} + \kappa \right),
\]  
(C.1)
is integer valued.

By substituting the truncated series, (36), into the governing equation (5) we obtain
\[
TT^2T + T^2 TR_N + 2TTTR_N + TT^2R_N + \cdots = T + R_N - \frac{1}{(1 + \epsilon)^{3/2}},
\]  
(C.2)
where the terms neglected are quadratic in \(R_N\). We can use the recurrence relation (9) to cancel terms of size \(O(\epsilon^{N-1})\) in (C.2). In particular, equation (C.2) can be rewritten, after rearrangement, as
\[
T^2 TR_N + 2TTTR_N + TT^2R_N - R_N \sim \epsilon^{2N}(4W_{0,3}^2 - 1)W_{2N},
\]  
(C.3)
as \(\epsilon \to 0\) and where the terms neglected are of order \(O(\epsilon^{2N+1}W_{2N+1})\) and terms quadratic in \(R_N\). Terms of these size are negligible compared to the terms kept in (C.3) as \(\epsilon \to 0\).

Away from the Stokes curve the inhomogeneous terms of equation (C.3) are negligible as \(\epsilon \to 0\). We may therefore apply a WKB analysis to the homogeneous version of (C.3) by setting \(R_{N,\text{hom}}(s) = \alpha(s) e^{\beta(s)/\epsilon}\) as \(\epsilon \to 0\). Using the WKB method, we find that the leading order equation gives the solution \(\beta = -\chi_3\). Continuing to the next order involves matching terms of \(O(\epsilon R_N)\). By collecting terms of this size in the homogeneous version of (C.3) we obtain the equation
\[
-2W_{0,3}^3 \sinh(\chi_3') \frac{\alpha'}{\alpha} - W_{0,3}^3 \chi_3'' \cosh(\chi_3') + 2W_{0,3}^2 W_{0,3}' \sinh(\chi_3') + 3\frac{W_1}{W_{0,3}} = 0,
\]  
(C.4)
where we have substituted the fact that $\beta = -\chi_3$. Comparing equations (C.4) and (28) show that $\alpha$ satisfies to same differential equation as the prefactor $U_3$ and hence $\alpha \propto U_3$. Hence, away from the Stokes curves the solution of (C.3) is given by

$$R_{N, \text{hom}}(s) \sim U_3(s)e^{-\chi_3(s)/\epsilon},$$

as $\epsilon \to 0$.

In order to determine the Stokes switching behavior associated with the optimally-truncated error, we set

$$R(s) = S_3(s)R_{N, \text{hom}}(s),$$

(C.5)

where $S_3(s)$ is the Stokes multiplier. We substitute (C.5) into (C.3) and use equations (25) and (28) to cancel terms. Doing this we find that the Stokes multiplier satisfies

$$\frac{dS_3}{ds} \sim \frac{e^{2N-1}(1 - 4W_{0,3}^3)W_{2N}}{2W_{0,3}^3 \sinh(\chi_3(s))e^{\chi_3(s)/\epsilon}} \sim e^{2N-1} \sqrt{1 - 4W_{0,3}^3} \frac{\Gamma(2N + \gamma_1)}{\chi_3^3} e^{\chi_3(s)/\epsilon},$$

as $\epsilon \to 0$.

Noting the form of $N$, we introduce polar coordinates by setting $\chi_3 = \rho e^{i\theta}$ where the fast and slow variables are $\theta$ and $\rho$, respectively. This transformation tells us that

$$\frac{d}{ds} = -i\frac{\chi_3'}{\rho} \frac{d}{d\theta},$$

and (C.1) becomes $N_{opt} = (\rho/\epsilon + \kappa)/2$. Under this change of variables equation (C.6) becomes

$$\frac{dS_3}{d\theta} \sim \frac{i\sqrt{2\pi\rho}\sqrt{1 - 4W_{0,3}^3}}{\chi_3^3} \exp \left( \frac{\rho}{\epsilon} (e^{i\theta} - 1 - i\theta) - i\theta(\kappa - 3/2) \right),$$

(C.8)

as $\epsilon \to 0$, where we have used Stirling’s approximation [1] of the Gamma function. For simplicity, we will let $H(s(\theta); \rho) = \sqrt{1 - 4W_{0,3}^3}/\chi_3$. From (C.8) we find that the right hand side is exponentially small everywhere except in the neighbourhood of $\theta = 0$, which is exactly where the Stokes curve lies (where $\chi_3$ is purely real and positive). We now rescale about the neighbourhood of the Stokes curve in order to study the switching behavior of $S_3$ by setting $\theta = \sqrt{\epsilon}\hat{\theta}$. Note that under this scaling, $H(s(\theta); \rho) \sim H(|\chi_3|)$ as $\epsilon \to 0$, which is therefore independent of $\theta$ to leading order. Applying the scaling $\theta = \sqrt{\epsilon}\hat{\theta}$ to (C.8) gives

$$\frac{1}{\sqrt{\epsilon}} \frac{dS_3}{d\hat{\theta}} \sim i\sqrt{2\pi\rho} H(|\chi_3|) \exp \left( -\frac{|\chi_3|\theta^2}{2} \right),$$

(C.9)

as $\epsilon \to 0$. Integrating (C.9) we find that

$$S_3 \sim i\sqrt{2\pi\rho} H(|\chi_3|) \int_{-\infty}^{\infty} e^{-x^2/2} dx = i\pi \sqrt{\epsilon} H(|\chi_3|) \left( \text{erf} \left( \sqrt{\frac{|\chi_3|}{2\epsilon}} \right) + C_3 \right),$$

(C.10)

where $C_3$ is an arbitrary constant. Thus, as Stokes curves are crossed, the Stokes multiplier changes in value by

$$\Delta S_3 \sim 2i\pi \sqrt{\epsilon} H(|\chi_3|),$$

and therefore
\[ \Delta R_N \sim 2i\pi \sqrt{\epsilon H(\chi_3)} U_3(s) e^{-\chi_3(s)/\epsilon}, \]
as \( \epsilon \to 0. \)

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