THE UNRAMIFIED BRAUER GROUPS OF NORMIC BUNDLES

DASHENG WEI

ABSTRACT. We produce a partial compactification of the variety given by $P(t) = N_{K/k}(z)$ whose Brauer group coincides with the unramified Brauer group, where $K$ is an étale $k$-algebra and $P(t) \in k[t]$ is a nonconstant polynomial. Then we obtain a systematic method to compute the unramified Brauer group for all such varieties.

1. Introduction

The present article focuses on the variety defined over the ground field $k$ by an equation

$$P(t) = N_{K/k}(z),$$

where $K$ is an étale $k$-algebra, $N_{K/k}$ denotes the norm map, $z$ is a “variable” in $K$ and $P(t) \in k[t]$ is a nonconstant polynomial.

Suppose $K/k$ is a finite extension of number fields, for a smooth and proper model of (1.1), Colliot-Thélène conjectured that its rational points are dense in its Brauer-Manin set (see [5]). Colliot-Thélène’s conjecture has been extensively studied. It is known in the case where $P(t)$ is constant [22]; if additionally $K/k$ is cyclic or of prime degree, the Hasse principle and weak approximation hold. Depending on Colliot-Thélène and Sansuc’s descent theory [9], other known cases of Colliot-Thélène’s conjecture, in some cases leading to a proof of the Hasse principle and weak approximation hold, include the class of Châtelet surfaces ([K : k] = 2 and deg($P(t)$) ≤ 4) [10, 11], a class of singular cubic hypersurfaces ([K : k] = 3 and deg($P(t)$) ≤ 3) [8], and the case where $K/k$ is arbitrary and $P(t)$ is split over $k$ with at most two distinct roots [6, 18, 23], and the case where $[K : k] = 4$ and deg($P(t)$) = 2 with $P(t)$ irreducible over $k$ and split over $K$ [1, 14], and the case where $k = \mathbb{Q}$, $K/\mathbb{Q}$ is arbitrary and deg($P(t)$) = 2 [14] and the case where $k = \mathbb{Q}$, $K/\mathbb{Q}$ is arbitrary and $P(t)$ is split over $\mathbb{Q}$ [2, 3, 16]. Finally, under Schinzel’s hypothesis, this conjecture is known for $P(t)$ arbitrary and either $K/k$ cyclic [12] or of prime degree and almost abelian [16, 26]. The integral points of the equation (1.1) have also been studied in [4, 15]. See [27, Example 3.12 and §3.3.4] for a more detailed discussion of these results.

To compute the Brauer-Manin set of (1.1), one must compute its unramified Brauer group. If $[K : k] = 2$, a smooth and proper model of (1.1) can been easily constructed, so we can compute its unramified Brauer group. Using a similar idea in the case $[K : k] = 2$, Várilly-Alvarado and Viray [24, 25] constructed a smooth and proper model of (1.1) for the special case when
$P(t)$ is a separable polynomial of degree $dn$ or $dn - 1$ and $K/k$ is a cyclic field extension of degree $n$. However, in generally, it is hard to construct a smooth and proper model of (1.1). In 2003, Colliot-Thélène, Harari and Skorobogatov [6] produced a partial compactification of (1.1). For this partial compactification, they gave a formula for its vertical Brauer group and the quotient of its Brauer group by the vertical Brauer group. Using this formula, they computed the unramified Brauer group for some special cases (e.g., $P(t)$ is split over $k$ with two distinct roots whose multiplicities are relatively prime). However, the Brauer group of the partial compactification is generally larger than the unramified Brauer group (e.g., see [26, Theorem 3.6 and Proposition 3.7]). Based on Colliot-Thélène, Harari and Skorobogatov’s work, the author [26] studied the case that $P(t)$ is irreducible and $K/k$ is abelian; Derenthal, Smeets and the author [14] studied the case that $\deg(P(t)) = 2$ and $[K : k] = 4$. The problem of producing partial compactifications of (1.1) whose Brauer group coincides with the unramified Brauer group remained open, the goal of this note is to solve this problem in full generality.

The paper is organized as follows. In Section 1, by blowing-up along some closed subsets of the singular locus, we construct a partial compactification of (1.1) and prove that its Brauer group is equal to the unramified Brauer group. In Section 2, applying this partial compactification, we give an explicit formula of the unramified Brauer group of the case $P(t)$ irreducible and $K/k$ Galois, furthermore, if either $K$ is the composite of two linearly disjoint fields or the Galois group $\text{Gal}(K/k)$ is abelian or dihedral, we compute the unramified Brauer group.

Terminology. Let $k$ be a field of characteristic zero and $\Gamma_k$ its absolute Galois group. Throughout this text, intersections of fields and composites of fields are taken inside the given algebraic closure $\bar{k}$. Let $Z$ be a variety over $k$. The Brauer group $\text{Br}(Z) = H^2_{et}(Z, \mathbb{G}_m)$ contains the algebraic Brauer group $\text{Br}_1(Z) = \text{Ker} (\text{Br}(Z) \to \text{Br}(Z \otimes_k \bar{k}))$, and the subgroup $\text{Br}_0(Z) = \text{Im}(\text{Br}(k) \to \text{Br}(Z))$ of constant classes. When $Z$ is smooth, we denote $\text{Br}_0(Z) \subset \text{Br}(Z)$ to be the unramified Brauer group, which coincides with the Brauer group of any smooth and proper model of $Z$.

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2. The construction of partial compactification

Let $k$ be a field of characteristic zero. Let $U_1$ be the affine variety in $\mathbb{A}^{n+1}_k$ defined by (1.1), where $P(t) \in k[t]$ is a polynomial of degree $m$, $K$ is an étale $k$-algebra of degree $n$ and $N_{K/k}$ is a norm form for the extension $K/k$.

We may assume $P(0) \neq 0$, otherwise we may replace $t$ by $t+a$, here $a \in k$ such that $P(a) \neq 0$. Let $0 \leq s < n, s \equiv -m \text{ mod } n$. Let $U_2$ be the affine variety defined by the equation

$$t^s \bar{P}(t') = N_{K/k}(z'),$$

(2.1)

where $\bar{P}(t') = t^{m}P(1/t')$. There is a birational map $U_2 \dashrightarrow U_1, (t', z) \mapsto \left(\frac{1}{t'}, \frac{z'}{t'}\right)$ with $j = (m + s)/n$. Let $U_0$ be the open subvariety of $U_2$ defined by $t' \neq 0$. This birational map yields
an open immersion $U_0 \rightarrow U_1$. We glue $U_1$ and $U_2$ along the open subset $U_0$, then we obtain the variety $U$ with a surjective projection $\pi : U \rightarrow \mathbb{P}^1$ by $(t, z) \mapsto t$. Let $\infty$ be the projection of the point $(0, z) \in U_1$, the fiber $\pi^{-1}(\infty)$ is the smooth variety defined by $0 \neq P(0) = N_{K/k}(z)$.

In this whole section, we will always assume $n \mid \text{deg}(P(t))$, otherwise we may replace $P(t)$ by $t^sP(t)$, $s \equiv -m \text{ mod } n$, $0 \leq s < n$, i.e., we replace (1.1) by (2.1). The variety $U$ has a surjective projection $\pi : U \rightarrow \mathbb{P}^1$ which is smooth at the point $t = \infty$.

We write $P(t) = cp_1(t)^{e_1} \cdots p_n(t)^{e_n}$, where $c \neq 0$ and $p_1(t), \ldots, p_n(t)$ are distinct irreducible polynomials. We may assume each $e_i < n$, otherwise we may replace $e_i$ by a positive integer $a$ with $a \equiv e_i \text{ mod } n$. Over $\bar{k}$, the affine variety $\bar{U}_1$ can be described as an equation of the form

$$P(t) = z_1 \cdots z_n.$$  

(2.2)

The singular locus of $\bar{U}_1$ is the closed subset

$$\bigcup_{e_i > 1 \mid j < \mu} \{(t, z_1, \ldots, z_n) : p_i(t) = z_j = z_\mu = 0\}.$$

For every exponent $e_i$ in $P(t)$, define $e'_i = \gcd(e_i, n)$. For each $e'_i > 1$, we define the set of integers

$$\{d \in \mathbb{Z} : d > 1 \text{ and } d \mid e'_i\}.$$

We list these integers in this set by $e'_i = d_{i,1} > d_{i,2} > \cdots > d_{i,t_i} > 1$, where $t_i$ is the numbers of factors of $e_i'$ which are larger than 1 and we denote $d_{i,t_i+1} = 1$.

We will construct the new variety by blowing-up along some ideal sheaves. By reordering the factors $p_i(t)^{e_i}$ of $P(t)$, we may assume

$$e'_i > 1 \text{ if } 1 \leq i \leq r; \ e'_i = 1 \text{ if } r < i \leq s.$$

For $1 \leq i \leq r$, we denote the closed subset of $X$ (defined over $k$) by

$$W_{i,j} = \bigcup_{1 \leq i_1 < i_2 < \cdots < i_j \leq n} \{p_i(t) = z_{i_1} = z_{i_2} = \cdots = z_{i_j} = 0\}.$$

For $i = 1$ and $d_{1,1} = e'_1$, we blow $V^{(0)} := U \setminus W_{1,e'_1+1}$ along the ideal sheaf $I_1$ defined by the ideal

$$\bigcap_{1 \leq i_1 < i_2 < \cdots < i_{e'_1} \leq n} (p_1(t)^{e'_1}, z_{i_1}, z_{i_2}, \ldots, z_{i_{e'_1}}),$$

then we get a variety $\tilde{V}^{(0)}$.

Over $\bar{k}$, the ideal sheaf $I_1$ on $V^{(0)}$ is the sum of the ideal sheaves of

$$(p_1(t)^{e'_1}, z_{i_1}, z_{i_2}, \ldots, z_{i_{e'_1}}),$$

whose support is the disjoint union of $\{t = \eta_j, z_{i_1} = z_{i_2} = \cdots = z_{i_{e'_1}} = 0\}$ with $p_1(\eta_j) = 0$. Since $V^{(0)}$ is geometrically integral, the variety $\tilde{V}^{(0)}$ is also geometrically integral by [17, Chapter 2, Proposition 7.16]. The blowing-up $\tilde{V}^{(0)}$ of $V^{(0)}$ along $(p_1(t)^{e'_1}, z_{i_1}, z_{i_2}, \ldots, z_{i_{e'_1}})$ is
the subvariety of $\mathbb{A}^{n+1} \times \mathbb{P}^{e_1}$ defined by equations (2.2) and

$$\frac{p_1(t)^{e_1/e_1'}}{t_0} = \frac{z_{i_1}}{t_1} = \cdots = \frac{z_{i_{e_1'}}}{t_{e_1'}};$$

where $(t, z_{i_1}, \ldots, z_{i_{e_1'}}) \in \mathbb{A}^{n+1}$ and $(t_0 : t_1 : \cdots : t_{e_1'}) \in \mathbb{P}^{e_1}$. Locally, we may set $t_0 = 1$, then $z_{ij} = p_1(t)^{e_1/e_1'}t_j$ for $1 \leq j \leq e_1'r$. Then the open subvariety $\{t_0 \neq 0\}$ of $\tilde{V}(0)$ is defined by

$$cp_2(t)^{e_2} \cdots p_s(t)^{e_s} = t_1 \cdots t_{e_1'} \prod_{j \notin \{i_1, i_2, \ldots, i_{e_1'}\}} z_j. \quad (2.3)$$

Let $t = \eta$ be a root of $p_1(t) = 0$. Set $t = \eta$ in (2.3) then we get an exceptional divisor of open subvariety $\{t_0 \neq 0\}$ of $\tilde{V}(0)$. It is clear that the intersection of the exceptional divisors in $\tilde{V}(0)$ with $\{t_0 = 0\}$ has dimension $n - 2$ (codimension 2 in $\tilde{V}(0)$), hence exceptional divisors in $\tilde{V}(0)$ are isomorphic to

$$\mathbb{Z}[L_1/k] \otimes \{M : M \subset Z_K, |M| = e_1'\}$$

as $\Gamma_k$-modules, here we denote $L_i = k[t]/(p_i(t))$, $\mathbb{Z}[L_i/k] = \mathbb{Z}[\Gamma_k/G_{L_i}]$ for any $i$ and $Z_K$ is the disjoint union of $\Gamma_k/G_{K_i}$ if we write $K = K_1 \times K_2 \times \cdots \times K_u$ with each $K_i$ is a field.

Let $V^{(1)}$ be the subvariety of $\tilde{V}(0)$ obtained by removing the intersection of the singular locus of $\tilde{V}(0)$ with its exceptional divisors. Obviously, $U \setminus W_{1,e_1'}$ can be viewed as an open subvariety of $V^{(1)}$. The intersection of the singular locus of $V^{(1)}$ with $\{p_1(t) = 0\}$ is just $W_{1,2} \setminus W_{1,e_1'}$. Therefore $W_{1,2} \setminus W_{1,e_1'}$ is a closed subset of $V^{(1)}$. Then $W_{1,d_1+1} \setminus W_{1,e_1'}$ is also a closed subset of $V^{(1)}$. Removing $W_{1,d_1+1} \setminus W_{1,e_1'}$ in $V^{(1)}$, we get an open subvariety of $V^{(1)}$, which is denoted by $V_{1,1}$.

Let $I_2$ be the ideal sheaf of $V_{1,1}$ which is uniquely determined by the $\Gamma_k$-invariant ideal

$$\bigcap_{i_1 < i_2 < \cdots < i_{d_1+1}} (p_1(t)^{e_1/d_1+1}, z_{i_1}, z_{i_2}, \ldots, z_{i_{d_1+1}}),$$

whose support is $W_{1,d_1+1} \setminus W_{1,d_1+1}$. We blow $V_{1,1}$ along the ideal sheaf $I_2$ and obtain a variety. Similarly, removing the intersection of the singular locus of this variety with exceptional divisors, then we get the variety denoting by $V^{(2)}$.

Over $\bar{k}$, the ideal sheaf $I_2$ on $V^{(1)}$ is also the sum of the ideal sheaves of

$$((t - \eta)^{e_1/d_1+1}, z_{i_1}, z_{i_2}, \ldots, z_{i_{d_1+1}}),$$

whose support is a disjoint union. By a similar argument as above, the divisor groups generated by exceptional divisors are isomorphic to $\mathbb{Z}[L_1/k] \otimes \{M : M \subset Z_K, |M| = d_1\}$ as $\Gamma_k$-modules.

Similarly, removing $W_{1,d_1+1} \setminus W_{1,d_1+1}$ in $V^{(2)}$, we get an open subvariety of $V^{(2)}$ denoted by $V_{1,2}$. For $d_1, d_2, \ldots, d_{d_1+1}$, we continue the process as above. Finally, we get a variety $V_{1,t_1}$.

Note that $U \setminus \{p_1(t) = 0\}$ can be viewed as an open subvariety of $V_{1,t_1}$, therefore, for $i = 2, \ldots, r$, i.e., $p_2(t)^{e_2}, \ldots, p_r(t)^{e_r}$, we continue the above process of blow-ups similar as $p_1(t)^{e_1}$, finally we get a variety $V_{r,t_r}$. Note that $V_{r,t_r}$ is a smooth variety over $k$ since we remove $W_{i,t_i+1} = W_{i,2}$ for each $i$. 
Denote $U' = U \setminus \{P(t) = 0\}$, any nonempty fiber of $\pi : U' \to \mathbb{P}^1$ is a principal homogeneous space of the torus $T$, here $T$ is defined by the equation $N_{K/k}(z) = 1$. Let $T^c$ be a fixed $T$-invariant smooth compactification of $T$ (see [7]). Let $U' \times^T T^c$ be the contracted product and it has an open subvariety $U' \times^T T \cong U'$. By our construction, $V_{r,t,c}$ is a system of blow-ups of $U$ and $U'$ can be viewed as an open subvariety of $V_{r,t,c}$. We glue $V_{r,t,c}$ and $U' \times^T T^c$ along the isomorphism $U' \times^T T \cong U'$, we obtain a new smooth scheme over $k$ which is separated over $k$.

In the sequel,

we shall denote $X'$ the variety thus constructed. \hfill (2.4)

We can also construct a simpler variety than $X'$. In the above blow-up for each $p_i(t)^{e_i}, 1 \leq i \leq r$, we only do the first step (and removing its singular locus), i.e., we do not blow-up for the smaller divisors of $e_i'$ than itself. Then we glue the variety similar as $V_{r,t,c}$ and $U' \times^T T^c$ along the isomorphism $U' \times^T T \cong U'$, we obtain a new smooth scheme over $k$. In the sequel,

we shall denote $X$ the variety thus constructed. \hfill (2.5)

It is clear that $X$ is an open subset of $X'$. Such $X$ will be useful when $K/k$ is a Galois field extension.

Recall that $U_1 \subset U$ is the affine variety defined by (1.1). Let $V$ be the maximal smooth open subset of $U_1$. We can compare $X'$ (or $X$) with the partial compactification $V^\text{CTHS}$ in [6]. Let $U'_1 = U_1 \setminus \{P(t) = 0\}$. Colliot-Thélène, Harari and Skorobogatov [6] constructed the partial compactification of (1.1) by gluing $U'_1 \times^T T^c$ and $V$ along $U'_1 \times^T T \cong U'_1$, where $T^c$ is the fixed $T$-invariant smooth compactification of $T$. They proved that $\bar{k}[V^\text{CTHS}]^\times = \bar{k}^\times$, Pic($V^\text{CTHS}$) is torsion free and Br($V^\text{CTHS}$) = 0, see [6, Proposition 2.3]. Obviously $V^\text{CTHS}$ is a Zariski open subset of $X$ (or $X'$), hence we have the following lemma.

**Lemma 2.1.** The groups Pic($\overline{X}$) and Pic($\overline{X}'$) are torsion free, $\bar{k}[X]^\times = \bar{k}[X']^\times = \bar{k}^\times$ and Br($\overline{X}$) = Br($\overline{X}'$) = 0.

We write $K = K_1 \times \cdots \times K_u$ where each $K_i$ is a field, recall that $Z_K$ is the disjoint union of $\Gamma_K \setminus \Gamma_{K_i}$. Now we consider the divisors in $\overline{X}/ \overline{U'}_1$ and $\overline{X}' \setminus \overline{U}'_1$:

i) horizontal divisors which are divisors of $\text{Div}(U' \times^T T')$ with support out of $\overline{T'}$, we denote it by $\text{Div}_h$. By [6, Lemma 2.1], $\text{Div}_h$ is isomorphic to $\text{Div}_{\overline{T}' \setminus \overline{T}}$ as $\Gamma_K$-modules.

ii) vertical divisors in $\overline{U} \setminus \overline{U}'_1$, which is isomorphic to $Z[K/k] \otimes Z_P \oplus Z$, where $Z$ comes from the divisor at $\infty$, $Z[K/k] := Z[Z_K]$ and $Z_P = \bigoplus_i Z[L_i/k]$ where $L_i = k[t]/(p_i(t))$.

iii) exceptional divisors, which is isomorphic to

$$S := \bigoplus_{i=1}^r Z[L_i/k] \otimes Z[\Omega_i],$$ \hfill (2.6)

where

$$\Omega_i = \bigcup_{d>1, d|e'_i} \{M : M \subset Z_K, \#M = d\}$$ \hfill (2.7)

for $\overline{X}'$, and

$$\Omega_i = \{M : M \subset Z_K, \#M = e'_i\}$$ \hfill (2.8)
Denote
\[ D = \mathbb{Z}_P \otimes \mathbb{Z}[K/k] \oplus \mathbb{Z} \oplus S, \]
(2.9)
\[ \text{Div}_{\overline{X/U_1}(X')} = \text{Div}_h \oplus \mathbb{D}. \]

Let \( N_K = \sum_{\sigma \in \Gamma_K} \sigma \) and \( \Upsilon = \sum_{i=1}^r \sum_{M \in \Omega_i} M \in S. \) We define a morphism \( f : \mathbb{Z}_P \to \mathbb{D} \) by sending any \( \tau \in \Gamma_{L_i}/\Gamma_{L_i} \) to \( (\tau \otimes N_K, -1, \tau \otimes \Upsilon). \) Let \( \widehat{T'} \) be the quotient of \( f \) which is torsion-free. Note that \( \text{Pic}(U'_1) = 0 \), we have the following commutative diagram with exact sequences (a similar commutative diagram has already appeared in [6, Proposition 2.2])

\[
\begin{array}{ccccccc}
0 & \to & \mathbb{Z}_P & \to & \mathbb{D} & \to & \widehat{T'} & \to & 0 \\
\downarrow & & \downarrow f & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & \mathbb{D} & \to & \text{Div}_{\overline{X/U_1}(X')} & \to & \text{Pic}(\overline{X'}) & \to & 0 \\
\downarrow & & \downarrow \text{Div}_h & & \downarrow & & \downarrow \text{Pic}(\overline{U'_1}) & & \downarrow & \\
0 & \to & \widehat{T} & \to & \text{Pic}(\overline{X'}) & \to & 0 & & \\
\end{array}
\]
(2.10)

In the diagram (2.10), we may replace \( X' \) by \( X \).

By the diagram (2.10), we may construct a morphism \( \widehat{T} \to \widehat{T'} \), such that the following diagram of exact sequences is commutative

\[
\begin{array}{cccccc}
0 & \to & \widehat{T} & \to & \text{Div}_h & \to & \text{Pic}(\overline{U'_1}) & \to & 0 \\
\downarrow j & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & \widehat{T'} & \to & \text{Pic}(\overline{X'}) & \to & 0 \\
\end{array}
\]
(2.11)

where \( j : \widehat{T} \to \widehat{T'} \) is induced by the map

\[ \mathbb{Z}[K/k] \to \mathbb{D}, \sigma \mapsto (-\sum_{i=1}^s e_i N_i \otimes \sigma, m/n, -S_\sigma) \]

where \( \sigma \in \Gamma_{L_i}/\Gamma_K, N_i = \sum_{\tau \in \Gamma_{L_i}} \tau \) and

\[ S_\sigma = \sum_{i=1}^r N_i \otimes \sum_{d>1,d|e'_i} \frac{e_i}{d} \sum_{\sigma \in M \subset Z_K} \sum_{\#M = d} M. \]
Proof. one obtains the exact sequence (2.2).

Lemma 2.2. We have exact sequences

\[ 0 \to H^1(k, \hat{T}) \to H^1(k, \text{Pic}(\overline{X})) \to \text{Ker}[\hat{\imath}^2_\omega(\hat{T}) \to \hat{\imath}^2_\omega(\hat{T}')] \to 0; \tag{2.12} \]

\[ 0 \to H^1(k, \hat{T}') \to H^1(k, \text{Pic}(\overline{X})) \to \text{Ker}[\hat{\imath}^2_\omega(\hat{T}) \to \hat{\imath}^2_\omega(\hat{T}')] \to 0. \tag{2.13} \]

Proof. By (2.11), we have the commutative diagram

\[
\begin{array}{cccc}
H^0(k, \text{Pic}(\overline{T})) & \longrightarrow & H^1(k, \hat{T}) & \longrightarrow & H^1(k, \text{Div}_h) & \longrightarrow & H^1(k, \text{Pic}(\overline{T})) & \longrightarrow & H^2(k, \hat{T}) \\
\downarrow & & \downarrow j^* & & \downarrow & & \downarrow & & \downarrow & \\
H^0(k, \text{Pic}(\overline{T})) & \longrightarrow & H^1(k, \hat{T}') & \longrightarrow & H^1(k, \text{Pic}(\overline{X})) & \longrightarrow & H^1(k, \text{Pic}(\overline{T})) & \longrightarrow & H^2(k, \hat{T}')
\end{array}
\]

Since \( \text{Div}_h \) is a permutation module, one has \( H^1(k, \text{Div}_h) = 0 \). Since \( H^1(k, \text{Pic}(\overline{T})) \cong \hat{\imath}^2_\omega(\hat{T}) \), one obtains the exact sequence (2.12) (similarly for (2.13)).

Remark 2.3. For any field \( k \subset F \subset \kbar \), we also have exact sequences

\[ 0 \to H^1(F, \hat{T}') \to H^1(F, \text{Pic}(\overline{X})) \to \text{Ker}[\hat{\imath}^2_\omega(\hat{T}_F) \to \hat{\imath}^2_\omega(\hat{T}'_F)] \to 0; \tag{2.14} \]

\[ 0 \to H^1(F, \hat{T}'_F) \to H^1(F, \text{Pic}(\overline{X})) \to \text{Ker}[\hat{\imath}^2_\omega(\hat{T}_F) \to \hat{\imath}^2_\omega(\hat{T}'_F)] \to 0. \tag{2.15} \]

Recall that \( U_1 \) is the affine variety defined by (1.1) and that \( U'_1 \subset U_1 \) is defined by \( P(t) \neq 0 \). We write \( P(t) = cp_1(t)^{e_1} \cdots p_m(t)^{e_m} \) is a product of irreducible polynomials over \( k \). Let \( L_i = k[t]/(P(t)), Z_P = \bigoplus_i Z[L_i/k] \) and \( \eta_i \) the class of \( t \) in \( L_i \). We define

\[ \alpha : Z_P \to \kbar[U'_1]^\times, \Gamma_{L_i} \in Z[L_i/k] \mapsto t - \eta_i. \tag{2.16} \]

Since \( H^2(k, Z_P) \cong \bigoplus_i H^1(L_i, \mathbb{Q}/\mathbb{Z}) \) and \( \text{Br}_1(U'_1) = H^2(k, \kbar[U'_1]^\times) \), \( \alpha \) induces a morphism

\[ \alpha' : \bigoplus_i H^1(L_i, \mathbb{Q}/\mathbb{Z}) \to \text{Br}_1(U'_1). \]

In fact, we do not need to assume that \( n \mid \text{deg}(P(t)) \) in the above argument and the following lemma.

Lemma 2.4. The map \( \alpha' \) sends \( (\chi_i)_{i=1}^r \in \bigoplus_i H^1(L_i, \mathbb{Q}/\mathbb{Z}) \) to

\[ \sum_i \text{Cor}_{L_i/k}(t - \eta_i, \chi_i) \in \text{Br}_1(U'_1). \]

Proof. We only need to show \( \alpha'(\chi_i) = \text{Cor}_{L_i/k}(t - \eta_i, \chi_i) \) for each \( i \). Let \( f : U'_1 \to \text{Spec}(k), f' : U'_1 \otimes_k L_i \to \text{Spec}(L_i) \) and \( \nu : U'_1 \otimes_k L_i \to U'_1 \) be natural morphisms. Let \( \gamma : Z \to \mathbb{G}_m \) be the morphism of étale sheaves on \( U'_1 \otimes_k L_i \) by sending \( 1 \) to \( t - \eta_i \in L_i[U'_1]^\times \). Applying the functors \( H^2(U_1, -) \) and \( H^2(k, f_* -) \) to the morphisms of étale sheaves on \( U'_1 \)

\[ \nu_* \mathbb{Z} \xrightarrow{\gamma_*} \nu_* \mathbb{G}_m \xrightarrow{N_{L/k}} \mathbb{G}_m, \]
we obtain a commutative diagram
\[
\begin{array}{ccc}
H^2(k, f_*\nu_*\mathbb{Z}) & \xrightarrow{\text{Res}(\alpha')} & H^2(k, f_*\mathbb{G}_m) \\
\downarrow & & \downarrow \\
H^1(L_i, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{f^*} & H^2(L_i, \mathbb{Z}) \\
\end{array}
\]

The lemma then follows from the commutativity of this diagram. \qed

Note that \(\text{Br}_1(X')/\text{Br}_0(X') \cong H^1(k, \text{Pic}(X'))\), hence (2.12) gives the morphism
\[
\beta : H^1(k, \hat{T}') \to \text{Br}_1(X')/\text{Br}_0(X').
\]

We denote
\[
\Psi_i := \text{Ker}[H^1(L_i, \mathbb{Q}/\mathbb{Z}) \to H^1(L_i \otimes K, \mathbb{Q}/\mathbb{Z}) \oplus H^1(F, \mathbb{Z}) \oplus H^2(F, S \otimes \mathbb{Q}/\mathbb{Z})].
\]

By the first row of the diagram (2.10), one obtains
\[
H^1(k, \hat{T}') \cong \text{Ker}[H^2(k, \mathbb{Z}_p) \to H^2(k, \mathbb{Z}_p \otimes \mathbb{Z}[K/k] \oplus \mathbb{Z} \oplus S)] \cong \bigoplus_{i=1}^s \Psi_i. \quad (2.17)
\]

Therefore, \(\beta\) induces the map \(\beta' : \bigoplus_{i=1}^s \Psi_i \to \text{Br}_1(X')/\text{Br}_0(X')\). In fact, the remark for [6, Proposition 2.5] gives the following explicit description of \(\beta'\) and here we provide a short proof.

**Lemma 2.5.** \(\beta'\) is defined by send any \((\chi_i)_i \in \bigoplus_{i=1}^s \Psi_i\) to
\[
\sum_i \text{Cor}_{L_i/k}(t - \eta_i, \chi_i) \in \text{Br}_1(X').
\]

**Proof.** For any field \(k\) with \(\text{char}(k) = 0\) and any smooth variety \(Z\), the classical exact sequence
\[
0 \to \bar{k}(Z)^*/\bar{k} \to \text{Div}(Z) \to \text{Pic}(Z) \to 0
\]
gives an exact sequence
\[
0 \to H^1(k, \text{Pic}(\overline{Z})) \to H^2(k, \bar{k}(Z)^*/\bar{k}) \to H^2(k, \text{Div}(\overline{Z})). \quad (2.18)
\]

Using Grothendieck’s purity theorem (cf. [13, Theorem 1.3.2]), (2.18) induces the natural map \(H^1(k, \text{Pic}(\overline{Z})) \to \text{Br}_1(Z)/\text{Br}_0(Z)\) which can be viewed as the inverse map in Hochschild-Serre’s spectral sequence by the observation 3 of [20, Section 2, p. 122].

The top two rows of the diagram (2.10) give the commutative diagram
\[
\begin{array}{ccc}
H^1(k, \hat{T}') & \xrightarrow{\alpha} & H^2(k, \mathbb{Z}_p) \\
\downarrow & & \downarrow \\
H^1(k, \text{Pic}(\overline{X}')) & \xrightarrow{\beta} & H^2(k, \bar{k}[U'_i]^*/\bar{k}),
\end{array}
\]

where the right-side vertical map is induced by \(\alpha\) in (2.16), the lemma then follows from the commutativity of the diagram (2.19) and Lemma 2.4. \qed
**Remark 2.6.** If we replace $X'$ by $X$, by the same arguments as above, Lemma 2.5 also holds for $X$.

Now we can prove the main theorem of this article.

**Theorem 2.7.** The Brauer group

$$Br_1(X') = Br(X') = Br_{un}(X').$$

If $K/k$ is a Galois field extension,

$$Br_1(X) = Br(X) = Br_{un}(X).$$

**Proof.** Since $Br(X) = Br(X') = 0$ by Lemma 2.1, we have $Br_1(X) = Br(X)$ and $Br_1(X') = Br(X')$.

Let $A$ be a discrete valuation ring containing $k$, and with the fraction field $k(X')$ and the residue field $\kappa_A$. Let $\partial_A : Br(k(X')) \to H^1(\kappa_A, \mathbb{Q}/\mathbb{Z})$ be the residue map. By Grothendieck’s purity theorem (cf. [13, Theorem 1.3.2]), we have

$$Br_{un}(X') = \bigcap_A \ker(\partial_A) \subset Br(k(X')),$$

where $A$ runs through all discrete valuation rings as above. Therefore, for any $B \in Br_1(X')$, we want to show $B \in Br_{un}(X')$, it suffices to prove the triviality of $\partial_A(B)$ for any such discrete valuation ring $A$, i.e., $\partial_A(B)(g) = 0$ for any $g \in \text{Gal}(\bar{\kappa}_A/\kappa_A)$.

We extend the embedding $k \subset \kappa_A$ to an embedding $\bar{k} \subset \bar{\kappa}_A$, then $g$ acts on $\bar{k}$. Let $F = \bar{k}^g$, then $\text{Gal}(\bar{k}/F)$ is pro-cyclic.

Since $X'$ is geometrically integral and $\text{char}(k) = 0$, the completion of $k(X')$ for the given valuation is isomorphic to $\kappa_A((\pi))$, where $\pi$ is a uniformizer. Considering the valuation given by $\pi$ on $(F, \kappa_A)((\pi))$ and using $F(X') = k(X) \otimes_k F$, this defines a discrete valuation on $F(X')$ with valuation ring $A_F$ whose residue field $\kappa_{A_F} = \kappa_A.F$, hence $g \in \text{Gal}(\bar{\kappa}_A/\kappa_{A_F})$, in order to show $\partial_A(B)(g) = 0$, we only need to show $\partial_{A_F}(B)(g) = 0$.

Over $F$, we have $\text{III}^2_\omega(\widehat{T}_F) = 0$ since $\bar{k}/F$ is pro-cyclic by the definition of $F$. By (2.14), we have

$$Br_1(X'_F)/Br_0(X'_F) \cong H^1(F, \text{Pic}(\overline{\mathbb{X}}')) \cong H^1(F, \widehat{T}_{\bar{\mathbb{X}}'})/j^*(H^1(F, \widehat{T}))/\text{Tor}.$$

Let $P(t) = cq_1(t)f_1 \cdots q_m(t)f_m$ be a product of irreducible polynomials over $F$ and let $F_i = F[t]/(q_i(t))$, then

$$H^1(F, \widehat{T}_{\bar{\mathbb{X}}'}) \cong \ker[H^2(F, \mathbb{Z}_P) \to H^2(F, \mathbb{Z}_P \otimes \mathbb{Z}[K/k] \oplus \mathbb{Z} \oplus S)]$$

$$\cong \ker[\sum_{i=1}^{s'} H^1(F_i, \mathbb{Q}/\mathbb{Z}) \to H^1(F \otimes K, \mathbb{Q}/\mathbb{Z}) \oplus H^1(F, \mathbb{Q}/\mathbb{Z}) \oplus H^1(F, S \otimes \mathbb{Q}/\mathbb{Z})].$$

Let $(\chi_i) \in \bigoplus_1 H^1(F_i, \mathbb{Q}/\mathbb{Z})$ be contained in the kernel (2.21), then we have $\sum_i \text{Cor}_{F_i/F}(\chi_i) = 0$. By Lemma 2.5 (replacing $k$ by $F$), such $(\chi_i)$, gives an element $B = \sum_i \text{Cor}_{F_i/F}(t - \eta_i, \chi_i) \in Br(X')$, we only need to show the triviality of the residue of $B$ by (2.20).

Let $A'$ be a discrete valuation of $F(X')$. Assume the valuation $\varphi_{A'}(t) = l < 0$, then $\varphi_{A'}(q_i(t)) = l \deg(q_i(t))$. Therefore, the residue

$$\partial_{A'}(B) = \sum_i \partial_{A'}(\text{Cor}_{F_i/F}(\frac{t - \eta_i}{t}, \chi_i)) + \sum_i \partial_{A'}(\text{Cor}_{F_i/F}(t, \chi_i)) = 0 + l \sum_i \text{Cor}_{F_i/F}(\chi_i) = 0.$$
Consider the case to show that the order of $\chi_{i_0}$ divides $v_i(t)$ since $v_{A'}(N_{K'/F_{i_0}}(z_1))$ is divided by $n'_i$, where $n'_i = [K'_i : F_{i_0}]$. In the following, we only need to show that the order of $\chi_{i_0}$ divides $\frac{\gcd(n'_1,\ldots,n'_m)}{\gcd(e_i,n_1,\ldots,n_m)}$, which implies $\partial_A(\mathcal{B}) = 0$. If we replace $X'$ by $X$, the above argument also holds.

(1) The case $X'$:
Let $d = (e_i,n_1',\ldots,n_m')$ be a factor of $e'_i = (e_i,n)$. For any $1 \leq j \leq m'$, let $H_j$ be the subgroup of order $d$ of the cyclic group $\text{Gal}(K'_j/F_{i_0})$. Then $H_j$ gives an element $\gamma_j$ of $\Omega_i$ (see (2.7) for definition), the $\gamma_j$ generates a permutation $\text{Gal}(K'_j/F_{i_0})$-module which is isomorphic to $\mathbb{Z}[G_j/H_j]$ and is a direct summand of $\mathbb{Z}[\Omega_i]$, where $G_j = \text{Gal}(K'_j/F_{i_0})$. By (2.21), the order of $\chi_{i_0}$ is a factor of $n'_j/d$. When $j$ runs through $1,2,\ldots,m'$, the order of $\chi_{i_0}$ is a factor of $\gcd(n'_1,\ldots,n'_m)/d$. Then we complete the proof for $X'$.

(2) The case $X$ (here $K/k$ is a Galois field extension):
Since $K/k$ is Galois, all $K'_i$ in (2.22) are isomorphic, hence $n' := n'_1 = \cdots = n'_m$. Let $d := \gcd(e_i,n_1',\ldots,n_m') = \gcd(e_i,n') = \gcd(e'_i,n')$. Let $H_i$ be the unique subgroup of order $d$ in $\text{Gal}(K'_i/F_{i_0})$ for $1 \leq i \leq r$. Let $r'_i := e'_i/d = e'_i/\gcd(e'_i,n')$, $r'_i$ is a factor of $n'/n'$, hence $r'_i \leq m'$. Then the disjoint union $\bigcup_{j=1}^{r'_i}H_i \subset \Gamma_k/\Gamma_K$ gives an element $\gamma$ of $\Omega_i$ (see (2.8)), the $\gamma$ generates a permutation $G'$-module which is isomorphic to $\mathbb{Z}[G'/H']$ and it is a direct summand of $\mathbb{Z}[\Omega_i]$, where $G' \cong \text{Gal}(K'_i/F_{i_0}) \cong \cdots \cong \text{Gal}(K'_{m'/F_{i_0}})$ and $H' \cong H_1 \cong \cdots \cong H_m$. By the definition of (2.21), the order of $\chi_{i_0}$ is a factor of $n'/d$. Then we complete the proof for $X$. \hfill \square

Remark 2.8. In fact, for any field $L$ over $k$, by similar arguments as above we can show
$$\text{Br}_1(X'_L) = \text{Br}(X'_L) = \text{Br}_{un}(X'_L);$$
if $K/k$ is a Galois field extension,
$$\text{Br}_1(X_L) = \text{Br}(X_L) = \text{Br}_{un}(X_L).$$
Combining Lemma 2.2 with Theorem 2.7, we have the following corollary.
Corollary 2.9. We have the exact sequence
\[ 0 \to H^1(k, \hat{T}')/j^*(H^1(k, \hat{T})) \to \text{Br}_{\text{un}}(X')/\text{Br}_{0}(X') \to \text{Ker}[\text{III}_2^2(\hat{T}) \to \text{III}_2^2(\hat{T}')] \to 0. \] (2.23)

If \( K/k \) is a Galois field extension, we have the exact sequence
\[ 0 \to H^1(k, \hat{T}')/j^*(H^1(k, \hat{T})) \to \text{Br}_{\text{un}}(X)/\text{Br}_{0}(X) \to \text{Ker}[\text{III}_2^2(\hat{T}) \to \text{III}_2^2(\hat{T}')] \to 0. \] (2.24)

Example 2.10. Let \( X \) be the variety (2.5) defined by the equation \( P_1(t)P_2(t) = N_{K/k}(z) \), where \( P_1(t) \) and \( P_2(t) \) are irreducible and of degree 2. Let \( L_i = k[t]/(P_i(t)) \). Suppose that \( L_1 \neq L_2 \) and \( K = L_1L_2 \). Then
\[ \text{Br}_{\text{un}}(X) = \text{Br}_{0}(X), \text{ but } \text{Br}(V^{\text{CHS}})/\text{Br}_{0}(V^{\text{CHS}}) = \mathbb{Z}/2. \]

Proof. It is clear that \( \text{Br}(V^{\text{CHS}})/\text{Br}_{0}(V^{\text{CHS}}) = \mathbb{Z}/2 \) by [6, Proposition 2.5].

Since \( P_1(t)P_2(t) \) is separable and of degree \( [K : k] = 4 \), we have \( S = 0 \) in (2.6). Let \( H_i = \text{Gal}(L_i/k), i = 1, 2 \) and \( G = \text{Gal}(K/k) = H_1 \times H_2 \). By Corollary 2.9, we only need to show that this map \( \text{III}_2^2(\hat{T}) \to \text{III}_2^2(\hat{T}') \) is injective. Since \( \hat{T} \) and \( \hat{T}' \) are split by \( K \), it suffices to show that \( H^2(G, \hat{T}) \to H^2(G, \hat{T}') \) is injective. By the Hochschild-Serre spectral sequence \( H^i(H_1, H^j(H_2, -)) \Rightarrow H^{i+j}(G, -) \), since \( E_2^{2,0}(\hat{T}) = E_2^{0,2}(\hat{T}) = 0 \), we have the commutative diagram
\[
\begin{array}{ccc}
H^2(G, \hat{T}) & \xrightarrow{g} & H^1(H_1, H^1(H_2, \hat{T})) \\
\downarrow & & \downarrow \\
H^2(G, \hat{T}') & \xrightarrow{g} & H^1(H_1, H^1(H_2, \hat{T}'))
\end{array}
\]
and the first horizontal map is injective. It is easy to show that \( H^1(H_2, \hat{T}) \cong H^1(H_2, \hat{T}') \cong \mathbb{Z}/2 \), which implies that \( g \) is injective. Therefore \( H^2(G, \hat{T}) \to H^2(G, \hat{T}') \) is injective. \( \square \)

If \( k = \mathbb{Q} \), \( K/\mathbb{Q} \) is an arbitrary extension of number fields and \( P(t) \) is split over \( \mathbb{Q} \), then the Brauer-Manin obstruction controls the rational points of (1.1) entirely (see [2, 3, 16]). The following result gives a simple description of the Brauer group in this direction.

Example 2.11. Let \( X \) be the variety (2.5) defined by the equation (1.1), where \( K/k \) is a finite Galois field extension of degree \( n \) and \( P(t) = c(t - a_1)^{e_1} \cdots (t - a_s)^{e_s} \) with all \( e_i \in k \) are distinct. Let \( e_i' = \text{gcd}(e_i, n) \). Suppose \( n | \deg(P(t)) \) and \( \text{gcd}(e_1, \ldots, e_s, n) = 1 \), then \( \text{Br}_{\text{un}}(X) \) is generated by
\[ \rho + \sum_{i=1}^{s} (t - a_i, \chi_i), \]
where \( \rho \in \text{Br}_{0}(X), \chi_i \in H^1(K/k, \mathbb{Q}/\mathbb{Z}) \) satisfies \( \sum_{i=1}^{s} \chi_i = 0 \) and \( \chi_i(g) = 0 \) for any \( g \in \text{Gal}(K/k) \) with \( g^{e_i'} = 1_G \).

Proof. The quotient of \( \hat{T}' \) by \( \mathbb{Z} \otimes S \) (in (2.9)) coincides with \( \hat{T} \otimes \mathbb{Z}_p \cong \hat{T}' \), then the map \( f : H^2(K/k, \hat{T}) \to H^2(K/k, \hat{T}') \) induces the map \( f' : H^2(K/k, \hat{T}) \to H^2(K/k, \hat{T})^s \) which sends \( \alpha \in H^2(K/k, \hat{T}) \) to \((-e_1\alpha, \cdots, -e_s\alpha)\). Since \( \text{gcd}(e_1, \ldots, e_s, n) = 1 \), one obtains \( f' \) is
injective, hence $f$ is also injective. Since $\hat{T}$ and $\hat{T}'$ is split by $K$, the injectivity of $f$ implies $\text{Ker}[\mathbb{I}^2(\hat{T}) \rightarrow \mathbb{I}^2(\hat{T}')] = 0$. The proof then follows from Corollary 2.9 and Lemma 2.5.

3. The case $P(t)$ irreducible and $K/k$ Galois

In the section, we always assume that $K/k$ is a Galois field extension of degree $n$ and that $P(t)$ is a separable polynomial of degree $m$. Let $X$ be the variety (2.5) defined by the equation (1.1). We write $P(t) = p_1(t) \cdots p_s(t)$ with all $p_i(t)$ are irreducible. Let $L_i = k[t]/(p_i(t))$ and $e' = \gcd(n, m)$. We do not assume $n \mid m$ in this section.

Suppose $n \nmid m$, let $U'_1$ be the open affine subvariety of $X$ defined by (1.1) and $P(t) \neq 0$. We may assume $P(0) \neq 0$. Let $0 \leq \delta < n, \delta \equiv -m \mod n$ and $\delta' = (m + \delta)/n$. Then $X$ contains an affine open subvariety $U_2$ defined by the equation

$$t^\delta \bar{P}(t') = N_{K/k}(z'),$$

where $\bar{P}(t') = t^{nm}P(1/t')$. Let $U'_2 \subset U_2$ be defined by $\bar{P}(t') \neq 0$.

In §2, in fact we obtain the diagram (2.10) by using the inclusion $U'_2 \subset X$ when $n \nmid m$; hence $\mathbb{D}$ in (2.9) contains the infinite divisor which is not very convenient when we compute cohomology groups of $\hat{T}'$. So, to obtain a new "simple" diagram similar with (2.10) we will use the inclusion $U'_1 \subset X$ (no matter whether $n \mid m$ or not) and redefine

$$\mathbb{D} = \mathbb{Z}_P \otimes \mathbb{Z}[K/k] \oplus S',$$ \hspace{1cm} (3.1)

where $S' = \mathbb{Z}[K/k] \oplus \mathbb{Z}[\Omega]$ with $\Omega = \{M : M \subset \Gamma_k/\Gamma_K, |M| = e'\}$ if $n \nmid m$; $S' = \mathbb{Z}$ if $n \mid m$. If $n \nmid m$, such $\mathbb{D}$ is simpler than (2.9). Similarly, the new diagram similar with (2.10) is also commutative, its rows and columns are exact. The $\Gamma_k$-module $\hat{T}'$ is the Galois lattice defined by the following exact sequence

$$0 \rightarrow \mathbb{Z}_P \xrightarrow{f} \mathbb{D} \rightarrow \hat{T}' \rightarrow 0$$ \hspace{1cm} (3.2)

where $f$ sends any $\tau_i \in \Gamma_k/\Gamma_{L_i}$ to

$$f(\tau_i) = \begin{cases} (\tau_i \otimes N_K, -N_K, -S_K) & \text{if } n \nmid m, \\ (\tau_i \otimes N_K, -1) & \text{if } n \mid m, \end{cases}$$

with

$$S_K = \sum_{M \subset \Gamma_k/\Gamma_K, \#M = e'} M \text{ and } N_K = \sum_{\sigma \in \Gamma_k/\Gamma_K} \sigma.$$
In the diagram (2.11), we redefine the morphism \( j : \hat{T} \to \hat{T}' \) induced by the morphism \( \mathbb{Z}[K/k] \to \mathbb{D} \) by sending \( \sigma \in \Gamma_k/\Gamma_K \) to \((-\sum_{i=1}^s N_i \otimes \sigma, j_\sigma)\), where

\[
j_\sigma = \begin{cases} 
(\delta^i N_K - \delta \sigma, \delta S_K - \delta e i S_\sigma) & \text{if } n \nmid m, \\
1 & \text{if } n \mid m;
\end{cases}
\]

(3.3)

\[
N_i = \sum_{\tau \in \Gamma_k/\Gamma_{L_i}} \tau \text{ and } S_\sigma = \sum_{\sigma \in M \subset \Gamma_k/\Gamma_K \# M = e'} M.
\]

Then we obtain a commutative diagram which is analogous to (2.11).

Combining with Theorem 2.7, by similar arguments as for (2.13) in the proof of Lemma (2.2), we obtain the exact sequence

\[
0 \to H^1(k, \hat{T}')/j^*(H^1(k, \hat{T})) \to \text{Br}_{un}(X)/\text{Br}_0(X) \to \text{Ker}[\Pi_2^\omega(\hat{T}) \to \Pi_2^\omega(\hat{T}')] \to 0.
\]

(3.4)

Let \( L'_i = L_i \cap K \), \( \mathbb{Z}_{P'} = \oplus_i \mathbb{Z}[L'_i/k] \) and

\[
\mathbb{D}' = \mathbb{Z}_{P'} \otimes \mathbb{Z}[K/k] \oplus S'.
\]

Let \( \hat{T}' \) be the lattice defined by the exact sequence

\[
0 \to \mathbb{Z}_{P'} \xrightarrow{f'} \mathbb{D}' \to \hat{T}' \to 0,
\]

(3.5)

where \( f' \) sends any \( \tau_i \in \Gamma_k/\Gamma_{L'_i} \) to \((\tau \otimes N_K, f'_{\tau_i})\), where

\[
f'_{\tau_i} = \begin{cases} 
-[L_i : L'_i], & \text{if } n \mid m, \\
(-[L_i : L'_i]N_K, -[L_i : L'_i]S_K), & \text{if } n \nmid m.
\end{cases}
\]

We can define \( \hat{T} \to \hat{T}' \) induced by the morphism

\[
\mathbb{Z}[K/k] \to \mathbb{D}', \sigma \longmapsto (-\sum_{i=1}^s N'_i \otimes \sigma, j_\sigma),
\]

where \( j_\sigma \) is defined in (3.3) and \( N'_i = \sum_{\tau' \in \Gamma_k/\Gamma_{L'_i}} \tau' \). We define \( \hat{T}' \to \hat{T}' \) induced by the morphism

\[
\mathbb{D}' \to \mathbb{D}, (\tau_i \otimes \sigma, a) \longmapsto \left( \sum_{\tau'_i \in \Gamma_{L'_i}/\Gamma_{L_i}} \tau_i \tau'_i \otimes \sigma, a \right) \text{ for any } \tau_i \in \Gamma_k/\Gamma_{L'_i} \text{ and } a \in S'.
\]

Therefore, one obtains the following commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathbb{Z} & \rightarrow & \mathbb{Z}[K/k] & \rightarrow & \hat{T} & \rightarrow & 0 \\
0 & \rightarrow & \mathbb{Z}_{P'} & \rightarrow & \mathbb{D}' & \rightarrow & \hat{T}' & \rightarrow & 0 \\
0 & \rightarrow & \mathbb{Z}_{P'} & \rightarrow & \mathbb{D} & \rightarrow & \hat{T}' & \rightarrow & 0.
\end{array}
\]

(3.6)
We can give a more amenable formula for the group \( \text{Ker}[\Omega^2_\omega(\hat{T}) \to \Omega^2_\omega(\hat{T}')] \), which is the difficult part of the formula (3.4).

**Lemma 3.1.** Suppose that \( K/k \) is a Galois field extension and that \( P(t) \) is a separable polynomial. Then \( \Omega^2_\omega(\hat{T}') \to \Omega^2_\omega(\hat{T}) \) is injective. In particular, we have

\[
\text{Ker}[\Omega^2_\omega(\hat{T}) \to \Omega^2_\omega(\hat{T}')] = \text{Ker}[\Omega^2_\omega(\hat{T}) \to \Omega^2_\omega(\hat{T})].
\]

**Proof.** The last two rows of (3.6) yield the exact sequence

\[
0 \to \hat{T}' \to \hat{T} \to U \to 0,
\]

where \( U \) is the Galois lattice defined by the exact sequence

\[
0 \to \mathbb{Z}_{p'} \otimes \hat{T} \to \mathbb{Z}_p \otimes \hat{T} \to U \to 0.
\]

From sequences (3.7) and (3.8) we derive the following commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{\cdot} & H^1(k, U) \xrightarrow{\cdot} H^2(k, \hat{T}') \xrightarrow{\cdot} H^2(k, \hat{T}) \\
\| \quad & \| \quad & \| \\
0 & \xrightarrow{\cdot} & H^1(k, U) \xrightarrow{\cdot} H^2(k, \mathbb{Z}_{p'} \otimes \hat{T}) \xrightarrow{\cdot} H^2(k, \mathbb{Z}_p \otimes \hat{T}),
\end{array}
\]

where the exactness of rows follows from the surjectivity of \( H^1(k, \hat{T}') \to H^1(k, \hat{T}) \) and \( H^1(k, \mathbb{Z}_{p'} \otimes \hat{T}) \to H^1(k, \mathbb{Z}_p \otimes \hat{T}) \). Hence, the injectivity of \( \Omega^2_\omega(\hat{T}') \to \Omega^2_\omega(\hat{T}) \) follows from the injectivity of \( \Omega^2_\omega(\mathbb{Z}_{p'} \otimes \hat{T}) \to \Omega^2_\omega(\mathbb{Z}_p \otimes \hat{T}) \).

Let

\[
A := \text{Ker}[\Omega^2_\omega(\mathbb{Z}_{p'} \otimes \hat{T}) \to \Omega^2_\omega(\mathbb{Z}_p \otimes \hat{T})] = \text{Ker}[\Omega^2_\omega(\mathbb{Z}_{p'} \otimes \hat{T}) \to H^2(k, \mathbb{Z}_p \otimes \hat{T})].
\]

For any Galois extension \( F'/F \) and \( \text{Gal}(F'/F) \)-module \( M \), we denote \( \Omega^2_\omega(F'/F, M) \) the subgroup of \( H^2(F'/F, M) \) consisting of those elements whose image in \( H^2(H, M) \) vanishes for every cyclic subgroup \( H \) of \( \text{Gal}(F'/F) \). Using Shapiro’s lemma, we have

\[
A \cong \bigoplus_i \text{Ker}[\text{Res}_{L_i'/L_i}(\bar{k}/L_i', \hat{T}) \to H^2(L_i, \hat{T})].
\]

We have the following commutative diagram

\[
\begin{array}{ccc}
\Omega^2_\omega(\bar{k}/L_i', \hat{T}) & \xrightarrow{\cdot} & H^2(L_i, \hat{T}) \\
\uparrow f & & \uparrow g \\
\Omega^2_\omega(K/L_i', \hat{T}) & \xrightarrow{\cdot} & H^2(K.L_i/L_i, \hat{T}),
\end{array}
\]

where \( f \) is an isomorphism since \( \hat{T} \) splits by \( K \), \( g \) is injective by Hochschild-Serre spectral sequence and \( H^1(K.L_i, \hat{T}) = 0 \). Since \( L_i \cap K = L_i' \), the second horizontal map of (3.9) is injective. Therefore

\[
A \cong \bigoplus_i \text{Ker}[\Omega^2_\omega(K/L_i', \hat{T}) \to H^2(K.L_i/L_i, \hat{T})] = 0. \quad \square
\]
In the remainder of this section, we always assume that $P(t)$ is an irreducible polynomial, $L = k[t]/(P(t))$ and that the fields $K$ and $L' := L \cap K$ are Galois over $k$. Let $G = \text{Gal}(K/k)$, $H = \text{Gal}(K/L')$ and $l = [L : L']$.

**Lemma 3.2.** The natural map

$$\text{Ker}[[\mathcal{I}^2_{L'}(\mathcal{T}') \to H^2(H, \mathcal{T}')] \to H^1(G/H, H^1(H, \mathcal{T}'))$$

is injective.

Let $g \in G$, $G' = \langle g \rangle$ and $H' = \langle g \rangle \cap H$. We will prove two lemmas before the proof of Lemma 3.2.

**Lemma 3.3.** The morphism $H^2(G'/H', \mathcal{T}'_{H'}) \to H^2(G', \mathcal{T}')$ is injective.

**Proof.** By Hochschild-Serre’s spectral sequence, we only need to show that $H^1(G', \mathcal{T}') \to H^1(H', \mathcal{T})_{G'/H'}$ is surjective.

Let $k' = K^{G'}$ and $P(t) = q_1(t)q_2(t) \cdots q_r(t)$ with all $q_i(t)$ irreducible over $k'$. Recall $l = [L : L']$. Let $L'_i = k'[t]/(q_i(t))$ for $1 \leq i \leq r$. All $\text{Gal}(K/K \cap L'_i)$ are isomorphic and we denote them by $H'$.

Let $\#H' = d_1, \#(G'/H') = d_2d_3$, where $d_3$ is the maximal factor of $\#(G'/H')$ such that $\gcd(d_1, d_3) = 1$, hence $\gcd(d_2, d_3) = 1$. Recall $e' = \text{gcd}(n, \deg P(t))$, hence $d_2d_3 \mid e'$. Let $e'' = e'/(d_2d_3)$. By (3.5), we have

$$H^1(G', \mathcal{T}') \cong \{(\chi_i)_{i \in H^1(H', \mathbb{Q}/\mathbb{Z})^r} : l \sum_{i=1}^r \text{Cor}_{H'/G'}(\chi_i)(g) = 0 \text{ for any } g \in G', g^{e'} = 1_{G'}\}$$

$$\cong \{(\widetilde{\chi}_i)_{i \in H^1(G', \mathbb{Q}/\mathbb{Z})^r} : l \sum_{i=1}^r d_2d_3\widetilde{\chi}_i(h) = 0 \text{ for any } h \in G', h^{e'} = 1_{G'}\}.$$ 

$$\cong \{(\widetilde{\chi}_i)_{i \in H^1(G', \mathbb{Q}/\mathbb{Z})^r} : l \sum_{i=1}^r \widetilde{\chi}_i(h) = 0 \text{ for any } h \in H', h^{e''} = 1_{H'}\}. \quad (3.10)$$

Similarly by (3.5), we have

$$H^1(H', \mathcal{T}') \cong \{ \sum_{\sigma \in G/H} \chi_{\sigma} \otimes \sigma \in H^1(H', \mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z}[G/H] : l \sum_{\sigma \in G/H} \chi_{\sigma}(h) = 0 \text{ for any } h \in H', h^{e'} = 1_{H'}\}.$$ 

Therefore

$$H^1(H', \mathcal{T}')^{G'/H'} \cong \{(\chi_i)_{i \in H^1(H', \mathbb{Q}/\mathbb{Z})^r} : ld_2d_3 \sum_{i=1}^r \chi_i(h) = 0 \text{ for any } h \in H', h^{e''} = 1_{H'}\}$$

$$\cong \{(\chi_i)_{i \in H^1(H', \mathbb{Q}/\mathbb{Z})^r} : l \sum_{i=1}^r \chi_i(h) = 0 \text{ for any } h \in H', h^{e''} = 1_{H'}\}. \quad (3.11)$$

By (3.10) and (3.11), the cyclicity of $G'$ implies that $H^1(G', \mathcal{T}') \to H^1(H', \mathcal{T}')^{H'}$ is surjective. \qed
Lemma 3.4. \( H^2(G/H, \widehat{\mathbb{T}}^H) \rightarrow \prod_{G' \text{ cyclic}} H^2(G'/H', \widehat{\mathbb{T}}^{H'}) \) is injective.

Proof. By the exact sequence (3.5), we have the exact sequence
\[
0 \rightarrow \mathbb{Z}_{p^r} \rightarrow \mathbb{D}^H \rightarrow \mathbb{T}^H \rightarrow 0.
\]
Hence
\[
H^2(G/H, \widehat{\mathbb{T}}^H) \cong H^2(G/H, \mathbb{D}^H) \cong H^2(G/H, S^H).
\] (3.12)
Similarly by the exact sequence (3.5), we have
\[
H^2(G'/H', \widehat{\mathbb{T}}^{H'}) \cong H^2(G'/H', \mathbb{D}'^{H'}) \cong H^2(G'/H', S'^{H'}) \text{.} \quad (3.13)
\]
By (3.12) and (3.13), it is enough to show that \( H^2(G/H, S'^{H}) \rightarrow \prod_{G' \text{ cyclic}} H^2(G'/H', S'^{H'}) \) is injective. The injectivity follows from the fact that \( S' \) is a \( G \)-permutation module and that \( S'^{H} \) is a direct factor of \( S'^{H'} \) as \( G'/H' \)-modules. \( \square \)

The proof of Lemma 3.2: By Hochschild-Serre's spectral sequence, we have the commutative diagram of exact sequences
\[
\begin{array}{ccc}
H^2(G/H, \widehat{\mathbb{T}}^H) & \longrightarrow & \text{Ker}[H^2(G, \widehat{\mathbb{T}}') \rightarrow H^2(H, \widehat{\mathbb{T}}')] \quad \psi \\
\downarrow f_1 & & \downarrow f_2 \\
\prod_g H^2(G'/H', \widehat{\mathbb{T}}^{H'}) & \longrightarrow & \text{Ker}[\prod_g H^2(G', \widehat{\mathbb{T}}') \rightarrow \prod_g H^2(H', \widehat{\mathbb{T}}')],
\end{array}
\]
where \( g \) runs through every element in \( G, G' = < g > \) and \( H' = < g > \cap H \).

Suppose \( \alpha \in \text{Ker} f_2 \cap \text{Ker} \psi \), then there exists an element \( \beta \in H^2(G/H, \widehat{\mathbb{T}}^H) \) such that \( \alpha \) is the image of \( \beta \). The map in the second row is injective by Lemma 3.3, hence \( f_1(\alpha) = 0 \). Since \( f_1 \) is injective by Lemma 3.4, we have \( \alpha = 0 \). Then we complete the proof of Lemma 3.2. \( \square \)

Let \( G \) be a finite group and \( H \) a normal subgroup of \( G \). Let \( [H, H] \) be the commutator subgroup of \( H \) and \( H^{ab} = H/[H, H] \). The group extension
\[
1 \rightarrow H^{ab} \rightarrow G/[H, H] \rightarrow G/H \rightarrow 1 \tag{3.14}
\]
defines a cohomology class \( u \in H^2(G/H, H^{ab}) \). Let \( A \) be a \( G \)-module with trivial \( H \)-action. For any \( p > 0 \), the cup product defines a morphism
\[
u \cup : H^{p-1}(G/H, H^1(H, A)) \rightarrow H^{p+1}(G/H, A).
\]
For any \( x \in H^{p-1}(G/H, H^1(H, A)) \), we have \( u \cup x = -d_2^{p-1,1}(x) \) by [21, Theorem 2.1.8], where \( d_2^{p-1,1} \) is the differential in Hochschild-Serre’s spectral sequence.

Let \( P(t) \) be an irreducible polynomial. Let \( L = k[t]/(P(t)) \). Suppose that the fields \( K \) and \( L' = L \cap K \) are Galois over \( k \). Let \( G = \text{Gal}(K/k), H = \text{Gal}(K/L') \). Recall that \( e' = \gcd(n, \deg P(t)) \) and \( l = [L : L'] \).
Denote
\[ C := \{ \chi \in H^1(H, \mathbb{Q}/\mathbb{Z}) : \text{Cor}_{L'/k}(\chi)(g) = 0 \text{ for any } g \in G \text{ with } g' = 1_G \}, \]
(3.15)
\[ C' := \{ \sum_{\sigma \in G/H} \chi_\sigma \otimes \sigma \in H^1(H, \mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z}[G/H] : l \sum_{\sigma} \chi_\sigma(h) = 0 \text{ for any } h \in H \text{ with } h' = 1_H \}. \]
(3.16)

**Theorem 3.5.** Let \( P(t) \) be an irreducible polynomial and \( L = k[t]/(P(t)) \). Suppose that the fields \( K \) and \( L' = L \cap K \) are Galois over \( k \). Let \( X \) be the variety (2.5) defined by the equation (1.1). Let \( G = \text{Gal}(K/k) \) and \( H = \text{Gal}(K/L') \). Then we have the exact sequences
\[
0 \to H^1(G/H, \mathbb{Q}/\mathbb{Z}) \to H^1(G, \mathbb{Q}/\mathbb{Z}) \to C \to \text{Br}_0(X)/\text{Br}_0(X) \to \text{Ker}[\pi_2^*(\hat{T}) \to \pi_2^*(\hat{T}')] \to 0,
\]
(3.17)
\[
0 \to H^1(G/H, \mathbb{Q}/\mathbb{Z}) \to H^1(G, \mathbb{Q}/\mathbb{Z}) \to H^1(H, \mathbb{Q}/\mathbb{Z})^{G/H} \to H^2(G/H, \mathbb{Q}/\mathbb{Z}) \to \text{Ker}[\pi_2^*(\hat{T})]
\]
\[ \to \pi_2^*(\hat{T}') \to H^1(G/H, H^1(H, \mathbb{Q}/\mathbb{Z})) \overset{(\text{Res},u\cup_1)}{\to} H^1(G/H, C') \oplus H^3(G/H, \mathbb{Q}/\mathbb{Z}), \]
(3.18)
where \( C \) and \( C' \) are defined by (3.15) and (3.16), and \( \text{Res} \) is induced by the diagonal map \( H^1(H, \mathbb{Q}/\mathbb{Z}) \to C' \).

**Proof.** Suppose \( e' \neq n, \) i.e., \( n \nmid \deg P(t) \). Then \( S' = \mathbb{Z}[K/k] \oplus \mathbb{Z}[\Omega], \) where \( \Omega = \{ M : M \subset G, |M| = e' \} \), see (3.1). As a \( G \)-module, \( \mathbb{Z}[\Omega] \) is the direct sum of submodules which are isomorphic to \( \mathbb{Z}[G/E] \) where \( E \) is a subgroup of \( G \) with degree dividing \( e' \); arbitrary such submodule \( \mathbb{Z}[G/E] \) must occur in \( \mathbb{Z}[\Omega] \). Hence, by (3.2) and (3.5), we have
\[
H^1(k, \hat{T}') \cong \text{Ker}[H^2(L, \mathbb{Z}) \to H^2(L \otimes K, \mathbb{Z}) \oplus H^2(k, S')]
\]
\[ = \text{Ker}[H^2(K,L/L, \mathbb{Z}) \to H^2(k, \mathbb{Z}[\Omega])] \cong C, \]
(3.19)
\[
H^1(H, \hat{T}') \cong \text{Ker}[f'^*: H^2(H, \mathbb{Z}) \otimes \mathbb{Z}[G/H] \to H^2(H, \mathbb{Z}[\Omega])] \cong C', \]
(3.20)
where \( f'^* \) is \( -l \) times the natural map. It is clear that (3.19) and (3.20) also hold when \( e' = n, \) i.e., \( n \mid \deg P(t) \). On the other hand, we have \( H^1(k, \hat{T}) \cong H^1(G, \mathbb{Q}/\mathbb{Z}) \) and \( H^1(H, \hat{T}) \cong H^1(H, \mathbb{Q}/\mathbb{Z}) \). The first exact sequence then follows from (3.4).

By Hochschild-Serre’s spectral sequence, we have the commutative diagram with exact sequences
\[
\begin{array}{cccccc}
H^2(G/H, \hat{T}^H) & \longrightarrow & \text{Ker}[H^2(G, \hat{T})] & \overset{f_1}{\longrightarrow} & H^2(H, \hat{T}) & \longrightarrow & H^1(G/H, H^1(H, \hat{T})) & \longrightarrow & H^3(G/H, \hat{T}^H) \\
\downarrow & & \downarrow f_2 & & \downarrow & & \downarrow & & \\
H^2(G/H, \hat{T}'^H) & \longrightarrow & \text{Ker}[H^2(G, \hat{T}')] & \overset{f_1'}{\longrightarrow} & H^2(H, \hat{T}') & \longrightarrow & H^1(G/H, H^1(H, \hat{T}')).
\end{array}
\]
(3.21)
The exact sequence (3.5) induces a morphism \( \hat{T}' \to \hat{T} \otimes \mathbb{Z}_p^\nu \) by sending \( S' \) to 0. Note that \( H^2(G, \hat{T} \otimes \mathbb{Z}_p^\nu) \cong H^2(H, \hat{T}) \), hence \( f_1 \) factors through \( f_2 \), so \( \text{Ker}(f_2) \subset \text{Ker}(f_1) \). By the exact sequence
\[ 0 \to \mathbb{Z} \to \mathbb{Z}[G] \to \hat{T} \to 0, \]
(3.22)
one obtains $\Pi^2_\omega(\hat{T}) = H^2(G, \hat{T})$. Therefore, by Lemma 3.2, the diagram (3.21) yields the commutative diagram

$$
\begin{array}{cccc}
H^2(G/H, \hat{T}^H) & \longrightarrow & \text{Ker}[\Pi^2_\omega(\hat{T})] & \longrightarrow & H^1(G/H, H^1(H, \hat{T})) & \longrightarrow & H^3(G/H, \hat{T}^H) \\
\downarrow f_2 & & \downarrow f_2 & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Ker}[\Pi^2_\omega(\hat{T}')] & \longrightarrow & H^1(G/H, H^1(H, \hat{T}')) & \longrightarrow & H^1(G/H, C').
\end{array}
$$

(3.23)

Let $A = (\mathbb{Q} \oplus \mathbb{Z}[G])/\mathbb{Z}$, then we have the morphisms of spectral sequences

$$
E(\mathbb{Q}/\mathbb{Z}) \leftarrow E(A) \rightarrow E(\hat{T}).
$$

(3.24)

By (3.24) and the exact sequence (3.22), we have the morphism of spectral sequences $E(\hat{T}) \rightarrow E(\mathbb{Q}/\mathbb{Z})$. By the exact sequence (3.5), the diagram (3.23) is equivalent to the following commutative diagram

$$
\begin{array}{cccc}
H^2(G/H, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \text{Ker}[\Pi^2_\omega(\hat{T})] & \longrightarrow & H^1(G/H, H^1(H, \mathbb{Q}/\mathbb{Z})) & \longrightarrow & H^3(G/H, \mathbb{Q}/\mathbb{Z}) \\
\downarrow f_2 & & \downarrow f_2 & & \downarrow & & \downarrow \text{Res} \\
0 & \longrightarrow & \text{Ker}[\Pi^2_\omega(\hat{T}')] & \longrightarrow & H^1(G/H, H^1(H, \hat{T}')) & \longrightarrow & H^1(G/H, C').
\end{array}
$$

(3.25)

By Lemma 3.1, $\text{Ker}[\Pi^2_\omega(\hat{T}) \rightarrow \Pi^2_\omega(\hat{T}')] = \text{Ker}(f_2) \subset \text{Ker}(f_1)$, the proof then follows from the commutativity of the diagram (3.25).

**Remark** 3.6. If $L \cap K = k$, one obtains $\text{Br}_\text{un}(X) = \text{Br}_0(X)$ by Theorem 3.5 and this result has been already proved in [26, Theorem 3.2].

For any finite group $M$, we denote $[M, M]$ the commutator subgroup of $M$ and $M^{ab} := M/[M, M]$.

**Corollary** 3.7. Let $X, G$ and $H$ be defined as in Theorem 3.5. Suppose $H^{ab} = 0$. Then we have $\text{Br}_\text{un}(X)/\text{Br}_0(X) \cong H^3(G/H, \mathbb{Z})$. Furthermore, if we assume that $G/H$ is cyclic, then $\text{Br}_\text{un}(X) = \text{Br}_0(X)$.

**Proof.** Since $H^{ab} = 0$, we have $C = 0$ and $H^1(H, \mathbb{Q}/\mathbb{Z}) = 0$. By Theorem 3.5, we have

$$
\text{Br}_\text{un}(X)/\text{Br}_0(X) \cong H^2(G/H, \mathbb{Q}/\mathbb{Z}) \cong H^3(G/H, \mathbb{Z}).
$$

If $G/H$ is cyclic, then $H^3(G/H, \mathbb{Z}) \cong H^1(G/H, \mathbb{Z}) = 0$.

**Lemma** 3.8. 1) Suppose that $G/[H, H]$ is abelian. Then $\text{Br}_\text{un}(X)/\text{Br}_0(X)$ is isomorphic to the subgroup $B$ of $H^2(G, \mathbb{Q}/\mathbb{Z})$ and the following exact sequence holds

$$
0 \rightarrow H^2(G/H, \mathbb{Q}/\mathbb{Z}) \rightarrow B \rightarrow H^1(G/H, H^1(H, \mathbb{Q}/\mathbb{Z}))
$$

$$
\rightarrow (\text{Res}, \text{u.l.}) H^1(G/H, C') \oplus H^3(G/H, \mathbb{Q}/\mathbb{Z});
$$

(3.26)

2) Let $P(t)$ be an irreducible polynomial and $L = k[t]/(P(t))$. Let $L' \subset L$ be a Galois extension of $k$. Let $F/k$ be a Galois field extension and $F \cap L = k$. Suppose $K = L'.F$. Then

$$
\text{Br}_\text{un}(X)/\text{Br}_0(X) \cong H^2(L'/k, \mathbb{Q}/\mathbb{Z}) \oplus \Delta,
$$

where $\Delta$ is the group of roots of unity in $L'/k$. 

**Proof.**
where $\Delta$ is the kernel of the map $\text{Res}: H^1(L'/k, H^1(F/k, \mathbb{Q}/\mathbb{Z})) \rightarrow H^1(L'/k, C')$.

**Proof.** For the case 2), we have $G = \text{Gal}(K/k) \cong \text{Gal}(F/k) \times \text{Gal}(L'/k)$, $H = \text{Gal}(K/L') \cong \text{Gal}(F/k)$, $G/H \cong \text{Gal}(L'/k)$ and $K \cap L = L'$. In both cases, the two maps $H^1(G, \mathbb{Q}/\mathbb{Z}) \rightarrow C$ in (3.10) and $H^1(G, \mathbb{Q}/\mathbb{Z}) \rightarrow H^1(H, \mathbb{Q}/\mathbb{Z})$ in (3.18) are surjective, hence the exact sequence (3.26) holds for both cases by Theorem 3.5. Now it remains to prove the case 2). Since $\text{Gal}(K/k) \cong \text{Gal}(F/k) \times \text{Gal}(L'/k)$, the cohomology class $\iota = 0$ since the sequence (3.14) is split and the third map in (3.26) is split by the main theorem of [19]. The proof for the case 2) then follows. □

**Lemma 3.9.** Suppose $G/H$ trivially acts on $H^1(H, \mathbb{Q}/\mathbb{Z})$ (e.g., $G/[H, H]$ is abelian). Let $\chi \in H^1(G/H, H^1(H, \mathbb{Q}/\mathbb{Z}))$ have order $r$. Then

$$2\text{Res}(\chi) = 0 \in H^1(G/H, C').$$

Furthermore, if $r$ is odd or 2 | $l\#\text{Ker}(\chi_h)$ for any $h \in H$ with $h^r = 1_H$, then

$$\text{Res}(\chi) = 0,$$

where $l = [L : L']$ and $\chi_h : G/H \rightarrow \mathbb{Q}/\mathbb{Z}$ by sending $\sigma \in G/H$ to $\chi(\sigma)(h)$.

**Proof.** Let $a := -\sum_{\sigma \in G/H} \chi(\sigma) \otimes \sigma \in H^1(H, \mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z}[G/H]$. For any $\tau \in G/H$, then

$$\partial^1(a)(\tau) := \tau a - a = -\sum_{\sigma \in G/H} \chi(\sigma) \otimes \tau \sigma + \sum_{\sigma \in G/H} \chi(\sigma) \otimes \sigma$$

$$= -\sum_{\sigma \in G/H} \chi(\tau^{-1}\sigma) \otimes \sigma + \sum_{\sigma \in G/H} \chi(\sigma) \otimes \sigma = \chi(\tau) \otimes \sum_{\sigma \in G/H} \sigma. \quad (3.28)$$

Therefore $\partial^1(a) = \text{Res}(\chi) \in C^1(G/H, H^1(H, \mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z}[G/H])$. To complete the proof, we only need to show $a \in C'$. For any $h \in H$, one has

$$l \sum_{\sigma \in G/H} \chi(\sigma)(h) = l\#\text{Ker}(\chi_h)r'(r' - 1)/2 \cdot \psi \in \mathbb{Q}/\mathbb{Z}, \quad (3.29)$$

where $\psi$ is a generator of the image of $\chi_h$ in $\mathbb{Q}/\mathbb{Z}$ which has order $r'$. Obviously $2a \in C'$, hence we proved (3.27). Since $r' | r$, if $r$ is odd or 2 | $[L : L']\#\text{Ker}(\chi_h)$, then the sum in (3.29) is zero, hence $a \in C'$, the proof then follows. □

Let $V$ be the maximal smooth open subset of (1.1). Let $V^{\text{CHS}}$ be the CTHS-partial compactification of $V$ (see §2 or [6]). By Theorem 3.5, we immediately obtain the following result which is slightly more general than [26, Theorem 3.6].

**Corollary 3.10.** Suppose $G/[H, H]$ is an abelian group. Then:

(a) The quotient $\text{Br}(V^{\text{CHS}})/\text{Br}_{\text{un}}(X)$ is 2-torsion.

(b) $\text{Br}_{\text{un}}(X) = \text{Br}(V^{\text{CHS}})$ if one of the following conditions holds:

1. either $H^{ab}$ or $G/H$ has odd order, or $G \cong \mathbb{Z}/2^i \times A$, where $A$ has odd order;
2. $[L : L \cap K]$ is even;
3. there exists $s \geq 1$ such that $2^s \mid \#(G/H)$ and $\#H^{ab} = 2^{s-1}d$, where $d$ is odd;
4. $L/k$ contains an abelian subfield $L''/k$ with $\text{Gal}(L''/k) \cong (\mathbb{Z}/2\mathbb{Z})^2$. 
Proof. Since $G/[H, H]$ is an abelian group, by [6, Proposition 2.5], one has
\[
\text{Br}(\text{V}^{\text{CTHS}})/\text{Br}_0(\text{V}^{\text{CTHS}}) \cong \text{Ker}[\mathbb{I}_2(\mathcal{T}) \to H^2(H, \mathcal{T}') \cong \text{Ker}[H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(H, \mathbb{Q}/\mathbb{Z})].
\]
(3.30)
Since $G/[H, H]$ is an abelian group, $H^1(G, \mathbb{Q}/\mathbb{Z}) \to C$ is surjective. By Theorem (3.5), the case (a) follows from Lemma 3.9. If $\text{Res} : H^1(G/H, H^1(H, \mathbb{Q}/\mathbb{Z})) \to H^1(G/H, C')$ is a zero map, comparing Theorem 3.5 with (3.30), we have $\text{Br}_{\text{un}}(X) = \text{Br}(\text{V}^{\text{CTHS}})$.

Let $L' = L \cap K$ and $l = [L : L']$. If $G \cong \mathbb{Z}/2^s \times A$ and $A$ has odd order, then $H^2(G, \mathbb{Q}/\mathbb{Z}) \cong H^2(A, \mathbb{Q}/\mathbb{Z})$ has odd order, hence the image of $B \to H^1(G/H, H^1(H, \mathbb{Q}/\mathbb{Z}))$ in (3.26) has odd order. In the following we will apply Lemma 3.9, it is clear that $r$ is odd for the case (b-1), $2 \mid l$ for the case (b-2) and $2 \mid \#\text{Ker}(\chi_h)$ for the case (b-3), the proof then follows from Lemma 3.9. For the case (b-4), if $L'' \not\subset L'$, then $2 \mid [L : L']$, the proof follows from the case (b-2), hence we may assume $L'' \subset L'$, then one obtains $2 \mid \#\text{Ker}(\chi_h)$, the proof then follows from Lemma 3.9.

Remark 3.11. We can compare this corollary with [26, Theorem 3.6]. Replacing the assumption that $G$ is abelian in [26, Theorem 3.6] we only need to assume that $G/[H, H]$ is abelian. In the case (b-1) we obtains the new case that $H^{ab}$ is odd order, and the case (b-4) is slightly more general than the case (b-5) of [26, Theorem 3.6] where the assumption needs $\text{Gal}(L''/k) \cong (\mathbb{Z}/2\mathbb{Z})^3$.

The following result gives an explicit description for the case 2) of Lemma 3.8.

Proposition 3.12. Let $P(t)$ be an irreducible polynomial and $L = k[t]/(P(t))$. Let $L' \subset L$ be a Galois extension over $k$. Let $F/k$ be a Galois field extension and $F \cap L = k$. Let $K = L'.F$ and $H = \text{Gal}(K/L') \cong \text{Gal}(F/k)$. Let $X$ be the variety (2.5) defined by the equation (1.1). Let $l = [L : L']$ and $\rho = \#\text{[Gal}(L'/k), \text{Gal}(L'/k)]$. We write
\[
H^{ab} = H_2 \times H_{\text{odd}}, \text{Gal}(L'/k)^{ab} = H_2' \times H_{\text{odd}}',
\]
where $H_2$ and $H_2'$ have 2-power order, $H_{\text{odd}}$ and $H_{\text{odd}}'$ have odd order.

Suppose that either $2 \mid l$ or $H_2$ is zero or non-cyclic. Then
\[
\text{Br}_{\text{un}}(X)/\text{Br}_0(X) \cong H^2(L'/k, \mathbb{Q}/\mathbb{Z}) \times H^1(L'/k, H^1(F/k, \mathbb{Q}/\mathbb{Z})).
\]

Suppose $2 \not\mid l$ and $H_2' \cong \mathbb{Z}/2^s$ with $s \geq 1$. Let $\tilde{H}_2 \times E$ be the image of $\{h \in H : h^{2^s} = 1_H\}$ by the projection $H \to H^{ab}$, where $\tilde{H}_2 \cong (\mathbb{Z}/2^s)^{u_2}$ and $E$ is killed by $2^{s-1}$. We write
\[
H_2 = \tilde{H}_2 \times (\mathbb{Z}/2^s)^{u_2} \times \prod_{i=1}^{v} \mathbb{Z}/2^{s_i} \times \prod_{j=1}^{w} \mathbb{Z}/2^{t_j}
\]
with $s_i < s$ and $t_j > s$ for any $i, j$. Then
\[
\text{Br}_{\text{un}}(X)/\text{Br}_0(X) \cong H^2(L'/k, \mathbb{Q}/\mathbb{Z}) \times H^1(H_{\text{odd}}, H^1(Q, \mathbb{Q}/\mathbb{Z})) \times (\mathbb{Z}/2^{s-1})^{u_1} \times (\mathbb{Z}/2^s)^{w+u_2} \times \prod_{i=1}^{v} \mathbb{Z}/2^{s_i}.
\]

Proof. By the case 2) of Lemma 3.8, we only need to compute the kernel $\Delta$ of the map
\[
\text{Res} : H^1(L'/k, H^1(F/k, \mathbb{Q}/\mathbb{Z})) \to H^1(L'/k, C').
\]
If \(2 \mid l\), then \(\text{Res}\) is a zero map by Lemma 3.9. If either \(2 \mid \rho\) or \(H_2'\) contains a subgroup \(\mathbb{Z}/2 \times \mathbb{Z}/2\), then \(2 \mid \#\text{Ker}(\chi_h)\) for any \(h \in H\); if \(H_2' = 0\), then \(r\) is odd; the map \(\text{Res}\) is zero for all above cases by Lemma 3.9. The proof of the case 1) then follows from the case 2) of Lemma 3.8.

Now it remains the case that \(2 \nmid l\rho\) and \(H_2' \cong \mathbb{Z}/2^s\) with \(s \geq 1\). It is clear that
\[
H^1(L'/k, H^1(F/k, \mathbb{Q}/\mathbb{Z})) = H^1(\mathbb{Z}/2^s, H^1(H_2, \mathbb{Q}/\mathbb{Z})) \times H^1(H_{\text{odd}}, H^1(\mathbb{Q}/\mathbb{Z})).
\]
Obviously \(\text{Res}\) restricting to \(H^1(H_{\text{odd}}, H^1(\mathbb{Q}/\mathbb{Z}))\) is zero by Lemma 3.9. So it remains to compute the kernel of \(\text{Res} |_{H^1(\mathbb{Q}/\mathbb{Z}, H^1(H_2, \mathbb{Q}/\mathbb{Z}))}\). We can write
\[
H^1(\mathbb{Z}/2^s, H^1(H_2, \mathbb{Q}/\mathbb{Z})) = A_1 \times A_2 \times A_3 \times A_4,
\]
where
\[
A_1 := H^1(\mathbb{Z}/2^s, H^1(H_2, \mathbb{Q}/\mathbb{Z})), A_2 := H^1(\mathbb{Z}/2^s, H^1(\mathbb{Z}/2^s, \mathbb{Q}/\mathbb{Z}))^{u_2},
\]
\[
A_3 := \prod_{i=1}^{u} H^1(\mathbb{Z}/2^s, H^1(\mathbb{Z}/2^s, \mathbb{Q}/\mathbb{Z})), A_4 := \prod_{j=1}^{w} H^1(\mathbb{Z}/2^s, H^1(\mathbb{Z}/2^j, \mathbb{Q}/\mathbb{Z})).
\]
Obviously \(A_1 \cong (\mathbb{Z}/2^s)^{u_1}, A_2 \cong (\mathbb{Z}/2^s)^{u_2}, A_3 \cong \prod_{i=1}^{u} \mathbb{Z}/2^s, \) and \(A_4 \cong (\mathbb{Z}/2^s)^w\).

If \(\chi \in A_3\), obviously \(2 \mid \#\text{Ker}(\chi_h)\) for any \(h \in H\) since \(s_i < s\), one obtains \(A_3 \subset \Delta\) by Lemma 3.9. In the following we will show that \(A_4 \subset \Delta\). It suffices that \(\text{Res}\) restricting to \(H^1(\mathbb{Z}/2^s, H^1(\mathbb{Z}/2^j, \mathbb{Q}/\mathbb{Z}))\) is zero. Let \(\chi \in H^1(\mathbb{Z}/2^s, H^1(\mathbb{Z}/2^j, \mathbb{Q}/\mathbb{Z}))\). By Lemma 3.9, we only need to show \(2 \mid \#\text{Ker}(\chi_h)\) for any \(h \in H\) with \(h^{2^j} = 1_H\). Let \(\tilde{h}\) be the image of \(h\) in \(H^{ab}\). Since \(2 \nmid l\), one obtains \(v_2(\epsilon') = s\). Since \(\text{Im}(\chi) \subset H^1(\mathbb{Z}/2^j, \mathbb{Q}/\mathbb{Z})\), it suffices to show \(2 \mid \#\text{Ker}(\chi_h)\) when \(\tilde{h} \in \mathbb{Z}/2^j\) satisfies \(\tilde{h}^{2^j} = 1_{H^{ab}}\). Let \(N = \max\{2s-t_j, 0\} < s\) since \(t_j > s\). Let \(\sigma\) be a generator of \(H_2'\) and \(\tau\) a generator of \(\mathbb{Z}/2^j\). We can write \(\tilde{h} = \tau^{2^a}\) with \(\alpha \geq t_j - s\). Therefore
\[
\chi(\sigma^{2^N})(\tilde{h}) = 2^{a+N}\chi(\sigma)(\tau) = 0
\]
since \(\alpha + N \geq t_j - s + 2s - t_j = s\) and the order of \(\chi\) is a factor of \(2^s\). One has \(\sigma^{2^N} \in \text{Ker}(\chi_h)\), it implies \(2 \mid \#\text{Ker}(\chi_h)\) since \(N < s\).

We may show that \(A_2 \subset \Delta\) by a similar argument as for \(A_4\). Let \(\chi \in H^1(\mathbb{Z}/2^s, H^1(\mathbb{Z}/2^s, \mathbb{Q}/\mathbb{Z}))\). By Lemma 3.9, we only need to show \(2 \mid \#\text{Ker}(\chi_h)\) for any \(h \in \mathbb{Z}/2^s \subset H\) with \(h^{2^j} = 1_H\). By our assumption, the generator \(\tau\) of \(\mathbb{Z}/2^s\) is not contained in the image of \(\{h \in H : h^{2^j} = 1_H\}\). Therefore, it suffices to show \(2 \mid \#\text{Ker}(\chi_h)\) for any \(\tilde{h} \in \mathbb{Z}/2^s \subset H^{ab}\) with \(\tilde{h}^{2^{s'}} = 1_{H^{ab}}, \) where \(s' < s\) is a non-negative integer. Let \(\tau\) be a generator of \(\mathbb{Z}/2^s \subset H\). We can write \(\tilde{h} = \tau^{2^{a'}}\) with \(\alpha \geq s - s'.\) Therefore
\[
\chi(\sigma^{2^{a'}})(\tilde{h}) = 2^{a'+s'}\chi(\sigma)(\tau) = 0
\]
since \(\alpha + s' \geq s - s' + s' = s\) and the order of \(\chi\) is a factor of \(2^s\). Therefore \(\sigma^{2^{a'}} \in \text{Ker}(\chi_h)\), it implies \(2 \mid \#\text{Ker}(\chi_h)\) since \(s' < s\).

For any \(\chi \in A_1\), one has \(2\chi \in \Delta\) by Lemma 3.9, it remains to show that \(\chi \not\in \Delta\) for any \(\chi\) has order \(2^s\). Let
\[
\Theta := \left\{ \sum_{\sigma \in H_2'} \tau_\sigma \otimes \sigma \in H^1(\tilde{H}_2, \mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z}[H_2'] : \sum_{\sigma \in H_2'} \tau_\sigma = 0 \right\}.
\]
The module $\Theta$ is a direct factor of $C'$ (see (3.16)) since $\widetilde{H}_2$ is contained in the image of \{ $h \in H : h^{2s} = 1_H$ \} in $H^{ab}$ and $\nu_2(c') = s$, hence we only need to show $\text{Res}(\chi) \neq 0 \in H^1(H'_2, \Theta)$.

From the exact sequence of $H'_2$-modules

$$0 \to \Theta \to H^1(\widetilde{H}_2, \mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z}[H'_2] \xrightarrow{j} H^1(\widetilde{H}_2, \mathbb{Q}/\mathbb{Z}) \to 0,$$

one derives the isomorphism

$$\text{Res}(\chi) \neq 0 \in H^1(H'_2, \Theta).$$

Let $a := -\sum_{\sigma \in H'_2} \chi(\sigma) \otimes \sigma \in H^1(\widetilde{H}_2, \mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z}[H'_2]$. Let $\gamma$ be the generator of $H'_2$ which has order $2^s$. Let $b := j(a) = -\sum_{\sigma \in H'_2} \chi(\sigma) = 2^{s-1}(2^s - 1)\chi(\gamma) \neq 0 \in H^1(\widetilde{H}_2, \mathbb{Q}/\mathbb{Z})$ since $\chi$ has order $2^s$. Therefore $\partial(b) \neq 0 \in H^1(H'_2, \Theta)$ since $\partial$ is an isomorphism. On the other hand, by a similar computation as in (3.28), one has $\partial(b) = \text{Res}(\chi)$, then $\text{Res}(\chi) \neq 0$ and the proof follows. \hfill $\square$

**Proposition 3.13.** Let $P(t)$ be an irreducible polynomial and $p$ a prime number. Let $L = k[t]/(P(t))$ and $L' = L \cap K$. Suppose that $L'/k$ is cyclic of order $p^s$ and that $G = \text{Gal}(K/k)$ is an abelian group. Let $X$ be the variety (2.5) defined by the equation (1.1). We can write $G = G_1 \times G_2$ satisfying $L' \subset K^{G_2}$, $G_1 = \mathbb{Z}/p^{s'}, s' \geq s$ and

$$G_2 = (\mathbb{Z}/p^s)^r \times \prod_{i=1}^{r_1} \mathbb{Z}/p^{e_i} \times \prod_{j=1}^{r_2} \mathbb{Z}/p^{\mu_j} \times H_1,$$

where $e_i < s$ and $\mu_j > s$ for any $i, j$ and $H_1$ has order prime to $p$.

1. If either $p$ is odd or $p = 2$ and $2 \mid [L : L']$, then

$$\text{Br}_{\text{un}}(X)/\text{Br}_0(X) \cong (\mathbb{Z}/p^s)^{r+r_2} \times \prod_{i=1}^{r_1} \mathbb{Z}/p^{e_i}.$$

2. If $p = 2$ and $2 \mid [L : L']$, then

$$\text{Br}_{\text{un}}(X)/\text{Br}_0(X) \cong (\mathbb{Z}/2^{s-1})^r \times (\mathbb{Z}/2^s)^{r_2} \times \prod_{i=1}^{r_1} \mathbb{Z}/2^{e_i}.$$

**Proof.** Let $u' \in H^2(\mathbb{Z}/p^s, \mathbb{Z}/p^{s'-s})$ be defined by the group extension

$$0 \to \mathbb{Z}/p^{s'-s} \to G_1 \to \mathbb{Z}/p^s \to 0.$$

Let $u \in H^2(\mathbb{Z}/p^s, H^{ab})$ be defined by (3.14), then

$$u = (u', 0) \in H^2(\mathbb{Z}/p^s, H^{ab}) \cong H^2(\mathbb{Z}/p^s, H^1(\mathbb{Z}/p^{s'-s}, \mathbb{Q}/\mathbb{Z})) \oplus H^2(\mathbb{Z}/p^s, G_2).$$

Since $H^2(G_1, \mathbb{Q}/\mathbb{Z}) = 0$, by Hochschild-Serre’s spectral sequence, it implies that the map

$$u' \cup : H^1(\mathbb{Z}/p^s, H^1(\mathbb{Z}/p^{s'-s}, \mathbb{Q}/\mathbb{Z})) \to H^3(\mathbb{Z}/p^s, \mathbb{Q}/\mathbb{Z})$$

is injective, hence the kernel of the map in Theorem 3.5

$$u \cup : H^1(\mathbb{Z}/p^s, H^1(H, \mathbb{Q}/\mathbb{Z})) \to H^3(\mathbb{Z}/p^s, \mathbb{Q}/\mathbb{Z})$$

coincides with $H^1(\mathbb{Z}/p^s, H^1(G_2, \mathbb{Q}/\mathbb{Z}))$. The proof of the case (1) then follows from Lemma 3.9 and the case 1) of Lemma 3.8.
Suppose \( p = 2 \) and \( 2 \nmid [L : L'] \). We need to determine the kernel of the map

\[
\text{Res} : H^1(\mathbb{Z}/2^s, H^1(G, \mathbb{Q}/\mathbb{Z})) \to H^1(\mathbb{Z}/2^s, C')
\]

which we denote by \( \Delta \). We can write \( H^1(\mathbb{Z}/2^s, H^1(G, \mathbb{Q}/\mathbb{Z})) = B_1 \times B_2 \times B_3 \), where

\[
\begin{align*}
B_1 & := H^1(\mathbb{Z}/2^s, H^1(\mathbb{Z}/2^s, \mathbb{Q}/\mathbb{Z})), \\
B_2 & := \prod_{i=1}^{r_1} H^1(\mathbb{Z}/2^s, H^1(\mathbb{Z}/2^{e_i}, \mathbb{Q}/\mathbb{Z})), \\
B_3 & := \prod_{j=1}^{r_2} H^1(\mathbb{Z}/2^s, H^1(\mathbb{Z}/2^{n_j}, \mathbb{Q}/\mathbb{Z})).
\end{align*}
\]

We have \( B_1 \cong (\mathbb{Z}/2^s)^r, B_2 \cong \prod_{i=1}^{r_1} \mathbb{Z}/2^{e_i} \) and \( B_3 \cong (\mathbb{Z}/2^s)^{r_2} \).

If \( \chi \in B_2 \), obviously \( 2 \nmid \# \ker(\chi_h) \) for any \( h \in H \) since \( e_i < s \), one obtains \( B_3 \subset \Delta \) by Lemma 3.9. By a similar argument as in the proof of Proposition 3.12 for \( A_4 \) in (3.31), we can show \( B_3 \subset \Delta \); by a similar argument as in the proof of Proposition 3.12 for \( A_1 \) in (3.31) we can show that \( 2A_1 \subset \Delta \) and \( \chi \not\in \Delta \) for any \( \chi \in A_1 \) has order \( 2^s \). The proof then follows from the case 1) of Lemma 3.8.

\[\square\]

**Remark** 3.14. In fact, if \( L'/k \) is not cyclic, by the case 1) of Lemma 3.8, we have an exact sequence

\[0 \to H^2(G/H, \mathbb{Q}/\mathbb{Z}) \to \text{Br}_n(X)/\text{Br}_0(X) \to \Lambda \to 0\]

where \( \Lambda \) is a subgroup of \( H^1(G/H, H^1(H, \mathbb{Q}/\mathbb{Z})) \) and can be explicitly determined.

**Lemma** 3.15. Let \( G = H \times C_2 \) is the dihedral group \( D_n \), where \( H \) is cyclic of order \( n \) and \( C_2 = \mathbb{Z}/2 \). For any character \( \chi \in H^1(H, \mathbb{Q}/\mathbb{Z}) \), \( \text{Cor}_{H/G}(\chi) = 0 \in H^1(G, \mathbb{Q}/\mathbb{Z}) \).

**Proof.** Note that \( G = HC_2 \), by [21, Corollary 1.5.7 and 1.5.8], we have

\[\text{Res}_{G/H}\text{Cor}_{H/G}(\chi) = N_{G/H}(\chi) = \chi + (-\chi) = 0, \quad \text{Res}_{G/C_2}\text{Cor}_{H/G}(\chi) = 0,\]

which implies \( \text{Cor}_{H/G}(\chi) = 0 \).

\[\square\]

**Proposition** 3.16. Let \( P(t) \) be an irreducible polynomial. Let \( L = k[t]/(P(t)) \) and \( L' = L \cap K \).

Suppose \( L'/k \) has order 2, \( K/k \) is Galois with \( G = \text{Gal}(K/k) = H \times \text{Gal}(L'/k) \) is the dihedral group \( D_{\tilde{n}} \), where \( H = \text{Gal}(K'/L') \) is cyclic of order \( \tilde{n} \). Let \( X \) be the variety (2.5) defined by the equation (1.1). Then,

1. if \( \tilde{n} \) is odd or \( 4 \nmid \tilde{n}[L : L'] \), then \( \text{Br}_n(X)/\text{Br}_0(X) \cong \mathbb{Z}/\tilde{n} \).
2. if \( \tilde{n} \) is even and \( 4 \nmid \tilde{n}[L : L'] \), then \( \text{Br}_n(X)/\text{Br}_0(X) \cong \mathbb{Z}/(\tilde{n}/2) \).

**Proof.** Suppose \( \tilde{n} \) is odd. By Lemma 3.15, the cokernel of the map \( H^1(G, \mathbb{Q}/\mathbb{Z}) \to C \) is \( \mathbb{Z}/\tilde{n} \). Since \( G/H \) is cyclic, one obtains \( H^2(G/H, \mathbb{Q}/\mathbb{Z}) = 0 \). Since \( H^1(G/H, H^1(H, \mathbb{Q}/\mathbb{Z})) = 0 \), by (3.18) in Theorem 3.5, we have \( \ker[\text{III}_2^k(\hat{T}) \to \text{III}_2^k(\hat{T})] = 0 \). Therefore,

\[\text{Br}_n(X)/\text{Br}_0(X) \cong \mathbb{Z}/\tilde{n} \]

In the following, we assume that \( \tilde{n} \) is even. The cokernel of the map \( H^1(G, \mathbb{Q}/\mathbb{Z}) \to C \) in (3.17) is \( \mathbb{Z}/(\tilde{n}/2) \) by Lemma 3.15. Since \( G = \text{Gal}(K/k) = H \times \mathbb{Z}/2 \), then the cohomology class \( u = 0 \) which is given by the extension (3.14), hence the map \( u \cup \) in Theorem 3.5 is a zero map.
Suppose $4 \nmid \tilde{n}[L : L']$. Then $v_2(\tilde{n}) = 1$ and $[L : L']$ is odd. Let $\tilde{H}$ be the unique subgroup of $H$ of order 2. Then $H^1(\tilde{H}, \mathbb{Q}/\mathbb{Z})$ is the 2-primary part of $H^1(\tilde{H}, \mathbb{Q}/\mathbb{Z})$. Let

$$
\tilde{C}' := \{(x_1, x_2) : x_1 + x_2 = 0, x_i \in H^1(\tilde{H}, \mathbb{Q}/\mathbb{Z}) \text{ for } i = 1, 2\} \cong H^1(\tilde{H}, \mathbb{Q}/\mathbb{Z}).
$$

(3.32)

The module $\tilde{C}'$ is the 2-primary part of $C'$ in (3.16). Since $G/H$ has order 2, one obtains $H^1(G/H, \mathbb{C}') \cong H^1(G/H, \tilde{C}')$. But the natural restriction map $H^1(G/H, H^1(\mathbb{Q}/\mathbb{Z})) \to H^1(G/H, \tilde{C}')$ is an isomorphism by (3.32), therefore the map Res in Theorem 3.5 is injective. So $\text{Br}_0(X)/\text{Br}_0(\tilde{X}) \cong \mathbb{Z}/(\tilde{n}/2)$.

Now it remains the case $4 \mid \tilde{n}[L : L']$. Let $B$ be the unique subgroup of $H^1(\mathbb{Q}/\mathbb{Z})$ of order 2. In the following we will show $B \otimes \mathbb{Z}[G/H] \subset C'$. Let $\chi \in B$, if $2 \mid [L : L']$ then $[L : L']\chi = 0$, hence $B \otimes \mathbb{Z}[G/H] \subset C'$; if $2 \nmid [L : L']$, then $4 \mid n$, it implies $\chi(h) = 0$ for any $h \in H$ with $h^2 = 1$, hence $B \otimes \mathbb{Z}[G/H] \subset C'$.

On the other hand, the natural map $H^{-1}(G/H, H^1(\mathbb{Q}/\mathbb{Z})) \to H^{-1}(G/H, B)$ is an isomorphism. Since $G/H$ is cyclic, so $H^1(G/H, H^1(\mathbb{Q}/\mathbb{Z})) \cong H^1(G/H, B)$, hence the map

$$
\text{Res} : H^1(G/H, H^1(\mathbb{Q}/\mathbb{Z})) \to H^1(G/H, C')
$$

factors through $H^1(G/H, B \otimes \mathbb{Z}[G/H]) = 0$. Therefore $\text{Ker}[\text{III}_n^2(\overline{T}) \to \text{III}_n^2(\overline{T}')] \cong \mathbb{Z}/2$ and the cardinality of $\text{Br}_0(X)/\text{Br}_0(\tilde{X})$ is $\tilde{n}$ by Theorem 3.5. In the following, we will show $\text{Br}_0(X)/\text{Br}_0(\tilde{X}) \cong \mathbb{Z}/\tilde{n}$ by showing that it is a subgroup of a cyclic group.

First, we will show the map $\text{Br}_0(X)/\text{Br}_0(\tilde{X}) \to \text{Br}_0(X_L)/\text{Br}_0(\tilde{X}_L)$ is injective, i.e., the map $H^1(k, \text{Pic}(\overline{X})) \to H^1(L, \text{Pic}(\overline{X}))$ is injective. It is sufficient to show that $H^1(L'/k, \text{Pic}(\overline{X})^{\Gamma_L'}) = 0$.

Let $U_1'$ be the open affine subvariety of $X$ defined by (1.1) and $P(t) \neq 0$. By the exact sequence

$$
0 \to \mathbb{Z} \to \mathbb{Z}[L/k] \oplus \mathbb{Z}[K/k] \to \overline{k}[U_1']^\times/\overline{k}^\times \to 0,
$$

(3.33)

it is clear that $H^1(L', \overline{k}[U_1']^\times/\overline{k}^\times) = 0$ since $L \cap K = L'$.

The exact sequence

$$
0 \to \overline{k}[U_1']^\times/\overline{k}^\times \to \mathbb{D} \to \text{Pic}(\overline{X}) \to 0
$$

yields the following exact sequence

$$
0 \to (\overline{k}[U_1']^\times/\overline{k}^\times)^{\Gamma_L'} \to \mathbb{D}^{\Gamma_L'} \to \text{Pic}(\overline{X})^{\Gamma_L'} \to H^1(L', \overline{k}[U_1']^\times/\overline{k}^\times) = 0,
$$

(3.34)

where $\mathbb{D}$ is defined by (3.1). Since $\mathbb{D}^{\Gamma_L'}$ is a permutation module, the sequence (3.34) yields the injectivity of the map $H^1(L'/k, \text{Pic}(\overline{X})^{\Gamma_L'}) \to H^2(L'/k, (\overline{k}[U_1']^\times/\overline{k}^\times)^{\Gamma_L'})$. By the exact sequence (3.33), we can show $H^2(L'/k, (\overline{k}[U_1']^\times/\overline{k}^\times)^{\Gamma_L'}) \cong H^3(L'/k, \mathbb{Z}) = 0$ since $L'/k$ is cyclic, hence $H^1(L'/k, \text{Pic}(\overline{X})^{\Gamma_L'}) = 0$.

Now it remains to show that $H^1(L', \text{Pic}(\overline{X}))$ is cyclic. Let $L \otimes_k L' = L_1 \oplus L_2$, by the definition of $\overline{T}'$, then

$$
H^1(L', \overline{T}') \subset \oplus_{i=1}^2 H^1(L_i, K/L_i, \mathbb{Q}/\mathbb{Z}) \cong \oplus_{i=1}^2 H^1(K/L_i', \mathbb{Q}/\mathbb{Z}).
$$

Since $H^1(L', \overline{T}) \cong H^1(K/L', \mathbb{Q}/\mathbb{Z})$, we have $H^1(L', \overline{T}')/j^*(H^1(L', \overline{T})) \subset H^1(K/L', \mathbb{Q}/\mathbb{Z})$ is a cyclic group by the cyclicity of $K/L'$. It is clear that $\text{III}_n^2(\overline{T}_L') = 0$ since $K/L'$ is cyclic, hence $H^1(L', \text{Pic}(\overline{X}))$ is cyclic by (3.4) (replacing $k$ by $L'$).
Remark 3.17. Let $\text{Br}_{\text{vert}}(X) := \text{Br}(k(t)) \cap \text{Br}_{\text{un}}(X)$. In fact, $\text{Br}_{\text{vert}}(X)/\text{Br}_0(X)$ coincides with the image of the first map in (2.24) by [6, Proposition 2.5-b], then the sequence (3.4) may be rewritten as the following exact sequence

$$0 \to \text{Br}_{\text{vert}}(X)/\text{Br}_0(X) \to \text{Br}_{\text{un}}(X)/\text{Br}_0(X) \to \text{Ker}[\Phi^2(\widehat{T}) \to \Phi^2(\widehat{T}')] \to 0.$$  (3.35)

If $4 \mid \bar{n}$, the case (1) of Proposition 3.16 shows that the sequence (3.35) needs not split.

References

[1] T. D. Browning and D. R. Heath-Brown, Quadratic polynomials represented by norm forms, Geom. Funct. Anal. 22 (2012), 1124–1190.
[2] T. D. Browning and L. Matthiesen, Norm forms for arbitrary number fields as products of linear polynomials, Ann. Sci. École Norm. Sup. (4) 50 (2017), 1383–1446.
[3] T. D. Browning, L. Matthiesen and A. N. Skorobogatov, Rational points on pencils of conics and quadrics with many degenerate fibers, Ann. of Math. (2) 180 (2014), 381–402.
[4] Y. Cao, D. Wei and F. Xu, Strong approximation for a family of norm varieties, arXiv:1803.11003v5.
[5] J.-L. Colliot-Thélène, Points rationnels sur les fibrations, in Higher dimensional varieties and rational points (Budapest, 2001), volume 12 of Bolyai Soc. Math. Stud., pages 171–221. Springer, Berlin, 2003.
[6] J.-L. Colliot-Thélène, D. Harari and A. N. Skorobogatov, Valeurs d’un polynôme à une variable représentées par une norme, in "Number Theory and Algebraic Geometry", ed. Miles Reid and Alexei Skorobogatov, London Math. Soc. Lecture Note Ser., 303, 2003, 69–89.
[7] J.-L. Colliot-Thélène, D. Harari and A. N. Skorobogatov, Compactification équivariante d’un tore (d’après Brylinski et Künnemann), Expo. Math. 23 (2005), 161–170.
[8] J.-L. Colliot-Thélène and P. Salberger, Arithmetic on some singular cubic hypersurfaces, Proc. London Math. Soc. (3) 58 (1989), 519–549.
[9] J.-L. Colliot-Thélène and J.-J. Sansuc, La descente sur les variétés rationnelles II, Duke Math. J. 54 (1987), 375–492.
[10] J.-L. Colliot-Thélène, J.-J. Sansuc and P. Swinnerton-Dyer, Intersections of two quadrics and Châtelet surfaces I, J. Reine Angew. Math. 373 (1987), 37–107.
[11] J.-L. Colliot-Colliot-Thélène, J.-J. Sansuc and P. Swinnerton-Dyer, Intersections of two quadrics and Châtelet surfaces II, J. Reine Angew. Math. 374 (1987), 72–168.
[12] J.-L. Colliot-Thélène, A. N. Skorobogatov and P. Swinnerton-Dyer, Rational points and zero-cycles on fibred varieties: Schinzel’s hypothesis and Salberger’s device, J. Reine Angew. Math. 495 (1998), 1–28.
[13] J.-L. Colliot-Thélène and P. Swinnerton-Dyer, Hasse principle and weak approximation for pencils of Severi-Brauer and similar varieties, J. Reine Angew. Math. 453 (1994), 49–112.
[14] U. Derenthal, A. Smeets and D. Wei, Universal torsors and values of quadratic polynomials represented by norms, Math. Ann. 361 (2014), 1021–1042.
[15] U. Derenthal and D. Wei, Strong approximation and descent, J. Reine Angew. Math. 731 (2017), 235–258.
[16] Y. Harpaz, A. N. Skorobogatov and O. Wittenberg, The Hardy-Littlewood conjecture and rational points, Compos. Math. 150 (2014), 2095–2111.
[17] R. Hartshorne, Algebraic geometry, GTM, vol. 52, Springer-Verlag, New York, 1977.
[18] D. R. Heath-Brown and A. Skorobogatov, Rational solutions of certain equations involving norms, Acta Math. 189 (2002), 161–177.
[19] U. Jannsen, The splitting of the Hochschild-Serre spectral sequence for a product of groups, Canad. Math. Bull. 33 (1990), 181–183.
[20] S. Lichtenbaum, Duality theorems for curves over p-adic fields, Invent. Math. 7 (1969), 120–136.
[21] J. Neukirch, A. Schmidt and K. Wingberg, Cohomology of Number Fields, Grundlehren der Math. 323, Springer, 2000.
[22] J.-J. Sansuc, *Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres*, J. Reine Angew. Math. 327 (1981), 12–80.
[23] M. Swarbrick Jones, *A note on a theorem of Heath-Brown and Skorobogatov*, Q. J. Math. 64 (2013), 1239–1251.
[24] A. Várilly-Alvarado and B. Viray, *Higher-dimensional analogs of Chatelet surfaces*, Bull. Lond. Math. Soc. 44 (2012), 125–135.
[25] A. Várilly-Alvarado and B. Viray, *Smooth compactifications of certain normic bundles*, Eur. J. Math. 1 (2015), 250–259.
[26] D. Wei, *On the equation $N_{K/k}(\Xi) = P(t)$*, Proc. Lond. Math. Soc. (3) 109 (2014), 1402–1434.
[27] O. Wittenberg, *Rational points and zero-cycles on rationally connected varieties over number fields*, Algebraic geometry: Salt Lake City 2015, 597–635, Proc. Sympos. Pure Math., 97, Amer. Math. Soc., Providence, RI, 2018.

HUA LOO-KENG KEY LABORATORY OF MATHEMATICS, ACADEMY OF MATHEMATICS AND SYSTEM SCIENCE, CAS, BEIJING 100190, P. R. CHINA and SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF CAS, BEIJING 100049, P.R.CHINA

*Email address: dshwei@amss.ac.cn*