Virasoro constraints in sheaf theory & Vertex algebras

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Theorem (Bojko-L-Moreira)

Virasoro constraints hold for moduli of torsion-free sheaves on any curves and surfaces with $H^1(\Theta_s) = H^2(\Theta_s) = 0$.

I. Virasoro constraints
II. Vertex algebras
III. Wall-crossing formulas
I. Virasoro constraints

\[ X \text{ smooth projective variety } / \mathbb{C} \]
\[ \alpha \in H^*(X, \mathbb{Q}) \]
\[ H \text{ ample line bundle} \]

\[ M_{x}^{ss}(\alpha) = M : \text{ moduli of } H\text{-semistable sheaves } F \]
\[ \text{s.t. } \text{ch}(F) = \alpha \]

very sensitive to \( X, \alpha \)

"less sensitive" to \( H \) \( \Rightarrow \) wall-crossing formulas

Assumptions

1. A strictly \( H\)-semistable sheaf in \( M \)
2. \( \text{Ext}^i(F, F) = 0 \) \( \forall F \in M, \forall i \geq 3 \)
3. \( \exists \) universal sheaf \( \mathcal{F} \) on \( M \times X \)

1 + 2 \( \Rightarrow \) virtual fundamental class

\[ [M]_{\text{vir}} \in A_\text{dim}(M) \xrightarrow{\chi} H_{2\text{dim}}(M, \mathbb{Z}) \xrightarrow{1 - \chi(F,F)} H_{2\text{dim}}(M^x) \]

\[ \exists \text{ natural cohomology classes } [\mathcal{F}]_{\mathcal{F}} : \text{ID}^X \rightarrow H^*(M, \mathbb{C}) \]
Def. Descendent algebra

\[ \text{Def. Descendent algebra} \quad \mathcal{D}^x := \left\langle \chi_i^H(x) \mid i \geq 0, \chi \in H^k(X, \mathbb{C}) \right\rangle \]

\[ \Rightarrow \text{formal symbols} \]

\[ \star \chi_i^H(x_1 + x_2 + x_2) = \chi_i^H(x_1) + \chi_2^H(x_2) \]

Realization homomorphism

\[ \exists \mathcal{F} : \mathcal{D}^x \rightarrow H^k(M, \mathbb{C}) \]

\[ \chi_i^H(x) \mapsto \pi_{ix} \left( \chi_i^\mathcal{F} \cdot \pi_{ix}^H(x) \right) \]

\[ i + \dim X - p \quad \text{if} \quad \chi \in H^k(X, \mathbb{C}) \]

\[ \Rightarrow \exists \mathcal{F}(\chi_i^H(x)) \in H^k(M, \mathbb{C}) \]

Virasoro operators

\[ L_n = R_n + T_n \subset \mathcal{D}^x, \quad n \geq -1 \]

\[ \star \quad R_n : \text{derivation s.t.} \]

\[ R_n \left( \chi_i^H(x) \right) = i (i+1) \cdots (i+n) \chi_i^H(x) \]

\[ \star \quad T_n : \text{multiplication by} \]

\[ T_n = \sum_{i+j=n} i ! j ! \sum_{s} (-1)^{\dim X - p} \chi_i^H(s) \chi_j^H(s) \]

where \[ \Delta_x + d(x) = \sum_{s} \chi^L_s \otimes \chi^R_s \]

Indeed, \[ [L_n, L_m] = (m-n) L_{n+m} \]
Wrong guess  \[ \sum [M]_{\text{vir}} \mathcal{E}_\text{H}(L_n(\partial)) = 0, \quad n \geq -1, \quad \forall D \in \text{ID}^X. \]

1) Introduce "correction operator"  

Conjecture ([Moreira-Oblomkov-Okounkov-Pandharipande], [M], [van Bree])  

\[ \sum [M]_{\text{vir}} \mathcal{E}_\text{H}((L_n + \frac{1}{r} S_n)(\partial)) = 0, \quad n \geq -1, \quad \forall D \in \text{ID}^X. \]

* [Moop, M]: \( M = \text{PT}_n(x, \beta), \ S^{\text{H}} \)

* [vB]: \( M = M_{\text{H}^{\text{HSS}}(r, c_1, c_2)} \) where \( \text{H}^{(\partial)} = \text{H}^{(\partial)} = 0 \)

2) Combine \( \sum L_n \) \( n \geq -1 \) to define  

\[ L_{\text{wt}_0} := \sum_{n = -1}^{\infty} \frac{(-1)^n}{(n+1)!} L_n \circ R_{-1}^{n+1} : \text{ID}^X \to \text{ID}^X, \]

\[ *= \mathcal{E}_\text{H} : \text{ID}^X_{\text{wt}_0} \to H^*(M, \mathbb{C}) \]

Conjecture (BLM)  

\[ \sum [M]_{\text{vir}} L_{\text{wt}_0}(D) = 0, \quad \forall D \in \text{ID}^X. \]

Proposition (BLM) 1) \( \iff \) 2) for old cases.

New cases: \( M = M_{\text{c}}(y, d) \), \( M_{\text{H}^{\text{HSS}}(r, \beta, \chi)} \), \( M_{\text{H}^{\text{HSS}}(r, c_1, c_2, c_3)} \)  

Fano 3-fold
Variation for pairs

\[ P^s_x(V, \sigma) = P: \text{moduli of "s-semistable" pairs} \]

\[ [V \xrightarrow{\phi} F] \text{ s.t. } \text{ch}(F) = d. \]

Assumptions

1. A strictly s-semistable pair in P
2. \( \text{Ext}^i([V \to F], F) = 0 \) unless \( i = 0, 1 \).
3. \exists! \text{ universal pair } \left[ \pi^* V \xrightarrow{\psi} F \right] \text{ on } P \times X

\[ \Rightarrow \left[ P \right]^\text{vir}, \quad \Xi_{(\pi^* V, F)} : \text{ID}^X \times \text{ID}^X \to H^*(P, \mathbb{C}) \]

Conjecture (BLM)

\[ \sum_{[P]^\text{vir}} \Xi_{(\pi^* V, F)} (L^\text{pair}_n (D)) = 0 \quad \forall n \geq 0, \forall D. \]

E.g. \( C^{[n]}, S^{[0, m]}, \text{Quot}_S(V, p, n), \quad P^t_{\text{oss}}(s) \)

\[ [\Theta_s \to F] \text{ is } \mu_\sigma \text{-semistable if} \]

* \( s \neq V \subset F \), \( \mu(G) \leq \mu(F) + \frac{t}{r(F)} \)
* \( s: \Theta_s \to V \subset F \), \( \mu(G) + \frac{t}{r(G)} \leq \mu(F) + \frac{t}{r(F)} \)
II. Vertex algebras

\[ \left( V, \, 10, \, T, \, Y(-,z) \right) \]

\( V \) - graded vector space
\( V = \bigoplus_{i \in \mathbb{Z}} V_i \)

\( 10 \) - vacuum vector
\( 10 \in V_0 \)

state-field correspondence
\( Y(a,z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1} \in \text{End}(V) \left[ [z,z^{-1}] \right] \)

transition map
\( T : V \to V_{+2} \)

Axioms for these data are related to 2d conformal field theory.

E.g. Locality: \( \forall a,b \in V. \, \exists N \text{ s.t. } (z-w)^N \left[ Y(a,z), Y(b,w) \right] = 0 \)

Example (free boson)

1. \( \mathcal{H} \) - vector space
2. \( \langle \cdot, \cdot \rangle \) symmetric non-degenerate form

1) \( V_0 := \text{Sym} \left[ \bigoplus_{k \geq 0} h t^k \right] \) where \( \text{deg}(t) = -2 \).
2) \( 10 = 1 \)
3) \( T = \text{derivation} \) s.t. \( T(a_{-k}) = k \cdot a_{-k-1} \)
4) \( \forall a \in \mathcal{H} \), \( Y(a_{-1},z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1} \in \text{End}(V_0) \left[ [z,z^{-1}] \right] \)
   \* \( \left[ a_{(n)}, b_{(m)} \right] = \delta_{n+m,0} \cdot n \cdot \langle a,b \rangle \forall n, m \in \mathbb{Z} \)
   \* \( a_{(-k-1)} = \cdot a_{-k-1} \), \( a_{(k)} |10\rangle = 0 \forall k \geq 0 \)
In many examples, \( \exists \text{ Virasoro algebra } \subset V_0 \) is a Lie algebra spanned by \( \{ \ln^z \}_{n \in \mathbb{Z}} \) & \( c \).

Recently, D. Joyce introduced sheaf theoretic vertex algebra.

\[(M_x, o, \Sigma, \Psi, \Theta) \mapsto V^{\text{Joyce}}_0 = H^\ast(M_x, \mathbb{C})\]

Crucially, \([M]_{\text{Vir}} \in H^\ast(M_x^\mathbb{C}, \mathbb{C}) = V^{\text{Joyce}}_0 := V_0^{+2}/T(V_0)\]

\( \exists \text{ Lie bracket operation } \circ \)

Def. \( w \in V_0 \) is a conformal element if

\[Y(w, z) = \sum_{n \in \mathbb{Z}} \ln^z z^{-n-2}\]

satisfies

1) \( [\ln, \ln^m] = (n-m)\ln^{n+m} + S_{m, 0} \cdot \frac{n^3 - n}{12} \cdot c \)

2) \( L_0 \subset V_0 \) diagonalizable.

3) \( L_{-1} = T \).

Example. \( V_0 = \text{Sym} \left[ \bigoplus_{k \geq 0} \mathbb{C} \right] \)

\[L_{-1} = \frac{1}{2} \sum_{a \in \mathbb{Z}} \hat{a}_- \cdot a_- \]

\( c = \dim(\mathfrak{h}) \) : central charge
Assume $X$ is either curve or surface w/ $H'(\Omega_\ast) = H'(\Omega_\ast) = 0$.

**Thm (BLM)** There is a natural conformal element

$$\mathbf{w} \in V_\ast = H_\ast(M_\times \times M_\times, \mathbb{C})$$

s.t. $H_\ast(M_a \times M_b) \otimes H_\ast(M_a \times M_b) \to \mathbb{C}$

$$\cup \cup \leftrightarrow \quad L^\ast_n: \text{Virasoro operator}$$

on $1D^\times \otimes 1D^\times$ dual from $\mathbf{w}$

**Remark** We use the computation of $H_\ast(M_a)$ by [Gross].

In particular, $1D^\times \otimes 1D^\times \overset{\cong}{\to} H_\ast(M_a \times M_b)$.

**Corollary**

1. $M$ satisfies Virasoro constraints
   $$\Leftrightarrow [M]^{\text{vir}} \in \mathfrak{P}_0 \subset \mathfrak{P}_0 (V_\ast, [\cdot, \cdot])$$

2. $P$ satisfies Virasoro constraints
   $$\Leftrightarrow L^\ast_n ([P]^{\text{vir}}) = 0 \in V_\ast, \quad \forall n > 0$$
   $$\Leftrightarrow [P]^{\text{vir}} \in \mathfrak{P}_0 \subset V_\ast$$

$\nabla\mathfrak{P}_0, \mathfrak{P}_0$: space of primary/physical states. $\mathfrak{P}_0$: Lie subalgebra $\mathfrak{P}_0$: Lie alg rep\textsuperscript{.}
III. Wall-crossing formulas

Thm (Joyce) "Wall-crossing formulas for $[M_{h}^{\text{H}-ss}(\alpha)]^{\text{inv}} \in (\tilde{\text{V}}_{\text{Joyce}}, [~,~])$ are explicitly written in terms of $[~,~]$.

Example (Simple wall-crossing)

$$F \in M_{x}^{H_{+}-ss}(\alpha) \setminus M_{x}^{H_{-}-ss}(\alpha)$$

iff $\circ \rightarrow F, \rightarrow F \rightarrow F_{2} \rightarrow \circ$, $F \in M_{x}^{H_{+}-ss}(\alpha_{i})$ 'irreducible

similarly for other complement

* with $F_{i}, F_{2}$ swapped

$$[M_{x}^{H_{+}-ss}(\alpha)]_{\text{vir}} \setminus [M_{x}^{H_{-}-ss}(\alpha)]_{\text{vir}} = \left[ [M_{x}^{H_{+}-ss}(\alpha_{i})]_{\text{vir}}, [M_{x}^{H_{-}-ss}(\alpha_{i})]_{\text{vir}} \right]$$

Theorem (Bojko-L-Moreira)

Virasoro constraints hold for moduli of torsion-free sheaves on any curves and surfaces with $H^{1}(\Theta_{s}) = H^{2}(\Theta_{s}) = 0$. 
key ideas of proof

* rank = 1 case: Virasoro constraints for $S^n$ [Moreira]

* wall-crossing [Joyce]

1) $P^0_S(\alpha) \leftrightarrow P^0_S(\alpha), \quad t \in (0, \infty)$

2) $\mathcal{S}_t = X(\alpha) \cdot \left[ M^{t*\alpha}_x(\alpha) \right]^{\text{inv}} + (\text{low rank } \alpha \text{'s})$

* Relation to Virasoro constraints [BLM]

1) wall-crossing compatibility ($\mathcal{P}_o, \mathcal{P}_o: \text{Lie algebra, repn}$)

2) projective bundle compatibility.

\[ P^0_S(\alpha) = \sum_{S^{[\alpha]}_p} \emptyset \quad \text{if } \text{rkd} > 1 \]

\[ P^0_S(\alpha) = \sum_{S^{[\alpha]}_p} \emptyset \quad \text{if } \text{rkd} = 1 \]

\[ \mathcal{S}_t \quad \text{(projective bundle compatibility)} \]

3) $\gamma_a := f_x \left( c_{t,\top}(T_f) \cap \left[ P^0_x(\alpha) \right]^{\text{inv}} \right)$

\[ = X(\alpha) \left[ M^{H_{s*\alpha}}_x(\alpha) \right]^{\text{inv}} + (\text{lower rank}) \]