Law of Large Numbers unifying Maxwell-Boltzmann, Bose-Einstein and Zipf-Mandelbrot distributions, and related fluctuations.

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Abstract

We consider a system composed of a fixed number of particles and having total energy smaller or equal to some prescribed value. The particles are non-interacting, distributed on a fixed number of energy levels. The energy levels are degenerate and degeneracy is a function of the number of particles. Three cases of the degeneration function is considered. It can increase with either the same rate as the number of particles or slower, or faster. We provide explicit points of maximum of entropy for all the cases. Depending on the total energy, the maximum can be in the interior of the system state space or on the boundary. On the boundary it can have further three cases depending on the degeneration function. The main result, Law of Large Numbers yields the most probable system states, which can become either Maxwell-Boltzmann statistics or Bose-Einstein statistics, or Zipf-Mandelbrot law. We also find the limiting laws for the fluctuations. These laws are different for various cases of the maximum point of the entropy. They can be mixture of a Normal, Exponential and Discrete distributions. Explicit rate of convergence is provided for all the theorems.

Key words — entropy, law of large numbers, Bose-Einstein statistics, Maxwell-Boltzmann statistics, Zipf-Mandelbrot law, fluctuations

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1 Introduction

The system under consideration is composed of $N$ non-interacting particles with total energy smaller or equal to $E$. The constant $E$ is the maximal average energy per particle. The particles are distributed over $m$ number of energy levels with occupations numbers given by the vector $(N_1, N_2, \ldots, N_m)$ and energies $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$. For the considered system the vector of the occupation numbers $(N_1, \ldots, N_m)$ is subject to the constraints

\begin{align*}
N &= N_1 + N_2 + \ldots + N_m, \quad (1) \\
EN &\geq \varepsilon_1 N_1 + \varepsilon_2 N_2 + \ldots + \varepsilon_m N_m. \quad (2)
\end{align*}
Each energy level has degeneracy, where \( i = 1, \ldots, m \). The particles are indistinguishable within a single level. The degeneracy influences the entropy of the system, which is

\[
S(N_1, \ldots, N_m) = \sum_{i=1}^{m} \ln \frac{(N_i + G_i - 1)!}{N_i!(G_i - 1)!}.
\]

The total degeneracy of the levels is \( G = \sum_{i=1}^{m} G_i \) and levels are ordered according to their energies i.e. \( \varepsilon_1 < \varepsilon_2 < \ldots < \varepsilon_m \). We assume that the degeneracy is an increasing function of the number of particles, i.e. \( G = G(N) \). We study three regimes of the behavior of \( G(N) \)

1) \( \lim_{N \to \infty} \frac{G(N)}{N} = \infty \),
2) \( \frac{G(N)}{N} = \alpha + \beta(N) \), where \( \alpha > 0 \), \( \lim_{N \to \infty} \beta(N) = 0 \),
3) \( \lim_{N \to \infty} \frac{G(N)}{N} = 0 \).

Moreover, for each \( N \) the components \( G_i \) are equally weighted and their number \( m \) remains constant. Which means that for all \( N \), \( G_i = g_i G(N) \), for \( i = 1, \ldots, m \) and some constants \( g_i \) such that \( 1^T g = 1 \) where \( g = (g_1, g_2, \ldots, g_m) \) and \( 1 = (1, 1, \ldots, 1) \). For \( G_i = g_i G(N) \) the entropy \( S(x, N) \) is given by

\[
S(x, N) = \sum_{i=1}^{m} \ln \frac{(x_i + g_i G(N) - 1)!}{(x_i N)! (g_i G(N) - 1)!},
\]

where \( x_i \)'s are weights of their distribution on the energy levels i.e. \( N_i = x_i N \).

We consider a discrete random vector \( X(N) = (X_1, X_2, \ldots, X_m) \), where \( X_i(N) = N_i/N \), \( i = 1, \ldots, m \) with sample space being systems states. Clearly, \( X(N) \in L_N \) where \( L_N := \left\{ \frac{x}{N}, x \in \mathbb{N}^m \right\} \). We define a set \( A_E \) to be the subset of \( \mathbb{R}^m \) such that the following constraints are valid

\[
1 = x_1 + x_2 + \ldots + x_m,
E \geq \varepsilon_1 x_1 + \varepsilon_2 x_2 + \ldots + \varepsilon_m x_m,
\]

\[
x_i \geq 0, \quad i = 1, \ldots, m.
\]

Therefore, the sample space of \( X(N) \) is the set \( A_E \cap L_N \). When \( E = \varepsilon_1 \), the set \( A_E \) is a single point. When \( E < \varepsilon_1 \), \( A_E \) is an empty set. Let us assume the parameter \( E \) is larger than \( \varepsilon_1 \).

We use a fundamental postulate of the statistical mechanics, see e.g. Pathria and Beale [2011] and Reif [2013], that system microstates are equally probable and define the pmf of \( X(N) \) to be

\[
P(X(N) = x) := \frac{e^{S(x, N)}}{\sum_{A_E \cap L_N} e^{S(y, N)}}.
\]

where the entropy \( S(x, N) \) is given by (5).

Our law of large numbers yields the most probable state of random vector \( X(N) \) as the number of particles goes to infinity. This state is the point of maximum of the entropy. When the maximum is in the interior of the domain, it is the point \( x^* = g \). When the maximum is on the boundary, for the three cases of \( G(N) \) we obtain points that are
the distributions: Maxwell-Boltzmann, Bose-Einstein and Zipf-Mandelbort. Our second result yields the distributions of fluctuations from the most probable system state. They are different for two cases of the entropy maximum. When the maximum is in the interior of the domain, fluctuations have Normal distribution. When the maximum is on the boundary, there can be further two cases depending on the degeneracy function. For the first and the second case in (4), the fluctuations distribution is Exponential in the direction orthogonal to the boundary on state space and Normal in other directions. For the third case fluctuations distribution is Discrete in direction orthogonal to the boundary and Normal in other directions. Explicit rate of convergence is provided for all the limit theorems.

The initial idea for this work was a part of the authors PhD thesis, Lapiński [2014]. There the considered system is introduced together with the draft of the systems entropy properties and the draft of the solution of the optimization problem. The thesis lacks the most important part, the rigorous approximation of the sum of states.

The same system, also with increasing degeneracy, was introduced in Maslov [2005a]. There the author stated existence of the two cases of maximum of the entropy without the proof. When the entropy maximum is on the boundary and when \( G(N) \) increases with the same rate as \( N \), optimization problem similar to ours was solved in Maslov [2004]. In Maslov [2005a] and Maslov [2005b] the author also proved convergence to the three distributions: Maxwell-Boltzmann, Bose-Einstein and Zipf-Mandelbort. However, for the situation when maximum is on the boundary only. The case when the maximum is in the interior was omitted. For the results on the fluctuations of related Bose-Einstein condensate see Chatterjee and Diaconis [2014]. Our results on the fluctuations are completely new. The proofs of the law of large numbers and the fluctuations theorems are completely different from one in Maslov [2005a]. They are based on the results from Lapiński [2015], which are summarized in the Appendix. There the methodology of the Laplace’s Method for integrals is developed for the sums.

## 2 Entropy properties

For some \( a > 0 \), let us define the set \( \mathcal{A}_E^{(a)} := \{ x : x \in \mathcal{A}_E, x_i > a; i = 1, \ldots, m \} \).

**Proposition 1.** The entropy \( S(x, N) \) given by (5) is three times differentiable and strictly concave on \( \mathcal{A}_E \). Moreover, for large enough \( N \), and on the set \( \mathcal{A}_E^{(a)} \) the following approximations hold

\[
S(x, N) = h(N)f(x, N) + C(N), \text{ with } f(x, N) = f(x) + \sigma(x, N)\epsilon(N),
\]

where

i) \( f(x, N) \) and \( f(x) \) are strictly concave and three times differentiable functions,

ii) \( h(N) \) and \( C(N) \) are some functions of \( N \),

iii) \( \sigma(x, N) \) and its derivatives up to third order are uniformly bounded.

Furthermore, for each case of \( G(N) \) given by (4) we have following explicit form of the above functions
\begin{enumerate}

1) \[ h(N) = N, \quad C(N) = N \left( \ln \frac{G(N)}{N} + 1 \right) - \frac{1}{2} \ln N, \]
\[ f(x) = \sum_{i=1}^{m} x_i \ln \frac{g_i}{x_i}, \]
\[ \epsilon(N) = \begin{cases} \frac{1}{N} & \text{if } \lim_{N \to \infty} \frac{N^2}{G(N)} = 0, \\ \frac{1}{|\beta(N)|} & \text{otherwise.} \end{cases} \]

2) \[ h(N) = N, \quad C(N) = -N \sum_{i=1}^{m} g_i \alpha \ln g_i \alpha - \frac{1}{2} \ln N, \]
\[ f(x) = \sum_{i=1}^{m} \left( (x_i + g_i \alpha) \ln(x_i + g_i \alpha) - x_i \ln x_i \right), \]
\[ \epsilon(N) = \begin{cases} \frac{1}{N} & \text{if } \lim_{N \to \infty} \beta(N)N = 0, \\ \frac{1}{|\beta(N)|} & \text{otherwise.} \end{cases} \]

3) \[ h(N) = G(N), \quad C(N) = G(N)(\ln N + 1) - \sum_{i=1}^{m} \left( g_i G(N) + \frac{1}{2} \right) \ln g_i G(N) - \ln N, \]
\[ f(x) = \sum_{i=1}^{m} g_i \ln x_i, \]
\[ \epsilon(N) = \begin{cases} \frac{1}{G(N)} & \text{if } \lim_{N \to \infty} \frac{G(N)^2}{N} = 0, \\ \frac{1}{G(N)} & \text{otherwise.} \end{cases} \]

**Proof.** Let us recall the Stirling’s approximation \[ \ln(\lambda!) = \left( \lambda + \frac{1}{2} \right) \ln \lambda - \lambda + \ln \sqrt{2\pi} + \frac{\omega(\lambda)}{\lambda}. \]

The remainder \( \omega \) is analytic function and \( \omega(\lambda) = O(1) \) as \( \lambda \to \infty \).

We use the Stirling’s approximation to obtain
\[ S(x, N) = \sum_{i=1}^{m} \left[ \left( x_i N + g_i G(N) - \frac{1}{2} \right) \ln(x_i N + g_i G(N)) - \left( x_i N + \frac{1}{2} \right) \ln x_i N \right. \]
\[ \left. - \left( g_i G(N) - \frac{1}{2} \right) \ln(g_i G(N)) + \omega(x_i, N) \right], \]

where the remainder term
\[ \omega(x_i, N) = \left( x_i N + g_i G(N) - \frac{1}{2} \right) \ln \left( 1 - \frac{1}{x_i N + g_i G(N)} \right) - \ln \sqrt{2\pi} \]
\[ - \left( g_i G(N) - \frac{1}{2} \right) \ln \left( 1 - \frac{1}{g_i G(N)} \right) + \frac{\omega_1(x_i, N)}{x_i N + g_i G(N)} - \frac{\omega_2(x_i, N)}{x_i N} - \frac{\omega_3(N)}{g_i G(N) - 1}. \]
More precisely, for \( a \in (0, 1) \) there exists \( N_0 \in \mathbb{Z}_+ \) such that this approximation holds for \( x_i \in (a, 1] \) and \( N \geq N_0 \), and hence for \( x \in \mathcal{A}_E^{(a)} \).

Let us perform calculations separately for each case of \( G(N) \) given by (4).

1) Using (9) the function \( S(x, N) \) can be presented as

\[
S(x, N) = Nf(x, N) + N \left( \ln \frac{G(N)}{N} + 1 \right) - \frac{1}{2} \ln N,
\]

where

\[
f(x, N) = \sum_{i=1}^{m} \left[ x_i \ln \frac{g_i}{x_i} + \left( x_i + \frac{g_i G(N)}{N} - \frac{1}{2N} \right) \ln \left( 1 + \frac{x_i N}{g_i G(N)} \right) - x_i - \frac{1}{2N} \ln x_i \right] + \omega(x_i, N).
\]

Then the function \( f(x, N) \) can be presented as

\[
f(x, N) = f(x) + \sigma(x, N)\epsilon(N),
\]

where

\[
f(x) = \sum_{i=1}^{m} x_i \ln \frac{g_i}{x_i},
\]

\[
\sigma(x, N) = \frac{1}{\epsilon(N)} \sum_{i=1}^{m} \left( x_i + \frac{g_i G(N)}{N} - \frac{1}{2N} \right) \ln \left( 1 + \frac{x_i N}{g_i G(N)} \right) - x_i - \frac{1}{2N} \ln x_i + \omega(x_i, N),
\]

\[
\epsilon(N) = \begin{cases} \frac{N}{G(N)} & \text{if } \lim_{N \to \infty} \frac{N^2}{G(N)} = 0, \\ \text{otherwise.} & \end{cases}
\]

By approximating the logarithm \( \ln(1 + x) = x + O(x^2), x \to 0 \), one can verify that \( \sigma(x, N) = O(1) \) as \( N \to \infty \), and \( \sigma(x, N) \) is uniformly bounded on \( \mathcal{A}_E^{(a)} \).

2) Analogical to 1), by (9),

\[
S(x, N) = Nf(x, N) - N \sum_{i=1}^{m} g_i \alpha \ln g_i \alpha - \frac{1}{2} \ln N,
\]

where

\[
f(x, N) = \sum_{i=1}^{m} \left[ \left( x_i + \frac{g_i G(N)}{N} + \frac{1}{2N} \right) \ln \left( x_i + \frac{g_i G(N)}{N} \right) - x_i \ln x_i - \frac{1}{2N} \ln x_i - \left( \frac{g_i G(N)}{N} - \frac{1}{2N} \right) \ln \frac{g_i G(N)}{N} + g_i \alpha \ln g_i \alpha + \frac{\omega(x_i, N)}{N} \right],
\]

and also

\[
f(x, N) = f(x) + \sigma(x, N)\epsilon(N),
\]
where
\[
\begin{align*}
  f(x) &= \sum_{i=1}^{m} (x_i + g_i \alpha) \ln(x_i + g_i \alpha) - x_i \ln x_i, \\
  \sigma(x, N) &= \frac{1}{\epsilon(N)} \sum_{i=1}^{m} \left[ (x_i + g_i \alpha) \ln \left( 1 + \frac{g_i \beta(N)}{x_i + g_i \alpha} \right) + \left( g_i \beta(N) + \frac{1}{2N} \right) \ln \left( x_i + \frac{g_i G(N)}{N} \right) \right] \\
  &\quad - \frac{1}{2N} \ln x_i - \left( \frac{g_i G(N)}{N} - \frac{1}{2N} \right) \ln \frac{g_i G(N)}{N} + g_i \alpha \ln g_i \alpha + \frac{\omega(x_i, N)}{N}, \\
  \epsilon(N) &= \begin{cases} 
    \frac{1}{N} & \text{if } \lim_{N \to \infty} \beta(N) N = 0, \\
    |\beta(N)| & \text{otherwise}.
  \end{cases}
\end{align*}
\]

3) Again, analogously to 1) and 2) by (9) we have
\[
S(x, N) = G(N) f(x, N) + G(N) (\ln N + 1) - \sum_{i=1}^{m} \left( g_i G(N) - \frac{1}{2} \right) \ln(g_i G(N)) - \ln N,
\]
where
\[
f(x, N) = \sum_{i=1}^{m} \left[ g_i \ln x_i + \left( \frac{x_i N}{G(N)} + g_i - \frac{1}{2G(N)} \right) \ln \left( 1 + \frac{g_i G(N)}{x_i N} \right) - g_i - \ln x_i \frac{\omega(x_i, N)}{G(N)} \right].
\]

Then
\[
f(x, N) = f(x) + \sigma(x, N) \epsilon(N),
\]
where
\[
f(x) = \sum_{i=1}^{m} g_i \ln x_i,
\]
\[
\sigma(x, N) = \frac{1}{\epsilon(N)} \sum_{i=1}^{m} \left[ \left( \frac{x_i N}{G(N)} + g_i - \frac{1}{2G(N)} \right) \ln \left( 1 + \frac{g_i G(N)}{x_i N} \right) - g_i - \ln x_i \frac{\omega(x_i, N)}{G(N)} \right],
\]
\[
\epsilon(N) = \begin{cases} 
    \frac{1}{G(N)} & \text{if } \lim_{N \to \infty} \frac{G(N)^2}{N} = 0, \\
    \frac{G(N)}{G(N)} & \text{otherwise}.
  \end{cases}
\]

For all the three cases, the functions \( S(x, N), f(x, N), f(x) \) are three times differentiable.

Now, denote the usual Gamma function \( \Gamma(\lambda) \) with \( \lambda \in \mathbb{R}_+ \). It is well know that \( \ln \Gamma(\lambda) \) is a strictly convex function. Its second derivative is obtained by differentiating the Digamma function \( \psi(\lambda) = \Gamma'(\lambda)/\Gamma(\lambda) \)
\[
\frac{d^2 \ln \Gamma(\lambda)}{d\lambda^2} = \frac{d\psi(\lambda)}{d\lambda} = \sum_{i=0}^{\infty} \frac{1}{(\lambda + i)^2} > 0.
\]

For the series representation of the Digamma function see Abramowitz and Stegun [1965] p. 259.

Functions \( S(x, N), f(x, N) \) and \( f(x) \) are the sums of \( n \) components, each depending only on single \( x_i \)'s, \( i = 1, \ldots, n \) hence their Hessians are a diagonal matrices. Additionally,
we use (10) to calculate the derivatives of $S(x, N)$ given by (5). This yields that for large enough $N$, the second derivatives of $S(x, N)$ w.r.t. $x_i$’s are negative. Moreover, $f(x)$ has also second derivatives of $x_i$’s negative. Therefore the Hessians of $S(x, N)$ and $f(x, N)$ are negative definite and those function are strictly concave. Since $f(x, N) = f(x) + \sigma(x, N)e(N)$ and $\sigma(x, N) = O(1)$ as $N \to \infty$, one can infer that $\sigma(x, N)e(N) \to 0$ as $N \to \infty$. Therefore, for large enough $N$, the function $f(x, N)$ is also strictly concave. 

Function $\sigma(x, N)$ is analytic as it is composed of analytic functions. It can be easily verified that $\sigma$ is also uniformly convergent as $N \to \infty$. Therefore, by the Theorem in Lang [1999] p. 157, one has that the derivative of $\sigma(x, N)$ converges uniformly as $N \to \infty$. Therefore, one can conclude that all the derivatives of $\sigma$ are uniformly bounded.

Proposition 2. The functions $f(x, N)$ and $f(x)$ from the Preposition have a unique maximum at $x^*(N)$ and $x^*$ respectively. Furthermore,

1) if $E > \varepsilon g$, then the points $x^*(N), x^*$ are in the interior of $A^{(a)}_E$ and $x^* = g$,

2) if $\varepsilon_1 < E < \varepsilon g$, then $x^*(N), x^*$ are noncritical points on the boundary $\varepsilon^T x^* = E$ of the set $A^{(a)}_E$. For each case of $G(N)$ given by (4), $x^*$ has the components

\[
1) \quad x^*_i = \frac{g_i}{e^{\lambda \varepsilon_i} + \nu}, \\
2) \quad x^*_i = \frac{g_i \alpha}{e^{\lambda \varepsilon_i} + \nu - 1}, \\
3) \quad x^*_i = \frac{g_i}{\lambda \varepsilon_i + \nu},
\]

for $i = 1, \ldots, m$, where the parameters $\lambda > 0$, $\nu$ are uniquely determined by the equations

\[
1^T x^* = 1, \quad \varepsilon^T x^* = E,
\]

3) if $E = \varepsilon^T g$ then the point $x^*$ is on the boundary. For the first case of $G(N)$, $x^*$ is a critical point of $f(x)$. For the second and third case $x^*$ is a noncritical point of $f(x)$.

Proof. Let us consider the following convex optimization problem on $\mathbb{R}^m$

\[
\begin{align*}
\text{maximize} & \quad f(x), \\
\text{subject to} & \quad \varepsilon^T x - E \leq 0, \\
& \quad 1^T x - 1 = 0, \\
& \quad x_i \geq 0, \quad i = 1, \ldots, m.
\end{align*}
\]

By convexity, problem (11) can have at most one solution. For $E > \varepsilon_1$, the Slater’s condition holds and the optimal vector exists. Therefore, the Karush-Kuhn-Tucker (KKT) conditions are necessary and sufficient for the existence and uniqueness of the optimal vector. For the details on above Optimization Theory see Boyd and Vandenberghe [2004], Chapter 5.

The KKT conditions for the problem (11) are the following

\[
\begin{align*}
\varepsilon^T x^* - E & \leq 0, \\
1^T x^* - 1 & = 0, \\
\lambda & \geq 0, \\
\lambda (\varepsilon^T x^* - E) & = 0, \\
Df(x^*) - \lambda \varepsilon - \nu & = 0.
\end{align*}
\]
The last condition for each case of $G(N)$ given by (4) yields
\begin{align}
1) \quad & x^*_i = \frac{g_i}{\varepsilon \lambda_i + \nu}, \\
2) \quad & x^*_i = \frac{g_i c}{\varepsilon \lambda_i + \nu - 1}, \\
3) \quad & x^*_i = \frac{g_i}{\lambda_i + \nu}.
\end{align}

For the analysis of the conditions (12) two cases can be distinguished, when $\lambda = 0$ and when $\lambda > 0$. In the first case we have
\begin{align}
\varepsilon^T x^* - E & \leq 0, \\
1^T x^* - 1 & = 0, \\
\lambda & = 0.
\end{align}

From (15) and (16) we get that the solution of the above system is $x^* = g$. Substituting that into (14) we obtain that $E \geq \varepsilon^T g$. When $E = \varepsilon^T g$, the maximum of $f$ is on the boundary $\varepsilon^T g = E$. For the first case of $G(N)$, maximum is at the critical point of $f$, since $(\lambda, \nu) = (0, 0)$ and $Df(x^*) = 0$. For the second and third case, the maximum of $f$ is at a noncritical point, as there $Df(x^*) \neq 0$. If $E > \varepsilon^T g$, then the maximum of $f$ is in the interior of $A_E^{(2)}$.

When $\lambda > 0$, we have the following system of equations
\begin{align}
\varepsilon^T x^* - E & = 0, \\
1^T x^* - 1 & = 0, \\
\lambda & > 0,
\end{align}

which has a unique solution when $\varepsilon_1 < E < \varepsilon^T g$. Substituting (13) into (17) and (18) we get the system of equations from which we can obtain the parameters $\lambda$ and $\nu$. From (17) we infer that the maximum is on the boundary. Furthermore, the maximum is at a noncritical point as $Df(x^*) \neq 0$.

Then we consider the optimization problem (11) for the function $f(x, N)$ instead of $f(x)$. The solution of this problem is analogous. The maximum $x^*(N)$ and the value of the parameter $E$ which separates the two cases of $\lambda$ depends on $N$. Those two values converges to $x^*$ and $\varepsilon^T g$ as $N \to \infty$.

\section{Limit theorems}

It is convenient to denote $X'(N) := (X_1(N), X_2(N), \ldots, X_{m-1}(N))$ and $x' := (x_1, x_2, \ldots, x_{m-1})$. The pmf of $X'(N)$ is the following
\begin{equation}
P(X'(N) = x') := \frac{e^{S(x', N)}}{\sum_{A_{E \cap L_N}} e^{S(y', N)}},
\end{equation}
where $S(x', N) := S((x_1, \ldots, x_{m-1}, 1 - \sum_{i=1}^{m-1} x_i), N)$. Now, let us introduce a preliminary result needed for the proofs of the limit theorems.
Proposition 3. For large enough \( N \), the following approximation of the pmf’s (8) and (20) holds

\[
P(X(N) = x) = P(X'(N) = x') =
\begin{cases}
\sum_{\mathcal{A}_E \cap L_N} e^{S(x,N)} & \text{if } X'(N) \in \mathcal{A}_E(a) \cap L_N, \\
O(1)e^{-h(N)\Delta} & \text{if } X'(N) \in (\mathcal{A}_E \setminus \mathcal{A}_E(a)) \cap L_N,
\end{cases}
\]

where \( \Delta > 0 \) is some constant, functions \( h, f \) are given by the Proposition 2 and \( f(x', N) := f((x_1, \ldots, x_{m-1}, 1 - \sum_{i=1}^{m-1} x_i), N) \).

Proof. We start by noticing that \( P(X'(N) = x') = P(X(N) = x) \) because \( X_m(N) \) can be represented in terms of the first \( m - 1 \) components of \( X(N) \), that is \( X_m(N) = 1 - \sum_{i=1}^{m-1} X_i(N) \) on \( \mathcal{A}_E \). Then, let us approximate the pmf (8) for \( X(N) \in \mathcal{A}_E(a) \cap L_N \).

First, we decompose the denominator

\[
\sum_{\mathcal{A}_E \cap L_N} e^{S(x,N)} = \sum_{\mathcal{A}_E(a) \cap L_N} e^{h(N)f(x,N)+C(N)} + \sum_{(\mathcal{A}_E \setminus \mathcal{A}_E(a)) \cap L_N} e^{S(x,N)}, \tag{21}
\]

where in the first sum we used the approximation of \( S(x, N) \) from the Proposition 1.

In the Proposition 2 the function \( f(x, N) \) is defined on \( \mathcal{A}_E(a) \). So, let us define \( f(x, N) \) on \( \mathcal{A}_E \setminus \mathcal{A}_E(a) \) by

\[
f(x, N) := \frac{1}{h(N)}(S(x, N) - C(N)),
\]

which is a strictly concave and differentiable function. Next, we apply the Taylor’s Theorem for \( f(x, N) \) at \( x \in (\mathcal{A}_E \setminus \mathcal{A}_E(a)) \cap L_N \)

\[
f(x, N) = f(x^*, N) + Df(x_\theta(N), N)(x^* - x),
\]

where \( x_\theta(N) \) is some point between \( x \) and \( x^* \). By the properties of the function \( f(x, N) \) we have

\[
0 < \Delta' < |Df(x_\theta(N), N)(x^* - x)| \leq \sup_{x \in \mathcal{A}_E, N \geq N_0} \|Df(x, N)\|\sqrt{m} \leq \Delta,
\]

where \( 0 < \Delta' < \Delta \) and \( N_0 \in \mathbb{Z}_+ \) are some constants. Therefore

\[
f(x, N) = f(x^*, N) - |Df(x_\theta(N), N)(x^* - x)| \geq f(x^*, N) - \Delta,
\]

and

\[
f(x, N) \leq f(x^*, N) - \Delta'. \tag{22}
\]

Then the second sum in (21) has a lower bound

\[
\sum_{(\mathcal{A}_E \setminus \mathcal{A}_E(a)) \cap L_N} e^{S(x,N)} \geq e^{h(N)f(x^*,N)+C(N) - h(N)\Delta} \sum_{(\mathcal{A}_E \setminus \mathcal{A}_E(a)) \cap L_N} 1.
\]

The sum on the r.h.s. is bounded

\[
\sum_{(\mathcal{A}_E \setminus \mathcal{A}_E(a)) \cap L_N} \sum_{\{x:0 < x_i \leq 1: i=1,\ldots,m\} \cap L_N} 1 = N^m.
\]
Hence
\[ \sum_{(A_E^o \cap A_E^s) \cap L_N} = O(1)N^m. \]

Combining above estimates yields
\[ \sum_{A_E \cap L_N} e^{S(x,N)} \geq \sum_{A_E^o \cap L_N} e^{h(N)f(x,N) + \omega(N)e^C(N)}, \] (23)

where
\[ \omega(N) := O(1)N^m e^{h(N)f(x,N) - h(N)\Delta}. \]

Hence the pmf for \( X(N) \in A_E^o \cap L_N \) can be estimated
\[ \frac{e^{S(x,N)}}{\sum_{A_E \cap L_N} e^{S(y,N)}} \leq \frac{e^{h(N)f(x,N)}}{\sum_{A_E^o \cap L_N} e^{h(N)f(y,N)}} \left( 1 - \frac{\omega(N)}{\sum_{A_E^o \cap L_N} e^{h(N)f(y,N)} + \omega(N)} \right). \] (24)

Now, using the Theorem 5 and 6 from the appendix we obtain
\[ \sum_{A_E^o \cap L_N} e^{h(N)f(x,N)} \geq O(1)N^m e^{h(N)f(x,N)} h(N)^{m+1}, \] (25)

and substituting that into (24) yields an estimate
\[
\left| \frac{e^{S(x,N)}}{\sum_{A_E \cap L_N} e^{S(y,N)}} - \frac{e^{h(N)f(x,N)}}{\sum_{A_E^o \cap L_N} e^{h(N)f(y,N)}} \right| \leq \frac{\omega(N)}{\sum_{A_E^o \cap L_N} e^{h(N)f(y,N)}} \leq \frac{\omega(N)h(N)^{m+1}}{\sum_{A_E^o \cap L_N} e^{h(N)f(y,N)}}
\]
\[
\leq \frac{e^{h(N)f(x,N)}}{\sum_{A_E^o \cap L_N} e^{h(N)f(y,N)}} O(1)e^{-h(N)\Delta''},
\]

where \( 0 < \Delta'' < \Delta' \) is some constant. Since \( f(x,N) = f((x_1, \ldots, x_{m-1}, 1 - \sum_{i=1}^{m-1} x_i), N) \), hence we obtain the first part of the Proposition.

When \( X(N) \in (A_E \setminus A_E^o) \cap L_N \) the pmf [3] can be approximated analogous. The lower bound of the denominator is also [23]. For the numerator, we use [22] and [25] to obtain an estimate
\[
\frac{e^{S(x,N)}}{\sum_{A_E \cap L_N} e^{S(y,N)}} \leq \frac{e^{h(N)f(x^*,N) - h(N)\Delta'}}{\sum_{A_E^o \cap L_N} e^{h(N)f(y,N) + \omega(N)}} \leq O(1)e^{-h(N)\Delta'} h(N)^{m+1} \leq O(1)e^{-h(N)\Delta''}.
\]

Hence the Proposition is proved.

Next, we prove the limit theorems where the vector \( x^* \) is given by the Proposition 2. In the results we include estimates valid for sufficiently large \( N \), where the parameter \( \delta \) has values in the interval \((0, \frac{1}{3(m+1)})\), and the functions \( f(x) \), \( h(N) \), \( \epsilon(N) \) are given by the Proposition 1.

**Theorem 1** (Weak law of large numbers). The random vector \( X(N) \) converges weakly to \( x^* \) and the following estimate of the mgf holds
\[
M_{X(N)}(\xi) = e^{\xi T x^*} \left( 1 + \frac{O(1)}{h(N)^{1/2-3\delta}} + O(1)\frac{h(N)^{1/2+(m+1)\delta}}{N} + O(1)\epsilon(N) \right).
\]
Remark 1. For this and the following limit theorems the convergence error term can be explicitly estimated using the results in [Lapinski, 2013].

For each case (i), let us assume $G(N)$ has following properties

\[ 1) \lim_{N \to \infty} \frac{N^\frac{3}{2}}{G(N)} = 0, \]
\[ 2) \lim_{N \to \infty} \frac{\beta(N)\sqrt{N}}{G(N)} = 0, \]
\[ 3) \lim_{N \to \infty} \frac{\sqrt{N}G(N)^\frac{3}{4}}{N} = 0. \]

Theorem 2 (Central limit theorem I). Let $E > \varepsilon^T g$. For $X'(N)$ the random vector $Z(N) = \sqrt{h(N)}(x^* - X'(N))$ converges weakly to $N(0, D^2_f(x^*)^{-1})$ and the following estimate of the mgf holds

\[ M_{Z(N)}(\xi) = \exp \left( \frac{1}{2} \xi^T D^2_f(x^*)^{-1} \xi \right) \left( 1 + \frac{O(1)}{h(N)^{1/2 - \delta}} + \frac{O(1)}{N\sqrt{h(N)}} \frac{h(N)^{1/2 +(m+1)\delta}}{N} \right). \]

Here let us introduce notation $\xi_y = (\xi_2, \ldots, \xi_{m-1})$, $Y = (X_2(N), \ldots, X_{m-1}(N))$ and $y^* = (x_2^*, \ldots, x_{m-1}^*)$.

Theorem 3 (Central limit theorem II). Let $\varepsilon_1 < E < \varepsilon^T g$ and consider the first or the second case of $G(N)$ in (i). For $X'(N)$ exists a subsequence of $N$ such that the random vector $Z(N) = (N(x_1^* - X_1(N)), \sqrt{N}(y^* - Y(N)))$ converges weakly to a discrete distribution with pmf

\[ P(Z_1(N) = i) = \exp \left( - \left| \frac{\partial f(x^*)}{\partial x_1} \right| i \right) \left( 1 - \exp \left( - \left| \frac{\partial f(x^*)}{\partial x_1} \right| \right) \right), \]

for $Z_1(N)$ and to $N(0, D^2_y f(x^*)^{-1})$ for $(Z_2(N), \ldots, Z_{m-1}(N))$.

Furthermore, the following estimate of the mgf holds

\[ M_{Z(N)}(\xi) = \frac{1 - \exp \left( - \left| \frac{\partial f(x^*)}{\partial x_1} \right| \right)}{1 - \exp \left( - \left| \frac{\partial f(x^*)}{\partial x_1} \right| - \xi_1 \right)} \exp \left( \frac{1}{2} \xi_y^T D^2_y f(x^*)^{-1} \xi_y \right) \times \left( 1 + \frac{O(1)}{N^{1/2 - \delta}} + \frac{O(1)}{N^{1/2 +(m+1)\delta}} + \frac{\epsilon(N)\sqrt{NO(1)}}{N} \right). \]

Theorem 4 (Central limit theorem III). Let $\varepsilon_1 < E < \varepsilon^T g$ and consider the third case of $G(N)$ in (i). For $X'(N)$ exists a subsequence of $N$ such that the random vector $Z(N) = (G(N)(x_1^* - X_1(N)), \sqrt{G(N)}(y^* - Y(N)))$ converges weakly to $\exp \left( \left| \frac{\partial f(x^*)}{\partial x_1} \right| \right)$ for $Z_1(N)$ and to $N(0, D^2_y f(x^*)^{-1})$ for $(Z_2(N), \ldots, Z_{m-1}(N))$. Furthermore, the following estimate of the mgf holds

\[ M_{Z(N)}(\xi) = \frac{\left| \frac{\partial f(x^*)}{\partial x_1} \right|}{\left| - \frac{\partial f(x^*)}{\partial x_1} - \xi_1 \right|} \exp \left( \frac{1}{2} \xi_y^T D^2_y f(x^*)^{-1} \xi_y \right) \times \left( 1 + \frac{O(1)}{G(N)^{1/2 - \delta}} + \frac{O(1)}{G(N)^{1/2 +(m+1)\delta}} + \frac{\epsilon(N)\sqrt{G(N)}}{N} \right). \]
Remark 2. Theorem 3 and 4 are valid only for particular subsequence of $N$, which can be found explicitly. For this subsequence the part of the boundary of $A_E$ on which is maximum of $f(x,N)$ is attained, coincide with the boundary points of the set $A_E \cap L_N$. For the details how to find this subsequence see Remark 4 in the Appendix.

Proof of Theorem 1, 2, 3, 4. Let us consider the random vector $X'(N)$. We approximate the pmf of $X'(N)$ given by (20) using Proposition 5. Then with use of the Proposition 1 and 2 for each Theorem we recognize the properties of the functions $h(x)$ and $f(x,N)$ in the approximated pmf.

The approximated pmf of $X'(N)$ differs from the pmf (29) or (30) by exponentially small term. Therefore, we can apply the Theorems 7, 8, 9, 10 for $X'$ respectively, Theorems 1, 2, 3, 4.

For the Theorem 1 we get the mgf of $X(N)$ from the mgf of $X'(N)$ in the following way

$$M_{X(N)}(\xi) = E[e^{\xi^T X(N) + \xi_m (1-X_1,\ldots,-X_{n-1})}] = E[e^{\xi^T X(N) + \xi_m}] = e^{\xi m} M_{X'(N)}(\xi^T - \xi_m),$$

and mgf of $x^*$ from $x^*_{\ast}$

$$e^{\xi^T x^*} = e^{\xi^T x^* + \xi_m x^*_{\ast}} = e^{\xi^T x^* + \xi_m (1-x_1^*,\ldots,-x_{n-1}^*)} = e^{\xi m} e^{(\xi^T - \xi_m) x^*}.$$

Hence we proved the Theorems.

\[ \square \]

A Appendix

Let us recall results from Lapiński [2015] with slightly relaxed assumptions. We consider a bounded, open set $A \subset \mathbb{R}^m$ and a lattice $L_N := \{x/N, x \in \mathbb{N}^m \}$ with $N \in \mathbb{Z}_+$. Then we introduce a function $f : A \times \mathbb{Z}_+ \rightarrow \mathbb{R}$ which have a unique maximum at $x^*(N)$ and is differentiable up to third order. We assume $\lim_{N \rightarrow \infty} x^*(N) = x^*$ and choose $x^*$ to be the origin of our coordinate system.

Additionally $f$ can be represented

$$f(x,N) = f(x) + \sigma(x,N)\epsilon(N), \quad \text{(26)}$$

where $f(x),\sigma(x,N)$ are three times differentiable w.r.t. $x$ and $\epsilon(N) > 0$, $\epsilon(N) \rightarrow 0$ as $N \rightarrow \infty$. Moreover, $\sigma(x,N)$ and its derivatives are uniformly bounded. Further, we introduce a positive, increasing function $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $\lim_{N \rightarrow \infty} h(N)/N = 0$ or $h(N) = N$ and a differentiable function $g : A \rightarrow \mathbb{R}$. Then

(a) for the sum

$$\Sigma(N) := \sum_{\mathcal{A}\cap L_N} g(x) e^{h(N)f(x,N)}, \quad \text{(27)}$$

we assume $f(\cdot,N)$ and $f(\cdot)$ has a unique nondegenerate maximum in the interior of $A$.

(b) for the sum

$$\Sigma(N) := \sum_{\mathcal{A}\cap L_N \cap \{x:x_1 \geq 0\}} g(x) e^{h(N)f(x,N)}, \quad \text{(28)}$$

we assume $f(\cdot,N)$ and $f(\cdot)$ has a unique maximum on the boundary $\{x : x_1 = 0\}$. Additionally, $\frac{\partial f(x^*(N),N)}{\partial x_1} < 0$, $\frac{\partial f(x^*)}{\partial x_1} < 0$ and w.r.t. coordinates $(x_2,\ldots,x_m)$ those functions have a nondegenerate maximum at $x^*(N)$ and $x^*$ respectively. We also assume that on every hyperplane parallel to boundary $\{x : x_1 = 0\}$, there is a unique nondegenerate maximum.
Remark 3. If the functions $f(x, N)$ and $f(x)$ are strictly concave then assumptions in the point (b) above reduces to having maximum at a noncritical point in the interior of the boundary $\{ x : x_1 = 0 \}$.

Remark 4. The situation when the boundary of the domain is $\{ x : x_1 = a \}$ with $a \in \mathbb{Q}_+$ can be reduced to the case of the boundary $\{ x : x_1 = 0 \}$ by considering $N$ such that $Na \in \mathbb{Z}$. This is because for those values of $N$, the lattice $L_N$ is preserved after appropriate shift of the coordinate system.

In the following theorems $\delta$ denote any number from the interval $(0, \frac{1}{3(m+1)})$.

Theorem 5. For the sum (27) as $N \to \infty$ the following approximation holds

$$
\sum_{A \subseteq L_N} g(x) e^{h(N)f(x,N)} = e^{h(N)f(x^*(N),N)} N^{m-1} \left( \frac{2\pi}{h(N)} \right)^{\frac{m}{2}} \times \left( \frac{g(x^*(N))}{\sqrt{\det D^2f(x^*(N),N)}} \right) + \omega_1(N) \frac{1}{h(N)^{1/2-3\delta}} + \omega_2(N) \frac{h(N)^{1/2+1+\delta}}{N},
$$

where $\omega_1(N) = O(1)$ and $\omega_2(N) = O(1)$.

Theorem 6. For the sum (28) as $N \to \infty$ the following approximation holds

$$
\sum_{A \subseteq L_N \cap \{x_1 : x_1 \geq 0\}} g(x) e^{h(N)f(x,N)} = e^{Nf(x^*(N),N)} N^{m-1} \left( \frac{2\pi}{h(N)} \right)^{\frac{m-1}{2}} \times \left( \frac{g(x^*(N))}{\sqrt{\det D^2f(x^*(N),N)}} \right) + \omega_1(N) \frac{1}{h(N)^{1/2-3\delta}} + \omega_2(N) \frac{h(N)^{1/2+1+\delta}}{N},
$$

where $\omega_1(N) = O(1)$, $\omega_2(N) = O(1)$.

Remark 5. The situation when the boundary is an arbitrary hyperplane with a rational coefficients can be reduced to the case with the boundary $\{ x : x_1 = 0 \}$. This is because after appropriate rotation of coordinate system, structure of the lattice essential for the application of the theorem is preserved. That is, all the points of the domain are on the equally spaced hyperplanes parallel to the boundary.

For large enough $N$, let $X(N)$ be a random vector with pmf defined using sums (27) and (28)

(a) $P(X(N) = x) := \frac{e^{h(N)f(x,N)}}{\sum_{A \subseteq L_N} e^{h(N)f(y,N)}}$, \hspace{1cm} (29)

(b) $P(X(N) = x) := \frac{e^{h(N)f(x,N)}}{\sum_{A \subseteq L_N \cap \{x_1 : x_1 \geq 0 \text{ or } x_1 \leq 0\}} e^{h(N)f(y,N)}}$, \hspace{1cm} (30)

Theorem 7 (Weak law of large numbers). As $N \to \infty$ the random vector $X(N)$ converges weakly to the constant $x^*$ and the following estimate of the mgf holds

$$
M_{X(N)}(\xi) = e^{\xi T x^*} \left( 1 + \frac{O(1)}{h(N)^{1/2-3\delta}} + \frac{O(1)}{h(N)^{1/2+1+\delta}} + \frac{O(1)}{N} \right).
$$
Remark 6. For this and the following limit theorems the convergence error term can be explicitly estimated by the previous results.

For $\epsilon(N) = o\left(\frac{1}{\sqrt{h(N)}}\right)$, $N \to \infty$ we have the following results

Theorem 8 (Central limit theorem I). Let $X(N)$ have distribution $\mathcal{N}(0, D^2 f(x^*)^{-1})$. As $N \to \infty$ the random vector $Z(N) = \sqrt{h(N)}(x^* - X(N))$ converges weakly to $\mathcal{N}(0, D^2 f(x^*)^{-1})$ and the following estimate of the mgf holds

$$M_{Z(N)}(\xi) = \exp\left(\frac{1}{2} \xi^T D^2 f(x^*)^{-1} \xi\right) \left(1 + \frac{O(1)}{h(N)^{1/2 - 3\delta}} + O(1)\frac{h(N)^{1/2 + (m+1)\delta}}{N}\right) + O(1)\epsilon(N)\sqrt{h(N)}.$$ 

Here let us introduce the notation $\xi_y = (\xi_2, \ldots, \xi_m)$, $Y = (X_2(N), \ldots, X_m(N))$ and $y^* = (x^*_2, \ldots, x^*_m)$.

Theorem 9 (Central limit theorem II). Let $X(N)$ have distribution $\mathcal{N}(0, D_y^2 f(x^*)^{-1})$ and assume $h(N) = N$. As $N \to \infty$ the random vector $Z(N) = (N(x^*_1 - X_1(N)), \sqrt{N}(y^* - Y(N)))$ converges weakly to a discrete distribution with pmf

$$P(Z_1(N) = i) = \exp\left(-\frac{1}{2} \left| \frac{\partial f(x^*)}{\partial x_1}\right|^2\right) \left(1 - \exp\left(-\left| \frac{\partial f(x^*)}{\partial x_1}\right|^2\right)\right),$$

for $Z_1(N)$ and to $\mathcal{N}(0, D_y^2 f(x^*)^{-1})$ for $(Z_2(N), \ldots, Z_m(N))$. Furthermore, the following estimate of the mgf holds

$$M_{Z(N)}(\xi) = \frac{1}{1 - \exp\left(-\left| \frac{\partial f(x^*)}{\partial x_1}\right|^2\right)} \exp\left(\frac{1}{2} \xi_y^T D_y^2 f(x^*)^{-1} \xi_y\right) \left(1 + \frac{O(1)}{N^{1/2 - 3\delta}} + \frac{O(1)\sqrt{N}}{N}\right) + O(1)\epsilon(N)\sqrt{h(N)}.$$ 

Theorem 10 (Central limit theorem III). Let $X(N)$ have distribution $\mathcal{N}(0, D_y^2 f(x^*)^{-1})$ and assume $\lim_{N \to \infty} \frac{h(N)}{N} = 0$. As $N \to \infty$ the random vector $Z(N) = (h(N)(x^*_1 - X_1(N)), \sqrt{h(N)}(y^* - Y(N)))$ converges weakly to $\mathcal{N}(0, D_y^2 f(x^*)^{-1})$ for $(Z_2(N), \ldots, Z_m(N))$. Furthermore, the following estimate of the mgf holds

$$M_{Z(N)}(\xi) = \frac{1}{\left| -\frac{\partial f(x^*)}{\partial x_1}\right|} \exp\left(\frac{1}{2} \xi_y^T D_y^2 f(x^*)^{-1} \xi_y\right) \left(1 + \frac{O(1)}{h(N)^{1/2 - 3\delta}} + \frac{O(1)\sqrt{h(N)}}{N}\right) + O(1)\epsilon(N)\sqrt{h(N)}.$$ 

Remark 7. In Lapinísk [2012] above results are proved for the situation when points of the domain $\mathcal{A} \cap L_N$ are above the boundary. So for example, if the boundary is $\{x : x_1 = 0\}$ all the points have $x_1 > 0$. Here we replaced $\frac{\partial f}{\partial x_1} = -\frac{\partial L}{\partial x_1}$ in the Theorems. By this modification, we included the case where the points can be below the boundary.

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