Lax-Halmos Type Theorems On $H^p$ Spaces

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Abstract. In this paper we characterize for $0 < p \leq \infty$, the closed subspaces of $H^p$ that are invariant under multiplication by all powers of a finite Blaschke factor $B$, except the first power. Our result clearly generalizes the invariant subspace theorem obtained by Paulsen and Singh [9] which has proved to be the starting point of important work on constrained Nevanlinna-Pick interpolation. Our method of proof can also be readily adapted to the case where the subspace is invariant under all positive powers of $B(z)$. The two results are in the mould of the classical Lax-Halmos Theorem and can be said to be Lax-Halmos type results in the finite multiplicity case for two commuting shifts and for a single shift respectively.

1. INTRODUCTION

In recent times a great deal of interest has been generated in the Banach algebra $H_1^\infty$ and subsequently also in a class of related algebras in the context of problems dealing with invariant subspaces and their use in solving Nevanlinna-Pick type interpolation problems for these algebras. We refer to [3], [4], [11], [15], [18], and [21]. Note that $H_1^\infty = \{ f(z) \in H^\infty : f(0) = 0 \}$ is a closed subalgebra of $H^\infty$, the Banach algebra of bounded analytic functions on the open unit disc. The starting point, in the sequence of papers cited above, is an invariant subspace theorem first proved by Paulsen and Singh in a special case [18, Theorem 4.3] and subsequently in more general forms in [4] and [17]. This invariant subspace result is crucial to the solutions of interpolation problems of the Pick-Nevanlinna type as presented in the papers cited above. This theorem characterizes the closed subspaces of the Hardy spaces that are left invariant under the action of every element of the algebra $H_1^\infty$. In this paper we present a far reaching generalization of this invariant subspace theorem by presenting a complete characterization of the invariant subspaces of the Banach algebra $H_1^\infty(B) = \{ f(B(z)) : f \in H_1^\infty ; B \text{ is a finite Blaschke product} \}$. In the special case where $B(z) = z$ we arrive at the first or original invariant subspace theorem mentioned above for the Banach algebra $H_1^\infty$. We also note that $H_1^\infty(B)$ stands for the closed subalgebra of $H^\infty$ generated by $B^2$ and $B^3$. Our paper also deals with a second and related problem of characterizing the invariant subspaces on $H^p$ of the algebra $H^\infty(B)$ which consists of the Banach algebra generated by $B$. For the case $p = 2$, this problem has been tackled in the far more general setting

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of de Branges spaces in [19] and in the same year, for the classical $H^p$ spaces for all values of $p \geq 1$, this problem has been tackled in [13]. However, for this second problem, we claim some novelty and completeness on two counts; for one we have shown that in the case when $0 < p < 1$ we have an explicit description of the invariant subspaces, and second, our proof is more elementary and different from that in [13] since we do not use their general inner-outer factorization theorem [13, page 112].

Finally, we wish to observe that the two main results presented in this paper can be interpreted as being in the mould of the classical Lax-Halmos Theorem in the case of finite multiplicity, see [8], [9], and [14], for two commuting shifts as represented by multiplication by $B^2$ and by $B^3$ and for a single shift represented by multiplication by $B$ except that unlike the classical versions of the Lax-Halmos theorem we work entirely in the scalar valued setting of the classical Hardy spaces and our characterisations are also inside this scalar setting.

Let $D$ denote the open unit disk, and let its boundary, the unit circle, be denoted by $T$. The Lebesgue space $L^p$ on the unit circle is the collection of complex valued functions $f$ on the unit circle such that $\int |f|^p dm$ is finite, where $dm$ is the normalized Lebesgue measure on $T$. The Hardy space $H^p$ is the closure in $L^p$ of the analytic polynomials. For $p \geq 1$, $H^p$ can be viewed as the following closed subspace of $L^p$:

$$\left\{ f \in L^p : \int f z^n dm = 0 \text{ for all } n \geq 1 \right\}.$$  

For $1 \leq p < \infty$, $H^p$ is a Banach space under the norm

$$\|f\|_p = \left( \int |f|^p dm \right)^{\frac{1}{p}}.$$  

$H^\infty$ is a Banach algebra under the essential supremum norm. The Hardy space $H^2$ turns out to be a Hilbert space under the inner product

$$\langle f, g \rangle = \int f \overline{g} dm.$$  

For a detailed account of $H^p$ spaces the reader can refer to [5], [7], [10], and [12]. By a finite Blaschke factor $B(z)$ we mean

$$B(z) = \prod_{j=1}^{n} \frac{z - \alpha_j}{1 - \overline{\alpha_j} z},$$

where $\alpha_j \in \mathbb{D}$. Throughout we shall assume $\alpha_1 = 0$ as this does not affect generality. The operator of multiplication by $B(z)$ denoted by $T_B$ is an isometric operator on $H^p$. We call a closed subspace $M$ of $H^p$ to be $B$-invariant if $T_B M \subset M$. By an invariant subspace $M$ (in $H^p$) of any subalgebra $A$ of $H^\infty$ we mean a closed subspace of $H^p$ such that $fg \in M$ for all $f$ in $A$ and for all $g \in M$.

Let $B_j(z)$ denote the product of the first $j$ factors in $B(z)$:

$$B_j(z) = \prod_{i=1}^{j} \frac{z - \alpha_i}{1 - \overline{\alpha_i} z}, \quad \alpha_j \in \mathbb{D}.$$
2. Preliminary Results

Note: We shall assume throughout this paper that the Blaschke product $B$ so indicated is fixed and has $n$ zeros that may not necessarily be distinct.

Theorem 1. ([19, Theorem 3.3]) The collection \( \{ e_{jm} = (1 - |\alpha_{j+1}|^2)^{1/2}(1 - \alpha_j z)^{-1}B_jB^m : 0 \leq j \leq n - 1, m = 0, 1, 2, \ldots \} \) is an orthonormal basis for $H^2$. Consequently $H^2 = \bigoplus_{j=0}^{n-1} e_{j0} H^2(B)$ where $H^2(B)$ stands for the closed linear span of $\{B^m : m = 0, 1, 2, \ldots \}$ in $H^2$.

Let $(\varphi_1, \ldots, \varphi_r)$ be an $r$ tuple of $H^\infty$ functions ($r \leq n$). Suppose each $\varphi_j$ has the representation

$$\varphi_j = \sum_{i=0}^{n-1} e_{ij}.\varphi_{ij}.$$ 

The matrix $A = (\varphi_{ij})_{n \times r}$ is called the $B-$ matrix of $(\varphi_1, \ldots, \varphi_r)$. The matrix $A$ is called $B$-inner if $A^*A = I$. Suppose an $H^\infty$ function $\psi$ has the representation

$$\psi = \sum_{j=0}^{n-1} e_{j0} \psi_j,$$

for some $\psi_0, \ldots, \psi_{n-1} \in H^2(B)$, then $\psi$ is called $B$-inner if

$$\sum_{j=0}^{n-1} |\psi_j|^2 = 1 \ a.e.$$

It has been proved in [19] that $\psi$ is $B$-inner if and only if $\{B^m \psi : m = 0, 1, \ldots \}$ is an orthonormal set in $H^2$.

Theorem 2. ([19, Theorem 4.1]) Let $M$ be a $B$-invariant subspace of $H^2$. Then there is an $r \leq n$ such that

$$M = \varphi_1 H^2(B) \oplus \varphi_2 H^2(B) \oplus \cdots \oplus \varphi_r H^2(B)$$

for some $B$-inner functions $\varphi_1, \ldots, \varphi_r$, and the $B$-matrix of $(\varphi_1, \ldots, \varphi_r)$ is $B$-inner. Further, this representation is unique in the sense that if

$$M = \psi_1 H^2(B) \oplus \psi_2 H^2(B) \oplus \cdots \oplus \psi_s H^2(B)$$

then $r = s$, and $\varphi_i = \sum_{j=1}^{r} \alpha_{ij} \psi_j$ for scalars $\alpha_{ij}$ such that the matrix $(\alpha_{ij})$ is unitary.

We shall also make use of the following results to establish certain key facts that are central to the proof of the main results.

Lemma 1. ([13, Proposition 3]) Let $1 \leq p \leq \infty$ and let $\varphi_1, \ldots, \varphi_k$ ($k \leq n$) be $B$-inner functions. Then for any $f \in H^\infty$ such that $f(z) = \sum_{i=1}^{k} \varphi_i(z) f_i(B(z))$, there exist constants $C_{i,p}$, $i = 1, \ldots, k$ such that $\|f\|_p \leq C_{i,p} \|f_i\|_p$.

Lemma 2. For $1 \leq p < 2$, we can write

$$H^p = e_{00} H^p(B) \oplus e_{10} H^p(B) \oplus \cdots \oplus e_{n-1,0} H^p(B),$$

where each $e_{j0}$ is as in Theorem 1 for each $j$. 


PROOF. It is trivial to note that
\[ e_{00}H^p(B) \oplus e_{10}H^p(B) \oplus \cdots \oplus e_{n-1,0}H^p(B) \subset H^p. \]

To establish the opposite inclusion take an arbitrary element \( f \in H^p \). Since \( H^\infty \) is dense in \( H^p \), there exists a sequence \( \{ f_k \} \) of \( H^\infty \) functions that converges to \( f \) in the norm of \( H^p \). In view of Theorem 1 we can write
\[ f_k = e_{00}f_k^{(1)} + e_{10}f_k^{(2)} + \cdots + e_{n-1,0}f_k^{(n)}, \]
where \( f_k^{(j)} \in H^2(B) \) for all \( j = 1, \ldots, n \). By Lemma 1, we have for all \( j = 1, \ldots, n \), the estimate
\[ \| f_k^{(j)} \|_p \leq D_{kj} \| f_k \|_p \]
for some constants \( D_{kj} \). Equation (2.2) implies that \( \{ f_k^{(j)} \} \) is a Cauchy sequence for all \( j = 1, \ldots, n \), and hence \( f_k^{(j)} \to f^{(j)} \) in \( H^p(B) \). Therefore \( f_k \to e_{00}f^{(1)} + e_{10}f^{(2)} + \cdots + e_{n-1,0}f^{(n)} \) as \( k \to \infty \) in \( H^p \). Hence \( f = e_{00}f^{(1)} + e_{10}f^{(2)} + \cdots + e_{n-1,0}f^{(n)} \). This completes the proof of the assertion.

LEMMA 3. ([16] Lemma 4.1) Suppose \( \{ f_n \}_{n=1}^\infty \) is a sequence of \( H^p \) functions, \( p > 2 \), which converges to an \( H^p \) function \( f \) in the \( H^2 \) norm. Then there exists a sequence \( \{ g_n \}_{n=1}^\infty \) of \( H^\infty \) functions such that \( g_nf_n \to f \) in the \( H^p \) norm (weak-star convergence when \( p = \infty \)). Further the sequence \( \{ g_n \}_{n=1}^\infty \) is uniformly bounded, and converges to the constant function \( 1 \) a.e.

LEMMA 4. Let \( p > 2 \). Suppose an \( H^p \) function \( f \) is of the form \( f = \varphi_1h_1 + \cdots + \varphi_nh_n \), where \( h_1, \ldots, h_n \in H^2(B) \), and \( \varphi_1, \ldots, \varphi_n \) are \( B \)-inner, then \( h_1, \ldots, h_n \) belong to \( H^p(B) \).

PROOF. Let
\[ f = \varphi_1h_1 + \varphi_2h_2 + \cdots + \varphi_rh_r, \]
where \( h_1, h_2, \ldots, h_r \in H^2(B) \), and \( \varphi_1, \ldots, \varphi_n \) are \( B \)-inner. Also \( f \) can be identified with a bounded linear functional \( F_f \in L^*_q \) \( (1 \leq q < 2) \) such that
\[ F_f(g) = \int fg \quad \text{and} \quad |F_f(g)| \leq \delta \|g\|_q \quad \text{for all} \quad g \in L^*_q \text{ for some} \quad \delta > 0. \]

Now for any \( l \in \text{span} \{ 1, B, B^2, \ldots \} \) such that \( l \) is a polynomial in \( B \), we have
\[ \left| \int f \overline{\varphi_1l} \right| \leq \delta \| \varphi_1l \|_q \leq \delta_1 \| l \|_q \]
for some $\delta_1 > 0$. Also note that

$$\left| \int f \varphi_1 l \right|$$

$$= \left| \int (\varphi_1 h_1 + \varphi_2 h_2 + \cdots + \varphi_r h_r) \varphi_1 l \right|$$

$$= \left| \int \varphi_1 h_1 \varphi_1 l + \int \varphi_2 h_2 \varphi_1 l + \cdots + \int \varphi_r h_r \varphi_1 l \right|$$

$$= \left| \int \varphi_1 h_1 \varphi_1 l \right|$$

$$= \left| \int h_1 l \right| .$$

Here $\int \varphi_j h_j \varphi_1 l = 0, j = 2, \ldots, r$ because $\varphi_1 H^2 (B) \perp \varphi_j H^2 (B)$, and $\int \varphi_1 h_1 \varphi_1 l = \int h_1 l$ because $\varphi_1$ is $B$–inner. Therefore,

$$\left| \int h_1 l \right| \leq \delta_1 \|l\|_q .$$

Now any analytic polynomial $k \in L_q$ can be written as

$$k = e_{00} k_1 (B) + \cdots + e_{r-10} k_r (B)$$

and

$$\left| \int h_1 k \right| = \left| \int h_1 k_1 + \int e_{10} h_1 k_2 + \cdots + \int e_{r-10} h_1 k_r \right|$$

$$\leq \left| \int h_1 k_1 \right| + \left| \int e_{10} h_1 k_2 \right| + \cdots + \left| \int e_{r-10} h_1 k_r \right|$$

$$= \left| \int h_1 k_1 \right| .$$

(2.3)

It is easily checked that all integrals in the above equation except the first integral shall be zero. We show this by examining one of the above integrals in question: Let $h_1 = \alpha_0 + \alpha_1 B + \alpha_2 B^2 + \cdots, k_2 = \beta_0 + \beta_1 B + \beta_2 B^2 + \cdots$.

Now,

$$\int e_{10} h_1 k_2 = \langle e_{10} h_1, k_2 \rangle$$

$$= \left\langle \frac{z}{1 - \alpha_0 z} (\alpha_0 + \alpha_1 B + \alpha_2 B^2 + \cdots), \beta_0 + \beta_1 B + \beta_2 B^2 + \cdots \right\rangle$$

$$= 0.$$
Let $k_1 = \gamma_0 + \gamma_1 B + \gamma_2 B^2 + \cdots$ so that
\[
\left| \int h_1 k_1 \right| = |\langle h_1, \overline{k_1} \rangle| \\
= |\alpha_0| |\gamma_0| \\
= \left| \int h_1 \right| \left| \int k_1 \right| \\
= \left| \int h_1 \right| \left| k \right| \\
\leq A \int |k| \leq A \|k\|_q
\]
where $A = \int |h_1|$. Thus $h_1$ acts as a bounded linear functional on the space of polynomials, and so it can be extended to a bounded linear functional on $L_q$. Call this extension as $F$. So $F(g) = \int h_1 g$ for all $g \in L_q$. But $F \in L^1_q$ implies that there exists $G \in L^p$ such that $F(g) = \int G g$ for all $g \in L_q$. In particular we have
\[
\int (G - h_1) z^n = 0 \text{ for all } n \in \mathbb{Z}.
\]
Hence $h_1 = G \in L^p$. In a similar fashion we get $h_2, \ldots, h_r \in L^p$. $\square$

3. The $B^2$ and $B^3$ invariant subspaces of $H^p$

**Theorem 3.** Let $M$ be a closed subspace of $H^p$, $0 < p \leq \infty$, such that $M$ is invariant under $H^\infty_1(B)$ but not invariant under $H^\infty(B)$. Then there exist $B$--inner functions $J_1, \ldots, J_r$ ($r \leq n$) such that
\[
M = \left( \sum_{j=1}^k \oplus \langle \varphi_j \rangle \right) \oplus \sum_{i=1}^r \oplus B^2 J_i H^p \ (B)
\]
where $k \leq 2r - 1$, and for all $j = 1, 2, \ldots, k$, $\varphi_j = (\alpha_{1j} + \alpha_{2j} B) J_1 + (\alpha_{3j} + \alpha_{4j} B) J_2 + \ldots + (\alpha_{2r-1,j} + \alpha_{2r,j} B) J_r$.

**Remark 1.** The proof shall also show that the matrix $\alpha = (\alpha_{ij})_{2r \times k}$ satisfies $A^* A = I$, and $\alpha_{st} \neq 0$ for some $(s, t) \in \{1, 3, \ldots, 2r - 1\} \times \{1, 2, \ldots, k\}$. Also, when $0 < p < 1$, the right hand side should be interpreted as being the closure of the sum in the $H^p$ metric.

**Proof.** The case $p = 2$. Using the initial line of argument as in [4] we define $M_1 = H^\infty(B) \cdot M_1$. It is easily seen that $M_1$ is a $B$--invariant subspace of $H^2$. Observe that
\[
M_1 \supset M \supset H^\infty_1(B) \cdot M \supset \overline{B^2 H^\infty(B) \cdot M} = B^2 M_1.
\]
Therefore
\[
B^2 M_1 \subset M \subset M_1
\]
Note that all containments in the above equation are strict. For if $M = M_1$ or $M = B^2 M_1$ it would then mean that $M$ is invariant under $H^\infty(B)$, which is a
contradiction. By Theorem 2, there exist $B$-inner functions $J_1, \ldots, J_r$ ($r \leq n$) such that
\[
M_1 = J_1 H^2(B) \oplus \cdots \oplus J_r H^2(B) = (\langle J_1 \rangle \oplus \langle BJ_1 \rangle \oplus B^2 J_1 H^2(B)) \oplus \cdots \oplus (\langle J_r \rangle \oplus \langle BJ_r \rangle \oplus B^2 J_r H^2(B)) = (\langle J_1 \rangle \oplus \langle BJ_1 \rangle \oplus \cdots \oplus \langle J_r \rangle \oplus \langle BJ_r \rangle) \oplus B^2 M_1.
\]
So $M_1 \ominus B^2 M_1$ has dimension $2r$, and hence $M \ominus B^2 M_1$ has dimension $k$, where $k \leq 2r - 1$. Let $\varphi_1, \ldots, \varphi_k$ be an orthonormal basis for $M \ominus B^2 M_1$. Now
\[
M = [M \ominus B^2 M_1] \oplus B^2 M_1
= \left( \sum_{j=1}^{k} \oplus \langle \varphi_j \rangle \right) \oplus B^2 J_1 H^2(B) \oplus \cdots \oplus B^2 J_r H^2(B).
\]
Since $\varphi_j \in M \ominus B^2 M_1 \subset M_1 \ominus B^2 M_1$, we see that each $\varphi_j$ is of the form $\alpha_{1,j} J_1 + \alpha_{2,j} J_2 + \alpha_{3,j} J_3 + \cdots + \alpha_{2r-1,j} J_{2r-1} + \alpha_{2r,j} J_r$. The conditions $\|\varphi_j\|_2 = 1$ and $\langle \varphi_j, \varphi_i \rangle = 0$ for $j \neq i$, imply that $|\alpha_{1,j}|^2 + |\alpha_{2,j}|^2 + \cdots + |\alpha_{2r-1,j}|^2 = 1$, and $\alpha_{1,j}\alpha_{1,i} + \alpha_{2,j}\alpha_{2,i} + \cdots + \alpha_{2r,j}\alpha_{2r,i} = 0$. In addition the $k$ tuples $(\alpha_{1,1}, \alpha_{2,1}, \ldots, \alpha_{2,1})$, $i = 1, 3, \ldots, 2r - 1$ cannot be simultaneously zero, otherwise $M$ would become $B$-invariant.

**The case $0 < p < 1$.** Observe that every $H^p$ function $f$, can be written as $f = IO$, where $I$ is an inner function and $O \in H^p$ is an outer function. Choose $n$ such that $2^n p > 2$, so that we can express $f$ as a product of $H^2$ functions:
\[
f = IO^\frac{1}{p} O^\frac{1}{p} \cdots O^\frac{1}{p}.
\]
We first show that $M \cap H^2 \neq [0]$. Let $0 \neq f \in M$. Then $f$ can be written as
\[
f = f_1 f_2 \cdots f_m,
\]
where $f_1, f_2, \ldots, f_m \in H^2$. In view of Theorem 11 we can express each $f_i$ as
\[
f_i = e_{00} g_1^{(i)} + \cdots + e_{r0} g_r^{(i)},
\]
for some $g_1^{(i)}, \ldots, g_r^{(i)} \in H^2(B^2)$. It is known that the operator $T : H^2 \rightarrow H^2$ defined by $Th = h(B^2(z))$ is an isometry, and its range is $H^2(B^2)$ (see [2]). So for each $g_j^{(i)}$, there exists $k_j^{(i)} \in H^2$ such that $g_j^{(i)} = k_j^{(i)}(B^2(z))$. Define
\[
q_{ji}(z) := \exp \left\{ -\frac{1}{2} \left[ k_j^{(i)}(z) - \bar{q}_j^{(i)}(z) \bar{k}_j^{(i)}(z) \right] \right\}. 
\]
Here $\sim$ denotes the harmonic conjugate. Then $|q_{ji}(z)| \leq 1$, and thus $h_i(z) := q_{1i}(z) q_{2i}(z) \cdots q_{ri}(z) \in H^\infty$.

Note that
\[
h_i(B^2(z)) f_i(z) = e_{00} h_i(B^2(z)) g_1^{(i)}(z) + \cdots + e_{r0} h_i(B^2(z)) g_r^{(i)}(z) = e_{00} h_i(B^2(z)) k_1^{(i)}(B^2(z)) + \cdots + e_{r0} h_i(B^2(z)) k_r^{(i)}(B^2(z)),
\]
which clearly belongs to $H^\infty$. This implies that $h_1(B^2(z)) \cdots h_m(B^2(z)) f = h_1(B^2(z)) f_1 \cdots h_m(B^2(z)) f_m \in H^\infty$. Since $h_1(B^2(z)) \cdots h_m(B^2(z)) \in H^\infty$ its Cesaro means $\{p_n(B^2)\}$, which is a sequence of polynomials, shall converge
to \( h_1 (B^2 (z)) \cdots h_m (B^2 (z)) \text{ a.e.} \). Hence, by the Dominated Convergence Theorem, we see that \( p_n (B^2) f \to h_1 (B^2 (z)) \cdots h_m (B^2 (z)) f \) in \( H^p \). Therefore, \( h_1 (B^2 (z)) \cdots h_m (B^2 (z)) f \in M \), because \( M \) is invariant under \( B^2 \). This establishes \( M \cap H^2 \neq \{0\} \). Next we claim that \( M \cap H^2 \) is dense in \( M \). The density will also imply that \( M \cap H^2 \) is not \( B^- \) invariant, otherwise it would force \( M \) to be \( B^- \) invariant, which is not possible. It is trivial to note that \( M \cap H^2 \subseteq M \) (bar denotes closure in \( H^p \)). Let \( f \in M \). We can express \( f \) as
\[
f = f_1 f_2 \cdots f_{2^m},
\]
where each \( f_l \in H^2 \), and \( m \) is chosen so that \( 2^m p > 2 \). As argued previously we can express each \( f_l \) as
\[
f_l = c_{\alpha_0} k_1^{(l)} (B^2 (z)) + \cdots + c_{\alpha_m} k_m^{(l)} (B^2 (z)),
\]
for certain \( k_1^{(l)}, \ldots, k_m^{(l)} \in H^2 \). Define
\[
q_n^{(j)} (z) = \exp \left( -\frac{|k_j^{(l)} (z)|}{|z|} \right)
\]
\((\sim \text{ denotes the harmonic conjugate which exists for } L^2 \text{ functions})\). Then \( q_n^{(j)} \in H^\infty \) and \( \left| q_n^{(j)} \right| \leq 1 \). For each \( l = 1, \ldots, 2^m \), the function \( h_n^{(l)} (z) = q_n^{(1)} (z) \cdots q_n^{(r_l)} (z) \) belongs to \( H^\infty \) and \( h_n^{(l)} (B^2 (z)) \) multiplies \( f_l \) into \( H^\infty \). This implies that
\[
h_n^{(1)} (B^2 (z)) \cdots h_n^{(2^m)} (B^2 (z)) f \in H^\infty.
\]
Since \( h_n^{(1)} (B^2 (z)) \cdots h_n^{(2^m)} (B^2 (z)) \to 1 \text{ a.e., we have} \)
\[
h_n^{(1)} (B^2 (z)) \cdots h_n^{(2^m)} (B^2 (z)) f \to f \text{ a.e.}
\]
so that
\[
\left| h_n^{(1)} (B^2 (z)) \cdots h_n^{(2^m)} (B^2 (z)) f - f \right|^p \to 0 \text{ a.e.}
\]
Moreover
\[
\left| h_n^{(1)} (B^2 (z)) \cdots h_n^{(2^m)} (B^2 (z)) f - f \right|^p \leq 2^p \left| f - f \right|^p
\]
so by the Dominated Convergence Theorem,
\[
h_n^{(1)} (B^2 (z)) \cdots h_n^{(2^m)} (B^2 (z)) f \to f \in H^p.
\]
We claim that \( h_n^{(1)} (B^2) \cdots h_n^{(2^m)} (B^2) f \in M \). To prove this claim we proceed as follows. For each \( n \), there exists a sequence of polynomials \( \{p^{(n)}_k\} \) such that
\[
p^{(n)}_k (B^2 (z)) \to h_n^{(1)} (B^2 (z)) \cdots h_n^{(2^m)} (B^2 (z))
\]
The sequence \( \{p^{(n)}_k\} \) is the Cesaro means of \( h_n^{(1)} \cdots h_n^{(2^m)} \) and it converges boundedly and pointwise. It is then easy to see by means of the Dominated Convergence Theorem that \( p^{(n)}_k (B^2) f \) converges to \( h_n^{(1)} (B^2) \cdots h_n^{(2^m)} (B^2) f \) in \( H^p \). The claim now follows in view of the fact that \( M \) is invariant under \( B^2 \) and the fact that \( p^{(n)}_k (B^2) f \in M \). By the validity of the result for the case \( p = 2 \), we have
\[
M \cap H^2 = \left( \bigoplus_{j=1}^{2^r} \langle \varphi_j \rangle \right) \oplus B^2 J_1 H^2 (B) \oplus \cdots \oplus B^2 J_n H^2 (B),
\]
and hence

\[
M = \left( \sum_{j=1}^{2r-1} \oplus \langle \varphi_j \rangle \right) \oplus B^2 J_1 H^2(B) \oplus \cdots \oplus B^2 J_r H^2(B)
\]

\[
= \left( \sum_{j=1}^{2r-1} \oplus \langle \varphi_j \rangle \right) \oplus B^2 \left( J_1 H^2(B) \oplus \cdots \oplus J_r H^2(B) \right)
\]

(bar denotes closure in \(H^p\)). We can easily see that \(B^2(\bigoplus J_i H^2(B)) \subset B^2 \left( J_1 H^2(B) \oplus \cdots \oplus J_r H^2(B) \right) \subset B^2 \left( J_1 H^p(B) + \cdots + J_r H^p(B) \right)\) and so upon taking the closure in \(H^p\) of all three subspaces we shall get equality throughout so that \(B^2 \left( J_1 H^2(B) \oplus \cdots \oplus J_r H^2(B) \right) = B^2 \left( J_1 H^p(B) + \cdots + J_r H^p(B) \right)\) and this gives us the characterisation for the case \(0 < p < 1\).

The case \(1 \leq p < 2\). The arguments and conclusions above in the case \(0 < p < 1\) are also valid for this case and so certainly

\[
M = \left( \sum_{j=1}^{2r-1} \oplus \langle \varphi_j \rangle \right) \oplus \left( J_1 H^p(B) + \cdots + J_r H^p(B) \right)
\]

where the bar denotes closure in \(H^p, 1 \leq p < 2\). Let \(N = \bigoplus J_i H^p(B)\), then as a closed subspace of \(H^p\), \(N\) is invariant under multiplication by \(B\). It can be verified that

\[
N \cap H^2 = J_1 H^2(B) \oplus \cdots \oplus J_r H^2(B).
\]

Any arbitrary \(g \in J_i H^p(B)\) can be written as \(g = J_i f\), for some \(f \in H^p(B)\). Then the Cesaro means of \(f\) denoted by the sequence of polynomials, \(\{p_n\}\), is such that \(p_n(z) \rightarrow f(z)\) in \(H^p\). Hence \(p_n(B) \rightarrow f(B)\) in \(H^p\). But \(J_i \in H^\infty\), so \(p_n(B)J_i \rightarrow J_i f(B)\) in \(H^p\). Because \(N\) is \(B\) invariant, we have \(p_n(B)J_i \in N\), and the fact that \(N\) is closed implies that \(J_i f \in N\). This establishes that \(J_1 H^p(B) + \cdots + J_r H^p(B) \subset N\). Now we establish the inclusion in the other direction. In a fashion, similar to as shown above, for any \(f \in N\), we can construct an outer function \(K \in H^\infty\) such that \(K f \in N \cap H^2\). Therefore,

\[
(3.3) \quad K f = J_1 h_1 + J_2 h_2 + \cdots + J_r h_r,
\]

for some uniquely determined \(h_1, h_2, \ldots, h_r \in H^2(B) \subset H^p(B)\). Since \(f \in H^p\), by Lemma 2, we can express it uniquely as

\[
(3.4) \quad f = e_{00} f_1 + e_{10} f_2 + \cdots + e_{n-1,0} f_n,
\]

for some \(f_1, \ldots, f_n \in H^p(B)\). Therefore,

\[
(3.5) \quad K f = e_{00} K f_1 + e_{10} K f_2 + \cdots + e_{n-1,0} K f_n.
\]

Because \(J_1, J_2, \ldots, J_r\) are \(B\) inner, we can write

\[
J_1 = e_{00} \phi_{10} + e_{10} \phi_{11} + \cdots + e_{n-1,0} \phi_{1,n-1}
\]

\[
J_2 = e_{00} \phi_{20} + e_{10} \phi_{21} + \cdots + e_{n-1,0} \phi_{2,n-1}
\]

\[
\vdots
\]

\[
J_r = e_{00} \phi_{r0} + e_{10} \phi_{r1} + \cdots + e_{n-1,0} \phi_{r,n-1},
\]
where the $B-$ matrix $(\varphi_{ij})_{r \times n}$ satisfies $(\varphi_{ij})_{r \times n} (\overline{\varphi_{ji}})_{n \times r} = I$. 
Equation 3.3 now becomes
\[
Kf = e_{00} (\varphi_{10} h_1 + \varphi_{20} h_2 + \cdots + \varphi_{r0} h_r) + \\
e_{10} (\varphi_{11} h_1 + \varphi_{21} h_2 + \cdots + \varphi_{r1} h_r) + \\
\cdots + \\
e_{n-1,0} (\varphi_{1,n-1} h_1 + \varphi_{2,n-1} h_2 + \cdots + \varphi_{r,n-1} h_r) 
\]
(3.6)
From equations (3.5) and (3.6) we see that
\[
Kf_1 = \varphi_{10} h_1 + \varphi_{20} h_2 + \cdots + \varphi_{r0} h_r \\
\vdots \\
Kf_n = \varphi_{1,n-1} h_1 + \varphi_{2,n-1} h_2 + \cdots + \varphi_{r,n-1} h_r.
\]
This in matrix form can be written as
\[
(Kf_i)_{1 \times n} = (h_i)_{1 \times r} (\varphi_{ij})_{r \times n}
\]
(3.7)
Taking the conjugate transpose we get
\[
(Kf_i)_{n \times 1} = (\overline{\varphi_{ji}})_{n \times r} (h_i)_{r \times 1}
\]
(3.8)
By multiplying equations (3.7) and (3.8) we get:
\[
\left| \frac{h_1}{K} \right|^2 + \cdots + \left| \frac{h_r}{K} \right|^2 = |f_1|^2 + \cdots + |f_n|^2 \\
\leq (|f_1| + \cdots + |f_n|)^2.
\]
Thus for $j = 1, \ldots, r$, we have
\[
\left| \frac{h_j}{K} \right| \leq |f_1| + \cdots + |f_n|
\]
and this clearly implies that $\frac{h_j}{K} \in L^p$. Because $K$ is outer, we have $\frac{h_j}{K} \in H^p$. 
Then from (3.3) we conclude that $f$ is in $J_1 H^p (B) \oplus \cdots \oplus J_r H^p (B)$ so that $N \subset J_1 H^p (B) \oplus \cdots \oplus J_r H^p (B)$ and so $N = J_1 H^p (B) \oplus \cdots \oplus J_r H^p (B)$ and this then implies that $M = \left( \sum_{j=1}^k \oplus \langle \varphi_j \rangle \right) \oplus \sum_{l=1}^r \oplus B^2 J_l H^p (B)$.

**The case** $1 < p \leq \infty$. Let $M_1 = \overline{M}$ denote the closure of $M$ in $H^2$. Suppose $M_1$ is invariant under multiplication by $B(z)$, then by Theorem 2, we can write
\[
M_1 = \varphi_1 H^2 (B) \oplus \cdots \oplus \varphi_n H^2 (B),
\]
for some $B$- inner functions $\varphi_1, \ldots, \varphi_n$. It follows that any element $f \in M$ can be written as
\[
f = \varphi_1 h_1 + \cdots + \varphi_n h_n,
\]
for some $h_1, \ldots, h_n \in H^2 (B)$. By Lemma 2, $h_j \in H^p$ (in fact $h_j \in H^p (B)$). We claim that $\Phi_k = \varphi_k h_k \in M$. Since $\Phi_k \in \overline{M}$, so there exists a sequence \( \{h_{i_l}^{(k)}\}_{l=1}^\infty \) \( \subset M \) such that $h_{i_l}^{(k)} \rightarrow \Phi_k$ in $H^2$ as $l \rightarrow \infty$. Moreover we can write
\[
h_{i_l}^{(k)} = e_{00} h_{i_l}^{(k,1)} + \cdots + e_{n-1,0} h_{i_l}^{(k,n)}
\]
and
\[
\Phi_k = e_{00} \Phi_k^{(1)} + \cdots + e_{n-1,0} \Phi_k^{(n)}.
\]
Therefore, $e_{j0}h^{(k,j)}_j(B) \rightarrow e_{j0}\Phi_k^{(j)}(B)$ in $H^2$. But multiplication by $e_{j0}$ is an isometry on $H^2(B)$, so we have $h^{(k,j)}_j(B) \rightarrow \Phi_k^{(j)}(B)$ in $H^2$. This implies that $h^{(k,j)}_j(z) \rightarrow \Phi_k^{(j)}(z)$ in $H^2$. By Lemma 2, there exists a sequence $\{g^{(k,j)}_l\}_{l=1}^\infty \subset H^\infty$ such that $g^{(k,j)}_l(z)h^{(k,j)}_l(z) \rightarrow \Phi_k^{(j)}(z)$ in $H^p$ as $l \rightarrow \infty$. Define

$$g^{(k)}_l = g^{(k,1)}_l(B^2) \cdots g^{(k,n)}_l(B^2)$$

so that $\{g^{(k)}_l\}_{l=1}^\infty$ is uniformly bounded and converges to 1 a.e. Consider

$$g^{(k)}_l h^{(k)}_l = \sum_{j=1}^n e_{j-1,0}g^{(k,j)}_l$$

$$= \sum_{j=1}^n e_{j-1,0}g^{(k,1)}_l(B^2) \cdots g^{(k,n)}_l(B^2) h^{(k,j)}_l$$

We now show that $g^{(k,1)}_l(B^2) \cdots g^{(k,n)}_l(B^2) h^{(k,j)}_l \rightarrow \Phi_k^{(j)}$ in $H^p$. Note that the sequence $\theta_l = g^{(k,1)}_l(B^2) \cdots g^{(k,j-1)}_l(B^2) g^{(k,j+1)}_l(B^2) \cdots g^{(k,n)}_l(B^2)$ is uniformly bounded and converges to 1 a.e., and the sequence $\psi_l = g^{(k,j)}_l h^{(k,j)}_l \rightarrow \Phi_k^{(j)}$ in $H^p$. It can be shown that $\theta_l \psi_l \rightarrow \Phi_k^{(j)}$ in $H^p$, and hence $g^{(k)}_l h^{(k)}_l \rightarrow \Phi_k$ in $H^p$. Now by the invariance of $M$ we have $\Phi_k \in M$. This means that $M$ can be written as

$$M = \varphi_1 N_1 \oplus \varphi_2 N_2 \oplus \cdots \oplus \varphi_n N_n,$$

where

$$N_j = \{h \in H^p(B) : \varphi_j h \in M\}$$

is a closed subspace of $H^p$. It is easy to see that $N_j$ is invariant under $B^2$ and $B^3$, and is also dense in $H^2(B)$. Note that all $N_j$'s cannot be $B$-invariant simultaneously. For if they are then it would imply that $M$ is also $B$-invariant which is not possible. Thus, some $N_j$ is not invariant under $B$. Without loss of generality assume that $N_1$ is not invariant. We show that this even is not possible. For any $f \in H^p(B)$, we can find a sequence $\{f_l\}$ in $N_1$ such that $f_l \rightarrow f$ in $H^2$ because $N_1$ is dense in $H^2(B)$. Once again we can write

$$f_l = e_{j0}f^{(1)}_l + \cdots + e_{n-1,0}f^{(n)}_l$$

and

$$f = e_{j0}f^{(1)} + \cdots + e_{n-1,0}f^{(n)}$$

so that $f^{(j)}_l \rightarrow f^{(j)}$ in $H^2$. Again by Lemma 3, there exists a sequence $\{g^{(j)}_l\}_{l=1}^\infty$ in $H^\infty$, such that $g^{(j)}_l f^{(j)}_l \rightarrow f^{(j)}$ in $H^p$. Taking $g_l = g^{(1)}_l(B) \cdots g^{(n)}_l(B)$, it follows that $B^m g_l f_l \rightarrow B^m f$ in $H^p$ for $m \geq 2$. Thus $B^m f \in N_1$ for $m \geq 2$. Let us choose $f = 1$. So we have $B^2 H^p(B) \subset N_1$. This inclusion must be strict as $N_1$ is not invariant under $B$. So $N_1$ is of the form

$$N_1 = A \oplus B^2 H^p(B),$$

where $A$ is a non zero subspace. We know that

$$A \nsubseteq N_1 \nsubseteq H^2(B).$$
If 1 and $B$ belong to $A$, then $N_1 = H^p(B)$ which is not possible. So $A = (\alpha + \beta B)$, where $\alpha \neq 0$. Again the density of $N_1$ implies that there exists a sequence $\{\alpha_n (\alpha + \beta B) + B^2 f_n\} \subset N_1$ that converges to 1 in $H^2$. This gives

$$
B^2 f_n \rightarrow 0,
$$

$$
\alpha \alpha_n \rightarrow 1, \text{ and}
$$

$$
\beta \alpha_n \rightarrow 0
$$

Therefore, $\beta = 0$, which means that there cannot be a sequence in $N_1$ that converges to $B$ in $H^2$ norm. This contradicts the fact that $N_1$ is dense in $H^2(B)$. This contradiction stems from the fact that $M_1$ is assumed to be invariant under $B$. Thus $M_1$ is invariant under $B^2$ and $B^3$ but not under $B$. Now by the validity of our result on $H^2$, there exist $B$-inner functions $J_1, \ldots, J_r$, $r \leq n$, such that

$$
M_1 = \left( \sum_{j=1}^{2r-1} \langle \varphi_j \rangle \right) \oplus B^2 J_1 H^2(B) \oplus \cdots \oplus B^2 J_r H^2(B)
$$

where $\varphi_j = \alpha_1^j J_1 + \alpha_2^j J_2 + \alpha_3^j J_3 + \cdots + \alpha_n^j J_n$. From the form of $\varphi_j$ it is clear that $\varphi_j \in H^p$. Using the arguments already used in the proof it can be shown that $\varphi_j \in M$. Also essentially repeating the arguments as in the previous case we can easily establish that $B^2 J_1, \ldots, B^2 J_r \in M$.

Thus

$$
\left( \sum_{j=1}^{2r-1} \langle \varphi_j \rangle \right) \oplus B^2 J_1 H^p(B) \oplus \cdots \oplus B^2 J_r H^p(B) \subset M.
$$

To establish the reverse inclusion consider any $f \in M$. By virtue of the characterization of $M_1$, $f = \alpha_1 \varphi_1 + \cdots + \alpha_{2n-1} \varphi_{2n-1} + B^2 J_1 h_1 + \cdots + B^2 J_n h_n$. Note that

$$
B^2 J_1 h_1 + \cdots + B^2 J_r h_r = f - \alpha_1 \varphi_1 - \cdots - \alpha_{2n-1} \varphi_{2n-1} \in H^p.
$$

Hence, by Lemma 4, $h_1, \ldots, h_n \in H^p(B)$. Hence $f \in \left( \sum_{j=1}^{2r-1} \langle \varphi_j \rangle \right) \oplus B^2 J_1 H^p(B) \oplus \cdots \oplus B^2 J_r H^p(B)$

so that $M \subset \left( \sum_{j=1}^{2r-1} \langle \varphi_j \rangle \right) \oplus B^2 J_1 H^p(B) \oplus \cdots \oplus B^2 J_r H^p(B)$. This completes the proof of the theorem.

4. The $B$-invariant subspaces of $H^p$

The ideas from the above proof can be applied to derive a new factorization free proof of the following invariant subspace theorem obtained in [13] for the cases $1 \leq p \leq \infty$, $p \neq 2$. In addition we have extended the theorem to the case $0 < p < 1$.

**Theorem 4.** Let $M$ be a closed subspace of $H^p$, $0 < p < \infty$, $p \neq 2$, such that $M$ is invariant under $H^\infty(B)$. Then there exist $B$-inner functions $J_1, \ldots, J_r$, $r \leq n$, such that

$$
M = J_1 H^p(B) \oplus \cdots \oplus J_r H^p(B).
$$

When $0 < p < 1$, then, as the proof will show, the right hand side is to be read as being dense in $M$ i.e. its closure in the $H^p$ metric is all of $M$.
LAX-HALMOS TYPE THEOREMS ON $H^p$ SPACES

PROOF. The idea of the proof is quite similar to the proof of the Theorem 3. We shall only sketch the details. Using the fact that every $0 \neq f \in H^p$ can be written as a product of an appropriate number of $H^2$ functions, we can construct an outer function $O(z)$, in a manner identical to the proof of Theorem 3, such that $O(B(z))f \in M \cap H^2$. Thereby establishing that $M \cap H^2 \neq \{0\}$. Now $M \cap H^2$ is a closed subspace of $H^2$ and invariant under $H^\infty(B)$, so by Theorem 2 there exist $B$–inner functions $J_1, \ldots, J_r$, with $r \leq n$, such that

$$M \cap H^2 = J_1H^2(B) \oplus \cdots \oplus J_rH^2(B).$$

Next we show that $M \cap H^2$ is dense in $M$. It is trivial to note that $\overline{M \cap H^2} \subseteq M$, where the bar denotes closure in $H^p$. In order to establish the reverse inequality, we follow the same arguments used in the proof of Theorem 3 to construct a sequence of outer functions $\{O_l(z)\}_{l=0}^\infty$, such that for any $f \in M$, $O_l(B(z))f \in M \cap H^2$, and $O_l(B(z))f \to f$ in $H^p$, as $l \to \infty$. The proof of Theorem 3 also establishes that, for $0 < p < 1$, $\overline{M \cap H^2}$ takes the form $J_1H^p(B) \oplus \cdots \oplus J_rH^p(B)$ and for $1 \leq p < 2$, $\overline{M \cap H^2} = J_1H^p(B) \oplus \cdots \oplus J_rH^p(B)$ thereby establishing the characterization for $M$ in these cases. Next we deal with the case when $2 < p \leq \infty$. As in the proof of Theorem 3, we consider $M_1 = \overline{M}$, the closure of $M$ in $H^2$. Since $M$ is $B$–invariant, we have $M_1$ is $B$–invariant. So by Theorem 2 there exist $B$–inner functions $J_1, \ldots, J_r$, $r \leq n$, such that

$$M_1 = J_1H^2(B) \oplus \cdots \oplus J_rH^2(B).$$

Thus any arbitrary $f \in M$ can be written as $f = J_1h_1 + \cdots + J_rh_r$, for some $h_1, \ldots, h_r \in H^2$. By Lemma 4, these $h_1, \ldots, h_r \in H^p$, and hence $M \subseteq J_1H^2(B) \oplus \cdots \oplus J_rH^2(B)$. To prove the inclusion in the reverse, we need to establish that, for each $k = 1, \ldots, r$, $J_kH^p(B) \subseteq M$. For an arbitrary $h \in H^p(B)$, consider $\Psi_k = J_kh$. Now proceeding in the same fashion as in the proof of Theorem 3 (where we show that $\Phi_k \in M$), it follows that $\Psi_k \in M$.

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