STABILITY OF GROUND STATE EIGENVALUES OF NON-LOCAL SCHRÖDINGER OPERATORS WITH RESPECT TO POTENTIALS AND APPLICATIONS

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Abstract. In a first part of this paper we investigate the continuity (stability) of the spectrum of a class of non-local Schrödinger operators on varying the potentials. By imposing conditions of different strength on the convergence of the sequence of potentials, we give either direct proofs to show the strong or norm resolvent convergence of the so-obtained sequence of non-local Schrödinger operators, or via Γ-convergence of the related positive forms for more rough potentials. In a second part we use these results to show via a sequence of suitably constructed approximants that the ground states of massive or massless relativistic Schrödinger operators with spherical potential wells are radially decreasing functions.

Key-words: Γ-convergence, Gagliardo seminorms, Bernstein functions, fractional Laplacian, potential well, massive and massless relativistic Schrödinger operator, moving planes

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1. Introduction

The basic motivation of this paper was to prove that, roughly, for a relativistic Schrödinger operator of the type \((-\Delta + m^2)^{1/2} - m - v1_{B_a}\), where \(m \geq 0\) is the rest mass of the particle, \(B_a\) is a \(d\)-dimensional ball of radius \(a\) centred in the origin, and \(v > 0\) is the depth of the spherical potential well (a coupling constant), the ground state of this Hamiltonian is a radially monotone decreasing function in space, whenever it exists. Such a behaviour is reasonable to expect but it turns out to require some work to show it rigorously, which is presented in this paper for a wider class of non-local Schrödinger operators. An immediate implication of this monotonicity is that precise local (i.e., non-asymptotic) estimates can be derived on the ground state, as presented in [3], which do not otherwise seem to come about by alternative methods currently available. Our approach is applicable to a range of further problems regarding the local behaviour of solutions of non-local equations.

To carry this proof through we need to vary the potentials, which requires a control on convergence, i.e., a study of the question whether for a given non-local operator \(L\) and a sequence \((V_n)_{n\in\mathbb{N}}\) of potentials, for which the corresponding non-local Schrödinger operators \(\mathcal{H}_n = L + V_n\) are assumed to have the ground state eigenvalues \(\lambda_n\) and eigenfunctions \(\varphi_n\), these objects converge to the eigendata of \(\mathcal{H} = \lim_{n \to \infty} \mathcal{H}_n\), when the limit is taken in an appropriate sense. While there are classic results using strong/norm resolvent convergence due to Kato [28, Ch.8, Sect.3] and Weidmann [43] guaranteeing this stability (or continuity) of the eigenvalues for a broad class of self-adjoint operators under conditions of varying strength, our approach here also aims to develop a framework based on \(\Gamma\)-convergence of positive quadratic forms. This has several advantages. One is that the conditions we draw are rather mild and can be expressed directly in explicit terms of the sequence of potentials, and by a verifiable criterion \(\Gamma\)-convergence implies the more conventional strong resolvent convergence. Secondly, our framework also accommodates “degenerate” cases when the sequence of potentials supported on full space \(\mathbb{R}^d\) replicates the Dirichlet exterior value problem in the limit, corresponding to the situation of having zero potential in a bounded domain of \(\mathbb{R}^d\) and infinite potential outside. Thirdly, a version of our approach also covers the cases when instead of varying the potentials, the kinetic terms are varied, i.e., then continuity of the eigenvalues of a sequence \(\mathcal{H}_n = L_n + V\) is considered, where, for instance, \(L_n = (-\Delta)^{s_n}\), and \((s_n)_{n\in\mathbb{N}} \subset (0, 2)\) is some sequence convergent to one of the endpoints of the interval, which will be explored elsewhere.

Tools relying on \(\Gamma\)-convergence proved to be rather powerful in various contexts in the literature. For instance, it has been used to show stability of variational eigenvalues for the fractional \(p\)-Laplacian by Brasco, Parini and Squassina [9]. Dell’Antonio [16, 17] studied the convergence of regularized three-body Hamiltonians to contact (or point) interactions, important in the understanding of the Efimov effect. Chen and Song used the related Mosco-convergence [35] of eigenvalues of generators of subordinate Lévy processes in domains [11], Song and Li to regular subspaces of one-dimensional diffusion processes [40], and Li, Uemura and Ying similarly to regular Dirichlet subspaces [31]. Ponce provided a \(\Gamma\)-convergence based proof of the convergence of integrals initially studied by Brézis, Bourgain and Mironescu [36].

The main objects of this paper are non-local Schrödinger operators of the form \(\mathcal{H} = \Phi(-\Delta) + V\), where the kinetic term is given by a Bernstein function \(\Phi\) of the Laplacian \(\Delta\), and the potential
V is a multiplication operator picked from a Kato-class defined with respect to an integrability condition relating with Φ. (For the details see Sections 2.1-2.3 below.) Such non-local Schrödinger operators have been first introduced and analyzed in [23], covering massless (fractional) and massive relativistic Schrödinger operators as well as others whose jump kernels may have a heavy or a light decay at infinity, and they have been the object of extensive study recently. Bernstein functions of the Laplacian and related non-local equations have been used since also in other directions such as a generalization of the Caffarelli-Silvestre extension technique [30], maximum principles [5, 6], the blow-up of solutions of stochastic PDE with white or coloured noise [18], or the theory of embedded eigenvalues and scattering [24, 25].

In this set-up, our work here has two main parts. In a first part we describe these operators and their related Dirichlet forms and form domains, introducing a space $H^\Phi$ as a counterpart of the fractional Sobolev space $H^s$, $s \in (0,1)$, for more general Bernstein functions $\Phi$ (Section 2), and a corresponding semi-norm $[u]_\Phi$ generalizing the Gagliardo semi-norm. We will establish that $H^\Phi$ is continuously embedded in $H^s$ (Proposition 2.2), allowing controls in terms of the different norms. Also, we will define Dirichlet forms associated with $\Phi (-\Delta)$ and with their perturbations by $\Phi$-Kato class potentials. Then, taking a sequence $(V_n)_{n \in \mathbb{N}}$ of suitably chosen potentials convergent to a potential $V$, we establish the strong resolvent convergence of $H_n = \Phi (-\Delta) + V_n$, $n \in \mathbb{N}$, to $H = \Phi (-\Delta) + V$ via the $\Gamma$-convergence of related positive quadratic forms (Theorem 3.1). This will, in particular, yield the stability of the spectra in the sense discussed above (Corollary 3.1).

Under two further sets of conditions we obtain strong resolvent convergence (Theorems 3.2) and norm resolvent convergence (Theorem 3.3) of $(H_n)_{n \in \mathbb{N}}$ directly, leading to the same conclusion.

In a second part (Section 4) we then turn to apply these stability results to the special case of the family

$$H = (\Delta + m^{2/\alpha})^{\alpha/2} - m - v 1_{B_a}, \quad 0 < \alpha < 2, \ m \geq 0, \ a, v > 0,$$

of fractional relativistic operators perturbed by spherical potential wells. By using appropriate mollifiers from the inside to the outside of the potential well, first we construct approximants of the ground state of $H$ (Section 4.1). We show that they are bounded smooth functions (Proposition 4.1) and replicate the ground state $\varphi$ of $H$ in the limit (Proposition 4.2). Making use of the sequence of approximants we then conclude that their images under the kinetic term $L_{m,\alpha} = (\Delta + m^{2/\alpha})^{\alpha/2} - m$ of the operator converge in $L^2$ sense to $L_{m,\alpha}\varphi$ (Theorem 4.1). Showing that the ground state $\varphi$ is rotationally symmetric (Proposition 4.3), we then prove that it is radially non-increasing (Theorems 4.3-4.4). A crucial technical step in order to achieve this relies on an estimate on $L_{m,\alpha}w$ at its minimum, where $w$ is $C^2$, bounded, and antisymmetric with respect to a hyperplane (Lemma 4.3), relating with a maximum principle for narrow domains.

2. Preliminaries

2.1. Bernstein functions and Gagliardo-type semi-norms

We start by a word on the notation. We write $\text{Spec}_{\text{ess}}(A)$ for the essential spectrum, $\text{Spec}_d(A)$ for the discrete spectrum, and $\text{Spec}(A)$ for the full spectrum of operator $A$. The Schwartz space of complex-valued functions on $\mathbb{R}^d$ will be denoted by $\mathcal{S}(\mathbb{R}^d; \mathbb{C})$, and we simply write $\mathcal{S}(\mathbb{R}^d)$ for the
similar object for real-valued functions. Also, we denote the space of square-integrable complex-valued functions on $\mathbb{R}^d$ by $L^2(\mathbb{R}^d, \mathbb{C})$. For the Fourier transform of a function $h$ we write $\hat{h} := \mathcal{F}[h]$ interchangeably, as simpler or convenient. Scalar product in $\mathbb{R}^d$ will be denoted by pointed brackets $\langle \cdot, \cdot \rangle$. For $L^p$ norms, we write for simplicity $\|u\|_p = \|u\|_{L^p(\mathbb{R}^d)}$, while spell out the subscript in detail when we use a set other than $\mathbb{R}^d$; below we will work with several different norms but no confusion will occur caused by this shorthand. A constant $C$ dependent on parameters $a, b, \ldots$ will be denoted by $C(a, b, \ldots)$. A ball of radius $r > 0$ centered in $x \in \mathbb{R}^d$ will be denoted by $B_r(x)$ and simply by $B_r$ when $x = 0$.

Recall that a Bernstein function is an infinitely differentiable non-negative function on the positive semi-axis, whose derivative is completely monotone, i.e., an element of the convex cone

$$
\mathcal{B} = \left\{ \Phi \in C^\infty((0, \infty)) : \Phi \geq 0 \text{ and } (-1)^n \frac{d^n\Phi}{dx^n} \leq 0, \text{ for all } n \in \mathbb{N} \right\}.
$$

In particular, Bernstein functions are increasing and concave. It is well-known that Bernstein functions have the following canonical (Lévy-Khintchine) representation: A function $\Phi : (0, \infty) \to \mathbb{R}$ belongs to $\mathcal{B}$ if and only if there exist $a_\Phi, b_\Phi \geq 0$ and a measure $\mu_\Phi$ on the positive semi-axis satisfying $\int_{(0,\infty)} (1 + t)\mu_\Phi(dt) < \infty$ such that for every $z > 0$ the expression

$$
\Phi(z) = a_\Phi + b_\Phi z + \int_0^\infty (1 - e^{-zt})\mu_\Phi(dt)
$$

holds. The triplet $(a_\Phi, b_\Phi, \mu_\Phi)$ determines $\Phi$ uniquely and vice versa. In particular, the measure $\mu_\Phi$ is a Lévy measure as it also satisfies $\int_{\mathbb{R}} (1 + t^2)\mu_\Phi(dt) < \infty$. Furthermore, a Bernstein function $\Phi$ is said to be complete if its Lévy measure $\mu_\Phi(ds) = m_\Phi(s)ds$ is such that $m_\Phi$ is a completely monotone function. The set $\mathcal{B}_C \subset \mathcal{B}$ of complete Bernstein functions is again a convex cone. Below we will restrict to the subset

$$
\mathcal{B}_0 = \{ \Phi \in \mathcal{B}_C : a_\Phi = b_\Phi = 0 \}.
$$

A standard reference on Bernstein functions is [38], in which many details and examples are presented.

Fix $d \in \mathbb{N}$, let $\Phi \in \mathcal{B}$, and define the space

$$
H^\Phi(\mathbb{R}^d) := \left\{ h \in L^2(\mathbb{R}^d) : \sqrt{\Phi(|\cdot|^2)} \hat{h} \in L^2(\mathbb{R}^d) \right\}.
$$

Furthermore, for any $\Phi \in \mathcal{B}_C$ we define the jump kernel

$$
j_\Phi(r) := \int_0^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{r^2}{4t}} m_\Phi(t)dt. \quad (2.1)
$$

We recall some properties of the function $j_\Phi$, for a proof see [2] Prop. 2.2].

**Lemma 2.1.** The jump kernel $j_\Phi$ satisfies the following properties:

1. $j_\Phi(r) < \infty$ for every $r > 0$;
2. $j_\Phi$ is a non-negative continuous and non-increasing function with $\lim_{r \to \infty} j_\Phi(r) = 0$;
3. the integrability relations $\int_0^1 r^{d+1} j_\Phi(r)dr < \infty$ and $\int_1^\infty r^{d-1} j_\Phi(r)dr < \infty$ hold.

Next we characterize the space $H^\Phi(\mathbb{R}^d)$ by reformulating an earlier result that first appeared in [26] Lem. 3.1], stated for Schwartz space functions.
Lemma 2.2. Let \( u : \mathbb{R}^d \to \mathbb{R} \) be a measurable function. Then \( u \in H^\Phi(\mathbb{R}^d) \) if and only if \( u \in L^2(\mathbb{R}^d) \) and
\[
[u]_\Phi := \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(x+h) - u(x)|^2 j_\Phi(|h|) dx dh < \infty.
\]

Proof. Let \( \psi(x) = 1/(4x) \). We determine a Radon measure \( \nu_\Phi \) on \((0, \infty)\) such that
\[
\mu_\Phi(ds) = (4\pi s)^{d/2} \psi(\nu_\Phi)(ds),
\]
where \( \psi(\nu_\Phi)(A) = \nu_\Phi(\psi^{-1}(A)) \) for every Borel set \( A \subset (0, \infty) \). Assume that \( \nu_\Phi \) is absolutely continuous with respect to Lebesgue measure and keep denoting its density still by \( \nu_\Phi \). Setting \( A = [t_1, t_2] \) with \( 0 < t_1 < t_2 \), we have
\[
\psi(\nu_\Phi)(A) = \int_{t_1}^{t_2} \nu_\Phi(s) ds = \int_{t_1}^{t_2} \nu_\Phi \left( \frac{1}{4s} \right) \frac{1}{4s^2} ds.
\]
Hence we get
\[
m_\Phi(s) = (4\pi s)^{d/2} \nu_\Phi \left( \frac{1}{4s} \right) \frac{1}{4s^2}
\]
and thus
\[
\nu_\Phi(s) = \frac{s^{d-2}}{4\pi^{d/2}} m_\Phi \left( \frac{1}{4s} \right).
\]
Computing its Laplace transform we obtain
\[
\int_0^\infty e^{-rs} \nu_\Phi(s) ds = \int_0^\infty e^{-rs \frac{d}{2} - 2\pi^{-d/2} 4^{-1} m_\Phi \left( \frac{1}{4s} \right)} ds
\]
\[
= \int_0^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{s}{4t} m_\Phi(t)} dt = j_\Phi(\sqrt{r}).
\]
By a combination with [26, Lem. 2.1, Lem. 3.1] the result follows.

In the following we will need to compare the semi-norm \([u]_\Phi\) with the standard Gagliardo semi-norm of order \( s \in (0, 1) \), defined by
\[
[[u]]_s := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x+y) - u(x)|^2}{|y|^{d+2s}} dy dx,
\]
giving rise to the fractional Sobolev space (for a useful survey see [19])
\[
H^s(\mathbb{R}^d) := \{ u \in L^2(\mathbb{R}^d) : [[u]]_s < \infty \}.
\]
The space \( H^s(\mathbb{R}^d) \) coincides with \( H^\Phi(\mathbb{R}^d) \) for \( \Phi(z) = z^s \), \( s \in (0, 1) \). For this case Lemma [22] has been shown to hold in an even more general anisotropic setting [33].

Clearly, \([\cdot]_\Phi\) is just a semi-norm on \( H^\Phi(\mathbb{R}^d) \) and we can define a norm by
\[
\|u\|_\Phi := \|u\|_2 + [u]_\Phi.
\]
Similarly, an equivalent norm on \( H^s(\mathbb{R}^d) \) is given by
\[
\|[u]\|_s := \|u\|_2 + [[u]]_s.
\]
In regard to this norm, we will need the following Sobolev-type inequalities (see, for instance, [19 Ths. 6.7, 6.10, 8.2] for a more general formulation).

Proposition 2.1. The following properties hold:
(1) Let \( d \geq 2 \), or \( d = 1 \) and \( s < \frac{1}{2} \). There exists a constant \( C = C_{d,s} > 0 \) such that
\[
\|u\|_{p^*} \leq C_{d,s} \|u\|_s,
\]
for every \( u \in H^s(\mathbb{R}^d) \), where \( p^* = p^*(d, s) = \frac{2d}{d - 2s} \).

(2) Let \( d = 1 \) and \( s = \frac{1}{2} \). There exists a constant \( C > 0 \) such that
\[
\|u\|_q \leq C \|u\|_s,
\]
for every \( u \in H^s(\mathbb{R}^d) \) and every \( 2 \leq q < \infty \).

(3) Let \( d = 1 \) and \( s > \frac{1}{2} \). There exists a constant \( C = C_s > 0 \) such that
\[
\|u\|_{C^\alpha, \eta(\mathbb{R}^d)} \leq C_s \|u\|_s,
\]
for every \( u \in H^s(\mathbb{R}^d) \), where \( \eta = s - \frac{1}{2} \).

In this paper we will work under the following standing assumptions.

**Assumption 2.1.** Let \( \Phi \in B_0 \) and \( j_\Phi \) be the jump kernel associated with \( \Phi \) as given in (2.1). The following hold:

1. For every \( t > 0 \) we require \( \int_{\mathbb{R}^d} e^{-t\Phi(|\xi|^2)} d\xi < \infty \).
2. We assume that there exist constants \( C > 0 \) and \( s \in (0, 1) \) such that
\[
j_\Phi(r) \geq \frac{C}{r^{d+2s}}, \quad r \in (0, 1).
\]

Condition (1) has also been used in [23, Assmp. 4.1]. Condition (2) is satisfied for any \( \Phi \) regularly varying at infinity, while the slowly varying \( \Phi(z) = \log(1 + z) \) does not, which can be checked by the jump kernel explicitly known in this case.

These conditions allow us to compare the spaces \( H^s(\mathbb{R}^d) \) and \( H^\Phi(\mathbb{R}^d) \) as laid out in the following statement.

**Proposition 2.2.** Let \( \Phi \in B_0 \) satisfy Assumption 2.1. There exists a constant \( C = C(d, s) > 0 \) such that for every \( u \in H^\Phi(\mathbb{R}^d) \) we have \( \|u\|_s \leq C \|u\|_\Phi \), i.e., \( H^\Phi(\mathbb{R}^d) \) is continuously embedded in \( H^s(\mathbb{R}^d) \).

**Proof.** Clearly, we only need to prove that \( [[u]]^2_s \leq C([[u]]^2_\Phi + \|u\|^2_2) \) for all \( u \in H^\Phi(\mathbb{R}^d) \). By using Assumption 2.1.2,
\[
[[u]]^2_\Phi \geq \int_{\mathbb{R}^d} \int_{|y| \leq 1} |u(x) - u(x + y)|^2 j_\Phi(|y|) dy dx
\geq \int_{\mathbb{R}^d} \int_{|y| \leq 1} |u(x) - u(x + y)|^2 |y|^{-d-2s} dy dx.
\]
On the other hand,
\[
\int_{\mathbb{R}^d} \int_{|y| \leq 1} |u(x) - u(x + y)|^2 |y|^{-d-2s} dy = [[u]]^2_s - \int_{\mathbb{R}^d} \int_{|y| \geq 1} |u(x) - u(x + y)|^2 |y|^{-d-2s} dy dx,
\]
so that
\[
[[u]]^2_s \leq [[u]]^2_\Phi + \int_{\mathbb{R}^d} \int_{|y| \geq 1} |u(x) - u(x + y)|^2 |y|^{-d-2s} dy dx.
\]
Furthermore, we have
\[ \int_{\mathbb{R}^d} \int_{|y| \geq 1} |u(x) - u(x + y)|^2 |y|^{-d-2s} dy \leq 2 \int_{\mathbb{R}^d} \int_{|x-y| \geq 1} (|u(x)|^2 + |u(y)|^2) |x-y|^{-d-2s} dy dx \]
\[ = 4 \|u\|_2^2 \int_{|y| \geq 1} |y|^{-d-2s} dy = \frac{4 \sigma_d}{2s} \|u\|_2^2, \]
where \( \sigma_d \) is the surface area of the \( d \)-dimensional unit sphere.

We will also make use of the following result.

**Proposition 2.3.** Let \( \Phi \in \mathcal{B}_0 \) satisfy Assumption 2.1 and define \( K_M = \{ u \in H^\Phi(\mathbb{R}^d) : \|u\|_\Phi \leq M \} \).

There exists a constant \( C \) such that
\[ \int_{\mathbb{R}^d} |u(x + h) - u(x)|^2 dx \leq C |h|^{2s}, \quad u \in K_M, \tag{2.2} \]
for every \( h \in \mathbb{R}^d \) with \( 0 \leq |h| \leq 1 \). Furthermore, if \((u_n)_{n \in \mathbb{N}} \subset H^\Phi(\mathbb{R}^d) \) with \( \sup_{n \in \mathbb{N}} \|u_n\|_\Phi < \infty \), then there exists a function \( u \in L^2(\mathbb{R}^d) \) and a non-relabelled subsequence \((u_n)_{n \in \mathbb{N}}\) such that the following hold:

1. \( u_n \rightharpoonup u \) in the weak topology of \( L^2(\mathbb{R}^d) \);
2. \( u_n 1_K \rightarrow u 1_K \) in the strong topology of \( L^2(\mathbb{R}^d) \), for every compact set \( K \subset \mathbb{R}^d \);
3. \( u_n(x) \rightarrow u(x) \) for almost every \( x \in \mathbb{R}^d \).

**Proof.** (1) By the definition of \( \|\cdot\|_\Phi \), for every \( u \in K_M \) we have \( \|u\|_2 \leq M \). Furthermore, by Proposition 2.2 there exists a constant \( C_1 \) such that for every \( u \in K_M \) the bound \( \|u\|_s \leq C_1 \) applies. Hence, by [3] Lem. A.1 we have that
\[ \sup_{0 \leq |h| \leq 1} \int_{\mathbb{R}^d} \frac{|u(x + h) - u(x)|^2}{|h|^{2s}} dx \leq C, \]
with a constant \( C \) independent of \( u \in K_M \). In particular, for every \( h \in \mathbb{R}^d \) with \( 0 \leq |h| \leq 1 \), it follows that
\[ \int_{\mathbb{R}^d} |u(x + h) - u(x)|^2 dx \leq C |h|^{2s}, \quad u \in K_M. \]

Next consider (2) and let \((u_n)_{n \in \mathbb{N}} \subset H^\Phi(\mathbb{R}^d) \) and \( M = \sup_{n \in \mathbb{N}} \|u_n\|_\Phi \). Then \( \|u_n\|_2 \leq M \) and there exists \( u \in L^2(\mathbb{R}^d) \) such that a non-relabelled subsequence \((u_n)_{n \in \mathbb{N}}\) is such that \( u_n \rightharpoonup u \) in the weak topology of \( L^2(\mathbb{R}^d) \). Furthermore, note that \((u_n)_{n \in \mathbb{N}} \subset K_M \), hence (2.2) holds. For any \( \ell \in \mathbb{N} \), consider the ball \( B_{\ell} \) and let \( \eta_{\ell} \in C_\infty(\mathbb{R}^d) \) be such that \( \eta_{\ell}(x) = 1 \) for all \( x \in B_{\ell} \), \( \eta_{\ell}(x) = 0 \) for all \( x \not\in B_{\ell+1} \) and \( 0 \leq \eta_{\ell}(x) \leq 1 \) for all \( x \). First, define \( \tilde{u}_n = \eta_{\ell} u_n \). Since they are all supported in \( B_2 \), the sequence is clearly equitight. Moreover, for every \( h \in \mathbb{R}^d \) with \( 0 \leq |h| \leq 1 \) we have
\[ \int_{\mathbb{R}^d} |\eta_{\ell}(x + h)u_n(x + h) - \eta_{\ell}(x)u_n(x)|^2 dx \]
\[ \leq \int_{\mathbb{R}^d} |\eta_{\ell}(x + h) - \eta_{\ell}(x)|^2 |u_n(x + h)|^2 dx + \int_{\mathbb{R}^d} |\eta_{\ell}(x)|^2 |u_n(x + h) - u_n(x)|^2 dx \]
\[ \leq \left( \int_{\mathbb{R}^d} |u_n(x + h)|^2 dx \right) \|\nabla \eta_{\ell}\|_\infty^2 |h| + \|\eta_{\ell}\|_\infty^2 \int_{\mathbb{R}^d} |u_n(x + h) - u_n(x)|^2 dx \]
\[ \leq C(|h| + |h|^{2s}), \tag{2.3} \]
where the constant $C$ is independent of $n$. Hence, by the Fréchet-Kolmogorov compactness theorem [10] Cor. 4.27 we obtain that there exists a subsequence $(\eta_n u_n^{(1)})_{n \in \mathbb{N}}$ convergent in the strong topology of $L^2(\mathbb{R}^d)$. In particular, this implies that $u_n^{(1)}$ converges in the strong topology of $L^2(B_1)$ and, since we already know that $u_n \rightarrow u$, we also have $u_n^{(1)} \rightarrow u$ in $L^2(B_1)$, i.e., $1_{B_1} u_n^{(1)} \rightarrow 1_{B_1} u$ in $L^2(\mathbb{R}^d)$. Now assume that we already defined $u^{(\ell)}_n$ for $\ell \in \mathbb{N}$, and consider $\tilde{u}^{(\ell)}_n = \eta_{\ell+1} u_n^{(\ell)}$. Clearly, (2.3) still holds if we substitute $\eta_1$ to $\eta_1$ and the sequence $\tilde{u}^{(\ell)}_n$ is still equitight as they are all supported in $B_{\ell+2}$. Thus there exists a convergent subsequence $(\eta_{\ell+1} u_n^{(\ell+1)})_{n \in \mathbb{N}}$ in the strong topology of $L^2(\mathbb{R}^d)$. By a similar argument, we have $1_{B_{\ell+1}} u_n^{(\ell+1)} \rightarrow 1_{B_{\ell+1}} u$ in $L^2(\mathbb{R}^d)$. Clearly, for any compact set $K \subset \mathbb{R}^d$, the sequence $(u_n)_{n \in \mathbb{N}}$ is by construction such that $1_K u_n^{(n)} \rightarrow 1_K u$ in $L^2(\mathbb{R}^d)$.

To obtain (3), let $(u_k)_{k \in \mathbb{N}}$ be a subsequence as constructed in part (2) and reset the label. For any $\ell \in \mathbb{N}$ we have $u_n \rightarrow u$ in $L^2(B_1)$ and thus there exists a set $E_1$ and a subsequence $(u_n^{(1)})_{n \in \mathbb{N}}$ such that $|B_1 \setminus E_1| = 0$ and $u_n^{(1)}(x) \rightarrow u(x)$ for all $x \in E_1$. Suppose we already defined the subsequence $u_n^{(\ell)}$. Then $u_n^{(\ell)} \rightarrow u$ in $L^2(B_{\ell+1})$ and thus there exists a set $E_{\ell+1}$ and a subsequence $u_n^{(\ell+1)}$ such that $|B_{\ell+1} \setminus E_{\ell+1}| = 0$ and $u_n^{(\ell+1)}(x) \rightarrow u(x)$ for all $x \in E_{\ell+1}$. The subsequence $(u_n^{(n)})_{n \in \mathbb{N}}$ satisfies all the properties in the statement.

Another property we will use is the following.

**Proposition 2.4.** The space $C_c^\infty(\mathbb{R}^d)$ is dense in $H^\Phi(\mathbb{R}^d)$.

This result follows from [23] Th. 3.3 but we provide a more constructive proof directly using $j_\Phi$ in Appendix 5.1 below. For later use we also note the following property.

**Lemma 2.3.** Let $\Phi \in \mathcal{B}_0$. If $u, v \in H^\Phi(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then $uv \in H^\Phi(\mathbb{R}^d)$. Moreover, the same holds if $v \in C_c^\infty(\mathbb{R}^d)$ and $u \in H^\Phi(\mathbb{R}^d) \cap L^\infty_{\text{loc}}(\mathbb{R}^d)$.

**Proof.** To show the first statement, observe that

$$[uv]^2_\Phi = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(x)v(x) - u(y)v(y)|^2 j_\Phi(|x - y|)dxdy$$

$$\leq 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(x)v(x) - u(x)v(y)|^2 j_\Phi(|x - y|)dxdy$$

$$+ 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(x)v(x) - u(y)v(x)|^2 j_\Phi(|x - y|)dxdy \leq 2(\|u\|_{L^\infty}^2 [v]_{\Phi}^2 + \|v\|_{L^\infty}^2 [u]_{\Phi}^2).$$

To prove the second, let $K$ be the support of $v$ and split up the integral like

$$[uv]^2_\Phi = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(x)v(x) - u(y)v(y)|^2 j_\Phi(|x - y|)dxdy$$

$$= \int_K \int_K |u(x)v(x) - u(y)v(y)|^2 j_\Phi(|x - y|)dxdy$$

$$+ 2 \int_{\mathbb{R}^d \setminus K} \int_K |u(x)v(x)|^2 j_\Phi(|x - y|)dxdy := I_1 + I_2.$$

Arguing as before, we get

$$I_1 \leq (\|u\|^2_{L^\infty(K)} [v]_{\Phi}^2 + \|v\|^2_{L^\infty(K)} [u]_{\Phi}^2).$$
To handle $I_2$, fix $\varepsilon > 0$ and let $K_\varepsilon = \{x \in \mathbb{R}^d : \text{dist}(x, K) \leq \varepsilon\}$. Then

$$I_2 = 2 \int_{\mathbb{R}^d \setminus K_\varepsilon} \int_K |u(x)v(x)|^2 j_\Phi(|x - y|)dxdy + 2 \int_{K_\varepsilon \setminus K} \int_K |u(x)v(x)|^2 j_\Phi(|x - y|)dxdy = I_3 + I_4.$$ 
Furthermore,

$$I_3 \leq 2 \|v\|_2^2 \int_K |u(x)|^2 \int_{\mathbb{R}^d \setminus K_\varepsilon} j_\Phi(|x - y|)dydx.$$

Clearly, if $x \in K$, we have $\mathbb{R}^d \setminus K_\varepsilon \subset \mathbb{R}^d \setminus B_\varepsilon(x)$ and then

$$I_3 \leq 2 \|v\|_2^2 \int_K |u(x)|^2 \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} j_\Phi(|x - y|)dydx \leq 2 \|v\|_2^2 \|u\|_2^2 \int_{\mathbb{R}^d \setminus K_\varepsilon} j_\Phi(|y|)dy < \infty.$$

Finally, observe that if $y \in K_\varepsilon \setminus K$, then $u(x)v(y) = 0$ and we have

$$I_4 = 2 \int_{K_\varepsilon \setminus K} \int_K |u(x)|^2 |v(x) - v(y)|^2 j_\Phi(|x - y|)dxdy$$

$$\leq 2 \|\nabla v\|_\infty^2 \int_K |u(x)|^2 \int_{K_\varepsilon \setminus K} |x - y|^2 j_\Phi(|x - y|)dydx.$$

Clearly, there exists $R > 0$ such that $K_\varepsilon \setminus K \subset B_R(x)$ and then

$$I_4 \leq 2 \|\nabla v\|_\infty^2 \|u\|_2^2 \int_{B_R(x)} |y|^2 j_\Phi(|y|)dy < \infty.$$

2.2. Bernstein functions of the Laplacian

Let $\Delta$ be the Laplacian and $\Phi \in \mathcal{B}_0$. For any measurable function $u : \mathbb{R}^d \to \mathbb{R}$ we define

$$\Phi(-\Delta)u(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} (u(x) - u(y)) j_\Phi(|x - y|)dy,$$

provided the involved quantities are finite. As $u \in H^1(\mathbb{R}^d)$, this expression is a direct consequence of the Bochner-Phillips subordination formula (see, for instance, [38, Ex. 11.6]). An alternative definition can be made via the following expression (for a proof see [2, Prop. 2.6]).

**Proposition 2.5.** Let $\Phi \in \mathcal{B}_0$, $x \in \mathbb{R}^d$ and a measurable function $u : \mathbb{R}^d \to \mathbb{R}$. Then

$$\Phi(-\Delta)u(x) = -\frac{1}{2} \int_{\mathbb{R}^d} (u(x + h) - 2u(x) + u(x - h)) j_\Phi(|h|)dh,$$

provided the integral at the right-hand side is convergent.

We want to establish a condition under which the integral at the right hand side of (2.4) is absolutely convergent. In [2] we considered a Hölder-Zygmund type condition. Here we aim to use a more classical approach in order to prove that whenever $u \in H^\Phi(\mathbb{R}^d)$, we can define $\Phi(-\Delta)u \in L^2(\mathbb{R}^d)$ by extension of a bounded linear operator. This property is well-known for the case of the fractional Laplacian corresponding to $\Phi(z) = z^\alpha$, see [19, Sect. 3]. First we prove the following technical lemma.
Lemma 2.4. If \( u \in C^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \), then the expression at the right-hand side of (2.4) is absolutely convergent for every \( x \in \mathbb{R}^d \).

Proof. Fix \( x \in \mathbb{R}^d \) and split up the integral in (2.4) like

\[
\int_{\mathbb{R}^d} |u(x+h)-2u(x)+u(x-h)|j_\Phi(|h|)dh = \left( \int_{B_1} + \int_{\mathbb{R}^d \setminus B_1} \right) |u(x+h)-2u(x)+u(x-h)|j_\Phi(|h|)dh
\]

Equation (2.4)

\[
= I_1 + I_2.
\]

To estimate \( I_1 \), we note that due to \( u \in C^2(\mathbb{R}^d) \) there exists \( C = C(x) > 0 \) such that

\[
|u(x+h)-2u(x)+u(x-h)| \leq C|h|^2, \quad h \in B_1,
\]

so that \( I_1 \leq C \int_{B_1} |h|^2 j_\Phi(|h|)dh < \infty \), as a consequence of Lemma 2.1(3). Also, we have \( I_2 \leq 4 \|u\|_\infty \int_{\mathbb{R}^d \setminus B_1} j_\Phi(|h|)dh < \infty \) by the same lemma.

Next we further use an equivalent formulation of \( \Phi(-\Delta) \), involving Schwartz space \( \mathcal{S}(\mathbb{R}^d) \). Taking \( u \in \mathcal{S}(\mathbb{R}^d) \), we have

\[
\mathcal{F}[\Phi(-\Delta)u](\xi) = \Phi(|\xi|^2)\hat{u}(\xi). \tag{2.5}
\]

This is a known result, but for the convenience of the reader we provide a proof in an appendix (Section 5.2 below), showing the precise relations with the set-up adopted in this paper. By a simple application of Plancherel’s theorem, Lemma 2.2 and Proposition 2.4, we have then the following consequence.

Corollary 2.1. If \( u \in \mathcal{S}(\mathbb{R}^d) \), then \( \|\sqrt{\Phi(-\Delta)}u\|_2 = |u|_\Phi \). Furthermore, \( \sqrt{\Phi(-\Delta)} \) can be extended uniquely to a self-adjoint bounded linear operator on \( H^s(\mathbb{R}^d) \) by setting

\[
\sqrt{\Phi(-\Delta)}u = \mathcal{F}^{-1}[\sqrt{\Phi(|\cdot|^2)}\hat{u}], \quad u \in H^s(\mathbb{R}^d).
\]

Using the representation (2.5) specifically for the space \( H^s(\mathbb{R}^d) \), \( s > 0 \), we also show the following for later use.

Lemma 2.5. Let \( u \in H^s(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \) and \( v \in C_\infty^\infty(\mathbb{R}^d) \). Then \( uv \in H^s(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \).

Proof. First observe that since \( v \in C_\infty^\infty(\mathbb{R}^d) \), then also \( v \in L^1(\mathbb{R}^d) \). Furthermore, since \( v \in C_\infty^\infty(\mathbb{R}^d) \), by the Hölder inequality we have \( uv \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \). By the convolution property \( \mathcal{F}[uv] = \tilde{u} \ast \tilde{v} \) of Fourier transform we have

\[
\int_{\mathbb{R}^d} |\xi|^{2s}|\mathcal{F}[uv](\xi)|^2d\xi = \int_{\mathbb{R}^d} |\xi|^{2s} \left| \int_{\mathbb{R}^d} \tilde{v}(y)\hat{u}(\xi - y)dy \right|^2 d\xi
\leq \|v\|_\infty^2 \int_{\text{supp}(v)} \int_{\mathbb{R}^d} |\xi|^{2s}\hat{u}(\xi - y)^2d\xi dy
\]

\[
= \|v\|_\infty^2 \int_{\text{supp}(v)} \int_{\mathbb{R}^d} |\xi + y|^{2s}\hat{u}(\xi)^2d\xi dy.
\]
Now let $R > 0$ be such that $\text{supp}(v) \subset B_R$. If $\xi \in \mathbb{R}^d \setminus B_{3R}$, then $\max_{y \in B_R} |\xi + y|^{2s} = (|\xi| + R)^{2s} \leq C|\xi|^{2s}$ for a constant $C > 0$. Then we have

$$
\int_{\mathbb{R}^d} |\xi|^{2s} |\mathcal{F}[uv](\xi)|^2 d\xi \leq \|v\|^2_2 \int_{B_R} \int_{B_{3R}} |\xi + y|^{2s} |\widehat{u}(\xi)|^2 d\xi dy
$$

$$
+ C \|v\|^2_2 \int_{B_R} \int_{\mathbb{R}^d \setminus B_{5R}} |\xi|^{2s} |\widehat{u}(\xi)|^2 d\xi dy
$$

$$
\leq \|v\|^2_2 \|\widehat{u}(\xi)\|_{L^\infty(B_{2R})}^2 R^{2(d+s)} 4^{2(d+s)} 3^d \omega_d^2
$$

$$
+ C \|v\|^2_2 R^d \omega_d \int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{u}(\xi)|^2 d\xi < \infty.
$$

2.3. Dirichlet forms, non-local Schrödinger operators and ground states

Clearly, the above description is not sufficient to guarantee the pointwise existence of $\Phi(-\Delta)u$ when $u \in H^\Phi(\mathbb{R}^d)$, thus to handle integro-differential equations involving these operators a weak formulation is needed. The operator $\Phi(-\Delta)$ naturally induces a Dirichlet form acting on $H^\Phi(\mathbb{R}^d) \times H^\Phi(\mathbb{R}^d)$ as

$$
\mathcal{E}_\Phi(u, v) = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(y) - u(x))(v(y) - v(x)) j_\Phi(|x - y|) dx dy.
$$

The fact that $\mathcal{E}_\Phi$ is well-defined on $H^\Phi(\mathbb{R}^d) \times H^\Phi(\mathbb{R}^d)$ is an easy consequence of Young’s inequality $2|ab| \leq a^2 + b^2$, since

$$
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(y) - u(x)||v(y) - v(x)| j_\Phi(|x - y|) dx dy \leq \frac{1}{2} (\|u\|^2_\Phi + \|v\|^2_\Phi).
$$

The link between the Dirichlet form $\mathcal{E}_\Phi$ and the operator $\Phi(-\Delta)$ can be seen by choosing two functions $u, v \in \mathcal{S}(\mathbb{R}^d)$ and observing that

$$
\int_{\mathbb{R}^d} \int_{|x - y| > \varepsilon} (u(x) - u(y))v(x) j_\Phi(|x - y|) dy dx
$$

$$
= \frac{1}{2} \int_{\mathbb{R}^d} \int_{|x - y| > \varepsilon} (u(x) - u(y))v(x) j_\Phi(|x - y|) dx dy
$$

$$
+ \frac{1}{2} \int_{\mathbb{R}^d} \int_{|x - y| > \varepsilon} (u(x) - u(y))v(x) j_\Phi(|x - y|) dx dy
$$

$$
= \frac{1}{2} \int_{\mathbb{R}^d} \int_{|x - y| > \varepsilon} (u(x) - u(y))v(x) j_\Phi(|x - y|) dx dy
$$

$$
- \frac{1}{2} \int_{\mathbb{R}^d} \int_{|x - y| > \varepsilon} (u(x) - u(y))v(y) j_\Phi(|x - y|) dx dy
$$

$$
= \frac{1}{2} \int_{\mathbb{R}^d} \int_{|x - y| > \varepsilon} (u(x) - u(y))(v(x) - v(y)) j_\Phi(|x - y|) dx dy.
$$

Taking the limit as $\varepsilon \to 0$, by an application of the dominated convergence theorem at the right-hand side we get

$$
\langle \Phi(-\Delta)u, v \rangle = \mathcal{E}_\Phi(u, v).
$$

Due to this identification, we can give the following alternative formulation of $\mathcal{E}_\Phi$. 
Proposition 2.6. For every \( u, v \in H^\Phi(\mathbb{R}^d) \) we have
\[
E_\Phi(u, v) = \int_{\mathbb{R}^d} \Phi(|\xi|^2) \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi.
\]

Proof. First consider \( u, v \in \mathcal{S}(\mathbb{R}^d) \). Then by Plancherel\'s theorem,
\[
E_\Phi(u, v) = (\Phi(-\Delta) u, v) = \int_{\mathbb{R}^d} \mathcal{F}(\Phi(-\Delta) u) \overline{\mathcal{F}(v)} d\xi = \int_{\mathbb{R}^d} \Phi(|\xi|^2) \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi.
\]
Next let \( u, v \in H^\Phi(\mathbb{R}^d) \) and take two sequences \( (u_n)_{n \in \mathbb{N}}, (v_n)_{n \in \mathbb{N}} \subset C^\infty_c(\mathbb{R}^d) \) such that \( u_n \to u \) and \( v_n \to v \) in \( H^\Phi(\mathbb{R}^d) \), which exist by Proposition 2.1. Clearly, \( E_\Phi \) is a symmetric non-negative defined bilinear form. However, \( E_\Phi(\cdot, \cdot) + \langle \cdot, \cdot \rangle \) is a scalar product on \( H^\Phi \) with induced norm given by \( \|u\|_\Phi \).
In particular, by the Schwarz inequality,
\[
|E_\Phi(u, v) - E_\Phi(u_n, v_n)| = |E_\Phi(u, v - v_n) + E_\Phi(u - u_n, v_n)|
\leq |E_\Phi(u, v - v_n)| + |E_\Phi(u - u_n, v_n)|
+ |\langle u, v - v_n \rangle| + |\langle u_n, v - v_n \rangle|
\leq \|u\|_\Phi \|v - v_n\|_\Phi + \|u - u_n\|_\Phi \|v_n\|_\Phi + \|u\|_2 \|v - v_n\|_2
+ \|u_n\|_2 \|v - v_n\|_2
\leq 2(\|u\|_\Phi \|v - v_n\|_\Phi + \|u - u_n\|_\Phi \|v_n\|_\Phi).
\]
Since \( v_n \to v \) in \( H^\Phi(\mathbb{R}^d) \), there exists a constant \( C > 0 \) such that for \( n \) large enough we have \( \|v_n\|_\Phi \leq C \). Hence for sufficiently large \( n \)
\[
|E_\Phi(u, v) - E_\Phi(u_n, v_n)| \leq C(\|v - v_n\|_\Phi + \|u - u_n\|_\Phi) \to 0, \quad n \to \infty,
\]
which implies \( \lim_{n \to \infty} E_\Phi(u_n, v_n) = E_\Phi(u, v) \). Now observe that
\[
E_\Phi(u_n, v_n) = \int_{\mathbb{R}^d} \Phi(|\xi|^2) \hat{u}_n(\xi) \overline{\hat{v}_n(\xi)} d\xi = \langle \sqrt{\Phi(|\cdot|^2)} \hat{u}_n, \sqrt{\Phi(|\cdot|^2)} \hat{v}_n \rangle_C, \tag{2.6}
\]
where \( \langle \cdot, \cdot \rangle_C \) denotes the scalar product on the space of square-integrable complex valued functions \( L^2(\mathbb{R}^d; \mathbb{C}) \). Again by Schwarz inequality
\[
|\langle \sqrt{\Phi(|\cdot|^2)} \hat{u}_n, \sqrt{\Phi(|\cdot|^2)} \hat{v}_n \rangle_C - \langle \sqrt{\Phi(|\cdot|^2)} \hat{u}, \sqrt{\Phi(|\cdot|^2)} \hat{v} \rangle_C|
\leq |\langle \sqrt{\Phi(|\cdot|^2)} \hat{u}, \sqrt{\Phi(|\cdot|^2)}(\hat{v} - \hat{v}_n) \rangle_C| + |\langle \sqrt{\Phi(|\cdot|^2)}(\hat{u} - \hat{u}_n), \sqrt{\Phi(|\cdot|^2)} \hat{v}_n \rangle_C|
\leq \left\| \sqrt{\Phi(|\cdot|^2)} \hat{u} \right\|_\Phi \left\| \sqrt{\Phi(|\cdot|^2)}(\hat{v} - \hat{v}_n) \right\|_\Phi + \left\| \sqrt{\Phi(|\cdot|^2)} \hat{v}_n \right\|_\Phi \left\| \sqrt{\Phi(|\cdot|^2)}(\hat{u} - \hat{u}_n) \right\|_\Phi
= [u]_\Phi[v - v_n]_\Phi + [v_n]_\Phi[u - u_n]_\Phi
\leq C([v - v_n] + [u - u_n]) \to 0 \quad \text{as} \quad n \to \infty,
\]
which implies
\[
\lim_{n \to \infty} \langle \sqrt{\Phi(|\cdot|^2)} \hat{u}_n, \sqrt{\Phi(|\cdot|^2)} \hat{v}_n \rangle_C = \langle \sqrt{\Phi(|\cdot|^2)} \hat{u}, \sqrt{\Phi(|\cdot|^2)} \hat{v} \rangle_C.
\]
Taking the limit on both sides of \( (2.6) \) we get the desired result. \( \square \)

Consider the heat kernel \( p_\Phi^t(x) \) of \( \Phi(-\Delta) \), defined via inverse Fourier transform as
\[
p_\Phi^t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-t\Phi(|\xi|^2)} d\xi, \quad x \in \mathbb{R}^d,
\]
and the 1-resolvent kernel 
\[ G^\Phi_1(x) = \int_0^\infty e^{-t} p^\Phi_t(x) dt, \quad x \in \mathbb{R}^d. \]

For \( \Phi \in \mathcal{B}_0 \), we know (see, for instance, [23, Sect. 3.1]) that there exists a probability measure \( g_\Phi(t, \cdot) \) on \((0, \infty)\) such that
\[ p^\Phi_t(x) = \frac{1}{(4\pi)^{d/2}} \int_0^\infty s^{-\frac{d}{2}} e^{-\frac{|x|^2}{4s}} g_\Phi(t, ds). \]

In particular, it is clear that \( p^\Phi_t \) is radially non-increasing for all \( t > 0 \) and \( \|p^\Phi_t\|_1 = 1 \). We will use a suitable class of functions that we will use as potentials, see [23, Sect. 4.1], [32, Def. 4.280].

**Definition 2.1.** We say that a non-negative function \( V \) belongs to \( \Phi\)-Kato class \( \mathcal{K}^\Phi(\mathbb{R}^d) \) whenever
\[ \lim_{|y| \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y|<\delta} G^\Phi_1(x-y)V(y)dy = 0. \]

Also, we say that a function \( V : \mathbb{R}^d \to \mathbb{R} \) is \( \Phi\)-Kato decomposable if \( V^- \in \mathcal{K}^\Phi(\mathbb{R}^d) \) and \( 1_KV^+ \in \mathcal{K}^\Phi(\mathbb{R}^d) \) for every compact set \( K \subset \mathbb{R}^d \), where \( V^+(x) = \max\{0, V(x)\} \) and \( V^-(x) = \max\{0, -V(x)\} \), \( x \in \mathbb{R}^d \). We denote the set of \( \Phi\)-Kato decomposable functions by \( \mathcal{K}_{\text{dec}}^\Phi(\mathbb{R}^d) \).

Clearly, if \( V \) is such that \( V^- \in L^\infty(\mathbb{R}^d) \) and \( V^+ \in L^\infty_{\text{loc}}(\mathbb{R}^d) \), then \( V \in \mathcal{K}_{\text{dec}}^\Phi(\mathbb{R}^d) \).

For any potential \( V \in \mathcal{K}_{\text{dec}}^\Phi(\mathbb{R}^d) \) we define the non-local Schrödinger operator \( \mathcal{H}_{\Phi,V} \) by
\[ \mathcal{H}_{\Phi,V} = \Phi(-\Delta) + V, \]
whose core is given by \( C_c^\infty(\mathbb{R}^d) \). Such an operator has a self-adjoint realization, see [23, Th. 4.8] and [32, Th. 4.285]).

We introduce the symmetric bilinear form
\[ A_{\Phi,V}(u,v) = \mathcal{E}_{\Phi}(u,v) + \langle V u, v \rangle, \]
with core \( C_c^\infty(\mathbb{R}^d) \times C_c^\infty(\mathbb{R}) \). Clearly, it is well-defined on \( \mathcal{D}(A_{\Phi,V}) \times \mathcal{D}(A_{\Phi,V}) \), where
\[ \mathcal{D}(A_{\Phi,V}) = \left\{ u \in H^\Phi(\mathbb{R}^d) : \int_{\mathbb{R}^d} V(x)u^2(x)dx < \infty \right\}. \]

Weak solutions of the Schrödinger equation \( \mathcal{H}_{\Phi,V}u = f \) are then defined as functions \( u \in H^\Phi(\mathbb{R}^d) \) such that
\[ A_{\Phi,V}(u,v) = \langle f, v \rangle, \quad v \in C_c^\infty(\mathbb{R}^d), \]
holds. Consider the respectively associated quadratic forms, extended to \( L^2(\mathbb{R}^d) \), defined by
\[ \mathcal{E}_{\Phi}[u] = \begin{cases} \mathcal{E}_{\Phi}(u,u) & \text{if } u \in H^\Phi(\mathbb{R}^d) \\ \infty & \text{if } u \notin H^\Phi(\mathbb{R}^d) \end{cases} \quad \text{and} \quad A_{\Phi,V}[u] = \begin{cases} A_{\Phi,V}(u,u) & \text{if } u \in \mathcal{D}(A_{\Phi,V}) \\ \infty & \text{if } u \notin \mathcal{D}(A_{\Phi,V}) \end{cases}. \]

Among potentials \( V \in \mathcal{K}_{\text{dec}}^\Phi(\mathbb{R}^d) \), there are two large subclasses of special interest:

- **confining potentials**: \( V \in L^1_{\text{loc}}(\mathbb{R}^d) \) such that \( V^- \in L^\infty(\mathbb{R}^d) \) and \( \lim_{|x| \to \infty} V(x) = \infty \); in this case, by a simple application of Rellich’s criterion [32, Th. 4.71] it follows that \( \text{Spec}(\mathcal{H}_{\Phi,V}) = \text{Spec}_{\text{d}}(\mathcal{H}_{\Phi,V}) \).

- **decaying potentials**: \( V \in L^{\infty,0}(\mathbb{R}^d) := \{ h \in L^\infty(\mathbb{R}^d) : \lim_{|x| \to \infty} h(x) = 0 \} \); in this case \( \mathcal{D}(A_{\Phi,V}) = H^\Phi(\mathbb{R}^d), \text{Spec}_{\text{ess}}(\mathcal{H}_{\Phi,V}) = [0, \infty) \) and \( \text{Spec}_{\text{d}}(\mathcal{H}_{\Phi,V}) \neq \emptyset \) as soon as there exists \( u \in H^\Phi(\mathbb{R}^d) \) such that \( A_{\Phi,V}[u] < 0. \)
Whenever \( \text{Spec}_d(\mathcal{H}_{\Phi,V}) \neq \emptyset \), the lowest lying eigenvalue \( \lambda_0 = \min \text{Spec}_d(\mathcal{H}_{\Phi,V}) \), called ground state eigenvalue (or ground state energy) plays a special role. Since the semigroup \( \{ e^{-t\mathcal{H}_{\Phi,V}} : t \geq 0 \} \) is positivity improving, by the Perron-Frobenius theorem \([32, \text{Th. 4.123}]\) the ground state eigenvalue \( \lambda_0 \) is simple and the uniquely corresponding eigenfunction \( \varphi_0 \) has a strictly positive version, which we will use throughout. Also, we choose \( \| \varphi_0 \|_2 = 1 \) by usual convention. Furthermore, as a direct consequence of the fact that \( e^{-t\mathcal{H}_{\Phi,V}} : L^p(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d), t > 0 \), is continuous for every \( 1 \leq p,q \leq \infty \), we know that \( \varphi_0 \in L^\infty(\mathbb{R}^d) \). Also, since \( p^*_t \Phi \in L^1(\mathbb{R}^d) \), it follows that \( \varphi_0 \in C(\mathbb{R}^d) \). For details we refer to \([32, \text{Prop. 4.291}]\).

Dirichlet forms and related operators can also be defined for cases when \( \mathbb{R}^d \) is replaced by a sufficiently regular bounded open set \( \Omega \subset \mathbb{R}^d \). The non-local Dirichlet Laplacian \( \Phi(-\Delta)_\Omega \) is then acting on functions \( u \in H^\Phi(\mathbb{R}^d) \) such that \( u(x) = 0 \) for every \( x \in \mathbb{R}^d \setminus \Omega \). Formally, the non-local Dirichlet Laplacian acts on the closure \( H^\Phi_0(\Omega) \) of \( C^\infty_c(\Omega) \) into \( H^\Phi(\mathbb{R}^d) \). The corresponding Dirichlet form \( \mathcal{E}_{\Phi,\Omega} \) can be extended to the whole space \( L^2(\mathbb{R}) \) by writing

\[
\mathcal{E}_{\Phi,\Omega}[u] = \begin{cases} 
\mathcal{E}_{\Phi,\Omega}(u,u) & u \in H^\Phi_0(\Omega) \\
\infty & u \notin H^\Phi_0(\Omega).
\end{cases}
\]

We can define the eigenvalues of the non-local Dirichlet Laplacian as the values \( \lambda \) such that

\[
\begin{align*}
\Phi(-\Delta)u(x) &= \lambda u(x) & x \in \Omega \\
u &= 0 & x \notin \Omega \quad \text{or equivalently} \quad \Phi(-\Delta)_\Omega u = \lambda u, \; u \in H^\Phi_0(\Omega).
\end{align*}
\]

For our purposes below, we will need \( \Omega = B_r, r > 0 \), and the corresponding principal Dirichlet eigenvalue \( \lambda_r > 0 \). Furthermore, denote by \( f_r \in L^2(\mathbb{R}^d) \) the corresponding eigenfunction, such that \( f_r \geq 0 \) and \( \| f_r \|_2 = 1 \), and by \( \mathcal{E}_{\Phi,B_r}(f_r,f_r) \) the associated Dirichlet form. It is known \([1, \text{Cor. 2.3}]\) that \( f_r \) is a radial, non-increasing function, and clearly \( \text{supp}(f_r) = B_r \).

We can use the Dirichlet eigenfunction to find a sufficient condition for the existence of a ground state of \( \mathcal{H}_{\Phi,V} \) as an operator on \( L^2(\mathbb{R}^d) \).

**Proposition 2.7.** Let \( V \in L^{\infty,0}(\mathbb{R}^d) \) and suppose that there exists \( r > 0 \) such that \( V(x) \leq v < 0 \) in \( B_r \) and \( \lambda_r - v < 0 \). Then \( \mathcal{H}_{\Phi,V} \) has a ground state.

**Proof.** Note that \( \mathcal{E}_\Phi(f_r,f_r) = \mathcal{E}_{\Phi,B_r}(f_r,f_r) = \lambda_r \), as \( f_r \in H^\Phi(\mathbb{R}^d) \). Hence we have

\[
\mathcal{A}_{\Phi,V}(f_r,f_r) = \lambda_r + \int_{B_r} V(x)f_r^2(x)dx \leq \lambda_r - v < 0.
\]

\[\square\]

In the remainder of this paper we will work under a second standing assumption as follows.

**Assumption 2.2.** There exists a unique, strictly positive ground state of \( \mathcal{H}_{\Phi,V} \).

We also note an alternative (variational) characterization of the ground state eigenvalue. If \( V \in K^{\Phi}_{\text{dec}}(\mathbb{R}^d) \), the operator \( \mathcal{H}_{\Phi,V} \) is self-adjoint on its domain, hence we can use Courant’s min-max theorem to write

\[
\lambda_0 = \min_{u \in H^\Phi(\mathbb{R}^d)} \frac{\mathcal{A}_{\Phi,V}(u,u)}{\| u \|_2^2}.
\]

\[(2.7)\]
2.4. Modes of convergence

Since the following notions of convergence will play a key role in this paper, we briefly recall some definitions.

Definition 2.2. Let $X$ be a metric space, $(F_n)_{n \in \mathbb{N}}$ a sequence of functionals $F_n : X \to \mathbb{R}$, $n \in \mathbb{N}$, and $F : X \to \mathbb{R}$ another functional. The sequence $(F_n)_{n \in \mathbb{N}}$ is said to be $\Gamma$-convergent to $F$, denoted $F = \Gamma - \lim_{n \to \infty} F_n$, whenever

1. for every sequence $(x_n)_{n \in \mathbb{N}} \subset X$ it follows that $F(x) \leq \liminf_{n \to \infty} F_n(x)$;
2. for every $x \in X$ there exists a sequence $(x_n)_{n \in \mathbb{N}} \subset X$ convergent to $x$ such that $F(x) \geq \limsup_{n \to \infty} F_n(x_n)$.

If $X = L^2(\mathbb{R}^d)$, we denote $F = s\Gamma - \lim_{n \to \infty} F_n$ to say that $(F_n)_{n \in \mathbb{N}}$ is $\Gamma$-convergent to $F$ in the strong topology of $L^2(\mathbb{R}^d)$, while we denote $F = w\Gamma - \lim_{n \to \infty} F_n$ if $(F_n)_{n \in \mathbb{N}}$ is $\Gamma$-convergent to $F$ in the weak topology of $L^2(\mathbb{R}^d)$.

We also recall the following concept.

Definition 2.3. Let $X$ be a metric space. A functional $F : X \to \mathbb{R}$ is called coercive if for every $M > 0$ there exists a compact set $K_M$ such that if $x \not\in K_M$, then $F(x) > M$ (or equivalently, if there exists a compact set $K_M$ such that if $F(x) \leq M$, then $x \in K_M$). A sequence of functionals $(F_n)_{n \in \mathbb{N}}$, $F_n : X \to \mathbb{R}$, is called equicoercive if for every $M > 0$ there exists a compact set $K_M$ (independent of $n$) such that, for all $n \in \mathbb{N}$, the relation $F_n(x) \leq M$ implies $x \in K_M$.

We will also make use of the following form of convergence of self-adjoint operators (see, for instance, [11], Sect. 6.6). Let $X$ be a Hilbert space and $A_n : \mathcal{D}(A_n) \subset X \to X$ be a sequence of closed self-adjoint operators with dense core $\mathcal{D}(A_n)$. Assume $A : \mathcal{D}(A) \subset X \to X$ is a closed self-adjoint operator with dense core $\mathcal{D}(A) \subseteq \mathcal{D}(A_n)$ for every $n \in \mathbb{N}$. Let $\rho(A_n), \rho(A)$ be the resolvents respectively of $A_n$ and $A$, and consider $R_{A_n} : \lambda \in \rho(A_n) \to (\lambda - A)^{-1} \in \mathcal{L}(X)$, where $\mathcal{L}(X)$ denotes the space of bounded linear operators of $X$ on itself.

Definition 2.4. Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of self-adjoint operators.

1. Convergence $A_n \to A$ holds in strong resolvent sense if $\rho(A) \cap (\cap_{n \in \mathbb{N}} \rho(A_n)) \neq \emptyset$ and there exists $\lambda \in \rho(A) \cap (\cap_{n \in \mathbb{N}} \rho(A_n))$ such that $R_{A_n}(\lambda)u \to R_A(\lambda)u$ in the strong topology of $X$, for every $u \in \mathcal{D}(A) \subseteq \mathcal{D}(A_n)$. We denote it as $A = \text{SR} - \lim_{n \to \infty} A_n$.

2. Convergence $A_n \to A$ holds in norm resolvent sense if $\rho(A) \cap (\cap_{n \in \mathbb{N}} \rho(A_n)) \neq \emptyset$ and there exists $\lambda \in \rho(A) \cap (\cap_{n \in \mathbb{N}} \rho(A_n))$ such that $R_{A_n}(\lambda) \to R_A(\lambda)$ in the strong topology of $\mathcal{L}(X)$. We denote it as $A = \text{NR} - \lim_{n \to \infty} A_n$.

Recall that if $A = \text{SR} - \lim_{n \to \infty} A_n$, then $R_{A_n}(\lambda)u \to R_A(\lambda)u$ in the strong topology of $X$, for all $u \in \mathcal{D}(A_n) \subseteq \mathcal{D}(A)$ and every $\lambda \in \rho(A) \cap (\cap_{n \in \mathbb{N}} \rho(A_n))$. Also, recall that if $A = \text{NR} - \lim_{n \to \infty} A_n$, then $R_{A_n}(\lambda) \to R_A(\lambda)$ in the strong topology of $\mathcal{L}(X)$ for every $\lambda \in \rho(A) \cap (\cap_{n \in \mathbb{N}} \rho(A_n))$. For a proof see [11], Cor. 6.32.

The relationship of interest in our context between these modes of convergence is summarized by the following; for a proof we refer to [14], Th. 13.6].
Proposition 2.8. Let $X$ be a Hilbert space with norm $\|\cdot\|_X$ induced by its scalar product. Also, let $(A_n)_{n\in\mathbb{N}}$ be a sequence of positive quadratic forms $A_n : L^2(\mathbb{R}^d) \to \mathbb{R}$, $n \in \mathbb{N}$, and $A : L^2(\mathbb{R}^d) \to \mathbb{R}$ another positive quadratic form. Furthermore let $(A_n)_{n\in\mathbb{N}}$ be the sequence of positive self-adjoint linear operators on $L^2(\mathbb{R}^d)$ correspondingly defined by the quadratic forms $(A_n)_{n\in\mathbb{N}}$, and $A$ be the positive self-adjoint linear operator on $L^2(\mathbb{R}^d)$ defined by the quadratic form $A$. Then the following two properties are equivalent:

1. There exists a constant $C_A > 0$ such that
   \[
   w\Gamma - \lim_{n \to \infty} (A_n + C_A \|\cdot\|_X^2) = s\Gamma - \lim_{n \to \infty} (A_n + C_A \|\cdot\|_X^2) = (A + C_A \|\cdot\|_X^2).
   \]

2. $(A_n)_{n\in\mathbb{N}}$ converges to $A$ in strong resolvent sense.

3. Continuity of ground state eigenvalues

In the following we prove the stability of the ground state eigenvalue with respect to suitable variations of the potential. We will make use of the connection between $\Gamma$-convergence and strong resolvent convergence as given in Proposition 2.8. We have the following result.

Theorem 3.1. Let $\Phi \in \mathcal{B}_0$ satisfy Assumption [2.1] for some $s \in (0,1)$. Also, let $(V_k)_{k\in\mathbb{N}} \subset \mathcal{K}_{\text{dec}}(\mathbb{R}^d)$ and $V \in \mathcal{K}_{\text{dec}}(\mathbb{R}^d)$ be such that

1. $\max\{\sup_{k \in \mathbb{N}} \|V_k(\cdot)\|_{\infty}, \|V(\cdot)\|_{\infty}\} =: C_V < \infty$;
2. $V_k(x) \to V(x)$ as $k \to \infty$ holds for almost every $x \in \mathbb{R}^d$;
3. for every $u \in D(A_{\Phi,V})$ we have $\limsup_{k \to \infty} \int_{\mathbb{R}^d} V_k(x)u^2(x)dx \leq \int_{\mathbb{R}^d} V(x)u^2(x)dx$.

Then $H_{\Phi,V} = \text{SR} - \lim_{k \to \infty} H_{\Phi,V_k}$.

Proof. For easing the notation, we write $A_{\Phi,k} := A_{\Phi,V_k}$, $A_{\Phi} := A_{\Phi,V}$, $H_{\Phi,k} := H_{\Phi,V_k}$ and $H_{\Phi} := H_{\Phi,V}$. We show that for every $\varepsilon > 0$

\[
\lim_{n \to \infty} \left( A_{\Phi,k} + (C_V + \varepsilon) \|\cdot\|_2^2 \right) = \lim_{n \to \infty} \left( A_\Phi + (C_V + \varepsilon) \|\cdot\|_2^2 \right).
\]

First we check part (2) of Definition 2.2. Consider $u \in L^2(\mathbb{R}^d)$. If $u \not\in D(A_{\Phi})$, then $A_{\Phi}[u] = \infty$ and the statement is trivial. Thus consider $u \in D(A_{\Phi}) \subseteq H^s(\mathbb{R}^d) \subseteq H^s(\mathbb{R}^d)$. Then part (2) follows by observing that

\[
\limsup_{k \to \infty} A_{\Phi,k}[u] + (C_V + \varepsilon) \|u\|_2^2 \leq \int_{\mathbb{R}^d} V_k(x)u^2(x)dx + (C_V + \varepsilon) \|u\|_2^2 \leq \int_{\mathbb{R}^d} \Phi + (C_V + \varepsilon) \|\cdot\|_2^2 = A_{\Phi}[u].
\]

To see part (1) of Definition 2.2, first consider $u_k \to u$ in the strong topology of $L^2(\mathbb{R}^d)$. The statement is clear as soon as $\liminf_{k \to \infty} (A_{\Phi,k}[u_k] + (C_V + \varepsilon) \|u_k\|_2^2) = \infty$. With no loss of generality, we can consider a (non-relabeled) subsequence $u_k$ such that (1) $u_k(x) \to u(x)$ a.e., (2) $\liminf_{k \to \infty} A_{\Phi,k}[u_k] = \lim_{k \to \infty} A_{\Phi,k}[u_k]$, and (3) $\liminf_{k \to \infty} \mathcal{E}_\Phi[u_k] = \lim_{k \to \infty} \mathcal{E}_\Phi[u_k]$. Then by
definition of $C_V$, we get $(V_k(x) + C_V + \varepsilon)u_k^2(x) \geq 0$ for a.e. $x \in \mathbb{R}^d$. Fatou’s Lemma then gives
\[
\liminf_{k \to \infty} \mathcal{A}_{\Phi,k}[u_k] + (C_V + \varepsilon)\|u_k\|_2^2 = \lim_{k \to \infty} \mathcal{E}_\Phi[u_k] + \liminf_{k \to \infty} \int_{\mathbb{R}^d} (V_k(x) + C_V + \varepsilon)u_k^2(x)dx
\]
\[
\geq \mathcal{E}_\Phi[u] + \int_{\mathbb{R}^d} (V(x) + C_V + \varepsilon)u^2(x)dx = \mathcal{A}_{\Phi}[u] + (C_V + \varepsilon)\|u\|_2^2.
\]
Thus $\sigma - \lim_{k \to \infty}(\mathcal{A}_{\Phi,k} + (C_V + \varepsilon)\|\cdot\|_2^2) = (\mathcal{A}_\Phi + (C_V + \varepsilon)\|\cdot\|_2^2)$. Next consider $u_k \to u$ in the weak topology of $L^2(\mathbb{R}^d)$. Again, the statement is straightforward whenever the left-hand side above is infinite. Assume then that for all $k \in \mathbb{N}$ we have $\mathcal{A}_{\Phi,k}[u_k] + (C_V + \varepsilon)\|u_k\|_2^2 \leq M$ and, without loss,
\[
\lim_{k \to \infty} \mathcal{A}_{\Phi,k}[u_k] + (C_V + \varepsilon)\|u_k\|_2^2 = \mathcal{A}_{\Phi}[u] + (C_V + \varepsilon)\|u\|_2^2
\]
and $\lim_{k \to \infty}[u_k]_\Phi^2 = [u]_\Phi^2$. This, along with the fact that $V_k(x) + C_V \geq 0$ a.e., implies that
\[
[u_k]_\Phi^2 + \varepsilon\|u_k\|_2^2 \leq \mathcal{A}_{\Phi,k}[u_k] + (C_V + \varepsilon)\|u_k\|_2^2 \leq M
\]
and then $\|u_k\|_\Phi \leq C$ where $C$ is independent of $k \in \mathbb{N}$. Proposition 2.3 says that there exists a (non-relabelled) subsequence $u_k$ converging almost everywhere to $u$. Thus by Fatou’s lemma we get with this subsequence (since $V_k + C_V + \varepsilon > 0$ for every $k \in \mathbb{N}$) that
\[
\liminf_{k \to \infty}(\mathcal{A}_{\Phi,k}[u_k] + (C_V + \varepsilon)\|u_k\|_2^2) = \liminf_{k \to \infty} \left( [u_k]_\Phi^2 + \int_{\mathbb{R}^d} (V_k + C_V + \varepsilon)|u_k(x)|^2dx \right)
\]
\[
= [u]_\Phi^2 + \liminf_{k \to \infty} \int_{\mathbb{R}^d} (V_k + C_V + \varepsilon)|u_k(x)|^2dx
\]
\[
\geq \mathcal{A}_{\Phi}[u] + (C_V + \varepsilon)\|u\|_2^2.
\]
Hence $\sigma - \lim_{k \to \infty}(\mathcal{A}_{\Phi,k} + (C_V + \varepsilon)\|\cdot\|_2^2) = (\mathcal{A}_\Phi + (C_V + \varepsilon)\|\cdot\|_2^2)$.

As a consequence of Theorem B.1 and [41] Th. 6.38 (see also [43]) we have the following result on the stability of the spectrum.

**Corollary 3.1.**

1. Under the assumptions of Theorem B.1, we have $\text{Spec}(\mathcal{H}_{\Phi,V}) \subseteq \lim_{k \to \infty}\text{Spec}(\mathcal{H}_{\Phi,V_k})$, in the sense that for any $\lambda \in \text{Spec}(\mathcal{H}_{\Phi,V})$ and for every $\varepsilon > 0$ there exists $l \in \mathbb{N}$ such that $(\lambda - \varepsilon, \lambda + \varepsilon) \cap \text{Spec}(\mathcal{H}_{\Phi,V_k}) \neq \emptyset$ for every $k \geq l$.

2. In particular, if $\text{Spec}_{\text{ess}}(\mathcal{H}_{\Phi,V_k}) = \text{Spec}_{\text{ess}}(\mathcal{H}_{\Phi,V})$ for all $k \in \mathbb{N}$, then for any $\lambda \in \text{Spec}_{\text{cl}}(\mathcal{H}_{\Phi,V})$ there exists a sequence $(\lambda_k)_{k \in \mathbb{N}}$ such that $\lambda_k \in \text{Spec}_{\text{cl}}(\mathcal{H}_{\Phi,V_k})$ and $\lambda_k \to \lambda$ as $k \to \infty$. Furthermore, for every $k \in \mathbb{N}$ there exists an eigenfunction $\varphi_k$ of $\mathcal{H}_{\Phi,V_k}$ at eigenvalue $\lambda_k$ such that $\varphi_k \to \varphi$ in $L^2(\mathbb{R}^d)$, where $\varphi$ is an eigenfunction of $\mathcal{H}_{\Phi,V}$ at eigenvalue $\lambda$.

**Remark 3.1.**

1. We give two sufficient conditions implying Condition (3) in Theorem B.1.

    (3') Condition (3) holds if for any $u \in \mathcal{D}(\mathcal{A}_\Phi)$ there exists $\bar{V} \geq 0$ such that $V_k \leq \bar{V}$ a.e. for all $k \in \mathbb{N}$ and $\int_{\mathbb{R}^d} \bar{V}(x)u^2(x)dx < \infty$. In this case, it is a direct consequence of Fatou’s Lemma.

    (3'') Condition (3) holds if $V_k - V \to 0$ in $L^p(\mathbb{R}^d)$, where $p > 1$ if $d = 1$ and $s \geq 1/2$, otherwise $p \geq \frac{d}{2s}$. Since $V^- \in L^\infty(\mathbb{R}^d)$ and $u \in \mathcal{D}(\mathcal{A}_\Phi)$, in this case indeed we have
\[
[u]_\Phi^2 \leq \mathcal{A}_{\Phi}[u] + \|V^-\|_\infty \|u\|_2^2 < \infty,
\]
which implies, due to Propositions 2.1, 2.2 and log-convexity of the $L^p$-norms, that $u \in L^{2q}(\mathbb{R}^d)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then
\[
\int_{\mathbb{R}^d} |V_k(x) - V(x)|u^2(x)dx \leq \|V_k - V\|_p \|u\|_{2q}^2 \to 0.
\]

(2) We note that the assumption $V \in \mathcal{K}_\text{dec}^\Phi(\mathbb{R}^d)$ is not necessary as long as there is an independent condition guaranteeing that $\mathcal{H}_\Phi$ is self-adjoint. For instance, if we consider $\mathcal{H}_\Phi$ as the non-local Dirichlet Laplacian of a smooth bounded open set $\Omega \subset \mathbb{R}^d$, we have $\mathcal{D}(\mathcal{E}_\Phi,\Omega) = H^0_0(\mathbb{R}^d)$. Consider
\[
V(x) = \begin{cases} 0 & x \in \Omega \\ \infty & x \notin \Omega 
\end{cases}
\]
and $\mathcal{D}(\mathcal{A}_{\Phi,V}) = H^0_0(\mathbb{R}^d)$. Thus the non-local Dirichlet Laplacian can also be seen as a Schrödinger operator with a degenerate potential. Consider the sequence of the anharmonic oscillators $V_k(x) = |x|^{2k}$. It is straightforward to see that $V_k \leq \tilde{V} := V + 1$ and $V_k \to V$ almost everywhere. Furthermore, $\int_{\mathbb{R}^d} \tilde{V}(x)u^2(x)dx = \|u\|_2^2$ for all $u \in H^0_0(\Omega)$. Hence we can use Theorem 3.1 and Corollary 3.1 to conclude that the eigenvalues of the sequence of anharmonic oscillators converge to the eigenvalues of the non-local Dirichlet Laplacian (and the same applies for the eigenfunctions).

(3) The proof of Theorem 3.1 also shows that $\mathcal{A}_{\Phi,k} + (C_V + \varepsilon) \|\cdot\|_2^2$ are equicoercive. In particular, along the same proof we can show that $\mathcal{A}_{\Phi,k}$ are equicoercive on any bounded subset of $L^2(\mathbb{R}^d)$ and, if $\inf_{k \in \mathbb{N}} \inf_{x \in \mathbb{R}^d} V_k(x) > 0$, then $\mathcal{A}_{\Phi,k}$ are equicoercive in $L^2(\mathbb{R}^d)$. The proof of Theorem 3.1 can be used to prove the existence of a ground state. Indeed, if for instance $V_k, V \in L^{\infty,0}(\mathbb{R}^d)$ and $\mathcal{H}_{\Phi,k}$ admit a sequence of ground state eigenvalues $\lambda_{0,k}$ with $\sup_{k \in \mathbb{N}} \lambda_{0,k} < 0$, then $\mathcal{H}_\Phi$ admits a ground state $\lambda_0 < 0$ defined as $\lim_{k \to \infty} \lambda_{0,k}$.

(4) Finally, observe that Corollary 3.1 can be used to prove the existence of a ground state. Indeed, if for instance $V_k, V \in L^{\infty,0}(\mathbb{R}^d)$ and $\mathcal{H}_{\Phi,k}$ admit a sequence of ground state eigenvalues $\lambda_{0,k}$ with $\sup_{k \in \mathbb{N}} \lambda_{0,k} < 0$, then $\mathcal{H}_\Phi$ admits a ground state $\lambda_0 < 0$ defined as $\lim_{k \to \infty} \lambda_{0,k}$.

(5) The same result holds also in the classical Schrödinger operators featuring the Laplacian, by substituting the space $H^\Phi(\mathbb{R}^d)$ by the Sobolev space $H^1(\mathbb{R}^d)$ and $|u|^2_\Phi$ by $\|\nabla u\|^2_{L^2(\mathbb{R}^d)}$.

(6) To highlight the importance of assumption (3) in Theorem 3.1 in the classical case, consider the following well-known example. Let $d = 1$, $V_k(x) = \omega x^2 + \frac{1}{k} |x|^{-3}$ and $V(x) = \omega x^2$. Clearly, $V_k(x) \to V(x)$ as $k \to \infty$, for all $x \in \mathbb{R} \setminus \{0\}$, and $V_k, V \geq 0$. However, with $\eta \in C^\infty_c(\mathbb{R})$ such that $\eta(x) = 1$ if $x \in [-1,1]$, $\eta(x) = 0$ if $x \notin [-2,2]$, and $0 \leq \eta(x) \leq 1$ in general, $\mathcal{D}(\mathcal{A}_{\Phi,V})$ and clearly $\int_\mathbb{R} V(x)\eta^2(x)dx < \infty$, while $\int_\mathbb{R} V_k(x)\eta^2(x)dx = \infty$. In the classical case, indeed it is well known that $\text{SR} - \lim_{k \to \infty} (\Delta + V_k) \neq -\Delta + V$. This is known as the Klauder phenomenon [15, 39], whose non-local counterpart we will discuss elsewhere.

The same conclusion of Theorem 3.1 can be obtained without Condition (3) under a stronger assumption on the convergence of $(V_k)_{k \in \mathbb{N}}$.

**Theorem 3.2.** Let $\Phi \in \mathcal{B}_0$ satisfy Assumption 2.1 for some $s \in (0,1)$. Also, let $(V_k)_{k \in \mathbb{N}} \subset \mathcal{K}_\text{dec}^\Phi(\mathbb{R}^d)$ and $V \in \mathcal{K}_\text{dec}^\Phi(\mathbb{R}^d)$ be such that for every compact set $K \subset \mathbb{R}^d$ we have $\lim_{k \to \infty} |V_k - V|1_K = 0$ in $L^2(\mathbb{R}^d)$. Then $\mathcal{H}_{\Phi,V} = \text{SR} - \lim_{k \to \infty} \mathcal{H}_{\Phi,V_k}$. 
Proof. Let \( u \in C_0^\infty(\mathbb{R}^d) \) and consider
\[
\|H_{\Phi} u - H_{\Phi, V_k} u\|_2 = \int_{\mathbb{R}^d} (V(x) - V_k(x))^2 u^2(x) dx.
\]
Write \( K = \text{supp}(u) \) and observe that
\[
\|H_{\Phi, V} u - H_{\Phi, V_k} u\|_2 = \int_K (V(x) - V_k(x))^2 u^2(x) dx \leq \|u^2\|_\infty \|(V - V_k)1_K\|_2^2.
\]
Taking the limit we have \( H_{\Phi, V} u \to H_{\Phi, V} u \) in \( L^2(\mathbb{R}^d) \). Since \( C_0^\infty(\mathbb{R}^d) \) is a core for \( H_{\Phi, V_k} \) and \( H_{\Phi, V} \), the proof follows by [41, Lem. 6.36].

**Remark 3.2.** Clearly, Corollary 3.1 continues to hold also under the assumptions of Theorem 3.2.

A further strong convergence condition on the potentials yields the following.

**Theorem 3.3.** Let \( \Phi \in B_0 \) satisfy Assumption 2.1 for some \( s \in (0, 1) \). Also, let \( (V_k)_{k \in \mathbb{N}} \subset K_{\Phi}^{\text{dec}}(\mathbb{R}^d) \) and \( V \in K_{\Phi}^{\text{dec}}(\mathbb{R}^d) \) be such that the convergence \( |V_k - V| \to 0 \) holds uniformly. Then \( H_{\Phi, V} = \text{NR} - \lim_{k \to \infty} H_{\Phi, V_k} \), in particular, \( \text{Spec}(H_{\Phi, V}) = \lim_{k \to \infty} \text{Spec}(H_{\Phi, V_k}) \).

**Proof.** Let \( u \in C_0^\infty(\mathbb{R}^d) \) and note that
\[
\|H_{\Phi, V} u - H_{\Phi, V_k} u\|_2 = \int_{\mathbb{R}^d} (V(x) - V_k(x))^2 u^2(x) dx \leq \|V - V_k\|_2 \|u\|_2^2.
\]
Since \( \|V - V_k\|_2 \to 0 \) by assumption, the proof is immediate by [41, Lem. 6.34, Th. 6.38].

In this case, Corollary 3.1 can be slightly improved by using the cited theorem in the above proof.

**Corollary 3.2.** Under the assumptions of Theorem 3.3 we have \( \lim_{k \to \infty} \text{Spec}(H_{\Phi, V_k}) = \text{Spec}(H_{\Phi, V}) \).

4. Regularity and monotonicity of ground states for relativistic Schrödinger operators with spherical potential wells

4.1. Approximant ground states

As an application of the previous stability results we now discuss important consequences on the properties of ground states of a key family of non-local Schrödinger operators. Let \( \alpha \in (0, 2) \) and \( m \geq 0 \), and define the function
\[
\Phi_{m, \alpha}(z) = (z + m^{2/\alpha})^{\alpha/2} - m, \quad z > 0.
\]
It is straightforward to show that \( \Phi_{m, \alpha} \in B_0 \). Furthermore, if \( m = 0 \), we have
\[
j_{0, \alpha}(r) := j_{\Phi_{0, \alpha}}(r) = \frac{2^\alpha \Gamma \left( \frac{d+\alpha}{2} \right)}{\pi^{d/2} \Gamma \left( -\frac{d}{2} \right)} \frac{1}{r^{d+\alpha}},
\]
while if \( m > 0 \), then
\[
j_{m, \alpha}(r) := j_{\Phi_{m, \alpha}}(r) = \frac{2^\alpha \Gamma \left( \frac{d+\alpha}{2} \right)}{\pi^{d/2} \Gamma \left( 1 - \frac{d}{2} \right)} r^{-\frac{d+\alpha}{2}} K_{\frac{d+\alpha}{2}}(m^{1/\alpha} r),
\]
where
\[
K_{\xi}(z) = \frac{1}{2} \left( \frac{z}{2} \right)^{-\xi} \int_0^\infty t^{-\xi} e^{-\frac{z^2}{4t}} dt, \quad z > 0, \quad \xi > -\frac{1}{2}
\]
is the modified Bessel function of the third kind. In the case \( m = 0 \), the operator \( \Phi_{0,\alpha}(-\Delta) \) is the fractional Laplacian and \( H^{\Phi_{0,\alpha}}(\mathbb{R}^d) = H^{\alpha/2}(\mathbb{R}^d) \) since then \( [u]_{\Phi_{0,\alpha}} = C_{\alpha,d}([u]_{\alpha/2}) \). In the case \( m > 0 \), the operator \( \Phi_{m,\alpha}(-\Delta) \) is called relativistic fractional Laplacian. This terminology is natural since for \( \alpha = 1 \) the operator coincides with the square-root Klein-Gordon operator (in conventional units) giving the kinetic term of a (semi-)relativistic particle. For a unified notation we use \( \Phi_{m,\alpha} \) and \( j_{m,\alpha} \) subsuming the case \( m = 0 \), and make the appropriate distinctions between massive and massless cases when necessary.

It is clear, either by direct computation or by using the fact that \( \Phi_{m,\alpha}(\lambda) \sim \lambda^{\alpha/2} \) as \( \lambda \to \infty \) and \([29]\) Th. 3.4, that \( j_{m,\alpha} \) satisfies Assumption \([2.1]\) with \( s = \frac{\alpha}{2} \) for all \( m \geq 0 \). Moreover, since for \( m > 0 \) we have \( j_{m,\alpha}(r) \sim Cr^{-d-\alpha} \) as \( r \to 0^+ \), for a constant \( C > 0 \) depending on \( d, m, \alpha \), it is not difficult to check that \( H^{\Phi_{m,\alpha}}(\mathbb{R}^d) = H^{\alpha/2}(\mathbb{R}^d) \) up to equivalence of norms. We denote \( E_{m,\alpha} := E_{\Phi_{m,\alpha}} \), which acts on \( H^{\alpha/2}(\mathbb{R}^d) \times H^{\alpha/2}(\mathbb{R}^d) \).

Furthermore, for the remainder of the paper we choose the specific potential

\[
V(x) = -v \mathbf{1}_{B_0}(x),
\]

where \( v, a > 0 \) are constants. By Proposition \([2.7]\) we can choose \( v \) large enough so that \( V \) satisfies Assumption \([2.2]\). For every \( \varepsilon > 0 \) let \( \eta_\varepsilon \) be a radially decreasing cutoff function for \( B_a \) with support contained in \( B_{a+\varepsilon} \). To construct such a function, we may take

\[
\varrho(x) = \begin{cases} 
C_\varrho e^{-\frac{|x|^2}{4d}} & \text{if } |x| \leq 1 \\
0 & \text{if } |x| > 1,
\end{cases}
\]

where \( C_\varrho \) is chosen in such a way that \( \int_{\mathbb{R}^d} \varrho(x)dx = 1 \). Then define \( \varrho_{\varepsilon/2}(x) = (2\varepsilon)^{-d} \varrho \left( \frac{2x}{\varepsilon} \right) \) and consider \( \eta_\varepsilon = \varrho_{\varepsilon/2} \ast \mathbf{1}_{B_{a+\frac{\varepsilon}{2}}} \). It is clear that \( \eta_\varepsilon(x) = 1 \) for every \( x \in B_a \) and \( \eta_\varepsilon(x) = 0 \) for all \( x \in \mathbb{R}^d \setminus B_{a+\varepsilon} \). To see that it is radially symmetric, let \( R \in \text{SO}(d) \) be any rotation on the space \( \mathbb{R}^d \) and observe that, since \( R \) is an isometry,

\[
\eta_\varepsilon(Rx) = \int_{\mathbb{R}^d} \varrho_{\varepsilon/2}(Rx-y) \mathbf{1}_{B_{a+\frac{\varepsilon}{2}}}(y)dy = \int_{\mathbb{R}^d} \varrho_{\varepsilon/2}(Rx-Ry) \mathbf{1}_{B_{a+\frac{\varepsilon}{2}}}(Ry)dy
= \int_{\mathbb{R}^d} \varrho_{\varepsilon/2}(x-y) \mathbf{1}_{B_{a+\frac{\varepsilon}{2}}}(y)dy = \eta_\varepsilon(x).
\]

Then taking the profile function \( \tilde{\eta}_\varepsilon : r \in [0, \infty) \mapsto \eta_\varepsilon(re_1) \in \mathbb{R} \), we have for every \( r > 0 \),

\[
\tilde{\eta}_\varepsilon'(r) = \frac{\partial \varrho_{\varepsilon/2}}{\partial x_1}(re_1) \ast \mathbf{1}_{B_{a+\frac{\varepsilon}{2}}}(re_1) \leq 0,
\]

(where \( e_1 \) is a unit vector in \( \mathbb{R}^d \)) implying that \( \eta_\varepsilon \) is radially decreasing. Hence we can define

\[
V_\varepsilon(x) = -v \eta_\varepsilon(x)
\]

so that \( V_\varepsilon \to V \) as \( \varepsilon \to 0 \) in \( L^p(\mathbb{R}^d) \) for any \( 1 \leq p < \infty \), and \( V_\varepsilon \) satisfies Assumption \([2.2]\) for every \( \varepsilon > 0 \). Furthermore, we also have \( V_\varepsilon \in C_\infty(\mathbb{R}^d) \). With these entries we then define

\[
\mathcal{H}_{m,\alpha} := \mathcal{H}_{\Phi_{m,\alpha},V} \quad \text{and} \quad \mathcal{H}_{m,\alpha}^{\varepsilon} := \mathcal{H}_{\Phi_{m,\alpha},V_\varepsilon}.
\]

To avoid multiple subscripts, in this section we denote the unique ground states and ground state eigenvalues of these non-local Schrödinger operators simply by

\[
\varphi, \varphi_\varepsilon \quad \text{and} \quad \lambda, \lambda_\varepsilon, \text{ respectively}.
\]
4.2. Regularity of the ground state

First we show that $\varphi_\varepsilon$ is regular to a high degree. Since the properties discussed just before introducing Assumption 2.2 also hold for the semigroup $\{e^{-tH_{m,\alpha}} : t \geq 0\}$, we have $\varphi_\varepsilon \in L^\infty(\mathbb{R}^d) \cap C(\mathbb{R}^d)$. Also, $\varphi_\varepsilon \in L^1(\mathbb{R}^d)$ by [22] Prop. 4.291.

**Proposition 4.1.** We have that $\varphi_\varepsilon \in C^\infty(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

**Proof.** By the above $\varphi_\varepsilon \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Also, we know by definition that $\varphi_\varepsilon \in H^{\Phi_{m,\alpha}}(\mathbb{R}^d)$ and

$$
\mathcal{E}_{m,\alpha}(\varphi_\varepsilon, v) = \lambda(\varphi_\varepsilon, v) + (V\varphi_\varepsilon, v), \quad v \in C_c^\infty(\mathbb{R}^d).
$$

By the continuity of the involved operators, the above equality holds also for $v \in H^{m,\alpha}(\mathbb{R}^d)$. Define the space

$$
\tilde{H}^{m,\alpha}(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d; \mathbb{C}) : [\Re(u)]^2 + [\Im(u)]^2 < \infty\}.
$$

On $\tilde{H}^{m,\alpha}(\mathbb{R}^d)$ we define the bilinear form

$$
\tilde{\mathcal{E}}_{m,\alpha}(u, v) = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(y) - u(x))(\bar{v}(y) - \bar{v}(x)) j_{m,\alpha}(|x - y|) dx dy.
$$

Assuming $u \in H^{\Phi}(\mathbb{R}^d)$ and noting that the real and imaginary parts of $v \in \tilde{H}^{m,\alpha}(\mathbb{R}^d)$ are such that $\Re(v), \Im(v) \in H^{m,\alpha}(\mathbb{R}^d)$, we have

$$
\tilde{\mathcal{E}}_{m,\alpha}(u, v) = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(y) - u(x))(\bar{v}(y) - \bar{v}(x)) j_{m,\alpha}(|x - y|) dx dy
$$

Choosing $u = \varphi_\varepsilon$ gives

$$
\tilde{\mathcal{E}}_{m,\alpha}(\varphi_\varepsilon, v) = \mathcal{E}_{m,\alpha}(u, \Re(v)) - i\mathcal{E}_{m,\alpha}(u, \Im(v))
$$

for all $v \in \tilde{H}^{m,\alpha}(\mathbb{R}^d)$, in particular, for all $v \in S(\mathbb{R}^d; \mathbb{C}) \subset \tilde{H}^{m,\alpha}(\mathbb{R}^d)$. By linearity of the Fourier transform, Proposition 2.6 holds also for $\tilde{\mathcal{E}}_{m,\alpha}$ and thus by Plancherel’s theorem we have

$$
\int_{\mathbb{R}^d} \Phi_{m,\alpha}(|\xi|^2) \widehat{\varphi_\varepsilon}(\xi) \overline{\varphi_\varepsilon}(\xi) d\xi = \lambda \int_{\mathbb{R}^d} \overline{\varphi_\varepsilon}(\xi) \overline{\varphi_\varepsilon}(\xi) d\xi + \int_{\mathbb{R}^d} \mathcal{F}[V\varphi_\varepsilon](\xi) \overline{\varphi_\varepsilon}(\xi) d\xi.
$$

In particular, since $C_c^\infty(\mathbb{R}^d) \subset S(\mathbb{R}^d; \mathbb{C})$ and $\mathcal{F}^{-1}(S(\mathbb{R}^d; \mathbb{C})) = S(\mathbb{R}^d; \mathbb{C})$, for every function $\psi \in C_c^\infty(\mathbb{R}^d)$, there exists a function $v \in S(\mathbb{R}^d; \mathbb{C})$ such that $\widehat{\psi} = \psi$ and then

$$
\int_{\mathbb{R}^d} (\Phi_{m,\alpha}(|\xi|^2) \widehat{\varphi_\varepsilon}(\xi) - \lambda \widehat{\varphi_\varepsilon}(\xi) - \mathcal{F}[V\varphi_\varepsilon](\xi)) \psi(\xi) d\xi = 0.
$$

By the fundamental lemma of variational calculus (see, e.g., [13] Th. 1.24), we obtain

$$
\Phi_{m,\alpha}(|\xi|^2) \widehat{\varphi_\varepsilon}(\xi) - \lambda \widehat{\varphi_\varepsilon}(\xi) - \mathcal{F}[V\varphi_\varepsilon](\xi) = 0, \quad \text{a.e. } \xi \in \mathbb{R}^d. \quad (4.1)
$$
This relation implies that $\Phi_{m,\alpha}(|\xi|^2)\widehat{\varphi}_\varepsilon \in L^2(\mathbb{R})$ and thus $\Phi_{m,\alpha}(-\Delta)\varphi_\varepsilon \in L^2(\mathbb{R})$. To prove that

$\varphi_\varepsilon \in C^\infty(\mathbb{R}^d)$, we only need to show that for any $n \in \mathbb{N}$ the function $|\xi|^{n\alpha}\widehat{\varphi}_\varepsilon(\xi)$ belongs to $L^2(\mathbb{R})$ (see, for instance, [20, Exercise 6, Sect. 2.3]). To do this, observe that $\Phi_{m,\alpha}(|\xi|^2) \sim |\xi|^n$ as $|\xi| \to \infty$. Combining this asymptotic behaviour with the fact that $\widehat{\varphi}_\varepsilon$ is uniformly continuous and $\Phi_{m,\alpha}(|\xi|^2)\widehat{\varphi}_\varepsilon \in L^2(\mathbb{R})$, we get that $|\xi|^{n\alpha}\widehat{\varphi}_\varepsilon \in L^2(\mathbb{R})$.

Now assume that $|\xi|^{n\alpha}\widehat{\varphi}_\varepsilon(\xi)$ belongs to $L^2(\mathbb{R})$ for some $n \in \mathbb{N}$. By Lemma 2.3, we also know that $|\xi|^{n\alpha}\mathcal{F}[V_\varepsilon\varphi_\varepsilon](\xi) \in L^2(\mathbb{R}^d)$. Note that $V_\varepsilon\varphi_\varepsilon \in L^1(\mathbb{R}^d)$ by the Hölder inequality, hence $\mathcal{F}[V_\varepsilon\varphi_\varepsilon](\xi)$ is uniformly continuous. Furthermore, we have

$$\widehat{\varphi}(\xi) = \frac{\mathcal{F}[V_\varepsilon\varphi_\varepsilon](\xi)}{\Phi_{m,\alpha}(|\xi|^2) - \lambda},$$

where, recalling that $\lambda < 0$, we have $\Phi_{m,\alpha}(|\xi|^2) - \lambda > 0$. Multiplying both sides by $|\xi|^{(n+1)\alpha}$, taking the square and integrating, we obtain

$$\int_{\mathbb{R}^d} |\xi|^{2(n+1)\alpha}|\widehat{\varphi}(\xi)|^2d\xi = \left(\int_{B_1} + \int_{\mathbb{R}^d \setminus B_1}\right) \frac{|\xi|^{2(n+1)\alpha}|\mathcal{F}[V_\varepsilon\varphi_\varepsilon](\xi)|^2}{(\Phi_{m,\alpha}(|\xi|^2) - \lambda)^2}d\xi \leq \frac{|||\mathcal{F}[V_\varepsilon\varphi_\varepsilon]|^2||_{L^\infty(B_1)}\omega_d}{-\lambda} + \frac{1}{C} \int_{\mathbb{R}^d \setminus B_1} |\xi|^{2n\alpha}|\mathcal{F}[V_\varepsilon\varphi_\varepsilon](\xi)|^2d\xi < \infty,$$

where $C = \inf_{|\xi| > 1} \frac{\Phi(|\xi|^2) - \lambda}{|\xi|^\alpha} > 0$. \hfill \Box

**Remark 4.1.** The argument we used to prove that $\varphi_\varepsilon \in C(\mathbb{R}^d)$ applies more generally. Indeed, we can prove in the same way that if $\Phi \in \mathcal{B}_0$ and $V \in L^{\infty,0}(\mathbb{R}^d)$ are such that Assumptions 2.1(1) and 2.2 are satisfied, the function $p_t^\Phi(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\xi \cdot x - \Phi(|\xi|^2)/2}d\xi$ is radially symmetric, and $\varphi$ is the ground state of $\mathcal{H}_{\Phi,V}$, then $\varphi \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. This can be further extended to $\Phi$-Kato-decomposable potentials.

Now that we know that $\varphi_\varepsilon \in C^\infty(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, Lemma 2.4 guarantees that $\Phi_{m,\alpha}(-\Delta)\varphi_\varepsilon$ is well defined as in (2.4). Using Theorem 3.1 we see that $\varphi_\varepsilon$ are in a sense approximants of $\varphi$.

**Proposition 4.2.** There exists a sequence $(\varepsilon_k)_{k \in \mathbb{N}}$, $\varepsilon_k \downarrow 0$, such that $\varphi_k := \varphi_{\varepsilon_k} \to \varphi$ both in $L^2(\mathbb{R}^d)$ and almost everywhere, and $\lambda_k := \lambda_{\varepsilon_k} \to \lambda$ as $k \to \infty$.

**Proof.** Let $\varepsilon_k \downarrow 0$ be any subsequence and set $V_k := V_{\varepsilon_k}$. Since $\sup_{k \geq 0} \|V_k\|_\infty = v < \infty$, we have $V_k \to V$ in any $L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$ and all $V_k$ and $V$ satisfy Assumption 2.2. Then we obtain the statement by Theorem 3.1 and taking a suitable subsequence. \hfill \Box

From now on, we denote by $\varphi_k$ the sequence identified by this result.

**Theorem 4.1.** We have $\Phi_{m,\alpha}(-\Delta)\varphi \in L^2(\mathbb{R}^d)$. Furthermore, $\Phi_{m,\alpha}(-\Delta)\varphi_k \to \Phi_{m,\alpha}(-\Delta)\varphi$ in $L^2(\mathbb{R}^d)$, as $k \to \infty$.

**Proof.** By a similar argument as in Proposition 4.1, we see that

$$\Phi_{m,\alpha}(|\xi|^2)\widehat{\varphi}(\xi) = \lambda\widehat{\varphi}(\xi) + \mathcal{F}[V\varphi](\xi),$$
for almost every $\xi \in \mathbb{R}^d$, hence $\Phi_{m,\alpha}(|\xi|^2)\tilde{\varphi}(\xi) \in L^2(\mathbb{R}^d)$, implying that $\Phi_{m,\alpha}(-\Delta)\varphi \in L^2(\mathbb{R}^d)$. Note that

\[
\int_{\mathbb{R}^d} |V_k(x)\varphi_k(x) - V(x)\varphi(x)|^2 \, dx \leq 2 \int_{\mathbb{R}^d} |V_k(x)|^2 |\varphi_k(x) - \varphi(x)|^2 \, dx
\]

\[
+ 2 \int_{\mathbb{R}^d} |V(x) - V_k(x)|^2 |\varphi(x)|^2 \, dx
\]

\[
\leq 2\nu^2 \|\varphi - \varphi\|_2^2 + 2 \|\varphi\|_\infty^2 \|V_k - V\|_2 \to 0,
\]

where we used the fact that $\varphi \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, as specified in Remark 4.1. Hence $V_k\varphi_k \to V\varphi$ in $L^2(\mathbb{R}^d)$. Clearly, this implies $F[V_k\varphi_k] \to F[V\varphi]$ in $L^2(\mathbb{R}^d)$, and thus we get $\Phi_{m,\alpha}(\cdot,|\cdot|^2)\tilde{\varphi}_k \to \Phi_{m,\alpha}(\cdot,|\cdot|^2)\tilde{\varphi}$, which in turn implies that $\Phi_{m,\alpha}(-\Delta)\tilde{\varphi}_k \to \Phi_{m,\alpha}(-\Delta)\tilde{\varphi}$ in $L^2(\mathbb{R}^d)$.  

\section{Radial decrease of the ground state}

The fact that the ground states are unique and the potentials are rotationally symmetric give us some information about the shape of $\varphi$.

\begin{proposition}
Let $\Phi \in \mathcal{B}_0$ and $V \in L^{\infty,0}(\mathbb{R}^d)$ be rotationally symmetric and satisfy Assumption \ref{as2}. Then the ground state $\varphi$ of $\mathcal{H}_{\Phi,V}$ is rotationally symmetric.
\end{proposition}

\begin{proof}
Consider an arbitrary rotation $R \in \text{SO}(d)$ and let $\tilde{\varphi}(x) = \varphi(Rx)$. Using that $\|\tilde{\varphi}\|_2 = 1$ and $\tilde{\varphi}(x) > 0$ for all $x \in \mathbb{R}^d$, we obtain

\[
\mathcal{A}_{\Phi,V}(\tilde{\varphi}, \tilde{\varphi}) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\tilde{\varphi}(x) - \tilde{\varphi}(y)|j_\Phi(|x - y|) \, dx \, dy + \int_{\mathbb{R}^d} V_k(x)\tilde{\varphi}^2(x) \, dx
\]

\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\varphi(Rx) - \varphi(Ry)|j_\Phi(|x - y|) \, dx \, dy + \int_{\mathbb{R}^d} V_k(x)\varphi(Rx) \, dx
\]

\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\varphi(Rx) - \varphi(Ry)|j_\Phi(|Rx - Ry|) \, dx \, dy + \int_{\mathbb{R}^d} V_k(Rx)\varphi(Rx) \, dx
\]

\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\varphi(x) - \varphi(y)|j_\Phi(|x - y|) \, dx \, dy + \int_{\mathbb{R}^d} V_k(x)\varphi(x) \, dx = \lambda.
\]

Hence $\tilde{\varphi}$ is an eigenfunction of $\mathcal{H}_{\Phi,V}$ at the same eigenvalue $\lambda$. By uniqueness we then have $\tilde{\varphi} = \varphi$, thus $\varphi$ is invariant to all rotations. 
\end{proof}

In the following we will need to work with antisymmetric functions. We will denote any $x \in \mathbb{R}^d$ as $x = (x_1, x')$, where $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{d-1}$. For every $\mu \in \mathbb{R}$ we write $x^\mu = (2\mu - x_1, x')$ and $\mathcal{U}_\mu = \{x \in \mathbb{R}^d : x_1 < \mu\}$. We say that a function $w$ is $\mu$-antisymmetric if $w(x^\mu) = -w(x)$ for every $x \in \mathcal{U}_\mu$. Recall the following result from \cite[Lem. 2.1]{12} for the fractional Laplacian.

\begin{lemma}
Let $w \in C^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be a $\mu$-antisymmetric function, and suppose that there exists $x \in \mathcal{U}_\mu$ such that $w(x) = \inf_{y \in \mathcal{U}_\mu} w(y) < 0$. Denote $\delta = \mu - x_1$. Then there exists a constant $C = C_{d,\alpha} > 0$ dependent only on $d$ and $\alpha$ such that

\[
\Phi_{0,\alpha}(-\Delta)w(x) \leq C_{d,\alpha} \left( \delta^{-\alpha}w(x) - \delta \int_{\mathcal{U}_\mu} \frac{(w(y) - w(x))(\mu - y_1)}{|x - y^\mu|^{d+\alpha+2}} \, dy \right). 
\]
\end{lemma}
We will extend this result to the operator $\Phi_{m,\alpha}(-\Delta)$. To do this, for every $m, r > 0$ we define
\[
\sigma_{m,\alpha}(r) = \frac{\alpha 2^{1-d+\alpha} \Gamma\left(1 - \frac{\alpha}{2}\right)}{\pi^{d/2} \Gamma\left(1 - \frac{\alpha}{2}\right)^{r^{d+\alpha}}} \left(\frac{2(d+\alpha)\Gamma\left(\frac{d+\alpha}{2}\right)}{r^{d+\alpha} m^{d+\alpha}} - \frac{m^{d+\alpha} K_{d+\alpha}(m^{1/\alpha} r)}{r^{d+\alpha}}\right)
\]
\[
= \frac{\alpha 2^{1-d+\alpha} \Gamma\left(1 - \frac{\alpha}{2}\right)}{\pi^{d/2} \Gamma\left(1 - \frac{\alpha}{2}\right)^{r^{d+\alpha}}} \int_0^{m^{1/\alpha} r} w^{d+\alpha} K_{d+\alpha-1}(w) \, dw,
\]
with the same Bessel function as used before. It has been shown in [37, Lem. 2] that $\int_{\mathbb{R}^d} \sigma_{m,\alpha}(|x|) \, dx = m$ and the decomposition
\[
j_{0,\alpha}(r) = \frac{\alpha 2^{1-d+\alpha} \Gamma\left(1 - \frac{\alpha}{2}\right)}{\pi^{d/2} \Gamma\left(1 - \frac{\alpha}{2}\right)^{r^{d+\alpha}}} \left(\frac{2(d+\alpha)\Gamma\left(\frac{d+\alpha}{2}\right)}{r^{d+\alpha} m^{d+\alpha}} - \frac{m^{d+\alpha} K_{d+\alpha}(m^{1/\alpha} r)}{r^{d+\alpha}}\right)
\]
holds. Due to this observation, we can define the operator $G_{m,\alpha} : L^\infty(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ given by
\[
G_{m,\alpha} f(x) = \frac{1}{2} \int_{\mathbb{R}^d} (f(x+h) - 2f(x) + f(x-h)) \sigma_{m,\alpha}(|h|) \, dh.
\]
In particular, for every $f \in C^\infty_c(\mathbb{R}^d)$ we have
\[
\Phi_{m,\alpha}(-\Delta) f = \Phi_{0,\alpha}(-\Delta) f - G_{m,\alpha} f.
\]
Clearly, this relation can be extended to any function $f$ such that both $\Phi_{m,\alpha}(-\Delta) f$ and $\Phi_{0,\alpha}(-\Delta) f$ are defined pointwise via (2.4). For the details and proof of such a decomposition, we refer to [2, Sect. 2.3.2].

Before proceeding with the extension, we determine the derivative of the jump kernel $j_{m,\alpha}$.

**Lemma 4.2.** We have
\[
j_{m,\alpha}'(r) = - \frac{\alpha 2^{1-d+\alpha} m^{d+\alpha+2} K_{d+\alpha+2}(m^{1/\alpha} r)}{\pi^{d/2} \Gamma\left(1 - \frac{\alpha}{2}\right)^{r^{d+\alpha}}} - \frac{\alpha 2^{1-d+\alpha} \Gamma\left(1 - \frac{\alpha}{2}\right)}{\pi^{d/2} \Gamma\left(1 - \frac{\alpha}{2}\right)^{r^{d+\alpha}}} \left(\frac{2(d+\alpha)\Gamma\left(\frac{d+\alpha}{2}\right)}{r^{d+\alpha} m^{d+\alpha}} - \frac{m^{d+\alpha} K_{d+\alpha}(m^{1/\alpha} r)}{r^{d+\alpha}}\right),
\]
\[
= - \frac{\alpha 2^{1-d+\alpha} m^{d+\alpha+2} K_{d+\alpha+2}(m^{1/\alpha} r)}{2 \pi^{d/2} \Gamma\left(1 - \frac{\alpha}{2}\right)^{r^{d+\alpha}}} \left((d+\alpha)K_{d+\alpha+2}(m^{1/\alpha} r) + m^{1/\alpha} r K_{d+\alpha+2}(m^{1/\alpha} r)\right).
\]

Since $2\xi K_{\xi}(r) + rK_{\xi-1}(r) = rK_{\xi+1}(r)$, see [42, §3.71(1)], a combination with (4.5) shows the claim.

**Lemma 4.3.** Let $m > 0$, $w \in C^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be a $\mu$-antisymmetric function, and suppose that there exists $x \in \mathcal{U}_\mu$ such that
\[
w(x) = \inf_{y \in \mathcal{U}_\mu} w(y) < 0.
\]
Denote $\delta = \mu - x_1$. Then there exists a constant $C = C(m, \alpha, d) > 0$ such that
\[
\Phi_{m,\alpha}(-\Delta) w(x) \leq C \left((\delta - m)w(x) - \delta \int_{\mathcal{U}_\mu} \frac{(w(y) - w(x))(\lambda - y_1) K_{d+\alpha+2}(m^{1/\alpha} |x-y|)}{|x-y|^{d+\alpha+2}} \, dy\right).
\]
Moreover, if $\delta \in (0, \delta_1]$, with $0 < \delta_1 < \infty$, then there exists $C = C(m, \alpha, d, \delta_1) > 0$ such that
\[
\Phi_{m,\alpha}(-\Delta)w(x) \leq C \left( \delta^{\frac{d+\alpha}{2}} w(x) - \delta \int_{U_\mu} \frac{(w(y) - w(x))(\lambda - y_1)K_{(d+\alpha+2)/2}(m^{1/\alpha}|x - y'|)}{|x - y'|^{d+\alpha+2}} dy \right).
\]  
(4.7)

**Proof.** We proceed through several steps.

**Step 1:** First observe that since $w \in C^2(\mathbb{R}^d)$, then $w(y) = 0$ for every $y \in \partial U_\mu$. Hence $d(x, \partial U_\mu) = \delta > 0$. Since the expression (2.4) applies for $w$, we have
\[
\Phi_{m,\alpha}(-\Delta)w(x) = \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} (w(x) - w(y)) j_{m,\alpha}(|x - y|) dy.
\]
Fix $\varepsilon > 0$ and choose it small enough to have $B_\varepsilon(x) \subset U_\mu$, which can be done since $\delta > 0$. We split up the integral in two parts as
\[
\int_{\mathbb{R}^d \setminus B_\varepsilon(x)} (w(x) - w(y)) j_{m,\alpha}(|x - y|) dy = \left( \int_{U_\mu \setminus B_\varepsilon(x)} + \int_{(\mathbb{R}^d \setminus U_\mu) \setminus B_\varepsilon(x)} \right) (w(x) - w(y)) j_{m,\alpha}(|x - y|) dy
\]
and apply the change of variable $y \rightarrow y^\mu$ which, using that $w$ is $\mu$-antisymmetric, gives
\[
\int_{\mathbb{R}^d \setminus B_\varepsilon(x)} (w(x) - w(y)) j_{m,\alpha}(|x - y|) dy = \int_{U_\mu \setminus B_\varepsilon(x)} (w(x) - w(y)) j_{m,\alpha}(|x - y|) dy + \int_{U_\mu \setminus B_\varepsilon(x)} (w(x) + w(y)) j_{m,\alpha}(|x - y^\mu|) dy
\]
\[
+ 2w(x) \int_{U_\mu \setminus B_\varepsilon(x)} j_{m,\alpha}(|x - y^\mu|) dy + \int_{B_\varepsilon(x)} (w(x) + w(y)) j_{m,\alpha}(|x - y^\mu|) dy.
\]
Since $w \in L^\infty(\mathbb{R}^d)$, we can take the limit $\varepsilon \downarrow 0$ giving
\[
\Phi_{m,\alpha}(-\Delta)w(x) = -\lim_{\varepsilon \downarrow 0} \int_{U_\mu \setminus B_\varepsilon(x)} (w(y) - w(x)) (j_{m,\alpha}(|x - y|) - j_{m,\alpha}(|x - y^\mu|)) dy
\]
\[
+ 2w(x) \int_{U_\mu} j_{m,\alpha}(|x - y^\mu|) dy.
\]  
(4.8)

By the decomposition formulae (4.3)-(4.4) we have
\[
L_{m,\alpha}w(x) = -\lim_{\varepsilon \downarrow 0} \int_{U_\mu \setminus B_\varepsilon(x)} (w(y) - w(x)) (j_{m,\alpha}(|x - y|) - j_{m,\alpha}(|x - y^\mu|)) dy
\]
\[
+ C_1(d,\alpha) 2w(x) \int_{U_\mu} \frac{1}{|x - y^\mu|^{d+\alpha}} dy - 2w(x) \int_{U_\mu} \sigma_m(|x - y^\mu|) dy,
\]  
(4.9)

where we denote $C_1(d,\alpha) = \frac{\alpha^2 - d}{\pi^2 2^{2-d}(1 - \frac{d}{2})}$.

**Step 2:** Next we estimate the integrals one by one, starting with the third. We have
\[
\int_{U_\mu} \sigma_m(|x - y^\mu|) dy = \int_{-\infty}^\mu \int_{\mathbb{R}^{d-1}} \sigma_m(|x' - y'|^2 - |2\mu - x_1 - y_1|^2)^{1/2}) dy' dy_1.
\]
Setting $z' = x' - y'$ and $z_1 = 2\mu - x_1 - y_1$ we have
\[
\int_{U_\mu} \sigma_m(|x - y^\mu|) dy = \int_{\mu - x_1}^\infty \int_{\mathbb{R}^{d-1}} \sigma_m(|z'|^2 - |z_1|^2)^{1/2}) dz' dz_1.
\]
With \( \tilde{U}_0 = \{ x \in \mathbb{R}^d : x_1 > 0 \} \), we have
\[
\int_{\tilde{U}_0} \sigma_{m, \alpha}(|x - y^\mu|)dy \leq \int_{\tilde{U}_0} \sigma_{m, \alpha}(|z|)dz = \frac{m}{2}.
\]

Next consider the second integral. We have
\[
\int_{\tilde{U}_0} \frac{dy}{|x - y^\mu|^{d+\alpha}} = \int_{-\infty}^{\mu} \int_{\mathbb{R}^{d-1}} \frac{dy_1 dy}{(|x'|^2 + |2\mu - x_1 - 1|^2)^{\frac{d+\alpha}{2}}}
= \frac{1}{(\mu - x_1)^{d+\alpha}}\int_{-\infty}^{\mu} \int_{\mathbb{R}^{d-1}} \frac{dy_1 dy}{(|x' - y'|^2 + |1 + \frac{\mu - y_1}{\mu - x_1}|^2)^{\frac{d+\alpha}{2}}}.
\]

Setting \( z' = \frac{x' - y'}{\mu - x_1} \) and \( z_1 = \frac{\mu - y_1}{\mu - x_1} \), we get
\[
\int_{\tilde{U}_0} \frac{dy}{|x - y^\mu|^{d+\alpha}} = \frac{1}{(\mu - x_1)^{d+\alpha}}\int_{0}^{\infty} \int_{\mathbb{R}^{d-1}} \frac{dz_1 dz}{(|z'|^2 + |1 + z_1|^2)^{\frac{d+\alpha}{2}}} = C_2(d, \alpha)\delta^{-\alpha},
\]
with constant \( C_2(d, \alpha) = \int_{0}^{\infty} \int_{\mathbb{R}^{d-1}} \frac{dz_1}{(|z'|^2 + |1 + z_1|^2)^{\frac{d+\alpha}{2}}}dz_1 < \infty.
\]

Finally consider the first integral. By Lagrange’s theorem there exists \( |x - y| < \theta(x, y) < |x - y^\mu| \) such that
\[
\int_{U_\delta \setminus B_\varepsilon(x)} (w(y) - w(x))(j_{m, \alpha}(|x - y|) - j_{m, \alpha}(|x - y^\mu|))dy = \int_{U_\delta \setminus B_\varepsilon(x)} (w(y) - w(x))j_{m, \alpha}(\theta(x, y))(|x - y| - |x - y^\mu|)dy
= C_3(d, m, \alpha) \int_{U_\delta \setminus B_\varepsilon(x)} (w(y) - w(x)) \frac{K_{(d+\alpha+2)/2}(m^{1/\alpha} \theta(x, y))}{\theta(x, y)^{\frac{d+\alpha}{2}}} (|x - y| - |x - y^\mu|)dy,
\]
where we used Lemma 4.2 and denoted \( C_3(d, m, \alpha) = \frac{\alpha^{\frac{d+\alpha}{2}} m^{\frac{d+\alpha+2}{\alpha}}}{\pi^{d/2} \Gamma(1 - \frac{d}{2})} \). With the choice of \( x \), recall that \( w(y) - w(x) \geq 0 \) as \( y \in \tilde{U}_\delta \setminus B_\varepsilon(x) \), which yields
\[
C_3(d, m, \alpha) \int_{U_\delta \setminus B_\varepsilon(x)} (w(y) - w(x)) \frac{K_{(d+\alpha+2)/2}(m^{1/\alpha} \theta)}{\theta^{\frac{d+\alpha}{2}}} (|x - y^\mu| - |x - y|)dy
\geq C_3(d, m, \alpha) \int_{U_\delta \setminus B_\varepsilon(x)} (w(y) - w(x)) \frac{K_{(d+\alpha+2)/2}(m^{1/\alpha} |x - y^\mu|)}{|x - y^\mu|^{\frac{d+\alpha}{2}}} (|x - y| - |x - y^\mu|)dy
= C_3(d, m, \alpha) \int_{U_\delta \setminus B_\varepsilon(x)} (w(y) - w(x)) \frac{K_{(d+\alpha+2)/2}(m^{1/\alpha} |x - y^\mu|)}{|x - y^\mu|^{\frac{d+\alpha}{2}}} (\mu - y_1)dy.
\]
Observe that \( |x - y^\mu| \geq \inf_{z \in \tilde{U}_\delta} |x - z| = \mu - x_1 = \delta \), and thus
\[
\int_{U_\delta \setminus B_\varepsilon(x)} (w(y) - w(x))(j_{m, \alpha}(|x - y|) - j_{m, \alpha}(|x - y^\mu|))dy
\geq 2C_3(d, m, \alpha) \int_{U_\delta \setminus B_\varepsilon(x)} (w(y) - w(x)) \frac{K_{(d+\alpha+2)/2}(m^{1/\alpha} |x - y^\mu|)}{|x - y^\mu|^{\frac{d+\alpha}{2}}} (\mu - y_1)dy \quad (4.10)
\geq C_3(d, m, \alpha)\delta \int_{U_\delta \setminus B_\varepsilon(x)} (w(y) - w(x)) \frac{K_{(d+\alpha+2)/2}(m^{1/\alpha} |x - y^\mu|)}{|x - y^\mu|^{\frac{d+\alpha+2}{2}}} (\mu - y_1)dy.
\]
Combining (4.9) and (4.10) and writing \( C = C(d, m, \alpha) = \min \{ 2C_1(d, \alpha), C_2(d, \alpha), 2C_3(d, m, \alpha) \} \) we obtain (4.6).
Step 3: To complete the proof, consider the second integral in (4.8). Using the explicit form of \(j_{m,\alpha}\), we obtain the expressions
\[
\int_{U_\mu} \frac{K_{(d+\alpha)/2}(m^{1/\alpha}|x - y|)}{|x - y|^{d+\alpha}} dy = \int_{-\infty}^\mu \int_{\mathbb{R}^{d-1}} K_{(d+\alpha)/2}(m^{1/\alpha}(|x' - y'|^2 + |2\mu - x - y_1|^2)^{1/2}) dy dy_1
\]
\[
= \frac{1}{\delta^{d+\alpha}} \int_{-\infty}^\mu \int_{\mathbb{R}^{d-1}} K_{(d+\alpha)/2}(m^{1/\alpha}\delta(|x' - y'|^2 + |1 + \frac{\mu - y_1}{\mu - x_1}|^2)^{1/2}) dy dy_1
\]
\[
\quad = \delta^{d-\alpha} \int_0^\infty \int_{\mathbb{R}^{d-1}} K_{(d+\alpha)/2}(m^{1/\alpha}\delta(|z'|^2 + |1 + z_1|^2)^{1/2}) dz' dz_1,
\]
where the last line is obtained by setting \(z' = \frac{x' - y'}{\mu - x_1}\) and \(z_1 = \frac{\mu - y_1}{\mu - x_1}\). Since \(K_{(d+\alpha)/2}\) is decreasing and \(\delta \in (0, \delta_1)\), we have
\[
\int_{U_\mu} \frac{K_{(d+\alpha)/2}(m^{1/\alpha}|x - y|)}{|x - y|^{d+\alpha}} dy \geq C_4(d, m, \alpha, \delta_1)\delta^{d-\alpha},
\]
with
\[
C_4(d, m, \alpha, \delta_1) = \int_0^\infty \int_{\mathbb{R}^{d-1}} K_{(d+\alpha)/2}(m^{1/\alpha}\delta_1(|z'|^2 + |1 + z_1|^2)^{1/2}) dz' dz_1 < \infty.
\]
Applying (4.10)-(4.11) to (4.8) (recall that \(w(x) < 0\)), we arrive at (4.7) with the constant \(C = \min\{2C_3(d, m, \alpha), C_4(d, m, \alpha, \delta_1)\}\).

Remark 4.2. Since \(K_\epsilon(|x|) \sim 2^{\xi-1}\Gamma(\xi)|x|^{-\xi}\) as \(|x| \downarrow 0\), we see that, apart possibly from the numerical prefactor, estimate (4.6) gives consistently back the estimate (4.2) in the limit \(m \downarrow 0\).

The following is a consequence of Lemmas 4.1 and 4.3, which we will use below.

Corollary 4.1. Let \(m \geq 0, w \in C^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\) be a \(\mu\)-antisymmetric function, and suppose that there exists \(x \in U_\mu\) such that \(w(x) = \inf_{y \in U_\mu} w(y) < 0\). Denote \(\delta = \mu - x_1\). Then
\[
\Phi_{m,\alpha}(-\Delta)w(x) < 0.
\]

Now we are ready to prove the following monotonicity result for \(\varphi_k\).

Theorem 4.2. Let \(\chi_k : [0, \infty) \to \mathbb{R}\) be such that \(\varphi_k(x) = \chi_k(|x|)\). Then \(\chi_k\) is non-increasing in \([a + \varepsilon_k, \infty)\).

Proof. For any \(\mu \leq 0\), let \(\varphi_k^\mu(x) = \varphi_k(x^\mu)\) and \(w^k_\mu(x) := \varphi_k^\mu(x) - \varphi_k(x)\). Clearly, since \(\varphi_k \in C^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\), also \(w^k_\mu \in C^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\), for every \(\mu \leq 0\). Furthermore, it is \(\mu\)-antisymmetric by construction. Also, \(\varphi_k(x) \to 0\) as \(|x| \to \infty\) (see, [27] Cor. 4.1, 4.3).

Let \(\mu \leq -(a + \varepsilon_k)\) and assume that \(\inf_{x \in \partial U_\mu} w^k_\mu(x) < 0\). Since \(w^k_\mu(x) \to 0\) as \(|x| \to 0\) and \(w^k_\mu\) is continuous, the infimum is actually a minimum, i.e., there exists \(x_\ast \in \partial U_\mu\) such that \(w^k_\mu(x_\ast) = \inf_{x \in \partial U_\mu} w^k_\mu(x)\). Obviously, \(x_\ast \notin \partial U_\mu\), otherwise \(w^k_\mu(x_\ast)\) and then \(\Phi_{m,\alpha}(-\Delta)w^k_\mu(x_\ast) < 0\) by Corollary 4.1. On the other hand,
\[
\Phi_{m,\alpha}(-\Delta)w^k_\mu(x_\ast) = \Phi_{m,\alpha}(-\Delta)\varphi^\mu_k(x_\ast) - \Phi_{m,\alpha}(-\Delta)\varphi_k(x_\ast)
\]
\[
= \Phi_{m,\alpha}(-\Delta)\varphi_k(x^\mu_\ast) - \Phi_{m,\alpha}(-\Delta)\varphi_k(x_\ast)
\]
\[
= (\lambda_k - V_k(x^\mu_\ast))\varphi_k(x^\mu_\ast) - (\lambda_k - V_k(x_\ast))\varphi_k(x_\ast)
\]
\[
= (\lambda_k - V_k(x_\ast))w^k_\mu(x_\ast) + (V_k(x_\ast) - V_k(x^\mu_\ast))\varphi_k(x_\ast).\]
Since \( x_* \in \mathcal{U}_\mu \), we have \( |x_*| \geq |x_1| > a + \varepsilon_k \) and \( V_k(x_* \mu) = 0 \). Hence we get
\[
\Phi_{m,\alpha}(-\Delta)w^k_\mu(x_*) = \lambda w^k_\mu(x_*) - V_k(x_*) \varphi_k(x_*) > 0,
\]
which is a contradiction. Hence \( \inf_{x \in \mathcal{U}_\mu} w^k_\mu(x) \geq 0 \) and thus \( w^k_\mu(x) \geq 0 \) for all \( x \in \mathcal{U}_\mu \).

Now let \( r_1 > r_2 \geq a + \varepsilon_k \) and consider \( x = -r_1 e_1 \) and \( y = -r_2 e_1 \), where \( e_1 = (1, 0, \ldots, 0) \). Furthermore, let \( \mu = -r_1^2 - r_2^2 \). Clearly, \( \mu \leq -(a + \varepsilon) \), \( x \in \mathcal{U}_\mu \) and \( y = x^\mu \). The previous argument guarantees that \( w_\mu(x) \geq 0 \), which implies that \( \chi_k(r_2) = \varphi_k(y) \geq \varphi_k(x) = \chi_k(r_1) \).

As a consequence of this and Theorem 5.1 we get the following monotonicity result for \( \varphi \).

**Theorem 4.3.** Let \( \chi : [0, \infty) \to \mathbb{R} \) be such that \( \varphi(x) = \chi(|x|) \). Then \( \chi \) is non-increasing on \([a, \infty)\).

**Proof.** Let \( \Omega = \{ x \in \mathbb{R}^d : \lim_{k \to \infty} \varphi_k(x) = \varphi(x) \} \) and observe that \( |\mathbb{R}^d \setminus \Omega| = 0 \). As a consequence, \( \Omega \) is dense in \( \mathbb{R}^d \). Let \( r_1 > r_2 > a \) and consider \( x = -r_1 e_1 \) and \( y = -r_2 e_2 \). Consider two sequences \((x^\ell)_{\ell \in \mathbb{N}}, (y^\ell)_{\ell \in \mathbb{N}} \subset \Omega\) with \( x^\ell \to x \) and \( y^\ell \to y \). Since \( |x| > |y| > a \), we can assume without loss of generality that \( |x^\ell| > |y^\ell| > a \) for every \( \ell \in \mathbb{N} \). Now fix \( \ell \in \mathbb{N} \) and observe that there exists \( k_\ell \in \mathbb{N} \) such that \( |y^\ell| > a + \varepsilon_k \) for every \( k \geq k_\ell \). By Theorem 4.2 we know that \( \varphi_k(x^\ell) \leq \varphi_k(y^\ell) \) for every \( k \geq k_\ell \). Taking the limit as \( k \to \infty \) and using the fact that \( x^\ell, y^\ell \in \Omega \), we obtain \( \varphi(x^\ell) \leq \varphi(y^\ell) \). Since \( \varphi \) is continuous, taking the limit \( \ell \to \infty \) we get \( \varphi(x) \leq \varphi(y) \). Finally, the case \( r_2 = a \) is obtained by continuity of \( \varphi \).

Next we prove that \( \chi \) is non-increasing also in \([0, a]\). To do this, we need a family of auxiliary functions. Specifically, for every \( \mu \leq 0 \) define \( w_\mu(x) = \varphi(x^\mu) - \varphi(x) \).

**Lemma 4.4.** Let \( \mu \in (-a, 0] \). If \( x \in \mathcal{U}_\mu \) is such that \( x^\mu \notin B_\alpha \), then \( w_\mu(x) \geq 0 \).

**Proof.** Let \( x \in \mathbb{R}^d \) be such that \( x^\mu \notin B_\alpha \). Since \( a < \mu < 0 \) we have that \( x \notin B_\alpha \). Note that \( x = (x_1, x') \) and \( x^\mu = (2\mu - x_1, x') \). Since \( x_1 < \mu < 0 \), it follows that \( |2\mu - x_1| \leq |\mu| + |x_1| = -\mu + \mu - x_1 = -x_1 = |x_1| \). Hence \( |x| \geq |x^\mu| \geq a \) and \( \varphi(x) \leq \varphi(x^\mu) \) by Proposition 4.3 which shows the claim.

The following result highlights the relation between the ground state eigenvalue and the principal Dirichlet eigenvalue of the support of the potential well.

**Lemma 4.5.** Let \( \lambda_\alpha \) be the principal Dirichlet eigenvalue of \( B_\alpha \). Then \( \lambda + v < \lambda_\alpha \).

**Proof.** Let \( f_\alpha \) be a Dirichlet eigenfunction with \( \|f_\alpha\|_2 = 1 \). Since \( \varphi \) is the minimizer of \( \mathcal{A}_{m,\alpha} := \mathcal{A}_{\Phi_{m,\alpha}} \) on the set \( \{ u \in H^{\alpha/2}(\mathbb{R}^d) : \|u\|_2 = 1 \} \), we have
\[
\lambda = \mathcal{A}_{m,\alpha}(\varphi, \varphi) < \mathcal{A}_{m,\alpha}(f_\alpha, f_\alpha),
\]
where the inequality is strict by the fact that \( \lambda \) is a simple eigenvalue. Since \( f_\alpha \) is supported in \( B_\alpha \), we have
\[
\mathcal{A}_{m,\alpha}(f_\alpha, f_\alpha) = \mathcal{E}_{m,\alpha}(f_\alpha, f_\alpha) + \int_{B_\alpha} V(x)f_\alpha^2(x)dx = \lambda_\alpha - v,
\]
which shows the claim.

Now we are finally ready to prove that \( \varphi_0 \) is radially decreasing.

**Theorem 4.4.** Let \( \chi : [0, \infty) \to \mathbb{R} \) be such that \( \varphi(x) = \chi(|x|) \). Then \( \chi \) is non-increasing.
Proof. As Theorem 4.3 guarantees that χ is non-increasing in \([a, \infty)\), we only need to prove that it is also non-increasing in \([0, a]\). To do this, let \(\mu \in (-a, 0]\) and consider \(w_\mu\). Let also \(w_\mu^{k}\) as in the proof of Theorem 4.2 and observe that by Theorem 4.1 we have \(\Phi_{m, \alpha}(-\Delta)w_\mu^k \to \Phi_{m, \alpha}(-\Delta)w_\mu\) in \(L^2(\mathbb{R}^d)\). Consider an arbitrary function \(u \in C_c^\infty(\mathbb{R}^d)\) and note that

\[
\langle \Phi_{m, \alpha}(-\Delta)w^{k}\mu, u \rangle = \langle (\lambda_k - V_k)w^{k}\mu, u \rangle + \langle (V_k - V^\mu)\varphi_k, u \rangle, \tag{4.13}
\]

where \(V^\mu_k(x) = V_k(x^\mu)\). Recall also that \(w_\mu^k \to w_\mu\) in \(L^2(\mathbb{R}^d)\) and \(\|w_\mu\|_\infty \leq 2 \|\varphi\|_\infty < \infty\). Thus we get

\[
\int_{\mathbb{R}^d}|(\lambda_k - V_k(x))w_\mu^k(x) - (\lambda - V(x))w_\mu(x)|^2 dx \leq 2 \int_{\mathbb{R}^d}|\lambda_k w_\mu^k(x) - \lambda w_\mu(x)|^2 dx
\]

\[
+ 2 \int_{\mathbb{R}^d}|V_k(x)w_\mu^k(x) - V(x)w_\mu(x)|^2 dx
\]

\[
\leq 2(|\lambda_k|^2 + v^2) \|w_\mu^k - w_\mu\|_2^2 + 2 \|w_\mu\|_2^2 |\lambda_k - \lambda|^2 + 4 \|\varphi\|_\infty^2 \|V_k - V\|_2^2,
\]

which implies \((\lambda_k - V_k)w_\mu^k \to (\lambda - V)w_\mu\) in \(L^2(\mathbb{R}^d)\). Arguing in the same way, we get \((V_k - V^\mu)\varphi_k \to (V - V^\mu)\varphi\) in \(L^2(\mathbb{R}^d)\), where \(V^\mu_k(x) = V_k(x^\mu)\). Hence by taking the limit \(k \to \infty\) in (4.13), we obtain

\[
\mathcal{E}_{m, \alpha}(w_\mu, u) = \langle \Phi_{m, \alpha}(-\Delta)w_\mu, u \rangle = \langle (\lambda - V)w_\mu, u \rangle + \langle (V - V^\mu)\varphi, u \rangle. \tag{4.14}
\]

Since \(u \in C_c^\infty(\mathbb{R}^d)\) is arbitrary, equality (4.14) holds also for \(u \in H^{a/2}(\mathbb{R}^d)\). At this point we borrow an argument from [21] Prop. 3.1. Since \(w_\mu\) is continuous and thus \(w_\mu(x) = 0\) for all \(x \in \partial U_\mu\), we may consider \(u = w^-_\mu 1_{U_\mu}\) as a test function (see, e.g., [34] Exercise 3.22), where \(w^-_\mu = \max\{-w_\mu, 0\}\), and observe that

\[
(w_\mu(x) - w_\mu(y))(u(x) - u(y)) = -(u(x) - u(y))^2 - u(x)(w_\mu(y) + u(y)) - u(y)(w_\mu(x) - u(x)).
\]

Hence we have

\[
\mathcal{E}_{m, \alpha}(w_\mu, u) = -\mathcal{E}_{m, \alpha}(u, u) - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(x)(w_\mu(y) + u(y))j_{m, \alpha}(|x - y|) dx dy
\]

\[
- \int_{U_\mu} \int_{\mathbb{R}^d \setminus U_\mu} u(x)(w_\mu(y) + u(y))j_{m, \alpha}(|x - y|) dy dx
\]

\[
- \int_{U_\mu} \int_{\mathbb{R}^d \setminus U_\mu} u(x)(w_\mu(y) + u(y))j_{m, \alpha}(|x - y|) dy dx
\]

\[
- \mathcal{E}_{m, \alpha}(u, u) - \int_{U_\mu} \int_{U_\mu} u(x)w_\mu^+(y)j_{m, \alpha}(|x - y|) dy dx + \int_{U_\mu} \int_{U_\mu} u(x)w_\mu(y)j_{m, \alpha}(|x - y^\mu|) dy dx
\]

\[
\leq -\mathcal{E}_{m, \alpha}(u, u).
\]

On the other hand, we have

\[
0 = \mathcal{E}_{m, \alpha}(w_\mu, u) - \langle (\lambda - V)w_\mu, u \rangle - \langle (V - V^\mu)\varphi, u \rangle
\]

\[
\leq -\mathcal{E}_{m, \alpha}(u, u) - \langle (\lambda - V)w_\mu, u \rangle - \langle (V - V^\mu)\varphi, u \rangle. \tag{4.15}
\]
By Lemma 4.4 we know that supp(\(u\)) \(\subset \{x \in \mathbb{R}^d : x^\mu \notin B_\alpha\}\). Hence
\[
\langle (\lambda - V)u, u \rangle = \int_{\mathbb{R}^d} (\lambda - V(x))u(x)dx = -\int_{\mathbb{R}^d} (\lambda - V(x))u(x)^2dx \geq -(\lambda + \nu)\|u\|^2_2. 
\]

Since the Dirichlet eigenvalues are translation-invariant, we also have
\[
E_{\alpha, \nu}(u, u) \geq C\|u\|^2. 
\]

Furthermore,
\[
\langle (V - V^\mu)\varphi, u \rangle = \int_{B_\alpha(2\mu e_1)} (V(x) - V^\mu(x))\varphi(x)u(x)dx = \int_{B_\alpha(2\mu e_1)} (V(x) + v)\varphi(x)u(x)dx \geq 0. \tag{4.16} 
\]

Combining (4.15)-(4.16) and using Lemma 4.5, we obtain
\[
0 \leq -(\lambda - (\lambda + \nu))\|u\|^2_2 \leq 0. 
\]

Thus necessarily \(u \equiv 0\), which implies \(w_\mu(x) \geq 0\) for every \(x \in U_\mu\). Since \(\mu \in (-a, 0]\) is arbitrary, we have that \(w_\mu(x) \geq 0\) for all \(\mu \in (-a, 0]\) and every \(x \in U_\mu\). Take any \(0 < r_1 < r_2 < a\) and let \(x = -r_1e_1\), \(y = -r_2e_1\) and \(\mu = -\frac{r_1 + r_2}{2}\). Clearly, \(y \in U_\mu\) and \(x = y^\mu\). We have \(w_\mu(y) \geq 0\), which implies
\[
\chi(r_1) - \chi(r_2) = \varphi(x) - \varphi(y) = \varphi(y^\mu) - \varphi(y) \geq 0. 
\]

We then conclude the proof by making use of the continuity of \(\varphi\). \(\square\)

5. Appendix

5.1. Denseness of \(C^\infty_c(\mathbb{R}^d)\) in \(H^\Phi(\mathbb{R}^d)\)

Here we provide a constructive proof of Proposition 2.4. First we show the following growth property of the second moment of the jump measure over balls.

**Lemma 5.1.** For every \(\Phi \in \mathcal{B}_0\) we have
\[
\lim_{R \to \infty} \frac{1}{R^2} \int_{B_R} |x|^2j_\Phi(|x|)dx = 0. 
\]

**Proof.** By using (2.1) we readily get
\[
\frac{1}{R^2} \int_{B_R} |x|^2j_\Phi(|x|)dx = \frac{\sigma_d}{R^2} \int_{0}^{R} r^{d+1}j_\Phi(r)dr = \frac{\sigma_d}{R^2} \int_{0}^{R} \int_{0}^{\infty} r^{d+1} e^{-\frac{r^2}{4t}} \mu_\Phi(t)dt \frac{dt}{d}\int_{0}^{\infty} t^{-\frac{d}{2}} \mu_\Phi(t) \left( \int_{0}^{R} r^{d+1} e^{-\frac{r^2}{4t}} dr \right) dt 
\]
\[
eq \frac{\sigma_d}{R^2} \int_{0}^{\infty} t^{\frac{d}{2}} \mu_\Phi(t) \left( \int_{0}^{R} r^{d+1} e^{-\frac{r^2}{4t}} dr \right) dt. 
\]
where \( \gamma(s; x) := \int_0^x z^{s-1} e^{-z} \, dz \) is the standard lower incomplete Gamma function. Writing \( s = \frac{R^2}{4\pi} \), we obtain
\[
\frac{1}{R^2} \int_{B_R} |x|^2 j_\Phi(|x|) \, dx = C_d R^2 \int_0^\infty s^{-3} \mu_\Phi \left( \frac{R^2}{4s} \right) \gamma \left( \frac{d}{2} + 1; s \right) \, ds
\]
\[
= C_d R^2 \left( \int_0^1 + \int_1^\infty \right) s^{-3} \mu_\Phi \left( \frac{R^2}{4s} \right) \gamma \left( \frac{d}{2} + 1; s \right) \, ds
\]
\[
= : I_1(R) + I_2(R).
\]
To handle \( I_1(R) \) we use that \( \gamma \left( \frac{d}{2} + 1; s \right) \rightarrow \frac{2}{\pi s} \) as \( s \rightarrow 0 \) and again make the substitution \( s = \frac{R^2}{4\pi} \) to obtain
\[
I_1(R) \leq C_d R^2 \int_0^1 s^{4-\frac{d}{2}} \mu_\Phi \left( \frac{R^2}{4s} \right) \, ds = C_d \int_{R^2/4}^\infty R^d t^{-\frac{d}{2}} \mu_\Phi(t) \, dt \leq C_d \int_{R^2/4}^\infty \mu_\Phi(t) \, dt \rightarrow 0,
\]
due to \( \int_1^\infty \mu(t) \, dt < \infty \).

Coming to \( I_2(R) \), we use that \( \gamma \left( \frac{d}{2} + 1; s \right) \rightarrow \Gamma \left( \frac{d}{2} + 1 \right) \) as \( s \rightarrow \infty \), leading to
\[
I_2(R) \leq C_d R^2 \int_1^\infty s^{-3} \mu_\Phi \left( \frac{R^2}{4s} \right) \, ds = C_d \int_0^{R^2/4} t \mu_\Phi(t) \, dt.
\]
Define \( \overline{\mu_\Phi}(t) := \int_t^\infty \mu_\Phi(\tau) \, d\tau \). The function \( \overline{\mu_\Phi} \in L^1(0, 1) \) and is non-increasing, hence \( \lim_{t \rightarrow 0} t \overline{\mu_\Phi}(t) = 0 \) by the monotone density theorem (see, e.g., [11, Th. 1.7.2]). Integration by parts gives
\[
I_2(R) \leq C_d \int_0^{R^2/4} \frac{R^2}{4} \frac{R^2}{4} \overline{\mu_\Phi}(t) \, dt = - \frac{C_d \overline{\mu_\Phi}(R^2/4)}{4} + C_d \int_0^{R^2/4} \overline{\mu_\Phi}(t) \, dt.
\]
Finally, again by the monotone density theorem and using that \( \overline{\mu_\Phi}(t) \rightarrow 0 \) as \( t \rightarrow \infty \), we obtain
\[
\lim_{R \rightarrow \infty} \frac{C_d}{R^2} \int_0^{R^2/4} \overline{\mu_\Phi}(t) \, dt = 0,
\]
which gives then \( \lim_{R \rightarrow \infty} I_2(R) = 0 \). \( \square \)

Now we are ready to prove that \( C_c^\infty(\mathbb{R}^d) \) is dense in \( H^\Phi(\mathbb{R}^d) \). Define \( g_n : \mathbb{R}^d \rightarrow \mathbb{R} \), \( n \in \mathbb{N} \), as a function in \( C_c^\infty(\mathbb{R}^d) \) such that \( g_n(x) = 1 \) for every \( x \in B_n \), \( g_n(x) = 0 \) for every \( x \in \mathbb{R}^d \setminus B_{2n} \), \( 0 \leq g_n(x) \leq 1 \) for every \( x \in B_{2n} \setminus B_n \) and \( \nabla g_n(x) \leq \frac{C}{n} \), with a constant \( C \) independent of \( n \). (This can be done by applying a mollifier to the characteristic function of \( B_n \), once we notice that \( \text{dist}(B_n, \partial B_{2n}) = n \) for all \( n \in \mathbb{N} \)). Furthermore, let \( (g_n)_{n \in \mathbb{N}} \) be a sequence of Friedrichs mollifiers of the form \( g_n(x) = n^d g(nx) \), \( \text{supp} \, g(x) = B_1 \). Fix \( u \in H^\Phi(\mathbb{R}^d) \) and define \( u_n := g_n * (g_n u) \). Clearly, \( u_n \in C_c^\infty(\mathbb{R}^d) \), moreover,
\[
\| u_n - u \|^2 = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} g_n(x-y) g_n(x) u(x) \, dx - u(y) \right|^2 \, dy
\]
\[
\leq 2 \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} g_n(x-y) g_n(x) u(x) \, dx - \int_{\mathbb{R}^d} g_n(x-y) u(x) \, dx \right|^2 \, dy
\]
\[
+ 2 \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} g_n(x-y) u(x) \, dx - u(y) \right|^2 \, dy =: 2 (I_{n}^{(1)} + I_{n}^{(2)}).
\]
It is well-known that \( I_{n}^{(2)} := \| g_n * u - u \|^2 \rightarrow 0 \) as \( n \rightarrow \infty \). Furthermore, by the generalized Minkowski inequality (see [22, ineq. 202]),
\[
I_{n}^{(1)} \leq \| g_n u - u \|^2 = \int_{B_{n+1} \setminus B_n} (1 - g_n(x))^2 |u(x)|^2 \, dx \leq \int_{B_{n+1} \setminus B_n} |u(x)|^2 \, dx \rightarrow 0.
\]
Hence \( u_n \to u \) in \( L^2(\mathbb{R}^d) \). Let now \( \overline{u}_n = g_n u \). We have

\[
[u_n - u]^2 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| u_n(y) - u(y) - u_n(x) + u(x) \right|^2 j_\Phi(|x - y|) \, dx \, dy \\
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| u_n(y) - \overline{u}_n(y) + \overline{u}_n(y) - u(y) - u_n(x) + \overline{u}_n(x) - \overline{u}_n(x) + u(x) \right|^2 j_\Phi(|x - y|) \, dx \, dy \\
\leq 2[u_n - \overline{u}_n]^2 + 2[\overline{u}_n - u]^2.
\]

First consider the second semi-norm. Define \( \overline{g}_n = 1 - g_n \) such that

\[
[\overline{u}_n - u]^2 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \overline{g}_n(x) u(x) - \overline{g}_n(y) u(y) \right|^2 j_\Phi(|x - y|) \, dx \, dy \\
= 2 \int_{B_n} \int_{B_{2n}\setminus B_n} |\overline{g}_n(y) u(y)|^2 j_\Phi(|x - y|) \, dx \, dy \\
+ 2 \int_{B_n} \int_{\mathbb{R}^d \setminus B_{2n}} |u(y)|^2 j_\Phi(|x - y|) \, dx \, dy \\
+ \int_{B_{2n}\setminus B_n} \int_{B_{2n}\setminus B_n} |\overline{g}_n(x) u(x) - \overline{g}_n(y) u(y)|^2 j_\Phi(|x - y|) \, dx \, dy \\
+ 2 \int_{B_{2n}\setminus B_n} \int_{\mathbb{R}^d \setminus B_{2n}} |\overline{g}_n(x) u(x) - u(y)|^2 j_\Phi(|x - y|) \, dx \, dy \\
+ \int_{\mathbb{R}^d \setminus B_{2n}} \int_{\mathbb{R}^d \setminus B_{2n}} |u(x) - u(y)|^2 j_\Phi(|x - y|) \, dx \, dy \\
= 2I_n^{(3)} + 2I_n^{(4)} + I_n^{(5)} + 2I_n^{(6)} + I_n^{(7)}.
\]

Since for every \( x \in B_n \) we have that \( \overline{g}_n(x) = 0 \) and \( \|\nabla \overline{g}_n\|_\infty \leq \frac{C}{n} \), we get

\[
I_n^{(3)} \leq \frac{C}{n^2} \int_{B_n} \int_{B_{2n}\setminus B_n} |u(y)|^2 |x - y|^2 j_\Phi(|x - y|) \, dx \, dy \leq \frac{C \|u\|^2}{n^2} \int_{B_{2n}} |x|^2 j_\Phi(|x|) \, dx.
\]

The second bound follows by the observation that since \( y \in B_{2n} \setminus B_n \), we have \( B_n \subset B_{3n}(y) \). Taking the limit as \( n \to \infty \), by Lemma [5.1] we have \( \lim_{n \to \infty} I_n^{(3)} = 0 \).

For the next integral we simply estimate

\[
I_n^{(4)} \leq \|u\|^2 \int_{\mathbb{R}^d \setminus B_{2n}} j_\Phi(|x|) \, dx,
\]

which on taking the limit gives \( \lim_{n \to \infty} I_n^{(4)} = 0 \), given that \( \int_{\mathbb{R}^d \setminus B_1} j_\Phi(|x|) \, dx < \infty \).

Next consider \( I_n^{(5)} \). Adding and subtracting \( \overline{g}_n(x) u(x) \) we have

\[
I_n^{(5)} \leq 2 \int_{B_{2n}\setminus B_n} \int_{B_{2n}\setminus B_n} |\overline{g}_n(x) - \overline{g}_n(y)|^2 |u(x)|^2 j_\Phi(|x - y|) \, dx \, dy \\
+ 2 \int_{B_{2n}\setminus B_n} \int_{B_{2n}\setminus B_n} |\overline{g}_n(x)|^2 |u(x) - u(y)|^2 j_\Phi(|x - y|) \, dx \, dy \\
\leq \frac{2C}{n^2} \int_{B_{2n}\setminus B_n} \int_{B_{2n}\setminus B_n} |x - y|^2 |u(x)|^2 j_\Phi(|x - y|) \, dx \, dy \\
+ 2 \int_{B_{2n}\setminus B_n} \int_{B_{2n}\setminus B_n} |u(x) - u(y)|^2 j_\Phi(|x - y|) \, dx \, dy = 2(I_n^{(8)} + I_n^{(9)}).
\]
Since \( x \in B_{2n} \), we have that \( B_{2n} \subset B_{4n}(x) \) and thus
\[
I_n^{(8)} \leq \frac{2C \|u\|_2}{n^2} \int_{B_{4n}} |y|^2 j_\Phi(|y|)dy \to 0,
\]
again by Lemma 5.1. On the other hand, \( \lim_{n \to \infty} I_n^{(9)} = 0 \) by dominated convergence, since \( |u|^2 \Phi < \infty \). Hence we have \( \lim_{n \to \infty} I_n^{(5)} = 0 \).

To estimate \( I_n^{(6)} \), we add and subtract \( u(x) = \bar{H}_n(y)u(x) \) using that \( y \in \mathbb{R}^d \setminus B_{2n} \), and further split off the integral giving
\[
I_n^{(6)} \leq 2 \int_{B_{2n} \setminus B_n} \int_{B_{3n} \setminus B_{2n}} |\mathcal{g}_n(x) - \mathcal{g}_n(y)|^2 |u(x)|^2 j_\Phi(|x - y|) dx dy
+ 2 \int_{B_{2n} \setminus B_n} \int_{\mathbb{R}^d \setminus B_{2n}} |\mathcal{f}_n(x) - \mathcal{f}_n(y)|^2 |u(x)|^2 j_\Phi(|x - y|) dx dy
+ 2 \int_{B_{2n} \setminus B_n} \int_{\mathbb{R}^d \setminus B_{2n}} |\mathcal{f}_n(x)|^2 |u(x) - u(y)|^2 j_\Phi(|x - y|) dx dy = 2(I_n^{(10)} + I_n^{(11)} + I_n^{(12)}).
\]

Concerning \( I_n^{(10)} \), using that \( x \in B_{3n} \setminus B_{2n} \) and so \( B_{2n} \subset B_{5n}(x) \), we have,
\[
I_n^{(10)} \leq \frac{C \|u\|_2}{n^2} \int_{B_{3n}} |y|^2 j_\Phi(|y|)dy \to 0,
\]
by Lemma 5.1. On the other hand, if \( x \in \mathbb{R}^d \setminus B_{3n} \) and \( y \in B_{2n} \setminus B_n \), we have \( |x - y| \geq 1 \) and thus
\[
I_n^{(11)} \leq 8 \left( \int_{\mathbb{R}^d \setminus B_{3n}} |u(x)|^2 dx \right) \left( \int_{\mathbb{R}^d \setminus B_1} j_\Phi(|y|)dy \right) \to 0
\]
since \( u \in L^2(\mathbb{R}^d) \). Furthermore, \( I_n^{(12)} \to 0 \) by dominated convergence, as \( |u|^2 \Phi < \infty \). Finally, \( I_n^{(7)} \to 0 \) simply by the dominated convergence theorem. Combining the limits \( I_n^{(i)} \to 0 \) for \( i = 3, \ldots, 7 \), we have \( \lim_{n \to \infty} \|\overline{u}_n - u\|_\Phi^2 = 0 \). Furthermore \( \|\overline{u}_n\| \leq \|u\| + \|\overline{u}_n - u\| \) and so \( \limsup_{n \to \infty} \|u_n\| \Phi \leq \|u\| \Phi < \infty \), which leads to \( \|\overline{u}_n\| \leq C \) for a constant \( C > 0 \) independent of \( n \).

Note that by the Jensen inequality
\[
[\overline{u}_n - \overline{u}_n \Phi]^2 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u_n(y) - \overline{u}_n(y) - u_n(x) + \overline{u}_n(x)|^2 j_\Phi(|x - y|) dx dy
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} g_n(z)\overline{u}_n(y - z)dz - \overline{u}_n(y) - \int_{\mathbb{R}^d} g_n(z)\overline{u}_n(x - z)dz + \overline{u}_n(x) \right|^2 j_\Phi(|x - y|) dx dy
\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} g_n(z)(\overline{u}_n(y - z) - \overline{u}_n(y) - \overline{u}_n(x - z) + \overline{u}_n(x))dz \right|^2 j_\Phi(|x - y|) dx dy
\leq 3 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g_n(z)(|\overline{u}_n(y - z) - u(y - z)|\overline{u}_n(x - z) + u(x - z) - u(y - z) + u(y) \overline{u}_n(x) + u(x))^2 j_\Phi(|x - y|) dz dx dy
\leq 6\|\overline{u}_n - u\|_\Phi^2 + 3 \int_{\mathbb{R}^d} g_n(z) \int_{\mathbb{R}^d \times \mathbb{R}^d} |(u_n(y) - u_n(x))|^2 j_\Phi(|x - y|) dy dz dx.
\]
Denote \( G(x, y) = (u(y) - u(x)) \sqrt{\| \Phi(\cdot - y) \| L_2(\mathbb{R}^d)} \in L^2(\mathbb{R}^d \times \mathbb{R}^d) \). By the above we then have
\[
[u_n - \Pi_n]^2 \leq 6|\Pi_n - u_n|^2 + 3 \int_{\mathbb{R}^d} \| G(\cdot - z, \cdot - z) - G \|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} |\eta(z)| dz
\]
\[
= 6|\Pi_n - u_n|^2 + 3n^d \int_{\mathbb{R}^d} \| G(\cdot - z, \cdot - z) - G \|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} |\eta(nz)| dz
\]
\[
= 6|\Pi_n - u_n|^2 + 3 \int_{\mathbb{R}^d} \| G(\cdot - \frac{z}{n}, \cdot - \frac{z}{n}) - G \|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} |\eta(z)| dz
\]
\[
\leq 6|\Pi_n - u_n|^2 + 3 \| \eta \|_{\infty} \int_{B_1} \| G(\cdot - \frac{z}{n}, \cdot - \frac{z}{n}) - G \|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} dz.
\]
Since \( G \in L^2(\mathbb{R}^d \times \mathbb{R}^d) \), the function \( z \in \mathbb{R}^d \mapsto \| G(\cdot - z, \cdot - z) - G \|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} \in \mathbb{R} \) is continuous and thus \( \sup_{z \in B_1} \| G(\cdot - z, \cdot - z) - G \|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} < \infty \). Hence we can use the dominated convergence theorem to obtain
\[
[u_n - \Pi_n]^2 \leq 6|\Pi_n - u_n|^2 + 3 \| \eta \|_{\infty} \int_{B_1} \| G(\cdot - \frac{z}{n}, \cdot - \frac{z}{n}) - G \|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} dz \to 0
\]as \( n \to \infty \).

We note that the strategy of the proof above follows [7, Th. 3.1]. In particular, in this proof we actually obtained a truncation result in the spirit of [7, Appx. B].

5.2. Equivalent expression

We prove relation (2.13). First of all, since \( S(\mathbb{R}^d) \subset C^2(\mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^d) \), Lemma [24] ensures that \( \Phi(-\Delta)u \) is well-defined. To see that we can exchange the order of Fourier transform and the integral in the definition of \( \Phi(-\Delta)u \), first we show that
\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(x + h) - 2u(x) + u(x - h)| j_\Phi(|h|) dh dx < \infty.
\]
Let \( D^2u \) be the Hessian matrix of \( u \) and denote \( |D^2u(x)| = \sup_{h \in \mathbb{R}^d \setminus \{0\}} \frac{|D^2u(x)|}{|h|} \). We split off the integral like
\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(x + h) - 2u(x) + u(x - h)| j_\Phi(|h|) dh dx = \int_{\mathbb{R}^d} \int_{B_1} |u(x + h) - 2u(x) + u(x - h)| j_\Phi(|h|) dh dx
\]
\[
+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d \setminus B_1} |u(x + h) - 2u(x) + u(x - h)| j_\Phi(|h|) dh dx
\]
\[
= I_1 + I_2.
\]
Note that there exists a function \( \theta : (x, h) \in \mathbb{R}^d \times \mathbb{R}^d \mapsto \theta(x, h) \in B_1(x) \) such that
\[
I_1 \leq \int_{\mathbb{R}^d} \int_{B_1} |D^2u(\theta(x, h))| |h|^2 j_\Phi(|h|) dh dx
\]
\[
= \int_{B_1} \int_{B_1} |D^2u(\theta(x, h))| |h|^2 j_\Phi(|h|) dh dx + \int_{\mathbb{R}^d \setminus B_2} \int_{B_1} |D^2u(\theta(x, h))| |h|^2 j_\Phi(|h|) dh dx
\]
\[
= I_3 + I_4.
\]
Concerning \( I_3 \), observe that if \( x \in B_2 \), then \( \theta(x, h) \in B_3 \) for every \( h \in B_1 \). Hence we have
\[
I_3 \leq 3^d \omega_d \| D^2u \|_{L^\infty(B_3)} \int_{B_1} |h|^2 j_\Phi(|h|) dh < \infty,
\]
where \( \omega_d \) is the Lebesgue measure of \( B_1 \). On the other hand, if \( x \in \mathbb{R}^d \setminus B_2 \), then \( \theta(x, h) \in \mathbb{R}^d \setminus B_{|x|^{-1}} \) and then we have

\[
I_4 \leq 3^d \omega_d \left( \int_{\mathbb{R}^d \setminus B_2} \| D^2 u \|_{L^\infty(\mathbb{R}^d \setminus B_{|x|^{-1}})} \, dx \right) \left( \int_{B_1} |h|^2 j_{\Phi}(|h|) \, dh \right).
\]

However, since \( u \in S(\mathbb{R}^d) \), it is clear that

\[
|z|^2 \| D^2 u(z) \| \leq d \sup_{z \in \mathbb{R}^d \setminus B_1} |z|^2 \max_{1 \leq i, j \leq d} \left| \frac{\partial^2 u}{\partial x_i \partial x_j} (z) \right| = d \max_{1 \leq i, j \leq d} \sup_{z \in \mathbb{R}^d \setminus B_1} |z|^2 \left| \frac{\partial^2 u}{\partial x_i \partial x_j} (z) \right| =: C_{S(\mathbb{R}^d)}(u) < \infty
\]

for all \( z \in \mathbb{R}^d \setminus B_{|x|^{-1}} \), which leads to

\[
|D^2 u(z)| \leq \frac{C_S(u)}{|z|^2} \leq \frac{C_S(u)}{(|x| - 1)^2}, \quad z \in \mathbb{R}^d \setminus B_{|x|^{-1}}.
\]

On taking the supremum on \( z \) we obtain

\[
\| D^2 u \|_{L^\infty(\mathbb{R}^d \setminus B_{|x|^{-1}})} \leq C_{S(\mathbb{R}^d)}(u) \leq \frac{C_{S(\mathbb{R}^d)}(u)}{(|x| - 1)^2}.
\]

In particular,

\[
\int_{\mathbb{R}^d \setminus B_2} \| D^2 u \|_{L^\infty(\mathbb{R}^d \setminus B_{|x|^{-1}})} \, dx \leq C_{S(\mathbb{R}^d)}(u) \int_{\mathbb{R}^d \setminus B_2} \frac{dx}{(|x| - 1)^2} < \infty,
\]

implying \( I_4 < \infty \). Finally, considering \( I_2 \) we have

\[
I_2 \leq 4 \| u \|_{\infty} \int_{\mathbb{R}^d \setminus B_1} j_{\Phi}(|h|) \, dh < \infty.
\]

Hence we can use Fubini’s theorem and the properties of the Fourier transform to obtain

\[
\mathcal{F}[\Phi(-\Delta) u](\xi) = -\frac{\tilde{u}(\xi)}{2} \int_{\mathbb{R}^d} (e^{i |h| \xi} + e^{-i |h| \xi} - 2 \tilde{u}(\xi)) j_{\Phi}(|h|) \, dh = \tilde{u}(\xi) \int_{\mathbb{R}^d} (1 - \cos(h \cdot \xi)) j_{\Phi}(|h|) \, dh.
\]

By a similar argument as in Lemma 2.2 and also using Lemma [26] Lem. 2.1, it follows then that

\[
\int_{\mathbb{R}^d} (1 - \cos(h \cdot \xi)) j_{\Phi}(|h|) \, dh = f(|\xi|^2),
\]

which completes the proof.

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