Symmetries of the Ricci Tensor of Static Space-times with Maximal Symmetric Transverse Spaces

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Abstract

Static space times with maximal symmetric transverse spaces are classified according to their Ricci collineations. These are investigated for non-degenerate Ricci tensor (det.(R_α) ≠ 0). It turns out that the only collineations admitted by these spaces can be ten, seven, six or four. Some new metrics admitting proper Ricci collineations are also investigated.

PACS numbers: 04.20.-q, 04.20.Jb
Keywords: Ricci collineation, maximal symmetric spaces, exact solutions of Einstein Field equations

1 Introduction

Let M be a four dimensional smooth, connected, hausdorff manifold admitting a smooth Lorentz metric g of signature (+ - - -). A Lie derivative along a vector field V is denoted by L_V, when component notation is used, a partial derivative and a covariant derivative with respect to the Levi-Civita connection Γ associated with g are denoted by a comma and a semi-colon respectively. In general relativity theory M plays the role of the space time and the geometrical objects g, Γ and the curvature tensor on M derived from Γ collectively describe the gravitational fields. Einstein’s equations

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu} \]  

(1)

provide physical restrictions on these objects. Where G_{μν} is the Einstein tensor, R_{μν} is the Ricci tensor, R = g^{\mu\nu} R_{\mu\nu} is the Ricci scalar and T_{μν} is the energy-momentum tensor. I have assumed here that the cosmological constant is zero. Using the Bianchi identity, it can easily be shown that G_{μν} = 0 ⇔ T_{μν} = 0. A space time is said to permit collineation if

\[ L_V \phi = \Lambda, \]  

(2)

where V is the collineation vector, φ is any of the quantities, g_{μν}, Γ^λ_μν, R_{μν}, R^λ_μν, and geometric objects derived from these quantities. Λ is a tensor with the same index symmetries as φ. Hence many types of symmetries structure of the space time are investigated in general theory of relativity in order to understand the natural relationship between geometry and matter furnished by the Einstein field equations [1], [2]. Isometries, Homothetic motions, conformal Isometries, affine and projective collineations and symmetries of curvature and related tensors are the examples of these symmetries [3]. These symmetries structure of the space times not only help us to

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obtain exact solutions of Einstein field equations but also provide invariant bases for classifying space-times. In recent years, a large number of exact solutions of Einstein field equations with different symmetry structures have been found [4] and classified space time according to their symmetries structure [5]. A space time is said to admit Ricci collineation vector field \( V \) on Lorentzian manifold \( M \) if the Lie derivative of \( R_{\mu\nu} \) along \( V \) is zero.

\[
(C_{\mu\nu}) = L_V R_{\mu\nu} = 0 \iff R_{\mu\nu;\lambda} V^\lambda + R_{\mu\lambda} V^\lambda_{;\nu} + R_{\nu\lambda} V^\lambda_{;\mu} = 0
\]  

The Ricci collinations are purely geometric in nature but like matter collinations provide physical information of space-times through Einstein field equations. In recent years, there has been much concern in the study of the various symmetries, particularly in matter and Ricci collinations. Green et al [6] and Nunez et al [7] have considered an example of Ricci collineation and the family of contracted Ricci collineation symmetries of Robertson-Walker metric. They have restricted their study to symmetries generated by the collineation vector field of the following form, respectively,

\[
V = V^4(t, r, \theta, \phi) \frac{\partial}{\partial t} \quad \text{and} \quad V = V^1(t, r) \frac{\partial}{\partial t} + V^4(t, r) \frac{\partial}{\partial r}
\]

Also, the relationship with the constants of motion between Ricci collineation and family of Contracted Ricci collineation has been investigated in references [8], [9], [10], [11]. Ricci and matter collinations for static spherically symmetric space times have been studied recently by various authors [12], [13], [14], [15]. Amir, Bokhari and Qadir has found the relationship between the Ricci collinations and Killing vectors for these space times [12]. Contreras et al [16] have investigated Ricci collinations for non-static spherically symmetric space times and they have confined their study to the symmetries generated by the vector field of the form,

\[
V = V^t(t, r) \frac{\partial}{\partial t} + V^r(t, r) \frac{\partial}{\partial r}
\]

Qadir, Saifullah, and Ziad [17] have considered cylindrically symmetric static space times and worked out complete classification according to their Ricci collinations. In addition, Caret et al [18] has studied matter collinations as a symmetric property of the energy-momentum tensor and Hall et al [19] have studied the Ricci and matter collinations. Recently Camci et al [20] have classified Bianchi types-1 and 111, and Kantowski-Sachs space times according to their Ricci collineation vector field. Petrov [21] has initiated an approach of finding information about the solutions of Einstein field equations without specifying the stress-energy tensor, instead of looking only at the space-time symmetries. This approach does not always provide specific metrics, or classes of metrics for a given isometry. In fact some symmetries were given in reference [22] for which there is no corresponding metric. Subsequently an approach was developed to ask for minimal isometry group and then classify completely all higher symmetry space times. This method provides complete classification for various space times.

The general line element for a static space times with maximal symmetric transverse spaces can be written as

\[
ds^2 = e^{A(r)} dt^2 - e^{B(r)} dr^2 - r^2 (d\theta^2 + f_k^2(\theta) d\phi^2)
\]

where \( f_k(\theta) \) is given by

\[
f_k(\theta) = \begin{cases} 
\sin \theta & \text{when } k = 1 \\
\theta & \text{when } k = 0 \\
\sinh \theta & \text{when } k = -1
\end{cases}
\]
The maximal symmetric transverse spaces are Einstein spaces. As the metric under consideration is diagonal and the metric coefficients depend on $r$ only, the non-zero components of Ricci tensor are

\[ R_{00} = \frac{A''}{2} - \frac{A'B'}{4} + \frac{A'^2}{4} + \frac{A'}{r}e^{A-B} \tag{5} \]

\[ R_{11} = \frac{-A''}{2} + \frac{A'B'}{4} - \frac{A'^2}{4} + \frac{B'}{r} \tag{6} \]

\[ R_{22} = \left(\frac{rB'}{2} - \frac{rA'}{2} - 1\right)e^{-B} + k \tag{7} \]

\[ R_{33} = f_k^2(\theta)R_{22} \tag{8} \]

Here prime indicates the derivative with respect to $r$. Let $R_{\alpha\alpha}$ is denoted by $R_{\alpha}$, where the Greek indices $\alpha$, $\beta$ will be used when it is needed to drop the summation convention. Further $R_{\alpha}$ are arbitrary function of $r$, for $\alpha = 0, 1, 2$ and $R_3 = R_2f_k^2(\theta)$. Using the components of Ricci tensor for equation (4) in Ricci collineation equations (3) I get the following independent equations

\[ (C_{\alpha\beta}) = R_{\alpha}V_{\beta,\beta} + R_{\beta}V_{\alpha,\alpha} = 0 \tag{9} \]

\[ (C_{00}) = R_0'V^1 + 2R_0V^0_0 = 0 \tag{10} \]

\[ (C_{11}) = R_1'V^1 + 2R_1V^1_1 = 0 \tag{11} \]

\[ (C_{22}) = R_2'V^1 + 2R_2V^2_2 = 0 \tag{12} \]

\[ (C_{33}) = R_2'V^1 + 2R_2[(\frac{f_k}{f_k^2})V^2 + V^3_{33}] = 0 \tag{13} \]

Equations (9) give six equations and equations (10-13) are four partial differential equations, these constitute together ten first order, non-linear coupled partial differential equations involving four components of the arbitrary Ricci collineation vector $V = (V^0, V^1, V^2, V^3)$, four components of Ricci tensor and their partial derivatives. The components $V^\mu$ with $\mu$ from 0 to 3 depend on $t, r, \theta$, and $\phi$ and the components of Ricci tensor on $r$ only. I solve these set of ten partial differential equations for the components of the Ricci collineation vector field. I consider first Ricci collineation equations (9) and solve them simultaneously to obtain the components of $V$ in terms of arbitrary functions of the coordinates. Substituting these values in other Ricci collineation equations I get conditions on these arbitrary functions. Solving these conditions and checking consistency with the Ricci collineation equations at every step until these functions are determined explicitly and the final form of $V$ involving arbitrary constants is obtained. While finding these solutions, the constraints on the components of the Ricci tensor are obtained. Solving these constraints will give the metrics of the space time. I evaluate Ricci collineations only for those cases which have non-degenerate Ricci tensor i.e. $\text{det}(R_{\mu\nu} \neq 0)$. The degenerate cases for these spaces have already been discussed in reference [23]. These equations (9 - 13) have been solved simultaneously for the components of collineation vector and get the solutions of these components in the form in which they become a known functions of $\theta$ and $\phi$ and unknown in these of $t$ and $r$, as given below

\[ V^0 = \left(\frac{-R_2}{R_0}\right)f_k^2[[\int \left(\frac{1}{f_k}\right)\int f_kd\theta]d\theta]A_1(t, r, 0\sin \phi - A_2(t, r, 0\cos \phi)] + \left(\frac{-R_2}{R_0}\right)A_3(t, r, 0\int f_kd\theta) + P(t, r) \tag{14} \]
\[ V^1 = \left( \frac{-R_2}{R_1} \right) f_k^2 \int \left( \frac{-1}{f_k^2} \right) \left( \int f_k d\theta \right) \left( A_1(t, r)_1 \sin \phi - A_2(t, r)_1 \cos \phi \right) + \left( \frac{-R_2}{R_1} \right) A_3(t, r)_1 \left( \int f_k d\theta \right) + Q(t, r) \]

(15)

\[ V^2 = \int f_k d\theta \left( A_1(t, r) \sin \phi - A_2(t, r) \cos \phi \right) + C_1 \sin \phi - C_2 \cos \phi + A_3(t, r) f_k \]

(16)

\[ V^3 = \left( \int \left( \frac{-1}{f_k^2} \right) \int f_k d\theta \right) \left[ A_1(t, r) \cos \phi + A_2(t, r) \sin \phi \right] + (C_1 \cos \phi + C_2 \sin \phi) \int \left( \frac{-1}{f_k^2} \right) d\theta + C_3 \]

(17)

where \( C_1, C_2 \) and \( C_3 \) are arbitrary constants. Here the partial derivatives with respect to 0 and 1, indicate the derivatives with respect to \( t \) and \( r \) coordinates, respectively. Replacing the values of \( V^\mu \) with \( \mu \) from 0 to 3 in Ricci collineation equations (9-13) show that \( (C_{01}), (C_{03}), (C_{12}), (C_{13}), \) and \( (C_{23}) \) are satisfied identically, whereas \( (C_{00}), (C_{02}), (C_{11}), (C_{22}), \) and \( (C_{33}) \) are satisfied subject to the following differential constraints on \( A_i(t, r), P(t, r) \) and \( Q(t, r) \), with \( C_4 = 0 \).

\[ R'_0 A'_i(t, r) + 2R_1 A_i(t, r)_0 = 0, \quad i = 1, 2, 3 \]

(18)

\[ \left[ \sqrt{\frac{R_2}{R_0}} A_i(t, r)_0 \right]' = 0 \]

(19)

\[ \left[ \frac{R_2}{\sqrt{R_1}} A'_i(t, r) \right]' = 0 \]

(20)

\[ \frac{R'_2}{2R_1} A'_i(t, r) - A_i(t, r) = 0 \]

(21)

\[ P(t, r)_0 + \frac{R'_0}{2R_0} Q(t, r) = 0 \]

(22)

\[ P'(t, r) + \frac{R_1}{R_0} Q(t, r)_0 = 0 \]

(23)

\[ (R_2 \sqrt{R_1} Q(t, r))' = 0 \]

(24)

\[ R'_2 Q(t, r) = 0 \]

(25)

where \( i = 1, ..., 3 \). In this article, The symmetry properties of static space times with maximal symmetric transverse spaces are investigated by considering Ricci collineation. Ricci collineation equations are solved for the components of collineation vector field by writing them in a form in which they become known functions of \( \theta \) and \( \phi \) and unknown in these of \( t \) and \( r \). Substituting these components into each of the collineation equations imply that some of these equations are identically satisfied, whereas others are not and get replaced by a set of constraint equations to be solved for classifying collineations. These collineations are explicitly derived in sec. 2. The solutions of the constraints on the components of Ricci tensor and metrics corresponding to these constraints are provided in sec. 3. Finally, the results are summarized in sec. 4.
2 Classification

The constraint equations (18 - 25) are used to classify the space times under consideration. The classification of the space times under consideration is started by using equation (25). This equation can be satisfied for three cases;

\[(I) : \quad R'_2 = 0 \quad \text{and} \quad Q(t, r) \neq 0\]

\[(II) : \quad R'_2 \neq 0 \quad \text{and} \quad Q(t, r) = 0\]

\[(III) : \quad R'_2 = 0 \quad \text{and} \quad Q(t, r) = 0\]

2.1 CASE (1) [when \(R'_2 = 0\) implies \(R_2 = \gamma \neq 0\) and \(Q(t, r) \neq 0\)]

In this case, it follows from equation (21) that \(A_i(t, r) = 0\). Replacing \(A_i(t, r) = 0\) into constraint equations(18-25), indicate that the equations (18-21) are satisfied identically while the others get replace into the constraints on \(P(t, r)\) and \(Q(t, r)\). Integrating equations (24) yields

\[Q(t, r) = f(t)/\gamma \sqrt{R_1}\]  \hspace{1cm} (26)

where \(f(t)\) is a function of integration with respect to \(r\). With this value of \(Q(t, r)\), I consider two more possibilities from equation (22), namely;

(a) \(R'_0 = 0\) and (b) \(R'_0 \neq 0\)

Case (1a): In this case, \(R_0 = \alpha\), where \(\alpha\) is a non-zero constant. Also it follows from equation (22) that \(P(t, r),_{0} = 0\) implies \(P \equiv P(r)\). Using these values in equation (23), yields

\[P(r) = \left(\frac{-f(t)}{\alpha \gamma}\right) \int \sqrt{R_1} dr + g(t)\]  \hspace{1cm} (27)

where \(g(t)\) is another function of integration. Since \(P(r) = 0\) implies \(f(t) = 0 = g(t)\), which lead to

\[f(t) = C_4 t + C_5 \quad \text{and} \quad g(t) = C_0\]

. Putting these results into the collineation equations (14-17), leads to six collineations

\[V^0 = \left(\frac{-C_4}{\alpha \gamma}\right) \int \sqrt{R_1} dr + C_0\]

\[V^1 = \frac{1}{\gamma \sqrt{R_1}}(C_4 t + C_5)\]

\[V^2 = C_1 \sin \phi - C_2 \cos \phi\]

\[V^3 = \int \left(\frac{-1}{f_k^2}\right)d\theta (C_1 \cos \phi + C_2 \sin \phi) + C_3\]

Case (1b): In this case, it follows from equations (22), (23) and (26) that

\[\gamma P(t, r),_{0} = \left(\frac{-R'_0}{2R_0 \sqrt{R_1}}\right)f(t)\]  \hspace{1cm} (28)

\[\gamma P(t, r),_{1} = \left(\frac{-\sqrt{R_1}}{R_0}\right)f(t),_{0}\]  \hspace{1cm} (29)

Differentiating equations (28) and (29) with respect to \(r\) and \(t\) respectively, and comparing them, one reaches

\[f(t),_{00} - \left(\frac{R_0}{\sqrt{R_1}}\right)(\frac{-R'_0}{2R_0 \sqrt{R_1}})' f(t) = 0\]  \hspace{1cm} (30)
The above equation suggests further two possibilities;

(1bi) \( f(t) = 0 \) and (1bii) \( f(t) \neq 0 \)

Case (1bi): In this case, it follows from equation (26), (22) and (23) that

\[ Q(t,r) = 0 \quad \text{and} \quad P(t,r) = C_0 \]

Using these results along with the fact that \( A_1(t,r) = 0 \), imply that the space times under consideration admit four collineations, given below

\[
\begin{align*}
V^0 &= C_0 \\
V^1 &= 0 \\
V^2 &= C_1 \sin \phi - C_2 \cos \phi \\
V^3 &= (C_1 \cos \phi + C_2 \sin \phi) \int \left( \frac{-1}{f_k^2} \right) d\theta + C_3
\end{align*}
\]

Case (1bii): Re-writing equation (30) in the form

\[ f(t,\theta) / f(t) - \left( \frac{R_0}{\sqrt{R_1}} \right) \left( \frac{R'_0}{2R_0 \sqrt{R_1}} \right)' = 0 \]

In this equation \( f(t) \) is a function of \( t \) only, whereas \( R_0 \) and \( R_1 \) are functions of \( r \) only. Hence this equation can be satisfied if and only if

\[ f(t,\theta) / f(t) = \left( \frac{R_0}{\sqrt{R_1}} \right) \left( \frac{R'_0}{2R_0 \sqrt{R_1}} \right)' = \lambda \]  \hspace{1cm} (31)

where \( \lambda \) is a separation constant. There are further three possibilities for the values of \( \lambda \)

(I): [when \( \lambda = 0 \)]: Equation (31) leads to

\[ f(t) = C_4 t + C_5 \]  \hspace{1cm} (32)

\[ R_0 = \alpha_2 \exp(2\alpha_1 \int \sqrt{R_1} dr) \]  \hspace{1cm} (33)

where \( \alpha_1 \) and \( \alpha_2 \) are constants of integration with \( \alpha_2 \neq 0 \). Using equations (32) and (33), (28) and (29) lead to two equations in \( P(t,r),_0 \) and \( P(t,r),_1 \) which on integration with respect to \( t \) and \( r \) give two values of \( P(t,r) \) along with two functions of integration depending on \( r \) and \( t \), respectively. Comparing these values of \( P(t,r) \) and requiring consistency fixes these functions to give

\[ P(t,r) = \left( \frac{-\alpha_1}{\gamma} \right) \left( \frac{C_4 t}{2} + C_5 \right) t - \left( \frac{C_4}{\gamma \alpha_2} \right) \int \left( (\sqrt{R_1}) (\exp(-2\alpha_1 \int \sqrt{R_1} dr)) \right) dr + C_0 \]  \hspace{1cm} (34)

In this case, it follows from equation (34) along with the previous results that there are six Ricci collineations, given by

\[
\begin{align*}
V^0 &= \left( \frac{-\alpha_1}{\gamma} \right) \left( \frac{C_4 t}{2} + C_5 \right) t - \left( \frac{C_4}{\gamma \alpha_2} \right) \int \left( (\sqrt{R_1}) (\exp(-2\alpha_1 \int \sqrt{R_1} dr)) \right) dr + C_0 \\
V^1 &= \left( \frac{1}{\gamma \sqrt{R_1}} \right) (C_4 t + C_5) \\
V^2 &= C_1 \sin \phi - C_2 \cos \phi
\end{align*}
\]
\[ V^3 = \int \left( -\frac{1}{f_k} \right) d\theta (C_1 \cos \phi + C_2 \sin \phi) + C_3 \]

(II): [When \( \lambda < 0 \) or \( \lambda > 0 \)] In these cases, from equation (31) the space times admit collineations, given below

\[ V^0 = \left( \frac{-R'_0}{2 \gamma R_0 \sqrt{\lambda R_1}} \right) (C_4 f^1_\lambda - C_5 f^2_\lambda) + C_0 / \gamma \]  \hspace{1cm} (35)

\[ V^1 = \left( \frac{1}{\gamma \sqrt{R_1}} \right) (C_4 f^2_\lambda + C_5 f^1_\lambda) \]  \hspace{1cm} (36)

\[ V^2 = C_1 \sin \phi - C_2 \cos \phi \]  \hspace{1cm} (37)

\[ V^3 = \int \left( -\frac{1}{f_k} \right) d\theta (C_1 \cos \phi + C_2 \sin \phi) + C_3 \]  \hspace{1cm} (38)

where the functions \( f^1_\lambda \) are defined by

\[ f^1_\lambda = \sin \sqrt{\lambda} t \text{ when } \lambda < 0 \]

and

\[ f^1_\lambda = \sinh \sqrt{\lambda} t \text{ when } \lambda > 0 \]

Similarly the values of the function \( f^2_\lambda \) are given by

\[ f^2_\lambda = \cos \sqrt{\lambda} t \text{ when } \lambda < 0 \]

and

\[ f^2_\lambda = \cosh \sqrt{\lambda} t \text{ when } \lambda > 0 \]

2.2 CASE (2): \[ \text{When } R'_2 \neq 0 \text{ and } Q(t, r) = 0 \]

In this case \( R'_2 \neq 0 \) implies through equation (2.41) that \( Q(t, r) = 0 \), however the equation (2.40) satisfied identically. Equations (2.38) and (2.49) yield \( P(t, r) = C_0 \) and solving equations (2.36) and (2.37) one gets

\[ A_i(t, r) = \left( \frac{R'_2}{2 R_2 \sqrt{R_1}} \right) f_i(t) \]  \hspace{1cm} (39)

where \( i = 1, 2, 3 \). Using this value of \( A_i \) in constraint equation (2.37) one reaches

\[ \left[ \frac{1}{R_2} - \left( \frac{1}{\sqrt{R_1}} \right) (\frac{R'_2}{2 R_2 \sqrt{R_1}})^' \right] f_i(t) = 0 \]  \hspace{1cm} (40)

This leads to two possibilities

**Case (2a):** \[ \left[ \frac{1}{R_2} - \left( \frac{1}{\sqrt{R_1}} \right) (\frac{R'_2}{2 R_2 \sqrt{R_1}})^' \right] = 0 \]

**Case (2b):** \[ \left[ \frac{1}{R_2} - \left( \frac{1}{\sqrt{R_1}} \right) (\frac{R'_2}{2 R_2 \sqrt{R_1}})^' \right] \neq 0 \]

From **Case (2b)**, one concludes \( f_i(t) = 0 \) which in turn leads that \( A_i(t, r) = 0 \) which is a case similar to **Case (1)**.

**Case (2a):** \[ \text{when } \left[ \frac{1}{R_2} - \left( \frac{1}{\sqrt{R_1}} \right) (\frac{R'_2}{2 R_2 \sqrt{R_1}})^' \right] = 0 \]

Equation (2.53) implies that \( f_i(t) \) are arbitrary functions of \( t \). Using equation (2.52) in equation (2.35) yields

\[ \left( \frac{R'_2}{\sqrt{R_0 R_1 R_2}} \right)^' f_i(t),_0 = 0 \]  \hspace{1cm} (41)
This suggests further two cases

(Case 2ai) \((\frac{R'_3}{\sqrt{R_0 R_1 R_2}})' \neq 0\)

(Case 2a(ii)) \((\frac{R'_3}{\sqrt{R_0 R_1 R_2}})' = 0\)

Consider Case (2ai), using equation (41) we get \(f_i(t),_0 = 0\) leads to

\[ f_i(t) = C_{i+3} \]

(42)

where \(i\) runs from 1 to 3. Using equation (39) and (21) yields

\[ A'_i(t, r) = (\frac{R'_2}{\sqrt{R_1 R_2}}) C_{i+3} \]

(44)

Using equations (42), (39) and (43) yield

\[ A'_i(t, r) = (\frac{R'_2}{\sqrt{R_1 R_2}}) C_{i+3} \]

(45)

Using the solutions of the constraint equations, the components of Ricci collineation vector field are given by

\[ V^0 = C_0 \]

\[ V^1 = (\frac{-1}{\sqrt{R_1}}) f^2_k [\int ((\frac{-1}{f^2_k}) (\int (f_k d\theta)) ) d\theta (C_4 sin\phi - C_5 cos\phi)] - C_6 (\frac{1}{\sqrt{R_1}}) \int f_k d\theta \]

\[ V^2 = (\frac{R'_2}{2R_2 \sqrt{R_1}}) \int f_k d\theta (C_1 sin\phi - C_2 cos\phi) + C_1 sin\phi - C_2 cos\phi + (\frac{R'_2}{2R_2 \sqrt{R_1}}) f_k C_6 \]

\[ V^3 = (\int ((\frac{-1}{f^2_k}) (\int (f_k d\theta)) ) d\theta) (\frac{R'_2}{2R_2 \sqrt{R_1}}) (C_4 cos\phi + C_5 sin\phi) + \int (\frac{-1}{f^2_k}) d\theta (C_1 cos\phi + C_2 sin\phi) + C_3 \]

In this case, I get seven Ricci Collineations.

(Case 2a(ii)): \((\frac{R'_3}{\sqrt{R_0 R_1 R_2}})' = 0\)

Equation (41) implies that \(f_i(t)\) are arbitrary of functions of \(t\) while equations (19) to (21) are satisfied identically, whereas equation (18) yields

\[ f_i(t),_0 / f_i(t) = (\frac{R'_0}{R'_2}) = \delta \]

(46)

where \(\delta\) is a separation constant. It contains further three possibilities depending on the value of \(\delta\). Hence the values of \(f_i(t)\) for different values of \(\delta\) can be written as

\[ f_i(t) = C_{i+3} cos\sqrt{\delta}t + C_{i+6} sin\sqrt{\delta}t \quad (\delta > 0) \]

(47)

\[ f_i(t) = C_{i+3} t + C_{i+6} \quad (\delta = 0) \]

(48)

\[ f_i(t) = C_{i+3} sinh\sqrt{-\delta}t + C_{i+6} sinh\sqrt{-\delta}t \quad (\delta < 0) \]

(49)

Using equations (38), (39) and (40) along with equations (2.52) and (2.56) implies that

\[ A_i(t, r) = (\frac{R'_2}{2R_2 \sqrt{R_1}}) [C_{i+3} cos\sqrt{\delta}t + C_{i+6} sin\sqrt{\delta}t] \quad (\delta > 0) \]

(50)
\[ A_i(t, r) = \left( \frac{R'_2}{2R_2 \sqrt{R_1}} \right)[C_{i+3}t + C_{i+6}] \quad (\delta = 0) \] (51)

\[ A_i(t, r) = \left( \frac{R'_2}{2R_2 \sqrt{R_1}} \right)[C_{i+3}cos\sqrt{-\delta}t + C_{i+6}sin\sqrt{-\delta}t] \quad (\delta < 0) \] (52)

\[ A'_i(t, r) = \left( \frac{\sqrt{R_1}R_2}{R_2} \right)[C_{i+3}cos\sqrt{\delta}t + C_{i+6}sin\sqrt{\delta}t] \quad (\delta > 0) \] (53)

\[ A'_i(t, r) = \left( \frac{\sqrt{R_1}}{R_2} \right)[C_{i+3}t + C_{i+6}] \quad (\delta = 0) \] (54)

\[ A'_i(t, r) = \left( \frac{\sqrt{R_1}}{R_2} \right)[C_{i+3}cos\sqrt{-\delta}t + C_{i+6}sin\sqrt{-\delta}t] \quad (\delta < 0) \] (55)

and one may get \( P(t, r) = C_0 \) and \( Q(t, r) = 0 \). Substituting these values in the Ricci collineation equations, the values of the components of the vector field are

\[ V^0 = \left( \frac{-R_2}{R_0} \right)f^2_k \left( \int \left( \frac{-1}{f_k^2} \int \frac{f_k d\theta}{(A_1(t, r)_0 \sin \phi - A_2(t, r)_0 \cos \phi)} + \frac{-R_2}{R_0}A_3(t, r)_0 \int f_k d\theta \right) + P(t, r) \right) \] (56)

\[ V^1 = \left( \frac{-R_2}{R_1} \right)f^2_k \left( \int \left( \frac{-1}{f_k^2} \int \frac{f_k d\theta}{(B_1(t, r)_1 \sin \phi - B_2(t, r)_1 \cos \phi)} + \frac{-R_2}{R_1}A_3(t, r)_1 \int f_k d\theta \right) + Q(t, r) \right) \] (57)

\[ V^2 = \int f_k d\theta (A_1(t, r) \sin \phi - A_2(t, r) \cos \phi) + C_1(t, r) \sin \phi - C_2(t, r) \cos \phi + A_3(t, r)f_k \] (58)

\[ V^3 = \int \left( \frac{-1}{f_k^2} \int f_k d\theta d\theta \right)(A_1(t, r) \cos \phi + A_2(t, r) \sin \phi) + (C_1 \cos \phi + C_2 \sin \phi) \int \left( \frac{-1}{f_k^2} \right) d\theta + C_3 \] (59)

where the values of \( A_i(t, r) \) and \( A'_i(t, r) \) are given above for different values of \( \delta \) and the values of \( A \) can be obtained by differentiating equations (41), (42) and (43) with respect to \( t \). In this case, there are ten collineations with a finite lie algebra of dimension ten.

2.3 CASE (3): \([R'_2 = 0 \text{ and } Q(t, r) = 0]\)

It can easily be verified that this case turn out to be the as case (1bii).

3 Exact solutions of Einstein equations Admitting Non-trivial Ricci Collineations

In the previous section, static space times with maximal symmetric transverse spaces are classified according to their Ricci collineation vectors. I worked out Ricci collineations for non-degenerate Ricci tensor and classified the space time, whereas degenerate case for these space times have already been worked out in reference [23]. While working out collineations, many constraints on the components of Ricci tensor are obtained. In this section I solve these constraints on Ricci tensor for non-degenerate tensor and present the explicit form of some metrics.
admitting non-trivial Ricci collineations. Equation (2.40) gives six Ricci collineations along with the constraint \( R_2 = \gamma \neq 0 \) and \( R_0' \sqrt{R_1} = \delta \). The simplest case could be \( \delta = 0 \) which implies \( R_0 = \alpha \). Assuming \( R_2 = k = R_0 \) and solving equations (5-7) for these values, yields

\[
\left( \frac{A''}{2} - \frac{A'B'}{4} + \frac{A'^2}{4} + \frac{A'}{r} \right) = ke^{B-A}
\]

where \( k = 0, \pm 1 \) and

\[
B' - A' = \frac{2}{r}
\]

Solving these equations simultaneously I get the following exact solution

\[
ds^2 = r^6 e^{\frac{bnr^2}{2}} (dt^2 - mr^2 dr^2) - r^2 (d\theta^2 + f_k^2 d\phi^2)
\]

where m and n are non-zero arbitrary constants of integration. These space times admit four Killing vectors and six Ricci collineations. One can consider \( B(r) = 0 \) so that the metric under consideration takes the form

\[
ds^2 = e^A dt^2 - dr^2 - r^2 (d\theta^2 + f_k^2 d\phi^2)
\]

along with the constraint \( R_2' = 0 \), I get the following metric

\[
ds^2 = a r^{(b-2k)} dt^2 - dr^2 - r^2 (d\theta^2 + f_k^2 d\phi^2)
\]

where \( a \) and \( b \) are constant of integrations with \( a \neq 0 \). This space times admit six Ricci collineations. Again for the case \( B = 0 \), considering \( R_2' = \alpha \neq 0 \), I get the metric for \( k = 0 \)

\[
ds^2 = e^{-2axr} dt^2 - dr^2 - r^2 (d\theta^2 + f_k^2 d\phi^2)
\]

This metric admits four Ricci collineations. If one assumes \( B = 0 \) along with a constraint \( R_2' = 0 \), I get the following exact solutions

\[
ds^2 = r^\alpha dt^2 - dr^2 - r^2 (d\theta^2 + f_k^2 d\phi^2)
\]

\[
ds^2 = \left( \frac{1}{ar^2 + 1} \right) dt^2 - dr^2 - r^2 (d\theta^2 + f_k^2 d\phi^2)
\]

These both metrics admit four Ricci collineations. The constraint equations corresponding to the case of ten Ricci collineations true in the case of the metric, given by

\[
ds^2 = (r^2/a^2 - 1) dt^2 - (\frac{1}{r^2/a^2 - 1}) dr^2 - r^2 (d\theta^2 + f_k^2 d\phi^2)
\]

which admits ten Ricci collineations. Here I presented few special cases, however one can solve these constraints equations on the components of Ricci tensor in general and may find more new exact solutions.

### 4 Summary and Conclusion

In this article, static space times with maximal symmetric transverse spaces are classified according to their Ricci collineations. The Ricci collineation equations have been solved for non-degenerate Ricci tensor \( det.(R_\alpha \neq 0) \). I have obtained the explicit form of collineation vectors along with constraints on the components of the Ricci tensor. Solving these constraints on Ricci tensor, the explicit forms of some exact solutions of the Einstein Field equations are obtained. In cases (1a) and (1bii), six collineations are obtained whereas the cases (1bi), (2ai) and (2aii) amounted to four, seven and ten collineations respectively. Hence it is concluded that the Lie algebras of the collineations for non-degenerate is finite i.e of dimensions ten, seven, six or four. It was found previously that the algebra for these spaces [23] when the Ricci tensor is degenerate, is not always finite. Note that the classification of these spaces also cover the classification of static spherically symmetric space times as a spacial case for \( k = 1 \).
References

[1] A. V. Aminova, Groups of transformations of Riemannian manifolds, J. Soviet. Maths., 55 (1991) 1995

[2] G. S. Hall, The mathematical study of symmetries in general relativity, Grav. and cosmology, 2(1996)270

[3] G. H. Katzin, J. Levine and W. R. Davis, J. Math. Phys. 10 (1969)617; 11 (1970) 1578; G. H. Katzin and W. R. Davis. Tensor(NS) 21 (1970) 64; L. H. Green, L. K. Norris, D. R. Oliver, and W. D. Davis, Gen. Rel. Grav. 8 (1977) 731

[4] A. Z. Petrov, Einstein Spaces (Pergamon, Oxford University Press,1973)

[5] H. Stephani, D. Kramer, M. A. H. MacCallum, C. Hoenselaers, and E. Hearlt, Exact Solutions of Einstein Field Equations (Cambridge University Press, 2003)

[6] L. H. Green, L. K. Norris, D. R. Oliver, and W. R. Davis, Gen. Rel. Grav. 8 (1977)731

[7] L. Nunez, U. Percoco, and V. M. Villalba, J. Math. Phys. 31 (1990)137

[8] D. R. Oliver and W. R. Davis, Gen. Rel. Grav. 8(1977)905

[9] S. Hojman, L. Nunez, A. Patino, and H. Rago, J. Math. Phys. 27 (1986) 281

[10] G. H. Katzin and J. Levine, Colloq. Math.(Poland) 26 (1972) 21

[11] G. H. Katzin and J. Levine, J. Math. Phys. 22 (1981)1878

[12] M. J. Amir, A. H. Bokhari, and A. Qadir, J. Math. Phys. 35 (1994) 3005

[13] J. Bin Farid, A. Qadir, and M. Ziad, J. Math. Phys. 36 (1995) 5812

[14] M. Sharif and Schar Aziz, Gen. Rel. Grav. 35 (2003) 1093

[15] M. Sharif, J. Math. Phys. 44 (2004) 5141

[16] G. Contreras, L. A. Nunez and U. Percoco, arXiv:gr-qc/9907075

[17] A. Qadir, K. Saifullah, and M. Ziad, Gen. Rel. Grav. 37 (2003) 1927

[18] J. Carot, J. da Costa and E. G. L. R. Vaz, J. Math. Phys. 35 (1994) 4832

[19] G. S. Hall, I. Roy, and E. G. L. R. Ven, Gen. Rel. Grav. 28 (1996)299

[20] U. Camci, I. Yavuz, H. Baysal, I. Tarhan, and I. Yilmaz, arXiv:gr-qc/0010057

[21] A. Z. Petrov, Einstein Spaces, Pergamon, New York (1969)

[22] P. Turkowski, J. Math. Phys. 29 (1988) 2130

[23] M. Akbar and R. G, Cai, Commun. Theor. Phys. 45 (2006) 95