COSMOLOGICAL k-ESSENCE CONDENSATION

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We consider a model of dark energy/matter unification based on a k-essence type of theory similar to tachyon condensate models. Using an extension of the general relativistic spherical model which incorporates the effects of both pressure and the acoustic horizon we show that an initially perturbative k-essence fluid evolves into a mixed system containing cold dark matter like gravitational condensate in significant quantities.

The most popular cosmological models such as ΛCDM model and a quintessence-CDM model assume that DM and DE are distinct entities. Another interpretation of the observational data is that DM/DE are different manifestations of a common structure. The first definite model of this type was proposed a few years ago, based upon the Chaplygin gas, a perfect fluid obeying the equation of state

\[ p = -\frac{A}{\rho}, \]  

which has been extensively studied for its mathematical properties. The general class of models, in which a unification of DM and DE is achieved through a single entity, is often referred to as quartessence. Among other scenarios of unification that have recently been suggested, interesting attempts are based on the so-called k-essence, a scalar field with noncanonical kinetic terms which was first introduced as a model for inflation.

All models that unify DM and DE face the problem of nonvanishing sound speed and the well-known Jeans instability. Soon after the appearance of and it was pointed out that the perturbative Chaplygin gas (for early work see and more recently) is incompatible with the observed mass power spectrum and microwave background. Essentially, these results are a consequence of a nonvanishing comov-
The perturbations whose comoving size $R$ is larger than $d_s$ grow as \[ \delta = \frac{(\rho - \bar{\rho})}{\bar{\rho}} \sim a. \] As soon as $R < d_s$, the perturbations undergo damped oscillations. For the Chaplygin gas we have $d_s \sim a^{7/2}/H_0$, where $H_0$ is the present day value of the Hubble parameter, reaching Mpc scales already at redshifts of order 10. However, as soon as $\delta \simeq 1$ the linear perturbation theory cannot be trusted. A significant fraction of initial density perturbations collapses in gravitationally bound structure - the condensate and the system evolves into a two-phase structure - a mixture of CDM in the form of condensate and DE in the form of uncondensed gas.

The simple Chaplygin gas does not exhaust all the possibilities for quartessence. A particular case of k-essence is the string-theory inspired tachyon Lagrangian \[ L = -V(\varphi) \sqrt{1 - g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu}}, \] (3) where 

\[ X = g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu}. \] (4) It may be shown that every tachyon condensate model can be interpreted as a 3+1 brane moving in a 4+1 bulk.\cite{14,16} Eq. (1) is obtained using the stress-energy tensor $T_{\mu\nu}$ derived from the Lagrangian (3) with $V(\varphi)$ replaced by a constant $\sqrt{A}$.

In a recent paper\cite{16} we have developed a fully relativistic version of the spherical model for studying the evolution of density perturbations even into the fully non-linear regime. The formalism is similar in spirit to\cite{17} and applicable to any k-essence model. The key element is an approximate method for treating the effects of pressure gradients. Here we give a brief description of our method and its application to a unifying model based on the Lagrangian (3) with a potential of the form 

\[ V(\varphi) = V_n \varphi^{2n}, \] (5)

where $n$ is a positive integer. In the regime where structure formation takes place, this model effectively behaves as the variable Chaplygin gas\cite{18} with the equation of state (1) in which $A \sim a^{6n}$. As a result, the much smaller acoustic horizon $d_s \sim a^{(7/2+3n)}/H_0$ enhances condensate formation by two orders of magnitude over the simple Chaplygin gas. Hence this type of model may salvage the quartessence scenario.

A minimally coupled k-essence model\cite{9,19} is described by

\[ S = \int d^4x \sqrt{-g} \left[ -\frac{R}{16\pi G} + \mathcal{L}(\varphi, X) \right], \] (6)

where $\mathcal{L}$ is the most general Lagrangian, which depends on a single scalar field $\varphi$ of dimension $m^{-1}$, and on the dimensionless quantity $X$ defined in (4). For $X > 0$ the energy momentum tensor obtained from (6) takes the perfect fluid form,

\[ T_{\mu\nu} = 2\mathcal{L}_X \varphi_{,\mu} \varphi_{,\nu} - \mathcal{L} g_{\mu\nu} = (\rho + p) u_{\mu} u_{\nu} - p g_{\mu\nu}, \] (7)
with $\mathcal{L}_X$ denoting $\partial \mathcal{L} / \partial X$ and 4-velocity

$$u_\mu = \text{sgn}(\varphi,0) \frac{\varphi_\mu}{\sqrt{X}}.$$ (8)

The sign of $u_\mu$ is chosen so $u_0$ is positive. The associated hydrodynamic quantities are

$$p = \mathcal{L}(\varphi,X); \quad \rho = 2X \mathcal{L}_X(\varphi,X) - \mathcal{L}(\varphi,X),$$ (9)

and the speed of sound is defined as

$$c_s^2 = \frac{\partial p}{\partial \rho}|_{s/n} = \frac{\mathcal{L}_X}{\mathcal{L}_X + 2X \mathcal{L}_{XX}}.$$ (10)

Two general conditions $\mathcal{L}_X \geq 0$ and $\mathcal{L}_{XX} \geq 0$ are required for stability and causality. Now, using (8)-(9) the $\varphi$ field equation can be expressed as

$$\dot{\rho} + 3H(\rho + p) + (\dot{\varphi} - \text{sgn}(\varphi,0) \sqrt{X}) \partial \mathcal{L} / \partial \varphi = 0.$$ (11)

Since the 4-velocity (8) is derived from a potential, the associated rotation tensor vanishes identically. The Raychaudhuri equation for the velocity congruence combined with Einstein’s equations and the Euler equation assumes a simple form

$$3\dot{H} + 3H^2 + \sigma_{\mu\nu} h^{\mu\nu} + 4\pi G(\rho + 3p) = \left( \frac{c_s^2 h^{\mu\nu} \rho_{\nu}}{p + \rho} \right)_{\mu},$$ (12)

where $\sigma_{\mu\nu}$ is the shear tensor and $h_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu$ is a projector onto the three-space orthogonal to $u^\mu$. The quantity $\mathcal{H}$ is the local Hubble parameter, defined as $3\dot{H} = w^\nu_{\nu}$. We thus obtain an evolution equation for $\mathcal{H}$ sourced by shear, density, pressure and pressure gradient. If $c_s = 0$, as for dust, Eq. (12) and the continuity equation comprise the spherical model. However, we are not interested in dust, since generally $c_s \neq 0$ and the right hand side of (12) is not necessarily zero.

In general, the 4-velocity $u^\mu$ can be decomposed as

$$u^\mu = \left( U^\mu + v^\mu \right) / \sqrt{1 - v^2},$$ (13)

where $U^\mu = \delta_0^\mu / \sqrt{g_{00}}$ is the 4-velocity of fiducial observers at rest, and $v^\mu$ is spacelike, with $v^\mu v_\mu = -v^2$ and $U^\mu v_\mu = 0$. In comoving coordinates $v^\mu = 0$.

In spherically symmetric spacetime it is convenient to write the metric in the form

$$ds^2 = N(t,r)^2 dt^2 - b(t,r)^2 (dr^2 + r^2 f(t,r) d\Omega^2),$$ (14)

where $N(t,r)$ is the lapse function, $b(t,r)$ is the local expansion scale, and $f(t,r)$ describes the departure from the flat space for which $f = 1$. We assume that $N$, $a$, and $f$ are arbitrary functions of $t$ and $r$ which are regular and different from zero at $r = 0$. Then, the local Hubble parameter and the shear are given by

$$\mathcal{H} = \frac{1}{N} \left( \frac{b_0}{b} + \frac{1}{3} \frac{f_0}{f} \right); \quad \sigma_{\mu\nu} \sigma^{\mu\nu} = \frac{2}{3} \left( \frac{1}{2N} \frac{f_0}{f} \right)^2.$$ (15)
In addition to the spherical symmetry we also require an FRW spatially flat asymptotic geometry, i.e., for $r \to \infty$ we demand

\[ N \to 1; \quad f \to 1; \quad b \to a(t), \quad (16) \]

where $a$ denotes the background expansion scale.

The righthand side of (12) is difficult to treat in full generality. As in\(^{17}\), we apply the “local approximation”. The density contrast $\delta = (\rho - \bar{\rho})/\bar{\rho}$ is assumed to be of fixed Gaussian shape of comoving size $R$ with time-dependent amplitude, so that

\[ \rho(t, r) = \bar{\rho}(t)[1 + \delta_R(t) e^{-r^2/(2R^2)}], \quad (17) \]

and the spatial derivatives are evaluated at the origin. This is in keeping with the spirit of the spherical model, where each region is treated as independent. Since $\partial_i \rho = 0$ at $r = 0$, naturally $\partial_i N = 0$ and $\partial_i b = 0$ at $r = 0$. Hence,

\[ N(t, r) = N(t, 0)(1 + O(r^2)); \quad b(t, r) = b(t, 0)(1 + O(r^2)). \quad (18) \]

Besides, one finds $f, \rho \to 0$ as $r \to 0$ which follows from Einstein’s equation $G^{10} = 0$.

From now on we denote by $H$, $b$, and $N$ the corresponding functions of $t$ and $r$ evaluated at $r = 0$, i.e., $H \equiv H(t, 0)$, $b \equiv b(t, 0)$ and $N \equiv N(t, 0)$. According to (15), the shear scalar $\sigma_{\mu\nu} \sigma^{\mu\nu}$ vanishes at the origin. Evaluating (12) at $r = 0$ yields our working approximation to the Raychaudhuri equation.

We will now apply our formalism to a particular subclass of $k$-essence unification models described by (3). The equation of state is then given by

\[ p = -\frac{V(\varphi)^2}{\rho}, \quad (19) \]

and the quantity $X$ may be expressed as

\[ X(\rho, \varphi) = 1 - \frac{V(\varphi)^2}{\rho^2} = 1 - c_s^2 = 1 + w. \quad (20) \]

The continuity equation, Eq. (11), and Eq. (12) evaluated at $r = 0$ determine the evolution of the density contrast. However, this set of equations is not complete as it must be supplemented by a similar set of equations for the background quantities $\bar{\rho}$ and $H$. The complete set of equations for $\bar{\rho}$, $H$, $\varphi$, $b$, $\rho$, and $H$ is

\[ \left( \frac{d\varphi}{dt} \right)^2 = X(\varphi, \bar{\rho}), \quad (21) \]

\[ \frac{d\bar{\rho}}{dt} + 3H(\bar{\rho} + \bar{\rho}) = 0, \quad (22) \]

\[ \frac{dH}{dt} + H^2 + \frac{4\pi G}{3}(\bar{\rho} + 3\bar{\rho}) = 0, \quad (23) \]

\[ \frac{db}{dt} = NbH, \quad (24) \]
\[
\frac{d\rho}{dt} + 3N \mathcal{H} (\rho + p) = 0, \tag{25}
\]

\[
\frac{dH}{dt} + N \left[ \frac{\mathcal{H}^2 + 4\pi G}{3} (\rho + 3p) - \frac{c_s^2 (\rho - \bar{\rho})}{b^2 R^2 (\rho + p)} \right] = 0, \tag{26}
\]

where \( \bar{\rho} = p(\bar{\rho}, \varphi) \) and \( N = \sqrt{X(\varphi, \bar{\rho})/X(\varphi, \rho)} \). Eqs. (21) and (24) follow from (11) and (15), respectively. Eqs. (11) and (25) are the continuity equations, and Eqs. (23) and (26) are the Raychaudhuri equations for the background and the spherical inhomogeneity, respectively.

Now we restrict our attention to the potential (5). In the high density regime we have \( X \approx 1 \), and (21) can be integrated yielding \( \varphi \approx \frac{2}{3} H \). Here \( H \approx H_0 \sqrt{\Omega}/a^{-3/2} \) with \( \Omega \) being the equivalent matter content at high redshift. Hence, \( V(\varphi)^2 \sim a^{6n} \), which leads to a suppression of \( 10^{-6} \) of the acoustic horizon at \( z = 9 \) for \( n = 1 \).

To proceed we require a value for the constant \( V_n \) in the potential (5). As the main purpose of this paper is to investigate the evolution of inhomogeneities we will not pursue the exact fitting of the background evolution. Instead, we estimate \( V_n \) as follows. We integrate (21) approximately with \( X = 1 + w(a) \approx 1 - \frac{\Omega}{\Omega_\Lambda + \Omega a^{-3}} \), \( \Omega + \Omega_\Lambda = 1 \), (27) as in a \( \Lambda \)CDM universe \(^{23} \) and we fix the pressure given by (3) to equal that of \( \Lambda \) at \( a = 1 \). In this way the naive background in our model reproduces the standard cosmology from decoupling up to the scales of about \( a = 0.8 \) and fits the cosmology today only approximately (figure 1(a)).

We solve our differential equations with \( a \) starting from the initial \( a_{\text{dec}} = 1/(z_{\text{dec}} + 1) \) at decoupling redshift \( z_{\text{dec}} = 1089 \) for a particular comoving size \( R \). The initial values for the background are given by

\[
\bar{\rho}_{\text{in}} = \rho_0 \frac{\Omega}{a_{\text{dec}}^3}; \quad H_{\text{in}} = H_0 \sqrt{\Omega a_{\text{dec}}^3}; \quad \varphi_{\text{in}} = \frac{2}{3H_{\text{in}}}, \tag{28}
\]

and for the initial inhomogeneity we take

\[
\rho_{\text{in}} = \bar{\rho}_{\text{in}} (1 + \delta_{\text{in}}), \quad \mathcal{H}_{\text{in}} = H_{\text{in}} \left( 1 - \frac{\delta_{\text{in}}}{3} \right), \tag{29}
\]

where \( \Omega = 0.27 \) represents the effective dark matter fraction and \( \delta_{\text{in}} = \delta_R(a_{\text{dec}}) \) is a variable initial density contrast, chosen arbitrarily for a particular \( R \).

In figure 1(b) the representative case of evolution of two initial perturbations starting from decoupling for \( R = 10 \) kpc is shown for \( n = 2 \). The plots represent two distinct regimes: the growing mode or condensation (dashed line) and the damped oscillations (solid line). In contrast to the linear theory, where for any \( R \) the acoustic horizon will eventually stop \( \delta_R \) from growing, irrespective of the initial value of the perturbation, here we have for an initial \( \delta_R(a_{\text{dec}}) \) above a certain threshold \( \delta_c(R) \), \( \delta_R(a) \to \infty \) at finite \( a \), just as in the dust model. Thus perturbations with
\( \delta_R(a_{\text{dec}}) \geq \delta_c(R) \) evolve into a *nonlinear* gravitational condensate that at low \( z \) behaves as pressureless super-particles. Conversely, for a sufficiently small \( \delta_R(a_{\text{dec}}) \), the acoustic horizon can stop \( \delta_R(a) \) from growing.

The crucial question now is what fraction of the tachyon gas goes into condensate. In\(^{25}\) it was shown that if this fraction was sufficiently large, the CMB and the mass power spectrum could be reproduced for the simple Chaplygin gas. To answer this question quantitatively, we follow the Press-Schechter procedure\(^{26}\) as in\(^{17}\). Assuming \( \delta_R(a_{\text{dec}}) \) is given by a Gaussian random field with dispersion \( \sigma(R) \), the condensate fraction at a scale \( R \) is given by

\[
F(R) = 2 \int_{\delta_c(R)}^\infty \frac{d\delta}{\sqrt{2\pi}\sigma(R)} \exp \left( -\frac{\delta^2}{2\sigma^2(R)} \right) = \text{erfc} \left( \frac{\delta_c(R)}{\sqrt{2}\sigma(R)} \right),
\]

(30)

where \( \delta_c(R) \) is the threshold shown in figure 2(a). In figure 2(a) we also exhibit the dispersion

\[
\sigma^2(R) = \int_0^\infty \frac{dk}{k} \exp(-k^2R^2)\Delta^2(k, a_{\text{dec}}),
\]

(31)

calculated using the variance of the concordance model\(^{27}\)

\[
\Delta^2(k, a) = \text{const} \left( \frac{k}{aH} \right)^4 T^2(k) \left( \frac{k}{7.5a_0H_0} \right)^{n_s-1}.
\]

(32)

In figure 2(b) we present \( F(R) \) for \( \text{const}=7.11 \times 10^{-9} \), the spectral index \( n_s=1.02 \), and the parameterization of Bardeen et al\(^{28}\) for the transfer function \( T(k) \) with \( \Omega_B=0.04 \). The parameters are fixed by fitting \( \Omega_B \) to the 2dFGRS power spectrum.
Fig. 2. (a) Initial value $\delta_R(\Delta_m)$ versus $R$ for $\Omega = 0.27$ and $h = 0.71$. The threshold $\delta_c(R)$ is shown by the line separating the condensation regime from the damped oscillations regime. The solid line gives $\sigma(R)$ calculated using the concordance model. (b) Fraction of the tachyon gas in collapsed objects using $\delta_c(R)$ and $\sigma(R)$ depicted in (a).

Our result demonstrates that the collapse fraction is about 70% for $n = 2$ for a wide range of the comoving size $R$ and peaks at about 45% for $n = 1$.

Albeit encouraging, these preliminary results do not in themselves demonstrate that the tachyon with potential (8) constitutes a viable cosmology. Such a step requires the inclusion of baryons and comparison with the full cosmological data. What has been shown is that it is not correct in an adiabatic model to simply pursue linear perturbations to the original background: the system evolves nonlinearly into a mixed system of gravitational condensate and residual k-essence so that the “background” at low $z$ is quite different from the initial one. Because of this one needs new computational tools for a meaningful confrontation with the data.

The tachyon k-essence unification remains to be tested against large-scale structure and CMB observations. An encouraging feature of the positive power-law potential is that it provides for acceleration as a periodic transient phenomenon which obviates the de Sitter horizon problem.

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