The algebraic area of closed lattice random walks

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Abstract
We propose a formula for the enumeration of closed lattice random walks of length \(n\) enclosing a given algebraic area. The information is contained in the Kreft coefficients which encode, in the commensurate case, the Hofstadter secular equation for a quantum particle hopping on a lattice coupled to a perpendicular magnetic field. The algebraic area enumeration is possible because it is split in \(2^{n/2-1}\) pieces, each tractable in terms of explicit combinatorial expressions.

Keywords: algebraic area, Hofstadter model, enumeration, closed random walks

(Some figures may appear in colour only in the online journal)

1. Introduction

The enumeration on a square lattice of closed random walks of length \(n\), with \(n\) then necessarily even, starting from a given point and enclosing a given algebraic area seems as far as we can see still a current issue. We propose a formula for the enumeration by splitting it in \(2^{n/2-1}\) pieces, where \(2^{n/2-1}\) is the number of partitions of \(n/2\) where partitions differing by the order of their parts are counted separately—e.g. \(4 = 2 + 1 + 1, 4 = 1 + 2 + 1\) and \(4 = 1 + 1 + 2\) each count. One then refers to compositions rather than to partitions.

The observation which allows for the algebraic area enumeration originates from the Hofstadter model for a quantum particle hopping on a square lattice and coupled to a perpendicular magnetic field.

The algebraic area is the area enclosed by a curve, weighted by its winding number: if the curve moves around a region in counterclockwise direction, its area counts as positive, otherwise negative. Moreover, if the curve winds around more than once, the area is counted with
multiplicity. We focus on the algebraic area of walks on a square lattice starting from a given point and at each step moving right, left, up or down with equal probability.

Suppose that a walk has moved \( m_1 \) steps right, \( m_2 \) steps left, \( l_1 \) steps up and \( l_2 \) steps down. If e.g. \( m_1 \geq m_2 \) and \( l_1 \geq l_2 \), we add \( l_1 - l_2 \) steps down followed by \( m_1 - m_2 \) steps left in order to close the walk and endow it with an algebraic area. Let \( C_{m_1,m_2,l_1,l_2}(A) \) be the number of such walks which enclose a given algebraic area \( A \). Finding the generating function for the \( C_{m_1,m_2,l_1,l_2}(A) \)'s

\[
Z_{m_1,m_2,l_1,l_2}(Q) = \sum_{A} C_{m_1,m_2,l_1,l_2}(A) Q^A
\]

is quite challenging. One restricts to closed lattice walks of length \( n \) (\( n \) is then necessarily even), i.e. walks with an equal number \( m \) of steps right/left and an equal number \( n/2 - m \) of steps up/down, \( m \in \{0,1,\ldots,n/2\} \), and focuses on their algebraic area generating function

\[
Z_n(Q) = \sum_{m=0}^{n/2} Z_{m,m,\frac{n}{2}-m,\frac{n}{2}-m}(Q) = \sum_{A} C_n(A) Q^A
\]

where \( C_n(A) \) enumerates closed walks of length \( n \) enclosing an algebraic area \( A \) (\( A \) is in between \(-|n^2/16|\) and \(|n^2/16|\) where \( \lfloor . \rfloor \) denotes the integer part; obviously \( C_n(A) = C_n(-A) \)).

There is a connection between the algebraic area distribution of curves and the quantum spectrum of a charged particle coupled to a perpendicular magnetic field. This connection arises for continuous closed Brownian curves and their algebraic area distribution given by Lévy’s law [1]

\[
P_s(A) = \frac{\pi}{2t} \frac{1}{\cosh(\pi A/t)^2},
\]

where \( t \) is the time of the Brownian motion. One notes that \( P_s(A) \) is nothing but the Fourier transform of the Landau partition function at inverse temperature \( t \) of a quantum planar particle coupled to a perpendicular magnetic field. It is not a surprise that a magnetic field should play a role since it indeed couples to the algebraic area spanned by paths in a path integral formulation. In the lattice case at hand, the mapping is on the quantum Hofstadter model [2] for a particle hopping on a two-dimensional lattice coupled to a magnetic field with flux \( \gamma \) per lattice cell, in unit of the flux quantum. \( Z_n(e^{i\gamma}) \) is mapped [3] on the \( n \)th moment \( \text{Tr} H_n^\gamma \) of the Hofstadter Hamiltonian \( H_\gamma \)

\[
Z_n(e^{i\gamma}) = \text{Tr} H_n^\gamma
\]

by virtue of which evaluating \( Z_n(e^{i\gamma}) \) for lattice walks gives an expression for the quantum trace \( \text{Tr} H_n^\gamma \), and vice versa. The coupling to a perpendicular magnetic field induces a non commuting lattice space which in turns allows for weighting discrete paths by their algebraic area [3].

\( \text{Tr} H_n^\gamma \) can be written [4] in terms of the Kreft coefficients [5] which encode the Schrödinger equation for the Hofstadter model. In the commensurate case with a rational flux \( \gamma = 2\pi p/q \rightarrow p, q \) coprime—the Schrödinger equation reduces to a \( q \times q \) matrix whose determinant, more precisely its momentum independent part, can be expressed in terms of the polynomial

\[
b_{p,q}(z) = -\sum_{j=0}^{\lfloor \frac{q}{2} \rfloor} a_{p,q}(2j) z^{2j}
\]

with the Kreft coefficients [5]

\[
a_{p,q}(2j) = (-1)^{j+1} \sum_{k_1=0}^{q-2j} \sum_{k_2=0}^{q-2j} \ldots \sum_{k_1=0}^{q-2j} \frac{1}{4 \sin^2 \left( \frac{\pi (k_1 + 2j - 1)p}{q} \right) 4 \sin^2 \left( \frac{\pi (k_1 + 2j - 3)p}{q} \right) \ldots 4 \sin^2 \left( \frac{\pi (k_1 + 1)p}{q} \right)}
\]

(4)
and \( a_{p,q}(0) = -1 \). We refer to e.g. [5], and also [4], where details can be found on how to arrive at (4). In [4] we obtained a closed expression for the Hofstadter trace \( \text{Tr}H_{2\pi p/q}^n \) in terms of the coefficients (4)

\[
\text{Tr}H_{2\pi p/q}^n = \frac{n}{q} \sum_{k=0}^{l} \sum_{\ell_1,\ell_2,\ldots,\ell_{[q/2]},2k} \left( \frac{\ell_1 + \ell_2 + \cdots + \ell_{[q/2]} + 2k}{\ell_1,\ell_2,\ldots,\ell_{[q/2]},2k} \right) (2k)^j \prod a_{p,q}(2j)^k.
\]

(5)

which in turn gave \( Z_\\alpha(e^{2\pi p/q}) \) via (3). However both (4) and (5) are somehow involved expressions which cannot be used in practice to reach the \( C_\alpha(A) \)'s in (1).

The observation which allows for the lattice walks algebraic area enumeration is that the \( C_\alpha(A) \)'s are contained in \([q]a_{p,q}(n)\) i.e.

\[
\frac{1}{n} \sum_A C_\alpha(A)e^{2\pi \alpha Ap/q} = [q]a_{p,q}(n)
\]

(6)

where \([q]a_{p,q}(n)\) stands for the coefficient of the first order term in the \( q \) expansion of the Kreft coefficient \( a_{p,q}(n) = q[q]a_{p,q}(n) + \cdots + q^n/[q^n]a_{p,q}(n) \), a polynomial in \( q \) of order \( n/2 \) with coefficients which are linear combinations of \( \cos(2\pi p/q) \) with \( A \in [0, [n^2/16]] \)—see e.g. (10), (16), (A.1) and (A.5). This in turn implies that the Hofstadter trace simplifies to

\[
\frac{1}{n} \text{Tr}H_{2\pi p/q}^n = [q]a_{p,q}(n)
\]

(7)

(\text{how}^1 \text{ to derive (6) or equivalently (7) will be addressed elsewhere [6].})

Likewise, the higher order terms in the \( q \) expansion of \( a_{p,q}(n) \) are given in terms of \([q]a_{p,q}(n-2), [q]a_{p,q}(n-4), \ldots\)

\[
a_{p,q}(2) = q[q]a_{p,q}(2)
\]

\[
a_{p,q}(4) = q[q]a_{p,q}(4) - \frac{q^2}{2!} ([q]a_{p,q}(2))^2
\]

\[
a_{p,q}(6) = q[q]a_{p,q}(6) - q^2 [q]a_{p,q}(2)[q]a_{p,q}(4) + \frac{q^3}{3!} ([q]a_{p,q}(2))^3
\]

etc, i.e.

\[
a_{p,q}(n) = - \sum_{j=0}^{n/2} \prod_{j=1}^{n/2} \left( -1 \right)^k \frac{1}{k!} \left( [q]a_{p,q}(2j) \right)^k
\]

which can be viewed, using (7),

\[
a_{p,q}(n) = - \sum_{j=0}^{n/2} \prod_{j=1}^{n/2} \left( -1 \right)^k \frac{1}{k!} \left( \frac{q}{2j} \text{Tr}H_{2\pi p/q}^j \right)^k
\]

as an inversion of (5).

\(^1\text{Or how to reduce (5) and (A.11)—see the appendix—to (7).}\)
In the LHS of (6) the algebraic area generating function \( \sum_A C_n(A)e^{2i\pi p/q} \) is defined for all \( n \) even and \( q \). It follows that in its RHS—and in the equations below—\( a_{p,q}(n) \) should be understood as well as defined for all \( n \) even and \( q \). However the coefficient \( a_{p,q}(2j) \) in (4) is not defined—in other words it trivially vanishes—as soon as \( q < 2j \). What is meant in (6) by \( a_{p,q}(n) \) is the coefficient (4) defined for \( q \geq 2j \) and extrapolated onto \( q < 2j \) in such a way that it obeys the same formula as for \( q \geq 2j \), rather than trivially vanishing. This is all what is needed in view of the algebraic area enumeration. Still, for the sake of completeness, we explain in the next section how to explicitly build this extrapolation.

2. Extrapolating the Kreft coefficients

Let us denote by \( \tilde{b}_{p,q}(k) \) the building block \( 4 \sin^2 \left( \frac{\pi p}{q} \right) = (1 - e^{\frac{2i\pi p}{q}})(1 - e^{-\frac{2i\pi p}{q}}) \) appearing in (4) so that

\[
a_{p,q}(2j) = (-1)^{j+1} \sum_{k_1=0}^{q-2j+1} \sum_{k_0=0}^{k_1} \prod_{i=1}^{k_1-1} \tilde{b}_{p,q}(k_1 + 2i - j) \tilde{b}_{p,q}(k_2 + 2j - 3) \cdots \tilde{b}_{p,q}(k_j + 1) \tag{8}
\]

where \( k_1 = q - 2j + 1 \) has been added to the summation because it does not contribute anyway. (8) is nonzero for \( q \geq 2j - 1 \), otherwise the outermost sum trivially vanishes by construction, in fact for \( q \geq 2j \), since when \( q = 2j - 1 \), i.e. \( k_1 = 0 \), the outermost \( \tilde{b}_{p,q}(k_1 + 2j - 1) \) vanishes.

2.1. Extrapolating \( a_{p,q}(2) \)

Shifting \( k_1 \) by 1 one rewrites \( a_{p,q}(2) \) as

\[
a_{p,q}(2) = \sum_{k_1=1}^{q} \tilde{b}_{p,q}(k_1)
= 2q - \sum_{k_1=1}^{q} \left( e^{\frac{2i\pi p}{q}} + e^{-\frac{2i\pi p}{q}} \right).
\tag{9}
\]

For any \( q > 1 \), the sum in the second line of (9) vanishes, being the sum of the \( q \)th roots of unity of power \( q \). However, for \( q = 1 \) this sum reduces to a single term equal to 1 + 1. Hence,

\[
a_{p,q}(2) = 2q \quad \text{when} \ q > 1, \quad a_{p,q}(2) = 0 \quad \text{when} \ q = 1.
\]

which is nothing but saying, accordingly to (4), that \( a_{p,q}(2) \) trivially vanishes when \( q = 1 \)—whereas it is equal to 2\( q \) when \( q > 1 \). The extrapolation amounts to extending the first equation for \( q > 1 \) onto the second one for \( q = 1 \), i.e. for any \( q \geq 1 \) one should end up with

\[
a_{p,q}(2) = 2q. \tag{10}
\]

For this to happen, it suffices to define

\[
\sum_{k=1}^{q} e^{\frac{2i\pi p}{q}} = 0 \tag{11}
\]

for all \( q \), including \( q = 1 \). Substituting this into (9) yields (10).
To extend this scheme onto any \( a_{p,q}(2j) \) it is necessary to generalize (11) to

\[
\sum_{k=1}^{q} e^{2\pi i k j} = 0
\]  

(12)

for any \( q \geq 1 \)—this sum is actually 0 when \( j \) is not a multiple of \( q \), being a sum of \( j \)th powers of \( q \)th roots of unity; however, when \( j \) is a multiple of \( q \) it is equal to \( q \). One has then to express \( a_{p,q}(2j) \) as combinations of sums of products of \( \tilde{b}_{p/q}(k) \) and use (12) to evaluate those sums. Specifically, as we have seen,

\[
\frac{1}{q} a_{p,q}(2) = \frac{1}{q} \sum_{k=1}^{q} \tilde{b}_{p/q}(k) = 2. \tag{13}
\]

Also

\[
\frac{1}{q} \sum_{k=1}^{q} \tilde{b}_{p/q}^2(k) = \frac{1}{q} \left( \sum_{k=1}^{q} \left( 2 - e^{2\pi i k / q} - e^{-2\pi i k / q} \right) \left( 2 - e^{2\pi i k / q} - e^{-2\pi i k / q} \right) \right) = 6, \tag{14}
\]

because when the parentheses are opened and (12) is used, only those terms survive where a constant, not an exponential, ends up being summed. They are \( 4 + 1 + 1 = 6 \) such terms.

Quite generally, following this line of reasoning one gets

\[
\frac{1}{q} \sum_{k=1}^{q} \tilde{b}_{p/q}^j(k) = \binom{2j}{j}, \tag{15}
\]

the equality becoming a strict equality for \( q > j \). Note that \( \binom{2j}{j} \) is the number of closed lattice walks of length \( 2j \) on a 1d lattice.

On the practical side one also remarks that this is precisely what one gets with Mathematica Simplify acting on (8) when \( 2j = 2 \) with output \( 2q \) or on \( \frac{1}{2} \sum_{k=1}^{q} \tilde{b}_{p/q}^j(k) \) with output \( \binom{2j}{j} \). In the sequel we will use, when needed, Mathematica Simplify to check results involving expressions with extrapolated coefficients.

One stresses again that for \( 2j = 2 \) starting from (8) defined for \( q \geq 2 \), obtaining \( a_{p,q}(2) = 2q \) and deciding that this expression is valid for all \( q \) including \( q = 1 \) is all what is needed in view of the algebraic area enumeration.

2.2. Extrapolating \( a_{p,q}(4) \)

Let us illustrate this scheme on \( a_{p,q}(4) \). On the one hand acting with Mathematica Simplify on \( a_{p,q}(4) \) as defined in (8) yields the output

\[
a_{p,q}(4) = q \left( 7 + 2 \cos \left( \frac{2\pi p q}{q} \right) - 2q \right). \tag{16}
\]

On the other hand one has, upon redefining the indices,

\[
a_{p,q}(4) = - \sum_{k_1=1}^{q} \sum_{k_2=1}^{q} \tilde{b}_{p/q}(k_1) \tilde{b}_{p/q}(k_2). \tag{17}
\]

Let us adjust the sum limits so as to end up with, among others, products of pieces of the form \( \sum_{k=1}^{q} \tilde{b}_{p/q}^j(k) \), which can then be directly evaluated using (15). One rewrites the double sum in (17) as
\[ \sum_{k_1=1}^{q} \sum_{k_2=1}^{q} \hat{b}_{p/q}(k_1) \hat{b}_{p/q}(k_2) = \sum_{k_1=1}^{q} \sum_{k_2=1}^{q} \bar{b}_{p/q}(k_1) \bar{b}_{p/q}(k_2) = \sum_{k_1=1}^{q} \bar{b}_{p/q}(k_1) \left( \sum_{k_2=1}^{q} \bar{b}_{p/q}(k_2) - \bar{b}_{p/q}(k_1 - 1) \right). \]

In the first term on the RHS of (18) one turns the triangular sum into a product of two sums, taking advantage of the fact that the summand is symmetric with respect to \( k_1 \) and \( k_2 \)

\[ \sum_{k_1=1}^{q} \sum_{k_2=1}^{q} \hat{b}_{p/q}(k_1) \hat{b}_{p/q}(k_2) = \sum_{k_1=1}^{q} \sum_{k_2=1}^{q} \bar{b}_{p/q}(k_1) \bar{b}_{p/q}(k_2) - \sum_{k_1=1}^{q} \sum_{k_2=1}^{q} \bar{b}_{p/q}(k_1) b_{p/q}(k_2) - \sum_{k_1=1}^{q} \sum_{k_2=1}^{q} \bar{b}_{p/q}(k_1) \bar{b}_{p/q}(k_2). \]

The last term is the first one with the opposite sign so

\[ \sum_{k_1=1}^{q} \sum_{k_2=1}^{q} \hat{b}_{p/q}(k_1) \hat{b}_{p/q}(k_2) = \frac{1}{2} \left( \left( \sum_{k=1}^{q} \hat{b}_{p/q}(k) \right)^2 - \sum_{k=1}^{q} \bar{b}_{p/q}^2(k) \right) \]

and one arrives at

\[ a_{p,q}(4) = -\frac{1}{2} \left( \left( \sum_{k=1}^{q} \hat{b}_{p/q}(k) \right)^2 - \sum_{k=1}^{q} \bar{b}_{p/q}^2(k) \right) + \frac{1}{2} \left( \left( \sum_{k=1}^{q} \hat{b}_{p/q}(k) \right)^2 - \sum_{k=1}^{q} \bar{b}_{p/q}^2(k) \right) + \sum_{k_1=1}^{q} \sum_{k_2=1}^{q} \bar{b}_{p/q}(k_1) \bar{b}_{p/q}(k_2). \]

For the first two terms in the RHS of (19), use (13) and (14), respectively; hence,

\[ -\frac{1}{2} \left( \left( \sum_{k=1}^{q} \hat{b}_{p/q}(k) \right)^2 - \sum_{k=1}^{q} \bar{b}_{p/q}^2(k) \right) = -\frac{1}{2} \left( 4q^2 - 6q \right). \]

For the last term, acting in the same way as in (14), one gets

\[ \sum_{k_1=1}^{q} \bar{b}_{p/q}(k) \bar{b}_{p/q}(k - 1) = \sum_{k=1}^{q} \left( 2 - e^{-\frac{2\pi k}{p}} - e^{-\frac{2\pi (k-1)}{p}} \right) \left( 2 - e^{-\frac{2\pi (k+1)}{p}} - e^{-\frac{2\pi (k+2)}{p}} \right) \]

\[ = q \left( 4 + e^{-\frac{2\pi}{p}} + e^{-\frac{2\pi}{p}} \right) = q \left( 2 + \cos \left( \frac{2\pi p}{q} \right) \right). \]

where, again, the parentheses have been opened and only those terms where a constant ends up being summed have been kept.

Combining (19)–(21) one finally gets

\[ a_{p,q}(4) = -2q^2 + 7q + 2q \cos \left( \frac{2\pi p}{q} \right) \]

i.e. (16). An exact calculation starting from (8) or (19) would yield the same result when \( q \geq 4 \) but, trivially, zero when \( q < 4 \). But the RHS of (22) does not trivially vanish when \( q < 4 \)—it does vanish when \( q = 3 \), but this is a non trivial vanishing—see (A.9) in the appendix.

Once again, postulating (12) results in \( a_{p,q}(4) \) getting extrapolated onto \( q < 4 \). In general for any \( j \) the same approach should be used: adjust the limits of sums, turn subdiagonal sums into products of independent sums, and resolve the sums of products of \( \hat{b}_{p/q}^j(k), \bar{b}_{p/q}^j(k) \bar{b}_{p/q}^j(k - 1) \), etc.

3. Algebraic area enumeration

3.1. At order \( q \): \( a_{p,q}(n) \)

Focusing now on \( a_{p,q}(4) \) at order \( q \), which is the part of interest for the algebraic area enumeration (6), one should discard in (19) the term which is a product of two sums since it contributes
at order $q^2$. The two other terms where only one sum appears contribute at order $q$. They are labelled by the $2^{n/2-1} = 2$ compositions of 2, 2 = 2 and 2 = 1 + 1, 

\[ q^n [q] a_{p,q}(4) = \frac{1}{2} \sum_{k=1}^{q} \tilde{b}_{p/q}^2(k) + \sum_{k=1}^{q} \tilde{b}_{p/q}(k)\tilde{b}_{p/q}(k-1). \]  

(23)

The algebraic area enumeration for walks of length 4 has thus narrowed down to reduce (17)–(23) and to compute the two terms in the RHS of the latter. This has been done as indicated above

\[ \frac{1}{q} \sum_{k=1}^{q} \tilde{b}_{p/q}^2(k) = 2(3) \]

\[ \frac{1}{q} \sum_{k=1}^{q} \tilde{b}_{p/q}(k)\tilde{b}_{p/q}(k-1) = 2(2 + \cos(\frac{2\pi p}{q})) \]

(24)

leading to the counting $4/2 \times 2(3) + 4 \times 2(2) = 28$ walks with algebraic area 0 and $4 \times 2 = 8$ = 4 + 4 walks with algebraic area ±1.

This pattern is easily seen to generalize to larger $n$’s as illustrated in the appendix in the case $n = 6$ and $n = 8$. Clearly when $n = 4, 6, 8$ the algebraic area enumeration has narrowed down to finding the coefficients in front of the $2^{n/2-1}$ terms in (23), (A.3) and (A.7), and those of their $\cos(2k\pi p/q)$ expansions in (24), (A.4) and (A.8).

When $q = 1$ the LHS of (6) counts the number $\binom{n}{a/2}^2$ of closed walks on a square lattice of length $n$, up to a factor $1/n$. One notes in the three cases above $n = 4, 6, 8$ that

- the coefficients in the cosine expansion of each of the $2^{n/2-1}$ terms contributing when $n = 4$ to (24), when $n = 6$ to (A.4) and when $n = 8$ to (A.8), add up to \( \binom{4}{2} \), \( \binom{6}{3} \) and \( \binom{8}{4} \) respectively.

- when $q = 1$ one can verify the $n = 4, 6, 8$ lattice walk countings

\[ \binom{4}{2}(1/2 + 1) = \binom{4}{2}^2/4 \Rightarrow 1/2 + 1 = \binom{4}{2}/4 \]

(25)

\[ \binom{6}{3}(1/3 + 1 + 1 + 1) = \binom{6}{3}^2/6 \Rightarrow 1/3 + 1 + 1 + 1 = \binom{6}{3}/6 \]

(26)

\[ \binom{8}{4}(1/4 + 1 + 3/2 + 2 + 1 + 1 + 1) = \binom{8}{4}^2/8 \Rightarrow 1/4 + 1 + 3/2 + 2 + 1 + 1 + 1 = \binom{8}{4}/8 \]

(27)

which hold because the coefficients in front of the said terms also add up to \( \binom{4}{2} \) up to a factor $1/4$, \( \binom{6}{3} \) up to a factor $1/6$ and \( \binom{8}{4} \) up to a factor $1/8$ respectively.

The countings (25)–(27) are particular cases of two general properties: by construction

- the coefficients in the cosine expansion of each of the $2^{n/2-1}$ terms contributing to the first order \([q] a_{p,q}(n)\) add up to \( \binom{n}{a/2} \)

- the coefficients in front of the said terms also add up to \( \binom{n}{a/2} \) —up to a $1/n$ factor.
\[ \binom{n}{n/2} \] counts the number of closed walks of length \( n \) on a 1d lattice. This hints to the fact that both set of coefficients, those in front of the terms contributing to \([q]a_{p,q}(n)\) as well as those appearing in their cosine expansions, might be expressable and interpretable in terms of properties of 1d closed random walks of length \( n \).

We now explain how to proceed\(^2\) for a general \( n \). We stress that the results to follow will in part be based on experimental observations and deductions and not actual derivations, which will remain to be built.

### 3.2. The coefficients of the \( 2^{n/2-1} \) terms

#### 3.2.1. Combinatorics

For a given \( n \), let us consider the \( 2^{n/2-1} \) compositions of \( n/2 = l_1 + l_2 + \ldots + l_{n/2} \), \( l_1 \geq 1, l_2, \ldots, l_{n/2} \geq 0 \). One infers from the cases and remarks above that in the reduction at order \( q \) of \( a_{p,q}(n) \) these compositions label the coefficients \( c(l_1, l_2, \ldots, l_{n/2}) \) of the \( \sum_{k=1}^q \hat{b}^{|k|}_{p/q}(k)\hat{b}^{|k-1|}_{p/q}(k-1) \ldots \hat{b}^{k/2}_{p/q}(k-n/2 + 1) \)'s, namely \([q]a_{p,q}(n)\) rewrites as

\[
[q]a_{p,q}(n) = \sum_{(l_1, l_2, \ldots, l_{n/2}) \text{composition of } n/2} c(l_1, l_2, \ldots, l_{n/2}) \frac{1}{q} \sum_{k=1}^q \hat{b}^{|k|}_{p/q}(k)\hat{b}^{|k-1|}_{p/q}(k-1) \ldots \hat{b}^{k/2}_{p/q}(k-n/2 + 1)
\]

where

\[
c(l_1, l_2, \ldots, l_{n/2}) = \frac{(l_1+l_2)}{l_1+l_2} \frac{(l_2+l_3)}{l_2+l_3} \ldots \frac{(l_{n/2-1}+l_{n/2})}{l_{n/2-1}+l_{n/2}}.
\]

In (29) \( c(l_1, l_2, \ldots, l_{n/2}) \) has been given in an explicitly symmetric form \( c(l_1, l_2, \ldots, l_{n/2}) = c(l_{n/2}, l_{n/2-1}, \ldots, l_1) \), paying attention to the fact that the last block

\[
\frac{l_{n/2-1}+l_{n/2}}{l_{n/2-1}+l_{n/2}}
\]

which is equal to 1 when \( l_{n/2} \) is equal to 0, so that \( c(l_1, l_2, \ldots, l_{n/2}) \) reduces to \( c(l_1, l_2, l_{n/2-1}, 0) \), should also be considered as equal to 1 when both \( l_{n/2} \) and \( l_{n/2-1} \) are equal to 0, these considerations extending to \( l_{n/2}, l_{n/2-1} \) and \( l_{n/2-2} \) equal to 0, etc.

One also checks that

\[
\sum_{(l_1, l_2, \ldots, l_{n/2}) \text{composition of } n/2} c(l_1, l_2, \ldots, l_{n/2}) = \frac{n}{n/2}
\]

as it should.

\(^2\) One remarks that mirror symmetric compositions give identical enumerations, see for example when \( n = 8 \) the compositions \( 4 = 2 + 1 + 1 \) and \( 4 = 1 + 1 + 2 \) with the same coefficient 1 and same output

\[
\sum_{k=1}^q \hat{b}^{|k|}_{p/q}(k)\hat{b}^{|k-1|}_{p/q}(k-2) = \sum_{k=1}^q \hat{b}^{|k|}_{p/q}(k)\hat{b}^{|k-1|}_{p/q}(k-2) = 2(12 + 14 \cos(2\pi p/q) + 8 \cos(4\pi p/q) + \cos(6\pi p/q)).
\]

One could then restrict to mirror-free compositions, with all compositions weighted twice except the palindromic ones. The number of mirror free compositions of \( n/2 \) is \((2^{n/2-1} + 2^{n/4})/2\), the number of palindromic compositions being \(2^{n/4}\).
3.2.2. Interpretation in terms of 1d lattice walking. It is sufficient to restrict to closed 1d lattice walks of length $n$ making the first step to the right: they are $\binom{n}{n/2}$ $2$ such walks each made of $n/2$ right steps and $n/2$ left steps, or, equivalently, $n/2$ right–left steps. Let us focus on the number of right–left steps which appear on top of each other along the $n$ steps made by the lattice walk.

For e.g. $n = 6$ the coefficients in

$$\sum_{k=1}^{q} \hat{b}_{p/q}^{4}(k) + 4 \sum_{k=1}^{q} \hat{b}_{p/q}^{3}(k) \hat{b}_{p/q}(k-1) + 4 \sum_{k=1}^{q} \hat{b}_{p/q}(k) \hat{b}_{p/q}(k-1) + 6 \sum_{k=1}^{q} \hat{b}_{p/q}^{2}(k) \hat{b}_{p/q}(k-1)$$

i.e. $1 + 3 + 3 = \binom{6}{1}/2$, count 1 lattice walk with the 3 right–left steps on top of each other $\hat{b}_{p/q}^{4}(k)$, 3 lattice walks with 2 right–left steps on top of each other followed by 1 right–left step $\hat{b}_{p/q}^{3}(k) \hat{b}_{p/q}(k-1)$, 3 lattice walks with 1 right–left step followed by 2 right–left steps on top of each other $\hat{b}_{p/q}(k) \hat{b}_{p/q}(k-1)$ and 3 lattice walks with 3 right–left steps following each other $\hat{b}_{p/q}^{2}(k) \hat{b}_{p/q}(k-1) \hat{b}_{p/q}(k-2)$—see figure 1.

For e.g. $n = 8$ the coefficients in

$$\sum_{k=1}^{q} \hat{b}_{p/q}^{6}(k) + 4 \sum_{k=1}^{q} \hat{b}_{p/q}^{5}(k) \hat{b}_{p/q}(k-1) + 4 \sum_{k=1}^{q} \hat{b}_{p/q}^{4}(k) \hat{b}_{p/q}^{2}(k-1) + 6 \sum_{k=1}^{q} \hat{b}_{p/q}^{3}(k) \hat{b}_{p/q}^{2}(k-1)$$

$$+ 8 \sum_{k=1}^{q} \hat{b}_{p/q}(k) \hat{b}_{p/q}^{2}(k-1) \hat{b}_{p/q}(k-2) + 4 \sum_{k=1}^{q} \hat{b}_{p/q}^{2}(k) \hat{b}_{p/q}^{2}(k-1) \hat{b}_{p/q}(k-3)$$

i.e. $1 + 4 + 4 + 6 + 8 + 4 + 4 + 4 = \binom{8}{1}/2$, count 1 lattice walk with the 4 right–left steps on top of each other $\hat{b}_{p/q}^{6}(k)$, 4 lattice walks with 3 right–left steps on top of each other followed by 1 right–left step $\hat{b}_{p/q}^{5}(k) \hat{b}_{p/q}(k-1)$, 4 lattice walks with 1 right–left step followed by 3 right–left steps on top of each other $\hat{b}_{p/q}(k) \hat{b}_{p/q}^{3}(k-1)$, 6 lattice walks with 2 right–left steps on top of each other followed by 2 right–left steps on top of each other $\hat{b}_{p/q}^{2}(k) \hat{b}_{p/q}^{2}(k-1)$, etc.

Quite generally the coefficient $c(l_1, l_2, \ldots, l_{n/2})$ in (28) with $l_1, l_2, \ldots, l_n/2$ a composition of $n/2$ does count, when multiplied by $n$, the number of closed lattice walks of length $n$ with $l_1$ right–left steps on top of each other followed by $l_2$ right–left steps on top of each other … followed by $l_{n/2}$ right–left steps on top of each other.

3.3. The coefficients in the cosine expansions

Combinatorial expressions for the coefficients of the cosine expansions of

$$\sum_{k=1}^{q} \hat{b}_{p/q}^{l_1}(k) \hat{b}_{p/q}^{l_2}(k-1) \ldots \hat{b}_{p/q}^{l_{n/2}}(k-n/2+1)$$

in (28) can be seen in simple cases to rewrite in term of products of deformed 1d lattice binomials $\binom{2l}{l}$ such as

$$\frac{1}{q} \sum_{k=1}^{q} \hat{b}_{p/q}^{l_1}(k) = \binom{2l_1}{l_1}$$

(31)
which can be checked to be, as it should,
\[ l_1, l_2, \ldots, l_j \rightarrow l_j, l_{j-1}, \ldots, l_1 \] symmetric—e.g. in (34) one redefines \( k_4 = k'_4 - k_3 - A \) followed by \( A \rightarrow -A \) and \( k_3 \rightarrow -k_3 \).

One observes that in (32)–(34), \(+A\) and \(-A\) always enter in the last two binomials respectively. With each additional binomial an additional summation enters which preserves the summations already present. The binomials have the following \( k_3, k_4, \ldots, k_j \) weight pattern.

\[ \begin{align*}
\frac{1}{q} \sum_{k=1}^{q} \tilde{b}_{p/q}^j k \tilde{b}_{p/q}^j (k-1) \\
= \sum_{A=-\infty}^{\infty} \cos \left( \frac{2A \pi p}{q} \right) \left( \frac{2l_1}{l_1 + A} \right) \left( \frac{2l_2}{l_2 - A} \right), \\
\frac{1}{q} \sum_{k=1}^{q} \tilde{b}_{p/q}^j k \tilde{b}_{p/q}^j (k-1) \tilde{b}_{p/q}^j (k-2) \\
= \sum_{A=-\infty}^{\infty} \cos \left( \frac{2A \pi p}{q} \right) \sum_{k_3=-\infty}^{\infty} \left( \frac{2l_1}{l_1 - k_3} \right) \left( \frac{2l_2}{l_2 + 2k_3 + A} \right) \left( \frac{2l_3}{l_3 - k_3 - A} \right), \\
\frac{1}{q} \sum_{k=1}^{q} \tilde{b}_{p/q}^j k \tilde{b}_{p/q}^j (k-1) \tilde{b}_{p/q}^j (k-2) \tilde{b}_{p/q}^j (k-3) \\
= \sum_{A=-\infty}^{\infty} \cos \left( \frac{2A \pi p}{q} \right) \sum_{k_4=-\infty}^{\infty} \sum_{k_3=-\infty}^{\infty} \left( \frac{2l_1}{l_1 - k_4 - k_3} \right) \left( \frac{2l_2}{l_2 + k_4 + 2k_3} \right) \left( \frac{2l_3}{l_3 + k_4 - k_3 + A} \right) \left( \frac{2l_4}{l_4 - k_4 - A} \right),
\end{align*} \]
One sees that for a given $k_j$ with $j \geq 3$

- the weights are symmetric and add up to 0
- when $j$ is even: $j - 2$ weights equal to $-1$ and 2 weights at the center equal to $j/2 - 1$
- when $j$ is odd: $j - 1$ weights equal to $-1$ and 1 weight at the center equal to $j - 1$.

One also notes that the $A$ summation upper and lower limits are opposite and bounded by \( \lfloor (l_1 + l_2 + \ldots + l_j)^2/4 \rfloor \) and \( -\lfloor (l_1 + l_2 + \ldots + l_j)^2/4 \rfloor \) respectively: for example in (32) and (37) the actual upper limit of summation is \( \min (l_1, l_2) \leq \lfloor (l_1 + l_2)^2/4 \rfloor \); likewise in (33) and (38) the actual upper limit of summation can be found to be \( l_1 + l_2 + l_3 - \max (|l_1 - l_3|, l_2) \leq \lfloor (l_1 + l_2 + l_3)^2/4 \rfloor \). It follows that in (32)–(34) the summation \( \sum_{A=-\infty}^{\infty} \) means at most \( \sum_{A=-\lfloor (l_1 + l_2 + \ldots + l_j)^2/4 \rfloor}^{\lfloor (l_1 + l_2 + \ldots + l_j)^2/4 \rfloor} \)—it is actually so when \( l_1 = l_2 = \ldots = l_j = 1 \), i.e. when the $l_i$'s do label a composition of $j$, the one whose parts are all equal to 1.

One infers in general that for $j \geq 2$—for $j = 1$ see (31)

\[
\frac{1}{q} \sum_{k=1}^{q} \hat{p}_{i/q}^j(k) \hat{p}_{i/q}(k-1) \ldots \hat{p}_{i/q}(k-j+1) = \sum_{A=-\infty}^{\infty} \cos \left( \frac{2A\pi p}{q} \right) \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_{j-1}=-\infty}^{\infty} \prod_{i=1}^{j} \left( l_i - k_{ij} + A(\delta_{ij-1} - \delta_{ij}) \right) \]

(35)

with $1 \leq i \leq j$

\[
k_{ij} = -\sum_{r=i}^{j} k_r + (i - 1)k_{2i-2} + (2i - 1)k_{2i-1} + ik_{2i}
\]

(36)

where the $k_{2i-2}, k_{2i-1}$ or $k_{2i}$ terms do materialize if $2i - 2, 2i - 1$ or $2i$ are lower or equal to $j$ respectively. One notes that the $k_1$ and $k_2$ terms in (36) cancel out. Also the fact that when $l_j = 0$ the RHS of (35) does reduce to the same form with $j$ replaced by $j - 1$ can be seen via some appropriate $k_i$ redefinitions.

When $q = 1$, (32)–(34) reduce as it should to the binomial countings.
\[
\binom{2(l_1 + l_2)}{l_1 + l_2} = \sum_{A = -\infty}^{\infty} \binom{2l_1}{l_1 + A} \binom{2l_2}{l_2 - A}
\]  
(37)

\[
\binom{2(l_1 + l_2 + l_3)}{l_1 + l_2 + l_3} = \sum_{A = -\infty}^{\infty} \sum_{k_3 = -\infty}^{\infty} \binom{2l_1}{l_1 - k_3} \binom{2l_2}{l_2 + 2k_3 + A} \binom{2l_3}{l_3 - k_3 - A}
\]  
(38)

\[
\binom{2(l_1 + l_2 + l_3 + l_4)}{l_1 + l_2 + l_3 + l_4} = \sum_{A = -\infty}^{\infty} \sum_{k_3 = -\infty}^{\infty} \sum_{k_4 = -\infty}^{\infty} \binom{2l_1}{l_1 - k_4 - k_3} \binom{2l_2}{l_2 + k_4 + 2k_3} \binom{2l_3}{l_3 + k_4 - k_3 + A} \binom{2l_4}{l_4 - k_4 - A}
\]  
(39)

and in general (35) reduces to the counting

\[
\binom{2(l_1 + l_2 + \ldots + l_j)}{l_1 + l_2 + \ldots + l_j} = \sum_{A = -\infty}^{\infty} \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \ldots \sum_{k_j = -\infty}^{\infty} \prod_{i=1}^{j} \binom{2l_i}{l_i - k_{ij} + A(\delta_{ij} - 1 - \delta_{ij})}
\]  
(40)

where, as well as in (35), the summation \(\sum_{A = -\infty}^{\infty}\) is understood as at most \(\sum_{A = -\left\lfloor \frac{j(j+1)}{2} \right\rfloor /4}^{\infty}\).

The binomial counting (40) can be easily checked by first summing over \(A\) using (A.12), redefining the \(k_i\)'s such that one arrives at the same expression now for \(l_{i-1} + l_i, l_{j-2} + \ldots, l_{t}\), i.e. by symmetry for \(l_1, l_2, \ldots, l_{j-1} + l_j\), and then repeating the procedure \(j - 2\) times to obtain the binomial \(\binom{2(l_1 + l_2 + \ldots + l_j)}{l_1 + l_2 + \ldots + l_j}\). For example in the case \(j = 5\) after summing over \(A\) one redefines \(2k_5 + k_4 = k_5', 4k_3 + k_4 - k_3 = k_5' + 2k_5'\) and \(k_3 = A, k_3' = A\), and so on.

One notes that an interpretation, if any, of the coefficients in (35) in terms of 1d lattice walks is lacking.

4. Conclusion

The algebraic area enumeration for lattice random walks follows from (28), (29) and (35). Using (6) one can then conjecture that for \(n \geq 4\) (for \(n = 2\) trivially \(C_2(A) = 4\delta_{A,0}\))

\[
C_n(A) = n \sum_{l_1, l_2, \ldots, l_n/2 \text{ composition of } n/2} \binom{l_1 + l_2}{l_1} \frac{l_1 + l_2}{l_2 + l_3} \ldots \binom{l_i + l_{i+1}}{l_{i+1}} \frac{l_{i+1}}{l_{i+1} + l_{i+2}} \ldots \binom{l_{n/2} - 1 + l_{n/2}}{l_{n/2} - 1 + l_{n/2}}
\]  
(41)

One checks that \(C_n(A)\) vanishes when \(A\) is not in between \(-n^2/16\) and \([n^2/16]\) and that the lattice walks counting \(\sum_A C_n(A) = \left(\frac{n}{2}\right)^2\) holds by using (40) and then (30).

The complexity of the formula (41) increases quickly with \(n\) since the number of compositions grows exponentially with \(n\). We have verified (41) for small \(n\) against complete enumeration—see e.g. [7] for a complete enumeration based on a recurrence relation which encodes the algebraic area combinatorics; see also [8] for recent efforts on the algebraic area enumeration

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of lattice walks. Larger $n$ verifications would gain in a better understanding of bounds on the $k_3, k_4, \ldots$ opposite upper and lower limits—clearly in (33) and (38) the $k_3$ summation goes from $-\min(l_1, l_2 + l_3)$ to $\min(l_1, l_2 + l_3); \text{ in (34) and (39) from } -\min(l_1 + l_2, l_3 + l_4)$ to $\min(l_1 + l_2, l_3 + l_4), \text{ etc}^3$.

An open issue concerns the $n \to \infty$ limit in (41), where one should, with an appropriate vanishing lattice spacing scaling, recover Lévy’s Brownian law (2).

We note that altogether with (1) and (3), the algebraic area enumeration (41) gives in $e^{2\pi \rho^2/q}$.

The irrational limit, where both $\rho$ and $q$ go to infinity, amounts to directly trade $2\pi \rho^2/q$ for $\gamma$ in $e^{2\pi \rho^2/q}$. It would be interesting to see if any insights on this limit are gained by doing so.

Finally in the appendix we generalize $C_n(A)$ in (41) to $C_{m,m,n/2-m,n/2-m}(A)$, the number of closed lattice walks of length $n$ with $m$ steps right, $m$ steps left, $n/2-m$ steps up, $n/2-m$ steps down enclosing the algebraic area $A$.

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Appendix

A.1. At order $q$: $[q] a_{p,q}(6)$ and $[q] a_{p,q}(8)$

The same logic at work for $a_{p,q}(4)$ should prevail in the case $n = 6$ with Mathematica output

$$a_{p,q}(6) = \frac{2}{q} \left( 58 + 36 \cos \left( \frac{2\pi p}{q} \right) + 6 \cos \left( \frac{4\pi p}{q} \right) - q(21 + 6 \cos \left( \frac{2\pi p}{q} \right)) + 2q^2 \right).$$

(A.1)

One has from (8)

$$a_{p,q}(6) = \sum_{k_1=0}^{q-5} \sum_{k_2=0}^{k_1} \sum_{k_3=0}^{k_2} \bar{b}_{p/q}(k_1 + 5) \bar{b}_{p/q}(k_2 + 3) \bar{b}_{p/q}(k_3 + 1)$$

i.e. upon redefining the indices,

$$C_n(A) = n \sum_{\text{compositions of } k_1/2} \frac{(l_1 + l_2)}{l_1 + l_2} \frac{(l_2 + l_3)}{l_2 + l_3} \cdots \frac{(l_{n/2 - 1} + l_{n/2})}{l_{n/2 - 1} + l_{n/2}} \sum_{l_{n/2} = 0}^{2l_{n/2}} \cdots \sum_{l_1 = 0}^{2l_1} \left( l_1 + A + \frac{2l_1}{\sum_{i=1}^{n/2} (i-1)(k_i - l_i)} \right) \left( l_2 + A - \frac{2l_2}{\sum_{i=3}^{n/2} (i-1)(k_i - l_i)} \right) \prod_{i=3}^{n/2} \left( l_i \right)$$

(42)

but with explicit upper and lower summation limits.

3 One can rewrite the multiple sum in (41) in the less symmetric form
\[ a_{p,q}(6) = \sum_{k_1=5}^{q} \sum_{k_2=3}^{q} \sum_{k_3=1}^{q} b_{p/q}(k_1) b_{p/q}(k_2) b_{p/q}(k_3) = \sum_{k_1=1}^{q} \sum_{k_2=1}^{q} \sum_{k_3=1}^{q} b_{p/q}(k_1) b_{p/q}(k_2) b_{p/q}(k_3). \]  

(A.2)

Similarly in the reduction of (A.2) at order \( q \) only the terms with only one sum contribute: there are four such terms labelled by the \( 26^{2-1} = 4 \) compositions of \( 3, 3 = 3, 3 = 2 + 1, 3 = 1 + 2 \) and \( 3 = 1 + 1 + 1 \)

\[ q[q] a_{p,q}(6) = + \frac{1}{3} \sum_{k=1}^{q} b_{p/q}^3(k) + \sum_{k=1}^{q} b_{p/q}^2(k) b_{p/q}(k - 1) + \sum_{k=1}^{q} b_{p/q}(k) b_{p/q}^2(k - 1) + \sum_{k=1}^{q} b_{p/q}(k) b_{p/q}(k - 1) b_{p/q}(k - 2) \]  

with output

\[
\begin{align*}
- \frac{1}{q} \sum_{k=1}^{q} b_{p/q}^3(k) &= 4(5) \\
- \frac{1}{q} \sum_{k=1}^{q} b_{p/q}^2(k) b_{p/q}(k - 1) &= 4(3 + 2 \cos(\frac{2\pi p}{q})) \\
- \frac{1}{q} \sum_{k=1}^{q} b_{p/q}(k) b_{p/q}^2(k - 1) &= 4(3 + 2 \cos(\frac{2\pi p}{q})) \\
- \frac{1}{q} \sum_{k=1}^{q} b_{p/q}(k) b_{p/q}(k - 1) b_{p/q}(k - 2) &= 4(2 + 2 \cos(\frac{2\pi p}{q}) + \cos(\frac{4\pi p}{q})).
\end{align*}
\]  

(A.3)

Likewise in the case \( n = 8 \) the Mathematica output is

\[ a_{p,q}(8) = \frac{1}{6} q \left( 1617 + 1512 \cos(\frac{2\pi p}{q}) + 462 \cos(\frac{4\pi p}{q}) + 72 \cos(\frac{6\pi p}{q}) + 12 \cos(\frac{8\pi p}{q}) + q(-617 - 372 \cos(\frac{2\pi p}{q}) - 54 \cos(\frac{4\pi p}{q})) + q^2(84 + 24 \cos(\frac{2\pi p}{q}) - 4q^2) \right). \]  

(A.4)

One has

\[ a_{p,q}(8) = - \sum_{k_1=1}^{q} \sum_{k_2=1}^{q} \sum_{k_3=1}^{q} b_{p/q}(k_1) b_{p/q}(k_2) b_{p/q}(k_3) b_{p/q}(k_4). \]  

(A.5)

In the reduction of (A.6) at order \( q \) there are eight terms with only one sum. They are labelled by the \( 28^{2-1} = 8 \) compositions of \( 4, 4 = 4, 4 = 3 + 1, 4 = 1 + 3, 4 = 2 + 2, 4 = 1 + 2 + 1, 4 = 2 + 1 + 1, 4 = 1 + 1 + 2, 4 = 1 + 1 + 1 \)


\[ q^4 [q]_{p,q} (8) = + \frac{1}{4} \sum_{k=1}^{q} \hat{b}_{p/q}^4 (k) \]
\[ + \sum_{k=1}^{q} \hat{b}_{p/q}^3 (k) \hat{b}_{p/q} (k - 1) \]
\[ + \sum_{k=1}^{q} \hat{b}_{p/q}^2 (k) \hat{b}_{p/q}^3 (k - 1) \]
\[ + \frac{3}{2} \sum_{k=1}^{q} \hat{b}_{p/q}^2 (k) \hat{b}_{p/q}^2 (k - 1) \]
\[ + 2 \sum_{k=1}^{q} \hat{b}_{p/q} (k) \hat{b}_{p/q}^2 (k - 1) \hat{b}_{p/q} (k - 2) \]
\[ + \sum_{k=1}^{q} \hat{b}_{p/q}^2 (k) \hat{b}_{p/q} (k - 1) \hat{b}_{p/q} (k - 2) \]
\[ + \sum_{k=1}^{q} \hat{b}_{p/q} (k) \hat{b}_{p/q} (k - 1) \hat{b}_{p/q}^2 (k - 2) \]
\[ + \sum_{k=1}^{q} \hat{b}_{p/q} (k) \hat{b}_{p/q} (k - 1) \hat{b}_{p/q} (k - 2) \hat{b}_{p/q} (k - 3) \]

(A.7)

with output

\[ \frac{1}{q} \sum_{k=1}^{q} \hat{b}_{p/q}^4 (k) = 2 (35) \]
\[ \frac{1}{q} \sum_{k=1}^{q} \hat{b}_{p/q}^3 (k) \hat{b}_{p/q} (k - 1) = 2 \left( 20 + 15 \cos \left( \frac{2 \pi p}{q} \right) \right) \]
\[ \frac{1}{q} \sum_{k=1}^{q} \hat{b}_{p/q}^2 (k) \hat{b}_{p/q}^3 (k - 1) = 2 \left( 20 + 15 \cos \left( \frac{2 \pi p}{q} \right) \right) \]
\[ \frac{1}{q} \sum_{k=1}^{q} \hat{b}_{p/q}^2 (k) \hat{b}_{p/q}^2 (k - 1) = 2 \left( 18 + 16 \cos \left( \frac{2 \pi p}{q} \right) + \cos \left( \frac{4 \pi p}{q} \right) \right) \]
\[ \frac{1}{q} \sum_{k=1}^{q} \hat{b}_{p/q} (k) \hat{b}_{p/q}^2 (k - 1) \hat{b}_{p/q} (k - 2) = 2 \left( 13 + 16 \cos \left( \frac{2 \pi p}{q} \right) + 6 \cos \left( \frac{4 \pi p}{q} \right) \right) \]
\[ \frac{1}{q} \sum_{k=1}^{q} \hat{b}_{p/q}^2 (k) \hat{b}_{p/q} (k - 1) \hat{b}_{p/q} (k - 2) = 2 \left( 12 + 14 \cos \left( \frac{2 \pi p}{q} \right) + 8 \cos \left( \frac{4 \pi p}{q} \right) + \cos \left( \frac{6 \pi p}{q} \right) \right) \]
\[ \frac{1}{q} \sum_{k=1}^{q} \hat{b}_{p/q} (k) \hat{b}_{p/q} (k - 1) \hat{b}_{p/q}^2 (k - 2) = 2 \left( 12 + 14 \cos \left( \frac{2 \pi p}{q} \right) + 8 \cos \left( \frac{4 \pi p}{q} \right) + \cos \left( \frac{6 \pi p}{q} \right) \right) \]
\[ \frac{1}{q} \sum_{k=1}^{q} \hat{b}_{p/q} (k) \hat{b}_{p/q} (k - 1) \hat{b}_{p/q} (k - 2) \hat{b}_{p/q} (k - 3) = 2 \left( 9 + 12 \cos \left( \frac{2 \pi p}{q} \right) + 9 \cos \left( \frac{4 \pi p}{q} \right) + 4 \cos \left( \frac{6 \pi p}{q} \right) + \cos \left( \frac{8 \pi p}{q} \right) \right). \]

(A.8)

A.2. On the extrapolation

One can obtain a closed expression for the extrapolated \( a_{p,q} (2j) \)'s when \( q < 2j \) in terms of the \( a_{p,q} (2j) \) defined in (4) when \( q \geq 2j \)
• when $j + 1 \leq q \leq 2j - 1$:

$$a_{pq}(2j) = 0 \quad (A.9)$$

• when $1 \leq q \leq j$:

$$a_{pq}(2j) = \sum_{l_{i}\geq 0} \sum_{\ell_{1} + \ell_{2} + \cdots + \ell_{\lfloor q/2 \rfloor} \geq 0} a_{1,1} (2(k + 1)) \left( \ell_{1} + \ell_{2} + \cdots + \ell_{\lfloor q/2 \rfloor}, 2k \right) \prod_{i=1}^{\lfloor q/2 \rfloor} a_{pq}(2i) \ell_{i}^{j} \quad (A.10)$$

with

$$a_{1,1}(2j) = - \sum_{\ell_{1} + 2\ell_{2} + \cdots + j\ell_{j} \geq 0} \frac{1}{\ell_{1}! \ell_{2}! \cdots \ell_{j}!} \left( -\frac{j}{2i} \right)^{2i} \ell_{i}^{j}.$$

In the RHS of (A.10) the Kreft coefficient $a_{pq}(2i)$ is the original one defined for $q \geq 2i$ in (4) whereas in the LHS of (A.10) the extrapolated coefficient $a_{pq}(2j)$ is computed for $1 \leq q \leq j$.

The two items above altogether with (4) coalesce to

$$a_{pq}(2j) = \sum_{j=0}^{\lfloor q/2 \rfloor} a_{1,1}(2k) \left[ z^{2j-q} \right] b_{pq}(z)^{1-2k}$$

where $b_{pq}(z) = - \sum_{k=0}^{\lfloor q/2 \rfloor} a_{pq}(2j) z^{2j}$.

One also note that the trace formula (5) reduces to a partition like formula

$$\frac{q}{n} \text{Tr} H_{2pq}^{a} = \sum_{l_{1} + l_{2} + \cdots + l_{\lfloor q/2 \rfloor} \geq 0} \frac{(l_{1} + l_{2} + \cdots + l_{\lfloor q/2 \rfloor}l_{1}, l_{2}, l_{3}, \ldots, l_{\lfloor q/2 \rfloor})^{n/2}}{l_{1} + l_{2} + \cdots + l_{\lfloor q/2 \rfloor} \prod_{j=1}^{\lfloor q/2 \rfloor} a_{pq}(2j) \ell_{j}^{j}} \quad (A.11)$$

when expressed in terms of the extrapolated coefficients.

### A.3. Additional binomial identities

The binomial countings (37)–(39) and (40) are particular cases of

$$\binom{l_{1} + l_{2} + l_{3}}{l_{1}'} + l_{2} = \binom{\infty}{A=-\infty} \binom{l_{1}}{l_{1}'} + A \binom{l_{2}}{l_{2} - A} \quad (A.12)$$

the Chu–Vandermonde identity,

$$\binom{l_{1} + l_{2}}{l_{1}'} + l_{3} = \sum_{A=-\infty}^{\infty} \sum_{k_{3}=-\infty}^{\infty} \binom{l_{1}}{l_{1}'} - k_{3} \binom{l_{2}}{l_{2} + 2k_{3} + A} \binom{l_{3}}{l_{3} - k_{3} - A},$$

$$\binom{l_{1} + l_{2} + l_{3} + l_{4}}{l_{1}'} + l_{2} + l_{3} = \sum_{A=-\infty}^{\infty} \sum_{k_{4}=-\infty}^{\infty} \sum_{k_{3}=-\infty}^{\infty} \binom{l_{1}}{l_{1}'} - k_{4} - k_{3} \binom{l_{2}}{l_{2} + k_{4} + 2k_{3}} \binom{l_{3}}{l_{3} + k_{4} - k_{3} + A} \binom{l_{4}}{l_{4} - k_{4} - A},$$

and in general of
\[
\left( l_1 + l_2 + \ldots + l_j \right) \\
\left( l_1' + l_2' + \ldots + l_j' \right)
= \sum_{A=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \ldots \sum_{k_j=-\infty}^{\infty} \prod_{i=1}^{j} \left( p_i' - k_{ij} + A(\delta_{i,j-1} - \delta_{ij}) \right).
\]

A.4. Additional identities

One has also found

\[
\frac{1}{q} \sum_{k=1}^{q} \tilde{b}_{p/q}^{l_i}(k) \tilde{b}_{p/q}^{l_i}(k-1) \tilde{b}_{p/q}^{l_i}(k-2) + \frac{1}{q} \sum_{k=1}^{q} \tilde{b}_{p/q}^{l_i}(k-1) \frac{1}{q} \sum_{k=1}^{q} \tilde{b}_{p/q}^{l_i}(k) \tilde{b}_{p/q}^{l_i}(k-2)
= \sum_{k=0}^{\infty} \left( \frac{2l_1}{l_1 - k_3} \right) \left( \frac{2l_2}{l_2 + 2k_3} \right) \left( \frac{2l_3}{l_3 - k_3} \right)
+ 2 \sum_{A=1}^{\infty} \cos \left( \frac{2A\pi p}{q} \right) \sum_{k_i = -\lfloor A/2 \rfloor}^{\infty} \left( \frac{2l_1}{l_1 - k_3 - A} \right) \left( \frac{2l_2}{l_2 + 2k_3 + A} \right) \left( \frac{2l_3}{l_3 - k_3} \right)
+ 2 \sum_{A=1}^{\infty} \cos \left( \frac{2A\pi p}{q} \right) \sum_{k_i = -\lfloor A/2 \rfloor}^{\infty} \left( \frac{2l_3}{l_3 - k_3 - A} \right) \left( \frac{2l_2}{l_2 + 2k_3 + A} \right) \left( \frac{2l_1}{l_1 - k_3} \right)
\]

and the binomial identity

\[
\left( \frac{2(l_1 + l_2 + l_3)}{l_1 + l_2 + l_3} \right) + \left( \frac{2l_2}{l_2 + 2k_3} \right) \left( \frac{2l_3}{l_3 - k_3} \right)
= 2 \sum_{k=0}^{\infty} \left( \frac{2l_1}{l_1 - k_3} \right) \left( \frac{2l_2}{l_2 + 2k_3} \right) \left( \frac{2l_3}{l_3 - k_3} \right)
+ 2 \sum_{A=1}^{\infty} \sum_{k_i = -\lfloor A/2 \rfloor}^{\infty} \left( \frac{2l_1}{l_1 - k_3 - A} \right) \left( \frac{2l_2}{l_2 + 2k_3 + A} \right) \left( \frac{2l_3}{l_3 - k_3} \right)
+ 2 \sum_{A=1}^{\infty} \sum_{k_i = -\lfloor A/2 \rfloor}^{\infty} \left( \frac{2l_3}{l_3 - k_3 - A} \right) \left( \frac{2l_2}{l_2 + 2k_3 + A} \right) \left( \frac{2l_1}{l_1 - k_3} \right)
\]

One derives from (32)

\[
\sum_{k=1}^{q} \tilde{b}_{p/q}^{l_i}(k) \tilde{b}_{p/q}^{l_i}(k - r) = \sum_{k=1}^{q} \tilde{b}_{p/q}^{l_i}(k) \tilde{b}_{p/q}^{l_i}(k-1) \text{ with } \cos \left( \frac{2A\pi p}{q} \right) \rightarrow \cos \left( \frac{2rA\pi p}{q} \right)
\]

and in general from (35)

\[
\sum_{k=1}^{q} \tilde{b}_{p/q}^{l_i}(k) \tilde{b}_{p/q}^{l_i}(k - r) \tilde{b}_{p/q}^{l_i}(k-2r) \ldots \tilde{b}_{p/q}^{l_i}(k - (j-1)r)
= \sum_{k=1}^{q} \tilde{b}_{p/q}^{l_i}(k) \tilde{b}_{p/q}^{l_i}(k-1) \tilde{b}_{p/q}^{l_i}(k-2) \ldots \tilde{b}_{p/q}^{l_i}(k - (j-1)) \text{ with } \cos \left( \frac{2A\pi p}{q} \right) \rightarrow \cos \left( \frac{2rA\pi p}{q} \right).
\]
A.5. Asymmetric probabilities

Considering the generalized coefficient $a_{p,q}^\lambda(n)$ obtained in [9] where $\lambda/2$ can be interpreted as the ratio between the lattice walk probabilities on the horizontal axis versus the vertical axis and following the same logic as in (6) one writes

$$[q]a_{p,q}^\lambda(n) = \frac{1}{n} \sum_A C_n^\lambda(A) e^{2\pi i p/q}$$

to get

$$C_n^\lambda(A) = n \sum_{l_1, l_2, \ldots} \left( \frac{l_1 + l_2}{l_1 + l_2 + l_3} \right) \cdots \left( \frac{l_{n/2-1} + l_{n/2}}{l_{n/2-1} + l_{n/2}} \right) \sum_{k_1, k_2, \ldots} \sum_{l_{n/2-1}, \ldots} \frac{1}{m} \prod_{i=1}^m \sum_{j_i} \left( \frac{l_i}{2} \right) \left( \frac{l_i - k_{i,n/2} + A(\delta_{i,n/2-1} - \delta_{i,n/2}) - j_i}{2} \right)^m.$$

(A.13)

$\lambda/2$ is a deformation parameter which encapsulates the relative weight of random steps on the horizontal versus the vertical axis. In (A.13) the $\lambda$-deformation has narrowed down to replace\textsuperscript{4} any binomial $\left( \frac{l_i}{2} \right)$ in (41) by $\sum_{j_i=0}^{l_i} \left( \frac{l_i}{2} \right) \left( \frac{l_i - k_{i,n/2} + A(\delta_{i,n/2-1} - \delta_{i,n/2}) - j_i}{2} \right)^m$ (which reduces to $\left( \frac{l_i}{2} \right)$ when $\lambda = 2$). Expanding $C_n^\lambda(A)$ in powers of $\lambda$

$$C_n^\lambda(A) = \sum_{m=0}^{n/2} C_{m,m,n/2-m,n/2-m}(A) \left( \frac{\lambda}{2} \right)^{2m}$$

yields $C_{m,m,n/2-m,n/2-m}(A)$, the number of closed lattice walks of length $n$ with $m$ steps right, $m$ steps left, $n/2 - m$ steps up, $n/2 - m$ steps down enclosing a given algebraic area $A$. It is obtained by extracting the coefficient at order $\left( \frac{\lambda}{2} \right)^{2m}$ in (A.13).

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