Compressible flows with a density-dependent viscosity coefficient

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Abstract

We prove the global existence of weak solutions for the 2-D compressible Navier-Stokes equations with a density-dependent viscosity coefficient ($\lambda = \lambda(\rho)$). Initial data and solutions are small in energy-norm with nonnegative densities having arbitrarily large sup-norm. Then, we show that if there is a vacuum domain at the initial time, then the vacuum domain will retain for all time, and vanishes as time goes to infinity. At last, we show that the condition of $\mu = \text{constant}$ will induce a singularity of the system at vacuum. Thus, the viscosity coefficient $\mu$ plays a key role in the Navier-Stokes equations.

1 Introduction

In this paper, we consider the following 2-D compressible Navier-Stokes equations

\[
\begin{cases}
\rho_t + \nabla(\rho u) = 0, \\
(\rho u)_t + \nabla(\rho u \otimes u) + \nabla P = \mu \Delta u + \nabla(\mu(\rho + \lambda(\rho))\nabla \rho) + \rho f,
\end{cases}
\]

for $x \in \mathbb{R}^2$ and $t > 0$, with the boundary and initial conditions

\[
\begin{align*}
&u(x, t) \to 0, \quad \rho(x, t) \to \hat{\rho} > 0, \quad \text{as } |x| \to \infty, \quad t > 0, \\
&(\rho, u)|_{t=0} = (\rho_0, u_0).
\end{align*}
\]

Here $\rho(x, t)$, $u(x, t)$ and $P = P(\rho)$ stand for the fluid density, velocity and pressure respectively, $f$ is a given external force, the dynamic viscosity coefficient $\mu$ is a positive constant, the second viscosity coefficient $\lambda = \lambda(\rho)$ is a function of $\rho$.

At first, we prove the global existence of weak solutions that are in an "intermediate" regularity class in which energies are small, but oscillations are arbitrarily large. Specifically, we fix a positive constant $\hat{\rho}$, assume that $(\rho_0 - \hat{\rho}, u_0)$ are small in $L^2$, and $(\rho_0, u_0) \in L^\infty \times H^1$ with no restrictions on their norms. Our existence result accommodates a wide class of pressures $P$, including pressures that are not monotone in $\rho$. Since the solutions may exhibit vacuum states and discontinuities in density and velocity gradient across hypersurfaces, our results are consequently much less regular and much more general than the well-known small-smooth theory, such as [4, 15]. This existence result generalizes and improves upon the earlier result of Vaigant-Kazhikhov [16] in two significant ways: the space domain is unbounded and the initial density may vanish in an open set. It also generalizes and improves upon earlier results of Hoff-Santos [6] and Hoff [7, 8, 9] in two significant ways: the second viscosity coefficient $\lambda = \lambda(\rho)$ is a function of the density $\rho$, and we omit the condition $\int (1 + |x|^2)^a (\rho_0|u_0|^2 + G(\rho_0)) \, dx \leq M_0$ with a constant $a > 0$.

We now give a precise formulation of our existence result.

Definition 1.1. We say that $(\rho, u)$ is a weak solution of (1.1)–(1.3), if $\rho$ and $u$ are suitably integrable and satisfy that
We also define the convective derivative
\[
\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi
\]
for all times \( t_2 \geq t_1 \geq 0 \) and all \( \phi \in C^1_0(\mathbb{R}^2 \times [t_1, t_2]) \),

\[
\int_{t_1}^{t_2} \rho \phi dt = \int_{t_1}^{t_2} \left( \rho \phi_t + \rho \mathbf{u} \cdot \nabla \phi \right) dt
\]
(1.4)

Concerning the pressure \( P \), viscosity coefficients \( \mu \) and \( \lambda \), we fix \( 0 < \bar{\rho} < \rho \) and assume that

\[
\begin{cases}
  P \in C^1([0, \bar{\rho}]), \quad \lambda \in C^2([0, \bar{\rho}]), \\
  \mu > 0, \quad \lambda(\rho) \geq 0, \quad \rho \in [0, \bar{\rho}], \\
  P(0) = 0, \quad P'(\rho) > 0, \\
  (\rho - \bar{\rho})P'(\rho) > 0, \quad \rho \neq \bar{\rho}, \quad \rho \in [0, \bar{\rho}], \\
  P \in C^2([0, \bar{\rho}]) \quad \text{or} \quad \frac{P(\rho)}{\rho^{\frac{\mu}{2(\mu + \lambda)}}} \quad \text{is a monotone function on} \quad [0, \bar{\rho}].
\end{cases}
\]
(1.6)

Let \( G \) be the potential energy density, defined by

\[
G(\rho) = \rho \int_0^\rho \frac{P(s) - P(\bar{\rho})}{s^2} ds.
\]
(1.7)

Then for any \( g \in C^2([0, \bar{\rho}]) \) with \( g(\bar{\rho}) = g'(\bar{\rho}) = 0 \), there is a constant \( C \) such that \( |g(\rho)| \leq CG(\rho) \), \( \rho \in [0, \bar{\rho}] \).

We measure the sizes of the initial data and the external force by

\[
C_0 = \int \left( \frac{1}{2} \rho_0 |u_0|^2 + G(\rho_0) \right) dx,
\]
(1.8)

\[
C_f = \sup_{t \geq 0} \| f(\cdot, t) \|^2_{L^2} + \int_0^\infty \left( \| f(\cdot, t) \|^2_{L^2} + \| f(\cdot, t) \|^2_{L^2} + \rho_f^2 \nabla f \|^4_{L^4} + \rho_f^4 \nabla f \|^4_{L^4} \right) dt,
\]
(1.9)

and

\[
M_q = \int_0^\infty \left( \rho^2 |f_F|^2_{L^2} + \rho^2 \nabla f \|^4_{L^4} + \rho^2 \nabla f \|^4_{L^4} + \rho^2 \nabla f \|^4_{L^4} \right) dt
\]

\[
+ \int \| \nabla u_0 \|^2_{L^2} dx + \sup_{t \geq 0} \| f \|_{L^{2+q}},
\]
(1.10)

where \( \sigma(t) = \min \{ 1, t \} \) and \( q \) is a constant satisfying

\[
q \in (0, 2) \quad \text{and} \quad q^2 < \frac{4\mu}{\mu + \lambda(\rho)}, \quad \forall \rho \in [0, \bar{\rho}].
\]

As in [3, 7], we recall the definition of the vorticity matrix \( \omega^{i,k} = \partial_k w^j - \partial_j w^k \), and define the function

\[
F = (\lambda + 2\mu)\text{div} u - P(\rho) + P(\bar{\rho}).
\]
(1.11)

We also define the convective derivative \( \frac{D}{Dt} \) by \( \frac{D}{Dt} \mathbf{w} = \dot{w} = w_t + \mathbf{u} \cdot \nabla w \), the Hölder norm

\[
< v >^\alpha_A = \sup_{x,y \in A, x \neq y} \frac{|v(x) - v(y)|}{|x - y|^\alpha},
\]

where \( A \) is a domain in \( \mathbb{R}^2 \).
and

\[< g >_{A \times [t_1, t_2]}^{\alpha, \beta} = \sup_{(x, t) \neq (y, s)} \frac{|g(x, t) - g(y, s)|}{|x - y|^{\alpha} + |t - s|^{\beta}}\]

where \( v : A \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2, g : A \times [t_1, t_2] \rightarrow \mathbb{R}^2 \) and \( \alpha, \beta \in (0, 1] \).

The following is the global existence result of this paper.

**Theorem 1.1.** Assume that conditions (1.6)–(1.10) hold. Then, for a given positive number \( M \) (not necessarily small) and \( \bar{\rho} \in (\hat{\rho}, \bar{\rho}) \), there are positive numbers \( \varepsilon, C \) and \( \theta \), such that, the Cauchy problem (1.1)–(1.3) with the initial data \((\rho_0, u_0)\) and external force \( f \) satisfying

\[0 \leq \inf \rho_0 \leq \sup \rho_0 \leq \bar{\rho}, \quad C_0 + C_f \leq \varepsilon, \quad M \leq M_\varepsilon \]

(1.12)

has a global weak solution \((\rho, u)\) in the sense of (1.14)–(1.15) satisfying

\[C^{-1} \inf \rho_0 \leq \rho \leq \bar{\rho}, \quad \text{a.e.} \]

(1.13)

\[\rho - \bar{\rho}, \quad \rho u \in C((0, \infty); H^{-1}), \quad \nabla u \in L^2(\mathbb{R}^2 \times [0, \infty)), \]

(1.14)

\[< u >_{\mathbb{R}^2 \times [\tau, \infty)}^{\alpha', \beta'} + \sup_{t \geq \tau} (\|\nabla F(t, \tau)\|_{L^2} + \|\nabla w(t)\|_{L^2}) \leq C(\alpha, \tau)(C_0 + C_f)^\theta, \]

(1.15)

\[\sup_{t > 0} \int |u|^2 dx + \int_{0}^{\infty} \int_{0}^{\infty} \sigma |\nabla u|^2 dx ds \leq C, \]

(1.16)

\[\int_{0}^{T} \|F(t)\|_{L^\infty} dt \leq C(T), \]

(1.17)

\[< F >_{\mathbb{R}^2 \times [\tau, T]}^{\alpha', \beta'} + \sup_{t \geq \tau} (\|u\|_{L^{\infty}} + \|\nabla u\|_{L^2}) \leq C(\tau, T), \]

(1.18)

where \( \alpha \in (0, 1), \tau > 0 \) and \( \alpha' \in (0, \frac{1}{2 + \gamma}) \), and

\[\sup_{t > 0} \int \left( \frac{1}{2} \rho|u|^2 + |\rho - \bar{\rho}|^2 + \sigma |\nabla u|^2 \right) dx\]

+ \[\int_{0}^{\infty} \int (|\nabla u|^2 + \sigma (\rho u)_t + \text{div}(\rho u \otimes u))^2 + \sigma^2 |\nabla u|^2) dx dt\]

\[\leq C(C_0 + C_f)^\theta. \]

(1.19)

In addition, in the case that \( \inf \rho_0 > 0 \), the term \( \int_{0}^{\infty} \sigma |\dot{u}|^2 dx dt \) may be included on the left hand side of (1.19).

**Remark 1.1.** Here, \( \theta \) is a universal positive constant (we choose \( \theta = \frac{1}{2} \) in this paper), \( \varepsilon \) and \( C \) depend on \( \hat{\rho}, \bar{\rho}, \bar{\rho}, P, \lambda, \mu, q \) and \( M \).

**Remark 1.2.** For example, we can choose that \( P = A \rho^\gamma \) and \( \lambda(\rho) = c \rho^\beta \) with \( \gamma \geq 1 \) and \( \beta \geq 2 \), where \( A \) and \( c \) are two positive constants. Also, we can choose that \( \lambda \) is a positive constant and \( P = A \rho^\gamma \) with \( \gamma \geq 1 \).

**Remark 1.3.** Considering the non-vacuum case, i.e., \( \rho_0 \geq 2 \rho \geq 0 \), we can replace the condition (1.6) by

\[\left\{ \begin{array}{l}
    P \in C^{1}([0, \bar{\rho}]), \quad \lambda \in C^{2}([\rho, \bar{\rho}]), \\
    \mu > 0, \quad \lambda(\rho) \geq 0, \quad P(0) = 0, \quad P'(\bar{\rho}) > 0, \\
    (\rho - \bar{\rho}) |P(\rho) - P(\bar{\rho})| > 0, \quad \rho \neq \bar{\rho}, \quad \rho \in [\rho, \bar{\rho}], \\
    P \in C^{2}([\rho, \bar{\rho}]) \quad \text{or} \quad \frac{P(\rho)}{\rho^{1}} \quad \text{is a monotone function on} \quad [\rho, \bar{\rho}],
\end{array} \right. \]

(1.20)

where \( 0 \leq 2 \rho < \rho < \bar{\rho} \). Then, we can choose that \( P = A \rho^\gamma \) and \( \lambda(\rho) = c \rho^\beta \) with \( \gamma \geq 1 \) and \( \beta \geq 0 \).
Remark 1.4. Using a similar argument as that in [10], one can obtain the uniqueness of the solution if the initial data satisfy \( \rho_0 \geq \rho > 0 \), and \((\rho_0, u_0) \in C^{1+\alpha} \times C^{2+\alpha} \) or \((\rho_0, u_0) \in W^{1,q} \times H^2 \), where \( \alpha \in (0, 1) \) and \( q > 2 \).

The proof of Theorem 1.1 consists in the derivation of a priori estimates for smooth solutions corresponding to mollified initial data, and the application of these estimates in extracting limiting weak solutions as the mollifying parameter goes to zero. Specifically, in Section 2, we fix a smooth, local in time solution for which \( 0 \leq \rho \leq \overline{\rho} \) and \( A_1 + A_2 \leq 2(C_0 + C_f)\theta \), where \( \theta \in (0, 1) \).

\[
A_1(T) = \sup_{0 < t \leq T} \sigma \int |\nabla u|^2 dx + \int_0^T \int \sigma \rho |u|^2 dx \, dt
\]

and

\[
A_2(T) = \sup_{0 < t \leq T} \sigma^2 \int \rho |u|^2 dx + \int_0^T \int \sigma^2 |\nabla u|^2 dx \, dt
\]

then obtain the estimate \( A_1 + A_2 \leq (C_0 + C_f)^\theta \), and prove that the density remains in a compact subset of \([0, \overline{\rho}]\). Using the classical continuation method, we can close these estimates, which are then applied in Section 3 to show the solution can be obtained in the limit as the mollifying parameter goes to zero.

Using the initial condition \( u_0 \in H^1 \), we can obtain pointwise bounds for \( F \) in Proposition 2.6, which is the key point of the a priori estimates. Because that the mass equation can be transformed to the following form,

\[
dt \Lambda(\rho(x(t), t)) + P(\rho(x(t), t)) - P(\tilde{\rho}) = -F(x(t), t),
\]

where \( \Lambda \) satisfies that \( \Lambda(\tilde{\rho}) = 0 \) and \( \Lambda'(\rho) = \frac{2(\alpha+1)}{\rho} \), a curve \( x(t) \) satisfies \( \dot{x}(t) = u(x(t), t) \), thus pointwise bounds for the density will therefore follow from pointwise bounds for \( F \). On the other hand, using a similar argument as that in [10] and the estimate \( \int_0^T \|F(\cdot, t)\|_{L^\infty} dt \leq C(T) \), we can obtain the strong limit of approximate densities \( \{\rho^\delta\} \), see Section 3.

Then, we study the propagation of singularities in solutions obtained in Theorem 1.1. We show that each point of \( \mathbb{R}^2 \) determines a unique integral curve of the velocity field at the initial time \( t = 0 \), and that this system of integral curves defines a locally bi-Hölder homeomorphism of any open subset \( \Omega \) onto its image \( \Omega^t \) at each time \( t > 0 \). Using this Lagrangean structure, we show that if there is a vacuum domain at the initial time, then the vacuum domain will exist for all time, and vanishes as time goes to infinity, see Theorem 1.4. Also, we show that, if the initial density has a limit at a point from a given side of a continuous hypersurface, then at each later time both the density and the divergence of the velocity have limits at the transported point from the corresponding side of the transported hypersurface, which is also a continuous manifold. If the limits from both sides exist, then the Rankine-Hugoniot conditions hold in a strict pointwise sense, showing that the jump in the \( (\lambda + 2\mu) \) divu is proportional to the jump in the pressure. This leads to a derivation of an explicit representation for the strength of the jump in \( \Lambda(\rho) \) in non-vacuum domain. These results generalize and improve upon the earlier results of Hoff-Santos [6] in a significant way: the domain may contain the vacuum states.

**Theorem 1.2.** Assume that the conditions of Theorem 1.1 hold.

1. For each \( t_0 \geq 0 \) and each \( \mathbf{x}_0 \in \mathbb{R}^2 \), there is a unique curve \( X(\cdot; \mathbf{x}_0, t_0) \in C^1((0, \infty); \mathbb{R}^2) \cap C^{1-\frac{\alpha}{2}}((0, \infty); \mathbb{R}^2) \), \( \alpha \in (0, 1) \), satisfying

\[
X(t; \mathbf{x}_0, t_0) = \mathbf{x}_0 + \int_{t_0}^t u(X(s; \mathbf{x}_0, t_0), s) \, ds.
\]

2. Denote \( X(t, \mathbf{x}_0) = X(t; \mathbf{x}_0, 0) \). For each \( t > 0 \) and any open set \( \Omega \subset \mathbb{R}^2 \), \( \Omega^t = X(t, \cdot)\Omega \) is open and the map \( \mathbf{x}_0 \mapsto X(t, \mathbf{x}_0) \) is a homeomorphism of \( \Omega \) onto \( \Omega^t \).

3. For any \( 0 \leq t_1, t_2 \leq T \), the map \( X(t_1, y) \mapsto X(t_2, y) \) is Hölder continuous from \( \mathbb{R}^2 \) onto \( \mathbb{R}^2 \). Specifically, for any \( y_1, y_2 \in \mathbb{R}^2 \),

\[
|X(t_2, y_2) - X(t_2, y_1)| \leq \exp(1 - e^{-C(1+T)})|X(t_1, y_2) - X(t_1, y_1)|e^{-C(1+T)}.
\]
(4) Let $\mathcal{M} \subset \mathbb{R}^2$ be a $C^\alpha$ 1-manifold, where $\alpha \in [0,1)$. Then, for any $t > 0$, $\mathcal{M}^t = X(t,\cdot)\mathcal{M}$ is a $C^\beta$ 1-manifold, where $\beta = \alpha e^{-C(1+t)}$.

**Theorem 1.3.** Assume that the conditions of Theorem 1.1 hold. Let $V$ be a nonempty open set in $\mathbb{R}^2$. If $\text{essinf}_{\mathcal{F}} \rho[V] \geq \underline{\rho} > 0$, then there is a positive number $\bar{\rho}$ such that,

$$
\rho(\cdot, t)|_{V^t} \geq \bar{\rho},
$$

for all $t > 0$, where $V^t = X(t,\cdot)V$.

**Theorem 1.4.** Assume that the conditions of Theorem 1.1 hold. Let $U$ be a nonempty open set in $\mathbb{R}^2$. Assume that $\rho_0|_U = 0$. Then,

$$
\rho(\cdot, t)|_{U^t} = 0,
$$

for all $t > 0$, where $U^t = X(t,\cdot)U$. Furthermore, we have

$$
\lim_{t \to -\infty} \int |\rho - \rho_0|^4(x,t)dx = 0, \quad (1.24)
$$

and

$$
\lim_{t \to -\infty} |\{x \in \mathbb{R}^2 | \rho(x,t) = 0\}| = 0. \quad (1.25)
$$

Theorems 1.2, 1.4 are proved in Section 4.

In the following theorem, applying the Lagrangean structure of Theorem 1.2, we establish a result concerning the transport by the velocity field of pointwise continuity of the density. Recall first that the oscillation of $g$ at $x$ with respect to $E$ is defined by (as in [6])

$$
\text{osc}(g; x, E) = \lim_{R \to 0} \left( \text{essup}_{x \in B_R(x)} g - \text{essinf}_{x \in B_R(x)} g \right),
$$

where $x \in E$ and $g$ maps an open set $E \subset \mathbb{R}^2$ into $\mathbb{R}$. We shall say that $g$ is continuous at an interior point $x$ of $E$, if $\text{osc}(g; x, E) = 0$.

**Theorem 1.5.** Assume that the conditions of Theorem 1.1 hold. Let $E \subset \mathbb{R}^2$ be open and $x_0 \in \partial E$. If $\text{osc}(\rho_0; x_0, E) = 0$, then $\text{osc}(\rho(\cdot, t); X(t, x_0), X(t, \cdot)E) = 0$. In particular, if $x_0 \in E$ and $\rho_0$ is continuous at $x_0$, then $\rho(\cdot, t)$ is continuous at $X(t, x_0)$.

Now, let $\mathcal{M}$ be a $C^0 1$-manifold in $\mathbb{R}^2$ and $x_0 \in \mathcal{M}$. Then there is a neighborhood $G$ of $x_0$ which is the disjoint union $G = (G \cap \mathcal{M}) \cup E_+ \cup E_-$, where $E_{\pm}$ are open and $x_0$ is a limit point of each. If $\text{osc}(g; x_0, E_{\pm}) = 0$, then the common value $g(x_0^{\pm}, t)$ is the one-sided limit of $g$ at $x_0$ from the plus-side of $\mathcal{M}$, and similar for the one-sided limit $g(x_0^{-}, t)$ from the minus-side of $\mathcal{M}$. If both of these limits exist, then the difference $|g(x_0)| := |g(x_0^+) - g(x_0^-)|$ is the jump in $g$ at $x_0$ with respect to $\mathcal{M}$. (see [6])

Now, we state our main results on the propagation of singularities in solutions.

**Theorem 1.6.** Let $(\rho, u)$ as in Theorem 1.1, $\mathcal{M}$ be a $C^0 1$-manifold and $x_0 \in \mathcal{M}$.

(a) If $\rho_0$ has a one-sided limit at $x_0$ from the plus-side of $\mathcal{M}$, then for each $t > 0$, $\rho(\cdot, t)$ and $\text{divu}(\cdot, t)$ have one-sided limits at $X(t, x_0)$ from the plus-side of the $C^0 1$-manifold $X(t, \cdot)\mathcal{M}$ corresponding to the choice $E_+^t = X(t, \cdot)E_+$. The map $t \mapsto \rho(X(t, x_0), +, t)$ is in $C^{\frac{3}{4}}([0,\infty)) \cap C^1((0,\infty))$ and the map $t \mapsto \text{divu}(X(t, x_0), +, t)$ is locally Hölder continuous on $(0,\infty)$.

(b) If both one-sided limits $\rho_0(x_0^{\pm})$ of $\rho_0$ at $x_0$ with respect to $\mathcal{M}$ exist, then for each $t > 0$, the jumps in $P(\rho(\cdot, t))$ and $\text{divu}(\cdot, t)$ at $X(t, x_0)$ satisfy the Rankine-Hugoniot condition

$$
[(2\mu + \lambda \rho(X(t, x_0), t)) \text{divu}(X(t, x_0), t)] = [P(\rho(X(t, x_0), t))]. \quad (1.26)
$$

(c) Furthermore, if $\rho_0(x_0) \geq \underline{\rho} > 0$, then the jump in $\Lambda(\rho)$ satisfies the representation

$$
[\Lambda(\rho(X(t, x_0), t))] = \exp \left( -\int_0^t \alpha(\tau, x_0) d\tau \right) [\Lambda(\rho_0(x_0))]. \quad (1.27)
$$

where $a(t, x_0) = \frac{[P(\rho(X(t, x_0), t))]}{[\Lambda(\rho(X(t, x_0), t))]}. $
Theorems 1.5-1.6 are proved in Section 5.

Remark 1.5. From Theorems 1.2-1.6, we have that if $[\rho_0]$ is nonzero at $x_0$, then $[\rho(\cdot, t)]$ is nonzero at $X(t; x_0, 0)$ for every $t > 0$. That is, singularities of this type persist for all time. On the other hand, if $P'(\rho) > 0$ with $\rho^- \leq \rho \leq \rho^+$, then $a$ is strictly positive and the jump in $\Lambda(\rho)$ in the non-vacuum domain decays exponentially in time. Thus, the jumps in $\rho$, in $\frac{P(\rho)}{\Lambda(\rho)+2\mu}$ and in $\text{div}\ u$ in the non-vacuum domain decay exponentially in time as well.

At last, we will show that the condition of $\mu = \text{constant}$ will induce a singularity of the system at vacuum in the following two aspects: 1) considering the special case where two fluid regions initially separated by a vacuum region, we show that the solution we obtained is a nonphysical weak solution in which separate kinetic energies of the two fluids need not to be conserved; 2) we show the blow-up of smooth solutions for the spherically symmetric system when the initial density is compactly supported. Thus, the viscosity coefficient $\mu$ plays a key role in the Navier-Stokes equations.

If we consider the special case where two fluid regions initially separated by a vacuum region, Theorem 1.4 shows that the solution obtained in Theorem 1.1 is a nonphysical weak solution in which the two fluids cannot collide independent of their initial velocities. In the following, we will show that the separate kinetic energies of the two fluids needn’t to be conserved.

If the initial data are spherically symmetric, i.e,

$$\rho_0(x) = \varrho_0(r), \quad u_0(x) = v_0(r) \frac{x}{r}, \quad r = |x|, \tag{1.28}$$

and the external force $f \equiv 0$, from Theorem 1.1, one can prove that the system (1.1)–(1.3) has a spherically symmetric solution $(\rho, u)$ satisfying

$$\rho(x, t) = \varrho(r, t), \quad u(x, t) = v(r, t) \frac{x}{r}, \quad r = |x|. \tag{1.29}$$

Then $(\varrho, v)$ is a solution of the following system

$$\begin{cases} \partial_t \varrho + \partial_r (\varrho v) + \frac{1}{r} \varrho v = 0, \\ \varrho \left( \partial_t v + v \partial_r v \right) + \partial_r P(\varrho) = \partial_r \left\{ \left( \lambda(\varrho) + 2\mu \right) \partial_r \left( v + \frac{\varrho}{r} \right) \right\} \end{cases}, \tag{1.30}$$

$$v(r, t) \to 0, \quad \varrho(r, t) \to \tilde{\rho} > 0, \quad \text{as } r \to \infty, \quad t > 0, \tag{1.31}$$

$$\varrho(r, t)|_{t=0} = (\varrho_0, v_0). \tag{1.32}$$

Furthermore, assume that there are two positive constant $0 < a < b$, such that

$$\varrho_0(r) = 0, \quad r \in (a, b),$$

and

$$\varrho_0(r) \geq \varrho > 0, \quad r \in (0, a) \cup (b, +\infty).$$

Then, from Theorems 1.2-1.4, we have there are two curves $a(t)$ and $b(t)$ satisfying

$$0 < a(t) < b(t) < \infty, \quad a(0) = a, \quad a'(t) = v(a(t), t), \quad b(0) = b, \quad b'(t) = v(b(t), t), \quad \varrho(r, t) = 0, \quad r \in (a(t), b(t)),$$

and

$$\varrho(r, t) \geq \varrho^- > 0, \quad r \in (0, a(t)) \cup (b(t), +\infty),$$

for some positive constant $\varrho^-$. Using a similar argument as that in [10], we can obtain the following theorem.
Theorem 1.7. Assume that the conditions of Theorem 1.1 hold, \( \int_0^1 s^{-2} P(s) ds < \infty, f = 0 \), and the initial data satisfy (1.23), then we have

\[
\frac{d}{dt} E(t) = 2(\lambda(0) + 2\mu)a(t)v(a(t), t)\frac{a(t)v(a(t), t) - b(t)v(b(t), t)}{a^2(t) - b^2(t)}, \quad t \geq 0, \tag{1.33}
\]

where

\[
E(t) = \int_0^{a(t)} \left( \frac{1}{2} \varrho v^2 + \mathcal{G}(\varrho) \right) r dr + \int_0^t \int_0^{a(s)} (\lambda + 2\mu)(v_r + \frac{v}{r})^2 r dr ds,
\]

and

\[
\mathcal{G}(\rho) = \rho \int_0^\rho \frac{P(s)}{s^2} ds.
\]

Remark 1.6. If the viscosity coefficient \( \mu \) is a function of the density and \( \lambda(0) = \mu(0) = 0 \), the equality (1.33) implies the separate kinetic energies of the two fluids are conserved. Thus, the main reason for the appearance of non-physical solutions comes from the viscosity coefficient \( \mu \) being independent of the density.

Remark 1.7. The physical solution of the (1.30)–(1.32) may be obtained by constructing separately the solutions for each of the fluids \( \varrho_0 \in [0, a] \) and \( \varrho_0 \in [b, \infty) \) with the boundary conditions

\[
(\lambda(\varrho) + 2\mu)(v_r + \frac{v}{r}) = P(\varrho), \quad r = a(t), \ b(t). \tag{1.34}
\]

When \( a(t) < b(t) \), one can obtain the composite solution \((\varrho, v)\). Observe that the kinetic energies are separately conserved because of the boundary conditions (1.34). When \( v_0 \) is large and positive on \([0, a]\), large and negative on \([b, \infty)\), a collision \( a(t) = b(t) \) may occur in finite time.

Finally, in Section 7 we will give a non-global existence theorem on smooth solutions for the spherically symmetric system when the initial density is of compact support. The corresponding theorem on compressible Navier-Stokes equations with constant viscosity and heat conductivity coefficients was obtained in [17]. Here we generalize the above theorem to the case when the second viscosity coefficient depends on the density for the isentropic gas flow.

Theorem 1.8. Suppose that \((\varrho, v) \in C^1([0, T]; H^k), k > 3\) is a spherically symmetric solution to the Cauchy problem (1.1) and (1.3) with \( f = 0 \). Assume that \( P(\varrho) = A\varrho^{\gamma} \) and \( \lambda(\varrho) = c\varrho^{\beta} \) with \( 1 < \beta \leq \gamma \) and \( A, c > 0 \). If the support of the initial density \( \varrho_0 \) is compact and \( \varrho_0 \not\equiv 0 \), then \( T \) must be finite.

We now briefly review some previous works about the Navier-Stokes equations with density-dependent viscosity coefficients. For the free boundary problem of one-dimensional or spherically symmetric isentropic fluids, there are many works, please see [1, 3, 11, 14, 18, 19] and the references cited therein. Under a special condition between \( \mu \) and \( \lambda, \lambda = 2\mu\varrho^2 - 2\mu \), there are some existence results of global weak solutions for the system with the Korteweg stress tensor or the additional quadratic friction term, see [12]. H. L. Li, J. Li and Z. P. Xin [12] showed a very interesting result that for any global entropy weak solution of the one-dimensional system, any vacuum state must vanish within finite time. Also see Lions [13] for multidimensional isentropic fluids.

We should mention that the methods introduced by Hoff-Santos in [5], Hoff in [8] and Vaigant-Kazhikhov in [16] will play a crucial role in our proof here.

## 2 A priori estimates

In this section, we derive some a priori estimates for local smooth solutions of the system (1.1)–(1.3) with strictly positive densities. Thus, we fix a smooth solution \((\varrho, u)\) of (1.1)–(1.3) on \( \mathbb{R}^2 \times [0, T] \) for some time \( T > 0 \), with smooth initial data \((\varrho_0, u_0)\) and smooth external force \( f \), satisfying

\[
0 \leq \varrho \leq \bar{\varrho} \tag{2.1}
\]

and

\[
A_1 + A_2 \leq 2(C_0 + C_f)^g. \tag{2.2}
\]


In this paper, we choose $\theta = \frac{1}{2}$ and assume that $\varepsilon \leq 1$.

Before proceeding, we remark that a careful application of the standard Rankine-Hugoniot condition to (1.1) shows that discontinuities in $\rho$, $P(\rho)$ and $\nabla u$ across hypersurfaces can be expected to persist for all time, but that the functions $F$ and $w$ should be relatively smooth in positive time reflecting a cancellation of singularities (for example, see [6, 7, 8, 9]). We can rewrite the momentum equation in the form,

$$\rho \dot{u}^j = \partial_j F + \mu \partial_k w^{j,k} + \rho f^j. \quad (2.3)$$

Thus $L^2$ estimates for $\rho \dot{u}$, immediately imply $L^2$ bounds for $\nabla F$ and $\nabla w$. Stated differently, the decomposition (2.3) implies that

$$\Delta F = \text{div} (\rho \dot{u} - \rho f). \quad (2.4)$$

These two relations (2.3)–(2.4) will play the important role in this section.

**Proposition 2.1.** There is a positive constant $C = C(\bar{\rho})$ such that, if $(\rho, u)$ is a smooth solution of (1.1)–(1.3) satisfying (2.1)–(2.2), then

$$\sup_{0 \leq t \leq T} \int \left[ \frac{1}{2} \rho |u|^2 + G(\rho) \right] dx + \int_0^T \int |\nabla u|^2 dxdt \leq C (C_0 + C_f) \quad (2.5)$$

**Proof.** Using the energy estimate, we can easily obtain (2.5), and omit the details. \qed

The following lemma contains preliminary versions of $L^2$ bounds for $\nabla u$ and $\rho \dot{u}$.

**Lemma 2.1.** If $(\rho, u)$ is a smooth solution of (1.1)–(1.3) as in Proposition 2.1, then there is a constant $C = C(\bar{\rho})$ such that

$$\sup_{0 < t \leq T} \sigma \int |\nabla u|^2 dx + \int_0^T \int \sigma |\dot{u}|^2 dxdt \leq C (C_0 + C_f + O_1), \quad (2.6)$$

where $O_1 = \int_0^T \int \sigma |\nabla u|^3 dxdt$, and

$$\sup_{0 < t \leq T} \sigma^2 \int \rho |\dot{u}|^2 dx + \int_0^T \int \sigma^2 |\nabla \dot{u}|^2 dxdt \leq C (C_0 + C_f + A_1(T)) + C \int_0^T \int \sigma^2 (|u|^4 + |\nabla u|^4) dxdt. \quad (2.7)$$

**Proof.** Multiplying (1.1) by $\sigma \dot{u}$, integrating it over $\mathbb{R}^2 \times [0, t]$, we obtain

$$\int_0^t \int \sigma \rho |\dot{u}|^2 dxds$$

$$= \int_0^t \int (-\sigma \dot{u} \cdot \nabla P + \mu \sigma \Delta u \cdot \dot{u} + \sigma \nabla ((\lambda + \mu) \text{div} u) \cdot \dot{u} + \sigma \rho \rho \cdot \dot{u}) dxds$$

$$:= \sum_{i=1}^4 J_i. \quad (2.8)$$

Using the integration by parts and Hölder’s inequality, we have

$$J_1 = -\int_0^t \int \sigma \dot{u} \cdot \nabla P dxds$$

$$= -\int_0^t \int (-\sigma (\text{div} u) P - (P(\rho)) + \sigma(u \cdot \nabla u) \cdot \nabla P) dxds$$

$$= \int \sigma \text{div} u (P - P(\rho)) dx$$

$$- \int_0^t \int (\sigma \text{div} u (P - P(\rho)) + \sigma P \text{div} u + \sigma(u \cdot \nabla u) \cdot \nabla P) dxds$$
where \( t \in [0, T] \). From (2.3), (2.4)–(2.8), we immediately obtain (2.9).

Taking the operator \( \partial_1 + \text{div}(u) \) in (2.11), multiplying by \( \sigma^2 \dot{u} \) and integrating, we obtain (2.12).

Using the integration by parts and Hölder’s inequality, we have

\[
K_2 = - \int_0^t \int \sigma^2 \dot{u}^j [\partial_j P_t + \text{div}(\partial_j P u)] \, dx \, ds
\]

\[
= \int_0^t \int \sigma^2 [\partial_j \dot{u}^j P_t + \partial_k \dot{u}^j \partial_j P u^k] \, dx \, ds
\]

\[
= \int_0^t \int \sigma^2 [-P \rho \text{div}u \partial_j \dot{u}^j + \partial_k (\partial_j \dot{u}^j u^k) P - P \partial_j (\partial_k \dot{u}^k u^j)] \, dx \, ds
\]

\[
\leq C \left( \int_0^t \int |\nabla u|^2 \, dx \, ds \right)^{\frac{1}{2}} \left( \int_0^t \int \sigma^2 |\nabla \dot{u}|^2 \, dx \, ds \right)^{\frac{1}{2}}
\]
\[
\begin{align*}
K_3 &= \int_0^t \int \sigma^2 \hat{u}^i \partial_i \partial_j ((\lambda + \mu) \text{div} u) + \text{div}(u \partial_j ((\lambda + \mu) \text{div} u)) dx ds \\
&= - \int_0^t \int \sigma^2 \mu \left[ \partial_i \hat{u}^i \partial_i \partial_j ((\lambda + \mu) \text{div} u) + D_{ij} \partial_k \partial_l u^k \partial_l \partial_j ((\lambda + \mu) \text{div} u) \right] dx ds \\
&= - \frac{1}{2} \int_0^t \int \sigma^2 \mu |\nabla \hat{u}|^2 dx ds + C \int_0^t \int \sigma^2 |\nabla u|^4 dx ds,
\end{align*}
\]

\[
K_4 = \int_0^t \int \sigma^2 \hat{u}^i [\partial_i ((\lambda + \mu) \text{div} u)] + \text{div}(u \partial_j ((\lambda + \mu) \text{div} u)) dx ds \\
= - \int_0^t \int \left( \sigma^2 \partial_j \hat{u}^i [\partial_i ((\lambda + \mu) \text{div} u)] + \text{div}(u \partial_j ((\lambda + \mu) \text{div} u)) \right) dx ds \\
+ \sigma^2 \hat{u}^i \partial_j ((\lambda + \mu) \text{div} u) dx ds \\
= - \int_0^t \int \left( \sigma^2 \partial_j \hat{u}^i [\partial_i ((\lambda + \mu) \text{div} u)] + \text{div}(u \partial_j ((\lambda + \mu) \text{div} u)) \right) dx ds \\
= - \int_0^t \int \sigma^2 \partial_j (\partial_i \hat{u}^i + u \cdot \nabla u^i) (\lambda + \mu) \frac{D}{Dt} \text{div} u dx ds + O_4 \\
= - \int_0^t \int \sigma^2 \partial_j (\partial_i \hat{u}^i + u \cdot \nabla u^i) (\lambda + \mu) \frac{D}{Dt} \text{div} u dx ds + O_4 \\
= - \int_0^t \int \sigma^2 (\lambda + \mu) \frac{D}{Dt} \text{div} u^2 dx ds + O_4,
\]

\[
K_5 = \int_0^t \int \sigma^2 \hat{u}^i (|\rho f|^2)_t + \text{div}(u \partial_j (\rho f^j)) dx ds \\
= \int_0^t \int \sigma^2 \hat{u}^i (|\rho f|^2 + \rho u \cdot \nabla f) dx ds \\
\leq C \int_0^t \int [\rho \hat{u}^2 + \sigma^2 |u|^4 + \sigma^4 |\nabla f|^4 + \sigma^6 |f|^2] dx ds,
\]

where \( O_4 \) denotes any term dominated by \( C \int_0^T \int \sigma^2 |\nabla u|^2 (|\nabla u| + \frac{|\rho f|^2}{L^2}) dx ds \), and \( t \in [0, T] \). From (2.13) – (2.17), we immediately obtain (2.18).

The following lemmas will be applied to bound the higher order terms occurring on the right hand sides of (2.6) – (2.7).

**Lemma 2.2.** If \((\rho, u)\) is a smooth solution of (1.1) – (1.5) as in Proposition 2.1, then there is a constant \( C = C(\bar{\rho}) \) such that,

\[
\begin{align*}
\|u\|_{L^p}^p &\leq C_p(C_0 + C_f) \|\nabla u\|_{L^2}^{p-2} + C_p(C_0 + C_f) \|\nabla u\|_{L^2}^p, \quad p \in [2, \infty), \\
\|\nabla u\|_{L^p} &\leq C_p(\|\nabla f\|_{L^p} + \|w\|_{L^p} + \|P - P(\bar{\rho})\|_{L^p}), \quad p \in (1, \infty), \\
\|\nabla F\|_{L^p} + \|\nabla w\|_{L^p} &\leq C_p(\|\rho u\|_{L^p} + \|f\|_{L^p}), \quad p \in (1, \infty),
\end{align*}
\]
Using the result of Proposition 2.1, we can immediately get (2.22).

Using the Galiardo-Nirenberg inequality, we have

\[ \text{Proof.} \]

Integrating, we get

\[ \text{applying (2.23) and Proposition 2.1, we get} \]

\[ \text{Since} \]

\[ \text{Combining it with (2.23), we get (2.18).} \]

Thus, we can obtain (2.20).

Also, for \( 0 \leq t_1 \leq t_2 \leq T, \ p \geq 2 \) and \( s \geq 0, \)

\[ \int_{t_1}^{t_2} \int \sigma^s |\rho - \tilde{\rho}|^p dxds \leq C \left( \int_{t_1}^{t_2} \int \sigma^s |F|^p dxds + C_0 + C_f \right). \] \hspace{1cm} (2.22)

\[ \text{Proof.} \]

Using the Galiardo-Nirenberg inequality, we have

\[ \|u\|_{L^p} \leq C_p\|u\|_{L^2}^{\frac{p}{2}}\|\nabla u\|_{L^2}^{\frac{p-2}{2}}, \quad p \in [2, \infty). \] \hspace{1cm} (2.23)

Since

\[ \tilde{\rho} \int |u|^2 dx \leq \int \rho |u|^2 dx + \left( \int |\rho - \tilde{\rho}|^2 dx \right)^{\frac{s}{2}} \left( \int |u|^4 dx \right)^{\frac{s}{4}}, \]

applying (2.23) and Proposition 2.1 we get

\[ \|u\|_{L^2}^2 \leq C(C_0 + C_f) + C(C_0 + C_f)^{\frac{s}{2}}\|u\|_{L^2}\|\nabla u\|_{L^2} \]

and

\[ \|u\|_{L^2}^2 \leq C(C_0 + C_f) + C(C_0 + C_f)\|\nabla u\|_{L^2}^2. \]

Combining it with (2.23), we get (2.18).

Since \( F = (\lambda + 2\mu)\text{div}u - P + P(\tilde{\rho}), \) we have

\[ \Delta u^j = \partial_j \left( \frac{F + P - P(\tilde{\rho})}{\lambda + 2\mu} \right) + \partial_i(w^{j,i}). \] \hspace{1cm} (2.24)

From the standard elliptic theory, we can get (2.19).

We compute from (1.12) that

\[ \mu \Delta w^{j,k} = \partial_k(\rho u^j) - \partial_j(\rho u^k) + \partial_j(\rho f^k) - \partial_k(\rho f^j). \]

Using the standard elliptic theory, we can get

\[ \|\nabla w\|_{L^p} \leq C_p(\|\rho \tilde{u}\|_{L^p} + \|f\|_{L^p}), \quad p \in (1, \infty). \]

From (2.3), we have

\[ \partial_j F = \rho \tilde{u}^j - \mu \partial_{k} w^{j,k} - \rho f^j \]

and

\[ \|\nabla F\|_{L^p} \leq C(\|\rho \tilde{u}\|_{L^p} + \|\nabla w\|_{L^p} + \|f\|_{L^p}), \quad p \in (1, \infty). \]

Thus, we can obtain (2.20).

From (2.20) with \( p = 2 \) and (2.23), we can immediately obtain (2.21).

Multiplying the mass equation (1.1) by \( p\sigma^s|\rho - \tilde{\rho}|^{p-1}\text{sgn}(\rho - \tilde{\rho}), \) we have

\[ \left( \partial_t + \text{div}(\mu) \right)(\sigma^s|\rho - \tilde{\rho}|^p) + \frac{\sigma^s}{\lambda + 2\mu}((p-1)\rho - \tilde{\rho})|\rho - \tilde{\rho}|^{p-1}|P - P(\tilde{\rho})| \]

\[ = s\sigma^{s-1}\sigma_t|\rho - \tilde{\rho}|^p - \frac{\sigma^s}{\lambda + 2\mu}((p-1)\rho - \tilde{\rho})\text{sgn}(\rho - \tilde{\rho})|\rho - \tilde{\rho}|^{p-1}F. \]

Integrating, we get

\[ \int_{t_1}^{t_2} \sigma^s|\rho - \tilde{\rho}|^p dx ds \leq C \left( \int_{t_1}^{t_2} \int \sigma^{s-1}|\rho - \tilde{\rho}|^p dxds \right). \] \hspace{1cm} (2.25)

Using the result of Proposition 2.1, we can immediately get (2.22). \[ \square \]
Similarly, we get

\begin{equation}
\sup_{0 < t \leq T} \int_0^T (\sigma |\nabla u|^2 + \sigma^2 \rho |\dot{u}|^2) \, dx + \int_0^T (\sigma \rho |\dot{u}|^2 + \sigma^2 |\nabla \dot{u}|^2) \, dxdt \leq (C_0 + C_f)^6. \tag{2.26}
\end{equation}

\textbf{Proof.} From Proposition 2.1 and Lemmas 2.1-2.2, we have

\begin{equation}
\text{From (2.5), (2.18) and (2.20)-(2.23), we obtain}
\end{equation}

\textbf{Proposition 2.2.} If \((\rho, u)\) is a smooth solution of \((1.1)-(1.3)\) as in Proposition 2.1 and \(\varepsilon\) is small enough, then we have

\begin{equation}
\text{Proof. From Proposition 2.1 and Lemmas 2.1-2.2, we have}
\end{equation}

\begin{equation}
\text{LHS of (2.20) \leq C(C_0 + C_f) + C \int_0^T \int (\sigma |\nabla u|^3 + \sigma^2 |\nabla u|^4) \, dxds.} \tag{2.27}
\end{equation}

From (2.19), we get

\begin{equation}
\int_0^T \int \sigma^2 |\nabla u|^4 \, dxds \leq \int_0^T \int \sigma^2 (|F|^4 + |w|^4 + |P - P(\tilde{\rho})|^4) \, dxds. \tag{2.28}
\end{equation}

From (2.20), (2.21) and (2.20)-(2.23), we obtain

\begin{equation}
\int_0^T \int \sigma^2 |F|^4 \, dxds
\leq C \int_0^T \sigma^2 \left( \int |F|^2 \, dx \right) \left( \int |\nabla F|^2 \, dx \right) ds
\leq C \sup_{0 \leq t \leq T} \int \sigma(|\nabla u|^2 + |\rho - \tilde{\rho}|^2) \, dx \int_0^T \int \sigma (\rho |\dot{u}|^2 + |\dot{f}|^2) \, dxds
\leq C(A_1 + C_0 + C_f)^2, \tag{2.29}
\end{equation}

\begin{equation}
\int_0^T \int \sigma^2 |w|^4 \, dxds
\leq C \int_0^T \sigma^2 \left( \int |w|^2 \, dx \right) \left( \int |\nabla w|^2 \, dx \right) ds
\leq C \sup_{0 \leq t \leq T} \int \sigma(|\nabla u|^2) \int_0^T \int \sigma (\rho |\dot{u}|^2 + |\dot{f}|^2) \, dxds
\leq C(A_1 + C_0 + C_f)^2, \tag{2.30}
\end{equation}

\begin{equation}
\int_0^T \int \sigma^2 |\rho - \tilde{\rho}|^4 \, dxds
\leq C \left( \int_0^T \int \sigma^2 |F|^4 \, dxds + C_0 + C_f \right)
\leq C(A_1 + C_0 + C_f)^2 + C(C_0 + C_f), \tag{2.31}
\end{equation}

\begin{equation}
\int_0^T \int \sigma^2 |u|^4 \, dxds
\leq C(C_0 + C_f) \int_0^T \sigma^2 (||\nabla u||^2_{L^2} + ||\nabla u||^2_{L^2}) \, ds
\leq CA_1(C_0 + C_f)^2 + C(C_0 + C_f)^2. \tag{2.32}
\end{equation}

From (2.25) - (2.28), we have

\begin{equation}
\int_0^T \int \sigma^2 (|u|^4 + |\nabla u|^4) \, dxds \leq CA_1^2 + C(C_0 + C_f). \tag{2.33}
\end{equation}

Similarly, we get

\begin{equation}
\int_0^T \int \sigma |\nabla u|^3 \, dxds \leq \int_0^T \int (\sigma^2 |\nabla u|^4 + |\nabla u|^2) \, dxds
\end{equation}
From (2.20)–(2.21), we obtain

\[ \text{LHS of (2.26)} \leq C(C_0 + C_f) + CA^2 \]
\[ \leq C(C_0 + C_f) + C(C_0 + C_f)^{2\theta} \]
\[ \leq (C_0 + C_f)^\theta, \tag{2.35} \]

when

\[ \varepsilon^{1-\theta} + \varepsilon^\theta \leq \frac{1}{2C}. \tag{2.36} \]

Then, we consider the Hölder continuity of \( u \) in the following lemma.

**Lemma 2.3.** When \( t \in (0,T] \) and \( \alpha \in (0,1) \), we have

\[ < u(\cdot,t) >^\alpha \leq C \left( \| \rho \dot{u} \|_{L^\alpha} (\| \nabla u \|_{L^2}^{1-\alpha} + (C_0 + C_f)^{\frac{1}{\alpha}}) + \| \nabla u \|_{L^2} + (C_0 + C_f)^{\frac{1}{\alpha}} \right). \tag{2.37} \]

**Proof.** Let \( p = \frac{2}{1-\alpha} \). From (2.19), (2.21) and Sobolev’s embedding theorem, we have

\[ < u(\cdot,t) >^\alpha \leq C \left[ \| \nabla u \|_{L^p} \right. \\
\leq C \| F \|_{L^p} + \| w \|_{L^p} + \| P - P(\tilde{\rho}) \|_{L^p} \]
\[ \leq C \left( \| \rho \dot{u} \|_{L^\infty} (\| \nabla u \|_{L^2}^{1-\alpha} + \| P - P(\tilde{\rho}) \|_{L^2}^{\frac{2}{1-\alpha}}) + \| \nabla u \|_{L^2} + \| f \|_{L^2} + \| P - P(\tilde{\rho}) \|_{L^2} \) \\
+ C \| \rho - \tilde{\rho} \|_{L^\infty} \| P - \tilde{P} \|_{L^\infty} \] \\
\leq C \left( \| \rho \dot{u} \|_{L^\infty} (\| \nabla u \|_{L^2}^{1-\alpha} + (C_0 + C_f)^{\frac{1}{\alpha}}) + \| \nabla u \|_{L^2} + (C_0 + C_f)^{\frac{1}{\alpha}} \right). \]

**Proposition 2.3.** If \( u_0 \in H^1 \), \( (\rho,u) \) is a smooth solution of (1.1)–(1.3) as in Proposition 2.1 then we have

\[ \sup_{0 \leq t \leq T} \int |\nabla u|^2 \, dx + \int_0^T \rho |\dot{u}|^2 \, dx \, dt \leq C(1 + M_q). \tag{2.38} \]

**Proof.** Using a similar argument as that in the proof of (2.10), we have

\[ \sup_{0 \leq t \leq T} \int |\nabla u|^2 \, dx + \int_0^T \rho |\dot{u}|^2 \, dx \, dt \leq C(C_0 + C_f + M_q) + C \int_0^T \int |\nabla u|^3 \, dx \, ds. \]

Without loss of generality, assume that \( T > 1 \). From (2.20) and (2.34), we get

\[ \sup_{0 \leq t \leq T} \int |\nabla u|^2 \, dx + \int_0^T \rho |\dot{u}|^2 \, dx \, dt \leq C(1 + M_q) + C \int_0^1 \int |\nabla u|^3 \, dx \, ds. \]

From (2.19) and (2.22), we have

\[ \sup_{0 \leq t \leq T} \int |\nabla u|^2 \, dx + \int_0^T \rho |\dot{u}|^2 \, dx \, dt \leq C(1 + M_q) + C \int_0^1 \int (|F|^3 + |w|^3) \, dx \, ds. \]

From (2.20)–(2.21) and (2.23), we obtain

\[ \int (|F|^3 + |w|^3) \, dx \]
Thus, from Proposition 2.1, we have
\[
\int |F|^2 dx \left( \int |\nabla F|^2 dx \right)^{\frac{1}{2}} + C \left( \int |w|^2 dx \right) \left( \int |\nabla w|^2 dx \right)^{\frac{1}{2}}
\]
\[
\leq C \int (|\nabla u|^2 + |\rho - \bar{\rho}|^2) dx \left( \int (|\rho|^2 + |f|^2) dx \right)^{\frac{1}{2}}.
\]
(2.39)

Using Gronwall’s inequality, we obtain
\[
\sup_{0 \leq t \leq T} \int |\nabla u|^2 dx + \int_0^T \rho |\bar{u}|^2 dx dt \leq C(1 + M_0) e^{K_0 T} \|\nabla u\|_{L^2}^2 \leq C(1 + M_0).
\]

\[\Box\]

**Proposition 2.4.** If \( u_0 \in H^1 \), \((\rho, u)\) is a smooth solution of (1.1) – (1.3) as in Proposition 2.1 then we have
\[
\sup_{0 < t \leq T} \sigma \int_0^T \left| \rho \left( |\bar{u}|^2 dx + \int_0^T \sigma |\nabla \bar{u}|^2 dx dt \leq C(1 + M_0). \right.
\]
(2.40)

**Proof.** Using a similar argument as that in the proof of (2.7), from (2.38), we have
\[
\sup_{0 < t \leq T} \sigma \int_0^T \left( \frac{1}{2} \rho |\bar{u}|^2 dx + \int_0^T \sigma |\nabla \bar{u}|^2 dx dt \leq C(1 + M_0) + \frac{1}{2} \int_0^T \sigma (|u|^4 + |\nabla u|^4) dx ds. \right.
\]
Without loss of generality, assume that \( T > 1 \). From (2.20) and (2.33), we get
\[
\sup_{0 < t \leq T} \sigma \int_0^T \left( \frac{1}{2} \rho |\bar{u}|^2 dx + \int_0^T \sigma |\nabla \bar{u}|^2 dx dt \leq C(1 + M_0) + \frac{1}{2} \int_0^1 \sigma (|u|^4 + |\nabla u|^4) dx ds. \right.
\]
From (2.5), (2.18) – (2.19), (2.22) and (2.38), we have
\[
\sup_{0 < t \leq T} \sigma \int_0^T \left( \frac{1}{2} \rho |\bar{u}|^2 dx + \int_0^T \sigma |\nabla \bar{u}|^2 dx dt \leq C(M_0) + \frac{1}{2} \int_0^1 \sigma (|F|^4 + |u|^4) dx ds. \right.
\]
From (2.5), (2.20), (2.29) and (2.38), we obtain
\[
\int_0^1 \sigma (|F|^4 + |u|^4) dx ds \leq C \int_0^1 \sigma \left( \int |F|^2 dx \right) \left( \int |\nabla F|^2 dx \right) ds + C \int_0^1 \sigma \left( \int |w|^2 dx \right) \left( \int |\nabla w|^2 dx \right) ds
\]
\[
\leq C \int_0^1 \sigma \left( \int (|\nabla u|^2 + |\rho - \bar{\rho}|^2) dx \right) \left( \int (|\rho|^2 + |f|^2) dx \right) ds
\]
\[
\leq C(1 + M_0) + \int_0^1 \sigma \left( \int |\nabla u|^2 dx \right) \left( \int |\rho|^2 + |f|^2 dx \right) ds.
\]
(2.41)

Using Gronwall’s inequality, we can finish the proof of this proposition. \[\Box\]
Lemma 2.4. For any \( p \in [2, \infty) \), we have

\[
\|\dot{u}\|_{L^p} \leq C_p \|\sqrt{\rho}\dot{u}\|_{L^2}^\frac{2}{p} \|\nabla\dot{u}\|_{L^2}^{1-\frac{2}{p}} + C_p \|\nabla\dot{u}\|_{L^2}. \tag{2.42}
\]

Proof. Since

\[
\bar{\rho} \int |\dot{u}|^2 \, dx \leq \int \rho |\dot{u}|^2 \, dx + \left( \int |\rho - \bar{\rho}|^2 \, dx \right)^\frac{1}{2} \left( \int |\dot{u}|^4 \, dx \right)^\frac{1}{2},
\]

applying (2.23) and Proposition 2.4, we get

\[
\|\dot{u}\|_{L^2}^2 \leq C \|\sqrt{\rho}\dot{u}\|_{L^2}^2 + C \|\nabla\dot{u}\|_{L^2}^2.
\]

From (2.23), we can immediately obtain (2.42).

Lemma 2.5. For any \( q \in (0, 2) \), we have

\[
\int_0^T \int \sigma^{\frac{4+q}{2}} \rho |\dot{u}|^{2+q} \, dx \, ds \leq C(1 + M_q^{1+\frac{q}{2}}). \tag{2.43}
\]

Proof. Let \( p = \frac{4}{2-q} \). Using Hölder’s inequality, (2.40) and (2.42), we have

\[
\begin{align*}
\int_0^T \int & \sigma^{\frac{4+q}{2}} \rho |\dot{u}|^{2+q} \, dx \, ds \\
& \leq C \int_0^T \int \sigma^{\frac{4+q}{2}} \|\sqrt{\rho}\dot{u}\|_{L^2}^{\frac{2p}{p-2q}} \|\dot{u}\|_{L^p}^{\frac{2p}{p-2q}} \, dx \, ds \\
& \leq C \int_0^T \sigma^{\frac{4+q}{2}} \|\sqrt{\rho}\dot{u}\|_{L^2}^{\frac{2p}{p-2q}} \left( \|\sqrt{\rho}\dot{u}\|_{L^2}^{2} \|\nabla\dot{u}\|_{L^2}^{1-\frac{2}{p}} + \|\nabla\dot{u}\|_{L^2}^{2} \right)^{\frac{2p}{p-2q}} \, ds \\
& \leq C \left( \int_0^T \sigma \|\nabla\dot{u}\|_{L^2}^{2} \, dt \right)^\frac{1}{2} \left( \int_0^T \|\sqrt{\rho}\dot{u}\|_{L^2}^{2} \, dt \right)^\frac{2p}{p-2q} \left( \sup_{t \in [0,T]} \sigma \left( \int \|\sqrt{\rho}\dot{u}\|_{L^2}^{2} \right) \right)^\frac{2p}{p-2q} \\
& \quad + C \left( \int_0^T \sigma \|\nabla\dot{u}\|_{L^2}^{2} \, dt \right)^\frac{1}{2} \left( \int_0^T \|\sqrt{\rho}\dot{u}\|_{L^2}^{2} \, dt \right)^\frac{2p}{p-2q} \left( \sup_{t \in [0,T]} \sigma \left( \int \|\sqrt{\rho}\dot{u}\|_{L^2}^{2} \right) \right)^\frac{2p}{p-2q} \\
& \leq C(1 + M_q^{1+\frac{q}{2}}). \quad \blacksquare
\end{align*}
\]

Proposition 2.5. If \( u_0 \in H^1 \), \((\rho, u)\) is a smooth solution of (1.1)–(1.3) as in Proposition 2.4 \( q \in (0, 2) \) and

\[
q^2 < \frac{4\mu}{\lambda(\rho)} + \mu, \quad \forall \rho \in [0, \bar{\rho}], \tag{2.44}
\]

then we have

\[
\sup_{0 < t < T} \sigma^{2+\frac{q}{2}} \int \rho |\dot{u}|^{2+q} \, dx + \int_0^T \int \sigma^{2+\frac{q}{2}} \|\nabla\dot{u}\|_{L^2}^2 \, dx \, dt \leq C(M_q). \tag{2.45}
\]

Proof. Using a similar argument as that in the proof of (2.13), we have

\[
\begin{align*}
\frac{1}{2+q} \int_0^t \int \rho |\dot{u}|^{2+q} \, dx \\
& = \int_0^t \int \left( \frac{4}{4+2q} \sigma^{1+\frac{q}{2}} \sigma' \rho |\dot{u}|^{2+q} - \sigma^{2+\frac{q}{2}} |\nabla\dot{u}|^2 (\partial_j P_t + \text{div}(\partial_j Pu)) \\
& \quad + \mu \sigma^{2+\frac{q}{2}} |\dot{u}|^2 u^j (\Delta u^i + \text{div}(u\Delta u^i)) \\
& \quad + \sigma^{2+\frac{q}{2}} |\nabla\dot{u}|^2 (\partial_j (\lambda + \mu) \text{div}u) + \text{div}(u\partial_j ((\lambda + \mu) \text{div}u)) \right) \, dx \, dt.
\end{align*}
\]
Using the integration by parts and Hölder’s inequality, we have

\[ H_2 = \int_0^t \left( \sigma^{2.5} |\dot{u}|^9 \dot{u}^j (\partial_j P_t + \text{div}(\partial_j Pu)) \right) dx ds \]

\[ = \int_0^t \left( \sigma^{2.5} |\dot{u}|^9 (\partial_j \dot{u}^j P' \rho_t + \partial_k \dot{u}^j \partial_j Pu^k) \right) dx ds \]

\[ + \int_0^t \left( \sigma^{2.5} (\partial_j |\dot{u}|^9 \dot{u}^j P' \rho_t + \partial_k |\dot{u}|^9 \dot{u}^j \partial_j Pu^k) \right) dx ds \]

\[ = \int_0^t \left( \sigma^{2.5} (-|\dot{u}|^9 P' \rho \text{div}\partial_j \dot{u}^j + \partial_k (|\dot{u}|^9 \partial_j \dot{u}^j u^k) P - P \partial_j (|\dot{u}|^9 \partial_k \dot{u}^j u^k)) \right) dx ds \]

\[ + \int_0^t \left( \sigma^{2.5} (|\dot{u}|^9 P' \rho \text{div}\partial_j \dot{u}^j + P \partial_k (|\dot{u}|^9 \partial_j \dot{u}^j u^k) - P \partial_j (|\dot{u}|^9 \partial_k \dot{u}^j u^k)) \right) dx ds \]

\[ \leq C \int_0^t \left( \int \sigma^{2.5} |\nabla \dot{u}| |\dot{u}|^9 |\nabla u| dx ds, \right) \tag{2.47} \]

\[ H_3 = \int_0^t \left( \int \mu \sigma^{2.5} |\dot{u}|^9 \dot{u}^j \left( \Delta u^j_t + \text{div}(u \Delta u^j) \right) \right) dx ds \]

\[ = \int_0^t \left( \int \sigma^{2.5} \mu \left( \partial_i (|\dot{u}|^9 \dot{u}^j) \partial_i u^j_t + \Delta u^j u \cdot \nabla (|\dot{u}|^9 \dot{u}^j) \right) \right) dx ds \]

\[ = \int_0^t \left( \int \sigma^{2.5} \mu \left( |\dot{u}|^7 |\nabla \dot{u}|^2 - \partial_i \dot{u}^j u^k \partial_k \partial_i u^j |\dot{u}|^9 - \partial_i \dot{u}^j |\dot{u}|^9 \partial_i u^k \partial_k u^j + \partial_i |\dot{u}|^9 \dot{u}^j u^j_t + \Delta u^j u \cdot \nabla (|\dot{u}|^9 \dot{u}^j) \right) \right) dx ds \]

\[ = \int_0^t \left( \int \sigma^{2.5} \mu \left( |\dot{u}|^7 |\nabla \dot{u}|^2 + \partial_i \dot{u}^j \text{div}\partial_i u^j |\dot{u}|^9 + \partial_i |\dot{u}|^9 \dot{u}^j \text{div}\partial_i u^j \right) + \partial_i \dot{u}^j u^k \partial_k u^j + \partial_i |\dot{u}|^9 \dot{u}^j \partial_i u^j - \partial_i \dot{u}^j \partial_i u^k \partial_k u^j + \partial_i \dot{u}^j \partial_i u^k \partial_k u^j \right) dx ds \]

\[ \leq \int_0^t \left( \int \sigma^{2.5} \mu |\dot{u}|^7 |\nabla \dot{u}|^2 dx ds \right) + \int_0^t \left( \int \frac{q}{4} \sigma^{2.5} \mu |\dot{u}|^9 |\nabla \dot{u}|^2 dx ds \right) \]

\[ + C \int_0^t \left( \int \sigma^{2.5} |\nabla u|^2 |\dot{u}|^9 |\nabla \dot{u}| dx ds, \right) \tag{2.48} \]

\[ H_4 = \int_0^t \left( \int \sigma^{2.5} |\dot{u}|^9 \dot{u}^j (\partial_j \partial_i ((\lambda + \mu) \text{div}u) + \text{div}(u \partial_j ((\lambda + \mu) \text{div}u))) dx ds \right) \]

\[ = \int_0^t \left( \int \left\{ -\sigma^{2.5} \partial_j (|\dot{u}|^9 \dot{u}^j) (\partial_i ((\lambda + \mu) \text{div}u) + \text{div}(u (\lambda + \mu) \text{div}u)) dx ds \right) \]
From (2.18)–(2.19), (2.21)–(2.22), (2.38), (2.40), (2.42)–(2.44) and (2.46)–(2.50), we have
\[
\mathcal{O}_5 \leq C\mathcal{O}_5
\]
denotes any term dominated by
\[
C\mathcal{O}_5
\]
\[
\begin{align*}
&\int_0^t \int \sigma^{2^{\frac{1}{2}}} |\tau|^{q\cdot\tau} \left( \partial_t u (\lambda + \mu) \right) d\tau dt + \int_0^t \int \sigma^{2^{\frac{1}{2}}} \left( \partial_t u \right)^2 d\tau dt \\
&\leq \sup_{0 < t \leq T} \sigma^{2^{\frac{1}{2}}} \int \rho |\dot{\mathcal{U}}|^2 d\tau + \int_0^T \int \sigma^{2^{\frac{1}{2}}} |\tau|^{q\cdot\tau} \left( \partial_t u \right)^2 d\tau dt \\
&\leq C(M_q) + C \int_0^T \int \sigma^{2^{\frac{1}{2}}} \left( |\tau|^{q\cdot\tau} |\nabla u|^2 + |\dot{\mathcal{U}}|^2 + |\nabla u|^2 + |u|^{4+2q} \right) d\tau dt \\
&\leq C(M_q) + C \left( \int_0^T \sigma \left\| \nabla u \right\|_{L^{4+q}}^2 d\tau \right)^{\frac{2}{2^{\frac{1}{2}}}} \left( \int_0^T \sigma \frac{16}{12-q} \right)^{\frac{1}{2^{\frac{1}{2}}}} + C \left( \int_0^T \sigma \left\| \nabla u \right\|_{L^{4+q}}^2 d\tau \right) \\
&\times (\int_0^T \sigma \frac{16}{12-q} \left\| \nabla u \right\|_{L^{4+q}}^2 d\tau) \left( \int_0^T \sigma \frac{16}{12-q} \right)^{\frac{1}{2^{\frac{1}{2}}}} + C \int_0^T \sigma^{2^{\frac{1}{2}}} \left( |\nabla u|^2 + |\nabla u|^2 + |u|^{4+2q} \right) d\tau dt \\
&\leq C(M_q) + C \int_0^T \sigma \left( |\sqrt{\rho} |\dot{\mathcal{U}}|_{L^2}^2 \left\| \nabla u \right\|_{L^{4+q}}^2 + \left\| \nabla \dot{\mathcal{U}} \right\|_{L^2}^2 \right) d\tau dt \\
&\quad + C \int_0^T \sigma^{2^{\frac{1}{2}}} \left( |\nabla u| + |\nabla u| + \left\| \nabla u \right\|_{L^{4+q}} \right) \left\| F \right\|_{L^{4+q}} d\tau dt \\
&\leq C(M_q) + C \int_0^T \sigma^{2^{\frac{1}{2}}} \left( |\nabla u| + |\nabla u| + \left\| \nabla u \right\|_{L^{4+q}} \right) \left\| F \right\|_{L^{4+q}} d\tau dt \\
&\quad + C \int_0^T \sigma^{2^{\frac{1}{2}}} \left( |\nabla u| + |\nabla u| + \left\| \nabla u \right\|_{L^{4+q}} \right) \left\| F \right\|_{L^{4+q}} d\tau dt
\end{align*}
\]
where $O_5$ denotes any term dominated by $C\int_0^1 \int \sigma^{2^{\frac{1}{2}}} (\rho \cdot \text{div}(u \cdot v)) d\tau dt$, and $t \in [0, T]$. From (2.18)–(2.19), (2.21)–(2.22), (2.38), (2.40), (2.42)–(2.44) and (2.46)–(2.50), we have
 Proposition 2.6. If $f \in L^\infty_t L^{2+q}_x$, $(\rho, u)$ is a smooth solution of (1.1)–(1.3) as in Proposition 2.5, then we have

$$\|F\|_{L^\infty} + \|w\|_{L^\infty} \leq C(\|\nabla u\|_{L^2} + \|\rho - \bar{\rho}\|_{L^2})^{\frac{2}{2+q}} (\|\rho\|_{L^{2+q}} + \|f\|_{L^{2+q}})^{\frac{2+q}{2+q}}$$

(2.51)

and

$$\int_0^T (\|F\|_{L^\infty} + \|w\|_{L^\infty}) ds \leq C(M_q)(C_0 + C_f)^{\frac{2+q}{2+q}} (1 + T).$$

(2.52)

Proof. From (2.20), (2.26), (2.45) and the Gagliardo-Nirenberg inequality, we have

$$\frac{d}{dt} (\sigma^{-\frac{q}{2}} (C_0 + C_f) \sigma^{-\frac{4+q}{2+q}} (\sigma^{-\frac{4+q}{2+q}})^{\frac{2+q}{2+q}} ds \leq C(M_q)(C_0 + C_f)^{\frac{2+q}{2+q}} (1 + T).$$

Similarly, we can obtain the same estimates for $w$. \qed

Then, we derive a priori pointwise bounds for the density $\rho$.

 Proposition 2.7. Given numbers $0 < \rho_1 < \rho_2 < \bar{\rho} < \bar{\rho}_1 < \bar{\rho}_2 < \bar{\rho}$, there is an $\varepsilon > 0$ such that, if $(\rho, u)$ is a smooth solution of (1.1)–(1.3) with $\bar{C}_0 + \bar{C}_f \leq \varepsilon$ and $0 < \rho_0 \leq \bar{\rho}$, then

$$0 < \rho \leq \bar{\rho}_2, \quad (x, t) \in \mathbb{R}^2 \times [0, T].$$

(2.53)

Similarly, if $\rho_0 \geq \bar{\rho}_1$, for all $x$, then $\rho \geq \bar{\rho}_1$ for all $x$ and $t$. Furthermore, the estimates in Propositions 2.1, 2.6 hold.

Proof. At first, we prove that if (2.1) and (2.2) hold, then estimate (2.6) holds.

We fix a curve $x(t)$ satisfying $\dot{x} = u(x(t), t)$ and $x(0) = x$. From (2.21), we have

$$\frac{d}{dt} \Lambda(\rho(x(t), t)) + P(\rho(x(t), t)) - P(\bar{\rho}) = -F(x(t), t),$$

(2.54)

where $\Lambda$ satisfies that $\Lambda(1) = 0$ and $\Lambda'(\rho) = \frac{2\mu + \lambda(\rho)}{\rho}$.

First for small time, we estimate the pointwise bounds of the density as follows. From (2.1) and (2.52), we have, for all $t \in [0, 1],

$$|\Lambda(\rho(x(t), t)) - \Lambda(\rho_0(x))| \leq C(M_q)(C_0 + C_f)^{\frac{2+q}{2+q}} + C.\]$$

When

$$2C(M_q)\varepsilon^{\frac{2+q}{2+q}} \leq \Lambda(\bar{\rho}_1 + \frac{1}{3}(\bar{\rho}_2 - \bar{\rho}_1)) - \Lambda(\bar{\rho}_1),$$

(2.55)
We get
\[ \Lambda(\rho(x(t), t)) \leq \Lambda(\bar{\rho}_1 + \frac{1}{3}(\bar{\rho}_2 - \bar{\rho}_1)), \quad t \in [0, \tau], \]
and
\[ \rho(x, t) \leq \bar{\rho}_1 + \frac{1}{3}(\bar{\rho}_2 - \bar{\rho}_1), \quad (x, t) \in \mathbb{R}^2 \times [0, \tau], \]
where \( \tau = \min\{1, \frac{1}{\C(\Lambda(\bar{\rho}_1) - \Lambda(\bar{\rho}_1))}\}. \) Since \( \rho_0 > 0 \), then we have
\[ \Lambda(\rho(x(t), t)) \geq \Lambda(\rho_0(x)) - C(M_q)(C_0 + C_f)^{\frac{2}{\alpha + \gamma}} - C\tau > -\infty, \quad t \in [0, \tau], \]
and
\[ \rho > 0, \quad (x, t) \in \mathbb{R}^2 \times [0, \tau]. \]
Similarly, if \( \rho_0 \geq \bar{\rho}_1 \) and
\[ 2C(M_q)^{\frac{2}{\alpha + \gamma}} \leq \Lambda(\bar{\rho}_1) - \Lambda(\bar{\rho}_2 - \frac{1}{3}(\bar{\rho}_2 - \bar{\rho}_1)), \]
we get
\[ \rho \geq \bar{\rho}_1 - \frac{1}{3}(\bar{\rho}_2 - \bar{\rho}_1), \quad (x, t) \in \mathbb{R}^2 \times [0, \tau], \]
where \( \tau_1 = \min\{\tau, \frac{1}{\C(\Lambda(\bar{\rho}_1) - \Lambda(\bar{\rho}_2 - \frac{1}{3}(\bar{\rho}_2 - \bar{\rho}_1)))}\}. \)
Then, for large time \( t \geq \tau \), we estimate the pointwise bounds of density as follows. From (2.56), (2.59), (2.60), (2.61), and (2.62), we have
\[ \frac{d\Lambda(\rho(x(t), t))}{dt} + P(\rho(x(t), t)) - P(\bar{\rho}) = O_5(t), \]
where
\[ |O_5(t)| \leq C(\tau, M_q)(C_0 + C_f)^{\frac{2}{\alpha + \gamma}}, \quad t \geq \tau. \]
Now, we apply a standard maximum principle argument to estimate the upper bounds of density. Let
\[ t_0 = \max\{t \in (\tau, T] | \Lambda(\rho(x(s), s)) \leq \Lambda(\bar{\rho}_2), \quad \text{for all } s \in [0, t]\}. \]
If \( t_0 < T \), we have
\[ \Lambda(\rho(x(t_0), t_0)) = \Lambda(\bar{\rho}_2), \]
\[ \frac{d\Lambda(\rho(x(t), t))}{dt} \bigg|_{t = t_0} \geq 0, \]
and
\[ \rho(x(t_0), t_0) = \bar{\rho}_2. \]
From (2.59), we have
\[ O_5(t_0) \geq P(\bar{\rho}_2) - P(\bar{\rho}). \]
On the other hand, when
\[ C(\tau, M_q)^{\frac{2}{\alpha + \gamma}} < P(\bar{\rho}_2) - P(\bar{\rho}), \]
we have
\[ O_5(t_0) < P(\bar{\rho}_2) - P(\bar{\rho}). \]
It is a contradiction. Thus, we have \( t_0 = T \) and
\[ \rho \leq \bar{\rho}_2, \quad (x, t) \in \mathbb{R}^2 \times [0, T]. \]
Similarly, we can obtain the lower bound of the density.
Using the classical continuation method, (2.20) and (2.63), we can finish the proof of this proposition.

Then, we can prove the global existence of smooth solutions to (1.1) – (1.3).
**Proposition 2.8.** Assume that \( \rho_0 - \tilde{\rho} \in W^{1,p} \cap C^{1+\alpha} \), \( u_0 \in H^2 \cap C^{2+\alpha} \), \( p > 2 \), \( \alpha \in (0, 1) \), \( \rho_0(x) \geq \rho \) with some \( \rho > 0 \) for all \( x \in \mathbb{R}^2 \), \( P, \lambda \in C^\infty([0, \tilde{\rho}]) \) and \( f \in C^\infty([0, \infty); C^\infty) \). Under the assumptions of Theorem 1.1, then there exists a solution \((\rho, u) \in C^{1+\alpha,1+\alpha/2} \times C^{2+\alpha,1+\alpha/2}(\mathbb{R}^2 \times [0, T])\) satisfying (1.1)–(1.3) and for which the bound estimates of Propositions 2.1–2.7 hold, for all \( T > 0 \).

**Proof.** Using similar arguments as that in [16] and in the proof of Proposition 3.2 in [8], one can obtain this proposition.

3 Proof of Theorem 1.1

Let \( j_\delta(x) \) be a standard mollifying kernel of width \( \delta \). Define the approximate initial data \((\rho_0^\delta, u_0^\delta)\) by

\[
\rho_0^\delta = j_\delta \ast \rho_0 + \delta, \quad u_0^\delta = j_\delta \ast u_0.
\]

Assuming that similar smooth approximations have been constructed for functions \( P, f \) and \( \lambda \), we may then apply Proposition 2.8 to obtain a global smooth solution \((\rho^\delta, u^\delta)\) of (1.1)–(1.3) with the initial data \((\rho_0^\delta, u_0^\delta)\), satisfying the bound estimates of Propositions 2.1–2.6 hold with constants independent of \( \delta \).

First, we obtain the strong limit of \( \{u^\delta\} \). From (2.20) and (2.27), we have

\[
\langle u^\delta(\cdot, t) \rangle > \alpha \leq C(\tau), \quad t \geq \tau > 0, \quad \alpha \in (0, 1).
\]

From (3.1), we have

\[
|u^\delta(x, t) - \frac{1}{|B_R(x)|} \int_{B_R(x)} u^\delta(y, t)dy| \leq C(\tau)R^\alpha, \quad t \geq \tau > 0.
\]

Taking \( R = 1 \), from (2.48) and (2.26), we have

\[
\|u^\delta\|_{L^\infty(\mathbb{R}^2 \times [\tau, \infty])} \leq C(\tau).
\]

Then, we need only to derive a modulus of Hölder continuity in time. For all \( t_2 \geq t_1 \geq \tau \), from (2.5), (2.26), (2.32) and (3.2), we have

\[
\begin{align*}
|u^\delta(x, t_2) - u^\delta(x, t_1)| &\leq \frac{1}{|B_R(x)|} \int_{t_1}^{t_2} \int_{B_R(x)} |u^\delta_y(y, s)|dyds + C(\tau)R^\alpha \\
&\leq CR^{-1}|t_2 - t_1|^\frac{\alpha}{2} \left( \int_{t_1}^{t_2} \int |u^\delta|^2dyds \right)^{\frac{1}{2}} + C(\tau)R^\alpha \\
&\leq CR^{-1}|t_2 - t_1|^\frac{\alpha}{2} \left( \int_{t_1}^{t_2} |u^\delta|^2 + |u^\delta \cdot \nabla u^\delta|^2dyds \right)^{\frac{1}{2}} + C(\tau)R^\alpha \\
&\leq C(\tau)(R^{-1}|t_2 - t_1|^\frac{\alpha}{2} + R^\alpha).
\end{align*}
\]

Choosing \( R = |t_2 - t_1|^\frac{1}{2+\alpha} \), we have

\[
\langle u^\delta \rangle > \alpha \frac{\tau}{\mathbb{R}^2 \times [\tau, \infty]} \leq C(\tau), \quad \tau > 0.
\]

From the Ascoli-Arzela theorem, we have (extract a subsequence)

\[
u^\delta \to u, \quad \text{uniformly on compact sets in } \mathbb{R}^2 \times (0, \infty).
\]

Second, we obtain the strong limits of \( \{F^\delta\} \) and \( \{w^\delta\} \). From (2.20), (2.21), (2.22), and (3.4), using similar arguments as that in the proof of (5.1)–(5.2), we have

\[
\langle F^\delta(\cdot, t) \rangle > \alpha' + \|F^\delta\|_{L^\infty(\mathbb{R}^2 \times [\tau, T])} + \langle w^\delta(\cdot, t) \rangle > \alpha' + \|w^\delta\|_{L^\infty(\mathbb{R}^2 \times [\tau, T])} \leq C(\tau, T),
\]

20
where $0 < \tau \leq t \leq T$ and $\alpha' \in (0, \frac{4}{2+\gamma}]$. The simple computation implies that

$$F^\delta_t = \int (2\mu + \lambda(\rho^\delta)) \left( \frac{F^\delta_t d}{ds} \left( \frac{1}{2\mu + \lambda(s)} \right) \right) \left|_{s=\rho^\delta} + \frac{d}{ds} \left( \frac{P(s) - P(\hat{\rho})}{2\mu + \lambda(s)} \right) \right|_{s=\rho^\delta} div \delta^u - (2\mu + \lambda(\rho^\delta)) \partial_j u^\delta_j \partial_i u^\delta_i \right) (3.6)$$

and

$$w^k_j = \int -u^\delta \cdot \nabla (w^\delta)^k_j + \partial_j u^\delta_k - \partial_k u^\delta_j - \partial_j u^\delta_i \partial_i u^\delta_j + \partial_k u^\delta_i \partial_i u^\delta_j. \right) (3.7)$$

Then, from (2.20), (2.26), (2.33), (3.2) and (3.5), we have

$$\|F^\delta_t\|_{L^2(\mathbb{R}^2 \times [\tau,T])} + \|w^\delta_t\|_{L^2(\mathbb{R}^2 \times [\tau,T])} \leq C(\tau, T), \ T > \tau > 0.$$  

Using a similar argument as that in the proof of (5.3), we obtain

$$< F^\delta >_{\mathbb{R}^2 \times [\tau,T]} > w^\delta >_{\mathbb{R}^2 \times [\tau,T]} \leq C(\tau, T), \ T > \tau > 0. \right) (3.8)$$

and (extract a subsequence)

$$F^\delta \to F, \ w^\delta \to w, \ \text{uniformly on compact sets in} \ \mathbb{R}^2 \times (0, \infty). \right) (3.9)$$

Third, we obtain the strong limit of $\{\rho^\delta\}$. From (2.53), we get (extract a subsequence)

$$\rho^\delta \rightharpoonup \rho, \ \text{weak-* in} \ L^\infty(\mathbb{R}^2).$$

Let $\Phi(s)$ be an arbitrary continuous function on $[0, \hat{\rho}]$. Then, we have that (extract a subsequence) $\Phi(\rho^\delta)$ converges weak-* in $L^\infty(\mathbb{R}^2)$. Denote the weak-* limit by $\tilde{\Phi}$:

$$\Phi(\rho^\delta) \rightharpoonup \tilde{\Phi}, \ \text{weak-* in} \ L^\infty(\mathbb{R}^2).$$

From the definition of $F$, we have

$$div u = \tilde{\nu} F + \tilde{P}_0, \right) (3.10)$$

where

$$\nu(\rho) = \frac{1}{2\mu + \lambda(\rho)}, \ P_0(\rho) = \nu(\rho)(P(\rho) - P(\hat{\rho})).$$

From (1.1), we have

$$\partial_t \rho \ln \rho + div(\rho \ln \rho u) + F_\rho \rho + \rho \bar{P}_0 = 0$$

and

$$\partial_t (\rho \ln \rho) + div(\rho \ln \rho u) + F \rho \rho + \rho \bar{P}_0 = 0.$$ 

Letting $\Psi = \rho \ln \rho - \rho \ln \rho \geq 0$, we obtain

$$\partial_t \Psi + div(\Psi u) + F_\rho \rho + F(\rho \ln \rho - \rho) + \rho \bar{P}_0 - \rho \bar{P}_0 = 0. \right) (3.11)$$

with the initial condition $\Psi|_{t=0} = 0$ almost everywhere in $\mathbb{R}^2$. Let $\phi(s) = s \ln s$. Since

$$\phi''(s) = \frac{1}{s}, \ s \geq \frac{1}{\hat{\rho}}, \ s \in [0, \hat{\rho}],$$

we get

$$\phi(\rho^\delta) - \phi(\rho) = \phi'(\rho)(\rho^\delta - \rho) + \frac{1}{2} \phi''(\rho + \xi(\rho^\delta - \rho))(\rho^\delta - \rho)^2, \ \xi \in [0, 1],$$

and

$$\lim_{\delta \to 0} \|\rho^\delta - \rho\|_{L^2} \leq C\|\Psi\|_{L^1}. \right) (3.12)$$
Similarly, every function \( f \in C^2([0, \bar{\rho}]) \) satisfies
\[
\left| \int g(f' - f(\rho))d\rho \right| \leq C \int |g|\Psi d\rho,
\]
where \( g \) is any function such that the integrations exist. Then, when \( \nu \in C^2([0, \bar{\rho}]) \), we have
\[
\left| \int F(\nu - \rho \nu)d\rho \right| \leq C \int |F|\Psi d\rho.
\]
and
\[
\left| \int \rho(\nu - \nu)d\rho \right| \leq C \int |F\nu|\Psi d\rho.
\]
When \( P_0 \in C^2([0, \bar{\rho}]) \), we have
\[
\left| \int (\rho P_0 - \rho \bar{P}_0)d\rho \right| \leq C \int |\rho - \bar{\rho}|\Psi d\rho.
\]
From (3.13–3.17), we obtain
\[
\int \Psi d\rho \leq \int (1 + |F|)\Psi d\rho.
\]
Using (2.52) and Gronwall's inequality, we get
\[
\Psi = 0, \ (t, x) \in [0, T] \times \mathbb{R}^2,
\]
and (extract a subsequence)
\[
\rho^\delta - \bar{\rho} \rightarrow \rho - \bar{\rho}, \ \text{strongly in} \ L^k(\mathbb{R}^2 \times [0, \infty)),
\]
for all \( k \in [2, \infty) \).
Thus, It is easy to show that the limit function \((\rho, u)\) are indeed a weak solution of the system (1.1–1.3). This finishes the proof of Theorem 1.1.

### 4 Lagrangean Structure

**Proof of Theorem 1.2.**

(1). Here, we consider the case \( t_0 = 0 \). The proof of the case \( t > 0 \) is similar, and omit the details.

We denote that \( X(t, x_0) = X(t; x_0, 0) \).

To prove the existence of the integral curve \( X(\cdot, x_0) \), we first assume that \( X^\delta(\cdot, x_0) \) is the corresponding integral curve of \( u^\delta \) with initial point \( x_0 \in \mathbb{R}^2 \),
\[
X^\delta(t, x_0) = x_0 + \int_0^t u^\delta(X^\delta(s, x_0), s)ds, \ s \in [0, \infty).
\]
From (2.18), (2.37), (2.38) and (2.40), using a similar argument as that in the proof of (3.2), we have
\[
\langle u^\delta(\cdot, t) \rangle^{\alpha} \leq C + C\alpha^{\dot{\omega}}, \ \alpha \in (0, 1),
\]
\[
\|u^\delta(t, \cdot)\|_{L^\infty} \leq C + C\alpha^{\dot{\omega}},
\]
and
\[
\int_{T_1}^T \|u^\delta(t, \cdot)\|_{L^\infty} dt \leq C(T - T_1 + T^{1-\dot{\omega}} - T_1^{1-\dot{\omega}}), \ 0 \leq T_1 < T < \infty.
\]
This implies that $X^{\delta}(\cdot, x_0)$ is Hölder continuous on $[0, \infty)$, uniformly in $\delta$. Therefore, there is a subsequence $X^{\delta_k}(\cdot, x_0)$ and a Hölder continuous map $X(\cdot, x_0)$ such that $X^{\delta_k}(\cdot, x_0) \rightarrow X(\cdot, x_0)$ uniformly on compacts sets in $[0, \infty)$. From this uniform convergence and (3.3), we have that $X(\cdot, x_0)$ satisfies (1.22).

Next we prove the uniqueness and continuous dependence for integral curves of $u$. Thus, let $X_1(\cdot, y_1)$ and $X_2(\cdot, y_2)$ be any two integral curves of $u$ with respective initial points $y_1, y_2 \in \mathbb{R}^2$ and define

$$
 g(t) = \frac{|u(X_2(t, y_2), t) - u(X_1(t, y_1), t)|}{m(|X_2(t, y_2) - X_1(t, y_1)|)},
$$

and

$$
 g^{\delta}(t) = \frac{|u^{\delta}(X_2(t, y_2), t) - u^{\delta}(X_1(t, y_1), t)|}{m(|X_2(t, y_2) - X_1(t, y_1)|)},
$$

for $t \in [0, \infty)$, where

$$
 m(x) = \begin{cases} 
 x(1 - \ln x), & 0 < x \leq 1, \\
 x, & 1 \leq x < \infty.
\end{cases}
$$

From (2.24) and Proposition 2.3.7 in [3], we have

$$
 g^{\delta}(t) \leq u^{\delta}(\cdot, t) >_{LL} \leq C\|u^{\delta}\|_{B^1_{\infty, \infty}} \leq C\|u^{\delta}\|_{L^2} + \|F^{\delta}\|_{L^\infty} + \|\rho^{\delta} - \tilde{\rho}\|_{L^\infty} + \|w^{\delta}\|_{L^\infty}).
$$

From (1.13), (4.6) and (2.32), we obtain

$$
 \int_0^T g^{\delta}(t) \leq C(1 + T).
$$

Using (3.4) and Fatou’s lemma, we have

$$
 \int_0^t g(s)ds \leq \liminf_{\delta \rightarrow 0} \int_0^t g^{\delta}(s)ds \leq C(1 + T).
$$

Using Gronwall inequality in (4.6), we get

$$
 |X_2(t, y_2) - X_1(t, y_1)| \leq \exp(1 - e^{-\int_0^t g^{\delta}ds})|y_2 - y_1|^{\exp(-\int_0^t g^{\delta}ds)} \leq \exp(1 - e^{-C(1+T)})|y_2 - y_1|^{\exp(-C(1+T))}.
$$

Thus, if $y_1 = y_2$, then $X_1 = X_2$. There is therefore exactly one integral curve originating from a given point at time $t = 0$. From this uniqueness, we obtain that the convergence $X^{\delta}(t, y_1) \rightarrow X(t, y_1)$ uniformly on compact sets in $[0, \infty)$ for entire sequence $\delta \rightarrow 0$, independently of $y_1$.

(2). We prove the injection of $X$ at first. Suppose that $X(t, y_1) = X(t, y_2)$ for some $t > 0$ and $y_1, y_2 \in \mathbb{R}^2$. Then for any $s \in [0, t)$,

$$
 |X(s, y_1) - X(s, y_2)| \leq \int_s^t <u(\cdot, \cdot) >_{LL} m(|X(\tau, y_1) - X(\tau, y_2)|)d\tau.
$$

Using a similar argument as that in the proof of (1.23), we have that $X(s, y_1) = X(s, y_2)$ for all $s \in [0, t]$.
Next we show that \( X(t, \cdot)|_{\Omega} \) is an open mapping. Let \( A \) be an open subset of \( \Omega \), \( y_1 \in A \) and \( B_{r_2}(y_1) \subset A \), \( 0 \leq s < t \), \( z_1 = X(t, y_1) \). From (1.22), using a similar argument as that in the proof of (4.6), we have
\[
|X(s, y_1) - X(s; z, t)| \leq \exp(1 - e^{-C(1 + t)})|z_1 - z|^{\exp(-C(1 + t))}.
\]
Thus, there is a sufficient small constant \( r_2 \) such that, if \( z \in B_{r_2}(z_1) \), then
\[
|y_1 - X(0; z, t)| < r_1.
\]
(4.7)
Thus \( X(0; z, t) \in B_{r_2}(y_1) \subset A \) if \( z \in B_{r_2}(z_1) \). From the uniqueness of the integral curves of \( u \), we obtain that \( z = X(t, X(0; z, t)) \in X(t, \cdot) A \). Therefore, \( B_{r_2}(z_1) \subset X(t, \cdot) A \), and \( X(t, \cdot)|_{\Omega} \) is an open mapping.

(3). Using a similar argument as that in the proof of (4.6), we have
\[
|X(t_2, y_2) - X(t_2, y_1)| \leq \exp(1 - e^{-\int_{t_1}^{t_2} gds})|X(t_1, y_2) - X(t_1, y_1)|^{\exp(-\int_{t_1}^{t_2} gds)} \leq \exp(1 - e^{-C(1 + T)})|X(t_1, y_2) - X(t_1, y_1)|^{\exp(-C(1 + T))},
\]
which proves part (3).

(4). Part (4) of Theorem 1.2 is an immediate consequence of part (3).

To prove Theorem 1.3 we need the following lemma.

Lemma 4.1. Given \( x \in X(t, \cdot) V \), say \( x = X(t, y) \) with \( y \in V \), there is a sequence \( \delta_j \to 0 \), which may depend on \( x \), such that \( X^{\delta_j}(t, \cdot)^{-1}(x) \) tends to \( y \) as \( \delta_j \to 0 \).

Proof. Using the argument as that in the proof of Part (1) of Theorem 1.2 we have that integral curves \( X^{\delta}(s; x, t) \) of the approximate velocity field \( u^{\delta} \), defined by
\[
X^{\delta}(s; x, t) = x - \int_s^t u^{\delta}(X^{\delta}(\tau; x, t), \tau) d\tau,
\]
are Hölder continuous in \( s \in [0, t] \), uniformly with respect to \( \delta \). Therefore, there is a sequence \( \delta_j \to 0 \) and a map \( \bar{X} \in C([0, t]; \mathbb{R}^2) \) such that \( X^{\delta_j}(\cdot; x, t) \) converges uniformly to \( \bar{X}(\cdot) \), which satisfies
\[
\bar{X}(s) = x - \int_s^t u(\bar{X}(\tau), \tau) d\tau, \quad 0 \leq s \leq t.
\]
From the uniqueness of integral curves proved in Part (1) of Theorem 1.2 we have that \( \bar{X}(t) = X(t, y) \). Taking \( s = 0 \), from (4.3) and (1.3), we get that \( y^{\delta_j} = X^{\delta_j}(t, \cdot)^{-1}(x) \) converges to \( y \) as \( \delta_j \) tends to zero. 

Proof of Theorem 1.3

Applying a standard maximum principle to the mass equation, using a similar argument as that in the proof of Proposition 2.7 we have
\[
\rho^{\delta}(X^{\delta}(t, y), t) \geq \underline{\rho},
\]
for any \( y \in V \), and all \( \delta \) sufficiently small. Let \( x = X(t, y) \in V^t \) and \( y^{\delta} = X^{\delta}(t, \cdot)^{-1}(x) \). From Lemma 4.1 we have that there is a sequence \( \delta_j \) tending to zero such that \( y^{\delta_j} \) tends to \( y \in V \). Then, for all sufficient small \( \delta_j \), we get
\[
y^{\delta_j} \in V \quad \text{and} \quad \rho^{\delta_j}(x, t) = \rho^{\delta_j}(X^{\delta_j}(t, y^{\delta_j}), t) \geq \underline{\rho}.
\]
From the convergence \( \rho^{\delta} \to \rho \) (which holds for a.a. \( x \)), we obtain that
\[
\rho(x, t) = \rho(X(t, y), t) \geq \underline{\rho},
\]
for all \( y \in V \). From Part (2) of Theorem 1.2 we can finish the proof of this theorem.

Proof of Theorem 1.4

For any \( y \in U \), there is a sufficient small \( \delta_0 \) such that
\[
\text{dist}(y, \partial U) \geq 2\delta_0.
\]
Let $U_{\delta_0} = \{ x \in U \mid \text{dist}(x, \partial U) \geq \delta_0 \}$. Then, we have
\[ \rho_{0|U_{\delta_0}} = \delta, \forall \delta \leq \delta_0. \]

From (2.20) and (2.21), we have
\[
\Lambda(\rho^\delta(X^\delta(t, z), t)) \leq \Lambda(\rho_0^\delta(z)) - \int_0^t P(\rho^\delta(X^\delta(s, z), s))ds + tP(\bar{\rho}) + \int_0^t \| F^\delta(\cdot, s) \|_{L^\infty}ds
\]
\[
\leq \Lambda(\delta) + C(T),
\]
for all $t \in [0, T]$, $\delta \leq \delta_0$ and all $z \in U_{\delta_0}$. Since $\Lambda(C\delta) - \Lambda(\delta) \geq 2\mu(\ln(C\delta) - \ln \delta) = 2\mu \ln C$, then we have
\[
\Lambda(\rho^\delta(X^\delta(t, z), t)) \leq \Lambda(\delta) + \Lambda(C(T)\delta) - \Lambda(\delta) = \Lambda(C(T)\delta),
\]
and
\[
\rho^\delta(X^\delta(t, z), t) \leq C(T)\delta \leq C(T)\delta_0,
\]
for all $t \in [0, T]$, $\delta \leq \delta_0$ and all $z \in U_{\delta_0}$. Let $x = X(t, y) \in U^t$ and $y^\delta = X^\delta(t, \cdot)^{-1}(x)$. From Lemma 4.1, we have that there is a sequence $\delta_j$ tending to zero such that $y^\delta$ tends to $y \in U_{\delta_0}$. Then, for all sufficient small $\delta_j$, we get
\[
y^\delta \in U_{\delta_0} \text{ and } \rho^\delta(x, t) = \rho^\delta(X^\delta(t, y^\delta), t) \leq C(T)\delta_0,
\]
for all $t \in [0, T]$. From the convergence $\rho^\delta \to \rho$ (which holds for a.a. $x$), we obtain that
\[
\rho(x, t) = \rho(X(t, y), t) \leq C(T)\delta_0,
\]
for all $t \in [0, T]$. Letting $\delta_0 \to 0$, we get that $\rho(X(t, y), t) = 0$ for all $t \in [0, T]$. From Part (2) of Theorem 1.2, we obtain $\rho(\cdot, t)|_{U^t} = 0$, $t \in [0, T]$.

From (2.22), (2.26) and (2.29), we have
\[
\int_1^\infty \int (|\rho^\delta - \bar{\rho}|^4 + |F^\delta|^4)dxdt \leq C.
\]

From the convergence of $\{\rho^\delta\}$ and $\{F^\delta\}$, we get
\[
\int_1^\infty \int (|\rho - \bar{\rho}|^4 + |F|^4)dxdt \leq C. \tag{4.8}
\]

From (2.25), we have
\[
\int |\rho^\delta - \bar{\rho}|^4(x, t)dx \leq \int |\rho^\delta - \bar{\rho}|^4(x, s)dx + C \int_N^{N+1} \int |F^\delta|^4dxdt,
\]
where $s, t \in [N, N + 1]$, $N > 1$. Integrating it with $s$ in $[N, N + 1]$, we obtain
\[
\sup_{t \in [N, N+1]} \int |\rho^\delta - \bar{\rho}|^4(x, t)dx \leq C \int_N^{N+1} \int (|\rho^\delta - \bar{\rho}|^4 + |F^\delta|^4)dxdt.
\]

From the convergence of $\{\rho^\delta\}$ and $\{F^\delta\}$, we get
\[
\sup_{t \in [N, N+1]} \int |\rho - \bar{\rho}|^4(x, t)dx \leq C \int_N^{N+1} \int (|\rho - \bar{\rho}|^4 + |F|^4)dxdt.
\]

Letting $N \to \infty$, using (4.8), we can easily obtain (1.24)–(1.25). \hfill \Box
5 Propagation of Singularities

Before proving Theorem 1.5, using the similar method in [6], we give the following three lemmas.

Lemma 5.1. Given \( x_0 \in \mathbb{R}^2 \) and \( R > 0 \), there are positive constants \( \delta_0 \) and \( r_0 \), and a subsequence \( \delta \rightarrow \delta_0 \), such that \( X^\delta(t, \cdot)^{-1}B_r(x_0) \subset B_R(x_0) \) for all \( \delta \leq \delta_0 \).

Proof. By Lemma 4.1 there is a sequence \( \delta \rightarrow 0 \) such that \( y_0^\delta := X^\delta(t, \cdot)^{-1}(x_0^\delta) \) tends to \( x_0 \) as \( \delta \rightarrow 0 \). Therefore, it suffices to show that there is a positive constant \( r_0 \) such that \( |X^\delta(t, \cdot)^{-1}(x) - y_0^\delta| < R \) for sufficiently small \( \delta \) and for all \( x \in B_{r_0}(x_0^\delta) \).

Letting \( y^\delta = X^\delta(t, \cdot)^{-1}(x) \), from (4.1), we have

\[
X^\delta(s; x, t) - X^\delta(s; x_0^\delta, t) = x - \int_s^t u^\delta(X^\delta(\tau; x, t), \tau)d\tau - x_0^\delta + \int_s^t u^\delta(X^\delta(\tau; x_0^\delta, t), \tau)d\tau,
\]

for any \( s \in [0, t) \). Using a similar argument as that in the proof (4.7), we have

\[
|\ y^\delta - y_0^\delta | \leq \exp(1 - e^{-C(1+t)})|x_0^\delta - x|\exp(-C(1+t)) < R
\]

of sufficiently small radius \( r_0 \). Then, we finish the proof.

Lemma 5.2. Let \( \delta \rightarrow 0 \) be the sequence of Lemma 5.1, then for all \( r_0 > 0 \), \( X^\delta(t, \cdot)^{-1} \rightarrow X(t, \cdot)^{-1} \) uniformly on \( B_{r_0}(x_0^\delta) \).

Proof. Let \( y^\delta := X^\delta(t, \cdot)^{-1}(x) \) and \( y := X(t, \cdot)^{-1}(x) \) for \( x \in B_{r_0}(x_0^\delta) \). For \( 0 \leq s \leq t \), we have

\[
|X^\delta(s, y^\delta) - X(s, y)| = \left| x - \int_s^t u^\delta(X^\delta(\tau, y^\delta), \tau)d\tau - x + \int_s^t u(X(\tau, y), \tau)d\tau \right|
\]

\[
\leq \int_s^t |u^\delta(X^\delta(\tau, y^\delta), \tau) - u^\delta(X(\tau, y), \tau)|d\tau + \int_s^t |u(X(\tau, y), \tau) - u^\delta(X(\tau, y), \tau)|d\tau
\]

\[
\leq \int_s^t \eta^\delta(\tau)\eta(|X^\delta(\tau, y^\delta) - X(\tau, y)|)d\tau + \int_s^t |u(X(\tau, y), \tau) - u^\delta(X(\tau, y), \tau)|d\tau.
\]

Using a similar argument as that in the proof (4.6), we obtain

\[
|y^\delta - y| = |X^\delta(0, y^\delta) - X(0, y)| \leq \exp(1 - e^{-C(1+T)}) \int_s^t |u(X(\tau, y), \tau) - u^\delta(X(\tau, y), \tau)|d\tau\exp(-C(1+T)).
\]

From (4.4) and the uniform convergence of \( u^\delta \) to \( u \) on compact sets in \( \mathbb{R}^2 \times (0, \infty) \), we can get that \( y^\delta(x) \rightarrow y(x) \) uniformly on \( B_{r_0}(x_0^\delta) \).

Lemma 5.3. Given \( r_0 \) sufficiently small and given \( t > 0 \), there is a nondecreasing function \( \eta : [0, \infty) \rightarrow \mathbb{R} \) satisfying \( \lim_{r \rightarrow 0} \eta(r) = 0 \) such that, for \( \delta \) as in Lemma 5.2 sufficient small,

\[
|X^\delta(t_2, y_2) - X^\delta(t_2, y_1)| \leq \eta(|X^\delta(t_1, y_2) - X^\delta(t_1, y_1)|),
\]

where \( t_1, t_2 \in [0, t] \) and \( X^\delta(t, y_1), X^\delta(t, y_2) \in B_{r_0}(x_0^\delta) \).

Proof. We have that, for \( 0 \leq t_1, t_2 \leq t \),

\[
|X^\delta(t_2, y_2) - X^\delta(t_2, y_1)| \leq |X^\delta(t_1, y_2) - X^\delta(t_1, y_1)| + \int_{t_1}^{t_2} |u^\delta(X^\delta(s, y_2^\delta), s) - u^\delta(X^\delta(s, y_2^\delta), s)|ds
\]

\[
\leq |X^\delta(t_1, y_2) - X^\delta(t_1, y_1)| + \int_0^t \eta^\delta(s)\eta(|X^\delta(s, y_2^\delta) - X^\delta(s, y_1^\delta)|)ds.
\]
Using a similar argument as that in the proof of Lemma 4.6, we obtain
\[ |X^\delta(t_2, y_2) - X^\delta(t_1, y_1)| \leq \exp(1 - e^{-C(1+t)})|X^\delta(t_1, y_2) - X^\delta(t_1, y_1)|\exp(-C(1+t)). \]

This finishes the proof of this lemma. \(\square\)

**Proof of Theorem 1.5.**

First, we show that
\[ \text{osc}(\rho; x_0^t, E^t) \leq \text{osc}(\rho; x_0^t, (E \cap B)^t) = 0, \quad (5.1) \]
for some open ball \( B \) centered at \( x_0 \), where \((E \cap B)^t := X(t, \cdot)(E \cap B)\). In deed, since the map \( X(t, \cdot)^{-1} : V^t \rightarrow V \) is continuous by Theorem 1.3 (2), there is a positive constant \( r_B \) such that \( B_{r_B}(x_0^t) \subset B^t \) for \( 0 < r_B \leq r_B \). This implies that \( E^t \cap B_{r_B}(x_0^t) \subset (E \cap B)^t \), so that, for any \( R > 0 \) and \( r_B \leq \min\{R, r_B\} \),
\[ (\text{esssup} - \text{essinf})\rho(\cdot, t)|_{E^t \cap B_{r_B}(x_0^t)} \leq (\text{esssup} - \text{essinf})\rho(\cdot, t)|_{(E \cap B)^t \cap B_{r_B}(x_0^t)}. \]

Letting first \( r_B \rightarrow 0 \) and then \( R \rightarrow 0 \), we can obtain (5.1).

**Case 1.** If \( \lim_{R \to 0} \text{essinf}\rho_0|_{(E \cap B)^t(x_0)} = 0 \), then for any \( \delta_0 > 0 \), there exists \( r_0 > 0 \) such that \( \rho_0|_{(E \cap B_{r_0}(x_0)} \leq \delta_0 \). Using a similar argument as that in the proof of Theorem 1.4, we have
\[ \rho_0|_{(E \cap B_{r_0}(x_0)} \leq C(T) \delta_0, \]
and
\[ \text{osc}(\rho(\cdot, t); x_0^t, (E \cap B_{r_0}(x_0))^t) \leq C(T) \delta_0, \quad t \in (0, T]. \]

From (5.1), we have
\[ \text{osc}(\rho(\cdot, t); x_0^t, E^t) \leq C(T) \delta_0, \quad 0 < t \leq T. \]

Letting \( \delta_0 \rightarrow 0 \), we get
\[ \text{osc}(\rho(\cdot, t); x_0^t, E^t) = 0. \]

**Case 2.** If \( \lim_{R \to 0} \text{essinf}\rho_0|_{(E \cap B)^t(x_0)} > 0 \), then there exist \( r_0 > 0 \) and \( \rho > 0 \) such that \( \rho_0|_{(E \cap B_{r_0}(x_0)} \geq \rho \).

Let \( B \) be an open ball centered at \( x_0 \) such that \( \overline{B} \subset B_{r_0}(x_0) \). By Theorem 1.2 (2) and Lemma 5.1, we can choose a positive constant \( r_1 \in (0, r_0) \) such that, if \( y_j = X(t, y_j) = X^\delta(t, y_j^t) \in B_{r_1}(x_0^t) \cap (E \cap B)^t \),
\[ j = 1, 2, \text{ then } y_1, y_2 \in B \cap E. \]
From the mass equation (1.1) for \((\rho^\delta, u^\delta)\), we obtain
\[ \frac{d}{dt}\Lambda(\rho^\delta(X^\delta(t; \cdot, 0), 0))^2_{y_1} = -P(\rho^\delta(X^\delta(t; \cdot, 0), 0))^2_{y_1} - F^\delta(X^\delta(t; \cdot, 0), 0)^2_{y_1} \]
\[ = a(t)\Lambda(\rho^\delta(X^\delta(t; \cdot, 0), 0))^2_{y_1} - F^\delta(X^\delta(t; \cdot, 0), 0)^2_{y_1}, \]
where \( a(t) = \frac{P(\rho^\delta(X^\delta(t; \cdot, 0), 0))^2_{y_1}}{\Lambda(\rho^\delta(X^\delta(t; \cdot, 0), 0))^2_{y_1}}. \)
From (1.13) and Theorem 4.3, we have \( |a(t)| \leq C. \) Using Gronwall's inequality, we get
\[ |\Lambda(\rho^\delta(x_2, t)) - \Lambda(\rho^\delta(x_1, t))| \leq C(T)|\Lambda(\rho^\delta(y_2)) - \Lambda(\rho^\delta(y_1))| + C(T)\int_0^t |F^\delta(X^\delta(s, y_2^t), s) - F^\delta(X^\delta(s, y_1^t), s)|ds \]
\[ \leq C(T)|\Lambda(\rho^\delta(y_2)) - \Lambda(\rho^\delta(y_1))| + C(T)\sup_{0 \leq s \leq t} |X^\delta(s, y_2^t) - X^\delta(s, y_1^t)| + \int_0^t g_{F^\delta}ds, \]
where \( g_{F^\delta}(s) := F^\delta(., s) > \int_0^\infty F^\delta(\cdot, s). \) Using a similar argument as that in Proposition 2.6, we have
\[ \int_0^T g_{F^\delta}ds \leq C(T). \quad (5.3) \]
From Lemma 5.3, we obtain
\[ \sup_{0 \leq s \leq t} |X^\delta(s, y_2^t) - X^\delta(s, y_1^t)|_{\mathbb{R}^2} \leq (\eta(|x_2 - x_1|))_{\mathbb{R}^2} \leq (\eta(2r_1))_{\mathbb{R}^2}. \quad (5.4) \]
Since \( y_j \in E \cap B \), there is a \( r_2 > 0 \) such that \( B_{r_2}(y_j) \subset E \cap B \). By Lemma 5.4, \( y_j^\delta \in B_{r_2}(y_j) \) for \( \delta \) sufficiently small. Thus, if \( \delta < \frac{r_2}{2} \), then \( |y - y_j| \leq |y - y_j^\delta| + |y_j^\delta - y_j| \leq \delta + \frac{r_2}{2} < r_2 \) for all \( y \in B_\delta(y_j^\delta) \); that is \( B_\delta(y_j^\delta) \subset B_{r_2}(y_j) \subset E \cap B \). Also, by Lemmas 5.2,5.3 for \( y \in B_\delta(y_j^\delta) \),

\[
|y - x_0| \leq |y - y_j^\delta| + |y_j^\delta - y_j| + |y_j - x_0| \leq \delta + |y_j - y_j^\delta| + \eta(r_1) \leq 2\eta(r_1),
\]

that is, \( B_\delta(y_j^\delta) \subset B_{2\eta(r_1)}(x_0) \) for \( \delta \) sufficiently small. Thus, \( B_\delta(y_j^\delta) \subset (E \cap B) \cap B_{2\eta(r_1)}(x_0) \). Then, since \( \rho_0^\delta(y_j^\delta) = \int_{B_\delta(y_j^\delta)} j_\delta(y_j^\delta - y_j) \rho_0^\delta(y)dy + \delta \), we have

\[
\text{essinf}_0 \rho_0|_{(E \cap B) \cap B_{2\eta(r_1)}(x_0)} \leq \rho_0^\delta(y_j^\delta) - \delta \leq \text{esssup}_0 \rho_0|_{(E \cap B) \cap B_{2\eta(r_1)}(x_0)}.
\]

From 5.2–5.5, we obtain

\[
|\Lambda(\rho^\delta(x_2,t)) - \Lambda(\rho^\delta(x_1,t))| \leq C(T)(\text{esssup} - \text{essinf})\rho_0|_{(E \cap B) \cap B_{2\eta(r_1)}(x_0)} + C(T)\eta(r_1)^{\frac{1}{p+1}},
\]

for \( x_1, x_2 \in (E \cap B)^t \cap B_{r_1}(x_0^\delta) \), sufficiently small \( r_1 > 0 \) and sufficiently small \( \delta > 0 \). Taking the limit as \( \delta \to 0 \), we get

\[
|\Lambda(\rho(x_2,t)) - \Lambda(\rho(x_1,t))| \leq C(T)(\text{esssup} - \text{essinf})\rho_0|_{(E \cap B) \cap B_{2\eta(r_1)}(x_0)} + C(T)\eta(r_1)^{\frac{1}{p+1}},
\]

for \( x_1, x_2 \in (E \cap B)^t \cap B_{r_1}(x_0^\delta) \) and sufficiently small \( r_1 > 0 \). Then, we have

\[
(\text{esssup} - \text{essinf})\rho|_{(E \cap B)^t \cap B_{r_1}(x_0)} \leq C(T)(\text{esssup} - \text{essinf})\rho_0|_{(E \cap B) \cap B_{2\eta(r_1)}(x_0)} + C(T)\eta(r_1)^{\frac{1}{p+1}},
\]

for sufficiently small \( r_1 > 0 \). Letting \( r_1 \to 0 \), using the condition \( \text{osc}(\rho_0; x_0, E \cap B) = 0 \), we obtain \( \text{osc}(\rho(\cdot,t); x_0^\delta(\cdot, E \cap B)^t) = 0 \). From 5.1, we complete the proof of Theorem 1.5.

Proof of Theorem 1.6.

It follows immediately from Theorem 1.5 that, under the conditions of Theorem 1.6 (a), \( \rho(\cdot,t) \) has a one-sided limit with respect to \( \mathcal{M}^t := X(t,\cdot) \mathcal{M} \) at the point \( X(t,x_0) \). Then, since \( F = (\lambda + \mu)\text{div} - P(\rho) + P(\bar{\rho}) \) is Hölder continuous for \( t > 0 \) by 1.18, \( \text{div}u(\cdot,t) \) has a one-sided limit with respect to \( \mathcal{M}^t := X(t,\cdot) \mathcal{M} \) at the point \( X(t,x_0) \). If these limits exist from both sides of \( \mathcal{M}^t \) at \( x_0 \), using the Hölder continuity of \( F \), then the Rankine-Hugoniot condition 1.20 holds at the point \( X(t,x_0) \).

To prove the regularity assertions in Theorem 1.6 (a), we consider the following two cases.

Case 1. If \( \rho_0(x_0^+ = 0) \), then using a similar argument as that in the proof of Theorem 1.4, we have \( \rho(X(t,x_0^+) = 0 \) for each \( t > 0 \). Thus, the maps \( t \mapsto \rho(X(t,x_0^+) \) and \( t \mapsto \rho(X(t,x_0^+) \) are in \( C^1([0,\infty)) \).

Case 2. If \( \rho_0(x_0^+) > 0 \), then there exist \( \rho > 0 \) and \( r_0 > 0 \) such that \( \rho_0|_{E_+ \cap B_{r_0}(x_0)} \geq \rho \). Denote \( A^\delta := \Lambda(\rho^\delta) \) and \( A := \Lambda(\rho) \). From 1.21, we have

\[
A^\delta + u^\delta \cdot \nabla A^\delta = -F^\delta - P(\rho^\delta) + P(\bar{\rho}).
\]

We will choose a sequence of smooth test functions \( \{\phi^{\delta,h}\}_{\delta,h>0} \) satisfying the equation \( \phi^{\delta,h}_t + \text{div}(\phi^{\delta,h}u^\delta) = 0 \), so that

\[
\int_0^t \phi^{\delta,h}_{x_0} dx \bigg|_0^t = -\int_0^t \int \phi^{\delta,h}_{x_0} (F^\delta + P(\rho^\delta) - P(\bar{\rho})) dx ds.
\]

To construct \( \phi^{\delta,h} \), we let \( \{x_h\}_{h>0} \) be a sequence in \( E_+ \) and \( \{r_h\} \) a sequence of positive numbers such that \( x_h \to x_0 \) and \( r_h \to 0 \) as \( h \to 0 \), and \( B_{2r_h}(x_h) \subset E_+ \cap B_{r_0}(x_0) \). Then, define \( \phi^{\delta,h} \) the solution of the equation

\[
\begin{cases}
\phi^{\delta,h}_t + \text{div}(\phi^{\delta,h}u^\delta) = 0, \\
\phi^{\delta,h}|_{t=0} = \phi^{h}_0,
\end{cases}
\]

where \( \phi^h_0 \) is a smooth function with support in \( B_{r_0}(x_0), \int \phi^h_0 dx = 1 \) and \( 0 \leq \phi^h_0 \leq C^h \). It follows that \( \phi^{\delta,h} \) has support in \( X^\delta(t,\cdot)B_{r_0}(x_0), \int \phi^{\delta,h}(x,t) dx = 1 \) for \( t > 0 \), and \( 0 \leq \phi^{\delta,h} \leq C^h(T) \) for \( 0 \leq t \leq T \). This last assertion is a consequence of 2.52.
In the following, we will take limits as $\delta$ and $h$ go to zero in (5.0). Form Lemma 5.2, we have $X^\delta$ converges uniformly to $X$ on $[0, t] \times B_{r_0}(x_0)$ as $\delta \to 0$. Then, we can easily prove that, for each $h > 0$, there is a $\delta_0(h) > 0$ such that

\[ X^\delta(s, \cdot)B_{r_0}(x_h) \subset X(s, \cdot)B_{r_0}(x_h), \quad 0 < \delta < \delta_0(h). \]  

(5.7)

We now obtain uniform bounds of the three terms in (5.0). For fixed $t > 0$, we get

\[
\int \phi^{\delta, h} \Lambda^\delta \, dx = \int \phi^{\delta, h}(\Lambda^\delta - \Lambda) \, dx + \int \phi^{\delta, h} \Lambda \, dx := I + II.
\]

From the convergence of $\rho^\delta$ and (1.13), we obtain that $I \to 0$ as $\delta \to 0$, and $I$ is bounded by $C^h(T)$. Also, from (5.7), we have

\[
\text{essinf} \Lambda(\cdot, t)|_{X(t, \cdot)B_{2r_0}(x_h)} \leq II \leq \text{esssup} \Lambda(\cdot, t)|_{X(t, \cdot)B_{2r_0}(x_h)},
\]

for sufficiently small $\delta$. Thus, there exists $\delta_0(h) > 0$

\[
\text{essinf} \Lambda(\cdot, t)|_{X(t, \cdot)B_{2r_0}(x_h)} - h \leq \int \phi^{\delta, h} \Lambda^\delta \, dx \leq \text{esssup} \Lambda(\cdot, t)|_{X(t, \cdot)B_{2r_0}(x_h)} + h,
\]

(5.8)

for $\delta \leq \delta_0(h)$.

Let $\Lambda(x_0^+, t) := \Lambda(\rho(x_0^+, t))$. Then,

\[
\Lambda(x_0^+, t) = \lim_{r' \to 0} \text{esssup} \Lambda(\cdot, t)|_{B_{r'}(x_0^+) \cap E^+} = \lim_{r' \to 0} \text{essinf} \Lambda(\cdot, t)|_{B_{r'}(x_0^+) \cap E^+}.
\]

For each $r' > 0$, we can choose $h_{r'} > 0$ such that $X(t, \cdot)B_{2r_0}(x_h) \subset B_{r'}(x_0^+) \cap E^+_t$ for all $h \leq h_{r'}$. Thus, for each $r' > 0$ and all $h \leq h_{r'}$,

\[
\text{essinf} \Lambda(\cdot, t)|_{B_{r'}(x_0^+) \cap E^+_t} \leq \text{essinf} \Lambda(\cdot, t)|_{X(t, \cdot)B_{2r_0}(x_h)} \leq \text{esssup} \Lambda(\cdot, t)|_{B_{r'}(x_0^+) \cap E^+_t}.
\]

Taking first the liminf and limsup as $h \to 0$ and then the limit as $r' \to 0$, we obtain that

\[
\lim_{h \to 0} \text{essinf} \Lambda(\cdot, t)|_{X(t, \cdot)B_{2r_0}(x_h)} = \Lambda(x_0^+, t).
\]

Similarly, we have

\[
\lim_{h \to 0} \text{esssup} \Lambda(\cdot, t)|_{X(t, \cdot)B_{2r_0}(x_h)} = \Lambda(x_0^+, t).
\]

From (5.8), we have

\[
\lim_{h \to 0} \int_{0}^{x_0^+} \int (\phi^{\delta, h} \Lambda^\delta)(x, t) \, dx = \Lambda(x_0^+, t).
\]

(5.9)

Similarly, we obtain

\[
\lim_{h \to 0} \int_{0}^{x_0^+} \int (\phi^{\delta, h} \Lambda^\delta)(x, 0) \, dx = \Lambda(x_0^+, 0),
\]

(5.10)

\[
\lim_{h \to 0} \int_{0}^{t} \int (\phi^{\delta, h} F^\delta)(x, s) \, dxds = \int_{0}^{t} F(x_0^+, s) \, ds,
\]

(5.11)

\[
\lim_{h \to 0} \int_{0}^{t} \int \phi^{\delta, h} \{ P(\rho^\delta) - P(\tilde{\rho}) \} \, dxds = \int_{0}^{t} \{ P(\rho(x_0^+, s)) - P(\tilde{\rho}) \} \, ds.
\]

(5.12)

From (5.6) and (5.9)–(5.12), we have

\[
\Lambda(x_0^+, t) - \Lambda(x_0^+, 0) = -\int_{0}^{t} \{ F(x_0^+, s) + P(\rho(x_0^+, s)) - P(\tilde{\rho}) \} \, ds.
\]

(5.13)

Since $F$ is locally Hölder continuous in $\mathbb{R}^2 \times (0, \infty)$, standard ODE theory implies that the map $t \mapsto \Lambda(x_0^+, t)$ is in $C([0, \infty)) \cap C^1((0, \infty))$. From (1.13), (1.16), (2.45) and (2.51), we have

\[
\int_{0}^{t} \| F(\cdot, s) \|_{L^\infty} \, ds \leq C \int_{0}^{t} (\sigma^{-2-\frac{2}{q}})^{\frac{1}{r-2}} \, ds \leq C t^{\frac{3q}{4 + 2q}}.
\]
Thus, map \( t \mapsto \Lambda(x_0^t, t) \) is in \( C^{3/2}_T((0, \infty)) \cap C^{1}((0, \infty)) \). From (1.13) and Theorem 1.3, the maps \( t \mapsto \rho(X(t, x_0^t), t) \) and \( t \mapsto P(\rho(X(t, x_0^t), t)) \) are in \( C^{3/2}_T((0, \infty)) \cap C^{1}((0, \infty)) \).

Since \( F \) is locally Hölder continuous in \( \mathbb{R}^2 \times (0, \infty) \), then the map \( t \mapsto \text{div} u(X(t, x_0^t), t) \) is locally Hölder continuous on \( (0, \infty) \). These finish the proof of Theorem 1.6 (a).

Then, we will prove Theorem 1.6 (c) as follows. From (5.13), we have

\[
\rho \left( \begin{array}{c} t, x \\ 0 \end{array} \right) = \Omega(0) \Rightarrow (x_0^t) \quad \text{locally } H^\alpha \quad \text{in } \mathbb{R}^2.
\]

We set

\[
\rho \left( \begin{array}{c} t, x \\ 0 \end{array} \right) = \Omega(0) \Rightarrow (x_0^t) \quad \text{locally } H^\alpha \quad \text{in } \mathbb{R}^2.
\]

Using Gronwall’s inequality, we can finish the proof of Theorem 1.6. \( \square \)

6 Nonphysical solution

Proof of Theorem 1.7.

First, from (1.5), we have

\[
\partial_r (\partial_r v + \frac{1}{r} v) = 0, \quad \text{in } \mathcal{D}'(U),
\]

where \( U = \{(r, t) \in \mathbb{R}^+ \times \mathbb{R}^+: r \in (a(t), b(t))\} \). From the regularity (1.15), we have

\[
v(r, t) = \alpha(t)r + \frac{\beta(t)}{r}, \quad \text{in } \mathcal{D}'(U), \tag{6.1}
\]

where \( \alpha, \beta \in L^1_{loc}([0, \infty)) \).

Then, from (1.30) and the regularity of the solution in Theorem 1.1, we have

\[
\frac{d}{dt} E(t) = \left\{ vr \left( (\lambda + 2\mu) (v_r + \frac{v}{r} ) - P(\rho) \right) \right\} \bigg|_{r=a(t)-0},
\]

where

\[
E(t) = \int_0^{a(t)} \left( \frac{1}{2} \rho v^2 + \overline{G}(\rho) \right) r dr + \int_0^t \int_0^{a(s)} (\lambda + 2\mu)(v_r + \frac{v}{r})^2 r dr ds.
\]

From (1.13) and (1.26), we get

\[
[v] = 0, \quad \left[ (\lambda(\rho) + 2\mu)(v_r + \frac{v}{r}) - P(\rho) \right] = 0, \quad \text{at } (a(t), t),
\]

and since \( \rho(a(t) + 0, t) = 0 \), we obtain

\[
\frac{d}{dt} E(t) = a(t)v(a(t), t)(\lambda(0) + 2\mu)(v_r + \frac{v}{r})(a(t) + 0, t). \tag{6.2}
\]

From (6.1), we have

\[
(v_r + \frac{v}{r})(a(t) + 0, t) = 2 \frac{a(t)v(a(t), t) - b(t)v(b(t), t)}{a^2(t) - b^2(t)}. \tag{6.3}
\]

From (6.2)–(6.3), we can immediately obtain (1.33). \( \square \)

7 Non-global Existence of Regular Solutions

Proof of Theorem 1.8.

Since the support of the initial spherical symmetric density \( \rho_0 \) is compact, we can assume that

\[
\text{supp} \rho_0 = \Omega(0) = \{ x \in \mathbb{R}^2 | |x| \leq R_0 \},
\]

for some \( R_0 > 0 \). We set

\[
\Omega(t) = \{ x = X(t, \alpha) | \alpha \in \Omega(0) \}, \quad t \in [0, T],
\]
where $X(t, \alpha)$ is defined in Theorem 1.2. From the transport equation (1.1), one can easily show that $\text{supp}\rho(x, t) = \Omega(t) = \{x \in \mathbb{R}^2 | x \leq R(t)\}$. From (6.1), we have

$$v(r, t) = \alpha(t)r + \frac{\beta(t)}{r}, \text{ in } \mathcal{D}'(U),$$

where $u(x, t) = v(r, t)\frac{x}{r}$, $|x| = r$, $U = \{x \in \Omega^c(t), t \in [0, T]\}$. Since $u \in C^1([0, T]; H^k)$, we have $u(x, t) \equiv 0$ in $x \in \Omega^c(t)$, and $\Omega(t) = \Omega(0)$ for all $0 < t < T$.

Now, we introduce the following functional as in [17, 18]:

$$\Omega\left(c_{x, t}\right) = \Omega(0),$$

From (7.1), we have

$$\text{H}^\prime(t) = \int (x - (1 + t)u)^2 \rho dx + \frac{2}{\gamma - 1} (1 + t)^2 \int A\rho^\gamma dx$$

Using (1.29)–(1.30), the Cauchy-Schwarz inequality and Hölder’s inequality, we obtain

$$\text{H}^\prime(t) = \frac{4(2 - \gamma)(1 + t)}{\gamma - 1} \int_0^\infty A\rho^\gamma r dr + 4(1 + t) \int_0^\infty \lambda(v_r + \frac{v}{r}) r dr$$

$- 2(1 + t)^2 \int_0^\infty (\lambda + 2(\mu) + \frac{v}{r})^2 r dr$$

$\leq \frac{4(2 - \gamma)(1 + t)}{\gamma - 1} \int_0^\infty A\rho^\gamma r dr + 2 \int_0^\infty c\rho^\beta r dr$$

$\leq \frac{4(2 - \gamma)(1 + t)}{\gamma - 1} \int_0^\infty A\rho^\gamma r dr + \frac{2c\beta}{\gamma} \int_0^\infty \rho^\gamma r dr + \frac{2c(\gamma - \beta)}{\gamma} \Omega(0), (7.2)$

where $t \in [0, T]$. From (7.1)–(7.2), we get

$$\text{H}^\prime(t) \leq \frac{2(2 - \gamma)}{1 + t} \text{H}(t) + \frac{2c\beta(\gamma - 1)}{A\gamma(1 + t)^2} \text{H}(t) + \frac{2c(\gamma - \beta)}{\gamma} \Omega(0)$$

and

$$\text{H}(t) \leq (1 + t)^{1 - 2\gamma} e^{-\frac{2c(\gamma - \beta)}{A\gamma(1 + t)^2}} \left(\text{H}(0) + \frac{2c(\gamma - \beta)}{\gamma} \Omega(0) \int_0^t (1 + s)^{2\gamma - 4} e^{\frac{2c(\gamma - \beta)}{A\gamma(1 + s)^2}} ds\right)$$

$$\leq (1 + t)^{1 - 2\gamma} \text{H}(0) + \frac{2c(\gamma - \beta)}{\gamma} \Omega(0) e^{\frac{2c\beta(\gamma - 1)}{A\gamma}} F(t),$$

where $t \in [0, T]$ and

$$F(t) = \left\{\begin{array}{l}
\frac{1}{2\gamma - 3}(1 - (1 + t)^{3 - 2\gamma}), \quad 2\gamma - 3 \neq 0, \\
(1 + t)^{1 - 2\gamma} \ln(1 + t), \quad 2\gamma - 3 = 0.
\end{array}\right.$$
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