Sufficiency for nondifferentiable multiobjective programming problem

Xiaoyan Gao\textsuperscript{1,3} and Ruijuan Li\textsuperscript{2}

\textsuperscript{1}College of Science, Xi’an University of Science and Technology, Xi’an 710054, China
\textsuperscript{2}College of Mathematics & Information Science, Langfang Normal University, Hebei 065000, China
\textsuperscript{3}Email: 1666465514@qq.com

Abstract. This paper introduces a class of new generalized functions of the concept of invex of $(\sigma, \rho)-V$ – type I for nondifferentiable locally Lipschitz functions by using the tools of Clarke subdifferential. These functions are used to derive the sufficient optimality conditions for a class of nondifferentiable multiobjective nonlinear programming problems with inequality constraints where the objective and constraint functions are locally Lipschitz.

1. Introduction
The field of multiobjective programming, also called vector programming, has grown remarkably in different directions in the settings of sufficient optimality conditions since the 1980s. Several authors have extended the basic theoretical results in multiobjective programming. The optimality conditions and duality in multiobjective programming have not only used in many theoretical and computational developments in mathematical programming itself but also used in economics, control theory, business problems and other diverse fields. For example, we can see in [1–4]. In particular, Agarwal et al. [5] presented the optimality and duality results for multiobjective optimization problems involving locally Lipschitz functions and type I invexity. Kim and Bae [6] formulated nondifferentiable multiobjective programs involving the support functions of a compact convex set. Also, Kim and Lee [7] introduced the nonsmooth multiobjective programming problems involving locally Lipschitz functions and support functions.

On the other hand, the invexity theory, previously introduced for differentiable functions, was generalized to locally Lipschitz functions with gradients replaced by the Clarke generalized gradient. (see for examples [8–10]). In [11] Antczak and Stasiak extended the concept of $(\phi, \rho)-V$– invexity for nondifferentiable optimization problems to the case of mathematical programming problems with locally Lipschitz functions. In [12], based upon the F-convexity and $\rho$-convexity, the authors defined the $(\phi, \rho)-V$– Type I Functions to consider a class of nonsmooth multiobjective programming problems. In [13], Izhar Ahmad and Suliman Al-Homidan introduced the concepts of strong invexity of order $\sigma$ for a locally Lipschitz function and dealt with several sufficient optimality conditions for higher order minimizers via introduced classes of functions.

In this paper, motivated by the above work, the sufficient optimality conditions are obtained for a class of nondifferentiable multiobjective programming problem under the assumptions of invex of order $\sigma (B, \varphi)-V$ – type I.
2. Preliminaries and definitions

Let $\mathbb{R}^n$ be the $n-$dimensional Euclidean space and let $X$ be a nonempty open subset of $\mathbb{R}^n$. For $x = (x_1, x_2, \ldots, x_n)^T, y = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n$, we denote:

\[ x = y \iff x_i = y_i, \quad i = 1, 2, L, n; \]
\[ x \equiv y \iff x_i \equiv y_i, \quad i = 1, 2, L, n; \]
\[ x \leq y \iff x \equiv y \text{ and } x \neq y; \]
\[ x < y \iff x_i < y_i, \quad i = 1, 2, L, n; \]
\[ x \in \mathbb{R}^n_+ \iff x \equiv 0. \]

Definition 2.1 [14]. The function $f : X \rightarrow \mathbb{R}$ is said to be locally Lipschitz at $x \in X$, if there exist scalars $k > 0$ and $\varepsilon > 0$, such that

\[ \left| f(y) - f(z) \right| \leq k \|y - z\|, \text{ for all } y, z \in B(x, \varepsilon). \]  

Where $B(x, \varepsilon)$ is the open ball of radius $\varepsilon$ about $x$.

Definition 2.2 [14]. The generalized directional derivative of a locally Lipschitz function $f$ at $x$ in the direction $d$, denoted by $f^0(x; d)$, is as follows:

\[ f^0(x; d) = \lim_{\lambda \to 0^+} \frac{f(y + \lambda d) - f(y)}{\lambda}. \]  

Definition 2.3 [14]. The generalized gradient of $f : X \rightarrow \mathbb{R}$ at $x \in X$, denoted by $\partial f(x)$, is defined as follows:

\[ \partial f(x) = \{ \xi \in \mathbb{R}^n : f^0(x; d) \geq \langle \xi, d \rangle, \forall d \in \mathbb{R}^n \}. \]  

Where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^n$.

In this paper, we consider the following multiobjective programming problem:

\[
\text{Minimize } f(x) = (f_1(x), f_2(x), \ldots, f_n(x)), \quad \tag{MP}
\]
\[
s.t. \quad g_j(x) \equiv 0, \quad x \in X.
\]

Where $f_j : X \rightarrow \mathbb{R}$ ($i \in K = \{1, 2, L, k\}$) and $g_j : X \rightarrow \mathbb{R}$ ($j \in M = \{1, 2, L, m\}$) are locally Lipschitz functions and $X$ is a convex set in $\mathbb{R}^n$. Let $X_0 = \{x \mid g_j(x) \equiv 0, j \in M\}$ be the set of feasible solutions of (MP) and let $I = \{j \mid g_j(x) = 0, j \in M\}$ for any $x \in X_0$ be the index active constraint set.

Definition 2.4. A point $x \in X_0$ is a local strict minimizer of order $\sigma$ for (MP) with respect to a nonlinear function $\psi : X \times X \rightarrow \mathbb{R}^n$, if for a constant $\rho \in \text{int } \mathbb{R}^k_+$, there exists no $x \in B(\bar{x}, \varepsilon) \cap X_0$, such that

\[ f(x) < f(\bar{x}) + \rho \|\psi(x, \bar{x})\|^\sigma. \]  

Definition 2.5. A point $\bar{x} \in X_0$ is a strict minimizer of order $\sigma$ for (MP) with respect to a nonlinear function $\psi : X \times X \rightarrow \mathbb{R}^n$, if for a constant $\rho \in \text{int } \mathbb{R}^k_+$, there exists no $x \in X_0$, such that

\[ f(x) < f(\bar{x}) + \rho \|\psi(x, \bar{x})\|^\sigma. \]  

Definition 2.6. $(f, g)$ is said to be in convex order of $\sigma$ $(B, \varphi) - V$-type $I$ at $\bar{x} \in X$, if there exist $\eta : X \times X \rightarrow \mathbb{R}^n; b_i, b_j : X \times X \rightarrow R; \varphi_i, \varphi_j : R \rightarrow R; \alpha_i, \beta_j : X \times X \rightarrow \mathbb{R} \setminus \{0\}, i \in K, j \in M;
\rho, \tau_j \in \mathbb{R} \setminus \{0\}$, such that for all $x \in X$ the following inequalities hold:

\[ b_i(x, \bar{x})\varphi_i[f_i(x) - f_i(\bar{x})] \leq \langle \alpha_i(x, \bar{x})\zeta_i(x, \bar{x}), \eta(x, \bar{x}) \rangle + \rho \|\psi(x, \bar{x})\|^\sigma, \forall \zeta_i \in \partial f_i(x), i \in K, \]
\[ -b_j(x, \bar{x})\varphi_j[g_j(x)] \leq \langle \beta_j(x, \bar{x})\zeta_j(x, \bar{x}), \eta(x, \bar{x}) \rangle + \tau_j \|\psi(x, \bar{x})\|^\sigma, \forall \zeta_j \in \partial g_j(x), j \in M. \]
Definition 2.7. \((f, g)\) is said to be strongly (pseudo, quasi) invex of order \(\sigma\) \((B, \varphi) - V - \text{type} I\) at \(\bar{x} \in X\), if there exist \(\eta : X \times X \to \mathbb{R}^n\); \(b_0, b_i : X \times X \to \mathbb{R}^n\); \(\varphi_0, \varphi_i : \mathbb{R} \to \mathbb{R}; \rho_i, \tau_j \in \mathbb{R}\setminus\{0\}\) \(\alpha_i, \beta_j : X \times X \to \mathbb{R}, \{0\}, i \in K, j \in M\), such that for all \(x \in X\) the following inequalities hold:

\[
b_0(x, \bar{x})\varphi_0[f_j(x) - f_j(\bar{x})] + \rho_i\|\psi(x, \bar{x})\| \leq 0 \Rightarrow \langle \alpha_j(x, \bar{x}), \eta_j(x, \bar{x}) \rangle < 0, \forall \xi_j \in \partial f_j(x), i \in K, \]  
\[
-\beta_j(x, \bar{x})\varphi_i[g_j(x) - g_j(\bar{x})] + \tau_j\|\psi(x, \bar{x})\| \leq 0, \forall \zeta_j \in \partial g_j(x), j \in M. \]  

3. **Sufficient optimality conditions**

Theorem 3.1. Let \(\bar{x} \in X_o\) be a feasible solution for (MP) and suppose that

(I) There exists scalars \(\lambda_i \geq 0, i \in K\), \(\sum_{i=1}^{k} \lambda_i = 1\) and \(\mu_j \geq 0, j \in M\), such that

\[
0 \geq \sum_{i=1}^{k} \lambda_i \partial f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \partial g_j(\bar{x}), \mu_j g_j(\bar{x}) = 0, j \in M; \]

(II) \((f, g)\) is invex of order \(\sigma\) \((B, \varphi) - V - \text{type} I\) at \(\bar{x} \in X_o\);

(III) \(b_0(x, \bar{x}) = 1, b_i(x, \bar{x}) = 0; \varphi_i(a) = a, \varphi_i(0) = 0; \alpha_i(x, \bar{x}) = 1, i \in K\).

Then \(\bar{x}\) is a strict minimizer of order \(\sigma\) with respect to \(\psi\) for (MP).

Proof: Let \(J = \{j \mid g_j(\bar{x}) < 0, j \in M\}\). Therefore \(I \cup J = M\). Also \(\mu_j \geq 0, g_j(\bar{x}) \geq 0\) and \(\mu_j g_j(\bar{x}) = 0, j \in M\) implies \(\mu_j = 0, j \in J\) and \(\mu_j \geq 0, j \in I\).

The condition (I) implies that there exist \(\xi_i \in \partial f_i(\bar{x})\) and \(\zeta_j \in \partial g_j(\bar{x})(i \in K, j \in M\) satisfying

\[
0 = \sum_{i=1}^{k} \lambda_i \xi_i + \sum_{j=1}^{m} \mu_j \zeta_j. \]

That is

\[
0 = \sum_{i=1}^{k} \lambda_i \xi_i + \sum_{j=1}^{m} \mu_j \zeta_j. \]

Now suppose that \(\bar{x}\) is not a strict minimizer of order \(\sigma\) with respect to \(\psi\) for (MP). Then, for \(\rho_i > 0, i \in K\), there exists \(x \in X_o\) such that

\[
f_i(x) < f_i(\bar{x}) + \rho \|\psi(x, \bar{x})\|^\sigma, i \in K. \]

Using \(\lambda_i \geq 0\) and \(\sum_{i=1}^{k} \lambda_i = 1\), the above inequality implies

\[
\sum_{i=1}^{k} \lambda_i f_i(x) < \sum_{i=1}^{k} \lambda_i f_i(\bar{x}) + \sum_{i=1}^{k} \lambda_i \rho_i \|\psi(x, \bar{x})\|^\sigma. \]

The hypothesis (II) yields

\[
b_0(x, \bar{x})\varphi_0[f_i(x) - f_i(\bar{x})] \leq \langle \alpha_i(x, \bar{x}), \eta_i(x, \bar{x}) \rangle + \rho_i \|\psi(x, \bar{x})\|^\sigma, \forall \xi_i \in \partial f_i(x), i \in K \]
\[
-\beta_i(x, \bar{x})\varphi_i[g_i(x) - g_i(\bar{x})] + \tau_i \|\psi(x, \bar{x})\|^\sigma, \forall \zeta_i \in \partial g_i(x), j \in I. \]

Using \(b_0(x, \bar{x}) = 1, \varphi_i(a) = a\) and \(\alpha_i(x, \bar{x}) = 1\), the inequality (14) yields

\[
f_i(x) - f_i(\bar{x}) \leq \langle \xi_i, \eta_i(x, \bar{x}) \rangle + \rho \|\psi(x, \bar{x})\|^\sigma, \forall \xi_i \in \partial f_i(x), i \in K. \]

For \(\lambda_i \geq 0\), we have

\[
\sum_{i=1}^{k} \lambda_i f_i(x) - \sum_{i=1}^{k} \lambda_i f_i(\bar{x}) \leq \langle \sum_{i=1}^{k} \lambda_i \xi_i, \eta_i(x, \bar{x}) \rangle + \sum_{i=1}^{k} \lambda_i \rho_i \|\psi(x, \bar{x})\|^\sigma. \]
Using (III), the inequality (15) follows
\[ 0 \geq \beta_j(x, \bar{x}) \xi_j(x, \bar{x}) \eta(x, \bar{x}) + \tau_j \|\psi(x, \bar{x})\|_2. \quad \forall \xi_j \in \partial g_j(\bar{x}), j \in I. \]  
(18)

For \( \mu_j \geq 0, \beta_j(x, \bar{x}) > 0, j \in I \), we get
\[ \left( \sum_{j=1}^{m} \mu_j \xi_j(x, \bar{x}) \right) + \sum_{j=1}^{m} \beta_j(x, \bar{x}) \|\psi(x, \bar{x})\|_2 \geq 0. \]
(19)

By adding (17) and (19), with the equality (11), we have
\[ \sum_{i=1}^{k} \lambda_i f_i(x) - \sum_{j=1}^{m} \lambda_j g_j(x) \geq \sum_{j=1}^{m} \lambda j \mu_j g_j(x), \quad j \in M. \]  
(20)

Which contradicts (13). Then \( \bar{x} \) is a strict minimizer of order \( \sigma \) with respect to \( \psi \) for (MP).

Theorem 3.2. Let \( \bar{x} \in X_0 \) be a feasible solution for (MP) and suppose that

(1) There exists scalars \( \lambda_i \geq 0, i \in K \), \( \sum_{i=1}^{k} \lambda_i = 1 \) and \( \mu_j \geq 0, j \in M \), such that
\[ 0 \leq \sum_{i=1}^{k} \lambda_i \partial f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \partial g_j(\bar{x}), \quad \mu_j g_j(\bar{x}) = 0, j \in M; \]
(II) \((f, g_j)\) is strongly (pseudo, quasi) invex of order \( (B, \varphi) - V \) - type I at \( \bar{x} \in X_0 \);
(III) \( b_0(x, \bar{x}) > 0, b_1(x, \bar{x}) \equiv 0, a < 0 \Rightarrow \varphi_0(a) < 0, \varphi_0(0) \equiv 0. \)

Then \( \bar{x} \) is a strict minimizer of order \( \sigma \) with respect to \( \psi \) for (MP).

Proof: From \( \mu_j g_j(\bar{x}) = 0, \mu_j \geq 0, j \in M \). We have \( \mu_j \geq 0, j \in I \) and \( \mu_j \geq 0, j \in J = \{ j \mid g_j(\bar{x}) < 0, j \in M \}. \)

The condition (I) implies that there exist \( \xi_i \in \partial f_i(\bar{x}) \) and \( \xi_j \in \partial g_j(\bar{x}) (i \in K, j \in M) \) satisfying
\[ 0 = \sum_{i=1}^{k} \lambda_i \xi_i + \sum_{j=1}^{m} \mu_j \xi_j. \]  
(21)

That is
\[ 0 = \sum_{i=1}^{k} \lambda_i \xi_i + \sum_{j=1}^{m} \mu_j \xi_j. \]  
(22)

Now suppose that \( \bar{x} \) is not a strict minimizer of order \( \sigma \) with respect to \( \psi \) for (MP). Then, for \( \rho_i > 0, i \in K \), there exists \( x \in X_0 \) such that
\[ f_i(x) < f_i(\bar{x}) + \rho_i \|\psi(x, \bar{x})\|_2, \quad i \in K. \]  
(23)

That is
\[ f_i(x) - f_i(\bar{x}) - \rho_i \|\psi(x, \bar{x})\|_2 < 0, \quad i \in K. \]  
(24)

By the hypothesis (III), the above inequality follows
\[ b_0(x, \bar{x}) \varphi_0[f_i(x) - f_i(\bar{x}) - \rho_i \|\psi(x, \bar{x})\|_2] < 0, \quad i \in K. \]  
(25)

Using the hypothesis (II), which implies
\[ \alpha_i(x, \bar{x}) \xi_i(x, \bar{x}) < 0, \quad \forall \xi_i \in \partial f_i(\bar{x}), i \in K. \]  
(26)

That is
\[ \alpha_i(x, \bar{x}) \xi_i(x, \bar{x}) < 0, \quad \forall \xi_i \in \partial f_i(\bar{x}), i \in K. \]  
(27)

For \( \lambda_i \geq 0 \) and \( \sum_{i=1}^{k} \lambda_i = 1 \), we have
\[
\left( \sum_{i=1}^{n} \lambda_i \xi_i, \eta(x, \bar{x}) \right) < 0 .
\]  
(28)

By the hypothesis (III), we have

\[-b_j(x, \bar{x})q_j[g_j(\bar{x})] \equiv 0, j \in I .\]  
(29)

Using the hypothesis (II), the above inequality yields

\[\left( \beta_j(x, \bar{x}) \zeta_j, \eta(x, \bar{x}) \right) + \tau_j \| \psi(x, \bar{x}) \|^\sigma \equiv 0, \forall \zeta_j \in \partial g_j(\bar{x}), j \in I .\]  
(30)

For \( \beta_j(x, \bar{x}) > 0, \mu_j \geq 0, j \in I \), which implies

\[\left( \sum_{j=1}^{m} \lambda_j \xi_j, \eta(x, \bar{x}) \right) + \sum_{j=1}^{m} \frac{\mu_j \tau_j}{\beta_j(x, \bar{x})} \| \psi(x, \bar{x}) \|^\sigma \equiv 0 .\]  
(31)

On adding (28) and (31), we have

\[\left( \sum_{i=1}^{n} \lambda_i \xi_i + \sum_{j=1}^{m} \lambda_j \xi_j, \eta(x, \bar{x}) \right) + \sum_{j=1}^{m} \frac{\mu_j \tau_j}{\beta_j(x, \bar{x})} \| \psi(x, \bar{x}) \|^\sigma < 0 .\]  
(32)

The above inequality along with (22) gives

\[\sum_{j=1}^{m} \frac{\mu_j \tau_j}{\beta_j(x, \bar{x})} \| \psi(x, \bar{x}) \|^\sigma < 0 .\]  
(33)

Which is not possible. Then \( \bar{x} \) is a strict minimizer of order \( \sigma \) with respect to \( \psi \) for (MP).

4. Conclusions
Throughout this paper, we have consider a class of nondifferentiable multiobjective programming problem in which the involved functions are locally Lipschitz. A new concept of invex of order \( \sigma (B, \phi) – V – \) type I have been introduced. By the assumption of the invexity, some sufficient optimality conditions for the multiobjective programming problem have been achieved and proved.

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