Rocollements and Hochschild theory

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Abstract

Two constructions of recollements of derived categories of algebras are provided. Triangulated functors in recollements of derived categories of algebras and tensor product algebras are realized as derived functors of the same forms. These results are applied to observe the relations between recollements of derived categories of algebras and smoothness and Hochschild (co)homology of algebras.

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1 Introduction

Recollements of triangulated categories are “short exact sequences” of triangulated categories, which were introduced by Beilinson-Bernstein-Deligne [5] and play an important role in algebraic geometry [5], representation theory [10, 35], etc. Let $k$ be a field and $\otimes := \otimes_k$. Throughout the paper, all algebras are assumed to be associative $k$-algebras with identity, and all modules are right unitary modules unless stated otherwise. Here, we focus on recollements of derived categories of algebras, i.e., all triangulated categories in the recollements are derived categories of algebras, which are closed related to tilting theory [1, 23, 32], (co)localization theory [31, 28], some important homological invariants of algebras such as global dimension [11, 22], finitistic dimension [17], Hochschild homology and cyclic homology [22], and so on.

In this paper, on one hand, in the interior of recollement theory, we shall provide two constructions of recollements of derived categories of algebras
and realize triangulated functors in recollements of derived categories of algebras and tensor product algebras as derived functors of the same forms in the spirit of [37]. On the other hand, as applications, we shall observe the relations between recollements of derived categories of algebras and smoothness, i.e., finiteness of Hochschild dimension, and Hochschild (co)homology of algebras. Note that the relations between (perfect) recollements of derived categories of algebras and Hochschild homology even cyclic homology had been clarified already in [22]. The paper is organized as follows: In section 2, we shall introduce perfect recollements of derived categories of algebras which correspond to “derived triangular matrix (differential graded) algebras”, and give a criterion for the derived category of an algebra to admit a perfect recollement of derived categories of algebras. In section 3, we shall construct (perfect) recollements of derived categories of tensor product algebras and opposite algebras respectively from a (perfect) recollement of derived categories of algebras. Applying the constructions provided in section 3, we shall show in section 4 that, in a perfect recollement of derived categories of algebras, or a recollement of derived categories of Noetherian algebras, the middle algebra is smooth if and only if so are the algebras on both sides. As a corollary, a triangular matrix algebra is smooth if and only if so are the algebras on diagonal. In section 5, we shall realize triangulated functors in recollements of derived categories of algebras and tensor product algebras as derived functors of the same forms. As applications, from a recollement of derived categories of algebras, we shall obtain in section 6 a triangle on Hochschild complexes due to Keller [22] which can induce a long exact sequence on Hochschild homologies of algebras, and in section 7 three triangles on Hochschild cocomplexes which can induce three long exact sequences on Hochschild cohomologies of algebras. Note that these long exact sequences on Hochschild cohomologies have been widely studied for one-point extensions [16, 14], triangular matrix algebras [8, 30, 15, 9, 6], stratifying ideals [25], homological epimorphisms [36, 39], etc.

2 (Perfect) Recollements

2.1 Recollements

Recall the definition of a recollement of triangulated categories:

Definition 1. (Beilinson-Bernstein-Deligne [5]) Let $\mathcal{T}_1$, $\mathcal{T}$ and $\mathcal{T}_2$ be trian-
gulated categories. A recollement of \( \mathcal{T} \) relative to \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) is given by

\[
\begin{array}{ccc}
\mathcal{T}_1 & \xrightarrow{i^*} & \mathcal{T} & \xleftarrow{j_*} & \mathcal{T}_2 \\
i_* = i_1 & \quad & \mathcal{T} & \quad & j^* = j_* \\
i_1 & \quad & j_* & \quad & j_1 = j
\end{array}
\]

and denoted by 9-tuple \((\mathcal{T}_1, \mathcal{T}, \mathcal{T}_2, i^*, i_*, i_1, i^!, i_1^!, j^!, j_*, j_1^!, j_!^!, j^*, j_*)\) such that

(R1) \((i^*, i_*), (i_1, i_1^!), (j_!, j^!), (j_*, j_1^!, j_!^!, j^*, j_*)\) are adjoint pairs of triangulated functors;
(R2) \(i_*, j_!^!, j_*\) and \(j^!, i_1^!, j_1^!\) are full embeddings;
(R3) \(j_!^! i_* = 0\) (and thus also \(i_1^! j_* = 0\) and \(i^* j_1 = 0\));
(R4) for each \(X \in \mathcal{T}\), there are triangles

\[
\begin{align*}
j_!^! j^! X & \rightarrow X \rightarrow i_* i^! X \rightarrow \\
i_1^! i^! X & \rightarrow X \rightarrow j_!^! j^* X \rightarrow 
\end{align*}
\]

From now on we focus on recollements of derived categories of algebras. Let \(A\) be an algebra. Denote by \(\text{Proj} A\) (resp. \(\text{proj} A\)) the category of projective (resp. finitely generated projective) \(A\)-modules. Denote by \(D(A)\) the unbounded derived category of complexes of \(A\)-modules. Denote by \(K^b(\text{Proj} A)\) (resp. \(K^- (\text{Proj} A)\)) the homotopy category of bounded (resp. right bounded) complexes of projective \(A\)-modules. Let \(X\) be an object in \(D(A)\). Denote by \(X^\perp\) the full subcategory of \(D(A)\) consisting of all objects \(Y \in D(A)\) such that \(\text{Hom}_{D(A)}(X, Y[n]) = 0, \forall n \in \mathbb{Z}\). Denote by \(\text{Tria} X\) the smallest full triangulated subcategory of \(D(A)\) which contains \(X\) and is closed under small coproducts. We say \(X\) is exceptional if \(\text{Hom}_{D(A)}(X, X[n]) = 0\) for all \(n \in \mathbb{Z}\setminus\{0\}\). We say \(X\) is compact if the functor \(\text{Hom}_{D(A)}(X, -)\) preserves small coproduct, or equivalently, \(X\) is perfect, i.e., isomorphic in \(D(A)\) to an object in \(K^b(\text{proj} A)\), the homotopy category of bounded complexes of finitely generated projective \(A\)-modules. We say \(X\) is self-compact if \(\text{Hom}_{D(A)}(X, -)\) preserves small coproducts in \(\text{Tria} X\). (ref. [20])

A very important criterion for the right bounded derived category of an algebra to admit a recollement is provided in [23] (cf. [34, Theorem 3]). It was extended and modified to suit for the unbounded derived categories of algebras (ref. [33, Corollary 3.4]), differential graded algebras (ref. [20, Theorem 3.3]) and differential graded categories (ref. [33, Corollary 3.4]).

**Theorem 1.** (König [23]; Jørgensen [20]; Nicolás-Saorín [33]) Let \(A_1, A\) and \(A_2\) be algebras. Then \(D(A)\) admits a recollement relative to \(D(A_1)\) and \(D(A_2)\) if and only if there are objects \(X_1\) and \(X_2\) in \(D(A)\) such that
An important example of recollements of derived categories of algebras is given by stratifying ideals:

**Example 1.** (Cline-Parshall-Scott [11]) Stratifying ideals. Let $A$ be an algebra, $e$ an idempotent of $A$, and $AeA$ a stratifying ideal of $A$, i.e., the multiplication in $A$ induces an isomorphism $Ae \otimes_{eAe} eA \cong AeA$ and $\text{Tor}_n^{eAe}(Ae, eA) = 0$ for all $n \geq 1$. Then there is a recollement $(D(A/AeA), D(A), D(eA), i^*, i_*, j^*, j_*)$ where

$$
\begin{align*}
    i^* &= - \otimes_A^L A/AeA, \\
    i_* &= i^* = - \otimes_{A/eA}^L A/eA, \\
    j^* &= j_1 = - \otimes_A^L eA, \\
    j_* &= j^* = - \otimes_A^L eA, \\
    i^1 &= \text{RHom}_A(A/eA, -), \\
    j^1 &= \text{RHom}_{eA}(eA, -).
\end{align*}
$$

**2.2 Perfect recollements**

Sometimes we work on a nicer class of recollements, i.e., the so-called perfect recollements, which correspond to “derived triangular matrix (differential graded) algebras”. More precisely, in a perfect recollement of derived categories of algebras, the middle algebra is derived equivalent to a triangular matrix (differential graded) algebra and the algebras on both sides are derived equivalent to the (differential graded) algebras on the diagonal [22].

**Definition 2.** Let $A_1$, $A$ and $A_2$ be algebras. A recollement $(D(A_1), D(A), D(A_2), i^*, i_*, j^*, j_*)$ is said to be perfect if $i_*, A_1$ is perfect.

From Theorem 1 we can obtain directly the following criterion for the derived category of an algebra to admit a perfect recollement of derived categories of algebras.

**Theorem 2.** Let $A_1$, $A$ and $A_2$ be algebras. Then $D(A)$ admits a perfect recollement relative to $D(A_1)$ and $D(A_2)$ if and only if there are objects $X_i, i = 1, 2$, in $D(A)$ such that

1. $\text{End}_{D(A)}(X_i) \cong A_i$ as algebras, $\forall i = 1, 2$;
2. $X_i$ is exceptional and perfect, $\forall i = 1, 2$;
3. $X_i \in X_i^+$;
4. $X_1^+ \cap X_2^+ = \{0\}$. 

(1) End$_{D(A)}(X_i) \cong A_i$ as algebras for $i = 1, 2$;
(2) $X_2$ (resp. $X_1$) is exceptional and compact (resp. self-compact);
(3) $X_1 \in X_2^+$;
(4) $X_1^+ \cap X_2^+ = \{0\}$. 

Now we provide some examples of either perfect or imperfect recollements of derived categories of algebras:

**Example 2.** (1) Derived equivalences. If the algebras \(A\) and \(B\) are derived equivalent then \(D(A)\) admits a perfect recollement relative to 0 and \(D(B)\), or to \(D(B)\) and 0. (ref. [24, Theorem 8.3.2])

(2) Triangular matrix algebras. Let \(A_1\) and \(A_2\) be algebras, \(M\) an \(A_2\text{-}A_1\)-bimodule, and \(A = \left[ \begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array} \right]\). Then \(X_1 := \left[ \begin{array}{c} 1 \end{array} \right] A_1 \otimes 0 \) and \(X_2 := \left[ \begin{array}{c} 0 \\ \hline 0 \end{array} \right] A_2 \) satisfy all conditions in Theorem 2. Thus there is a perfect recollement of \(D(A)\) relative to \(D(A_1)\) and \(D(A_2)\).

(3) Perfect stratifying ideals. Let \(A\) be an algebra, \(e\) an idempotent of \(A\), and \(AeA\) a perfect stratifying ideal of \(A\), i.e., a stratifying ideal which is perfect in \(D(A)\). Then there is a perfect recollement of \(D(A)\) relative to \(D(A/eA)\) and \(D(eAe)\). Note that a triangular matrix algebra always has a projective stratifying ideal.

(4) Imperfect recollement. Let \(A\) be the infinite Kronecker algebra \(\left[ \begin{array}{c|c} V & 0 \\ \hline 0 & k \end{array} \right]\), where \(V\) is an infinite-dimensional \(k\)-vector space. Choose \(X_2\) to be the simple projective \(A\)-module and \(X_1\) the other simple \(A\)-module. Then \(X_1\) and \(X_2\) satisfy all conditions in Theorem 2. Thus \(D(A)\) admits a recollement relative to \(D(k)\) and \(D(k)\), which is not perfect since \(X_1\) is isomorphic to an object in \(K^b(\text{Proj}A)\) but not in \(K^b(\text{proj}A)\). (ref. [23, Example 9])

### 3 Constructions of (perfect) recollements

In this section, we shall provide two constructions of (perfect) recollements: one is via tensor product algebras, the other is via opposite algebras.

#### 3.1 Tensor product algebras

From a recollement of derived categories of algebras, we can obtain recollements of derived categories of tensor product algebras.

**Theorem 3.** Let \(A_1, A_2\) be algebras, and \(D(A)\) admit a (perfect) recollement relative to \(D(A_1)\) and \(D(A_2)\). Then, for each algebra \(B\), \(D(B \otimes A)\) admits a (perfect) recollement relative to \(D(B \otimes A_1)\) and \(D(B \otimes A_2)\).

**Proof.** By Theorem 2 there are objects \(X_i, i = 1, 2\), in \(D(A)\) such that they satisfy all conditions in Theorem 1. Let \(Z_i := B \otimes X_i\) for \(i = 1, 2\). Now we show that \(Z_1\) and \(Z_2\) satisfy all conditions in Theorem 1 for tensor product algebras.
Since $X_2$ is perfect in $D(A)$, $Z_2$ is perfect in $D(B \otimes A)$. Since $X_2$ is perfect and exceptional and $\text{End}_{D(A)}(X_2) \cong A_2$ as algebras, we have

$$
\text{Hom}_{D(B \otimes A)}(Z_2, Z_2[n]) \cong H^n(\text{RHom}_{B \otimes A}(B \otimes X_2, B \otimes X_2)) \\
\cong H^n(\text{RHom}_B(B, \text{RHom}_A(X_2, B \otimes X_2))) \\
\cong H^n(\text{RHom}_A(X_2, B \otimes X_2)) \\
\cong \text{Hom}_{D(A)}(X_2, B \otimes X_2[n]) \\
\cong B \otimes \text{Hom}_{D(A)}(X_2, X_2[n]) \\
\cong \begin{cases} B \otimes A_2, & \text{if } n = 0; \\
0, & \text{otherwise.} \end{cases}
$$

Thus $Z_2$ is exceptional and $\text{End}_{D(B \otimes A)}(Z_2) \cong B \otimes A_2$ as algebras.

Since $X_1$ is self-compact, for any index set $A$, we have

$$
\text{Hom}_{D(B \otimes A)}(Z_1, Z_1^{(A)}[n]) \cong H^n(\text{RHom}_{B \otimes A}(B \otimes X_1, (B \otimes X_1)^{(A)})) \\
\cong H^n(\text{RHom}_A(X_1, (B \otimes X_1)^{(A)})) \\
\cong \text{Hom}_{D(A)}(X_1, (B \otimes X_1[n])^{(A)}) \\
\cong \text{Hom}_{D(A)}(X_1, B \otimes X_1[n])^{(A)} \\
\cong H^n(\text{RHom}_A(X_1, B \otimes X_1))^{(A)} \\
\cong H^n(\text{RHom}_{B \otimes A}(B \otimes X_1, B \otimes X_1))^{(A)} \\
\cong \text{Hom}_{D(B \otimes A)}(Z_1, Z_1[n])^{(A)}.
$$

Thus $Z_1$ is self-compact.

Since $X_1$ is self-compact and exceptional and $\text{End}_{D(A)}(X_1) \cong A_1$ as algebras, we have

$$
\text{Hom}_{D(B \otimes A)}(Z_1, Z_1[n]) \cong H^n(\text{RHom}_{B \otimes A}(B \otimes X_1, B \otimes X_1)) \\
\cong H^n(\text{RHom}_A(X_1, B \otimes X_1)) \\
\cong \text{Hom}_{D(A)}(X_1, B \otimes X_1[n]) \\
\cong B \otimes \text{Hom}_{D(A)}(X_1, X_1[n]) \\
\cong \begin{cases} B \otimes A_1, & \text{if } n = 0; \\
0, & \text{otherwise.} \end{cases}
$$

Thus $Z_1$ is exceptional and $\text{End}_{D(B \otimes A)}(Z_1) \cong B \otimes A_1$ as algebras.

Since $X_2$ is perfect and $X_1 \in X_2^\perp$, we have

$$
\text{Hom}_{D(B \otimes A)}(Z_2, Z_1[n]) \cong H^n(\text{RHom}_{B \otimes A}(B \otimes X_2, B \otimes X_1)) \\
\cong H^n(\text{RHom}_A(X_2, B \otimes X_1)) \\
\cong \text{Hom}_{D(A)}(X_2, B \otimes X_1[n]) \\
\cong B \otimes \text{Hom}_{D(A)}(X_2, X_1[n]) = 0
$$

for all $n \in \mathbb{Z}$. Thus $Z_1 \in Z_2^\perp$. 

6
For any $Z \in Z_1^\perp \cap Z_2^\perp$, we have

$$0 = \text{Hom}_{D(B \otimes A)}(Z, Z[n])$$

$$\cong H^n(\text{RHom}_{B \otimes A}(B \otimes X_i, Z))$$

$$\cong H^n(\text{RHom}_A(X_i, Z))$$

$$\cong \text{Hom}_{D(A)}(X_i, Z[n])$$

for all $n \in \mathbb{Z}$ and $i = 1, 2$. Thus $Z \in X_1^\perp \cap X_2^\perp = \{0\}$. Hence $Z_1^\perp \cap Z_2^\perp = \{0\}$.

Now we have shown that $Z_1$ and $Z_2$ satisfy all conditions in Theorem 1 for tensor product algebras. By Theorem 1, we are done. The statement on perfect recollement can be obtained simultaneously. □

### 3.2 Opposite algebras

From a perfect recollement of derived categories of algebras, we can obtain a perfect recollement of derived categories of opposite algebras.

**Theorem 4.** Let $A_1$, $A$ and $A_2$ be algebras, and $(D(A_1), D(A), D(A_2), i^*, i, i^* = i, i^* = j, j^* = j^*, j^*), j^*), a perfect recollement. Then $D(A^\text{op})$ admits a perfect recollement relative to $D(A_2^\text{op})$ and $D(A_1^\text{op})$.

**Proof.** The objects $X_1 := i_* A_1$ and $X_2 := j_* A_2$ in $D(A)$ satisfy all conditions in Theorem 2. Let $Z_i = \text{RHom}_A(X_i, A), i = 1, 2$. Now we show that $Z_1$ and $Z_2$ satisfy all conditions in Theorem 2 for opposite algebras.

Since $X_i$ is perfect in $D(A)$, $Z_i$ is perfect in $D(A^\text{op})$. Since $X_i$ is perfect and exceptional and $\text{End}_{D(A)}(X_i) \cong A_i$ as algebras, we have

$$\text{Hom}_{D(A^\text{op})}(Z_i, Z_i[n]) \cong H^n(\text{RHom}_{A^\text{op}}(\text{RHom}_A(X_i, A), \text{RHom}_A(X_i, A)))$$

$$\cong H^n(\text{RHom}_A(X_i, X_i))$$

$$\cong \text{Hom}_{D(A)}(X_i, X_i[n])$$

$$\cong \begin{cases} A_i, & \text{if } n = 0; \\ 0, & \text{otherwise}. \end{cases}$$

Thus $Z_i$ is exceptional and $\text{End}_{D(A^\text{op})}(Z_i) \cong A_i^\text{op}$ as algebras for $i = 1, 2$.

Since $X_1$ and $X_2$ are perfect and $X_1 \in X_2^\perp$, we have

$$\text{Hom}_{D(A^\text{op})}(Z_1, Z_2[n]) \cong H^n(\text{RHom}_{A^\text{op}}(\text{RHom}_A(X_1, A), \text{RHom}_A(X_2, A)))$$

$$\cong H^n(\text{RHom}_A(X_2, X_1))$$

$$\cong \text{Hom}_{D(A)}(X_2, X_1[n]) = 0$$

for all $n \in \mathbb{Z}$. Thus $Z_2 \in Z_1^\perp$. 7
For all $Z \in Z_1^+ \cap Z_2^+$ and $n \in \mathbb{Z}$, since $X_i, i = 1, 2$, are perfect, we have

$$0 = \text{Hom}_{D(A^\text{op})}(Z, Z[n])$$

$$\cong H^n(\text{RHom}_{A^\text{op}}(\text{RHom}_A(X_i, A), Z))$$

$$\cong H^n(X_i \otimes_A Z)$$

i.e., $X_i \otimes_A Z = 0$ in $D(k)$. Thus, for all $W \in D(k)$ and $n \in \mathbb{Z}$, we have

$$H^n(i^! \text{RHom}_k(Z, W)) \cong H^n(\text{RHom}_{A_1}(A_1, i^! \text{RHom}_k(Z, W)))$$

$$\cong \text{Hom}_{D(A_1)}(A_1, i^! \text{RHom}_k(Z, W)[n])$$

$$\cong \text{Hom}_{D(A)}(i_! A_1, \text{RHom}_k(Z, W)[n])$$

$$= \text{Hom}_{D(A)}(X_1, \text{RHom}_k(Z, W)[n])$$

$$\cong \text{Hom}_{D(k)}(X_1 \otimes_A^L Z, W[n]) = 0,$$

i.e., $i^! \text{RHom}_k(Z, W) = 0$ in $D(A_1)$. Similarly, we have

$$H^n(j^* \text{RHom}_k(Z, W)) \cong H^n(\text{RHom}_{A_2}(A_2, j^* \text{RHom}_k(Z, W)))$$

$$\cong \text{Hom}_{D(A_2)}(A_2, j^* \text{RHom}_k(Z, W)[n])$$

$$\cong \text{Hom}_{D(A)}(j_! A_2, \text{RHom}_k(Z, W)[n])$$

$$= \text{Hom}_{D(A)}(X_2, \text{RHom}_k(Z, W)[n])$$

$$\cong \text{Hom}_{D(k)}(X_2 \otimes_A^L Z, W[n]) = 0,$$

i.e., $j^* \text{RHom}_k(Z, W) = 0$ in $D(A_2)$. In the triangle

$$i_! i^! \text{RHom}_k(Z, W) \to \text{RHom}_k(Z, W) \to j_! j^* \text{RHom}_k(Z, W) \to,$$

both sides are zero. Thus $\text{RHom}_k(Z, W) = 0$ for all $W \in D(k)$. Hence $Z = 0$ in $D(A^\text{op})$. Therefore, $Z_1^+ \cap Z_2^+ = \{0\}$.

Now we have shown that $Z_1$ and $Z_2$ satisfy all conditions in Theorem 2 for opposite algebras. By Theorem 2, we are done.

4 Recollements and smoothness

In this section, we shall apply two constructions provided in section 3 to study the relations between recollements of derived categories of algebras and smoothness of algebras, i.e., finiteness of Hochschild dimensions of algebras. For this, we need to know the relation between recollements of derived categories of algebras and global dimensions of algebras.
4.1 Recollements and global dimensions

The following result is due to König:

**Theorem 5.** (König [23, Corollary 5]) Let $A_1, A$ and $A_2$ be algebras, and $D(A)$ admit a perfect recollement relative to $D(A_1)$ and $D(A_2)$. Then $A$ is of finite global dimension if and only if so are $A_1$ and $A_2$.

**Proof.** Denote by $D^{-}(A)$ the derived category of complexes of $A$-modules with right bounded cohomologies. By Theorem 2 and [23, Theorem 1] or [34, Theorem 3], we have a restricted recollement of $D^{-}(A)$ relative to $D^{-}(A_1)$ and $D^{-}(A_2)$. By [23, Corollary 5], we are done.

Recently, the following more general result is obtained:

**Theorem 6.** (Angeleri Hügel-König-Liu-Yang [2]) Let $A_1, A$ and $A_2$ be algebras, and $D(A)$ admit a recollement relative to $D(A_1)$ and $D(A_2)$. Then $A$ is of finite global dimension if and only if so are $A_1$ and $A_2$.

4.2 Recollements and smoothness

Let $A$ be an algebra and $A^e := A^{op} \otimes A$ its enveloping algebra. The *Hochschild dimension* of $A$ is the projective dimension of $A$ as a left or right $A^e$-module. The Hochschild dimensions of algebras were studied very earlier [7]. An algebra $A$ is of Hochschild dimension 0 if and only if $A^e$ is semisimple [7, Theorem 7.9]. In case $A$ is finitely generated, $A$ is of Hochschild dimension 0 if and only if $A$ is separable [7, Theorem 7.10]. The algebras of Hochschild dimension $\leq 1$ are called quasi-free or formally smooth [12, 26]. An algebra $A$ is said to be smooth if it has finite Hochschild dimension, i.e., $\text{pd}_{A^e} A < \infty$ (ref. [10]), or equivalently, $A$ is isomorphic to an object in $K^b(\text{Proj}A^e)$. It follows from [7, Chap. IX, Proposition 7.5, 7.6] and [13, Proposition 2] that $A$ is smooth if and only if $\text{gl.dim} A^e < \infty$.

**Remark 1.**

1. *Sometimes* $\text{gl.dim} A < \infty \Leftrightarrow \text{gl.dim} A^e < \infty$: Let $A$ be either a commutative Noetherian algebra over a perfect field $k$, or a finite-dimensional $k$-algebra such that the factor algebra $A/J$ of $A$ modulo its Jacobson radical $J$ is separable. Then $\text{gl.dim} A < \infty$ if and only if $\text{gl.dim} A^e < \infty$. (ref. [18, Theorem 2.1] and [3, Theorem 16])

2. *In general* $\text{gl.dim} A < \infty \not\Rightarrow \text{gl.dim} A^e < \infty$: Let $A$ be a finite inseparable field extension of an imperfect field $k$. Then $\text{gl.dim} A = 0$. However, $\text{gl.dim} A^e = \infty$, since $A \otimes_k A$ is not semisimple. (ref. [4, Page 65, Remark])
Let the algebras $A$ and $B$ be derived equivalent. Then by \[37\] Proposition 9.1 and \[38\] Theorem 2.1 we know $A^e$ and $B^e$ are derived equivalent. Thus $\text{gl.dim} A^e < \infty$ if and only if $\text{gl.dim} B^e < \infty$ (ref. \[23\] p.37). Hence, $A$ is smooth if and only if so is $B$, i.e., the smoothness of algebras is invariant under derived equivalences. More general, we have the following result:

**Theorem 7.** Let $A_1$, $A$ and $A_2$ be algebras, and $D(A)$ admit a perfect recollement relative to $D(A_1)$ and $D(A_2)$. Then $A$ is smooth if and only if so are $A_1$ and $A_2$.

**Proof.** By Theorem 3 we have a perfect recollement of $D(A^\text{op} \otimes A)$ relative to $D(A^\text{op} \otimes A_1)$ and $D(A^\text{op} \otimes A_2)$. It follows from Theorem 5 that $\text{gl.dim} A^e < \infty$ if and only if $\text{gl.dim} A^e \otimes A_i < \infty$ for all $i = 1, 2$. By Theorem 4 and Theorem 3 we have a perfect recollement of $D(A^\text{op} \otimes A_i)$ relative to $D(A^\text{op} \otimes A_i)$. It follows from Theorem 5 that $\text{gl.dim} A^e < \infty$ if and only if $\text{gl.dim} A^e_i < \infty$ for all $i = 1, 2$. Therefore, $\text{gl.dim} A^e < \infty$ if and only if $\text{gl.dim} A^e_i < \infty$ for all $i = 1, 2$, by \[13\] Proposition 2.

Theorem 7 can be applied to judge the smoothness of some algebras or construct some smooth algebras. For instance, when applied to triangular matrix algebras, we have the following result:

**Corollary 1.** Let $A_1$ and $A_2$ be algebras, $M$ an $A_2$-$A_1$-bimodule, and $A = \begin{bmatrix} A_1 & 0 \\ M & A_2 \end{bmatrix}$. Then $A$ is smooth if and only if so are $A_1$ and $A_2$.

**Proof.** It follows from Example 2 (2) and Theorem 7.

**Remark 2.** An algebra $A$ is said to be *homotopically smooth* if $A$ is compact in $D(A^e)$, i.e., $A$ is isomorphic in $K^b(\text{proj} A^e)$ (ref. \[27\]). By Corollary 1 we know the infinite Kronecker algebra (ref. Example 2 (4)) is smooth. However, it is not homotopically smooth, because the finitely generated projective $A^e$-module resolution of $A$ would induce a finitely generated projective $A$-module resolution of the nonprojective simple $A$-module. Hence, Corollary 1 thus Theorem 7 is not correct for homotopically smoothness.

We don’t know whether Theorem 7 holds for any recollement of derived categories of algebras? However, it does hold when restricted to Noetherian algebras.

**Theorem 8.** Let $A_1$, $A$ and $A_2$ be Noetherian algebras, and $D(A)$ admit a recollement relative to $D(A_1)$ and $D(A_2)$. Then $A$ is smooth if and only if so are $A_1$ and $A_2$. 

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Proof. Necessity. Since $A$ is smooth, by [7, Chap. IX, Proposition 7.6], we have $\dim A < \infty$. It follows from Theorem [B] that $\dim A_i < \infty$ for all $i = 1, 2$. Since $A_i$ is Noetherian, by [3, Corollary 5], we have $\dim A_i < \infty$ for all $i = 1, 2$. It follows from [13, Proposition 2] that $\dim A_i^\text{op} \otimes A < \infty$ for all $i = 1, 2$. By Theorem [5], we have a recollement of $D(A_i^\text{op} \otimes A)$ relative to $D(A_i^\text{op} \otimes A_1)$ and $D(A_i^\text{op} \otimes A_2)$. It follows from Theorem [C] that $\dim A_i^\text{op} \otimes A_j < \infty$ for all $i, j = 1, 2$. Therefore, $\dim A_i^\text{op} < \infty$, i.e., $A_i$ is smooth, for all $i = 1, 2$.

Sufficiency. Since $A_i$ is smooth, by [7, Chap. IX, Proposition 7.6], we have $\dim A_i < \infty$ for all $i = 1, 2$. It follows from Theorem [B] that $\dim A < \infty$. Since $A$ is Noetherian, by [3, Corollary 5], we have $\dim A^\text{op} = \dim A < \infty$. It follows from [13, Proposition 2] that $\dim A^\text{op} \otimes A_i < \infty$ for all $i = 1, 2$. By Theorem [5], we have a recollement of $D(A^\text{op} \otimes A)$ relative to $D(A^\text{op} \otimes A_1)$ and $D(A^\text{op} \otimes A_2)$. It follows from Theorem [6] that $\dim A^\text{op} \otimes A < \infty$, i.e., $A$ is smooth. \hfill $\square$

5 Recollements and derived functors

In this section, we shall realize triangulated functors in recollements of derived categories of algebras and tensor product algebras as derived functors of the same forms in the spirit of [38].

Lemma 1. Let $A_1, A$ and $A_2$ be algebras, and $D(A)$ admit a recollement relative to $D(A_1)$ and $D(A_2)$. Then

(1) there is a recollement $(D(A_1), D(A), D(A_2), i^*, i_*, j^!, j_!)$ such that

\begin{align*}
i_* &= i_1 = - \otimes_{A_1}^L Y_1, & j_! &= j_1 = - \otimes_{A_1}^L Y_2, \\
i^! &= \text{RHom}_{A_1}(Y_1, -), & j^! &= \text{RHom}_{A_2}(\text{RHom}_A(Y_2, -), -),
\end{align*}

and

(2) for each algebra $B$, there is a recollement $(D(B \otimes A_1), D(B \otimes A), D(B \otimes A_2), I^*, I_*, J^!, J_!)$ such that

\begin{align*}
I_* &= I_1 = - \otimes_{A_1}^L Y_1, & J_! &= J_1 = \text{RHom}_{A_1}(Y_2, -), \\
I^! &= \text{RHom}_{A_1}(Y_1, -), & J^! &= \text{RHom}_{A_2}(\text{RHom}_A(Y_2, -), -),
\end{align*}

for some $Y_i \in D(A_i^\text{op} \otimes A)$, $i = 1, 2$. 

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Proof. It follows from Theorem 1 that there are objects $X_i, i = 1, 2,$ in $D(A)$ such that they satisfy all conditions in Theorem 1. Clearly, we may assume that $X_1$ and $X_2$ are homotopically projective. Since $X_i$ is exceptional, it follows from [24, 8.3.1] that there exists $Y_i \in D(A_1^{op} \otimes A)$ such that the derived tensor functor $- \otimes_{A_1}^L Y_i : D(A_1) \to D(A)$ sends $A_i$ to $X_i$ for all $i = 1, 2$.

(1) By [32] Theorem 2.8, we have a recollement $(X_2^+, D(A), D(A_2), i^*, i'_* = i_i^*, j_i, j_i = j_i^*, j_*)$ where $j_i = - \otimes_{A_2}^L Y_2, j_i = j_i^* = \text{RHom}_A(Y_2, -), j_* = \text{RHom}_{A_2}(\text{RHom}_A(Y_2, A), -)$, and $i'_* = i_i'$ is the natural embedding.

Since $X_1$ is self-compact, it is a compact generator in $\text{Tria} X_1$. It follows from [21] Lemma 4.2] that the functor $- \otimes_{A_1}^L Y_1 : D(A_1) \to D(A)$ is fully faithful and its essential image is $\text{Tria} X_1$. From $X_1 \in X_2^+$ and $X_2^+ \cap X_2 = \{0\}$ we can obtain $\text{Tria} X_1 = X_2^+$ (ref. [20] Proof of Theorem 3.3]). Thus we have a recollement $(D(A_1), D(A), D(A_2), i^*, i_* = i_i, j_i, j_0 = j_i^*, j_*)$ with described five triangulated functors.

(2) Repeat the proof above, mutatis mutandis, we can obtain a recollement $(D(B \otimes A_1), D(B \otimes A), D(B \otimes A_2), I^*, I_* = I_i, I', j_i, j_i^0 = j_i^*, j_*)$ such that

\[
\begin{align*}
J_i &= - \otimes_{B_1}^L (B \otimes Y_2), \\
I_* &= I_1 = - \otimes_{B_1}^L (B \otimes Y_1), \\
J_i &= J_i^* = \text{RHom}_{B_2}(B \otimes Y_2, -), \\
I' &= \text{RHom}_{B_2}(B \otimes Y_1, -), \\
J_* &= \text{RHom}_{B_2}(\text{RHom}_{B_2}(B \otimes Y_2, B \otimes A), -).
\end{align*}
\]

Note that all conditions on $B \otimes X_1$ needed here have been checked already in the proof of Theorem 3.

Similarly, $J_i \cong J_i = - \otimes_{B_2}^L A_2, Y_2$. We also have $I_i = \text{RHom}_{B_2}(B \otimes Y_1, -) \cong \text{RHom}_A(Y_1, -)$. Similarly, $J_i = J_i^* \cong \text{RHom}_A(Y_2, -)$. Moreover, since $Y_2$ is perfect in $D(A)$, we have

\[
\begin{align*}
J_* &= \text{RHom}_{B_2}(\text{RHom}_{B_2}(B \otimes Y_2, B \otimes A), -) \\
&\cong \text{RHom}_{B_2}(\text{RHom}_A(Y_2, B \otimes A), -) \\
&\cong \text{RHom}_{B_2}(B \otimes \text{RHom}_A(Y_2, A), -) \\
&\cong \text{RHom}_{A_2}(\text{RHom}_A(Y_2, A), -).
\end{align*}
\]

Thus we can obtain a recollement $(D(B \otimes A_1), D(B \otimes A), D(B \otimes A_2), I^*, I_* = I_i, I', j_i, j_i^0 = j_i^*, j_*)$ with described triangulated functors. \qed

Remark 3. Two recollements of triangulated categories $(\mathcal{T}_1, \mathcal{T}, \mathcal{T}_2, i^*, i_* = i_i, i_i', j_i, j_i' = j_i^*, j_i)$ and $(\mathcal{T}'_1, \mathcal{T}, \mathcal{T}'_2, i'^*, i'_* = i_i', i_i'^0, j_i'^0 = j_i'^0)$ are said to be equivalent if $(\text{Im} j_i, \text{Im} i_i, \text{Im} j_i) = (\text{Im} j_i', \text{Im} i_i', \text{Im} j_i')$, where $\text{Im}$ denotes the
essential image of the functor \([33]\). It is easy to see that the resulting recollement in the first part of Lemma 1 is equivalent to the known one. Indeed, \(\text{Im} j_! = \text{Tria} X_2 = \text{Tria} Y_2 = \text{Im}(- \otimes_{A_2}^L Y_2)\), \(\text{Im}_* = \text{Tria} X_1 = \text{Tria} Y_1 = \text{Im}(- \otimes_{A_1}^L Y_1)\) and \(\text{Im} j_* = X_1^+ = Y_1^+ = \text{Im} (\text{RHom}_{A_2} (\text{RHom}_{A_2} (Y_2, A), -))\).

**Lemma 2.** Let \(\mathcal{T}\) be a triangulated category with translation functor \([1]\), \(F_i, G_i : \mathcal{T} \rightarrow \mathcal{T}, i = 1, 2, 3\), six triangulated functors, \(\alpha_j : F_j \rightarrow F_{j+1}\), \(\beta_j : G_j \rightarrow G_{j+1}\), \(\gamma_j : F_j \rightarrow G_j\), \(j = 1, 2\), six natural transformations, such that \(\gamma_2 \alpha_1 = \beta_1 \gamma_1\), \(F_1 X \xrightarrow{\alpha_1 X} F_2 X \xrightarrow{\alpha_2 X} F_3 X\), \(G_1 X \xrightarrow{\beta_1 X} G_2 X \xrightarrow{\beta_2 X} G_3 X\) are triangles for all \(X \in \mathcal{T}\), and \(\text{Hom}_\mathcal{T}(F_1, G_3[-1]) = 0\). Then there is a natural transformation \(\gamma_3 : F_3 \rightarrow G_3\) such that \(\gamma_3 \alpha_2 = \beta_2 \gamma_2\).

**Proof.** For each \(X \in \mathcal{T}\), by assumption, we have a commutative diagram

\[
\begin{array}{ccc}
F_1 X & \xrightarrow{\alpha_1 X} & F_2 X & \xrightarrow{\alpha_2 X} & F_3 X \\
\downarrow{\gamma_1 X} & & \downarrow{\gamma_2 X} & & \\
G_1 X & \xrightarrow{\beta_1 X} & G_2 X & \xrightarrow{\beta_2 X} & G_3 X
\end{array}
\]

By an axiom of triangulated category, there exists a morphism \(\gamma_3 X : F_3 X \rightarrow G_3 X\) such that \(\gamma_3 X \alpha_2 X = \beta_2 X \gamma_2 X\).

For each morphism \(f : X \rightarrow Y\) in \(\mathcal{T}\), we have commutative diagrams

\[
\begin{array}{ccc}
F_1 X & \xrightarrow{\alpha_1 X} & F_2 X & \xrightarrow{\alpha_2 X} & F_3 X \\
\downarrow{G_1(f) \gamma_1 X} & & \downarrow{G_2(f) \gamma_2 X} & & \downarrow{G_3(f) \gamma_3 X} \\
G_1 Y & \xrightarrow{\beta_1 Y} & G_2 Y & \xrightarrow{\beta_2 Y} & G_3 Y
\end{array}
\]

and

\[
\begin{array}{ccc}
F_1 Y & \xrightarrow{\alpha_1 Y} & F_2 Y & \xrightarrow{\alpha_2 Y} & F_3 Y \\
\downarrow{\gamma_1 Y F_1(f)} & & \downarrow{\gamma_2 Y F_2(f)} & & \downarrow{\gamma_3 Y F_3(f)} \\
G_1 Y & \xrightarrow{\beta_1 Y} & G_2 Y & \xrightarrow{\beta_2 Y} & G_3 Y
\end{array}
\]

Since \(G_1(f) \gamma_1 X = \gamma_1 Y F_1(f)\) and \(G_2(f) \gamma_2 X = \gamma_2 Y F_2(f)\), by \([33]\) Proposition 1.1.9] we have \(G_3(f) \gamma_3 X = \gamma_3 Y F_3(f)\).

Thus \(\gamma_3\) is a natural transformation such that \(\gamma_3 \alpha_2 = \beta_2 \gamma_2\). \(\square\)
Now we can realize all six triangulated functors in a recollement of derived categories of algebras or tensor product algebras as derived functors:

**Theorem 9.** Let $A_1, A$ and $A_2$ be algebras, and $D(A)$ admit a recollement relative to $D(A_1)$ and $D(A_2)$. Then

1. there is a recollement $(D(A_1), D(A), D(A_2), i^*, i_*, i^! = i_!, j_!, j^! = j^*, j_*)$ such that
   
   \[
   \begin{align*}
   i^* &= - \otimes_A^L Y, & j_! &= - \otimes_A^R Y_2, \\
   i_* &= i_! = - \otimes_A^L Y_1, & j^! &= j^* = \text{RHom}_A(Y_2, -), \\
   i^! &= \text{RHom}_A(Y_1, -), & j_* &= \text{RHom}_{A_2}(\text{RHom}_A(Y_2, A), -),
   \end{align*}
   \]

   and

2. for each algebra $B$, there is a recollement $(D(B \otimes A_1), D(B \otimes A), D(B \otimes A_2), I^*, I_*, I^! = I_!, J_!, J^! = J^*, J_*)$ such that
   
   \[
   \begin{align*}
   I^* &= - \otimes_A^L X, & J_! &= - \otimes_A^R X_2, \\
   I_* &= I_! = - \otimes_A^L Y_1, & J^! &= J^* = \text{RHom}_A(Y_2, -), \\
   I^! &= \text{RHom}_A(Y_1, -), & J_* &= \text{RHom}_{A_2}(\text{RHom}_A(Y_2, A), -),
   \end{align*}
   \]

   for some $Y \in D(A \text{op} \otimes A_1)$ and $Y_i \in D(A_i \text{op} \otimes A)$, $i = 1, 2$.

**Proof.** (1) By Lemma 4 we may assume that all triangulated functors but $i^*$ in the recollement are as required. For each $X \in D(A)$, we have a triangle in $D(A)$:

\[
X \xrightarrow{\varepsilon} j_j^* X \xrightarrow{\eta} i_i^* X \rightarrow
\]

where $\varepsilon$ is the counit of the adjoint pair $(j_i, j^i)$ and $\eta$ is the unit of the adjoint pair $(i^*, i_*)$.

By Lemma 5 we have a recollement $(D(A_{\text{op}} \otimes A_1), D(A_{\text{op}} \otimes A), D(A_{\text{op}} \otimes A_2), I^*, I_*, I^! = I_!, J_!, J^! = J^*, J_*)$ such that

\[
\begin{align*}
J_! &= - \otimes_A^R Y_2, \\
I_* &= I_! = - \otimes_A^L Y_1, & J^! &= J^* = \text{RHom}_A(Y_2, -), \\
I^! &= \text{RHom}_A(Y_1, -), & J_* &= \text{RHom}_{A_2}(\text{RHom}_A(Y_2, A), -),
\end{align*}
\]

for some $Y_i \in D(A_i \text{op} \otimes A)$, $i = 1, 2$.

Let $Y := I^* A \in D(A_{\text{op}} \otimes A_1)$. Then we have a triangle in $D(A_{\text{op}})$:

\[
J_i J^! A = \text{RHom}_A(Y_2, A) \otimes_{A_2}^L Y_2 \xrightarrow{E} A \xrightarrow{H} I_* I^* A = Y \otimes_{A_1}^L Y_1 \rightarrow
\]

where $E$ is the counit of the adjoint pair $(J_i, J^i)$ and $H$ is the unit of the adjoint pair $(I^*, I_*)$. 

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torus, which are given by three bimodule complexes of algebras or tensor product algebras are realized as derived functors such that we show that

\textbf{Definition 3.} A recollement $(\mathcal{D}(A_1), \mathcal{D}(A), \mathcal{D}(A_2), i^*, i_*, j^*, j_*)$ is said to be \textit{standard} and \textit{given by $Y \in \mathcal{D}(A^\text{op} \otimes A_1)$ and $Y_2 \in \mathcal{D}(A^\text{op}_2 \otimes A)$ if $i^* \cong - \otimes_A^L Y$ and $j_1 \cong - \otimes_{A_2}^L Y_2$.}

By Theorem \ref{thm:standard} and Remark \ref{rem:standard} we know any recollement of derived categories of algebras is equivalent to a standard one.

\textbf{Theorem 10.} Let $A_1, A$ and $A_2$ be algebras, and $(\mathcal{D}(A_1), \mathcal{D}(A), \mathcal{D}(A_2), i^*, i_*, j^*, j_*)$ a standard recollement given by $Y \in \mathcal{D}(A^\text{op} \otimes A_1)$ and $Y_2 \in \mathcal{D}(A^\text{op}_2 \otimes A)$. Then

\begin{equation}
\begin{aligned}
i^* &\cong - \otimes_A^L Y, \\
i_* &\cong i_1 \cong \text{RHom}_{A_1}(Y, -), \\
j^* &\cong \text{RHom}_{A}(\text{RHom}_{A_1}(Y, A_1), -), \\
j_* &\cong \text{RHom}_{A_2}(\text{RHom}_{A_1}(Y, A_2), -),
\end{aligned}
\end{equation}

and
(2) for each algebra $B$, there is a recollement $(D(B \otimes A_1), D(B \otimes A), D(B \otimes A_2), I^*, I, I^!, J_1, J)$ such that

\[
I^* \cong - \otimes_A^{L_1} Y, \quad J_1 \cong - \otimes_A^{L_2} Y_2, \\
I_s = I_1 \cong \mathcal{RHom}_{A_1}(Y, -), \quad J^* = J^* \cong \mathcal{RHom}_A(Y_2, -), \\
I^! \cong \mathcal{RHom}_A(\mathcal{RHom}_{A_1}(Y, A_1), -), \quad J_s \cong \mathcal{RHom}_{A_2}(\mathcal{RHom}_A(Y_2, A), -).
\]

Proof. (1) Since $i_*$ is a left adjoint of $i^!$, it commutes with small coproduct. The functor $- \otimes_A^L Y \cong i^*$ has a right adjoint $i_*$ which commutes with small coproduct, thus $Y$ is perfect in $D(A_1)$. Therefore, $\mathcal{RHom}_{A_1}(Y, -) \cong - \otimes_A^{L_1} \mathcal{RHom}_{A_1}(Y, A_1)$ (ref. [32, Lemma 2.6]). Since the right adjoint is unique up to natural isomorphism, we can take $i_*=i_!$ and $i^!=i^!$ as required. Similar for $j^*=j^!$ and $j_s$.

(2) The proof of Lemma 11 implies that, for each algebra $B$, there is a recollement $(D(B \otimes A_1), D(B \otimes A), D(B \otimes A_2), I^*, I_s = I_1, I^!, J, J^* = J^*, J_s)$ such that

\[
J^* \cong - \otimes_A^{L_2} Y_2, \\
I_s = I_1 \cong - \otimes_A^{L_1} \mathcal{RHom}_{A_1}(Y, A_1), \quad J^* = J^* \cong \mathcal{RHom}_A(Y_2, -), \\
I^! \cong \mathcal{RHom}_A(\mathcal{RHom}_{A_1}(Y, A_1), -), \quad J_s \cong \mathcal{RHom}_{A_2}(\mathcal{RHom}_A(Y_2, A), -).
\]

Since $I^*$ is a left adjoint of $- \otimes_A^{L_1} \mathcal{RHom}_{A_1}(Y, A_1) \cong \mathcal{RHom}_{A_1}(Y, -)$ and the left adjoint is unique up to natural isomorphism, we have $I^* \cong - \otimes_A^{L_1} Y$. 

6 Recollements and Hochschild homology

In this section, we shall apply the results obtained in section 5 to study the relation between recollements of derived categories of algebras and Hochschild homology of algebras, which had been clarified by Keller in [22]. Recall that the $n$-th Hochschild homology of an algebra $A$ is $HH_n(A) := \text{Tor}_n^A(A, A) \cong H^{-n}(A \otimes_A^{L} A)$. Note that in $D(k)$ the complex $A \otimes_A^{L} A$ is isomorphic to the Hochschild complex of $A$. The following result is due to Keller, which is a corollary of [22, Theorem 3.1] (ref. [22, Remarks 3.2 (a)]). Here, we apply Theorem 9 to give a direct proof.

**Theorem 11.** (Keller [22]) Let $A$, $A_1$ and $A_2$ be algebras, and $D(A)$ admit a recollement relative to $D(A_1)$ and $D(A_2)$. Then there is a triangle in $D(k)$:

$$A_2 \otimes_{A_2}^{L} A_2 \rightarrow A \otimes_A^{L} A \rightarrow A_1 \otimes_{A_1}^{L} A_1 \rightarrow .$$
Proof. Applying Theorem 9 to the case $B = A^\text{op}$, we obtain a recollement $(D(A^\text{op} \otimes A_1), D(A^\text{op} \otimes A), D(A^\text{op} \otimes A_2), I^*, I^\dagger, J, J^! = J^*, J^!_*)$. Furthermore, we have a triangle in $D(A^e)$:

$$J^! J^1 A = \text{RHom}_A(Y_2, A) \otimes_{A_2}^L Y_2 \to A \to I^! I^* A = Y \otimes_{A_1}^L Y_1 \to .$$

The derived tensor functor $- \otimes_{A^e}^L A : D(A^e) \to D(k)$ sends this triangle to a triangle in $D(k)$:

$$(\text{RHom}_A(Y_2, A) \otimes_{A_2}^L Y_2) \otimes_{A^e}^L A \to A \otimes_{A^e}^L A \to (Y \otimes_{A_1}^L Y_1) \otimes_{A^e}^L A \to .$$

Its left hand side

$$(\text{RHom}_A(Y_2, A) \otimes_{A_2}^L Y_2) \otimes_{A^e}^L A \cong (Y_2 \otimes_{A}^L \text{RHom}_A(Y_2, A)) \otimes_{A_2}^L A_2$$

$$\cong J^! J^1 A_2 \otimes_{A_2}^L A_2$$

and its right hand side

$$(Y \otimes_{A_1}^L Y_1) \otimes_{A^e}^L A \cong (Y_1 \otimes_{A_2}^L Y) \otimes_{A_1}^L A_1 \cong I^* I^! A_1 \otimes_{A_1}^L A_1 \cong A_1 \otimes_{A_1}^L A_1,$$

equivalently we have a triangle in $D(k)$:

$$A_2 \otimes_{A_2}^L A_2 \to A \otimes_{A^e}^L A \to A_1 \otimes_{A_1}^L A_1 \to .$$

From the triangle in Theorem 11 by taking cohomologies, we can obtain a long exact sequence on the Hochschild homologies of the algebras:

$$\cdots \to \text{HH}_{n+1}(A_1) \to \text{HH}_n(A_2) \to \text{HH}_n(A) \to \text{HH}_n(A_1) \to \cdots .$$

For perfect recollements of derived categories of algebras, we have the following stronger conclusion:

**Theorem 12.** (Keller [22, Remarks 3.2 (b)]) Let $A, A_1$ and $A_2$ be algebras, and $D(A)$ admit a recollement relative to $D(A_1)$ and $D(A_2)$. Then there is a long exact sequence on the Hochschild homologies of these algebras:

$$\cdots \to \text{HH}_{n+1}(A_1) \to \text{HH}_n(A_2) \to \text{HH}_n(A) \to \text{HH}_n(A_1) \to \cdots .$$

Indeed, perfect recollements of derived categories of algebras correspond to “derived triangulated matrix (differential graded) algebras”. Then Kadijon’s method works here (ref. [29, 1.2.15]).
7 Recollements and Hochschild cohomology

In this section, we shall apply the results obtained in section 5 to observe the relations between recollements of derived categories of algebras and Hochschild cohomology of algebras. Recall that the $n$-th Hochschild cohomology of an algebra $A$ is $HH^n(A) := \text{Ext}^n_{A^{op}}(A, A) \cong H^n(\text{RHom}_{A^{op}}(A, A))$. Note that in $D(k)$ the complex $\text{RHom}_{A^{op}}(A, A)$ is isomorphic to the Hochschild cochain complex or Hochschild cocomplex of $A$. From a recollement of derived categories of algebras, we can obtain three triangles on Hochschild cocomplexes of these algebras, which can induce three long exact sequences on their Hochschild cohomologies.

Lemma 3. Let $A$ be an algebra and $X \xrightarrow{u} Y \xrightarrow{v} Z \rightarrow$ a triangle in $D(A)$ such that $\text{RHom}_A(X, Z) = 0$ in $D(k)$. Then there are three triangles in $D(k)$:

1. $\text{RHom}_A(Y, X) \rightarrow \text{RHom}_A(Y, Y) \xrightarrow{\phi} \text{RHom}_A(Z, Z) \rightarrow$
2. $\text{RHom}_A(Z, Y) \rightarrow \text{RHom}_A(Y, Y) \xrightarrow{\psi} \text{RHom}_A(X, X) \rightarrow$
3. $\text{RHom}_A(Z, X) \rightarrow \text{RHom}_A(Y, Y) \xrightarrow{\varphi} \text{RHom}_A(X, X) \oplus \text{RHom}_A(Z, Z) \rightarrow$.

Moreover, $\phi$ (resp. $\psi, \varphi$) induces a homomorphism of graded rings $\bar{\phi}$ (resp. $\bar{\psi}, \bar{\varphi}$) between the corresponding cohomology rings.

Proof. Applying the bifunctor $\text{RHom}_A(-, -)$ to the triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \rightarrow$, we have the following commutative diagram:

\[
\begin{array}{cccccc}
\text{RHom}_A(X[1], Z[-1]) & \rightarrow & \text{RHom}_A(X[1], X) & \rightarrow & \text{RHom}_A(X[1], Y) & \rightarrow & \text{RHom}_A(X[1], Z) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{RHom}_A(Z, Z[-1]) & \rightarrow & \text{RHom}_A(Z, X) & \rightarrow & \text{RHom}_A(Z, Y) & \rightarrow & \text{RHom}_A(Z, Z) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{RHom}_A(Y, Z[-1]) & \rightarrow & \text{RHom}_A(Y, X) & \rightarrow & \text{RHom}_A(Y, Y) & \rightarrow & \text{RHom}_A(Y, Z) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{RHom}_A(X, Z[-1]) & \rightarrow & \text{RHom}_A(X, X) & \rightarrow & \text{RHom}_A(X, Y) & \rightarrow & \text{RHom}_A(X, Z) \\
\end{array}
\]

in which the four corners are zero by the assumption $\text{RHom}_A(X, Z) = 0$ in $D(k)$. It follows two triangles (1) and (2).

By Octahedral axiom, we have the following commutative diagram:

\[
\begin{array}{cccccc}
\text{RHom}_A(Z, Z[-1]) & = & \text{RHom}_A(Z, Z[-1]) \\
\downarrow & & \downarrow \\
\text{RHom}_A(Z, X) & \rightarrow & \text{RHom}_A(Y, X) & \rightarrow & \text{RHom}_A(X, X) & \rightarrow & \text{RHom}_A(Z[-1], X) \\
\| & & \downarrow & & \downarrow & & \| \\
\text{RHom}_A(Z, X) & \rightarrow & \text{RHom}_A(Y, Y) & \rightarrow & \text{RHom}_A(X, X) \oplus \text{RHom}_A(Z, Z) & \rightarrow & \text{RHom}_A(Z[-1], X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{RHom}_A(Z, Z) & = & \text{RHom}_A(Z, Z) \\
\end{array}
\]
where the morphism $\text{RHom}_A(Z, Z[-1]) \to \text{RHom}_A(X, X)$ is zero. It follows the triangle (3).

For the last statement, it is enough to note that $\phi$ induces a map
\[
\tilde{\phi} : \oplus_{n \in \mathbb{Z}} \text{Hom}_{D(A)}(Y, Y[n]) \to \oplus_{n \in \mathbb{Z}} \text{Hom}_{D(A)}(Z, Z[n])
\]
sending $f_n \in \text{Hom}_{D(A)}(Y, Y[n])$ to $\tilde{\phi}(f_n) \in \text{Hom}_{D(A)}(Z, Z[n])$ such that $\bar{\phi}(f_n) \circ v = v[n] \circ f_n$, i.e., the following diagram in $D(A)$ is commutative:

\[
\begin{array}{ccc}
Y & \xrightarrow{\phi} & Z \\
\downarrow f_n & & \downarrow \tilde{\phi}(f_n) \\
Y[n] & \xrightarrow{v[n]} & Z[n],
\end{array}
\]
which is clearly a homomorphism of graded rings. Similar for $\tilde{\psi}$ and $\bar{\phi}$.

Let $A$ and $B$ be algebras. Denote by $\text{rep}(B, A)$ the full subcategory of $D(B^{\text{op}} \otimes A)$ consisting of all complexes of $B$-$A$-bimodules which are perfect when restricted to complexes of $A$-modules.

**Lemma 4.** Let $A$ and $B$ be algebras. Then the derived functor $\text{RHom}_A(\cdot, A) : D(B^{\text{op}} \otimes A) \to D(A^{\text{op}} \otimes B)$ induces a dual from $\text{rep}(B, A)$ to $\text{rep}(B^{\text{op}}, A^{\text{op}})$.

**Proof.** The functor $\text{RHom}_A(\cdot, A)$ (resp. $\text{RHom}_{A^{\text{op}}}(\cdot, A)$) is a contravariant functor from $D(B^{\text{op}} \otimes A)$ to $D(A^{\text{op}} \otimes B)$ (resp. from $D(A^{\text{op}} \otimes B)$ to $D(B^{\text{op}} \otimes A)$). Let $I$ be a homotopically injective resolution of $A$ as $A$-$A$-bimodule. Then the functors $\text{RHom}_A(\cdot, A)$ (resp. $\text{RHom}_{A^{\text{op}}}(\cdot, A)$) and $\text{Hom}_A(\cdot, I)$ (resp. $\text{Hom}_{A^{\text{op}}}(\cdot, I)$) are natural isomorphisms.

If $X \in \text{rep}(B, A)$ then, in $D(A^{\text{op}})$, $\text{RHom}_A(X, A) \cong \text{Hom}_A(X_A, I) \cong \text{RHom}_A(X_A, A)$ is perfect. Thus the functor $\text{RHom}_A(\cdot, A)$ can be restricted to a functor from $\text{rep}(B, A)$ to $\text{rep}(B^{\text{op}}, A^{\text{op}})$. Analogously, $\text{RHom}_{A^{\text{op}}}(\cdot, A)$ can be restricted to a functor from $\text{rep}(B^{\text{op}}, A^{\text{op}})$ to $\text{rep}(B, A)$.

We have clearly a natural transformation $\phi : 1 \to \text{Hom}_{A^{\text{op}}}(\text{Hom}_A(\cdot, I), I)$ between these two functors from $\text{rep}(B, A)$ to $\text{rep}(B, A)$. Now it is enough to show that $\phi_X$ is an isomorphism in $\text{rep}(B, A)$ or $D(B^{\text{op}} \otimes A)$, i.e., $\phi_X$ is a quasi-isomorphism of complexes of $B$-$A$-bimodules. If we can prove that $\phi_X$ is a quasi-isomorphism of complexes of $A$-modules, we will be done. This is clear, since $\text{Hom}_{A^{\text{op}}}(\text{Hom}_A(X, I), I)_A = \text{Hom}_{A^{\text{op}}}(\text{Hom}_A(X_A, I), I) \cong \text{RHom}_{A^{\text{op}}}(\text{RHom}_A(X_A, A), A) \cong X_A$ in $D(A)$.

The main result in this section is the following:
Theorem 13. Let \(A_1, A\) and \(A_2\) be algebras, and \((D(A_1), D(A), D(A_2), i^*, i_*) = i, i', i, j^* = j^*, j_*)\) a standard recollement given by \(Y \in D(A_2^{op} \otimes A_1)\) and \(Y_2 \in D(A_1^{op} \otimes A)\). Then there are three triangles in \(D(k)\):

(1) \(\text{RHom}_{A_2}(A, \text{RHom}_A(Y_2, A) \otimes_{A_2} Y_2)\) \\
\(\rightarrow \text{RHom}_{A_2}(A, A) \xrightarrow{\psi^*} \text{RHom}_{A_1}(A_1, A_1) \rightarrow\) \\
(2) \(\text{RHom}_{A_2}(\text{RHom}_A(Y, A), Y)\) \\
\(\rightarrow \text{RHom}_{A_2}(A, A) \xrightarrow{\phi} \text{RHom}_{A_1}(A_2, A_2) \rightarrow\) \\
(3) \(\text{RHom}_{A_2}(\text{RHom}_A(Y, Y), \text{RHom}_A(Y_2, A) \otimes_{A_2} Y_2)\) \\
\(\rightarrow \text{RHom}_{A_2}(A, A) \xrightarrow{\psi^*} \text{RHom}_{A_1}(A_1, A_1) \oplus \text{RHom}_{A_2}(A_2, A_2) \rightarrow\).

Moreover, \(\phi\) (resp. \(\psi, \varphi\)) induces a homomorphism of graded rings \(\tilde{\phi}\) (resp. \(\tilde{\psi}, \tilde{\varphi}\)) between Hochschild cohomology rings.

Proof. By Theorem 10, we have a recollement \((D(A_2^{op} \otimes A_1), D(A^{op} \otimes A), D(A_2^{op} \otimes A_2), I^*, I_* = I, I', J_*, J^* = J^*, J_*)\) such that

\[
\begin{align*}
I^* &\cong - \otimes_{A} Y, \\
J_1 &\cong - \otimes_{A_2} Y_2, \\
I_* &\cong \text{RHom}_{A_1}(Y, -), \\
J^* &\cong \text{RHom}_{A}(Y_2, -), \\
I' &\cong \text{RHom}_{A}(\text{RHom}_{A_1}(Y, A_1), -), \\
J_* &\cong \text{RHom}_{A_2}(\text{RHom}_{A}(Y_2, A), -).
\end{align*}
\]

Thus we obtain a triangle \(J_1 J^* A \rightarrow A \rightarrow I_* I^* A \rightarrow\) in \(D(A_2^{op} \otimes A)\). By Lemma 3 we have three triangles in \(D(k)\):

(1) \(\text{RHom}_{A^*}(A, J_1 J^* A) \rightarrow \text{RHom}_{A^*}(A, A) \rightarrow \text{RHom}_{A^*}(I_* I^* A, I_* I^* A) \rightarrow\) \\
(2) \(\text{RHom}_{A^*}(I_* I^* A, A) \rightarrow \text{RHom}_{A^*}(A, A) \rightarrow \text{RHom}_{A^*}(J_1 J^* A, J_1 J^* A) \rightarrow\) \\
(3) \(\text{RHom}_{A^*}(I_* I^* A, J_1 J^* A) \rightarrow \text{RHom}_{A^*}(A, A) \rightarrow \text{RHom}_{A^*}(J_1 J^* A, J_1 J^* A) \oplus \text{RHom}_{A^*}(I_* I^* A, I_* I^* A) \rightarrow\).

By Lemma 4 and Theorem 10 we have

\[
\begin{align*}
\text{RHom}_{A^*}(J_1 J^* A, J_1 J^* A) &\cong \text{RHom}_{A^* \otimes A_2}(J_1^* A, J_1^* A) \\
&\cong \text{RHom}_{A^* \otimes A_2}(\text{RHom}_{A}(Y_2, A), \text{RHom}_{A}(Y_2, A)) \\
&\cong \text{RHom}_{A_2^* \otimes A_2}(Y_2, Y_2) \\
&\cong \text{RHom}_{A_2^*}(A_2, A_2)
\end{align*}
\]

and

\[
\begin{align*}
\text{RHom}_{A^*}(I_* I^* A, I_* I^* A) &\cong \text{RHom}_{A^* \otimes A}(I_* A, I_* A) \\
&\cong \text{RHom}_{A^* \otimes A}(Y, Y) \\
&\cong \text{RHom}_{A_2^* \otimes A}(\text{RHom}_{A_1}(Y, A_1), \text{RHom}_{A_1}(Y, A_1)) \\
&\cong \text{RHom}_{A_1^*}(A_1, A_1).
\end{align*}
\]
Thus there are three triangles in $D(k)$ as required.

For the last statement, it is enough to note that $\phi$ induces a map

$$\bar{\phi} : \oplus_{n \in \mathbb{Z}} \text{Hom}_{D(A^i)}(A, A[n]) \to \oplus_{n \in \mathbb{Z}} \text{Hom}_{D(A^f)}(A_1, A_1[n])$$

sending $f_n \in \text{Hom}_{D(A^i)}(A, A[n])$ to $\bar{\phi}(f_n) \in \text{Hom}_{D(A_1^f)}(A_1, A_1[n])$ such that the following diagram in $D(A_1^{op} \otimes A)$ is commutative:

$$\begin{array}{ccc}
\text{RHom}_{A_1}(Y, A_1) & \xrightarrow{\text{RHom}_{A_1}(Y, \bar{\phi}(f_n))} & \text{RHom}_{A_1}(Y, A_1[n]) \\
\downarrow \cong & & \downarrow \cong \\
\text{RHom}_{A_1}(Y[n], A_1[n]) & \xrightarrow{\text{RHom}_{A_1}(f_n \otimes \bar{Y}, A_1[n])} & \text{RHom}_{A_1}(Y, A_1[n]),
\end{array}$$

which is clearly a homomorphism of graded rings. Similar for $\bar{\psi}$ and $\bar{\phi}$. \qed

From the three triangles in Theorem 13 by taking cohomologies, we can obtain three long exact sequences on the Hochschild cohomologies of the algebras:

**Corollary 3.** Let $A_1$, $A$ and $A_2$ be algebras, and $(D(A_1), D(A), D(A_2), i^*, i_*) = i^*, j_1, j_2 = j^*, j_*$) a standard recollement given by $Y \in D(A^{op} \otimes A_1)$ and $Y_2 \in D(A_2^{op} \otimes A)$. Then there are three long exact sequences:

1. $\cdots \to \text{Ext}^n_{A_i}(A, \text{RHom}_A(Y_2, A) \otimes_{A_2} Y_2) \to HH^n(A) \xrightarrow{\phi} HH^n(A_1) \to \cdots$
2. $\cdots \to \text{Ext}^n_{A_i}(\text{RHom}_{A_i}(Y, Y), A) \to HH^n(A) \xrightarrow{\psi} HH^n(A_2) \to \cdots$
3. $\cdots \to \text{Ext}^n_{A_i}(\text{RHom}_{A_i}(Y, Y), \text{RHom}_A(Y_2, A) \otimes_{A_2} Y_2) \to HH^n(A) \xrightarrow{\varphi} HH^n(A_1) \oplus HH^n(A_2) \to \cdots$.

Moreover, $\oplus_{n \in \mathbb{N}}(\phi_n \text{ resp. } \oplus_{n \in \mathbb{N}}(\psi_n, \oplus_{n \in \mathbb{N}}(\varphi_n))$ is a homomorphisms of graded rings between Hochschild cohomology rings.

Applying Corollary 3 to Example 1 we can obtain the following result due to König and Nagase:

**Corollary 4.** (König-Nagase \cite{König-Nagase}) Let $A$ be an algebra, $e$ an idempotent of $A$ and $eAe$ a stratifying ideal of $A$. Then there are three long exact sequences:

1. $\cdots \to \text{Ext}^n_{A_i}(A, eAe) \to HH^n(A) \xrightarrow{\psi} HH^n(A/eAe) \to \cdots$
2. $\cdots \to \text{Ext}^n_{A_i}(A/eAe, A) \to HH^n(A) \xrightarrow{\phi} HH^n(eAe) \to \cdots$
3. $\cdots \to \text{Ext}^n_{A_i}(A/eAe, eAe) \to HH^n(A) \xrightarrow{\varphi} HH^n(A/eAe) \oplus HH^n(eAe) \to \cdots$. 

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Moreover, $\oplus_{n \in \mathbb{N}} \phi_n$ (resp. $\oplus_{n \in \mathbb{N}} \psi_n$, $\oplus_{n \in \mathbb{N}} \varphi_n$) is a homomorphisms of graded rings between Hochschild cohomology rings.

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References

[1] L. Angeleri Hügel, S. König and Q.H. Liu, Recollements and tilting objects, J. Pure Appl. Algebra 215 (2011), 420–438.
[2] L. Angeleri Hügel, S. König, Q.H. Liu and D. Yang, Derived simple algebras and restrictions of recollements of derived module categories, Preprint.
[3] M. Auslander, On the dimension of modules and algebras, III : Global dimension, Nagoya Math. J. 9 (1955), 67–77.
[4] M. Auslander, On the dimension of modules and algebras, VI : Comparison of global and algebra dimension, Nagoya Math. J. 11 (1957), 61–65.
[5] A.A. Beilinson, J. Bernstein and P. Deligne, Faisceaux pervers, Astérisque 100 (1982).
[6] B. Benddifaalah and D. Guin, Cohomologie de l’algèbre triangulaire et applications, J. Algebra 282 (2004), 513–537.
[7] H. Cartan and S. Eilenberg, Homological algebra, Princeton University Press, 1956.
[8] C. Cibils, Tensor Hochschild homology and cohomology, in : Interactions between ring theory and representations of algebras, Lecture Notes in Applied Mathematics, Vol. 210, Dekker, New York, 2000, pp. 35–51.
[9] C. Cibils, E. Marcos, M.J. Redondo and A. Solotar, Cohomology of split algebras and trivial extensions, Glasgow Math. J. 45 (2003), 21–40.
[10] E. Cline, B. Parshall and L. Scott, Finite-dimensional algebras and highest weight categories, J. Reine Angew. Math. 391 (1988), 85–99.
[11] E. Cline, B. Parshall and L. Scott, Stratifying endomorphism algebras, Mem. Amer. Math. Soc. 591 (1996), 1–119.
[12] J. Cuntz and D. Quillen, Algebra extensions and nonsingularity, J. Amer. Math. Soc. 8 (1995), 251–289.
[13] S. Eilenberg, A. Rosenberg and D. Zelinsky, On the dimension of modules and algebras, VIII : Dimension of tensor products, Nagoya Math. J. 12 (1957), 71–93.
[14] E.L. Green, E.N. Marcos and N. Snashall, The Hochschild cohomology ring of a one point extension, Comm. Algebra 31 (2003), 357–379.
[15] E.L. Green and Ø. Solberg, Hochschild cohomology rings and triangular rings, in : Representations of algebras, Vol. II, Beijing Normal University Press, Beijing, 2002, pp. 192–200.
[16] D. Happel, Hochschild cohomology of finite-dimensional algebras, in: Séminaire d’Algèbre Paul Dubreil et Marie-Paul Malliavin, 39ème Année (Paris, 1987/1988), Lecture Notes in Math., Vol. 1404, Springer, Berlin, 1989, pp. 108–126.

[17] D. Happel, Reduction techniques for homological conjectures, Tsukuba J. Math. 17 (1993), 115–130.

[18] G. Hochschild, B. Kostant and A. Rosenberg, Differential forms on regular affine algebras, Trans. Amer. Math. Soc. 102 (1962), 383–408.

[19] K. Igusa and D. Zacharia, On the cyclic homology of monomial relation algebras, J. Algebra 151 (1992), 502–521.

[20] P. Jørgensen, Recollement for differential graded algebras, J. Algebra 299 (2006), 589–601.

[21] B. Keller, Deriving DG categories, Ann. Sci. École Norm. Sup. 27 (1994), 63–102.

[22] B. Keller, Invariance and localization for cyclic homology of DG algebras, J. Pure Appl. Algebra 123 (1998), 223–273.

[23] S. König, Tilting complexes, perpendicular categories and recollements of derived module categories of rings, J. Pure Appl. Algebra 73 (1991), 211–232.

[24] S. König and A. Zimmermann, Derived equivalences for group rings, Lecture Notes in Math. 1685, Springer-Verlag, 1998.

[25] S. König and H. Nagase, Hochschild cohomology and stratifying ideals, J. Pure Appl. Algebra 213 (2009), 886–891.

[26] M. Kontsevich and A. Rosenberg, Noncommutative smooth spaces, The Gelfand Mathematical Seminars, 1996–1999, 85–108, Birkhäuser Boston, 2000.

[27] M. Kontsevich and Y. Soibelman, Notes on $A_\infty$-algebras, $A_\infty$-categories and noncommutative geometry. I, arXiv:math.RA/0606241.

[28] H. Krause, Localization theory for triangulated categories, arXiv:0806.1324 [math.CT].

[29] J.L. Loday, Cyclic homology, Grundlehren 301, Springer, Berlin, 1992.

[30] S. Micheletta and M.I. Platzeck, Hochschild cohomology of triangular matrix algebras, J. Algebra 233 (2000), 502–525.

[31] J. Miyachi, Localization of triangulated categories and derived categories, J. Algebra 141 (1991), 463–483.

[32] J. Miyachi, Recollement and tilting complexes, J. Pure Appl. Algebra 183 (2003), 245–273.

[33] P. Nicolás and M. Saorin, Parametrizing recollement data for triangulated categories, J. Algebra 322 (2009), 1220–1250.

[34] P. Nicolás and M. Saorin, Lifting and restricting recollement data, Appl. Categor. Struct., DOI 10.1007/s10485-009-9198-z.
[35] B. Parshall and L. Scott, Derived categories, quasi-hereditary algebras and algebraic groups, Carlton Univ. Math. Notes 3 (1988), 1–104.

[36] J.A. de la Peña and C.C. Xi, Hochschild cohomology of algebras with homological ideals, Tsukuba J. Math. 30 (2006), 61–79.

[37] J. Rickard, Morita theory for derived categories, J. London Math. Soc. 39 (1989), 436–456.

[38] J. Rickard, Derived equivalences as derived functors, J. London Math. Soc. 43 (1991), 37–48.

[39] M. Suárez-Alvarez, Applications of the change-of-rings spectral sequence to the computation of Hochschild cohomology, [arXiv:0707.3210](https://arxiv.org/abs/0707.3210)

[40] M. Van den Bergh, A relation between Hochschild homology and cohomology for Gorenstein rings, Proc. Amer. Math. Soc. 126 (1998), 1345–1348. Erratum, Proc. Amer. Math. Soc. 130 (2002), 2809–2810.

[41] A. Wiedemann, On stratifications of derived module categories, Canad. Math. Bull. 34 (1991), 275–280.