Trace regularity for biharmonic evolution equations with Caputo derivatives

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Abstract
Our goal is to establish a hidden regularity result for solutions of time fractional Petrovsky systems. The order \( \alpha \) of the Caputo fractional derivative belongs to the interval \((1, 2)\). We achieve such result for a suitable class of weak solutions.

Keywords Caputo fractional derivative (primary) · Fractional Petrovsky systems · Riemann–Liouville fractional integral · Hidden regularity

Mathematics Subject Classification 26A33 (primary) · 35D30

1 Introduction

In the recent years the study of viscoelastic materials has been addressed by several authors. In particular, Petrovsky systems model the free vibrations of beams and plates. The problem to show a trace regularity for weak solutions of a non fractional Petrovsky system has been studied in [4].

Slowing processes can be globally described using fractional calculus. Nowadays, fractional calculus has attracted an increasing interest in modeling evolution equations. We refer to Mainardi and Gorenflo [12] for a survey devoted to the theory of relaxation processes governed by linear differential equations of fractional order, where the fractional derivatives are intended both in the sense of Riemann-Liouville and in the sense of Caputo.

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To our best knowledge a fractional formulation for Petrovsky systems has not been studied yet. In this paper we concentrate our analysis on establishing hidden regularity for weak solutions of

\[
\begin{aligned}
\partial_t^\alpha u + \Delta^2 u &= 0 \quad \text{in} \quad (0, T) \times \Omega, \\
u &= \Delta u = 0 \quad \text{on} \quad (0, T) \times \partial \Omega,
\end{aligned}
\]  

(1.1)

where \(\partial_t^\alpha u\) is the Caputo derivative of order \(\alpha \in (1, 2)\).

The linear vibrations of elastic body occur at discrete frequencies depending on the geometry and material systems. These phenomena are important in applications such as engineer structures. They are ruled by frequencies that are eigenvalues and displacement fields that are the eigenfunctions. By the linearity assumptions the superposition principle applies and the solution can be expressed by a series. In the fractional framework the Mittag-Leffler functions have a crucial role (see [3]) and take place of the exponential functions in the series.

A first open question we intend to investigate is the existence of weak solutions for problem (1.1) in appropriate Sobolev spaces. We also analyse if trace regularity holds for weak solutions. For non fractional hyperbolic partial differential equations trace regularity is also known as hidden regularity, a concept introduced by J.L. Lions in [6], see also [5].

One can consider plate equations with different boundary conditions. Nevertheless, in the literature two classes of boundary conditions are mainly discussed. The first one involves the function and its normal derivative on the boundary (Dirichlet-Neumann) and it describes the physical model of vibration of guided beams and plates. The second type of boundary conditions includes the function and its Laplacian on the boundary: it depicts the physical model of vibration of hinged beams and plates, see [8] and also [4]. Both problems have been studied in control theory to get the exact controllability by acting on (eventually on a part of) the boundary, see [7].

To our best knowledge, the time-fractional Petrovsky system has not been studied from the point of view of the hidden regularity for weak solutions. This paper would be a first attempt to solve some questions about trace regularity.

A first difficulty is an appropriate definition for weak solutions. To understand the importance of a proper request for the regularity of solutions, we refer to [18], where the author points out that the second time derivative of the solutions can blow up at \(t = 0\) even in presence of regular initial and boundary data. Due to such singularities, shown in some examples, it is not convenient to make the \(a \text{ priori}\) assumption that second order time derivatives of the solutions are smooth on the closure of the domain, otherwise one obtains solutions belonging to a very restricted subclass.

To overcome the above problem, in [10] we introduce the definition of \(H^2\)-solutions. If we consider the operator \(A\) defined below in (2.1), for a \(H^2\)-solution of the fractional boundary value problem (1.1) we mean a function \(u\) belonging to the space \(C([0, T]; H^2(\Omega) \cap H^1_0(\Omega))\) with \(u_t \in L^2(0, T; L^2(\Omega)) \cap C([0, T]; D(A^{-\theta}))\), \(\theta \in (0, 1)\), and satisfying for any \(v \in H^2(\Omega) \cap H^1_0(\Omega)\)

\[
\int_\Omega I^{2-\theta}(u_t(\cdot, x) - u_t(0, x))v(x) \, dx \in C^1([0, T])
\]
and
\[
\frac{d}{dt} \int_{\Omega} I^{2-\alpha}(u_t(t,x) - u_t(0,x))v(x) \, dx + \int_{\Omega} \Delta u(t,x) \Delta v(x) \, dx = 0, \quad t \in (0,T).
\]

The notion of $H^2$-solution is suggested by the analysis of the stationary case given in [14].

As a consequence of the existence and uniqueness result for $H^2$-solutions, the normal derivative of the solution on the boundary is well defined. Our purpose is to give a meaning to the normal derivative of weaker solutions belonging to $H^1_0(\Omega)$.

A function $u \in C([0,T]; H^1_0(\Omega))$ is called a $H^1$-solution of (1.1) if,

\[
\int_{\Omega} \partial^\alpha_t w(t,x)v(x) \, dx - \int_{\Omega} \nabla \Delta w(t,x) \cdot \nabla v(x) \, dx = 0 \quad v \in H^1_0(\Omega), \quad t \in (0,T).
\]

Thanks to this setting we are able to handle the question of well-posedness. Indeed, we establish the following result.

**Theorem 1** Let $u_0 \in H^1_0(\Omega)$ and $u_1 \in H^{-1}(\Omega)$. Then the function

\[
u(t,x) = \infty \sum_{n=1}^{\infty} \left[ (u_0, e_n) E_\alpha(-\lambda_n t^\alpha) + (u_1, e_n) E_{\alpha,2}(-\lambda_n t^\alpha) \right] e_n(x)
\]

is the $H^1$-solution of (1.1) satisfying the initial conditions

\[
u(0, \cdot) = u_0, \quad \nu_t(0, \cdot) = u_1. \tag{1.2}
\]

The next step is to prove the hidden regularity for $H^1$-solutions. Although the result is similar to [4, pag. 29], several technical steps need a change in order to avoid integration by parts with respect to the time. This difficulty also appears in [9] and we borrow from [9] the way to deal with some identities useful in the proof of the hidden regularity result.

**Theorem 2** For $u_0 \in H^1_0(\Omega)$ and $u_1 \in H^{-1}(\Omega)$, if $u$ is the $H^1$-solution of (1.1)–(1.2) then for any $T > 0$ we have

\[
\int_0^T \int_{\partial\Omega} \partial_t u|^2 \, d\sigma \, dt \leq C \left( \|u_0\|_{L^1(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2 \right),
\]

for some constant $C = C(T)$ independent of the initial data.

The paper is organized as follows. In Section 2 we give a mathematical background on biharmonic operators and fractional derivatives. In Section 3 we introduce $H^1$ and $H^3$ solutions for fractional Petrovsky systems and provide existence theorems. In Section 4 we give a result of trace regularity for biharmonic evolution equations with Caputo derivatives. Section 5 is devoted to conclusions and further questions.
2 Preliminaries

Here we collect some notations, definitions and known results that we use to prove our main results.

2.1 Biharmonic operators

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded open set with sufficiently smooth boundary denoted by $\partial \Omega$. As usual, we consider $L^2(\Omega)$ endowed with the inner product and norm defined by

$$\langle u, v \rangle = \int_{\Omega} u(x)v(x) \, dx, \quad \|u\|_{L^2(\Omega)} = \left( \int_{\Omega} |u(x)|^2 \, dx \right)^{1/2} \quad u, v \in L^2(\Omega).$$

We define the operator $A$ in $L^2(\Omega)$ by

$$D(A) = \{u \in H^4(\Omega) : u = \Delta u = 0 \text{ on } \partial \Omega\}$$

$$(Au)(x) = \Delta^2 u(x), \quad u \in D(A), \ x \in \Omega. \quad (2.1)$$

The spectrum of the operator $A$ consists of a sequence $\{\lambda_n\}$ tending to $+\infty$. Moreover, we assume that the eigenvalues $\lambda_n$ are all distinct numbers, and hence the eigenspace generated by $\lambda_n$ has dimension one. For special domains this assumption is fulfilled, e.g. if $\Omega$ is a ball in $\mathbb{R}^N$.

In addition, we denote by $e_n$ the eigenfunctions of $A$ ($A e_n = \lambda_n e_n$) that constitute an orthonormal basis of $L^2(\Omega)$.

The biharmonic operator given by (2.1) is self-adjoint, positive and the domain $D(A)$ is dense in $L^2(\Omega)$.

The fractional powers $A^\theta$ are defined for $\theta > 0$, see e.g. [11, 15].

The domain $D(A^\theta)$ of $A^\theta$ consists of those functions $u \in L^2(\Omega)$ such that

$$\sum_{n=1}^{\infty} \lambda_n^{2\theta} |\langle u, e_n \rangle|^2 < +\infty$$

and

$$A^\theta u = \sum_{n=1}^{\infty} \lambda_n^\theta \langle u, e_n \rangle e_n, \quad u \in D(A^\theta).$$

Moreover, $D(A^\theta)$ is a Hilbert space with the norm given by

$$\|u\|_{D(A^\theta)} = \|A^\theta u\|_{L^2(\Omega)} = \left( \sum_{n=1}^{\infty} \lambda_n^{2\theta} |\langle u, e_n \rangle|^2 \right)^{1/2} \quad u \in D(A^\theta), \quad (2.2)$$

and for any $0 < \theta_1 < \theta_2$ we have $D(A^{\theta_2}) \subset D(A^{\theta_1})$. 

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In particular, \( D(A^{\frac{3}{4}}) = H^2(\Omega) \cap H^1_0(\Omega) \) and \( D(A^{\frac{1}{4}}) = H^1_0(\Omega) \) with the respective norms given by
\[
\|u\|_{D(A^{\frac{1}{2}})} = \left( \sum_{n=1}^{\infty} \lambda_n |\langle u, e_n \rangle|^2 \right)^{1/2} = \|\Delta u\|_{L^2(\Omega)} \quad u \in H^2(\Omega) \cap H^1_0(\Omega), \tag{2.3}
\]
\[
\|u\|_{D(A^{\frac{1}{4}})} = \left( \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}} |\langle u, e_n \rangle|^2 \right)^{1/2} = \|\nabla u\|_{L^2(\Omega)} \quad u \in H^1_0(\Omega). \tag{2.4}
\]

We also note that \( D(A^{\frac{3}{4}}) = \{ u \in H^3(\Omega) : u = \Delta u = 0 \text{ on } \partial \Omega \} \) and
\[
\|u\|_{D(A^{\frac{3}{4}})} = \left( \sum_{n=1}^{\infty} \lambda_n^{\frac{3}{2}} |\langle u, e_n \rangle|^2 \right)^{1/2} = \|\nabla \Delta u\|_{L^2(\Omega)} \quad u \in D(A^{\frac{3}{4}}), \tag{2.5}
\]

see [4, Lemma 1.7]. If we identify the dual \((L^2(\Omega))'\) with \(L^2(\Omega)\) itself, then we have \(D(A^\theta) \subset L^2(\Omega) \subset (D(A^\theta))'\). From now on we set
\[
D(A^{-\theta}) := (D(A^\theta))', \tag{2.6}
\]
whose elements are continuous linear functionals on \(D(A^\theta)\). If \(u \in D(A^{-\theta})\) and \(\varphi \in D(A^\theta)\) the value \(u(\varphi)\) is denoted by
\[
\langle u, \varphi \rangle_{-\theta, \theta} := u(\varphi). \tag{2.7}
\]
According to the Riesz-Fréchet representation theorem, \(D(A^{-\theta})\) is a Hilbert space with the norm given by
\[
\|u\|_{D(A^{-\theta})} = \left( \sum_{n=1}^{\infty} \lambda_n^{-2\theta} |\langle u, e_n \rangle_{-\theta, \theta}|^2 \right)^{1/2} \quad u \in D(A^{-\theta}). \tag{2.8}
\]
Moreover,
\[
A^{-\theta} u = \sum_{n=1}^{\infty} \lambda_n^{-\theta} \langle u, e_n \rangle_{-\theta, \theta} e_n, \quad \|u\|_{D(A^{-\theta})} = \|A^{-\theta} u\|_{L^2(\Omega)} \quad u \in D(A^{-\theta}),
\]
and for any \(0 < \theta_1 < \theta_2\) we have \(D(A^{-\theta_1}) \subset D(A^{-\theta_2})\). We also recall that
\[
\langle u, \varphi \rangle_{-\theta, \theta} = \langle u, \varphi \rangle \quad \text{for } u \in L^2(\Omega), \ \varphi \in D(A^\theta), \tag{2.9}
\]
see e.g. [1, Ch. V]. We note that $D(A^{-\frac{1}{4}}) = H^{-1}(\Omega)$ and

$$
\|u\|_{D(A^{-\frac{1}{4}})} = \left( \sum_{n=1}^{\infty} \lambda_n^{-\frac{1}{2}} |(u, e_n)_{-\frac{1}{4}, \frac{1}{4}}|^2 \right)^{1/2} = \|u\|_{H^{-1}(\Omega)} \quad u \in H^{-1}(\Omega).
$$

(2.10)

2.2 Fractional derivatives

**Definition 1** We denote the Riemann–Liouville fractional integral operator of order $\beta > 0$ by

$$
I^\beta(f)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau) \, d\tau, \quad f \in L^1(0, T), \text{ a.e. } t \in (0, T),
$$

(2.11)

where $T > 0$ and $\Gamma(\beta) = \int_0^\infty t^{\beta-1} e^{-t} \, dt$ is the Euler Gamma function.

The Caputo fractional derivative of order $\alpha \in (1, 2)$ is given by

$$
\partial_t^\alpha f(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} \frac{d^2 f}{d\tau^2}(\tau) \, d\tau,
$$

(2.12)

and in terms of the Riemann–Liouville integral operator $I^{2-\alpha}$ can be written as

$$
\partial_t^\alpha f(t) = I^{2-\alpha}\left(\frac{d^2 f}{dt^2}\right)(t).
$$

(2.13)

Note also that

$$
\partial_t^\alpha f(t) = \frac{d}{dt} I^{2-\alpha}(f'(0))(t),
$$

(2.14)

when $f'$ is absolutely continuous.

For arbitrary constants $\alpha, \beta > 0$, we denote the Mittag–Leffler functions by

$$
E_{\alpha, \beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak + \beta)}, \quad z \in \mathbb{C}.
$$

(2.15)

The power series $E_{\alpha, \beta}(z)$ is an entire function of $z \in \mathbb{C}$. The Mittag–Leffler function $E_{\alpha, 1}(z)$ is usually denoted by $E_{\alpha}(z)$. Observe that $E_{\alpha}(0) = 1$.

The proof of the following result can be found in [16, p. 35], see also [17, Lemma 3.1]. In the following we denote the Laplace transform of a function $f(t)$ by the symbol

$$
\mathcal{L}[f(t)](z) := \int_0^\infty e^{-zt} f(t) \, dt \quad z \in \mathbb{C}.
$$

(2.16)
Lemma 1 1. Let $\alpha \in (1, 2)$ and $\beta > 0$ be. Then for any $\mu \in \mathbb{R}$ such that $\pi \alpha / 2 < \mu < \pi$ there exists a constant $C = C(\alpha, \beta, \mu) > 0$ such that

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1 + |z|}, \quad z \in \mathbb{C}, \mu \leq |\arg(z)| \leq \pi. \quad (2.17)$$

2. For $\alpha, \beta, \lambda > 0$ one has

$$L\left[t^{\beta-1}E_{\alpha,\beta}(-\lambda t^\alpha)\right](z) = \frac{z^{\alpha-\beta}}{z^\beta + \lambda}, \quad \Re z > \lambda. \quad (2.18)$$

3. If $\alpha, \lambda > 0$, then we have

$$\frac{d}{dt} E_{\alpha}(-\lambda t^\alpha) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha), \quad t > 0, \quad (2.19)$$

$$\frac{d}{dt} \left(t^k E_{\alpha,k+1}(-\lambda t^\alpha)\right) = t^{k-1} E_{\alpha,k}(-\lambda t^\alpha), \quad k \in \mathbb{N}, \quad t \geq 0, \quad (2.20)$$

$$\frac{d}{dt} \left(t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)\right) = t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha), \quad t \geq 0. \quad (2.21)$$

We recall an elementary result that is useful in our estimates.

Lemma 2 For any $0 < \beta < 1$ the function $x \rightarrow x^{\beta} + \frac{1}{1+x}$ gains its maximum on $[0, +\infty[$ at point $\frac{\beta}{1-\beta}$ and the maximum value is given by

$$\max_{x \geq 0} \frac{x^\beta}{1+x} = \beta \beta(1 - \beta)^{1-\beta}, \quad \beta \in (0, 1). \quad (2.22)$$

We now remind the definition of fractional vector-valued Sobolev spaces. For $\beta \in (0, 1), T > 0$ and a Hilbert space $H$, endowed with the norm $\| \cdot \|_H$, $H^\beta(0, T; H)$ is the space of all $u \in L^2(0, T; H)$ such that

$$[u]_{H^\beta(0, T; H)} := \left(\int_0^T \int_0^T \frac{\|u(t) - u(\tau)\|_H^2}{|t - \tau|^{1+2\beta}} \, dt \, d\tau\right)^{1/2} < +\infty, \quad (2.23)$$

that is $[u]_{H^\beta(0, T; H)}$ is the so-called Gagliardo semi-norm of $u$. $H^\beta(0, T; H)$ is endowed with the norm

$$\| \cdot \|_{H^\beta(0, T; H)} := \| \cdot \|_{L^2(0, T; H)} + [ \cdot ]_{H^\beta(0, T; H)}. \quad (2.24)$$

The following extension of a known result (see [2, Theorem 2.1]) to the case of vector valued functions is crucial in the proof of Theorem 6. We use the symbol $\sim$ between norms to indicate two equivalent norms.

Theorem 3 Let $H$ be a separable Hilbert space.
(i) The Riemann–Liouville operator $I^\beta : L^2(0, T; H) \to L^2(0, T; H)$, $0 < \beta \leq 1$, is injective and the range $\mathcal{R}(I^\beta)$ of $I^\beta$ is given by

$$
\mathcal{R}(I^\beta) = \begin{cases} 
H^\beta(0, T; H), & 0 < \beta < \frac{1}{2}, \\
\left\{ v \in H^\frac{1}{2}(0, T; H) : \int_0^T t^{-1}|v(t)|^2 dt < \infty \right\}, & \beta = \frac{1}{2}, \\
0H^\beta(0, T; H), & \frac{1}{2} < \beta \leq 1,
\end{cases}
$$

where $0H^\beta(0, T) = \{ u \in H^\beta(0, T) : u(0) = 0 \}$.

(ii) For the Riemann–Liouville operator $I^\beta$ and its inverse operator $I^{-\beta}$ the norm equivalences

$$
\|I^\beta(u)\|_{H^\beta(0,T;H)} \sim \|u\|_{L^2(0,T;H)}, \quad u \in L^2(0, T; H),
$$

$$
\|I^{-\beta}(v)\|_{L^2(0,T;H)} \sim \|v\|_{H^\beta(0,T;H)}, \quad v \in \mathcal{R}(I^\beta),
$$

hold true.

3 Existence and regularity of solutions

First, we recall the definition of $H^2$-solutions and strong solutions, see [10].

**Definition 2** Let $\alpha \in (1, 2)$ and $T > 0$.

1. A function $u$ is called a $H^2$-solution of the fractional boundary value problem

$$
\begin{cases}
\partial_t^\alpha u + \Delta^2 u = 0 & \text{in } (0, T) \times \Omega, \\
u = \Delta u = 0 & \text{on } (0, T) \times \partial \Omega,
\end{cases}
$$

if $u \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega))$, $u_t \in L^2(0, T; L^2(\Omega))$, for some $\theta \in (0, 1)$ and $u_t \in C([0, T]; D(A^{-\theta}))$ and for any $v \in H^2(\Omega) \cap H_0^1(\Omega)$ one has

$$
\int_\Omega I^{2-\alpha}(u_t(t, x) - u_t(0, x)) v(x) \, dx \in C^1([0, T]) \text{ and for } t \in [0, T]
$$

$$
\frac{d}{dt} \int_\Omega I^{2-\alpha}(u_t(t, x) - u_t(0, x)) v(x) \, dx + \int_\Omega \Delta u(t, x) \Delta v(x) \, dx = 0.
$$

2. A function $u$ is called a strong solution if $u \in C([0, T]; D(A))$, $u \in C^1([0, T]; L^2(\Omega))$, $\partial_t^\alpha u \in C([0, T]; L^2(\Omega))$ and satisfies (3.1).

**Remark 1** A strong solution is also a $H^2$-solution.

We recall the following existence result, see [10, Theorem 4.3].

**Theorem 4** (i) If $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$, then the function

$$
u(t, x) = \sum_{n=1}^{\infty} \left[ (u_0, e_n) E_\alpha(-\lambda_n t^\alpha) + (u_1, e_n) t E_{\alpha,2}(-\lambda_n t^\alpha) \right] e_n(x)
$$

(3.3)
is the unique $H^2$-solution of (3.1) satisfying the initial conditions

$$u(0, \cdot) = u_0, \quad u_t(0, \cdot) = u_1.$$  (3.4)

In addition,

$$u_t(t, x) = \sum_{n=1}^{\infty} \left[ -\lambda_n \langle u_0, e_n \rangle t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^{\alpha}) + \langle u_1, e_n \rangle t E_{\alpha, 2}(-\lambda_n t^{\alpha}) \right] e_n(x),$$  (3.5)

and $u_t \in C([0, T]; D(A^{-\theta}))$ for $\theta \in \left(\frac{2-\alpha}{2\alpha}, \frac{1}{2}\right)$.

(ii) For $u_0 \in \{u \in H^4(\Omega) : u = \Delta u = 0 \text{ on } \partial\Omega\}$ and $u_1 \in H^2(\Omega) \cap H^1_0(\Omega)$ the $H^2$-solution given by (3.3) is a strong one and

$$\partial_t^\alpha u(t, x) = -\sum_{n=1}^{\infty} \left[ \lambda_n \langle u_0, e_n \rangle E_{\alpha, 1}(-\lambda_n t^{\alpha}) + \lambda_n \langle u_1, e_n \rangle t E_{\alpha, 2}(-\lambda_n t^{\alpha}) \right] e_n(x).$$  (3.6)

**Proposition 1** For $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$ and $u_1 \in L^2(\Omega)$ if $u$ is the $H^2$-solution given by (3.3), then $A^{-\frac{1}{2}} u$ is a strong solution.

**Proof** To prove the statement we set $w = A^{-\frac{1}{2}} u$. So, we have $w \in D(A^{\frac{1}{2}}) = H^2(\Omega) \cap H^1_0(\Omega)$, and in particular $w = 0$ on $\partial\Omega$. Moreover,

$$-\Delta w = A^{\frac{1}{2}} A^{-\frac{1}{2}} u = u \text{ on } \Omega.$$  

Since $u \in H^2(\Omega) \cap H^1_0(\Omega)$, we get $w \in H^4(\Omega)$ and $\Delta w = 0$ on $\partial\Omega$. In addition, thanks to Theorem 4 we have $A^{\frac{1}{2}} w = u \in C^1([0, T]; D(A^{-\theta}))$ for $\theta \in \left(\frac{2-\alpha}{2\alpha}, \frac{1}{2}\right)$. Therefore

$$\|w_t(t, \cdot)\|_{L^2(\Omega)}^2 = \sum_{n=1}^{\infty} \lambda_n^{\beta-1} \lambda_n^{\frac{1-2\beta}{2}} |\langle w_t, e_n \rangle|^2 \leq C \sum_{n=1}^{\infty} \lambda_n^{1-2\beta} |\langle w_t, e_n \rangle|^2 = C \|A^{\frac{1}{2}} w_t(t, \cdot)\|_{D(A^{-\theta})}^2,$$

and hence $A^{-\frac{1}{2}} u = w \in C^1([0, T]; L^2(\Omega))$.

To complete the proof we have to show that

$$\frac{d}{dt} t^{2-\alpha} (w_t - A^{-\frac{1}{2}} u_1) = \partial_t^\alpha w.$$  (3.7)
To this end, thanks to (2.14) it is sufficient to prove that $w_t$ is absolutely continuous. Indeed, by (3.5) we get

$$w_t(t, \cdot) = A^{-\frac{1}{2}} u_t(t, \cdot)$$

$$= \sum_{n=1}^{\infty} \lambda_n^{-\frac{1}{2}} \left[ -\lambda_n \langle u_0, e_n \rangle t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle E_{\alpha,\alpha}(-\lambda_n t^\alpha) \right] e_n.$$ 

The series of the derivatives with respect to the variable $t$, thanks to (2.21) and (2.19), is given by

$$- \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}} \left[ \langle u_0, e_n \rangle t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) \right] e_n. \tag{3.8}$$

In virtue of (2.17) we note that

$$\int_0^T \left\| \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}} \left[ \langle u_0, e_n \rangle t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) \right] e_n \right\|_{L^2(\Omega)}^2 dt$$

$$\leq C \left( t^{\alpha-1} \| A^{\frac{1}{2}} u_0 \|_{L^2(\Omega)}^2 + T^{\frac{\alpha}{2}} \| u_1 \|_{L^2(\Omega)} \right),$$

that is the series (3.8) belongs to $L^1(0, T; L^2(\Omega))$, and hence $w_t$ is absolutely continuous.

In conclusion, if we substitute in formula (3.2) $u$ with $A^{\frac{1}{2}} w$, thanks to (3.17) we obtain

$$\int_{\Omega} A^{\frac{1}{2}} \left( \partial_t^\alpha w(t, x) + \Delta^2 w(t, x) \right) v(x) \, dx = 0 \quad t \in [0, T], \quad v \in H^2(\Omega) \cap H_0^1(\Omega),$$

that is $\partial_t^\alpha w + \Delta^2 w = 0$. \hfill \Box

**Definition 3.** 1. A function $u$ is called a $H^3$-solution of the fractional boundary value problem

$$\left\{ \begin{array}{ll}
\partial_t^\alpha u + \Delta^2 u = 0 & \text{in } (0, T) \times \Omega, \\
u = \Delta u = 0 & \text{on } (0, T) \times \partial \Omega,
\end{array} \right. \tag{3.9}$$

if $u \in C([0, T]; D(A^{\frac{3}{2}})), u_t \in L^2(0, T; H^1(\Omega)) \cap C([0, T]; H^{-1}(\Omega)), \partial_t^\alpha u \in L^2([0, T]; L^2(\Omega))$ and for any $v \in H_0^1(\Omega)$ one has

$$\int_{\Omega} \partial_t^\alpha u(t, x) v(x) \, dx - \int_{\Omega} \nabla \Delta u(t, x) \cdot \nabla v(x) \, dx = 0 \quad t \in (0, T). \tag{3.10}$$
2. A function $u \in C([0, T]; H^1_0(\Omega))$ is called a $H^1$-solution of (3.9) if $A^{-\frac{1}{2}}u$ is a $H^3$-solution.

**Theorem 5** (i) Let $u_0 \in D(A^{\frac{3}{4}})$ and $u_1 \in H^1_0(\Omega)$. Then the $H^2$-solution

$$u(t, x) = \sum_{n=1}^{\infty} \left[ \langle u_0, e_n \rangle E_\alpha(-\lambda_n t^{\alpha}) + \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^{\alpha}) \right] e_n(x)$$  \hspace{1cm} (3.11)

of (3.9) with initial conditions

$$u(0, \cdot) = u_0, \quad u_t(0, \cdot) = u_1,$$  \hspace{1cm} (3.12)

is a $H^3$-solution of (3.9) and

$$\|\partial_t^2 u\|_{L^2(0, T; L^2(\Omega))} \leq C \left( \|\nabla \Delta u_0\|_{L^2(\Omega)} + \|\nabla u_1\|_{L^2(\Omega)} \right), \quad (C > 0).$$  \hspace{1cm} (3.13)

In addition, for $\theta \in (0, \frac{1}{2})$ we have $\nabla u \in L^2(0, T; D(A^\theta))$ and

$$\|\nabla \Delta u\|_{L^2(0, T; D(A^\theta))} \leq C \left( \|\nabla u_0\|_{L^2(\Omega)} + \|\nabla u_1\|_{L^2(\Omega)} \right), \quad (C > 0).$$  \hspace{1cm} (3.14)

(ii) Let $u_0 \in H^1_0(\Omega)$ and $u_1 \in H^{-1}(\Omega)$. Then the function

$$u(t, x) = \sum_{n=1}^{\infty} \left[ \langle u_0, e_n \rangle E_\alpha(-\lambda_n t^{\alpha}) + \langle u_1, e_n \rangle t^{\frac{1}{4}} E_{\alpha,2}(-\lambda_n t^{\alpha}) \right] e_n(x)$$  \hspace{1cm} (3.15)

is a $H^1$-solution of (3.9).

**Proof** (i) First, we note that for any $t \in [0, T]$ we have $u(t) \in D(A^{\frac{3}{4}})$. Indeed, in view of (2.5) we have

$$\|\nabla \Delta u(t)\|_{L^2(\Omega)}^2 \leq 2 \sum_{n=1}^{\infty} \lambda_n^\frac{3}{2} |\langle u_0, e_n \rangle E_\alpha(-\lambda_n t^{\alpha})|^2 + 2 \sum_{n=1}^{\infty} \lambda_n^\frac{3}{2} |\langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^{\alpha})|^2.$$  

Thanks to (2.17) we get

$$\lambda_n^\frac{3}{2} |\langle u_0, e_n \rangle E_\alpha(-\lambda_n t^{\alpha})|^2 \leq C \lambda_n^\frac{3}{2} |\langle u_0, e_n \rangle|^2,$$

$$\lambda_n^\frac{3}{2} |\langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^{\alpha})|^2 \leq C t^{2-\alpha} \lambda_n^\frac{1}{2} |\langle u_1, e_n \rangle|^2 \frac{\lambda_n t^{\alpha}}{(1 + \lambda_n t^{\alpha})^2} \leq C t^{2-\alpha} \lambda_n^\frac{1}{2} |\langle u_1, e_n \rangle|^2,$$
whence, being $\alpha < 2$, we obtain

$$\| \nabla u(t) \|^2_{L^2(\Omega)} \leq C \| \nabla u_0 \|^2_{L^2(\Omega)} + C T^{2-\alpha} \| \nabla u_1 \|^2_{L^2(\Omega)}. \quad (3.16)$$

Following the same arguments used to prove (3.16), we get for any $n \in \mathbb{N}$

$$\left\| \nabla \sum_{k=n}^{\infty} \left[ (u_0, e_k) E_{\alpha}(-\lambda_k t^\alpha) + (u_1, e_k) t E_{\alpha,2}(-\lambda_k t^\alpha) \right] e_k \right\|_{L^2(\Omega)}^2 \leq C \sum_{k=n}^{\infty} \lambda_k^\frac{3}{2} |(u_0, e_k)|^2 + C T^{2-\alpha} \sum_{k=n}^{\infty} \lambda_k^\frac{1}{2} |(u_1, e_k)|^2,$$

and hence

$$\lim_{n \to \infty} \sup_{t \in [0, T]} \left\| \nabla \sum_{k=n}^{\infty} \left[ (u_0, e_k) E_{\alpha}(-\lambda_k t^\alpha) + (u_1, e_k) t E_{\alpha,2}(-\lambda_k t^\alpha) \right] e_k \right\|_{L^2(\Omega)} = 0.$$

As a consequence, the series

$$\sum_{n=1}^{\infty} \left[ (u_0, e_n) E_{\alpha}(-\lambda_n t^\alpha) + (u_1, e_n) t E_{\alpha,2}(-\lambda_n t^\alpha) \right] e_n$$

is convergent in $D(A^{\frac{3}{4}})$ uniformly in $t \in [0, T]$, so $u \in C([0, T]; D(A^{\frac{3}{4}}))$. Now we have to prove that

$$\frac{d}{dt} I^{2-\alpha} (u_t - u_1) = \partial_t^{\alpha} u. \quad (3.17)$$

To this end, thanks to (2.14) it is sufficient to show that $u_t$, given by (3.5), is absolutely continuous. Indeed, the series of the derivatives with respect to the variable $t$, thanks to (2.21) and (2.19), is given by

$$- \sum_{n=1}^{\infty} \lambda_n \left[ (u_0, e_n) t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n t^\alpha) + (u_1, e_n) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) \right] e_n. \quad (3.18)$$

In virtue of (2.17) we have

$$\int_0^T \left\| \sum_{n=1}^{\infty} \lambda_n \left[ (u_0, e_n) t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n t^\alpha) + (u_1, e_n) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) \right] e_n \right\|_{H^{-1}(\Omega)} dt \leq C (T^{\alpha-1} \| \nabla u_0 \|_{L^2(\Omega)} + T^{\frac{\alpha}{2}} \| \nabla u_1 \|_{L^2(\Omega)}),$$
that is the series (3.18) belongs to $L^1(0, T; H^{-1}(\Omega))$, and hence $u_t$ is absolutely continuous. So, thanks to (3.17) one can verify (3.10).

To prove (3.13) we note that, thanks to (3.6) and (2.17), we have

$$
\| \partial_\alpha t u(t, \cdot) \|_{L^2(\Omega)}^2 = \sum_{n=1}^{\infty} \left| \lambda_n \langle u_0, e_n \rangle E_\alpha(-\lambda_n t^\alpha) + \lambda_n \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha) \right|^2
$$

$$
\leq C \sum_{n=1}^{\infty} \lambda_n^{3/2} |\langle u_0, e_n \rangle|^2 \frac{\lambda_n^{1/2}}{(1 + \lambda_n t^\alpha)^2} + C \sum_{n=1}^{\infty} \lambda_n^{3/2} |\langle u_1, e_n \rangle|^2 \frac{\lambda_n^{1/2} t^2}{(1 + \lambda_n t^\alpha)^2}.
$$

(3.19)

Observing that

$$
\frac{\lambda_n^{1/2}}{(1 + \lambda_n t^\alpha)^2} = \left( \frac{(\lambda_n t^\alpha)^{1/2}}{1 + \lambda_n t^\alpha} \right)^2 t^{-\frac{\alpha}{2}}, \quad \frac{\lambda_n^{3/2} t^2}{(1 + \lambda_n t^\alpha)^2} = \left( \frac{(\lambda_n t^\alpha)^{3/2}}{1 + \lambda_n t^\alpha} \right)^2 t^{-\frac{3}{2} \alpha},
$$

from (2.22) and (3.19) we deduce

$$
\| \partial_\alpha t u(\cdot, t) \|_{L^2(\Omega)}^2 \leq Ct^{-\frac{\alpha}{2}} \| \nabla \Delta u_0 \|_{L^2(\Omega)}^2 + Ct^{2-\frac{3}{2} \alpha} \| \nabla u_1 \|_{L^2(\Omega)}^2.
$$

Therefore, since $\alpha < 2$ we have

$$
\| \partial_\alpha t u \|_{L^2(0, T; L^2(\Omega))} \leq CT^{1-\frac{\alpha}{2}} \| \nabla \Delta u_0 \|_{L^2(\Omega)}^2 + CT^{3-\frac{3}{2} \alpha} \| \nabla u_1 \|_{L^2(\Omega)}^2,
$$

so $\partial_\alpha u \in L^2(0, T; L^2(\Omega))$ and (3.13) holds.

Now, we fix $\theta \in \left(0, \frac{1}{2\alpha}\right)$. Thanks to (3.3) and (2.17), we get

$$
\| \nabla \Delta u(t, \cdot) \|_{D(A^\theta)}^2 = \sum_{n=1}^{\infty} \lambda_n^{3+2\theta} |\langle u_0, e_n \rangle E_\alpha(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha) |^2
$$

$$
\leq C \sum_{n=1}^{\infty} \lambda_n^{3+2\theta} |\langle u_0, e_n \rangle|^2 \frac{\lambda_n^{2\theta}}{(1 + \lambda_n t^\alpha)^2} + C \sum_{n=1}^{\infty} \lambda_n^{3+2\theta} |\langle u_1, e_n \rangle|^2 \frac{\lambda_n^{1+2\theta} t^2}{(1 + \lambda_n t^\alpha)^2}.
$$

Since

$$
\frac{\lambda_n^{2\theta}}{(1 + \lambda_n t^\alpha)^2} = \left( \frac{(\lambda_n t^\alpha)^{\theta}}{1 + \lambda_n t^\alpha} \right)^2 t^{-2\alpha \theta},
$$

$$
\frac{\lambda_n^{1+2\theta} t^2}{(1 + \lambda_n t^\alpha)^2} = \left( \frac{(\lambda_n t^\alpha)^{1+2\theta}}{1 + \lambda_n t^\alpha} \right)^2 t^{-\alpha(1+2\theta)},
$$
and $0 < \theta < \frac{1}{2}$, we can apply (2.22) to have
\[
\|\nabla \Delta u(t, \cdot)\|_{D(\Lambda_\theta)}^2 \leq C t^{-2\alpha \theta} \|\nabla \Delta u_0\|_{L^2(\Omega)}^2 + C t^{2-\alpha (1+2\theta)} \|\nabla u_1\|_{L^2(\Omega)}^2.
\]

Taking into account that $\theta \in \left(0, \frac{1}{2} \alpha \right)$ we have $\nabla \Delta u \in L^2(0, T; D(\Lambda_\theta))$ and (3.14) follows.

(ii) If $u_0 \in H^1_0(\Omega)$ and $u_1 \in H^{-1}(\Omega)$, then $A^{-\frac{1}{2}} u_0 \in D(A^{\frac{3}{4}})$ and $A^{-\frac{1}{2}} u_1 \in H^1_0(\Omega)$. Therefore, by (i) the function
\[
w(t, x) = \sum_{n=1}^{\infty} \left[ \langle A^{-\frac{1}{2}} u_0, e_n \rangle E_\alpha(-\lambda_n t^{\alpha}) + \langle A^{-\frac{1}{2}} u_1, e_n \rangle t E_{\alpha, 2}(-\lambda_n t^{\alpha}) \right] e_n(x)
\]
is a $H^3$-solution of (3.9). So, by definition $A^{\frac{1}{2}} w$ is a $H^1$-solution of (3.9). Moreover,
\[
A^{\frac{1}{2}} w(t, x)
= \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}} \left[ \langle A^{-\frac{1}{2}} u_0, e_n \rangle E_\alpha(-\lambda_n t^{\alpha}) + \langle A^{-\frac{1}{2}} u_1, e_n \rangle t E_{\alpha, 2}(-\lambda_n t^{\alpha}) \right] e_n(x)
= \sum_{n=1}^{\infty} \left[ \langle u_0, e_n \rangle E_\alpha(-\lambda_n t^{\alpha}) + \langle u_1, e_n \rangle t E_{\alpha, 2}(-\lambda_n t^{\alpha}) \right] e_n(x),
\]
that is, if $u$ is given by (3.15), then $u = A^{\frac{1}{2}} w$. The proof is complete.

\[
\boxright
\]

4 Hidden regularity results

We develop a procedure similar to that followed in [4] for Petrovsky systems. First, we prove some identities that are useful in the proof of the main theorem.

**Lemma 3** If $w \in H^4(\Omega)$ with $\Delta w = 0$ on $\partial \Omega$ and $h : \overline{\Omega} \to \mathbb{R}^N$ is a vector field of class $C^1$, then
\[
2 \int_\Omega \Delta^2 w \, h \cdot \nabla \Delta w \, dx = \int_{\partial \Omega} h \cdot v |\partial_\nu \Delta w|^2 \, d\sigma - 2 \sum_{i,j=1}^{N} \int_\Omega \partial_i h_j \partial_i \Delta w \partial_j \Delta w \, dx + \int_\Omega \sum_{j=1}^{N} \partial_j h_j \, |\nabla \Delta w|^2 \, dx. \tag{4.1}
\]

**Proof** First, we observe that by $w \in H^4(\Omega)$ it follows that the normal derivative $\partial_\nu \Delta w$ is well defined on $\partial \Omega$. 

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\]
Integrating by parts we get
\[
\int_{\Omega} \Delta^2 w \cdot \nabla \Delta w \; dx = \int_{\partial \Omega} \frac{\partial}{\partial \nu} \Delta w \cdot \nabla (h \cdot \nabla \Delta w) \; d\sigma - \int_{\Omega} \nabla \Delta w \cdot \nabla (h \cdot \nabla \Delta w) \; dx.
\]
Since \( \Delta w = 0 \) on \( \partial \Omega \) we have
\[
\nabla \Delta w = (\frac{\partial}{\partial \nu} \Delta w) \nu \quad \text{on} \quad \partial \Omega,
\]
(see e.g. [13, Lemma 2.1] for a detailed proof) and hence
\[
\int_{\Omega} \Delta^2 w \cdot \nabla \Delta w \; dx = \int_{\partial \Omega} h \cdot \nu |\nabla \Delta w|^2 \; d\sigma - \int_{\Omega} \nabla \Delta w \cdot \nabla (h \cdot \nabla \Delta w) \; dx.
\]
(4.2)

We note that
\[
\int_{\Omega} \nabla \Delta w \cdot \nabla (h \cdot \nabla \Delta w) \; dx = \sum_{i,j=1}^{N} \int_{\Omega} \frac{\partial}{\partial i} \Delta w \; \frac{\partial}{\partial j} (h_j \frac{\partial}{\partial j} \Delta w) \; dx
\]
\[
= \sum_{i,j=1}^{N} \int_{\Omega} \frac{\partial}{\partial i} \Delta w \; \frac{\partial}{\partial j} h_j \frac{\partial}{\partial j} \Delta w \; dx + \sum_{i,j=1}^{N} \int_{\Omega} h_j \frac{\partial}{\partial i} \Delta w \frac{\partial}{\partial j} (\frac{\partial}{\partial i} \Delta w) \; dx.
\]
We estimate the last term on the right-hand side again by an integration by parts, so we obtain
\[
\sum_{i,j=1}^{N} \int_{\Omega} h_j \frac{\partial}{\partial i} \Delta w \frac{\partial}{\partial j} (\frac{\partial}{\partial i} \Delta w) \; dx = \frac{1}{2} \sum_{j=1}^{N} \int_{\Omega} h_j \frac{\partial}{\partial j} \left( \sum_{i=1}^{N} (\frac{\partial}{\partial i} \Delta w)^2 \right) \; dx
\]
\[
= \frac{1}{2} \int_{\partial \Omega} h \cdot \nu |\nabla \Delta w|^2 \; d\sigma - \frac{1}{2} \int_{\Omega} \sum_{j=1}^{N} \frac{\partial}{\partial j} h_j \; |\nabla \Delta w|^2 \; dx.
\]
In conclusion, putting the above identities into (4.2) we deduce (4.1).

We prove the next result for strong solutions, see Definition 2–2, whose existence is given by Theorem 4–(ii). We recall that \( I^\beta \) is the Riemann–Liouville operator of order \( \beta > 0 \), see (2.11).

**Lemma 4** Suppose \( u \) is a strong solution of
\[
\begin{align*}
\frac{\partial}{\partial \tau} u + \Delta^2 u &= 0 \quad \text{in} \quad [0, T] \times \Omega, \\
u = \Delta u &= 0 \quad \text{on} \quad [0, T] \times \partial \Omega.
\end{align*}
\]
(4.3)

For a vector field \( h : \overline{\Omega} \to \mathbb{R}^N \) of class \( C^1 \) and \( \beta \in (0, 1) \) we have

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\[ \int_{\partial\Omega} h \cdot v |I^\beta(\partial_\nu \Delta u)(t)|^2 d\sigma = -2 \int_{\Omega} I^\beta(\partial_i^\alpha u)(t)h \cdot I^\beta(\nabla \Delta u)(t) dx \]

\[ +2 \sum_{i,j=1}^N \int_{\Omega} \partial_i h j I^\beta(\partial_i \Delta u)(t)I^\beta(\partial_j \Delta u)(t) dx \]

\[ - \int_{\Omega} \sum_{j=1}^N \partial_j h j |I^\beta(\nabla \Delta u)(t)|^2 dx, \quad t \in [0, T] \]

(4.4)

\[ \int_{\partial\Omega} h \cdot v |I^\beta(\partial_\nu \Delta u)(t) - I^\beta(\partial_\nu \Delta u)(\tau)|^2 d\sigma \]

\[ = -2 \int_{\Omega} (I^\beta(\partial_i^\alpha u)(t) - I^\beta(\partial_i^\alpha u)(\tau))h \cdot (I^\beta(\nabla \Delta u)(t) - I^\beta(\nabla \Delta u)(\tau)) dx \]

\[ +2 \sum_{i,j=1}^N \int_{\Omega} \partial_i h j (I^\beta(\partial_i \Delta u)(t) - I^\beta(\partial_i \Delta u)(\tau))(I^\beta(\partial_j \Delta u)(t) - I^\beta(\partial_j \Delta u)(\tau)) dx \]

\[ - \sum_{j=1}^N \int_{\Omega} \partial_j h j |I^\beta(\nabla \Delta u)(t) - I^\beta(\nabla \Delta u)(\tau)|^2 dx, \quad t, \tau \in [0, T]. \]

(4.5)

**Proof** First, we apply the operator $I^\beta$, $\beta \in (0, 1)$, to equation (4.3):

\[ I^\beta(\partial_\nu \Delta u)(t) = -I^\beta(\nabla \Delta u)(t) \quad t \in [0, T]. \]

(4.6)

By means of the scalar product in $L^2(\Omega)$ we multiply (4.6) by

\[ 2h \cdot \nabla I^\beta(u)(t), \]

that is

\[ 2 \int_{\Omega} I^\beta(\partial_i^\alpha u)(t)h \cdot \nabla I^\beta(u)(t) dx = -2 \int_{\Omega} \Delta^2 I^\beta(u)(t)h \cdot \nabla I^\beta(u)(t) dx \] (4.7)

To evaluate the term

\[ 2 \int_{\Omega} \Delta^2 I^\beta(u)(t)h \cdot \nabla I^\beta(u)(t) dx, \]

we apply Lemma 3 to the function $w(t, x) = I^\beta(u)(t)$, so from (4.1) we deduce

\[ 2 \int_{\Omega} \Delta^2 I^\beta(u)(t)h \cdot \nabla I^\beta(u)(t) dx = \int_{\partial\Omega} h \cdot v |I^\beta(\nabla \Delta u)(t)|^2 d\sigma \]

\[ -2 \sum_{i,j=1}^N \int_{\Omega} \partial_i h j I^\beta(\partial_i \Delta u)(t)I^\beta(\partial_j \Delta u)(t) dx + \int_{\Omega} \sum_{j=1}^N \partial_j h j |I^\beta(\nabla \Delta u)(t)|^2 dx. \]
In conclusion, plugging the above formula into (4.7), we obtain (4.4).

The proof of (4.5) is alike: we start from

\[ I^\beta(\partial_t^\alpha u)(t) - I^\beta(\partial_t^\alpha u)(\tau) = -(I^\beta(\Delta^2 u)(t) - I^\beta(\Delta^2 u)(\tau)) \quad t, \tau \in [0, T], \]

and we multiply both terms by \( 2h \cdot \nabla (I^\beta(u)(t) - I^\beta(u)(\tau)) \). Then, applying Lemma 3 to the function \( w(t, \tau, x) = I^\beta(u)(t) - I^\beta(u)(\tau) \) we get the identity (4.5). \( \square \)

**Remark 2** We observe that the proof of the identities (4.4) and (4.5) cannot be done for a general function \( w \) and then applied to \( w = I^\beta(u) \), since

\[ \partial_t^\alpha I^\beta(u) \neq I^\beta(\partial_t^\alpha u), \]

as one can conclude from (2.12).

**Theorem 6** Let \( u_0 \in \{ u \in H^4(\Omega) : u = \Delta u = 0 \text{ on } \partial \Omega \} \) and \( u_1 \in H^2(\Omega) \cap H^1_0(\Omega) \).

Then, for \( T > 0 \) the strong solution \( u \) of problem

\[
\begin{align*}
\begin{cases}
\partial_t^\alpha u + \Delta^2 u = 0 & \text{in } [0, T] \times \Omega, \\
u = \Delta u = 0 & \text{on } [0, T] \times \partial \Omega \\
u(0, \cdot) = u_0, \quad u_t(0, \cdot) = u_1.
\end{cases}
\end{align*}
\]

satisfies the inequality

\[ \int_0^T \int_{\partial \Omega} |\partial_v \Delta u|^2 d\sigma dt \leq C \left( \| \nabla \Delta u_0 \|^2_{L^2(\Omega)} + \| \nabla u_1 \|^2_{L^2(\Omega)} \right), \]

for some constant \( C = C(T) > 0 \).

**Proof** We apply Theorem 3 to prove the statement. Indeed, for \( H = L^2(\partial \Omega) \) and \( \beta \in (0, 1) \) we can apply (2.26) to have

\[ \| \partial_v \Delta u \|_{L^2(0,T;L^2(\partial \Omega))} \sim \| I^\beta(\partial_v \Delta u) \|_{H^\beta(0,T;L^2(\partial \Omega))}. \]

So, the inequality (4.9) is equivalent to

\[ \| I^\beta(\partial_v \Delta u) \|_{H^\beta(0,T;L^2(\partial \Omega))} \leq C \left( \| \nabla \Delta u_0 \|_{L^2(\Omega)} + \| \nabla u_1 \|_{L^2(\Omega)} \right). \]

To prove (4.11), keeping in mind (2.24) we have to estimate two terms: \( \| I^\beta(\partial_v \Delta u) \|_{L^2(0,T;L^2(\partial \Omega))} \) and \( [I^\beta(\partial_v \Delta u)]_{H^\beta(0,T;L^2(\partial \Omega))} \). To this end we employ the two identities in Lemma 4 by means of a suitable choice of the vector field \( h \). Indeed, we consider a vector field \( h \in C^1(\overline{\Omega}; \mathbb{R}^N) \) satisfying the condition

\[ h = v \quad \text{on } \partial \Omega \]
(see e.g. [4] for the existence of such vector field $h$). If we integrate (4.4) over $[0, T]$, then we obtain

$$
\int_0^T \int_{\partial \Omega} |I^\beta (\partial_v \Delta u)(t)|^2 \, d\sigma \, dt = -2 \int_0^T \int_{\Omega} I^\beta (\partial^\alpha u)(t) h \cdot I^\beta (\nabla \Delta u)(t) \, dx \, dt \\
+ 2 \sum_{i,j=1}^N \int_0^T \int_{\Omega} \partial_i h_j I^\beta (\partial_i \Delta u)(t) I^\beta (\partial_j \Delta u)(t) \, dx \, dt \\
- \int_0^T \int_{\Omega} \sum_{j=1}^N \partial_j h_j |I^\beta (\nabla \Delta u)(t)|^2 \, dx \, dt.
$$

Since $h \in C^1(\bar{\Omega}; \mathbb{R}^N)$ from the above inequality we get

$$
\|I^\beta (\partial_v \Delta u)\|_{L^2(0,T;L^2(\partial \Omega))} \leq C \left( \|I^\beta (\partial^\alpha u)\|_{L^2(0,T;L^2(\Omega))} + \|I^\beta (\nabla \Delta u)\|_{L^2(0,T;L^2(\Omega))} \right), \tag{4.13}
$$

for some constant $C > 0$.

We have to evaluate the Gagliardo semi-norm $[I^\beta (\partial_v \Delta u)]_{H^\beta(0,T;L^2(\partial \Omega))}$ see (2.23). To this end we multiply both terms of (4.5) by $\frac{1}{|t - \tau|^{1+2\beta}}$: taking into account (4.12) we have

$$
\frac{1}{|t - \tau|^{1+2\beta}} \int_{\partial \Omega} |I^\beta (\partial_v \Delta u)(t) - I^\beta (\partial_v \Delta u)(\tau)|^2 \, d\sigma \\
= -2 \int_{\Omega} \frac{(I^\beta (\partial^\alpha u)(t) - I^\beta (\partial^\alpha u)(\tau)) h \cdot (I^\beta (\nabla \Delta u)(t) - I^\beta (\nabla \Delta u)(\tau))}{|t - \tau|^{1+2\beta}} \, dx \\
+ 2 \sum_{i,j=1}^N \int_{\Omega} \frac{\partial_i h_j (I^\beta (\partial_i \Delta u)(t) - I^\beta (\partial_i \Delta u)(\tau)) (I^\beta (\partial_j \Delta u)(t) - I^\beta (\partial_j \Delta u)(\tau))}{|t - \tau|^{1+2\beta}} \, dx \\
- \sum_{j=1}^N \int_{\Omega} \frac{\partial_j h_j |I^\beta (\nabla \Delta u)(t) - I^\beta (\nabla \Delta u)(\tau)|^2}{|t - \tau|^{1+2\beta}} \, dx. \tag{4.14}
$$

After an integration over $[0, T] \times [0, T]$, the first term on the right-hand side can be estimated as follows

$$
\int_0^T \int_0^T \left( I^\beta (\partial^\alpha u)(t) - I^\beta (\partial^\alpha u)(\tau) \right) h \cdot \left( I^\beta (\nabla \Delta u)(t) - I^\beta (\nabla \Delta u)(\tau) \right) \, dx \, dt \, dtd\tau \\
\leq C \left( \left[ I^\beta (\partial^\alpha u)^2 \right]_{H^\beta(0,T;L^2(\Omega))} + \left[ I^\beta (\nabla \Delta u)^2 \right]_{H^\beta(0,T;L^2(\Omega))} \right).
$$
Therefore if we integrate (4.14) over \([0, T] \times [0, T]\), we can deduce
\[
\left[ I^\beta (\partial_t \Delta u) \right]_{H^\beta(0,T;L^2(\partial \Omega))} \leq C \left( \left[ I^\beta (\partial_t^\alpha u) \right]_{H^\beta(0,T;L^2(\Omega))} + \left[ I^\beta (\nabla \Delta u) \right]_{H^\beta(0,T;L^2(\Omega))} \right). \tag{4.15}
\]

Combining (4.13) and (4.15) we obtain
\[
\| I^\beta (\partial_t \Delta u) \|_{H^\beta(0,T;L^2(\partial \Omega))} \leq C \left( \| I^\beta (\partial_t^\alpha u) \|_{H^\beta(0,T;L^2(\Omega))} + \| I^\beta (\nabla \Delta u) \|_{H^\beta(0,T;L^2(\Omega))} \right). \tag{4.16}
\]

Thanks again to Theorem 3 we have
\[
\| I^\beta (\partial_t^\alpha u) \|_{H^\beta(0,T;L^2(\Omega))} \sim \| \partial_t^\alpha u \|_{L^2(0,T;L^2(\Omega))},
\]
\[
\| I^\beta (\nabla \Delta u) \|_{H^\beta(0,T;L^2(\Omega))} \sim \| \nabla \Delta u \|_{L^2(0,T;L^2(\Omega))}.
\]

Since a strong solution is also a \(H^3\)-solution we have that (3.13) and (3.14) hold. Finally, by (4.16) we can deduce (4.11). The proof is complete. \(\square\)

\textbf{Theorem 7} Let \(u_0 \in H^1_0(\Omega)\) and \(u_1 \in H^{-1}(\Omega)\). If \(u\) is the \(H^1\)-solution of
\[
\begin{aligned}
\partial_t^\alpha u + \Delta^2 u &= 0 \quad \text{in } (0, T) \times \Omega, \\
u &= \Delta u = 0 \quad \text{on } (0, T) \times \partial \Omega, \\
0 \leq u(0, \cdot) = u_0, \quad u_t(0, \cdot) = u_1,
\end{aligned}
\]
then we define the normal derivative \(\partial_\nu u\) of \(u\) such that for any \(T > 0\) we have
\[
\int_0^T \int_{\partial \Omega} |\partial_\nu u|^2 d\sigma dt \leq C \left( \| u_0 \|_{H^1_0(\Omega)}^2 + \| u_1 \|_{H^{-1}(\Omega)}^2 \right),
\]
for some constant \(C = C(T)\) independent of the initial data.

\textbf{Proof} First we assume that \(u_0 \in H^2(\Omega) \cap H^1_0(\Omega), u_1 \in L^2(\Omega)\) and consider the \(H^2\)-solution \(u\) of (4.17). By Proposition 1 \(A^{-\frac{1}{2}} u\) is the strong solution of (4.17) with the initial conditions \(u_0\) and \(u_1\) replaced by \(A^{-\frac{1}{2}} u_0 \in D(A)\) and \(A^{-\frac{1}{2}} u_1 \in H^2(\Omega) \cap H^1_0(\Omega)\). Therefore, by Theorem 6 we can apply inequality (4.9) to \(A^{-\frac{1}{2}} u\) to get
\[
\int_0^T \int_{\Omega} |\partial_\nu \Delta A^{-\frac{1}{2}} u|^2 d\sigma dt \leq C \left( \| \nabla \Delta A^{-\frac{1}{2}} u_0 \|_{L^2(\Omega)}^2 + \| \nabla A^{-\frac{1}{2}} u_1 \|_{L^2(\Omega)}^2 \right),
\]
for any \(T > 0\). Thanks to (2.4) and (2.10) we have
\[
\| \nabla A^{-\frac{1}{2}} u_1 \|_{L^2(\Omega)} = \| u_1 \|_{H^{-1}(\Omega)},
\]
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so we obtain
\[
\int_0^T \int_{\partial \Omega} |\partial_{\nu} u|^2 d\sigma dt \leq C \left( \|\nabla u_0\|_{L^2(\Omega)}^2 + \|u_1\|_{H^{-1}(\Omega)}^2 \right).
\]

By density there exists a unique continuous linear map
\[
\mathcal{D}_v : H^1_0(\Omega) \times H^{-1}(\Omega) \to L^2(0, T; L^2(\partial\Omega))
\]
such that
\[
\mathcal{D}_v(u_0, u_1) = \partial_{\nu} u \quad \forall (u_0, u_1) \in H^2(\Omega) \cap H^1_0(\Omega) \times L^2(\Omega)
\]
and
\[
\int_0^T \int_{\partial \Omega} |\mathcal{D}_v(u_0, u_1)|^2 d\sigma dt \leq C \left( \|u_0\|_{H^2_0(\Omega)}^2 + \|u_1\|_{H^{-1}(\Omega)}^2 \right) \quad \forall (u_0, u_1) \in H^1_0(\Omega) \times H^{-1}(\Omega).
\]

In conclusion, given \( u_0 \in H^1_0(\Omega) \) and \( u_1 \in H^{-1}(\Omega) \) we define the normal derivative of the \( H^1 \)-solution \( u \) of (4.17) as \( \mathcal{D}_v(u_0, u_1) \) and use the standard notation \( \partial_{\nu} u \) instead of \( \mathcal{D}_v(u_0, u_1) \) in such a way that inequality (4.18) holds. \( \square \)

**Remark 3** Theorem 7 does not follow from the strong trace theorems of the Sobolev spaces. For this reason it can be called a hidden regularity result. The corresponding inequality (4.18) is often called a direct inequality.

## 5 Conclusions and further questions

In this paper we prove existence and hidden regularity for weak solutions of the fractional boundary value problem
\[
\begin{cases}
\partial_t^\alpha u + \Delta^2 u = 0 & \text{in } (0, T) \times \Omega, \\
u = \partial_{\nu} u = 0 & \text{on } (0, T) \times \partial\Omega.
\end{cases}
\]

To conclude our analysis, we direct reader’s attention to the following open problems:

1. Investigate the hidden regularity in the case of Dirichlet–Neumann boundary conditions
\[
\begin{cases}
\partial_t^\alpha u + \Delta^2 u = 0 & \text{in } (0, T) \times \Omega, \\
u = \partial_{\nu} u = 0 & \text{on } (0, T) \times \partial\Omega,
\end{cases}
\]

with \( \alpha \in (1, 2) \). This study needs a new setting of the spaces and also a trick in order to apply the multiplier method without using the integration by parts.
2. A generalization of our research is dealing with polyharmonic operator $\Delta^{2m}$ of order $2m$ with boundary conditions

$$u = \Delta u = \ldots = \Delta^{2m-1} u = 0.$$  

The corresponding problem is

$$\begin{cases}
\partial_t^\alpha u + \Delta^{2m} u = 0 & \text{in } (0, T) \times \Omega, \\
u = \Delta u = \ldots = \Delta^{2m-1} u = 0 & \text{on } (0, T) \times \partial \Omega.
\end{cases}$$

To carry out this study, first one has to understand in which spaces the solution $u$ lives, and then to analyze how the methods used in this paper may be applied.

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**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

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