Self-dual bending theory for vesicles

Jérôme Benoit¹, Elizabeth von Hauff² and Avadh Saxena³

¹ Physics Department, University of Crete and Foundation for Research and Technology-Hellas, PO Box 2208, GR-71003 Heraklion, Crete, Greece
² Faculty of Physics, Department of Energy and Semiconductor Research, University of Oldenburg, D-26111 Oldenburg, Germany
³ Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545, USA

E-mail: jgmbenoit@wanadoo.fr

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Abstract
We present a self-dual bending theory that may enable a better understanding of highly nonlinear global behaviour observed in biological vesicles. Adopting this topological approach for spherical vesicles of revolution allows us to describe them as frustrated sine-Gordon kinks. Finally, to illustrate an application of our results, we consider a spherical vesicle globally distorted by two polar latex beads.

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1. Introduction
Our primary motivation is to contribute to the understanding of highly nonlinear global behaviour observed in vesicles. Such cases include closed vesicles with low genus exhibiting spontaneous conformal transformation [1–3], spherical vesicles globally distorted by a single latex bead [4], and spherical vesicles with several arms [5]. Whereas the difference in scale between a vesicle’s membrane thickness and its overall size allows the vesicle to be described as an embedded surface, the understanding of vesicle morphology is currently founded on the concept of bending elasticity [6–11]. To investigate vesicles, biophysicists widely invoke a harmonic bending Hamiltonian, historically introduced by Helfrich [10], and perform polynomial approximations to describe matter interactions. However, within the context of exploring nonlinear global behaviours, such an approach may appear inappropriate to a geometric topologist. In this paper an attempt to construct a more suitable covariant field theory for the bending behaviour of vesicles, outlined in a previous work [12], is illustrated through spherical vesicles of revolution.
Customarily, the Monge representation (surface equations) of the vesicle shape [7,8] leads to a local characterization of the surface by its two principal curvatures [13–15], and thereby to an expression for the bending energy as an expansion in the principal curvatures up to a given order and with respect to some desired symmetries: (i) the even symmetric expansion up to second order corresponds to the curvature energy density introduced by Canham [9], which has one phenomenological parameter; (ii) the most general symmetric expansion up to second order is the well-known Helfrich curvature energy density, which contains three phenomenological parameters [6–8, 11]; (iii) the antisymmetric expansion up to second order fits the deviatoric bending contribution density suggested by Fischer [16, 17], which has two phenomenological parameters; (iv) expansion of higher order can be envisaged likewise [18].

By assumption, the general symmetric expansions in the principal curvatures can be formulated as expansions in the mean curvature (mean of the principal curvatures) and in the Gaussian curvature (product of the principal curvatures): the harmonic Helfrich curvature energy density is quadratic in the mean curvature and linear in the Gaussian curvature [6–8, 11]. Commonly, as long as the vesicle remains in the same topological class, the Gaussian curvature term is dropped out [6–8] by invoking the Gauss–Bonnet theorem [13–15], which claims that the total integral of the Gaussian curvature over the vesicle surface depends only on the topology of the shape—the total Gaussian curvature measures the genus of the shape. From the perspective of a geometric topologist, ignoring such a global scale property may be undesirable and may essentially favour the notion of bending elasticity on the local scale. Although this local approach allows a large variety of phenomena to be understood [5–8, 19, 20], it may certainly preclude a deep understanding of some global bending phenomena as it fails to reveal the topological classification of vesicles.

In contrast, the fundamental theorem of surface theory [13–15, 21] leads to a description of the surface by a prescribed metric tensor and a prescribed shape tensor related to each other by integrability conditions. We should therefore inquire how bending elasticity may be formulated within a covariant field theory for vesicles, and ultimately determine which principle may dictate bending elasticity. With this in mind, the observation of closed vesicles with low genus exhibiting spontaneous conformal transformation [1–3] strongly suggests imposing a covariant functional that reveals the underlying topology and that is globally invariant under smooth conformal transformations as a bending Hamiltonian. In general, symmetries and conservation laws are connected by the Noether theorem [22–24]. Nevertheless, in this context, the conserved entity emerges from topology while the pertinent transformations are restricted by topology: two vesicles of different genus cannot be continuously deformable into one another—they are topologically distinct. As a matter of fact, topology gives rise to new physics [24, 25] with some of the following relevant features: (i) metastable configurations (mostly solitons) fall into distinct classes, of which the trivial configuration (vacuum) belongs only to one class; (ii) there exists no superposition principle, i.e. resultant configurations form complicated ones; (iii) frustration phenomena may arise from topological obstruction, leading to both global and localized effects [26–29].

Noether-like calculational machineries exist to study topological systems: vesicles were shown [12] to be subject to a technique known as the Bogomol’nyi decomposition, which successfully treats various topological models in fields ranging from condensed matter physics to high energy physics [24, 30–33]. Concisely, the Bogomol’nyi technique applies to the total integral of the contracted self-product of the shape tensor, known to be globally invariant under conformal transformations [34, 35], revealing the topological nature of bending phenomena by both identifying the nontrivial metastable bending configurations and splitting bending configurations into topological classes according to their genus/end. The nontrivial metastable bending configurations are the round sphere and the minimal surfaces up to a
conformal transformation of the ambient space (e.g. catenoid and Lawson surfaces [36]), in agreement with observations [1,6–8,37]. Furthermore, it is worth noting that any deformation of the nontrivial metastable bending configurations spontaneously matches, for vesicles of spherical topology, the deviatoric bending contribution described by Fischer [16, 17] and, for vesicles of nonspherical topology, the mean curvature bending contribution (up to a conformal transformation of the ambient space) described by Helfrich [6–8, 11]. For completeness, we note that the involved total integral corresponds to the covariant form of the bending energy proposed by Canham [9]. Clearly, at least to study global bending phenomena, the topological approach is very appealing. Nevertheless, the mathematics involved here may discourage some readers. To overcome this, we show how this unorthodox approach leads to a description of spherical vesicles of revolution in terms of ‘frustrated’ sine-Gordon kinks.

2. Self-dual approach

Before focusing on spherical vesicles of revolution, we first succinctly expose how topology in vesicles is revealed by the Bogomol’nyi decomposition [24,30–33]. Following the fundamental theorem of surface theory, we represent the vesicle shape by a pair of tensors coupled to each other by integrability conditions, namely a prescribed first fundamental form (the metric tensor) $g_{ij}$ coupled to a prescribed second fundamental form (the shape tensor) $b_{ij}$ with respect to the equation of Gauss and the equations of Codazzi and Peterson [13–15]. For the bending Hamiltonian, we consider the covariant functional

$$H_b[S] \equiv \frac{1}{2} k \int_S dS \, b_{ij}^{ij}, \quad (1)$$

which depends on the vesicle shape $S$ through the prescribed pair $(g_{ij}, b_{ij})$ and on the phenomenological parameter $k$ describing the bending rigidity. The summation convention has been adopted, while customary notations have been used: the integral runs over the surface manifold $S$ with surface element $dS = dx^2 \sqrt{|g|}$, where $|g|$ represents the determinant $\det(g_{ij})$ and $x$ the set of arbitrary intrinsic coordinates. Observe that the suggested covariant functional (1) corresponds to the covariant form of the bending energy invoked by Canham [9] since the scalar $b_{ij}^{ij}$ is equal to the trace of the square of the mixed shape tensor $b'$, and thus to the sum of its squared eigenvalues, namely the sum of the squared principal curvatures. Next, we define the dual tensor $\tilde{a}_{ij}$ of an arbitrary tensor $a_{ij}$ by

$$\tilde{a}_{ij} \equiv \epsilon_{ik} \epsilon_{jl} a_{kl}, \quad (2)$$

where $\epsilon_{mn}$ denotes the totally antisymmetric tensor associated with the surface $S$. It is easily checked that the dual transformation $a_{ij} \rightarrow \tilde{a}_{ij}$ is an involution, i.e.

$$\tilde{\tilde{a}}_{ij} = a_{ij}. \quad (3)$$

While there exists no such relation between the shape tensor $b_{ij}$ and its dual $\tilde{b}_{ij}$ the metric tensor $g_{ij}$ satisfies the property of self-duality; that is

$$\tilde{g}_{ij} = g_{ij}. \quad (4)$$

Nevertheless the bending Hamiltonian (1) remains locally invariant under the dual transformation as we have, after straightforward index rearrangements,

$$\tilde{b}_{ij} b^{ij} = b_{ij} b^{ij}. \quad (5)$$

Most significantly, the extrinsic curvature $G$, which yields

$$G = \frac{1}{2} b_{ij} b^{ij}, \quad (6)$$
remains clearly unchanged under the dual transformation according to the involutive relation (3). The interesting feature is that the extrinsic curvature $G$ coincides with the intrinsic curvature (the Gaussian curvature) $K$ when the embedding ambient space is flat: its total integral then measures the topology of the vesicle shape.

More precisely, the Gauss–Bonnet theorem [13, 38] claims that

$$\int_{S_{g,e}} \text{d}S \, G = -4\pi(g + e - 1),$$  \hspace{1cm} (7)

where $S_{g,e}$ is a surface manifold embedded in a flat ambient space and topologically equivalent to a closed surface manifold of genus $g$ less $e$ points (ends). These considerations lead to the construction of a self-dual theory by introducing the self-dual/anti-self-dual tensors

$$B_{ij}^\pm \equiv \frac{1}{\sqrt{2}} [b_{ij} \pm \tilde{b}_{ij}],$$  \hspace{1cm} (8)

which manifestly verify the property of self-duality/anti-self-duality according to the involutive relation (3):

$$\tilde{B}_{ij}^\pm = \pm B_{ij}^\pm.$$  \hspace{1cm} (9)

Contracting the self-product of the self-dual/anti-self-dual tensors (8) and using the equalities (5) and (6) readily show that the bending Hamiltonian density $H_b$ extracted from the bending Hamiltonian (1) decomposes as

$$H_b = \frac{1}{2}k B_{ij}^\pm B^{\pm ij} \mp kG.$$  \hspace{1cm} (10)

Integrating this decomposition (10) over the surface manifold $S$ and recognizing the total curvature (7) lead to rewriting the bending Hamiltonian (1) in each topological class specified by the pair $(g, e)$ in the form

$$\mathcal{H}_b[S_{g,e}] = \frac{1}{2}k \int_{S_{g,e}} \text{d}S \, B_{ij}^\pm B^{\pm ij} \pm 4\pi k (g + e - 1).$$  \hspace{1cm} (11)

Since the tensors $B_{ij}^\pm$ yield the precious inequalities

$$B_{ij}^\pm B^{\pm ij} \geq 0,$$  \hspace{1cm} (12)

the decompositions (11) saturate when the tensors $B_{ij}^\pm$ vanish. Henceforth, since the shape tensor is self-dual ($B_{ij}^\pm = 0$) only for the round sphere $S^2$, the decomposition with sign $(-)$ in (11) is relevant only for the vesicle surfaces $S_{0,e}$ topologically equivalent to the round sphere $S^2$: we get

$$\mathcal{H}_b[S_0] = \frac{1}{2}k \int_{S_0} \text{d}S \, B_{ij}^- B^{-ij} + 4\pi k.$$  \hspace{1cm} (13)

Similarly, as the shape tensor is anti-self-dual ($B_{ij}^\pm = 0$) only when the surface manifold is a minimal surface, the decomposition with sign $(+)$ in (11) is pertinent only for the vesicle surfaces $S_{g,e}$ topologically equivalent to a minimal surface of genus $g$ with $e$ ends. Since within flat space there is no closed minimal surface ($e = 0$) whereas minimal surfaces of genus $g$ ($g \geq 0$) with $e$ ends ($e \geq 2$) do exist [38], we write

$$\mathcal{H}_b[S_{g,e}] = \frac{1}{2}k \int_{S_{g,e}} \text{d}S \, B_{ij}^+ B^{+ij} + 4\pi k (g + e - 1),$$  \hspace{1cm} (14)

which is valid only when $g \geq 0$ and $e \geq 2$. Notice that the minimal surface of genus zero with two ends is the catenoid. Tedious calculations allow one to expand the previous decomposition (14) to closed surfaces of arbitrary genus ($g \geq 1$) [12]. Besides bringing out the inherent topology, the Bogomol’nyi decompositions (13) and (14) offer a new perspective on bending
phenomena: when the shape tensor obeys the local property of self-duality/anti-self-duality, then the bending Hamiltonian is saturated. Conversely, when the shape tensor violates the local property of self-duality/anti-self-duality, then the bending Hamiltonian is frustrated, i.e. an extra energy contribution is spontaneously created [12, 26–29] that tends to vanish globally in the system. Before pursuing this matter any further, we turn our attention to the spherical vesicles of revolution, which are of pertinent interest.

3. Spherical vesicles of revolution

Now, we concentrate on vesicle shapes that are both smoothly transformable to the round sphere \( S^2 \) and can be generated by rotating a two-dimensional curve (a profile) \( P \) about an axis, i.e. on spherical vesicles of revolution. For our purpose, the azimuthal isometric coordinates \((u, \varphi)\) appear to be a suitable choice of coordinates as follows. First, adopting isometric coordinates (‘isothermal’ coordinates) allows us to write the metric tensor \( g_{ij} \) in the form \[ g_{uu} = g_{\varphi\varphi} = e^{2\sigma(u)}, \tag{15} \] where the local Weyl gauge field \( \sigma \) depends only on the isometric coordinate \( u \) because of the azimuthal symmetry. Moreover, for any surface of revolution, analytical manipulations show that the shape tensor \( b_{ij} \) in azimuthal isometric coordinates \((u, \varphi)\) yields \[ b_{uu} = -e^{\sigma(u)} \partial_u \Omega(u) \quad \text{and} \quad b_{\varphi\varphi} = -e^{\sigma(u)} \sin \Omega(u), \tag{16} \] where \( \Omega \) corresponds to the polar angle of the unit normal vector of the surface. However, in order for the two tensors \( g_{ij} \) and \( b_{ij} \), cast respectively in the forms (15) and (16), to be the first and second fundamental forms for a surface \( S \) in \( \mathbb{R}^3 \), they must satisfy the Gauss equation and the Codazzi and Peterson equations [13–15]: in this case, after easy computations, this set of integrability conditions reduces to the equation \[ \partial_u \sigma(u) = \cos \Omega(u). \tag{17} \] Conversely, simple calculations show that the Gauss–Weingarten equations [13–15] associated to the first form (15) and the second form (16) under the integrability condition (17) give a unique surface of revolution, except for its position and scale in space. To summarize, any pair of tensors \((g_{ij}, b_{ij})\) that verifies the formulae (15), (16) and (17) in azimuthal isometric coordinates \((u, \varphi)\) is equivalent to a surface of revolution \( S \) in \( \mathbb{R}^3 \). Substituting (15) and (16) into (8) leads to \[ B_{uu}^\pm = B_{\varphi\varphi}^\pm = -\frac{1}{\sqrt{2}} e^{\sigma(u)} \left[ \partial_u \Omega(u) \pm \sin \Omega(u) \right]. \tag{18} \] Henceforth, by applying (13), the bending Hamiltonian Bogomol’nyi decomposition for the spherical surfaces of revolution \( S_0 \) reads \[ \mathcal{H}_b[S_0] = \pi k \int_P du \left[ \partial_u \Omega(u) - \sin \Omega(u) \right]^2 + 4\pi k, \tag{19} \] where the integral runs along the profile \( P \) of the spherical surface \( S_0 \). The bending Hamiltonian (19) is saturated when the polar angle \( \Omega \) satisfies the equation \[ \partial_u \Omega(u) = \sin \Omega(u), \tag{20} \] whose centred solution is the sine-Gordon kink \[ \Omega(u) = \pi - \arccos(\tanh(u)). \tag{21} \]
Resolving the Gauss–Weingarten equations [13–15] with respect to (15), (16), (17), and (21) uniquely gives (except for its position and scale in space) the surface, in cylindrical parametrization,

\[ r(u) = \text{sech}(u), \quad z(u) = -\tanh(u), \] (22)

which is the round sphere \( S^2 \) in azimuthal isometric coordinates \((u, \varphi)\), \( u \) running from \(-\infty\) to \(+\infty\). Hence, in accordance with the existence theorem previously invoked, the spherical vesicle of revolution that saturates the bending Hamiltonian (1) is the round sphere \( S^2 \). However the key result is contained in the fact that the metastable spherical vesicle of revolution can be described by a saturated sine-Gordon kink: any spherical vesicle of revolution may be thereby envisioned as a frustrated (i.e. unsaturated) sine-Gordon kink, and highly nonlinear global physics may henceforth emerge from the model [26–29]. Therefore, without loss of generality, the initial bidimensional system has been reduced to a well-known unidimensional nonlinear system. Next we illustrate our approach through an illuminating example.

4. Latex bead on a spherical vesicle

Grafting a latex bead on a spherical vesicle induces a global distortion characterized by two polar concave regions: a pinched concave region around the bead faces a smooth one as if a second bead were diametrically opposite [4]. Accordingly, we assume a bare spherical vesicle of revolution with two identical round latex beads that are respectively attached at the north and south poles. By bare, we mean that only the bending Hamiltonian (1) is considered. To begin with, substituting (15) and (16) into (1) gives the Hamiltonian of our system in the canonical form:

\[ \mathcal{H}_b[S] = \pi k \int \rho \left[ \partial_u \Omega(u)^2 + \sin^2 \Omega(u) \right]. \] (23)

The Euler–Lagrange equation derived from (23) (or from (19)) is the sine-Gordon equation,

\[ \partial_{uu} \Omega(u) = \sin \Omega(u) \cos \Omega(u). \] (24)

The solution of (24) extending (21),

\[ \Omega(u) = \pi - \arccos \left( \frac{u}{\sqrt{m}} \right), \quad m \in (0, 1], \] (25)

Figure 1. North pole sketch of a spherical vesicle of revolution distorted by two identical round latex beads grafted at its poles as deduced by the self-dual bending theory. The dashed circle profiles the bead, the bold arc the polar cap imposed by it. The arrows indicate the nomenclature: the latex bead dimensionless radius \( \rho \) the encapsulated dimensionless radius \( \hat{\rho} \) and the encapsulation angle \( \hat{\alpha} \). Dimensionless latex bead radius \( \rho = 0.15 \); relative encapsulated radius \( \hat{\rho}/\rho = \frac{7}{8} \).
Figure 2. North hemispheric profiles of a spherical vesicle of revolution distorted by two identical round latex beads grafted at its poles with respect to different encapsulated radii \( \hat{\rho} \) as deduced by the self-dual bending theory. The bold arcs correspond to the profiles of the polar cap imposed by the bead. Dimensionless latex bead radius \( \rho = 0.15 \); relative encapsulated radius \( \frac{\hat{\rho}}{\rho} \) from outside to inside: 0, 1/4, 1/2, 3/4, 7/8, 1.

corresponds to the surface of revolution

\[
\begin{align*}
    r(u) &= \frac{1}{1 + \sqrt{m}} \left[ \text{dn} \left( \frac{u}{\sqrt{m}} \bigg| m \right) + \sqrt{m} \text{cn} \left( \frac{u}{\sqrt{m}} \bigg| m \right) \right], \\
    z(u) &= -\frac{1}{1 + \sqrt{m}} \left[ \sqrt{m} \text{sn} \left( \frac{u}{\sqrt{m}} \bigg| m \right) + \text{Dn} \left( \frac{u}{\sqrt{m}} \bigg| m \right) - \left( 1 - m \right) u \right],
\end{align*}
\]

which smoothly joins\(^4\) the concavely bounded spherical polar caps to the vesicle surface when the parameter \( m \) satisfies

\[
m = \left[ \frac{\rho (1 - \hat{\rho}^2)}{\rho (1 + \hat{\rho}^2) + 2\hat{\rho}^2} \right]^2 0 \leq \hat{\rho} \leq \rho < 1,
\]

\( \rho \) and \( \hat{\rho} \) being, respectively the latex bead and encapsulated dimensionless radii—see figures 1 and 2. The isometric coordinate \( u \) runs from \(-\hat{u}\) to \(+\hat{u}\) with \( \hat{u} = \sqrt{m} \left[ 2K(m) - F(\cos \hat{\alpha} | m) \right] \), where the encapsulation angle \( \hat{\alpha} \) yields \( \sin \hat{\alpha} = \hat{\rho} / \rho \). The reader is encouraged to check that, when the encapsulated dimensionless radius \( \hat{\rho} \) vanishes, the parameter \( m \) effectively tends towards 1 and thus the surface (26) tends to the round sphere \( S^2 \) (22) as expected. Now the extra energy \( \Delta \hat{E} \) spontaneously generated by the polar latex beads can be computed. We have

\[
B_{uu} = -B_{\varphi \varphi} = -\frac{1}{\sqrt{2}} \frac{1 - \sqrt{m}}{\sqrt{m}},
\]

from which we obtain

\[
\Delta \hat{E} = \frac{4\pi k}{\sqrt{m}} \left[ 2 \left( E(m) - \frac{1 - m}{2} K(m) \right) - \left( E(\cos \hat{\alpha} | m) - \frac{1 - m}{2} F(\cos \hat{\alpha} | m) - \sqrt{m} \cos \hat{\alpha} \right) \right].
\]

\(^4\) The junction conditions are the ones implicitly assumed in [4]: at the junction points, the normal lines of the vesicle surface and of the latex bead surface are imposed to coincide—see figure 1. Such an assumption is reasonable at the vesicle scale; nevertheless at the membrane scale an adequate treatment may be needed since the curvature experiences a discontinuity at the junction points—as far as we know there is no such treatment in the literature.
Figure 3. The relative extra bending energy $\Delta \hat{E} / 4\pi k$ versus the relative encapsulated radius $\hat{\rho} / \rho$ for different radii $\rho$ of the grafted latex beads: the radius $\rho$ increases in the direction of the arrow. Dimensionless latex bead radius from right to left: 0.01, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6.

Standard notations have been adopted$^5$ [41–43]. Observing that, according to (27), the parameter $m$ decreases strictly from 1 to 0 with respect to the encapsulated radius $\hat{\rho}$ allows an effortless description: when the encapsulated radius $\hat{\rho}$ vanishes, the underlying soliton (25) reaches its saturated configuration (21); when the encapsulated radius $\hat{\rho}$ increases, the underlying soliton (25) becomes frustrated and the bending energy is accordingly altered—see figure 3. Finally, our naive model system demonstrates at least one weakness of the traditional (local) approach for the description of global bending phenomena, as follows: since its mean curvature $H$ or the half trace of the shape tensor, is constant,

$$H = \frac{1}{2} \frac{1 + \sqrt{m}}{\sqrt{m}},$$

the surface of revolution (26) causes the harmonic Helfrich bending Hamiltonian density either to vanish if a phenomenological spontaneous curvature is introduced or, if not, to be proportional to the square of the mean curvature [6–8, 11], irrespective of the value of the encapsulated radius $\hat{\rho}$. Consequently, the traditional approach is partially insensitive to the encapsulation mechanism experienced by our system, in the sense that an observer somewhere on the surface outside the grafted regions cannot determine whether latex beads are grafted at the poles or not just from the harmonic Helfrich bending Hamiltonian density, whereas the same observer can obtain a quantitative answer by computing the extra bending Hamiltonian density. In the former case, what occurs globally is not known because in every scenario the harmonic Helfrich bending Hamiltonian density takes the same form; in the latter, the extra bending Hamiltonian density vanishes when there are no latex beads but it measures the deformation otherwise. Nevertheless it may be objected that, if no phenomenological spontaneous curvature is introduced, computing the harmonic Helfrich bending Hamiltonian density allows for an answer that is quantitatively similar, provided that the expression is known for the particular

$^5$ The functions sn, cn, and dn are the Jacobian elliptic functions, $F$ and $K$ are the incomplete and complete elliptic integrals of the first kind, $E$ is the incomplete and complete (depending on the argument) elliptic integrals of the second kind, and the function $Dn$ denotes the Jacobian Epsilon function defined as $Dn(u \mid m) = \int_0^u \frac{dt}{\sqrt{(1 - t^2)(1 - m^2 t^2)}}$ [41].
state, the one without latex beads for example. Such exact information, required for the former approach but superfluous for the topological approach, may not be so easily identified in a more realistic case. Second, the topological approach actually provides a finer response as the deformation measurement is really enabled by the anti-self-dual tensor, which can be conceived as a bending deformation tensor.

5. Conclusion

In conclusion, the results of our investigations not only provide a novel viewpoint regarding bending phenomena, but also demonstrate the possibility of describing spherical vesicles of revolution by frustrated sine-Gordon kinks as a method to investigate the essence of highly nonlinear global behaviour observed in vesicles. The geometry of vesicles has been studied quite extensively in recent years [44–47] but it was not considered in conjunction with the global topological aspects. Of course, this model is too naive to capture the complexity of biological vesicles; in particular it should be complemented by the inclusion of material fields and constraints [6–8]. Still, as far as bending phenomena are concerned, the self-dual bending theory should provide a powerful theoretical framework to study global behaviour observed in vesicles or, in general, other deformable systems.

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References

[1] Michalet X, Jülicher F, Fourcade B, Seifert U and Bensimon D 1994 *La Recherche* 25 1012
[2] Michalet X and Bensimon D 1995 *Science* 269 666
[3] Michalet X and Bensimon D 1995 *J. Phys. II* 5 263
[4] Koltover I, Rädler J O and Safinya C R 1999 *Phys. Rev. Lett.* 82 1991
[5] Wintz W, Döbereiner H G and Seifert U 1996 *Europhys. Lett.* 33 403
[6] Seifert U and Lipowsky R 1995 *Handbook of Biological Physics* vol 1 (Amsterdam: Elsevier) pp 403–62
[7] Peliti L 1996 *Les Houches* vol LXII (Amsterdam: Elsevier) pp 195–285
[8] Seifert U 1997 *Adv. Phys.* 46 13
[9] Canham P B 1970 *J. Theoret. Biol.* 26 61
[10] Helfrich W 1973 *Z. Naturforsch.* C 28 693
[11] Helfrich W 1974 *Z. Naturforsch.* C 29 510
[12] Benoit J, Saxena A and Lookman T 2001 *J. Phys. A: Math. Gen.* 34 9417
[13] Struik D J 1961 *Lectures on Classical Differential Geometry* 2nd edn (New York: Dover)
[14] Gray A 1997 *Modern Differential Geometry of Curves and Surfaces with Mathematica* 2nd edn (Boca Raton, FL: CRC Press)
[15] Frankel T 1999 *The Geometry of Physics: An Introduction* (Cambridge: Cambridge University Press)
[16] Fischer T M 1992 *J. Phys. II* 2 337
[17] Fischer T M 1993 *J. Phys. II* 3 1795
[18] Goetz R and Helfrich W 1996 *J. Phys. II* 6 215
[19] Jülicher F, Seifert U and Lipowsky R 1993 *Phys. Rev. Lett.* 71 452
[20] Jülicher F 1996 *J. Phys. II* 6 1797
[21] Bishop R L and Crittenden R J 1964 *Geometry of manifolds Pure and Applied Mathematics* vol XV (New York: Academic)
[22] Noether E A 1918 *Nachr. v. d. Ges. d. Wiss. Göttingen* 2 235
[23] Noether E A and Tavel M A 1971 *Transport Theory Stat. Phys.* 1 183
[24] Ryder L H 1996 *Quantum Field Theory* 2nd edn (Cambridge: Cambridge University Press)
[25] Nakahara M 1990 *Geometry, Topology and Physics* Graduate Student Series in Physics (Bristol: Adam Hilger)
[26] Dandoloff R, Villain-Guillot S, Saxena A and Bishop A R 1995 *Phys. Rev. Lett.* **74** 813
[27] Villain-Guillot S, Dandoloff R, Saxena A and Bishop A R 1995 *Phys. Rev. B* **52** 6712
[28] Benoit J and Dandoloff R 1998 *Phys. Lett. A* **248** 439
[29] Benoit J, Dandoloff R and Saxena A 2000 *Int. J. Mod. Phys. B* **14** 2093
[30] Hlousek Z and Spector D 1993 *Nucl. Phys. B* **397** 173
[31] Belavin A A and Polyakov A M 1975 *JETP Lett.* **22** 245
[32] Bogomol'nyi E B 1976 *Sov. J. Nucl. Phys.* **24** 449
[33] Felsager B 1987 *Geometry, Particles and Fields* 4th edn (Gylling: Odense University Press)
[34] Chen B Y 1973 *Proc. Am. Math. Soc.* **40** 563
[35] Weiner J L 1978 *Indiana Univ. Math. J.* **27** 19
[36] Lawson H B 1970 *Ann. Math.* **92** 335
[37] Fourcade B, Mutz M and Bensimon D 1992 *Phys. Rev. Lett.* **68** 2551
[38] Hoffman D and Karcher H 1997 Geometry V *Encyclopædia of Mathematical Sciences* ed R Osserman vol 90 (part 1) (Berlin: Springer) pp 7–93
[39] Chandrasekhar S 1998 *The Mathematical Theory of Black Holes* Oxford Classic Texts (Oxford: Oxford University Press)
[40] Fulton T, Rohrlich F and Witten L 1962 *Rev. Mod. Phys.* **34** 442
[41] Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions* (New York: Dover)
[42] Whittaker E T and Watson G N 1927 *A Course of Modern Analysis* 4th edn (Cambridge: Cambridge University Press)
[43] Byrd P F and Friedman M D 1971 *Handbook of Elliptic Integrals for Engineers and Scientists* (Lecture Notes in Mathematics vol 67) 2nd edn (Berlin: Springer)
[44] Bloor M I G and Wilson M J 2000 *Phys. Rev. E* **61** 4218
[45] Capovilla R and Guven J 2002 *Phys. Rev. E* **66** 041604
[46] Lenz P and Nelson D R 2002 *Phys. Rev. E* **67** 031502
[47] Lin C H, Lo M H and Tsai Y C 2003 *Prog. Theor. Phys.* **109** 591