Research Article
Some Properties of Generalized Einstein Tensor for a Pseudo-Ricci Symmetric Manifold

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1. Introduction

In the late twenties, because of the important role of symmetric spaces in differential geometry, Cartan [1], who, in particular, obtained a classification of those spaces, established Riemannian symmetric spaces. The notion of the pseudosymmetric manifold was introduced by Chaki [2] and Deszcz [3]. Recently, some necessary and sufficient conditions for a Chaki pseudosymmetric (respectively, pseudo-Ricci symmetric [4]) manifold to be Deszcz pseudosymmetric (respectively, Ricci-pseudo symmetric [5]) have been examined in [6].

A nonflat n-dimensional Riemannian manifold \((M, g)\), \((n > 3)\) is called a pseudo-Ricci symmetric manifold if the Ricci tensor \(S\) of type \((0,2)\) is not identically zero and satisfies the condition [4]

\[
(\nabla_Z S)(X,Y) = 2\pi(Z)S(X,Y) + \pi(X)S(Z,Y) + \pi(Y)S(X,Z),
\]

where \(\pi\) is a nonzero 1-form, \(\rho\) is a vector field by

\[
g(X, \rho) = \pi(X)
\]

for all vector fields \(X\), and \(\nabla\) denotes the operator of covariant differentiation with respect to the metric \(g\). Such a manifold is denoted by \((\text{PRS})_n\). The 1-form \(\pi\) is called the associated 1-form of the manifold. If \(\pi = 0\), then the manifold reduces to a Ricci symmetric manifold or covariantly constant

\[
(\nabla_Z S)(X,Y) = 0.
\]

The notion of pseudo-Ricci symmetry is different from that of Deszcz [3].

The pseudo-Ricci symmetric manifolds have some importance in the general theory of relativity. So, pseudo-Ricci symmetric manifolds on some structures have been studied by many authors (see, e.g., [7, 8]).

A nonflat Riemannian manifold \((M, g)\), \((n > 2)\), is called generalized recurrent if the Ricci tensor \(S\) is nonzero and satisfies the condition

\[
(\nabla_Z S)(X,Y) = A(Z)S(X,Y) + B(Z)g(X,Y),
\]

where \(A\) and \(B\) are nonzero 1-forms [9]. If the associated 1-form \(B\) becomes zero, then the manifold reduces to Ricci recurrent, i.e.,
\[(\nabla_Z S)(X, Y) = A(Z)S(X, Y).\]

A Riemannian manifold \((M, g), (n \geq 2)\), is said to be an Einstein manifold if the following condition:

\[S = \frac{r}{n}g,\]

holds on \(M\), where \(S\) and \(r\) denote the Ricci tensor and scalar curvature of \((M, g)\), respectively. According to [10], equation (6) is called the Einstein metric condition. Also, Einstein manifolds form a natural subclass of various classes of Riemannian manifolds by a curvature condition imposed on their Ricci tensor [10]. For instance, every Einstein manifold belongs to the class of Riemannian manifolds \((M, g)\) realizing the following relation:

\[S(X, Y) = ag(X, Y) + bA(X)A(Y),\]

where \(a, b\) are real numbers and \(A\) is a nonzero 1-form such that

\[g(X, U) = A(X)\]

for all vector fields \(X\).

A nonflat Riemannian manifold \((M, g), (n > 2)\), is defined to be a quasi-Einstein manifold if its Ricci tensor \(S\) of type \((0, 2)\) is not identically zero and satisfies the condition (7).

2. Recurrent Generalized Einstein Tensor

\[G(X, Y)\] in \((PRS)_n\)

It is well known that the Einstein tensor \(E(X, Y)\) for a Riemannian manifold is defined by

\[E(X, Y) = S(X, Y) - rg(X, Y),\]

where \(S(X, Y)\) and \(r\) are, respectively, the Ricci tensor and the scalar curvature of the manifold, playing an important part in Einstein’s theory of gravitation as well as in proving some theorems in Riemannian geometry [10]. Moreover, the Einstein tensor can be obtained from Yano’s tensor of concircular curvature. In [11], by using this approach, some generalizations of the Einstein tensor were achieved.

In this section, we consider the generalized Einstein tensor

\[G(X, Y) = S(X, Y) - \kappa rg(X, Y),\]

where \(\kappa\) is constant [12].

Now, we assume that our manifold \((PRS)_n\) has nonzero \(G(X, Y)\)-Einstein tensor. By taking the covariant derivative of (10), in the local coordinates, we get

\[\nabla_k G_{ij} = \nabla_k S_{ij} - \kappa g_{ij} \nabla_k r.\]

If we contract (1) over \(X\) and \(Y\), then we obtain

\[\nabla_k r = 2\pi \kappa r + 2\pi^h S_{hk}.\]

Substituting (1) and (12) into (11), we achieve

\[\nabla_k G_{ij} = 2\pi \kappa S_{ij} + \pi \kappa S_{jk} + \pi \kappa S_{ik} - \left(2\pi \kappa r + 2\pi^h S_{hk}\right) \kappa g_{ij}.\]

Now, contracting (13) with respect to \(i\) and \(k\), we obtain

\[\text{div } G_{ij} = \nabla_k G^k_{ij} = (3 - 2\kappa) \pi^h S_{bj} + (1 - 2\kappa) r \kappa g_{ij}.\]

If we assume that \(G(X, Y)\) is conservative [13], i.e., \(\text{div } G = 0\), then from (14), we have

\[(3 - 2\kappa) P_j + (1 - 2\kappa) r \kappa g_j = 0,\]

where \(P_j = \pi^h S_{bj}\).

If \((1 - 2\kappa) r\) is an eigenvalue of the Ricci tensor \(S\) corresponding to the eigenvector \(\pi(X)\), then \((3 - 2\kappa)\) is an eigenvalue of the Ricci tensor \(S\) corresponding to the eigenvector \(P_j\). Conversely, if equation (15) holds, then the form (14) is the generalized Einstein tensor \(G(X, Y)\) is conservative. We have thus proved the following.

Theorem 1. For a \((PRS)_n\) manifold, the necessary and sufficient condition of the generalized Einstein tensor \(G(X, Y)\) be conservative is that \((1 - 2\kappa) r\) and \((3 - 2\kappa)\) be eigenvalues of the Ricci tensor \(S\) corresponding to the eigenvectors \(\pi_j\) and \(P_j = \pi^h S_{bj}\), respectively.

Let \(G(X, Y)\) be recurrent, i.e., from (5),

\[\nabla_k G_{ij} = A_k G_{ij}.\]

Substituting equations (10) and (13) into equation (16) yields

\[2\pi \kappa S_{ij} + \pi \kappa S_{jk} + \pi \kappa S_{ik} - \left(2\pi \kappa r + 2\pi^h S_{hk}\right) \kappa g_{ij} = A_k \left(S_{ij} - r^h g_{ij}\right).\]

If we contract (17) over \(i\) and \(k\), then we have

\[(3 - 2\kappa) P_j + (1 - 2\kappa) r \kappa g_j = A_k S_{kj}^h - r^h A_j.\]

This leads to the following result:

Theorem 2. In a \((PRS)_n\) manifold, let us assume that the generalized Einstein tensor \(G(X, Y)\) is recurrent with the recurrence vector generated by the 1-form \(A\). Then, the recurrence vector \(A\) satisfies equation ((18)).
Now, we assume that the generalized Einstein tensor \( G(X, Y) \) is conservative. From (15) and (18), we get
\[
Q_j - r\kappa A_j = 0, \tag{19}
\]
where \( Q_j = A_k S^k_j \).

Then, the following theorem holds true:

**Theorem 3.** In a \((PRS)_n\) manifold, let the generalized Einstein tensor \( G(X, Y) \) be recurrent with the recurrence vector generated by the 1-form \( A \). If the generalized Einstein tensor \( G(X, Y) \) is also conservative, then the vectors \( Q_j \) and \( A_j \) are linearly dependent.

Let \( G(X, Y) \) be a generalized recurrent. Then from (4),
\[
\nabla_i G_{ij} = A_k G_{ij} + B_k g_{ij}. \tag{20}
\]

Using (1) and (10), we get
\[
2\pi_k S_{ij} + \pi_i S_{kj} + \pi_j S_{ik} + \kappa g_{ij} \left( 2\pi_k r + 2\pi^h S_{kh} \right) = A_k \left( S_{ij} + \kappa g_{ij} \right) + B_k g_{ij}. \tag{21}
\]

If we contract (21) over \( i \) and \( j \), then we have
\[
(1 + \kappa n)(2\pi_k r + 2P_k - A_k r) = nB_k. \tag{22}
\]
If \( 1 + \kappa n = 0 \), then \( B_k = 0 \).

This leads to the following result:

**Theorem 4.** If \( \kappa = -1/n \), a \((PRS)_n\) manifold admitting the generalized Einstein tensor \( G(X, Y) \) which is the generalized recurrent cannot exist.

### 3. Birecurrent Generalized Einstein Tensor

**\( G(X, Y) \) in \((PRS)_n\)**

In this section, we examine some properties of the generalized Einstein tensor \( G(X, Y) \) in \((PRS)_n\), which is birecurrent. If the generalized Einstein tensor \( G(X, Y) \) satisfies the condition
\[
\nabla_i \nabla_j G_{ij} = \mu_{ik} G_{ij}, \tag{23}
\]
for some nonzero covariant tensor field \( \mu_{ik} \), then we call \( G_{ij} \) as birecurrent generalized Einstein tensor.

It is easy to see that a recurrent generalized Einstein tensor \( G(X, Y) \) is birecurrent. In fact, by taking the covariant derivative of (16) with respect to \( U^l \), we get
\[
\nabla_i \nabla_k G_{ij} = (\nabla_i A_k + A_k A_l) G_{ij}, \tag{24}
\]
with \( \mu_{ik} = \nabla_i A_k + A_k A_l \).

Now, we assume that \((PRS)_n\) admitting the generalized Einstein tensor \( G(X, Y) \) satisfies (24), but not (16). Changing the order of indices \( i \) and \( k \) in (23) and subtracting the expression so obtained from (23), we have
\[
\nabla_i \nabla_k G_{ij} = \mu_{ik} G_{ij}, \tag{25}
\]
where the bracket indicates antisymmetrization. If \( \mu_{ik} \) is a symmetric tensor, then \( \nabla_i \nabla_k G_{ij} = 0 \), and vice versa.

Thus, we have the following result:

**Lemma 5.** The birecurrency tensor of the generalized Einstein tensor \( G(X, Y) \) is symmetric if and only if the equation
\[
\nabla_i \nabla_k G_{ij} = 0 \tag{26}
\]
holds.

Now, by taking the covariant derivative of (13), we obtain
\[

\nabla_i \nabla_k G_{ij} = (4\pi_i \pi_j + 2\pi_i \pi_k) \left( S_{ij} - r\kappa g_{ij} \right) + 3\pi_k \pi_i S_{lj} + 3\pi_k \pi_j S_{li} + (\nabla_i \pi_j + 2\pi_i \pi_j) S_{ik} + (\nabla_j \pi_i + 2\pi_i \pi_i) S_{jk} + (\nabla_i \pi_j + 2\pi_i \pi_i) S_{ik} - 4\pi_i \pi_k \kappa g_{ij} - 2\nabla_i P_k \kappa g_{ij}, \tag{27}
\]
where \( P_k = \pi_k S_k^l \).

The covariant derivative of \( P_k \) is
\[
\nabla_i P_k = \nabla_i \left( \pi_k S_k^l \right) = (\nabla_i \pi_k) S_k^l + \pi_k \left( \nabla_i S_k^l \right). \tag{28}
\]

Writing (1) as
\[
\nabla_i S_k^l = 2\pi_i S_k^l + \pi_k S_k^l + \pi^l S_{kl}, \tag{29}
\]
using (28) and (29), we achieve
\[
\nabla_i P_k = (\nabla_i \pi_k) S_k^l + 2\pi_i P_k + \pi_k P_l + \pi S_{kl}. \tag{30}
\]
\[
(\nabla_i \pi_k) S_k^l = \nabla_i P_k - 2\pi_i P_k - \pi_k P_l - \pi S_{kl}. \tag{31}
\]

Now, we apply Lemma 5, and by using equation (26), we obtain
\[
2(\nabla_i \pi_k - \nabla_k \pi_i) \left( S_{ij} - r\kappa g_{ij} \right) + (\pi_i \pi_k - \nabla_k \pi_i) S_{lj} - (\pi_i \pi_l - \nabla_l \pi_i) S_{jk} + 3\pi_k \pi_j - \nabla_j \pi_k \pi_i) S_{lj} - 3\pi_i \pi_j - \nabla_j \pi_i \pi_k) S_{lk} + 4(\pi_i P_k - \pi_k P_l) \kappa g_{ij} - 2(\nabla_i P_k - \nabla_k P_l) \kappa g_{ij} = 0. \tag{32}
\]
Contracting (32) with respect to \( i \) and \( j \), we get
\[
r(\nabla_i \pi_k - \nabla_k \pi_i) (1 - \kappa n) + 2(1 + \kappa n) (\pi_i P_k - \pi_k P_l) + (\nabla_i \pi_k) S_k^l - (\nabla_k \pi_i) S_l^k - (\nabla_i P_k - \nabla_k P_l) \kappa n = 0. \tag{33}
\]
Substituting (31) into (33) yields

$$
(1 - kn) \left[ r(\nabla_\pi_k - \nabla_l \pi_l) + (\nabla_k P_k - \nabla_k P_l) \right] + (1 + 2kn) (\pi_k P_k - \pi_k P_l) = 0.
$$

(34)

If $\kappa = 1/n$, the generalized Einstein tensor $G(X, Y)$ reduces to the Einstein tensor $E(X, Y)$. So, we can state the following:

**Theorem 6.** In $(PRS)_n$, the birecurrency tensor of Einstein tensor $E(X, Y)$ is symmetric if and only if the vector fields $\pi_k$ and $P_k$ are linearly dependent.

Let us now recall that a $\varphi(\text{Ric})$ vector field was introduced by Hinterleitner and Kiosak as a vector field satisfying the condition $\nabla \varphi = \mu \text{Ric}$ [14], where $\mu$ is some constant, Ric is the Ricci tensor, and $\nabla$ is the Levi-Civita connection.

If $\kappa = -1/2n$, then it follows from (34) that

$$
r(\nabla_\pi_k - \nabla_k \pi_l) + (\nabla_k P_k - \nabla_k P_l) = 0.
$$

(35)

It is evident that $\pi_k$ and $P_k$ are closed or $\pi(\text{Ric})$ and $P(\text{Ric})$ vector fields. Therefore, we have

**Theorem 7.** In $(PRS)_n$, the birecurrency tensor of generalized Einstein tensor $G(X, Y)$ with $\kappa = -1/2n$ is symmetric if and only if the vector fields $\pi_k$ and $P_k$ are closed or $\pi(\text{Ric})$ and $P(\text{Ric})$.

**Theorem 8.** In $(PRS)_n$, the birecurrency tensor of generalized Einstein tensor $G(X, Y)$ with $\kappa \neq -1/2n$ is symmetric if and only if the vector fields $\pi_k$ and $P_k$ are linearly dependent, and the vector field $\pi_k$ is closed or $\pi(\text{Ric})$.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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