5-choosability of graphs with crossings far apart

Zdeněk Dvořák† Bernard Lidický‡ Bojan Mohar§¶

Abstract

We give a new proof of the fact that every planar graph is 5-choosable, and use it to show that every graph drawn in the plane so that the distance between every pair of crossings is at least 15 is 5-choosable. At the same time we may allow some vertices to have lists of size four only, as long as they are far apart and far from the crossings.

Thomassen [5] gave a strikingly beautiful proof that every planar graph is 5-choosable. To show this claim, he proved the following more general statement:

Theorem 1. Let $G$ be a plane graph with the outer face $F$, $xy$ an edge of $F$, and $L$ a list assignment such that $|L(v)| \geq 5$ for $v \in V(G) \setminus V(F)$, $|L(v)| \geq 3$ for $v \in V(F) \setminus \{x,y\}$, $|L(x)| = |L(y)| = 1$ and $L(x) \neq L(y)$. Then $G$ is $L$-colorable.

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†Computer Science Institute of Charles University, Prague, Czech Republic. E-mail: rakdver@iuuk.mff.cuni.cz. Supported by Institute for Theoretical Computer Science (ITI), project 1M0545 of Ministry of Education of Czech Republic, and by project GA201/09/0197 (Graph colorings and flows: structure and applications) of Czech Science Foundation.
‡Charles University, Prague, Czech Republic. E-mail: bernard@kam.mff.cuni.cz.
§Department of Mathematics, Simon Fraser University, Burnaby, B.C. V5A 1S6. E-mail: mohar@sfu.ca. Supported in part by an NSERC Discovery Grant (Canada), by the Canada Research Chair program, and by the Research Grant P1–0297 of ARRS (Slovenia).
¶On leave from: IMFM & FMF, Department of Mathematics, University of Ljubljana, Ljubljana, Slovenia.
Let us note that the lists of $x$ and $y$ of size 1 give a precoloring of a path of length 1 in the outer face of $G$. Unfortunately, this statement is rather tight, and almost any attempt to alter it (e.g., by allowing more than two vertices to be precolored, allowing lists of size 2 subject to some constraints, allowing some crossings in the drawing, etc.) fails with infinitely many counterexamples. We give a proof of a different version of Theorem 1 (see Theorem [3]), that turns out to be more robust with respect to some strengthenings of the planar 5-choosability theorem. Our proof is inspired by Thomassen's proof [6] of 3-choosability of planar graphs of girth 5. Using this technique, we give the proof of our main result:

**Theorem 2.** Let $G$ be a graph drawn in the plane with some crossings and let $N \subseteq V(G)$ be a set of vertices such that the distance between any pair of crossed edges is at least 15, the distance between any crossing and a vertex in $N$ is at least 13, and the distance between any two vertices in $N$ is at least 11. Then $G$ is $L$-colorable for any list assignment $L$ such that $|L(v)| = 4$ for $v \in N$ and $|L(v)| \geq 5$ for $v \in V(G) \setminus N$.

Let us recall that a list assignment $L$ for $G$ is a function that assigns to each vertex of $G$ a set $L(v)$, called the list of admissible colors for $v$. An $L$-coloring is a choice of a color $c(v) \in L(v)$ for each $v \in V(G)$ such that no two adjacent vertices receive the same color. The graph is $k$-choosable if it admits an $L$-coloring for every list assignment $L$ with $|L(v)| \geq k$ for every $v \in V(G)$.

Our main result, Theorem 2, clearly implies the following special case.

**Theorem 3.** Every graph drawn in the plane so that the distance between every pair of crossings is at least 15 is 5-choosable.

Some distance condition on the crossings is necessary, even if we would allow only three crossings, as shown by $K_6$. On the other hand, it was proved in [4] and independently also in [2] that the distance requirement is not needed, if we have at most two crossings. The inductive proof of Theorem 2 involves a stronger inductive hypothesis that is stated later as Theorem 7 and in particular also implies the above-mentioned result from [2, 4].

**Theorem 4 ([4, 2]).** Every graph whose crossing number is at most two is 5-choosable.

The proof of Theorem 4 is given at the end of the paper. Another special case of Theorem 2 is the following.
Theorem 5. Let $G$ be a planar graph and $N \subseteq V(G)$ a set of vertices such that the distance between any pair of vertices in $N$ is at least 11. Then $G$ is $L$-colorable for any list assignment $L$ such that $|L(v)| = 4$ for $v \in N$ and $|L(v)| \geq 5$ for $v \in V(G) \setminus N$.

The last result is motivated by the result of Voigt [7] showing that not all planar graphs are 4-choosable. Furthermore, it is related to the following problem of Albertson [1]:

Problem 1. Does there exist a constant $d$ such that whenever $G$ is a planar graph with list assignment $L$ that gives a list of size one or five to each vertex and the distance between any pair of vertices with list of size one is at least $d$, then $G$ is $L$-colorable?

Starting with a similar technique as used in this paper, we were able to give a positive answer to this problem (which we present in a separate paper [3]).

We start with giving the proof of the basic statement for planar graphs in Section 1. In Section 2 we then generalize it by allowing crossings and 4-lists subject to distance constraints, obtaining the proof of Theorem 2.

1 Planar graphs

Let $P$ be a path or a cycle. The length $\ell(P)$ of $P$ is the number of its edges, i.e., a path of length $l$ has $l + 1$ vertices and a cycle of length $l$ has $l$ vertices. Given a graph $G$ and a cycle $K \subseteq G$, an edge $uv$ of $G$ is a chord of $K$ if $u, v \in V(K)$, but $uv$ is not an edge of $K$. For an integer $k \geq 2$, a path $v_0v_1 \ldots v_k$ is a $k$-chord if $v_0, v_k \in K$ and $v_1, \ldots, v_{k-1} \notin V(K)$. We define a chord to be a 1-chord. If $G$ is a plane graph, let $\text{Int}_K(G)$ be the subgraph of $G$ consisting of the vertices and edges drawn inside the closed disc bounded by $K$, and $\text{Ext}_K(G)$ the subgraph of $G$ obtained by removing all vertices and edges drawn inside the open disc bounded by $K$. In particular, $K = \text{Int}_K(G) \cap \text{Ext}_K(G)$. Note that each $k$-chord of $K$ belongs to exactly one of $\text{Int}_K(G)$ and $\text{Ext}_K(G)$. If the cycle $K$ is the outer face of $G$ and $Q$ is a $k$-chord of $K$, let $C_1$ and $C_2$ be the two cycles in $K \cup Q$ that contain $Q$. Then the subgraphs $G_1 = \text{Int}_{C_1}(G)$ and $G_2 = \text{Int}_{C_2}(G)$ are the $Q$-components of $G$. As we have mentioned earlier, Thomassen’s Theorem does not extend to the case when we have a precolored path of length two. However, if we
strengthen the condition on the list sizes of the other vertices on the outer face, such an extension is possible.

**Theorem 6.** Let $G$ be a plane graph with the outer face $F$, $P$ a subpath of $F$ of length at most two and $L$ a list assignment such that the following conditions are satisfied:

(i) $|L(v)| \geq 5$ for $v \in V(G) \setminus V(F)$,
(ii) $|L(v)| \geq 3$ for $v \in V(F) \setminus V(P)$,
(iii) $|L(v)| = 1$ for $v \in V(P)$,
(iv) no two vertices with lists of size three are adjacent in $G$,
(v) $L$ gives a proper coloring to the subgraph induced by $V(P)$, and
(vi) if $P = uvw$ has length two and $x$ is a common neighbor of $u$, $v$ and $w$, then $L(x) \neq L(u) \cup L(v) \cup L(w)$.

Then $G$ is $L$-colorable.

**Proof.** Suppose for a contradiction that the claim is false, and let $G$ be a counterexample with $|V(G)| + |E(G)|$ the smallest possible, and subject to that, with the longest path $P$ and with the minimum size of the lists (while satisfying (i)–(vi)). It is clear that $G$ is connected and that every vertex $v \in V(G)$ satisfies $\deg(v) \geq |L(v)|$.

Furthermore, $G$ is 2-connected: otherwise, let $v$ be a cut-vertex and let $G_1$ and $G_2$ be subgraphs of $G$ such that $G_1 \cup G_2 = G$, $V(G_1) \cap V(G_2) = \{v\}$ and $|V(G_1)|, |V(G_2)| > 1$. If $v \in V(P)$, then by the minimality of $G$ there exist $L$-colorings of $G_1$ and $G_2$, and these colorings together give an $L$-coloring of $G$. Otherwise, we may assume by symmetry that $P \subseteq G_1$. Consider an $L$-coloring $\varphi$ of $G_1$. Let $L_2$ be the list assignment for $G_2$ such that $L_2(u) = L(u)$ for $u \neq v$ and $L_2(v) = \{\varphi(v)\}$. By the minimality of $G$, $G_2$ is $L_2$-colorable, and this coloring together with $\varphi$ gives an $L$-coloring of $G$.

Every triangle $T$ in $G$ bounds a face: otherwise, first color the subgraph $\text{Ext}_T(G)$ and then extend the coloring to $\text{Int}_T(G)$. A similar argument shows that $G$ contains no separating 4-cycles; otherwise, consider such a 4-cycle $K = k_1k_2k_3k_4$, and let $\varphi$ be an $L$-coloring of $\text{Ext}_K(G)$. Let $G' = \text{Int}_K(G)$. Since $K$ is separating, we have $V(G') \neq V(K)$, and since every triangle bounds a face, we conclude that $K$ has no chord in $G'$. Let $L'$ be the list assignment for $G' - k_1$ such that $L'(z) = \{\varphi(z)\}$ for $z \in \{k_2, k_3, k_4\}$, $L'(z) = L(z) \setminus \{\varphi(k_1)\}$ if $z \notin \{k_2, k_4\}$ is a neighbor of $k_1$ and $L'(z) = L(z)$ if $z$ is any
other vertex. By the minimality of \(G\), the graph \(G' - k_1\) is \(L'\)-colorable, and this coloring together with \(\varphi\) gives an \(L\)-coloring of \(G\).

Since \(G\) is 2-connected, its outer face is bounded by a cycle, which we denote by \(F\) as well. Next, we show that \(F\) has no chords. Otherwise, let \(uv\) be a chord of \(F\) and let \(G_1\) and \(G_2\) be the \(uv\)-components of \(G\). If \(P \subseteq G_1\), then we first color \(G_1\) and then extend the coloring to \(G_2\). The case that \(P \subseteq G_2\) is symmetric. It follows that \(P\) has length two and all the chords of \(F\) are incident with its middle vertex. Let \(P = z_1u z_2\), where \(z_i \in V(G_i)\) for \(i \in \{1, 2\}\). Let \(\varphi\) be an \(L\)-coloring of \(G_1\) and let \(L_2\) be the list assignment for \(G_2\) such that \(L_2(z) = L(z)\) for \(z \neq v\) and \(L_2(v) = \{\varphi(v)\}\). Since \(G\) is not \(L\)-colorable, \(G_2\) is not \(L_2\)-colorable. By the minimality of \(G\), either \(v\) is adjacent to \(z_2\), or \(u\), \(v\) and \(z_2\) have a common neighbor \(w\) with list of size three (which means, in particular, that \(w \in V(F)\)). Since every chord of \(G\) is incident with \(u\), the edge \(vz_2\) or \(vw\) belongs to \(F\). Since every triangle bounds a face, we conclude that \(v\) has degree two in \(G_2\). By symmetry, \(v\) has degree two in \(G_1\) as well, and thus \(v\) has degree three in \(G\). It follows that \(|L(v)| = 3\), and thus \(v\) cannot be adjacent to any other vertex with list of size three. In particular, we cannot have the case with the vertex \(w\). We conclude that \(v\) is adjacent to \(z_1\) and \(z_2\) and \(V(G) = \{u, v, z_1, z_2\}\). However, \(L(v) \neq L(u) \cup L(z_1) \cup L(z_2)\) by (vi), and thus \(G\) is \(L\)-colorable. This contradiction proves that \(F\) has no chords.

Similarly, we have the following property:

1. Let \(uvw\) be a 2-chord of \(F\) and let \(G_1\) and \(G_2\) be \(uvw\)-components of \(G\). If \(P \subseteq G_1\), then either \(u\) and \(w\) are adjacent and \(G_2\) is equal to the triangle \(uvw\), or there exists a vertex \(x\) such that \(V(G_2) = \{u, v, w, x\}\), \(|L(x)| = 3\) and \(x\) is adjacent to \(u\), \(v\) and \(w\).

If \(\ell(P) < 2\), then it is easy to see that we can precolor \(2 - \ell(P)\) more vertices of \(F\) without violating (vi). Thus, we may assume that \(\ell(P) = 2\). Let \(P = p_0p_1p_2\). Suppose that \(p_0\), \(p_1\) and \(p_2\) have a common neighbor \(v\). If \(v \in V(F)\), then \(V(G) = \{p_0, p_1, p_2, v\}\) and \(G\) is \(L\)-colorable. If \(v \notin V(F)\), then \(v\) has degree at most four in \(G\) by (1) and thus \(\deg(v) < |L(v)|\), which is a contradiction. Therefore, \(p_0\), \(p_1\) and \(p_2\) have no common neighbor.

Furthermore, \(\ell(F) \geq 6\): If \(\ell(F) = 3\), then we remove one vertex of \(F\) and remove its color from the lists of all its neighbors, and observe that the resulting graph is a smaller counterexample to Theorem 6. In the case when \(\ell(F) = 4\), then similarly color and remove the vertex of \(V(F) \setminus V(P)\). Finally, suppose that \(\ell(F) = 5\). Let \(\varphi\) be an arbitrary \(L\)-coloring of \(F = p_2p_1p_0v_1v_2\).
Remove $v_1$ and $v_2$ from $G$ and remove their colors according to $\varphi$ from the lists of their neighbors, obtaining a graph $G'$ with the list assignment $L'$. Since every triangle in $G$ bounds a face, at most one vertex in $G'$ has list of size three. Since $p_0$, $p_1$ and $p_2$ have no common neighbor and $p_0$ is not adjacent to $p_2$, $G'$ with the list assignment $L'$ is a smaller counterexample to Theorem 6, which is a contradiction.

Let $F = p_2p_1p_0v_1v_2v_3v_4\ldots$. If $\ell(F) = 6$, then we set $v_4 = p_2$. We may assume that $|L(v_1)| = 3$ or $|L(v_2)| = 3$, since otherwise we can remove a color from the list of $v_1$. Let us consider a set $X \subseteq V(F) \setminus V(P)$ and a partial $L$-coloring $\varphi$ of $X$ that are defined as follows:

(X1) If $|L(v_1)| = 3$ and $|L(v_3)| \neq 3$, then $X = \{v_1\}$ and $\varphi(v_1) \in L(v_1) \setminus (p_0)$ is chosen arbitrarily.

(X2) If $|L(v_1)| = 3$ and $|L(v_3)| = 3$, then $X = \{v_1, v_2\}$ and $\varphi$ is chosen so that $\varphi(v_2) \in L(v_2) \setminus L(v_3)$ and $\varphi(v_1) \in L(v_1) \setminus (L(p_0) \cup \{\varphi(v_2)\})$.

(X3) If $|L(v_2)| = 3$, and either $|L(v_4)| \neq 3$ or $|L(v_3)| \geq 5$, then $X = \{v_2\}$ and $\varphi(v_2) \in L(v_2)$ is chosen arbitrarily.

(X4) If $|L(v_2)| = 3$, $|L(v_4)| = 4$ and $|L(v_3)| = 3$, then:

(X4a) If $v_1$, $v_2$ and $v_3$ do not have a common neighbor or $|L(v_1)| \geq 5$, then $X = \{v_2, v_3\}$ and $\varphi$ is chosen so that $\varphi(v_3) \in L(v_3) \setminus L(v_4)$ and $\varphi(v_2) \in L(v_2) \setminus \{\varphi(v_3)\}$.

(X4b) If $v_1$, $v_2$ and $v_3$ have a common neighbor and $|L(v_1)| = 4$, then $X = \{v_1, v_2, v_3\}$ and $\varphi$ is chosen so that $\varphi(v_3) \in L(v_3) \setminus L(v_4)$, $\varphi(v_1) \in L(v_1) \setminus L(p_0)$ and either at least one of $\varphi(v_1)$ and $\varphi(v_3)$ does not belong to $L(v_2)$, or $\varphi(v_1) = \varphi(v_3)$. The vertex $v_2$ is left uncolored.

For later reference, Figure 1 shows the subcases used in the definition of $X$ and $\varphi$.

Let $G' = G - X$ and let $L'$ be the list assignment obtained from $L$ by removing the colors of the vertices of $X$ according to $\varphi$ from the lists of their neighbors (if a vertex of $X$ is not colored according to $\varphi$, we do not remove any colors for it). Observe that any $L'$-coloring of $G'$ can be extended to an $L$-coloring of $G$, thus $G'$ is not $L'$-colorable. By the minimality of $G$, this implies that $G'$ violates the assumptions of Theorem 6. Since $F$ has no
Figure 1: Subcases in the definition of $X$. Triangle vertices represent lists of size 3, square vertices list of size $\geq 4$. Encircled vertices are in $X$. 
chords, the choice of $X$ and $\varphi$ implies that every vertex of $V(G') \setminus V(P)$ has list of size at least three. Since $p_0$ is not adjacent to $p_2$ and $p_0$, $p_1$ and $p_2$ do not have a common neighbor in $G$, it follows that the conditions (v) and (vi) are satisfied by $G'$ with the list assignment $L'$. We conclude that (iv) is false, i.e., $G'$ contains adjacent vertices $u$ and $v$ such that $|L'(u)| = |L'(v)| = 3$.

Since $F$ has no chords, the choice of $X$ ensures that at most one of $u$ and $v$ belongs to $V(F)$; hence, we can assume that $v \not\in V(F)$ and $v$ has two neighbors in $X$. In particular, $X$ was chosen according to the cases (X2) or (X4). Since $G$ contains no separating cycles of length at most 4, we conclude that $u$ has at most one neighbor in $X$, and thus $u \in V(F)$. Let $x \in X$ be the neighbor of $v$ such that the distance between $u$ and $x$ in $F - P$ is as large as possible. By (1) applied to the 2-chord $xvu$, we conclude that the $xvu$-component of $G$ that does not contain $P$ consists of $xvu$ and a vertex $z$ adjacent to $x, v$ and $u$ with $|L(z)| = 3$. It follows that $|L(u)| > 3$, and since $|L'(u)| = 3$, we have $z \in X$ and $|L(u)| = 4$. The inspection of the choice of $X$ shows that (X4) holds, i.e., $u = v_1, z = v_2$ and $x = v_3$. However, note that the condition of (X4b) holds; hence $u \in X$, contrary to the assumption that $u \in V(G')$. This completes the proof of Theorem 6.

2 Near-planar graphs

In this section, we aim to show that graphs drawn in the plane with crossings far apart are 5-choosable. For the purposes of the induction, it will be useful to allow other kinds of irregularities (adjacent vertices with list of size three, as well as vertices with list of size four not incident with the outer face, which arise when some vertices incident with a crossing are colored and their color is removed from the lists of their neighbors), subject to distance constraints.

Let us first introduce some terminology. Let $G$ be a graph. For two subgraphs $H_1, H_2 \subseteq G$, the distance $d(H_1, H_2)$ between $H_1$ and $H_2$ is the minimum of the distances between the vertices of $H_1$ and $H_2$, i.e., the minimum $k$ such that there exists a path $v_0v_1 \ldots v_k$ in $G$ with $v_0 \in V(H_1)$ and $v_k \in V(H_2)$. A drawing $G$ of $G$ in the plane consists of a set $V = \{p_v \mid v \in V(G)\}$ of distinct points in the plane and a set of simple polygonal curves $E = \{c_e \mid e \in E(G)\}$ such that

- if $uv \in E(G)$, then $p_u$ and $p_v$ are the endpoints of $c_{uv}$,
- no internal point of any $c_e \in E$ belongs to $V$, and
at most two of the curves in $\mathcal{E}$ contain any point that does not belong to $\mathcal{V}$, and any two curves in $\mathcal{E}$ have at most one point in common.

A crossing of $\mathcal{G}$ is a point in the plane that belongs to two of the curves in $\mathcal{E}$, but not to $\mathcal{V}$. An edge $e$ is incident with the crossing $x$ if $x \in c_e$. An edge $e$ is crossed if it is incident with some crossing, and non-crossed otherwise. For a crossing $x$, we define $G_x$ to be the graph consisting of the two edges incident with $x$. Two vertices of $G$ are crossing-adjacent if they belong to $G_x$ for some crossing $x$ and are not adjacent in $G_x$. Removal of $\bigcup \mathcal{E}$ splits the plane into several connected subsets, which we call faces of $\mathcal{G}$. By a slight abuse of terminology, we sometimes identify a face with its boundary and hence speak about the vertices, edges and crossings of the face.

Let $\mathcal{G}$ be a drawing of a graph $G$, let $P$ be a path of length at most three contained in the boundary of the outer face $F$ of $\mathcal{G}$ (where in particular, no edge of $P$ is crossed), $N$ a subset of $V(G)$ and $M$ a subset of $E(G)$, and let $L$ be a list assignment for $G$. We say that $L$ is $(P, N, M)$-valid if the following conditions are satisfied:

(S) $|L(v)| \geq 5$ for $v \in V(G) \setminus (V(F) \cup N)$, $|L(v)| \geq 3$ for $v \in V(F) \setminus V(P)$ and $|L(v)| = 1$ for $v \in V(P)$,

(N) $|L(v)| \geq 4$ for $v \in N \setminus V(F)$,

(M) if $|L(u)| = |L(v)| = 3$ and $u$ and $v$ are adjacent, then $uv \in M$,

(P) $L$ gives a proper coloring to the subgraph induced by $V(P)$,

(T) if a vertex $v$ has three neighbors $w_1, w_2, w_3$ in $V(P)$, then $L(v) \neq L(w_1) \cup L(w_2) \cup L(w_3)$, and

(C) if $x$ is a crossing and $G_x$ contains a vertex with list of size three, then all other vertices of $G_x$ have lists of size 1 or $\geq 5$.

We define some subgraphs $H$ of $G$ to be special, and assign a rank $r(H)$ to each such subgraph (see Figure 2). Specifically, $H$ is special if it falls into one of the following cases:

- $H$ consists of the two edges incident with a crossing. In this case, its rank is 4.
- $P$ has length three and $H$ consists of the middle edge of $P$; the rank of $H$ is 3.
Figure 2: Special subgraphs and their ranks

- $H$ is equal to a vertex of $N$, and $r(H) = 2$.
- $H$ is equal to an edge of $M$, and $r(H) = 0$.

The drawing $G$ is $(P, N, M)$-distant if $d(H_1, H_2) \geq r(H_1) + r(H_2) + 7$ for every pair $H_1 \neq H_2$ of special subgraphs of $G$. We shall occasionally refer to the $(P, N, M)$-distant requirement as the distance condition. The purpose of the introduced rank function is the following. In our inductive arguments, we will occasionally construct a smaller graph $G'$ and introduce a new special subgraph $H'$ in a vicinity of a special subgraph $H$ that would no longer exist in $G'$. If $H'$ has smaller rank than $H$, the distance condition for special subgraphs in $G'$ would still hold, and the induction hypothesis can be applied.

A subgraph $O \subseteq G$ is an obstruction if $O$ is isomorphic to one of the graphs drawn in Figure 3 and sizes of the lists of its vertices match those prescribed by the figure, where the full-circle vertices have list of size one, triangle vertices have list of size three, square vertices have list of size four and pentagonal vertices have list of size five. Let us remark that if the distance condition holds, then $G$ can contain at most one of the obstructions. For further reference we exhibit in Figure 4 all possible list assignments for which the obstructions are not colorable. In particular, observe that the following holds:

1. Let $O_M$ be one of the obstructions and let $Q$ be the path in $H$ consisting of full-circle vertices. Suppose that $Q$ has length two and that $H$ is neither $O_M$ nor $O_C$. Let $q$ be the middle vertex of $Q$ and let $L$ be a list assignment such that each vertex $v$ drawn by a $k$-gon has $|L(v)| = k$, while the vertices of $Q$ have lists consisting of all possible colors. Then there exists a color $b$ such that every $L$-coloring $\psi$ of $Q$ with $\psi(q) \neq b$ extends to an $L$-coloring of $H$.

We prove the following claim, which obviously implies our main result, Theorem 2.
Figure 3: The obstructions used in Theorem 7.
Figure 4: The lists for which the obstructions cannot be colored. Colors represented by different letters may be equal to each other if they do not occur in the same list for a particular obstruction.
**Theorem 7.** Let $G$ be a graph drawn in the plane, $P$ a path of length at most three contained in the boundary of the outer face $F$ of $G$ and $L$ a list assignment for $G$. Suppose that there exist sets $N \subseteq V(G)$ and $M \subseteq E(G)$ such that $L$ is $(P,N,M)$-valid and the drawing of $G$ is $(P,N,M)$-distant. If

(O) every obstruction in $G$ is $L$-colorable,

then $G$ is $L$-colorable.

Before giving the proof of this statement, let us give a quick outline. Essentially, we follow the proof of Theorem 6. First, we show that the outer face of a hypothetical minimal counterexample $G$ has no chords and then we also restrict its 2-chords. This is somewhat more complicated due to the presence of crossings and the condition (O). Next, we find the set $X$ and its partial coloring $\varphi$ defined in the same way as in the proof of Theorem 6 and use it to construct the graph $G'$ with the list assignment $L'$. By the minimality of $G$, we conclude that $G'$ violates one of the assumptions of the theorem. A straightforward case analysis shows that (O) holds, and the conditions (S), (P) and (T) follow in the same way as in the proof of Theorem 6, but (M), (N) and (C) can be violated in ways which do not enable us to obtain a contradiction directly. However, we observe that in such a case, there is a special subgraph $S$ near to $X$. In this situation, we apply the symmetric argument on the other side of the path $P$, and obtain another set $X'$ and a special subgraph $S'$ close to it. By the distance condition, we have $S = S'$, and thus there exists a short path from $X$ to $X'$ passing through $S$. In this situation, we consider all the possible combinations of $X$, $X'$ and their positions relatively to $S$, and obtain a contradiction similarly to the way we deal with 2-chords.

Let us note that the assumption (C) is a product of a somewhat delicate tradeoff. We believe the claim still essentially holds even without this assumption, and avoiding it would greatly reduce the number of possible bad cases and simplify the last part of the proof. However, the list of obstructions in (O) would be significantly larger, making the first part of the proof longer and more complicated. Moreover, if we omit (C) completely, then there exists an obstruction with a precolored path of length one (see Figure 5(a)), which would be a major problem (we could not easily get rid of chords of $F$). One could consider excluding Figure 5(a) by forbidding vertices with lists of sizes three or four joined by a crossed edge. This would still simplify the last part of the proof a lot. However, in addition to having more than 10
new obstructions, we do not see a way how to reduce the 2-chord depicted in Figure 5(b), which would need to be dealt with somehow.

Proof of Theorem 7. We follow the outline of the proof of Theorem 6. We assume that $G$ is a counterexample to Theorem 7 with $|V(G)| + |E(G)|$ as small as possible, and subject to that, with the minimum sizes of its lists. Let $k = \ell(P) \leq 3$, and let $P = p_0p_1\ldots p_k$. By applying the same kind of reductions as used in the proof of Theorem 6 together with the minimality of $G$, one can show:

(3) The following properties are satisfied:

(a) Every vertex $v \in V(G)$ satisfies $\deg(v) \geq |L(v)|$.
(b) $G$ is 2-connected and $\ell(P) \geq 1$.
(c) Every non-crossed chord of $F$ is incident with exactly one internal vertex of $P$.
(d) If $K$ is a triangle in $G$ and no edge of $K$ is crossed, then $K$ is not separating. If $K$ is a separating 4-cycle without crossed edges, then $\text{Int}_K(G) - V(K)$ is either a vertex in $N$ or a complete graph on 4 vertices involving a crossing.
(e) Every vertex $v \in V(G)$ satisfies $|L(v)| \leq 5$.
(f) If $v \in V(G) \setminus V(P)$ is adjacent to a vertex $p \in V(P)$, then $L(p) \subseteq L(v)$.

Most properties in (3) are easy to argue about; they are left to the reader. Property (e) is achieved by removing colors from lists of size 6 or more. The only problem that may arise is that we obtain an obstruction; however, inspection of bad lists for the obstructions exhibited in Figure 4 shows that
we can always remove one of the colors so that (O) still holds. The only remaining nontrivial claim is the property of separating 4-cycles in part (d). To prove that, we first color the subgraph of \( G \) consisting of \( \text{Ext}_K(G) \) and all chords of \( K \) and then consider \( G' = \text{Int}_K(G) \). Let \( K = v_1v_2v_3v_4 \), and let \( c_i \) be the color of \( v_i \) as used in the coloring of \( \text{Ext}_K(G) \). Suppose that \( c_1 \neq c_3 \). In that case we consider the list assignment \( L' \) for \( G' \) given by setting \( L'(v) = L(v) \) if \( v \notin V(K) \), \( L'(v_i) = \{c_i\} \) for \( i = 1, 2, 3 \), and \( L'(v_4) = \{c_1, c_3, c_4\} \). Since any \( L' \)-coloring of \( G' \) yields an \( L \)-coloring of \( G \), we conclude that \( G' \) does not satisfy all assumptions of the theorem. It is easy to see that the only possible violation is that \( G' \) contains an obstruction. Note that this obstruction contains the whole path \( v_1v_2v_3 \) and that the only vertices whose lists have size 3 or 4 are \( v_4 \) and possibly a vertex in \( N \). If a vertex in \( N \) is present, there is no crossing by the distance condition. The only obstructions with these properties are \( O_{N_1} \) and \( O_{C_5} \), yielding the outcome of the claim (a similar argument shows that \( V(\text{Int}_K(G)) \) only consists of the vertices of the obstruction; see (4) below for details). The remaining case to consider is when \( c_1 = c_3 \). In this case we replace the color \( c_3 \) in the list of \( v_3 \) and in the lists of all its neighbors by a new color \( c'_3 \) that does not occur elsewhere, and then apply the same argument as in the previous case. It is to be observed that the color \( c'_3 \) will only be used for \( v_3 \), and the color \( c_3 \) will not be used on any of the neighbors of \( v_3 \). Thus a coloring with the revised lists gives rise to an \( L \)-coloring of \( G \) also in this case. This completes the proof of (3).

Let \( T = v_1v_2v_3 \) be a triangle in \( G \). Suppose that the edge \( v_1v_2 \) is crossed by an edge \( uw \), where \( w \) belongs to \( G_2 = \text{Int}_T(G) \) and \( w \neq v_3 \). Let \( \varphi \) be an \( L \)-coloring of \( \text{Ext}_T(G) \) and let \( L_2 \) be the list assignment such that \( L_2(v_i) = \{\varphi(v_i)\} \) for \( 1 \leq i \leq 3 \), \( L_2(w) = L(w) \setminus \{\varphi(u)\} \) and \( L_2(z) = L(z) \) otherwise. Note that \( |L_2(w)| \geq 4 \), that \( G_2 \) is not \( L_2 \)-colorable and that it is \((v_1v_2v_3, (N \setminus V(G_2)) \cup \{w\}, \emptyset)\)-distant. Observe that \( G_2 \) satisfies the validity conditions (S)–(C), and also satisfies (O). Hence it is a counterexample to Theorem 7 contradicting the minimality of \( G \). Similarly, if \( w = v_3 \), then we conclude that no vertex is drawn in the open disc bounded by \( T \). Together with (3), we obtain the following conclusion:

(4) If \( T \) is a triangle in \( G \) distinct from \( F \), then \( V(\text{Int}_T(G)) = V(T) \).

Suppose now that \( G \) contains one of the obstructions from Figure 3. Note that each of the obstructions contains a special subgraph. By the distance condition, none of them has further crossed edges and (4) implies that no such obstruction \( H \) appears in \( G \), as otherwise we would have \( G = H \) and \( G \)
would be $L$-colorable by the assumptions.

Furthermore, analogous arguments as used in the proof of (3)(d) show that the following conditions hold:

(5) If $K$ is a 4-cycle in $G$ distinct from $F$ and $V(\text{Int}_K(G)) \neq V(K)$, then either $\text{Int}_K(G) - V(K)$ is $K_4$, or there exists a vertex $z$ such that $V(\text{Int}_K(G)) = V(K) \cup \{z\}$, $z$ is adjacent to all vertices of $K$ and $z$ either belongs to $N$ or is incident with an edge crossing an edge of $K$.

(6) If $K$ is a 5-cycle in $G$ distinct from $F$, $V(\text{Int}_K(G)) \neq V(K)$, no edge of $K$ is crossed and there exists a special subgraph $S \subseteq \text{Ext}_K(G)$ such that $d(S, K) \leq 1$, then $V(\text{Int}_K(G)) = V(K) \cup \{z\}$ for a vertex $z$ adjacent to all vertices of $K$.

Some explanation concerning the proof of (6) is needed: Again, we first color $\text{Ext}_K(G)$ and then consider $\text{Int}_K(G)$ with the 5-cycle precolored. By the previous results, $K$ has no chords, since the outcomes of (5) would yield a special subgraph too close to $S$. Let $uv$ be an edge of $K$, and let $G' = \text{Int}_K(G) - \{u, v\}$. By removing the colors of $u$ and $v$ from the lists of their neighbors, we obtain another instance of a list coloring problem with a precolored path of length 2. Since any coloring of $G'$ gives rise to a coloring of $G$, we conclude that one of the assumptions of the theorem is violated. By the distance condition, the only one that may not hold is the assumption (T). Since the common neighbor $w$ of the three vertices on the path has list of size 3 (and it had list of size 5 in $G$), it is adjacent to $u$ and $v$ in $G$, thus proving the claim.

Our next goal is to show that $F$ does not have chords. Let $uv$ be a non-crossed chord of $F$. By (3)(c), $u$ is an internal vertex of $P$, say $u = p_1$, while $v \notin V(P)$. Let $G_1$ and $G_2$ be the $uv$-components for $G$ such that $p_0 \in V(G_1)$, and let $P_1 = p_0 p_1 v$ and $P_2 = v p_1 \ldots p_k$. For each color $c \in L(v) \setminus L(u)$, let $L_c$ be the list assignment such that $L_c(v) = \{c\}$ and $L_c(z) = L(z)$ if $z \neq v$. Since $G$ is not $L$-colorable, either $G_1$ or $G_2$ is not $L_c$-colorable. Furthermore, since both $G_1$ and $G_2$ are $L$-colorable (by the minimality of $G$), there exist distinct colors $c_1$ and $c_2$ such that $G_1$ is not $L_{c_1}$-colorable and $G_2$ is not $L_{c_2}$-colorable. Since $G$ is a minimal counterexample, the assumptions of our theorem fail for $G_1$ and $G_2$ with respect to these list assignments. In the sequel we discuss what can go wrong.

All special subgraphs in $G$ that do not contain $v$ remain special in $G_1$ or $G_2$ and no new special subgraphs arise. Thus, $G_1$ is $(P_1, N \cap V(G_1), M \cap E(G_1))$-
Similarly, by exchanging the roles of two from distant and role of special subgraphs in the boundary of $G$. Since no of $p$ could consider the chord $\ell(T)$, or $(O)$ fails. If conditions $(S)$, $(N)$, $(M)$, and $(C)$ hold for both graphs. Thus one of $(P)$, $(T)$, or $(O)$ fails. If $G_1$ contains an obstruction, then it contains a special subgraph whose distance to $p_1$ is at most two. In that case, we conclude that $\ell(P) \leq 2$ and that $G_2$ contains no obstruction, since the distance between special subgraphs in $G$ is more than four; also, no edge at distance at most two from $p_1$ in $G_2$ is crossed. Since $\ell(P) \leq 2$, we may in this case exchange the role of $G_1$ and $G_2$ and henceforth assume that $G_1$ contains no obstructions. Similarly, by exchanging the roles of $G_1$ and $G_2$ if necessary, we may assume that no edge in $G_1$ at distance at most 2 from $p_1$ is crossed. Furthermore, if $G_1$ violates $(T)$, then since no edge in $G_1$ incident with $p_1$ is crossed, we could consider the chord $p_1z$ instead of $p_1v$, where $z$ is the common neighbor of $p_0$, $p_1$ and $v$. Therefore, we can assume that $G_1$ satisfies $(T)$ and $(O)$. Since no $L$-coloring of $G_2$ extends to an $L$-coloring of $G_1$, we conclude that $G_1$ violates $(P)$, and thus $v$ is adjacent to $p_0$. Since $vp_0$ is neither crossed nor incident with an internal vertex of $P$, we conclude that $vp_0$ is part of the boundary of $F$, and hence $G_1$ is equal to the triangle $p_0p_1v$. Suppose now that $G_2$ contains an obstruction $H$; by (3) and (4), we have $G_2 = H$. However, the inspection of the obstructions shows that $G$ would either be $L$-colorable or an obstruction. Therefore, $G_2$ satisfies $(O)$. Furthermore, by the absence of $O_{P_1}$ and property $(T)$ of $G$, we conclude that there exists a color $c \in L(v) \setminus (L(p_0) \cup L(p_1))$ such that $G_2$ satisfies $(P)$. Since this coloring does not extend to an $L$-coloring of $G_2$, it follows that $G_2$ violates $(T)$, i.e., there exists a vertex $w$ adjacent to $v$ and to vertices $p, p' \in V(P) \setminus \{p_0\}$ such that $L(w) = L(p) \cup L(p') \cup \{c\}$. Since we cannot choose $c$ so that $G_2$ satisfies both $(P)$ and $(T)$, it follows that either $G$ contains $O_{P_2}$, or $vw \in M$ (in which case $\ell(P) = 2$), and $G$ contains $O_{M_1}$. This is a contradiction, thus every chord of $F$ is crossed.

Consider now a (crossed) chord $uv$ of $F$ that is not incident with an internal vertex of $P$. Let $e$ be the edge crossing $uv$ and let $G_1$ and $G_2$ be the $uv$-components of $G - e$ such that $P \subseteq G_1$. Let $e = x_1x_2$, where $x_1 \in V(G_1)$ and $x_2 \in V(G_2)$. By the minimality of $G$, there exists an $L$-coloring $\varphi$ of $G_1$. Since $\varphi(u) \neq \varphi(v)$, we can assume that $\varphi(x_1) \neq \varphi(u)$. Let $G'$ be the graph obtained from $G_2 - uv$ by adding new vertices $y_1$ and $y_2$, edges of the path $P' = uy_1y_2v$ and the edge $y_1x_2$. Let $L'$ be the list assignment for $G'$ such that $L'(u) = \{\varphi(u)\}$, $L'(v) = \{\varphi(v)\}$, $L'(y_1) = \{\varphi(x_1)\}$, $L'(y_2) = \{c\}$ for a new color $c$ that does not appear in any of the lists and $L'(z) = L(z)$ for any other vertex $z$. Note that $G'$ has a new special subgraph consisting

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of the edge $y_1y_2$ and that $G'$ is $(P', N \cap V(G'), M \cap E(G'))$-distant, since the crossing of $G$ incident with $x_2$ does not belong to $G'$ and any path from a special subgraph in $G'$ to $y_1y_2$ passes through one of the vertices $u, v, x_2$ of the crossing in $G$. Furthermore, $G'$ is not $L'$-colorable, and by the minimality of $G$, it violates (T) or (O). The latter is not possible, since $y_2$ has degree two, thus (T) does not hold in $G'$. This implies that $x_2$ has list of size three and it is adjacent to $u$ and $v$. By (3) and (4), we have $V(G_2) = \{u, v, x_2\}$.

Note that by (C), we conclude that each of $G$ for each subgraph of $G$ is \(\phi\) or (P,N). Let $a$ be a color in $L(x_2)$ distinct from the colors of its neighbors in $P$, which exists by (T). Let $G'' = G - x_2$ with the list assignment $L''$ such that $L''(z) = L(z) \setminus \{a\}$ for $z \in \{u, v, x_1\}$ and $L''(z) = L(z)$ otherwise. Note that $G''$ is $(P, N \cup \{x_1\}, M)$-distant and $L''$ is $(P, N \cup \{x_1\}, M)$-valid. By the minimality of $G$, we have that $G''$ violates (O). The obstruction involved is $L$-colorable, so it must contain one of the vertices whose lists have been changed. Since none of these vertices has list of size 3 or 4 in $L$ and all special subgraphs are far away from the crossing, we conclude that $\ell(P) = 2$, $|L''(x_1)| = 4$ and $G''$ contains one of $O_{N1}$, $O_{N2}$ or $O_{N3}$, in which the interior vertex with list of size 4 is $x_1$. However, inspection of these graphs shows that $|L''(u)| = 3$ or $|L''(v)| = 3$, which is a contradiction.

Finally, consider a crossed chord $uv$ of $F$ incident with an internal vertex $u$ of $P$. Since $G$ is $(P, N, M)$-distant, we have $\ell(P) = 2$, thus $u = p_1$. Let $e$ be the edge crossing $uv$ and let $G_1$ and $G_2$ be the $uv$-components of $G - e$ such that $p_0 \in V(G_1)$ and $p_2 \in V(G_2)$. Let $P_1 = p_0p_1v$ and $P_2 = p_2p_1v$, and let $e = x_1x_2$, where $x_i \in V(G_i)$ for $i \in \{1, 2\}$. Note that $G_i$ is $(P_i, (N \cap V(G_i)) \cup \{x_1\}, M \cap E(G_i))$-distant. If $G_i$ contains an edge $f$ different from $p_0p_1, p_0v, p_1v$, then by the minimality of $G$ there exists an $L$-coloring $\varphi_{3-i}$ of $G - f \supseteq G_{3-i} + x_1x_2$. If additionally $|L(x_i)| \in \{1, 5\}$, then define $L_i$ to be the list assignment for $G_i$ such that $L_i(v) = \{\varphi_{3-i}(v)\}$, $L_i(x_1) = L(x_1) \setminus \{\varphi_{3-i}(x_{3-i})\}$, and $L_i(z) = L(z)$ for any other vertex $z$. Observe that $G_i$ is not $L_i$-colorable, and we conclude that it violates (P), (T) or (O). (For (S) to hold, we add $x_i$ to $N$. Since $\varphi_{3-i}$ is a coloring of $G - f$, (P) is satisfied for $G_i$. Since $G$ is $(P, N, M)$-distant and contains no non-crossed chord, it follows that $G_i$ satisfies (T). Thus, $G_i$ violates (O). The corresponding obstruction is $O_{N1}$ since all others either have a special subgraph of $G$ that would violate the distance condition in $G$, or have a non-crossed chord incident with $p_1$. Together with (3) and (4), we have that for each $i \in \{1, 2\}$, one of the following holds:
\[ a < b < c \]

\[ G \text{ holds for } O \]

Let \( L \) of size three, \( G \) is either \( L \)-colorable or equal to \( O_{C1} \). Therefore, \(|L(x_2)| \in \{3,4\} \).

Since we already excluded all chords not incident with \( p_1 \), at most one of \( x_1 \) and \( x_2 \) has list of size three or four. By symmetry assume that \(|L(x_1)| \in \{1,5\}\). If \(|L(x_2)| \in \{1,5\} \), then all the possible combinations of such graphs \( G_1 \) and \( G_2 \) are either \( L \)-colorable or equal to \( O_{C1} \).

Since every chord of \( F \) is crossed and incident with an internal vertex of \( P \), we have \( x_1 \notin V(F) \), thus \( G_1 \) is \( O_{N1} \). Let \( w \) be the vertex of \( G_1 \) with list of size three, \( G' = G - \{w,p_0\} - p_1v \) and \( L' \) the list assignment such that \( L'(x_1) = \{\varphi_1(x_1)\} \), \( L'(v) = \{\varphi_1(v)\} \) and \( L'(z) = L(z) \) otherwise. Note that \( G' \) is \((p_2,p_1,x_1,v,N \cap V(G'),M \cap E(G'))\)-distant and not \( L' \)-colorable. If \( v \) has degree at least five in \( G \), then \( G \) has degree at least three in \( G' \). Together with \([3]\), this implies that \( x_2 \) is not adjacent to \( v \), hence \((T) \) holds. If \( v \) has degree at most four, then \(|L(v)| \leq 4 \), and by \((C)\), \(|L(x_2)| = 4 \), and again \((T) \) holds.

Therefore, \( G' \) violates \((O)\). Since \( x_1 \) has degree three in \( G' \) and it is adjacent to a vertex with list of size three or four, \( G' \) contains \((and \ by \ [3]\), is equal to) \( O_{P1} \) or \( O_{P2} \). However, then \( G \) is \( L \)-colorable. Therefore, we obtain the following conclusion:

\[(7) \quad F \text{ has no chords.}\]

An easy corollary is that

\[(8) \quad \text{no vertex of } P \text{ is incident with a crossed edge.}\]

Indeed, if \( vp \) were a crossed edge with \( p \in V(P) \), then by \([7]\), we have \( v \notin V(F) \). Furthermore, since \( P \) is incident with a crossing, we have \( \ell(P) \leq 2 \).

Let \( L' \) be the list assignment such that \( L'(v) = L(v) \setminus L(p) \) and \( L' \) matches \( L \) on the rest of the vertices of \( G \). Note that \( G - vp \) is not \( L' \)-colorable, and by the minimality of \( G \), we conclude that \( G - vp \) contains \( O_{N1} \), \( O_{N2} \) or \( O_{N3} \), whose internal vertex with list of size four is \( v \). It cannot contain \( O_{N1} \), since \( v \) is not adjacent to all vertices of \( P \) in \( G - vp \). Similarly, it cannot contain \( O_{N3} \), since the edge \( vp \) would be crossed twice. If \( G - vp \) contains \( O_{N2} \), then \( G \) contains \( O_{C1} \). Comparison of bad lists for \( O_{N2} \) and \( O_{C1} \) in Figure \([4]\) shows that \( O_{C1} \) is not \( L \)-colorable, which is a contradiction to the assumption that \((O) \) holds for \( G \) and \( L \).

Consider now a vertex \( v \) with three neighbors \( p_a, p_b, p_c \in V(P) \), where \( a < b < c \). Let \( K \) be the cycle \( p_ap_{a+1} \ldots p_cv \), and note that \( K \) has a chord
case, we have \( \ell \) is either \( O \) or \( P \). Suppose that \( v \notin V(F) \), and \( v \notin N \) if \( \ell(P) = 3 \), there is a color in \( L(v) \) that does not appear in the lists of vertices in \( P \). Observe that \( G_2 \) is \( \ell \)-colorable. However, then \( G_2 \) is either \( O_{N_1} \) or \( O_{P_6} \). In the latter case, we have \( \ell(P) = 2 \) by the distance condition, and \( \ell \) implies that \( G_2 \) is either \( O_{M_1} \) or \( O_{C_1} \). If \( G_2 \) is \( O_{M_1} \), then \( G \) is \( O_{M_2} \), and if \( G_2 \) is \( O_{C_1} \), then \( G \) is \( L \)-colorable.

Finally, consider the case that \( \ell(P) = 3 \) and \( v \) is adjacent to \( p_0, p_1 \) and \( p_2 \) is not adjacent to \( p_3 \). If \( L(p_0) = L(p_2) \), then \( G - vp_2 \) is a counterexample to Theorem 7 contradicting the minimality of \( G \). Therefore, \( L(p_0) \neq L(p_2) \). Since the edges \( vp_0, vp_1, \) and \( vp_2 \) are not crossed, the degree of \( p_1 \) is three. Let \( G' = G - p_1 + p_0p_2 \), with the list assignment \( L' \) such that \( L'(v) = L(v) \setminus L(p_1) \) and \( L'(z) = L(z) \) otherwise. Note that \( G' \) is \( (p_0p_2p_3, N \cup \{v\}, M) \)-distant, since the rank of the special subgraph \( p_1p_2 \) in \( G \) is greater than the rank of the special subgraph \( v \) (in \( G' \)), and any path \( Q \) between two special subgraphs \( S_1 \) and \( S_2 \) that uses the new edge \( p_0p_2 \) gives rise to paths between \( S_1 \) or \( S_2 \) and the middle edge \( p_1p_2 \) in \( P \) in \( G \), thus implying \( \ell(Q) \geq 14 + r(S_1) + r(S_2) + 2r(p_1p_2) - 1 > 7 + r(S_1) + r(S_2) \). We conclude that \( G' \) violates \( O \) and contains \( O_{N_1}, O_{N_2} \) or \( O_{N_3} \) that is not \( L' \)-colorable; however, then \( G \) contains a non-\( L \)-colorable obstruction \( O_{P_6}, O_{P_4} \) or \( O_{P_5} \), respectively. Therefore, we have:

\[(9) \] Every vertex has at most two neighbors in \( P \).

Suppose now that \( uv \) and \( xy \) are edges crossing each other and \( u, x \in V(F) \). By \( \ell \), neither \( u \) nor \( x \) belongs to \( P \). Let \( c \) be the curve formed by the part of the edge \( uv \) between \( u \) and the crossing and the part of the edge \( xy \) between the crossing and \( x \). If \( c \) is not part of the boundary of \( F \), then let \( G_2 \) be the subgraph of \( G \) drawn inside the closed disc bounded by \( c \) and the part of the boundary of \( F \) between \( u \) and \( x \) that does not contain \( P \). Note that there are two possible situations, depending on whether \( G_2 \) includes the vertices \( v \) and \( y \) or not. In any case, we can write \( G = G_1 \cup G_2 \), where \( G_1 \cap G_2 \) consists only of vertices \( u \) and \( x \). Let \( G'_2 \) be the graph obtained from \( G_2 \) by adding a common neighbor \( w \) of \( u \) and \( x \). No \( L \)-coloring of \( G_1 \) extends to an \( L \)-coloring of \( G'_2 \) (where \( w \) is assigned an arbitrary color different from the
colors of $u$ and $x$). By the minimality of $G$, we conclude that $G'_2$ (with the precolored path $uwx$) violates (P), thus $ux \in E(G)$. The conclusion is:

(10) If $u$ and $x$ are crossing-adjacent and $u, x \in V(F)$, then either $ux \in E(G)$, or the crossing incident with $u$ and $x$ belongs to the boundary of $F$.

Similarly, we derive the following property:

(11) Suppose that $Q = x_1x_2\ldots x_{t-1}x_t$ is a path in $G$, where $t \leq 6$ and $x_1x_2$ crosses $x_{t-1}x_t$. Let $c$ be the closed curve consisting of the path $x_2\ldots x_{t-1}$ and parts of the edges $x_1x_2$ and $x_{t-1}x_t$, and let $X$ be the set of vertices of $G$ drawn in the open disc bounded by $c$. If $x_1 \notin X$, then $X = \emptyset$.

The proof of (11) proceeds as follows. Observe first that the curve $c$ is not crossed since all its edges are close to a crossing. If the path $x_2\ldots x_{t-1}$ is induced in $G$, then the subgraph of $G$ drawn inside the closed disc bounded by $c$, with the precolored path $x_2x_3\ldots x_{t-1}$, would be a counterexample to Theorem 7 smaller than $G$ (the distance constraints are satisfied even if $t = 6$, since the middle edge of the path $x_2x_3x_4x_5$ has smaller rank then the crossing, whose distance to $x_3x_4$ in $G$ is one). If $x_2\ldots x_{t-1}$ contains a chord $x_ix_j$ ($i < j$), then we first consider $Q' = x_1x_2\ldots x_ix_j\ldots x_t$ and then apply (3)(d) to show that no vertices are contained in the interior of the cycle $x_1x_{i+1}\ldots x_jx_i$.

Now, we shall consider the 2-chords of $F$.

(12) Let $uvw$ be a 2-chord of $F$ such that $vw$ is not crossed. Let $c$ be the closed curve bounding the outer face of $G$ and $q$ the curve corresponding to the 2-chord $uvw$. Let $c_1$ and $c_2$ be the simple closed curves in $c \cup q$ distinct from $c$, and let $G_1$ and $G_2$ be the subgraphs of $G$ drawn inside $c_1$ and $c_2$, respectively, so that $G_1 \cap G_2 = uvw$ and $G_1 \cup G_2$ is equal to $G$ if the edge $uw$ is not crossed, and is equal to $G$ minus $xy$ if $uv$ is crossed by the edge $xy$. If neither $u$ nor $w$ is an internal vertex of $P$ and $P \subseteq G_1$, then one of the following holds (cf. Figure 6):

- $V(G_2) = \{u,v,w\}$, and either $uv$ is not crossed and $uw \in E(G)$, or $uv$ is crossed by an edge incident with $w$; in the latter case, $uw$ may or may not be an edge.
- $V(G_2) = \{u,v,w,z\}$ for a vertex $z$ with list of size three, and either $uw$ is not crossed and $uz,vz,wz \in E(G)$, or $uv$ is crossed by an edge incident with $z$, $zw \in E(G)$ and at least one of $uz$ and $vz$ is an edge.
Figure 6: Possible cases for $G_2$ for a 2-chord $uvw$

• $V(G_2) = \{u, v, w, z\}$ for a vertex $z$ with list of size four adjacent to $u$, $v$, $w$ and incident with an edge crossing $uv$.

Proof. Let us consider a 2-chord $uvw$ that does not satisfy the conclusion of the claim such that $G_2$ is maximal. First, suppose that $uv$ is not crossed. An $L$-coloring of $G_1$ does not extend to an $L$-coloring of $G_2$, hence $G_2$ (with the precolored path $uvw$) violates (P), (T) or (O). If $G_2$ violates (P) or (T), then by (4) and (7) the outcome of the claim holds. Therefore, we conclude that $G_2$ violates (O). Since the obstruction in $G_2$ violating (O) contains a special subgraph with a vertex distinct from $v$ and $v \not\in V(F)$, we conclude that $v \not\in N$, and hence $|L(v)| = 5$. By (3) and (4) we also conclude that $G_2$ is the obstruction. Let $S$ be the set of $L$-colorings of $uvw$ that do not extend to an $L$-coloring of $G_2$. The inspection of the non-colorable obstructions with $\ell(P) = 2$ in Figure 4 shows that one of the following holds:

(R1) there exists a set $A$ of at most two colors and $S$ contains only colorings $\psi$ such that $\psi(v) \in A$, and furthermore, if $|A| = 2$ then neither $u$ nor $w$ has list of size three; or,
Indeed, by (2), all obstructions except for $O_{M1}$ and $O_{C1}$ satisfy (R1) with $|A| = 1$. If $G_2$ is $O_{M1}$ or $O_{C1}$, then neither $u$ nor $w$ has list of size three, by (M) together with the distance condition and by (C). The inspection of the colorings shows that if $G_2$ is $O_{C1}$, then (R1) holds with $|A| = 2$, and if $G_2$ is $O_{M1}$, then either (R1) holds with $|A| = 2$, or (R2) holds (the latter is the case when the two lists of size 3 are equal, i.e., $a = c$ in Figure 4).

If (R1) holds, then let $G' = G_1$, with the list assignment $L'$ such that $L'(v) = L(v) \setminus A$ and $L'(z) = L(z)$ for $z \neq w$. Note that if $|A| = 2$, then $v$ has no neighbor in $G_1$ with list of size three by (R1) and by the maximality of $G_2$. If (R2) holds, let $G' = G_1 + uw$ with the list assignment $L' = L$. In either case, $G'$ is not $L'$-colorable and it is $(P, N \cap V(G'), M \cap E(G'))$-distant (in the latter case, any path $Q$ between special subgraphs $H_1$ and $H_2$ using the added edge $uw$ gives rise to paths from $H_1$ and $H_2$ to the special subgraph of $G_2$, and thus $\ell(Q) \geq 14 + r(H_1) + r(H_2) - 3$). Furthermore, $G'$ satisfies (T) by (7) and (9), and if $G'$ violated (C) or (O), then $v$ or $uw$ would have to belong to a crossing or to an obstruction in $G'$, and the distance between its special subgraph and the special subgraph of $G_2$ would be at most 4. Note that $G'$ cannot violate (P), as otherwise $u, w \in V(P)$ and $G_2$ is $O_{M1}$, and by (3) and (9), $v$ would have degree four and list of size five. Therefore, $G'$ is a counterexample to Theorem 7 smaller than $G$, which is a contradiction.

Suppose now that $uw$ is crossed by an edge $xy$, where $x \in V(G_1)$ and $y \in V(G_2)$. If $y = w$, then the conclusion of the claim holds by (10), hence assume that $y \neq w$. Furthermore, $x \neq w$ by (10), and $uw \notin E(G)$ by (1). Let $G'_1$ be the graph obtained from $G_1$ by adding the edges $ux$ and $vx$ (if they are not present already). Note that this can be done without introducing any new crossings. Since $u, v$ and $x$ are incident with a crossing in $G$, $G'_1$ is $(P, N \cap V(G'_1), M \cap E(G'_1))$-distant. Furthermore, $G'_1$ does not contain any obstruction, as its special subgraph would be at distance at most 2 from the crossing. By (7), $u$ has at most one neighbor in $P$ within $G$, hence $G'_1$ satisfies (T). By (8), $u$ and $x$ cannot belong to $P$, hence by the minimality of $G$, there exists an $L$-coloring $\varphi$ of $G'_1$. Let $G'_2$ be the graph obtained from $G_2 - uw$ by adding the vertex $x$ and edges $ux, vx, yx$. Consider the list assignment $L'_2$ for $G'_2$ such that $L'_2(z) = \{\varphi(z)\}$ for $z \in \{u, v, w, x\}$ and $L'_2(z) = L(z)$ otherwise. Note that $G'_2$ is not $L'_2$-colorable and that it is $(uxvw, N \cap V(G'_2), M \cap E(G'_2))$-distant.
Since \( y \neq w \) and since \( uw \notin E(G'_2) \), the graph \( G'_2 \) satisfies (P) for the \( \varphi \)-colored path \( uxvw \). If \( G'_2 \) violates (T), then by (4) we have that \( |L(y)| = 3 \) and \( y \) is adjacent to at least two of \( u, v \) and \( w \). In particular, \( y \in V(F) \).

Observe that if \( vy \in E(G) \), then the \( yvw \)-component \( K \) that does not contain \( P \) can only be a triangle (since otherwise any \( L \)-coloring of the other \( yvw \)-component \( K' \) would extend to \( K \), and \( K' \) would contradict the minimality of \( G \)). Thus \( yw \in E(G) \). By (11) applied to the path \( xwyvu \), we have \( V(G^2) = \{ u, v, w, y \} \) and the conclusion of the claim holds. Let us now consider the remaining case that \( G'_2 \) satisfies (T), and thus by the minimality of \( G \), it violates (O). By (3) and (4), \( G'_2 \) is equal to one of \( O_{P1}, \ldots, O_{P6} \), but not \( O_{P3} \) since \( x \) has degree 3 in \( G'_2 \).

If \( G'_2 \) is \( O_{P1} \), then the conclusion of the claim holds. Otherwise, let us define \( S \) as the set of colorings \( \psi \) of the path \( uxvw \) that do not extend to an \( L \)-coloring of \( G'_2 \) and satisfy \( \psi(u) \neq \psi(v) \). The inspection of the obstructions and their problematic list assignments displayed in Figure 4 shows that either (R1) or one of the following holds:

(R3) \( G'_2 \) is \( O_{P2} \) and there exists a color \( c \) such that \( S \) contains only colorings \( \psi \) such that either \( \psi(u) = c \) and \( \psi(x) = \psi(w) \), or \( \psi(x) = c \) and \( \psi(u) = \psi(w) \). Moreover, \( |L(u)| \neq 3 \) and \( |L(w)| \neq 3 \).

(R4) \( G'_2 \) is \( O_{P4} \) and there exists a color \( c \) such that \( S \) contains only colorings \( \psi \) satisfying either \( \psi(v) = c \) or \( \psi(x) = c \). Moreover, \( |L(u)| \neq 3 \).

Again, the conclusions that the specified vertices do not have lists of size three follow in all applicable cases by noting that otherwise either (C) or the distance condition would be violated. Let us remark that for \( O_{P2} \) we have (R1) if the colors \( a, b, c, d \) in Figure 4 are different; we have (R3) if \( b = d \) or \( a = d \). To argue for \( O_{P4}, O_{P5}, O_{P6} \) we observe that \( \psi(x) \) and \( \psi(v) \) should be taken from the difference of the lists of the two neighbors of \( u \) (so these are colors \( b, c \) in Figure 4). This yields (R1) with the only exception in the case of \( O_{P4} \), where we cannot argue about \( |L(w)| \neq 3 \), so we need (R4) in this (and only this) case.

The condition in (R3) that the lists of \( u \) and \( w \) do not have precisely three elements is argued as follows. Since \( x \) has degree 3 in \( G'_2 \), the vertex \( z \) of \( O_{P2} \) with list of size 3 is not the vertex \( y \), and \( v, w \) are both adjacent to \( z \). Since \( |L(z)| = 3 \) and the edge \( wz \) is close to a crossing in \( G \), we conclude that \( wz \notin M \) and hence \( |L(w)| \neq 3 \). Since \( |L(y)| = 4 \), (C) implies that \( |L(u)| \neq 3 \).
Now, the case when (R1) holds is handled in the same way as the case when \( uv \) was not crossed. If (R3) holds, then we let \( G'_1 = G_1 + uv \) with the list assignment obtained from \( L \) by removing \( c \) from the list of \( u \) (note that \( |L(u)| \neq 1 \) by \([8]\)); we may need to add an edge incident with \( u \) to \( M \) in order to satisfy (M). If (R4) holds and \( |L(x)| = 5 \), then let \( G'_1 = G_1 \) with the list assignment obtained by removing \( c \) from the lists of \( x \) and \( v \) (and adding \( x \) to \( N \)). In all the cases, \( G'_1 \) satisfies the assumptions of the theorem. Indeed, (P) is trivial, since \( u \not\in V(P) \) by \([8]\). Similarly, (T) follows by \([7]\) and \([9]\). Finally, (O) holds since by the distance condition, we could only create \( O_{M1}, O_{M2}, O_{N1}, O_{N2} \) or \( O_{N3} \), and each of them is excluded by \([7]\) or \([9]\). Therefore, \( G'_1 \) contradicts the minimality of \( G \), since its coloring would extend to an \( L \)-coloring of \( G \).

Finally, consider the case that (R4) holds and \( |L(x)| \in \{3, 4\} \). By \([10]\), all neighbors of \( u \) distinct from \( x \) belong to \( G_2 \). By \([8]\), we have \( u \not\in V(P) \), \( \deg(u) \geq |L(u)| \geq 4 \), and thus \( u \) is adjacent to \( x \) and \( |L(u)| = 4 \). Since \( G'_2 \) is \( O_{P4} \), every coloring of \( x, v \) and \( w \) extends to an \( L \)-coloring of \( G_2 \), hence \( G_1 \) contradicts the minimality of \( G \). This completes the proof of \([12]\).

Similarly, one can prove the following:

(13) Let \( u, w \in V(F) \) be distinct vertices, neither of which is an internal vertex of \( P \). Suppose that \( v \not\in V(F) \) is a vertex adjacent to \( w \) and crossing-adjacent to \( u \). Let \( c \) be the closed curve not containing \( P \) consisting of \( vw \), parts of the crossed edges incident with \( u \) and \( v \), and a part of the boundary of \( F \) between \( u \) and \( w \) that does not contain \( P \), and let \( G_2 \) be the subgraph of \( G \) drawn in the closed disc bounded by \( c \). Then \( G_2 \) does not contain the crossing and satisfies one of the following:

(a) \( V(G_2) = \{u, v, w\} \text{ and } uw \in E(G) \), or

(b) \( V(G_2) = \{u, v, w, z\} \), \( |L(z)| = 3 \) and \( z \) is adjacent to \( u, v \) and \( w \).

Proof. By \([12]\), it suffices to consider the case that \( uv \not\in V(G) \). Let \( G'_1 \) be the graph obtained from \( G_1 \) as follows: If \( uw \in E(G) \), then we add the edge \( uw \). If \( u, v \) and \( w \) have a common neighbor \( z \) with list of size three, then we add \( z \) and incident edges. If \( V(G'_1) = V(G) \), then (a) or (b) holds. Otherwise, there exists an \( L \)-coloring \( \psi \) of \( G'_1 \) by the minimality of \( G \). Let \( L' \) be the list assignment such that \( L'(v) = \{\psi(v)\}, L'(w) = \{\psi(w)\}, L'(u) = \{c\} \) for a new color \( c \), \( L'(x) = (L(x) \setminus \{\psi(u)\}) \cup \{c\} \) for each neighbor \( x \) of \( u \) distinct from
v and w and \( L'(x) = L(x) \) for all other vertices \( x \). Note that \( G'_2 = G_2 + uv \) is not \( L' \)-colorable, and by the minimality of \( G \), one of the assumptions of the theorem is violated in \( G'_2 \). By the construction of \( G'_1 \) and the choice of \( \psi \), (P) and (T) hold. By the distance condition, the only obstruction that can appear in \( G'_2 \) is \( O_{C1} \). However, then the 2-chord \( wvt \) (where \( t \) is the neighbor of \( u \) in \( G_2 \) with list of size four) contradicts (12).

Let us now introduce a way of defining list assignments that will be used throughout the rest of the paper. Let \( \phi \) be any proper partial \( L \)-coloring of \( G \) such that \( \phi(v) \notin L(p) \) for every pair of adjacent vertices \( v \in \text{dom}(\phi) \) and \( p \in V(P) \). For each vertex \( z \in V(G) \), let

\[
R_z = \bigcup_{p \in V(P) \setminus \text{dom}(\phi), zp \in E(G)} L(p).
\]

We define \( L_\phi \) to be the list assignment such that

\[
L_\phi(z) = \left(L(z) \setminus \{\phi(x) : x \in \text{dom}(\phi), xz \in E(G)\}\right) \cup R_z.
\]

Let us also define \( G_\phi = G - \text{dom}(\phi) \). Consider any \( L_\phi \)-coloring \( \psi \) of \( G_\phi \). We claim that the combination of \( \phi \) with \( \psi \) is a proper \( L \)-coloring of \( G \). Indeed, for any \( z \in V(G_\phi) \), we clearly have \( \psi(z) \notin R_z \), and thus \( \psi(z) \in L_\phi(z) \) is different from the colors of the neighbors of \( z \) in \( \text{dom}(\phi) \). Since \( G \) is not \( L \)-colorable, we conclude that \( G_\phi \) is not \( L_\phi \)-colorable.

Suppose now that \( G \) contains a subgraph \( H \) isomorphic to one of the graphs drawn in Figure 3 such that the subgraph of \( H \) corresponding to full-circle vertices is equal to \( P \), triangle vertices have lists of size at least three, square vertices have lists of size at least four and pentagonal vertices have lists of size five. Then we say that \( H \) is a near-obstruction.

**Proof.** By (9), \( H \) is isomorphic to one of \( O_{M1}, O_{N2}, O_{N3} \) or \( O_{P3} \). Furthermore, \( |(V(H) \cap V(F)) \setminus V(P)| \leq 1 \), and if \( (V(H) \cap V(F)) \setminus V(P) \neq \emptyset \), then \( H \) is \( O_{N2} \) or \( O_{N3} \).

(14) **If** \( H \) **is a near-obstruction, then** \( H \) **is isomorphic to one of** \( O_{M1}, O_{N2}, O_{N3}, O_{P3} \). **Furthermore,** \( |(V(H) \cap V(F)) \setminus V(P)| \leq 1 \), **and if** \( (V(H) \cap V(F)) \setminus V(P) \neq \emptyset \), **then** \( H \) **is** \( O_{N2} \) **or** \( O_{N3} \).**

**Proof.** By (9), \( H \) is isomorphic to one of \( O_{M1}, O_{N2}, O_{N3}, O_{C2}, O_{C3}, O_{C4}, O_{C5}, O_{P3} \).

By (12), if \( H \) is \( O_{C5} \), then \( V(G) = V(H) \cup \{z\} \), where \( z \) is a vertex of degree three adjacent to \( p_0, p_2 \) and the vertex \( w \notin V(P) \) in the outer face of
$H$. However, the distance condition implies that $w \notin N$, so that $|L(w)| = 5$. This implies that $G$ is $L$-colorable, which is a contradiction.

If $H$ is $O_{C_2}$, then let $p_0w_1w_2p_2$ be the path in the outer face of $H$. If $w_1,w_2 \in V(F)$, then $V(G) = V(H)$ and $G$ is $L$-colorable by (O). Thus assume that $w_2 \notin V(F)$; hence $|L(w_2)| = 5$. If $w_1 \in V(F)$, then since $w_2$ has degree at least 5, by \[12\] we have that $V(G) = V(H) \cup \{z\}$, where $z$ is adjacent to $w_1$, $w_2$ and $p_2$. However, then $G$ is $L$-colorable. Therefore, $w_1 \notin V(F)$. Let $\varphi$ be an $L$-coloring of $H$ and $G_2$ the $p_0w_1w_2p_2$-component of $G$ that does not contain $P$. Since $\varphi$ does not extend to an $L$-coloring of $G_2$, it follows that $G_2$ with the precolored path $p_0w_1w_2p_2$ violates (P), (T) or (O). Since both $w_1$ and $w_2$ have degree at least 5, it follows that $p_0w_2 \notin E(G)$ and $w_1p_2 \notin E(G)$, hence (P) holds. Suppose that $G_2$ violates (T). Then a vertex $z$ with list of size three is adjacent to three vertices among $p_0$, $w_1$, $w_2$ and $p_2$. If it is adjacent to all four of them, then $G$ contains $O_{C_5}$ which has already been excluded. Otherwise, since $w_1$ and $w_2$ have degree at least 5, by \[5\] $z$ cannot be adjacent to $p_0$, $w_1$, and $p_2$. By symmetry, we may assume that $z$ is adjacent to $p_0,w_1,w_2$. Then \[12\] applied to the 2-chord $zw_2p_2$ shows that there is a vertex $z'$ adjacent to $z$ whose list has size 3, and we see that the special edge $zz'$ gives a contradiction. Finally, if $G_2$ violates (O), then the obstruction is equal to one of $O_{P_1}$, $O_{P_2}$, $O_{P_3}$, $O_{P_4}$, $O_{P_5}$ or $O_{P_6}$, and now it is easy to see (by comparing bad lists for the obstructions) that $G$ is $L$-colorable.

If $H$ is $O_{C_3}$, then let $w_1$ be the vertex of $H$ drawn by the triangle and $w_2$ the vertex of $P$ that is not adjacent to it in $G$. If $H$ is $O_{C_4}$, then let $w_1$ and $w_2$ be the vertices of $H$ drawn by triangles. By symmetry, we can assume that $w_1$ is the neighbor of $p_2$. Let $w_1x_1x_2w_2$ be the path in $H$ formed by neighbors of $p_1$. Note that $|L(w_1)| \in \{1,5\}$ by \[7\]. Choose an $L$-coloring $\varphi$ of the subgraph of $G$ induced by $V(P) \cup \{w_1,w_2\}$ such that $\varphi(w_1) \neq \varphi(w_2)$ and either $|L_\varphi(x_1)| \geq 4$ or $L_\varphi(x_1) \neq L_\varphi(x_2)$. Note that this is possible since $|L(w_1)| = 5$. Let $G' = G - \{p_1,x_1,x_2\} + \{w_1w_2\}$ with the list assignment $L'$ such that $L'_\varphi(z) = \{\varphi(z)\}$ for $z \in \{w_1,w_2\}$ and $L'_\varphi(z) = L_\varphi(z)$ otherwise. Observe that $G'$ is not $L'$-colorable (since every $L'$-coloring of $G'$ extends to an $L$-coloring of $G$) and that it satisfies (P) for the precolored path $w_2w_1p_2$ or $p_0w_2w_1p_2$. By the minimality of $G$, $G'$ violates (T) or (O). In the former case, by symmetry we can assume that there exists a vertex $z \in V(G)$ such that $|L(z)| = 3$ and $z$ is adjacent to $p_2$, $w_1$ and either $w_2$ or $p_0$. It follows that $G$ contains a separating 4-cycle formed by non-crossed edges, and by \[3\] the interior of this 4-cycle contains $K_4$. By \[7\] and \[12\], there are no other
vertices in $G$. Now, it is easy to see that the resulting graph is $L$-colorable. Therefore, $G'$ violates (O). Since $G$ is $(P,N,M)$-distant, this is only possible if $H$ is $O_{C_4}$. In this case the obstruction in $G'$ is one of $O_{P_1}$. Note that the edge $w_1w_2$ is contained in a triangle; let $z$ be their common neighbor. By [3], the 4-cycle $w_1zw_2p_1$ surrounds $K_4$ in $G$. However, the graphs obtained from the obstructions $O_{P_1}$ with the precolored path $p_0w_2w_1p_2$ by adding the vertex $p_1$ joined to the vertices $p_0, w_1, w_2, p_2$, deleting the edge $w_1w_2$, and adding $K_4$ inside the 4-cycle $w_1zw_2p_1$ are easily seen to be $L$-colorable.

The remaining obstructions are $O_{M_1}, O_{N_2}, O_{N_3}$, and $O_{P_3}$. If $H$ is $O_{M_1}$ or $O_{P_3}$, then none of the vertices in $V(H) \setminus V(P)$ belongs to $F$ since this would contradict [7]. In the other cases, at most one of the vertices of $H$ can belong to $F$ by the same reason.

Observe that $\ell(P) \geq 2$, since if $\ell(P) < 2$, then we can precolor $2 - \ell(P)$ more vertices of $F$ (by [8], we can extend $P$ in the boundary of $F$). Let $p_k \ldots p_1p_0v_1v_2 \ldots v_s$ be the vertices contained in the boundary of $F$ in the cyclic order around it. We either have $v_iv_{i+1} \in E(G)$, or $v_i$ and $v_{i+1}$ are crossing-adjacent, for each $i$. As we already observed, $p_0v_1, p_kv_s \in E(G)$. We also define $v_{s+1} = p_k, v_{s+2} = p_{k-1}, \ldots$.

If $s = 0$, then let $\varphi$ be the $L$-coloring of $p_0$. Then $G_\varphi$ with the list assignment $L_\varphi$ is a smaller counterexample to Theorem [6]. This contradiction shows that $s \geq 1$.

Suppose that $s = 1$ and let $\varphi$ be the partial coloring that assigns a color in $L(v_1) \setminus (L(p_0) \cup L(p_k))$ to $v_1$. Note that if $v_1$ is adjacent to a vertex $x$ by a crossed edge, then $G_\varphi$ is $(P,N \cup \{x\}, \emptyset)$-distant, otherwise $G_\varphi$ is $(P,N,\emptyset)$-distant. By the minimality of $G$, $G_\varphi$ with the list assignment $L_\varphi$ violates (O) and contains an obstruction $H$, which by [14] is one of $O_{M_1}, O_{N_2}, O_{N_3}$ or $O_{P_3}$. Note that if $z \in V(H)$ is a vertex with list of size three according to $L_\varphi$, then $z$ is adjacent to $v_1$ and belongs to $N$; but $z$ is at distance at most one from the special subgraph in the obstruction, contradictory to the assumption that $G$ is $(P,N,M)$-distant. Therefore, $s \geq 2$.

If $v_1$ is not adjacent to $v_2$ (i.e., $v_1$ and $v_2$ are crossing-adjacent), then let $\varphi$ be a coloring that assigns a color from $L(v_1) \setminus L(p_0)$ to $v_1$ and the color from $L(p_0)$ to $p_0$. Note that $\ell(P) = 2$ by the distance condition. Let $y$ be the vertex adjacent to $v_1$ by the crossed edge, and note that $G_\varphi$ is $(p_1p_2, N \cup \{y\}, M)$-distant. By the minimality of $G$ together with [3], [7] and [9], we conclude that $G_\varphi$ with the list assignment $L_\varphi$ violates (N) or (M). In the former case, we have $|L_\varphi(y)| = 3$ and since $|L(y)| = 5$, it follows that
y is adjacent to \( p_0 \). However, by (4), \( v_2 \) would be adjacent to \( p_0 \), contrary to (7). In the latter case, \( p_0 \) and \( v_1 \) have a common neighbor \( u \neq y \) adjacent to a vertex \( w \) with \( |L(w)| = 3 \). This contradicts (12). Therefore, \( v_1v_2 \in E(G) \), and by symmetry, \( v_{s-1}v_s \in E(G) \).

Suppose now that \( s = 2 \). By symmetry, assume that if \( v_2 \) is incident with a crossed edge, then \( v_1 \) is incident with a crossed edge as well. If \( v_1v_2 \in M \), then let \( \varphi \) be an \( L \)-coloring of \( v_1 \) and \( v_2 \) such that \( \varphi(v_1) \notin L(p_0) \) and \( \varphi(v_2) \notin L(p_k) \). Otherwise, let \( \varphi \) be a coloring of \( v_1 \) by a color in \( L(v_1) \setminus L(p_0) \) such that if \( |L(v_2)| = 3 \), then \( \varphi(v_1) \notin L(v_2) \setminus L(p_k) \). Note that this is possible by (3)(f). Let us remark that when \( |L(v_2) \setminus \{\varphi(v_1)\}| = 2 \), then \( L(p_k) = \{\varphi(v_1)\} \) and \( L\varphi(v_2) = L(v_2) \) by the definition of \( L\varphi \), and thus we always have \( |L\varphi(v_2)| \geq 3 \). If \( G\varphi \) with the list assignment \( L\varphi \) violated (C), then \( v_2 \) would have to be incident with a crossing, and by the choice of \( v_1 \), the vertex \( v_1 \) would be incident with the same crossing, which then would not appear in \( G\varphi \). Therefore, \( G\varphi \) satisfies (C). If \( v_1 \) is incident with a crossed edge \( v_1x \), then let \( N' = N \cup \{x\} \); if \( v_1 \) is adjacent to a vertex \( y \in N \), then let \( N' = N \setminus \{y\} \); otherwise let \( N' = N \). If \( v_1 \) and \( v_2 \) have a common neighbor \( z \) belonging to \( N \), then let \( M' = M \cup \{v_2z\} \); otherwise let \( M' = M \setminus \{v_1v_2\} \). Observe that \( G\varphi \) is \((P, N', M')\)-distant and that it satisfies (S), (N) and (M). By the minimality of \( G \), \( G\varphi \) violates (O) and thus \( G \) contains a near-obstruction \( H \). By (14), \( H \) is \( O_{M_1}, O_{N_2}, O_{N_3} \) or \( O_{P_3} \). Observe that \( v_1v_2 \notin M \), since otherwise the distance between \( v_1v_2 \) and the special subgraph of \( H \) (which is also special in \( G \)) is at most 3. Every vertex with list of size three according to \( L\varphi \) either belongs to \( N \) or is equal to \( v_2 \). If \( v_2 \notin V(H) \), then \( H \) contains only one vertex with list of size three, hence \( H \) is \( O_{N_2} \). However, then \( N \) contains two adjacent vertices, which is a contradiction. Similarly, we exclude the case that \( v_2 \in V(H) \) and \( H \) is \( O_{N_3} \) or \( O_{P_3} \). Therefore \( v_2 \in V(H) \) and \( H \) is \( O_{M_1} \) or \( O_{N_2} \). The former is excluded by (7). If \( H \) is \( O_{N_2} \), then we have \( V(G) = V(H) \cup \{v_1\} \) by (4). If \( v_1 \) is incident with a crossed edge, then \( G \) contains \( O_{c_2} \). On the other hand, if \( v_1 \) is not incident with a crossed edge, then \( |L(v_1)| = 3, |L(v_2)| = 4, |N| = 1 \) and \( G \) is \( L \)-colorable. This is a contradiction, hence \( s \geq 3 \).

Next, observe that if \( v_1 \) and \( v_2 \) are not crossing-adjacent, then \( |L(v_1)| = 3 \) or \( |L(v_2)| = 3 \). Otherwise, we could remove a color from the list of \( v_1 \). If the edges \( v_1x \) and \( e \) cross, then \( |L(x)| = 5 \) by (7), and both vertices incident with \( e \) have list of size five by (8) and (10), hence \( G \) with the new list assignment satisfies (C). By (14), no obstruction arises (since all vertices with lists of size three or four in the new list assignment are contained in \( V(F) \)). Thus \( G \) with
the reduced lists satisfies (O) and contradicts the minimality assumption. Similarly, if \(v_1\) and \(v_2\) are crossing-adjacent and \(|L(v_1)| > 3\) and \(|L(v_2)| > 3\), then we can assume that \(|L(v_1)| = |L(v_2)| = 4\).

If \(|L(v_1)| = 3\) or \(|L(v_2)| = 3\) and furthermore \(v_1v_2,v_2v_3 \not\in M\), then let the set \(X \subseteq V(F) \setminus V(P)\) and its partial \(L\)-coloring \(\varphi\) be defined as in (X1)–(X4) in the proof of Theorem 6. Let us add two more cases for the situation when \(v_1\) and \(v_2\) are crossing-adjacent:

(X5) If \(|L(v_1)| = |L(v_2)| = 4\) and \(|L(v_3)| \neq 3\), then \(X = \{v_1\}\) and \(\varphi(v_1) \in L(v_1) \setminus L(p_0)\) is chosen arbitrarily.

(X6) If \(|L(v_1)| = |L(v_2)| = 4\) and \(|L(v_3)| = 3\), then \(X = \{v_2\}\) and \(\varphi(v_2) \in L(v_2) \setminus L(v_3)\) is chosen arbitrarily.

Let \(m\) be the largest index such that \(v_m \in X\). Let us note that \(m = 1\) in (X1) and (X5), \(m = 3\) in (X4), and \(m = 2\) otherwise. Also, \(X = \text{dom}(\varphi)\) in all cases except for (X4b), when \(X = \{v_1,v_2,v_3\}\) and \(\text{dom}(\varphi) = \{v_1,v_3\}\).

(15) One of the following cases holds:

(A1) \(v_1v_2 \in M\) or \(v_2v_3 \in M\).

(A2) Either \(v_1\) and \(v_2\) or two distinct vertices in \(\text{dom}(\varphi)\) have a common neighbor in \(N\).

(A3) There exists a crossing \(q\) and two crossing-adjacent vertices \(w_1,w_2 \in V(G_q)\) such that \(V(G_q) \cap X = \emptyset\), \(w_1\) has a neighbor in \(\text{dom}(\varphi)\) and \(w_2\) has two neighbors in \(\text{dom}(\varphi)\).

(A4) \(v_mv_{m+1} \in E(G)\), there exists a crossing \(q\) such that \(V(G_q) \cap X = \emptyset\) and \(v_{m+1},v_{m+2} \in V(G_q)\), and either \(|L(v_{m+1})| = |L(v_{m+2})| = 4\) or \(|L(v_{m+1})| = 5\) and \(|L(v_{m+2})| = 3\).

(A5) \(v_mv_{m+1} \in E(G), |L(v_{m+1})| \in \{3,4\}\) and there exists a crossing \(q\) such that \(V(G_q) \cap X = \emptyset\), \(v_{m+1} \in V(G_q)\) and a neighbor \(w \not\in V(F)\) of \(v_m\) is crossing-adjacent to \(v_{m+1}\).

(A6) \(v_1 \not\in X\) and there exists a crossing \(q\) such that \(V(G_q) \cap X = \emptyset\), \(v_1 \in V(G_q)\) and a neighbor \(w \not\in V(F)\) of \(v_2\) is crossing-adjacent to \(v_1\).

(A7) \(|X| \geq 2\) and there exists a path \(v_{m-1}xyv_{m+1}\), where \(x\) and \(y\) are neighbors of \(v_m\) and \(y \in N\).
Figure 7: Possible outcomes of (15)
Proof. See Figure [7] for the illustration of the possibilities. Note that if (A1) does not hold, then \( X \) and \( \varphi \) are defined.

Suppose now for a contradiction that none of (A1)–(A7) holds. Let us consider the graph \( G' = G - \text{dom}(\varphi) \) with the list assignment \( L_{\varphi} \), and let \( G' \) be the graph obtained from \( G'_0 \) by repeatedly removing vertices whose list is larger than their degree. If \( \text{dom}(\varphi) \neq X \), then we have case (X4b) and \( X \setminus \text{dom}(\varphi) = \{v_2\} \). If \( v_2 \) is not incident with a crossing, then its degree in \( G'_0 \) is 1, and since \( |L_{\varphi}(v_2)| \geq 2 \), it is not present in \( G' \). On the other hand, if \( v_2 \) were incident with a crossing, then the fact that \( |L(v_1)| = |L(v_3)| = 4 \) would contradict (C). This shows that \( G' \subset G - X \). Observe also that \( G' \) is not \( L_{\varphi} \)-colorable.

Next, we argue that \( G' \) satisfies the assumptions of Theorem [7] (with the sets \( N' \subset V(G') \) and \( M' \subset E(G') \) defined as the minimal sets such that (S), (N) and (M) hold), thus contradicting the minimality of \( G \). The property (P) holds trivially, (T) holds by [9]. To verify (S), note first that the only vertices not in the outer face of \( G' \) with list of size less than five are those belonging to \( N \), or incident with a crossed edge joining them in \( G \) to a vertex in \( \text{dom}(\varphi) \); and the vertices of the latter kind have list of size four. Thus, they have been added into \( N' \) without violating the distance condition since the rank of special vertices in \( N' \) is smaller than the rank of the crossing.

Next, suppose that a vertex \( v \in V(G') \setminus V(P) \) satisfies \( |L_{\varphi}(v)| \leq 2 \). Note that \( v \notin V(F) \) by the choice of \( X \) and \( \varphi \) and by [7]. It follows that \( v \in N \) and \( v \) has two colored neighbors in \( X \), thus (A2) holds. This confirms that (S) is satisfied.

Now, let us consider property (C). Let \( q \) be a crossing in \( G' \) and suppose that (C) is violated at \( q \), i.e., there exist distinct \( u, v \in V(G_q) \) such that \( |L_{\varphi}(u)| = 3 \) and \( |L_{\varphi}(v)| \in \{3, 4\} \). If both \( u \) and \( v \) belong to \( F \), then by [7] and [10] we have that \( u \) and \( v \) are crossing-adjacent, \( \{u, v\} = \{v_{m+1}, v_{m+2}\} \) and \( L(v_{m+2}) = L_{\varphi}(v_{m+2}) \). It follows that \( |L(v_{m+1})| \neq 3 \) and that (A4) holds. If \( u \in V(F) \) and \( v \notin V(F) \) and \( u \) and \( v \) are not crossing-adjacent, then since \( V(G_q) \cap X = \emptyset \), [12] implies that (A4) holds. If \( u \in V(F) \) and \( v \notin V(F) \) and \( u \) and \( v \) are crossing-adjacent, then we apply [13]. The outcome (a) of [13] gives (A5) or (A6). The outcome (b) gives a vertex \( w \in X \) that is adjacent to \( v \) and a vertex \( z \) with \( |L(z)| = 3 \) that is adjacent to \( u, v \) and \( w \). Therefore, \( |L(u)| \neq 3 \), so \( u \) has a neighbor in \( X \). This is only possible in the subcase (X4a) of the definition of \( X \), where \( z = v_2, w = v_3 \) and \( u = v_1 \), thus obtaining (A6). If \( u \notin V(F) \) and \( v \in V(F) \), then \( u \) has two neighbors in \( \text{dom}(\varphi) \). Since \( V(G_q) \cap X = \emptyset \), [12] implies that \( u \) and \( v \) are crossing-adjacent. By [13],
one of the neighbors of \( u \) in \( X \) is also adjacent to \( v \) and has list of size three, and by the choice of \( X \), we conclude that (A6) holds. Finally, if \( u, v \notin V(F) \), then they are crossing-adjacent by \([4]\) and the fact that \( V(G_q) \cap X = \emptyset \), and (A3) holds.

Therefore, we can assume that \( G' \) satisfies (C). Let us now consider the newly created special subgraphs in \( G' \). If \( v \in N' \setminus N \), then \( v \) is adjacent to a vertex of \( X \) by an edge containing a crossing \( q \), and no other vertex of \( G_q \) belongs to \( X \). Therefore, there is at most one such vertex. Consider now an edge \( xy \in M' \setminus M \); we will show that either there exists a crossing \( q \) such that \( \{x, y\} = V(G_q) \setminus \{X\} \), or at least one of \( x \) and \( y \) belongs to \( N \). Note that \( xy \) has been added to \( M' \) because \( |L_{\varphi}(x)| = |L_{\varphi}(y)| = 3 \). Suppose that \( x, y \notin N \). If \( x, y \notin V(F) \), then both \( x \) and \( y \) have two neighbors in \( \text{dom}(\varphi) \). It is easy to see using \([4]\) and \([5]\) that this implies that \( x \) and \( y \) are crossing-adjacent in \( G \) via the edges joining \( x, y \) with \( X \). If \( x, y \in V(F) \), then by \([7]\) we can assume that \( x = v_{m+1} \) and \( y = v_{m+2} \); but then \( |L_{\varphi}(x)| \neq 3 \) or \( |L_{\varphi}(y)| \neq 3 \) by the choice of \( X \), which is a contradiction. Finally, suppose that say \( x \in V(F) \) and \( y \notin V(F) \); then \( y \) has two neighbors in \( \text{dom}(\varphi) \) and, in particular, we have cases (X2) or (X4). By \([12]\), we have \( x \in \{ v_1, v_{m+1} \} \). If \( x = v_1 \), then \( y \) would be a common neighbor of \( v_1, v_2 \) and \( v_3 \), contradicting the choice of \( X \) (assumptions of (X4b) are satisfied, hence we would have \( v_1 \in X \)). If \( x = v_{m+1} \), then \( |L(v_m)| = 4 \) and therefore one of the edges \( v_{m-1}y \) and \( v_{m+1}y \) is crossed since \( \text{deg}(v_m) \geq 4 \). However, by the choice of \( X \) we have \( |L(v_{m-1})| = |L(v_{m+1})| = 3 \), contradicting (C).

It follows that \( d(S_1, S_2) \geq 7 + r(S_1) + r(S_2) \) whenever \( S_1 \) is a special subgraph of \( G \) that is also special in \( G' \) and \( S_2 \) is any special subgraph of \( G' \). Suppose now that \( S_1 \) and \( S_2 \) are both distinct newly created special subgraphs in \( G' \). Note that \( |N' \setminus N| \leq 1 \) and if \( N' \setminus N \neq \emptyset \), then \( M' \setminus M = \emptyset \). It follows that \( S_1, S_2 \in M' \setminus M \). As proved in the previous paragraph, each edge in \( M' \setminus M \) is incident with a special subgraph in \( G \) that is adjacent to \( X \). By the distance condition, we conclude that there exists a path \( xyz \) in \( G' \) such that \( |L_{\varphi}(x)| = |L_{\varphi}(y)| = |L_{\varphi}(z)| = 3 \) and \( y \in N \). Note that at most one of \( x \) and \( z \) can have two neighbors in \( \text{dom}(\varphi) \), as otherwise \( G \) would contain a crossing at distance at most one from \( y \); thus we may assume that \( x \in V(F) \). By \([12]\), \( x \in \{ v_1, v_{m+1}, v_{m+2} \} \). If \( x = v_{m+2} \), then we would have \( |L(v_{m+1})| = |L(x)| = 3 \) and \( v_{m+1}x \in M \) would be at distance one from \( y \in N \), which is a contradiction; therefore, \( x \neq v_{m+2} \). If \( x = v_1 \), then (A2) holds, hence \( x = v_{m+1} \) and \( z \notin V(F) \) has two neighbors in \( \text{dom}(\varphi) \). However, then \( |L(v_m)| = 4 \), hence \( \text{deg}(v_m) \geq 4 \) and \( v_m \) is adjacent to \( y \) and (A7) holds.
We conclude that $G'$ is $(P, N', M')$-distant.

Finally, suppose that $G'$ violates condition (O), and thus $G$ contains a near-obstruction $H$. By (14), $H$ is one of $O_{M1}$, $O_{N2}$, $O_{N3}$ or $O_{P3}$.

- If $H$ is $O_{M1}$, then let $xy$ be the edge of $H$ that belongs to $M'$, where $x$ is adjacent to $p_2$. Note that $x, y \notin V(F)$ by (7) and $xy \notin M$. If $x \notin N$, then $x$ has two neighbors $v_i$ and $v_j$ in $\text{dom}(\varphi)$, where $i < j$. By (12) applied to $p_2v_i$, we have $j = i + 1$ and by the choice of $X$, $|L(v_j)| = 4$; hence $v_j$ is incident with a crossing and thus $y \notin N$. Consequently, $y$ is also adjacent to $v_i$ and $v_j$. However, note that $|L(v_i)| = 3$, contradicting (C) for $G$. Therefore, $x \in N$ is adjacent to $v_j$, and $y$ is adjacent to both $v_i$ and $v_j$. By (12) applied to $p_0yv_j$, we have $i = 1$, $j = 2$ and $|L(v_1)| = 3$, and by (12) applied to $p_2xv_2$, we have that $s = 3$ and $|L(v_3)| = 3$. However, then $G$ is $L$-colorable.

- Next, suppose that $H$ is $O_{N2}$ and let $x$ and $y$ be the vertices in the outer face of $H$ such that $|L_{\varphi}(x)| = 3$ and $|L_{\varphi}(y)| = 4$. By (7), $y \notin V(F)$. If $x \in V(F)$, then by (12) we have $s \leq 2$, which is a contradiction, hence $x \notin V(F)$. Thus $x$ has two neighbors in $\text{dom}(\varphi)$ and $y$ has one, and by (12) we conclude that $s = 3$ and $|L(v_1)| = |L(v_3)| = 3$. It follows that $X = \{v_1, v_2\}$, $x$ is adjacent to $v_1$ and $v_2$, and $y$ is adjacent to $v_2$. There are two cases, either $v_2$ is incident with a crossed edge or $|N| = 1$; in both of them, $G$ is $L$-colorable.

- If $H$ is $O_{N3}$, then let $xyz$ be the path in the outer face of $H$ such that $|L_{\varphi}(x)| = |L_{\varphi}(z)| = 3$, $|L_{\varphi}(y)| = 4$ and $z$ is adjacent to $p_1$. By (7), $z \notin V(F)$, thus $z$ has two neighbors $w_1, w_2 \in \text{dom}(\varphi)$, and by (12), we can assume that the neighbors of $w_1$ are $w_2$, $z$ and an endvertex of $P$, and that $|L(w_1)| = 3$. Since $y \notin V(F)$, $y$ is adjacent to $w_2$. Since $x$ cannot have more than one neighbor in $\text{dom}(\varphi)$, we have $x \in V(F)$. If $xw_2 \notin E(G)$, then (12) implies that $x$ is adjacent to a vertex with list of size three, and thus $|L_{\varphi}(x)| = |L(x)| > 3$. This is a contradiction, hence $xw_2 \in E(G)$. By the choice of $X$, $|L(x)| = 3$. Again, we distinguish two cases depending on whether $w_2$ is incident with a crossed edge (in this case $|L(w_2)| = 5$ by (C)) or $|N| = 1$. In both cases, $G$ is $L$-colorable.

- Therefore, $H$ is $O_{P3}$. But then two of the vertices of $H$ have two neighbors in $\text{dom}(\varphi)$, hence $G$ contains a crossing at distance at most one from $P$, contradictory to the assumption that $G$ is $(P, N, M)$-distant.
We have shown that $G'$ satisfies all conditions of Theorem 7 for the list assignment $L_\varphi$. This gives a contradiction to the minimality of $G$ and proves claim (15).

Each case among (A1)–(A7) in (15) contains a special subgraph. Thus, $G$ contains a special subgraph $S$ whose distance from $p_0$ is at most $2 + r(S)$. Consequently, $\ell(P) = 2$. Next, we consider the set $X' \subseteq \{v_s, v_{s-1}, v_{s-2}\}$ defined symmetrically to $X$ and conclude that there exists a special subgraph $S'$ (satisfying one of (A1)–(A7) with $v_i$ replaced by $v_{s+1-i}$) whose distance to $p_2$ is at most $2 + r(S')$. It follows that $d(S, S') \leq 6 + r(S) + r(S')$, and since $G$ is $(P, N, M)$-distant, we have $S = S'$.

Next, we show that (16) $S$ consists of two edges incident with a crossing.

Proof. If not, then either $S \in M$ or $S \in N$. Suppose first that $S \in M$. Then (A1) holds and $s \leq 4$. Since $s \geq 3$, we can by symmetry assume that $S = v_2v_3$. If $v_2, v_3$ and $v_i$ have no common neighbor for $i \in \{1, 4\}$ ($i = 1$ if $s = 3$), then let $\varphi$ be an arbitrary $L$-coloring of $S$ (such that $\varphi(v_3) \notin L(p_2)$ if $s = 3$). Observe that $G_\varphi$ cannot contain an obstruction since its special subgraph would be a special subgraph in $G$, too close to the special edge $v_2v_3$. Now it is easy to check using previously proved properties of $G$ that $G_\varphi$ satisfies all conditions of Theorem 7. (The same reasoning will be applied in the sequel without repeating it.) Therefore, $G_\varphi$ with the list coloring $L_\varphi$ is a counterexample to Theorem 7, contradicting the minimality of $G$. Thus, by symmetry, we may assume that $v_1, v_2$ and $v_3$ have a common neighbor $w$. In that case, $w$ is not adjacent to $v_4$ by (12). Let $\varphi$ be an $L$-coloring of $v_1$ and $v_3$ such that $\varphi(v_1) \notin L(p_0), \varphi(v_3) \notin L(p_2)$ and $|L_\varphi(v_2)| \geq 2$. Then $G' = G - \{v_1, v_2, v_3\}$ with the list assignment $L_\varphi$ contradicts the minimality of $G$ since any $L_\varphi$-coloring of $G'$ can be extended to $v_2$ by using a color in $L_\varphi(v_2)$, and can henceforth be extended to $G$.

Let us now consider the case that $S \in N$, hence (A2) or (A7) holds. Let $i$ and $j$ be the smallest and the largest integer, respectively, such that $S$ is adjacent to $v_i$ and $v_j$. By (12), we have $j \in \{i + 1, i + 2\}$. We consider the two possible values of $j$ separately:

- Suppose first that $j = i + 1$. If $|X| \geq 2$, then $|L(v_m)| \geq 4$ and $|L(v_{m+1})| = 3$, hence (A7) cannot hold for both $X$ and $X'$. If both $X$ and $X'$ satisfy (A2), then since $s \geq 3$, we can assume that $v_2, v_3 \in X$. 


have a common neighbor in \( \mathcal{N} \). By the choice of \( X \), we have \(|L(v_4)| = 3\), hence \( s = 4 \) and \( v_2, v_3 \in X' \). However, then \(|L(v_1)| \geq 4\) by the choice of \( X \) and \(|L(v_1)| = 3\) by the choice of \( X' \), which is a contradiction.

Hence, we can assume that (A7) holds for \( X \) and (A2) for \( X' \); then we either have \( s = m + 1 \), or we have \( s = m + 2 \) and \( X' = \{v_m, v_{m+1}\} \). If there exists an \( L \)-coloring \( \varphi \) of \( v_{m-1} \) and \( v_{m+1} \) such that their colors are distinct from the colors of their neighbors in \( P \) and \(|L_{\varphi}(v_m)| \geq 3\), then \( G' = G - \{v_m, v_{m+1}\} \) with the list assignment \( L_{\varphi} \) contradicts the minimality of \( G \). (Observe that \( G' \) satisfies (O), since no special subgraph of \( G \) is at distance at most two from \( S \).) A new special subgraph would appear in \( G' \) only if \( S \) would be adjacent to \( v_{m+2} \), which is not the case since \( j = i + 1 \).

We conclude that no such coloring exists, hence both \( v_{m-1} \) and \( v_{m+1} \) have a neighbor in \( P \) and \( s = 3 \). Furthermore, \(|L(v_i)| = 3 \) and \( L(v_i) \setminus L(p_0) < L(v_2) \). Let \( w' \) be the common neighbor of \( S \) and \( v_1 \). Suppose that there exists a color \( c \in L(w') \) different from the colors of the neighbors of \( w' \) in \( P \) such that either \( c \notin L(v_2) \), or \( v_1 \) has degree three and \( c \notin L(v_1) \setminus L(p_0) \). In this case, we let \( \varphi \) be the partial coloring such that \( \varphi(w') = c \) and let \( G' = G - \{w', v_2\} \) if \( c \notin L(v_2) \) and \( G' = G - \{w', v_1, v_2\} \) if \( c \in L(v_2) \). Observe that \( G' \) is not \( L_{\varphi} \)-colorable. Furthermore, it satisfies the assumptions of Theorem [7] with the edge \( Sv_3 \) belonging to \( M \) (the condition (O) holds by [14], the distance condition and (7)). This contradicts the minimality of \( G \), and thus no such color \( c \) exists. Since \(|L(w')| > |L(v_2)|\), it follows that \( w' \) has a neighbor in \( P \). By [12], \( w' \) is not adjacent to \( p_2 \), hence it is adjacent to \( p_0 \) or \( p_1 \). However, then [5] and [7] imply that \( v_1 \) has degree three, and since \(|L(v_1) \setminus L(p_0)| = 2\) and \( w' \) has at most two neighbors in \( P \), the color \( c \) exists. This is a contradiction.

- It remains to consider the case when \( j = i + 2 \). In this case \( S \) is adjacent to \( v_i \) and \( v_{i+2} \), and by [12] we conclude that \( v_{i+1} \) is a vertex of degree 3 with neighbors \( v_i, v_{i+2} \), and \( S \). Thus, \(|L(v_{i+1})| = 3\). Suppose first that both \( X \) and \( X' \) satisfy (A7). If there exists a coloring \( \varphi \) of \( S \) by a color different from the colors of its neighbors in \( P \) such that \( \varphi(S) \notin L(v_i) \cap L(v_{i+1}) \cap L(v_{i+2}) \), then \( G' = G - \{S, v_i, v_{i+1}, v_{i+2}\} \) with the list assignment \( L_{\varphi} \) is a counterexample contradicting the minimality of \( G \) (since in this case any \( L_{\varphi} \)-coloring of \( G' \) extends to an \( L \)-coloring).
of \( G \). Otherwise, note that \( S \) is not adjacent to \( p_0 \) or \( p_2 \) by \([12]\), hence \( S \) is adjacent to \( p_1 \) and \( L(S) \setminus L(p_1) = L(v_{i+1}) \subseteq L(v_i) \cap L(v_{i+2}) \). However, in this case we let \( \varphi \) be the \( L \)-coloring of \( X \) as chosen in the proof of Theorem \([6]\) and note that \( \varphi(v_i) \notin L(v_{i+1}) = L(S) \setminus L(p_1) \). Then \( G - X \) with the list assignment \( L_\varphi \) for any other vertex \( z \) is a counterexample contradicting the minimality of \( G \).

Hence, we can assume that say \( X' \) satisfies (A2). Let us first consider the case that \( X \) satisfies (A2) as well. Note that \( v_{i+2} \notin X \), as otherwise \( |L(v_{i+3})| = 3 \) by the choice of \( X \), and thus \( v_{i+1} \notin X' \), contradictory to the assumption that \( X' \) satisfies (A2). Symmetrically, \( v_i \notin X' \). Since \( |L(v_{i+1})| = 3 \), we cannot have \( \{v_i, v_{i+1}\} \subseteq X \), thus \( i = 1 \), and by symmetry, \( s = 3 \). Observe that we cannot color \( S \) by a color \( \varphi(S) \notin L(v_{i+1}) \), as otherwise \( G - \{S, v_{i+1}\} \) with the list assignment \( L_\varphi \) would contradict the minimality of \( G \). Therefore, \( S \) has a neighbor in \( P \), and by \([12]\), this neighbor is \( p_1 \). By \([5]\), the 4-cycle \( p_1p_0v_1S \) is not separating, and by \([7]\), \( v_1 \) has degree three. This is a contradiction, since \( |L(v_1)| > 3 \).

Therefore, \( X \) satisfies (A7). Note that \( v_{i+1} \) cannot be the element of \( X' \) with the smallest index, thus \( i + 2 = s \). As before, we exclude the case that \( S \) can be colored by a color not belonging to \( L(v_i) \cap L(v_{i+1}) \), hence \( S \) has a neighbor in \( P \). By \([12]\), \( S \) is not adjacent to \( p_0 \) or \( p_2 \), hence \( S \) is adjacent to \( p_1 \). However, by \([5]\), the 4-cycle \( p_1Sv_{i+2}p_2 \) is not separating, and by \([7]\), \( v_{i+2} \) is not adjacent to \( p_1 \). Thus, \( v_{i+2} \) has degree three and list of size at least four, which is a contradiction.

This completes the proof of the case when \( S \notin N \).

Therefore, \([16]\) holds and \( S \) consists of two edges incident with a crossing \( q \). We conclude that each of \( X \) and \( X' \) satisfies one of (A3), (A4), (A5) or (A6). If one of them satisfies (A4), then \( |V(G_q) \cap V(F)| = 2 \) by \([7]\). If it satisfies (A6), then by \([7]\), \([8]\) and \([10]\) we have \( |V(G_q) \cap V(F)| = 1 \). If it satisfies (A3), then similarly \( |V(G_q) \cap V(F)| \leq 1 \), and if it satisfies (A5) then \( 1 \leq |V(G_q) \cap V(F)| \leq 2 \).

(17) Neither \( X \) nor \( X' \) satisfies (A3).

Proof. Suppose for a contradiction that \( X \) satisfies (A3). Let \( w_1 \) and \( w_2 \) be as in the description of (A3). Note that \( w_2 \) is adjacent to \( v_{m-1} \) and \( v_m \) (even if \( v_{m-1} \notin \text{dom}(\varphi) \), in the case (X4b)) and that \( |L(v_{m-1})| = |L(v_{m+1})| = 3 \).
Let us first consider the case that \(|V(G_q) \cap V(F)| = 0\). In this case \(X'\) satisfies (A3) as well, i.e., there exists \(w'_2 \in V(G_q)\) adjacent to \(v_b\) and \(v_{b+1}\), where \(b\) is the smallest index of an element of \(X'\), and another vertex \(w'_1\) of \(G_q\) that has one neighbor in \(X'\). Since \(|L(v_b)| \neq 3\), we have \(b \notin \{m - 1, m + 1\}\). Consequently, \(|X \cap X'| \leq 1\), and \(w'_2 \neq w_2\) by (12).

We now distinguish two cases regarding whether \(w_2\) is adjacent or crossing-adjacent to \(w'_2\) in \(G_q\).

- Suppose that \(w_2w'_2\) is a crossed edge. Then \(b \neq m\) by (4) and the assumption that \(G_q\) is disjoint with \(F\); thus \(b \geq m+2\). Let \(G_1\) and \(G_2\) be the subgraphs of \(G\) intersecting in \(v_mw_2w'_2v_b\), such that \(G_1 \cup G_2 = G - e\), where \(e\) is the edge crossing \(w_2w'_2\), and \(P \subset G_1\). By (12), we have that \(w_1 \neq w'_2, w'_1 \neq w_2\) and that if \(w_1 = w'_1\), then \(w_1\) belongs to \(G_2\). By symmetry, assume that \(w_1\) belongs to \(G_2\). If \(w_1\) is adjacent to \(v_b\), then \(b = m + 2\) by (12). Let \(T = \{v_m, v_{m+1}, v_{m+2}, w_1\}\). By using (11) it is easy to see that \(|L(t)| = \deg(t)\) for each \(t \in T \setminus \{w_1\}\) and that \(\deg(w_1) \leq 6\). By the minimality of \(G\), there exists an \(L\)-coloring \(\varphi\) of \(G - T\). Consider the subgraph \(G'\) of \(G\) induced by \(T\) with the list assignment \(L_{\varphi}\). We have \(|L_{\varphi}(v_{m+1})| \geq 3\) and \(|L_{\varphi}(z)| \geq 2\) for \(z \in T \setminus \{v_m+1\}\). If \(L_{\varphi}(w_1) \neq L_{\varphi}(v_m)\), then we color \(v_1\) by a color in \(L_{\varphi}(w_1) \setminus L_{\varphi}(v_m)\) and extend this coloring to the rest of \(G'\). Similarly, \(G'\) is \(L_{\varphi}\)-colorable if \(L_{\varphi}(w_1) \neq L_{\varphi}(v_{m+2})\). If \(L_{\varphi}(v_m) = L_{\varphi}(w_1) = L_{\varphi}(v_{m+2})\), then we color \(v_{m+1}\) by a color in \(L_{\varphi}(v_{m+1}) \setminus L_{\varphi}(w_1)\) and again we can extend this to an \(L_{\varphi}\)-coloring of \(G'\). It follows that \(G\) is \(L\)-colorable, which is a contradiction.

Therefore, \(w_1\) is not adjacent to \(v_b\), and in particular \(w_1 \neq w'_1\) and \(w'_1 \in V(G_1)\). Let \(\varphi\) be an \(L\)-coloring of \(G_1\), which exists by the minimality of \(G\). Since \(w_1\) is not adjacent to \(v_b\), note that \(w_1\) has at most three neighbors in \(G_1\) different from \(w'_1\). Hence, we can additionally choose a color \(\varphi(w_1)\) for \(w_1\) different from the colors of its neighbors in \(G_1\) so that \(\varphi(w_1) \neq \varphi(w'_1)\). Let \(G'_2 = G_2 - w_2 + w_1w'_2\). Note that \(G'_2\) gives an instance of Theorem 7 with the precolored path \(P' = v_mw_1w'_2v_b\), since the added edge \(w_1w'_2\) can be drawn without crossings following the crossed edges of \(G\) that are no longer in \(G'_2\). It is clear that \(G'_2\) satisfies validity and distance constraints. Note that \(\varphi\) does not extend to an \(L\)-coloring of \(G'_2\). Thus \(G'_2\) violates (T) or (O). In the former case, the vertex violating (T) must be \(v_{m+1}\) and we would have \(b = m + 2\). Consequently, \(v_b\) would have degree at most three, which is a
contradiction. In the latter case, since $|L(v_{m+1})| = |L(v_{b-1})| = 3$ and $v_b$ has degree at least three in $G_2'$, we have that $G_2'$ is equal to $O_{p5}$ or $O_{p6}$. In both cases, any $L$-coloring of $G_1 - \{v_m, v_b\}$ would extend to an $L$-coloring of $G$, a contradiction.

- Suppose now that $w_2$ is crossing-adjacent to $w'_2$. Let $G_1$ and $G_2$ be the subgraphs of $G$ intersecting in $\{v_b, w'_2, w_2, v_m\}$, where $P \subset G_1$ and $G_1 \cup G_2$ is equal to $G$ without the crossed edges. We have two subcases: either $b > m$ or $b = m$.

  - If $b > m$, then (12) implies that $w'_2$ has no neighbor in $X$, and thus $w_1 \neq w'_2$. Symmetrically, $w'_1 \neq w_2$. Considering the drawing of $G$ in the plane, we conclude that the edges of $G_2'$ are $w_1w'_2$ and $w'_1w_2$.

If $w_1, w'_1 \notin V(G_1)$ (see Figure 8(a)), then $w_1v_m, w'_1v_b \in E(G)$. Let $\varphi$ be an $L$-coloring of $G_1 + \{w_1w'_2, w'_1w_2, w_1w'_1\}$ which exists by the minimality of $G$, and note that $\varphi$ does not extend to an $L$-coloring of $G_2' = G_2 + w_1w'_1$. Observe that $G_2'$ provides an instance for Theorem 7 with the precolored path $v_mw_1w'_1v_b$. It is easy to see that we can choose the colors of $w_1$, $w'_1$, $v_m$ and $v_b$ so that $G_2'$ satisfies the assumptions of the theorem (once the coloring of $G_1 - \{v_m, v_b\}$ is fixed, we still have two possible choices for the colors of $v_m$ and $v_b$ and three possible choices for the colors of $w_1$.
and \( w'_1 \). This is a contradiction. The case that \( w_1, w'_1 \in V(G_1) \) (see Figure 8(b)) is excluded similarly.

- If \( b = m \), then let \( w_2z \) and \( w'_2z' \) be the edges of \( G_q \) (note that we have \( w_1 = w'_2 \) and \( w'_1 = w_2 \)). Suppose that \( z, z' \in V(G_2) \). Note that \( V(G_2) \neq \{ z, z', w_2, w'_2, v_m \} \), since otherwise \( z \) would have degree at most four and \( |L(z)| = 5 \). Therefore, the subgraph of \( G \) induced by \( V(G_1) \cup \{ z, z' \} \) has an \( L \)-coloring \( \psi \) by the minimality of \( G \). Let \( L' \) be the list assignment for \( G'_2 = G_2 - \{ z, z' \} \) obtained from \( L \) by removing the colors of \( z \) and \( z' \) according to \( \psi \) from the lists of their neighbors and by setting \( L'(w_2) = \{ \psi(w_2) \} \), \( L'(v_m) = \{ \psi(v_m) \} \) and \( L'(w'_2) = \{ \psi(w'_2) \} \). Note that \( G'_2 \) satisfies \( (O) \) by the distance condition and \( (P) \) by the choice of \( \psi \), and since \( G \) is not \( L \)-colorable, we conclude that \( G'_2 \) violates \( (T) \). Therefore, \( G_2 \) contains a vertex adjacent to \( w_2, w'_2, v_m, z \) and \( z' \), and by \([4]\), \( z \) and \( z' \) have degree at most four. This is a contradiction.

Therefore, we have \( z, z' \in V(G_1) \) (see Figure 8(c)), and by \([11]\), \( \deg(v_m) = 4 \). Let \( S_1 = L(v_2) \) if \( m = 3 \) and \( S_1 = L(v_1) \setminus L(p_0) \) if \( m = 2 \). Note that \( S_1 \subset L(v_m) \), as otherwise we consider the partial coloring \( \varphi \) with \( \varphi(v_{m-1}) \in S_1 \setminus L(v_m) \) and conclude that \( G'_2 \) with the list assignment \( L_\varphi \) contradicts the minimality of \( G \). Suppose that there exists a color \( c \in L(w_2) \setminus L(v_m) \), or that \( \deg(v_{m-1}) = 3 \) and there exists a color \( c \in L(w_2) \setminus S_1 \), such that this color \( c \) is distinct from the colors of the neighbors of \( w_2 \) in \( P \). Let \( G' = G - \{ w_2, v_m \} \) if \( \deg(v_{m-1}) > 3 \) and \( G' = G - \{ w_2, v_m, v_{m-1} \} \) if \( \deg(v_{m-1}) = 3 \), with the list assignment \( L' \) obtained from \( L \) by removing \( c \) from the lists of neighbors of \( w_2 \) and setting \( L'(v_{m-1}) = L(v_{m-1}) \) if \( v_{m-1} \) belongs to \( V(G') \) (observe that \( c \notin S_1 \) and that in any \( L' \)-coloring of \( G' \), the color of \( v_{m-1} \) must belong to \( S_1 \)). Note that \( L' \) is \( (P, N \cup \{ z \}, M) \)-valid. Every \( L' \)-coloring of \( G' \) would extend to an \( L \)-coloring of \( G \), thus \( G' \) is not \( L \)-colorable. By the minimality of \( G \), we conclude that \( G' \) violates \( (O) \), and by \([14]\) and the distance condition, \( G' \) contains \( O_{N2} \) or \( O_{N3} \). However, then \( z \) is adjacent to two vertices of \( P \) and to \( z' \) and \( w'_2 \), and at least one of \( z' \) and \( w'_2 \) has a list of size three according to \( L' \), which is a contradiction since \( |L(z')| = |L(w'_2)| = 5 \).

We conclude that there exists no such color \( c \). Since \( |L(v_m)| = 4 \) and \( |L(w_2)| = 5 \), we conclude that \( w_2 \) has a neighbor in \( P \). By
Figure 9: Subcase combining (A3) and (A5).

$w_2$ is not adjacent to $p_2$, and if it were adjacent to $p_0$, then we would have $m = 2$, $\deg(v_1) = 3$ and there would exist a color $c \in L(w_2) \setminus (S_1 \cup L(p_0) \cup L(p_1))$. Therefore, $w_2$ is adjacent to $p_1$. By symmetry, $w'_2$ is adjacent to $p_1$ as well. However, the edges $w_2p_1$ and $w'_2p_1$ are not crossed by (8), and thus the crossing is contained inside the 4-cycle $v_mw_2p_1w'_2$, contrary to (5).

We conclude that $V(G_q) \cap V(F) \neq \emptyset$. By (7), $w_2 \notin V(F)$. Let $w$ be the vertex joined to $w_2$ by a crossed edge, and let $w_1w'$ be the other crossing edge. Since $V(G_q) \cap X = \emptyset$, by (12) we have $w \notin V(F)$. Since $v_m$ has degree at least four, we cannot have $w_1 = v_{m+1}$, thus by (7) and (8), we have $w_1 \notin V(F) \setminus \{v_1\}$. If $w_1 \notin V(F)$ and $x \in X$ is a neighbor of $w_1$, then the 2-chord $xw_1w'$ separates $P$ from either $w_2$ or $w$, and neither $w_2$ nor $w$ belongs to $F$, contrary to (12). We conclude that $w_1 = v_1$ and $V(G_q) \cap V(F) = \{v_1\}$, hence $v_1 \notin X$ and $X$ was chosen according to (X4a).

Since $|V(G_q) \cap V(F)| = 1$, $X'$ must satisfy (A3), (A5) or (A6). If $X'$ satisfied (A3), the conclusions of the preceding paragraph would apply symmetrically and we would have $v_1 = v_b$, which is a contradiction. Similarly, $X'$ cannot satisfy (A6). The remaining possibility is that (A5) holds for $X'$. Then $v_1 = v_{b-1}$ and $v_b = v_2$. The situation is shown in Figure 9. Since $X$ was chosen according to (X4a), we have $|L(v_b)| = |L(v_{b+1})| = 3$; in particular, $s \geq 4$ and $b \leq s - 2$. This is only possible if $X'$ has been chosen according to (X4), but then $|L(v_b)| > 3$. This is a contradiction, showing that (A3) does not occur.

Next, we claim that

$$|V(G_q) \cap V(F)| = 1.$$  

Proof. Since $X$ does not satisfy (A3), if $|V(G_q) \cap V(F)| \neq 1$ then $|V(G_q) \cap V(F)| = 2$ and each of $X$ and $X'$ satisfies (A4) or (A5). By (7) and (10).
$V(G_q) \cap V(F) = \{v_{m+1}, v_{m+2}\}$ and $v_{m+1}$ is crossing-adjacent to $v_{m+2}$. Let $v_{m+1}w$ and $v_{m+2}w'$ be the crossed edges. By symmetry, we can assume that $|L(v_{m+1})| \geq |L(v_{m+2})|$. By (C), either $|L(v_{m+1})| \geq |L(v_{m+2})| \geq 4$ or $|L(v_{m+1})| = 5$ and $|L(v_{m+2})| = 3$. Therefore, $X$ was chosen according to the rules (X1) or (X3) and $|L(v_m)| = 3$.

If $L(v_{m+2}) \neq L(v_{m+1})$, then let $c$ be a color in $L(v_{m+1}) \setminus L(v_{m+2})$. If $v_{m+1}$ is not adjacent to $v_{m+2}$, then let $c$ be an arbitrary color in $L(v_{m+1})$. In both cases, let $\varphi$ be an $L$-coloring of $v_m$ and $v_{m+1}$ such that $\varphi(v_{m+1}) = c$ and if $m = 1$, then $\varphi(v_1) \notin L(p_0)$. It is easy to see that $L_\varphi$ is a $(P, N \cup \{w\}, M)$-valid list assignment for $G_\varphi$. Therefore, $G_\varphi$ violates (O). By (14), $G_\varphi$ contains $O_{N2}$ or $O_{N3}$. It follows that $w$ is adjacent to $p_1$ and to $p_0$ or $p_2$. However, if $w$ is adjacent to $p_0$, then by (12), $v_{m+2}$ is incident with a chord of $F$, contradicting (7). If $w$ is adjacent to $p_2$, then $v_{m+2}$ has degree at most three in $G_\varphi$ by (12), and since $|L_\varphi(v_{m+2})| \geq 4$, $G_\varphi$ contains neither $O_{N2}$ nor $O_{N3}$. This is a contradiction, implying that $L(v_{m+1}) = L(v_{m+2})$ (and in particular, $|L(v_{m+1})| = |L(v_{m+2})| = 4$), and $v_{m+1}v_{m+2} \in E(G)$. By the choice of $X'$, we have $|L(v_{m+3})| = 3$.

Suppose now that $w'v_m \in E(G)$. Note that $v_{m+1}$ has degree at least four, so it is adjacent to $w'$. Let $S_1 = L(v_m)$ if $m \neq 1$ and $S_1 = L(v_m) \setminus L(p_0)$ if $m = 1$. Note that $S_1 \subseteq L(v_{m+1})$, as otherwise we can choose an $L$-coloring $\varphi$ of $v_m$ such that $\varphi(v_m) \in S_1 \setminus L(v_{m+1})$, and $G_1 = G - \{v_m, v_{m+1}\}$ with the list assignment $L_\varphi$ is a counterexample contradicting the minimality of $G$ (note that $G_1$ cannot contain an obstruction since no internal vertex in $G_1$ has its list decreased and $q$ is not a crossing in $G_1$). Since $L(v_{m+1}) = L(v_{m+2})$, we conclude that $S_1 \subseteq L(v_{m+2})$. Let $G'$ be the graph obtained from $G - v_{m+1}$ by identifying $v_m$ with $v_{m+2}$, and give the resulting vertex $z$ the list of $v_m$. Note that $G'$ satisfies the validity and the distance conditions of Theorem 7 (with the edge $zv_{m+3}$ added to $M$). Since every coloring of $G'$ gives rise to an $L$-coloring of $G$, condition (O) is violated in $G'$ by the minimality of $G$. However, $G'$ contains neither $O_{M1}$ nor $O_{M2}$ (and the exclusion of other obstructions is obvious). Therefore, $w'v_m \notin E(G)$, and by symmetry, $wv_{m+3} \notin E(G)$.

Let $S_2 = L(v_{m+3})$ if $m + 3 \neq s$ and $S_2 = L(v_{m+3}) \setminus L(p_2)$ if $m + 3 = s$. Suppose now that there exists an $L$-coloring $\varphi$ of $v_{m+1}$ and $v_{m+2}$ such that $\varphi(v_{m+1}) \notin S_1$ and $\varphi(v_{m+2}) \notin S_2$. Then $L_\varphi$ is a $(P, N, \{ww'\})$-valid list assignment for $G_\varphi$, and by the minimality of $G$, $G_\varphi$ violates (O). By (14), $G_\varphi$ contains $O_{M1}$ (the other cases are easily excluded: $O_{N2}$ and $O_{N3}$ since no internal vertex gets a reduced list and $O_{P3}$ since $\ell(P) = 2$). But then $w'$
Figure 10: A subcase in the proof when \( X \) and \( X' \) satisfy (A4) or (A5). The dotted edges may or may not be present.

is adjacent to \( p_0 \), and the 2-chord \( p_0w'v_{m+2} \) contradicts (12). Therefore, no such coloring \( \varphi \) exists. It follows that \( |S_1| = |S_2| = 3 \) and \( S_1 \subseteq L(v_{m+1}) \). Since \( L(v_{m+1}) = L(v_{m+2}) \), we also have that \( S_1 = S_2 \). Since \( |S_1| = |S_2| = 3 \), claim (3)(f) implies that \( m = 2 \) and \( s = 6 \). Similarly, we conclude that \( L(v_1) = L(p_0) \cup L(v_2) \) and \( L(v_6) = L(p_2) \cup L(v_5) \), as otherwise we can color and remove \( v_1 \) or \( v_6 \).

Let us now consider the case that \( v_2, v_3 \) and \( w' \) have no common neighbor. If \( v_1, v_2 \) and \( v_3 \) have no common neighbor, then let \( \varphi \) be an \( L \)-coloring of \( v_2, v_3 \) and \( v_4 \) such that \( \varphi(v_4) \notin L(v_5) \). Otherwise, let \( \varphi \) be an \( L \)-coloring of \( v_1, v_3 \) and \( v_4 \) such that \( \varphi(v_4) \notin L(v_5) \) and \( \varphi(v_1) = \varphi(v_3) \). In the former case, let \( G' = G_\varphi \), in the latter case let \( G' = G_\varphi - v_2 \). Observe that \( L_\varphi \) is a valid list assignment for \( G' \) (after possibly adding the edge \( ww' \) into \( M \)) and that any \( L_\varphi \)-coloring of \( G' \) extends to \( G \). Furthermore, \( G' \) satisfies (O) by (14), since \( w' \) cannot be adjacent to \( p_0 \). Hence, \( G_\varphi \) contradicts the minimality of \( G \). Therefore, \( v_2, v_3 \) and \( w' \) have a common neighbor \( x' \), and by symmetry, \( v_4, v_5 \) and \( w \) have a common neighbor \( x \) (see Figure 10).

By (12), we have \( x \neq x' \) and \( x \) is adjacent neither to \( p_0 \) nor to \( p_2 \). Furthermore, if \( xp_1 \in E(G) \), then consider the cycle \( K = p_1p_2v_5v_3x \). Since \( v_6 \) has degree at least four, we conclude by (6) that \( K \) has two chords incident with \( v_6 \). However, that contradicts (7). Therefore, \( x \) (and symmetrically \( x' \)) has no neighbor in \( P \). By (12), neither \( w \) nor \( w' \) is adjacent to \( p_0 \) or \( p_2 \). Claims (4) and (5) imply that \( x'w, xw', xx' \notin E(G) \). Since both \( w \) and \( w' \) have degree at least 5, we conclude that each of them has a neighbor that is different from all vertices shown in Figure 10. Suppose that \( w'p_1 \notin E(G) \).
Then let \( \varphi \) be an \( L \)-coloring of \( x \) and \( w' \) such that \( \varphi(x), \varphi(w') \notin L(v_4) \) (note that these colors do not belong to the lists of \( v_2, v_3 \) and \( v_5 \), as well as to \( L(v_1) \setminus L(p_0) \) and \( L(v_6) \setminus L(p_2) \)). Let \( G' = G - \{ x, w', v_3, v_4 \} \) if \( \deg(w) > 5 \) and \( G' = G - \{ x, w', v_3, v_4, w \} \) if \( \deg(w) = 5 \). Note that \( G' \) is not \( L_{\varphi} \)-colorable since any \( L_{\varphi} \)-coloring of \( G' \) extends to \( G \). Furthermore, the only possible vertices with list of size three in \( G' \) are \( v_2, v_5, w \) and a common neighbor \( u \) of \( x \) and \( w' \) distinct from \( w \) and \( v_4 \), if such a vertex exists. By [5], if \( u \) exists, then \( \deg(w) = 5 \) and \( w \notin V(G') \). Furthermore, by [5], \( u \) and \( w \) are not adjacent to \( v_2 \) and \( v_5 \). Therefore, \( L_{\varphi} \) is a valid list assignment, the distance condition implies that \( G' \) satisfies (O), and thus \( G' \) contradicts the minimality of \( G \).

We conclude that \( w'p_1 \in E(G) \). Let \( G_1 \) and \( G_2 \) be the \( p_1w'v_4 \)-components of \( G \), where \( G_1 \) contains \( p_0 \). Consider an \( L \)-coloring of \( G_2 \). Note that \( v_3 \) has only two neighbors in \( G_2 - w' \), thus the coloring of \( G_2 \) can be extended to \( v_3 \) in such a way that its color is different from the color of \( w' \). Then \( G_1 - v_4 + w'v_3 \) (with the precolored path \( p_0p_1w'v_3 \)) violates (O). Observe that only \( v_1 \) and \( v_2 \) have list of size at most four and that \( x' \) is a common neighbor of \( v_3 \) and \( w' \). Therefore, \( x' \) is a vertex in the corresponding obstruction \( K \), and \( v_2 \) is a vertex in \( K \) with list of size 3. It follows that \( K \) is equal to \( O_{P_4} \). However, then \( v_1p_1 \in E(G) \), contradicting [7].

Therefore, \(|V(G_2) \cap V(F)| = 1 \), and thus each of \( X \) and \( X' \) satisfies (A5) or (A6). Since \( s \geq 3 \), we can assume that \( X' \) satisfies (A5). Suppose first that \( X \) satisfies (A6), and thus \( b = 2 \). Since \( v_1 \notin X \), the inspection of possible cases for \( X \) and \( X' \) shows that we have \( |L(v_2)| = 3 \), \( X' = \{ v_2 \} \), and \( s = 3 \). If \( v_1, v_2 \) and \( v_3 \) have no common neighbor, then consider any \( L \)-coloring \( \varphi \) of \( v_1 \) and \( v_2 \) such that \( \varphi(v_1) \notin L(p_0) \), and observe that \( G_{\varphi} \) with the list assignment \( L_{\varphi} \) is a counterexample contradicting the minimality of \( G \): since \( v_1, v_2, v_3 \) do not have a common neighbor, we do not get adjacent vertices with lists of size 3; but we may need to add the neighbor \( z \) of \( v_1 \) along the crossed edge into the set \( N \). The resulting graph satisfies (O), since \( z \) is not adjacent to \( p_0 \) and \( p_2 \) by [12] and [8]. Hence, we can assume that \( v_1, v_2 \) and \( v_3 \) have a common neighbor \( w \), and thus \( \deg(v_2) = 3 \). Similarly, we conclude that \( L(v_1) = L(p_0) \cup L(v_2) \) (if not, we color \( v_1 \) with a color in \( L(v_1) \setminus (L(p_0) \cup L(v_2)) \)) and then consider \( G' = G - \{ v_1, v_2 \} \) and that \( L(v_3) = L(p_2) \cup L(v_2) \) (if not, we can color \( v_3 \) by a color in \( L(v_3) \setminus (L(p_2) \cup L(v_2)) \)) and then consider \( G' = G - \{ v_2, v_3 \} \). By [5], [5] and [12], \( w \) has no neighbor in \( P \). Let \( u \) be the vertex adjacent to \( w \) by the crossed edge, let \( \varphi \) be an \( L \)-coloring of \( w \) such that \( \varphi(w) \notin L(v_2) \) and let \( G' = G - \{ v_2, w \} \) Note that \( L_{\varphi} \) is a
If $m \in \{\phi \}_m$ colors with the property (b) and no colors with the property (a). If $m \phi \mid \mid$ and at most one color with the property (a). It follows that $y$ chosen by (X2) or (X4) and we have $\mid \mid$ is a contradiction. Therefore, $\mid \mid$ but in that case we have $\mid \mid$ of $\mid \mid$ -valid list assignment for $\mid \mid$ by (14) and contradicts the minimality of $\mid \mid$.

Therefore, both $X$ and $X'$ satisfy (A5) and $b = m + 2$. Moreover, \[5\] implies that the neighbor $w'$ of $v$ in $V(G_q) \{v_{m+1}\}$ is different from $w$ (the neighbor of $v_m$). Let $y$ be the vertex joined to $v_{m+1}$ by a crossed edge. If $|L(v_{m+1})| \neq 3$, then both $X$ and $X'$ are chosen by cases (X1) or (X3) and at most one color with the property (a). It is possible that $v$ is adjacent neither to $p_1$, $v_m$ and $v_{m+1}$, and by symmetry, the only possible neighbors of $w'$ in $F$ are $p_1$, $v_m$ and $v_{m+1}$, and by symmetry, the only possible neighbors of $w'$ in $F$ are $p_1$, $v_m$ and $v_{m+1}$, and by symmetry, the only possible neighbors of $w'$ in $F$ are $p_1$, $v_m$ and $v_{m+1}$.

Let $S_1 = L(v_{m-1})$ if $m = 3$ and $S_1 = L(v_{m-1}) \{p_0\}$ if $m = 2$. Let $S_2 = L(v_{b+1})$ if $b = s - 2$ and $S_2 = L(v_{b+1}) \{p_2\}$ if $b = s - 1$. By symmetry, we can assume that if $m = 2$, then $b = s - 1$. Let $S$ be the set of colors $c \in L(v_{m+1})$ such that either

(a) $L(v_{m+2}) = S_2 \cup \{c\}$, or

(b) $|L(v_m)| = 4$, $c \not\in S_1$ and $S_1 \cup \{c\} \subseteq L(v_m)$.

If $m = 2$, then we have $b = s - 1$, $|S_1| = |S_2| = 2$, there are at most two colors with the property (b) and no colors with the property (a). If $m = 3$, then $|S_1| = 3$ and $|S_2| \leq 3$, there is at most one color with the property (b) and at most one color with the property (a). It follows that $|S| \leq 2$. Let $\varphi$ be an $L$-coloring of $v_{m-1}$, $v_{m+1}$ and $v_{m+2}$ chosen so that $\varphi(v_{m+2}) \not\in S_2$, $\varphi(v_{m+1}) \not\in S$, $\varphi(v_{m-1}) \in S_1$ and $|L(v_m) \{\varphi(v_{m-1}), \varphi(v_{m+1})\}| \geq 3$. Note that the choices for $\varphi(v_{m+2})$ and $\varphi(v_{m+1})$ are possible, since $\varphi(v_{m+1})$ does not satisfy (a) and (b), respectively. Consider $G' = G - \{v_{m-1}, v_{m+1}, v_{m+2}\}$ with the list assignment $L_{\varphi}$. By \[12\], $v_{m-1}$ has no common neighbor with $v_{m+1}$ and none with $v_{m+2}$, and the only common neighbor of $v_{m+1}$ and $v_{m+2}$ is $w'$. Therefore, the only vertices with list of size three are $v_i$ if $m = 3$, $v_m$, $v_{m+3}$ and $w'$. Since $w'$ is not adjacent to $v_{m+3}$, $L_{\varphi}$ is $(P, N \cup \{y\}, \emptyset)$-valid. Furthermore, $y$ is adjacent neither to $p_0$ nor to $p_2$ by \[12\], hence $G'$ satisfies (O) by \[14\] and contradicts the minimality of $G$. This completes the proof of Theorem \[7\].
Proof of Theorem 4. Let $G$ be a graph with crossing number at most two. We may assume that $G$ is nonplanar. Consider a drawing of $G$ in the plane with one or two crossings and let $L$ be a list assignment such that each vertex has five admissible colors. Let $xy$ and $uv$ be two edges crossing each other at the crossing $q$. Suppose first that the edges $xy$ and $uv$ do not participate in another crossing. Now remove the two edges and add the edges $xu, uy, yv,$ and $vx$ (if they are not already present). This gives rise to a graph $G'$ with at most one crossing, and we can redraw it so that the cycle $xuyv$ bounds the outer face. Now we $\varphi$-precolor the path $xuy$ such that $\varphi(x) \neq \varphi(y)$, and give $v$ the list $L(v) \setminus \{\varphi(u)\}$. Theorem 7 now implies that $G'$ has a list coloring which in turn shows that $G$ is $L$-colorable.

If the edge $uv$ participates in another crossing, then $xy$ does not participate in another one. Suppose that the segment of $uv$ from $u$ to the crossing $q$ does not contain the other crossing. Then we proceed similarly as above: we remove the edges $xy$ and $uv$ and add edges $xu$ and $uy$. The resulting graph is planar and the path $P = xuy$ is part of a facial walk. Thus we may $\varphi$-precolor the path so that $\varphi(x) \neq \varphi(y)$ and then remove $\varphi(u)$ from the list of $v$. Now, we apply Theorem 7 with $N = \{v\}$ to obtain a coloring that again confirms that $G$ is $L$-colorable.

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