Velocity and acceleration statistics in particle-laden turbulent swirling flows

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We present a comparison of different particles’ velocity and acceleration statistics in two paradigmatic turbulent swirling flows: the von Kármán flow in a laboratory experiment, and the Taylor-Green flow in direct numerical simulations. Tracers, as well as inertial particles, are considered. Results indicate that, in spite of the differences in boundary conditions and forcing mechanisms, scaling properties and statistical quantities reveal similarities between both flows, pointing to new methods to calibrate and compare models for particles dynamics in numerical simulations, as well as to characterize the dynamics of particles in simulations and experiments.

I. INTRODUCTION

Turbulent flows are common in nature and industrial applications. One of their main properties is the enhancement of the mixing of quantities transported by the flow, and in recent years, significant advancements have been made in the study of turbulent particle-laden flows [1–3]. The modeling of such flows requires a wide variety of approximations, and their study in the laboratory has important consequences for flow characterization as well for practical applications. Examples of such applications include cloud dynamics and droplet formation [4], aerosol and pollution dispersion in the atmosphere [5], and nutrient transport in the oceans [6] among others [7]. In many cases, the study of particle-laden flows has focused on the paradigmatic case of isotropic and homogeneous turbulence, a landmark in the study of turbulence. In such case, experiments of particle-laden flows are often carried out in wind tunnels, while numerical simulations use delta-correlated random forcing to sustain the turbulence against dissipation [8–11].

However, in recent years significant advancements were made in the study of turbulent swirling flows in the laboratory using an experimental setup that allows generation of flows with high Reynolds numbers and with strong turbulent fluctuations superimposed on a well-defined mean flow. This includes experiments in water and air, and in different configurations (open or closed domains) and geometries (cylindrical or square cells), between two counter-rotating propellers, in a setup that generates a von Kármán flow [12–14]. The turbulence generated using this setup is not isotropic and homogeneous, and as a result it has been sometimes called “axisymmetric turbulence” [15]. Experiments using this setup have been employed to study statistics of turbulence [16], turbulent transport [17, 18], bistability and long-term memory [19], the emergence of singularities [20], and even dynamo action when conducting flows are used [21].

Studies of tracers and inertial particles in such experiments have confirmed that turbulence in this flow displays some specific properties, and that under some circumstances transport is dominated by the strain in the center of the domain, in what have been called “stagnation point turbulence” [22]. Evidence of anisotropy in the flow has also been reported, as well as some common behavior with observations of particle-laden flows in isotropic and homogeneous turbulence when the mean flow is statistically removed [17].

The von Kármán flow shares some similarities with a paradigmatic flow in the study of tur-
bulence in periodic boundary conditions: the Taylor-Green flow [23]. This flow, that displays multiple symmetries [24], has been used to study turbulence [24, 25], singularities of the Euler equation [26, 27], and dynamo action [28]. In studies of magnetohydrodynamic dynamo action, it was successfully used to compare with von Kármán experiments, reproducing many features observed in the laboratory except for those directly associated with the different boundary conditions in both flows [28, 29]. It is thus worth pointing out that while some recent numerical studies consider flows with more realistic mechanical forcing and boundaries (in comparison to the von Kármán experiments) [30, 31], the similarities between Taylor-Green and von Kármán flows still allow for interesting comparisons when it comes to attaining the largest possible Reynolds number at a fixed spatial resolution, as periodic boundary conditions are amenable to powerful and high order numerical methods.

In spite of these similarities, there are very few comparisons of Taylor-Green and von Kármán dynamics in the case of particle-laden flows. With this motivation, in this work we present a comparison of particles’ velocity and acceleration statistics in these two paradigmatic turbulent swirling flows, considering on one hand laboratory experiments, and on the other direct numerical simulations. Tracers and Lagrangian particles are compared, as well as the particular case of a large inertial particle. Numerically, particles are modeled using the Maxey-Riley equation [32] in the small particle approximation. The main objective is to characterize similarities and differences between the two approaches and to evaluate the possibility of using such a comparison to validate models for particle dynamics. We consider spectra, correlation functions, single and two-times statistics, and structure functions for the particles’ velocities and accelerations. In spite of the differences in boundary conditions and the forcing mechanisms, scaling properties and statistical quantities share interesting similarities between both flows, and also display a clear effect of the mean flow on particle dynamics which affect turbulent statistical properties. The comparison also allows us to disentangle contributions of the turbulence and of the mean flow to Lagrangian statistics of the particles, in particular for the often reported poor inertial range scaling of the second order Lagrangian structure function.

II. EXPERIMENTAL AND NUMERICAL SETUPS

A. The von Kármán flow experiment

The experimental setup comprises two facing disks of diameter $D = 19$ cm, separated by a vertical distance of $H = 20$ cm, and each fitted with 8 straight blades. The impellers are contained in a cell of square cross-section, with side $h = 20$ cm, giving access to an experimental volume of $(20 \times 20 \times 20)$ cm$^3$ where the flow can be measured. The total size of the cell is $(20 \times 20 \times 50)$ cm$^3$, leaving space on the back of the impellers for shafts that connect the impellers to motors, and for refrigeration coils that allow heat removal if needed. Each impeller is driven by an independent brushless rotary servomotor (Yaskawa SMGV-20D3A61, 1.8 kW), controlled by a servo controller (Yaskawa SGDV-8R4D01A) which provides access to the instantaneous velocity and torque of the motor. The cell is filled with distilled water from a double pass reverse osmosis system, to remove ions and dissolved or suspended solid particles from the working fluid. The setup is similar to those considered in previous laboratory studies of von Kármán flows (see, e.g., [18]).

In all configurations considered in this study, the two disks rotate in opposite directions with angular velocity $\pm \Omega_0$, stirring the working fluid in the cell. This generates two large counter-rotating circulation cells producing, on average, a strong shear layer at the mid-plane between the disks. (Its detailed dynamics, however, is rather complex and it has been shown that this shear
FIG. 1. (a) Experimental setup, with a schematic representation of the mean large-scale flow in the von Kármán experiment. $H$ is the separation between the two impellers, $D$ is the diameter of the impellers, $h$ is the horizontal length of the cell, and $f_0$ denotes the rotation frequency of the impellers. (b) Schematic top view of the setup, with the measuring configuration (not to scale). The cell is illuminated with two LED panels, and a fast camera captures the position of the particles.

layer gets deformed and fluctuates between different configurations over time [33]). A secondary circulation in the axial direction is also generated by the impellers, resulting in an full three-dimensional turbulent flow [34]. As a result, the flow has a mean macroscopic structure which is anisotropic: the large-scale structures in the directions parallel to the plane of the disks are larger than the structures in the axial direction. A schematic visualization of the setup and the mean flow generated is depicted in Fig. 1(a).

For each individual experimental run we seed the flow with either tracer or inertial particles, and stir the flow employing different values of the angular velocity $\Omega_0 = 2\pi f_0$ (expressed in rad/s), with $f_0$ being the frequency (in Hz). For practical reasons, we will also refer to the disks’ rotational velocity as measured in revolutions per minute (rpm), which will be herein denoted by $f_0'$, with $f_0' = 60 f_0$.

The tracer particles are neutrally-buoyant polyethylene micro-spheres (density equal to 1 g cm$^{-3}$) of diameter $d = 250 - 300$ µm, commercially available from Cospheric. These particles are commonly used in experiments as Lagrangian tracers [35]. Prior to suspension, these particles were coated with a biocompatible surfactant (Tween 80) in order to ensure proper placement in suspension. For these particles we explored three different rotation velocities; namely: $f_0' = 25, 50$ and 100 rpm (corresponding to $f_0 = 0.42, 0.83$ and 1.66 Hz, respectively). The particles verify $d/\eta \lesssim 5$, $\eta$ being the Kolmogorov dissipation length of the flow.

The inertial particles are 6 mm plastic spheres with density 0.92 g cm$^{-3}$. In this case, these inertial particles were 3D-printed using a thermoplastic polymer (acrylonitrile butadiene styrene, or ABS) and were injected in the flow generated by stirring exclusively at $f_0' = 50$ rpm.

In all cases, measurements of particles’ dynamics are carried out using shadow particle tracking velocimetry (SPTV) [36]. For this technique, the cell is illuminated from two adjacent sides using two $(25 \times 25)$ cm$^2$ LED panels (each 1880 lm, 22 W) and a fast camera captures the particle’s shadow projection over a bright background. Exploiting the discrete $\pi/2$ rotational symmetry of the setup about the $z$ axis, imposed by the square cross-section the cell, only a projection of the
U is the r.m.s. value of $\eta$ in the experiments, and measured directly from the injected power in the simulations. The Kolmogorov characteristic fluid velocity and length, respectively. Generally, $\ell$ is the frequency in s$^{-1}$.

The experiments can be characterized by two dimensionless numbers, one pertaining to the flow and another related to the particles’ dynamics. An integral Reynolds number for the experiment can be defined as

$$Re_{int} = \frac{2\pi f_0 (D/2)^2}{\nu},$$

(1)

where $\nu$ is the kinematic viscosity of water. The other important dimensionless parameter for the inertial particles is the Stokes number, which is usually defined as [37]

$$St = \frac{2R^2 (\rho_p / \rho + 1/2) u}{9 \nu} = T_p u / \ell,$$

(2)

where $R$ is the particle radius, $\rho_p$ is the particle density, $\rho$ is the fluid density, and $u$ and $\ell$ are a characteristic fluid velocity and length, respectively. Generally, $\ell$ and $u$ are chosen so that their

| Dataset | $f_0$ | $U$ | $L$ | $\tau_{int} f_0$ | $\varepsilon$ | $\eta$ | $Re_{int}$ | $Re_{part}$ | $R_{\lambda}$ | $\tau_{L}^{(\chi)} / \tau_{L}^{(\ell)}$ |
|---------|------|-----|-----|-----------------|--------------|-------|-------------|-------------|-------------|------------------|
| EXP25   | 25   | 0.034 | 0.19 | 0.34           | 7 x 10^{-4}  | 195   | 2.4 x 10^4  | 0.6 x 10^4  | 170         | 1.23             |
| EXP50   | 50   | 0.083 | 0.19 | 0.31           | 9.3 x 10^{-3} | 100   | 4.7 x 10^4  | 1.6 x 10^4  | 275         | 1.28             |
| EXP100  | 100  | 0.170 | 0.19 | 0.31           | 7.8 x 10^{-2} | 60    | 9.5 x 10^4  | 3.2 x 10^4  | 400         | 1.27             |
| DNS     | 1/2\pi | 0.904 | 2\pi | 0.35           | 2.4 x 10^{-1} | 4.4 x 10^{-3} | -       | 1.3 x 10^4  | 305         | 1.27             |

TABLE I. Values of the parameters for both experiments and simulations with tracers. DNS values are dimensionless. For the experiments, $f_0'$ corresponds to the rpm frequency of the disks ($f_0' = 60 f_0$, where $f_0$ is the frequency in s$^{-1}$). For the DNS, $f_0$ is the frequency associated to a large-scale eddy turn over time. $U$ is the r.m.s. value of $v_x$, the x component of the particles’ velocity, $L$ is the flow integral scale, and $\tau_{L}^{(\chi)}$ is the particle velocity autocorrelation time based on $v_x$. The energy injection rate is given by $\varepsilon = 1/2 U^2 / \tau_{L}^{(\chi)}$ in the experiments, and measured directly from the injected power in the simulations. The Kolmogorov dissipation scale is $\eta = (v^3 / \varepsilon)^{1/4}$. $Re_{int}$ and $Re_{part}$ are respectively the integral and tracer-based Reynolds numbers. The Taylor-based Reynolds number is $R_{\lambda} = \sqrt{15 U^4 / \nu \varepsilon}$, and $\tau_{L}^{(\chi)} / \tau_{L}^{(\ell)}$ is the ratio of the particles’ autocorrelation times based on $v_x$ and $v_z$.

Trajectories in the $xz$-plane is registered. Individual particles are tracked in this plane using a high-speed Photron FASTCAM SA3 camera with a resolution of $(1024 \times 1024)$ px$^2$ and 12-bit color depth. The camera is placed in front of the cell so that the region of observation $(15 \times 15 \times 15$ cm$^3$) covers nearly the whole experimental volume while warranting minimal optical distortion. Datasets are obtained using a frame rate of $f_s = 500$ Hz, which is adequate since we are not interested in fully resolving the dissipative time scales in the experiments.

For the tracers, there are approximately 100 particles being detected by our tracking algorithm simultaneously in each individual frame, resulting in $O(10^3)$ trajectories captured after multiple realizations of the experiment, with a mean duration per trajectory of $0.45 / f_0$. For the inertial particles, in contrast, only one sphere is present in the cell in each experimental run, so as to avoid possible interactions that would result from the presence of other particles. In this case its motion within the whole experimental volume ($20 \times 20 \times 20$ cm$^3$) is tracked using the fast single camera (but with a sampling frequency of $f_s = 125$ Hz). Multiple realizations of the experiment result in $O(10^3)$ trajectories with a mean duration per trajectory of $1 / f_0$. Finally, and irrespective of their nature (tracer or inertial), each particle instantaneous velocity is derived from its individual trajectory after applying a Gaussian filter.

The experiments can be characterized by two dimensionless numbers, one pertaining to the flow and another related to the particles’ dynamics. An integral Reynolds number for the experiment can be defined as

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$$St = \frac{2R^2 (\rho_p / \rho + 1/2) u}{9 \nu} = T_p u / \ell,$$

(2)

where $R$ is the particle radius, $\rho_p$ is the particle density, $\rho$ is the fluid density, and $u$ and $\ell$ are a characteristic fluid velocity and length, respectively. Generally, $\ell$ and $u$ are chosen so that their
quotient $\ell/u = \tau_\eta = (\nu/\varepsilon)^{1/2}$ is the Kolmogorov time scale of the flow; in that case we will use the notation $St_\eta$. Another possible choice is to use $\ell = L$ and $u = U$, both associated to the large scale motion; the Stokes number resulting from this choice will be denoted as $St_{\text{int}}$. Alternatively, an effective Stokes number may be also defined as

$$St_R = \frac{\tau_R}{\tau_\eta},$$

where $\tau_R$ is the turbulent turn over time associated with a scale $\ell$ of the order of the particle size, i.e., $\ell = R$, and therefore

$$\tau_R = \left(\frac{R^2}{\varepsilon}\right)^{1/3}.$$  

**B. Taylor-Green direct numerical simulations**

We performed direct numerical simulations (herein, DNSs) of the incompressible Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{F},$$

where $\mathbf{u}$ is the solenoidal fluid velocity field ($\nabla \cdot \mathbf{u} = 0$), $p$ is the pressure, $\nu$ is again the kinematic viscosity, and $\mathbf{F}$ is an external volumetric mechanical forcing. Equations are written in dimensionless units based on a unit length $L_0$ and a unit velocity $U_0$, and solved in a three-dimensional $2\pi$-periodic cubic box using a parallel pseudo-spectral method with the GHOST code [38]. A fixed spatial resolution of $N^3 = 768^3$ grid points is used. To mimic the geometry of the large-scale flow in the von Kármán experiments, the external forcing $\mathbf{F}$ is given by the Taylor-Green flow [25],

$$\begin{cases}
F_x &= F_0 \sin(k_F x) \cos(k_F y) \cos(k_F z), \\
F_y &= -F_0 \cos(k_F x) \sin(k_F y) \cos(k_F z), \\
F_z &= 0,
\end{cases}$$

with forcing wavenumber $k_F = 1$. Note that this forcing corresponds to a periodic array of counter-rotating large-scale vortices, which in the domain $[0, \pi] \times [0, \pi] \times [0, \pi]$ reduces to just two counter-rotating vortices separated vertically by a shear layer.

Such forcing is anisotropic: it is similar in the $x$ and $y$ directions (in fact, it has a discrete $\pi/2$ rotation symmetry about the $z$ axis as the von Kármán flow), and it injects no energy directly into the $z$ component of the velocity. As a result, the generated flow presents symmetries in a statistical sense (see [24, 25] for more details) while keeping its anisotropy. Also as a result of these symmetries, the Taylor-Green flow in the full $(2\pi)^3$ domain can be split into 8 cells, each consisting of two counter-rotating eddies akin to those generated in the von Kármán flow as mentioned above (see Fig. 2). As in the case of the von Kármán flow, the resulting Taylor-Green flow in this cell has a layer of strong shear in which the forcing is zero, and which separates two planes of maximum forcing. The instantaneous streamlines of the flow in a cell are shown in Figure 2(b). As it evolves in time, the flow generates a secondary circulation in the axial direction ($\hat{z}$), driven by pressure gradients (unlike its experimental counterpart, in which Ekman pumping plays a crucial role). As mentioned in the Introduction, due to both the geometrical similarities between the Taylor-Green and von Kármán flows and the convenience of having a similar flow with periodic boundary conditions for the numerical study of turbulence, this forcing has been
FIG. 2. (a) Schematic of the $2\pi$-periodic 3D domain used in the DNSs. When $z$ changes in $\pi$, the Taylor-Green vortices change their sense of rotation. A region of the domain is highlighted, where the flow has a large-scale structure reminiscent of the one in the laboratory experiment (although with different boundary conditions). (b) Instantaneous streamlines in a sub-region of the solving domain. Colors denote intensity of the horizontal ($x$) component of the (Eulerian) velocity field. Note the two large-scale counter-rotating eddies.

used in many cases to compare simulations with experimental data obtained from von Kármán setups [29, 39], specially for magnetohydrodynamic dynamo studies.

In our DNSs, the flow is first evolved until a turbulent steady state is reached. From that instant on, point particles are injected and evolved in time together with the flow, while computing their instantaneous position, velocity, and acceleration. Particles do not interact with each other, and their dynamics do not affect the flow evolution. Multiple simulations with different particles are then performed, in each case with $10^6$ particles.

We study the dynamics of two types of particles. Firstly, we consider Lagrangian tracers which evolve according to

$$\frac{dx_p}{dt} = u(x_p, t),$$

where $x_p(t)$ is the position of the Lagrangian tracer at time $t$, and $u(x_p, t)$ is the velocity of the fluid element at position $x_p(t)$. Secondly, we also examine the evolution of inertial, neutrally buoyant (point) particles under the Maxey-Riley equation [32, 37]

$$\frac{dx_p}{dt} = v(t), \quad \frac{dv}{dt} = \frac{1}{\tau_p} (u(x_p, t) - v(t)),$$

where $v(t)$ and $\tau_p$ represent the particle velocity and Stokes time, respectively. Both for the tracers and the inertial particles, integration of the corresponding dynamical equations (either Eq.(7) or Eq. (8)) is performed using a high-order Runge-Kutta time stepping scheme combined with a high-order three-dimensional spatial spline interpolation to obtain the fluid velocity $u(x_p, t)$ at the position of the particles [40].

For the inertial particles the Stokes number $St$ is defined as the ratio of two characteristic times: the relaxation time of the particle, $\tau_p$, and some characteristic time of the fluid $\tau_\ell$ at scale $\ell$, so that $St = \tau_p / \tau_\ell$. Just as in Sec. II A, when $\tau_\ell$ is the the Kolmogorov dissipation time scale $\tau_\eta$, we obtain $St_\eta = \tau_p / \tau_\eta$. When $\tau_\ell$ is evaluated at the flow integral scale $L$, we obtain $St_{int} = \tau_p / \tau_L$. And
finally, it is worth noting that Eq. (4) allows us to estimate an effective radius $R$ for the (otherwise point) inertial particles in the simulations, as

$$ R = \tau^3 / 2 \epsilon^{1/2}. $$

(9)

From this relation, we can also compute an effective Stokes number $St_R$ using Eq. (3).

III. LAGRANGIAN TRACERS

A. Velocity autocorrelation functions

We start by characterizing tracers’ dynamics by means of the Lagrangian autocorrelation function of particles’ velocities. For a single Cartesian component of the tracer velocity $v_i$, the Lagrangian (normalized) autocorrelation function is given by

$$ R_{L}^{(i)}(\tau) \equiv \frac{C_{v}^{(i)}(\tau)}{C_{v}^{(i)}(0)} = \frac{\langle v_{i}(t)v_{i}(t+\tau) \rangle}{\langle v_{i}^{2}(t) \rangle} $$

(10)

where the brackets $\langle \cdot \rangle$ denote averages over the time $t$ and over all trajectories, and where $\tau$ is the time lag. To compute this magnitude, experimental velocity tracks with a duration longer than $1/f_0$ were included in the ensemble averaging. No significant bias was observed by doing so. The resulting autocorrelation functions for both the experiments and the simulations are shown in Fig. 3, for the horizontal ($i = x$) and axial ($i = z$) velocity components.

For the sake of clarity and consistency in the graphical representation of our results, we shall adopt the following convention throughout the rest of the paper. Symbols (lines) are used to identify experimental (numerical) results. Full symbols or continuous lines (depending on whether the data is experimental or numerical) denote the horizontal component ($x$), whereas empty symbols or dashed lines represent the axial component ($z$). In the case of the experiments, runs performed at $f_0' = 25, 50, \text{ and } 100\text{ rpm}$ are symbolized by circles, squares, and upside triangles respectively.

Firstly it is worth noting that, both in the experiments and in the simulations, the Lagrangian autocorrelation function becomes negative for long times and does not converge rapidly to zero afterwards. This can be interpreted as an effect of the mean flow in the system. Although the effects
of the mean flow in the statistics can be partially alleviated by studying particles’ statistics in a smaller subdomain of the von Kármán cell or by carefully removing mean flow components (see, e.g., [17, 41]), in our case we wish to compare quantities in simulations and experiments without removing the effects associated with the large-scale flow, so as to identify global similarities and differences between both setups. Secondly, we observe that for short times the autocorrelation of the axial velocity component, $R_L^{(z)}$, decays faster with $\tau$ than its horizontal counterpart, $R_L^{(x)}$, and that this behavior is also common to both the experiments (for all values of the Reynolds number considered in this study) and the simulation.

From Fig. 3 we can estimate $\tau_L^{(i)}$, the tracers’ component-wise velocity autocorrelation time, as the instant corresponding to the first zero-crossing of the corresponding autocorrelation function. The differences in the decay of $R_L^{(x)}$ and $R_L^{(z)}$ with $\tau$ show that the ratio $\tau_L^{(x)}/\tau_L^{(z)} > 1$, which is a signature of the flow anisotropy and of the effect of the large-scale circulation. In particular, its value (shown in the right-most column of Table I) is practically the same for all datasets, numerical or experimental. Moreover, $\tau_L^{(i)} f_0$ seems to become less sensitive to the Reynolds number as the latter is increased (see Table I), and is comparable for the both the experiments and the DNS. A related point to consider is the fact that the autocorrelation functions for $f_0' = 50$ rpm and $f_0' = 100$ rpm (i.e., those with the highest values of the Reynolds numbers considered) collapse component-wise for nearly all time lags. This all could indicate that the main contributor to the decorrelation of particles’ velocities is the mean flow, which is expected to vary less as the Reynolds number becomes sufficiently large.

The Lagrangian autocorrelation time, together with measurements of the tracers’ velocities, lead us to define quantities that will allow for comparisons of experiments and simulations on equal footing. We start by estimating the energy injection rate $\varepsilon$ in the experiments from the r.m.s. value of the horizontal component of the particles’ velocity, $v_x$, denoted herein by $U$, and from the
This estimation of the energy injection rate is an increasing fraction (with increasing \( f_0 \)) of the energy injection rate measured from the power consumption of the motors (as estimated, e.g., in [42, 43]), but in our experiments the former does not scale as expected with \( f_0 \). This behavior could be a consequence of moderate Reynolds number effects (arising, e.g., from energy losses associated to friction and boundary layer effects) but could also be attributed to specific properties of von Kármán flows in square cells, such as those discussed in [22]. Most certainly, the former effects could be explained by the fact that, in order to facilitate comparisons with numerical simulations, we employ values of \( f_0 \) lying on the lower range of those considered in previous studies of von Kármán flows. The latter effects result from the choice of using a square cell, which also facilitates comparisons with the simulations. It is also interesting that estimating \( \varepsilon \) from the zero crossing of the acceleration autocorrelation function, as it is done in [22], yields values similar to those obtained by employing the definition of \( \varepsilon \) given by Eq. (11). In contrast, and as previously mentioned, the numerical simulations provide us with direct access to \( \varepsilon \) from the computation of the power injected by the forcing.

Using the r.m.s. velocity \( U \) we can define a Reynolds number based on the particles’ velocity as

\[
Re_{\text{part}} = \frac{UL}{\nu},
\]

(12)

where \( L \) is associated to the forcing scale and is provided in Table I. For the simulations, \( Re_{\text{int}} \approx Re_{\text{part}} \). Using \( U \) and \( \varepsilon \), the Taylor-based Reynolds number can be also defined as

\[
Re_\lambda = \sqrt{\frac{15U^4}{\nu\varepsilon}}.
\]

(13)

All the relevant parameters for the experimental and numerical datasets are listed in Table I. Note that from the values of \( Re_{\text{part}} \) and \( Re_\lambda \), EXP50 (i.e., the experiment with \( f_0 = 50 \) rpm) and the DNS share comparable values of the Reynolds number (note other dimensionless numbers are also comparable between this experiment and the simulation). In the following, and in light of these similarities, comparisons between experiments and simulations for Lagrangian tracers will focus on these two cases.

B. Velocity power spectra

The power spectra of the tracers’ velocities for the horizontal and axial velocity components in the three experiments and in the DNS are depicted in Fig. 4. For the experiments, Fig. 4(a) shows that spectra are compatible with a scaling law over a frequency range exceeding a decade. For low frequencies (i.e., long time scales associated with large scale motions) the power law exponent is close to \(-5/3\) (see the inset for compensated spectra). For intermediate frequencies, the power law may be compatible with a \(-2\) scaling, albeit in a shorter range of wavenumbers (see also the inset). This behavior is somehow unexpected: a Kolmogorov scaling of the Eulerian energy spectrum \( E(k) \sim k^{-5/3} \) is expected to yield a Lagrangian spectrum \( E(f) \sim f^{-2} \), as Lagrangian trajectories are not expected to be affected by sweeping (see, for example, [44, 45] and references therein, and the discussion in Sec. III C) which results in the \( \sim f^{-5/3} \) Eulerian frequency spectrum. As will be
confirmed by the second order Lagrangian structure functions and the acceleration spectrum, our data indicates that sweeping by the large-scale flow plays a relevant role in the particle evolution even in the Lagrangian frame.

In Fig. 4(b) we compare the power spectra of the tracer’s velocities from EXP50 (the experiment with \(f_0 = 50\) rpm) with those resulting from the simulations. As mentioned before, these datasets have the closest matching values of \(Re_{\text{part}}\) and \(Re_\lambda\). All curves collapse for nearly all time scales (including forcing and dissipative time scales), and scaling ranges compatible with both power laws (\(-5/3\) and \(-2\)) are again identifiable in both datasets. The DNS spectra are slightly more anisotropic at the largest scales, as the curves for the \(x\) and \(z\) velocity components have a larger relative difference in their amplitudes. Except for this difference, the power spectra of the tracers’ velocities in the experiment and the simulation show good agreement.
FIG. 7. Logarithmic derivatives of the particle velocity structure functions of order $p = 1$ and $p = 2$ (see labels in the inset). Each individual panel gathers both experimental and numerical results for a given Cartesian component of the velocity; $x$ component in panel (a), and $z$ component in panel (b). Values of $1/3$ and $2/3$ are indicated as references by the horizontal dashed lines.

C. Structure functions

The tracers’ (component-wise) second order velocity structure function is given by

$$S_2^{(i)}(\tau) = \langle [v_i(t + \tau) - v_i(t)]^2 \rangle, \quad (14)$$

where again the index $i$ denotes the Cartesian component of the velocity considered, and $\tau$ is the time lag. In Fig. 5 we show these structure functions for each component, both for the experiments and the DNS. While for very small time lags we can expect $S_2^{(i)} \sim \tau^2$ from the regularity of the velocity field, for intermediate time lags (i.e., in the turbulent inertial range) the prediction for isotropic and homogeneous turbulence is $S_2^{(i)} \sim \varepsilon \tau$ [46]. Such a behavior is compatible with the prediction for the Lagrangian energy spectrum $E(f) \sim f^{-2}$. As a result, in Fig. 5 the structure functions are compensated by $\tau \varepsilon$. A very short range with constant $S_2^{(i)}/(\tau \varepsilon)$ is seen, slightly more clearly for EXP100 with $f_0' = 100$ rpm. The lack of clear scaling has already been pointed out in the literature, for both numerical and experimental data [47–49]. It has been reported that the reason for this can be that this quantity mixes low-frequency and inertial-range fluctuations [50], and that it converges very slowly towards its asymptotic value, reaching a plateau only for $Re_\lambda \gtrsim 3 \times 10^4$ [51].

In the von Kármán and Taylor-Green flows, the anomaly in the scaling of the Lagrangian structure function may be also associated to the effect of the mean flow (which has low-frequency components). Indeed, the mean flow can introduce a decorrelation time associated with a large-scale
FIG. 8. Particle acceleration spectra normalized by the energy injection rate $\varepsilon$, in (a) experiments, and (b) EXP50 and the DNS. The dashed horizontal line indicates the amplitude $C_0 = 6.9$, which corresponds to the asymptotic value expected for the amplitude of the acceleration spectrum in the Lagrangian inertial range. In panel (b) a $\omega^2$ slope is also indicated as a reference.

In Fig. 6 we show the second order Lagrangian structure functions $S_2(\tau)$ compensated by this prediction, both for all experimental datasets, as well as comparing EXP50 data with the DNS. A range of time lags with approximately constant compensated structure functions can be seen; its width being approximately independent of the Reynolds number. This is consistent with the interpretation that the observed scaling of $S_2(\tau)$ is associated with large-scale flow effects, as the structure of the mean flow is fixed by the geometry of the setup in the experiments or by the turnover time, resulting from the sweeping of the tracers by the largest eddies. This argument, which is often considered in the Eulerian frame [52, 53], may be also important in Lagrangian measurements as indicated by recent models of Lagrangian dispersion [54, 55]. Kraichnan already noted this when developing the Lagrangian-History Direct Interaction Approximation (LHDIA), and introduced mixed Eulerian-Lagrangian correlations (based on his so-called “generalized velocity”) to fully remove the advection of eddies by a large-scale flow [56, 57]. In a similar bridging effort between the Eulerian and Lagrangian descriptions, Belinicher and L’Vov [58] showed that sweeping effects can be eliminated by using a reference frame (termed the quasi-Lagrangian reference frame) shared by all fluid points inside a large eddy (see also [59, 60], and references therein).

If sweeping by the large-scale eddies is indeed affecting the second order Lagrangian statistics, we can estimate its effect in the context of Kolmogorov’s theory. In the inertial range, the second order Eulerian structure function scales as $S_2(\ell) \sim (\varepsilon \ell)^{2/3}$. If instead of assuming that decorrelation in the measurements is controlled by the local turnover time we consider the decorrelation from the large-scale eddies, then $\tau \sim \ell/U$ instead of $\tau \sim \ell/u_\ell$ (where $u_\ell$ is the characteristic velocity at scale $\ell$). As a result, it follows that

$$S_2(\tau) \sim (U \varepsilon \tau)^{2/3}.$$  \hspace{1cm} (15)
volumetric forcing in the DNS. Indeed, energy is injected at approximately the same scales in all experimental runs, and the mean flow has little dependence with the Reynolds number when turbulence has reached a fully developed state, as already discussed. The observed plateau is slightly wider for the horizontal velocity component, which is also consistent with the anisotropy generated by the mean flow at the largest scales (large-scale vortices in von Kármán and Taylor-Green flows display larger correlation length and correlation time in the horizontal than in the vertical direction). Both the experiments and the DNS show similar levels of anisotropy (although, as before, the DNS results are slightly more anisotropic). All these observations reinforce the idea that advection of the tracers by the mean Eulerian flow affect second order Lagrangian statistical measurements, and could also be the source of the very limited $S_2(\tau) \sim \epsilon \tau$ scaling reported in previous experiments [12, 49].

The anomalous behavior of the second order Lagrangian structure function, and the impact of the mean flow, can be also observed in other statistical moments of the tracers’ velocity. Statistics of the tracers’ for other orders can be accessed, e.g., considering the Lagrangian structure function of order $p$,

$$S_p^{(i)}(\tau) = \langle |v_i(t+\tau) - v_i(t)|^p \rangle.$$ (16)

In order to asse the statistical properties locally, we compute the local scaling exponent (LSE) given by the logarithmic derivative of the corresponding structure function:

$$\xi_p^{(i)}(\tau) = \frac{d \log S_p^{(i)}(\tau)}{d \log(\tau)},$$ (17)

for $p = 1$ and 2. LSEs are more conducive to analysing scaling properties scale-by-scale as they are expected to remove large-order non-universal contributions coming from the overall prefactors in the structure functions (see, e.g., [61]). In particular, the case $p = 2$ provides us with an alternative way to analyze the behavior of $S_2^{(i)}(\tau)$. The results for all the experiments and the simulation are shown in Fig. 7, each panel corresponding to a different Cartesian velocity component. In the Lagrangian inertial range and in the absence of intermittency, $\xi_{p} \approx p/2$ is expected, whereas in the sweeping-dominated range we anticipate $\xi_{p} \approx 3p$. For $\tau / \tau_{L}^{(x)} \approx 0.1$ the local slopes cross the Lagrangian inertial range prediction, while for $\tau / \tau_{L}^{(x)} \lesssim 1$ the local slopes are consistent with sweeping for both $p = 1$ and 2. Moreover, and as noted for other quantities, a better scaling is seen (for both ranges) in the horizontal component of the velocity, both for the experiments and the DNS, with clear differences between the two velocity components due to anisotropy.

### D. Acceleration spectra

As discussed above, the reason for the anomalous behavior of the energy spectra $E(f)$ and of the structure functions $S_2(\tau)$ observed in the literature and in our datasets may be contamination of the scaling by mean flow effects. It has already been proposed by other authors (albeit not identifying the source of contamination to sweeping) that second order structure functions mix low frequency fluctuations with inertial range fluctuations [50], and that once the large scale contamination is removed, a plateau should emerge even at $Re\lambda \approx 400$ [62]. As a result, computation of the so-called acceleration spectrum was proposed as a solution, as this spectrum is expected to better disentangle the contributions from different time scales, and as a result show a clearer Lagrangian inertial range scaling [51]. Recently, the acceleration spectrum was computed for data
from von Kármán flows \cite{41}, and it was found that the anisotropy of the flow is indeed contained in low frequencies, and that the spectra are isotropic for frequencies in the dissipative range.

The tracers’ one-dimensional acceleration spectrum, computed from the particles’ acceleration autocorrelation function \( C_A^{(i)}(\tau) \), is defined as

\[
\phi_A^{(i)}(\omega) = \frac{2}{\pi} \int_0^\infty C_A^{(i)}(\tau) \cos(\omega \tau) d\tau = \frac{2}{\pi} \omega^2 \int_0^\infty C_v^{(i)}(\tau) \cos(\omega \tau) d\tau, \tag{18}
\]

where \( \omega = 2\pi f \) as before. Note the second expression allows computation of \( \phi_A^{(i)}(\omega) \) directly from the velocity autocorrelation function \( C_v^{(i)}(\tau) \). This spectrum is expected to scale as \( \pi \phi_A(\omega)/\varepsilon \sim \omega^2 \) for \( \omega \to 0 \), and as \( \pi \phi_A(\omega)/\varepsilon = C_0 \) in the inertial range, i.e., for \( \tau_L^{-1} \ll \omega \ll \tau_\eta^{-1} \). The value of \( C_0 \), estimated by extrapolating the peak in \( S_2(\tau)/\varepsilon \tau \), is found to be \( C_0 \approx 6.9 \) (see \cite{51}). As mentioned before, in previous studies it was noted that this spectrum converges much faster to a plateau than \( S_2(\tau) \), and thus it should be a better indicator of the existence of a Lagrangian “Kolmogorov-like” inertial range.

The normalized acceleration spectra associated to our experimental datasets are shown in Fig. 8(a), while the DNS data is compared with results from EXP50 in Fig. 8(b). For both the experiments and the simulation, we obtain a plateau spanning almost one decade in frequency range with an amplitude close to the predicted value for \( C_0 \). The extension of the plateau appears to grow with the Reynolds number. For the DNS, where larger times have been sampled, at small frequencies \( \omega \tau_L^{(x)} \approx 1 \) the spectrum grows as \( \omega^2 \) as expected. More importantly, the numerical and experimental data show a remarkable collapse, sharing characteristics such as the growth at low frequencies, a comparable span of time scales for the inertial range, and a similar drop at high frequencies. The large-scale anisotropy of the flow can be also identified in these curves: the plateau is wider for the horizontal component although, as in all previous cases, the anisotropy at large scales is more pronounced in our DNS than in the experiments.

### IV. INERTIAL PARTICLES

The good agreement between the statistical properties of tracers’ velocity and acceleration in the von Kármán experiments and in the Taylor-Green simulations, even when considering large scales associated to the mean flow and in spite of the conspicuous differences in the boundary conditions, encourages us to perform comparisons between inertial particles in the laboratory flow and the simulations. The motivation behind the comparisons in this section is to explore the possibility of validating models for particles (and their parameters) using statistical comparisons between two reminiscent flows. Thus, we consider a large particle in the experiment, and we compare it with a point particle in simulations looking for effective values of the Stokes numbers that make the statistical properties between the two cases comparable. In future works, other models (e.g., considering finite particle size effects \cite{32}) can be used to quantify their effectiveness.

On the one hand, the dynamics of a large particle (with diameter 6 mm) is studied in the experiment, using \( f_0' = 50 \) rpm. As shown in the previous section, this rotation frequency generates an experimental flow with a \( Re_{\text{part}} \) value similar to that reached in the simulations. On the other hand, in the simulations four values of \( \tau_p \) are explored; namely: \( \tau_p = 0.2, 0.5, 1.5, \) and \( 3.0 \) (in dimensionless units). For ease of reference, simulations for each of these values of \( \tau_p \) will be termed DNS0.2, DNS0.5, DNS1.5, and DNS3.0, respectively. Inertial particles were evolved in the same turbulent flow employed in the study of Lagrangian tracers described in Section III.
Panel (a) presents the results for the four numerical simulations considered. As for the experiments and isotropic turbulence [63], inertial/finite-sized particles in experiments [13], and in numerical simulations of homogeneous tracers, deviations resulting, e.g., in sub-Gaussian statistics have been reported in the case of the velocity of the inertial particles. Even though a Gaussian distribution is expected in the case of datasets.

\[ R_{L}(\tau) \text{ is the ratio of the correlation times } \tau_{L}^{(1)} / \tau_{L}^{(2)} \text{, obtained from the zero-crossing of the normalized autocorrelation functions } R_{L}^{(1)}(\tau). \text{ Then inertial particles' response to the experimental data. Using Eqs. (2) and (3) are shown in Table II. As is usually the case in studies of particle-laden turbulent flows, the values of the Stokes numbers in the experiment and in the simulations can be vastly different depending on the definition used. The particle Stokes time } T_{p} \text{ computed as in Eq. (2) yields } T_{p} = 2840 \text{ ms, resulting in Stokes numbers } St_{f} \text{ and } St_{int} \text{ with differences of several orders of magnitude between the experiment and the DNS data (even though, as will be shown next, statistical results between the experiment and the simulations are compatible in many cases). Instead, the Stokes number } St_{R} \text{ defined as in Eqs. (3) and (4) (based on } \tau_{R} = 100 \text{ ms for the experiments) results in values which are comparable with those of the simulations showing closest agreement to the experimental data. Using } St_{R}, \text{ values of the experimental data lay between DNS0.2 and DNS0.5, the numerical simulations with } \tau_{p} = 0.2 \text{ and 0.5 respectively. Note that the ratio } R/\eta, \text{ as well as other dimensionless numbers in Table II, are also of the same order of magnitude for these datasets.}

### A. Velocity probability density functions

We begin by studying the probability density functions (PDFs) of the Cartesian components of the velocity of the inertial particles. Even though a Gaussian distribution is expected in the case of tracers, deviations resulting, e.g., in sub-Gaussian statistics have been reported in the case of inertial/finite-sized particles in experiments [13], and in numerical simulations of homogeneous and isotropic turbulence [63].

Probability density functions, normalized by their standard deviation, are shown in Figure 9. Panel (a) presents the results for the four numerical simulations considered. As for the experiments

| Dataset    | $f_{0}$ | $\tau_{p}^{*}$ | $R/\eta$ | $St_{f}$ | $St_{int}$ | $St_{R}$ | $\tau_{L}^{(x)}$ | $\tau_{L}^{(x)}/\tau_{L}^{(z)}$ | $T^{(x)} f_{0}$ | $T^{(x)} / T^{(z)}$ |
|------------|---------|----------------|----------|----------|-----------|-----------|------------------|----------------|----------------|----------------|
| EXP6       | 1/2π    | 0.2            | 10.2     | 4.7      | 0.029     | 4.7       | 0.43             | 1.19           | 0.18           | 1.39           |
| DNS0.2     | 1/2π    | 0.2            | 10.2     | 4.7      | 0.029     | 4.7       | 0.43             | 1.19           | 0.18           | 1.39           |
| DNS0.5     | 1/2π    | 0.5            | 37.1     | 11.1     | 0.069     | 11.1      | 0.48             | 1.22           | 0.19           | 1.26           |
| DNS1.5     | 1/2π    | 1.5            | 201.0    | 34.3     | 0.22      | 34.3      | 0.58             | 1.09           | 0.27           | 1.15           |
| DNS3.0     | 1/2π    | 3.0            | 574.1    | 69.1     | 0.43      | 69.1      | 0.72             | 1.00           | 0.36           | 1.09           |
both the Stokes number based on the particle radius, \( S_t R \), and the ratio \( R/\eta \), lie between the corresponding values of simulations DNS0.2 and DNS0.5 (see Table II), Fig. 9(b) offers a comparison between these two simulations and the experiment.

We observe a sub-Gaussian tendency in the distributions obtained from the experimental data, which is well captured by the inertial point-particles in the simulations DNS0.2 and DNS0.5 (see Fig. 9(b)). The values of the kurtosis \( \kappa \) for those datasets are \( \kappa^{DNS0.2}_x = 2.62 \), \( \kappa^{DNS0.5}_x = 2.62 \), and \( \kappa^{EXP6}_x = 2.56 \) for the horizontal (x) component; whereas for the axial (z) component we obtain \( \kappa^{DNS0.2}_z = 2.96 \), \( \kappa^{DNS0.5}_z = 3.14 \), and \( \kappa^{EXP6}_z = 3.02 \). Incidentally, the flatness values for the horizontal component of the velocity are very close to those reported in [13]. The effects of the large-scale anisotropy of the flow are evident in the PDFs: while the distribution for the horizontal component is sub-Gaussian, the statistics in the axial direction presents a kurtosis close to the Gaussian value. For the numerical data, it is also interesting that as the particle relaxation time is further increased (or equivalently, the Stokes numbers are increased), tails in the distributions become heavier, as is clearly seen in Fig. 9(a). This behavior could be explained by a preferential sampling of the flow by the inertial particles, further enhanced by the particle increasing inertia [3, 64]. However, these tails in the PDFs of simulations DNS1.5 and DNS3.0 (with larger Stokes numbers) deviate significantly from the data from EXP6.

B. Velocity power spectra

The inertial particles’ velocity power spectrum was computed for the x and z components of the velocity. The results for the DNSs are shown in Fig. 10(a). A power law compatible with a \(-5/3\) exponent is observed in the data corresponding to DNS0.2 and DNS0.5 in a short frequency range. Moreover, as \( \tau_p \) increases in the simulations, no clear power law is observed. This might be linked to the fact that as the particle response time grows, the particle becomes less sensitive to fluctuations in frequencies \( f \gg 1/\tau_p \). To illustrate this we can consider fields in Eq. (8) as random
variables, and Fourier transform this equation. Taking the absolute value results in

$$|\hat{v}_f|^2 \leq |\hat{u}_f|^2 \frac{1}{1 + (2\pi f \tau_p)^2},$$

(19)

where $\hat{v}_f$ and $\hat{u}_f$ are respectively the Fourier transforms of $v$ and $u$. Note this equation can be interpreted as a filter: fluctuations in the fluid velocity $u$ (at the particle position) with frequency $f \gg 1/\tau_p$ are attenuated in $v$. Moreover, assuming that the particle samples the flow in the same way as a tracer (which is not entirely correct, as preferential flow sampling has been observed for particles with inertia and finite-size [2]), then $u(x_p, t)$ should have the same spectral properties as the power spectrum of the tracer’s velocity. Under these hypothesis, $v$ should have a power spectrum similar to the spectrum in Fig. 4 at low frequencies, and damped amplitudes for large frequencies (compared with $1/\tau_p$). Increasing $\tau_p$ should also result in a stronger filter acting at at smaller frequencies. Even though these arguments neglect the effect of preferential sampling, the conclusions are in qualitative agreement with the observed spectra.

Figure 10(b) exhibits the experimental data compared with the data from DNS0.2 and DNS0.5. The experimental data presents a power law compatible with $-5/3$ for almost a decade of frequencies. The $-2$ power law, which was present in the tracers measurements, appears here only for a very short range. The particle’s finite-size effectively filters the fluctuations at intermediate frequencies corresponding to the inertial range. Consequently, the $-2$ range shortens significantly when compared with the tracers. Both at low and at intermediate frequencies (up to $f/f_0 \approx 8$) the DNSs and the experimental data are in good agreement (slightly better when EXP6 is compared with DNS0.2). This again indicates, as with the PDFs of the particles’ velocities, that numerical simulations of point particles in a Taylor-Green flow can mimic some statistical properties observed in the von Kármán periments when the Stokes number is estimated using the particles’ effective response time $\tau_R$.

Moreover, note that even though the numerical datasets DNS0.2 and DNS0.5 have similar $Str_R$ and $R/\eta$, the “filtering” of the spectrum observed in the experiment is sharper: while the DNS and
experimental spectra coincide up to $f/f_0 \approx 8$, for larger frequencies the experimental spectrum decays much faster than in the simulations. This is an indication that to fully capture the behavior of the particle in the experiments other effects need to be considered in the simulations, such as buoyancy, added mass effects, or other effects related to the finite size of the particles \cite{32}. However, the behavior observed in the experiments can be also mimicked if the filtering of the fluid velocity by the particle is of the form

$$|\hat{v}_f|^2 = \frac{|\hat{u}_f|^2}{1 + (2\pi f \tau_p)^\alpha},$$

(20)

where $\alpha$ controls how abruptly the inertial particle power spectrum deviates from the fluid velocity power spectrum. As $\alpha$ grows, the spectrum $|\hat{v}_f|^2$ retains the behavior of $|\hat{u}_f|^2$ better up to $f \approx 1/\tau_p$, and decays faster for $f \gg 1/\tau_p$. Note the model in Eq. (8) can be also modified to result in such a sharper decay.

C. Velocity autocorrelation functions

The inertial particles’ velocity autocorrelation function is computed using the definition in Eq. (10). The curves for the DNSs are shown in Fig. 11(a). It can be seen that, the larger $\tau_p$, the smaller the anisotropy between the $x$ and $z$ coordinates: the zero-crossing times of the two velocity components become more similar (see values in Table II). At the same time, as $\tau_p$ increases,
the particles decorrelate more slowly. This is in agreement with the simple “filter” model discussed in the previous section. Since the velocity autocorrelation function is related to the energy spectrum via the Wiener-Khinchin theorem, the Fourier transform of the expression in Eq. (19) is the velocity autocorrelation function $C_v(\tau)$ associated to that spectrum. By numerical computation of the Fourier transform of Eq. (19) (using the spectrum $|\hat{u}_f|^2$ obtained from the tracers), we can confirm that the autocorrelation decays faster for smaller values of $\tau_p$, which is consistent with the behavior observed in the data.

In Fig. 11(b), the experimental data is compared with the numerical datasets DNS0.2 and DNS0.5. A similar decay is observed in the EXP6 and DNS0.5 curves, specially for time lags $\tau f_0 \leq 0.3$. The zero-crossing times $\tau^{(i)}_L$ (for $i = x$ or $z$) are again comparable between the experiment and simulations DNS0.2 and DNS0.5 (see also Table II). An even better agreement is obtained if instead of $\tau^{(i)}_L$, we compute a component-wise “particle integral time” as

$$T^{(i)} = \int_0^{\tau_{05}} R^{(i)}_L(\tau) \, d\tau,$$

where the time $\tau_{05}$ is the time needed for $R^{(i)}_L(\tau)$ to decrease by 95%. This definition was introduced by Machicoane and Volk [17] to better quantify decorrelation times of inertial particles in von Kármán experiments; here we use the same definition but, following our motivation to compare bulk flows, we do not apply any specific method to try to reduce the effects of mean flow contributions. The values for this time are also shown in Table II; we find that the product $T^{(x)} f_0$, as well as the ratio $T^{(x)} / T^{(x)}$, are similar for EXP6 and DNS0.5, accounting for the similar decay of $R_L(\tau)$, and confirming once again certain statistical agreement between the behavior of inertial particles in the von Kármán experiment, and of the modeled particles in the Taylor-Green simulation.
D. Structure functions

The one-dimensional second order velocity structure function is calculated as in Eq. (14) for the inertial particles. The numerical data, compensated by the Lagrangian prediction for the inertial range, is shown in Fig. 12(a). No plateau is present, and the amplitude of the curves decreases with increasing $\tau_p$. The absence of a plateau (which is already barely visible in tracers' measurements), is also a consequence of the particles’ insensitivity to fluctuations in time scales $\tau < \tau_p$, combined with the slow convergence of $S_2(\tau)$ towards its asymptotic value in the inertial range discussed in Sec. III. On the other hand, as $S_2(\tau) = 2(C_v(0)) - C_v(\tau)$, the numerical estimation of $S_2(\tau)$ from Eq. (19) indicates that increasing $\tau_p$ has the same effect observed in our data: the amplitude of the curves decreases as $\tau_p$ grows. Since $S_2(\tau)$ is a measure of how disperse the particle’s velocity increments are for a given time lag $\tau$, a decrease in its amplitude can be also thought of as a smoothing of the velocity signal.

In Figure 12(b), the compensated data from EXP6 is compared with the numerical data from DNS0.2 and DNS0.5. The overall shape of the curves is similar, notably they reach their maximum value at nearly the same value for the ratio $\tau / T(x)$. Note in this case the time axis is normalized by $T(x)$ instead of using $\tau_L$, as it was shown that the former time captures better the similarities in the decay of the velocity autocorrelation functions. The experimental data, as in the case of the tracers, displays a plateau for a narrow range of frequencies, and is less anisotropic than that obtained from the simulations.

E. Acceleration spectra

Finally, the inertial particles’ acceleration spectrum is computed from the velocity autocorrelation function using the definition in Eq. (18). The spectra, normalized as for the case of the tracers, is plotted in Figure 13(a) for the DNSs. For DNS0.2, the dataset with the smallest value of $\tau_p$, a narrow plateau may be present in the vicinity of $\omega T(x) \approx 3$. No plateau is observed in the rest of the curves, which is consistent with the absence of clear scaling laws in the velocity power spec-
trum of the inertial particles. At the same time, the amplitude of the curves decreases as $\tau_p$ grows. Since we can estimate $\sqrt{\langle a^2(\tau) \rangle} \sim \langle v(t+\tau) - v(t) \rangle_t / \tau$, a smaller amplitude of the acceleration spectrum is compatible with a smaller $\sigma_{v\tau}$, the dispersion of the velocity differences at a given $\tau$, which was already observed to decrease with increasing $\tau_p$ in the second order structure functions.

In Figure 13(b) the experimental data is compared with DNS0.2 and DNS0.5. Even though the DNS data is more anisotropic, the amplitude and overall shape of the experimental spectra are similar to those of DNS0.5. However, the experimental curves do display a plateau for almost a decade of frequencies, but with an amplitude smaller than that of the tracers. This difference between the simulations and the experiments further suggests that additional effects need to be considered in the simulations in order to achieve a more detailed description of the behavior of the particle dynamics in the experiments.

V. DISCUSSION

In this work we presented a comparison of tracers’ and finite-size inertial particles’ velocity and acceleration statistics in two paradigmatic turbulent swirling flows: an experimental von Kármán flow, and a numerical flow obtained by imposing a Taylor-Green mechanical forcing. For the simulations of the inertial particles, a simple point particle model was used, with an effective particle response time.

In spite of the differences in boundary conditions and in the forcing mechanisms, scaling and statistical properties of tracers share similarities between both flows, and also display a clear effect of the mean flow on particle dynamics which in turn affect turbulent statistical properties. Results from the experiments and the simulations show good agreement in the decorrelation times of the tracers’ velocity, and also in the power laws observed in the velocity power spectrum. We find two different power law exponents: a range compatible with $-2$ for sufficiently small frequencies, and one compatible with $-5/3$ for frequencies associated to the large-scale motion of the flow. In both the numerical simulations and the experiments, the compensated tracers’ velocity second order structure function has a short plateau corresponding to the Lagrangian inertial range. A more clear scaling range is also seen, associated to the effect of the mean flow. Similar results are obtained in the experiments and the simulations when the acceleration spectrum is considered, up to high frequencies that can be associated to the dissipative range. Remarkably, our results for the second order Lagrangian structure functions and the acceleration spectra indicate that sweeping by the large-scale flow plays a relevant role in the particle evolution even in the Lagrangian frame.

Having found good agreement between the experiment and the numerical simulations from the tracer’s dynamics viewpoint, we compare the velocity and acceleration statistics of an inertial particle. We find that a suitable way to compare finite-size particles in the experiments with inertial point-particles in the numerics is via an effective Stokes number based on the eddy turnover time at the particle radius. Even though in our simulations only viscous drag is considered, several statistical quantities in the experiments are well captured by the simulations; these include the behavior of the velocity probability distribution function, the velocity power spectrum, the amplitude of the second order velocity structure function, and the acceleration spectrum. However, the “filtering” of fast fluctuations in the flow by the inertial particles is more accentuated in the experiment than in the DNSs. Such a deficiency can be corrected by a simple modification to the equation for the evolution of the particles.

The comparison presented here between experimental data and numerical simulations also led us to introduce definitions of dimensionless numbers that allow comparisons between the von Kármán and Taylor-Green flows. In particular, definitions of Reynolds numbers, Taylor-based
Reynolds numbers, dimensionless autocorrelation times, and Stokes numbers were provided that are in good agreement between the experiments and the simulations that display similar behavior for the different particles studied.

Finally, in spite of the similarities in the results, there are important discrepancies and limitations that are worth mentioning, and that open new paths for future studies. Firstly, only one inertial particle was used in the experiments, and a detailed exploration of particles with different masses and radius is needed to better calibrate effective dimensionless numbers for the simulations. Secondly, the large-scale flow in the simulations displays a stronger anisotropy in its velocity components than the flow in the experiments. To improve comparisons between experiments and simulations, the geometry of the blades in the impellers could be changed, to change the ratio of the vertical to horizontal velocities in the von Kármán flow (as was done before, e.g., in von Kármán dynamo experiments, see [21]). And thirdly, our numerical model for the particles is based on a model for point particles that only considers the Stokes drag, and as a result our particles response times can only be interpreted in an effective way. Other effects, such as added mass effects, buoyancy, or finite size effects such as the Basset-Boussinesq history term or the Faxén corrections [32] should be taken into account to improve particle modeling, and to properly consider the spatial variation of the flow in the vicinity of the finite-size particles.

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