Monotonicity axioms in approval-based multi-winner voting rules

Luis Sánchez-Fernández
Universidad Carlos III
de Madrid, Spain
luiss@it.uc3m.es

Jesús A. Fisteus
Universidad Carlos III
de Madrid, Spain
jaf@it.uc3m.es

October 13, 2017

Abstract

In this paper we study several monotonicity axioms in approval-based multi-winner voting rules. We consider monotonicity with respect to the support received by the winners and also monotonicity in the size of the committee. Monotonicity with respect to the support is studied when the set of voters does not change and when new voters enter the election. For each of these two cases we consider a strong and a weak version of the axiom. We observe certain incompatibilities between the monotonicity axioms and well-known representation axioms (extended/proportional justified representation) for the voting rules that we analyze and provide formal proofs of incompatibility between some of these axioms and perfect representation.

1 Introduction

There are many situations in which it is necessary to aggregate the preferences of a group of agents to select a finite set of alternatives. Typical examples are the election of representatives in indirect democracy, shortlisting candidates for a position [Elkind et al., 2017, Barberá and Coelho, 2008], selection by a company of the group of products that is going to offer to its customers [Lu and Boutilier, 2011], selection of the web pages that should be shown to a user in response of a given query [Dwork et al., 2001, Skowron et al., 2016] or recommender systems [Elkind et al., 2017, Naamani-Dery et al., 2014]. The typical mechanism for such preference aggregations is the use of multi-winner voting rules.

The use of axioms for analyzing voting rules is well established in social choice and dates back to the works of Arrow [1951]. However, the axiomatic study of multi-winner voting rules has not been studied much so far. In particular, we can cite the work of Dummet [1984], Woodall [1994], and Elkind et al. [2017] for multi-winner elections that make use of ranked ballots. For approval-based multi-winner elections the concept of representation has been recently axiomatized in two
works by Aziz et al. [2017] (they proposed two axioms called justified representation and extended justified representation) and Sánchez-Fernández et al. [2017] (they proposed a weakening of extended justified representation that they called proportional justified representation).

In this paper we complement these previous works with the study of monotonicity axioms for approval-based multi-winner voting rules. First of all, we consider monotonicity in the support received by the winners. Informally, the idea of monotonicity in the support is that if a subset of the winners in an election view their support increased and the support of all the other candidates remains the same, then such candidates must still be in the set of winners. We note that this axiom is desirable in basically all cases in which multi-winner voting rules are applied. Monotonicity with respect to the support is studied when the set of voters does not change and when new voters enter the election. For each of these two cases we define a strong and a weak version of the axiom. Secondly, we study monotonicity in the size of the committee. This type of monotonicity is usually considered of minor importance, although it is a desirable property in scenarios like short-listing [Elkind et al., 2017, Barberá and Coelho, 2008]. Then, we analyze several well-known voting rules with these axioms. We observe certain incompatibilities between the monotonicity axioms and extended/proportional justified representation for the voting rules that we analyze and provide formal proofs of incompatibility between some of these axioms and perfect representation (another axiom proposed by Sánchez-Fernández et al. [2017]).

The rest of this paper is organized as follows. In the next section we formalize the concept of approval-based multi-winner elections and describe briefly the concept of perfect representation and the rules that we are going to study. The next two sections are devoted to give formal definitions of support monotonicity and committee monotonicity (for committee monotonicity we use the definition of Elkind et al. [2017]) and to study several voting rules with these axioms. Then, we observe that none of the rules that we study satisfies simultaneously certain strong notions of representation and one of the strong support monotonicity axioms. Further, we prove incompatibilities between perfect representation and some of the axioms that we study in this paper. We finish with some conclusions and future lines of research.

2 Preliminaries

We consider elections in which a fixed number $k$ of candidates or alternatives must be chosen from a set of candidates $C$. We assume that $|C| \geq k \geq 1$. The set of voters is represented as $N = \{1, \ldots, n\}$, and thus $n$ is the total number of voters that participate in the election. Each voter $i$ that participates in the election casts a ballot $A_i$ that consists of the subset of the candidates that the voter approves of (that is, $A_i \subseteq C$). We refer to the ballots cast by the voters that participate in the election as the ballot profile $A = (A_1, \ldots, A_n)$. An approval-based multi-winner
election $\mathcal{E}$ is therefore represented by $\mathcal{E} = (N, C, A, k)$. The set of voters $N$ and the set of candidates $C$ will be omitted when they are clear from the context.

Given a voting rule $R$, for each election $\mathcal{E} = (A, k)$, we say that $R(\mathcal{E})$ is the output of the voting rule $R$ for such election. Ties may happen in the voting rules that we are going to consider. To take this into account, given an election $\mathcal{E}$ and a voting rule $R$ we say that the value of $R(\mathcal{E})$ is the set of size at least one composed of all the possible sets of winners outputed by rule $R$ and election $\mathcal{E}$. We say that a candidates subset $W$ of size $k$ is a set of winners for election $\mathcal{E}$ and rule $R$ if $W$ belongs to $R(\mathcal{E})$. We stress that our results are to a large extent independent of how ties are broken.

The following definition is due to Sánchez-Fernández et al. [2017].

**Definition 1. Perfect representation (PR)** Consider a ballot profile $A$ over a candidate set $C$, and a target committee size $k$, $k \leq |C|$, such that $k$ divides $n$ (we recall that $n$ is the number of voters that participate in the election). We say that a set of candidates $W$, $|W| = k$, provides perfect representation (PR) for $(A, k)$ if it is possible to partition the set of voters in $k$ pairwise disjoint subsets $N_1, \ldots, N_k$ of size $\frac{n}{k}$ each, such that each candidate $w$ in $W$ can be assigned to one (and only one) different subset $N_i$ so that for all pairs $(w, N_i)$ all the voters in $N_i$ approve of their assigned candidate $w$. We say that an approval-based voting rule satisfies perfect representation (PR) if for every election $(A, k)$ it does not output any winning set of candidates $W$ that does not provide PR for $(A, k)$ if at least one set of candidates $W'$ that provides PR for $(A, k)$ exists.

We now introduce the voting rules that we are going to consider in this study. First of all, we describe the following voting rules, surveyed by Kilgour [2010].

**Approval Voting (AV).** Under AV, the winners are the $k$ candidates that receive the largest number of votes. Formally, for each approval-based multi-winner election $(A, k)$, the approval score of a candidate $c$ is $|\{i : c \in A_i\}|$. The $k$ candidates with higher approval scores are chosen.

**Satisfaction Approval Voting (SAV).** A voter’s satisfaction score is the fraction of her approved candidates that are elected. SAV maximizes the sum of the voters’ satisfaction scores. Formally, for each approval-based multi-winner election $\mathcal{E} = (A, k)$:

$$\text{SAV}(\mathcal{E}) = \arg\max_{W \subseteq C : |W| = k} \sum_{i \in N} \frac{|A_i \cap W|}{|A_i|}. \quad (1)$$

**Minimax Approval Voting (MAV).** MAV selects the set of candidates $W$ that minimizes the maximum Hamming distance [Hamming, 1950] between $W$ and the voters’ ballots. Let $d(A, B) = |A \setminus B| + |B \setminus A|$, for each pair of candidates subsets $A$ and $B$. Then, for each approval-based multi-winner election $\mathcal{E} = (A, k)$:

$$\text{MAV}(\mathcal{E}) = \arg\min_{W \subseteq C : |W| = k} \left( \max_{i \in N} d(W, A_i) \right). \quad (2)$$
Since we are interested in the compatibility between representation axioms and monotonicity axioms we are going to study also several rules that satisfy some of the above mentioned representation axioms.

**Chamberlin and Courant and Monroe rules.** The voting rules proposed by [Chamberlin and Courant, 1983] and [Monroe, 1995] select sets of winners that minimize the missrepresentation of the voters (the number of voters represented by a candidate that they do not approve of). The difference between the rule of Chamberlin and Courant (CC) and the rule of Monroe is that in CC each candidate may represent an arbitrary number of voters while in the Monroe rule each candidate must represent at least $\left\lfloor \frac{n}{k} \right\rfloor$ and at most $\left\lceil \frac{n}{k} \right\rceil$ voters. For each approval-based multi-winner election $E = (A, k)$:

$$\text{CC}(E) = \arg\min_{W \subseteq C : |W| = k} |\{ i : A_i \cap W = \emptyset \}|. \quad (3)$$

Given an election $E = (A, k)$ and a candidates subset $W$ of size $k$ let $M_{N,W}$ be the set of all mappings $\pi : N \to W$ such that for each candidate $c$ in $W$ it holds that $\left\lfloor \frac{n}{k} \right\rfloor \leq |\{ i : \pi(i) = c \}| \leq \left\lceil \frac{n}{k} \right\rceil$. Then,

$$\text{Monroe}(E) = \arg\min_{W \subseteq C : |W| = k} \min_{\pi \in M_{N,W}} |\{ i : \pi(i) \notin A_i \}|. \quad (4)$$

**PAV and SeqPAV** The Proportional Approval Voting (PAV) and the Sequential Proportional Approval Voting (SeqPAV) were proposed by the Danish mathematician [Thiele, 1895] in the late 19th century. Given an election $E = (A, k)$ and a candidates subset $W$ of size $k$, the PAV-score of a voter $i$ is 0 if such voter does not approve any of the candidates in $W$ and $\sum_{j=1}^{L_k} 1$ if the voter approves of some of the candidates in $W$. PAV selects the sets of winners that maximize the sum of the PAV-scores of the voters.

$$\text{PAV}(E) = \arg\max_{W \subseteq C : |W| = k} \sum_{i : A_i \cap W \neq \emptyset} \frac{|A_i \cap W|}{\sum_{j=1}^{L_k} 1}.$$ 

The Sequential Proportional Approval Voting (SeqPAV) is an iterative algorithm in which at each iteration the candidate with highest SeqPAV score is added to the set of winners. The SeqPAV score of a candidate $c$ at iteration $j$ is computed as follows:

$$\text{SeqPAV}(c) = \sum_{i : c \in A_i} \frac{1}{1 + |A_i \cap W_{j-1}|}. \quad (6)$$

Here $W_{j-1}$ is the set of the first $j - 1$ candidates added by SeqPAV to the set of winners.

**Phragmén rules** Phragmén rules were proposed by the Swedish mathematician [Phragmén, 1894, 1895, 1896, 1899] in the late 19th century. In this paper we consider two of these rules that are known to satisfy some of the representation
axioms we have mentioned before. We refer to the survey by Janson [2016] for an extensive discussion of the rules proposed by Phragmén.

Phragmén voting rules are based on the concept of load. Each candidate in the set of winners incurs in one unit of load, that should be distributed among the voters that approve of such candidate. The goal is to choose the set of winners such that the total load is distributed as evenly as possible between the voters.

Formally, given an election $E = (A, k)$ and a candidates subset $W \subseteq C$, $|W| = k$, a load distribution is a two dimensional array $x = (x_{i,c})_{i \in N, c \in W}$, that satisfies the following 4 conditions:

$$0 \leq x_{i,c} \leq 1 \quad \text{for all } i \in N \text{ and } c \in W,$$

$$x_{i,c} = 0 \quad \text{if } c \notin A_i,$$

$$\sum_{i \in N} \sum_{c \in W} x_{i,c} = k,$$

$$\sum_{i \in N} x_{i,c} = 1 \quad \text{for all } c \in W.$$

Given a load distribution $x$, the load of each voter $i$ is defined as $x_i = \sum_{c \in W} x_{i,c}$. Then, given an election $E$, the rule max-Phragmén outputs the set of winners $W$ that minimizes the maximum voter load.

The rule seq-Phragmén is a greedy algorithm for max-Phragmén. The load of each voter changes (increases) at each iteration as candidates are added to the set of winners. For $j = 0, \ldots, k$, let $x^{(j)}_i$ be the load of voter $i$ after $j$ iterations of seq-Phragmén. The initial load $x^{(0)}_i$ of each voter $i$ is set to 0.

At each iteration $j + 1$ the load $s^{(j+1)}_c$ associated to each candidate $c$ is computed as:

$$s^{(j+1)}_c = \frac{1 + \sum_{i \in A_c} x^{(j)}_i}{|\{i : c \in A_i\}|}.$$  

The underlying idea of this expression is to distribute equally between all the voters that approve of candidate $c$ the unit of load corresponding to such candidate plus the load that each of such voters had after the first $j$ iterations. Then, at each iteration the candidate $w$ with the lowest load is added to the set of winners and the loads of the voters are updated as follows: for each voter $i$ that approves of candidate $w$, we have $x^{(j+1)}_i = s^{(j+1)}_w$, while the load of each voter $h$ that does not approve of candidate $w$ does not change: $x^{(j+1)}_h = x^{(j)}_h$.

### 3 Support monotonicity

The first set of axioms that we are going to consider is support monotonicity. We consider two types of support monotonicity and, for each of these types a strong and a weak version.
Definition 2. Consider an approval-based multi-winner election \( E = (N, C, A, k) \), with \( N = \{1, \ldots , n\} \), and \( A = (A_1, \ldots , A_n) \). Given a non-empty candidates subset \( G \) we define \( E_{\Delta G} \), as the election obtained by adding to election \( E \) one voter that approves of only the candidates in \( G \). That is, \( E_{\Delta G} = (N_{\Delta G} = \{1, \ldots , n, n+1\}, C, A_{\Delta G} = (A_1, \ldots , A_n, G), k) \). Given a non-empty candidates subset \( G \) and a voter \( i \in N \) such that she does not approve of any of the candidates in \( G \) we define \( E_{i+G} \), as the election obtained if voter \( i \) decides to approve of all the candidates in \( G \) in addition to the candidates in \( A_i \). That is, \( E_{i+G} = (N, C, A_{i+G} = (A_1, \ldots , A_{i-1}, A_i \cup G, A_{i+1}, \ldots , A_n), k) \).

We say that a rule \( R \) satisfies strong support monotonicity with population increase (respectively, weak support monotonicity with population increase) if for each election \( E \), for each set of candidates \( W \) that belongs to \( R(E) \) and for each non-empty subset \( G \) of \( W \), there exists a set of candidates \( W' \) that belongs to \( R(E_{\Delta G}) \) such that \( G \subseteq W' \) (respectively, \( G \cap W' \neq \emptyset \)).

We say that a rule \( R \) satisfies strong support monotonicity without population increase (respectively, weak support monotonicity without population increase) if for each election \( E \), for each set of candidates \( W \) that belongs to \( R(E) \), for each non-empty subset \( G \) of \( W \), and for each voter \( i \) such that \( A_i \cap G = \emptyset \) there exists a set of candidates \( W' \) that belongs to \( R(E_{i+G}) \) such that \( G \subseteq W' \) (respectively, \( G \cap W' \neq \emptyset \)).

Previous works that consider support monotonicity in approval-based multi-winner voting rules, e.g. [Lackner and Skowron, 2017], mostly consider support monotonicity when \( |G| = 1 \) (notable exceptions are the works of Mora and Oliver [2015] and Janson [2016] for Phragmén rules). In contrast, we believe that it is important to know what happens when the support of several of the candidates in the set of winners is incremented simultaneously. Moreover, our results show that for each of the rules that we consider that satisfies any of the support monotonicity axioms (with or without population increase) for \( |G| = 1 \), such rule also satisfies the corresponding weak support monotonicity axiom, which is slightly stronger, and therefore provides more information about the behaviour of the rule.

From now on we will refer to support monotonicity with population increase as SMWPI and to support monotonicity without population increase as SMWOPI. Table I summarizes the results we have obtained in this paper. With respect to the support monotonicity axioms we use the keys “Str.” when the rule satisfies the strong version of the axiom, “Wk.” when the rule satisfies the weak version of the axiom and “No” when the rule does not satisfy any of them.

For completeness, we also include previous results related to the computational complexity of the rules and the representation axioms that they satisfy, including pointers to the appropriate references. The column entitled “JR/PJR/EJR” provides information about which of the following representation axioms are satisfied by the rules considered in this study: justified representation (JR), proportional justified representation (PJR) and extended justified representation (EJR). Roughly speaking, JR establishes requirements on when a group of agents deserves at least
one representative, while EJR establishes requirements on when a group of agents deserves several representatives. Finally, PJR is a weakening of EJR. In summary, every rule that satisfies EJR also satisfies PJR and every rule that satisfies PJR also satisfies JR. The table shows for each rule the strongest of these axioms satisfied by the rule. The next column says which rules satisfy PR. Most of the results related with JR/PJR/EJR and with PR are already known. Missed cases are discussed in the appendix of this study.

| Rule          | Complexity | JR/PJR/EJR | PR     | SMWPI | SMWOPI | Comm. Mon. |
|---------------|------------|------------|--------|-------|--------|------------|
| AV            | P          | No         | No     | Str.  | Str.   | Yes        |
| SAV           | P          | No         | No     | Str.  | Str.   | Yes        |
| MAV           | NP-hard    | No         | No     | Str.  | Wk.    | No         |
| CC            | NP-complete | JR          | Yes    | Str.  | Wk.    | No         |
| Monroe        | NP-complete | JR          | Yes    | No    | Wk.    | No         |
| PAV           | NP-complete | EJR         | No    | Str.  | Wk.    | No         |
| SeqPAV        | P          | No         | No    | Wk.   | Wk.    | Yes        |
| max-Phragmén  | NP-complete | PJR        | Yes   | Wk.   | Wk.    | No         |
| seq-Phragmén  | P          | No         | No    | Wk.   | Wk.    | Yes        |

* Monroe satisfies PJR if $k$ divides $n$ [Sánchez-Fernández et al., 2017].
* max-Phragmén satisfies PJR when combined with certain tie-breaking rule [Brill et al., 2017].
* Results taken from [Aziz et al., 2015] and [Skowron et al., 2016].
* Results taken from [LeGrand et al., 2006].
* Results taken from [Procaccia et al., 2008].
* Results taken from [Aziz et al., 2017].
* Results taken from [Sánchez-Fernández et al., 2017].
* Results taken from [Phragmén, 1896].
* Results taken from [Thiele, 1895].
* This paper.

An important type of rules in approval-based multiwinner elections are approval-based multi-winner counting rules, which, as discussed by Lackner and Skowron [2017], can be seen as analogous to the class of committee scoring rules introduced by Elkind et al. [2017] for ranked-based multiwinner elections.

**Definition 3.** A counting function $f : \{1, \ldots, k\} \times \{1, \ldots, |C|\} \to \mathbb{R}$ is a function that satisfies that $f(x, y) \geq f(x', y)$ whenever $x > x'$. Intuitively, a counting function $f$ defines the score $f(x, y)$ that a certain counting rule $r_f$ assigns to a voter $i$ that approves of $x$ candidates in the set of winners $W$ and $y$ candidates in total. Given a counting function $f$, and an election $E = (A, k)$, the total score of a candidates subset $W$ for counting function $f$ is
\[ s_f(W, E) = \sum_{i \in N} f(|A_i \cap W|, |A_i|), \]

and the counting rule \( r_f \) associated to counting function \( f \) is defined as follows:

\[ r_f(E) = \arg\max_{W \subseteq \mathcal{C} : |W| = k} s_f(W, E). \]

As discussed by Lackner and Skowron [2017] several of the voting rules that we have presented in the previous section are counting rules. In particular, we have \( f_{\text{AV}}(x, y) = x \) for AV, \( f_{\text{SAV}}(x, y) = \frac{x}{y} \) for SAV, \( f_{\text{CC}}(x, y) = 1 \) if \( x > 0 \) and \( f_{\text{CC}}(0, y) = 0 \) for CC, and \( f_{\text{PAV}}(x, y) = \sum_{j=1}^{x} \frac{1}{y} \) if \( x > 0 \) and \( f_{\text{PAV}}(0, y) = 0 \) for PAV.

For counting rules we have the following results with respect to support monotonicity.

**Theorem 1.** Every counting rule satisfies strong SMWPI.

**Proof.** Consider an election \( E = (A, k) \), a counting function \( f \) and its associated rule \( r_f \), a set of winners \( W \) outputted by \( r_f \) for election \( E \) and a non-empty subset \( G \) of \( W \). We are going to prove that \( W \) belongs also to \( r_f(E_{\Delta G}) \). The theorem follows from that immediately.

Consider any other candidates subset \( W' \) of size \( k \). We simply have to observe that the total score of \( W \) for election \( E_{\Delta G} \) under rule \( r_f \) is \( \sum_{i \in N} f(|A_i \cap W'|, |A_i|) + f(|G \cap W'|, |G|) \), that \( \sum_{i \in N} f(|A_i \cap W'|, |A_i|) \geq \sum_{i \in N} f(|A_i \cap W, |A_i|) \) (because \( W \) is a set of winners for rule \( r_f \) and election \( E \)), and that \( f(|G \cap W'|, |G|) = f(|G|, |G|) \geq f(|G \cap W'|, |G|) \) (by the definition of counting function). 

We can also prove weak SMWPOI introducing an slight restriction to the counting functions that is satisfied by all the counting rules that we consider in this paper.

**Theorem 2.** Consider a counting function \( f \). If \( f \) holds that \( f(x, y) \geq f(x, y') \) whenever \( y \leq y' \), and that for each positive integer \( z \) it holds that \( f(x + z, y + z) \geq f(x, y) \), then its associated rule \( r_f \) satisfies weak SMWPOI.

**Proof.** Consider an election \( E = (A, k) \), a counting function \( f \) and its associated rule \( r_f \), a set of winners \( W \) outputted by \( r_f \) for election \( E \), a non-empty subset \( G \) of \( W \), and a voter \( i \) such that she does not approve of the candidates in \( G \).

We observe first that because \( A_i \) and \( G \) are disjoint, for each candidates subset \( W' \) it holds that \( f(|A_i \cup G) \cap W'|, |A_i \cup G|) = f(|A_i \cap W'| + |G \cap W'|, |A_i| + |G|) \) and that \( s_f(W', E_{i+G}) - s_f(W', E) = f(|A_i \cap W'| + |G \cap W'|, |A_i| + |G|) - f(|A_i \cap W'|, |A_i|) \).

Suppose that \( f \) satisfies that \( f(x, y) \geq f(x, y') \) whenever \( y \leq y' \), and that for each positive integer \( z \) it holds that \( f(x + z, y + z) \geq f(x, y) \), and consider any candidates subset \( W' \) of size \( k \) such that \( W' \cap G = \emptyset \). Then, \( s_f(W, E_{i+G}) - \)
Theorem 3. AV and SAV satisfy strong SMWOPI.

Proof. The counting functions of AV and SAV hold that $f(x, y) = xf(1, y).$ This makes it possible to assign each candidate a score $s_f(c, E) = \sum_{i \in A, \mathcal{f}(1, |A_i|)}$ irrespective of which other candidates are in the set of winners $W$ so that $s_f(W, E) = \sum_{c \in W} s_f(c, E)$. Therefore, the winners in AV and SAV are the $k$ candidates with higher candidate score.

Each candidate that belongs to $G$ increases her score in election $\mathcal{E}_{i+G}$ in $f(1, |A_i| + |G|)$ with respect to her score in election $\mathcal{E}$. Each candidate that does not belong to $A_i \cup G$ has the same score in election $\mathcal{E}_{i+G}$ as in election $\mathcal{E}$. Finally, each candidate that belongs to $A_i$ has a score in election $\mathcal{E}_{i+G}$ equal to (for AV) or less than (for SAV) the score that she has in election $\mathcal{E}$. Therefore, if the candidates in $G$ were some of the $k$ candidates with higher score for election $\mathcal{E}$, they should also be some of the $k$ candidates with higher score for election $\mathcal{E}_{i+G}$.

The following examples prove that PAV and CC fail strong SMWOPI.

Example 1. Let $k = 4$ and $C = \{c_1, \ldots, c_7\}$. 131 voters cast the following ballots: for $i, j = 1$ to 3, 3 voters approve of $\{c_i, c_{j+4}\}$, 100 voters approve of $\{c_4\}$, 1 voter approves of $\{c_1, c_2\}$, 1 voter approves of $\{c_1, c_2, c_3\}$, and 2 voters approve of $\{c_5, c_6\}$. For this election PAV outputs one set of winners: $\{c_1, c_2, c_3, c_4\}$, with a PAV score of 301/3. However, if the voter that approves of $\{c_1, c_2\}$ decides to approve of $\{c_1, c_2, c_3, c_4\}$, then PAV outputs only $\{c_4, c_5, c_6, c_7\}$, with a PAV score of 131. Intuitively, this example works as follows. First, the 100 voters that approve of $\{c_4\}$ force that $c_4$ has to be in the set of winners. Second, the first 27 votes force that either $\{c_1, c_2, c_3\}$ or $\{c_5, c_6, c_7\}$ are in the set of winners. The last 4 votes break the tie between $\{c_1, c_2, c_3, c_4\}$ and $\{c_4, c_5, c_6, c_7\}$ in the two cases considered.

Example 2. Let $k = 3$ and $C = \{a, b, c, d, e\}$. 13 voters cast the following ballots: 2 voters approve of $\{a, d\}$, 2 voters approve of $\{a, e\}$, 2 voters approve of $\{c, d\}$, 2 voters approve of $\{c, e\}$, 2 voters approve of $\{b\}$, 2 voters approve of $\{a\}$, and 1 voter approves of $\{d\}$. For this election CC outputs one set of winners: $\{a, b, c\}$ (one voter misrepresented). Now, we consider two consecutive increases of support of $\{b, c\}$, where, in each increase one of the voters that approve of $\{a\}$ decides to approve of $\{a, b, c\}$. Then, after the first increase of support of $\{b, c\}$, CC outputs $\{a, b, c\}$ and $\{b, d, e\}$ (one voter misrepresented), and after the second increase of support of $\{b, c\}$ CC outputs only $\{b, d, e\}$ (0 voters misrepresented). Observe that this example proves that CC fails strong SMWOPI even if combined
with any tie breaking rule, because if the tie breaking rule selects \{a, b, c\} after the first increase of support, then strong SMWOPI is violated in the second increase of support and if the tie breaking rule selects \{b, d, e\} after the first increase of support, then strong SMWOPI is violated in the first increase of support.

Let us now turn to analyze the remaining voting rules.

**Theorem 4.** MAV satisfies strong SMWPI and weak SMWOPI.

**Proof.** Consider first an election \(E = (A, k)\), a set of winners \(W\) outputted by MAV for election \(E\), and a non-empty subset \(G\) of \(W\). Since \(G \subseteq W\), we have \(d(W, G) = |W \setminus G| + |G \setminus W| = k - |G|\). For each candidate set \(W'\) of size \(k\) such that a candidate \(c\) exists that belongs to \(G\) but not to \(W'\), we have \(d(W', G) = |W' \setminus G| + |G \setminus W'| \geq (k - |G| + 1) + 1 = k - |G| + 2\). Therefore, \(\max\{\max_{i \in N} d(W, A_i), d(W, G)\}\) has to be less than or equal to \(\max\{\max_{i \in N} d(W', A_i), d(W', G)\}\). This proves that MAV satisfies strong population monotonicity with population increase.

Consider now an election \(E = (A, k)\), a set of winners \(W\) outputted by MAV for election \(E\), a non-empty subset \(G\) of \(W\), and a voter \(i\) that does not approve of any of the candidates in \(G\). We observe first that \((A_i \cup G) \setminus W = A_i \setminus W\), and that \(|W \setminus (A_i \cup G)| = |W \setminus A_i| - |G|\), and therefore, \(d(W, A_i \cup G) = d(W, A_i) - |G|\). For each candidate set \(W'\) of size \(k\) such that \(W' \cap G = \emptyset\), we have \(d(W', A_i \cup G) = |W' \setminus (A_i \cup G)| + |(A_i \cup G) \setminus W'| = |W' \setminus A_i| + |A_i \setminus W'| + |G| = d(W', A_i) + |G|\). It follows immediately that for each candidate set \(W'\) of size \(k\) such that \(W' \cap G = \emptyset\), the maximum Hamming distance between \(W'\) and the voters in election \(E_{i+G}\) does not decrease with respect to the maximum Hamming distance between \(W'\) and the voters in election \(E\), and therefore, that \(W\) or another set of candidates that includes some of the candidates in \(G\) must be output by MAV for election \(E_{i+G}\). \(\Box\)

However, the following example shows that MAV fails strong SMWOPI.

**Example 3.** Let \(k = 5\) and \(C = \{c_1, \ldots, c_7\}\). 9 voters cast the following ballots:
1 voter approves of \(\{c_1\}\), 1 voter approves of \(\{c_2\}\), 1 voter approves of \(\{c_3\}\), 1 voter approves of \(\{c_5\}\), for \(i, j = 0, 1\), 1 voter approves of \(\{c_1, c_{4+i}, c_{6+j}\}\), and 1 voter approves of \(\{c_1, c_6\}\). For this election the only set of winners output by MAV is \(\{c_1, c_2, c_3, c_4, c_5\}\). The Hamming distance between \(\{c_1, c_2, c_3, c_4, c_5\}\) and the ballot profile of each voter is always less than or equal to 5 (in particular, the Hamming distance between \(\{c_1, c_2, c_3, c_4, c_5\}\) and \(\{c_1, c_6\}\) is 5). We show now that for any other candidate subset of size 5 there is a ballot profile with distance 6 to such candidate subset. First, for each \(j = 1, 2, 3\), and for each candidate subset \(W\) of size 5 such that \(c_3\) does not belong to \(W\), the Hamming distance between \(W\) and \(\{c_j\}\) is 6. There exist 6 candidates subsets of size 5 that contain \(c_1, c_2,\) and \(c_4\) (one of them is \(\{c_1, c_2, c_3, c_4, c_5\}\)). For \(i, j = 0, 1\), the Hamming distance between \(\{c_1, c_2, c_3, c_{4+i}, c_{6+j}\}\) and \(\{c_1, c_{5-i}, c_{7-j}\}\) is 6. The only remaining candidates subset is \(\{c_1, c_2, c_3, c_6, c_7\}\), that has a Hamming distance with \(\{c_5\}\) of 6. Observe that the Hamming distance between \(\{c_1, c_2, c_3, c_6, c_7\}\) and all the other
ballot profiles is always less than or equal to 4. Now, if the voter that approves of \{c_3\} decides to approve of \{c_1, c_2, c_3, c_4, c_5\}, then the Hamming distance between the ballot profile of such voter and \{c_1, c_2, c_3, c_6, c_7\} fails to 4, and therefore, in that case MAV would output only \{c_1, c_2, c_3, c_6, c_7\}.

**Theorem 5.** The Monroe rule satisfies weak SMWPOI.

**Proof.** Consider an election \(E = (A, k)\), a set of winners \(W\) outputted by Monroe for election \(E\), a non-empty subset \(G\) of \(W\), and a voter \(i\) that does not approve of any of the candidates in \(G\). Let \(\pi_W\) be a mapping that minimizes the missrepresentation of \(W\) for election \(E\). Clearly the missrepresentation of \(W\) with mapping \(\pi_W\) for election \(E_{i+G}\) is the same as for election \(E\) if the candidate \(\pi_W(i)\) assigned by \(\pi_W\) to voter \(i\) does not belong to \(G\) and is equal to the missrepresentation of \(W\) with mapping \(\pi_W\) for election \(E\) minus one if \(\pi_W(i)\) belongs to \(G\). Further, for each candidates set \(V\) such that \(V \cap G = \emptyset\), and for each mapping \(\pi_V\) of the voters in \(N\) to the candidates in \(V\) it holds that the candidate \(\pi_V(i)\) assigned by \(\pi_V\) to voter \(i\) belongs to \(A_i \cup G\) if and only if such candidate belongs to \(A_i\), and therefore, the missrepresentation values of \(V\) with mapping \(\pi_V\) are the same for election \(E_{i+G}\) and for election \(E\). It follows immediately that Monroe must output \(W\) or other candidates set that contains some of the candidates in \(G\) for election \(E_{i+G}\). \(\square\)

Examples 4 and 5 prove that Monroe fails weak SMWPOI and strong SMWOPI, respectively. As in the case of CC, these examples prove that Monroe fails weak SMWPOI and strong SMWOPI even if combined with any tie breaking rule.

**Example 4.** Let \(k = 4\) and \(C = \{a, b, c, d, e, f, g, h\}\). 33 voters cast the following ballots: 5 voters approve of \{a, e\}, 4 voters approve of \{a, g\}, 5 voters approve of \{b, e\}, 4 voters approve of \{b, h\}, 5 voters approve of \{c, f\}, 4 voters approve of \{c, g\}, 3 voters approve of \{d, f\}, and 3 voters approve of \{d, h\}. For this election Monroe outputs only \{c, e, f, g, h\} (missrepresentation 1 due to one of the voters that approve of \(e\) being represented by \(h\)). We now consider two consecutive voters that enter the election, such that each of the new voters approves of \{e\}. Then, after the first new voter enters the election Monroe outputs \{a, b, c, d\} and \{e, f, g, h\} (missrepresentation 2) and, after the second new voter enters the election, Monroe outputs only \{a, b, c, d\} (missrepresentation 2: the new voters would be represented by candidate \(d\)).

**Example 5.** Let \(k = 3\) and \(C = \{a, b, c, d, e\}\). 18 voters cast the following ballots: 2 voters approve of \{a\}, 2 voters approve of \{a, d\}, 2 voters approve of \{a, e\}, 4 voters approve of \{b\}, 1 voter approves of \{b, e\}, 4 voters approve of \{c, d\}, and 3 voters approve of \{c, e\}. For this election Monroe outputs only \{a, b, c\} (missrepresentation 1 due to one of the voters that approve of \(c\) being represented by candidate \(b\)). Now, we consider two consecutive increases of support of \{b, c\}, where, in each increase one of the voters that approve of \(a\) decides to approve of \{a, b, c\}. Then, after the first increase of support of \{b, c\}, Monroe outputs \{a, b, c\}.
Lemma 1. Consider an election $E = (A, k)$, a set of winners $W$ outputted by seq-Phragmén for election $E$, a non-empty subset $G$ of $W$, and a voter $i$ that does not approve of any of the candidates in $G$. Let $h$ be the first iteration in which a candidate that belongs to $G$ is added to the set of winners by seq-Phragmén, and let $c_h$ be such candidate. Then, it holds that $s_{ch}^{(h)} \geq \frac{1 + x_i^{(h-1)} + \sum_{r \in A_r} x_r^{(h-1)}}{1 + \| \{ r \in A_r \} \|}$.

Proof. Brill et al. [2017] prove that for each election $(A, k)$, and for each $1 \leq j \leq k$, it holds that $s^{(1)} \leq \ldots \leq s^{(k)}$, where $s^{(j)}$ is the load $s_{cj}$ of the candidate $c_j$ elected at iteration $j$. Therefore, $s_{cj}^{(j)} \geq x_i^{(j-1)}$ for each iteration $j$ and each voter $r$.

Thus, in the case of election $E$ and iteration $h$ we have $s_{ch}^{(h)} = \frac{1 + \sum_{r \in A_r} x_r^{(h-1)}}{1 + \| \{ r \in A_r \} \|} \geq \frac{1 + x_i^{(h-1)} + \sum_{r \in A_r} x_r^{(h-1)}}{1 + \| \{ r \in A_r \} \|}$. \hfill \square

The following theorem has already been proved by Phragmén [1896] and Janson [2016] for seq-Phragmén in the case in which $|G| = 1$. Our proof follows the same ideas.

Theorem 6. SeqPAV and seq-Phragmén satisfy weak SMWPI and weak SMWOPI.

Proof. Consider an election $E = (A, k)$, a set of winners $W$ outputted by SeqPAV (respectively, by seq-Phragmén) for election $E$, a non-empty subset $G$ of $W$, and a voter $i$ that does not approve of any of the candidates in $G$. Let $h$ be the first iteration in which a candidate that belongs to $G$ is added to the set of winners by SeqPAV (respectively, by seq-Phragmén) and let $c_h$ be such candidate. We observe first that while no candidate that belongs to $G$ is added to the set of winners, the SeqPAV score (for SeqPAV) and the load (for seq-Phragmén) of each candidate is the same for elections $E$, $E_{\Delta G}$, and $E_{i+G}$. For each of $E_{\Delta G}$ and $E_{i+G}$ there are therefore two possibilities: either a candidate that belongs to $G$ is added to the set of winners by SeqPAV (respectively, by seq-Phragmén) in the first $h - 1$ iterations (in that case, the theorem holds) or the first $h - 1$ candidates added to the set of winners by SeqPAV (respectively, by seq-Phragmén) for elections $E_{\Delta G}$ and $E_{i+G}$ are the same (and selected in the same order) as the first $h - 1$ candidates added to the set of winners by SeqPAV (respectively, by seq-Phragmén) for election $E$. We now simply observe that for $E_{\Delta G}$ and $E_{i+G}$ if the first $h - 1$ candidates added to the set of winners by SeqPAV (respectively, by seq-Phragmén) are the same and in the same order as those added for election $E$, then at iteration $h$ the SeqPAV score of candidate $c_h$ increases with respect to her SeqPAV score for election $E$ and the load of candidate $c_h$ (for seq-Phragmén) decreases with respect her load for election $E$ (in the case of election $E_{i+G}$ this follows from lemma[1]), while the SeqPAV score
and the load of all the candidates that do not belong to $G$ does not change. This proves that the candidate elected at iteration $h$ both by SeqPAV and seq-Phragmén in elections $E_{i+G}$ and $E_{i+G}$ must belong to $G$.

The following example proves that SeqPAV and seq-Phragmén fail both strong SMWPI and strong SMWOPI.

\textbf{Example 6.} Let $k = 4$ and $C = \{a, b, c, d, e\}$. 19 voters cast the following ballots: 7 voters approve of $\{a, b, d\}$, 4 voters approve of $\{a, b, e\}$, 3 voters approve of $\{a, c, d\}$, and 5 voters approve of $\{a, c, e\}$. For this election both SeqPAV and seq-Phragmén output only $\{a, b, c, d\}$ (the candidates are added to the set of winners in this order). Now, if an additional voter enters the election and approves of only $\{c, d\}$, then both SeqPAV and seq-Phragmén output only $\{a, d, e, b\}$ (the candidates are added to the set of winners in this order). This proves that both SeqPAV and seq-Phragmén fail strong population monotonicity with population increase. To prove that SeqPAV and seq-Phragmén fail strong population monotonicity without population increase we simply add an additional candidate $f$ to the original election and a voter that approves of $\{f\}$. This does not make any difference and the set of winners both with SeqPAV and seq-Phragmén will be again $\{a, b, c, d\}$. Now, if this new voter decides to approve of $\{c, d, f\}$, then both SeqPAV and seq-Phragmén output only $\{a, d, e, b\}$. Mora and Oliver [2015] have previously observed that seq-Phragmén fails the strong support monotonicity axioms. This fact has also been discussed by Janson [2016] (they use different examples from the one presented here). The previous example is included for completeness and as a counterexample for SeqPAV.

We study now support monotonicity for max-Phragmén. Phragmén [1896] proved that max-Phragmén satisfies support monotonicity when $|G| = 1$. We follow the same ideas to prove that max-Phragmén satisfies weak SMWPI and weak SMWOPI.

\textbf{Theorem 7.} max-Phragmén satisfies weak SMWPI and weak SMWOPI.

\textbf{Proof.} Consider an election $E = (A, k)$, a set of winners $W$ outputted by max-Phragmén for election $E$, a non-empty subset $G$ of $W$, and a voter $i$ that does not approve of any of the candidates in $G$. Let $x^{opt} = (x_{i',c}^{opt})_{i' \in N, c \in W}$ be a load distribution that minimizes the maximum voter load for election $E$ and candidates subset $W$, and let $m_E$ be the maximum voter load for load distribution $x^{opt}$, that is, $m_E = \max_{i' \in N} x_{i',c}^{opt}$.

Observe that $x^{opt}$ is a valid, possibly non-optimal, load distribution for election $E_{i+G}$ and candidates subset $W$. In particular, for each candidate $c$ that belongs to $G$, since voter $i$ does not approve of $c$ it holds that $x_{i',c}^{opt} = 0$. We can also build a, possibly non-optimal, load distribution $y$ for candidates subset $W$ and election $E_{DG}$ using $x^{opt}$. We simply set $y_{i',c} = x_{i',c}^{opt}$ for each voter $i' \in N$ and each candidate $c \in W$ and $y_{n+1, c} = 0$ for the additional voter in $E_{DG}$ and each candidate $c \in W$. Clearly, $y$ is a valid load distribution for $W$ and election $E_{DG}$.
(that is, it satisfies equations (7) to (10)). Also, it is easy to see that the maximum voter load for load distribution $y$ is also $m_E$.

Consider now any candidates subset $W'$ of size $k$ such that $W' \cap G = \emptyset$. Observe first that for the candidates subset $W'$ the set of valid load distributions for election $E_{i+G}$ are the same as the set of valid load distributions for election $E_i$. In particular, for voter $i$, the candidates for which $x_{i,c}$ can be greater than 0 are $A_i \cap W'$ both in election $E$ and in election $E_{i+G}$. It follows immediately that the minimum maximum voter load for candidates subset $W'$ is the same in elections $E$ and $E_{i+G}$. In the second place, for each valid load distribution $x$ for election $E_{\Delta G}$ and candidates subset $W'$, for each candidate $c$ that belongs to $W'$, according to equation (8) it must hold that $x_{n+1,c} = 0$, because $c$ does not belong to $G$. All the valid load distributions for election $E_{\Delta G}$ and candidates subset $W'$ are therefore also valid load distributions $x$ for election $E$ and candidates subset $W'$, extended with $x_{n+1,c} = 0$ for each candidate $c$ in $W'$. It follows again immediately that the minimum maximum voter load for candidates subset $W'$ is the same in elections $E$ and $E_{\Delta G}$.

Since the minimum maximum voter load for the candidates subset $W$ does not increase in elections $E_{\Delta G}$ and $E_{i+G}$ with respect to election $E$ and, for each candidates subset $W'$ such that $W' \cap G = \emptyset$ the minimum maximum voter load for the candidates subset $W'$ is the same in elections $E_{\Delta G}$, $E_{i+G}$, and $E$, it follows that $W$ or some candidates subset that contains some of the candidates in $G$ must be outputted by max-Phragmén for elections $E_{\Delta G}$ and $E_{i+G}$.

The following example proves that max-Phragmén fails both strong SMWPI and strong SMWOPI.

Example 7. Let $k = 6$ and $C = \{a, b, c_1, \ldots, c_5\}$. 18 voters cast the following ballots: 13 voters approve of $\{c_1, \ldots, c_5\}$, 2 voters approve of $\{a, b\}$, 2 voters approve of $\{a\}$, and 1 voter approves of $\{b\}$. For this election max-Phragmén outputs only one set of winners: $\{a, c_1, \ldots, c_5\}$. The minimum maximum load for this election is achieved as follows: for each voter $i$ that approves of $\{c_1, \ldots, c_5\}$ and each candidate $c$ in $\{c_1, \ldots, c_5\}$ we have $x_{i,c} = \frac{1}{13}$, and for each voter $i'$ that approves of $a$ we have $x_{i',a} = \frac{1}{3}$. Then, the load of the voters that approve of $\{c_1, \ldots, c_5\}$ is $\frac{5}{13}$ and the load of the voters that approve of $a$ is $\frac{1}{3}$. The maximal voter load for this example is therefore $\frac{5}{13}$. Now, if a new voter enters the election and approves of precisely $\{a, c_1, \ldots, c_5\}$, then the sets of winners outputted by max-Phragmén consist of $\{a, b\}$ plus 4 candidates from $\{c_1, \ldots, c_5\}$. In this case the minimum maximum voter load is achieved by assigning again $x_{i,c} = \frac{1}{13}$ for each voter $i$ that approves of $\{c_1, \ldots, c_5\}$ and each candidate $c$ in $\{c_1, \ldots, c_5\}$, assigning $x_{i,a} = \frac{1}{3}$ to the new voter and the voters that approve of $\{a\}$, and assigning $x_{i,b} = \frac{1}{3}$ to all the voters that approve of candidate $b$. This leads to a maximum voter load of $\frac{4}{3}$. Observe that in this case the minimum maximum voter load for the set $\{a, c_1, \ldots, c_5\}$ would be obtained by $x_{i,c} = \frac{1}{13}$ for each voter $i$ that approves of $\{c_1, \ldots, c_5\}$ and the new voter which leads to a maximum voter load of $\frac{5}{13}$, greater than $\frac{4}{3}$. This example
proves that max-Phragmén fails strong population monotonicity with population increase.

To prove that max-Phragmén fails strong population monotonicity without population increase we use the same strategy as in example 6. We simply add an additional candidate \( d \) to the original election and a voter that approves of \{d\}. This does not make any difference and the set of winners will be again \{a, c_1, \ldots, c_5\}. Now, if this new voter decides to approve of \{a, c_1, \ldots, c_5, d\}, then the sets of winners outputted by max-Phragmén consist of \{a, b\} plus 4 candidates from \{c_1, \ldots, c_5\}.

4 Committee monotonicity

We turn now to the study of committee monotonicity. The following definition, due to Elkind et al. [2017], was given in the context of multi-winner voting rules that make use of ranked ballots but it can also be directly used for approval-based multi-winner voting rules.

Definition 4. We say that a voting rule \( R \) satisfies committee monotonicity if for every set of voters \( N \), every set of candidates \( C \), every ballot profile \( A \) and every \( k \in \{1, \ldots, |C| - 1\} \), the following conditions hold:

1. for each \( W \) in \( R(N, C, A, k) \) there exists a \( W' \) in \( R(N, C, A, k + 1) \) such that \( W \subseteq W' \), and

2. for each \( W \) in \( R(N, C, A, k + 1) \) there exists a \( W' \) in \( R(N, C, A, k) \) such that \( W' \subseteq W \).

It is easy to see that committee monotonicity is satisfied by those rules that consist of an iterative algorithm such that at each iteration the candidate that is added to the set of winners does not depend on the target committee size. This holds for AV, SAV, SeqPAV and seq-Phragmén. The remaining rules fail committee monotonicity. Thiele [1895] and Mora and Oliver [2015] have already proved that PAV and max-Phragmén, respectively, fail committee monotonicity. For completeness we give counterexamples for all the rules that fail committee monotonicity.

Example 8. Let \( C = \{a, b, c\} \). 4 voters cast the following ballots: 1 voter approves of \{b\}, 1 voter approves of \{c\}, 1 voter approves of \{a, b\}, and 1 voter approves of \{a, c\}. For this set of candidates and this ballot profile, for \( k = 1 \) MAV outputs only \{a\} (with a maximum Hamming distance of 2). For \( k = 2 \), MAV outputs only \{b, c\} (also with a maximum Hamming distance of 2).

Example 9. Let \( C = \{a, b, c\} \). 10 voters cast the following ballots: 3 voters approve of \{a, b\}, 3 voters approve of \{a, c\}, 2 voters approve of \{b\} and 2 voters approve of \{c\}. For this set of candidates and this ballot profile, for \( k = 1 \) both CC and Monroe output only \{a\}. For \( k = 2 \), both CC and Monroe output only \{b, c\}.
Example 10. Let $C = \{a, b, c\}$. 13 voters cast the following ballots: 3 voters approve of $\{a, b\}$, 3 voters approve of $\{b, c\}$, 3 voters approve of $\{a\}$, 1 voter approves of $\{b\}$ and 3 voters approve of $\{c\}$. For this set of candidates and this ballot profile, for $k = 1$ both PAV and max-Phragmén output only $\{b\}$. For $k = 2$, both PAV and max-Phragmén output only $\{a, c\}$.

5 Compatibility of axioms

In many applications it would be interesting to use voting rules that satisfy both support monotonicity and representation axioms. While all the voting rules that we have analyzed that satisfy PJR (or EJR) also satisfy the weak support monotonicity axioms, the situation changes when we require the strong axioms. In particular, none of the rules analyzed that satisfy PJR also satisfy strong SMWOPI, and only PAV (which has the additional difficulty of being NP-hard to compute) satisfies strong SMWPI. Whether it is possible to develop a voting rule that satisfies strong SMWPI and PJR at the same time is left open.

In contrast, we can formally prove that PR and strong SMWPI are incompatible axioms. 

Theorem 8. No rule can satisfy PR and strong SMWPI at the same time.

Proof. Consider the following election. Let $k = 3$ and $C = \{c_1, \ldots, c_5\}$. 12 voters cast the following ballots: 2 voters approve of $\{c_1, c_4\}$, 2 voters approve of $\{c_1, c_5\}$, 3 voters approve of $\{c_2, c_4\}$, one voter approves of $\{c_2, c_5\}$, 2 voters approve of $\{c_3, c_5\}$, and 2 voters approve of $\{c_3\}$. For this election any voting rule that satisfies PR has to output $\{c_1, c_2, c_3\}$. Now, suppose that 3 new voters enter the election, and that all these new voters approve of $\{c_1, c_3\}$. For this extended election a voting rule that satisfies PR has to output only $\{c_3, c_4, c_5\}$.

There is an apparent contradiction between this theorem and table 1 because table 1 says that CC satisfies both PR and strong SMWPI. The reason for this apparent contradiction is that, as explained in footnote 8, CC satisfies PR only if ties are broken in favour of the sets of candidates that provide PR. The example of theorem 8 illustrates this. For the initial election CC outputs $\{c_1, c_2, c_3\}$ and $\{c_3, c_4, c_5\}$. However, if ties are broken in favour of the sets of candidates that provide PR, then CC (with this tie-breaking rule) will output only $\{c_1, c_2, c_3\}$. Now, after adding 3 new voters that approve of $\{c_1, c_3\}$, strong SMWPI requires that both $c_1$ and $c_3$ are in the set of winners while PR requires that the set of winners is $\{c_3, c_4, c_5\}$.

Whether strong SMWOPI and PR are compatible axioms is unclear. We observe that if certain candidate subset $W$ provides PR for a certain election $E = (N, C, A, k)$, then for each non-empty candidates subset $G$ of $W$, and for each voter $i$ such that $A_i \cap G = \emptyset$, it holds that $W$ also provides PR for $E_{i+G}$. It is enough to observe that the same assignment between candidates and voters that works for $E$
will also work for $E_{i+G}$. In particular if voter $i$ approved of her assigned candidate $w$ in election $E$, this means that $w \in A_i$, and therefore voter $i$ approves of $w$ also in election $E_{i+G}$.

However, PR and committee monotonicity are also incompatible.

**Theorem 9.** No rule can satisfy PR and committee monotonicity at the same time.

*Proof.* Consider the following election. Let $k = 3$ and $C = \{c_1, \ldots, c_5\}$. 6 voters cast the following ballots. For $i = 1$ to 3, and for $j = 1$ to 2, one voter approves of $\{c_i, c_{3+j}\}$. If the target committee size is 2, a voting rule that satisfies PR has to output only $\{c_4, c_5\}$, but if the target committee size is 3, a voting rule that satisfies PR has to output only $\{c_1, c_2, c_3\}$.

### 6 Conclusions and future work

In this paper we have complemented previous works on the axiomatic study of multi-winner voting rules with the study of monotonicity axioms for rules that make use of approval ballots. Our results show that support monotonicity in approval-based multi-winner voting rules is more tricky than it may seem at first glance. While the weak support monotonicity axioms are satisfied in almost all the cases analyzed in this study (only Monroe fails one of these) the situation changes completely when we look to the strong axioms. Of the 9 rules analyzed only 5 satisfy strong SMWPI and only 2 satisfy SMWOPI. The analysis of committee monotonicity is more straightforward and its scope of application is more limited. However, we believe that it is interesting to include this axiom in a review of the properties satisfied by approval-based multi-winner voting rules.

This study, as others that we have already mentioned, is focused on the definition of axioms and the analysis of voting rules with such axioms. It should be mentioned that there exist several other papers related to the axiomatic characterization of multi-winner voting rules. The idea of axiomatic characterization is to identify a voting rule as the only voting rule (possibly within certain family of voting rules) that possesses simultaneously a certain combination of axioms. Examples of works in this direction are those of Skowron et al. [2016b] and Freeman et al. [2014] for ranked ballots and Lackner and Skowron [2017] for approval ballots.

We stress that none of the rules that have been analyzed satisfy both PJR (or EJR) and strong SMWOPI. As we said in the introduction, in many applications of multi-winner voting rules it is desired that the set of winners represent the preferences of voters and support monotonicity is generally considered a desirable property. Therefore, we believe that it would be very interesting to find a voting rule that satisfies PJR (or EJR) and both strong SMWPI and strong SMWOPI. Other open issues would be to find a rule that satisfies both EJR and committee monotonicity and to find a rule that satisfies strong SMWOPI and PR.

\footnote{A similar idea can be used to characterize a subfamily of voting rules within a family.}
7 Acknowledgements

This research was supported in part by the Spanish Ministerio de Economía y Competitividad (project AUDACity TIN2016-77158-C4-1-R) and by the Autonomous Community of Madrid (project e-Madrid S2013/ICE-2715).

References

Kenneth J. Arrow. *Social choice and individual values*. John Wiley and sons, 1951.

H. Aziz, M. Brill, V. Conitzer, E. Elkind, R. Freeman, and T. Walsh. Justified representation in approval-based committee voting. *Social Choice and Welfare*, 48(2):461–485, 2017.

Haris Aziz, Serge Gaspers, Joachim Gudmundsson, Simon Mackenzie, Nicholas Mattei, and Toby Walsh. Computational aspects of multi-winner approval voting. In *Proceedings of the 2015 International Conference on Autonomous Agents and Multiagent Systems*, pages 107–115. International Foundation for Autonomous Agents and Multiagent Systems, 2015.

Salvador Barberá and Danilo Coelho. How to choose a non-controversial list with k names. *Social Choice and Welfare*, 31(1):79–96, 2008.

M. Brill, R. Freeman, S. Janson, and M. Lackner. Phragmén’s voting methods and justified representation. In *31st AAAI Conference on Artificial Intelligence (AAAI)*, pages 406–413. AAAI Press, 2017.

John R. Chamberlin and Paul N. Courant. Deliberations and representative decisions: Proportional representation and the Borda rule. *The American Political Science Review*, 77(3):718–733, 1983.

Michael Dummet. *Voting Procedures*. Oxford University Press, 1984.

Cynthia Dwork, Ravi Kumar, Moni Naor, and Dandapani Sivakumar. Rank aggregation methods for the web. In *Proceedings of the 10th international conference on World Wide Web*, pages 613–622. ACM, 2001.

E. Elkind, P. Faliszewski, P. Skowron, and A. Slinko. Properties of multiwinner voting rules. *Social Choice and Welfare*, 48(3):599–632, 2017.

Rupert Freeman, Markus Brill, and Vincent Conitzer. On the axiomatic characterization of runoff voting rules. In *Twenty-Eighth AAAI Conference on Artificial Intelligence (2014)*, pages 675–681, 2014.

Richard W. Hamming. Error detecting and error correcting codes. *Bell System technical journal*, 29(2):147–160, 1950.
S. Janson. Phragmén’s and Thiele’s election methods. ArXiv e-prints, November 2016. arXiv:1611.08826 [math.HO].

D. Marc. Kilgour. Approval balloting for multi-winner elections. In Jean-François Laslier and M. Remzi Sanver, editors, Handbook on Approval Voting, pages 105–124. Springer, 2010.

M. Lackner and P. Skowron. Consistent Approval-Based Multi-Winner Rules. ArXiv e-prints, April 2017. arXiv:1704.02453 [cs.GT].

Rob LeGrand, Evangelos Markakis, and Aranyak Mehta. Approval voting: local search heuristics and approximation algorithms for the minimax solution. In First International Workshop on Computational Social Choice (COMSOC 2006), pages 276–289, Amsterdam, Netherlands, 2006.

Tyler Lu and Craig Boutilier. Budgeted social choice: From consensus to personalized decision making. In Twenty-Second International Joint Conference on Artificial Intelligence, pages 280–286. AAAI Press, July 2011.

Burt L. Monroe. Fully proportional representation. The American Political Science Review, 89(4):925–940, 1995.

Xavier Mora and Maria Oliver. Eleccions mitjancant el vot d’aprovació. el mètode de phragmén i algunes variants. Butlletí de la Societat Catalana de Matemàtiques, 30(1):57–101, 2015.

Lihi Naamani-Dery, Meir Kalech, Lior Rokach, and Bracha Shapira. Preference elicitation for narrowing the recommended list for groups. In Proceedings of the 8th ACM Conference on Recommender Systems, RecSys ’14, pages 333–336, New York, NY, USA, 2014. ACM. ISBN 978-1-4503-2668-1. doi: 10.1145/2645710.2645760. URL http://doi.acm.org/10.1145/2645710.2645760

Edvard Phragmén. Sur une méthode nouvelle pour réaliser, dans les élections, la représentation proportionnelle des partis. Öfversigt af Kongliga Vetenskaps-Akademins Förhandlingar, 51(3):133–137, 1894.

Edvard Phragmén. Proportionella val. en valtekinsk studie. In Svenska spörsmål 25. Lars Hökersbergs förlag, Stockholm, 1895.

Edvard Phragmén. Sur la théorie des élections multiples. Öfversigt af Kongliga Vetenskaps-Akademins Förhandlingar, 53:181–191, 1896.

Edvard Phragmén. Till frågan om en proportionell valmetod. Statsvetenskaplig Tidskrift, 2(2):297–305, 1899. URL http://cts.lub.lu.se/ojs/index.php/st/article/view/1949
Ariel D. Procaccia, Jeffrey S. Rosenschein, and Aviv Zohar. On the complexity of achieving proportional representation. *Social Choice and Welfare*, 30(3):353–362, 2008.

L. Sánchez-Fernández, E. Elkind, M. Lackner, N. Fernández, J. A. Fisteus, P. Basanta Val, and P. Skowron. Proportional Justified Representation. In *31st AAAI Conference on Artificial Intelligence (AAAI)*, pages 670–676. AAAI Press, 2017.

P. Skowron, M. Lackner, M. Brill, D. Peters, and E. Elkind. Proportional Rankings. *ArXiv e-prints*, December 2016. arXiv:1612.01434 [cs.GT].

Piotr Skowron, Piotr Faliszewski, and Jérôme Lang. Finding a collective set of items: From proportional multirepresentation to group recommendation. *Artificial Intelligence*, 241:191–216, 2016a.

Piotr Skowron, Piotr Faliszewski, and Arkadii Slinko. Axiomatic characterization of committee scoring rules. In *Sixth International Workshop on Computational Social Choice (2016)*, 2016b.

Thorvald N. Thiele. Om flerfoldsvalg. *Oversigt over det Kongelige Danske Videnskabernes Selskabs Forhandlinger*, pages 415–441, 1895.

D. R. Woodall. Properties of preferential election rules. *Voting matters*, 3:8–15, 1994.
A PJR and PR: omitted proofs and counterexamples

First of all we review the definition of proportional justified representation (PJR).

Definition 5. Proportional justified representation Consider a ballot profile $A = (A_1, \ldots, A_n)$ over a candidate set $C$ and a target committee size $k$, $k \leq |C|$. Given a positive integer $\ell \in \{1, \ldots, k\}$, we say that a set of voters $N^* \subseteq N$ is $\ell$-cohesive if $|N^*| \geq \frac{\ell}{k}$ and $|\bigcap_{i \in N^*} A_i| \geq \ell$. We say that a set of candidates $W$, $|W| = k$, provides proportional justified representation (PJR) for $(A, k)$ if for every $\ell \in \{1, \ldots, k\}$ and every $\ell$-cohesive set of voters $N^* \subseteq N$ it holds that $|W \cap (\bigcup_{i \in N^*} A_i)| \geq \ell$. We say that an approval-based voting rule satisfies proportional justified representation (PJR) if for every ballot profile $A$ and every target committee size $k$ it outputs a set of winners that provides PJR for $(A, k)$.

Aziz et al. [2017] prove that CC satisfies JR but fails EJR. We use here the same example used by Sánchez-Fernández et al. [2017] to prove that CC fails also PJR.

Example 11. Let $k = 7$ and $C = \{c_1, \ldots, c_8\}$. 10 agents cast the following ballots: 6 agents approve of $\{c_1, c_2, c_3, c_4\}$, 1 agent approves only of candidate $c_5$, 1 agent approves only of candidate $c_6$, 1 agent approves only of candidate $c_7$, and 1 agent approves only of candidate $c_8$. The 6 agents that approve of $\{c_1, c_2, c_3, c_4\}$ form a 4-cohesive group of agents ($6\frac{2}{5} = 4.2 \geq 4$). In this example it is possible to find a set of winners such that all the agents approve of some of the winners (this is the optimal case for the Chamberlin and Courant rule). But this requires candidates $c_5$, $c_6$, $c_7$, and $c_8$ to be in the set of winners, and therefore only 3 of $c_1, c_2, c_3$ and $c_4$ can be in the set of winners outputted by the Chamberlin and Courant rule for this election.

We turn now to the study of PR. Sánchez-Fernández et al. [2017] proved that every rule that satisfies PR is NP-hard to compute. Therefore, we should not expect that any rule computable in polynomial time satisfies PR, as the following examples show.

Example 12. Let $k = 3$ and $C = \{a_1, a_2, a_3, b_1, b_2, b_3\}$. 3 voters cast the following ballots: 2 voters approve of $\{a_1, a_2, a_3\}$ and 1 voter approves of $\{b_1, b_2, b_3\}$. In this example both AV and SAV output $\{a_1, a_2, a_3\}$. However, all the candidates subsets that provide PR for this example have to include one candidate from $\{b_1, b_2, b_3\}$.

Example 13. Let $k = 2$ and $C = \{a, b, c\}$. 6 voters cast the following ballots: 2 voters approve of $\{a, b\}$, 2 voters approve of $\{a, c\}$, 1 voter approves of $\{b\}$ and 1 voter approves of $\{c\}$. The only candidates subset that provides PR for this example is $\{b, c\}$. However, both SeqPAV and max-Phragmén add candidate $a$ (the most approved of one) to the set of winners in the first iteration.

MAV also fails PR as the following example shows.

21
**Example 14.** Let \( k = 3 \) and \( C = \{a_1, a_2, b_1, b_2, b_3\} \). 3 voters cast the following ballots: 2 voters approve of \( \{a_1, a_2\} \) and 1 voter approves of \( \{b_1, b_2, b_3\} \). In this example the sets of winners outputted by MAV contain 1 candidate from \( \{a_1, a_2\} \) and 2 candidates from \( \{b_1, b_2, b_3\} \). However, all the candidates subsets that provide PR for this example have to include \( \{a_1, a_2\} \) and one candidate from \( \{b_1, b_2, b_3\} \).

The last case is CC. It is evident that for each election \( \mathcal{E} = (A, k) \) in which candidates subsets that provide PR exist, for each candidates subset \( W \) that provides PR for election \( \mathcal{E} \) it holds that all the voters that participate in the election approve of some of the candidates in \( W \). It follows immediately that \( W \) would have missrepresentation 0 according to the rules in CC, and therefore that \( W \) is outputted by CC for election \( \mathcal{E} \). It may happen, however, that CC outputs sets of winners that do not provide PR even if sets of winners that provide PR exist.

**Example 15.** We consider again the election in example 12. One of the sets of winners that CC outputs for this election is \( \{a_1, b_1, b_2\} \). However, all the candidates subsets that provide PR for this example have to include two candidates from \( \{a_1, a_2, a_3\} \).

In summary, CC satisfies PR only if ties are broken always in favour of the candidates subsets that provide PR.