EQUATIONS OF THE FORM \( t(x + a) = t(x) \) AND \( t(x + a) = 1 - t(x) \) FOR THUE-MORSE SEQUENCE

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Abstract. For every \( a \geq 1 \) we give a recursion algorithm of building of set of solutions of equations of the form \( t(x + a) = t(x) \) and \( t(x + a) = 1 - t(x) \), where \( \{t(n)\} \) is Thue-Morse sequence. We pose an open problem and two conjectures.

1. Introduction and main results

The Thue-Morse (or Prouhet-Thue-Morse [1]) sequence \( \{t_n\}_{n \geq 0} \) is one of the most known and useful \((0, 1)\)-sequences. By the definition, \( t_n = 0 \), if the binary expansion of \( n \) contains an even number of \( 1 \)'s, and \( t_n = 1 \) otherwise. It is sequence A010060 in OEIS [8]. Numerous applications of this sequence and a large bibliography one can find in [1] (see also the author’s articles [6]-[7] and especially the applied papers [4]-[5] in combinatorics and [3] and [11] in informative theory, in which the Thue-Morse sequence plays a key role in their constructions). Let \( \mathbb{N}_0 \) be the set of nonnegative integers. For \( a \in \mathbb{N} \), consider on \( \mathbb{N}_0 \) equations

\[
(1) \quad t(x + a) = t(x), \\
(2) \quad t(x + a) = 1 - t(x).
\]

Denote \( C_a \) and \( B_a \) the sets of solutions of equations (1) and (2) correspondingly. Evidently we have

\[
(3) \quad B_a \cup C_a = \mathbb{N}_0, \quad B_a \cap C_a = \emptyset.
\]

The following lemma is proved straightforward (cf.[8], A079523, A121539).

Lemma 1. \( B_1 \) (\( C_1 \)) consists of nonnegative integers the binary expansion of which ends in an even (odd) number of \( 1 \)'s.

For a set of integers \( A = \{a_1, a_2, \ldots\} \) let us introduce a translation operator

\[
(4) \quad E_h(A) = \{a_1 - h, a_2 - h, \ldots\}.
\]

One of our main result is the following.

Theorem 1. \( B_a \) and \( C_a \) are obtained by a finite set of operations of translation, union and intersection over \( B_1 \) and \( C_1 \).
It is well known that the Thue-Morse sequence is not periodic (a very attractive proof of this fact is given in [9]). Nevertheless, it is trivial to note that for every \( n \in \mathbb{N}_0 \) there exists \( x = 1, 2 \) or \( 3 \) such that \( t(n + x) = t(n) \). Indeed, as is well-known, the Thue-Morse sequence does not contain configurations of the form 000 or 111. Therefore, if to suppose that the equalities \( t(n + x) = 1 - t(n) \), \( x = 1, 2, 3 \), are valid simultaneously, then we have a contradiction. In connection with this, it is natural to pose the following problem.

\textbf{Question 1.} For which numbers \( a, b, c \) one can state that for every \( n \in \mathbb{N}_0 \) there exists \( x = a, b \) or \( c \) such that \( t(n + x) = t(n) \)?

In conclusion of introduction, we pose a quite another conjecture. Recall that \( n \in \mathbb{N}_0 \) is called evil (odious) if the number of 1’s in its binary expansion is even (odd). Thus, by the above definition of Thue-Morse sequence, for evil (odious) \( n \) we have \( t_n = 0 \) \( (t_n = 1) \). Denote \( \{B_a(n)\}\{C_a(n)\} \) the sequence of elements of \( B_a(C_a) \) in the increasing order. Denote, furthermore, \( \{\beta_a(n)\}\{\gamma_a(n)\} \) \((0, 1)\)-sequence, which is obtained from \( \{B_a(n)\}\{C_a(n)\} \) by replacing the odious terms by 1’s and the evil terms by 0’s.

\textbf{Conjecture 1.} 1) Sequence \( \{\gamma_a(n)\} \) is periodic; 2) if \( 2^m | a \), then the minimal period has \( 2^{m+1} \) terms, moreover, 3) if \( a \) is evil, then the minimal period contains the first \( 2^{m+1} \) terms of Thue-Morse sequence \( \{t_n\} \), otherwise, it contains the first \( 2^{m+1} \) terms of sequence \( \{1 - t_n\} \); 4) \( \beta_a(n) + \gamma_a(n) = 1 \).

Below we prove the conjecture in case of \( a = 2^m \).

\textbf{2. Some formulas for} \( B_a \) \textbf{and} \( C_a \)

\textbf{Theorem 2.}

(5) \[ B_{a+1} = (C_a \cap E_a(B_1)) \cup (B_a \cap E_a(C_1)) \]

(6) \[ C_{a+1} = (C_a \cap E_a(C_1)) \cup (B_a \cap E_a(B_1)) \]

\textbf{Proof.} Denote the right hand sides of (5) and (6) via \( B^*_a \) and \( C^*_a \) correspondingly. Show that \( B^*_a \cup C^*_a = \mathbb{N}_0 \). Indeed, using (3)-(6), we have \( B^*_a \cup C^*_a = (C_a \cap (E_a(B_1) \cup E_a(C_1))) \cup (B_a \cap (E_a(C_1) \cup E_a(B_1))) = \)
Now it is sufficient to show that $B_{a+1}^*$ contains only solutions of (2) for $a := a+1$, while $C_{a+1}^*$ contains only solutions of (1) for $a := a+1$. Indeed, let $x \in B_{a+1}^*$. Distinguish two cases: 1) $x \in C_a \cap E_a(B_1)$ and 2) $x \in B_a \cap E_a(C_1)$. In case 1) (1) is valid and $x + a \in B_1$. Thus
\[ t(x + a + 1) + t(x + a) = 1, \]
or, taking into account (1), we have
\[ t(x + a + 1) = 1 - t(x). \]
In case 2) (2) is valid and $x + a \in C_1$. Thus
\[ t(x + a + 1) = t(x + a), \]
or, taking into account (2), we have
\[ t(x + a + 1) = 1 - t(x). \]

Let now $x \in C_{a+1}^*$. Again distinguish two cases: 1) $x \in C_a \cap E_a(C_1)$ and 2) $x \in B_a \cap E_a(B_1)$. In case 1) (1) is valid and $x + a \in C_1$. Thus
\[ t(x + a + 1) = t(x + a), \]
or, taking into account (1), we have
\[ t(x + a + 1) = t(x). \]
In case 2) (2) is valid and $x + a \in B_1$. Thus
\[ t(x + a + 1) = 1 - t(x + a), \]
or, taking into account (2), we have
\[ t(x + a + 1) = t(x). \]

Consequently, $B_{a+1}^* \cap C_{a+1}^* = \emptyset$ and $B_{a+1}^* = B_{a+1}$, $C_{a+1}^* = C_{a+1}$. ■

**Example 1.** (cf. A081706[8]; this sequence is closely connected with sequence of Allouche et al. [2], A003159[8])

According to Theorem 2, we have
\[ C_2 = (C_1 \cap E_1(C_1)) \cup (B_1 \cap E_1(B_1)). \]
Since, evidently, $C_1 \cap E_1(C_1) = \emptyset$, then we obtain a representation
\[ C_2 = B_1 \cap E_1(B_1). \]
Example 2. (cf. our sequences A161916, A161974 in [8]) Denote $C_3^{(0)}$ the subset of $C_3$ such that for $n \in C_3^{(0)}$ we have: $\min\{x : t(n + x) = t(x)\} = 3$. The following simple formula is valid:

$$C_3^{(0)} = E_1(C_1).$$

Proof. Using (7), consider the following partition of $\mathbb{N}_0$:

$$\mathbb{N}_0 = C_1 \cup B_1 = C_1 \cup (B_1 \cap E_1(B_1)) \cup (B_1 \cap \overline{E_1(B_1)}) = C_1 \cup C_2 \cup D,$$

where

$$D = B_1 \cap \overline{E_1(B_1)}$$

Evidently,

$$D \cap C_1 = \emptyset, \quad D \cap C_2 = D \cap (B_1 \cap E_1(B_1)) = \emptyset.$$

Thus $D = C_3^{(0)}$. On the other hand, we have

$$D = B_1 \cap \overline{E_1(B_1)} = B_1 \cap E_1(C_1) = E_1(C_1).$$

By the same way one can prove the following more general results.

Theorem 3. (A generalization) Let $l + m = a + 1$. Then we have

$$B_{a+1} = (C_l \cap E_l(B_m)) \cup (B_l \cap E_l(C_m)),$$

$$C_{a+1} = (C_l \cap E_l(C_m)) \cup (B_l \cap E_l(B_m)).$$

In particular together with (5)-(6) we have

$$B_{a+1} = (C_l \cap E_l(B_a)) \cup (B_l \cap E_l(C_a)),$$

$$C_{a+1} = (C_l \cap E_l(C_a)) \cup (B_l \cap E_l(B_a)).$$

Further, for a set of integers $A = \{a_1, a_2, \ldots\}$, denote $hA$ the set $A = \{ha_1, ha_2,\ldots\}$.

Theorem 4. For $m \in \mathbb{N}$ we have

$$B_{2^m} = \bigcup_{k=0}^{2^m-1} E_{-k}(2^m B_1),$$

$$C_{2^m} = \bigcup_{k=0}^{2^m-1} E_{-k}(2^m C_1).$$

Proof. It is sufficient to consider numbers of the form

$$n = \ldots 011\ldots 1 \times \ldots \times,$$

where the $m$ last digits are arbitrary. The theorem follows from a simple observation that the indicated in (14) series of 1’s contains an odd (even) number of 1’s if and only if $n \in C_{2^m} \ (n \in B_{2^m}).$
Example 3.

\begin{equation}
C_2 = (2C_1) \cup E_{-1}(2C_1).
\end{equation}

Comparison with (7) leads to an identity
\begin{equation}
(2C_1) \cup E_{-1}(2C_1) = B_1 \cap E_1(B_1).
\end{equation}

On the other hand, the calculating $B_2$ by Theorems 3,5 leads to another identity
\begin{equation}
(2B_1) \cup E_{-1}(2B_1) = C_1 \cup E_1(C_1).
\end{equation}

**Corollary 1.** For $a = 2^n$, Conjecture 1 is true.

**Proof.** In view of the structure of formulas (12)-(13), it is sufficient to prove that in sequences $\{B_1(n)\}, \{C_1(n)\}$ odious and evil terms alternate. Indeed, in the mapping $\{B_{2m}(n)\}(\{C_{2m}(n)\})$ on $\{\beta_{2m}(n)\}(\{\gamma_{2m}(n)\})$ correspondingly, for any $x \in B_1(n)$ the ordered subset
\begin{equation}
\bigcup_{k=0}^{2^{m-1}} E_{-k}(2^m x)
\end{equation}
of $B_{2m}$ (12) maps on the first $2^m$ terms of sequence $\{t_n\}$ or $\{1-t_n\}$ depending on the number $x$ is evil or odious. Therefore, if odious and evil terms of $B_1(n)$ alternate, then we obtain the minimal period $2^{m+1}$ for $\{\beta_{2m}(n)\}$. By the same way we prove that if odious and evil terms of $C_1(n)$ alternate, then we obtain the minimal period $2^{m+1}$ for $\{\gamma_{2m}(n)\}$. Now we prove that odious and evil terms of, e.g., $C_1(n)$, indeed, alternate. If the binary expansion of $n$ ends in more than 1 odd 1’s, then the nearest following number from $\{C_1(n)\}$ is $n + 2$, and it is easy to see that the relation $t(n+2) = 1 - t(n)$ satisfies; if the binary expansion of $n$ ends in one isolated 1, and before it we have a series of more than 1 0’s, then the nearest following number from $\{C_1(n)\}$ is $n + 4$, and it is easy to see that the relation $t(n+4) = 1 - t(n)$ again satisfies; at last, if the binary expansion of $n$ ends in one isolated 0, i.e. $n$ has the form $\ldots 011\ldots 101$, then we distinguish two cases: the series of 1’s before two last digits 01 contains a)odd and b)even 1’s. In case a) the nearest following number from $\{C_1(n)\}$ is $n + 2$, with the relation $t(n+2) = 1 - t(n)$, while in case b) it is $n + 4$ with the relation $t(n+4) = 1 - t(n)$. Thus odious and evil terms of $\{C_1(n)\}$, indeed, alternate. For $\{B_1(n)\}$ the statement is proved quite analogously.\[\blacksquare\]
Theorem 5. (Formulas of complement to power of 2) Let \( 2^{m-1} + 1 \leq a \leq 2^m \). Then we have
\[
B_a = (C_{2^m} \cap E_a(B_{2^m-2})) \cup (B_{2^m} \cap E_a(C_{2^m-2})) ,
\]
\[
C_a = (B_{2^m} \cap E_a(B_{2^m-2})) \cup (C_{2^m} \cap E_a(C_{2^m-2})�)
\]

Proof. Denote the right hand sides of the formulas being proved via \( B^{**}_a \) and \( C^{**}_a \) correspondingly. Show that \( B^{**}_a \cup C^{**}_a = \mathbb{N}_0 \). Indeed,
\[
B^{**}_a \cup C^{**}_a = (C_{2^m} \cap (E_a(B_{2^m-2}) \cup E_a(C_{2^m-2})) \cup (B_{2^m} \cap (E_a(C_{2^m-2}) \cup E_a(B_{2^m-2})))) = (E_a(B_{2^m-2}) \cup E_a(C_{2^m-2})) \cup (B_{2^m} \cup C_{2^m})) = E_a(\mathbb{N}_0) \cap \mathbb{N}_0 = \mathbb{N}_0 .
\]

Now, by the same way as in proof of Theorem 3, it is easy to show that \( B^{**}_a \) contains only solutions of \( (2) \) , while \( C^{**}_a \) contains only solutions of \( (1) \). Then \( B^{**}_a \cap C^{**}_a = \emptyset \) and \( B^{**}_a = B_a \), \( C^{**}_a = C_a \). \( \blacksquare \)

3. An approximation of Thue-Morse constant

Let \( T_m \) (\( U_m \)) be the number which is obtained by the reading the period of \( \{ \beta_a(n) \} \{ \gamma_a(n) \} \) as \( 2^{m+1} \)–bits binary number. Note that \( U_m = T_m \), i.e. \( U_m \) is obtained from \( T_m \) by replacing 0’s by 1’s and 1’s by 0’s. Therefore,
\[
T_m + U_m = 2^{2m+1} - 1 .
\]

Denote \( U_m \vee T_m \) the concatenation of \( U_m \) and \( T_m \). Then, using (18), we have
\[
U_0 = 1, \text{ for } m \geq 0 ,
\]
(19) \( U_{m+1} = U_m \vee T_m = 2^{2m+1}U_m + 2^{2m+1} - U_m - 1 = (2^{2m+1} - 1)(U_m + 1). \)

Consider now the infinite binary fraction corresponding to sequence \( \{ \gamma_a(n) \} : \)
\[
\tau_m = . U_m \vee U_m \vee \ldots = U_m/(2^{2m+1} - 1).
\]

Lemma 2. If \( F_n = 2^{2n+1} \) is \( n \)-th Fermat number, then we have a recursion:
(21) \( F_{m+1} \tau_{m+1} = 1 + (F_{m+1} - 2) \tau_m, \quad m \geq 0 \)

with \( \tau_0 \) defined as the binary fraction
(22) \( \tau_0 = .010101\ldots = 1/3. \)

Proof. Indeed, according to (19)-(20), we have
\[
\tau_{m+1} = U_{m+1} \vee U_{m+1} \vee \ldots = U_{m+1}/(2^{2m+2} - 1) = (2^{2m+1} - 1)(U_m + 1)/(2^{2m+2} - 1) = (U_m + 1)/(2^{2m+1} + 1) = (1 + \tau_m(2^{2m+1} - 1))/(2^{2m+1} + 1) = \]

\[(1 + \tau_m(F_{m+1} - 2))/F_{m+1},\]
and the lemma follows. ■

So, by (21)-(22) for \(m = 0, 1, \ldots\) we find
\[
\tau_1 = 2/5, \quad \tau_2 = 7/17, \quad \tau_3 = 106/257, \\
\tau_4 = 27031/65537, \quad \tau_5 = 1771476586/4294967297, \ldots .
\]

It follows from (21) that the numerators \(\{s_n\}\) of these fractions satisfy the recursion
\[(23) \quad s_1 = 2, \quad s_{n+1} = 1 + (2^{2n} - 1)s_n, \quad n \geq 1,
\]
while the denominators are \(\{F_n\}\). Of course, by its definition, the sequence \(\{\tau_n\}\) very fast converges to the Thue-Morse constant
\[
\tau = \sum_{n=1}^{\infty} \frac{t_n}{2^n} = 0.4124540336401\ldots.
\]
E.g., \(\tau_5\) approximates \(\tau\) up to \(10^{-9}\).

**Conjecture 2.** For \(n \geq 1\), the fraction \(\tau_n = s_n/F_n\) is a convergent corresponding to the continued fraction for \(\tau\).

Note that, the first values of indices of the corresponding convergents, according to numeration of A085394 and A085395 [8] are: 3, 5, 7, 13, 23, ...

Note also that the binary fraction corresponding to sequence \(\{\beta_a(n)\}\):
\[
\bar{\tau}_m = .T_m \lor T_m \lor ...
\]
satisfies the same relation (21) but with
\[
\bar{\tau}_0 = .101010\ldots = 2/3,
\]
and converges to \(1 - \tau\).

4. **Comparison with the Weisstein approximations**

Now we want to show that Conjecture 2 is very plausible. As is well known, if the fraction \(p/q, \quad q > 0\), is a convergent (beginning the second one) corresponding to the continued fraction for \(\alpha\), then \(p/q\) is the best approximation to \(\alpha\) between all fractions of the form \(x/y, \quad y > 0\), with \(y \leq q\).

Weisstein [10] considered the approximations of \(\tau\) of the form:
\[
a_0 = 0.02; \quad a_1 = 0.012; \quad a_2 = 0.01102; \quad a_3 = 0.011010012; \\
a_4 = 0.0110100110010101102; \ldots
\]
If to keep the "natural" denominators $F_n - 1$ (without cancelations), then, denoting $w_n$ the numerators of these fractions, we have

\[(25)\quad w_{n+1} = U_n, \quad n \geq 0.\]

Since, according to (19),

\[U_n = 2^{2^n} U_{n-1} + 2^{2^n} - U_{n-1} - 1, \quad n \geq 1,\]

then

\[(26)\quad w = 1, \quad w_{n+1} = 2^{2^n} - 1 + (2^{2^n} - 1)w_n, \quad n \geq 1.\]

**Theorem 6.** We have

\[(27)\quad w_n/(F_n - 1) < s_n/F_n < \tau\]

**Proof.** It is easy to see that $s_n/F_n < \tau$. Indeed, since $t_{2^n} = 1$, then $(2^n + 1)$-th binary digit of $\tau$ after the point is 1, while the period of $\tau_n$ begins from 0. Let us now prove the left inequality. To this end, let us prove by induction that

\[(28)\quad s_n - w_n = 1.\]

Indeed, if (28) is true for some $n$, then, subtracting (26) from (23), we find

\[s_{n+1} - w_{n+1} = -2^{2^n} + 2 + (2^{2^n} - 1)(s_n - w_n) = 1.\]

Thus finally we have

\[w_n/(F_n - 1) = (s_n - 1)/(F_n - 1) < s_n/F_n < \tau.\]

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