Two coupled, driven Ising spin systems working as an Engine

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(Dated: September 23, 2016)

Miniaturized heat engines constitute a fascinating field of current research. They are being studied theoretically as well as experimentally, with experiments involving colloidal particles and harmonic traps and even bacterial baths acting like thermal baths. They are interesting to study because usual equilibrium thermodynamic notions can not be applied directly to these systems. These systems are micron sized or even smaller and they are subjected to laud thermal fluctuations. Thus one needs to study the behavior of such systems in terms of these fluctuations. Average thermodynamic quantities like work done, heat exchanged, efficiency lose meaning unless otherwise supported by their full probability distributions. Earlier studies on micro-engines are concerned with applying Carnot or Stirling engine protocols to miniaturized systems, where system undergoes typical two isothermal and two adiabatic changes. Unlike these models we for the first time, study a prototype system of two classical Ising spins driven by time dependent, phase different, external magnetic fields. These spins are simultaneously in contact with two heat reservoirs at different temperatures for the full duration of the driving protocol. Performance of the model as an engine or a refrigerator depends only on a single parameter namely the phase between two external drivings. We study this system in terms of fluctuations in efficiency and coefficient of performance (COP). We also find full distributions of these quantities numerically and also study the tails of these distributions. We also study reliability of this engine. We find the fluctuations dominate mean values of efficiency and COP and their probability distributions are broad with power law tails.

Introduction: After Feynman’s theoretical construction of his famous Ratchet and Pawl machine in [1], due to advancement in nano science, it is now possible to realize such miniaturized engines experimentally [2–9]. Study of these micro-engines theoretically is also fascinating because many of the experiments are actually based on theoretical predictions namely the fluctuation theorems which put bounds on thermodynamic quantities of interests like efficiency of such miniature engines [8, 7]. Thermodynamic engines like Carnot or Stirling, where fluctuations are usually ignored and most of the physics comes out of average values of work and heat [8]. These notions however fail in case of microscopic engines. Micro-engines behave differently and the main reason behind this odd behavior are the laud thermal fluctuations they experience. These thermal fluctuations cause energy exchanges of the order of \( k_B T \), where \( k_B \) is the Boltzmann constant, \( T \) ambient temperature. Thus one can not just rely on mean values of work and heat or as a matter of fact any thermodynamic quantity, but one has to look at full probability distributions. To deal with such systems one needs to use a framework of stochastic thermodynamics [9–11]. Many studies on such small scale engines have shown that fluctuations in thermodynamic quantities dominate over mean values even in the quasistatic limit [12–19]. Many studies have also dealt with looking at full distributions of efficiency [13–14] and also the large deviation functions [20]. Models with feedback control both instantaneous and delayed have also been investigated [21–24].

Most of the earlier studies both theoretical and experimental were based on applying the thermodynamic engine protocols like Carnot or Stirling to a colloidal particle placed in an harmonic trap. The trap strength is then modified time dependently to mimic isothermal expansion, compression and adiabatic expansion, compression steps [2, 3, 13–14]. We would like to point out that there are in fact no detailed studies which deal with systems which are simultaneously in contact with several heat baths. These systems show many novel features not seen in earlier studied models.

In this work we have studied a model of classical heat engine and a pump where two Ising spins are independently kept in contact with two heat baths at different temperatures. These spins are externally driven by time dependent magnetic fields with a phase difference [25], see Fig. 1. During full driving protocol system is never isolated from the heat baths. Amazingly this phase dif-

FIG. 1. Cartoon of the model discussed in the text.
ference is the only parameter which decides whether system works as a heat engine or a refrigerator. This model has been studied earlier by us in [25] but the focus of that study was primarily based on working of the model as a heat engine or a pump. Fluctuations in work and heat exchanged were not studied. Here we extend the work to address rich features this model exhibits in terms of phase diagrams of engine and pump performance. Fluctuations in efficiency, COP and their probability distributions, and also reliability of the model to work either as an engine or a refrigerator.

Model: Consider two classical Ising spins interacting with each other through the interaction energy $J$, driven by time dependent external magnetic fields of type $h_1(t) = h_0 \cos(\omega t)$ and $h_2(t) = h_0 \cos(\omega t + \phi)$, where $\phi$ is the phase difference and $\omega$ the driving frequency, as shown in Fig. 1. The Hamiltonian for this system is written as:

$$
\mathcal{H} = -J\sigma_1\sigma_2 - h_1(t)\sigma_1 - h_2(t)\sigma_2, \quad \sigma_{1,2} = \pm 1
$$

Left and right spins are in contact with heat baths at temperature $T_L$ and $T_R$ respectively. Interaction of spins with the respective heat baths is modeled by Glauber dynamics [26]. We define heat currents coming from left(right) baths $\dot{Q}_L(\dot{Q}_R)$ and work done on left(right) spin $\dot{W}_L(\dot{W}_R)$ to be positive. The total work done is nothing but $\dot{W} = \dot{W}_L + \dot{W}_R$. If $P(\sigma_1, \sigma_2, t)$ represents the probability to have state of the spins $\{\sigma_1, \sigma_2\}$ at time $t$ then the heat exchange rates can be written as:

$$
\dot{Q}_L = \sum_{\sigma_1, \sigma_2} P(\sigma_1, \sigma_2, t) r_{L,\sigma_1,\sigma_2}^L \Delta E_1(\sigma_1, \sigma_2)
$$

$$
\dot{Q}_R = \sum_{\sigma_1, \sigma_2} P(\sigma_1, \sigma_2, t) r_{R,\sigma_1,\sigma_2}^R \Delta E_2(\sigma_1, \sigma_2)
$$

$$
\dot{W}_L = -\langle \sigma_1 \rangle \dot{h}_1(t) = \sum_{\sigma_1, \sigma_2} \sigma_1 P(\sigma_1, \sigma_2, t)
$$

$$
\dot{W}_R = -\langle \sigma_2 \rangle \dot{h}_2(t) = \sum_{\sigma_1, \sigma_2} \sigma_2 P(\sigma_1, \sigma_2, t)
$$

where $r_{L,R}^{L,R} = r(1 - \gamma_{L,R}\sigma_1\sigma_2)(1 - \delta_{L,R}\sigma_{1,2})$ are the modified Glauber spin flip rates to compensate for two heat reservoirs, with $\gamma_{L,R} = \tanh(J/k_B T_{L,R})$, $\delta_{L,R} = \tanh(h_{1,2}/k_B T_{L,R})$, also $\Delta E_1 = 2(J\sigma_1\sigma_2 + h_1(t)\sigma_1)$ and $\Delta E_2 = 2(J\sigma_1\sigma_2 + h_2(t)\sigma_2)$ are energy changes associated with left or right spin flips respectively, with $r$ being a rate constant. In all the numerical calculations we put $J = 1.0$ and $k_B = 1$, and all energies are measured in these units. Expressions in Eq. 2 can be easily obtained from the master equation satisfied by $P(\sigma_1, \sigma_2, t)$, see [25] for details. It is easy to show that the average energy $U = \langle \mathcal{H} \rangle = \sum_{\sigma_1, \sigma_2} \mathcal{H}(\sigma_1, \sigma_2) P(\sigma_1, \sigma_2, t)$ and $\dot{U} = \dot{Q}_L + \dot{Q}_R + \dot{W}_L + \dot{W}_R$ from above expressions. Since external driving is time dependent, probability $P(\sigma_1, \sigma_2, t)$ remains time dependent even in the steady state. We also need time averaged quantities namely $\dot{q}_L = \dot{Q}_L$ and $\dot{q}_R = \dot{Q}_R$ and $\eta = \dot{W}/\dot{w}$, where $\tau = 2\pi/\omega$ is the time period of the external driving. If the heat is coming into the system from left or right baths then it is taken to be positive. Also the work done on the system is positive.

Once these definitions are set, we define stochastic efficiency $\epsilon$ and stochastic coefficient of performance (COP) $\eta$ as:

$$
\epsilon = \frac{\dot{w}}{\dot{q}_L}, \quad \eta = \frac{\dot{q}_R}{\dot{w}}.
$$

FIG. 2. (a) Engine mode of operation. $\dot{q}_L > 0$, $\dot{q}_R < 0$ and $\dot{w} < 0$ for certain values of the phase $\phi$. Parameter values are $h_0 = 0.25$, $\tau = 190$, $T_L = 1.0$, $T_R = 0.1$. At $\phi = 0.7\pi$ maximum work is extracted from the system (inset). In all the results discussed further these parameters are considered to be for engine mode of operation. (b) Pump/refrigerator mode of operation. Parameter values are $h_0 = 0.25$, $\tau = 225$, $T_L = 0.5$, $T_R = 0.5$. See for example at $\phi = 0.7\pi$ we have $\dot{q}_R > 0$, $\dot{w} > 0$, and $\dot{q}_L < 0$ implying heat is taken from the right bath, work is done on the system and heat is dumped into the left bath. In all the results discussed further these parameters are considered to be for pump/refrigerator mode of operation.
We note that due to large thermal fluctuations two efficiencies
\[ \bar{\epsilon} = \langle \dot{w} \rangle / \langle -\dot{q}_L \rangle \quad \text{and} \quad \langle \epsilon \rangle = \left\langle \frac{\dot{w}}{-\dot{q}_L} \right\rangle, \]
are in general not equal that is \( \langle \epsilon \rangle \neq \bar{\epsilon} \) similarly \( \langle \eta \rangle \neq \bar{\eta} \). The symbol \( \langle \cdot \rangle \) represents the ensemble average. For completeness we reproduce results from reference [25] to show how the phase \( \phi \) for given parameter values determines the engine or pump behavior. In Fig. 2 (a), (b), we plot \( \dot{w} \), \( \dot{q}_L \) and \( \dot{q}_R \) as a function of the phase \( \phi \) for engine and pump mode of operation respectively. Similar results, as in Fig. 2 (b) are obtained if the left bath is slightly colder showing one can transfer heat from colder to hotter bath acting like a refrigerator, see [25].

The average quantities can easily be obtained by solving the master equation numerically. But to study fluctuations and distributions of these quantities we have to rely on Monte-Carlo simulations which we now describe. **Simulations:** To study the dynamics of the system and for evaluating different heat currents, we perform Monte-Carlo simulations. We discretize the magnetic field sweep which consists of \( \sim 10^4 \) time steps such that each time step \( dt = \tau / 10^4 \) with \( \tau = 2\pi/\omega \) where \( \tau \) is the time period of external driving. We also fix the Boltzmann constant \( k_B = 1 \), and the interaction energy \( J = 1.0 \). Simulation follows usual Monte-Carlo steps in which first or second spin is chosen at random and it flips with probabilities given by Glauber dynamics discussed earlier. At each step, if a spin flips, heat is exchanged between the left (right) spin and left (right) bath. We calculate these rates of heat exchange between the system and the left and right baths. We also calculate the rate of work done on the first and second spin.

For our systems there are only four thermodynamically possible machines. Working principle of these modes depend on the positive or negative values of heat exchanges namely \( \dot{q}_L \), \( \dot{q}_R \) and the total work done \( \dot{w} \). If we consider \( T_L \geq T_R \) these four modes of operation are Engine, Heater 1, Heater 2 and Refrigerator. We now describe in detail how these modes operate. Let’s consider first the engine mode of operation in which \( \langle \dot{q}_L \rangle > 0 \), \( \langle \dot{q}_R \rangle < 0 \), \( \langle \dot{w} \rangle < 0 \) implying heat flows from left bath into the system, which is used by the working substance to do work on the external agent and remaining heat is dissipated into the right bath. Second mode is that of Heater 1 where \( \langle \dot{q}_L \rangle < 0 \), \( \langle \dot{q}_R \rangle < 0 \), \( \langle \dot{w} \rangle > 0 \). In this case external agent dissipates large amount of heat in form of work into system and this heat is then dissipated in both left and right reservoirs. In case of Heater 2 operation \( \langle \dot{q}_L \rangle > 0 \), \( \langle \dot{q}_R \rangle < 0 \), \( \langle \dot{w} \rangle > 0 \) heat flows from the left bath, as well as work is done on the system, hence a large amount of heat in dissipated in the right bath. Fourth mode is that of refrigeration where \( \langle \dot{q}_L \rangle < 0 \), \( \langle \dot{q}_R \rangle > 0 \), \( \langle \dot{w} \rangle > 0 \) such that heat is taken from the right bath which is at a lower temperature than the left bath, work is done on the system, this results in spending of heat in the left bath [13]. In our model the phase difference \( \phi \) alone can determine different modes of operation as can be seen from Fig. 2 (a), (b).

We are also interested in studying fluctuations in heat exchanged and work done as well as to study how sensitive is the performance of the model on these parameter values. Hence we construct phase diagram for both mode of operations as a function of the phase \( \phi \) and the temperature \( T_L \) keeping \( T_R = 0.1 \) for Engine mode and \( T_R = 0.5 \) for pump mode of operation. These phase diagrams are shown in Fig. 3 (a) and (b) respectively. Fig. 3 (a) shows

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**FIG. 3.** (a) Shows the phase diagram as a function of the phase difference \( \phi \) for engine mode of operation. (b) Phase diagram for the refrigerator mode. Different modes are indicated on the figure.
how Engine mode of operation depends on the phase \( \phi \) and the temperature \( T_L \) for fixed \( T_R = 0.1 \) and \( \tau = 190 \). It has two distinct domains namely Engine and Heater 2. For \( \pi/4 < \phi < \pi \) and \( T_L - T_R > 1 \), Engine behavior is observed (\( \langle \dot{q}_L \rangle > 0, \langle \dot{q}_R \rangle < 0, \langle \dot{w} \rangle < 0 \)). Other part of the diagram is dominated by Heater 2 operation. In Fig. 3(b) we plot phase diagram for the Refrigerator mode of operation where \( T_R = 0.5, \tau = 225 \). It is equally dominated by Heater 1, Refrigerator and Heater 2 modes. with refrigerator mode occurring in a narrow strip between \( \pi/2 < \phi < \pi \), and very small temperature differences \( T_L - T_R \leq 0.005 \).

We now examine how different modes of operations depend on different parameters in the model other than the phase \( \phi \). To this end we construct the phase diagram where we keep the phase \( \phi = 0.7\pi \), temperature \( T_R = 0.1 \) fixed and vary \( T_L \) for different time periods of driving \( \tau \). This phase diagram is shown in Fig. 3(a). We see that for small \( \tau \sim 50 \), work done \( \langle \dot{w} \rangle < 0 \) with \( \langle \dot{q}_L \rangle > 0, \langle \dot{q}_R \rangle < 0 \), system works as an Engine independent of the temperature difference \( T_L - T_R \). For large \( \tau > 50 \) engine behavior persists but only for the moder-
The phase diagram is mainly dominated by the Heater 2 ability distribution of efficiency . This can be understood as follows. Since the temperature time period of external driving \( \tau \) changes, the Heater 2 mode \( (\langle \dot{q}_L \rangle < 0, \langle \dot{q}_R \rangle < 0, \langle \dot{w} \rangle > 0) \) occurs in a thin band for \( \tau \geq 100 \) for temperature differences \( T_L - T_R \sim 0.005 \). Other regions of the phase diagram are mainly dominated by Heater 1 \( (\langle \dot{q}_L \rangle < 0, \langle \dot{q}_R \rangle < 0, \langle \dot{w} \rangle > 0) \), for \( \tau < 100 \) and \( T_L - T_R > 0.005 \). Heater 2 mode \( (\langle \dot{q}_L \rangle > 0, \langle \dot{q}_R \rangle < 0, \langle \dot{w} \rangle > 0) \) appears for larger values of \( \tau > 200 \) and larger temperature differences. We also plot the distribution of COP namely \( P(\eta) \) in Fig. 4(d). We see that distribution is broad with many distinct minima and long power law tails (inset Fig. 4(b)) with exponent \( \sim -2 \).

We also look at the behavior of different average heat currents namely \( \langle \dot{q}_L \rangle, \langle \dot{q}_R \rangle, \langle \dot{w} \rangle \) as a function of the driving period \( \tau \). This is crucial in order to understand how this engine performs when compared to the Carnot engine. In Fig. 4(a) we plot these currents for the engine mode, where as expected \( \langle \dot{q}_L \rangle > 0, \langle \dot{q}_R \rangle < 0 \) for all \( \tau \) values they saturate to some finite value in the quasistatic limit \( \tau \to \infty \), however work done is negative only for a short interval when \( \tau \sim 100 \) (inset of Fig. 3(a)). Fig. 5(b) shows the Refrigerator mode where behavior changes from Heater 1 for \( \tau \sim 10 \), then to Refrigerator \( \tau \sim 50 \) then to Heater 1 for \( \tau \sim 100 \). Refrigerator mode recovers for \( \tau \sim 500 \) before all heat currents vanish in the quasistatic limit. Lastly in Fig. 5(c) we plot average efficiency \( \bar{\epsilon} \) as a function of \( \tau \) where for \( \tau < 100 \) system is in the Heater 2 mode \( (\langle \dot{q}_L \rangle > 0, \langle \dot{q}_R \rangle < 0, \langle \dot{w} \rangle > 0) \) (see Fig. 5(a)), reaches a maximum value \( \bar{\epsilon} \sim 0.025 \) at \( \tau \sim 190 \) and then vanishes as \( \tau \to \infty \), in quasistatic limit. This is consistent with the fact that though \( \langle \dot{q}_L \rangle \) is finite at large \( \tau \) (see Fig. 5(a)), work done actually approaches zero in the quasistatic limit (inset of Fig. 5(a)). This behavior is absent in usual colloidal engines where efficiency actually approaches Carnot efficiency in the quasistatic limit, distinguishing our model from earlier models \[13, 14\]. Finally in Fig. 6 we plot Power \( \langle \dot{w} \rangle / \tau \), as a function of the time period \( \tau \) for fixed \( T_L, T_R \) and \( \phi \). As expected, for \( \tau \sim 1 \) finite amount of power is generated but it approaches zero as \( \tau \) is increased.

Conclusion. To conclude we have studied a novel model for different values of the time period \( \tau \).
of two classical Ising spin interacting simultaneously with two heat baths and driven by time dependent, phase different magnetic fields. Unlike earlier models the working substance is in contact with heat baths for the full duration of the driving protocol. The model works as an engine or a refrigerator and this behavior solely depends on one parameter which is the phase difference between two magnetic fields. We also found that the performance of the system as an engine or a pump is highly affected by thermal fluctuations. For usual heat engines e.g. colloidal particles in contact with multiple baths, one expects that the efficiency should approach Carnot limit $1 - T_c/T_h$ in the quasistatic or under zero power generation limit [1]. In our model however efficiency never reaches the Carnot limit, even in the quasistatic limit. In fact it goes to zero as the time period $\tau \to \infty$ as seen Fig. 5(c). This points to the fact that for systems simultaneously in contact with several heat baths and for the full duration of the driving protocol, usual definitions of Carnot efficiency do not seem to work. Reliability of the engine is an important technological issue. Here reliability implies for how many cycles out of the total cycles, over which the averages are calculated, actually performed as an engine. We found that for optimal parameters in engine mode of operation $\tau = 190$ the reliability was about 75%. It was also seen that for $\tau \sim 10$ reliability was about 35%. As the time period increased $\tau \sim 2000$ reliability almost reached 100% showing similar behavior as that of macroscopic engines. This again points to the fact that fluctuations largely affect the behavior. One interesting issue would be to look for possible ways to optimize the power and efficiency, on which we are currently working. To quantify fluctuations more concretely, we also numerically obtained probability distribution functions for efficiency $P(\epsilon)$ and COP $P(\eta)$. We found distributions to be very broad with power law tails, with exponent $-2$. This points to the fact that RMS fluctuations are much larger than unity. Finally we would like to mention that our model is realizable experimentally with driven nano particles coupled to two separate heat baths.

RM thanks DST, India for financial support. DB, JN, RM thank the IIT Delhi HPC facility for computational resources. AMJ also thanks DST, India for J. C. Bose National Fellowship.

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