ON APPROXIMATION OF $p$-ADIC NUMBERS BY $p$-ADIC ALGEBRAIC NUMBERS

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1. Introduction

Throughout $p \geq 2$ is a fixed prime number, $\mathbb{Q}_p$ is the field of $p$-adic numbers, $|\omega|_p$ is the $p$-adic valuation of $\omega \in \mathbb{Q}_p$, $\mu(S)$ is the Haar measure of a measurable set $S \subset \mathbb{Q}_p$, $\mathbb{A}_{p,n}$ is the set of algebraic numbers of degree $n$ lying in $\mathbb{Q}_p$, $\mathbb{A}_p$ is the set of all algebraic numbers, $\mathbb{Q}_p^*$ is the extension of $\mathbb{Q}_p$ containing $\mathbb{A}_p$. There is a natural extension of $p$-adic valuation from $\mathbb{Q}_p$ to $\mathbb{Q}_p^*$ [Cas86, Lut55]. This valuation will also be denoted by $|\cdot|_p$. The disc in $\mathbb{Q}_p$ of radius $r$ centered at $\alpha$ is the set of solutions of the inequality $|x - \alpha|_p < r$.

Throughout, $\mathbb{R} > a = \{x \in \mathbb{R} : x > a\}$, $\mathbb{R}^+ = \mathbb{R}_0^+$ and $\Psi : \mathbb{N} \to \mathbb{R}^+$ is monotonic.

Given a polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \in \mathbb{Z}[x]$ with $a_n \neq 0$, $\deg P = n$ is the degree of $P$, $H(P) = \max_{0 \leq i \leq n} |a_i|$ is the usual height of $P$. Also $H(\alpha)$ will stand for the usual height of $\alpha \in \mathbb{A}_p$, i.e. the height of the minimal polynomial for $\alpha$. The notation $X \ll Y$ will mean $X = O(Y)$ and the one of $X \asymp Y$ will stand for $X \ll Y \ll X$.

In 1989 V. Bernik [Ber89] proved A. Baker’s conjecture by showing that for almost all $x \in \mathbb{R}$ the inequality $|P(x)| < H(P)^{-n+1} \Psi(H(P))$ has only finitely many solutions in $P \in \mathbb{Z}[x]$ with $\deg P \leq n$ whenever and the sum

$$\sum_{h=1}^{\infty} \Psi(h)$$

(1)

converges. In 1999 V. Beresnevich [Ber99] showed that in the case of divergence of (1) this inequality has infinitely many solutions.

We refer the reader to [BBKM02, BD99, Ber02, BKM01, Spr79] for further development of the metric theory of Diophantine approximation. In this paper we establish a complete analogue of the aforementioned results for the $p$-adic case.

**Theorem 1.** Let $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ be monotonically decreasing and $M_n(\Psi)$ be the set of $\omega \in \mathbb{Q}_p$ such that the inequality

$$|P(\omega)|_p < H(P)^{-n} \Psi(H(P))$$

(2)

has infinitely many solutions in polynomials $P \in \mathbb{Z}[x]$, $\deg P \leq n$. Then $\mu(M_n(\Psi)) = 0$ whenever the sum (1) converges and $M_n(\Psi)$ has full Haar measure whenever the sum (1) diverges.
The following is a $p$-adic analogue of Theorem 2 in [Ber99].

**Theorem 2.** Let $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ be monotonically decreasing and $\mathbb{A}_{p,n}(\Psi)$ be the set of $\omega \in \mathbb{Q}_p$ such that the inequality

$$|\omega - \alpha|_p < H(\alpha)^{-n}\Psi(H(\alpha))$$

(3)

has infinitely many solutions in $\alpha \in \mathbb{A}_{p,n}$. Then $\mu(\mathbb{A}_{p,n}(\Psi)) = 0$ whenever the sum (1) converges and $\mathbb{A}_{p,n}(\Psi)$ has full Haar measure whenever the sum (1) diverges.

2. Reduction of Theorem 1

We are now going to show that the convergence part of Theorem 1 follows from the following two theorems. Also we show that the divergence part of Theorem 1 follows from Theorem 2.

**Proposition 1.** Let $\delta, \xi \in \mathbb{R}_+, \xi < 1/2$, $Q \in \mathbb{R}_{>1}$ and $K_0$ be a finite disc in $\mathbb{Q}_p$. Given a disc $K \subset K_0$, let $E_1(\delta, Q, K, \xi)$ be the set of $\omega \in K$ such that there is a non-zero polynomial $P \in \mathbb{Z}[x]$, deg $P \leq n$, $H(P) \leq Q$ satisfying the system of inequalities

$$\begin{align*}
|P(\omega)|_p < \delta Q^{-n-1}, \\
|P'(\alpha_{\omega,P})|_p > H(P)^{-\xi},
\end{align*}$$

(4)

where $\alpha_{\omega,P} \in \mathbb{A}_p$ is the root of $P$ nearest to $\omega$ (if there are more than one root nearest to $\omega$ then we choose any of them). Then there is a positive constant $c_1$ such that for any finite disc $K \subset K_0$ there is a sufficiently large number $Q_0$ such that $\mu(E_1(\delta, Q, K, \xi)) \leq c_1\delta \mu(K)$ for all $Q \geq Q_0$ and all $\delta > 0$.

**Proposition 2.** Let $\xi, C \in \mathbb{R}_+$, $K_0$ be a finite disc in $\mathbb{Q}_p$ and let $E_2(\xi, C, K_0)$ be the set of $\omega \in \mathbb{Q}_p$ such that there are infinitely many polynomials $P \in \mathbb{Z}[x]$, deg $P \leq n$ satisfying the system of inequalities

$$\begin{align*}
|P(\omega)|_p < C H(P)^{-n-1}, \\
|P'(\alpha_{\omega,P})|_p < H(P)^{-\xi},
\end{align*}$$

(5)

Then $\mu(E_2(\xi, C, K_0)) = 0$.

**Proof of the convergence part of Theorem 1 modulo Propositions 1 and 2.** Let the sum (1) converges. Then it is readily verified that

$$\sum_{t=1}^{\infty} 2^t \Psi(2^t) < \infty$$

(6)

and

$$\Psi(h) = o(h^{-1})$$

(7)

as $h \to \infty$. For the proofs of (6) see Lemma 5 in [Ber99]. The arguments for (7) can be found in the proof of Lemma 4 in [Ber99].

Fix any positive $\xi < 1/2$. By (7), $H(P)^{-n}\Psi(H(P)) < H(P)^{-n-1}$ for all but finitely many $P$. Then, by Proposition 2 to complete the proof of the convergence part of Theorem 1 it remains to show that for any finite disc $K$ in $\mathbb{Q}_p$, the set $E_1(\xi, \Psi)$ consisting
of \( \omega \in \mathbb{Q}_p \) such that there are infinitely many polynomials \( P \in \mathbb{Z}[x], \deg P \leq n \) satisfying the system of inequalities

\[
\begin{align*}
|P(\omega)|_p &< H(P)^{-n} \Psi(H(P)), \\
|P'(\alpha, p)|_p &\geq H(P)^{-\xi}
\end{align*}
\]

has zero measure.

The system (8) implies

\[
\begin{align*}
|P(\omega)|_p &< (2t)^{-n-1} 2^t \Psi(2^t), \\
|P'(\alpha, p)|_p &\geq H(P)^{-\xi},
\end{align*}
\]

where \( t = t(P) \) with \( 2^t \leq H(P) < 2^{t+1} \), which means that \( \omega \in E_1(2^{n+1} 2^t \Psi(2^t), 2^{t+1}, K, \xi) \). The system (9) holds for infinitely many \( t \) whenever (8) holds for infinitely many \( P \). Therefore,

\[ E_1(\xi, \Psi) \subset \limsup_{t \to \infty} E_1(2^{n+1} 2^t \Psi(2^t), 2^{t+1}, K, \xi). \]

By Proposition 1, \( \mu(E_1(2^{n+1} 2^t \Psi(2^t), 2^{t+1}, K, \xi)) \ll 2^t \Psi(2^t) \). Taking into account (9), the Borel-Cantelli lemma completes the proof. \( \square \)

Next, we are going to show that the divergence part of Theorem 2 is a consequence of Theorem 1.

**Proof of the divergence part of Theorem 1 modulo Theorem 2.** Fix any finite disc \( K \) in \( \mathbb{Q}_p \). Then there is a positive constant \( C > 0 \) such that \( |\omega|_p \leq C \) for all \( \omega \in K \). Let \( \Psi : \mathbb{R}_+ \to \mathbb{R}_+ \) be a given monotonic function such that the sum (11) diverges. Then the function \( \tilde{\Psi}(h) = |n|_p C^{1-n} \Psi(h) \) is also monotonic and the sum \( \sum_{h=1}^{\infty} \tilde{\Psi}(h) \) diverges. By Theorem 2 for almost every \( \omega \in K \) there are infinitely many \( \alpha \in \mathbb{A}_{p,n} \) satisfying

\[ |\omega - \alpha|_p < H(\alpha)^{-n} \tilde{\Psi}(H(\alpha)). \]

As \( \Psi \) decreases, the right hand side of (11) is bounded by a constant. Then we can assume that \( |\omega - \alpha|_p \leq C \) for the solutions of (11). Then \( |\alpha|_p = |\alpha - \omega + \omega|_p \leq \max\{|\alpha - \omega|_p, |\omega|_p\} \leq C \).

Let \( P_\alpha \) denote the minimal polynomial for \( \alpha \). Since \( P^{(i)}_\alpha \) is a polynomial with integer coefficients of degree \( n - i \), we have \( |P^{(i)}_\alpha(\alpha)|_p \leq \max_{0 \leq j \leq n-i} |\alpha|_p^j \leq C^{n-i} \). Then

\[
\begin{align*}
|P_\alpha(\omega)|_p &= |\omega - \alpha|_p \left| \sum_{i=1}^{n} i! P^{(i)}_\alpha(\omega - \alpha)^{i-1} \right|_p \\
&\leq |\omega - \alpha|_p \cdot \max_{1 \leq i \leq n} |i! P^{(i)}_\alpha(\omega - \alpha)^{i-1}|_p \\
&\leq |\omega - \alpha|_p \cdot |n|_p^{-1} C^{n-1} |\omega - \alpha|_p.
\end{align*}
\]

Therefore (11) implies

\[ |P_\alpha(\omega)|_p < H(\alpha)^{-n} \tilde{\Psi}(H(\omega))|n|_p^{-1} C^{n-1} = H(\alpha)^{-n} \Psi(H(\omega)) = H(P_\alpha)^{-n} \Psi(H(P_\alpha)). \]

Inequality (11) has infinitely many solutions for almost all \( \omega \in K \) and so has (11). As \( \omega \) is almost every point of \( K \), the proof is completed. \( \square \)
3. Reduction of Theorem 2

Proof of the convergence part of Theorem 2. Given an \( \alpha \in A_{p,n} \), let \( \chi(\alpha) \) be the set of \( \omega \in \mathbb{Q}_p \) satisfying (3). The measure of \( \chi(\alpha) \) is \( \ll H(\alpha)^{-n}\Psi(H(\alpha)) \). Then

\[
\sum_{\alpha \in A_{p,n}} \mu(\chi(\alpha)) = \sum_{h=1}^{\infty} \sum_{\alpha \in A_{p,n}, H(\alpha)=h} \mu(\chi(\alpha)) \ll \sum_{h=1}^{\infty} \sum_{\alpha \in A_{p,n}, H(\alpha)=h} h^{-n}\Psi(h) \ll \sum_{h=1}^{\infty} \Psi(h) < \infty.
\]

Here we used the fact that the quantity of algebraic numbers of height \( h \) is \( \ll h^n \). The Borel-Cantelli lemma completes the proof. \( \square \)

The proof of the divergence part of Theorem 2 will rely on the regular systems method of \cite{Ber99}. In this paper we give a generalization of the method for the \( p \)-adic case.

**Definition 1.** Let a disc \( K_0 \) in \( \mathbb{Q}_p \), a countable set of \( p \)-adic numbers \( \Gamma \) and a function \( N : \Gamma \to \mathbb{R}_+ \) be given. The pare \((\Gamma, N)\) is called a regular system of points in \( K_0 \) if there is a constant \( C > 0 \) such that for any disc \( K \subset K_0 \) for any sufficiently large number \( T \) there exists a collection \( \gamma_1, \ldots, \gamma_t \in \Gamma \cap K \) satisfying the following conditions

\[
N(\gamma_i) \leq T \quad (1 \leq i \leq t),
\]

\[
|\gamma_i - \gamma_j|_p \geq T^{-1} \quad (1 \leq i < j \leq t),
\]

\[
t \geq CT \mu(K).
\]

**Proposition 3.** Let \((\Gamma, N)\) be a regular system of points in \( K_0 \subset \mathbb{Q}_p \), \( \tilde{\Psi} : \mathbb{R}_+ \to \mathbb{R}_+ \) be monotonically decreasing function such that \( \sum_{h=1}^{\infty} \tilde{\Psi}(h) = \infty \). Then \( \Gamma_{\tilde{\Psi}} \) has full Haar measure in \( K_0 \), where \( \Gamma_{\tilde{\Psi}} \) consists of \( \omega \in K_0 \) such that the inequality

\[
|x - \gamma|_p < \tilde{\Psi}(N(\gamma))
\]

has infinitely many solutions \( \gamma \in \Gamma \).

This theorem is proved in \cite{BK03}. The proof is also straightforward the ideas of the proof of Theorem 2 in \cite{Ber99}.

**Proposition 4.** The pare \((\Gamma, N)\) of \( \Gamma = A_{p,n} \) and \( N(\alpha) = H(\alpha)^{n+1} \) is a regular system of points in any finite disc \( K_0 \subset \mathbb{Q}_p \).

Proof of the divergence part of Theorem 2 modulo Propositions 3 and 4. Let \( \Psi : \mathbb{R}_+ \to \mathbb{R}_+ \) be a monotonic function and the sum (11) diverges. Fix any finite disc \( K_0 \subset \mathbb{Q}_p \).

Let \((\Gamma, N)\) be a regular system defined in Proposition 3 and let \( \Psi \) be a monotonic function such that the sum (11) diverges. Define a function \( \tilde{\Psi} \) by setting \( \tilde{\Psi}(x) = x^{-n/(n+1)}\Psi(x^{1/(n+1)}) \). Using the monotonicity of \( \Psi \), we obtain

\[
\sum_{h=1}^{\infty} \tilde{\Psi}(h) = \sum_{t=1}^{\infty} \sum_{(t-1)^{n+1} < h \leq t^{n+1}} \tilde{\Psi}(h) \geq \sum_{t=1}^{\infty} \sum_{(t-1)^{n+1} < h \leq t^{n+1}} t^{-n} \Psi(t) =
\]
Given a \( \# | \alpha \) has infinitely many solutions in \( \mathbb{A}_{p,n} \). The proof is completed. \( \square \)

4. Proof of Proposition 1

Fix any finite \( K \subset K_0 \) in \( \mathbb{Q}_p \). Let \( \chi(P) \) be the set of \( \omega \in K \) satisfying (11) and let \( \mathcal{P}_n(Q, K) \) be the set of non-zero polynomials \( P \) with integer coefficients, \( \deg P \leq n \), \( H(P) \leq Q \) and with \( \chi(P) \neq \emptyset \). We will use the following

Lemma 1. Let \( \alpha_{\omega,P} \) be the nearest root of a polynomial \( P \) to \( \omega \in \mathbb{Q}_p \). Then

\[
|\omega - \alpha_{\omega,P}|_p \leq |P(\omega)|_p |P'(\alpha_{\omega,P})|_p^{-1}.
\]

For the proof see [Spr69] p. 78.

Given a polynomial \( P \in \mathcal{P}_n(Q, K) \), let \( \mathcal{Z}_P \) be the set of roots of \( P \). It is clear that \( \# \mathcal{Z}_P \leq n \). Given an \( \alpha \in \mathcal{Z}_P \), let \( \chi(P, \alpha) \) be the subset of \( \chi(P) \) consisting of \( \omega \) with \( |\omega - \alpha|_p = \min \{ |\alpha' - \omega|_p : \alpha' \in \mathcal{Z}_P \} \).

By Lemma 1 for any \( P \in \mathcal{P}_n(Q, K) \) and any \( \alpha \in \mathcal{Z}_P \) one has

\[
\mu(\chi(P, \alpha)) \ll \delta Q^{-n-1} |P'(\alpha)|_p^{-1}. \tag{14}
\]

Given a \( P \in \mathcal{P}_n(Q, K) \) and an \( \alpha \in \mathcal{Z}_P \), define the disc

\[
\tilde{\chi}(P, \alpha) = \left\{ \omega \in K : |\omega - \alpha|_p \leq \left( 4Q|P'(\alpha)|_p \right)^{-1} \right\}. \tag{15}
\]

It is readily verified that if \( \tilde{\chi}(P, \alpha) \neq \emptyset \) then \( \mu(\tilde{\chi}(P, \alpha)) \gg \left( 4Q|P'(\alpha)|_p \right)^{-1} \). Using (14) we get

\[
\mu(\chi(P, \alpha)) \ll \delta Q^{-n-1} \mu(\tilde{\chi}(P, \alpha)) \tag{16}
\]

with the implicit constant depending on \( p \) only.

Fix any \( P \in \mathcal{P}_n(Q, K) \) and an \( \alpha \in \mathcal{Z}_P \) such that \( \chi(P, \alpha) \neq \emptyset \). Let \( \omega \in \tilde{\chi}(P, \alpha) \). Then

\[
P(\omega) = P'(\alpha)(\omega - \alpha) + (\omega - \alpha)^2 \left( \sum_{i=2}^{n} P^{(i)}(\alpha)(\omega - \alpha)^{i-2} \right). \tag{17}
\]

By the inequalities \( |P'(\alpha)|_p \geq H(P)^{-\xi} \) and \( H(P) \leq Q \), we have \( |P'(\alpha)|_p^{-1} \leq Q^\xi \). Then by (15), \( |\omega - \alpha|_p \leq Q^{-1+\xi} \). Next, as \( \omega \in K \) and \( K \) is finite, it is readily verified that \( |P^{(i)}(\alpha)|_p \ll 1 \), where the constant in this inequality depends on \( K \). Then

\[
|\omega - \alpha|^2 \left( \sum_{i=2}^{n} P^{(i)}(\alpha)(\omega - \alpha)^{i-2} \right) \ll Q^{-2+2\xi}. \tag{18}
\]
By (15), we have $|P'(\alpha)(\omega - \alpha)|_p \leq (4Q)^{-1}$. Using this inequality, (18) and $\xi < 1/2$, we conclude that

$$|P(\omega)|_p \leq (4Q)^{-1}, \quad \omega \in \chi(P, \alpha)$$  \hspace{1cm} (19)

if $Q$ is sufficiently large.

Assume that $P_1, P_2 \in \mathcal{P}_n(Q,K)$ satisfy $P_1 - P_2 \in \mathbb{Z}_{\neq 0}$ and assume that there is an $\omega \in \bar{\chi}(P_1) \cap \bar{\chi}(P_2)$. Then $\omega \in \chi(P_1, \alpha) \cap \chi(P_2, \beta)$ for some $\alpha \in \mathbb{Z}_{P_1}$ and $\beta \in \mathbb{Z}_{P_2}$. Then, by (19), $|P_1(\omega) - P_2(\omega)|_p < (4Q)^{-1}$. On the other $P_1(\omega) - P_2(\omega)$ is an integer not greater than $2Q$ in absolute value. Therefore, $|P_1(\omega) - P_2(\omega)|_p \geq (2Q)^{-1}$ that leads to a contradiction. Hence there is no such an $\omega$ and $\chi(P_1) \cap \chi(P_2) = \emptyset$. Therefore

$$\sum_{P \in \mathcal{P}_n(Q,K,a_n,\ldots,a_1)} \mu(\chi(P)) \leq \mu(K),$$

(20)

where $\mathcal{P}_n(Q,K)$ is the subset of $\mathcal{P}_n(Q,K)$ consisting of $P$ with fixed coefficients $a_n, \ldots, a_1$.

By (16) and (20), $\sum_{P \in \mathcal{P}_n(Q,K,a_n,\ldots,a_1)} \mu(\chi(P)) \ll \delta Q^{-n} \mu(K)$. Summing this over all $(a_n, \ldots, a_1) \in \mathbb{Z}^n$ with coordinates at most $Q$ in absolute value gives

$$\sum_{P \in \mathcal{P}_n(Q,K)} \mu(\chi(P)) \ll \delta \mu(K).$$

(21)

It is obvious that

$$E_1(\delta, Q, K, \xi) = \bigcup_{P \in \mathcal{P}_n(Q,K)} \chi(P).$$

(22)

As the Haar measure is subadditive (21) and (22) imply the statement of Proposition 1.

5. Reduction to irreducible primitive leading polynomials in Proposition 2

The following lemma shows us that there is no loss of generality in neglecting reducible polynomials while proving Proposition 2.

Lemma 2 (Lemma 7 in [BDY99]). Let $\delta \in \mathbb{R}_+$ and $E(\delta)$ be the set of $\omega \in \mathbb{Q}_p$ such that the inequality

$$|P(\omega)|_p < H(P)^{-n-\delta}$$

has infinitely many solutions in reducible polynomials $P \in \mathbb{Z}[x]$, deg $P \leq n$. Then

$$\mu(E(\delta)) = 0.$$

Also, by Sprindžuk’s theorem [Spr69] there is no loss of generality in assuming that $\deg P = n$. From now on, $\mathcal{P}$ will denote the set of irreducible polynomials $P \in \mathbb{Z}[x]$ with $\deg P = n$.

Next, a polynomial $P \in \mathbb{Z}[x]$ is called primitive if the gcd (greatest common divisor) of its coefficients is 1. To perform the reduction to primitive polynomials we fix an $\omega$ such that the system (5) has infinitely many solutions in polynomials $P \in \mathcal{P}$ and show that either $\omega$ belongs to a set of measure zero or (5) holds for infinitely many primitive $P \in \mathcal{P}$.
Define \( a_P = \gcd(a_n, \ldots, a_1, a_0) \in \mathbb{N} \). Given a \( P \in \mathcal{P} \), there is a uniquely defined primitive polynomial \( P_1 \) (i.e. \( a_{P_1} = 1 \)) with \( P = a_PP_1 \). Then \( H(P) = a_PH(P_1) \). Let \( P \in \mathcal{P} \) be a solution of (5). By (5), \( P_1 \) satisfies the inequalities

\[
\begin{align*}
|a_P| \cdot |P(\omega)|_p &= |P(\omega)|_p \ll H(P)^{-n-1} = (a_PH(P_1))^{-n-1}, \\
|a_P| \cdot |P'(\omega, P)|_p &= |P'(\omega, P)|_p < H(P)^{-\xi} = (a_PH(P_1))^{-\xi}.
\end{align*}
\]

(23)

As \( |a_P|^{-1} \leq a_P \), (23) implies

\[
|P_1(\omega)|_p \ll H(P_1)^{-n-1} a_P^{-n}, \quad |P'(\omega, P)|_p < H(P)^{-\xi} a_P^{-\xi}.
\]

(24)

If (24) takes place only for a finite number of different polynomials \( P_1 \in \mathcal{P} \), then there exists one of them such that (5) has infinitely many solutions in polynomials \( P \) with the same \( P_1 \). It follows that \( \omega \) is a root of \( P_1 \) and thus belongs to a set of measure zero. Further we assume that there are infinitely many \( P_1 \) satisfying (24).

If \( \xi \geq 1 \) then the reduction to primitive polynomials is obvious as \( a_P \in \mathbb{N} \). Let \( \xi < 1 \). Then, if (5) holds for infinitely many polynomials \( P \in \mathcal{P} \) such that \( a_P > H(P_1)^{\xi} \), where \( \xi' = \xi/(2-2\xi) \), then the first inequality in (24) implies that \( |P_1(\omega)|_p \ll H(P_1)^{-n-1} a_P^{-n} \ll H(P_1)^{-n-1-n\xi'} \) holds for infinitely many polynomials \( P_1 \in \mathcal{P} \). By Sprindzhuk’s theorem [Spr69], the set of those \( \omega \) has zero measure.

If (5) holds for infinitely many polynomials \( P \in \mathcal{P} \) such that \( a_P < H(P_1)^{\xi'} \) then (24) implies that the system of inequalities

\[
|P_1(\omega)|_p \ll H(P_1)^{-n-1}, \quad |P'(\omega, P)|_p < H(P_1)^{-\xi+(1-\xi)\xi'} < H(P_1)^{-\xi/2}
\]

holds for infinitely many polynomials \( P_1 \). Thus, we get the required statement with a smaller \( \xi \).

A polynomial \( P \in \mathbb{Z}[x] \) with the leading coefficient \( a_n \) will be called leading if

\[
a_n = H(P) \quad \text{and} \quad |a_n|_p > p^{-n}.
\]

(25)

Let \( \mathcal{P}_n(H) \) be the set of irreducible primitive leading polynomials \( P \in \mathbb{Z}[x] \) of degree \( n \) with the height \( H(P) = H \). Also define

\[
\mathcal{P}_n = \bigcup_{H=1}^{\infty} \mathcal{P}_n(H).
\]

(26)

Reduction to leading polynomials is completed with the help of

**Lemma 3.** Let \( \Omega \) be the set of points \( \omega \in \mathbb{Q}_p \) for which (5) has infinitely many solutions in irreducible primitive polynomials \( P \in \mathbb{Z}[x] \), deg \( P = n \). Let \( \Omega_0 \) be the set of points \( \omega \in \mathbb{Q}_p \) for which (5) has infinitely many solutions in polynomials \( P \in \mathcal{P}_n \), where \( \mathcal{P}_n \) is defined in (26). If \( \Omega \) has positive measure then so has \( \Omega_0 \) with probably a different constant \( C \) in (5).

Proof of this lemma is very much the same as the one of Lemma 10 in [Spr69] and we leave it as an exercise.

Every polynomial \( P \in \mathcal{P}_n \) has exactly \( n \) roots, which can be ordered in any way: \( \alpha_{P_1}, \ldots, \alpha_{P_n} \). The set \( E_2(\xi, C, K_0) \) can be expressed as a union of subsets \( E_{2,k}(\xi, C, K_0) \) with \( 1 \leq k \leq n \), where \( E_{2,k}(\xi, C, K_0) \) is defined to consist of \( \omega \in K_0 \) such that (5) holds infinitely often with \( \alpha_{\omega,P} = \alpha_{P,k} \). To prove Proposition 2 it suffices to show that
We assume that \( \rho \) the lemma. \( \square \)

\[ \begin{aligned}
E_{2,k}(\xi, C, K_0) \text{ has zero measure for every } k. \text{ The consideration of these sets will not depend on } k. \text{ Therefore we can assume that } k = 1 \text{ and omit this index in the notation of } E_{2,k}(\xi, C, K_0). \text{ Also whenever there is no risk of confusion we will write } \alpha_1, \ldots, \alpha_n \text{ for } \alpha_{P,1}, \ldots, \alpha_{P,n}.
\end{aligned} \]

6. Auxiliary statements and classes of polynomials

**Lemma 4.** Let \( \alpha_1, \ldots, \alpha_n \) be the roots of \( P \in \mathcal{P}_n \). Then \( \max_{1 \leq i \leq n} |\alpha_i|_p < p^n \).

For the proof see [Spr69] p. 85.

For the roots \( \alpha_1, \ldots, \alpha_n \) of \( P \) we define the sets

\[ S(\alpha_i) = \{ \omega \in \mathbb{Q}_p : |\omega - \alpha_i|_p = \min_{1 \leq j \leq n} |\omega - \alpha_j|_p \} \quad (1 \leq i \leq n). \]

Let \( P \in \mathcal{P}_n \). As \( \alpha_1 \) is fixed, we reorder the other roots of \( P \) so that \( |\alpha_1 - \alpha_2|_p \leq |\alpha_1 - \alpha_3|_p \leq \ldots \leq |\alpha_1 - \alpha_n|_p \). We can assume that there exists a root \( \alpha_m \) of \( P \) for which \( |\alpha_1 - \alpha_m|_p \leq 1 \) (see [Spr69] p. 99). Then we have

\[ |\alpha_1 - \alpha_2|_p \leq |\alpha_1 - \alpha_3|_p \leq \ldots \leq |\alpha_1 - \alpha_m|_p \leq 1 \leq \ldots \leq |\alpha_1 - \alpha_n|_p. \quad (27) \]

Let \( \varepsilon > 0 \) be sufficiently small, \( d > 0 \) be a large fixed number and let \( \varepsilon_1 = \varepsilon/d, \)
\[ T = [\varepsilon_1^{-1}] + 1. \]

We define real numbers \( \rho_j \) and integers \( l_j \) by the relations

\[ |\alpha_1 - \alpha_j|_p = H^{-\rho_j}, \quad \frac{l_j - 1}{T} \leq \rho_j < \frac{l_j}{T} \quad (2 \leq j \leq m). \quad (28) \]

It follows from (27) and (28) that \( \rho_2 \geq \rho_3 \geq \ldots \geq \rho_m \geq 0 \) and \( l_2 \geq l_3 \geq \ldots \geq l_m \geq 1 \).

We assume that \( \rho_j = 0 \) and \( l_j = 0 \) if \( m < j \leq n \).

Now for every polynomial \( P \in \mathcal{P}_n(H) \) we define a vector \( \vec{l} = (l_2, \ldots, l_n) \) having non-negative components. In [Spr69] p. 99–100 it is shown that the number of such vectors is finite and depends on \( n, p \) and \( T \) only. All polynomials \( P \in \mathcal{P}_n(H) \) corresponding to the same vector \( \vec{l} \) are grouped together into a class \( \mathcal{P}_n(H, \vec{l}) \). We define

\[ \mathcal{P}_n(\vec{l}) = \bigcup_{H=1}^{\infty} \mathcal{P}_n(H, \vec{l}). \quad (29) \]

Let \( K_0 = \{ \omega \in \mathbb{Q}_p : |\omega|_p < p^n \} \) be the disc of radius \( p^n \) centered at 0. Define

\[ r_j = r_j(P) = (l_j + 1 + \ldots + l_n)/T \quad (1 \leq j \leq n - 1). \]

**Lemma 5.** Let \( \omega \in S(\alpha_1) \) and \( P \in \mathcal{P}_n(H) \). Then

\[ H^{-r_1} \ll |P'(|\alpha_1|)|_p \ll H^{-r_1 + (m-1)\varepsilon_1}, \]

\[ |P^{(j)}(|\alpha_1|)|_p \ll H^{-r_j + (m-j)\varepsilon_1} \quad \text{for} \quad 2 \leq j \leq m, \]

\[ |P^{(j)}(|\alpha_1|)|_p \ll 1 \quad \text{for} \quad m < j \leq n. \]

**Proof.** From (25) we have \( p^{-n} < |H|_p \leq 1 \). Then, on differentiating the identity \( P(\omega) = H(\omega - \alpha_1) \cdots (\omega - \alpha_n) \) \( j \) times \( (1 \leq j \leq n) \) and using (27), (28) we get the statement of the lemma. \( \square \)
Lemma 6. Let $\delta \in \mathbb{R}_+$, $\sigma \in \mathbb{R}_+$, $n \geq 2$ be a natural number and $H = H(\delta, n)$ be a sufficiently large real number. Further let $P$, $Q$ in $\mathbb{Z}[x]$ be two relatively prime polynomials of degree at most $n$ with $\max(H(P), H(Q)) \leq H$. Let $K(\alpha, p^{-t})$ be a disc of radius $p^{-t}$ centered at $\alpha$ where $t$ is defined by the inequalities $p^{-t} \leq H^{-\sigma} < p^{-t+1}$. If there exists a number $\tau > 0$ such that for all $\omega \in K(\alpha, p^{-t})$ one has
\[
\max(|P(\omega)|_p, |Q(\omega)|_p) < H^{-\tau}
\]
then $\tau + 2 \max(\tau - \sigma, 0) < 2n + \delta$.

For the proof see Lemma 5 in [BDY99].

7. Proof of Proposition 2

As in the previous section $K_0 = \{\omega \in \mathbb{Q}_p : |\omega|_p < p^n\}$. Let $A(\mathbf{l}, \xi)$ be the set of points $\omega \in K_0$ for which
\[
\begin{cases}
|P(\omega)|_p < CH(P)^{-n-1}, \\
|P'(\alpha_1)|_p < H(P)^{-\xi}
\end{cases}
\]
has infinitely many solutions in polynomials $P \in \mathcal{P}_n(\mathbf{l})$, where $\mathcal{P}_n(\mathbf{l})$ is defined in (29). It follows from the previous discussion that to prove Proposition 2 it suffices to show that $A(\mathbf{l}, \xi)$ has zero measure for all possible vectors $\mathbf{l}$.

The following investigation essentially depends on the value of $r_1 + l_2/T$. According to Lemma 5 we have $|P'(\alpha_1)|_p \gg H^{-r_1}$. It follows from this and the second inequality of (30) that $H^{-r_1} \leq cH^{-\xi}$, i.e.
\[
r_1 \geq \xi - \ln c / \ln H > \xi / 2 \quad \text{for} \quad H \geq H_0.
\]
Further we assume that $r_1$ satisfies (31). Further we set $\varepsilon$ to be $\xi / 2$.

Lemma 7. If $r_1 + l_2/T > n$ then the set of points $\omega \in K_0$ for which the inequality
\[
|P(\omega)|_p < H(P)^{-n-\varepsilon}
\]
holds for infinitely many polynomials $P \in \mathcal{P}_n(\mathbf{l})$ has zero measure.

For the proof see Proposition 3 in [Spr69], p. 111.

The proof of Proposition 2 is divided into 3 cases, each corresponding to one of the propositions of this section (see below).

Let $\chi(P) = \{\omega \in K_0 \cap S(\alpha; 1) : |P(\omega)|_p < H^{-n-1}\}$. Thus, we investigate the set of $\omega$ that belong to infinitely many $\chi(P)$.

Proposition 5. If $n - 1 + 2n\varepsilon_1 < r_1 + l_2/T$ then $\mu(A(\mathbf{l}, \xi)) = 0$.

Proof. Let $r_1 + l_2/T > n$. Using Lemma 7 with $\varepsilon < 1$ we get $\mu(A(\mathbf{l}, \xi)) = 0$.

Let $n - 1 + 2n\varepsilon_1 < r_1 + l_2/T \leq n$ and $t$ be a sufficiently large fixed natural number. We define the set
\[
\mathcal{M}_t(\mathbf{l}) = \bigcup_{2^t \leq H < 2^{t+1}} \mathcal{P}_n(H, \mathbf{l}).
\]
We divide the set $K_0$ into the discs of radius $2^{-t}\sigma$, where $\sigma = n + 1 - r_1 - \varepsilon_1$. 
First, we consider the polynomials $P \in \mathcal{M}_t(\tilde{I})$ such that there is one of the introduced discs, say $K$, such that $\chi(P) \cap K \neq \emptyset$ and $\chi(Q) \cap K = \emptyset$ for $Q \in \mathcal{M}_t(\tilde{I}) \setminus \{P\}$. The number of the discs and respectively the number of the polynomials is at most $p^{n}2^{\sigma}$. From Lemmas 1 and 5 we get

$$\mu(\chi(P)) \ll |P(\omega)|_p|P'(\alpha_1)|_p^{-1} \ll 2^{-t(n+1-r_1)}$$

and thus summing the measures of $\chi(P)$ for the polynomials $P$ of this class leads to

$$\sum_P \mu(\chi(P)) \ll 2^{t(n+1-r_1-\varepsilon_1-n-1+r_1)} = 2^{-t\varepsilon_1}.$$ 

The latest gives the convergent series and, by the Borel-Cantelli lemma, completes the proof in this case.

Now we consider the other type of polynomials. Let $P$ and $Q$ be different polynomials of $\mathcal{M}_t(\tilde{I})$ such that $\chi(P)$ and $\chi(Q)$ intersect the same disc $D$ introduced above. Then there exist the points $\omega_1$ and $\omega_2$ belonging to $D$ such that

$$\max(|P(\omega_1)|_p, |Q(\omega_2)|_p) \ll 2^{-t(n+1)}.$$  

(32)

Let $\alpha_{P,1}$ and $\alpha_{Q,1}$ be the nearest roots of $P$ and $Q$ to $\omega_1$ and $\omega_2$ respectively. By (32), Lemmas 1 and 5 we get

$$\max(|\omega_1 - \alpha_{P,1}|_p, |\omega_2 - \alpha_{Q,1}|_p) \ll 2^{-t(n+1-r_1)}.$$ 

Hence, according to the definition of the $\sigma$ we have

$$|\alpha_{P,1} - \alpha_{Q,1}|_p \leq \max(|\alpha_{P,1} - \omega_1|_p, |\omega_1 - \omega_2|_p, |\alpha_{Q,1} - \omega_2|_p) \ll \max(2^{-t(n+1-r_1)}, 2^{-t\sigma}) = 2^{-t\sigma}.$$ 

Now we estimate $|\alpha_{P,1} - \alpha_{Q,i}|_p$ $(2 \leq i \leq m)$. Since $r_1 + l_2/T \leq n$ it follows that

$$|\alpha_{P,1} - \alpha_{Q,i}|_p \leq \max(|\alpha_{P,1} - \alpha_{Q,i}|_p, |\alpha_{Q,1} - \alpha_{Q,i}|_p) \ll \max(2^{-t\sigma}, 2^{-t\rho}) \leq \max(2^{-t\sigma}, 2^{-t(l_1-1)/T}) \leq 2^{-t(l/T-\varepsilon_1)}.$$ 

Hence

$$\prod_{i=1}^{m} |\alpha_{P,1} - \alpha_{Q,i}|_p \ll 2^{-t(\sigma+(l_2+\ldots+l_m)/T-(m-1)\varepsilon_1)} = 2^{-t(\sigma+r_1-(m-1)\varepsilon_1)}.$$ 

Similarly we obtain

$$\prod_{i=1}^{m} |\alpha_{P,2} - \alpha_{Q,i}|_p \ll \prod_{i=1}^{m} \max(|\alpha_{P,2} - \alpha_{P,1}|_p, |\alpha_{P,1} - \alpha_{Q,1}|_p, |\alpha_{Q,1} - \alpha_{Q,i}|_p) \leq \max(2^{-t\rho_2}, 2^{-t\sigma}) \prod_{i=2}^{m} \max(2^{-t\rho_2}, 2^{-t\sigma}, 2^{-t\rho}) \ll 2^{-t(l_2/T-\varepsilon_1)} \prod_{i=2}^{m} 2^{-t(l_i/T-\varepsilon_1)} = 2^{-t(l_2/T-\varepsilon_1+(l_2+\ldots+l_m)/T-(m-1)\varepsilon_1)} = 2^{-t(l_2/T+r_1-m\varepsilon_1)}.$$ 

Let $R(P, Q)$ be the resultant of $P$ and $Q$, i.e.

$$|R(P, Q)|_p = |H'_p|^{2n} \prod_{1 \leq i, j \leq n} |\alpha_{P,i} - \alpha_{Q,j}|_p.$$
By the previous estimates for \(i = 1, 2\) and the trivial estimates \(|\alpha_{P,i} - \alpha_{Q,j}|_p \ll p^n\) for \(3 \leq i \leq n\) we get
\[
|R(P, Q)|_p \ll 2^{-t(\sigma_1 r_1 + r_1 - (m-1)\epsilon_1 + l_2 / T + r_1 - m\epsilon_1)} \ll 2^{-t(\sigma_1 + l_2 / T - (2n-1)\epsilon_1)} < 2^{t(2n + \delta')}
\]
where \(\delta' > 0\). On the other hand we have \(|R(P, Q)|_p \gg 2^{-2nt}\) as \(P\) and \(Q\) have not common roots. The last inequalities lead to a contradiction. \(\square\)

**Proposition 6.** If
\[
2 - \varepsilon / 2 < r_1 + l_2 / T \leq n - 1 + 2n\epsilon_1
\]
then \(\mu(A(\bar{l}, \xi)) = 0\).

**Proof.** Let
\[
\theta = n + 1 - r_1 - l_2 / T.
\]
Let \([\theta]\) and \(\{\theta\}\) be the integral and the fractional parts of \(\theta\) respectively.

At first we consider the case \(\{\theta\} \geq \varepsilon\). We define
\[
\beta = [\theta] - 1 + 0, 2\{\theta\} - 0, 1\varepsilon,
\]
\[
\sigma_1 = l_2 / T + 0, 8\{\theta\} + (m + 1)\varepsilon_1,
\]
\[
d = [\theta] - 1.
\]
Fix any sufficiently large integer \(H\) and divide the set \(K_0\) into the discs of radius \(H^{-\sigma_1}\).

The number of these discs is estimated by \(\ll H^{\sigma_1}\). We shall say that the disc \(D\) contains the polynomial \(P \in \mathcal{P}_n(H, \bar{l})\) and write \(P < D\) if there exists a point \(\omega_0 \in D\) such that \(|P(\omega_0)|_p < H^{-n-1}\).

Let \(B_1(H)\) be the collection of discs \(D\) such that \(\#\{P \in \mathcal{P}_n(H, \bar{l}) : P < D\} \leq H^\beta\).

By Lemmas 4 and 5 (55) and (56) we have
\[
\sum_{P \in B_1(H)} \mu(\chi(P)) \ll H^\beta H^{\sigma_1} H^{-n-1+1} = H^{\sigma_1-1+r_1+l_2 / T - 0, 1\varepsilon + (m+1)\varepsilon_1 - n-1}.
\]

From (34) we get
\[
\sum_{P} \mu(\chi(P)) \ll \sum_{H} H^{-1-\varepsilon / 20} < \infty.
\]

By Borel-Cantelli lemma the set of those \(\omega\), which belong to \(\chi(P)\) for infinitely many \(P \in \bigcup_{H} B_1(H)\), has zero measure.

Let \(B_2(H)\) be the collection of the discs that do not belong to \(B_1(H)\) and thus contain more than \(H^\beta\) polynomials \(P \in \mathcal{P}_n(H, \bar{l})\). Let \(D \in B_2(H)\). We divide the set \(\{P \in \mathcal{P}_n(H, \bar{l}) : P < D\}\) into classes as follows. Two polynomials
\[
P_1(x) = H x^n + a_{n-1}^{(1)} x^{n-1} + \ldots + a_1^{(1)} x + a_0^{(1)},
\]
\[
P_2(x) = H x^n + a_{n-1}^{(2)} x^{n-1} + \ldots + a_1^{(2)} x + a_0^{(2)}
\]
are in one class if
\[
a_{n-1}^{(1)} = a_{n-1}^{(2)}, \ldots, a_{n-d}^{(1)} = a_{n-d}^{(2)},
\]
where \(d\) is defined in (37). It is clear that the number of different classes is less than \((2H+1)^d\) and the number of polynomials under consideration is greater than \(H^\beta\). By the pigeon-hole principle, there exists a class \(M\) which contains at least \(cH^{\beta-d}\) polynomials where \(c > 0\) is a constant independent of \(H\). The classes containing less than \(cH^{\beta-d}\) polynomials are considered in a similar way as above, with the Borel-Cantelli arguments.
Further, we denote polynomials from $M$ by $P_1(x), \ldots, P_{s+1}(x)$ and consider $s$ new polynomials

$$R_1(x) = P_2(x) - P_1(x), \ldots, R_s(x) = P_{s+1}(x) - P_1(x).$$

By (37), we get

$$\deg R_i \leq n - d - 1 = n - [\theta] \quad (1 \leq i \leq s). \quad (38)$$

Using Taylor’s formula for $P$ there exists a point $\omega_0 \in D$ such that $|P(\omega_0)|_p < H^{-n-1}$. Let $\alpha_{i1}$ be the root nearest to $\omega_0$. By Lemmas 1 and 5 we have $|\omega_0 - \alpha_{i1}|_p < H^{-n-1+r_1}$ and

$$|\omega - \alpha_{i1}|_p \leq \max(|\omega - \omega_0|_p, |\omega_0 - \alpha_{i1}|_p) \ll \max(H^{-\sigma_1}, H^{-n-1+r_1})$$

for any $\omega \in D$. It follows from (39) and the right-hand side of (33) that

$$\sigma_1 \leq n - 1 - r_1 + 2n\varepsilon_1 + 0,8\{\theta\} + (m+1)\varepsilon_1 < n + 1 - r_1.$$

Therefore $|\omega - \alpha_{i1}|_p \ll H^{-\sigma_1}$. By Lemma 1 we have

$$|P_i^{(j)}(\alpha_{i1})(\omega - \alpha_{i1})^j|_p \ll H^{-r_1+(m-j)\varepsilon_1-j\sigma_1} \quad \text{for} \quad 1 \leq j \leq m,$$

$$|P_i^{(j)}(\alpha_{i1})(\omega - \alpha_{i1})^j|_p \ll H^{-j\sigma_1} \quad \text{for} \quad m < j \leq n.$$

Using Taylor’s formula for $P_i(\omega) (1 \leq i \leq s+1)$ in the disc $|\omega - \alpha_{i1}|_p \ll H^{-\sigma_1}$ and the previous estimates, we obtain

$$|R_i(\omega)|_p \ll H^{-(n+1-\theta)-0,8\{\theta\}-2\varepsilon_1} = H^{-\tau} \quad (1 \leq i \leq s) \quad (40)$$

for any $\omega \in D$. There are the following three cases:

1) Suppose that for each $i (1 \leq i \leq s), R_i(x) = b_i R(x)$ with $b_i \in \mathbb{Z}$. Since the $R_i$ are all different so are the $b_i$. Let $b = \max_{1 \leq i \leq s} |b_i| = |b_1|$, so that $b > s/2$. As $bH(R) \leq 2H, s \gg H^{\beta-d} = H^{0,2\{\theta\}-0,1\varepsilon}$ and $\{\theta\} \gg \varepsilon$, we get

$$H(R) \ll H^{1-0,2\{\theta\}+0,1\varepsilon} \quad \text{and} \quad b \gg H^{0,2\{\theta\}-0,1\varepsilon}. \quad (41)$$

Using (40) and $H(R_1) = bH(R)$ we have

$$|R_1(\omega)|_p = |b|R(\omega)|_p \ll H(R_1)^{-\tau} = H(R)^{-\tau}b^{-\tau}$$

and

$$|R(\omega)|_p \ll H(R)^{-\tau}|b|^{-\tau}|b|^{-1} \leq H(R)^{-\tau}b^{-\tau+1}.$$

From this and (41) we find

$$|R(\omega)|_p \ll H(R)^{-\lambda}, \quad (42)$$
where
\[ \lambda = \tau + (\tau - 1)(0, 2\{\theta\} - 0, 1\varepsilon)(1 - 0, 2\{\theta\} + 0, 1\varepsilon)^{-1}. \]

By the definition of the \( \tau \) in \((10)\), the condition \( \{\theta\} \geq \varepsilon \), \((38)\) and \((39)\) we get
\[ \lambda > n - [\theta] + 1 \geq \deg R + 1. \]

It follows from \((42)\) that
\[ |R(\omega)|_p \ll H(R)^{-\deg R - 1 - \delta'} \]
for all \( \omega \in D' \), where \( \delta' > 0 \). By Sprindžuk’s theorem [Spr69, p.112], the set of \( \omega \) for which there are infinitely many polynomials \( R \) satisfying the previous inequality has zero measure.

2) Suppose that some of polynomials \( R_i \) are reducible. By \((38)\) we have \((10)\) with \( \tau \geq \deg R_i + \delta \) where \( \delta = 1 - 0, 2\{\theta\} + \varepsilon_1 > 0 \). Then Lemma \(2\) shows that the set of \( \omega \) for which there are infinitely many such polynomials has zero measure.

3) Suppose that all polynomials \( R_i \) are irreducible and that at least two are relatively prime (otherwise use case 1). Then Lemma \(6\) can be used on two of polynomials, \( R_1 \) and \( R_2 \), say. We have \( \deg R_i \leq n - [\theta] \) \( (i = 1, 2) \). It follows from \((10)\), \((34)\) and \((36)\) that
\[ \tau = n + 1 - \theta + 0, 8\{\theta\} + 2\varepsilon_1 = r_1 + l_2/T + 0, 8\{\theta\} + 2\varepsilon_1, \]
\[ \tau - \sigma_1 = r_1 - (m - 1)\varepsilon_1 = (l_2 + \ldots + l_m)/T - (m - 1)/T \geq T^{-1} > 0, \]
\[ \tau + 2(\tau - \sigma_1) = 3r_1 + l_2/T + 0, 8\{\theta\} - 2(m - 2)\varepsilon_1, \]
\[ 2(n - [\theta]) + \delta = -2 + 2r_1 + 2l_2/T + 2\{\theta\} + \delta. \]

As \( r_1 \geq l_2/T \) then \( \tau + 2(\tau - \sigma_1) > 2(n - [\theta]) + \delta \) if \( 0 < \delta < \varepsilon \). The last inequality contradicts Lemma \(6\).

In the case of \( \{\theta\} < \varepsilon \) we set
\[ \beta = [\theta] - 1 + \varepsilon, \quad \sigma_1 = l_2/T + \{\theta\} +(m+1)\varepsilon_1 - (1, 5 + \varepsilon')\varepsilon, \quad \varepsilon' = \varepsilon/(9n+2), \quad d = [\theta] - 1 \]
and apply the same arguments as above. \( \square \)

**Proposition 7.** If
\[ \varepsilon \leq r_1 + l_2/T \leq 2 - \varepsilon/2 \]
then \( \mu(A(I, \xi)) = 0 \).

**Proof.** All polynomials \( P(\omega) = H\omega^n + a_{n-1}\omega^{n-1} + \ldots + a_1x + a_0 \in P_n(H, \mathbf{7}) \) corresponding to the same vector \( \overline{a} = (a_{n-1}, \ldots, a_2) \) are grouped together into a class \( P_n(H, \mathbf{7}, \overline{a}) \). Let
\[ B(P) = \{ \omega \in K_0 \cap S(\alpha_1) : |\omega - \alpha_1|_p \leq H^{-n-1}|P'(\alpha_1)|^{-1}_p \}, \]
\[ B_1(P) = \{ \omega \in K_0 \cap S(\alpha_1) : |\omega - \alpha_1|_p \leq H^{-2+\varepsilon'}|P'(\alpha_1)|^{-1}_p \}, \]
where \( \varepsilon' = \varepsilon/6 \). It is clear that \( B(P) \subset B_1(P) \),
\[ \mu B(P) = c_1(p)H^{-n-1}|P'(\alpha_1)|^{-1}_p, \quad \mu B_1(P) = c_2(p)H^{-2+\varepsilon'}|P'(\alpha_1)|^{-1}_p \]
and
\[ \mu B(P) = c_3(p)H^{-n+1-\varepsilon'}\mu B_1(P), \]
where \( c_i(p) > 0 \) \( (i = 1, 2, 3) \) are the constants dependent on \( p \). Now we estimate \( |P(\omega)|_p \) when \( P \in P_n(H, \mathbf{7}, \overline{a}) \) and \( \omega \in B_1(P) \). It follows from the definition of \( B_1(P) \) that
|P'_{i}(\alpha_1)(\omega - \alpha_1)|_p < H^{-2+\varepsilon'}. By the right-hand side of \([43]\) and the definition of the \(r_j\) \((2 \leq j \leq m)\) we have

\[ jr_1 - r_j = (j - 1)r_1 + r_1 - r_j = (j - 1)r_1 + (l_2 + \ldots + l_j)/T \leq (j - 1)(2 - \varepsilon/2). \]

From this, Lemma \([5]\) and the definition of \(B_1(P)\) we find

\[ |P^{(j)}(\alpha_1)(\omega - \alpha_1)^j|_p < H^{-r_j + (m-j)\varepsilon_1} H^{-2(\varepsilon')j + jr_1} \leq H^{-j(2-\varepsilon'-r_1)} < H^{-2-\varepsilon/3} \]

for \(m < j \leq n\). By Taylor’s formula and the previous estimates we get

\[ |P(\omega)|_p \ll H^{-2+\varepsilon'} \quad (45) \]

for any \(\omega \in B_1(P)\). Further we use essential and inessential domains introduced by Sprindžuk \([Spr69]\). The disc \(B_1(P)\) is called inessential if there exists a polynomial \(Q \in \mathcal{P}_n(H,\overline{\mathbb{F}})\) such that \(\mu(B_1(P) \cap B_1(Q)) \geq \frac{1}{2}\mu B_1(P)\) and essential otherwise.

Let the disc \(B_1(P)\) be inessential and \(D = B_1(P) \cap B_1(Q)\). Then

\[ \mu D \geq \frac{1}{2}\mu B_1(P) = c_4(p)H^{-2+\varepsilon'|P'(\alpha_1)|_p}^{-1} \]

where \(c_4(p) > 0\) is a constant dependent on \(p\). By \([45]\) the difference \(R(\omega) = P(\omega) - Q(\omega) = b_1 \omega + b_0\), where \(\max(|b_0|, |b_1|) \leq 2H\), satisfies

\[ |R(\omega)|_p = |b_1 \omega - b_0|_p \ll H^{-2+\varepsilon'} \quad (46) \]

for any \(\omega \in B_1(P)\). Note that \(b_1 \neq 0\) since if \(b_1 = 0\), then \(|b_0|_p \ll H^{-2+\varepsilon'}. It is contradicted to \(|b_0|_p \geq |b_0|^{-1} \gg H^{-1}. It follows from \([46]\) that

\[ |\omega - b_0/b_1|_p \ll H^{-2+\varepsilon'|b_1|^{-1}_p} \quad (47) \]

Let \(D_1 = \{\omega \in K_0 \cap S(\alpha_1) : \text{ the inequality } (47) \text{ holds }\}. Then \(D \subseteq D_1\) and \(\mu D_1 = c_5(p)H^{-2+\varepsilon'|b_1|^{-1}_p}\), where \(c_5(p) > 0\) is a constant dependent on \(p\). We have

\[ c_4(p)H^{-2+\varepsilon'|P'(\alpha_1)|_p}^{-1} \leq \mu D \leq \mu D_1 \ll H^{-2+\varepsilon'|b_1|^{-1}_p}. \]

Hence

\[ |b_1|_p \ll |P'(\alpha_1)|_p \quad (48) \]

From \([48]\) and Lemma \([5]\) we get

\[ |b_1|_p \ll |P'(\alpha_1)|_p \ll H^{-r_1 + (m-1)\varepsilon_1}. \]

Since \(r_1 \geq l_2/T\) the left-hand side of \([43]\) implies \(r_1 \geq \varepsilon/2\). Now we find \(|b_1|_p \ll H^{-\varepsilon/3}\) for \(\varepsilon_1 \leq \varepsilon/(2n)\). It follows from \([46]\) that \(|b_0|_p \ll H^{-\varepsilon/3}. \) Suppose that \(s\) is defined by the inequalities \(p^s \leq H < p^{s+1}\). We have \(H^{\varepsilon/3} < p^{\varepsilon/3}\) for sufficiently large \(H\). Hence \(b_1 \gg p^{\varepsilon/3}b_{11}\) and \(b_0 \gg p^{\varepsilon/3}b_{01}\) where \(b_{11}, b_{01}\) are integers. We have

\[ b_1 \omega + b_0 \gg p^{\varepsilon/3}(b_{11} \omega + b_{01}) \quad \text{with max}(|b_{11}|, |b_{01}|) \ll H^{1-\varepsilon/3}. \quad (49) \]
Let $R_1(\omega) = b_{11}\omega + b_{01}$. Then $H(R_1) \ll H^{1-\epsilon/3}$. It follows from (46) and (49) that 
\[ |b_{11}\omega + b_{01}|_p \ll p^{\epsilon/3}H^{-2+\epsilon'} \ll H^{-2+\epsilon'+\epsilon/3} = H(R_1)^{-2-\epsilon/(6-2\epsilon)}. \]

Using Khintchine’s theorem in $\mathbb{Q}_p$ [Spr69, p. 94], we get that the set of $\omega$ belonging to infinitely many discs $B_1(P)$ has zero measure.

Let the disc $B_1(P)$ be essential. By the property of $p$-adic valuation every point $\omega \in K_0$ belong to no more than one essential disc. Hence 
\[ \sum_{P \in \mathcal{P}(H \not\equiv \pi)} \mu B_1(P) \ll p^n. \]

It follows from (44) that 
\[ \sum_{H} \sum_{P \in \mathcal{P}(H \not\equiv \pi)} \mu B(P) = \sum_{H} \sum_{\prod P \in \mathcal{P}(H \not\equiv \pi)} \mu B(P) \ll \]
\[ \ll \sum_{H} H^{n-2} \sum_{P \in \mathcal{P}(H \not\equiv \pi)} H^{-n+1-\epsilon'} \mu B_1(P) \ll \sum_{H} H^{-1-\epsilon'} < \infty. \]

The Borel-Cantelli lemma completes the proof. \qed

8. Proof of Proposition 4

First of all we impose some reasonable limitation on the disc $K_0$ that appear in the statement of Proposition 4. To this end we notice the following two facts.

**Remark 1.** Let $\omega_0, \theta_0 \in \mathbb{Q}_p$. It is a simple matter to verify that if $(\Gamma, N)$ is a regular system in a disc $K_0$ then $(\tilde{\Gamma}, \tilde{N})$ is regular in $\theta_0 K_0 + \omega_0$, where $\tilde{\Gamma} = \{\delta_0 \gamma + \omega_0 : \gamma \in \Gamma\}$, $\tilde{N}(\delta_0 \gamma + \omega_0) = N(\gamma)$ and $\theta_0 K_0 + \omega_0 = \{\theta_0 \omega + \omega_0 : \omega \in K_0\}$.

**Remark 2.** One more observation is that if $c > 0$ is a constant and $(\Gamma, N)$ is a regular system in a disc $K_0$ then $(\Gamma, cN)$ is also a regular system in $K_0$.

The proofs are easy and left as exercises. Now we notice that for any disc $K_0$ in $\mathbb{Q}_p$ we can choose two numbers $\omega_0, \theta_0 \in \mathbb{Q}$ such that $\theta_0 \mathbb{Z}_p + \omega_0 = K_0$. It is clear that the map $\omega \mapsto \theta_0 \omega + \omega_0$ sends $\mathbb{A}_{p,n}$ to itself. Moreover, there is a constant $c_1 > 0$ such that for any $\alpha \in \mathbb{Z}_p \cap \mathbb{A}_{p,n}$ one has $H(\theta_0 \alpha + \omega_0) \leq c_1 H(\alpha)$. Hence, if we will succeed to prove Proposition 4 for the disc $\mathbb{Z}_p$ then in view of the Remarks above it will be proved for $K_0$. Thus without loss of generality we assume that $K_0 = \mathbb{Z}_p$.

In the proof of Proposition 4 we will refer to the following statement known as Hensel’s Lemma (see [BD99, p. 134]).

**Lemma 8.** Let $P$ be a polynomial with coefficients in $\mathbb{Z}_p$, let $\xi = \xi_0 \in \mathbb{Z}_p$ and $|P(\xi)|_p < |P'(\xi)|_p^2$. Then as $n \to \infty$ the sequence 
\[ \xi_{n+1} = \xi_n - \frac{P(\xi_n)}{P'(\xi_n)} \]
tends to some root $\alpha \in \mathbb{Z}_p$ of the polynomial $P$ and 
\[ |\alpha - \xi|_p \leq \frac{|P(\xi)|_p}{|P'(\xi)|_p^2} < 1. \]
**Proposition 8.** Let $\delta > 0$, $Q \in \mathbb{R}_{>1}$. Given a disc $K \subset \mathbb{Z}_p$, let
\[
E(\delta, Q, K) = \bigcup_{P \in \mathbb{Z}[x], \deg P \leq n, H(P) \geq Q} \{\omega \in K : |P(\omega)|_p < \delta Q^{-n-1}\}. \tag{50}
\]
Then there is a positive constant $c$ such that for any finite disc $K \subset \mathbb{Z}_p$ there is a sufficiently large number $Q_0$ such that $\mu(E(\delta, Q, K)) \leq c\delta \mu(K)$ for all $Q \geq Q_0$.

**Proof.** The set $E(\delta, Q, K)$ can be expressed as follows
\[
E(\delta, Q, K) \subset E_1(\delta, Q, K, 1/3) \cup E_2(Q, K) \cup E_4(),
\]
where $E_1(\delta, Q, K, 1/3)$ is introduced in Proposition 1.

\[
E_3(Q, K) = \bigcup_{P \in \mathbb{Z}[x], \deg P \leq n, H(P) \geq \log Q} \chi(P),
\]
$\chi(P)$ is the set of solutions of (53) lying in $K$ with $\xi = 1/3$ and $C = \delta$,
\[
E_4(Q, K) = \bigcup_{P \in \mathbb{Z}[x], \deg P \leq n, H(P) \leq \log Q} \{\omega \in K : |P(\omega)|_p < \delta Q^{-n-1}\}.
\]

By Proposition 2,
\[
\mu(E_3(Q, K)) \to 0 \text{ as } Q \to \infty. \tag{51}
\]
By Proposition 1,
\[
\mu(E_1(\delta, Q, K, 1/3)) \leq c_1 \delta \mu(K) \text{ for sufficiently large } Q. \tag{52}
\]
Now to estimate $\mu(E_4(Q, K))$ we first estimate the measure of $\{\omega \in K : |P(\omega)|_p < \delta Q^{-n-1}\}$ for a fixed $P$. If $\alpha_{\omega,P}$ is the nearest root to $\omega$ then $|a_n(\omega - \alpha_{\omega,P})^n|_p < Q^{-n-1}$. Since $|a_n|_p \geq Q^{-1}$, we get $|\omega - \alpha_{\omega,P}|_p < Q^{-1}$. It follows that
\[
\mu\{\omega \in K : |P(\omega)|_p < \delta Q^{-n-1}\} \ll Q^{-1}.
\]
Hence $\mu(\mu(E_4(Q, K))) \ll (\log Q)^{n+1} Q^{-1} \to 0$ as $Q \to \infty$. Combining this with (51) and (52) completes the proof. \[\square\]

**Proof of Proposition 4.** Fix any disc $K \subset \mathbb{Z}_p$ and let $Q > 0$ be a sufficiently large number. Let $\omega \in K$. Consider the system
\[
\begin{cases}
|P(\omega)|_p < \delta^2 C Q^{-n-1}, & P(\omega) = a_n \omega^n + \cdots + a_1 \omega + a_0, \\
|a_j| \leq \delta^{-1} Q, & j = 0, n, \\
|a_j|_p \leq \delta, & j = 2, n.
\end{cases} \tag{53}
\]

By Dirichlet’s principle, it is easy to show that there is an absolute constant $C > 0$ such that for any $\omega \in K$ the system (53) has a non-zero solution $P \in \mathbb{Z}[x]$. Fix such a solution $P$.

If $|P'(\omega)| < \delta$, then, by (53),
\[
|a_1|_p = |P'(\omega)| - \sum_{k=2}^{n} k a_k \omega^{k-1} |_p \leq \max\{|P'(\omega)|_p, |2a_2 \omega^1|_p, \ldots, |n a_n \omega^{n-1}|_p\} < \delta.
\]
Also, if $Q$ is sufficiently large, then
\[
|a_0|_p = |P(\omega) - \sum_{k=1}^{n} a_k \omega^k|_p \leq \max\{|P(\omega)|_p, |a_1 \omega^1|_p, \ldots, |a_n \omega^n|_p\} < \delta.
\]
Therefore, the coefficients of $P$ have a common multiple $d$ with $\delta/p \leq |d|_p < \delta$. It follows that $d^{-1} \leq \delta$. Define $P_1 = P/d \in \mathbb{Z}[x]$. Obviously $H(P_1) \leq Q$. Also, by (53),

$$|P_1(\omega)|_p = |P(\omega)|_p |d|^{-1}_p \leq |P(\omega)|_p \times \delta^{-1} p < \delta C p Q^{-n-1}.$$ 

This implies $\omega \in E(\delta C p, Q, K)$. By Proposition 8, $\mu(E(\delta C p, Q, K)) \leq e \delta C p \mu(K)$ for sufficiently large $Q$. Put $\delta = (2cpC)^{-1}$. Then $\mu(K \setminus E(\delta C p, Q, K)) \geq \frac{1}{2} \mu(K)$. If now we take $\omega \in K \setminus E(\delta C p, Q, K)$ then we get

$$|P'(\omega)|_p \geq \delta.$$

By Hensel’s lemma there is a root $\alpha \in \mathbb{Z}_p$ of $P$ such that $|\omega - \alpha|_p < CQ^{-n-1}$. If $Q$ is sufficiently large then $\alpha \in K$. The height of this $\alpha$ is $\leq \delta^{-1} Q$.

Let $\alpha_1, \ldots, \alpha_t$ be the maximal collection of algebraic numbers in $K \cap \mathbb{A}_{p,n}$ satisfying $H(\alpha_j) \leq \delta^{-1} Q$ and

$$|\alpha_i - \alpha_j|_p \geq Q^{-n-1} \quad (1 \leq i < j \leq t).$$

By the maximality of this collection, $|\omega - \alpha_j|_p < CQ^{-n-1}$ for some $j$. As $\omega$ is arbitrary point of $E(\delta C p, Q, K)$, we get

$$E(\delta C p, Q, K) \subseteq \bigcup_{j=1}^{t} \{ \omega \in \mathbb{Z}_p : |\omega - \alpha_j|_p < CQ^{-n-1} \}.$$

Next,

$$\frac{1}{2} \mu(K) \leq \mu(E(\delta C p, Q, K)) \ll Q^{-n-1} t,$$

whence $t \gg Q^{n+1} \mu(K)$. Taking $T = \delta^{-1} Q^{n+1}$ one readily verifies the definition of regular systems. The proof is completed. 

\begin{flushright}
\square
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**Acknowledgements.** The research was supported by Belorussian Fond of Fundamental research (Project 00-249) and by INTAS (project 00-429).

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