GOLD ALIGNMENT AND INTERNAL DISSIPATION

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ABSTRACT

The measures of mechanical alignment are obtained for both prolate and oblate grains whose temperatures are comparable to the grain kinetic energy divided by $k$, the Boltzmann constant. For such grains, the alignment of angular momentum, $J$, with the axis of maximal inertia, $a$, is only partial, which substantially alters the mechanical alignment as compared with the results obtained by Lazarian and Roberge, Hanany, & Messinger under the assumption of perfect alignment. We also describe Gold alignment when the Barnett dissipation is suppressed and derive an analytical expression that relates the measure of alignment to the parameters of grain nonsphericity and the direction of the gas-grain drift. This solution provides the lower limit for the measure of alignment, while the upper limit is given by the method derived by Lazarian. Using the results of a recent study of incomplete internal relaxation by Lazarian & Roberge, we find measures of alignment for the whole range of ratios of grain rotational energy to $kT$, where $T$ is the grain temperature. To describe alignment for mildly supersonic drifts, we suggest an analytical approach that provides good correspondence with the results of direct numerical simulations by Roberge, Hanany, & Messinger. We also extend our approach to account for simultaneous action of the Gold and Davis-Greenstein mechanisms.

Subject headings: dust, extinction — ISM: clouds — MHD — polarization — waves

1. INTRODUCTION

Understanding the observed alignment of the ISM grains is as yet an unsolved astrophysical problem (see Hildebrand 1988; Whittet 1992; Goodman et al. 1995; Roberge 1996). This unknown factor limits the use of polarimetry data for studying interstellar magnetic fields.

The mechanism of mechanical alignment of thermally rotating grains was pioneered by Gold (1951) in the same year that the classic paper by Davis & Greenstein (1951) introducing paramagnetic alignment was published. Originally, Gold suggested that grain alignment arises from cloud-cloud collisions (Gold 1951, 1952), but it was shown in Davis (1955) that such collisions can align only an insignificant fraction of interstellar grains. Therefore a further study of the mechanism was devoted mainly to the alignment in the vicinity of bright sources, where radiation pressure can drive grains to supersonic velocities (see Purcell 1969; Aitken et al. 1995). A new, important idea, put forward by Roberge & Hanany (1990), that grains can be aligned by ambipolar diffusion made the Gold alignment a more promising mechanism (see also Roberge, Hanany, & Messinger 1995, hereafter RH). Our study in Lazarian (1994, hereafter Paper I, and 1995a) showed that the role of mechanical processes had been underestimated. It was found that pervasive MHD waves can produce grain alignment even in ideal MHD theory.

The goal of this paper is to provide an analytical, quantitative description of the mechanical alignment of thermally rotating grains, referred to as “Gold alignment.” We distinguish between Gold alignment and the mechanical alignment of suprathermally rotating grains (see Lazarian 1995a, 1995b; Lazarian & Efroimsky 1996; Lazarian, Efroimsky, & Ozik 1996).

Although the expressions for the measure of alignment corresponding to Gold alignment were obtained in Paper I, the shortcoming of that study was that only perfect alignment of angular momentum, $J$, with the the principal axis of the maximal moment of inertia, $a$ (hereafter the “major axis of inertia”), was considered. Although this condition is valid for the highly supersonic motions discussed in Paper I, the wealth of ISM conditions presents us with a wide range of other options.

It is well known from theoretical mechanics that internal dissipation of energy cannot change grain angular momentum and that, for a grain with fixed angular momentum, the minimal energy corresponds to rotation about the major axis of inertia. For suprathermally rotating grains, efficient internal dissipation of energy, leading to nearly perfect alignment of angular momentum with the grain major axis of inertia, was discovered by Purcell (1979) (see also Spitzer & McGlynn 1979). For thermally rotating grains, the alignment is only partial (Paper I; Lazarian & Roberge 1997, henceforth LR97).

To relate polarimetry observations, which are influenced by the alignment of the long grain axis, to theory, which deals with the alignment of the angular momentum $J$, one has to describe the alignment of $J$ not only with respect to the magnetic field, but also with respect to the grain axes. These alignments will be called “external” and “internal” alignment, respectively. Internal alignment arises both from the difference in grain moments of inertia (“Maxwellian alignment”) and from the Barnett relaxation (“Barnett alignment”).

Below we use the statistical approach introduced in LR97. The gist of it is to describe deviations of $J$ from the

1 Present address: Department of Astrophysical Sciences, Princeton University, Princeton, NJ 08544.
2 Throughout this work, when speaking of MHD waves we mean both Alfvénic and magnetosonic waves.
3 A magnetic field can act as an axis of alignment because the timescale of Larmor precession is usually much shorter than the timescale of alignment. We remind the reader that the rapid Larmor precession arises from magnetic moments of grains; those moments are caused by the Barnett effect (Dolginov & Mitrofanov 1976).
4 As a rule, Barnett relaxation dominates internal relaxation (see Purcell 1979).
major axis of inertia, \( a \), as thermal fluctuations from the position of static equilibrium on the assumption that the time of internal relaxation is much shorter than that of the gaseous damping. By introducing the grain rotational temperature, which depends on both the temperature of the ambient gas and on the drift velocity, we show that \( J \) is parallel to \( a \) when the ratio of this temperature to the temperature of grain material tends to infinity, which justifies our approach in Paper I. For finite ratios, however, the deviations of \( J \) become important for the alignment calculation.

To describe Gold alignment for mildly supersonic velocities, we must account for the rms motions of gas atoms. We assume that grains are subjected to a superposition of fluxes within a small angle of the direction of the initial flux and that the value of this angle depends on the ratio of the flux velocity to the mean velocity of thermal motion. This procedure enables us to obtain an asymptotic solution that provides fair agreement with the numerical calculations in Paper I. 

Within the same model of perfect alignment between \( J \) and \( a \). Within the same model of perfect alignment, an agreement with the numerical values in RHM is obtained for the joint action of Gold and Davis-Greenstein mechanisms. A comparison between numerical and analytical approaches for the general case of incomplete internal alignment is planned, and we anxiously follow the progress of the numerical method in this direction.

The structure of the paper is as follows. First of all, we remind our reader of the concept of internal relaxation and describe how grain drift influences internal alignment (§ 2). Then, in § 3, we obtain analytic solutions for the alignment measure when internal relaxation is negligible and compare these results with the other extreme case, namely the one in which \( J \) is perfectly parallel to \( a \). The treatment of the problem for an arbitrary degree of internal alignment is given in § 4. In § 5, we compare our results with numerical results and argue that our analytic approach provides an adequate description of the alignment both for mildly supersonic velocities and for Gold and Davis-Greenstein processes acting together. A short summary of results is presented in § 6.

2. INTERNAL ALIGNMENT

As pointed out above, to relate theory and observations it is essential to describe the alignment of grain long axes with respect to the magnetic field. This alignment is determined uniquely by the alignment of grain angular momentum \( J \) with the major axis of inertia \( a \). Further on we study the general problem of partial internal alignment.

Theoretical studies deal with the measure of alignment of angular momentum \( J \):

\[
Q_J = \frac{1}{2} \left( \cos^2 \theta - \frac{1}{2} \right),
\]

where \( \theta \) is the angle between \( J \) and the direction of the magnetic field, while polarimetry provides us with data on the Rayleigh reduction factor (Greenberg 1968),

\[
R = \frac{1}{2} \left( \cos^2 \theta - \frac{1}{2} \right),
\]

where \( \theta \) is the angle between the direction of the magnetic field and the symmetry axis of a spheroid approximating the grain. The question of how to relate these two quantities was on the astrophysical agenda from the very beginning of research in this area. First, in Jones & Spitzer (1967), it was assumed that the distribution function of \( J \) in the grain reference frame is independent of the alignment of \( J \) in respect to magnetic field, a natural assumption to start with. Later, Spitzer (1978) showed that additional alignment of \( J \) with respect to the major axis of inertia should be present because of paramagnetic relaxation in the external magnetic field. This effect will be called Spitzer relaxation to distinguish it from the Barnett relaxation discovered by Purcell (1979).

Here we assume that the ratio of the gas damping time to the time of internal relaxation is much greater than unity (see estimates in Roberge, DeGraff, & Flaherty 1993). In this case, the deviations of \( J \) from the major axis of inertia \( a \) can be described thermodynamically (LR97).

For a grain with fixed angular momentum \( J \equiv |J| \) and with components of the moment of inertia related by \( I_x < I_y < I_z \), the kinetic energy is

\[
E = \frac{1}{2} \left( \frac{1}{I_x} - \frac{1}{I_y} \right) \sin^2 \theta \sin^2 \phi + \left( \frac{1}{I_y} - \frac{1}{I_z} \right) \sin^2 \theta + \frac{1}{I_z},
\]

where \( \xi \) is the azimuthal angle in the \( x-y \) plane and \( \theta \) is the angle between \( J \) and the \( z \)-axis. Although the equilibrium position of a grain with fixed \( J^2 \) corresponds to \( \theta \equiv 0 \), thermal fluctuations cause deviations from that position. These deviations can be described by the Boltzmann factor \( \exp \left[ -E/(kT) \right] \) (LR97).

A complex precession of \( J \) arising from these fluctuations modifies both the grain interaction with the gaseous flow and the dichroic absorption. However, it seems reasonable to assume that the behavior of such an asymmetric grain can be approximated by the behavior of a spheroidal grain with a moment of inertia somewhere between \( I_x \) and \( I_y \) (hereafter we denote this mean moment by \( I \)). Here we adopt this approximation and hope to compare the results for spheroidal and irregular grains elsewhere.

For a spheroidal grain with temperature \( T \), equation (3) can be simplified, and the distribution function of the angular momentum in the grain reference frame is (LR97)

\[
f_{r_J}(\theta) = \text{const} \times \sin \theta \exp \left\{ -\frac{J^2}{2I kT} \left[ (\hbar - 1) \sin^2 \theta + 1 \right] \right\},
\]

where \( \hbar \) is the angle between \( J \) and the rotational symmetry axis of the spheroid, \( h = I_y/I_x \), and the value of \( \theta \) is different for different grains of the ensemble. The fact that the probability of a particular angular momentum depends on the angle \( \theta_2 \) complicates the study. However, computations in LR97 show that, to a sufficient degree of accuracy, it is possible to substitute the rms value of \( J \) for a Maxwellian angular momentum distribution in equation (4) to get an approximate measure of alignment for an ensemble of
Below we describe the Gold alignment when the constraint seem to be universally applicable to interstellar grains. For gas and grains have similar temperatures and for mildly grain environment. For instance, in molecular clouds function. This result, however, may not be true for a different of $J$ by a factor of 3, The corresponding distribution of grains exceeds the velocity of atoms in a diffuse cloud of course drift velocities, $T_{\text{eff}}$ of the same order as $T_s$.

In short, perfect alignment between $J$ and $a$ does not seem to be universally applicable to interstellar grains. Below we describe the Gold alignment when the constraint of perfect $J$-$a$ alignment is lifted.

3. GOLD ALIGNMENT FOR $J$ NOT PARALLEL TO $a$

3.1. Analytic Solution for the Rayleigh Reduction Factor in the Absence of Internal Relaxation

The distribution of angular momentum can be characterized by the function $f(n, J)$, where $n$ is the number of grain-atom collisions. In general, the direction of $J$ should be defined by angles $\theta_1$ and $\varphi_1$ in the “gas reference frame” and by $\theta_2$ and $\varphi_2$ in the “grain reference frame” (see Fig. 2). Angle $\varphi_1$ describes the precession of $J$ about the magnetic field and angle $\varphi_2$ describes the precession of the spheroid’s axis of rotational symmetry about $J$. Hereafter grains will be approximated by spheroids with semi-axes $a$ and $b$.

Since the change of grain angular momentum in the course of an individual collision is small, the alignment of $J$ can be described by the Fokker-Planck equation (see Reichl 1980, p. 168; Roberge et al. 1993). Then, following Dolginov & Mitrofanov (1976), we can write

$$\frac{\partial f(x, n)}{\partial n} = \sum_{i=0}^{2} a_i(x) \frac{\partial f(x, n)}{\partial x_i} + \sum_{k=0}^{2} b_{ik}(x) \frac{\partial^2 f(x, n)}{\partial x_i \partial x_k},$$

where $x$ is a vector in the phase space with coordinates $J, \cos \theta_1, \cos \theta_2, \varphi_1,$ and $\varphi_2$, and the coefficients $a_i$ and $b_{ik}$, obtained under the assumption of supersonic grain drift, can be found in Appendix A. The solution of equation (7) in the limit of hypersonic drift is (Dolginov & Mitrofanov 1976)

$$f(J, \cos \theta_1, \cos \theta_2, n) = \frac{\text{const}}{n^{3/2}} \times \exp \left[ \frac{-J^2(1 + g \cos^2 \theta_1 + s \cos^2 \theta_1)}{2nb^2p^2(1 + g + s)} \right],$$

where

$$s = -\frac{1}{2} \frac{\langle p^2 \rangle - 3\langle p_z^2 \rangle}{\langle p^2 \rangle - \langle p_z^2 \rangle}$$

is the external flux anisotropy and

$$g = \frac{a^2 - b^2}{2b^2}$$

is the grain nonsphericity. Note that $\langle p^2 \rangle$ and $\langle p_z^2 \rangle$ are the averaged squared momentum transferred to a grain in an individual collision and the square of the $Z_1$ component of that momentum (see Fig. 2). Both $g$ and $s$ can vary from $-0.5$ to $\infty$. It is easy to see that $g = -0.5$ corresponds to flakes and $g \to \infty$ to needles; $s = -0.5$ corresponds to a
flux perpendicular to magnetic field, while \( s \to \infty \) corresponds to a flux parallel to the field. The fluxes are measured in the grain reference frame, and therefore a gaseous flux with the velocity \( u \) is equivalent to grain drift with the velocity \(-u\) with respect to the ambient gas.

Spherical grains \((g = 0)\) correspond to \( a = b, \) while isotropic fluxes \((s = 0)\) correspond to \( \langle p^2 \rangle = 3 \langle p_z^2 \rangle. \) We expect changes in grain alignment when \( s = 0 \) and/or \( g = 0. \)

To have a picture that is easy to visualize, we refer to a flux (or drift) corresponding to \( s < 0 \) as a "flux (drift) at a large angle to the magnetic field" and to those corresponding to \( s > 0 \) as a "flux (drift) at a small angle to the magnetic field." Note that "small" angles lie in the interval \([0, \arccos(1/3)]\) and "large" angles in \([\arccos(1/3), \pi/2]\), if we limit our discussion to the first quadrant. Obviously, oblate and prolate grains correspond to \( g < 0 \) and \( g > 0, \) respectively.

Calculations in Dolginov & Mitrofanov (1976) show that the solution given by equation (8) is accurate up to terms of order 0.25 \( |g|/s | \) for \( |s| < 1 \) and \( |g| < 1. \) The accuracy tends to the order of \( s^{-2} \) as \( s \to \infty \) for \( |g| < 0, \) whereas it tends to the order of \( g^{-2} \) as \( g \to \infty \) for \( s < 0. \) If both \( g \) and \( s \) are large, the accuracy is of the order of \( \max \left[ g^{-1}, s^{-1}\right]. \)

The angle \( \theta \) can be found from simple geometric considerations (see Fig. 2) as

\[
\cos \theta = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos (\phi_1 - \phi_2). \tag{11}
\]

Averaging out the dependences on \( \phi_1 \) and \( \phi_2 \) gives (see eq. [108] of Davis & Greenstein 1951)

\[
\langle \cos^2 \theta \rangle = 0.5(1 - \cos^2 \theta_1 - \cos^2 \theta_2 + 3 \cos^2 \theta_1 \cos^2 \theta_2). \tag{12}
\]

To calculate the Rayleigh reduction factor given by equation (2), it is necessary to find \( \langle \cos^2 \theta \rangle. \) After averaging equation (8) over \( J, \) the required distribution function is found to be

\[
W_0(\theta_1, \theta_2) = C(g, s)(1 + g \cos^2 \theta_2 + s \cos^2 \theta_1)^{-3/2}, \tag{13}
\]

where \( C(g, s) \) is the normalization constant for any fixed \( s \) and \( g. \) Then

\[
\langle \cos^2 \theta \rangle = \frac{1}{2gs} \left[ 1 + g + s + gs - C(g, s)\sqrt{1 + g + s} \right]. \tag{14}
\]

The necessary calculations are performed in Appendix B, and here we present the final expression,

\[
\langle \cos^2 \theta \rangle = \begin{cases} 
\sqrt{gs} \left( \arctan \sqrt{\frac{gs}{1 + g + s}} \right)^{-1}, & gs > 0, \\
\sqrt{gs} \left( \arctanh \sqrt{\frac{|g|}{1 + g + s}} \right)^{-1}, & gs < 0,
\end{cases} \tag{15}
\]

which is valid for all possible values of \( s \) and \( g. \) This expression can be compared with the analytic expression found for the case of strong relaxation (see Appendix C).

The corresponding Rayleigh reduction factor is

\[
R = \frac{1}{4} + \frac{3}{4gs} \left[ \sqrt{1 + g + s} - C(g, s) \right]. \tag{17}
\]

This measure enters the formulas for polarization of radiation by dichroic absorption. Equation (17) encompasses a variety of circumstances that are explored below.

3.2. Comparison with Paper I

Here we discuss the alignment for various values of \( s \) and \( g. \) When internal dissipation makes \( J \) and the major axis of inertia perfectly parallel, equations (12), (2) and (1) give the following relation between the Rayleigh reduction factor and the measure of alignment of angular momentum, \( \sigma_J, \) used in Paper I:

\[
R = \begin{cases} 
\sigma_J, & \text{oblate grains,} \\
-0.5 \sigma_J, & \text{prolate grains.} \tag{18}
\end{cases}
\]

It is easy to see that prolate and oblate grains produce polarization of the same sign despite the fact that their Rayleigh reduction factors have opposite signs. This fact becomes clear if one recalls that oblate grains align their short axes with respect to the magnetic field, while prolate grains align their long axes.

First, consider oblate grains \((g < 0)\) subjected to drift at a large angle to the magnetic field \((s < 0). \) The corresponding measure of alignment is shown in Figure 3. A comparison with Figure 3 of Paper I reveals the decrease of alignment associated with suppression of internal dissipation. For instance, for drift perpendicular to magnetic field lines, the measure of alignment for flake-like grains is 0.25 if the Barnett alignment is absent, while it reaches 1 (complete alignment) if the internal dissipation is efficient (see Fig. 4). A cross section of the plot for \( g = -0.5 \) (see Fig. 5) shows that flakes are only marginally aligned. This result is in contrast with the case of intense internal dissipation in which flakes are well aligned and the measure of alignment approaches \(-0.5\) if the flux and magnetic field directions coincide (Lazarian 1994). The alignment measure for oblate grains drifting at small angles to the magnetic field \((s > 0)\) is shown in Figure 6. Comparison with Figure 4 of Paper I shows an order of magnitude decrease of the measure. Because of the symmetry of the distribution function given by equation (8) under the simultaneous interchange \( \cos \theta_1 \leftrightarrow \cos \theta_2, s \leftrightarrow g, \) we can claim that needles are marginally aligned if the grain drift is perpendicular to the field, i.e., \( \mathbf{n} \perp \mathbf{B}. \) Indeed, the corresponding measure of alignment for prolate grains is negligible (see Fig. 7). In brief, if the inter-
Rayleigh reduction factor, $R$, for oblate grains ($g < 0$) under MHD waves ($s = -0.5$). The alignment is most efficient for flakes ($g = -0.5$). Because of the intrinsic symmetry of the problem, the same plot represents the Rayleigh reduction factor for flakes ($g = -0.5$) when $s$ varies from $-0.5$ to $0$ along the $x$-axis.

The measure of alignment for both $s$ and $g$ positive is shown in Figure 8. One can see that the measure of alignment tends to $0.25$ when $s$ and/or $g$ tend to infinity. In terms of $\sigma_j$ (see eq. [18]) this result is equivalent to $\sigma_j = -0.5$, as obtained in Paper I. This correspondence has a simple explanation, as for a needle the angular momentum
should be directed along the major axis of inertia even in the absence of internal dissipation.

The requirement of large $s$ and $g$ places stringent constraints on the efficiency of alignment for typical ISM conditions. Indeed, let the grain drift velocity components perpendicular and parallel to magnetic field be $u_\perp$ and $u_\parallel$, respectively. Then if $u_\perp$ is much greater than the rms velocity of gas atoms $v_{\text{rms}}$,

$$s = \frac{2 - w^2}{2w^2},$$  \hspace{1cm} (19)

where $w \approx u_\perp/u_\parallel$. If $u_\perp \ll v_{\text{rms}}, w \approx v_{\text{rms}}/v_\parallel$ should be used in equation (19). In any case, $w$ is unlikely to be less than 0.1, and therefore $s$ is not likely to be greater than 100. We do not believe that the ratio of the axis ratio of a typical prolate grain exceeds 10, either, so that $g$ is likely to be less than 50. The measure of alignment for $w \in [0.2, 0.7]$ and the axis ratio $y \equiv a/b \in [1, 10]$ is shown in Figure 9. The cross sections of the plot for $y = 5$ and $y = 10$ are shown in Figure 10.

To summarize, in the absence of internal dissipation, the measure of alignment drops. Oblate grains are most aligned for $u \perp B$, and prolate grains are most aligned for $u \parallel B$.

4. THE GENERALIZED PROBLEM

Up to now, two extreme cases have been discussed: strong internal dissipation, for which the angular momentum is parallel to the major axis of inertia, and weak internal dissipation, for which the residual alignment of angular momentum occurs because of the differences between the maximal and minimal moments of inertia. In both cases analytic solutions have been found. These solutions provide the upper and lower bounds for the measure of alignment. For instance, Figure 11 shows these two bounds for $u \perp B$, when the alignment is caused by ambipolar diffusion. A conspicuous feature of this particular figure is that the upper bound does not equal the lower bound for ideal spheres, i.e., when no axis alignment is expected. Equation (4) shows that for $h \rightarrow 1$ all positions of $J$ become equally probable. This fact was ignored in the simplified approach adopted in Paper I. Figure 11 also shows that the internal dissipation strongly influences grain alignment, as the spread in Rayleigh reduction factors for the two extreme cases is wide.

To account for the incomplete alignment of $J$ in the grain reference frame, one has to incorporate internal dissipation in the Fokker-Planck equation. An analytical study in this case seems formidable, and a numerical approach, e.g., similar to the one used by Robere et al. (1993), may be advantageous. For such a study both the analytical solutions obtained above and those derived in Paper I should serve as benchmarks.

A less rigorous but less laborious way to account for the incomplete alignment is to follow Jones & Spitzer (1967). Let the distribution function of $J$ in the presence of internal dissipation, $W_{GD}$, be the product of the distribution functions $W_G$ and $W_D$ given by equations (13) and (4), respectively. Then, after expressing $h$ in terms of $g$ and $J^2$ in terms of $T_{\text{eff}}$ using equations (5) and (6), respectively, we get

$$W_{GD} \approx \text{const} \times \sin \theta_2 (1 + g \cos^2 \theta_2 + s \cos^2 \theta_1)^{-3/2} \times \exp \left[ \frac{T_{\text{eff}}}{T_e} \frac{g(2g + 3)}{g + 1} \frac{\sin^2 \theta_2}{2} \right].$$  \hspace{1cm} (20)

Using $W_{GD}$ in equation (14), we have calculated $\langle \cos^2 \theta \rangle$ and the Rayleigh reduction factor for $T_{\text{eff}}/T_e$ equal to 100, 10, and 1.1. For example, Figure 12 shows the Rayleigh reduction factor as a function of grain oblateness. The comparison between Figures 12 and 11 shows that, while for efficient internal relaxation corresponding to $T_{\text{eff}}/T_e > 100$ the approximation of perfect coupling is appropriate, in the case of $T_{\text{eff}} \approx T_e$ this approximation no longer holds, and the analytic solutions that disregard internal relaxation provide a better fit. Similar conclusions are valid for other values of $s$ and $g$, with the exception of $g \rightarrow 0$. In this last case, as discussed earlier, the angular momentum can have any direction.

The alignment of prolate grains by ambipolar diffusion is marginal, and the comparison between Figures 13 and 7 shows that the internal dissipation does not change the alignment much. In contrast, Figures 14 and 5 show that the internal dissipation drastically changes the alignment of oblate grains because of the radiation pressure. If the alignment is marginal when the internal relaxation is suppressed...
(see Fig. 5), it becomes substantial as soon as the internal relaxation is present. This difference in the susceptibility to internal dissipation is easy to understand if one recalls that, without internal dissipation, \( J \) is only marginally aligned with the major axis of inertia for oblate grains and the alignment is substantial in the case of prolate grains. Therefore Figure 14 shows a marginal difference in grain alignment for different values of internal dissipation as grains become sufficiently prolate.

To summarize, our results show that Gold alignment is modified by internal relaxation and that it is essential to estimate the ratio of the effective rotational grain temperature to grain material temperature to predict the alignment accurately. Small temperature ratios usually occur when the grain drift and the thermal velocities are comparable. Then it is essential to account for the fact, that, in the grain reference frame, atoms move at the drift velocity, which is modified by thermal motion. In other words, grain-gas collisions formally correspond to a range of values of \( s \). To account for this effect, we suggest integrating over the corresponding range of \( s \) (see Lazarian 1995a). Since there are no direct numerical calculations of the measure of alignment for the case of incomplete internal relaxation, in the following we compare our results with numerical calculations only for model grains with \( J \) along the major axis of inertia.

5. COMPARISON WITH RHM

A comprehensive numerical study of the Gold alignment for perfect Barnett relaxation was done in RHM. The authors presented a detailed study of grain alignment for a range of drift velocities from subsonic velocity up. Paramagnetic relaxation was also included in their model. Here we briefly discuss how to improve our model to include both subsonic drift and paramagnetic relaxation.

5.1. Subsonic Drift

The model adopted here assumes that the grain drift is essentially hypersonic. In other words, we have ignored the rms velocity of gaseous atoms as compared to the drift

\footnote{We disregard the rather artificial case of hot grains drifting in cold gas.}
velocity and assumed that atoms hit the grain from a single
direction defined by
\[
\phi = \arcsin \sqrt{\frac{u_x^2 + u_y^2}{u_x^2 + u_y^2 + u_z^2}},
\]
(21)
where \(u_i, i = x, y, z\), are the components of the atom drift
velocity in the grain reference frame. Obviously, the anisot-
ropy parameter \(s\) can be expressed as a function of \(\phi\):
\[
s = \frac{1 - 3 \cos^2 \phi}{1 - \cos^2 \phi},
\]
(22)
and hence the Rayleigh reduction can also be written as a
function of \(\phi\).
When the drift is subsonic, atoms are seen as approaching
from various directions in the grain reference frame, with any particular atom approaching at angle
\[
\phi = \arcsin \sqrt{\frac{(u_x + v_x)^2 + (u_y + v_y)^2}{(u_x + v_x)^2 + (u_y + v_y)^2 + (u_z + v_z)^2}},
\]
(23)
where \(v_i, i = x, y, z\), are the components of rms velocity of
the atom. According to equation (23), the angle \(\phi\) varies
from atom to atom because of variations in \(v_i\). For each of the subgroups, \(\phi\) is constant; hence the measure of alignment can be obtained by averaging \(R(u, v)\) over a Maxwell-Boltzmann distribution of \(v\). Elsewhere we hope to test the applicability of such
an approach for a considerable range of drift velocities by
comparing our predictions with direct numerical simulations.
At the moment, we want to show that our estimates are in a reasonable agreement with the data presented in
RHM.
RHM assume that the angular momentum is perfectly
parallel to the major axis of inertia, which corresponds to
\(T/T_{eff} = 0\) in our model. The effect of the spread of atom
velocities in the grain reference frame should not depend
upon the position of the angular momentum with respect to
the major axis of inertia. Indeed, the velocities of particles
on the grain surface are much smaller than the velocities of
incoming atoms, and therefore, if we obtain agreement
between our predictions and those of RHM, we may hope
that our treatment is applicable when the internal align-
ment is incomplete.
Figure 8 of RHM shows the Rayleigh reduction factor for
oblate grains drifting with different velocities with respect to
the gas. The calculations were performed only up to Mach
number 4, but the saturation of the alignment is obvious from
their figure. Therefore we will consider that Mach numbers of \(\sim 3-4\) can be treated as hypersonic velocities and define \(R(\phi_0)\), where \(\phi_0\), in the case of ambipolar diffusion
considered in RHM, is \(\pi/2\). For our simplified estimates we observe that for large Mach numbers \(M\), equation
(23) shows
\[
\phi \approx \arcsin (1 - 0.5M^{-2}) \approx \frac{\pi}{2} - \frac{1}{2} M^{-2} - \frac{\sqrt{2}}{24} M^{-3},
\]
(24)
where the first term corresponds to \(\phi_0\) and the rest can be
interpreted as \(\delta \phi\). Therefore
\[
\langle R \rangle_\phi \approx \frac{1}{2 \delta \phi} \int_{\phi_0 - \delta \phi}^{\phi_0 + \delta \phi} R(\phi) d\phi \approx R(\phi_0) + 2 \frac{dR}{ds} \frac{ds}{d\phi} \delta \phi,
\]
(25)
where
\[
R(\phi_0) = -3.5 - 3g + 3\sqrt{1 + 2g(1 + g)} \arcsin \frac{1}{2\sqrt{1 + g}},
\]
(26)
and
\[
\frac{dR}{ds}\bigg|_{s=-0.5} = \frac{3 + 3g}{\sqrt{1 + 2g\sqrt{3 + 4g}}} \left[-1 - 2g - 2\sqrt{1 + 2g\sqrt{3 + 4g}} + \sqrt{3 + 4g(4 + 6g)} \arcsin \left(\frac{1}{2\sqrt{1 + g}}\right)\right],
\]
(27)
with \(ds/dx \approx 0.032\) when \(x = \phi = \pi/2\).
For instance, RHM find that at large Mach numbers the
Rayleigh reduction factor for grains with the axis ratio 0.5 is
\(\approx 0.25\). This axis ratio corresponds to \(g = 1.7\), which, when plugged into equation (25), gives \(\langle R \rangle_\phi \approx 0.21\) for \(M = 2\). This number is comparable with the value \(\approx 0.19\) that follows from Figure 8 of RHM. Similarly, for the axis ratio 0.25 corresponding to \(g = 7.5\), \(\langle R \rangle_\phi \approx 0.27\) for \(M = 2\), which is of the same order as the result in RHM \(\approx 0.26\).
Again we take the values of the Rayleigh reduction factor
obtained for \(M = 4\) in RHM and substitute in our formulas;
for the axis ratio 0.25, this value is \(\approx 0.39\). As the values of the Rayleigh reduction factor obtained in RHM for high 
Mach numbers essentially correspond to the values
obtained in the analytic treatment in Paper I (see Fig. 12 of
RHM) it is possible to see that a purely analytic treatment is
appropriate for at least some values of subsonic drift veloci-
ties. The entire range of velocities will be treated elsewhere.
Our estimates above were obtained using the analytical
solutions obtained in Paper I for perfect alignment of \(J\) with
the major axis of inertia. Evidently our approach is also
applicable to describe alignment at low Mach numbers
using the analytic and semianalytic results obtained in §§ 3
and 4, respectively. We are looking forward to progress in
numerical techniques that will allow us to compare our
predictions with the results of direct numerical simulations.

5.2. Mechanical and Paramagnetic Alignment
So far we have only discussed mechanical alignment and
have completely disregarded paramagnetic alignment. This
approximation is justifiable only if the Davis-Greenstein
alignment is negligible. In general, the Davis-Greenstein
alignment must be accounted for; to perform this task we
propose a simple formula.

---
*8 Since alignment time is much shorter than the damping time (see Paper I), each of the subgroups may by itself cause alignment.*
If we denote the Rayleigh reduction factor of the Gold alignment by \( R_G \) and that of the Davis-Greenstein alignment by \( R_{DG} \), and write the measure of the internal alignment as \( Q_x \), it is possible to estimate the measure of the overall alignment as

\[
R^2 \approx Q_x \frac{Q_x \sigma_G + R_G R_{DG} + Q_x R_{DG}}{Q_x^2 + 2R_G R_{DG}}. \tag{29}
\]

To obtain the expression above, we used the expression for \( J \)-alignment given by equation (45) of Lazarian (1995a) and the approximation

\[
R_i \approx Q_{J(0)} \times Q_x, \tag{30}
\]

where we write \( R_i \), the Rayleigh reduction factor and \( Q_{J(0)} \) the measure of \( J \) alignment relative to the magnetic field considering the \( i \)th mechanism acting alone. The latter approximation follows from spherical trigonometry (see eq. [12]). Indeed, if two processes are independent, then

\[
\langle \cos^2 \theta_1 \cos^2 \theta_2 \rangle \approx \langle \cos^2 \theta_1 \rangle \langle \cos^2 \theta_2 \rangle. \tag{31}
\]

The approximation above was proved to be sufficiently accurate for the Davis-Greenstein process by Lazarian (1995a), but may be much less accurate for the Gold alignment. In any case, we treat equation (29) only as a conjecture to be tested in the future.

A study of the simultaneous action of paramagnetic and mechanical alignment was performed recently in RHM for \( T_d/T_g = 0 \). In this case, \( \sigma_B \equiv 1 \) and equation (29) reduces to the formula derived in Lazarian (1995a).

First of all, to make the comparison between the two types of alignment we must define the measure of alignment for the Davis-Greenstein process when grains are subjected to a supersonic flow. Naturally, as a grain drifts faster, the rate at which atoms impact its surface increases, and therefore the gaseous damping time decreases. For our rough estimate of the Davis-Greenstein alignment subjected to the supersonic flow, we simply substitute the overall velocity instead of the thermal velocity into the expressions for the diffusion coefficients for gaseous bombardment (see Roberge et al. 1993). This method changes both \( T_{\text{eff}} \) and the ratio of the gaseous and magnetic damping times \( \delta_1 \) (see eq. [25] of Lazarian 1995c). The first change is not relevant here, because, in order to compare our estimates with the calculations in RHM, we assume \( T_u/T_{\text{eff}} = 0 \). On the other hand, the second change rescales the value of \( \delta_1 \) calculated for the nondrifting grain (the nondrifting value is shown in Figs. 4 and 9 of RHM). Let the rescaled value be \( \delta/M \). To obtain \( a_G \), we use the first approximation of Lazarian (1995c).

For spherical grains with \( \delta_1 = 10 \), RHM obtained \( \sigma \approx 0.40 \) (see Fig. 4 in RHM). Our approximation gives \( \sigma \approx 0.38 \). It is easy to see from Figure 9 of RHM that for \( \delta_1 = 1 \) an axis ratio of 0.5 gives \( \sigma \approx 0.32 \), whereas an axis ratio of 0.25 gives \( \sigma \approx 0.48 \). Our approach gives \( \sigma \approx 0.29 \) and 0.46, respectively. This approximate correspondence lets us hope that our simplified analytical treatment reproduces essential features of the alignment.

6. DISCUSSION

In short, we have shown that the alignment of \( J \) in the grain reference frame that arises from the difference between the grain material and rotational temperatures is essential for the Gold alignment. When velocities of grain drift are hypersonic, we can assume perfect alignment of \( J \) and the major axis of inertia. However, this assumption fails for mildly supersonic drifts, in which the gas and grain temperatures are comparable. Such conditions are expected, e.g., in molecular clouds undergoing ambipolar diffusion. In those cases, incomplete alignment between \( J \) and the major axis of inertia should be accounted for.

Our results also show that the effect of incomplete relaxation is more striking for oblate grains than for prolate ones. This result is a consequence of the fact that for sufficiently prolate grains the alignment of \( J \) with respect to the grain major axis of inertia is manifest even without internal relaxation.

The analytic results obtained above and those derived in Paper I provide the lower and upper bounds for Gold alignment. In the general case of incomplete internal alignment, we suggest a semianalytic approach.

To provide a quantitative description of Gold alignment for drift velocities comparable with the thermal velocities of gaseous atoms, we suggest an expansion of Rayleigh reduction factor in a series over the drift Mach number, for which we obtained a fair correspondence with the results of direct numerical computations.

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APPENDIX A

THE COEFFICIENTS OF THE FOKKER-PLANCK EQUATION

The coefficients \( a_i \) and \( b_{ik} \) of the Fokker-Planck equation are determined by the change of the grain angular momentum by grain-atom collisions,

\[
a_0 = \left\langle \frac{1}{2} \Delta x_2 \frac{\partial \Delta x_1}{\partial x_2} + \frac{1}{2} \Delta \phi_1 \frac{\partial \Delta x_1}{\partial \phi_1} - \Delta x_1 \right\rangle, \tag{A1}
\]

\[
a_m = \left\langle \frac{1}{2} \Delta x_1 \frac{\partial \Delta x_m}{\partial x_1} + \frac{1}{2} \Delta x_m \frac{\partial \Delta x_1}{\partial x_m} + \frac{1}{2} \Delta \phi_m \frac{\partial \Delta x_m}{\partial \phi_m} - \Delta x_m \right\rangle, \tag{A2}
\]

\[
\bar{b}_{ik} = \left\langle \Delta x_i \Delta x_k \right\rangle. \tag{A3}
\]
For a uniform representation, one can also use inverse hyperbolic sine for
where 
\[ \Delta x_j = (e_j \cdot J) - (e_j \cdot J)\Delta J , \]
\[ \Delta x_0 = j \cdot \Delta J , \]
\[ \Delta \varphi_j = (e_j \times j) \cdot \Delta J [J(1 - x_j^2)]^{-1} , \]
where \( j = 1, 2, e_1 = H/|H| \) is a unit vector along the magnetic field, \( e_2 = a/|a| \) is a unit vector along the Z_1-axis of the grain, \( j = J/|J| \) is a unit vector along \( J \), and \( \Delta J = r \times p \), with \( r \) and \( p \) the point of impact and the momentum of the atom, respectively.

To find coefficients \( a_k \), (\( k = 0, 1, 2 \)) and \( b_{ijk} \), (\( i, k = 0, 1, 2 \)) (see eq. [5] of Dolginov & Mitrofanov 1976), one must substitute \( \Delta x_j, \Delta x_0, \Delta \varphi_j \) into equations (A1), (A2), and (A3) and perform the necessary averaging. Apart from averaging over the surface area exposed to the flux, one has to average over the angles of precession of \( J \) about \( e_2 \) and \( m \).

\[ \text{APPENDIX B} \]

\[ \text{COMPUTATION OF INTEGRALS} \]

Equation (14) involves integration of equation (13) over the distribution function \( W_{\text{c}}(\theta_1, \theta_2) \). We find the result
\[ \langle \cos^2 \theta \rangle = 0.5(1 - \langle \cos^2 \theta_1 \rangle - \langle \cos^2 \theta_2 \rangle + 3\langle \cos^2 \theta_1 \cos^2 \theta_2 \rangle) , \]
where the explicit expression for the first term is
\[ \langle \cos^2 \theta_1 \rangle = C(g, s) \int_0^{\pi/2} \int_0^{\pi/2} \cos^2 \theta_1 \sin \theta_1 \sin \theta_2 (1 + s \cos^2 \theta_1 + g \cos^2 \theta_2)^{3/2} d\theta_2 . \]
After integrating over \( \theta_2 \) and making the change of variables \( \sin \theta_1 = x \), we have
\[ \langle \cos^2 \theta_1 \rangle = C(g, s) \int_0^1 \frac{x^2 dx}{(1 + sx^2)\sqrt{1 + sx^2 + g}} , \]
where
\[ C(g, s) \int_0^1 \frac{dx}{(1 + sx^2)\sqrt{1 + sx^2 + g}} \equiv 1 . \]
Using the identity
\[ \frac{1}{s} (1 + sx^2) - \frac{1}{s} = x^2 , \]
Equation (B3) reduces to
\[ \langle \cos^2 \theta_1 \rangle = C(g, s) \frac{1}{s} \int_0^1 \frac{dx}{\sqrt{1 + sx^2 + g}} - \frac{1}{s} , \]
which can be solved (Gradshtein & Ryzhik 1965, p. 105) to give
\[ i_1 = \int_0^1 \frac{dx}{\sqrt{(1 + g) + sx^2}} = \begin{cases} \frac{1}{\sqrt{s}} \ln \left( \frac{s + \sqrt{s + 1} + g}{s + 1} \right) , & s > 0 , \\ \frac{1}{\sqrt{-s}} \arcsinh \left( \frac{s}{1 + g} \right) , & s < 0 . \end{cases} \]
For a uniform representation, one can also use inverse hyperbolic sine for \( s < 0 \), giving
\[ i_1 = \frac{1}{\sqrt{s}} \arcsinh \sqrt{\frac{s}{1 + g}} , \]
where
\[ \text{arcsinh } z = \ln(z + \sqrt{z^2 + 1}) = \frac{1}{i} \text{arcsin } iz . \]
To find $C(g, s) = i_2^{-1}$, we calculate

$$i_2 = \int_0^1 \frac{dx}{(1 + sx^2)/(1 + s^2) + g}.$$  \hfill (B10)

By substituting $u = x^2 + s$ into equation (B10) and evaluating the resulting integral (Gradshtein & Ryzhik 1965), we have

$$i_2 = \begin{cases} \frac{1}{2} \ln \frac{\sqrt{1 + s + g + \sqrt{-gs}}}{\sqrt{1 + s + g - \sqrt{-gs}}} & , \quad gs < 0 , \\ \frac{1}{\sqrt{gs}} \arctan \frac{\sqrt{-gs}}{\sqrt{1 + s + g}} & , \quad gs > 0 , \end{cases}$$  \hfill (B11)

Using the inverse hyperbolic tangent

$$\arctanh z = \frac{1}{2} \ln \frac{1 + z}{1 - z} = \arctan iz ,$$  \hfill (B12)

the expression for $i_2$ can be rewritten as

$$i_2 = \frac{1}{\sqrt{-gs}} \arctanh \frac{\sqrt{-gs}}{\sqrt{1 + s + g}} , \quad gs < 0 .$$  \hfill (B13)

Finally, we obtain, for $s < 0$ and $g < 0$,

$$\langle \cos^2 \theta_1 \rangle = \frac{\sqrt{-g} \arcsin \sqrt{-s/(1 + g)}}{s \arctan \sqrt{gs/(1 + s + g)}} - \frac{1}{s} ,$$  \hfill (B14)

for $s < 0$ and $g > 0$,

$$\langle \cos^2 \theta_1 \rangle = \frac{\sqrt{g} \arcsin \sqrt{-s/(1 + g)}}{s \arctanh \sqrt{-gs/(1 + s + g)}} - \frac{1}{s} ,$$  \hfill (B15)

for $s > 0$ and $g < 0$,

$$\langle \cos^2 \theta_1 \rangle = -\frac{\sqrt{g} \arcsinh \sqrt{s/(1 + g)}}{s \arctan \sqrt{sg/(1 + s + g)}} - \frac{1}{s} ,$$  \hfill (B16)

and for $s > 0$ and $g > 0$,

$$\langle \cos^2 \theta_1 \rangle = \frac{\sqrt{g} \arcsinh \sqrt{s/(1 + g)}}{s \arctan \sqrt{sg/(1 + s + g)}} - \frac{1}{s} ,$$  \hfill (B17)

which covers all cases. Similarly, for the second term in equation (B1) we obtain

$$\langle \cos^2 \theta_2 \rangle = \begin{cases} C(g, s) \frac{1}{g} \arcsin \frac{\sqrt{-g}}{\sqrt{1 + s}} & , \quad g < 0 , \\ C(g, s) \frac{1}{g} \arcsinh \frac{\sqrt{g}}{\sqrt{1 + s}} & , \quad g > 0 . \end{cases}$$  \hfill (B18)

The last term in equation (B1) is

$$\langle 3 \cos^2 \theta_1 \cos^2 \theta_2 \rangle = 3C(g, s) \int_0^{\pi/2} \int_0^{\pi/2} \frac{\cos^2 \theta_1 \cos^2 \theta_2 \sin \theta_1 \sin \theta_2 \, d\theta_1 \, d\theta_2}{(1 + s \cos^2 \theta_1 + g \cos^2 \theta_2)^{3/2}} ,$$  \hfill (B19)

which, after obvious substitutions, takes the form

$$\langle 3 \cos^2 \theta_1 \cos^2 \theta_2 \rangle = 3C(g, s) \int_0^1 x^2 \, dx \int_0^1 \frac{y^2 \, dy}{(1 + sx^2 + gy^2)^{3/2}} .$$  \hfill (B20)

Let $s < 0$ and $g < 0$; then

$$s = -b^2 ,$$

$$g = -d^2 .$$  \hfill (B21)
for some nonzero $b$ and $d$, which gives, for the inner integral in equation (B20),

$$G_1 = \int_0^1 \frac{y^2 \, dy}{(1 - b^2 x^2 - d^2 y^2)^{3/2}} = \left[ \frac{y}{d^2 \sqrt{1 - b^2 x^2 - d^2 y^2}} - \frac{1}{d^3} \arcsin \frac{y d}{\sqrt{1 - b^2 x^2}} \right]_0^1.$$ 

Therefore

$$\langle 3 \cos^2 \theta_1 \cos^2 \theta_2 \rangle = \frac{3C(g, s)}{d^2} \left( \int_0^1 \frac{x^2 \, dx}{\sqrt{1 - d^2 - b^2 x^2}} - \frac{1}{d} \int_0^1 x \, \arcsin \frac{d}{\sqrt{1 - b^2 x^2}} \, dx \right). \tag{B23}$$

The second integral in the square brackets can be integrated by parts:

$$G_2 = \frac{1}{d} \int_0^1 x \, \arcsin \frac{d}{\sqrt{1 - b^2 x^2}} \, dx = \frac{1}{3} \left[ \frac{1}{d} \arcsin \frac{d}{\sqrt{1 - b^2 x^2}} - \frac{b^2 d}{d} \int_0^1 x^4 \, dx \right], \tag{B24}$$

where

$$\frac{d}{dx} \left( \arcsin \frac{d}{\sqrt{1 - b^2 x^2}} \right) = \frac{db^2 x}{(1 - b^2 x^2) \sqrt{1 - d^2 - b^2 x^2}} \tag{B25}$$

was taken into account. The last integral in equation (B24) can be evaluated using the identity

$$-b^2 x^4 = -x^2(b^2 x^2 - 1) - x^2. \tag{B26}$$

Hence

$$-\int_0^1 \frac{b^2 x^4 \, dx}{(1 - b^2 x^2) \sqrt{1 - d^2 - b^2 x^2}} = \int_0^1 \frac{x^2 \, dx}{\sqrt{1 - d^2 - b^2 x^2}} \int_0^1 \frac{x^2 \, dx}{(1 - b^2 x^2) \sqrt{1 - d^2 - b^2 x^2}}. \tag{B27}$$

Thus

$$\langle 3 \cos^2 \theta_1 \cos^2 \theta_2 \rangle = \frac{C(g, s)}{d^2} \left[ 2 \int_0^1 \frac{x^2 \, dx}{\sqrt{1 - d^2 - b^2 x^2}} - \frac{1}{d} \, \arcsin \frac{d}{\sqrt{1 - b^2 x^2}} + \int_0^1 \frac{x^2 \, dx}{(1 - b^2 x^2) \sqrt{1 - d^2 - b^2 x^2}} \right], \tag{B28}$$

which gives

$$\langle 3 \cos^2 \theta_1 \cos^2 \theta_2 \rangle = C(g, s) \left[ -\sqrt{1 - d^2 - b^2} + \frac{1}{b^3} \, \arcsin \frac{b}{\sqrt{1 - d^2}} - \frac{1}{d} \, \arcsin \frac{d}{\sqrt{1 - b^2}} - \frac{b^2}{b^3} \, \arcsin \frac{b}{\sqrt{1 - d^2}} + \frac{1}{C(g, s) b^2} \right]. \tag{B29}$$

Finally,

$$\langle 3 \cos^2 \theta_1 \cos^2 \theta_2 \rangle = C(g, s) \left[ -\sqrt{1 - d^2 - b^2} \frac{d^2 b^2}{d^2 b^2} + \frac{1}{b^3} \, \arcsin \frac{b}{\sqrt{1 - d^2}} - \frac{1}{d} \, \arcsin \frac{d}{\sqrt{1 - b^2}} + \frac{1}{C(g, s) b^2} \right], \tag{B30}$$

which is symmetric with respect to the interchange $g \leftrightarrow s$ if both parameters are negative. The symmetry breaks if the parameters have opposite signs. Let

$$s = b^2,$$

$$g = -d^2; \tag{B31}$$

then one must calculate

$$I_3 = C(g, s) \left[ 2 \int_0^1 \frac{x^2 \, dx}{\sqrt{1 - d^2 + b^2 x^2}} + \int_0^1 \frac{x^2 \, dx}{(1 + b^2 x^2) \sqrt{1 - d^2 + b^2 x^2}} - \frac{1}{d} \, \arcsin \frac{d}{\sqrt{1 + b^2}} \right]. \tag{B32}$$

The result is

$$I_3 = C(g, s) \left[ \sqrt{1 - d^2 + b^2} - \frac{1}{b^3} \ln \left| \frac{b + \sqrt{1 - d^2 + b^2}}{\sqrt{1 - d^2}} \right| + \frac{1}{b^3} \ln \left| \frac{b + \sqrt{1 - d^2 + b^2}}{\sqrt{1 - d^2}} \right| - \frac{1}{b^2 C(g, s)} - \frac{1}{d} \, \arcsin \frac{d}{\sqrt{1 + b^2}} \right]. \tag{B33}$$
If \( s \) is negative while \( g \) is positive, the integral is
\[
I_3 = \frac{C(g, s)}{b^2} \times \left[ -\frac{\sqrt{1 + d^2 - b^2}}{d^2} - \frac{1 - b^2}{d^3} \ln \left| \frac{d + \sqrt{1 + d^2 - b^2}}{\sqrt{1 - d^2}} \right| + \frac{1}{d^3} \ln \frac{d + \sqrt{1 + d^2 - b^2}}{\sqrt{1 - d^2}} - \frac{1}{d^2 C(g, s)} - \frac{1}{b} \arcsin \frac{b}{\sqrt{1 + d^2}} \right].
\]

(B34)

For \( s \) and \( g \) both positive, one can perform the integral by changing the inverse trigonometric functions to inverse hyperbolic functions:
\[
I_3 = C(g, s) \left[ -\frac{\sqrt{1 + d^2 + b^2}}{b^2 d^2} + \frac{1}{b^3} \arcsinh \frac{b}{\sqrt{1 + d^2}} + \frac{1}{d^3} \arcsinh \frac{d}{1 + b^2} + \frac{1}{C(g, s) d^2 b^2} \right].
\]

(B35)

Note that this expression is also symmetric with respect to \( s \leftrightarrow g \) interchange. The following formula, valid for all values of \( s \) and \( g \), sums up our results:
\[
\langle \cos^2 \theta \rangle = \frac{1}{2gs} \left[ 1 + g + s + gs - C(g, s) \sqrt{1 + g + s} \right].
\]

(B36)

Note that \( C(g, s) \), given by equation (16), completely defines the solution for \( \langle \cos^2 \theta \rangle \).

APPENDIX C

ANALYTIC SOLUTION FOR PERFECT \( J-a \) ALIGNMENT

Analytical solutions for the alignment measure corresponding to perfect alignment of \( J \) with the major axis of inertia were obtained in Paper I. Here we write down those solutions in a form convenient for comparison with the solutions obtained in the main body of the present paper.

For oblate grains,
\[
\sigma = -\frac{3(1 + g)}{2s} \left( 1 - \sqrt{1 + s + g} \arcsin \sqrt{\frac{s}{1 + g}} \right) + \frac{1}{2}, \quad s < 0,
\]

(C1)

and
\[
\sigma = -\frac{3(1 + g)}{4s^{3/2}} \left[ 2s^{1/2} + (1 + s + g)^{1/2} \times \left\{ \ln (1 + g) - 2 \ln \left[ s^{1/2} + (1 + s + g)^{1/2} \right] \right\} - \frac{1}{2} \right], \quad s > 0.
\]

(C2)

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