PROFINE GROUPS WITH MANY ELEMENTS OF BOUNDED ORDER

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Abstract. Lévai and Pyber proposed the following as a conjecture:
Let $G$ be a profinite group such that the set of solutions of the equation $x^n = 1$
has positive Haar measure. Then $G$ has an open subgroup $H$ and an element $t$ such that all elements of the coset $tH$
have order dividing $n$ (see Problem 14.53 of [The Kourovka Notebook, No. 19, 2019]).
The validity of the conjecture has been proved in [Arch. Math. (Basel) 75
(2000) 1-7] for $n = 2$. Here we confirm the conjecture for $n = 3$.

1. Introduction and Results

Let $G$ be a Hausdorff compact group. Then $G$ has a unique normalized Haar
measure denoted by $m_G$. In general, the question of weather the interior of every non-empty measurable subset of $G$
with positive Haar measure is non-empty has negative answer even if $G$ is profinite (see e.g. [3]). However the same question for
subsets defined by words is still open. In [3] the following conjecture is proposed.

Conjecture 1.1. (Conjecture 3 of [3], Problem 14.53 of [7]) Let $G$ be a profinite
group such that the set $X_n(G)$ of solutions of the equation $x^n = 1$ in $G$ has positive
Haar measure. Then $G$ has an open subgroup $H$ and an element $t$ such that all
elements of the coset $tH$ have order dividing $n$.

The validity of Conjecture 1.1 has been proved in [3] for $n = 2$. In [5] it is shown
that the conjecture is valid for $n = 2$ even if $G$ is Hausdorff compact. It is also
proved in [5] that if $X_3(G)$ has positive Haar measure in a compact group $G$, then $G$
contains an open normal subgroup which is 2-Engel. Here we confirm Conjecture
1.1 for $n = 3$. To do so, we first show that Conjecture 1.1 is equivalent to the
following one. We need the following notation in the statement of the conjecture.
For an arbitrary group $K$ and an automorphism $\phi$ of $K$ of order dividing a positive
integer $n$, define

$$X_{n,\phi}(K) := \{x \in K \mid x x^\phi x^{\phi^2} \cdots x^{\phi^{n-1}} = 1\}.$$

The automorphism group of $K$ will be denoted by $\text{Aut}(K)$.

Conjecture 1.2.

$$\sup\left\{\frac{|X_{n,\phi}(H)|}{|H|} : H \text{ is a finite group and } \phi \in \text{Aut}(H), \phi^n = \text{id}\right\} \setminus \{1\} < 1.$$

It is known that Conjecture 1.2 is valid for $n = 2$ and the supremum is $\frac{3}{4}$ (see
[4]).

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2. Profinite groups

We denote the normalized Haar measure of a compact group $G$ by $m_G$, and we will simply write $m$ if there is no ambiguity.

The following easy lemma will be used in the sequel without referring to it.

**Lemma 2.1.** (cf. [1, Lemma 2.5]) Let $G$ be a compact group and $A \subseteq G$ be a measurable subset. Assume that $m(A) \geq 1 - \epsilon$, then $m \left( \bigcap_{k=1}^n g_k A \right) \geq 1 - n\epsilon$ for all $g_1, \ldots, g_n \in G$. The similar result with strict inequalities holds.

**Proof.** By induction on $n$, we prove the result. For $n = 1$, it holds as the measure is left-invariant. Assume that the result is true for $n$; therefore

$$m \left( \bigcap_{k=1}^{n+1} g_k A \right) = m \left( \bigcap_{k=1}^n g_k A \right) + m(g_{n+1} A) - m \left( \bigcap_{k=1}^n g_k A \cup (g_{n+1} A) \right)$$

$$\geq (1 - n\epsilon) + (1 - \epsilon) - 1, \quad \text{by the induction hypothesis},$$

$$= 1 - (n + 1)\epsilon. \quad \square$$

**Lemma 2.2.** Let $G$ be a compact group and $\phi$ be a continuous automorphism of $G$ of order dividing $n$. Denote by $G \rtimes \langle \phi \rangle$ the semidirect product of $G$ by $\langle \phi \rangle$. Then:

(i) $X_n,\phi(G)$ has nonempty interior if and only if $X_n(G \rtimes \langle \phi \rangle)$ has nonempty interior.

(ii) If $X_n,\phi(G)$ has positive Haar measure then $X_n(G \rtimes \langle \phi \rangle)$ has positive Haar measure.

**Proof.** It follows from the equality $X_n(G \rtimes \langle \phi \rangle) \cap \phi^{-1}(G) = X_n,\phi(G)\phi^{-1}$. \quad \square

**Proposition 2.3.** Conjecture 1.1 implies Conjecture 1.2.

**Proof.** If $n$ is such that the Conjecture 1.2 is not valid, then there exist sequences $(G_k)$ of finite groups and $(\phi_k) \in \prod_{k=1}^\infty \text{Aut}(G_k)$ such that $\phi_k^n = 1$ and

$$0 < \prod_{k=1}^\infty \frac{|X_n,\phi_k(G_k)|}{|G_k|} < 1.$$

Consider the cartesian product $G = \prod_k G_k$ which is clearly profinite. Then $(\phi_k)$ is an automorphism of $G$ of order dividing $n$. It is clear that the measure of $X_n,\phi(G)$ is equal to $\prod_k \frac{|X_n,\phi_k(G_k)|}{|G_k|}$ and its interior is empty, so by Lemma 2.2, $X_n(G \rtimes \langle \phi \rangle)$ has positive Haar measure and empty interior showing that Conjecture 1.2 is not valid. \quad \square

The following lemma will be used in the proof that “Conjecture 1.2 implies Conjecture 1.3”. We write “$N \unlhd G$” whenever $N$ is a normal and open subgroup of $G$.

**Lemma 2.4.** Let $A$ be a closed subset of a profinite group with positive Haar measure. Then

$$\sup \left\{ \frac{m(Ng \cap A)}{m(N)} : g \in G, N \unlhd G \right\} = 1$$
Proof. Let $N$ be an open normal subgroup of index $r$ of $G$. If $s$ is the number of cosets of $N$ which intersect $A$, then
\begin{equation}
(r-s)m(N) \leq 1 - m(A).
\end{equation}
On the other hand, assume that $m(Nx \cap A) = \max \{m(Ng \cap A) : g \in G\}$, so
\begin{equation}
m(A) \leq sm(Nx \cap A)
\end{equation}
It follows from inequalities (2.1) and (2.2) that
\[
\frac{m(A)}{1-m(A)} \frac{r-s}{s} \leq \frac{m(Nx \cap A)}{m(N)}
\]
Since $A$ is closed, $m(A) = \inf \left\{ \frac{|AN/N|}{|G/N|} : N \leq_o G \right\}$. The result now follows from the last inequality. \hfill \square

**Proposition 2.5.** Assume $c_n < 1$. Let $G$ be a profinite group and $g \in G$. Let $M$ be a normal open subgroup of $G$ and $\phi$ be a continuous automorphism of $M$ such that $N^\phi \subseteq N$ for all normal open subgroups $N$ of $G$ contained in $M$. Then $m_M(X_{n,\phi}(M)) \leq c_n$ if $X_{n,\phi}(M) \neq M$.

**Proof.** Seeking a contradiction, let us suppose $m_M(X_{n,\phi}(M)) > c_n$. Let $N$ be a normal open subgroup of $G$. Consider the following well-defined homomorphism of $M/(M \cap N)$,
\[
\bar{\phi} : \frac{M}{M \cap N} \to \frac{M}{M \cap N}, \quad \bar{x}^\phi := x^\phi
\]
We have
\[
m_M(X_{n,\phi}(M)) \leq \frac{|X_{n,\phi}(M/(M \cap N))|}{|M/(M \cap N)|}
\]
the inequality holds because if $x \in X_{n,\phi}(M)$, then $\bar{x} \in X_{n,\phi}(M/(M \cap N))$. Since $c_n < 1$, so $X_{n,\phi}(M/(M \cap N)) = M/(M \cap N)$, whence $\prod_{k=0}^{n-1} x^\phi = N$ for all $x \in M$. Since $\bigcap \{N \leq_o G : N \leq M\} = \{1\}$, $\prod_{k=0}^{n-1} x^\phi = 1$ for all $x \in M$, i.e. $X_{n,\phi}(M) = M$. \hfill \square

We are now in a position to prove the following:

**Proposition 2.6.** Conjecture [1,2] implies Conjecture [1,4]

**Proof.** Let $G$ be a profinite group such that $m_G(X_n(G)) > 0$. By Lemma [2.4] there exist a normal open subgroup $M$ and $g \in X_n(G)$, such that
\begin{equation}
c_n < \frac{m_G(Mg^{-1} \cap X_n(G))}{m_G(M)} = \frac{m_G(M \cap X_n(G))}{m_G(M)}.
\end{equation}
Put $\phi : M \to M$, $x \mapsto g^{-1}xg$. Since $g^n = 1$,
\[
M \cap X_n(G)g = X_{n,\phi}(M)
\]
and the inequality (2.3) means that $m_M(X_{n,\phi}(M)) > c_n$, so by Proposition 2.5 $X_{n,\phi}(M) = M$, whence $Mg^{-1} \subseteq X_n(G)$. \hfill \square
3. Compact groups with splitting automorphisms of order 3

In this section we prove $c_3 < 1$.

**Theorem 3.1.** Let $G$ be a compact group and $\alpha$ be an automorphism of $G$ such that $\alpha^3 = \text{id}$ and the set $X = \{ x \in G \mid xx^\alpha x^{\alpha^2} = 1 \}$ is measurable with $m(X) > \frac{15}{16}$. Then $X = G$.

**Proof.** First we prove that $G$ is 2-Engel. The proof is similar to an argument used in the proof of [5, Theorem 4.4]. We give the proof for the reader’s convenience. For any $a, b \in G$ we must prove that $[a, b, b] = 1$. Consider the set

$$M := X \cap b^{-1}X \cap aX \cap a^{-1}X \cap ba^{-1}X \cap ab^{-1}X \cap abX \cap b^{-1}a^{-1}X.$$ 

Since $m(M) > \frac{1}{2}$, there exists $x \in X$ such that

$$1 = (x\alpha^{-1})^3 = (bx\alpha^{-1})^3 = (ax\alpha^{-1})^3 = (a^{-1}x\alpha^{-1})^3 = (ab^{-1}x\alpha^{-1})^3$$

$$= (ba^{-1}x\alpha^{-1})^3 = (ab\alpha^{-1})^3 = (b^{-1}a^{-1}x\alpha^{-1})^3,$$

where we are writing in the semidirect product $G \rtimes \langle \alpha \rangle$ by noting that $g \in X$ if and only if $(g\alpha^{-1})^3 = 1$ in $G \rtimes \langle \alpha \rangle$. Now [5, Lemma 4.1] implies that $[a, b, b] = 1$.

Let $X^{-1} = \{ x^{-1} \mid x \in X \}$ and $Y = X \cap X^{-1}$. Then $m(Y) > \frac{7}{8}$. Note that for all $x \in Y$,

$$\langle x, x^\alpha, x^{\alpha^2} \rangle = \langle x^\alpha, x^{\alpha^2} \rangle = \langle x, x^{\alpha^2} \rangle$$

are all abelian. (**)

By [6, Theorem 7.15 (iv)],

$$[a, b, c] = [a, c, b]^{-1} \text{ for all } a, b, c \in G.$$ (***)

It follows from (** and (***) that

$$[a, b, c] = 1 \text{ for all } a, b, c \in \{ x, x^\alpha, x^{\alpha^2}, h \}, \text{ for all } h \in G. \text{ (1)}$$

Let $g$ be an arbitrary element of $G$. Consider the set $Z = Y \cap g^{-1}Y$. Since $m(Z) > \frac{3}{4}$, $Z \neq \emptyset$ and for all $x \in Z$ we have

$$xx^\alpha x^{\alpha^2} = gx(gx)^\alpha(gx)^{\alpha^2} = 1. \text{ (2)}$$

Using (1) and (2) and since $G$ is nilpotent of class at most 3 (see e.g. [7, Corollary 3 page 45]), it follows that

$$(gg\alpha)g^{\alpha^2})^{-1} = [x, g\alpha][x, g\alpha, g^{\alpha^2}][g^{\alpha^2}, x^{\alpha^7}] \text{ for all } x \in Z \text{ (3)}$$

Now consider $W = Z \cap x_0^{-1}Z$ for some $x_0 \in Z$. Since $m(W) > \frac{1}{2}$, $W$ is nonempty and for all $y \in W$ we have

$$(gg\alpha)g^{\alpha^2})^{-1} = [x_0y, g\alpha][x_0y, g\alpha, g^{\alpha^2}][g^{\alpha^2}, (x_0y)^{\alpha^2}] = [y, g\alpha][y, g\alpha, g^{\alpha^2}][g^{\alpha^2}, y^{\alpha^2}].$$

Note that since $G$ is nilpotent of class at most 3, commutators of weight 3 are central. It follows from (3) that

$$(gg\alpha)g^{\alpha^2})^{-1} = [x_0, g\alpha, y][g^{\alpha^2}, x_0^\alpha, y^{\alpha^2}] \text{ for all } y \in W. \text{ (4)}$$

Now consider $T = W \cap y_0^{-1}W$ for some $y_0 \in W$. Since $m(T) > 0$, there exists $z \in W$ such that $y_0z \in W$. It follows from (4) that

$$(gg\alpha)g^{\alpha^2})^{-1} = [x_0, g\alpha, y_0z][g^{\alpha^2}, x_0^\alpha, (y_0z)^{\alpha^2}] = [x_0, g\alpha, y_0][g^{\alpha^2}, x_0^\alpha, y_0^{\alpha^2}].$$
Since commutators of weight 3 are central in $G$,
\[ [x_0, g^\alpha, y_0 z][g^{\alpha^2}, x_0^{\alpha^2}, (y_0 z)^{\alpha^2}] = [x_0, g^\alpha, y_0][g^{\alpha^2}, x_0^{\alpha^2}, y_0^{\alpha^2}][x_0, g^\alpha, z][g^{\alpha^2}, x_0^{\alpha^2}, z^{\alpha^2}]. \]
Therefore,
\[ [x_0, g^\alpha, z][g^{\alpha^2}, x_0^{\alpha^2}, z^{\alpha^2}] = 1 \]
and so, as $z \in W$, it follows from (4) that $gg^\alpha g^{\alpha^2} = 1$. This completes the proof. □

**Remark 3.2.** Using the first part of our proof of Theorem 3.1 and applying **[1, Theorem 6.5]** (cf **[2, Théorème 5]**) the number $\frac{15}{16}$ can be reduced to $\frac{7}{8}$.

**Theorem 3.3.** $c_3 < 1$.

*Proof.* It follows from Theorem 3.1 □

We now see that Conjecture 1.1 of Lévi and Pyber is true for $n = 3$.

**Theorem 3.4.** Let $G$ be a profinite group such that the equation $x^3 = 1$ holds on a set with positive Haar measure. Then the solutions set of the equation $x^3 = 1$ has non-empty interior.

*Proof.* It follows from Theorem 3.3 and Proposition 2.6 □

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