IMPROVEMENTS OF PLACHKY-STEINEBACH THEOREM

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Abstract. We show that the conclusion of Plachky-Steinebach theorem holds true for intervals of the form \( [T_r(\lambda), y_1] \), where \( T_r(\lambda) \) is the right derivative (but not necessarily a derivative) of the generalized log-moment generating function \( T \) at some \( \lambda > 0 \) and \( y \in [T_r(\lambda), +\infty] \), under the only two following conditions: (a) \( T_r(\lambda) \) is a limit point of the set \( \{ T_r(t) : t > \lambda \} \), (b) \( T_r(\lambda) \) is a limit with \( t \) belonging to some decreasing sequence converging to \( \sup \{ t > \lambda : T_r(t) \} \). By replacing \( T_r(\lambda) \) by \( T_r(\lambda^+) \), the above result extends verbatim to the case \( \lambda = 0 \) (replacing (a) by the right continuity of \( T \) at zero when \( T_r(0^+) = -\infty \)). No hypothesis is made on \( T_{[-\infty, \lambda]} \) (e.g. \( T_{[-\infty, \lambda]} \) may be the constant \( -\infty \) when \( \lambda = 0 \)); \( \lambda \geq 0 \) may be a non-differentiability point of \( T \) and moreover a limit point of non-differentiability points of \( T \). \( \lambda = 0 \) may be a left and right discontinuity point of \( T \). The map \( T_{[\lambda, \lambda+\varepsilon]} \) may fail to be strictly convex for all \( \varepsilon > 0 \). If we drop the assumption (b), then the same conclusion holds with upper limits in place of limits. Furthermore, the foregoing is valid for general nets \((\mu_\alpha, c_\alpha)\) of Borel probability measures and powers (in place of the sequence \((\mu_\alpha, n^{-1})\)) and replacing the intervals \( [T_r(\lambda^+), y] \) by \( [x_\alpha, y_\alpha] \) or \( [x_\alpha, y_\alpha], \) where \((x_\alpha, y_\alpha)\) is any net such that \((x_\alpha)\) converges to \( T_r(\lambda^+) \) and \( \lim \inf_\alpha y_\alpha > T_r(\lambda^+) \).

1. Introduction and statement of the results

Let \((\mu_\alpha, c_\alpha)\) be a net where \( \mu_\alpha \) is a Borel probability measure on \( \mathbb{R} \), \( c_\alpha > 0 \) and \((c_\alpha)\) converges to zero. Let \( \overline{T} \) be the generalized log-moment generating function associated with \((\mu_\alpha, c_\alpha)\), defined for each \( t \in \mathbb{R} \) by

\[
\overline{T}(t) = \limsup_\alpha c_\alpha \log \int_{-\infty}^\infty e^{c_\alpha t x} \mu_\alpha(dx);
\]

when the above upper limit is a limit we write \( L(t) \) in place of \( \overline{T}(t) \). Let \( \lambda > 0 \) and assume that the derivative map \( L' \) exists and is strictly monotone on a neighbourhood of \( \lambda \) (in particular, \( L(t) \) exists in \( \mathbb{R} \) for all \( t \) in this neighbourhood). When \((\mu_\alpha, c_\alpha)\) is a sequence of the form \((\mu_\alpha, n^{-1})\), Plachky-Steinebach theorem ([10]) asserts that for each sequence \((x_n)\) of real numbers converging to \( L'(\lambda) \) we have

\[
\lim_n n^{-1} \log \mu_n(|x_n|, +\infty) = L(\lambda) - \lambda L'(\lambda).
\]

(Note that \( L(\lambda) - \lambda L'(\lambda) = -T_r(L'(\lambda)) = \inf_{t > 0} \{ T_r(t) - tL'(\lambda) \} \), where \( T_r \) denotes the Legendre-Fenchel transform of \( T \).)

In this paper, we improve the above result in various ways: First, we weaken all the hypotheses by (a) replacing the differentiability of \( T \) on a neighbourhood of \( \lambda \) by the condition that \( T_r(\lambda) \) is a limit point of the set \( \{ T_r(t) : t > \lambda \} \), where \( T_r \) denotes the right derivative map of \( T \), (b) removing entirely the strict convexity hypothesis, (c) allowing \( \lambda = 0 \) (with the only hypothesis of right continuity of \( T \) at zero when \( T_r(0^+) = -\infty \)), (d) requiring the existence of \( L \) only on a suitable sequence \((\mu_\alpha, c_\alpha, x_n)\) in place of the sequence \((\mu_n, n^{-1}, x_n)\); second, we strengthen the conclusion by allowing a more general
class of intervals in the left hand side of (2); third, we generalize all the above by giving a version with upper limits.

For each $\lambda \geq 0$, our result covers all situations except when the condition mentioned in (a) or in (c) fails, i.e. excluding the following ones (cf. Lemma 1):

(i) The map $T_r]|\lambda, +\infty[$ takes only infinite values (equivalently, either $L|]-\infty, +\infty[$ is improper, or $T_r(t) = +\infty$ for all $t > \lambda$).

(ii) There exists $T \in ]\lambda, +\infty]$ such that $T|]\lambda, T[$ is affine, with $T_r'$(+*) $< T_r(T) < +\infty$ when $T < +\infty$.

(iii) $\lambda = 0$, $T_r(0^+)$ $= -\infty$ and $T$ is right discontinuous at zero.

(In particular, in the most common case where $L|]0, +\infty[$ is real-valued and continuous for some $\varepsilon > 0$, only (ii) is excluded.) Equivalently, Theorem 1 applies in (and only in) all the cases where $T_r(\lambda^+) = \lim T_r(\lambda_i)$ for some sequence $(\lambda_i)$ fulfilling eventually

$$T_r(\lambda^+) < T_r(\lambda_i) < +\infty,$$

with $T(0^+)$ $= 0$ when $\lambda = 0$ and $T_r(0^+)$ $= -\infty$ (note that the above condition is satisfied under the hypotheses of Plachky-Steinebach theorem); then, there is a unique real number $\lambda \geq \lambda$ fulfilling $T_r(\lambda^+) = T_r(\lambda^+)$ and $T_r(t) > T_r(\lambda^+)$ for all $t > \lambda$; in fact, $\lambda = \sup\{t > \lambda : T_r|]\lambda, t[$ is affine$\} = \lim T_r(\lambda_i)$ for every sequence as above (cf. Lemma 1).

Let us emphasize that no hypothesis is made on the map $T|]-\infty, \lambda[$. The proof deals only with the properties of the right derivative map $T_r]|\lambda, \lambda +\varepsilon[$: in particular, the following special situations may arise:

- $\lambda$ may be a non-differentiability point of $T$, and moreover a limit point of non-differentiability points of $T$ (e.g. when $\lambda = \lambda$ and eventually $\lambda_i$ is a non-differentiability point of $T_i$).
- $T$ may be left discontinuous at $\lambda$ when $\lambda = 0$ (which is the case when $T_r(0) = -\infty$) and moreover right discontinuous at 0 (when $T_r(0) = -\infty < T_r(0^+)$); cf. Remark 1
- $T|]0, \lambda +\varepsilon[$ may not be strictly convex for all $\varepsilon > 0$ (e.g. when $\lambda = \lambda$ and eventually $T_i|]0, \lambda_i[$ is affine); when $\lambda < \lambda$, $T$ is not strictly convex in any neighbourhood of $\lambda$ or $\lambda$.
- As regards the strong version with limits, for each $\lambda < \lambda$, $T_r(t)$ may not be a limit.

The foregoing contrasts sharply with the hypotheses of Plachky-Steinebach theorem, which require $\lambda > 0$ and some $\varepsilon > 0$ such that $L(t)$ exists in $\mathbb{R}$ for all $t \in [\lambda - \varepsilon, \lambda + \varepsilon]$ and $L|]0, \lambda - \varepsilon, \lambda + \varepsilon[$ is strictly convex and differentiable.

Theorem 1 below constitutes the first general result giving the upper limit (and a fortiori, the limit) of $(c_n \log \mu_\alpha([x_n, y_n]))$ or $(c_n \log \mu_\alpha([x_n, y_n]))$ in terms of $T$, when $\lim_{x_n} x_n$ is not a value of the derivative map of the generalized log-moment generating function (cf. Remark 2).

**Theorem 1.** Let $\lambda \geq 0$. When $T_r(\lambda^+) > -\infty$ we assume that $T_r(\lambda^+)$ is a limit point of the set $\{T_r(t) : t > \lambda\}$; when $T_r(\lambda^+) = -\infty$ we assume that $T_r(\lambda) = T_r(\lambda^+)$. For each net $(x_n, y_n)$ in $[-\infty, +\infty]^2$ such that $(x_n)$ converges to $T_r(\lambda^+)$ and $\liminf y_n > T_r(\lambda^+)$ we
have
\[
\limsup_{\alpha} c_\alpha \log \mu_\alpha ([x_\alpha, y_\alpha]) = \limsup_{\alpha} c_\alpha \log \mu_\alpha ([x_\alpha, y_\alpha]) = L_\lambda^+(\lambda^+) - \lambda L'_\lambda(\lambda^+)
\]
\[
= \begin{cases} 
-\inf_{t > 0} \{ L_\lambda(t) - t L'_\lambda(\lambda^+) \} & \text{if } L'_\lambda(\lambda^+) \neq -\infty, \\
-\lim_\alpha L'_\lambda(x_\alpha) & \text{if } L'_\lambda(\lambda^+) = -\infty.
\end{cases}
\]

Furthermore, if \( L(t_i) \) exists for a sequence \( (t_i) \) in \( ]\lambda, +\infty[ \) converging to \( \lambda \), then the above upper limits are limits (with \( \lambda = \sup \{ t > \lambda : L_\lambda(t) \) is affine\}).

It is possible that for some \( \varepsilon > 0 \), \( \lambda \) is the only point in \( [0, \lambda + \varepsilon[ \) where Theorem 1 applies (e.g. when \( \lambda = \lambda \), \( L \) is not differentiable at \( \lambda \), and eventually \( \lambda_i \) appearing in (3) is a non-differentiability point of \( L \) and \( L_{\lambda_i, 1, \lambda_i} \) is affine; although this may seem to be an extreme case, we show in Appendix A that there are plenty of such examples); in the light of the above, Theorem 1 may be thought of as a pointwise version of Plachky-Steinebach theorem.

Hereafter, we focus on the opposite situation, motivated by the following observation: If \( L \) exists as a differentiable and strictly convex map on some open interval containing \( \lambda > 0 \) and zero (say \( [-\varepsilon, \lambda + \varepsilon[ \)), then the hypotheses of Plachky-Steinebach theorem hold for all \( t \in [0, \lambda + \varepsilon[ \) (in place of \( \lambda \)), and thus it applies to every sequence converging to \( L(t) \) for all \( t \in ]0, \lambda + \varepsilon[ \); since \( L_{]\lambda, 0, \lambda[} \) is this case an increasing homeomorphism, the limit obtained, as a function of \( L(t) \), is exactly \( -L^*_{]\lambda, L(0), L'(\lambda + \varepsilon)[} \) (where \( L^* \) denotes the left derivative map of \( L \)); a weak version of this (with the constant sequence \( (x_n) \equiv L(t) \)) has been widely used in the context of dynamical systems (e.g. [8], Lemma 13.2; [2], Lemma 6.2; [3], Corollary 4.3; [4], Theorem 2.2; [1], Proposition 7).

A much stronger result can be derived from Theorem 1 as shows the following Corollary 1: aside the extension with upper limits and the possibility to consider nets in place of sequences, the improvements are obtained first, by allowing a more general class of intervals in the conclusion, and second, by weakening the hypotheses as follows:

- The differentiability of \( L_{]\lambda, -\varepsilon, \lambda + \varepsilon[} \) is replaced by the differentiability of \( L_{]\lambda, \lambda + \varepsilon[} \).
- The strict convexity of \( L_{]-\varepsilon, \lambda + \varepsilon[} \) is replaced by the condition that \( L_{]\lambda, \lambda + \varepsilon[} \) does not attain its supremum (equivalently, for each \( t \in ]\lambda, \lambda + \varepsilon[ \) the map \( L_{]t, \lambda + \varepsilon[} \) is not affine).

(Note first, that this condition is obviously fulfilled when \( L_{]t, \lambda + \varepsilon[} \) is strictly convex for some \( t \in ]\lambda, \lambda + \varepsilon[ \), and second, it is far weaker than strict convexity: There may be an infinite countable set \( \{ S_i : i \in \mathbb{N} \} \) of non-empty mutually disjoint intervals included in \( ]\lambda, \lambda + \varepsilon[ \) on which \( L \) is affine; when sup\( i \) sup\( S_i = \lambda + \varepsilon \), the above condition implies sup\( i S_i < \lambda + \varepsilon \) for all \( i \in \mathbb{N} \).)

- The possibility to take \( \lambda = 0 \), including the cases \( L'(0^+) = -\infty \) and \( L'(0) = -\infty < L'(0^+) \).
- The version with limits only requires the existence of \( L \) on a sequence \( (t_i) \) in \( ]\lambda, +\infty[ \) converging to \( \lambda \), for all \( t \in ]\lambda, \lambda + \varepsilon[ \) contains all the open intervals on which \( L \) is affine.

**Corollary 1.** Let \( \lambda \geq 0 \) and let \( \varepsilon > 0 \). We assume that \( L_{]\lambda, \lambda + \varepsilon[} \) is differentiable and sup \( \{ L'(t) : t \in ]\lambda, \lambda + \varepsilon[ \} \) is not a maximum; when \( \lambda = 0 \) and \( L'(0^+) = -\infty \), we further assume that \( L' \) is right continuous at zero.

a) For each \( z \in ]L'(\lambda^+), L'_{\lambda + \varepsilon}[ \), there exists \( t_z \in ]\lambda, \lambda + \varepsilon[ \) fulfilling \( L'(t_z^+) = z \), and for each such \( t_z \), for each net \( (x_{\alpha, z}) \) in \([\lambda, +\infty[ \) converging to \( z \), and for each net
\[(y_{α, z}) \text{ in } [−∞, +∞] \text{ fulfilling } \lim \inf_{α} y_{α, z} > z, \text{ we have }\]
\[\limsup_{α} c_α \log \mu_α(x_{α, z}, y_{α, z}) = \limsup_{α} c_α \log \mu_α(x_{α, z}, y_{α, z}) = 1 \quad \text{if } z \neq -∞\]
\[- \lim_{α} \mathcal{T}_r(x_{α, z}) = 0 \quad \text{if } z = -∞ \quad (i.e., \ λ = 0 \text{ and } z = \mathcal{T}_r(0^+) = -∞).\]

Furthermore, if \(\mathcal{L}(t_{z, i})\) exists for a sequence \((t_{z, i})\) in \(]t_z, +∞[\) converging to \(t_z\), then the above upper limits are limits \((\text{with } t_z = \sup\{t > t_z : \mathcal{L}[t_{z, i}] \text{ is affine}\})\).

b) The map \(\mathcal{T}_r(\lambda^+)\), \(\mathcal{T}_r(\lambda + ε)\) \(\exists z \rightarrow \mathcal{T}_r(t_z^+) - t_z z\) is \(-∞, 0]\)-valued, strictly decreasing, continuous, unbounded below if and only if \(\mathcal{T}_r(\lambda + ε) = +∞\), and vanishes at \(\mathcal{T}_r(\lambda^+)\) if and only if either \(\lambda > 0\) and \(\mathcal{T}_r\) is differentiable at \(\lambda\) and linear on \([0, \lambda]\), or \(\lambda = 0\) and \(\mathcal{T}\) is right continuous at zero; its restriction to \(\mathcal{T}_r(\lambda^+), \mathcal{T}_r(\lambda + ε)\) is strictly concave, and it is furthermore differentiable when \(\mathcal{T}_r[\tilde{λ}, \lambda + ε]\) is strictly convex \((\text{with } \tilde{λ} = \sup\{t > \lambda : \mathcal{T}_r[t_{\tilde{λ}, λ}] \text{ is affine}\})\).

Remark 1. We have \(\mathcal{T}_r(\lambda^+) = -∞\) only if \(\lambda = 0\), in which case the hypothesis of right continuity of \(\mathcal{T}\) at zero in Theorem 1 and Corollary 1 cannot be removed; indeed, suppose that \(\mathcal{T}_r(0^+) \neq 0\), and let \(c \in ]0, 1[\). Since for each index \(α\) the measure \(μ_α\) is tight, there exists a net \((x_{α})\) of real numbers converging to \(-∞\) such that \(μ_α([x_α, -x_α]) > c\), hence
\[\lim_{α} c_α \log \mu_α(x_{α}, +∞[) = 0 \neq \mathcal{L}(0^+),\]
and the conclusion does not hold with \((x_{α})\). The hypothesis for the case \(\mathcal{T}_r(\lambda^+) > -∞\) also holds when \(\mathcal{T}_r(\lambda^+) = -∞\) (cf. Lemma 1).

Remark 2. The upper limits in Theorem 1 vanish if and only if one of the following cases occurs:

- \(\lambda > 0\), \(\mathcal{T}\) is differentiable at \(\lambda\) and linear on \([0, \lambda]\);
- \(\lambda = 0\) and \(\mathcal{T}\) is right continuous at zero.

Remark 3. Under the hypotheses of Theorem 1, the equality \(\mathcal{T}_r(\mathcal{T}_r(\lambda^+)) = \lim_{α} \mathcal{T}_r(x_α)\) fails when eventually \(x_α\) does not belong to the effective domain of \(\mathcal{T}_r\), which happens if and only if \(\mathcal{L}[0, +∞[\) is linear with slope \(\mathcal{T}_r(\lambda^+)\) (equivalently, \(\mathcal{T}_r(\lambda^+)\) is the left end-point of the effective domain of \(\mathcal{T}_r\)) and eventually \(x_α < \mathcal{T}_r(\lambda^+)\).

Remark 4. Regarding the proof as well as the wording of Theorem 1, the main difference between the cases \(\lambda > 0\) and \(\lambda = 0\) stems from the fact that when the map \(\mathcal{L}[0, +∞[\) is proper (which is implied by the hypotheses, cf. Lemma 1), 0 is the only nonnegative real number \(\lambda\) in the effective domain of \(\mathcal{L}[0, +∞[\) for which:

- \(\mathcal{T}_r(\lambda)\) or \(\mathcal{T}_r(\lambda^+)\) may take the value \(-∞\);
- \(\mathcal{T}_r(\lambda)\) may differ from \(\mathcal{T}_r(\lambda^+)\);
- \(\mathcal{T}_r(\lambda^+) < +∞\) and \(\mathcal{T}\) may be right discontinuous at \(\lambda\).

(cf. Lemma 2 and Lemma 3 for instance, the conclusion in the last assertion of Lemma 2 is not true for \(\lambda = 0\)). Theorem 1 and Corollary 1 include the case \(\lambda = 0\) and \(\mathcal{T}_r(0) = -∞ < \mathcal{T}_r(0^+)\), which implies the left and right discontinuities of \(\mathcal{T}\) at zero (cf. Lemma 3): the case \(\lambda = 0\) and \(\mathcal{T}_r(0) = -∞\) implies the left discontinuity of \(\mathcal{T}\) at zero (in both cases, we have \(\mathcal{L}[−∞, 0[ \equiv +∞\)).

Remark 5. The standard version of Gärtnner-Ellis theorem is unworkable when \(\mathcal{T}\) is not differentiable at \(\lambda\); Indeed, the main hypothesis (i.e. essentially smoothness) implies the differentiability of \(\mathcal{T}\) on the interior of its effective domain (beside requiring the existence
of $L$ on an open interval containing 0); the same applies to the variant of Gärtner-Ellis theorem given by Theorem 5.1 of [3] (which allows $L$ to exist only on some open interval not necessarily containing 0) since it also requires the essential smoothness. Corollary 1 of [3] strengthens both above versions, but, although weaker than essential smoothness, the general hypothesis is still a global condition requiring the existence of $L$ on some open interval and relating the range of the one-sided derivatives of $\mathcal{T}$ with the effective domain of $\mathcal{T}'$, and thus it is of no use here.

2. Proofs

Recall that $\mathcal{T}$ is a $[\infty, +\infty]$-valued convex function ([7], [3]), and such a function is said to be proper when it is $]-\infty, +\infty]$-valued and takes at least one finite value; note that $\mathcal{T}(0) = 0$. We denote by $\mathcal{L}$ and $\mathcal{L}_r$ (resp. $\mathcal{L}_l$) respectively the Legendre-Fenchel transform and right (resp. left) derivative map of $\mathcal{T}$, where for each $t \geq 0$ we put $\mathcal{L}_r(t) = +\infty$ (resp. $\mathcal{L}_l(t) = -\infty$) when $\mathcal{T}(s) = +\infty$ for all $s > t$ (resp. $\mathcal{T}(s) = -\infty$ for some $s > t$); note that $\mathcal{L}$ is $[0, +\infty]$-valued since $\mathcal{T}(0) = 0$. The basic properties of the map $\mathcal{L}_{r|[0, +\infty]}$ are summarized in Lemmas [1], Lemma [2] and Lemma [3] in particular, $\mathcal{L}_{r|[0, +\infty]}$ is non-decreasing so that the quantity $\mathcal{L}_r(\lambda^+) = \lim_{t\to\lambda^+, t>\lambda} \mathcal{L}_r(t)$ is well-defined for all $\lambda \geq 0$.

Let $\lambda \geq 0$. All the lemmas below hold for any net $(\mu_\alpha, c_\alpha)$ as in [1]; the hypotheses of Theorem [1] are made only in the last part of the proof.

**Lemma 1.** 

a) The map $\mathcal{L}_{r|[0, +\infty]}$ is improper if and only if there exists $T \in [0, +\infty)$ such that $\mathcal{L}_{r|[0,T]} = \mathcal{L}_{r|[0,T]} = -\infty$ and $\mathcal{L}_{r|[T, +\infty]} = \mathcal{L}_{r|[T, +\infty]} = +\infty$.

b) If $\mathcal{L}_{r|[0, +\infty]}$ is proper, then one and only one of the following cases holds:

(i) There exists a sequence $(\mathcal{L}_r(\lambda_i))$ in $[\mathcal{L}_r(\lambda^+), +\infty]$ converging to $\mathcal{L}_r(\lambda^+)$. 

(ii) There exists $\varepsilon > 0$ such that $\mathcal{L}_{r|[\lambda, \lambda+\varepsilon]}$ is affine and $\mathcal{L}_r(\lambda^+) < \mathcal{L}_r(\lambda + \varepsilon) < +\infty$. 

(iii) $\mathcal{L}_{r|[\lambda, +\infty]}$ is affine. 

(iv) $\mathcal{L}(\lambda) \in \mathbb{R}$ and $\mathcal{L}(t) = +\infty$ for all $t > \lambda$; 

(v) $\mathcal{L}(t) = +\infty$ for all $t \geq \lambda$.

In particular, (i) or (ii) or (iii) holds if and only if there exists $\varepsilon > 0$ such that $\mathcal{L}_{r|[\lambda, \lambda+\varepsilon]}$ is real-valued, bounded, and $\mathcal{L}_{r|[\lambda, +\infty]}$ (resp. $\mathcal{L}_{r|[\lambda, \lambda+\varepsilon]}$ when $\lambda > 0$) continuous. Furthermore, (i) holds if and only if there exists $\lambda \in [\lambda, +\infty]$ fulfilling $\mathcal{L}_r(\lambda^+) = \mathcal{L}_r(\lambda^+)$ and $\mathcal{L}_r(t) > \mathcal{L}_r(\lambda^+)$ for all $t > \lambda$; such a $\lambda$ is unique and given by

$$\lambda = \sup\{t > \lambda : \mathcal{L}_{r|[\lambda,t]} \text{ is affine}\} = \lim_{i} \lambda_i$$

for every sequence $(\lambda_i)$ as in (i).

**Proof.** a) It is a direct consequence of the definitions together with the convexity of $\mathcal{T}$ and the fact that $\mathcal{T}(0) = 0$.

b) Assume that $\mathcal{L}_{r|[0, +\infty]}$ is proper. If $\mathcal{L}(\lambda) = +\infty$, then $\mathcal{L}(t) = +\infty$ for all $t \geq \lambda$ (because $\mathcal{T}$ is convex and $\mathcal{T}(0) = 0$) and (v) holds. Assume now that $\mathcal{T}(\lambda) < +\infty$. Then $\mathcal{L}(\lambda) \in \mathbb{R}$ because $\mathcal{L}_{r|[0, +\infty]}$ is proper. Since $\mathcal{T}$ is convex and $\mathcal{T}(0) = 0$, we have either $\mathcal{L}(t) = +\infty$ for all $t > \lambda$ and (iv) holds, or $\mathcal{L}_{r|[\lambda, +\infty]}$ is real-valued for some $\delta > 0$, in which case $\mathcal{L}_r(t) \in \mathbb{R}$ for all $t \in [\lambda, \lambda+\delta]$. Assume that (i) does not hold, i.e. $\mathcal{L}_r(\lambda^+) < \inf\{\mathcal{L}_r(t) \in [\mathcal{L}_r(\lambda^+), +\infty] : t > \lambda\}$. If $\mathcal{L}_r(t) \in [\mathcal{L}_r(\lambda^+), +\infty] : t > \lambda = \emptyset$, then $\mathcal{L}_r(\lambda^+) = \mathcal{L}_r(\lambda^+)$ for all $t > \lambda$, and (iii) holds. If $\mathcal{L}_r(t) \in [\mathcal{L}_r(\lambda^+), +\infty] : t > \lambda \neq \emptyset$, then there exists $\varepsilon \in [0, \delta]$ such that $\mathcal{L}_r(\lambda^+) < \mathcal{L}_r(\lambda + \varepsilon)$ and $\mathcal{L}_r(\lambda^+) = \mathcal{L}_r(t)$ for all $t \in [\lambda, \lambda+\varepsilon]$, so that (ii) holds. The first assertion is proved.
Assume that either (i), (ii), or (iii) holds. There exists $\varepsilon > 0$ such that $T_{r,\lambda,\lambda+\varepsilon}$ is real-valued, hence $T_{r,\lambda,\lambda+\varepsilon}$ is real-valued. The map $T_{r,\lambda,\lambda+\varepsilon/2}$ (resp. $T_{r,\lambda,\lambda+\varepsilon/2}$ when $\lambda > 0$) is continuous since $[\lambda, \lambda + \varepsilon/2]$ (resp. $[\lambda, \lambda + \varepsilon/2]$ when $\lambda > 0$) belongs to the interior of the effective domain of $T_{r,\lambda,\lambda+\varepsilon/2}$. The boundness of $T_{\lambda,\lambda+\varepsilon/2}$ follows from the continuity when $\lambda > 0$, and the boundness of $T_{\lambda,\lambda+\varepsilon/2}$ when $\lambda = 0$ follows from the convexity and the fact that $\tilde{T}(0) = 0$. Conversely, if $T_{r,\lambda,\lambda+\varepsilon}$ is real-valued for some $\varepsilon > 0$, then the first assertion implies that either (i), (ii), or (iii) holds. The second assertion is proved.

Assume that there exists $s \in [\lambda, +\infty]$ fulfilling $T_r(\lambda^+) = \tilde{T}_r(s^+)$ and $\tilde{T}_r(t) > T_r(\lambda^+)$ for all $t > s$. We have $\tilde{T}_r(\lambda^+) = \lim_{t \to s, t > s} \tilde{T}_r(t)$ hence (i) holds.

Assume that (i) holds. Put $\tilde{\lambda} = \sup\{t > \lambda : T_{r,\lambda,t} \text{ is affine}\}$. If $\tilde{\lambda} = +\infty$, then $\tilde{T}_r(t) = T_r(\lambda^+)$ for all $t > \lambda$, which contradicts (i), hence $\tilde{\lambda} \in [\lambda, +\infty]$. Let $t > \tilde{\lambda}$. Since $T_{r,\lambda,t}$ is not affine, we have $\tilde{T}_r(t) > T_r(\lambda^+)$. Suppose that $\tilde{T}_r(\lambda^+) < T_r(\lambda^+)$; by (i) there exists $\tilde{T}_r(\lambda_i)$ such that

$$\tilde{T}_r(\lambda^+) < \tilde{T}_r(\lambda_i) < \tilde{T}_r(\tilde{\lambda} + \varepsilon),$$

which implies $\lambda < \lambda_i < \tilde{\lambda}$ and contradicts the fact that $T_{r,\lambda,\tilde{\lambda}}$ is affine; therefore, $\tilde{T}_r(\lambda^+) = T_r(\lambda^+)$. Let $(\lambda_i)$ be a sequence as in (i). For each $\varepsilon > 0$ we have eventually

$$\tilde{T}_r(\lambda^+) = \tilde{T}_r(\tilde{\lambda}) < \tilde{T}_r(\lambda_i) < \tilde{T}_r(\tilde{\lambda} + \varepsilon)$$

hence eventually $\tilde{\lambda} < \lambda_i < \lambda + \varepsilon$, which implies $\tilde{\lambda} = \lim_{i \to \infty} \lambda_i$.

The equalities $T_r(\lambda^+) = \tilde{T}_r(\lambda^+) = \tilde{T}_r(s^+)$ implies $T_{r,\lambda,s}$ affine, hence $s \leq \tilde{\lambda}$ by definition of $\tilde{\lambda}$. Since $\tilde{T}_r(t) > \tilde{T}_r(\lambda^+) = \tilde{T}_r(\lambda^+)$ for all $t > s$, we have $\tilde{T}_r(s^+) > \tilde{T}_r(\alpha^+)$ hence $s > \tilde{\lambda}$; therefore, $s = \tilde{\lambda}$. The proof of the third assertion is complete. \qed

**Lemma 2.** The following statements are equivalent:

(i) $T_{r,\lambda,\lambda+\varepsilon}$ is proper;
(ii) $T_{r,\lambda,\lambda+\varepsilon}$ is $-\infty, +\infty$-valued, non-decreasing and right continuous.

When the above holds and $\lambda > 0$, we have $\tilde{T}_r(\lambda) \in \mathbb{R}$ if and only if $\tilde{T}(t) < +\infty$ for some $t > \lambda$.

**Proof.** (ii) $\Rightarrow$ (i) follows from the definitions since $\tilde{T}_r$ takes the value $-\infty$ when $T_{r,\lambda,\lambda+\varepsilon}$ is improper (cf. Lemma 1). Assume that (i) holds. Since $\tilde{T}$ is convex and $\tilde{T}(0) = 0$, the map $T_{r,\lambda,\lambda+\varepsilon}$ is $-\infty, +\infty$-valued. Let $\alpha > 0$. If $T_r(\lambda) = +\infty$, then $\lambda$ fulfills one of the last two cases of Lemma 1 so that $\tilde{T}_r$ takes only the value $+\infty$ on $[\lambda, +\infty]$ hence is non-decreasing and right continuous on $[\lambda, +\infty]$. Assume that $\tilde{T}_r(\lambda) < +\infty$. Then $\lambda$ fulfills one of the first three cases of Lemma 1 in particular, there exists $\varepsilon > 0$ such that $T_{r,\lambda,\lambda+\varepsilon}$ is real-valued and continuous. Extend $T_{r,\lambda,\lambda+\varepsilon}$ to a lower semi-continuous function $f$ on $\mathbb{R}$ by putting $f(x) = +\infty$ for all $x \in \mathbb{R} \setminus [\lambda, \lambda + \varepsilon]$. The right derivative map of $f$ is non-decreasing and right continuous on $\mathbb{R}$ by Theorem 24.1 of [11] so that $\tilde{T}_r$ is non-decreasing on $[\lambda, \lambda + \varepsilon]$ and right continuous on $[\lambda, \lambda + \varepsilon]$; therefore, (ii) holds and the first assertion is proved. If $\tilde{T}(t) < +\infty$ for some $t > \lambda$, then $\tilde{T}_r(\lambda) < +\infty$ (by definition) hence $\tilde{T}_r(\lambda) \in \mathbb{R}$ since $\lambda > 0$ and $\tilde{T}_{r,\lambda,\lambda+\varepsilon}$ is proper; the converse is obvious. \qed

**Lemma 3.** The following statements are equivalent:

(i) $\tilde{T}_r$ is not right continuous at 0;
(ii) $T_{r,\lambda,\lambda+\varepsilon}$ is not lower semi-continuous at 0, $\tilde{T}_r(0) = -\infty$ and $\tilde{T}_r(0^+) > -\infty$.
(iii) $T_{r,\lambda,\lambda+\varepsilon}$ is not lower semi-continuous at 0 and $\tilde{T}_r(0^+) \in \mathbb{R}$.
(iv) $\tilde{T}_r(0) = -\infty$ and $\tilde{T}_r(0^+) > -\infty$. 


The above equivalence hold verbatim replacing $\bar{L}(0^+) = -\infty$ by $\bar{L}(0^+) \in \mathbb{R}$. In particular, $\bar{L}(0) \in \mathbb{R}$ if and only if $\bar{L}(0) = \bar{L}(0^+) \in \mathbb{R}$.

**Proof.** Assume that (i) holds. By Lemma [1] the map $\bar{L}|_{[0, +\infty[}$ is proper, and by Lemma [1] there exists $\varepsilon > 0$ such that $\bar{L}|_{[0, \varepsilon[}$ is real-valued and $\bar{L}|_{[0, \varepsilon[}$ is continuous. If $\bar{L}|_{[0, +\infty[}$ is lower semi-continuous at $0$, then the extension of $\bar{L}|_{[0, \varepsilon[}$ to a lower semi-continuous function on $\mathbb{R}$ yields the right continuity of $\bar{L}$ at $0$ (by Theorem 24.1 of [11]), which gives a contradiction; therefore, $\bar{L}|_{[0, +\infty[}$ is not lower semi-continuous at $0$; since $\bar{L}$ is convex and $\bar{L}(0) = 0$ it follows that $\bar{L}(0^+) \in \mathbb{R}$ exists as a negative number, and consequently, $\bar{L}(0^+) = -\infty$. Since $\bar{L}(t), t \in \mathbb{R}$ for all $t \in [0, \varepsilon[$, and $\bar{L}(t), t \in [0, +\infty[)$ is non-decreasing by Lemma [2] $\bar{L}(0^+) \in \mathbb{R}$ exists in $[-\infty, +\infty[$, hence $\bar{L}(0^+) \in \mathbb{R}$ (since $\bar{L}$ is not right continuous at $0$ by hypothesis) and (ii) holds.

Assume that (ii) holds. Then, $\bar{L}(0^+) \in \mathbb{R}$ (otherwise $\bar{L}(0^+) = +\infty$ would imply $\bar{L}(0) = +\infty$ and a contradiction) hence (iii) holds.

Assume that (iii) holds. By Lemma [1] there exists $\varepsilon > 0$ such that $\bar{L}|_{[0, \varepsilon[}$ is real-valued and $\bar{L}|_{[0, \varepsilon[}$ is continuous, hence (since $\bar{L}$ is convex and $\bar{L}(0) = 0$), $\bar{L}(0^+) \in \mathbb{R}$ exists as a negative number, which implies $\bar{L}(0) = -\infty$, and gives (iv). Since the implication (iv) $\Rightarrow$ (i) is obvious, the proof of the first two assertions is complete; the last assertion is a direct consequence of them. $\square$

Let $l_0$ be the function defined on $\mathbb{R}$ by

$$
\forall x \in \mathbb{R}, \quad l_0(x) = -\lim_{\varepsilon \to 0, \varepsilon > 0} \limsup_{\alpha} c_{\alpha} \log \mu_{\alpha}([x - \varepsilon, x + \varepsilon[);
$$

note that $l_0 \in [0, +\infty[\text{-valued and lower semi-continuous.}

**Lemma 4.** We have

$$
\forall x \in \mathbb{R}, \quad l_0(x) \geq \bar{L}'(x).
$$

**Proof.** Since for each real number $\lambda \neq 0$ the set $\{x \in \mathbb{R} : a \leq e^{\lambda x} \leq b\}$ is compact for all $(a, b) \in \mathbb{R}^2$ with $a \leq b$, Theorem 1 of [11] yields

$$
\forall \lambda \in \mathbb{R} \setminus \{0\}, \quad \bar{L}(\lambda) \geq \sup_{y \in \mathbb{R}} \{\lambda y - l_0(y)\}.
$$

Since $l_0$ is a $[0, +\infty[\text{-valued function and } \bar{L}(0) = 0$, the above inequality is true with $\lambda = 0$ so that

$$
\forall (\lambda, x) \in \mathbb{R}^2, \quad \bar{L}(\lambda) - \lambda x \geq \sup_{y \in \mathbb{R}} \{\lambda y - l_0(y)\} - \lambda x \geq -l_0(x),
$$

hence

$$
\forall x \in \mathbb{R}, \quad l_0(x) \geq \sup_{\lambda \in \mathbb{R}} \{\lambda x - \bar{L}(\lambda)\} = \bar{L}'(x).
$$

$\square$

**Lemma 5.** Assume that $\bar{L}|_{[0, +\infty[}$ is proper and $\bar{L}(t) < +\infty$ for some $t > \lambda$. There exists $\varepsilon > 0$ such that for each $t \in [\lambda, \lambda + \varepsilon[\text{ we have}

$$
\bar{L}(t) = \sup_{x \in \mathbb{R}} \{tx - l_0(x)\} = tx_t - l_0(x_t)
$$

for some $x_t \in \mathbb{R}$.

**Proof.** By Lemma [1] there exists $\varepsilon > 0$ such that $\bar{L}|_{[\lambda, \lambda + \varepsilon]}$ is real-valued, hence

$$
\forall t \in [\lambda, \lambda + \varepsilon[\text{, } \lim_{M \to +\infty} \limsup_{\alpha} c_{\alpha} \log \int_{[M/t, +\infty[} e^{\varepsilon t z} \mu_{\alpha}(dz) = -\infty
$$

(Lemma 4.3.8 of [11]). Part (b) of Theorem 1 of [11] yields for each $t \in [\lambda, \lambda + \varepsilon[\text{ and for each } M$ large enough

$$
\bar{L}(t) = \sup_{x \leq M/t} \{tx - l_0(x)\}. \quad (4)
$$
Let $t \in ]\lambda, \lambda + \varepsilon[.$ For each integer $n \geq 1$ there exists $x_n \leq M/t$ such that

$$\mathcal{T}(t) \geq tx_n - l_0(x_n) > \mathcal{T}(t) - 1/n \geq \mathcal{T}(t) - 1$$

hence $x_n \in [\mathcal{T}(t)/t - 1/t, M/t].$ Therefore, the sequence $(x_n)$ has a subsequence $(x_{n_m})$ converging to some $x \in [\mathcal{T}(t)/t - 1/t, M/t],$ so that letting $n \to +\infty$ in (5) yields

$$\mathcal{T}(t) = \lim_m (tx_{n_m} - l_0(x_{n_m})) = tx - \lim_m l_0(x_{n_m}) \leq tx - l_0(x),$$

where the last inequality follows from the lower semi-continuity of $l_0.$ From the above expression and (4), we get $\mathcal{T}(t) = tx - l_0(x),$ which proves the lemma. \hfill \Box

**Lemma 6.** Assume that $\mathcal{T}_{\varepsilon}$ is proper and $\mathcal{T}(t) < +\infty$ for some $t > \lambda.$ There exists $\varepsilon > 0$ such that

$$\forall t \in ]\lambda, \lambda + \varepsilon[,$$

$$l_0(\mathcal{T}_r(t)) = \mathcal{T}_r(t).$$

When $\mathcal{T}_r(\lambda^+) \neq -\infty,$ the above equality is true with $t = \lambda,$ and $\mathcal{T}_r(\lambda^+)$ in place of $\mathcal{T}_r(t).$

**Proof.** By Lemma [3] there exists $\varepsilon_0 > 0$ such that $\mathcal{T}_{|\lambda, \lambda + \varepsilon_0]}$ is real-valued bounded and $\mathcal{T}_{|\lambda, \lambda + \varepsilon_0]}$ continuous. Let $t \in ]\lambda, \lambda + \varepsilon_0[.$ For each $\varepsilon \in [0, \lambda + \varepsilon_0]$ let $\partial \mathcal{T}(t + \varepsilon)$ denote the set of subgradients of $\mathcal{T}$ at $t + \varepsilon.$ Note that

$$\sup \partial \mathcal{T}(t) = \mathcal{T}_r(t) \in \mathbb{R}$$

and

$$\forall \varepsilon \in [0, \lambda + \varepsilon_0 - t[,$$ $\quad \partial \mathcal{T}(t + \varepsilon) = \left[\mathcal{T}(t + \varepsilon), \mathcal{T}(t + \varepsilon)\right].$

For each $\varepsilon \in [0, \lambda + \varepsilon_0 - t[,$ Lemma [5] yields some $x_{t+\varepsilon} \in \mathbb{R}$ such that

$$\mathcal{T}(t + \varepsilon) = (t + \varepsilon)x_{t+\varepsilon} - l_0(x_{t+\varepsilon})$$

hence by Lemma [4]

$$\mathcal{T}(t + \varepsilon) \leq (t + \varepsilon)x_{t+\varepsilon} - \mathcal{T}^*(x_{t+\varepsilon}) \leq \mathcal{T}^*(t + \varepsilon);$$

since $\mathcal{T} \geq \mathcal{T}^*$ we obtain $l_0(x_{t+\varepsilon}) = \mathcal{T}^*(x_{t+\varepsilon});$ in particular, $x_{t+\varepsilon} \in \partial \mathcal{T}(t + \varepsilon).$ Putting $x = \mathcal{T}_r(t)$ we have

$$\forall \varepsilon \in [0, \lambda + \varepsilon_0 - t[,$$ $\quad x = \sup \partial \mathcal{T}(t) \leq x_{t+\varepsilon} \leq \mathcal{T}_r(t + \varepsilon)$$

hence $\lim_{\varepsilon \to 0} x_{t+\varepsilon} = x$ (because $\lim_{\varepsilon \to 0} \mathcal{T}_r(t + \varepsilon) = x$ by Lemma [2]) and

$$x = \lim_{\varepsilon \to 0^+} \frac{\mathcal{T}(t + \varepsilon) - \mathcal{T}(t)}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{(t + \varepsilon)x_{t+\varepsilon} - l_0(x_{t+\varepsilon}) - \mathcal{T}(t)}{\varepsilon} = x + \lim_{\varepsilon \to 0^+} \frac{tx_{t+\varepsilon} - l_0(x_{t+\varepsilon}) - \mathcal{T}(t)}{\varepsilon},$$

which implies

$$0 = \lim_{\varepsilon \to 0^+} \frac{(tx_{t+\varepsilon} - l_0(x_{t+\varepsilon}) - \mathcal{T}(t))}{\varepsilon} \leq tx - l_0(x) - \mathcal{T}(t) \leq 0,$$

where the first inequality follows from the lower semi-continuity of $l_0,$ and the second inequality follows from Lemma [5] therefore, $\mathcal{T}(t) = tx - l_0(x)$ hence $l_0(x) = \mathcal{T}^*(x)$ since $\mathcal{T}(t) = tx - \mathcal{T}^*(x)$ (because $x \in \partial \mathcal{T}(t);)$ this proves the first assertion.

When $\mathcal{T}_r(\lambda^+) \neq -\infty,$ the hypotheses imply $\mathcal{T}_r(\lambda^+)$ is proper and the last assertion follows noting that the above proof works verbatim with $t = \lambda$ and $\mathcal{T}_r(\lambda^+)$ (resp. $\mathcal{T}(\lambda^+)$) in place of $\mathcal{T}_r(t)$ (resp. $\mathcal{T}(t)).$ \hfill \Box

**Lemma 7.** Assume that $\mathcal{T}_{\varepsilon}$ is proper and $\mathcal{T}(t) < +\infty$ for some $t > 0.$ Let $(t_i)$ be a sequence of positive numbers converging to 0. We have

$$\lim_i \mathcal{T}_r(\mathcal{T}_r(t_i)) = -\mathcal{T}(0^+) \in [0, +\infty[.$$
If furthermore $\mathcal{T}_r(0^+) > -\infty$, then
$$\lim_{t \to 0^+} \mathcal{T}_r'(\mathcal{T}_r(t)) = \mathcal{T}_r'(\mathcal{T}_r(0^+)) = \sup_{t > 0}\{t\mathcal{T}_r(0^+) - \mathcal{T}(t)\}.$$ 

Proof. By Lemma 4 (applied with $\lambda = 0$), there exists $\varepsilon > 0$ such that $\mathcal{T}_{[0,\varepsilon]}$ is real-valued, bounded and continuous, hence eventually $\mathcal{T}_r(t_i) \in \mathbb{R}$,
$$\mathcal{T}_r'(\mathcal{T}_r(t_i)) = t_i\mathcal{T}_r'(t_i) - \mathcal{T}(t_i) = \sup_{t > 0}\{t\mathcal{T}_r(t) - \mathcal{T}(t)\},$$
and $\mathcal{T}(0^+)$ exists in $]-\infty,0]$; furthermore, $\mathcal{T}_r'(0^+)$ exists in $]-\infty, +\infty]$ by Lemma 2. Since $\mathcal{T}$ is continuous on its effective domain, and $\mathcal{T}(0^+)$ is a non-positive real number, (6) ensures the existence of $\lim_{t \to 0^+} \mathcal{T}_r'(\mathcal{T}_r(t_i))$ in $[0, +\infty[$.

First assume that $\mathcal{T}_r'(0^+) > -\infty$. Then $\mathcal{T}_r'(0^+) \in \mathbb{R}$ and (6) yields
$$\mathcal{T}(0^+) = \lim_i \mathcal{T}_r(t_i) = \lim_i (t_i\mathcal{T}_r'(t_i) - \mathcal{T}(t_i)) = -\lim_i \mathcal{T}_r'(\mathcal{T}_r(t_i)) = -\mathcal{T}_r'(\mathcal{T}_r(0^+)),$$
which proves the first assertion and the first equality of the second assertion; we have
$$\mathcal{T}_r'(\mathcal{T}_r(0^+)) \geq \sup_{t > 0}\{t\mathcal{T}_r(0^+) - \mathcal{T}(t)\} \geq \lim_i \{t_i\mathcal{T}_r(0^+) - \mathcal{T}(t_i)\} = -\mathcal{T}(0^+),$$
and the second equality of the second assertion follows.

Assume that $\mathcal{T}_r'(0^+) = -\infty$. For each $i$ large enough there exists $\lambda_i > 0$ such that $\lambda_i\mathcal{T}_r(t_i) = \mathcal{T}(\lambda_i)$, hence
$$\sup_{t \to \lambda_i}\{t\mathcal{T}_r(t) - \mathcal{T}(t)\} = 0$$
and
$$\lim_{t \to \lambda_i} \sup_{t \in [0,\lambda_i]}\{t\mathcal{T}_r(t) - \mathcal{T}(t)\} = -\mathcal{T}(0^+),$$
so that the first assertion follows from (6) since $-\mathcal{T}(0^+) \geq 0$. \hfill \Box

Lemma 7 shows that when $\mathcal{T}_r'(0^+) = -\infty$, the map $\mathcal{T}$ extends by continuity to a $]-\mathcal{T}(0^+), +\infty]-valued map on $]-\infty, +\infty[$ by putting $\mathcal{T}(-\infty) = -\mathcal{T}(0^+)$; in what follows, we implicitly use this extension.

Lemma 8. We assume that $\mathcal{T}_{[\lambda, \lambda+\varepsilon]}$ is differentiable for some $\varepsilon > 0$.

a) $\mathcal{T}_{[\lambda, \lambda+\varepsilon]}$ extends to a non-decreasing continuous surjection between $[\lambda, \lambda + \varepsilon]$ and $[\mathcal{T}_r(\lambda^+), \mathcal{T}_r(\lambda + \varepsilon)]$.

b) $\mathcal{T}_{[\mathcal{T}_r(\lambda^+), \mathcal{T}_r(\lambda+\varepsilon)]}$ is $[0, +\infty]-valued and extends to a $[0, +\infty]-valued, strictly increasing and continuous map on $[\mathcal{T}_r(\lambda^+), \mathcal{T}_r(\lambda + \varepsilon)]$, which takes the value $+\infty$ at $\mathcal{T}_r(\lambda + \varepsilon)$ if and only if $\mathcal{T}_r(\lambda + \varepsilon) = +\infty$; it vanishes at $\mathcal{T}_r(\lambda^+)$ if and only if either $\lambda > 0$, $\mathcal{T}$ is differentiable at $\lambda$ and linear on $[0, \lambda]$, or $\lambda = 0$ and $\mathcal{T}$ is right continuous at zero. The map $\mathcal{T}_{[\mathcal{T}_r(\lambda^+), \mathcal{T}_r(\lambda+\varepsilon)]}$ is positive and strictly convex; if furthermore, $\lambda < \lambda + \varepsilon$ and $\mathcal{T}_{[\mathcal{T}_r(\lambda^+), \mathcal{T}_r(\lambda+\varepsilon)]}$ is strictly convex, then $\mathcal{T}_{[\mathcal{T}_r(\lambda^+), \mathcal{T}_r(\lambda+\varepsilon)]}$ is differentiable (with $\hat{\lambda} = \sup\{t > \lambda : \mathcal{T}_r(t) \text{ is affine}\}$).

Proof. a) Since $\mathcal{T}_{[\lambda, \lambda+\varepsilon]}$ is differentiable, the map $\mathcal{T}_{[\lambda, \lambda+\varepsilon]}$ is continuous by Corollary 25.5.1 of [11]. Since $\mathcal{T}_r(\lambda + \varepsilon) = \lim_{t \to \lambda+\varepsilon} \mathcal{T}_r(\lambda + t)$, the map $\mathcal{T}_{[\lambda, \lambda+\varepsilon]}$ is a non-decreasing continuous surjection onto $[\mathcal{T}_r(\lambda^+), \mathcal{T}_r(\lambda + \varepsilon)]$, which can be extended by continuity to a non-decreasing continuous surjection between $[\lambda, \lambda + \varepsilon]$ and $[\mathcal{T}_r(\lambda^+), \mathcal{T}_r(\lambda + \varepsilon)]$. 

b) Each \( t \in ]\lambda, \lambda + \varepsilon[ \) is a subgradient of \( \mathcal{T}^* \) at \( \mathcal{T}(t) \) so that \( \mathcal{T}^*|_{[\mathcal{T}(\lambda^+), \mathcal{T}(\lambda + \varepsilon)]^c} \) is \([0, +\infty[\)-valued and strictly increasing, hence \( \mathcal{T}^*|_{[\mathcal{T}(\lambda^+), \mathcal{T}(\lambda + \varepsilon)]} \) is \([0, +\infty[\)-valued and strictly increasing; in particular, \( \mathcal{T}^*|_{[\mathcal{T}(\lambda^+), \mathcal{T}(\lambda + \varepsilon)]} \) is positive. Since \( \mathcal{T}^* \) is lower semi-continuous, it is continuous on its effective domain so that \( \mathcal{T}^*|_{[\mathcal{T}(\lambda^+), \mathcal{T}(\lambda + \varepsilon)]} \) is continuous, and thus extends to a \([0, +\infty[\)-valued, strictly increasing and continuous map on \([\mathcal{T}(\lambda^+), \mathcal{T}(\lambda + \varepsilon)]\); this extended map can takes the value \(+\infty\) only at \( \mathcal{T}(\lambda + \varepsilon) \); since \( \mathcal{T}^* \) is lower semi-continuous, \( \mathcal{T}^*(\mathcal{T}(\lambda + \varepsilon)) \) is finite when \( \mathcal{T}(\lambda + \varepsilon) \) is finite; conversely, if \( \mathcal{T}(\lambda + \varepsilon) = +\infty \), then \( \mathcal{T}^*(\mathcal{T}(\lambda + \varepsilon)) = +\infty \) is included in the effective domain of \( \mathcal{T}^* \), hence \( \lim_{t \to \lambda^+ + \varepsilon} \mathcal{T}^*(\mathcal{T}(t)) = +\infty \). Let \( g \) be the extension of \( \mathcal{T}^*|_{\lambda, \lambda + \varepsilon} \) to \( \mathbb{R} \) defined by putting

\[
g(t) = \begin{cases} 
+\infty & \text{if } t \in ]-\infty, \lambda[, \text{ and } \mathcal{T}_r(\lambda^+) = -\infty \\
(t - \lambda)\mathcal{T}_r(\lambda) + g(\lambda) & \text{if } t \in ]-\infty, \lambda[, \text{ and } \mathcal{T}_r(\lambda^+) > -\infty \\
\mathcal{T}_r(\lambda^+) & \text{if } t = \lambda \\
\mathcal{T}_r(t) & \text{if } t \in ]\lambda, \lambda + \varepsilon[ \\
\mathcal{T}_r((\lambda + \varepsilon)^-) & \text{if } t = \lambda + \varepsilon \\
(t - (\lambda + \varepsilon))\mathcal{T}_r(\lambda) + g(\lambda + \varepsilon) & \text{if } t \in ]\lambda + \varepsilon, +\infty[ \text{ and } \mathcal{T}_r(\lambda) + \varepsilon < +\infty \\
{+\infty} & \text{if } t \in ]\lambda + \varepsilon, +\infty[ \text{ and } \mathcal{T}_r(\lambda + \varepsilon) = +\infty, 
\end{cases}
\]

so that \( g \) is convex, lower semi-continuous, differentiable on the interior of its effective domain, but not sub-differentiable at each point in the complement of the interior of its effective domain; therefore, the Legendre-Fenchel transform \( g^* \) of \( g \) is strictly convex on the interior of its effective domain \((\text{[12], Theorem 11.13}), \) hence \( \mathcal{T}_r^*|_{\mathcal{T}(\lambda^+), \mathcal{T}(\lambda + \varepsilon)} \mathcal{T}_r^* \) is strictly convex. Since \( g'(t) = \mathcal{T}_r(t) \) for all \( t \in ]\lambda, \lambda + \varepsilon[, \) we obtain \( g^*|_{\mathcal{T}(\lambda^+), \mathcal{T}(\lambda + \varepsilon)} \mathcal{T}_r^* \) which proves the strict convexity property. The first part of the first assertion, and the first part of the second assertion are proved. The continuity and strict increasingness imply that \( \inf_{\mathcal{T}(\lambda^+), \mathcal{T}(\lambda + \varepsilon)} \mathcal{T}_r^* \) is a minimum, which is attained at the unique point \( \mathcal{T}_r(\lambda^+) \); this proves the second part of the first assertion since \( \mathcal{T}_r^*(\mathcal{T}(\lambda^+)) = \lambda \mathcal{T}_r^*(\lambda^+) - \mathcal{T}(\lambda^+) \) (using Lemma \([17\) when \( \lambda = 0 \)). Assume furthermore that \( \mathcal{T}_r^*|_{\lambda, \lambda + \varepsilon} \) is strictly convex. The map \( g \) is not sub-differentiable at some \( t \in ]\lambda, \lambda + \varepsilon[ \) and only if either \( t = \lambda = 0 \) and \( g^*(0^+) = -\infty \), or \( t = \lambda + \varepsilon \) and \( g^*(\lambda + \varepsilon) = +\infty \), hence \( \lambda, \lambda + \varepsilon \subseteq \{ t \in ]\lambda, \lambda + \varepsilon[ : g \text{ is sub-differentiable at } t \} \). Therefore, the map \( g_1|_{\lambda, \lambda + \varepsilon} + \infty 1_{R\setminus[\lambda, \lambda + \varepsilon]} \) is strictly convex on every convex subset of the set \( \{ t \in ]\lambda, \lambda + \varepsilon[ : g_1|_{\lambda, \lambda + \varepsilon} + \infty 1_{R\setminus[\lambda, \lambda + \varepsilon]} \text{ is sub-differentiable at } t \} \), hence its Legendre-Fenchel transform is differentiable on the interior of its effective domain \((\text{[12], Theorem 11.13}) \); consequently, \( \mathcal{T}_r^*|_{\mathcal{T}(\lambda^+), \mathcal{T}(\lambda + \varepsilon)} \mathcal{T}_r^* \) is strictly convex. Assume that \( \lambda < \lambda + \varepsilon \) and \( \mathcal{T}_r^*|_{\lambda, \lambda + \varepsilon} \) is strictly convex. The above proof and the proof of part a) work verbatim replacing \( ]\lambda, \lambda + \varepsilon[ \) by \( ]\lambda, \lambda + \varepsilon[. \) Since \( \mathcal{T}_r^* \) is differentiable at \( \lambda \), we have \( \mathcal{T}_r^*(\lambda^+) = \mathcal{T}_r^*(\lambda^+) \), which proves the second part of the second assertion.

**Proof of Theorem 4** Put \( x = \mathcal{T}_r(\lambda^+) \). The hypotheses imply that \( \mathcal{T}_r|_{[0, +\infty[} \) is proper (otherwise, by Lemma \([11\)), \( \mathcal{T}_r(t) \in (-\infty, +\infty] \) for all \( t > 0 \), which contradicts the hypothesis when \( x > -\infty \), and \( \mathcal{T}_r|_{[0,t]} = -\infty \) for all \( t > 0 \) small enough, which contradicts the hypothesis when \( x = -\infty \), and the case (i) of Lemma \([10\) holds; in particular, \( x < +\infty \) and there exists \( \varepsilon > 0 \) such that \( \mathcal{T}_r|_{[\lambda, \lambda + \varepsilon]} \) is real-valued and \( \mathcal{T}_r|_{[\lambda, \lambda + \varepsilon]} \) is continuous. Let \( (x_\alpha, y_\alpha) \) be a net in \([-\infty, +\infty]^2 \) such that \( \lim_{\alpha} x_\alpha = x \) and \( \liminf_{\alpha} y_\alpha > x \). Put \( y = \liminf_{\alpha} y_\alpha \).

- First assertion, the case \( x \in \mathbb{R} \): For each \( t \in ]\lambda, \lambda + \varepsilon[, \) Chebyshev’s inequality yields eventually

\[
c_\alpha \log \int_{\mathbb{R}} e^{-c_\alpha^{\lambda^+}} \mu_\alpha(dz) \geq c_\alpha \log(e^{c_\alpha^{\lambda^+}} \mu_\alpha([x_\alpha, y_\alpha])) = t x_\alpha + c_\alpha \log \mu_\alpha([x_\alpha, y_\alpha])
\]
where the last equality follows from Lemma 7 when $\lambda = 0$. Suppose that
\[ \lim_{i} c_{\alpha} \log \mu_{\alpha}([x_{\alpha}, y_{\alpha}]) < -\overline{T}'(x). \]
The hypothesis together with the continuity of $\overline{T}'$ on its effective domain implies the existence of $t > \lambda$ and $\delta > 0$ such that
\[ x + 2\delta < \overline{T}'(t) < y - 2\delta \]
and
\[ \limsup_{\alpha} c_{\alpha} \log \mu_{\alpha}([x_{\alpha}, y_{\alpha}]) < -\overline{T}'(\overline{T}'(t)). \] (8)
Since eventually
\[ x_{\alpha} + \delta < \overline{T}'(t) < y_{\alpha} - \delta, \]
we have eventually
\[ \mu_{\alpha}([x_{\alpha}, y_{\alpha}]) \geq \mu_{\alpha}([\overline{T}'(t) - \delta, \overline{T}'(t) + \delta]) \]

hence
\[ \limsup_{\alpha} c_{\alpha} \log \mu_{\alpha}([x_{\alpha}, y_{\alpha}] \geq -l_{0}(\overline{T}'(t)), \]
which contradicts (8) since $l_{0}(\overline{T}'(t)) = \overline{T}'(\overline{T}'(t))$ by Lemma 6. Therefore, we have
\[ \limsup_{\alpha} c_{\alpha} \log \mu_{\alpha}([x_{\alpha}, y_{\alpha}] \geq -\overline{T}'(x), \]
which together with (7) proves the first three equalities of the first assertion; the last equality is obvious when $\lambda > 0$ (definition of $\overline{T}'$), and follows from Lemma 7 when $\lambda = 0$.

- First assertion, the case $x = -\infty$: Lemma 2 implies $\lambda = 0$. Let $(t_{i})$ be a sequence of positive numbers converging to 0, so that eventually
\[ l_{0}(\overline{T}'(t_{i})) = \overline{T}'(\overline{T}'(t_{i})) \]
by Lemma 6. Since $\lim_{i} \overline{T}'(t_{i}) = \overline{T}'(\overline{T}'(t_{i})) = \overline{T}'(0^{+}) = 0$, the first two equalities of the first assertion follow from the hypothesis of right continuity of $\overline{T}$ at zero. The first two equalities of the first assertion follow from the above expression together with Lemma 7 (recall that by convention, $0 \cdot (-\infty) = 0$). For each $i \in \mathbb{N}$, $t_{i}$ is a subgradient of $\overline{T}'$ at $\overline{T}'(t_{i})$ so that $\overline{T}'$ is non-decreasing on $[\overline{T}'(t_{i})]$. Since $\lim_{\alpha} x_{\alpha} = -\infty$, eventually $x_{\alpha}$ belongs to the effective domain of $\overline{T}'$ and fulfills
\[ 0 \leq \overline{T}'(x_{\alpha}) \leq \overline{T}'(\overline{T}'(t_{i})), \]
hence
\[ 0 \leq \liminf_{\alpha} \overline{T}'(x_{\alpha}) \leq \limsup_{\alpha} \overline{T}'(x_{\alpha}) \leq \overline{T}'(\overline{T}'(t_{i})). \]

letting $i \to +\infty$ gives $\lim_{\alpha} \overline{T}'(x_{\alpha}) = 0$, which proves the last two equalities of the first assertion. The proof of the first assertion is complete.
Since \( \beta \), we have all the above inequalities are equalities, which gives
\[
\mathcal{L}^{(\mu_\beta,c_\beta)}(t) = \lim_{\beta} e^{c_\beta t} \mu_\beta(dx).
\]
Since \( \mathcal{L}^{(\mu_\beta,c_\beta)}(t) = L(t) \) for all \( i \in \mathbb{N} \), we have
\[
\mathcal{L}^{(\mu_\beta,c_\beta)}(\lambda^+) = \lim_{i} \mathcal{L}^{(\mu_\beta,c_\beta)}(t) = \lim_{i} L(t) = L(\lambda^+) \tag{9}
\]

hence
\[
\mathcal{L}^{(\mu_\beta,c_\beta)}(\lambda^+) = \lim_{i} \mathcal{L}^{(\mu_\beta,c_\beta)}(t) = \lim_{i} L(t) - L(\lambda^+) = \mathcal{T}_r(\lambda^+) = x, \tag{10}
\]
where the last equality follow from Lemma (11). The inequality \( \mathcal{L}^{(\mu_\beta,c_\beta)}(\lambda^+) \leq \mathcal{T}_r(\lambda^+) \) together with (10) implies
\[
\mathcal{L}^{(\mu_\beta,c_\beta)}(\lambda^+) \leq x. \tag{11}
\]

We have \( \mathcal{T}_r(t) > \mathcal{T}_r(\lambda^+) \) for all \( i \in \mathbb{N} \) (by definition of \( \lambda \)) and
\[
x = \lim_{i} \mathcal{T}_r(t), \tag{12}
\]
so that \( (\mathcal{L}_r(t)) \) has a strictly decreasing subsequence \( (\mathcal{T}_r(t)) \) converging to \( \mathcal{T}_r(\lambda^+) \), which implies eventually \( t_{j+1} < t_{j} < t_{j-1} \), hence
\[
x = \lim_{j} \mathcal{T}_r(t_{j}), \tag{13}
\]
and
\[
\forall j \in \mathbb{N}, \quad \mathcal{T}_r(t_{j}) > x. \tag{14}
\]

Put \( \check{\lambda}^{(\mu_\beta,c_\beta)} = \sup\{t > \lambda : \mathcal{L}^{(\mu_\beta,c_\beta)}(t) \text{ is affine}\} \). The inequality \( \mathcal{L}^{(\mu_\beta,c_\beta)} \leq \mathcal{L} \) together with (9) implies
\[
\check{\lambda}^{(\mu_\beta,c_\beta)} \geq \lambda. \tag{15}
\]

Claim 1. When \( \mathcal{L}^{(\mu_\beta,c_\beta)}(\lambda^+) > -\infty \), the hypothesis of Theorem (12) holds with the net \( (\mu_\beta,c_\beta) \) in place of \( (\mu_\alpha,c_\alpha) \). Furthermore, we have \( \mathcal{L}^{(\mu_\beta,c_\beta)}(\lambda^+) = \mathcal{L}(\lambda^+) \) and \( \mathcal{L}^{(\mu_\beta,c_\beta)}(\lambda^+) = \mathcal{L}(\lambda^+) \).

Proof of Claim 1. Assume \( \mathcal{L}^{(\mu_\beta,c_\beta)}(\lambda^+) > -\infty \). Since \( x > -\infty \) by (11) we have
\[
x(\check{\lambda} - \lambda) = \mathcal{L}(\check{\lambda}) - \mathcal{L}(\lambda^+) \leq \mathcal{L}(\check{\lambda}) - \mathcal{L}^{(\mu_\beta,c_\beta)}(\lambda^+) = \mathcal{L}^{(\mu_\beta,c_\beta)}(\check{\lambda}) - \mathcal{L}^{(\mu_\beta,c_\beta)}(\lambda^+)
\]
where the second equality follows from (9) and the last equality follows from (10), therefore, all the above inequalities are equalities, which gives
\[
\mathcal{L}^{(\mu_\beta,c_\beta)}(\lambda^+) = \mathcal{L}(\lambda^+), \tag{16}
\]

which together with (9) implies
\[
\mathcal{L}^{(\mu_\beta,c_\beta)}(\lambda) = \mathcal{L}(\lambda) \tag{17}
\]

Since \( \mathcal{L}^{(\mu_\beta,c_\beta)} \leq \mathcal{L} \) with \( \mathcal{L} \) convex, (17) implies
\[
\check{\lambda}^{(\mu_\beta,c_\beta)} = \check{\lambda}. \tag{18}
\]
We have
\[
\lim_j \mathcal{L}^\prime(t_{ij}) \leq \liminf_j \mathcal{L}^{\mu_3,c_3}(t_{ij}) \leq \liminf_j \mathcal{L}^{\mu_3,c_3}(r(t_{ij})) \leq \limsup_j \mathcal{L}^{\mu_3,c_3}(r(t_{ij})) \leq \lim_j \mathcal{L}^\prime(t_{ij}),
\]
hence by (12) and (13), all the above inequalities are equalities, which together with (10) yields
\[
\forall j \in \mathbb{N}, \quad \mathcal{L}^{\mu_3,c_3}(r(t_{ij})) = \mathcal{L}^{\mu_3,c_3}(r(t_{ij})) = \mathcal{L}^{\mu_3,c_3}(r(t_{ij})) = x,
\]
where the strict inequality follows from (14). The above expression together with (10) and (18) gives
\[
\forall j \in \mathbb{N}, \quad \mathcal{L}^{\mu_3,c_3}(r(t_{ij})) = \mathcal{L}^{\mu_3,c_3}(r(t_{ij})) = \mathcal{L}^{\mu_3,c_3}(r(t_{ij})) = x,
\]
which proves the first assertion of the claim.

The first equality of the second assertion is given by (16); the second equality of the second assertion is given by (10) when \( \lambda = \tilde{\lambda} \). Assume \( \lambda < \tilde{\lambda} \). Then, (17) and (18) yield
\[
\mathcal{L}^{\mu_3,c_3}(\tilde{\lambda}^+) = \mathcal{L}^{\mu_3,c_3}(\lambda^+),
\]
(20) Since \( \mathcal{L} \) is differentiable at \( \tilde{\lambda} \) when \( \tilde{\lambda} > \lambda \), we have
\[
\mathcal{L}^{\mu_3,c_3}(\lambda^+) \leq x = \mathcal{L}^\prime(\tilde{\lambda}^+) \leq \mathcal{L}^{\mu_3,c_3}(\tilde{\lambda}^+) \leq \mathcal{L}^{\mu_3,c_3}(\lambda^+),
\]
(20) This last equality together with (9) yields \( \mathcal{L}^{\mu_3,c_3}(\lambda^+) = 0 \), which proves the claim. \( \Box \)

Claim 2. When \( \mathcal{L}^{\mu_3,c_3}(r)(\lambda^+) = -\infty \), the hypothesis of Theorem 1 holds with the net \( (\mu_3,c_3) \) in place of \( (\mu_3,c_3) \); in particular, \( \mathcal{L}^{\mu_3,c_3}(\lambda^+) = \mathcal{L}(\lambda^+) \) and \( \mathcal{L}^{\mu_3,c_3}(r)(\lambda^+) = x \).

Proof of Claim 2. The hypothesis implies \( \lambda = 0 \) and \( \tilde{\lambda} = 0 \) by (15); therefore, \( x = -\infty \) by (16), which implies \( \mathcal{L}(\lambda^+) = 0 \) by hypothesis of Theorem 1 this last equality together with (9) yields \( \mathcal{L}^{\mu_3,c_3}(\lambda^+) = 0 \), which proves the claim. \( \Box \)

By Claim 1 and Claim 2, we can apply the first assertion of Theorem 1 with the net \( (\mu_3,c_3,\beta,\alpha) \) in place of \( (\mu_3,c_3,\alpha) \), which gives
\[
\limsup_\beta c_3 \log \mu_3(\beta,\alpha) = \limsup_\beta c_3 \log \mu_3(\beta,\alpha) = \mathcal{L}^{\mu_3,c_3}(\lambda^+) = \mathcal{L}(\lambda^+) - \lambda x.
\]
Since the right hand side of the above last equality does not depend on \( (\mu_3,c_3,\beta,\alpha) \), the second assertion follows. \( \Box \)

Proof of Corollary 1. Let \( t \in [\lambda,\lambda + \varepsilon] \) and put \( \tilde{t} = \sup \{ s < t : \mathcal{L}(t,s) \text{ is affine} \} \) (note that \( \tilde{t} \) is a maximum since \( \mathcal{L}(\lambda,\lambda + \varepsilon) \) is continuous). Since (by hypothesis) \( \sup \{ \mathcal{L}(t) : t \in [\lambda,\lambda + \varepsilon] \} \) is not a maximum, we have \( \tilde{t} < \lambda + \varepsilon \). For each \( \delta > 0 \), the map \( \mathcal{L}(\tilde{t},\tilde{t}+\delta) \) is not affine (otherwise, since \( \mathcal{L}(\tilde{t},\tilde{t}+\delta) \) is affine at \( \tilde{t} \), the slope of \( \mathcal{L}(\tilde{t},\tilde{t}+\delta) \) would be the same as the one of \( \mathcal{L}(\tilde{t}) \), which would contradict the definition of \( \tilde{t} \)). Therefore, we have \( \mathcal{L}(s) > \mathcal{L}(\tilde{t}^+) \) for all \( s > \tilde{t} \); since \( \mathcal{L}(t^+) = \mathcal{L}(\tilde{t}^+) \) (because \( \mathcal{L}(\tilde{t}) \) is affine and \( \mathcal{L}(\tilde{t}) \) is differentiable at \( \tilde{t} \)) we get \( \mathcal{L}(s) > \mathcal{L}(t^+) \) for all \( s > \tilde{t} \). Consequently, by the last assertion of Lemma 1, the case (i) of Lemma 1 holds, hence the hypotheses of Theorem 1 holds for \( t \) (i.e. with \( \lambda = t \) in
Theorem 1, and in particular when $T_t(t^+)>-\infty$. If $L_t(t^+)=-\infty$, then $t=\lambda=0$ by Lemma 2 since in this case $L$ is assumed to be right continuous at zero, the hypotheses of Theorem 1 hold for $t$. Lemma 8) ensures that for each $z \in [T_t(\lambda^+),T_t(\lambda+\varepsilon)]$ there exists $t_z \in [\lambda,\lambda+\varepsilon]$ such that $z = T_t(t_z^+)$, hence part a) follows applying Theorem 1 for all $t \in [\lambda,\lambda+\varepsilon]$. Part b) follows from Lemma 8 b). □

APPENDIX

Let $\lambda \geq 0$, let $\varepsilon > 0$ and let $f$ be a real-valued strictly convex and continuous function on $[0, \lambda + \varepsilon]$ such that $f(0) = 0$. In what follows, we show how from such a function $f$ it is possible to build a generalized log-moment generating function $L_f$ with effective domain $[0, \lambda + \varepsilon]$, such that $L_f(t)$ is a limit for all $t \in \mathbb{R}$, $L_f$ is not differentiable at $\lambda$, $L_f|_{[0,\lambda]}$ is affine, $\lambda$ is a limit of a decreasing sequence $(\lambda_i)_{i \geq 1}$ of non-differentiability points of $L_f|_{[\lambda,\lambda+\varepsilon]}$ and $L_f|_{[\lambda_i,\lambda_{i+1}]}$ is affine for all $i \geq 1$ (with $\lambda_0 = \lambda + \varepsilon$). Therefore, the hypotheses of Theorem 1 hold for $\lambda$, but the usual version of Plachky-Steinebach theorem does not apply (either because $L$ is not differentiable in a neighbourhood of $\lambda$, or because $L$ is not strictly convex in a neighbourhood of $\lambda$); however, its conclusion remains true (i.e. both conclusions of Theorem 1 hold); the case $\lambda = 0$ and $L_f(t^+) = -\infty$ is included.

Note that $\lambda$ is the only point in $[0, \lambda + \varepsilon]$ to which Theorem 1 applies, since (a) every $t \in [0, \lambda + \varepsilon]$ fulfils the case (ii) of Lemma 1, hence the excluded case (ii) mentioned in Section 1 and (b) $L_f'(\lambda + \varepsilon) = +\infty$, i.e. $\lambda + \varepsilon$ fulfils the case (ii) of Lemma 1 hence the excluded case (i) mentioned in Section 1.

First, we extend $f$ to a convex lower semi-continuous function on $\mathbb{R}$ by putting $f(t) = +\infty$ for all $t \notin [0, \lambda + \varepsilon]$. Put $\lambda_0 = \lambda + \varepsilon$. Let $(\lambda_i)_{i \geq 1}$ be a decreasing sequence in $[\lambda, \lambda + \varepsilon]$ converging to $\lambda$. Draw a line segment $D_i$ between $(\lambda_i, f(\lambda_i))$ and $(\lambda_{i+1}, f(\lambda_{i+1}))$ for all $i \in \mathbb{N}$, and draw a line segment $D_i$ between $(0,0)$ and $(\lambda, f(\lambda))$. Let $L_f$ be the function whose graph coincides with the graph of $f$ (resp. $D_i$, $D$) on $]-\infty,0[ \cup [\lambda_0, +\infty[ \cup ]\lambda_i, +\infty[ \cup ]\lambda_{i+1}, \lambda_i[$ for all $i \in \mathbb{N}$, $[0, \lambda]$). Clearly, $L_f$ is a convex function on $\mathbb{R}$, continuous on its effective domain $[0, \lambda + \varepsilon]$, affine on $[0, \lambda]$ and $[\lambda_i+1, \lambda_i]$ for all $i \in \mathbb{N}$, and fulfils $L_f(0) = 0$. The strict convexity of $f$ ensures for each $i \geq 1$ the existence of some $t_i \in [\lambda_i+1, \lambda_i]$ such that

$$f_r'(\lambda) < f_r'(t_{i+1}) < L_f'(\lambda_{i+1}) < f_r'(t_i) < L_f'(\lambda_i),$$

(21)

hence

$$L_f'(\lambda) = f_r'(\lambda) = \lim_{i \to \infty} L_f'(\lambda_i)$$

(22)

and $L_f'(\lambda) < L_f'(\lambda)$ when $L_f'(\lambda) > -\infty$. Therefore, $L_f$ is not differentiable at $\lambda$, and not differentiable at $\lambda_i$ for all $i \in \mathbb{N}$, but (21) and (22) show that $L_f$ fulfils the hypotheses of Theorem 1.

It remains to show that $L_f$ is a generalized log-moment generating function such that $L_f(t)$ is a limit for all $t \in \mathbb{R}$. Let $\{z_k : k \geq 1\}$ be a countable set dense in the effective domain $L_f^*$ of $L_f$, and put

$$\forall n \geq 1, \quad \mu_{n,f} = \frac{\sum_{k=1}^{n} e^{-nL_f(z_k)}}{\sum_{k=1}^{n} e^{-nL_f(z_k)}} \delta_{z_k}.$$

Since $L_f$ is lower semi-continuous, we have $L_f = L_f^{**}$ (11), Theorem 23.5) i.e.

$$\forall t \in \mathbb{R}, \quad L_f(t) = \sup_{z \in \mathbb{R}} \{tz - L_f^*(z)\} = \sup_{z \in \text{dom } L_f^*} \{tz - L_f^*(z)\},$$

(23)
\[ \forall n \geq 1, \quad \max_{k=1}^{n} \{ t z_k - L_f^* (z_k) \} \leq n^{-1} \log \sum_{k=1}^{n} e^{n (t z_k - L_f^* (z_k))} \]

\[ \leq n^{-1} \log (n \max_{k=1}^{n} \{ e^{n (t z_k - L_f^* (z_k))} \}) = n^{-1} \log n + \max_{k=1}^{n} \{ t z_k - L_f^* (z_k) \} \leq n^{-1} \log n + L_f(t) \]

and letting \( n \to +\infty \),

\[ \sup_{k=1}^{\infty} \{ t z_k - L_f^* (z_k) \} \leq \liminf_{n \to \infty} \frac{1}{n} \log \sum_{k=1}^{n} e^{n (t z_k - L_f^* (z_k))} \]

\[ \leq \limsup_{n \to \infty} \frac{1}{n} \log \sum_{k=1}^{n} e^{n (t z_k - L_f^* (z_k))} \leq L_f(t). \]

Since the set \( \{ z_k : k \geq 1 \} \) is dense in \( \text{dom} \ L_f^* \) and \( L_f^* \) is continuous on \( \text{dom} \ L_f^* \), we get for each \( t \in \mathbb{R} \) and for each \( z \in \text{dom} \ L_f^* \),

\[ tz - L_f^* (z) \leq \liminf_{n \to \infty} \frac{1}{n} \log \sum_{k=1}^{n} e^{n (t z_k - L_f^* (z_k))} \leq \limsup_{n \to \infty} \frac{1}{n} \log \sum_{k=1}^{n} e^{n (t z_k - L_f^* (z_k))} \leq L_f(t), \]

hence

\[ \forall t \in \mathbb{R}, \quad \sup_{z \in \text{dom} \ L^*} \{ tz - L^* (z) \} \leq \liminf_{n \to \infty} \frac{1}{n} \log \sum_{k=1}^{n} e^{n (t z_k - L_f^* (z_k))} \]

\[ \leq \limsup_{n \to \infty} \frac{1}{n} \log \sum_{k=1}^{n} e^{n (t z_k - L_f^* (z_k))} \leq L_f(t), \]

which together with (23) gives

\[ \forall t \in \mathbb{R}, \quad \lim_{n \to \infty} \frac{1}{n} \log \sum_{k=1}^{n} e^{n (t z_k - L_f^* (z_k))} = L_f(t). \] (24)

Since

\[ n^{-1} \log \int_{\mathbb{R}} e^{ntz} \mu_{n,f} (dz) = n^{-1} \left( \log \sum_{k=1}^{n} e^{n (t z_k - L_f^* (z_k))} - \log \sum_{k=1}^{n} e^{-n L_f^* (z_k)} \right) \]

for all \((t, n) \in \mathbb{R} \times \mathbb{N} \setminus \{0\}\), it follows from (24) that the generalized log-moment generating function associated with \((\mu_{n,f}, n^{-1})\) is a limit for all \( t \in \mathbb{R} \), and coincides with \( L_f \). Consequently, both conclusions of Theorem 1 hold, i.e. for every sequence \((x_n)\) converging to \( L_f^* (\lambda) \), and for every sequence \((y_n)\) fulfilling \( \liminf \inf_{n} y_n > L_f^* (\lambda) \), we have

\[ \lim_{n \to \infty} n^{-1} \log \mu_{n,f} ([x_n, y_n]) = \lim_{n \to \infty} n^{-1} \log \mu_{n,f} ([x_n, y_n]) = L_f (\lambda) - \lambda L_f^* (\lambda) \]

\[ = \begin{cases} -L_f^* (L_f^* (\lambda)) & \text{if } \lambda > 0 \text{ or } (\lambda = 0 \text{ and } L_f^* (0) > -\infty) \\ \lim_{n \to \infty} L_f^* (x_n) = 0 & \text{if } \lambda = 0 \text{ and } L_f^* (0) = -\infty. \end{cases} \]

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