CONSTRUCTION AND CLASSIFICATION OF
COMBINATORIAL WEA VING DIAGRAMS

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ABSTRACT. This paper introduces a new systematic algorithm for constructing periodic Euclidean weaving diagrams with combinatorial arguments. It is shown that such a weaving diagram can be considered as a specific type of four-regular periodic planar tiling with over or under information at each vertex. Therefore, a weaving diagram can be constructed using two sets of cycles, one to build a tiling, and a second to define the crossing information. However, this construction method does not guarantee the uniqueness of the diagram, so we define the notion of equivalence classes of weaving diagrams using the concept of crossing-matrices. Finally, we present a classification of our periodic structures according to the minimum number of crossings on a unit cell.

1. INTRODUCTION

Complex entangled networks have recently become an important topic in mathematics and research on this area is also motivated by its relevance to materials science. The control of the topology of threaded rings allows for example the synthesis of giant virus capsids which mimic nature, as recently shown by T. Sawada, Y. Inomata, K. Shimokawa, et al. in [16]. A particular case of entangled frameworks is that of three-dimensional weaves, characterized by infinite threads, which are sets homeomorphic to \( \mathbb{R} \), woven together with respect to a set of crossing sequences. An interesting example of application is the case weaving carbon nanotube wires, which allows for example the development of supercapacitors for energy storage, as seen in the study of M. K. Jha, K. Hata, and C. Subramaniam [11]. Although these objects have been known and investigated for so many years by materials scientists among others and inspire new mathematical developments, as mentioned in the work of O.M. Yaghi and his coauthors [12], we still do not have a universal study about weaves to identify and classify them. Many very interesting attempts have been made to consider these objects from a mathematical point of view, for example through tiling theory with the work of S.T. Hyde et al. [2, 3, 4, 10, 18], as well as E.D. Miro et al. [14, 15], or through knot theory as considered by S. Grishanov et al. [6, 7, 8], but the community has not agreed to a formal definition of weavings yet, as well as a way to distinguish them from other entangled networks. This paper is the second in a series of articles which aim to contribute to this objective by proposing a new systematic approach, based on low-dimensional topology, combinatorics and geometric principles, to describe, construct, and classify weaves. Such
an investigation is an attempt towards a better understanding of their structure, that is often associated with the physical and functional properties of a material.

A formal definition of weaving networks has been stated in our previous work [13], inspired by related structures in materials science. The general idea of our study is to start from simple models and generalize to more complex frameworks. Next, the strategy is to focus on the unit cells of a planar weaving diagram, instead of its more complex associated three-dimensional weave. Such a diagram is the projection of a weave on the Euclidean or the hyperbolic plane, as a periodic planar graph with an “over or under” information at each crossing, representing the entanglements of the three-dimensional structure. In this way, it is possible to extend some results of knot theory to study their topological properties, such as Tait’s First and Second conjectures [13].

The primary aim of this paper is to focus on the mathematical construction of weaving diagrams. A first concept using tilings and entitled “polygonal links” was developed in our first article [13], but due to the restrictions considered in this approach, we now introduce a new systematic algorithm, which allows a more efficient and powerful construction of Euclidean weaving diagrams, using combinatorial arguments. An extension to the hyperbolic case is planned for future work. Recall that the two main criteria which characterize a weaving diagram are its number of sets of threads \(N\), meaning the number of “directions” in which the threads are organized, as well as the set of crossing sequences, which gives the entanglement information at each intersection between two distinct threads. As with our previous construction method, we can see a weaving diagram as being a particular type of periodic tiling by convex polygons, together with an over or under information at each vertex. With this in mind, we decompose our construction method into two main steps. Let \(i, j, k \in \{1, \cdots , N\}\). First we build such a particular tiling, that we call a thread-tiling, using a set of cycles \(\Sigma' = \{\sigma'_1, \cdots , \sigma'_N\}\), with each \(\sigma'_i\) called a vertex-cycle, and whose elements \(v_{i,j}\) are intersections between two threads belonging to two distinct sets of threads \(T_i\) and \(T_j\). It this way, we can construct a periodic four-regular and connected graph in terms of nearest neighbor vertices, which can be seen as a regular projection of a three-dimensional weave. Second, we add a crossing information at each vertex using another set of cycles \(\Sigma = \{\sigma_1, 2, \cdots , \sigma_{1,N}, \sigma_{2,3}, \cdots , \sigma_{2,N}, \cdots , \sigma_{N-1,N}\}\), with each \(\sigma_{i,j}\) called a crossing-cycle, and whose elements are a sequence of \(+1\) (over) and/or \(-1\) (under). Such a sequence gives the entanglement behavior of the threads of one set of threads with respect to the threads of another one, once again in terms of nearest neighbor vertices, using the first set of vertex-cycles. This algorithm guarantees the construction of infinitely many weaving diagrams, but not the uniqueness. A complete statement is given in Theorem 3.7.

**Theorem 1.1. (Construction of Combinatorial Weaving Diagrams)** Let \(i, j, k \in \{1, \cdots , N\}\) be distinct integers. Given a set of vertex-cycles \(\Sigma' = \{\sigma'_1, \cdots , \sigma'_N\}\), with \(\sigma'_i = (v_{i,j}, \cdots , v_{i,k})\), and a set of crossing-cycles \(\Sigma = \{\sigma_1, 2, \cdots , \sigma_{1,N}, \sigma_{2,3}, \cdots , \sigma_{2,N}, \cdots , \sigma_{N-1,N}\}\), with each element of \(\sigma_{i,j}\) in \(\pm 1\), then, they generate a weaving diagram with \(N\) sets of threads, namely the planar projection of a weave, with \(\Sigma'\) and \(\Sigma\) as its sets of vertex-cycles and crossing-cycles, respectively.
The second objective of this paper is to classify our combinatorial Euclidean weaving diagrams according to their crossing number, which is the minimum number of crossings that can be found on a unit cell of the infinite periodic weaving diagram, referred to here as a torus-diagram, as in [6, 7, 8]. We have indeed seen with Tait’s First and Second conjectures [13], that the number of crossings is in particular a suitable concept for studying and classifying torus-diagrams since it characterizes the complexity of the structure. However, in this previous work, we also concluded that it is difficult to find such a unit cell having the minimum number of crossings, called a minimal diagram, since this number of crossings depends on the scale chosen for the unit flat torus. Our new construction methodology allows us meet this challenge of finding the minimal diagram. Indeed, by using the two types of cycles mentioned above, we can deduce the minimum number of necessary crossings \( C_{i,j} \), to read the crossing sequence on each thread of each pair of sets \((T_i, T_j)\), meaning to respect the periodicity of the structure. We call \( C_{i,j} \) the \((i,j)\)-pairwise crossing number as defined in Lemma 4.1, and we can compute the total crossing number of a weaving diagram using all its pairwise crossing numbers, as stated in the following reduced theorem (See Theorem 4.2 for the complete version).

**Theorem 1.2. (Weaving Diagram Crossing Number)** Let \( D_{W_0} \) be a weaving diagram with \( N \) sets of threads defined by the pair \((\Sigma', \Sigma)\). Let \( i, j \in \{1, \ldots, N\} \) distinct, \( C_{i,j} \) be the \((i,j)\)-pairwise crossing numbers and \( C'_{i,j} = k_l C_{i,j} \), a multiple of \( C_{i,j} \), with \( l \in \{1, \ldots, \frac{N(N-1)}{2} \} \). Let \( a_m, b_m \in \{-1, 0, 1\} \) or \( a_m \) and \( b_m \) be coprime integers, such that all pairs \((a_m, b_m)\) are distinct. Then, the total crossing number of \( D_{W_0} \) is given by,

\[
C = \sum_{i<j=1}^{N} k_l C_{i,j}
\]

such that for all \( i \) and \( j \), the numbers \( k_l \) are the smallest positive integers such that the system of equations \((S_{min})\), given in Theorem 4.2, has a solution minimizing each \( a_m \) and \( b_m \), where \(|a_i' b_j' - a_j' b_i'| = k_l |a_i b_j - a_j b_i'|\), for all \( i, j, l \). We say that it is the minimal solution of \((S_{min})\).

Thus, taking into account the minimum number of simple closed curves representative of each set of threads, with respect to the two sets of cycles, we systematically obtain a minimal diagram of an associated weaving diagram, which can be described by this pair.

However, as mentioned previously, two different weaving diagrams can be defined by the same pair \((\Sigma', \Sigma)\) of sets of cycles, and generally do not have the same crossing number. This motivates the definition equivalent classes of weaving diagrams, by constructing a new parameter which would make it possible to distinguish between two of these diagrams. To do this, we were inspired by the concept of block matrix used in materials science, and more particularly in the textile industry, to classify the weaves with two sets of threads, perpendicular to each other. The mathematician E. Lucas used this concept to classify a particular class of such weaves, called satin, using arithmetic arguments [1]. We extend this idea to the other classes of weaves with two sets of threads, namely twill and basket square weaves, and generalize it to weaves with \( N > 2 \) sets of threads, by creating a set of crossing-matrices.
which is in direct correspondence with the set of crossing-cycles, and gives the configuration of the entanglement information of two of the sets of threads in a torus-diagram, also in terms of nearest neighbor vertices. This allows us to conclude that if two weaving diagrams having the same pair \((\Sigma', \Sigma)\) of sets of permutations have non-equivalent crossing matrices, then they are not equivalent (see Theorem 4.4 and Theorem 4.6).

We can deduce at this stage that if a weaving diagram is unique for a given pair \((\Sigma', \Sigma)\), then its total crossing number is given by Theorem 4.2. However, if it is not the case, then we find the next smallest solutions of the system of equations \((I_{\min})\), such that the associated crossing matrices are not equivalent to the ones given for the minimal solution of \((I_{\min})\). This will give the crossing number of the non-equivalent weaving diagram and allow the classification of our structure.

Finally, we apply our new results to classify Euclidean weaving diagrams for the cases of two and three sets of threads, and such that the crossing sequences allow at most four consecutive over- or under-crossings. Our classification tables, from page 24, specifies the number of sets of threads, the set of crossing sequences, the set of crossing matrices, the total crossing number and the writhe of each weaving diagram. A picture of the minimal torus-diagram is also given for each case.

This paper starts with some preliminaries, to recall the main definitions of weaves and weaving diagrams from our previous work. In Section 3, we explain our construction algorithm to build a periodic tiling and add information to each vertex using combinatorial arguments. Then, in Section 4, we discuss the concept of equivalent classes of weaving diagrams and characterize their crossing number in order to define a minimal torus-diagram. Finally in the last section, we apply our new systematic algorithm to build and classify simple square and kagome weaving diagrams by hand, according to their total crossing number.

2. Preliminaries: Weaves and their Planar Projections

A weave \(W\) has been defined as a 3-dimensional object, embedded in the ambient space \(\mathbb{X}^3 = \mathbb{E}^2 \times I\), with \(I = [-1, 1]\). In this paper, we only consider periodic Euclidean weaves. We start this subsection by recalling our main definitions [13].

\textbf{Definition 2.1.} A weave \(W\) in \(\mathbb{X}^3\) is an infinite union of threads, belonging to \(N \geq 2\) disjoint sets of threads \(T_1, \cdots, T_N\) of infinite cardinal,

- \(a\) thread \(t\) in \(\mathbb{X}^3\) is a set homeomorphic to \(\mathbb{R}\), and a strand \(s \subseteq t\) is any subset of \(t\).

- \(a\) crossing \(c\) is an intersection between the projections of two distinct threads onto \(\mathbb{E}^2\) with an over or under information.
Moreover, we say that two threads belong to two different sets of threads if their respective projections onto $\mathbb{E}^2$, also called threads, have distinct directions.

In this paper, we will only consider weaving diagrams whose crossings are intersections between threads from different sets, and will not allow any crossing, or twist, between two threads of a same set. By definition of a thread as being homeomorphic to a real line, we can say that every thread is homotopic to a unique geodesic thread. This notion of geodesic threads, or strands, will be used throughout this study.

**Definition 2.2.** Two threads in $\mathbb{E}^2$ have the same direction if their corresponding geodesic threads are related by a translation of the plane $\mathbb{E}^2$, up to isotopy.

![Weave and components](image)

**Figure 1.** A weave (left) and its components (right)

In these two definitions, by ”projection” we mean a planar projection on $\mathbb{E}^2$ by a map $\pi : \mathbb{R}^3 \rightarrow \mathbb{E}^2$, $(x,y,z) \mapsto (x,y,0)$. Thus, by projecting a weave $W$ on the Euclidean plane by $\pi$, we obtain a planar connected graph $W_0$ with all vertices having degree four, by definition of a crossing. If $Q$ is a point of intersection of $W_0$, then the inverse image $\pi^{-1}(Q)$ of $Q$ in $W$ has exactly two points: so, $Q$ is a double point of $W_0$. We say that $W_0$ is a regular projection. Then, by recording the extra information of which arc is over and which is under at each intersection point of $W_0$, we define an infinite periodic weaving diagram $D_{W_0}$, and we call a unit cell of such a periodic $D_{W_0}$ a torus-diagram $D_W$.

**Definition 2.3.** Let $i$ and $j$ be distinct integers such that $i, j \in \{1, \cdots, N\}$. Let $p$ and $q$ be two positive integers. Let $W$ be a weave with $N$ sets of threads and call $D_{W_0}$ its infinite weaving diagram. The crossing sequence $C_{i,j} = (p,q)$ associated to the sets of threads $T_i$ and $T_j$ of $W$, or $D_{W_0}$, is defined such that if one travels along any strand $s_i \in T_i$ of a unit cell of $D_{W_0}$ with the two following conditions satisfied:

1. starting from a crossing of $s_i$, such that at least one of two nearest neighbor crossings of $s_i$ with strands of $T_j$ has a different position (over: $+1$ or under: $-1$),
2. walking on this strand in the opposite direction of this different crossing.
Then there are cyclically $p$ crossings in which this strand is over the other strands of $T_j$, followed by $q$ crossings in which it is under. Moreover, if there is an integer $k \in \{1, \cdots, N\}$, $k \neq i$, such that any strand of $T_i$ is always over, or under, all the strands belonging to $T_k$, meaning that one of $p$ or $q$ is equal to zero, we say in this case that this strand, or thread, does not cross the set $T_k$, or that there are no crossings between $T_i$ and $T_k$. In this case, we use as convention the crossing sequence $(1, 0)$ or $(0, 1)$.

In this paper, we will consider that if $C_{i,j} = (p, q)$, then $C_{j,i} = (q, p)$, so only one of these two crossing sequences is sufficient to describe the entanglements of the structure. A generalization to more complex crossing sequences is under study.

By definition of the projection $W_0$ of a weave $W$ and of a crossing, we notice that $W_0$ is isotopic to a particular type of periodic topological four-regular polygonal tiling with convex polygons, meaning that each vertex has degree four. As defined by B. Grünbaum and G.C. Shephard in *Tilings and Patterns* (Chapter 4) [9], from a topological point of view, a tiling by irregular polygons is equivalent to a tiling by regular polygons if the application of a homeomorphism to such a tiling preserves the degree of the vertices and the number of adjacent and neighbors of each tile. We define this particular class of tilings related to weaves for our study.

**Definition 2.4.** A thread-tiling, composed of $N \geq 2$ sets of threads, is a periodic topological planar four-regular edge-to-edge tiling by convex polygons such that the following conditions are satisfied,

1. each edge of these polygons belongs to a unique thread and two adjacent edges belong to two threads from different sets;
2. each thread is an infinite union of such edges, such that it is homotopic to an infinite straight line.

Note that the Definition 2.4 implies that each vertex of a thread-tiling is the intersection of two threads belonging to two different sets, which is a key characteristic of the weaves.

**Figure 2. Weaving Diagram**
considered in this paper. Moreover, it is important to remember at this stage that the topological quality of our weaves allows some deformations of the structures, with respect to the Reidemeister Theorem for Weaves [6,13].

**Theorem 2.5. (Reidemeister Theorem for Weaves [6])** Two weaves in $\mathbb{R}^3$ are ambient isotopic if and only if their torus-diagrams can be obtained from each other by a sequence of Reidemeister moves $\Omega_1$, $\Omega_2$, and $\Omega_3$, isotopies on the surface of the torus, and torus-twists.

![Reidemeister moves](image)

**FIGURE 3. Reidemeister moves**

It is interesting to notice at that point that for the case geodesic weaving diagrams, which is the situation that we will consider in the following sections, the Reidemeister moves $\Omega_1$ (self-intersection of a thread) and $\Omega_2$ (intersection of two threads of a same set) do not occur according to Definition 2.1. It is therefore necessary to ensure that such deformations preserve the properties of the weave, that is to say the number of sets of threads and the crossing sequence of each set. To do this, we must first define how threads are blocked in a weave.

**Definition 2.6.** Let $D_{W_0}$ be a weaving diagram with $N$ sets of threads $T_1, \cdots, T_N$, and $i, j, k \in \{1, \cdots, N\}$, distinct positive integers. We say that a thread $t_i \in T_i$ is blocked by,

- another thread $t'_i$, if $t'_i$ is a nearest neighbor thread of $t_i$ such that $t'_i \in T_i$;
- a crossing $c = t_j \cap t_k$, with $t_j \in T_j$ and $t_k \in T_k$, if $c$ is a nearest neighbor crossing of $c' = t_i \cap t_j$ (resp. $c'' = t_i \cap t_k$) and such that the common thread $t_j$ (resp. $t_k$) has different position on these two crossings $c$ and $c'$ (resp. $c''$): if $t_j$ (resp. $t_k$) is over $t_k$ (resp. $t_j$) on $c$, then it is under $t_i$ on $c'$ (resp. $c''$) and conversely.

![Blocking thread and crossing](image)

**FIGURE 4. Blocking thread and crossing**
Remark 2.7. Let $D_{W0}$ be a weaving diagram with $N$ sets of threads $T_1, \ldots, T_N$, and $i, j, k \in \{1, \ldots, N\}$, distinct positive integers. Let $t_i \in T_i$, $t_j \in T_j$ and $t_k \in T_k$, and consider $T_j$ and $T_k$ interchangeable. If a thread $t_i$ can “jump” a crossing $c = t_j \cap t_k$, by a Reidemeister move of type $\Omega_3$, then it means that the geodesic isotopic to $t_i$ is not bounded by a crossing whose distance is inferior to its distance with $c$.

We can conclude at this stage that if a set of threads crosses another set, then there must be at least two crossings of opposite type between strands of these two sets on a torus-diagram (for each of the strands of these sets). If not, then a single crossing may suffice if it satisfies the previous condition with the sets that it crosses.

3. Construction of Combinatorial Weaving Diagrams

On the previous section, we saw that an infinite weaving diagram $D_{W0}$ can be described by a periodic thread-tiling, which encodes the number of sets of threads, together with an over or under information at each vertex, which is given by a set of crossing sequences. Now our objective is to build such a diagram by a systematic method using combinatorial arguments, which would be beneficial for the classification of our structures. This approach consists of two main steps. First, given a fixed number of set of threads, we construct a periodic thread-tiling from a set of vertex-cycles describing to which sets of threads each the nearest neighbor vertices of a given vertex belong. Second, we give an over or under information at each vertex of this tiling, with respect to another set of cycles consisting of symbols $+1$ and $-1$. The starting point is to assign an information to an arbitrary vertex of this tiling, and to walk along both threads which cross on it in order to attribute an information at each of the closest neighboring vertices according to these crossing-cycles. Then the process is repeated for each vertex having an information until we obtain a torus-diagram.

3.1. Thread-tiling constructed by vertex-cycles.

A thread-tiling has been defined as a particular type of periodic tiling by convex polygons, which implies its $k$-uniformity, with $k$ a positive integer, meaning that it has $k$ equivalence classes of vertices [9] (Chapter 2). Such a $k$-uniform topological tiling can therefore be described in a combinatorial way. Consider $N$ sets of geodesic threads $T_1, \ldots, T_N$ arbitrarily labelled, meaning that all threads are assumed to be straight lines.

Definition 3.1. Let $i, j, k \in \{1, \ldots, N\}$, be distinct positive integers, and $v_{i,j} = v_{j,i} = t_i \cap t_j$ be a vertex between two threads from distinct sets $t_i \in T_i$ and $t_j \in T_j$. Consider the cycle of minimal order consisting in vertices $\sigma_i^j = (v_{i,j_1} \ v_{i,j_2} \ \cdots \ v_{i,j_k})$, which indicates the order of the closest neighboring vertices on any thread $t_i \in T_i$, for all $i \in \{1, \ldots, N\}$. This means that by walking along such a thread, there exits a vertex $v_{i,j_1}$ whose nearest neighbor vertices will be $v_{i,j_2}$ on one side, and $v_{i,j_k}$ on the other side. Moreover, this thread will periodically intersects threads belonging to $T_{j_1}, T_{j_2}, \ldots, T_{j_k}$, up to cyclic or countercyclical permutations. We call $\sigma_i^j$ a vertex-cycle.
Then, we can prove that a periodic thread-tiling can be defined in terms of interdependent cycles consisting of such vertices.

Let us start with the case of two sets of threads $T_1$ and $T_2$, that is to say, $N = 2$.

**Example 3.2. (Square Tiling)** So, we only have one type of vertex $v_{1,2} = v_{2,1}$ since a thread of $T_1$ can only meet threads of $T_2$. This means that by walking along any thread of $T_1$, we can describe these intersections by a vertex-cycle composed of a single element, $\sigma'_1 = (v_{1,2}) = \sigma'_2$. Therefore, any vertex $v_{1,2}$ has for closest neighbors vertices of the same type $v_{1,2}$ on its two threads, which defines a topological thread-tiling with a single type of vertex, being the intersection of four topological squares, which is the definition of a topological square-tiling.

Now, we can study the case of three sets of threads $T_1$, $T_2$ and $T_3$, i.e. $N = 3$.

**Example 3.3. (Kagome Tiling)** In this situation, we can have three types of vertices $v_{1,2}$, $v_{1,3}$ and $v_{2,3}$. A possible set of vertex-cycles associated to these sets of threads is,

$$\sigma'_1 = (v_{1,2} \ v_{1,3}) ; \sigma'_2 = (v_{2,1} \ v_{2,3}) ; \sigma'_3 = (v_{3,1} \ v_{3,2}).$$

Then, we start from any arbitrary intersection, say $v_{1,2}$, which is a vertex between two geodesics not related by a planar translation $t_1^1 \in T_1$ and $t_1^2 \in T_2$, meaning that this vertex is the unique intersection between them (Figure 5, (A)). According to the definition of $\sigma'_1$, the two neighboring vertices closest to $v_{1,2}$ belonging to $t_1^1$ are of type $v_{1,3}$ (Figure 5, (B)). So at one of these vertices $v_{1,3}$ (resp. the other), the thread $t_1^1$ intersects a thread $t_1^3 \in T_3$ (resp. $t_3^2 \in T_3$), which is not related by a planar translation, or parallel, to the threads of $T_1$ and $T_2$. On the other hand, by definition of a set of threads, $t_2^1$ and $t_3^2$ are parallel. Next, notice that any vertex $v_{i,j}$ belongs to both $\sigma'_i$ and $\sigma'_j$ by definition, for any $i, j \in \{1, \cdots, N\}$. Thus, according to the definition of $\sigma'_2$, the two neighboring vertices closest to $v_{1,2}$ belonging to $t_2^1$ are of type $v_{2,3}$, therefore at one of these vertices (resp. the other), $t_2^1$ intersects a thread $t_2^3 \in T_3$ (resp. $t_3^j \in T_3$), with $i$ and $j$ natural numbers. It is immediate to prove that $i = 1$ and $j = 2$ (or conversely), using absurd logic and Thales’ Theorem (Figure 5, (C) and (D)). So any triple of three nearest neighbor vertices $(v_{1,2}, v_{1,3}, v_{2,3})$ forms a topological triangle. And finally, since each vertex is the intersection of at least two triangles and has degree four, we have the definition of a topological trihexagonal tiling, also called kagome tiling.

The situation can be generalized for any $N \geq 2$ and any set of cycles by the following proposition.

**Proposition 3.4. (Algorithm to construct tread-tilings)** Let $T_1, \cdots, T_N$ be $N$ sets of threads arbitrarily labelled. Then, a set of interdependent vertex-cycles $\Sigma' = \{\sigma'_1, \cdots, \sigma'_N\}$, generates a topological thread-tiling with $N$ sets of threads.

**Proof.** A vertex-cycle consists of vertices which are the intersection between two threads. Therefore, $\Sigma'$ generates a planar graph such that each of its vertices has degree four. Since each of these threads is homotopic to a unique geodesic, we can consider the case where all
FIGURE 5. Construction of the topological kagome-tiling. Starting from an arbitrary vertex \( v_{1,2} \), the closest neighboring vertices are \( v_{1,3} \) on \( t_1 \) according to \( \sigma'_1 \), and \( v_{2,3} \) on \( t_2 \) according to \( \sigma'_2 \). Each such vertex has degree four and is the intersection of two topological triangles.

the threads are straight lines for simplicity. Let \( i, j \in \{1, \cdots, N\} \) distinct. If we start from an arbitrary vertex \( v_{i,j} = t_i \cap t_j \), with \( t_i \in T_i \) and \( t_j \in T_j \) and walk along the thread \( t_i \) such that the next vertex given by \( \sigma'_i \) is \( v_{i,k} \), where \( k \in \{1, \cdots, N\} \), then walking along the thread \( t_k \in T_k \) leads to two scenarios,

1. if \( k = j \), then \( t_k \) is parallel to \( t_j \), and by reading the vertex-cycle \( \sigma'_j \) from \( v_{j,i} \), the next intersection can be either a similar vertex \( v_{j,k} \), where \( k \in \{1, \cdots, N\} \), then walking along the thread \( t_i \in T_i \) or \( t_m \in T_m \) will intersect \( t_j \) at some point, since two non-parallel lines intersect once. We thus obtain a polygon.

2. if \( k \neq j \), then \( t_k \) and \( t_j \) will intersect and we come to the same conclusion.

We therefore obtain a tiling which satisfies the definition of a thread-tiling.

Notice that even though this algorithm often allows the construction of a single topological thread-tiling, which is a regular projection of a three-dimensional weave, it does not mean that this projection is unique. A deformation of the weave such as described in Remark 2.7 can modify the configuration of the polygons around a vertex, and can therefore generate a different thread-tiling after projection.

3.2. Crossing information on a thread-tiling defined by cycles.

Consider a thread-tiling \( \mathcal{T} \) defined as in Proposition 3.4 by a set of \( N \) interdependent vertex-cycles \( \{\sigma'_1, \cdots, \sigma'_N\} \). Now, we would like to give an over or under information at each vertex of \( \mathcal{T} \). Since a vertex is an intersection between two distinct threads \( t \) and \( t' \), saying that \( t \) is over (resp. under) \( t' \) also means that \( t' \) is under (resp. over) \( t \) at this same crossing. This time we can use a set of independent cycles, associated to the \( N \) sets of threads of \( \mathcal{T} \), in order to indicate the information at each crossing, following the strategy of working with closest neighboring vertices of same type. We use the convention that an over crossing is given by the symbol \( +1 \), while an under crossing is given by the symbol \( -1 \). Let
$i,j \in \{1, \ldots , N\}$ and $\sigma_{i,j} = \{+1 \ \cdots \ +1 \ -1 \ \cdots \ -1\}$ be the cycle consisting of $p$ symbols $+1$ and $q$ symbols $-1$, $p$ and $q$ being positive integers. We can create a direct correspondence between this cycle $\sigma_{i,j}$ and the crossing sequence $C_{i,j} = (p,q)$ of Definition 2.3, which means that there are cyclically $p$ crossings in which the threads of $T_i$ are over the threads of $T_j$, followed by $q$ crossings in which they are under.

**Definition 3.5.** Let $i,j \in \{1, \ldots , N\}$ distinct, $m$ and $n$ be distinct strictly positive integers. For all $k \in 1, \ldots , n$, let $c_k = \pm 1$. Consider the cycle of minimal order $\sigma_{i,j} = (c_1 , \cdots , c_n)$, which gives the crossing sequence of $T_i$ and $T_j$: $(p,q)$ with $p = m$ such that $c_1 , \cdots , c_m = +1$ and $q = n - m +1$ such that $c_{m+1} , \cdots , c_n = -1$. If one of $p$ or $q$ equal zero, it means that the sets $T_i$ and $T_j$ do not cross since the information is the same at every crossing between these two sets. Otherwise, for any thread $t_i \in T_i$, there exists a crossing such that at least one of the two nearest neighbor crossings on $t_i$ with other threads of $T_j$ have a different position (over: +1 or under: −1). Then walking on $t_i$ in the opposite direction of this different crossing, there are periodically $p$ crossings in which this strand is over the other threads, followed by $q$ crossings in which it is under. We call $\sigma_{i,j}$ the crossing-cycle associated to the sets $T_i$ and $T_j$, from $T_i$.

Note that if $\sigma_{i,j}$ contains $p$ symbols $+1$ and $q$ symbols $-1$, this also implies that $\sigma_{j,i}$, the crossing-cycle associated to the sets $T_i$ and $T_j$, from $T_j$, contains $q$ symbols $+1$ and $p$ symbols $-1$, as for the corresponding crossing sequence. Therefore, we also need only one of $\sigma_{i,j}$ or $\sigma_{j,i}$ for each such pair of sets $T_i$ and $T_j$ crossing each other, to define the entanglements of the weave. We can conclude at this point that a weaving diagram can be constructed in a combinatorial way using a set $\Sigma'$ of vertex-cycles and a set $\Sigma$ of crossing-cycles. We will first use the previous kagome thread-tiling as an example to apply our methodology.

**Example 3.6. (Kagome Weaving)** In the previous subsection, we built a topological kagome thread-tiling $\mathcal{T}$ from the set of cycles $\Sigma' = \{\sigma'_1 , \sigma'_2 , \sigma'_3\}$, with $\sigma'_1 = (v_{1,2} \ v_{1,3})$, $\sigma'_2 = (v_{2,1} \ v_{2,3})$, and $\sigma'_3 = (v_{3,1} \ v_{3,2})$. Since $N = 3$, a set of three crossing-cycles

$\Sigma = \{\sigma_{1,2} , \sigma_{2,3} , \sigma_{3,1}\}$, would be necessary to construct a weaving diagram from this thread-tiling, assuming here that every set crosses the two others. We choose arbitrarily the following crossing-cycles,

$\sigma_{1,2} = \sigma_{2,3} = \sigma_{3,1} = \{+1 \ -1 \ -1\}$

We start by choosing an arbitrary vertex $v_{1,2} = v_{2,1}$ of $\mathcal{T}$, and we choose an arbitrary information, say the first in $\sigma_{1,2}$, meaning that the thread $t_1 \in T_1$ is over the thread $t_2 \in T_2$ at this vertex. Then, we can walk along $t_1$ in one way and give an under information to the two next vertices of type $v_{1,2}$, followed again by one over and two under, and so on, cyclically reading the cycle $\sigma_{1,2}$. Walking in the other direction of $t_1$ implies a countercyclical reading of $\sigma_{1,2}$. Then, returning to the first $v_{1,2}$, this time we can walk on the thread $t_2$ and put the information on the vertices of same type $v_{1,2}$, following the induced complementary crossing-cycle $\sigma_{2,1} = \{+1 \ +1 \ -1\}$. We repeat the process starting from any vertex of type $v_{1,2}$ having an information and walking along the associated threads of $T_1$ and $T_2$ which cross at these vertices, until all have an information. Then we start the procedure again from the
beginning, starting from an arbitrary vertex \( v_{1,3} = v_{3,1} \), and give an information to each of these vertices with respect to \( \sigma_{1,3} \) and its complementary crossing-permutation \( \sigma_{3,1} \). Finally, we do it with vertices of type \( v_{2,3} = v_{3,2} \) and the complementary crossing-permutations \( \sigma_{2,3} \) and \( \sigma_{3,2} \). By following this process for all the vertices of the thread-tiling, we generate a weaving diagram. Recall that we saw in a previous section that there exist weaves in which a set of threads could not cross all the sets of the weave. In this case, the cycle between two such non-crossing sets is given by a single element +1 or −1.

Once again, this idea can be generalized for any \( N \geq 2 \) and any set of crossing-cycles by the following Theorem.

**Theorem 3.7. (Construction of Combinatorial Weaving Diagrams)** Let \( i, j, k \in \{1, \cdots, N\} \) be distinct integers. Given a set of vertex-cycles \( \Sigma' = \{\sigma'_1, \cdots, \sigma'_N\} \), with \( \sigma'_i = (v_{i,j}, \cdots, v_{i,k}) \), and a set of crossing-cycles \( \Sigma = \{\sigma_{1,2}, \cdots, \sigma_{1,N}, \sigma_{2,3}, \cdots, \sigma_{2,N}, \cdots, \sigma_{N-1,N}\} \), with each element of \( \sigma_{i,j} \) in \( \pm 1 \), then, they generate a weaving diagram with \( N \) sets of threads, namely the planar projection of a weave, with \( \Sigma' \) and \( \Sigma \) as its sets of vertex-cycles and crossing-cycles, respectively. More precisely, if \( \mathcal{T} \) is the thread-tiling generated by \( \Sigma' \), then a weaving diagram is constructed by the following steps,

1. start from an arbitrary vertex \( v_{i,j} \in \mathcal{T}, i, j \in \{1, \cdots, N\} \) distinct;
2. assign to this vertex the information corresponding to the first element of \( \sigma_{i,j} \);
3. give an over or under information to each vertex of same type \( v_{i,j} \), while walking along the two threads which intersect at the first vertex \( v_{i,j} \), with respect to \( \sigma_{i,j} \) and the induced complementary crossing-cycle \( \sigma_{j,i} \);
4. repeat step (3) starting from any other vertex of type \( v_{i,j} \) having an information, until all \( v_{i,j} \in \mathcal{T} \) have an information.

**Proof.** This is a direct consequence of the definition of a weaving diagram and the previous Proposition 3.2. Indeed, a weaving diagram can be defined as a thread-tiling, generated by
a set of vertex-cycles $\Sigma'$, together with an over or under information at each of its vertices, given by a set of crossing-cycles $\Sigma'$ as follows,

1. start from an arbitrary vertex $v_{1,2}$ to which we assign the value $(c_{1})_{1,2}$;
2. walk along the thread $t_1 \in T_1$ of this vertex and assign the value to each next nearest neighbor vertex of type $v_{1,2}$ by reading $\sigma_{1,2}$ cyclically in one direction, and countercyclical in the other direction;
3. walk along the thread $t_2 \in T_2$ of this same vertex and assign the value to each next nearest neighbor vertex of type $v_{1,2}$ by reading $\sigma_{2,1}$ starting from an arbitrary $(c_{1})_{2,1} = -(c_{1})_{1,2}$, cyclically in one direction, and countercyclical in the other direction.
4. Repeat the process for all the other vertices $v_{1,i} \in T_1$, and $v_{2,i} \in T_2$, $i \in 3, \ldots, N$.
5. Repeat the previous steps for all the vertices of the other sets of threads $T_i$ without information.

This theorem guarantees the construction of a weaving diagram, but not its uniqueness. We will indeed see in the next section that two different weaving diagrams can be constructed from a same pair $(\Sigma', \Sigma)$.

4. Equivalence of Weaving Diagrams and Minimal Torus-Diagram

The construction method described in the previous section generates a systematic way of constructing a minimal diagram, meaning a torus-diagram with the minimal number of possible crossings associated to a given infinite weaving diagram. As we saw in our previous work, this is one of the main challenges for the classification of our structures [13]. To do this, let us recall that each element of the pair $(\Sigma', \Sigma)$, related to a given set of threads $T_i$, indicates the cyclic behaviors of every thread $t_i \in T_i$, with $i$ a positive integer. This means that if we start by walking along $t_i$ from an arbitrary crossing $v_{i,j}$, we can deduce from the two associated cycles $\sigma_{i,j}'$ and $\sigma_{i,j}$ the number of crossings $C$ that are required to read the crossing sequence $C_{i,j}$.

**Lemma 4.1. (Pairwise Crossing Number)** Let $D_{W0}$ be a weaving diagram with $N$ sets of threads defined by the pair $(\Sigma', \Sigma)$. Let $i, j, k \in \{1, \ldots, N\}$ be distinct integers, and $T_{i}, T_{j}$ be two sets of threads of $D_{W0}$. Then, to ensure the periodicity on a torus-diagram of $D_{W0}$, the minimum number of necessary crossings $c_{i,j}$ on a thread $t_i \in T_i$ and $t_j \in T_j$ is given by

$$c_{i,j} = \text{lcm}\left\{\left|v_{i,j}\right|_{\max} \frac{\prod_{k=1}^{\pm 1} o_{\sigma_{i,j}'} \left|v_{i,j}\right|_{\max}}{\gcd\left(\{o_{\sigma_{i,j}} \mid l \in \{1, \ldots, N\}\}\right)} \mid v_{i,j}\right\}_{\max} \frac{\prod_{k=1}^{\pm 1} o_{\sigma_{i,j}} \left|v_{i,j}\right|_{\max}}{\gcd\left(\{o_{\sigma_{i,j}} \mid l \in \{1, \ldots, N\}\}\right)}\}$$

Where for any pair of distinct integers $m, n \in \{1, \ldots, N\}$ $o_{\sigma_{m,n}}$ is the order of the cycle $\sigma_{m,n}$, and $\left|v_{i,j}\right|_{\max} = \max\{\left|v_{i,j}\right| \mid v_{i,j} \in \sigma_{i,j}' \text{ or } \sigma_{i,j}'\}$, with $|v_{i,j}|$ the number of vertices $v_{i,j}$. We call $c_{i,j}$ the $(i, j)$-pairwise crossing number.
Figure 7. Examples of minimum number of crossings for each set of threads a torus-diagram.

**Proof.** To read the crossing sequence between two sets of threads $T_i$ and $T_j$ which cross on a thread $t_i \in T_i$ or $t_j \in T_j$ and ensure the periodicity of the weaving diagram $D_{W_0}$, we must first consider the minimum number of vertices $v_{i,j}$ necessary on $t_i$ and $t_j$ to satisfy the definition of the thread-tiling given by $\Sigma'$. This number is the maximal number of times a vertex $v_{i,j}$ appears on $\sigma'_{i}$ or $\sigma'_{j}$, since these cycles have minimal length (order) according to Definition 3.1. This number is denoted by $|v_{i,j}|_{\text{max}}$. Then, we must multiply this number by the minimum number of crossings $c_{i,j}$ necessary to read the crossing-cycle $\sigma'_{i,j}$, that is to say its order, which is minimal according to Definition 3.3, divided by $|v_{i,j}|_{\text{max}}$. However, as on a thread $t_i$ or $t_j$ there are not only crossings of type $c_{i,j}$ when $|T| > 2$, we must also
multiply by the minimum number of necessary crossings of type \(c_{i,k}\) on \(t_i\) to read the other crossing-cycle \(\sigma_{i,k}\), and \(c_{j,k}\) on \(t_j\) to read the crossing cycle \(\sigma_{j,k}\), as defined for the pair \((i, j)\). Finally, the global minimality is ensure by dividing the total by the greatest common divisor of all the \(\sigma_{i,l}\) for \(t_i\), and all the \(\sigma_{j,l}\) for \(t_j\), to avoid repetitions of the crossings when these cycles are not coprime, which gives us the desired formula, considering that we must obtain the same number of crossing on each thread of \(T_i\) and \(T_j\) on the torus-diagram.

We can note in particular that if there exists \(i\) and \(k\) such that \(|v_{i,k}|\max = 0\), then this means that there is no crossings between the sets \(T_i\) and \(T_k\), nor vertices between these two sets in the corresponding thread-tiling. Therefore \(\sigma_{i,k}\) does not exist in this case and do not appear in the formula.

Then, since a torus-diagram is defined in term of simple closed curves [5] on a torus, this same number of crossings \(C_{i,j}\) must belong to the same representative simple closed curve for each set \(T_i\) and \(T_j\), to fulfill the definition of such a torus-diagram. We can therefore conclude that this number of crossings corresponds to the geometric intersection number between two simple closed curves, defined for free homotopy classes of simple closed curves, which is the minimum number of intersections between a representative of each set of threads [5]. In our case, since we are interested in the number of intersections between two simple closed curves on a torus defined by \((a_i, b_i)\) and \((a_j, b_j)\), with \(a_i, a_j, b_i\), and \(b_j\) four integers, this geometric intersection number is given by the formula:

\[ |a_i b_j - a_j b_i| = \mathcal{C}_{i,j} \]

We choose these four integers satisfying this relation such that their absolute value is minimal. However, this condition must be satisfied to all pairs of sets of threads, which leads to solving a system of such equation, as described in the following Theorem 4.2.

**Theorem 4.2. (Total Crossing Number)** Let \(D_{W_0}\) be a weaving diagram with \(N\) sets of threads defined by the pair \((\Sigma^\prime, \Sigma)\). Let \(i, j \in \{1, \ldots, N\}\) distinct, \(\mathcal{C}_{i,j}\) be the \((i, j)\)-pairwise crossing numbers and \(\mathcal{C}'_{i,j} = k_l \mathcal{C}_{i,j}\), a multiple of \(\mathcal{C}_{i,j}\), with \(l \in \{1, \ldots, \frac{N(N-1)}{2}\}\). Let \(a_m, b_m \in \{-1, 0, 1\}\) or \(a_m\) and \(b_m\) be coprime integers, such that all pairs \((a_m, b_m)\) are distinct. Then, the total crossing number of \(D_{W_0}\) is given by,

\[ \mathcal{C} = \sum_{i<j=1}^{N} k_l \mathcal{C}_{i,j} \]

such that for all \(i\) and \(j\), the numbers \(k_l\) are the smallest positive integers such that the system of equations \((\mathcal{S}_{\min})\) has a solution minimizing each \(a_m\) and \(b_m\), where \(|a'_i b'_j - a'_j b'_i| = k_l |a_i b_j - a_j b_i|\), for all \(i, j, l\).
We say that it is the minimal solution of \((S_{\text{min}})\).

**Proof.** Recall that a unit cell of such a periodic weaving diagram is a torus-diagram, meaning a set of simple closed curves which cross on the surface of a torus. Therefore, the geometric number of intersection between a simple closed curve of a set \(T_i\) and a simple closed curve of a set \(T_j\) defines the number of crossings of type \(c_{i,j}\) on a torus-diagram. Define such a simple closed curve of a set \(T_m\) by the pair \((a_m, b_m)\) such that \(a_m, b_m \in \{-1,0,1\}\) or \(a_m\) and \(b_m\) are coprime integers. Therefore, from Lemma 4.1, we can deduce that the minimal number of crossings between two sets \(T_i\) and \(T_j\) necessary to assure the periodicity in a torus-diagram is given by the equation

\[
|a_i b_j - a_j b_i| = C_{i,j}
\]

Moreover, since this condition must be satisfied for all distinct pairs of sets of threads \((T_i, T_j)\), we conclude that to find the minimal number of crossings on the torus-diagram, we must solve the system of equations \((S_{\text{min}})\) with the smallest positive integers \(k_l\) and such that all the integers \(a_m\) and \(b_m\) are as small as possible in absolute value. \(\square\)

Recall then that a simple closed curve corresponds to many parallel strands on the flat torus. To find the number of such parallel strands corresponding to a single \((a,b)\)-simple closed curve on the torus, with \(a,b \in \{-1,0,1\}\) or coprime integers, and such that none of these segments intersects a corner of the flat torus, we use the fact from [5], that for a square flat torus, \((|a| + |b|)\) parallel segments correspond to a \((a,b)\)-simple closed curve on the torus. We can indeed use a square unit cell for our flat torus since we have seen that all unit parallelograms representing a unit cell of a two-periodic weaving diagram contain the same number of crossings [13]. The classification tables of Section 5 below show examples of this process of constructing minimal diagrams for the cases of square and kagome weaving diagrams.

This means that by taking \(\alpha_i = \frac{a_i}{a_l} = \frac{b_i}{b_l}\) representative simple closed curves for each set of threads \(T_i\), characterized by the pairs \((a_l, b_l)\), and satisfying Theorem 4.2, we can construct a minimal diagram on a square unit cell, whose associated infinite weaving diagrams can be defined by the pair \((\Sigma', \Sigma)\). However, as discussed in the previous section, non-equivalent weaving diagrams can be characterized by this same pair. Moreover, their respective minimal diagram can have a different crossing numbers. We must therefore find a way to distinguish them.
Weaving diagrams are mainly characterized by a number of sets of threads as well as a set of crossing sequences, as seen in Definition 2.1. Therefore, the new method described in the two previous subsections enables a systematic construction of such structures according to these two principal parameters. The first set of cycles $\Sigma'$ determines the organizations of the different threads into a topological thread-tiling, while the set of crossing-cycles $\Sigma$ attributes the over or under information at each vertex. However, we noticed from some examples of woven frameworks in materials science that two distinct weaving diagrams, representing different three-dimensional networks, can be defined by the same pair of sets of cycles $(\Sigma', \Sigma)$. The most simple cases which illustrate this fact are the diagrams related to the basket weave $(2,2)$ and the twill weave $(2,2)$, showed in the Figure 8 below. Both diagrams have the same pair $(\Sigma', \Sigma)$ with $\Sigma' = \{(v_{1,2})\}$, and $\Sigma = \{(+1, +1, -1, -1)\}$, which corresponds to a square thread-tiling such that every thread of both sets are two times over the other threads, followed by two times under. Nevertheless, these two woven materials have different physical properties, and it would be interesting to make this distinction with our mathematical model.

This motivates the study of equivalence classes of weaving diagrams and the development of a new parameter $\Pi$, such that any weaving diagram constructed from the triple $(\Sigma', \Sigma, \Pi)$ would be unique, up to equivalence. To do this, we are inspired by the concept of block-matrix used in materials science to characterize the order and position of the crossings on a torus-diagram, with respect to the crossing sequences for the cases of square weaves, meaning when $N = 2$. More information about these block matrices can be found in [17]. Examples of such matrices are given on the Figure 8 for the basket and twill weave cases mentioned above. To read these matrices, we must consider that the rows represent the horizontal strands of a unit cell, the columns the vertical strands, and each box represent a crossing. Then, a grey box corresponds to a crossing where the horizontal strand is under the vertical strand, and conversely for a black box. Notice in the figure that the grey and black boxes alternate diagonally for the case of the twill weave, which is not the case on the basket weave. Moreover, it is immediate that a cyclic permutation of the rows or the columns of the matrix correspond to a translation of the unit cell on the periodic infinite diagram. Thus, we can consider that the elements of the matrix are in direct correspondence with the crossing information of the torus-diagram. These different types of organization of the crossings on a
flat torus-diagram justify the different physical properties of the corresponding woven materials.

Square weaves are organized into three classes, depending on their crossing sequences. A square weave with a crossing sequence \((p, q)\) and module \(m = p + q\) is called,

- A **basket weave** if \(p = q\), with plain configuration, including the particular case of the alternating weave \((1, 1)\).
- A **twill weave** if \(p = q\) or \(p \neq q\), with diagonal configuration.
- A **satin weave** if \(p = 1\) or \(q = 1\). The alternating weave \((1, 1)\) and the twill weave \((p, 1)\) or \((1, q)\) are particular cases of satin.

**THE THREE CLASSIC WEAVES**

![Figure 9. Basket, Twill and Satin.](image)

The mathematician Edouard Lucas studied the construction and equivalence of satin square weaves using arithmetic arguments: “The general problem of the construction of the armor of the regular satin comes down to placing in the cases of the square chessboard of \(m^2\) squares, pawns such that two of them are not in the same horizontal or vertical row, and so that, with respect to any one of these pawns (assuming the chessboard repeated endlessly in all directions), the other pawns are always placed in the same way.” (E. Lucas 1867), [1]

Then, the conditions for achieving a square satin weave of module \(m\) are satisfied by the choice of a shift \(a\), such that, \(a < m\) and \(gcd(a, m) = 1\), which allows the construction of two sequences,

- the sequence of column indices of the matrix \(x = 0, 1, 2, 3, \ldots, k, \ldots, m - 1\);
- the sequence of row indices of the matrix \(y = 0, a, 2a, \ldots, (m - 1)a\), calculated in \(\mathbb{Z}_p\).

Then, an arithmetic theorem [Gauss 1801] guarantees that the remainders \((\text{mod } m)\) of the numbers \(y\) thus obtained constitute a permutation of the numbers 0, 1, 2, \(\ldots, m - 1\). The black boxes are therefore colored on the block-matrix at the intersection of the column \(x = k\) and
the row \( y = ka \ (mod\ p) \). Lucas proved that the weaves with shift \( a, m - a, a' \) and \( m - a' \) are equivalent, with \( aa' = 1 \ mod(p) \).

These block-matrices were created and are very useful for woven materials consisting of two sets of perpendicular threads only, and therefore are not sufficient to create our parameter \( \Pi \). We must therefore generalize this concept to our weaving diagrams with \( |T| \geq 2 \) sets of threads, so that it can describe the topological order and position of the crossings for each pair of sets of threads that cross on a unit cell. The idea is to create a set of crossing matrices associated with a flat torus-diagram which would allow the distinction between structures characterized by the same pair \((\Sigma', \Sigma)\), and thus become a weaving invariant for the infinite diagram \( D_{W_0} \). Our concept of crossing matrices is directly related to the crossing sequences of a weaving diagram \( D_{W_0} \), which means that each matrix is associated to a pair of distinct sets of threads of the diagrams. The elements of a crossing matrix are the symbols \( +1 \) representing an over crossing, or \(-1\) representing an under crossing. In particular, we will work with square matrices associated with such a pair \((T_i, T_j)\), having size \( m = p + q \), and associated to the crossing sequence \( C_{i,j} = (p, q) \). This implies that a torus-diagram \( D_W \) will contain \( C_{i,j} \) simple closed curves for each set \( T_i \) and \( T_j \). Notice that in general, such a torus-diagram is not a minimal. We will see that once again only one matrix for each pair will suffice, so for a weaving diagram with \( N \) sets of threads, we can define \( \sum_{k=1}^{N-1} 2k \) matrices for \( D_W \), such that each matrix encodes the crossing information between two sets of threads, from the point of view of one of them. This means that at an arbitrary crossing between two strands \( s_1 \in T_1 \) and \( s_2 \in T_1 \), with \( T_1 \) and \( T_2 \) two disjoint sets of threads, \( s_1 \) is over (resp. under) \( s_2 \), if we analyze the position of the strands of \( T_1 \) with respect to the strands of \( T_2 \), or conversely that \( s_2 \) is under (resp. over) \( s_1 \), if we analyze this time the position of the strands of \( T_2 \) with respect to the strands of \( T_1 \).

We first start to extend Lucas’s result to the two other classes of square-weaves, giving the definition of the crossing-matrices for each cases.

**Definition 4.3.** Let \( W \) be a square weave with \( N = 2 \) directions. Let \( x \) represents the column indices of its crossing matrix and \( y \) the row indices. Let \( m = p + q \) be the module of the weave, with \( p \) and \( q \) be strictly positive integers.

- A satin weave with crossing sequence \((p,1)\) or \((1,p)\) is defined by a \((m \times m)\) satin crossing matrix \( M = (m_{x,y})_{0 \leq x,y \leq m-1} \), with \( m = p + 1 \), and such that the symbols \(+1\) are positioned at the element \( m_{x,y} \) satisfying the system
  \[
  \begin{align*}
  x &= k, \quad \text{with } k \in \{0,1,\ldots,p\} \\
  y &= ak \ mod (m), \quad \text{with } a < m \text{ fixed, } \gcd(a,m) = 1.
  \end{align*}
  \]

- A basket weave with crossing sequence \((p,p)\) is defined by a \((m \times m)\) basket crossing matrix \( M = (m_{x,y})_{0 \leq x,y \leq m-1} \), with \( m = 2p \), and such that the symbols \(+1\) are positioned at the element \( m_{x,y} \) satisfying the system
  \[
  \begin{align*}
  x_k &= k, \quad \text{with } k \in \{0,1,\ldots,2p-1\} \\
  y_k,j &= y_{k,j-1} \pm 1, \quad \text{if } k \neq p \ mod (m), \text{ or} \\
  y_k,j &= y_{k,j-1} \pm p, \quad \text{otherwise, } j \in \{1,\ldots,p-1\}.
  \end{align*}
  \]
  with the first column given by,
\[
\begin{align*}
&\begin{aligned}
x_0 &= 0 \\
y_{0,0} &= 0 \text{ and } y_{0,j} = y_{0,j-1} \pm 1, j \in \{1, \cdots, p-1\}.
\end{aligned}
\end{align*}
\]

- A twill weave with crossing sequence \((p, q)\) is defined by a \((m \times m)\) twill crossing matrix \(M = (m_{x,y})_{0 \leq x,y \leq m-1}\), with \(m = p + q\), and such that the symbols \(+1\) are positioned at the element \(m_{x,y}\) satisfying the system
\[
\begin{align*}
x &= k, \text{ with } k \in 0, 1, \cdots, p + q - 1 \\
y &= ak \pm i \text{ mod}(m), \text{ with } a = \pm 1 \text{ fixed}, i \in \{1, \cdots, p-1\}.
\end{align*}
\]

We can now characterize the equivalent classes for all the square weaves having a crossing sequence of type \((p, q)\).

**Theorem 4.4.** Consider the square weaving diagrams with \(N = 2\) directions and with crossing matrices of module \(m\) and shift \(a\).

1. The satin weaving diagrams with shift equal to \(a, m - a, a'\) and \(m - a'\), such that \(ad' = 1 \text{ mod}(m)\), are equivalent \([1]\).
2. For each crossing sequence, there exists a unique basket weaving diagrams and a unique twill weaving diagrams.
3. A weaving diagram with crossing sequence \((p, p)\) is either a basket or a twill diagram and both are non equivalent.

**Proof.** The first point has already been proven by E. Lucas, in 1867. Let us now prove the other two points simultaneously. For a crossing sequence \((p, q)\), we must have \(p \geq 2\) consecutive symbols \(+1\) in every row and column of the crossing matrix. By calling \(x\) the column indices and \(y\) the row indices as before, the condition is satisfied on each column if by putting a symbol \(+1\) at an arbitrary element, the \(p - 1\) consecutive element the closer to the top of it (resp. at the bottom) are also \(+1\) symbols. Thus for a column \(x = k\), the \(p\) elements with symbol \(+1\) must be given by the equations \(y = ak + i \text{ mod}(p)\) or \(y = ak - i \text{ mod}(p)\), with \(i \in \{0, 1, \cdots, p-1\}\) and \(a < m\) integer. Then, to satisfy the same condition on each row, we only have two options,

- if \(m_0,0 = m_{0,1} = m_{1,0} = m_{1,1} = +1\) or \(m_0,0 = m_{0,m-1} = m_{1,0} = m_{1,m-1} = +1\), then the \(p\) consecutive columns must be the same, and satisfy \(x = k\) and \(y_k = y_{k-1}\), unless if \(k = p \text{ mod}(m)\), then \(y_k = y_{k-1} \pm p \text{ mod}(m)\) because of the periodicity of the crossing sequence. It also implies that the \(m - p\) next consecutive columns must also be the same for similar reasons, which implies that \(p = q\). Indeed, if \(p > q\), then because the \(p\) first columns are equal, it would implies that there exists a row in which we have \(p\) consecutive symbol \(-1\), which is a contradiction. This is the definition of a basket weave and therefore it is unique.

- if \(m_0,0 \neq m_{1,0}\) and \(m_0,0 = m_{0,1} = m_{1,1} = +1\) or \(m_0,0 = m_{0,m-1} = m_{1,0} = m_{1,m-1} = +1\). Then on each row, the \(p\) consecutive symbol \(+1\) must organized in a diagonal configuration, meaning that the first symbol \(+1\) on each columns is shifted of one element from the first symbol of its closest left column: if \(m_{i-1,j}\) is the first \(+1\) symbol of the column \(i - 1\), then either \(m_{i,j+1}\) or \(m_{i,j-1}\) is the first \(+1\) symbol of the column \(i\). This is the definition of a twill weave and the two possible matrices obtained mirror
one another, up to a cyclic permutation of the columns, and here represent the same weave.

Finally, it is noted that a basket weaving crossing matrix have rank 1 while a twill weaving crossing matrix have rank \( p \), which confirms the non equivalence of the two structures. Notice that we consider that two matrices are equivalent up to cyclic or countercyclical permutations of rows and columns here.

Remark 4.5. According to the definition of a crossing-matrix, we can interpret that the shift \( a \) in a crossing-matrix means that starting from a symbol \(+1\) in an arbitrary column, we go up \( a \) times and to the right once from it to have the first symbol \(+1\) on the next column on its right. Therefore, the condition of having \( p \) consecutive symbols \(+1\) on each row is only given by a shift \( a = \pm 1 \mod(p) \). However, the other shifts generate new type of weavings such that if one set of threads has a crossing sequence \((p,q)\), the other set does not have for crossing sequence \((q,p)\). This could be interesting to study in a future work.

We now want to generalize this approach to the cases of weaving diagrams with \( N > 2 \) sets of threads. Therefore, for each distinct pair of sets of threads \((T_i, T_j)\), considering that \((T_i, T_j) = (T_j, T_i)\), we will have a crossing-matrix \( M_{i,j} \), as defined above in Definition 4.3, which would define the entanglement information of the strands of two distinct sets of threads \( T_i \) and \( T_j \), from the viewpoint of \( T_i \). We can also deduce that the information at the crossing between \( s_i \) and \( s_j \) must be opposite on the two matrices \( M_{i,j} \) and \( M_{j,i} \), describing the position of the two sets of threads considered, since if \( s_i \) is over \( s_j \) at a crossing, then the corresponding element of the matrix from the viewpoint of \( T_i \) will be \(+1\), while it will be \(-1\) from the point of view of \( T_j \), which explains why only one of the two matrices associated to a pair of sets of threads is enough to characterize our structures. The crossing-matrix \( M_{j/i} \) is in fact directly deduced by transposing \( M_{i/j} \) and changing the \(+1\) values into \(-1\) and conversely: \( M_{j/i} = -(M_{i/j})^T \). See Figure 10 below for an illustration using a kagome weaving diagram.

Twill Kagome Weave \((3,2)\)_3: module \( m = 5 \) with shift \( a = 1 \)

\[
\begin{align*}
\sigma_1' &= (v_{1,2} \ v_{1,3}) ; \sigma_2' &= (v_{2,1} \ v_{2,3}) ; \sigma_3' &= (v_{3,1} \ v_{3,2}) \\
\sigma_{12} - \sigma_{23} &= \sigma_{13} = (+1 \ +1 \ +1 \ -1 \ -1)
\end{align*}
\]

In Figure 10 below we illustrate how this concept of crossing-matrices, defined on Definition 4.3, makes it possible to distinguish two weaving diagrams characterized by the same pair \((\Sigma', \Sigma)\), by assigning them a fixed
sequence of crossing matrices $\Pi = \{M_{i,j} \mid i, j \in \{1, \ldots, N\}\}$, with $M_{i,j}$ the crossing matrix defined above, and such that the sets of threads are indexed on $W_0$ in order to compare the strands having the same direction on the two diagrams. So now, it is possible to characterize the notion of equivalence classes for weaving diagrams of $\mathbb{E}^2$, using the triple $(\Sigma', \Sigma, \Pi)$.

Recall that the Reidemeister Theorem for Weaves given in Theorem 2.5 define the notion of equivalent weaves through the equivalence of their corresponding torus-diagrams, using the Reidemeister moves and torus-twists.

Theorem 4.6. Let $D_{W_1}$ and $D_{W_2}$ be two torus-diagrams with $N > 2$ sets of threads, indexed such that their regular projections are identical, and constructed from the same pair of sets of cycles $(\Sigma', \Sigma)$. Then, they are equivalent if and only if their crossing-matrices are pairwise equivalent, up to the indices, meaning that if at least one of the two conditions is satisfied:

- all the matrices of $D_{W_2}$ are equivalent to the respective matrices of $D_{W_1}$, up to a same cyclic or countercyclical permutations of all the rows and/or columns;
- all the matrices of $D_{W_2}$ are equivalent to the respective matrices of $D_{W_1}$, up to a same equivalent shift $a$.

Proof. Without loss of generality, we assume that our weaving diagrams are geodesic. Then, by the definition of geodesic weaving diagrams, Reidemeister moves of type I and type II do not occur. For a Reidemeister move of type III, we consider three threads from three different sets. However, recall that each crossing matrix is obtained from the crossing information of a pair of sets of threads only. This means that a Reidemeister move of type III does not change any of the crossing matrices. Now, let $p$ be the intersection point of preferred meridian-longitude pair $(\mu, \lambda)$ such that the flat torus-diagram is obtained by cutting along this pair. Let $p' \neq p$ be a point on the longitude $\lambda$ and $p'' \neq p$ be a point on the meridian $\mu$. Then we obtain new preferred meridian-longitude pairs $(\mu, \lambda')$ and $(\mu', \lambda)$. The operation of cyclical/countercyclical permutations of the rows of the crossing matrices corresponds to the change of the pair $(\mu, \lambda)$ into $(\mu', \lambda)$ for cutting the flat diagram, and the operation of cyclical/countercyclical permutations of the columns of the crossing matrix corresponds to change the pair $(\mu, \lambda)$ into $(\mu, \lambda')$. Then, recall that a torus twist can be characterized by a matrix of $SL_2(\mathbb{Z})$, as seen in [13], and notice that two matrices having equivalent shift $a$ are obtained from each other by a rotation of $\frac{\pi}{4}$ or $\frac{\pi}{2}$ degree, which can be described by a rotational matrix belonging to $SL_2(\mathbb{Z})$ too, and therefore a torus twist, which conclude our proof.

 Remark 4.7. The case of crossing matrices of $D_{W_2}$, corresponding to cyclic or countercyclical permutations of all the rows and/or columns of the set of matrices of $D_{W_1}$ is immediate, since the flat torus-diagram of $D_{W_2}$ constructed from such a set of crossing matrices will correspond to a translation of the torus-diagram of $D_{W_1}$ on the infinite diagram. Indeed, on the proof above, we recognize that a permutation of the rows of the matrices corresponds to a vertical translation on the flat torus-diagram, while a permutation of the columns corresponds to a horizontal translation. So any translation of a unit cell in the infinite diagram can be described as a composition of these two translations. Next, the case of equivalent shift is a generalization of Theorem 4.4, since two matrices with equivalent shifts are related by a rotation. Therefore, by applying such a same transformation to all the matrices of $D_{W_1}$
to obtain the matrices of $D_{W_2}$, we define an involution, which justifies the equivalence. However, for both cases, if one of the crossing matrix of $D_{W_2}$ has a different permutation or shift than the others matrices, then $D_{W_1}$ and $D_{W_2}$ are not equivalent. Indeed, there would exist of thread $t_i$ of $D_{W_2}$ for which the order of all its crossings will be different than its representative in $D_{W_1}$. This means that there would exist a crossing $c_{i,j}$, in $D_{W_2}$, with $T_i$ and $T_j$ two sets of threads, such that by walking on the thread $t_i$, one of the nearest neighboring crossing $c_{i,k}$, of $c_{i,j}$, with a different set of threads $T_k$ will have a different type than the corresponding one in $D_{W_1}$. So the two weaving-diagrams cannot be superimposed and are therefore not equivalent.

**Twill Kagome Weave (3,2)$_3$ : module $m = 5$**

![Diagram of Twill Kagome Weave (3,2)$_3$](image)

**Figure 11. Non Equivalent Kagome Matrices**

One of the other great interests of these crossing matrices for the study of equivalence classes of weaving diagrams is that by defining any arbitrary weaving diagram by a triple $(\Sigma', \Sigma, \Pi)$, we can find non equivalent structures with the same pair $(\Sigma', \Sigma)$ just by modifying the shift or by doing a cyclic or countercyclical permutation of the columns or the rows of one or two of the matrices of $\Pi$, with respect the related crossing permutations. This is very useful for the construction an classification of our diagrams.

At this point, we can conclude that if a set of crossing matrices, corresponding to a minimal diagram constructed by the Theorem 4.2 is unique up to equivalence, then the associated weaving diagram characterized by the pair $(\Sigma', \Sigma)$ has its crossing number given by the number of crossings of this torus-diagram and is unique for this pair. However, if a non-equivalent crossing matrix is found, it means that there exist a non-equivalent weaving diagram defined by $(\Sigma', \Sigma)$, and with a different crossing number, if this number is not equal to the sum of the elements of the crossing matrices, meaning that its minimal diagram cannot be constructed using the minimal solution of the system $(S_{min})$ of Theorem 4.2. Therefore, to construct the associated minimal diagram of such a non-equivalent weaving diagram, we must solve the system $(S_{min})$ with the next smallest solutions which generate this new weaving diagrams.

We can illustrate this method with the same example of basket and twill square (2,2) weaving diagrams.
**Example 4.8.** The minimal diagram for the twill case is obtained with one representative simple closed curve for each set of twist. Recall that each of these curves must have four crossings, two over and two under, so its crossing number is $C = 4$. This can be done with a $(2,1)$-curves for one set and a $(-2,1)$-curves for the other set. For the basket case, we find that $k_{1,2} = 2$ is the next smallest solution of $(s_{\text{min}})$, with the four integers satisfying this relation and such that their absolute value is minimal, given by two $(1,1)$-curves for one set and two $(-1,1)$-curves for the other set. Finally since we can organize the crossings on an associated unit cell with eight crossings, such that the corresponding crossing matrix is not equivalent to the one of the twill case, we can confirm that the crossing number of the plain square $(2,2)$ weaving diagram is eight.

5. **Classification of Euclidean Weaving Diagrams**

The purpose of this section is to apply our new systematic algorithm to build and classify some simple square and kagome weaving diagrams by hand.
### Classification Square Weaving Diagrams: $|T| = 2$

| Set of Crossing Sequences | Crossing number (Writhe) | Minimal Diagram | Set of Crossing Matrices | Matrices | Number of Crossings by S.C.C. | Number of S.C.C. for Each Set on the Minimal Diagram | Name |
|---------------------------|-------------------------|----------------|--------------------------|----------|-----------------------------|-----------------------------------------------------|------|
| (4,1)                     | 5                       | ![Minimal Diagram for (4,1) with 5 Crossings] | $\begin{pmatrix} 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & 1 \end{pmatrix}$ | Rank = 5  
  “Diagonal configuration” | 5 | 1 | Ex: (1,1) and (-3,2) | Twill Square Weaving (4,1) |
| (4,1)                     | 25                      | ![Minimal Diagram for (4,1) with 25 Crossings] | $\begin{pmatrix} 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & 1 \end{pmatrix}$ | Rank = 5  
  “Diagonal configuration” | 5 | 5 | Ex: (1,1) and (-1,1) | Satin Square Weaving (4,1) |
| (4,2)                     | 6                       | ![Minimal Diagram for (4,1) with 6 Crossings] | $\begin{pmatrix} 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & 1 \end{pmatrix}$ | Rank = 5  
  “Diagonal configuration” | 6 | 1 | Ex: (3,1) and (-3,1) | Twill Square Weaving (4,2) |
| (4,3)                     | 7                       | ![Minimal Diagram for (4,1) with 7 Crossings] | $\begin{pmatrix} 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & 1 \end{pmatrix}$ | Rank = 7  
  “Diagonal configuration” | 7 | 1 | Ex: (4,1) and (-3,1) | Twill Square Weaving (4,3) |
| (4,4)                     | 8                       | ![Minimal Diagram for (4,1) with 8 Crossings] | $\begin{pmatrix} 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & 1 \end{pmatrix}$ | Rank = 4  
  “Diagonal configuration” | 5 | 1 | Ex: (2,1) and (-3,2) | Twill Square Weaving (4,4) |
| (4,4)                     | 32                      | ![Minimal Diagram for (4,1) with 32 Crossings] | $\begin{pmatrix} 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & 1 \end{pmatrix}$ | Rank = 1  
  “Diagonal configuration” | 5 | 4 | Ex: (1,1) and (-1,1) | Plain Square Weaving (4,4) |

**Figure 13.** Table Classification Square Weavings 2
### CLASSIFICATION KAGOME WEAVING DIAGRAMS: |T| = 3 (each set meet the 2 others)

| Set of Crossing Sequences | Crossing number (Wrihe) | Minimal Diagram | Set of Crossing Matrices | Matrices | Number of Crossings by S.C.C. (pair) | Number of S.C.C. for Each Set on the Minimal Diagram | Name |
|---------------------------|-------------------------|-----------------|--------------------------|----------|--------------------------------------|---------------------------------------------------------------|------|
| ((1,0), (1,0), (1,0))     | 3 (1)                  | ![Diagram](1)   | [1]:[1]:[1]             | Rank = 1 | ≥ 2                                  | 1                                                             | Kagome Weaving (1,0) |
| ((1,0), (1,0), (1,1))     | 6 (0)                  | ![Diagram](2)   | [1]:[1]: [−1]  | Rank = 1 | ≥ 2 for (1,0)                        | 1                                                             | Kagome Weaving (1,0), (1,1) |
| ((1,0), (1,1), (1,1))     | 12 (4)                 | ![Diagram](3)   | [1]:[−1]:[+1]         | Rank = 1 | ≥ 2 for (1,0)                        | 2                                                             | Kagome Weaving (1,0), (1,1) |
| ((1,0), (1,1), (2,1))     | 12 (0)                 | ![Diagram](4)   | [1]:[+1]: [−1]         | Rank = 1 | ≥ 2 for (1,0)                        | 2                                                             | Kagome Weaving (1,0), (2,1) |
| ((1,0), (1,1), (2,1))     | 9 (1)                  | ![Diagram](5)   | [1]:[+1]: [−1]         | Rank = 1 | ≥ 4                                  | 2                                                             | Kagome Weaving (1,0), (2,1) |
| ((1,0), (2,1), (2,1))     | 27 (9)                 | ![Diagram](6)   | [1]:[+1]: [−1]         | Rank = 1 | ≥ 2 for (1,0)                        | 3                                                             | Kagome Weaving (1,0), (2,1) |
| ((2,1), (2,1), (2,1))     | 27 (3)                 | ![Diagram](7)   | [1]:[+1]: [−1]         | Rank = 1 | ≥ 4                                  | 3                                                             | Kagome Weaving (2,1) |

**Figure 14. Table Classification Kagome Weavings**

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