A NOTE ON THE PARTIAL DATA INVERSE PROBLEM FOR A NONLINEAR MAGNETIC SCHRÖDINGER OPERATOR ON RIEMANN SURFACE

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Abstract. We recover a nonlinear magnetic Schrödinger potential from measurement on an arbitrarily small open subset of the boundary on a compact Riemann surface. We assume that the magnetic potential satisfies suitable analytic properties, in which case the recovery can be obtained by a linearisation argument. The proof relies on the complex analytic methods introduced in [15].

Keywords: Calderón Problem, Nonlinear magnetic Schrödinger operator, Riemann surface

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1. Introduction

Let \((M, g)\) be a compact Riemann surface with smooth boundary \(\partial M\). For a complex parameter \(\zeta \in \mathbb{C}\) and some \(\alpha > 0\), we let \((X(\zeta, \cdot), V(\zeta, \cdot))\) be in \(C^{2,\alpha}(T^*M) \times C^{1,\alpha}(M)\) depending analytically in \(\zeta\), in the sense that

\[
X(\zeta, p) \overset{\text{def}}{=} \sum_{k \geq 0} \frac{\zeta^k}{k!} X_k(p) \quad \text{and} \quad V(\zeta, p) \overset{\text{def}}{=} \sum_{k \geq 0} \frac{\zeta^k}{k!} V_k(p)
\]

with convergence respectively in the \(C^{2,\alpha}(T^*M)\) and \(C^{1,\alpha}(M)\) topologies and \(p \in M\). For each \(k \geq 0\), the pair \((X_k, V_k)\) belongs to \(C^{2,\alpha}(T^*M) \times C^{1,\alpha}(M)\), and we may consider a differential operator

\[
L_{X,V}u \overset{\text{def}}{=} -*(d* + iX(u, p) \wedge *)(d + iX(u, p))u + +V(u, p)
\]

where \(*\) denotes the Hodge star operator on \((M, g)\). One refers to \(L_{X,V}\) as the magnetic Schrödinger operator with magnetic potential \((X, V)\). It is known from [4] that if \(X_0 = 0\) and \(V_0 = V_1 = 0\), then there exists \(\delta > 0\) small and a constant \(C > 0\) such that if we set

\[
U_\delta \overset{\text{def}}{=} \{ f \in C^{2,\alpha}(\partial M) / \|f\|_{C^{2,\alpha}(\partial M)} < \delta \} \quad \text{and}
\]

\[
V_{C,\delta} \overset{\text{def}}{=} \{ u \in C^{2,\alpha}(M) / \|u\|_{C^{2,\alpha}(M)} < C\delta \},
\]

then for every \(f \in U_\delta\) there exists \(u_f \in V_{C,\delta}\) with \(u_f|_{\partial M} = f\) such that \(L_{X,V}u_f = 0\). As such we may define the Dirichlet-Neumann map by

\[
\Lambda_{X,V} : \begin{cases} 
U_\delta \to \gamma \nabla^X_{\nu}(V_{C,\delta}), \\
\quad f \mapsto \nabla^X_{\nu} u_{f|_{\partial M}},
\end{cases}
\]

where \(\nabla^X u = du + iuX(u, p)\), \(\nabla^X_{\nu} u = \nabla^X u(\nu)\) by definition where \(\nu\) is the outward point unit normal, and \(\gamma\) is the boundary trace map. The inverse problem asks whether \(\Lambda_{X,V}\) uniquely determines \((X, V)\). However, in practice one might only have access to
measurements on a small portion of the boundary. Hence if \( \Gamma \subset \partial M \) is an arbitrary non-empty open subset, one could asks whether the partial data Dirichlet-Neumann map
\[
\Lambda^\Gamma_{X,V} : \left\{ \begin{array}{l}
U_\delta \cap \mathcal{E}'(\Gamma) \to \gamma \nabla^X_\nu (V_{C,\delta}) \cap \mathcal{E}'(\Gamma) \\
f \mapsto \nabla^X_\nu u_f|_\Gamma
\end{array} \right.
\]
determines \((X, V)\). In this article we will give positive answer to this inverse problem, the precise formulation of which is the following:

**Theorem 1.1.** Let \((X_j(\zeta, \cdot), V_j(\zeta, \cdot)) \in C^{2,\alpha}(T^* M) \times C^{1,\alpha}(M)\), \(j = 1, 2\) be pairs depending holomorphically on a complex parameter \(\zeta \in \mathbb{C}\) in the sense that
\[
X_j(\zeta, p) \overset{\text{def}}{=} \sum_{k \geq 0} \frac{\zeta^k}{k!} X_{j,k}(p) \quad \text{and} \quad V_j(\zeta, p) \overset{\text{def}}{=} \sum_{k \geq 0} \frac{\zeta^k}{k!} V_{j,k}(p).
\]
Assume that \(X_{j,0} = 0\) and \(V_{j,0} = V_{j,1} = 0\). If \(\Lambda^\Gamma_{X_1,V_1} = \Lambda^\Gamma_{X_2,V_2}\) where \(\Gamma\) is an arbitrarily small non-empty open subset of \(\partial M\), then \((X_1, V_1) = (X_2, V_2)\).

The Calderón problem for elliptic equations has been intensively studied since the groundbreaking result \([13]\) of Sylvester-Uhlmann. Since them, much attentions have been paid to the study of Calderón problem for the linear magnetic Schrödinger operator. See \([11, 10, 1, 3, 12]\) and in particular \([16, 14]\) for results on Riemann surface. The list is not exhaustive. In the case where \(X = 0\) and a nonlinear potential \(V(u, p)\) is present. Lassas-Liimatainen-Lin-Salo observed in \([6]\) that recovery is possible provided \(V(u, p) = u^k V(p)\), \(k \geq 2\) is of power type and the underlying geometry is Euclidean with dimension \(n \geq 2\). By linearising the Dirichlet-Neumann map, they showed that the recovery of \(V\) amounts to solving the linearised Calderón problem. By taking successive linearisations, Krupchyk-Uhlmann in \([5]\) gave positive answer to the partial data inverse problem of a nonlinear \(V(u, p)\) satisfying condition \([11]\) with \(V_0 = V_1 = 0\). The same was done on Stein manifolds with Kähler metric by Ma-Tzou in \([9]\). If in addition \(V_2\) is known, then Feizmohammadi-Oksanen showed in \([2]\) that the full data inverse problem for \(V(u, p)\) can be solved on conformally anisotropic manifolds (CTA) with admissible transversed manifolds.

In the case where \(X \neq 0\) and is nonlinear, the boundary integral identities obtained after linearising the Dirichlet-Neumann maps involve first order derivatives which is different from the standard linearised Calderón problem. Under the condition of vanishing lower order terms, Lai-Zhou in their recent article \([7]\) recovered \((X, V)\) from partial data in the Euclidean settings for all dimensions \(n \geq 2\). In another result \([4]\), Krupchyk-Uhlmann solved the relevant full data problem on CTA manifolds of dimensions \(n \geq 3\). Since the Calderón problem for the linear Schrödinger equation in two dimensions has historically been reliant on complex analytic techniques (see \([15]\) and the references therein), it is encouraging to see whether these powerful methods can be applied to the current problem and in particular fill the gap for the case of two dimensional non-Euclidean geometry. As it turns out, in this case the key obstacle encountered by \([4, 7]\) is resolved by an incredibly short argument.

### 2. Recovery from Integral Identities

In this section we state the main recovery results to be used in the proof of Theorem 1.1.

**Proposition 2.1.** Let \(X,Q \in C^{1,\alpha}(T^* M)\) be such that
\[
\int_M u_1 u_2 X \wedge * du_3 = \int_M u_1 u_2 u_3 Q
\]
for all harmonic functions \(u_1, u_2\) and \(u_3\) which vanish on \(\partial M \setminus \Gamma\), then \(X = Q = 0\).

Our starting point is the following result on the linearised Calderón problem.
Proposition 2.2. Let \( f \in C^{1,\alpha}(M) \) be such that
\[
\int_M u_1 u_2 f \, dv_g = 0
\]
for all harmonic functions \( u_1, u_2 \) which vanish on \( \partial M \setminus \Gamma \), then \( f = 0 \).

We refer to [15, Proposition 5.1] for a proof of this claim. Applying the above to (4), we deduce immediately
\[ X \wedge \star du = uQ \]
for all harmonic function \( u \) which vanish on \( \partial M \setminus \Gamma \). We also remark that

Lemma 2.1 (Riemann-Lebesgue). Let \( \psi \in C^\infty(M) \) be a real valued function for some \( N \) large. For every \( f \in L^1(M) \) we have as \( \lambda \to \infty \)
\[
\int_M e^{i\lambda \psi} f \, dv_g = o(1).
\]
The proof is standard and can be found in [15, Lemma 5.3]. Now we prove

Lemma 2.2. Let \( F \) be a holomorphic function which is purely real on \( \partial M \setminus \Gamma \). Let \( A, B \) be respectively the holomorphic and anti-holomorphic parts of \( X \), then we have
\[ A \wedge \partial F = 0 \text{ and } B \wedge \bar{\partial} F = 0. \]

Proof. We only prove the case of \( A \) since the other claim follows from conjugation. Let \( v \) be any \( L^1(M) \) function and \( \lambda > 0 \). Since \( e^{\lambda F} \) is purely real on \( \partial M \setminus \Gamma \) and holomorphic, the function \( e^{\lambda F} - e^{\bar{\lambda} \bar{F}} \) is harmonic and vanish on \( \partial M \setminus \Gamma \). Using (5) we deduce that
\[
\int_M v e^{-\lambda F} (e^{\lambda F} - e^{\bar{\lambda} \bar{F}}) Q = \int_M v e^{-\lambda F} X \wedge \star (e^{\lambda F} - e^{\bar{\lambda} \bar{F}})
= \int_M v e^{-\lambda F} A \wedge i \partial e^{\lambda F} + v e^{-\lambda F} B \wedge i \bar{\partial} e^{\lambda F}
= \int_M \lambda v A \wedge i \partial F + \lambda v e^{-\lambda (F - \bar{F})} B \wedge i \bar{\partial} F.
\]
On the other hand
\[
\int_M v e^{-\lambda F} (e^{\lambda F} - e^{\bar{\lambda} \bar{F}}) Q = \int_M v Q - v e^{-\lambda (F - \bar{F})} Q,
\]
thus by Lemma 2.1 we can divide (6) by \( i \lambda \) everywhere to get that
\[
\int_M v A \wedge \partial F = o(1) + \frac{O(1)}{\lambda}
\]
as \( \lambda \to \infty \). Taking this limit gives us
\[
\int_M v A \wedge \partial F = 0.
\]
Since \( v \) is an arbitrary function in \( L^1(M) \) we must have \( A \wedge \partial F = 0 \).

Proof of Proposition 2.2. The existence of a holomorphic function \( F \) with the required boundary condition is ensured by [15, Proposition 2.1]. Since locally in a holomorphic coordinate we can write \( A = \alpha d\bar{z} \) and \( |A \wedge \partial F| = |\alpha| |\partial_{\bar{z}} F| \) where \( \partial_{\bar{z}} F \) is harmonic, we can use the result of [8] which state that the zeros of harmonic functions has measure zero to conclude from Lemma 2.2 that \( A = 0 \). Similarly we can deduce \( B = 0 \). Now we have \( uQ = 0 \) for any harmonic function \( u \) which vanish on \( \partial M \setminus \Gamma \), but the same reason we must have \( Q = 0 \).

We will also need the following result on boundary determination which follows from a rather standard calculation.
Proposition 2.3. Let \( X \in C^{2,\alpha}(T^*M) \), \( Q \in L^\infty(T^*M) \) be such that

\[
\int_M u_1 X \wedge \ast du_2 = \int_M u_1 u_2 Q
\]

for all harmonic functions \( u_1 \) and \( u_2 \) which vanish on \( \partial M \setminus \Gamma \). Let \( \tau \) be the unit tangential vector field along \( \partial M \), then \( X(\nu) = X(\tau) = 0 \) along \( \Gamma \).

Let \( \alpha = (i, -1) \) be chosen such that \( \alpha \cdot \alpha = 0 \). Near an arbitrary \( p_0 \in \Gamma \) we find conformal boundary normal coordinate \( z = x + iy \) centered at \( p_0 \) where \( |z| \leq 1 \), \( y > 0 \) and the boundary is given by \( \{ x = 0 \} \). In such a chart the metric is given by \( g = e^{2\rho}|dz|^2 \) for some smooth function \( \rho \) and \( (\partial^2_z + \partial^2_y)e^{\alpha z/h} = 0 \). Hence the conformal covariance of the Laplacian ensures that \( \Delta g e^{\alpha z/h} = 0 \). Let \( \eta \) be a smooth cut-off function that is identically 1 for \( |z| < 1/2 \) and 0 for \( |z| > 3/4 \). Set \( \eta_h(z) = \eta(z/\sqrt{h}) \) and \( \psi_h(z) = \eta_h e^{\alpha z/h} \).

Lemma 2.3. Let \( \psi_h \) be defined as above, then there exists \( (r_1, r_2) \in L^2(M) \times H^1(M) \) such that \( \| r_1 \|_{L^2(M)} \leq Ch^{5/4} \) and \( \| r_2 \|_{H^1(M)} \leq Ch^{1/4} \). Moreover, the functions \( u_1 = v_h + r_1 \) and \( u_2 = v_h + r_2 \) are harmonic. In particular, \( u_1 \) and \( u_2 \) vanish on \( \partial M \setminus \Gamma \).

Proof. The construction of \( r_1 \) is covered in [15, Lemma 7.2]. We extend their argument to construct \( r_2 \). First we have the computation

\[
\Delta_g \psi_h(z) = \frac{1}{h} e^{\alpha z/h}(\Delta_g \eta)(z/\sqrt{h}) + \frac{2}{h^{3/2}} e^{\alpha z/h} \langle d\eta(z/\sqrt{h}), \alpha dz \rangle_g.
\]

Let \( w \in H^1_0(M) \). For \( \chi = -i\partial_x \eta + \partial_y \eta \) and \( \chi_h(z) = \chi(z/\sqrt{h}) \) we have

\[
\frac{2}{h^{3/2}} \int_{|z| < \sqrt{h}} e^{\alpha z/h} \langle d\eta(z/\sqrt{h}), \alpha dz \rangle_g w d\nu_g = \frac{1}{h^{1/2}} \int_{|z| < \sqrt{h}} \chi_h w (i\partial_x + \partial_y) e^{\alpha z/h} dxdy
\]

\[
= -\frac{1}{h^{1/2}} \int_{|z| < \sqrt{h}} e^{\alpha z/h} \chi_h (i\partial_x + \partial_y) (\chi_h e^{2\rho} dxdy)
\]

there is no boundary term in the integration by parts since \( w_{|_{\partial M}} = 0 \). Integral (11) can be easily estimate by \( Ch^{1/4}\| w \|_{H^1} \). However, the differential operator in (12) will cause the appearance of an extra order \( h^{-1/2} \) in front of the integral, so we have only improved the growth from \( h^{-3/2} \) to \( h^{-1} \). But as long as \( w \) is not differentiated, we can apply the same integration by parts argument to obtain improvement as above. By doing so we can get that (12) is bounded by \( Ch^{1/4}\| w \|_{H^1} \) as well. The other term in (10) can be estimated in a similar fashion. This shows that \( \| \Delta_g \psi_h \|_{H^{-1}} \leq Ch^{1/4} \). Now by the standard Lax-Milgram argument we can find \( r_2 \in H^1_0(M) \) such that \( \Delta_g r_2 = -\Delta_g \psi_h \) and \( \| r_2 \|_{H^1} \leq C\| \Delta_g \psi_h \|_{H^{-1}} \) and so our claim is proved.

Proof of Proposition 2.3. By an elementary calculation we have \( \| v_h \|_{L^2} \leq Ch^{3/4} \) and \( \| dv_h \|_{L^2} \leq Ch^{-1/4} \). Let \( u_1 \) and \( u_2 \) be constructed as in Lemma 2.3, then we easily get

\[
\int_M u_1 X \wedge \ast du_2 = \int_M v_h X \wedge \ast dv_h + o(h^{1/2}) = \frac{1}{h^{1/2}} \int_{|z| < \sqrt{h}} e^{-2y/h} \eta^2_h X \wedge \ast \alpha dz + o(h^{1/2})
\]

On the other hand, it is clear that

\[
\int_M u_1 X \wedge \ast du_2 = \int_M u_1 \bar{u}_2 Q = o(h^{1/2}).
\]

Moreover, we can do the same by replacing \( u_1, u_2 \) with their conjugates, so we have

\[
\frac{1}{h} \int_{|z| < \sqrt{h}} e^{-2y/h} \eta^2_h X \wedge \ast \alpha dz = o(h^{1/2}) \quad \text{and} \quad \frac{1}{h} \int_{|z| < \sqrt{h}} e^{-2y/h} \eta^2_h X \wedge \ast \alpha dz = o(h^{1/2}).
\]
Let $f = e^{2i\langle X, dy \rangle}$. Adding the two equations in (12) together and a standard integration by parts calculation gives us

$$0 = \frac{2}{h} \int_{|x|<\sqrt{h}} \int_{0}^{\sqrt{\eta}} e^{-2y/h} h^2 f \, dy \, dx = 2 \int_{|x|<1} \int_{0}^{1} e^{-2y/\sqrt{\eta}} h^2 f(h^{1/2}x) \, dy \, dx$$

$$= -h^{1/2} \int_{|x|<1} \eta(x) f(h^{1/2}x) \, dx + o(h^{1/2}) = Ch^{1/2} f(p_0) + o(h^{1/2})$$

for some $C > 0$. Dividing over by $h^{1/2}$ and taking the limit as $h \to 0$ yields $f(p_0) = 0$. Since $p_0 \in \Gamma$ is chosen arbitrarily we must have $X(\nu)$ along $\Gamma$. Taking the difference of (11) and applying the same argument also shows $X(\tau) = 0$ along $\Gamma$. □

3. PROOF OF THE THEOREM

In this section we give the proof of Theorem [14].

Proof. We will proceed via induction. For all $k \geq 1$ we set

$$X_k \overset{\text{def}}{=} X_{k,1} - X_{k,2} \quad \text{and} \quad V_k \overset{\text{def}}{=} V_{k,1} - V_{k,2}.$$  

3.1. The Set Up. Let $f_1$ and $f_2$ be elements of $C^{2,\alpha}(\partial M) \cap C^\alpha(\Gamma)$, and for $\epsilon = (\epsilon_1, \epsilon_2) \in C^\infty$ we may set $f_\epsilon = \epsilon_1 f_1 + \epsilon_2 f_2$. It follows that if $|\epsilon|$ sufficiently small, then $f_\epsilon$ belongs to the domain of $\Lambda_{X_{j,V_j}}$, $j = 1, 2$. Let $u_{j,f_\epsilon}$ be the unique small solution to $L_{X_{j,V_j}} u_{j,f_\epsilon} = 0$ with boundary condition $f_\epsilon$. Differentiating $u_{j,f_\epsilon}$ with respect to $\epsilon_\ell$ and evaluating at $\epsilon = 0$ yields $\Delta_g \partial_{\epsilon_\ell} u_{j,f_\epsilon|\epsilon=0} = 0$ and $\partial_{\epsilon_\ell} u_{j,f_\epsilon|\epsilon=0} = f_\ell$ on $\partial M$. By uniqueness of harmonic functions we may denote $u_\ell = \partial_{\epsilon_\ell} u_{1,f_\epsilon|\epsilon=0} = \partial_{\epsilon_\ell} u_{2,f_\epsilon|\epsilon=0}$. Since $u_{j,f_\epsilon|\epsilon=0} = 0$ on $\partial M$, any term involving positive powers of $u_{j,f_\epsilon}$ in $\partial_{\epsilon_1} \partial_{\epsilon_2} (L_{X_{j,V_j}} - \Delta_g) u_{j,f_\epsilon|\epsilon=0}$ must vanish as well. Thus by a direct calculation we now get

$$\begin{cases} 
\Delta_g \partial_{\epsilon_1} \partial_{\epsilon_2} u_{j,f_\epsilon|\epsilon=0} = 3i \star X_{j,1} \wedge \star d(u_1 u_2) - u_1 u_2(2i d^* X_{j,1} + V_{j,2}) \quad \text{in} \quad \partial M \\
\partial_{\epsilon_1} \partial_{\epsilon_2} u_{j,f_\epsilon|\epsilon=0} = 0 \quad \text{on} \quad \partial M.
\end{cases}$$

Similarly, one has

$$\partial_{\epsilon_1} \partial_{\epsilon_2} i u_{j,f_\epsilon}(X_j(u_{j,f_\epsilon},p),\nu)_{|\epsilon=0} = 2iu_1 u_2 X_{j,1}(\nu) \quad \text{on} \quad \partial M.$$

Hence if $u_3$ is any harmonic function which vanishes on $\partial M \setminus \Gamma$, then linearising the Dirichlet-Neumann maps yields the integral identity

$$0 = \int_{\partial M} u_3 \partial_{\epsilon_1} \partial_{\epsilon_2} (\Lambda_{X_{1,V_1}} - \Lambda_{X_{2,V_2}}) f_{\epsilon|\epsilon=0} \, ds_g$$

$$= \int_{\partial M} u_3 \partial_{\nu} \partial_{\epsilon_1} \partial_{\epsilon_2} (u_{1,f_\epsilon} - u_{2,f_\epsilon})_{|\epsilon=0} \, ds_g + \int_{\partial M} 2iu_1 u_2 u_3 X_{1}(\nu) \, ds_g. \quad (15)$$

Integration by parts combined with (13) shows

$$\int_{\partial M} u_3 \partial_{\nu} \partial_{\epsilon_1} \partial_{\epsilon_2} (u_{1,f_\epsilon} - u_{2,f_\epsilon})_{|\epsilon=0} \, ds_g$$

$$= \int_{\partial M} (du_3, d\partial_{\epsilon_1} \partial_{\epsilon_2} (u_{1,f_\epsilon} - u_{2,f_\epsilon})_{|\epsilon=0})_g \, dv_g - \int_{\partial M} u_3 \Delta_g \partial_{\epsilon_1} \partial_{\epsilon_2} (u_{1,f_\epsilon} - u_{2,f_\epsilon})_{|\epsilon=0} \, dv_g$$

$$= \int_{\partial M} -3iu_3 X_1 \wedge \star d(u_1 u_2) + u_1 u_2 u_3 * (2d^* X_1 + V_2), \quad (16)$$

where we have used that $\partial_{\epsilon_1} \partial_{\epsilon_2} (u_{1,f_\epsilon} - u_{2,f_\epsilon})_{|\epsilon=0}$ have zero Dirichlet boundary condition and $u_3$ is harmonic.
3.2. **Boundary Determination.** Now we claim that

\[(17) \quad \int_M u_1 u_2 u_3 X_1 \wedge *d = \int_M u_1 u_2 u_3 \wedge *d u_3\]

for all harmonic functions \(u_1, u_2\) and \(u_3\) which vanish on \(\partial M \setminus \Gamma\). Indeed, combining (15) and (16) we see that

\[(18) \quad \int_M 3i u_3 X_1 \wedge *d(u_1 u_2) = \int_M u_1 u_2 u_3 \wedge (2id^* X_1 + V_2) + \int_{\partial M} 2i u_1 u_2 u_3 X_1(\nu) d_s g.\]

By permuting the roles of \(u_2\) and \(u_3\), we also get

\[(19) \quad \int_M 3i u_2 X_1 \wedge *d(u_1 u_3) = \int_M u_1 u_2 u_3 \wedge (2id^* X_1 + V_1) + \int_{\partial M} 2i u_1 u_2 u_3 X_1(\nu) d_s g.\]

Taking the difference of (18) and (19), we have

\[
0 = \int_M 3i u_3 X_1 \wedge *d(u_1 u_2) - 3i u_2 X_1 \wedge *d(u_1 u_3) = \int_M 3i u_1 u_3 X_1 \wedge *d u_2 - 3i u_1 u_2 X_1 \wedge *d u_3
\]

as claimed. Hence we can apply Proposition 2.3 to conclude that \(u X_1(\tau) = u X_1(\nu) = 0\) along \(\Gamma\) for any harmonic function \(u\) vanishing on \(\partial M \setminus \Gamma\). By choosing \(u\) to be non-vanishing on the requires sets thus implies \(X_1(\tau) = X_1(\nu) = 0\) along \(\Gamma\).

3.3. **Interior Determination.** By boundary determination, the difference of the linearised Dirichlet-Neumann maps reduces to

\[(20) \quad \partial_{f_1} \partial_{f_2}(\Lambda_{X_1, V_1}^\Gamma - \Lambda_{X_2, V_2}^\Gamma) f_{\epsilon |_{\Gamma}} = \partial_{f_1} \partial_{f_2}(u_{1,f_1} - u_{2,f_2})|_{\Gamma},\]

and so the integral identity coming from (15) becomes

\[
\int_M 3i u_3 X \wedge *d(u_1 u_2) = \int_M u_1 u_2 u_3 \wedge (2id^* X_1 + V_1).
\]

Since \(X(\nu) = 0\) along \(\Gamma\), integrating by parts shows that

\[
\int_M 3i u_1 u_2 u_3 X \wedge *d = \int_M u_1 u_2 u_3 \wedge (id^* X_1 - V_2).
\]

Proposition 2.4 now implies \(X_1 = 0\) and \(id^* X_1 = V_2\). In particular \(V_2 = 0\).

3.4. **The Induction Step.** Now we assume that \(X_{k'<k} = 0\) and \(V_{k'} = 0\) for all \(k' < k\) and \(k > 2\). We will show that \(X_{k-1} = 0\) and \(V_k = 0\) via the same strategy as above. Let \(\epsilon = (\epsilon_1, ..., \epsilon_k) \in \mathbb{C}^k\) and \(f_{\epsilon} = \epsilon_1 f_1 + + ... + \epsilon_k f_k\) for \(|\epsilon|\) sufficiently small and \(f_1, ..., f_k\) in \(\mathcal{C}^{2,\alpha}(\partial M) \cap \mathcal{E}(\Gamma)\). Let \(u_{f_{\epsilon}}\) be the unique small solution to \(\Lambda_{X_1, V_1} u_{f_{\epsilon}} = 0\) with boundary condition \(f_{\epsilon}\). A direct calculation as before shows that \(\partial_{f_1} \cdots \partial_{f_k} u_{f_{\epsilon}}|_{\Gamma = 0} = 0\) on \(\partial M\) and

\[
\Delta \partial_{f_1} \cdots \partial_{f_k} (u_{f_{\epsilon}} - u_{f_{\epsilon}})|_{\Gamma = 0} = (k+1)iX_{k-1} \wedge *d(u_1 \cdots u_k) - u_1 \cdots u_k (kid^* X_{k-1} + V_k)
\]

where \(u_1, ..., u_k\) are harmonic functions with Dirichlet conditions \(u_k = f_{\epsilon}\). Likewise, we have

\[
\partial_{f_1} \cdots \partial_{f_k} (X_1(u_{f_{\epsilon}}, p), \nu) = (k+1)iX_{k-1} \wedge *d(u_1 \cdots u_k) - u_1 \cdots u_k (kid^* X_{k-1} + V_k)\]

where \(u_{f_{\epsilon}}\) is any harmonic function which vanish on \(\partial M \setminus \Gamma\). Integration by parts now gives

\[
0 = \int_{\partial M} u_{k+1}(\Lambda_{X_1, V_1}^\Gamma - \Lambda_{X_2, V_2}^\Gamma) f_{\epsilon |_{\Gamma}} d_s g = \int_M (k+1)i u_{k+1} X_{k-1} \wedge *d(u_1 \cdots u_k)
\]

\[
+ \int_M u_1 \cdots u_k u_{k+1} (kid^* X_{k-1} + V_k) + \int_{\partial M} u_1 \cdots u_k u_{k+1} X_{k-1}(\nu) d_s g.
\]

The same permutation argument leads to

\[
\int_M u_1 \cdots u_{k-1} u_{k+1} X_{k-1} \wedge *d u_k = \int_M u_1 \cdots u_{k-1} u_{k} X_{k-1} \wedge *d u_{k+1}.
\]
Hence by Proposition 2.3 we conclude that $X_{k-1}(\tau) = X_{k-1}(\nu) = 0$ along $\Gamma$. The boundary integral identity now reduces to

$$\int_M (k+1)iu_{k+1}X_{k-1} \wedge \star d(u_1 \cdots u_k) = \int_M u_1 \cdots u_k u_{k+1} \star (kid^* X_{k-1} + V_k).$$

Integration by parts lead to

$$\int_M (k+1)iu_1 \cdots u_k X_{k-1} \wedge \star d u_{k+1} = \int_M u_1 \cdots u_k u_{k+1} \star (id^* X_{k-1} - V_k).$$

Hence Proposition 2.4 combined with the result of [8] implies $X_{k-1} = 0$ and $V_k = 0$ as before. By induction, $X_k = 0$ and $V_k = 0$ for all $k \geq 1$. It follows that $(X_1, V_1) = (X_2, V_2)$ and we are done.

□

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