COMPUTATION OF THE CLASSIFYING RING OF FORMAL GROUPS WITH
COMPLEX MULTIPLICATION.

ANDREW SALCH

Abstract. Machinery is developed for computing the classifying ring $L^A$ of one-dimensional
formal groups with complex multiplication by $A$, for a given commutative ring $A$. The ma-
chinery is then applied to compute $L^A$ for various number rings and cyclic group rings $A$.
The ring $L^A$ has been computed, for certain choices of $A$, by M. Lazard, V. Drinfeld, and
M. Hazewinkel, but in those cases $L^A$ is always isomorphic to a polynomial algebra. In the
present paper, $L^A$ is computed in many cases in which it fails to be a polynomial algebra,
leading to a qualitatively different moduli theory and a different presentation for the moduli
stack of formal $A$-modules.

Contents

1. Introduction and review of some known facts. 1
1.1. Introduction. 1
1.2. Review of some known facts. 4
2. $U$-homology. 6
2.1. $U$-homology as the obstruction to $L^A$ being a polynomial algebra. 6
2.2. Local properties of $U$-homology. 14
2.3. Global consequences. 17
3. Computations of $L^A$ for certain classes of ring $A$. 19
3.1. Number rings. 19
3.2. Group rings. 26
References 27

1. Introduction and review of some known facts.

1.1. Introduction. This paper is about the computation of the classifying rings $L^A$ and
classifying Hopf algebroids $(L^A, L^A B)$ of formal $A$-modules. I ought to explain what this
means. When $A$ is a commutative ring, a formal $A$-module is a formal group law $F$ over
an $A$-algebra $R$, which is additionally equipped with a ring map $\rho : A \to \text{End}(F)$ such that
$\rho(a)(X) \equiv aX$ modulo $X^2$. In other words, a formal $A$-module is a formal group law with
complex multiplication by $A$. A recap of classical facts about formal modules and their
moduli theory, as well as some of the areas in which they have found fruitful applications

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This paper is the second in a series about formal groups with complex multiplication, but makes no assump-
tion that the reader has read the first paper, [14]. The connections between this material and homotopy theory do
not become significant until the later papers in the series, e.g. [13] and [11].
(algebraic geometry, number theory, algebraic topology), can be found in [14]. Higher-dimensional formal modules exist, but all formal modules in this paper will be implicitly assumed to be one-dimensional.

It is easy to show (see [5]) that there exists a classifying ring \( L^A \) for formal \( A \)-modules, i.e., a commutative \( A \)-algebra \( L^A \) such that \( \text{hom}_{A\text{-alg}}(L^A, R) \) is in natural bijection with the set of formal \( A \)-modules over \( R \). The hard part is actually computing this ring \( L^A \). M. Lazard proved (see [9]) that

\[
L^A \cong A[x_1, x_2, \ldots],
\]

a polynomial algebra on countably infinitely many generators, when \( A = \mathbb{Z} \). The ring \( L^Z \) is consequently often called the Lazard ring. In [5], V. Drinfeld proved that isomorphism 1.1.1 also holds when \( A \) is the ring of integers in a local nonarchimedean field (e.g. a \( p \)-adic number field). Finally, in [7], M. Hazewinkel proved that 1.1.1 is an isomorphism when \( A \) is a discrete valuation ring or a global number rings of class number one.

Hazewinkel also makes the observation, in 21.3.3A of [7], that the same result cannot possibly hold for arbitrary global number rings. Specifically, when \( A \) is the ring of integers in \( \mathbb{Q}(\sqrt{\nu-18}) \), then Hazewinkel shows that the sub-\( A \)-module of \( L^A \) consisting of elements of grading degree 2 (see [14] for a description of where this grading comes from) is not a free \( A \)-module, which could not occur if \( L^A \) were polynomial. Hazewinkel does not, however, attempt to compute \( L^A \).

In fact, it seems that there are no computations of \( L^A \) in the literature whatsoever except in the cases that \( L^A \) turns out to be polynomial. This matters especially because there are qualitative features of formal \( A \)-modules which depend on whether \( L^A \) is polynomial. Hazewinkel does not, however, attempt to compute \( L^A \).

In the present paper I compute \( L^A \) for certain classes of commutative ring \( A \). These are apparently the first known full computations of \( L^A \) in which \( L^A \) fails to be polynomial. Specifically, the computations I make are as follows:

- In Theorem 2.3.2, I prove the following: let \( A \) be a commutative ring whose underlying abelian group is finitely generated and free, and let \( S \) be a set of prime numbers such that the ring \( A[S^{-1}] \) is hereditary. (If, for example, \( A \) is already hereditary, then we can let \( S \) be the empty set.)

Then the commutative graded ring \( L^A \) is, after inverting \( S \), isomorphic to a tensor product of (graded) Rees algebras:

\[
L^A[S^{-1}] \cong \left( \text{Rees}_h^2(\mathfrak{I}_2^A) \otimes_A \text{Rees}_h^3(\mathfrak{I}_2^A) \otimes_A \text{Rees}_h^5(\mathfrak{I}_2^A) \otimes_A \text{Rees}_h^8(\mathfrak{I}_2^A) \otimes_A \ldots \right)[S^{-1}]
\]

where \( \mathfrak{I}_2^A \) is the ideal in \( A \) generated by \( \nu(n) \) and by all elements of the form \( a - a^\nu \), and where \( \nu(n) \) is defined to be \( p \) if \( n \) is a power of a prime number \( p \), and \( \nu(n) = 1 \) if \( n \) is not a prime power.
Hence we have Drinfeld’s lifting and extension properties for formal $A$-modules over $A[S^{-1}]$-algebras (since, even though $L^A[S^{-1}]$ is not necessarily polynomial, it is isomorphic to a symmetric algebra on a projective module!).

- In Theorem 3.1.1, I prove the following: let $A$ be the ring of integers in a finite extension $K/Q$, let $1, \alpha_1, \ldots, \alpha_j$ be a $\mathbb{Z}$-linear basis for $A$, and let $J^n_A$ be the ideal $(\nu(n), a_1 - \alpha_1^n, a_2 - \alpha_2^n, \ldots, a_j - \alpha_j^n)$ of $A$. Let $P$ denote the set of integers $> 1$ which are prime powers, and let $R$ denote the set of integers $> 1$ which are not prime powers. Then we have an isomorphism of commutative graded $A$-algebras:

$$L^A \cong \left( \bigotimes_{A}^{\nu \in P} \text{Rees}^{2n-2}(J^n_A) \right) \otimes_A A[x_{n-1} : n \in R],$$

with $x_{n-1}$ in grading degree $2(n-1)$.

- For quadratic number rings, I prove Theorem 3.1.5: let $K$ be a quadratic extension of the rational numbers, and let $A = \mathbb{Z}[\alpha]$ be the ring of integers of $K$. Let $\Delta$ denote the discriminant of $K/Q$. For each prime number $p$ which divides $\Delta$, let $\mathfrak{m}_p$ be the (unique, since $p$ ramifies totally in $A$) maximal ideal of $A$ over $p$. Let $R$ be the set of primes $p$ which divide $\Delta$ and which have the property that $J^n_{\mathfrak{m}_p} = (p, \alpha - \alpha^p)$ is nonprincipal for some positive integer $m$, and let $S$ be the set of integers $> 1$ which are not powers of primes contained in $R$. Then we have an isomorphism of commutative graded $A$-algebras:

$$L^A \cong A[[x_{n-1} : n \in S]] \otimes_A \left( \bigotimes_{A}^{\nu \in R} \text{Rees}^{2p-2}(J^\nu_A) \otimes_A \text{Rees}^{2p-2}(J^\nu_{\mathfrak{m}_p}) \otimes_A \ldots \right)$$

with each polynomial generator $x_{n-1}$ in grading degree $2(n-1)$.

Consequently, we have an isomorphism of commutative graded $A[R^{-1}]$-algebras:

$$L^A[R^{-1}] \cong A[R^{-1}][x_1, x_2, \ldots],$$

with each $x_i$ in grading degree $2i$.

- As an example computation, I provide Theorem 3.1.11, where I fully work out the ring $L^A$ in the case where $A$ is the ring of integers in the number field $\mathbb{Q}(\sqrt{-18})$. This was Hazewinkel’s original example of a number ring $A$ in which $L^A$ could not possibly be a polynomial ring (but Hazewinkel’s computation stopped at grading degree 2). The full result is: let $S$ denote the set of all integers $> 1$ which are not powers of 2 or 3. Then we have an isomorphism of commutative graded $A$-algebras

$$L^A \cong A[[x_{n-1} : n \in S]] \otimes_A A[x_1, y_1]/(2x_1 - (\alpha - \alpha^2)y_1)$$

$$\otimes_A \left( A[x_{2n-1}, y_{2n-1}]/(2x_{2n-1} - \alpha y_{2n-1}) \right)$$

$$\otimes_A \left( A[x_{3n-1}, y_{3n-1}]/(3x_{3n-1} - \alpha y_{3n-1}) \right),$$

where $\alpha = \sqrt{-18} \in A$, and where the polynomial generators $x_i$ and $y_i$ are each in grading degree $2i$.

Consequently, we have an isomorphism of commutative graded $A[\frac{1}{6}]$-algebras:

$$L^A[\frac{1}{6}] \cong A[\frac{1}{6}][x_1, x_2, \ldots].$$
with each $x_i$ in grading degree $2i$.

- In Theorem 3.2.2, I prove the following: let $C_n$ be the cyclic group of order $n$, let $P$ be the set of integers $>1$ which are prime powers relatively prime to $n$, and let $S$ be the set of integers $>1$ not contained in $P$. Then we have an isomorphism of graded rings

$$L^Z(C_n)[\frac{1}{n}] \cong \bigotimes_{\mathbb{Z}[\frac{1}{n}][C_n]} \left( \mathbb{Z}[\frac{1}{n}][C_n][x_{i-1}, y_{i-1}] / (px_{i-1} - (1 - \sigma)y_{i-1}) \right) \bigotimes_{\mathbb{Z}[\frac{1}{n}][C_n]} \mathbb{Z}[\frac{1}{n}][C_n][x_i : i \in S]$$

where $\sigma$ denotes a generator of $C_n$, and where the polynomial generators $x_{i-1}$ and $y_{i-1}$ are each in grading degree $2(i-1)$.

The description of $L^Z(C_n)[\frac{1}{n}]$ in Theorem 3.2.2 ought to be compared to the classifying ring $L^C_n$ of “$C_n$-equivariant formal groups” as in [6] and [4], whose relationship to formal $\mathbb{Z}[C_n]$-modules is presently not known. This “should” have some bearing on Greenlees’ conjecture; see Remark 3.2.1 for more about this.

In all the above cases, $L^A$ is a tensor product of Rees rings, so even when $L^A$ fails to be a polynomial algebra, $L^A$ is still a subalgebra of a polynomial algebra.

In [14] I showed that the classifying ring $L^A B$ for formal $A$-modules is always a polynomial algebra over $L^A$, hence the Hopf algebroid $(L^A, L^A B)$ is isomorphic to $(L^A, L^A [b_1, b_2, \ldots])$. The stack associated to the groupoid scheme $(\text{Spec } L^A, \text{Spec } L^A B)$ is the moduli stack of formal $A$-modules, so the reader who is so inclined can regard the computations in this paper as computations of presentations for this moduli stack.

Producing these computations of $L^A$ for various $A$ requires significant preliminary work, some of which is worth something in its own right. In section 2 I define a certain homology theory on rings, “$U$-homology,” by means of a certain cyclic bar-type construction. In homological degrees 0 and 1, this $U$-homology is the obstruction to $L^A$ being a polynomial algebra; these and other general properties of $U$-homology are worked out in section 2.1, while local properties of $U$-homology are in section 2.2, including a spectral sequence (Lemma 2.2.2) for $U$-homology and a resulting rigidity theorem for $U$-homology, Theorem 2.2.3, which plays an essential role in everything that follows; in section 2.3 I derive the various consequences, most importantly Theorem 2.3.2, which is the essential result for the computations of $L^A$ for number rings and group rings $A$, all of which appear in section 3.

In the previous paper in this series, [14], I computed the classifying ring $L^A$ of formal $A$-modules, modulo torsion, for all characteristic zero Dedekind domains $A$; in the present paper, I get stronger results, computing $L^A$ without reducing modulo torsion, but in a different level of generality, requiring $A$ to be finitely generated as an abelian group so that the rigidity theorem, Theorem 2.2.3, applies. Consequently the results of the present paper do not render those of [14] obsolete, and the results of [14] do not render those of the present paper obsolete.

I am grateful to D. Ravenel for teaching me a great deal about formal modules and homotopy theory when I was a graduate student. I also found the SAGE computer algebra package quite helpful when I was preparing the number-theoretic results in section 3.1.

1.2. Review of some known facts. Proposition 1.2.1 appears as Proposition 1.1 in [5].

**Proposition 1.2.1.** Let $A$ be a commutative ring, let $n$ be an integer, and let $L^A_n$ be the grading degree $2n$ summand in $L^A$. Let $D^A_n$ be the sub-$A$-module of $L^A$ generated by all products of the form $xy$ where $x, y$ are homogeneous elements of $L^A$ of grading degree $< 2n$. If $n \geq 2$, then $L^A_{n-1}/D^A_{n-1}$ is isomorphic to the $A$-module generated by symbols $d$ and
\{ c_a : a \in A \}, that is, one generator \( c_a \) for each element \( a \) of \( A \) along with one additional generator \( d \), modulo the relations:

\begin{align*}
(1.2.1) & \quad d(a - a^p) = \nu(n)c_a \text{ for all } a \in A \\
(1.2.2) & \quad c_{a+b} - c_a - c_b = d^\nu(n)(a + b)^n - (a + b)^n \text{ for all } a, b \in A \\
(1.2.3) & \quad ac_b + b^c c_a = c_{ab} \text{ for all } a, b \in A.
\end{align*}

I will call this Drinfeld’s presentation for \( L_{n-1}^A/D_{n-1}^A \).

The grading degrees in Proposition 1.2.1 are twice what they are in Drinfeld’s statement of the result; this is to match the gradings that occur in algebraic topology, where the generator of \( L^Z = MU \), classifying an extension of a formal group \( n \)-bud to a formal group \((n+1)\)-bud is in grading degree \( 2n \) rather than \( n \).

One fairly easy application of Proposition 1.2.1 is Proposition 1.2.2; see [14] for proof, but the result also appears in [8].

\textbf{Proposition 1.2.2.} Let \( A \) be a commutative \( \mathbb{Q} \)-algebra. Suppose that \( A \) is additively torsion-free, i.e., if \( m \in \mathbb{Z} \) and \( a \in A \) and \( ma = 0 \), then either \( m = 0 \) or \( a = 0 \). Then the classifying ring \( L^A \) of formal \( A \)-modules is isomorphic, as a graded \( A \)-algebra, to \( A[x_1, x_2, \ldots, ] \), with \( x_n \) in grading degree \( 2n \), and with each \( x_n \) corresponding to the generator \( d \) of \( A \cong L^A_0/D^A_0 \) in the Drinfeld presentation for \( L^A_{n-1}/D^A_{n-1} \).

There are two facts I will use which were proven in [14], Theorem 1.2.3 and Definition-Proposition 1.2.4:

\textbf{Theorem 1.2.3.} Let \( A \) be a commutative ring and let \( S \) be a multiplicatively closed subset of \( A \). Then the homomorphism of graded rings \( L^A[S^{-1}] \to L^{A[S^{-1}]} \) is an isomorphism. Even better, the homomorphism of graded Hopf algebroids

\begin{equation}
(\mu^A(S^{-1}), L^A[S^{-1}]) \to (\mu^{A[S^{-1}]}, L^{A[S^{-1}]})
\end{equation}

is an isomorphism of Hopf algebroids.

\textbf{Definition-Proposition 1.2.4.} Let \( A \) be a commutative ring and let \( n \) be a positive integer. Recall from Proposition 1.2.1 that \( L^A_{n-1}/D^A_{n-1} \) is described by Drinfeld’s presentation: it is generated, as an \( A \)-module, by elements \( d \) and \( \{ c_a \}_{a \in A} \), subject to the relations 1.2.1, 1.2.2, and 1.2.3.

Let \( M^A_{n-1} \) denote the \( A \)-module generated by elements \( d \) and \( \{ c_a \}_{a \in A} \), subject only to the relations 1.2.1. Let \( q^A_{n-1} : M^A_{n-1} \to L^A_{n-1}/D^A_{n-1} \) denote the obvious \( A \)-module quotient map.

By the \( n \)th fundamental functional of \( A \), I mean the \( A \)-module homomorphism

\begin{equation}
\sigma_n : L^A_{n-1}/D^A_{n-1} \to A
\end{equation}

given by

\begin{align*}
\sigma_n(d) & = \nu(n), \\
\sigma_n(c_a) & = a - a^p,
\end{align*}

where \( \nu(n) = p \) if \( n \) is a power of a prime number \( p \), and \( \nu(n) = 1 \) if \( n \) is not a prime power.

If \( n > 1 \), then the kernel of the composite map \( \sigma_n \circ q^A_{n-1} : M^A_{n-1} \to A \) is exactly the set of \( \nu(n) \)-torsion elements of \( M^A_{n-1} \). Furthermore, the kernel of \( \sigma_n \) and the kernel of \( q^A_{n-1} \) are each annihilated by multiplication by \( \nu(n) \). Furthermore, if \( n \) is not a prime power, then \( \sigma_n \) and \( q^A_{n-1} \) are both isomorphisms of \( A \)-modules.
It is the injectivity of the fundamental functional, for certain rings $A$, which makes the computation of $L^A$ possible using the methods of this paper. The entire apparatus of $U$-homology, which is defined and developed later in this paper, is not much more than a tool for showing that the fundamental functional is injective.

2. $U$-HOMOLOGY.

2.1. $U$-homology as the obstruction to $L^A$ being a polynomial algebra. In this subsection I introduce “$U$-homology,” an invariant of commutative rings. In Proposition 2.1.4 and in Proposition 2.1.9 I demonstrate the main properties of $U$-homology: in dimension 1, it is the obstruction to injectivity of the fundamental functional; in dimension 0, it is the obstruction to surjectivity of the fundamental functional; and the vanishing of $U^A_0(n)$ and $U^A_1(n)$ for all $n$ is equivalent to $L^A$ being isomorphic to a polynomial algebra by a certain fundamental comparison map.

**Definition 2.1.1.** When $A$ is a commutative ring and $n > 1$ an integer, let $F_n(A)$ denote the free $A/n(n)$-module generated by the underlying abelian group of $A$, i.e., $F_n(A)$ is the $A/n(n)$-module with one generator $c_a$ for each element $a \in A$ and subject to the relation $c_0 = 0$ and the relation $c_{a+b} = c_a + c_b$ for each $a, b \in A$.

**Definition-Proposition 2.1.2.** Suppose $A$ is a commutative ring and $n > 1$ an integer which is a power of a prime number $\nu(n)$. Let $\mathcal{T}^A(n)_\bullet$ denote the simplicial $A/n(n)$-module given as follows:

- $\mathcal{T}^A(n)_0 = A/n(n)$,
- $\mathcal{T}^A(n)_m = F_n(A)^{\otimes m}$, i.e., the $m$-fold tensor product, over $A$, of $F_n(A)$ with itself,
- the face map $d_0 : \mathcal{T}^A(n)_1 = F_n(A) \to A/n(n) = \mathcal{T}^A(n)_0$ is given by $d_0(c_a) = a$,
- the face map $d_1 : \mathcal{T}^A(n)_1 = F_n(A) \to A/n(n) = \mathcal{T}^A(n)_0$ is given by $d_1(c_a) = a^n$,
- if $m \geq 1$, the face map $d_0 : \mathcal{T}^A(n)_{m+1} = F_n(A)^{\otimes (m+1)} \to F_n(A)^{\otimes m} = \mathcal{T}^A(n)_m$ is given by $d_0(c_{a_1} \otimes \cdots \otimes c_{a_{m+1}}) = a(c_{a_2} \otimes \cdots \otimes c_{a_{m+1}})$,
- if $m \geq 1$ and $1 \leq i \leq m$, the face map $d_i : \mathcal{T}^A(n)_{m+1} = F_n(A)^{\otimes (m+1)} \to F_n(A)^{\otimes m} = \mathcal{T}^A(n)_m$ is given by $d_i(c_{a_1} \otimes \cdots \otimes c_{a_{m+1}}) = c_{a_i} \otimes \cdots \otimes c_{a_{m+1}}$,
- the degeneracy map $s_0 : A/\nu(n) = \mathcal{T}^A(n)_0 \to \mathcal{T}^A(n)_1 = F_n(A)$ is given by $s_0(a) = ac_1$,
- if $m \geq 1$ and $0 \leq i \leq m$, the degeneracy map $s_i : \mathcal{T}^A(n)_m = F_n(A)^{\otimes m} \to F_n(A)^{\otimes (m+1)} = \mathcal{T}^A(n)_{m+1}$ is given by $s_i(c_{a_1} \otimes \cdots \otimes c_{a_m}) = c_{a_1} \otimes \cdots \otimes c_{a_{i-1}} \otimes c_1 \otimes \cdots \otimes c_{a_m}$.

Let $U^A(n)_\bullet$ denote the chain complex of $A/n(n)$-modules which is the alternating sign chain complex of $\mathcal{T}^A(n)_\bullet$. Let $U^A_i(n)$ denote the $i$th homology group $H_i(U^A_i(n)_\bullet)$ of the chain complex $U^A(n)_\bullet$, respectively. I will call these homology groups $U$-homology.

**Proof.** I need to show that $\mathcal{T}^A(n)_\bullet$ is actually a simplicial $A/n(n)$-module, i.e., that the simplicial identities are satisfied. (It is easy to check that the face and degeneracy maps given in the definition of $\mathcal{T}^A(n)_\bullet$ are indeed well-defined, although it is worth noting that the well-definedness of these maps does rely on the fact that $n$ is a power of the prime $\nu(n)$ that we have quotiented out by in the definition of $F_n(A)$.) This is routine, but I prefer to provide the details, for the reader who does not find this kind of thing routine:
\begin{align*}
& \text{if } i < j: \quad d_{i}d_{j} = d_{i-1}d_{j}, \quad i = 0, j = 1 : d_{0}d_{1}(c_{a_{1}} \otimes \cdots \otimes c_{a_{m}}) \\
& \quad = a_{1}a_{2}c_{a_{1}} \otimes \cdots \otimes c_{a_{m}} \\
& \quad = d_{0}d_{0}(c_{a_{1}} \otimes \cdots \otimes c_{a_{m}}) \\
& i = 0, 1 < j < m : d_{0}d_{j}(c_{a_{1}} \otimes \cdots \otimes c_{a_{m}}) \\
& \quad = a_{1}(c_{a_{1}} \otimes \cdots \otimes c_{a_{j-1}} \otimes c_{a_{j+1}} \otimes \cdots \otimes c_{a_{m}}) \\
& \quad = d_{j-1}d_{0}(c_{a_{1}} \otimes \cdots \otimes c_{a_{m}}) \\
& i = 0, j = m : d_{0}d_{m}(c_{a_{1}} \otimes \cdots \otimes c_{a_{m}}) \\
& \quad = a_{1}a_{m}(c_{a_{1}} \otimes \cdots \otimes c_{a_{m-1}}) \\
& \quad = d_{m-1}d_{0}(c_{a_{1}} \otimes \cdots \otimes c_{a_{m}}) \\
& 0 < i < j < m : d_{i}d_{j}(c_{a_{1}} \otimes \cdots \otimes c_{a_{m}}) \\
& \quad = c_{a_{1}} \otimes \cdots \otimes c_{a_{i-1}} \otimes c_{a_{i}} \otimes \cdots \otimes c_{a_{j-1}} \otimes \cdots \otimes c_{a_{m}} \\
& \quad = d_{j-1}d_{i-1}(c_{a_{1}} \otimes \cdots \otimes c_{a_{m}}) \\
& 0 < i < m, j = m : d_{i}d_{m}(c_{a_{1}} \otimes \cdots \otimes c_{a_{m}}) \\
& \quad = a_{m}(c_{a_{1}} \otimes \cdots \otimes c_{a_{i-1}} \otimes c_{a_{i+1}} \otimes c_{a_{i+2}} \otimes \cdots \otimes c_{a_{m-1}}) \\
& \quad = d_{m-1}d_{i-1}(c_{a_{1}} \otimes \cdots \otimes c_{a_{m}}). \\
& d_{i}s_{j} = s_{j-1}d_{i} \text{ if } i < j: \\
& \quad i = 0, 0 < j \leq m : d_{0}s_{j}(c_{a_{1}} \otimes \cdots \otimes c_{a_{m}}) \\
& \quad = a_{1}(c_{a_{1}} \otimes \cdots \otimes c_{a_{j}} \otimes c_{1} \otimes \cdots \otimes c_{a_{m}}) \\
& \quad = s_{j-1}d_{0}(c_{a_{1}} \otimes \cdots \otimes c_{a_{m}}) \\
& 0 < i < j \leq m : d_{i}s_{j}(c_{a_{1}} \otimes \cdots \otimes c_{a_{m}}) \\
& \quad = c_{a_{1}} \otimes \cdots \otimes c_{a_{i}} \otimes c_{a_{j}} \otimes \cdots \otimes c_{a_{m}} \\
& \quad = s_{j-1}(d_{i-1}(c_{a_{1}} \otimes \cdots \otimes c_{a_{m}}). \\
& d_{i}s_{j} = i \text{d} \text{ if } i = j \text{ or } i = j + 1: \\
& \quad 0 \leq i \leq m, i \leq j \leq i + 1 : d_{i}s_{j}(c_{a_{1}} \otimes \cdots \otimes c_{a_{m}}) \\
& \quad = c_{a_{1}} \otimes \cdots \otimes c_{a_{m}} \\
& \quad = d_{j+1}s_{i}(c_{a_{1}} \otimes \cdots \otimes c_{a_{m}}). \\
& d_{i}s_{j} = s_{j-1}d_{i-1} \text{ if } i > j + 1: \\
& \quad 0 \leq j, j + 1 < i \leq m + 1 : d_{i}s_{j}(c_{a_{1}} \otimes \cdots \otimes c_{a_{m}}) \\
& \quad = c_{a_{1}} \otimes \cdots \otimes c_{a_{j}} \otimes c_{a_{j+1}} \otimes \cdots \otimes c_{a_{m-2}} \otimes c_{a_{m-1}} \otimes \cdots \otimes c_{a_{m}} \\
& \quad = s_{j}d_{i-1}(c_{a_{1}} \otimes \cdots \otimes c_{a_{m}}) \\
& 0 \leq j < m, i = m + 1 : d_{m+1}s_{j}(c_{a_{1}} \otimes \cdots \otimes c_{a_{m}}) \\
& \quad = a_{m}(c_{a_{1}} \otimes \cdots \otimes c_{a_{j}} \otimes c_{a_{j+1}} \otimes \cdots \otimes c_{a_{m-1}}) \\
& \quad = s_{j}d_{m}(c_{a_{1}} \otimes \cdots \otimes c_{a_{m}}). \\
& s_{i}s_{j} = s_{j+1}s_{i} \text{ if } i \leq j: \\
& \quad 0 \leq i \leq j \leq m : s_{i}s_{j}(c_{a_{1}} \otimes \cdots \otimes c_{a_{m}}) \\
& \quad = c_{a_{1}} \otimes \cdots \otimes c_{a_{i}} \otimes c_{a_{i+1}} \otimes \cdots \otimes c_{a_{j}} \otimes \cdots \otimes c_{a_{m}} \\
& \quad = s_{j+1}s_{i}(c_{a_{1}} \otimes \cdots \otimes c_{a_{m}}). 
\end{align*}
So, for example, in low degrees, the chain complex $U^A(n)_*$ is
\[ \cdots \to F_n(A) \otimes A F_n(A) \xrightarrow{\delta_1} F_n(A) \xrightarrow{\delta_0} A/v(n) \to 0 \]
with $\delta_0$ and $\delta_1$ defined by:
\[ \delta_0(c_a) = a - a^n, \]
\[ \delta_1(c_a \otimes c_b) = ac_b - c_{ab} + b^n c_a. \]
There are two reasons for defining $U$-homology beginning with a simplicial $A$-module in Definition-Proposition 2.1.2, and not just defining the chain complex directly: first, the alternating sign chain complex of a simplicial module is a chain complex, and checking the simplicial identities is (at least to a certain mindset) more conceptually satisfying than checking the condition $\delta \circ \delta = 0$ to verify that a sequence of modules is indeed a chain complex; and second, simplicial constructions generalize to nonabelian settings (e.g. spectra) much more readily than definitions that directly involve an alternating sum. I do not know whether any such generalizations of $\cdots$ are useful.

**Definition-Proposition 2.1.3.** Let $n > 1$ be an integer. Let $P^A_{n-1}$ denote the cokernel of the $A$-module homomorphism $A \to L^A_{n-1}/D^A_{n-1}$ sending 1 to $d$ (see Proposition 1.2.1 for the element $d$). Clearly $P^A_{n-1}$ is functorial in the choice of commutative ring $A$.

For any commutative ring $A$, the natural map of $A$-modules
\[ P^A_{n-1} \to P^A_{n-1}/v(n) \]
is an isomorphism.

**Proof.** After reducing modulo $d$, the Drinfeld relations 1.2.1, 1.2.2, and 1.2.3 become
\[ \begin{align*}
0 &= v(n)c_a, \\
c_{a+b} &= c_a + c_b, \\
c_{ab} &= ac_b + b^n c_a.
\end{align*} \]
In particular, $c_1 = 0$ and hence $c_{v(n)} = v(n)c_1 = 0$, and so $c_{v(n)a} = v(n)c_a + a^n c_{v(n)} = 0$. Hence $P^A_{n-1}$ is isomorphic, as an $A$-module, to the $A/v(n)$-module with one generator $c_a$ for each $a \in A/v(n)$, subject to the modulo $d$ Drinfeld relations above; this is exactly the modulo $d$ Drinfeld presentation for $L^A_{n-1}/D^A_{n-1}$, i.e., this is $P^A_{n-1}/v(n)$.

**Proposition 2.1.4.** Let $A$ be a commutative ring and let $n > 1$ be an integer. Suppose that $A$ is $v(n)$-torsion-free, i.e., if $a \in A$ and $v(n)a = 0$ then $a = 0$. Then $U^A_1(n) \cong 0$ if and only if the fundamental functional $\sigma_n : L^A_{n-1}/D^A_{n-1} \to A$ is injective. Furthermore, $U^A_1(n)$ is isomorphic to the cokernel of $\sigma_n$.

**Proof.** I will write $\delta_1, \delta_0$ for the differentials in $U^A(n)_*$, defined in Definition-Proposition 2.1.2. Clearly (see the proof of Definition-Proposition 2.1.3) the cokernel of $\delta_1$ is $P^A_{n-1}$, since $\text{coker} \delta_1$ and $P^A_{n-1}$ are $A/v(n)$-modules with the same set of generators as one another, and the same set of relations as one another. We have the commutative diagram of $A$-modules with exact rows
\[ \begin{array}{cccccc}
\text{ker } d & \to & A & \xrightarrow{d} & L^A_{n-1}/D^A_{n-1} & \to & P^A_{n-1} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A & \xrightarrow{\text{id}} & A & \xrightarrow{\text{id}} & A & \xrightarrow{\delta_0} & 0.
\end{array} \]
Since \( U^A_1(n) \equiv \ker \delta_0 / \text{im } \delta_1 \), vanishing of \( U^A_1(n) \) is equivalent to injectivity of \( \delta_0 \). The “four lemma” from homological algebra then tells us that injectivity of \( \delta_0 \) is equivalent to injectivity of \( \sigma_n \).

For the claim about \( U^A_0(n) \): we have the commutative square of \( A \)-modules with exact columns and rows

\[
\begin{array}{ccc}
A[d] \oplus \bigsqcup_{a \in A} A[c_a] & \xrightarrow{\sigma_n} & A \\
\pi' & & \text{coker } \sigma_n \to 0 \\
\bigsqcup_{a \in A} A[c_a] \otimes_A F_n(A) & \xrightarrow{\delta_n} & A/\nu(n) \\
\pi & & U^A_0(n) \to 0 \\
0 & & 0
\end{array}
\]

where \( \pi'(d) = 0 \) and \( \pi' \) is the modulo \( \nu(n) \) reduction map on each of the summands \( A[c_a] \), and \( \pi \) is the modulo \( \nu(n) \) reduction map. So \( U^A_0(n) \) is the reduction modulo \( \nu(n) \) of \( \text{coker } \sigma_n \), but \( \nu(n) \) is already zero in \( \text{coker } \sigma_n \) since \( \sigma_n(d) = \nu(n) \). \( \square \)

I do not know if there is any practical purpose for \( U^A_j(n) \) for \( j > 1 \). In the present paper I only ever have need of \( U^A_0(n) \) for \( j = 0, 1 \). The construction of \( U^A(n)_* \) resembles the construction of the cyclic bar complex, whose homology computes the Hochschild homology of a ring, but since \( A/\nu(n) \) is typically not a field, \( U^A(n)_* \) is typically not a special case of the usual cyclic bar complex that computes Hochschild homology. Instead the homology groups of \( U^A(n)_* \) are a kind of Shukla homology with twisted coefficients; see [15]. In Theorem 2.2.4 we see that when \( A \) is a field, the \( U \)-homology of \( A \) is indeed isomorphic to certain Hochschild homology groups.

This is an opportune time to introduce both the symmetric algebras and the Rees algebras, both of which are classical constructions, but for which we will need graded versions as well, which are slightly less classical:

**Definition 2.1.5.** Let \( A \) be a commutative ring, \( I \) an ideal of \( A \).

- By the Rees algebra of \( I \), written \( \text{Rees}_A(I) \), I mean the commutative \( A \)-algebra \( \bigsqcup_{n \ge 0} I^n[t^r] \subseteq A[t] \).
- Let \( j \) be an integer. By the \( j \)-suspended Rees algebra of \( I \), written \( \text{Rees}_A^j(I) \), I mean the commutative graded \( A \)-algebra whose underlying commutative \( A \)-algebra is \( \text{Rees}_A(I) \), and which is equipped with the grading in which the summand \( I^n[t^r] \) is in grading degree \( jn \).

Now, more generally, let \( A \) be a commutative ring and let \( M \) be an \( A \)-module.

- By the symmetric algebra of \( M \), written \( \text{Sym}_A(M) \), I mean the commutative \( A \)-algebra \( \bigsqcup_{n \ge 0} (M^{\otimes n})_{\Sigma_n} \), where \( (M^{\otimes n})_{\Sigma_n} \) is the orbit module under the action of the symmetric group \( \Sigma_n \) on \( M^{\otimes n} \) given by permuting the tensor factors.
- Let \( j \) be an integer. By the \( j \)-suspended symmetric algebra of \( M \), written \( \text{Sym}_A^j(M) \), I mean the commutative graded \( A \)-algebra whose underlying commutative \( A \)-algebra is \( \text{Sym}_A(M) \), and which is equipped with the grading in which the summand \( (M^{\otimes n})_{\Sigma_n} \) is in grading degree \( jn \).

**Definition 2.1.6.** Let \( A \) be a commutative ring. I will say that \( A \) satisfies the fundamental comparison condition if the \( A \)-module \( L^A_{n-1}/D^A_{n-1} \) is projective for all integers \( n > 1 \).

If \( A \) satisfies the fundamental comparison condition, then the projection \( A \)-module map \( L^A_{n-1} \to L^A_{n-1}/D^A_{n-1} \) splits for all integers \( n > 1 \). Choose such a splitting \( A \)-module map.
SymA \left( \bigsqcup_{n>1} L_{n-1}^A / D_{n-1}^A \right) \xrightarrow{s} L^A \xleftarrow{\beta} \text{SymA} \left( \bigsqcup_{n>1} A \right) \xrightarrow{\alpha} A[x_1, x_2, \ldots]

where \( s = \text{SymA} \left( \bigsqcup_{n>1} \sigma_n \right) \), the symmetric algebra functor applied to the coproduct of the fundamental functionals. (Note that we need \( A \) to satisfy the fundamental comparison condition in order to define the fundamental comparison triangle.) Finally, I will say that \( L^A \) is polynomial by the fundamental comparison if each of the \( A \)-algebra homomorphisms in the fundamental comparison triangle are isomorphisms.

Remark 2.1.7. The fundamental comparison triangle is, at least \textit{a priori}, not natural in the choice of \( A \), since it involves making choices of the splitting maps \( \{ i_{n-1} \} \), and when one has a homomorphism from one split short exact sequence to another, there is an (often nontrivial) obstruction to the existence of a \textit{compatible} splitting of the two short exact sequences; see \cite{12} for this material.

Remark 2.1.8. Although \( \text{SymA} \) preserves epimorphisms, it typically does not preserve monomorphisms; see section 6.2 of chapter III of \cite{3}.

Proposition 2.1.9. Suppose that \( A \) is a commutative ring satisfying the fundamental comparison condition, and suppose that \( A \) is additively torsion-free, i.e., if \( a \in A \) and \( ma = 0 \) for some \( m \in \mathbb{Z} \) then either \( a = 0 \) or \( m = 0 \). Then \( L^A \) is polynomial by the fundamental comparison if and only if \( U_1^A(n) = 0 \) and \( U_0^A(n) = 0 \) for all integers \( n > 1 \).

Proof. From Theorem 1.2.3 we know that the natural commutative graded \( \mathbb{Q} \otimes \mathbb{Z} \)-algebra homomorphism \( \mathbb{Q} \otimes \mathbb{Z} L^A \to \mathbb{L}^{\mathbb{Q} \otimes \mathbb{A}} \) is an isomorphism. We can fit the fundamental comparison triangle for \( A \) together with the fundamental comparison triangle for \( \mathbb{Q} \otimes \mathbb{Z} A \) to get the diagram

In light of Remark 2.1.7, I ought to explain why we have a map of fundamental comparison triangles in this situation. The reason is that, when we choose a splitting \( i_{n-1} \) of the projection \( \pi_{n-1} : L_{n-1}^A \to L_{n-1}^A / D_{n-1}^A \), we can tensor that splitting map over \( A \) with \( \mathbb{Q} \otimes \mathbb{Z} A \).
to get a splitting map $\mathbb{Q} \otimes_\mathbb{Z} L^A_{n-1} \to \mathbb{Q} \otimes_\mathbb{Z} L^A_{n-1} / D^A_{n-1}$, and hence, using Theorem 1.2.3, a commutative diagram of $A$-modules

\[
\begin{array}{ccc}
L^A_{n-1} / D^A_{n-1} & \xrightarrow{id} & L^A_{n-1} / D^A_{n-1} \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
• $L^A$ is isomorphic, as a graded $A$-algebra, to the tensor product of the suspended symmetric algebras of the ideals $I_2^A, I_3^A, \ldots$ of $A$:

$$L^A \cong \bigotimes_{n>1} \text{Sym}_{A}^{2(n-1)}(I_n^A)$$

$$\cong \text{Sym}_{A}(I_2^A) \otimes_{A} \text{Sym}_{A}(I_3^A) \otimes_{A} \text{Sym}_{A}(I_4^A) \otimes_{A} \ldots,$$

and

• $L^A$ is isomorphic, as a graded $A$-algebra, to the tensor product of the suspended Rees algebras of the ideals $I_2^A, I_3^A, \ldots$ of $A$:

$$L^A \cong \bigotimes_{n>1} \text{Rees}_{A}^{2(n-1)}(I_n^A)$$

$$\cong \text{Rees}_{A}(I_2^A) \otimes_{A} \text{Rees}_{A}(I_3^A) \otimes_{A} \text{Rees}_{A}(I_4^A) \otimes_{A} \ldots.$$  

Proof. 

• Since $\sigma_n$ is injective for all $n$, the $A$-module $L^A_{n-1}/D^A_{n-1}$ is a sub-$A$-module of a free $A$-module for all $n$, and $L^A_{n-1}/D^A_{n-1}$ is projective for all $n$, by assumption. So $A$ satisfies the fundamental comparison condition.

• I claim that the inclusion $I_n^A \subseteq A$, regarded as an $A$-module map, induces a monomorphism $\text{Sym}_{A}(I_n^A) \to \text{Sym}_{A}(A)$ after applying $\text{Sym}_{A}$, since $I_n^A$ was assumed to be projective (although this is not generally true without making some assumption, like projectivity or flatness, on $I_n^A$, see Remark 2.1.8). The argument for this is as follows: we have the short exact sequence of $A$-modules

$$0 \to I_n^A \to A \to A/I_n^A \to 0,$$

and on tensoring with $I_n^A$, the exact sequence

$$0 \to I_n^A \otimes_{A} I_n^A \to I_n^A \otimes_{A} A,$$

since $\text{Tor}_1(I_n^A, A/I_n^A) \cong 0$ by projectivity of $I_n^A$. So the map $I_n^A \otimes_{A} I_n^A \to I_n^A$, which agrees with multiplication, is injective. It is also surjective on to $(I_n^A)^2 \subseteq I_n^A$, since any given element $j \in (I_n^A)^2$ is the image of $1 \otimes j$. So the multiplication map $I_n^A \otimes_{A} I_n^A \to (I_n^A)^2$ is an isomorphism of $A$-modules. A completely analogous argument shows easily that the multiplication map $(I_n^A)^\otimes_m \to (I_n^A)^m$ is an isomorphism of $A$-modules; since $(I_n^A)^m = (I_n^A)^m_{\Sigma_m} = (I_n^A)^m_{\Sigma_m}$, we now have that $(I_n^A)^m_{\Sigma_m}$, hence that the natural map of $A$-algebras $\text{Sym}_{A}(I_n^A) \to \text{Rees}_{A}(I_n^A)$ is a levelwise isomorphism. Of course the natural map $\text{Rees}_{A}(I_n^A) \to \text{Rees}_{A}(A)$ is a monomorphism by definition.

Now we need to know why the map $s$ is injective and why $\hat{\ell}$ is an isomorphism. The map $s$ is $\text{Sym}_{A}$ applied to the coproduct $\coprod_{n>0} \sigma_n$, and so we have just shown that $s$ is injective. Now we have the commutative diagram:

$$\begin{array}{ccc}
\text{Sym}_{A}(\coprod_{n>1} L^A_{n-1}/D^A_{n-1}) & \xrightarrow{\hat{\ell}} & \text{Sym}_{A}(L^A) \\
Q \otimes_{A} \text{Sym}_{A}(\coprod_{n>1} L^A_{n-1}/D^A_{n-1}) & \xrightarrow{s} & \text{Sym}_{A}(\coprod_{n>1} A) \\
\end{array}$$
true: if $\nu$ by is additively torsion-free, and suppose that the fundamental functional ideal in $R$, then every formal $A$-module over $R$ extends to a formal $A$-module. Furthermore, if $R$ is a commutative $A$-algebra and $I$ is an injective, and has image a projective module, for all $n$. Then every formal $A$-module $n$-bud pose that $A$ is additively torsion-free, and suppose that the fundamental functional

\begin{corollary}
\label{cor:lifting-and-extensions}

Corollary 2.1.12. (Lifting and extensions.) Suppose that $A$ is a hereditary commutative ring, suppose that $A$ is additively torsion-free, and suppose that the fundamental functional $\sigma_n$ is injective, and has image a projective module, for all $n$. Then every formal $A$-module $n$-bud extends to a formal $A$-module. Furthermore, if $R$ is a commutative $A$-algebra and $I$ is an ideal in $R$, then every formal $A$-module over $R/I$ is the reduction modulo $I$ of a formal $A$-module over $R$.  

The suspended symmetric and suspended Rees algebras were defined in Definition 2.1.5. The fundamental comparison maps $s$ and $\overline{s}$ were defined in Definition 2.1.6.

The fundamental comparison works especially well for hereditary rings $A$:

\begin{corollary}
\label{cor:lifting-and-extensions}

Suppose that $A$ is a hereditary commutative ring, suppose that $A$ is additively torsion-free, and suppose that the fundamental functional $\sigma_n$ is injective for all $n$. For each integer $n > 1$, let $I_n^4$ denote the image of $\sigma_n$, i.e., $I_n^4$ is the ideal in $A$ generated by $v(n)$ and by all elements of the form $a - a^n \in A$. Then the following statements are all true:

- $A$ satisfies the fundamental comparison condition,
- of the fundamental comparison maps $s : \text{Sym}_A(\coprod_{n>1} I_{n-1}^4/D_{n-1}^4) \to \text{Sym}_A(\coprod_{n>1} A)$ and $i : \text{Sym}_A(\coprod_{n>1} L_{n-1}^4/D_{n-1}^4) \to L^A$, the map $s$ is injective and $i$ is an isomorphism (and consequently, $L^A$ is isomorphic to a sub-$A$-algebra of a polynomial $A$-algebra),
- $L^A$ is isomorphic, as a graded $A$-algebra, to the tensor product of the suspended symmetric algebras of the ideals $I_3^4, I_4^4, \ldots$ of $A$:

$$L^A \cong \bigotimes_{n>1} \text{Sym}_A^{2n-1}(I_n^4)$$

$$\cong \text{Sym}_A^2(I_2^4) \otimes_A \text{Sym}_A^4(I_3^4) \otimes_A \text{Sym}_A^6(I_4^4) \otimes_A \ldots,$$

and

- $L^A$ is isomorphic, as a graded $A$-algebra, to the tensor product of the suspended Rees algebras of the ideals $I_3, I_4, \ldots$ of $A$:

$$L^A \cong \bigotimes_{n>1} \text{Rees}_A^{2n-1}(I_n^4)$$

$$\cong \text{Rees}_A^2(I_2^4) \otimes_A \text{Rees}_A^4(I_3^4) \otimes_A \text{Rees}_A^6(I_4^4) \otimes_A \ldots.$$
from Theorem 2.1.10 we have a commutative square

\[
\begin{array}{ccc}
L_{\text{uf}} & \xrightarrow{\iota} & L_{\text{uf}} \\
\cong & & \cong \\
\text{Sym}_A \left( \bigoplus_{i \in \mathbb{N}} I_m^A \right) & \xrightarrow{\text{Sym}_A(\kappa)} & \text{Sym}_A \left( \bigoplus_{i \in \mathbb{N}} I_m^A \right)
\end{array}
\]

where \( \kappa \) is the inclusion of the summand \( \kappa : \bigoplus_{i \in \mathbb{N}} I_m^A \xrightarrow{\cong} \bigoplus_{i \in \mathbb{N}} I_m^A \). By the universal property of \( \text{Sym}_A \), every morphism of commutative \( A \)-algebras \( L_{\text{uf}}^A \to R \) extends over \( \iota \) to a morphism of commutative \( A \)-algebra \( L_{\text{uf}}^A \to R \), hence every formal \( A \)-module \( n \)-bud extends to a formal \( A \)-module.

Furthermore, by the universal property of \( \text{Sym}_A \) and the lifting property of projective modules, every morphism of commutative \( A \)-algebras \( L_{\text{uf}}^A \to R/I \) lifts to a morphism \( L_{\text{uf}}^A \to R \). Hence every formal \( A \)-module over \( R/I \) is the reduction modulo \( I \) of a formal \( A \)-module over \( R \). \( \square \)

2.2. Local properties of \( U \)-homology.

**Lemma 2.2.1.** Let \( A \) be a commutative ring, \( n > 1 \) an integer, \( I \) a finitely generated ideal of \( A \) containing \( \nu(n) \). Let \( m \) be a positive integer. Define a valuation function \( \nu : F_n(A)^{\otimes m} \to \mathbb{N} \) by letting \( \nu(\alpha c_{a_1} \otimes \ldots \otimes c_{a_m}) = \nu_1(\alpha) + \sum_{j=1}^m \nu_1(a_{a_j}) \).

Topologize the \( m \)-fold tensor power \( F_n(A)^{\otimes m} \) by letting a neighborhood basis of zero be

\[
F_n(A)^{\otimes m}_i = (F_n(A)^{\otimes m}_i)^0 \supseteq (F_n(A)^{\otimes m}_i)^1 \supseteq (F_n(A)^{\otimes m}_i)^2 \supseteq \ldots
\]

where \( (F_n(A)^{\otimes m}_i)^j \) is the sub-\( A \)-module of \( (F_n(A)^{\otimes m}_i)^j \) consisting of elements of valuation \( \geq j \). Call this the filtration topology on \( F_n(A)^{\otimes m}_i \). This filtration (and its associated topology) induces a filtration (and associated topology) on the chain complex \( U_A^n(n)_i \). I will write \( \left( U_A^n(n)_i \right) \) for the completion of \( U_A^n(n)_i \) with respect to this topology.

Suppose that the underlying abelian group of \( A \) is finitely generated. Then the natural map \( U_A^n(n) \to H_1 \left( \left( U_A^n(n)_i \right) \right) \) coincides with the \( I \)-adic completion map \( U_A^n(n) \to \left( U_A^n(n)_i \right) \).

**Proof.** Since the underlying abelian group of \( A \) is finitely generated, \( A \) is Noetherian, and furthermore \( F_n(A) \) is a finitely generated \( A \)-module. Hence \( F_n(A)^{\otimes i} \cong U_A^n(n)_i \) is a finitely generated \( A \)-module for all integers \( i \). By a standard Artin-Rees argument, completion with respect to the valuation filtration on \( F_n(A)^{\otimes i} \) coincides with completion with respect to the \( I \)-adic topology.

Hence we have an isomorphism of chain complexes of \( A \)-modules

\[
\left( U_A^n(n)_i \right) \cong U_A^n(n)_i \otimes_A \hat{A}_I,
\]

and since \( \hat{A}_I \) is a flat \( A \)-module (for this fact and many others which I am using in this proof, see chapter 10 of [1]), we have isomorphisms of \( A \)-modules

\[
H_* \left( \left( U_A^n(n)_i \right) \right) \cong H_* \left( U_A^n(n)_i \otimes_A \hat{A}_I \right) \cong H_* \left( U_A^n(n)_i \right) \otimes_A \hat{A}_I \cong H_* \left( U_A^n(n)_i \right)_I.
\]
Recall that, in J. P. May’s 1964 doctoral thesis, [10], May filtered the Steenrod algebra by powers of its maximal homogeneous ideal, then studied the spectral sequence arising in Ext from the induced filtration on the bar complex of the Steenrod algebra. In Lemma 2.2.2 I construct a similar spectral sequence, but for \( U \)-homology rather than for Ext groups:

**Lemma 2.2.2.** (The May spectral sequence for \( U \)-homology.) Let \( A \) be a commutative ring, let \( n > 1 \) be an integer, and suppose that \( I \) is an ideal of \( A \) containing \( v(n) \). Suppose that \( A \) is \( I \)-adically separated. Equip the chain complex \( U^A(n)_* \), with the decreasing filtration defined in Lemma 2.2.1.

Then there exists a conditionally convergent spectral sequence

\[
E^1_{s,t} = H_s(\mathcal{E}_0(U^A(n)_*)) \Rightarrow U^A(n)_t
\]

where \( \mathcal{E}_0(U^A(n)_*) \) is the associated graded chain complex of the given filtration on \( U^A(n)_* \), and \( H_s(\mathcal{E}_0(U^A(n)_*)) \) is its homology in homological degree \( s \) and internal grading degree \( t \).

**Proof.** This is just the spectral sequence of a filtered chain complex, as in Theorem 9.3 of [2]. From the general theory of such spectral sequences, one knows that it converges to the homology of the completion \( (U^A(n)_*)^\wedge \) with respect to the given filtration; by Lemma 2.2.1, the homology of this completion is the \( I \)-adic completion of \( U^A(n)_* \).

\[ \square \]

**Theorem 2.2.3.** (Rigidity of \( U \)-homology.) Let \( n > 1 \) be an integer, and let \( A \) be a commutative ring with a maximal ideal \( \mathfrak{m} \) containing \( v(n) \). Suppose that the underlying abelian group of \( A \) is finitely generated. Then the reduction map \( A \to A/\mathfrak{m} \) induces isomorphisms of \( A \)-modules

\[
(2.2.1) \quad U^A(n)_* \otimes A/\mathfrak{m} \to U^A(n)/\mathfrak{m}
\]

\[
(2.2.2) \quad U^A(n)_* \otimes A/\mathfrak{m} \to U^A(n)/\mathfrak{m}
\]

**Proof.** First, since \( A \) is finitely generated as an abelian group, it is \( \mathfrak{m} \)-adically separated. Lemma 2.2.2 applies to this situation. We need to identify the \( E^1_{1,0} \)-term of the May spectral sequence. First, since \( U^A(n)_*/F^1U^A(n)_* \) is isomorphic to \( U^{A/\mathfrak{m}}(n)_*/F^1U^{A/\mathfrak{m}}(n)_* \), the \( t = 0 \) line \( E^1_{1,0} = H_{-1}(\mathcal{E}_0(U^A(n)_*)) \) is the homology of the chain complex \( U^{A/\mathfrak{m}}(n)_* \).

Now I claim that \( E^1_{1,0} \equiv 0 \) if \( t > 0 \) and \( 0 \leq s \leq 1 \). The proof is as follows: when \( t > 0 \), \( E^1_{0,t} \) is the homology of the chain complex

\[
F^i(F_n(A)/F^{i+1}(F_n(A))) \xrightarrow{\delta_1} F^iA/F^{i+1}A \to 0,
\]

where \( \delta_1(\alpha c_0) = \alpha a \) if \( a \in I \) and \( \delta_1(\alpha c_0) = \alpha(a - d^n) \) if \( a \notin I \). Consequently every element in \( F^iA/F^{i+1}A \) for \( t > 0 \) is in the image of \( \delta_1 \), and consequently \( E^1_{0,t} \equiv 0 \) for \( t > 0 \).

Now when \( t > 0 \), \( E^1_{1,t} \) is the homology of the chain complex

\[
F^i(F_n(A)^{\oplus 2})/F^{i+1}(F_n(A)^{\oplus 2}) \xrightarrow{\delta_2} F^i(F_n(A))/F^{i+1}(F_n(A)) \xrightarrow{\delta_1} F^iA/F^{i+1}A,
\]

where \( \delta_2(\alpha c_a \otimes c_b) = \alpha a c_b - \alpha c_{ab} + a b^n c_a \). Suppose \( x \) is a 1-cycle in \( F^i(F_n(A))/F^{i+1}(F_n(A)) \). Using the boundary formula

\[
\delta_2(\alpha c_a \otimes c_b) = \alpha a c_b - \alpha c_{ab} + a b^n c_a,
\]
if \( b \in I \), then \( aac_b \) is homologous to \( \alpha c_{ab} \), so choose a set of generators \( i_1, \ldots, i_m \) for \( I \), and then up to boundaries we can write \( x \) as

\[
x = \sum_{j=1}^{m} \alpha_j c_{i_j} + \sum_{s \in S, i_s \notin I} \beta_s c_{a_s}.
\]

I will also make use of the boundary formula

\[
\tilde{\delta}_1(c \beta, \otimes c_{a_i} - c \sigma_i \otimes c \rho_i) = (\beta_s - \delta^p_s) c_{a_s} - c a \beta_i + c \sigma_i \beta_i
\]

for \( a_i \in I \) and \( \beta_i \in A \). Since \( x \) is a cycle, we have

\[
\tilde{\delta}_1(x) = \sum_{j=1}^{m} \alpha_j j + \sum_{s \in S, i_s \notin I} \beta_s (a_s - a_i^p),
\]

so up to boundaries, we have

\[
\sum_{j=1}^{m} \alpha_j j = \sum_{j=1}^{m} \alpha_j j,
\]

\[
= c_{\sum_{j=1}^{m} \alpha_j j},
\]

\[
= c_{\sum_{s \in S, i_s \notin I} \beta_s (a_s - a_i^p)}
\]

\[
= \sum_{s \in S, i_s \notin I} (c \beta, a_s - c \rho_i)
\]

\[
= \sum_{s \in S, i_s \notin I} (-\beta_s + \delta^p_s) c_{a_s},
\]

and now I use the fact that, if \( s \in S \), then \( a_s \notin I \) and hence \( \beta_s \in I \) so that \( \beta_s c_{a_s} \) is in filtration \( t > 0 \); so \( \beta^p_s = 0 \), and hence

\[
x = \sum_{j=1}^{m} \alpha_j c_{i_j} + \sum_{s \in S, i_s \notin I} \beta_s c_{a_s} = 0
\]

modulo boundaries. Hence \( E^1_{1,t} \cong 0 \) whenever \( t > 0 \).

Hence the \( s = 0 \) and \( s = 1 \) rows of the May spectral sequence vanish at \( E^1 \) whenever \( t > 0 \). Hence there is no room for differentials or extension problems in the spectral sequence when \( s = 0 \) and \( s = 1 \). Hence the maps 2.2.1 and 2.2.2 are isomorphisms. □

**Proposition 2.2.4.** Let \( k \) be a finite field, and let \( n > 1 \) be a prime power. Then \( U^k_i(n) \cong 0 \) for all positive integers \( i \). Furthermore, \( U^k_0(n) \cong 0 \) if \( n \) is not a power of the number of elements in \( k \), and \( U^k_0(n) \cong k \) if \( n \) is a power of the number of elements in \( k \).

**Proof.** Let \( k = \mathbb{F}_{p^n} \). Since \( k \) is a field, \( t^k(n)_* \) is just the cyclic bar construction on \( F_n(k) \) with coefficients in the \( F_n(k) \)-module \( \mathbb{I}_{k_n} \), where \( \mathbb{I}_{k_n} \) is a one-dimensional \( k \)-vector space with \( F_n(k) \) acting on \( \mathbb{I}_{k_n} \) on the left by \( e_a \cdot x = ax \) and acting on the right by \( x \cdot c_{a} = a^t x \). Consequently \( U^k_1(n) \cong HH_1(F_n(k); \mathbb{I}_{k_n}) \) for all \( i \).

The underlying abelian group of \( k \) is \( \mathbb{F}_{p^n} \), hence \( F_n(k) \) is an \( m \)-fold product of copies of \( k \), with ring structure (given by letting \( \alpha c_a \cdot \beta c_b = \alpha \beta c_{ab} \)) coinciding with that on \( \mathbb{F}_{p^n} \otimes \mathbb{F}_{p^n} \), i.e., since \( \mathbb{F}_{p^n} / \mathbb{F}_p \) is Galois, we have an isomorphism of rings \( \mathbb{F}_{p^n} \otimes \mathbb{F}_{p^n} \cong \text{hom}_{\mathbb{F}_{p^n}}(\mathbb{F}_{p^n} / \mathbb{F}_p, \mathbb{F}_{p^n}), \) the \( \mathbb{F}_{p^n} \)-linear dual Hopf algebra to the \( \mathbb{F}_{p^n} \)-group algebra of the Galois group \( \text{Gal}(\mathbb{F}_{p^n} / \mathbb{F}_p) \). In particular, \( F_n(k) \) is a separable \( k \)-algebra, hence \( HH_1(F_n(k); \mathcal{M}) \) vanishes for all positive integers \( i \) and all \( F_n(k) \)-bimodules \( \mathcal{M} \), hence \( U^k_i(n) \) vanishes for all positive integers \( i \).
Lemma 2.2.5. Let $A$ be a commutative ring and let $M$ be an $A$-module, The canonical $A$-module homomorphism

$$f : M \to \prod_m M_m,$$

with the product taken over all maximal ideals $m$ of $A$, is injective.

Proof. If $m \in M$ satisfies $f(m) = 0$, then $m$ maps to zero under each localization map $M \to M_m$, and consequently, for each maximal ideal $m$ of $A$ there exists some element $w \in A$ not in $m$ such that $wm = 0$. Hence the annihilator ideal $\text{Ann}_A(m)$ of $m$ is not contained in any single maximal ideal of $A$, hence $\text{Ann}_A(m) = A$, hence $1m = 0$ and hence $m = 0$. So $f$ is injective.

Theorem 2.2.6. Let $n > 1$ be a power of a prime $p$, and let $A$ be a commutative ring. Suppose that the underlying abelian group of $A$ is finitely generated. Then the fundamental functional $\sigma_n : L^{n-1}_{\mathbb{Z}}/D^{n-1}_{\mathbb{Z}} \to A$ is injective. Furthermore, $\sigma_n$ is surjective if, for all maximal ideals $m$ of $A$ containing $\wp(n)$, $n$ is not a power of the number of elements in $A/m$.

If we assume additionally that $A$ is a local ring with maximal ideal $m$ containing $\wp(n)$, then $\sigma_n$ is surjective if and only if $n$ is not a power of the number of elements in $A/m$.

Proof. Theorem 2.2.3 and Proposition 2.2.4 together imply that, for each maximal ideal $m$ of $A$ containing $(p)$, the $A$-module $(U^A_i(n))_m$ is trivial, and that $(U^A_0(n))_m \equiv 0$ if and only if $n$ is not a power of the number of elements in $A/m$.

Here is the argument for triviality of $U^A_i(n)$; the argument for $U^A_0(n)$ is completely analogous. Under the assumptions stated in the statement of the theorem, we know that $(U^A_1(n))_m \equiv 0$ for all maximal ideals $m$ of $A$ containing $(p)$, hence that $0 = U^A_1(n)/mU^A_1(n)$ for all such ideals, hence that $m(U^A_1(n)) = (U^A_1(n))_m$: here I am writing $(U^A_1(n))_m$ for the localization of $U^A_1(n)$ at the maximal ideal $m$ of $A$. Since $A$ is finitely generated as an abelian group, we know that $F^A(n)$ is a finitely generated $A$-module, hence that $(F^A(n))^{\otimes i} = U^A_i(n)$ is a finitely generated $A$-module for each positive integer $i$, hence that the homology of the chain complex $U^A_i(n)$ consists of finitely generated $A$-modules. In particular $(U^A_1(n))_m$ is a finitely generated $A_m$-module, so the equality $(U^A_1(n))_m = m(U^A_1(n))_m$ implies $(U^A_1(n))_m \equiv 0$, by Nakayama’s lemma. Hence $U^A_1(n)$ is an $A/(p)$-module whose localization $(U^A_1(n))_m$ is trivial for all maximal ideals $m$ of $A/(p)$. Hence, by Lemma 2.2.5, the $A/(p)$-module $U^A_1(n)$ vanishes. Now Proposition 2.1.4 implies the injectivity of $\sigma_n$.

2.3. Global consequences. It is easy to show that $U$-homology commutes with localizations that invert integers (although showing that taking the the $U$-homology of $A$ commutes with inverting elements in $A$, not in $\mathbb{Z}$, is a trickier question):

Proposition 2.3.1. (Localization for $U$-homology.) Let $A$ be a commutative ring, let $S$ be a multiplicatively closed subset of $\mathbb{Z}$, and let $n > 1$ be an integer. Then, for each nonnegative integer $i$, there exist a natural isomorphism of $A[S^{-1}]$-modules

$$U^A_i(n)[S^{-1}] \cong U^A_i(n[S^{-1}]).$$
Proof. We have the forgetful functor $U : \text{Mod}(A) \to \text{Mod}(\mathbb{Z})$ and the free functor $F : \text{Mod}(\mathbb{Z}) \to \text{Mod}(\mathbb{Z}/\mathfrak{p}(n))$, and both of these functors preserve all small colimits, including the colimit that defines the localization inverting $S$. The functor $F_n$ is simply the composite $F \circ U$, so the $A$-module $F_n(A)$ has the property that $F_n(A)[S^{-1}] \cong F_n(A)[S^{-1}]$.

Since $(F_n(A)[S^{-1}])^\otimes A \cong (F_n(A)[S^{-1}])^\otimes \mathbb{Z} \cong (F_n(A))^\otimes \mathbb{Z} \cong \mathbb{Z}[S^{-1}]$
and since these isomorphisms are compatible with the face and degeneracy maps defining $\tau$ (as in Definition-Proposition 2.1.2), we have $\tau L^A(g)(n) \cong \tau L^A(g)(S^{-1})$, and since $\mathbb{Z}[S^{-1}]$ is a flat $\mathbb{Z}$-module, we have exactly the isomorphism 2.3.1.

Theorem 2.3.2. Let $A$ be a commutative ring whose underlying abelian group is finitely generated and free. Let $S$ be a set of prime numbers such that the ring $A[S^{-1}]$ is hereditary. (If, for example, $A$ is already hereditary, then we can let $S$ be the empty set.)

Then the commutative graded ring $L^A$ is, after inverting $S$, isomorphic to a tensor product of ("suspended," i.e., graded) Rees algebras:

$$L^A[S^{-1}] \cong \left(\text{Rees}_S^2(I^A_n) \otimes_A \text{Rees}_A^1(I^A_n) \otimes_A \text{Rees}_A^0(I^A_n) \otimes_A \text{Rees}_A^0(I^A_n) \otimes_A \text{Rees}_A^0(I^A_n) \otimes_A \ldots\right)[S^{-1}]$$

where $I^A_n$, for each positive integer $n$, is the ideal of $A$ defined as follows:

- if $n$ is not a prime power or $n$ is a power of a prime in $S$ then $I^A_n = A$, i.e., $\text{Rees}_A^{2n-2}(I^A_n) \cong A[x_{2n-2}], x_{2n-2}$ in grading degree $2n-2$.
- if $n$ is a power of a prime $p$ not contained in $S$, then $I^A_n$ is the ideal generated by $p$ and by all elements of the form $a - d^i$ in $A$.

Proof. Theorem 2.2.6 tells us that, for each power $n$ of a prime not contained in $S$, the fundamental functional $\sigma_n : L^A_n / D^A_n \to A$ is injective and has image $I^A_n$, and so $L^A_n / D^A_n \cong (L^A_n / D^A_n)[S^{-1}] \to A[S^{-1}]$ is injective with image $I^A_n[S^{-1}]$. For the integers $n$ such that $n$ is not a prime power, we have $L^A_n / D^A_n \cong A$ by Definition-Proposition 1.2.4. and for the integers $n$ such that $n$ is a power of a prime in $S$, we have that $L^A_n / D^A_n \cong A[S^{-1}]$ since $L^A_n / D^A_n$ is an $A/\mathfrak{p}(n)$-module, and $L^A_n / D^A_n \cong A[S^{-1}]$ by Proposition 2.1.14. In particular, Proposition 2.1.4 implies that the fundamental functional $\sigma_n : L^A_n / D^A_n \to A[S^{-1}]$ is injective for all $n > 1$.

Now $A[S^{-1}]$ is assumed to be hereditary, so injectivity of the fundamental functionals for $A[S^{-1}]$ along with Proposition 2.1.11 together imply that we have isomorphisms of commutative graded $A[S^{-1}]$-algebras

$$L^A[S^{-1}] \cong \text{Sym}_A^2(I_2[S^{-1}]) \otimes_A \text{Sym}_A^1(I_3[S^{-1}]) \otimes_A \text{Sym}_A^0(I_4[S^{-1}]) \otimes_A \text{Sym}_A^0(I_5[S^{-1}]) \otimes_A \ldots$$

$$\cong \text{Rees}_A^2(I_2[S^{-1}]) \otimes_A \text{Rees}_A^1(I_3[S^{-1}]) \otimes_A \text{Rees}_A^0(I_4[S^{-1}]) \otimes_A \text{Rees}_A^0(I_5[S^{-1}]) \otimes_A \ldots$$

$$\cong \left(\text{Rees}_A^2(I_2) \otimes_A \text{Rees}_A^1(I_3) \otimes_A \text{Rees}_A^0(I_4) \otimes_A \text{Rees}_A^0(I_5) \otimes_A \ldots\right)[S^{-1}],$$

and finally Theorem 1.2.3 implies that the natural map $L^A[S^{-1}] \to L^A[S^{-1}]$ is an isomorphism.

Corollary 2.3.3. Let $A$ be a commutative ring and $S$ a set of prime numbers such that $A$ and $S$ satisfy the assumptions of Theorem 2.3.2. Then the following statements are all true:

- $L^A[S^{-1}]$ is a commutative graded sub-$A$-algebra of $A[S^{-1}][x_1, x_2, \ldots]$, with $x_i$ in grading degree $2i$.
- $L^A[S^{-1}]$ is not Noetherian, but for every integer $n$, the sub-$A$-algebra of $L^A[S^{-1}]$ generated by all elements of grading degree $\leq n$ is Noetherian.
• If $A[S^{-1}]$ is an integral domain, then the underlying $A[S^{-1}]$-module of $L^A[S^{-1}]$ is torsion-free.
• If $U^A(n)[S^{-1}]$ is trivial for all $n$, then $L^A[S^{-1}]$ is polynomial by the fundamental comparison condition.
• All formal module buds extend: Every formal $A$-module $n$-bud over a commutative $A[S^{-1}]$-algebra extends to a formal $A$-module.
• All formal modules lift: Since $A[S^{-1}]$ is hereditary, the image of $\sigma_n[S^{-1}]$ in $A[S^{-1}]$ is a projective $A[S^{-1}]$-module, and consequently $L^A[S^{-1}]$ is a symmetric $A[S^{-1}]$-algebra on a projective $A[S^{-1}]$-module. Consequently, if $R$ is a commutative $A[S^{-1}]$-algebra and $I$ is an ideal of $R$, then every formal $A$-module over $R/I$ is the modulo-$I$ reduction of a formal $A$-module over $R$.

Proof. Theorem 2.3.2 together with Corollary 2.1.12.

Corollary 2.3.4. Let $K/\mathbb{Q}$ be a finite field extension with ring of integers $A$. Then every formal $A$-module $n$-bud extends to a formal $A$-module. Furthermore, if $R$ is a commutative $A$-algebra and $I$ is an ideal in $R$, then every formal $A$-module over $R/I$ is the reduction modulo $I$ of a formal $A$-module over $R$.

Remark 2.3.5. It may be useful to have a slightly smaller and consequently more concrete description of the ideal $I^R_n$ of $A$ appearing in Theorem 2.3.2, when $A$ is generated as a commutative ring by a single element $t \in A$, i.e., the ring homomorphism $\mathbb{Z}[t] \to A$, sending $t$ to $t$, is surjective. Then $I^A_n = (\nu(n), t^\nu(n) - t)$. The proof is elementary: if $a$ is an element of $A$, write $a$ as a polynomial $a = \sum_{j \geq 0} a_j t^j$, where each $a_j$ is an integer. Then

$$a^{\nu(n)} - a = \left( \sum_{j \geq 0} a_j t^j \right)^{\nu(n)} - \sum_{j \geq 0} a_j t^j$$

$$\equiv \sum_{j \geq 0} a_j^{\nu(n)} t^{\nu(n)} - \sum_{j \geq 0} a_j t^j \mod \nu(n)$$

$$\equiv \sum_{j \geq 0} a_j \left( t^{\nu(n)} - t^j \right) \mod \nu(n)$$

$$\equiv 0 \mod (\nu(n), t^{\nu(n)} - t),$$

since $\nu(n)$ is always either 1 or a prime number.

Remark 2.3.6. It seems reasonable to conjecture that the finite generation condition in the statement of Theorem 2.3.2 can be removed, at least if $A$ is a Dedekind domain of characteristic zero. I do not know a proof of this, though. It would simply require extending the rigidity theorem, Theorem 2.2.3, to the case where $A$ is not finitely generated. This in turns amounts to solving the following problem: when $A$ is not finitely generated, the May spectral sequence of Lemma 2.2.2 converges to the homology of the completion of the chain complex $U^A(n)_\ast$ with respect to the filtration defined in the lemma. One needs to know that, if this completion is acyclic, then the original chain complex $U^A(n)_\ast$ is acyclic. Despite some effort I did not find a way to prove this in general (nor did I find a counterexample).

3. Computations of $L^A$ for certain classes of ring $A$.

3.1. Number rings.

Theorem 3.1.1. Let $A$ be the ring of integers in a finite extension $K/\mathbb{Q}$, let $1, \alpha_1, \ldots, \alpha_j$ be a $\mathbb{Z}$-linear basis for $A$, and let $J^A_n$ be the ideal $(\nu(n), \alpha_1 - \alpha_1^n, \alpha_2 - \alpha_2^n, \ldots, \alpha_j - \alpha_j^n)$ of $A$. 
Let $P$ denote the set of integers $> 1$ which are prime powers, and let $R$ denote the set of integers $> 1$ which are not prime powers. Then we have an isomorphism of commutative graded $A$-algebras:

$$L^A \cong \left( \bigotimes_A \text{Rees}_A^{2n-2}(J^A_n) \right) \otimes_A A[x_{n-1} : n \in R],$$

with $x_{n-1}$ in grading degree $2(n-1)$.

**Proof.** We use Theorem 2.3.2. The ideal $I_n^A$ is principal, and hence $\text{Rees}_A^{2n-2}(I_n^A) \cong A[x_{2n-2}]$, if $n$ is not a prime power. If $n = p^m$ for some prime number $p$, then recall that $I_n^A$ is the ideal generated by $p$ and by all elements of the form $a - a^{p^m}$ with $a \in A$. One checks very easily that, if an ideal contains $p$ as well as $\alpha - \alpha^{p^m}$ for every element $\alpha$ in some $\mathbb{Z}$-linear basis for $A$, then that ideal contains $a - a^{p^m}$ for all $a \in A$. So $J_n^A = I_n^A$.

Some (but not all) rings of integers can be written in the form $A = \mathbb{Z}[\alpha]$ for some element $\alpha$. Remark 2.3.5 gives us an even more compact description of $L^A$ in that case:

**Corollary 3.1.2.** Let $A = \mathbb{Z}[\alpha]$ be the ring of integers in a finite extension $K/\mathbb{Q}$, and let $K_n^A$ be the ideal $(\nu(n), \alpha - \alpha^n)$ of $A$. Then we have an isomorphism of commutative graded $A$-algebras:

$$L^A \cong \text{Rees}_{\Delta}^2(K_2^A) \otimes_A \text{Rees}_{\Delta}^4(K_4^A) \otimes_A \text{Rees}_{\Delta}^6(K_6^A) \otimes_A \text{Rees}_{\Delta}^8(K_8^A) \otimes_A \ldots$$

**Proof.** This is just Theorem 3.1.1 together with Remark 2.3.5 to get a small set of generators for the ideals $I_n^A$. \qed

Here is another corollary of Theorem 3.1.1:

**Corollary 3.1.3.** Let $A$ be the ring of integers in a finite extension $K/\mathbb{Q}$. Let $P$ denote the set of integers $> 1$ which are prime powers, and let $R$ denote the set of integers $> 1$ which are not prime powers. Then we have an isomorphism of commutative graded $A$-algebras:

$$L^A \cong \left( \bigotimes_A A[x_{n-1}, y_{n-1}]/(f_n(x_{n-1}, y_{n-1})) \right) \otimes_A A[x_{n-1} : n \in R],$$

for some set of polynomials $\{f_n\}_{n \in P}$, with each $f_n \in A[x, y]$, and with $x_{n-1}$ and $y_{n-1}$ in grading degree $2(n-1)$.

**Proof.** Every ideal in $A$ can be generated by two elements, so the ideals $J_n^A$ appearing in Theorem 3.1.1 can each be generated by two elements, and with a single relation between them. Hence $\text{Rees}_A(J_n^A) \cong A[x, y]/f(x, y)$ with $f(x, y)$ the relation between the two generators of $J_n^A$. \qed

**Remark 3.1.4.** It seems have already been essentially known to Hazewinkel in 1978 that, when $A$ is the ring of integers in a finite extension of $\mathbb{Q}$, the $A$-module $L_{n-1}^A/D_{n-1}^A$ is isomorphic to the ideal of $A$ generated by $\nu(n)$ and by all elements of the form $a - a^n$; see Example 21.3.3A of [7], where this is almost (but not quite) stated in these terms. The full description of $L^A$ given in Theorem 3.1.2 is, on the other hand, new.

**Theorem 3.1.5.** Let $K$ be a quadratic extension of the rational numbers, and let $A = \mathbb{Z}[\alpha]$ be the ring of integers of $K$. Let $\Delta$ denote the discriminant of $K/\mathbb{Q}$. For each prime number $p$ which divides $\Delta$, let $\nu_p$ be the (unique, since $p$ ramifies totally in $A$) maximal ideal of $A$ over $p$. Let $R$ be the set of prime numbers $p$ which divide $\Delta$ and which have the property that $I_{\nu_p}^A = (p, \alpha - \alpha^{p^n})$ is nonprincipal for some positive integer $m$, and let $S$ be the set of
integers > 1 which are not powers of primes contained in \( R \). Then we have an isomorphism of commutative graded \( A \)-algebras:

\[
L^A \cong A[[x_{n-1} : n \in \mathbb{N}]] \otimes_{A} \bigoplus_{x \in \mathbb{P}} \left( \operatorname{Rees}_A^{2p-2}(I_p^A) \otimes_{A} \operatorname{Rees}_A^{2p^2-2}(I_p^A) \otimes_{A} \operatorname{Rees}_A^{2p^3-2}(I_p^A) \otimes_{A} \ldots \right)
\]

with each polynomial generator \( x_{n-1} \) in grading degree \( 2n-1 \).

Consequently, we have an isomorphism of commutative graded \( A[R^{-1}] \)-algebras:

\[
L^A[R^{-1}] \cong A[R^{-1}][x_1, x_2, \ldots],
\]

with each \( x_n \) in grading degree \( 2n \).

**Proof.** Every quadratic extension \( K \) of \( \mathbb{Q} \) can be written as \( K = \mathbb{Q}(\sqrt{d}) \) for some square-free integer \( d \). We now break into cases:

**If \( d \) is congruent to 2 or 3 modulo 4:** Then \( A = \mathbb{Z}[\sqrt{d}] \), and the primes dividing the discriminant \( \Delta \) are 2 and the primes dividing \( d \). Let \( p \) be a prime number. Recall that, for each prime power \( p^m \), we write \( I_{p^m}^A \) for the ideal of \( A \) generated by \( p \) and by all elements of \( A \) of the form \( a - a^{p^m} \). By Remark 2.3.5, \( I_{p^m}^A = (p, \sqrt{d} - \sqrt{d^{p^m}}) \).

If \( p \) is odd and does not divide \( d \), then the ideal \( I_{p^m}^A \) contains

\[
(\sqrt{d} - \sqrt{d^{p^m}})^2 = (\sqrt{d}(1 - d^{\frac{p^m-1}{2}}))^2 = d(1 - d^{\frac{p^m-1}{2}})^2.
\]

If \( p \) does not divide \( 1 - d^{\frac{p^m-1}{2}} \), then \( p \) is coprime to \( d(1 - d^{\frac{p^m-1}{2}})^2 \) and hence \( I_{p^m}^A = (1) \), which is certainly a principal ideal. If \( p \) does divide \( 1 - d^{\frac{p^m-1}{2}} \), then \( I_{p^m}^A = (p, \sqrt{d} - \sqrt{d^{p^m}}) = (p, \sqrt{d}(1 - d^{\frac{p^m-1}{2}})) = (p) \), which is again principal.

**If \( d \) is congruent to 1 modulo 4:** If we write \( \alpha = \frac{1}{2} + \frac{\sqrt{d}}{2} \), Then \( A = \mathbb{Z}[\alpha] \), and the primes dividing the discriminant \( \Delta \) are exactly the primes dividing \( d \). If \( p \) is odd and does not divide \( d \), then it is still the case that \( \sqrt{d} \in A \) (even though \( A \) is not equal to \( \mathbb{Z}[\sqrt{d}] \)), so the ideal \( I_{p^m}^A \) still contains \( (\sqrt{d} - \sqrt{d^{p^m}})^2 = d(1 - d^{\frac{p^m-1}{2}})^2 \), hence, if \( p \) does not divide \( 1 - d^{\frac{p^m-1}{2}} \), \( I_{p^m}^A = (1) \), which is principal. If \( p \) instead divides \( 1 - d^{\frac{p^m-1}{2}} \), then \( \sqrt{d} - \sqrt{d^{p^m}} = \sqrt{d}(1 - d^{\frac{p^m-1}{2}}) \) is divisible by \( p \), so \( \sqrt{d^{p^m}} = \sqrt{d} \) in \( A/(p) \), and consequently

\[
\alpha - \alpha^{p^m} = \left( \frac{1}{2} + \frac{\sqrt{d}}{2} \right) - \left( \frac{1}{2} + \frac{\sqrt{d}}{2} \right)^{p^m} \\
\equiv \left( \frac{1}{2} + \frac{\sqrt{d}}{2} \right) - \left( \frac{1}{2} \right)^{p^m} + \left( \frac{\sqrt{d}}{2} \right)^{p^m} \mod p \\
\equiv 0 \mod p,
\]

since \( d \equiv (\sqrt{d})^{p^m} \) modulo \( p \) and hence \( \left( \frac{\sqrt{d}}{2} \right)^{p^m} = \frac{\sqrt{d}}{2} \). So \( I_{p^m}^A = (p, \alpha - \alpha^{p^m}) = (p) \), which is again principal.
The last case to consider is when \( p = 2 \). The minimal polynomial of \( \alpha \in A \) is
\[
\alpha^2 - \alpha + \frac{1}{4} \equiv 0 \mod 2,
\]
so modulo 2, we have
\[
\alpha - \alpha^2 \equiv \alpha - \left( \alpha + \frac{d - 1}{4} \right)^{2^{n-1}}
\equiv -\left( \frac{d - 1}{4} \right)^{2^{n-1}} + \alpha - \alpha^{2^{n-1}}
\equiv -\left( \frac{d - 1}{4} \right)^{2^{n-1}} + \alpha - \left( \alpha + \frac{d - 1}{4} \right)^{2^{n-2}}
\equiv -\left( \frac{d - 1}{4} \right)^{2^{n-1}} - \left( \frac{d - 1}{4} \right)^{2^{n-2}} + \alpha - \alpha^{2^{n-2}}
\equiv \ldots
\equiv -\sum_{i=0}^{m-1} \left( \frac{d - 1}{4} \right)^{2^i},
\]
which is an integer, consequently is either 0 or 1 modulo 2. If it is congruent to 0 modulo 2, then \( A/I_p^n \cong A/(2) \) and hence \( I_p^n \equiv (2) \), which is principal; if it is congruent to 1 modulo 2, then \( A/I_p^n \equiv 0 \) and hence \( I_p^n \equiv (1) \), which is principal.

Hence, when \( p \) does not divide \( d \), the ideal \( I_p^n \) is principal. The theorem as stated now follows from Theorem 3.1.2.

\[\square\]

**Corollary 3.1.6.** Let \( K \) be a quadratic extension of the rational numbers, let \( A \) be the ring of integers of \( K \), and let \( \Delta \) denote the discriminant of \( K/\mathbb{Q} \). Then we have an isomorphism of commutative graded \( A[\mathbb{R}^{-1}] \)-algebras:
\[
L^A \left[ \frac{1}{\Delta} \right] \cong A \left[ \frac{1}{\Delta} \right] \left[ x_1, x_2, \ldots \right],
\]
with each \( x_i \) in grading degree \( 2i \).

**Corollary 3.1.7.** Let \( A \) be the ring of integers in a quadratic extension \( K/\mathbb{Q} \), and let \( \Delta \) be the discriminant of \( K/\mathbb{Q} \). Then every formal \( A \)-module \( n \)-bud over a commutative \( A[\mathbb{R}^{-1}] \)-algebra extends to a formal \( A \)-module. Furthermore, if \( R \) is a commutative \( A[\mathbb{R}^{-1}] \)-algebra and \( I \) is an ideal of \( R \), then every formal \( A \)-module over \( R/I \) is the modulo-\( I \) reduction of a formal \( A \)-module over \( R \).

**Remark 3.1.8.** Corollaries 3.1.6 and 3.1.7 do not remain true if we simply remove the word “quadratic” from their statements; it is not the case that, for the ring of integers \( A \) in an arbitrary finite extension \( K/\mathbb{Q} \), the ring \( L^A \) becomes polynomial after inverting the discriminant of \( K/\mathbb{Q} \). For example, the cubic field \( \mathbb{Q}(\sqrt[3]{7}) \) has the property that, in its ring of integers \( \mathbb{Z}[\sqrt[3]{7}] \), the ideals \( I_2 \), \( I_3 \), \( I_5 \), \( I_7 \), \( I_{11} \), \( I_{17} \), \( I_{23} \), \( I_{37} \), \( I_{47} \), \( I_{53} \), and probably \( I_p \) for many other \( p \), are nonprincipal; however, 3 and 7 are the only primes dividing the discriminant of \( \mathbb{Q}(\sqrt[3]{7})/\mathbb{Q} \), so inverting the discriminant does not make \( \mathbb{Z}[\sqrt[3]{7}] \) isomorphic to a polynomial algebra. It seems reasonable to guess that, in fact, there is no finite collection of primes one can invert to make \( \mathbb{Z}[\sqrt[3]{7}] \) isomorphic to a polynomial algebra, but I have not tried to prove that.

**Example 3.1.9.** Let \( A \) be the ring of integers in \( \mathbb{Q}(\sqrt{-5}) \). By Theorem 3.1.5, the only primes \( p \) such that \( I_p^n \) is possibly nonprincipal are those primes \( p \) that ramify in \( A \), i.e.,
2 and 5. Let $\alpha$ denote a square root of $\sqrt{-5}$ in $A$. By direct computation one finds that $I^A_{\alpha^n} = (2, \alpha - 1)$ for all $m$, which is nonprincipal, and that $I^A_{\alpha^m} = (\alpha)$ for all $m$, which is of course principal. Consequently:

$$L^A \equiv A[x_1, y_1, x_3, y_3, x_7, y_7, x_{15}, y_{15}, \ldots]/((\alpha - 1) x_{2^n - 1} = 2 y_{2^n - 1} \forall i \geq 1) \otimes_A A[x_j : j \neq 2^n - 1],$$

where each $x_j$ and each $y_j$ is in grading degree $2j$.

**Remark 3.1.10.** In the proof of Theorem 3.1.5, I show that the ideal $I^A_{\alpha^n}$ in $A$ is principal for all $n$ not ramifying in $A$. It is worth mentioning that this means that $I^A_{\alpha^n}$ is often principal even when the factors of $(p)$ in $A$ are not principal. For example, in the case of Example 3.1.9 (i.e., $A = \mathbb{Z}[\sqrt{-5}]$), the prime numbers 3, 7, 23, 43, 47, 67, 83, 103, and many others, all split as products of distinct nonprincipal primes, despite the class number of $\mathbb{Z}[\sqrt{-5}]$ being only 2. But $I^A_{\alpha^n} \subseteq \mathbb{Z}[\sqrt{-5}]$ is still principal for those primes $p$ and for all positive integers $m$.

Something similar happens in the case of Theorem 3.1.11, i.e., $A$ the ring of integers in $\mathbb{Q}(\sqrt{-18})$: the prime numbers 17, 41, 59, 107, 137, 179, 227, and many others split in $A$ and have nonprincipal prime factors, but $I^A_{\alpha^n} \subseteq A$ is still principal for those primes $p$ and for all positive integers $m$.

In 21.3A of [7], Hazewinkel explains that the extension $K = \mathbb{Q}(\sqrt{-18})$ of $\mathbb{Q}$ has the property that the ideal $I^A_2$ of its ring of integers $A$ generated by 2 and by all elements of the form $\alpha - \alpha^2$ is nonprincipal, and consequently $L^A$ is not a polynomial ring. Hazewinkel does not attempt a computation of $L^A$, however. We now compute $L^A$ explicitly:

**Theorem 3.1.11.** Let $K = \mathbb{Q}(\sqrt{-18})$, and let $A$ be the ring of integers of $K$. Let $S$ denote the set of all integers $> 1$ which are not powers of 2 or of 3. Then we have an isomorphism of commutative graded $A$-algebras

$$L^A \cong A[[x_0 : n \in S]] \otimes_A A[x_1, y_1]/(2x_1 - (\alpha - \alpha^2)y_1)$$

$$\otimes_A \bigotimes_{m \geq 2} (A[x_{2^n - 1}, y_{2^n - 1}]/(2x_{2^n - 1} - \alpha y_{2^n - 1}))$$

$$\otimes_A \bigotimes_{m \geq 1} (A[x_{3^n - 1}, y_{3^n - 1}]/(3x_{3^n - 1} - \alpha y_{3^n - 1})),
$$

where $\alpha = \sqrt{-18} \in A$, and where the polynomial generators $x_i$ and $y_i$ are in grading degree $2i$.

Consequently, we have an isomorphism of commutative graded $A[\frac{1}{2}]$-algebras:

$$L^A \cong A[[\frac{1}{6}]] \cong A[[\frac{1}{6}]][x_1, x_2, \ldots].$$

with each $x_i$ in grading degree $2i$.

**Proof.** Write $\alpha$ for $\sqrt{-18}$, and then the ring of integers of $A$ is the ring of $\mathbb{Z}$-linear combinations of $\alpha$ and $\frac{1}{3} \alpha^2$ and $\frac{1}{3} \alpha^3$. Consequently

$$I^A_{\alpha^n} = \left( p, \alpha - \alpha^{p^n}, \left( \frac{1}{3} \alpha^2 \right)^{p^n} - \left( \frac{1}{3} \alpha^2 \right) \alpha^{p^n}, \frac{1}{3} \alpha^3 - \left( \frac{1}{3} \alpha^3 \right) \alpha^{p^n} \right),$$

but if $p \neq 3$, then $A/I^A_{\alpha^n} \cong A/(p, \alpha - \alpha^{p^n})$, so $I^A_{\alpha^n} = (p, \alpha - \alpha^{p^n})$ for $p \neq 3$.

Let $p > 3$ be a prime number and let $m$ be a positive integer. I claim that the ideal $I^A_{\alpha^n}$ is principal. The proof requires that we break into cases:
If \( p^m - 1 \) is congruent to 0 modulo 4: Then:

\[
(3.1.1) \quad \alpha - \alpha^{p^m} = \alpha \left( 1 - (-18)^{\frac{p^m-1}{2}} \right),
\]

and consequently,

\[
(\alpha - \alpha^{p^m})^4 = -18 \left( 1 - (-18)^{\frac{p^m-1}{2}} \right)^4,
\]

so since we assumed \( p > 3 \), if \( p \) does not divide \( 1 - (-18)^{\frac{p^m-1}{2}} \) then \( p \) is coprime to the integer \((\alpha - \alpha^{p^m})^2 \in I_{p^m}^A\). Hence \( I_{p^m}^A \) contains two coprime integers, hence \( I_{p^m}^A = (1) \), which is principal.

So suppose instead that \( p > 3 \) but \( p \) does divide \( 1 - (-18)^{\frac{p^m-1}{2}} \). Then \( \alpha - \alpha^{p^m} \in I_{p^m}^A \), by equality 3.1.1, so \( I_{p^m}^A = (p) \), which is again principal.

If \( p^m - 1 \) is congruent to 2 modulo 4: This case takes more work. Then

\[
(3.1.2) \quad \alpha - \alpha^{p^m} = \alpha \left( 1 - \alpha^2(-18)^{\frac{p^m-1}{2}} \right),
\]

and consequently

\[
(\alpha - \alpha^{p^m})^2 = \alpha^2 \left( 1 + (-18)^{\frac{p^m-1}{2}} - 2\alpha^2(-18)^{\frac{p^m-1}{2}} \right),
\]

so if \( p > 3 \) and \( p \) divides \( 1 + (-18)^{\frac{p^m-1}{2}} \), then \( 2(-18)^{\frac{p^m-1}{2}} \) is congruent to 0 modulo \( I_{p^m}^A \). The integer \( 2(-18)^{\frac{p^m-1}{2}} \) is coprime to \( p \), so \( A/I_{p^m}^A \equiv 0 \), so \( I_{p^m}^A = (1) \), which is principal.

So suppose instead that \( p > 3 \) and \( p \) does not divide \( 1 + (-18)^{\frac{p^m-1}{2}} \). Then I claim that \( p \) splits in \( \mathbb{Z}[\sqrt{-2}] \). The proof is as follows: by the criterion of Euler for being a square modulo \( p \), \( (-18)^{\frac{p^m-1}{2}} \) is congruent to the Legendre symbol \( \left( \frac{-18}{p} \right) \) modulo \( p \), and one easily computes the Legendre symbol:

\[
(3.1.3) \quad \left( \frac{-18}{p} \right) = \left( \frac{-1}{p} \right) \left( \frac{2}{p} \right)
\]

\[
(3.1.4) \quad = \begin{cases} 1 & \text{if } p \equiv 1 \text{ or } 3 \mod 8 \\ -1 & \text{if } p \equiv 5 \text{ or } 7 \mod 8. \end{cases}
\]

We have the standing assumption that \( p^m - 1 \equiv 2 \mod 4 \), so \( p \equiv 3 \mod 4 \), so we can restrict our attention to the cases where \( p \) is congruent to 3 or to 7 modulo 8. Consequently,

\[
(-18)^{\frac{p^m-1}{2}} = \left( (-18)^{\frac{p^m-1}{2}} \right)^{p^{m-1}+p^{m-2}+\cdots+p+1}
\]

\[
\equiv \begin{cases} 1 & \text{if } 2 \mid m \text{ or } p \equiv 3 \mod 8 \\ -1 & \text{if } 2 \not{\mid} m \text{ and } p \equiv 7 \mod 8 \end{cases}
\]

modulo \( p \). The assumption that \( p^m - 1 \equiv 2 \mod 4 \) implies that \( m \) is odd. Hence, if \( p \equiv 7 \mod 8 \), then \( (-18)^{\frac{p^m-1}{2}} \equiv -1 \mod p \), i.e., \( p \) divides \( 1 + (-18)^{\frac{p^m-1}{2}} \), which contradicts our assumption that \( p \) does not divide \( 1 + (-18)^{\frac{p^m-1}{2}} \). Consequently \( p \equiv 3 \mod 8 \).

Similarly, \( p \) splits in \( \mathbb{Z}[\sqrt{-2}] \) if and only if \( \mathbb{F}_p \) contains a square root of \( -2 \), i.e., if and only if the Legendre symbol \( \left( \frac{-2}{p} \right) \) is equal to 1. By the computation 3.1.4 above, this happens (again assuming \( p \equiv 3 \mod 4 \)) if and only if \( p \) is congruent
to 3 modulo 8. Hence, if \( p > 3 \) and \( p^n - 1 \equiv 2 \) modulo 4 and \( p \) does not divide \( 1 + (-18)^{m/2} \), then \( p \) splits in \( \mathbb{Z}[\sqrt{-2}] \).

Hence there exist positive integers \( c, d \) such that \( 2c^2 + d^2 = p \). I will need to use these integers \( c, d \) momentarily. First, observe that \( (-18)^{m/2} \equiv -18 \) modulo \( p \), and since we assumed that \( p > 3 \), we have that 18 is a unit modulo \( p \), so

\[
0 \equiv 1 - (-18)^{m-1} \mod p
\]

\[
= (1 - (-18)^{m-1})(1 + (-18)^{m-1})
\]

\[
\equiv (1 + (-18)^{m-1})(1 + \frac{3d}{c}(-18)^{m-1})(1 - \frac{3d}{c}(-18)^{m-1}) \mod p,
\]

since \( 2c^2 \equiv -d^2 \) modulo \( p \). We assumed that \( p \) does not divide \( 1 + (-18)^{m-1} \), so either \( p \mid (1 + \frac{3d}{c}(-18)^{m-1}) \) or \( p \mid (1 - \frac{3d}{c}(-18)^{m-1}) \). We handle the two cases separately:

If \( p \mid (1 - \frac{3d}{c}(-18)^{m-1}) \): Let \( z = \frac{2c}{3} - d \in A \). I claim that \( I_{pm}^A = (z) \). We can prove this as follows:

\[
\left(\frac{-ca^2}{3} - d\right) = p, \text{ so } p \in (z), \text{ and }
\]

\[
α - α^{p^n} = α \left(1 - α^2(-18)^{m-1}\right)
\]

\[
\equiv α \left(1 - \frac{3d}{c}(-18)^{m-1}\right) \mod (z) \equiv 0 \mod (z),
\]

since \( p \mid \left(1 - \frac{3d}{c}(-18)^{m-1}\right) \) and \( p \in (z) \). So \( I_{pm}^A = (p, α - α^{p^n}) \subseteq (z) \). Conversely, \( \frac{(-18)^{m/2}}{c} z \equiv 1 - \frac{3d}{c}(-18)^{m-1} \equiv 0 \) modulo \( I_{pm}^A \), so \((z) \subseteq I_{pm}^A \), so \((z) = I_{pm}^A \) and consequently \( I_{pm}^A \) is principal.

If \( p \mid (1 + \frac{3d}{c}(-18)^{m-1}) \): Let \( y = \frac{2c}{3} + d \in A \). I claim that \( I_{pm}^A = (y) \). We can prove this as follows:

\[
\left(\frac{-ca^2}{3} + d\right) = p, \text{ so } p \in (y), \text{ and }
\]

\[
α - α^{p^n} = α \left(1 - α^2(-18)^{m-1}\right)
\]

\[
\equiv α \left(1 + \frac{3d}{c}(-18)^{m-1}\right) \mod (y) \equiv 0 \mod (y),
\]

since \( p \mid \left(1 + \frac{3d}{c}(-18)^{m-1}\right) \) and \( p \in (y) \). So \( I_{pm}^A = (p, α - α^{p^n}) \subseteq (y) \). Conversely, \( \frac{(-18)^{m/2}}{c} y \equiv 1 + \frac{3d}{c}(-18)^{m-1} \equiv 0 \) modulo \( I_{pm}^A \), so \((y) \subseteq I_{pm}^A \), so \((y) = I_{pm}^A \) and consequently \( I_{pm}^A \) is principal.

Since \( p^n \) is odd, the cases \( p^n - 1 \equiv 0 \) and \( p^n - 1 \equiv 2 \) modulo 4 are all the possible cases. So, if \( p > 3 \), then \( I_{pm}^A \) is principal.

Now for the primes \( p = 2 \) and \( p = 3 \). For \( p = 2 \), we have \( I_2^A = (2, α - α^2) \) and

\[
I_{2m}^A = (2, α - α^{2m}) = (2, α - (-18)^{2m-2}) = (2, α)
\]

for \( m > 1 \), and \( I_{pm}^A \) is nonprincipal for all \( m \geq 1 \). For \( p = 3 \), I claim that \( I_{3m}^A = (3, α) \) for all \( m \geq 1 \), which is again nonprincipal. The proof is as follows: first, observe that
\(-2 \equiv 1 \mod 3, \text{ so } 1 - (-2)^{\frac{m-1}{3}} \text{ is divisible by } 3. \text{ Next, observe that } \frac{m-1}{3} \text{ is congruent to } \mod 1 \text{ modulo } 3, \text{ so } (-2)^{\frac{m-1}{3}} \text{ is congruent to } 7 \mod 9, \text{ so } 1 - (-2)^{\frac{m-1}{3}} \text{ is not divisible by } 9. \text{ Consequently, }
\begin{align*}
P^A_{3^m} &= \left( 3, \alpha - \alpha^2 \frac{\alpha^2}{3} - \left( \frac{\alpha^2}{3} \right)^{3^m} \right) \\
&= \left( 3, \alpha - \alpha^2 \frac{\alpha^2}{3^{3^m}} \left( 3^{3^m-1} - (-2)^{\frac{m-1}{3}} \right) \right) \\
&= \left( 3, \alpha - \alpha^2 \frac{\alpha^2}{3^{3^m}} \left( 1 - (-2)^{\frac{m-1}{3}} \right) \right) \\
&= \left( 3, \alpha - \alpha^2 \frac{\alpha^2}{3} \right) \\
&= \left( 3, \alpha \right)
\end{align*}
(3.1.5)
with equality 3.1.5 because \( \nu_3(1 - (-2)^{\frac{m-1}{3}}) = 1, \) so \( \frac{1}{3} \left( 1 - (-2)^{\frac{m-1}{3}} \right) \) is a unit modulo 3.

The theorem as stated now follows from Theorem 3.1.2. \( \square \)

3.2. **Group rings.** Suppose \( G \) is a finite group. In [4] (see also [6] for a nice survey) a theory of “\( G \)-equivariant formal group” is developed, which is designed in order to admit a classifying ring \( L^G \) for \( G \)-equivariant formal groups with a canonical comparison map with the \( G \)-equivariant complex bordism ring \( MU^G \); indeed, this comparison map exists, is known to be surjective, and is conjectured to be an isomorphism (this is Greenlees’ conjecture; see [6] and [16]).

In order to make this comparison map work well, these \( G \)-equivariant formal groups are much more complicated than just a formal group \( F \) equipped with a choice of group homomorphism \( G \to \text{Aut}(F) \), and even the definition of a \( G \)-equivariant formal group is rather involved. On the other hand, one expects that \( G \)-equivariant formal groups ought to have some relationship with the simple notion of a formal group equipped with an action by the group \( G \); this is essentially just a formal \( \mathbb{Z}[G] \)-module (clearly, to specify the structure map \( \rho : \mathbb{Z}[G] \to \text{End}(F) \) of a formal \( \mathbb{Z}[G] \)-module, we could just as well have specified a group homomorphism \( G \to \text{Aut}(F) \); the only point to mention here is the tangency axiom in the definition of a formal module, i.e., that \( \rho(g)(X) \equiv gX \mod X^2 \), meaning that we need to have an action of \( G \) on the coefficient ring of \( F \). But if we begin with a group homomorphism \( G \to \text{Aut}(F) \), we can typically choose an action of \( G \) on the coefficient ring of \( F \) so that the tangency axiom is satisfied. So the distinction between a \( \mathbb{Z}[G] \)-module and a formal group with an action by \( G \) is very slight.).

In Theorem 3.2.2 I compute the classifying ring \( L^{\mathbb{Z}[G]} \) of formal \( \mathbb{Z}[C_n] \)-modules, after inverting \( n \), so that \( \mathbb{Z}[C_n][\frac{1}{n}] \) is hereditary and Theorem 2.3.2 applies. The resulting ring \( L^{\mathbb{Z}[G]}[\frac{1}{n}] \) is not polynomial but has a relatively tractable (although infinite) presentation.

**Remark 3.2.1.** The ring \( L^{\mathbb{Z}[G]}[\frac{1}{n}] \) ought to be compared to the localized classifying ring \( L^C[\frac{1}{n}] \) of \( C_n \)-equivariant formal groups, which at present has only been computed in the case \( n = 2 \), by Strickland in [16], whose presentation for \( L^C_2 \) is algebraically complicated enough that it is still not known whether it is isomorphic to the \( C_2 \)-equivariant complex bordism ring \( MU^{C_2} \), despite both rings having known presentations; that is, the \( G = C_2 \) case of Greenlees’ conjecture remains open, simply because the two rings conjectured to be isomorphic are given by such complicated presentations. The main application I have in mind for Theorem 3.2.2 is to compare \( L^{\mathbb{Z}[G]}[\frac{1}{n}] \) to Strickland’s presentation for \( L^C[\frac{1}{n}] \), in order to establish the relationship between the moduli theory of \( C_2 \)-equivariant
formal groups and the (much simpler) moduli theory of \( \mathbb{Z}[C_2] \)-formal modules. Owing to the complexity of both the theory of equivariant formal groups and also of Strickland’s presentation for \( L^C_2 \), however, the task of producing and computing a natural map between \( L^C_2[\frac{1}{2}] \) and \( \mathbb{Z}[C_2][\frac{1}{2}] \) is beyond the scope of the present paper.

**Theorem 3.2.2.** Let \( C_n \) be the cyclic group of order \( n \). Let \( P \) be the set of integers > 1 which are prime powers relatively prime to \( n \). Let \( S \) be the set of integers > 1 not contained in \( P \). Then we have an isomorphism of graded rings

\[
L^\mathbb{Z}[C_2][\frac{1}{n}] \cong \bigotimes_{a \in P} \left( \mathbb{Z}[\frac{1}{n}][C_n][x_{i-1}, y_{i-1}] / (px_{i-1} - (1 - \sigma)y_{i-1}) \right) \otimes_{\mathbb{Z}[\frac{1}{n}][C_n]} \mathbb{Z}[\frac{1}{n}][C_n][x_i : i \in S],
\]

where \( \sigma \) denotes a generator of \( C_n \) and where the polynomial generators \( x_{i-1} \) and \( y_{i-1} \) are each in grading degree \( 2(i - 1) \).

**Proof.** This is another special case of Theorem 2.3.2, since \( \mathbb{Z}[C_n] \) is finitely generated as an abelian group and since the ring \( \mathbb{Z}[\frac{1}{n}][C_n] \) is hereditary. Recall that \( \mathbb{Z}[\frac{1}{n}][C_n] \) denotes the ideal in \( \mathbb{Z}[C_n] \) generated by \( p \) and by all elements of the form \( a - a^p \). If \( p \) divides \( n \), then clearly this ideal becomes principal after inverting \( n \), hence Rees\( \mathbb{Z}[\frac{1}{n}][C_n] \)\( (1 - \frac{1}{n}) \) \(\mathbb{Z}[\frac{1}{n}][C_n][a] \) if \( p \) divides \( n \).

If \( p \) does not divide \( n \), then \( p^m \) is relatively prime to \( n \), so \( \mathbb{Z}[C_n] / (p, \sigma - \sigma^p) \cong \mathbb{F}_p \), so \( p, \sigma - \sigma^p \) is a maximal ideal of \( \mathbb{Z}[C_n] \) contained in \( \mathbb{I}_{p^m} \), and \( \mathbb{I}_{p^m} \) is a proper ideal. So \( (p, \sigma - \sigma^p) = \mathbb{I}_{p^m} \). Furthermore the projection \( \mathbb{Z}[C_n] \to \mathbb{Z}[C_n] / (p, \sigma - \sigma^p) \) sends \( p \) to zero and \( \sigma \) to \( 1 \), i.e., the kernel of the projection is the ideal \( (p, 1 - \sigma) \). So \( \mathbb{I}_{p^m} = (p, 1 - \sigma) \). Now the claim follows from Theorem 2.3.2.

**Corollary 3.2.3.** Let \( C_n \) be the cyclic group of order \( n \). Then every formal \( \mathbb{Z}[\frac{1}{n}][C_n] \)-module \( m \)-bud extends to a formal \( \mathbb{Z}[\frac{1}{n}][C_n] \)-module.

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