Nonlinear dynamics of soft boson excitations in hot QCD plasma I: plasmon-plasmon scattering

Yu.A. Markov* and M.A. Markova*

Institute of System Dynamics and Control Theory Siberian Branch of Academy of Sciences of Russia, P.O. Box 1233, 664033 Irkutsk, Russia

Abstract

On the basis of pure gauge sector of Blaizot-Iancu equation, we derive kinetic equation of Boltzmann type, taking into account $2n + 2$-colorless plasmon decay processes, $n = 1, 2, \ldots$. Using so-called Tsytovich correspondence principle, a direct connection between matrix elements of the plasmon decay processes and certain effective current, generating these processes, is established. The procedure of calculation of matrix element for simplest four-plasmon decay is considered comprehensively. The limiting value of the plasmon occupation number ($\sim 1/g^2$, where $g$ is a strong coupling) wherein all plasmon decays with $n \geq 1$, contribute to the right-hand side of the Boltzmann equation, is defined. The iterative method of calculation of matrix elements for higher decay processes ($n > 1$), is proposed, and a problem of their gauge-invariance is discussed. Proceed from the general reasons the problem of extension of suggested approach to the case of color plasmons, is considered. The explicit form of linearized Boltzmann equation for color plasmons is written out, and it is shown that this equation covariantly conserves the color current, resulting from color-plasmon number density.

PACS: 12.38.Mh, 24.85.+p, 11.15.Kc

*e-mail: markov@icc.ru
1 Introduction

This work consisting of three parts, is concerned with in-depth analysis of dynamics of boson excitations in hot QCD plasma at the soft momentum scale ($\sim gT$, where $T$ is a temperature of a system), within the hard thermal loop (HTL) effective theory. Although this range of momentum was studied in detail in the literature (see review of Blaizot and Iancu [1]), but nevertheless the dynamical processes of higher order in the coupling $g$ connected with high-loop diagrams of an effective perturbation theory [2], remain to be practical analyzed. To such processes in particular we carry over the processes of the following type: the scattering of soft modes among themselves and the processes of scattering of two and more plasma waves by hard thermal particle. This and accompanying papers are concerned with research of the processes within a real-time formalism based on kinetic equation for soft excitations. For sufficiently high energy level of the soft plasma excitations (exact estimations will be given below) these higher processes of interaction play an important role identical to simplest scattering process including one soft collective quantum and hard particle, and is called nonlinear Landau damping. In the case of non-Abelian plasma, for boson excitations, the nonlinear Landau damping was studied in detail in Ref. [3]. In [3] it was shown that the nonlinear Landau damping rate is closely related to the damping rate obtained in Ref. [1] in one-loop approximation of an effective perturbation theory. In the second part of our paper [5] the extension of an approach developed in Ref. [3] to the case of scattering of arbitrary number of waves by hard thermal particle will be introduced, and exact estimations for oscillation amplitudes, wherein these higher processes are of the same order in $g$, will be given. The application of the theory developed there to the problem of energy loss of a color charged particle traversing the hot QCD plasma will be considered. In the third part the plasmon production by bremsstrahlung and processes of multiple scattering with emission of soft boson waves (longitudinal or transverse) will be studied, what can be useful for research of Landau-Pomeranchuk-Migdal effect in hot gauge plasma.

Here, in Paper I we have restricted ourselves to study of the processes connected with self-interaction of the soft excitations. The study of these processes is to be preceded by consideration of more complicated scattering processes including soft modes and hard particles which contain self-interaction of the soft modes as a required constituent element.

In our previous paper [1] a first attempt of research of dynamical process of nonlinear self-interaction of longitudinal colorless excitations (colorless plasmons) based on the Boltzmann equation for plasmon number density $N_l$, was presented. The process of elastic scattering of two colorless plasmons for which the appropriate scattering probability was calculated, is considered. Unfortunately, method of derivation of collision term proposed there, is rather cumbersome and gives no way of its direct extension to calculation of collision term taking into account the scattering processes involved arbitrary (even) number of colorless plasmons. The present work is based on use of some different approach allowing immediately to reduce the problem of calculation of the collision term to the problem of calculation of the scattering probabilities for appropriate scattering
processes. Above-mentioned assertion that for certain energy level of the soft excitations all processes of self-interaction of the plasma collective modes are of the same order in $g$, is stated more concrete in terms of the plasmon number density, and the equation of the Boltzmann type defines its space-time evolution. In present paper it shown that with the condition when the plasmon occupation number is of order $1/g^2$, the right-hand side of the Boltzmann equation represents a sum of an infinity number of terms having a sense of $(2n + 2)$-plasmon collision integrals, $n = 1, 2, \ldots$. The last circumstance is in reality a different restatement of well-known [7, 8] very nontrivial fact that for strong field $A_\mu(X) \sim T$, all $N$-gluon amplitudes with $N \geq 2$ contribute to the induced current $j^{\text{ind}}_\mu[A]$ at the same order in the coupling. The HTL-amplitudes here represent coefficient functions in integrands for functional expansion of $j^{\text{ind}}_\mu[A]$ in powers of gauge field $A_\mu$. In this paper by introducing notions of interacting and free soft gauge fields, we somewhat deepen analysis of this circumstance. In particular, it shown that the induced current has the more physical content if it will be represented as an expansion not in interacting field $A_\mu$ (as it actually takes place in Refs. [7, 8]), but in free field $A_\mu^{(0)}$. The coefficient functions of this expansion, called effective amplitudes, represent rather complicated combinations of HTL-amplitudes and an equilibrium propagator for a soft gluon. They preserve many features of usual HTL-amplitudes, but have more direct physical meaning – these functions set on mass-shell plasma excitations representing the scattering amplitudes of soft collective modes.

At present there are a few methods of construction of relativistic kinetic equations (see, e.g. review in Ref. [9]). A more powerful and more convenient tool to derive relativistic Boltzmann equations from exact field Schwinger-Dyson equations, the so-called closed-time-path (CTP) formalism of nonequilibrium quantum-field theory, is considered. This general method is suitable to the same extent for deriving transport equations both hard modes and soft ones of plasma medium. From most papers close to the subject of our research, where for derivation of the Boltzmann equations for soft gluon longitudinal and transverse quasiparticles the CTP formalism is used, it should be mentioned the work by Niégawa [9]. Our view on the problem of derivation of the Boltzmann equations for soft plasma modes is quite different from a view presented in Ref. [6]. Since we are interested only with the processes of nonlinear interaction of the soft excitations among themselves and with hard thermal particles, the use of quasiclassical approximation is justified. By virtue of this fact we consider that it has no need to use tools of nonequilibrium field theory which are general, but for our particular problems, are needlessly complicated.

Our approach is based on the fundamental system of dynamical equations derived by Blaizot and Iancu [8], which is a local formulation of the HTL-equations of motion for soft fluctuating gluon and quark fields and their induced sources. These equations are obtained by means of a truncation of the Schwinger-Dyson hierarchy and already contain the maximum comprehensive information at leading order in $g$ about dynamics of hot QCD plasma at soft momentum scale\footnote{However it should be noted that the approximation used here may be unsuitable in attempt of calculation of effective amplitude of elastic plasmon-plasmon collision in the limit of vanishing momentum}. Therefore this system is the most convenient and
optimal start point for derivation of transport equations for soft plasma excitations. In our previous paper [6] it was shown how starting from similar dynamic equations one can derive collision term for elastic scattering of two colorless plasmons. However, as was mentioned above, the direct approach presented there, is very cumbersome and complicated for its extension to the processes of collisions with account of many colorless plasmons. For this reason in the present paper and accompanying ones, proceed from quasiclassical character of a problem, we initially assume that a structure of the collision terms are determined by Fermi’s golden rule and thus we focus our efforts on calculation exclusively of the scattering probabilities, which in fact determine all dynamics of interaction of soft boson excitations.

The following step is to extend analysis carried out in these papers on the fermion degree of freedom of plasma excitations in hot QCD plasma. Here, the series of new specific moments, which will be fully considered in the subsequent works, is arisen.

The paper I is organized as follows. In section 2 preliminary comments, with regard to derivation of the Boltzmann equation, describing $(2n+2)$-plasmon decay processes, are given. In section 3 all the conventions and the notations used in this paper, are summarized, and the nonlinear integral equation for gauge potential $A_\mu$, playing key position in our subsequent research, is written out. Section 4 is devoted to explanation of the correspondence principal that enables one to establish a direct connection of the matrix elements of elastic collision of $n+1$ colorless plasmons with certain effective current, generating these processes. Section 5 presents a detailed consideration of the Boltzmann equation for four-plasmon decay. In section 6 we estimate the typical value of plasmon occupation numbers, wherein one can restricted the consideration to taking into account only contribution from four-plasmon decay or it should be considered all higher decay processes. In section 7 a complete algorithm of succesive calculation of the matrix elements defining $(2n+2)$-plasmon decay processes in the temporal gauge, is presented. In section 8 we discuss a problem of a gauge independence of these matrix elements. In section 9 we give some speculation concerning extension of an approach proposed in this work to research of dynamics of colorless plasmons to the case of color plasmons. In closing section, when the main results were outlined, we briefly discuss the additional difficulties associated with appearance of the nonlinear frequency shift of longitudinal oscillations generated by plasmon collective interactions.

2 Preliminaries

The main part of our work will be concerned with the construction of an effective kinetic theory for soft colorless longitudinal excitations, propagating in a purely gluonic plasma, with no quarks. In other words we assume that a localized number density of plasmons transfer, near to "forward scattering".
\( N^l(p, x) \equiv (N^{l \alpha \beta}_p) \) is diagonal in color space

\[
N^{l \alpha \beta}_p = \delta^{\alpha \beta} N^l_p,
\]

where \( a, b = 1, \ldots, N_c^2 - 1 \) for \( SU(N_c) \) gauge group, and we consider the change of the number density of the colorless plasmons \( N^l_p \) as a result of their interactions among themselves. In section 9 only using some general reasonable assumptions, we made an attempt of an extension of the obtained results to the case of color excitations with non-diagonal plasmon number density.

The dispersion relation for plasmons \( \omega^l = \omega^l(p) \equiv \omega^l_p \) is defined by

\[
\Re^* \Delta^{-1l}(\omega, p) = 0,
\]

where

\[
^* \Delta^{-1l}(\omega, p) = p^2 \left( 1 + \frac{3\omega_{pl}^2}{p^2} \left[ 1 - F\left( \frac{\omega}{|p|} \right) \right] \right),
\]

\[
F(x) \equiv \frac{x}{2} \left[ \ln \left| \frac{1 + x}{1 - x} \right| - i\pi \theta(1 - |x|) \right]
\]

is an inverse resummed longitudinal propagator and \( \omega_{pl}^2 = g^2 N_c T^2 / 9 \) is a plasma frequency square. From physical considerations it is clear that the notions of the plasmon and accordingly, plasmon number density have a meaning only with use of the condition

\[
\omega^l \tau^l \gg 1,
\]

i.e. for the plasmon frequency that is much more than inverse lifetimes of plasmon \( 1/\tau^l \). Generally speaking, this time is defined by just as the processes of plasmons scattering off each other, so by the scattering processes including soft modes and hard thermal particles (hard thermal gluons in our case). For sufficiently large intensity of plasma excitations (exact estimations will be given in Sec. 6 of this paper and in [3]), the processes of a first type can play the same important role in dynamics of a system as the second ones and even become dominant. Under this condition, plasma may be considered as involving two interacting subsystems: the subsystem of hard thermal gluons and the subsystem of soft plasmons, which exchange energy among themselves. As was mentioned in Introduction, here, we have restricted ourselves to detailed consideration of the processes of interactions in the plasmon subsystem. This fact is embodied in the structure of collision term in the plasmon kinetic equation proposed below.

We expect the time-space evolution of scalar function \( N^l_p \) to be described by

\[
\frac{\partial N^l_p}{\partial t} + V^l_p \cdot \frac{\partial N^l_p}{\partial x} = -N^l_p \Gamma_d[N^l_p] + (1 + N^l_p) \Gamma_i[N^l_p],
\]

where \( V^l_p = \partial \omega^l_p / \partial p \) is a group velocity of the longitudinal oscillations. The generalized decay rate \( \Gamma_d \) and inverse decay rate \( \Gamma_i \) in general case are (non-linear) functionals dependent on the plasmon number density. From here on such a functional dependence is denoted by argument of a function in square brackets.
We shall consider that the decay rate $\Gamma_d$ and regenerating rate $\Gamma_i$ can be formally represented in the form of functional expansion in powers of the plasmon number density

$$\Gamma_d \{ N_p \} = \sum_{n=1}^{\infty} \Gamma_d^{(2n+1)} \{ N_p \}, \quad \Gamma_i \{ N_p \} = \sum_{n=1}^{\infty} \Gamma_i^{(2n+1)} \{ N_p \},$$

(2.3)

where

$$\Gamma_d^{(2n+1)} \{ N_p \} = \int dT^{(2n+1)} w_{2n+2} (p, p_1, \ldots, p_{n}; p_{n+1}, \ldots, p_{2n+1}) N_p^l \cdots N_{p_n}^l \quad (2.4)$$

$$\times (1 + N_{p_{n+1}}^l) \cdots (1 + N_{p_{2n+1}}^l),$$

$$\Gamma_i^{(2n+1)} \{ N_p \} = \int dT^{(2n+1)} w_{2n+2} (p, p_1, \ldots, p_{n}; p_{n+1}, \ldots, p_{2n+1}) (1 + N_{p_1}^l) \cdots (1 + N_{p_n}^l) \quad (2.5)$$

$$\times N_{p_{n+1}}^l \cdots N_{p_{2n+1}}^l.$$

Here, $w_{2n+2} (p, p_1, \ldots, p_{n}; p_{n+1}, \ldots, p_{2n+1})$ is a scattering probability for process of elastic collision of $n + 1$ plasmons, and phase-space measure is

$$\int dT^{(2n+1)} \equiv \int (2\pi)^4 \delta^{(3)} (p_{\text{in}} - p_{\text{out}}) \delta (E_{\text{in}} - E_{\text{out}}) \prod_{k=1}^{2n+1} \frac{dp_k}{(2\pi)^3},$$

(2.6)

where

$$p_{\text{in}} = p + p_1 + \ldots + p_n, \quad E_{\text{in}} = \omega_p^l + \omega_{p_1}^l + \ldots + \omega_{p_n}^l,$$

and

$$p_{\text{out}} = p_{n+1} + \ldots + p_{2n+1}, \quad E_{\text{out}} = \omega_{p_{n+1}}^l + \ldots + \omega_{p_{2n+1}}^l.$$

The Dirac $\delta$-functions in Eq. (2.4) expresses the momentum and energy conservation in the collision process. In writing the expressions (2.4) and (2.5), we have used the fact that the probabilities of direct and reverse processes are identical.

We make some comments concerning Eqs. (2.3) – (2.5). By virtue of the fact that processes of nonlinear interaction of odd number of the plasmons are kinematically forbidden by the conservation laws, in expansions of generalized rates (2.3) we leave only odd terms in powers $N_p^l$. Furthermore a structure of the terms of the expansions (2.4) and (2.5) is chosen such that it taken into account the scattering processes only with equal number of the plasmons prior to interaction and upon it, i.e. the scattering processes of the following type:

$$g^* + g_1^* \leftarrow g_2^* + g_3^*, \quad \text{for } n = 1,$$

$$g^* + g_1^* + g_2^* \leftarrow g_3^* + g_4^* + g_5^*, \quad \text{for } n = 2,$$

$$\ldots,$$

(2.7)

\footnote{The term the decay processes will be also used for the processes of (2.7) type.}
where $g^*, g_1^*, \ldots$ are the plasmon collective excitations. Such a choose of the structure of the decay and regenerating rates assumes that decay processes with unequal even number of incoming and outgoing soft external legs of the type

$$g^* + g_1^* \rightarrow g_2^* + g_3^* + g_4^* + g_5^*,$$

for $n = 2$,

etc, are kinematically suppressed as compared with “elastic” scattering processes

(speaking more exactly, kinematic regions of momentum variables in “elastic” and “inelastic” scattering accessible by conservation laws, are not be covered each other, and a contribution of the last processes to the nonlinear plasmon dynamics to the order of interest is not important).

The scattering probability $w_{2n+2}$ must satisfy the symmetry relations over permutations of external momenta

$$w_{2n+2}(p, p_1, \ldots, p_n; p_{n+1}, \ldots, p_{2n+1}) = w_{2n+2}(p_1, p, \ldots, p_n; p_{n+1}, \ldots, p_{2n+1}) = \ldots$$

$$= w_{2n+2}(p_n, p_1, \ldots, p; p_{n+1}, \ldots, p_{2n+1}) = w_{2n+2}(p_{n+1}, \ldots, p_2, p; p_1, \ldots, p_n),$$

which are a consequence of an indistinguishability of the colorless plasmons. The special consequences of symmetry properties (2.8) are the conservation laws of energy momentum and total number of colorless plasmons in the decay process (2.7)

$$E \equiv \int \frac{dP}{(2\pi)^3} \omega^l p^l N_l = \text{const},$$

$$K \equiv \int \frac{dP}{(2\pi)^3} p N_l = \text{const},$$

$$N \equiv \int \frac{dP}{(2\pi)^3} N^l p^l = \text{const}.$$

3 Blaizot-Iancu equations. Correlation function of the boson excitations

We adopt conventions of Blaizot and Iancu [3]. We use the metric $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$, choose units such that $c = k_B = 1$ and note $X = (X_0, X)$, $p = (p_0 \equiv \omega, p)$. The gauge field $A_\mu = A_\mu^a t^a$ with $N_c^2 - 1$ Hermitian generators in the fundamental representation obeys the field equation

$$[D^\nu, F_{\mu\nu}(X)] - \xi_0^{-1} n_{\mu\nu} A_\nu(X) = j_\mu(X),$$

where $D_\mu = \partial_\mu + ig A_\mu(X)$ is a covariant derivative, $F_{\mu\nu} = F_{\mu\nu}^a t^a$ is a field strength tensor with $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c$, $[,]$ denotes a commutator and $\xi_0$ is a gauge
parameter fixing a temporal gauge. In what follows 4-vector $n_\mu$ will be identified with global 4-velocity $u_\mu$ of the plasma.

The color current $j_\mu$ on the right-hand side of field equation (3.1) can be written as

$$ j_\mu(X) = 2gN_c \int \frac{d|k|}{(2\pi)^3} v_\mu \delta N(k, X), \quad v = (1, v), \quad v = k/|k|, \quad (3.2) $$

where $\delta N(k, X) = \delta N^a t^a$ is a soft fluctuation in the gluon color density. We suppose that there is no external color current and/or mean color field in the system, and therefore an expectation value of the induced color current (3.2) over the off-equilibrium ensemble equals zero, i.e. $\langle j_\mu(X) \rangle = 0$. The consequence of this fact is $\langle \delta N(k, X) \rangle = 0$.

On a space-time scale $(gT)^{-1}$ the soft fluctuation $\delta N(k, X)$ satisfies the following equation:

$$ [v \cdot D_X, \delta N(k, X)] = ig\{[v \cdot A(X), \delta N(k, X)] \} - g\{ v \cdot E(X) - \langle v \cdot E(X) \rangle \} \frac{dN(\epsilon_k)}{d\epsilon_k}. \quad (3.3) $$

Here, $E(X) = E^a(X)t^a$ is a chromoelectric field, $E^i = F^{i0}$; $N(\epsilon_k) = 1/(\exp(\epsilon_k/T) - 1)$ is a boson occupation factor and $\epsilon_k \equiv |k|$. On the right-hand side of Eq. (3.3) we have entered the additional average terms, so that Eq. (3.3) will be equal zero for the expectation value taken over the off-equilibrium ensembles. The self-consistent system of Eqs. (3.1) – (3.3) for a soft fluctuating field $A_\mu$ and their induced current presents a pure gauge sector of general system of self-consistent equations first obtained by Blaizot and Iancu in Ref. [8].

The equations (3.1) – (3.3) are solved by the approximation scheme method – the weak-field expansion. For this purpose first of all we expand the soft fluctuation of the gluon color density

$$ \delta N = \sum_{s=1}^{\infty} \delta N^{(s)}, \quad (3.4) $$

where index $s$ shows that $\delta N^{(s)}$ is proportional to the $s$th power of $A_\mu$. The expansion of a color current, corresponding to (3.4) has the form

$$ j_\mu = \sum_{s=1}^{\infty} j_\mu^{(s)}, \quad j_\mu^{(s)} = 2gN_c \int \frac{d|k|}{(2\pi)^3} v_\mu \delta N^{(s)}. \quad (3.5) $$

Now we turn to the field equation (3.1), connecting the gluon soft field with induced color current $j_\mu$. Let us rewrite this equation, explicitly separating free parts in (3.1) from interacting terms. We have

$$ \partial' (F_{\mu\nu})_L - \xi_0^{-1} n_\mu n_\nu A_\nu - j_\mu^{(1)} = j_{NL\mu} - ig \{ \partial' [A_\mu, A_\nu] - \langle \partial' [A_\mu, A_\nu] \rangle \} - 

- ig \{ [A_\nu, (F_{\mu\nu})_L] - \langle [A_\nu, (F_{\mu\nu})_L] \rangle \} + g^2 \{ [A_\nu, [A_\mu, A_\nu]] - \langle [A_\nu, [A_\mu, A_\nu]] \rangle \}. \hspace{1cm} (3.6) $$

Here, indices $L$ and $NL$ denote linear and nonlinear parts of strength tensor and the color induced current with respect to $A_\mu$. As in the case of dynamical equation (3.3), we enter
the additional averaged terms on the right-hand side of Eq. (3.8), which ensure vanishing left and right parts when an expectation value is taken.

Substituting the expansion [3.4] into (3.3) and collecting the terms of the same order in \( A_\mu \), we derive the system of equations for \( \delta N^{(s)}, s = 1, 2, \ldots \), whose solution is easily obtained by Fourier transformation. Substituting such a derived expression we obtain the explicit form of the terms in a color current expansion (3.5)

\[
j^{(s)}_\mu(X) = t^a \int dp j^{(s)\mu}_\mu(p) e^{-ipX}, s = 1, 2, \ldots, \tag{3.7}
\]

where

\[
\begin{align*}
j^{(1)\mu}_\mu(p) &= \Pi_{\mu\nu}(p)A^{\mu\nu}(p), \\
j^{(2)\mu}_\mu(p) &= \frac{1}{2!} g \int \delta\Gamma^{a_1 a_2}_{\mu_1 \mu_2}(p, -p_1, -p_2) \left\{ A^{a_1 \mu_1}(p_1)A^{a_2 \mu_2}(p_2) - \langle A^{a_1 \mu_1}(p_1)A^{a_2 \mu_2}(p_2) \rangle \right\} \\
&\quad \times \delta(p - p_1 - p_2) dp_1 dp_2, \\
&\quad \ldots, \\
j^{(s)\mu}_\mu(p) &= \frac{1}{s!} g^{s-1} \int \delta\Gamma^{a_1 a_2 \ldots a_s}_{\mu_1 \mu_2 \ldots \mu_s}(p, -p_1, \ldots, -p_s) \left\{ A^{a_1 \mu_1}(p_1)A^{a_2 \mu_2}(p_2) \ldots A^{a_s \mu_s}(p_s) - A^{a_1 \mu_1}(p_1)\langle A^{a_2 \mu_2}(p_2) \ldots A^{a_s \mu_s}(p_s) \rangle - \ldots - \langle A^{a_1 \mu_1}(p_1)A^{a_2 \mu_2}(p_2) \ldots A^{a_s \mu_s}(p_s) \rangle \right\} \\
&\quad \times \delta(p - \sum_{i=1}^s p_i) \prod_{i=1}^s dp_i, \ldots.
\end{align*}
\]

Here, \( \Pi_{\mu\nu}(p) \) is a soft-gluon self-energy and \( \delta\Gamma^{a_1 \ldots a_s}_{\mu_1 \ldots \mu_s} \) are usual HTL-functions [2, 8]. Rewriting equation (3.3) in momentum space and taking into account (3.7) we lead to the nonlinear integral equation for gauge potential \( A_\mu \), playing key position in our subsequent research

\[
*\bar{D}^{-1\mu\nu}(p)A^\nu_\nu(p) = -J^\mu_{NL}[A] \equiv -\sum_{s=2}^\infty J^{(s)\mu}(A, \ldots, A), \tag{3.8}
\]

where members of a series on the righthand side of Eq. (3.8) have a structure

\[
\begin{align*}
J^{(s)\mu}(A, \ldots, A) &= \frac{1}{s!} g^{s-1} \int *\Gamma^{a_1 \ldots a_s}_{\mu_1 \ldots \mu_s}(p, -p_1, \ldots, -p_s) \left\{ A^{a_1 \mu_1}(p_1)A^{a_2 \mu_2}(p_2) \ldots A^{a_s \mu_s}(p_s) - A^{a_1 \mu_1}(p_1)\langle A^{a_2 \mu_2}(p_2) \ldots A^{a_s \mu_s}(p_s) \rangle - \ldots - \langle A^{a_1 \mu_1}(p_1)A^{a_2 \mu_2}(p_2) \ldots A^{a_s \mu_s}(p_s) \rangle \right\} \\
&\quad \times \delta(p - \sum_{i=1}^s p_i) \prod_{i=1}^s dp_i.
\end{align*}
\]

Here, the coefficient functions \( *\Gamma^{a_1 \ldots a_s}_{\mu_1 \ldots \mu_s} \) for \( s = 3 \) and \( s = 4 \) represent a sum of a bare vertices and corresponding HTL-corrections, and for \( s > 4 \) we rename \( *\Gamma^{(s)} \equiv \delta\Gamma^{(s)} \).

\(*\bar{D}^{\mu\nu}(p)\) is a medium modified (retarded) gluon propagator in a temporal gauge

\[
*\bar{D}^{\mu\nu}(p) = -P^{\mu\nu}(p) \Delta^\mu(p) - Q^{\mu\nu}(p) \Delta^\mu(p) + \xi_0 \frac{\beta^2}{(p \cdot u)^2} D^{\mu\nu}(p), \tag{3.10}
\]
where \( *\Delta^{t,l}(p) = 1/(p^2 - \Pi^{t,l}(p)) \), \( \Pi(p) = \frac{1}{2}\Pi^\mu^\nu(p)P^\mu\nu(p) \), \( \Pi'(p) = \Pi^\mu^\nu(p)\tilde{Q}^\mu\nu(p) \). The Lorentz matrices in (3.10) are defined by

\[
P_{\mu\nu}(p) = g_{\mu\nu} - u_\mu u_\nu - \frac{(p \cdot u)^2}{p^2}Q_{\mu\nu}(p), \quad \tilde{Q}_{\mu\nu}(p) = \frac{\bar{u}_\mu(p)\bar{u}_\nu(p)}{u^2(p)}, \quad D_{\mu\nu}(p) = \frac{p_\mu p_\nu}{p^2}, \quad (3.11)
\]

\[
\bar{u}_\mu(p) = \frac{p^2}{(p \cdot u)}(p_\mu - u_\mu(p \cdot u)), \quad \bar{u} = p^2 u_\mu - p_\mu(p \cdot u). \quad (3.12)
\]

Let us assume that we are in a rest frame of a heat bath, so that \( u_\mu = (1, 0, 0, 0) \).

At the end of this section we would like to introduce the correlation function of the soft-bosonic excitations,

\[
I_{\mu\nu}^{ab}(p', p) = \langle A^a_\mu(p')A^b_\nu(p) \rangle. \quad (3.13)
\]

The asterisk denotes the complex conjugate. The considered soft-gluon excitations are necessarily colorless by virtue of the fact that mean field \( \langle A^a_\mu(p) \rangle \) or associated mean induced color current are assumed to be vanishing. Therefore, for the physical situation of interest, the off-equilibrium two-point function (3.13) is diagonal in a color space.

For the conditions of stationary and homogeneous of hot gluon plasma we have

\[
I_{\mu\nu}^{ab}(p', p) = \delta^{ab}\delta(\mu')\delta(p' - p). \quad (3.14)
\]

The off-equilibrium perturbations which are slowly varying in space and time lead to a \( \delta \)-function broadering, and \( I_{\mu\nu}(p', p) \) depends on both arguments \( p \) and \( p' \). Let us introduce \( I_{\mu\nu}(p', p) = I_{\mu\nu}(p, \Delta p) \), \( \Delta p = p' - p \) with \( |\Delta p/p| \ll 1 \), and insert the correlation function in the Wigner form

\[
I_{\mu\nu}(p, x) = \int I_{\mu\nu}(p, \Delta p) e^{-i\Delta p \cdot x} d\Delta p, \quad (3.15)
\]

slowly depending on \( x = (t, \mathbf{x}) \). In global equilibrium hot QCD plasma the oscillations of two types: the longitudinal and transverse ones can be extended. In this connection we define the Wigner function \( I_{\mu\nu}(p, x) \) in the form of an expansion

\[
I_{\mu\nu}(p, x) = P_{\mu\nu}(p)I^t_p + \tilde{Q}_{\mu\nu}I^l_p, \quad I^{(t,l)}_p \equiv I^{(t,l)}(p, x). \quad (3.16)
\]

4 The correspondence principle

The main purpose of this section is a derivation in an explicit form of the scattering probability \( w_{2n+2} \) for process of elastic collision of \( n + 1 \) plasmons in hot gluon plasma. In our previous paper [6] we have derived the probability \( w_4 \) for a two-to-two scattering process of colorless plasmons. It was calculated by simple extracting all possible contributions to this process. However, use of such a direct approach in a general case of the
process of elastic scattering of \( n + 1 \) plasmons becomes ineffective because of awkwardness and complicated computations. In this case the method developed by Tsytovich [10, 11] in the theory of nonlinear processes in electron-ion plasma, known as the correspondence principle, is more convenient for deriving explicit form of the scattering probability \( w_{2n+2} \). We have already mentioned it in Ref. [6]. Here, this approach will be developed as applied to general case of \((2n + 2)\)-plasmon decay. For non-Abelian plasma this approach is especially effective in the temporal gauge, when we have a closer correspondence with the electrodynamics of an ordinary plasma. The gist of this method is as follows.

The change in the colorless plasmon numbers, caused by spontaneous processes of plasmon decays only, is

\[
\left( \frac{\partial N^l_k}{\partial t} + V^l_p \cdot \frac{\partial N^l_p}{\partial x} \right)^{sp} = \sum_{n=1}^{\infty} \int dT^{(2n+1)} \omega_{2n+2} p_1, \ldots, n; p_{n+1}, \ldots, p_{2n+1}) N^l_{p_1} \cdots N^l_{p_{2n+1}}.
\]

This equation follows from (2.2) in the limit of a small intensity \( N^l_p \to 0 \) and using the fact that the occupation numbers \( N^l_{p_i} \) are larger than one, \( N^l_{p_i} + 1 \approx N^l_{p_i} \). In this case the change of energy of the longitudinal excitations is

\[
\left( \frac{dE}{dt} \right)^{sp} = \frac{d}{dt} \left( \int \frac{dp}{(2\pi)^3} \omega_p N^l_p \right)
\]

(4.1)

On the other hand the value \( \left( \frac{dE}{dt} \right)^{sp} \) represents the emitted radiant power of the longitudinal waves \( I^l \), which in turn is equal to the work done by the radiation field with the color current, creating it, per unit time

\[
I^l = \lim_{\tau, V \to \infty} \frac{1}{\tau V} \int_{-\tau/2}^{\tau/2} \int dx dt \langle E^a(x, t) \cdot J^a(x, t) \rangle = \lim_{\tau, V \to \infty} \frac{(2\pi)^4}{\tau V} \int dp d\omega \langle E^a(p, \omega) \cdot J^a(p, \omega) \rangle.
\]

(4.2)

Here, \( V \) is a spatial volume of integration, \( E^{ai}(x, t) = -\partial A^{ai}(x, t)/\partial t \) is a chromoelectric field in the temporal gauge, \( E^{ai}(p, \omega) \) and \( J^{ai}(p, \omega) \) are the Fourier components of field and current, correspondingly. Averaging over the time was made for elimination of oscillating terms in \( I^l \). For a transformation (4.2) we use a relation

\[
\lim_{\tau \to \infty} \int_{-\tau/2}^{\tau/2} e^{i\omega t} dt = 2\pi \delta(\omega).
\]

The Fourier-component of a field \( E^a(p, \omega) = p E^a(p, \omega)/|p| \) is associated with \( J^a(p, \omega) \) by the field equation

\[
E^a(p, \omega) = i \frac{p^2}{\omega} \Delta^l(p, \omega) \frac{\langle p \cdot J^a(p, \omega) \rangle}{|p|}.
\]

(4.3)
For weak-absorption medium, when $\text{Im}^* \Delta^{-1}(p, \omega) \to 0$, the propagator $^*\Delta^l(p, \omega)$ can be approximated in the following way

$$^*\Delta^l(p) = \frac{1}{p^2 - \Pi^l(p)} \approx \frac{P}{\text{Re}(p^2 - \Pi^l(p))} - i\pi \text{sign}(\omega) \delta(\text{Re}^* \Delta^{-1l}(p)).$$

Here, we consider that $\text{sign}(\text{Im}^* \Delta^{-1l}(p)) = \text{sign}(\omega)$. The symbol $P$ denotes a principal value. Substituting Eq. (4.3) into (4.2) and taking into account reality of $\mathcal{I}^l$, we derive

$$\mathcal{I}^l = -\pi \lim_{\tau, V \to \infty} \frac{(2\pi)^4}{\tau V} \int dp \omega \text{sign}(\omega) \tilde{Q}^{\mu\nu}(p) \langle J^* a_{\mu}(p) J^a_{\nu}(p) \rangle \delta(\text{Re}^* \Delta^{-1l}(p)). \quad (4.4)$$

In derivation of the last expression, we present a longitudinal projector $p \otimes p / p^2$ in a Lorentz covariant form (Eqs. (3.11) and (3.12))

$$\frac{p \otimes p}{p^2} \rightarrow -\frac{\omega^2}{p^2} \tilde{Q}^{\mu\nu}(p). \quad (4.5)$$

Since the nonlinear part of a current only is responsible for processes of plasmon decays, then it is necessary to set $J^a_{\mu} = J^a_{NL}$ in the expression (4.4), where $J^a_{NL}$ is defined by equation (3.8). We perceive the $\delta$-function of the real part of inverse longitudinal propagator in the ordinary sense

$$\delta(\text{Re}^* \Delta^{-1l}(p, \omega)) = \frac{1}{2\omega_p} Z_l(p) [\delta(\omega - \omega_p) + \delta(\omega + \omega_p)], \quad (4.6)$$

where $Z_l(p)$ is the residue of the effective gluon propagator at the plasmon pole.

Substituting the nonlinear current expansion (3.8) into a correlation function in integrand of Eq. (4.4) we face the product of two series. However in this product it is necessary to leave only a sum of a product of terms having the same order in power of a potential $A^a_{\mu}$. Then we have equal number of complex-conjugate potentials and the non-conjugate ones between the inside of the angular brackets of statistical averaging. The necessity of such a choose of terms was demonstrated by a direct calculation of the Boltzmann equation for four-plasmon decay in Ref. [6]. In this case only a desired $\delta$-functions entering to integration measure $\int dT^{(3)}$, expressing the energy and the momentum conservation of the four-plasmon decay, arise. We assume this for general rule, which is valid for decay process with arbitrary number of plasmons. Thus the emitted radiant power (4.4), taking into account emission caused by plasmon decay processes only, can be presented in the form of expansion

$$\mathcal{T}^l = \sum_{s=2}^{\infty} \mathcal{T}^{l(s)}, \quad (4.7)$$

where

$$\mathcal{T}^{l(s)} = -\pi \lim_{\tau, V \to \infty} \frac{(2\pi)^4}{\tau V} \int dp \omega \text{sign}(\omega) \tilde{Q}^{\mu\nu}(p) \langle J^{(s)a}_{NL}(p) J^{(s)a}_{NL}(p) \rangle \delta(\text{Re}^* \Delta^{-1l}(p)). \quad (4.8)$$
In order to define the scattering probability \( w_{2n+2} \), the correlation function on the right-hand side of Eq. (4.8) has to contain terms of \( 2(2n+2) \)th order in a free field \( A_\mu^{(0)a} \). The required \( 2(2n+2) \)-order correlator yields the color current \( J^{(s)a}_\mu (3.9) \) for \( s = 2n+1 \), where the interacting field should be replaced by free field: \( A_\mu^{(a)} \rightarrow A^{(0)a}_\mu \). However, here, it is also necessary to take into account the effects which arise from iteration of the currents \( J^{(s')a}_\mu \) of the lower order, \( 2 \leq s' < 2n+1 \). They give a contribution to the process of \((2n+2)\)-plasmon decay of the same order in coupling constant, as current \( J^{(2n+1)a}_\mu \). The consideration of all similar contributions can be effectively presented as a replacement of a current \( J^{(s)a}_\mu [A] \) by certain effective current \( \tilde{J}^{(s)a}_\mu [A^{(0)}] \):

\[
J^{(s)a}_\mu (A, \ldots, A) \rightarrow \tilde{J}^{(s)a}_\mu (A^{(0)}, \ldots, A^{(0)})
\]

\[
\equiv \frac{1}{s!} g^{s-1} \int \Gamma^{aa_{1}\ldots a_s}_{\mu_1\ldots \mu_s} (p_1, -p_1, \ldots, -p_s) \{ A^{(0)a_1\mu_1}(p_1) A^{(0)a_2\mu_2}(p_2) \ldots A^{(0)a_s\mu_s}(p_s) - A^{(0)a_1\mu_1}(p_1) A^{(0)a_2\mu_2}(p_2) \ldots A^{(0)a_s\mu_s}(p_s) \} \times \delta(p - \sum_{i=1}^{s} p_i) \prod_{i=1}^{s} dp_i,
\]

where the coefficient functions \( \Gamma^{aa_{1}\ldots a_s}_{\mu_1\ldots \mu_s} \) (which in the subsequent discussion we shall call effective amplitudes as distinct from usual HTL–amplitudes \( \Gamma^{aa_{1}\ldots a_s}_{\mu_1\ldots \mu_s} \)) represent highly nontrivial combinations of HTL-amplitudes and an equilibrium propagator for a soft gluon. Below an algorithm of their successive definition will be presented. Here, the fact that a general structure of effective current \( \tilde{J}^{(s)a}_\mu \) is similar to a structure of initial expression (3.9), is only important for us.

Substituting (3.9) into Eq. (4.8) we obtain a sum of products of the correlators of different orders: 2th-order correlator, product of pair correlators multiplied by \( 2(2n-2) \)-th-order correlators etc. The next step is the correlation decoupling of higher correlators in terms of pairs and next expressing \( A^{(0)} A^{(0)} \) in terms of plasmon number density \( N^I \). However as was shown in Ref. [7] the terms in the decomposition do not all associated with the processes of plasmon decays. The decomposition of averaging free field amplitudes into the correlators containing the pair of complex-conjugate potentials or one of the non-conjugate potentials between the inside of the angular brackets of statistical averaging, does not give a contribution to the processes of interest to us. The reason of this fact is similar to reason, by which we have dropped all crossed terms in a product of two series in initial expression (4.4). Here, it is not appeared the required \( \delta \)-functions in the phase-space measure, expressing the energy and the momentum conservation of the decay processes of plasmons. For this reason in substituting the expression (4.9) into (4.8) we can drop all the terms in braces in (4.9) containing the averaging and lead to the following expression, instead of (4.8):

\[
\mathcal{I}^{(s)} = -\pi \lim_{\tau, V \rightarrow \infty} \frac{(2\pi)^4}{\tau V} \frac{1}{(s!)^2} g^{2s-2} \int dp \omega \operatorname{sign}(\omega) \delta^{aa'} \tilde{Q}^{\mu_1 \ldots \mu_s}(p, -p_1, \ldots, -p_s)
\]
× i \tilde{\Gamma}^{a' \alpha'_{1} \ldots a'_{s'}}(p_{1} - p'_{1}, \ldots, -p'_{s'}) \langle A^{(0)\alpha_{1}\mu_{1}}(p_{1}) \ldots A^{(0)\alpha_{s}\mu_{s}}(p_{s})A^{(0)\alpha'_{1} \mu'_{1}}(p'_{1}) \ldots A^{(0)\alpha'_{s'} \mu'_{s'}}(p'_{s'}) \rangle
\times \delta^{(4)}(p - \sum_{i=1}^{s} p_{i}) \delta^{(4)}(p - \sum_{i=1}^{s} p'_{i}) \delta(\text{Re} \, \Delta^{-1}(p)) \prod_{i=1}^{s} dp_{i} dp'_{i}. \quad (4.10)

In order not to overburden equations by symbol \( \dagger \) we use sometimes a dagger \( \dagger \) to denote complex conjugation. We write out decoupling of the 2nd-order correlator on the right-hand side of Eq. (4.10) in terms of pair correlators, which gives a contribution to the processes of interest to us. Suppressing color and Lorentz indices and employing a condensed notion, \( A_{1} \equiv A^{(0)\alpha_{1}}(p_{1}) \), we have

\[ \langle A_{1}^{*}(0) \ldots A_{s}^{*}(0) A_{1'}^{(0)} \ldots A_{s'}^{(0)} \rangle = \left\{ \left[ \langle A_{1}^{*}(0) A_{1'}^{(0)} \rangle \ldots \langle A_{s}^{*}(0) A_{s'}^{(0)} \rangle + (\text{perms. of } 1', 2', \ldots, s') \right] 
\quad + (\text{perms. of } 1, 2, \ldots, s) \right\} = s! \left\{ \langle A_{1}^{*}(0) A_{1'}^{(0)} \rangle \ldots \langle A_{s}^{*}(0) A_{s'}^{(0)} \rangle + (\text{perms. of } 1', 2', \ldots, s') \right\}. \]

All terms on the right-hand side of the latter equality, that is obtained from the first one by all possible permutations of arguments \( (p'_{i}, a'_{i}, \mu'_{i}), \quad i = 1, \ldots, s \), give the same contribution to the emitted radiant power \( \mathcal{I}^{(s)}(x) \) and therefore, here, we can write

\[ \langle A_{1}^{*}(0) \ldots A_{s}^{*}(0) A_{1'}^{(0)} \ldots A_{s'}^{(0)} \rangle = (s!)^{2} \langle A_{1}^{*}(0) A_{1'}^{(0)} \rangle \ldots \langle A_{s}^{*}(0) A_{s'}^{(0)} \rangle \]

\[ = (s!)^{2} \prod_{i=1}^{s} \delta^{a_{i}a'_{i}} \tilde{Q}^{\mu_{i}\mu'_{i}}(p_{i}) I^{l}(p_{i}) \delta(p_{i} - p'_{i}). \]

Here, in the last line we use a definition of the correlation function \( \tilde{Q}^{\mu_{i}\mu'_{i}}(p_{i}) \) in the equilibrium, Eq. (3.14), and leave only the longitudinal part in its expansion (3.16). Now we substitute the obtained correlation decoupling into Eq. (4.10) and perform an integration over \( \prod_{i=1}^{s} dp'_{i} \),

\[ \mathcal{I}^{(s)} = -\pi \lim_{\tau, V \to \infty} \frac{1}{\tau V} g^{2s-2} \int dp \int \prod_{i=1}^{s} dp_{i} (2\pi)^{4} \left[ \delta^{(4)}(p - \sum_{i=1}^{s} p_{i}) \right]^{2} \omega \text{sign}(\omega) \tilde{Q}^{\mu_{i}\mu'_{i}}(p) \quad (4.11) \]

\[ \times \prod_{i=1}^{s} \tilde{Q}^{\mu_{i}\mu'_{i}}(p_{i})^{*} \tilde{\Gamma}^{\alpha_{1} \ldots \alpha_{s}}_{\mu_{1} \ldots \mu_{s}}(p_{1} - p_{1}, \ldots, -p_{s})^{*} \Gamma^{\alpha_{1} \ldots \alpha_{s}}_{\mu'^{1} \ldots \mu'^{s}}(p_{1} - p_{1}, \ldots, -p_{s}) \prod_{i=1}^{s} I^{l}(p_{i}) \delta(\text{Re} \, \Delta^{-1}(p_{i})). \]

By a \( \delta \)-functions squared here, we mean as usual \[ \text{II} \]

\[ \left[ \delta^{(4)}(p - \sum_{i=1}^{s} p_{i}) \right]^{2} = \frac{1}{(2\pi)^{4}} \tau V \delta(p - \sum_{i=1}^{s} p_{i}). \]

To take into account weakly inhomogeneous and weakly non-stationary in the medium it is sufficient, within the accepted accuracy, to replace equilibrium spectral densities \( I^{l}(p_{i}) \) by off-equilibrium ones in the Wigner form (3.15): \( I^{l}(p_{i}) \to I^{l}(p_{i}, x_{i}) \), slowly depending on \( x \). Furthermore, we take the functions \( I^{l}(p_{i}, x_{i}) \) in the form of the quasiparticle approximation

\[ I^{l}(p_{i}, x_{i}) = I^{l}_{p_{i}} \delta(\omega_{i} - \omega_{p_{i}}^{l}) + I^{l}_{-p_{i}} \delta(\omega_{i} + \omega_{p_{i}}^{l}), \quad I_{p_{i}}^{l} \equiv I^{l}(p_{i}, x_{i}) \quad (4.12) \]
and goes over from functions $I_{p_i}^l$ to the plasmon number density

$$N_{p_i}^l = -(2\pi)^3 2\omega_{p_i}^l Z_l^{-1}(p_i) I_{p_i}^l.$$  \hfill (4.13)

Since external soft-gluon legs lie on the plasmon mass-shell, then account must be taken of the fact that conservation laws of energy and momentum (see, section 2) admit decay processes with even number of plasmons, i.e. it is necessary to set $s = 2n+1$, $n = 1, 2, \ldots$

Taking into account above-mentioned, and Eqs. (3.12), (4.6), (4.12) and (4.13) we obtain instead of (4.11)

$$I^{(2n+1)} = \frac{1}{2} \int d\omega \prod_{i=1}^{2n+1} d\omega_i \int \frac{dP}{(2\pi)^3} \int \frac{dP_i}{(2\pi)^3} (2\pi)^4 \delta^{(4)}(p - \sum_{i=1}^{2n+1} p_i) \omega \text{sign}(\omega)

\times M^{a_1 \ldots a_{2n+1}}(p, -p_1, \ldots, -p_{2n+1}) M^{a_1 \ldots a_{2n+1}}(p, -p_1, \ldots, -p_{2n+1})

\times \{ \delta(\omega - \omega_p^l) + \delta(\omega + \omega_p^l) \} \prod_{i=1}^{2n+1} \{ N_{p_i}^l \delta(\omega_i - \omega_p^l) + N_{-p_i}^l \delta(\omega_i + \omega_p^l) \}.$$  \hfill (4.14)

Here, we introduce

$$M^{a_1 \ldots a_{2n+1}}(p, -p_1, \ldots, -p_{2n+1}) = g^{2n} \left( \frac{Z_l(p)}{2\omega_p^l} \right)^{1/2} \left( \frac{\tilde{u}^\mu(p)}{\sqrt{\tilde{u}^2(p)}} \right)$$

$$\times \prod_{i=1}^{2n+1} \left( \frac{Z_l(p_i)}{2\omega_p^l} \right)^{1/2} \left( \frac{\tilde{u}^\mu(p_i)}{\sqrt{\tilde{u}^2(p_i)}} \right) \right|_{\text{on-shell}}$$

representing the interaction matrix element for $(2n + 2)$-plasmon decay. 4-vectors

$$\left( \frac{Z_l(p)}{2\omega_p^l} \right)^{1/2} \frac{\tilde{u}^\mu(p)}{\sqrt{\tilde{u}^2(p)}} \equiv \frac{1}{\sqrt{2\omega_p^l}} \epsilon^\mu(p, \lambda), \quad \lambda = l$$

on the right-hand side of (4.13) represent usual wave functions of longitudinal physical gluon in temporal gauge, where factor $Z_l^{1/2}(p)$ provides renormalization of gluon wave function by thermal effects.

Furthermore we multiply out terms in curly brackets in integrand of Eq. (4.14). In terms containing the factor of $N_{p_i}^l \delta(\omega_i + \omega_p^l)$ type we replace $p_i \rightarrow -p_i$ (in so doing we follow the rule $\omega_{-p_i}^l \rightarrow -\omega_{p_i}^l$). By arguments presented in section 2 it should be left only the terms responsible for elastic plasmon scattering of (2.7) type, and drop the remaining ones in obtained expression. In this case integrand in Eq. (4.14) has the form

$$2(2\pi)^4 \omega_p^l \left\{ \delta^{(4)}(p + p_1 + \ldots + p_n - p_{n+1} - \ldots - p_{2n+1}) M^{*\{a\}}(p, p_1, \ldots, p_n, -p_{n+1}, \ldots, -p_{2n+1}) \right\} \prod_{i=1}^{2n+1} N_{p_i}^l.$$  \hfill (4.16)
Here, for brevity we enter multi-index notation \( \{a\} = (a, a_1, \ldots, a_{2n+1}) \) and fix a sign of a first argument of the matrix element \( M^{(a)} \). The factor 2 takes into account a contribution of the term \( \delta(\omega + \omega'_{p}) \). The permutation over sign means all possible placements of \( n + 1 \) minus signs among external momenta \( p_1, p_2, \ldots, p_{2n+1} \). As the last step it should be performed a replacement of momenta such that arguments of \( \delta \)-functions for all terms in braces of Eq. (4.14) become equal to argument of \( \delta \)-function entering into the first term. This enables us put outside the brackets of \( \delta \)-function as a common factor. Confronting an obtained expression for \( T^{(2n+1)} \) and corresponding term in expansion \( (\text{L.1}) \), one identifies the required probability \( w_{2n+2} \):

\[
w_{2n+2}(p, p_1, \ldots, p_n; p_{n+1}, \ldots, p_{2n+1}) = M^{(a)}(p, p_1, \ldots, p_n, -p_{n+1}, \ldots, -p_{2n+1})M^{(a)}(p, p_1, \ldots, p_n, -p_{n+1}, \ldots, -p_{2n+1}) + \sum_{1 \leq i_1 \leq n} M^{(a)}M^{(a)} + \ldots + \sum_{(p_{i_1}, \ldots, p_n)} M^{(a)}M^{(a)},
\]

where on the last line \( M^{(a)} \equiv M^{(a)}(p, p_1, \ldots, p_n, -p_{n+1}, \ldots, -p_{2n+1}) \). The summing symbol \( \sum_{1 \leq i_1 \leq n} \) denotes summing over all possible momentum interchange \( p_{i_1}, 1 \leq i_1 \leq n \) by momentum \( -p_{j_1} \), \( n + 1 \leq j_1 \leq 2n + 1 \). The summing symbol \( \sum_{1 \leq i_1 < i_2 \leq n} \) analogously denotes a summing over all possible interchange of momenta pair \( (p_{i_1}, p_{i_2}) \), \( 1 \leq i_1 < i_2 \leq n \) by momenta pair \( (-p_{j_1}, -p_{j_2}) \), where \( n + 1 \leq j_1 < j_2 \leq 2n + 1 \) etc. Let us consider as an example a case of four-plasmon decay, that corresponds to \( n = 1 \). By Eq. (4.17) the scattering probability \( w_4 \) has a form

\[
w_4(p, p_1, p_2, p_3) = M^{(a)}(p, p_1, -p_2, -p_3)M^{(a)}(p, p_1, -p_2, -p_3) + M^{(a)}(p, -p_2, p_1, -p_3)M^{(a)}(p, -p_2, p_1, -p_3) + M^{(a)}(p, -p_3, -p_2, p_1)M^{(a)}(p, -p_3, -p_2, p_1).
\]

Here, three terms imply the existence of three possible channels of this process, which change the plasmon number density \( N^p \)

\[g^* + g_1^* \rightleftharpoons g_2^* + g_3^*, \quad g^* + g_2^* \rightleftharpoons g_1^* + g_3^*, \quad g^* + g_3^* \rightleftharpoons g_1^* + g_2^*.
\]

The expression (4.18) is suitable for checking on the symmetry conditions (2.8), which is imposed on the plasmon-plasmon scattering probability.

5 The Boltzmann equation for four-colorless plasmon decays

In this section, we review the main features of the Boltzmann equation for four-colorless plasmon decays which recently derived in Ref. [3]. There are no new results to be reported.
by the effective amplitude formula (4.15) the matrix element \( M \) here, and they are given for coherent and complete subsequent discussion. By general definition of explicit form of \( \tilde{\Gamma} \) three and four-gluon HTL-amplitudes accordingly.

The solution of this equation, which we denote by \( \tilde{J}_\mu^{(3)a} [A^{(0)}] \) (Eq. (5.1)), is solved by the approximation scheme method. Discarding the nonlinear terms in \( A \) on the right-hand side of Eq. (5.1), we obtain in the first approximation

\[
* \tilde{D}^{-1} \mu \nu (p) A_\nu^a(p) = - J^{(2)a\mu} (A, A) - J^{(3)a\mu} (A, A, A),
\]

where

\[
J^{(2)a}_\mu (A, A) = \frac{1}{2!} \int_{\mu \mu_1 \mu_2} * \Gamma^{a a_1 a_2}_{\mu \mu_1 \mu_2 \mu_3} (p, -p_1, -p_2) \{ A^{a_1 \mu_1} (p_1) A^{a_2 \mu_2} (p_2) - \langle A^{a_1 \mu_1} (p_1) A^{a_2 \mu_2} (p_2) \rangle \} \times \delta (p - p_1 - p_2) dp_1 dp_2,
\]

and

\[
J^{(3)a}_\mu (A, A, A) = \frac{1}{3!} \int_{\mu \mu_1 \mu_2} * \Gamma^{a a_1 a_2}_{\mu \mu_1 \mu_2 \mu_3} (p, -p_1, -p_2, -p_3) \{ A^{a_1 \mu_1} (p_1) A^{a_2 \mu_2} (p_2) A^{a_3 \mu_3} (p_3) - A^{a_1 \mu_1} (p_1) A^{a_2 \mu_2} (p_2) A^{a_3 \mu_3} (p_3) \} \delta (p - p_1 - p_2 - p_3) dp_1 dp_2 dp_3.
\]

The function

\[
* \Gamma^{a a_1 a_2}_{\mu \mu_1 \mu_2} (p, -p_1, -p_2) = - i f^{a a_1 a_2} * \Gamma^{a_1 \mu_1 \mu_2} (p, -p_1, -p_2),
\]

are three and four-gluon HTL-amplitudes accordingly.

The nonlinear integral equation (5.1) is solved by the approximation scheme method. Discarding the nonlinear terms in \( A \) on the right-hand side of Eq. (5.1), we obtain in the first approximation

\[
* \tilde{D}^{-1} \mu \nu (p) A_\nu^a(p) = 0.
\]

The solution of this equation, which we denote by \( A^{(0)a}_\mu (p) \), is the solution for a free field.

Furthermore keeping the term, quadratic in field on the right-hand side of Eq. (5.1), we derive

\[
* \tilde{D}^{-1} \mu \nu (p) A_\nu^a(p) = - J^{(2)a\mu} (A^{(0)}, A^{(0)}),
\]

where on the right-hand side we substitute free fields instead of interacting ones. The general solution of the last equation is given in the form

\[
A_\mu^a (p) = A^{(0)a}_\mu (p) - * \tilde{D}_\mu \nu (p) \tilde{J}^{(2)a\nu} (A^{(0)}, A^{(0)}),
\]

where \( \tilde{J}^{(2)a\nu} (A^{(0)}, A^{(0)}) \equiv J^{(2)a\nu} (A^{(0)}, A^{(0)}) \).
The following term in the expansion of the interacting field is defined from equation

\[ *\tilde{D}^{-1\mu\nu}(p) A^\sigma_\mu(p) = -J^{(2)\alpha\mu}(A^{(0)}, A^{(0)}, A^{(0)}) \]

\[ - J^{(2)\alpha\mu}(A^{(0)}, -*\tilde{D} J^{(2)}(A^{(0)}, A^{(0)})) - J^{(3)\alpha\mu}(A^{(0)}, A^{(0)}, A^{(0)}). \]

The first two terms on the right-hand side of this equation represent iteration of the currents \( J^{(s')\alpha} \) of lower order (in this case, \( s' = 2 \)) mentioned in previous section, whose contribution is of the same order in the coupling \( g \) as \( J^{(3)\alpha\mu}(A^{(0)}, A^{(0)}, A^{(0)}) \). Using explicit expressions for currents \( (5.2) \), after cumbersome algebraic transformations, we obtain the form of interacting field from the Eq. \( (5.3) \) with accuracy required for our further calculations

\[ A^\alpha_\mu(p) = A^{(0)}_\mu(p) - *\tilde{D}^{\mu\nu}(p) \tilde{J}^{(2)\alpha\nu}(A^{(0)}, A^{(0)}) - *\tilde{D}^{\mu\nu}(p) \tilde{J}^{(3)\alpha\nu}(A^{(0)}, A^{(0)}, A^{(0)}). \]

Here, third-order color current on the right-hand side is defined by expression

\[ \tilde{J}^{(3)\alpha}_\mu(A^{(0)}, A^{(0)}, A^{(0)}) \]

\[ = \frac{1}{3!} g^2 \int *\bar{\Gamma}^{a_1 a_2 a_3}_{\mu\mu_1 \mu_2 \mu_3}(p, -p_1, -p_2, -p_3) \{ A^{(0)}(p_1) A^{(0)}(p_2) A^{(0)}(p_3) \}
\]

\[ - A^{(0)}(p_1) A^{(0)}(p_2) A^{(0)}(p_3) \}

\[ \times \delta(p - p_1 - p_2 - p_3) dp_1 dp_2 dp_3, \]

where the effective four-gluon amplitude \( *\bar{\Gamma}^{a_1 a_2 a_3}_{\mu\mu_1 \mu_2 \mu_3}(\equiv *\bar{\Gamma}^{(4)}) \) is defined by expression

\[ *\bar{\Gamma}^{a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3}(p, -p_1, -p_2, -p_3) = -f^{a_1 b} f^{b a_2 a_3} *\tilde{\Gamma}^{\mu_1 \mu_2 \mu_3}(p, -p_1, -p_2, -p_3) \]

\[ - f^{a_2 b} f^{b a_1 a_3} *\tilde{\Gamma}^{\mu_2 \mu_1 \mu_3}(p, -p_2, -p_1, -p_3). \]

The color factors in Eq. \( (5.3) \) are multiplied by purely kinematical coefficients, which we shall call partial effective amplitudes or effective subamplitudes, and defined as follows

\[ *\tilde{\Gamma}^{\mu_1 \mu_2 \mu_3}(p, -p_1, -p_2, -p_3) \equiv *\Gamma^{\mu_1 \mu_2 \mu_3}(p, -p_1, -p_2, -p_3) \]

\[ - *\Gamma^{\mu_1 \nu}(p, -p_1, -p + p_1) *\tilde{D}^{\nu\mu}(p_2 + p_3) *\Gamma^{\nu \mu_2 \mu_3}(p_2 + p_3, -p_2, -p_3) \]

\[ - *\Gamma^{\mu_2 \nu}(p, -p_2, -p + p_2) *\tilde{D}^{\nu\mu}(p_1 + p_2) *\Gamma^{\nu \mu_1 \mu_3}(p_1 + p_2, -p_2, -p_1). \]

For complete picture we leave the term \( \langle A^{(0)}(0) A^{(0)} \rangle \) in braces under the integral sign on the right-hand side of Eq. \( (5.4) \). It is clear that this term is equal to zero, because \( A^{(0)} \) represents the amplitudes of entirely uncorrelated waves. The expression \( (5.4) \) represents

---

\( ^3 \)Here, we use the terminology accepted in theory of multi-parton hard processes (see, e.g. \( [5.3] \)), where the expansion of \( (5.5) \) type in calculation of the tree QCD amplitudes in the high-energy limit, are also arizen.
desired effective current $\tilde{j}^{(3)\alpha}_\mu [A^{(0)}]$, whose coefficient function by a formula (4.13) defines the matrix element for four-plasmon decay

$$M^{a_1a_2a_3}(p, -p_1, -p_2, -p_3) = \beta^2 \left( \frac{Z_l(p)}{2\omega^l_{p}} \right)^{1/2} \left( \frac{\bar{u}_\mu(p)}{\sqrt{\bar{u}^2(p)}} \right) \prod_{i=1}^3 \left( \frac{Z_l(p_i)}{2\omega^l_{p_i}} \right)^{1/2} \left( \frac{\bar{u}_\mu(p_i)}{\sqrt{\bar{u}^2(p_i)}} \right) \tilde{\Gamma}^{a_1a_2a_3}_{\mu\mu\mu\mu}(p, -p_1, -p_2, -p_3) \bigg|_{\text{on-shell}}. \tag{5.7}$$

The obtained matrix element has a simple diagrammatic representation drawn in Fig. 1. The black square here, denotes an effective amplitude $\tilde{\Gamma}^{(4)}$. The first term on the right-hand side Fig. 1 represents a direct interaction of four plasmons, that induced by usual four-gluon HTL-amplitude. The remaining terms are connected with plasmons interaction, induced by three-gluon HTL-amplitudes with intermediate virtual oscillations representing $s$- and $t$-channel contributions respectively (effective subamplitude $\tilde{\Gamma}^{(3)}_{\mu\nu\rho\mu\nu\rho}(p, -p_2, -p_1, -p_3)$ in Eq. (5.8) contains also $u$-channel contribution, which we have not drawn in Fig. 1).

With the matrix element (5.7) in hand it is not difficult to write the Boltzmann equation describing the plasmon-plasmon scattering. Here, we have restricted our consideration to linearized version of this equation. For this purpose we assume that the off-equilibrium fluctuation is perturbative small and write the number density of colorless plasmons as

$$N^l_p = N^l_{eq}(p) + \delta N^l_p,$$

where $N^l_{eq}(p) = (e^{\omega^l_p/T^*} - 1)^{-1}$ is the Planck distribution function and $T^*$ is a certain constant, which can be interpreted as a plasmon gas temperature in the statistical equilibrium state. Parametrizating off-equilibrium fluctuation of the occupation number $\delta N^l_p$ as follows

$$\delta N^l_p \equiv -\frac{dN^l_{eq}(p)}{dp} \omega^l_p = \frac{1}{T^*} N^l_{eq}(p)(N^l_{eq}(p) + 1), \tag{5.8}$$

we derive from Eqs. (2.2) – (2.5) for $n = 1$, and Eqs. (4.18) and (5.7), after simple color algebra, a linearized Boltzmann equation for function $W^l_p$

$$\frac{\partial W^l_p}{\partial t} + V^l_p \cdot \frac{\partial W^l_p}{\partial x} = -\int \frac{dp_1}{(2\pi)^3} \frac{dp_2}{(2\pi)^3} \frac{dp_3}{(2\pi)^3} \left( \frac{(2\pi)^4}{2!} \delta(\omega^l_p + \omega^l_{p_1} - \omega^l_{p_2} - \omega^l_{p_3}) \right) \times$$

$$\times \delta(p + p_1 - p_2 - p_3) \frac{N^l_{eq}(p_1)(N^l_{eq}(p_2) + 1)(N^l_{eq}(p_3) + 1)}{(N^l_{eq}(p) + 1)}$$

$$\times w(p, p_1; p_2, p_3) \{ W^l_p - W^l_{p_2} + W^l_{p_1} - W^l_p \}, \tag{5.9}$$

where the function

$$w(p, p_1; p_2, p_3) = 3g^4 N^2_c \left( \frac{Z_l(p)}{2\omega^l_{p} \bar{u}^2(p)} \right) \prod_{i=1}^3 \left( \frac{Z_l(p_i)}{2\omega^l_{p_i} \bar{u}^2(p_i)} \right) \left| \tilde{\Gamma}(p, -p_2, p_1, -p_3) \right|^2 \tag{5.10}$$
\[ + |\tilde{\Gamma}(p, p_1, -p_3, -p_2)|^2 + \text{Re} \left( \tilde{\Gamma}(p, -p_2, p_1, -p_3) \tilde{\Gamma}^\dagger(p, p_1, -p_3, -p_2) \right) \big|_{\text{on-shell}} \]  \tag{5.10}

is the probability of four-colorless plasmon decay. Here, we denote

\[ \tilde{\Gamma}(p, p_1, p_2, p_3) \equiv \tilde{\Gamma}_{\mu\nu\lambda\mu_2\mu_3}(p, p_1, p_2, p_3) \bar{u}_\mu(p) \bar{u}_{\mu_1}(p_1) \bar{u}_{\mu_2}(p_2) \bar{u}_{\mu_3}(p_3). \]  \tag{5.11}

The structure of the expression on the right-hand side of Eq. \((5.9)\) is just as the structure of the expression for the case of the Boltzmann equation for hard gluons \([14]\). However, here, we have more complicated and cumbersome expression for scattering probability of two soft quasiparticles involving the resummed vertices. Besides, the momentum transfer \(q = p - p_2 = p_3 - p_1\) is of the same order as the momentum of the soft quasiparticles. In the last case it is impossible to expand functions \(W_{p_2=p-q}^{(3)}\) and \(W_{p_3=p_1+q}^{(3)}\) as power series in momentum transfer \(q\), as it takes place for hard gluons \([14]\).

Using the linearized Boltzmann equation \((5.9)\) it is not difficult to estimate the order of the lifetimes of plasmon \(\tau^l\) caused by the process of four-plasmon decay. Considering the plasmon gas in thermal equilibrium with hard particles from the heat bath, i.e. \(T^* \simeq T\), and using Eqs. \((5.9)\) and \((5.10)\), we obtain

\[ \frac{1}{\tau^l} \sim g^3 N_c^2 T. \]

Thus necessary condition \(\omega^l \tau^l \gg 1\) (section 2), such that excited state of plasma can be described in terms of the plasmon occupation number \(N_p^l\) for \(g \ll 1\), is high accurate, since here, we have \(\omega^l \tau^l \sim 1/g^2\).

At the end of this section we also represent some properties of the effective subamplitude \(\tilde{\Gamma}_{\mu\nu\lambda\mu_2\mu_3}(p, -p_1, -p_2, -p_3)\) defined by Eq. \((5.6)\). In particular, deriving Eqs. \((5.9)\) and \((5.10)\) we have used two relations which satisfy an effective subamplitude

\[
\begin{align*}
\tilde{\Gamma}_{\mu\nu\lambda\mu_2\mu_3}(p, -p_1, -p_2, -p_3) + \tilde{\Gamma}_{\mu_2\mu_1\lambda\mu_3}(p, -p_2, -p_1, -p_3) + \tilde{\Gamma}_{\mu_1\mu_3\mu_2}(p, -p_1, -p_3, -p_2) &= 0, \\
\tilde{\Gamma}_{\mu\nu\lambda\mu_2\mu_3}(p, -p_1, -p_2, -p_3) &= \tilde{\Gamma}_{\mu_3\mu_2\mu_1}(p, -p_3, -p_2, -p_1).
\end{align*}
\]  \tag{5.12} \tag{5.13}

Their correctness may be verified by direct calculation using known properties of HTL-amplitudes \([2]\), entering into the definition of \(\tilde{\Gamma}^{(4)}\). These relations show that the effective amplitude \(\tilde{\Gamma}^{(4)}\) possesses some properties of usual four-gluon HTL-amplitude. As we shall show below this statement takes place for the effective amplitudes \(\tilde{\Gamma}^{(s)}\) for any values \(s \geq 4\) (the case when \(s = 3\) is trivial by virtue of \(\tilde{\Gamma}^{(3)} \equiv \Gamma^{(3)}\)). Here, the exception, for example, is the Ward identity, which for \(\tilde{\Gamma}^{(4)}\) has not a standard form.

### 6 Characteristic amplitudes of the soft gluon field

In this section we shall estimate for which typical amplitude of the soft gluon field, the contribution of four-plasmon decay process to a generalized decay and regeneration rates,
$\Gamma_d[N^l_p]$ and $\Gamma_i[N^l_p]$, will be leading and for which value all terms in the expansions (2.3) will be of the same order in $g$. At the last case two problem arise: the computation of explicit form of matrix elements of decay processes of high orders, for $n > 1$, and the problem of summing (2.3). Recall that in this work we consider the collective processes only at the soft momentum scale for $\omega, |p| \sim gT$.

First of all we estimate an order of matrix element $M^{aa_1...a_{2n+1}}$. We use general expression (4.15) connecting matrix element for $(2n+2)$-plasmon decay with the effective amplitude $^*\bar{\Gamma}^{2n+2}$ in which the kinematical factors corresponding to external soft-gluon legs are single out. In the soft region of the momentum scale the following estimation results from this expression

$$M^{aa_1...a_{2n+1}} \sim (gT)^{-(n+1)} ^*\bar{\Gamma}^{aa_1...a_{2n+1}}_{\mu\mu_1...\mu_{2n+1}}.$$

Here, the symbol $\sim$ denotes that the order of value from the left in coupling constant equals to the order of value from the right. The order of effective amplitude $^*\bar{\Gamma}^{aa_1...a_{2n+1}}_{\mu\mu_1...\mu_{2n+1}}$ can be estimated from arbitrary tree diagram with amputate $2n+2$ soft external legs. Let us consider, for example, the diagram drawn in Fig. 2. Four-gluon vertices are of order $g^2$, and HTL-resummed propagators are of order $1/(gT)^2$. From simple power counting of the diagram it follows an estimation

$$^*\bar{\Gamma}^{aa_1...a_{2n+1}}_{\mu\mu_1...\mu_{2n+1}} \sim g^{2n} N_c^{\frac{2}{3}} \frac{1}{(gT)^{2(n-1)}}.$$

Furthermore integration measure has an estimation

$$dT^{(2n+1)} \sim (gT)^{3n-1}.$$  

(6.2)

Let us define an order in $g$ of the plasmon density. For this purpose we use initial relations connecting the plasmon number density $N^l_p$ with two-point correlation function (3.13) of soft-gluon field. Here we have

$$N^l_p = - (2\pi)^3 2\omega^l_p Z^{-1}_l(p) I^l_p,$$

where in turns the spectral density $I^l_p$ is defined from relation $I^l_p = I^l_p(\omega - \omega^l_p) + I^l_p(\omega + \omega^l_p)$, and finally function $I^l_p$ is defined by two-point correlation function

$$\langle A_\mu(p')A_\nu(p) \rangle \sim \bar{Q}_{\mu\nu}(p') I^l_p(\delta(p' - p)).$$

Taking into account above-mentioned, we derive the following expression for estimation of the plasmon number density

$$N^l_p \sim \omega^l_p Z^{-1}_l(p) \delta(\omega - \omega^l_p) \delta(p' - p) \langle A_\mu(p')A_\nu(p) \rangle. $$

(6.3)

Let us consider more typical two values of amplitude of a gauge field $A_\mu$ at the soft momentum scale. Using Eq. (3.3) we have:
1. let $|A_\mu(X)| \sim T \, (|A_\mu(p)| \sim 1/gT^3)$, then $N^l_p \sim \frac{1}{g^2}$, \hspace{1cm} (6.4)

2. let $|A_\mu(X)| \sim \sqrt{gT} \, (|A_\mu(p)| \sim 1/\sqrt{gT})$, then $N^l_p \sim \frac{1}{g}$. \hspace{1cm} (6.5)

For latter value of amplitude of the gauge field, when $g \ll 1$, the occupation number $N^l_p$ is large, and the use of classical description is justified. The value of the amplitude of a gauge field ($\sim \sqrt{gT}$) is more probably at the momentum and energy scale of the plasmon model.\footnote{Blaizot and Iancu \cite{1} showed in the special case when the soft fields were thermal fluctuations at soft scale $gT$, that their typical amplitudes would be of the order $|A_\mu(X)| \sim \sqrt{gT}$. A similar estimation of a value of the classical fields at the soft scale was shown by Nauta and van Weert in Ref. \cite{15} based upon a somewhat different context.} Power counting of the decay and regenerating rates, (2.4) and (2.5) with regard to (6.1) and (6.2) gives the following estimation

$$
\Gamma_{d,i}^{(2n+1)aa'} \sim g^{4n}(gT)N^c_2 (N^l_p)^{2n+1}.
$$

If we now set $N^l_p \sim \frac{1}{g^\alpha}$, $\alpha > 0$, then from the last expression it follows

$$
\Gamma_{d,i}^{(2n+1)aa'} \sim g^{2n(2-\alpha)}g^{1-\alpha}N^c_2 T.
$$

For small value of oscillation amplitude (6.5) we have

$$
\left(\Gamma_{d,i}^{(2n+1)aa'}\right)_{A \sim \sqrt{gT}} \sim g^{2n}N^c_2 T.
$$

From this estimation it can be seen that each subsequent term in the functional expansions (2.3) is suppressed by more power of $g^2$ and here, we can only restrict ourselves to first leading term, describing four-plasmon decay process. Besides, here, the use of the linearized Boltzmann equation for colorless plasmons, i.e. Eq. (5.9) is justified, such the Planck distribution, relative of which the departure of the plasmon number density is defined, is of order

$$
N^l_{eq}(p) \sim \frac{T}{\omega_p} \sim \frac{1}{g}.
$$

In this case it is said that a theory of a plasmon-plasmon interaction for small amplitude of soft excitations is linear (linear amplitude regime), and nonlinear effects connected with nonlinear terms over off-equilibrium fluctuations of plasmon occupation number $\delta N^l_p$ can be treated as perturbations.

A situation is qualitatively changed, when a system is highly excited. In a limiting case of a strong field, $A_\mu \sim T$, for $\alpha = 2$ from the estimation (6.6) it follows that functions $\Gamma_{d,i}^{(2n+1)aa'}$ is not to depend on $n$. All terms in the expansions (2.3) become of the same order in magnitude, and the problem of resummation of all relevant contributions
arises. It is evident that a procedure of a linearization of a kinetic equation for plasmon occupation number $N_P$ in this case becomes unacceptable and here, we lead to truly nonlinear interaction theory of the soft excitations in hot QCD plasma.

It is noted an interesting analogy with phenomenon of so-called parton saturation, which is of great interest in the physics of nuclear and hadronic processes in the regime where Bjorken’s $x$ becomes very small \[16, 17\]. Saturation is expected at gluon phase-space density of order $1/g^2$ (this is just maximum value of the plasmon occupation number that in our case corresponds to maximum value of an oscillation amplitude of soft gluon fields: $A_\mu \sim T$). Using this fact McLerran and Venugopalan \[18\] suggested a classical effective theory to describe the gluon distribution in large nuclei and valid for some range of $x$. The consequence of a development of this theory is a construction by Jalilian-Marian, Kovner, Leonidov and Weigert of renormalization group equation \[19\] (the JKLW equation) that predicts the evolution of the gluon distribution function. In the saturation regime, where $A_\mu \sim 1/g$, a contributions of all orders in the strong background fields which represent the condensate, should be taken into account. In the low density, or weak-field limit, this equation is linearized and reduced to known equations. In our case the role of the JKLW equation fulfils a kinetic equation (2.2) that is complete nonlinear in a limiting case of a highly excitations of a soft-gluon field $A_\mu \sim T$ and is reduced to a linearized Boltzmann equation (5.9) in the weak-field limit.

## 7 The matrix elements for $(2n+2)$-plasmon decays

As shown in section 4, a calculation of matrix elements $M^{a_1...a_{2n+1}}$ which are responsible for processes of $(2n+2)$-plasmon decays reduces to calculation of certain effective currents $J^{(2n+1)}_\mu$ being functionals of $2n + 2$ free soft-gluon fields $A^{(0)}_\mu, n = 1, 2, \ldots$. The coefficient functions in integrand of effective currents (4.9) by equation (4.13) define matrix elements of decay processes in tree approximation, which allowed by conservation laws of energy and momentum. In section 5 an example of construction of such an effective current in simplest case of four-plasmon decay was presented. However as we have shown in previous section, it becomes necessary to take into account the contributions of higher decay processes, i.e. subsequent terms in expansions (2.3) when the energy level of the soft plasma excitations increases. Here, we consider the problem of determination of explicit form of matrix elements for higher decay processes leaving aside more subtle questions connected with summing and convergence of a series (2.3).

As we have seen in section 5 the effective currents appear in the solution of the non-linear field equation (3.3), that defines interacting soft-gluon field $A_\mu$ in the form of a functional expansion in free field $A^{(0)}_\mu$. Here, we shall extend this approach for determination of the effective amplitudes $\tilde{\Gamma}^{(s)}$ for $s$ arbitrary values. Let us rewrite here, the field
equation (3.8) for convenience of further references

\[ \ast \mathcal{D}^{-1 \mu \nu}(p) A^a_\mu(p) = -J^{a\mu}_{NL}[A] \equiv - \sum_{s=2}^{\infty} J^{(s)a\mu}(A, \ldots, A), \quad (7.1) \]

where

\[ J^{(s)a}(A, \ldots, A) = \frac{1}{s!} g^{s-1} \int \Gamma_{\mu_{s+1} \ldots \mu_s} A^{a_{s+1}}(p_1) A^{a_{s+2}}(p_2) \ldots A^{a_s}(p_s) \times \delta(p - \sum_{i=1}^{s} p_i) \prod_{i=1}^{s} dp_i. \quad (7.2) \]

On the right-hand side of Eq. (7.2) we discard all terms, containing the thermal averaging (see discussion following by Eq. (4.9) in section 4). The nonlinear field equation (7.1) can be formally solved perturbatively (at least in the weak-field limit) order by order in the coupling constant. We expand

\[ A^a_\mu(p) = \sum_{s=1}^{\infty} A^{(s-1)a}(p), \quad (7.3) \]

where \( A^{(s-1)a}(p) \) is a contribution of order \( g^{s-1} \) to the soft-gluon field. For \( s = 1 \) we have a solution for a free field. Substituting an expansion (7.3) into Eq. (7.1) we obtain iterative solutions of higher order in \( s \):

\[ A^{(1)a}_\mu(p) = -\ast \mathcal{D}_{\mu \nu}(p) \tilde{J}^{(2)a \nu}(A^{(0)}, A^{(0)}), \]
\[ A^{(2)a}_\mu(p) = -\ast \mathcal{D}_{\mu \nu}(p) \tilde{J}^{(3)a \nu}(A^{(0)}, A^{(0)}, A^{(0)}), \]
\[ A^{(3)a}_\mu(p) = -\ast \mathcal{D}_{\mu \nu}(p) \tilde{J}^{(4)a \nu}(A^{(0)}, A^{(0)}, A^{(0)}, A^{(0)}), \quad (7.4) \]
\[ \ldots \]
\[ A^{(s-1)a}_\mu(p) = -\ast \mathcal{D}_{\mu \nu}(p) \tilde{J}^{(s)a \nu}(A^{(0)}, \ldots, A^{(0)}), \quad \ldots \]

Here, the effective current \( \tilde{J}^{(3)a \nu} \) is defined by Eqs. (5.4) – (5.6), the current \( \tilde{J}^{(4)a \nu} \) is defined by next iteration

\[ \tilde{J}^{(4)a \nu}(A^{(0)}, A^{(0)}, A^{(0)}, A^{(0)}) \equiv J^{(4)a \nu}(A^{(0)}, A^{(0)}, A^{(0)}, A^{(0)}) \]
\[ - \left\{ J^{(3)a \nu}(\ast \mathcal{D} \tilde{J}^{(2)}(A^{(0)}, A^{(0)}), A^{(0)}, A^{(0)}), A^{(0)}) + J^{(3)a \nu}(A^{(0)}, \ast \mathcal{D} \tilde{J}^{(2)}(A^{(0)}, A^{(0)}), A^{(0)}) \right\} \]
\[ + J^{(3)a \nu}(A^{(0)}, A^{(0)}, \ast \mathcal{D} \tilde{J}^{(2)}(A^{(0)}, A^{(0)))) \] \( (7.5) \)
\[ - \left\{ J^{(2)a \nu}(\ast \mathcal{D} \tilde{J}^{(3)}(A^{(0)}, A^{(0)}, A^{(0)}), A^{(0)}) + J^{(2)a \nu}(A^{(0)}, \ast \mathcal{D} \tilde{J}^{(3)}(A^{(0)}, A^{(0)}, A^{(0)})) \right\} \]
\[ - J^{(2)a \nu}(\ast \mathcal{D} \tilde{J}^{(3)}(A^{(0)}, A^{(0)}), \ast \mathcal{D} \tilde{J}^{(3)}(A^{(0)}, A^{(0)})) \]
etc. A similar iterative procedure of computing perturbative solutions of the classical Yang-Mills equation in various modifications is often used in solving specific problems. From more closely works to subject of our research mention may be made of the works of Kovchegov and Rischke [20], Matinyan, Müller and Rischke [21] concerned with the problem of the classical gluon radiation in ultrarelativistic nucleus-nucleus collisions. Furthermore, in Ref. [22] Bödeker used iterative solutions of the classical Yang-Mills equations for soft modes of the gluon field in deriving the Boltzmann equation for hard transverse gluons. Finally in the construction of the correlation functions for ultrasoft fields \((\omega, |\mathbf{p}| \sim g^2 T)\), a similar iterative procedure was used by Guerin in Ref. [23].

However such a direct approach for determination of an explicit form of the higher effective amplitudes \(\tilde{\Gamma}^{(s)}\), \(s > 3\), becomes very complicated and as consequence ineffective. In particular this associated with a necessity of additional symmetrization of integrands, which becomes extremely intricate with increasing order \(s\). It is easily seen, for example, in attempt to determine the effective amplitude \(\tilde{\Gamma}^{(5)}\) immediately from the right-hand side of Eq. (7.5). Here we proposed a somewhat different approach of calculation of \(\tilde{\Gamma}^{(s)}\). It is not transparent as the approach based on iterations (7.4), but it allows us to avoid many intermediate operations and somewhat automate calculation procedure.

The calculating algorithm is based on a simple idea. The nonlinear current \(J_{NL}^{a\mu}[A]\) has two representations: by means of free and interacting fields, which must be equal each other

\[
J_{NL}^{a\mu}[A] = \sum_{s=2}^{\infty} J^{(s)a\mu}(A^{(0)}, \ldots, A^{(0)}) = \sum_{s=2}^{\infty} J^{(s)a\mu}(A, \ldots, A). \tag{7.6}
\]

Here, the interacting fields on the right-hand side of the last equation are defined by expansion

\[
A^{a\mu}(p) = A^{(0)a\mu}(p) - \tilde{\mathcal{D}}^{a\mu}(p) \sum_{s=2}^{\infty} J^{(s)a\mu}(A^{(0)}, \ldots, A^{(0)}), \tag{7.7}
\]

where

\[
J^{(s)a\mu}(A^{(0)}, \ldots, A^{(0)}) = \frac{1}{s!} g^{-1} \int \tilde{\Gamma}^{a\mu_{a_1}\ldots a_s}(p, -p_1, \ldots, -p_s) A^{(0)a_1\mu_1}(p_1) \ldots A^{(0)a_s\mu_s}(p_s) \tag{7.8}
\]

\[
\times \delta(p - \sum_{i=1}^{s} p_i) \prod_{i=1}^{s} dp_i.
\]

Substitution of Eqs. (7.7) and (7.8) into (7.3) turns this equation into identity. Now we functionally differentiate left- and right-hand sides of equality (7.6) with respect to free field \(A^{(0)a\mu}\) considering Eq. (7.8) for differentiation on the left-hand side and Eqs. (7.2), (7.7) and (7.8) for differentiation on the right-hand side, and set \(A^{(0)a\mu} = 0\) after all calculations. The required effective amplitudes \(\tilde{\Gamma}^{(s)}\) will appear both on the left-hand side and on the right-hand side. However main effect is that the effective amplitudes on the right-hand side will have at least one less external soft leg than on the left-hand side. This enables us to calculate them in a recurrent way. Below we shall give a few examples.
The second differentiation of a nonlinear current yields

\[
\frac{\delta^2 J_{NL}^a [A](p)}{\delta A^{(0) a_1 \mu_1}(p_1) \delta A^{(0) a_2 \mu_2}(p_2)}\bigg|_{A^{(0)}=0} = *\Gamma_{\mu_1 \mu_2}^{a_1 a_2} (p, -p_1, -p_2) \delta^{(4)} (p - p_1 - p_2)
\]

\[
= *\Gamma_{\mu_1 \mu_2}^{a_1 a_2} (p, -p_1, -p_2) \delta^{(4)} (p - p_1 - p_2).
\]

This example shows that the effective amplitude of three-plasmon decay coincide with usual three-gluon HTL-amplitude. This decay process is kinematically forbidden and hence the \(\delta\)-function no support on the plasmon mass-shell.

The first interesting case arises in calculation of the next derivative. It defines the effective amplitude for four-plasmon decay

\[
\frac{\delta^3 J_{NL}^a [A](p)}{\delta A^{(0) a_1 \mu_1}(p_1) \delta A^{(0) a_2 \mu_2}(p_2) \delta A^{(0) a_3 \mu_3}(p_3)}\bigg|_{A^{(0)}=0}
\]

\[
= *\tilde{\Gamma}_{\mu_1 \mu_2 \mu_3}^{a_1 a_2 a_3} (p, -p_1, -p_2, -p_3) \delta^{(4)} (p - p_1 - p_2 - p_3) = \left[ *\Gamma_{\mu_1 \mu_2 \mu_3}^{a_1 a_2 a_3} (p, -p_1, -p_2, -p_3)
\right.
\]

\[
- \left\{ \Gamma_{\mu \nu \mu_2}^{a b a_2} (p, -p+p_2, -p_2) *\tilde{D}^{\mu \nu} (p_1+p_3) *\tilde{\Gamma}_{\nu \mu_1 \mu_3}^{b a_1 a_3} (p_1+p_3, -p_1, -p_3) + (\text{circular perms. of } 1, 2, 3) \right\}
\]

\[
\times \delta^{(4)} (p - p_1 - p_2 - p_3).
\]

By using Jacobi identity for anti-symmetric structure constants, the right-hand side of the last expression can be led to the expression (7.5) obtained before. It can be viewed as an expansion of an effective amplitude \( *\tilde{\Gamma}_{\mu_1 \mu_2 \mu_3}^{a_1 a_2 a_3} \) in terms of two independent combinations of a product of two structure constants

\[
f^{a a_1} f^{b a_2 a_3}, \quad f^{a a_2} f^{b a_1 a_3},
\]

(7.9)

as it occurs for usual four-gluon HTL-amplitude \([2]\). A similar expansions take place also for higher \(*\tilde{\Gamma}^{(s)}\) functions. Let us consider the fourth derivative of a nonlinear current with respect to \(A^{(0) a_4}\), defining a five-point effective amplitude. After rather cumbersome but direct calculations we obtain

\[
\frac{\delta^4 J_{NL}^a [A](p)}{\delta A^{(0) a_1 \mu_1}(p_1) \ldots \delta A^{(0) a_4 \mu_4}(p_4)}\bigg|_{A^{(0)}=0}
\]

\[
= *\tilde{\Gamma}_{\mu_1 \ldots \mu_4}^{a_1 \ldots a_4} (p, -p_1, \ldots, -p_4) \delta^{(4)} (p - \sum_{i=1}^{4} p_i) = \left[ *\Gamma_{\mu_1 \ldots \mu_4}^{a_1 \ldots a_4} (p, -p_1, \ldots, -p_4)
\right.
\]

\[
- \left\{ \Gamma_{\mu \nu \mu_2 \mu_3}^{a b a_2 a_3} (p, -p_1 - p_4, -p_2, -p_3) *\tilde{D}^{\mu \nu} (p_1 + p_4) *\tilde{\Gamma}_{\nu \mu_1 \mu_3}^{b a_1 a_3} (p_1 + p_4, -p_1, -p_4)
\right.
\]

\[
+ \Gamma_{\mu \nu \mu_1 \mu_3}^{a_1 b a_3} (p, -p_1, -p_2 - p_4, -p_3) *\tilde{D}^{\mu \nu} (p_2 + p_4) *\tilde{\Gamma}_{\nu \mu_2 \mu_3}^{b a_2 a_4} (p_2 + p_4, -p_2, -p_4)
\]

\[
+ (\text{circular perms. of } 2, 3, 4) \right\}
\]

\[
- \left\{ \Gamma_{\mu \nu \mu_1}^{a b a_1} (p, -p + p_1, -p_1) *\tilde{D}^{\mu \nu} (p_2 + p_3 + p_4) *\tilde{\Gamma}_{\nu \mu_2 \mu_3}^{b a_2 a_3 a_4} (p - p_1, -p_2, -p_3, -p_4)
\right\}
\]
\[ + \text{(circular perms. of } 1, 2, 3, 4) \} \]
\[ + \left\{ \Gamma_{\mu\nu\lambda}^{abc}(p, -p_1 - p_3, -p_2 - p_4) * \tilde{D}^{\mu\nu\lambda}(p + p_3) * \tilde{D}_{\mu_1\mu_2\mu_3}^{\nu\alpha\beta}(p_1 + p_3, -p_1, -p_3) \right\} \]
\[ \times * \tilde{D}^{\lambda\mu\nu}(p_2 + p_4) * \tilde{D}_{\lambda\mu_2\mu_3}^{\alpha\beta\gamma}(p_2 + p_4, -p_2, -p_4) + \text{(circular perms. of } 2, 3, 4) \} \}
\[ \delta^{(4)}(p - \sum_{i=1}^{4} p_i). \]

The obtained expression as it stands is very complicated and has involved color structure. By direct algebraic transformations, using the Jacobi identity for structure constants and relations (5.12) and (5.13) it can be represented in the form of an expansion in terms of six independent combinations of a product of three structure constants, which we are choose as follows
\[ f_{aa_1} b_1 f_{b_1 a_2} b_2 f_{b_2 a_3} a_4, \quad f_{aa_1} b_1 f_{b_1 a_2} b_2 f_{b_2 a_3} a_4, \quad f_{aa_3} b_1 f_{b_1 a_2} b_2 f_{b_2 a_3} a_4, \quad f_{aa_3} b_1 f_{b_1 a_2} b_2 f_{b_2 a_3} a_4. \]

(7.11)

Here, as in basis (7.3), we fixed the first and the last color indices. The expansion \( \ast \Gamma^{(5)} \) in terms of a basis (7.11) reads
\[ \ast \tilde{\Gamma}_{\mu_1...\mu_4}^{aa_1...a_4}(p, -p_1, \ldots, -p_4) = f_{aa_1} b_1 f_{b_1 a_2} b_2 f_{b_2 a_3} a_4 \ast \tilde{\Gamma}_{\mu_1...\mu_4}^{aa_1...a_4}(p, -p_1, \ldots, -p_4) \]
\[ + \text{(perms. of } 1, 2, 3), \]

where a sum is over all 3! permutations of the external gluons 1, 2 and 3, and effective subamplitude is defined as
\[ \ast \tilde{\Gamma}_{\mu_1...\mu_4}(p, -p_1, \ldots, -p_4) \equiv \ast \Gamma_{\mu_1...\mu_4}(p, -p_1, \ldots, -p_4) \]
\[ - \left\{ \Gamma_{\mu_1\mu_2\nu}(p, -p_1, -p_2, -p_3 - p_4) * \tilde{D}^{\nu\mu\lambda}(p + p_3) * \tilde{\Gamma}_{\nu_1\nu_2\nu_3}^{\mu_1\mu_2\mu_3}(p_3 + p_4, -p_3, -p_4) \right\} \]
\[ + \text{(circular perms. of } 2, 3, 4) \} \}
\[ - \left\{ \Gamma_{\mu_1\nu_1}(p, -p_1, -p + p_1) * \tilde{D}^{\nu\mu}(p - p_1) * \tilde{\Gamma}_{\nu_1\mu_2\nu_3\mu_4}^{\mu_1\mu_2\mu_3}(p_1 + p_2, -p_1, -p_2, -p_3) \right\} \]
\[ + \left\{ \Gamma_{\mu_1\nu_2}(p, -p + p_1, -p_2, -p_3 - p_4) * \tilde{D}^{\nu\mu}(p + p_4) * \tilde{\Gamma}_{\nu_2\nu_3\mu_1\mu_2}(p_2 + p_4, -p_2, -p_1, -p_2) \right\} \]
\[ \times * \tilde{D}^{\lambda\mu\nu}(p_3 + p_4) * \tilde{\Gamma}_{\lambda\mu_3\mu_4}^{\mu_1\mu_2}(p_3 + p_4, -p_3, -p_4). \]

The examples considered above suggest that a color structure of the effective amplitudes \( \ast \tilde{\Gamma}^{(5)} \) entirely coincides with color structure of usual s-gluon HTL-amplitudes\[1\] derived by Braaten and Pisarski\[24\] for an arbitrary number of external soft-gluon legs.

\[5\] It is clear that if this statement is really true (for the present we cannot rigorously prove this fact) then the algorithm of calculation of \( \ast \tilde{\Gamma}^{(5)} \) is not optimal. It takes its additional modification, which enables one to derive at once the effective subamplitudes instead of the total effective amplitudes.
Besides, by direct calculation it can be shown that an effective subamplitude \((7.13)\) satisfies two relations which hold for usual five-gluon HTL-amplitude, namely

\[
\begin{align*}
\ast \tilde{\Gamma}_{\mu_1 \mu_2 \mu_3 \mu_4} (p, -p_1, -p_2, -p_3, -p_4) + \ast \tilde{\Gamma}_{\mu_3 \mu_4 \mu_1 \mu_2} (p, -p_3, -p_1, -p_2, -p_4) = 0, \\
\ast \tilde{\Gamma}_{\mu_1 \mu_2 \mu_3 \mu_4} (p, -p_1, -p_3, -p_2, -p_4) + \ast \tilde{\Gamma}_{\mu_1 \mu_2 \mu_4 \mu_3} (p, -p_1, -p_2, -p_4, -p_3) = 0, \\
\ast \tilde{\Gamma}_{\mu_1 \mu_2 \mu_3 \mu_4} (p, -p_1, -p_2, -p_3, -p_4) = -\ast \tilde{\Gamma}_{\mu_4 \mu_3 \mu_2 \mu_1} (p, -p_4, -p_3, -p_2, -p_1).
\end{align*}
\]

Note that the order of the spacetime indices in relations \((7.14)\) (as well as in relations \((5.12)\) and \((5.13)\)) is important. Unfortunately, for the present we cannot prove similar properties for an arbitrary effective subamplitude \(\ast \tilde{\Gamma}^{(s)}\), since we have not in hand general analytic expression for \(\ast \tilde{\Gamma}^{(s)}\). However relations \((5.12)\), \((5.13)\), and \((7.14)\) for the first nontrivial \(s = 4, 5\) provide a reason for assumption that they hold for a general case in a form, suggested by Braaten and Pisarski (Eqs. (2.3) and (2.8) in Ref. [24]) for an arbitrary \(s\)-gluon HTL-amplitude.

However as was mentioned at the end of section 5 the Ward identities for effective amplitudes have not take place in the form as for usual HTL-amplitudes. The relations between higher and lower effective amplitudes can be obtained by direct contraction of the expression \((5.6)\) and \((7.13)\) type with momentum variable \(p\). But it is more convenient to derive similar relations in approach suggested by Blaizot and Iancu in Ref. [8] through corresponding differentiation of the conservation law for the color current, which in the momentum representation has a form

\[
p^\mu J_\mu (p) = -ig f_{abc} \int A^{\mu}(p_1) J_c^c (p_2) \delta(p - p_1 - p_2) dp_1 dp_2.
\]

A gauge field \(A^{\mu}(p_1)\) on the right-hand side of Eq. \((7.15)\) is considered as interacting one, i.e. specified by a series \((7.7)\). It is convenient to represent the color current \(J_\mu^a (p)\) entering into Eq. \((7.15)\) as a sum of two parts: linear and nonlinear ones with respect to interacting field \(A_\mu^a (p)\)

\[
J_\mu^a (p) = J^a_\mu^L (p) + J^a_\mu^NL (p).
\]

By relation \((7.0)\) a nonlinear part of the current can be represented in the form of expansion in terms of a free field

\[
J^a_\mu^NL [A](p) \rightarrow \tilde{J}^a_\mu^NL [A^{(0)}](p) = \sum_{s=2}^{\infty} \tilde{j}^{(s)a\mu} (A^{(0)}, \ldots, A^{(0)}).
\]

By virtue of the fact that a current \(J^a_\mu^NL [A](p)\) involves also bare three- and four-gluon vertices into its definition, for consistency one need to take a linear part of the current in the form

\[
J^a_\mu^L (p) = -\ast \mathcal{D}^{\mu\nu} (p) A^a_\nu (p),
\]

i.e. a coefficient of proportionality between \(J^a_\mu^L\) and \(A_\mu^a\) is here the inverse resumm gluon propagator (without gauge fixing term), and not the polarization tensor, as in usual
definition. Substituting (7.10) into (7.13) with regard to (7.14) and (7.18) we result in the following expression instead of (7.15)

\[ p_\mu \tilde{\eta}_{NL}^\mu [A^{(0)}](p) = -ig f^{abc} \int A_\mu(p_1) \tilde{D}^{\mu
u}(p_2) A_\nu(p_2) \delta(p - p_1 - p_2) dp_1 dp_2 \]  

(7.19)

Here, on the left-hand side we taken into account orthogonality of an inverse propagator \( p_\mu \tilde{D}^{-1\mu
u}(p) = 0 \). Differentiating Eq. (7.19) with respect to free field \( A^{(0)a}_\mu \), considering (7.7) and (7.8), one obtains relations between effective amplitudes \( \tilde{\Gamma}^{(s)} \). A contraction momentum \( p \) with the effective amplitude \( \tilde{\Gamma}^{(s)} \) is expressed here in the form of combination of the same effective amplitudes of lower order, \( \tilde{\Gamma}^{(s')}, s' < s \).

8 Gauge invariance of the effective amplitudes

The properties (5.12), (5.13) and (7.14) for effective amplitudes are trivial in the sense that they separately hold for every typical group of terms, formed effective amplitudes. These properties do not reflect more deep connection between different terms involved into different groups. Here, we consider one further property of the effective amplitudes, for which such a connection completely manifests.

In proving the gauge-invariance of the nonlinear Landau damping rate, we have shown that the function (5.11) entering into the definition of the matrix element (5.7) can be introduced in its simplest reduced form,

\[ \tilde{\Gamma}(p, -p_1, -p_2, -p_3) = p_1^2 p_2^2 p_3^2 \Gamma_{\mu
u\rho\sigma}(p_1 + p_2 + p_3) \Gamma_{\rho\sigma00}(p_2 + p_3, -p_2, -p_3) \]  

(8.1)

The proof of validity of this reduction is based on the use of the Ward effective identities for HTL-amplitudes and the plasmon mass-shell condition. The right-hand side of last expression, as shown in Ref. [3] is identical both in temporal and covariant gauges. The latter is obtained from Eq. (3.11) by the following replacements of projector and propagator, (see Eqs. (3.10) – (3.12))

\[ \tilde{u}_\mu(p) \rightarrow \tilde{u}_\mu(p), \]

\[ *\tilde{D}_{\mu\nu}(p) \rightarrow *D_{\mu\nu}(p) = -P_{\mu\nu}(p) *\Delta^l(p) - Q_{\mu\nu}(p) *\Delta^r(p) + \xi D_{\mu\nu}(p) \Delta^0(p), \]  

(8.2)

In the case of four-plasmon function (5.4) we look at two groups of terms differ in structure: four-gluon HTL-amplitude and group consisting of two terms with three-gluon HTL-amplitude. For five-point function (7.13) we have already four groups of terms with essentially different structure.
where

\[ P_{\mu\nu}(p) = g_{\mu\nu} - D_{\mu\nu}(p) - Q_{\mu\nu}(p), \quad Q_{\mu\nu}(p) = \frac{\bar{u}_\mu(p)\bar{u}_\nu(p)}{\bar{u}^2(p)}, \quad \Delta^0(p) = \frac{1}{p^2}, \]

and \( \xi \) is a gauge parameter in the covariant gauge. All terms on the right-hand side of Eq. (8.1), which include gauge parameter, vanish on mass-shell.

We can assume that a similar reduction holds for an arbitrary effective amplitude. However a proof of this statement is impossible in the general case by reason of absence of general expression for \( \Gamma^{(s)} \), when a proof by induction is allowed. The only thing that we can make is to consider a contraction similar to (8.1) for five-point effective amplitude, exact form of which (7.13) is known. It should be noted that as distinct from (8.1) given contraction is of no any physical meaning on the plasmon mass-shell, because the process of five-plasmon decay is kinematically forbidden. Since here, the intermediate calculations are cumbersome, we give only the net result. Using effective Ward identities for HTL-amplitudes [3, 24] and mass-shell condition, we derive

\[ \Gamma_{\mu_1...\mu_4}(p, -p_1, ..., -p_4) \frac{\bar{u}^\mu(p)\bar{u}^{\mu_1}(p_1)\ldots\bar{u}^{\mu_4}(p_4)}{\bar{u}^2(p)} \bigg|_{\text{on-shell}} = -p^2 p_1^2 p_2^2 p_3^2 p_4^2 \]

\[ \Gamma_{00...0}(p, -p_1, ..., -p_4) \left\{ \Gamma_{0000}(p, -p_1, -p_2, -p_3, -p_4) \frac{\bar{D}^{\nu\nu'}(p_3+p_4)\tilde{\Gamma}_{\nu'00}(p_3+p_4, -p_3, -p_4)}{\bar{D}^{\nu\nu'}(p_3+p_4)\tilde{\Gamma}_{\nu'00}(p_3+p_4, -p_3, -p_4)} \right\} \]

\[ + \left\{ \Gamma_{0000}(p, -p_1, -p_1, -p_2, -p_3, -p_4) \frac{\bar{D}^{\nu\nu'}(p_1+p_2)\tilde{\Gamma}_{\nu'00}(p_1+p_2, -p_1, -p_1, -p_2)}{\bar{D}^{\nu\nu'}(p_1+p_2)\tilde{\Gamma}_{\nu'00}(p_1+p_2, -p_1, -p_1, -p_2)} \right\} \]

\[ + \left\{ \Gamma_{0000}(p, -p_1, -p_1, -p_2, -p_3, -p_4) \frac{\bar{D}^{\nu\nu'}(p_1+p_2)\tilde{\Gamma}_{\nu'00}(p_1+p_2, -p_1, -p_1, -p_2)}{\bar{D}^{\nu\nu'}(p_1+p_2)\tilde{\Gamma}_{\nu'00}(p_1+p_2, -p_1, -p_1, -p_2)} \right\} \]

As we see, the result is similar to (8.1). It is also easily to check that all terms with a gauge parameter in Eq. (8.3) vanish on mass-shell. If we perform a replacements (8.2) on the left-hand side (i.e. define a matrix element of five-plasmon decay in a covariant gauge), then after analogous computations we lead to the same expression on the right-hand side of (8.3) with only distinction in a common sign. This fact of a gauge non-invariance immediately points to the fact that odd effective amplitudes taken on mass-shell of plasma excitations are of no physical meaning, as said above. All decay processes involving odd numbers of plasmons are kinematically forbidden. Therefore in an expansion of the nonlinear current (7.17) all effective currents \( \tilde{J}^{(s)}_{\mu} \) with even number of free fields \( A^{(0)}_\mu \) (correspondingly containing odd effective amplitudes) by \( \delta \)-functions are equal to zero on mass-shell.

At the end of this section we note that in spite of the fact that result (8.3) is of only pure methodological meaning, nevertheless two examples (8.1) and (8.3) provide a reason
to use of considerably simple expressions of \( (8.1) \) type for all \((2n + 2)\)-matrix elements \( M^{a_1 \ldots a_{2n+1}} \) in particular calculations.

9 The Vlasov-Boltzmann equation for color plasmons

In a previous sections we have considered the problem of deriving kinetic equation of a Boltzmann type, describing decay processes involving colorless plasmons. The number density of plasmons here has a trivial color structure: \( N^l_{p} = \delta^{ab} N^l_{p} \). Such a structure of the plasmon number density takes place, when there is no external color current or/and color mean field in the system. Now we assume that a time-space dependent external perturbation (e.g. external color current \( j^\mu_{\text{ext}}(x) \)) starts acting on the system. In the presence of the external color perturbation, the soft gauge field develops an expectation value \( \langle A^\mu_l(x) \rangle \equiv A^\mu_l(x) \neq 0 \), and the number density of the plasmons acquires a non-diagonal color structure. Thereby, we lead to the problem of construction of more complicated kinetic theory for color plasmons. The kinetic (matrix) equation in this theory will take into account such purely non-Abelian effect as the precession of the color charge of the plasmon, the existence of which qualitatively differs non-Abelian plasma from Abelian one, where the plasmons do not carry electric charge. Our further consideration will be to a certain extent of phenomenological character, being only a first step towards to construction of a kinetic theory for color plasmons in hot QCD plasma. For rigorous justification of all assumption it should be invoked the methods of an off-equilibrium field theory.

The assumption that a structure of the kinetic equation for color plasmons is similar to the structure of the kinetic equation for hard transverse gluons in the form proposed by Arnold, Son and Yaffe in Ref. [25], will be our initial theoretical premise to construction of above-mentioned transport theory. More precisely, we expect the time-space evolution of \( N^l_{p} = (N^{l\ ab}) \) to be described by

\[
\left( D_t + V^l_p \cdot D_x \right) N^l_p - \frac{1}{2} \left\{ \left( \mathcal{E}(x) + (V^l_p \times B(x)) \right)_i, \nabla_p N^l_i \right\} = - C \left[ N^l_p \right], \tag{9.1}
\]

where \( D_\mu \) is a covariant derivative acting as

\[
D_\mu N^l_p \equiv \partial_\mu + ig[A_\mu(x), N^l_p],
\]

\( A^a_\mu \) is mean soft-gluon field expressed in terms of Hermitian generators in adjoint representation \( T^a \), \((T^a)^{bc} = -if^{abc}, \ \text{tr}(T^a T^b) = N_c \delta^{ab})\), with \([,] \) and \(\{,\} \) denoting the commutator and anticommutator in color space, respectively; \( \mathcal{E}^i(x) \) and \( B^i(x) \) are the mean chromoelectric and chromomagnetic fields. Futhermore we consider that the collision term \( C \left[ N^l_p \right] \) has a following structure [25]

\[
C \left[ N^l_p \right] = \frac{1}{2} \left\{ N^l_p, \Gamma_d[N^l_i] \right\} - \frac{1}{2} \left\{ (1 + N^l_p), \Gamma_i[N^l_i] \right\} - \ldots, \tag{9.2}
\]
where \( \Gamma_d[N^l_p] = (\Gamma^a_{a}^d[N^l_p]) \) and \( \Gamma_i[N^l_p] = (\Gamma^a_{a}^i[N^l_p]) \) represent a generalized decay and re-generating rates of color plasmons, respectively. These rates can be formally represented, like a colorless case in the form of functional expansion in powers of the number density of color plasmons

\[
\Gamma_d^{(2n+1)aa'}[N^l_p] = \sum_{n=1}^{\infty} \Gamma_d^{(2n+1)aa'}[N^l_p], \quad \Gamma_i^{(2n+1)aa'}[N^l_p] = \sum_{n=1}^{\infty} \Gamma_i^{(2n+1)aa'}[N^l_p],
\]

(9.3)

where

\[
\Gamma_d^{(2n+1)aa'}[N^l_p] = \int dT^{(2n+1)} w_{2n+2}^{\{a';a\}}(p, p_1, \ldots, p_n; p_{n+1}, \ldots, p_{2n+1}) N^l_{p_1} a_1 \ldots N^l_{p_n} a_n
\]

\[
\times (1 + N^l_{p_{n+1}}) a_n+1 a_{n+1} \ldots (1 + N^l_{p_{2n+1}}) a_{2n+1},
\]

(9.4)

\[
\Gamma_i^{(2n+1)aa'}[N^l_p] = \int dT^{(2n+1)} w_{2n+2}^{\{a';a\}}(p, p_1, \ldots, p_n; p_{n+1}, \ldots, p_{2n+1}) (1 + N^l_{p_1}) a_1 \ldots (1 + N^l_{p_n}) a_n
\]

\[
\times N^l_{p_{n+1}} a_{n+1} \ldots N^l_{p_{2n+1}} a_{2n+1}.
\]

(9.5)

Here, \( w_{2n+2}^{\{a';a\}} = w^{a'a'_{2n+1}a_{2n+1}}(p, p_1, \ldots, p_n; p_{n+1}, \ldots, p_{2n+1}) \) is the scattering probability defined as

\[
= M^{a'}(p, p_1, \ldots, p_n, -p_{n+1}, \ldots, -p_{2n+1}) M^{(a)}(p, p_1, \ldots, p_n, -p_{n+1}, \ldots, -p_{2n+1})
\]

\[
+ \sum_{1 \leq i < j \leq n} M^{a'}(p, p_{i}, a_j; p_{n+1}, a_{i+1} p_{j+1}) + \sum_{1 \leq i < j \leq n+1} M^{a'}(p, p_{i}, a_j; p_{n+2}, a_{i+1} p_{j+1}) + \ldots
\]

\[
+ \sum_{1 \leq i < j \leq n} M^{a'}(p, p_{i}, a_j; p_{2n+1}, a_{i+1} p_{j+1}).
\]

with the same matrix elements \( M^{(a)}(p, p_1, \ldots, p_n, -p_{n+1}, \ldots, -p_{2n+1}) \) as was defined by us in colorless case by general relation (1.13) (see, however the end of this section). Here, the summing symbol \( \sum_{1 \leq i < j \leq n} \) designates that besides summing over all possible interchange of momenta \( p_i \leftrightarrow -p_j \), where indices \( i, j \) run \( 1 \leq i \leq n, n + 1 \leq j \leq 2n + 1 \), it needs to interchange and color indices \( a_i \leftrightarrow a_j, a_i' \leftrightarrow a_j' \) at one time. The other summing symbols are taken in a similar fashion. In arrangement of color indices in Eqs. (9.4) and (9.3) we follow rule proposed by Arnold, Son and Yaffe in Ref. [23] in the context of deriving collision term for hard color particles. The dots on the right-hand side of Eq. (9.2) is referred to contributions containing commutators of \([ \text{Re} \Pi_R, N^l_p] \) type, where \( \Pi_R \) is retarded soft-gluon self-energy.

In the case of four-plasmon decay, the scattering probability \( w_4^{\{a';a\}} \) by general expression has a form

\[
w_4^{\{a';a\}}(p, p_1; p_2, p_3) = M^{a'a'_{a_1}a_2}(p, -p_2, -p_3) M^{a'_{a_2}a_1}(p, -p_1, -p_3)
\]

\[
+ M^{a'a'_{a_1}a_2}(p, -p_2, p_1, -p_3) M^{a'_{a_2}a_1}(p, -p_2, p_1, -p_3)
\]

\[
+ M^{a'a'_{a_1}a_2}(p, p_3, -p_2, p_1) M^{a'_{a_2}a_1}(p, p_3, -p_2, p_1).
\]

32
By direct calculation, using an explicit expression (5.3) for color structure of an effective amplitude $i\Gamma_{a|\mu}^{i|\mu}$ and properties for subamplitude $i\Gamma_{a|\mu}^{\mu}$ (5.12), (5.13), it can be shown that a sum of two last terms in this expression is equal to doubled value of a first term.

It needs to supplement the Vlasov-Boltzmann equation (9.1) by mean field equation, defining a change of mean field in a system in self-consistent manner

$$D^\nu(x)F_{\mu\nu}(x) = j^\text{plasm}_\mu(x) + j^\text{ext}_\mu(x),$$

where induced current $j^\text{plasm}_\mu(x) = (j^\text{plasm}_0(x), j^\text{plasm}(x))$,

$$j^\text{plasm}_0(x) = gT^a \int \frac{dp}{(2\pi)^3} \text{tr}(T^a N^l_p), \quad j^\text{plasm}(x) = gT^a \int \frac{dp}{(2\pi)^3} \partial_\mu \omega^l_p \text{tr}(T^a N^l_p)$$

is the color current resulting from color-plasmon number density and $j^\text{ext}_\mu$ is the external current plays a part of initial color perturbation.

It is clear that dynamics of nonlinear interaction of soft color excitations is considerably more intricate as compared with colorless ones. To form a certain notion of this fact we consider in detail the linearized version of Vlasov-Boltzmann equation (9.1), keeping only the first term, $n = 1$, in expansions (9.3). We write the number density of color plasmons as

$$N^{l,ab}_p = N^{l,eq}_p \delta^{ab} + \delta N^{l,ab}_p,$$

and parametrize the off-equilibrium fluctuation of the occupation number similar to Eq. (5.8), where now the function $\lambda^{l}_p$ is a color matrix in the adjoint representation: $\lambda^{l}_p = \lambda^{la}_p T^a$. After some cumbersome algebraic transformations, we derive a linearized kinetic equation for the color plasmons from Eqs. (9.1) - (9.3)

$$\left(D_t + V^l_p \cdot \mathbf{D}_x\right) \lambda^{l}_p = -g \left(V^l_p \cdot \mathbf{E}(x)\right) - \delta C [\lambda^{l}_p],$$

where linearized collision term $\delta C$ has a following form

$$\delta C [\lambda^{l}_p] = \int \frac{dp_1}{(2\pi)^3} \frac{dp_2}{(2\pi)^3} \frac{dp_3}{(2\pi)^3} (2\pi)^4 \delta (\omega^l_{p_1} + \omega^l_{p_2} - \omega^l_{p_3} - \omega^l_{p_4})$$

$$\times \delta (p_1 + p_2 - p_3) \frac{N^{l,eq}_p(N^{l,eq}_p + 1)(N^{l,eq}_p + 1)}{(N^{l,eq}_p + 1)}$$

$$\times \left(|\mathcal{T}_1|^2 \{\lambda^{l}_p - \frac{1}{2} \lambda^{l}_p \lambda^{l}_p - \frac{1}{4} (\lambda^{l}_p \lambda^{l}_p + \lambda^{l}_p \lambda^{l}_p)\} + |\mathcal{T}_2|^2 \{\lambda^{l}_p - \frac{1}{2} \lambda^{l}_p \lambda^{l}_p - \frac{1}{4} (\lambda^{l}_p \lambda^{l}_p + \lambda^{l}_p \lambda^{l}_p)\}\right) + \text{Re} \left(\mathcal{T}_1 \mathcal{T}_2^* \{\lambda^{l}_p - \frac{1}{2} (\lambda^{l}_p \lambda^{l}_p + \lambda^{l}_p \lambda^{l}_p)\}\right).$$

On the right-hand side of Eq. (9.8) we introduce for brevity a notation

$$\mathcal{T}_1 \equiv \frac{3}{2} g^2 N_c \left(\frac{Z_i(p)}{2\omega^2_p u^2(p)}\right)^3 \prod_{i=1} \left(\frac{Z_i(p_i)}{2\omega^2_p u^2(p_i)}\right)^{*\Gamma(p, p_1, -p_3, -p_2)},$$

$$\mathcal{T}_2 \equiv \frac{3}{2} g^2 N_c \left(\frac{Z_i(p)}{2\omega^2_p u^2(p)}\right)^3 \prod_{i=1} \left(\frac{Z_i(p_i)}{2\omega^2_p u^2(p_i)}\right)^{*\Gamma(p, p_2, -p_1, -p_3)}.$$
Here, the function $\tilde{\Gamma}^*$ is defined by Eq. (5.11), and a function $T_2$ is defined from $T_1$ by a replacement

$$\tilde{\Gamma}(p, p_1, -p_3, -p_2) \rightarrow \tilde{\Gamma}(p, -p_2, p_1, -p_3).$$

In deriving (9.8) we use the following identities:

$$\text{tr}(T^a T^b T^c) = \frac{i}{2} N_c f^{abc}, \quad (T^a T^b) \text{tr}(T^a T^c T^b) = \frac{1}{4} N_c^2 T^c, \quad (T^a T^b T^c T^a T^b) = 0, \text{ etc.}$$

Since the equilibrium plasmon number density is proportional to the identity, the commutator terms on the right-hand side of Eq. (9.2) vanish. Therefore if we assume that the off-equilibrium function $\delta N^l$ is perturbative small, then a linearized Boltzmann equation (9.7), (9.8) in a certain sense is exact.

The linearized collision term (9.8) for color plasmons has noteworthy structure (cp. collision term for colorless case (5.9), (5.10)). The expressions in curly brackets behind squared moduli $|T_1|^2$ and $|T_2|^2$ coincide in a structure with a similar expression entering into a linearized collision term for color fluctuations of number density of hard gluons in form which was proposed by Blaizot and Iancu [14]. Besides, here additional ”interference” term $\text{Re}(T_1 T_2^*)$ appears. Going immediately after an expression in curly braces does not coincide with none of known expressions. In spite of some complexity of obtained expression for $\delta C[W^l_p]$ it provides remark internal symmetry, directly connected with existence of relation (5.12). Let us rewrite this relation in terms of $T$ functions (9.9). For this purpose we introduce a third function $T_3$, that is defined from $T_1$, by replacement

$$\tilde{\Gamma}(p, p_1, -p_3, -p_2) \rightarrow \tilde{\Gamma}(p, p_1, -p_2, -p_3).$$

Then from (5.12) it follows

$$T_1 + T_2 + T_3 = 0.$$ 

Now we set, for example, a function $T_2 = -T_1 - T_3$ and substitute it into expression in round brackets in integrand (9.8). After simple algebraic transformations we have

$$|T_1|^2\left\{W_p^l - \frac{1}{2} W_{p_2}^l + \frac{3}{4} (W_{p_1}^l + W_{p_3}^l)\right\} + |T_3|^2\left\{W_p^l - \frac{1}{2} W_{p_3}^l + \frac{3}{4} (W_{p_1}^l + W_{p_2}^l)\right\} + \text{Re}(T_1 T_3^*)\left\{W_p^l - \frac{1}{2} (W_{p_2}^l + W_{p_3}^l)\right\}$$

(9.10)

The structure of the last expression with an accuracy of replacement of momenta fully coincides with a structure of Eq. (9.8). The linearized collision integral (9.8) may be rewritten in terms of functions $T_2$ and $T_3$ by setting $T_1 = -T_2 - T_3$. We obtain similar expression with another permutation of momenta. Adding such obtained three expressions and dividing by 3, we result in an expression that is more symmetric relative to the external soft momenta and will be used somewhat below

$$Q(p, p_1; p_2, p_3) \equiv \frac{1}{3} \left\{|T_1|^2\left\{2W_p^l - \frac{3}{2} W_{p_3}^l + \frac{3}{4} (W_{p_1}^l + W_{p_2}^l)\right\}\right.$$
By direct calculations we show that the color current induced by off-equilibrium fluctuations of the color plasmons number density \( \delta N_{lab} \), covariantly conserves. In terms of functions \( \mathcal{W}_{p}^{ia} \), the expressions for components of color current (9.10) is rewritten in the form

\[
\begin{align*}
    j_{0}^{\text{plasm}}(x) &= -gT^{a}\int \frac{dp}{(2\pi)^3} \frac{\partial N_{eq}(p)}{\partial \omega_{p}^{a}} \mathcal{W}_{p}^{ia}, \\
    j_{\mu}^{\text{plasm}}(x) &= -gT^{a}\int \frac{dp}{(2\pi)^3} \frac{\partial N_{eq}(p)}{\partial p_{\mu}} \mathcal{W}_{p}^{ia}.
\end{align*}
\]  

(9.12)

By linearized Boltzmann equation (9.7) we have

\[
\begin{align*}
    \mathcal{D}_{\mu}^{\text{plasm}} \cdot \mathcal{J}_{\mu}^{\text{plasm}}(x) &= -\frac{1}{3} \beta g N_{c}^{2} \int \frac{dp}{(2\pi)^3} \prod_{i=1}^{3} \frac{dp_{i}}{(2\pi)^3} (2\pi)^4 \delta(p + p_{1} - p_{2} - p_{3}) \delta(\omega_{p}^{1} + \omega_{p_{1}}^{1} - \omega_{p_{2}}^{1} - \omega_{p_{3}}^{1}) \\
    &\times N_{eq}(p) N_{eq}(p_{1})(1 + N_{eq}(p_{2}))(1 + N_{eq}(p_{3})) Q(p, p_{1}; p_{2}, p_{3}),
\end{align*}
\]  

(9.13)

where the function \( Q \) on the right-hand side is defined by Eq. (9.11). Now we show that integral on the right-hand side of the last expression equals to zero. As a first step we symmetrize integrand in Eq. (9.13) with respect to permutation \( p \leftrightarrow p_{1} \). Using definitions of functions \( T_{1}, T_{2} \) and \( T_{3} \) and property of invariance of the function \( \tilde{\Gamma}(p, -p_{1}, -p_{2}, -p_{3}) \) (Eq. (5.13)), when the momenta order is reversed, we obtain laws of transformation of these functions with replacement \( p \leftrightarrow p_{1} \)

\[
T_{1} \rightarrow T_{3}, \quad T_{2} \rightarrow T_{2}, \quad T_{3} \rightarrow T_{1}.
\]  

(9.14)

Furthermore, we replace \( p \leftrightarrow p_{1} \) in expression (9.11) with regard to (9.14). Adding (9.11) with such an obtained expression and dividing by 2, we lead to the expression symmetric over replacement \( p \leftrightarrow p_{1} \):

\[
\begin{align*}
    \frac{1}{2} \left( \frac{5}{4} |T_{1}|^2 + \frac{3}{2} |T_{2}|^2 + \frac{5}{4} |T_{3}|^2 + \frac{1}{2} \text{Re}(T_{1}T_{2}^*) + \frac{1}{2} \text{Re}(T_{1}T_{3}^*) + \frac{1}{2} \text{Re}(T_{2}T_{3}^*) \right) \\
    \times \left\{ \mathcal{W}_{p}^{i} + \mathcal{W}_{p_{1}}^{i} - \mathcal{W}_{p_{2}}^{i} - \mathcal{W}_{p_{3}}^{i} \right\}.
\end{align*}
\]  

(9.15)

As we see, this expression is automatically symmetric and with respect to permutation \( p_{2} \leftrightarrow p_{3} \) (here, transformation laws of functions \( T \) coincide with (9.14)).

Let us consider now a crossed symmetry \( p \leftrightarrow p_{2}, p_{1} \leftrightarrow p_{3} \). The statistical factor in integrand (9.13) is symmetric by virtue of identity

\[
N_{eq}(p) N_{eq}(p_{1})(1 + N_{eq}(p_{2}))(1 + N_{eq}(p_{3})) = (1 + N_{eq}(p))(1 + N_{eq}(p_{1}))(1 + N_{eq}(p_{2}))(1 + N_{eq}(p_{3})).
\]

35
The transformation laws of functions $\mathcal{T}_i$ are trivial in this case

$$\mathcal{T}_1 \to \mathcal{T}_1, \quad \mathcal{T}_2 \to \mathcal{T}_2, \quad \mathcal{T}_3 \to \mathcal{T}_3.$$ 

By using these transformation laws and expression (9.13) we see that integrand on the right-hand side of Eq. (9.13) is odd for replacements $p \leftrightarrow p_2, \; p_1 \leftrightarrow p_3$. By virtue of this fact the integral equals zero and a color current (9.12) is covariantly conserved.

Closing this section note that without rigorous transport theory for color plasmons in hand here, we given no any estimations for the amplitude of mean gauge field $A_\mu$. We can only made proposal that for sufficiently strong background field, a structure of the kinetic equation (9.1) can be more involved. The interaction matrix elements $M_{aa_1...a_{2n+1}}$ (or effective amplitudes), for example, can depend by itself in a highly nontrivial way on mean field. In other words, they can contain insertions of external mean field, in general, of arbitrary order. This leads to the fact that mean gluon field will be “involved” in the dynamics of color plasmons not only through the left-hand side of the kinetic equation, but through a collision term.

10 Conclusion

On the basis of pure gauge sector of the Blaizot-Iancu equations we have obtained the transport equation for the colorless plasmons, taking into account $(2n+2)$-plasmon decays. The algorithm of successive calculation of the probabilities of the decay processes is proposed. The limiting value of the plasmon occupation number $N_p^l (\sim 1/g^2)$ is defined, whereby the decay processes with an arbitrary number of soft external legs give contributions of the same order in the coupling constant to the change of $N_p^l$ and transport theory becomes essentially nonlinear. Here, it should be pointed to certain difficulty of a principal character, which arises when we going over from low excited state ($A_\mu(X) \sim \sqrt{gT}$) to highly excited one ($A_\mu(X) \sim T$) of system.

The kinetic equation (2.2) describes the processes of plasmon scattering. However moreover, there is a collective plasmon interaction resulting in frequency shift of plasmons

$$\omega_p^l \to \tilde{\omega}_p^l = \omega_p^l + \Delta \omega_p^l.$$ 

Here, $\tilde{\omega}_p^l$ is the frequency renormalized due to the interaction. In the linear plasma theory, i.e. in the theory with infinitesimal amplitudes of plasma oscillations, the frequency spectrum represents the function of values ($T, g, \ldots$), not dependent on waves energy (or their occupation number). In the framework of nonlinear theory, the nonlinear dispersion correction represents in general case a functional of plasmon number density $N_p^l$. To

8The usual HTL-amplitudes in the presence of a background gauge field, proposed by Blaizot and Iancu in Ref. [8] can be an example of such a dependence.
lowest nontrivial order in $N_p^l$ the nonlinear frequency shift of longitudinal oscillations is defined by the formula

$$
\Delta \omega_p^l = g^2 N_c \left\{ \frac{dp_1}{(2\pi)^3} \right\} N_p^l \left( \frac{Z_l(p)}{2\omega_p^l} \right) \left( \frac{Z_l(p_1)}{2\omega_p^{l_1}} \right) \left[ \frac{1}{\bar{u}^2(p)\bar{u}^2(p_1)} \text{Re} \hat{\Gamma}(p, p_1, -p, -p_1) \right]_{\text{on-shell}} + O((N_p^l)^2) + \ldots, \tag{10.1}
$$

where the function $\hat{\Gamma}(p, p_1, -p, -p_1)$ is defined by Eq. (5.11). At the soft momentum scale, for the thermal fluctuations of a gauge field we have an estimation for the correction (10.1)

$$
\Delta \omega_p^l \sim g^2 N_c T \ll \omega_p^l.
$$

Such for low excited state the frequency shift is perturbative small. In a limiting case of a highly excited state (6.4) from Eq. (10.1) we have the following estimation

$$
\Delta \omega_p^l \sim g N_c T \sim \omega_p^l,
$$

i.e. the nonlinear frequency shift becomes of the same order as a linear part of a spectrum $\omega_p^l$. The spectrum broadening over frequency takes place, and for strong field $A_\mu \leq T$ a dependence $I^l(p, x)$ on a frequency (which in this work is defined in the form of the quasiparticle approximation (1.12)) has no connection with $\delta$-function. Therefore, the kinetic equation in the form (2.2) in this limiting case becomes in general inappropriate for description of nonlinear dynamics of soft excitations, and here, it should be derived more general equation (or system of equations) for initial Wigner function $I^l(p, x)$, taking into account the spectrum broadening effects. The examples of contruction of such an equations for usual plasma can be found in Ref. [26]. Nevertheless, we suppose that somewhat simplified approach to kinetic description of highly excited state of hot gluon plasma considered in this paper will be useful in at least qualitative analysis of complicated dynamics of interaction of soft boson excitations taking place in the extremal medium state.

At the end of this section we note that at present we study a possibility of an alternative way of description of nonlinear plasmons dynamics based on a Hamiltonian formalism. The cornerstone of this approach is a fundamental fact that the equations describing a collisionless hot gluon plasma in HTL-approximation, possess a Hamiltonian structure, which was proposed by Nair, Blaizot and Iancu [7, 8, 27]. The last circumstance enables us to developed (at least for low excited states) independent method of derivation of kinetic equation for soft gluon plasma excitations. Within the framework of Hamiltonian approach, matrix elements of $(2n + 2)$-plasmon decays are obtained as result of a special canonical transformation, simplifying the interaction part of the plasmon Hamiltonian. The exact consideration of this approach will be subject of a separate publication.

9In this case to be sure in the right-hand side of Eq. (10.1) all terms of higher power in $N_p^l$ are of the same order in $g$, as the first term (similar to expansion of generalized rates (2.3)).
Acknowledgments

This work was supported by Grant INTAS (No. 2000-15) and Grant for Young Scientist of Russian Academy of Sciences (No. 1999-80). The authors are grateful Alexander N. Vall for valuable discussions.
References

[1] J.-P. Blaizot and E. Iancu, Phys. Rept. 359, 355 (2002).

[2] E. Braaten and R.D. Pisarski, Nucl. Phys. B337, 569 (1990); J. Frenkel and J.C. Taylor, Nucl. Phys. B334, 199 (1990).

[3] Yu.A. Markov and M.A. Markova, J. Phys. G 26, 1581 (2000).

[4] E. Braaten and R.D. Pisarski, Phys. Rev. D 42, 2156 (1990).

[5] Yu.A. Markov and M.A. Markova, “Nonlinear dynamics of soft boson excitations in hot QCD plasma II: plasmon – hard particle scattering”, in preparation.

[6] Yu.A. Markov and M.A. Markova, J. Phys. G 27, 1869 (2001).

[7] R. Efraty and V.P. Nair, Phys. Rev. D 47, 5601 (1993); R. Jackiw and V.P. Nair, *ibid.* 48, 4991 (1993); V.P. Nair, *ibid.* 48, 3432 (1993); *ibid.* 50, 4201 (1994); J.-P. Blaizot and E. Iancu, Nucl. Phys. B434, 662 (1995).

[8] J.-P. Blaizot and E. Iancu, Nucl. Phys. B417, 608 (1994).

[9] A. Niégawa, Phys. Rev. D 64, 036004 (2001).

[10] A. Gailitis and V.N. Tsytovich, Zh. Eksp. Teor. Fiz. 47, 1469 (1964) [Sov. Phys.-JETP 20, 987 (1965)].

[11] V.N. Tsytovich, *Non-linear Effects in Plasma* (Pergamon, Oxford, 1970); V.N. Tsytovich, *Theory of a Turbulent Plasma* (Consultant Bureau, New York, 1977).

[12] J.D. Bjorken and S.D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, London, 1965); A.I. Akhiezer and V.B. Berestetsky, *Quantum Electrodynamics* (Interscience Publishers, New York, 1965).

[13] Z. Bern, L. Dixon, and D.A. Kosover, Ann. Rev. Nucl. Part. Sci. 46, 109 (1996); V. Del Duca, A. Frizzo, and F. Maltoni, Nucl. Phys. B568, 211 (2000); V. Del Duca, L. Dixon, and F. Maltoni, *ibid.* B571, 51 (2000).

[14] J.-P. Blaizot and E. Iancu, Nucl. Phys. B557, 183 (1999); *ibid.* B570, 326 (2000).

[15] B.J. Nauta and Ch.G. van Weert, ITFA-98-44, [hep-ph/9901213](https://arxiv.org/abs/hep-ph/9901213) (unpublished).

[16] J. Jalilian-Marian, A. Kovner, L. McLerran and H. Weigert, Phys. Rev. D 55, 5414 (1997); Yu.V. Kovchegov and A.H. Mueller, Nucl. Phys. B529, 451 (1998).

[17] E. Iancu, A. Leonidov, and L. McLerran, Nucl. Phys. A692, 583 (2001); E. Ferreiro, E. Iancu, A. Leonidov, and L. McLerran, Nucl. Phys. A703, 489 (2002).
[18] L. McLerran and R. Venugopalan, Phys. Rev. D 49, 2233 (1994); 49, 3352 (1994); 50, 2225 (1994).

[19] J. Jalilian-Marian, A. Kovner, A. Leonidov, and H. Weigert, Nucl. Phys. B504, 415 (1997); Phys. Rev. D 59, 014014 (1999).

[20] Yu.V. Kovchegov and D.H. Rischke, Phys. Rev. C 56, 1084 (1997).

[21] S.G. Matinyan, B. Müller, and D.H. Rischke, Phys. Rev. C 56, 2191 (1997); ibid. 57, 1927 (1998).

[22] D. Bödeker, Nucl. Phys. B559, 502 (1999).

[23] F. Guerin, Nucl. Phys. B629, 233 (2002).

[24] E. Braaten and R.D. Pisarski, Nucl. Phys. B339, 310 (1990);

[25] P. Arnold, D.T. Son, and L.G. Yaffe, Phys. Rev. D 59, 105020 (1999).

[26] B.B. Kadomtsev, Plasma Turbulence (Academic, New York, 1965).

[27] E. Iancu, SACLAY-T97/123, hep-ph/9710543 (unpublished).
FIGURES

FIG. 1. The matrix element for four-plasmon decay. The wave lines denote soft quasiparticles (plasmons) and the blob stands for HTL resummation.

FIG. 2. The typical tree-level Feynman diagram for \((2n+2)\)-plasmon decay with amputate external legs.
FIG. 1.
