q-difference equations for the composite 2D q-Appell polynomials and their applications

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Abstract: The main aim of this article is to introduce a new class of composite 2D q-Appell polynomials and to study their properties. The generating function, series definition and some explicit relations for these polynomials are derived. These polynomials are studied from determinantal view point and their q-recurrence relations and q-difference equations are established. The composite 2D q-Bernoulli, q-Euler and q-Genocchi and composite q-Bernoulli–Euler, q-Bernoulli–Genocchi and q-Euler–Genocchi polynomials are studied as particular members of this class. Certain interesting examples are considered in terms of these members to give the applications of main results.

Subjects: Applied Mathematics; Mathematical Modeling; Mathematical Physics

Keywords: 2D q-Appell polynomials; composite 2D q-Appell polynomials; determinant; q-difference equations

AMS subject classifications: 33D45; 33D99; 33E20

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PUBLIC INTEREST STATEMENT

The studies on introducing mixed type polynomials via operational techniques and establishing their determinantal forms, recurrence relations and differential equations via factorization method have been done (Araci, Riyasat, Wani, & Khan 2017; Khan & Riyasat, 2016a). The q-series and q-polynomials have many applications in different fields of mathematics, physics and engineering. Recently, a new replacement technique has been adopted to introduce mixed type q-special polynomials and a different method is used to establish their q-recurrence relations and q-difference equations (Khan & Riyasat, 2016b; Srivastava, Khan, & Riyasat, in press). Motivated by this, in this article, we introduce a composite class of 2D q-Appell polynomials and studied its several properties. The generating function, series definition, explicit relations, determinantal definition, q-recurrence relations and q-difference equations are established. Certain interesting examples are framed in terms of the composite 2D q-Bernoulli, q-Euler and q-Genocchi and composite q-Bernoulli–Euler, q-Bernoulli–Genocchi and q-Euler–Genocchi polynomials to give the applications of main results. This process can be used to establish further quite a wide variety of formulas for several other q-special polynomials and can be extended to derive new relations for conventional and generalized q-polynomials.
1. Introduction and preliminaries

The subject of $q$-calculus started appearing in the nineteenth century due to its applications in various fields of mathematics, physics and engineering. Recently, it seems to have more usefulness in combinatorics and fluid mechanics, quantum mechanics, having an intimate connection with commutativity relations and Lie algebra (Bohner & Ünal, 2005; Ernst, 2017; Floreanini & Vinet, 1992, 1993a, 1993b; Katriel, 1998; Miller, 1970). The definitions and notations of $q$-calculus reviewed here are taken from Andrews, Askey, and Roy (1999).

The $q$-analogue of the shifted factorial $(a)_n$ is defined by

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{m=0}^{n-1} (1 - q^m a), \quad n \in \mathbb{N}. \quad (1.1)$$

The $q$-analogues of a complex number $a$ and of the factorial function are defined by

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad q \in \mathbb{C} - \{1\}; \quad a \in \mathbb{C}, \quad (1.2)$$

$$[n]_q! = \prod_{m=1}^{n} [m]_q = [1]_q [2]_q \cdots [n]_q = \frac{(q^n)_{n}}{(1-q)^n}, \quad q \neq 1; \quad n \in \mathbb{N}, \quad [0]_q! = 1, \quad q \in \mathbb{C}; \quad 0 < q < 1. \quad (1.3)$$

The $q$-binomial coefficient \[ \binom{n}{k} \] is defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \quad k = 0, 1, \ldots, n. \quad (1.4)$$

The $q$-analogue of the function $(x + y)^n$ is defined as:

$$(x + y)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q q^{n-k} x^{k} y^{k}, \quad n \in \mathbb{N}_0. \quad (1.5)$$

The exponential functions are defined as:

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{((1 - q)x)_q}, \quad 0 < |q| < 1; \quad |x| < |1 - q|^{-1}, \quad (1.6)$$

$$E_q(x) = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]_q!} = (-1 - q)x)_q, \quad 0 < |q| < 1; \quad x \in \mathbb{C}. \quad (1.7)$$

The functions $e_q(x)$ and $E_q(x)$ satisfy the following properties:

$$D_q e_q(x) = e_q(x), \quad D_q E_q(x) = E_q(qx). \quad (1.8)$$

The $q$-derivative $D_q f$ of a function $f$ at a point $0 \neq z \in \mathbb{C}$ is defined as:

$$D_q f(z) = \frac{f(qz) - f(z)}{qz - z}, \quad 0 < |q| < 1. \quad (1.9)$$

For any two arbitrary functions $f(z)$ and $g(z)$, the $q$-derivative satisfy
\[ D_{q,t}(f(z)g(z)) = f(z)D_{q,t}g(z) + g(qz)D_{q,t}f(z). \] (1.10)

The \( q \)-Appell polynomials \( A_{n,q}(x) \) are introduced and studied from different approaches, see Al-Salam (1967), Eini Keleshteri and Mahmudov (2015a) and Mahmudov (2014). These polynomials arise in numerous problems of applied mathematics, theoretical physics, approximation theory and several other mathematical branches. The interest in Appell polynomials and their applications in different fields has significantly increased. The recent applications of Appell polynomials in probability theory and statistics are considered in Anshelevich (2009) and Salminen (2011). The generalized Appell polynomials as tools for approximating 3D-mappings were introduced for the first time in Malonek and Falcão (2006) in combination with Clifford analysis methods. The representation theoretical results like those of Brackx, De Schepper, Lávička, and Soucek (2010) and Lávička (2010) provide new examples of applications of Appell polynomials and gave evidence to the central role of Appell polynomials as orthogonal polynomials. Representation theory is also the tool for their applications in quantum physics as explained in Weinberg (1995).

**Definition 1.1** For \( q \in \mathbb{C} \), \( 0 < |q| < 1 \), the \( q \)-Appell polynomials \( A_{n,q}(x) \) are defined by the following generating function (Al-Salam, 1967):

\[ A_{q,t}(x)e_{x}(t) = \sum_{n=0}^{\infty} A_{n,q}(x) \frac{t^n}{[n]_{q}!}, \] (1.11)

where \( A_{q,t}(x) \) is analytic at \( t = 0 \) and \( A_{n,q} := A_{n,q}(0) \).

Based on different selection for function \( A_{q}(t) \) different members belonging to the family of \( q \)-Appell polynomials can be obtained. These members are mentioned in Table 1.

The \( q \)-Bernoulli, \( q \)-Euler and \( q \)-Genocchi polynomials play an important role in various expansions and approximation formulae, which are useful both in analytic theory of numbers and in classical and numerical analysis for this see Ryoo (2007), Ryoo and Kim (2008) and Ryoo (2008). These polynomials provide solutions to various problems of engineering and physics.

The special polynomials of two variables are important from the point of view of applications. These polynomials allow the derivation of a number of useful identities and relations in a fairly straightforward way and help in introducing new families of special polynomials. These polynomials are frequently used in different branches of pure and applied mathematics and physics. Recently, the 2D \( q \)-Appell polynomials \( A_{n,q}(x, y) \) are introduced and studied from different viewpoint (Eini Keleshteri & Mahmudov, 2015a, 2015b).

**Definition 1.2** For \( q \in \mathbb{C} \), \( 0 < |q| < 1 \), the 2D \( q \)-Appell polynomials \( A_{n,q}(x, y) \) are defined by the following generating function (2015):

\[ A_{q,t}(x)e_{x}(t)E_{y}(t) = \sum_{n=0}^{\infty} A_{n,q}(x, y) \frac{t^n}{[n]_{q}!}, \] (1.12)

where \( A_{q,t}(x) \) is analytic at \( t = 0 \) and \( A_{n,q} := A_{n,q}(0, 0) \).

| S.No. | Name of the q-polynomial | \( A_{q,t}(x) \) | Generating function and related number |
|-------|--------------------------|----------------|--------------------------------------|
| I.    | \( q \)-Bernoulli polynomials (Al-Salam, 1959; Ernst, 2006) | \( \frac{1}{[q]_{q}^{0}} \) | \( A_{n,q}(x) = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{[n]_{q}!} \) (\( B_{n,q} := B_{n,q}(0) \)) |
| II.   | \( q \)-Euler polynomials (Ernst, 2006; Mahmudov, 2013) | 0 | \( A_{n,q}(x) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{[n]_{q}!} \) (\( E_{n,q} := E_{n,q}(0) \)) |
| III.  | \( q \)-Genocchi polynomials (Araci et al., 2014; Mahmudov, 2013) | \( \frac{2}{[q]_{q}^{0+1}} \) | \( A_{n,q}(x) = \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{[n]_{q}!} \) (\( G_{n,q} := G_{n,q}(0) \)) |
Different members belonging to the family of the 2D $q$-Appell polynomials $A_{n,q}(x, y)$ can be obtained by choosing appropriate $A_q(t)$. These polynomials are mentioned in Table 2.

The $q$-Bernoulli, $q$-Euler and $q$-Genocchi numbers have deep connections with number theory and occur in combinatorics. We give first few values of $q$-Bernoulli numbers $B_{n,q}$ (Ernst, 2006), $q$-Euler numbers $E_{n,q}$ (Ernst, 2006) and $q$-Genocchi numbers $G_{n,q}$ (Araci, Acikgoz, Jolany, and He, 2014) in Table 3, which will be used later.

For decades, various families of $q$-polynomials have been investigated rather widely and extensively due to their potentially usefulness in such wide variety of fields as theory of partitions, number theory, combinatorial analysis, finite vector spaces, Lie theory, etc. for this see Ernst (2017), Bohner and Ünal (2005), Floreanini and Vinet (1992), Floreanini and Vinet (1993a), Miller (1970).

The study of differential and difference equations is a wide field in pure and applied mathematics, physics and engineering. The problems arising in different areas of science and engineering are usually expressed in terms of differential or difference equations, which in most of the cases have special functions as their solutions. During the past three decades, the development of non-linear analysis, dynamical systems and their applications to science and engineering has stimulated renewed enthusiasm for the theory of differential or difference equations. The difference equations are important as they deal with the discrete and differential equations deal with the continuous and both our mathematical and physical universes are inherently discrete. The prominence of the role of the stability properties makes the difference among the numerical analysis and other branches of mathematics which also use the difference equations as a main tool. For example, in combinatorics, difference equations are very important. Differential equations play an important role in modelling virtually every physical, technical or biological process, from celestial motion, to bridge design, to interactions between neurons. Many fundamental laws of physics and chemistry can be formulated as differential equations. In biology and economics, differential equations are used to model the behaviour of complex systems (Johnson, 1913).

### Table 2. Certain members belonging to the 2D $q$-Appell family

| S.No. | Name of the $q$-polynomial | $A_q(t)$ | Generating function and related number |
|-------|---------------------------|---------|----------------------------------------|
| I.    | $2D$ $q$-Bernoulli polynomials (Mahmudov, 2013) | $\frac{1}{q^t t!} \left( \frac{1}{q^t t!} \right)^t$ | $e_q(x) e_q(y) = \sum_{m=0}^{\infty} B_{n,m} (x, y) \frac{t^m}{m!} (B_{n,q} : = B_{n,q}(0, 0))$ |
| II.   | $2D$ $q$-Euler polynomials (Mahmudov, 2013) | $\frac{1}{q^t t!} \left( \frac{1}{q^t t!} \right)^t$ | $e_q(x) e_q(y) = \sum_{m=0}^{\infty} E_{n,m} (x, y) \frac{t^m}{m!} (E_{n,q} : = E_{n,q}(0, 0))$ |
| III.  | $2D$ $q$-Genocchi polynomials (Mahmudov & Momenzadeh, 2014) | $\frac{1}{q^t t!} \left( \frac{1}{q^t t!} \right)^t$ | $e_q(x) e_q(y) = \sum_{m=0}^{\infty} G_{n,m} (x, y) \frac{t^m}{m!} (G_{n,q} : = G_{n,q}(0, 0))$ |

### Table 3. Values of first four $B_{n,q}$, $E_{n,q}$ and $G_{n,q}$

| $n$ | 0 | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|---|
| $B_{n,q}$ | 1 | $-(1 + q)^{-1}$ | $q^t((3l)^{2t}-1)$ | $(1 - q)q^t((2l)^{2t}-(t+1)^{2t})^{-1}$ | $q^t((1 - q^2 - 2q^2 - q^3 + q^4)(2l)^{2t}-(t+1)^{2t})^{-1}$ |
| $E_{n,q}$ | 1 | $-\frac{1}{2}$ | $\frac{1}{2}(-1 + q)$ | $\frac{1}{2}(-1 + 2q + 2q^2 - q^3)$ | $\frac{1}{8}(q - 1)^{3l}q(q^2 - 4q + 1)$ |
| $G_{n,q}$ | 0 | $\frac{1}{4}$ | $\frac{1}{4}((3l)^{2t}-(t+1)^{2t})^{2t}$ | $-\frac{1}{4}(3l)^{2t}((3l)^{2t}-(t+1)^{2t})^{2t}$ | $-\frac{1}{8}(3l)^{2t}((3l)^{2t}-(t+1)^{2t})^{2t}$ |

Notes: It is to be noted that the $q$-Genocchi polynomials $G_{n,q}(x)$ (Table 1 (III)) do not fulfill all requirements of $q$-Appell sequences as for instance the degree of $G_{n,q}(x)$ is $n - 1$ which can be seen from Table 3 that $G_{n,q} = 0 \Rightarrow G_{n,q}(x) = 0$, however the degree of all other $q$-Appell polynomials is $n$. Therefore, $G_{n,q}(x)$ is considered in the class of polynomial sequences which are not $q$-Appell in the strong sense.
The article is organized as follows. In Section 2, a composite class of the 2D q-Appell polynomials is introduced by means of generating function, series definition and determinantal definition. The q-recurrence relations and q-difference equations for these polynomials are established. In Section 3, certain examples of the members of the composite 2D q-Appell polynomials are considered as applications.

2. Composite 2D q-Appell polynomials
To introduce the Composite 2D q-Appell polynomials (C2DqAP), we consider two different sets of the 2D q-Appell polynomials such that

\[ A_q^I(t)e_q(xt)E_q(yt) = \sum_{n=0}^{\infty} A_{n,q}^I(x, y) \frac{t^n}{[n]_q!}, \quad (2.1) \]

\[ A_q^H(t)e_q(xt)E_q(yt) = \sum_{n=0}^{\infty} A_{n,q}^H(x, y) \frac{t^n}{[n]_q!}, \quad (2.2) \]

where

\[ A_q^I(t) = \sum_{n=0}^{\infty} A_{n,q}^I \frac{t^n}{[n]_q!}, \quad A_q^I(t) \neq 0 \quad (2.3) \]

and

\[ A_q^H(t) = \sum_{n=0}^{\infty} A_{n,q}^H \frac{t^n}{[n]_q!}, \quad A_q^H(t) \neq 0 \quad (2.4) \]

respectively.

In order to give the generating function for the composite 2D q-Appell polynomials, we prove the following result:

**Theorem 2.1** The composite 2D q-Appell polynomials \( A_{n,q}(x, y) \) are defined by the following generating function:

\[ A_q^I(t)A_q^H(t)e_q(xt)E_q(yt) = \sum_{n=0}^{\infty} A_{n,q}(x, y) \frac{t^n}{[n]_q!}. \quad (2.5) \]

**Proof** In view of Equations (1.5)–(1.7), (2.1) is written as

\[ A_q^I(t) \sum_{n=0}^{\infty} (x+y)^n \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} A_{n,q}(x, y) \frac{t^n}{[n]_q!}. \quad (2.6) \]

Expanding the summation and then replacing the powers 1, \( (x+y)_q^1 \), \( (x+y)_q^2 \), \( (x+y)_q^3 \) by the polynomials \( A_{1,q}(x, y) \), \( A_{2,q}(x, y) \), \( A_{3,q}(x, y) \), \( A_{1,q}(x, y) \) in the l.h.s. and \( (x+1)_q^1 \) by the polynomial \( A_{1,q}(x, 0) \) in the r.h.s. of above equation, we have

\[ A_q^I(t) \left[ 1 + A_{1,q}(x, y) \frac{t}{[1]_q!} + A_{2,q}(x, y) \frac{t^2}{[2]_q!} + \cdots + A_{n,q}(x, y) \frac{t^n}{[n]_q!} + \cdots \right] = \sum_{n=0}^{\infty} A_{n,q}(A_{n,q}(x, 0), y) \frac{t^n}{[n]_q!}, \quad (2.7) \]

which on summing up the series and using Equation (2.2) in the l.h.s. and denoting the resultant C2DqAP in the r.h.s. by \( A_{n,q}(x, y) \) yields assertion (2.5).

**Remark 2.1** It is remarked that, for \( y = 0 \), the composite 2D q-Appell polynomials \( A_{n,q}(x, y) \) reduce to the composite q-Appell polynomials (Khan & Riyasat, 2016), such that
Using Equation (1.6) in the l.h.s. of generating function (2.5) gives
\[ \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{q^n k!} = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!}, \]
which on applying the Cauchy product rule in the r.h.s. and then comparing the coefficients of same powers of \( t \) in both sides of resultant equation yields assertion (2.10) follows.

Theorem 2.2: The composite 2D q-Appell polynomials \( \alpha A_{n,q}(x, y) \) are defined by the following series definition:
\[ \alpha A_{n,q}(x, y) = \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{[n]_q!} \alpha A_{k,q}(0, 0) x^k y^n. \] (2.10)

Proof: Using Equations (2.1) and (2.4) in generating function (2.5) and then using Cauchy product rule in the l.h.s. of the resultant equation, we have
\[ \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{[n]_q!} \alpha A_{k,q}(x, y) A_{n-k,q}(x, y) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!}. \] (2.11)
Equating the coefficients of same powers of \( t \) in both sides of Equation (2.11), assertion (2.10) follows.

Theorem 2.3: The composite 2D q-Appell polynomials \( \alpha A_{n,q}(x, y) \) satisfy the following relations:
\[ \alpha A_{n,q}(x, y) = \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{[n]_q!} \alpha A_{k,q}(0, 0) x^k y^n. \] (2.12)
\[ \alpha A_{n,q}(x, y) = \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{[n]_q!} \alpha A_{k,q}(x, y) x^n y^k. \] (2.13)
\[ \alpha A_{n,q}(x, y) = \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{[n]_q!} \alpha A_{k,q}(x, y) x^k y^n. \] (2.14)

Proof: Using Equation (1.5) in the l.h.s. of generating function (2.5) gives
\[ \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} (x+y)^n \alpha A_{k,q}(0, 0) \frac{t^k}{[k]_q!}. \] (2.15)
which on applying the Cauchy product rule in the r.h.s. and then comparing the coefficients of same powers of \( t \) in both sides of resultant equation yields assertion (2.12).

Using Equation (1.7) in the l.h.s. generating function (2.5) gives
\[ \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} q^{-n} (x+y)^n \alpha A_{k,q}(0, 0) \frac{t^k}{[k]_q!}. \] (2.16)
which on applying the Cauchy product rule in the r.h.s. and then comparing the coefficients of same powers of \( t \) in both sides of resultant equation yields assertion (2.13).

Using Equation (1.6) in the l.h.s. of generating function (2.5) gives
\[ \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} x^n \alpha A_{k,q}(0, 0) \frac{t^k}{[k]_q!}. \] (2.17)
which on applying the Cauchy product rule in the r.h.s. and then comparing the coefficients of same powers of t in both sides of resultant equation yields assertion (2.14).

In order to give the determinantal definition for the composite 2D $q$-Appell polynomials $\mathcal{A}_{n,q}(x, y)$, the following result is proved:

**Theorem 2.4**  The composite 2D $q$-Appell polynomials $\mathcal{A}_{n,q}(x, y)$ of degree $n$ are defined by

$$
\mathcal{A}_{0,q}(x, y) = \frac{1}{\rho_{0,q}},
\begin{bmatrix}
1 & A_{1,q}(x, y) & A_{2,q}(x, y) & \ldots & A_{n-1,q}(x, y) & A_{n,q}(x, y) \\
\rho_{0,q} & \rho_{1,q} & \rho_{2,q} & \ldots & \rho_{n-1,q} & \rho_{n,q} \\
0 & \rho_{0,q} & \left[\begin{array}{c}2 \\ 1 \end{array}\right] & \rho_{1,q} & \ldots & \left[\begin{array}{c}n-1 \\ 1 \end{array}\right] & \rho_{n-2,q} & \left[\begin{array}{c}n \\ 1 \end{array}\right] & \rho_{n-1,q} \\
0 & 0 & \rho_{0,q} & \ldots & \left[\begin{array}{c}n-1 \\ 2 \end{array}\right] & \rho_{n-3,q} & \left[\begin{array}{c}n \\ 2 \end{array}\right] & \rho_{n-2,q} & \ldots & \ldots \\
0 & 0 & 0 & \ldots & \rho_{0,q} & \left[\begin{array}{c}n \\ n-1 \end{array}\right] & \rho_{1,q} & \ldots & \ldots & \ldots \\
\end{bmatrix}
$$

(2.18)

*Proof*  We recall the following determinantal definition for the 2D $q$-Appell polynomials (Eini Keleshteri & Mahmudov, 2015a):

$$
A_{0,q}(x, y) = \frac{1}{\rho_{0,q}},
\begin{bmatrix}
1 & (x+y)_q & (x+y)_q^2 & \ldots & (x+y)^{n-1}_q & (x+y)^n_q \\
\rho_{0,q} & \rho_{1,q} & \rho_{2,q} & \ldots & \rho_{n-1,q} & \rho_{n,q} \\
0 & \rho_{0,q} & \left[\begin{array}{c}2 \\ 1 \end{array}\right] & \rho_{1,q} & \ldots & \left[\begin{array}{c}n-1 \\ 1 \end{array}\right] & \rho_{n-2,q} & \left[\begin{array}{c}n \\ 1 \end{array}\right] & \rho_{n-1,q} \\
0 & 0 & \rho_{0,q} & \ldots & \left[\begin{array}{c}n-1 \\ 2 \end{array}\right] & \rho_{n-3,q} & \left[\begin{array}{c}n \\ 2 \end{array}\right] & \rho_{n-2,q} & \ldots & \ldots \\
0 & 0 & 0 & \ldots & \rho_{0,q} & \left[\begin{array}{c}n \\ n-1 \end{array}\right] & \rho_{1,q} & \ldots & \ldots & \ldots \\
\end{bmatrix}
$$

(2.19)

Replacing the powers $1, (x+y)_q, \ldots, (x+y)^n_q$ by the corresponding polynomials $1, A_{1,q}(x, y), \ldots, A_{n-1,q}(x, y), A_{n,q}(x, y)$ in the r.h.s. and $(x+0)_q^1$ by the polynomial $A_{2,q}(x, 0)$ and then using the appropriate notation $\mathcal{A}_{n,q}(x, y)$ in the l.h.s. of Equation (2.19) yields assertion (2.18).

In order to derive the $q$-recurrence relations and $q$-difference equations for the composite 2D $q$-Appell polynomials, we first prove the following lemma:

**Lemma 2.1**  The composite 2D $q$-Appell polynomials $\mathcal{A}_{n,q}(x, y)$ satisfy the following operational relations:
respectively.

The \( k \)-times lowering operators are given by

\[
D_{q,x}^k A_n(x, y) = \frac{[n]_q!}{[n-k]_q!} A_{n-k,q}(x, y),
\]

(2.20)

\[
D_{q,y}^k A_n(x, y) = \frac{[n]_q!}{[n-k]_q!} A_{n-k,q}(x, q^ky),
\]

(2.21)

Proof Differentiating generating function (2.5) \( k \)-times with respect to \( x \) and \( y \) and using the fact that it follows that which on using Cauchy product rule and then equating the coefficients of same powers of \( t \) in both sides of resultant equations yields assertions (2.20) and (2.21).

Since the following operational relations holds:

\[
D_{q,x} A_n(x, y) = [n]_q A_{n-1,q}(x, y),
\]

(2.28)

\[
D_{q,y} A_n(x, y) = [n]_q A_{n-1,q}(x, qy),
\]

(2.29)

which gives the lowering operators as:

\[
\Phi_{n,q_x} = \frac{1}{[n]_q} D_{q,x},
\]

(2.30)

\[
\Phi_{n,q_y} = \frac{1}{[n]_q} D_{q,y}.
\]

(2.31)

Therefore, in view of Equations (2.20)–(2.21) and (2.30)–(2.31), the \( k \)-times lowering operators for the
C2DqAP \( A_{n,q}(x, y) \) are given by Equations (2.22) and (2.23).

To derive the \( q \)-recurrence relations for the composite 2D q-Appell polynomials, the following result is proved:

**Theorem 2.5** For two different sets of 2D q-Appell polynomials \( A_{n,q}^I(x, y) \) and \( A_{n,q}^{II}(x, y) \) and with \( A_q^I(t) \) and \( A_q^{II}(t) \) defined by Equations (2.3) and (2.4), assume that

\[
\frac{D_q A_q^I(t)}{A_q^I(qt)} = \sum_{n=0}^{\infty} a_n \frac{t^n}{[n]_q!},
\]

(2.32)

\[
\frac{D_q A_q^{II}(t)}{A_q^{II}(qt)} = \sum_{n=0}^{\infty} \beta_n \frac{t^n}{[n]_q!},
\]

(2.33)

and

\[
A_q^{II}(t) = \sum_{n=0}^{\infty} \gamma_n \frac{t^n}{[n]_q!},
\]

(2.34)

\[
A_q^I(t) = \sum_{n=0}^{\infty} \delta_n \frac{t^n}{[n]_q!},
\]

(2.35)

respectively.

Then, the following linear homogeneous recurrence relations for the composite 2D q-Appell polynomials \( A_{n,q}(x, y) \) holds true for \( n \geq 1 \):

\[
\begin{align*}
A_{n,q}(qx, y) &= \frac{1}{[n]_q} \sum_{k=0}^{n} \binom{n}{k}_q q^{n-k} \left( \sum_{s=1}^{k} \binom{k}{s}_q a_{k-s} \delta_s \right) + \beta_k \\
&\quad + y \frac{1}{[k]_q} \sum_{s=0}^{k-1} \binom{k-1}{s}_q \gamma_{k-1-s} \delta_s A_{n-k,q}(x, y) + xq^n A_{n-1,q}(x, y),
\end{align*}
\]

(2.36)

\[
\begin{align*}
A_{n,q}(qx) &= q^n(x + (a_0 \delta_0 + \beta_0 + yq_0 \delta_0)q^{-1}) A_{n-1,q}(x, y) + \frac{1}{[n]_q} \sum_{k=1}^{n-1} \binom{n}{k}_q q^{n-k} \left( \sum_{s=1}^{k} \binom{k}{s}_q \gamma_{n-k-s} \delta_s \right) A_{n-k-1,q}(x, y) \\
&\quad + a_{n-k-1,s} + y \frac{1}{[n-k+1]_q} \sum_{s=0}^{n-k-1} \binom{n-k}{s}_q (n-k-s) \gamma_{n-k-s} \delta_s A_{n-k-1,q}(x, y).
\end{align*}
\]

(2.37)

and

\[
\begin{align*}
A_{n,q}(qx) &= q^n(x + (a_0 \delta_0 + \beta_0 + yq_0 \delta_0)q^{-1}) A_{n-1,q}(x, y) + \frac{1}{[n]_q} \sum_{k=1}^{n-1} \binom{n}{k}_q q^{n-k} \left( \sum_{s=1}^{k} \binom{k}{s}_q \gamma_{n-k-s} \delta_s \right) A_{n-k-1,q}(x, y) \\
&\quad + a_{n-k-1,s} + y \frac{1}{[n-k+1]_q} \sum_{s=0}^{n-k-1} \binom{n-k}{s}_q (n-k-s) \gamma_{n-k-s} \delta_s A_{n-k-1,q}(x, y).
\end{align*}
\]

(2.38)

Proof Taking \( x \rightarrow qx \) and then \( q \)-derivative with respect to \( t \) on both sides of generating function (2.5) using formula (1.10) gives

\[
\begin{align*}
qx A_q^I(qt) & A_{n,q}(qt)e_q(tqx) E_q(tqx) + y A_q^I(t)A_q^{II}(qt)e_q(tqx) E_q(tqx) + A_q^{II}(t)D_q(t)A_q^I(t) e_q(tqx) \\
E_q(tqx) + A_q^I(qt)D_q(t)A_q^{II}(t)e_q(tqx) E_q(tqx) &= \sum_{n=0}^{\infty} A_{n+1,q}(qx, y) \frac{t^n}{[n]_q!},
\end{align*}
\]

(2.39)

which on multiplying by \( t \) on both sides and then simplifying the resultant equation yields
Replacing both sides of the above equation yields assertion (2.36).

\( (2.36) \) yields assertion (2.44) and (2.45), respectively.

Using Equations (2.32)–(2.35) with generating function (2.5) gives

\[\sum_{n=0}^{\infty} [n]_q A_{n,q}(x, y) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} q^n A_{n,q}(x, y) \frac{t^n}{[n]_q!} \sum_{k=0}^{n} \sum_{s=0}^{k} \frac{\delta_k}{[k]_q} \frac{t^s}{[s]_q!} + \sum_{k=0}^{n} \beta_k \frac{t^k}{[k]_q!} \]

(2.41)

which on rearranging the summations using Cauchy product rule in the r.h.s. becomes

\[\sum_{n=0}^{\infty} [n]_q A_{n,q}(x, y) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left[ \binom{n}{k} [n]_q q^{n-k} \sum_{s=0}^{k} \frac{\delta_k}{[k]_q} \frac{t^s}{[s]_q!} + \sum_{k=0}^{n} \beta_k \frac{t^k}{[k]_q!} \right] A_{k-q,n-k}(x, y) \frac{t^n}{[n]_q!} + y \sum_{n=1}^{\infty} \sum_{k=0}^{n} \left[ \binom{n-1}{k} [n]_q q^{n-k-1} \sum_{s=0}^{k-1} \frac{\delta_k}{[k]_q} \frac{t^s}{[s]_q!} \right] A_{k-q,n-k}(x, y) \frac{t^n}{[n]_q!}. \]

(2.42)

The above equation can also be written as

\[\sum_{n=0}^{\infty} [n]_q A_{n,q}(x, y) \frac{t^n}{[n]_q!} = (\alpha_0 \delta_0 + \beta_0) A_{0,q}^{(2)}(x, y) + \sum_{n=1}^{\infty} \sum_{k=0}^{n} \binom{n}{k} [n]_q q^{n-k} \sum_{s=0}^{k} \frac{\delta_k}{[k]_q} \frac{t^s}{[s]_q!} A_{k-q,n-k}(x, y) \frac{t^n}{[n]_q!} + y \sum_{n=1}^{\infty} \sum_{k=0}^{n} \binom{n-1}{k} [n]_q q^{n-k-1} \sum_{s=0}^{k-1} \frac{\delta_k}{[k]_q} \frac{t^s}{[s]_q!} A_{k-q,n-k}(x, y) \frac{t^n}{[n]_q!}. \]

(2.43)

which on using the fact that \( \alpha_0 = \beta_0 = 0 \) and then equating the coefficients of same powers of \( t \) in both sides of the above equation yields assertion (2.36).

Solving the summation for \( k = 1 \) in the first term of the r.h.s. of Equation (2.36) yields assertion (2.37).

Replacing \( k \) by \( n - k + 1 \) in Equation (2.37) and then simplifying the resultant equation yields assertion (2.38).

**Theorem 2.6** The composite 2D q-Appell polynomials \( A_{n,q}(x, y) \) satisfy the following q-difference equations:

\[\sum_{k=1}^{n} \frac{q^{k-1}}{[k]_q!} \sum_{s=0}^{k-1} \frac{\delta_k}{[k]_q} \frac{t^s}{[s]_q!} A_{k-q,n-k}(x, y) \frac{t^n}{[n]_q!} = \frac{q^n}{[n]_q!} A_{0,q}(x, y) - [n]_q A_{n,q}(x, y) = 0. \]

(2.44)

\[\sum_{k=0}^{n} \frac{q^{k-1}}{[k]_q!} \sum_{s=0}^{k-1} \frac{\delta_k}{[k]_q} \frac{t^s}{[s]_q!} A_{k-q,n-k}(x, y) \frac{t^n}{[n]_q!} + q^n A_{n,q}(x, y) = 0. \]

(2.45)

**Proof** Using Equations (2.20) with (2.28) and (2.21) with (2.29), respectively, in the r.h.s. of Equation (2.36) yields assertion (2.44) and (2.45), respectively.
In Section 3, certain examples are considered as applications of the results derived above.

3. Applications

The generating function, series definition, q-recurrence relations and q-difference equations of some members of the C2DqAP \( A_{n,q}(x, y) \) are derived by considering the following examples:

By making suitable selection for the functions \( A_{1,q}^1(t) \) and \( A_{1,q}^2(t) \), the members belonging to the family of composite 2D q-Appell polynomials \( A_{n,q}(x, y) \) can be obtained. The generating functions and other results for these polynomials are given in Table 4.

| S.No. | \( A_{1,q}^1(t) \) | \( A_{1,q}^2(t) \) | Name of the resultant C2DqAP and related number | Generating function, series definition and other relations |
|-------|-----------------|-----------------|-----------------------------------------------|--------------------------------------------------|
| I.    | \( \frac{1}{q^k} \) | \( \frac{1}{q^k} \) | \( q \)-Bernoulli polynomials | \( \left( \frac{1}{q^{k+1}} \right)^2 e_{q}(xt)E_q(yt) = \sum_{n=0}^{\infty} B_{n,q}(x, y) \frac{t^n}{n!q^n} \) |
|       |                 |                 | \( B_{n,q}(x, y) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] B_{n-k,q}(x, y) q^{k} \) |                                                   |
|       |                 |                 | C2DqBP                                        |                                                   |
|       |                 |                 | \( B_{n,q}(x, y) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] B_{n-k,q}(0, 0)(x + y)^{n-k} \) |                                                   |
| II.   | \( \frac{1}{q^k} \) | \( \frac{1}{q^k} \) | \( q \)-Euler polynomials                     | \( \left( \frac{1}{q^{k+1}} \right)^2 e_{q}(xt)E_q(yt) = \sum_{n=0}^{\infty} E_{n,q}(x, y) \frac{t^n}{n!q^n} \) |
|       |                 |                 | \( E_{n,q}(x, y) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] E_{n-k,q}(x, y) q^{k} \) |                                                   |
|       |                 |                 | C2DqEP                                        |                                                   |
|       |                 |                 | \( E_{n,q}(x, y) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] E_{n-k,q}(0, 0)(x + y)^{n-k} \) |                                                   |
|       |                 |                 | \( E_{n,q}(x, y) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] E_{n-k,q}(0, 0)y^{n-k} \) |                                                   |
| III.  | \( \frac{1}{q^k} \) | \( \frac{1}{q^k} \) | \( q \)-Genocchi polynomials                  | \( \left( \frac{1}{q^{k+1}} \right)^2 e_{q}(xt)E_q(yt) = \sum_{n=0}^{\infty} G_{n,q}(x, y) \frac{t^n}{n!q^n} \) |
|       |                 |                 | \( G_{n,q}(x, y) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] G_{n-k,q}(x, y) \) |                                                   |
|       |                 |                 | C2DqGP                                        |                                                   |
|       |                 |                 | \( G_{n,q}(x, y) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] G_{n-k,q}(0, 0)(x + y)^{n-k} \) |                                                   |
|       |                 |                 | \( G_{n,q}(x, y) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] G_{n-k,q}(0, 0)y^{n-k} \) |                                                   |

(Continued)
Continued

| S.No. | \( A_q^n(t); A_q^n(t) \) | Name of the resultant C2DqAP and related number | Generating function, series definition and other relations |
|-------|-----------------|---------------------------------|--------------------------------------------------|
| IV.   | \( \frac{t}{e_q^{(n-1)} - e_q^{(n-1)}} \) \( e_q^{(n-1)} \) | \( B_{n,q}(x, y) \) : Composite 2D q-Bernoulli-Euler | \( \sum_{k=0}^{n} \frac{n}{k} B_{n,k,q}(x, y) x^{n-k} \) |
|      | \( \frac{t}{e_q^{(n-1)} - e_q^{(n-1)}} \) \( e_q^{(n-1)} \) | \( B_{n,q}(x, y) \) : Composite 2D q-Bernoulli-Euler | \( \sum_{k=0}^{n} \frac{n}{k} B_{n,k,q}(x, y) x^{n-k} \) |
|      | \( \frac{t}{e_q^{(n-1)} - e_q^{(n-1)}} \) \( e_q^{(n-1)} \) | \( B_{n,q}(x, y) \) : Composite 2D q-Bernoulli-Euler | \( \sum_{k=0}^{n} \frac{n}{k} B_{n,k,q}(x, y) x^{n-k} \) |
| V.    | \( \frac{t}{e_q^{(n-1)} - e_q^{(n-1)}} \) \( e_q^{(n-1)} \) | \( B_{n,q}(x, y) \) : Composite 2D q-Bernoulli-Euler | \( \sum_{k=0}^{n} \frac{n}{k} B_{n,k,q}(x, y) x^{n-k} \) |
|      | \( \frac{t}{e_q^{(n-1)} - e_q^{(n-1)}} \) \( e_q^{(n-1)} \) | \( B_{n,q}(x, y) \) : Composite 2D q-Bernoulli-Euler | \( \sum_{k=0}^{n} \frac{n}{k} B_{n,k,q}(x, y) x^{n-k} \) |
|      | \( \frac{t}{e_q^{(n-1)} - e_q^{(n-1)}} \) \( e_q^{(n-1)} \) | \( B_{n,q}(x, y) \) : Composite 2D q-Bernoulli-Euler | \( \sum_{k=0}^{n} \frac{n}{k} B_{n,k,q}(x, y) x^{n-k} \) |
| VI.   | \( \frac{t}{e_q^{(n-1)} - e_q^{(n-1)}} \) \( e_q^{(n-1)} \) | \( B_{n,q}(x, y) \) : Composite 2D q-Bernoulli-Euler | \( \sum_{k=0}^{n} \frac{n}{k} B_{n,k,q}(x, y) x^{n-k} \) |
|      | \( \frac{t}{e_q^{(n-1)} - e_q^{(n-1)}} \) \( e_q^{(n-1)} \) | \( B_{n,q}(x, y) \) : Composite 2D q-Bernoulli-Euler | \( \sum_{k=0}^{n} \frac{n}{k} B_{n,k,q}(x, y) x^{n-k} \) |
|      | \( \frac{t}{e_q^{(n-1)} - e_q^{(n-1)}} \) \( e_q^{(n-1)} \) | \( B_{n,q}(x, y) \) : Composite 2D q-Bernoulli-Euler | \( \sum_{k=0}^{n} \frac{n}{k} B_{n,k,q}(x, y) x^{n-k} \) |
|      | \( \frac{t}{e_q^{(n-1)} - e_q^{(n-1)}} \) \( e_q^{(n-1)} \) | \( B_{n,q}(x, y) \) : Composite 2D q-Bernoulli-Euler | \( \sum_{k=0}^{n} \frac{n}{k} B_{n,k,q}(x, y) x^{n-k} \) |
|      | \( \frac{t}{e_q^{(n-1)} - e_q^{(n-1)}} \) \( e_q^{(n-1)} \) | \( B_{n,q}(x, y) \) : Composite 2D q-Bernoulli-Euler | \( \sum_{k=0}^{n} \frac{n}{k} B_{n,k,q}(x, y) x^{n-k} \) |
|      | \( \frac{t}{e_q^{(n-1)} - e_q^{(n-1)}} \) \( e_q^{(n-1)} \) | \( B_{n,q}(x, y) \) : Composite 2D q-Bernoulli-Euler | \( \sum_{k=0}^{n} \frac{n}{k} B_{n,k,q}(x, y) x^{n-k} \) |

In order to give the applications of Theorems 2.6 and 2.7, we consider the following examples:

**Example 3.1** We know that for \( A_q^n(t) = A_q^n(t) = \left( \frac{t}{e_q^{(n-1)} - e_q^{(n-1)}} \right) \), the C2DqAP \( B_{n,q}(x, y) \) reduce to the C2DqBP \( B_{n,q}(x, y) \), therefore in view of Equations (2.32)–(2.35), we find \( a_n = \beta_n = 0; \quad \alpha_1 = \beta_1 = 0; \quad y_n = \delta_n = 0; \quad y_0 = 0; \quad \gamma_0 = 1 \), and \( \gamma_n = \delta_n = 1 \). Using these values in Theorems 2.6 and 2.7, we conclude the following:
Corollary 3.1 The following linear homogeneous recurrence relation for the composite 2D q-Bernoulli polynomials $B_{n,q}(x,y)$ holds true for $n \geq 1$:

\[
aB_{n,q}(x,y) = q^\left( x - \frac{1}{q} + \frac{y}{q} \right)B_{n-1,q}(x,y) + \frac{1}{[n]_q} \sum_{k=0}^{n} \binom{n}{k} \frac{1-q^{n-k+1}}{q^{k+1}B_{k-1,q}} \sum_{s=0}^{n-k} \binom{n-k+1}{s} \frac{1}{[s]_q} B_{l,q} - \frac{1}{q} B_{k,q}
\]

\[
B_{n+1,q}(x,y) = -[n]_q B_{n,q}(x,y) - \frac{1}{[n+1]_q} \sum_{s=0}^{n} \binom{n}{s} \frac{1-q^{n-s+1}}{q^{s+1}B_{s-1,q}} \sum_{l=0}^{s} \binom{n-s+1}{l} \frac{1}{[l]_q} B_{m,q}
\]

(3.1)

Corollary 3.2 The composite 2D q-Bernoulli polynomials $B_{n,q}(x,y)$ satisfy the following q-difference equations:

\[
q^n(x - \frac{1}{q} + \frac{y}{q}) + \frac{y}{q}D_{q,x} = \frac{1}{[n-1]_q} \sum_{k=2}^{n} \binom{n}{k} \frac{1-q^{n-k+1}}{q^{k+1}B_{k-1,q}} \sum_{s=0}^{n-k} \binom{n-k+1}{s} \frac{1}{[s]_q} B_{l,q} - \frac{1}{q} B_{k,q}
\]

\[
+ \frac{y}{q} q^k [k]_q \sum_{s=0}^{k-1} \binom{k-1}{s} \frac{1-q^{n-k+1}}{q^{s+1}B_{s-1,q}} \sum_{l=0}^{s} \binom{n-k+1}{l} \frac{1}{[l]_q} B_{m,q} D_{q,y} - \frac{1}{q} B_{k,q}
\]

(3.2)

\[
q^n(x - \frac{1}{q} + \frac{y}{q}) + \frac{y}{q} D_{q,y} = \frac{1}{[n-1]_q} \sum_{k=2}^{n} \binom{n}{k} \frac{1-q^{n-k+1}}{q^{k+1}B_{k-1,q}} \sum_{s=0}^{n-k} \binom{n-k+1}{s} \frac{1}{[s]_q} B_{l,q} - \frac{1}{q} B_{k,q}
\]

(3.3)

To show the graphical representation of the C2DqBP $B_{n,q}(x,y)$ for an index $n = 4$ and $q = \frac{1}{2}$ ($0 < q < 1$), we require the first few expressions of the composite q-Bernoulli polynomials $B_{n,q}(x)$ (Khan & Riyasat, 2016). These expressions are given in Table 5.

Using the expressions of first five $B_{n,1/2}(x)$ from Table 5 in Equation (4.1), we find

\[
aB_{n,1/2}(x,y) = \frac{1}{64}y^4 + \frac{15}{64}xy^3 - \frac{5}{16}y^3 + \frac{35}{32}x^2y^2 - \frac{35}{16}y^2x + \frac{15}{16}y^2 + \frac{15}{8}x^3y - \frac{35}{8}xy^2 + \frac{45}{16}xy - \frac{1}{3}y + x^4 - \frac{5}{2}x^3 \quad \text{for } q = \frac{1}{2}
\]

(3.4)

With the help of Matlab and using above expression, the surface plot for the C2DqBP is drawn, for this see Figure 1.

The determinantal definition for the C2DqBP $B_{n,q}(x,y)$ can be obtained by taking $\beta_{0,q} = 1$, $\beta_{i,q} = \frac{1}{1+q^{i+1}}$ ($i = 1, 2, \ldots, n$) for which determinantal definition of $A_{n,q}(x)$ reduces to $B_{n,q}(x)$ and $A_{n,q}(x,y) = B_{n,q}(x,y)$ ($n = 0, 1, 2, \ldots, n$) in determinantal definition (2.18) of the C2DqAP $A_{n,q}(x,y)$.

| n   | 0  | 1  | 2  | 3  | 4  |
|-----|----|----|----|----|----|
| $aB_{n,1/2}(x)$ | 1  | $x - \frac{1}{3}$ | $x^2 - 2x + \frac{8}{7}$ | $x^3 - \frac{2}{3}x^2 + \frac{1}{2}x - \frac{8}{15}$ | $x^4 - \frac{5}{2}x^3 + \frac{35}{16}x^2 - \frac{45}{16}x - \frac{221}{7812}$ |
Example 3.2 We know for $A_n(t) = A_0^n(t) = \left( \frac{t}{e^{qt}+1} \right)$, the C2DqAP $A_{n,q}(x, y)$ reduce to the C2DqEP $E_{n,q}(x, y)$, therefore in view of Equations (2.32)-(2.35), we find $a_n = \beta_n = \frac{1}{2}E_{n-1,q}; \quad a_0 = \beta_0 = 0; \quad a_1 = \beta_1 = -\frac{1}{2} \text{ and } \gamma_n = \delta_n = \frac{n-q+1}{2}$. Using these values in Theorems 2.6 and 2.7, we conclude the following:

**COROLLARY 3.3** The following linear homogeneous recurrence relation for the composite 2D $q$-Euler polynomials $E_{n,q}(x, y)$ holds true for $n \geq 1$:

$$E_{n,q}(x, y) = q^n \left( x - \frac{1}{2q} + \frac{y(q+1)^2}{4q} \right) D_{n,q} \left( x, y \right) + \frac{1}{q} \sum_{k=0}^{n} q^{n-k} \left( \sum_{i=1}^{k} \left[ k \atop s \right] \frac{q-1}{q} E_{k-s-1,q} \sum_{m=0}^{s} \left[ s \atop m \right] E_{m,q} \right) D_{n,q} \left( x, y \right) + \frac{y}{q} \sum_{k=0}^{n} q^{n-k} \left( \sum_{i=1}^{k} \left[ k \atop s \right] \frac{q-1}{q} E_{k-s-1,q} \sum_{m=0}^{s} \left[ s \atop m \right] E_{m,q} \right) D_{n,q} \left( x, y \right) - \left[ n \right]_{q} E_{n,q}(x, y) = 0,$$

(3.5)

**COROLLARY 3.4** The composite 2D $q$-Euler polynomials $E_{n,q}(x, y)$ satisfy the following $q$-difference equations:

$$E_{n,q}(x, y) = q^n \left( x - \frac{1}{2q} + \frac{y(q+1)^2}{4q^2} \right) D_{n,q} \left( x, y \right) + \frac{1}{q} \sum_{k=0}^{n} q^{n-k} \left( \sum_{i=1}^{k} \left[ k \atop s \right] \frac{q-1}{q} E_{k-s-1,q} \sum_{m=0}^{s} \left[ s \atop m \right] E_{m,q} \right) D_{n,q} \left( x, y \right) + \frac{y}{q} \sum_{k=0}^{n} q^{n-k} \left( \sum_{i=1}^{k} \left[ k \atop s \right] \frac{q-1}{q} E_{k-s-1,q} \sum_{m=0}^{s} \left[ s \atop m \right] E_{m,q} \right) D_{n,q} \left( x, y \right) - \left[ n \right]_{q} E_{n,q}(x, y) = 0.$$

(3.6)

To show the graphical representation of the C2DqEP $E_{n,q}(x, y)$ for an index $n = 4$ and $q = \frac{1}{2}$ $(0 < q < 1)$, we require the first few expressions of the composite $q$-Euler polynomials $E_{n,q}(x, y)$ (Khan & Riyasat, 2016). These expressions are given in Table 6.

Using the expressions of first five $E_{n,1/2}(x)$ from Table 6 in Equation (4) (III), we find

$$E_{n,1}(x, y) = \frac{1}{64} y^4 + \frac{15}{64} x y^3 - \frac{15}{64} y^3 x + \frac{35}{32} y^2 x^2 - \frac{35}{28} y^2 x + \frac{15}{8} y^2 x^2 - \frac{105}{32} x^2 y + \frac{15}{256} y^2 x^2 +\ldots$$

(3.8)

With the help of Matlab and using above expression, the surface plot for the C2DqEP is drawn, for this see Figure 2.

### Table 6. Expressions of first five $E_{n,1/2}(x)$

| $n$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $E_{n,1/2}(x)$ | $x - 1$ | $x^2 - \frac{5}{2} x + \frac{1}{16}$ | $x^3 - \frac{7}{4} x^2 + \frac{3}{16} x + \frac{1}{16}$ | $x^4 - \frac{15}{8} x^3 + \frac{15}{128} x^2 + \frac{125}{1024} x + \frac{71}{1024}$ |
The determinantal definition for the C2DqEP $\mathcal{E}_{n,q}(x,y)$ can be obtained by taking $\beta_{n,q} = 1$, $\beta_{i,q} = \frac{1}{q} (i = 1, 2, \ldots, n)$ (for which determinantal definition of $A_{n,q}(x,y)$ reduce to $E_{n,q}(x,y)$ and $A_{n,q}(x,y) = E_{n,q}(x,y)$) in determinant definition (2.18) of the C2DqAP $\mathcal{A}_{n,q}(x,y)
$.

**Example 3.3** We know for $A_n^{\alpha}(t) = A_n^{\beta}(t) = \left(\frac{x^{k-1}}{G_n^{\alpha}(1+q^{-1})} q^{\frac{k-1}{2q+1}} \right)$ the C2DqAP $\mathcal{A}_{n,q}(x,y)$ reduce to the C2DqGP $\mathcal{A}_{n,q}(x,y)$, therefore in view of Equations (2.32)–(2.35), we find $\alpha_n = \frac{1}{20} G_{n,q}; \alpha_0 = \frac{1}{2}; \alpha_1 = \frac{1}{4}$ and $\gamma_n = \frac{1}{2q} n = \frac{n-2}{2q} \sum_{k=0}^{n} k \quad G_{n,q} n \geq 1$; $\gamma_0 = \delta_0 = \frac{1}{2}$. Using these values in Theorems 2.6 and 2.7, we conclude the following:

**COROLLARY 3.5** The following linear homogeneous recurrence relation for the composite 2D q-Genocchi polynomials $\mathcal{G}_{n,q}(x,y)$ holds true for $n \geq 1$:

$$\mathcal{G}_{n,q}(qx,y) = q^n (x + \frac{(q-1)(x+q-1)}{2q^2+1} - \frac{1}{q^2} + y) \mathcal{G}_{n,q-1}(x,y) + \frac{1}{[m]_q} \sum_{s=0}^{m-1} \left( \sum_{k=0}^{n-1} \left( \begin{array}{c} n-k+1 \\ s \end{array} \right) \sum_{q=0}^{n-k-1} \left( \begin{array}{c} n-k-s \\ q \end{array} \right) \right) \mathcal{G}_{m,q} \left( \begin{array}{c} s \\ m \end{array} \right) \mathcal{G}_{n,q-1}(x,y).$$

(3.9)

**COROLLARY 3.6** The composite 2D q-Genocchi polynomials $\mathcal{G}_{n,q}(x,y)$ satisfy the following q-difference equations:

$$q^n \left( x - \frac{(q-1)(x+q-1)}{2q^2+1} - \frac{1}{q^2} + y \right) \mathcal{G}_{n,q}(x,y) + \frac{1}{q^2} \mathcal{G}_{n,q-1}(x,y) + \frac{q-1}{2q^2} \mathcal{G}_{n,q-2}(x,y) \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) \mathcal{G}_{i,q} x^{n-i} \mathcal{G}_{i,q-1}(x,y) = 0,$$

(3.10)

$$q^n \left( x - \frac{(q-1)(1-2q)}{2q^2+1} - \frac{1}{q^2} + y \right) \mathcal{G}_{n,q}(x,y) + \frac{q-1}{2q^2} \mathcal{G}_{n,q-1}(x,y) \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) \mathcal{G}_{i,q} x^{n-i} \mathcal{G}_{i,q-1}(x,y) = 0.$$

(3.11)

To show the graphical representation of the C2DqGP $\mathcal{G}_{n,q}(x,y)$ for an index $n = 4$ and $q = \frac{1}{2}$ ($0 < q < 1$), we require the first few expressions of the composite q-Genocchi polynomials $\mathcal{G}_{n,q}(x,y)$ (Khan & Riyasat, 2016). These expressions are given in Table 7.

Using the expressions of first five $\mathcal{G}_{n,1/2}(x)$ from Table 7 in Equation (4 (III)), we find

| $n$ | 0 | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|---|
| $\mathcal{G}_{n,1/2}(x)$ | 0 | 0 | $\frac{s}{2}$ | $\frac{s}{8} x - \frac{s}{2}$ | $\frac{100s^3}{32768} - \frac{125s^2}{1536} x + \frac{5s}{48}$ |

Notes: From Table 7, it can be seen that degree of composite q-Genocchi polynomials is $n - 2$. Hence, the polynomials $\mathcal{G}_{n,q}(x)$ are considered in the class of polynomial sequences which are not composite q-Appell in the strong sense.
The determinantal definition for the C2DqGP $$G_{n,q}(x,y)$$ can be obtained by taking $$\beta_{0,q} = 1, \beta_{i,q} = \frac{1}{2^{|i|-1}}$$ ($$i = 1, 2, \ldots, n$$) (for which determinantal definition of $$A_{n,q}(x)$$ reduce to $$G_{n,q}(x)$$) and $$A_{n,q}(x,y) = G_n(q,x(y))$$ ($$n = 0, 1, 2, \ldots, n$$) in determinantal definition (2.18) of the C2DqAP $$A_{n,q}(x,y)$$.

**Example 3.4** We know that for $$A_t^t(t) = \left( \frac{t}{e_q(t-1)} \right)$$ and $$A_t^2(t) = \left( \frac{2}{e_q(t+1)} \right)$$, the C2DqAP $$A_{n,q}(x,y)$$ reduce to the C2DqBEP $$E_{n,q}(x,y)$$, therefore in view of Equations (2.32)–(2.35), we find

$$\alpha_n = \frac{1}{q}B_{n,q}; \alpha_0 = 0; \alpha_1 = -\frac{1}{q^2}; \alpha_2 = \frac{1}{2}E_{n-1,q}; \beta_0 = 0; \beta_1 = -\frac{1}{q}; \gamma_n = \frac{1}{2} \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] B_{k,q}; n \geq 1; \gamma_0 = 1$$ and

$$\delta_n = \frac{q}{2} \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] E_{k,q}, \ n \geq 1; \delta_0 = \frac{1}{2}. \text{ Using these values in Theorems 2.6 and 2.7, we conclude the following:}

**Corollary 3.7** The following linear homogeneous recurrence relation for the composite 2D q-Bernoulli–Euler polynomials $$E_{n,q}(x,y)$$ holds true for $$n \geq 1$$:

$$E_{n,q}(q,x,y) - q^q(x - \frac{1}{2} + \frac{q^q + 1}{2q})D_{q,x} + \sum_{k=2}^{n} \frac{q^q - k}{[k]_q} \binom{k}{s} \sum_{l=0}^{s} \left[ \begin{array}{c} s \\ l \end{array} \right] \frac{1 - q}{q} B_{k,s,q} E_{l,q} + \frac{1}{2} E_{k-1,q}$$

$$\left( q - \frac{1}{\sqrt{2q}} \right)^2 E_{k-1,q}(x,y).$$

**Corollary 3.8** The composite 2D q-Bernoulli–Euler polynomials $$E_{n,q}(x,y)$$ satisfy the following q-difference equations:

$$q^q(x - \frac{1}{2} + \frac{q^q + 1}{2q})D_{q,x}E_{q,x} + \sum_{k=2}^{n} \frac{q^q - k}{[k]_q} \binom{k}{s} \sum_{l=0}^{s} \left[ \begin{array}{c} s \\ l \end{array} \right] \frac{1 - q}{q} B_{k,s,q} E_{l,q} + \frac{1}{2} E_{k-1,q}$$

$$\left( q - \frac{1}{\sqrt{2q}} \right)^2 \sum_{l=0}^{s} \left[ \begin{array}{c} s \\ l \end{array} \right] B_{l,q} \sum_{m=0}^{s} \left[ \begin{array}{c} s \\ m \end{array} \right] E_{m,q} D_{q,x} E_{q,x} = 0,$$

$$- [n]_{q} q E_{n,q}(q,x,y) = 0.$$
To show the graphical representation of the C2DqBEP $qE_{n,q}(x,y)$ for an index $n = 4$ and $q = \frac{1}{2}$ $(0 < q < 1)$, we require the first few expressions of the composite $q$-Bernoulli–Euler polynomials $qE_{n,q}(x)$ (Khan & Riyasat, 2016). These expressions are given in Table 8.

Using the expressions of first five $qE_{n,1/2}(x)$ from Table 8 in Equation (4 (IV)), we find

$$qE_{n,1/2}(x, y) = \frac{1}{64} y^6 + \frac{15}{64} y^3 x^3 - \frac{35}{128} y^3 x^3 - \frac{35}{32} y^2 x^2 + \frac{105}{64} y^2 x^2 + \frac{255}{768} y^2 x^2 + \frac{15}{8} x^3 y - \frac{245}{64} x^2 y + \frac{395}{256} x^2 y + \frac{379}{1536} y x^4 + \frac{35}{16} y x^4 - \frac{445}{384} y^2 x^2 - \frac{461}{1536} x - \frac{402305}{999960}.$$  \hspace{1cm} (3.16)

With the help of Matlab and using above expression, the surface plot for the C2DqBEP is drawn, for this see Figure 4.

The determinantal definition for the C2DqBEP $qE_{n,q}(x, y)$ can be obtained by taking $\beta_{0,q} = 1$, $\beta_{1,q} = \frac{1}{2}$, $\beta_{2,q} = \frac{1}{2}$, $\ldots$, $\beta_{n,q} = 0$; and for an index $n \geq 1$; $\gamma_0 = 0; \gamma_1 = 1$. Using these values in Theorems 2.6 and 2.7, we conclude the following:

COROLLARY 3.9 The following linear homogeneous recurrence relation for the composite 2D $q$-Bernoulli–Genocchi polynomials $qG_{n,q}(x, y)$ holds true for $n \geq 1$:

$$qG_{n,q}(x, y) = q^n \left( x - \frac{1}{2} y + y \frac{1}{2} q \right)^n G_{n-1,q}(x, y) + \frac{1}{m!} \sum_{k=0}^{n-1} \begin{pmatrix} n-1 \cr k \end{pmatrix} G_{n-k,q}(x, y) + \frac{1}{2} \sum_{s=1}^{n-k} \begin{pmatrix} n-k+1 \cr s \end{pmatrix} \left( \frac{1-q}{2q^2} \right)^s B_{n-k,s,q} \sum_{l=0}^{n-k-s} \begin{pmatrix} n-k-s \cr l \end{pmatrix} B_{l,q} \sum_{m=0}^{n-k-s-l} \begin{pmatrix} n-k-s-l \cr m \end{pmatrix} G_{m,q}.$$ \hspace{1cm} (3.17)

COROLLARY 3.10 The composite 2D $q$-Bernoulli–Genocchi polynomials $qG_{n,q}(x, y)$ satisfy the following $q$-difference equations:

$$q^n \left( x - \frac{1}{2} y + y \frac{1}{2} q \right)^n D_{n,q} + \frac{n}{k!} \sum_{k=1}^{n} q^n \left( \frac{k}{q} \right)^n \begin{pmatrix} k \cr s \end{pmatrix} \frac{1-q}{2q^2} B_{k-s,q} \sum_{l=0}^{s} \begin{pmatrix} s \cr l \end{pmatrix} G_{l,q} + \frac{1}{2q} G_{k-1,q}$$

$$+ y \frac{k-1}{k} \sum_{j=0}^{k-1} \begin{pmatrix} k-1 \cr s \end{pmatrix} \left( \frac{q-1}{\sqrt{2q}} \right)^{2k-1-s} B_{l,q} \sum_{m=0}^{s} \begin{pmatrix} s \cr m \end{pmatrix} G_{m,q}.$$ \hspace{1cm} (3.18)
\[ q^n \left( x - \frac{1}{q^2} + y \frac{1}{q} \right) D_{q,y}^n G_{n,q}(x, y) + \sum_{k=0}^{n} \frac{q^{n-k}}{[q]_k!} \left( \sum_{s=1}^{k} \left[ \begin{array}{c} k \\ s \end{array} \right]_{q} \frac{1-q}{2q^2} B_{k-s,q} \sum_{l=0}^{s} \left[ \begin{array}{c} s \\ l \end{array} \right]_{q} \right) q^n G_{l,q} \] (3.19)

\[ + \frac{1}{2q^2} G_{k-1,q} + \frac{y}{q^3} \sum_{s=0}^{k-1} \left[ \begin{array}{c} k-1 \\ s \end{array} \right]_{q} \left( \frac{q-1}{\sqrt{2q}} \right)^2 \sum_{m=0}^{k-1-s} \left[ \begin{array}{c} k-1-s \\ m \end{array} \right]_{q} B_{m,q} \sum_{l=0}^{s} \left[ \begin{array}{c} s \\ m \end{array} \right]_{q} G_{m,q} \right) \]

To show the graphical representation of the C2DqBG is drawn, for 
reduce to the C2DqEGP

\[ \text{Table 9. Expressions of first five } G_{n,1/2}(x) \]

\[ \begin{array}{cccccc}
 n & 0 & 1 & 2 & 3 & 4 \\
 G_{n,1/2}(x) & 0 & 1 & \frac{1}{2}x - 2 & \frac{1}{2}x^2 - \frac{1}{2}x + 1 & \frac{1}{2}x^3 - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{13}{4} \\
 \end{array} \]

Notes: From Table 9, it can be seen that degree of composite q-Bernoulli-Genocchi polynomials is n - 1. Hence, the polynomials G_{n,q}(x) are considered in the class of polynomial sequences which are not composite q-Appell in the strong sense.
\[
\left[ q^n \left( x - \frac{1}{q^2} + y q + \frac{1}{2q^2} \right) D_{q, x} + \sum_{k=2}^{n} q^{n-k} \left( \sum_{j=0}^{k} \left[ \frac{k}{q^2} \right] \right) \frac{q-1}{q^2} E_{q, x, 1} \sum_{j=0}^{k} \left[ \begin{array}{c} s \\ l \end{array} \right] G_{s, l, q} \right] + \frac{1}{2q} G_{k-1, q} + \frac{\beta_{k-1, q}}{q^k} \sum_{s=0}^{k-1} \left[ k - 1 \\ s \right] \frac{q-1}{q^2} \left( \sum_{j=0}^{k} \left[ \begin{array}{c} s \\ l \end{array} \right] E_{l, q} \sum_{m=0}^{s} \left[ \begin{array}{c} m \\ q \end{array} \right] G_{m, q} \right) \right] - [n]_{q^n} G_{n, q}(q, y) = 0.
\]

To show the graphical representation of the C2DqEGP \( G_{n, q}(x, y) \) for an index \( n = 4 \) and \( q = \frac{1}{2} \) (0 < \( q < 1 \)), we require the first few expressions of the composite \( q \)-Euler-Genocchi polynomials \( G_{n, q}(x) \) (Khan & Riyasat, 2016). These expressions are given in Table 10.

Using the expressions of first five \( q \)-EG polynomials from Table 10 in Equation (Table 4 (VI)), we find

\[
G_{4, 1/2}(x, y) = \frac{1}{64} y^4 + \frac{15}{64} x^2 y^2 + \frac{35}{128} y^4 + \frac{35}{32} x^4 + \frac{256}{64} y^3 x + \frac{675}{768} x^2 y^2 + \frac{15}{8} x^3 y - \frac{455}{64} x^2 y^2 + \frac{1025}{256} y^4 x + \frac{3027}{4608} x^2 y^3 + \frac{65}{16} x^4 + \frac{1025}{384} x^3 y^2 + \frac{315}{512} x^2 y^3 + \frac{256231}{1999872}.
\]

With the help of Matlab and using above expression, the surface plot for the C2DqEGP is drawn, for this see Figure 6.

The determinantal definition for the C2DqEGP \( G_{n, q}(x, y) \) can be obtained by taking \( \beta_{0, q} = 1, \beta_{1, q} = \frac{1}{2q+1} \) (\( i = 1, 2, \ldots, n \)) and \( A_{n, q}(x, y) = E_{n, q}(x, y) \) (\( n = 0, 1, 2, \ldots, n \)) in determinantal definition (2.18) of the C2DqAP \( A_{n, q}(x, y) \).

### 3.1. Graphical Interpretation

The 3D plots typically display a surface defined by a function in two variables, \( z = f(x, y) \). The 3D plots are more informative and better for analysis. A surface plot contains the following elements:

- Predictors on the \( x \) - and \( y \) -axes.
- A continuous surface that represents the response values on the \( z \)-axis. The peaks and valleys correspond with combinations of \( x \) and \( y \) that produce local maxima or minima. We display a 3D plot of a set of data points and also show a contour plot under a surface plot of peak function. We can compute the polynomials values according to define arrays. These graph show that the height along \( z \)-axis, is a single valued function over a geometrically rectangular grid. The \( z \) -axis specifies the value of the polynomials. We can click any number of points in our graphs and analyze the maxima and minima.

### Table 10. Expressions of first five \( q \)-EG polynomials

| \( n \) | 0 | 1 | 2 | 3 | 4 |
|-------|---|---|---|---|---|
| \( G_{n, 1/2}(x) \) | 0 | 1 | \( \frac{x-\frac{1}{2}}{2} \) | \( \frac{x^2 - \frac{9}{10} x - \frac{11}{60} \) | \( \frac{11}{30} x^3 - \frac{29}{60} x^2 + \frac{143}{768} x + \frac{331}{25760} \) |

Notes: From Table 10, it can be seen that degree of composite \( q \)-Euler-Genocchi polynomials is \( n - 1 \). Hence, the polynomials \( G_{n, q}(x) \) are considered in the class of polynomial sequences which are not composite \( q \)-Appell in the strong sense.
From Figures 1–6, we interpret that the polynomials \( B_{n,q}(x, y) \), \( E_{n,q}(x, y) \), \( G_{n,q}(x, y) \), \( B_{n,q}(x, y) \), \( E_{n,q}(x, y) \), \( G_{n,q}(x, y) \) and \( G_{n,q}(x, y) \) achieve the maximum value at \( x = y = -4 \) and minimum at \( x = 4 \) and \( y > -4 \).

**Figure 1.** The surface plot for the \( 2\mathrm{D}_q \) BP.

**Figure 2.** The surface plot for the \( 2\mathrm{D}_q \) EP.

**Figure 3.** The surface plot for the \( 2\mathrm{D}_q \) GP.
Figure 4. The surface plot for the C2DqBEP.

Figure 5. The surface plot for the C2DqBGP.

Figure 6. The surface plot for the C2DqEGP.
4. Concluding remarks

Al-Salam showed that the class of all q-Appell polynomials is a maximal commutative subgroup of the group of all polynomial sets, i.e. the class of all q-Appell sequences is closed under the operation of q-umbral composition of polynomial sequences. It can be seen from the fact that if \( A_{n,q}(x) = \sum_{k=0}^{n} a_{n,k} q_{k} x^{k} \) and \( B_{n,q}(x) = \sum_{k=0}^{n} b_{n,k} q_{k} x^{k} \) are sequences of q-polynomials, then the q-umbral composition of \( A_{n,q}(x) \) with \( B_{n,q}(x) \) is defined to be the sequence

\[
(A_{n,q} \circ B_{q})(x) = \sum_{k=0}^{n} a_{n,k} q_{k} B_{n,q}(x) = \sum_{0 \leq l \leq n} a_{n,l} q_{l} x^{l}.
\]

(4.1)

and under this operation the set of all q-Appell sequences is an abelian group. By taking \( \lim_{n \to \infty} \), the reduced set of Appell sequences also forms an abelian group under the umbral composition of polynomial sequences and is a subgroup of Sheffer group which is non-abelian (Roman, 1984, 1985).

By taking \( \lim_{q \to 1} \) in generating function (2.5) of the C2DqAP \( A_{n,q}(x, y) \), we have

\[
A^{(t)} A^{(t)} e^{x+y} = \sum_{n=0}^{\infty} A_{n}(x+y) \frac{t^{n}}{n!},
\]

(4.2)

which gives

\[
A_{n}^{(t)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} A_{k}^{(t)}(x) A_{n-k}^{(t)}(y).
\]

(4.3)

Hence, the C2DqAP \( A_{n,q}(x, y) \) for \( \lim_{q \to 1} \) is a group of sequences of binomial type if it satisfy identity (4.3). The group of sequences of binomial type is not a normal subgroup, which shows that the group of C2DqAP \( A_{n,q}(x, y) \) for \( \lim_{q \to 1} \) is also non-abelian normal subgroup of composite 2D Sheffer sequences (Khan & Riyasat, 2015).

Acknowledgements

The authors are thankful to the reviewer(s) for several useful comments and suggestions towards the improvement of this paper. Of-ce Memo No.2/40(38)/2016/R&D-II/1063

Funding

This work is partially funded by Post-Doctoral [grant number Of-ce Memo No.2/40(38)/2016/R&D-II/1063] awarded to Mumtaz Riyasat by the National Board of Higher Mathematics, Department of Atomic Energy, Government of India, Mumbai.

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Citation information

Cite this article as: q-difference equations for the composite 2D q-Appell polynomials and their applications, Mumtaz Riyasat, Subuhi Khan & Tabinda Nahid, Cogent Mathematics (2017), 4: 1376972.

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