Upper Bound on the Capacity of the Nonlinear Schrödinger Channel

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Abstract—It is shown that the capacity of the channel modeled by (a discretized version of) the stochastic nonlinear Schrödinger (NLS) equation is upper-bounded by log(1 + SNR) with SNR = \( P_0/\sigma^2(z) \), where \( P_0 \) is the average input signal power and \( \sigma^2(z) \) is the total noise power up to distance \( z \). The result is a consequence of the fact that the deterministic NLS equation is a Hamiltonian energy-preserving dynamical system.

I. INTRODUCTION

Half a century after the introduction of the optical fiber, the problem of determining its capacity remains open. This holds even for the single-user point-to-point channel subject to a power and bandwidth constraint. There is also a lack of general upper bounds, as well as lower bounds in the high-power regime. The asymptotic capacity when power \( P \to \infty \) is also unknown.

Numerical simulations of the optical fiber channel with additive white Gaussian noise (AWGN) seem to indicate that the data rates that can be achieved using current methods are below \( \log(1 + \text{SNR}) \), the capacity of an AWGN channel with signal-to-noise ratio SNR. In this paper, we prove this conjecture, namely, we show that

\[
C \leq \log(1 + \text{SNR}),
\]

where SNR(\( z \)) \( \triangleq P_0/\sigma^2(\( z \)) \), in which \( P_0 \) is the average input signal power and \( \sigma^2(\( z \)) \) is the total noise power up to the distance \( z \). Here \( C \) is the capacity of the point-to-point channel per complex degree of freedom.

Motivated by recent developments suggesting that the nonlinearity can be constructively taken into account in the design of communication schemes to potentially address the capacity bottleneck problem in optical fiber [1]-[3], it has been speculated that data rates above \( \log(1 + \text{SNR}) \) may even be achievable. While the nonlinearity can be exploited, as for instance in [1]-[6], the upper bound [1] shows that it does not offer any gain in capacity relative to the linear channel. All one can hope for is to embrace nonlinearity in the communication design so that it does not penalize the capacity at high powers. This is expected in the (closed) conservative system [2], which does not include any gain (amplification) mechanism.

Throughout this paper, lower and upper case letters represent, respectively, deterministic and random variables. Row vectors are denoted by underline, e.g., \( Q^n \triangleq (Q_1, \ldots, Q_n) \). As usual, \( \mathbb{R} \), resp. \( \mathbb{C} \), denotes the set of real, resp. complex, numbers. The imaginary unit is denoted by \( j = \sqrt{-1} \).

II. CONTINUOUS-TIME CHANNEL MODEL AND ITS DISCRETIZATION

Let \( Q(t, z) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{C} \) be a function of time \( t \) and space \( z \). Signal propagation in optical fiber is described by the stochastic nonlinear Schrödinger (NLS) equation [1] Eq. 3

\[
j \partial_t Q = \partial_{zz} Q + 2|Q|^2 Q + W(t, z).
\]

Here \( W(t, z) \) is space-time white circularly symmetric complex Gaussian noise with constant power spectral density \( \sigma_0^2 \) and bandlimited to \([-B/2, B/2] \), i.e.,

\[
\mathbb{E}(W(t, z)W^*(t', z')) = \sigma_0^2 \delta_B(t-t') \delta(z-z'),
\]

where \( \delta_B(x) \triangleq B \text{sinc}(Bx) \), \( \text{sinc}(x) \triangleq \sin(\pi x)/\pi x \), and \( \delta(x) \) is the Dirac delta function. The transmitted signal power is limited so that

\[
\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |Q(t, 0)|^2 dt \leq P_0.
\]

We discretize the continuous-time model [2] by considering the partial differential equation (PDE) [2] with periodic boundary conditions

\[
Q(t + T, z) = Q(t, z), \quad \forall t, z,
\]

where \( T \) is the signal period. Substituting the two-dimensional Fourier series (see [7] Sections III and VI)

\[
Q(t, z) = \sum_{k=-\infty}^{\infty} Q_k(z)e^{j(k\omega_0 t + k^2 \omega_0^2 z)},
\]

into the NLS equation [2], we obtain

\[
j \partial_z Q_k(z) = -2j \sum_{l,m} e^{j\Omega_{lmnk}^2} Q_l(z)Q_m(z)Q_n^*(z) \delta_{lmnk} + W_k(z),
\]

where \( \delta_{lmnk} \triangleq \delta[l + m - n - k] \), \( \delta[k] \) is the Kronecker delta function, and

\[
\Omega_{lmnk} \triangleq \omega_0^2 (l^2 + m^2 - n^2 - k^2), \quad \omega_0 \triangleq 2\pi/T.
\]

We assume that \( T \to \infty \) so that the discrete model [4] captures the infinitely many signal degrees-of-freedom in the continuous model [2] in a one-to-one manner. As a result,
$W_k$ are uncorrelated circularly symmetric complex Gaussian random variables, with

$$E(W_k(z)W_k^*(z')) = f_0\sigma_0^2\delta(k-k')\delta(z-z'), \quad f_0 \triangleq \omega_0/2\pi.$$ 

The coupled stochastic ordinary differential equation (ODE) system \textsuperscript{4} defines a discrete vector communication channel in the frequency domain $Q^n(0) \rightarrow Q^n(z)$. For notational convenience, we limit to positive frequencies so that vector indices start from one. We denote the action of the stochastic ODE system \textsuperscript{4} on input $Q^n(0)$ by $S_z$, i.e., $Q^n(z) = S_z(Q^n(0))$. We denote the action of the deterministic (noiseless) system (where $W^n = 0$) on input $Q^n(0)$ by $T_z$, i.e., $Q^n(z) = T_z(Q^n(0))$. The power constraint \textsuperscript{3} is discretized to $P(0) \leq P_0$, where

$$P(z) \triangleq \sum_{k=1}^n E(Q_k(z))^2. \quad (5)$$

In this paper, we assume $n = m$ and study the capacity of the discretized channel $S_z$, instead of the original continuous-time channel \textsuperscript{2}. See Remark \textsuperscript{1} for the case $n \neq m$.

The upper bound \textsuperscript{1} on the capacity of $S_z$ is obtained as follows. The transformation $T_z$ is energy-preserving, implying that the output power in $S_z$ is $P(0) + \sigma^2(z)$, $\sigma^2(z) \triangleq B\sigma_0^2$. Consequently, the output (differential) entropy rate is upper-bounded, from the maximum entropy theorem, by

$$C_n + \log(P_0 + \sigma^2(z)).$$

Combining these two results, $C \leq \log(1 + \text{SNR})$. In what follows, we establish these two steps.

The use of the EPI in bounding the conditional entropy rate is an important step in our proof. It is therefore worth elaborating on the EPI briefly, to see why entropy should increase at least by a constant amount at each point that noise is added along the link. In Appendix \textsuperscript{A} we briefly review this interesting inequality.

III. Upper Bound

A. Upper Bound on the Output Entropy

**Lemma 1** (Monotonicity of the Power in $S_z$). Let $B$ be the common signal and noise (passband) bandwidth from input to output. The output average power in $S_z$ is

$$P(z) = P(0) + \sigma^2(z). \quad (6)$$

**Proof:** Since the signal and noise are commonly bandlimited to $B$, $Q_k$ and $W_k$ are supported in $1 \leq k \leq n$ for all $z$, $n = B/f_0$. Taking the derivative with respect to $z$ in \textsuperscript{5} and using \textsuperscript{4}, we obtain

$$\frac{dP(z)}{dz} = 4\mathbb{E}\left( \sum_{l=0}^n E(Q_l Q_{n-l} Q_{n-l}^*, e^{j\Omega_{l,m} z^2}) \right)$$

$$+ \sum_{k=1}^n E(Q_k W_k + Q_k W_k^*)$$

$$= \sum_{k=1}^n \left( E(Q_k^*(z)W_k(z) + Q_k(z)W_k^*(z)) \right), \quad (7)$$

where we used the fact that the nonlinear term is real-valued, since $\Omega_{l,m} = -\Omega_{m,k}$. We now integrate \textsuperscript{7} in distance. From \textsuperscript{4}, $Q_k(z)$ contains a term depending on $W_k(l)$, $l < z$, and a Brownian motion term $B_k(z) = \int_0^z W_k(l)dl$. The first term is independent of $W_k(z)$; from the second term we get

$$E(\frac{z}{0} (Q_k^*(l)W_k(l) + c.c.) dl) = E(\frac{z}{0} (B_k^*(l)dB_k(l) + c.c.))$$

$$= E(B_k(z))^2$$

$$= f_0\sigma_0^2\pi.$$ 

where c.c. stands for complex conjugate. Summing over $1 \leq k \leq n$, we obtain \textsuperscript{6}.

Using Lemma \textsuperscript{1}, the output entropy rate can be upper bounded as follows:

$$\frac{1}{n} h(Q^n(z)) \overset{(a)}{\leq} \frac{1}{n} \log((\pi e)^n \det K(z))$$

$$= \log \pi e + \frac{1}{n} \log(\det K(z))$$

$$\overset{(b)}{\leq} \log \pi e + \frac{1}{n} \sum_{k=1}^n \log(K_{kk}(z))$$

$$\overset{(c)}{\leq} \log \pi e + \frac{1}{n} \sum_{k=1}^n \log(E|Q_k(z)|^2)$$

$$\overset{(d)}{=} \log \pi e + \log\left(\frac{1}{n} \sum_{k=1}^n |Q_k(z)|^2\right)$$

$$\overset{(e)}{\leq} C_n + \log(P_0 + \sigma^2(z)), \quad (8)$$

where $K(z) > 0$ is the covariance matrix of $Q^n(z)$ with entries $K_{kk}(z)$. Step \textsuperscript{(a)} is due to the maximum entropy theorem. Step \textsuperscript{(b)} follows from Hadamard’s inequality. For step \textsuperscript{(c)}, note that in \textsuperscript{3}, power was defined as average energy in time interval $T$ divided by $T$. As a result, a non-zero constant signal has non-zero power. In the covariance matrix, in contrast to \textsuperscript{3} and \textsuperscript{5}, the mean of the random variable is subtracted as $K_{kk}(z) = E(Q_k(z))^2 - E(Q_k(z))^2$. Unlike \textsuperscript{5}, the mean term $\sum_{k=1}^n |Q_k(z)|^2$ is not preserved in the noise-free channel. Furthermore, a zero-mean signal at the input may not have zero mean at $z > 0$. Nevertheless, step \textsuperscript{(e)} holds since $K_{kk}(z) \leq E(Q_k(z))^2$. Steps \textsuperscript{(d)} and \textsuperscript{(e)} follow, respectively, from the concavity of the log function and \textsuperscript{6}. In steps \textsuperscript{(b)}, \textsuperscript{(c)} and \textsuperscript{(e)}, we also used the fact that log is an increasing function.
B. Lower Bound on the Conditional Entropy

Lemma 2 (Volume Preservation in $T_z$). Let $\Omega = (\ell^2, \mathcal{E}, \mu)$ be a measure space, where $\ell^2 \triangleq \{g^n \mid \sum |g_k|^2 < \infty\}$ and

$$\mu(A) = \text{vol}(A) = \int_A \left( \prod_{k=1}^n dq_k dq_k^* \right), \quad \forall A \in \mathcal{E},$$

is the Lebesgue measure. Transformation $T_z$, as a dynamical system on $\Omega$, is measure-preserving. That is to say

$$\mu(T_z^{-1}(A)) = \mu(A), \quad \forall A \in \mathcal{E}.$$  

Proof: We note that, when $W_k = 0$, the ODE system (4) is Hamiltonian, i.e., it permits an alternative formulation

$$\dot{x}_k = \frac{\partial H}{\partial y_k}, \quad \dot{y}_k = -\frac{\partial H}{\partial x_k}, \quad k = 1, \cdots, n,$$  

(9)

where dot represents $\frac{d}{dt}$. $(x_k, y_k) = (q_k, \dot{q}_k)$ and the Hamiltonian function $H$ is given by

$$H(x^n, y^n) = \sum_{k=1}^n \omega_k^2 k^2 x_k y_k$$

$$- \sum_{a=1}^n \sum_{b=1}^n \sum_{c=1}^n \sum_{d=1}^n x_a x_b y_c y_d \exp(j \omega_{abcd} z) \delta_{abcd}.$$  

Liouville’s theorem asserts that Hamiltonian systems preserve the Lebesgue measure [8]. This is indeed easy to see. Let $d\mu = \prod_{k=1}^n dx_k dy_k$. Then

$$d\bar{\mu} = (d\dot{x}_1 dy_1 + dx_1 dy_1) \prod_{k=2}^n dx_k dy_k + \cdots$$

$$= 0,$$  

where we substituted (9). It follows that $T_z$ is a volume-preserving transformation (in the sense of ergodic theory [9]).

Lemma 3 (Entropy Preservation in $T_z$). The flow of $T_z$ is entropy-preserving, i.e., $h(T_z^{-1}(Q^n)) = h(Q^n).$

Proof: From Lemma 2, $T_z$ is a measure-preserving transformation; therefore it has unit (determinant) Jacobian, det $J = 1$, where $J$ is the $2n \times 2n$ Jacobian matrix. Since $T_z$ is also invertible

$$h(T_z^{-1}(Q^n)) = h(Q^n) = E \log |\det J| = h(Q^n).$$

Note that with $J$ as a $C^{n \times n}$ matrix, there would be a factor 2 in front of the log.

In the example of the NLS channel (2), the dispersion and nonlinear parts are separable and can be solved in simple forms. In such examples, it might be possible to directly check that the flow of the equation has unit Jacobian. Note that the dispersion operator, being a unitary transformation, has unit Jacobian. One can also verify that the nonlinear part of the NLS equation (2) has unit Jacobian too. Consider

$$Y = X \exp(j f(|X|)), \quad X, Y \in \mathbb{C},$$  

(10)

for any differentiable function $f(X)$. In (2), $f(X) = zX^2$, $X = Q(t, 0)$ and $Y = Q(t, z)$. Linearizing at $X = 0$, $dY = dX$. More formally, in polar coordinates

$$R_Y = R_X, \quad \Phi_Y = \Phi_X + f(R_X),$$

where $(R_X, \Phi_X)$ and $(R_Y, \Phi_Y)$ are coordinates of $X$ and $Y$, respectively. Clearly det $J = 1$, which can be seen is the same in the Cartesian coordinates because $|Y| = |X|$. Since the transformation from the NLS equation (2) in the time domain to the ODE system (4) in the discrete frequency domain is also unitary and unit Jacobian, $T_z$ has unit Jacobian.

Finally, it is also possible to check that $T_z$ is entropy-preserving using the elementary properties of the entropy. It is obvious that the dispersion operator is entropy-preserving. In the continuous model (2), the nonlinear transformation in each time sample is given by (10). Using the chain rule for entropy

$$h(R_Y, \Phi_Y) = h(R_X) + h(\Phi_Y | R_Y)$$

$$= h(R_X) + h(\Phi_X + f(R_X) | R_X)$$

$$= h(R_X) + h(\Phi_X | R_X)$$

$$= h(R_X, \Phi_X).$$

Note that the entropy of a complex random variable is defined as the joint entropy of the real and imaginary parts. Changing variables to the Cartesian coordinate system shifts the entropy by $E \log |\det J| = E \log R_Y = E \log R_X$. Thus $h(R_Y \exp(j \Phi_Y)) = h(R_X \exp(j \Phi_X))$. The result also holds for the vector version of (10) as well. Because the Fourier transform is also entropy-preserving, so is $T_z$.

The last two approaches, however, depend on details of the example at hand. For some equations the nonlinear part is not an additive term to dispersion, and even if it may be not simply solvable like (10). For instance, the nonlinear part of the Korteweg-de Vries (KdV) equation is Burgers’ equation, which is not easily solvable as (10), so as to examine entropy preservation directly. However, it is quite easy to show that the KdV equation, and indeed a large number of evolution equations, are Hamiltonian.

Lemma 4 (Monotonicity of the Entropy in $S_z$). The conditional entropy rate in $S_z$ is lower-bounded by the noise entropy rate, i.e.,

$$\frac{1}{n} h(S_z(Q^n(0))|Q^n(0)) \geq C_n + \log \sigma^2(z).$$

Proof: In a small interval $\Delta z$ in (4)

$$S_{z+\Delta z}(Q^n(z)) = T_{\Delta z}(S_z(Q^n(z))) + W^n(z)\sqrt{\Delta z}. ($$  

(11)

The two terms in the right hand side of (11) are independent. Applying the EPI (14),

$$2^{\frac{1}{n} h(S_{z+\Delta z}(Q^n))} \geq 2^{\frac{1}{n} h(T_{\Delta z}(S_z(Q^n)))} + 2^{\frac{1}{n} h(W^n(z)\sqrt{\Delta z})}$$

$$= 2^{\frac{1}{n} h(S_z(Q^n))} + 2 C_n \sigma^2(\Delta z),$$
The upper bound (1) on the SE holds if its spectral broadening does not increase data rate or the SE.

Summary, nonlinearity is entropy-preserving and the effect of the frequency band.

be (nearly) unbounded by exploiting the (nearly) noise-free normalizing by the chain rule. Furthermore, let 

$\max_z B(z)$ would only decrease the SE relative to the linear dispersive channel (where $B(z) = B(0)$). In summary, nonlinearity is entropy-preserving and the effect of its spectral broadening does not increase data rate or the SE.

The upper bound (1) on the SE holds if $n \neq m$.

Throughout the paper, we assumed that noise bandwidth is larger than the signal bandwidth. Otherwise, capacity can be (nearly) unbounded by exploiting the (nearly) noise-free frequency band.

The upper bound (1) is indeed simple. In this paper, we discussed it in the context of a general Hamiltonian channel with continuous evolution. In particular, it also holds for a discrete concatenation of energy- and entropy-preserving systems with additive white Gaussian noise.

A different account of the upper bound (1) is given in [10] using the split-step Fourier method.

IV. CONCLUSION

It is shown that the capacity of the point-to-point optical fiber channel, modeled via the stochastic nonlinear Schrödinger equation (2), and subject to a power and bandwidth constraint, is upper-bounded by $\log(1 + \text{SNR})$.

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APPENDIX A

THE ENTROPY POWER INEQUALITY

Lemma 5 (Entropy Power Inequality). Let $X, Y \in \mathbb{R}^n$ be independent random variables. Define the entropy power of a random variable $X \in \mathbb{R}^n$ as

$$\sigma^2_e(X) = \frac{1}{2\pi e} 2^{\frac{1}{2} h(X)}.$$  

Then

$$\sigma^2_e(X + Y) \geq \sigma^2_e(X) + \sigma^2_e(Y).$$  

Equality holds if and only if $X$ and $Y$ are Gaussian with proportional covariance matrices.

Proof: By now there are many proofs of the EPI. A simple proof is given in [11, Section 17.8]. It can be explained as follows.

Consider $n = 1$. We are looking for an inequality involving the convolution $f_X(x) * f_Y(y)$. The well-known Young’s inequality for $f_X(x) \in L^p(\mathbb{R})$ and $f_Y(y) \in L^q(\mathbb{R})$ states

$$\|f_X(x) * f_Y(y)\|_a \leq C \|f_X(x)\|_p \|f_Y(y)\|_q,$$  

where $1/p + 1/q = 1/a + 1 (p, q, a \geq 1)$, and $C = \sqrt{C_p C_q / C_a}$, $C_x = x^{\frac{a}{2}} / x^{\frac{a}{2}}$, where $x'$ is conjugate to $x$, i.e., $1/x + 1/x' = 1$. When $p, q \neq 1$, the equality holds if and only if $f_X(x)$ and $f_Y(y)$ are Gaussian. On the other hand, entropy and norm of a probability density $f_X(x)$ are related via $h(X) = -\int_a \log \|f_X(x)\|_a^a$ at $a = 1$. However differentiating both sides of an inequality does not preserve the sense of the inequality. Nevertheless, using L’Hôpital’s rule we can convert differentiation to a limit

$$h(X) = \lim_{a \rightarrow 1} \frac{1}{1 - a} \log \|f_X(x)\|_a^a.$$  

This in turn gives $\sigma_e(X) \leq \|f_X(x)\|_a^a(\mu_a)^{1 - (1/a)}$. At $a = 1 + \epsilon$ ($\epsilon \rightarrow 0$), the left side of (15) gives $\sigma_e(X + Y)$. For a given $a$, there is one free parameter in the right hand side of (15). By choosing the free parameter such that the right side of (15) is maximized, we obtain the EPI. The case $n > 1$ is obtained by replacing entropy with entropy rate (and using a version of (15) in $\mathbb{R}^n$ to find conditions of equality). The equality in (14) results from the equality in (15).

The EPI, in some sense, is the derivative of the Young’s inequality.

Several remarks are in order now.

a) Bound on conditional discrete entropy: Let $A$ and $B$ be finite discrete sets (alphabets). Since not all elements of $A + B$ are distinct, we have the sunset inequality

$$\mu(A + B) \leq \mu(A) \mu(B),$$  

where $\mu$ denotes set cardinality. This in turn gives

$$H(X + Y) \leq H(X) + H(Y),$$  

where $X$ and $Y$ are independent discrete random variables taking values, respectively, in alphabets $A$ and $B$, and $H$ is discrete entropy. For uniform random variables (17) is
just the sumset inequality (16); non-uniform distributions can (almost) be converted to uniform distributions via the asymptotic equipartition theorem (11). The inequality (17) reflects the fact that the sum of independent discrete random variables typically does not tend to a uniform random variable (maximum entropy). In fact, in a sense, \( X + Y \) is “less uniform” than \( X \) and \( Y \). In sharp contrast, the (normalized) sum of independent continuous random variables tends to a Gaussian random variable (maximum entropy)—however, the increase in randomness is measured in entropy power, not the entropy itself.

The inequality (17) seems to indicate that as noise is added along the optical fiber, the conditional entropy of the signal does not increase. Two distinct pairs \((q_1, y_1)\) and \((q_2, y_2)\) can have the same sum \( q_1^2 + w_1^2 = q_2^2 + w_2^2 \), making \( Q^2 + W^2 \) potentially “less random,” so to speak. This is, however, true only in a discrete-state model in which \( q^2 \) is quantized in a finite set. It follows that, the entropy bounds in this paper may not be valid in discrete-state models, due to important differences between the differential and discrete entropies. This difference stems from the properties of the cardinality (volume) in discrete (continuous) sets.

b) Growth of the effective variance in evolution: For a Gaussian random variable with variance \( \sigma^2 \), \( \sigma^2(X) = \sigma^2 \). Thus one may think of \( \sigma^2(X) \) as the effective variance of \( X \) or the squared radius of the support of \( X \) (hence the notation).

A family of fascinating metric inequalities analogous to (14) exist in geometry and analysis, where the squared radius (13) is defined differently (12). Notably, in one of its facets, the Brunn-Minkowski inequality (BMI) for compact regions \( A, B \subset \mathbb{R}^n \) states

\[
\mu^\frac{1}{p}(A + B) \geq \mu^\frac{1}{p}(A) + \mu^\frac{1}{p}(B), \tag{18}
\]

where \( \mu \) is the Lebesgue measure (volume) and \( A + B \) is the Minkowski sum of \( A \) and \( B \). The BMI looks like the EPI with \( \sigma^2(X) \equiv \mu^\frac{1}{p}(A) \). Let \( A^m, B^m \) and \( C^m \) be, respectively, the \( \epsilon \)-typical sets of random variables \( X, Y \in \mathbb{R}^n \) and \( Z = X + Y \). From the concentration of measure \( \mu(A^m) \to 2^{-\epsilon m} \), or \( \mu(A^m) \to 2^{-\epsilon m} / h(X) \), as \( \epsilon \to 0 \). Applying the BMI to \( A^m \) and \( B^m \), we obtain the EPI with factor one in the exponent in (13) instead of two. The result is not the desired BMI inequality. This is because \( C^m \neq A^m + B^m \). In fact, again from the concentration of measure, \( Z \) concentrates on a smaller set \( C^m \subset A^m + B^m \), i.e., \( \mu(C^m) < \mu(A^m) \mu(B^m) \). To obtain the EPI from the BMI, and thus to give the EPI a geometric meaning, we need a probabilistic version of the Minkowski sum, where the volume is defined as the size of high probability sequences. Define the \( \Omega \)-restricted Minkowski sum of two sets \( A, B \subset \mathbb{R}^n \)

\[
A +_\Omega B \triangleq \{ a + b \mid (a, b) \in \Omega \subset A \times B \}. \tag{19}
\]

The restricted BMI states that, if \( \mu(\Omega \setminus (1 - \delta) \mu(A) \mu(B) \) for some \( \delta > 0 \), then (18) holds but with exponent \( 2/n \) (Theorem 1.2, with large \( n \)]. Furthermore, the restricted BMI is sharp, regardless of how close \( \Omega \) is to \( A \times B \), i.e., \( \delta \to 0 \). That is to say, even a small uncertainty in the size of \( A \times B \) would increase the exponent in the BMI by a factor of two. The inequality is best seen for Gaussian random variables where typical sets can be imagined as spherical shells (13).

Applying the restricted BMI to \( A^m \) and \( B^m \) with \( \Omega = \{ (a, b) \mid a \in A^m, b \in B^m, a + b \in C^m \} \), we successfully obtain the EPI.

With the geometric interpretation of the BMI for typical sequences, the upper bound (1) is trivial. The output typical set \( A^m(\Omega(z)) \) is covered in the sphere \( S_{2nm}g_n^m(z), \sqrt{m(\mu + \sigma^2(z))} \), centered at some \( g_n^m(z) \). For a particular input sequence \( g^m(0) \), as the typical set of the signal and noise are overlapped in the optical link, the resulting region can be packed by a sphere \( S_{2nm}(g_n^m(z), \sqrt{m\sigma^2(z)}) \), centered at some \( g_n^m(z) \). The capacity sphere-packing interpretation gives (1).

Inequalities in the family to which (14) and (15) belong appear intimately connected; however, it seems difficult to deduce them all from one master inequality, due to important differences among them. There is substantial work on this type of inequality; see (12) and references in (11).

REFERENCES

[1] M. I. Yousefi and F. R. Kschischang, “Information transmission using the nonlinear Fourier transform, Part I: Mathematical tools,” IEEE Trans. Inf. Theory, vol. 60, no. 7, pp. 4312–4328, Jul. 2014, Also published on arXiv. Feb. 2012. [Online]. Available: http://arxiv.org/abs/1202.3653.

[2] ——, “Information transmission using the nonlinear Fourier transform, Part II: Numerical methods,” IEEE Trans. Inf. Theory, vol. 60, no. 7, pp. 4329–4345, Jul. 2014, Also published on arXiv, Apr. 2012. [Online]. Available: http://arxiv.org/abs/1204.0830.

[3] ——, “Information transmission using the nonlinear Fourier transform, Part III: Spectrum modulation,” IEEE Trans. Inf. Theory, vol. 60, no. 7, pp. 4346–4369, Jul. 2014, Also published on arXiv, Feb. 2013. [Online]. Available: http://arxiv.org/abs/1302.2875.

[4] S. Wahl and H. V. Poor, “Fast numerical nonlinear Fourier transforms,” Feb. 2014, arXiv:1402.1605v2. [Online]. Available: http://arxiv.org/abs/1402.1605v2.

[5] Q. Zhang, T. H. Chan, and A. Grant, “Spatially periodic signals for fiber channels,” in 2014 IEEE Int. Symp. Inf. Theory (ISIT 2014), Jun. 29–Jul. 4, 2014, pp. 2804–2808.

[6] J. E. Prilepsky, S. A. Derevyanko, and S. K. Turitsyn, “Nonlinear spectral management: Linearization of the lossless fiber channel,” Opt. Exp., vol. 21, no. 20, pp. 24344–24367, Oct. 2013.

[7] M. I. Yousefi, F. R. Kschischang, and G. Kramer, “Kolmogorov-Zakharov model for optical fiber communications,” Dec. 2014, arXiv:1411.6550. [Online]. Available: http://arxiv.org/abs/1411.6550.

[8] V. I. Arnold, Mathematical Aspects of Classical and Celestial Mechanics, 2nd ed., ser. Graduate Texts in Math. New York, NY, USA: Springer Science & Business Media, 2010, vol. 60, translated by K. Vogtmann and A. Weinstein.

[9] P. Walters, An Introduction to Ergodic Theory, ser. Graduate Texts in Math. New York, NY, USA: Springer-Verlag, 2000, vol. 79.

[10] G. Kramer, M. I. Yousefi, and F. R. Kschischang, “Upper bound on the capacity of a cascade of nonlinear and noisy channels,” in 2015 IEEE Info. Theory Workshop (ITW 2015), Jerusalem, Israel, Apr. 26–May 1, 2015. [Online]. Available: http://arxiv.org/abs/1503.07652v2.

[11] T. M. Cover and J. A. Thomas, Elements of Information Theory, 2nd ed. New Jersey, NY, USA: John Wiley & Sons, Inc., 2006.

[12] R. Gardner, “The Brunn-Minkowski inequality,” Bull. Amer. Math. Soc., vol. 39, no. 3, pp. 355–405, Apr. 2002.

[13] S. J. Szarek and D. Voiculescu, “Volumes of restricted Minkowski sums and the free analogue of the entropy power inequality,” Commun. Math. Phys., vol. 178, no. 3, pp. 563–570, Jul. 1996.