MOMENT-ANGLE COMPLEXES AND COMBINATORICS OF SIMPLICIAL MANIFOLDS

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Abstract. Let \( \rho : (D^2)^m \to I^m \) be the orbit map for the diagonal action of the torus \( T^m \) on the unit poly-disk \((D^2)^m \subset \mathbb{C}^m, I^m = [0,1]^m \) is the unit cube. Let \( C \) be a cubical subcomplex in \( I^m \). The moment-angle complex \( ma(C) \) is a \( T^m \)-invariant bigraded cellular decomposition of the subset \( \rho^{-1}(C) \subset (D^2)^m \) with cells corresponding to the faces of \( C \). Different combinatorial problems concerning cubical complexes and related combinatorial objects can be treated by studying the equivariant topology of corresponding moment-angle complexes. Here we consider moment-angle complexes defined by canonical cubical subdivisions of simplicial complexes. We describe relations between the combinatorics of simplicial complexes and the bigraded cohomology of corresponding moment-angle complexes. In the case when the simplicial complex is a simplicial manifold the corresponding moment-angle complex has an orbit consisting of singular points. The complement of an invariant neighbourhood of this orbit is a manifold with boundary. The relative Poincaré duality for this manifold implies the generalized Dehn–Sommerville equations for the number of faces of simplicial manifolds.

1. Introduction

The classical Dehn–Sommerville equations for simplicial convex \( n \)-dimensional polytope \( P^n \) give a set of linear relations among the numbers \( f_i \) of \( i \)-dimensional faces of \( P^n \). The integer vector \( f(P^n) = (f_0, f_1, \ldots, f_{n-1}) \) is called the \( f \)-vector of \( P^n \). We put \( f_{-1} = 1 \). The Dehn–Sommerville equations were established by Dehn for \( n \leq 5 \), and by Sommerville in the general case in 1927 (see [So]). Their original form is as follows

\[
\begin{equation}
 f_k = \sum_{j=k}^{n-1} (-1)^{n-1-j} \binom{j+1}{k+1} f_j, \quad k = 0, \ldots, n - 1. 
\end{equation}
\]

Define the \( h \)-vector from the equation

\[
\begin{equation}
 h_0 t^n + \ldots + h_{n-1} t + h_n = (t-1)^n + f_0(t-1)^{n-1} + \ldots + f_{n-1}.
\end{equation}
\]
Obviously, the $f$-vector and the $h$-vector determine each other by means of linear equations (note that $h_0 = 1$). For instance,

\begin{equation}
    h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{n-i}{k-i} f_{i-1}.
\end{equation}

The notion of $h$-vector gives rise to the simplest and the most elegant form of the Dehn–Sommerville equations (1) for simplicial polytopes:

\begin{equation}
    h_i = h_{n-i}, \quad i = 0, \ldots, n.
\end{equation}

The Dehn–Sommerville equations can be generalized quite widely. In [Kl] V. Klee reproved the Dehn–Sommerville equations in the form (1) in a more general context of Eulerian manifolds. In particular, it turns out that equations (1) hold for any simplicial manifold (i.e. triangulated topological manifold) of dimension $n - 1$. Analogues of equations (1) were obtained by Bayer and Billera [BaBi] (for Eulerian partially ordered sets) and Chen and Yan [CY] (for arbitrary polyhedra).

It follows directly from (4) that the affine hull of $f$-vectors $(f_0, \ldots, f_{n-1})$ of simplicial polytopes in $n$-dimensional space is an $\frac{n}{2}$-dimensional plane.

The same is true for the affine hull of vectors $(b_0, b_1, \ldots, b_n)$ of Betti numbers $b_i = \dim H_i(M)$ of orientable connected (simplicial) $n$-dimensional manifolds $M^n$ due to the Poincaré duality:

\begin{equation}
    b_i(M^n) = b_{n-i}(M^n), \quad i = 0, \ldots, n.
\end{equation}

This similarity between $f$-vectors of simplicial polytopes and Betti numbers of orientable manifolds was pointed out by Klee in [Kl]. Moreover, it turns out that the “combinatorial” duality (4) can be interpreted in terms of the “topological” duality (5). Given an $n$-dimensional simplicial polytope $P$ (or, more generally, a complete simplicial fan $\Sigma$), one may construct the toric variety $M_P$ (or $M_\Sigma$) of (real) dimension $2n$ (see [Da], [Fu]). This variety is not necessarily a manifold, however its homology (with rational coefficients in general) satisfies the Poincaré duality, and its even Betti numbers equal the components of $h$-vector of $P$: $b_{2i}(M_P) = h_i(P)$. This gives a “topological” proof of the Dehn–Sommerville equations (4).

The dual (or polar) to any simplicial polytope $P$ is a simple polytope, which we denote $P^*$. Given an $n$-dimensional simple polytope $P^*$ with $m$ facets, Davis and Januszkiewicz defined in [Dj] a manifold $Z_{P^*}$ of dimension $m + n$ acted on by the torus $T^m$. This manifold depends only on the combinatorial type (i.e., the face lattice) of a polytope, and the orbit space for the $T^m$-action is combinatorially $P^*$. The manifolds $Z_{P^*}$ establish the bridge between topology of manifolds and combinatorics of polytopes (or more general objects, such as simplicial spheres). Various connections between topology of $Z_{P^*}$ and toric geometry, symplectic geometry, subspace arrangements, and combinatorial theory of $f$-vectors were studied in [BP1], [BP2], [BP3]. Any smooth toric variety (or Hamiltonian $T^n$-manifold) defined by a simple polytope of combinatorial type $P^*$ is the quotient of $Z_{P^*}$.
by a freely acting toric subgroup $T^{m-n} \subset T^m$. This is also true for topological analogues of smooth toric varieties, which we call \textit{quasitoric manifolds}, also introduced in [13]. For more information about quasitoric manifolds and their topology see [8, 9, 10, 11, 12, 13, 14]. We showed in [15, 16] that the cohomology algebra of $Z_{P^*}$ with coefficients in any field $k$ is isomorphic to the Tor-algebra Tor$_{k[u_1, \ldots, u_m]}(k(P), k)$, where $P$ is polar to $P^*$, $m = f_0$ is the number of vertices of $P$, and $k(P)$ is the \textit{Stanley–Reisner face ring} of $P$. The Koszul resolution gives then a very simple model $H[k(P) \otimes \Lambda[u_1, \ldots, u_m], d]$ for the cohomology algebra of $Z_{P^*}$. In particular, the cohomology of $Z_{P^*}$ acquires a \textit{bigraded} algebra structure, and the bigraded Poincaré duality implies the Dehn–Sommerville equations (4).

The boundary complex of a simplicial polytope is a simplicial sphere. However, now it is well known that not any triangulation of a topological sphere can be obtained in such way. First examples of such “non-polytopal” spheres were found by Grünbaum, and the smallest non-polytopal sphere can be obtained in such way. First examples of such “non-polytopal” spheres were found by Grünbaum, and the smallest non-polytopal sphere can be obtained in such way. First examples of such “non-polytopal” spheres were found by Grünbaum, and the smallest non-polytopal sphere can be obtained in such way. First examples of such “non-polytopal” spheres were found by Grünbaum, and the smallest non-polytopal sphere can be obtained in such way. First examples of such “non-polytopal” spheres were found by Grünbaum, and the smallest non-polytopal sphere can be obtained in such way.

We endow the cone cone($K$) with a structure of cubical complex and embed it into the boundary complex of $m$-dimensional unit cube $I^m$ (here $m = f_0(K)$ is the number of vertices of $K$). Then we view $I^m$ as the orbit space for the diagonal action of the torus $T^m = S^1 \times \cdots \times S^1$ on the unit poly-disk $(D^2)^m$ in $m$-dimensional complex space $\mathbb{C}^m$. Hence, the above cubical embedding cone($K$) $\rightarrow I^m$ is covered by a $T^m$-equivariant embedding $Z_K \rightarrow (D^2)^m$, where $Z_K$ is a cellular complex canonically decomposed into the union of blocks $(D^2)^n \times T^{m-n}$ with $n = \dim K + 1$. We call this $Z_K$ the \textit{moment-angle complex associated to the simplicial complex} $K$. If $K$ is a simplicial sphere, then $Z_K$ is a manifold. (If, moreover, $K$ is polytopal, i.e. $K = \partial P$, then $Z_K$ coincides with the above described manifold $Z_{P^*}$, where $P^*$ is polar to $P$.) However, in general, $Z_K$ has more complicated structure. In [15, 16] we showed that our moment-angle complex $Z_K$ is homotopy equivalent to the complement of complex coordinate subspace arrangement defined by $K$, and its cohomology algebra is isomorphic, as in the polytopal case, to the Tor-algebra Tor$_{k[v_1, \ldots, v_m]}(k(K), k)$. We note that the Betti numbers of the complement of a \textit{real} coordinate subspace arrangement were calculated in terms of resolution of the Stanley–Reisner ring in [17].

In the case when $K$ is a simplicial sphere, the bigraded Poincaré duality in the cohomology algebra of $Z_K$ gives a “topological proof” of the Dehn–Sommerville equations (4) for simplicial spheres. In this paper we construct a cellular chain complex that calculates the homology of $Z_K$ and gives a very transparent characterization of homology classes of $Z_K$ (and of the complement of a coordinate subspace arrangement). This chain complex is dual to a certain cochain subcomplex of the Koszul complex $[k(K) \otimes \Lambda[u_1, \ldots, u_m], d]$ for the Tor-algebra Tor$_{k[v_1, \ldots, v_m]}(k(K), k)$. 
From the topological viewpoint it is very interesting to study the case when $K$ is a simplicial manifold. The moment-angle complex $Z_K$ here is not a manifold, however, its singularities are tractable. It turns out that $Z_K$ contains the product $|\text{cone}(K)| \times T^m$ of (polyhedron corresponding to) the cone over $K$ and torus $T^m$. The closure $W_K = \overline{Z_K \setminus |\text{cone}(K)| \times T^m}$ of the complement of this singular part is a manifold with boundary $|K| \times T^m$. This $W_K$ is homotopically equivalent to another moment-angle complex $W_K$ which covers equivariantly the restriction to $K \subset \text{cone}(K)$ of the cubical embedding $\text{cone}(K) \to I^m$. We construct an appropriate cellular decomposition of $W_K$, which allows to calculate the homology efficiently. The Poincaré duality for manifolds with boundary gives in this case

$$H_i(W_K) \cong H^{m+n-i}(W_K, |K| \times T^m), \quad i = 0, \ldots, m + n.$$  

This duality again regards the bigraded structure. As a consequence, we obtain a topological proof of the Dehn–Sommerville equations for simplicial complexes in the following nice form:

$$(6) \quad h_{n-i} - h_i = (-1)^i \left( \chi(K^{n-1}) - \chi(S^{n-1}) \right) \binom{n}{i}, \quad i = 0, 1, \ldots, n.$$  

where $\dim K^{n-1} = n - 1$, and $\chi(\cdot)$ denotes the Euler number. We have $\chi(K^{n-1}) = f_0 - f_1 + \ldots + (-1)^{n-1}f_{n-1} = 1 + (-1)^{n-1}h_n$ and $\chi(S^{n-1}) = 1 + (-1)^n$. This generalizes equations to the case of arbitrary simplicial manifolds. In particular, if $K$ is an odd-dimensional simplicial manifold, one has $h_{n-i} - h_i$. In [Pa, (7.11)] Pachner proved by means of his bistellar flip theorem that the value $h_{n-i} - h_i$ is a topological invariant of a PL-manifold (i.e. is independent on a PL-triangulation). The formula (6) calculates this invariant exactly for any simplicial (not necessarily PL) manifold.

Our moment-angle complexes enable to reformulate many combinatorial statements and hypotheses concerning $f$-vectors in topological terms. We have already mentioned the Dehn–Sommerville relations for both simplicial spheres and simplicial manifolds, however this is only the first and the simplest example. The most intriguing open problem here is the so-called $g$-theorem (or McMullen’s inequalities) for simplicial spheres [St3]. It includes the Generalized Lower Bound hypothesis for simplicial spheres, which asserts the monotonicity property for the $h$-vector:

$$(7) \quad h_0 \leq h_1 \leq h_2 \leq \cdots \leq h_{\lfloor \frac{n}{2} \rfloor}.$$  

For polytopal spheres $g$-theorem was proved by Stanley [St1] (necessity) and Billera, Lee [BiLe] (sufficiency) in 1980. The first inequality $h_0 \leq h_1$ is equivalent to $1 \leq m - n$, which is obvious. The second ($h_1 \leq h_2, n \geq 4$) is equivalent to the lower bound $f_1 \leq nf_0 - \binom{n+1}{2}$ for the number of edges, which is also known for simplicial spheres. The next inequality $h_2 \leq h_3$ is still open (for simplicial spheres). For the history of $g$-theorem and related questions see [St2], [St3], [Zi]. As we mentioned in [BP2], the inequality $h_1 \leq h_2$ is equivalent to the upper bound $h_3(Z_K) \leq \binom{m-n}{2}$ for the third Betti
number of manifold $Z_K$ (we note that $Z_K$ is always 2-connected). Other inequalities from \([8]\) also acquire such topological interpretation, however, in general case in terms of \textit{bigraded} Betti numbers of $Z_K$. We hope that such inequalities can be deduced from the equivariant topology of $Z_K$.

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2. Cubical complexes determined by a simplicial complex

Let $K^{n-1}$ be an $(n - 1)$-dimensional simplicial complex on the vertex set $[m] = \{1, \ldots, m\}$ (hence, any simplex of $K$ has at most $n$ vertices). As usual, we view $K$ as a set of subsets of $[m]$, that is, $K \subset 2^{[m]}$. We would assume that $K$ is not the $m$-simplex $\Delta^m = 2^{[m]}$, so $n < m$. If $I = \{i_1, \ldots, i_k\} \subset [m]$ is a simplex of $K$, then we would write $I \in K$. By definition, the barycentric subdivision of $K$ is the simplicial complex $bs(K)$ whose simplices are chains $I_1 \subset I_2 \subset \cdots \subset I_p$ of embedded simplices of $K$. In particular, the vertices of $bs(K)$ are in one to one correspondence with simplices of $K$ of all dimensions. We denote the geometric realization of $K$ (as a polyhedron) by $|K|$. Let $I^m$ be the standard unit $m$-dimensional cube in $\mathbb{R}^m$:

$$I^m = \{(y_1, \ldots, y_m) \in \mathbb{R}^m : 0 \leq y_i \leq 1, i = 1, \ldots, m\}.$$ 

Every face of $I^m$ has the form

$$F_{I \subset J} = \{(y_1, \ldots, y_m) \in I^m : y_i = 0 \text{ for } i \in I, y_j = 1 \text{ for } j \notin J\},$$

were $I \subset J$ are two (possibly empty) subsets of $[m]$. Now assign to each subset $I = \{i_1, \ldots, i_k\} \subset [m]$ the vertex $v_I = F_{I \subset I}$ of the cube $I^m$. One has $v_I = (\varepsilon_1, \ldots, \varepsilon_m)$, where $\varepsilon_i = 0$ if $i \in I$ and $\varepsilon_i = 1$ otherwise. Viewing $I$ as a vertex of the barycentric subdivision $bs(\Delta^m)$ of an $m$-simplex, we see that the correspondence $I \mapsto v_I$ extends to a piecewise linear embedding $i_c$ of the polyhedron $|bs(\Delta^m)|$ into the (boundary complex of) unit cube $I^m$. Under this embedding, the vertices of $\Delta^m$ are mapped to the vertices $(1, \ldots, 1, 0, 1, \ldots, 1)$ of the cube $I^m$, while the point in the interior of $|\Delta^m|$ (viewed as a vertex of $bs(\Delta^m)$) is mapped to the vertex $(0, \ldots, 0)$ of $I^m$. Hence, the whole image of $|bs(\Delta^m)|$ is the set of $m$ facets of $I^m$ meeting at the vertex $(0, \ldots, 0)$. Moreover, given a pair $I, J$ of non-empty subsets of $[m]$ such that $I \subset J$, all simplices of $bs(\Delta^m)$ of the form $I = I_1 \subset I_2 \subset \cdots \subset I_k = J$ are mapped to the same face $F_{I \subset J}$ of $I^m$ (see \([3]\)).

Our simplicial complex $K^{n-1}$ can be viewed as a subcomplex in $\Delta^m$. Hence, the above constructed map $i_c : |bs(\Delta^m)| \to I^m$ embeds $|bs(K)|$ piecewise linearly into the boundary complex of $I^m$. The image $i_c(|bs(K)|)$ is a certain $(n - 1)$-dimensional cubical complex, which we denote $\text{cub}(K)$. Thus, we have proved the following statement.

**Proposition 2.1.** There is a piecewise linear embedding $i_c$ of $|bs(\Delta^m)|$ into the boundary complex of $I^m$ such that for any simplicial complex $K \subset \Delta^m$
on $m$ vertices the image $i_c(|bs(K)|) =: cub(K)$ is the union of faces corresponding to all pairs $I \subset J$ of embedded simplices of $K$. □

Remark. Cubes of the cubical subdivision $cub(K)$ of the polyhedron $|K|$ are formed by simplices of the barycentric subdivision $bs(K)$. This cubical subdivision was employed in some previous researches for different purposes (see, e.g., [DJ, p. 434]). The above cubical embedding $i_c : cub(K) \to I^m$ was used previously in [SS] to study which cubical complexes can be embedded into the standard cubical lattice.

The map $i_c : |bs(\Delta^m)| \to I^m$ can be extended to a piecewise linear map $cone(i_c) : |cone(bs(\Delta^m))| \to I^m$ by taking the vertex of the cone to the vertex $(1, \ldots, 1)$ of the cube $I^m$. (Note that the cone over the barycentric subdivision of a $k$-simplex is identified with the standard triangulation of a $(k+1)$-cube.) Now the image of $|cone(bs(\Delta^m))|$ is the whole $I^m$, so $cone(i_c)$ is a PL homeomorphism. The image $i_c(|cone(bs(K))|) \subset I^m$ is another, this time $n$-dimensional, cubical subcomplex of $I^m$, which we denote $cc(K)$. This cubical complex is explicitly described by the following proposition.

**Proposition 2.2.** For any simplicial complex $K$ on $m$ vertices there is a piecewise linear embedding of the polyhedron $|cone(bs(K))|$ into the boundary complex of $I^m$ such that its image $cc(K)$ is the union of faces

$$F_J = \{(y_1, \ldots, y_m) \in I^m : y_j = 1 \text{ for } j \notin J\} \subset I^m$$

and all their subfaces, where $J$ ranges over all simplices of $K$. □

According to (8), $F_J = F_{\emptyset \subset J}$. Hence, any subface of $F_J$ is $F_{I \subset J}$ for some (possibly empty) $I \subset J$. It follows that

$$cub(K) = \bigcup_{J \in K} F_I \subset J, \quad cc(K) = \bigcup_{J \in K} F_I \subset J.$$

Remark. Viewed as a topological space, $|cub(K)|$ is homeomorphic to $|K|$, while $|cc(K)|$ is homeomorphic to $|cone(K)|$. Viewing $cone(K)$ as a simplicial complex, one may construct the cubical complex $cub(cone(K))$, which is also homeomorphic to $|cone(K)|$. However, as cubical complexes, $cc(K)$ and $cub(cone(K))$ differ.

The cubical complex $cc(K)$ was introduced in [BP2] and studied in [BP3], [BP4] in connection with simple (and simplicial) polytopes and subspace arrangements.

**Example 2.3.** The cubical complex $cub(K)$ for $K$ a disjoint union of 3 vertices and the boundary complex of a 2-simplex is shown on Figure 1 a) and b) respectively. The corresponding cubical complexes $cc(K)$ are indicated on Figure 2 a) and b).
Let \((D^2)^m\) denote the unit poly-disk in the complex space:
\[
(D^2)^m = \{(z_1, \ldots, z_m) \in \mathbb{C}^m : |z_i| \leq 1, \quad i = 1, \ldots, m\}.
\]
The unit cube \(I^m\) can be viewed as the orbit space for the standard action of \(m\)-dimensional torus \(T^m\) on \((D^2)^m\) by coordinatewise rotations. The orbit map \(\rho : (D^2)^m \to I^m\) can be given by \((z_1, \ldots, z_m) \mapsto (|z_1|^2, \ldots, |z_m|^2)\). For each face \(F_{I \subset J}\) of \(I^m\) (see (8)) define
\[
B_{I \subset J} := \rho^{-1}(F_{I \subset J}) = \{(z_1, \ldots, z_m) \in (D^2)^m : z_i = 0 \text{ for } i \in I, \quad |z_j| = 1 \text{ for } j \notin J\}.
\]
It follows that if \(#I = i, \#J = j\), then \(B_{I \subset J} \cong (D^2)^{j-i} \times T^{m-j}\), where disk factors \(D^2 \subset (D^2)^{j-i}\) correspond to elements from \(J \setminus I\), while circle factors \(S^1 \subset T^{m-j}\) correspond to elements from \([m] \setminus J\). Introducing the polar coordinate system in \((D^2)^m\), we see that \(B_{I \subset J}\) is parametrized by \((j - i)\) radial (or moment) and \((m - i)\) angle coordinates. Here we come to the following definition.
Figure 3. Cellular decomposition of $D^2$.

**Definition 3.1.** Let $C$ be a cubical subcomplex of $I^m$. The *moment-angle complex* $\text{ma}(C)$ corresponding to $C$ is the $T^m$-invariant decomposition of $\rho^{-1}(|C|)$ to “moment-angle” blocks $B_{I \cup J}$ (see (10)) corresponding to faces $F_{I \cup J} \subset |C| \subset I^m$. Hence, $\text{ma}(C)$ is defined from the commutative diagram

$$
\begin{array}{ccc}
\text{ma}(C) & \longrightarrow & (D^2)^m \\
\downarrow & & \downarrow \rho \\
|C| & \longrightarrow & I^m
\end{array}
$$

It follows that the torus $T^m$ acts on $\text{ma}(C)$ with orbit space $|C|$.

The moment-angle complexes corresponding to the introduced above cubical complexes $\text{cub}(K)$ and $\text{cc}(K)$ (see propositions 2.1 and 2.2) will be denoted $W_K$ and $Z_K$ respectively. Thus, we have

$$
\begin{array}{ccc}
W_K & \longrightarrow & (D^2)^m \\
\downarrow & & \downarrow \rho \\
|\text{cub}(K)| & \longrightarrow & I^m
\end{array}
\quad \quad \quad
\begin{array}{ccc}
Z_K & \longrightarrow & (D^2)^m \\
\downarrow & & \downarrow \rho \\
|\text{cc}(K)| & \longrightarrow & I^m
\end{array}
$$

(11)

where horizontal arrows are embeddings, while vertical ones are orbit maps for certain $T^m$-actions. It follows that $\dim Z_K = m + n$ and $\dim W_K = m + n - 1$.

Let us consider the cellular decomposition of $D^2$ with one 2-cell $D$, two 1-cells $I$, $T$, and two 0-cells 0, 1 (see Figure 3). It defines a $T^m$-invariant cellular decomposition of the poly-disk $(D^2)^m$ with $5^m$ cells. Each cell of this decomposition is the product of cells of 5 different types: $D_i$, $I_i$, $0_i$, $T_i$, and $1_i$, $i = 1, \ldots, m$. We will encode cells of $(D^2)^m$ by “words” of type $D_I I_J 0_L T_P 1_Q$, where $I, J, L, P, Q$ are pairwise disjoint subsets of $[m]$ such that $I \cup J \cup L \cup P \cup Q = [m]$. Sometimes we would drop the last factor $1_Q$, so in our notations $D_I I_J 0_L T_P = D_I I_J 0_L T_P 1_{[m] \setminus (I \cup J \cup L \cup P)}$. It follows that the closure of $D_I I_J 0_L T_P 1_Q$ is homeomorphic to the product of $\# I$ disks, $\# J$ segments, and $\# P$ circles. The constructed cellular decomposition of $(D^2)^m$ allows to identify moment-angle complexes as certain cellular subcomplexes in $(D^2)^m$. 
Lemma 3.2. For any cubical subcomplex $C$ of $I^m$ the corresponding moment-angle complex $\text{ma}(C)$ is a $T^m$-invariant cellular subcomplex of $(D^2)^m$.

Proof. Since $\text{ma}(C)$ is a union of “moment-angle” blocks $B_{I \subseteq J}$ (see (11)), and each $B_{I \subseteq J}$ is obviously $T^m$-invariant, the whole $\text{ma}(C)$ is also $T^m$-invariant. In order to show that $\text{ma}(C)$ is a cellular subcomplex of $(D^2)^m$ (with respect to the above constructed cellular decomposition) we just mention that $B_{I \subseteq J}$ is the closure of cell $D_J \setminus I \emptyset T_{[m] \setminus J} \emptyset$. □

4. Cohomology of $Z_K$, subspace arrangements, and numbers of faces of $K$

Here we study the moment-angle complex $Z_K$ corresponding to the cubical complex $cc(K) \subset I^m$ (see (11)). Remember that $K$ is an $(n - 1)$-dimensional simplicial complex on $m$ vertices, and $cc(K)$ topologically is the cone over $K$. By definition, $Z_K$ is a union of certain moment-angle blocks $B_{I \subseteq J} \subset (D^2)^m$ with $I = \emptyset$ (see Proposition 2.2). In analogy with (11), we put

$$B_J := B_{\emptyset \subseteq J} = \rho^{-1}(F_J) = \{(z_1, \ldots, z_m) \in (D^2)^m : |z_j| = 1 \text{ for } j \notin J\}.$$ 

Hence, $Z_K = \bigcup_{J \in K} B_J$, where $B_J \cong (D^2)^{j} \times T^{m-j}$, $j = \# J$. This remark allows to simplify the cellular decomposition constructed in the previous section for $\text{ma}(C)$ in the case when $\text{ma}(C) = Z_K$. To do this we replace the union of cells $0, I, D$ (see Figure 3) by one 2-dimensional cell. To simplify notations we denote this 2-cell by $D$ throughout this section. The resulting $T^m$-invariant cellular decomposition of $(D^2)^m$ now has 3$^m$ cells, each of which is the product of 3 different types of cells: $D_t$, $T_i$, and $1_i$, $i = 1, \ldots, m$.

In this section we encode these cells of $(D^2)^m$ as $D_I T_P 1_Q$, where $I, P, Q$ are pairwise disjoint subsets of $[m]$ such that $I \cup P \cup Q = [m]$. Usually we would denote the cell $D_I T_P 1_Q$ just by $D_I T_P$, so $D_I T_P = D_I T_P 1_{[m] \setminus I \cup P}$. Since $B_J = B_{\emptyset \subseteq J}$ is the closure of cell $D_J T_{[m] \setminus J} \emptyset$, the moment-angle complex $Z_K$ is a $T^m$-invariant cellular subcomplex of $(D^2)^m$ (with respect to the new 3$^m$-cell decomposition). The complex $Z_K$ consists of all cells $D_I T_P \subset (D^2)^m$ such that $I$ is a simplex of $K$.

Remark. Note that for general $C$ the moment-angle complex $\text{ma}(C)$ is not a cellular subcomplex for the 3$^m$-cell decomposition of $(D^2)^m$.

The cohomology ring of $Z_K$ was described in [BP2], [BP3] (in the case when $K$ is a polytopal sphere) and in [BP4] (for general $K$). Before going further, we review some results of these papers.

Throughout the rest of this paper we work over some field $k$, referred to as the ground field. Let $k[v_1, \ldots, v_m]$ be the polynomial algebra, and $\Lambda[u_1, \ldots, u_m]$ the exterior algebra over $k$ on $m$ generators. We make both algebras graded by putting $\deg(v_i) = 2$, $\deg(u_i) = 1$.

Definition 4.1. The face ring (or the Stanley–Reisner ring) $k(K)$ of simplicial complex $K$ is the quotient ring $k[v_1, \ldots, v_m]/I$, where the ideal $I$ is
generated by all square-free monomials \( v_{i_1} \cdots v_{i_s}, 1 \leq i_1 < \cdots < i_s \leq m, \) such that \( \{i_1, \ldots, i_s\} \) is not a simplex of \( K. \)

For any subset \( I = \{i_1, \ldots, i_k\} \subseteq [m] \) denote by \( L_I \) the coordinate plane in \( \mathbb{C}^m \) consisting of points whose \( i_1, \ldots, i_k \) coordinates vanish:

\[
L_I = \{ (z_1, \ldots, z_m) \in \mathbb{C}^m : z_{i_1} = \cdots = z_{i_k} = 0 \}.
\]

Each simplicial complex \( K \) on \( m \) vertices defines a complex coordinate subspace arrangement \( \mathcal{A}(K). \) The latter is the set of all planes \( L_I \) such that \( I \) is not a simplex of \( K. \)

\[
\mathcal{A}(K) = \{ L_I : I \notin K \}.
\]

Define the support of \( \mathcal{A}(K) \) as \( |\mathcal{A}(K)| = \bigcup_{I \notin K} L_I \subset \mathbb{C}^m \) and the complement \( U(K) = \mathbb{C}^m \setminus |\mathcal{A}(K)|, \) that is

\[
U(K) = \mathbb{C}^m \setminus \bigcup_{I \notin K} L_I.
\]

It can be easily seen that the complement of any coordinate subspace arrangement in \( \mathbb{C}^m \) (i.e., the complement of any set of planes \((12)\)) is \( U(K) \) for some \( K. \) Note that \( U(K) \) is invariant with respect to the standard \( T_m \)-action on \( \mathbb{C}^m. \) The following lemma establishes the connection between moment-angle complexes and complements of coordinate subspace arrangements.

**Lemma 4.2.** The complement \( U(K) \) is \( T_m \)-equivariantly homotopy equivalent to the moment-angle complex \( Z_K. \)

**Proof.** See \cite[Lemma 2.13]{BP4} \qed

The next theorem describes the cohomology algebra of \( Z_K. \)

**Theorem 4.3.** The following isomorphisms of graded algebras holds:

\[
H^*(Z_K) \cong \text{Tor}_{k[v_1, \ldots, v_m]}(k(K), k) \cong H[k(K) \otimes \Lambda[u_1, \ldots, u_m], d],
\]

where \( k(K) = k[v_1, \ldots, v_m]/I \) is the face ring, and the differential \( d \) is defined by \( d(v_i) = 0, d(u_i) = v_i, i = 1, \ldots, m. \)

**Proof.** See \cite[theorems 3.2 and 3.3]{BP4} \qed

Due to Lemma \( 12, \) isomorphism \((14)\) also holds for the cohomology algebra of the complement of a coordinate subspace arrangement in \( \mathbb{C}^m. \) The first isomorphism of \((14)\) is proved by applying the Eilenberg–Moore spectral sequence to some \( T^m \)-bundles. The second isomorphism follows from the Koszul complex for the \( k[v_1, \ldots, v_m] \)-module \( k(K). \)

**Remark.** As we mentioned in the introduction, the Betti numbers of the complement of a real coordinate subspace arrangement were calculated in terms of resolution of the Stanley–Reisner ring in \cite{GPW}. The latter paper also contains the formulation of the multiplicative isomorphism \((14)\) for complex coordinate subspace arrangements (see \cite[Thm. 3.6]{GPW}) with a reference to yet unpublished paper by Babson and Chan. We note also, that,
as it was observed in [GPW], there is no isomorphism between the cohomology algebra of a real coordinate subspace arrangement and the corresponding Stanley–Reisner Tor-algebra.

The Tor-algebra from [L4] is naturally a bigraded algebra with bideg$(v_i) = (0,2)$, bideg$(u_i) = (-1,2)$, and differential $d$ adding $(1,0)$ to bidegree. Since the differential does not change the second grading, the whole algebra is decomposed into the sum of differential algebras consisting of elements with fixed second degree. Below we deduce some important consequences of this bigraded structure.

**Remark.** Note that according to our agreement the first degree in the Tor-algebra is non-positive. This corresponds to numerating the terms of Koszul $k[v_1, \ldots, v_m]$-free resolution of $k$ by non-positive integers. In such notations the Koszul complex $[k(K) \otimes \Lambda[u_1, \ldots, u_m], d]$ becomes a cochain complex, and $\text{Tor}^*[v_1, \ldots, v_m](k(K), k)$ is its cohomology, not the homology as usually regarded. This is the standard trick used for applying the Eilenberg–Moore spectral sequence, see [Sm]. It also explains why we write $\text{Tor}^*[v_1, \ldots, v_m](k(K), k)$ instead of usual $\text{Tor}^*_k[v_1, \ldots, v_m](k(K), k)$.

Following [BP2], define the subcomplex $C^*(K)$ of the cochain complex $[k(K) \otimes \Lambda[u_1, \ldots, u_m], d]$ (see [L4]) as follows. As a $k$-module, $C^*(K) = \bigoplus_{q=0}^n C^{-q}(K)$, where $C^{-q}(K)$ is generated by monomials $u_{j_1} \cdots u_{j_q}$ and $v_{i_1} \cdots v_{i_p} u_{j_1} \cdots u_{j_q}$ such that $\{i_1, \ldots, i_p\}$ is a simplex of $K$ and $\{i_1, \ldots, i_p\} \cap \{j_1, \ldots, j_q\} = \emptyset$. Since $d(u_i) = v_i$, we have $d(C^{-q}(K)) \subset C^{-q+1}(K)$ and, therefore, $C^*(K)$ is a cochain subcomplex. Moreover, $C^*(K)$ inherits the bigraded module structure from $k(K) \otimes \Lambda[u_1, \ldots, u_m]$, with differential $d$ adding $(1,0)$ to bidegree. Hence, we have the additive inclusion (i.e., the monomorphism of bigraded modules) $i : C^*(K) \hookrightarrow k(K) \otimes \Lambda[u_1, \ldots, u_m]$. Finally, $C^*(K)$ can be viewed as an algebra in obvious way, however this time this is not a subalgebra of $k(K) \otimes \Lambda[u_1, \ldots, u_m]$ (since, for instance, $v_i^2 = 0$ in $C^*(K)$ but not in $k(K) \otimes \Lambda[u_1, \ldots, u_m]$). However, we have the multiplicative projection (i.e., the epimorphism of bigraded algebras) $j : k(K) \otimes \Lambda[u_1, \ldots, u_m] \to C^*(K)$. The additive inclusion $i$ and the multiplicative projection $j$ obviously satisfy $j \cdot i = id$.

**Lemma 4.4.** Cochain complexes $[k(K) \otimes \Lambda[u_1, \ldots, u_m], d]$ and $[C^*(K), d]$ have same cohomology. Hence, the following isomorphism of bigraded $k$-modules holds:

$$H[C^*(K), d] \cong \text{Tor}^*_k[v_1, \ldots, v_m](k(K), k).$$

**Proof.** See [BP2, Lemma 5.3].

In the sequel we denote (square-free) monomials $v_{i_1} \cdots v_{i_p} u_{j_1} \cdots u_{j_q} \in k(K) \otimes \Lambda[u_1, \ldots, u_m]$ by $v_I u_J$, where $I = \{i_1, \ldots, i_p\}$, $J = \{j_1, \ldots, j_q\}$ are multiindices.

Now we recall our cellular decomposition of $Z_K$, whose cells are $D IT_J$, where $I, J \subset [m]$, $I$ is a simplex of $K$, and $I \cap J = \emptyset$. Let $C_*(Z_K)$ and $C^*(Z_K)$
In the sequel we would not distinguish cochain complexes.
Both complexes (or differential algebras) \( C^* \) and \( C^* \) have same cohomology \( H^*(Z_K) \).
Cochain complex \( C^*(Z_K) \) has basis consisting of elements \( (D_i T_j)^* \) dual to \( D_i T_j \in C_*(Z_K) \) (the latter is viewed as a cellular chain).
Note that the cochain algebra \( C^*(Z_K) \) is multiplicatively generated by the elements \( T_i, D_i, i, j = 1, \ldots, m \), (of dimension 1 and 2 respectively), while \( C^*(K) \) is multiplicatively generated by \( u_i, v_i, i, j = 1, \ldots, m \). The following theorem shows that these two algebras are the same.

**Theorem 4.5.** The correspondence \( v_j u_j \mapsto (D_i T_j)^* \) establishes a canonical isomorphism of differential graded algebras \( C^*(K) \) and \( C^*(Z_K) \).

**Proof.** It follows directly from the definitions of \( C^*(K) \) and \( C^*(Z_K) \) that the constructed map is an isomorphism of graded algebras. So, it remains to prove that it commutes with differentials. Let \( d, d_c \) and \( \partial_c \) denote the differentials in \( C^*(K) \), \( C^*(Z_K) \) and \( C_*(Z_K) \) respectively. Since \( d(v_i) = 0, \)
\( d(u_i) = v_i \), we need to show that \( d_c(D_i^*) = 0, d_c(T_i^*) = D_i^* \). We have \( \partial_c(D_i) = T_i, \partial_c(T_i) = 0 \). Since any 2-cell of \( Z_K \) is either \( D_j \) or \( T_{jk}, k \neq j \), it follows that
\[
(d_c T_i^*, D_j) = (T_i^*, \partial_c D_j) = (T_i^*, T_j) = \delta_{ij}, \quad (d_c T_i^*, T_{jk}) = (T_i^*, \partial_c T_{jk}) = 0,
\]
where \( \delta_{ij} = 1 \) if \( i = j \) and \( \delta_{ij} = 0 \) otherwise. Hence, \( d(T_i^*) = D_i^* \). Further, since any 3-cell of \( Z_K \) is either \( D_j T_k \) or \( T_{jk}, j \neq j \), it follows that
\[
(d_c D_i^*, D_j T_k) = (D_i^*, \partial_c (D_j T_k)) = (D_i^*, T_{jk}) = 0,
\]
\[
(d_c D_i^*, T_{j,k,j,k}) = (D_i^*, \partial_c T_{j,k,j,k}) = 0.
\]
Hence, \( d_c(D_i^*) = 0 \). \( \Box \)

In the sequel we would not distinguish cochain complexes \( C^*(K) \) and \( C^*(Z_K) \) and their elements \( u_i \) and \( T_i^* \), \( v_i \) and \( D_i^* \). The above theorem provides two methods for calculating the (co)homology of \( Z_K \): either by means of the differential (bi)graded algebra \( \mathcal{C}^*(K), d \), where \( \mathcal{C}^*(K) \subset k(K) \otimes \Lambda[u_1, \ldots, u_m] \) (as modules), or using the cellular chain complex \( [C_*(Z_K), \partial_c] \).

Now we recall that the algebra \( \mathcal{C}^*(K), d \) is bigraded. Theorem 4.5 shows that the cellular chain complex \( [C_*(Z_K), \partial_c] \) can be also made bigraded by setting
\[
bideg(D_i) = (0, 2), \quad bideg(T_i) = (-1, 2), \quad bideg(1) = (0, 0).\]
The differential \( \partial_c \) adds \((-1, 0)\) to bidegree, and the cellular homology of \( Z_K \) also acquires a bigraded structure. Let us assume now that the ground field \( k \) is of zero characteristic (e.g., \( k = \mathbb{Q} \) is the field of rational numbers). Define the bigraded Betti numbers
\[
b_{q,2p}(Z_K) = \dim H_{q, 2p}[C_*(Z_K), \partial_c], \quad q, p = 0, \ldots, m.
\]
Theorem 4.5 and Lemma 4.4 show that
\[
b_{q,2p}(Z_K) = \dim \text{Tor}^{-q}_{k[v_1, \ldots, v_m]}(k(K), k)
\]
We use the cochain complex $H^*[k(K) \otimes \Lambda[u_1, \ldots, u_m], d]$. For the ordinary Betti numbers $b_k(Z_K)$ holds

$$b_k(Z_K) = \sum_{-q+2p=k} b_{-q,2p}(Z_K), \quad k = 0, \ldots, m+n.$$ 

Below we describe some basic properties of bigraded Betti numbers. Let $I$ be a $(n-1)$-dimensional simplicial complex with $m = f_0$ vertices and $f_1$ edges, and let $Z_K$ be the corresponding moment-angle complex, $\dim Z_K = m+n$. Then

(a) $b_{0,0}(Z_K) = b_0(Z_K) = 1$, $b_{0,2p}(Z_K) = 0$ if $p > 0$;

(b) $b_{-q,2p} = 0$ if $p > m$ or $q > p$;

(c) $b_1(Z_K) = b_2(Z_K) = 0$;

(d) $b_3(Z_K) = b_{-1,4}(Z_K) = (\frac{1}{2}) - f_1$;

(e) $b_{-q,2p}(Z_K) = 0$ if $q \geq p > 0$ or $p > 0$;

(f) $b_{m+n}(Z_K) = b_{-(m-n),2m}(Z_K)$.

**Proof.** We use the cochain complex $C^*(K) \subset k(K) \otimes \Lambda[u_1, \ldots, u_m]$ for calculations. The module $C^*(K)$ has basis consisting of monomials $v_I u_J$ with $I \in K$ and $I \cap J = \emptyset$. Since $\text{bideg} v_i = (0,2)$, $\text{bideg} u_j = (-1,2)$, the bigraded component $C^{-q,2p}(K)$ is generated by monomials $v_I u_J$ with $\# I = p-q$ and $\# J = q$. In particular, $C^{-q,2p}(K) = 0$ if $p > m$ or $q > p$, whence the assertion (b) follows. To prove (a) we mention that $C^0,0(K)$ is generated by 1, while $v_I \in C^0,2p(K), p > 0$, is a coboundary, hence, $H^{-q,2p}(K) = 0, p > 0$.

Now we are going to prove the assertion (e). Since any $v_I u_J \in C^{-q,2p}(K)$ has $I \in K$, and any simplex of $K$ is at most $(n-1)$-dimensional, it follows that $C^{-q,2p}(K) = 0$ for $q > p$. It follows from (b) that $b_{-q,2p}(Z_K) = 0$ for $q > p$, so it remains to prove that $b_{-q,2p}(Z_K) = 0$ for $q > 0$. The module $C^{-q,2p}(K)$ is generated by monomials $u_J$, $\# J = q$. Since $d(u_i) = v_i$, it follows easily that there are no cocycles in $C^{-q,2p}(K)$. Hence, $H^{-q,2p}(Z_K) = 0$.

To prove (c) we mention that $H^1(Z_K) = H^1,2(K)$ and $H^2(Z_K) = H^{-2,4}(Z_K)$ (this follows from (a) and (b)). Hence, the assertion (c) follows from (e).

As for (d), it follows from (a), (b) and (e) that $H^3(Z_K) = H^{-1,4}(Z_K)$. The module $C^{-1,4}(K)$ is generated by monomials $v_i u_j, i \neq j$. We have $d(v_i u_j) = v_i v_j$ and $d(u_i u_j) = v_i u_j - v_j u_i$. It follows that $v_i u_j$ is a cocycle if and only if $\{i, j\}$ is not a 1-simplex in $K$; in this case two cocycles $v_i u_j$ and $v_j u_i$ are cohomological. The assertion (d) now follows easily.

The remaining assertion (f) follows from the fact that the monomial $u_I v_J \in C^*(K)$ of maximal total degree $m+n$ necessarily has $\# I + \# J = m$, $\# J = n$, $\# I = m-n$.

The above lemma shows that non-zero bigraded Betti numbers $b_{-q,2p}(Z_K)$, $r \neq 0$ appear only in the "strip" bounded by the lines $r = -(m-1), r = -1$, $p + r = 1$ and $p + r = n$ in the second quadrant (see Figure 4 (a)).
Let us consider now the bigraded cellular chain complex $[C_{*,*}(\mathbb{Z}_K), \partial_c]$. The homogeneous component $C_{-q,2p}(\mathbb{Z}_K)$ consists of cellular chains $D_I T_J$ with $I \in K$, $\# I = p - q$, $\# J = q$. It follows that

\[
\dim C_{-q,2p}(\mathbb{Z}_K) = f_{p-q-1} \binom{m-p+q}{q}
\]

(with usual agreement $\binom{0}{j} = \binom{i}{0} = 0$ if $i < j$ or $j < 0$), where $f_i$ is the number of $i$-simplices of $K^{n-1}$ and $f_{-1} = 1$. Since the differential $\partial_c$ does not change the second degree, i.e.

$\partial_c : C_{-q,2p}(\mathbb{Z}_K) \rightarrow C_{-q-1,2p}(\mathbb{Z}_K)$,

the chain complex $C_{*,*}(\mathbb{Z}_K)$ splits into the sum of chain complexes as follows:

$[C_{*,*}(\mathbb{Z}_K), \partial_c] = \bigoplus_{p=0}^{m} [C_{*,2p}(\mathbb{Z}_K), \partial_c]$.

The similar decomposition holds also for the cellular cochain complex $[C^{*,*}(\mathbb{Z}_K), d_c] \cong [C^{*,*}(K), d]$. Let $\chi_p(\mathbb{Z}_K)$ denote the Euler characteristic of complex $[C_{*,2p}(\mathbb{Z}_K), \partial_c]$, i.e.

\[
\chi_p(\mathbb{Z}_K) = \sum_{q=0}^{m} (-1)^q \dim C_{-q,2p}(\mathbb{Z}_K) = \sum_{q=0}^{m} (-1)^q b_{-q,2p}(\mathbb{Z}_K).
\]

Define the generating polynomial $\chi(\mathbb{Z}_K, t)$ as

$\chi(\mathbb{Z}_K, t) = \sum_{p=0}^{m} \chi_p(\mathbb{Z}_K)t^{2p}$.
The following theorem calculates this polynomial in terms of the numbers of faces of $K$. It was firstly proved in [BP2] in the particular case of polytopal $K$.

**Theorem 4.7.** For any $(n - 1)$-dimensional simplicial complex $K$ with $m$ vertices holds
\[
\chi(Z_K, t) = (1 - t^2)^{m-n}(h_0 + h_1 t^2 + \cdots + h_n t^{2n}),
\]
where $(h_0, h_1, \ldots, h_n)$ is the $h$-vector of $K$ (see (2)).

**Proof.** It follows from (18) and (17) that
\[
\chi_p(Z_K) = \sum_{j=0}^{m} (-1)^{p-j} f_{j-1} \binom{m-j}{p-j},
\]
Then
\[
\chi(Z_K, t) = \sum_{p=0}^{m} \chi_p(Z_K) t^{2p} = \sum_{p=0}^{m} \sum_{j=0}^{m} t^{2j} t^{2(p-j)} (-1)^{p-j} f_{j-1} \binom{m-j}{p-j}
= \sum_{j=0}^{m} f_{j-1} t^{2j} (1 - t^2)^{m-j} = (1 - t^2)^m \sum_{j=0}^{n} f_{j-1} (t^2 - 1)^{-j}.
\]
Denote $h(t) = h_0 + h_1 t + \cdots + h_n t^n$. Then it follows from (2) that
\[
t^n h(t^{-1}) = (t - 1)^n \sum_{i=0}^{n} f_{i-1} (t - 1)^{-i}.
\]
Substituting above $t^{-2}$ for $t$, we finally obtain from (21)
\[
\frac{\chi(Z_K, t)}{(1 - t^2)^m} = \frac{t^{-2n} h(t^2)}{(1 - t^2)^m} = \frac{h(t^2)}{(1 - t^2)^n},
\]
which is equivalent to (13). \hfill \square

The above theorem allows to express the numbers of faces of a simplicial complex in terms of bigraded Betti numbers of the corresponding moment-angle complex $Z_K$. The first important corollary of this is as follows.

**Corollary 4.8.** For any simplicial complex $K$ the Euler number of the corresponding moment-angle complex $Z_K$ is zero.

**Proof.** We have
\[
\chi(Z_K) = \sum_{p,q=0}^{m} (-1)^{-q+2p} b_{-q,2p}(Z_K) = \sum_{p=0}^{m} \chi_p(Z_K) = \chi(Z_K, 1)
\]
Now the statement follows from (13). \hfill \square

**Remark.** Another way to prove the above corollary is to mention that the diagonal subgroup $S^1 \subset T^m$ always acts freely on the moment-angle complex $Z_K$ (see [BP2]). Hence, there exists a principal $S^1$-bundle $Z_K \to Z_K/S^1$, which implies $\chi(Z_K) = 0$. 

Corollary 4.9. The Euler number of the complement of a complex coordinate subspace arrangement is zero.

Proof. This follows from the previous corollary and Lemma 4.4. □

By definition (see Proposition 2.2), the cubical complex $\text{cc}(K)$ always contains the vertex $(1, \ldots, 1) \in I^n$. Hence, the torus $T^n = \rho^{-1}(1, \ldots, 1)$ is contained in $Z_K$. Here $\rho : (D^2)^n \to I^n$ is the orbit map for the $T^n$-action (see (11)).

Lemma 4.10. The inclusion $T^n = \rho^{-1}(1, \ldots, 1) \hookrightarrow Z_K$ is a cellular map homotopical to the map to a point, i.e. the torus $T^n = \rho^{-1}(1, \ldots, 1)$ is a contractible cellular subcomplex of $Z_K$.

Proof. To prove that $T^n = \rho^{-1}(1, \ldots, 1)$ is a cellular subcomplex of $Z_K$ we just mention that this $T^n$ is the closure of the $m$-cell $D_2T_{[m]} \subset Z_K$. So, it remains to prove that $T^n$ is contractible within $Z_K$. To do this we show that the embedding $T^n \subset (D^2)^m$ is homotopical to the map to the point $(1, \ldots, 1) \in T^n \subset (D^2)^m$. On the first step we note that $Z_K$ contains the cell $D_1T_2, \ldots, m$, whose closure contains $T^n$ and is homeomorphic to $D^2 \times T^{m-1}$. Hence, our $T^n$ can be contracted to $1 \times T^{m-1}$ within $Z_K$. On the second step we note that $Z_K$ contains the cell $D_2T_3, \ldots, m$, whose closure contains $1 \times T^{m-1}$ and is homeomorphic to $D^2 \times T^{m-2}$. Hence, $1 \times T^{m-2}$ can be contracted to $1 \times 1 \times T^{m-2}$ within $Z_K$, and so on. On the $k$th step we note that $Z_K$ contains the cell $D_kT_{k+1}, \ldots, m$, whose closure contains $1 \times \cdots \times 1 \times T^{m-k+1}$ and is homeomorphic to $D^2 \times T^{m-k}$. Hence, $1 \times \cdots \times 1 \times T^{m-k+1}$ can be contracted to $1 \times \cdots \times 1 \times T^{m-k}$ within $Z_K$. We end up at the point $1 \times \cdots \times 1$ to which the whole torus $T^n$ can be contracted. □

Corollary 4.11. For any simplicial complex $K$ the moment-angle complex $Z_K$ is simply connected.

Proof. Indeed, the 1-skeleton of our cellular decomposition of $Z_K$ is contained in the torus $T^n = \rho^{-1}(1, \ldots, 1)$. □

The cohomology of cellular pair $(Z_K, T^n)$ also can be calculated by means of the cochain complex $C^*(K)$. The cellular cochain subcomplex $C^*(T^n) \subset C^*(K)$ consists of monomials $u_I$ (i.e. monomials that do not contain $v_i$’s). This, of course, is just the exterior algebra $\Lambda[u_1, \ldots, u_m]$. Hence,

$$C^*(Z_K, T^n) = C^*(Z_K) / \Lambda[u_1, \ldots, u_m]$$

(as complexes, not as algebras). We can also introduce relative bigraded Betti numbers

$$b_{-q,2p}(Z_K, T^n) = \dim H^{-q,2p}[C^*(Z_K, T^n), d], \quad q, p = 0, \ldots, m,$$
define the $p$th relative Euler characteristic $\chi_p(Z_K, T^m)$ as the Euler number of complex $C^{*,2p}(Z_K, T^m)$:
\begin{equation}
\chi_p(Z_K, T^m) = \sum_{q=0}^{m} (-1)^q \dim C^{-q,2p}(Z_K, T^m) = \sum_{q=0}^{m} (-1)^q b_{-q,2p}(Z_K, T^m),
\end{equation}
and define the generating polynomial $\chi(Z_K, T^m, t)$ as
\begin{equation}
\chi(Z_K, T^m, t) = \sum_{p=0}^{m} \chi_p(Z_K, T^m) t^{2p}.
\end{equation}
We will use the following theorem in the next section.

**Theorem 4.12.** For any $(n-1)$-dimensional simplicial complex $K$ with $m$ vertices holds
\begin{equation}
\chi(Z_K, T^m, t) = (1 - t^2)^{m-n}(h_0 + h_1 t^2 + \cdots + h_n t^{2n}) - (1 - t^2)^m.
\end{equation}

**Proof.** Since $C^*(T^m) = \Lambda[\mu_1, \ldots, \mu_m]$, and $\text{bideg} u_i = (-1,2)$, we have
\[\dim C^{-q}(T^m) = \dim C^{-q,2q}(T^m) = \binom{m}{q}.\]

It follows from (22), (18) and (24) that
\[\chi_p(Z_K, T^m) = \chi_p(Z_K) - (-1)^p \dim C^{-p,2p}(T^m).\]
Hence,
\[\chi(Z_K, T^m, t) = \chi(Z_K, t) - \sum_{p=0}^{m} (-1)^p \binom{m}{p} t^{2p} = (1 - t^2)^{m-n}(h_0 + h_1 t^2 + \cdots + h_n t^{2n}) - (1 - t^2)^m,\]
by (15). \hfill \Box

At the end of this section we review the most important additional properties of $Z_K$ in the case when $|K| \cong S^{n-1}$, i.e. $K$ is a simplicial sphere.

**Lemma 4.13.** If $K$ is a simplicial sphere, i.e. $|K| = S^{n-1}$, then $Z_K$ is an $(m+n)$-dimensional (closed) manifold. \hfill \Box

In [Dj, p. 434] the authors considered the manifold $Z$ defined for any simple $n$-polytope $P^*$ with $m$ facets as $Z = (T^m \times P^*) / \sim$, where $\sim$ is a certain equivalence relation. We showed in [BP2] that if $K$ is a polytopal sphere, i.e. $K = \partial P^n$ for some simplicial polytope $P^n$, then our moment-angle complex $Z_K$ coincides with the manifold $Z$ defined by simple polytope $P^*$ dual to $P$. For the case of general simplicial sphere $K$, see [BP3], [BP4].

**Theorem 4.14.** Let $K$ be an $(n-1)$-dimensional simplicial sphere, and let $Z_K$ be the corresponding moment-angle manifold. The fundamental cohomological class of $Z_K$ is represented by any monomial $\pm v_I u_J \in C(K)$ of bidegree $(-m-n, 2m)$ such that $I$ is an $(n-1)$-simplex of $K$ and $I \cap J = \emptyset$. The choice of sign depends on the orientation of $Z_K$. 
Proof. We have \( \dim Z_K = m + n \). It follows from Lemma 4.6 (f) that \( H^{m+n}(Z_K) = H^{-(m-n),2m}(Z_K) \). By definition, the module \( \mathcal{C}^{-(m-n),2m}(Z_K) \) is spanned by monomials \( v_1u_j \) such that \( I \in K^{n-1}, \# I = n, J = [m] \setminus I \), and all these monomials are cocycles. Suppose that \( I, I' \) are two \((n - 1)\)-simplices of \( K^{n-1} \) sharing a common \((n - 2)\)-face. Then the corresponding cocycles \( v_1u_J, v_1u_{J'} \), where \( J = [m] \setminus I, J' = [m] \setminus I' \), are cohomological (up to sign). Indeed, let \( v_1u_J = v_i \cdots v_{i_n}u_{j_1} \cdots u_{j_{m-n}} \), \( v_1u_{J'} = v_i \cdots v_{i_n}u_{j_1}u_{j_2} \cdots u_{j_{m-n}} \). Since any \((n - 2)\)-face of \( K \) is contained in exactly two \((n - 1)\)-faces, the identity

\[
d(v_i \cdots v_{i_n}u_{j_1} \cdots u_{j_{m-n}}) = v_i \cdots v_{i_n}u_{j_1} \cdots u_{j_{m-n}} - v_i \cdots v_{i_n-1}u_{j_1}u_{j_2} \cdots u_{j_{m-n}}
\]

holds in \( \mathcal{C}(K) \subset k(K) \otimes \Lambda[u_1, \ldots, u_m] \), hence, \( v_1u_J \) and \( v_1u_{J'} \) are cohomological. Since \( K^{n-1} \) is a simplicial sphere, any two \((n - 1)\)-simplices of \( K^{n-1} \) can be connected by a chain of simplices such that any two successive simplices share a common \((n - 2)\)-face. Thus, any two cocycles in \( \mathcal{C}^{-(m-n),2m}(Z_K) \) are cohomological, and we can take any one as a representative for the fundamental cohomological class of \( Z_K \) (after a proper choice of sign).

\[\blacksquare\]

Remark. In the proof of the above theorem we have used two combinatorial properties of \( K^{n-1} \). The first one is that any \((n - 2)\)-face is contained in exactly two \((n - 1)\)-faces, and the second one is that any two \((n - 1)\)-simplices can be connected by a chain of simplices such that any two successive simplices share a common \((n - 2)\)-face. Both properties hold for any simplicial manifold. Hence, for any simplicial manifold \( K^{n-1} \) one has \( b_{m+n}(Z_K) = b_{-(m-n),2m}(Z_K) = 1 \) and the generator of \( H^{m+n}(Z_K) \) can be chosen as in Theorem 4.14.

\[\blacksquare\]

Corollary 4.15. The Poincaré duality for the moment angle manifold \( Z_K \) defined by a simplicial sphere \( K^{n-1} \) regards the bigraded structure in the (co)homology, i.e.

\[ H^{-q,2p}(Z_K) \cong H^{-(m-n)+q,2(m-p)}(Z_K). \]

In particular,

\[ b_{-q,2p}(Z_K) = b_{-(m-n)+q,2(m-p)}(Z_K). \]

\[\blacksquare\]

Corollary 4.16. Let \( K^{n-1} \) be an \((n - 1)\)-dimensional simplicial sphere, and let \( Z_K \) be the corresponding moment-angle complex, \( \dim Z_K = m + n \). Then

(a) \( b_{-q,2p}(Z_K) = 0 \) if \( q \geq m - n \), with only exception \( b_{-(m-n),2m} = 1 \);

(b) \( b_{-q,2p}(Z_K) = 0 \) if \( p - q \geq n \), with only exception \( b_{-(m-n),2m} = 1 \).

\[\blacksquare\]

It follows that if \( K^{n-1} \) is a simplicial sphere, then non-zero bigraded Betti numbers \( b_{r,2p}(Z_K), r \neq 0, r \neq m - n \), appear only in the “strip” bounded by the lines \( r = -(m - n - 1), r = -1, p + r = 1 \) and \( p + r = n - 1 \) in the second
quadrant (see Figure 4 (b)). Compare this with Figure 4 (a) corresponding to the case of general $K$.

It follows from (18) and (26) that for any simplicial sphere $K$ holds

$$\chi_p(Z_K) = (-1)^{m-n} \chi_{m-p}(Z_K).$$

From this and (19) we get

$$\frac{h_0 + h_1 t^2 + \cdots + h_n t^{2n}}{(1-t^2)^n} = (-1)^{m-n} \frac{\chi_0 + \chi_1 t^{-2} + \cdots + \chi_m t^{-2m}}{(1-t^{-2})^m} = (-1)^n \frac{h_0 + h_1 t^{-2} + \cdots + h_n t^{-2n}}{(1-t^{-2})^n} = \frac{h_0 t^{2n} + h_1 t^{2(n-1)} + \cdots + h_n}{(1-t^2)^n}.$$

Hence, $h_i = h_{n-i}$. Thus, we have deduced the Dehn–Sommerville equations as a corollary of the bigraded Poincaré duality (26).

The identity (19) also allows to interpret different inequalities for the numbers of faces of simplicial spheres (resp. simplicial manifolds) in terms of topological invariants (bigraded Betti numbers) of the corresponding moment-angle manifolds (resp. complexes) $Z_K$.

**Example 4.17.** It follows from Lemma 4.6 that for any $K$ holds

$$\chi_0(K) = 1,$$
$$\chi_1(K) = 0,$$
$$\chi_2(K) = -b_{-1,4}(Z_K) = -b_3(Z_K),$$
$$\chi_3(K) = b_{-2,6}(Z_K) - b_{-1,6}(Z_K)$$

(note that $b_4(Z_K) = b_{-2,6}(Z_K)$, while $b_5(Z_K) = b_{-1,6}(Z_K) + b_{-3,8}(Z_K)$).

Now, identity (19) shows that

$$h_0 = 1,$$
$$h_1 = m - n,$$
$$h_2 = \binom{m-n+1}{2} - b_3(Z_K),$$
$$h_3 = \binom{m-n+2}{3} - (m-n)b_{-1,4}(Z_K) + b_{-2,6}(Z_K) - b_{-1,6}(Z_K).$$

It follows that the inequality $h_1 \leq h_2$, $n \geq 4$, from the Generalized Lower Bound hypothesis (7) for simplicial spheres is equivalent to the following inequality:

$$b_3(Z_K) \leq \binom{m-n}{2}.$$

The next inequality $h_2 \leq h_3$, $n \geq 6$, from (7) is equivalent to the following inequality for the bigraded Betti numbers of $Z_K$:

$$\binom{m-n+1}{3} - (m-n-1)b_{-1,4}(Z_K) + b_{-2,6}(Z_K) - b_{-1,6}(Z_K) \geq 0.$$
Thus, we see that the combinatorial Generalized Lower Bound inequalities are interpreted as “topological” inequalities for the (bigraded) Betti numbers of a certain manifold. So, one can try to use topological methods (such as the equivariant topology or Morse theory) to prove inequalities like (27) or (28). Such topological approach to the hypotheses like g-theorem or Generalized Lower Bound has an advantage of being independent on whether the simplicial sphere $K$ is polytopal or not. Indeed, all known proofs of the necessity of $g$-theorem for simplicial polytopes (including the original one by Stanley [St2], McMullen’s proof [McM], and the recent proof by Timorin [Ti]) follow the same scheme. Namely, the numbers $h_i$, $i = 1, \ldots, n$, are interpreted as the dimensions of graded components $A_i$ of a certain graded algebra $A$ satisfying the Hard Lefschetz Theorem. The latter means that there is an element $\omega \in A^1$ such that the multiplication by $\omega$ defines a monomorphism $A^i \to A^{i+1}$ for $i < \left[ \frac{n}{2} \right]$. This implies $h_i \leq h_{i+1}$ for $i < \left[ \frac{n}{2} \right]$. However, such element $\omega$ is lacking for non-polytopal $K$, which means that one should develop a new technique for proving the $g$-theorem (or, better to say, $g$-conjecture) for simplicial spheres. Certainly, it may happen that the $g$-theorem fails to be true for simplicial spheres, however, many recent efforts of computer-aided seek for counter examples were unsuccessful (see, e.g., [BjLu]).

It can be easily seen that our moment-angle complex $Z_K$ is a manifold provided that the cone cone($K$) is non-singular. This is equivalent to the condition that the suspension $\Sigma |K|$ is a (topological) manifold, which implies (due to the suspension isomorphism and Poincaré duality in the homology) that $|K|$ is a homology sphere. An important class of simplicial homology spheres is known in combinatorics as Gorenstein* complexes (see, e.g., [St3] for definition). As it was pointed out by Stanley in [St3], the Gorenstein* complexes are the most general objects appropriate for generalizing the $g$-theorem (they include polytopal spheres, PL-spheres and simplicial spheres as particular cases). In our terms, the Gorenstein* complexes $K$ (see [St1, p.75]) can be characterized by the condition that the Tor-algebra $\text{Tor}_{k[v_1, \ldots, v_m]}(k(K), k)$ satisfies the bigraded duality (29), i.e., $Z_K$ is a Poincaré duality space (not necessarily a manifold). In particular, the Dehn–Sommerville relations $h_i = h_{n-i}$ continue to hold for Gorenstein* complexes.

5. Homology of $W_K$ and Generalized Dehn–Sommerville equations

Here we assume that $K^{n-1}$ is a triangulation of a manifold, i.e., a simplicial manifold. In this case the moment-angle complex $Z_K$ is not a manifold, however, its singularities can be easily treated. Indeed, $|\text{cc}(K)|$ is homeomorphic to $|\text{cone}(K)|$, and the vertex of the cone is the point $p = (1, \ldots, 1) \in |\text{cc}(K)| \subset I^n$. Let $U_{\varepsilon}(p) \subset |\text{cc}(K)|$ be a small neighbourhood of $p$ in $|\text{cc}(K)|$. Then the closure of $U_{\varepsilon}(p)$ is also homeomorphic to $|\text{cone}(K)|$. It follows from the definition of $Z_K$ (see [11]) that
\(U_\varepsilon(T^m) = \rho^{-1}(U_\varepsilon(p)) \subset \mathcal{Z}_K\) is a small invariant neighbourhood of the torus \(T^m = \rho^{-1}(p)\) in \(\mathcal{Z}_K\). Here \(\rho : (D^2)^m \to I^m\) is the orbit map. Then for small \(\varepsilon\) the closure of the neighbourhood \(U_\varepsilon(T^m)\) is homeomorphic to \(|\text{cone}(K)| \times T^m\). Taking \(U_\varepsilon(T^m)\) away from \(\mathcal{Z}_K\) we obtain a manifold with boundary, which we denote \(W_K\). Hence, we have

\[
W_K = \overline{\mathcal{Z}_K} \setminus |\text{cone}(K)| \times T^m, \quad \partial W_K = |K| \times T^m.
\]

Note that since the neighbourhood \(U_\varepsilon(T^m)\) is \(T^m\)-invariant, the torus \(T^m\) acts on \(W_K\).

**Theorem 5.1.** The manifold with boundary \(W_K\) is equivariantly homotopy equivalent to the moment-angle complex \(W_K\) (see (11)). There is a canonical relative isomorphism of pairs \((W_K, \partial W_K) \to (\mathcal{Z}_K, T^m)\).

*Proof.* To prove the first assertion we construct homotopy equivalence \(|\text{cc}(K)| \setminus U_\varepsilon(p) \to |\text{cub}(K)|\) (see Proposition 2.1) as it is shown on Figure 5. This homotopy equivalence is covered by a \(T^m\)-invariant homotopy equivalence \(W_K = \overline{\mathcal{Z}_K} \setminus U_\varepsilon(T^m) \to W_K\), as needed. The second assertion follows easily from the definition of \(W_K\). \(\square\)

According to Lemma 3.2 the moment-angle complex \(W_K \subset (D^2)^m\) has a cellular structure with 5 different cell types \(D_i, I_i, 0_i, T_i, 1_i, \ i = 1, \ldots, m\), (see Figure 3). The homology of \(W_K\) (and of \(W_K\)) can be calculated by means of the corresponding cellular chain complex, which we denote \([\mathcal{C}_*(W_K), \partial_*]\). In comparison with the moment-angle complex \(\mathcal{Z}_K\) studied in the previous section the complex \(W_K\) has more types of cells (remember that \(\mathcal{Z}_K\) has only 3 cell types \(D_i, T_i, 1_i\)). However, the wonderful thing is that the cellular chain complex \([\mathcal{C}_*(W_K), \partial_*]\) can be canonically made bigraded. Namely, the following statement holds (compare with (15)).

**Lemma 5.2.** Put

\[
\text{bideg } D_i = (0, 2), \quad \text{bideg } T_i = (-1, 2), \quad \text{bideg } I_i = (1, 0),
\]

\[
\text{bideg } 0_i = \text{bideg } 1_i = (0, 0), \quad i = 1, \ldots, m.
\]
This makes the cellular chain complex \([C_*(W_K), \partial_c]\) a bigraded differential module with differential \(\partial_c\) adding \((-1,0)\) to bidegree. The original grading of \(C_*(W_K)\) by dimensions of cells corresponds to the total degree (i.e., the dimension of a cell equals the sum of its two degrees).

**Proof.** We need only to check that the differential \(\partial_c\) adds \((-1,0)\) to bidegree. This follows from (29) and

\[
\partial_c D_i = T_i, \quad \partial_c I_i = 1_i - 0_i, \quad \partial_c T_i = \partial_c 1_i = \partial_c 0_i = 0.
\]

\(\square\)

Note that, unlike bigraded structure in \(C_*(Z_K)\), elements of \(C_*,*(W_K)\) may have **positive** first degree (due to the positive first degree of \(I_i\)'s). However, as in the case of \(Z_K\), the differential \(\partial_c\) does not change the second degree, which allows to split the bigraded complex \(C_*,*(W_K)\) to the sum of complexes \(C_{*,2p}(W_K), \ p = 0, \ldots, m.\)

In the same way as we have done this for \(Z_K\) and for the pair \((Z_K, T^m)\) define the bigraded Betti numbers

\[
(30) \quad b_{q,2p}(W_K) = \dim H_{q,2p}[C_*,*(W_K), \partial_c], \quad -m \leq q \leq m, \ 0 \leq p \leq m
\]

(note that \(q\) may be both positive and negative), the \(p\)th Euler characteristic \(\chi_p(W_K)\) as the Euler number of complex \(C_{*,2p}(W_K)\):

\[
(31) \quad \chi_p(W_K) = \sum_{q=-m}^{m} (-1)^q \dim C_{q,2p}(W_K) = \sum_{q=-m}^{m} (-1)^q b_{q,2p}(W_K),
\]

and the generating polynomial \(\chi(W_K, t)\) as

\[
\chi(W_K, t) = \sum_{p=0}^{m} \chi_p(W_K) t^{2p}.
\]

The following theorem provides the exact formula for generating polynomial \(\chi(W_K, t)\) and is analogous to theorems 4.7 and 1.12.

**Theorem 5.3.** For any \((n-1)\)-dimensional simplicial complex \(K\) with \(m\) vertices holds

\[
\chi(W_K, t) = (1 - t^2)^{m-n} (h_0 + h_1 t^2 + \cdots + h_n t^{2n}) + (\chi(K) - 1)(1 - t^2)^m
= (1 - t^2)^{m-n} (h_0 + h_1 t^2 + \cdots + h_n t^{2n}) + (-1)^{n-1} h_n (1 - t^2)^m,
\]

where \(\chi(K) = f_0 - f_1 + \cdots + (-1)^{n-1} f_{n-1} = 1 + (-1)^{n-1} h_n\) is the Euler number of \(K\).

**Proof.** Lemma 3.2 shows that every moment-angle complex \(\text{ma}(C)\) is a cellular subcomplex of \((D^2)^m\), and each cell is the product of cells of 5 different types: \(D_i, I_i, 0_i, T_i, 1_i, i = 1, \ldots, m.\) These products are encoded by words \(D_I J_0 T P Q,\) where \(I, J, L, P, Q\) are pairwise disjoint subsets of \([m]\) such that \(I \cup J \cup K \cup P \cup Q = [m]\). In the case \(\text{ma}(C) = W_K = \text{ma}(\text{cub}(K))\) the definition of \(\text{cub}(K)\) (see Proposition 2.1)
shows that the cell $D_I J_0 T_P 1_Q$ belongs to $W_K$ if and only if the following two conditions are satisfied:

1. The set $I \cup J \cup L$ is a simplex of $K^{n-1}$.
2. $\# L \geq 1$.

Let $c_{i,j,p,q}(W_K)$ denote the number of cells $D_I J_0 T_P 1_Q \subset W_K$ with $i = \# I$, $j = \# J$, $l = \# L$, $p = \# P$, $q = \# Q$, $i + j + l + p + q = m$. It follows that

$$c_{i,j,p,q}(W_K) = f_{i+j+l−1} \binom{i+j+l}{i} \binom{j+l}{1} \binom{m-i-j-l}{p-q}.$$  

(32)

where $(f_0, \ldots, f_{n-1})$ is the $f$-vector of $K$ (we also assume $f_{-1} = 1$ and $f_k = 0$ for $k < -1$ or $k > n - 1$). By (29),

$$\text{bideg}(D_I J_0 T_P 1_Q) = (j - p, 2(i + p)).$$

Now we calculate $\chi_r(W_K)$ as it is defined by (11), using (32):

$$\chi_r(W_K) = \sum_{i,j,p,q \geq 0, i + j \geq l} (-1)^{i+j+l-1} f_{i+j+l-1} \binom{i+j+l}{i} \binom{j+l}{1} \binom{m-i-j-l}{p-q}.$$  

Substituting $s = i + j + l$, $i = r - p$ above, we obtain

$$\chi_r(W_K) = \sum_{s, p \geq 0, \sum_{l \geq 1} (-1)^{s-r-l} f_{s-1} \binom{s}{r-p} \binom{s-r+p}{l} \binom{m-s}{p}.$$  

By (28),

$$\text{bideg}(D_I J_0 T_P 1_Q) = (j - p, 2(i + p)).$$

Since

$$\sum_{l \geq 1} (-1)^l \binom{s-r+p}{l} = \begin{cases} -1, & s > r - p, \\ 0, & s \leq r - p, \end{cases}$$

we obtain

$$\chi_r(W_K) = -\sum_{s, p \geq 0, s > r-p} (-1)^{s-r} f_{s-1} \binom{s}{r-p} \binom{m-s}{p}.$$  

$$= -\sum_{s, p \geq 0} (-1)^{r-s} f_{s-1} \binom{s}{r-p} \binom{m-s}{p} + \sum_{s} (-1)^{r-s} f_{s-1} \binom{m-s}{r-s}.$$  

The second sum in the above formula is exactly $\chi_r(Z_K)$ (see (20)). To calculate the first sum, we mention that

$$\sum_{p} \binom{s}{r-p} \binom{m-s}{p} = \binom{m}{r}$$

(this follows from calculating the coefficient of $\alpha^r$ in both sides of $(1+\alpha)^s(1+\alpha)^{m-s} = (1+\alpha)^m$). Hence,

$$\chi_r(W_K) = -\sum_{s} (-1)^{r-s} f_{s-1} \binom{m}{r} + \chi_r(Z_K) = (-1)^r \binom{m}{r} (\chi(K) - 1) + \chi_r(Z_K),$$
Corollary 5.4. The following relations hold for the $h$-vector $(h_0, h_1, \ldots, h_n)$ of any simplicial manifold $K^{n-1}$:

$$h_{n-i} - h_i = (-1)^i \left( \chi(K^{n-1}) - \chi(S^{n-1}) \right) \binom{n}{i}, \quad i = 0, 1, \ldots, n,$$

where $\chi(S^{n-1}) = 1 + (-1)^{n-1}$ is the Euler number of an $(n-1)$-sphere.

Proof. Theorem 5.1 shows that $H^{n+n-k}(W_K, \partial_c W_K) = H^{m+n-k}(Z_K, T^m)$ and $H_k(W_K) = H_k(W_K)$. Moreover, it can be seen in the same way as in Corollary 4.1 that relative Poincaré duality isomorphisms (33) regard the bigraded structures in the (co)homology of $W_K$ and $(Z_K, T^m)$. It follows that

$$b_{-q,2p}(W_K) = b_{-(m-n)+q,2(m-p)}(Z_K, T^m).$$

Hence,

$$\chi_p(W_K) = (-1)^{m-n} \chi_{m-p}(Z_K, T^m),$$

and

$$\chi(W_K, t) = (-1)^{m-n} \sum_p \chi_{m-p}(Z_K, T^m) t^{2p}$$

Using (25), we calculate

$$(-1)^{m-n} t^{2m} \chi(Z_K, T^m, \frac{1}{t})$$

$$= (-1)^{m-n} t^{2m} (1 - t^{-2})^{m-n}(h_0 + h_1 t^{-2} + \cdots + h_n t^{-2n})$$

$$- (-1)^{m-n} t^{2m} (1 - t^{-2})^{m}$$

$$= (1 - t^2)^{m-n} (h_0 t^{2n} + h_1 t^{2n-2} + \cdots + h_n) + (-1)^{n-1} (1 - t^2)^m.$$

Substituting the formula for $\chi(W_K, t)$ from Theorem 5.1 and the above expression into formula (34), we obtain

$$(1 - t^2)^{m-n} (h_0 + h_1 t^2 + \cdots + h_n t^{2n}) + (\chi(K) - 1) (1 - t^2)^m$$

$$= (1 - t^2)^{m-n} (h_0 t^{2n} + h_1 t^{2n-2} + \cdots + h_n) + (-1)^{n-1} (1 - t^2)^m.$$
Calculating the coefficient of $t^2i$ in both sides after dividing the above identity by $(1 - t^2)^m - n$, we obtain $h_{n-i} - h_i = (-1)^i(\chi(K^{n-1}) - \chi(S^{n-1}))(\binom{n}{i})$, as needed.

Corollary 5.4 generalize the Dehn–Sommerville equations (4) for simplicial spheres. If $|K| = S^{n-1}$ or $(n - 1)$ is odd, Corollary 5.4 gives just $h_{n-i} = h_i$.

**Corollary 5.5.** The following relations hold for any simplicial manifold $K^{n-1}$:

$$h_{n-i} - h_i = (-1)^i(h_n - 1)(\binom{n}{i}), \quad i = 0, 1, \ldots, n.$$  

**Proof.** Since $\chi(K^{n-1}) = 1 + (-1)^{n-1}h_n$, $\chi(S^{n-1}) = 1 + (-1)^{n-1}$, we have

$$\chi(K^{n-1}) - \chi(S^{n-1}) = (-1)^{n-1}(h_n - 1) = (h_n - 1)$$

(the coefficient $(-1)^{n-1}$ can be dropped since for odd $(n - 1)$ the left side is zero).

**Corollary 5.6.** For any $(n - 1)$-dimensional simplicial manifold the numbers $h_{n-i} - h_i$, $i = 0, 1, \ldots, n$, are homotopy invariants. In particular, they are independent on a triangulation.

In the particular case of PL-manifolds the topological invariance of numbers $h_{n-i} - h_i$ was firstly observed by Pachner in [Pa (7.11)].

**Example 5.7.** Consider triangulations of the 2-torus $T^2$, so $n = 3$, $\chi(T^2) = 0$. Since for any $K^{n-1}$ holds $\chi(K^{n-1}) = 1 + (-1)^{n-1}h_n$, in our case we have $h_3 = -1$. Then Corollary 5.4 gives

$$h_3 - h_0 = -2, \quad h_2 - h_1 = 6.$$  

For instance, the triangulation on Figure 6 has $f_0 = 9$ vertices, $f_1 = 27$ edges and $f_2 = 18$ triangles. The corresponding $h$-vector is $(1, 6, 12, -1)$.
6. Concluding remarks

The main goal of the present paper was to establish new connections between topology of manifolds and cellular complexes and combinatorics by means of the notion of a moment-angle complex, introduced by the authors in previous papers [BP2], [BP3], [BP4]. As we have seen, the combinatorics of simplicial manifolds and related objects (polytopes, simplicial spheres, simplicial complexes, coordinate subspace arrangements) can be effectively described by means of topological invariants of bigraded equivariant moment-angle complexes. One of the main properties of a moment-angle complex is the existence of a torus action all of whose isotropy subgroups are coordinate ones. This, in particular, allows to introduce an additional grading to the cohomology ring of the moment-angle complex. On this point one can observe that there is a natural $\mathbb{Z}/2\mathbb{Z}$-analogue of almost all constructions presented in this paper. The first step is to replace the torus $T^m$ by its “real analogue”, namely, the group $(\mathbb{Z}/2\mathbb{Z})^m$. Then the unit cube $I^m = [0, 1]^m$ is the orbit space for the action of $(\mathbb{Z}/2\mathbb{Z})^m$ on the bigger cube $[-1, 1]^m$, the “real analogue” of the poly-disk $(D^2)^m \subset \mathbb{C}^m$. Now, starting from any cubical subcomplex $C \subset I^m$ one can construct another $(\mathbb{Z}/2\mathbb{Z})^m$-symmetrical cubical complex embedded into $[-1, 1]^m \subset \mathbb{R}^m$, just in the same way as we did it in Definition 3.1. In particular, for any simplicial complex $K$ on $m$ vertices one can construct $(\mathbb{Z}/2\mathbb{Z})^m$-symmetrical cubical complexes $Z^R_K$, $W^R_K$, the “real analogues” of moment-angle complexes $Z_K$, $W_K$, see (11). The analogue of Lemma 4.2 holds for real coordinate subspace arrangements. Namely, the complement $U(A^R(K))$ of the real coordinate subspace arrangement $A^R(K)$ defined by $K$ in the same way as in (13) is $(\mathbb{Z}/2\mathbb{Z})^m$-equivariantly homotopy equivalent to $Z^R_K$. However, the situation with the cohomology algebra of $Z^R_K$ is more subtle: as we have already mentioned, the analogue of Theorem 4.3 does not hold for $Z^R_K$, at least for $\mathbb{Q}$-coefficients (this is a usual thing in topology: the cohomology of “real” objects is more complicated than that of “complex” ones). At the same time $Z^R_K$ is an $m$-dimensional manifold provided that $K$ is a simplicial sphere. So, for any simplicial sphere $K$ with $m$ vertices we have a $(\mathbb{Z}/2\mathbb{Z})^m$-symmetric manifold with $(\mathbb{Z}/2\mathbb{Z})^m$-invariant cubical complex structure. This class of cubical manifolds may be useful in the cubical analogue of the combinatorial theory of $f$-vectors of simplicial complexes.

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