A classical history theory: Geometrodynamics and general field dynamics regained.

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Abstract

Assuming that the Hamiltonian of a canonical field theory can be written in the form $NH + N^i H_i$, and using as the only input the actual choice of the canonical variables, we derive: (i) The algebra satisfied by $H$ and $H_i$, (ii) any constraints, and (iii) the most general canonical representation for $H$ and $H_i$. This completes previous work by Hojman, Kuchař and Teitelboim who had to impose a set of additional postulates, among which were the form of the canonical algebra and the requirement of path-independence of the dynamical evolution. A prominent feature of the present approach is the replacement of the equal-time Poisson bracket with one evaluated at general times. The resulting formalism is therefore an example of a classical history theory—an interesting fact, especially in view of recent work by Isham et al.
1 Introduction.

We discuss the issue concerning the transition from a general canonical Hamiltonian of the form $NH + N^i H_i$ to a specific canonical representation. The form of the Hamiltonian is general enough to incorporate a large variety of canonical field theories—including general relativity—while the information that distinguishes one theory from the other comes, solely, through the choice of canonical variables. Changing the geometric interpretation of the functions $N$ and $N^i$ changes the form of the algebra of the canonical generators $H$ and $H_i$.

Some of this has been discussed already by Hojman, Kuchař and Teitelboim[1] who succeeded in deriving the canonical form of covariant spacetime theories from a few simple postulates. As stated by the authors themselves, however, a reduction of these postulates to the minimum was not attempted and some redundancy was left in the system. A couple of superfluous requirements were pointed out at the end of their paper and were not used in a subsequent one[2] but, still, the exact relationship between the remaining postulates was not clarified completely, and a further reduction seemed to be possible. We show that this is indeed the case, and that one can derive the complete set of postulates in [1] from just the minimum requirement that the canonical Hamiltonian is of the form $NH + N^i H_i$.

Just as interesting as this result, however, is the ensuing conclusion that the reduction of the postulates to the minimum could never have been achieved in the framework used by Kuchař et al due to their use of equal-time Poisson brackets. This is because, in an equal-time formalism, Poisson brackets that involve the time derivatives of the canonical variables cannot be defined—at least, not without the addition of further structure. Seen from a spacetime perspective, however, these brackets ought to be treated in an equivalent way, in which case they would give important information about the theory’s kinematics. In the equal-time formalism the missing information is precisely recovered by the additional postulates imposed in [1], most notably that of the Dirac algebra.

The present approach, on the other hand, is based on a Hamiltonian formalism whose phase space includes the fields at general times, i.e., is defined over the space of classical histories. An exact correspondence with the spacetime picture is thus established from the beginning and the reduction of the postulates comes as a direct consequence. In fact, the effectiveness of the history formalism could suggest that the latter is genuinely superior to its equal-time counterpart, although this depends on whether it is possible, or not, to establish a direct link between the equal-time and the history approaches. This is also discussed by Isham et al[3] in the context of continuous-time histories.

This paper is organized as follows. In section 2 we review the existing work on the subject and emphasize the main issues. The aim of this section is twofold: first, to act as an introduction for the reader who is not familiar with the subject and, second, to give the motivation for the construction that follows. In section 3 we present the framework for passing from the Lagrangean to a Hamiltonian formalism defined over the space of classical histories. The formalism incorporates both constrained and unconstrained systems and—at least for the issues of interest—is a simpler alternative to the Dirac method.

In section 4 we apply the history formalism to transform the postulated form $NH + N^i H_i$ of the canonical Hamiltonian to a set of kinematical conditions on the canonical generators.
and, in section 5, we use these conditions to derive the additional postulates imposed by Kuchař et al. Having established the connection between our approach and the approach in [1] we can be certain that the expression $NH + N^i H_i$, alone, is enough to determine the system. This includes the canonical algebra, any constraints, as well as the most general canonical representation of the generators.

2 Motivation.

The canonical decomposition of Hilbert’s action brings the theory of general relativity into the Hamiltonian form

$$S = \int d^3x dt [p^{ij} \dot{g}_{ij} - NH^g_r - N^i H^r_i]. \quad (2.1)$$

The lapse function $N$ and the shift vector $N^i$ acquire the meaning of Lagrange multipliers and, as a result, the canonical generators

$$H^g_r = \frac{1}{2} (g_{ik} g_{jl} + g_{il} g_{jk} - g_{ij} g_{kl}) p^{ij} p^{kl} - g^{\frac{1}{2}(3)} R + g^{\frac{1}{2}} \Lambda, \quad (2.2)$$

$$H^g_i = -2 p^j_{i | j}, \quad (2.3)$$

are constrained to vanish:

$$H^g_r \simeq 0, \quad (2.4)$$

$$H^g_i \simeq 0. \quad (2.5)$$

The generators can be shown to satisfy the Dirac algebra,[5]

$$\{H(x), H(x')\} = g^{ij}(x) H_i(x) \delta_j(x, x') - (x \leftrightarrow x') \quad (2.6)$$

$$\{H(x), H_i(x')\} = H(x) \delta_i(x, x') + H_i(x) \delta(x, x') \quad (2.7)$$

$$\{H_i(x), H_j(x')\} = H_j(x) \delta_{i,j}(x, x') - (ix \leftrightarrow jx'), \quad (2.8)$$

that is also satisfied by the canonical generators of a parametrized field theory. For a field theory that is not parametrized, the canonical algebra is not very different from the Dirac one and can always be recognized as a suitably modified version of it.

The Dirac algebra and the principle of path independence. This universality implies that the Dirac algebra is connected with a very general geometric property of spacetime which is independent of the specific dynamics of the canonical theory. The fact that the Dirac algebra is merely a kinematical consistency condition was shown by Teitelboim[6], who proceeded to derive it from a simple geometric argument corresponding to the integrability of Hamilton’s equations.

This consistency argument, termed by Kuchař[7] “the principle of path independence of the dynamical evolution”, ensures that the change in the canonical variables during the
evolution from a given initial surface to a given final surface is independent of the particular sequence of intermediate surfaces used in the actual evaluation of this change.

To be precise, besides the assumption of path independence—which applies regardless of the specific form of the canonical Hamiltonian—Teitelboim’s derivation also explicitly involved the assumption that the Hamiltonian is decomposable according to the lapse-shift formula written in equation (2.1). Using these two postulates, together, he was then led to the conclusion that in order for the theory to be consistent the phase space should be restricted by the initial value equations (2.4-2.5) while the canonical generators should satisfy the Dirac algebra (2.6-2.8).

Strictly speaking, the very last statement is not true due to a mistake in Teitelboim’s reasoning concerning the fact that the system is constrained. The correct algebra—as it arises from the requirement of path independence—is nonetheless very similar to the Dirac one but modified by certain terms $G, G_i$ and $G_{ij}$ whose first partial derivatives with respect to the canonical variables vanish on the constraint surface:

\begin{align}
\{H(x), H(x')\} &= g^{ij}(x)H_i(x)\delta_{,j}(x, x') + G(x, x') - (x \leftrightarrow x'), \\
\{H(x), H_i(x')\} &= H_i(x)\delta_{,i}(x, x') + H_i(x)\delta(x, x') + G_i(x, x'), \\
\{H_i(x), H_j(x')\} &= H_j(x)\delta_{,i}(x, x') + G_{ij}(x, x') - (ix \leftrightarrow jx').
\end{align}

(2.9)

(2.10)

(2.11)

A derivation of the above set of relations—which we call the weak Dirac algebra—is given in section 5.

The problem of deriving a physical theory from just the canonical algebra. The principle of path independence was an indication that a Hamiltonian theory need not be based exclusively on the canonical decomposition of some given spacetime action, but may also have an independent status. However, one finds that some of the necessary information is missing when one tries to construct specific canonical theories via the principle of path independence alone. The reason is that the weak Dirac algebra—which expresses the principle in the canonical language—allows a vast variety of representations to exist whose physical relevance is doubtful.

To give an example, we consider the case when the canonical variables are the spatial metric and its conjugate momentum, and we take the limit of the algebra (2.9-2.11) when all the terms $G, G_i$ and $G_{ij}$ are identically zero, i.e., we take the usual Dirac case. We let the $H_i$ generator be the super-momentum of the gravitational field,

$$H_i(x) = H^\text{grav}_i(x),$$

(2.12)

and require that the normal generator $H(x)$ be a scalar density of weight one. Under these conditions, the second and third Dirac relations (2.7-2.8) are satisfied, and the Dirac algebra—which can be seen as a set of coupled differential equations for the canonical generators—decouples completely. One is left with a single first-order equation for $H(x)$, equation (2.6), which is normally expected to admit an infinite number of distinct solutions.

In particular, one can take $H(x)$ to have the form

$$H(x) = g^{\frac{1}{2}}W[h, f](x),$$

(2.13)
where the weight-zero quantities $h$ and $f$ are defined by

$$h = g^{-\frac{1}{2}}H^g,$$

$$f = g^{-1}g^{ij}H^i_i H^g_j.$$

The resulting equation for the function $W[h, f]$ is

$$\frac{1}{2} W \frac{\partial W}{\partial f} = f \left( \frac{\partial W}{\partial f} \right)^2 - \frac{1}{4} \left( \frac{\partial W}{\partial h} \right)^2 + \frac{1}{4},$$

(2.15)

and can be shown to admit a family of solutions that is parametrized by an arbitrary function of one variable.

The super-Hamiltonian of general relativity, arising when $W[h, f] = h$, is the only one of these solutions that is ultralocal in the field momenta. The ultralocality is actually related to the geometric meaning of the canonical variables, but this will be discussed properly in section 5. For the moment, note that if one uses the weak Dirac algebra (2.9-2.11) as the starting point of the above calculation—which is the correct thing to do—one is forced to solve a set of coupled differential equations whose actual form is unknown!

**Selecting the physical representations of the Dirac algebra.** Deriving geometrodynamics from plausible first principles, Hojman, Kuchař and Teitelboim chose to lay the stress on the concept of infinite dimensional groups, and placed the strong Dirac algebra at the centre of their approach. They expected that the closing relations (2.6-2.8) themselves carry enough information about the system to uniquely select a physical representation, but they were unable to extract this information directly from them. We now know that the existence of solutions like (2.13) was the reason why.

What the authors of [1] did instead, was to follow an indirect route and select the physically relevant representations by supplementing the strong Dirac algebra with four additional conditions. Specifically, they introduced the tangential and normal generators of hypersurface deformations, defined respectively by

$$H^D_i(x) := \mathcal{X}_i^\alpha(x) \frac{\delta}{\delta \lambda^\alpha}(x),$$

(2.16)

$$H^D(x) := n^\alpha(x) \frac{\delta}{\delta \lambda^\alpha}(x),$$

(2.17)

and acted with these on the spatial metric:

$$H^D_k(x') g_{ij}(x) = g_{ki}(x) \delta_j(x, x') + g_{kj}(x) \delta_i(x, x') + g_{ij,k}(x) \delta(x, x'),$$

(2.18)

$$H^D(x') g_{ij}(x) = 2n_{\alpha;\beta}(x) \mathcal{X}_i^\alpha(x) \mathcal{X}_j^\beta(x') \delta(x, x').$$

(2.19)

Then, they required that equations (2.18-2.19)—which are purely kinematical and hold in an arbitrary Riemannian spacetime—should also be satisfied by the canonical generators,

$$\{g_{ij}(x), H_k(x')\} = g_{ki}(x) \delta_j(x, x') + g_{kj}(x) \delta_i(x, x') + g_{ij,k}(x) \delta(x, x'),$$

(2.20)

$$\{g_{ij}(x), H(x')\} \propto \delta(x, x'),$$

(2.21)
so that any dynamics in spacetime would arise as a different canonical representation of the universal kinematics. Note that only the ultralocality of the second Poisson bracket was actually used. The justification and geometric interpretation of equations (2.20) and (2.21) can be found in [1].

The strong Dirac algebra with the conditions (2.20), (2.21) results in a unique representation for the generators $H$ and $H_i$, corresponding to the super-Hamiltonian (2.2) and super-momentum (2.3) of general relativity. The requirement of path independence—which was imposed as an additional postulate to the algebra—enforces the initial value constraints (2.4-2.5) and, hence, the complete set of Einstein’s equations is recovered. The most general scalar field Lagrangean with a non-derivative coupling to the metric was derived along similar lines[2].

The precise assumptions used by the authors were summarized at the end of their paper. They are written here in an equivalent form and, in the case of pure gravity, they are the following:

(i) The evolution postulate: The dynamical evolution of the theory is generated by a Hamiltonian that is decomposed according to the lapse-shift formula, equation (2.1).

(ii) The representation postulate: The canonical generators must satisfy the closing relations (2.6-2.8) of the strong Dirac algebra.

(iii) Initial data reshuffling: The Poisson bracket (2.20) between the super-Hamiltonian and the configuration variable $g_{ij}$ must coincide with the kinematical relation (2.18).

(iv) Ultralocality: The Poisson bracket (2.21) between the super-momentum and the configuration variable $g_{ij}$ must coincide with the kinematical relation (2.19).

(v) Reversibility: The time-reversed spacetime must be generated by the same super-Hamiltonian and super-momentum as the original spacetime.

(vi) Path independence: The dynamical evolution predicted by the theory must be such that the change in the canonical variables during the evolution from a given initial surface to a given final one is independent of the actual sequence of intermediate surfaces used in the evaluation of this change.

**The need for a detailed understanding of the selection postulates.** The above assumptions comprise a set of natural first principles on which the canonical formulation of a theory can be based. There is a certain sense, however, in which they are not completely satisfying. First, they do not correspond to a minimum set and, second, the connection between them is not very clear. The authors themselves pointed out the redundancy of the reversibility postulate (v) as well as the fact that the third closing relation of the representation postulate (ii) is made redundant by the reshuffling requirement (iii). They stressed the need for understanding the precise reason why some equations hold strongly while others hold only weakly and, in particular, for clarifying the relationship between the strong representation postulate (ii) and the weak requirement of path independence (vi).

The revised form of Teitelboim’s argument makes such a clarification an even more important issue since, now, the strong representation requirement—which is at the very heart of the approach in [1]—seems to be unjustified. Adding to that, one can repeat
Teitelboim’s argument in the reverse order and show that the dynamical evolution of the theory should also hold weakly, in contrast to the strong equalities in postulates (iii) and (iv). On the other hand, we already know that any attempt to replace these equalities by weak ones would result in a situation where the actual form of the differential equations would not be known and no further progress would be made. Even if one justifies postulates (ii), (iii) and (iv) by assuming that general relativity just happens to exist on the strong limit of path independence, one will not be able to justify postulate (vi) whose weak imposition is necessary in order for the theory to be consistent.

Putting the issue of the weak equalities aside, an understanding of the exact relationship between the postulates is also needed if the method of \[1\] is to be applied to the case of an arbitrary canonical algebra. The reason is that, in the existing formulation of the postulates, the overall consistency is only made certain by the fact that the reshuffling and ultralocality assumptions (iii) and (iv) are respected by the dynamical law of the theory (i). On the other hand, nothing in the remaining postulates ensures that assumptions (iii) and (iv) are the only ones compatible with this law. If different compatible assumptions are used as supplementary conditions to the algebra, the above method will yield different canonical representations. Note, however, that the dynamical law of the theory is the only assumption—besides the principle of path independence—that enters the derivation and geometric interpretation of the algebra. It follows that if the existing formulation of the postulates is used as an algorithm for passing from the interpretation of an algebra to its physical representations, it will be highly ambiguous.

We basically have in mind the interpretation of the genuine Lie algebra that was discovered by Brown and Kuchař [10]. There have been some interesting approaches in this subject [8, 11, 12], but they all have revolved round the abstract algebra, thus ignoring the actual procedure that led from the Dirac algebra to general relativity. For example, the solutions found by Markopoulou [8] are essentially the equivalent of the solutions (2.13) of the Dirac algebra. They do not depend on anything but the algebra and, as such, they are expected to contain certain unphysical representations among them. An unambiguous formulation of the algorithm in [1] will find here a most natural application.

As a final note, we point out an asymmetry in the formulation of the postulates that actually provides the main motivation for the paper. It concerns the kinematic equations (2.18) and (2.19) on which postulates (iii) and (iv) are based. Namely, if the identification of the canonical generators with the generators of normal and hypersurface deformations is to be taken as a fundamental principle in the canonical theory, one anticipates that it will hold for both the canonical variables. However, in an equal-time formalism one can neither confirm nor reject this conjecture simply because the action of the deformation generators on the canonical momenta cannot be defined without additional structure. Marolf [13] used the Hamiltonian as an additional structure to extend the Poisson bracket from a Lie bracket on phase space to a Lie bracket on the space of histories. What we do, instead, is to ignore completely the equal-time formalism, and proceed with a phase space whose Poisson bracket is defined over the space of histories from the beginning.
3 The history formalism.

3.1 The unconstrained Hamiltonian.

Consider the theory described by the canonical action

\[ S[q^A, p_A] = \int d^3x \, dt \left( p_A \dot{q}^A - \mathcal{H} \right), \]

\[ \mathcal{H} = NH + N^i H_i, \]  
(3.1)

where \( N \) and \( N^i \) are prescribed functions of space and time. The generators \( H \) and \( H_i \) are given functions of the canonical fields \( (q^A, p_A) \) and may also depend on additional prescribed fields \( c^K \). The index \( A \) runs from 1 to half the total number of canonical variables, while \( K \) runs from 1 to the total number of prescribed fields.

One can generalize the phase space to include the canonical fields at all times by introducing the space of histories,

\[ (q^A(x, t), p_A(x, t)), \]  
(3.2)

and defining on it the Poisson bracket

\[ \{ q^A(x, t), p_B(x', t') \} = \delta^A_B \delta(x, x') \delta(t, t'). \]  
(3.3)

The quantum analogue of the canonical fields in (3.2) is the one-parameter family of Schrödinger operators introduced by Isham et al in their study of continuous time consistent histories[3][14].

Using the bracket (3.3)—which turns the space of histories into a Poisson manifold—the variation of the canonical action can be concisely written in the form

\[ \{ S, q^A(x, t) \} \simeq 0 \]  
(3.4)

\[ \{ S, p_A(x, t) \} \simeq 0, \]  
(3.5)

and defines a constraint surface on this space. The physical fields are defined to satisfy these relations for each value of \( x \) and \( t \). For the particular form (3.1) of the canonical action, the weak equations (3.4-3.5) become

\[ \dot{q}^A(x, t) \simeq \int d^3x' \, dt' \{ q^A(x, t), \mathcal{H}(x', t') \} \equiv \int d^3x' \frac{\delta \mathcal{H}}{\delta p_A}(x', t) \delta(x, x') \]  
(3.6)

\[ \dot{p}_A(x, t) \simeq \int d^3x' \, dt' \{ p_A(x, t), \mathcal{H}(x', t') \} \equiv \int d^3x' \frac{\delta \mathcal{H}}{\delta q^A}(x', t) \delta(x, x'), \]  
(3.7)

1 Throughout this paper, the functional derivative \( \frac{\delta F}{\delta q^A} \) is defined by \( \frac{\delta F}{\delta q^A} = \frac{\partial F}{\partial q^A} + \frac{\partial F}{\partial q^A_{ij}} \partial_i + \frac{\partial F}{\partial q^A_{iij}} \partial_{ij} + \ldots etc. \) We will call \( F \) a functional, although we essentially mean a local function of the canonical variables and a finite number of their derivatives.
which can be recognised as Hamilton’s equations in the usual equal-time sense. This follows
from the fact that the Hamiltonian in equation (3.1) is by construction independent of any
time derivatives and, hence, one can integrate trivially over \( \int dt' \delta(t, t') \).

The weak equality sign is a reminder of the fact that Hamilton’s equations—and hence
the actual theory—are not preserved under a general Poisson bracket. In the equal-time
formalism this presents no problem because the canonical velocities are only defined ex-
ternally but, here, they are equally included in the phase space. As a result, the Poisson
bracket between a field velocity and its conjugate momentum can be evaluated to give a
time derivative of the \( \delta \)-function, which is not the result one will get if the corresponding
Hamilton equation is used to replace the field velocity before the commutation. Nonetheless,
since the theory is about time evolution only, it is sufficient that Hamilton’s equations
are preserved weakly under the Poisson bracket with the Hamiltonian.

In the unconstrained theory (3.1) this follows automatically from Hamilton’s equations
and the definition of the general time Poisson bracket (3.3) without any reference to the
specific form of the Hamiltonian. Before checking this explicitly, however, we need to extend
the definition of the Hamiltonian in order to incorporate the trivial dynamical evolution
of the prescribed functions \( c^K, N \) and \( N^i \). This is also appropriate for the completeness of
the formalism.

3.2 Incorporating the fixed functions.

One defines the extended unconstrained action by

\[
S[q^A, p_A, \omega_K, \omega, \omega_i] = \int d^3x dt \left( p_A \dot{q}^A + \omega_K \dot{c}^K + \omega \dot{N} + \omega_i \dot{N}^i - \mathcal{H}^{ext} \right),
\]

\[
\mathcal{H}^{ext} = NH + N^i H_i + \omega_K \dot{c}^K + \omega \dot{N} + \omega_i \dot{N}^i,
\] (3.8)

where the momenta \( \omega_K, \omega \) and \( \omega_i \) are defined through the Poisson bracket relations

\[
\{ c^K(x, t), \omega_L(x', t') \} = \delta^K_L \delta(x, x') \delta(t, t'),
\]

\[
\{ N(x, t), \omega(x', t') \} = \delta(x, x') \delta(t, t'),
\]

\[
\{ N^i(x, t), \omega_j(x', t') \} = \delta^i_j \delta(x, x') \delta(t, t').
\] (3.9)

These momenta are not assumed to have any direct physical significance or interpreta-
tion, and the whole purpose of their introduction is to allow the time derivative of the fixed
functions to be calculated inside the Poisson bracket formalism. Restricting our attention
to functionals of the canonical and the fixed variables one gets

\[
\{ F(x, t), \int d^3x' dt' \mathcal{H}^{ext}(x', t') \} = \frac{\delta F}{\delta q^A(x, t)} \{ q^A(x, t), \int d^3x' dt' \mathcal{H}^{ext}(x', t') \}
+ \frac{\delta F}{\delta p_A}(x, t) \{ p_A(x, t), \int d^3x' dt' \mathcal{H}^{ext}(x', t') \} + \frac{\delta F}{\delta c^K}(x, t) c^K(x, t)
+ \frac{\delta F}{\delta N}(x, t) N(x, t) + \frac{\delta F}{\delta N^i}(x, t) N^i(x, t) \simeq \dot{F}(x, t),
\] (3.10)
which implies that the extended Hamiltonian can be seen as the canonical representation of the total time derivative operator.

Equivalently one may observe that, when acting on $F$, the kinematical half of the extended action
\[
\int d^3t \left( p_A \dot{q}^A + \omega_K \dot{c}^K + \omega \dot{N} + \omega_i \dot{N}^i \right)
\]
produces the time derivative of $F$ in the strong sense. A weakly vanishing result on the other hand arises, by definition, when the total extended action acts on any $F$. One concludes that the remaining half of the action—i.e., the dynamical half corresponding to the integral of the extended Hamiltonian—produces the total time derivative of $F$ in the weak sense.

Using either of the above methods, one can prove that Hamilton’s equations are automatically preserved under the dynamical evolution of the theory. Indeed, if $F$ is any functional of the canonical and the fixed variables that vanishes on the constraint surface, it follows that its total time derivative will also vanish on the same surface. Since this derivative is weakly equal to the commutation of $F$ with the integral of the extended Hamiltonian, it follows that all weakly vanishing functionals remain weakly zero under this commutation. Choosing these $F$s to be Hamilton’s equations themselves shows that the constraint surface is preserved. This completes the treatment of systems that are unconstrained in the usual sense.

### 3.3 The constrained Hamiltonian.

The extended form of the action, equation (3.8), arises naturally when the functions $N$ and $N^i$ are either constrained canonical variables or acquire the meaning of Lagrange multipliers. An example of the first case is the history formulation of general relativity, where one does not use the Dirac procedure for passing to the Hamiltonian but, instead, follows the usual Legendre definition without replacing the non-invertible velocity terms. An example of the second case is the history formulation of parametrized theories.

We present both these cases in their most general form by considering the canonical action
\[
S[q^A, p_A, N, \omega, N^i, \omega_i, \omega_K] = \int d^3t \left( p_A \dot{q}^A + \omega_K \dot{c}^K + \omega \dot{N} + \omega_i \dot{N}^i - \mathcal{H} \right),
\]
\[
\mathcal{H} = NH + N^i H_i + \omega_K \dot{c}^K + \omega \dot{N} + \omega_i \dot{N}^i,
\]
which is now additionally varied with respect to the functions $N$ and $N^i$. The fields $c^K$ are still treated as fixed.

The variation of (3.12) leads to the same equations as before, namely
\[
\{S, q^A(x, t)\} = 0 \Leftrightarrow \dot{q}^A(x, t) \simeq \int d^3x' \left( N \frac{\delta \mathcal{H}}{\delta p_A} + N^i \frac{\delta H_i}{\delta p_A} \right)(x', t)\delta(x, x'),
\]
\[
\{S, p_A(x, t)\} = 0 \Leftrightarrow \dot{p}_A(x, t) \simeq \int d^3x' \left( N \frac{\delta \mathcal{H}}{\delta q^A} + N^i \frac{\delta H_i}{\delta q^A} \right)(x', t)\delta(x, x'),
\]
\[
\{S, c^K(x, t)\} = 0 \Leftrightarrow \dot{c}^K(x, t) = c^K(x, t) \Leftrightarrow 0 = 0,
\]
\{S, N(x, t)\} = 0 \iff \dot{N}(x, t) = 0 = 0, \quad (3.16)
\{S, N^i(x, t)\} = 0 \iff \dot{N}^i(x, t) = 0 = 0, \quad (3.17)

subject to the additional equations
\{S, \omega(x, t)\} \simeq 0 \iff \dot{\omega}(x, t) \simeq \omega(x, t) + H(x, t) \simeq 0, \quad (3.18)
\{S, \omega^i(x, t)\} \simeq 0 \iff \dot{\omega}^i(x, t) \simeq \omega^i(x, t) + H_i(x, t) \simeq 0, \quad (3.19)

arising from the variation of the action with respect to \(N\) and \(N^i\).

For a functional \(F[q^A, p_A, c^K, N, N^i]\) the proof of the previous section still applies,
\[ \{F(x, t), \int d^3x' dt' H(x', t')\} \simeq \dot{F}(x, t), \quad (3.20) \]
with the weak equality referring to Hamilton’s equations (3.13-3.14). It follows that if \(F\) is any functional that vanishes on the surface defined by Hamilton’s equations, its time derivative will also vanish on this surface, and by taking \(F\) to be Hamilton’s equations themselves one can deduce that (3.13-3.17) are weakly preserved under the dynamical evolution of the theory. On the other hand, if \(F\) vanishes on the surface defined by the constraint equations (3.18-3.19), its time derivative will still vanish on this surface but, now, it does not follow that this time derivative will be the one generated by the Hamiltonian of the theory.

One must ensure that the time derivatives of the fields calculated by differentiating equations (3.18-3.19) are compatible with the time derivatives of the same fields calculated from Hamilton’s equations. If the constraints (3.18-3.19) do not depend on the prescribed fields \(c^K\)—which is the case for most of the physical theories—this compatibility condition results in the requirement that the algebra of \(H\) and \(H_i\) must close weakly under the general-time Poisson bracket. Since \(H\) and \(H_i\) are by construction independent of any time derivatives, the weak closure of the algebra only refers to the constraint equations (3.18-3.19).

4 The evolution postulate.

4.1 The inverse procedure and the evolution postulate.

The aim is to invert the above argument, and derive the general canonical Hamiltonian of a theory from a set of first principles. The requirement for these principles to be minimal implies that the appropriate starting point of the derivation is the form (3.12) of the canonical action. This form, which is the only prerequisite for the existence of a canonical algebra in the theory, is valid for both constrained and unconstrained systems, and is present in both the approaches in [1] and [6] as the so-called “evolution postulate”. In case that this postulate turns out to be insufficient to determine the theory completely, the plan is that any supplementary conditions that may be added must be such that the connection between them remains clear throughout the derivation.
The evolution postulate is to be understood as follows. Initially, one looks for the most general canonical representation of the Hamiltonian that satisfies the unconstrained version of the postulate,

\[
\frac{\partial}{\partial t} q^A(x, t) \simeq \int d^3 x' dt' \{q^A(x, t), (NH + N^i H_i)(x', t')\},
\]

(4.1)

\[
\frac{\partial}{\partial t} p_A(x, t) \simeq \int d^3 x' dt' \{p_A(x, t), (NH + N^i H_i)(x', t')\},
\]

(4.2)

where the functions \(N\) and \(N^i\) can take arbitrary values and the canonical generators can also depend on some fixed fields \(c^K\). It should be mentioned here that when we say "can take arbitrary values" we essentially mean that the formalism allows \(N\) and \(N^i\) to take arbitrary values, although in practice \(N\) and \(N^i\) will be required to be positive.

If such a Hamiltonian cannot be found, one resorts to the alternative possibility of varying the action with respect to the functions \(N\) and \(N^i\). Equations (4.1-4.2) must then be supplemented by the constraint equations

\[
H(x, t) \simeq 0,
\]

(4.3)

\[
H_i(x, t) \simeq 0,
\]

(4.4)

which have to be preserved under the dynamical evolution of the theory. This consistency requirement—amounting to the weak closure of the algebra—is supposed to be included among the evolution postulate for constrained systems. Note that since the time derivatives of \(N\) and \(N^i\) do not appear in the equations of motion (4.3-4.4), \(N\) and \(N^i\) can still take arbitrary numerical values. The evolution postulate for both constrained and unconstrained systems can then be stated as the requirement that the canonical action is of the form (3.12) with \(N\) and \(N^i\) arbitrary.

4.2 The evolution postulate in an equivalent form.

At first sight, conditions (4.1-4.2) seem to be rather too loose for something definite to be drawn out of them. It seems that the canonical representations can be chosen at will, and that any constrained theory can be created by just requiring the closure of the resulting algebra. This view changes radically, however, when one realizes that the canonical fields have a precise geometric meaning that has to be respected by the Hamiltonian system. In a scalar field theory, for example, the field \(\phi(x, t)\) is not merely a spatial scalar but is also by definition the pull-back of a spacetime scalar field. Below, the evolution postulate is transformed to an equivalent condition on the canonical generators that is more appropriate for the exploitation of this fact.

The functions \(N\) and \(N^i\) are chosen as the lapse function and the shift vector, and this will be the case henceforth unless stated otherwise. Decomposing the time derivative operator in equations (4.1-4.2) according to the lapse-shift formula,

\[
\frac{\partial}{\partial t} = Nn^\alpha \frac{\partial}{\partial X^\alpha} + N^i \mathcal{X}^\alpha i \frac{\partial}{\partial X^\alpha},
\]

(4.5)
and introducing the momentum $\mathcal{P}_\alpha$ conjugate to the embedding, i.e.,

$$\{\mathcal{X}^\alpha(x, t), \mathcal{P}_\beta(x', t')\} = \delta^\alpha_\beta \delta(x, x') \delta(t, t')$$

(4.6)

one can bring (4.1-4.2) into the form:

$$\{q^A(x, t), H(x', t')\} \simeq \{q^A[\mathcal{X}(x, t)], \mathcal{P}_\beta(x', t')\} n^\beta(x', t'),$$

(4.7)

$$\{q^A(x, t), H_i(x', t')\} \simeq \{q^A[\mathcal{X}(x, t)], \mathcal{P}_\beta(x', t')\} \mathcal{X}^{\beta}_i(x', t'),$$

(4.8)

$$\{p_A(x, t), H(x', t')\} \simeq \{p_A[\mathcal{X}(x, t)], \mathcal{P}_\beta(x', t')\} n^\beta(x', t'),$$

(4.9)

$$\{p_A(x, t), H_i(x', t')\} \simeq \{p_A[\mathcal{X}(x, t)], \mathcal{P}_\beta(x', t')\} \mathcal{X}^{\beta}_i(x', t').$$

(4.10)

Note that the arbitrariness of $N$ and $N_i$ was used to eliminate the integration.

On the right side of the above equations the explicit dependence of the fields on the spacetime embedding is fully taken into account. For the configuration fields this is just the dependence arising from the definition of the fields as geometric objects in spacetime. For the conjugate fields the situation is more complicated, and equation (4.1) is assumed to be inverted to express the momenta as functionals of the configuration variables, the lapse, the shift, and the prescribed fields $c^K$. All the latter have a definite dependence on the spacetime embedding which is then conveyed to the conjugate canonical fields.

Equation (4.1) is always invertible for the momenta because the system is by construction constrained only in the quantities $N$ and $N_i$ at the very most. There is one exception to this rule when the action is not derivable from a spacetime Lagrangian but, instead, is brought into the form (3.12) through the introduction of Lagrange multipliers, as in the case of parametrized theories. Nonetheless, the momentum can still be defined as a functional of the spacetime embedding as will be shown elsewhere[15].

For the purposes of performing actual calculations, the evolution postulate is to be used in the following way. Any time derivatives of the canonical variables that arise on the right side of equations (4.7-4.10) are replaced by the original Hamilton’s equations (4.1-4.2). When the theory is unconstrained, this results in a coupled system of four functional differential equations for $H$ and $H_i$, whose solution—if it exists—corresponds to the general canonical representation compatible with the evolution postulate. When the theory is constrained, on the other hand, the resulting conditions on $H$ and $H_i$ are not proper differential equations since it is sufficient that they only hold on the constraint surface (4.3-4.4).

If the constraints (4.3-4.4) implied that the canonical variables can not be treated as independent in these conditions for $H$ and $H_i$, the evolution postulate for constrained systems would not make any sense at all. However, by construction of the canonical formalism the constraints must be imposed only after the Poisson brackets have been evaluated. It follows therefore that—even for constrained systems—the differential equations for $H$ and $H_i$ must be solved treating the canonical variables as independent and imposing the constraints (4.3-4.4) only at the end. Equivalently, one simply adds on each of the differential equations an arbitrary term whose value is required to vanish on the constraint surface. We shall see exactly how this works in the following section.
Finally note that the replacement of the field velocities in equations (4.7-4.10) by the original and equivalent equations (4.1-4.2) does not lead to cyclic identities as one might have expected. The reason is that—due to the arbitrariness of $N$ and $N^i$—the latter equations hold in integrated form while the former hold at every point in space and time. The information incorporated in these equations is actually so rich that it determines the canonical theory completely.

5 The canonical theory regained.

That no further assumptions are needed in order to recover a canonical theory from first principles can be shown in an indirect way, by starting from the evolution postulate and deriving the additional postulates of Kuchař et al. For constrained systems, it turns out that these postulates have to be imposed weakly, which is also predicted from a revised version of Teitelboim’s argument. The new solutions arising from this modification are displayed here in the case of gravity.

5.1 Derivation of the reshuffling and ultralocality postulates.

On the right side of equations (4.7-4.8) the configuration fields are treated as functionals of the embedding relative to which the decomposition of the spacetime theory has been performed. The reshuffling and ultralocality postulates follow immediately from equations (4.7-4.8) once the geometric meaning of the configuration variable is taken into account. This is also recognized in [1] although, there, the emphasis is given on the compatibility of the postulates with the dynamical law (4.1-4.2) rather than on the fact that the postulates are uniquely determined by this law. Referring to the corresponding comment in section 2, it is only because of this fact that the method in [1] can be used unambiguously as an algorithm for finding the physically relevant representations of a general canonical algebra.

Below, we write down the ultralocality and reshuffling postulates for the physical examples that one usually considers. The relevant calculations can be found in the appendix. Note that a strong equality sign is used, with the understanding that all canonical velocities have been eliminated through the corresponding Hamilton’s equations. This is consistent with our general plan, according to which we originally look for an unconstrained representation of the evolution postulate. If a theory is proved to be constrained we will revise the following equations accordingly.

Scalar field theory. The configuration variable is the pullback of a spacetime scalar field,

$$\phi(x, t) = \phi[X](x, t),$$

and, as such, is an ultralocal function of the embedding. Equations (4.7-4.8) become

$$\{\phi(x, t), H^\beta(x', t')\} = \phi,_{\beta}(x, t)n^\beta(x, t)\delta(x, x')\delta(t, t')$$  \hspace{1cm} (5.2)

$$\{\phi(x, t), H^\beta_i(x', t')\} = \phi,_{i}(x, t)\delta(x, x')\delta(t, t'),$$  \hspace{1cm} (5.3)
which can be recognized as the history analogues of the reshuffling and ultralocality conditions in [3]. Indeed, the $\delta(t, t')$ function indicates that the canonical generators are independent of the field velocities, the ultralocality of the first equation implies that the super-Hamiltonian is an ultralocal function of the momenta, while the form of the second equation ensures that the super-momentum just reshuffles the data on the hypersurface.

**General relativity.** The configuration variable is the pullback of the spacetime metric,

$$g_{ij}(x, t) = \gamma_{\alpha\beta}[\mathcal{X}](x, t)\mathcal{X}^\alpha_i(x, t)\mathcal{X}^\beta_j(x, t).$$

and equations (4.7-4.8) result in the following conditions on the canonical generators,

$$\{g_{ij}(x, t), H^g_{\alpha\beta}(x', t')\} = g_{ki}(x, t)\delta_{ij}(x, x')\delta(t, t') + g_{kj}(x, t)\delta_i(x, x')\delta(t, t'),$$

$$\{g_{ij}(x, t), H^g_{\alpha\beta}(x', t')\} = 2n_{\alpha\beta}(x, t)\mathcal{X}^\alpha_i(x, t)\mathcal{X}^\beta_j(x, x')\delta(x, x')\delta(t, t').$$ (5.5)

These are indeed equivalent to the reshuffling and ultralocality postulates (2.20-2.21) that are used in [4].

**Deformation and parametrized theories.** For the theory of hypersurface deformations, the configuration variable is the embedding itself. Equations (5.7-5.8) become

$$\{\mathcal{X}^\alpha(x, t), H^D(x', t')\} = n^\alpha(x, t)\delta(x, x')\delta(t, t'),$$

$$\{\mathcal{X}^\alpha(x, t), H^D_i(x', t')\} = \mathcal{X}^\alpha_i\delta(x, x')\delta(t, t'),$$ (5.8)

which are the reshuffling and ultralocality conditions for the deformation theory. Putting equations (5.7-5.8) and (5.2-5.3) together, one gets the corresponding conditions for a parametrized scalar field theory.

### 5.2 Derivation of the super-momentum constraint, of the representation postulate, and of the principle of path independence.

This is the revised version of Teitelboim’s argument that was mentioned in section 2, so some of the following results can be found in [3][4] and are only stated here for completeness. Besides the revision of the argument for constrained systems, the other main difference between this approach and the approach in [4] is that the present argument does not rely on the principle of path independence but derives it.

For the representation postulate the derivation starts from the following two Jacobi identities,

$$\{\{H_j(x', t'), F(x'', t''), H_i(x, t)\}, H_j(x', t')\} + \{\{F(x'', t''), H_i(x, t)\}, H_j(x', t')\}$$

$$+ \{\{H_i(x, t), H_j(x', t')\}, F(x'', t'')\} = 0,$$ (5.9)

$$\{\{H^D_j(x', t'), F(x'', t''), H^D_i(x, t)\}, H^D_j(x', t')\} + \{\{F(x'', t''), H^D_i(x, t)\}, H^D_j(x', t')\}$$

$$+ \{\{H^D_i(x, t), H^D_j(x', t')\}, F(x'', t'')\} = 0,$$ (5.10)
that hold, respectively, on the canonical and on the deformation history phase space. The arbitrary functional \( F \) depends on both the canonical variables \( q^i \) and \( p_A \), while the action of the deformation generators on these variables is defined as in section 4. The notation for the normal and tangential projections of \( P_\alpha \) is chosen to coincide with the equal-time definitions \((2.10, 2.17)\).

We consider only the case when the canonical Hamiltonian is independent of the prescribed fields \( c^K \), which is the relevant case for general relativity. When prescribed fields are present in the Hamiltonian the following derivation still applies but depends on the actual character of the prescribed fields and—for simplicity—is avoided. An extensive account of such systems can be found in [19].

Having restricted \( H, H_i \) and \( F \) to be pure functionals of the canonical variables, we compare the first terms in the identities \((5.9)\) and \((5.10)\). The evolution postulate implies that

\[
\{ H_j(x', t'), F(x'', t'') \} = \{ H^D_j(x', t'), F(x'', t'') \},
\]

where both brackets depend solely on the canonical variables because of the restrictions imposed. The use of the strong sign is due to the replacement of the field velocities, as it has been already explained. A further application of the evolution postulate then gives

\[
\{ H_j(x', t'), F(x'', t'') \}, H_i(x, t) \} = \{ H^D_j(x', t'), F(x'', t'') \}, H^D_i(x, t) \},
\]

where we have used the fact that Hamilton’s equations are preserved under the commutation with the Hamiltonian.

Repeating this argument when comparing the second terms in the identities \((5.9)\) and \((5.10)\), we get that

\[
\{ F(x'', t''), H_i(x, t) \}, H_j(x', t') \} = \{ F(x'', t''), H^D_i(x, t) \}, H^D_j(x', t') \},
\]

which implies that the remaining terms in the identities should also be equal,

\[
\{ H_i(x, t), H_j(x', t') \}, F(x'', t'') \} = \{ H^D_i(x, t), H^D_j(x', t') \}, F(x'', t'') \} = 0.
\]

The commutation between the two deformation generators is calculated to give the history analogue of the Dirac relation \((2.8)\),

\[
\{ H^D_i(x, t), H^D_j(x', t') \} = H^D_j(x, t)\delta_i(x, x')\delta(t, t') - (ix \leftrightarrow jx'),
\]

and then the evolution postulate is used once more to give an equation that holds exclusively on the canonical phase space,

\[
\left\{ \left[ H_i(x, t), H_j(x', t') \right] - \left( H_j(x, t)\delta_i(x, x')\delta(t, t') - (ix \leftrightarrow jx') \right) \right\}, F(x'', t'') \} = 0.
\]

Since it holds for any choice of the functional \( F \), the following relation for the supermomenta arises,

\[
\{ H_i(x, t), H_j(x', t') \} = H_j(x, t)\delta_i(x, x')\delta(t, t') + C_{ij}[x, t; x', t'] - (ix \leftrightarrow jx'),
\]
where \( C_{ij} \) is just a constant term.

The same argument can be applied to the mixed Jacobi identities

\[
\{\{H_j(x', t'), F(x'', t'')\}, H(x, t)\} + \{\{F(x'', t''), H(x, t)\}, H_j(x', t')\}
+ \{\{H(x, t), H_j(x', t')\}, F(x'', t'')\} = 0,
\]

resulting in the relation

\[
\{H(x, t), H_i(x', t')\} = H(x, t)\delta_i(x, x')\delta(t, t') + H_i(x, t)\delta(x, x')\delta(t, t') + C_i[x, t; x', t'],
\]

with \( C_i \) being constant.

The situation changes considerably, when the argument is applied to the remaining identities between the super-Hamiltonians,

\[
\{\{H(x', t'), F(x'', t'')\}, H(x, t)\} + \{\{F(x'', t''), H(x, t)\}, H(x', t')\}
+ \{\{H(x, t), H(x', t')\}, F(x'', t'')\} = 0,
\]

This leads to the relation

\[
\{\{H(x, t), H(x', t')\}, F(x'', t'')\} = \{\{H^D(x, t), H^D(x', t')\}, F(x'', t'')\},
\]

whose left and right side is evaluated on the canonical and on the deformation phase space, respectively.

Considering the Poisson bracket between the deformation generators one has to deal with the fact that the Dirac algebra is not a genuine Lie algebra but depends explicitly on the spatial metric,

\[
\{H^D(x, t), H^D(x', t')\} = g^{ij}(x, t)H^D_i(x, t)\delta_j(x, x')\delta(t, t') - (x \leftrightarrow x').
\]

Since the theory is by assumption independent of any prescribed fields, it follows that the metric has to be a canonical variable in order to appear in equation (5.23).

Using the evolution postulate and the fact that the metric is a canonical variable, one writes equation (5.23) exclusively on the canonical phase space,

\[
\begin{align*}
\{\{H(x, t), H(x', t')\} &- \left( g^{ij}(x, t)H_i(x, t)\delta_j(x, x')\delta(t, t') - (x \leftrightarrow x') \right)\}, F(x'', t'') \}
= -\left( H_i(x, t)\delta_j(x, x')\delta(t, t')\left( g^{ij}(x, t), F(x'', t'') \right) - (x \leftrightarrow x') \right).
\end{align*}
\]

The term on the right side is the compensation needed in order for the metric to be taken inside the Poisson brackets in the canonical phase space.
Because equation (5.25) is a linear first order equation holding for an arbitrary choice of functional \( F \), it cannot be generally satisfied unless the super-momenta are constrained to vanish,

\[
H_i(x, t) \simeq 0.
\]  

(5.26)

The proof follows from the fact that one can expand both sides of equation (5.25) in terms of the spatial derivatives of the \( \delta \)-functions and then always find—due to the linearity and the actual form of the equation—particular choices of functionals \( F \) that will violate at least one of the terms in the expansion.

The constraint (5.26) leads to

\[
\{ \{ H(x, t), H(x', t') \} - \left( g^{ij}(x, t) H_i(x, t) \delta_j(x, x') \delta(t, t') - (x \leftrightarrow x') \right) \}, F(x'', t'') \} \simeq 0,
\]  

(5.27)

which must also hold for every choice of functional \( F \). Teitelboim argued[6] that the weak equation (5.27)—which in the equal-time approach is derived from the principle of path independence—is enough to imply that the expression

\[
\{ H(x, t), H(x', t') \} - \left( g^{ij}(x, t) H_i(x, t) \delta_j(x, x') \delta(t, t') - (x \leftrightarrow x') \right)
\]  

(5.28)

vanishes strongly. Specifically, he argued that (5.28) must not depend on any canonical variables because, if it did, particular choices of functionals \( F \) could always be found to violate equation (5.27), in a process similar to the one described above. The quantity (5.28) should therefore be equal to a constant function, which is zero[6] because of the requirement that the algebra is weakly closed. The requirement of closure actually implies that the constant terms \( C_{ij} \) and \( C_i \) in equations (5.17) and (5.20) should also be zero[6] and, hence, the history analogue of the strong Dirac algebra is derived.

This argument is not generally true, however, because in a constrained system one must additionally ensure that all the terms in equation (5.27) are well-defined on the constraint surface. If any of the first partial derivatives of (5.28) does not vanish on the constraint surface Teitelboim’s argument can be applied indeed, and leads to the conclusion that the expression (5.28) is strongly zero. On the other hand, if both partial derivatives of (5.28) vanish weakly, one cannot find well-defined choices for functionals \( F \) that violate equation (5.27) because, to do so, would require the first partial derivatives of any such \( F \) to have an infinite value on the constraint surface. Consequently, the most general expression for the algebra between the super-Hamiltonians is the weak Dirac relation mentioned in section 2,

\[
\{ H(x, t), H(x', t') \} = g^{ij}(x, t) H_i(x, t) \delta_j(x, x') \delta(t, t') + G(x, t; x', t') - (x \leftrightarrow x'),
\]  

(5.29)

where both the first derivatives of \( G \) vanish on the constraint surface (5.26). Note that any constant terms are absorbed in this definition of \( G \).

One now has to go back and re-examine the validity of the steps that led to equations (5.17), (5.21) and (5.29), taking into account the fact that the system is constrained. The only requirement for the consistency of the previous procedure is the preservation
of any weak equality under commutation with the canonical generators. However, this is already included in the definition of the evolution postulate for constrained systems and, therefore, one simply has to replace any strong equality signs with weak ones. The complete history analogue of the weak Dirac algebra \( (2.9-2.11) \) is therefore obtained, as well as the weak reshuffling and ultralocality postulates and the rest of the weak evolution postulate. Note that although the term “weak” currently refers to the constraint surface \( (5.20) \) the arguments that we have used do not depend on the actual definition of this surface and, therefore, in case that the super-Hamiltonian is proved to be constrained all our conclusions will remain valid.

Finally, let us mention again that the path independence of the dynamical evolution does not need to be assumed separately in the above argument, but is a consequence of the evolution postulate. In particular, one starts from the derived weak Dirac algebra and the evolution postulate and repeats Teitelboim’s argument in the reverse order. It follows immediately that the change in the canonical variables during the dynamical evolution of the theory will be independent of the path used in their actual evaluation. This is of course to be expected when realising that the principle of path independence is a consequence of the integrability of Hamilton’s equations. The evolution postulate is just another name for these equations and, hence, any solution of the postulate will lead automatically to a path-independent dynamical evolution.

### 5.3 Derivation of the super-Hamiltonian constraint.

When the representation postulate is imposed in the weak sense, the super-Hamiltonian constraint does not follow immediately from the closure of the Dirac algebra—as in \[ 3 \]—but it is also necessary to take into account the actual form of equations \( (4.7-4.10) \). We consider these equations in the case of general relativity or, more accurately, in the case when the configuration variable is the pullback of the spacetime metric.

Refering to the corresponding comment at the end of section 4, the most general form of the weak evolution postulate is the following:

\[
\{ g_{ij}(x, t), H(x', t') \} = 2n_{\alpha;\beta}(x, t) \mathcal{X}_i^\alpha(x, t) \mathcal{X}_j^\beta(x, t) \delta(x, x') \delta(t, t') + V_{ij}(x, t; x', t'),
\]

\[
\{ g_{ij}(x, t), H_k(x', t') \} = g_{ki}(x, t) \delta_j(x, x') \delta(t, t') + g_{kj}(x, t) \delta_i(x, x') \delta(t, t') + V_{ijk}(x, t; x', t'),
\]

\[
\{ p^{ij}(x, t), H(x', t') \} = \{ p^{ij}[\mathcal{X}(x, t)], H^P(x', t') \} + W^{ij}(x, t; x', t'),
\]

\[
\{ p^{ij}(x, t), H_k(x', t') \} = \{ p^{ij}[\mathcal{X}(x, t)], H^P_k(x', t') \} + W^{ij}_k(x, t; x', t').
\]

The tensors \( V_{ij}, V_{ijk}, W^{ij} \) and \( W^{ij}_k \) depend on the canonical fields and are required to vanish on the constraint surface \( H_i \approx 0 \). Because of the existence of the additional terms, the general solution of the coupled set \( (5.30-5.33) \) cannot be found explicitly. Nevertheless, the form of the evolution postulate allows some definite conclusions to be drawn, a part of which can be used to prove that the Hamiltonian is constrained. A full discussion can be found in \[ 14 \].
The important observation[1] is that the conjugate momentum $p^{ij}$ must be a tensor density of weight one, in order that the form $p^{ij} \delta g_{ij}$ that appears in the canonical action be coordinate independent. As a result, the Poisson brackets between the tangential density of weight one, in order that the form

$$p^{ij}(x, t), H_k(x', t') = \delta^j_k \delta^{im}(x, t) \delta_{m}(x, x') \delta(t, t') + \delta^i_k \delta^{pm}(x, t) \delta_{m}(x, x') \delta(t, t')$$

$$-p^{ij}(x, t) \delta_k(x, x') \delta(t, t') - p^{ij}(x, t) \delta(x, x') \delta(t, t')$$

$$+ W^{ij}_k(x, t; x', t').$$

(5.34)

Consider therefore a solution $(H, H_i)$ of the system (5.31-5.33), taking into account equation (5.34). By the assumption of existence of such a solution, the left sides of equations (5.31) and (5.34) must satisfy the integrability condition

$$\{\{g_{ij}(x, t), H_k(x', t')\}, p^{mn}(x'', t'')\} = \{\{p^{mn}(x'', t''), H_k(x', t')\}, g_{ij}(x, t)\}. \quad (5.35)$$

Because the non-vanishing terms in equations (5.31) and (5.34) are integrable[1], the weakly vanishing terms in the same equations should also be integrable,

$$\{V_{ijk}(x, t; x', t'), p^{mn}(x'', t'')\} = \{W^{mn}_{ik}(x'', t''; x', t'), \{g_{ij}(x, t)\}, \quad (5.36)$$

and, hence, one can always find some functionals $H^*_{i}$ and $K_i$ satisfying

$$\{g_{ij}(x, t), H^*_{k}(x', t')\} = g_{ki}(x, t) \delta_j(x, x') \delta(t, t') + g_{kj}(x, t) \delta_{i}(x, x') \delta(t, t')$$

$$+ g_{ij}(x, t) \delta(x, x') \delta(t, t'), \quad (5.37)$$

$$\{p^{ij}(x, t), H^*_{k}(x', t')\} = \delta^j_k \delta^{im}(x, t) \delta_{m}(x, x') \delta(t, t') + \delta^i_k \delta^{pm}(x, t) \delta_{m}(x, x') \delta(t, t')$$

$$-p^{ij}(x, t) \delta_k(x, x') \delta(t, t') - p^{ij}(x, t) \delta(x, x') \delta(t, t'), \quad (5.38)$$

$$\{g_{ij}(x, t), K_{k}(x', t')\} = V_{ijk}(x, t; x', t'), \quad (5.39)$$

$$\{p^{ij}(x, t), K_{k}(x', t')\} = W^{ij}_k(x, t; x', t'). \quad (5.40)$$

It follows from equations (5.34-5.40) that every solution $H_i$ of the weak evolution postulate can be written as the sum of two terms,

$$H_i = H^*_{i} + K_i. \quad (5.41)$$

Furthermore, the form of $H^*_{i}$ is uniquely fixed by equations (5.37) and (5.38), and corresponds to the super-momentum of general relativity[1],

$$H^*_{i} = H^{gr}_{i}, \quad (5.42)$$

written explicitly in equation (2.3).

We can now show that the super-Hamiltonian of the theory is constrained. As in [1], this follows from the preservation of the super-momentum constraint under the dynamical evolution, resulting in the condition

$$\{H(x, t), H_i(x', t')\} \simeq 0. \quad (5.43)$$
Using equations (5.41) and (5.42), this condition can be written as
\[
\{H(x, t), \left[ H^{gr}_i(x', t') + K_i(x', t') \right] \} \simeq 0 \quad (5.44)
\]
or, equivalently, as
\[
\{H(x, t), H^{gr}_i(x', t') \} \simeq 0. \quad (5.45)
\]
To obtain equation (5.44) we used equations (5.39-5.40) and the fact that \( W_{ijk} \) and \( V_{ijk} \) vanish on the constraint surface (5.26).

The form (2.3) of the gravitational super-momentum is such that the left side of equation (5.45) depends only on the weight of the super-Hamiltonian—necessarily being one,[1]
\[
\{H(x, t), H^{gr}_i(x', t') \} = H(x, t)\delta_{i}(x, x')\delta(t, t') + H_i(x, t)\delta(x, x')\delta(t, t'), \quad (5.46)
\]
and hence the constraint \( H \simeq 0 \) is proved. Recall that the actual definition of the constraint surface does not affect the validity of any of the above arguments, and hence the procedure just described remains consistent under the additional constraint.

6 Final comments.

We have shown that the procedure devised by the authors of [1] corresponds to the requirement that the canonical action is of the form (3.12). For unconstrained systems the correspondence is exact, and the strong reshuffling, ultralocality and representation postulates determine the form of the canonical theory completely. For systems subject to constraints the correspondence is not exact unless the strong postulates are replaced by weak ones, in which case new canonical representations arise.

Although the understanding of the relationship between the strong and the weak equations is no longer an issue—recall that in the revised version no strong equations are used—a corresponding issue still exists, and concerns the relation between the “strong” and “weak” solutions of the evolution postulate. In particular, there is need to understand exclusively in terms of the weak evolution postulate how the standard representation of general relativity arises, and also to find out if the new representations are physically equivalent to the standard one. By “physically equivalent” we mean to generate weakly the same equations of motion and to lead to the same constraint surface.

A preliminary examination of this issue was actually carried out in the previous section, when proving that the super-Hamiltonian of the theory is constrained. Indeed, equation (5.41) shows that the standard representation of the super-momentum can be derived from the evolution postulate as the special case \( K_i = 0 \). In addition, equations (5.31) and (5.34) show that all solutions \( H_i \) generate weakly the same equations of motion, while—starting from equation (5.41)—it can also be shown[13] that the constraints \( H_i \) and \( H^{gr}_i \) imply each other. The representations \( H_i \) and \( H^{gr}_i \) are therefore physically equivalent, and the privileged position occupied by \( H^{gr}_i \) is merely because the standard description of the system is minimal.
On the other hand, whether the same is true for the representations of the super-Hamiltonian cannot be said without further examination. The complication arises because of the inversion of equation (4.1) in order to define the momenta as functionals of the embedding, and also because of the replacement of the field velocities on the right side of equations (5.30) and (5.32) by use of (4.1-4.2). The former procedure involves numerous calculations because the inversion can only be achieved implicitly, while the latter implies that the right side of equations (5.30) and (5.32) will not be the same for all representations and, therefore, makes the issue of the physical equivalence rather unclear. It would be certainly interesting if representations could be found that are not equivalent to the standard super-Hamiltonian, but this possibility is rather remote considering the restrictions imposed on the spacetime character of any such representations by Lovelock's theorem[17]. We hope to be able to say more about the new representations in the future[15].

7 Acknowledgements.

My thanks go to C. Isham, K. Garth, C. Anastopoulos and K. Savvidou for their help. I would also like to thank the “Alexander S. Onassis Public Benefit Foundation” for their financial support.

A Poisson brackets in the history phase space.

In the following we denote \( \delta(x, x') \delta(t, t') \) by \( \delta \delta \), \( \partial \delta(x, x') \delta(t, t') \) by \( \delta \delta \), and \( \partial \delta(x, x') \delta(t, t') \) by \( \delta \delta \). If some expressions are calculated at \((x', t')\) they will be simply primed.

\[
\{X^\alpha, P_\beta\} = \delta^\alpha_\beta \delta \\
\{\gamma_{\alpha\epsilon}, P_\beta\} = \gamma_{\alpha\epsilon\beta} \delta \\
\{\gamma^{\alpha\epsilon}, P_\beta\} = \gamma^{\alpha\epsilon\beta} \delta \\
\{\delta^\alpha, P_\beta\} = 0 \\
\{X^\alpha_i, P_\beta\} = \delta^\alpha_\beta \delta_i \\
\{\gamma_{\alpha\beta} i, P_\beta\} = \gamma_{\alpha\beta\epsilon} \delta + \gamma_{\alpha\mu,\beta} X^\mu_i \delta \\
\{X^\alpha, P_\beta\} = \delta^\alpha_\beta \delta \\
\{n^\alpha, P_\beta\} = -n_\beta X^{\alpha m} \delta_m \delta - \frac{1}{2} \gamma_{\mu\nu,\beta} n^\mu n^\nu n^\alpha \delta - \gamma_{\mu\nu,\beta} n^\mu n^{\alpha\nu} \delta \delta \\
\{n_\alpha, P_\beta\} = -n_\beta X^{\alpha m} \delta_m \delta - \frac{1}{2} \gamma_{\mu\nu,\beta} n^\mu n^\nu n_\alpha \delta \\
\{g_{ij}, P_\beta\} = X^\alpha_{i} \gamma_j \delta_m \delta + X^\alpha_{j} \gamma_i \delta_m \delta + \gamma_{\mu\nu,\beta} X^\mu_i X^\nu_j \delta \delta \\
\{g^{ij}, P_\beta\} = -X^\beta_i g^{im} \delta_m \delta - X^\beta_j g^{im} \delta_m \delta - \gamma_{\mu\nu,\beta} X^{\mu i} X^{\nu j} \delta \delta \\
\{\delta^i, P_\beta\} = 0 \\
\{X^{\alpha i}, P_\beta\} = -n_\beta n^\alpha g^{im} \delta_m \delta - X^{\alpha m} X^\beta_i \delta_m \delta - \gamma_{\mu\nu,\beta} X_{\alpha m} X^{\nu m} X^{\mu i} \delta \delta
\]
\{X^i, P_\beta\} = -n_\alpha n_\beta g^{im} \delta_m \delta_i - \alpha^m X^i_\beta \delta_m \delta_i - \gamma_{\mu\nu, \beta} n_\alpha n_\nu X^{i\mu} \delta \delta_i \\
\{g, P_\beta\} = 2gX^m_\beta \delta_m \delta_i + g\gamma_{\mu\nu, \beta} X^{\mu m} \delta \delta_i \\
\{N, P_\beta\} = -n_\beta \delta \delta_\beta + n_\beta N^m \delta_m \delta_i - \frac{1}{2} N\gamma_{\mu\nu, \beta} n_\mu n_\nu \delta \delta_i \\
\{N^i, P_\beta\} = X^i_\beta \delta \delta_\beta + Nn_\beta g^{im} \delta_m \delta_i - N^m X^i_\beta \delta_m \delta_i + N\gamma_{\mu\nu, \beta} n_\mu X^{i\nu} \delta \delta_i \\
(A.1)
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