Kantorovich problems and conditional measures depending on a parameter

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Abstract. We study measurable dependence of measures on a parameter in the following two classical problems: constructing conditional measures and the Kantorovich optimal transportation. We obtain broad sufficient conditions for the existence of conditional probabilities measurably depending on a parameter in the case of parametric families of measures and mappings. A particular emphasis is made on the Borel measurability (which cannot be always achieved). Our second main result gives sufficient conditions for the Borel measurability of optimal transports and transportation costs with respect to a parameter in the case where marginal measures and cost functions depend on a parameter. As a corollary we obtain the Borel measurability with respect to the parameter for conditional measures of optimal plans. Finally, we show that the Skorohod parametrization of measures by mappings can be also made measurable with respect to a parameter.

Keywords: Kantorovich problem, conditional measure, weak convergence, measurable dependence on a parameter, Skorohod representation

AMS MSC 2010: 28C15, 60G57, 46G12

1. Introduction

This paper was motivated by several questions posed by Sergey Kuksin about measurable dependence of Kantorovich optimal transportation plans on a parameter in optimal transportation problems depending on a parameter.

We recall that, given two probability spaces \((X, \mathcal{B}_X, \mu)\) and \((Y, \mathcal{B}_Y, \nu)\) and a non-negative \(\mathcal{B}_X \otimes \mathcal{B}_Y\)-measurable function \(h\) on \(X \times Y\) (called a cost function), the associated Kantorovich problem is to find the infimum of the integral

\[
I_h(\sigma) := \int h \, d\sigma
\]

over all probability measures \(\sigma\) on \(\mathcal{B}_X \otimes \mathcal{B}_Y\) with projections \(\mu\) and \(\nu\) on the factors. This infimum is denoted by

\[
K_h(\mu, \nu)
\]

and called the transportation cost for \(h, \mu, \nu\). If this infimum is attained (is a minimum, which happens under broad assumptions), then the minimizing measures are called optimal measures (and also optimal plans or optimal transports). The measures \(\mu\) and \(\nu\) are called marginal distributions.

Suppose now that \((T, \mathcal{T})\) is a measurable space and for each \(t\) we have marginal probability measures \(\mu_t\) and \(\nu_t\) (which depend on \(t\) measurably in the sense that the functions \(t \mapsto \mu_t(A)\) are \(\mathcal{T}\)-measurable for all \(A \in \mathcal{B}_X\) and similarly for \(\nu_t\)) and that also the cost function depends on the parameter \(t\), i.e.,

\[
h: T \times X \times Y \to [0, +\infty)
\]

is a \(\mathcal{T} \otimes \mathcal{B}_X \otimes \mathcal{B}_Y\)-measurable function. We set

\[
h_t(x, y) := h(t, x, y).
\]

Thus, we obtain a Kantorovich problem with a parameter. Dependence on a parameter appears even for a single cost function if only marginal distributions depend
on $t$. The question is whether the infimum depends measurably on $t$ and there are optimal plans $\sigma_t$ measurably depending on $t$.

Several results have already been obtained in this situation. Villani [37] considered the situation where only the marginal distributions depend on a parameter (and are Borel measures on Polish spaces), but the cost function does not. Dedecker, Prieur and Raynaud De Fitte [14] studied the case of metric-type cost functions (such that $h(x, y) = \sup |u(x) - u(y)|$, where sup is taken over bounded continuous functions $u$ with $|u(x) - u(y)| \leq h(x, y)$) on rather general spaces (including completely regular Souslin spaces) and established the existence of a measurable selection of an optimal measure and the measurability of the Kantorovich minimum, however, this measurability is with respect to the $\sigma$-algebra of universally measurable sets, not with respect to the Borel $\sigma$-algebra. Similar results are also contained in [14, Sections 3.4 and 7.1]. Zhang [38] gave a result for continuous cost functions on Polish spaces $X$ and $Y$ and an arbitrary measurable space $T$, but the justification contains a gap and the really proved fact is this: if we consider the space $M = C(X \times Y)$ with the Borel $\sigma$-algebra corresponding to the metric

$$d_M(f, g) = \sum_{n=1}^{\infty} 2^{-n} \min(1, \sup_{z \in B_n} |f(z) - g(z)|),$$

where $\{B_n\}$ is a fixed sequence of increasing balls with the union $X \times Y$, and to every triple $(h, \mu, \nu)$ with a nonnegative function $h \in M$ we associate the set $\text{Opt}(h, \mu, \nu)$ of all optimal measures, then there is a selection of an optimal measure measurable with respect to the $\sigma$-algebras $B(M) \otimes B(\mathcal{P}(X)) \otimes B(\mathcal{P}(Y))$ and $B(\mathcal{P}(X \times Y))$. However, this does not imply the measurability claimed in [38] (the measurability with respect to $\mathcal{T}$ for a general $\sigma$-algebra $\mathcal{T}$), because the mapping $t \mapsto h(t, \cdot, \cdot)$ can fail to be measurable with respect to $\mathcal{T}$ and $B(M)$ under the only assumption that $h$ is measurable on $T \times X \times Y$. The point is that for a noncompact space $Z$ the Borel $\sigma$-algebra of the space $C_b(Z)$ with its sup-norm is not generated by evaluation functionals $z \mapsto f(z)$ (see Remark 5.12 below). A consequence of this in the situation of [38] is that the assumed measurability of the cost function is not sufficient for the applicability of the established selection result. However, it will be shown below in Theorem 4.2 that the main result of [38] is valid. Moreover, we show that optimal transports can be made Borel measurable with respect to the parameter for lower semicontinuous cost functions in place of continuous ones, provided that $T$ is a Souslin space with its Borel $\sigma$-algebra.

In the study of optimal plans one often deals with conditional measures. It is, of course, a question of independent interest to study conditional measures depending on a parameter (and this question was also suggested to us by Sergey Kuksin). The general framework for conditional measures is this: given a measure $\mu$ on a space $X$ and a measurable mapping $f$ of $X$ onto another measurable space $Y$, we are looking for measures $\mu^y$ concentrated on the level sets $f^{-1}(y)$ for $y \in Y$ such that $\mu$ has the form

$$\mu = \int_Y \mu^y \nu(dy),$$

where $\nu = \mu \circ f^{-1}(dy)$ is the image of $\mu$ under $f$ (or some other natural measure on $Y$). Below we recall a precise definition. Again, once $\mu$ and $f$ depend on a parameter $t$, the question is whether one can pick conditional measures $\mu_t^y$ measurably depending on $t$. A positive result was obtained in [28] (where a sketch of the
proof is given), but, as above, it is in terms of measurability with respect to the extensions of Borel \(\sigma\)-algebras generated by Souslin sets. Below (see Theorem 3.4) we provide all technical details for a more general result and complement this result by sufficient conditions for the Borel measurability (Theorem 3.5). Moreover, the existence of jointly (i.e., in both variables) Borel measurable conditional measures depending on a parameter is shown (see Proposition 3.10) to be equivalent to the existence of jointly Borel measurable right inverse mappings, similarly to the result of Blackwell and Ryll-Nardzewski [6] in the case of measures and mappings without parameters. It is worth noting that although sets from the \(\sigma\)-algebra generated by Souslin sets remain measurable with respect to all Borel measures, their weak point is that continuous images (say, projections) of such sets can fail to be measurable. This is one of motivations to desire the Borel measurability.

Both problems (dependence on a parameter for optimal plans and conditional measures) have some common features and are strongly connected with measurable choice theorems. It will be more convenient to start with conditional measures, which is done in Section 3. In Section 4 we discuss optimal plans and formulate our main results, which are proved in Section 5 along with a number of auxiliary results.

Finally, in Section 6 we consider along the same lines the classical result going back to Skorohod and giving a parametrization of Borel probability measures \(\mu\) on a Polish space \(X\) by Borel mappings \(\xi_\mu: [0,1] \to X\) such that \(\mu\) is the image of Lebesgue measure \(\lambda\) under \(\xi_\mu\) and measures \(\mu_n\) converge weakly to \(\mu\) if and only if the mappings \(\xi_{\mu_n}\) converge to \(\xi_\mu\) almost surely. We show that there is a version of \(\xi_\mu\) such that the function \((\mu,t) \mapsto \xi_\mu(t)\) is Borel measurable on \([0,1] \times \mathcal{P}(X)\). It follows that for any family of measures \(\mu_\omega\) measurably depending on a parameter \(\omega\), the function \((\omega,t) = \xi_{\mu_\omega}(t)\) is jointly Borel measurable.

2. Notation and Terminology

We shall consider Borel measures on complete separable metric spaces and in some results on Souslin spaces. So we briefly recall these concepts and some related objects.

Let \(X\) be a topological space. Its Borel \(\sigma\)-algebra (the smallest \(\sigma\)-algebra containing all open sets) is denoted by \(\mathcal{B}(X)\). A real function \(f\) on \(X\) is called Borel measurable if the sets \(\{x : f(x) < c\}\) are Borel for all \(c\).

The space of bounded continuous functions on \(X\) with its sup-norm is denoted by \(C_b(X)\). The space of bounded Borel measurable functions with the same norm is denoted by \(B_b(X)\).

If \((T, \mathcal{T})\) is a measurable space (i.e., \(\mathcal{T}\) is a \(\sigma\)-algebra), then a mapping \(f: T \to X\) is called \(T\)-measurable (or \((\mathcal{T}, \mathcal{B}(X))\)-measurable) if \(f^{-1}(B) \in \mathcal{T}\) for all \(B \in \mathcal{B}(X)\).

A space homeomorphic to a complete separable metric space is called Polish. A Hausdorff space that is the image of a complete separable metric space under a continuous mapping is called Souslin or analytic (see, e.g., [7], [25]). If such a mapping can be found one-to-one, then \(X\) is called a Luzin space. A Hausdorff space \(X\) is completely regular if for every point \(x \in X\) and every open set \(U\) containing \(x\) there is a continuous function \(f: X \to [0,1]\) such that \(f(x) = 1\) and \(f = 0\) outside \(U\).

Borel sets in Polish spaces are Souslin (and even Luzin) spaces; Borel sets in Souslin spaces are also Souslin. However, unlike the case of Borel sets, the complement of a Souslin set \(A\) in a Polish space is not always Borel, moreover, it can
be Borel only if $A$ itself is Borel. For this reason, the $\sigma$-algebra $\sigma(S(X))$ generated by the class $S(X)$ of all Souslin sets in $X$ is much larger than the Borel $\sigma$-algebra (although its cardinality is the continuum for infinite spaces); for example, in typical cases it is not countably generated (see [7, Example 6.5.9]).

Souslin sets belong to the Lebesgue completion of the Borel $\sigma$-algebra for every Borel measure on a Souslin space (i.e., they are universally measurable), hence the same is true for the generated $\sigma$-algebra $\sigma(S(X))$. However, this $\sigma$-algebra is not stable under the Souslin operation, unlike the completion of the Borel $\sigma$-algebra (see [7, p. 66]) and unlike the $\sigma$-algebra of universally measurable sets. The images and preimages of Souslin sets under Borel mappings are Souslin. For Borel sets, only preimages are Borel: it was shown by Souslin that the projection of a Borel set in $\mathbb{R}^2$ can fail to be Borel. Next, the preimages of sets in $\sigma(S(X))$ under Borel mappings are also in $\sigma(S(X))$. This is not true for their images even under continuous mappings: the projection of the complement of a Souslin set need not belong to $\sigma(S(X))$ (for example, the projection of the complement of a Souslin set need not be Lebesgue measurable).

Borel measures are finite (possibly, signed) measures on $\mathcal{B}(X)$. A signed Borel measure $\mu$ can be written as $\mu = \mu^+ - \mu^-$, where $\mu^+$ and $\mu^-$ are mutually singular nonnegative Borel measures. The measure $|\mu| = \mu^+ + \mu^-$ is called the total variation of $\mu$ and the number $\|\mu\| = |\mu|(X)$ is called the total variation norm or the variation norm. We mostly deal with probability measures.

A Borel measure $\mu$ is called Radon if for every Borel set $B$ and every $\epsilon > 0$ there is a compact set $K_\epsilon \subset B$ such that $|\mu|(B \setminus K_\epsilon) < \epsilon$. On a Souslin space all Borel measures are Radon.

The image of a measure $\mu$ on $X$ under a measurable mapping $f: X \to Y$ is denoted by $\mu \circ f^{-1}$ and defined by the equality

$$(\mu \circ f^{-1})(E) = \mu(f^{-1}(E)), \ E \in \mathcal{B}(Y).$$

Let $\mathcal{P}(X)$ be the space of all Borel probability measures on a completely regular space $X$. The space of measures is considered with the weak topology and the corresponding Borel structure. Recall that the weak topology on the whole space $\mathcal{M}(X)$ of signed Borel measures is generated by duality with $C_b(X)$, i.e., is defined by means of seminorms

$$\mu \mapsto \left| \int_X f \, d\mu \right|,$$

where $f \in C_b(X)$.

If $X$ is a completely regular Souslin space, then $\mathcal{M}(X)$ and $\mathcal{P}(X)$ are also completely regular Souslin spaces; if $X$ is a Polish space, then $\mathcal{P}(X)$ is also Polish (but $\mathcal{M}(X)$ is not in nontrivial cases) and if $X$ is a Luzin space, then so is $\mathcal{P}(X)$. These facts can be found in [7, Chapter 8] or in [10, Chapter 5].

We shall employ Prohorov’s condition for compactness in $\mathcal{M}(X)$: a set $M$ has compact closure in $\mathcal{M}(X)$ if it is bounded in variation and uniformly tight, which means that for every $\epsilon > 0$ there is a compact set $K \subset X$ such that $|\mu|(X \setminus K) \leq \epsilon$ for all $\mu \in M$. If $X$ is a Polish space, then this condition is also necessary.

For a completely regular Souslin space $X$, a mapping $m: (\Omega, \mathcal{E}) \to \mathcal{P}(X)$ from a measurable space $(\Omega, \mathcal{E})$ is measurable precisely when all functions

$$\omega \mapsto \int_X \varphi(x) \ m(\omega)(dx), \ \varphi \in C_b(X)$$
are measurable with respect to \( \mathcal{E} \). This is equivalent to the \( \mathcal{E} \)-measurability of all functions
\[
\omega \mapsto \int_X \varphi_n(x) \, m(\omega)(dx)
\]
for a fixed countable family of functions \( \varphi_n \in C_b(X) \) of the form \( \varphi_n = p(f_1, \ldots, f_k) \), where \( p \) is a polynomial on \( \mathbb{R}^k \) with rational coefficients and \( \{f_j\} \subset C_b(X) \) is a sequence separating the points in \( X \) (such sequence exists under our assumptions, see [7, Theorem 6.7.7]). Recall that any sequence of Borel functions separating points of a Souslin space generates the Borel \( \sigma \)-algebra of this space (see [7, Theorem 6.8.9]). It is worth noting that this measurability yields (by the monotone class theorem) that all functions
\[
\omega \mapsto m(\omega)(B), \ B \in \mathcal{B}(X),
\]
are \( \mathcal{E} \)-measurable (actually, this is an equivalent condition). Indeed, the class of bounded Borel measurable functions \( \psi \) on \( X \) for which the function
\[
\omega \mapsto \int_X \psi(x) \, m(\omega)(dx)
\]
is \( \mathcal{E} \)-measurable is closed under taking uniform limits and limits of monotonically increasing uniformly bounded sequences. Hence this class coincides with the class of all bounded Borel functions provided it contains \( C_b(X) \) (see [7, Theorem 6.7.7]).

Recall that a mapping \( \Psi \) from a measurable space \((T, \mathcal{T})\) to the set of nonempty subsets of a topological space \( X \) is called measurable if for every open set \( U \subset X \) the set \( \{t \in T \mid \Psi(t) \cap U \neq \emptyset\} \) belongs to \( \mathcal{T} \).

The space \( \mathcal{K}(X) \) of nonempty compact subsets of a complete metric space \( X \) is equipped with the Hausdorff distance
\[
d_H(A, B) = \inf\{r > 0 \mid \text{dist}(a, B) < r, \ \text{dist}(b, A) < r \ \forall \ a \in A, b \in B\}.
\]
It is known that this space is complete and separable (and is compact if \( X \) is compact), see [15].

3. **Conditional measures depending on a parameter**

We first address the problem of conditional measures. A general discussion can be found in [7, Chapter 10]; diverse aspects of the theory of conditional measures are also discussed in [11, 8, 21, 23, 24, 32], and [34], where additional references can be found. Connections of conditional measures with surface measures are considered in [13 and 9].

It is known that, for every Borel probability measure \( \mu \) on a Souslin space \( X \) and every Borel mapping \( f \) from \( X \) to a Souslin space \( Y \), the level sets \( f^{-1}(y) \) can be equipped with Borel probability measures \( \mu^y \), called conditional measures generated by \( f \), possessing the following properties:

1) the measure \( \mu^y \) is concentrated on \( f^{-1}(y) \) for each \( y \in f(X) \), i.e.,
\[
\mu^y(f^{-1}(y)) = 1, \ y \in f(X),
\]

2) the functions
\[
y \mapsto \mu^y(B), \ B \in \mathcal{B}(X),
\]
are measurable with respect to the \( \sigma \)-algebra \( \sigma(S(X)) \) generated by the class of Souslin sets in \( X \),
3) for all Borel sets $B \subset X$ and $E \subset Y$, there holds the equality

$$
\mu(B \cap f^{-1}(E)) = \int_E \mu^y(B) \, \mu \circ f^{-1}(dy).
$$

Conditional measures with these properties are called regular proper conditional probabilities.

The equality in 3) is equivalent to the following: for each bounded Borel function $\varphi$ on $X$ and every Borel set $E \subset Y$ we have

$$
\int_{f^{-1}(E)} \varphi \, d\mu = \int_E \int_X \varphi(x) \, \mu^y(dx) \, \mu \circ f^{-1}(dy) = \int_{f^{-1}(E)} \int_X \varphi(x) \, \mu^y(dx) \, \mu(du).
$$

Therefore, for the conditional expectation of $\varphi$ with respect to the $\sigma$-algebra $\mathcal{B}^f$ generated by $f$ (which is the class of sets $f^{-1}(A)$ with $A \in \mathcal{B}(Y)$) one can take the function

$$
E(\varphi|\mathcal{B}^f)(u) = \int_X \varphi(x) \, \mu^y(dx).
$$

Moreover, conditional measures are defined uniquely up to a redefinition for $y$ from a set of measure zero with respect to the induced measure $\mu \circ f^{-1}$.

Conditions 1) and 2) can be modified as follows: one can obtain the Borel measurability of all functions in 2) at the expense of relaxing condition 1) to the condition that it holds for $\mu \circ f^{-1}$-almost all $y$. It is known that in general it is impossible to guarantee the Borel measurability of functions in 2) requiring that 1) be valid for each $y$; counter-examples exist even if $X$ is a Borel subset of $[0,1]$ and $f$ is an infinitely differentiable function (see [6] or [7, p. 430]). If $X$ is a Polish space, then a necessary (but not sufficient) condition in order to ensure the Borel measurability in 2) while 1) is kept is that $f(X)$ must be a Borel set. A necessary and sufficient condition is this: there is a mapping $F$: $X \to X$ that is measurable with respect to the pair of $\sigma$-algebras $\mathcal{B}^f$ and $\mathcal{B}(X)$ and satisfies the equality $f(F(x)) = f(x)$. If $f$ is surjective, this is equivalent to the existence of a Borel mapping $g$: $Y \to X$ that is right inverse to $f$: $f(g(y)) = y$. Indeed, since $F$ is $\mathcal{B}^f$-measurable, it must be of the form $F(x) = g(f(x))$ for some Borel mapping $g$: $Y \to X$, hence $f(g(f(x))) = f(x)$, whence $f(g(y)) = y$ for all $y \in Y$. Conversely, if such $g$ exists, we can take $F(x) = g(f(x))$. See also [31] concerning various measurability difficulties related to proper conditional measures.

Let us now assume that the measure $\mu$ and the mapping $f$ depend measurably on a parameter from some other Souslin space $Z$. Can we pick conditional measures in such a way that they will depend on this parameter measurably? This question can be of interest for applications, in particular, in parametric statistics and in optimal transportation where conditional measures for different measures and mappings naturally arise (see, e.g., [29], [12], and [19]). We prove below that the answer is positive for an appropriate concept of measurability. It is worth noting that regular conditional expectations depending on a parameter were studied in [37], where a different problem in a different setting was considered (in particular, the basic probability measure was fixed).

We shall assume throughout that $X,Y,Z$ are completely regular Souslin spaces (in some results certain stronger assumptions are used).

**Lemma 3.1.** Suppose that $\psi$: $X \times Z \to \mathbb{R}$ is a bounded function measurable with respect to the $\sigma$-algebra $\sigma(S(X)) \otimes \sigma(S(Z))$. Let $z \mapsto \mu_z$, $X \to \mathcal{P}(X)$ be Borel
measurable or, more generally, \((\sigma(S(Z)), B(P(X)))\)-measurable. Then the function
\[
h(z) = \int_X \psi(x, z) \mu_z(dx)
\]
is \(\sigma(S(Z))\)-measurable on \(Z\). If \(\psi\) and \(z \mapsto \mu_z\) are Borel measurable, then the function \(h\) is Borel measurable as well.

Proof. Indeed, in the first case the class \(\mathcal{H}\) of all bounded \(\sigma(S(X)) \otimes \sigma(S(Z))\)-measurable functions \(\psi\) with this property is obviously closed under taking uniform limits and limits of increasing uniformly bounded sequences. In addition, this class contains all functions of the form
\[
\varphi_1(x)\psi_1(z) + \cdots + \varphi_n(x)\psi_n(z)
\]
with bounded functions \(\varphi_i\) and \(\psi_i\) on \(X\) and \(Z\) measurable with respect to \(\sigma(S(X))\) and \(\sigma(S(Z))\), respectively. By the monotone class theorem \(\mathcal{H}\) coincides with the set of all bounded \(\sigma(S(X)) \otimes \sigma(S(Z))\)-measurable functions (see [7, Theorem 2.12.9]). In the case of Borel measurability, the same reasoning applies to the class \(\mathcal{H}\) of bounded Borel measurable functions for which the claim is true.

Remark 3.2. It follows from the lemma that if we have a family of Borel sets \(B_z\) such that the function \(I_{B_z}(x)\) is \(\sigma(S(X)) \otimes \sigma(S(Z))\)-measurable, then the function \(z \mapsto \mu_z(B_z \cap B)\) is \(\sigma(S(Z))\)-measurable for all \(B \in \sigma(S(X))\) and similarly in the Borel case.

Lemma 3.3. Suppose that we have a mapping \((y, z) \mapsto \nu^y_z\) from \(Y \times Z\) to \(P(X)\) that is measurable with respect to \(\sigma(S(Y \times Z))\) and \(B(P(X))\) and a mapping
\[
(z, x) \mapsto f_z(x), \quad Z \times X \to Y
\]
that is measurable with respect to \(\sigma(S(Y \times Z))\) and \(B(Y)\). Then the set
\[
S = \{(y, z) \in Y \times Z : \nu^y_z(f_z^{-1}(y)) = 1\}
\]
belong to \(\sigma(S(Y \times Z))\). If both mappings are Borel measurable, then \(S\) is also Borel.

Proof. Indeed, the membership in this set is equivalent to the equality
\[
\nu^y_z \circ f_z^{-1} = \delta_y,
\]
which, in turn, is equivalent to the equality
\[
\int_X \psi_j(f_z(x)) \nu^y_z(dx) = \psi_j(y)
\]
for a countable collection of continuous functions \(\psi_j\) on \(Y\) separating Borel measures (as recalled above, such collection exists, since \(Y\) is a completely regular Souslin space). Since the right-hand side is Borel measurable in \(y\), the desired measurability in both cases follows by Lemma 3.1 (now applied to the product \(Y \times Z\)).

We now prove the existence of conditional measures measurably depending on a parameter.

Theorem 3.4. Suppose that we are given a Borel mapping
\[
f : (x, z) \mapsto f_z(x), \quad X \times Z \to Y.
\]
Suppose also that for every \(z \in Z\) we are given a Borel probability measure \(\mu_z\) on \(X\) such that the mapping
\[
z \mapsto \mu_z, \quad Z \to P(X)
\]
is Borel measurable or, more generally, \((\sigma(S(Z)), \mathcal{B}(\mathcal{P}(X)))\)-measurable. Then, for all pairs \((\mu_z, f_z)\), there exist proper conditional measures \(\{\mu^y_z\}_{y \in Y}\) on \(X\) such that, for each Borel set \(B\) in \(X\), the function

\[ (y, z) \mapsto \mu^y_z(B) \]

on \(Y \times Z\) is measurable with respect to the \(\sigma\)-algebra \(\sigma(S(Y \times Z))\), or, equivalently, the mapping

\[ (y, z) \mapsto \mu^y_z, \quad Y \times Z \to \mathcal{P}(X) \]

is measurable when \(Y \times Z\) is equipped with the \(\sigma\)-algebra \(\sigma(S(Y \times Z))\) and \(\mathcal{P}(X)\) is equipped with the Borel \(\sigma\)-algebra.

**Proof.** Set

\[ \sigma_z := \mu_z \circ f_z^{-1}. \]

For every \(z \in Z\), let us take an increasing sequence of finite algebras \(\mathcal{B}_{z,n}\) whose union generates the preimage

\[ \mathcal{B}_z := f_z^{-1}(\mathcal{B}(Y)) \]

of the whole Borel \(\sigma\)-algebra of \(Y\) under the mapping \(f_z\). We can assume that \(\mathcal{B}_{z,n}\) is generated by a finite partition of \(X\) into disjoint sets

\[ A_{z,n,1} = f_z^{-1}(B_{n,1}), \ldots, A_{z,n,m_n} = f_z^{-1}(B_{n,m_n}), \]

where \(B_{n,1}, \ldots, B_{n,m_n}\) is a finite partition of \(Y\) into disjoint Borel sets such that the union of all these sets generates the Borel \(\sigma\)-algebra of \(Y\). For example, if \(Y = [0, 1]\), we can take the sets \(A_{z,n,i} = f_z^{-1}([i/n, (i + 1)/n))\). In the general case one can take an injective continuous mapping \(T\) from \(Y\) to the compact metric space \([0, 1]^\infty\) by finitely many balls of radius \(1/n\) and then take the preimages under \(T\) of the sets in the obtained finite partition.

It is easy to find explicitly the conditional measures for \(\mu_z\) corresponding to the \(\sigma\)-algebra \(\mathcal{B}_{z,n}\):

\[ \mu^y_{z,n}(A) = \sum_{i=1}^{m_n} \frac{\mu_z(A \cap A_{z,n,i})}{\mu_z(A_{z,n,i})} I_{B_{n,i}}(y), \]

where in case \(\mu_z(A_{z,n,i}) = 0\) we set \(\mu_z(A \cap A_{z,n,i})/\mu_z(A_{z,n,i}) = 0\). By Lemma 3.1 the functions \((y, z) \mapsto \mu^y_{z,n}(A)\) are \(\sigma(S(Z)) \otimes \sigma(S(Z))\)-measurable (and Borel measurable if \(z \mapsto \mu_z\) is Borel), because \(I_{A_{z,n,i}}(x) = I_{B_{n,i}}(f_z(x)), I_{A \cap A_{z,n,i}} = I_A I_{A_{z,n,i}}\). It is readily seen that \(\mu^y_{z,n}(A)\) coincides with the elementary conditional expectation of \(I_A\) with respect to the \(\sigma\)-algebra \(\mathcal{B}_{z,n}\) and the measure \(\mu_z\). Similarly, for every bounded Borel measurable function \(\varphi\) on \(X\) we obtain that the conditional expectation \(E_z(\varphi | \mathcal{B}_{z,n})\) of \(\varphi\) with respect to the \(\sigma\)-algebra \(\mathcal{B}_{z,n}\) and the measure \(\mu_z\) coincides with the integral

\[ \int_X \varphi(x) \mu^y_{z,n}(dx). \]

By the martingale convergence theorem the functions \(E_z(\varphi | \mathcal{B}_{z,n})\) converge \(\sigma_z\)-almost everywhere and in \(L^1(\sigma_z)\) to the conditional expectation \(E_z(\varphi | \mathcal{B}_z)\) of the function \(\varphi\) with respect to the \(\sigma\)-algebra \(\mathcal{B}_z\) and the measure \(\mu_z\).

On the other hand, it is known that the existing conditional measures \(\mu^y_z\) can be also used to produce the same conditional expectations. Unfortunately, this does not solve the problem, since these relations hold almost everywhere and the corresponding measure zero sets can depend on \(\varphi\) (and certainly depend on \(z\)). Therefore, we need an effective procedure to select our conditional measures (which,
in turn, are defined not uniquely, but merely uniquely up to measure zero sets depending on \( z \).

To this end we define the desired version of conditional measures by considering the set where the sequence of measures \( \mu_{z,n}^y \) converges in a suitable sense and the limiting measure is concentrated on \( f_z^{-1}(y) \). Since there is a sequence of bounded continuous functions \( h_j \) on \( X \) separating points, we can assume that \( X \) is continuously and injectively mapped into \( I := [0,1]^{\infty} \), the countable power of \([0,1]\) with the product topology. Hence we can assume that \( X \) is a Souslin subset of \( I \) (equipped with a stronger topology, of course). Let \( \{ \varphi_j \} \) be the countable collection of polynomials in the coordinate functions that are rational linear combinations of monomials of the form \( x_{i_1}^{k_1} \cdots x_{i_m}^{k_m} \).

Let \( \Omega \) be the set of all points \((y, z) \in Y \times Z\) such that the sequence

\[
\int_X \varphi_j(x) \mu_{z,n}^y(dx)
\]

has a finite limit (as \( n \to \infty \)) for every function \( \varphi_j \). Since each integral determines an \( \sigma(S(Y \times Z)) \)-measurable function on \( Y \times Z \), the set \( \Omega \) belongs to \( \sigma(S(Y \times Z)) \) (and is Borel in \( Y \times Z \) if \( z \mapsto \mu_z \) is Borel measurable).

It follows by the compactness of \( I \) that for every pair \((y, z) \in \Omega \), the measures \( \mu_{z,n}^y \) converge weakly to a Borel probability measure \( \nu_z^y \) on \( I \) (not necessarily concentrated on \( X \)).

According to Lemma [3.3], the subset

\[
\Omega_0 := \{(y, z) \in \Omega: \nu_z^y(f_z^{-1}(y)) = 1\}
\]

belongs to \( \sigma(S(Y)) \otimes \sigma(S(Z)) \) (and is Borel if \( z \mapsto \mu_z \) is Borel). For each \((y, z) \in \Omega_0\) the measure \( \nu_z^y \) is obviously concentrated on \( X \). Let

\[
\Omega_1 = \{(y, z) \in Y \times Z: y \in f_z(X)\},
\]

which is the projection of the graph of \( f \) under the mapping

\[
X \times Z \times Y \to Y \times Z, \quad (x, z, y) \mapsto (y, z),
\]

hence is a Souslin set in \( Y \times Z \). If each \( f_z \) is a surjection, then \( \Omega_1 = Y \times Z \). According to the measurable choice theorem (see [7, Theorem 6.9.2]), there is a mapping

\[
g: (y, z) \mapsto g_z(y), \quad \Omega_1 \to X,
\]

measurable with respect to the pair of \( \sigma \)-algebras \( \sigma(S(Y)) \otimes \sigma(S(Z)) \) (more precisely, its restriction to \( \Omega_1 \)) and \( B(X) \) such that \( g_z(y) \in f_z^{-1}(y) \) for all \( z \in Z \), \( y \in f_z(X) \). In order to apply the cited theorem we consider on \( \Omega_1 \) the multivalued mapping \( \Psi: (y, z) \mapsto f_z^{-1}(y) \) with values in the class of non-empty subsets of \( X \). The graph of \( \Psi \) is the set

\[
\{(y, z, u): (y, z) \in \Omega_1, u \in f_z^{-1}(y)\} = \{(y, z, u): (y, z) \in \Omega_1, f_z(u) = y\},
\]

which is a Souslin set, since the mappings \((y, z, u) \mapsto f_z(u)\) and \((y, z, u) \mapsto y\) are Borel measurable.

For every pair \((y, z) \) that is not in \( \Omega_0 \) we set \( \nu_z^y := \delta_{g_z(y)} \) if \( y \in f_z(X) \) (equivalently, \((y, z) \in \Omega_1 \)) and \( \nu_z^y := \delta_{x_0} \) if \( y \not\in f_z(X) \), where \( x_0 \in X \) is a fixed point common for all \( y \) and \( z \). The family of measures \( \nu_z^y \) is measurable with respect to the \( \sigma \)-algebra \( S(Y \times Z) \), because this is true for its restriction to \( \Omega_0 \) (when we even have the measurability with respect to \( \sigma(S(Y)) \otimes \sigma(S(Z)) \)) and is also true for the restrictions to the sets \((Y \times Z) \cap \Omega_1 \setminus \Omega_0 \) and \((Y \times Z) \setminus (\Omega_0 \cup \Omega_1) \) belonging to \( \sigma(S(Y \times Z)) \). The
measurability on the first of these sets is clear from the fact that the integral of \( f \in C_b(X) \) with respect to \( \delta_{g_z(y)} \) is \( f(g_z(y)) \), which is measurable with respect to \( \sigma(S(Y \times Z)) \) by the measurability of \( g \).

Let us show that the obtained family of measures \( \nu^y_z \) can be taken for regular conditional probabilities with the desired measurability properties.

By construction \( \nu^y_z \) is concentrated on \( f_z^{-1}(y) \) for all \( y \in Y \) and \( z \in Z \) and the function

\[
(y, z) \mapsto \int_X \varphi_j(x) \nu^y_z(dx)
\]

is measurable with respect to the \( \sigma \)-algebra \( \sigma(S(Y \times Z)) \) for every function \( \varphi_j \), which means the desired measurability of \( \{\nu^y_z\} \). Now it suffices to verify that, for each \( z \in Z \), we have

\[
\nu^y_z = \mu^y_z \quad \text{for } \sigma_z\text{-almost every } y,
\]

where \( \mu^y_z \) are arbitrary regular conditional measures for \( \mu_z \) (without measurability in \( z \)). Indeed, let \( z \) be fixed. To prove our claim it suffices to show that, for every function \( \psi \) from a countable collection of bounded continuous functions on \( Y \) separating Borel measures, the integrals of \( \psi \) with respect to \( \nu^y_z \) and \( \mu^y_z \) coincide \( \sigma_z\)-almost everywhere. This is indeed true, because we know that both integrals after substituting \( y = f_z(x) \) give versions of the conditional expectation of \( \psi \) with respect to the \( \sigma \)-algebra \( B_z \) and the measure \( \mu_z \).

\[\square\]

**Theorem 3.5.** Suppose that in Theorem 3.4, for each \( z \) the mapping \( f_z : X \to Y \) is a Borel surjection possessing a right inverse mapping \( g_z \) such that \((y, z) \mapsto g_z(y)\) is Borel measurable (or, more generally, the set \( \bigcup_z \{f_z(X) \times \{z\}\} \) is Borel in \( Y \times Z \) and the mapping \((y, z) \mapsto g_z(y)\) is Borel measurable); for example, the mapping \( f : X \to Y \) does not depend on \( z \) and is a Borel surjection possessing a Borel right inverse mapping \( g \). If also \( z \mapsto \mu_z \) is Borel measurable, then there exists a jointly Borel measurable version of conditional measures \( \mu_z^y \).

In particular, this is true if \( X \) is the product of two Souslin spaces \( X_1 \) and \( X_2 \), \( f \) is the standard projection onto \( X_2 \), and \( z \mapsto \mu^z \) is Borel measurable.

**Proof.** This follows from our reasoning above (taking into account the notes about Borel measurability), since under stronger assumptions of this theorem we already have jointly Borel measurable right inverse mappings \( g_z \) without any measurable choice theorems. The corresponding Dirac measures \( \delta_{g_z(y)} \) defined for all \( y, z \) from the complement of the Borel set \( \Omega_0 \) are also jointly Borel measurable and \( \Omega_1 = Y \times Z \) in the surjective case. Similarly we consider the case where \( f_z \) is not surjective and \( \bigcup_z \{f_z(X) \times \{z\}\} \) is Borel.

\[\square\]

Note that in the case of the product-space \( X = X_1 \times X_2 \) and the projection \( \pi_{X_2} \) on \( X_2 \) it is sometimes more convenient to consider conditional measures on the common space \( X_1 \) in place of the slices \( X_1 \times \{x_2\} = \pi_{X_2}^{-1}(x_2) \subset X_1 \times X_2 \). Both representations are equivalent and it is easy to pass from one to the other.

The assertion with a single mapping \( f \) not depending on the parameter admits an obvious generalization.

**Corollary 3.6.** Let \( X_1 \) and \( X_2 \) be completely regular Souslin spaces, let \((T, \mathcal{T})\) be a measurable space, and let \( t \mapsto \mu_t \) be a mapping from \( T \) to \( \mathcal{P}(X_1 \times X_2) \) that is measurable with respect to \( \mathcal{T} \) and \( \mathcal{B}(\mathcal{P}(X_1 \times X_2)) \). Then there is a mapping

\[
(t, x_2) \mapsto \mu^x_2 \in \mathcal{P}(X_1),
\]
measurable with respect to $\mathcal{T} \otimes \mathcal{B}(X_2)$ and $\mathcal{B}(\mathcal{P}(X_1))$, such that the measures $\mu_t^{x_2}$ serve as conditional measures for $\mu_t$ and the projection on $X_2$.

**Proof.** The previous theorem can be applied with the space $Z = \mathcal{P}(X_1 \times X_2)$ as a parameter space, which gives a Borel mapping $(\mu, x_2) \mapsto P_{\mu}^{x_2}$ such that $P_{\mu}^{x_2}$ serve as conditional measures for $\mu$. Then the mapping $(t, x_2) \mapsto \mu_t^{x_2} := P_{\mu_t}^{x_2}$ is measurable with respect to $\mathcal{T} \otimes \mathcal{B}(X_2)$.

The following parametric version of the so-called gluing lemma (see [37]) has been noted in [27, Theorem 7.3] (for Polish spaces).

**Corollary 3.7.** Let $X_1, X_2, X_3$ be completely regular Souslin spaces, let $(T, \mathcal{T})$ be a measurable space, and let $t \mapsto \mu_{1,2,t}$, $T \mapsto \mathcal{P}(X_1 \times X_2)$ and $t \mapsto \mu_{2,3,t}$, $T \mapsto \mathcal{P}(X_2 \times X_3)$ be $\mathcal{T}$-measurable mappings such that, for each $t$, the projections of $\mu_{1,2,t}$ and $\mu_{2,3,t}$ on $X_2$ coincide. Then there is a $\mathcal{T}$-measurable mapping $t \mapsto \eta_t$ from $T$ to the space $\mathcal{P}(X_1 \times X_2 \times X_3)$ such that, for each $t$, the projection of $\eta_t$ on $X_1 \times X_2$ is $\mu_{1,2,t}$ and the projection on $X_2 \times X_3$ is $\mu_{2,3,t}$.

**Proof.** It suffices to recall the usual construction of the measure on $X_1 \times X_2 \times X_3$ for every fixed $t$ via conditional measures (see [10, Lemma 3.3.1] or [36]): using disintegrations

$$
\mu_{1,2,t}(dx_1 dx_2) = \mu_{1,2,t}^{x_2}(dx_1) \pi_t(dx_2), \quad \mu_{2,3,t}(dx_2 dx_3) = \mu_{2,3,t}^{x_2}(dx_3) \pi_t(dx_2),
$$

where $\pi_t$ is the common projection of $\mu_{1,2,t}$ and $\mu_{2,3,t}$ on $X_2$, $\mu_{1,2,t}^{x_2}$ and $\mu_{2,3,t}^{x_2}$ are the corresponding conditional measures measurably depending on $t$, we set

$$
\eta_t(dx_1 dx_2 dx_3) = \mu_{1,2,t}^{x_2}(dx_1) \mu_{2,3,t}^{x_2}(dx_3) \pi_t(dx_2).
$$

This means that for each bounded Borel function $f$ on $X_1 \times X_2 \times X_3$ we have the following equality:

$$
\int f \, d\eta_t = \int_{X_2} \int_{X_3} \int_{X_1} f(x_1, x_2, x_3) \, \mu_{1,2,t}^{x_2}(dx_1) \mu_{2,3,t}^{x_2}(dx_3) \pi_t(dx_2).
$$

The measurability of the mapping $t \mapsto \eta_t$ follows by the measurability of conditional measures and the projection. The fact that $\eta_t$ has the prescribed projections is verified directly (see [10, Lemma 3.3.1]).

**Remark 3.8.** The assumption that the space of parameters $Z$ is Souslin is quite natural in the situation of Theorem 3.4. However, in the situation of Theorem 3.5 for $Z$ we can take an arbitrary measurable space $(Z, \mathcal{Z})$ without any topology. The same reasoning shows that if $(z, x) \mapsto f_z(x)$ is $Z \otimes \mathcal{B}(X)$-measurable, each $f_z$ is a surjection that admits a right inverse mapping $g_z$ for which $(z, y) \mapsto g_z(y)$ is $Z \otimes \mathcal{B}(Y)$-measurable, and $\mu_z$ is $\mathcal{Z}$-measurable, then there are conditional measures $\mu_z^{y}$, measurable with respect to $Z \otimes \mathcal{B}(Y)$.

**Remark 3.9.** (i) It is worth recalling that the $\sigma$-algebra $\sigma(S(X))$ is not countably generated for an uncountable Polish space $X$ (see [4, Example 6.5.9]). This causes obvious technical problems when one attempts to employ standard tools related to measurable selection theorems. If the spaces $X, Y, Z$ are compact metric, the mappings $f_z$ are continuous and there exist weakly continuous regular conditional measures $y \mapsto \mu_z^y$, then one can consider a multi-valued mapping $z \mapsto M(z)$ with values in non-empty subsets of the separable metric space $C(Y, \mathcal{P}(X))$ of continuous
mappings from $Y$ to $\mathcal{P}(X)$ (where $\mathcal{P}(X)$ with the weak topology also becomes a compact metric space). In that case, one can apply a measurable selection theorem. However, in the general case, too many non-separable objects arise.

(ii) It should be also noted that the problem of finding conditional expectations measurable with respect to a parameter (when both the basic measure and the $\sigma$-algebra depend on the parameter) is easier, since in this case we need not care about property 1) of proper conditional measures. In addition, a less restrictive problem of constructing disintegrations (see [7]) measurably depending on a parameter can be solved under broader assumptions. Continuous modifications of conditional expectations were considered in [22].

(iii) One can show that if $Y$ and $Z$ are uncountable Souslin spaces, then the product $\sigma$-algebra $\sigma(S(Y) \otimes S(Z))$ is strictly smaller than the $\sigma$-algebra $\sigma(S(Y \times Z))$ of the product-space.

We now see that the existence of jointly Borel conditional measures depending on the parameter $z$ implies some restrictions on the mappings $f_z$, so that such joint Borel measurability cannot be always guaranteed. The next proposition is a parametrized version of the known result of Blackwell and Ryll-Nardzewski [6] for single measures.

**Proposition 3.10.** Let $X,Y,Z$ be Polish spaces. Suppose that there is a jointly Borel measurable version of conditional measures $\mu^y_z$ concentrated on the sets $f_z^{-1}(y)$ for all $y \in Y$ and $z \in Z$. Then there is a Borel mapping $g: Z \times Y \to X$ such that $f_z(g(z,y)) = y$ for all $y \in Y$ and $z \in Z$.

**Proof.** We shall use the following result of Blackwell and Ryll-Nardzewski [6] (see also [25, Corollary 18.7] or [7, Exercise 10.10.47], where the hint contains the proof). For our convenience we change their notation of spaces. Suppose that $U$ and $X$ are Borel sets in Polish spaces, $\mathcal{A}$ is a countably generated sub-$\sigma$-algebra in $\mathcal{B}(U)$ and for each $u \in U$ there is a measure $\mu^u \in \mathcal{P}(X)$ such that the function $u \mapsto \mu^u(B)$ is $\mathcal{A}$-measurable for every set $B \in \mathcal{B}(X)$. Let $S \subset X \times U$ be a set such that $\mu^u(S_u) > 0$ for all $u \in U$, where $S_u = \{x \in X: (x,u) \in S\}$. Then $S$ contains the graph of an $(\mathcal{A},\mathcal{B}(X))$-measurable mapping $F: U \to X$.

We apply this result in the situation where $U = Z \times X$, $\mathcal{A}$ is the sub-$\sigma$-algebra in $\mathcal{B}(Z \times X)$ generated by the mapping

$$h: Z \times X \to Z \times Y, \quad (z, x) \mapsto (z, f_z(x)),$$

$$\mu^u = \mu^u_z(x), \quad u = (z, x),$$

and

$$S = \{(z, x, v) \in Z \times X \times X: f_z(v) = f_z(x)\}.$$

The section $S_u$ is defined by

$$S_u = S_{z,x} = \{v \in X: f_z(v) = f_z(x)\} = f_z^{-1}(f_z(x)),$$

hence $\mu^u(S_u) = \mu^u_z(f_z^{-1}(f_z(x))) = 1$. By the cited result there is a mapping $F: Z \times X \to X$ with the graph in $S$ such that $F$ is $\mathcal{A}$-measurable. The latter means that there is a Borel mapping $g: Z \times Y \to X$ such that $F(z, x) = g(h(z, x))$. Since the graph of $F$ is contained in $S$, by the definition of $h$ we obtain

$$f_z(g(z, f_z(x))) = f_z(x) \quad \forall x \in X, z \in Z.$$

It follows that $f_z(g(z, y)) = y$ for all $z \in Z$ and $y \in Y$. \qed
It is known that in general there is no $g$ with the stated properties (see, e.g., [1, §6.9]). A sufficient condition for the existence of $g$ is this: for each $y \in Y$ and $z \in Z$ the set $f_z^{-1}(y)$ is a countable union of compact sets. Indeed, we consider again the Borel mapping $h: (z, x) \mapsto (z, f_z(x))$ and observe that the sets $h^{-1}(z, y)$ are countable unions of compact sets. Hence by a classical result (see Theorem C in the next section) there is a Borel mapping $g: Z \times Y \to X$ with the graph contained in the set $\{(z, y, x): f_z(x) = y\}$.

4. Kantorovich problems with a parameter

We now turn to Kantorovich optimal plans depending on a parameter.

Let $X$ and $Y$ be completely regular Souslin spaces (for example, Polish spaces). The corresponding spaces of Borel probability measures $P(X)$ and $P(Y)$ will be equipped with their weak topologies (making them Souslin or Polish spaces, respectively). By $\pi_X$ and $\pi_Y$ we denote the projections of $X \times Y$ on $X$ and $Y$.

For any pair of measures $\mu \in P(X)$ and $\nu \in P(Y)$, the set

$$\Pi(\mu, \nu) = \{\sigma \in P(X \times Y): \sigma \circ \pi_X^{-1} = \mu, \sigma \circ \pi_Y^{-1} = \nu\}$$

is convex and compact in the weak topology, which follows from Prohorov’s theorem. This set is not empty: it always contains the product of $\mu$ and $\nu$.

Recall that a function $f$ is lower semicontinuous if the sets $\{f \leq c\}$ are closed. It is known (see [10, Corollary 4.3.4]) that if $f$ is a bounded lower semicontinuous function on $X$ and Borel probability measures $\mu_n$ on $X$ converge weakly to $\mu$, then

$$\int_X f \, d\mu \leq \liminf_{n \to \infty} \int_X f \, d\mu_n.$$

Given a lower semicontinuous cost function function $h \geq 0$ on $X \times Y$ and a pair of measures $\mu \in P(X)$ and $\nu \in P(Y)$, in the aforementioned Kantorovich problem of finding the infimum of $K_h(\mu, \nu)$ of the quantity $I_h(\sigma)$ over all measures $\mu \in \Pi(\mu, \nu)$ the minimum is attained if there is a measure $\sigma$ with $I_h(\sigma) < \infty$ (which is always true if $h$ is bounded).

Let $(T, \mathcal{T})$ be a measurable space. In the case where $T$ is a topological space we assume that $\mathcal{T}$ is its Borel $\sigma$-algebra $\mathcal{B}(T)$.

Assume also that

$$h: T \times X \times Y \to [0, +\infty)$$

is a $\mathcal{T} \otimes \mathcal{B}(X) \otimes \mathcal{B}(Y)$-measurable function such that $h_t: (x, y) \mapsto h(t, x, y)$ is lower semicontinuous for each $t$.

Thus, we obtain a Kantorovich problem with a parameter. Dependence on a parameter appears even for a single cost function if marginal distributions depend on $t$. We consider the case where both marginals and the cost function depend on $t$.

Let $t \mapsto \mu_t$, $T \to P(X)$ be a $(\mathcal{T}, \mathcal{B}(P(X)))$-measurable mapping and let $t \mapsto \nu_t$, $T \to P(Y)$ be a $(\mathcal{T}, \mathcal{B}(P(Y)))$-measurable mapping.

**Theorem 4.1.** Suppose that the transportation costs $K(t) := K_{h_t}(\mu_t, \nu_t)$ are finite and the cost functions $h_t: (x, y) \mapsto h(t, x, y)$ are continuous. Then the function $K$ is $(\mathcal{T}, \mathcal{B}(P(X \times Y)))$-measurable. In addition, one can choose optimal measures $\sigma_t$ such that the mapping $t \mapsto \sigma_t$ is measurable with respect to $\sigma(\mathcal{S}(\mathcal{T}))$ and $\mathcal{B}(P(X \times Y))$.

In the next theorem we remove the assumption of continuity of cost functions and reinforce the conclusion by the existence of Borel measurable selections, but $T$ is required to be a Souslin space.
Theorem 4.2. Suppose that $T$ is a Souslin space, $t \mapsto \mu_t$ and $t \mapsto \nu_t$ are Borel mappings with values in the spaces $\mathcal{P}(X)$ and $\mathcal{P}(Y)$, respectively, and the corresponding transportation costs $K_{h_t}(\mu_t, \nu_t)$ are finite. Then the function $t \mapsto K_{h_t}(\mu_t, \nu_t)$ is Borel measurable and there is a mapping $t \mapsto \sigma_t$, $T \to \mathcal{P}(X \times Y)$, measurable with respect to $\mathcal{B}(T)$ and $\mathcal{B}(\mathcal{P}(X \times Y))$, such that for all $t \in T$ we have

$$\sigma_t \in \Pi(\mu_t, \nu_t), \quad \int h(t, x, y) \sigma_t(dxdy) = K_{h_t}(\mu_t, \nu_t).$$

Corollary 4.3. In the previous theorem, there is a sequence of Borel measurable mappings $\Phi_n : T \to \mathcal{P}(X \times Y)$ such that, for every $t \in T$, the sequence $\{\Phi_n(t)\}$ is dense in the convex compact set $M_t$ of $h_t$-optimal measures in $\Pi(\mu_t, \nu_t)$.

Corollary 4.4. In the previous theorem, the optimal plans $\sigma_t$ admit disintegrations

$$\sigma_t = \int_Y \sigma_t^y \nu_t(dy)$$

with Borel probability measures $\sigma_t^y$ on $X$ that are Borel measurable in $(t, y)$.

For Souslin spaces $X$ and $Y$ we have the following result.

Theorem 4.5. Let $X$ and $Y$ be completely regular Souslin spaces and let $T$ be a Souslin space. Let $(x, y) \mapsto h(t, x, y)$ be continuous for every $t$ and let $t \mapsto \mu_t$ and $t \mapsto \nu_t$ be Borel measurable. Then the function $t \mapsto K(t)$ is measurable with respect to $\sigma(S(T))$.

Note that if in the last theorem the function $t \mapsto K(t)$ is Borel measurable, then there is a sequence of mappings $\Phi_n : T \to \mathcal{P}(X \times Y)$, measurable with respect to $\sigma(S(T))$, $\mathcal{B}(\mathcal{P}(X \times Y))$, such that, for every $t \in T$, the sequence $\{\Phi_n(t)\}$ is dense in the convex compact set $M_t$ of $h_t$-optimal measures in $\Pi(\mu_t, \nu_t)$.

5. Auxiliary results and proofs

The following general version of the Kantorovich duality for finite nonnegative lower semicontinuous cost functions holds:

$$K_{h}(\mu, \nu) = \sup \left\{ \int \varphi\;d\mu + \int \psi\;d\nu : \varphi \in C_b(X), \psi \in C_b(Y), \varphi(x) + \psi(y) \leq h(x, y) \right\}. \quad (5.1)$$

See [3, 4, 26, 30], and [37] for a discussion of this duality; a short derivation of the general case from the case of bounded continuous cost functions is given in [4]. Hence for each $\varepsilon > 0$ there are functions $\varphi \in C_b(X)$ and $\psi \in C_b(Y)$ such that

$$\varphi(x) + \psi(y) \leq h(x, y) \quad \text{for all } x \text{ and } y$$

and

$$K_{h}(\mu, \nu) \leq \int \varphi\;d\mu + \int \psi\;d\nu + \varepsilon.$$

Moreover, for bounded $h$, in the right-hand side of (5.1) one can take the supremum over $\varphi$ and $\psi$ such that $|\varphi| \leq \|h\|_{\infty}$, $|\psi| \leq \|h\|_{\infty}$. This is explained in [37] Remark 1.13], but for the reader’s convenience we give a straightforward justification. We can assume that $\|h\|_{\infty} = 1$. If a pair $\varphi, \psi$ satisfies the indicated bound, then, for any number $t$, the pair $\varphi + t, \psi - t$ also satisfies this bound and the sum of
the corresponding integrals is the same. Hence we can assume that \( \sup_x \varphi(x) = 1 \). Hence \( \psi(y) \leq 0 \). Next, we replace \( \varphi \) by \( \varphi_1 = \max(\varphi, 0) \) and obtain a pair with \( \varphi_1(x) + \psi(y) \leq h(x, y) \) and \( 0 \leq \varphi_1 \leq 1 \) for which the integral of \( \varphi_1 \) is not less than that of \( \varphi \). Finally, we replace \( \psi \) by \( \psi_1 = \max(\psi, -1) \), which keeps the upper bound by \( h \) and increases the integral. Hence we obtain a pair with \( 0 \leq \varphi_1 \leq 1 \), \(-1 \leq \psi_1 \leq 0\) and \( \varphi_1(x) + \psi_1(y) \leq h(x, y) \) for which the sum of the respective integrals dominates the original sum. The next lemma is an immediate corollary of this bound.

**Lemma 5.1.** Let \( h \leq 1 \). Then for all \( \mu_1, \mu_2 \in \mathcal{P}(X) \) and \( \nu_1, \nu_2 \in \mathcal{P}(Y) \) we have
\[
|K_h(\mu_1, \nu_1) - K_h(\mu_2, \nu_2)| \leq \|\mu_1 - \mu_2\| + \|\nu_1 - \nu_2\|.
\]

**Proof.** We can assume that \( K_h(\mu_1, \nu_1) > K_h(\mu_2, \nu_2) \). Let \( \varepsilon > 0 \). There are functions \( \varphi, \psi \in C_b(X) \) with \( \varphi(x) + \psi(y) \leq h(x, y), |\varphi| \leq 1, |\psi| \leq 1 \) such that
\[
K_h(\mu_1, \nu_1) < \int \varphi \, d\mu_1 + \int \psi \, d\nu_1 + \varepsilon.
\]
Since
\[
K_h(\mu_2, \nu_2) \geq \int \varphi \, d\mu_2 + \int \psi \, d\nu_2,
\]
we have
\[
K_h(\mu_1, \nu_1) - K_h(\mu_2, \nu_2) \leq \varepsilon + \int \varphi \, d(\mu_1 - \mu_2) + \int \psi \, d(\nu_1 - \nu_2),
\]
whence our claim follows with the extra term \( \varepsilon \) on the right, so it remains to let \( \varepsilon \to 0 \). \( \square \)

**Lemma 5.2.** Let \((T, \mathcal{T})\) be a measurable space, \( Z \) a Polish space, and let \( t \mapsto \mu_t \) be a mapping from \( T \) to \( \mathcal{P}(Z) \) measurable with respect to \( \mathcal{T} \) and \( \mathcal{B}(\mathcal{P}(Z)) \). Then there is a sequence of increasing compact sets \( Z_n(t) \subseteq Z \) such that the sets \( \bigcup_t \{t\} \times Z_n(t) \) are in \( \mathcal{T} \otimes \mathcal{B}(Z) \), the set-valued mapping \( t \mapsto Z_n(t) \) is \( \mathcal{T} \)-measurable, the normalized restrictions \( \mu_t^n \) of \( \mu_t \) to \( Z_n(t) \) define mappings \( t \mapsto \mu_t^n \) from \( T \) to \( \mathcal{P}(Z) \) measurable in the same sense and \( \|\mu_t^n - \mu_t\| \to 0 \).

The same is true if \( Z \) is a completely regular Luzin space, hence this is true if \( Z \) is a Borel set in a Polish space.

**Proof.** It suffices to introduce the parameter \( t \) in the standard proof of Ulam’s theorem. We consider \( Z \) with a complete separable metric and take a dense countable set \( \{z_j\} \subseteq Z \). Let \( n \in \mathbb{N} \). For each \( k \) and \( m \) in \( \mathbb{N} \) let \( A_{k,m} \) be the union of \( m \) closed balls of radius \( 2^{-k} \) centered at \( z_1, \ldots, z_m \). Then \( \mu_t(A_{k,m}) \to 1 \) as \( m \to \infty \). Let
\[
N_{n,k}(t) = \min\{m : \mu_t(A_{k,m}) > 1 - 2^{-n-k}\},
\]
\[
Z_n(t) = \bigcap_{k \geq 1} A_{k,N_{n,k}}(t).
\]
The sets \( A_{k,N_{n,k}}(t) \) are closed. Hence the sets \( Z_n(t) \) are also closed. In addition, each \( Z_n(t) \) is contained in finitely many balls of radius \( 2^{-k} \) for each \( k \). Hence \( Z_n(t) \) is compact. By construction,
\[
\mu_t(X \setminus Z_n(t)) < \sum_{k=1}^{\infty} 2^{-n-k} = 2^{-n}.
\]
Let $\mu^a_t$ be the normalized restriction of $\mu_t$ to $Z_n(t)$. Then $\|\mu_t - \mu^a_t\| < 2^{-n}(1 - 2^{-n})^{-1}$. We have $Z_n(t) \subset Z_{n+1}(t)$, since $N_{n,k}(t) \leq N_{n,k+1}(t)$, so $A_{k,N_{n,k}}(t) \subset A_{k,N_{n,k+1}}(t)$.

The functions $t \mapsto N_{n,k}(t)$ are $\mathcal{T}$-measurable, since the set $N_{n,k}^{-1}(q)$ is the intersection of the sets $\{t: \mu_t(A_{k,j}) \leq 1 - 2^{-n-k}\}$ with $j < q$ and $\{t: \mu_t(A_{k,q}) > 1 - 2^{-n-k}\}$ that are $\mathcal{T}$-measurable, which readily follows from the measurability of $t \mapsto \mu_t$. In order to show the measurability of $\mu^a_t$ it suffices to show the measurability of the mapping $t \mapsto \mu_t|_{Z_n(t)}$. This mapping is the limit of restrictions of $\mu_t$ to the decreasing sets $\bigcap_{k=1}^m A_{k,N_{n,k}}(t)$. Such restrictions $\mu^a_{t,m}$ are $\mathcal{T}$-measurable. Indeed, the sets $N_{n,k}^{-1}(q)$ are $\mathcal{T}$-measurable, hence so are their finite intersections, but $\mu^a_{t,m}$ has countably many values assumed on such intersections.

Every set $\bigcup \{\{t\} \times Z_n(t)\}$ belongs to $\mathcal{T} \otimes \mathcal{B}(Z)$, because it is the intersection of the sets $\bigcup \{\{t\} \times \bigcap_{k=1}^m A_{k,N_{n,k}}(t)\}$, which are in $\mathcal{T} \otimes \mathcal{B}(Z)$, since they are countable unions of sets of the form $T_{k,n,m} \times A_{k,m}$ with $T_{k,n,m} = \{t: N_{k,n}(t) = m\}$. Let us show that the set-valued mapping $t \mapsto Z_n(t)$ is $\mathcal{T}$-measurable. It suffices to show that for every $x \in X$ the real function $t \mapsto \text{dist}(x, Z_n(t))$ is $\mathcal{T}$-measurable, see [15, Theorem III.9]. Let $D_{n,m}(t) = \bigcap_{k=1}^m A_{k,N_{n,k}}(t)$. We observe that

$$d_B(Z_n(t), D_{n,m}(t)) \to 0 \quad \text{and} \quad \text{dist}(x, D_{n,m}(t)) \to \text{dist}(x, Z_n(t)) \quad \text{as} \quad m \to \infty.$$ 

Indeed, for every fixed $\varepsilon > 0$ there is $m$ such that $D_{n,m}(t)$ is contained in the $\varepsilon$-neighborhood of $Z_n(t)$, because otherwise there is a sequence of points $x_m \in D_{n,m}(t)$ with $\text{dist}(x_m, Z_n(t)) \geq \varepsilon$. Each $D_{n,m}(t)$ is a union of finitely many balls of radius $2^{-k}$, hence $\{x_m\}$ is precompact and has a limit point $x_0$. This point must belong to all $D_{n,m}(t)$, hence to $Z_n(t)$, which is impossible, since $\text{dist}(x_0, Z_n(t)) \geq \varepsilon$. This proves the first relation, the second is its corollary.

The case of Luzin spaces follows from the considered case, because $Z$ admits a stronger Polish topology that generates a stronger Polish topology on $\mathcal{P}(Z)$ with the same Borel sets as in the original topology, so the measurability of $\mathcal{P}(Z)$-valued mappings remains the same. Finally, we recall than any Borel set in a Polish space is the image of a Polish space under a continuous injective mapping (see [7, Corollary 6.8.5]).

**Remark 5.3.** Under a stronger condition that $t \mapsto \mu_t(A)$ is $\mathcal{T}$-measurable for every Souslin set $A$ (which does not follow automatically) the previous assertion extends to the case of a Souslin subspace $Z$ in a Polish space $E$ and gives increasing compact sets $Z_n(t)$ such that the functions $(t, x) \mapsto I_{Z_n(t)}(x)$ are $\mathcal{T} \otimes \sigma(\mathcal{S}(Z))$-measurable and $\mu_t(Z_n(t)) > 1 - 2^{-n}$. To this end, we first take such compact sets $Z_n(t)$ in $E$ and then consider a parametric version of the standard proof of measurability of sets obtained by means of the Souslin operation (see [7, Theorem 1.10.5]). Recall that $Z$ can be written as

$$Z = \bigcup_{(n_i)_{k=1}}^{\infty} \bigcap_{k=1}^m E_{n_1,\ldots,n_k},$$

where $\{E_{n_1,\ldots,n_k}\}$ is a certain monotone table of closed balls of rational radii centered at points of a fixed countable dense set and the union is taken over all natural sequences $(n_i)$. For every collection $m_1, \ldots, m_k$ of natural numbers, denote by $D_{m_1,\ldots,m_k}$ the union of the sets $E_{n_1,\ldots,n_k}$ over all $n_1 \leq m_1, \ldots, n_k \leq m_k$. This is a closed set. It is clear from the proof of the cited theorem (taking into account Remark 3.2) that one can find numbers $m_k(t)$ measurably depending on $t$ such that

$$\mu_t(D_{m_1(t),\ldots,m_k(t)} \cap Z_n^1(t)) > 1 - 2^{-n}.$$
Then
\[ \mu_t \left( \bigcap_{k=1}^{\infty} D_m(t),...,m_k(t) \cap Z^1_n(t) \right) \geq 1 - 2^{-n}. \]

It is verified in that proof that \( \bigcap_{k=1}^{\infty} D_m(t),...,m_k(t) \) is contained in \( Z \). It is clear that this set is closed, so its intersection with \( Z^1_n(t) \) is compact.

**Lemma 5.4.** Suppose that \((T, \mathcal{T})\) is a measurable space, \( X \) and \( Y \) are Polish (or Luzin) spaces, \( t \mapsto \mu_t \) and \( t \mapsto \nu_t \) are \( \mathcal{T} \)-measurable mappings with values in \( \mathcal{P}(X) \) and \( \mathcal{P}(Y) \), correspondingly. Let \((x,y) \mapsto h(t,x,y)\) be lower semicontinuous and \( K_{h_t}(\mu_t,\nu_t) < \infty \) for each \( t \). Then for the measures \( \mu^n_t \) and \( \nu^n_t \) from the previous lemma applied to \( \mu_t \) and \( \nu_t \) we have

\[ K_{h_t}(\mu_t,\nu_t) = \lim_{n \to \infty} K_{h_t}(\mu^n_t,\nu^n_t) \quad \forall t \in T. \]

**Proof.** Let \( t \) be fixed. We have

\[ \mu^n_t \leq p_n(t)\mu^{n+1}_t \quad \text{and} \quad \mu^n_t \leq q_n(t)\mu_t, \]

where \( p_n(t) > 1 \) and \( q_n(t) > 1 \) are numbers converging to 1. Hence there is a finite limit \( \lim_{n \to \infty} K_{h_t}(\mu^n_t,\nu^n_t) \leq K_{h_t}(\mu_t,\nu_t) \). We now prove the opposite inequality. Let \( \sigma^n_t \in \Pi(\mu^n_t,\nu^n_t) \) be optimal measures for \( h_t \). Both sequences \( \{\mu^n_t\} \) and \( \{\nu^n_t\} \) are uniformly tight, hence \( \{\sigma^n_t\} \) is also uniformly tight and contains a weakly convergent subsequence, which we denote by the same indices. Let \( \sigma_t \) be its limit. Clearly, \( \sigma_t \in \Pi(\mu_t,\nu_t) \). The integral of \( h_t \) against \( \sigma_t \) does not exceed the liminf of the integrals of \( h_t \) against the measures \( \sigma^n_t \) (see [10, Corollary 4.3.4]), which is exactly the limit of \( K_{h_t}(\mu^n_t,\nu^n_t) \).

Let us recall the following classical result going back to Novikoff and Kunugui, see [17, p. 224, 225] (or [25, Theorem 18.18]), where \( X \) is a standard Borel space.

**Theorem A.** Let \( X \) be a Souslin space, \( Y \) a Polish space, and \( B \subset X \times Y \) a Borel set such that for all \( x \in X \) the sections \( B_x \) are \( \sigma \)-compact (countable unions of compact sets). Then \( B \) admits a Borel uniformization, which means that the projection \( \pi_X(B) \) of \( B \) on \( X \) is a Borel set and there is a Borel mapping

\[ f : \pi_X(B) \to Y \]

whose graph is contained in \( B \).

There is also another classical result with somewhat different assumptions (see, e.g., [7, Theorem 6.9.3 and Corollary 6.9.4]).

**Theorem B.** Let \((T, \mathcal{T})\) be a general measurable space, let \( E \) be a Polish space, and let \( \Psi \) be a mapping on \( T \) with values in the set of nonempty closed subsets of \( E \) that is measurable in the following sense: for every open set \( U \subset E \), the projection of the set \( \{(t,x) : x \in \Psi(t) \cap U\} \) on \( T \) belongs to \( \mathcal{T} \). Then there is a \((\mathcal{T}, \mathcal{B}(E))\)-measurable mapping \( \zeta : T \to E \) with \( \zeta(t) \in \Psi(t) \) for all \( t \), i.e., a \((\mathcal{T}, \mathcal{B}(E))\)-measurable selection. Moreover, there is a sequence of \((\mathcal{T}, \mathcal{B}(E))\)-measurable mappings \( \zeta_n : T \to X \) such that the sequence \( \{\zeta_n(t)\} \) is dense in \( \Psi(t) \) for each \( t \).

The difference between the two theorems is that in the latter the space \( T \) is more general, but the hypotheses include the measurability of the aforementioned projections, while in the former this measurability follows from other assumptions (here we consider \( \Psi(x) = B_x \) in order to compare the settings). Indeed, to see
this we observe that it suffices to verify the required measurability for closed sets $U$ (since any open set in a Polish space is some countable union of closed sets). But then the sections of $B \cap (X \times U)$ remain $\sigma$-compact, so the projection remains Borel. Note that in Theorem A there is also a sequence of Borel mappings $f_n : \pi_X(B) \to Y$ such that $\{f_n(x)\}$ is dense in $B_x$ for each $x \in \pi_X(B)$.

Thus, Theorem B is formally more general (but to see this we need Theorem A), however, practically the most difficult part is to verify the measurability of projections (and the proof of Theorem A is more difficult). So our main tool will be Theorem A. It should be noted that Theorem A is not valid for arbitrary measurable spaces in place of Souslin spaces (it fails even for co-analytic sets in $[0, 1]$ and single-valued sections).

Finally, let us mention yet another known result (see [7, Theorem 6.9.5]) in which the assumptions are weaker, but also the guaranteed measurability of selections is weaker.

**Theorem C.** Suppose that $T$ and $E$ are Souslin spaces. Let $\Psi$ be a multivalued mapping from $T$ to the set of nonempty subsets of $E$ such that its graph

$$\Gamma_\Psi = \{(t, u) : t \in T, u \in \Psi(t)\}$$

is a Souslin set in $T \times E$. Then, there exists a sequence of selections $\zeta_n$ that are measurable as mappings from $(T, \sigma(S(T)))$ to $(E, \mathcal{B}(E))$ and, for every $t \in T$, the sequence $\{\zeta_n(t)\}$ is dense in the set $\Psi(t)$.

In our situation, a typical application of these results is this.

The set-valued mapping

$$(\mu, \nu) \mapsto \Pi(\mu, \nu)$$

from $\mathcal{P}(X) \times \mathcal{P}(Y)$ to the set of nonempty compact subsets of $\mathcal{P}(X \times Y)$ is measurable in the aforementioned sense. Alternatively, we can apply Theorem A by using the easy fact that the set $B$ of triples $(\mu, \nu, \sigma)$ in $\mathcal{P}(X) \times \mathcal{P}(Y) \times \mathcal{P}(X \times Y)$ such that $\sigma \circ \pi_X^{-1} = \mu$ and $\sigma \circ \pi_Y^{-1} = \nu$ is Borel and its sections $B_{\mu, \nu}$ are compact. Hence there is a sequence of Borel mappings

$$\Phi_n : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y)$$

such that the sequence $\{\Phi_n(\mu, \nu)\}$ is dense in $\Pi(\mu, \nu)$ for all $\mu$ and $\nu$.

Let $t \mapsto \mu_t$ and $t \mapsto \nu_t$ be measurable mappings from $(T, T)$ to the spaces $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ of Borel probability measures on Polish spaces $X$ and $Y$. Then there is a sequence of measurable mappings $\Psi_n : T \to \mathcal{P}(X \times Y)$ such that $\Psi_n(t) \in \Pi(\mu_t, \nu_t)$ and the sequence $\{\Psi_n(t)\}$ is dense in $\Pi(\mu_t, \nu_t)$ for each $t$. To this end, we set $\Psi_n(t) := \Phi_n(\mu_t, \nu_t)$.

Suppose now that $(x, y) \mapsto h(t, x, y)$ is continuous for each fixed $t \in T$ (the case of Theorem 1.1.1). Then the function

$$K(t) = K_{h_t}(\mu_t, \nu_t)$$

is measurable on $T$, which proves the first assertion of Theorem 1.1.1. Indeed,

$$K(t) = \inf_n \int_{X \times Y} h(t, x, y) \Psi_n(t)(dxdy).$$

Let now

$$M_t = \left\{ \sigma \in \Pi(\mu_t, \nu_t) : \int h(t, x, y) \sigma(dxdy) = K(t) \right\}.$$
Each set $M_t$ is compact in $\Pi(\mu_t, \nu_t) \subset \mathcal{P}(X \times Y)$. Once we have the measurability of the set-valued mapping $t \mapsto M_t$ we can use selection theorems. However, the problem is to verify this measurability. This will be done below for Souslin spaces $T$ in order to have the Borel measurability. However, if we are satisfied with the measurability with respect to the $\sigma$-algebra $\sigma(S(T))$ on $T$, then we can apply Theorem B to this larger $\sigma$-algebra. The hypothesis of Theorem B is fulfilled. Indeed, let $U$ be an open set in $\mathcal{P}(X \times Y)$. Since the set of pairs $(t, \sigma)$ in $T \times \mathcal{P}(X \times Y)$ such that $\sigma \in \Pi(\mu_t, \nu_t)$ and the integral of $h_t$ against $\sigma$ is contained in $T \otimes \mathcal{B}(\mathcal{P}(X \times Y))$ by the $T$-measurability of $K$, its intersection with $T \times U$ is also in $T \otimes \mathcal{B}(\mathcal{P}(X \times Y))$. Hence the projection of this intersection belongs to $S(T)$ by a known result (see [2, Corollary 6.10.10]).

Finally, the proof of Theorem 4.5 is completely analogous, the only difference is that now we apply Theorem C: the set of pairs $(t, \sigma)$ in $T \times \mathcal{P}(X \times Y)$ such that $\sigma \in \Pi_t(\mu_t, \nu_t)$ is Borel as above. Hence there is a sequence of $S(T)$-measurable mappings $\Psi_n : T \rightarrow \mathcal{P}(X \times Y)$ such that the sequence $\{\Psi_n(t)\}$ is dense in $\Pi_t(\mu_t, \nu_t)$, so $K(t)$ equals the infimum of the sequence of integrals of $h_t$ against $\Psi_n(t)$. Once we know that $K(t)$ is Borel measurable, the same reasoning applies to the set of pairs $(t, \sigma)$ with the additional restriction that the integral of $h_t$ against $\sigma$ equals $K(t)$, but this restriction determines a Borel set.

**Lemma 5.5.** Suppose that $Z$ is a Borel set in a complete separable metric space with a metric $d$, $T$ is a Souslin space, and $h : T \times Z \rightarrow [0, 2]$ is a Borel function that is lower semicontinuous in the second variable and has the following property: for every $t$ there is a compact set $Z_t \subset Z$ such that $h(t, z) \in [0, 1]$ for all $z \in Z_t$ and $h(t, z) = 2$ for all $z \in Z \setminus Z_t$. Then there is a sequence of Borel mappings $\psi_j : T \rightarrow Z$ such that

$$\inf \{h(t, z) + d(x, z) : z \in Z\} = \inf \{h(t, \psi_j(t)) + d(x, \psi_j(t))\} \quad \forall x \in Z, t \in T. \tag{5.2}$$

**Proof.** We consider the sets

$$S_{k,m} = \{(t, z) \in T \times B_m : h(t, z) \in U_k\},$$

where $\{U_k\}$ is the sequence of all rational semiclosed intervals $(a, b]$ in $[-1, 1]$ and $\{B_m\}$ is the sequence of all closed balls with positive rational radii centered at the points of a fixed countable dense set $\{z_l\}$ in $Z$. The sets $S_{k,m}$ are Borel. We take into account only nonempty sets $S_{k,m}$. Note that if $(t, z) \in S_{k,m}$, then $z$ must belong to $Z_t$, since $U_k \subset [-1, 1]$ and $h(t, \cdot) = 2$ outside $Z_t$. For each $t \in T$, the section

$$S'_{k,m} = \{x : (t, x) \in S_{k,m}\}$$

is the difference of two compact sets by the lower semicontinuity of $h$ in the second argument and the inclusion $S'_{k,m} \subset Z_t$. Hence this section is $\sigma$-compact. Therefore, by Theorem A stated above, the projection of $S_{k,m}$ onto $T$, denoted by $T_{k,m}$, is a Borel set and there is a Borel mapping $\psi_{k,m} : T_{k,m} \rightarrow Z$ such that $\psi_{k,m}(t) \in S'_{k,m}$ for each $t \in T_{k,m}$. Outside $T_{k,m}$ we set $\psi_{k,m}(t) = z_1$. Let us add to this sequence the countable family of constant mappings with values in $\{z_l\}$. Finally, we renumber the obtained collection by using a single index $j$.

We now verify (5.2). Since both sides of (5.2) are Lipschitz in $x$, it suffices to show that they coincide for all $x \in \{z_l\}$. Fix $t \in T$, $x = z_l$ and $\varepsilon > 0$. Take $z \in Z$ for which $h(t, z) + d(x, z) - \varepsilon/2$ is less than the left-hand side of (5.2). If $x \notin Z_t$, then either the left-hand side equals 2 and the minimum is attained at $z = z_l$, so the corresponding constant function works, or $z \in Z_t$, because $h(t, z) = 2$ outside $Z_t$. If
that the left-hand side of (5.2) is larger than $h(t, \psi_{k,m}(t)) + d(x, \psi_{k,m}(t)) - \varepsilon$.

To this end, we find $k$ and $m$ for which $h(t, z) \in U_k$ and $z \in B_m$, moreover, we pick $k$ and $m$ such that the length of $U_k$ and the diameter of $B_m$ are less than $\varepsilon/8$. Then for the corresponding $\psi_{k,m}(t)$ we have $\psi_{k,m}(t) \in B_m$, $h(t, \psi_{k,m}(t)) \in U_k$, so that

\[ h(t, \psi_{k,m}(t)) + d(x, \psi_{k,m}(t)) < h(t, z) + d(x, z) + \varepsilon/4 < \inf\{h(t, z) + d(x, z) : z \in Z\} + \varepsilon, \]

which completes the proof.

**Lemma 5.6.** Under the hypotheses of the previous lemma, there is a sequence of Borel functions $h_n : T \times Z \to [0, 2]$ such that $h_n \leq h_{n+1}$, $h(t, z) = \lim_{n \to \infty} h_n(t, z)$, and the functions $z \mapsto h_n(t, z)$ are bounded Lipschitz for each $t$.

**Proof.** There is a classical construction for approximations:

\[ h_n(t, z) = \inf\{h(t, y) + nd(z, y) : y \in Z\}. \]

The function $h_n$ is Lipschitz in $z$ and $h_n \leq h$. Its Borel measurability in $t$ follows by the previous lemma applied to the metric $nd$, so $h_n$ is jointly Borel measurable. □

**Remark 5.7.** The assumption that $T$ is a Souslin space has been used in the previous two lemmas to cover the case of lower semicontinuous functions. If the functions $h_t$ are continuous for each $t$ and $h$ is measurable on $T \times Z$ (not necessarily bounded), then both lemmas are valid for arbitrary measurable spaces $(T, \mathcal{T})$, since the approximations

\[ h_n(t, z) = \inf_k [h(t, y_k) + nd(z, y_k)], \]

where $\{y_k\}$ is a fixed sequence dense in $Z$, coincide with the functions defined above by the infimum over the whole space and are Lipschitz. Replacing them by $\min(h_n, n)$ we obtain bounded Lipschitz functions increasing to $h$ and measurable on $T \times Z$.

**Lemma 5.8.** Suppose that lower semicontinuous cost functions $h_n \geq 0$ increase pointwise to a function $h$ for which $K_h(\mu, \nu) < \infty$. Let $\pi_n \in \Pi(\mu, \nu)$ be optimal measures for $h_n$ converging weakly to a Radon measure $\pi$. Then $\pi$ is an optimal measure for the triple $h, \mu, \nu$. In addition, $K_h(\mu, \nu) = \lim_{n \to \infty} K_{h_n}(\mu, \nu) = \lim_{n \to \infty} I_{h_n}(\pi_n)$.

**Proof.** For continuous cost functions this assertion is simple. For the reader’s convenience, we include the proof. Clearly, $\pi \in \Pi(\mu, \nu)$. The sequence $\{\pi_n\}$ is uniformly tight, so, given $\varepsilon > 0$, there is a compact set $K$ with $\pi(K) > 1 - \varepsilon$, $\pi_n(K) > 1 - \varepsilon$ for all $n$. Enlarging $K$ we can assume that the integral of $h$ over the complement of $K$ with respect to $\pi$ is less than $\varepsilon$. On $K$ convergence is uniform by Dini’s theorem. Then

\[ |I_h(\pi) - K_{h_n}(\mu, \nu)| \leq 2\varepsilon \]

for large $n$. Hence the numbers $K_{h_n}(\mu, \nu)$ increase to $I_h(\pi)$. Since

\[ K_{h_n}(\mu, \nu) \leq K_h(\mu, \nu) \leq I_h(\pi), \]

we have $I_h(\pi) = K_h(\mu, \nu)$. This reasoning also applies to the case where only the function $h$ is continuous, but all $h_n$ are lower semicontinuous (to apply Dini’s theorem, we need the upper semicontinuity of the functions $h - h_n$).
Our next step is to observe that for lower semicontinuous \( h \) the quantity \( K_h(\mu, \nu) \) coincides with the supremum of \( K_w(\mu, \nu) \) over bounded continuous cost functions \( w \geq 0 \) such that \( w(x,y) \leq h(x,y) \) for all \( x \) and \( y \). This follows by the Kantorovich duality: for each \( \varepsilon > 0 \) there are functions \( \varphi \in C_b(X) \) and \( \psi \in C_b(Y) \) such that
\[
\varphi(x) + \psi(y) \leq h(x,y)
\]
for all \( x \) and \( y \) and
\[
\int \varphi \, d\mu + \int \psi \, d\nu \geq K_h(\mu, \nu) - \varepsilon.
\]
We now take \( w(x,y) = \max(\varphi(x) + \psi(y), 0) \). Since \( w(x,y) \geq \varphi(x) + \psi(y) \), the integral of \( h \) against \( \mu, \nu \) is at least \( K_h(\mu, \nu) - \varepsilon \). Hence we have
\[
K_w(\mu, \nu) \geq K_h(\mu, \nu) - \varepsilon.
\]
It follows that there is a pointwise increasing sequence of nonnegative functions \( w_n \in C_b(X \times Y) \) such that \( w_n(x,y) \leq h(x,y) \) and \( K_{w_n}(\mu, \nu) \rightarrow K_h(\mu, \nu) \). Such functions can be found converging to \( h \), since there is a sequence of bounded continuous functions \( u_n \geq 0 \) increasing to \( h \), so we can take \( \max(w_n, u_n) \) and observe that \( K_{w_n}(\mu, \nu) \leq K_{\max(w_n, u_n)}(\mu, \nu) \leq K_h(\mu, \nu) \).

Let us show that there is no gap between \( K_h(\mu, \nu) \) and the limit of \( K_{w_n}(\mu, \nu) \) in the general case. Let \( \varepsilon > 0 \). Take a function \( w \in C_b(X \times Y) \) with \( 0 \leq w \leq h \) and \( K_w(\mu, \nu) \geq K_h(\mu, \nu) - \varepsilon \).

The sequence of bounded lower semicontinuous functions \( v_n = \min(w, h_n) \) increases pointwise to the bounded continuous function \( w \). Hence by the previous step
\[
K_{v_n}(\mu, \nu) \rightarrow K_w(\mu, \nu) \geq K_h(\mu, \nu) - \varepsilon.
\]
Since \( K_h(\mu, \nu) \geq K_{v_n}(\mu, \nu) \), we conclude that \( K_{v_n}(\mu, \nu) \rightarrow K_h(\mu, \nu) \).

It remains to show that \( K_h(\mu, \nu) \) coincides with \( I_h(\pi) \). Otherwise for some \( \delta > 0 \) we have \( I_h(\pi) > K_h(\mu, \nu) + \delta \). Using the functions \( w_n \) constructed above, we obtain a number \( N \) such that
\[
\int w_N \, d\pi > K_h(\mu, \nu) + \delta/2.
\]
Hence
\[
\int w_N \, d\pi_n > K_h(\mu, \nu) + \delta/2
\]
for all \( n \) large enough. Since \( w_N \) is bounded and \( \{\pi_n\} \) is uniformly tight, there is a compact set \( K \) such that
\[
\int_K w_N \, d\pi_n > K_h(\mu, \nu) + \delta/4
\]
for all \( n \) large enough. The functions \( \min(h_n, w_N) \) are lower semicontinuous and increase to the continuous function \( w_N \). Hence convergence is uniform on \( K \). Therefore,
\[
\int_K \min(h_n, w_N) \, d\pi_n > K_h(\mu, \nu) + \delta/8
\]
for all \( n \) large enough. This yields the bound
\[
K_h(\mu, \nu) = \int_{X \times Y} h_n \, d\pi_n \geq \int_K h_n \, d\pi_n \geq \int_K \min(h_n, w_N) \, d\pi_n > K_h(\mu, \nu) + \delta/8,
\]
which is a contradiction.
Lemma 5.9. Suppose that in the situation of Theorem 4.2 the measurability of \( t \mapsto K_{h_t}(\mu_t, \nu_t) \) is given in advance. Then the assertion about the existence of a Borel version of \( \sigma_t \) is true.

Proof. Now by assumption the function
\[
K(t) = K_{h_t}(\mu_t, \nu_t)
\]
is measurable on \( T \). Let
\[
M_t := \left\{ \sigma \in \Pi(\mu_t, \nu_t) : \int h(t, x, y) \sigma(dxdy) = K(t) \right\}.
\]
Each set \( M_t \) is compact in \( \Pi(\mu_t, \nu_t) \subset P(X \times Y) \), because if measures \( \sigma_n \in M_t \) converge weakly to a measure \( \sigma \), then \( \sigma \in \Pi(\mu_t, \nu_t) \) and the integral of \( h_t \) against \( \sigma \) cannot be larger than \( K(t) \) by the lower semicontinuity of \( h_t \), but obviously it cannot be smaller than \( K(t) \) by the definition of \( K(t) \).

By the Borel measurability of the function \( t \mapsto K(t) \) and the Borel measurability of the function
\[
(t, \sigma) \mapsto \int_{X \times Y} h(t, x, y) \sigma(dxdy)
\]
on \( T \times \mathcal{P}(X \times Y) \), which follows by the joint measurability of \( h \) (see [10, Theorem 5.8.4]), the set
\[
B = \left\{ (t, \sigma) : \sigma \in \mathcal{P}(X \times Y), \sigma \in \Pi(\mu_t, \nu_t), \int h(t, x, y) \sigma(dxdy) = K(t) \right\}
\]
is Borel in \( T \times \mathcal{P}(X \times Y) \) and \( M_t \) is its section at \( t \). Hence again Theorem A applies. \( \square \)

Lemma 5.10. Let \((T, T)\) be a measurable space, let \( E \) be a completely regular Souslin space, and let \( u_n : T \to E \) be a sequence of \( T \)-measurable mappings such that the sequence \( \{u_n(t)\} \) has compact closure for every fixed \( t \in T \). Then there is a sequence of \( T \)-measurable functions \( t \mapsto \eta_k(t) \) with values in \( \mathbb{N} \) such that, for every \( t \), the numbers \( \eta_k(t) \) increase to infinity and the sequence \( \{u_{\eta_k(t)}(t)\} \) converges to some point \( u(t) \) such that the mapping \( t \mapsto u(t) \) is \( T \)-measurable.

Proof. There is a continuous injection of \( E \) into \([0, 1]^\infty\), so we can consider \( E \) as a set in \([0, 1]^\infty\) with a stronger Souslin topology. Points of \([0, 1]^\infty\) will be written as \( x = (x^1, x^2, \ldots) \). It suffices to pick increasing numbers \( \eta_k(t) \) measurably in \( t \) in such a way that, for each \( j \) and \( t \), the sequence of numbers \( u_{\eta_k(t)}^j(t) \) will converge. Indeed, this convergence implies that the sequence \( \{u_{\eta_k(t)}(t)\} \) cannot have different limit points, but by the compactness of the closure this sequence must have limit points, so it follows that the whole sequence converges.

We construct \( \eta_k(t) \) inductively. By the measurability of \( u_n \) the functions
\[
L_j(t) = \limsup_{n \to \infty} u_n^j(t)
\]
are \( T \)-measurable. Let \( \eta_k^1(t) \) be the minimal number \( n \) such that
\[
|u_{\eta_k^1(t)}^1(t) - L_1(t)| < 1.
\]
This number measurably depends on $t$, because
\[ \left\{ t \in T : \eta_1^k(t) = m \right\} = \left\{ t : |u_n^1(t) - L_1(t)| \geq 1, n = 1, \ldots, m, |u_m^1(t) - L_1(t)| < 1 \right\}. \]
Assuming that $\eta_k^1(t)$ is already defined and $T$-measurable, we take for $\eta_{k+1}^1(t)$ the minimal number $n$ such that $n > \eta_k^1(t)$ and
\[ |u_n^1(t) - L_1(t)| < \frac{1}{k+1}. \]
As above, the function $\eta_{k+1}^1$ is $T$-measurable. It follows that the first coordinates of $u_{\eta_k^1(t)}$ converge to $L_1(t)$.

The second step is to pick a subsequence in $\{\eta_k^1(t)\}$ for which the second coordinates will converge to $L_2(t)$. To this end, we take for $\eta_2^2(t)$ the minimal number $n > \eta_1^1(t)$ among the numbers $\eta_k^2(t)$ such that
\[ |u_n^2(t) - L_2(t)| < 1. \]
We have
\[ \{ t \in T : \eta_1^2(t) = m \} = \{ t : |u_n^2(t) - L_2(t)| \geq 1, n = 1, \ldots, m, |u_m^2(t) - L_2(t)| < 1 \}, \]
which shows that $\eta_1^2$ is $T$-measurable. We proceed inductively and find $T$-measurable functions $\eta_k^2$ such that $\eta_2^2(t)$ is the minimal number in $\{\eta_1^k(t)\}$ for which the difference between $L_2(t)$ and the second coordinate of $u_{\eta_1^k(t)}(t)$ becomes less than $1/k$.

We continue this process inductively and obtain embedded subsequence $\{\eta_k^m(t)\}$ such that the functions $\eta_k^m$ are $T$-measurable and the $m$th coordinates of $u_{\eta_k^m(t)}(t)$ converge to $L_m(t)$. For the diagonal sequence $\eta_k^k(t)$ we have convergence of all coordinates, which proves convergence of all $u_{\eta_k^k(t)}(t)$.

**Corollary 5.11.** Let $(T, T)$ be a measurable space, let $X$ be a completely regular Souslin space, and let $t \mapsto \mu_{t,n}, T \to M(X)$ be a sequence of $T$-measurable mappings such that the sequence of measures $\{\mu_{t,n}\}$ has weakly compact closure (for example, is uniformly tight) for every fixed $t \in T$. Then there is a sequence of $T$-measurable functions $t \mapsto \eta_k(t)$ with values in $\mathbb{N}$ such that, for every $t$, the numbers $\eta_k(t)$ increase to infinity and the sequence of measures $\mu_{t,\eta_k(t)}$ converges to some measure $\mu_t$ such that $t \mapsto \mu_t$ is $T$-measurable.

**Proof.** The previous lemma applies, since the space of measures on $X$ with the weak topology is also Souslin. \(\square\)

**Proof of Theorem 4.7.** By Corollary 5.11 for completing the proof of Theorem 4.7 it suffices to find approximate $T$-measurable solutions $\sigma_{t,n}$ with $I_h(\sigma_{t,n}) \to K(t)$ for each $t$. To this end, we find $T$-measurable solutions $\pi_{t,n}$ for bounded Lipschitz cost functions $h_n$ increasing to $h$ and constructed according to Remark 5.7. Therefore, the general case reduces to the case in which every function $h_t$ is bounded by 1 and Lipschitz with constant 1. Moreover, by Lemma 5.2 and Lemma 5.4 it suffices to consider the case in which the measures $\mu_t$ and $\nu_t$ have compact supports, so that for each $t$ there is a compact set $S_t$ on which all measures from $\Pi(\mu_t, \nu_t)$ are concentrated and $S_t$ depends on $t$ measurably.
Let us consider the space $K(X \times Y)$ of nonempty compact subsets of $X \times Y$ with the Hausdorff distance $d_H$ introduced in Section 2. This space is separable, hence there is a sequence of compacts sets $Q_j$ dense in the union of $S_t$. Let fix $n$ and consider the sets

$$T_j = \{ t \in T : \text{dist}_H(S_t, Q_j) \leq 1/n \}.$$  

Note that $T_j \in \mathcal{T}$ (this follows from the proof of Lemma 5.2). The set of 1-Lipschitz functions on $Q_j$ with values in $[0,1]$ is compact in the sup-norm, hence there is a sequence $h_{j,m}$ dense in it. Each function $h_{j,m}$ has an extension (denoted by the same symbol) to all of $X \times Y$ with values in $[0,1]$ and 1-Lipschitz.

We further define the sets

$$T_{j,m} = \{ t \in D_j : \sup_{(x,y) \in Q_j} |h_t(x,y) - h_{j,m}(x,y)| \leq 1/n \}.$$  

The supremum can be taken over a countable set dense in $Q_j$, hence $T_{j,m} \in \mathcal{T}$. Using these sets we obtain a partition of $T$ into nonempty disjoint sets $D_k \subset T$ with the following property: for each $D_k$ there are numbers $j$ and $m$ such that $\text{dist}_H(S_t, Q_j) \leq 1/n$ and $\sup_{(x,y) \in Q_j} |h_t(x,y) - h_{j,m}(x,y)| \leq 1/n$ for all $t \in D_k$. In every set $D_k$ take a point $t_k$. The cost function $h_{t_k}$ differs from any other cost function $h_t$ with $t \in D_k$ by at most $3/n$ on the set $S_t$. Indeed, if $(x,y) \in S_t$, then we can find $(u,v) \in Q_j$ with $d((x,y),(u,v)) \leq 1/n$. Since on $Q_j$ the functions $h_t$ and $h_{t_k}$ differ by at most $1/n$, we have

$$|h_t(x,y) - h_{t_k}(x,y)| \leq |h_t(x,y) - h_t(u,v)| + |h_t(u,v) - h_{t_k}(u,v)| + |h_{t_k}(u,v) - h_{t_k}(x,y)| \leq 3n^{-1}.$$  

Finally, on each $D_k$ we solve the Kantorovich problem with the cost function $h_{t_k}$ independent of $t$ and the original marginals. Hence there is a solution $\pi^k_t \in \Pi(\mu_t, \nu_t)$ that is $\mathcal{T}$-measurable. Clearly, $|K_{h_t}(\mu_t, \nu_t) - K_{h_k}(\mu_t, \nu_t)| \leq 3/n$ for all $t \in D_k$. Therefore, on all of $T$ we obtain the desired approximation. \hfill $\square$

**Proof of Theorem 4.2.** By Lemma 5.9 it suffices to prove the Borel measurability of the transportation cost $K_t = K_{h_t}(\mu_t, \nu_t)$. Using Lemma 5.8 and the truncations $\min(h_t, N)$ we can pass to uniformly bounded cost functions. So we can assume that $h_t < 1$. Lemma 5.4 reduces the assertion to the case of measures $\mu^n_t$ and $\nu^n_t$ with compact supports $Z^n_1(t)$ and $Z^n_2(t)$. The value of the cost does not change if we redefine $h_t$ outside $Z^n_1(t) \times Z^n_2(t)$ by the value 2. Since the set $\bigcup_t (\{t\} \times Z^n_1(t) \times Z^n_2(t))$ belongs to $\mathcal{T} \otimes \mathcal{B}(X) \otimes \mathcal{B}(Y)$ by Lemma 5.2, this new cost function is Borel. It is readily seen that it is lower semicontinuous.

Now we are in the situation of Lemma 5.6. Therefore, Lemma 5.8 further reduces everything to continuous cost functions. This case is covered by the first (and easy) part of Theorem 4.1. \hfill $\square$

**Remark 5.12.** As already noted in the introduction, Zhang [38] proved that if cost functions $h_t$ are continuous, then the space $M$ of nonnegative continuous cost functions can be regarded as a parametric space and equipped with its natural Borel $\sigma$-algebra (generated by the metric introduced above) and the set-valued mapping $(h, \mu, \nu) \mapsto \text{Opt}(h, \mu, \nu)$ has a Borel measurable selection.

However, it is not clear how this can be applied to the assertion announced in [38] that a measurable selection exists for any parametric measurable space $(T, \mathcal{T})$. The point is that the mapping $t \mapsto h_t$ with values in $M$ generated by a function $h$ Borel measurable in $t$ can fail to be measurable when $M$ is equipped with the
Borel σ-algebra. For example, this happens if $T = C_b(B \times B)$ with its sup-norm, where $B$ is the unit ball in $l^2$, $\mathcal{T}$ is generated by evaluation functionals $t \mapsto t(x, y)$, $X = Y = B$, and $h(t, x, y) = t(x, y)$. Here $h$ is bounded and continuous in $(x, y)$ and $\mathcal{T}$-measurable, but $t \mapsto h_t$ is not measurable with values in $C_b(B \times B)$ equipped with the Borel σ-algebra.

To see this, let us observe that the Borel σ-algebra of the space $C_b(\mathbb{N})$ is not countably generated, because its cardinality is greater than that of the continuum. Indeed, this space contains a closed discrete set of cardinality of the continuum; all subsets of this set are also closed. It follows that $B(C_b(\mathbb{N}))$ is not generated by the evaluation functionals $f \mapsto f(n)$. The same is true for any metric space containing a discrete countable subset, hence for any noncompact metric space. Similarly, the Borel σ-algebra of the metric space $M = C(X \times Y)$ mentioned in the introduction is not generated by evaluation functionals if the balls in $X \times Y$ are not compact.

However, for any compact metric space $K$ the Borel σ-algebra of the space $C_b(K)$ is generated by the evaluation functionals $f \mapsto f(k)$, because this space is separable and these functionals separate its points. Hence the proof in [38] for general $(T, \mathcal{T})$ is correct if $X$ is a locally compact Polish space. We have not succeeded to fix the general case in a simple way and needed several steps. Recall also that for lower semicontinuous cost functions we still assume that $X$ spaces it is possible to impose the following stronger condition on $\mathcal{T}$-algebras.

Remark 5.13. The question also arises whether Theorem 4.2 extends to Souslin spaces $X$ and $Y$. A major problem is to extend Lemma 5.4 to Souslin spaces $X$. Suppose that for a lower semicontinuous $h_t$ and Souslin spaces $X$ and $Y$ we know that there are measurable set-valued mappings $t \mapsto Z_n(t)$ as in Lemma 5.4. We take a bounded continuous metric $d$ on $X \times Y$ and observe that the functions $h_k(t, x, y) = \inf \{h(t, u, v) + kd((x, y), (u, v)) : (u, v) \in Z_n(t)\}$ increase on $Z_n(t)$ to $h(t, x, y)$, because on $Z_n(t)$ the topology of $X \times Y$ is metrizable by $d$ by compactness. Moreover, the assumed measurability of $Z_n(t)$ ensures (for each fixed $n$) the existence of a sequence of measurable mappings $\xi_n : T \times X \times Y$ such that $\xi_n(t) \subseteq Z_n(t)$ and $Z_n(t)$ is the closure of $\{\xi_n(t)\}$. So the infimum defining $h_k(t, x, y)$ can be evaluated over $\{\xi_n(t)\}$, which shows the measurability of $h_k(t, x, y)$.

Of course, if we agree to leave the safe area of Borel measurability, for Souslin spaces it is possible to impose the following stronger condition on $\mu_t$ and $\nu_t$: let these mappings be measurable when $X$ and $Y$ are equipped with the σ-algebras $\sigma(S(X))$ and $\sigma(S(Y))$. Then $K(t)$ is $\sigma(S(T))-measurable and $\sigma_t$ can be made $\sigma(S(T))-measurable. Indeed, there are continuous surjections $g_1 : \mathbb{R}^\infty \to X$, $g_1 : \mathbb{R}^\infty \to Y$. Hence we obtain families $\mu_t^1 = \mu_t \circ g_1^{-1}$, $\nu_t^2 = \nu_t \circ g_2^{-1}$ of measures on $\mathbb{R}^\infty$ that are $\sigma(S(T))-measurable. The obtained results apply to these measures and the cost function $h^0(t, u, v) = h(t, g_1(u), g_2(u))$, which satisfies our hypotheses. The corresponding transportation cost and optimal measures will be $\sigma(S(T))-measurable. Then we take the images of optimal measures under the mapping $(g_1, g_2)$.

6. The Skorohod parametrization with a parameter

In this short section we consider another parametric problem in the same circle of ideas. It was shown by Skorohod [33] that for any weakly convergent sequence of Borel probability measures $\mu_n$ on a complete separable metric space $X$ there is a sequence of Borel mappings $\xi_n : [0, 1] \to X$ with $\mu_n = \lambda \circ \xi_n^{-1}$, where $\lambda$ is Lebesgue measure, converging almost everywhere. This important result was generalized by Blackwell and Dubins [5] and Fernique [20], who proved that for every measure...
\( \mu \in \mathcal{P}(X) \) there is a Borel mapping \( \xi_\mu : [0, 1] \to X \) such that \( \mu \) is the image of Lebesgue measure \( \lambda \) under \( \xi_\mu \) and measures \( \mu_n \) converge weakly to \( \mu \) if and only if the mappings \( \xi_{\mu_n} \) converge to \( \xi_\mu \) almost everywhere. A topological proof of this result along with some generalizations was given in [11] (see also [2], [7], and [10] on this topic). The purpose of this section is to verify that this topological proof actually yields the following result.

**Theorem 6.1.** Let \( X \) be a complete separable metric space. For every measure \( \mu \in \mathcal{P}(X) \) there is a Borel mapping \( \xi_\mu : [0, 1] \to X \) with \( \mu = \lambda \circ \xi_\mu^{-1} \) such that the mapping \( (\mu, t) \mapsto \xi_\mu(t) \) is Borel measurable on \( \mathcal{P}(X) \times [0, 1] \) and measures \( \mu_n \) converge weakly to \( \mu \) if and only if the mappings \( \xi_{\mu_n} \) converge to \( \xi_\mu \) almost everywhere.

Therefore, for any family of measures \( \mu_\omega \in \mathcal{P}(X) \) measurable depending on a parameter \( \omega \) from a measurable space \( (\Omega, \mathcal{A}) \), the mapping \( (\omega, t) = \xi_{\mu_\omega}(t) \) with values in \( X \) is \( A \otimes \mathcal{B}[0, 1] \)-measurable.

**Proof.** We verify that the proof suggested in [11] and also presented in [7, §8.5] and [10, §2.6] gives the desired version. This proof is very simple. First we explicitly define the desired mapping for the space \( X = [0, 1] \):

\[
\xi_\mu(t) = \sup \{ x \in [0, 1] : \mu([0, x]) \leq t \}.
\]

It is shown in [10, Theorem 2.6.4] that this is the desired parametrization. We only need to show that \( \xi_\mu(t) \) is jointly Borel measurable on \( \mathcal{P}([0, 1]) \times [0, 1] \). Note that \( \xi_\mu(t) \) is increasing and right-continuous in \( t \). It is known that if a function \( \xi_\mu(t) \) is increasing and right-continuous in \( t \) for every fixed \( \mu \) and is Borel measurable in \( \mu \) for each fixed \( t \), then it is jointly Borel measurable. Indeed, it suffices to observe that it is the limit of the decreasing sequence of functions \( \xi_n(\mu, t) \) defined as follows: for each \( n \), we partition \( [0, 1] \) into \( 2^n \) intervals \( I_1 = [0, 2^{-n}) \), \( I_2 = [2^{-n}, 2^{1-n}) \), \ldots, \( I_{2^n} = [1 - 2^{-n}, 1] \) and set \( \xi_n(\mu, t) = \xi_\mu(r_k) \) if \( t \in I_k \) and \( r_k \) is the right end of \( I_k \).

The next step is to observe that once this theorem is established for some space \( X \), it remains valid for every Borel subspace \( E \subset X \). Indeed, every measure \( \mu \in \mathcal{P}(E) \) extends to a measure on \( X \) by letting \( \mu(X \setminus E) = 0 \). We take a jointly Borel measurable mapping \( (\mu, t) \mapsto \xi_\mu(t) \) for \( X \) and for measures concentrated on \( E \) redefine it by \( \eta_\mu(t) = \xi_\mu(t) \) if \( \xi_\mu(t) \in E \) and \( \eta_\mu(t) = x_0 \) if \( \xi_\mu(t) \notin E \), where \( x_0 \in E \) is a fixed element. Since \( \xi_\mu(t) \in E \) for almost all \( t \) for \( \mu \) concentrated on \( E \), we do not change the image of Lebesgue measure. The obtained mapping is obviously Borel measurable and gives the desired parametrization for \( \mathcal{P}(E) \).

It follows from the previous step that the theorem is true for the Cantor set \( C \). It is known that every compact metric space is the image of \( C \) under some continuous mapping, in particular, there is a continuous surjection \( h : C \to [0, 1]^\infty \). Then the induced mapping \( H : \mathcal{P}(C) \to \mathcal{P}([0, 1]^\infty) \) defined by \( H(\mu) = \mu \circ h^{-1} \) is also a continuous surjection. By the Milyutin theorem (see [10, §2.6] for details) there is a continuous affine mapping \( G : \mathcal{P}([0, 1]^\infty) \to \mathcal{P}(C) \) that is a right inverse for \( H \), i.e., \( H(G(\nu)) = \nu \) for all \( \nu \in \mathcal{P}([0, 1]^\infty) \). Therefore, using a jointly Borel measurable parametrization \( \xi_\mu(t) \) for \( \mathcal{P}(C) \) we obtain a jointly Borel measurable parametrization \( h(\xi_G(\mu)(t)) \) for \( \mathcal{P}([0, 1]^\infty) \). Hence the desired parametrization exists for every Borel subspace in \( [0, 1]^\infty \), but every Polish space is homeomorphic to a \( G_\delta \)-set in \( [0, 1]^\infty \), see [13] Theorem 4.2.10, Theorem 4.3.24, Corollary 4.3.25, which completes the proof. \( \square \)
Remark 6.2. A drawback of convergence almost everywhere is that there is no topology in which convergent sequences are precisely the sequences converging almost everywhere. For this reason it may be more convenient to consider on the space of Borel mappings from $[0,1]$ to $X$ the semimetric of convergence in measure defined by
\[ d_0(\xi, \eta) = \int_0^1 \min(d(\xi(t), \eta(t)), 1) \, dt, \]
where $d$ is a complete metric on $X$. The corresponding quotient space is also complete separable. It is clear that for the obtained parametrization convergence of mappings in this semimetric is equivalent to weak convergence of their laws. Actually, this parametrization gives a homeomorphism of the quotient space $L^0(\lambda, X)$ of $X$-valued mappings with convergence in measure and the space $\mathcal{P}(X)$.

This work has been supported by the Russian Science Foundation Grant 17-11-01058 at Lomonosov Moscow State University. We are very grateful to Sergey Kuksin and Armen Shirikyan for inspiring discussions and useful comments.

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