ON THE RELATIVISTIC PRECESSION AND OSCILLATION FREQUENCIES OF TEST PARTICLES AROUND RAPIDLY ROTATING COMPACT STARS

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Abstract

Whether or not analytic exact vacuum (electrovacuum) solutions of the Einstein (Einstein–Maxwell) field equations can accurately describe the exterior space-time of compact stars still remains an interesting open question in relativistic astrophysics. As an attempt to establish their level of accuracy, the radii of the innermost stable circular orbits (ISCOs) of test particles given by analytic exterior space-time geometries have been compared with those given by numerical solutions for neutron stars (NSs) obeying a realistic equation of state (EOS). It has been so shown that the six-parametric solution of Pachón et al. (PRS) more accurately describes the NS ISCO radii than other analytic models do. We propose here an additional test of accuracy for analytic exterior geometries based on the comparison of orbital frequencies of neutral test particles. We compute the Keplerian, frame-dragging, and precession and oscillation frequencies of the radial and vertical motions of neutral test particles for the Kerr and PRS geometries and then compare them with the numerical values obtained by Morris & Stella for realistic NSs. We identify the role of high-order multipole moments such as the mass quadrupole and current octupole in the determination of the orbital frequencies, especially in the rapid rotation regime. The results of this work are relevant to cast a separatrix between black hole and NS signatures and to probe the nuclear-matter EOS and NS parameters from the quasi-periodic oscillations observed in low-mass X-ray binaries.

Key words: black hole physics – celestial mechanics – stars: neutron – stars: oscillations – X-rays: binaries

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1. INTRODUCTION

One of the greatest challenges of the general theory of relativity has been the construction of solutions to the Einstein–Maxwell field equations representing the gravitational field of compact stars such as neutron stars (NSs). Stationary axially symmetric space-times satisfy basic properties one expects for rotating objects, namely, time symmetry and reflection symmetry with respect to the rotation axis (see, e.g., Manko & Sanabria-Gómez 2006). The simplest stationary axially symmetric exact exterior vacuum solution describing a rotating configuration is the well-known Kerr metric (Kerr 1963). The Kerr metric is fully described by two free parameters: the mass $M$ and the angular momentum $J$ of the object. However, it is known from numerical models that the quadrupole moment of rotating NSs deviates considerably from the one given by the Kerr solution $Q_{Kerr} = -J^2/(Mc^2)$ (for details see, e.g., Laarakkers & Poisson 1999).

In the mean time, a considerable number of analytic exterior solutions with a more complex multipolar structure than that of the Kerr solution have been developed (see, e.g., Manko et al. 1995, 2000; Stephani et al. 2003). Whether analytic exterior solutions are accurate or not at describing the gravitational field of compact stars is an interesting and very active topic of research (see, e.g., Stute & Camenzind 2002; Berti & Stergioulas 2004; Pachón et al. 2006, and references therein).

The accuracy of analytic solutions for describing the exterior geometry of a realistic rotating compact star has been tested by comparing physical properties, e.g., the radius of the innermost stable circular orbit (ISCO) on the equatorial plane and the gravitational redshift (for details see Sibigatullin & Sunyaev 1998; Berti & Stergioulas 2004; Pachón et al. 2006). In order to make such a comparison, the free parameters (i.e., the lowest multipole moments) of the analytic exterior space-time are fixed to the corresponding lowest multipole moments given by numerical interior solutions of the Einstein equations, for NS realistic models (see, e.g., Berti & Stergioulas 2004).

Following such a procedure, the solution of Manko et al. (2000) has been compared by Stute & Camenzind (2002) and Berti & Stergioulas (2004) with the numerical solutions for NSs calculated by Cook et al. (1994) and with those derived by Berti & Stergioulas (2004), respectively. However, being a generalization of the solution of Tomimatsu & Sato (1972), it cannot describe slowly rotating compact stars (see, e.g., Berti & Stergioulas 2004), but the dynamics of astrophysical objects with anisotropic stresses (for details see Dubeibe et al. 2007).

Following a similar procedure, based on tests of the ISCO radii on the equatorial plane of the rotating NSs obtained by Berti & Stergioulas (2004), it has been shown that the six-parametric solution of Pachón et al. (2006; hereafter PRS solution; see Section 2 for details) is more accurate than the model of Manko et al. (2000). In addition, being a generalization of the Kerr solution, this solution can be used for arbitrary rotation rates. Besides the ISCO radii, there are additional physical properties that can be computed with analytic and numerical models and it is thus useful to compare and contrast the accuracy of analytic exact models. The aim of this article is to analyze the properties of orbital frequencies of neutral test particles in the PRS and in the Kerr geometries with particular focus on the Keplerian $v_K$, frame-dragging (Lense–Thirring) $v_{LT}$, and precession (oscillation) frequencies of the radial and vertical motions, $v_ρ^p$, $v_ρ^O$, and $v_z^O$, respectively.

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The relevance of these frequencies relies on the fact that they are often invoked to explain the quasi-periodic oscillations (QPOs) observed in some relativistic astrophysical systems such as low-mass X-ray binaries (LMXBs), binary systems harboring either an NS or a black hole (BH) accreting matter from a companion star. For instance, within the relativistic precession model (RPM) introduced by Stella & Vietri (1998, 1999), Morsink & Stella (1999), and Stella et al. (1999), the kilohertz QPOs are interpreted as a direct manifestation of the modes of the relativistic epicyclic motion of blobs arising at various radii \( r \) in the inner parts of the accretion disk around the compact object (see Section 6 for details).

In addition to the RPM, the Keplerian, precession, and oscillation frequencies are used in other QPO theoretical models (see, e.g., Lin et al. 2011, for a recent comparison of the existing models). Due to the influence of general relativistic effects in the determination of such frequencies, an observational confirmation of any of the models might lead to an outstanding test of general relativity in the strong-field regime. In this line, it is of interest to compare and contrast the orbital frequencies given by the Kerr solution and by the PRS solution (see Section 3), which help to establish the differences between possible BH and NS signatures. We emphasize in this article the major role of the quadrupole moment as well as of the octupole moment of the object, whose possible measurement can be used as a tool to test the no-hair theorem of BHs (see, e.g., Johannsen & Psaltis 2011) and to discriminate between the different theoretical models proposed to explain the physics at the interior and exterior of the NSs. Additionally, in the case of NSs, the interpretation of QPOs as the manifestation of orbital motion frequencies might lead to crucial information of the NS parameters such as mass, angular momentum (see, e.g., Stella & Vietri 1998; Török et al. 2010), and quadrupole moment (see, e.g., Morsink & Stella 1999). These parameters reveal, at the same time, invaluable information about the equation of state (EOS) of nuclear matter.

The article is organized as follows. In Section 2 we recall the properties of the PRS solution. The computation of the orbital frequencies and the comparison of their features in the Kerr and PRS space-times are shown in Section 3. In Section 4 we study the accuracy of the analytic formulæ of the periastron frequencies and the comparison of their features in the Kerr properties of the PRS solution. The computation of the orbital frequencies and the comparison of their features in the Kerr

\[ g_{\mu\nu} \] are

\[ g_{\phi\phi} = \frac{\rho^2}{f(\rho, z)} - f(\rho, z)\omega(\rho, z)^2, \]

\[ g_{tt} = - f(\rho, z), \]

\[ g_{t\phi} = f(\rho, z)\omega(\rho, z), \]

\[ g_{zz} = \frac{\varepsilon^2 f(\rho, z)}{f(\rho, z)} = \frac{1}{g^{zz}} = \frac{1}{g^{\rho\rho}}. \]

Using the above line element, the Einstein–Maxwell equations can be reformulated, via Ernst’s procedure in terms of two complex potentials \( E(\rho, z) \) and \( \Phi(\rho, z) \) (Ernst 1968a, 1968b). By means of Shibatullin’s integral method (Shibatullin 1991; Manko & Shibatullin 1993), this system of equations can be solved via

\[ E(z, \rho) = \int_{-1}^{1} \frac{d\sigma}{\pi} \frac{e(\xi)\mu(\sigma)}{\sqrt{1 - \sigma^2}}, \]

\[ \Phi(z, \rho) = \int_{-1}^{1} \frac{d\sigma}{\pi} \frac{f(\xi)\mu(\sigma)}{\sqrt{1 - \sigma^2}}, \]

where \( e(\varepsilon) := E(z, \rho = 0) \) and \( f(\varepsilon) := \Phi(z, \rho = 0) \).

The unknown function \( \mu(\sigma) \) must satisfy the singular integral equation

\[ \int_{-1}^{1} \frac{\mu(\sigma)[e(\xi) + \bar{e}(\eta) + 2f(\xi)\bar{f}(\eta)]d\sigma}{(\sigma - \tau)\sqrt{1 - \sigma^2}} = 0 \]

and the normalizing condition

\[ \int_{-1}^{1} \frac{\mu(\sigma)d\sigma}{\sqrt{1 - \sigma^2}} = \pi, \]

where \( \xi = z + i\rho \sigma, \eta = z + i\rho \tau, \rho, \text{and} z \) is the Weyl–Papapetrou quasi-cylindrical coordinates, \( \sigma, \tau \in [-1, 1], e(\eta) := e(\bar{\eta}), \bar{f}(\eta) := \bar{f}(\bar{\eta}) \), and the overbar stands for complex conjugation. In Pachón et al. (2006), the Ernst potentials were chosen as

\[ e(\varepsilon) = \frac{z^3 - z^2(m + ia) - kz + is}{z^3 + z^2(m - ia) - kz + is}, \]

\[ f(\varepsilon) = \frac{q z^2 + i \mu z}{z^3 + z^2(m - ia) - kz + is}. \]

We calculate the multipole moments following the procedure of Hoenselaers & Perjes (1990). We denote the mass multipoles by \( M_i \) and the current (rotation) multipoles by \( S_i \). The electric multipoles are denoted by \( Q_i \) and the magnetic ones by \( B_i \). Thus, for the PRS solution we have

\[ M_0 = m, \quad M_2 = mk - ma^2, \quad \ldots \]

\[ S_1 = ma, \quad S_3 = -ma^2 + 2mak - ms, \quad \ldots \]

\[ Q_0 = q, \quad Q_2 = -a^2q - a \mu + kq, \quad \ldots \]

\[ B_1 = \mu + aq, \quad B_3 = -a^2\mu + \mu k - a^2q + 2akq - qs, \quad \ldots \]
This allows us to identify $m$ as the total mass and $a$ as the total angular moment per unit mass ($a = J/m$, $J$ being the total angular moment), while $k$, $s$, $q$, and $\mu$ are associated with the mass-quadrupole moment $M_2$, current octupole $S_3$, electric charge, and magnetic dipole, respectively.

The potentials (10) can be written in an alternative way, namely,

$$e(z) = 1 + \sum_{i=3}^{3} \frac{e_i}{z - \beta_i}, \quad f(z) = \sum_{i=3}^{3} \frac{f_i}{z - \beta_i}, \quad (13)$$

with

$$e_j = (-1)^j \frac{2mb_j^2}{(\beta_j - \beta_k)(\beta_j - \beta_l)}, \quad (14)$$

$$f_j = (-1)^{j+1} \frac{i\mu + dB_j}{(\beta_j - \beta_k)(\beta_j - \beta_l)}, \quad i, k \neq j. \quad (15)$$

Then, using Equations (6) and (10), we obtain the Ernst potentials

$$E = \frac{A + B}{A - B}, \quad \Phi = \frac{C}{A - B^2} \quad (16)$$

and the metric functions in the whole space-time

$$f = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{(A - B)(A - B)}, \quad e^{2\gamma} = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{K\bar{K}}, \quad (17)$$

$$\omega = \frac{\text{Im}[(A + B)\bar{H} - (\bar{A} + \bar{B})G - C\bar{I}]}{A\bar{A} - B\bar{B} + C\bar{C}}, \quad (18)$$

where the functions $A$, $B$, $C$, $H$, $G$, $K$, and $I$ can be found in Appendix A.

The PRS electrovacuum exact solution belongs to the extended $N$-soliton solution of the Einstein–Maxwell equations derived by Ruiz et al. (1995), in the particular case $N = 3$. In addition, the functional form of the metric functions resembles the one derived previously by Bretón et al. (1999). Besides the limiting cases discussed in Pachón et al. (2006), it is worth mentioning that, in the vacuum case $q = 0$ and $\mu = 0$, for $s = 0$ this solution reduces to the solution of Manko et al. (1995) under the same physical conditions, namely, $q = 0$, $c = 0$, and $b = 0$ in Manko et al. (1995).

3. ORBITAL MOTION FREQUENCIES ON THE EQUATORIAL PLANE

Although for the case of compact stars contributions from the magnetic field could be relevant (see, e.g., Bakala et al. 2010, 2012; Sanabria-Gómez et al. 2010), we focus in this work on the frequencies of neutral particles orbiting a neutral compact object. We calculate here the Keplerian $v_K = \Omega_K/(2\pi)$, frame-dragging (Lense-Thirring) $v_f = \Omega_f/(2\pi)$, radial oscillation and precession $v^{\text{OS}}_\rho = \Omega^{\text{OS}}_\rho/(2\pi)$ and $v^{p}_\rho = \Omega^{p}_\rho/(2\pi)$, and vertical oscillation and precession frequencies $v^{\text{OS}}_z = \Omega^{\text{OS}}_z/(2\pi)$ and $v^{p}_z = \Omega^{p}_z/(2\pi)$, respectively.

The geodesic motion of test particles along the radial coordinate, on the equatorial plane $z = 0$, is governed by the effective potential (see, e.g., Ryan 1995)

$$V(\rho) = 1 - \frac{E^2g_{\phi\phi} + 2ELg_{\theta\phi} + L^2g_{tt}}{g^\phi_{\phi} - g_{tt}g_{\phi\phi}}, \quad (19)$$

where, for circular orbits, the energy $E$ and angular momentum $L$ are determined by the conditions $V = 0$ and $dV/d\rho = 0$ (see Equations (22) and (23)). The frequencies at the ISCO’s location (determined by the additional condition $d^2V/d\rho^2 = 0$) are of particular interest. Thus, before starting the discussion of the frequencies, it is important to explore the ISCO parametric dependence. We report here, as standard in the literature, the physical ISCO radius given by $\sqrt{\rho_{\text{ISCO}}}$ evaluated at the root of Equation (19) that gives the coordinate ISCO radius. In the upper panel of Figure 1 we plotted contours of constant ISCO radii as a function of $j$ and the difference $M_{2,\text{PRS}} - M_{2,\text{Kerr}}$. The quadrupole moment difference is comprised in the range $-2.7 \times 10^{47}$ g cm$^2$ $\leq M_2 \leq 6.8 \times 10^{47}$ g cm$^2$.  

Figure 1. Upper panel: contours of constant ISCO radius as a function of the dimensionless angular momentum parameter $j = J/M_2^2$ and the quadrupole moment $M_2$ for the PRS solution, for a compact object with mass $M_0 = m = 1.88 M_\odot = 2.78$ km. Contours are labeled by the corresponding value of the ISCO radius in km. Negative values of $j$ depict the counter-rotating case, and negative values of the quadrupole moment $M_2$ correspond to oblate configurations. The values of $M_2$ are in the range $0 \leq M_2 \leq 20$ km$^3$ that corresponds in CGS units to $0 \leq M_2 \leq 2.7 \times 10^{47}$ g cm$^2$, which covers the typical range of fast-rotating NSs. Lower panel: contours of constant ratio $M_{2,\text{PRS}}/M_{2,\text{Kerr}}$ as a function of $j$ and the difference $M_{2,\text{PRS}} - M_{2,\text{Kerr}}$. The quadrupole moment difference is comprised in the range $-2.7 \times 10^{47}$ g cm$^2$ $\leq M_2 \leq 6.8 \times 10^{47}$ g cm$^2$.  

the horizontal axis allows us, qualitatively, to relate deviations of the contour lines from vertical lines to the influence of the quadrupole moment. We can see that the ISCO radius decreases for increasing $j$ and decreasing $M$. A quantitative measurement of this influence could be derived from the effective slope of the contour lines. We are interested in the comparison with the Kerr geometry, so in the lower panel, we plotted contours of constant ratio $r_{ISCO,PRS}/r_{ISCO,Kerr}$ as a function of $j$ and the difference between the quadrupole moment of the PRS solution $M_{2,PRS}$ and the Kerr quadrupole $M_{2,Kerr} = -ma^2$, i.e., $M_{2,PRS} = M_{2,Kerr} + ma^2 = mk$ (see Equation (11)). Deviations from the Kerr geometry are evident. Negative values of the angular momentum correspond to the radii of the counter-rotating orbits obtained here through the change $g_{\phi} \rightarrow -g_{\phi}$ (see discussion below).

We stress that the accuracy of the PRS solution for describing the ISCO radius of realistic NSs was already shown to be higher with respect to other analytic models (for details see Pachón et al. 2006). In Table 1 we compare the ISCO radius for two rapidly rotating NSs, models 20 and 26, of Table VI of Pappas & Apostolatos (2012) for the EOS L. The lowest multipole moments of the analytic models are fixed to the numerical values obtained by Pappas & Apostolatos (2012). In the case of the Kerr solution, only $M_0$ and $J$ can be fixed, while $M_2$ and $S_3$ have values that depend on $M_0$ and $J$ and therefore cannot be fixed. For the PRS solution with $s = 0$, $M_0$, $J$, and $M_2$ can be fixed while $S_3$ remains induced by the lower moments. We also present the ISCO radius obtained by fixing $M_0$, $J$, $M_2$, and $S_3$ in the PRS analytic exact model.

In Figures 1–6, we have fixed as an example $M_0 = m = 1.88 M_\odot = 2.78$ km and $s = 0$. We recall that the quadrupole moment in the geometric units used here (km$^3$) is related to the one in CGS units by $M_{CGS}^2 = (10^{15}c^2/G)M_{geo}^2 = 1.35 \times 10^{33}(M_{geo}^2/$km$^3$) g cm$^2$, and the mass of the Sun is $M_\odot^3 = 1.477$ km. The dimensionless angular momentum $j$ is obtained from the CGS values of $J$ and $M_0$ as $j = cJ/(GM_0^2)$.

It is appropriate to compare the range of values of $j = J/M_0^2$ and $M_2$ used in Figures 1–6 with typical values of an NS. For the used mass $M_0 = 1.88 M_\odot$, Morsink & Stella (1999) obtained a quadrupole moment $M_2 = -5.3 \times 10^{33}$ g cm$^2$, with the latter value in geometric units, for an NS of angular rotation frequency $\nu_c = 290$ Hz (rotation period of 3.45 ms), corresponding to a dimensionless angular momentum $j = J/M_0^2 = 0.19$, for the EOS L. For a fixed mass the quadrupole moment is an increasing function of $j$ because an increase of the angular momentum at fixed mass results in an increase of the oblateness (eccentricity) of the star and thus the quadrupole moment. Based on this fact, it is clear that not all the values of the $(M_2, j)$ pairs of quadrupole and angular momentum depicted in, e.g., Figure 1 are physically meaningful. The maximum rotation rate of an NS taking into account both the effects of general relativity and deformations has been found to be $\nu_{\text{max}} = 1045(M_0/M_\odot)^{1/2}(10$ km$/R)^{3/2}$ Hz, largely independent of the EOS (for details see Lattimer & Prakash 2004). Corresponding to this maximum rotation rate, the angular momentum is $J_{\text{max}} = 2\pi \nu_{\text{max}} I \sim 6.60 \times 10^{48} I_{45}$ g cm$^2$s$^{-1}$, and $j_{\text{max}} = G J_{\text{max}}/(c M_0^2) \sim 0.74 I_{45}/(M_0/M_\odot)^2$, where $I_{45}$ is the moment of inertia of the NS in units of $10^{45}$ g cm$^2$. The fastest observed pulsar is PSR J1748-2246ad, with a rotation frequency of 716 Hz (Hessels et al. 2006), which constrains the mass of the NS to $M_0 \gtrsim 0.47 (R/10$ km$)^3 M_\odot$, and $j \sim 0.51 I_{45}/(M_0/M_\odot)^2$, which becomes $j \sim 0.26 I_{45}$ for a canonical NS of $M_0 = 1.4 M_\odot$.
Now we turn to the frequency analysis. For stationary axially symmetric space-times, the frequency of Keplerian orbits is given by (see, e.g., Ryan 1995)

$$\Omega_K = \frac{-g_{\phi\rho} \pm \sqrt{g_{\phi\phi,\rho} - g_{\phi\phi,\rho} g_{tt,\rho}}}{g_{\phi\phi,\rho}},$$

(20)

where a comma stands for partial derivative with respect to the indicated coordinate and “+” and “−” stand for corotating and counter-rotating orbits, respectively.

For the case of static space-times, i.e., for $$\omega = 0$$ and therefore $$g_{\phi\phi} = 0$$, $$\Omega_K = \pm \sqrt{-g_{\phi\phi,\rho} g_{tt,\rho}}/g_{\phi\phi,\rho}$$ and the energy $$E$$ and angular momentum $$L$$ per mass $$\mu$$ of the test particle can be expressed in terms of the metric tensor components (see, e.g., Ryan 1995),

$$\frac{E}{\mu} = \sqrt{-g_{tt} - g_{\phi\phi} \Omega_K^2},$$

(21)

$$\frac{L}{\mu} = \frac{g_{\phi\phi} \Omega_K}{\sqrt{-g_{tt} - g_{\phi\phi} \Omega_K^2}}.$$

From here, it is clear that taking the negative branch of the root for $$\Omega_K$$ in Equation (20) is equivalent to studying a particle with opposite angular momentum, i.e., $$L_{\text{count-rot}} = -L_{\text{co-rot}}$$. Thus, in the static case the magnitudes of the energy and angular momentum are invariant under the change $$\Omega_K \rightarrow -\Omega_K$$.

Now we consider the case of stationary space-times, given by (see, e.g., Ryan 1995)

$$\frac{E}{\mu} = \sqrt{-g_{tt} - g_{\phi\phi} \Omega_K^2},$$

(22)

$$\frac{L}{\mu} = \frac{g_{\phi\phi} \Omega_K}{\sqrt{-g_{tt} - g_{\phi\phi} \Omega_K^2}}.$$
angular momentum of the star, \( J \rightarrow -J \), but all the rotational multipolar moments. For the Kerr metric this change is obtained by changing the sign of the parameter \( a \) (see Appendix B), while in the PRS solution we also need to change the sign of the parameter \( s \) associated with differential rotation, i.e., by changing \( a \rightarrow -a \) and \( s \rightarrow -s \).6

Once we have clarified this important issue about the co-rotating and counter-rotating orbits, we proceed to analyze the functional dependence of the Keplerian frequency on the multipole moments. In the upper panel of Figure 2 we plotted contours of constant Keplerian frequency for the PRS solution, \( \nu_{K,PRS} = \Omega_{K,PRS}/(2\pi) \), as a function of the dimensionless angular momentum parameter \( j \) and the quadrupole moment \( M_{2,PRS} \), at the ISCO radius. It can be seen that the influence of the quadrupole moment is non-negligible, as evidenced from the departure of the contour lines from vertical lines. The Keplerian frequency grows for increasing \( J \) and \( M_2 \). In the lower panel, we plotted contours of constant ratio \( \nu_{K,PRS}/\nu_{K,Kerr} \) as a function of \( j \) and the difference between the quadrupole moment of the PRS solution, \( M_{2,PRS} \), and the Kerr quadrupole, \( M_{2,Kerr} \).

It is appropriate to recall here that because the Keplerian and the other frequencies calculated below are evaluated using formulae in the coordinate frame (see, for instance, Equation (20)), they must be evaluated at coordinate radii \( \rho \) and not at physical radii given by \( \sqrt{g_{\phi\phi}} \). In the specific case of the ISCO the frequencies are evaluated at the radius that simultaneously solves the equations \( V = 0 \), \( dV/d\rho = 0 \), and \( d^2V/d\rho^2 = 0 \), where \( V \) is the effective potential Equation (19).

### 3.2. Oscillation and Precession Frequencies

The radial and vertical oscillation (or epicyclic) frequencies are the frequencies at which the periastron and orbital plane of a circular orbit oscillates if we apply slightly radial and vertical perturbations to it, respectively. According to Ryan (1995), in stationary axially symmetric vacuum space-times described by the Weyl–Papapetrou metric Equation (1), the radial and vertical epicyclic frequencies can be obtained as

\[
\nu_{a}^{\text{OS}} = \frac{1}{2\pi} \left\{ \frac{g_{t\phi} + g_{\phi\phi}\Omega_{K}}{\sqrt{-g_{tt} - 2g_{t\phi}\Omega_{K} - g_{\phi\phi}\Omega_{K}^2}} \right\}^{1/2},
\]

and the corresponding periastron (\( \nu_{p}^{\text{P}} \)) and nodal (\( \nu_{p}^{\text{N}} \)) precession frequencies as

\[
\nu_{p}^{\text{P}} = \nu_{K} - \nu_{a}^{\text{OS}},
\]

where \( \alpha = \{ \rho, z \} \), respectively, and \( \nu_{K} = \Omega_{K}/(2\pi) \) is the Keplerian orbital frequency with \( \Omega_{K} \) given by Equation (20).

In the upper panel of Figure 3, we plotted contours of constant nodal precession frequency \( \nu_{p}^{\text{N}} \) at the ISCO radius as a function of \( j = J/M_{2}^{2} \) and \( M_{2} \) for the PRS solution, at the ISCO radius. We can see now that the influence of the quadrupole moment is quite important. The nodal precession frequency increases for increasing \( J \) and decreasing \( M_{2} \), at fixed \( M_{0} \). In the lower panel

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6 For the vacuum case, in the solution by Manko et al. (2000), the sign change of \( g_{t\phi} \) is obtained after performing simultaneously the replacements \( a \to -a \) and \( b \to -b \).
we plotted contours of constant ratio \( \nu^P_{\text{PRS}}/\nu^P_{\text{Kerr}} \), at the ISCO radius, as a function of \( j \) and the difference \( M^2_{2,\text{PRS}} - M^2_{2,\text{Kerr}} \), in order to provide evidence for deviations from the Kerr solution. The radial oscillation frequency \( \nu^\rho_{\text{OS}} \) vanishes at the ISCO radius, and therefore at such a location the radial precession frequency equals the Keplerian frequency, whose contours have been plotted in Figure 2.

In Figures 4 and 5 we plotted the nodal precession frequency \( \nu^p_0 \) and the radial oscillation frequency \( \nu^\rho_{\text{OS}} \) as a function of the Keplerian frequency \( \nu_K \), respectively, for both the Kerr and PRS solutions. As an example, we have shown the results for rotating NS models 20 and 26 of Table VI of Pappas & Apostolatos (2012) for the EOS L. The lowest multipole moments of the PRS solution \( M_0, J, M_2, \) and \( S_3 \) have been fixed to the numerical values obtained by Pappas & Apostolatos (2012). In the case of the Kerr solution, only \( M_0 \) and \( J \) can be fixed, while \( M_2 \) and \( S_3 \) have values induced by the lower moments \( M_0 \) and \( J \). For the PRS solution with \( s = 0 \), \( M_0, J, \) and \( M_2 \) can be fixed while \( S_3 \) cannot be fixed and depends on the lower moments. The results for the PRS analytic model obtained by fixing \( M_0, J, M_2, \) and \( S_3 \) are also shown.

The deviations of the quadrupole and current octupole moments given by the Kerr solution from the numerical values of Pappas & Apostolatos (2012) can be used to show the low accuracy of the Kerr solution to describe fast-rotating NSs. The accuracy of the PRS solution in describing the ISCO radii of these two models has been shown in Table 1.

In Figures 4 and 5 we can see the differences of the \( \nu^P_0 - \nu_K \) and \( \nu^\rho_{\text{OS}} - \nu_K \) relations between the Kerr and PRS solutions for realistic NS models. The deviations of the Kerr solution, especially at fast rotation rates, are evident because of the influence of the deformation (quadrupole \( M_2 \)) of the star and although in less proportion, of the octupole current \( S_3 \). In general, we observe that the larger the angular momentum, the poorer the performance of the predictions of the Kerr solution.

We have also shown in Figures 4 and 5 the influence of the current octupole \( S_3 \) in the determination of the precession and oscillation frequencies. We found that the effect of \( S_3 \) is only appreciable for the fastest models. The minor influence, in this case, of the current octupole \( S_3 \) is expected from the small values of the parameter \( s \) needed to fit the numerical values of Pappas & Apostolatos (2012). Clearly, larger values of the parameter \( s \) needed to fit realistic values of \( S_3 \) will also enhance deviations from the Kerr space-time.

The effects of a multipolar structure that deviates from the one of the Kerr geometry on the various quantities analyzed here are relevant, for instance, in the RPM of the QPOs observed in LMXBs (see, e.g., Stella & Vietri 1998, 1999; Morsink & Stella 1999; Stella et al. 1999, and Section 6 for details).

3.3. Dragging of Inertial Frames

It is known that a prediction of general relativity is that a rotating object makes a zero angular momentum test particle orbit around it, namely, it drags the particle into the direction of its rotation angular velocity; such an effect is called dragging of inertial frames or the Lense–Thirring effect. Consequently, oblique particle orbit planes with respect to the source equatorial plane will precess around the rotation axis of the object. In stationary axially symmetric space-times described by the metric Equation (1) the frame-dragging precession frequency is given by (see, e.g., Ryan 1995)

\[
\nu_{\text{LT}} = -\frac{1}{2\pi} \frac{g_{\phi\phi}}{g_{\rho\rho}}.
\]

Many efforts have been made to test the predictions of general relativity around the Earth such as the analysis of the periastron precession of the orbits of the LAser GEOdynamics Satellites, LAGEOS and LAGEOS II (see, e.g., Lucchesi & Peron 2010), and the relativistic precession of the gyroscopes on board the Gravity Probe B satellite (for details see Everett et al. 2011). The latter experiment measured a frame-dragging effect within an accuracy of 19% with respect to the prediction of general relativity.

The smallness of this effect around the Earth makes such measurements quite difficult and has represented a multi-year challenge for astronomy. The frame-dragging precession increases with the increase of the angular momentum of the rotating object, and therefore a major hypothetical arena for the search of more appreciable Lense–Thirring precession is the space-time around compact objects such as BHs and NSs. The much stronger gravitational field of these objects with respect to the Earth allows them to attain much faster angular rotation rates and thus larger angular momentum.

Stella & Vietri (1998) showed how, in the weak-field slow rotation regime, the vertical precession frequency \( \nu^P_0 \) (orbital plane precession frequency) can be divided into one contribution due to the Lense–Thirring precession and another one due to the deformation (nonzero quadrupole moment) of the rotating object, both of which are comparable from the quantitative point of view. These frequencies could in principle be related to the motion of the matter in the accretion disks around BHs and NSs and thus particularly applicable to LMXBs. For fast-rotating NSs and BHs the frame-dragging precession frequency can reach values of the order of tens of Hz (see, e.g., Stella & Vietri 1998 and Figures 3 and 4).

Thus, it is clear that an observational confirmation of the relativistic precession of matter around either an NS or a BH will lead to an outstanding test of the general relativity in the strong-field regime and, at the same time, an indirect check of the large effects of the frame dragging in the exterior space-time of compact objects (for details see, e.g., Morsink & Stella 1999).

Although making independent measurements of the frame-dragging effect around BHs and NSs is a very complicated task, it is important to know the numerical values of the precession frequency due to the frame dragging with respect to other relativistic precession effects, e.g., geodetic precession. In addition, it is important to know the sensitivity of the precession frequency to object parameters such as mass, angular momentum, quadrupole, and octupole moment.

In the upper panel of Figure 6 we plotted contours of constant frame-dragging frequency \( \nu_{\text{LT}} \) for the PRS solution, at the ISCO radius, as a function of the angular momentum per unit mass \( J/M_0 \) and the quadrupole moment \( M_2 \), for a compact object mass \( M_0 = m = 1.88 M_\odot \). Correspondingly, in the lower panel of Figure 6, we show the differences between the frame-dragging precession frequency as predicted by the Kerr and PRS solutions, at the ISCO radius, as a function of \( j = J/M_0^2 \) and the difference between the quadrupole moments, \( M^2_{2,\text{PRS}} - M^2_{2,\text{Kerr}} \).

The influence of the quadrupole moment in the determination of the frame-dragging frequency is evident; the frequency \( \nu_{\text{LT}} \) given by an NS is generally smaller than the one given by a BH, as can be seen from the value of the ratio \( \nu_{\text{LT},\text{PRS}}/\nu_{\text{LT},\text{Kerr}} < 1 \) obtained for configurations with a quadrupole moment that
deviates with respect to the one given by the Kerr solution, namely, for \(M_{2,\text{PRS}} - M_{2,\text{Kerr}} = M_{2,\text{PRS}} + ma^2 = mk \neq 0\) (see Equation (11)).

It is also worth mentioning that frame-dragging precession can also be affected by the presence of electromagnetic fields (see Herrera et al. 2006), and further research in this respect deserves due attention.

4. ACCURACY OF RYAN’S ANALYTIC FORMULAE

Following a series expansion procedure in powers of \(1/\rho\), Ryan (1995) found that the periastron (radial) and nodal (vertical) precession frequencies, \(\nu^P_{\nu K}\) and \(\nu^V_{\nu K}\), given by Equation (24), can be written as a function of the Keplerian frequency \(\nu_{\nu K}\) as

\[
\frac{\nu^P_{\nu K}}{\nu_{\nu K}} = 3\nu^3 - 4\frac{S_1}{M_0^2}\nu^5 + \left(\frac{9}{2} - \frac{3}{2}\frac{M_2}{M_0^3}\right)\nu^7 + \left(-10\frac{S_1}{M_0^2} - 5\frac{S_1M_2}{M_0^3}\right)\nu^9 + \ldots,
\]

and

\[
\frac{\nu^V_{\nu K}}{\nu_{\nu K}} = 2\frac{S_1}{M_0^2}\nu^3 + 3\frac{M_2}{2M_0^3}\nu^5 + \left(\frac{7}{2}\frac{S_1^2}{M_0^4} + \frac{5}{2}\frac{M_2^2}{M_0^5}\right)\nu^7 + \left(\frac{11}{8}\frac{S_1M_2}{M_0^4} - \frac{6}{5}\frac{S_3}{M_0^5}\right)\nu^9 + \ldots,
\]

where \(\nu = (2\pi M_0\nu_{\nu K})^{1/3}\), \([M_0, M_2, M_4]\) are the lowest three mass moments, and \([S_1, S_3]\) are the lowest two current moments.

For the PRS solution in the vacuum case, \(M_4 = m(a^4 - 3a^2 + k^2 + 2as)\).

The above formulae are approximate expressions of the periastron and nodal precession frequencies in the weak-field (large distances from the source) and slow-rotation regimes. We should therefore expect that they become less accurate at distances close to the central object, e.g., at the ISCO radius, and for fast-rotating objects. However, such formulae are an important tool for understanding the role of the lowest multipole moments on the values of the relativistic precession frequencies, such as the importance of the higher multipole moments at short distances and high frequencies as can be seen from Equations (27) and (28).

At high frequencies, for instance, of the order of kHz, deviations from the above scaling laws are appreciable. In Figures 7 and 8 we compare the radial precession and vertical oscillation frequencies, \(\nu^P_{\nu K}\) and \(\nu^{OS}_{\nu K}\), as a function of the Keplerian frequency \(\nu_{\nu K}\), as shown by the full expressions (24) for the PRS solution and by the approximate formulae (27) and (28), respectively.\(^7\) The lowest multipole moments \(M_0, J, M_2,\) and \(S_3\) of the PRS solution have been fixed to the values of two models of Table VI of Pappas & Apostolatos (2012): model 20 with \(M_0 = 2.071\) km (1.402 \(M_\odot\)), \(J = 0.194\), \(M_2 = -2.76\) km\(^3\) (3.73 \(10^3\) g cm\(^{-3}\)), and \(S_3 = -2.28\) km\(^2\).

\(^7\) Because the scale of the \(\nu^P_{\nu K}\) and \(\nu^{OS}_{\nu K}\) frequencies is very similar, we decided to plot in Figure 7 \(\nu^P_{\nu K}\) and \(\nu^{OS}_{\nu K}\), whose scales are different, allowing a more clear comparison with the PRS solution in a single figure.
Table 2

| Model  | $M_0/M_\odot$ | $j$ | $M_2/Q_0$ | $r_s^{\text{MS}}$ | $v_r^{\text{K}}$ | $v_r^{\text{PRS}}$ | $v_s^{\text{MS}}$ | $v_s^{\text{PRS}}$ | $\nu_1^{\text{K}}$ | $\nu_1^{\text{PRS}}$ | $\nu_2^{\text{K}}$ | $\nu_2^{\text{PRS}}$ |
|--------|---------------|-----|-----------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| M1     | 1.88          | 0.19 | -5.3      | 15.4             | 14.90            | 15.42            | 1.31             | 1.363            | 1.304            | 39.7             | 42.248           | 38.476           |
| M2     | 2.71          | 0.14 | -3.0      | 22.2             | 22.16            | 22.23            | 0.90             | 0.906            | 0.902            | 19.6             | 19.676           | 19.493           |
| M3     | 1.94          | 0.24 | -8.2      | 15.6             | 14.89            | 15.63            | 1.29             | 1.380            | 1.296            | 49.8             | 57.001           | 49.804           |
| M4     | 2.71          | 0.18 | -4.8      | 21.8             | 21.62            | 21.74            | 0.93             | 0.937            | 0.931            | 26.1             | 27.245           | 26.833           |
| M5     | 2.07          | 0.40 | -23.1     | 16.3             | 14.18            | 16.06            | 1.26             | 1.514            | 1.289            | 84.3             | 125.75           | 88.905           |
| M6     | 2.72          | 0.30 | -13.9     | 20.6             | 20.05            | 20.45            | 1.01             | 1.041            | 1.015            | 53.5             | 57.467           | 54.391           |
| M7     | 2.17          | 0.51 | -39.4     | 17.0             | 13.58            | 16.53            | 1.22             | 1.637            | 1.269            | 106.7            | 201.52           | 116.25           |
| M8     | 2.73          | 0.39 | -24.4     | 19.8             | 18.85            | 19.65            | 1.06             | 1.136            | 1.077            | 78.8             | 90.821           | 80.895           |

Note. The quadrupole moment $M_2$ has been normalized for convenience to the value $Q_0 = 10^{15}$ g cm$^{-2}$.

For the analysis of the $v_p^{\text{PRS}}-v_{\text{K}}$ relation we followed the same procedure as described above. In this case, Ryan’s expressions tend from the top to the exact result, the continuous thick black curve, represented by the PRS solution. It is interesting to see that the introduction of the octupole moment (thick light red dashed line) makes the approximation deviate from the exact result; however, by including more terms, the accuracy is enhanced. As can be seen from Figures 7 and 8, the quantitative accuracy of Ryan’s approximate formulae in the periastron precession frequency $v_p^{\text{PRS}}$ is less than the one obtained in the vertical oscillation frequency $v_z^{\text{OS}}$.

The importance of the high-order multipole moments such as the quadrupole and the octupole moments is evident in the high-frequency regime. This is in line with the results shown in Figures 2–5 and in Figures 4–5. We can see from Figures 7 and 8 that Ryan’s approximate formulae describe model 2 more accurately than model 20. The reason for this is that, as we mentioned above, we should expect a better accuracy of the series expansions from low to moderate rotation rates, and consequently the same to occur for the quadrupole deformations. It is clear that there are appreciable differences in both rotation and deformation between the two selected models; we also recall that the rotation frequency of the star can be expressed as a function of the dimensionless $j$ parameter as $v_r = Gj/M_0^2/(2\pi c I) = 1.4(M/M_\odot)c^3/I_{45}$ kHz.

It is noteworthy that we have checked that Ryan’s series expansions, Equations (27) and (28), fit quite accurately the exact results if taken up to order $v_{10}^{10}$. In particular, the values of the vertical oscillation and precession frequencies are fit better than the corresponding radial ones. For model 2 the radial oscillation frequency is well fitted by Ryan’s expression up to Keplerian frequencies of order ~1.2 kHz, while, for model 20, the approximate formulae break down at a lower value ~0.7 kHz. These results are of particular relevance because it makes possible the extraction of the object parameters (for instance, the lowest multipoles up to $S_1$) by fitting the of the observed QPO frequencies in LMXBs, provided that they are indeed related to the precession and oscillation frequencies of matter in the accretion disk (see Section 6 for details) and for Keplerian motion not exceeding a few kHz of frequency.

5. ACCURACY OF PRS SOLUTION

We now analyze the behavior of the Kerr and PRS solutions in predicting results for the Keplerian, frame-dragging, and vertical oscillation frequencies, for realistic NSs. In particular, we compare their predictions with the frequencies calculated by Morsink & Stella (1999). Since Morsink & Stella (1999) did not include the values of the octupole current moment $S_3$, here we set $s = 0$ in Equation (10) for the PRS solution. For the sake of comparison, we choose the results derived by Morsink & Stella (1999) for the EOS L, because for this EOS the highest rotating parameter $j$ and quadrupole moment $M_2$ were found. In addition, the stiffness of such an EOS allows the maximum mass of the NS to be larger than the highest observed NS mass, $M_0 = 1.97 \pm 0.04 M_\odot$, corresponding to the 317 Hz (3.15 ms rotation period) pulsar J1614-2230 (for details see Demorest et al. 2010).

This regime of high $j$ and $M_2$ in realistic models is particularly interesting to test the deviations of the Kerr solution in the description of NS signatures, as well as to explore the accuracy of the PRS solution. In Table 2, we present the results for four different sets of the star spin frequency $v_r$, namely, $v_r = 290$ Hz (M1 and M2), $v_r = 360$ Hz (M3 and M4), $v_r = 580$ Hz (M5 and M6), and $v_r = 720$ Hz (M7 and M8).

In Table 2, we clearly observe that the results predicted by the PRS$_{\text{K0}}$ solution for the Keplerian and frame-dragging frequencies are in excellent agreement with those calculated by Morsink & Stella (1999) for even highly massive, rotating, and deformed models such as model M7 with $M_0 = 2.17 M_\odot$, $j = 0.51$, and $M_2 = -39.4Q_0$. We note that Morsink & Stella (1999) reported some configurations with negative values of $v_r$ (see Table 2). We advance the possibility that this is due to instabilities of the numerical code that occur when the ISCO radius is located very close to or inside the surface of the object. Thus, the values of the frequencies given by the analytic solution in these cases are to be considered predictions to be tested for future numerical computations. This fact can be checked within the calculations of Morsink & Stella (1999) by exploring the properties of counter-rotating orbits that produce in general ISCO radii larger than those of the corotating ones. In Table 3, we depicted the results in the counter-rotating case, where we note an improvement of the accuracy of the PRS solution with respect to the corotating case.

In line with this, we consider it worth performing numerical computations of the precession and oscillation frequencies of particles around realistic NSs in a wider space of parameters and using up-to-date numerical techniques, which will certainly help to establish and elucidate more clearly the accuracy of analytic models. It is also appropriate to recall the recent results of Pappas & Apostolatos (2012) on the computation of the general relativistic multipole moments in axially symmetric space-times.
6. THE RELATIVISTIC PRECISION MODEL

The X-ray light curves of LMXBs show a variability from which a wide variety of QPOs have been measured, expanding from relatively low ~Hz frequencies all the way up to high kHz frequencies (for details see, e.g., van der Klis 1995). In particular, such frequencies usually come in pairs (often called twin peaks), the lower and upper frequencies, $v_l$ and $v_h$, respectively. BHs and NSs with similar masses can show similar signatures, and therefore the identification of the compact object in an LMXB is not a simple task. If the QPO phenomena observed in these systems are indeed due to relativistic motion of accretion disk matter, the knowledge of the specific behavior of the particle frequencies (e.g., rotation, oscillation, precession) in the exterior geometry of NSs and BHs becomes essential as a tool for the identification of the nature of the compact object harbored by an LMXB.

It is not within the scope of this work to test a particular model for the QPO phenomenon in LMXBs, but instead to show the influence of the high multipole moments on the orbital motion of test particles, especially the role of the quadrupole moment, which is of particular interest to differentiate an NS from a BH. There are several models in the literature that describe the QPOs in LMXBs through the frequencies of particles around the compact object, and for a recent review and comparison of the different models we refer to the recent work of Lin et al. (2011). In order to show here the main features and differences between the Kerr and the PRS solutions, we shall use the RPM.

The RPM model identifies the lower and higher (often called twin peaks) kHz QPO frequencies, $v_l$ and $v_h$, with the periastron precession and Keplerian frequencies, namely, $v_l = v_p^K$ and $v_h = v_K$, respectively. The so-called horizontal branch oscillations (HBOs), which belong to the low-frequency QPOs observed in high-luminosity Z-sources (for details see, e.g., van der Klis 1995), are related within the RPM model to the nodal precession frequency $v_p^N$ of the same orbits (see Morsink & Stella 1999). We will use here in particular the realistic NS models of Morsink & Stella (1999) for the EOS L.

One of the salient features of the RPM model is that in the case of the HBO frequencies, the relations inferred from the first term of the expansions (27) and (28) are

$$v_K = 3^{-3/5}(2\pi)^{-2/5}m^{-2/5}(v_p^K)^{3/5}, \quad (29)$$

$$v_p^N = (2/3)^{6/5}\pi^{-1/5}m^{1/5}(v_p^K)^{6/5}, \quad (30)$$

which implies that a nodal precession frequency proportional to the square of the Keplerian frequency has been observed in some sources, for instance, in the LMXB 4U 1728–34 (for details see Ford & van der Klis 1998). In addition, a 6/5 power law relating the nodal and periastron precession frequencies can explain (see Stella et al. 1999) the correlation between two of the observed QPO frequencies found in the fluxes of NS and BH LMXBs (for details see Psaltis et al. 1999). This fact provides, at the same time, a significant test of Ryan’s analytic expressions.

It is interesting to analyze the level of predictability of the precession and oscillation frequencies on particular astrophysical sources. In Figure 9 we show the $v_l - v_h$ relation within the RPM.
model, namely, \( \nu_p^P \) versus \( \nu_K \) for models M1–M8 of Table 2. In the upper panel we show the results for the PRS solution, while in the lower panel we present the results for the Kerr solution. We have indicated the QPO frequencies observed in sources GX 5–1 (see, e.g., Wijnands et al. 1998; Jonker et al. 2002), 4U 1735–44 (see, e.g., Ford et al. 1998), 4U 1636–53 (see, e.g., Wijnands et al. 1997), Sco X1 (see, e.g., van der Klis et al. 1996), GX 17–2 (see, e.g., Homan et al. 2002), GX 340+0 (see, e.g., Jonker et al. 2000), Cir X1 (see, e.g., van der Klis et al. 1996), 4U 0614+091 (see, e.g., Ford et al. 1997), and 4U 1728–34 (see, e.g., Strohmayer et al. 1996).

Both the upper and lower panels of Figure 9 have been plotted using the same frequency scales in order to aid the identification of the differences between the Kerr and the PRS solutions. One can note that all the dark blue curves in the Kerr solution (lower panel of Figure 9) are outside the range of the observed QPO frequencies exemplified, while all dashed and solid curves of the PRS are inside the QPO range. It is then clear that making a fit of the observed QPO frequencies of the selected LMXBs of Figure 9 will necessarily require a different choice of parameters in the Kerr and PRS solutions. Therefore, conclusions, for instance, on the NS parameters (e.g., mass, angular momentum, quadrupole deformation) based on fitting QPOs using the Kerr geometry will deviate from the actual extractable parameters (for details see, e.g., Laarakkers & Poisson 1999), more reliably from a more complex geometry, such as the PRS one, that allows a better estimate, for instance, of the quadrupole moment of a compact star.

In Figure 9 we show the relation \( \nu_p^P \) versus \( \nu_K \) for models M1–M8 of Table 2. For the sake of comparison we show the low-frequency branch observed in the LMXB 4U 1728–34 (for details see Ford & van der Klis 1998). From the analysis of the pulsating X-ray flux it turns out that very likely the spin frequency of the NS in 4U 1728–34 is \( \sim 363 \) Hz (for details see Strohmayer et al. 1996). Thus, models M3 \(( M_0 = 1.94 M_\odot, \ j = 0.24 \) and M4 \(( M_0 = 2.71 M_\odot, \ j = 0.18 \) in Table 2 that correspond to an NS of spin frequency 360 Hz are of particular interest for the analysis of this source. It was suggested by Stella et al. (1999) and Stella & Vietri (1999) that the low frequency observed in 4U 1728–34 is likely to be due to excitations of the second harmonic of the vertical motion, and therefore a better fit of the lower-higher QPO frequencies of 4U 1728–34 (and of similar sources) will be obtained for the relation \( 2 \nu_p^P – \nu_K \). The thick black curves in Figure 10 indicate the \( 2 \nu_p^P – \nu_K \) relation for models M3 and M4 (dark blue and light red curves, respectively) following the above suggestion. Although the improvement of the fit is evident, we note that the NS parameters that correctly reproduce the features of 4U 1728–34 are likely in between models M3 and M4.

7. CONCLUDING REMARKS

We have made an extensive comparison of the orbital motion of neutral test particles in the PRS and Kerr space-time geometries. In particular, we have emphasized the Keplerian and frame-dragging frequencies, as well as the precession and oscillation frequencies of the radial and vertical motions.

We have provided evidence for the differences in this respect between the Kerr and PRS solution, especially in the rapid ~kHz rotation regime. Such differences are the manifestation of the influence of the high-order multipole moments such as the quadrupole and octupole.

The analysis of the deviations between the Kerr and PRS features for given mass and angular momentum of a source studied in this work is useful to distinguish the signatures between BHs and NSs, which are relevant to establish a separatrix for the identification of the compact objects harbored in X-ray binaries. In the case of BH candidates, these results might become important for testing the no-hair theorem of BHs (see, e.g., Johannsen & Psaltis 2011). Equally important, the application of the precession and oscillation frequencies to the explanation of QPOs in LMXBs possessing an NS can unveil information on the NS parameters, leading to a possible identification of the behavior of the nuclear matter EOS at supranuclear densities. In this line, the identification of the rotation frequency of NSs in LMXBs from the pulsating X-ray flux \( \nu_{\text{burst}} \), e.g., the case of 4U 1728–34 (Ford & van der Klis 1998), 4U 1916–053 (Galloway et al. 2001), and more recently IGR J17191–2821 (Altamirano et al. 2010), will certainly help to constrain QPO models as well as the NS parameters. Additional information coming from the photospheric radius expansion phenomena observed in these systems (for details see, e.g., Munoz et al. 2001) during their transient activity with super-Eddington emission can become of paramount importance if combined with the QPO information.

The generalization of the present work to the electrovacuum case is important to establish the influence of the magnetic dipole and quadrupole moments on the orbital motion of particles around compact objects (see, e.g., Bakala et al. 2010, 2012; Sanabria-Gómez et al. 2010).
Interesting effects on the epicyclic frequencies due to the presence of the magnetic dipole have been pointed out recently by Bakala et al. (2010, 2012). These effects were predicted after neglecting the contribution of the electromagnetic field to the curvature; for \( j = 0 \) see Bakala et al. (2010) and for \( j \neq 0 \) Bakala et al. (2012). Bakala et al. (2010) assumed the model of curvature; for \( j \) neglecting the contribution of the electromagnetic field to the overcame, at least for the Kerr case, by considering the simplest influence of the electromagnetic field on the curvature could be the star as a dipole magnetic field superposed on a Schwarzschild BH. In the second work, they studied the case of magnetized slowly rotating NSs; to build the model, they superpose a dipolar magnetic field on the Lense–Thirring geometry. The lack of influence of the electromagnetic field on the curvature could be overcome, at least for the Kerr case, by considering the simplest generalization of the Kerr–Newman derived by Manko (1993; for a typo-free version see Pachón & Dubiebe 2011). The effects of the magnetic dipole on the location of the ISCO, within the PRS solution, have been investigated by Sanabria-Gómez et al. (2010).

A complete analysis of the effects due to the emergence of electromagnetic structure on the orbital motion of charged particles is therefore of interest and deserves the appropriate attention. Recent observations have shown that for stars with strong magnetic fields the quadrupole and octupole magnetic terms make significant contributions to the magnetic field (Donati et al. 2006), which indicates that arbitrary higher order multipole components might be required in a realistic model.

The presence of a magnetic quadrupole demands the breaking of the reflection symmetry (for details see Pachón & Sanabria-Gómez 2006), by means of a slight change to the Ernst electric potential over the symmetry axis:

\[
f(z) = \frac{qz^2 + i\mu z + i\zeta}{z^3 + z^2(m - ia) - kz + is};
\]

a quadrupolar magnetic component \( B_2 = \zeta \) can be introduced to the PRS solution. Such a change generates just a redefinition of the coefficients \( f_i \) in Equation (15). In this way the PRS solution can be readily used to explore the effect of strong magnetic fields with non-dipolar structure.

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**APPENDIX A**

**METRIC FUNCTIONS**

The functions \( A, B, C, H, G, K \), and \( I \) used to express the metric functions (17) are given by

\[
A = \sum_{1 \leq i < j < k \leq 6} a_{ijk} r_i r_j r_k, \quad B = \sum_{1 \leq i < j \leq 6} b_{ij} r_i r_j, \quad C = \sum_{1 \leq i < j \leq 6} c_{ij} r_i r_j, \quad K = \sum_{1 \leq i < j < k \leq 6} a_{ijk}, \quad (A1)
\]

\[
H = z A - (\beta_1 + \beta_2 + \beta_3) B + \sum_{1 \leq i < j < k \leq 6} h_{ijk} r_i r_j r_k + \sum_{1 \leq i < j \leq 6} (\alpha_i + \alpha_j) b_{ij} r_i r_j, \quad (A2)
\]

\[
G = - (\beta_1 + \beta_2 + \beta_3) A + z B + \sum_{1 \leq i < j \leq 6} g_{ij} r_i r_j + \sum_{1 \leq i < j < k \leq 6} (\alpha_i + \alpha_j + \alpha_k) a_{ijk} r_i r_j r_k, \quad (A3)
\]

\[
I = (f_1 + f_2 + f_3)(A - B) + (\beta_1 + \beta_2 + \beta_3 - z) C + \sum_{1 \leq i < j < k \leq 6} p_{ijk} r_i r_j r_k + \sum_{1 \leq i < j \leq 6} p_i r_i + \sum_{1 \leq i < j \leq 6} [p_{ij} - (\alpha_i + \alpha_j) c_{ij}] r_i r_j, \quad (A4)
\]

with

\[
r_i = \sqrt{\rho^2 + (z - \alpha_i)^2}, \quad a_{ijk} = (-1)^{j+i+1} \Delta_{ijk} \Gamma_{[lmn]}, \quad b_{ij} = (-1)^{i+j} \lambda_{ij} H_{[mpn]},
\]

\[
c_{ij} = (-1)^{i+j} \lambda_{ij} \left[ f(\alpha_i) \Gamma_{[mnp]} - f(\alpha_m) \Gamma_{[ipn]} + f(\alpha_p) \Gamma_{[mip]} - f(\alpha_p) \Gamma_{[imp]} \right],
\]

\[
h_{ijk} = (-1)^{j+i+k} \delta_{ij} \left[ e^1 \delta_{23}[lmn] + e^2 \delta_{31}[lmn] + e^3 \delta_{12}[lmn] \right], \quad g_{ij} = (-1)^{i+j} \lambda_{ij} \left[ (\alpha_i \Gamma_{[mnp]} - \alpha_m \Gamma_{[npi]} + \alpha_n \Gamma_{[imp]} - \alpha_p \Gamma_{[imn]} \right],
\]

\[
p_i = (-1)^i D_i \left[ f(\alpha_i) H_{[mpn]} - f(\alpha_m) H_{[npi]} + f(\alpha_p) H_{[imp]} - f(\alpha_p) H_{[imn]} \right],
\]

\[
p_{ij} = (-1)^{i+j} \lambda_{ij} \left[ e^1 \Psi_{23}[mn] + e^2 \Psi_{31}[mn] + e^3 \Psi_{12}[mn] \right], \quad p_{ijk} = (-1)^{i+j+k} \delta_{ij} \left[ e^1 \Psi_{23}[mn] + e^2 \Psi_{31}[mn] + e^3 \Psi_{12}[mn] \right],
\]

\[
\lambda_{ij} = (\alpha_i - \alpha_j) D_i D_j, \quad \Delta_{ijk} = (\alpha_i - \alpha_j)(\alpha_i - \alpha_k)(\alpha_j - \alpha_k) D_i D_j D_k,
\]

\[
D_i = \frac{1}{(\alpha_i - \beta_1)(\alpha_i - \beta_2)(\alpha_i - \beta_3)}, \quad \Gamma_{[lmn]} = H_3(\alpha_i) \Delta_{12}[mn] + H_3(\alpha_m) \Delta_{12}[nl] + H_3(\alpha_n) \Delta_{12}[lm],
\]
and

\[
\Delta_{\text{Im}lp} = H_l(\alpha_n) H_m(\alpha_p) - H_l(\alpha_p) H_m(\alpha_n), \quad H_l(\alpha_n) = \frac{2 \prod_{p \neq n} (\alpha_p - \beta_p^*)}{\prod_{k=1}^{l-1} (\beta_k^* - \beta_k)} - \sum_{k=1}^{l} \frac{2 f_k^* f_k}{(\beta_k^* - \beta_k)(\alpha_n - \beta_k)},
\]

\[
\delta_{\text{Im}pr} = \Delta_{\text{Im}lp} + \Delta_{\text{Im}pl} + \Delta_{\text{Im}rr}, \quad h_{\text{Im}lnp} = H_l(\alpha_i) \delta_{\text{Im}lp}.
\]

\[
\Psi_{\text{Im}lnpr} = f(\alpha_n) \Delta_{\text{Im}lp} + f(\alpha_p) \Delta_{\text{Im}lr} + f(\alpha_i) \Delta_{\text{Im}ln},
\]

\[
\gamma_{\text{Im}lnprs} = f(\alpha_n) \delta_{\text{Im}lp} + f(\alpha_p) \delta_{\text{Im}lr} + f(\alpha_i) \delta_{\text{Im}ln},
\]

where \( \alpha \) values are the roots of the Sibgatullin equation (Sibgatullin 1991; Manko & Sibgatullin 1993)

\[
e(z) + \tilde{e}(z) + 2 \tilde{f}(z) f(z) = 0.
\]

(A5)

**APPENDIX B**

**KERR’S METRIC IN WEYL–PAPAPETROU QUASI-CYLINDRICAL COORDINATES**

In order to keep comparisons safe, we consider it useful to display the Kerr solution in Weyl–Papapetrou quasi-cylindrical coordinates. For this case,

\[
f = \frac{A \bar{A} - B \bar{B}}{(A - B)(A - B)}, \quad e^{2\gamma} = \frac{A \bar{A} - B \bar{B}}{K \prod_{n=1}^{2} r_n}, \quad \omega = \text{Im}[(A + B) \bar{H} - (\bar{A} + \bar{B}) G] / (A \bar{A} - B \bar{B}),
\]

(B1)

where for our own convenience we do not present the definition of each term, but present the final combination of them, i.e.,

\[
A \bar{A} - B \bar{B} = -8(a^2 - m^2)^3 \rho^2 z^2 (m^2 - a^2 - a^2 + m^2 + \rho^2 + z^2 \sqrt{m^2 - a^2} + z^2 + \rho^2 + z^2) - 2a^2 \sqrt{2z \sqrt{m^2 - a^2} - a^2 + m^2 + \rho^2 + z^2 + \sqrt{2z \sqrt{m^2 - a^2} - a^2 + m^2 + \rho^2 + z^2 + 3m^2}}
\]

\[
(A - B)(A - B) = -8(m^2 - a^2)^3
\]

\[
\times \rho^2 (2m \sqrt{-2z \sqrt{m^2 - a^2} - a^2 + m^2 + \rho^2 + z^2 + 2z \sqrt{m^2 - a^2} - a^2 + m^2 + \rho^2 + z^2 + 3m^2})
\]

\[
+ \sqrt{-2z \sqrt{m^2 - a^2} - a^2 + m^2 + \rho^2 + z^2 + 2z \sqrt{m^2 - a^2} - a^2 + m^2 + \rho^2 + z^2})
\]

\[
\sum_{n=1}^{2} r_n = 16(m^2 - a^2)^4 \rho^2 z^2 \sqrt{z - (m^2 - a^2)^2} + \rho^2 \sqrt{m^2 - a^2} + \rho^2 + z^2)
\]

(B4)

\[
\text{Im}[(A + B) \bar{H} - (\bar{A} + \bar{B}) G] = 16a m (m^2 - a^2)^3 \rho^2 z^2 (-m^2 \sqrt{-2z \sqrt{m^2 - a^2} - a^2 + m^2 + \rho^2 + z^2})
\]

\[
- m \sqrt{-2z \sqrt{m^2 - a^2} - a^2 + m^2 + \rho^2 + z^2} \sqrt{m^2 - a^2 + z^2 + \rho^2}
\]

\[
+ a^2 \sqrt{-2z \sqrt{m^2 - a^2} - a^2 + m^2 + \rho^2 + z^2 - z \sqrt{m^2 - a^2} - z \sqrt{m^2 - a^2} - a^2 + \rho^2}
\]

\[
- m^2 \sqrt{m^2 - a^2} \rho^2 + a^2 \sqrt{m^2 - a^2 + z^2 + \rho^2}
\]

\[
+ z \sqrt{m^2 - a^2} \rho^2 \sqrt{m^2 - a^2 + z^2 + \rho^2 + a^2 m - m^3 + m \rho^2 + m z^2}.
\]

(B5)

From here, it is clear how changing \( a \rightarrow -a \) will cause only a global change in the sign of the metric function \( \omega \) and therefore only a change in the \( g_{\phi} \) metric component.

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