Construction and analysis of polar and concatenated polar codes: practical approach

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Abstract—We consider two problems related to polar codes. First is the problem of polar codes construction and analysis of their performance without Monte-Carlo method. The formulas proposed are the same as those in [Mori-Tanaka], yet we believe that our approach is original and has clear advantages. The resulting computational procedure is presented in a fast algorithm form which can be easily implemented on a computer. Secondly, we present an original method of construction of concatenated codes based on polar codes. We give an algorithm for construction of such codes and present numerical experiments showing significant performance improvement with respect to original polar codes proposed by Arıkan. We use the term concatenated code not in its classical sense (e.g. [Forney]). However we believe that our usage is quite appropriate for the exploited construction. Further, we solve the optimization problem of choosing codes minimizing the block error of the whole concatenated code under the constraint of its fixed rate.

I. INTRODUCTION

Research related to construction of coding systems whose performance is close to Shannon limit while encoding and decoding algorithms are of low complexity, goes for more than 60 years. 

A significant modern example of such system is linear codes with low-density parity checks (LDPC). Usually these are binary linear block codes with sparse parity-check matrix. Decoding is performed via iterative algorithms whose convergence is described by quite a few theoretical results and in general, these algorithms are quite good. In practice, LDPC codes have good performance for high noise levels. There exist experimentally constructed codes approaching Shannon limit very closely (e.g. [8]). However for high bandwidth region, short LDPC codes exhibit so-called “error floor”, i.e. significant slowdown in decrease of decoding error probability corresponding to channel improvement, occurring due to decoding algorithm features.

Polar codes were invented by E. Arıkan in 2008. They are the first coding system possessing, on the theorem level, the convergence to Shannon limit for code length $N \to \infty$, as well as fast encoding/decoding algorithms with complexity bound $O(N \log_2 N)$. Thus polar codes are a significant theoretical result.

On the other hand, the performance of polar codes in their initial form presented by Arıkan, is considerably inferior, for a fixed code length, to other coding systems. To date there exist many proposals for improvement of polar codes performance (e.g. [11], [12]), yet work in this direction seems to be very promising.

In this paper we consider two problems related to polar codes. First is the problem of polar codes construction and analysis of their performance for various types of binary symmetric channel without Monte Carlo method. The formulas proposed are the same as those in [10], yet we believe that our approach is original and has clear advantages. Moreover, the resulting computational procedure is presented in a fast algorithm form which can be easily implemented on a computer. Secondly, we present an original method of construction of concatenated codes based on polar codes. We give an algorithm for construction of such codes and present numerical experiments showing significant performance improvement with respect to original polar codes proposed by Arıkan. It should be noted that we use the term concatenated code not in its classical sense (e.g. [7]). However we believe that our usage is quite appropriate for the exploited construction. Our idea is simple. It is known that approaching the Shannon limit is possible only with sufficiently large code length. Increasing the code length however makes the problem of code construction with large minimum distance and efficient ML decoder very hard. The situation is different for low-noise channel. Here codes of moderate length are sufficient so that ML decoder complexity is not too large. In order to obtain those low-noise channels we employ the polarization effect observed by E. Arıkan in polar codes. Further, we solve the optimization problem of choosing codes minimizing the block error of the whole concatenated code under the constraint of its fixed rate.

Unfortunately, we do not have a theorem on asymptotic optimality of our approach or just on its clear advantage with respect to known approaches, like e.g. [11]. Yet the simplicity of our approach, its flexibility and further possibilities of its improvement make it hopefully interesting.

Other examples of concatenated and generalized concatenated codes based on polar codes can be found in e.g. [11], [12].

A word on the channels considered here. We assume that the channel is defined by input alphabet $\mathcal{X}$, output alphabet $\mathcal{Y}$
and transition function

$$W(y | x) : \mathcal{Y} \times \mathcal{X} \rightarrow [0, 1]$$

defined for any pair $x \in \mathcal{X}$, $y \in \mathcal{Y}$. The function $W(y | x)$ defines the probability (or its density) that symbol $y$ is received under the condition that symbol $x$ was sent. For the sake of simplicity and in order to avoid generalized distributions we restrict our discussion to finite output alphabet. All formulas can be easily transplanted to the case of continuous channel by replacing the probabilities by the probability densities and replacing some sums by integrals. Note also that most frequently used channel models can be approximated by discrete ones. Moreover, data transmission systems used in practice represent output symbols with some fixed accuracy which is equivalent to some discrete channel model.

Besides, we consider only symmetric channels with binary input $[1]$. In such channels, the input alphabet contains two symbols:

$$\mathcal{X} = \{0, 1\} = \text{GF}(2),$$

Output alphabet $\mathcal{Y}$ is a subset of real numbers, and the function $W(y | x)$ possesses the following symmetry,

$$W(y | 0) = W(-y | 1).$$

The rest of this paper contains 5 sections. In section III we consider the problem of obtaining the optimal statistical estimate of a bit variable restricted by a linear system. Results of this auxiliary section are well known and belong to factor graph theory. These results are used in obtaining relations which determine the probability of erroneous bit decoding for ML decoder in section III. Our derivation essentially differs from original one proposed by E. Arkan. It is based on explicit representation of factor graph of polar code, its interpretation as a set of trees and application of density evolution method. Note also that for polar codes we consider two types of factor graphs: encoder graph and decoder graph. In section IV we describe the polar code construction method in the form of fast algorithms taking on input discrete probability function defined by the channel. Presented also is the analysis of obtained codes and numerical simulation for polar codes of different length.

In section V we discuss the possibility of polar code construction using polarization kernels other than $G_2$ which was introduced in Arkan’s paper. Finally, in section VI we introduce a class of concatenated codes based on polar codes and present numerical comparison of concatenated and classical polar codes performance.

II. Problem of bit variable estimation

Before proceeding directly to polar codes, consider the problem of estimation of one random bit entering as a variable in a linear system. To this end we investigate two simpler problems: estimation of sum of two random bits transmitted through the channels and estimation of one random bit for which we have several independent sources of information. Actually, this section contains short presentation of factor graph theory which is widely used in the modern coding theory [1].

A. Estimation of sum of bits

Let the values of two independent random bits $x_1$ and $x_2$ taking the values 0 and 1 equiprobably, were transmitted through channels $W_1$ and $W_2$, respectively, which resulted in received symbols $y = [y_1, y_2]$. Using the channel model we compute the logarithmic likelihood ratios (LLRs)

$$L(x_i) = \ln l(x_i) = \ln \frac{\Pr \{ y_{i} | x_{i} = 0 \}}{\Pr \{ y_{i} | x_{i} = 1 \}} = \ln \frac{W_i(y_i | 0)}{W_i(y_i | 1)}, \quad i = 1, 2.$$  

Assume the following quantity is required

$$L(x_1 + x_2) = \ln l(x_1 + x_2) = \ln \frac{\Pr \{ y | x_1 + x_2 = 0 \}}{\Pr \{ y | x_1 + x_2 = 1 \}},$$

i.e. estimate the sum of two bits provided $L(x_1)$ and $L(x_2)$ are known. Considering two possible equiprobable cases, we get

$$\Pr \{ y | x_1 + x_2 = 0 \} = \frac{1}{2} \Pr \{ y | x_1 = 0, x_2 = 0 \} + \frac{1}{2} \Pr \{ y | x_1 = 1, x_2 = 1 \}.$$  

Since the bits $y_1$ and $y_2$ are transmitted independently,

$$\Pr \{ y | x_1 = 0, x_2 = 0 \} = \Pr \{ y_1 | x_1 = 0 \} \Pr \{ y_2 | x_2 = 0 \},$$  

$$\Pr \{ y | x_1 = 1, x_2 = 1 \} = \Pr \{ y_1 | x_1 = 1 \} \Pr \{ y_2 | x_2 = 1 \}.$$  

Hence

$$\Pr \{ y | x_1 + x_2 = 0 \} = \frac{1}{2} \Pr \{ y_1 | x_1 = 0 \} \Pr \{ y_2 | x_2 = 0 \} + \frac{1}{2} \Pr \{ y_1 | x_1 = 1 \} \Pr \{ y_2 | x_2 = 1 \}.$$  

In a similar way we get

$$\Pr \{ y | x_1 + x_2 = 1 \} = \frac{1}{2} \Pr \{ y_1 | x_1 = 0 \} \Pr \{ y_2 | x_2 = 1 \} + \frac{1}{2} \Pr \{ y_1 | x_1 = 1 \} \Pr \{ y_2 | x_2 = 0 \}.$$  

Inserting the last two formulas in likelihood ratio $l(x_1 + x_2)$ and cancelling the factor $\frac{1}{2}$, we obtain

$$l(x_1 + x_2) = \frac{\Pr \{ y_1 | x_1 = 0 \} \Pr \{ y_2 | x_2 = 0 \} + \Pr \{ y_1 | x_1 = 1 \} \Pr \{ y_2 | x_2 = 1 \}}{\Pr \{ y_1 | x_1 = 0 \} \Pr \{ y_2 | x_2 = 1 \} + \Pr \{ y_1 | x_1 = 1 \} \Pr \{ y_2 | x_2 = 0 \}}.$$  

Divide the numerator and the denominator by $\Pr \{ y_1 | x_1 = 1 \}$:

$$l(x_1 + x_2) = \frac{\Pr \{ y_1 | x_1 = 0 \} \Pr \{ y_2 | x_2 = 0 \} + \Pr \{ y_1 | x_2 = 1 \}}{\Pr \{ y_1 | x_1 = 0 \} \Pr \{ y_2 | x_2 = 1 \} + \Pr \{ y_1 | x_2 = 0 \}}.$$  

Pr \{ y_2 | x_2 = 1 \}:
Using the likelihood ratios \( l(x_1) \) and \( l(x_2) \), rewrite the last formula as follows,
\[
l(x_1 \oplus x_2) = \frac{l(x_1)l(x_2) + 1}{l(x_1) + l(x_2)},
\]
or using logarithms,
\[
L(x_1 \oplus x_2) = \ln \frac{e^{L(x_1) + L(x_2)} + 1}{e^{L(x_1)} + e^{L(x_2)}} = 2 \tanh^{-1} \left( \tanh \left( \frac{L(x_1)}{2} \right) \tanh \left( \frac{L(x_2)}{2} \right) \right).
\]

For convenience introduce the binary operation
\[
a \odot b = 2 \tanh^{-1} \left( \tanh \left( \frac{a}{2} \right) \tanh \left( \frac{b}{2} \right) \right).
\]
Note some useful properties of the \( \odot \) operation:
\[
as \odot b = b \odot a, \quad \forall a, b \in \mathbb{R}
\]
\[
as \odot (b \odot c) = (a \odot b) \odot c, \quad \forall a, b, c \in \mathbb{R},
\]
\[
as \odot 0 = 0, \quad \forall a \in \mathbb{R},
\]
\[
-(a) \odot b = -(a \odot b), \quad \forall a, b \in \mathbb{R},
\]
\[
as \odot +\infty = a, \quad \forall a \in \mathbb{R},
\]
\[
as \odot -\infty = -a, \quad \forall a \in \mathbb{R},
\]
\[
|a \odot b| \leq \min(|a|, |b|), \quad \forall a, b \in \mathbb{R},
\]
\[
\text{sgn}(a \odot b) = \text{sgn}a \cdot \text{sgn}b, \quad \forall a, b \in \mathbb{R}.
\]

We now extend the problem to three bits \( x_1, x_2, x_3 \). Let these quantities be transmitted via channels \( W_1, W_2, W_3 \), respectively, and symbols \( y = [y_1, y_2, y_3] \) be received. Assume the following quantity is required
\[
L(x_1 \oplus x_2 \oplus x_3) = \ln l(x_1 \oplus x_2 \oplus x_3) = \ln \frac{\Pr \{ y_1 | x_1 \oplus x_2 \oplus x_3 = 0 \}}{\Pr \{ y_1 | x_1 \oplus x_2 \oplus x_3 = 1 \}}.
\]

Introduce new variable \( t \) taking values 0 and 1 equiprobably:
\[
t = x_1 \oplus x_2.
\]
We can assume that \( t \) was transmitted via channel with the following transition function,
\[
W(y_1 y_2 | t) = \Pr \{ y_1 y_2 | x_1 \oplus x_2 = t \},
\]
and write its LLR value as
\[
L(t) = \ln \frac{\Pr \{ y_1 y_2 | x_1 \oplus x_2 = 0 \}}{\Pr \{ y_1 y_2 | x_1 \oplus x_2 = 1 \}} = L(x_1 \oplus x_2) = L(x_1) \odot L(x_2).
\]

Then
\[
L(x_1 \oplus x_2 \oplus x_3) = L(t \oplus x_3) = L(t) \odot L(x_3) = (L(x_1) \odot L(x_2)) \odot L(x_3).
\]

Since the \( \odot \) operation is associative, drop the parentheses:
\[
L(x_1 \oplus x_2 \oplus x_3) = L(x_1) \odot L(x_2) \odot L(x_3).
\]

Using induction, we obtain formula for arbitrary number of variables:
\[
L(x_1 \oplus x_2 \oplus \ldots \oplus x_n) = L(x_1) \odot L(x_2) \odot \ldots \odot L(x_n).
\]

We now proceed to estimation of bit transmitted independently via several channels.

\textbf{B. Estimation of bit transmitted several times}

Let random bit \( x \) taking values 0 and 1 equiprobably be transmitted via \( n \) different channels \( W_1, W_2, W_3, \ldots \) receiving symbols \( y = [y_1, y_2, y_3] \). One can compute LLRs relying only on one channel:
\[
L_i(x) = \ln l_i(x) = \ln \frac{\Pr \{ y_i | x = 0 \}}{\Pr \{ y_i | x = 1 \}} = \ln \frac{W_i(y_i | 0)}{W_i(y_i | 1)}, \quad i = 1, n.
\]

We need to estimate \( x \), that is
\[
L(x) = \ln l(x) = \ln \frac{\Pr \{ y | x = 0 \}}{\Pr \{ y | x = 1 \}}.
\]

Since channels transmit symbols independently,
\[
\Pr \{ y | x = \alpha \} = \prod_{i=1}^{n} \Pr \{ y_i | x = \alpha \}, \quad \alpha = 0, 1.
\]

Inserting the last formula in expression for \( l(x) \), we have
\[
l(x) = \prod_{i=1}^{n} \frac{\Pr \{ y_i | x = 0 \}}{\Pr \{ y_i | x = 1 \}} = \prod_{i=1}^{n} l_i(x),
\]
or taking logarithms,
\[
L(x) = \sum_{i=1}^{n} L_i(x).
\]

Obtained formula gives the estimate of a bit for which we have several independent sources of information.

\textbf{C. Estimation of a bit entering a linear system}

Let random vector variable \( x = [x_1, x_2, \ldots, x_n]^T \) whose components take values 0 and 1 equiprobably, satisfy the linear system
\[
A x = 0,
\]
where the matrix \( A \in \text{GF}(2)^{m \times n} \) is exactly known. Assume that the quantities \( x_1, x_2, \ldots, x_n \) are transmitted via channels \( W_1, \ldots, W_n \), and received symbols are \( y = [y_1, \ldots, y_n] \). Then initial LLRs are known
\[
\lambda_i = \ln \frac{W_i(y_i | 0)}{W_i(y_i | 1)}, \quad i = 1, n.
\]

Assume the following LLR is required
\[
L(x_1) = \ln \frac{\Pr \{ y | x_1 = 0 \}}{\Pr \{ y | x_1 = 1 \}},
\]
without knowledge of \( x \).
Note that if some component \( x_i \) is exactly known, we can assume that it is transmitted via binary symmetric channel with zero error probability, and
\[
\lambda_i = \begin{cases} 
+\infty, & x_i = 0, \\
-\infty, & x_i = 1.
\end{cases}
\]

Vice versa, if some bit \( x_i \) is not transmitted, we can assume that it is transmitted via absolutely noisy channel with the transition function
\[
W(0|a) = 1, \quad a = 0, 1.
\]
It is easy to see that in this case
\[
\lambda_i = 0.
\]

If such bit enters only one equation, we can remove this bit and respective equation. If some equation contains only exactly known bits (with \( \lambda_i = \pm \infty \)), this equation also can be removed.

Associate matrix \( A \) with a bipartite undirected graph by the following rule. Each matrix row (i.e. each equation) is associated with a square vertex. Each matrix column (i.e. each bit variable) is associated with a round vertex. A round vertex and a square vertex are connected by an edge if respective matrix row and matrix column intersect at value one (i.e. if the respective variable enters respective equation). Such a graph is referred to as Tanner graph for the matrix \( A \).

We focus on just one bit variable, say \( x_1 \). If the Tanner graph is disconnected, remove all connected components save one containing the vertex \( x_1 \). Now if removing some bit \( x_p \) results in emerging of \( q \) graph components \( A_1, A_2, \ldots, A_q \), our problem of estimating \( x_1 \) is split into \( q \) smaller problems. Let \( Y_i \) be a subvector of \( y \) containing only those components which arise in transmission of bits entering the subgraph \( A_i \). Assume \( p \neq 1 \), and let \( x_1 \in A_i \). We can assume that \( x_p \) is transmitted via \( q - 1 \) different channels with transition functions
\[
\hat{W}_i(Y_i | a) = \Pr \{ Y_i | x_p = a \}, \quad i = \overline{1,q}.
\]

Compute LLRs of \( x_p \) considering only channel \( i \),
\[
L_i(x_p) = \ln \frac{\hat{W}_i(Y_i | 0)}{\hat{W}_i(Y_i | 1)}.
\]
Then the subgraphs \( A_2, A_3, \ldots, A_q \) may be removed with updating the initial \( \lambda_p \) estimate to
\[
\hat{\lambda}_p = \lambda_p + \sum_{i=1}^{q} L_i(x_p).
\]

Note that for each \( i \) the problem of computation of \( L_i(x_p) \) is also a bit estimation problem formulated on a smaller graph \( A_i \) augmented by the vertex \( x_p \).

Now assume that the Tanner graph is a tree, i.e. it is connected and acyclic. Assume also that every equation contains at least two variables. Choose the vertex \( x_1 \) as a tree root vertex. The leaf vertices will be some subset of \( x_2, \ldots, x_n \).

Let the vertex \( x_1 \) be incident to equations \( c_1, c_2, \ldots, c_q \), and let each vertex \( c_i \), \( i = \overline{1,q} \) be incident to variables \( x_{k_{i,1}}, \ldots, x_{k_{i,v_i}} \), not counting \( x_1 \). Let \( T_i \) be the subtree with root at \( x_{k_{i,1}} \), not counting the root itself (see fig. 1).

Let \( I_i^c \subset \{ 1, 2, \ldots, n \} \) be an index set for variables entering the subgraph \( T_i^c \). For all \( i,j \) define the set
\[
Y_i^j = \{ y_r : r \in I_i^c \} \cup \{ y_{k_{i,j}} \},
\]
i.e. the set of all symbols obtained via transmitting variables entering the subtree rooted at \( x_{k_{i,j}} \). Assume that for each pair \( i,j \) we know the LLRs
\[
L(x_{k_{i,j}}) = \ln \frac{\Pr \{ Y_i^j | x_{k_{i,j}} = 0 \}}{\Pr \{ Y_i^j | x_{k_{i,j}} = 1 \}},
\]
i.e. bit \( x_{k_{i,j}} \) estimates based only on tree rooted at the vertex \( x_{k_{i,j}} \). This can be interpreted as transmitting each such bit via the channel with the following transition function,
\[
W_i^j(Y_i^j | a) = \Pr \{ Y_i^j | x_{k_{i,j}} = a \}.
\]
Write each equation \( c_i \) in the following way:
\[
x_1 = x_{k_{i,1}} + x_{k_{i,2}} + \ldots + x_{k_{i,v_i}}, \quad i = \overline{1,q}.
\]

We can assume that \( x_1 \) was transmitted via independent channels \( \hat{W}_i \) with transition functions
\[
\hat{W}_i(Y_i, Y_i^2, \ldots, Y_i^q | a) = \Pr \{ Y_i, Y_i^2, \ldots, Y_i^q | x_1 = a \}, \quad i = \overline{1,q}.
\]

Then LLRs based on these channels have the form
\[
L_i(x_1) = \ln \frac{\Pr \{ Y_i, Y_i^2, \ldots, Y_i^q | x_1 = 0 \}}{\Pr \{ Y_i, Y_i^2, \ldots, Y_i^q | x_1 = 1 \}}.
\]

Inserting (2), we get
\[
L_i(x_1) = \ln \frac{\Pr \{ Y_i, Y_i^2, \ldots, Y_i^q | x_{k_{i,1}}, \ldots, x_{k_{i,v_i}} = 0 \}}{\Pr \{ Y_i, Y_i^2, \ldots, Y_i^q | x_{k_{i,1}}, \ldots, x_{k_{i,v_i}} = 1 \}}.
\]
Taking into account (1) and using result of section 1-A we obtain
\[
L_i(x_1) = L(x_{k_{i,1}}) \oplus L(x_{k_{i,2}}) \oplus \ldots \oplus L(x_{k_{i,v_i}}) = L(x_{k_{i,1}}) \oplus \ldots \oplus L(x_{k_{i,v_i}}).
\]

![Fig. 1. Tree-like Tanner graph given in the rooted tree form](image-url)
Finally assuming $x_1$ be transmitted via the channels $W_1, W_2, \ldots, W_g$ and also via $W_1$, write

$$L(x_1) = \lambda_1 + \sum_{i=1}^{g} L_i(x_1).$$

In order to compute $L(x_{k_i})$, we can apply the same reasoning to the subtree rooted at $x_{k_i}$. Thus we have a recursive algorithm computing $L(x_1)$. It is essentially equivalent to the algorithm known as “belief propagation”.

### III. POLAR CODES

In this section we consider polar codes in exactly that form which they were presented in originally [2], but take a slightly different look. Let $u^0$ and $u^1$ be two independent random bits taking values 0 and 1 equiprobably. Define two more bits

$$x[0] = u^0 \oplus u^1,$$

$$x[1] = u^1.$$ \hspace{1cm} (3)

In matrix notation,

$$[x[0], x[1]] = [u^0, u^1] \cdot G_2, \quad G_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$  

Note that bits $x[0], x[1]$ also take the values 0 and 1 equiprobably. Construct the Tanner graph for the system (3) and denote it as encoder graph (see fig. 2).

![Encoder Graph](image)

Fig. 2. The encoder graph

Since $G_2^{-1} = G_2$, we can rewrite the system (3) in equivalent form

$$u^0 = x[0] \oplus x[1], \quad u^1 = x[1].$$

Construct the Tanner graph for this system also and denote it as decoder graph (see fig. 3).

![Decoder Graph](image)

Fig. 3. The decoder graph

Let bits $x[0]$ and $x[1]$ be transmitted via a given channel $W$ receiving symbols $y = [y^0, y^1]$. Assume the following LLR is required

$$L(u^0) = \ln \Pr \{ y \mid u^0 = 0 \} / \Pr \{ y \mid u^0 = 1 \}.$$  

Using results of the section [2] we have

$$L(u^0) = L(x[0]) \square L(x[1]),$$

where

$$L(x[i]) = \ln \frac{W(y^i \mid 0)}{W(y^i \mid 1)}, \quad i = 0, 1.$$  

Now assume that the value of $u^0$ is exactly known and that we need

$$L(u^1) = \ln \frac{\Pr \{ y \mid u^1 = 1, u^0 = 0 \}}{\Pr \{ y \mid u^1 = 0, u^0 \}}.$$  

Using again section [2] we get

$$L(u^1) = L(x[1]) + (L(x[0]) \square L(u^0)) = L(x[1]) + (-1)^0 L(x[0]).$$

We proceed to recursive construction of the larger system and then to similar problem of determining of one bit.

#### A. Hierarchical graph construction

Double the encoder graph taking two copies of each variable and of each equation. Now let $u_0^0, u_1^0, u_2^0, u_3^0$ be the transmitted random bits while $u_0^0[0], u_0^0[1], u_1^0[0], u_1^0[1]$ are their functions:

$$u_0^0[0] = u^0 \oplus u^1,$$

$$u_0^0[1] = u^1,$$

$$u_1^0[0] = u^2 \oplus u^3,$$

$$u_1^0[1] = u^3.$$ \hspace{1cm} (4)

From the other hand,

$$u^0 = u_0^0[0] \oplus u_0^0[1],$$

$$u^1 = u_1^0[1],$$

$$u^2 = u_1^0[0] \oplus u_1^0[1],$$

$$u^3 = u_1^0[1].$$

For consistency, set $u_0^0[0] \equiv u^i$. Figure 4 gives the graph for the system (4).

![Graph for system of equations](image)

Fig. 4. Graph for the system of equations (4)

Introduce four new variables (fig. 5),

$$u_0^0[0] \quad u_0^0[1] \quad u_1^0[0] \quad u_1^0[1] \quad u_0^2[0] \quad u_0^2[1] \quad u_1^2[0] \quad u_1^2[1] \quad u_0^3[0] \quad u_0^3[1] \quad u_1^3[0] \quad u_1^3[1] \quad u_2^0[0] \quad u_2^0[1] \quad u_2^1[0] \quad u_2^1[1] \quad u_2^2[0] \quad u_2^2[1] \quad u_2^2[2] \quad u_2^2[3] \quad u_2^3[0] \quad u_2^3[1] \quad u_2^3[2] \quad u_2^3[3].$$

In matrix notation,

$$[u_0^0[0], u_0^0[1]] = [u_0^0[0], u_1^0[0]] \cdot G_2,$$

$$[u_0^2[0], u_0^2[1]] = [u_0^2[0], u_1^2[0]] \cdot G_2.$$
Again employ the relation $G_{2}^{-1} = G_{2}$ and express old variables in terms of new ones (fig. 6).

$$u_{0}^{0}[0] = u_{0}^{0}[0] \oplus u_{0}^{0}[1],$$
$$u_{1}^{0}[0] = u_{1}^{0}[1],$$
$$u_{0}^{0}[1] = u_{0}^{0}[2] \oplus u_{0}^{0}[3],$$
$$u_{1}^{0}[1] = u_{2}^{0}[3].$$

(6)

We call the graph displayed on fig. 5 the encoder graph and the graph displayed on fig. 6 the decoder graph. Now we are able to repeat the whole operation, double the graph and introduce eight new variables $u_{0}^{0}[i]$, $i = 0, 7$, and proceed further. We can express new variables in terms of old ones,

$$[u_{k}^{i+1}[2j], u_{k}^{i+1}[2j + 1]] = [u_{k}^{i}[j], u_{k}^{2i+1}[j]] \cdot G_{2},$$

and vice versa:

$$[u_{k}^{2i}[j], u_{k}^{2i+1}[j]] = [u_{k}^{i+1}[2j], u_{k}^{i+1}[2j + 1]] \cdot G_{2}.$$

Assume we make $n$ steps and stop at introducing new variables $u_{0}^{0}[i]$. Indices in the expression $u_{k}^{i}[j]$ have the following interpretation based on decoder graph. Lower index $k$ specifies the vertical “layer” of the graph of $2^{n}$ variables where the given vertex is, if we count layers right to left. Bracketed index $j$ specifies the independent group of variables in a layer. Upper index $i$ specifies the variable inside a group.

Increasing layer number by one doubles the number of independent groups $J_{k}$, i.e.

$$J_{k} = 2J_{k-1}, \quad k = 1, n.$$
On the decoder graph, the subvector \( Y_k[j] \) may be interpreted as components of \( y \) corresponding to those bits \( u_{n}[l] \) which are strictly on the left of variables of group \( j \), layer \( k \). For example, if \( n = 4 \) (fig. 7), then group 1 of layer 2 consists of variables \( u_0^2[1], u_1^2[1], u_2^2[1], u_3^2[1] \), while the subvector \( Y_2[1] \) is obtained via transmission of the bits \( u_0^2[4], u_1^2[5], u_2^2[6], u_3^2[7] \). Introduce the following notation,

\[
L(u_k^j[j]) = \ln \left( \frac{\Pr \left\{ Y_k[j] \mid u_k^j[j] = 0, u_k^j[j], l = 0, i = 1 \right\}}{\Pr \left\{ Y_k[j] \mid u_k^j[j] = 1, u_k^j[j], l = 0, i = 1 \right\}} \right).
\]

This means that \( L(u_k^j[j]) \) is the LLR for the bit \( u_k^j[j] \) provided that \( Y_k[j] \) is received and that the quantities \( u_0^j, u_1^j, \ldots, u_{k-1}^j \) are exactly known. Note that the formula (9) is a special case of (9) for \( k = 0 \), and that for \( k = n \) the formula (9) takes the form

\[
L(u_n^j[j]) = \ln \left( \frac{\Pr \left\{ y_j \mid u_n^j[j] = 0 \right\}}{\Pr \left\{ y_j \mid u_n^j[j] = 1 \right\}} \right) = \lambda_j.
\]

Our goal is to obtain recursive formula for \( L(u_k^j[j]) \) in terms of \( L(u_{k+1}^j[1]) \). If we have it, we can compute the required \( L(u_0^n[0]) \) using (10) as a recursion base.

Denote the subgraph consisting of single vertex \( u_k^0[j] \) by \( A_n[j] \). By induction, let \( A_k[j] \) for \( j = 0, J_k = 1 \) be the union of subgraphs \( A_{k+1}[2j], A_{k+1}[2j+1] \), of all vertices of group \( j \), layer \( k \) and of incident equations. On the graph drawing we can interpret \( A_{k+1}[j] \) as a subgraph whose vertices are all bits of group \( j \), layer \( k \) and all vertices on the left of these. For example, if \( n = 4 \), the subgraph \( A_2[1] \) consists of the variables

\[
u_0^2[4], u_1^2[5], u_0^2[6], u_1^2[7], u_0^2[2], u_1^2[2], u_0^2[3], u_1^2[3], u_0^2[1], u_1^2[1], u_0[2], u_1[2], u_0[3], u_1[3]
\]

and incident equations (fig. 7). Note that the subgraph \( A_k[j] \) contains those and only those variables of layer \( n \), whose transmission results in the vector \( Y_k[j] \).

C. Recursive formulas

Here we find the expression for \( L(u_k^j[j]) \) under the constraint \( k < n \). All variables of layer 0 enter only one equation, and for \( k > 0 \) we do not have any immediate information for them, thus we remove them and the incident equation. Now for \( k > 1 \) the same can be done for layer 1 etc. Finally we retain only layers with index at least \( k \). The graph will be divided in \( J_k \) connected components \( A_k[j] \). Remove all components save one containing \( u_k^j[j] \), which means that we keep only the component \( A_k[j] \). Denote \( q = \lfloor i/2 \rfloor \). If bits \( u_k^j[j], u_k^j[j], \ldots, u_{k-1}^j[j] \) are exactly known, then according to the equation (7), exactly known are also the quantities \( u_{k+1}^j[2j], u_{k+1}^j[2j+1], l = 0, q-1 \).

Hence the equations incident to \( u_k^0[j], u_k^1[j], \ldots, u_{k-1}^j[j] \) and to (11) may be removed: these equations contain only known quantities. After the removal the vertices

\[
u_k^j[0], u_k^j[1], \ldots, u^{2q-1}_k[j]
\]

become isolated and also may be removed. The vertices

\[
u^{2q+2}_k[j], u^{2q+3}_k[j], \ldots, u^{S_k-1}_k[j]
\]

are not transmitted and do not have any estimates, thus also may be removed with corresponding equations (one per vertex). After these transformations the graph will have the form depicted on fig. 8.

Since removing the vertex \( u_k^j[2j+1] \) divides the graph into two components one of which is \( A_{k+1}[2j] \), we can assume that the bit \( u_k^j[2j+1] \) is transmitted via the channel with the following transition function

\[
\tilde{W}_0(Y_k+1[2j] \mid a) = \Pr \{ Y_k+1[2j] \mid u_k^j[2j] = a; u_{k+1}^j[2j+1], l = 0, q-1 \}.
\]

Also we can substitute the subgraph \( A_{k+1}[2j] \) by the single vertex \( u_k^j[2j] \), with the following initial likelihood ratio,

\[
\lambda_k^q[2j] = \ln \frac{\tilde{W}_0(Y_k+1[2j] \mid 0)}{\tilde{W}_0(Y_k+1[2j] \mid 1)}.
\]

Comparing (9) with (12), we conclude that

\[
\lambda_k^q[2j] = L(u_k^j[2j]).
\]

Similarly, the vertex \( u_k^j[2j+1] \) can be substituted for the whole subgraph \( A_{k+1}[2j+1] \), if we set

\[
\lambda_k^q[2j+1] = L(u_k^j[2j+1]).
\]

Resulting graph is displayed on fig. 9. We know already that for such graph, \( L(u_k^j[2j]) = L(u_k^j[2j]) \cup L(u_k^j[2j+1]), L(u_k^{2q+1}[j]) = L(u_k^{2q+1}[j]) + (-1)^{q-1}u_k[j]L(u_k^j[2j]) \).

Thus we have obtained the recursive formulas giving \( L(u_0^i[0]) \) for all \( i \) using (10) as a recursion base.
D. Successive cancellation method

Let the vector

\[ u = [u_0^0[0], u_1^0[0], u_2^0[0], \ldots, u_n^{2^n-1}[0]] \]

be the message required for transmission. Starting from \( u \) we can compute

\[ x = [u_0^0[0], u_1^0[1], u_0^0[2], \ldots, u_n^{2^n-1}[2^n-1]], \]

using the formulas (7). The vector \( x \) will be considered as a codeword and transmitted componentwise via given symmetric channel \( W \) producing vector \( y \) at the receiver. We want to recover \( u \) using \( y \). We do it sequentially bit by bit. First we compute \( L(u_0^0[0]) \) and estimate the bit \( u_0^0[0] \) as follows,

\[ \hat{u}_0^0[0] = \begin{cases} 0, & L(u_0^0[0]) > 0, \\ 1, & L(u_0^0[0]) < 0, \\ \text{choose randomly,} & L(u_0^0[0]) = 0. \end{cases} \]

Now assuming \( u_0^0[0] \) exactly known, we compute \( L(u_1^0[0]), \) estimate \( u_1^0[0] \) etc. Each bit is estimated using the rule

\[ \hat{u}_i^0[0] = \begin{cases} 0, & L(u_i^0[0]) > 0, \\ 1, & L(u_i^0[0]) < 0, \\ \text{choose randomly,} & L(u_i^0[0]) = 0. \end{cases} \]

Finally we produce some estimate of the initial message \( u \). This decoding method is called successive cancellation. Of course, presented coding system is useless since the redundancy is missing.

E. Polar codes

Choose some index set \( F \subset \{0, 1, 2, \ldots, 2^n - 1\} \). Denote \( K = 2^n - |F| \). Make a convention that the only possible messages are those with bits from \( F \) equal to zero. We call these bits frozen, and other ones information bits. Again we use the successive cancellation however with modified bit estimation rule:

\[ \hat{u}_i^k[0] = \begin{cases} 0, & i \in F, \\ \text{choose using (14),} & \text{otherwise.} \end{cases} \]

Since the admissible messages form the linear space of dimension \( K \) and codewords \( x \) depend linearly on \( u \), the set of all codewords is also a linear space. In other words, we have linear block code of length \( 2^n \) and of rate \( K/2^n \). It can be shown that its generator is obtained by deleting rows with indices in \( F \) from the matrix

\[ G_2^{2^n} \cdot R_n, \]

where \( G_2^{2^n} \) denotes the \( n \)-th Kronecker power of the matrix \( G_2 \) and \( R_n \) the bit reverse permutation matrix. The rule describing this permutation is as follows: let binary representation of index \( i \) be \( \alpha_{n-1} \alpha_{n-2} \ldots \alpha_2 \alpha_1 \alpha_0 \), then the element \( i \) is swapped with the element indexed \( \alpha_n \alpha_1 \alpha_2 \ldots \alpha_{n-2} \alpha_{n-1} \), in binary representation. Thus constructed code is called the polar code.

How to choose the set \( F \) of frozen bits? Denote the probability of erroneous detection of the bit \( u_0^0[0] \) using the successive cancellation method provided that all previous bits are detected correctly and \( F = \emptyset \), by \( E_i, i = 0, 2^n - 1 \). The probability of block error \( P_E \) with \( F \) fixed may be estimated from above as a sum of probability errors for each information bit, i.e.

\[ P_E \leq \sum_{i \not\in F} E_i. \] (15)

The set \( F \) may contain indices of bits with maximal error probabilities, which will minimize the upper bound (15) of the block error probability. To this end, one has to compute the probabilities \( E_i \), which is discussed in section IV.

F. Complexity of encoding and decoding

Polar codes would not have any practical value without fast algorithms of encoding and decoding. The encoding process is carried out by recursive formulas (7)

\[ [u_{k+1}^0[2j], u_{k+1}^0[2j + 1]] = [u_{k}^2[j], u_{k}^{2+1}[j]] \cdot G_2 \]

and requires \( n \) sequential steps \( k = 1, 2, 3, \ldots, n \). On each of these steps, all variables of layer \( k \) are defined. Since each layer contains \( 2^n \) bits, the overall encoding complexity is \( O(n \cdot 2^n) \) operations, which is \( O(N \cdot \log_2 N) \) if we introduce the code length \( N = 2^n \).

Decoding by successive cancellation method using the recursive formulas (13) requires computation of \( n \cdot 2^n \) different quantities \( L(u_k^i[j]) \) and of \( n \cdot 2^n \) quantities \( u_k^i[j] \) in a more complex order. Hence the decoding complexity also is \( O(N \cdot \log_2 N) \) operations.

IV. CONSTRUCTION AND ANALYSIS OF POLAR CODES

Construction of polar code of given length \( N = 2^n \) and rate \( K \) for a given channel \( W \) amounts to choosing the set \( F \) of \( N - K \) frozen bits. The choice which minimizes the block error probability \( P_E \) would be optimal. However computation of \( P_E \) is complicated and it is reasonable to substitute its upper bound in the minimization problem,

\[ \min_F \sum_{i \not\in F} E_i, \]

where \( E_i \) is the probability of erroneous detection of bit \( i \) by successive cancellation under assumption that all previous bits are detected without error. In this formulation, it is sufficient to choose \( N - K \) indices corresponding to maximal values of \( E_i \) as the set \( F \) provided that \( E_i \) are known for \( i = 0, 2^n - 1 \). Thus the polar code construction problem is reduced to computation of quantities \( E_i \).

A. Likelihood ratios as random variables

Since the channel is symmetric and the code is linear, in computation of \( E_i \) we can assume that all-zeros codeword is transmitted. In this case the probability to receive the vector \( y \) is

\[ p(y) = \prod_{i=0}^{N-1} W(y_i | 0). \] (16)
By definition of $E_i$ we assume all bits $u_0, u_1, \ldots, u_{i-1}$ zero. In this case the recursion \cite{13} takes the form

$$L(u^q_{2j}[j]) = L(u^q_{k+1}[2j]) \square L(u^q_{k+1}[2j + 1]),$$
$$L(u^q_{2j+1}[j]) = L(u^q_{k+1}[2j + 1]) + L(u^q_{k+1}[2j]). \quad (17)$$

The recursion base \cite{10} remains unchanged:

$$L(u^q_0[j]) = \lambda_j = \ln \frac{W(y_j | 0)}{W(y_j | 1)}.$$

Now the quantities $L(u^q_{i,j})$ depend only on $y$ and do not depend on $u = [u^0_0[0], \ldots, u^0_{N-1}[0]]$, thus we consider $L(u^q_{i,j})$ as random variables defined on probability space $\mathcal{Y}^N$ with probability measure \cite{16}.

According to \cite{16}, the quantities $y_0, y_1, \ldots, y_{N-1}$ are mutually independent. Hence the quantities $L(u^0_0[0]), L(u^0_0[1]), \ldots, L(u^0_0[N - 1])$ are also mutually independent, because every quantity $L(u^0_{i,j})$ depends on only one symbol $y_j$. The quantities $L(u^q_{i,j})$ and $L(u^q_{i,j'})$ are independent for all $k > 0$, $i'$, $i''$ and $j' \neq j''$, because they are defined by recursive formulas \cite{17} via non-intersecting sets of $L(u^q_{i,j})$.

Following the hard decision rule \cite{14} we see that the bit $i$ is detected erroneously in all cases when $L(u^q_0[0]) < 0$ and in half of cases when $L(u^q_0[0]) = 0$. In other words, $E_i$ is \cite{14}

$$E_i = \Pr \{ L(u^q_0[0]) < 0 \} + \frac{1}{2} \Pr \{ L(u^q_0[0]) = 0 \}.$$

Extend the problem of computation of $E_i$ to computation of distributions of random variables $L(u^q_0[0])$. Denote by $f^q_k[j]$ the probability function \cite{14} of the random variable $L(u^q_{k,j})$:

$$f^q_k[j](z) = \Pr \{ L(u^q_{k,j}) = z \}.$$

From the channel model we have $f^q_n[j], j = 0, N-1$:

$$f^q_n[j](z) = \Pr \left\{ \ln \frac{W(y_j | 0)}{W(y_j | 1)} = z \mid u^q_0[j] = 0 \right\} = \sum_{b \mid \ln \frac{W(b | 0)}{W(b | 1)} = z} W(b | 0).$$

We see that the distributions $f^q_n[j]$ of random variables $L(u^q_n[j])$ are the same for all $j$. Formulas \cite{17} imply that for $k < n$ the distributions of $L(u^q_{k,j})$ also do not depend on $j$, i.e. $f^q_k[j'] = f^q_k[j’’]$ \forall $i, k, j’, j''$. Therefore in what follows, we drop the square brackets in the notation $f^q_k[j]$.

\section*{B. Recurrent relations for the distributions}

Here we show that the distributions $f^q_k$ satisfy the recurrent relations analogous to the formulas \cite{17}. We start from random variables with odd indices:

$$L(u^q_{2j+1}[j]) = L(u^q_{k+1}[2j + 1]) + L(u^q_{k+1}[2j]).$$

If $L(u^q_0[0])$ has continuous distribution, the term $\frac{1}{2} \Pr \{ L(u^q_0[0]) = 0 \}$ should be deleted.

In the continuous case, $f^q_n[j]$ will be the probability density function.

Since $L(u^q_{k+1}[2j])$ and $L(u^q_{k+1}[2j + 1])$ are i.i.d.,

$$f^q_{2k+1}(z) = \Pr \{ L(u^q_{k+1}[2j]) = z \} = \sum_{a,b \in \text{supp} f^q_k} f^q_k(a) f^q_k(b),$$

where $\text{supp} f^q_k = \{ v : f^q_k(v) \neq 0 \}$. Rewrite the sum so that it will go over one index only:

$$f^q_{2k+1}(z) = \sum_{a \in \text{supp} f^q_k} f^q_k(a) f^q_k(z - a). \quad (18)$$

If $\text{supp} f^q_k$ is a uniform mesh, the formula \cite{18} is nothing else but discrete convolution of a sequence with itself. In the continuous case, the formula \cite{18} will have the form

$$f^q_{2k+1}(z) = \int_{-\infty}^{+\infty} f^q_k(a) f^q_k(z - a) da,$$

which is also convolution of the function $f^q_k$ with itself. Hence it is quite natural to call the probability function $h$ defined by the formula

$$h(z) = \sum_{a \in \text{supp} f} f(a) g(z - a),$$

the convolution of the functions $f$ and $g$ with the notation

$$h = f \ast g.$$

Thus in new notation

$$f^q_{2k+1} = f^q_k \ast f^q_k.$$

Random variables with even indices are treated analogously. Using the corresponding recursive formula

$$L(u^q_{k+1}[j]) = L(u^q_{k+1}[2j]) \square L(u^q_{k+1}[2j])$$

we write

$$f^q_{2k}(z) = \Pr \{ L(u^q_{2k}[j]) = z \} = \sum_{a,b \in \text{supp} f^q_k} f^q_k(a) f^q_k(b).$$

Introduce the notation

$$f \boxtimes g(z) = \sum_{a \in \text{supp} f, b \in \text{supp} g} f(a) g(b)$$

and rewrite the recursion as

$$f^q_{2k}(z) = f^q_{k+1} \boxtimes f^q_{k+1}.$$

While the operation $\boxtimes$ is not a convolution in the usual sense, we will still use this term. Now we have the recurrent formulas \cite{19} and \cite{20} which give the required distributions of random variables $L(u^q_0[0])$ if the initial probability function

$$f^q_n(z) = \sum_{b \in \text{supp} f} W(b | 0)$$

is used as the recursion base.
The probability error $E_i$ is obtained from the probability function $f_0^i$:

$$E_i = \sum_{a \in \text{supp } f_0^i; a < 0} f_0^i(a) + \frac{1}{2} f_0^i(0).$$

In what follows we consider a simple case when the convolutions $\ast$ and $\boxplus$ of two functions are reduced to simple operations on pairs of numbers.

C. The case of binary erasure channel (BEC)

The problem of polar codes construction for BEC was solved in that very article where the polar codes were introduced, however in a different formulation \cite{[3]}. The BEC scheme is shown on fig. \[10\]

![Fig. 10. Schematic view of binary erasure channel](image)

Binary erasure channel is a binary input symmetric channel, possibly simplest one in terms of decoding: if the received symbol is 1 or −1, the transmitted symbol is unconditionally known. Initial probability function has the support of two values,

$$f_n^0(z) = B_p(z) = \begin{cases} 1 - p, & z = +\infty, \\ p, & z = 0, \\ 0, & \text{otherwise}. \end{cases}$$

The quantity $p \in [0, 1]$ is termed the erasure probability. Consider the convolution of functions $B_p$ and $B_r$ for some $p, r \in [0, 1]$. The sum $a + b$ of all possible pairs $(a, b), a \in \text{supp } B_p, b \in \text{supp } B_r$ has the form

$$0 + 0 = 0, \quad 0 + (+\infty) = +\infty, \quad (+\infty) + 0 = +\infty, \quad (+\infty) + (+\infty) = +\infty.$$ 

Hence $\text{supp}(B_p \ast B_r) = \{0, +\infty\}$. Value of zero is obtained only with $a = 0, b = 0$,

$$(B_p \ast B_r)(0) = pr,$$

therefore

$$(B_p \ast B_r)(+\infty) = 1 - pr$$

and

$$B_p \ast B_r = B_{pr}.$$

Now we turn to the convolution $B_p \boxplus B_r$. Again consider all possible pairs $(a, b)$ and the output of $a \boxplus b$:

$$0 \boxplus 0 = 0, \quad 0 \boxplus (+\infty) = 0, \quad (+\infty) \boxplus 0 = 0, \quad (+\infty) \boxplus (+\infty) = +\infty.$$ 

Again $\text{supp}(B_p \boxplus B_r) = \{0, +\infty\}$. Since the value of $+\infty$ is obtained only with $a = +\infty, b = +\infty$,

$$(B_p \boxplus B_r)(+\infty) = (1 - p)(1 - r).$$

Therefore

$$(B_p \boxplus B_r)(0) = 1 - (1 - p)(1 - r) = p + r - pr$$

and

$$B_p \boxplus B_r = B_{p + r - pr}.$$

Thus the formulas (19) and (20) take the form

$$p_{k+1}^{2q+1} = 2p_k^q - (p_k^q)^2,$$ 

$$p_k^q = (p_k^q)^2,$$ 

$$f_k^i = B_{p_k^i}.$$

The recursion base will be $p_0^0 = p$, the erasure probability of the channel. It only remains to note that

$$E_i = \frac{1}{2}f_i^0, \quad i = 0, 2^n - 1.$$ 

The case of general symmetric channel is considered in the next subsection.

D. The general case

In the general case the complexity of exact computation of convolutions becomes too high since the support cardinality (and memory requirements) for the probability function grows exponentially with the code length. Approximation of the probability functions using a uniform grid is quite natural. Denote by $\delta$ the grid step and by $\Omega_i$ the grid cell number $i$ with $i = -Q, Q$:

$$\Omega_0 = \left( -\frac{\delta}{2}, -\frac{\delta}{2} \right),$$ 

$$\Omega_i = \left[ i\delta - \frac{\delta}{2}, i\delta + \frac{\delta}{2} \right], \quad i = 1, Q - 1,$$ 

$$\Omega_i = \left[ i\delta - \frac{\delta}{2}, i\delta + \frac{\delta}{2} \right], \quad i = -Q + 1, -1,$$ 

$$\Omega_Q = \left[ Q\delta - \frac{\delta}{2}, +\infty \right),$$ 

$$\Omega_{-Q} = \left( -\infty, -Q\delta + \frac{\delta}{2} \right).$$

where $Q$ is a positive integer which will be called the number of quantization levels. Thus the grid consists of $2Q + 1$ cells. The points $i\delta$ will be called the grid nodes. All cells except for extreme ones have grid nodes as centers.
Algorithm 1 Computation of the projection $\hat{f}$ of $g \ast h$ with arguments supported at grid nodes

1. $\hat{f} \leftarrow g \ast h$
2. $f(x) \leftarrow 0 \ \forall x$
3. for $i \leftarrow -Q, \ldots, Q$ do
4. \hspace{5pt} $\hat{f}(i\delta) \leftarrow f(i\delta)$
5. \hspace{5pt} end for
6. for $i \leftarrow Q + 1, \ldots, 2Q$ do
7. \hspace{5pt} $f(Q\delta) \leftarrow \hat{f}(Q\delta) + f(i\delta)$
8. \hspace{5pt} end for
9. for $i \leftarrow -Q, \ldots, Q$ do
10. \hspace{5pt} end for

Algorithm 2 Computation of the projection $\hat{f}$ of $g \otimes h$ with arguments supported at grid nodes

1. $\hat{f}(x) \leftarrow 0 \ \forall x$
2. for $i \leftarrow -Q, \ldots, Q$ do
3. \hspace{5pt} for $j \leftarrow -Q, \ldots, Q$ do
4. \hspace{5pt} \hspace{5pt} $k \leftarrow \text{nearest}(i\delta \sqcap j\delta)$
5. \hspace{5pt} \hspace{5pt} $\hat{f}(k\delta) \leftarrow \hat{f}(k\delta) + g(i\delta)h(j\delta)$
6. \hspace{5pt} end for
7. end for

Define the grid projection operator. Let $f$ be some probability function. For each grid node, sum all nonzero values of $f$ which belong to the corresponding grid cell

$$\hat{f}(i\delta) = \sum_{z \in \Omega_i \cap \text{supp } f} f(z). \quad (21)$$

Let $g$ and $h$ be two functions supported at grid nodes. The convolutions $g \ast h$ and $g \otimes h$ can take nonzero values outside the set of grid nodes. Use the projection (21) to restrict the resulting function to the grid.

Note that the convolution $g \ast h$ can have nonzero values only at points $i\delta$ with $i$ from $-2Q$ up to $2Q$. Among these points, only those with $|i| > Q$ are not grid nodes. If $i > Q$, these points belong to the rightmost cell $\Omega_Q$, and if $i < -Q$, to the leftmost cell $\Omega_{-Q}$. Thus the projection operation for the convolution result consists in summing the values outside the interval $[-Q\delta, Q\delta]$. The convolution $g \otimes h$, on the contrary, is not supported outside the interval $[-Q\delta, Q\delta]$ because $|a \sqcap b| \leq \min(|a|, |b|)$.

Denote by nearest($x$) the index of the cell $\Omega_i$ containing the point $x$. In other words, nearest($x$) is the index of the grid node closest to $x$. Then the approximate computation of convolutions $\ast$ and $\otimes$ described above corresponds to algorithms 1 and 2 respectively.

Since the grid is uniform, the convolution $f \leftarrow g \ast h$ from the first step of the algorithm 1 may be computed in $\mathcal{O}(Q \log_2 Q)$ operations using FFT. The rest of the algorithm takes only $\mathcal{O}(Q)$ operations, hence the overall complexity of the algorithm 1 is $\mathcal{O}(Q \log_2 Q)$ operations. The complexity of the algorithm 2 is $\mathcal{O}(Q^2)$, which is much worse.

Convolutions $\ast$ and $\otimes$ arise also in the problem of optimizing the weight distributions for rows and columns of the LDPC check matrix. Results from this area may be used for the design of fast version of the algorithm 2, namely the algorithm from [1]. It is based on the following inequalities for the quantity $k$ appearing in the line 4 of the algorithm 2

$$\min(|i|, |j|) - \left(\frac{\ln 2}{\delta} - \frac{1}{2}\right) < |k| < \min(|i|, |j|).$$

Also, $\text{sgn } k = \text{sgn } i \cdot \text{sgn } j$, i.e. the quantity $\text{sgn } i \cdot \text{sgn } j \cdot \min(|i|, |j|)$ estimates $k$ with an error not exceeding $M(\delta) = \left\lceil \frac{\ln 2}{\delta} - \frac{1}{2}\right\rceil$. This observation helps to reduce the complexity of the algorithm to $\mathcal{O}(Q \cdot M(\delta))$ operations. However taking finer grid makes $M(\delta)$ larger, and the speedup smaller. However the speedup is noticeable. Let $A = \delta Q$ be the rightmost grid node and $[-A, A]$ the segment containing all grid nodes. Typical values used in our numerical experiments were $A = 60$, $Q = 2^{13}$. In this case $\delta = \frac{A}{Q} \approx 0.007324$ and $M(\delta) = 95$.

Thus making the grid projection of the initial probability function $f_{i_0, n}$ and substituting approximations for the exact computations which use formulas (19) and (20) we obtain a numerical method for computation of probability errors $E_i$. While the accuracy analysis for this method remains an open question, our numerical experiments show that good accuracy can be achieved without refining the grid too much.

E. Performance analysis

Construction procedure described above implies that the polar code is built for a concrete channel. In practice, channel properties may change with time, therefore it is important to analyze the performance of the constructed code for channel models with different noise levels. For most modern coding systems, in particular for low-density parity check codes, the only available tool is the Monte-Carlo simulation.

For polar codes, such analysis is available in much less expensive way. To obtain the upper bound for the block error probability, one can compute the error probabilities $E_i$ by the method used in code construction and sum these quantities over indices of information bits. Numerical experiments show that this estimate is quite accurate.

F. Numerical experiments

It is instructive to check the quality of the estimates given by the described performance analysis method. To this end, one can compare the Monte-Carlo simulation results and the obtained estimate for some concrete code. Using random number generator, form “received” vector $y$ satisfying the channel model and decode it. For large number of trials $N_T$, the decoder will make $N_E$ errors. We can estimate the block error probability as follows,

$$P_E \approx \frac{N_E}{N_T}.$$ 

According to the central limit theorem, with the probability of some 95% this estimate belongs to the confidence interval of
Fig. 11. Performance of the polar code of length 1024, rate $\frac{1}{2}$ on binary symmetric channel estimated by Monte-Carlo simulations and proposed analysis method.

\[
\sigma^2 = P_E (1 - P_E)
\]

and while it is unknown, we can estimate it using the sample variance formula

\[
\sigma^2 \approx \frac{N_T}{N_T - 1} \cdot \frac{N_T}{N_E} \left( 1 - \frac{N_T}{N_E} \right)
\]

Thus the Monte-Carlo method has the accuracy of the order $N_T^{-\frac{1}{2}}$ which implies large computational costs. For $P_E \ll 1$, obtaining a 50% confidence estimate requires according to formula (22), some $4 \cdot 1.96^2 \cdot P_E^{-1}$ trials. For example, if $P_E = 10^{-7}$, one will need $15 \cdot 10^7$ trials. Further, with 10% confidence level the number of trials increases up to $100 \cdot 1.96^2 \cdot P_E^{-1}$. If $P_E = 10^{-7}$, this number will be approximately $3.8 \cdot 10^9$. Therefore in Monte-Carlo simulations we restrict the noise level to interval corresponding to $P_E$ exceeding $10^{-5}$.

For this experiment we constructed a polar code of length 1024 and rate $\frac{1}{2}$ for binary symmetric channel with error probability 0.06. Monte-Carlo simulation was run for binary symmetric channels with various error probabilities. Also, an estimate of block error probability was computed using the proposed analysis method. Obtained graphs are shown on fig. 11. One can see that the results produced independently in two different ways are very close.

Consider now a different channel model, an AWGN channel with binary input and additive normal noise. Output alphabet for this channel is the real axis, while the transition function has the form

\[
W(y \mid x) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp \left( -\frac{(y - (1 - 2x))^2}{2\sigma^2} \right), \quad x \in \{0, 1\}.
\]

In other words, transmission over such channel amounts to mapping input bits 0 and 1 to symbols 1 and -1, respectively, and adding afterwards normal noise with zero average and variance $\sigma^2$. Note that this channel has continuous output alphabet and does not fit to previous sections theory. However a similar numerical experiment is perfectly possible for a discrete approximation of this channel. Instead of $\sigma^2$, on the horizontal axis we plot the signal/noise ratio in decibels

\[
\text{SNR (dB)} = 10 \log_{10} \frac{1}{\sigma^2}.
\]

We constructed a polar code of length 1024 and rate $\frac{1}{2}$ for the noise level 3dB. Monte-Carlo simulation was run for various noise levels. Also, an estimate was computed using the proposed analysis method. The results are shown in fig. 12. One can see that the graphs again are almost identical.

Fig. 13 shows performance graphs for polar codes of rate $\frac{1}{2}$ and of lengths $2^{13} = 8192$, $2^{16} = 65536$ and $2^{18} = 262144$ for binary symmetric channel. For code rate $\frac{1}{2}$ and binary symmetric channel, the Shannon limit corresponds to $p = 0.11$. One can see that the convergence to Shannon limit is rather slow. Next section is devoted to generalization of polar codes which allows to increase the convergence rate.
V. POLARIZATION KERNELS

For the definition of polar codes, the following matrix was used in section III
\[ G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}. \]

One can use another invertible matrix \( G \) of arbitrary order \( l \).
Then the hierarchical graph construction will involve taking \( l \) copies of encoder graph, instead of doubling. New variables will be expressed in terms of old ones by the formula
\[
[u_{k+1}[l], u_{k+1}[j + 1], \ldots, u_{k+1}[l + l - 1]] = [u_k[l], u_k[l+1], \ldots, u_k[l+l-1]] \cdot G,
\]
and old variables in terms of new ones, by the formula
\[
[u_k[l], u_k[l+1], \ldots, u_k[l+l-1]] = [u_{k+1}[l], u_{k+1}[j + 1], \ldots, u_{k+1}[j+l-1]] \cdot G^{-1}.
\]

After \( n \) steps of graph construction, the code length will be \( l^n \) and the decoder graph will consist of \( n \) layers, each having \( J_k = l^k \) groups of \( S_k = l^{n-k} \) variables. The vector of output symbols \( Y_k[j] \) corresponding to group \( j \) of layer \( k \), still will consist of contiguous bits from index \( j \cdot S_k \) up to index \( (j+1) \cdot S_k - 1 \) inclusively. The problem of computation of quantities
\[
L(u_0^i[j]), L(u_0^{i+1}[j]), \ldots, L(u_0^{i+l-1}[j])
\]

using the values of
\[
L(u_{k+1}[l]), L(u_{k+1}[l+1]), \ldots, L(u_{k+1}[l+l-1])
\]
leads to a graph analogous to the shown in the fig. 9, this time isomorphic to the Tanner graph of the matrix \( G^{-1} \). In general this problem cannot be reduced to belief propagation algorithm working on a tree and requires exponential in \( l \) number of operations. Computation of \( L(u_{k+1}^{i+m}[j]) \) is always possible by enumeration of all possible events. For brevity, denote \( u_m = u_{k+1}^{i+m}[j], x_m = u_{k+1}[l+j+m], Y = Y_k[j], x = [x_0, x_1, \ldots, x_{l-1}] \).

Then
\[
L(u_m) = \ln \frac{\Pr(Y | u_m = 0; u_0, u_1, \ldots, u_{m-1})}{\Pr(Y | u_m = 1; u_0, u_1, \ldots, u_{m-1})}\]
(23)

Let \( X_a, a = 0, 1 \) be the set of all vectors \( x \) such that
\[
[u_0, u_1, \ldots, u_{m-1}, a, **] = xG^{-1},
\]
where \( ** \) stands for \( l - m - 1 \) arbitrary bits. Then
\[
\Pr(Y | u_m = a; u_0, u_1, \ldots, u_{m-1}) = \frac{1}{|X_a|} \sum_{x \in X_a} \Pr(Y | x) = \frac{1}{|X_a|} \sum_{x \in X_a} \prod_{t=0}^{l-1} \Pr(Y^t | x_t),
\]
where \( Y^t \) is component \( t \) of \( Y \). Inserting the last equality for \( a = 0, 1 \) into numerator and denominator of (23), we get
\[
L(u_m) = \ln \frac{\sum_{x \in X_0} \prod_{t=0}^{l-1} \Pr(Y^t | x_t)}{\sum_{x \in X_1} \prod_{t=0}^{l-1} \Pr(Y^t | x_t)},
\]
Dividing the numerator and denominator by \( \prod_{t=0}^{l-1} \Pr(Y^t | x_t) \), we get
\[
L(u_m) = \ln \frac{\prod_{x \in X_0} \Pr(Y^t | x_t)}{\prod_{x \in X_1} \Pr(Y^t | x_t)},
\]
where
\[
l(x_t) = \frac{\Pr(Y^t | x_t = 0)}{\Pr(Y^t | x_t = 1)} = e^{L(x_t)}.
\]
Using the last equality, write
\[
L(u_m) = \ln \frac{\prod_{x \in X_0} \exp(\sum_{t=0}^{l-1} (1 + x_t) L(x_t))}{\prod_{x \in X_1} \exp(\sum_{t=0}^{l-1} (1 + x_t) L(x_t))},
\]
(24)

This gives the recursive formula for computation of \( L(u_{k+1}^{i+m}[j]) \) in the case of arbitrary matrix \( G \), however involving sums with exponential in \( l \) number of terms.

A. Obtaining the recursive formulas

For some polarization kernels, the formulas (24) may be replaced by simpler relations containing familiar operations + and \( \circ \). For example, consider the matrix
\[
G_3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.
\]

Note that \( G_3^{-1} = G_3 \) and draw the Tanner graph analogous to the shown in the fig. 9 but corresponding to \( G_3^{-1} \) (fig. 14).

We now find the expression for \( L(u_k^{3q}[j]) \). The vertices \( u_k^{3q+1}[j], u_k^{3q+2}[j] \) may be removed from the graph together with incident equations. We obtain the graph shown in the fig. 15. Using results of section II one can write
\[
L(u_k^{3q}[j]) = L(u_{k+1}^{3q}[j]) \circ L(u_{k+1}^{3q}[j+1]) \circ L(u_{k+1}^{3q}[j+2]).
\]

In order to find \( L(u_k^{3q+1}[j]) \), the quantity \( u_k^{3q}[j] \) should be considered known exactly. We have to remove the vertex \( u_k^{3q+2}[j] \) from the graph in fig. 14 and the incident equation (fig. 16). From the last graph we conclude that
\[
L(u_k^{3q+1}[j]) = L(u_{k+1}^{3q}[j+1]) + (-1)^{u_k^{3q}[j]}(L(u_{k+1}^{3q}[j]) \circ L(u_{k+1}^{3q}[j+2])).
\]

Fig. 14. Graph for the recurrent relations for the matrix \( G_3 \)

Fig. 15. Graph from fig. 14 after removal of excessive vertices
Similarly we can write the third formula:
\[
L(u^q_{k+1}(j)) = L(u^q_{k+1}(3j+2)) + (-1)^{u^q_{k+1}(j+2)}L(u^q_{k+1}(3j)).
\]

In an analogous way, one can try to obtain recurrent formulas for an arbitrary polarization kernel. If the Tanner graph for some fixed index is not tree-like, one can try to amend it by adding some equations to other ones. Further, if a cycle contains an exactly known bit, the cycle can be broken by doubling the respective vertex. Unfortunately, starting from \( l = 5 \) the tree-like graph can be obtained only for some of polarization kernels.

Those kernels which admit simple formulas also admit simple code construction and analysis. For instance, in the example considered above the recurrent formulas for probability functions have the form
\[
\begin{align*}
 f_{k}^q &= f_{k+1}^q \boxtimes f_{k+1}^q \boxtimes f_{k+1}^q, \\
 f_{k}^{q+1} &= f_{k+1}^q \ast (f_{k+1}^q \boxtimes f_{k+1}^q), \\
 f_{k}^{q+2} &= f_{k+1}^q \ast f_{k+1}^q.
\end{align*}
\]

For polarization kernels which do not admit simple recurrent formulas, the problem of code construction is open. In general, the computation of probability functions \( f_k^q \) via \( f_{k+1}^q \) is a multidimensional summation problem. Possible solutions are Monte-Carlo method and approximations by normal distribution.

VI. CONCATENATED POLAR CODES

In this section we consider a method of performance improvement for polar codes in which short classic error correcting codes are used together with polar codes.

Let \( C_1, C_2, \ldots, C_q \) be a set of linear codes of equal length \( M \). Let \( K_i \) be the number of information bits in the code \( C_i \). Let \( V \) be some \( M \times N \) matrix each of whose elements is 0 or 1. Denote by \( v_{ij} \) with \( 0 \leq i < N \) and \( 0 \leq j < M \) the elements of \( V \), by \( v^j \) its row \( j \) and by \( v_i \) its column \( i \). For all 0 \( i = 0, N-1 \) choose some integer \( a_i \) in the range 1 to \( q \). We consider only such matrices \( V \) whose columns \( v_i \) are codewords of \( C_{a_i} \), i.e.
\[
v_i \in C_{a_i}, \quad i = 0, N-1.
\]

Consider an arbitrary polar code of length \( N \) and rate 1, i.e. without redundancy, with matrix generator \( G \in \text{GF}(2)^{N \times N}. \) Encode each row of \( V \) with this polar code obtaining a new matrix \( X \in \text{GF}(2)^{M \times N}: \)
\[
X = VG.
\]

If the matrix \( X \) is “reshaped” into a row, one can consider the set of all such possible rows subject to restriction \( (25) \) as a linear code of length \( M \cdot N \) and rate
\[
\frac{K}{MN} = \frac{1}{MN} \sum_{i=0}^{N-1} K_{a_i}.
\]

Thus obtained linear code we will call the concatenated polar code. Let \( Y \) be the matrix received after the transmission of \( X \) through the channel and let \( y^j \) be its row \( j \). The decoder works by applying alternatively the steps of successive cancellation method for rows of \( Y \) and maximum likelihood decoder for its columns.

In order to decode the column \( v_0 \), compute for each row of \( Y \) independently the logarithmic likelihood ratios
\[
L(v_{j,0}) = \ln \frac{\Pr\{y_{j,0} = 0\}}{\Pr\{y_{j,0} = 1\}},
\]
just like in the usual successive cancellation method. Then the values \( L(v_{j,0}), j = 0, M - 1 \) gathered in a vector \( y \) are given as input to ML-decoder for the code \( C_{a_0} \). The most likely codeword \( w \in C_{a_0} \) produced on output is taken as an estimate of \( v_0 \).

Next we compute the estimate of \( v_1 \). Assuming \( v_0 \) already known, again compute for each row independently the LLRs
\[
L(v_{j,1}) = \ln \frac{\Pr\{y_{j,1} = 0; v_{j,0}\}}{\Pr\{y_{j,1} = 1; v_{j,0}\}},
\]
concatenate the values \( L(v_{j,1}) \) into a vector \( y \), which will be the input of ML-decoder for the code \( C_{a_1} \). The obtained codeword is taken as an estimate of \( v_1 \). Next, assuming \( v_0 \) and \( v_1 \) exactly known, compute the estimate of \( v_2 \) etc.

Note that the polar codes are a special case of concatenated polar codes for \( M = 1, q = 2 \) and \( C_1 = \{0\}, C_2 = \{0, 1\} \). In this case, bit \( i \) is frozen if \( a_i = 1 \), and it is information bit, if \( a_i = 2 \).

A. Code construction and analysis

Let \( E_i \) be the error probability for estimation of column \( i \) under the constraint that all previous columns were estimated error-free. Write again the upper bound for block error probability:
\[
P_E \leq \sum_{i=0}^{N-1} E_i. \tag{27}
\]

Fix some symmetric channel \( W \), set of codes \( C_1, \ldots, C_q \) of length \( M \), polar code of length \( N \) and rate 1. We require to construct a concatenated polar code of given rate \( k/N \), i.e. choose numbers \( a_0, a_1, \ldots, a_{N-1} \) such that
\[
\sum_{i=0}^{N-1} K_{a_i} = K. \tag{28}
\]

We will choose these numbers so as to minimize the upper bound \( (27) \). Denote by \( E_i^K \) the error probability for estimation of the column \( v_i \) under the constraint that all previous columns were estimated error-free and \( a_i = k \). Note that \( E_i^K \) does
not depend on \( a_j \) for all \( j \neq i \). For a concrete choice of \( a_0, a_1, \ldots, a_{N-1} \) we can write the following upper bound for \( P_E \),

\[
P_E \leq \sum_{i=0}^{N-1} E_i^{a_i}. \tag{29}
\]

Assume for now that for all \( i = 0, N-1 \) and \( k = 0, q \) we can compute \( E_i^k \). In order to choose the optimal \( a_0, a_1, \ldots, a_{N-1} \), we will use the dynamic programming method. Let \( F(s, t) = 0, N-1, t = 0, K \) be the minimal possible value of the sum

\[
\sum_{i=0}^{s} E_i^{a_i}
\]

under the constraint

\[
\sum_{i=0}^{s} K_{a_i} = t, \tag{30}
\]

or let \( F(s, t) = +\infty \), if there is no sets of \( a_i \) satisfying (30). For convenience, set \( F(s, t) = +\infty \) for \( t < 0 \). It is easy to note that

\[
F(s, t) = \min_i \{ F(s - 1, t - K_i) + E_i^k \}. \tag{31}
\]

Introduce the notation

\[
A(s, t) = \arg \min_i \{ F(s - 1, t - K_i) + E_i^k \}.
\]

To make the formula (31) correct also for \( s = 0 \), let

\[
F(-1, 0) = 0, \quad F(-1, t) = +\infty, \quad t \neq 0.
\]

Now using (31) for sequential computation of \( F(s, t) \) for \( s = 0, 1, 2, \ldots \) and all \( t \), and saving the corresponding quantities \( A(s, t) \), one can compute \( F(N - 1, K) \), which by definition is the minimal possible value of the sum (29) under the constraint (28). If \( F(N - 1, K) = +\infty \), there is no set of \( a_0, a_1, \ldots, a_{N-1} \) satisfying the constraint (28).

Let \( F(N - 1, K) \neq +\infty \). In order to recover the sequence \( a_0, a_1, \ldots, a_{N-1} \) delivering the minimum to the sum (29), we make a pass in reverse order utilizing the saved quantities

\[
A(s, t),
\]

\[
\begin{align*}
a_{N-1} &= A(N - 1, K), \\
a_{N-2} &= A(N - 2, K - K_{a_{N-1}}), \\
a_{N-3} &= A(N - 3, K - K_{a_{N-1}} - K_{a_{N-2}}), \\
&\vdots \\
a_i &= A(i, K - \sum_{l=i+1}^{N-1} K_{a_l}), \\
&\vdots \\
a_0 &= A(0, K - \sum_{l=1}^{N-1} K_{a_l}).
\end{align*}
\]

\(^4\)If the minimum is attained for several values of \( l \), any of those can serve as \( A(s, t) \).

Now we get back to the problem of estimating \( E_i^k \). Since the channel is symmetric and the code is linear, we assume the all-zero codeword is sent. Suppose that the columns \( v_0, v_1, \ldots, v_{l-1} \) have been estimated correctly and the decoder is to estimate \( v_l \). Next the ML-decoder for the code \( C_k \) takes input the vector

\[
\lambda = [L(v_{0,i}), L(v_{1,i}), L(v_{2,i}), \ldots, L(v_{M-1,i})].
\]

For convenience, introduce the notation \( \lambda_j \equiv L(v_{j,i}) \). The components of \( \lambda \) are i.i.d. random variables. Their probability function (or pdf) \( f \) can be computed approximately using the method described in section [IV] We can assume that the column \( v_i \) is transmitted via some symmetric channel with LLR distribution \( f_i \). Thus the problem of computing \( E_i^k \) is reduced to the estimation of error probability for the ML-decoder on a channel with given probability function \( f_i \). As stated in the introduction, the ML-decoder minimizes the linear functional

\[
\phi(c) = \sum_{j=0}^{M-1} c_j \lambda_j,
\]

where \( c = [c_0, c_1, \ldots, c_{M-1}] \) runs over all codewords of the code \( C_k \). For the all-zero codeword the functional \( \phi \) is zero. Hence if the decoding error occurs, there necessarily exists some codeword \( c' \) such that \( \phi(c') \leq 0 \). The last inequality can be rewritten as the sum of \( w_H(c') \) terms,

\[
\sum_{j \in \text{supp} c'} \lambda_j \leq 0.
\]

Some nonzero codeword \( c' \) will be strictly more preferable than \( 0 \) if \( \phi(c') < 0 \) and in this case the decoder error will surely occur. If \( \phi(c') = 0 \), the decoder may choose the correct codeword among those which zero the functional \( \phi \). For simplicity assume that \( \phi(c') = 0 \) also implies the decoder error. Write the probability of the event that for a fixed \( c' \) the inequality \( \phi(c') \leq 0 \) holds as

\[
\Pr \left\{ \sum_{j \in \text{supp} c'} \lambda_j \leq 0 \right\}.
\]

The sum consists of \( w_H(c') \) i.i.d. random variables with the probability function \( f_i \), therefore the probability function of the sum is

\[
f_i^{w_H(c')} \equiv f_i * f_i * \ldots * f_i
\]

\( w_H(c') \) times

It follows that the probability of the event \( \phi(c') \leq 0 \) depends only on the weight \( w \) of the codeword \( c' \) and it can be written as

\[
P(f, w) = \sum_{x \in \text{supp} f^w, x \leq 0} f^w(x).
\]

The main contribution in the error probability is made by codewords of minimal weight. Let \( d_k^* \) be the code distance of the code \( C_k \), and let \( m_k \) be the number of different codewords
of weight $d_k$ in the code $C_k$. Then the probability $E_i^k$ may be estimated as

$$E_i^k \approx m_k \cdot P(f_i, d_k). \quad (32)$$

Experiments show that this value is likely to overestimate the real error probability (computed by a Monte-Carlo simulation) by a constant factor which does not depend on the channel. For this reason in experiments reported in this paper a simple empiric technique was used to correct the multiplier $m_k$. Each of the codes $C_1, C_1, \ldots, C_q$ was simulated on an AWGN channel with different SNR ratios to obtain its FER curve. The number $m_k$ was chosen so that the estimate (32) fitted the experimental curve best. We do not have a theoretical justification of this procedure, however, the results of numerical experiments show its high accuracy.

In a similar way one can estimate the FER of a concrete concatenated polar code on a given channel. It is sufficient to approximate numerically the sum

$$\sum_{i=0}^{N-1} E_{i}^{a_i} \quad (33)$$

and take it as an upper bound for block error rate.

### B. Numerical experiments

For the construction of concatenated polar codes consider a set of 26 different codes of length 32. Numbers of information bits $K_i$ and code distances $d_i$ for each code are given in the table I.

| $i$ | $K_i$ | $d_i$ |
|-----|-------|-------|
| 0   | 1     | 32    |
| 1   | 2     | 31    |
| 2   | 3     | 30    |
| 3   | 4     | 29    |
| 4   | 5     | 28    |
| 5   | 6     | 27    |
| 6   | 7     | 26    |
| 7   | 8     | 25    |

**Table I**

Code distances of the codes of length 32 used for the construction of concatenated polar codes

The first interesting question is the accuracy of the block error estimate (33) which is computed approximately. In the fig. 17 we show the performance graph of the concatenated polar code of length 1024 and rate $\frac{1}{2}$ on an AWGN channel. The code was constructed for the channel with SNR = 2.5dB. The solid curve represents the Monte-Carlo estimate, the dotted curve is the estimate (33) computed approximately. One can see that the curves are practically identical within the limit of applicability of the Monte-Carlo method.

Fig. 17. Performance of the concatenated polar code of length 1024 = $32 \times 32$ and rate $\frac{1}{2}$ on an AWGN channel estimated using Monte-Carlo method and using the proposed method

Fig. 18. Comparison of performance of usual polar code and concatenated polar code of length 1024 and rate $\frac{1}{2}$ on an AWGN channel

Fig. 19. Comparison of performance of usual polar code and concatenated polar code of length 8192 and rate $\frac{1}{2}$ on an AWGN channel
It is also interesting to compare the performance of polar code and of concatenated polar code of the same length and rate. Fig. 18 shows the performance graph of the concatenated polar code of length 1024 and rate $\frac{1}{2}$ which was already presented above together with the polar code of the same length and rate. Both codes were constructed for an AWGN channel with \( \text{SNR} = 2.5 \text{dB} \). One can see that the concatenated code outperforms the usual one by an order of magnitude. The figures 19 and 20 show analogous comparative graphs for the codes of length 8192 and rates $\frac{1}{2}$ and $\frac{3}{4}$, respectively. Similar to the previous example, concatenated polar codes also outperform the usual ones approximately by an order of magnitude.

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