GEOMETRY OF BI-WARPED PRODUCT SUBMANIFOLDS IN SASAKIAN MANIFOLDS

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Abstract. In this paper, we study bi-warped product submanifolds of Sasakian manifolds, which are the natural generalizations of single warped products and Riemannian products. We show that if $M$ is a bi-warped product submanifold of the form $M = N_T \times f_1 N_L \times f_2 N_\theta$ of a Sasakian manifold $\tilde{M}$, where $N_T$, $N_L$ and $N_\theta$ are invariant, anti-invariant and proper pointwise slant submanifolds of $\tilde{M}$, respectively then the second fundamental form of $M$ satisfies a general inequality:

$$\|h\|^2 \geq 2n_1(\|\bar{\nabla}(\ln f_1)\|^2 + 1) + 2n_2(1 + 2\cot^2 \theta)\|\bar{\nabla}(\ln f_2)\|^2 + 2n_3(1 + \cos^4 \theta),$$

where $n_1 = \dim(N_L)$, $n_2 = \dim(N_\theta)$ and $h$ is the second fundamental form and $\nabla(\ln f_i)$ are the gradient components along $N_L$ and $N_\theta$, respectively. Some applications of this inequality are given and we provide some non-trivial examples of bi-warped products.

1. Introduction

Bi-warped product submanifolds are natural generalizations of warped product submanifolds and Riemannian product manifolds, which are introduced by B.-Y. Chen and F. Dillen [9] as a multiply CR-warped product submanifold $M = N_T \times f_i \times N$ in an arbitrary Kaehler manifold $\tilde{M}$, where $N = f_1 N_L \times f_2 N_{L_2} \times \cdots \times f_k N_{L_k}$, a product of $k$-totally real submanifolds and $N_T$ is a holomorphic submanifold of $\tilde{M}$. They obtained a sharp inequality $\|h\|^2 \geq 2\sum_{i=1}^k n_i \|\nabla(\ln f_i)\|^2$, where $n_i = \dim(N_L)$, for each $i = 1, \ldots, k$ and discussed the equality case of this inequality, also provided some non-trivial examples to illustrate the inequality.

Recently, we studied bi-warped product submanifolds of locally product Riemannian manifolds [21] and bi-warped product submanifolds of Kenmotsu manifolds [35]. We provided non-trivial examples of bi-warped products in locally product Riemannian manifolds as well as Kenmotsu manifolds. We have obtained a sharp inequality for each case and discussed the equality cases. Bi-warped submanifolds were also appeared in (for instance, see [1, 28]). Notice that bi-warped products are spacial classes of multiply warped products which were introduced by S. Nolker [23], B. Unal [37] and B.-Y. Chen and F. Dillen [9].

In this paper, we studied bi-warped product submanifolds of the form $N_T \times f_1 N_L \times f_2 N_\theta$ of a Sasakian manifold $\tilde{M}$, where $N_T$, $N_L$ and $N_\theta$ are invariant, anti-invariant and pointwise slant submanifolds of $\tilde{M}$, respectively. As a main result, we establish a sharp relationship for the squared norm of the second fundamental form.

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in terms of the warping functions. The equality is discussed and some applications of this inequality are given. Further, we provide two non-trivial examples of bi-warped product submanifolds.

The paper is organised as follows: Section 2 is concerned with some basic notations, formulas, definitions and necessary preliminaries results. In Section 3 we recall some basic facts about bi-warped products. In this section, we derive some useful results on bi-warped products which are useful to the next section. In Section 4, we establish a sharp inequality and discuss the equality case. In Section 5, we give some applications of our results. In Section 6, we provide some non-trivial examples.

2. Preliminaries

A \((2m+1)\)-dimensional Riemannian manifold \((\tilde{M}^{2m+1}, g)\) is said to be a Sasakian manifold if it admits a \((1, 1)\) tensor field \(\varphi\) on its tangent bundle \(TM^{2m+1}\), a vector field \(\xi\) and a 1-form \(\eta\), satisfying:

\[
\begin{align*}
\varphi^2 &= -\text{Id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi \xi = 0, \quad \eta \circ \varphi = 0, \\
g(\varphi X, \varphi Y) &= g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(\varphi X, \xi), \\
(\tilde{\nabla}_X \varphi)Y &= g(X, Y)\xi - \eta(Y)X, \quad \tilde{\nabla}_X \xi = -\varphi X,
\end{align*}
\]

(2.1)

for any vector fields \(X, Y\) on \(\tilde{M}^{2m+1}\), where \(\tilde{\nabla}\) denotes the Riemannian connection with respect to the Riemannian metric \(g\).

Let \(M^n\) be an \(n\)-dimensional submanifold of a Sasakian manifold \(\tilde{M}^{3m+1}\) such that the structure vector field \(\xi\) tangent to \(M^n\) with induced metric \(g\). Let \(\Gamma(TM^n)\) be the Lie algebra of vector fields of \(M^n\) in \(\tilde{M}^{2m+1}\) and \(\Gamma(T^\perp M^{2m+1-n})\), set of all vector fields normal to \(M^n\). Then, the Gauss and Weingarten formulas are given respectively by (see, for instance, \([6, 13, 38]\))

\[
\begin{align*}
\tilde{\nabla}_X Y &= \nabla_X Y + h(X, Y), \\
\tilde{\nabla}_X N &= -A_N X + \nabla^\perp_X N,
\end{align*}
\]

(2.2) \hspace{2cm} (2.3)

for any vector fields \(X, Y \in \Gamma(TM^n)\) and \(N \in \Gamma(T^\perp M^{2m+1-n})\), where \(\nabla\) and \(\nabla^\perp\) are the induced connections on the tangent and normal bundles of \(M^n\), respectively, and \(h\) denotes the second fundamental form, \(A\) the shape operator of the submanifold. The second fundamental form \(h\) and the shape operator \(A\) are related by (see, \([10, 15, 38]\))

\[
g(h(X, Y), N) = g(A_N X, Y).
\]

(2.4)

Let \(p \in M^n\) and \(e_1, \cdots, e_n, e_{n+1}, \cdots, e_{2m+1}\) are orthonormal basis of the tangent space \(M^{2m+1}\) such that restricted to \(M^n\), the vectors \(e_1, \cdots, e_n\) are tangent to \(M^n\) at \(p\) and hence \(e_{n+1}, \cdots, e_{2m+1}\) are normal to \(M^n\). Let \(\{h^r_{ij}\}, i, j = 1, \cdots, n; r = n + 1, \cdots, 2m+1\), denote the coefficients of the second fundamental form \(h\) with respect to the local frame field. Then, we have

\[
h^r_{ij} = g(h(e_i, e_j), e_r) = g(A_{e_r} e_i, e_j), \quad \|h\|^2 = \sum_{i,j=1}^n g(h(e_i, e_j), h(e_i, e_j)).
\]

(2.5)
The mean curvature vector $H$ is defined by $H = \frac{1}{n} \text{trace } h = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i)$. A submanifold $M$ is called totally geodesic, if $h = 0$; totally umbilical if $h(X, Y) = g(X, Y)H$ and minimal, if $H = 0$.

For submanifolds tangent to the structure vector field $\xi$, there are different classes of submanifolds depend on the behaviour the tangent spaces under the action of $\varphi$. Some of them, we mention the following:

(i) A submanifold $M^{2n+1}$ tangent to $\xi$ is called an invariant submanifold if $\varphi$ preserves any tangent space of $M^{2n+1}$, that is, $\varphi(T_pM^{2n+1}) \subseteq T_pM^{2n+1}$, for any $p \in M^{2n+1}$.

(ii) A submanifold $M^n$ tangent to $\xi$ is called an anti-invariant submanifold if $\varphi$ maps any tangent space of $M^n$ into the normal space, that is, $\varphi(T_pM^n) \subseteq T_pM^{2n+1} - n$, for any $p \in M^n$.

(iii) A submanifold $M^{2n+1}$ tangent to $\xi$ is called a slant submanifold if for any vector field $X \in T_pM^{2n+1}$, which is linearly independent to $\xi_p$, the angle between $\varphi X$ and $T_pM^{2n+1}$ is constant, which is independent of the choice of $p \in M^{2n+1}$ and $X \in T_pM^{2n+1}$, for any $p \in M^{2n+1}$.

For any $X \in \Gamma(TM^n)$, we write $\varphi X = TX + FX$, where $TX$ is the tangential component of $\varphi X$ and $FX$ is the normal component of $\varphi X$. If $F = 0$, then $M^n$ is an invariant submanifold and if $T = 0$, then it is anti-invariant. Similarly, for any vector field $N$ normal to $M^n$, we put $\varphi N = tN + fN$, where $tN$ and $fN$ are the tangential and normal components of $\varphi N$, respectively.

As an extension of slant submanifolds, F. Etayo [17] introduced the notion of pointwise slant submanifolds under the name of quasi-slant submanifolds. Later, these submanifolds of almost Hermitian manifolds were studied by B.-Y. Chen and O.J. Garay in [11]. On the similar line of B.-Y. Chen, the pointwise slant and pointwise semi-slant submanifolds of Sasakian manifolds were appeared in (for instance, see [24, 36]).

A submanifold $M^n$ of an almost contact metric manifold $\tilde{M}^{2m+1}$ is said to be pointwise slant, if for a nonzero vector $X \in T_pM^n$ at $p \in M$, which is linearly independent to $\xi_p$, the angle $\theta(X)$ between $\varphi X$ and $T^n M = T_pM^n - \{0\}$ is independent of the choice of nonzero vector $X \in T^n M$. In this case, $\theta$ can be regarded as a function on $M^n$, which is called the slant function of the pointwise slant submanifold. Notice that a pointwise slant submanifold $M$ is slant, if its slant function $\theta$ is globally constant on $M$. Moreover, invariant and anti-invariant submanifolds are pointwise slant submanifolds with slant function $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A pointwise slant submanifold is proper if it is neither invariant nor anti-invariant. Non-trivial examples of pointwise slant submanifolds are given in (see, [36, 24]).

Now, we recall the following useful characterization given in [36].

**Theorem 2.1.** Let $M^n$ be a submanifold of an almost contact metric manifold $\tilde{M}^{2m+1}$ such that $\xi \in \Gamma(TM^n)$. Then, $M^n$ is pointwise slant if and only if

$$T^2 = \cos^2 \theta \left( -\text{Id} + \eta \otimes \xi \right),$$

(2.6)

for some real valued function $\theta$ defined on the tangent bundle $TM^n$ of $M^n$.

Following relations are straightforward consequence of (2.6)

$$g(TX, TY) = \cos^2 \theta |g(X, Y) - \eta(X)\eta(Y)|,$$

(2.7)
(2.8) \[ g(FX,FY) = \sin^2 \theta [g(X,Y)\eta(X)\eta(Y)] , \]

for any vector fields \( X, Y \) tangent to \( M^n \).

Also, for a pointwise slant submanifold, we have the following useful relations.

(2.9) \[ tFX = \sin^2 \theta (-X + \eta(X)\xi) , \quad fFX = -FTX \]

for any \( X \in \Gamma(TM^n) \).

3. SOME RESULTS ON BI-WARPED PRODUCTS

Let \( N_1, N_2, N_3 \) be Riemannian manifolds and let \( M = N_1 \times_{f_1} N_2 \times_{f_2} N_3 \) be the product manifold of \( N_1, N_2, M_3 \) such that \( f_1, f_2 : N_i \to \mathbb{R}^+ \) are positive real valued functions. For each \( i \), denote by \( \pi_i : M \to N_i \) the canonical projection of \( M \) onto \( N_i \), \( i = 1, 2, 3 \). Then, the metric on \( M \), called a bi-warped product metric is given by

\[
g(X,Y) = g(\pi_1 X, \pi_1 Y) + (f_1 \circ \pi_1)^2 g(\pi_2 X, \pi_2 Y) + (f_2 \circ \pi_1)^2 g(\pi_3 X, \pi_3 Y)
\]

for any \( X, Y \) tangent to \( M \) and \(*\) denotes the symbol for tangent maps. The product manifold \( M \) endowed with this metric denoted by \( \theta \) is a bi-warped product manifold. In this case, \( f_1, f_2 \) are non-constant functions, called warping functions on \( M \). It is clear that if both \( f_1, f_2 \) are constant on \( M \), then \( M \) is simply a Riemannian product manifold and if anyone of the function is constant, then \( M \) is a single warped product manifold. Also, if neither \( f_1 \) nor \( f_2 \) is constant, then \( M \) is a proper bi-warped product manifold.

Remark 3.1. If \( M = B \times_{f} F \) is a warped product manifold, then, \( B \) is totally geodesic in \( M \) and \( F \) is totally umbilical in \( M \) (for instance, see, [2, 6]).

Let \( M = N_1 \times_{f_1} N_2 \times_{f_2} N_3 \) be a bi-warped product submanifold of a Riemannian manifold \( \bar{M} \). Then, we have

(3.1) \[ \nabla_X Z = \sum_{i=1}^{2} \langle X(\ln f_i) \rangle Z^i \]

for any \( X \in \mathfrak{D}_1 \), the tangent space of \( N_1 \) and \( Z \in \Gamma(TN) \), where \( N = N_1 \times_{f_2} N_2 \times_{f_3} N_3 \) and \( Z^j \) is \( N_i \)-component of \( Z \) for each \( i = 2, 3 \) and \( \nabla \) is the Levi-Civita connection on \( M \) (for instance, see [32]).

In this paper, we study bi-warped product submanifolds \( N_{T}^{2p+1} \times_{f_1} N_{1}^{n_1} \times_{f_2} N_{\theta}^{n_2} \) of a Sasakian manifold \( \bar{M}^{2m+1} \), where \( N^{2p+1} \) is a \( (2p+1) \)-dimensional invariant submanifold tangent to the structure vector field \( \xi \), \( N_{1}^{n_1} \), a \( n_1 \)-dimensional anti-invariant submanifold and \( N_{\theta}^{n_2} \) is a \( n_2 \)-dimensional pointwise slant submanifold. Throughout this paper, we denote the tangent spaces of \( N_T, N_1 \) and \( N_\theta \) by \( \mathfrak{D}, \mathfrak{D}^\perp \) and \( \mathfrak{D}^\theta \), respectively.

Let \( M^n = N_{T}^{2p+1} \times_{f_1} N_{1}^{n_1} \times_{f_2} N_{\theta}^{n_2} \) be a bi-warped product submanifold of a Sasakian manifold \( \bar{M}^{2m+1} \). Then, the tangent and normal spaces of \( M^n \), respectively are given by

(3.2) \[ TM^n = \mathfrak{D} \oplus \mathfrak{D}^\perp \oplus \mathfrak{D}^\theta \oplus \langle \xi \rangle \]

and

(3.3) \[ T^\perp M^n = \varphi \mathfrak{D}^\perp \oplus F \mathfrak{D}^\theta \oplus \mu, \]
where $\mu$ is the invariant normal subbundle of $T^\perp M^n$.

First, we have the following result.

**Theorem 3.1.** Let $M^n = N_T \times f_1 N_\perp \times f_2 N_\theta$ be a bi-warped product submanifold of a Sasakian manifold $\tilde{M}^{2m+1}$ such that $N_T$, $N_\perp$ and $N_\theta$ are invariant, anti-invariant and proper pointwise slant submanifolds of $\tilde{M}^{2m+1}$, respectively. Then, we have

(i) $f_1$ is constant on $M^n$, if $\xi \in \Gamma(\mathcal{D}^\perp)$;
(ii) $f_2$ is constant on $M^n$, if $\xi \in \Gamma(\mathcal{D}^\theta)$.

**Proof.** If $\xi$ is tangent to $N_\perp$, then for any $X \in \Gamma(\mathcal{D})$, we have $\tilde{\nabla}_X \xi = -\varphi X$. Using (2.2) and (3.1), we derive $X(\ln f_1)\xi = -\varphi X$. Taking the inner product with $\xi$, we get $X \ln f_1 = 0$, i.e., $f_1$ is constant. Similarly, when $\xi$ is tangent to $N_\theta$, we find that $f_2$ is constant, hence the result is proved. \hfill \Box

From the above theorem it is clear that there do not exist any proper bi-warped product submanifolds of the form $N_T \times f_1 N_\perp \times f_2 N_\theta$ in Sasakian manifolds, if the structure vector field $\xi$ is tangent to any fiber.

Now, we consider the structure vector field $\xi \in \Gamma(\mathcal{D})$ and it is noted that from (2.1) and (3.1), we have

(3.4) \hspace{1cm} \xi(\ln f_1) = 0 = \xi(\ln f_2)

Next, we obtain the following result.

**Lemma 3.1.** Let $M^n = N_T \times f_1 N_\perp \times f_2 N_\theta$ be a bi-warped product submanifold of a Sasakian manifold $\tilde{M}^{2m+1}$ such that $N_T$, $N_\perp$ and $N_\theta$ are invariant, anti-invariant and proper pointwise slant submanifolds of $\tilde{M}^{2m+1}$, respectively. Then, we have

(i) $g(h(X,Y),\varphi Z) = 0 = g(h(X,Y), FV)$
(ii) $g(h(X,Z), \varphi W) = -\varphi X(\ln f_1) + \eta(X)) g(Z,W)$,\hspace{1cm} (iii) $g(h(X,U), FV) = -X(\ln f_2) g(U,TV) - (\varphi X(\ln f_2) + \eta(X)) g(U,V)$

for any $X, Y \in \Gamma(\mathcal{D}), Z, W \in \Gamma(\mathcal{D}^\perp)$ and $U, V \in \Gamma(\mathcal{D}^\theta)$.

**Proof.** For any $X, Y \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\perp)$, we have

$g(h(X,Y), \varphi Z) = g(\tilde{\nabla}_X Y, \varphi Z) = g((\tilde{\nabla}_X \varphi)Y, Z) - g(\tilde{\nabla}_X \varphi Y, Z)$.

Using (2.1) and the fact that $N_T$ is totally geodesic in $M^n$ (see, Remark 3.1) with the orthogonality of vector fields, we find that $g(h(X,Y), \varphi Z) = 0$, which is the first equality of (i). Similarly, we can prove the second equality of (i). For (ii), we have

$g(h(X,Z), \varphi W) = g(\tilde{\nabla}_X Z, \varphi W) = g((\tilde{\nabla}_X \varphi)Z, W) - g(\tilde{\nabla}_X \varphi Z, W)$.

Using (2.1) and (3.1) with the orthogonality of vector fields, we get (ii). In the similar fashion, we can derive (iii). \hfill \Box

Particularly, if $X = \xi$ in Lemma 3.1 (ii), then we have

(3.5) \hspace{1cm} g(h(\xi, Z), \varphi W) = -g(Z, W).

Similarly, if $X = \xi$ in Lemma 3.1 (iii), then by using (3.4), we derive

(3.6) \hspace{1cm} g(h(\xi, U), FV) = -g(U, V).
Interchanging $X$ by $\varphi X$, for any $X \in \Gamma(\mathcal{D})$ in Lemma 3.1 (ii) and using (3.4), we find
\begin{equation}
(3.7) \quad g(h(\varphi X, Z), \varphi W) = X(ln f_1)g(Z, W).
\end{equation}
Similarly, if we interchange $X$ by $\varphi X$ and $U$ by $TU$ and $V$ by $TV$ in Lemma 3.1 (iii), we derive the following useful relations
\begin{align}
(3.8) & \quad g(h(\varphi X, U), FV) = X(ln f_2)g(U, V) - \varphi X(ln f_2)g(U, TV), \\
(3.9) & \quad g(h(X, TU), FV) = (\varphi X(ln f_2) + \eta(X))g(U, TV) - \cos^2 \theta X(ln f_2)g(U, V), \\
(3.10) & \quad g(h(X, U), FTV) = \cos^2 \theta X(ln f_2)g(U, V) - (\varphi X(ln f_2) + \eta(X))g(U, TV) \\
(3.11) & \quad g(h(\varphi X, TU), FV) = -X(ln f_2)g(U, TV) - \cos^2 \varphi X(ln f_2)g(U, V), \\
(3.12) & \quad g(h(\varphi X, U), FTV) = X(ln f_2)g(U, TV) + \cos^2 \varphi X(ln f_2)g(U, V), \\
(3.13) & \quad g(h(X, TU), FTV) = -\cos^2 \theta (\varphi X(ln f_2) + \eta(X))g(U, V), \\
(3.14) & \quad g(h(\varphi X, TU), FTV) = \cos^2 \theta X(ln f_2)g(U, V) - \cos^2 \varphi X(ln f_2)g(U, TV).
\end{align}

**Lemma 3.2.** Let $M^n = N_T \times f_1, N_\perp \times f_2 N_\theta$ be a bi-warped product submanifold of a Sasakian manifold $M^{2m+1}$. Then, we have
\begin{equation}
(3.15) \quad g(h(X, Z), FV) = g(h(X, V), \varphi Z) = 0,
\end{equation}
for any $X \in \Gamma(\mathcal{D})$, $Z \in \Gamma(\mathcal{D}^\perp)$ and $V \in \Gamma(\mathcal{D}^\theta)$.

**Proof.** For any $X \in \Gamma(\mathcal{D})$, $Z \in \Gamma(\mathcal{D}^\perp)$ and $V \in \Gamma(\mathcal{D}^\theta)$, we have
\begin{equation}
\begin{split}
g(h(X, Z), FV) &= g(\overline{\nabla}_X Z - \nabla_X Z, \varphi V - TV) \\
&= g((\overline{\nabla}_X \varphi) Z, V) - g(\overline{\nabla}_X \varphi Z, V) - X(ln f_1)g(Z, TV).
\end{split}
\end{equation}
Then, from (3.1), (2.3) and the orthogonality of vector fields, we get the first equality of (3.15). For, the second equality, we have
\begin{equation}
\begin{split}
g(h(X, Z), FV) &= g(\overline{\nabla}_Z X - \nabla_Z X, \varphi V - TV) \\
&= g((\overline{\nabla}_Z \varphi) X, V) - g(\overline{\nabla}_Z \varphi X, V) - X(ln f_1)g(Z, TV) \\
&= -\eta(X)g(Z, V) + \varphi X(ln f_2)g(Z, V) - X(ln f_1)g(Z, TV).
\end{split}
\end{equation}
By the orthogonality of the distributions, all terms in the right hand side are identically zero and hence, we achieve the second equality. \hfill \square

**Theorem 3.2.** Let $M^n = N_T \times f_1, N_\perp \times f_2 N_\theta$ be a bi-warped product submanifold of a Sasakian manifold $M^{2m+1}$ such that $h(\mathcal{D}, \mathcal{D}^\perp) \perp \mu$. Then, $M^n$ is single warped product, i.e., $f_1$ is constant on $M$ if and only if $M^n$ is $\mathcal{D} - \mathcal{D}^\perp$ mixed totally geodesic in $M^{2m+1}$. 
Proof. If $M^n$ is $\mathcal{D} - \mathcal{D}^{\perp}$ mixed totally geodesic, then, from (3.7), we find that $f_1$ is constant on $M^n$.

Conversely, if $f_1$ is constant on $M^n$, then from Lemma 3.2 we have

(3.16) \quad g(h(\mathcal{D}, \mathcal{D}^{\perp}), F\mathcal{D}^\theta) = 0.

On the other hand, from (3.7), we find

(3.17) \quad g(h(\mathcal{D}, \mathcal{D}^{\perp}), \varphi\mathcal{D}^{\perp}) = 0.

From the hypothesis of the theorem, we have

(3.18) \quad g(h(\mathcal{D}, \mathcal{D}^{\perp}), \mu) = 0.

Then, from (3.16)-(3.18), we derive

(3.19) \quad g(h(\mathcal{D}, \mathcal{D}^{\perp}), N) = 0, \quad \forall N \in \Gamma(T^{\perp}M^n),

that is $M^n$ is $\mathcal{D} - \mathcal{D}^{\perp}$ mixed totally geodesic. Hence, the theorem proof is complete. □

**Theorem 3.3.** Let $M^n = N_T \times f_1 N_\perp \times f_2 N_\theta$ be a bi-warped product submanifold of a Sasakian manifold $\tilde{M}^{2m+1}$ such that $h(\mathcal{D}, \mathcal{D}^\theta) \perp \mu$. Then, $M^n$ is again a single warped product submanifold, i.e., $f_2$ is constant on $M$ if and only if $M^n$ is a $\mathcal{D} - \mathcal{D}^\theta$ mixed totally geodesic submanifold of $\tilde{M}^{2m+1}$.

Proof. If $M^n$ is $\mathcal{D} - \mathcal{D}^\theta$ mixed totally geodesic, then, from (3.8), we get

(3.19) \quad X(\ln f_2) g(U, V) - \varphi X(\ln f_2) g(U, TV) = 0.

Also, from (3.9), we have

(3.20) \quad (\varphi X(\ln f_2) + \eta(X)) g(U, TV) - \cos^2 \theta X(\ln f_2) g(U, V) = 0.

Then, from (3.19) and (3.20), we derive

(3.21) \quad \sin^2 \theta X(\ln f_2) g(U, V) + \eta(X) g(U, TV) = 0.

Interchanging $X$ by $\varphi X$, we find that $\sin^2 \theta \varphi X(\ln f_2) g(U, V) = 0$. Since, $N_\theta$ is a proper pointwise slant submanifold and $g$ is Riemannian, thus, we find that $\varphi X(\ln f_2) = 0$, that is $f_2$ is constant.

Conversely, if $f_2$ is constant on $M^n$, then from (3.8), we have

(3.22) \quad g(h(\mathcal{D}, \mathcal{D}^\theta), F\mathcal{D}^\theta) = 0.

Again, from the second equality of Lemma 3.2 we derive

(3.23) \quad g(h(\mathcal{D}, \mathcal{D}^\theta), \varphi\mathcal{D}^{\perp}) = 0.

Also, by the hypothesis of the theorem, we have

(3.24) \quad g(h(\mathcal{D}, \mathcal{D}^\theta), \mu) = 0.

Thus, by (3.22)-(3.24), we find $g(h(\mathcal{D}, \mathcal{D}^\theta), N) = 0, \quad \forall N \in \Gamma(T^{\perp}M^n)$, that is, $M^n$ is a $\mathcal{D} - \mathcal{D}^\theta$ mixed totally geodesic submanifold, which proves the theorem completely. □
4. A General Inequality for Bi-warped Products

In this section, we establish a relationship between the warping functions and the squared norm of the second fundamental of the bi-warped product $N_T \times_{f_1} N_\perp \times_{f_2} N_\theta$. We discuss the equality case and give some applications.

Let $M^n = N_T \times_{f_1} N_\perp \times_{f_2} N_\theta$ be an $n = (2p + 1 + n_1 + n_2)$-dimensional bi-warped product submanifold of a Sasakian manifold $\tilde{M}^{2m+1}$. We choose the orthonormal frames of the tangent subbundles $\mathcal{D}$, $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ of $TM^n$, respectively as follows:

\[
\mathcal{D} = \text{Span}\{e_1, \cdots, e_p, e_{p+1} = \varphi e_1, \cdots, e_{2p} = \varphi e_p, e_{2p+1} = \xi\},
\]

\[
\mathcal{D}^\perp = \text{Span}\{e_{2p+2} = \bar{e}_1, \cdots, e_{2p+1+n_1} = e_{n_1}\}
\]

and

\[
\mathcal{D}^\theta = \text{Span}\{e_{2p+2+n_1} = e_1^s, \cdots, e_{2p+1+n_1+n_2} = e_{s_1}^s, e_{2p+2+n_1+n_2} = e_{s_1+1}^s = \sec \theta Te_1^s, \cdots, e_n = e_{2\alpha}^s = \sec \theta Te_s^s\}.
\]

Then the orthonormal frames of the normal subbundles $\varphi \mathcal{D}^\perp$, $F \mathcal{D}^\theta$ and $\mu$, respectively are

\[
\varphi \mathcal{D}^\perp = \text{Span}\{e_{n+1} = \varphi e_1, \cdots, e_{n+n_1} = \varphi e_n\},
\]

\[
F \mathcal{D}^\theta = \text{Span}\{e_{n+n_1+1} = \hat{e}_1 = \csc \theta Fe_1^s, \cdots, e_{n+n_1+n_2} = \hat{e}_n = \csc \theta Fe_n^s, e_{n+n_1+n_2+1} = \hat{e}_{s+1} = \csc \theta \sec \theta FT e_1^s, \cdots, e_{n+n_1+n_2} = \hat{e}_n = \csc \theta \sec \theta FT e_s^s\}
\]

and

\[
\mu = \text{Span}\{e_{n+n_1+n_2+1} = \bar{e}_1, \cdots, e_{2m+1} = \bar{e}_{2m+1-n-n_1-n_2}\}.
\]

It is clear that $\dim(\mathcal{D}) = 2p + 1$, $\dim(\mathcal{D}^\perp) = n_1$ and $\dim(\mathcal{D}^\theta) = 2s = n_2$. On the other hand, the dimensions of the normal subspaces are $\dim(\varphi \mathcal{D}^\perp) = n_1$, $\dim(F \mathcal{D}^\theta) = 2s = n_2$ and $\dim \mu = 2m + 1 - n - n_1 - n_2$.

Now, we prove the main result of this section.

**Theorem 4.1.** Let $M^n = N_T \times_{f_1} N_\perp \times_{f_2} N_\theta$ be an $n$-dimensional bi-warped product submanifold of a $(2m+1)$-dimensional Sasakian manifold $\tilde{M}^{2m+1}$. Then, we have

(i) The squared norm of the second fundamental from $\|h\|^2$ satisfying the following inequality

\[
\|h\|^2 \geq 2n_1(\|\bar{\nabla}(\ln f_1)\|^2 + 1) + 2n_2(1 + 2 \cot^2 \theta) \|ar{\nabla}(\ln f_2)\|^2
\]

(4.1)

where $n_1 = \dim(N_\perp)$, $n_2 = \dim(N_\theta)$ and $\bar{\nabla}(\ln f_1)$ and $\bar{\nabla}(\ln f_2)$ are gradients of the warping functions on $M$ along $N_\perp$ and $N_\theta$, respectively.

(ii) If the equality sign holds in (4.1), then $N_T$ is totally geodesic in $\tilde{M}^{2m+1}$ and $N_\perp$ and $N_\theta$ are totally umbilical submanifolds of $\tilde{M}^{2m+1}$. Moreover, $M$ is a $\mathcal{D}^\perp - \mathcal{D}^\theta$ mixed totally geodesic submanifold but never be $\mathcal{D} - \mathcal{D}^\perp$ and $\mathcal{D} - \mathcal{D}^\theta$ mixed totally geodesic.

**Proof.** From (2.5), we have

\[
\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i,e_j),h(e_i,e_j)) = \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^{n} g(h(e_i,e_j),e_r)^2.
\]
Then, from (4.2), the above equation takes the form
\[
\|h\|^2 = \sum_{r=n+1} \sum_{i,j=1} g(h(e_i, e_j), e_r)^2 + \sum_{r=n+1} \sum_{i,j=1} g(h(\bar{e}_i, \bar{e}_j), e_r)^2
\]
\[+ \sum_{r=n+1} \sum_{i,j=1} g(h(e_i, e_j)^*, e_r)^2 + 2 \sum_{r=n+1} \sum_{i=1}^{n_2} g(h(e_i, \bar{e}_j), e_r)^2
\]
\[+ 2 \sum_{r=n+1} \sum_{i=1}^{n_2} g(h(e_i, e_j)^*, e_r)^2 + 2 \sum_{r=n+1} \sum_{i=1}^{n_2} g(h(\bar{e}_i, \bar{e}_j), e_r)^2.
\]
Again, from (4.3), we derive
\[
\|h\|^2 = \sum_{r=1} \sum_{i,j=1} g(h(e_i, e_j), \varphi \bar{e}_r)^2 + \sum_{r=1} \sum_{i,j=1} g(h(e_i, e_j), \bar{e}_r)^2
\]
\[+ \sum_{r=1} \sum_{i,j=1} g(h(e_i, e_j), \hat{e}_r)^2 + \sum_{r=1} \sum_{i,j=1} g(h(\bar{e}_i, \bar{e}_j), \varphi \bar{e}_r)^2
\]
\[+ \sum_{r=1} \sum_{i,j=1} g(h(\bar{e}_i, \bar{e}_j), \bar{e}_r)^2 + \sum_{r=1} \sum_{i,j=1} g(h(e_i, e_j)^*, \varphi \bar{e}_r)^2
\]
\[+ \sum_{r=1} \sum_{i,j=1} g(h(e_i, e_j)^*, \bar{e}_r)^2 + 2 \sum_{r=1} \sum_{i=1}^{n_2} g(h(e_i, e_j), \varphi \bar{e}_r)^2
\]
\[+ 2 \sum_{r=1} \sum_{i=1}^{n_2} g(h(e_i, e_j), \bar{e}_r)^2 + 2 \sum_{r=1} \sum_{i=1}^{n_2} g(h(e_i, e_j)^*, \hat{e}_r)^2
\]
\[+ 2 \sum_{r=1} \sum_{i=1}^{n_2} g(h(e_i, e_j)^*, \bar{e}_r)^2 + 2 \sum_{r=1} \sum_{i=1}^{n_2} g(h(\bar{e}_i, \bar{e}_j)^*, \varphi \bar{e}_r)^2
\]
\[+ 2 \sum_{r=1} \sum_{i=1}^{n_2} g(h(\bar{e}_i, \bar{e}_j)^*, \bar{e}_r)^2 + 2 \sum_{r=1} \sum_{i=1}^{n_2} g(h(\bar{e}_i, \bar{e}_j)^*, \hat{e}_r)^2.
\]
(4.2)

As we couldn’t find any relation for bi-warped products in terms of the \(\mu\)-component, so leaving the positive third, sixth, ninth, twelfth, fifteenth and eighteenth terms in the right hand side of (4.2). Also, we couldn’t find any relations for the fourth, fifth, seventh, eighth, sixteenth and seventeenth terms, we also leave these positive terms. On the other hand, by using Lemma 3.1 (i), the first and second vanish identically. Similarly, eleventh and thirteenth terms are identically zero by using Lemma 3.2. Hence, with remaining tenth and fourteenth terms, the
From (4.6) and (4.11), we find
\[ (4.11) \]
also, from the leaving sixteenth and seventeenth terms in the right hand side
\[ (4.10) \]
of (4.2), we arrive at
\[ (4.12) \]
Similarly, from the leaving seventh and eighth terms in the right hand side of (4.2), we find
\[ (4.9) \]
Thus from (4.6) and (4.7), we obtain
\[ (4.7) \]
Thus, from the definition of gradient, we achieve the inequality (i). For the equality case, from the leaving \( \mu \)-components terms in (4.2), we have
\[ (4.6) \]
From the leaving fourth and fifth terms in (4.2), we find
\[ (4.5) \]
Since \( \xi (\ln f_i) = 0, \forall i = 1, 2 \) (from (3.4)), then above relation will be
\[ (4.4) \]
Using Lemma 3.1 (ii)-(iii) and relations (3.3)-(3.14) with orthonormality of vector fields, we derive
\[ (4.3) \]
above expression takes the form.
\[ ||h||^2 \geq 2 \sum_{r=1}^{n_1} \sum_{i=1}^{p} \sum_{j=1}^{n_1} g(h(e_i, e_j), \varphi \bar{e}_r)^2 + 2 \sum_{r=1}^{n_1} \sum_{i=1}^{p} \sum_{j=1}^{n_1} g(h(\varphi e_i, \bar{e}_j), \varphi \bar{e}_r)^2 + 2 \sum_{r=1}^{n_2} \sum_{i=1}^{p} \sum_{j=1}^{n_2} g(h(e_i, e_j^*), \bar{e}_r)^2 + 2 \sum_{r=1}^{n_2} \sum_{i=1}^{p} \sum_{j=1}^{n_2} g(h(\varphi e_i, e_j^*), \bar{e}_r)^2. \]
(4.3)
Thus from (4.6) and (4.9), we conclude that
\[ (4.10) \]
Similalry, from the leaving sixteenth and seventeenth terms in the right hand side of (4.2), we arrive at
\[ (4.11) \]
From (4.6) and (4.11), we find
\[ (4.12) \]
On the other hand, from the vanishing first and second terms, we get
\[ h(D, D) \perp \varphi D^\perp \text{ and } h(D, D) \perp F D^\theta. \]
Then, from (4.6) and (4.13), we obtain
\[ h(D, D) = 0. \]
And, from the vanishing eleventh term of (4.2) and (4.6), we obtain
\[ h(D, D^\perp) \subset \varphi D^\perp. \]
Similarly, from the vanishing thirteenth term in (4.2) with (4.6), we find
\[ h(D, D^\theta) \subset F D^\theta. \]
Since \( N_T \) is totally geodesic in \( M \) (see Remark 3.1), using this fact with (4.8), (4.10) and (4.14), again, from (4.12), (4.15) and (4.16) with Remark 3.1, we conclude that \( N^\perp \) and \( N^\theta \) are totally umbilical in \( \tilde{M} \), while; using all conditions with (4.12), \( M \) is a \( D^\perp - D^\theta \) mixed totally geodesic submanifold of \( \tilde{M} \) but not \( D - D^\perp \) and \( D - D^\theta \) mixed totally geodesic.

5. Some Applications like B.-Y. Chen’s inequalities

We have the following applications of our derived inequality. First we obtain B.-Y. Chen’s type inequality for multiply contact CR-warped product submanifolds of Sasakian manifolds.

As a particular case of Theorem 4.1, when we consider \( \dim(N_\theta) = 0 \), then the bi-warped product reduces to a contact CR-warped product submanifold \( M = N_T \times f_1 N_\perp \) studied by Hasegawa and Mihai. In that case the inequality (4.6), takes the from
\[ \|h\|^2 \geq 2n_1 \left( \|\nabla(\ln f_1)\|^2 + 1 \right), \]
which is inequality (2.4) of [19] obtained as a Theorem 2.2 of [19]. Hence, Theorem 2.2 of [19] is a particular case of Theorem 4.1.

On the other hand, if \( \dim(N_\perp) = 0 \) in Theorem 4.1, then the bi-warped product reduces to warped product pointwise semi-slant submanifold \( M = N_T \times f_\theta N_\perp \). In that case, the inequality (4.1) will be
\[ \|h\|^2 \geq 2n_2(csc^2\theta + cot^2\theta)\|\nabla(\ln f_\theta)\|^2 + 2n_2(1 + \cos^4\theta), \]
which is inequality (10.1) of [24] derived as a main result. Thus, Theorem 10.1 of [24] is a particular case of Theorem 4.1.

Now, we have another application of Theorem 4.1 as follows.

**Theorem 5.1.** Let \( M = N_T \times f_\theta N_1 \times \cdots \times f_k N_k \) be a multiply CR-warped product submanifold in a Sasakian manifold \( M^{2m+1} \) such that the structure vector field \( \xi \) is tangent to \( N_T \). Then, the squared norm of the second fundamental form \( \|h\|^2 \) and the warping functions \( f_1, \cdots, f_k \) satisfy
\[
\|h\|^2 \geq 2 \sum_{i=1}^{k} n_i \left( \|\nabla(\ln f_i)\|^2 + 1 \right),
\]
where \( N_T \) is an invariant submanifold and each \( N_i, i = 1, \cdots, k \) is an anti-invariant submanifold of dimension \( n_i \) in \( M^{2m+1} \).
The equality sign in (5.1) holds identically if and only if the following statements hold:

(i) $N_T$ is a totally geodesic submanifold of $\tilde{M}^{2m+1}$.
(ii) For each $i \in \{1, \cdots , k\}$, $N_i$ is totally umbilical submanifold of $\tilde{M}^{2m+1}$.
(iii) $f_1N_1 \times \cdots \times f_kN_k$ is immersed as mixed totally geodesic submanifold in $\tilde{M}^{2m+1}$

Proof. In Theorem 4.1, if $\dim(N_\theta) = 0$, then the warped products takes the from $N_T \times f_1N_1$, where $N_1$ is an anti-invariant submanifold and in that case, the inequality (4.1) will be

\[ \|h\|^2 \geq 2n_1 \left( \|\vec{\nabla}(\ln f_1)\|^2 + 1 \right), \]  

which means that the inequality (5.1) holds for $k = 1$.

On the other hand, if $\theta = \frac{\pi}{2}$ in Theorem 4.1, then the bi-warped products take the form $N_T \times f_1N_1 \times f_2N_2$, where $N_1$ and $N_2$ are anti-invariant submanifolds. Then, from (4.1), we have

\[ \|h\|^2 \geq 2 \sum_{i=1}^{2} n_i \left( \|\vec{\nabla}(\ln f_i)\|^2 + 1 \right), \]  

which implies that (5.1) holds identically for $k = 2$. Hence, the result follows by mathematical induction. The equality case holds identically if and only if the given statements (i)-(iii) satisfy. For the equality case, we use the similar arguments as in Theorem 4.1.

Another motivation of inequality (4.1) is to express the Dirichlet energy of the warping functions $f_1$ and $f_2$, which is a useful tools in physics. The Dirichlet energy of any function $\psi$ on a compact manifold $M$ is defined as follows

\[ E(\psi) = \frac{1}{2} \int_M \|\nabla \psi\|^2 dV \]  

where $\nabla \psi$ is the gradient of $\psi$ and $dV$ is the volume element.

Then, for a compact invariant submanifold $N_T$ on a bi-warped product submanifold $N_T \times f_1N_\perp \times f_2N_\theta$ in a Sasakian manifold $\tilde{M}^{2m+1}$, we write the expression of Dirichlet energy of the warping functions $f_1$ and $f_2$ as follows:

**Theorem 5.2.** Let $M^n = N_T \times f_1N_\perp \times f_2N_\theta$ be a bi-warped product submanifold of a Sasakian manifold $\tilde{M}^{2m+1}$ with compact $N_T$ and for each $p \in N_\perp$ and $q \in N_\theta$. Then, we have the following inequality for the Dirichlet energy $E$ of $f_1$ and $f_2$:

\[ n_1E(\ln f_1) + n_2(\csc^2 \theta + \cot^2 \theta)E(\ln f_2) \leq \frac{1}{4} \int_{N_T \times \{p\} \times \{q\}} \|h\|^2 dV_T - \frac{1}{2} [n_1 + n_2(1 + \cos^2 \theta)] \text{Vol}(N_T), \]  

where $\text{vol}(N_T)$ is the volume of $N_T$ and $n_1$ and $n_2$ are dimensions of $N_\perp$ and $N_\theta$, respectively.

Proof. Integrating (4.1) over $N_T \times \{p\} \times \{q\}$, for any $p \in N_\perp$ and $q \in N_\theta$, we derive the required inequality. \[ \square \]

The following corollaries are immediate consequences of Theorem 5.2.
Corollary 5.1. Let $M = N_T \times_f N_\perp$ be a contact CR-warped product submanifold of a Sasakian manifold $M^{2m+1}$ such that $N_T$ is compact and any $q \in N_\perp$. Then, we have

\begin{equation}
E(\ln f_1) \leq \frac{1}{4n_1} \int_{N_T \times \{q\}} \|h\|^2 \, dV_T - \frac{1}{2} Vol(N_T).
\end{equation}

Similarly, if $\dim(N_\perp) = 0$ in Theorem 5.2, then we have

Corollary 5.2. Let $M = N_T \times_f N_\perp$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $M^{2m+1}$ such that $N_T$ is compact and any $s \in N_\perp$. Then, we have

\begin{equation}
E(\ln f_2) \leq \frac{1}{4n_2(1 + 2\cot^2 \theta)} \int_{N_T \times \{s\}} \|h\|^2 \, dV_T - \frac{(1 + \cos^4 \theta)}{2(1 + 2\cot^2 \theta)} Vol(N_T).
\end{equation}

6. Examples

In this section, we construct some non-trivial examples of bi-warped product submanifolds of the form $N_T \times_f N_\perp \times_f N_\theta$ in Euclidean spaces.

Example 6.1. Consider a submanifold of $\mathbb{R}^{11}$ with the cartesian coordinates $(x_1, y_1, \ldots, x_5, y_5, z)$ and the almost contact structure

\[ \varphi \left( \frac{\partial}{\partial x_i} \right) = \frac{\partial}{\partial y_i}, \quad \varphi \left( \frac{\partial}{\partial y_j} \right) = \frac{\partial}{\partial x_j}, \quad \varphi \left( \frac{\partial}{\partial z} \right) = 0, \quad 1 \leq i, j \leq 5. \]

Let us consider the submanifold $M$ of $\mathbb{R}^{11}$ defined by the immersion $\psi : \mathbb{R}^6 \rightarrow \mathbb{R}^{11}$ as follows

\[ \psi(u, v, t, s, r, z) = (u, v, -\frac{t^2 \cos \theta}{2}, \frac{t^2 \sin \theta}{2}, r, s, r, 0, s, \frac{r^2}{2}, \frac{s^2}{2}, z) \]

for all non-zero variables and $\theta \in (0, \frac{\pi}{2})$.

\[
\begin{align*}
Z_1 &= \frac{\partial}{\partial x_1}, \quad Z_2 = \frac{\partial}{\partial y_1}, \quad Z_3 = -t \cos \theta \frac{\partial}{\partial x_2} + t \sin \theta \frac{\partial}{\partial y_2}, \\
Z_4 &= \frac{\partial}{\partial x_3} + s \frac{\partial}{\partial x_4} + r \frac{\partial}{\partial x_5}, \quad Z_5 = \frac{\partial}{\partial y_3} + r \frac{\partial}{\partial x_4} + s \frac{\partial}{\partial y_5}, \quad Z_6 = \frac{\partial}{\partial z}.
\end{align*}
\]

Then, we find

\[
\begin{align*}
\varphi Z_1 &= -\frac{\partial}{\partial y_1}, \quad \varphi Z_2 = \frac{\partial}{\partial x_1}, \quad \varphi Z_3 = t \cos \theta \frac{\partial}{\partial y_2} + t \sin \theta \frac{\partial}{\partial x_2}, \\
\varphi Z_4 &= -\frac{\partial}{\partial y_3} - s \frac{\partial}{\partial y_4} - r \frac{\partial}{\partial y_5}, \quad \varphi Z_5 = \frac{\partial}{\partial x_3} - r \frac{\partial}{\partial y_4} + s \frac{\partial}{\partial x_5}.
\end{align*}
\]

It is easy to see that $\varphi Z_1$ is orthogonal to $TM$ and $\mathcal{D} = \text{Span}\{Z_1, Z_2\}$ is an invariant distribution, $\mathcal{D}^\perp = \text{Span}\{Z_3\}$ is an anti-invariant distribution and $\mathcal{D}^\theta = \text{Span}\{Z_4, Z_5\}$ is a pointwise slant distribution with slant angle $\theta = \arccos(\frac{1 + r s}{\sqrt{1 + r^2 + s^2}})$. Hence, we conclude that $TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \mathcal{D}^\theta \oplus (\xi)$. It is easy to observe that $\mathcal{D}$, $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ are integrable distributions. Denoting the integral manifolds of $\mathcal{D}$, $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ by $M_T$, $M_\perp$ and $M_\theta$, respectively. Then the induced metric tensor $g$ of $M = M_T \times_f M_\perp \times_f M_\theta$ is given by

\[ ds^2 = du^2 + dv^2 + t^2 dt^2 + (1 + r^2 + s^2)(dr^2 + ds^2) = g_{M_T} + f_1^2 g_{M_\perp} + f_2^2 g_{M_\theta} \]
Thus $M$ is a bi-warped product submanifold of $\mathbb{R}^11$ with the warping functions $f_1 = t$ and $f_2 = \sqrt{1 + r^2 + s^2}$.

**Example 6.2.** Consider a submanifold of $\mathbb{R}^19$ with the cartesian coordinates $(x_1, y_1, \ldots, x_9, y_9, z)$ and the almost contact structure
\[
\varphi \left( \frac{\partial}{\partial x_i} \right) = -\frac{\partial}{\partial y_i}, \quad \varphi \left( \frac{\partial}{\partial y_j} \right) = \frac{\partial}{\partial x_j}, \quad \varphi \left( \frac{\partial}{\partial z} \right) = 0, \quad 1 \leq i, j \leq 9.
\]
Let $M$ be a submanifold of $\mathbb{R}^19$ defined by the immersion $\psi$ as follows
\[
\psi(u, v, s, t, r, z) = (u \cos w, v \cos w, u \cos s, v \cos s, u \sin w, v \sin w, u \sin s, v \sin s, u \cos t, v \cos t, u \cos r, v \cos r, u \sin t, v \sin t, u \sin r, v \sin r, k(r - t), -k(r + t), z)
\]
where $u \neq 0, v \neq 0$ and $k \in \mathbb{R} - \{0\}$. If the tangent space $TM$ is spanned by the following vector fields
\[
Z_1 = \cos w \frac{\partial}{\partial x_1} + \cos s \frac{\partial}{\partial x_2} + \sin w \frac{\partial}{\partial x_3} + \sin s \frac{\partial}{\partial x_4} + \cos t \frac{\partial}{\partial x_5} + \cos r \frac{\partial}{\partial x_6} + \sin t \frac{\partial}{\partial x_7} + \sin r \frac{\partial}{\partial x_8},
\]
\[
Z_2 = \cos w \frac{\partial}{\partial y_1} + \cos s \frac{\partial}{\partial y_2} + \sin w \frac{\partial}{\partial y_3} + \sin s \frac{\partial}{\partial y_4} + \cos t \frac{\partial}{\partial y_5} + \cos r \frac{\partial}{\partial y_6} + \sin t \frac{\partial}{\partial y_7} + \sin r \frac{\partial}{\partial y_8},
\]
\[
Z_3 = -u \sin w \frac{\partial}{\partial x_1} - v \sin w \frac{\partial}{\partial y_1} + u \cos w \frac{\partial}{\partial x_3} + v \cos w \frac{\partial}{\partial y_3},
\]
\[
Z_4 = -u \sin s \frac{\partial}{\partial x_2} - v \sin s \frac{\partial}{\partial y_2} + u \cos s \frac{\partial}{\partial x_4} + v \cos s \frac{\partial}{\partial y_4},
\]
\[
Z_5 = -u \sin t \frac{\partial}{\partial x_5} - v \sin t \frac{\partial}{\partial y_5} + u \cos t \frac{\partial}{\partial x_7} + v \cos t \frac{\partial}{\partial y_7} - k \frac{\partial}{\partial x_9} - k \frac{\partial}{\partial y_9},
\]
\[
Z_6 = -u \sin r \frac{\partial}{\partial x_6} - v \sin r \frac{\partial}{\partial y_6} + u \cos r \frac{\partial}{\partial x_8} + v \cos r \frac{\partial}{\partial y_8} + k \frac{\partial}{\partial x_9} - k \frac{\partial}{\partial y_9},
\]
\[
Z_7 = \frac{\partial}{\partial z}
\]
then, we obtain
\[
\varphi Z_1 = -\cos w \frac{\partial}{\partial y_1} - \cos s \frac{\partial}{\partial y_2} - \sin w \frac{\partial}{\partial y_3} - \sin s \frac{\partial}{\partial y_4} - \cos t \frac{\partial}{\partial y_5} - \cos r \frac{\partial}{\partial y_6} - \sin t \frac{\partial}{\partial y_7} - \sin r \frac{\partial}{\partial y_8},
\]
\[
\varphi Z_2 = \cos w \frac{\partial}{\partial x_1} + \cos s \frac{\partial}{\partial x_2} + \sin w \frac{\partial}{\partial x_3} + \sin s \frac{\partial}{\partial x_4} + \cos t \frac{\partial}{\partial x_5} + \cos r \frac{\partial}{\partial x_6} + \sin t \frac{\partial}{\partial x_7} + \sin r \frac{\partial}{\partial x_8},
\]
\[
\varphi Z_3 = u \sin w \frac{\partial}{\partial y_1} - v \sin w \frac{\partial}{\partial x_1} - u \cos w \frac{\partial}{\partial y_3} + v \cos w \frac{\partial}{\partial x_3},
\]
\[
\varphi Z_4 = u \sin s \frac{\partial}{\partial y_2} - v \sin s \frac{\partial}{\partial x_2} - u \cos s \frac{\partial}{\partial y_4} + v \cos s \frac{\partial}{\partial x_4}.
\]
\[ \varphi Z_5 = u \sin t \frac{\partial}{\partial y_5} - v \sin t \frac{\partial}{\partial x_5} - u \cos t \frac{\partial}{\partial y_7} + v \cos t \frac{\partial}{\partial x_7} + k \frac{\partial}{\partial y_9} - k \frac{\partial}{\partial x_9}, \]
\[ \varphi Z_6 = u \sin r \frac{\partial}{\partial y_6} - v \sin r \frac{\partial}{\partial x_6} - u \cos r \frac{\partial}{\partial y_8} + v \cos r \frac{\partial}{\partial x_8} - k \frac{\partial}{\partial y_9} - k \frac{\partial}{\partial x_9}, \]
\[ \varphi Z_7 = 0. \]

Clearly, we find that \( \varphi Z_3 \) and \( \varphi Z_3 \) are orthogonal to \( TM = \text{Span}\{Z_1, \cdots, Z_7\} \) and thus we consider \( D = \text{Span}\{Z_1, Z_2\} \) is an invariant distribution, \( D^\perp = \text{Span}\{Z_3, Z_4\} \) is an anti-invariant distribution and \( D^\theta = \text{Span}\{Z_5, Z_6\} \) is a pointwise slant distribution with slant angle \( \theta = \arccos\left(\frac{2k^2}{u^2 + v^2 + 2k^2}\right) \). It is easy to see that \( D, D^\perp \) and \( D^\theta \) are integrable distributions on \( M \). If we denote the integral manifolds of \( D, D^\perp \) and \( D^\theta \) by \( M_T, M_L \) and \( M_\theta \), respectively, then the induced metric tensor \( g \) of \( M = M_T \times f_1, M_L \times f_2, M_\theta \) is given by
\[ dS^2 = 4(du^2 + dv^2) + (u^2 + v^2)(dw^2 + ds^2) + (u^2 + v^2 + 2k^2)(dt^2 + dr^2) \]
\[ = g_{M_T} + f_1^2 g_{M_L} + f_2^2 g_{M_\theta}. \]

Hence, \( M \) is a bi-warped product submanifold of \( \mathbb{R}^{19} \) with the warping functions \( f_1 = \sqrt{u^2 + v^2} \) and \( f_2 = \sqrt{u^2 + v^2 + 2k^2} \).

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