ON THE CAMERON-PRAEGER CONJECTURE

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Abstract. This paper takes a significant step towards confirming a long-standing and far-reaching conjecture of Peter J. Cameron and Cheryl E. Praeger. They conjectured in 1993 that there are no non-trivial block-transitive 6-designs. We prove that the Cameron-Praeger conjecture is true for the important case of non-trivial Steiner 6-designs, i.e. for 6-(v, k, λ) designs with λ = 1, except possibly when the group is $PGL(2, p^e)$ with $p = 2$ or 3, and $e$ is an odd prime power.

1. INTRODUCTION

The characterization of combinatorial or geometric structures in terms of their groups of automorphisms has attracted considerable interest in the last decades and is now commonly viewed as a natural generalization of Felix Klein’s Erlangen program (1872). There has been recent progress in particular on the characterization of Steiner t-designs which admit groups of automorphisms with sufficiently strong symmetry properties: The author classified all flag-transitive Steiner t-designs with $t > 2$ (see [14, 15, 16, 17, 18] and [20] for a monograph). In particular, he showed in [17] that no non-trivial flag-transitive Steiner 6-design can exist. These results answer a series of 40-year-old problems and generalize theorems of J. Tits [36] and H. Lüneburg [31]. Previously, F. Buekenhout, A. Delandtsheer, J. Doyen, P. Kleidman, M. Liebeck, and J. Saxl [4, 10, 26, 29, 33] had characterized all flag-transitive Steiner 2-designs, up to the 1-dimensional affine case. All these classification results rely on the classification of the finite simple groups.

In 1993, P. J. Cameron and C. E. Praeger ([7, Conj. 1.2]) conjectured that there are no non-trivial block-transitive 6-designs.¹ Our main result is as follows:

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¹see also Kourovka Notebook [25, Problem 11.45], and Peter Cameron’s conjectures online at http://www.maths.qmw.ac.uk/~pjc/cameronconjs.html.
Main Theorem. Let $D = (X, B, I)$ be a non-trivial Steiner $t$-design. Then $G \leq \text{Aut}(D)$ cannot act block-transitively on $D$, except possibly when $G = P\Gamma L(2, p^e)$ with $p = 2$ or $3$ and $e$ is an odd prime power.

The result has been announced (without proof) in a recent paper [19] on the existence problem for Steiner $t$-designs for large values of $t$. The proof makes use of the classification of the finite 3-homogeneous permutation groups, which in turn relies on the classification of the finite simple groups. It will be given in Section 4. Preliminary results which are important for the remainder of the paper are collected in Section 3.

2. Definitions and Notations

For positive integers $t \leq k \leq v$ and $\lambda$, we define a $t-(v, k, \lambda)$ design to be a finite incidence structure $D = (X, B, I)$, where $X$ denotes a set of points, $|X| = v$, and $B$ a set of blocks, $|B| = b$, with the following regularity properties: each block $B \in B$ is incident with $k$ points, and each $t$-subset of $X$ is incident with $\lambda$ blocks. A flag of $D$ is an incident point-block pair $(x, B) \in I$ with $x \in X$ and $B \in B$.

For historical reasons, a $t-(v, k, \lambda)$ design with $\lambda = 1$ is called a Steiner $t$-design (sometimes also a Steiner system). We note that in this case each block is determined by the set of points which are incident with it, and thus can be identified with a $k$-subset of $X$ in a unique way. If $t < k < v$, then we speak of a non-trivial Steiner $t$-design. There are many infinite classes of Steiner $t$-designs for $t = 2$ and 3, however for $t = 4$ and 5 only a finite number are known. For a detailed treatment of combinatorial designs, we refer to [1, 8, 13, 21, 35]. In particular, [1, 8] provide encyclopedic accounts of key results and contain existence tables with known parameter sets.

In what follows, we are interested in $t$-designs which admit groups of automorphisms with sufficiently strong symmetry properties such as transitivity on the blocks or on the flags. We consider automorphisms of a $t$-design $\mathcal{D}$ as pairs of permutations on $X$ and $B$ which preserve incidence, and call a group $G \leq \text{Aut}(\mathcal{D})$ of automorphisms of $\mathcal{D}$ block-transitive (respectively flag-transitive, point $t$-transitive, point $t$-homogeneous) if $G$ acts transitively on the blocks (respectively transitively on the flags, $t$-transitively on the points, $t$-homogeneously on the points) of $\mathcal{D}$. For short, $\mathcal{D}$ is said to be, e.g., block-transitive if $\mathcal{D}$ admits a block-transitive group of automorphisms.

For $\mathcal{D} = (X, B, I)$ a Steiner $t$-design with $G \leq \text{Aut}(\mathcal{D})$, let $G_x$ denote the stabilizer of a point $x \in X$, and $G_B$ the setswise stabilizer of a block $B \in B$. For $x, y \in X$ and $B \in B$, we define $G_{xy} = G_x \cap G_y$.

3. Preliminary Results

3.1. Combinatorial Results.

Basic necessary conditions for the existence of $t$-designs can be obtained via elementary counting arguments (see, for instance, [1]):
Proposition 1. Let $\mathcal{D} = (X, \mathcal{B}, I)$ be a $t$-$(v, k, \lambda)$ design, and for a positive integer $s \leq t$, let $S \subseteq X$ with $|S| = s$. Then the total number of blocks incident with each element of $S$ is given by

$$\lambda_s = \lambda \frac{(v-s)}{(t-s)} \frac{(k-s)}{(t-s)}.$$ 

In particular, for $t \geq 2$, a $t$-$(v, k, \lambda)$ design is also an $s$-$(v, k, \lambda_s)$ design.

It is customary to set $r := \lambda_1$ denoting the total number of blocks incident with a given point.

Corollary 2. Let $\mathcal{D} = (X, \mathcal{B}, I)$ be a $t$-$(v, k, \lambda)$ design. Then the following holds:

(a) $bk = vr$.

(b) $\binom{v}{t} \lambda = b \binom{k}{t}$.

(c) $r(k-1) = \lambda_2(v-1)$ for $t \geq 2$.

Corollary 3. Let $\mathcal{D} = (X, \mathcal{B}, I)$ be a $t$-$(v, k, \lambda)$ design. Then

$$\lambda \binom{v-s}{t-s} \equiv 0 \pmod{\binom{k-s}{t-s}}$$

for each positive integer $s \leq t$.

For non-trivial Steiner $t$-designs lower bounds for $v$ in terms of $k$ and $t$ can be given (see P. Cameron [5, Thm. 3A.4], and J. Tits [36, Prop. 2.2]):

Proposition 4. If $\mathcal{D} = (X, \mathcal{B}, I)$ is a non-trivial Steiner $t$-design, then the following holds:

(a) (Tits 1964): $v \geq (t+1)(k-t+1)$.

(b) (Cameron 1976): $v - t + 1 \geq (k - t + 2)(k - t + 1)$ for $t > 2$. If equality holds, then $(t, k, v) = (3, 4, 8), (3, 6, 22), (3, 12, 112), (4, 7, 23)$, or $(5, 8, 24)$.

In the case when $t = 6$, we deduce from Part (b) the following upper bound for the positive integer $k$.

Corollary 5. Let $\mathcal{D} = (X, \mathcal{B}, I)$ be a non-trivial Steiner $t$-design with $t = 6$. Then

$$k \leq \left\lfloor \sqrt{v - \frac{19}{4} + \frac{9}{2}} \right\rfloor.$$
3.2. Highly Symmetric Designs.

We will now focus on \( t \)-designs which admit groups of automorphisms with sufficiently strong symmetry properties. One of the reasons for this consideration of highly symmetric designs is a general view that, while the existence of combinatorial objects is of interest, they are even more fascinating when they have a rich group of symmetries.

One of the early important results regarding highly symmetric designs is due to R. Block [2, Thm. 2]:

**Proposition 6.** (Block 1965). Let \( D = (X, B, I) \) be a non-trivial \( t \)-(\( v, k, \lambda \)) design with \( t \geq 2 \). If \( G \leq \text{Aut}(D) \) acts block-transitively on \( D \), then \( G \) acts point-transitively on \( D \).

For a \( 2 \)-(\( v, k, 1 \)) design \( D \), it is elementary that the point 2-transitivity of \( G \leq \text{Aut}(D) \) implies its flag-transitivity. For \( 2 \)-(\( v, k, \lambda \)) designs, this implication remains true if \( r \) and \( \lambda \) are relatively prime (cf. [11, Chap. 2.3, Lemma 8]). However, for \( t \)-(\( v, k, \lambda \)) designs with \( t \geq 3 \), it can be deduced from Proposition 6 that always the converse holds (see [3] or [14, Lemma 2]):

**Proposition 7.** Let \( D = (X, B, I) \) be a non-trivial \( t \)-(\( v, k, \lambda \)) design with \( t \geq 3 \). If \( G \leq \text{Aut}(D) \) acts flag-transitively on \( D \), then \( G \) acts point \( 2 \)-transitively on \( D \).

Investigating highly symmetric \( t \)-designs for large values of \( t \), P. Cameron and C. Praeger [7, Thm. 2.1] derived from Proposition 6 and a combinatorial result of D. Ray-Chaudhuri and R. Wilson [32, Thm. 1] the following assertion:

**Proposition 8.** (Cameron & Praeger 1993). Let \( D = (X, B, I) \) be a \( t \)-(\( v, k, \lambda \)) design with \( t \geq 2 \). Then, the following holds:

(a) If \( G \leq \text{Aut}(D) \) acts block-transitively on \( D \), then \( G \) also acts point \([t/2]\)-homogeneously on \( D \).

(b) If \( G \leq \text{Aut}(D) \) acts flag-transitively on \( D \), then \( G \) also acts point \([ (t+1)/2 \)]\)-homogeneously on \( D \).

As for \( t \geq 7 \) the flag-transitivity, respectively for \( t \geq 8 \) the block-transitivity of \( G \leq \text{Aut}(D) \) implies at least its point 4-homogeneity, they obtained the following restrictions as a consequence of the finite simple group classification (cf. [7, Thm. 1.1]):

**Theorem 9.** (Cameron & Praeger 1993). Let \( D = (X, B, I) \) be a \( t \)-(\( v, k, \lambda \)) design. If \( G \leq \text{Aut}(D) \) acts block-transitively on \( D \) then \( t \leq 7 \), while if \( G \leq \text{Aut}(D) \) acts flag-transitively on \( D \) then \( t \leq 6 \).

Moreover, they formulated the following far-reaching conjecture (cf. [7, Conj. 1.2]):

**Conjecture 1.** (Cameron & Praeger 1993). There are no non-trivial block-transitive 6-designs.
3.3. Finite 3-homogeneous Permutation Groups.

In order to investigate all block-transitive Steiner 6-designs, we can as a consequence of Proposition 8 (a) make use of the classification of all finite 3-homogeneous permutation groups, which itself relies on the classification of all finite simple groups (cf. [6, 12, 23, 28, 30]).

Let $G$ be a finite 3-homogeneous permutation group on a set $X$ with $|X| \geq 4$. Then $G$ is either of

(A) **Affine Type:** $G$ contains a regular normal subgroup $T$ which is elementary Abelian of order $v = 2^d$. If we identify $G$ with a group of affine transformations $x \mapsto x^g + u$ of $V = V(d, 2)$, where $g \in G_0$ and $u \in V$, then one of the following occurs:

1. $G \cong AGL(1, 8)$, $AGL(1, 32)$
2. $G_0 \cong SL(d, 2)$, $d \geq 2$
3. $G_0 \cong A_7$, $v = 2^4$

or

(B) **Almost Simple Type:** $G$ contains a simple normal subgroup $N$, and $N \leq G \leq \text{Aut}(N)$. In particular, one of the following holds, where $N$ and $v = |X|$ are given as follows:

1. $A_v$, $v \geq 5$
2. $PSL(2, q)$, $q > 3$, $v = q + 1$
3. $M_v$, $v = 11, 12, 22, 23, 24$ (Mathieu groups)
4. $M_{11}$, $v = 12$

We note that if $q$ is odd, then $PSL(2, q)$ is 3-homogeneous for $q \equiv 3 \pmod{4}$, but not for $q \equiv 1 \pmod{4}$, and hence not every group $G$ of almost simple type satisfying (2) is 3-homogeneous on $X$. For required basic properties of the listed groups, we refer, e.g., to [9], [22], [27, Ch. 2, 5].

**Remark 10.** If $G \leq \text{Aut}(\mathcal{D})$ acts block-transitively on any Steiner $t$-design $\mathcal{D}$ with $t \geq 6$, then by Proposition 8 (a), $G$ acts point 3-homogeneously and in particular point 2-transitively on $\mathcal{D}$. Applying Corollary 2 (b) yields the equation

$$b = \binom{v}{2} \frac{v(v-1)|G_{xy}|}{|G_B|},$$

where $x$ and $y$ are two distinct points in $X$ and $B$ is a block in $\mathcal{B}$. 
4. Proof of the Main Theorem

Let $\mathcal{D} = (X, \mathcal{B}, I)$ be a non-trivial Steiner 6-design with $G \leq \text{Aut}(\mathcal{D})$ acting block-transitively on $\mathcal{D}$ throughout the proof. We recall that due to Proposition 8 (a), we may restrict ourselves to the consideration of the finite 3-homogeneous permutation groups listed in Section 3. Clearly, in the following we may assume that $k > 6$ as trivial Steiner 6-designs are excluded.

4.1. Groups of Automorphisms of Affine Type.

Case (1): $G \cong \text{AGL}(1, 8), \text{AGL}(1, 32), \text{A}_\Gamma L(1, 32)$.

If $v = 8$, then Corollary 5 yields $k \leq 6$, a contradiction. For $v = 32$, Corollary 5 implies that $k = 7, 8$ or $9$; for each of these values, $29$ divides $b$, and so divides $|G|$ by block-transitivity, a contradiction since $29$ does not divide $|A_\Gamma L(1, 32)|$.

Case (2): $G_0 \cong \text{SL}(d, 2), d \geq 2$.

Here $v = 2^d > k > 6$. For $d = 3$, we have $v = 8$, already ruled out in Case (1). So, we may assume that $d > 3$. Any six distinct points being non-coplanar in $AG(d, 2)$, they generate an affine subspace of dimension at least 3. Let $\mathcal{E}$ be the 3-dimensional vector subspace spanned by the first three basis vectors $e_1, e_2, e_3$ of the vector space $V = V(d, 2)$. Then the pointwise stabilizer of $\mathcal{E}$ in $\text{SL}(d, 2)$ (and therefore also in $G$) acts point-transitively on $V \setminus \mathcal{E}$. If the unique block $B \in \mathcal{B}$ which is incident with the 6-subset $\{0, e_1, e_2, e_3, e_1 + e_2, e_2 + e_3\}$ contains some point outside $\mathcal{E}$, then $B$ contains all points of $V \setminus \mathcal{E}$, and so $k \geq v - 2$, a contradiction to Corollary 5. Hence $B$ lies completely in $\mathcal{E}$, and so $k \leq 8$. On the other hand, for $\mathcal{D}$ to be a block-transitive 6-design admitting $G \leq \text{Aut}(\mathcal{D})$, we deduce from [7, Prop. 3.6 (b)] the necessary condition that $2^d - 3$ must divide $\binom{v}{6}$, and hence it follows for each respective value of $k$ that $d = 3$, contradicting our assumption.

Case (3): $G_0 \cong A_7, v = 2^4$.

For $v = 2^4$, we have $k \leq 7$ by Corollary 5, contradicting Proposition 1 since $r = \lambda_1$ is not an integer.

4.2. Groups of Automorphisms of Almost Simple Type.

Case (1): $N = A_v, v \geq 5$.

Since $\mathcal{D}$ is non-trivial with $k > 6$, we may assume that $v \geq 8$. Then $A_v$, hence also $G$, is 6-transitive on $X$, and so cannot act on any non-trivial Steiner 6-design by [24, Thm. 3].

Case (2): $N = \text{PSL}(2, q), v = q + 1, q = p^e > 3$.

Here $\text{Aut}(N) = \text{PGL}(2, q)$, and $|G| = (q + 1)q(q - 1)n$ with $n = (2, q - 1)$ and $a \mid ne$. We may again assume that $v = q + 1 \geq 8$.

We will first assume that $N = G$. Then, by Remark 10, we obtain
\begin{equation}
(1) \quad (q - 2)(q - 3)(q - 4) | \text{PSL}(2, q)_B | n = k(k - 1)(k - 2)(k - 3)(k - 4)(k - 5).
\end{equation}
In view of Proposition 4 (b), we have
\[(2)\quad q - 4 \geq (k - 4)(k - 5).\]
It follows from Equation (1) that
\[(3)\quad (q - 2)(q - 3) |PSL(2, q)_B| n \leq k(k - 1)(k - 2)(k - 3).\]
If we assume that \(k \geq 21\), then obviously
\[k(k - 1)(k - 2)(k - 3) < 2[(k - 4)(k - 5)]^2,\]
and hence
\[(q - 2)(q - 3) |PSL(2, q)_B| n < 2(q - 4)^2\]
in view of Inequality (2). Clearly, this is only possible when \(|PSL(2, q)_B| 
\cdot n = 1\). In particular, \(q\) has to be even. But then the right hand side
of Equation (1) is always divisible by 16 but never the left hand side, a
contradiction. If \(k < 21\), then the few remaining possibilities for \(k\) can easily
be ruled out by hand using Equation (1), Inequality (2), and Corollary 3.

Now, let us assume that \(N < G \leq \text{Aut}(N)\). We recall that \(q = p^k \geq 7,\)
and will distinguish in the following the cases \(p > 3\), \(p = 2\), and \(p = 3\).

First, let \(p > 3\). We define \(G^* = G \cap (PSL(2, q) \rtimes \langle \tau_\alpha \rangle)\)
with \(\tau_\alpha \in \text{Sym}(GF(p^e) \cup \{\infty\}) \cong S_v\) of order \(e\) induced by the Frobenius automorphism
\(\alpha : GF(p^e) \to GF(p^e), x \mapsto x^p\). Then, by Dedekind’s law, we can write
\[(4)\quad G^* = PSL(2, q) \times (G^* \cap \langle \tau_\alpha \rangle).\]

Defining \(P\Sigma L(2, q) = PSL(2, q) \times \langle \tau_\alpha \rangle\), it can easily be calculated that
\(P\Sigma L(2, q)_{0,1,\infty} = \langle \tau_\alpha \rangle\), and \(\langle \tau_\alpha \rangle\) has precisely \(p + 1\) distinct fixed points
(cf., e.g., [11, Ch. 6.4, Lemma 2]). As \(p > 3\), we conclude therefore that
\(G^* \cap \langle \tau_\alpha \rangle \leq G^*_B\) for some appropriate, unique block \(B \in \mathcal{B}\) by the definition
of Steiner 6-designs. Furthermore, clearly \(PSL(2, q) \cap (G^* \cap \langle \tau_\alpha \rangle) = 1\). Hence, we have
\[
\begin{align*}
|B^{G^*}| &= |G^*: G^*_B| \\
&= [PSL(2, q) \times (G^* \cap \langle \tau_\alpha \rangle) : PSL(2, q)_B \times (G^* \cap \langle \tau_\alpha \rangle)] \\
&= [PSL(2, q) : PSL(2, q)_B] \\
&= |B^{PSL(2,q)}|.
\end{align*}
\]
Thus, if we assume that \(G^* \leq \text{Aut}(\mathcal{D})\) acts already block-transitively on \(\mathcal{D},\)
then we obtain \(|B^{G^*}| = |B^{PSL(2,q)}| = b\) in view of Remark 10. Hence, \(PSL(2, q)\) must also act block-transitively on \(\mathcal{D},\) and we may proceed as in
the case when \(N = G\). Therefore, let us assume that \(G^* \leq \text{Aut}(\mathcal{D})\) does not
act block-transitively on \(\mathcal{D}\). Then, we conclude that \([G : G^*] = 2\) and \(G^*\) has
exactly two orbits of equal length on the set of blocks. Thus, by Equation (5),
we obtain for the orbit containing the block \(B\) that \(|B^{G^*}| = |B^{PSL(2,q)}| = b_2\).
As it is well-known the normalizer of \(PSL(2, q)\) in \(\text{Sym}(X)\) is \(P\Sigma L(2, q),\)
and hence in particular \(PSL(2, q)\) is normal in \(G\). It follows therefore that
we have under \(PSL(2, q)\) also precisely one further orbit of equal length on
the set of blocks. Then, proceeding similarly to the case \( N = G \) for each orbit on the set of blocks, we have (representative for the orbit containing the block \( B \)) that
\[
\frac{(q - 2)(q - 3)(q - 4)}{2} \mid_{\text{PSL}(2,q)_B} n = k(k - 1)(k - 2)(k - 3)(k - 4)(k - 5),
\]
which gives
\[
(q - 2)(q - 3)(q - 4) \mid_{\text{PSL}(2,q)_B} = k(k - 1)(k - 2)(k - 3)(k - 4)(k - 5),
\]
as here \( n = 2 \). Using again
\[
q - 4 \geq (k - 4)(k - 5),
\]
we obtain
\[
(q - 2)(q - 3) \mid_{\text{PSL}(2,q)_B} \leq k(k - 1)(k - 2)(k - 3).
\]
If we assume that \( k \geq 21 \), then again
\[
k(k - 1)(k - 2)(k - 3) < 2[(k - 4)(k - 5)]^2,
\]
and thus
\[
(q - 2)(q - 3) \mid_{\text{PSL}(2,q)_B} < 2(q - 4)^2,
\]
which is only possible when \( \mid_{\text{PSL}(2,q)_B} = 1 \). But, involutions in \( \text{PSL}(2,q) \) have precisely two fixed points on the points of the projective line for \( q \equiv 1 \pmod{4} \) and are fixed point free for \( q \equiv 3 \pmod{4} \). Hence, each involution always fixes a unique block by the definition of Steiner 6-designs, a contradiction. The few remaining possibilities for \( k < 21 \) can again easily be ruled out by hand.

Now, let \( p = 2 \). Then, clearly \( N = \text{PSL}(2,q) = \text{PGL}(2,q) \), and we have \( \text{Aut}(N) = \Sigma \Sigma L(2,q) \). If we assume that \( \langle \tau_\alpha \rangle \leq \Sigma \Sigma L(2,q)_B \) for some appropriate, unique block \( B \in \mathcal{B} \), then, using the terminology of (4), we have \( G^* = G = \Sigma \Sigma L(2,q) \) and as clearly \( \text{PSL}(2,q) \cap \langle \tau_\alpha \rangle = 1 \), we can apply Equation (5). Thus, \( \text{PSL}(2,q) \) must also be block-transitive, which has already been considered. Therefore, we may assume that \( \langle \tau_\alpha \rangle \not\leq \Sigma \Sigma L(2,q)_B \). Let \( s > 2 \) be a prime divisor of \( e = \mid \langle \tau_\alpha \rangle \mid \). As the normal subgroup \( H := (\Sigma \Sigma L(2,q)_{0,1,\infty})^s \leq \langle \tau_\alpha \rangle \) of index \( s \) has precisely \( p^s + 1 \) distinct fixed points (see, e.g., [11, Ch.6.4, Lemma 2]), we have \( G \cap H \leq G_B \) for some appropriate, unique block \( B \in \mathcal{B} \) by the definition of Steiner 6-designs. It can then be deduced that \( e = s^u \) for some \( u \in \mathbb{N} \), since if we assume for \( G = \Sigma \Sigma L(2,q) \) that there exists a further prime divisor \( \pi > 2 \) of \( e \) with \( \pi \neq s \), then \( H := (\Sigma \Sigma L(2,q)_{0,1,\infty})^\pi \leq \langle \tau_\alpha \rangle \) and \( H \) are both subgroups of \( \Sigma \Sigma L(2,q)_B \) by the block-transitivity of \( \Sigma \Sigma L(2,q) \), and hence \( \langle \tau_\alpha \rangle \leq \Sigma \Sigma L(2,q)_B \), a contradiction. Furthermore, as \( \langle \tau_\alpha \rangle \not\leq \Sigma \Sigma L(2,q)_B \), we may, by applying Dedekind’s law, assume that
\[
G_B = \text{PSL}(2,q)_B \rtimes (G \cap H).
\]
Thus, by Remark 10, we obtain
\[(q-2)(q-3)(q-4) | \text{PSL}(2, q)_B | G \cap H | = k(k-1)(k-2)(k-3)(k-4)(k-5) | G \cap \langle \tau \rangle |.
\]
More precisely:
(A) if \( G = \text{PSL}(2, q) \rtimes (G \cap H) \):
\[(q-2)(q-3)(q-4) | \text{PSL}(2, q)_B | = k(k-1)(k-2)(k-3)(k-4)(k-5)
\]
(B) if \( G = \text{PSL}(2, q) \):
\[(q-2)(q-3)(q-4) | \text{PSL}(2, q)_B | = k(k-1)(k-2)(k-3)(k-4)(k-5)s.
\]
As far as condition (A) is concerned, we may argue exactly as in the earlier case \( N = G \). Thus, only condition (B) remains. If \( e \) is a power of 2, then Remark 10 gives
\[(q-2)(q-3)(q-4) | G_B | = k(k-1)(k-2)(k-3)(k-4)(k-5)a
\]
with \( a \mid e \). In particular, \( a \) must divide \( |G_B| \), and we may proceed similarly as in the case \( N = G \), yielding a contradiction.

The case \( p = 3 \) may be treated, mutatis mutandis, as the case \( p = 2 \).

Case (3): \( N = M_v, v = 11, 12, 22, 23, 24 \).

By Corollary 5, we get \( k = 7 \) for \( v = 11 \) or 12, and \( k = 7 \) or 8 for \( v = 22, 23 \) or 24, and the very small number of cases for \( k \) can easily be eliminated by hand using Corollary 3 and Remark 10.

Case (4): \( N = M_{11}, v = 12 \).

As in Case (3), for \( v = 12 \), we have \( k = 7 \) in view of Corollary 5, a contradiction since no 6-(12, 7, 1) design can exist by Corollary 3.

This completes the proof of the Main Theorem.

Remark 11. The cases excluded from the Main Theorem remain elusive. One can slightly reduce the possible open cases by some sophisticated and lengthy work on condition (B) and the corresponding one for \( p = 3 \). This includes a detailed consideration of the orbit-lengths from the action of subgroups of \( \text{PSL}(2, q) \) on the points of the projective line (cf. [18]). More precisely, we obtain the equality
\[(q-2)(q-3)(q-4)6c = k(k-1)(k-2)(k-3)(k-4)(k-5)s,
\]
where \( q = p^s \), \( p = 2 \) or 3, \( s \) some odd prime power, \( s > 6c, c = 1, 2, 4 \) or 5. By Siegel’s classical theorem [34] on integral points on algebraic curves only a finite number of solutions are possible for fixed \( s \). However, with regard to the additional arithmetical conditions that are imposed in these cases, it seems to be very unlikely that admissible parameter sets of Steiner 6-designs can be found.
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