Loop cluster on the discrete circle

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Abstract

The loop clusters of Poissonian ensemble of Markov loops on a finite or countable graph have been studied in [LJL12]. In the present article, we study the loop clusters associated with a rotation invariant nearest neighbor walk on the n-th discrete circle $G(n)$. We show that the closed edges form a conditioned renewal process given that a particular edge is closed. Then we prove the convergence in the sense of Skorokhod of those conditioned renewal processes towards a conditioned subordinator as $n$ tends to infinity. Then, we present a description of the conditional distribution of loop clusters given that the number of clusters is strictly greater than 1 in the scaling limit. We also calculate the probability that the number of clusters is equal to 1 in the scaling limit. In the final section, we give another informal explanation of the above convergence results.

1 Introduction

Consider a discrete circle $G(n)$ with $n$ vertices $1, \ldots, n$ and $2n$ directed edges

$$E(n) = \{(1, 2), (2, 3), \ldots, (n - 1, n), (n, 1), (2, 1), (3, 2), \ldots, (n, n - 1), (1, n)\}.$$ 

Define the clockwise edges set $E^+(n) = \{(1, 2), (2, 3), \ldots, (n - 1, n), (n, 1)\}$ and the counter clockwise edges set $E^-(n) = E(n) \setminus E^+(n)$. Consider a Markovian generator $L(n)$ such that for any $e = (e-, e+) \in E^+(n)$, $(L(n))_{e+} = p_n, (L(n))_{e-} = 1 - p_n, (L(n))_{e-} = -(1 + c_n)$ and $L(n)$ is null elsewhere. Set $(Q(n))_g^x = 1_{x \neq g} \frac{(L(n))_g^x}{(L(n))_g^-}$. As [LJ11] and [Szn12], we define a loop measure and a Poissonian loop ensemble associated with $L(n)$. The non-trivial

*The main part of the work is done during my PHD in the department of mathematics of Université Paris-Sud, Orsay, France
A pointed loop 1 measure is given by the following expression of the mass of the pointed loop \( l = (x_1, \ldots, x_k) \) \((k \geq 2)\):

\[
\mu_n(l = (x_1, \ldots, x_k)) = \frac{1}{k} (Q(n))^{x_1} \cdots (Q(n))^{x_{k-1}} (Q(n))^{x_k}.
\]  

(1)

A loop is an equivalence class of pointed loops. The loop measure is the corresponding push-forward measure on the space of loops. For simplicity of notations, we use the same notation \( l \) for a loop and \( \mu \) for a loop measure.

Denote by \( L^{\alpha}_n \) the Poisson ensemble (or “loop soup”) of non-trivial loops with intensity measure \( \alpha \mu_n \) where \( \alpha > 0 \) is a fixed parameter. As in [LJL12], we define the loop cluster model as follows: two vertices are in the same cluster iff. they are linked through a sequence of connecting non-trivial loops. Another equivalent way is to define the closed edges: an undirected edge \{x, y\} is closed iff. there is no loop in the loop soup \( L^{\alpha}_n \) which covers \((x, y)\) or \((y, x)\). Otherwise, we say the undirected edge \{x, y\} is open. Then the loop cluster is just the cluster connected by open edges. In [LJL12], as one of the examples, it is shown that the closed edges in loop cluster model on \( \mathbb{N} \) form a renewal process. Moreover, the scaling limit of that process is a subordinator with potential density \( U(x, y) = \left(\frac{\sqrt{\kappa}}{1 - e^{-2|x-y|\sqrt{\kappa}}}\right)^{\alpha} \), see Section 3 of [LJL12] in which this is proved in the finite marginal sense.

In this article, we consider the loop cluster \( C^{\alpha}_n \) in the discrete circle \( G^{(n)} \). To be more precise, \( C^{\alpha}_n \) is the collection of discrete arc separated by the closed edges.

In Section 2, we calculate the probability that an edge is closed. Conditionally on \{\{1, n\} is closed\}, we show that the closed edges in this model form a conditioned renewal process. It has the same distribution as closed edges in the model of a discrete interval. Thus, similar conditional result of the closed edges holds given that the loops avoid a particular vertex.

In Section 3, we identify the scaling limit of the conditioned renewal process as a subordinator conditioned to approach 1 continuously.

In Section 4, we strengthen the above convergence result to a convergence in the sense of Skorokhod, see Theorem 4.5.

In Section 5, we calculate the limit distribution of the cluster containing a particular vertex. By combining with the previous result of the limit distribution of the closed edges in

\[1\] A pointed loop \( l = (x_1, \ldots, x_k) \) is a bridge on the graph from \( x_1 \) back to itself: \( x_1 \to x_2 \to \cdots \to x_k \to x_1 \). For a pointed loop \( l = (x_1, \ldots, x_k) \), \( k \) is called the length of this pointed loop. A loop is non-trivial iff. \( k \geq 2 \).

\[2\] Two pointed loop are equivalent iff. they are the same under a circular permutation.
a discrete interval, we give a description of the limit conditional distribution of clusters given that the number of clusters is strictly larger than 1. Under the assumption that 
\[ \lim_{n \to \infty} n^2 c_n = \epsilon, \]
we calculate the probability that there is only one cluster in the scaling limit. We summarize these results in Theorem 5.10.

In the final section, we provide an informal explanation for the convergence results from the point of view of the convergence of Poissonian loop ensembles.

Finally, we briefly present the difficulties and the techniques in the following. We would like to make use of the convergence result of the unconditioned processes proved in [LJL12]. Thus, we firstly calculate the Radon-Nikodym derivatives of the conditioned renewal process (resp. subordinator) with respect to the unconditioned renewal process (resp. subordinator) on a family of sub-\(\sigma\)-fields. Then, we show the convergence of the Radon-Nikodym derivatives on these sub-\(\sigma\)-fields. Together with the convergence result in [LJL12], we conclude the convergence of the corresponding conditioned processes. In order to strengthen the convergence result to a convergence in the sense of Skorokhod, we need the tightness of the family of conditioned renewal process which is equivalent to a uniform control of the càdlàg modulus of continuity. Thanks to the stationary and independent increments, we get the tightness of the unconditioned process as an application of Aldous’ criteria, see Lemma 3.1. The difficulty of passing through the unconditioned processes to the conditioned processes is due to the absence of the Radon-Nikodym derivatives between the conditioned processes and the corresponding unconditioned processes on the whole path. Then, we have to cut the whole path into two parts and then prove the tightness for each part. For the first half part of the paths, the Radon-Nikodym derivatives exist and hence the tightness is established. For the second half part of the paths, we use time reversal to get back to the first half part. For that purpose, we study in the end of Section 3 the left limit at the lifetime and the time reversal distribution of the conditioned subordinator. Finally, in order to get the limit distribution of the clusters in the multi-clusters case, we have to calculate the limit distribution of the cluster containing a particular vertex, e.g. \(x_0\). More precisely, recall that the edges uncovered by those loops avoiding the particular vertex \(x_0\) is described by the closure of the range of a conditioned subordinator in the scaling limit. Now, we consider the loops passing through the vertex \(x_0\) which are not too large to cover the whole space. This cluster might cover

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3The law of the subordinator and that of the conditioned one are singular to each other. That is the reason we calculate the derivatives on some sub-\(\sigma\)-fields.

4Theorem 3.21 in [JS03] explains this equivalence.

5In fact, we prove the Skorokhod convergence of the first half part by showing the convergence of the derivatives. The tightness comes from the Skorokhod convergence.
some edges which are not covered by the loops avoiding \( x_0 \). Accordingly, we erase a part of the range of the conditioned subordinator. Then, the remaining part of the range of the subordinator represents the closed edges in the scaling limit. For this part, the key is the independence between the loops avoiding \( x_0 \) and those loops passing through \( x_0 \) which is guaranteed by the Poisson loop soup construction. In order to express the results explicitly, we calculate the Lévy measure of the subordinator in Lemma 5.6 by inverse Laplace transform.

2 Closed edges as a conditioned renewal process

In order to calculate the probability that a particular edge is closed, we need the following classical result on the determinant of Toeplitz matrices. Please refer to Proposition 2.2 and Example 2.8 in [BG05].

Lemma 2.1 ([BG05]). Let \( T_{3,n} \) be a \( n \times n \) tri-diagonal Toeplitz matrix of the following form:

\[
\begin{bmatrix}
  a & b & 0 & \cdots & 0 \\
  c & a & b & \ddots & \\
  0 & \ddots & \ddots & \ddots & 0 \\
  \vdots & \ddots & c & a & b \\
  0 & \cdots & 0 & c & a
\end{bmatrix}
\]

Let \( S_n \) be the following circulant \( n \times n \) matrix:

\[
\begin{bmatrix}
  a & b & 0 & c \\
  c & a & b & \ddots \\
  0 & \ddots & \ddots & \ddots & 0 \\
  \vdots & \ddots & c & a & b \\
  b & 0 & c & a
\end{bmatrix}
\]

Let \( x_1, x_2 \) be the roots of \( x^2 - ax + bc = 0 \). Then,

- \( \det(T_{3,n}) = \frac{x_1^{n+1} - x_2^{n+1}}{x_1 - x_2} \),
- \( \det(S_n) = x_1^n + x_2^n + (-1)^{n+1}(b^n + c^n) \).

One can check the following result from the definition of pointed loop measure.

Lemma 2.2. Define a modified generator \( \tilde{L}^{(n)} \) from \( L^{(n)} \) by replacing \( (L^{(n)})^1_1 \) and \( (L^{(n)})^n_1 \) by 0. Let \( \tilde{\mu}_n \) be the pointed loop measure associated with \( \tilde{L}^{(n)} \). Then,

\[
\mu_n(N^1_n(l) + N^n_1(l) = 0, dl) = \tilde{\mu}_n(dl).
\]
As a corollary, conditionally on that the edge \( \{1, n\} \) is closed, the loop soup \( \mathcal{L}^{(n)}_{\alpha} \) is the Poisson loop ensemble of intensity measure \( \alpha \tilde{\mu}_n \).

Next, we present some useful properties which are frequently used throughout the paper.

**Definition 2.1.** Let \( F \) be a subset of the state space \( S \) and \( l \) is a loop on \( S \). We say \( l \) is inside \( F \) if \( l \) does not visit any vertex in \( S \setminus F \), which is denoted by \( l \subset F \).

**Lemma 2.3.** Let \( \mu \) be the Markovian loop measure associated with a generator \( L \) on a state space \( S \). Let \( F \) be a finite subset of the state space \( S \). Then, \( \mu(l \text{ is non-trivial}, l \subset F, dl) \) is the Markovian loop measure associated with the generator \( L|_{F \times F} \). Moreover,

\[
\mu(l \text{ is non-trivial and } l \subset F) = -\log \det(-L|_{F \times F}) + \sum_{x \in F} \log(-L^x_x).
\]

**Proof.** One can check from definition that \( \mu(l \subset F, dl) \) is the Markovian loop measure associated with the generator \( L|_{F \times F} \). It remains to prove that for a Markovian loop measure \( \mu \) associated with generator \( L \) on a finite state space \( S \),

\[
\mu(\text{non-trivial loops}) = -\log \det(-L) + \sum_{x \in S} \log(-L^x_x).
\]

We see that

\[
\mu(\text{non-trivial loops}) = \sum_{k \geq 2} \frac{1}{k} \text{Tr} Q^k = \sum_{k \geq 1} \frac{1}{k} \text{Tr} Q^k = -\text{Tr} \log(I - Q)
\]

\[
= -\log \det(I - Q) = -\log \det(-L) + \sum_{x \in S} \log(-L^x_x)
\]

where \( Q^x_y \triangleq \begin{cases} -\frac{L^x_y}{L^y_y} & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases} \)

Then, we give in the following proposition the probability that a particular edge is closed.

**Proposition 2.4.** Set

\[
x_1 = \frac{1}{2} (1 + c + \sqrt{(1 + c)^2 - 4p(1 - p)}),
\]

\[
x_2 = \frac{1}{2} (1 + c - \sqrt{(1 + c)^2 - 4p(1 - p)}).
\]

Then,

\[
\mathbb{P}[\text{closed}] = \left( \frac{x_1^{n+1} - x_2^{n+1}}{(x_1 - x_2)(x_1^n + x_2^n - p^n - (1 - p)^n)} \right)^{-\alpha}.
\]

**Proof.** For a loop \( l \), denote by \( N^i_j(l) \) the number of jumps from \( i \) to \( j \). Then, for two adjacent vertices \( i \) and \( j \), a loop \( l \) visits both \( i \) and \( j \) iff. \( N^i_j(l) + N^j_i(l) > 0 \). Recall that
an undirected edge $\{x, y\}$ is closed iff. there is no loop jumping between $x$ and $y$ among the Poissonian loop soup. Then, we have
\[
\mathbb{P}\{1, n\ \text{is closed}\} = \exp(-\alpha \mu_n(1)(N^n_1(l) + N^n_1(l) > 0)) = \exp\{-\alpha(\mu_n(1) - \mu_n(N^n_1(l) + N^n_1(l) = 0))\}.
\]

By Lemma 2.2,
\[
\mathbb{P}\{1, n\ \text{is closed}\} = \exp\{-\alpha(\mu_n(1) - \tilde{\mu}_n(1))\}.
\]
where $\tilde{\mu}_n$ is the same as in Lemma 2.2. Then, by Lemma 2.3 and Lemma 2.1,
\[
\mathbb{P}\{1, n\ \text{is closed}\} = \exp\{-\alpha \left(\frac{x_1^{n+1} - x_2^{n+1}}{(x_1 - x_2)(x_1^n + x_2^n - p^n - (1-p)^n)}\right)^{-\alpha}\}
\]
where $x_1 = \frac{1}{2}(1+c+\sqrt{(1+c)^2 - 4p(1-p)})$ and $x_2 = \frac{1}{2}(1+c-\sqrt{(1+c)^2 - 4p(1-p)})$.

In the following context, we consider the loop cluster conditionally on $\{1, n\}$ being closed. By deleting the edge $\{1, n\}$ and adding $\{0, 1\}$ and $\{n, n+1\}$, we get a discrete segment with vertices $\{0, 1, \ldots, n, n+1\}$ and undirected edges $\{\{0, 1\}, \ldots, \{n, n+1\}\}$. We see that it turns out to be the loop cluster model on that discrete segment.

**Proposition 2.5.** Conditionally on $\{1, n\}$ being closed, the left points of these closed edges, together with the left points of $\{0, 1\}$ and $\{n, n+1\}$, form a renewal process conditioned to jump at $n$.

**Proof.** Among the Poissonian loop ensemble, the ensemble of loops crossing the edge $\{1, n\}$ is independent of its complement in the Poissonian loop ensemble. Therefore, the law $Q$ of the loops avoiding the edge $\{1, n\}$ conditionally on (no loop crosses $\{1, n\}$) is exactly the same as the unconditioned law. Consider another Poissonian loop ensembles on $\mathbb{Z}$ associated with the following generator:
\[
L_m^m = -(1 + c_n), L_{m+1}^m = p_n, L_{m-1}^m = 1 - p_n \text{ for all } m \in \mathbb{Z}, \text{ and } L \text{ is null elsewhere}.
\]

Then, $Q$ equals the conditional law of the loop ensembles contained in $\{1, \ldots, n\}$ given the condition that $\{0, 1\}$ and $\{n, n+1\}$ are closed. It is known that the loop measure is invariant under Doob’s $h$-transform, see Remark 1.1 of [LJL12]. Thus, we can perform on $L$ a harmonic transform and modify $L$ as follows:
\[
L_m^m = -(1 + c_n), L_{m+1}^m = L_{m-1}^m = \sqrt{p_n(1 - p_n)} \text{ for all } m \in \mathbb{Z}, \text{ and } L \text{ is null elsewhere}.
\]

According to Proposition 3.1 in [LJL12], in the case of loop cluster model on $\mathbb{Z}$, conditionally on $\{\{0, 1\} \text{ is closed}\}$, the left points of the closed edges form a renewal process. Finally, in our situation, conditionally on $\{1, n\}$ being closed, we can identify the left end points of closed edges as a renewal process conditioned to jump at $n$. \qed
Remark 2.1. It is not hard to find the correspondence between the killing parameter $\kappa$ in [LJL12] and our parameters $c$ and $p$:

$$\kappa = \frac{1 + c - 2\sqrt{p(1-p)}}{\sqrt{p(1-p)}}.$$  

3 Finite-marginal convergence towards a conditioned subordinator

In the following context, we always assume the following relation between the parameters $c$, $p$ and $\kappa$:

$$\kappa = \frac{1 + c - 2\sqrt{p(1-p)}}{\sqrt{p(1-p)}}.$$  

As mentioned in the proof of Proposition 2.5, conditionally on $\{1, n\}$ is closed, we can go back to the symmetric loop cluster model on $\mathbb{Z}$ by Doob’s $h$-transform. More precisely, we use the following modified model: consider a pure-jump Markov process on $\{1, \ldots, n, \ldots\}$ with generator $L$ such that $L_{m+1}^m = L_{m}^m = 1/2$, $L_m^m = -(1 + \kappa/2)$ for $m \in \mathbb{N}_+$ and $L$ is null elsewhere. Then, associated with this $L$, we have a loop measure $\mu^{(\kappa)}$ and a Poisson point process of loops of intensity $\alpha \mu^{(\kappa)}$. The corresponding loop probability depends on $\kappa$ and we will denote it by $P^{(\kappa)}$.

In the following context, we always assume the following whenever the domain of $\alpha$ is not specified.

Hypothesis 3.1. Assume $\alpha \in [0, 1]$.

For a càdlàg process $X$ and a subset $A$ of the state space, denote by $T_A$ the entrance time of $A$: $T_A \overset{\text{def}}{=} \{ t \geq 0 : X_t \in A \}$. We denote by $X_{<}$ the left hand limit $\lim_{s\uparrow t} X_s$.

It has been showed in [LJL12] that the left points of the closed edges form a renewal process $(S_m^{(\kappa)}, m \geq 0)$ $(S_0^{(\kappa)} = 0)$, see Proposition 3.1 in [LJL12]. Moreover, in Proposition 3.1 of [LJL12], it has been proved that $(\varepsilon S_{[\varepsilon a^{-1}]}^{(\kappa)}, t \geq 0)$ converges to a subordinator $(X_t^{(\kappa)}, t \geq 0)$ with potential density $U(x, y) = 1_{\{y > x\}} \left( \frac{2\sqrt{\kappa}}{1 - e^{-2\sqrt{\kappa}(y - x)}} \right)^\alpha$ in the sense of finite marginals distribution. Once we show the tightness in the sense of Skorokhod, we could replace the finite marginals convergence by the convergence in law in the sense of Skorokhod.

Lemma 3.1 (Tightness of $(\varepsilon S_{[\varepsilon a^{-1}]}^{(\kappa)}, t \geq 0)$). The distribution $(\varepsilon S_{[\varepsilon a^{-1}]}^{(\kappa)}, t \geq 0)$ is tight in the Skorokhod space. Therefore, $(\varepsilon S_{[\varepsilon a^{-1}]}^{(\kappa)}, t \geq 0)$ converges to a subordinator $(X_t^{(\kappa)}, t \geq 0)$ in the sense of Skorokhod.
Let $\mathcal{F}_t^{(\epsilon)}$ be a renewal process, for $\mathcal{F}_t^{(\epsilon)}$ conditioned to jump at the left point of a renewal process, for $\mathcal{F}_t^{(\epsilon)}$ conditioning on the event $\{n, n+1\}$ being closed, the left points of the closed edges together with the left point of $\{0,1\}$ form a renewal process conditioned to jump at $n$. Define a conditioned loop probability as follows: $\mathbb{P}^{n,\kappa}[\cdot] = \mathbb{P}^{(\kappa/n^2)}[\cdot|\{n, n+1\} \text{ is closed}].$ Let $\mathbb{Q}^{n,\kappa}$ be the law of $Z^{(\kappa/n^2)}$ under $\mathbb{P}^{n,\kappa}$. As $n$ tends to infinity, under $\mathbb{Q}^{n,\kappa}$, $(Z^{(\kappa/n^2)}_{n+1-\epsilon} / n, t \geq 0)$ converges in law to $(X^{(\kappa)}_{T_{1,\infty}^-} = 1)$ in the sense of finite marginal convergence.

Before proving this, let us describe the law of $(X^{(\kappa)}_t, t < T_{1,\infty})$ conditionally on the event $\{X^{(\kappa)}_{T_{1,\infty}^-} = 1\}$ in the following lemma. Recall that the potential density $U(x, y)$ of $X^{(\kappa)}_t$ equals $1_{\{y>x\}} \left(\frac{2v\kappa}{1-e^{-2v\kappa(y-x)}}\right)^\alpha$. Set $u(x) = U(0, x)$ and $h(x) = U(x, 1) = u(1-x)$.

Lemma 3.3.
1. For all positive functions \( f \), we have

\[
\mathbb{E}^0[f(X_s^{(s)}, s \in [0, t])1_{\{t < T_{1, \infty}\}}, \ X_{T_{1, \infty}}^{(s)} \in db] = \mathbb{E}^0[X_{T_{1, \infty}}^{(s)} \in db] \mathbb{E}^0[f(X_s^{(s)}, s \in [0, t])1_{\{t < T_{1, \infty}\}} u(b - X_t^{(s)}) / u(b) \].
\]

2. The conditioned process \( Y^{(s)} \) is a \( h \)-transform of the original subordinator with respect to the excessive function \( x \rightarrow u(1 - x) \). To be more precise, for \( y \in [x, 1] \), its semi-group is given by

\[
Q_t^{(s)}(x, dy) = \frac{u(1 - y)}{u(1 - x)} P_t^{(s)}(x, dy).
\]

Let \( \mathbb{Q}^x \) stand for the law of the Markov process with sub-Markovian semi-group \( Q_t^{(s)}(x, dy) = \frac{u(1 - y)}{u(1 - x)} P_t^{(s)}(x, dy) \) and initial state \( x \). (We choose the càdlàg version of \( Y^{(s)} \).)

3. Denote by \( \zeta \) the lifetime of the conditioned process \( Y^{(s)} \). Then, \( Y_{\zeta -}^{(s)} = 1 \).

4. The semi-group \( Q \) is a Feller semi-group.

5. The time reversal from the lifetime of the process \( Y^{(s)} \) is the left-continuous modification of \( 1 - Y^{(s)} \) under \( \mathbb{Q}^0 \).

**Proof.**

1. The subordinator \( (X_t^{(s)}, t \geq 0) \) has the potential density \( U(x, y) = \left(\frac{2\sqrt{y}}{1 - e^{-2\sqrt{y}(y - x)}}\right)^\alpha \) for \( y > x \). When \( y \) tends to \( x \), \( U(x, y) \) tends to \( \infty \). As a consequence, the drift coefficient \( d = 0 \), see Proposition 1.7 in [Ber99]. It is proved by H. Kesten [Kes69] that for a fixed \( x > 0 \), \( x \) does not belong to the range of the subordinator with probability 1, see Proposition 1.9 in [Ber99]. By applying the strong Markov property to any stopping time \( S \),

\[
\mathbb{E}^0[f(X_s^{(s)}, s \in [0, S])1_{\{S < T_{1, \infty}\}}, \ X_{T_{1, \infty}}^{(s)} \in db] = \mathbb{E}^0[f(X_s^{(s)}, s \in [0, S])1_{\{S < T_{1, \infty}\}}] \mathbb{E}^X_s[X_{T_{1, \infty}}^{(s)} \in db].
\]

Then, we use Lemma 1.10 in [Ber99]:

- \( \mathbb{E}^X_s[X_{T_{1, \infty}}^{(s)} \in db] = \Pi(1 - b)u(b - X_s^{(s)}) \, db \)
- \( \mathbb{E}^0[X_{T_{1, \infty}}^{(s)} \in db] = \Pi(1 - b)u(b) \, db = \frac{u(b)}{u(b - X_s^{(s)})} \mathbb{E}^X_s[X_{T_{1, \infty}}^{(s)} \in db].\)

\(^6\)More precisely, the process defined by the probability \( \mathbb{E}^0[f(X_s^{(s)}, s \in [0, t])1_{\{t < T_{1, \infty}\}} u(b - X_s^{(s)}) / u(b)], \)

\(^7\)Here, \( \Pi \) represents the tail of the characteristic measure of the subordinator.
Then, we see that
\[
\mathbb{E}^0 \left[ f(X^x_s, s \in [0, S]) 1_{\{s < T_{1, \infty}\}} \mathbb{E}^{X^x_s} \left[ X^{(x)}_{T_{1, \infty}| -} \in db \right] \right] 
= \mathbb{E}^0[X^{(x)}_{T_{1, \infty}| -} \in db] \mathbb{E}^0 \left[ f(X^x_s, s \in [0, S]) 1_{\{s < T_{1, \infty}\}} \frac{u(b - X^x_s)}{u(b)} \right].
\]

In particular, for fixed time \( t \):
\[
\mathbb{E}^0[f(X^x_s, s \in [0, t]) 1_{\{t < T_{1, \infty}\}}] X^{(x)}_{T_{1, \infty}| -} = 1
= \mathbb{E}^0 \left[ f(X^x_s, s \in [0, t]) 1_{\{t < T_{1, \infty}\}} \frac{u(1 - X^x_s)}{u(1)} \right]. \quad (*)
\]

2. It is enough for us to show that \( x \to u(1 - x) = U(x, 1) \) is excessive. Then, the rest will follow from the classical results on the \( h \)-transform, see Chapter 11 of [CW05].

Take a positive function \( g \), we have \( P_t U g = \int_0^\infty P_t g ds \) and \( U g = \int_0^\infty P_t g ds \). Then, for all positive function \( g \), we have \( P_t U g \leq U g \) and \( P_t U g \) increases to \( U g \) as \( t \) decreases to 0. As a consequence, except for a set \( N \) of \( z \) of zero Lebesgue measure, \( y \to u(y, z) \) is an excessive function, i.e.

- \( \int P_t(x, d) u(y, z) \leq u(x, z) \),
- \( \lim_{t \to 0} P_t(x, d) u(y, z) = u(x, z) \).

Take a decreasing sequence \( (z_n)_n \) with limit 1 which is outside of the negligible set \( N \). As the increasing limit of a sequence of excessive functions \( y \to u(y, z_n) \), \( y \to u(y, 1) \) is excessive.

3. Before providing the proof, we would like to give a short explanation. From the symmetry of the loop model on the discrete segment, the graphs of the conditional renewal processes are centrosymmetric. Then, as the scaling limit, the conditional subordinator has a centrosymmetric graph. Thus, \( Y^{(x)}_{\zeta^-} = 1 \) is equivalent to \( Y^{(x)}_{0+} = 0 \) which is obviously true.

In the following, we will not use the discrete approximation described above. Instead, we will prove that \( Q^x[X_{T_{1-\delta, \infty}} \in [0, 1]] = Q^x[X_{T_{0+}} \in [0, 1]] \) which is motivated by the idea of time reversal.

Let’s begin to prove \( Y^{(x)}_{\zeta^-} = 1 \). In order to prove this, it is enough to show that
\[
Q^x[T_{1-\delta, \infty} < \zeta] = 1 \text{ for all } \delta > 0.
\]

By applying Theorem 11.9 of [CW05] to the stopping time \( T_{1-\delta, \infty} \),
\[
Q^x[T_{1-\delta, \infty} < \zeta] = P^x \left[ T_{1-\delta, \infty} < T_{1, \infty}, \frac{u(1 - X_{T_{1-\delta, \infty}})}{u(1 - x)} \right].
\]
If $X$ follows the law $\mathbb{P}^0$, then $X + x$ has the law $\mathbb{P}^x$. Therefore, the above quantity equals

$$\mathbb{P}^0 \left[ T_{1-x-\delta,\infty} < T_{1-x,\infty}, \frac{u(1-x - X_{T_{1-x-\delta,\infty}})}{u(1-x)} \right].$$

By Lemma 1.10 in [Ber99], for $0 \leq a < 1 - x - \delta \leq a + b$, we have

$$\mathbb{P}^0 [ X_{T_{1-x-\delta,\infty}} - X_{T_{1-x-\delta,\infty}} \in da, X_{T_{1-x-\delta,\infty}} - X_{T_{1-x-\delta,\infty}} \in db ] = u(a) da \Pi(db).$$

Consequently,

$$Q^x [ T_{1-\delta,\infty} < \zeta ] = \int_{0 < a < 1 - x - \delta < a + b < 1 - x} \frac{u(1-x - a - b)}{u(1-x)} u(a) da \Pi(db).$$

Performing the change of variable $c = 1 - x - a - b$:

$$Q^x [ T_{1-\delta,\infty} < \zeta ] = \int_{0 < c < 1 - x - \delta < c + b < 1 - x} \frac{u(c)}{u(1-x)} u(1-x - c - b) dc \Pi(db)$$

$$= \mathbb{P}^0 \left[ T_{\delta,\infty} < T_{1-x,\infty}, \frac{u(1-x - X_{T_{\delta,\infty}})}{u(1-x)} \right]$$

$$= \mathbb{P}^x \left[ T_{1-x+\delta,\infty} < T_{1,\infty}, \frac{u(1-x - X_{T_{1-x+\delta,\infty}})}{u(1-x)} \right]$$

$$= Q^x [ T_{1-x+\delta,\infty} < \zeta ].$$

Then, we have that

$$Q^x [ T_{1-\delta,\infty} < \zeta ] = Q^x [ T_{1-x+\delta,\infty} < \zeta ].$$

By the right-continuity of the path,

$$\lim_{\delta \to 0} Q^x [ T_{1-x+\delta,\infty} < \zeta ] = 1.$$

Then,

$$\lim_{\delta \to 0} Q^x [ T_{1-\delta,\infty} < \zeta ] = \lim_{\delta \to 0} Q^x [ T_{1-x+\delta,\infty} < \zeta ] = 1.$$

Since $a \to Q^x [ T_{1-x+a,\infty} < \zeta ]$ is non-increasing, we must have

$$Q^x [ T_{y,\infty} < \zeta ] = 1 \text{ for } y \in [x, 1[.$$

4. We know that $P_t^{(x)}$ is a Feller semi-group. For $f \in C_K([0, 1])$, $x \to Q_t f(x)$ belongs to $C_K([0, 1])$. ($C_K([0, 1])$ denotes the collection of compact supported continuous functions over $[0, 1]$ and

$$C_0([0, 1]) = \{ f : [0, 1] \to \mathbb{R} : f \text{ is continuous and } \lim_{x \to 1} f(y) = 0 \}.$$

)
By the Markov property of the semi-group $Q_t$, $\|Q_t f\|_{\infty} \leq \|f\|_{\infty}$ for every $f \in C_0([0, 1])$. Thus, we have $Q_t f = \lim_{n \to \infty} Q_i(f|_{[0,1-1/n]}) \in C_0([0, 1])$. For fixed $x \in [0, 1]$ and $f \in C_0([0, 1])$,

$$
\lim_{t \to 0} Q_t f(x) = \lim_{t \to 0} \mathbb{P}_x \left[ 1_{\{t<T\}} f(X_t^{(\kappa)}) \frac{u(1-X_t^{(\kappa)})}{u(1-x)} \right] \text{ dominated convergence } f(x).
$$

Then we see that the semi-group $(Q_t, t \geq 0)$ is Feller.

5. By a classical result about time reversal, the reversed process is a moderate Markov process, its semi-group $\hat{Q}_t(x, dy)$ is given by the following formula:

$$
\langle g, Q_t f \rangle_G = \langle \hat{Q}_t g, f \rangle_G.
$$

where $Q_t(x, dy) = \frac{U(y,1)}{U(x,1)} P_t(x, dy)$ and $G(dx) = \int_0^\infty Q_t(0, dx) dt = \frac{U(0,x)U(x,1)}{U(0,1)} dx$. Denote by $(\hat{P}_t, t \geq 0)$ the dual semi-group of $(P_t, t \geq 0)$ (or the semi-group of $-X^{(\kappa)}$ equivalently). Denote by $u(x)$ the function $U(0, x)$ and by $h(x)$ the function $U(x, 1)$.

$$
\langle g, Q_t f \rangle_G = \int_0^1 \frac{P_t(hf)(x)}{h(x)} g(x) \frac{u(x)h(x)}{u(1)} dx.
$$

Then we use the duality between $P_t$ and $\hat{P}_t$:

$$
\langle g, Q_t f \rangle_G = \int_0^1 f(x) \frac{\hat{P}_t(ug)}{u(x)} \frac{u(x)h(x)}{u(1)} dx = \langle f, \frac{\hat{P}_t(ug)}{u} \rangle_G.
$$

This implies that the semi-group $(\hat{Q}_t, t \geq 0)$ associated with the reversed process of $Y$ is given by

$$
\hat{Q}_t(x, dy) = \hat{P}_t(x, dy) \frac{U(0,y)}{U(0,x)} = \hat{P}_t(x, dy) \frac{U(1-y,1)}{U(1-x,1)}.
$$

Then, by a change of variable, we find that it equals to the semi-group of $1 - Y^{(\kappa)}$. By result 3 in this lemma, the reversed process starts from 1. Then, it is exactly the left-continuous modification of $1 - Y^{(\kappa)}$ for $Y^{(\kappa)}$ starting from 0.

\[\square\]

The above proposition gives the Radon-Nikodym derivative between the subordinator and its bridge on a sub-$\sigma$-field. We will prove Proposition 3.2 by showing the convergence of the Radon-Nikodym derivatives from the discrete case to the continuous case.
Proof of Proposition 3.2. Recall that $\mathbb{P}(\kappa/n^2)$ is the law of the loop model associated with the following generator on $\mathbb{N}$:

$$
\begin{bmatrix}
-1 - \frac{\kappa}{n^2} & 1/2 \\
1/2 & \ddots \\
\vdots & \ddots & \ddots
\end{bmatrix}.
$$

Define $f^{n,\kappa}(k) = \mathbb{P}(\kappa/n^2)[S_1^{(\kappa/n^2)} = k]$, $g^{n,\kappa}(k) = \mathbb{P}(\kappa/n^2)[S_1^{(\kappa/n^2)} \geq k]$ and $C^{\kappa/n^2}(k) = \mathbb{P}(\kappa/n^2)[\{k, k+1\} \text{ is closed}]$. Let $W_i^{(\kappa)} = S_i^{(\kappa/n^2)} - S_{i-1}^{(\kappa/n^2)}$ for $i \in \mathbb{N}_+$. As mentioned above, $(W_i, i \in \mathbb{N}_+)$ is a sequence of i.i.d. variables under $\mathbb{P}(\kappa/n^2)$. Then,

$$
d \hat{Q}^{n,\kappa} = \frac{f^{n,\kappa}(n - S_{T_{n-1}}^{(\kappa/n^2)})}{C^{\kappa/n^2}(n)g^{n,\kappa}(n - S_{T_{n-1}}^{(\kappa/n^2)})}
$$

where $T_p = \inf\{l \in \mathbb{N} : S_l^{(\kappa/n^2)} \geq p\}$. Let $\mathcal{F}_m = \sigma(S_0^{(\kappa/n^2)}, \ldots, S_m^{(\kappa/n^2)})$ and $\mathcal{G}_m = \sigma(Z_0^{(\kappa/n^2)}, \ldots, Z_m^{(\kappa/n^2)})$. Then, $\mathcal{G}_m \cap \{m < T_{n}\} = \mathcal{F}_m \cap \{m < T_{n}\}$ and $\mathcal{G}_m \cap \{T_{n} \leq m\} = \mathcal{F}_{T_{n-1}} \cap \{T_{n} \leq m\}$.

Under $\mathbb{P}(\kappa/n^2)$, $S^{(\kappa/n^2)}$ has stationary independent increments. Therefore, conditionally on $\mathcal{G}_m$, $(S_{m+1} - S_m, S_{m+2} - S_m, \ldots) \overset{\text{law}}{=} (S_1, S_2, \ldots)$. For that reason,

$$
\mathbb{E}(\kappa/n^2) \left[ 1_{\{T_{n} \geq m+1\}} \frac{f^{n,\kappa}(n - S_{m}^{(\kappa/n^2)} - (S_{T_{n-1}}^{(\kappa/n^2)} - S_{m}^{(\kappa/n^2)}))}{g^{n,\kappa}(n - S_{m}^{(\kappa/n^2)} - (S_{T_{n-1}}^{(\kappa/n^2)} - S_{m}^{(\kappa/n^2)}))} \bigg| \mathcal{G}_m \right] = 1_{\{T_{n} \geq m+1\}} C^{\kappa/n^2}(n - S_m^{(\kappa/n^2)})
$$

Thus,

$$
d \hat{Q}^{n,\kappa} \bigg|_{\mathcal{G}_m} = 1_{\{T_{n} \leq m\}} \frac{f^{n,\kappa}(n - S_{T_{n-1}}^{(\kappa/n^2)})}{C^{\kappa/n^2}(n)g^{n,\kappa}(n - S_{T_{n-1}}^{(\kappa/n^2)})} + 1_{\{T_{n} \geq m+1\}} \frac{C^{\kappa/n^2}(n - S_{m}^{(\kappa/n^2)})}{C^{\kappa/n^2}(n)}.
$$
Define \( \mathcal{G}^{(n)}_t = \mathcal{G}_{[n^{1-\alpha}t]} \) for \( t \geq 0 \). Then,

\[
\frac{d\tilde{Q}^{n,\alpha}}{d\tilde{Q}^{n,\alpha}} \bigg|_{\mathcal{G}^{(n)}_t} 1_{\{S^{(n/\alpha^2)}_{[n^{1-\alpha}t]/n<1}\}} = \frac{d\tilde{Q}^{n,\alpha}}{d\tilde{Q}^{n,\alpha}} \bigg|_{\mathcal{G}^{(n)}_t} 1_{\{S^{(n/\alpha^2)}_{[n^{1-\alpha}t]/n<1}\}} = 1_{\{S^{(n/\alpha^2)}_{[n^{1-\alpha}t]/n<1}\}} \frac{C^{\alpha/n^2}(n - S^{(n/\alpha^2)}_{[n^{1-\alpha}t]/n<1})}{C^{\alpha/n^2}(n)}.
\]

In the proof of Proposition 3.1 of [LJL12], it has been showed that

\[
C^{\alpha/n^2}(m) = \left( \frac{1 - (1 + \frac{\kappa}{2n^2} + \sqrt{\frac{\kappa^2}{n^4} + \frac{n^2}{3n^4}})^{-2}}{1 - (1 + \frac{\kappa}{2n^2} + \sqrt{\frac{\kappa^2}{n^4} + \frac{n^2}{3n^4}})^{-2}} \right)^\alpha
\]

(In fact, \( C^{\alpha/n^2}(m) \) is denoted by \( q^{\alpha/n^2}(m) \) there.) We deduce the following estimation for \( C^{\alpha/n^2}([bn]) \) as \( n \) tends to \( \infty \):

\[
C^{\alpha/n^2}([bn]) \sim \begin{cases} (\frac{2\sqrt{\kappa}}{1 - e^{-2\sqrt{\kappa}s}})^\alpha n^{-\alpha}, & \kappa > 0, \\ (bn)^{-\alpha}, & \kappa = 0. \end{cases}
\]

Moreover, for any compact subset \( K \subset [0, \infty] \), \( (C^{\alpha/n^2}([bn]))_{n>0} \) converges uniformly. From Lemma 3.1, we know that the sequence of renewal processes \( S^{(n/\alpha^2)} \) converges towards the subordinator \( X^{(\kappa)} \) in the sense of Skorokhod. By the coupling theorem of Skorokhod and Dudley, we can suppose that \( S^{(n/\alpha^2)} \) converges to \( X^{(\kappa)} \) almost surely as long as our result only depends on the law. The Proposition 3.1 in [LJL12] gives the density of the renewal measure of the subordinator \( (X^{(\kappa)}_t, t \geq 0) \):

\[
u(s) = U(0, s) = \left( \frac{2\sqrt{\kappa}}{1 - e^{-2\sqrt{\kappa}s}} \right)^\alpha \] for \( s > 0 \).

Since \( U(x, x+) = \infty \), the drift of the subordinator is zero, see Theorem 5 in Chapter 3 of [Ber96]. Then, by Theorem 4 in Chapter 3 of [Ber96], for any \( x > 0 \), \( X^{(\kappa)}_{T^{(\kappa)}_{[x, \infty)-}} > x > X^{(\kappa)}_{T^{(\kappa)}_{[1, \infty)-}} \) holds with probability 1. Thus, \( S^{(n/\alpha^2)}_{n^{-1}} \) converges to \( X^{(\kappa)}_{T^{(\kappa)}_{[1, \infty)-}} \) as \( n \) tends to infinity. Moreover, if we fix \( t > 0 \), \( 1_{\{S^{(n/\alpha^2)}_{[n^{1-\alpha}t]/n<1}\}} \) converges to \( 1_{\{t > 0\}} \frac{u(1 - X^{(\kappa)}_t)}{u(1)} \) almost surely. Consequently,

\[
\frac{d\tilde{Q}^{n,\alpha}}{d\tilde{Q}^{n,\alpha}} \bigg|_{\mathcal{G}^{(n)}_t} 1_{\{S^{(n/\alpha^2)}_{[n^{1-\alpha}t]/n<1}\}} \text{ converges to } 1_{\{t > 0\}} \frac{u(1 - X^{(\kappa)}_t)}{u(1)} \text{ almost surely.}
\]

Let \( Q^x \) stand for the law of the Markov process with sub-Markovian semi-group \( Q_t(x, dy) = \frac{u(1-y)}{u(1-x)} P_t(x, dy) \) and initial state \( x \). By Lemma 3.3,

\[
Q^x[1_A, t < \zeta] = \mathbb{P}^x \left[ 1_A \frac{u(1 - X^{(\kappa)}_t)}{u(1)} 1_{(t < \zeta)} \right] \text{ for } A \in \mathcal{F}_t
\]

where \( \mathbb{P}^x \) is the law of the process \( X^{(\kappa)} \) starting from \( x \). We fix any \( \delta > 0 \).
convergence out of the fixed time discontinuity set. Under Lemma 3.3. Thus, it is quasi-left continuous and therefore has no fixed time of discontinuity. We will strengthen convergence in the sense of Skorokhod.

Taking any bounded continuous function \( f \), by the coupling assumption and dominated convergence\(^8\), we have

\[
\lim_{n \to \infty} \hat{Q}^{n,\kappa} \left[ 1 - \frac{1}{n} Z^{(\kappa/n^2)}_{[n^{1-\alpha} t]} \right] = 1 = Q^0[X_t^{(\kappa)} < 1].
\]

Therefore, for any fixed \( t \), \( \frac{1}{n} Z^{(\kappa/n^2)}_{[n^{1-\alpha} t]} \) converges in law towards \( X_t^{(\kappa)} \) (under the law \( \hat{Q}^{n,\kappa} \) and \( Q^0 \) respectively) as \( n \) tends to infinity. In particular, we have

\[
\lim_{n \to \infty} Q^{n,\kappa} \left[ \left\{ \left\lfloor \frac{1}{n} S^{(\kappa/n^2)}_{[n^{1-\alpha} t]} \right\rfloor < 1 \right\} \frac{C^{\kappa/n^2}(n - S^{(\kappa/n^2)}_{[n^{1-\alpha} t]})}{C^{\kappa/n^2}(n)} \right] = \lim_{n \to \infty} \hat{Q}^{n,\kappa} \left[ 1 - \frac{1}{n} Z^{(\kappa/n^2)}_{[n^{1-\alpha} t]} \right] = Q^0[X_t^{(\kappa)} < 1] = P^0 \left[ X_t^{(\kappa)} < 1, \frac{u(1 - X_t^{(\kappa)})}{u(1)} \right].
\]

Equivalently,

\[
\lim_{n \to \infty} Q^{n,\kappa} \left[ f \left( \frac{1}{n} S^{(\kappa/n^2)}_{[n^{1-\alpha} s]} s \in [0, t] \right) \frac{C^{\kappa/n^2}(n - S^{(\kappa/n^2)}_{[n^{1-\alpha} t]})}{C^{\kappa/n^2}(n)} \right] = P^0 \left[ f(X_s^{(\kappa)}, s \in [0, t]) \frac{u(1 - X_t^{(\kappa)})}{u(1)}, X_t^{(\kappa)} < 1 \right].
\]

Therefore, we have the finite marginals convergence.

\[\square\]

4 Convergence in the sense of Skorokhod

We will strengthen\(^9\) the finite marginals convergence to the convergence in the sense of Skorokhod.

\[^8\]The dominating sequence is \( \left( 1_{\{S^{(\kappa/n^2)}_{[n^{1-\alpha} t]} < 1 \}} \frac{C^{\kappa/n^2}(n - S^{(\kappa/n^2)}_{[n^{1-\alpha} t]})}{C^{\kappa/n^2}(n)}, n > 0 \right). \)

\[^9\]By Proposition 3.14 in Chapter 6 of [JS03], the Skorokhod convergence implies the finite marginals convergence out of the fixed time discontinuity set. Under \( Q^0 \), \( (X_s^{(\kappa)}, s \in [0, T_1]) \) is a Feller process, see Lemma 3.3. Thus, it is quasi-left continuous and therefore has no fixed time of discontinuity.

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Definition 4.1. Define a càdlàg process $(H_t^{(n)}, t \geq 0)$ by

$$H_t^{(n)} = \begin{cases} \frac{1}{n} Z^{(n/n^2)}_{\lfloor n^{1-\alpha} t \rfloor} & \text{for } t < T_{\lfloor 1/2, \infty \rfloor}^{(n)}; \\ -1 & \text{for } t \geq T_{\lfloor 1/2, \infty \rfloor}^{(n)}. \end{cases}$$

Define a càglàd process $(R_t^{(n)}, t \geq 0)$ by

$$R_t^{(n)} = \begin{cases} 1 - \frac{1}{n} Z^{(n/n^2)}_{\lfloor n^{1-\alpha} (T_t^{(n)} - t) \rfloor} & \text{for } t < T_{\lfloor 1/2, \infty \rfloor}^{(n)}; \\ -1 & \text{for } t \geq T_{\lfloor 1/2, \infty \rfloor}^{(n)}. \end{cases}$$

Define $(H_t, t \geq 0)$ by

$$H_t = \begin{cases} Y_t^{(\kappa)} & \text{for } t < T_{\lfloor 1/2, \infty \rfloor}; \\ -1 & \text{for } t \geq T_{\lfloor 1/2, \infty \rfloor}. \end{cases}$$

where the sub-Markovian process $Y_t^{(\kappa)}$ is the conditioned subordinator defined in Lemma 3.3.

Remark 4.1. The purpose of taking the value “$-1$” is to ensure that the last jump is larger than a strictly positive constant.

By Theorem 3.21 of [JS03], $(Z^{(n/n^2)}_{\lfloor n^{1-\alpha} t \rfloor}/n, t \geq 0)$ is tight iff. we have tightness for $(H^{(n)}, n \geq 1)$ and $(R^{(n)}, n \geq 1)$. Roughly speaking, the tightness is equivalent to a uniform control of the càdlàg modulus of continuity. The tightness of $(H^{(n)}, n \geq 1)$ gives the uniform control of the càdlàg modulus of continuity on the first half parts of the paths. Whereas the tightness of $(R^{(n)}, n \geq 1)$ gives the uniform control of the càdlàg modulus of continuity on the second half parts. In this way, we can find a uniform control over the whole path and then we can conclude the tightness of $((Z^{(n/n^2)}_{\lfloor n^{1-\alpha} t \rfloor}/n, t \geq 0), n \geq 1)$. The details are presented in the proof of Lemma 4.4 in the following context.

For the purpose of self-containedness, we will explain several notations and state a criterion of tightness for stochastic process. They are taken from [JS03]. See also [EK86].

Definition 4.2. For $\theta > 0$, $I$ an interval of $\mathbb{R}_+$, $N \in \mathbb{N}$ and a càdlàg function $\alpha$ in the Skorokhod space $\mathbb{D}_{\lfloor 0, \infty \rfloor}(\mathbb{R})$, define

- $\omega(\alpha, I) = \sup_{s, t \in I} \{ |\alpha(s) - \alpha(t)| \}$;
- $\omega'_{N}(\alpha, \theta) = \inf_{i \leq r} \max_{t_i \leq t \leq t_{i+1}} \{ \omega(\alpha, [t_i, t_{i+1}]) : 0 = t_0 \leq \cdots \leq t_r = N, \inf_{i < r} (t_i - t_{i-1}) > \theta \}$.

Theorem 4.1 (Theorem 3.21 in [JS03]). The sequence of càdlàg stochastic process $X^{(n)}$ is tight iff.

Notice that there is no restriction on the last interval.
1. for all \(N \in \mathbb{N}_+\),
\[
\lim_{n \to \infty} \mathbb{P}^{(n)}[\sup_{t \leq N} |X^n_t| > K] = 0;
\]

2. for all \(N \in \mathbb{N}_+, \eta > 0, \epsilon > 0\), there exists \(n_0 \in \mathbb{N}\) and \(\theta \in [0,1]\) such that
\[
\mathbb{P}^{(n)}[\omega_N(X^{(n)}, \theta) \geq \eta] < \epsilon.
\]

Then we turn to prove tightness for \((H^{(n)}, n \geq 1)\) and \((R^{(n)}, n \geq 1)\).

**Lemma 4.2.** As \(n \to \infty\), the law of \(H^{(n)}\) under \(\hat{Q}^{n,n}\) converges in the sense of Skorokhod to the law of \(H\) under \(Q^0\). As a result, the family of stochastic càdlàg processes \((H^{(n)}, n \geq 1)\) is tight.

**Remark 4.2.** Note that Lemma \(4.2\) implies the following result:

Kill the process \(Z^{(\kappa/n^2)}\) when it enters \([1/2, \infty[\), and normalize the time to 1. Then, we have Skorokhod convergence for the normalized killed process in \(D_{[0,1]}(\mathbb{R})\).

**Proof of Lemma 4.2.** Let \(T^{(n)}_{1/2, \infty[}\) be the first time that the process \(\left(\frac{1}{n}Z^{(\kappa/n^2)}_{[n^{1-\alpha}], s \geq 0}\right)\) exceeds \(\frac{1}{2}\). It is enough to prove that for any bounded continuous \(f\),
\[
\lim_{n \to \infty} \hat{Q}^{n,\kappa}[f(H_s^{(n)}, s \in [0, \infty[)] = Q^0[f(H_s, s \in [0, \infty[)].
\]

The proof is very similar to that of Proposition 3.2. Thus, we merely provide a stretch of proof here.

\[
\frac{d\hat{Q}^{n,\kappa}}{dQ^{n,\kappa}}_{T^{(n)}_{1/2, \infty[}} G^{(n)}_{T^{(n)}_{1/2, \infty[}} \frac{1}{\{Z^{(\kappa/n^2)}_{[n^{1-\alpha}], s \geq 1}]\}} = 1_{\{S^{(\kappa/n^2)}_{[n^{1-\alpha}], s \geq 1}]\}} \frac{C^{n/2}(n - S^{(\kappa/n^2)}_{[n^{1-\alpha}], s \geq 1])}}{C^{n/2}(n)},
\]

By the coupling theorem of Skorokhod and Dudley, we can suppose that the renewal process \(S^{(\kappa/n^2)}\) converges to the subordinator \(X^{(\kappa)}\) almost surely as long as our result only depends on the law. Since for any \(x > 0, X^{(\kappa)}_{[1, \infty]} > x > X^{(\kappa)}_{[1/2, \infty[}\) holds with probability \(1, \frac{1}{n}S^{(\kappa/n^2)}_{[n^{1-\alpha}], s \geq 1}]\) converges to \(X^{(\kappa)}_{[1, \infty]}\). Consequently, the quantity \((***)\) converges to \(1\) almost surely. For fixed \(\delta > 0\), the quantity \((***)\) is uniformly bounded from above as long as \(\frac{1}{n}Z^{(\kappa/n^2)}_{[n^{1-\alpha}], s \geq 1}] < 1 - \delta\). (Recall that \(Z = S \wedge n\)).

---

\(^{11}\)In fact, the convergence \([\square]\) implies that \(\lim_{n \to \infty} \hat{Q}^{n,\kappa}\left[\frac{1}{n}Z^{(\kappa/n^2)}_{[n^{1-\alpha}], s \geq 1}] < 1\right] = 1\). Thus, we can remove \(\frac{1}{n}Z^{(\kappa/n^2)}_{[n^{1-\alpha}], s \geq 1}] < 1\) on the left hand side of \([\square]\).
Then, by dominated convergence, for all $\delta > 0$, we have
\[
\lim_{n \to \infty} \tilde{Q}^{n,\kappa} \left[ \frac{1}{n} Z_{n^{1-\alpha}T_{1/2,\infty}^{(n)}}^{(\kappa/n^2)} < 1 - \delta \right] = Q^0 \left[ X_{T_{1/2,\infty}}^{(\kappa)} < 1 - \delta \right].
\]

It implies that
\[
\lim_{n \to \infty} Q^{n,\kappa}\left[ (\text{***}) \right] = \lim_{n \to \infty} \tilde{Q}^{n,\kappa} \left[ \frac{1}{n} Z_{n^{1-\alpha}T_{1/2,\infty}^{(n)}}^{(\kappa/n^2)} < 1 \right] = Q^0 \left[ X_{T_{1/2,\infty}}^{(\kappa)} < 1 \right] \text{Lemma 3.3}.
\]

Taking any bounded continuous $f$, by the coupling assumption and dominated convergence\(^{12}\),
\[
\lim_{n \to \infty} \tilde{Q}^{n,\kappa} \left[ f \left( H_s^{(n)}, s \in [0, \infty) \right), \frac{1}{n} Z_{n^{1-\alpha}T_{1/2,\infty}^{(n)}}^{(\kappa/n^2)} < 1 \right]
= \lim_{n \to \infty} Q^{n,\kappa} \left[ f \left( H_s^{(n)}, s \in [0, \infty) \right) \cdot (\text{***}) \right]
= \mathbb{P}^0 \left[ f(H_s, s \in [0, \infty]), \frac{u(1) - X_{T_{1/2,\infty}}^{(\kappa)}}{u(1)}, X_{T_{1/2,\infty}}^{(\kappa)} < 1 \right]
= Q^0 \left[ f(H_s, s \in [0, \infty]), X_{T_{1/2,\infty}}^{(\kappa)} < 1 \right]
= Q^0 \left[ f(H_s, s \in [0, \infty]) \right].
\]

\[\Box\]

**Lemma 4.3.** As $n \to \infty$, the right-continuous modification of the càglàd processes $(R_t^{(n)}, t \geq 0)$ (under $\tilde{Q}^{n,\kappa}$) converges in law (in the sense of Skorokhod) to $(H_t, t \geq 0)$ (under $Q^0$).

**Proof.** By the loop model interpretation, the positions of closed edges are symmetric. Thus, the càglàd process $(R_t^{(n)}, t \geq 0)$ defined above is the left-continuous modification of $(H_t^{(n)}, t \geq 0)$. Then, the desired result comes from Lemma 4.2. \[\Box\]

**Lemma 4.4.** The conditioned renewal processes $(\frac{1}{n} Z_{n^{1-\alpha}T_{1/2,\infty}^{(n)}}^{(\kappa/n^2)}, t \geq 0)$ under $\tilde{Q}^{n,\kappa}$ is tight.

**Proof.** By Theorem 4.1 (Theorem 3.21 in \cite{JS03}), it is enough to verify two conditions in that criterion where the first one can be deduced from the finite marginal convergence. It remains to verify the second condition. (It is enough to verify this for $\eta \in [0, 1[).) Fix $N \in \mathbb{N}_+$ and $\theta \in [0, 1[$. By Lemma 4.2 there exist $n_0 \in \mathbb{N}_+$ and $\theta \in [0, 1[\] such that for $n \geq n_0$,
\[
\tilde{Q}^{\kappa/n^2}[\omega_{N+1}^{(n)}(H^{(n)}, \theta) \geq \eta] < \epsilon/2.
\]

Consequently, there exists $n_0 \in \mathbb{N}_+$ and $\theta > 0$ such that for $n \geq n_0$, there exists a partition $0 = t_0 \leq \cdots \leq t_r = N + 1$, $\inf_{i < r}(t_i - t_{i-1}) > \theta$ of $[0, N + 1]$ such that
\[
\tilde{Q}^{\kappa/n^2}[\max_{i < r} \omega(H^{(n)}, [t_{i-1}, t_i]) \geq \eta] < \epsilon/2.
\]

\(^{12}\)The dominating sequence is (***).
(Here, \( r \) depends on \((N, \eta, n_0, \theta)\).) At the hitting time \( T_{[1/2, \infty]}^{(n)} \) of \((\frac{1}{n} Z_{[n^{-1} \cdot t]}^{\kappa/n^2}, t \geq 0)\), the jump of the process \((H_t^{(n)}, t \geq 0)\) is strictly greater than 1. Therefore, \( T_{[1/2, \infty]}^{(n)} \) must belong to \( \{t_0, \ldots, t_r\} \) mentioned above. Thus, we have a partition of \([0, T_{[1/2, \infty]}^{(n)}]\): \( 0 = s_0 \leq \cdots \leq s_k = T_{[1/2, \infty]}^{(n)} \), \( \inf_{i=1, \ldots, k} (s_i - s_{i-1}) > \theta \) such that

\[
\bar{Q}_{\kappa/n^2} \left[ \max_{i=1, \ldots, k} \omega' \left( \left( \frac{1}{n} Z_{[n^{-1} \cdot t]}^{\kappa/n^2}, t \geq 0 \right) \right), [s_{i-1}, s_i] \right] \geq \eta \right] < \epsilon/2.
\]

Since the process \((\frac{1}{n} Z_{[n^{-1} \cdot t]}^{\kappa/n^2}, t \geq 0)\) is càdlàg, we can replace \([s_{i-1}, s_i]\) by \([s_{i-1}, s_i] \). By Lemma 4.3, we can find a partition of \([T_{[1/2, \infty]}^{(n)}], T_{[1/2, \infty]}^{(n)} = u_0 \leq \cdots \leq u_\ell = T_{1}^{(n)} \), \( \inf_{i=1, \ldots, \ell} (u_i - u_{i-1}) > \hat{\theta} \) such that for \( n \geq n_0 \),

\[
\bar{Q}_{\kappa/n^2} \left[ \max_{i=1, \ldots, \ell} \omega' \left( \left( \frac{1}{n} Z_{[n^{-1} \cdot t]}^{\kappa/n^2}, t \geq 0 \right) \right), [u_{i-1}, u_i] \right] \geq \eta \right] < \epsilon/2.
\]

Notice that \((\frac{1}{n} Z_{[n^{-1} \cdot t]}^{\kappa/n^2}, t \geq 0)\) equals 1 after \( T_{1}^{(n)} \). By collecting these two partitions together, one can check that the second condition in Theorem 4.4 is fulfilled for \( \theta \land \hat{\theta} \) and \( n_0 \lor \hat{n}_0 \).

Finally, we could replace the finite-marginals convergence by Skorokhod convergence in the Proposition 3.2.

**Theorem 4.5.** The convergence result in Proposition 3.2 can be strengthened to Skorokhod convergence.

**Remark 4.3.** All the convergence results remain correct if we replace the killing measure \( \kappa/n^2 \) by \( \kappa^{(n)} \) such that \( \lim_{n \to \infty} n^2 \kappa^{(n)} = \kappa \).

**Remark 4.4.** Notice that Theorem 4.5 is actually a convergence result for the closed edges given by Poisson loop ensemble in a discrete interval.

## 5 The limit distribution of the loop clusters.

In this section, we would like to have a description of the loop clusters together with its scaling limit. We will always assume the following assumption when we talk about its scaling limit:

\[
\kappa^{(n)} \overset{\text{def}}{=} \frac{1 + c_n - \sqrt{p_n(1 - p_n)}}{\sqrt{p_n(1 - p_n)}} \sim \kappa/n^2.
\]

We have seen in the above sections that the conditioned renewal processes (formed by the closed edges conditionally on the closeness of a particular edge) converges towards a

---

\( {T}_{[1/2, \infty]}^{(n)} \) and \( T_{1}^{(n)} \) are hitting times of the process \((\frac{1}{n} Z_{[n^{-1} \cdot t]}^{\kappa/n^2}, t \geq 0)\).
conditioned subordinator. In the proof, we use the independence of the Poisson ensemble structure: the ensemble of loops crossing the edge \( \{1, n\} \) (in either direction) is independent of the loops avoiding \( \{1, n\} \). It implies that the conditional distribution of the closed edges (given that the edge \( \{1, n\} \) is closed) is equal in law to the edges uncovered by the ensemble of loops avoiding \( \{1, n\} \). Similarly to Theorem 4.5, we have the following proposition.

**Proposition 5.1.** The closed edges associated with the ensemble of loops avoiding the vertex 1 can be viewed as a conditioned renewal process. The limit of these conditioned renewal processes is a conditioned subordinator described in Lemma 3.3.

Thus, it remains to study the ensemble of loops passing the vertex 1. For this part, we need to introduce several notations:

We know that \( Z \) is a covering space of the discrete circle \( G^{(n)} \) under the following mapping \( \pi^{(n)} \):

\[
\pi^{(n)}(i + kn) = i + 1 \quad \text{for} \quad k \in \mathbb{Z} \quad \text{and} \quad i = 0, \ldots, n - 1.
\]

**Definition 5.1.** For a pointed loop \( l \) on \( G^{(n)} \), there is a unique bridge from \( \text{Ini}^{(n)}(l) \) to \( \text{End}^{(n)}(l) \) as the lift of \( l \) on \( Z \) associated with the covering map \( \pi^{(n)} \) under the condition that \( \text{Ini}^{(n)}(l) \in \{0, \ldots, n - 1\} \). Define the rotation number \( \text{Rot}(l) \) of pointed loop \( l \) as \( \text{End}^{(n)}(l) - \text{Ini}^{(n)}(l) \) mod \( n \). Since two equivalent pointed loops have the same rotation number, it is well-defined for loops.

**Definition 5.2.** Define a 0-1 valued function \( \Psi^{(n)}(l) \) on non-trivial loops on \( G^{(n)} \): \( \Psi^{(n)}(l) = 1 \) iff. the following conditions are all fulfilled:

- \( \text{Rot}(l) = 0 \),
- \( l \) passes through the vertex 1 in the discrete circle \( G^{(n)} \),
- suppose \( l^{pt,1} \) and \( l^{pt,2} \) are both in the equivalence class \( l \) and they both start from the vertex 1 in \( G^{(n)} \). Denote by \( \tilde{l}^{pt,i} \) \((i = 1, 2)\) the unique pointed loop on \( Z \) starting from 0 as the lift of \( l^{pt,i} \) \((i = 1, 2)\). Then, \( \tilde{l}^{pt,1} \) and \( \tilde{l}^{pt,2} \) are equivalent pointed loops.

For a loop \( l \) such that \( \Psi^{(n)}(l) = 1 \), we choose a representative pointed loop \( l^{pt} \) in the equivalence class \( l \). Denote by \( \tilde{l}^{pt} \) the unique pointed loop on \( Z \) starting from 0 as the lift of \( l^{pt} \). Then, we define the lift of the loop \( l \) to be the loop \( \text{Lift}(l) \) which is the equivalence class of \( \tilde{l}^{pt} \).

Next, we define a partition \( (O_i^{(n)}, i = 1, 2, 3, 4) \) of possible non-trivial loops on \( n \)-th discrete circle \( G^{(n)} \).

---

\(^{14}\)The precise description is as same as that in the above sections.
Definition 5.3. Let $O_1^{(n)}$ be the ensemble of non-trivial loops avoiding 1. Let $O_2^{(n)}$ be the ensemble of non-trivial loops passing 1 with non-zero rotation numbers. Let $O_3^{(n)}$ be

$$\{l \text{ is non-trivial : } \Phi^{(n)}(l) = 1\}.$$ 

Let $O_4^{(n)}$ be \{non-trivial loops on $G^{(n)}\} \setminus (\bigcup_{i=1}^{3} O_i^{(n)}).$ Let $O_{\text{cov}}^{(n)}$ be the ensemble of non-trivial loops which cover all the vertices in the discrete circle $G^{(n)}$. Define $L_{\alpha,i}^{(n)} = L_{\alpha}^{(n)} \cap O_i^{(n)}$ for $i = 1, 2, 3, 4$ where $L_{\alpha}^{(n)}$ is the Poissonian loop ensemble of intensity $\alpha \mu_n$.

Lemma 5.2. Let $\mu_{n,\mathbb{Z}}$ be the non-trivial loop measure on $\mathbb{Z}$ associated with the following generator $L$:

$$L^i_j = \begin{cases} \frac{1}{2} & |i - j| = 1, \\ 1 + \frac{\kappa(n)}{2} & i = j, \\ 0 & \text{otherwise}. \end{cases}$$

Then, the push forward measure $\text{Lift } \mu_n (\Phi^{(n)}(l) = 1, dl)$ equals

$$\mu_{n,\mathbb{Z}}(l \text{ visits 0 and } l \subset [1 - n, n - 1], dl).$$

Lemma 5.2 can be proven by comparing the weights of each particular loop under these two measures.

Remark 5.1. The Markovian loop measure is invariant under Doob’s h-transform, see e.g. Remark 1.1 in [LJL12]. Thus, the Markovian loop measure remains the same if we replace the above generator by the following one

$$L^i_j = \begin{cases} -(1 + c_n) & \text{if } i = j, \\ p_n & \text{if } j = i + 1, \\ 1 - p_n & \text{if } j = i - 1, \\ 0 & \text{otherwise}. \end{cases}$$

By the definition of the Poisson random measure, $(L^{(n)}_{\alpha,i}, i = 1, 2, 3, 4)$ are independent. We see that $L^{(n)}_{\alpha,1} \cap O^{(n)}_{\text{cov}} = \emptyset$, $L^{(n)}_{\alpha,2} \cup L^{(n)}_{\alpha,4} \subset O^{(n)}_{\text{cov}} \subset L^{(n)}_{\alpha,2} \cup L^{(n)}_{\alpha,3} \cup L^{(n)}_{\alpha,4}$. It is sufficient to study $L^{(n)}_{\alpha,1}$ and $L^{(n)}_{\alpha,3}$ if we are interested in the conditional distribution of the clusters given that there exists at least two clusters. From the definition of the non-trivial loop measure $\mu_n$, the law of $L^{(n)}_{\alpha,1}$, $L^{(n)}_{\alpha,3}$ and $L^{(n)}_{\alpha,4}$ do not change if we replace $(p_n, 1 - p_n, 1 + c_n)$ by $(\sqrt{p_n}(1 - p_n), \sqrt{p_n}(1 - p_n), 1 + c_n)$ (or $\left(\frac{1}{2}, \frac{1}{2}, 1 + \frac{\kappa(n)}{2}\right)$ equivalently). The non-symmetry only affects the distribution of $L^{(n)}_{\alpha,2}$. Therefore, we will use the symmetrized parameter $\left(\frac{1}{2}, \frac{1}{2}, 1 + \frac{\kappa(n)}{2}\right)$ when we study the distribution of $L^{(n)}_{\alpha,3}$ in the following context.
For a loop $l$ in $L_{\alpha,3}^{(n)}$, Lift($l$) is a loop on $\mathbb{Z}$ passing through 0 but never reaching $-n$ nor $n$. By Lemma 5.2, Lift($L_{\alpha,3}^{(n)}$) $\equiv \{\text{Lift}(l) : l \in L_{\alpha,3}^{(n)}\}$ is the Poisson ensemble of loops on $\mathbb{Z}$ with intensity measure $\alpha \mu_{n,\mathbb{Z}}$. They cover a discrete random sub-interval $[-A_n, B_n]$ of $[-n + 1, \ldots, n - 1]$ which contains 0. When $A_n + B_n < n - 1$, there is a correspondence between $[-A_n, B_n]$ and the random discrete arc covered by $L_{\alpha,3}^{(n)}$. We give the distribution of $[-A_n, B_n]$ in the following proposition.

**Proposition 5.3.** For a fixed sub-interval of $[-m_n, M_n]$ in $[1 - n, n - 1]$,

$$
P([A_n, B_n] \subset [-m_n, M_n]) = \left( \frac{(x_1^n - x_2^n)(x_1^n - x_2^n)(x_1^{m_n+M_n+2} - x_2^{m_n+M_n+2})}{(x_1^{M_n+1} - x_2^{M_n+1})(x_1^{m_n+1} - x_2^{m_n+1})(x_1^{2} - x_2^{2})} \right)^{\alpha},
$$

where $x_1, x_2$ are the roots of the polynomial $x^2 - (1 + \kappa^{(n)})x + \frac{1}{4}$. As $n$ tends to infinity, under the assumption that $\lim_{n \to \infty} n^2 \kappa^{(n)} = \kappa$, the pair of random variables $(\frac{A_n}{n}, \frac{B_n}{n})$ converges in distribution towards $(A, B) \in [0, 1]^2$ where

$$
P[A \leq a, B \leq b] = \left( \frac{2 \cosh(\sqrt{\kappa})}{\sinh(\sqrt{\kappa})} \right)^{\alpha} \left( \frac{\sinh(\sqrt{\kappa}a) \sinh(\sqrt{\kappa}b)}{\sinh(\sqrt{\kappa}(a + b))} \right)^{\alpha}.
$$

**Proof.** We fix a sub-interval of $[-m_n, M_n]$ in $[1 - n, n - 1].$

Then,

$$
P([-A_n, B_n] \subset [-m_n, M_n]) = \exp\{-\alpha \mu_{n,\mathbb{Z}}(l \text{ visits 0, } l \text{ is in } [1 - n, n - 1] \text{ but not in } [-m_n, M_n])\}
$$

where $\mu_{n,\mathbb{Z}}$ is defined in Lemma 5.2. By inclusion-exclusion principle, for $m_n, M_n > 0$, we have

$$
\mu_{n,\mathbb{Z}}(l \text{ is a non-trivial loop passing through 0, } l \text{ is contained in } [-m_n, M_n])
= \mu_{n,\mathbb{Z}}(l \text{ is a non-trivial loop inside } [-m_n, M_n])
- \mu_{n,\mathbb{Z}}(l \text{ is a non-trivial loop inside } [-m_n, -1])
- \mu_{n,\mathbb{Z}}(l \text{ is a non-trivial loop inside } [1, M_n]).
$$

By Lemma 2.3, we see that

$$
\mu_{n,\mathbb{Z}}(l \text{ is a non-trivial loop inside } [-m_n, M_n]) = -\log \det(-L_{[-m_n,M_n]}) + \sum_{x=-m_n}^{M_n} \log(-L_x^z)
$$

where $L$ is the generator defined in Lemma 5.2. Similar expressions hold for

$$
\mu_{n,\mathbb{Z}}(l \text{ is a non-trivial loop inside } [-m_n, -1])
$$

and

$$
\mu_{n,\mathbb{Z}}(l \text{ is a non-trivial loop inside } [1, M_n]).
$$
Thus,

\[
\mu_{n,\mathcal{Z}}(l) \text{ is a non-trivial loop passing through 0, } l \text{ is contained in } [-m_n, M_n] \\
= \log(-L^0) - \log \det(-L|[-m_n, M_n]) + \log \det(-L|[-m_n, -1]) + \log \det(-L|[1, M_n]).
\]

We calculate the above determinants by using Lemma 2.1

\[
\det(-L|[-m_n, M_n]) = \frac{x_1^{m_n+m_n+2} - x_2^{m_n+m_n+2}}{x_1 - x_2},
\]

\[
\det(-L|[-m_n, -1]) = \frac{x_1^{m_n+1} - x_2^{m_n+1}}{x_1 - x_2},
\]

\[
\det(-L|[1, M_n]) = \frac{x_1^{m_n+1} - x_2^{m_n+1}}{x_1 - x_2},
\]

where \(x_1\) and \(x_2\) are the roots of the polynomial \(x^2 - \left(1 + \frac{\kappa(n)}{2}\right)x + \frac{1}{4}\). Therefore,

\[
\mu_{n,\mathcal{Z}}(l) \text{ is a non-trivial loop passing through 0, } l \text{ is contained in } [-m_n, M_n] \\
= \log(1 + \frac{\kappa(n)}{2}) + \log \left( \frac{x_1^n - x_2^n}{x_1^n x_2^n} \right).
\]

In particular,

\[
\mu_{n,\mathcal{Z}}(l) \text{ is a non-trivial loop passing through 0, } l \text{ is contained in } [-n + 1, n - 1] \\
= \log(1 + \frac{\kappa(n)}{2}) + \log \left( \frac{x_1^n - x_2^n}{x_1^n x_2^n} \right).
\]

Therefore,

\[
\mu_{n,\mathcal{Z}}(l) \text{ passes through 0, } l \subset [-n + 1, n - 1], \text{ but } l \text{ is not contained in } [-m_n, M_n] \\
= \log \left( \frac{x_1^n - x_2^n}{x_1^n x_2^n} \right).
\]

Then,

\[
\mathbb{P}([-A_n, B_n] \subset [-m_n, M_n]) = \left( \frac{x_1^n - x_2^n}{x_1^{M_n+1} - x_2^{M_n+1}} \right)^\alpha,
\]

where \(2x_1 = 1 + \frac{\kappa(n)}{2} + \sqrt{\kappa(n) + \frac{(\kappa(n))^2}{4}}\) and \(2x_2 = 1 + \frac{\kappa(n)}{2} - \sqrt{\kappa(n) + \frac{(\kappa(n))^2}{4}}\). We see that \(2x_1 \sim 1 + \frac{\sqrt{n}}{n}\) and \(2x_2 \sim 1 - \frac{\sqrt{n}}{n}\) if \(\kappa(n) \sim \frac{\sqrt{n}}{n}\). Finally, we get the convergence result for \((-\frac{A_n}{n}, \frac{B_n}{n})\) under the assumption that \(\lim_{n \to \infty} n^2\kappa(n) = \kappa\).

We are mostly interested in the scaling limit\(^{15}\) of

\[
\mathbb{P}[(\text{random clusters given by } \mathcal{L}_{\mathcal{A},1}^{(n)} \cup \mathcal{L}_{\mathcal{A},3}^{(n)}) \in \cdot| \text{there exists at least two clusters}].
\]

\(^{15}\)We normalize the discrete intervals to have length 1.
(As we have explained in the paragraph below Remark 5.1, it is as same as the distribution of random clusters given that there exists at least 2 clusters.) We have described in Proposition 5.3 the distribution of the random interval covered by the loops in \( \text{Lift}(\alpha, 3) \).

Moreover, we also identify the clusters of \( \text{L}(\alpha, 1) \) as the complement of the closure of the range of a conditioned subordinator in the scaling limit, see Proposition 5.1. By combining these two results together, we get the distribution of the closed edges in the scaling limit.

Proposition 5.4. Let \( S_0^{(n)} < S_1^{(n)} < \cdots < S_{k(n)}^{(n)} \) be the left end points of closed edges in \( G^{(n)} \). Define \( Z_t^{(n)} = \frac{1}{n} S_{\lfloor n^{1-\alpha} t \rfloor} \) for \( t \in [0, n^{\alpha-1} (k(n) + 1)] \). Let \( Y^{(\kappa)} \) be a subordinator bridge described in Lemma 3.3. Let \((A, B)\) be a pair of real valued variables independent of \( Y^{(\kappa)} \) and with the following distribution:

\[
P[A \leq a, B \leq b] = \left( \frac{2 \cosh(\sqrt{\kappa})}{\sinh(\sqrt{\kappa})} \right)^{\alpha} \left( \frac{\sinh(\sqrt{\kappa} a) \sinh(\sqrt{\kappa} b)}{\sinh(\sqrt{\kappa} (a + b))} \right)^{\alpha}.
\]

For \( A + B \leq 1 \), define \( T_1 = \inf \{ t \geq 0 : Y_t^{(\kappa)} \geq B \} \) and \( T_2 = \inf \{ t \geq 0 : Y_t^{(\kappa)} \geq 1 - A \} \).

Then,

\[
\lim_{n \to \infty} P[(Z_t^{(n)}, t \in [0, n^{\alpha-1} (k(n) + 1)]) : k(n) \geq 1] = P[(Y_t^{(\kappa)}, t \in [T_1, T_2]) \in \cdot | A + B \leq 1]
\]

in the sense of Skorokhod.

There is another way to express the scaling limit. Firstly, we could calculate the limit distribution of the cluster containing the vertex \( x_0 \) for a fixed point \( x_0 \) in the discrete circle. (By rotation invariance of the model, we previously take \( x_0 \) to be 1.) In the discrete model, this cluster is a discrete arc containing \( x_0 \). We map it to \( Z \) such that \( x_0 \) is mapped to 0. When there exists a closed edge, we identify this cluster as a random interval \([-G_n, D_n]\) such that \( G_n, D_n \geq 0 \) and \( G_n + D_n \leq n - 1 \). We will give the limit distribution of \((G_n/n, D_n/n)\) given that the cluster at 0 does not contain all the vertices. Next, we calculate the limit distribution of other clusters given the cluster containing the vertex \( x_0 \). From the construction of the discrete model, given the cluster containing the vertex \( x_0 \), the closed edges have the same conditional law as the closed edges in the random discrete interval \( J_{x_0} \) where \( J_{x_0} \) is the complement of the cluster at \( x_0 \). As a result, we can identify the limit of closed edges. The details will be presented in Theorem 5.10. For this alternative way to express the scaling limit, it is enough for us to state the limit joint distribution of \( P[G_n/n \in dx, D_n/n \in dy] \).

In fact, we have the following proposition:

\[\text{See Remark 4.4}\]
Proposition 5.5.

\[
\lim_{n \to \infty} \mathbb{P}[\text{All the edges are not covered by the loops in } L_{\alpha,1}(n) \cup L_{\alpha,3}(n)] = \frac{(2 \cosh{\sqrt{\kappa}})^\alpha \sinh{\sqrt{\kappa}(1 - \alpha)}}{\sinh{\sqrt{\kappa}}}.
\]

Conditionally on the existence of closed edges, \((G_n/n, D_n/n)\) converges in distribution towards \((G, D)\) where the density \(q(x, y)\) of \((G, D)\) is given by

\[
\mathbb{P}\left[G \in dx, D \in dy \right] = \frac{\sin(\alpha \pi)}{\pi} \frac{2^{\alpha-2}(1 - \alpha) \kappa \sinh{\sqrt{\kappa}}}{\sinh(\sqrt{\kappa}(1 - \alpha)) [\sinh(\sqrt{\kappa}(1 - x - y))]^2 \sinh(\sqrt{\kappa}(x + y))^{2-\alpha}}.
\]

Remark 5.2. Proposition 5.5 implies that the following probability goes to 0 as \(n\) tends to infinity:

\[
\mathbb{P}[\exists \text{ a unique closed edge in } G^{(n)}].
\]

The reason is as follows:

\[
\mathbb{P}[\exists \text{ a unique closed edge in } G^{(n)}] \leq \mathbb{P}[G_n + D_n = n - 1 | \exists \text{ closed edges}].
\]

Since \(\mathbb{P}[(G_n/n, D_n/n) \in \cdot | \exists \text{ closed edges}]\) converges towards \((G, D)\) and the distribution of \(G + D\) has no atom, we must have

\[
\lim_{n \to \infty} \mathbb{P}[G_n + D_n = n - 1 | \exists \text{ closed edges}] = 0.
\]

To prove Proposition 5.5, we need the following lemmas.

Lemma 5.6. The Lévy measure \(\Pi\) of the subordinator of the renewal density \(u(x) = (\frac{2\sqrt{\pi}}{1 - e^{-2\sqrt{\kappa}x}})^\alpha\) is given by the following expression:

\[
\Pi(dt) = dt \cdot \frac{1}{\pi} (1 - \alpha) \sin(\alpha \pi) e^{2\sqrt{\pi}(\alpha - 1)t} \left( \frac{2\sqrt{\kappa}}{1 - e^{-2\sqrt{\kappa}t}} \right)^{2-\alpha}.
\]

Proof. The Lévy measure \(\Pi\) and the renewal density \(u\) are related through the Laplace exponent of the subordinator as follows:

\[
\frac{1}{\Phi(\alpha)} = \int_0^\infty e^{-\lambda x} u(x) \, dx
\]

\[
\Phi(\lambda) = \int_0^\infty (1 - e^{-\lambda x}) \Pi(dx) = \lambda \int_0^\infty e^{-\lambda t} \bar{\Pi}(t)
\]

where \(\bar{\Pi}\) is the tail mass of \(\Pi\). Then, we compute \(\Phi(\lambda)\) from \(u\):

\[
\frac{1}{\Phi(\lambda)} = \int_0^\infty e^{-\lambda x} u(x) \, dx
\]
\[
\int_0^\infty \left( \frac{2\sqrt{\kappa}}{1 - e^{-2\sqrt{\kappa}x}} \right)^\alpha e^{-\lambda x} \, dx.
\]

We change the variable \( x \) by \( \log(1 - s) - 2\sqrt{\kappa} \):

\[
\frac{1}{\Phi(\lambda)} = \int_0^1 \left( \frac{2\sqrt{\kappa}}{s} \right)^\alpha e^{-\lambda \log(1 - s) - 2\sqrt{\kappa}} \frac{1}{2\sqrt{\kappa}(1 - s)} \, ds
\]

\[
=(2\sqrt{\kappa})^{\alpha-1} \int_0^1 s^{-\alpha} (1 - s)^{\frac{\lambda}{2\sqrt{\kappa}} - 1} \, ds
\]

\[
=(2\sqrt{\kappa})^{\alpha-1} \text{Beta} \left( \frac{\lambda}{2\sqrt{\kappa}}, 1 - \alpha \right).
\]

Then, we use the following equality\( ^{17} \)

\[
\text{Beta}(x, y) \cdot \text{Beta}(x + y, 1 - y) = \frac{\pi}{x \sin(\pi y)}.
\]

We see that

\[
\Phi(\lambda) = (2\sqrt{\kappa})^{1-\alpha} \cdot \frac{1}{\text{Beta} \left( \frac{\lambda}{2\sqrt{\kappa}}, 1 - \alpha \right)}
\]

\[
=(2\sqrt{\kappa})^{1-\alpha} \cdot \frac{\lambda}{2\sqrt{\kappa}} \cdot \frac{\sin(\alpha\pi)}{\pi} \cdot \text{Beta} \left( \frac{\lambda}{2\sqrt{\kappa}} + 1 - \alpha, \alpha \right)
\]

\[
= \frac{1}{\pi} \lambda (2\sqrt{\kappa})^{-\alpha} \sin(\alpha\pi) \int_0^1 y^{\frac{\lambda}{2\sqrt{\kappa}}-\alpha} (1 - y)^{\alpha-1} \, dy
\]

Now, we change the variable \( y \) by \( e^{-2\sqrt{\kappa}u} \):

\[
\Phi(\lambda) = \lambda \cdot \frac{1}{\pi} (2\sqrt{\kappa})^{-\alpha} \sin(\alpha\pi) \int_0^\infty e^{-\lambda u} e^{2\alpha\sqrt{\kappa}u} (1 - e^{-2\sqrt{\kappa}u})^{\alpha-1} \cdot 2\sqrt{\kappa} e^{-2\sqrt{\kappa}u} \, du
\]

\[
= \lambda \cdot \frac{1}{\pi} \sin(\alpha\pi) (2\sqrt{\kappa})^{1-\alpha} \int_0^\infty e^{-\lambda u} (e^{2\sqrt{\kappa}u} - 1)^{\alpha-1} \, du.
\]

Thus,

\[
\bar{\Pi}(t) = \frac{1}{\pi} \sin(\alpha\pi) (2\sqrt{\kappa})^{1-\alpha} (e^{2\sqrt{\kappa}t} - 1)^{\alpha-1}.
\]

Finally, we find \( \Pi(dt) \) by calculating the derivative of \( \bar{\Pi} \):

\[
\Pi(dt) = dt \cdot \frac{1}{\pi} (1 - \alpha) \sin(\alpha\pi) e^{2\sqrt{\kappa}(\alpha-1)t} \left( \frac{2\sqrt{\kappa}}{1 - e^{-2\sqrt{\kappa}t}} \right)^{2-\alpha}.
\]

\[\text{This is implied by Euler’s reflection principle: } \Gamma(\alpha)\Gamma(1 - \alpha) = \frac{\pi}{\sin(\alpha\pi)}.\]
Lemma 5.7. For the subordinator $X^{(\kappa)}$ of potential density $u(x) = \left(\frac{2\sqrt{\pi}}{1-e^{-2\sqrt{\pi}x}}\right)^{\alpha}$, we have

$$\mathbb{P}^0[X_{T_{[a,\infty]}}^{(\kappa)} \in dx]/dx = \int_{z \in [x-a,x]} u(x-z)\Pi(dz) = \frac{\sqrt{\kappa}}{\pi} \sin(\alpha \pi) e^{\alpha \sqrt{\kappa}x} \frac{(\sinh(\sqrt{\kappa}a))^{1-\alpha}}{\sinh(\sqrt{\kappa}x)(\sinh(\sqrt{\kappa}(x-a)))^{1-\alpha}}.$$ 

Proof. According to Lemma 1.10 in [Ber99],

$$\mathbb{P}^0[X_{T_{[a,\infty]}}^{(\kappa)} \in dx]/dx = \int_{z \in [x-a,x]} u(x-z)\Pi(dz).$$

By Lemma 5.6

$$\Pi(dz) = dz \cdot \frac{1}{\pi} (1-\alpha) \sin(\alpha \pi) e^{2\sqrt{\kappa}(1-a)z} \left(\frac{2\sqrt{\kappa}}{1-e^{-2\sqrt{\kappa}x}}\right)^{2-\alpha}.$$ 

Thus,

$$\mathbb{P}^0[X_{T_{[a,\infty]}}^{(\kappa)} \in dx]/dx = \frac{(2\sqrt{\kappa})^\alpha}{(1-e^{-2\sqrt{\kappa}(x-z)})^\alpha} \times \frac{1-\alpha}{\pi} \sin(\alpha \pi) e^{2\sqrt{\kappa}(1-a)z} \frac{(2\sqrt{\kappa})^{2-\alpha}}{(1-e^{-2\sqrt{\kappa}x})^{2-\alpha}} dz.$$ 

By performing the change of variable $t = \frac{1-e^{-2\sqrt{\kappa}x}}{1-e^{-2\sqrt{\kappa}(x-a)}}$, we see that

$$\mathbb{P}^0[X_{T_{[a,\infty]}}^{(\kappa)} \in dx]/dx = \frac{2\sqrt{\kappa}}{\pi} (1-\alpha) \sin(\alpha \pi) (1-e^{-2\sqrt{\kappa}x})^{\alpha-2} (1-e^{-2\sqrt{\kappa}})^{-\alpha} \times \int_{1-e^{-2\sqrt{\kappa}(x-a)}}^{1} (t^{-1} - 1)^{-\alpha} t^{-2} dt$$

$$= \frac{\sqrt{\kappa}}{\pi} \sin(\alpha \pi) e^{\alpha \sqrt{\kappa}x} \frac{(\sinh(\sqrt{\kappa}a))^{1-\alpha}}{\sinh(\sqrt{\kappa}x)(\sinh(\sqrt{\kappa}(x-a)))^{1-\alpha}}.$$

\[\square\]

Lemma 5.8. Consider the càdlàg subordinator bridge $Y^{(\kappa)}$ introduced\footnote{We get this process by applying the Doob's harmonic transform to a subordinator with potential density $U(x,y) = 1_{(y>x)} \left(\frac{2\sqrt{\pi}}{1-e^{-2\sqrt{\pi}y}}\right)^{\alpha}$ with respect to the excessive function $x \to U(x,1)$. This is the process conditioned to approach 1 continuously and killed when it exceeds 1.} in the above sections, see Lemma 5.5. Let $\mathbb{Q}^0$ stand for the law of this Markov process starting from 0. Fix a positive measurable function $f : [0,1]^2 \to \mathbb{R}_+$ and $0 < a, b < 1$ such that $a + b < 1$.

Then,

$$\mathbb{Q}^0 \left[ f(Y_{T_{[a,\infty]}}^{(\kappa)} \left| 1 - Y_{T_{[1-b,\infty]}}^{(\kappa)} \right| \left| 1_{Y_{T_{[a,\infty]}}^{(\kappa)} < 1-b} \right) \right]$$
\[= \int f(x, y) \frac{\kappa}{\pi^2} (\sinh(\sqrt{\kappa}(1 - x - y)))^\alpha \sinh(\sqrt{\kappa}x) \sinh(\sqrt{\kappa}y) \times \left( \frac{\sinh(\sqrt{\kappa}a) \sinh(\sqrt{\kappa}b)}{\sinh(\sqrt{\kappa}(x - a)) \sinh(\sqrt{\kappa}(y - b))} \right)^{1-\alpha} \, dx \, dy.\]

**Proof.** Let \(X^{(\kappa)}\) be the subordinator with the potential density \(U(x, y) = 1_{\{y > x\}} \left( \frac{2\sqrt{x}}{e^{2\sqrt{x(y-x)}}} \right)^\alpha\). Let \(\mathbb{P}^0\) stand for its law with starting state 0. We have seen that this subordinator has zero drift. Consequently, for any fixed \(a > 0\),

\[\mathbb{P}^0[a \text{ belongs to the closure of the range of } X^{(\kappa)}] = 0\]

which is frequently used to replace “\(\leq\)" or “\(\geq\)" by strict inequalities “\(<\)" or “\(>\)". Set \(u(x) = U(0, x)\) and denote by \(\Pi\) its Lévy measure. According to Lemma 1.10 in [Ber99], for \(0 \leq x < a \leq x + y\),

\[\mathbb{P}^0[X^{(\kappa)}_{T_{[a,\infty]-}} \in dx, X^{(\kappa)}_{T_{[a,\infty]-}} - X^{(\kappa)}_{T_{[a,\infty]-}} \in dy] = u(x) dx \Pi(dy).\]

By using the strong Markov property at time \(T_{[a,\infty]}\) for the subordinator \(X^{(\kappa)}\), we see that

\[\mathbb{P}^0 \left[ \phi(X^{(\kappa)}_{T_{[a,\infty]-}}, X^{(\kappa)}_{T_{[a,\infty]-}}) X^{(\kappa)}_{T_{[1-b,\infty]-}}, X^{(\kappa)}_{T_{[1-b,\infty]-}} \right] 1_{\{X^{(\kappa)}_{T_{[a,\infty]-}} < 1-b\}} = \int \phi(z_1, z_1 + z_2, z_1 + z_2 + z_3, z_1 + z_2 + z_3 + z_4) u(z_1) dz_1 \Pi(dz_2) u(z_3) dz_3 \Pi(dz_4)\]

where \(\phi\) is a positive measurable function. Therefore, for a positive measurable function \(\phi\), we have

\[\mathbb{P}^0 \left[ \phi(X^{(\kappa)}_{T_{[a,\infty]-}}, X^{(\kappa)}_{T_{[a,\infty]-}}) X^{(\kappa)}_{T_{[1-b,\infty]-}}, X^{(\kappa)}_{T_{[1-b,\infty]-}} \right] 1_{\{X^{(\kappa)}_{T_{[a,\infty]-}} < 1-b\}} = \int \phi(z_1, z_1 + z_2, z_1 + z_2 + z_3, z_1 + z_2 + z_3 + z_4) u(z_1) dz_1 \Pi(dz_2) u(z_3) dz_3 \Pi(dz_4) \times \frac{u(1 - z_1 - z_2 - z_3 - z_4)}{u(1)}\]

As a result,

\[\mathbb{P}^0 \left[ f(Y^{(\kappa)}_{T_{[a,\infty]-}} 1 - Y^{(\kappa)}_{T_{[1-b,\infty]-}}) 1_{\{Y^{(\kappa)}_{T_{[a,\infty]-}} < 1-b\}} \right] = \int f(z_1 + z_2, 1 - z_1 - z_2 - z_3) u(z_1) dz_1 \Pi(dz_2) u(z_3) dz_3 \Pi(dz_4) \times \frac{u(1 - z_1 - z_2 - z_3 - z_4)}{u(1)}\]
By performing the change of variables $x = z_1 + z_2, y = 1 - z_1 - z_2 - z_3$:

\[
Q^0 \left[ f(Y^{(κ)}_{T_{[a,∞]}}, 1 \cdot Y^{(κ)}_{T_{[1-b,∞]}(-)}) \right] = \int_{0<x-z_2<y<1} \frac{u(1-x-y)}{u(1)} dx dy \cdot u(y-z_4) u(x-z_2)\Pi(dz_2)\Pi(dz_4)
\]

Finally, the calculation is finished by using Lemma 5.7.

\[\Box\]

**Proof of Proposition 5.3.** Firstly, we deduce from Proposition 5.3 the density $\rho(a, b)$ of $A, B$:

\[
\rho(a, b) = \kappa \alpha (\alpha + 1) \left( \frac{2 \cosh(\sqrt{\kappa})}{\sinh(\sqrt{\kappa})} \right)^\alpha \left( \frac{\sinh(\sqrt{\kappa}a) \sinh(\sqrt{\kappa}b)}{(\sinh(\sqrt{\kappa}(a + b)))^{\alpha + 2}} \right)
\]

Take an independent subordinator bridge $Y^{(κ)}$ defined in Lemma 3.3. Set

\[(g, d) = (Y^{(κ)}_{T_{[a,∞]}}, 1 - Y^{(κ)}_{T_{[1-b,∞]}(-)})\]

By Proposition 5.3, Proposition 5.1, and the independence of $L^{(n)}_{α,1}$ and $L^{(n)}_{α,3}$,

\[\lim_{n \to \infty} \mathbb{P}[\text{All the edges are not covered by the loops in } L^{(n)}_{α,1} \cup L^{(n)}_{α,3}] = \mathbb{P}[g + d < 1].\]

Moreover, conditionally on the existence of closed edges, $(G_n/n, D_n/n)$ converges in distribution towards $(G, D)$ whose density equals

\[1_{\{x>0,y>0,x+y<1\}} \frac{\mathbb{P}[g \in dx, d \in dy] / dx dy}{\mathbb{P}[g + d < 1]}.
\]

By Lemma 5.8 for $x > 0, y > 0, x + y < 1$,

\[\mathbb{P}[g \in dx, d \in dy | A = a, B = b] = dx dy \cdot 1_{\{a<x<1-y<1-b\}} \frac{\kappa^2}{\pi^2} \left( \frac{\sin^2(\alpha\pi)(\sinh(\sqrt{\kappa}))^\alpha}{(\sinh(\sqrt{\kappa}(1-x-y)))^{\alpha} \sinh(\sqrt{\kappa}x) \sinh(\sqrt{\kappa}y)} \right) \times \left( \frac{\kappa \alpha (\alpha + 1) \left( \frac{2 \cosh(\sqrt{\kappa})}{\sinh(\sqrt{\kappa})} \right)^\alpha \left( \frac{\sinh(\sqrt{\kappa}a) \sinh(\sqrt{\kappa}b)}{(\sinh(\sqrt{\kappa}(a + b)))^{\alpha + 2}} \right)}{(\sinh(\sqrt{\kappa}(x-a))) \sinh(\sqrt{\kappa}(y-b))} \right)^{1-\alpha}.
\]

Therefore,

\[\mathbb{P}[g \in dx, d \in dy] = \int_{0<a<x,0<b<y} \rho(a, b) \mathbb{P}[g \in dx, d \in dy | A = a, B = b] da db = dx dy \cdot \int_{0<a<x,0<b<y} \frac{\kappa^2}{\pi^2} \frac{\alpha (\alpha + 1) \sin^2(\alpha\pi)(2 \cosh(\sqrt{\kappa}))^\alpha \sinh(\sqrt{\kappa}x) \sinh(\sqrt{\kappa}y)}{(\sinh(\sqrt{\kappa}(1-x-y)))^{\alpha} \sinh(\sqrt{\kappa}x) \sinh(\sqrt{\kappa}y)}
\]

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\begin{align*}
\times \frac{\sinh(\sqrt{\kappa a}) \sinh(\sqrt{\kappa b}) \, da \, db}{(\sinh(\sqrt{\kappa(a+b)})\alpha + 2)(\sinh(\sqrt{\kappa(x-a)}) \sinh(\sqrt{\kappa(y-b)})\alpha + 2).}
\end{align*}

We make a change of variable as follows:

\[ p = \frac{1 - e^{-2\sqrt{\kappa a}}(1 - e^{-2\sqrt{\kappa x}})}{e^{-2\sqrt{\kappa a}} - e^{-2\sqrt{\kappa x}}} \quad \text{and} \quad q = \frac{1 - e^{-2\sqrt{\kappa b}}(1 - e^{-2\sqrt{\kappa y}})}{e^{-2\sqrt{\kappa b}} - e^{-2\sqrt{\kappa y}}}. \]

Then,

\[ \mathbb{P}[g \in dx, d \in dy] = dx dy \cdot \frac{2^{\alpha + 2}\kappa^{\alpha}(\alpha + 1) \sin^{2}(\alpha)(\cosh(\sqrt{\kappa}))^{\alpha}}{\pi^{2}(\sinh(\sqrt{\kappa(1-x-y)})^{\alpha}(\sinh(\sqrt{\kappa(x)}) \sinh(\sqrt{\kappa(y)})^{\alpha + 2}}
\]

\[ \times \int_{p,q>0} \frac{pq \cdot dp \, dq}{p^{\alpha + 2}(1-e^{-2\sqrt{\kappa(x+y)}})(1-e^{-2\sqrt{\kappa(y)}})pq + p + q}^{\alpha + 2}. \]

For the simplicity of notation, set \( \delta = \frac{1 - e^{-2\sqrt{\kappa(x+y)}}}{(1-e^{-2\sqrt{\kappa(x)}})(1-e^{-2\sqrt{\kappa(y)}})} \). By performing the change of variable \( z = \frac{p}{\delta pq + p + q} \),

\[ \int_{p,q>0} \frac{pq \cdot dp \, dq}{p^{\alpha + 2}(1-e^{-2\sqrt{\kappa(x+y)}})(1-e^{-2\sqrt{\kappa(y)}})pq + p + q}^{\alpha + 2} = \int_{p,q>0} \frac{pq dp \, dq}{(p + q + \delta pq)^{\alpha + 2}} \]

\[ = \int_{p>0, \delta \in [0,1]} \frac{p^{1-\alpha}}{(1+\delta)^{\alpha}(1-\delta)^{\alpha}z^{\alpha}(1-z)} \, dp \, dz \]

\[ = \frac{1}{\alpha(\alpha+1)} \int_{0}^{\infty} \frac{p^{1-\alpha}}{(1+\delta)^{\alpha}(1-\delta)^{\alpha}} \, dp. \]

We take \( w = \frac{1}{1+\delta} \):

\[ \int_{p,q>0} \frac{pq dp \, dq}{(p + q + \delta pq)^{\alpha + 2}} = \frac{1}{\alpha(\alpha+1)\delta^{2-\alpha}} \int_{0}^{1} w^{\alpha-1}(1-w)^{1-\alpha} \, dw \]

\[ = \frac{1}{\alpha(\alpha+1)\delta^{2-\alpha}} \text{Beta}(2-\alpha, \alpha) \]

\[ = \frac{1-\alpha}{\alpha(\alpha+1)\delta^{2-\alpha}} \text{Beta}(1-\alpha, \alpha). \]

By Euler’s reflection formula, \( \text{Beta}(1-\alpha, \alpha) = \frac{\pi}{\sin(\pi\alpha)} \). Thus, for \( x > 0, y > 0, x + y < 1 \),

\[ \mathbb{P}[g \in dx, d \in dy] = dx dy \cdot \frac{\sin(\alpha\pi)}{\pi} \frac{2^{\alpha}(1-\alpha)\kappa(\cosh(\sqrt{\kappa}))^{\alpha}}{[\sinh(\sqrt{\kappa(1-x-y)})]^{\alpha}[\sinh(\sqrt{\kappa(x+y)})]^{\alpha + 2}}. \]

Then we have that

\[ Pr \overset{\text{def}}{=} \int_{x>0,y>0,x+y<1} \mathbb{P}[g \in dx, d \in dy] \]

\[ = \int_{x>0,y>0,x+y<1} \frac{\sin(\alpha\pi)}{\pi} \frac{\kappa(1-\alpha)(2 \cosh(\sqrt{\kappa}))^{\alpha} \, dx \, dy}{[\sinh(\sqrt{\kappa(1-x-y)})]^{\alpha}[\sinh(\sqrt{\kappa(x+y)})]^{\alpha + 2}} \]

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\[
\begin{align*}
\int_0^{1-e^{-2\sqrt{\kappa}}} & \frac{\sin(\alpha\pi)}{\pi} \frac{\kappa(1-\alpha)(2\cosh\sqrt{\kappa})^\alpha dx dz}{[\sinh(\sqrt{\kappa}(1-z))]^\alpha[\sinh(\sqrt{\kappa}z)]^{2-\alpha}} \\
= & \int_0^{1} \frac{\sin(\alpha\pi)}{\pi} \frac{\kappa(1-\alpha)(2\cosh\sqrt{\kappa})^\alpha z dz}{[\sinh(\sqrt{\kappa}(1-z))]^\alpha[\sinh(\sqrt{\kappa}z)]^{2-\alpha}}.
\end{align*}
\]

Take \( s = \frac{1-e^{-2\sqrt{\kappa}}}{1-e^{-2\sqrt{\kappa}}} \):

\[
Pr = \frac{\sin(\alpha\pi)}{\pi} (1-\alpha)(2\cosh\sqrt{\kappa})^\alpha \frac{e^{-\alpha\sqrt{\kappa}}}{1-e^{-2\sqrt{\kappa}}} \times \int_0^{1} s^{\alpha-2}(1-s)^{-\alpha} \left( -\log(1 - (1-e^{-2\sqrt{\kappa}})s) \right) ds.
\]

Since \(-\log(1 - (1-e^{-2\sqrt{\kappa}})s) = \sum_{n=1}^{\infty} \frac{1}{n} (1-e^{-2\sqrt{\kappa}})^n s^n \),

\[
Pr = \frac{\sin(\alpha\pi)}{\pi} (1-\alpha)(2\cosh\sqrt{\kappa})^\alpha \frac{e^{-\alpha\sqrt{\kappa}}}{1-e^{-2\sqrt{\kappa}}} \sum_{n=1}^{\infty} \left( \frac{1-e^{-2\sqrt{\kappa}}}{n} \right)^n s^{-2+n}(1-s)^{-\alpha} ds
\]

\[
= \frac{\sin(\alpha\pi)}{\pi} (1-\alpha)(2\cosh\sqrt{\kappa})^\alpha \frac{e^{-\alpha\sqrt{\kappa}}}{1-e^{-2\sqrt{\kappa}}} \sum_{n=1}^{\infty} \left( \frac{1-e^{-2\sqrt{\kappa}}}{n} \right)^n \text{Beta}(\alpha + n - 1, 1 - \alpha)
\]

\[
= \frac{\sin(\alpha\pi)}{\pi} (1-\alpha)(2\cosh\sqrt{\kappa})^\alpha \frac{e^{-\alpha\sqrt{\kappa}}}{1-e^{-2\sqrt{\kappa}}} \sum_{n=1}^{\infty} \left( \frac{1-e^{-2\sqrt{\kappa}}}{n} \right)^{n-1} \Gamma(\alpha + n - 1) \Gamma(1 - \alpha).
\]

We have

\[
\sum_{n=1}^{\infty} \frac{(1-e^{-2\sqrt{\kappa}})^{n-1}}{n!} \Gamma(\alpha + n - 1) = \sum_{n=1}^{\infty} \frac{(1-e^{-2\sqrt{\kappa}})^{n-1}}{n!} \int_0^{\infty} e^{-t} t^{\alpha-2+n} dt
\]

\[
= \int_0^{\infty} e^{-t} t^{\alpha-1} \left( \sum_{n=1}^{\infty} \frac{(1-e^{-2\sqrt{\kappa}})^{n-1} t^{n-1}}{n!} \right) dt
\]

\[
= \frac{1}{1-e^{-2\sqrt{\kappa}}} \int_0^{\infty} (e^{-e^{-2\sqrt{\kappa}}t} - e^{-t}) t^{\alpha-2} dt.
\]

By integration by parts,

\[
\sum_{n=1}^{\infty} \frac{(1-e^{-2\sqrt{\kappa}})^{n-1}}{n!} \Gamma(\alpha + n - 1) = \frac{1}{1-e^{-2\sqrt{\kappa}}} (e^{-e^{-2\sqrt{\kappa}}t} - e^{-t}) \left. t^{\alpha-1} \right|_0^{\infty}
\]

\[
- \frac{1}{1-e^{-2\sqrt{\kappa}}} \left[ \int_0^{\infty} (e^{-t} - e^{-2\sqrt{\kappa}e^{-2\sqrt{\kappa}}t}) t^{\alpha-1} dt \right.
\]

\[
= \frac{\Gamma(\alpha)(1-e^{-2\sqrt{\kappa}})}{(1-\alpha)(1-e^{-2\sqrt{\kappa}})}.
\]

Then,

\[
P[g + d < 1] = \frac{(2 \cosh \sqrt{\kappa})^\alpha \sinh(\sqrt{\kappa}(1-\alpha))}{\sinh \sqrt{\kappa}}.
\]

Finally, one can deduce the distribution of \((G, D)\).  

\[\square\]
Next, we will calculate the probability that there is no closed edge. For that part, we need the following lemma.

**Lemma 5.9.** Set

\[ x_1 = \frac{1}{2} \left( 1 + c_n + \sqrt{(1 + c_n)^2 - 4p_n(1 - p_n)} \right), \]

\[ x_2 = \frac{1}{2} \left( 1 + c_n - \sqrt{(1 + c_n)^2 + 4p_n(1 - p_n)} \right) \]

and

\[ r_n = \log \left( \frac{x_1}{\sqrt{p_n(1 - p_n)}} \right) \]

\[ = \log \left( \frac{1 + c_n}{2\sqrt{p_n(1 - p_n)}} + \sqrt{\frac{(1 + c_n)^2}{4p_n(1 - p_n)} - 1} \right) \]

\[ = \log \left( 1 + \frac{\kappa(n)}{2} + \sqrt{\kappa(n) + \frac{(\kappa(n))^2}{4}} \right). \]

Then,

\[ a) \]

\[ \mu_n(l \ \text{visits} \ 1) = \log \left( \frac{1 + c_n}{\sqrt{(1 + c_n)^2 - 4p_n(1 - p_n)}} \right) + \log(\sinh(nr_n)) \]

\[ - \log \left( \cosh(nr_n) - \cosh(n \log(p_n/(1 - p_n))/2) \right). \]

\[ b) \]

\[ \mu_n(O^{(n)}_3) = \log \left( \frac{1 + c_n}{\sqrt{(1 + c_n)^2 - 4p_n(1 - p_n)}} \right) + \log(\tanh(nr_n)). \]

\[ c) \]

\[ \mu_n(O^{(n)}_2 \cup O^{(n)}_4) = \log \left( \frac{\cosh(nr_n)}{\cosh(nr_n) - \cosh(n \log(p_n/(1 - p_n))/2)} \right) \]

and

\[ P[L^{(n)}_{\alpha,2} \cup L^{(n)}_{\alpha,4} = \phi] = \left( \frac{\cosh(nr_n)}{\cosh(nr_n) - \cosh(n \log(p_n/(1 - p_n))/2)} \right)^{-\alpha}. \]

\[ d) \] If \ \lim_{n \to \infty} n^2\kappa^{(n)} = \kappa \ \text{and} \ \lim_{n \to \infty} n^2c_n = \epsilon \in [0, \kappa/2], \ then

\[ \lim_{n \to \infty} P[L^{(n)}_{\alpha,2} \cup L^{(n)}_{\alpha,4} = \phi] = \left( \frac{\cosh(\sqrt{\kappa}) - \cosh(\sqrt{\kappa - 2\epsilon})}{\cosh(\sqrt{\kappa})} \right)^{\alpha}. \]

**Proof.**
a) By Lemma 2.3 we have that
\[
\mu_n(1) = -\log \det(-L^{(n)}) + \sum_{i=1}^{n} \log(-L^{(n)})_i^n
\]
and that
\[
\mu_n(l \text{ is inside } \{2, \ldots, n\}) = \log \det(-L^{(n)}|_{\{2, \ldots, n\}}^2) - \sum_{i=2}^{n} \log(-L^{(n)})_i^n.
\]

Thus,
\[
\mu_n(l \text{ visits } 1) = \mu_n(1) - \mu_n(l \text{ is inside } \{2, \ldots, n\})
\]
\[
= \log(-L^{(n)})_1^n + \log(\det(-L^{(n)}|_{\{2, \ldots, n\}}^2)) - \log(\det(-L^{(n)}))
\]

By Lemma 2.1 the above quantity equals
\[
\log(1 + c_n) + \log(x_1^n - x_2^n) - \log(x_1 - x_2) - \log(x_1^n + x_2^n - p_n^n - (1 - p_n)^n).
\]

Then,
\[
\mu_n(l \text{ visits } 1) = \log \left( \frac{1 + c_n}{\sqrt{(1 + c_n)^2 - 4p_n(1 - p_n)}} \right) + \log(\sinh(nr_n))
\]
\[
- \log(\cosh(nr_n) - \cosh(n \log(p_n/(1 - p_n))/2))
\]

b) By Lemma 5.2
\[
\mu_n(\mathcal{O}_3^{(n)}) = \mu_{n,3}(l \text{ visits } 0 \text{ and } l \subset [1 - n, n - 1])
\]
\[
= \mu_{n,3}(l \subset [1 - n, n - 1]) - \mu_{n,3}(l \subset [1 - n, -1]) - \mu_{n,3}(l \subset [1, n - 1])
\]
where \(\mu_{n,3}\) is defined in Lemma 5.2. By Lemma 2.3 and Lemma 2.1
\[
\mu_n(\mathcal{O}_3^{(n)}) = \log \left( \frac{1 + c_n}{\sqrt{(1 + c_n)^2 - 4p_n(1 - p_n)}} \right) + \log(\tanh(nr_n))
\]

c) We have that \(\mu_n(\mathcal{O}_2^{(n)} \cup \mathcal{O}_4^{(n)}) = \mu_n(l \text{ visits } 1) - \mu_n(\mathcal{O}_3^{(n)})\). As a corollary of part a) and b),
\[
\mu_n(\mathcal{O}_2^{(n)} \cup \mathcal{O}_4^{(n)}) = \log \left( \frac{\cosh(nr_n)}{\cosh(nr_n) - \cosh(n \log(p_n/(1 - p_n))/2)} \right).
\]
Consequently,
\[
\mathbb{P}[\mathcal{L}_{\alpha,2}^{(n)} \cup \mathcal{L}_{\alpha,4}^{(n)} = \phi] = \exp\{-\alpha \mu_n(\mathcal{O}_2^{(n)} \cup \mathcal{O}_4^{(n)})\}
\]
\[
= \left( \frac{\cosh(nr_n)}{\cosh(nr_n) - \cosh(n \log(p_n/(1 - p_n))/2)} \right)^{-\alpha}.
\]
d) Under the assumptions, we have
\[ p_n = 1 - \frac{2\sqrt{\kappa - 2\epsilon}}{n} + o(1/n) \text{ and } r_n = \frac{\sqrt{\kappa}}{n} + o(1/n). \]

Thus,
\[ \lim_{n \to \infty} P[L(n^2)_{\alpha,2} \cup L(n^2)_{\alpha,4} = \emptyset] = \left( \frac{\cosh(\sqrt{\kappa}) - \cosh(\sqrt{\kappa - 2\epsilon})}{\cosh(\sqrt{\kappa})} \right)^\alpha. \]

Finally, we summarize the above results and present the limit distribution of \( C_n^{(\alpha)} \) in the following theorem.

**Theorem 5.10.** Suppose \( \lim_{n \to \infty} n^2 \kappa^{(n)} = \kappa \) where \( \kappa^{(n)} \overset{\text{def}}{=} \frac{1+c_n-2\sqrt{p_n(1-p_n)}}{\sqrt{p_n(1-p_n)}} \). Suppose that \( \lim_{n \to \infty} n^2 c_n = \epsilon \in [0, \kappa/2] \). Let \( C^{(n)}_\alpha \) be the partition given by loop clusters on discrete circle which is defined in the introduction. If \( C^{(n)}_\alpha \) is not a single partition, then there exists \( G_n \geq 0, D_n \geq 0 \) such that \( D_n + G_n < n \) and that
\[ \{-G_n + n + 1, \ldots, n, 1, \ldots, 1 + D_n\} \]
is the cluster containing 1. In this case, let
\[ 1 + D_n = S^{(n)}_0 < S^{(n)}_1 < \cdots < S^{(n)}_{k^{(n)}} = n - G_n \]
be all the left end points of the closed edges. Define
\[ Z_t^{(n)} = \frac{1}{n-1-G_n-D_n} (S^{(n)}_{(n-1-G_n-D_n)^{1-\alpha} t} - S^{(n)}_0). \]

Let \( (G, D) \) be a pair of variables with the following density \( p(x, y) \)
\[ 1_{\{x, y > 0, x+y < 1\}} \frac{\sin(\alpha \pi)}{\pi} \frac{2^{\alpha-2}(1-\alpha)\kappa \sinh(\sqrt{\kappa})}{\sinh(\sqrt{\kappa}(1-\alpha)) [\sinh(\sqrt{\kappa}(1-x-y))]^{\alpha} [\sinh(\sqrt{\kappa}(x+y))]^{2-\alpha}}. \]

Let \( Y^{(\kappa)} \) be a subordinator bridge described in Lemma 3.3.

- For \( \alpha \geq 1 \), \( \lim_{n \to \infty} P[C^{(n)}_\alpha \text{ is a single partition}] = 1. \) For \( \alpha \in ]0, 1[ \),
\[ \lim_{n \to \infty} P[C^{(n)}_\alpha \text{ is not a single partition}] \]
\[ = \frac{1}{\sinh(\sqrt{\kappa})}\frac{2^{\alpha} \sinh(\sqrt{\kappa}(1-\alpha)) \cosh(\sqrt{\kappa}) - \cosh(\sqrt{\kappa - 2\epsilon})}{(\cosh(\sqrt{\kappa}) - \cosh(\sqrt{\kappa - 2\epsilon}))^\alpha}. \]

- Fix \( \alpha \in ]0, 1[ \). Conditionally on that \( C^{(n)}_\alpha \) is not a single partition, \( (\frac{G_n}{n}, \frac{D_n}{n}, Z^{(n)}) \)
converges in distribution to \( (G, D, M) \). Conditionally on \( (G, D) \), \( M \) has the same distribution of \( Y^{(\kappa(1-G-D)^2)} \).
Proof.

- By Proposition 5.5, Lemma 5.9 (see part d)) and Remark 5.2 for \( \alpha \in [0, 1[ , \]
  \[
  \lim_{n \to \infty} \mathbb{P}[\mathcal{C}_n^{(n)} \text{ is not a single partition}]
  = \lim_{n \to \infty} \mathbb{P}[\exists \text{ at least a closed edge}]
  = \lim_{n \to \infty} \mathbb{P}[\mathcal{L}_{\alpha,2}^{(n)} \cup \mathcal{L}_{\alpha,4}^{(n)} \text{ is empty}]
  \times \lim_{n \to \infty} \mathbb{P}[\exists \text{ an edge which is not covered by } \mathcal{L}_{\alpha,1}^{(n)} \cup \mathcal{L}_{\alpha,3}^{(n)}]
  = \left( \frac{\cosh(\sqrt{\kappa}) - \cosh(\sqrt{\kappa - 2\epsilon})}{\cosh(\sqrt{\kappa})} \right)^\alpha \left( 2 \cosh(\sqrt{\kappa})^\alpha \sinh(\sqrt{\kappa}(1 - \alpha)) \right)
  = \frac{1}{\sinh(\sqrt{\kappa})} 2^\alpha \sinh(\sqrt{\kappa}(1 - \alpha))(\cosh(\sqrt{\kappa}) - \cosh(\sqrt{\kappa - 2\epsilon}))^\alpha.
\]

Since \( \mathcal{L}_{\alpha}^{(n)} \) increases as \( \alpha \) increases, \( \mathbb{P}[\mathcal{C}_n^{(n)} \text{ is a single partition}] \) is a non-decreasing function of \( \alpha \). Therefore, for \( \alpha \geq 1, \lim_{n \to \infty} \mathbb{P}[\mathcal{C}_n^{(n)} \text{ is a single partition}] = 1. \]

- By Proposition 5.5,
  \[
  \lim_{n \to \infty} \mathbb{P}[(G_n/n, D_n/n) \in \cdot|\exists \text{ a closed edge}] = \mathbb{P}[(G, D) \in \cdot].
  \]

Then, by Remark 5.2 and the first half part of the proof, we also have
  \[
  \lim_{n \to \infty} \mathbb{P}[(G_n/n, D_n/n) \in \cdot|\exists \text{ more than two clusters}] = \mathbb{P}[(G, D) \in \cdot].
  \]

From the construction of the discrete model, given the cluster containing the vertex \( x_0 = 1 \), the closed edges have the same conditional law as the closed edges in the random discrete interval \( J_x \) where \( J_x \) is the complement of the cluster at \( x_0 \). One can find \( (G_n', n, D_n'/n, Z_n')_n \) and \( (G', D') \) such that

- \( (G', D') \overset{\text{law}}{=} (G, D), \)
- \( (G_n', n, D_n'/n, Z_n') \overset{\text{law}}{=} (G_n/n, D_n/n, Z_n) \) for all \( n, \)
- \( \lim_{n \to \infty} (G_n'/n, D_n'/n) = (G, D). \)

Conditionally on \( (G_n', D_n'), Z_n' \) is the process of left end points of closed edges\(^{19}\) on discrete interval \( \{ 1, \ldots, n-1-G_n'-D_n' \} \) with killing measure \( \kappa_n \). By assumptions, \( \lim_{n \to \infty} \kappa_n n^2 = \kappa. \) Therefore, \( \lim_{n \to \infty} (n-1-G_n'-D_n')^2 \kappa_n = (1-G'+D')^2 \kappa. \) By Theorem 4.5 and its following remarks,

\[
\lim_{n \to \infty} \mathbb{P}[Z_n' \in \cdot|G_n', D_n'] = \mathbb{P}[Y_{(1-G'+D')^2} \in \cdot|G', D'],
\]

\[^{19}\text{More accurately, we add egdes } \{0, 1 \} \text{ and } \{n-1-G_n'-D_n', n-G_n'-D_n' \} \text{ to the collection of closed edges.} \]
6 Informal relation with convergence of loop soups

Finally, we would like to give informal remarks of the previous results from the point of view of the scaling limit of the loop soup. Please refer to [Lup13] for the Markovian loop soup of one dimensional diffusions.

Firstly, let us give an informal explanation of the convergence result for the closed edges in the loop cluster model on $N$ which is proved in [LJL12].

It is known that the Brownian loop soup is the scaling limit of simple random walk loop soup. Intuitively, the scaling limit of the closed edges probably has some relationship with the zero set of the occupation field of the Brownian loop. In fact, as an application of Proposition 3.4 of [Lup13], the occupation field of Brownian loop soup with killing rate $\frac{\kappa}{2}$ within $[0, \infty]$, is a homogeneous branching process with immigration. It is the solution of the following SDE:

$$dX_t = 2\sqrt{X_t}dB_t - 2\sqrt{\kappa}X_t dt + 2\alpha dt, t \in [0, \infty[$$

where $B$ is a Brownian motion and $X_0 = 0$. (It belongs to the Cox-Ingersoll-Ross (CIR) family of diffusions which could be viewed as a generalization of squared Bessel process. More precisely, it is a radial Ornstein-Uhlenbeck process of dimension $2\alpha$ with parameter $-\sqrt{\kappa}$, see [GY03].) To be more precise, when we apply Proposition 3.4 in [Lup13], we take the non-increasing positive harmonic function to be $u_\downarrow(x) = e^{-\sqrt{\kappa}x}$ and take the non-decreasing positive harmonic function to be $u_\uparrow(x) = \frac{2}{\sqrt{\kappa}}\sinh(\sqrt{\kappa}x)$ such that the Green function density with respect to the Lebesgue measure $G(x, y)$ is given by $G(x, y) = u_\downarrow(x)u_\downarrow(y)$ for $x \leq y$. (This normalization is required in order to apply Proposition 3.4 in [Lup13].) We see that $w(x) = \text{Wronskian}(u_\downarrow, u_\uparrow) = 2$. One can check that the zero set is given by the range of the subprocessor with potential density $U(x, y) = 1_{\{y>x\}}\left(\frac{2\sqrt{\kappa}}{1-e^{-2\sqrt{\kappa}(y-x)}}\right)^\alpha$.

Next, we consider the loop cluster over a discrete interval which is considered in this article. If we suppose the approximation by Brownian loop soup within $]0, 1[$ works, then we expect that the limit distribution of the closed edges is the zero set of the occupation field of this Brownian loop soup. By Proposition 3.4 of [Lup13], we know that the occupation field over the interval $]0, 1[$ indexed by the position $t \in ]0, 1[$ is the solution of the following SDE:

$$dY_t = 2\sqrt{Y_t}dB_t + \left(2\alpha - \frac{\cosh(\sqrt{\kappa}(1-t))}{\sinh(\sqrt{\kappa}(1-t))}Y_t\right) dt, t \in [0, 1].$$

In fact, it is the bridge of a square radial OU process of dimension $2\alpha$ of parameter $-\lambda$ from 0 to 0 of fixed time duration $1$. Please refer to [FPY93] for Markovian bridge and

\footnote{There is not an immediate consequence of convergence of loop soup. That’s why our explanation stays informal.}
refer to [GJY03] for the transition density of square radial OU process and its relationship with squared Bessel process. Let $D_t$ be the first time of hitting 0 after time $t$. Then, the Radon-Nikodym derivative of the bridge process over the square radial OU process is

$$1_{\{D_t < 1\}} \left( \frac{1 - e^{-\sqrt{\kappa} t}}{1 - e^{-2\sqrt{\kappa}(1-D_t)}} \right)^{\alpha}$$

restricted on the sub-$\sigma$-field up to time $D_t$. This is exactly the same as $\frac{U(D_t, 1)}{U(0, 1)}$ which is used to construct our subordinator bridge. Then, one can check that the zero set of the bridge of the square radial OU process agrees with the range of the conditioned subordinator defined in Lemma 3.3.

Finally, we would like to point out the way to get the limit distribution of $(A, B)$ in Lemma 5.3 from the point of view of Brownian loops. By the structure of Poisson random measure, it is enough to check this for $\alpha = 1$. In this case, there is a connection between the loops passing through a fixed point and the Poisson point process of excursions at the same point, see e.g. [LJ11], [Lup13]. For $\alpha = 1$, they agree with each other. Then, the distribution of $[-A, B]$ is exactly the random interval covered by these excursions under the condition that they don’t cover $-1$ nor $1$. The condition of avoiding $-1$ and $1$ only affects the joint density of $(A, B)$ up to a normalization constant. Therefore, we could remove this restriction for the moment. The total local time at $0$ is an exponential variable with expectation $G(x, x) = 1/\sqrt{\kappa}$ as a result of Poisson point process nature of the excursions at $0$. The occupation time (total local time) indexed by the position $x \in ]-\infty, \infty[$ forms a two-sided process, the part on the left hand side of $0$ is denoted by $(U_{-x}, x \geq 0)$ under time reversal and the right part is denoted by $(V_x, x \geq 0)$. By Ray-Knight theorem for diffusions, conditioned on the total local time at $0$, $U$ and $V$ are two independent copies of square radial OU processes of dimension zero and parameter $-\sqrt{\kappa}$, see e.g. Proposition 3.1 [Lup13]. Thus, it is enough to compute the first hitting time of $0$, and then integrate them with respect to the total local time. The density of the first hitting time of $0$ for our square radial OU process is given by

$$t \to \frac{x^2}{2} \left( \frac{\sqrt{\kappa}}{\sinh(\sqrt{\kappa}t)} \right)^2 \exp\{\frac{\sqrt{\kappa}}{2} x^2 (1 - \coth(\sqrt{\kappa}t))\},$$

see e.g. [ELY99] (Corollary 3.19). Finally, we get the joint density of the first hitting times of $0$ for $U$ and $V$. We see that it is exactly the same density as the limit distribution of $(A_n/n, B_n/n)$ as $n \to \infty$ up to a normalization constant, see Lemma 5.3.
References

[Ber96] Jean Bertoin, *Lévy processes*, Cambridge Tracts in Mathematics, vol. 121, Cambridge University Press, Cambridge, 1996. MR 1406564 (98e:60117)

[Ber99] ________, *Subordinators: examples and applications*, Lectures on probability theory and statistics (Saint-Flour, 1997), Lecture Notes in Math., vol. 1717, Springer, Berlin, 1999, pp. 1–91. MR 1746300 (2002a:60001)

[BG05] Albrecht Böttcher and Sergei M. Grudsky, *Spectral properties of banded Toeplitz matrices*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2005. MR 2179973 (2006k:47054)

[CW05] Kai Lai Chung and John B. Walsh, *Markov processes, Brownian motion, and time symmetry*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 249, Springer, New York, 2005. MR 2152573 (2006j:60003)

[EK86] Stewart N. Ethier and Thomas G. Kurtz, *Markov processes*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons Inc., New York, 1986, Characterization and convergence. MR 838085 (88a:60130)

[ELY99] K. D. Elworthy, Xue-Mei Li, and M. Yor, *The importance of strictly local martingales; applications to radial Ornstein-Uhlenbeck processes*, Probab. Theory Related Fields 115 (1999), no. 3, 325–355. MR 1725406 (2001b:60064)

[FPY93] Pat Fitzsimmons, Jim Pitman, and Marc Yor, *Markovian bridges: construction, Palm interpretation, and splicing*, Seminar on Stochastic Processes, 1992 (Seattle, WA, 1992), Progr. Probab., vol. 33, Birkhäuser Boston, Boston, MA, 1993, pp. 101–134.

[GJY03] Anja Göing-Jaeschke and Marc Yor, *A survey and some generalizations of Bessel processes*, Bernoulli 9 (2003), no. 2, 313–349. MR 1997032 (2004g:60098)

[JS03] Jean Jacod and Albert N. Shiryaev, *Limit theorems for stochastic processes*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 288, Springer-Verlag, Berlin, 2003.

[Kes69] Harry Kesten, *Hitting probabilities of single points for processes with stationary independent increments*, Memoirs of the American Mathematical Society, No.
[LJ11] Yves Le Jan, *Markov paths, loops and fields*, Lecture Notes in Mathematics, vol. 2026, Springer, Heidelberg, 2011, Lectures from the 38th Probability Summer School held in Saint-Flour, 2008, École d'Été de Probabilités de Saint-Flour. [Saint-Flour Probability Summer School].

[LJL12] Yves Le Jan and Sophie Lemaire, *Markovian loop clusters on graphs*, Preprint, [http://arxiv.org/abs/1211.0300](http://arxiv.org/abs/1211.0300) [math.PR], 2012.

[Lup13] Titus Lupu, *Poissonian ensembles of loops of one-dimensional diffusions*, Preprint, [http://http://arxiv.org/abs/1302.3773](http://http://arxiv.org/abs/1302.3773) [math.PR], 2013.

[Szn12] Alain-Sol Sznitman, *Topics in occupation times and Gaussian free fields*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2012.