Estimation of Complexity for the Ohya-Masuda-Volovich SAT Algorithm

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Abstract
Ohya and Volovich have been proposed a new quantum computation model with chaos amplification to solve the SAT problem, which went beyond usual quantum algorithm. In this paper we study the complexity of the SAT algorithm by counting the steps of computation algorithm rigorously, which was mentioned in the paper [1, 2, 3, 5, 7]. For this purpose, we refine the quantum gates treating the SAT problem step by step.

1 Introduction
The problem, asking whether NP-complete problem can be solved in polynomial time, is one of the most important problems in the computation theory. If the computational methods are based only on the classical Turing machines, it seems that the difficulty attached to this problem cannot be removed.

Ohya and Volovich [1, 2, 3] have been proposed a new quantum computation algorithm with chaos amplifier to solve the SAT problem [4], which went beyond usual quantum algorithm. This quantum chaos algorithm has enabled us to solve the SAT problem in a polynomial time [1, 2, 3, 5], and moreover alternative solution of the SAT problem is given in the stochastic limit by Accardi and Ohya [6].

In this paper, we study the computational complexity of the SAT algorithm mentioned in [1, 2, 5, 7] by counting the steps of computation accurately. For this purpose, we show explicitly how to construct the Ohya-Masuda algorithm from the elementary gates. In Section 2, the definition of the SAT problem is explained according to Ohya-Masuda and Accardi-Sabbadini [5, 7]. In Section 3, mathematical basis of quantum computation is given. In Section 4, we determine the number of dust qubits required in the Ohya-Masuda algorithm exactly, and we construct the unitary operator needed for computation of the SAT problem. In Section 5, the chaos amplifier introduced by Ohya-Volovich algorithm is explained. In Section 6, we discuss the computational complexity of their SAT algorithm.
2 The SAT Problem

In this section we review the SAT problem according to Ohya-Masuda, Accardi-Ohya and Accardi-Sabaddini[5, 6, 7]. Through this paper, $\mathbb{N}$ denotes the set of all positive integers, and $\{0, 1\}$ denotes the simplest Boolean lattice with the meet-operation $\land$, the join-operation $\lor$ and the negation-operation $\lnot$. Let $n$ be a positive integer and let $X$ be a set consisting of $n$ Boolean variables, which is denoted by $\{x_1, \ldots, x_n\}$. Then, $\bar{X}$ and $X'$ denote the two sets consisting of Boolean variables, which are defined as $\{\bar{x}_1, \ldots, \bar{x}_n\}$ and $X \cup \bar{X}$, respectively, where $\bar{x}$ means the negation of $x$. For any subset $C$ of $X'$, the truth value of $C$, which is denoted by $t(C)$, is defined as

$$t(C) \equiv \left\{ \bigvee_{x_i \in C \cap X} t(x_i) \right\} \lor \left\{ \bigvee_{\bar{x}_j \in C \cap \bar{X}} t(\bar{x}_j) \right\}, \quad (2.1)$$

where $t(x_i)$ and $t(\bar{x}_j)$ are the Boolean values of $x_i$ and $\bar{x}_j$. $C$ is called a clause, and $t(C)$ is called the truth value of $C$. Let $m$ be a positive integer and let $\mathcal{C}$ be a set consisting of $m$ clauses. Then, the truth value of $\mathcal{C}$, which is denoted by $t(\mathcal{C})$, is defined as

$$t(\mathcal{C}) = \land_{i=1}^m t(C_i), \quad (2.2)$$

where $C_i$ is an element of $\mathcal{C}$. It is a matter of course that the truth value of $\mathcal{C}$ can be exactly determined by $\{\varepsilon_i \equiv t(x_i); i = 1, \ldots, n\}$. Therefore, under the above notations, the SAT problem is the problem asking whether, for a given set of clauses $\mathcal{C}$, there exists an assignment $(\varepsilon_1, \ldots, \varepsilon_n)$ belonging to $\{0, 1\}^n$ and satisfying that $t(\mathcal{C}) = 1$ holds. Here, $\mathcal{C}$ is called satisfiable if there exists a solution $(\varepsilon_1, \ldots, \varepsilon_n)$ satisfying $t(\mathcal{C}) = 1$. Here, we can illustrate the following example:

**Example 1** Let $x_1, x_2, x_3$ and $x_4$ be four Boolean variables, and $C_1, C_2$ and $C_3$ be three clauses defined as $\{x_1, \bar{x}_2, \bar{x}_3\}, \{\bar{x}_1, x_2, x_4\}$ and $\{x_1, x_3, \bar{x}_4\}$, respectively. Then, $(0, 0, 0, 1)$ is a solution of the SAT problem, because, if we take 0 as values of $t(x_1), t(x_2)$ and $t(x_3)$, and 1 as a value of $t(x_4)$, then, the following three equalities:

$$t(C_1) = t(x_1) \lor t(\bar{x}_2) \lor t(\bar{x}_3) = 1$$
$$t(C_2) = t(\bar{x}_1) \lor t(x_2) \lor t(x_4) = 1$$
$$t(C_3) = t(\bar{x}_1) \lor t(x_3) \lor t(\bar{x}_4) = 1 \quad (2.3)$$

hold. Therefore, we can obtain

$$t(C_1) \land t(C_2) \land t(C_3) = 1. \quad (2.4)$$

These equalities show that $\{C_1, C_2, C_3\}$ is satisfiable.
3 Elements of Quantum Computation

In this section, we review the foundation of quantum computation (see for instance, [3]). Let \( \mathbb{C} \) be the set of all complex numbers, and \( |0\rangle \) and \( |1\rangle \) be the two unit vectors \( (10) \) and \( (01) \), respectively. Then, for any two complex numbers \( \alpha \) and \( \beta \) satisfying \( |\alpha|^2 + |\beta|^2 = 1 \), \( \alpha |0\rangle + \beta |1\rangle \) is called a qubit. For any positive integer \( N \), let \( \mathcal{H} \) be the tensor product Hilbert space defined as \( (\mathbb{C}^2)^\otimes N \) and let \( \{|e_i\rangle; 0 \leq i \leq 2^N - 1\} \) be the basis whose elements are defined as

\[
|e_0\rangle = |0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle \equiv |0,0,\cdots,0\rangle,
|e_1\rangle = |1\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle \equiv |1,0,\cdots,0\rangle,
|e_2\rangle = |0\rangle \otimes |1\rangle \otimes \cdots \otimes |0\rangle \equiv |0,1,\cdots,0\rangle,
\]

\[
\vdots
|e_{2^N-1}\rangle = |1\rangle \otimes |1\rangle \otimes \cdots \otimes |1\rangle \equiv |1,1,\cdots,1\rangle, \tag{3.1}
\]

respectively. For any two qubits \( |x\rangle \) and \( |y\rangle \), \( |x, y\rangle \) and \( |x\rangle \otimes |y\rangle \) is defined as \( |x\rangle \otimes |y\rangle \) and \( |x\rangle \otimes \cdots \otimes |x\rangle \), respectively.

The quantum computation can be formulated mathematically as the multiplication by unitary operators. Let \( U_{\text{NOT}} \), \( U_{\text{CN}} \) and \( U_{\text{CCN}} \) be the three unitary operators defined as

\[
U_{\text{NOT}} \equiv |1\rangle \langle 0| + |0\rangle \langle 1|,
U_{\text{CN}} \equiv |0\rangle \langle 0| \otimes I + |1\rangle \langle 1| \otimes U_{\text{NOT}},
U_{\text{CCN}} \equiv |0\rangle \langle 0| \otimes I \otimes I + |1\rangle \langle 1| \otimes |0\rangle \langle 0| \otimes I + |1\rangle \langle 1| \langle 1| \otimes |1\rangle \langle 1| \otimes U_{\text{NOT}}. \tag{3.2}
\]

\( U_{\text{NOT}} \), \( U_{\text{CN}} \) and \( U_{\text{CCN}} \) are called the NOT-gate, the Controlled-NOT gate and the Controlled-Controlled-NOT gate, respectively. Moreover, Hadamard transformation \( H \) is defined as the transformation on \( \mathbb{C}^2 \) such that

\[
H |0\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle),
H |1\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle). \tag{3.3}
\]

The four operators \( U_{\text{NOT}} \), \( U_{\text{CN}} \), \( U_{\text{CCN}} \) and \( H \) are called the elementary gates. For any \( k \in \mathbb{N} \), \( U_H^{(N)}(k) \) denotes the \( k \)-tuple Hadamard transformation on \( (\mathbb{C}^2)^\otimes N \) defined as

\[
U_H^{(N)}(k) |0^N\rangle = \frac{1}{2^{k/2}} (|0\rangle + |1\rangle)^\otimes k |0^{N-k}\rangle
= \frac{1}{2^{k/2}} \sum_{i=0}^{2^k-1} |e_i\rangle \otimes |0^{N-k}\rangle. \tag{3.4}
\]
These unitary operators can be used for the construction of the following three unitary operators on \((\mathbb{C}^2)^{\otimes N}\):

\[
U_{\text{NOT}}^{(N)}(n) = I \otimes I^{n-1} \otimes \ket{0} \bra{1} + \ket{1} \bra{0} I^{\otimes N-u-1}
\]

\[
U_{\text{AND}}^{(N)}(u,v) = I \otimes I^{n-1} \otimes \ket{0} \bra{0} I^{\otimes N-u-1} + I \otimes I^{v-u-1} \otimes I \otimes U_{\text{NOT}} \otimes I^{\otimes N-v-1}
\]

\[
U_{\text{CCN}}^{(N)}(u,v,w) = I \otimes I^{n-1} \otimes \ket{0} \bra{0} I^{\otimes N-u-1} + I \otimes I^{v-u-1} \otimes \ket{1} \bra{0} I^{\otimes w-t-1} \otimes U_{\text{NOT}} \otimes I^{\otimes N-w-1},
\]

where \(u, v\) and \(w\) be a positive integers satisfying \(1 \leq u < v < w \leq N\). \(U_{\text{NOT}}^{(N)}(u), U_{\text{AND}}^{(N)}(u,v), U_{\text{CCN}}^{(N)}(u,v)\) and \(U_{\text{DEF}}^{(N)}(k)\) are called \(N\)-qubit elementary gates. When no confusion may arise, we identify the \(N\)-qubit elementary gates with the elementary gates itself.

Next, three unitary operators \(U_{\text{AND}}, U_{\text{OR}}\) and \(U_{\text{COPY}}\) are called the logical gates, defined as [7]

\[
U_{\text{AND}} = \sum_{\varepsilon_1, \varepsilon_2 \in \{0,1\}} \{ \ket{\varepsilon_1, \varepsilon_2, \varepsilon_1 \land \varepsilon_2} \bra{\varepsilon_1, \varepsilon_2, 0} + \ket{\varepsilon_1, \varepsilon_2, 1 - \varepsilon_1 \land \varepsilon_2} \bra{\varepsilon_1, \varepsilon_2, 1} \}
\]

\[
= \ket{0,0,0} \bra{0,0,0} + \ket{0,0,1} \bra{0,0,1} + \ket{1,0,0} \bra{1,0,0} + \ket{1,0,1} \bra{1,0,1} + \ket{0,1,0} \bra{0,1,0} + \ket{0,1,1} \bra{0,1,1} + \ket{1,1,0} \bra{1,1,0} + \ket{1,1,1} \bra{1,1,1}.
\]

\[
U_{\text{OR}} = \sum_{\varepsilon_1, \varepsilon_2 \in \{0,1\}} \{ \ket{\varepsilon_1, \varepsilon_2, \varepsilon_1 \lor \varepsilon_2} \bra{\varepsilon_1, \varepsilon_2, 0} + \ket{\varepsilon_1, \varepsilon_2, 1 - \varepsilon_1 \lor \varepsilon_2} \bra{\varepsilon_1, \varepsilon_2, 1} \}
\]

\[
= \ket{0,0,0} \bra{0,0,0} + \ket{0,0,1} \bra{0,0,1} + \ket{1,0,0} \bra{1,0,0} + \ket{1,0,1} \bra{1,0,1} + \ket{0,1,0} \bra{0,1,0} + \ket{0,1,1} \bra{0,1,1} + \ket{1,1,0} \bra{1,1,0} + \ket{1,1,1} \bra{1,1,1}.
\]

\[
U_{\text{COPY}} = \sum_{\varepsilon_1 \in \{0,1\}} \{ \ket{\varepsilon_1, \varepsilon_1} \bra{\varepsilon_1, 0} + \ket{\varepsilon_1, 1 - \varepsilon_1} \bra{\varepsilon_1, 1} \}
\]

\[
= \ket{0,0} \bra{0,0} + \ket{0,1} \bra{0,1} + \ket{1,0} \bra{1,0} + \ket{1,1} \bra{1,1}.
\]

\(U_{\text{AND}}, U_{\text{OR}}\) and \(U_{\text{COPY}}\) are called the AND gate, the OR gate and the COPY gate, respectively. Finally the unitary operators on \((\mathbb{C}^2)^{\otimes N}\), which are denoted by \(U_{\text{AND}}^{(N)}, U_{\text{OR}}^{(N)}\) and \(U_{\text{COPY}}^{(N)}\), can be defined as
unitary operator satisfying the following equation.

\[ U_{\text{AND}}^{(N)}(u, v, w) = \sum_{\varepsilon_1, \varepsilon_2 \in \{0, 1\}} I^{\otimes u-1} \otimes |\varepsilon_1\rangle \langle \varepsilon_1| I^{\otimes v-u-1} \otimes |\varepsilon_2\rangle \langle \varepsilon_2| \]
\[ + I^{\otimes w-v-u-1} \otimes |\varepsilon_1 \wedge \varepsilon_2\rangle \langle 0| I^{\otimes N-v-u+1} \]
\[ + I^{\otimes u-1} \otimes |\varepsilon_1\rangle \langle \varepsilon_1| I^{\otimes v-u-1} \otimes |\varepsilon_2\rangle \langle \varepsilon_2| \]
\[ + I^{\otimes w-v-u-1} \otimes |1 - \varepsilon_1 \wedge \varepsilon_2\rangle \langle 1| I^{\otimes N-v-u}. \] (3.10)

\[ U_{\text{OR}}^{(N)}(u, v, w) = \sum_{\varepsilon_1, \varepsilon_2 \in \{0, 1\}} I^{\otimes u-1} \otimes |\varepsilon_1\rangle \langle \varepsilon_1| I^{\otimes v-u-1} \otimes |\varepsilon_2\rangle \langle \varepsilon_2| \]
\[ + I^{\otimes w-v-u-1} \otimes |\varepsilon_1 \lor \varepsilon_2\rangle \langle 0| I^{\otimes N-v-u+1} \]
\[ + I^{\otimes u-1} \otimes |\varepsilon_1\rangle \langle \varepsilon_1| I^{\otimes v-u-1} \otimes |\varepsilon_2\rangle \langle \varepsilon_2| \]
\[ + I^{\otimes w-v-u-1} \otimes |1 - \varepsilon_1 \lor \varepsilon_2\rangle \langle 1| I^{\otimes N-v-u}. \] (3.11)

\[ U_{\text{COPY}}^{(N)}(u, v) = \sum_{\varepsilon_1 \in \{0, 1\}} I^{\otimes u-1} \otimes |\varepsilon_1\rangle \langle \varepsilon_1| I^{\otimes v-u-1} \otimes |0\rangle \langle 0| I^{\otimes N-v-u} \]
\[ + I^{\otimes u-1} \otimes |\varepsilon_1\rangle \langle \varepsilon_1| I^{\otimes v-u-1} \otimes |1 - \varepsilon_1\rangle \langle 1| I^{\otimes N-v-u}. \] (3.12)

where \( u, v \) and \( w \) are positive integers satisfying \( 1 \leq u < v < w \leq N \). These operators can be represented, in terms of elementary gates, as

\[ U_{\text{OR}}^{(N)}(u, v, w) = U_{\text{CN}}^{(N)}(u, v) \cdot U_{\text{CN}}^{(N)}(v, w) \cdot U_{\text{CN}}^{(N)}(u, v, w), \]
\[ U_{\text{AND}}^{(N)}(u, v, w) = U_{\text{CCN}}^{(N)}(u, v, w), \]
\[ U_{\text{COPY}}^{(N)}(u, v) = U_{\text{CN}}^{(N)}(u, v). \] (3.13)

4 Quantum Computational Model of the Ohya-Masuda Algorithm

In this section, we explain the computation method which has been developed by Ohya-Masuda and Accardi-Sabbadini[5, 7]. The quantum algorithm is described by a combination of the unitary operators on a Hilbert space \( \mathcal{H} \). Throughout this section, let \( n \) be the total number of Boolean variables used in the SAT problem. Let \( \mathcal{C} \) be a set of clauses whose cardinality is equal to \( m \). Following the method of the Ohya-Masuda algorithm[5], let \( \mathcal{H} = (\mathbb{C}^2)^{\otimes n+\mu+1} \) be a Hilbert space and \( |v_{in}\rangle \) be the initial state \( |v_{in}\rangle = |0^0, 0^\mu, 0\rangle \), where \( \mu \) is the number of dust qubits which is determined by the following theorem 2. Let \( U_{\mathcal{C}}^{(n)} \) be a unitary operator satisfying the following equation.

\[ U_{\mathcal{C}}^{(n)} |v_{in}\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |e_1, x^\mu, t_{e_1}(\mathcal{C})\rangle \]
\[ = |v_{out}\rangle \] (4.1)
where \( x^\mu \) denotes a \( \mu \) strings of binary symbols and \( t_{e_i}(C) \) is a truth value of \( C \) with \( e_i \). In [7], a method to construct \( U_C^{(n)} \) is discussed. Let \( \{s_k; k = 1, \ldots , m\} \) be the sequence defined as

\[
\begin{align*}
    s_1 &= n + 1, \\
    s_2 &= s_1 + \text{card}(C_1) + \delta_{1, \text{card}(C_1)} - 1, \\
    s_i &= s_{i-1} + \text{card}(C_{i-1}) + \delta_{1, \text{card}(C_{i-1})}, \quad 3 \leq i \leq m, \\
\end{align*}
\]

where \( \text{card}(C_i) \) means the cardinality of a clause \( C_i \). And let \( s_f \) be a number defined as

\[
s_f = s_m - 1 + \text{card}(C_m) + \delta_{1, \text{card}(C_m)}. \tag{4.3}
\]

Then we can prove the following:

**Theorem 2** For \( m \geq 2 \), the total number of dust qubits \( \mu \) is

\[
\mu = s_f - 1 - n = \sum_{k=1}^{m} \text{card}(C_k) + \delta_{1, \text{card}(C_k)} - 2. \tag{4.4}
\]

**Proof.** If \( \text{card}(C_k) \) is greater than 1, it is required to use the join operation \((\text{card}(C_k) - 1)\) times to obtain the value of \( t(C_k) \). If \( \text{card}(C_k) \) is equal to 1, we prepare one qubit to make a copy of a Boolean variable included in \( C_k \). Here, assume that there exists a qubit where \( \bigwedge_{i=1}^{k-1} t(C_i) \) is stored. Then, one more qubit is required to store \((\bigwedge_{i=1}^{k-1} t(C_i)) \land t(C_k) \). These results imply that \( \text{card}(C_k) - 1 + \delta_{1, \text{card}(C_k)} + 1 \) qubits are required to compute \( t(C_k) \) and \( \bigwedge_{i=1}^{k-1} t(C_i) \). Therefore, we can obtain

\[
s_{k+1} - s_k = \text{card}(C_k) + \delta_{1, \text{card}(C_k)}. \tag{4.5}
\]

Finally, the total number of dust qubits \( \mu \) which are required to compute \( \bigwedge_{i=1}^{m} t(C_i) \) is

\[
\mu = \sum_{k=1}^{m} s_{k+1} - s_k
= s_2 - s_1 + \sum_{k=2}^{m} \text{card}(C_k) + \delta_{1, \text{card}(C_k)}
= \sum_{k=1}^{m} \text{card}(C_k) + \delta_{1, \text{card}(C_k)} - 2
= s_f - 1 - n \tag{4.6}
\]

Determining \( \mu \) and the work spaces for computing \( t(C_k) \), we can construct \( U_C^{(n)} \) concretely.
Theorem 3  The unitary operator $U^{(n)}_c$, is represented as

$$U^{(n)}_c = \prod_{i=m-1}^1 U^{(n+\mu+1)}_{AN\Delta}(i) \prod_{j=m}^1 U^{(n+\mu+1)}_{OR}(j) U^{(n+\mu+1)}_{H}(n). \quad (4.7)$$

Proof. For any positive integers $u, v$ and $w$ satisfying $1 \leq u < v < w \leq n + \mu + 1$, if $C_k \cap \{x_u, \bar{x}_u\} \neq \phi$ and $C_k \cap \{x_v, \bar{x}_v\} \neq \phi$ hold, $U^{(N)}_{OR}(u, v, w)$ is defined as

$$U^{(N)}_{OR}(u, v, w) = \left\{\begin{array}{ll}
U^{(N)}_{NOT}(u) U^{(N)}_{NOT}(u, v, w) U^{(N)}_{NOT}(u), & x_u, x_v \in C_k, \\
U^{(N)}_{NOT}(v) U^{(N)}_{NOT}(u, v, w) U^{(N)}_{NOT}(v), & x_u, \bar{x}_v \in C_k, \\
U^{(N)}_{NOT}(u) U^{(N)}_{NOT}(v) U^{(N)}_{OR}(u, v, w) U^{(N)}_{NOT}(v), & x_u, \bar{x}_v \in C_k.
\end{array} \right. \quad (4.8)$$

If the cardinality of $C_k$ is equal to one, then there exists a Boolean variable $x_u$ satisfying $C_k = \{x_u\}$ or $C_k = \{\bar{x}_u\}$. Therefore, $U^{(N)}_{OR}(k)$ is defined as

$$U^{(N)}_{OR}(k) = \left\{\begin{array}{ll}
U^{(N)}_{COPY}(u, s_k), & x_u \in C_k, \\
U^{(N)}_{NOT}(s_k) U^{(N)}_{COPY}(u, s_k), & \bar{x}_u \in C_k.
\end{array} \right. \quad (4.9)$$

If the cardinality of $C_k$ is equal to two, then there exists two Boolean variables $x_u$ and $x_v$ satisfying that either $x_u \in C_k$ or $\bar{x}_u \in C_k$ holds, and moreover, either $x_v \in C_k$ or $\bar{x}_v \in C_k$ holds. Therefore, $U^{(N)}_{OR}(k)$ is defined as

$$U^{(N)}_{OR}(k) = U^{(N)}_{OR}(u, v, s_k). \quad (4.10)$$

If the cardinality of $C_k$ is greater than 2, $U^{(N)}_{OR}(k)$ can be defined by the way as above, namely, this operator is defined as

$$U^{(N)}_{OR}(k) = \prod_{i=\text{card}(C_k)-2}^1 U^{(N)}_{OR}(w, s_k + i - 1, s_k + i) U^{(N)}_{OR}(u, v, s_k). \quad (4.11)$$

If the cardinality of $C$ is equal to one, then $U^{(N)}_{AN\Delta}(1)$ is defined as

$$U^{(N)}_{AN\Delta}(1) = U^{(N)}_{COPY}(s_1 + \text{card}(C_1) + \delta_{1,\text{card}(C_1)} - 1, s_1 + \text{card}(C_1) + \delta_{1,\text{card}(C_1)}). \quad (4.12)$$

If the cardinality of $C$ is greater than one, then $U^{(N)}_{AN\Delta}(k)$ is defined as

$$U^{(N)}_{AN\Delta}(k) = \left\{\begin{array}{ll}
U^{(N)}_{AN\Delta}(s_k + 1, s_k + 2, s_k + 1) - 1, & 1 \leq k \leq m - 2, \\
U^{(N)}_{AN\Delta}(s_m - 1, s_m + \text{card}(C_m) + \delta_{1,\text{card}(C_m)} - 2), & k = m - 1.
\end{array} \right. \quad (4.13)$$

It is clear that $U^{(N)}_{AN\Delta}(k)$ can compute $\land_{i=1}^{k-1} (C_i)$. We can construct the unitary operator $U^{(n)}_c$ from $\{U^{(n+\mu+1)}_{OR}(i); 1 \leq i \leq m\}$ and $\{U^{(n+\mu+1)}_{AN\Delta}(i); 1 \leq i \leq m - 1\}$.
as follows:

\[ U^{(n)}_C = \prod_{i=m-1}^{1} U^{(n+\mu+1)}_{\text{AND}}(i) \prod_{j=m}^{1} U^{(n+\mu+1)}_{\text{OR}}(j) U^{(n+\mu+1)}_{H}(n). \]  \tag{4.14}

The following theorem is shown in Accardi-Ohya [6].

**Theorem 4** C is SAT if and only if

\[ P_{n+\mu,1}U^{(n)}_C |v_{in}\rangle \neq 0 \]  \tag{4.15}

where \( P_{n+\mu,1} \) denotes the projector

\[ P_{n+\mu,1} \equiv I^\otimes n+\mu \otimes |1\rangle \langle 1| \]  \tag{4.16}

onto the subspace of \( \mathcal{H} \) spanned by the vectors \( |\varepsilon^n, \varepsilon^{\mu}, 1\rangle \).

**4.1 Example**

For example, Let \( x_1, x_2, x_3 \) and \( x_4 \) be four Boolean variables, and \( C_1, C_2, C_3 \) and \( C_4 \) be four clauses defined as \( \{x_1, x_4, \bar{x}_2\}, \{x_2, x_3, x_4\}, \{x_1, \bar{x}_3\} \) and \( \{x_3, \bar{x}_1, \bar{x}_2\} \), respectively. Let \( \mathcal{C} \) be the set of clauses consisting of \( C_1, C_2, C_3 \) and \( C_4 \). First, we calculate \( s_1, s_2, s_3, s_4 \) and \( s_f \). According to Theorem 6.7, we obtain

\[
\begin{align*}
    s_1 &= n + 1 = 5, \\
    s_2 &= s_1 + \text{card}(C_1) + \delta_{1,\text{card}(C_1)} - 1 \\
        &= 5 + 3 + 0 - 1 \\
        &= 7, \\
    s_3 &= s_2 + \text{card}(C_2) + \delta_{1,\text{card}(C_2)} \\
        &= 7 + 3 + 0 \\
        &= 10, \\
    s_4 &= s_3 + \text{card}(C_3) + \delta_{1,\text{card}(C_3)} \\
        &= 10 + 2 \\
        &= 12, \\
    s_f &= s_4 + \text{card}(C_4) + \delta_{1,\text{card}(C_4)} - 1 \\
        &= 12 + 3 - 1 \\
        &= 14. \tag{4.17}
\end{align*}
\]

Then we construct OR and AND gates following Theorem 3. We have
\[ U_{OR}^{(14)} (1) = U_{NOT}^{(14)} (2) U_{OR}^{(14)} (2, 5, 6) U_{NOT}^{(14)} (2) U_{OR}^{(14)} (1, 4, 5), \]
\[ U_{OR}^{(14)} (2) = U_{OR} (4, 7, 8) U_{OR} (2, 3, 7), \]
\[ U_{OR}^{(14)} (3) = U_{NOT}^{(14)} (3) U_{OR}^{(14)} (1, 3, 10) U_{NOT}^{(14)} (3), \]
\[ U_{OR}^{(14)} (4) = U_{NOT}^{(14)} (2) U_{OR}^{(14)} (2, 12, 13) U_{NOT}^{(14)} (2) U_{NOT}^{(14)} (1) U_{OR}^{(14)} (3, 1, 12) U_{NOT}^{(14)} (1), \]

\[ (4.18) \]

\[ \begin{align*}
U_{AND}^{(14)} (1) &= U_{AND}^{(14)} (6, 8, 9), \\
U_{AND}^{(14)} (2) &= U_{AND}^{(14)} (9, 10, 11), \\
U_{AND}^{(14)} (3) &= U_{AND}^{(14)} (11, 13, 14).
\end{align*} \]

\[ (4.19) \]

Thus, we obtain the unitary gate \( U_c^{(4)} \) by the combination of the above gates as

\[ \begin{align*}
U_c^{(4)} &= U_{AND}^{(14)} (11, 13, 14) U_{AND}^{(14)} (9, 10, 11) U_{AND}^{(14)} (6, 8, 9) \\
&\quad \cdot U_{NOT}^{(14)} (2) U_{OR}^{(14)} (2, 12, 13) U_{NOT}^{(14)} (2) U_{NOT}^{(14)} (1) U_{OR}^{(14)} (3, 1, 12) U_{NOT}^{(14)} (1) \\
&\quad \cdot U_{NOT}^{(14)} (3) U_{OR}^{(14)} (1, 3, 10) U_{NOT}^{(14)} (3) \\
&\quad \cdot U_{OR}^{(14)} (4, 7, 8) U_{OR}^{(14)} (2, 3, 7) \\
&\quad \cdot U_{NOT}^{(14)} (2) U_{OR}^{(14)} (2, 5, 6) U_{NOT}^{(14)} (2) U_{OR}^{(14)} (1, 4, 5). \\
\end{align*} \]

\[ (4.20) \]

Let \( |v_{in}\rangle \) be the initial state \( |v_{in}\rangle = |0^4, 0^{10}, 0\rangle \). Applying \( U_{DFT}^{(14)} (4) \) to \( |v_{in}\rangle \), we have

\[ \begin{align*}
|v_{in}\rangle &\equiv U_{DFT}^{(14)} (4) |0^4, 0^9, 0\rangle \\
&= \frac{1}{\sqrt{2^4}} \sum_{i=0}^{2^4-1} |\varepsilon_i, 0^9, 0\rangle \\
&= \frac{1}{\sqrt{2^4}} \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{0, 1\}} |\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, 0^9, 0\rangle \\
&\equiv |v\rangle. \\
\end{align*} \]

\[ (4.21) \]

Next, applying \( \prod_{k=4}^{1} U_{OR}^{(14)} (k) \) to \( |v\rangle \), we obtain
\[ U^{(14)}_{OR} (4) U^{(14)}_{OR} (3) U^{(14)}_{OR} (2) U^{(14)}_{OR} (1) |\psi \rangle = \frac{1}{(\sqrt{2})^4} U^{(14)}_{OR} (4) U^{(14)}_{OR} (3) U^{(14)}_{OR} (2) U^{(14)}_{OR} (1) \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{0, 1\}} |\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, 0^8, 0\rangle \\
= \frac{1}{(\sqrt{2})^4} U^{(14)}_{OR} (4) U^{(14)}_{OR} (3) U^{(14)}_{OR} (2) \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{0, 1\}} |\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_1 \lor \varepsilon_4, \varepsilon_1 \lor \varepsilon_4 \lor \varepsilon_2, 0^8, 0\rangle \\
= \frac{1}{(\sqrt{2})^4} \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{0, 1\}} |\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_1 \lor \varepsilon_4, \varepsilon_1 \lor \varepsilon_4 \lor \varepsilon_2, \varepsilon_2 \lor \varepsilon_3, \varepsilon_2 \lor \varepsilon_3 \lor \varepsilon_4, 0, 0\rangle \\
\equiv |v'\rangle. \quad (4.22) \]

Finally, applying AND gates to \(|v'\rangle\), we have

\[ U^{(14)}_{\text{AND}} (m) |v'\rangle = U^{(14)}_{\text{AND}} (11, 14, 15) U^{(14)}_{\text{AND}} (9, 10, 11) U^{(14)}_{\text{AND}} (6, 8, 9) |v'\rangle \]

\[ = \frac{1}{(\sqrt{2})^4} \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{0, 1\}} |\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_1 \lor \varepsilon_4, \varepsilon_1 \lor \varepsilon_4 \lor \varepsilon_2, \varepsilon_2 \lor \varepsilon_3, \varepsilon_2 \lor \varepsilon_3 \lor \varepsilon_4, \varepsilon_1 \lor \varepsilon_3, \varepsilon_1 \lor \varepsilon_3 \lor \varepsilon_2, \varepsilon_1 \lor \varepsilon_3 \lor \varepsilon_2, t \varepsilon (C)\rangle. \quad (4.23) \]

After the measurement of the last qubit, we obtain the final state

\[ \rho' = \frac{7}{16} |1\rangle \langle 1| + \frac{9}{16} |0\rangle \langle 0|. \quad (4.24) \]

## 5 Chaos Amplification of the SAT algorithm

Let us explain the chaos amplifier introduced by Ohya-Volovich [1, 2]. Let \( T (C) \) be the set of all the elements in \( \{0, 1\}^m \) satisfying \( t (C) = 1 \) and \(|T (C)|\) be the cardinality of \( T (C) \). After the quantum computation due to the Ohya-Masuda algorithm, the quantum computer will be in the state

\[ |v_{\text{out}}\rangle = \sqrt{1 - q^2} |\varphi_0\rangle \otimes |0\rangle + q |\varphi_1\rangle \otimes |1\rangle, \quad (5.1) \]

where \(|\varphi_0\rangle\) and \(|\varphi_1\rangle\) are normalized \(n\) qubit states and \(q = \sqrt{|T (C)|/2^n}\). It is useful to quantum computing in which the result probability of unitary computation is very small. Let \( E_0 \) and \( E_1 \) be a projection operators \( E_0 = |0\rangle \langle 0| \) and \( E_1 = |1\rangle \langle 1| \). According to the Ohya-Volovich algorithm[1, 2], we transform the state \(|v_{\text{out}}\rangle\) into the density matrix of the form

\[ \rho = q^2 E_1 + (1 - q^2) E_0. \quad (5.2) \]

The logistic map which is given by the equation

\[ x_{n+1} = ax_n (1 - x_n) \equiv f_a (x_n), \quad x_n \in [0, 1]. \quad (5.3) \]
The properties of this map depend on the parameter $a$. Then the density matrix $\rho$ above is interpreted as the initial data $\rho_0$, and Ohya-Volovich applied the logistic map to the state $\rho$ as

$$\rho_m = \frac{(I + f_a^m(\rho_0)\sigma_3)}{2}, \quad (5.4)$$

where $I$ is the identity matrix and $\sigma_3$ is the $z$-component of Pauli matrix on $\mathbb{C}^2$. Finally the value of $\sigma_3$ is measured in the state $\rho_m$

$$M_m \equiv tr\rho_m\sigma_3. \quad (5.5)$$

The following theorems 5, 6 and 7 are proven in [1, 2].

**Theorem 5**

$$\rho_m = \frac{(I + f_a^m(q^2)\sigma_3)}{2}, \quad (5.6)$$

**Theorem 6** For the logistic map $x_{n+1} = ax_n (1 - x_n)$ with $a \in [0, 4]$ and $x_0 \in [0, 1]$, let $x_0$ be $\frac{1}{2}$ and a set $J$ be $\{0, 1, 2, \ldots, n, \ldots, 2n\}$. If $a$ is 3.71, then there exists an integer $m$ in $J$ satisfying $x_m > \frac{1}{2}$.

**Theorem 7** Let $a$ and $n$ be the same in above proposition. If there exists $m$ in $J$ such that $x_m > \frac{1}{2}$, then $m > \frac{n - 1}{\log_2 3.71}$.

From these theorems, we have

**Corollary 8** Let $\rho$ be the initial state of the Ohya-Masuda algorithm and $\rho_0$ be the initial data of the chaos amplifier correspond to $\rho$. For all $m$, it holds

$$M_m \begin{cases} = 0 & \text{iff $C$ is not SAT} \\ > 0 & \text{iff $C$ is SAT} \end{cases} \quad (5.7)$$

**Corollary 9** Let $x_0 \equiv \frac{r}{2}$ with $r \equiv |T(C)|$. From Theorem 7, there exists $m$ satisfying the following inequation if $C$ is SAT.

$$\left[ \frac{n - 1 - \log_2 r}{\log_2 3.71 - 1} \right] \leq m \leq \left[ \frac{5}{4} (n - 1) \right]. \quad (5.8)$$

We plot $q^2$ as a function of $m$ with $n = 12$ and $l = 1$. In Figure 1, it is shown that $q^2$ increases in the first six steps. We can see that $q^2$ becomes more than $\frac{1}{2}$ within $\frac{n - 1}{\log_2 3.71 - 1}$ steps.

If we apply the chaos amplifier to $\rho_0$ $\frac{n - 1 - \log_2 r}{\log_2 3.71 - 1}$ times, then the maximum value of $q^2$ can be achieved as we can see Figure 2. In other words, Figure 2 shows that, $\frac{n - 1 - \log_2 r}{\log_2 3.71 - 1}$ times application of the chaos amplifier can obtain the maximum value of $q^2$ which is greater than or equal to $\frac{1}{2}$. 

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Figure 1: Amplification process, $n = 12$ and $l = 1$.

Figure 2: Maximum value of $q^2$
6 Computational Complexity of the OMV SAT Algorithm

In this section, we define the computational complexity of the OMV SAT algorithm including the chaos amplifier. First, we define the computational complexity of the quantum part, Ohya-Masuda algorithm. The computational complexity of a quantum algorithm is determined by the number of elementary gates in the algorithm. Since the unitary operator $U^{(n)}_C$ which has been used in the Ohya-Masuda algorithm is constructed by the product of elementary gates (see Theorem 3), we define the computational complexity of $U^{(n)}_C$ as the number of the elementary gates.

**Definition 10** The computational complexity of the unitary operator $U$ consisting of the elementary gates, denoted by $T_Q(U)$, is defined as

$$T_Q(U) = |U|,$$

where $|U|$ denotes the number of elementary gates which are the components of $U$.

Next, we define the computational complexity of the chaos amplifier as follows.

**Definition 11** For any positive integer $n$, we define the computational complexity of the chaos amplifier, denoted by $T_C(n)$, is defined as

$$T_C(n) = \max \left\{ m; m = \min \left\{ l; M_l \left( q^2 \right) \geq \frac{1}{2} \right\} , q^2 \in \left\{ \frac{1}{2}, \frac{2}{2n}, \ldots, \frac{2^{n-1}}{2n} \right\} \right\}.$$

**Corollary 12** The computational complexity of the chaos amplifier of the SAT algorithm with $n$ Boolean variables can be obtained as

$$T_C(n) = \left\lceil \frac{5}{4} (n-1) \right\rceil.$$

In the SAT algorithm with $n$ Boolean variables, according to Corollary 9, there exists a proper $m$ satisfying $M_m \geq \frac{1}{2}$ within $m \leq \left\lceil \frac{5}{4} (n-1) \right\rceil$. Since it is impossible to know the value of $m$ satisfying $M_m \geq \frac{1}{2}$ before the computation of the chaos amplifier, we have to compute again when we chose $m$ not satisfying $M_m \geq \frac{1}{2}$. It implies that we must repeat the quantum computation $\frac{5}{4} (n-1)$ times at worst. Thus, we define the computational complexity of the SAT algorithm as the product of $T_Q \left( U^{(n)}_C \right)$ and $T_C(n)$.

**Definition 13** The computational complexity of the SAT algorithm is defined as

$$T_Q \left( U^{(n)}_C \right) T_C(n).$$
Theorem 14  For a set of clauses $C$ and $n$ Boolean variables, the computational complexity of the SAT algorithm including the chaos amplifier, denoted by $T(C,n)$, is obtained as follows.

$$T(C,n) = T_Q \left( U_c^{(n)} \right) T_C(n)$$

$$= \left\{ 3 \sum_{k=1}^{m} (\text{card}(C_k) - 1) + \sum_{k-1}^{m} 2\text{card}(C_k \cap \{\bar{x}_1, \ldots, \bar{x}_n\}) ight\}$$

$$+ m - 1 + n \left\lfloor \frac{5}{4} (n - 1) \right\rfloor$$

$$\leq (8mn - 2m - 1) \left\lfloor \frac{5}{4} (n - 1) \right\rfloor$$

$$= \mathcal{O}(\text{poly}(n)),$$

where $\text{poly}(n)$ denotes a polynomial of $n$.

Proof. Since $T_Q \left( \prod_{k=m}^{1} U^{(n+\mu+1)}_{OR} (k) \right)$, $T_Q \left( \prod_{k=m-1}^{1} U^{(n+\mu+1)}_{AND} (k) \right)$, and $T_Q \left( U^{(n+\mu+1)}_{DFT} (n) \right)$ can be estimated by the next inequalities:

$$T_Q \left( \prod_{k=m}^{1} U^{(n+\mu+1)}_{OR} (k) \right) = 3 \sum_{k=1}^{m} (\text{card}(C_k) - 1) + \sum_{k-1}^{m} 2\text{card}(C_k \cap \{\bar{x}_1, \ldots, \bar{x}_n\})$$

$$\leq 3m (2n - 1) + 2mn,$$

$$T_Q \left( \prod_{k=m-1}^{1} U^{(n+\mu+1)}_{AND} (k) \right) = m - 1,$$

$$T_Q \left( U^{(n+\mu+1)}_{DFT} (n) \right) = n. \quad (6.6)$$

Then, $T_Q \left( U_c^{(n)} \right)$ can be obtained by the next inequalities:

$$T_Q \left( U_c^{(n)} \right) = T_Q \left( \prod_{k=m}^{1} U^{(n+\mu+1)}_{OR} (k) \right) + T_Q \left( \prod_{k=m-1}^{1} U^{(n+\mu+1)}_{AND} (k) \right)$$

$$+ T_Q \left( U^{(n+\mu+1)}_{DFT} (n) \right)$$

$$= 3 \sum_{k=1}^{m} (\text{card}(C_k) - 1) + \sum_{k-1}^{m} 2\text{card}(C_k \cap \{\bar{x}_1, \ldots, \bar{x}_n\})$$

$$+ m - 1 + n$$

$$\leq 8mn - 2m - 1.$$

Since $T_C(n) = \left\lfloor \frac{5}{4} (n - 1) \right\rfloor$, we obtain (6.5). □
7 Conclusion

In this paper, we have determined the number of dust qubits \( \mu \) exactly and constructed \( U^n_C \) step by step. Moreover, we set the computational complexity of the Ohya-Masuda algorithm and Ohya-Volovich algorithm, and computed \( T(C, n) \) accurately. Therefore, the Ohya-Volovich algorithm can give us a practical method to solve the SAT problem.

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