Quantitative Chevalley-Weil Theorem for Curves

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Abstract

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1 Introduction

The Chevalley-Weil theorem is one of the most basic principles of the Diophantine analysis. Already Diophantus of Alexandria routinely used reasoning of the kind “if \( a \) and \( b \) are ‘almost’ co-prime integers and \( ab \) is a square, then each of \( a \) and \( b \) is ‘almost’ a square”. The Chevalley-Weil theorem provides a general set-up for this kind of arguments.

**Theorem 1.1 (Chevalley-Weil)** Let \( \bar{V} \xrightarrow{\phi} V \) be a finite étale covering of normal projective varieties, defined over a number field \( \mathbb{K} \). Then there exists a non-zero integer \( T \) such that for any \( P \in V(\mathbb{K}) \) and \( \bar{P} \in \bar{V}(\bar{\mathbb{K}}) \) such that \( \phi(\bar{P}) = P \), the relative discriminant of \( \mathbb{K}(\bar{P})/\mathbb{K}(P) \) divides \( T \).

There is also a similar statement for coverings of affine varieties and integral points. See [11, Section 2.8] for more details.

The Chevalley-Weil theorem is indispensable in the Diophantine analysis, because it reduces a Diophantine problem on the variety \( V \) to that on the covering variety \( \bar{V} \), which can often be simpler to deal. In particular, the Chevalley-Weil theorem is used, sometimes implicitly, in the proofs of the great finiteness theorems of Mordell-Weil, Siegel and Faltings.

In view of all this, a quantitative version of the Chevalley-Weil theorem, at least in dimension 1, would be useful to have. One such version appears in Chapter 4 of [1], but it is not explicit in

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all parameters; neither is the version recently suggested by Draziots and Poulakis [3, 6], who also make some other restrictive assumptions (see Remark 1.4 below).

In the present article we present a version of the Chevalley-Weil theorem in dimension 1, which is explicit in all parameters and considerably sharper than the previous versions. Our approach is different from that of [3, 6] and goes back to [1, 2].

To state our principal results, we have to introduce some notation. Let $K$ be a number field, $C$ an absolutely irreducible smooth projective curve $C$ defined over $K$, and $x \in K(C)$ a non-constant $K$-rational function on $C$. We also fix a covering $\tilde{C} \rightarrow C$ of $C$ by another smooth irreducible projective curve $\tilde{C}$; we assume that both $C$ and the covering $\phi$ are defined over $K$. We consider $K(C)$ as a subfield of $\tilde{K}(\tilde{C})$; in particular, we identify the functions $x \in K(C)$ and $x \circ \phi \in K(\tilde{C})$.

We also fix one more rational function $y \in K(C)$ such that $K(\tilde{C}) = K(x, y)$ (existence of such $y$ follows from the primitive element theorem). Let $f(X, Y) \in K[X, Y]$ be the $K$-irreducible polynomial such that $f(x, y) = 0$ (it is well-defined up to a constant factor). Since $C$ is absolutely irreducible, so is the polynomial $f(\tilde{x}, \tilde{y})$. We put $m = \deg_X f$ and $n = \deg_Y f$.

Similarly, we fix a function $\tilde{y} \in K(\tilde{C})$ such that $K(\tilde{C}) = K(x, \tilde{y})$. We let $\tilde{f}(X, \tilde{Y}) \in K[X, \tilde{Y}]$ be an irreducible polynomial such that $\tilde{f}(x, \tilde{y}) = 0$. We put $\tilde{m} = \deg_X \tilde{f}$ and $\tilde{n} = \deg_Y \tilde{f}$. We denote by $\nu$ the degree of the covering $\phi$, so that $\tilde{n} = \nu n$.

**Remark 1.2** Notice that equations $f(X, Y) = 0$ and $\tilde{f}(X, \tilde{Y}) = 0$ define affine plane models of our curves $C$ and $\tilde{C}$; we do not assume these models non-singular.

In the sequel, $h_\nu(\cdot)$ and $h_\lambda(\cdot)$ denote the projective and the affine absolute logarithmic heights, respectively, see Section 2 for the definitions. We also define normalized logarithmic discriminant $\partial_{L/K}$ and the height $h(S)$ of a finite set of places $S$ as

$$\partial_{L/K} = \frac{\log N_{K/Q}[D_{L/K}]}{[L : Q]}, \quad h(S) = \sum_{v \in S} \log N_{K/Q}(v)$$

see Section 2 for the details.

Put

$$\Omega = 200mn^3 \log n(h_\nu(f) + 2m + 2n),$$

$$\tilde{\Omega} = 200\tilde{m}\tilde{n}^3 \log \tilde{n}(h_\lambda(\tilde{f}) + 2\tilde{m} + 2\tilde{n}),$$

$$\Upsilon = 2\tilde{n}(m h_\nu(f) + m h_\nu(\tilde{f})).$$

**Theorem 1.3 (“projective” Chevalley-Weil theorem)** In the above set-up, assume that the covering $\tilde{C} \rightarrow C$ is unramified. Then for every $P \in C(\tilde{K})$ and $\tilde{P} \in \tilde{C}(\tilde{K})$ such that $\phi(\tilde{P}) = P$ we have

$$\partial_{\tilde{K}(\tilde{P})/K(P)} \leq 2(\Omega + \tilde{\Omega} + \Upsilon).$$

**Remark 1.4** Draziots and Poulakis [3, Theorem 1.1], assume that $C$ is a non-singular plane curve (which is quite restrictive) and that $P \in C(\tilde{K})$. Their set-up is slightly different, and the two estimates cannot be compared directly. But it would be safe to say that their estimate is not sharper than

$$\partial_{\tilde{K}(\tilde{P})/K(P)} \leq cm^3N^{30N^{13}}(h_\nu(f) + h_\nu(\tilde{f})) + C,$$

where $N = \deg f$, $\tilde{N} = \deg \tilde{f}$, the constant $c$ is absolute and $C$ depends of $N$, $\tilde{N}$ and the degree $[K : Q]$.

Now let $S$ be a finite set of places of $K$, including all the archimedean places. A point $P \in C(\tilde{K})$ will be called $S$-integral if for any $v \in M_K \setminus S$ and any extension $\bar{v}$ of $v$ to $\tilde{K}$ we have $|x(P)|_{\bar{v}} \leq 1$.

**Theorem 1.5 (“affine” Chevalley-Weil theorem)** In the above set-up, assume that the covering $\tilde{C} \rightarrow C$ is unramified outside the poles of $x$. Then for every $S$-integral point $P \in C(\tilde{K})$ and $\tilde{P} \in \tilde{C}(\tilde{K})$ such that $\phi(\tilde{P}) = P$ we have

$$\partial_{\tilde{K}(\tilde{P})/K(P)} \leq \Omega + \tilde{\Omega} + \Upsilon + h(S).$$
Again, Draziotis and Poulakis [5, Theorem 1.1] obtain a less sharp result under more restrictive assumptions.

It might be also useful to have a statement free of the defining equations of the curves $\mathcal{C}$ and $\tilde{\mathcal{C}}$. Using the result of [3], we obtain versions of Theorems 1.3 and 1.5 which depend only on the degrees and the ramification points of our curves over $\mathbb{P}^1$. For a finite set $A \subset \mathbb{P}^1(K)$ we define $h_a(A)$ as the affine height of the vector whose coordinates are the finite elements of $A$.

**Theorem 1.6** Let $A$ be a finite subset of $\mathbb{P}^1(K)$ such that the covering $\mathcal{C} \to \mathbb{P}^1$ is unramified outside $A$. Put

$$
\delta = [K(A) : K], \quad g = g(\tilde{\mathcal{C}}), \quad \Lambda = ((g + 1)\bar{n})^{25(\bar{n}+1)} + 2(\delta - 1).
$$

1. Assume that the covering $\phi : \tilde{\mathcal{C}} \to \mathcal{C}$ is unramified. Then for every $P \in \mathcal{C}(\bar{K})$ and $\tilde{P} \in \tilde{\mathcal{C}}(\bar{K})$ such that $\phi(\tilde{P}) = P$ we have

$$
\partial_{\bar{K}(\tilde{P})/K(P)} \leq \Lambda(h_a(A) + 1).
$$

2. Assume that the covering $\phi : \tilde{\mathcal{C}} \to \mathcal{C}$ is unramified outside the poles of $x$, and let $S$ be as above. Then for every $S$-integral point $P \in \mathcal{C}(\bar{K})$ and $\tilde{P} \in \tilde{\mathcal{C}}(\bar{K})$ such that $\phi(\tilde{P}) = P$ we have

$$
\partial_{\bar{K}(\tilde{P})/K(P)} \leq h(S) + \Lambda(h_a(A) + 1).
$$

**2 Preliminaries**

Let $K$ be any number field and let $M_K = M_K^0 \cup M_K^\infty$ be the set of its places, with $M_K^0$ and $M_K^\infty$ denoting the sets of finite and infinite places, respectively. For every place $v \in M_K$ we normalize the corresponding valuation $| \cdot |$ so that its restriction to $\mathbb{Q}$ is the standard infinite or $p$-adic valuation. Also, we let $K_v$ be the $v$-adic completion of $K$, (in particular, $K_v$ is $\mathbb{R}$ or $\mathbb{C}$ when $v$ is infinite).

**Heights** For a vector $\alpha = (\alpha_1, \ldots, \alpha_N) \in \bar{Q}^N$ we define, as usual, the *absolute logarithmic projective height* and the *absolute logarithmic affine height* (in the sequel simply *projective* and *affine heights*) by

$$
h_p(\alpha) = \frac{1}{[K : Q]} \sum_{v \in M_K} [K_v : Q_v] \log \|\alpha\|_v, \quad h_a(\alpha) = \frac{1}{[K : Q]} \sum_{v \in M_K} [K_v : Q_v] \log^+ \|\alpha\|_v, \tag{3}
$$

where $K$ is any number field containing the coordinates of $\alpha$.

$$
\|\alpha\|_v = \max\{|\alpha_0|_v, \ldots, |\alpha_N|_v\}
$$

and $\log^+ = \max\{\log, 0\}$. With our choice of normalizations, the right-hand sides in (3) are independent of the choice of the field $K$. For a polynomial $f$ with algebraic coefficients we denote by $h_p(f)$ and by $h_a(f)$ the projective height and the affine height of the vector of its coefficients respectively, ordered somehow.

**Logarithmic discriminant** Given an extension $L/K$ of number fields, we denote by $\partial_{L/K}$ the *normalized logarithmic relative discriminant*:

$$
\partial_{L/K} = \frac{\log N_{L/K}D_{L/K}}{[L : Q]},
$$

where $D_{L/K}$ is the usual relative discriminant and $N_{K/Q}$ is the norm map. The properties of this quantity are summarized in the following proposition.

---

1In the definition of the projective height we assume that at least one coordinate of $\alpha$ is non-zero.
Proposition 2.1  
1. (additivity in towers) If $\mathbb{K} \subset \mathbb{L} \subset \mathbb{M}$ is a tower of number fields, then 
$$\partial_{\mathbb{M}/\mathbb{K}} = \partial_{\mathbb{L}/\mathbb{K}} + \partial_{\mathbb{M}/\mathbb{L}}.$$ 
2. (base extension) If $\mathbb{K}'$ is a finite extension of $\mathbb{K}$ and $\mathbb{L}' = \mathbb{L}\mathbb{K}'$ then $\partial_{\mathbb{L'}/\mathbb{K}'} \leq \partial_{\mathbb{L}/\mathbb{K}}.$ 
3. (triangle inequality) If $\mathbb{L}_1$ and $\mathbb{L}_2$ are two extensions of $\mathbb{K}$, then $\partial_{\mathbb{L}_1\mathbb{L}_2/\mathbb{K}} \leq \partial_{\mathbb{L}_1/\mathbb{K}} + \partial_{\mathbb{L}_2/\mathbb{K}}.$ 

These properties will be used without special reference.

**Height of a set of places**  Given a number field $\mathbb{K}$ and finite set of places $S \subset M_{\mathbb{K}}$, we define the absolute logarithmic height of this set as 
$$h(S) = \sum_{v \in S} \log \mathcal{N}_{\mathbb{K}/\mathbb{Q}}(v),$$ 
where the norm $\mathcal{N}_{\mathbb{K}/\mathbb{Q}}(v)$ of the place $v$ is the norm of the corresponding prime ideal if $v$ is finite, and is set to be 1 when $v$ is infinite. The properties of this height are summarized in the following proposition.

Proposition 2.2  
1. (field extension) If $\mathbb{L}$ is an extension of $\mathbb{K}$ and $S_{\mathbb{L}}$ is the set of extensions of the places from $S$ to $\mathbb{L}$, then $h(S_{\mathbb{L}}) \leq h(S) \leq [\mathbb{L} : \mathbb{K}]h(S_{\mathbb{L}})$.
2. (denominators and numerators) For $\alpha \in \bar{\mathbb{K}}$ let the sets $\text{Den}_\mathbb{K}(\alpha)$ and $\text{Num}_\mathbb{K}(\alpha)$ consist of all $v \in M_\mathbb{K}$ having an extension $\bar{v}$ to $\bar{\mathbb{K}}$ such that $\|\alpha\|_{\bar{v}} > 1$, respectively, $\|\alpha\|_{\bar{v}} < 1$. Then 
$$h(\text{Den}_\mathbb{K}(\alpha)) \leq [\mathbb{K}(\alpha) : \mathbb{K}]h_\alpha(\alpha),$$ 
$$h(\text{Num}_\mathbb{K}(\alpha)) \leq [\mathbb{K}(\alpha) : \mathbb{K}](h_\alpha(\alpha) - h_p(\alpha)) \quad (\alpha \neq 0).$$

In particular, for $\alpha \in \bar{\mathbb{K}}^*$ we have $h(\text{Num}_\mathbb{K}(\alpha)) \leq [\mathbb{K}(\alpha) : \mathbb{K}]h_\alpha(\alpha)$.

This will also be used without special reference.

**Sums over primes**  We shall systematically use the following estimates from [12].

$$\sum_{p \leq x} 1 \leq 1.26 \frac{x}{\log x}, \quad (4)$$

$$\sum_{p \leq x} \log p \leq 1.02x, \quad (5)$$

$$\sum_{p \leq x} \frac{\log p}{p - 1} \leq 2 \log x. \quad (6)$$

See [12], Corollary 1 of Theorem 2 for (4), Theorem 9 for (5), and (6) follows easily from the Corollary of Theorem 6.

### 3 Auxiliary Material

In this section we collect miscellaneous facts, mostly elementary and/or well-known, to be used in the article.

#### 3.1 Integral Elements

In this subsection $\mathbb{R}$ is an integrally closed integral domain and $\mathbb{K}$ its quotient field.

**Lemma 3.1** Let $\mathbb{L}$ be a finite separable extension of $\mathbb{K}$ of degree $n$ and $\bar{\mathbb{R}}$ the integral closure of $\mathbb{R}$ in $\mathbb{L}$. Let $\omega_1, \ldots, \omega_n \in \bar{\mathbb{R}}$ form a base of $\mathbb{L}$ over $\mathbb{K}$. We denote by $\Delta$ the discriminant of this basis: 
$$\Delta = \left( \det |\sigma_{ij}(\omega_j)| \right),$$
where $\sigma_1, \ldots, \sigma_n : \mathbb{L} \hookrightarrow \bar{\mathbb{K}}$ are the distinct embeddings of $\mathbb{L}$ into $\bar{\mathbb{K}}$. Then $\bar{\mathbb{R}} \subset \Delta^{-1}(\mathbb{R}\omega_1 + \cdots + \mathbb{R}\omega_n).$
Proof This is standard. Write \( \beta \in \hat{R} \) as \( \beta = a_1 \omega_1 + \cdots + a_n \omega_n \) with \( a_i \in \mathbb{K} \). Solving the system of linear equations
\[
\sigma_i(\beta) = a_1 \sigma_i(\omega_1) + \cdots + a_n \sigma_i(\omega_n) \quad (i = 1, \ldots, n)
\]
using the Kramer rule, we find that the numbers \( \Delta a_i \) are integral over \( R \). Since \( R \) is integrally closed, we have \( \Delta a_i \in R \).

Corollary 3.2 Let \( f(T) = f_0 T^n + f_1 T^{n-1} + \cdots + f_n \in R[T] \) be a \( \mathbb{K} \)-irreducible polynomial, and \( \alpha \in \bar{\mathbb{K}} \) one of its roots. Let \( R \) be the integral closure of \( R \) in \( \mathbb{K}(\alpha) \). Then \( R \subset \Delta(f)^{-1}R[\alpha] \). where \( \Delta(f) \) is the discriminant of \( f \).

Proof It is well-known that the quantities
\[
\omega_1 = 1, \quad \omega_2 = f_0 \alpha, \quad \omega_3 = f_0 \alpha^2 + f_1 \alpha, \quad \ldots \quad \omega_n = f_0 \alpha^{n-1} + f_1 \alpha^{n-2} + \cdots + f_{n-2} \alpha
\]
are integral over \( R \); see, for example, [14, page 183]. Applying Lemma 3.1 to the basis \( \omega_1, \ldots, \omega_n \), we complete the proof.

3.2 Local Lemmas

In this subsection \( \mathbb{K} \) is a field of characteristic 0 supplied with a discrete valuation \( v \). We denote by \( \mathcal{O}_v \) the local ring of \( v \). We say that a polynomial \( F(X) \in \mathbb{K}[X] \) is \( v \)-monic if its leading coefficient is a \( v \)-adic unit.

Lemma 3.3 Let \( F(X) \in \mathcal{O}_v[X] \) be a \( v \)-monic polynomial, and let \( \eta \in \bar{\mathbb{K}} \) be a root of \( F \). (We do not assume \( F \) to be the minimal polynomial of \( \eta \) over \( \mathbb{K} \), because we do not assume it \( \mathbb{K} \)-irreducible.) Assume that \( v \) ramifies in the field \( \mathbb{K}(\eta) \). Then \( |F'(\eta)|_v < 1 \) (for any extension of \( v \) to \( \mathbb{K}(\eta) \)).

Proof We may assume that \( \mathbb{K} \) is \( v \)-complete, and we let \( \mathfrak{d} = \mathfrak{d}_{\mathbb{K}(\eta)/\mathbb{K}} \) be the different of the extension \( \mathbb{K}(\eta)/\mathbb{K} \). Since \( v \) ramifies in \( \mathbb{K}(\eta) \), the different is a non-trivial ideal of \( \mathcal{O}_v \).

Since \( \eta \) is a root of a \( v \)-monic polynomial, it is integral over \( \mathcal{O}_v \). Let \( G(X) \in \mathcal{O}_v[X] \) be the minimal polynomial of \( \eta \). Then the different \( \mathfrak{d} \) divides \( G'(\eta) \), which implies that \( |G'(\eta)|_v < 1 \).

Write \( F(X) = G(X) H(X) \). By the Gauss lemma, \( H(X) \in \mathcal{O}_v[X] \). Since \( F'(\eta) = G'(\eta) H(\eta) \), we obtain \( |F'(\eta)|_v \leq |G'(\eta)|_v < 1 \), as wanted.

Given a polynomial \( F(X) \) over some field of characteristic 0, we define by \( \hat{F}(X) \) the radical of \( F \), that is, the separable polynomial, having the same roots and the same leading coefficient as \( F \):
\[
\hat{F}(X) = f_0 \prod_{F(\alpha) = 0} (X - \alpha),
\]
where \( f_0 \) is the leading coefficient of \( F \) and the product runs over the distinct roots of \( F \) (in an algebraic closure of the base field).

Lemma 3.4 Assume that \( F(X) \in \mathcal{O}_v[X] \). Then the radical \( \hat{F}(X) \) is in \( \mathcal{O}_v[X] \) as well. Also, if \( |F(\xi)|_v < 1 \) for some \( \xi \in \mathcal{O}_v \), then we have \( |\hat{F}(\xi)|_v < 1 \) as well.

\(^2\)We say that \( \alpha \) is a \( v \)-adic unit if \( |\alpha|_v = 1 \).
Proof Let $F(X) = P_1(X)^{\alpha_1} \cdots P_k(X)^{\alpha_k}$ be the irreducible factorization of $F$ in $\mathbb{K}[X]$. The Gauss Lemma implies that we can choose $P_i(X) \in \mathcal{O}_v[X]$ for $i = 1, \ldots, k$. Since the characteristic of $\mathbb{K}$ is 0, every $P_i$ is separable. Obviously, the leading coefficient of the separable polynomial $P_1(X) \cdots P_k(X)$ divides that of $F(X)$ in the ring $\mathcal{O}_v$. Hence $\hat{F}(X) = \gamma P_1(X) \cdots P_k(X)$ with some $\gamma \in \mathcal{O}_v$, which proves the first part of the lemma. The second part is obvious: if $|F(\xi)|_v < 1$ then $|P_i(\xi)|_v < 1$ for some $i$, which implies $|\hat{F}(\xi)|_v < 1$. \qed

Lemma 3.5 Let $F(X) \in \mathcal{O}_v[X]$ and $\xi \in \mathcal{O}_v$ satisfy $|F(\xi)|_v < 1$ and $|F'(\xi)|_v = 1$. Let $\bar{v}$ be an extension of $v$ to $\bar{K}$. Then there exists exactly one root $\alpha \in \bar{K}$ of $F$ such that $|\xi - \alpha|_{\bar{v}} < 1$.

Proof This is a consequence of Hensel's lemma. Extending $\mathbb{K}$, we may assume that it contains all the roots of $F$. Hensel's lemma implies that there is exactly one root $\alpha$ in the $v$-adic completion of $\mathbb{K}$ with the required property. This root must belong to $\mathbb{K}$. \quad \qed

Lemma 3.6 Let $F(X), G(X) \in \mathcal{O}_v[X]$ and $\alpha, \xi \in \mathcal{O}_v$ satisfy

$F(X) = (X - \alpha)^m G(X), \quad G(\alpha) \neq 0, \quad |\xi - \alpha|_v < |G(\alpha)|_v$

with some non-negative integer $m$. Expand the rational function $F(X)^{-1}$ into the Laurent series at $\alpha$. Then this series converges at $X = \xi$.

Proof Substituting $X \mapsto \alpha + X$, we may assume $\alpha = 0$, in which case the statement becomes obvious. \quad \qed

3.3 Heights

Recall that, for a polynomial $f$ with algebraic coefficients, we denote by $h_p(f)$ and by $h_a(f)$, respectively, the projective height and the affine height of the vector of its coefficients ordered somehow. More generally, the height $h_a(f_1, \ldots, f_s)$ of a finite system of polynomials is, by definition, the affine height of the vector formed of all the non-zero coefficients of all these polynomials.

Lemma 3.7 Let $f_1, \ldots, f_s$ be polynomials in $\mathbb{Q}[X_1, \ldots, X_r]$ and put

$N = \max\{\deg f_1, \ldots, \deg f_s\}, \quad h = h_a(f_1, \ldots, f_s)$.

Let also $g$ be a polynomial in $\mathbb{Q}[Y_1, \ldots, Y_s]$. Then

1. $h_a(\prod_{i=1}^s f_i) \leq \sum_{i=1}^s h_a(f_i) + \log(r + 1) \sum_{i=1}^{s-1} \deg f_i$,
2. $h_p(\prod_{i=1}^s f_i) \geq \sum_{i=1}^s h_p(f_i) - \sum_{i=1}^{s-1} \deg f_i$,
3. $h_a(g(f_1, \ldots, f_s)) \leq h_a(g) + (h + \log(s + 1) + N \log(r + 1)) \deg g$.

Notice that we use the projective height in item 1 and the affine height in the other items.

Proof Item 2 is the famous Gelfond inequality, see, for instance, Proposition B.7.3 in [10]. The rest is an immediate consequence of Lemma 1.2 from [10]. \quad \qed

Remark 3.8 If in item 3 we make substitution $Y_i = f_i$ only for a part of the indeterminates $Y_i$, say, for $t$ of them, where $t \leq s$, then we may replace $\log(s + 1)$ by $\log(t + 1)$, and $\deg g$ by the degree with respect to these indeterminates:

$h_a(g(f_1, \ldots, f_t, Y_{t+1}, \ldots, Y_s)) \leq h_a(g) + (h + \log(t + 1) + N \log(r + 1)) \deg_{Y_{t+1}, \ldots, Y_s} g$. 

6
Remark 3.9 When all the \( f_i \) are just linear polynomials in one variable, item 2 can be refined as follows: let \( F(X) \) be a polynomial of degree \( \rho \), and \( \beta_1, \ldots, \beta_\rho \) are its roots (counted with multiplicities); then
\[
h_a(\beta_1) + \cdots + h_a(\beta_\rho) \leq h_p(F) + \log(\rho + 1).
\]
This is a classical result of Mahler, see, for instance, [13, Lemma 3].

Corollary 3.10 Let \( f \) and \( g \) be polynomials with algebraic coefficients such that \( f \) divides \( g \). Let also \( a \) be a non-zero coefficient of \( f \). Then
1. \( h_p(f) \leq h_p(g) + \deg g \),
2. \( h_a(f) \leq h_p(g) + h_a(a) + \deg g \).

Proof Item 1 is a direct consequence of item 2 of Lemma 3.7. For item 2 remark that one of the coefficients of \( f/a \) is 1, which implies that
\[
h_a(f/a) = h_p(f/a) = h_p(f) \leq h_p(g) + \deg g.
\]
Since \( h_a(f) \leq h_a(a) + h_a(f/a) \), the result follows. \( \square \)

Corollary 3.11 Let \( \alpha \) be an algebraic number and \( f \in \bar{\mathbb{Q}}[X,Y] \) be a polynomial with algebraic coefficients, let also \( f^{(\alpha)}(X,Y) = f(X + \alpha, Y) \) then
\[
h_a(f^{(\alpha)}) \leq h_a(f) + m h_a(\alpha) + 2m \log 2,
\]
where \( m = \deg_X f \).

Proof This is a direct application of item 3 of Lemma 3.7 together with Remark 3.8. \( \square \)

In one special case item 3 of Lemma 3.7 can be refined.

Lemma 3.12 Let
\[
F_{ij}(X) \in \bar{\mathbb{Q}}[X] \quad (i,j = 1, \ldots, s)
\]
be polynomials of degree bounded by \( \mu \) and of affine height bounded by \( h \); then
\[
h_a(\det(F_{ij})) \leq sh + s(\log s + \mu \log 2).
\]
For the proof see [10], end of Section 1.1.1.

We also need an estimate for both the affine and the projective height of the \( Y \)-resultant \( R_f(X) \) of a polynomial \( f(X,Y) \in \bar{\mathbb{Q}}[X,Y] \) and its \( Y \)-derivative \( f'_Y \), in terms of the affine (respectively, projective) height of \( f \).

Lemma 3.13 Let \( f(X,Y) \in \bar{\mathbb{Q}}[X,Y] \) be of \( X \)-degree \( m \) and \( Y \)-degree \( n \). Then
\[
h_a(R_f) \leq (2n - 1)h_a(f) + (2n - 1) \left( \log(2n^2) + m \log 2 \right), \quad (7)
\]
\[
h_p(R_f) \leq (2n - 1)h_p(f) + (2n - 1) \log \left((m + 1)(n + 1)\sqrt{n}\right), \quad (8)
\]

Proof Estimate (8) is due to Schmidt [13, Lemma 4]. To prove (7), we invoke Lemma 3.12. Since \( R_f(X) \) can be presented as a determinant of dimension \( 2n - 1 \), whose entries are polynomials of degree at most \( m \) and of affine height at most \( h_a(f) + \log n \), the result follows after an obvious calculation. \( \square \)

Remark 3.14 Estimate (7) holds true also when \( m = 0 \). We obtain the following statement: the resultant \( R_f \) of a polynomial \( f(X) \) and its derivative \( f'(X) \) satisfies
\[
h_a(R_f) \leq (2 \deg f - 1)h_a(f) + (2 \deg f - 1) \log (2(\deg f)^2).
\]
3.4 Discriminants

We need some estimates for the discriminant of a number field in terms of the heights of its generators. In this subsection \( K \) is a number field, \( d = [K : \mathbb{Q}] \) and \( \mathcal{N}(\cdot) = \mathcal{N}_{K/\mathbb{Q}}(\cdot) \). The following result is due to Silverman [16, Theorem 2].

**Lemma 3.15** Let \( \underline{a} = (a_1, \ldots, a_k) \) be a point in \( \overline{\mathbb{Q}}^k \). Put \( \nu = [K(\underline{a}) : K] \). Then
\[
\partial_{K(\underline{a})/K} \leq 2(\nu - 1)h_a(\underline{a}) + \log \nu.
\]

This has the following consequence.

**Corollary 3.16** Let \( F(X) \in K[X] \) be a polynomial of degree \( N \). Then
\[
\sum_{F(\alpha) = 0} \partial_{K(\alpha)/K} \leq 2(N - 1)h_p(F) + 3N \log N,
\]
the sum being over the roots of \( F \).

**Proof** Since for any root \( \alpha \) we have \([K(\alpha) : K] \leq N\), we estimate the left-hand side of (9) as
\[
2(N - 1) \sum_{F(\alpha) = 0} h_a(\alpha) + N \log N.
\]
Remark 3.14 allows us to bound the sum on the right by \( h_p(F) + \log(N + 1) \). Now, to complete the proof, just remark that \((N - 1) \log(N + 1) \leq N \log N\).

We shall also need a bound for the discriminant of a different nature, known as the Dedekind-Hensel inequality (see [4, page 397] for historical comments and further references). This inequality gives an estimate of the relative discriminant of a number field extension in terms of the ramified places.

**Lemma 3.17** Let \( K \) be a number field of degree \( d \) over \( \mathbb{Q} \), and \( L \) an extension of \( K \) of finite degree \( \nu \), and let \( \text{Ram}(L/K) \) be the set of places of \( K \) ramified in \( L \). Then
\[
\partial_{L/K} \leq \nu \frac{1}{\nu} h(\text{Ram}(L/K)) + 1.26 \nu.
\]

This is Proposition 4.2.1 from [2] (though the notation in [2] is different, and the quantity estimated therein is \( \nu \partial L/K \) in our notation), the only difference being that the error term is now explicit. The proof is the same as in [2], but in the very last line one should use the estimate \( \sum_{p \leq \nu} \frac{1}{p} \leq 1.26 \nu / \log \nu \), which is (11).

A similar estimate was obtained by Serre [15, Proposition 4]. However, (10) is more suitable for our purposes.

It is useful to have an opposite estimate as well. The following is obvious.

**Lemma 3.18** In the set-up of Lemma 3.17 we have \( h(\text{Ram}(L/K)) \leq \nu \partial_{L/K} \).

4 Power Series

Our main technical tool in the quantitative Eisenstein theorem, based on the work of Dwork, Robba, Schmidt and van der Poorten [7, 8, 13]. Let
\[
y = \sum_{k=-k_0}^{\infty} a_k x^{k/e}
\]
be an algebraic power series with coefficients in \( \overline{\mathbb{Q}} \), where we assume \( k_0 \geq 0 \) and \( a_{-k_0} \neq 0 \) when \( k_0 > 0 \). The classical Eisenstein theorem tells that the coefficients of this series belong to some number field, that for every valuation \( v \) of this field \( |a_k|_v \) grows at most exponentially in \( k \), and for all but finitely many \( v \) we have \( |a_k|_v \leq 1 \) for all \( k \). We need a quantitative form of this statement, in terms of an algebraic equation \( f(x, y) = 0 \) satisfied by \( y \).
4.1 Eisenstein Theorem

Thus, let \( f(X, Y) \in \mathbb{K}(X, Y) \) be a polynomial over a number field \( \mathbb{K} \). We put

\[
d = [\mathbb{K} : \mathbb{Q}], \quad m = \deg_X f, \quad n = \deg_Y f.
\]

(12)

Write

\[
f(X, Y) = f_0(X)Y^n + f_1(X)Y^{n-1} + \ldots
\]

(13)

and put

\[
u = \text{ord}_0 f_0.
\]

(14)

If the series \( y \), written as (11), satisfies the equation \( f(x, y) = 0 \), then

\[
k_0/e \leq u \leq m.
\]

Also, if \( \mathbb{L} \) is the extension of \( \mathbb{K} \) generated by all the coefficients \( a_k \) of the series \( y \), then it is well-known that

\[
[\mathbb{L} : \mathbb{K}] \leq n.
\]

Finally, for \( v \in M_\mathbb{K} \) we denote by \( d_v \) its local degree over \( \mathbb{Q} \), and by \( \mathcal{N}v \) its absolute norm:

\[
d_v = [\mathbb{K}_v : \mathbb{Q}_v], \quad \mathcal{N}v = \mathcal{N}_{\mathbb{K}/\mathbb{Q}}(v).
\]

(15)

The following theorem is a combination of the ideas and results from [7, 8, 13].

**Theorem 4.1** Let \( \mathbb{K} \) be a number field and \( f(X, Y) \in \mathbb{K}(X, Y) \). We use notation (12–15). For every \( v \in M_\mathbb{K} \) there exist real numbers \( A_v, B_v \geq 1 \), with \( A_v = B_v = 1 \) for all but finitely many \( v \), such that the following holds. First of all,

\[
d - 1 \sum_{v \in M_\mathbb{K}} d_v \log A_v \leq (2n - 1)h_p(f) + 6n^2 + 2n \log m,
\]

(16)

\[
d - 1 \sum_{v \in M_\mathbb{K}} d_v \log B_v \leq h_p(f) + \log(2n).
\]

(17)

Further, for any non-archimedean valuation \( v \) of \( \mathbb{K} \) we have \( d_v \log B_v / \log Nv \in \mathbb{Z} \) and

\[
h \left( \left\{ v \in M_\mathbb{K}^0 : \frac{d_v \log A_v}{\log Nv} \notin \mathbb{Z} \right\} \right) \leq (2n - 1)h_p(f) + n(2 \log m + 3 \log n + 5).
\]

(18)

Finally, let \( y \) be an algebraic power series, written as in (11), and satisfying \( f(x, y) = 0 \). Let \( \mathbb{L} \) be the number field generated over \( \mathbb{K} \) by the coefficients of \( y \). Then for any valuation \( w|v \) of \( M_\mathbb{L} \) and for all \( k \geq -k_0 \) we have

\[
|a_k|_w \leq B_v A_v^{u+k/e},
\]

(19)

where \( u \) is defined in (14).

**Proof** This is, essentially, [2, Theorem 2.1], with the error terms made explicit. (Warning: we denote by \( B_v \) the quantity denoted in [2] by \( A'_v \).) Below we briefly review the proof from [2], indicating the changes needed to get explicit error terms.

Denote by \( R_f(X) \) the \( Y \)-resultant of \( f \) and \( f'_Y \), and put

\[
\mu = \text{ord}_0 R_f(X),
\]

(20)

(notice that \( \mu \) has a slightly different meaning in [2]). We may normalize the polynomial \( f \) to have \( f_0(X) = X^uf_0^*(X) \) with \( f_0^*(0) = 1 \). We also write \( R_f(X) = AX^uR^*(X) \) with \( R^*(0) = 1 \).
Let \( \alpha_1, \ldots, \alpha_t \) be the roots of \( R^*(X) \). For every valuation \( v \) in \( K \) fix an extension to \( K(\alpha_1, \ldots, \alpha_t) \) and put 
\[ \sigma_v = \min(1, |\alpha_1|_v, \ldots, |\alpha_t|_v). \]
Clearly \( \sigma_v \) does not depend on the fixed prolongation. Schmidt \[13\] Lemma 5] proves that 
\[d^{-1} \sum_{v \in M_K} d_v \log(1/\sigma_v) \leq (2n - 1)h_p(f) + 2n \log \left( (m + 1)(n + 1) \sqrt{n} \right). \] (21)
(Notice that Schmidt uses a different normalization of valuations).

For every valuation \( v \) of \( K \) we define real numbers \( A_v, B_v \geq 1 \) as follows:
\[ A_v = \begin{cases} 2^n/\sigma_v, & p(v) = \infty, \\ 1/\sigma_v, & n < p(v) < \infty, \\ (np(v)^{-1})^{n/\sigma_v}, & p(v) \leq n, \end{cases} \]
\[ B_v = \begin{cases} 2n|f|_v, & p(v) = \infty, \\ |f|_v, & p(v) < \infty, \end{cases} \]
where \( p(v) \) is the underlying prime. Notice that \( A_v = B_v = 1 \) for all but finitely many \( v \).

Inequality \[13\] is established in \[2, \text{Section 2.3}\]. Inequality \[17\] is immediate from the definition of \( B_v \). Further,
\[d^{-1} \sum_{p(v) \leq n} d_v \log \left( np(v)^{-1} \right)^n = d^{-1} n \sum_{p \leq n} \sum_{p(v)=p} d_v \left( \log n + \frac{\log p}{p - 1} \right)\]
\[= n \sum_{p \leq n} \left( \log n + \frac{\log p}{p - 1} \right)\]
\[\leq 1.26 n^2 + 2n \log n,\]
where we used \[14\] and \[16\] for the last estimate. Combining this with \[21\], we obtain
\[d^{-1} \sum_{v \in M_K} d_v \log A_v \leq (2n - 1)h_p(f) + n \log 2 + 1.26 n^2 + 2n \log n + 2n \log \left( (m + 1)(n + 1) \sqrt{n} \right),\]
which implies \[16\] after a routine calculation.

The definition of \( B_v \) implies that \( \frac{d_v \log B_v}{\log N_v} \in \mathbb{Z} \). We are left with \[18\]. Write the set on the left of \[18\] as \( S_1 \cup S_2 \), where \( S_1 \) consists of \( v \) with \( p(v) \leq n \), and \( S_2 \) of those with \( p(v) > n \). Obviously,
\[h(S_1) \leq \sum_{p \leq n} \log p \leq 1.02 n, \] (22)
where we use \[15\]. For \( S_2 \) we have the estimate (see \[2, \text{second displayed equation on page 134}\])
\[h(S_2) \leq h_p(R^*) + \log(1 + \deg R^*) \]
\[\leq (2n - 1)h_p(f) + (2n - 1) \log \left( (m + 1)(n + 1) \sqrt{n} \right) + \log(2mn),\]
where we use the property \( h_p(R^*) = h_p(R) \) and Lemma \[3,13\] for the last inequality. Together with \[22\], this implies \[18\] after an easy calculation. \( \square \)

Here is one consequence that we shall use.

**Corollary 4.2** In the set-up of Theorem \[4.1\] let \( T \) be the subset of \( v \in M_K \) such that one of the inequalities \( A_v > 1 \) or \( B_v > 1 \) holds. Then \( h(T) \leq (4n - 1)h_p(f) + 13n^2 + 4n \log m. \)

**Proof** For \( v \in T \) we have either \( \log N_v < d_v \log A_v \) or \( \log N_v < d_v \log B_v \) or \( \frac{d_v \log A_v}{\log N_v} \notin \mathbb{Z} \). Partitioning the set into three sets and using \[10\], \[17\] and \[18\], we obtain the result after an easy calculation. \( \square \)
4.2 Fields Generated by the Coefficients

We also need to bound the discriminant of the number field, generated by the coefficients of an algebraic power series. Such a bound is obtained in [2, Lemma 2.4.2]. Here we obtain a similar statement, explicit in all parameters.

Proposition 4.3 In the set-up of Theorem 4.1 let \( L \) be the number field generated by the coefficients of the series \( y \). Put \( \nu = [L : \mathbb{K}] \), and define \( u \) and \( \mu \) as in (14) and (20). Then

\[
\partial_{\nu L/K} \leq (4n\mu + 4nu
u + 2\nu) h_p(f) + (\mu + \nu) \left( 12n^2 + 4n \log m \right) + 3\nu \log(2n).
\] (23)

Proof As we have seen in [2, Lemma 2.4.2], the field \( L \) is generated over \( K \) by \( a_{-k_0}, \ldots, a_{\kappa} \), where \( \kappa \leq e\mu/\nu \) (we recall that in [2] the quantity \( \mu \) has a slightly different definition and, is less than or equal to our \( \mu \)). Using Theorem 4.1 we estimate the height of the vector \( a = (a_{-k_0}, \ldots, a_{\kappa}) \) as follows:

\[
h_a(u) \leq d^{-1} \sum_{v \in \text{M}_L} d_v \log B_v + d^{-1} \left( u + \frac{\kappa}{\nu} \right) \sum_{v \in \text{M}_L} d_v \log A_v
\]

\[
\leq h_p(f) + \log(2n) + \left( \frac{\mu}{\nu} + u \right) \left( (2n - 1) h_p(f) + 6n^2 + 2n \log m \right)
\]

\[
\leq \left( 2n \left( \frac{\mu}{\nu} + u \right) + 1 \right) h_p(f) + \left( \frac{\mu}{\nu} + u \right) \left( 6n^2 + 2n \log m \right) + \log(2n).
\]

Applying now Lemma 3.15 we obtain

\[
\partial_{\nu L/K} \leq (4n\mu + 4nu\nu + 2\nu) h_p(f) + (\mu + \nu\nu) \left( 12n^2 + 4n \log m \right) + 2\nu \log(2n) + \log \nu,
\]

which is even sharper than (23).

Let now \( \{y_1, \ldots, y_n\} \) be the set of all power series roots of \( f \) at 0, that is,

\[
f(x, Y) = f_0(x)(Y - y_1) \cdots (Y - y_n),
\]

and let \( L_i \) be the field generated by the coefficients of \( y_i \). Summing up and estimating each degree \([L_i : \mathbb{Q}]\) by \( n \), we obtain the following consequence of Proposition 4.3.

Corollary 4.4 In the previous set-up, the following inequality holds

\[
\sum_{i=1}^{n} \partial_{\nu L_i/K} \leq (4n^2\mu + 4n^3u + 2n^2) h_p(f) + (\mu + \nu u) \left( 12n^2 + 4n \log m \right) + 3n^2 \log(2n).
\]

4.3 The “Essential” Coefficients

In this subsection the letter \( q \) always denotes a prime number. Let us assume now that the series (11) has exact ramification \( e \); that is, it cannot be written as a series in \( x^{1/e'} \) with \( e' < e \). Then for every \( q \) dividing \( e \) there exist at least one \( k \) such that \( q \nmid k \) and \( a_k \neq 0 \). We denote by \( \kappa(q) \) the smallest \( k \) with this property, and we call \( a(q) = a_{\kappa(q)} \) the \( q \)-essential coefficient of the series (11). We want to estimate the height of the \( q \)-essential coefficients. We again put \( \nu = [L : \mathbb{K}] \), and define \( u \) and \( \mu \) as in (14) and (20).

Proposition 4.5 In the above set-up, the following inequality holds:

\[
\sum_{q \mid e} h_a(a(q)) \leq \log_2 e \left( u + \frac{\mu}{\nu} \right) \left( 2nh_p(f) + 6n^2 + 2n \log m \right).
\] (24)
We can bound the number $\kappa(q)$ by
\[ \kappa(q) \leq \frac{e\mu}{\nu(q-1)} - 1 \]
by [2 Lemma 2.4.4]. We shall use the trivial bounds
\[ \sum_{q \mid e} \frac{1}{q-1} \leq \sum_{q \mid e} 1 \leq \log_2 e. \] (25)

Now, Theorem 4.1 gives us the explicit bound
\[ h_a(a(q)) \leq \left( u + \frac{\mu}{\nu(q-1)} \right) \left( 2n h_p(f) + 6n^2 + 2n \log m \right) \]
for the height of the $q$-essential coefficient $h_a(a_{\kappa(q)})$. Summing up over all primes $q$ dividing $e$, and simplifying by (25) we obtain the result. \[ \square \]

**Corollary 4.6** Define $y_i$ as in the previous section and let $a_i(q)$ denote the $q$-essential coefficient of the series $y_i$. Then
\[ \sum_{i=1}^{n} \sum_{q \mid e} h_a(a_i(q)) \leq (u + \mu) \left( 2n^2 h_p(f) + 6n^3 + 2n^2 \log m \right) \log_2 n. \]

**Proof** Let $e_i$ be the ramification of the series $y_i$. Summing up and using $\log_2 e_i \leq \log_2 n$ and $1/\nu \leq 1$, we obtain the result. \[ \square \]

## 5 Proximity and Ramification

This section is the technical heart of the article. We consider a covering $C \to \mathbb{P}^1$, a point $Q$ on $C$ and a non-archimedean place $v$, and show, that in a certain $v$-adic neighborhood of $Q$, the $v$-ramification is the same and is determined by the ramification of $Q$ over $\mathbb{P}^1$. Roughly speaking, “geometric ramification defines arithmetic ramification”. It is not difficult to make a qualitative statement of this kind, but it is a rather delicate task to make everything explicit.

Thus, in this section we fix, once and for all:

- a number field $K$;
- an absolutely irreducible smooth projective curve $C$ defined over $K$;
- a non-constant rational function $x \in K(C)$;
- one more rational function $y \in K(C)$ such that $K(C) = K(x, y)$ (existence of such $y$ follows from the primitive element theorem).

Let $f(X, Y) \in K[X, Y]$ be the $K$-irreducible polynomial such that $f(x, y) = 0$ (it is well-defined up to a constant factor). Since $C$ is absolutely irreducible, so is the polynomial $f(X, Y)$.

We put $m = \deg_X f$, $n = \deg_Y f$, and write
\[ f(X, Y) = f_0(X)Y^m + f_1(X)Y^{m-1} + \cdots + f_n(X). \] (26)

Let $Q \in C(\mathbb{K})$ be a $\mathbb{K}$-point of $C$, which is not a pole of $x$. We let $\alpha = x(Q)$ and we denote by $e_Q$ the ramification index of $x$ at $Q$ (that is, $e = \ord_Q(x - \alpha)$). When it does not cause a confusion
(in particular, everywhere in this section) we write \( e \) instead of \( e_Q \). Fix a primitive \( e \)-th root of unity \( \zeta = \zeta_e \). Then there exist \( e \) equivalent Puiseux expansions of \( y \) at \( Q \):

\[
y_i^{(Q)} = \sum_{k=-\kappa^{(Q)}}^{\infty} a_k^{(Q)} \zeta^{ik} (x - \alpha)^{k/e} \quad (i = 0, \ldots, e - 1),
\]

where \( \kappa^{(Q)} = \max \{0, -\text{ord}_Q(y)\} \).

Let \( \bar{v} \) be a valuation of \( \bar{K} \). We say that the series (27) \( \bar{v} \)-adically converge at \( \xi \in \bar{K} \), if, for a fixed \( e \)-th root \( \sqrt[\xi - \alpha]{} \), the \( e \) numerical series

\[
\sum_{k=-\kappa^{(Q)}}^{\infty} a_k^{(Q)} \left( \zeta^i \sqrt[\xi - \alpha]{} \right)^k \quad (i = 0, \ldots, e - 1)
\]

converge in the \( \bar{v} \)-adic topology. We denote by \( y_i^{(Q)}(\xi) \), with \( i = 0, \ldots, e - 1 \), the corresponding sums. While the individual sums depend on the particular choice of the root \( \sqrt[\xi - \alpha]{} \), the very fact of convergence, as well as the set \( \{y_0^{(Q)}(\xi), \ldots, y_{e-1}^{(Q)}(\xi)\} \) of the sums, are independent of the choice of the root.

Now we are ready to introduce the principal notion of this section, that of proximity of a point to a different point with respect to a given valuation \( \bar{v} \in M_{\bar{K}} \).

**Definition 5.1** Let \( P \in \mathcal{C}(\bar{K}) \) be a \( \bar{K} \)-point of \( \mathcal{C} \), not a pole of \( x \), and put \( \xi = x(P) \). We say that \( P \) is \( \bar{v} \)-adically close to \( Q \) if the following conditions are satisfied:

- \( |\xi - \alpha|_{\bar{v}} < 1 \);
- the \( e \) series (27) \( \bar{v} \)-adically converge at \( \xi \), and one of the sums \( y_i^{(Q)}(\xi) \) is equal to \( y(P) \).

**An important warning:** the notion of proximity just introduced is not symmetric in \( P \) and \( Q \): the proximity of \( P \) to \( Q \) does not imply, in general, the proximity of \( Q \) to \( P \). Intuitively, one should think of \( Q \) as a “constant” point, and of \( P \) as a “variable” point.

To state the main results of this section, we have to define a finite set \( Q \) of \( \bar{K} \)-points of the curve \( \mathcal{C} \), and certain finite sets of non-archimedean places of the field \( K \). Let \( R(X) = R_f(X) \in K[X] \) be the \( Y \)-resultant of \( f(X,Y) \) and \( f_0(X,Y) \), and let \( A \) be the set of the roots of \( R(X) \):

\[
A = \{ \alpha \in \bar{K} : R(\alpha) = 0 \}.
\]

We define \( Q \) as follows:

\[
Q = \{ Q \in \mathcal{C}(\bar{K}) : x(Q) \in A \}.
\]

It is important to notice that \( Q \) contains all the finite ramification points of \( x \) (and may contain some other points as well). Also, the set \( Q \) is Galois-invariant over \( K \): every point belongs to it together with its Galois orbit over \( K \).

Now let us define the finite sets of valuations of \( K \) mentioned above. First of all we assume (as we may, without loss of generality) that

the polynomial \( f_0(X) \), defined in (20), is monic. \hfill (28)

In particular, \( f \) has a coefficient equal to 1, which implies equality of the affine and the projective heights of \( f \):

\[
h_a(f) = h_p(f).
\]

Now, we define

\[
T_1 = \{ v \in M_{\bar{K}}^0 : \text{the prime below } v \text{ is } \leq n \},
\]

\[
T_2 = \{ v \in M_{\bar{K}}^0 : |f|_v > 1 \}.
\]
Further, let $r_0$ be the leading coefficient of $R(X)$. We define
\[ T_3 = \{ v \in M_K^0 : |r_0|_v < 1 \} . \]

Next, we let $\Delta$ be the resultant of $\tilde{R}(X)$ and $\tilde{R}'(X)$, where $\tilde{R}$ is the radical of $R$, see Subsection \[S2\].

Since the polynomial $\tilde{R}(X)$ is separable, we have $\Delta \in \mathbb{K}^*$. Now we define the set $T_4$ as follows:
\[ T_4 = \{ v \in M_K^0 : |\Delta|_v < 1 \} . \]

Now fix $Q \in \mathcal{C}(\mathbb{K})$ and define three sets $T_5^{(Q)}$, $T_6^{(Q)}$ and $T_7^{(Q)}$, using the Puiseux expansions of $y$ at $Q \in \mathcal{Q}$. As in \[S4\], we denote by $a_k^{(Q)}$ the coefficients of these expansions. Now define
\[ T_5^{(Q)} = \{ v \in M_K^0 : |a_k^{(Q)}|_v > 1 \text{ for some } k \text{ and some } \hat{v} \text{ extending } v \} , \quad T_5 = \bigcup_{Q \in \mathcal{Q}} T_5^{(Q)} . \]

The Eisenstein theorem implies that the set $T_5^{(Q)}$ is finite. Further, the coefficients $a_k^{(Q)}$ generate a finite extension $\mathbb{K}^{(Q)}$ of $\mathbb{K}(\alpha)$ (where, as above, $\alpha = x(Q)$); more precisely, $[\mathbb{K}^{(Q)} : \mathbb{K}(\alpha)] \leq n$. Now we define
\[ T_6^{(Q)} = \{ v \in M_K^0 : v \text{ ramifies in } \mathbb{K}^{(Q)} \} , \quad T_6 = \bigcup_{Q \in \mathcal{Q}} T_6^{(Q)} . \]

Finally, for any prime divisor $q$ of $e = e_Q$ let $a^{(Q)}(q)$ be the $q$-essential coefficient of the series $y^{(Q)}$, as defined Subsection \[L3\]. Now put
\[ T_7^{(Q)} = \{ v \in M_K^0 : |a^{(Q)}(q)|_v < 1 \text{ for some } q|e_Q \text{ and some } \hat{v} \text{ extending } v \} , \quad T_7 = \bigcup_{Q \in \mathcal{Q}} T_7^{(Q)} . \]

Finally, for $P,Q \in \mathcal{C}(\mathbb{K})$ and a finite valuation $\bar{v} \in M_{\mathbb{K}}$, we let $v$ be the restriction of $\bar{v}$ to $\mathbb{K}$ and $\pi$ a primitive element of the local ring $\mathcal{O}_v$, and define
\[ \ell(P,Q,\bar{v}) = \frac{\log |\xi - \alpha|_v}{\log |\pi|_v} , \quad (30) \]
where, as above, $\xi = x(P)$ and $\alpha = x(Q)$.

Now we are ready to state the principal results of this section. Call a point $P \in \mathcal{C}(\mathbb{K})$ semi-defined over $\mathbb{K}$ if $\xi = x(P) \in \mathbb{K}$.

**Proposition 5.2** Let $\mathcal{Q}$ be the set defined above. Then for any point $P \in \mathcal{C}(\mathbb{K}) \setminus \mathcal{Q}$ semi-defined over $\mathbb{K}$, and for any finite valuation $v \in M_{\mathbb{K}}$, at least one of the following conditions is satisfied (we again put $\xi = x(P)$).

- $|\xi|_v > 1$.
- $v \in T_2 \cup T_3 \cup T_4 \cup T_5$.
- $v$ is not ramified in the field $\mathbb{K}(P)$.
- For any $\bar{v} \in M_{\mathbb{K}}$, extending $v$, our point $P$ is $\bar{v}$-adically close to some $Q \in \mathcal{Q}$, which is well-defined when $\bar{v}$ is fixed. Moreover, the integers $e_Q$ and $\ell(P,Q,\bar{v})$ are independent of the particular choice of $\bar{v}$.

**Proposition 5.3** Assume that $P \in \mathcal{C}(\mathbb{K})$ is semi-defined over $\mathbb{K}$, and let $P$ be $\bar{v}$-adically close to some $Q \in \mathcal{C}(\mathbb{K})$ for some finite valuation $\bar{v} \in M_{\mathbb{K}}$. Let $v$ and $w$ be the restrictions of $\bar{v}$ to $\mathbb{K}$ and $\mathbb{K}(P)$, respectively. Assume that $v$ does not belong to $T_1 \cup T_5^{(Q)} \cup T_6^{(Q)} \cup T_7^{(Q)}$. Then the ramification index of $w$ over $v$ is equal to $e_Q/(\gcd(e_Q, \ell))$, where $\ell = \ell(P,Q,\bar{v})$ is defined in \[S30\].

(Intuitively, the last condition means that the arithmetic ramification comes from the geometric ramification.)

**Proposition 5.4** Put $T = T_1 \cup T_2 \cup \ldots \cup T_7$. Then
\[ h(T) \leq 150mn^3 \log n(h_p(f) + 2m + 2n) . \]
5.1 Proof of Proposition 5.2

We fix, once and for all, a finite valuation \( v \in M_\mathbb{K} \), its extension \( \bar{v} \in M_\mathbb{K} \), and a point \( P \in C(\mathbb{K}) \) semi-defined over \( \mathbb{K} \) and such that \( \xi = x(P) \neq A \). We assume that \( |\xi|_v \leq 1 \), that \( v \notin T_2 \cup \ldots \cup T_5 \) and that \( v \) is ramified in \( \mathbb{K}(P) \), and we shall prove that \( P \) is \( \bar{v} \)-adically close to a unique \( Q \in \mathbb{Q} \), and that the numbers \( e_Q \) and \( \ell(P, Q, \bar{v}) \) are independent of the selected \( \bar{v} \).

Since \( v \notin T_2 \cup T_3 \), the polynomial \( R(X) \) belongs to \( O_v[X] \) and is \( v \)-monic. Lemma 3.4 implies that so is its radical \( \hat{R}(X) \). Also, every root \( \alpha \) of \( R \) is a \( v \)-adic integer.

Put \( \eta = y(P) \). Since \( \xi \notin A \), the point \((\xi, \eta)\) of the plane curve \( f(X, Y) = 0 \) is non-singular, which implies that \( \mathbb{K}(P) = \mathbb{K}(\xi, \eta) = \mathbb{K}(\eta) \) (recall that \( \xi \in \mathbb{K} \)). Now Lemma 3.3 implies that \( |f_1'(\xi, \eta)|_v < 1 \). It follows that \( |\hat{R}(\xi)|_v < 1 \), which implies that \( |R(\xi)|_v < 1 \) by Lemma 3.4.

Next, since \( v \notin T_4 \), we have \( |\hat{R}(\xi)|_v = 1 \). Lemma 3.5 implies now that there exists a unique \( \alpha \in A \) such that \( |\xi - \alpha|_v < 1 \). The uniqueness of \( \alpha \) implies that the couple \((\alpha, \bar{v})\) is well-defined up to the Galois action of \( \text{Gal}(\mathbb{K}/\mathbb{K}) \). Hence, while \( \alpha \) depends on the choice of \( v \), the quantity \( |\xi - \alpha|_\bar{v} \) is independent of \( \bar{v} \).

Fix this \( \alpha \) from now on. There is \( \sum_{x(Q)=\alpha} e_Q = n \) Puiseux expansions of \( y \) at the points \( Q \) above \( \alpha \), and they satisfy

\[
\hat{R}(\xi) = \hat{R}(\xi) \prod_{x(Q)=\alpha} e_Q^{-1} \left( Y - y_i^{(Q)}(\xi) \right).
\]

Since \( v \notin T_5 \), each of the series \( y_i^{(Q)} \) has \( v \)-adic convergence radius at least 1. Since \( |\xi - \alpha|_\bar{v} < 1 \), all them \( \bar{v} \)-adically converge at \( \xi \). Moreover, the convergence is absolute, because \( \bar{v} \) is non-archimedean. Hence

\[
f(\xi, Y) = f_0(\xi) \prod_{x(Q)=\alpha} e_Q^{-1} \left( Y - y_i^{(Q)}(\xi) \right).
\]

Since \( R(\xi) \neq 0 \), we have \( f_0(\xi) \neq 0 \) as well. Hence we have on the left and on the right polynomials of degree \( n \) in \( Y \), the polynomial on the left having \( \eta = y(P) \) as a simple root (here we again use that \( R(\xi) \neq 0 \)). Hence exactly one of the sums \( y_i^{(Q)}(\xi) \) is equal to \( \eta \). We have proved that \( P \) is \( \bar{v} \)-adically close to exactly one \( Q \in \mathbb{Q} \).

The uniqueness of \( Q \) implies that \( e_Q \) is independent of the particular choice of \( \bar{v} \). Indeed, if we select a different \( \bar{v} \), then \( Q \) will be replaced by a conjugate over \( \mathbb{K} \), and conjugate points have the same ramification. Also, as we have seen above \(|\xi - \alpha|_\bar{v} \) is independent of the choice of \( \bar{v} \) as well; hence so is \( \ell(P, Q, \bar{v}) \).

\( \square \)

5.2 Proof of Proposition 5.3

We may assume, by re-defining the root \( \sqrt[n]{\xi - \alpha} \) that \( \eta = y(P) \) is the sum of \( y_0^{(Q)} \) at \( \xi \). In the sequel we omit reference to \( Q \) (when it does not lead to confusion) and write \( e \) for \( e_Q \), \( a_k \) for \( a_k^{(Q)} \), etc. Thus, we have, in the sense of \( \bar{v} \)-adic convergence,

\[
\eta = \sum_{k=-k^{(Q)}}^{\infty} a_k \left( \sqrt[n]{\xi - \alpha} \right)^k. \tag{31}
\]

We have to show that the ramification index of any extension of \( v \) to \( \mathbb{K}(P) \) is \( e' = e / \gcd(e, \ell) \), where \( \ell \) is defined in (30). Since \( v \notin T_6 \), it is unramified in the field, generated over \( \mathbb{K} \) by \( \alpha \) and the coefficients \( a_k \). Hence we may extend \( \mathbb{K} \) and assume that all of the latter belong to it.

Let \( \mathbb{K}_v \) be a \( v \)-adic completion of \( \mathbb{K} \). We consider \( \mathbb{K}_v \) as its algebraic closure, and the fields \( \mathbb{K}_v(P) = \mathbb{K}_v(\eta) \) and \( \mathbb{K}_v \left( \sqrt[n]{\xi - \alpha} \right) \) as subfields of the latter. According to (31), we have \( \mathbb{K}_v(\eta) \subset \mathbb{K}_v \left( \sqrt[n]{\xi - \alpha} \right) \). The latter field has ramification \( e' \) over \( \mathbb{K}_v \); see, for instance, Proposition 3.3 from [2]. (Here and below we use the fact that all our ramifications are tame, which
follows from the assumption $v \notin T_1$. If the ramification of $\mathbb{K}_v(\eta)$ is not $e'$, then it must divide $e'/q$, where $q$ is a prime divisor of $e'$. We want to show that this is impossible.

Let $\kappa = \kappa^{(Q)}(q)$ be as defined in Subsection 3.3. Then the $q$-essential coefficient $a^{(Q)}(q)$ is equal to $a_{\kappa}$. Put

$$\theta = \eta - \sum_{k=k_0(Q)}^{\kappa-1} a_k \left( \sqrt[\kappa]{\xi - \alpha} \right)^k = a_{\kappa} \left( \sqrt[\kappa]{\xi - \alpha} \right)^\kappa + \sum_{k=\kappa+1}^{\infty} a_k \left( \sqrt[\kappa]{\xi - \alpha} \right)^k$$

By the definition of $\kappa$, we have $\theta \in \mathbb{K}_v(\eta, \sqrt[\kappa]{\xi - \alpha})$. The ramification of $\mathbb{K}_v(\sqrt[\kappa]{\xi - \alpha})/\mathbb{K}_v$ is $(\ell'/q) / \gcd(e'/q, \ell) = e'/q$ (since $q$ divides $e'$, it cannot divide $\ell' = \ell / \gcd(e, \ell)$, and we have $\gcd(e'/q, \ell) = \gcd(e, \ell)$). Hence the ramification of $\mathbb{K}_v(\theta)/\mathbb{K}_v$ divides $e'/q$. But, since $v \notin T_2 \cup T_7$, we have $|a_k|_v \leq 1$ for all $k$ and $|a_{\kappa}|_v = 1$, which implies that $|\theta|_v = |\left( \sqrt[\kappa]{\xi - \alpha} \right)^\kappa|_v$. If $\pi$ and $\Pi$ are primitive elements of the local fields $\mathbb{K}_v$ and $\mathbb{K}_v(\theta)$, respectively, then ord,$\Pi$ divides $e'/q$. On the other hand, ord,$\theta = (\kappa\ell'/e) \cdot \text{ord}_\Pi \in \mathbb{Z}$, which implies that $(\kappa\ell'/e) \cdot (e'/q) = \kappa\ell'/q \in \mathbb{Z}$. But $q$ does not divide any of the numbers $\kappa$ and $\ell'$, a contradiction. □

5.3 Proof of Proposition 5.4

The proposition is a direct consequence of the estimates

$$h(T_1) \leq 1.02n,$$  \hspace{1cm} (32)

$$h(T_2) \leq h_p(f),$$  \hspace{1cm} (33)

$$h(T_3) \leq (2n - 1)(h_p(f) + m \log 2 + \log(2n^2)),$$  \hspace{1cm} (34)

$$h(T_4) \leq 16mn^2(h_p(f) + 2m + 2 \log n),$$  \hspace{1cm} (35)

$$h(T_5) \leq 16mn^2(h_p(f) + 2m + 2n),$$  \hspace{1cm} (36)

$$h(T_6) \leq 40mn^3(h_p(f) + 2m + 2n),$$  \hspace{1cm} (37)

$$h(T_7) \leq 18mn^3 \log n(h_p(f) + 2m + 2n).$$  \hspace{1cm} (38)

Remark 5.5 Estimates (37) and (38) can probably be refined, to have the main term of the form $O(mn^2h_p(f))$, which would result in the similar main term in Proposition 5.4.

Proof of (32) Obviously, $h(T_1) \leq \sum_{p \leq n} \log p$, which is bounded by $1.02n$ according to 5. □

Proof of (33) Item 2 of Proposition 2.2 implies that $h(T_2) \leq h_a(f)$. Since $h_a(f) = h_p(f)$ by 20, the result follows. □

Proof of (34) Item 2 of Proposition 2.2 and Lemma 3.13 imply that

$$h(T_3) \leq h_a(r_0) \leq h_a(R) \leq (2n - 1)h_a(f) + (2n - 1)(\log(2n^2) + m \log 2).$$  \hspace{1cm} (39)

Again using 20, we have the result. □

Proof of (35) We have deg $\tilde{R} \leq \deg R \leq (2n - 1)m$. Further, using Corollary 3.10 and inequalities (39), we find

$$h_a(R) \leq h_p(R) + h_a(r_0) + \deg R \leq (4n - 2)h_a(f) + (8n - 4)(\log n + m).$$

Finally, using Remark 3.14 and the previous estimates, we obtain

$$h(T_4) \leq h_a(\Delta) \leq (2\deg \tilde{R} - 1) \left( h_a(\tilde{R}) + \log(2(\deg \tilde{R})^2) \right) \leq 16mn^2h_a(f) + 32mn^2(\log n + m).$$

Using (20), we obtain the result. □
Preparation for the proofs of \((36-38)\)  
Recall that we denote by \(R(X)\) the \(Y\)-resultant of \(f(X,Y)\) and \(f'_Y(X,Y)\) and by \(A\) the set of the roots of \(R(X)\). If we denote by \(\mu_\alpha\) the order of \(\alpha\) as the root of \(R(X)\), then we have

\[
|A| \leq \sum_{\alpha \in A} \mu_\alpha \leq \deg R(X) \leq m(2n - 1),
\]

where for \((41)\) we use Remark 3.9 and Lemma 3.13. Denoting by \(u_\alpha\) order of \(\alpha\) as the root of \(f_\alpha(X)\), we have, obviously,

\[
\sum_{\alpha \in A} u_\alpha \leq m,
\]

\[
\sum_{\alpha \in A} u_\alpha h_\alpha(\alpha) \leq h_p(f_\alpha) + \log(m + 1) \leq h_p(f) + \log(m + 1).
\]

Using the notation

\[
f^{(\alpha)}(X,Y) = f(X + \alpha, Y)
\]

and Corollary 3.11 we obtain the following inequalities:

\[
\sum_{\alpha \in A} h_\alpha(f^{(\alpha)}) \leq \sum_{\alpha \in A} \mu_\alpha h_\alpha(f^{(\alpha)}) \leq 4mn h_p(f) + 7m^2n + 3mn \log n,
\]

\[
\sum_{\alpha \in A} u_\alpha h_\alpha(f^{(\alpha)}) \leq 2mh_p(f) + 3m^2.
\]

Proof of \((36)\)  
For \(\alpha \in A\) and \(v \in M_{\mathbb{K}(\alpha)}\) let \(A_\nu(\alpha)\) and \(B_\nu(\alpha)\) be the quantities of Theorem 4.11 but for the polynomial \(f^{(\alpha)}\) instead of \(f\). Put

\[
T_5^{(\alpha)} = \{ v \in M_{\mathbb{K}(\alpha)} : A_\nu^{(\alpha)} > 1 \text{ or } B_\nu^{(\alpha)} > 1 \},
\]

Corollary 4.2 implies that

\[
h(T_5^{(\alpha)}) \leq (4n - 1) h_p(f^{(\alpha)}) + 13n^2 + 4n \log m.
\]

Now let \(A'\) be a maximal selection of \(\alpha \in A\) pairwise non-conjugate over \(K\). Then every place from \(T_5\) extends to some place from \(T_5^{(\alpha)}\) for some \(\alpha \in A'\). Item 1 of Proposition 4.2 implies that

\[
h(T_b) \leq \sum_{\alpha \in A'} [\mathbb{K}(\alpha) : \mathbb{K}] h(T_5^{(\alpha)}).
\]

Using \((45)\), we obtain

\[
h(T_b) \leq (4n - 1) \sum_{\alpha \in A'} [\mathbb{K}(\alpha) : \mathbb{K}] h_p(f^{(\alpha)}) + (13n^2 + 4n \log m) \sum_{\alpha \in A'} [\mathbb{K}(\alpha) : \mathbb{K}]
\]

\[
= (4n - 1) \sum_{\alpha \in A} h_p(f^{(\alpha)}) + (13n^2 + 4n \log m) |A|.
\]

Using \((40)\) and \((43)\), we obtain \((36)\) after an easy calculation.

Proof of \((37)\)  
We again let \(A'\) be a maximal selection of \(\alpha \in A\) pairwise non-conjugate over \(K\), and for any \(\alpha \in A\) we let \(Q_\alpha\) be a maximal selection of points \(Q\) with \(x(Q) = \alpha\), pairwise non-conjugate over \(\mathbb{K}(\alpha)\). A place \(v \in M_{\mathbb{K}}\) belongs to \(T_6\) in one of the following cases: either \(v\) ramifies
in \( \mathbb{K}(\alpha) \) for some \( \alpha \in A' \), or an extension of \( v \) to some \( \mathbb{K}(\alpha) \) ramifies in \( \mathbb{K}(Q) \) for some \( Q \in Q_\alpha' \).

Item 1 of Proposition 2.2 implies that

\[
h(T_6) \leq \sum_{\alpha \in A'} h\left( \text{Ram}(\mathbb{K}(\alpha)/\mathbb{K}) \right) + \sum_{\alpha \in A'} \sum_{Q \in Q_\alpha'} [\mathbb{K}(\alpha) : \mathbb{K}] h\left( \text{Ram}(\mathbb{K}(Q)/\mathbb{K}(\alpha)) \right),
\]

where \( \text{Ram}(\cdot) \) is defined in Lemma 3.17. Lemma 3.18 implies that

\[
\sum_{\alpha \in A'} h\left( \text{Ram}(\mathbb{K}(\alpha)/\mathbb{K}) \right) \leq \sum_{\alpha \in \mathcal{A}} [\mathbb{K}(\alpha) : \mathbb{K}] \partial_{\mathbb{K}(\alpha)/\mathbb{K}} = \sum_{\alpha \in \mathcal{A}} \partial_{\mathbb{K}(\alpha)/\mathbb{K}}.
\]

The latter sum can be easily estimated by Corollary 3.16 and Lemma 3.13:

\[
\sum_{\alpha \in \mathcal{A}} \partial_{\mathbb{K}(\alpha)/\mathbb{K}} \leq 4mn h_p(R) + 6mn \log(2mn) \leq 8mn^2 h_p(f) + 18mn^2 \log(3mn).
\]

To estimate the second sum in (46), we again use Lemma 3.18:

\[
\sum_{\alpha \in A'} \sum_{Q \in Q_\alpha'} [\mathbb{K}(\alpha) : \mathbb{K}] h\left( \text{Ram}(\mathbb{K}(Q)/\mathbb{K}(\alpha)) \right) \leq \sum_{\alpha \in \mathcal{A}} [\mathbb{K}(\alpha) : \mathbb{K}] \sum_{Q \in Q_\alpha'} [\mathbb{K}(Q) : \mathbb{K}(\alpha)] \partial_{\mathbb{K}(Q)/\mathbb{K}(\alpha)}
\]

\[
= \sum_{\alpha \in \mathcal{A}} \sum_{x(Q) = \alpha} \partial_{\mathbb{K}(Q)/\mathbb{K}(\alpha)}
\]

Corollary 4.3 implies that

\[
\sum_{x(Q) = \alpha} \partial_{\mathbb{K}(Q)/\mathbb{K}(\alpha)} \leq \left( 4n^2 \mu_\alpha + 4u_\alpha n^3 + 2n^2 \right) h_p(f(\alpha)) + \left( \mu_\alpha n + u_\alpha n^2 \right) (12n^2 + 4n \log m) + 3n^2 \log(2n).
\]

Summing up over \( \alpha \in \mathcal{A} \) and using (40), (43) and (44), we obtain

\[
\sum_{\alpha \in \mathcal{A}} \sum_{x(Q) = \alpha} \partial_{\mathbb{K}(Q)/\mathbb{K}(\alpha)} \leq 32mn^3 h_p(f) + 60mn^3 (m + n),
\]

which, together with (47), implies (37). \( \square \)

**Proof of (38)** For \( Q \in Q \) denote by \( \Sigma_Q \) the sum of the heights of all the essential coefficients of the Puiseux expansion at \( Q \). Keeping the notation \( A' \) and \( Q'_\alpha \) from the previous proof, and using item 2 of Proposition 2.2 we obtain

\[
h(T_7) \leq \sum_{\alpha \in A'} [\mathbb{K}(\alpha) : \mathbb{K}] \sum_{Q \in Q_\alpha'} [\mathbb{K}(Q) : \mathbb{K}(\alpha)] \Sigma_Q = \sum_{\alpha \in \mathcal{A}} \sum_{x(Q) = \alpha} \Sigma_Q.
\]

Corollary 4.3 estimates the inner sum by

\[
(u_\alpha + \mu_\alpha) \left( 2n^2 h_p(f(\alpha)) + 6n^3 + 2n^2 \log m \right) \log_2 n.
\]

Applying inequalities (40), (43) and (44) once again, we obtain

\[
h(T_7) \leq 12mn^3 \log_2 n h_p(f) + 24mn^3 (m + n) \log_2 n.
\]

Since \( \log_2 n \leq 1.5 \log n \), the result follows. This completes the proof of Proposition 5.4. \( \square \)
6 A Tower of $\bar{\mathbb{K}}$-Points

In this section we retain the set-up of Section 5 that is, we fix a number field $\mathbb{K}$, a curve $C$ defined over $\mathbb{K}$ and rational functions $x, y \in \mathbb{K}(C)$ such that $\mathbb{K}(C) = \mathbb{K}(x, y)$. Again, let $f(X, Y) \in \mathbb{K}[X, Y]$ be the $\mathbb{K}$-irreducible polynomial of $X$-degree $m$ and $Y$-degree $n$ such that $f(x, y) = 0$, and we again assume that $f_0(X)$ in (26) is monic. We again define the polynomial $R(X)$, the sets $\mathcal{A} \subset \bar{\mathbb{K}}$, $Q \subset \mathbb{C}(\mathbb{K})$ and $T_1, \ldots, T_7 \subset M_\mathbb{K}$, etc.

We also fix a covering $\bar{C} \to C$ of $C$ by another smooth irreducible projective curve $\bar{C}$; we assume that both $C$ and the covering $\phi$ are defined over $\mathbb{K}$. We consider $\mathbb{K}(C)$ as a subfield of $\mathbb{K}(\bar{C})$; in particular, we identify the functions $x \in \mathbb{K}(C)$ and $x \circ \phi \in \mathbb{K}(\bar{C})$. We fix a function $\bar{y} \in \mathbb{K}(\bar{C})$ such that $K(\bar{C}) = \mathbb{K}(x, \bar{y})$. We let $\bar{f}(X, \bar{Y}) \in \mathbb{K}[X, \bar{Y}]$ be an irreducible polynomial of $X$-degree $\bar{m}$ and $\bar{Y}$-degree $\bar{n}$ such that $\bar{f}(x, \bar{y}) = 0$; we write

$$\bar{f}(X, \bar{Y}) = \bar{f}_0(X)\bar{Y}^\bar{n} + \bar{f}_1(X)\bar{Y}^{\bar{n}-1} + \cdots + \bar{f}_\bar{n}(X)$$

and assume that the polynomial $\bar{f}_0(X)$ is monic. We define in the similar way the polynomial $\bar{R}(X)$, the sets $\mathcal{A} \subset \bar{\mathbb{K}}$, $\bar{Q} \subset \mathbb{C}(\bar{\mathbb{K}})$ and $\bar{T}_1, \ldots, \bar{T}_7 \subset M_\bar{\mathbb{K}}$, etc. We also define the notion of proximity on the curve $\bar{C}$ exactly in the same way as we did it for $C$ in Definition 5.1 and we have the analogues of Propositions 5.2, 5.3 and 5.4.

In addition to all this, we define one more finite set of places of the field $\mathbb{K}$ as follows. Write $\bar{R}(X) = \bar{R}_1(X)\bar{R}_2(X)$, where the polynomials $\bar{R}_1(X), \bar{R}_2(X) \in \mathbb{K}(X)$ are uniquely defined by the following conditions:

- the roots of $\bar{R}_1(X)$ are contained in the set of the roots of $f_0(X)$;
- the polynomial $\bar{R}_2(X)$ has no common roots with $f_0(X)$ and is monic.

Now let $\Theta$ be the resultant of $f_0(X)$ and $\bar{R}_2(X)$. Then $\Theta \neq 0$ by the definition of $\bar{R}_2(X)$, and we put

$$U = \{v \in M_\mathbb{K} : |\Theta|_v < 1\}.$$

**Proposition 6.1** Let $P \in C(\bar{\mathbb{K}})$ be semi-defined over $\mathbb{K}$ (that is, $\xi = x(P) \in \mathbb{K}$), and let $\bar{P} \in \bar{C}(\bar{\mathbb{K}})$ be a point above $P$ (that is, $\phi(\bar{P}) = P$). Let $v$ be a finite place of $\mathbb{K}$, and $\bar{v}$ an extension of $v$ to $\bar{\mathbb{K}}$. Assume that $\bar{P}$ is $\bar{v}$-close to some $\bar{Q} \in \bar{\mathbb{Q}}$. Then we have one of the following options.

- $|\xi|_v > 1$.
- $v \in T \cup \bar{T} \cup U$.
- $P$ is $\bar{v}$-adically close to the $Q \in C(\mathbb{K})$ which lies below $\bar{Q}$.

For the proof we shall need a simple lemma.

**Lemma 6.2** In the above set-up, there exists a polynomial $\Phi(X, \bar{Y}) \in \mathbb{K}[X, \bar{Y}]$ such that

$$y = \frac{\Phi(x, \bar{y})}{f_0(x)\bar{R}(x)}$$

**Proof** Since $f_0(x)y$ is integral over $\mathbb{K}[x]$, Corollary 5.2 implies that $f_0(x)y \in \bar{R}(x)^{-1}\mathbb{K}[x, \bar{y}]$, whence the result. $\square$
Since both $f$ and $g$ we may replace the affine heights by the projective heights. Further, we have the estimates

$$ h(\tilde{y}(\tilde{Q})) = \sum_{k=-k(\tilde{Q})}^{\infty} \alpha_k(\tilde{Q}) \tilde{z}^k(x - \alpha)^{k/\tilde{c}} \quad (i = 0, \ldots, \tilde{c} - 1), \quad (48) $$

be the equivalent Puiseux expansions of $\tilde{y}$ at $\tilde{Q}$ (here $\tilde{c}$ is a primitive $\tilde{c}$-th root of unity). Since $\tilde{P}$ is $\tilde{v}$-close to $\tilde{Q}$, we have $|\xi - \alpha|_{\tilde{v}} < 1$ and the $\tilde{c}$ series (48) converge at $\xi$, with one of the sums being $\tilde{y}(\tilde{P})$.

Now let $\Phi(X, \tilde{Y})$ be the polynomial from Lemma 6.2. Then the $\tilde{c}$ series

$$ \frac{\Phi(x, \tilde{y}(\tilde{Q}))}{f_0(x) \tilde{R}(x)} \quad (i = 0, \ldots, \tilde{c} - 1) \quad (49) $$

contain all the equivalent Puiseux series of $y$ at $Q = \phi(\tilde{Q})$. More precisely, if the ramification of $Q$ over $P$ is $e$, then every of the latter series occurs in (49) exactly $\tilde{c}/e$ times.

Write $f_0(X) \tilde{R}(X) = (X - \alpha)^{\nu} g(X)$ with $g(\alpha) \neq 0$. The assumption $v \notin T_2 \cup \tilde{T}_2 \cup \tilde{T}_3 \cup \tilde{T}_4 \cup U$ implies that $|g(\alpha)|_v = 1$. Now Lemma 3.6 implies that the Laurent series at $\alpha$ of the rational function $(f_0(x) \tilde{R}(x))^{-1}$ converges at $\xi$. Hence all the series (49) converge at $\xi$, and among the sums we find

$$ \frac{\Phi(x, \tilde{y}(\tilde{Q}))}{f_0(x(P)) \tilde{R}(x(P))} = y(P). $$

Hence $P$ is $v$-close to $Q$. $\square$

We shall also need a bound for $U$ similar to that of Proposition 5.4.

**Proposition 6.3** We have $h(U) \leq \Upsilon + \Xi$, where $\Upsilon$ is defined in (1) and

$$ \Xi = 2m\tilde{m}(2\tilde{m} + 3 \log \tilde{n}) + (m + 2\tilde{m}\tilde{n}) \log (m + 2\tilde{m}\tilde{n}). \quad (50) $$

**Proof** Item 2 of Proposition 2.2 implies that $h(U) \leq h_\alpha(\Theta)$, where $\Theta$ is the resultant of $f_0(X)$ and $\tilde{R}_2(X)$. Expressing $\Theta$ as the familiar determinant, we find

$$ h_\alpha(\Theta) \leq \deg \tilde{R}_2 h_\alpha(f_0) + \deg f_0 h_\alpha(\tilde{R}_2) + (\deg f_0 + \deg \tilde{R}_2) \log (\deg f_0 + \deg \tilde{R}_2) \quad (51) $$

Since both $f_0$ and $\tilde{R}_2$ are monic polynomials (by the convention 28 and the definition of $\tilde{R}_2$), we may replace the affine heights by the projective heights. Further, we have the estimates

$$ \deg f_0 \leq m, \quad \deg \tilde{R}_2 \leq \tilde{m}(2\tilde{n} - 1), \quad h_\alpha(f_0) \leq h_\alpha(f), \quad h_\alpha(\tilde{R}_2) \leq (2\tilde{n} - 1)h_\alpha(\tilde{f}) + (2\tilde{n} - 1) \left(2\tilde{m} + \log ((\tilde{n} + 1)\sqrt{\tilde{n}})\right), $$

the latter estimate being a consequence of Corollary 3.10 and Lemma 3.13. Substituting all this to 51, we obtain the result. $\square$

### 7 The Chevalley-Weil Theorem

Now we may to gather the fruits of our hard work. In this section we retain the set-up of Section 6. Here is our principal result, which will easily imply all the theorems stated in the introduction.

**Theorem 7.1** Assume that the covering $\phi$ is unramified outside the poles of $x$. Let $P \in \tilde{C}(\mathbb{K})$ be semi-defined over $\mathbb{K}$, and let $\tilde{P} \in \tilde{C}(\mathbb{K})$ be a point above $P$. As before, we put $\xi = x(P) = x(\tilde{P})$. Then for every non-archimedean $v \in M_\mathbb{K}$ we have one of the following options.
• $|\xi|_v > 1$.
• $v \in T \cup \tilde{T} \cup U$.
• Any extension of $v$ to $K(P)$ is unramified in $K(\tilde{P})$.

**Proof** Let $v \in M_\nu$ be a non-archimedean valuation such that $|\xi|_v \leq 1$ and $v \notin T \cup \tilde{T} \cup U$. Fix an extension $\tilde{v}$ of $v$ to $\tilde{K}$, and let $\tilde{w}$ and $w$ be the restrictions of $\tilde{v}$ to $K(\tilde{P})$ and $K(P)$, and $\tilde{e}$ and $e$ their ramification indexes over $v$, respectively.

If $\tilde{P} \notin \tilde{Q}$ then $\tilde{v}$ is unramified in $K(\tilde{P})$ because $\tilde{v} \notin \tilde{T}_1 \cup \tilde{T}_0$. From now on, we assume that $\tilde{P} \notin \tilde{Q}$. Proposition 5.2 implies that $\tilde{P}$ is $\nu$-adically close to some $\tilde{Q} \in \tilde{Q}$, and Proposition 5.3 implies that $\tilde{e} = e_{\tilde{Q}}/\gcd(e_{\tilde{Q}}, \ell)$. Let $Q$ be the point of $C$ lying under $\tilde{Q}$. Put $\alpha = x(\tilde{Q}) = x(Q)$. If $\alpha \notin A$ then the covering $C \mapsto P^1$ does not ramify at $Q$. Since $\phi$ is unramified outside the poles of $x$, the covering $C \mapsto P^1$ does not ramify at $Q$, that is, $e_{\tilde{Q}} = 1$. Hence $\tilde{e} = 1$, which means that $v$ is not ramified in $K(\tilde{P})$.

Now assume that $\alpha \in A$. Proposition 5.1 implies that $P$ is $\nu$-adically close to $Q$. Now notice that $e_{\tilde{Q}} = e_{\tilde{Q}}$, again because $\phi$ is unramified. Also, $\ell(P, Q, \tilde{v}) = \ell(\tilde{P}, \tilde{Q}, \tilde{v}) = \ell$, just by the definition of this quantity. Again using Proposition 5.3 we obtain that $e = e_Q/\gcd(e_Q, \ell) = \tilde{e}$. This shows that $w$ is unramified over $v$, completing the proof. \[\square\]

We also need an estimate for $h(T \cup \tilde{T} \cup U)$.

**Proposition 7.2** We have

$$h(T \cup \tilde{T} \cup U) \leq \Omega + \tilde{\Omega} + \Upsilon.$$  \hspace{1cm} (52)

where $\Omega$, $\tilde{\Omega}$ and $\Upsilon$ are defined in \[\Omega\].

**Proof** Combining Propositions 5.4 and 6.3, we obtain the estimate

$$h(T \cup \tilde{T} \cup U) \leq \frac{3}{4}(\Omega + \tilde{\Omega}) + \Upsilon + \Xi,$$

where $\Xi$ is defined in \[\Xi\]. A routine calculation show that $\Xi \leq (\Omega + \tilde{\Omega})/4$, whence the result. \[\square\]

Now we can prove the theorems from the introduction.

**Proof of Theorem 1.3** We may replace $K$ by $K(P)$ and assume that $P \in C(K)$. Put $\xi = x(P)$ and let $R$ be the set of places of $K$ having an extension to $K(P)$ that ramifies in $K(\tilde{P})$. Theorem \[\text{1.4}\] and estimate \[\text{52}\] imply that

$$h(\{v \in R : |\xi|_v \leq 1\}) \leq \Omega + \tilde{\Omega} + \Upsilon.$$  \hspace{1cm} (52)

Replacing $x$ by $x^{-1}$ and the polynomials $f$, $\tilde{f}$ by $X^m f(X^{-1}, Y)$ and $X^{\tilde{m}} \tilde{f}(X^{-1}, Y)$, respectively, we obtain the estimate

$$h(\{v \in R : |\xi|_v \geq 1\}) \leq \Omega + \tilde{\Omega} + \Upsilon.$$  \hspace{1cm} (52)

Thus,

$$h(R) \leq 2(\Omega + \tilde{\Omega} + \Upsilon),$$

and Lemma \[\text{3.17}\] implies that

$$\partial_{K(\tilde{P})/K} \leq \frac{\nu - 1}{\nu}(\Omega + \tilde{\Omega} + \Upsilon) + 1.26\nu \leq 2(\Omega + \tilde{\Omega} + \Upsilon).$$

The theorem is proved. \[\square\]

\[\text{The field } K(\tilde{P}) \text{ coincides with the field } K(\tilde{P}), \text{ generated by the coefficients of the Puiseux series at } \tilde{P}, \text{ if } x \text{ is unramified at } \tilde{P}; \text{ if there is ramification } e, \text{ we have } K(\tilde{P}, \xi) = K(\tilde{P})(\xi_e). \text{ Hence any } v \notin \tilde{T}_1 \cup \tilde{T}_0 \text{ is unramified at any } \tilde{P} \in \tilde{Q}.\]

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Proof of Theorem 1.5 Let $S'$ be set of places of the field $\mathbb{K}(P)$ extending the places from $S$. The right-hand side of (2) will not increase (see item 1 of Proposition 2.2) if we replace $\mathbb{K}$ by $\mathbb{K}(P)$ and $S$ by $S'$. Thus, we may assume that $P \in \mathcal{C}(\mathbb{K})$. Again using Theorem 7.1 and (52), we obtain

$$h(R \prec S) \leq \Omega + \tilde{\Omega} + \Upsilon.$$ 

We again complete the proof, applying Lemma 3.17.

To prove Theorem 1.6 we need the following result from [3].

Theorem 7.3 Let $x : \mathcal{C} \to \mathbb{P}^1$ be a finite covering of degree $n \geq 2$, defined over $\mathbb{K}$ and unramified outside a finite set $A \subset \mathbb{P}^1(\bar{\mathbb{K}})$. Put $h = h_a(A)$ and $\Lambda' = (2(g + 1)n^2)^{10gn+12n}$, where $g = g(\mathcal{C})$. Then there exists a rational function $y \in \bar{\mathbb{K}}(\mathcal{C})$ such that $\bar{\mathbb{K}}(\mathcal{C}) = \bar{\mathbb{K}}(x, y)$ and the rational functions $x, y \in \bar{\mathbb{K}}(\mathcal{C})$ satisfy the equation $f(x, y) = 0$, where $f(X, Y) \in \bar{\mathbb{K}}[X, Y]$ is an absolutely irreducible polynomial satisfying

$$\deg_X f = g + 1, \quad \deg_Y f = n, \quad h_p(f) \leq \Lambda'(h + 1). \quad (53)$$

Moreover, the number field $\mathbb{L}$, generated over $\mathbb{K}$ by the set $A$ and by the coefficients of $f$ satisfies $\partial_{\mathbb{L}/\mathbb{K}(A)} \leq \Lambda'(h + 1)$.

Proof of Theorem 1.6 We shall prove the “projective” case (that is, item 1) of this theorem. The “affine” case is proved similarly.

We define $\Lambda'$ in the same way as $\Lambda'$ in Theorem 7.3 but with $n$ and $g$ replaced by $\bar{n}$ and $\bar{g}$. We use Theorem 7.3 to find functions $y \in \bar{\mathbb{K}}(\mathcal{C})$ and $\bar{y} \in \bar{\mathbb{K}}(\mathcal{C})$, and polynomials $f(X, Y) \in \bar{\mathbb{K}}[X, Y]$ and $\bar{f}(X, Y) \in \bar{\mathbb{K}}[X, Y]$. Denoting by $L$ the field generated by the set $A$ and the coefficients of both the polynomials, we find $\partial_{\mathbb{L}/\mathbb{K}(A)} \leq (\Lambda' + \bar{\Lambda}')(h + 1)$ with $h = h_a(A)$. Using Lemma 3.15 we estimate $\partial_{\mathbb{L}/\mathbb{K}} \leq 2(\delta - 1)h + \log \delta$. Hence

$$\partial_{\mathbb{L}/\mathbb{K}} \leq (\Lambda' + \bar{\Lambda}' + 2(\delta - 1))(h + 1).$$

We define the quantities $\Omega$, $\bar{\Omega}$ and $\Upsilon$ as in the introduction. Then, applying Theorem 1.3 but over the field $L$ rather than $K$, we find $\partial_{\mathbb{L}(\bar{P})/\mathbb{L}(P)} \leq 2(\Omega + \bar{\Omega} + \Upsilon)$. We have

$$\partial_{\mathbb{K}(\bar{P})/\mathbb{K}(P)} \leq \partial_{\mathbb{L}(\bar{P})/\mathbb{L}(P)} \leq \partial_{\mathbb{L}(P)/\mathbb{K}(P)} \leq \partial_{\mathbb{L}(\bar{P})/\mathbb{L}(P)} + \partial_{\mathbb{L}/\mathbb{K}}.$$ 

The last sum is bounded by

$$2(\Omega + \bar{\Omega} + \Upsilon) + (\Lambda' + \bar{\Lambda}' + 2(\delta - 1))(h + 1),$$

which, obviously, does not exceed $\Lambda(h + 1)$, as wanted.

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