Redshift factor and diffusive equilibrium of unbound neutrons in the single nucleus model of accreting neutron star crust

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March 2, 2020

Abstract. Using a Wigner-Seitz approximation with spherical cells, we re-analyze a widely used single nucleus model of accreting neutron star crust. We calculate beta disequilibrium within the crust, which is sizable, and implies that neutron and baryon chemical potentials, $\mu_n$ and $\mu_b$, are not equal. We include also non-equilibrium reactions, driven by matter compression, and proceeding in the reaction layers. The constancy of $e^\Phi \mu_n$, where the spacetime metric component $g_{00} = e^{2\Phi}$, in the shells between the reaction layers is not applicable, because single electron captures are blocked, so that the neutron fraction is fixed, and therefore neutrons are not an independent component of the crust matter. The absence of neutron diffusion in the shells between the reaction layers, stems from the constancy of the neutron fraction (concentration) in these shells. In the reaction layers, the outward force resulting from neutron fraction gradient is balanced by the inward gravitational force acting on unbound neutrons. Neglecting the thickness of the reaction layers compared to the shell thickness, we obtain condition $e^{\Phi(r)} f_Q(r) g(r) =$constant, where $g$ is Gibbs energy per nucleon, undergoing discontinuous drops on the reaction surfaces, and $f_Q(r) g(r) = \tilde{g}(r)$ is a continuous function, due to the factor $f_Q(r)$ canceling the discontinuities (drops) in $g(r)$. The function $f_Q(r)$ is calculated using the Tolman-Oppenheimer-Volkov equations from $f_Q(P)$ and $g(P)$ obtained from the equation of state (EOS) with discontinuities. The constancy of of $e^{\Phi(r)} \tilde{g}(r)$ is an extension of the standard relation $e^{\Phi(r)} \mu_b(r) =$constant, valid in hydrostatic equilibrium for catalyzed crust.
1 Introduction

A model of the matter with only one nuclear species present at a given pressure yields a simplest approximation of the neutron star crust, called frequently a single nucleus model (SNM). The crust is treated there as a one component plasma. The popularity of the SNM stems from its simplicity. SNM was used in the calculation of the accreted crust structure \[9, 11\] and crustal heating in accreting neutron stars, generated by the non-equilibrium nuclear processes induced by the compression of the crust matter \[10, 12, 14\]. In spite of its simplicity and some seemingly unrealistic features, including sharply localized heat sources, SNM yields cumulated deep crustal heating consistent with advanced numerical simulations involving multi-component plasma and large nuclear reaction networks \[15, 16, 17\]. Recently, the validity of the SNM for calculating the crust structure and the cumulated deep crust heating in accreting neutron stars has been questioned \[2\]. This was argued to result in neutron diffusion, and a strong decrease of the deep crustal heating.

In the present paper we address the above mentioned problems. Additionally, we show using the Tolman-Oppenheimer-Volkov (TOV) equations, that for Gibbs energy per nucleon \(g(r)\) with discontinuous drops associated with heat release, the standard constancy relation \(e^{\Phi(r)\mu_b(r)} = \text{constant}\) is replaced by \(e^{\Phi(r)\tilde{g}(r)} = \text{constant}\), with \(\tilde{g}(r) = f_Q(r)g(r)\). Here, \(f_Q(r)\) is a step-like function making \(\tilde{g}(r)\) continuous, calculated using the TOV equation that yield pressure profile \(P(r)\) and \(\Phi(r)\) for an assumed EOS. The constancy of \(e^{\Phi(r)\tilde{g}(r)}\) holds throughout both the outer (no free neutrons, but still \(g(r)\) discontinuities present!) and the inner crust.

In Sect.2 we review essential features of the SNM of the crust, and applications of this model to the simulations of an accreting neutron star crust. We point out the differences between the accreted and catalyzed crusts, and illustrate the applications of the SNM to both cases, using a nuclear model for the nucleon component of the matter. In the last subsection of Sect.2 we demonstrate diffusive equilibrium of unbound neutrons in the accreted crust.

In Sect.3 we consider the hydrostatic equilibrium of an accreted crust and derive an extension of \(e^{\Phi(r)\mu_b(r)} = \text{constant}\) theorem to the case of of \(g(r)\) with discontinuities (sharp drops) characteristic of SNM of accreting neutron star crust. Section 4 presents discussion of our results and conclusion.
2 Single-nucleus model of the inner accreted crust

In the present paper we obtain strong and electromagnetic interaction equilibrium of the crust by putting \( Z \) protons and electrons, and \( N_{\text{cell}} = A_{\text{cell}} - Z \) neutrons into a spherical Wigner-Seitz (W-S) cell under pressure \( P \) and calculating the ground state of the system using an approximate solution of the nuclear many-body theory. Temperature effects are neglected and \( T = 0 \) approximation is used. The ground state has a proton cluster at the cell center, neutrons bound to the proton cluster, and beyond neutron drip pressure, \( P_{\text{ND}} \), also a fraction of \( N_{\text{cell}} \) unbound and filling the W-S cell. The possibility of the non-spherical pasta phases at the bottom of the crust will not be considered. By construction, \( A_{\text{cell}} \) is an integer number. We also calculate the neutron chemical potential \( \mu_n \), as well as \( \mu_p \) and \( \mu_e \) (all \( \mu \)-s include rest energies of particles). The Gibbs free energy per nucleon (equal to baryon chemical potential \( \mu_b \)) is

\[
g = \frac{[Z(\mu_p + \mu_e) + N_{\text{cell}}\mu_n]}{A_{\text{cell}}} = x_p(\mu_p + \mu_e) + (1 - x_p)\mu_n ,
\]

where the proton fraction \( x_p = Z/A_{\text{cell}} \). Expression for \( g \), Eq.(1), can be rewritten as

\[
g = x_p(\mu_p + \mu_e - \mu_n) + \mu_n = x_p\Delta\mu + \mu_n .
\]

A strict beta equilibrium between \( n, p, \) and \( e \) corresponds to \( \Delta\mu = 0 \) and \( \mu_b = \mu_n \).

2.1 Accreted crust

For the sake of simplicity, we will limit to the case of a stationary fully accreted crust. The structure of such a crust, calculated using the SNM, was derived in numerous papers [9 11 10 12 14]. It is obtained by simulating a compressional evolution of a W-S cell \( (A_{\text{cell}}, Z; P) \) with \( P \) increasing due to the weight of accreted matter, and taking into account possible electron captures, neutron emissions and absorptions, and at density \( \rho > 10^{12} \) g cm\(^{-3}\), also pycnonuclear fusion of neighbouring nucleon clusters, driven by the quantum zero point oscillations of the clusters and penetration of the Coulomb barriers between them.

In the process of formation of a fully accreted crust, nuclear ashes at \( \sim 10^8 \) g cm\(^{-3}\), resulting from an explosive thermonuclear burning of a freshly accreted plasma which generates the X-ray bursts in a LMXB, are compressed up to \( 10^{14} \) g cm\(^{-3}\) after reaching the bottom of the crust. This compressional evolution can be followed within the SNM by simulating compression of a single initial W-S cell \( (A_{\text{cell}}^{(\text{in})}, Z^{(\text{in})}) \) under increasing pressure \( P \), inducing electron captures
decreasing $Z$ at constant $A_{\text{cell}}$, then neutron drip combined with electron captures for densities exceeding $5 \times 10^{11}$ g cm$^{-3}$, still at constant $A_{\text{cell}} = A_{\text{cell}}^{\text{(in)}}$, and finally including also pycnonuclear fusions accompanied by the electron captures and neutron emissions and absorptions, resulting in a doubling of $A_{\text{cell}}$. As a final result, one gets an evolutionary track given as $(A_{\text{cell}}, Z)_{\text{AC}}$ as a function of $P$. At $T = 0$ reactions start at specific threshold values $P = P_j$. The evolutionary trajectory in the $A_{\text{cell}}, Z$ plane follows the local minimum of $g(A_{\text{cell}}, Z; P)$. The minimization is done at fixed $A_{\text{cell}}$, with an option $A_{\text{cell}} \rightarrow 2A_{\text{cell}}$ open if the timescale of the pycnonuclear fusion (usually taking place after electron captures) becomes shorter than a local compression timescale.

Let us remind, that in the case of the crust made of cold catalysed matter, its composition calculated using the SNM corresponds to an absolute (global) minimum at a given $P$, i.e., to the ground state (GS), and complete thermodynamic equilibrium of the crust $(A_{\text{cell}}, Z; P)_{\text{GS}}$. The track $(A_{\text{cell}}, Z)_{\text{GS}}$ is very different from the $(A_{\text{cell}}, Z)_{\text{AC}}$ one, with $Z_{\text{AC}}(P)$ values being significantly lower (by a factor $\sim 2 - 3$) than $Z_{\text{GS}}(P)$.

Both GS and AC crusts have an onion-like structure consisting of shells $(A^{(j)}, Z^{(j)})$, $j = 1, \ldots, j_{\text{max}}$. The transition pressure $P_j$ between $j$ and $j + 1$ shell is associated with a density discontinuity (drop) much smaller in the GS crust than in the AC crust. As $P_{j_{\text{AC}}}$ corresponds to the threshold for an equilibrium electron capture followed by a second non-equilibrium electron capture, and possibly emission of neutrons, there is not only a density drop at $P_{j_{\text{AC}}}$, but also a $\Delta g(P_{j_{\text{AC}}})$ drop associated with a heat release per one accreted nucleon.

Another important feature of an AC crust, differing it from the GS crust, is a (significant) metastability of the local equilibrium state, between the consecutive reaction surfaces $P_i$, where $P_i$ is a threshold pressure for a single electron capture initiating transition to a different (lower) local minimum of $g$. For $P_{i-1} < P < P_i$ the W-S cell is in a metastable state, because single electron capture is blocked by the energy barrier, while the double electron capture - obviously leading to a lower $g$ because of the nucleon paring - is assumed to be too slow to proceed.

The integer parameters $Z$ and $A_{\text{cell}}$ (and therefore neutron and proton fractions within nucleons) stay constant within shells $P_{i-1} < P < P_i$. At each $P = P_i$, the value of $\mu_b$ undergoes a drop by $Q_i$, while $P$ changes from $P_i^- = P_i - 0$ to $P_i^+ = P_i + 0$. So $P_i$ determines a surface of discontinuity of $\mu_b$. This spherical surface is an idealization of the heating (reaction) layer in an accreting crust.
Specific features of an EOS.AC of a fully accreted crust are illustrated in Figs. [12]. The calculations were done for Mackie-Baym (MB) model of the nucleon sector of the crust plasma [21].

### 2.2 Diffusive equilibrium of unbound neutrons in the inner accreted crust

Protons cannot diffuse because they are bound (confined) in the localized clusters, which in turn are localized at the lattice sites of the Coulomb crystal. The electrons are coupled to protons by the electromagnetic forces. Neutrons do not diffuse within shells between reaction surfaces, because neutron fraction per nucleon within a $j$-th shell, $x_n^{(j)} = 1 - x_p^{(j)}$, is constant, $\frac{dx_n^{(j)}}{dr} = 0$ [7] (we neglect corrections of the order of $(R - r)/R$, where $R$ is the NS radius).

In reality, the changes of matter composition due to reactions triggered by the electron capture take place in a layer separating the two shells. In this reaction layer we are dealing with a two-
Figure 2: Baryon chemical potential $\mu_b$ versus pressure (blue line) in the inner crust. The drops in $\mu_b$ correspond to thin heating layers. In red - $\tilde{g}(P)$ defined in Sect. 3.1. Green curve - $\mu_{no}$ - neutron chemical potential of the neutron gas. Calculations performed for the MB model of the nucleon sector.

component plasma of proton clusters, $Z^{(j)}$ and $Z^{(j+1)}$, neutrons, and electrons. The neutron fraction per nucleon grows with depth $z = R - r$ from $x_n^{(j-1)}$ to $x_n^{(j)}$, and $dx_n/dz > 0$, but resulting generalized force $d\mu_n/dz$ acting outwards is balanced by the gravitational pull $g_z m_n$. In the Newtonian approximation

$$\frac{d\mu_n}{dz} = m_n g_z$$

which can be integrated over the reaction layer $[z_j, z_{j+1}]$,

$$\mu_n(z) = m_n g_r \cdot (z - z_j) + \mu_n^{(j)} ,$$

where $\mu_i^j = \mu_n(P_i) , i = j, j + 1$ This is diffusive equilibrium condition within the $j$th reaction layer between $j$th and $j + 1$th shells: neutron diffusion outwards is blocked by the gravitational pull acting on a neutron inwards.
Denoting by $d_j$ the thickness of the $j$th reaction layer, we get from Eq. (4) an estimate

$$d_j = \frac{\Delta \mu^j_n}{m_n g_z} \simeq 10 \text{ m},$$

(5)

where $\Delta \mu^j_n = \mu^{j+1}_n - \mu^j_n$. This allows for an estimate of a time for an element of matter to cross the reaction layer, due to accretion onto the NS surface,

$$\tau_{\text{cross}} = \frac{4\pi r^2 \rho}{\dot{M}} d_j.$$

(6)

In the reasoning presented above we did not consider timescale of neutron diffusion process stemming from the scattering of neutrons on nucleon clusters ("nuclei"), $\tau_{\text{diff}}$, and its interplay with two other timescales, $\tau_{\text{cross}}$, and single electron capture timescale, $\tau_{\text{cap}}$, which strongly depends on $\mu_e(z) - W$ where $W$ is the energy threshold. Self-consistent treatment of all three processes - unbound neutron diffusion, electron capture, and the inward matter flow due to accretion is beyond the scope of the present paper, and will be presented in our forthcoming publication.

In the simplest version of the SNM the reaction layer is replaced by a surface with appropriate boundary conditions on both its sides. Actually, SNM of accreted crust can be extended to treat a finite thickness of the reaction layer [19]. Using the $\mu_n(r)$ profile and the reaction layer thickness $\Delta r$, one can estimate heating intensity $dQ(j)/dr$ profile within $j$-th reaction layer, and then obtain the cumulated deep crustal heating

$$Q(r) = \int_{r'_1}^{r} dr \sum_{(j)} \frac{dQ(j)}{dr} = \int_{P'_1}^{P} dP \sum_{(j)} \frac{dQ(j)}{dP}.$$

(7)

The cumulated heating due to continuous sources is reasonably well reproduced by the simple SNM with infinitely thin sources and $dQ(j)/dr = Q_j \delta(r-r_j)$ [14].

The commonly used diffusive equilibrium condition $e^\Phi \mu_n = \text{constant}$ applies only when neutrons constitute an independent component of the crust [8]. This is not the case for the shells between the reaction layers, where beta processes are blocked and $x_n = 1 - x_p$ is fixed. Within the shells there is only one independent density, $n_b(P)$, and the corresponding chemical potential is $\mu_b(P)$ (see next section). To determine thermodynamic equilibrium in a $j$-th shell with fixed $x_n^{(j)}$ it is sufficient to know $P$ or $n_b$. In the next section, an extension of $e^\Phi \mu_n = \text{constant}$ to the case of accreted crust, is derived from the TOV equation relating equation of state (EOS) of accreted crust and $\Phi(r)$. 

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Diffusion of unbound neutrons in neutron star crust was studied in [6] in the context of heating of the envelopes of single X-ray sources. An initial large excess of unbound neutrons was a leftover from a hot state of a newly born neutron star, where a significant fraction of unbound neutrons existed at densities below cold neutron drip point (see, e.g., Fig. 3.1 in [1]). A rapid cooling led to formation of a non-equilibrium layer with a sizable neutron excess. Neutron diffusion inwards was then driven mostly by the gravitational pull and generated heating of the crust during some $10^4$ yr after which neutron equilibrium was reached. The scenario considered in [6] is different from that associated with accreting neutron stars.

3 Hydrostatic equilibrium, EOS, and metric function $\Phi$

The TOV equation for $\Phi(r)$ is

$$\frac{d\Phi}{dr} = \frac{1}{\mathcal{E} + P} \frac{dP}{dr},$$

(8)

where $\mathcal{E}$ is the energy density of the matter (including rest energies of the matter constituents).

The dimensionless pseudoenthalpy $H(P)$ is defined by (e.g., [1])

$$H(P) = \int_0^P \frac{dP'}{\mathcal{E}(P') + P'}. $$

(9)

As we assume that remaining TOV equations have been integrated, we can use $P(r)$ profile corresponding to the hydrostatic equilibrium of NS. Notice that $H(P)$ and $P(r)$ are smooth (differentiable), even when $\mathcal{E}(P)$ and $n_b(r)$ are not. Therefore, we can rewrite Eq. (9) as

$$\frac{dH}{dr} = \frac{1}{\mathcal{E}(P) + P} \frac{dP}{dr},$$

(10)

so that

$$\frac{d}{dr} [H(r) + \Phi(r)] = 0,$$

(11)

which results in

$$H(r) + \Phi(r) = \text{constant} = \Phi(R),$$

(12)

where $R$ is the radius of NS.

In the case of complete thermodynamic equilibrium (cold catalyzed matter) $g = \mu_b$ and we obtain well known constancy of $e^{\Phi(r)}\mu_b(r)$ [18], valid within NS built of cold catalyzed matter. Simultaneously, $\mu_b$ can be replaced by $\mu_n$ (Sect. 2), resulting in equilibrium condition for neutrons $e^{\Phi(r)}\mu_n(r) = \text{constant}$. However, an accreted crust is off beta equilibrium. Moreover, simplest version of SNM yields sharp discontinuities in $\mu_b$ and $\mu_n$. 


Before reconsidering the problems related to the SNM of accreted crust, let us remind the derivation of $e^\Phi \mu_n = \text{constant}$ for uniform npe matter in beta equilibrium. In this case Gibbs free energy per baryon $g(P) = (\mathcal{E} + P)/n_b = \mu_b = \mu_n$, $dP = n_b \, dg$, and the integral in $H(P)$ can be taken to yield

$$H(P) = \ln \left[ \frac{g(P)}{g(0)} \right].$$

Then Eq. (12) could be written as

$$\mu_n(r) e^{\Phi(r)} = \text{constant},$$

valid over the whole core region composed of npe matter in beta equilibrium.

### 3.1 Accreting neutron star crust

Let us consider a simplest SNM of a fully accreted crust. It consists of shells with fixed $(A_{\text{cell}}, Z)_{AC}$, separated from the neighboring shells by spherical surfaces $P = P_{i-1}$, $P_i$. While $P$ is continuous across these surfaces, $g$, $\mathcal{E}$, $n_b$ suffer a discontinuity there. Therefore we have to precisely define the boundary conditions on both sides of the $P_i$ surfaces. Let $P^+_i = P_{i-1} + 0$, $P^-_i = P_i$. Within $P^-_{i-1} \leq P \leq P^+_i$ function $g(P)$ is continuous. Define also $g_j = g(P^-_j)$. Then $H(P)$ can be split into a sum of integrals of continuous functions,

$$H(P) = \int_0^{P^-_1} \frac{dP'}{n_b(P')g(P')} + \int_{P^+_1}^{P^-_2} \frac{dP'}{n_b(P')g(P')} + \ldots + \int_{P^+_j}^{P^-_j+1} \frac{dP'}{n_b(P')g(P')}.$$

Relation between $dP$ and $dg$ for a cell with fixed $A_{\text{cell}}$ is

$$dg = dP/n_b + \sum_{i=e,n,p} \mu_i dx_i.$$

As on the continuous segments of $g(P)$ fractions $x_i$ are constant, we get $dP = n_b \, dg$, and all integrations in Eq. (15) are easily taken, leading to

$$H(P) = \ln \left[ \frac{g(P)}{g_0} \cdot \frac{g_1}{g_1 - Q_1} \cdots \frac{g_j}{g_j - Q_j} \right],$$

where $Q_j$ is energy release per one accreted nucleon on the $j$-th reaction surface and $P_j < P < P_{j+1} < P_{j_{\text{max}}}$. Equation (17) can be rewritten in the form

$$H(P) = \ln \left[ \frac{g(P)}{g_0} \prod_{j=1}^{j_{\text{max}}} \left( \frac{g_j}{g_j - Q_j} \right)^{\Theta(P - P_j)} \right].$$
where the Heaviside function $\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$ otherwise. Let us denote

$$f_Q(P) = \prod_{j=1}^{j_{\text{max}}} \left( \frac{g_j}{g_j - Q_j} \right)^{\Theta(P - P_j)}.$$  \hfill (19)

The product of two discontinuous functions, $g(P)$ and $f_Q(P)$, is a continuous function $\tilde{g}$,

$$\tilde{g}(P) = f_Q(P)g(P).$$  \hfill (20)

We can call $\tilde{g}(P)$ as regularized EOS for accreted crust. It is represented by red lines in Figs. 11-12. It was already introduced in [12] as a continuous envelope of a set of discontinuous EOS’s for accreted crust. It was demonstrated in [13] that function $f_Q(P)$ describes relative difference in thickness of the accreted and catalyzed crust.

Using our expression for $H(P)$, together with pressure profile $P(r)$ within the hydrostatic static configuration of NS, we can rewrite Eq.(12) in the form

$$\Phi(r) + H(r) = \ln g(r) - \ln g_0 + \sum_{j=1}^{j_{\text{max}}} \ln \left( \frac{g_j}{g_j - Q_j} \right) \Theta(r - r_j) + \ln(e^{\Phi(r)}) - \ln(e^{\Phi_0}) ,$$  \hfill (21)

where $\Phi_0 = \Phi(R)$ and $g_0 = g(R)$. This leads to

$$f_Q(r)g(r)e^{\Phi(r)} = \text{constant} = g(R)e^{\Phi(R)},$$  \hfill (22)

where

$$f_Q(r) = \prod_{j=1}^{j_{\text{max}}} \left( \frac{g_j}{g_j - Q_j} \right)^{\Theta(r - r_j)}. \hfill (23)$$

The above condition is an extension of the standard relation for a one-parameter EOS $\mu_b e^\Phi = \text{constant}$ with a continuous $\mu_b(r)$, derived for cold catalyzed matter [18]. In the strict beta equilibrium $\mu_b = \mu_n$ and with all $Q_i$ vanishing, we get $\mu_n e^\Phi = \text{constant}$. For the SNM of fully accreted crust, $g(r)$ undergoes discontinuous drops at $r_j$, and a continuous $\tilde{g}(r) = f_Q(r)\mu_b(r)$, fulfills $e^{\Phi(r)}\tilde{g}(r) = \text{constant}$.

Deviation of $e^{\Phi(r)}\mu_n(r)$ from constancy in fully accreted inner crust fluctuates around mean value with maximum amplitude $\sim 0.4\text{MeV}$ reached after the neutron drip point, and then decreasing with density and depth. This is to be contrasted with constancy of $e^{\Phi(r)}\tilde{g}(r)$, as visualized in Fig. 3.

The case of the accreted outer crust deserves a comment. There, all $N_{\text{cell}}$ neutrons are bound in nuclei, so that $\mu_n < m_ne^2$, where $m_n$ is the neutron mass in vacuum. Beta equilibrium is not
Figure 3: The quantities $e^{\Phi(z)}\mu_n(z)$ (black line), $e^{\Phi(z)}g(z)$ (red line) and $e^{\Phi(z)}\tilde{g}(z)$ (blue line) vs. depth $z = R - r$ within a fully accreted inner crust for NS mass $1.4 M_\odot$. Calculations performed for the MB model of the nucleon sector. A barely visible minute deviation of the $e^{\Phi(z)}\tilde{g}(z)$ line from constancy results from limitations of numerical precision of solving relevant nonlinear equations. The lowest (green) line $m_n c^2 e^\Phi(z)$ shows the overwhelming importance of the $z$-dependence of the redshift factor.

fulfilled, and $\mu_b \neq \mu_n$. Let us notice, that a minor breaking of equality $\mu_b = \mu_n$ occurs also in the SNM of cold catalyzed matter, and results there from the discreteness of $N$ and $Z$. This has been noted in the classical paper [5].

Let us also notice that for accreted crusts $Q_j/g_j \lesssim 10^{-3}$. Therefore an approximation of the regularizing factor $f_Q$ in Eq. (22) by $1 + \sum_i Q_i/g_i \Theta(P > P_j^+)$ is precise within $10^{-5}$.

4 Discussion and conclusion

We reconsidered the simplest version of the single nucleus model for fully accreted NS crust, with reactions induced by matter compression, acting on reaction surfaces determined by the electron
capture threshold. As we have shown, this simplest SNM does not violate neither the general relativistic equations of hydrostatic equilibrium, nor diffusive equilibrium of unbound neutrons. It fulfils a generalized constancy condition $e^{Φ}g = \text{constant}$ where $g$ is $f_Q \cdot g$ with regularizing factor $f_Q$ calculated from the EOS with discontinuities. We considered also a more realistic SNM with finite thickness of the reaction layer [20], and we have found diffusive equilibrium of the unbound neutrons within this layer.

The simplest SNM of accreted crust looks very crude, and it does not represent details of the deep crust heating. The step-like cumulated (integrated) heating looks very schematic. However, it is sufficient to model the thermal state of NS in LMXB in quiescence.

Some extentions of the SNM are to be made. This includes modeling of the finite thickness of the reaction layers [19], with possible overlapping of them, and inclusion of the temperature effects, that allow for the sub-threshold electron captures [20, 19]. We do not think that these extensions of the SNM will affect the integrated heat and therefore the predicted thermal state of NS in LMXB in quiescence.

We stress that pairing of the bound protons and neutrons within the clusters, as well as the proton shell effects are crucial to get a correct $Q_{\text{tot}} = \int_{\text{crust}} drdQ/dr = \int_{\text{crust}} dPdQ/dP$. Without these effects only pycnonuclear fusion could generate heat. Then the many-body model is the Extended Thomas-Fermi one [4], and $Q_{\text{tot}}$ decreases to one third ($\sim 0.5$ MeV) of the actual value [4]. While pairing for bound nucleons is so important, the effect of the superfluidity of unbound neutrons still remains to be considered.

**Acknowledgements** This work was supported in part by the National Science Centre, Poland, grant 2018/29/B/ST9/02013.

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