BLOCK TRIANGULAR MINIVERSAL DEFORMATIONS OF MATRICES AND MATRIX PENCILS

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Abstract. For each square complex matrix, V. I. Arnold constructed a normal form with the minimal number of parameters to which a family of all matrices $B$ that are close enough to this matrix can be reduced by similarity transformations that smoothly depend on the entries of $B$. Analogous normal forms were also constructed for families of complex matrix pencils by A. Edelman, E. Elmroth, and B. Kågström, and contragredient matrix pencils (i.e., of matrix pairs up to transformations $(A, B) \to (S^{-1}AR, R^{-1}BS)$) by M. I. Garcia-Planas and V. V. Sergeichuk. In this paper we give other normal forms for families of matrices, matrix pencils, and contragredient matrix pencils; our normal forms are block triangular.

Key words. canonical forms, matrix pencils, versal deformations, perturbation theory

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1. Introduction. The reduction of a matrix to its Jordan form is an unstable operation: both the Jordan form and the reduction transformations depend discontinuously on the entries of the original matrix. Therefore, if the entries of a matrix are known only approximately, then it is unwise to reduce it to Jordan form. Furthermore, when investigating a family of matrices smoothly depending on parameters, then although each individual matrix can be reduced to its Jordan form, it is unwise to do so since in such an operation the smoothness relative to the parameters is lost.

For these reasons, Arnold [1] constructed a miniversal deformation of any Jordan canonical matrix $J$; that is, a family of matrices in a neighborhood of $J$ with the minimal number of parameters, to which all matrices $M$ close to $J$ can be reduced by similarity transformations that smoothly depend on the entries of $M$ (see Definition 2.1).

Miniversal deformations were also constructed for:

(i) the Kronecker canonical form of complex matrix pencils by Edelman, Elmroth, and Kågström [9]; another miniversal deformation (which is simple in the sense of Definition 2.2) was constructed by Garcia-Planas and Sergeichuk [10];

(ii) the Dobrovol’skaya and Ponomarev canonical form of complex contragredient matrix pencils (i.e., of matrices of counter linear operators $U \rightleftharpoons V$) in [10].

Belitskii [4] proved that each Jordan canonical matrix $J$ is permutationally similar to some matrix $J^\#$, which is called a Weyr canonical matrix and possesses the property: all matrices that commute with $J^\#$ are block triangular. Due to this property, $J^\#$ plays a central role in Belitskii’s algorithm for reducing the matrices of any system of linear mappings to canonical form, see [5, 11].

In this paper, we find another property of Weyr canonical matrices: they possess block triangular miniversal deformations (in the sense of Definition 2.2). Therefore, if we consider, up to smooth similarity transformations, a family of matrices that are
close enough to a given square matrix, then we can take it in its Weyr canonical form $J^\#$ and the family in the form $J^\# + E$, in which $E$ is block triangular.

We also give block triangular miniversal deformations of those canonical forms of pencils and contragredient pencils that are obtained from (i) and (ii) by replacing the Jordan canonical matrices with the Weyr canonical matrices.

All matrices that we consider are complex matrices.

2. Miniversal deformations of matrices.

**Definition 2.1** (see [1 2 3]). A deformation of an $n$-by-$n$ matrix $A$ is a matrix function $\mathcal{A}(\alpha_1, \ldots, \alpha_k)$ (its arguments $\alpha_1, \ldots, \alpha_k$ are called parameters) on a neighborhood of $\vec{0} = (0, \ldots, 0)$ that is holomorphic at $\vec{0}$ and equals $A$ at $\vec{0}$. Two deformations of $A$ are identified if they coincide on a neighborhood of $\vec{0}$.

A deformation $\mathcal{A}(\alpha_1, \ldots, \alpha_k)$ of $A$ is versal if all matrices $A + E$ in some neighborhood of $A$ reduce to the form

$$\mathcal{A}(h_1(E), \ldots, h_k(E)) = S(E)^{-1}(A + E)S(E), \quad S(0) = I_n,$$

in which $S(E)$ is a holomorphic at zero matrix function of the entries of $E$.

A versal deformation with the minimal number of parameters is called miniversal.

**Definition 2.2.** Let a deformation $\mathcal{A}$ of $A$ be represented in the form $A + B(\alpha_1, \ldots, \alpha_k)$.

- If $k$ entries of $B(\alpha_1, \ldots, \alpha_k)$ are the independent parameters $\alpha_1, \ldots, \alpha_k$ and the others are zero then the deformation $\mathcal{A}$ is called simple.

- A simple deformation is block triangular with respect to some partition of $A$ into blocks if $B(\alpha_1, \ldots, \alpha_k)$ is block triangular with respect to the conformal partition and each of its blocks is either 0 or all of its entries are independent parameters.

If $\mathcal{A}(\alpha_1, \ldots, \alpha_k)$ is a miniversal deformation of $A$ and $S^{-1}AS = B$ for some nonsingular $S$, then $S^{-1}\mathcal{A}(\alpha_1, \ldots, \alpha_k)S$ is a miniversal deformation of $B$. Therefore, it suffices to construct miniversal deformations of canonical matrices for similarity.

Let

$$J(\lambda) := J_{n_1}(\lambda) \oplus \cdots \oplus J_{n_t}(\lambda), \quad n_1 \geq n_2 \geq \cdots \geq n_t, \quad (2.1)$$

be a Jordan canonical matrix with a single eigenvalue equal to $\lambda$; the unites of Jordan blocks are written over the diagonal:

$$J_{n_i}(\lambda) := \begin{bmatrix} \lambda & 1 & 0 \\ & \ddots & \ddots \\ 0 & & \lambda \end{bmatrix} (n_i\text{-by-}n_i).$$

Arnold’s miniversal definitions presented in Theorem 2.3 are simple. Moreover, by [10 Corollary 2.1] the set of matrices of any quiver representation (i.e., of any finite system of linear mappings) over $\mathbb{C}$ or $\mathbb{R}$ possesses a simple miniversal deformation.
For each natural numbers \( p \) and \( q \), define the \( p \times q \) matrix

\[
\mathcal{T}_{pq} := \begin{cases} 
\ast \ 0 \ \ldots \ 0 & \text{if } p < q, \\
\vdots & \\
\ast \ 0 \ \ldots \ 0 & \\
0 \ \ldots \ 0 & \text{if } p \geq q,
\end{cases}
\]

in which the stars denote independent parameters (alternatively, we may take \( \mathcal{T}_{pq} \) with \( p = q \) as in the case \( p < q \)).

**Theorem 2.3** ([3, §30, Theorem 2]). (i) Let \( J(\lambda) \) be a Jordan canonical matrix of the form (2.1) with a single eigenvalue equal to \( \lambda \). Let \( \mathcal{H} := [\mathcal{T}_{n_i,n_j}] \) be the parameter block matrix partitioned conformally to \( J(\lambda) \) with the blocks \( \mathcal{T}_{n_i,n_j} \) defined in (2.2). Then

\[
J(\lambda) + \mathcal{H}
\]

is a simple miniversal deformation of \( J(\lambda) \).

(ii) Let

\[
J := J(\lambda_1) \oplus \cdots \oplus J(\lambda_r), \quad \lambda_i \neq \lambda_j \text{ if } i \neq j,
\]

be a Jordan canonical matrix in which every \( J(\lambda_i) \) is of the form (2.1), and let \( J(\lambda_i) + \mathcal{H}_i \) be its miniversal deformation (2.3). Then

\[
J + \mathcal{K} := (J(\lambda_1) + \mathcal{H}_1) \oplus \cdots \oplus (J(\lambda_r) + \mathcal{H}_r)
\]

is a simple miniversal deformation of \( J \).

**Definition 2.4** ([13]). The Weyr canonical form \( J^\# \) of a Jordan canonical matrix \( J \) (and of any matrix that is similar to \( J \)) is defined as follows.

(i) If \( J \) has a single eigenvalue, then we write it in the form (2.1). Permute the first columns of \( J_{n_1}(\lambda) \), \( J_{n_2}(\lambda) \), \ldots, and \( J_{n_k}(\lambda) \) into the first \( l \) columns, then permute the corresponding rows. Next permute the second columns of all blocks of size at least \( 2 \times 2 \) into the next columns and permute the corresponding rows; and so on. The obtained matrix is the Weyr canonical form \( J(\lambda)^\# \) of \( J(\lambda) \).

(ii) If \( J \) has distinct eigenvalues, then we write it in the form (2.4). The Weyr canonical form of \( J \) is

\[
J^\# := J(\lambda_1)^\# \oplus \cdots \oplus J(\lambda_r)^\#.
\]

Each direct summand of (2.4) has the form

\[
J(\lambda)^\# = \begin{bmatrix} \lambda I_{s_1} & I_{s_2} & 0 \\ 0 & \lambda I_{s_2} & \ddots \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 0 & \lambda I_{s_k} \end{bmatrix},
\]
in which \( s_i \) is the number of Jordan blocks \( J_i(\lambda) \) of size \( l \geq i \) in \( J(\lambda) \). The sequence \((s_1, s_2, \ldots, s_k)\) is called the Weyr characteristic of \( J \) (and of any matrix that is similar to \( J \)) for the eigenvalue \( \lambda \), see [12]. By [4] or [11, Theorem 1.2], all matrices commuting with \( J^\# \) are block triangular.

In the next lemma we construct a miniversal deformation of \( J^\# \) that is block triangular with respect to the most coarse partition of \( J^\# \) for which all diagonal blocks have the form \( \lambda I \) and each off-diagonal block is 0 or \( I \). This means that the sizes of diagonal blocks of (2.7) with respect to this partition form the sequence obtained from

\[
\begin{align*}
& s_k, s_k-1 - s_k, \ldots, s_2 - s_3, s_1 - s_2, \\
& s_k, s_k-1 - s_k, \ldots, s_2 - s_3, \\
& \ldots \ldots \ldots \\
& s_k, s_k-1 - s_k, \\
& s_k 
\end{align*}
\]

by removing the zero members.

**Theorem 2.5.** (i) Let \( J(\lambda) \) be a Jordan canonical matrix of the form (2.1) with a single eigenvalue equal to \( \lambda \). Let \( J(\lambda) + \mathcal{H} \) be its miniversal deformation (2.3). Denote by

\[
J(\lambda)^\# + \mathcal{H}^\# \tag{2.8}
\]

the parameter matrix obtained from \( J(\lambda) + \mathcal{H} \) by the permutations described in Definition 2.4(i). Then \( J(\lambda)^\# + \mathcal{H}^\# \) is a miniversal deformation of \( J(\lambda)^\# \) and its matrix \( \mathcal{H}^\# \) is lower block triangular.

(ii) Let \( J \) be a Jordan canonical matrix represented in the form (2.4) and let \( J^\# \) be its Weyr canonical form. Let us apply the permutations described in (i) to each of the direct summands of miniversal deformation (2.8) of \( J \). Then the obtained matrix

\[
J^\# + K^\# := (J(\lambda_1)^\# + \mathcal{H}_1^\#) \oplus \cdots \oplus (J(\lambda_t)^\# + \mathcal{H}_t^\#) \tag{2.9}
\]

is a miniversal deformation of \( J^\# \), which is simple and block triangular (in the sense of Definition 2.2).

Let us prove this theorem. The form of \( J(\lambda)^\# + \mathcal{H}^\# \) and the block triangularity of \( \mathcal{H}^\# \) become clearer if we carry out the permutations from Definition 2.4(i) in two steps.

**First step.** Let us write the sequence \( n_1, n_2, \ldots, n_t \) from (2.1) in the form

\[
\underbrace{m_1, \ldots, m_1}_{r_1 \text{ times}}, \underbrace{m_2, \ldots, m_2}_{r_2 \text{ times}}, \ldots, \underbrace{m_t, \ldots, m_t}_{r_t \text{ times}}
\]

where

\[
m_1 > m_2 > \cdots > m_t. \tag{2.10}
\]

Partition \( J(\lambda) \) into \( t \) horizontal and \( t \) vertical strips of sizes

\[
r_1 m_1, r_2 m_2, \ldots, r_t m_t
\]

(each of them contains Jordan blocks of the same size), produce the described permutations within each of these strips, and obtain

\[
J(\lambda)^+ := J_{m_1}(\lambda I_{r_1}) \oplus \cdots \oplus J_{m_t}(\lambda I_{r_t}), \tag{2.11}
\]

\[
J(\lambda)^\# + \mathcal{H}^\# := (J_{m_1}(\lambda I_{r_1}) + \mathcal{H}_{1, r_1}^\#) \oplus \cdots \oplus (J_{m_t}(\lambda I_{r_t}) + \mathcal{H}_{t, r_t}^\#). \tag{2.12}
\]

By (2.11), (2.12), and Theorem 2.5(i), \( J(\lambda)^\# + \mathcal{H}^\# \) is a miniversal deformation of \( J^\# \) and its matrix \( \mathcal{H}^\# \) is lower block triangular.
in which
\[
J_m(\lambda I_{r_i}) := \begin{bmatrix}
\lambda I_{r_i} & I_{r_i} & 0 \\
\lambda I_{r_i} & \cdots & \\
0 & \cdots & \lambda I_{r_i}
\end{bmatrix}
\] (m_i diagonal blocks).

By the same permutations of rows and columns of \(J(\lambda) + \mathcal{H}\), reduce \(\mathcal{H}\) to
\[
\mathcal{H}^+ := [\tilde{T}_{m_1,m_2}(r_i,r_j)],
\]
in which every \(\tilde{T}_{m_1,m_2}(r_i,r_j)\) is obtained from the matrix \(T_{m_1,m_2}\) defined in (2.2) by replacing each entry 0 with the \(r_i \times r_j\) zero block and each entry * with the \(r_i \times r_j\) block
\[
* := \begin{bmatrix}
* & \cdots & * \\
\vdots & \ddots & \vdots \\
* & \cdots & *
\end{bmatrix}
\] (2.12)

For example, if
\[
J(\lambda) = J_4(\lambda) \oplus \cdots \oplus J_4(\lambda) \oplus J_2(\lambda) \oplus \cdots \oplus J_2(\lambda)
\]
p times q times
(2.13)

then
\[
J(\lambda)^+ = J_4(\lambda p) \oplus J_2(\lambda q)
\]
(2.14)

A strip is indexed by \((i, j)\) if it contains the \(j\)-th strip of \(J_m(\lambda I_{r_i})\). Correspondingly,
\[
\mathcal{H}^+ = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & *
\end{bmatrix}
\] (2.15)

Second step. We permute in \(J(\lambda)^+\) the first vertical strips of
\[
J_{m_1}(\lambda I_{r_1}), J_{m_2}(\lambda I_{r_2}), \ldots, J_{m_t}(\lambda I_{r_t})
\]
into the first \(t\) vertical strips and permute the corresponding horizontal strips, then permute the second vertical strips into the next vertical strips and permute the corresponding horizontal strips; continue the process until \(J(\lambda)^\#\) is achieved. The same permutations transform \(\mathcal{H}^+\) to \(\mathcal{H}^\#\).
which is is ordered lexicographically. Rearranging the pairs by the columns of (2.18):

\[
J(\lambda)^\# = \begin{bmatrix}
(1,1) & (1,2) & (1,3) & (1,4) \\
\lambda I_p & 0 & I_p & 0 \\
0 & \lambda I_q & 0 & I_q \\
0 & 0 & \lambda I_p & 0 \\
0 & 0 & 0 & \lambda I_p \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

and

\[
H^\# = \begin{bmatrix}
(1,1) & (1,2) & (1,3) & (1,4) \\
0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & * & * & * \\
* & * & * & * \\
\end{bmatrix}
\]

Proof of Theorem 2.7. (i) Following (2.14), we index the vertical (horizontal) strips of \(J(\lambda)^+\) in (2.11) by the pairs of natural numbers as follows: a strip is indexed by \((i, j)\) if it contains the \(j\)-th strip of \(J_{m_i}(\lambda I_{r_i})\). The pairs that index the strips of \(J(\lambda)^+\) form the sequence

\[
(1, 1), (1, 2), \ldots, (1, m_1), \ldots, (1, m_2), \ldots, (1, m_1), (2, 1), (2, 2), \ldots, (2, m_1), \ldots, (2, m_2),
\]

\[
(1, 1), (1, 2), \ldots, (t, 1); \ldots; (1, m_1), (2, m_1), \ldots, (t, m_1); \ldots; (1, m_1)
\]

(i.e., as in lexicographic ordering but starting from the second elements of the pairs) and making the same permutation of the corresponding strips in \(J(\lambda)^+\) and \(H^+\), we obtain \(J(\lambda)^\#\) and \(H^\#\); see examples (2.16) and (2.17).

The \(((i, j), (i', j'))\)-th entry of \(H^+\) is a star if and only if either \(i \leq i'\) and \(j = m_i\), or \(i > i'\) and \(j' = 1\).

By (2.14), in these cases \(j \geq j'\) and if \(j = j'\) then either \(j = j' = m_i\) and \(i = i'\), or \(j = j' = 1\) and \(i > i'\). Therefore, \(H^\#\) is lower block triangular.

(ii) This statement follows from (i) and Theorem 2.8(ii).

Remark 2.6. Let \(J(\lambda)\) be a Jordan matrix with a single eigenvalue, let \(m_1 > m_2 > \cdots > m_t\) be the distinct sizes of its Jordan blocks, and let \(r_i\) be the number of Jordan blocks of size \(m_i\). Then the deformation \(J(\lambda)^\# + H^\#\) from Theorem 2.5 can be formally constructed as follows:

- \(J(\lambda)^\#\) and \(H^\#\) are matrices of the same size; they are conformally partitioned into horizontal and vertical strips, which are indexed by the pairs (2.19).
- The \(((i, j), (i, j))\)-th diagonal block of \(J(\lambda)^\#\) is \(\lambda I_{r_i}\), its \(((i, j), (i, j + 1))\)-th block is \(I_{r_i}\), and its other blocks are zero.
- The \(((i, j), (i', j'))\)-th block of \(H^+\) has the form (2.22) if and only if (2.20) holds; its other blocks are zero.
3. Miniversal deformations of matrix pencils. By Kronecker’s theorem on matrix pencils (see [6, Sect. XII, §4]), each pair of $m \times n$ matrices reduces by equivalence transformations

$$(A, B) \mapsto (S^{-1}AR, S^{-1}BR), \quad S \text{ and } R \text{ are nonsingular},$$

to a Kronecker canonical pair $(A_{kr}, B_{kr})$ being a direct sum, uniquely determined up to permutation of summands, of pairs of the form

$$(I_r, J_r(\lambda)), \quad (J_r(0), I_r), \quad (F_r, G_r), \quad (F_r^T, G_r^T),$$

in which $\lambda \in \mathbb{C}$ and

$$F_r := \begin{bmatrix} 1 & 0 \\ 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & 1 \\ 0 & 0 \end{bmatrix}, \quad G_r := \begin{bmatrix} 0 & 0 \\ 1 & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 1 \end{bmatrix} \quad (3.1)$$

are matrices of size $r \times (r-1)$ with $r \geq 1$.

Definitions 2.1 and 2.2 are extended to matrix pairs in a natural way.

Miniversal deformations of $(A_{kr}, B_{kr})$ were obtained in [9, 10]. The deformation obtained in [10] is simple; in this section we reduce it to block triangular form by permutations of rows and columns. For this purpose, we replace in $(A_{kr}, B_{kr})$

- the direct sum $(I, J)$ of all pairs of the form $(I_r, J_r(\lambda))$ by the pair $(I, J^\#)$, and
- the direct sum $(J(0), I)$ of all pairs of the form $(J_r(0), I_r)$ by the pair $(J(0)^\#, I)$,

in which $J^\#$ and $J(0)^\#$ are the Weyr matrices from Definition 2.4. We obtain a canonical matrix pair of the form

$$\bigoplus_{i=1}^{l} (F_{p_i}^T, G_{p_i}^T) \oplus (I, J^\#) \oplus (J(0)^\#, I) \oplus \bigoplus_{i=1}^{r} (F_{q_i}, G_{q_i}); \quad (3.2)$$

in which we suppose that

$$p_1 \leq \ldots \leq p_l, \quad q_1 \geq \ldots \geq q_r. \quad (3.3)$$

(This special ordering of direct summands of (3.2) admits to construct its miniversal deformation that is block triangular.)

Denote by

$$0^\uparrow := \begin{bmatrix} * & \cdots & * \\ 0 \end{bmatrix}, \quad 0^\downarrow := \begin{bmatrix} 0 & \cdots & * \\ * \end{bmatrix}, \quad 0^\leftarrow := \begin{bmatrix} * \\ \vdots \\ 0 \end{bmatrix}, \quad 0^\rightarrow := \begin{bmatrix} 0 \\ * \end{bmatrix}$$

the matrices, in which the entries of the first row, the last row, the first column, and the last column, respectively, are stars and the other entries are zero, and write

$$Z := \begin{bmatrix} * & \cdots & * & 0 & \cdots & 0 \\ 0 & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$
(the number of zeros in the first row of $Z$ is equal to the number of rows). The stars denote independent parameters.

In the following theorem we give a simple miniversal deformation of (3.2) that is block triangular with respect to the partition of (3.2) in which $J^#$ and $J(0)^#$ are partitioned as in Theorem 2.5 and all blocks of $(F^T_{p_i}, G^T_{p_i})$ and $(F_{q_i}, G_{q_i})$ are 1-by-1.

**Theorem 3.1.** Let $(A, B)$ be a canonical matrix pair of the form (3.2), satisfying (3.3). One of the block triangular simple miniversal deformations of $(A, B)$ has the form $(A, B)$, in which

\[
A := \begin{bmatrix}
F^T_{p_1} & F^T_{p_2} & \cdots & 0 \\
0 & 0 & \cdots & F^T_{p_i} \\
0 & 0 & \cdots & I \\
0 & 0 & \cdots & J(0)^# + H^# \\
0 & 0 & \cdots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}
\]

and

\[
B := \begin{bmatrix}
G^T_{p_1} & Z^T & G^T_{p_2} & \cdots & \cdots & \cdots & G^T_{p_i} \\
Z^T & \cdots & Z^T & G^T_{p_i} & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & \cdots & \cdots & J^# + K^# \\
0 & 0 & \cdots & I
\end{bmatrix}
\]

where $J(0)^# + H^#$ and $J^# + K^#$ are the block triangular miniversal deformations (2.8) and (2.9).

**Proof.** The following miniversal deformation of matrix pairs was obtained in [10].

The matrix pair (3.2) is equivalent to its Kronecker canonical form

\[
(A_{kr}, B_{kr}) := \bigoplus_{i=1}^r (F_{q_i}, G_{q_i}) \oplus (I, J) \oplus (J(0), I) \oplus \bigoplus_{i=1}^l (F^T_{p_i}, G^T_{p_i}).
\]

By [10] Theorem 4.1], one of the simple miniversal deformations of $(A_{kr}, B_{kr})$ has the
form \((\mathcal{A}_{kr}, \mathcal{B}_{kr})\), in which

\[
\mathcal{A}_{kr} := \begin{bmatrix}
F_{q_r} & 0 & \cdots & 0 \\
F_{q_{r-1}} & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & F_1 & I
\end{bmatrix} + \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
J(0) + \mathcal{H} & \mathcal{J} & \cdots & 0
\end{bmatrix}
\]

\[
\mathcal{B}_{kr} := \begin{bmatrix}
G_{q_r} & Z & \cdots & Z \\
G_{q_{r-1}} & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & G_{q_1} & 0
\end{bmatrix}
\]

In view of Theorem 2.5 the deformation \((\mathcal{A}_{kr}, \mathcal{B}_{kr})\) is permutationally equivalent to the deformation \((\mathcal{A}, \mathcal{B})\) from Theorem 3.1. (The blocks \(\mathcal{H}\) and \(\mathcal{K}\) in \((\mathcal{A}_{kr}, \mathcal{B}_{kr})\) are lower block triangular; because of this we reduce \((\mathcal{A}_{kr}, \mathcal{B}_{kr})\) to \((\mathcal{A}, \mathcal{B})\), which is lower block triangular.)

**Remark 3.2.** Constructing \(J(\lambda)\#\), we for each \(r\) join all \(r\)-by-\(r\) Jordan blocks \(J_r(\lambda)\) of \(J(\lambda)\) in \(J_r(\lambda I)\); see (2.11). We can join analogously pairs of equal sizes in (3.2) and obtain a pair of the form

\[
\bigoplus_{i=1}^{r'} (\hat{F}_{p_i}^T, \hat{G}_{p_i}^T) \oplus (J(0)^#, I) \oplus \bigoplus_{i=1}^{r'} (\hat{F}_{q_i}^T, \hat{G}_{q_i}^T),
\]

in which \(p'_1 < \cdots < p'_r\) and \(q'_1 < \cdots < q'_r\). This pair is permutationally equivalent to (3.2). Producing the same permutations of rows and columns in (3.2) and (3.3), we join all \(F_p^T, G_p^T, F_q^T, G_q^T, F_{q'}^T, G_{q'}^T, 0, 0^+, 0^-, 0^-\), \(\hat{Z}\) in \(\hat{0}, \hat{0}^+, \hat{0}^-, \hat{0}^+, \hat{Z}\) which consist of blocks 0 and \(\ast\) defined in (2.12); the obtaining pair is a block triangular miniversal deformation of (3.6).

4. **Miniversal deformations of contragredient matrix pencils.** Each pair of \(m \times n\) and \(n \times m\) matrices reduces by transformations of contragredient equivalence

\[
(A, B) \mapsto (S^{-1} AR, R^{-1} BS), \quad S \text{ and } R \text{ are nonsingular,}
\]
to the Dobrovol’skaya and Ponomarev canonical form \([7]\) (see also \([8]\)) being a direct sum, uniquely determined up to permutation of summands, of pairs of the form

\[(I_r, J_r(\lambda)), \ (J_r(0), I_r), \ (F_r, G^T_r), \ (F^T_r, G_r), \quad (4.1)\]

in which \(\lambda \in \mathbb{C}\) and the matrices \(F_r\) and \(G_r\) are defined in \((3.1)\).

For each matrix \(M\), define the matrices

\[M_\Delta := \begin{bmatrix} 0 & \cdots & 0 \\ M \end{bmatrix}, \quad M_\triangledown := \begin{bmatrix} M \\ 0 \end{bmatrix}\]

that are obtained by adding the zero row to the top and the zero column to the right, respectively. Each block matrix whose blocks have the form \(T_\Delta\) (in which \(T\) is defined in \((2.2)\)) is denoted by \(H_\Delta\). Each block matrix whose blocks have the form \(T_\triangledown\) is denoted by \(H_\triangledown\).

**Theorem 4.1.** Let

\[(I, J) \oplus (A, B) \quad (4.2)\]

be a canonical matrix pair for contragredient equivalence, in which \(J\) is a nonsingular Jordan canonical matrix,

\[(A, B) := \bigoplus_{i=1}^{l} (F_{p_i}, G_{p_i}^T) \oplus (I, J(0)) \oplus (J'(0), I) \oplus \bigoplus_{i=1}^{r} (F_{q_i}^T, G_{q_i}),\]

\(J(0)\) and \(J'(0)\) are Jordan matrices with the single eigenvalue 0, and

\[p_1 \geq p_2 \geq \ldots \geq p_l, \quad q_1 \leq q_2 \leq \ldots \leq q_r.\]

Then one of the simple miniversal deformations of \((4.2)\) has the form

\[(I, J + \mathcal{K}) \oplus (A, B), \quad (4.3)\]

in which \(J + \mathcal{K}\) is the deformation \((2.5)\) of \(J\) and \((A, B)\) is the following deformation of \((A, B)\):

\[
\begin{bmatrix}
F_{p_1} & T & \cdots & T \\
F_{p_2} & \ddots & \ddots & \ddots \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots \\
& & & & \ddots \\
& & & & & \ddots \\
0 & & & & & & I \\
& & & & & & \mathcal{H} \\
& & & & & & \mathcal{H} \\
& & & & & & \mathcal{H} \\
& & & & & & \mathcal{H} \\
& & & & & & \mathcal{H} \\
& & & & & & \mathcal{H} \\
& & & & & & \mathcal{H} \\
& & & & & & \mathcal{H} \\
& & & & & & \mathcal{H} \\
& & & & & & \mathcal{H} \\
& & & & & & \mathcal{H} \\
& & & & & & \mathcal{H} \\
\end{bmatrix}
\]
and

\[
B := \begin{bmatrix}
G_{p_1}^T + \mathcal{T} \\
\mathcal{T} & \ldots & \ldots \\
\vdots & \ddots & \vdots \\
\mathcal{T} & \ldots & G_{p_k}^T + \mathcal{T}
\end{bmatrix}
\begin{bmatrix}
J(0) + \mathcal{H} \\
\mathcal{H} \\
\mathcal{H}_\Delta
\end{bmatrix}
\begin{bmatrix}
F_{q_1} + \mathcal{T} \\
\mathcal{T} & \ldots & \ldots \\
\vdots & \ddots & \vdots \\
\mathcal{T} & \ldots & F_{q_r} + \mathcal{T}
\end{bmatrix},
\]

Proof. The following simple miniversal deformation of (4.2) was obtained in Theorem 5.1: up to obvious permutations of strips, it has the form

\[(I, J + \mathcal{K}) \oplus (\mathcal{A}', \mathcal{B}'),\]

in which \(J + \mathcal{K}\) is (2.5),

\[
\mathcal{A}' := \begin{bmatrix}
F_{p_1} + \mathcal{T} & \mathcal{T} & \ldots & \mathcal{T} \\
\mathcal{T} & \mathcal{T} & \ldots & \mathcal{T} \\
\vdots & \ddots & \ddots & \vdots \\
\mathcal{T} & \ldots & \mathcal{T} & G_{q_1}^T \\
0 & 0 & \mathcal{H} & G_{q_1}^T \\
0 & 0 & 0 & G_{q_2}^T \\
0 & 0 & \mathcal{H} & \ldots & \mathcal{T} \\
0 & 0 & \mathcal{H} & \ldots & G_{q_r}^T
\end{bmatrix},
\]

and

\[
\mathcal{B}' := \begin{bmatrix}
G_{p_1} & \ldots & \ldots & \mathcal{H} \\
\mathcal{T} & \ldots & \ldots & \mathcal{H} \\
\vdots & \ddots & \vdots & \vdots \\
\mathcal{T} & \ldots & \mathcal{T} & \mathcal{H}
\end{bmatrix}
\begin{bmatrix}
0 \\
\mathcal{H} \\
\mathcal{H}_\Delta
\end{bmatrix}
\begin{bmatrix}
F_{q_1} + \mathcal{T} \\
\mathcal{T} & \ldots & \ldots \\
\vdots & \ddots & \vdots \\
\mathcal{T} & \ldots & F_{q_r} + \mathcal{T}
\end{bmatrix},
\]

Let \((C, D)\) be the canonical pair (4.2), and let \((\mathcal{P}, \mathcal{Q})\) be any matrix pair of the same size in which each entry is 0 or *. By Theorem 2.1, see also the beginning of
the proof of Theorem 5.1 in [10]. \((C + \mathcal{P}, D + Q)\) is a versal (respectively, miniversal) deformation of \((C, D)\) if and only if for every pair \((M, N)\) of size of \((C, D)\) there exist square matrices \(S\) and \(R\) and a pair (respectively, a unique pair) \((P, Q)\) obtained from \((\mathcal{P}, Q)\) by replacing its stars with complex numbers such that
\[
(M, N) + (CR - SC, DS - RD) = (P, Q).
\] (4.5)

The matrices of \((C, D)\) are block diagonal:
\[
C = C_1 \oplus C_2 \oplus \ldots \oplus C_t, \quad D = D_1 \oplus D_2 \oplus \ldots \oplus D_t,
\]
in which \((C_i, D_i)\) are of the form \([4.1]\). Partitioning conformally the matrices of \((M, N)\) and \((\mathcal{P}, Q)\) and equating the corresponding blocks in \([4.5]\), we find that \((C + \mathcal{P}, D + Q)\) is a versal deformation of \((C, D)\) if and only if

for each pair of indices \((i, j)\) and every pair \((M_{ij}, N_{ij})\) of the size of \((P_{ij}, Q_{ij})\) there exist matrices \(S_{ij}\) and \(R_{ij}\) and a pair \((P_{ij}, Q_{ij})\) obtained from \((P_{ij}, Q_{ij})\) by replacing its stars with complex numbers such that
\[
(M_{ij}, N_{ij}) + (C_iR_{ij} - S_{ij}C_j, D_iS_{ij} - R_{ij}D_j) = (P_{ij}, Q_{ij}).
\] (4.6)

Let \((C + \mathcal{P}', D + Q')\) be the deformation \([4.4]\) of \((C, D)\). Since it is versal, for each pair of indices \((i, j)\) and every pair \((M_{ij}, N_{ij})\) of the size of \((P_{ij}', Q_{ij}')\) there exist matrices \(S_{ij}\) and \(R_{ij}\) and a pair \((P_{ij}', Q_{ij}')\) obtained from \((P_{ij}', Q_{ij}')\) by replacing its stars with complex numbers such that
\[
(M_{ij}, N_{ij}) + (C_iR_{ij} - S_{ij}C_j, D_iS_{ij} - R_{ij}D_j) = (P_{ij}', Q_{ij}').
\] (4.7)

Let \((C + \mathcal{P}, D + Q)\) be the deformation \([4.3]\). In order to prove that it is versal, let us verify the condition \([4.6]\). If \((P_{ij}, Q_{ij}) = (P_{ij}', Q_{ij}')\) then \([4.6]\) holds by \([4.7]\).

Let \((P_{ij}, Q_{ij}) \neq (P_{ij}', Q_{ij}')\) for some \((i, j)\). Since the condition \([4.7]\) holds, it suffices to verify that for each \((P_{ij}', Q_{ij}')\) obtained from \((P_{ij}', Q_{ij}')\) by replacing its stars with complex numbers there exist matrices \(S\) and \(R\) and a pair \((P_{ij}, Q_{ij})\) obtained from \((P_{ij}, Q_{ij})\) by replacing its stars with complex numbers such that
\[
(P_{ij}', Q_{ij}') + (C_iR - SC_j, D_iS - RD_j) = (P_{ij}, Q_{ij}).
\] (4.8)

The following 5 cases are possible.

Case 1: \((C_i, D_i) = (F_p, G_i^T)\) and \(i = j\). Then
\[
(P_{ii}', Q_{ii}') = (T, 0) = \begin{pmatrix} \alpha_1 & \cdots & \alpha_{p-1} & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}
\]
(we denote by \(T\) any matrix obtained from \(T\) by replacing its stars with
complex numbers). Taking
\[
S := \begin{bmatrix} 0 & \cdots & 0 \\ \alpha_1 & \cdots & \alpha_{p-1} \\ \vdots & \ddots & \vdots \\ \alpha_2 & \cdots & \alpha_{p-1} \end{bmatrix}, \quad R := \begin{bmatrix} 0 & \cdots & 0 \\ \alpha_{p-1} & \cdots & 0 \\ \alpha_1 & \cdots & \alpha_{p-1} \\ \alpha_2 & \cdots & \alpha_{p-1} \end{bmatrix}
\]

in (4.3), we obtain

\[
(P_{i\alpha}, Q_{i\alpha}) = \begin{pmatrix}
0, \\
\alpha_{p-1} \\
\vdots \\
\alpha_1
\end{pmatrix} = (0, T).
\]

Case 2: \((C_1, D_1) = (F_p, G_p^T)\) and \((C_2, D_2) = (I_m, J_m(0))\). Then \((P'_{ij}, Q'_{ij}) = (0, T)\).

Taking \(S := -T_5\) and \(R := 0\) in (4.8), we obtain \((P_{ij}, Q_{ij}) = (T_1, 0)\).

Case 3: \((C_1, D_1) = (I_m, J_m(0))\) and \((C_2, D_2) = (J_n(0), I_n)\). Then \((P'_{ij}, Q'_{ij}) = (0, T)\).

Taking \(S := 0\) and \(R := T\) in (4.8), we obtain \((P_{ij}, Q_{ij}) = (T, 0)\).

Case 4: \((C_1, D_1) = (I_m, J_m(0))\) and \((C_2, D_2) = (G_q, F_q)\). Then \((P'_{ij}, Q'_{ij}) = (0, T)\).

Taking \(S := 0\) and \(R := T_b\) in (4.8), we obtain \((P_{ij}, Q_{ij}) = (T_b, 0)\).

Case 5: \((C_1, D_1) = (J_n(0), I_n)\) and \((C_2, D_2) = (F_p, G_p^T)\). Then \((P'_{ij}, Q'_{ij}) = (T, 0)\).

Taking \(S := T_b\) and \(R := 0\) in (4.8), we obtain \((P_{ij}, Q_{ij}) = (0, T_b)\).

We have proved that the deformation (4.3) is versal. It is miniversal since it has the same number of parameters as the miniversal deformation (4.4). \qed

Remark 4.2. The deformation \((I, J + K) \oplus (A, B)\) from Theorem 4.1 can be made block triangular by the following permutations of its rows and columns, which are transformations of contragredient equivalence:

- First, we reduce \((I, J + K)\) to the form \((I, J^\# + K^\#)\), in which \(J^\# + K^\#\) is defined in (2.9).
- Second, we reduce the diagonal block \(J(0) + \mathcal{H}\) in \(B\) to the form \(J(0)^\# + \mathcal{H}^\#\) (defined in (2.3)) by the permutations of rows and columns of \(B\) described in Definition 2.3. Then we make the contragredient permutations of rows and columns of \(A\).
- Finally, we reduce the diagonal block \(J'(0)^\# + \mathcal{H}^\#\) (defined in (2.3)) by the permutations of rows and columns of \(A\) described in Definition 2.3 and make the contragredient permutations of rows and columns of \(B\). The obtained deformation \(J'(0)^\# + \mathcal{H}^\#\) is lower block triangular, we make it upper block triangular by transformations

\[
P(J'(0)^\# + \mathcal{H}^\#)P, \quad P := \begin{pmatrix}
0 \\
\ddots \\
1 \\
1 \\
0
\end{pmatrix}
\]

(i.e., we rearrange in the inverse order the rows and columns of \(A\) that cross \(J'(0)^\# + \mathcal{H}^\#\) and make the contragredient permutations of rows and columns of \(B\)).

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