A Review of Learning Vector Quantization Classifiers

David Nova · Pablo A. Estévez

Abstract In this work we present a review of the state of the art of Learning Vector Quantization (LVQ) classifiers. A taxonomy is proposed which integrates the most relevant LVQ approaches to date. The main concepts associated with modern LVQ approaches are defined. A comparison is made among eleven LVQ classifiers using one real-world and two artificial datasets.

Keywords Learning Vector Quantization · Supervised Learning · Neural Networks · Margin Maximization · Likelihood Ratio Maximization

1 Introduction

Learning Vector Quantization (LVQ) is a family of algorithms for statistical pattern classification, which aims at learning prototypes (codebook vectors) representing class regions. The class regions are defined by hyperplanes between prototypes, yielding Voronoi partitions. In the late 80’s Teuvo Kohonen introduced the algorithm LVQ1 [36, 38], and over the years produced several variants. Since their inception LVQ algorithms have been researched by a small but active community. A search on the ISI Web of Science in November, 2013, found 665 journal articles with the keywords “Learning Vector Quantization” or “LVQ” in their titles or abstracts. This paper is a review of the progress made in the field during the last 25 years.

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LVQ algorithms are related to other competitive learning algorithms such as self-organizing maps (SOMs) [38] and c-means. Competitive learning algorithms are based on the winner-take-all learning rule, and variants in which only certain elements or neighborhoods are updated during learning. The original LVQ algorithms and most modern extensions use supervised learning for obtaining class-labeled prototypes (classifiers). However, LVQ can also be trained without labels by unsupervised learning for clustering purposes [6, 17, 32, 33, 34, 47]. In this paper we will focus our review only on LVQ classifiers.

LVQ classifiers are particularly intuitive and simple to understand because they are based on the notion of class representatives (prototypes) and class regions usually in the input space (Voronoi partitions). This is an advantage over multilayer perceptrons or support vector machines (SVMs), which are considered to be black boxes. Moreover, support vectors are extreme values (those having minimum margins) of the datasets, while LVQ prototypes are typical vectors. Another advantage of LVQ algorithms is that they are simple and fast, as a result of being based on Hebbian learning. The computational cost of LVQ algorithms depends on the number of prototypes, which are usually a fixed number. SVMs depend on the number of training samples instead, because the number of support vectors is a fraction of the size of the training set. LVQ has been shown to be a valuable alternative to SVMs [19, 27].

LVQ classifiers try to approximate the theoretical Bayesian border, and can deal directly with multi-class problems. The initial LVQ learning rules were heuristic, and showed sensitivity to initialization, slow convergence problems and instabilities. However, two main approaches have been proposed defining explicit cost functions from which to derive learning rules via steepest descent or ascent [56, 63, 64], and solving the problem of convergence of the original LVQ algorithms. The first model is a generalization of LVQ called Generalized Learning Vector Quantization (GLVQ) [56]. In GLVQ a cost function is defined in such a way that a learning rule is derived via the steepest descent. This cost function has been shown to be related to a minimization of errors, and a maximization of the margin of the classifier [7]. The second approach is called Robust Soft-LVQ (RSLVQ) [63, 64], in which a statistical objective function is used to derive a learning rule by gradient ascent. The probability density function (pdf) of the data is assumed to be a Gaussian mixture for each class. Given a data point, the logarithm of the ratio of the pdf of the correct class versus the pdf's of the incorrect classes serves as a cost function to be maximized.

Other LVQ improvement deals with the initialization sensitivity of the original LVQ algorithms and GLVQ [21, 31, 33, 54]. Recent extensions of the LVQ family of algorithms substitute the Euclidean distance with more general metric structures such as: weighted Euclidean metrics [18], adaptive relevance matrix metrics [61], pseudo-Euclidean metrics [22], and similarity measures in kernel feature space that lead to kernelized versions of LVQ [52].

There are thousands of LVQ applications such as in: image and signal processing [4, 8, 55, 40, 44, 68, 72], the biomedical field and medicine [2, 11]
In this paper we present a comprehensive review of the most relevant supervised LVQ algorithms developed since the original work of Teuvo Kohonen. We introduce a taxonomy of LVQ classifiers, and describe the main algorithms. We compare the performance of eleven LVQ algorithms empirically on artificial and real-world datasets. We discuss the advantages and limitations of each method depending on the nature of the datasets.

The remainder of this paper is organized as follows: In section 2 a taxonomy of LVQ classifiers is presented. In section 3 the main LVQ learning rules are described. In section 4 the results obtained with eleven different LVQ methods in three different datasets are shown. In section 5 some open problems are presented. Finally, in section 6 conclusions are drawn.

2 A Taxonomy of LVQ Classifiers

2.1 Learning Vector Quantization

Let $X = \{(x_i, y_i) \subset \mathbb{R}^D \times \{1, \ldots, C\} | i = 1, \ldots, N\}$ be a training data set, where $x = (x_1, \ldots, x_D) \in \mathbb{R}^D$ are $D$-dimensional input samples, with cardinality $|X| = N$; $y_i \in \{1, \ldots, C\}$ $i = 1, \ldots, N$ are the sample labels, and $C$ is the number of classes. The neural network consists of a number of prototypes, which are characterized by vectors $w_i \in \mathbb{R}^D$, for $i = 1, \ldots, M$, and their class labels $c(w_i) \in \{1, \ldots, C\}$ with $Y = \{c(w_j) \in \{1, \ldots, C\} | j = 1, \ldots, M\}$. The classification scheme is based on the best matching unit (BMU) (winner-takes-all strategy). The receptive field of prototype $w_i$ is defined as follows:

$$R^i = \{x \in X | \forall w_j (j \neq i) \rightarrow d(w_i, x) \leq d(w_j, x)\}, \quad (1)$$
where $d(w, x)$ is a distance measure. Learning aims at determining the weight vectors (prototypes), so that the training data samples are mapped to their corresponding class labels.

Fig. 1 shows a taxonomy of the most relevant LVQ classifiers developed since the pioneer work of Kohonen. LVQ methods are decomposed into 3 families: Kohonen’s LVQ methods (middle branch: heuristic), methods based on margin maximization (left branch), and methods based on likelihood ratio maximization (right branch).

The original LVQ algorithm does not have an associated cost function to ensure convergence. Some improvements such as LVQ2.1, LVQ3 or OLVQ [37] aim at achieving higher convergence speed or better approximation of the Bayesian borders. All these LVQ versions are based on Hebbian learning, i.e., are heuristic.

The original LVQ1 [36, 38] corrected only the winner prototype. This algorithm pulled the prototypes away from the class borders. LVQ1 assumes a good initial state of the network, i.e., it requires a preprocessing method. It also shows sensitivity to overlapping data sets and in it, some neurons never learn the training patterns. The LVQ2 algorithm updates two vectors at each step, the winner and the runner-up. The purpose is to estimate differentially the decision border towards the theoretical Bayes decision border. But this algorithm makes corrections that are dependent on error only, and present some instabilities. LVQ3 corrects the LVQ2 convergence problem consisting of the location of prototypes changing in continued learning by adding a stability factor.

2.2 Margin Maximization

In [12] a margin analysis of the original LVQ algorithm was performed. There are two definitions of margin. The first one is the sample-margin, which corresponds to the quantification of samples which can travel through space without changing the classification rate of the classifier. This is the definition used by SVMs. The second definition is called the hypothesis-margin, which corresponds to the quantification of the distance that the classifier (e.g. a hyperplane) can be altered without changing the classification rate. This is the definition used by Adaboost.

In the context of LVQ, the sample margin is hard to compute and numerically unstable [12]. This is because small repositionings of the prototypes might create large changes in the sample margin. Crammer et al. [12] showed that the decision borders of the original LVQ algorithms are hypothesis margin maximizers. The margin is associated with generalization error bounds. Therefore, maximizing the margin is equivalent to minimizing the generalization error. Interestingly, the bound is dimension free, but depends on the number of prototypes.

The left branch of our taxonomy shown in Fig. 1 corresponds to the LVQ methods based on a margin maximization approach. The GLVQ [56] proposed
a cost function that aims at margin maximization. This approach solves some limitations of the original LVQ algorithms such as: slow convergence, initialization sensitivity, limitations in multidimensional data where correlations exist between dimensions, just to name a few.

The GLVQ cost function is defined as follows:

\[ E_{GLVQ} = \sum_{i=1}^{N} \phi (\mu(x_i)), \]

(2)

where \( \phi(\cdot) \) is the logistic sigmoid function, and \( \mu \) is the relative distance difference

\[ \mu(x_i, W) = \frac{d^+ - d^-}{d^+ + d^-}, \]

(3)

where \( d^+ = d(x_i, w^+) \) is the Euclidean distance of data point \( x_i \) from its closest prototype \( w^+ \) having the same class label, and \( d^- = d(x_i, w^-) \) is the Euclidean distance from the closest prototype \( w^- \) having a different class label. The term \( d^+(x_i) - d^-(x_i) \) constitutes the hypothesis margin of an LVQ classifier according to the winner-takes all rule \[12, 20\]. Note that since GLVQ includes this margin in its cost function, it can be described as a margin optimization learning algorithm. Generalization bounds show that the larger the margin, the better the generalization ability \[20\]. The cost function in Eq. \[2\] has been extended to other distance metrics. In \[20\] a generalization bound is derived for Generalized Relevance LVQ (GRLVQ), which uses an adaptive metric.

2.3 Likelihood Ratio Maximization

In this section we describe in detail the cost function proposed by Robust Soft-Learning Vector Quantization (RSLVQ) \[63, 64\]. This cost function gives origin to the right branch of the taxonomy illustrated in Fig. \[1\] which is based on a Gaussian probabilistic model of data in forms of mixture models. It is assumed that the probability density function (pdf) of the data is described by a Gaussian mixture model for each class. The pdf of a sample that is generated by the Gaussian mixture model of the correct class is compared to the pdf of this sample that is generated by the Gaussian mixture models of the incorrect classes. The logarithm of the ratio between the correct mixture Gaussian model and the incorrect mixture Gaussian models of probability densities is maximized. Let \( W \) be a set of labeled prototype vectors. The probability density of the data is given by

\[ p(x|W) = \sum_{y=1}^{C} \sum_{\{j : c(w_j) = y\}} p(x|j) P(j), \]

(4)

where \( C \) is the number of classes and \( y \) is the class label of the data points generated by component \( j \). Also, \( P(j) \) is the probability that data points are
generated by component $j$ of the mixture and it can be chosen identically for each prototype $w_j$. $p(x|j)$ is the conditional pdf that the component $j$ generates a particular data point $x$ and it is a function of prototype $w_j$. The following likelihood ratio is proposed as a cost function to be maximized:

$$E_{RSLVQ} = \sum_i^N \log \left( \frac{p(x_i, y|W)}{p(x_i|W)} \right)$$

(5)

where $p(x, y|W)$ is the pdf of a data point $x$ that is generated by the mixture model for the correct class $y$, and $p(x|W)$ is the total probability density of the data point $x$. These probabilities are defined as follows:

$$p(x_i, y_i|W) = \sum_{j: c(w_j) = y} p(x_i|j) P(j)$$

(6)

$$p(x_i|W) = \sum_j p(x_i|j) P(j).$$

(7)

The conditional pdfs are assumed to be of the normalized exponential form $p(x|j) = K(j) \cdot \exp f(x, w_j, \sigma_j^2)$, with

$$f(x, w_j, \sigma_j^2) = -\frac{d(x, w)}{2\sigma^2},$$

(8)

where $d(x, w)$ is the Euclidean distance measure. Note that Eq. (8) provides a way to extend LVQ to other distance metrics by changing the distance measure $d$.

2.4 Distance Learning

The original LVQ, GLVQ and RSLVQ methods rely on the Euclidean distance. This choice of metrics assumes a spherical receptive field of prototypes, as shown in Fig. 2a. However, for heterogeneous datasets, there can be different scaling and correlations of the dimensions; and for high-dimensional datasets,
the estimate errors accumulate and can disrupt the classification \cite{61}. In distance learning the distance measure is adaptive during training, allowing us to get receptive fields for prototypes such as those shown in Figs. \ref{fig:2} and \ref{fig:2c}. In the generalized matrix LVQ (GMLVQ) \cite{60, 61} a generalized distance metric is proposed as

$$d^A(w, x) = (x - w)^T \Lambda (x - w),$$

(9)

where $\Lambda$ is a full $D \times D$ matrix. To get a valid metric $\Lambda$ must be positive (semi-) definite. This is achieved by substituting $\Lambda = \Omega^T \Omega$,

(10)

which yields $u^T \Lambda u = u^T \Omega^T \Omega u = (\Omega^T \Omega)^2 \geq 0$ for all $u$, where $\Omega \in \mathbb{R}^{D \times D}$. The receptive field of prototype $w_i$ becomes

$$R^i_A = \{x \in X | \forall w_j (j \neq i) \rightarrow d^A(w_i, x) \leq d^A(w_j, x)\}.$$ (11)

In this way arbitrary Euclidean metrics can be realized, in particular correlations of dimensions and rotations of axes can be taken into account. If $\Lambda$ is restricted to being diagonal then Eq. (9) is reduced to

$$d^\lambda(x, w) = ||x - w||^2 = \sum_{j=1}^{D} \lambda_j (x_j - w_j)^2.$$ (12)

This simplification gives origin to what are called Generalized Relevance approaches to LVQ (GRLVQ)\cite{18, 60, 65}.

2.5 Kernelization

LVQ classifiers have been kernelized \cite{52, 59}. A mapping function $\Phi(\cdot)$ is defined in order to realize a nonlinear transformation from the data space $\mathbb{R}^D$ to a higher dimensional possibly linearly separable feature space $F$, such as follows

$$\Phi : \mathbb{R}^D \rightarrow F, \ x \rightarrow \Phi(x).$$ (13)

A kernel function \cite{62}, can be represented as a dot product and is usually chosen as a Gaussian kernel,

$$k(x_i, x_j) = \Phi(x_i) \cdot \Phi(x_j) = \exp \left(-\frac{||x_i - x_j||^2}{2\sigma^2}\right).$$ (14)

The LVQ classifiers can then be applied in feature space, where the prototypes are represented by an implicit form

$$w_j^F = \sum_{m=1}^{N} \gamma_{jm} \Phi(x_m),$$ (15)
where $\gamma_j \in \mathbb{R}^N$ are the combinatorial coefficient vectors. The distance in feature space between an image $\Phi(x_i)$ and a prototype vector $w_j^F$ can be directly computed using kernels:

$$
d^F(x_i, w_j) = \|\Phi(x) - w_j^F\|^2 = \sqrt{\|\Phi(x) - \sum_{m=1}^{N} \gamma_{jm} \Phi(x_m)\|^2} = k(x_i, x_i) - 2 \sum_{m=1}^{N} \gamma_{jm} k(x_i, x_m) + \sum_{s,t=1}^{N} \gamma_{js} \gamma_{jt} k(x_s, x_t). \tag{16}
$$

This is another form of the kernel trick, in which there is no need of knowing the non-linear mapping $\Phi(\cdot)$. This approach is called kernel GLVQ (KGLVQ) [52].

2.6 Dis-/similarities

Some problems do not allow a vectorial representation, e.g. alignment of symbolic strings. In these cases the data can be represented by pairwise similarities $s_{ij} = s(x_i, x_j)$ or dissimilarities $d_{ij} = d(x_i, x_j)$ and their corresponding matrices $S$ and $D$, respectively. These matrices are symmetric, i.e. $S = S^T$ and $D = D^T$, with zero diagonals. It is easy to turn similarities into dissimilarities and vice-versa, as shown in [48]. This corresponds to what is called ‘relational data’ representation. Data represented by pairwise dissimilarities with the previously mentioned restrictions can always be embedded in a pseudo-Euclidean space [22, 23, 48]. A pseudo-Euclidean space is a vector space equipped with a symmetric bilinear form composed of two parts: An Euclidean one (positive eigenvalues), and a correction (negative eigenvalues) [48]. The bilinear form is expressed as

$$
\langle x, y \rangle_{p,q} = x^t I_{p,q} y, \tag{17}
$$

where $I_{p,q}$ is a diagonal matrix with $p$ entries 1 and $q$ entries $-1$. This pseudo-Euclidean space is characterized by the signature $(p, q, N - p - q)$, where the first $p$ components are Euclidean, the next $q$ components are non-Euclidean, and $N - p - q$ components are zeros. The prototypes are assumed to be linear combinations of data samples:

$$
w_j = \sum_m \alpha_{jm} x_m, \quad \text{with} \quad \sum_m \alpha_{jm} = 1, \tag{18}
$$

where $\alpha_j = (\alpha_{j1}, ..., \alpha_{jN})$ is the vector of coefficients, that describes the prototype $w_j$ implicitly. Unlike kernel approaches, dis-/similarities approaches do not assume that the data is Euclidean. The distances between all data points and prototypes are computed based on pairwise data similarities as follows:

$$
d^D(x_i, w_j) = \|x_i - w_j\|^2 = \|D \cdot \alpha_j\|_1 = \frac{1}{2} \alpha_j^T D \alpha_j. \tag{19}
$$
Fig. 3 Schematic of the LVQ2.1 updating rule. The circles represent the prototypes; the square represents an input sample; and the colors indicate different class labels. In this example, the sample is incorrectly classified and the nearest prototype of the correct class moves towards the sample, while the nearest prototype of the incorrect class moves away from the sample.

3 LVQ Learning Rules

3.1 LVQ 2.1

Among the initial variants proposed by Kohonen, the most popular is LVQ2.1 [37], which is described in detail below. As in LVQ2, two prototypes are updated at each step, the winner and the runner-up. One of them, \( w^+ \), belongs to the correct class, while the other, \( w^- \), belongs to the incorrect class. But now either the winner or the runner-up have the same class label as the sample. Furthermore, the current input \( x_i \) must fall within a window defined around the mid-plane of vectors \( w^+ \) and \( w^- \) (see Fig. 3). The updating rule is as follows:

\[
\begin{align*}
w^+(t+1) &= w^+(t) + \epsilon(t) \cdot (x_i - w^+(t)), \quad \text{if } c(w^+) = y \\
w^-(t+1) &= w^-(t) - \epsilon(t) \cdot (x_i - w^-(t)), \quad \text{if } c(w^-) \neq y
\end{align*}
\]

(20)

where \( w^+ (w^-) \) is the closest prototype to the input sample \( x_i \), with the same (different) class label as the sample class label \( y \), and \( \epsilon \in [0,1] \) is the learning rate. The prototypes, however, are changed only if the data point \( x \) is close to the classification boundary, i.e., if it lands within a window of relative width \( \omega \) defined by

\[
\min \left( \frac{d(x, w^-)}{d(x, w^+)} , \frac{d(x, w^+)}{d(x, w^-)} \right) < s, \quad s = \frac{1 - \omega}{1 + \omega},
\]

(21)

where \( d(\cdot) \) is the Euclidean distance measure and \( \omega \in [0,1] \) (typically set to \( \omega = 0.2 \) or \( \omega = 0.3 \)). A didactic scheme of the LVQ2.1 update learning rule is illustrated in Fig. 3.
Fig. 4 Schematic of the GLVQ learning rule which analyzes the behavior of the two closest prototypes belonging to different classes. The circles represent the prototypes; the square represents the input sample; and the colors indicate the different class labels. (a) Rule behavior when an input sample with a red class label is presented. (b) Rule behavior when an input sample with a blue class label is presented.

3.2 LVQ Methods Based on Margin Maximization

Using stochastic gradient descent to minimize Eq. (2), the following learning rules for GLVQ are obtained:

\[
\begin{align*}
w^+(t+1) &= w^+(t) + 2 \cdot \epsilon \cdot \phi'(\mu(x_i)) \cdot \mu^+ \cdot (x_i - w^+) \\
w^-(t+1) &= w^-(t) - 2 \cdot \epsilon \cdot \phi'(\mu(x_i)) \cdot \mu^- \cdot (x_i - w^-)
\end{align*}
\]

where \( \mu^+ = \frac{2 \cdot d^-}{(d^-+d^+)^2} \), \( \mu^- = \frac{2 \cdot d^+}{(d^-+d^+)^2} \), and \( \epsilon \in [0,1] \) is the learning rate. A didactic scheme of the GLVQ learning rule is shown in Fig. 4.

Early LVQ versions assumed a fixed number of codebook vectors per class and their initial values were set using an ad-hoc method. In order to overcome these problems, alternative supervised versions of Neural Gas (NG) [42] and Growing Neural Gas (GNG) [16] have been developed. These extensions have been called Supervised Neural Gas (SNG) [21] and Supervised Growing Neural Gas (SGNG) [31].

SNG adds neighborhood cooperativity to GLVQ solving the dependency on the initialization. All prototypes \( w_j \) having the same class as the current input sample \( x_i \) are adapted towards the data point according to their ranking, and the closest prototype of a different class label is moved away from the data sample. SGNG incorporates plasticity by selecting an adaptive number of prototypes during learning. Another method, the Harmonic to Minimum LVQ (H2MLVQ) [53] allows adapting all prototypes simultaneously in each iteration. All prototypes having the same class label as the current sample are updated. Likewise, all prototypes with a different label from that of the sample are adjusted too, in which \( d^- \) and \( d^+ \) in Eq. (3) are replaced by harmonic average distances [74]. For more details the reader is referred to [54].
The previously mentioned methods tackle the initialization sensitivity problem associated with the initial position of the prototypes.

Other margin maximization methods are based on Information Theoretic Learning (ITL) \[39, 66, 67\]. These methods use a new divergence measure proposed by Principe et al. \[51\], based on the Cauchy-Schwarz inequality. Together with a consistently chosen Parzen-estimator for the densities, it gives a numerically well-behaved approach of information optimization prototype-based vector quantization called Cauchy-Schwarz Divergence LVQ (CSDLVQ) \[69, 70\]. These methods use fuzzy class labels to train the classifier.

### 3.2.1 Distance Learning Approaches

In the case of GRLVQ \[13\], the relevance factors can be determined by gradient descent, as follows:

\[
\lambda_m(t + 1) = \lambda_m(t) - \epsilon_\phi \cdot (\mu^+(x_m - w^+_m)^2 - \mu^-(x_m - w^-_m)^2).
\]

The adaptation formulas are obtained by computing the derivatives with respect to \(w\) and \(\lambda\) using the following relative distance difference:

\[
\mu(x_i, W, \lambda) = \frac{d^\lambda(x_i, w^+) - d^\lambda(x_i, w^-)}{d^\lambda(x_i, w^+)}
\]

with \(\mu^+ = \frac{2d^\lambda(x_i, w^-)}{(d^\lambda(x_i, w^-) + d^\lambda(x_i, w^+))^2}\) and \(\mu^- = \frac{2d^\lambda(x_i, w^+)}{(d^\lambda(x_i, w^-) + d^\lambda(x_i, w^+))^2}\), where \(d^\lambda\) was defined in Eq. (12). The updating rules for \(w\) can be obtained from Eq. (22) by replacing \(\mu^+\) and \(\mu^-\) with those obtained from Eq. (24) above.

Analogously, the relative distance difference for GMLVQ is as follows:

\[
\mu(x_i, W, \lambda) = \frac{d^\lambda(x_i, w^+) - d^\lambda(x_i, w^-)}{d^\lambda(x_i, w^+)}
\]

The update rules for GMLVQ are obtained by minimizing the cost function defined in Eq. (2) with the generalized distance metric defined in Eq. (9). The update rules for \(w\) and \(\Omega\) are as follows:

\[
\begin{align*}
w^+(t + 1) &= w^+(t) + \epsilon \cdot 2 \cdot \phi' \cdot \mu^+ \cdot \Lambda \cdot (x - w^+(t)) \\
w^-(t + 1) &= w^-(t) - \epsilon \cdot 2 \cdot \phi' \cdot \mu^- \cdot \Lambda \cdot (x - w^-(t)),
\end{align*}
\]

and

\[
\begin{align*}
\Omega(t + 1) &= \Omega(t) - \tilde{\epsilon} \cdot 2 \cdot \phi' \\
&\cdot \left((x_m - w^+_m)[\Omega(x - w^+)]_l)
\end{align*}
\]

\[
-\mu^- \cdot ((x_m - w^-_m)[\Omega(x - w^-)]_l),
\]

where \(l\) and \(m\) specify matrix components, \(\mu^+ = \frac{2d^\lambda(x_i, w^-)}{(d^\lambda(x_i, w^-) + d^\lambda(x_i, w^+))^2}\) and \(\mu^- = \frac{2d^\lambda(x_i, w^+)}{(d^\lambda(x_i, w^-) + d^\lambda(x_i, w^+))^2}\), and the distance \(d^\lambda(\cdot)\) is described in Eq. (9).
3.2.2 Kernel Approaches

An extension of GLVQ is the Kernel GLVQ (KGLVQ) [52]. KGLVQ projects the data space non-linearly into a higher dimensional feature space, and then applies the GLVQ algorithm in this newly formed feature space. The prototypes are represented in an implicit form as shown by Eq. (15). The updating rules of the GLVQ algorithm in Eq. (22) can be generalized from the data space into the feature space \( F \) by using the following relative distance difference:

\[
\mu(\Phi(x_i), W^F) = \frac{d^F(\Phi(x_i), w^+) - d^F(\Phi(x_i), w^-)}{d^F(\Phi(x_i), w^+) + d^F(\Phi(x_i), w^-)},
\]

and its derivatives \( \mu^+ = \frac{2d^F(x_i, w^+)}{(d^F(x_i, w^+)+d^F(x_i, w^-))^2} \) and \( \mu^- = \frac{2d^F(x_i, w^-)}{(d^F(x_i, w^+)+d^F(x_i, w^-))^2} \)

where the metric \( d^F \) is defined in Eq. (16).

The equivalent updating rules in terms of adjusting parameters \( \gamma \) in Eq. (15) are:

\[
\gamma^+_s(t + 1) = \begin{cases} 
1 - \epsilon \cdot \phi' \cdot \mu^+ & \text{if } x_s = x_i, \\
1 - \epsilon \cdot \phi' \cdot \mu^+ & \text{if } x_s \neq x_i,
\end{cases}
\]

\[
\gamma^-_s(t + 1) = \begin{cases} 
1 + \epsilon \cdot \phi' \cdot \mu^- & \text{if } x_s = x_i, \\
1 + \epsilon \cdot \phi' \cdot \mu^- & \text{if } x_s \neq x_i,
\end{cases}
\]

where \( s \) specifies combinatorial coefficient vectors of \( \gamma \), \( \gamma^+ \) is the coefficient vector associated with the nearest prototype \( w^+ \) having the same class label as \( x_i \), and \( \gamma^- \) is the coefficient vector associated with the nearest prototype \( w^- \) having a different class label from \( x_i \).

Furthermore, in [59] KGLVQ is extended to get the Nystrom-Approximation Generalized LVQ (AKGLVQ), where sparsity is imposed and a Nystrom approximation technique is used to reduce the learning complexity in large data sets.

3.2.3 Dis-/similarities Approaches

In [22] a new method was proposed using dis-/similarity data by representing it in an implicit form. This approach is extended to the margin maximization technique giving origin to the relational GLVQ (RGLVQ).

An implicit representation of the prototypes is assumed as described through a model by Eq. (18). Once again the starting point is the GLVQ updating rule (Eq. 22) where the relative distance difference is now computed as

\[
\mu(x_i, W, D) = \frac{d^D(x_i, w^+)}{d^D(x_i, w^+)} - \frac{d^D(x_i, w^-)}{d^D(x_i, w^-)},
\]

where \( d^D(\cdot) \) is the distance defined in Eq. (19), and \( \mu^+ = \frac{2d^D(x_i, w^-)}{(d^D(x_i, w^+)+d^D(x_i, w^-))^2} \) and \( \mu^- = \frac{2d^D(x_i, w^+)}{(d^D(x_i, w^+)+d^D(x_i, w^-))^2} \) are the respective partial derivatives of
the relative distance difference. The $d_{ij}$ are dissimilarity matrix elements of $D \in \mathbb{R}^{N \times N}$; $\alpha_{mi}^+$ and $\alpha_{mi}^-$ are the coefficient vector elements associated with the nearest prototype $w^+$ having the same class label as $x_i$, and the nearest prototype $w^-$ having a different class label from $x_i$, respectively. The equivalent updating rules for the $\alpha$ parameters are:

$$
\alpha_{mi}^+(t+1) = \alpha_{mi}^+(t) - \epsilon \cdot \phi^+ \cdot (d_{im} - \sum_l d_{lm} \alpha_{li}^-)
$$

$$
\alpha_{mi}^-(t+1) = \alpha_{mi}^-(t) + \epsilon \cdot \phi^- \cdot (d_{im} - \sum_l d_{lm} \alpha_{li}^+).
$$

(31)

3.3 Methods Based on Likelihood Ratio Maximization

In RSLVQ [63, 64], the updating rule for the prototypes is derived from the likelihood ratio Eq. (5) by maximizing this cost function through gradient ascent,

$$
\mathbf{w}_j(t+1) = \mathbf{w}_j(t) + \epsilon(t) \frac{\partial}{\partial \mathbf{w}_j} \log \left( \frac{p(x_i, y_i|\mathbf{W})}{p(x_i|\mathbf{W})} \right).
$$

Eq. (32) can be re-written as follows:

$$
\mathbf{w}_j(t+1) = \mathbf{w}_j(t) + \epsilon \frac{1}{\sigma^2} \left\{ \begin{array}{ll}
(P_y(j|x) - P(j|x))(x - \mathbf{w}_j), & \text{if } c(\mathbf{w}_j) = y, \\
-P(j|x)(x - \mathbf{w}_j), & \text{if } c(\mathbf{w}_j) \neq y,
\end{array} \right.
$$

(33)

where $\epsilon \in [0, 1]$ is the learning rate, $y$ is the class label of the data points generated by component $j$, and $P_y(j|x)$ and $P(j|x)$ are assignment probabilities described as follows:

$$
P_y(j|x) = \frac{\sum_{i: c(\mathbf{w}_j) = y} p(i) \exp \left( f(x, \mathbf{w}_i, \sigma_i^2) \right)}{\sum_{i=1}^M p(i) \exp \left( f(x, \mathbf{w}_i, \sigma_i^2) \right)}.
$$

(34)

$$
P(j|x) = \frac{p(j) \exp \left( f(x, \mathbf{w}_j, \sigma_j^2) \right)}{\sum_{i=1}^M p(i) \exp \left( f(x, \mathbf{w}_i, \sigma_i^2) \right)}.
$$

(35)

Note that Eq. (33) provides a way to extend LVQ to other distance measures by changing $f(x, \mathbf{w}, \sigma^2)$ in Eqs. (34-35).

3.3.1 Distance Learning Approaches

The matrix learning scheme was applied to RSLVQ [61] obtaining what was named the Matrix RSLVQ (MRSLVQ) and its local version, Local MRSLVQ (LMRSLVQ). The latter consists of a local distance adaptation for each prototype, which in these cases is a full matrix of relevance factors. As it is based on RSLVQ, this approach uses the conditional probability density function $p(x|j)$ in which component $j$ generates a particular data point $x$. It is a function of prototype $\mathbf{w}_j$ and it is assumed to have a normalized exponential form
\[ p(x|j) = K(j) \cdot \exp f(x, w_j, \sigma_j^2). \] In matrix learning Eq. (8) is substituted with Eq. (36) where distance \( d^A \), defined in Eq. (9), is used:

\[ f(x, w_j, \sigma_j^2, A) = -\frac{d^A(x, w_j)}{2\sigma^2}. \quad (36) \]

The updating rules for \( w \) and \( \Omega \) are obtained in a way similar to RSLVQ but using the adaptive distance \( d^A \), as follows:

\[ w_j(t + 1) = w_j(t) + \frac{\epsilon}{\sigma^2} \left\{ \begin{array}{ll}
(P_y(j|x) - P(j|x))A(x - w_j), & \text{if } c(w_j) = y \\
-P(j|x)A(x - w_j), & \text{if } c(w_j) \neq y
\end{array} \right., \quad (37) \]

where \( y \) is the class label of the data points generated by component \( j \), and

\[ \Omega_{lm}(t + 1) = \Omega_{lm}(t) - \frac{\epsilon}{\sigma^2} \cdot \sum \left[ (\delta_{y,l}(P_y(j|x) - P(j|x)) - (1 - \delta_{y,l})P(j|x)) \cdot ([\Omega(x - w_j)]_l(x_m - w_{jm})) \right], \quad (38) \]

where \( l \) and \( m \) are matrix elements. This approach can be extended to the local matrix for every prototype as proposed in [61].

### 3.3.2 Kernel Approaches

LVQ methods based on likelihood ratio maximization have been extended to feature space by using kernels. A kernelized version of RSLVQ was proposed in [27] and called Kernel Robust Soft Learning Vector Quantization (KRSLVQ). The prototypes are represented as a linear combination of data images in the feature space as described in Eq. (15). As in RSLVQ, a Gaussian mixture model is assumed, but the Euclidean distance is measured in feature space, using the distance \( d^F \) defined in Eq. (16). Consequently Eq. (8) is substituted with

\[ f(\Phi(x), w_j^F, \sigma_j^2) = -\frac{d^F(\Phi(x), w_j^F)}{2\sigma^2}. \quad (39) \]

The updating rules of KRSLVQ are

\[ w_j(t + 1) = w_j(t) + \frac{\epsilon}{\sigma^2} \left\{ \begin{array}{ll}
-(P_y(j|\Phi(x_i)) - P(j|\Phi(x_i)))\gamma_{jm}, & \text{if } x_m \neq x_i, \ c(w_j) = y \\
(P_y(j|\Phi(x_i)) - P(j|\Phi(x_i)))(1 - \gamma_{jm}) & \text{if } x_m = x_i, \ c(w_j) = y \\
(P(j|\Phi(x_i))\gamma_{jm}) & \text{if } x_m \neq x_i, \ c(w_j) \neq y \\
-P(j|\Phi(x_i))(1 - \gamma_{jm}) & \text{if } x_m = x_i, \ c(w_j) \neq y
\end{array} \right., \quad (40) \]
3.3.3 Dis-/similarities Approaches

Dis-/similarities approaches allow extending the likelihood-ratio maximization LVQ methods to a pseudo-Euclidean space giving origin to relational RSLVQ [23]. The prototypes are represented in an implicit form by a linear combination of the data points as in Eq. (18). Remember that $D$ represents a dissimilarity matrix which is symmetric and with diagonal elements $d_{ii} = 0$. The dissimilarities are computed by using Eq. (19). In this approach the argument of the normalized exponential takes the following form:

$$f(x, w_j, \sigma^2) = -\frac{d(x, w_j)}{2\sigma^2} = -\frac{[D \cdot \alpha_j] - \frac{1}{2} \cdot \alpha_j^T D \alpha_j}{2\sigma^2}.$$  (41)

Using stochastic gradient ascent the following updating rules are obtained:

$$\alpha_{jm}(t + 1) = \alpha_{jm}(t) - \frac{1}{2\sigma^2} \begin{cases} 
\sum_{j: c(w_j) = y} p(x_i | j) - \frac{1}{\sum_{j: c(w_j) = y} p(x_i | j)} d_{im} - \sum_j d_{im} \alpha_{jm} & \text{if } c(w_j) = y, \\
\sum_{j: c(w_j) = y} p(x_i | j) - \frac{1}{\sum_{j: c(w_j) = y} p(x_i | j)} d_{im} - \sum_j d_{im} \alpha_{jm} & \text{if } c(w_j) \neq y
\end{cases}$$  (42)

where $p(x_i | j) = K \cdot \exp \left( -\frac{[D \cdot \alpha_j] - \frac{1}{2} \cdot \alpha_j^T D \alpha_j}{2\sigma^2} \right)$ and $\alpha_j$ is a coefficient vector which implicitly describes $w_j$ through Eq. (18).

4 Results

In this section the results of three different experiments are presented. The first one employs the Multi-modal dataset [53], which is used for studying the sensitivity to the initialization or initial position of the prototypes. The second dataset is Image Segmentation [15], which is used for comparing the performance of the different methods when the nature of the features in the data is heterogeneous. Finally, the third dataset is USPS [25], which allows us to compare the performance of the LVQ classifiers of the different methods in a real-world problem.

The multi-modal dataset [53] has three classes C1, C2 and C3, with 1200 training samples per class. The training samples in class C3 are distributed in three clusters, while those in classes C1 and C2 have multi-modal distributions. Class C1 consists of 15 sub-clusters and the number of samples per cluster are 50, 50, 50, 50, 50, 50, 150, 150, 150, 100, 100 and 100, respectively. Class C2 is composed of 12 sub-clusters and the number of samples per cluster are 100, 100, 100, 50, 50, 50, 50, 50, 50, 200, 200 and 200, respectively.

The USPS dataset consists of 9298 images of handwritten digits 0-9 (10 classes) of 16x16 pixels in gray scale which are split into 7291 training set images and 2007 test set images [25]. In this experiment we used the original
Table 1: Summary of 11 LVQ Classifiers

| Name               | Characteristics                          | Parameters | Distance                  | Constraint |
|--------------------|------------------------------------------|------------|----------------------------|------------|
| LVQ 2.1 Heuristic  | $\{\epsilon, \omega, W\}$              | Euclidean  | $d(x, w) = \|x - w\|$     | $s = \frac{d(x, w^+) - d(x, w^-)}{d(x, w^+)} < s$ |
| GLVQ Margin Maximization | $\{\epsilon, W\}$                   | Euclidean  | $d(x, w) = \|x - w\|$     | no         |
| RSLVQ Likelihood Ratio Maximization | $\{\epsilon, \sigma, W\}$        | Euclidean  | $d(x, w) = \|x - w\|$     | no         |
| SNG Margin Maximization | $\{\epsilon, W\}$                   | Euclidean  | $d(x, w) = \|x - w\|$     | no         |
| SGNG Margin Maximization | $\{\epsilon, W\}$                   | Euclidean  | $d(x, w) = \|x - w\|$     | no         |
| H2MLVQ Margin Maximization | $\{\epsilon, W\}$                   | Harmonic to minimum distance | no         |
| GRLVQ Margin Maximization | $\{\epsilon, W\}$                   | Adaptive distance | $d^a(x, w) = \sum \lambda_i (x_i - w_i)^2$ | $\sum \lambda_i = 1$ |
| GMLVQ Margin Maximization | $\{\epsilon, A, W\}$                | Adaptive distance | $d^a(x, w) = (x - w)^T A (x - w)$ | Matrix $A = \Omega \Omega^T$ with $\sum A_{ii} = 1$ |
| LGRLVQ Margin Maximization | $\{\epsilon, W\}$                   | Adaptive distance | $d^a(x, w) = \sum \lambda_i (x_i - w_i)^2$ | $\sum \lambda_i = 1$ |
| LGMLVQ Margin Maximization | $\{\epsilon, A, W\}$                | Adaptive distance | $d^a(x, w) = (x - w)^T A (x - w)$ | Matrix $A = \Omega \Omega^T$ with $\sum A_{ii} = 1$ |
| KRSLVQ Likelihood Ratio Maximization | $\{\epsilon, \sigma, \sigma, \gamma\}$ | EUclidean distance in feature space | $F$ | no |

dataset and also a subset of 2000 images, which is named USPS*. This smaller dataset is used for comparison purposes with other works published in the literature.

The Image Segmentation dataset consists of 2100 samples having 12 features which correspond to 3x3 pixel regions extracted from outdoor images. There are 7 classes which are: brick-face, sky, foliage, cement, window, path and grass [15]. Because features 3-5 are constants, they were eliminated.

10-fold cross validation was used for comparing the performance of the different LVQ algorithms. In addition, a multi-comparison statistical test was used to compare the means of all pairs of LVQ classifiers. This test involves comparing many group means, i.e. pairs of simple t-test comparisons are realized and then a Bonferroni adjustment is done to compensate the critical value used in a multiple comparison procedure [26, 55].

For all LVQ classifiers the following learning rate was used:

$$\epsilon(t) = \epsilon_0 \left(1 + \tau \cdot (t - t_0)\right)^{-\frac{1}{\gamma}},$$

where $\epsilon_0$ is the initial value of the learning rate, $\tau = 0.0001$ for Multi-modal, $\tau = 0.001$ for the Image Segmentation dataset, and $\tau = 0.001$ for USPS; and $t_0$ is the start time for the learning rate with $T_{max} = 2000$ as the maximum number of training epochs for all experiments.

Table 1 shows a summary of 11 LVQ classifiers chosen for comparison. All of them are compared using the 3 datasets described above.
Table 2 shows the parameter values used for the 11 LVQ classifiers for the three datasets. The expression “id” stands for equal values to those shown in the first column.

### 4.1 Multi-modal dataset

Table 3 (the second column) shows the results obtained for the multi-modal dataset using 10-fold cross validation. The initialization of prototypes was random in the mean of the whole dataset. This setting allows quantifying the sensitivity to the initial conditions of each algorithm under study. The number of prototypes per class was set to $N_p = 15$. As shown in Table 2 (second
Table 3 Average Classification Errors obtained by using 10-fold Cross Validation. (The standard deviation is shown within brackets.)

| Algorithm   | Multi-modal | Image Segmentation | USPS* | USPS |
|-------------|-------------|--------------------|-------|------|
| LVQ 2.1     | 0.3289 (0.0400) | 0.2886 (0.0387) | 0.2390 (0.0361) | 0.2700 (0.0954) |
| GLVQ        | 0.0669 (0.0141) | 0.1205 (0.1497) | 0.0570 (0.0173) | 0.0831 (0.0036) |
| RSLVQ       | 0.1583 (0.0346) | 0.2124 (0.0539) | **0.0415 (0.0076)** | 0.0566 (0.0141) |
| SNG         | 0.0678 (0.0141) | 0.1205 (0.0300) | 0.0570 (0.0141) | **0.0410 (0.0714)** |
| SGNG        | 0.0732 (0.1873) | 0.2200 (0.2121) | 0.0815 (0.2141) | 0.0922 (0.2213) |
| H2MLVQ      | **0.0294 (0.0141)** | 0.1743 (0.0387) | 0.0330 (0.0632) | 0.0455 (0.0917) |
| GMLVQ       | 0.1183 (0.0224) | 0.0881 (0.0141) | 0.0970 (0.0245) | 0.0970 (0.1497) |
| GMLVQ       | 0.1142 (0.0141) | 0.0890 (0.0300) | 0.1165 (0.0224) | 0.1321 (0.0837) |
| LGMLVQ      | 0.1031 (0.0361) | 0.0531 (0.0224) | 0.1040 (0.0632) | 0.1091 (0.0200) |
| LGMLVQ      | 0.0955 (0.0458) | **0.0357 (0.0224)** | 0.1012 (0.0265) | 0.1074 (0.0224) |
| KRSLVQ      | 0.1134 (0.0387) | 0.1587 (0.0342) | 0.0448 (0.0346) | 0.0514 (0.0224) |

The algorithms that update more than one prototype per iteration such as H2M-LVQ, SNG and SGNG, obtained the best results together with GLVQ. A multi-comparison statistical test was performed which showed that H2M-LVQ is significantly better than 6 other algorithms (LVQ 2.1, RSLVQ, GMLVQ, LGRLVQ, LGMLVQ, KRSLVQ) with a 95% confidence interval of the mean difference. The statistical test indicates that there are 4 algorithms that are not significantly different from H2M-LVQ: SGNG, SNG, GLVQ and GRLVQ. However, if we take any of these 4 algorithms as a reference, the statistical test shows that they are statistically significantly different from only 3 LVQ classifiers instead of 6 as was done by H2MLVQ. The results in Table 3 show that the performance of LVQ 2.1 is inferior to all other algorithms. This is because LVQ 2.1 does not have a associated functional and it is very sensitive to the initial condition of the prototypes. In practice LVQ 2.1 needs a pre-processing technique for finding a good initial position of the prototypes. For methods using the adaptive (local) metric, such as: GRLVQ, LGRLVQ, GMLVQ and LGMLVQ, the classification error is higher that obtained by GLVQ because these methods are similar to GLVQ in the early iterations, and are prone to over-fitting the adaptive metric while trying to find a good performance.

4.2 Image Segmentation dataset

In this experiment the prototypes were initialized in the mean of each class to avoid initialization sensitivity. The number of prototypes per class was set to $N_p = 1$. The algorithms based on distance learning (matrix or relevance learning) reached a better performance, as shown in the third column of Table 3. The mean (std) using 10-fold cross validation are shown in this table. The lowest classification error was obtained by LGMLVQ with a mean value of 0.0357. This is significantly better than 6 other algorithms (KRSLVQ, RSLVQ, LGRLVQ, SGNG, H2M-LVQ and LVQ) with a 95% confidence interval of the mean difference according to the multi-comparison statistical test using Bon-
ferroni adjustment. On the other hand, LGMLVQ is not significantly different from GLVQ, GRLVQ, SNG and GMLVQ. But, as discussed in the previous section, the means of these 4 algorithms are significantly different from only three algorithms instead of 6 as was done by LGMLVQ. Between GMLVQ and GRLVQ there is no statistically significant difference, with a 95% confidence interval of the mean difference. In the case of local distance learning (LGM-LVQ and LGRLVQ), the matrix method obtained better performance than the relevance version, which is significantly different with a confidence interval of 95%. The methods based on distance learning obtained better performance due to their capacity of modifying the shape of the receptive field of prototypes, locally or globally, being robust against a dataset with heterogeneous features such as the Image Segmentation dataset.

4.3 USPS dataset

For both USPS and USPS* datasets the prototypes were initialized in the mean of each class and the number of prototypes per class was set to $N_p = 3$. The methods based on likelihood ratio maximization, RSLVQ and KRSLVQ, obtained higher performance than the methods based on margin maximization. For USPS* both RSLVQ and KRSLVQ reached average classification errors that are significantly better than the other 6 algorithms with a 95% confidence interval. The best performance was obtained by RSLVQ with a mean of 0.0415. The multi-comparison statistical test indicates that the means obtained by RSLVQ and KRSLVQ are not significantly different with a 95% confidence interval. The algorithms based on likelihood ratio maximization achieved higher performance in this dataset because the best prototypes are not necessarily located in the centroids of each cluster, which allows obtaining better performance in datasets that are very overlapped. In the case of USPS the best performance was obtained by SNG followed by H2M-LVQ, but the methods based on likelihood-ratio maximization RSLVQ and KRSLVQ appear closely in the third and fourth places as is shown in Table 3 (the fifth column).

5 Open Problems

In this section some open problems and challenges in the field of LVQ classifiers are presented.

1. **Principled Approach to LVQ.** Although GLVQ provides a cost function and it can be shown to be a margin maximizer method, the cost function is not derived from first principles, such as probabilistic or information theory.
2. **Sparsity.** The real world datasets are becoming larger and larger, and the computational cost of applying a prototype-based method, such as the LVQ classifier, keeps growing. For this reason, recently several sparsity approaches have been extended to LVQ classifiers which allow obtaining a linear training time without losing classification performance [28, 71].
3. **Semi-supervised learning.** In the real world due to the increasing size of datasets it is no longer possible to put labels on all samples. For this reason, it is necessary to adapt the LVQ classifiers to semi-supervised learning or an active learning framework [10].

4. **Visualization of LVQ classifiers.** Prototype-based methods such as LVQ classifiers allow the prototypical representation of the data in the input space. This is an advantage because when the prototypes and data can be visualized, the classifier can be interpreted easily. But when the LVQ classifiers work in a space different from the data space, such as in kernelized and relational variants, the classifiers are no longer easily visualized and interpreted. Not losing this natural capacity of the early LVQ classifiers is of interest [24, 46].

5. **Active learning.** For improving the generalization ability of the prototype-based methods such as LVQ classifiers, active learning can be used. This method gives the learner the capability of selecting samples during training. Furthermore, using the active learning approach, the generalization ability of the model can be increased as well as its learning speed [5, 43, 58].

### 6 Conclusions

We have presented a review of the most relevant LVQ classifiers developed in the last 25 years. We introduced a taxonomy of LVQ classifiers as a general framework. Two different main cost functions have been proposed in the literature: margin maximization and likelihood ratio maximization. LVQ classifiers based on margin maximization have been demonstrated to have good performance in many problems. These methods put the prototypes in the centroid of the data which makes them less flexible with overlapping data. LVQ classifiers based on a likelihood ratio maximization are an alternative that uses a probabilistic approach, in which the prototypes are not put in the centroid of the classes, which gives them flexibility in the case of overlapping data. On the other hand, the LVQ classifiers based on an adaptive metric have reached the best performance in heterogeneous feature datasets. Also, LVQ classifiers which update more than one prototype per iteration are less sensitive to initial conditions and get better performance in multi-modal datasets.

Recently, LVQ classifiers such as kernelized or relational LVQ classifiers have been based on data representation for improving the performance of the classifiers when the data is more complex. With relational LVQ classifiers there is a more general representation of the data which allows working with non-Euclidean spaces. In this sense, the recent approaches based on dis-/similarities capture the inherent data structure naturally which should improve the performance of the classifier. The experiments done in this work have shown that there is no free lunch; each method has its own pros and cons. The different LVQ methods were designed for dealing with specific problems and datasets. From a more general point of view, LVQ classifiers have been demonstrated to
be very competitive classifiers, and further research is needed to achieve the greatest success in pattern recognition tasks.

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References

1. Ahn KK, Nguyen HTC (2007) Intelligent switching control of a pneumatic muscle robot arm using learning vector quantization neural network. Mechatronics 17(4):255–262
2. Anagnostopoulos C, Anagnostopoulos J, Vergados D, Kayafas E, Lounos V, Theodoropoulos G (2001) Training a learning vector quantization network for biomedical classification. In: Proceedings of the International Joint Conference on Neural Networks, Natl. Tech. Univ. of Athens (NTUA), Electrical and Computer Eng. Dept., vol 4, pp 2506–2511
3. Bashyal S, Venayagamoorthy GK (2008) Recognition of facial expressions using gabor wavelets and learning vector quantization. Engineering Applications of Artificial Intelligence 21(7):1056–1064
4. Bassiuny A, Li X, Du R (2007) Fault diagnosis of stamping process based on empirical mode decomposition and learning vector quantization. International Journal of Machine Tools and Manufacture 47(15):2298–2306
5. Baum EB (1991) Neural net algorithms that learn in polynomial time from examples and queries. IEEE Transactions on Neural Networks, 2(1):5–19
6. Bezdek JC, Pal NR (1995) Two soft relatives of learning vector quantization. Neural networks 8(5):729–743
7. Biehl M, Hammer B (2007) Dynamics and Generalization Ability of LVQ Algorithms 8:323–360
8. Blume M, Ballard DR (1997) Image annotation based on learning vector quantization and localized Haar wavelet transform features. In: Rogers SK (ed) Society of Photo-Optical Instrumentation Engineers (SPIE) Conference Series, Society of Photo-Optical Instrumentation Engineers (SPIE) Conference Series, vol 3077, pp 181–190
9. Chang CY, Chang CH, Li CH, Der Jeng M (2007) Learning vector quantization neural networks for led wafer defect inspection. In: Innovative Computing, Information and Control, 2007. ICICIC’07. Second International Conference on, IEEE, pp 229–229
10. Chapelle O, Schölkopf B, Zien A (eds) (2006) Semi-supervised learning, vol 2. MIT press Cambridge
11. Chen CY (2012) Accelerometer-based hand gesture recognition using fuzzy learning vector quantization. Advanced Science Letters 9(1):38–44
12. Crammer K, Gilad-Bachrach R, Navot A, Tishby A (2002) Margin analysis of the LVQ algorithm. Advances in neural information processing systems 15:462–469
13. Dieterle F, Muller-Hagedorn S, Liebich HM, Gauglitz G (2003) Urinary nucleosides as potential tumor markers evaluated by learning vector quantization. Artificial intelligence in medicine 28(3):265–280
14. Dutta S, Chatterjee A, Munshi S (2011) Identification of ecg beats from cross-spectrum information aided learning vector quantization. Measurement 44(10):2020–2027
15. Frank A, Asuncion A (2010) UCI machine learning repository. URL http://archive.ics.uci.edu/ml
16. Fritzke B, et al (1995) A growing neural gas network learns topologies. Advances in neural information processing systems 7:625–632
17. González AI, Grana M, D’Anjou A (1995) An analysis of the glvq algorithm. Neural Networks, IEEE Transactions on 6(4):1012–1016
18. Hammer B, Villmann T (2002) Generalized relevance learning vector quantization. Neural Networks 15(8-9):1059–1068
19. Hammer B, Strickert M, Villmann T (2004) Relevance lvq versus svm. In: Rutkowski L, Siekmann J, Tadeusiewicz R, Zadeh L (eds) Artificial Intelligence and Soft Computing (ICAISC 2004), Lecture Notes in Artificial Intelligence, vol 3070, Springer Verlag, Berlin,Heidelberg, pp 592–597
20. Hammer B, Strickert M, Villmann T (2005) On the generalization ability of grlvq networks. Neural Processing Letters 21(2):109–120
21. Hammer B, Strickert M, Villmann T (2005) Supervised neural gas with general similarity measure. Neural Processing Letters 21(1):21–44
22. Hammer B, Mokbel B, Schleif FM, Zhu X (2011) Prototype-based classification of dissimilarity data. Advances in Intelligent Data Analysis X pp 185–197
23. Hammer B, Schleif FM, Zhu X (2011) Relational extensions of learning vector quantization. In: Neural Information Processing, Springer, pp 481–489
24. Hammer B, Gisbrecht A, Schulz A (2013) How to visualize large data sets? In: Estévez PA, Príncipe JC, Zegers P (eds) Advances in Self-Organizing Maps, Advances in Intelligent Systems and Computing, vol 198, Springer Berlin Heidelberg, pp 1–12
25. Hastie T, Tibshirani R, Friedman JJH (2001) The elements of statistical learning, vol 1. Springer New York
26. Hochberg Y, Tamhane AC (1987) Multiple Comparison Procedures. NJ: John Wiley & Sons
27. Hofmann D, Hammer B (2012) Kernel Robust Soft Learning Vector Quantization. Lecture Notes in Artificial Intelligence 7477:14–23
28. Hofmann D, Gisbrecht A, Hammer B (2013) Efficient approximations of kernel robust soft lvq. In: Estévez PA, Príncipe JC, Zegers P (eds) Advances in Self-Organizing Maps, Advances in Intelligent Systems and Computing, vol 198, Springer Berlin Heidelberg, pp 183–192
29. Hung WL, Chen DH, Yang MS (2011) Suppressed fuzzy-soft learning vector quantization for mri segmentation. Artificial intelligence in medicine 52(1):33–43
30. Jeng JY, Mau TF, Leu SM (2000) Prediction of laser butt joint welding parameters using back propagation and learning vector quantization networks. Journal of Materials Processing Technology 99(1):207–218

31. Jirayusakul A, Aumawanmongkol S (2007) A supervised growing neural gas algorithm for cluster analysis. Int J Hybrid Intell Syst 4(2):129–141

32. Karayiannis NB (1997) A methodology for constructing fuzzy algorithms for learning vector quantization. IEEE Transactions on Neural Networks 8(3):505–518

33. Karayiannis NB (1999) An axiomatic approach to soft learning vector quantization and clustering. IEEE Transactions on Neural Networks 10(5):1153–1165

34. Karayiannis NB, Pai PI (1996) Fuzzy algorithms for learning vector quantization. IEEE Transactions on Neural Networks 7(5):1196–1211

35. Karayiannis NB, Zervos N (2000) Entropy-constrained learning vector quantization algorithms and their application in image compression. Journal of Electronic Imaging 9(4):495–508

36. Kohonen T (1988) An introduction to neural computing. Neural networks 1(1):3–16

37. Kohonen T (1990) Improved versions of learning vector quantization. In: Neural Networks, 1990., 1990 IJCNN International Joint Conference on, IEEE, pp 545–550

38. Kohonen T (1997) Self-organizing maps. Springer-Verlag New York, Inc., Secaucus, NJ, USA

39. Lehn-Schiøler T, Hegde A, Erdogmus D, Principe JC (2005) Vector quantization using information theoretic concepts. Natural Computing 4(1):39–51

40. Lendasse A, Verleysen M, De Bodt E, Cottrell M, Grégoire P (1998) Forecasting time-series by kohonen classification. In: Proc. of European Symposium on Artificial Neural Networks, pp 221–226

41. Lieberman MA, Patil RB (1997) Evaluation of learning vector quantization to classify cotton trash. Optical Engineering 36(3):914–921

42. Martinetz TM, Berkovich SG, Schulten KJ (1993) Neural-gas’ network for vector quantization and its application to time-series prediction. IEEE Transactions on Neural Networks 4(4):558–569

43. Mitra P, Murthy C, Pal SK (2004) A probabilistic active support vector learning algorithm. IEEE Transactions on Pattern Analysis and Machine Intelligence, 26(3):413–418

44. Nanopoulos A, Alcock R, Manolopoulos Y (2001) Feature-based classification of time-series data. International Journal of Computer Research pp 49–61

45. Neural Networks Research Centre Helsinki University of Technology (2005) Bibliography on the self-organizing map (som) and learning vector quantization (lvq). URL [http://liinwww.ira.uka.de/bibliography/Neural/SOM.LVQ.html](http://liinwww.ira.uka.de/bibliography/Neural/SOM.LVQ.html)

46. Nova D, Estévez PA (2013) Online visualization of prototypes and receptive fields produced by lvq algorithms. In: Estévez PA, Príncipe JC,
Zegers P (eds) Advances in Self-Organizing Maps, Advances in Intelligent Systems and Computing, vol 198, Springer Berlin Heidelberg, pp 173–182

47. Pal NR, Bezdek JC, Tsao EK (1993) Generalized clustering networks and kohonen’s self-organizing scheme. IEEE Transactions on Neural Networks 4(4):549–557

48. Pękalska E, Duin RP (2005) The dissimilarity representation for pattern recognition: foundations and applications. 64, World Scientific, Singapore

49. Pesu L, Helisto P, Ademovic E, Pesquet J, Saarinen A, Sovijärvi A (1998) Classification of respiratory sounds based on wavelet packet decomposition and learning vector quantization. Technology and Health Care 6(1):65–74

50. Pradhan N, Sadasivan P, Arunodaya G (1996) Detection of seizure activity in eeg by an artificial neural network: a preliminary study. Computers and Biomedical Research 29(4):303–313

51. Principe JC, Xu D, Fisher J (2000) Information theoretic learning. In: Haykin S (ed) Unsupervised adaptive filtering, Wiley, New York, NY

52. Qin AK, Suganathan P (2004) A novel kernel prototype-based learning algorithm. In: Pattern Recognition, 2004. ICPR 2004. Proceedings of the 17th International Conference on, vol 4, pp 621–624

53. Qin AK, Suganathan PN (2005) Initialization insensitive LVQ algorithm based on cost-function adaptation. Pattern recognition 38(5):773–776

54. Qin AK, Suganathan P, Liang JJ (2004) A new generalized lvq algorithm via harmonic to minimum distance measure transition. In: 2004 IEEE International Conference on Systems, Man and Cybernetics, vol 5, pp 4821–4825

55. Salzberg SL (1997) On comparing classifiers: Pitfalls to avoid and a recommended approach. Data Mining and knowledge discovery 1(3):317–328

56. Sato A, Yamada K (1996) Generalized Learning Vector Quantization. In: Touretzky DS, Mozer MC, Hasselmo ME (eds) Advances in Neural Information Processing Systems, MIT Press, vol 8, pp 423–429

57. Savio A, García-Sebastián M, Hernández C, Graña M, Villanúa J (2009) Classification results of artificial neural networks for alzheimer’s disease detection. Intelligent Data Engineering and Automated Learning-IDEAL 2009 pp 641–648

58. Schleif FM, Hammer B, Villmann T (2007) Margin-based active learning for LVQ networks. Neurocomputing 70(7-9):1215–1224

59. Schleif FM, Villmann T, Hammer B, Schneider P (2011) Efficient Kernelized Prototype Based Classification. International Journal of Neural Systems 21(06):443

60. Schneider P, Biehl M, Hammer B (2009) Adaptive relevance matrices in learning vector quantization. Neural computation 21(12):3532–61

61. Schneider P, Biehl M, Hammer B (2009) Distance learning in discriminative vector quantization. Neural Computation 21(10):2942–69

62. Scholkopf B, Mika S, Burges CJ, Knirsch P, Muller KR, Ratsch G, Smola AJ (1999) Input space versus feature space in kernel-based methods. IEEE Transactions on Neural Networks 10(5):1000–1017
63. Seo S, Obermayer K (2003) Soft learning vector quantization. Neural Computation 15(7):1589–1604
64. Seo S, Bode M, Obermayer K (2003) Soft nearest prototype classification. IEEE transactions on neural networks 14(2):390–8
65. Strickert M, Bojer T (2001) Generalized relevance LVQ for time series. In: Artificial Neural Networks - ICANN'2001, pp 677–683
66. Torkkola K (2003) Feature extraction by non parametric mutual information maximization. J Mach Learn Res 3:1415–1438
67. Torkkola K, Campbell WM (2000) Mutual information in learning feature transformations. In: In Proceedings of the 17th International Conference on Machine Learning, Morgan Kaufmann, pp 1015–1022
68. Tse P, Wang DD, Xu J (1995) Classification of image texture inherited with overlapped features using learning vector quantization. In: Proceedings of the Second International Conference on Mechatronics and Machine Vision in Practice. M/sup 2/VIP ‘95, City Univ. Hong Kong, Hong Kong, pp 286–90
69. Villmann T, Haase S (2011) Divergence-based vector quantization. Neural computation 23(5):1343–92
70. Villmann T, Hamner B, Schleif FM, Hermann W, Cottrell M (2008) Fuzzy classification using information theoretic learning vector quantization. Neurocomputing 71(16-18):3070–3076
71. Williams C, Seeger M (2001) Using the nystrom method to speed up kernel machines. In: Advances in Neural Information Processing Systems 13, MIT Press, pp 682–688
72. Xuan J, Adali T (1995) Learning tree-structured vector quantization for image compression. In: Proc. WCNN’95, World Congress on Neural Networks, INNS, vol I, pp 756–759
73. Yang HT, Liao CC, Chou JH (2001) Fuzzy learning vector quantization networks for power transformer condition assessment. Dielectrics and Electrical Insulation, IEEE Transactions on 8(1):143–149
74. Zhang B, Hsu M, Dayal U (1999) K-harmonic means—a data clustering algorithm. Hewllet-Packard Research Laboratory Technical Report HPL-1999-124