Research article

Chen-Ricci inequality for biwarped product submanifolds in complex space forms

Amira A. Ishan¹,* and Meraj Ali Khan²

¹ Department of Mathematics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia.
² Department of Mathematics, University of Tabuk, Tabuk, Saudi Arabia

* Correspondence: Email: a.ishan@tu.edu.sa.

Abstract: The main objective of this paper is to achieve the Chen-Ricci inequality for biwarped product submanifolds isometrically immersed in a complex space form in the expressions of the squared norm of mean curvature vector and warping functions. The equality cases are likewise discussed. In particular, we also derive Chen-Ricci inequality for CR-warped product submanifolds and point wise semi slant warped product submanifolds.

Keywords: Ricci curvature; biwarped product submanifolds; complex space form; CR-warped product submanifolds; semi slant warped product submanifolds
Mathematics Subject Classification: 53C25, 53C40, 53C42, 53D15

1. Introduction

The accomplishment of warped product manifolds came into existent after the study of Bishop and O’Neill [1] on the manifolds of negative curvature. Examining the fact that a Riemannian product of manifolds can not have negative curvature, they constructed the model of warped product manifolds for the class of manifolds of negative (or non positive) curvature which is defined as follows:

Let \((U_1, g_1)\) and \((U_2, g_2)\) be two Riemannian manifolds with Riemannian metrics \(g_1\) and \(g_2\) respectively and \(\psi\) be a positive differentiable function on \(U_1\). If \(\xi : U_1 \times U_2 \to U_1\) and \(\eta : U_1 \times U_2 \to U_2\) are the projection maps given by \(\xi(p, q) = p\) and \(\eta(p, q) = q\) for every \((p, q) \in U_1 \times U_2\), then the warped product manifold is the product manifold \(U_1 \times U_2\) equipped with the Riemannian structure such that

\[
g(V_1, V_2) = g_1(\xi_\ast V_1, \xi_\ast V_2) + (\xi \circ \pi)^2 g_2(\eta_\ast V_1, \eta_\ast V_2),
\]
for all \( V_1, V_2 \in TU \). The function \( \psi \) is called the warping function of the warped product manifold. If the warping function is constant, then the warped product is trivial i.e., simply Riemannian product. On the basis of the fact that warped product manifolds admit a number of applications in Physics and theory of relativity [2], this has been a topic of extensive research. Warped products provide many fundamental solutions to Einstein field equations [2]. The concept of modelling of space-time near black holes adopts the idea of warped product manifolds [3]. Schwartschild space-time is an example of warped product \( U \times_r K^2 \), where the base \( U = R \times R^+ \) is a half plane \( r > 0 \) and the fibre \( K^2 \) is the unit sphere. Under certain conditions, the Schwartschild space-time becomes the black hole. A cosmological model to represent the universe as a space-time known as Robertson-Walker model is a warped product [4].

In [1] authors have studied some fundamental features of warped product manifolds. An extrinsic study on warped product submanifolds of the kaehler manifolds was performed by B. Y. Chen ([5,6]). Since then, many geometers have explored warped product manifolds in different settings like almost complex and almost contact manifolds and various existence results have been investigated (see the survey article [7]).

In 1999, Chen [8] discovered a relationship between Ricci curvature and squared mean curvature vector for an arbitrary Riemannian manifold. On the line of Chen a series of articles have been appeared to formulate the relationship between Ricci curvature and squared mean curvature in the setting of some important structures on Riemannian manifolds (see [9–14]). Recently, Mustafa et al. [15] proved a relationship between Ricci curvature and squared mean curvature for warped product submanifolds of a semi-slant submanifold of Kenmotsu space forms.

In this paper, our aim is to obtain a relationship between Ricci curvature and squared mean curvature for biwarped product submanifolds in the setting of complex space forms.

2. Preliminaries

Let \( \bar{U} \) be an almost Hermitian manifold with an almost complex structure \( J \) and a Hermitian metric \( g \), i.e., \( J^2 = -I \) and \( g(JV_1, JV_2) = g(V_1, V_2) \), for all vector fields \( V_1, V_2 \) on \( \bar{U} \). If \( J \) is parallel with respect to the Levi-Civita connection \( \bar{D} \) on \( \bar{U} \), that mean

\[
(\bar{D}_i J)V_2 = 0, \tag{2.1}
\]

for all \( V_1, V_2 \in T\bar{U} \), then \( (\bar{U}, J, g, \bar{D}) \) is called a Kaehler manifold. A Kaehler manifold \( \bar{U} \) is called a complex space form if it has constant holomorphic sectional curvature denoted by \( \bar{U}(c) \). The curvature tensor of the complex space form \( \bar{U}(c) \) is given by

\[
\bar{R}(V_1, V_2, V_2, V_4) = \frac{c}{4}[g(V_2, V_3)g(V_1, V_4) - g(V_1, V_3)g(V_2, V_4) + g(V_1, JV_3)g(JV_2, V_4) - g(V_2, JV_3)g(JV_1, V_4) + 2g(V_1, JV_2)g(JV_3, V_4)], \tag{2.2}
\]

for any \( V_1, V_2, V_3, V_4 \in T\bar{U} \).

Let \( U \) be an \( n \)-dimensional Riemannian manifold isometrically immersed in a \( m \)-dimensional Riemannian manifold \( \bar{U} \). Then the Gauss and Weingarten formulas are \( \bar{D}_i V_2 = D_i V_2 + h(V_1, V_2) \) and \( \bar{D}_i V_1 \xi = -A_2 V_1 + D_i V_1 \xi \) respectively, for all \( V_1, V_2 \in TU \) and \( \xi \in T^{\perp}U \). Where \( D \) is the induced Levi-civita connection on \( U \), \( \xi \) is a vector field normal to \( U \), \( h \) is the second fundamental form of \( U \).
$D^+$ is the normal connection in the normal bundle $T^\perp U$ and $A_\xi$ is the shape operator of the second fundamental form. The second fundamental form $h$ and the shape operator are associated by the following formula

$$g(h(V_1, V_2), \xi) = g(A_\xi V_1, V_2).$$

(2.3)

The equation of Gauss is given by

$$R(V_1, V_2, V_3, V_4) = \bar{R}(V_1, V_2, V_3, V_4) + g(h(V_1, V_4), h(V_2, V_3)) - g(h(V_1, V_3), h(V_2, V_4)),$$

(2.4)

for all $V_1, V_2, V_3, V_4 \in TU$. Where, $\bar{R}$ and $R$ are the curvature tensors of $\bar{U}$ and $U$ respectively.

For any $V \in TU$ and $N \in T^\perp U$, $JV_1$ and $JN$ can be decomposed as follows

$$JV_1 = PV_1 + FV_1$$

(2.5)

and

$$JN = tN + fN,$$

(2.6)

where $PV_1$ (resp. $tN$) is the tangential and $FV_1$ (resp. $fN$) is the normal component of $JV_1$ (resp. $JN$).

For any orthonormal basis $\{e_1, e_2, \ldots, e_k\}$ of the tangent space $T_x U$, the mean curvature vector $H(x)$ and its squared norm are defined as follows

$$H(x) = \frac{1}{n} \sum_{i=1}^{k} h(e_i, e_i), \quad ||H||^2 = \frac{1}{k^2} \sum_{i,j=1}^{k} g(h(e_i, e_i), h(e_j, e_j)),$$

(2.7)

where $k$ is the dimension of $U$. If $h = 0$ then the submanifold is said to be totally geodesic and minimal if $H = 0$. If $h(V_1, V_2) = g(V_1, V_2)H$ for all $V_1, V_2 \in TU$, then $U$ is called totally umbilical.

The scalar curvature of $\bar{U}$ is denoted by $\bar{\tau}(\bar{U})$ and is defined as

$$\bar{\tau}(\bar{U}) = \sum_{1 \leq p < q \leq m} \bar{k}_{pq},$$

(2.8)

where $\bar{k}_{pq} = \bar{\kappa}(e_p \wedge e_q)$ and $m$ is the dimension of the Riemannian manifold $\bar{M}$. Throughout this study, we shall use the equivalent version of the above equation, which is given by

$$2\bar{\tau}(\bar{U}) = \sum_{1 \leq p < q \leq m} \bar{k}_{pq}.$$  

(2.9)

In a similar way, the scalar curvature $\bar{\tau}(L_x)$ of a $L$–plane is given by

$$\bar{\tau}(L_x) = \sum_{1 \leq p < q \leq m} \bar{k}_{pq}.$$  

(2.10)

Let $\{e_1, \ldots, e_k\}$ be an orthonormal basis of the tangent space $T_x U$ and if $e_r$ belongs to the orthonormal basis $\{e_{k+1}, \ldots, e_m\}$ of the normal space $T^\perp U$, then we have

$$h'_{pq} = g(h(e_p, e_q), e_r)$$

(2.11)

and

$$||h||^2 = \sum_{p,q=1}^{n} g(h(e_p, e_q), h(e_p, e_q)).$$

(2.12)
Let $\kappa_{pq}$ and $\bar{\kappa}_{pq}$ be the sectional curvatures of the plane sections spanned by $e_p$ and $e_q$ at $x$ in the submanifold $U^k$ and in the Riemannian space form $\tilde{U}^m(c)$, respectively. Thus by Gauss equation, we have

$$\kappa_{pq} = \bar{\kappa}_{pq} + \sum_{r=k+1}^m (h'_{pp}h'_{qq} - (h'_{pq})^2). \quad (2.13)$$

The global tensor field for orthonormal frame of vector field $\{e_1, \ldots, e_k\}$ on $U^k$ is defined as

$$\bar{T}(V_1, V_2) = \sum_{i=1}^k \{g(\bar{R}(e_i, V_1)V_2, e_i)\}, \quad (2.14)$$

for all $V_1, V_2 \in T_xU^k$. The above tensor is called the Ricci tensor. If we fix a distinct vector $e_u$ from $\{e_1, \ldots, e_k\}$ on $U^k$, which is governed by $\chi$. Then the Ricci curvature is defined by

$$R(\chi) = \sum_{p=1, p\neq u}^k \kappa(e_p \wedge e_u). \quad (2.15)$$

For a smooth function $\psi$ on a Riemannian manifold $U$ with Riemannian metric $g$, the gradient of $\psi$ is denoted by $\nabla \psi$ and is defined as

$$g(\nabla \psi, U_1) = U_1 \psi, \quad (2.16)$$

for all $U_1 \in TU$.

Let the dimension of $U$ is $k$ and $\{e_1, e_2, \ldots, e_k\}$ be a basis of $TU$. Then as a result of (2.16), we get

$$\|\nabla \psi\|^2 = \sum_{i=1}^k (e_i(\psi))^2. \quad (2.17)$$

The Laplacian of $\psi$ is defined by

$$\Delta \psi = \sum_{i=1}^k [(\nabla_i e_i)\psi - e_i e_i \psi]. \quad (2.18)$$

3. Biwarped product submanifolds of a Kaehler manifold

B. Y. Chen and F. Dillen [16] generalize the definition of warped product submanifold to multiply warped product manifolds as follows.

Let $\{U_i\}, \quad i = 1, 2, \ldots, k$ be Riemannian manifolds with respective Riemannian metrics $\{g_i\}_{i=1,2,\ldots,k}$ and $\{\psi\}_{i=2,3,\ldots,k}$ are positive valued functions on $U_1$. Then the product manifold $U = U_1 \times U_2 \times \cdots \times U_k$ endowed with the Riemannian metric $g$ given by

$$g = h_i^*(g_1) + \sum_{i=2}^k (\psi_i \circ h_1)^2 h_i^*(g_i)$$

is called multiply warped product manifold and denoted by $U = U_1 \times f_1 U_2 \times \cdots \times f_k U_k$ where $h_i(i = 1, 2, \ldots, k)$ are the projection maps of $U$ onto $U_i$ respectively. The functions $f_i$ are known as the
Lemma 3.1. [17] Let $U = U_0 \times f_1 U_1 \times f_2 U_2$ be a biwarped product manifold. Then we have

$$D_{V_1} V_2 = D_{V_2} V_1 = V_1 (\ln f_1) V_2$$ \hspace{1cm} (3.1)

for $V_1 \in T U_0$ and $V_2 \in T U_i$, for $i = 1, 2$.

Recently, H. M. Tastan [18] studied biwarped submanifolds in the Kaehler manifolds and this was followed by M. A. Khan and K. Khan [19]. Basically, M. A. Khan and K. Khan explored biwarped product submanifolds of the type $U = U_T \times f_1 U_1 \times f_2 U_2$ with the Levi-civita connection of $U_i$ for $i = 0, 1, 2$. Now, we have the following result for biwarped product submanifold.

Lemma 3.2. Let $U = U_0 \times f_1 U_1 \times f_2 U_2 \times f_3 U_3$ be a biwarped product manifold isometrically immersed in a Kaehler manifold $\bar{U}$. Then

(i) $g(h(V_1, V_2), F V_3) = 0$,
(ii) $g(h(V_1, V_2), J F V_3) = 0$,
(iii) $g(h(J V_1, J V_1), N) = -g(h(V_1, V_1), N),$

for any $V_1, V_2 \in T U_0$, $V_4 \in T U_1$, $V_3 \in T U_2$ and $N$ belongs to invariant subbundle of $T^1 U$. 

Proof. By using Gauss and Weingarten formulae in Eq (2.1), we have

$$D_{V_1} PV_3 + h(V_1, PV_3) - A_{F V_1} V_1 + D_{V_1}^1 F V_3 + J D_{V_1} V_3 + D_{V_1} J V_1 + J h(V_1, V_3) = 0,$$

taking inner product with $V_2$ and using 3.1, we get the required result. In a similar way, we can prove the part (ii).

To prove (iii), for any $V_1 \in T U_T$ we have

$$\tilde{D}_{V_1} J V_1 = J \tilde{D}_{V_1} V_1,$$
using Gauss formula and (2.1), we get

\[ D_V(JV_1 + h(JV_1, V_1)) = JD_VV_1 + Jh(V_1, V_1), \]

taking inner product with \( JN \), above equation yields

\[ g(h(JV_1, V_1), JN) = g(h(V_1, V_1), N), \]  

(3.3)

interchanging \( V_1 \) by \( JV_1 \) the above equation gives

\[ g(h(JV_1, V_1), JN) = -g(h(JV_1, JV_1), N). \]  

(3.4)

From (3.3) and (3.4), we get the required result. \( \Box \)

**Definition 3.1.** The warped product \( U_1 \times_{f_1} U_2 \times_{f_2} U_3 \) isometrically immersed in a Riemannian manifold \( \bar{U} \) is called \( U_i \) totally geodesic if the partial second fundamental form \( h_i \) vanishes identically. It is called \( U_i \)-minimal if the partial mean curvature vector \( H^j \) becomes zero for \( i = 1, 2, 3 \).

Assume that the distributions corresponding to the submanifolds \( U^k_1, U^k_2 \) and \( U^k_\theta \) are \( S, S^\perp \) and \( S^\theta \) respectively. From the Lemma 3.2 it is evident that the isometric immersion \( U^k_1 \times_{f_1} U^k_2 \times_{f_2} U^k_\theta \) into a Kaehler manifold is \( D \)-minimal. The \( S \)-minimality property provides us a useful relationship between the biwarped product submanifold \( U^k_1 \times_{f_1} U^k_2 \times_{f_2} U^k_\theta \) and the equation of Gauss.

Let \( \{e_1, \ldots, e_p, e_{p+1} = Je_1, \ldots, e_{k_1} = Je_{p+1}, e_{k_1+1}, \ldots, e_{k_2}, e_{k_2+1} = e^1, \ldots, e_{k_2+q} = e^q, e_{k_2+q+1} = e^{q+1} = \sec \theta Pe^1, \ldots, e_{(k_2-2)q} = e^{k_2} = \sec \theta Pe^q \} \) be a local orthonormal frame of vector fields on the biwarped product submanifold \( U^k_1 \times_{f_1} U^k_2 \times_{f_2} U^k_\theta \) such that the set \( \{e_1, \ldots, e_p, e_{p+1} = Je_1, \ldots, e_{k_1} = Je_{p+1} \} \) is tangent to \( U^k_1 \), the set \( \{e_{k_1+1}, \ldots, e_{k_2} \} \) is tangent to \( U^k_2 \) and the set \( \{e_{k_2+1}, \ldots, e_{k_2+q} \} \) is tangent to \( U^k_\theta \). Moreover, \( \{e_{k_1+1} = Je_{k_1+1}, \ldots, e_{k_2} = Je_{k_2}, e_{k_2+1} = \csc \theta Fe^1, \ldots, e_{k_2+q} = \csc \theta Fe^q, e_{k_2+q+1} = e^1, \ldots, e_m = e^q \} \) is a basis for the normal bundle \( T^\perp U \), such that the sets \( \{e_{k_2+1} = Je_{k_2+1}, \ldots, e_{k_2+q} = Je_{k_2} \} \) is tangent to \( JS^\perp \), \( \{e_{k_1+1} = \csc \theta Fe^1, \ldots, e_{k_2} = \csc \theta Fe^q \} \) is tangent to \( FS^\theta \) and \( \{e^1, \ldots, e^q \} \) is tangent to the complementary invariant subbundle \( \mu \) with even dimension \( l \).

From Lemma 3.2, it is easy to conclude that

\[ \sum_{r=k+1}^{m} \sum_{i,j=1}^{k_1} g(h(e_i, e_j), e_r) = 0. \]  

(3.5)

Thus it follows that the trace of \( h \) due to \( U^k_1 \) becomes zero. Hence in view of the Definition 3.1, we obtain the following important result.

**Theorem 3.3.** Let \( U^k = U^k_1 \times_{f_1} U^k_2 \times_{f_2} U^k_\theta \) be a biwarped product submanifold isometrically immersed in a Kaehler manifold. Then \( U^k \) is \( S \)-minimal.

So, it is easy to conclude the following

\[ \|H\|^2 = \frac{1}{k^2} \sum_{r=k+1}^{m} (h^r_{k_1+1} + \cdots + h^r_{k_2} + \cdots + h^r_{k_\theta})^2, \]  

(3.6)

where \( \|H\|^2 \) is the squared mean curvature.
4. Ricci curvature for biwarped product submanifold

In this section, we investigate Ricci curvature in terms of the squared norm of mean curvature and the warping functions as follows.

**Theorem 4.1.** Let $U^k = U_T^{k_1} \times f_1 U_\perp^{k_2} \times f_2 U_\theta^{k_3}$ be a biwarped product submanifold isometrically immersed in a complex space form $\bar{U}(c)$. Then for each orthogonal unit vector field $\chi \in T_x U$, either tangent to $U_T^{k_1}$, $U_\perp^{k_2}$ or $U_\theta^{k_3}$, we have

(1) The Ricci curvature satisfy the following inequalities

(i) If $\chi$ is tangent to $U_T^{k_1}$, then

$$\frac{1}{4} k^2 ||H||^2 \geq R(\chi) + \frac{k_2 \Delta f_2}{f_2} + \frac{k_3 \Delta f_3}{f_3} + \frac{c}{4} (k - k_1 k_2 - k_2 k_3 - k_1 k_3 - \frac{1}{2}).$$  \hspace{1cm} (4.1)

(ii) If $\chi$ is tangent to $U_\perp^{k_2}$, then

$$\frac{1}{4} k^2 ||H||^2 \geq R(\chi) + \frac{k_2 \Delta f_2}{f_2} + \frac{k_3 \Delta f_3}{f_3} + \frac{c}{4} (k - k_1 k_2 - k_2 k_3 - k_1 k_3 + 1).$$  \hspace{1cm} (4.2)

(iii) $\chi$ is tangent to $U_\theta^{k_3}$, then

$$\frac{1}{4} k^2 ||H||^2 \geq R(\chi) + \frac{k_2 \Delta f_2}{f_2} + \frac{k_3 \Delta f_3}{f_3} + \frac{c}{4} (k - k_1 k_2 - k_2 k_3 - k_1 k_3$$

$$+ 1 - \frac{3}{2} \cos^2 \theta).$$

(2) If $H(x) = 0$, then each point $x \in U^k$ there is a unit vector field $\chi$ which satisfies the equality case of (1) if and only if $U^k$ is mixed totally geodesic and $\chi$ lies in the relative null space $N_x$ at $x$.

(3) For the equality case we have

(a) The equality case of (4.1) holds identically for all unit vector fields tangent to $U_T^{k_1}$ at each $x \in U^k$ if and only if $U^k$ is mixed totally geodesic and $S$–totally geodesic biwarped product submanifold in $\bar{U}(c)$.

(b) The equality case of (4.2) holds identically for all unit vector fields tangent to $U_\perp^{k_2}$ at each $x \in U^k$ if and only if $U$ is mixed totally geodesic and either $U^k$ is $S^1$–totally geodesic biwarped product or $U^k$ is a $S^1$–totally umbilical in $\bar{U}(c)$ with dim $S^1 = 2$.

(c) The equality case of (4.3) holds identically for all unit vector fields tangent to $U_\theta^{k_3}$ at each $x \in U^k$ if and only if $U$ is mixed totally geodesic and either $U^k$ is $S^\theta$–totally geodesic biwarped product submanifold or $U^k$ is a $S^\theta$–totally umbilical in $\bar{U}(c)$ with dim $S^\theta = 2$.

(d) The equality case of (1) holds identically for all unit tangent vectors to $U^k$ at each $x \in U^k$ if and only if either $U^k$ is totally geodesic submanifold or $U^k$ is a mixed totally geodesic totally umbilical and $S$–totally geodesic submanifold with dim $U_\theta = 2$ and dim $U_\perp = 2$

where $k_1$, $k_2$, and $k_3$ are the dimensions of $U_T^{k_1}$, $U_\perp^{k_2}$, and $U_\theta^{k_3}$ respectively.

**Proof.** Suppose that $U^k = U_T^{k_1} \times f_1 U_\perp^{k_2} \times f_2 U_\theta^{k_3}$ be a biwarped product submanifold of a complex space form. From Gauss equation, we have

$$k^2 ||H||^2 = 2\tau(U^k) + ||h||^2 - 2\bar{\tau}(U^k).$$  \hspace{1cm} (4.4)
Let \( \{e_1, \ldots, e_{k_1}, e_{k_1+1}, \ldots, e_{k_2}, \ldots e_k\} \) be a local orthonormal frame of vector fields on \( U^k \) such that \( \{e_1, \ldots, e_{k_1}\} \) are tangent to \( U^k_T \), \( \{e_{k_1+1}, \ldots, e_{k_2}\} \) are tangent to \( U^k_\perp \) and \( \{e_{k_2+1}, \ldots, e_k\} \) are tangent to \( U^k_0 \).

So, the unit tangent vector \( \chi = e_A \in \{e_1, \ldots, e_k\} \) can be expanded (4.4) as follows

\[
k^2||H||^2 = 2\tau(U^k) + \frac{1}{2} \sum_{r=k+1}^{m} \{(h'_{11} + \ldots h'_{k_2k_2} + \cdots + h'_{kk})^2 + (h_{AA}')^2\}
- \sum_{r=k+1}^{m} \sum_{1 \leq i < j \leq k} h'_{ij}h'_{jj} - 2\tau(U^k).
\]

The above expression can be written as follows

\[
k^2||H||^2 = 2\tau(U^k) + \frac{1}{2} \sum_{r=k+1}^{m} \{(h'_{11} + \ldots h'_{k_2k_2} + \cdots + h'_{kk})^2 + (2h_{AA}' - (h'_{11} + \cdots + h'_{kk}))^2\} + 2 \sum_{r=k+1}^{m} \sum_{1 \leq i < j \leq k} (h'_{ij})^2
- 2 \sum_{r=k+1}^{m} \sum_{1 \leq i < j \leq k} h'_{ij}h'_{jj} - 2\tau(U^k).
\]

In view of the Lemma 3.2, the preceding expression takes the form

\[
k^2||H||^2 = 2\tau(U^k) + \frac{1}{2} \sum_{r=k+1}^{m} \{(h'_{1,k_1+1} + \ldots h'_{k_2k_2} + \cdots + h'_{kk})^2 + (2h_{AA}' - (h'_{11} + \cdots + h'_{kk}))^2\}
+ \sum_{r=k+1}^{m} \sum_{1 \leq i < j \leq k} (h'_{ij})^2 - 2 \sum_{r=k+1}^{m} \sum_{1 \leq i < j \leq k} h'_{ij}h'_{jj} - 2\tau(U^k).
\]

Considering unit tangent vector \( \chi = e_A \), we have three choices \( \chi \) is either tangent to the base manifold \( U^k_T \) or to the fibers \( U^k_\perp \) and \( U^k_0 \).

**Case 1:** If \( \chi \) is tangent to \( U^k_T \), then we need to choose a unit vector field from \( \{e_1, \ldots, e_{k_1}\} \). Let \( \chi = e_1 \).
Then from (2.14) and (3.5) we have

\[
\begin{align*}
    k^2\|H\|^2 & \geq R(\chi) + \frac{1}{2} \sum_{r=k+1}^{m} ((h'_{k1} + h'_{k2}) + \ldots + h'_{kk})^2 + \frac{k_2 \Delta f_2}{f_2} \\
    & + \frac{k_3 \Delta f_3}{f_3} + \frac{1}{2} \sum_{r=k+1}^{m} (2h'_{r1} - (h'_{r1} + h'_{r2} + \ldots + h'_{rk}))^2 \\
    & + \sum_{r=k+1}^{m} \sum_{1 \leq a < b \leq k_1} (h'_{aa} - h'_{bb})^2 \\
    & + \sum_{r=k+1}^{m} \sum_{1 \leq p < q \leq k_2} (h'_{pp} h'_{qq} - (h'_{pq})^2) \\
    & + \sum_{r=k+1}^{m} \sum_{1 \leq r \leq k} (h'_{rr} - (h'_{rr})^2) \\
    & + \sum_{r=k+1}^{m} \sum_{1 \leq i < j \leq k} (h'_{ij})^2 - \sum_{r=k+1}^{m} \sum_{2 \leq i < j \leq k} (h'_{ii} h'_{jj}) \\
    & - 2\bar{\tau}(U) + \sum_{2 \leq i < j \leq k} \bar{\kappa}(e_i, e_j) + \bar{\tau}(U_{T}^{k_1}) + \bar{\tau}(U_{L}^{k_2}) + \bar{\tau}(U_{\theta}^{k_3}).
\end{align*}
\]

Putting \( V_1, V_4 = e_i \) and \( V_2, V_3 = e_j \) in the formula (2.2), we have

\[
2\bar{\tau}(U) = \frac{c}{4} [k(k - 1) + 3k_1 + 3k_3 \cos^2 \theta] \quad (4.8)
\]

\[
\sum_{2 \leq i < j \leq k} \bar{\kappa}(e_i, e_j) = \frac{c}{8} [(k - 1)(k - 2) + 3(k_1 - 1) + 3k_3 \cos^2 \theta]
\]

\[
\bar{\tau}(U_T^{k_1}) = \frac{c}{8} [k_1(k_1 - 1) + 3k_1]
\]

\[
\bar{\tau}(U_L^{k_2}) = \frac{c}{8} [k_2(k_2 - 1)]
\]

\[
\bar{\tau}(U_\theta^{k_3}) = \frac{c}{8} [k_3(k_3 - 1) + 3k_3 \cos^2 \theta].
\]
Using these values in (4.7), we get

\[
k^2||H||^2 \geq R(\chi) + \frac{1}{2} k^2 ||H||^2 + \frac{1}{2} \sum_{r=k+1}^{m} (2h_{11}' - (h_{k+1+1}' + \cdots + h_{kk}')^2
\]

\[
+ \frac{k_2 \Delta f_2}{f_2} + \frac{k_3 \Delta f_3}{f_3} + \frac{m}{r=k+1} k_{1j} k_{j+1} (h_{ij}')^2
\]

\[
+ \sum_{r=k+1}^{m} k_{1j} k_{j+1} (h_{ik}')^2 + \sum_{r=k+1}^{m} \sum_{j=k+1}^{k} (h_{ij}')^2
\]

\[
- \sum_{r=k+1}^{m} \sum_{j=k+1}^{k} (h_{ij}')^2 = \sum_{r=k+1}^{m} \sum_{i=1}^{k} \sum_{j=k+1}^{k} (h_{ij}')^2
\]

\[
- \sum_{r=k+1}^{m} \sum_{i=1}^{k} \sum_{j=k+1}^{k} (h_{ij}')^2 - \sum_{r=k+1}^{m} \sum_{j=k+1}^{k} (h_{ij}')^2 + \sum_{r=k+1}^{m} \sum_{j=k+1}^{k} (h_{ij}')^2
\]

\[
+ \frac{c}{4} (k - k_1k_2 - k_2k_3 - k_3k_1 - \frac{1}{2}).
\]

In view of Lemma 3.1

\[
\sum_{r=k+1}^{m} \sum_{i=1}^{k} \sum_{j=k+1}^{k} (h_{ij}')^2 = \sum_{r=k+1}^{m} \sum_{i=1}^{k} \sum_{j=k+1}^{k} (h_{ij}')^2
\]

Utilizing in (4.9), we have

\[
k^2||H||^2 \geq R(\chi) + \frac{1}{2} k^2 ||H||^2 + \frac{1}{2} \sum_{r=k+1}^{m} (2h_{11}' - (h_{k+1+1}' + \cdots + h_{kk}')^2
\]

\[
+ \frac{k_2 \Delta f_2}{f_2} + \frac{k_3 \Delta f_3}{f_3} + \frac{m}{r=k+1} k_{1j} k_{j+1} (h_{ij}')^2
\]

\[
+ \sum_{r=k+1}^{m} k_{1j} k_{j+1} (h_{ik}')^2 + \sum_{r=k+1}^{m} \sum_{j=k+1}^{k} (h_{ij}')^2
\]

\[
- \sum_{r=k+1}^{m} \sum_{j=k+1}^{k} (h_{ij}')^2 = \sum_{r=k+1}^{m} \sum_{i=1}^{k} \sum_{j=k+1}^{k} (h_{ij}')^2
\]

\[
- \sum_{r=k+1}^{m} \sum_{i=1}^{k} \sum_{j=k+1}^{k} (h_{ij}')^2 - \sum_{r=k+1}^{m} \sum_{j=k+1}^{k} (h_{ij}')^2 + \sum_{r=k+1}^{m} \sum_{j=k+1}^{k} (h_{ij}')^2
\]

\[
+ \frac{c}{4} (k - k_1k_2 - k_2k_3 - k_3k_1 - \frac{1}{2}).
\]

The third term on the right hand side can be written as

\[
= 2 \sum_{r=k+1}^{m} (h_{11}')^2 + \frac{1}{2} k^2 ||H||^2 - 2 \sum_{r=k+1}^{m} \sum_{j=k+1}^{k} (h_{11}', h_{jj}')
\]

\[
+ \sum_{n=k+1}^{k} (h_{11}', h_{nn}')
\]
Combining above two expressions, we have

\[
\frac{1}{2}k^2||H||^2 \geq R(\chi) + \frac{1}{4} \sum_{r=k+1}^{m} \left( h_{i1}'^2 - \sum_{r=k+1}^{m} \sum_{j=k+1}^{k} h_{i1}' h_{jj}' \right) \\
+ \frac{1}{2} \sum_{r=k+1}^{m} (h_{k_{1}+1k_{1}+1}' + \cdots + h_{k_{k}k_{k}}' + \cdots + h_{kk}')^2 \\
+ \sum_{r=k+1}^{m} \sum_{i=1}^{k_{1}} \sum_{j=k_{1}+1}^{k} (h_{ij}')^2 + \frac{k_{2}f_{2}}{f_{2}} + \frac{k_{3}f_{3}}{f_{3}} \\
+ \frac{c}{4} (k - k_{1}k_{2} - k_{2}k_{3} - k_{3}k_{1} - 1/2),
\]

(4.12)

Or equivalently

\[
\frac{1}{4}k^2||H||^2 \geq R(\chi) + \frac{1}{4} \sum_{r=k+1}^{m} \left( 2h_{i1}' - (h_{k_{1}+1k_{1}+1}' + \cdots + h_{k_{k}k_{k}}' + \cdots + h_{kk}')^2 ight) \\
+ \sum_{r=k+1}^{m} \sum_{i=1}^{k_{1}} \sum_{j=k_{1}+1}^{k} (h_{ij}')^2 + \frac{k_{2}f_{2}}{f_{2}} + \frac{k_{3}f_{3}}{f_{3}} \\
+ \frac{c}{4} (k - k_{1}k_{2} - k_{2}k_{3} - k_{3}k_{1} - 1/2),
\]

(4.13)

which gives the inequality (i) of (1).

**Case 2.** If \( \chi \) is tangent to \( U_{e_{k}}' \), we chose the unit vector from \( \{e_{k_{1}}, \ldots, e_{k_{k}}\} \). Suppose \( \chi = e_{k_{2}} \), then from (4.6), we deduce

\[
k^2||H||^2 \geq R(\chi) + \frac{1}{2} \sum_{r=k+1}^{m} (h_{k_{1}+1k_{1}+1}' + \cdots + h_{k_{k}k_{k}}' + \cdots + h_{kk}')^2 + \frac{k_{2}f_{2}}{f_{2}} \\
+ \frac{k_{3}f_{3}}{f_{3}} + \frac{1}{2} \sum_{r=k+1}^{m} (h_{k_{1}+1k_{1}+1}' + \cdots + h_{k_{k}k_{k}}' + \cdots + h_{kk}')^2 \\
- \sum_{r=k+1}^{m} \sum_{1 \leq i < q \leq k} (h_{ip}'h_{jq}') - 2\bar{\tau}(U) + \sum_{1 \leq i < j \leq k} (h_{ij}')^2 \\
+ \bar{\tau}(U_{j_{1}^k}) + \bar{\tau}(U_{j_{2}^k}) + \bar{\tau}(U_{j_{3}^k}).
\]

From (2.2) by putting \( V_{1}, V_{4} = e_{i} \) and \( V_{2}, V_{3} = e_{j} \), one can compute

\[
\frac{1}{8} \sum_{1 \leq i < j \leq k} (k - 1)(k - 2) + 3k_{1} + 3k_{3} \cos^2 \theta
\]
\[ \bar{\tau}(U_{\ell}^{k_1}) = \frac{c}{8}[k_1(k_1 - 1) + 3k_1] \]
\[ \bar{\tau}(U_{\ell}^{k_2}) = \frac{c}{8}[k_2(k_2 - 1)] \]
\[ \bar{\tau}(U_{\ell}^{k_3}) = \frac{c}{8}[k_3(k_3 - 1) + 3k_3 \cos^2 \theta]. \]

Using these values together with (4.8) in (4.14) and applying similar techniques as in Case 1, we obtain

\[ k^2 \|H\|^2 \geq R(\chi) + \frac{1}{2} \sum_{r=k+1}^{m} ((h_{k_1+1}^{r} + \cdots + h_{k_2}^{r}) - 2h_{k_2}^{r})^2 \]
\[ + \frac{1}{2} k^2 \|H\|^2 + \frac{k_2 \Delta f_2}{f_2} + \frac{k_3 \Delta f_3}{f_3} + \sum_{r=k+1}^{m} \sum_{\ell \leq j \leq k} (h_{\ell}^{r})^2 \]
\[ + \sum_{r=k+1}^{m} \sum_{i=k_1+1}^{k_1-1} \sum_{j=k_1+1}^{k_2-1} h_{k_3k_2}^{i} h_{i}^{r} + \sum_{i=k_2+1}^{k} h_{k_3k_2}^{i} h_{i}^{r} \]
\[ + \frac{c}{4} (k - k_1 k_2 - k_2 k_3 - k_3 k_1 + 1). \]

By the Lemma 3.1, one can conclude
\[ \sum_{r=1}^{m} \sum_{i=1}^{k_1} \sum_{j=k_1+1}^{k_2-1} h_{i}^{r} h_{j}^{r} + \sum_{i=k_2+1}^{k} h_{i}^{r} h_{i}^{r} = 0. \]

The second and seventh terms on right hand side of (4.15) can be solved as follows

\[ \frac{1}{2} \sum_{r=k+1}^{m} ((h_{k_1+1}^{r} + \cdots + h_{k_2}^{r}) - 2h_{k_2}^{r})^2 + \sum_{r=k+1}^{m} \sum_{i=k_1+1}^{k_1-1} \sum_{j=k_1+1}^{k_2-1} h_{k_3k_2}^{i} h_{i}^{r} + \sum_{i=k_2+1}^{k} h_{k_3k_2}^{i} h_{i}^{r} \]
\[ = \frac{1}{2} \sum_{r=k+1}^{m} (h_{k_1+1}^{r} + \cdots + h_{k_2}^{r})^2 + 2 \sum_{r=k+1}^{m} (h_{k_3k_2}^{r})^2 \]
\[ - 2 \sum_{r=k+1}^{m} \sum_{j=k_1+1}^{k} h_{k_3k_2}^{j} h_{j}^{r} + \sum_{r=k+1}^{m} \sum_{i=k_1+1}^{k_1-1} \sum_{j=k_1+1}^{k_2-1} h_{k_3k_2}^{i} h_{i}^{r} - \sum_{r=k+1}^{m} (h_{k_3k_2}^{r})^2 \]
\[ = \frac{1}{2} \sum_{r=k+1}^{m} (h_{k_1+1}^{r} + \cdots + h_{k_2}^{r})^2 + \sum_{r=k+1}^{m} (h_{k_3k_2}^{r})^2 \]
\[ - \sum_{r=k+1}^{m} \sum_{j=k_1+1}^{k} h_{k_3k_2}^{j} h_{j}^{r}. \]
Utilizing these two values in (4.15), we arrive

\[
\frac{1}{2}k^2\|H\|^2 \geq R(\chi) + \sum_{r=k+1}^{m} (h'_{k_1+k_1+1})^2 + \sum_{r=k+1}^{m} \sum_{i=k+1}^{k} h'_{ik}h'_{ij}
\]

\[
+ \frac{1}{2} \sum_{r=k+1}^{m} (h'_{k_1+k_1+1} + \cdots + h'_{k_2})^2 + \frac{1}{2}k^2\|H\|^2 + \frac{k_2\Delta f_2}{f_2} + \frac{k_3\Delta f_3}{f_3}
\]

(4.17)

By using similar steps as in Case 1, the above inequality can be written as

\[
\frac{1}{4}k^2\|H\|^2 \geq R(\chi) + \frac{1}{4} \sum_{r=k+1}^{m} (2h'_{k_3+k_2} - (h'_{k_1+k_1+1} + \cdots + h'_{k_2}))^2
\]

\[
+ \frac{k_2\Delta f_2}{f_2} + \frac{k_3\Delta f_3}{f_3} + \frac{c}{4}(k - k_1k_2 - k_2k_3 - k_3k_1 + 1).
\]

(4.18)

The last inequality leads to inequality (ii) of (1).

**Case 3.** If \(\chi\) is tangent to \(U_{k_3}\), then we choose the unit vector field from \(\{e_{k_3+1}, \ldots, e_k\}\). Suppose the vector \(\chi\) is \(e_k\). Then from (4.6)

\[
k^2\|H\|^2 \geq R(\chi) + \frac{1}{2} \sum_{r=k+1}^{m} (h'_{k_1+k_1+1} + \cdots + h'_{k_2})^2 + \frac{k_2\Delta f_2}{f_2}
\]

\[
+ \frac{k_3\Delta f_3}{f_3} + \frac{1}{2} \sum_{r=k+1}^{m} ((h'_{k_1+k_1+1} + \cdots + h'_{k_2}) - 2h'_{k})^2
\]

\[
+ \sum_{i=1}^{m} \sum_{1 \leq s \leq k_1} (h'_{i_1} - h'_{i_2})^2 + \sum_{r=k+1}^{m} \sum_{1 \leq s \leq k_2} (h'_{i_3} - h'_{i_4})^2
\]

(4.19)

\[
+ \sum_{r=k+1}^{m} \sum_{1 \leq s \leq k_3} (h'_{i_5} - h'_{i_6})^2 + \sum_{1 \leq s \leq k_7} \tilde{k}(e_i, e_j)
\]

\[
+ \tilde{\tau}(U_{k_1}^{i_1}) + \tilde{\tau}(U_{k_2}^{i_2}) + \tilde{\tau}(U_{k_3}^{i_3}).
\]

From (2.2), one can compute

\[
\sum_{1 \leq s \leq k_7} \tilde{k}(e_i, e_j) = \frac{c}{8}[(k - 1)(k - 2) + 3k_1 + 3(k_3 - 1) \cos^2 \theta]
\]

\[
\tilde{\tau}(U_{k_1}^{i_1}) = \frac{c}{8}[k_1(k_1 - 1) + 3k_1]
\]

\[
\tilde{\tau}(U_{k_2}^{i_2}) = \frac{c}{8}[k_2(k_2 - 1)]
\]
\[
\tau(U^k_{\theta}) = \frac{c}{8} [k_3(k_3 - 1) + 3k_3 \cos^2 \theta].
\]

By usage of these values together with (4.8) in (4.19) and analogous to case 1 and case 2, we obtain

\[
k^2||H||^2 \geq R(\chi) + \frac{4}{2} k^2 ||H||^2 + \frac{1}{2} \sum_{r=k+1}^{m} (h_{r_{k_1+1}} + \ldots h_{r_{k_3+2}} + \ldots + h_{r_{k_k}} - 2h_{r_{k_k}})^2
\]

\[
+ \frac{k_2 \Delta f_2}{f_2} + \frac{k_3 \Delta f_3}{f_3} + \sum_{r=k+1}^{m} \sum_{1 \leq i < j \leq k} (h_{ij}^r)^2 + \sum_{r=k+1}^{m} \sum_{q=k+1}^{m} \sum_{i=k+1}^{k-1} h_{q_q}^r h_{ij}^r
\]

\[
+ \frac{c}{4} (k - k_1 k_2 - k_2 k_3 - k_1 k_3 + 1 - \frac{3}{2} \cos^2 \theta).
\]

On applying the Lemma 3.1, it is easy to verify

\[
\sum_{r=k+1}^{m} \sum_{i=k+1}^{k-1} \sum_{j=k+1}^{k-1} h_{ij}^r h_{ij}^q = 0. \tag{4.21}
\]

Using in (4.20), we obtain

\[
k^2||H||^2 \geq R(\chi) + \frac{4}{2} k^2 ||H||^2 + \frac{1}{2} \sum_{r=k+1}^{m} (h_{r_{k_1+1}} + \ldots h_{r_{k_3+2}} + \ldots + h_{r_{k_k}} - 2h_{r_{k_k}})^2
\]

\[
+ \frac{k_2 \Delta f_2}{f_2} + \frac{k_3 \Delta f_3}{f_3} + \sum_{r=k+1}^{m} \sum_{1 \leq i < j \leq k} (h_{ij}^r)^2 + \sum_{r=k+1}^{m} \sum_{q=k+1}^{m} \sum_{i=k+1}^{k-1} h_{q_q}^r h_{ij}^r \tag{4.22}
\]

\[
+ \frac{c}{4} (k - k_1 k_2 - k_2 k_3 - k_1 k_3 + 1 - \frac{3}{2} \cos^2 \theta).
\]

The third and seventh terms on the right hand side of (4.22) in a similar way as in case 1 and case 2 can be simplified as

\[
\frac{1}{2} \sum_{r=k+1}^{m} (h_{r_{k_1+1}} + \ldots h_{r_{k_3+2}} + \ldots + h_{r_{k_k}} - 2h_{r_{k_k}})^2 + \sum_{r=k+1}^{m} \sum_{q=k+1}^{m} h_{q_q}^r h_{ij}^r
\]

\[
\frac{1}{2} \sum_{r=k+1}^{m} (h_{r_{k_1+1}} + \ldots h_{r_{k_3+2}} + \ldots + h_{r_{k_k}})^2 + \sum_{r=k+1}^{m} (h_{r_{r_{k_k}}}^r)^2 \tag{4.23}
\]

\[
- \sum_{r=k+1}^{m} \sum_{j=k+1}^{k} h_{r_{k_k}} h_{ij}^r.
\]

By combining (4.22) and (4.23) and using similar techniques as used in case 1 and case 2, we can
derive

\[ \frac{1}{4} k^2 \| H \|^2 \geq R(\chi) + \frac{1}{4} \sum_{r=1}^{m} (2h_{r}^{kk} - (h_{r}^{k} - \sum_{k=1}^{m} h_{kk}^{r}))^2 \]

\[ + \frac{k_2 \Delta f_2}{f_2} + \frac{k_3 \Delta f_3}{f_3} + \frac{c}{4} (k - k_1 k_2 - k_2 k_3 - k_1 k_3 + 1) \quad (4.24) \]

The last inequality leads to inequality (iii) in (1).

Next, we explore the equality cases of (1). First, we redefine the notion of the relative null space \( N_x \) of the submanifold \( U^k \) in the complex space form \( \bar{U}^m(c) \) at any point \( x \in U^k \), the relative null space was defined by B. Y. Chen [8], as follows

\[ N_x = \{ V_1 \in T_x U^k : h(V_1, V_2) = 0, \forall V_2 \in T_x U^k \}. \]

For \( A \in \{1, \ldots, k\} \) a unit vector field \( e_A \) tangent to \( U^k \) at \( x \) satisfies the equality sign of (4.1) identically if and only if

\[ (i) \sum_{p=1}^{k_1} \sum_{q=k_1+1}^{k} h_{pq}^r = 0 \quad (ii) \sum_{b=1}^{n} \sum_{A=1}^{k} h_{bA}^r = 0 \quad (iii) 2h_{pp}^r = \sum_{q=k_1+1}^{k} h_{qq}^r. \quad (4.25) \]

holds for \( r \in \{k + 1, \ldots, m\} \), which implies that \( U^k \) is mixed totally geodesic biwarped product submanifold. Combining statements (ii) and (iii) with the fact that \( U^k \) is biwarped product submanifold, we get that the unit vector field \( \chi = e_A \) belongs to the relative null space \( N_x \). The converse is trivial, this proves statement (2).

For a biwarped product submanifold, the equality sign of (4.1) holds identically for all unit tangent vector belong to \( U_T \) at \( x \) if and only if

\[ (i) \sum_{p=1}^{k_1} \sum_{q=k_1+1}^{k} h_{pq}^r = 0 \quad (ii) \sum_{b=1}^{n} \sum_{A=1}^{k} h_{bA}^r = 0 \quad (iii) 2h_{pp}^r = \sum_{q=k_1+1}^{k} h_{qq}^r. \quad (4.26) \]

where \( p \in \{1, \ldots, k_1\} \) and \( r \in \{k + 1, \ldots, m\} \). Since \( U^k \) is biwarped product submanifold, the third condition implies that \( h_{pp}^r = 0, \quad p \in \{1, \ldots, k_1\} \). Using this in the condition (ii), we conclude that \( U^k \) is \( S \)-totally geodesic biwarped product submanifold in \( \bar{U}^m(c) \) and mixed totally geodesicness follows from the condition (ii). Which proves (a) in the statement (3).

For a biwarped product submanifold, the equality sign of (4.2) holds identically for all unit tangent vector fields tangent to \( U_\perp \) at \( x \) if and only if

\[ (i) \sum_{p=1}^{k_1} \sum_{q=k_1+1}^{k} h_{pq}^r = 0 \quad (ii) \sum_{b=1}^{n} \sum_{A=k_1+1}^{k} h_{bA}^r = 0 \quad (iii) 2h_{KK}^r = \sum_{q=k_1+1}^{k} h_{qq}^r, \quad (4.27) \]

such that \( K \in \{k_1 + 1, \ldots, k_2\} \) and \( r \in \{k + 1, \ldots, m\} \). From the condition (iii) two cases emerge, that is

\[ h_{LL}^r = 0, \quad \forall L \in \{k_1 + 1, \ldots, k_2\} \quad \text{and} \quad r \in \{k + 1, \ldots, m\} \quad \text{or} \quad \dim U_\perp = 2. \quad (4.28) \]
If the first case of (4.27) satisfies, then by virtue of condition (ii), it is easy to conclude that $U^k$ is a $S^\perp$–totally geodesic biwarped product submanifold in $\bar{U}^m(c)$. This is the first case of part (b) of statement (3).

For a biwarped product submanifold, the equality sign of (4.3) holds identically for all unit tangent vector fields tangent to $U_{\theta}^{k_3}$ at $x$ if and only if

\begin{equation}
(i) \sum_{p=1}^{k_1} \sum_{q=k_1+1}^{k} h_{pq}^r = 0 \quad (ii) \sum_{b=1}^{n} \sum_{A=k_2+1}^{k_1} h_{bA}^r = 0 \quad (iii) 2h_{LL}^r = \sum_{q=k_1+1}^{n} h_{qq}^r
\end{equation}

such that $L \in \{k_2 + 1, \ldots, k\}$ and $r \in \{k + 1, \ldots, m\}$. From the condition (iii) two cases arise, that is

\begin{equation}
h_{LL}^r = 0, \quad \forall L \in \{k_2 + 1, \ldots, n\} \quad \text{and} \quad r \in \{k + 1, \ldots, m\} \quad \text{or} \quad \dim U_\theta = 2.
\end{equation}

If the first case of (4.29) satisfies, then by virtue of condition (ii), it is easy to conclude that $U^k$ is a $S^\theta$–totally geodesic biwarped product submanifold in $\bar{U}^m(c)$. This is the first case of part (c) of statement (3).

For the other case, assume that $U^k$ is not $S^\theta$–totally geodesic biwarped product submanifold and $\dim U_\theta = 2$. Then condition (ii) of (4.29) implies that $U^k$ is $S^\theta$–totally umbilical biwarped product submanifold in $\bar{U}(c)$, which is second case of this part. This verifies part (c) of (3).

To prove (d) using parts (a), (b) and (c) of (3), we combine (4.26), (4.27) and (4.29). For the first case of this part, assume that $\dim U_\perp \neq 2$ and $\dim U_\theta \neq 2$. Since from parts (a), (b) and (c) of statement (3) we conclude that $U^k$ is $S$–totally geodesic, $S^\perp$–totally geodesic and $S^\theta$–totally geodesic submanifolds in $\bar{U}^m(c)$. Hence $U^k$ is a totally geodesic submanifold in $\bar{U}^m(c)$.

For another case, suppose that first case does not satisfy. Then parts (a), (b) and (c) provide that $U^k$ is mixed totally geodesic and $S$–totally geodesic submanifold of $\bar{U}^m(c)$ with $\dim U_\perp = 2$ and $\dim U_\theta = 2$. From the conditions (b) and (c) it follows that $U^k$ is $S^\perp$– and $D^\theta$–totally umbilical biwarped product submanifolds and from (a) it is $S$–totally geodesic, which is part (d). This proves the theorem. □

If $U_{\perp}^{k_3} = \emptyset$, then the biwarped product submanifold becomes the Point wise semi-slant warped product submanifold that is $U^k = U_T^{k_1} \times_f U_\theta^{k_3}$. Now, we have the following corollary which can be deduced from the Theorem 4.2.

**Corollary 4.2.** Let $U^k = U_T^{k_1} \times_f U_\theta^{k_3}$ be a pointwise semi-slant warped product submanifold isometrically immersed in a complex space form $\bar{U}(c)$. Then for each orthogonal unit vector field $\chi \in T_xU$, either tangent to $U_T^{k_1}$ or $U_\theta^{k_3}$, we have

1. The Ricci curvature satisfy the following inequalities
   
   (i) If $\chi$ is tangent to $U_T^{k_1}$, then
      \begin{equation}
      \frac{1}{4}k^2\|H\|^2 \geq R(\chi) + \frac{k_3\Delta f_3}{f_3} + \frac{c}{4}(k - k_1k_3 - \frac{1}{2}).
      \end{equation}

   (ii) $\chi$ is tangent to $U_\theta^{k_3}$, then
      \begin{equation}
      \frac{1}{4}k^2\|H\|^2 \geq R(\chi) + \frac{k_3\Delta f_3}{f_3} + \frac{c}{4}(k - k_1k_3 + 1 - \frac{3}{2}\cos^2 \theta).
      \end{equation}
(2) If $H(x) = 0$, then each point $x \in U^k$ there is a unit vector field $\chi$ which satisfies the equality case of (1) if and only if $U^k$ is mixed totally geodesic and $\chi$ lies in the relative null space $N_\chi$ at $x$.

(3) For the equality case we have

(a) The equality case of (4.31) holds identically for all unit vector fields tangent to $U_T$ at each $x \in U^k$ if and only if $U^k$ is mixed totally geodesic and $S$–totally geodesic point wise semi slant warped product submanifold in $\tilde{U}^m(c)$.

(b) The equality case of (4.32) holds identically for all unit vector fields tangent to $U^k_{\theta^k}$ at each $x \in U^k$ if and only if $S$ is mixed totally geodesic and either $U^k$ is $D^\theta$–totally geodesic point wise semi slant warped product submanifold or $U^k$ is a $S^\theta$ totally umbilical in $\tilde{U}^m(c)$ with $\dim S^\theta = 2$.

(c) The equality case of (1) holds identically for all unit tangent vectors to $U^k$ at each $x \in U^k$ if and only if either $U^k$ is totally geodesic submanifold or $U^k$ is a mixed totally geodesic totally umbilical and $S$–totally geodesic submanifold with $\dim U_\theta = 2$.

where $k_1$ and $k_2$ are the dimensions of $U^k_{\theta^k}$ and $U^k_{\theta^2}$ respectively.

Now, we have another case that is if $U^k_{\theta^2} = \emptyset$ then the biwarped product submanifold becomes the CR-warped product submanifold. In this case we have the following corollary.

**Corollary 4.3.** Let $U^k = U^{k_1}_T \times_{\theta^2} U^{k_2}_\perp$ be a CR-warped product submanifold isometrically immersed in a complex space form $\tilde{U}^m(c)$. Then for each orthogonal unit vector field $\chi \in T_x U$, either tangent to $U^{k_1}_T$ or $U^{k_2}_\perp$, we have

(1) The Ricci curvature satisfy the following inequalities

(i) If $\chi$ is tangent to $U^{k_1}_T$, then

$$\frac{1}{4}k_1^2\|H\|^2 \geq R(\chi) + \frac{U_2\Delta f_2}{f_2} + \frac{c}{4}(k_1k_2 - \frac{1}{2}).$$  \hfill (4.33)

(ii) If $\chi$ is tangent to $U^{k_2}_\perp$, then

$$\frac{1}{4}k_2^2\|H\|^2 \geq R(\chi) + \frac{k_2\Delta f_2}{f_2} + \frac{c}{4}(k_1k_2 + 1).$$  \hfill (4.34)

(2) If $H(x) = 0$, then each point $x \in U^k$ there is a unit vector field $\chi$ which satisfies the equality case of (1) if and only if $U^k$ is mixed totally geodesic and $\chi$ lies in the relative null space $N_\chi$ at $x$.

(3) For the equality case we have

(a) The equality case of (4.33) holds identically for all unit vector fields tangent to $U_T$ at each $x \in U^k$ if and only if $U^k$ is mixed totally geodesic and $S$–totally geodesic CR-warped product submanifold in $\tilde{U}^m(c)$.

(b) The equality case of (4.34) holds identically for all unit vector fields tangent to $U^{k_2}_\perp$ at each $x \in U^k$ if and only if $U$ is mixed totally geodesic and either $U^k$ is $S^\perp$– totally geodesic biwarped product or $U^k$ is a $S^\perp$ totally umbilical in $\tilde{U}^m(c)$ with $\dim S^\perp = 2$.

(c) The equality case of (1) holds identically for all unit tangent vectors to $U^k$ at each $x \in U^k$ if and only if either $U^k$ is totally geodesic submanifold or $U^k$ is a mixed totally geodesic totally umbilical and $S$–totally geodesic submanifold with $\dim U_\perp = 2$.

where $k_1$ and $k_2$ are the dimensions of $U^{k_1}_T$ and $U^{k_2}_\perp$ respectively.
Acknowledgments

The authors are highly thankful to anonymous referees and the handling editor for their valuable suggestions and comments which have improved the contents of the paper. This work was supported by Taif University Researchers Supporting Project number (TURSP-2020/223), Taif University, Taif, Saudi Arabia.

Conflict of interest

The authors declare that they have no any conflict of interest.

References

1. R. L. Bishop, B. O’Neil, Manifolds of negative curvature, *T. Am. Math. Soc.*, 145 (1969), 1–9.
2. J. K. Beem, P. Ehrlich, T. G. Powell, Warped product manifolds in relativity, In: Th. M Rassias, G. M. Rassias, Editor, *Selected Studies: Physics-Astrophysics, Mathematics, History of Science*, Amsterdam: North-Holland, 1982, 41–56.
3. S. W. Hawkings, G. F. R. Ellis, *The large scale structure of space-time*, Cambridge: Cambridge Univ. Press, 1973.
4. B. O’Neill, *Semi-Riemannian geometry with application to relativity*, Academic Press, 1983.
5. B. Y. Chen, CR-submanifolds of a Kaehler manifold I, *J. Differ. Geom.*, 16 (1981), 305–323.
6. B. Y. Chen, Geometry of warped product CR-submanifolds in Kaehler manifolds I, *Monatsh Math.*, 133 (2001), 177–195.
7. B. Y. Chen, A survey on geometry of warped product submanifolds, *arXiv:1307.0236*.
8. B. Y. Chen, Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimension, *Glasg. Math. J.*, 41 (1999), 33–41.
9. K. Arslan, R. Ezentas, I. Mihai, C. Ozgur, Certain inequalities for submanifolds in \((k,\mu)\)–contact space form, *B. Aust. Math. Soc.*, 64 (2001), 201–212.
10. D. Cioroboiu, B. Y. Chen, Inequalities for semi-slant submanifolds in Sasakian space forms, *IJMMS*, 27 (2003), 1731–1738.
11. A. Mihai, C. Ozgur, Chen inequalities for submanifolds of real space forms with a semi-symmetric metric connection, *Taiwan. J. Math.*, 14 (2010), 1465–1477.
12. D. W. Yoon, Inequality for Ricci curvature of slant submanifolds in cosymplectic space forms, *Turk. J. Math.*, 30 (2006), 43–56.
13. M. Aquib, J. W. Lee, G. E. Vîlcu, W. Yoon, Classification of Casorati ideal Lagrangian submanifolds in complex space forms, *Differ. Geom. Appl.*, 63 (2019), 30–49.
14. M. M. Tripathi,Improved Chen-Ricci inequality for curvature-like tensors and its applications, *Differ. Geom. Appl.*, 29 (2011), 685–698.
15. A. Mustafa, S. Uddin, F. R. Al-Solamy, Chen-Ricci inequality for warped products in Kenmotsu space forms and its applications, *RACSAM*, 113 (2019), 1–18.
16. B. Y. Chen, F. Dillen, Optimal inequality for multiply warped product submanifolds, *IEJG*, 1 (2008), 1–11.

17. J. P. Baker, *Twice warped products*, M. Sc. Thesis, University of Missouri-Columbia, Columbia, 1997.

18. H. M. Tastan, Biwarped product submanifolds of a Kaehler manifold, *Filomat*, 32 (2018), 2349–2365.

19. M. A. Khan, K. Khan, Biwarped product submanifolds of complex space forms, *Int. J. Geom. Methods Mod. Phys.*, 16 (2019), 1950072.