A conformal-type energy inequality on hyperboloids and its application to quasi-linear wave equation in $\mathbb{R}^{3+1}$

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Abstract

In the present work, we will develop a conformal inequality in the hyperbolic foliation context which is analogous to the conformal inequality in the classical time-constant foliation context. Then as an application, we will apply this a priori estimate to the problem of global existence of quasi-linear wave equations in three spatial dimensions under null condition. With the aid of this inequality, we can establish more precise decay estimates on the global solution.

1 Introduction

In this article we will develop a conformal inequality in the hyperbolic foliation context analogue to the classical time-constant foliation context. Then we will apply this estimate to the problem of global existence of regular solution associated to small regular initial data (also called the global stability for short in the following text) for quasi-linear wave equation in three spatial dimensions. More precisely, we are going to regard the following quasi-linear wave equation:

\begin{equation}
\Box u + Q^{\alpha\beta\gamma} \partial_{\gamma} u \partial_{\alpha} \partial_{\beta} u = 0
\end{equation}

where $Q$ is supposed to be a null cubic form.

One can also consider a system with semi-linear terms such as:

\begin{equation}
\Box u + Q^{\alpha\beta\gamma} \partial_{\gamma} u \partial_{\alpha} \partial_{\beta} u = N^{\alpha\beta} \partial_{\alpha} u \partial_{\beta} u
\end{equation}

with $N$ a null quadratic form. But through an algebraic observation one can show that by a change of known

\[v = u + \frac{\sigma}{2} u^2, \quad \sigma = N^{00}\]

the semi-linear term can be eliminated with some high-order correction terms, which are negligible in dimension three.

1.1 The conformal inequality on hyperboloids

In the classical flat foliation context, the conformal energy inequality is a well-known $L^2$ type estimate which controls more quantities than the classical energy inequality. We recall that for

\[\Box u = f,\]

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For the linear equation
\[ H (1.7) \]
the following estimate holds:
\[ u (1.5) \]
written as
\[ □ \]
are bounded by (1.4)
\[ \text{method supplies better decay bounds on the solution} \]
Compared with the classical energy-vector field argument, this conformal energy-vector field
\[ \text{are bounded. Then by some estimates on commutators and the Klainerman-Sobolev type inequal-} \]
\[ (1.3) \]
from the above estimates we can see the following facts. For the homogeneous wave equation
\[ □ u = 0, \] if we differentiate the equation with respect to \( ∂^I L^J \),
\[ □ ∂^I L^J u = 0 \]
and by (1.3), we see that \( E_{\text{con}}(t, \partial^I L^J u) \) is bounded by the initial conformal energy. This leads to the fact that
\[ (s/t) ((t^2 + r^2) \partial_r + 2x^a t \partial_a + 2t) u \]
where
\[ s = \sqrt{t^2 - r^2} \] is the hyperbolic time. In the flat (Minkowski metric) case, this energy is
\[ E_{\text{con}}(s, u) = \int_{\mathbb{R}^3} |s(s/t)^2 \partial_t u|^2 + (s/t)^2 \sum_a |L_a u|^2 + (s/t) u^2 \text{dx}. \]
Here \( \mathcal{K}_s = \{ x \in \mathbb{R}^{1+3} | t = \sqrt{s^2 + r^2} \} \), and for a \( f \) defined in \( \mathbb{R}^{1+3} \), we define
\[ \int_{\mathcal{K}_s} f \text{dx} := \int_{\mathbb{R}^3} f(\sqrt{s^2 + |x|^2}, x) \text{dx}. \]
For the linear equation
\[ □ u = f \]
the following estimate holds:
\[ (1.7) \]
For the homogeneous linear wave equation $\Box u = 0$, the above quantity is conserved. For the quasi-linear wave equation (1.1), we will prove that this energy is also bounded up to the highest order.

Combined with the global Sobolev’s inequality, the above energy bounds leads to the following terms bounded by $C(t + 1)^{-3/2}$:

$$(s/t)L_\alpha u, \quad s(s/t)^2 \partial_t (s/t)u,$$

and these bounds gives the following decay rate:

$$u \sim (t + 1)^{-1}(|t - r| + 1)^{-1/2}, \quad \partial_\alpha u \sim (t + 1)^{-1}(|t - r| + 1)^{-3/2}$$

which coincides with the classical conformal energy bounds. We will prove that in the quasi-linear case, these decay rates still hold.

### 1.2 The global existence result for quasi-linear wave equation

The problem on the stability of quasi-linear wave equations or systems has attracted lots of attentions of the mathematical community. Our method to be presented belongs to the “vector-field method” which was introduced by S. Klainerman for wave equation (11) and for Klein-Gordon equation (5). This method is then extended to many different cases.

The global existence of (1.1) has been established by S. Klainerman in [4]. His method is based on the time-constant foliation and standard energy inequalities.

The hyperbolic foliation and hyperbolic variables are introduced by S. Klainerman firstly for the analysis on Klein-Gordon equation in [5], see also [3] for the “alternative energy method”. This method is then revisited and applied by P. LeFloch et al. in [6] on system composed by wave equations and Klein-Gordon equations. The application of the hyperbolic foliation has the advantage that it does not require the scaling invariance of the system (for example, the wave-Klein-Gordon system).

However, the scaling invariance of the wave equation does supply more conserved quantities. The conformal inequality (1.3) is essentially due to the scaling invariance of the wave equation (i.e. it does not hold for Klein-Gordon equation, which does not enjoy the scaling invariance), and it leads to better decay estimates (1.5) compared with (1.6) supplied by the classical energy). In this article, we show that this conformal energy inequality on hyperboloids also leads to the global stability of (1.1) and it gives more precise decay rate for the global solution.

### 2 The conformal inequality on hyperboloids: flat case

#### 2.1 The hyperbolic variables and hyperbolic frame

In this subsection we briefly recall the notion of the hyperbolic frame.

We denote by $\mathcal{K} = \{(t, x) \in \mathbb{R}^{1+3} | t > |x| + 1\}$ the interior of the light cone. In this region we introduce the following parametrization by the hyperbolic variables:

$$\bar{x}^0 := s := \sqrt{t^2 - |x|^2}, \quad \bar{x}^a := x^a.$$  

The canonical frame associate to this parametrization is called the hyperbolic frame, denoted by:

$$\bar{\partial}_0 := \partial_s = \frac{s}{t} \partial_t, \quad \bar{\partial}_a := \frac{x^a}{t} \partial_t + \partial_a.$$  

The transition matrix between the hyperbolic frame and the natural frame is

$$\bar{\partial}_\alpha = \Phi_\alpha^\beta \partial_\beta$$
with

\[ (\Psi^0_{\alpha\beta}) := \begin{pmatrix} s/t & 0 & 0 & 0 \\ x^1/t & 1 & 0 & 0 \\ x^2/t & 0 & 1 & 0 \\ x^3/t & 0 & 0 & 1 \end{pmatrix} \]

and its inverse is

\[ (\Psi^3_{\alpha\beta}) = \begin{pmatrix} t/s & 0 & 0 & 0 \\ -x^1/s & 1 & 0 & 0 \\ -x^2/s & 0 & 1 & 0 \\ -x^3/s & 0 & 0 & 1 \end{pmatrix} \]

thus

\[ \partial_\alpha = \Psi^3_{\alpha\beta} \partial_\beta. \]

Now we introduce the following notation. Let \( T \) be a two-tensor, we recall that it can be written in the natural frame \( \{\partial_\alpha\} \) or in the hyperbolic frame \( \{\bar{\partial}_\alpha\} \). That is

\[ T = T^{\alpha\beta} \partial_\alpha \otimes \partial_\beta = T^{\alpha\beta} \bar{\partial}_\alpha \otimes \bar{\partial}_\beta \]

with

\[ T^{\alpha\beta} = T^{\alpha\beta'} \Psi^0_{\alpha\beta'} \Psi^3_{\beta'\delta}. \]

For a cubic form \( Q \), we also have similar notation:

\[ Q = Q^{\alpha\beta\gamma} \partial_\alpha \otimes \partial_\beta \otimes \partial_\gamma = Q^{\alpha\beta\gamma} \bar{\partial}_\alpha \otimes \bar{\partial}_\beta \otimes \bar{\partial}_\gamma, \]

and

\[ Q^{\alpha\beta\gamma} = Q^{\alpha'\beta'\gamma'} \Psi^0_{\alpha\beta'} \Psi^3_{\beta'\gamma'}. \]

We also recall the dural co-frame to the hyperbolic frame:

\[ \bar{\omega}^0 = (t/s) dt - \sum_a (x^a/s) dx^a, \quad \bar{\omega}^a = dx^a. \]

We recall the Minkowski metric \( m^{\alpha\beta} \) written in the hyperbolic frame:

\[ m^{\alpha\beta} = \begin{pmatrix} 1 & x^1/s & x^2/s & x^3/s \\ x^1/s & -1 & 0 & 0 \\ x^2/s & 0 & -1 & 0 \\ x^3/s & 0 & 0 & -1 \end{pmatrix}. \]

And we remark that for a second order differential operator \( g^{\alpha\beta} \partial_\alpha \partial_\beta \), we see that

\[ g^{\alpha\beta} \partial_\alpha \partial_\beta = g^{\alpha\beta} \bar{\Psi}_{\alpha\beta} \bar{\partial}_\alpha \bar{\partial}_\beta + g^{\alpha\beta} \partial_\alpha \left( \Psi^3_{\beta} \right) \bar{\partial}_{\beta'} = \bar{g}^{\alpha\beta} \bar{\partial}_\alpha \bar{\partial}_\beta + g^{\alpha\beta} \partial_\alpha \left( \Psi^3_{\beta} \right) \bar{\partial}_{\beta'}. \]

Here we list out \( \partial_\alpha \left( \Psi^3_{\beta} \right) \):

(2.1a) \[ \partial_t \left( \Psi^3_0 \right) = \frac{-t^2}{s^3} = -s^{-1} \Psi^0_0 \Psi^0_0 + s^{-1}, \quad \partial_t \left( \Psi^3_a \right) = \frac{x^a t}{s^3} = -s^{-1} \Psi^0_a \Psi^0_a \]

and

(2.1b) \[ \partial_\alpha \left( \Psi^3_0 \right) = \frac{tx^a}{s^3} = -s^{-1} \Psi^0_0 \Psi^0_a, \quad \partial_\alpha \left( \Psi^3_b \right) = -\frac{\delta_{ab}}{s^3} - s^{-1} \Psi^0_a \Psi^0_b. \]

The rest components of \( \partial_\alpha \left( \Psi^3_{\beta'} \right) \) are zero.

Now we recall that

\[ \Box = m^{\alpha\beta} \partial_\alpha \partial_\beta \]

thus in hyperbolic frame:

(2.2) \[ \Box = m^{\alpha\beta} \bar{\partial}_\alpha \bar{\partial}_\beta + m^{\alpha\beta} \partial_\alpha \left( \Psi^3_{\beta} \right) \bar{\partial}_{\beta'} = \bar{\partial}_\alpha \bar{\partial}_\alpha + 2(x^a/s) \bar{\partial}_a \bar{\partial}_a - \sum_a \bar{\partial}_a \bar{\partial}_a + \frac{3}{s} \bar{\partial}_a. \]
2.2 Hyperbolic decomposition of \( \Box \)

We write (2.2) into the following form:

\[
\Box u = \partial_s (\partial_s + (2x^a/s)\partial_a) u - \sum_a \partial_a \partial_s u + \frac{2x^a}{s^2} \partial_a u + \frac{3}{s} \partial_s u
\]

(2.3)

Thus

\[
= \partial_s (\partial_s + (2x^a/s)\partial_a) u - \sum_a \partial_a \partial_s u + s^{-1} (\partial_s + (2x^a/s)\partial_a) u + \frac{2}{s} \partial_s u
\]

\[
= s^{-1} \partial_s (s \partial_s u + 2x^a \partial_a u) - \sum_a \partial_a \partial_s u + \frac{2}{s} \partial_s u
\]

We denote by

\[
K u := (s \partial_s + 2x^a \partial_a) u = \frac{l^2 + r^2}{t} \partial_t u + 2r \partial_r u
\]

2.3 The energy identity

We make the following calculation:

\[
sK u \cdot \Box u = (s (\partial_s + (2x^a/s)\partial_a) u) \partial_s (s (\partial_s + (2x^a/s)\partial_a) u)
\]

\[
- \sum_a \partial_a (s^2 (\partial_s u + (2x^b/s)\partial_b u) \cdot \partial_a u) + s^2 \sum_a \partial_a ((\partial_s u + (2x^b/s)\partial_b u)) \partial_s u
\]

\[
+ 2s \partial_s u \cdot (\partial_s u + (2x^b/s)\partial_b u) - \sum_a \partial_a (s^2 (\partial_s u + (2x^b/s)\partial_b u) \cdot \partial_a u)
\]

(2.4)

\[
= \frac{1}{2} \partial_s (|K u|^2) + \sum_a \partial_s (|s \partial_a u|^2)
\]

\[
+ 2s \left( \partial_s u \cdot (\partial_s u + (2x^b/s)\partial_b u) - \sum_a |\partial_a u|^2 \right)
\]

\[
+ \sum_a \partial_a (sx^b \partial_a u) - \sum_a \partial_a (s^2 (\partial_s u + (2x^b/s)\partial_b u) \cdot \partial_a u)
\]

We also remark that

\[
u \cdot \Box u = \partial_s (u \partial_s u) - (\partial_s u)^2 + s^{-1} \partial_s (2x^a u \partial_a u) - (2x^a / s) \partial_s u \partial_s u
\]

\[
- \sum_a \partial_a (u \partial_a u) + \sum_a (\partial_a u)^2 + \frac{n}{2} s^{-1} \partial_s (u^2)
\]

\[
= s^{-1} \partial_s (su \partial_s u) + s^{-1} \partial_s (u^2) + s^{-1} \partial_s (2x^a u \partial_a u) - \sum_a \partial_a (u \partial_a u)
\]

\[
- \left( |\partial_s u|^2 + (2x^a / s) \partial_s u \partial_s u - \sum_a |\partial_a u|^2 \right)
\]

thus

\[
su \cdot \Box u = \partial_s (su \partial_s u) + \partial_s (u^2) + \partial_s (2x^a u \partial_a u) - \sum_a \partial_a (su \partial_a u)
\]

\[
- s \left( (\partial_s u)^2 + 2(x^a / s) \partial_s u \partial_s u - \sum_a (\partial_a u)^2 \right).
\]
So we see that

\[
2su \cdot \Box u + 2s \left( (\partial_s u)^2 + 2(\bar{x}^a/s)\partial_s u \partial_{\bar{x}^a} u - \sum_a (\partial_a u)^2 \right) 
\]

\[
= \partial_s \left( 2su \partial_s u + 4ux^a \partial_{\bar{x}^a} u + 2u^2 \right) - \sum_a \partial_a \left( 2su \partial_a u \right).
\]

Then we combine (2.4) and (2.5),

\[
2s (Ku + 2u) \square u = \partial_s \left( |Ku|^2 + \sum_a |s\partial_a u|^2 + 4uKu + 4u^2 \right) + \partial_a (v^a)
\]

with

\[
v^a := -4su \partial_a u + 2sx^a \sum_b |\partial_b u|^2 - 2s\partial_a u \cdot Ku
\]

We define the flat energy

\[
E_{\text{con}}(s, u) := \int_{\mathcal{M}_s} (|Ku|^2 + \sum_a |s\partial_a u|^2 + 4uKu + 4u^2) \, dx
\]

\[
= \int_{\mathcal{M}_s} \left( \sum_a (s\partial_a u)^2 + (Ku + 2u)^2 \right) \, dx
\]

Then (2.6) leads to

\[
2 \int_{\mathcal{M}_s} s (K + 2) u \Box u \, dx = \frac{d}{ds} E_{\text{con}}(s, u)
\]

Remark that \(\|(Ku + 2u)\|_{L^2(\mathcal{M}_s)} \leq E_{\text{con}}(s, u)^{1/2}\),

\[
\frac{d}{ds} E_{\text{con}}(s, u)^{1/2} \leq 2s\|\Box u\|_{L^2(\mathcal{M}_s)}
\]

which leads to

\[
E_{\text{con}}(s, u)^{1/2} \leq E_{\text{con}}(s_0, u)^{1/2} + 2 \int_{s_0}^s \tau \|\Box u\|_{L^2(\mathcal{M}_s)} \, d\tau.
\]

We conclude the above calculation by the following conformal energy estimate in flat case

**Proposition 2.1.** Let \(u\) be a function defined in \(\mathcal{K}_{[s_0, s_1]}\), sufficiently regular and vanishes near the conical boundary \(\partial \mathcal{K} := \{(t, x) | t = r + 1\}\). Then the estimate (2.8) holds.

### 2.4 Analysis on the flat energy

It is clear that the flat energy can control the following quantities:

\[
\|Ku + 2u\|_{L^2(\mathcal{M}_s)}, \quad \|\partial_s u\|_{L^2(\mathcal{M}_s)}
\]

and in this subsection we will prove the following property:

**Proposition 2.2.** Let \(n = 3\) and \(u\) be a sufficiently regular function defined in the region \(\mathcal{K}_{[s_0, s_1]}\) and vanishes near the conical boundary. Then the following inequality holds for \(s_0 \leq s \leq s_1\):

\[
\|(s/r)u\|_{L^2(\mathcal{M}_s)} \leq 2E_{\text{con}}(s, u)^{1/2},
\]

\[
\|s^2t^{-1}\partial_s u\|_{L^2(\mathcal{M}_s)} \leq CE_{\text{con}}(s, u)^{1/2}
\]

and

\[
\|s(s/t)^2\partial_s u\|_{L^2(\mathcal{M}_s)} \leq CE_{\text{con}}(s, u)^{1/2}
\]

with \(C\) a universal constant.
Proof. For (2.9), we recall the Hardy’s inequality for \( \mathbb{R}^3 \). Let \( w \) be a sufficiently regular function defined in \( \mathbb{R}^3 \) and decreases sufficiently fast at infinity, then

\[
\|w/r\|_{L^2(\mathbb{R}^3)} \leq 2 \sum_{\alpha} \|\partial_\alpha w\|_{L^2(\mathbb{R}^3)}.
\]

We define (for a fixed \( s \)) \( \tilde{u}_s : \mathbb{R}^3 \to \mathbb{R} \):

\[
\tilde{u}_s(x) := su(\sqrt{s^2 + r^2}, x).
\]

We see that

\[
\partial_\alpha \tilde{u}_s = s \bar{\partial}_\alpha u
\]

and we apply (2.12) with \( w = \tilde{u}_s \) (remark that \( \tilde{u}_s \) is compactly supported), this leads to

\[
\|su/r\|_{L^2(H^s)} \leq 2 \sum_{\alpha} \|s \bar{\partial}_\alpha u\|_{L^2(H^s)} \leq 2E_{\text{con}}(s, u)^{1/2}
\]

which is (2.9).

To establish (2.10), we remark that

\[
Ku + 2u = s \bar{\partial}_s u + 2x^\alpha \bar{\partial}_\alpha u + 2u
\]

which leads to

\[
(s^2/t)\bar{\partial}_s u \leq (s/t)|Ku + 2u| + 2 \sum_{\alpha} |(x^\alpha/t)s \bar{\partial}_\alpha u| + 2|s/tu|
\]

thus

\[
\|(s^2/t)\bar{\partial}_s u\|_{L^2(H^s)} \leq \|(s/t)(Ku + 2u)\|_{L^2(H^s)} + 2 \sum_{\alpha} \|s \bar{\partial}_\alpha u\|_{L^2(H^s)} + 2\|(s/r)u\|_{L^2(H^s)}
\]

\[
\leq 7E_{\text{con}}(s, u)^{1/2}.
\]

For the third one, we remark that

\[
s(s/t)^2 \partial_s u = s(s/t)^2 \bar{\partial}_s u - s(s/t)^2 (x^\alpha/t) \partial_\alpha u
\]

which leads to

\[
\|s(s/t)^2 \partial_s u\|_{L^2(H^s)} \leq \|s \bar{\partial}_s u\|_{L^2(H^s)} + \|s(s/t) \bar{\partial}_s u\|_{L^2(H^s)} \leq CE_{\text{con}}(s, u)^{1/2}.
\]

\[\square\]

Remark 2.3. We see that when \( n = 3 \) and when the flat energy is satisfies the following increasing condition:

\[
E_{\text{con}}(s, u)^{1/2} \leq Cs^\delta
\]

then

\[
\|sr^{-1}u\|_{L^2(H^s)} \leq Cs^\delta.
\]

3 The conformal inequality on hyperboloids: curved case

3.1 Differential identity

We suppose that \( g^{\alpha\beta} \) is a (symmetric) Lorentzian metric defined in \( \mathcal{K} \), sufficiently regular, and coincides with the Minkowski metric near the light cone \( \partial \mathcal{K} \). Then we remark the following calculation:

\[
g^{\alpha\beta} \bar{\partial}_\alpha \bar{\partial}_\beta u = g^{\alpha\beta} \bar{\partial}_\alpha \bar{\partial}_\beta u + g^{\alpha\beta} \bar{\partial}_\alpha \left( \overline{\nabla}_\beta \right) \bar{\partial}_\beta u
\]

\[
= g^{\alpha\beta} \bar{\partial}_\alpha \bar{\partial}_\beta u + g^{\alpha\beta} \bar{\partial}_\alpha \left( \overline{\nabla}_\beta \right) \bar{\partial}_\beta u.
\]
and we thus have
\[
g^{\alpha\beta} \partial_{\alpha} \partial_{\beta} u = \partial_u \left( g^{00} \partial_0 u + 2 g^{0a} \partial_a u \right) - \partial_a g^{0a} \partial_u u - 2 \partial_u g^{00} \partial_a u \\
+ \bar{g}^{ab} \partial_a \partial_b u + \bar{g}^{\alpha\beta} \partial_u \left( \bar{\nabla}_{\beta} \right) \partial_{\alpha} u \\
= \partial_u \left( g^{00} \partial_0 u + 2 g^{0a} \partial_a u \right) + s^{-1} \left( g^{00} \partial_0 u + 2 g^{0a} \partial_a u \right) + \bar{g}^{ab} \partial_a \partial_b u \\
- \partial_a \bar{g}^{0a} \partial_u u - 2s^{-1} \left( \bar{g}^{00} + s \partial_u g^{0a} \right) \partial_a u + \left( \bar{g}^{\alpha\beta} \partial_u \left( \bar{\nabla}_{\beta} \right) - s^{-1} \bar{g}^{00} \right) \partial_{\alpha} u \\
= s^{-1} \partial_u \left( s \left( g^{00} \partial_0 u + 2 g^{0a} \partial_a u \right) \right) + \bar{g}^{ab} \partial_a \partial_b u \\
- \partial_a \bar{g}^{0a} \partial_u u - 2s^{-1} \left( \bar{g}^{00} + s \partial_u g^{0a} \right) \partial_a u + \left( \bar{g}^{\alpha\beta} \partial_u \left( \bar{\nabla}_{\beta} \right) - s^{-1} \bar{g}^{00} \right) \partial_{\alpha} u. 
\]

We denote by
\[
\mathcal{K}_g = s \left( g^{00} \partial_0 + 2 g^{0a} \partial_a \right).
\]

and we calculate the following relations:
\[
g^{0a} + s \partial_a g^{0a} = s \bar{\nabla}_{\alpha} \partial_\alpha g^{0a} + (s/t) g^{0a}
\]

and
\[
\partial_a g^{00} = \bar{\nabla}_{\alpha} \bar{\nabla}_{\beta} \partial_a g^{\alpha\beta} - 2s^{-3} \left( r^2 g^{00} + x^a x^b g^{ab} \right) - 2g^{0a} \bar{\nabla}_{\alpha} \bar{\nabla}^{\alpha} \frac{t^2 + r^2}{ts^2}.
\]

and
\[
g^{a\beta} \partial_u \bar{\nabla}^{\alpha} = - s^{-1} \left( g^{00} (r/s)^2 - 2 g^{0a} \frac{t}{s} \frac{x^a}{s} + g^{ab} \left( \frac{x^a}{s} \frac{x^b}{s} + \partial_u \right) \right)
\]
\[
= - s^{-1} \left( g^{00} \frac{(t/s)^2}{s} - 2 g^{0a} \frac{t}{s} \frac{x^a}{s} + g^{ab} \frac{x^a x^b}{s^2} \right) + s^{-1} \left( g^{00} - \sum_a g^{aa} \right)
\]
\[
= - s^{-1} g^{00} + s^{-1} \left( g^{00} - \sum_a g^{aa} \right).
\]

Then as the flat case, we use the multiplier \( s \mathcal{K}_g u \) and see that
\[
s \mathcal{K}_g u \cdot s^{-1} \partial_u \left( \mathcal{K}_g u \right) = \frac{1}{2} \partial_u (| \mathcal{K}_g u |^2)
\]
\[
s \mathcal{K}_g u \cdot \bar{g}^{ab} \partial_a \partial_b u = s \partial_u \left( \mathcal{K}_g u \right) \bar{g}^{ab} \partial_a \partial_b u - s \partial_u \left( \mathcal{K}_g u \right) \bar{g}^{ab} \partial_u \partial_b u - s \partial_u \bar{g}^{ab} \cdot \mathcal{K}_g u \partial_b u
\]
where
\[
s \partial_u \left( \mathcal{K}_g u \right) \bar{g}^{ab} \partial_a \partial_b u = s^2 \partial_u \left( g^{00} \partial_0 u + 2 g^{0a} \partial_a u \right) \cdot \bar{g}^{ab} \partial_b u
\]
\[
= s^2 \partial_u g^{00} \partial_0 \partial_a \partial_b u \cdot \bar{g}^{ab} \partial_b u + 2 s^2 g^{0a} \partial_0 \partial_a \partial_a \partial_b u \cdot \bar{g}^{ab} \partial_b u \\
+ s^2 \partial_a g^{00} \cdot \bar{g}^{ab} \partial_a \partial_b u + 2 s^2 \partial_a \bar{g}^{00} \cdot \bar{g}^{ab} \partial_a \partial_b u \\
- \frac{s^2}{2} \partial_u \left( g^{00} \bar{g}^{ab} \partial_a \partial_b u \right) + s^2 \partial_u \left( g^{00} \bar{g}^{ab} \partial_a \partial_b u \right) \\
- \frac{s^2}{2} \partial_u \left( \bar{g}^{00} \bar{g}^{ab} \partial_a \partial_b u \right) - s \partial_r \left( \bar{g}^{00} \bar{g}^{ab} \partial_a \partial_b u \right) \\
+ s^2 \partial_a \partial_0 g^{00} \cdot \bar{g}^{ab} \partial_a \partial_b u + 2 s^2 \partial_u g^{00} \cdot \bar{g}^{ab} \partial_a \partial_b u \\
= \frac{s^2}{2} \partial_u \left( \bar{g}^{00} \bar{g}^{ab} \partial_u \partial_b u \right) + \partial_u \left( s^2 \bar{g}^{00} \bar{g}^{ab} \partial_a \partial_b u \right) \\
- s g^{00} \bar{g}^{ab} \partial_a \partial_b u - s \partial_u \left( g^{00} \bar{g}^{ab} \partial_a \partial_b u \right) - s \partial_r \left( \bar{g}^{00} \bar{g}^{ab} \partial_a \partial_b u \right) \\
+ s^2 \partial_a \partial_0 g^{00} \cdot \bar{g}^{ab} \partial_a \partial_b u + 2 s^2 \partial_u g^{00} \cdot \bar{g}^{ab} \partial_a \partial_b u,
\[ s \mathcal{X}_g u \cdot (g^\alpha{}^\beta \partial_\alpha \left( \Psi_\beta^0 \right) - s^{-1} g^{00}) \partial_\alpha u = s \left( sg^{\alpha\beta} \partial_\alpha \left( \Psi_\beta^0 \right) - g^{00} \right) \left( g^{00} \partial_\alpha u \partial_\alpha u + 2g^{ab} \partial_a u \partial_b u \right). \]

Thus we see that
\begin{equation}
\tag{3.2}
s \mathcal{X}_g u \cdot g^{\alpha\beta} \partial_\alpha \partial_\beta u = \frac{1}{2} \partial_\alpha \left( |\mathcal{X}_g u|^2 - s^2 g^{00} g^{ab} \partial_\alpha u \partial_\beta u \right) + \partial_\alpha (v^\alpha_\alpha)
+ s \left( g^{00} g^{ab} + \partial_\alpha \left( g^{00} g^{ab} \right) - 2s \partial_\alpha g^{00} \cdot \partial_\alpha \partial_\beta u \right) \partial_\alpha u \partial_\beta u
+ \left( sg^{\alpha\beta} \partial_\alpha \left( \Psi_\beta^0 \right) - g^{00} \right) \mathcal{X}_g u \cdot \partial_\alpha u
+ \frac{s^2}{2} \partial_\alpha \left( g^{00} g^{ab} \right) \partial_\alpha u \partial_\beta u - s^2 \partial_\alpha \partial_\alpha g^{00} \cdot \partial_\alpha \partial_\beta u \partial_\beta u
- s \mathcal{X}_g u \cdot \partial_\alpha g^{00} \partial_\alpha u - 2 \mathcal{X}_g u \cdot \partial_\alpha \left( sg^{00} \right) \partial_\alpha u - s \partial_\alpha g^{ab} \cdot \partial_\alpha u \partial_\beta u, \]
\end{equation}
with
\[ v^\alpha_\alpha = s \mathcal{X}_g \cdot g^{ab} \partial_\alpha u - s^2 g^{00} g^{ab} \partial_\alpha u \partial_\beta u, \]
\[ su \cdot g^{\alpha\beta} \partial_\alpha \partial_\beta u = su \cdot g^{\alpha\beta} \partial_\alpha \partial_\beta u + su \cdot g^{\alpha\beta} \partial_\alpha \left( \Psi_\beta^0 \right) \partial_\beta u \]
\[ = \partial_\alpha \left( u \cdot \mathcal{X}_g u + \frac{1}{2} \left( sg^{\alpha\beta} \partial_\alpha \left( \Psi_\beta^0 \right) - \partial_\alpha \left( sg^{00} \right) \right) u^2 \right) + \partial_\alpha \left( su g^{ab} \partial_\beta u \right) \]
\[ - sg^{\alpha\beta} \partial_\alpha u \partial_\beta u + \frac{1}{2} \partial_\alpha \left( \partial_\alpha \left( sg^{00} \right) - sg^{\alpha\beta} \partial_\alpha \left( \Psi_\beta^0 \right) \right) u^2 - 2 \partial_\alpha \left( sg^{ab} \right) u \partial_\alpha u - \partial_\alpha \left( sg^{00} \right) u \partial_\beta u. \]

We denote by
\[ N_g := sg^{\alpha\beta} \partial_\alpha \left( \Psi_\beta^0 \right) - \partial_\alpha \left( sg^{00} \right) \]
and by (3.1) we see that
\[ N_g = g^{00} - \sum_a g^{aa} - 2g^{00} - s \partial_\alpha g^{00}. \]
When \( g^{\alpha\beta} = m^{\alpha\beta} \), we see that \( N_m = n - 1 \).
Then we see that
\[ \begin{aligned}
N_g su \cdot g^{\alpha\beta} \partial_\alpha \partial_\beta u &= -N_g sg^{\alpha\beta} \partial_\alpha u \partial_\beta u + \partial_\alpha \left( N_g u \cdot \mathcal{X}_g u + \frac{1}{2} N_g u^2 \right) + \partial_\alpha \left( N_g \cdot su g^{ab} \partial_\beta u \right) \\
& - \partial_\alpha N_g \cdot \left( u \cdot \mathcal{X}_g u + \frac{1}{2} N_g u^2 \right) - \partial_\alpha N_g \cdot su g^{ab} \partial_\beta u \\
& - \frac{1}{2} \partial_\alpha N_g \cdot N_g \cdot u^2 - N_g \left( 2 \partial_\alpha \left( sg^{ab} \right) u \partial_\alpha u + \partial_\alpha \left( sg^{ab} \right) u \partial_\beta u \right). 
\end{aligned} \]
\[ \tag{3.3} \]
We combine (3.2) and (3.3), and see that
\[ \begin{aligned}
s \left( \mathcal{X}_g u + N_g u \right) \cdot g^{\alpha\beta} \partial_\alpha \partial_\beta u &= \frac{1}{2} \partial_\alpha \left( |\mathcal{X}_g u + N_g u|^2 - s^2 g^{00} g^{ab} \partial_\alpha u \partial_\beta u \right) + \partial_\alpha (u^\alpha_\alpha) \\
& + R_g \left( \nabla u, \nabla u \right) + S_g \left[ \nabla u \right] \cdot \left( \mathcal{X}_g + N_g \right) u + T_g [u] 
\end{aligned} \]
\[ \tag{3.4} \]
where \( R_g (\nabla u, \nabla u) \) is a quadratic form acting on the gradient of \( u \):
\[ \begin{aligned}
R_g (\nabla u, \nabla u) := & s \left( g^{00} g^{ab} + \partial_\alpha \left( g^{00} g^{ab} \right) - 2s \partial_\alpha g^{00} \cdot \partial_\alpha \partial_\beta u \right) \partial_\alpha u \partial_\beta u \\
& + \left( sg^{\alpha\beta} \partial_\alpha \left( \Psi_\beta^0 \right) - \partial_\alpha \left( sg^{00} \right) \right) \mathcal{X}_g u \cdot \partial_\alpha u - N_g sg^{\alpha\beta} \partial_\alpha u \partial_\beta u, \\
& + \frac{s^2}{2} \partial_\alpha \left( g^{00} g^{ab} \right) \partial_\alpha u \partial_\beta u - s^2 \partial_\alpha g^{00} g^{ab} \partial_\alpha u \partial_\beta u \\
= & L_g \partial_\alpha u \partial_\beta u + N_g \mathcal{X}_g u \cdot \partial_\alpha u - N_g sg^{\alpha\beta} \partial_\alpha u \partial_\beta u \\
& + \frac{s^2}{2} \partial_\alpha \left( g^{00} g^{ab} \right) \partial_\alpha u \partial_\beta u - s^2 \partial_\alpha g^{00} g^{ab} \partial_\alpha u \partial_\beta u 
\end{aligned} \]
with
\[ L_g^{ab} := g^{00}g^{ab} + 2s\partial_c (g^{0c}g^{ab}) - 2s\partial_c\bar{g}^{0a} \cdot \bar{g}^{cb}, \]
\[ \mathcal{K}_g u \cdot S_g[\nabla u] := - (\mathcal{K}_g + N_g)u \cdot (2\partial_c(s\bar{g}^{0a})\partial_a u + s\partial_a\bar{g}^{ab}\partial_b u) \]
and
\[ T_g[u] := -\partial_c N_g \cdot u (\mathcal{K}_g + N_g) u - su \cdot \bar{g}^{ab}\partial_a N_g \partial_b u. \]

In (3.3),
\[ w_g = v_g^0 + N_g su \cdot \bar{g}^{ab}\partial_b u = s\mathcal{K}_g \cdot \bar{g}^{ab}\partial_b u - s^2 \bar{g}^{0a}g^{cb}\partial_a u \partial_b u + N_g su \cdot \bar{g}^{ab}\partial_b u. \]

We analyse the structure of \( R_g(\nabla u, \nabla u) \): remark the coefficient \( L_g^{ab} \)
\[ L_m^{ab} = 2\bar{m}^{ab}. \]

We also see that
\[ L_g^{ab} - L_m^{ab} = \bar{g}^{00}g^{ab} - \bar{m}^{ab} = s\partial_c (\bar{g}^{0c}g^{ab} - \bar{m}^{0c}m^{ab}) - 2s\partial_c\bar{g}^{0a}\bar{g}^{cb} + 2s\partial_a\bar{m}^{0a}m^{rb} \]
\[ = \bar{g}^{00}\bar{h}^{ab} + \bar{h}^{00}m^{ab} + s\partial_c (\bar{h}^{0c}g^{ab}) + s\partial_c (\bar{h}^{ab}m^{0c}) - 2s (\partial_c\bar{g}^{0a}\bar{h}^{cb} + \partial_c\bar{h}^{00}m^{ab}) \]

On the other hand, we see that
\[ N_m = 2 \]
and
\[ N_g - N_m = h^{00} - \sum_a h^{aa} = 2h^{00} - s\bar{\partial}_a\bar{h}^{00} \]

and
\[ sN_g \bar{g}^{0a}\partial_a u \partial_b u = sN_g (\bar{g}^{0a}\partial_a u + 2\bar{g}^{00}\partial_a u) \bar{\partial}_a u + sN_g \bar{g}^{ab}\partial_a u \partial_b u = N_g \mathcal{K}_g u \partial_a u + sN_g \bar{g}^{ab}\partial_a u \partial_b u. \]

Thus we see that
\[ R_g(\nabla u, \nabla u) = s (L_g^{ab} - L_m^{ab}) \bar{\partial}_a u \partial_b u - s (N_g \bar{g}^{ab} - 2\bar{m}^{ab}) \bar{\partial}_a u \partial_b u \]
\[ + \frac{s^2}{2} \partial_c (\bar{g}^{00}g^{ab}) \bar{\partial}_a u \partial_b u - s^2 \bar{\partial}_c \bar{g}^{00}g^{ab} \bar{\partial}_a u \partial_b u \]
\[ = s (L_g^{ab} - L_m^{ab}) \bar{\partial}_a u \partial_b u - s(N_g - N_m)g^{ab}\bar{\partial}_a u \partial_b u - 2s\bar{\partial}_a u \partial_b u \]
\[ + \frac{s^2}{2} \partial_c (\bar{g}^{00}g^{ab}) \bar{\partial}_a u \partial_b u - s^2 \bar{\partial}_c \bar{g}^{00}g^{ab} \bar{\partial}_a u \partial_b u. \]

### 3.2 Energy estimate in curved case

We integrate the identity (3.3) in the region \( \mathcal{K}_{[\kappa_0, \kappa]} \), remark that we suppose that \( u \) is sufficiently regular and vanishes near the conical boundary. By Stokes formula:
\[ \int_{\mathcal{K}_{[\kappa_0, \kappa]}} s (\mathcal{K}_g u + N_g u) \cdot g^{03}\partial_a \partial_b u \ dx ds = \frac{1}{2} (E_{con, g}(s, u) - E_{con, g}(s_0, u)) \]
\[ + \int_{\mathcal{K}_{[\kappa_0, \kappa]}} (R_g(\nabla u, \nabla u) + \mathcal{K}_g u \cdot S_g[\nabla u] + T_g[u]) \ dx ds \]
with
\[ E_{con, g} := \int_{\mathcal{K}_{\kappa}} (|\mathcal{K}_g u + N_g u|^2 - s^2 \bar{g}^{ab}\partial_a u \partial_b u) \ dx ds, \]
called the curved energy.
Thus, then

Recall the structure of the curved energy, we consider first the term \( \bar{\kappa} \)

Proof.

We suppose that there exists a \( \kappa \geq 1 \) such that

\[
(3.8) \quad \kappa^{-1} E_{\text{con},g}(s,u)^{1/2} \leq E_{\text{con}}(s,u)^{1/2} \leq \kappa E_{\text{con},g}(s,u)^{1/2},
\]

and

\[
(3.9) \quad \| R_g(\nabla u, \nabla u) + \mathcal{K} u \cdot S_g[u] + T_g[u] \|_{L^1(\mathcal{I}_s)} \leq E_{\text{con}}(s,u)^{1/2} M_g(s,u).
\]

Then we see that from (3.7)

\[
\frac{d}{ds} E_{\text{con},g}(s,u)^{1/2} \leq \| s g^{\alpha \beta} \partial_\alpha \partial_\beta u \|_{L^2(\mathcal{I}_s)} + M_g(s,u)
\]

Thus

\[
(3.10) \quad E_{\text{con},g}(s,u)^{1/2} \leq E_{\text{con},g}(s_0,u)^{1/2} + \int_{s_0}^{s} \left( \tau \| g^{\alpha \beta} \partial_\alpha \partial_\beta u \|_{L^2(\mathcal{I}_\tau)} + M_g(\tau,u) \right) d\tau
\]

which (combined with (3.8)) leads to

\[
(3.11) \quad E_{\text{con}}(s,u)^{1/2} \leq \kappa E_{\text{con}}(s_0,u)^{1/2} + \kappa \int_{s_0}^{s} \left( \tau \| g^{\alpha \beta} \partial_\alpha \partial_\beta u \|_{L^2(\mathcal{I}_\tau)} + M_g(\tau,u) \right) d\tau.
\]

which is the conformal energy estimate in curved case.

### 3.3 Analysis on curved energy

In this subsection we will analyse the structure of the curved energy and more precisely, we will give a sufficient condition for (3.8).

**Proposition 3.1.** There exists a constant \( \varepsilon_s \) (with the \( s \) stands for “structure”) which depends only on \( u \) such that when

\[
\begin{align*}
|\tilde{h}^{00}| &\leq (s/t)\varepsilon_s, \quad |s\partial_\alpha \tilde{h}^{00}| \leq \varepsilon_s(s/t), \\
|h^{\alpha \beta}| &\leq (s/t)\varepsilon_s.
\end{align*}
\]

then (3.8) holds.

**Proof.** Recall the structure of the curved energy, we consider first the term \( g^{ab} \bar{\partial}_a \bar{\partial}_b u \). We see that

\[
\bar{g}^{ab} = g^{\alpha \beta} \bar{\nabla}_\alpha \bar{\nabla}_\beta = g^{ab},
\]

in the same way \( \tilde{h}^{ab} = h^{ab} \). Thus there exists a positive constant \( \varepsilon_s \) such that if \( |h^{ab}| \leq \varepsilon_s \),

\[
\left\| s^2 \bar{g}^{ab} \partial_\alpha \bar{\partial}_b u - \sum_a |s \partial_\alpha u|^2 \right\|_{L^2(\mathcal{I}_s)} \leq C \varepsilon_s \left\| \sum_a |s \partial_\alpha u|^2 \right\|_{L^2(\mathcal{I}_s)} \leq C \varepsilon_s E_{\text{con}}(s,u)^{1/2}.
\]
Then we regard the first term in $E_{\text{con},g}$. We first remark that

$$N_g - 2 = g^{00} - \sum_a g^{aa} - 2g^{00} - s\partial_a g^{00} - \left(m^{00} - \sum_a m^{aa} - 2\bar{m}^{00} - s\bar{\partial}_a \bar{m}^{00}\right) = \Delta h^{00} - \sum_a h^{aa} - 2\bar{h}^{00} - s\bar{\partial}_a \bar{h}^{00}.$$ 

Then we see that (taking (3.12) with $\varepsilon_s$ sufficiently small)

$$|N_g| \leq 2 + \left| \Delta h^{00} - \sum_a h^{aa} - 2\bar{h}^{00} - s\bar{\partial}_a \bar{h}^{00} \right| \leq 3.$$ 

We see that

$$\mathcal{K} g^{uu} + N_g u - \bar{g}^{00} (Ku + 2u) = 2s \left(\bar{g}^{00} - \bar{g}^{00} m^{00}\right) \bar{\partial}_a u + (N_g - 2\bar{g}^{00}) u$$

(3.15)

$$= 2s \left(\bar{g}^{00} - \bar{m}^{00} + \bar{m}^{00} m^{00} - \bar{g}^{00} \bar{m}^{00}\right) \bar{\partial}_a u + (N_g - 2\bar{g}^{00}) u$$

$$= 2s \left(\bar{h}^{00} + 2\bar{h}^{00} m^{00}\right) \bar{\partial}_a u + (N_g - 2\bar{g}^{00}) u$$

where we remark that

$$\bar{h}^{\alpha\beta} = \bar{g}^{\alpha\beta} - \bar{m}^{\alpha\beta}, \quad \text{and especially} \quad \bar{h}^{00} = \bar{g}^{00} - 1.$$ 

Thus when $|\bar{h}^{00}| \leq \varepsilon_s$ and $|\bar{h}^{00}| \leq \varepsilon_s (s/t)$, we see that for the first term in right-hand-side of (3.15):

$$2s |\bar{h}^{00} + \bar{h}^{00} m^{00}| \bar{\partial}_a u \leq \varepsilon_s \sum_a |s \bar{\partial}_a u|.$$ 

We remark that

$$\bar{h}^{00} = h^{00} - h^{00} u / s = h^{00} (t/s) - h^{ab} (x^b / s)$$

thus by (3.16), $|\bar{h}^{00}| \leq \varepsilon_s$.

On the other hand, we see that by (3.1),

$$(N_g - 2\bar{g}^{00}) u = \left(-2\bar{h}^{00} + \bar{g}^{00} - \sum_a h^{aa} - 2\bar{h}^{00} - s\bar{\partial}_a \bar{h}^{00}\right) u$$

$$= \left(-4\bar{h}^{00} + \bar{g}^{00} - \sum_a h^{aa} - s\bar{\partial}_a \bar{h}^{00}\right) u.$$ 

We see that under the assumption (3.12),

$$\left| (N_g - 2\bar{g}^{00}) u \right| \leq \varepsilon_s (s/t)|u|$$

thus

$$\left| \mathcal{K} g^{uu} + N_g u - \bar{g}^{00} (Ku + 2u) \right| \leq \varepsilon_s \sum_a \left| s \bar{\partial}_a u \right| + \varepsilon_s (s/t)|u|.$$ 

Then we see that

$$\left| (\mathcal{K} g^{uu} + N_g u) - (Ku + 2u) \right| \leq \left| (\mathcal{K} g^{uu} + N_g u) - \bar{g}^{00} (Ku + 2u) \right| + |\bar{h}^{00}| |Ku + 2u|$$

(3.16)

$$\leq \varepsilon_s \sum_a \left| s \bar{\partial}_a u \right| + \varepsilon_s (s/t)|u| + \varepsilon_s |Ku + 2u|$$

This leads to (recall (2.39)):

$$\| (\mathcal{K} g^{uu} + N_g u) - (Ku + 2u) \|_{L^2(\Omega_t)} \leq \varepsilon_s E_{\text{con} \cdot (s, u)}^{1/2}$$

(3.17)

Then we see that (by (3.13) and (3.17))

$$1 - \varepsilon_s E_{\text{con} \cdot (s, u)}^{1/2} \leq E_{\text{con} \cdot (s, u)}^{1/2} \leq (1 + \varepsilon_s) E_{\text{con} \cdot (s, u)}^{1/2}$$

(3.18)

Then we take $\varepsilon_s$ sufficiently small and the desired result is established.
4 Commutators and decay estimate

4.1 Global Sobolev inequality on hyperboloids

In this subsection we recall the Klainerman-Sobolev inequality on hyperboloids due to Hörmander [3]:

**Proposition 4.1.** Let \( u \) be a sufficiently regular function defined in \( \mathcal{K} \) and vanishes near the conical boundary \( \partial \mathcal{K} \). Then the following estimate holds:

\[
\sup_{\mathcal{K}_c} (t^{3/2} |u|) \leq C \sum_{|I|+|J| \leq 2} \| \partial^I L^J u \|_{L^2(\mathcal{K}_c)}.
\]

Here \( C \) is a positive constant independent of \( u \).

This result is essentially due to Hörmander (see in detail [3](1997), Lemma 7.6.1). For this slightly improved version, see [5] section 5.1. This inequality helps us to get decay estimate via \( L^2 \) norm. So in the following we need to control on hyperboloids the \( L^2 \) norm of the following quantities for \( |I'| + |J'| \leq 2 \):

\[
\begin{align*}
&\partial^I L^{I'} (s^2/t) \partial_\alpha \partial^I L^J u, \quad \partial^I L^{J'} ((s^2/t) \partial^I L^J u), \\
&\partial^I L^{I'} (s \partial_\alpha \partial^I L^J u), \quad \partial^I L^{J'} (s \partial^I L^J u), \\
&\partial^I L^{I'} ((s^2/t) \partial^I L^J \partial_\alpha u), \quad \partial^I L^{J'} ((s^2/t) \partial^I L^J u), \\
&\partial^I L^{I'} (s^2 \partial^I L^J \partial_\alpha u), \quad \partial^I L^{J'} (s \partial^I L^J \partial_\alpha u).
\end{align*}
\]

These are to be bounded by the conformal energies \( E_{\text{con}}(s, \partial^I L^J u) \) with \( |I'| + |J'| \leq |I| + |J| + 2 \).

To do so, we need some estimates on commutators, which are studied in the following sections.

4.2 Commutators I

In this section we calculate the following quantities:

\[
[L_a, \partial_\alpha], \quad [L_a, \partial^I], \quad [L^J, \partial^I],
\]

and

\[
[L_a, \partial_\alpha], \quad [\partial^I, \partial_\alpha].
\]

We begin with the first group. It is easy to see that

\[
[L_a, \partial_\alpha] = -\partial_\alpha, \quad [L_a, \partial_b] = -\delta_{ab} \partial_b
\]

and we denote by

\[
[L_a, \partial_\alpha] = \theta^{\alpha}_{\alpha \beta} \partial_\beta
\]

where \( \theta^{\alpha}_{\alpha \beta} \) are constants.

**Important convention:** in the following we often make summation over multi-indices of order less than an integer. For the convenience of expression, we only give the upper bound of the order of a multi-index and omit the lower bound. For example when we write

\[
\sum_{|I| \leq N}
\]

we always mean

\[
\sum_{0 \leq |I| \leq N}
\]
which is a sum over all multi-index of order from zero to \( N \). In certain case, we take the sum from a positive order. In this case we will write

\[
\sum_{n \leq |I| \leq N}.
\]

We make the convention that when \( n > N \), this sum is taken as zero.

Then we have the following decompositions:

**Lemma 4.2.** Let \( u \) be a function defined in \( \mathcal{K} \), sufficiently regular. Then the following identities hold:

1. \( [L_a, \partial^I]u = \sum_{|I'| = |I|} \theta^I_{I'} \partial^{I'} u, \) \hspace{1cm} (4.2)
2. \( [L^J, \partial^I]u = \sum_{|I'| = |I|, |J'| < |J|} \theta^I_{J'} \partial^{I'} L^{J'} u, \) \hspace{1cm} (4.3)
3. \( [L^J, \partial_a]u = \sum_{|J'| < |J|} \theta^I_{J'} \partial_a L^{J'} u, \) \hspace{1cm} (4.4)

where \( \theta^I_{I'}, \theta^I_{J'} \), and \( \theta^I_{aJ'} \) are constants.

**Proof.** These are by induction. For the first it is an induction on \(|I|\). When \(|I| = 1\) it is already proved. We suppose that (4.2) holds for \(|I| \leq m\). Then we consider

\[
[L_a, \partial^I]u = [L_a, \partial^I u + \partial_a L_a, \partial^I]u = \theta^I_{I'a} \partial_{aI} u + \sum_{|I'| = |I|} \partial_a \left( \theta^I_{I'} \partial^{I'} u \right) = \theta^I_{aI} \partial_a \partial^I u + \sum_{|I'| = |I|} \theta^I_{I'} \partial_a \partial^{I'} u.
\]

which concludes (4.2) for \(|I| = m + 1\).

For (4.3), it is an induction on \(|J|\). For \(|J| = 1\) it is already proved. Suppose that (4.3) holds for \(|J| \leq m\). Then we consider

\[
[L^J, L_a, \partial^I]u = [L^J, L_a, \partial^I u + L^J, \partial^I L_a]u = L^J \left( \sum_{|I'| = |I|} \theta^I_{I'} \partial^{I'} u \right) + \sum_{|I'| = |I|, |J'| < |J|} \theta^I_{J'} \partial^{I'} L^{J'} L_a u
\]

\[
= \sum_{|I'| = |I|} \theta^I_{aI} L^J \partial^{I'} u + \sum_{|I'| = |I|, |J'| < |J|} \theta^I_{J'a} \partial^{I'} L^{J'} L_a u
\]

\[
= \sum_{|I'| = |I|} \theta^I_{aI} L^J \partial^{I'} u + \sum_{|I'| = |I|} \theta^I_{aI} [L^J, \partial^{I'}]u + \sum_{|I'| = |I|, |J'| < |J|} \theta^I_{J'a} \partial^{I'} L^{J'} L_a u
\]

\[
= \sum_{|I'| = |I|} \theta^I_{aI} L^J \partial^{I'} u + \sum_{|I'| = |I|} \theta^I_{aI} \sum_{|I''| = |I'|} \theta^{I''}_{J'a} \partial^{I''} L^{J''} L_a u + \sum_{|I'| = |I|, |J'| < |J|} \theta^I_{J'a} \partial^{I'} L^{J'} L_a u
\]

This concludes (4.3) for \(|J| = m + 1\).

For (4.4) is a direct result of (4.4).

Here we also state a simple but frequently used result:
Lemma 4.3. The following identity holds:

\begin{equation}
\partial^i L^j \partial^k L^j u = \sum_{|I| = |I| + |J|} \zeta_{i J}^1 \zeta_{i J}^2 \partial^i L^j u
\end{equation}

where \( \zeta_{i J}^1 \zeta_{i J}^2 \) are constants

Proof. This is by (4.3). We see that

\[
\partial^i L^j \partial^k L^j u = \partial^i \partial^k L^j L^j u + \partial^i ([L^j, \partial^k] L^j u)
\]

where \( \theta_{i J}^1 \theta_{i J}^2 \) are constants. This proves the desired result.

Now we consider the commutator of \([\partial_a, \partial_b]\). For the convenience of expression, we introduce the following notion of homogeneous function. Let \( u \) be a \( C^\infty \) function defined in \( \{ r < t \} \) and satisfies the following condition:

\begin{equation}
u(\lambda, \lambda x) = \lambda^n u(1, x), \quad n \in \mathbb{Z}.
\end{equation}

for all \( \lambda > 0 \) and \( \partial^i u(1, x) \) are bounded in \( \{|x| \leq 1\} \). Such function \( u \) is called a homogeneous function of degree \( n \). For example \( x^a/t \) is a homogeneous function of degree zero. We state the following property of a homogeneous function of degree \( n \):

Lemma 4.4. Let \( u \) be a homogeneous function of degree \( n \). Then \( \partial^i L^j u \) is homogeneous of degree \( n - |I| \), and the following estimates holds in \( \mathcal{K} \):

\begin{equation}
|\partial^i L^j u| \leq C t^{n-|I|}
\end{equation}

with \( C \) a constant determined by \( u, I \) and \( J \).

Proof. We remark that

\begin{equation}
|\partial^i u| \leq C t^{n-|I|}
\end{equation}

which is checked directly:

\[
u(t, x) = t^n u(1, x/t) \Rightarrow \partial_a u(t, x) = nt^{n-1} u(1, x/t) + t^n (-x^a/t^2) \partial_a u(1, x/t) = t^{n-1} (nu(1, x/t) - (x^a/t) \partial_a u(1, x/t))
\]

and

\[
\partial_a u = t^n (-x_a/t^2) \partial_a u(1, x/t) = -t^{n-1} (x^a/t) \partial_a u(1, x/t)
\]

where we remark that

\[
nu(1, x/t) - (x^a/t) \partial_a u(1, x/t), \quad (x^a/t) \partial_a u(1, x/t)
\]

are homogeneous of degree zero. Thus \( \partial_a u \) is homogeneous of degree \( n - 1 \). Then by recurrence we see that \( \partial^i u \) is homogeneous of degree \( n - |I| \). This leads to (4.8).

Then we prove that

\begin{equation}
L^j u \quad \text{is homogeneous of degree } n
\end{equation}

This is also checked directly by

\[
L_a u(t, x) = (t \partial_a + x^a \partial_t) u(t, x) = -t^n (x^a/t) \partial_a u(1, x/t) + t^n (x^a/t) (nu(1, x/t) - (x^a/t) \partial_a u(1, x/t))
\]

which is homogeneous of degree \( n \). Thus \( L^j u \) is also homogeneous of degree zero, which leads to (4.9).

The desired result is a combination of (4.8) and (4.9).
Now we see that
\[ [\partial_t, \bar{\partial}_\alpha] = -\frac{x^a}{t^2} \partial_t, \quad [\partial_a, \bar{\partial}_b] = \frac{\delta_{ab}}{t} \partial_t, \]

Then we denote by
\[ [\partial_a, \bar{\partial}_b] = \sigma_{ab} \partial_t \]

with \(\sigma_{ab}\) homogeneous functions of degree \(-1\). Then we establish the following result:

**Lemma 4.5.** For \(|I| \geq 1\),
\begin{equation}
[\partial^I, \bar{\partial}_a] = \sum_{1 \leq |J| \leq |I|} \sigma_{aJ}^I \partial^J
\end{equation}

with \(\sigma_{aJ}^I\) a homogeneous function of degree \(|J| - |I| - 1\).

**Proof.** This is by induction on \(|I|\). For \(|I| = 1\) we see that holds. We suppose that (4.10) holds for \(|I| \leq m\), we consider
\[ [\partial_a \partial^I, \bar{\partial}_a] = \partial_a \left( \sum_{1 \leq |J| \leq |I|} \sigma_{aJ}^I \partial^J \right) + [\partial_a, \bar{\partial}_a] \partial^I \]
\[ = \sum_{1 \leq |J| \leq |I|} \sigma_{aJ}^I \partial_a \partial^J + \sum_{1 \leq |J| \leq |I|} \partial_a \sigma_{aJ}^I \partial^J + \sigma_{aa} \partial_t \partial^I. \]

We see that \(\partial_a \partial^I\) is of order \(|J| + 1\), \(\partial^I \partial^I\) is of order \(|I| + 1\); we recall that \(\partial_a \sigma_{aJ}^I\) is homogeneous of order \(|J| - |I| - 2 = |J| - (|I| + 1) - 1\) (by the assumption of induction combined with lemma 4.4), and \(\sigma_{aa}\) is homogeneous of degree \(-1\). Thus we see that (4.10) is proved in the case \(|I| = m + 1\).

Now we calculate:
\[ [L_a, \bar{\partial}_b] = -\frac{x^b}{t} \partial_a = \nabla_b \bar{\partial}_a. \]

We denote by
\[ [L_a, \bar{\partial}_b] = \eta_{ab} \bar{\partial}_a \]

where \(\eta_{ab}\) is a homogeneous function of degree zero. Now we establish the following result:

**Lemma 4.6.** Let \(|I| = n\), then
\begin{equation}
[L^I, \bar{\partial}_a] = \sum_{|J| < |I|} \eta_{aJ}^b \bar{\partial}_b L^J
\end{equation}

where \(\eta_{aJ}^b\) are homogeneous functions of degree zero.

**Proof.** This is by induction on \(|I|\). We see that for \(|I| = 1\) (4.11) holds. Suppose that (4.11) holds for \(|I| \leq m\), now we consider
\[ [L_a L^I, \bar{\partial}_b] = L_a ([L^I, \bar{\partial}_b]) + [L_a, \bar{\partial}_b] L^I = L_a \left( \sum_{|I'| < |I|} \eta_{aI'}^b \bar{\partial}_b L^{I'} \right) + \eta_b \partial_a L^I \]
\[ = \sum_{|I'| < |I|} L_a \eta_{aI'}^b \partial_b L^{I'} + \sum_{|I'| < |I|} \eta_{aI'}^b L_a \bar{\partial}_b L^{I'} + \eta_b \partial_a L^I \]
\[ = \sum_{|I'| < |I|} L_a \eta_{aI'}^b \partial_b L^{I'} + \sum_{|I'| < |I|} \eta_{aI'}^b \partial_b L^I + \sum_{|I'| < |I|} \eta_{aI'} L_a \bar{\partial}_b L^{I'} + \eta_b \partial_a L^I \]
\[ = \sum_{|I'| < |I|} L_a \eta_{aI'}^b \partial_b L^{I'} + \sum_{|I'| < |I|} \eta_{aI'}^b \bar{\partial}_b L^I + \sum_{|I'| < |I|} \eta_{aI'} L_a \partial_b L^{I'} + \eta_b \partial_a L^I \]
\[ = \sum_{|I'| < |I|} L_a \eta_{aI'}^b \partial_b L^{I'} + \sum_{|I'| < |I|} \eta_{aI'} L_a L^{I'} + \sum_{|I'| < |I|} \eta_{aI'} \partial_b L^{I'} + \eta_b \partial_a L^I. \]
We recall that $\eta^{Ic}_{bI}$ is homogeneous of degree zero so $L^J \eta^{Ic}_{bI}$ is again homogeneous of degree zero. This concludes (4.11) for the case $|I| = m + 1$.

Now we are ready to establish the following result:

Lemma 4.7.

\[(4.12) \quad [\partial^J L^J, \bar{\partial}_a] = \sum_{|I'| \leq |J|} \rho^{I_J}_{aI'} \bar{\partial}_a \partial^{I'} L^J + \sum_{1 \leq |I'| \leq |I|} \rho^{I'}_{aI'} \partial^{I'} L^J \]

where $\rho^{I_J}_{aI'}$ are homogeneous functions of degree $(|I'| - |J|)$ and $\rho^{I'}_{aI'}$ are homogeneous functions of degree $(|I'| - |I| - 1)$. Furthermore, in $X$ for a function $u$ sufficiently regular, we have

\[(4.13) \quad \| \partial^J L^J, \bar{\partial}_a | u \| \leq C \sum_{c, |J| < |I|} | \bar{\partial}_a \partial^{I'} L^J u | + C t^{-1} \sum_{1 \leq |I'| \leq |I|} \| \partial^{I'} L^J u \| . \]

Proof. We remark that

\[
[\partial^J L^J, \bar{\partial}_a] = \partial^J \left( \sum_{|J| \leq |I|} \eta^{Ic}_{bI} \bar{\partial}_a L^J \right) + \sum_{1 \leq |I'| \leq |I|} \sigma^{I_J}_{aI'} \partial^{I'} L^J
\]

\[
= \sum_{t_1 + t_2 = t, |J| \leq |I|} \partial^{I_1} \eta^{Ic}_{aI_1} \cdot \partial^{I_2} \bar{\partial}_a L^J + \sum_{1 \leq |I'| \leq |I|} \sigma^{I_J}_{aI'} \partial^{I'} L^J
\]

\[
= \sum_{t_1 + t_2 = t, |J| \leq |I|} \partial^{I_1} \eta^{Ic}_{aI_1} \cdot \partial^{I_2} \bar{\partial}_a L^J + \sum_{t_1 + t_2 = t, |I'| \leq |I|} \partial^{I_1} \eta^{Ic}_{aI_1} \left( \sum_{1 \leq |I'| \leq |I|} \sigma^{I_2}_{aI_2} \partial^{I'} L^J \right)
\]

\[
+ \sum_{1 \leq |I'| \leq |I|} \sigma^{I_J}_{aI'} \partial^{I'} L^J
\]

\[
= \sum_{t_1 + t_2 = t, |J| \leq |I|} \partial^{I_1} \eta^{Ic}_{aI_1} \cdot \partial^{I_2} \bar{\partial}_a L^J + \sum_{t_1 + t_2 = t, |I'| \leq |I|} \partial^{I_1} \eta^{Ic}_{aI_1} \cdot \sigma_{aI_2}^{I_2} \partial^{I'} L^J
\]

\[
+ \sum_{1 \leq |I'| \leq |I|} \sigma^{I_J}_{aI'} \partial^{I'} L^J.
\]

Now we recall that $\partial^{I_1} \eta^{Ic}_{aI_1}$ is homogeneous of degree $-|I_1| = |I_2| - |I|$, $\partial^{I_1} \eta^{Ic}_{aI_1} \cdot \sigma_{aI_2}^{I_2}$ is homogeneous of degree $-|I_1| + |I_2| - |I_2| - 1 = |I_2| - |I| - 1$ and $\sigma^{I_J}_{aI'}$ is homogeneous of degree $|I'| - |I| - 1$. Thus the desired result is established.

(4.13) is direct by (4.12).

4.3 Commutators II

In this subsection we consider the following quantities:

$$\partial^J L^J(s/t), \quad \partial^J L^J s.$$

And then based on these calculation, we analyse $[\partial^J, \bar{\partial}_a]$.

We first remark the following result:

\[(4.14) \quad \partial_t(s/t) = \frac{\sigma^2}{t^2} s^{-1}, \quad \bar{\partial}_a(s/t) = -\frac{\sigma^a}{t} s^{-1}.\]
We denote by
\[ \partial_\alpha (s/t) = \pi_\alpha s^{-1} \]
with \( \pi_\alpha \) a homogeneous function of degree zero.

(4.15) \[ \partial_s = t/s, \quad \partial_a s = -t^a \quad \frac{t}{s}, \]
We denote by
\[ \partial_\alpha s = \rho_\alpha (s/t)^{-1} \]
with \( \rho_\alpha \) a homogeneous function of degree zero. We also recall
\[ L_a (s/t) = -\frac{x^a}{t} (s/t), \quad L_a s = 0. \]

We first establish the following relation:

**Lemma 4.8.**

(4.16) \[ L^J (s/t) = \lambda^J (s/t) \]
with \( \lambda^J \) a homogeneous function of degree zero.

**Proof.** This is by induction. It is clear that (4.16) holds for \(|J| = 1\). Then we consider

\[ L_a L^J (s/t) = L_a (\lambda^J (s/t)) = L_a \lambda^J \cdot (s/t) + \lambda^J L_a (s/t) = (L_a \lambda^J - (x^a/t)) (s/t). \]

We see that by (4.14), \( L_a \lambda^J \) is homogeneous of degree zero. Furthermore \( x^a/t \) is also homogeneous of degree zero. Thus \( (L_a \lambda^J - (x^a/t)) \) is homogeneous of degree zero.

Then we establish the following result:

**Lemma 4.9.** For \(|I| \geq 1\),

(4.17) \[ \partial^I (s/t) = \sum_{1 \leq k \leq |I|} \pi_k^I (s/t)^{-k+1} s^{-k} \]
with \( \pi \) a sum of finite many homogeneous functions of degree \((k - |I|)\). Furthermore, in \( \mathcal{X} \),

(4.18) \[ |\partial^I (s/t)| \leq C s^{-1}. \]

**Proof.** This is also by induction on \(|I|\). For \(|I| = 1\), we see that it is established by (4.14). We
suppose that \(4.17\) holds for \(|I| \leq m\), and we consider (where we use \(4.14\) and \(4.15\))

\[
\partial_{\alpha} \partial^J s = \partial_{\alpha} \left( \sum_{1 \leq k \leq |I|} \pi_k^J (s/t)^{-k+1} s^{-k} \right)
= \sum_{1 \leq k \leq |I|} \partial_{\alpha} (\pi_k^J) \cdot (s/t)^{-k+1} s^{-k} \cdot \partial_{\alpha} + \sum_{1 \leq k \leq |I|} \pi_k^J \partial_{\alpha} ((s/t)^{-k+1}) \cdot s^{-k} \\
+ \sum_{1 \leq k \leq |I|} \pi_k^J (s/t)^{-k+1} \partial_{\alpha} (s^{-k})
= \sum_{1 \leq k \leq |I|} \partial_{\alpha} (\pi_k^J) \cdot (s/t)^{-k+1} s^{-k} + \sum_{1 \leq k \leq |I|} \pi_k^J (s/t)^{-k+1+k} \partial_{\alpha} (s/t) \cdot s^{-k} \\
+ \sum_{1 \leq k \leq |I|} \pi_k^J (s/t)^{-k+1-k} \partial_{\alpha} s
= \sum_{1 \leq k \leq |I|} \partial_{\alpha} (\pi_k^J) \cdot (s/t)^{-k+1} s^{-k} + \sum_{1 \leq k \leq |I|} \pi_k^J (s/t)^{-k+1-k} \partial_{\alpha} s \\
- \sum_{1 \leq k \leq |I|} k \pi_k^J (s/t)^{-k+1-k} \rho_{\alpha}
= \partial_{\alpha} (\pi_k^J) s^{-1} + \sum_{2 \leq k \leq |I|} \left( \partial_{\alpha} (\pi_k^J) + (1-k) \pi_k^J \rho_{\alpha} + (1-k) \pi_k^J \rho_{\alpha} \right) (s/t)^{-k+1-k} \rho_{\alpha}
+ ((1 - |I|) \pi_{I} - |I| \rho_{\alpha}) \pi_{I}^J (s/t)^{-((|I|+1) - 1)} s^{-((|I|+1) - 1)}.
\]

WE check that for each term the coefficients are homogeneous of degree \((k - (|I| + 1))\), and this concludes the case where \(|I| = m + 1\).

For \(4.18\), we see that in \(4.17\),

\[
|\pi_k^J (s/t)^{-k+1} s^{-k}| \leq C t^{-1} s^{-2k+1} = C t^{-1} s^{-2k+2} s^{-1} = C (t/s^2)^{k-1} s^{-1}.
\]

We remark that in \(K\), \(t/s^2\) is bounded. Then \(4.15\) is established. \(\square\)

Now we observe the quantity \(\partial^J s\) (for \(|I| \geq 1\)). We see that

\[
\partial^J (s) = \partial^J (t \cdot (s/t)) = \sum_{I_1 + I_2 = I} \partial^{I_1} t \cdot \partial^{I_2} (s/t).
\]

We see that for \(|I_1| \geq 2\), \(\partial^{I_1} t = 0\). Thus we see

\[
\sum_{I_1 + I_2 = I} \partial^{I_1} t \cdot \partial^{I_2} (s/t) = t \partial^I (s/t) + \sum \partial^{I_1} t \cdot \partial^{I_2} (s/t)
\]

where the second term does not exist when \(|I| \leq 0\). Then combined with \(4.18\), we see that

\[
|\partial^I s| \leq \begin{cases} C (t/s), & |I| \geq 1 \\ s, & |I| = 0. \end{cases}
\]

**Remark 4.10.** In the following application, we see that in \(K\), because \(s^2 \geq t\), we have \(|\partial^I s| \leq C s\). Furthermore, we see that when \(|J| \geq 1\),

\[
\partial^J L^J s = 0.
\]

Combine \(4.16\) and \(4.17\), we see that
Lemma 4.11. In $\mathcal{K}$,

\[(4.20)\]
\[
|\partial^J L^J(s/t)| \leq \begin{cases} 
Cs^{-1}, & |I| \geq 1, \\
Cs/t, & |I| = 0. 
\end{cases}
\]

Let $n \in \mathbb{N}^*$. Then

\[(4.21)\]
\[
|\partial^J L^J((s/t)^n)| \leq \begin{cases} 
Cs^{-1}(s/t)^{n-1}, & |I| \geq 1, \\
C(s/t)^n, & |I| = 0. 
\end{cases}
\]

Proof. If $|I| = 0$, by (4.16), (4.20) is established. When $|I| \geq 1$, we denote by $\partial^J = \partial^I \partial^{I'}$ with $|I'| \geq 0$. Then

\[
\partial^J L^J(s/t) = \partial^I \partial^{I'} L^J(s/t) = \partial^I \partial^{I'} (\lambda^J(s/t)) = \partial^I \left( \sum_{I_1 + I_2 = I'} \partial^{I'} \lambda^J \cdot \partial^{I_2}(s/t) \right) = \sum_{I_1 + I_2 = I'} \partial^{I_1} \lambda^J \cdot \partial^{I_2}(s/t) + \sum_{I_1 + I_2 = I'} \partial^{I_1} \lambda^J \cdot \partial^{I_2}(s/t).
\]

Now we recall that

\[
|\partial^J(s/t)| \leq \begin{cases} 
Cs^{-1}, & |I| \geq 1, \\
Cs/t, & |I| = 0. 
\end{cases}
\]

and the fact that $|\partial^I \partial^{I'} \lambda^J| \leq C t^{-1} \leq Cs^{-1}$. Thus (4.20) is established.

For (4.21), we see that

\[
\partial^J L^J((s/t)^n) = \sum_{I_1 + \ldots+ I_n = I} \partial^{I_1} L^{J_1}(s/t) \cdots \partial^{I_n} L^{J_n}(s/t).
\]

We apply (4.20) on each factor, (4.21) is established.

Then we make the following estimate:

\[
\partial^J L^J(s^2/t) = \partial^J L^J(s(s/t)) = \sum_{I_1 + I_2 = I} \partial^{I_1} L^{J_1}(s) \cdot \partial^{I_2} L^{J_2}(s/t).
\]

Then we see that by when $|J_1| \geq 1$, we see that $\partial^{I_1} L^{J_1}(s) = 0$. So we see that

\[
\partial^J L^J(s^2/t) = \sum_{I_1 + I_2 = I} \partial^{I_1} s \cdot \partial^{I_2} L^J(s/t).
\]

Now we see that when $|I_1| \geq 1$, we see that

\[
|\partial^{I_1} s \cdot \partial^{I_2} L^J(s/t)| \leq C.
\]

When $|I_1| = 0$, suppose that $|I| \geq 1$, thus

\[
|s \cdot \partial^J L^J(s/t)| \leq C.
\]

When $|I| = 0$, we see that

\[
|sL^J(s/t)| \leq Cs^2/t.
\]

Thus we conclude that

\[(4.22)\]
\[
|\partial^J L^J(s^2/t)| \leq \begin{cases} 
C, & |I| \geq 1, \\
Cs^2/t, & |I| = 0. 
\end{cases}
\]

Then we also establish the following result:
Lemma 4.12. In $\mathcal{K}$ the following bound holds:

\begin{equation}
|\partial I(s^{-n})| \leq \begin{cases} 
Cts^{-n-2}, & |I| \geq 1, \\
s^{-n}, & |I| = 0.
\end{cases}
\end{equation}

and

\begin{equation}
|\partial I L^J(t/s)^n| \leq \begin{cases} 
C(t/s)^{n+1}s^{-1}, & |I| \geq 1, \\
(t/s)^n, & |I| = 0.
\end{cases}
\end{equation}

Proof. This is by the identity of Faà di Bruno. We denote by

$$f : \mathbb{R}^+ \to \mathbb{R}^+, \quad x \to x - n.$$

Then we see that for $|I| \geq 1$,

$$\partial I f(s) = \sum_{1 \leq k \leq |I|} \sum_{I_1 + I_2 + \cdots + I_k = I} f^{(k)}(s) \partial I_1 s \partial I_2 s \cdots \partial I_k s.$$

We see that in the above expression,

$$|f^{(k)}(s)| \leq Cs^{-n-k}, \quad |\partial I_1 s| \leq C(t/s)$$

This we see that

$$|f^{(k)}(s)\partial I_1 s \partial I_2 s \cdots \partial I_k s| \leq Cs^{-n-k}(t/s)^k \leq Cs^{-n}(t/s^2)^k.$$

Recall that $k \geq 1$ and in $\mathcal{K}$, $s^2 \geq t$, (4.23) is established.

For (4.24), we see that

$$\partial I L^J(t^n s^{-n}) = \sum_{I_1 + I_2 = I} \partial I_1 L^J_{1} t^n \partial I_2 L^J_{2}(s^{-n}).$$

Then we see that $t^n$ is homogeneous of degree $n$ thus $|\partial I_1 L^J_{1} t^n| \leq Ct^{n-|I_1|}$. Thus by (4.23), (4.24) is proved.

For simplicity of expression, we introduce the following notation:

$$\Lambda^{IJ} := \partial I L^J(s/t).$$

Then by (4.5), the following estimate is direct:

\begin{equation}
|\partial I L^J \Lambda^{IJ'}| \leq \begin{cases} 
C s^{-1}, & |I| + |I'| \geq 1, \\
C(s/t), & |I| + |I'| = 0.
\end{cases}
\end{equation}

Now we are ready to calculate the commutator $[\partial I L^J, \partial_s]$. We have the following result:

Lemma 4.13. Let $u$ be a function defined in $\mathcal{K}$, sufficiently regular and vanishes near the conical boundary. Then the following estimate holds:

\begin{equation}
|[\partial I L^J, \partial_s]u| \leq C s^{-1} \sum_{|I'| < |I|} |\partial I \partial I' L^J u| + C(s/t) \sum_{|J'| < |J|} |\partial_{\alpha} \partial I' L^{J'} u|.
\end{equation}

where $C$ is a constant determined by $I, J$. 

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Proof. We remark that
\[
[\partial^J L^J, \partial_\alpha] = [\partial^J L^J, (s/t)\partial_\alpha] = \sum_{t_1 + t_2 = t, J_1 + J_2 = J} \Lambda^{I_1;J_1} \partial^{I_2} L^{J_2} \partial_{I_2} + \frac{(s/t)}{t} [\partial^J L^J, \partial_\alpha]
\]
\[
= \sum_{t_1 + t_2 = t, J_1 + J_2 = J} \Lambda^{I_1;J_1} \partial_I \partial^{I_2} L^{J_2} + \sum_{t_1 + t_2 = t, J_1 + J_2 = J} \Lambda^{I_1;J_1} \partial^{I_2} [L^{J_2}, \partial_\alpha] + \frac{(s/t)}{t} [L^J, \partial_\alpha]
\]
\[
= \sum_{t_1 + t_2 = t, J_1 + J_2 = J} \Lambda^{I_1;J_1} \partial_I \partial^{I_2} L^{J_2} + \sum_{t_1 + t_2 = t, J_1 + J_2 = J} \Lambda^{I_1;J_1} \partial^{I_2} [L^{J_2}, \partial_\alpha]
\]
We decompose the first term in right-hand-side and see that
\[
[\partial^J L^J, \partial_\alpha] = \sum_{t_1 + t_2 = t, J_1 + J_2 = J} \Lambda^{IO} \partial_I \partial^{I_2} L^{J_2} + \sum_{t_1 + t_2 = t, J_1 + J_2 = J} \Lambda^{I_1;J_1} \partial^{I_2} [L^{J_2}, \partial_\alpha] + \sum_{t_1 + t_2 = t, J_1 + J_2 = J} \Lambda^{I_1;J_1} \partial^{I_2} L^{J_2} \partial_\alpha.
\]
Were \(\Lambda^{IO}\) means in \(\partial^J(s/t)\) where \(|J| = 0\).
Recall the bound on \(\Lambda^{I;J}\), the desired result is direct.

Finally we establish the following estimates:

**Lemma 4.14.** Let \(u\) be a function defined in \(\mathcal{K}\), sufficiently regular, then
\[
|[(\partial^J L^J, \partial_\alpha)]_u| \leq C \sum_{\alpha, \beta, \gamma \in \mathcal{J} \setminus \{J\}, |\gamma| < |\alpha| + |\beta|} |(s/t)^2 \partial_\alpha \partial_\beta \partial^{I} \partial^J u| + C \sum_{\alpha, \beta, \gamma \in \mathcal{J} \setminus \{J\}, |\gamma| < |\alpha| + |\beta|} |t^{-1} \partial_\alpha \partial^{I} \partial^J u|.
\]

**Proof.** We remark that
\[
\tilde{\partial}_I \tilde{\partial}_\alpha = (s/t)^2 \partial_\alpha \partial_I + t^{-1}(r/t)^2 \partial_I.
\]
Then we see that
\[
[(\partial^J L^J, \partial_\alpha)]_u = [(\partial^J L^J, (s/t)^2 \partial_\alpha \partial_I)]_u + [(\partial^J L^J, t^{-1}(r/t)^2 \partial_I)]_u =: T_1 + T_2
\]
We see that
\[
[\partial^J L^J, (s/t)^2 \partial_I \partial_\alpha]_u = \sum_{t_1 + t_2 = t, J_1 + J_2 = J} \partial^{I_1} L^{J_1} (s/t)^2 \partial^{I_2} L^{J_2} \partial_\alpha \partial_I + (s/t)^2 [\partial^J L^J, \partial_\alpha \partial_I]_u.
\]
We see that by (4.28):
\[
[\partial^J L^J, \partial_\alpha \partial_I]_u = \partial_\alpha \left( [\partial^J L^J, \partial_I]_u + [\partial^J L^J, \partial_I]_u \partial_\alpha \partial_I \right)
\]
\[
= \sum_{|J| < |I|} \theta_{I,J}^0 \partial_\alpha \partial_I \partial^J \partial^I u + \sum_{|J| < |I|} \theta_{I,J}^0 \partial_\alpha \partial^{I_2} L^{J_2} \partial_\alpha \partial_I \partial^I u
\]
\[
= 2 \sum_{|J| < |I|} \theta_{I,J}^0 \partial_\alpha \partial_\beta \partial^J \partial^I u + \sum_{|J| < |I|} \theta_{I,J}^0 \partial_\alpha \partial_\beta \partial^J \partial^I \partial^I u
\]
\[
= 2 \sum_{|J| < |I|} \theta_{I,J}^0 \partial_\alpha \partial_\beta \partial^J \partial^I u + \sum_{|J| < |I|} \theta_{I,J}^0 \theta_{J,J}^0 \partial_\alpha \partial_\beta \partial^J \partial^I \partial^I u.
\]
This leads to

\begin{equation}
\sum_{|\alpha|<|\beta|} |\partial_t^\alpha \partial_j \partial_k u| \leq C \sum_{|\alpha|<|\beta|} |\partial_t^\alpha \partial_j \partial_k L^\gamma u| \quad \quad \tag{4.29}
\end{equation}

By the above inequality we also see that:

\[|\partial_t^\alpha \partial^i L^J \partial_j u| \leq |\partial_t \partial_j \partial_t^i L^J u| + |\partial_t^i \partial^j L^J \partial_j u| \leq C \sum_{|\alpha|<|\beta|} |\partial_t^\alpha \partial_j \partial^i L^J u|\]

Thus we see that (recall \(|\partial_t^i L^J (s/t)^2| \leq C(s/t)^2\)

\[|T_1| \leq C \sum_{|\alpha|<|\beta|} |(s/t)^2 \partial_t^\alpha \partial^i L^J u| \]

For the term \(T_2\), by \(1.12\) and the fact that \(r^2/t^3\) is homogeneous of degree \(-1:\)

\[|T_2| \leq \sum_{|\alpha|<|\beta|} |\partial_t^\alpha \partial^i L^J (r^2/t^3) \cdot \partial_t^i \partial^j L^J \partial_j u| + |(r^2/t^3)|\partial_t^i L^J, \partial_t u| \]

\[\leq \sum_{|\alpha|<|\beta|} |r^{-1} \partial_t^\alpha \partial^i L^J u|.
\]

The bounds on \(T_1\) and \(T_2\) concludes the desired result.

**Lemma 4.15.** Let \(u\) be a function defined in \(\mathcal{K}\) and sufficiently regular. Then the following estimate holds:

\[|\partial_t^i L^J \partial^j \partial_k u| \leq C \sum_{|\alpha|<|\beta|} \left( t^{-1} |\partial_t^\alpha \partial^i L^J u| + (s/t^2) |\partial_t^i L^J u| \right) \]

\[+ C(s/t^2) \sum_{|\alpha|<|\beta|} |\partial_t^\alpha \partial^i L^J u|. \quad \quad \tag{4.30}\]

**Proof.**

\[|\partial_t^i L^J \partial^j \partial_k u| = |\partial_t^i L^J \partial^j \partial_k u + \partial_t (|\partial_t^i L^J \partial_k u|)\]

For the first term in right-hand-side of the above equation, we see that by \(1.26\),

\[|\partial_t^i L^J \partial^j \partial_k u| \leq C \sum_{|\alpha|<|\beta|} |\partial_t \partial^i L^J \partial^j \partial_k u| + C(s/t) \sum_{|\alpha|<|\beta|} |\partial_t^\alpha \partial^i L^J \partial^j \partial_k u| \]

\[= C \sum_{|\alpha|<|\beta|} |\partial_t \partial^i L^J (t^{-1} L_k u) | + C(s/t) \sum_{|\alpha|<|\beta|} |\partial_t^\alpha \partial^i L^J (t^{-1} L_k u) | \]

\[\leq C \sum_{|\alpha|<|\beta|} |\partial_t \partial^i L^J L_k u| + C(s/t^2) \sum_{|\alpha|<|\beta|} |\partial_t^\alpha \partial^i L^J L_k u| \]

\[\leq C(s/t^2) \sum_{|\alpha|<|\beta|} |\partial_t^\alpha \partial^i L^J u| \]

For the term \(\partial_t (|\partial_t^i L^J \partial_k u|)\), we see that by applying \(1.12\),

\[\partial_t (|\partial_t^i L^J \partial_k u|) = \sum_{|\alpha|<|\beta|} \partial_t \left( \rho_{\alpha}^i \partial_k u \right) + \sum_{|\alpha|<|\beta|} \partial_t \left( \rho_{\alpha}^i \partial^i L^J u \right) \quad \quad \tag{4.31}\]

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Now we for the first term in the right-hand-side, we see that
\[ \partial_s \left( \tilde{\rho}^{I,J}_{a I,J} \partial_s \partial^I U^J u \right) = \partial_s \tilde{\rho}^{I,J}_{a I,J} \cdot \partial_s \partial^I U^J u + \tilde{\rho}^{I,J}_{a I,J} \partial_s \partial_s \partial^I U^J u \]
We see that
\[ |\partial_s \tilde{\rho}^{I,J}_{a I,J}| \leq C(s/t^2) \]
thus
\[ (4.32) \quad \left| \partial_s \tilde{\rho}^{I,J}_{a I,J} \cdot \partial_s \partial^I U^J u \right| \leq C(s/t^2) |\partial_s \partial^I U^J u|, \]
Now we consider \( \tilde{\rho}^{I,J}_{a I,J} \partial_s \partial^I U^J u \). This is by the following calculation (by (4.2)): \[ \partial_s \partial_s \partial^I U^J u = \partial_s \left( t^{-1} L_c \partial^I U^J u \right) + \sum_{|I'|=1} \tilde{\partial}_s \left( t^{-1} \theta_{I'} \partial^{I''} U^J u \right) \]
Thus we see that
\[ \left| \tilde{\rho}^{I,J}_{a I,J} \partial_s \partial^I U^J u \right| \leq C \sum_{|I'| \leq |I|} \left( (s/t^2) |\partial^{I''} U^J u| + t^{-1} |\partial_s \partial^I U^J u| \right) \]
For the second term in (4.31), we see that \[ \partial_s \left( \rho^{I,J}_{a I,J} \partial^I U^J u \right) = \partial_s \rho^{I,J}_{a I,J} \cdot \partial^I U^J u + \rho^{I,J}_{a I,J} \partial_s \partial^I U^J u \]
and recall that \( \rho^{I,J}_{a I,J} \) is homogeneous of degree \( \leq -1 \). Thus we see that \( |\partial_s \rho^{I,J}_{a I,J}| \leq C s/t^2 \). So we see that
\[ |\partial_s \left( \rho^{I,J}_{a I,J} \partial^I U^J u \right) | \leq C \sum_{|I'| \leq |I|} \left( t^{-1} |\partial_s \partial^{I'} U^J u| + (s/t^2) |\partial^{I''} U^J u| \right) \]
Thus we see that the desired estimate is established.

\begin{lemma}
Let \( u \) be a function defined in \( K \) sufficiently regular. Then the following estimate holds:
\[ (4.33) \quad \left| |\partial^I U^J, \partial_s \partial_h u| \right| \leq C \sum_{|I'| \leq |I|} \left( t^{-1} |\partial_s \partial^{I'} U^J u| + t^{-2} |\partial^{I''} U^J u| \right). \]
\end{lemma}
\begin{proof}
We remark that
\[ (4.34) \quad |\partial^I U^J, \partial_s \partial_h u| = |\partial^I U^J, \partial_s \partial_h u + \partial_s (|\partial^I U^J, \partial_h u|) \]
For the first term in right-hand-side of the above equation, we see that by (4.12), \[ |\partial^I U^J, \partial_s \partial_h u| = \sum_{|I'| \leq |I| \atop |J'| < |J|} \rho_{a I', J'} \partial_s \partial^{J'} \partial^I U^J \sum_{1 \leq |I'| \leq |I|} \rho_{a I', J} \partial^{I'} U^J \partial_h u \]
We see that (by homogeneity of \( \partial^{I'} U^J \partial_h u \) and \( t^{-1} \))
\[ |\rho_{a I', J'} \partial_s \partial^{J'} \partial^I U^J \partial_h u| = |\rho_{a I', J'} \partial_s \partial^{J'} \partial^I U^J \partial_h u| \leq C \sum_{c \mid |I'| \leq |I| \atop |J'| < |J|} \left( t^{-1} |\partial_s \partial^{I''} U^J u| + t^{-2} |\partial^{I''} U^J u| \right). \]
\end{proof}
By (4.13)
\[ |\rho_n(j)^L, \partial^t L^I \partial^n u| \leq C t^{-1} \sum_{\substack{|J'| \leq |J| \\ |J'| \leq |J|}} |\partial_n \partial^t L^I u| + C t^{-2} \sum_{1 \leq |J'| \leq |J|} |\partial^t L^I u| \]

For the second term in right-hand-side of (4.34), we see that by (4.12)
\[ \partial_n ([\partial^t L^I, \partial^n u] = \sum_{\substack{|J'| \leq |J| \\ |J'| \leq |J|}} \partial_n \left( \rho_n(j)^L, \partial_n \partial^t L^I u \right) + \sum_{1 \leq |J'| \leq |J|} \partial_n \left( \rho_n(j)^L \partial^t L^I u \right). \]

By homogeneity, we see that
\[ |\partial_n ([\partial^t L^I, \partial^n u]| \leq C \sum_{\substack{|J'| \leq |J| \\ |J'| \leq |J|}} \left( |\partial_n \partial_n \partial^t L^I u| + t^{-1} |\partial_n \partial^t L^I u| \right) \]
\[ + C \sum_{1 \leq |J'| \leq |J|} \left( t^{-1} |\partial_n \partial^t L^I u| + t^{-2} |\partial^t L^I u| \right). \]

Now we see that
\[ |\partial_n \partial_n \partial^t L^I u| = |\partial_n \left( t^{-1} L_c \partial_n \partial^t L^I u \right) | \leq C t^{-2} |L_c \partial^t L^I u| + C t^{-1} |\partial_n L_c \partial^t L^I u| \]
\[ \leq C \sum_{|J'| \leq |J|} \left( t^{-2} |\partial^t L^I u| + t^{-1} |\partial_n \partial^t L^I u| \right). \]

The above bounds conclude the desired result.

4.4 Estimates based on commutators I

In this subsection, we will control the following terms
\[(4.35) \quad \| (s^2/t) \partial^t L^I \partial_n u \|_{L^2(\mathcal{K})}, \quad \| s \partial^t L^I \partial_n u \|_{L^2(\mathcal{K})}, \quad \| (s/t) \partial^t L^I u \|_{L^2(\mathcal{K})} \]
where $|I| + |J| \leq N$.

\[(4.36) \quad \| s^2 \partial^t L^I \partial_n \partial_n u \|_{L^2(\mathcal{K})}, \quad \| s t \partial^t L^I \partial_n \partial_n u \|_{L^2(\mathcal{K})} \]
where $|I| + |J| \leq N - 1$. We have the following result

**Lemma 4.17.** Let $u$ be a function defined in $\mathcal{K}$, sufficiently regular. Then the terms in (4.35) and (4.36) are bounded by
\[ \sum_{|I| + |J| \leq N} E_{\text{com}}(s, \partial^t L^I u)^{1/2}. \]

**Proof.** These are by apply the decomposition of commutators. For the first term in (4.35),
\[ \| (s^2/t) \partial^t L^I \partial_n u \| \leq \| (s^2/t) \partial_n \partial^t L^I u \| + \| (s^2/t) \partial^t L^I, \partial_n u \| \]
\[ \leq \| (s^2/t) \partial_n \partial^t L^I u \| + C(s/t) \sum_{|J'| \leq |I|} \| \partial_n \partial^t L^I u \| + C s(t)^2 \sum_{|J'| \leq |I|} |\partial_n \partial^t L^I u| \]
\[ \leq \| (s^2/t) \partial_n \partial^t L^I u \| + C \sum_{|J'| \leq |I|} |(1 + s^2/t) \partial_n \partial^t L^I u| \]
\[ + C \sum_{a \neq |J'| \leq |I|} |s(\partial^t)^2 \partial_n \partial^t L^I u| \]
Thus (remark that for the second term, \( s^2/t \geq 1 \geq s/t \) in \( \mathcal{K} \))

\[
\| (s^2/t) \partial^I \bar{L}^J \partial_\alpha u \|_{L^2(\mathcal{K}_t)} \leq C \sum_{|I|+|J| \leq N} E_{\text{con}}(s, \partial^I \bar{L}^J u)^{1/2}.
\]

For the second term we apply (4.13), we omit the detail. The third is guaranteed by (2.9). For the first term in (4.36), we see that

\[
\| s^2 \partial^I \bar{L}^J \partial_\alpha u \|_{L^2(\mathcal{K}_t)} \leq \| s^2 \partial \partial^I \bar{L}^J u \|_{L^2(\mathcal{K}_t)} + \| s^2 \partial^I \bar{L}^J, \partial_\alpha \bar{L}^J u \|_{L^2(\mathcal{K}_t)}
\]

By lemma (4.18) we see that

\[
\| s^2 \partial^I \bar{L}^J, \partial_\alpha \partial^I \bar{L}^J u \|_{L^2(\mathcal{K}_t)} \leq \sum_{|I'|+|J'| \leq N} E_{\text{con}}(s, \partial^I \bar{L}^J u)^{1/2}.
\]

\[
\| s^2 \partial_\alpha \partial^I \bar{L}^J u \|_{L^2(\mathcal{K}_t)} = \| s^2 \partial_\alpha (t^{-1} L_\alpha \partial^I \bar{L}^J u) \|_{L^2(\mathcal{K}_t)}
\]

\[
\leq \| (s^2/t) \partial_\alpha t L_\alpha \partial^I \bar{L}^J u \|_{L^2(\mathcal{K}_t)} + \| (s/t)^3 L_\alpha \partial^I \bar{L}^J u \|_{L^2(\mathcal{K}_t)}
\]

\[
\leq \| (s^2/t) \partial_\alpha \partial^I \bar{L}^J u \|_{L^2(\mathcal{K}_t)} + \| (s/t)^3 \partial^I \bar{L}^J u \|_{L^2(\mathcal{K}_t)}
\]

Then by (4.2), we see that

\[
\| s^2 \partial_\alpha \partial^I \bar{L}^J u \|_{L^2(\mathcal{K}_t)} \leq \sum_{|I'| \leq |I|+1} \| (s^2/t) \partial_\alpha \partial^I \bar{L}^J u \|_{L^2(\mathcal{K}_t)} + \| (s/t)^3 \partial^I \bar{L}^J u \|_{L^2(\mathcal{K}_t)}
\]

and this proved the bound on the first term of (4.36).

Now we regard the term \( \partial^I \bar{L}^J \partial_\alpha \partial_\beta u \).

(4.37) \[
\| s t \partial^I \bar{L}^J \partial_\alpha \partial_\beta u \|_{L^2(\mathcal{K}_t)} = \| s t \partial_\alpha \partial_\beta \partial^I \bar{L}^J u \|_{L^2(\mathcal{K}_t)} + \| s t [\partial^I \bar{L}^J, \partial_\alpha \partial_\beta u] \|_{L^2(\mathcal{K}_t)}.
\]

For the first term in the right-hand-side of the above equation, we see that

\[
\| s t \partial_\alpha \partial_\beta \partial^I \bar{L}^J u \|_{L^2(\mathcal{K}_t)} = \| s t \partial_\alpha (t^{-1} L_\alpha \partial^I \bar{L}^J u) \|_{L^2(\mathcal{K}_t)}
\]

\[
\leq \| s \partial_\alpha L_\alpha \partial^I \bar{L}^J u \|_{L^2(\mathcal{K}_t)} + \| (s/t) L_\alpha \partial^I \bar{L}^J u \|_{L^2(\mathcal{K}_t)}
\]

\[
\leq \| s \partial_\alpha \partial^I \bar{L}^J u \|_{L^2(\mathcal{K}_t)} + \| (s/t) \partial^I \bar{L}^J u \|_{L^2(\mathcal{K}_t)}
\]

Then, also by (4.2),

\[
\| s t \partial_\alpha \partial_\beta \partial^I \bar{L}^J u \|_{L^2(\mathcal{K}_t)} \leq C \sum_{|I'| \leq |I|+1} E_{\text{con}}(s, \partial^I \bar{L}^J u)^{1/2}.
\]

For the second term in right-hand-side of (4.37), by applying lemma (4.16) we see that it is also bounded by

\[
C \sum_{|I|+|J| \leq N} E_{\text{con}}(s, \partial^I \bar{L}^J u)
\]

Thus the desired bound is established. 

We also establish a rough bound on \( \partial^I \bar{L}^J \partial_\alpha \partial_\beta u \):
Lemma 4.18. Let $u$ be a function defined in $\mathcal{K}$, sufficiently regular and vanishes near the conical boundary. Then the following bound holds for $|I| + |J| \leq N - 1$:

$$
\|s \partial_s \bar{\partial}_s L^J u\|_{L^2(\mathcal{K}_s)} + \|s \partial^J L^J \bar{\partial}_s u\|_{L^2(\mathcal{K}_s)} \leq C \sum_{|I'| + |J'| \leq N} E_{\text{con}}(s, \partial^I L^J u)^{1/2}.
$$  \hspace{1cm} (4.38)

Proof. For the first term, we recall that

$$
\sum_{|I'| + |J'| \leq N} E_{\text{con}}(s, \partial^I L^J u)^{1/2}.
$$  \hspace{1cm} (4.39)

We see that for the first term in right-hand-side, by (4.20),

$$
\|s \partial_s ((s/t) \partial_t L^J u)\|_{L^2(\mathcal{K}_s)} \leq C \|((s/t) \partial_t L^J u)\|_{L^2(\mathcal{K}_s)} \leq C \sum_{|I'| + |J'| \leq N} E_{\text{con}}(s, \partial^I L^J u)^{1/2}.
$$  \hspace{1cm} (4.40)

For the second term in right-hand-side of (4.39), We remark the following calculation (where we apply (4.39)):

$$
\|s \partial_s ((s/t) \partial_t L^J u)\|_{L^2(\mathcal{K}_s)} \leq C \|((s/t) \partial_t L^J u)\|_{L^2(\mathcal{K}_s)} \leq C \sum_{|I'| + |J'| \leq N} E_{\text{con}}(s, \partial^I L^J u)^{1/2}.
$$  \hspace{1cm} (4.41)

Then by applying (4.20) and the bounds on terms in (4.41), we see that the desired result is proved.

4.5 Estimates based on commutators II

In view of the global Sobolev inequality (4.1), to turn the $L^2$ bounds (supplied by the energy) into $L^\infty$ bounds, we need to bound some terms. To do so, we need some preparations. Through out this subsection, we denote by $u$ a sufficiently regular function defined in $\mathcal{K}$, and the following estimates are valid in $\mathcal{K}$.

Lemma 4.19. Let $u$ be a sufficiently regular function defined in $\mathcal{K}$. Then the following estimates hold:

$$
|\partial^I L^J \partial_s L^I u| \leq C \sum_{|I'| \leq |I| + |J|, \ |J'| \leq |J| + |J'|} |\partial_s \partial^I L^J u| + C \sum_{|I'| \leq |I| + |J|, \ |J'| \leq |J| + |J'|} \|\partial^I L^J u\|_{L^2(\mathcal{K}_s)}.
$$  \hspace{1cm} (4.42)

Proof. For the first term, we see that

$$
\partial^I L^J \partial_s L^I u = \partial_s \partial^I L^J L^I u + [\partial^I L^J, \partial_s] \partial^I L^J u =: T_1 + T_2.
$$

Then we see that by (4.20)

$$
T_1 = \sum_{|I'| \leq |I| + |J|, \ |J'| \leq |J| + |J'|} \zeta_{J} L^J L^I \partial_s \partial^I L^J u
$$

which leads to

$$
|T_1| \leq C \sum_{|I'| \leq |I| + |J|, \ |J'| \leq |J| + |J'|} |\partial_s \partial^I L^J u|.
$$

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On the other hand,

\[ |T_2| \leq Cs^{-1} \sum_{|I|<|J|} |\partial_t \partial^T L^J \partial^I L^j u| + C(s/t) \sum_{\alpha, |I|^\alpha \leq |J|} |\partial_\alpha \partial^T L^J L^I u| \]

= T_3 + T_4.

We see that for each term of T_3, by (4.5) and the fact that in K s^2 \geq t,

\[ s^{-1} |\partial_t \partial^T L^J \partial^I L^j u| \leq C(t/s^2) \sum_{|I|^\alpha \leq |J|} |(s/t) \partial_\alpha \partial^T L^J L^I u| \]

\[ \leq C \sum_{|I|^\alpha \leq |J|} |\partial_\alpha \partial^T L^J L^I u|. \]

Also, for T_4, when \( \alpha = 0, \partial_\alpha = \partial_t, \) then

\[ (s/t) |\partial_t \partial^T L^J \partial^I L^j u| \leq C \sum_{|I|^\alpha \leq |J|} |(s/t) \partial_\alpha \partial^T L^J L^I u| \]

\[ \leq C \sum_{|I|^\alpha \leq |J|} |\partial_\alpha \partial^T L^J L^I u|. \]

For \( \alpha > 0, \) we denote by \( \alpha = a, \) then

\[ (s/t) |\partial_a \partial^T L^J \partial^I L^j u| \leq C \sum_{|I|^\alpha \leq |J|} |(s/t) \partial_a \partial^T L^J L^I u| \]

This leads to

\[ |T_2| \leq C \sum_{|I|^\alpha \leq |J|} |\partial_\alpha \partial^T L^J u| + C \sum_{|I|^\alpha \leq |J|} |(s/t) \partial_\alpha \partial^T L^J L^I u|. \]

The bounds of T_1 and T_2 leads to (4.11).

For (4.12), we see that

\[ \partial^T L^J \partial_\alpha L^I u = \partial_\alpha \partial^T L^J \partial^I L^j u + \partial^T L^J, \partial_\alpha \partial^T L^I u = T_3 + T_4. \]

We see that by (4.5)

\[ |T_3| \leq C \sum_{|I|^\alpha \leq |J|} |\partial_\alpha \partial^T L^J u|. \]

For T_4, we apply (4.13):

\[ |T_4| \leq C \sum_{|I|^\alpha \leq |J|} |\partial_\alpha \partial^T L^J \partial^I L^j u| + C t^{-1} \sum_{1 \leq |I|^\alpha \leq |J|} |\partial^T L^J \partial^I L^j u|. \]

Also by (4.5):

\[ |T_4| \leq C \sum_{c|I|^\alpha \leq |J|} |\partial_\alpha \partial^T L^J u| + C t^{-1} \sum_{1 \leq |I|^\alpha \leq |J|} |\partial^T L^J u|. \]

\[ = C \sum_{c|I|^\alpha \leq |J|} |\partial_\alpha \partial^T L^J u| + Cs^{-1} \sum_{c|I|^\alpha \leq |J|} |(s/t) \partial^T L^J u|. \]

Now the bounds on |T_3| and |T_4| leads to (4.12).
Then the following bounds are direct:

**Lemma 4.20.** The following bounds are direct:

\[(4.43) \quad \| (s^2/t) \partial^{J''} L^{J'} \partial_s \partial^J L^J u \|_{L^2(\mathcal{H}_t)} \leq C \sum_{|J''| \leq |J'| + |J|} \| C_{J''} \|_{L^2(\mathcal{H}_t)} \| \partial^{J''} \partial_s L^{J'} \partial^J L^J u \|_{L^2(\mathcal{H}_t)} \]

are bounded by

\[C \sum_{|J' + J| \leq N} E_{con}(s, \partial^{J''} L^{J''} u)^{1/2}.\]

Now we regard the following terms:

\[(4.44) \quad \| \partial^{J''} L^{J'} ((s^2/t) \partial_s \partial^J L^J u) \|_{L^2(\mathcal{H}_t)} \leq C \sum_{|J''| \leq |J'| + |J|} E_{con}(s, \partial^{J''} L^{J''} u)^{1/2},\]

where \(|I'| + |J'| \leq 2\) and \(|I| + |J| \leq N - 2\).

\[(4.45) \quad \| \partial^{J''} L^{J'} (s^2 \partial_s \partial_s \partial^J L^J u) \|_{L^2(\mathcal{H}_t)} \leq C \sum_{|I''| \leq |I| + |J|} E_{con}(s, \partial^{J''} L^{J''} u)^{1/2},\]

where \(|I'| + |J'| \leq 2\) and \(|I| + |J| \leq N - 3\). We have the following results:

**Lemma 4.21.** The terms in (4.44) and (4.45) are bounded by

\[C \sum_{|I' + J| \leq N} E_{con}(s, \partial^{J''} L^{J''} u)^{1/2}.\]

**Proof.** For the first term in (4.44), we see that by (4.22), (4.31) and lemma 4.20

\[
\left\| \partial^{J''} L^{J'} ((s^2/t) \partial_s \partial^J L^J u) \right\|_{L^2(\mathcal{H}_t)} \leq C \sum_{I'' + J'' = J'} \left\| \partial^{J''} L^{J'} (s^2/t) \partial^J L^J u \right\|_{L^2(\mathcal{H}_t)}
\]

\[
\leq C \sum_{|I'| + |J'| \leq N} E_{con}(s, \partial^{J''} L^{J''} u)^{1/2}.
\]

The second term of (4.44) is controlled by (4.49) (combined with the fact that in \(\mathcal{K}, s \geq t/s\), (4.22) and lemma 4.20)

\[
\left\| \partial^{J''} L^{J'} (s^2 \partial_s \partial^J L^J u) \right\|_{L^2(\mathcal{H}_t)} \leq C \sum_{J'' + J' + J'' = I'} \left\| \partial^{J''} L^{J'} s \cdot \partial^J L^J \partial_s \partial^J L^J u \right\|_{L^2(\mathcal{H}_t)}
\]

\[
\leq C \sum_{|J''| \leq |J|} E_{con}(s, \partial^{J''} L^{J''} u)^{1/2}.
\]

The first term in (4.45) is bounded as following. First we remark that

\[
\partial^{J''} L^{J'} (s^2 \partial_s \partial_s \partial^J L^J u) = \partial^{J''} L^{J'} (s^2 \partial_s (t^{-1} L_a \partial^J L^J u))
\]

\[
= \partial^{J''} L^{J'} ((s^2/t) \partial_s L_a \partial^J L^J u) - \partial^{J''} L^{J'} ((s/t) \partial_s \partial_s \partial^J L^J u)
\]

\[= T_1 + T_2.
\]
For $T_1$, we see that by (4.22) and lemma [4.20] we see that
\[
\|T_1\| \leq C \sum_{|I' + J'| = |I|} \partial^{I'} L^{J'} (s^2/t) \cdot \partial^{I} L^{J} \partial_a \partial_l L^j u
\leq C \sum_{|I' + J'| = |I|} \| (s^2/t) \partial^{I'} L^{J'} L^a \partial_l L^j u \|_{L^2(\Omega_s)}
\leq C \sum_{|I' + J'| \leq N} E_{\text{con}} (s, \partial^{I''} L^{J''} u)^{1/2}.
\]
Here by For the term $T_2$, we see that by (4.3) and lemma [4.20]
\[
\|T_2\| \leq C \sum_{|I' + J'| = |I|} \| (s^2/t) \partial^{I'} L^{J'} L^a \partial_l L^j u \|_{L^2(\Omega_s)} 
\leq C \sum_{|I' + J'| \leq N} E_{\text{con}} (s, \partial^{I''} L^{J''} u)^{1/2}.
\]
The bounds on $T_1$ and $T_2$ give the bounds of the first term of (4.46).

Now we regard the second term in (4.46):
\[
\partial^{I'} L^{J'} (s \partial_a \partial_b \partial_l L^j u) = \partial^{I'} L^{J'} (s \partial_a (t^{-1} L^a \partial_l L^j u))
= \partial^{I'} L^{J'} ((s x^a /t^2) L^a \partial_l L^j u)
= \sum_{|I' + J'| = |I|} \partial^{I'} L^{J'} (s \partial_a \partial_l L^j u) - \partial^{I'} L^{J'} ((s x^a /t^2) L^a \partial_l L^j u)
= T_3 + T_4.
\]
For $T_3$, we see that by (4.43) together with the fact that in $\mathcal{X}$, $t/s \leq s$ and [4.20]
\[
\|T_3\| \leq C \sum_{|I' + J'| = |I|} \| (s \partial_a \partial_l L^j u) \|_{L^2(\Omega_s)} \leq C \sum_{|I' + J'| \leq N} E_{\text{con}} (s, \partial^{I''} L^{J''} u)^{1/2}.
\]
For $T_4$, we see that by (4.20) and lemma [4.23] ($x^a /t$ is homogeneous of degree zero)
\[
\| \partial^{I'} L^{J'} (s \partial_a \partial_l L^j u) \|_{L^2(\Omega_s)} \leq C \sum_{|I' + J'| \leq N} \| (s \partial_a \partial_l L^j u) \|_{L^2(\Omega_s)}
\leq C \sum_{|I' + J'| \leq N} E_{\text{con}} (s, \partial^{I''} L^{J''} u)^{1/2}
\]
The bounds on $T_3$ and $T_4$ concludes the second term in (4.46).

Now we list out another group of terms:
\[
\| \partial^{I'} L^{J'} ((s^2/t) \partial^I L^j \partial_a u) \|_{L^2(\Omega_s)}, \quad \| \partial^{I'} L^{J'} (s \partial^I L^j \partial_a u) \|_{L^2(\Omega_s)}, \quad \| \partial^{I'} L^{J'} ((s/t)^I L^j u) \|_{L^2(\Omega_s)}
\]
where $|I'| + |J'| \leq 2, |I| + |J| \leq N - 2$ and
\[
\| \partial^{I'} L^{J'} (s \partial^I L^j \partial_a u) \|_{L^2(\Omega_s)}, \quad \| \partial^{I'} L^{J'} ((s/t)^I L^j u) \|_{L^2(\Omega_s)}
\]
where $|I'| + |J'| \leq 2, |I| + |J| \leq N - 3$ with $N$ is an integer. In general we have the following estimates:
**Lemma 4.22.** In $\mathcal{K}$, the terms in (4.40) and (4.41) are bounded by

$$C \sum_{|I'|+|J'| \leq N} E_{\text{con}}(s, \partial^{I'} L^{J'} u)^{1/2}.$$ 

*Proof.* These are by lemma 4.17 and the corresponding estimates of commutators. For the first term in (4.40), by (4.29)

$$\partial^{I'} L^{J'} \left( (s^2/t) \partial L J \partial_s u \right) = \sum_{I_1^t + I_2^s = I'} \partial^{I_1} L^{I_2} (s^2/t) \cdot \partial L^{I_2} L^{I_2} \partial_s u$$

$$= \sum_{I_1^t + I_2^s = I'} \sum_{J_1^J = J} \partial^{I_1} L^{I_2} (s^2/t) \cdot \partial L^{I_2} L^{I_2} \partial_s u.$$

Then by (4.22) and lemma 4.17, the desired bound is direct.

The second term in (4.40) is bounded similarly by (4.30), (4.19) and lemma 4.17. We omit the detail.

The third term in (4.40) is by (4.19) combined with lemma 4.17. We also omit the detail.

The first term in (4.41) is by (4.19) combined with lemma 4.17.

$$\|\partial^{I'} L^{J'} (s^2 \partial L J \partial_s u) \|_{L^2(\mathcal{X}_s)} \leq \sum_{I_1^t + I_2^s = I'} \|\partial^{I_1} L^{I_2} s \cdot \partial L^{I_2} s \cdot \partial L^{I_2} \partial_s u \|_{L^2(\mathcal{X}_s)}$$

$$\leq C \sum_{|I'|+|J'| \leq N} \|\partial^{I_1} L^{I_2} \partial_s u \|_{L^2(\mathcal{X}_s)}$$

which is bounded by $C \sum_{|I'|+|J'| \leq N} E_{\text{con}}(s, \partial^{I'} L^{J'} u)^{1/2}$.

The second term in (4.41) is by (4.19) combined with lemma 4.17. We omit the detail.  

\[\Box\]

### 4.6 Decay bounds from global Sobolev inequality and commutators

In this section by applying lemma 4.21 and lemma 3.22 combined with (1.1), we will establish a series decay estimates. In general we have the following results:

**Proposition 4.23** (Decay estimates by energy bounds). Let $u$ be a function defined in $\mathcal{K}$, sufficiently regular. Then for $|I|+|J| \leq N-2$, the following terms:

\[
\text{(4.48a)} \quad \sup_{\mathcal{X}_s} \left\{ s^{1/2} \partial^{I'} L^{J'} u \right\}, \quad \sup_{\mathcal{X}_s} \left\{ s^{1/2} \partial \partial_s L^{J'} u \right\}, \quad \sup_{\mathcal{X}_s} \left\{ s^{1/2} L \partial_s L^{J'} u \right\},
\]

and

\[
\text{(4.48b)} \quad \sup_{\mathcal{X}_s} \left\{ s^{1/2} \partial L^{I'} L \partial_s u \right\}, \quad \sup_{\mathcal{X}_s} \left\{ s^{1/2} L \partial_s L^{I'} u \right\},
\]

are bounded by

$$C \sum_{|I|+|J| \leq N} E_{\text{con}}(s, \partial^{I'} L^{J'} u)^{1/2}. $$

For $|I|+|J| \leq N-3$, the following terms:

\[
\text{(4.49a)} \quad \sup_{\mathcal{X}_s} \left\{ s^{3/2} \partial \partial_s L^{I'} L^{J'} u \right\}, \quad \sup_{\mathcal{X}_s} \left\{ s^{5/2} \partial \partial_b L^{I'} L^{J'} u \right\}
\]

and

\[
\text{(4.49b)} \quad \sup_{\mathcal{X}_s} \left\{ s^{3/2} \partial L^{I'} \partial_s L^{J'} u \right\}, \quad \sup_{\mathcal{X}_s} \left\{ s^{5/2} \partial L^{J'} \partial_s \partial_b u \right\}
\]
are bounded by
\[ C \sum_{|I|+|J| \leq N} E_{\text{con}}(s, \partial^I L^J u)^{1/2}. \]

Furthermore, we have the following rough decay: For $|I| + |J| \leq N - 3$
\[ (4.50) \quad \sup_{\mathcal{N}} \{s^{3/2} \partial^I L^J \partial_a \partial_b u\} \leq C \sum_{|I|+|J| \leq N} E_{\text{con}}(s, \partial^I L^J u)^{1/2}. \]

This is by (4.40) combined with (4.20) and (4.48b).

5 Estimates on Hessian form

5.1 Objective and algebraic preparation

The purpose of this section is to give better $L^2$ and decay bounds on the following terms:
\[ \partial_\alpha \partial_\beta \partial^I L^J u, \quad \partial^I L^J \partial_a \partial_\beta u \]

We first make the following identities:
\[ \partial_t \partial_t = (t/s)^2 \partial_s \partial_s - s^{-1}(r/s)^2 \partial_r, \]
\[ \partial_t \partial_a = (t/s) + (x^a/s) \partial_s \partial_a = -\frac{tx^a}{s^2} \partial_s \partial_a + s^{-1} L_a \partial_s + \frac{tx^a}{s^3} \partial_a \]
and
\[ \partial_a \partial_b = (\partial_a - (x^a/s) \partial_s) (\partial_b - (x^b/s) \partial_s) \]
\[ = \frac{x^a x^b}{s^2} \partial_s \partial_a + t^{-1} L_a \partial_b - \frac{x^a}{st} L_b \partial_s - \frac{x^a}{s^2} L_b \partial_s - s^{-1} \partial_a - \frac{x^a x^b}{s^3} \partial_a. \]

These identities show the fact that in the Hessian form, the component $\partial_\beta \partial_a$ has an essential contribution, because the rest terms as at least $s^{-1}$ as a supplementary decay factor. So we will concentrate on the bounds of $\partial_a \partial_a$. More precisely the above identities lead to the following result:

**Lemma 5.1.** For $u$ a function defined in $\mathcal{K}$, sufficiently regular, the following estimate holds:
\[ (5.1) \quad \|s^2(t) \partial_a \partial_\beta u\|_{L^2(\mathcal{N}, \partial \mathcal{N})} \leq C \|s^2(t) \partial_s \partial_a u\|_{L^2(\mathcal{N}, \partial \mathcal{N})} + C \sum_{|J| \leq 1} E_{\text{con}}(s, L^J u)^{1/2}. \]

**Proof.** This is by direct calculation. We see that
\[ \|s^2(t) \partial_a \partial_\beta u\|_{L^2(\mathcal{N}, \partial \mathcal{N})} \leq \|s^2(t) \partial_s \partial_a u\|_{L^2(\mathcal{N}, \partial \mathcal{N})} + \|s(t) \partial_s u\|_{L^2(\mathcal{N}, \partial \mathcal{N})} \]
\[ \leq \|s^2(t) \partial_s \partial_a u\|_{L^2(\mathcal{N}, \partial \mathcal{N})} + C E_{\text{con}}(s, u)^{1/2}. \]

For $\partial_\beta \partial_a u$, we need to apply lemma 4.1.17.
\[ \|s^2(t) \partial_\beta \partial_a u\|_{L^2(\mathcal{N}, \partial \mathcal{N})} \leq \|s^2(t) \partial_s \partial_\beta u\|_{L^2(\mathcal{N}, \partial \mathcal{N})} + \|s(t) \partial_\beta u\|_{L^2(\mathcal{N}, \partial \mathcal{N})} + \|s(t) \partial_s u\|_{L^2(\mathcal{N}, \partial \mathcal{N})} \]
\[ \leq \|s^2(t) \partial_\beta \partial_a u\|_{L^2(\mathcal{N}, \partial \mathcal{N})} + C \sum_{|J| \leq 1} E_{\text{con}}(s, L^J u)^{1/2}. \]

The term $\partial_\beta \partial_a u$ is also by (4.1.17):
\[ \|s^2(t) \partial_\beta \partial_a u\|_{L^2(\mathcal{N}, \partial \mathcal{N})} \leq \|s^2(t) \partial_s \partial_a u\|_{L^2(\mathcal{N}, \partial \mathcal{N})} + \|s(t) \partial_\beta u\|_{L^2(\mathcal{N}, \partial \mathcal{N})} + \|s(t) \partial_s u\|_{L^2(\mathcal{N}, \partial \mathcal{N})} \]
\[ \leq \|s^2(t) \partial_\beta \partial_a u\|_{L^2(\mathcal{N}, \partial \mathcal{N})} + C \sum_{|J| \leq 1} E_{\text{con}}(s, L^J u)^{1/2}. \]

\[ \Box \]
Then we remark the following identity (see (2.2)):
\[
\Box = \partial_t \partial_s + 2(x^a/s) \partial_t \partial_a - \sum_a \partial_a \partial_a + \frac{3}{s} \partial_s \\
= \partial_t \partial_s + (2x^a/s)t^{-1}L_a \partial_a - t^{-1}L_a \partial_a + 3s^{-1} \partial_s
\]
This leads to
\[
(5.2) \quad \partial_t \partial_s u = \Box u + H_1[u].
\]
where
\[
H_1[u] := -(2x^a/s)t^{-1}L_a \partial_a + t^{-1}L_a \partial_a - 3s^{-1} \partial_s
\]

5.2 $L^2$ bounds

We remark that from (5.2),
\[
(s^3/t) \partial_t \partial_s \partial^J L^J u = (s^3/t) \Box \partial^J L^J u + (s^3/t)H_1[\partial^J L^J u]
\]
We remark the following property:

**Lemma 5.2.** Let $u$ be a function defined in $\mathcal{K}$, sufficiently regular. Then the following estimate holds:
\[
(5.3) \quad \| (s^3/t)H_1[\partial^J L^J u] \|_{L^2(\mathcal{K}_t)} \leq C \sum_{|I'|+|J'| \leq |I|+|J|+1} E_{\text{con}}(s, \partial^I L^J u)^{1/2}.
\]

**Proof.** We see that
\[
(5.2) \quad \partial_t \partial_s \partial^J L^J u \leq |(s^2/t)L_a \partial_t \partial^J L^J u| + |sL_a \partial_t \partial^J L^J u| + 3|s^2/t)\partial_s \partial^J L^J u|.
\]
Then by lemma 4.20, the result is proved. □

On the other hand,
\[
(s^3/t) \partial^J L^J \partial_t \partial_s u = (s^2/t) \partial^J L^J \Box u + \partial^J L^J (H_1[u])
\]
So we establish the following result:

**Lemma 5.3.** Let $u$ be a function defined in $\mathcal{K}$, sufficiently regular. Then the following estimate holds:
\[
(5.4) \quad \| (s^3/t)H_1[u] \|_{L^2(\mathcal{K}_t)} \leq C \sum_{|I'|+|J'| \leq |I|+|J|+1} E_{\text{con}}(s, \partial^I L^J u)^{1/2}.
\]

**Proof of lemma 5.3.** For the first term in $H_1$, we see that
\[
(s^3/t) \partial^J L^J ((x^a/t)s^{-1}L_a \partial_t u) = \sum_{i_1+i_2+i_3=J} (s^3/t) \partial^{I_1} L^{J_1} (x^a/t) \partial^{I_2} L^{J_2} (s^{-1}) \partial^{I_3} L^{J_3} L_a \partial_t u.
\]
Recall that (4.23) and the fact that $x^a/t$ is homogeneous of degree zero, we see that (by lemma 4.17)
\[
\| (s^3/t) \partial^J L^J ((x^a/t)s^{-1}L_a \partial_t u) \|_{\mathcal{K}_t} \leq C \sum_{|I'| \leq |I|} \| (s^2/t) \partial^{I_1} L^{J_1} L_a \partial_t u \|_{L^2(\mathcal{K}_t)}
\leq C \sum_{|I'| \leq |I|+1} E_{\text{con}}(s, \partial^I L^J u)^{1/2}.
\]
The rest terms are bounded similarly and we omit the detail. □
Finally we conclude by the following estimate:

**Proposition 5.4.** Let $u$ be a function defined in $\mathcal{K}$, sufficiently regular. Then the following bounds hold:

$$
\| (s^3/t) \partial^J L^J \partial_s \partial_s u \|_{L^2(\mathcal{K}_s)} + \| (s^3/t) \partial_s \partial_s \partial^J L^J u \|_{L^2(\mathcal{K}_s)} \leq C \| (s^3/t) \Box \partial^J L^J u \|_{L^2(\mathcal{K}_s)} + C \sum_{|J'| \leq |J| + 1} E_{\text{con}}(s, \partial^{J'} L^{J'} u)^{1/2}.
$$

(5.5)

5.3 $L^\infty$ bounds

The $L^\infty$ bounds are based on the above $L^2$ bounds and the global Sobolev inequality \[\text{(4.2)}\]. We first establish the following results:

**Lemma 5.5.** Let $u$ be a function defined in $\mathcal{K}$, sufficiently regular. Then the following bounds hold:

$$
\| \partial^J L^J (s^3/t) \partial^J L^J \partial_s \partial_s u \|_{L^2(\mathcal{K}_s)} \leq C \sum_{|J''| \leq |J| + 1} \| (s^3/t) \Box \partial^{J''} L^{J''} u \|_{L^2(\mathcal{K}_s)}
$$

(5.6)

$$
+ C \sum_{|J''| \leq |J| + 1} E_{\text{con}}(s, \partial^{J''} L^{J''} u)^{1/2}.
$$

$$
\| \partial^{J'} L^{J'} (s^3/t) \partial_s \partial_s \partial^J L^J u \|_{L^2(\mathcal{K}_s)} \leq C \sum_{|J''| \leq |J|} \| (s^3/t) \Box \partial^{J''} L^{J''} u \|_{L^2(\mathcal{K}_s)}
$$

(5.7)

$$
+ C \sum_{|J''| \leq |J| + 1} E_{\text{con}}(s, \partial^{J''} L^{J''} u)^{1/2}.
$$

To prove this we first remark the following bound:

$$
\| \partial^J L^J (s^3/t) \| \leq C(s^3/t).
$$

(5.8)

This is checked by applying \[\text{(4.19)}\] and \[\text{(4.22)}\]:

$$
| \partial^J L^J (s^3/t) | = \partial^J L^J (s \cdot s^2/t) \leq C \sum_{J_1 + J_2 = J} | \partial^{J_1} L^{J_1} s \partial^{J_2} L^{J_2} (s^3/t) | \leq Cs(s^2/t) \leq Cs^3/t.
$$

**Proof of lemma 5.5.** For (5.6), we see that by (5.8):

$$
\| \partial^{J'} L^{J'} (s^3/t) \partial^J L^J \partial_s \partial_s u \|_{L^2(\mathcal{K}_s)} \leq C \sum_{|J''| \leq |J| + 1} \| \partial^{J''} L^{J''} (s^3/t) \partial^{J''} L^{J''} \partial^J L^J \partial_s \partial_s u \|_{L^2(\mathcal{K}_s)}
$$

$$
\leq C \sum_{|J''| \leq |J|} \| (s^3/t) \partial^{J''} L^{J''} L^{J''} \partial^J \partial_s \partial_s u \|_{L^2(\mathcal{K}_s)}
$$

$$
\leq C \sum_{|J''| \leq |J| + 1} \| (s^3/t) \partial^{J''} L^{J''} \partial_s \partial_s u \|_{L^2(\mathcal{K}_s)}
$$

where in the last inequality we have applied lemma \[\text{(4.3)}\]. Then by (5.5), we see that (5.6) is established.

(5.7) is more complicated. We see that by (5.8),

$$
\| \partial^{J'} L^{J'} (s^3/t) \partial_s \partial_s \partial^J L^J u \|_{L^2(\mathcal{K}_s)} \leq C \sum_{|J''| \leq |J| + 1} \| (s^3/t) \partial^{J''} L^{J''} \partial_s \partial_s \partial^J L^J u \|_{L^2(\mathcal{K}_s)}.
$$
To bound this term we observe that by \[4.12\]
\[
\| (s^3/t) \partial^I J^I \partial_s \partial_s J^I u \|_{L^2(\mathcal{K}_s)} \leq C \sum_{|J| \leq |I| + 1} \| s^2 (s/t)^3 \partial_s \partial_s J^I u \|_{L^2(\mathcal{K}_s)} + C \sum_{|J| \leq |I| + 1} \| s (s/t)^2 \partial_s J^I u \|_{\mathcal{K}_s}.
\]

Then we apply \[5.3\] and see that by lemma \[5.1\]
\[
\| (s^3/t) \partial^I J^I \partial_s \partial_s J^I u \|_{L^2(\mathcal{K}_s)} \leq C \sum_{|J| \leq |I| + 1} \| s^2 (s/t)^3 \partial_s \partial_s J^I u \|_{L^2(\mathcal{K}_s)} + C \sum_{|J| \leq |I| + 1} \| s (s/t)^2 \partial_s J^I u \|_{\mathcal{K}_s}.
\]

Then combined with \[5.3\] the desired result is established.

Now, combined with the global Sobolev’s inequality \[4.11\], we have the following decay estimates:

**Proposition 5.6.** The \(L^\infty\) bounds are based on the above \(L^2\) bounds and the global Sobolev inequality \[4.11\]. We first establish the following results:

\[5.9\]
\[
\sup_{\mathcal{K}_s} \| s^{3/2} \partial^I J^I \partial_s \partial_s J^I u \| + \sup_{\mathcal{K}_s} \| s^3 \partial^I J^I \partial_s \partial_s J^I u \| \leq C \sum_{|J| \leq |I| + 1} \| (s^3/t) \partial J^I u \|_{L^2(\mathcal{K}_s)} + C \sum_{|J| \leq |I| + 1} E_{\text{con}}(s, \partial J^I u)^{1/2}.
\]

## 6 Null condition in hyperbolic frame

### 6.1 Objective and basic calculations

The objective of this section is to give a first analysis on the following terms:

\[\partial^I J^I (Q^{\alpha\beta\gamma} \partial_s \partial_s \partial_s u), \quad \partial^I J^I (Q^{\alpha\beta\gamma} \partial_s \partial_s \partial_s u)\]

which will play essential role in the following sections. In this section we always suppose that \(u\) is a function defined in \(\mathcal{K}\), sufficiently regular and vanishes near the conical boundary \(\partial \mathcal{K} = \{ t = |x| + 1 \}\).

The first two subsections are preparations for the last one. Here we analyse the property of the quantity \(r/t\) in the region \(\{ t/2 < |x| < t \}\). We establish the following bounds:

**Lemma 6.1.** In the region \(\mathcal{K} \cap \{ t/2 < |x| < t \}\), the following bounds hold with a constant \(C\) determined by \(I, J\):

\[6.1\]
\[
\| \partial^I J^I (r/t) | \leq C t^{-|I|}, \quad \| \partial^I J^I (t^{1/2}(r + t)^{-1/2}) | \leq C t^{-|I|}.
\]

\[6.2\]
\[
\| \partial^I J^I (t - r)^{1/2}(r + t)^{-1/2} \| \leq \begin{cases} C s^{-1}, & |I| \geq 1, \\ C s/t, & |I| = 0. \end{cases}
\]
Proof. We recall that $r^2/t^2$ is homogeneous of degree zero. For the convenience of expression, we denote by

$$f : (1/4, +\infty) \to \mathbb{R}$$

$$x \to x^{1/2}$$

and $v := (r/t)^2$. Thus we see that $r/t = f(v)$. Then (let $|I| + |J| = N \geq 1$)

$$\partial^I L^J(r/t) = \partial^I L^J(f(v)) = \sum_{1 \leq k \leq N} \sum_{\substack{j_1 + j_2 + \ldots + j_k = t \leq t \leq \frac{t}{t+1} \leq J}} f^{(n)}(v) \cdot \partial^{i_1} L^{j_{i_1}} \ldots \partial^{i_k} L^{j_k} v$$

Thus we see that because $v$ is homogeneous of degree zero, and the fact that $f^{(n)}$ is bounded on $(1/4, +\infty)$ by a constant $C$ (determined by $n \geq 1$), we see that for $1/2 < r/t < 1$, $\partial^I L^J(r/t)$ is bounded by $C t^{-|I|}$. For $N = 0$, we see that $r/t < 1$. Thus the first term is correctly bounded.

For the second term in (6.1), we see that

$$\left( \frac{t}{t + r} \right)^{1/2} = (1 + r/t)^{-1/2} = f(1 + r/t).$$

Then we see that

$$\partial^I L^J(f(1 + r/t)) = \sum_{1 \leq k \leq N} \sum_{\substack{j_1 + j_2 + \ldots + j_k = t \leq t \leq \frac{t}{t+1} \leq J}} f^{(n)}(1 + (r/t)) \cdot \partial^{i_1} L^{j_{i_1}} (1 + (r/t)) \cdots \partial^{i_k} L^{j_k} (1 + (r/t)).$$

We see that $3/2 < 1 + r/t < 2$, and by the bounds on $r/t$, we see that the second bound is established.

For the third term, we observe that

$$\partial^I L^J \left((t - r)^{1/2}(t + r)^{-1/2}\right) = \partial^I L^J \left((s/t) \frac{t}{t + r}\right) = \sum_{\substack{j_1 + j_2 = t \leq t \leq \frac{t}{t+1} \leq J}} \partial^{i_1} L^{j_{i_1}}(s/t) \cdot \partial^{i_2} L^{j_2} \frac{t}{t + r}.$$}

We recall (6.20) and for the second factor, we see that

$$\partial^I L^J \left(\frac{t}{t + r}\right) = \partial^I L^J \left((1 + (r/t))^{-1}\right) = \sum_{\substack{j_1 + j_2 = t \leq t \leq \frac{t}{t+1} \leq J}} \partial^{i_1} L^{j_{i_1}} \left((1 + (r/t))^{-1/2}\right) \partial^{i_2} L^{j_2} \left(1 + (r/t)^{-1/2}\right).$$

Then by the bound of the second term in (6.1), the desired bound is established. 

\[\Box\]

Corollary 6.2. By lemma 6.1 for an integer $k$, in the region $1/2 \leq r/t \leq 1$,

$$\left| \partial^I L^J \left(\frac{t - r}{t + r}\right)^k\right| \leq \begin{cases} C(s/t)^{k-1}s^{-1}, & |I| \geq 1. \\ C(s/t)^k, & |I| = 0 \end{cases}$$

$$\left| \partial^I L^J \left(\frac{t}{t + r}\right)^k\right| \leq C t^{-|I|}.$$

6.2 Estimates on null forms

Recall the transition relation between $T^{\alpha\beta}, \bar{T}^{\alpha\beta}$ and $T^{\alpha\beta}, Q^{\alpha\beta\gamma}$, we see that the following terms are homogeneous of degree zero:

$$T^{\alpha\beta}, \quad (s/t)T^{\alpha\beta}, \quad (s/t)\bar{T}^{\alpha\beta}, \quad (s/t)^2\bar{T}^{00}$$

and

$$\bar{Q}^{abc}, \quad (s/t)\bar{Q}^{0bc}, \quad (s/t)\bar{Q}^{00}, \quad (s/t)^2\bar{Q}^{00}, \quad (s/t)^3\bar{Q}^{00}.$$
Lemma 6.3. In \( \mathcal{K} \), the following quantities are bounded by a constant \( C \) which is determined by \( I, J \):

\[
\begin{align*}
\partial^I L^J \overline{Q}^{abc}, \quad &\partial^I L^J \overline{T}^{ab} \\
(s/t)\partial^I L^J \overline{Q}^{0bc}, \quad & (s/t)\partial^I L^J \overline{Q}^{00c}, \quad (s/t)\partial^I L^J \overline{T}^{0b}, \quad (s/t)\partial^I L^J \overline{T}^{00}, \\
(s/t)^2 \partial^I L^J \overline{Q}^{00c}, \quad & (s/t)^2 \partial^I L^J \overline{Q}^{000}, \quad (s/t)^2 \partial^I L^J \overline{T}^{00}, \quad (s/t)^2 \partial^I L^J \overline{T}^{000}, \\
(s/t)^3 \partial^I L^J \overline{Q}^{000}, \quad &
\end{align*}
\]

and

\[
\begin{align*}
t\partial_a \partial^I L^J \overline{Q}^{abc}, \quad & t\partial_a \partial^I L^J \overline{T}^{ab} \\
s(s/t)^2 \partial_a \partial^I L^J \overline{Q}^{0bc}, \quad & s(s/t)^2 \partial_a \partial^I L^J \overline{Q}^{00c}, \quad s(s/t)^2 \partial_a \partial^I L^J \overline{Q}^{000}, \\
s(s/t)^2 \partial_a \partial^I L^J \overline{T}^{0b}, \quad & s(s/t)^2 \partial_a \partial^I L^J \overline{T}^{00}, \quad s(s/t)^3 \partial_a \partial^I L^J \overline{Q}^{00c}, \\
s(s/t)^3 \partial_a \partial^I L^J \overline{Q}^{000}, \quad &
\end{align*}
\]

Proof. This is by applying (4.24) and the fact that the terms in (6.5) and (6.6) are homogeneous of degree zero. We remark the following calculation: let

\[
I, J
\]

are homogeneous of degree zero where \( m, n \)

and

by applying (4.24) on the first factor, we see that

\[
\text{Lemma 6.4.}
\]

This concludes the desired result.

Remark the relation

\[
\bar{\partial}_a = t^{-1} L_a, \quad \bar{\partial}_s = (s/t) \partial_t
\]

the following bounds are direct:

Lemma 6.4. In \( \mathcal{K} \), the following terms are bounded by \( C \):

\[
\begin{align*}
t(t/s)\bar{\partial}_s \overline{Q}^{abc}, \quad & t(t/s)\bar{\partial}_s \overline{T}^{ab} \\
s(s/t)\bar{\partial}_s \overline{Q}^{abc}, \quad & (s/t)\bar{\partial}_s \overline{Q}^{0bc}, \quad (s/t)\bar{\partial}_s \overline{Q}^{00c}, \quad (s/t)\bar{\partial}_s \overline{T}^{0b}, \quad (s/t)\bar{\partial}_s \overline{T}^{00}, \\
s(s/t)^2 \bar{\partial}_s \overline{Q}^{00c}, \quad & (s/t)^2 \bar{\partial}_s \overline{Q}^{000}, \quad (s/t)^2 \bar{\partial}_s \overline{T}^{00}, \quad (s/t)^2 \bar{\partial}_s \overline{T}^{000}, \\
s(s/t)^3 \bar{\partial}_s \overline{Q}^{000}, \quad &
\end{align*}
\]

and

\[
\begin{align*}
t\bar{\partial}_a \overline{Q}^{abc}, \quad & t\bar{\partial}_a \overline{T}^{ab} \\
s\bar{\partial}_a \overline{Q}^{abc}, \quad & s\bar{\partial}_a \overline{Q}^{0bc}, \quad s\bar{\partial}_a \overline{Q}^{00c}, \quad s\bar{\partial}_a \overline{T}^{0b}, \quad s\bar{\partial}_a \overline{T}^{00}, \\
s\bar{\partial}_a \overline{Q}^{00c}, \quad & s\bar{\partial}_a \overline{Q}^{000}, \quad s\bar{\partial}_a \overline{T}^{00}, \quad (s/t)\bar{\partial}_a \overline{Q}^{000}, \\
s\bar{\partial}_a \overline{Q}^{000}, \quad &
\end{align*}
\]

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Now we introduce the following notion of the null form. Let $T$ be a quadratic form defined in $\mathfrak{K}$ with constant coefficient (with respect to the canonical frame). We call $T$ a null quadratic form, if for any $\xi \in \mathbb{R}^4$ satisfying

\[(6.9) \quad \xi_3^2 - \sum_{a=1}^{3} \xi_a^2 = 0\]

the following equation holds:

\[(6.10) \quad T^{\alpha\beta} \xi_\alpha \xi_\beta = 0.\]

We can also define the null condition for a cubic form: let $Q$ be a constant cubic form defined in $\mathfrak{K}$ and for any $\xi$ satisfying (6.9), the following condition holds:

\[(6.11) \quad Q^{\alpha\beta\gamma} \xi_\alpha \xi_\beta \xi_\gamma = 0.\]

Then we establish the following important result:

**Proposition 6.5** (Null condition in hyperbolic frame). Let $T$ and $Q$ be null quadratic and cubic form respectively. Then the following bounds hold:

\[(6.12) \quad |\partial^j L^j T^{00}| \leq C, \quad |\partial^j L^j Q^{000}| \leq C(t/s)\]

and

\[(6.13) \quad |\partial_a \partial^j L^j T^{00}| \leq C s^{-1}, \quad |\partial_a \partial^j L^j Q^{000}| \leq C(t/s)^2 s^{-1}\]

Furthermore, the following estimates hold:

\[(6.14) \quad |\partial_a T^{00}| \leq C t^{-1}, \quad |\partial_a Q^{000}| \leq C(t/s)s^{-1}, \quad |\partial_a Q^{000}| \leq C s^{-1}.\]

**Proof of proposition 6.5.** We observe that in the region $\{(t, x) \in \mathbb{R}^4 \mid |x| \leq t/2\}$, this is a direct result of the fact that $s/t)^3 Q^{000}$ and $(s/t)^2 T^{00}$ are homogeneous of degree zero. To see this, we denote by

\[f := (s/t)^3 Q^{000}, \quad g := (s/t)^2 T^{00}.\]

Then we see that

\[\partial^j L^j Q^{000} = \partial^j L^j ((t/s)^3 f) = \sum_{l_1+l_2+j=3} \partial^{l_1} L^{l_1} (t/s)^3 \cdot \partial^{l_2} L^{l_2} f\]

then recalling that $|\partial^j L^j f| \leq C$, and by (6.2), we see that $|\partial^j L^j Q^{000}| \leq C(t/s)^3$. Remark that in the region $\{(t, x) \in \mathbb{R}^4 \mid |x| \leq t/2\}$, $t/s \leq 4/3$. Thus the desired result is established. For $T^{00}$ the argument is similar and we omit the detail.

Then we discuss the region $\mathfrak{K} \cap \{t/2 < |x| < t\}$. Let

\[\zeta_\alpha := \Psi_\alpha, \quad \xi = (r/t, x^1/t, x^2/t, x^3/t).\]

We see that $\xi$ satisfies (6.9). Furthermore,

\[\nu := \zeta - (t/s) \xi = ((t - r)/s, 0, 0, 0).\]

Now we see that

\[T^{00} = T^{\alpha\beta} \Psi_\alpha \Psi_\beta = T^{\alpha\beta} (\nu_\alpha + (t/s) \xi_\alpha) (\nu_\beta + (t/s) \xi_\beta) = T^{\alpha\beta} \nu_\alpha \nu_\beta + (t/s) T^{\alpha\beta} \nu_\alpha \xi_\beta + (t/s) T^{\alpha\beta} \nu_\beta \xi_\alpha + (t/s)^2 T^{\alpha\beta} \xi_\alpha \xi_\beta = T^{\alpha\beta} \nu_\alpha \nu_\beta + (t/s) T^{\alpha\beta} \nu_\alpha \xi_\beta + (t/s) T^{\alpha\beta} \nu_\beta \xi_\alpha = \frac{t - r}{t + r} T^{00} + \frac{t}{t + r} T^{0\beta} \xi_\beta + \frac{t}{t + r} T^{0\alpha} \xi_\alpha.\]
where we have applied the null condition \( T^{\alpha\beta} \xi_\alpha \nu_\beta = 0 \). Recall that \( \xi_\alpha \) are homogeneous of degree zero, combined with (6.3),

\[
|\partial^I J^T|^0 \leq \begin{cases} 
C s^{-1}, & |I| \geq 1, \\
C, & |I| = 0.
\end{cases}
\]

Then we regard the cubic form. We see that similar to the quadratic case:

\[
\overline{Q}^0 = \overline{Q}^\alpha_{\beta\gamma} \nu_\alpha \nu_\beta \nu_\gamma + (t/s) \overline{Q}^\alpha_{\beta\gamma} (\nu_\alpha \nu_\beta \xi_\gamma + \nu_\alpha \xi_\beta \nu_\gamma + \xi_\alpha \nu_\beta \nu_\gamma) + (t/s)^2 \overline{Q}^\alpha_{\beta\gamma} (\nu_\alpha \xi_\beta \xi_\gamma + \xi_\alpha \nu_\beta \xi_\gamma + \xi_\alpha \xi_\beta \nu_\gamma) + (t/s)^3 \overline{Q}^\alpha_{\beta\gamma} \xi_\alpha \xi_\beta \xi_\gamma
\]

where we have applied the null condition. We see that

\[
(t/s) \overline{Q}^\alpha_{\beta\gamma} (\nu_\alpha \nu_\beta \xi_\gamma + \nu_\alpha \xi_\beta \nu_\gamma + \xi_\alpha \nu_\beta \nu_\gamma) = \frac{t(t-r)}{s^2} f = \frac{t}{t+r} f
\]

where \( f \) is a homogeneous function of degree zero. Also,

\[
(t/s)^2 \overline{Q}^\alpha_{\beta\gamma} (\nu_\alpha \xi_\beta \xi_\gamma + \xi_\alpha \nu_\beta \xi_\gamma + \xi_\alpha \xi_\beta \nu_\gamma) = \frac{t^2}{(t+r)^2} \left( \frac{t+r}{t-r} \right)^{1/2} f.
\]

Then also by (6.3), the bound (6.12) is established.

For (6.14), it is by direct calculation and the following relation in \( K \cap \{ r \geq t/2 \} \):

\[
\left| \partial_s \left( \frac{t-r}{t+r} \right) \right| \leq C(s/t) t^{-1}, \quad \left| \partial_s \left( \frac{t-r}{t+r} \right) \right| \leq C(s/t^2) t^{-1}.
\]

and

\[
\left| \partial_s \left( \frac{r}{t+r} \right) \right| \leq C t^{-1}.
\]
The rest component of $\partial_\alpha \Psi_\beta$ are zero. Thus we see that

$$Q^{\alpha\beta\gamma} \partial_\alpha \Psi_\beta \partial_\gamma u = Q^{\alpha\beta\gamma} \partial_\alpha \Psi_\beta \partial_\gamma u$$

$$= -s^{-1}Q^{\alpha\beta\gamma} \Psi_\beta \partial_\gamma u + s^{-1}Q^{\alpha\beta\gamma} \partial_\alpha \Psi_\beta \partial_\gamma u$$

$$= -s^{-1}Q^{\alpha\beta\gamma} \Psi_\beta \partial_\gamma u + s^{-1}Q^{\alpha\beta\gamma} \partial_\alpha \Psi_\beta \partial_\gamma u$$

that is

(6.19) $Q^{\alpha\beta\gamma} \partial_\alpha \Psi_\beta \partial_\gamma u = -s^{-1}Q^{\alpha\beta\gamma} \Psi_\beta \partial_\gamma u + s^{-1}Q^{\alpha\beta\gamma} \partial_\alpha \Psi_\beta \partial_\gamma u - s^{-1} \sum_a Q^{\alpha\beta\gamma} \partial_\gamma u_a$

We first concentrate on $\partial^I L^J \left( Q^{\alpha\beta\gamma} \partial_\gamma u, \partial_\alpha, \partial_\beta \right)$. Applying lemma 6.3 and proposition 6.5, we see that by (6.17) and (6.19), it is bounded by the sum of the following terms (modulo a constant determined by $I, J$):

(6.20)

$$\left| \partial^I L^J \right| \left( Q^{\alpha\beta\gamma} \partial_\gamma u, \partial_\alpha, \partial_\beta \right) \leq \sum \left| \partial^I L^J \right| \left( Q^{\alpha\beta\gamma} \partial_\gamma u, \partial_\alpha, \partial_\beta \right)

\text{where} \quad |I_1| + |I_2| \leq |I| \quad \text{and} \quad |J_1| + |J_2| \leq |J|.

For the last term we have applied the fact that $\partial_\alpha \Psi_\beta$ is homogeneous of degree $-1$.

Then we regard $\partial^I L^J \left( Q^{\alpha\beta\gamma} \partial_\gamma u, \partial_\alpha, \partial_\beta \right)$. We see that by (6.17), it is the sum of the following terms:

$$\left| \partial^I L^J \right| \left( Q^{\alpha\beta\gamma} \partial_\gamma u, \partial_\alpha, \partial_\beta \right) \leq \sum \left| \partial^I L^J \right| \left( Q^{\alpha\beta\gamma} \partial_\gamma u, \partial_\alpha, \partial_\beta \right)

\text{where} \quad |I_1| + |I_2| \leq |I| \quad \text{and} \quad |J_1| + |J_2| \leq |J|.

In the section 8 we will make $L^2$ estimates on these terms based on the bootstrap bounds. As an preparation, we establish the following bounds:

**Lemma 6.6.** Let $u$ be a function defined in $X$, sufficiently regular. Then the following estimates hold:

$$\left\| s^{2/3} (s/t) \left| \partial^I L^J \right| \partial_\alpha \partial_\beta \right\|_{L^2(2)} \leq C \sum \left| \partial^I L^J \right| \left( Q^{\alpha\beta\gamma} \partial_\gamma u, \partial_\alpha, \partial_\beta \right)

\text{and} \quad |I_1| + |I_2| \leq |I| \quad \text{and} \quad |J_1| + |J_2| \leq |J|$$

$$\left| \partial^I L^J \right| \left( Q^{\alpha\beta\gamma} \partial_\gamma u, \partial_\alpha, \partial_\beta \right) \leq \sum \left| \partial^I L^J \right| \left( Q^{\alpha\beta\gamma} \partial_\gamma u, \partial_\alpha, \partial_\beta \right)

\text{where} \quad |I_1| + |I_2| \leq |I| \quad \text{and} \quad |J_1| + |J_2| \leq |J|.

$$\left| \partial^I L^J \right| \left( Q^{\alpha\beta\gamma} \partial_\gamma u, \partial_\alpha, \partial_\beta \right) \leq \sum \left| \partial^I L^J \right| \left( Q^{\alpha\beta\gamma} \partial_\gamma u, \partial_\alpha, \partial_\beta \right)

\text{where} \quad |I_1| + |I_2| \leq |I| \quad \text{and} \quad |J_1| + |J_2| \leq |J|.$$
We see first that by (4.27) and lemma 5.1:

\[ s_{\text{con}}(s, \partial^I J^J u) \leq C \sum |I'|+|J'| \leq |I|+|J| E_{\text{con}}(s, \partial^I J^J u)^{1/2} \]

and

\[ s_{\text{con}}(s, \partial^I J^J u) \leq C \sum |I'|+|J'| \leq |I|+|J| E_{\text{con}}(s, \partial^I J^J u)^{1/2} \]

Proof. We see first that by (4.27) and lemma 5.1:

\[ \| s_{\text{con}}(s, \partial^I J^J u) \|_{L^2(\mathcal{C}_s)} \]

\[ \leq C \sum \alpha |I'| \leq |I| \leq |J| \| s_{\text{con}}(s, \partial^I J^J u) \|_{L^2(\mathcal{C}_s)} + C \sum |I'|+|J'| \leq |I|+|J| E_{\text{con}}(s, \partial^I J^J u)^{1/2} \]

Then by proposition 5.4 we see that

\[ \| s_{\text{con}}(s, \partial^I J^J u) \|_{L^2(\mathcal{C}_s)} \leq C \sum |I'|+|J'| \leq |I|+|J| E_{\text{con}}(s, \partial^I J^J u)^{1/2} \]

The rest to estimates are direct by lemma 4.13 and lemma 4.16, we omit the detail.

7 Global existence: bootstrap argument

7.1 The bootstrap bounds

We consider the main equation of interest together with initial data:

\[ \begin{cases}
\Box u + Q^\alpha\beta \gamma \partial_\alpha \partial_\beta u = 0 \\
u|_{\mathcal{C}_s} = u_0, \quad \partial_t u|_{\mathcal{C}_s} = u_1.
\end{cases} \tag{7.1} \]

where \( u_0 \) are sufficiently regular functions defined on the hyperboloid \( \mathcal{H}_2 \) and supported in \( \mathcal{H}_2 \cap \mathcal{K} \).

Remark 7.1. The fact that the initial data are posed on hyperboloid is not a standard but we can see it in the following way: we pose the initial data set on the hyperplane \( \{ t = 2 \} \) and supported in the unit disc. Then by standard local existence result, the associated local solution extends to region \( \{(t, x)| 2 \leq t \leq \sqrt{4 - x^2} \}\cap \mathcal{K}. \) Then we can restrict the solution on \( \mathcal{H}_2 \). Thus we for global result we can pose our initial data on \( \mathcal{H}_2 \). For more detail, see for example [9] or [8].

We will apply the so-called bootstrap argument, explained in detail here: Let \( \epsilon \) be the local-in-time solution associated to (7.1). Assume that the largest hyperbolic time interval of existence is \( \mathcal{X}_{[2, s^*]} \).

We define the bootstrap bounds for \( s \in [2, s^*] \):

\[ \sum_{|I|+|J| \leq N} E_{\text{con}}(s, \partial^I J^J u)^{1/2} \leq C_1 \epsilon \tag{7.2} \]

with \( (C_1, \epsilon) \) a pair of positive constant to be determined. Then we define \( s_1 \) to be the largest (hyperbolic) time where \( u \) satisfies this condition, that is,

\[ s_1 := \sup \{ s^* > s \geq 2 | (7.2) \text{ holds on } [s_0, s] \} \]
We suppose that

\[(7.3) \quad \sum_{|I|+|J| \leq N} E_{\text{con}}(2, \partial^I L^J u)^{1/2} \leq C_0 \varepsilon.\]

which can be guaranteed by the smallness of the initial data. We see that when taking \(C_1 > C_0\), by continuity, \(s_1 > 2\).

To argue by contradiction, we suppose that \(s_1 < s^*\). If we could deduce, for a suitable pair \((C_1, \varepsilon_0)\), an improved bound for all \(0 < \varepsilon \leq \varepsilon_0\):

\[(7.4) \quad \sum_{|I|+|J| \leq N} E_{\text{con}}(s, \partial^I L^J u)^{1/2} \leq \frac{1}{2} C_1 \varepsilon \quad \text{for} \ s \in [2, s_1].\]

On the other hand, we see that by continuity,

\[(7.5) \quad \sum_{|I|+|J| \leq N} E_{\text{con}}(s_1, \partial^I L^J u)^{1/2} = C_1 \varepsilon.\]

This contradiction leads to the fact that \(s_1 = s^*\), then

\[(7.6) \quad \sum_{|I|+|J| \leq N} E_{\text{con}}(s^*, \partial^I L^J u)^{1/2} \leq C_1 \varepsilon.\]

Then by standard local-in-time theory (with \(N\) sufficiently large), we see that \(s^*\) could not be finite. This leads to the global existence result.

Now we state the main result of this article:

**Theorem 7.2.** There exists a constant \(\varepsilon_0 > 0\), determined only by the system \((7.1)\), such for all \(0 \leq \varepsilon \leq \varepsilon_0\), if

\[(7.7) \quad \|u_0\|_{H^{N+1}(\mathcal{K}_\varepsilon)} + \|u_1\|_{H^N(\mathcal{K}_\varepsilon)} \leq \varepsilon\]

holds for \(N\) sufficiently large (\(N \geq 9\) is enough), then the associated local-in-time solution extends to time infinity.

Based on the above discussion on bootstrap argument, we see that the above result is deduced from the following proposition:

**Proposition 7.3.** There exists a pair of positive constant \((C_1, \varepsilon_0)\), determined only by the system \((7.1)\) such that if the initial data set satisfies \((7.3)\) with \(0 < \varepsilon \leq \varepsilon_0\), then \((7.2)\) leads to \((7.4)\).

The following sections from 7 to 9 are devoted to the proof of this proposition.

### 7.2 Basic bounds

The following bounds hold in the region \(\mathcal{K}_{[2,s_1]}\).

The following terms are bounded by \(CC_1 \varepsilon\) for \(|I| + |J| \leq N:\)

\[(7.8) \quad \|s \partial_a \partial^I L^J u\|_{L^2(\mathcal{K}_\varepsilon)} , \quad \|(s^2/t) \partial_a \partial^I L^J u\|_{L^2(\mathcal{K}_\varepsilon)} , \quad \|s \partial^I L^J u\|_{L^2(\mathcal{K}_\varepsilon)} .\]

Then by \((7.35), (7.36)\), and \((7.38)\), the following bounds are also bounded by \(CC_1 \varepsilon\):

\[(7.9) \quad \|s \partial^I L^J \partial_a u\|_{L^2(\mathcal{K}_\varepsilon)} , \quad \|(s^2/t) \partial^I L^J \partial_a u\|_{L^2(\mathcal{K}_\varepsilon)} , \quad \|s \partial^I L^J \partial_a u\|_{L^2(\mathcal{K}_\varepsilon)} , \quad \|s \partial^I L^J \partial_a \partial_a u\|_{L^2(\mathcal{K}_\varepsilon)} .\]
By proposition 4.23 and the global Sobolev inequality, for $|I| + |J| \leq N - 2$, the following terms are bounded by $CC_1\varepsilon$:

\[(7.8) \sup_{\mathcal{N}_s} \left\{ t^{1/2} s \partial_\alpha \partial^I L^J u \right\}, \quad \sup_{\mathcal{N}_s} \left\{ t^{1/2} s^2 \partial_\alpha \partial^I L^J u \right\}, \quad \sup_{\mathcal{N}_s} \left\{ t^{1/2} s \partial^I L^J u \right\},\]

\[(7.9) \sup_{\mathcal{N}_s} \left\{ t^{3/2} s \partial^I L^J \partial_\alpha u \right\}, \quad \sup_{\mathcal{N}_s} \left\{ t^{1/2} s^2 \partial^I L^J \partial_\alpha u \right\},\]

and for $|I| + |J| \leq N - 3$, the following terms are bounded by $CC_1\varepsilon$:

\[(7.10a) \sup_{\mathcal{N}_s} \left\{ s^2 t^{3/2} \partial_\alpha \partial^I L^J u \right\}, \quad \sup_{\mathcal{N}_s} \left\{ st^{5/2} \partial_\alpha \partial^I L^J u \right\}\]

\[(7.10b) \sup_{\mathcal{N}_s} \left\{ s^2 t^{3/2} \partial^I L^J \partial_\alpha u \right\}, \quad \sup_{\mathcal{N}_s} \left\{ st^{5/2} \partial^I L^J \partial_\alpha u \right\}\]

and

\[(7.10c) \sup_{\mathcal{N}_s} \left\{ st^{3/2} \partial^I L^J \partial_\alpha \partial_\beta u \right\}.
\]

where for the last term in the above list we applied (4.50).

8 Global existence: refined bounds

8.1 Estimates on Hessian form

We combine (7.2) together with proposition (5.3):

\[(8.1) \| (s^3/t) \partial^I L^J \partial_\alpha \partial_\beta u \|_{L^2(\mathcal{N}_s)} + \| (s^3/t) \partial^I L^J u \|_{L^2(\mathcal{N}_s)} \leq C \| (s^3/t) \partial^I L^J u \|_{L^2(\mathcal{N}_s)} + C C_1\varepsilon.\]

Similar bounds hold for the combination of (7.2) with (5.9). Thus we need to control $\| (s^3/t) \partial^I L^J u \|_{L^2(\mathcal{N}_s)}$. This is by the following lemma:

**Lemma 8.1.** Under the bootstrap bound (7.2), the following estimate holds for $|I| + |J| \leq N - 1$: \n
\[(8.2) \quad \| (s^3/t) \partial^I L^J (Q^\alpha \partial_\gamma \partial_\beta u_\alpha \partial_\beta u_\gamma) \|_{L^2(\mathcal{N}_s)} \leq C(C_1\varepsilon)^2.\]

*Proof.* This is based on the $L^2$ bounds and $L^\infty$ bounds established in the last section. We need to bound each term in the list (6.20).

For each term in (6.20), for $|I_1| + |J_1| \leq N - 2$, we apply the decay bounds (7.9) on the first factor and the apply the $L^2$ bounds (7.7) on the second factor. We can check that for each term, the $L^2$ norm is bounded as

$$\| (s^3/t) X \|_{L^2(\mathcal{N}_s)} \leq C(C_1\varepsilon)^2$$

where $X$ represents a term in (7.9).

When $|I_1| + |J_1| \geq N - 1$, we see that $|I_2| + |J_2| \leq N - 3$. Thus in the similar way, we apply the decay estimates of (7.10a), (7.10c) on the second factor and (7.7) (the first two terms) on the first factor. \hfill \Box

Now we are ready to establish the refined bound on $\partial^I L^J \partial_\alpha \partial_\beta u$.

**Lemma 8.2.** Under the bootstrap assumption, we see that

\[(8.3) \quad \| (s^3/t) \partial^I L^J \partial_\alpha \partial_\beta u \|_{L^2(\mathcal{N}_s)} + \| (s^3/t) \partial_\alpha \partial^I L^J u \|_{L^2(\mathcal{N}_s)} \leq CC_1\varepsilon\]

and

\[(8.4) \quad \sup_{\mathcal{N}_s} \left\{ t^{1/2} s^3 \partial^I L^J \partial_\alpha \partial_\beta u \right\} + \sup_{\mathcal{N}_s} \left\{ s^{1/2} \partial_\alpha \partial^I L^J u \right\} \leq CC_1\varepsilon\]

for $|I| + |J| \leq N - 1$. 

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Proof. This is by using the equation. We see that
\[ \Box \partial^I L^J u = \partial^I L^J \left( Q^\alpha^\beta^\gamma \partial_\alpha u \partial_\beta u \right). \]

Thus by (8.2)
\begin{equation}
(8.5) \quad \| (s^3/t) \Box \partial^I L^J u \|_{L^2(\Omega_k)} \leq C(1)^2.
\end{equation}

Now we apply (8.5) together with the above bounds and the bootstrap bound on energy, and we see that the desired bounds is established. \( \square \)

### 8.2 Estimates on null form

In this section we concentrate on the \( L^2 \) bounds on \(|\partial^I L^J, Q^\alpha^\beta^\gamma \partial_\alpha u \partial_\beta u| \). To get started we combine the bootstrap bounds with lemma 6.6 and we see that the following terms are bounded by \( CC_1 \varepsilon \):
\begin{equation}
(8.6) \quad \| s^2 (s/t) [\partial^I L^J, \partial_\alpha u \partial_\beta u] \|_{L^2(\Omega_k)} \leq C(1) \varepsilon^2 s^{-2}.
\end{equation}

Proof. For \((\alpha, \beta, \gamma) = (0, 0, 0)\), we see that by (6.12), (7.9) and (8.5) (recall that in \( \mathcal{K}, s^2 \geq t \)):
\begin{equation}
\| s^2 \Box^\alpha^\beta^\gamma \partial_\alpha u \partial_\beta u \|_{L^2(\Omega_k)} \leq CC_1 \varepsilon \| s^2 (s/t) t^{1/2} s^{-2} \cdot (t/s) s^{-2} \cdot (s^3/t) [\partial^I L^J, \partial_\alpha u \partial_\beta u] \|_{L^2(\Omega_k)} \\
\leq C(1) \varepsilon^2 s^{-2}.
\end{equation}

For the rest components, we apply lemma 8.3, (7.9) and (8.5). We omit the detail. \( \square \)

**Lemma 8.3.** Under the bootstrap assumption and assume that \( Q^\alpha^\beta^\gamma \) be a null cubic form, then the following bounds hold:
\begin{equation}
\| s^2 (s/t) [\partial^I L^J, \partial_\alpha u \partial_\beta u] \|_{L^2(\Omega_k)} \leq C(1) \varepsilon^2 s^{-2}.
\end{equation}

**Lemma 8.4.** Under the bootstrap assumption, the \( L^2 \) norm of the following term
\[ s^2 \partial^{I_1} L^{J_1} Q^{\alpha^\beta^\gamma} \partial^{I_2} L^{J_2} \partial_\alpha \partial^{I_3} L^{J_3} \partial_\beta u \]
is controlled by \( C(1) \varepsilon^2 s^{-2} \), where
\[ I_1 + I_2 + I_3 = I, \quad J_1 + J_2 + J_3 = J, \quad |I_3| + |I_3| < |I| + |J| \leq N. \]

Proof. This is also by applying (6.12) (for \((\alpha, \beta, \gamma) = (0, 0, 0)\)) or lemma 6.3 (for \((\alpha, \beta, \gamma) \neq (0, 0, 0)\)). We see that when \(|I_1| + |I_1| \leq N - 2\), we apply (7.9) on the factor \( \partial^{I_2} L^{J_2} \partial_\alpha u \), (7.7) (the third and forth term) on the factor \( \partial^{I_3} L^{J_3} \partial_\alpha \partial_\beta u \) and (8.3) on the factor \( \partial^{I_3} L^{J_3} \partial_\alpha \partial_\beta u \). For example for the component \((0, 0, 0)\), we see that
\begin{equation}
\| s^2 \partial^{I_1} L^{J_1} Q^{\alpha^\beta^\gamma} \partial^{I_2} L^{J_2} \partial_\alpha \partial^{I_3} L^{J_3} \partial_\beta u \|_{L^2(\Omega_k)} \leq CC_1 \varepsilon \| s^2 (s/t) t^{-1/2} s^{-2} \cdot (t/s) s^{-2} \cdot (s^3/t) [\partial^I L^J, \partial_\alpha \partial_\beta u] \|_{L^2(\Omega_k)} \leq C(1) \varepsilon^2 s^{-2}.
\end{equation}

The rest components are verified similarly and we omit detail.

When \(|I_1| + |I_2| \geq N - 1\), we see that \(|I_1| + |I_3| \leq N \leq 3\). Thus we apply (7.9) (the first two terms) on the factor \( \partial^{I_2} L^{J_2} \partial_\alpha u \), (7.10b) on the factor \( \partial^{I_3} L^{J_3} \partial_\alpha \partial_\beta u \) and (8.3) for \( \partial^{I_3} L^{J_3} \partial_\alpha \partial_\beta u \). For example for the component \((0, 0, 0)\):
\begin{equation}
\| s^2 \partial^{I_1} L^{J_1} Q^{\alpha^\beta^\gamma} \partial^{I_2} L^{J_2} \partial_\alpha \partial^{I_3} L^{J_3} \partial_\beta u \|_{L^2(\Omega_k)} \leq CC_1 \varepsilon \| s^2 (s/t) t^{-1/2} s^{-2} \cdot (t/s) s^{-2} \cdot (s^3/t) [\partial^I L^J, \partial_\alpha \partial_\beta u] \|_{L^2(\Omega_k)} \leq C(1) \varepsilon^2 s^{-2}.
\end{equation}

The rest components are verified similarly and we omit detail. \( \square \)
Lemma 8.5. Under the bootstrap assumption, the following estimate holds:

\begin{equation}
\|s|\partial^I L^J| Q^{\alpha \beta \gamma} \partial_\alpha u \partial_\beta \bar{\partial}_\gamma| u \|_{L^2(\Omega_\epsilon)} \leq C(C_1 \epsilon)^2 s^{-3/2}.
\end{equation}

Proof. We recall (6.22). For the first term in right-hand-side, we see that

\begin{equation}
\|s|\partial^I L^J| s^{-1} Q^{\alpha \beta \gamma} \partial_\alpha u \partial_\beta \bar{\partial}_\gamma| u \|_{L^2(\Omega_\epsilon)} \leq C(C_1 \epsilon)^2 s^{-3/2}.
\end{equation}

Then applying (4.23) together with lemma 6.3 (for \( \gamma > 0 \)) or (6.12) (for \( \gamma = 0 \)):

\begin{equation}
|T_1| \leq \sum_{|I| + |J| \leq |I|} |\partial^I L^J| Q^{\alpha \beta \gamma} \partial_\alpha u \partial_\beta \bar{\partial}_\gamma| u |, \gamma = 0,
\end{equation}

\begin{equation}
C s^{-1} (t/s)^2 \sum_{|I| + |J| \leq |I|} |\partial^I L^J| Q^{\alpha \beta \gamma} \partial_\alpha u \partial_\beta \bar{\partial}_\gamma| u |, \gamma = a > 0.
\end{equation}

Now, for \(|I| + |J| \leq N - 2\), we apply decay estimate (7.9) on the first factor and \( L^2 \) bounds (7.7) (the first two terms) on the second factor. When \(|I| + |J| \geq N - 1\), we see that \(|I| + |J| \leq 1 \leq N - 2\). In this case we apply (7.9) on the second factor and (7.10) on the first factor. This leads to:

\begin{equation}
\|sT_1\|_{L^2(\Omega_\epsilon)} \leq C(C_1 \epsilon)^2 s^{-2}.
\end{equation}

For the term \( T_2 \), we see that by (7.9) applied on \( \partial_\alpha u \), lemma 6.3 (for \( \gamma > 0 \)) or (6.12) (for \( \gamma = 0 \))

\begin{equation}
|T_2| \leq \sum_{|I| + |J| \leq |I|} |\partial^I L^J| Q^{\alpha \beta \gamma} \partial_\alpha u \partial_\beta \bar{\partial}_\gamma| u |, \gamma = 0,
\end{equation}

\begin{equation}
C s^{-1} (t/s)^2 \sum_{|I| + |J| \leq |I|} |\partial^I L^J| Q^{\alpha \beta \gamma} \partial_\alpha u \partial_\beta \bar{\partial}_\gamma| u |, \gamma = a > 0.
\end{equation}

Then we combine (4.3), (4.26) together with (7.10), the following bound is established

\begin{equation}
\|sT_2\|_{L^2(\Omega_\epsilon)} \leq C(C_1 \epsilon)^2 s^{-2}.
\end{equation}

The rest terms in right-hand-side of (6.22) are bounded similarly, we omit the detail.

We are ready to conclude the following result:

Proposition 8.6. Under the bootstrap assumption, for \(|I| + |J| \leq N\), the following estimate holds:

\begin{equation}
\|s|\partial^I L^J| Q^{\alpha \beta \gamma} \partial_\alpha u \partial_\beta \bar{\partial}_\gamma| u \|_{L^2(\Omega_\epsilon)} \leq C(C_1 \epsilon)^2 s^{-2}.
\end{equation}

9 Global existence: conclusion of bootstrap argument

Now we are ready to prove proposition 7.3. To do so we need to guarantee (3.3) and the bounds on \( M_\delta(s) \) (with the notion in (3.1)). We first remark that by the notation in subsection 3.1

\begin{equation}
\bar{h}^{\alpha \beta} = Q^{\alpha \beta \gamma} \bar{\partial}_\gamma u.
\end{equation}

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For the convenience of discussion, we list out the following bounds on \( \bar{h}^{\alpha\beta} \). These are by (7.9) combined with lemma (6.3) and (6.12):

\[
\begin{align*}
|\bar{h}^{00}| & \leq CC_1 \varepsilon t^{1/2} s^{-3}, & |\bar{h}^{a0}| & \leq CC_1 \varepsilon t^{3/2} s^{-4}, \\
|\bar{h}^{ab}| & \leq CC_1 \varepsilon t^{1/2} s^{-3}, & |\bar{h}^{\alpha\beta}| & \leq CC_1 \varepsilon t^{1/2} s^{-3}.
\end{align*}
\]

Furthermore, we see that

\[
\partial_t \bar{h}^{\alpha\beta} = \partial_t Q^{\beta\gamma} \cdot \partial_x u + Q^{\alpha\beta} \partial_x \partial_s u.
\]

We apply, lemma (6.3) and (6.12) combined with (7.9), (7.10b) and (8.4).

\[
\begin{align*}
|\partial_t \bar{h}^{00}| & \leq C(C_1 \varepsilon) t^{1/2} s^{-4}, & |\partial_t \bar{h}^{a0}| & \leq C(C_1 \varepsilon) t^{3/2} s^{-5}, & |\partial_t \bar{h}^{ab}| & \leq C(C_1 \varepsilon) t^{1/2} s^{-4}, \\
|\partial_t \bar{h}^{00}| & \leq C(C_1 \varepsilon) t^{-1/2} s^{-3}, & |\partial_t \bar{h}^{a0}| & \leq C(C_1 \varepsilon) t^{-1/2} s^{-4}, & |\partial_t \bar{h}^{ab}| & \leq C(C_1 \varepsilon) t^{-1/2} s^{-3}.
\end{align*}
\]

We also remark that

\[
\partial_t \bar{h}^{\alpha\beta} = Q^{\alpha\beta\gamma} \partial_t \partial_s u
\]

that is

\[
\begin{align*}
\partial_t \bar{h}^{\alpha\beta} &= Q^{\alpha\beta\gamma} \partial_t \partial_s u + Q^{\alpha\beta c} \partial_t \partial_s u \\
&= Q^{\alpha\beta\gamma} \partial_t (t/s) \partial_s u + Q^{\alpha\beta c} \partial_t (\partial_s u - (x^c/s) \partial_s u) \\
&= \left( (t/s) Q^{\alpha\beta\gamma} - (x^c/s) Q^{\alpha\beta c} \right) \partial_t \partial_s u + Q^{\alpha\beta c} \partial_t \partial_s u + \left( (x^c/s^2) Q^{\alpha\beta c} - \frac{r^2}{t^2} \right) \partial_x u,
\end{align*}
\]

which leads to

\[
|\partial_t \bar{h}^{\alpha\beta}| \leq C(t/s)|\partial_t \partial_s u| + C \sum \alpha |\partial_t \partial_s u| + C(t/s^2)|\partial_s u|.
\]

Similar calculation shows that

\[
|\partial_t \bar{h}^{\alpha\beta}| \leq C(t/s)|\partial_t \partial_s u| + C \sum \alpha |\partial_t \partial_s u| + s^{-1}|\partial_s u|.
\]

Then combined with (8.4), (7.10a) and (7.8), we see that

\[
|\partial_t \bar{h}^{\alpha\beta}| \leq Ct^{1/2} s^{-4}, & |\partial_t \bar{h}^{\alpha\beta}| \leq Ct^{-1/2} s^{-3}.
\]

Then we establish the following bounds:

**Lemma 9.1.** Under the bootstrap assumption (7.2),

\[
|\partial_t N_g| \leq CC_1 \varepsilon t^{1/2} s^{-4}.
\]

**Proof.** We recall that

\[
N_g - 2 = h^{00} - \sum \alpha h^{\alpha\alpha} - 2\bar{h}^{00} - s\partial_t \bar{h}^{00}
\]

thus

\[
|\partial_t N_g| \leq 2|\partial_t \bar{h}^{00}| + \sum \alpha |\partial_t h^{\alpha\alpha}| + |\partial_t \bar{h}^{00}| + |\partial_t (s\partial_s \bar{h}^{00})|
\]

The first three terms are bounded by (7.9) and (7.11). For the last term, we remark the following relation:

\[
|\partial_t (s\partial_s \bar{h}^{00})| \leq C|\partial_t \bar{h}^{00}| + C|\partial_t \partial_s Q^{00} \partial_s u| + C|\partial_t \bar{Q}^{00} \partial_s u| + C|\bar{Q}^{00} \partial_s \partial_s u| + C|\bar{Q}^{00} \partial_s \partial_s u| + C|\bar{Q}^{00} \partial_s \partial_s u| + C|\bar{Q}^{00} \partial_s \partial_s u|
\]
and

\begin{align}
\left| \bar{\partial}_a (s \bar{\partial}_s \check{h}^{00}) \right| &= s \left| \bar{\partial}_a \bar{\partial}_s \check{h}^{00} \right| + C |\bar{\partial}_a \bar{\partial}_s \check{Q}^{00} \bar{\partial}_s u| + C |\bar{\partial}_a \check{Q}^{00} \bar{\partial}_a \bar{\partial}_s u| + C |\bar{\partial}_a \check{Q}^{00} \bar{\partial}_s u| + C |\bar{\partial}_a \check{Q}^{00c} \bar{\partial}_s u| + C |\bar{\partial}_a \check{Q}^{00c} \bar{\partial}_a \bar{\partial}_s u|.
\end{align}

(9.8)

And we see that by lemma 6.3 and proposition 6.5, we see that

\begin{align}
|\bar{\partial}_a \bar{\partial}_s \check{Q}^{00c}| \leq (s/t) |\bar{\partial}_a \bar{\partial}_s \check{Q}^{00c}| + s^{-1} |\bar{\partial}_a \check{Q}^{00c}| \leq C C_1 \varepsilon^{-1} s^{-1},
\end{align}

(9.9)

and by lemma 6.4:

\begin{align}
|\bar{\partial}_a \bar{\partial}_s \check{Q}^{00c}| \leq (s/t) |\bar{\partial}_a \bar{\partial}_s \check{Q}^{00c}| + s^{-1} |\bar{\partial}_a \check{Q}^{00c}| \leq C C_1 (t/s) s^{-1}.
\end{align}

(9.10)

Similar calculation shows that

\begin{align}
|\bar{\partial}_a \bar{\partial}_s \check{Q}^{00c}| \leq C C_1 \varepsilon s^{-2},
\end{align}

(9.11)

We also see that by (8.4):

\begin{align}
|\bar{\partial}_a \bar{\partial}_s \check{Q}^{00c}| \leq C C_1 \varepsilon t^{-3/2} s^{-2},
\end{align}

(9.12)

and by (7.10a),

\begin{align}
|\bar{\partial}_a \bar{\partial}_s \check{Q}^{00c}| \leq C C_1 \varepsilon t^{-5/2} s^{-2}.
\end{align}

(9.13)

We now substitute the bounds (9.9), (9.10) together with (9.9), (9.10), lemma 6.3, proposition 6.5, 9.11, 9.12, 8.3, 7.105 and 7.8, we see that

\begin{align}
|\bar{\partial}_a (s \bar{\partial}_s \check{h}^{00})| \leq C C_1 \varepsilon t^{1/2} s^{-4},
\end{align}

(9.13)

Now combine (9.3), (9.13) and (9.13), we see that the desired bound is established.

Then we establish the following bound:

Lemma 9.2. Under the bootstrap assumption with $\varepsilon$ sufficiently small, (8.5) holds for a $\kappa > 1$.

Proof. This is by verifying proposition 3.1. We see that by (9.1) and (9.3), (9.12) is verified with $\varepsilon \leq C C_1$.

Lemma 9.3. Under the bootstrap assumption, we have

\[ M_\varepsilon (s, \partial^I L^J u) = C (C_1 \varepsilon)^2 s^{-2} \]

where $M_\varepsilon$ is defined as in (8.9).

Proof. Recall (3.5), we see that

\[ \| R_a (\bar{\nabla} \partial^I L^J u, \bar{\nabla} \partial^I L^J u) \|_{L^1 (\mathcal{C}_\varepsilon)} \leq s^{-1} \left( L_a^{ab} - L_a^{ab} \right) \|_{L^\infty (\mathcal{C}_\varepsilon)} \| s^2 \bar{\partial}_a \partial^I L^J u \bar{\partial}_b \partial^I L^J u \|_{L^1 (\mathcal{C}_\varepsilon)} + \| s^{-1} (N_a - N_m) \bar{g}^{ab} \|_{L^\infty (\mathcal{C}_\varepsilon)} \| s^2 \bar{\partial}_a \partial^I L^J u \bar{\partial}_b \partial^I L^J u \|_{L^1 (\mathcal{C}_\varepsilon)} + 2 \| s^{-1} L_a^{ab} \|_{L^\infty (\mathcal{C}_\varepsilon)} \| s^2 \bar{\partial}_a \partial^I L^J u \bar{\partial}_b \partial^I L^J u \|_{L^1 (\mathcal{C}_\varepsilon)} + \frac{1}{2} \| \bar{\partial}_a (s \bar{\partial}_s \check{g}^{00} g^{ab}) \|_{L^\infty (\mathcal{C}_\varepsilon)} \| s^2 \bar{\partial}_a \partial^I L^J u \bar{\partial}_b \partial^I L^J u \|_{L^1 (\mathcal{C}_\varepsilon)} + \| \bar{\partial}_a (s \bar{\partial}_s \check{g}^{00} g^{ab}) \|_{L^\infty (\mathcal{C}_\varepsilon)} \| s^2 (s/t) \bar{\partial}_a \partial^I L^J u \bar{\partial}_b \partial^I L^J u \|_{L^1 (\mathcal{C}_\varepsilon)} \leq M_1 E_{\text{con}} (s, \partial^I L^J u) \]

where

\[ M_1 := \left( \| L_a^{ab} \|_{L^\infty (\mathcal{C}_\varepsilon)} + \| s^{-1} (N_a - N_m) \bar{g}^{ab} \|_{L^\infty (\mathcal{C}_\varepsilon)} + \| L_a^{ab} \|_{L^\infty (\mathcal{C}_\varepsilon)} + \| \bar{\partial}_a (s \bar{\partial}_s \check{g}^{00} g^{ab}) \|_{L^\infty (\mathcal{C}_\varepsilon)} + \| \bar{\partial}_a (s \bar{\partial}_s \check{g}^{00} g^{ab}) \|_{L^\infty (\mathcal{C}_\varepsilon)} \right). \]
We remark the following bounds (by (9.3)):

\[ \| L^{ab}_g - L^{ab}_m \|_{L^\infty(\Sigma_t)} \leq CC_1 \varepsilon s^{-2}, \quad \| N_g - N_m \|_{L^\infty(\Sigma_t)} \leq CC_1 \varepsilon s^{-2}, \]
\[ \| h^{ab} \|_{L^\infty(\Sigma_t)} \leq CC_1 \varepsilon s^{-2}, \]
\[ \| \partial_\alpha (\tilde{g}^{ab}_0 \hat{g}^{ab}) \|_{L^\infty(\Sigma_t)} \leq CC_1 \varepsilon s^{-3}, \quad \| (t/s) \partial_\alpha \tilde{g}^{ab}_0 \hat{g}^{ab} \|_{L^\infty(\Sigma_t)} \leq CC_1 \varepsilon s^{-3} \]

then we see that
\[ M_1 \leq CC_1 \varepsilon s^{-3}. \]

To analysis the term \((\mathcal{K}_g + N_g) \partial^J L^J u \cdot S_g \nabla \partial^J L^J u\), we see that
\[ \| (\mathcal{K}_g + N_g) \partial^J L^J u \cdot S_g \nabla \partial^J L^J u \|_{L^1(\Sigma_t)} \leq C \| S_g \nabla \partial^J L^J u \|_{L^2(\Sigma_t)} \cdot \text{Econ}(s, \partial^J L^J u)^{1/2}. \]

Then we see that by (9.5) and especially the bound on \(h^{ab}, \partial_\alpha h^{ab}\)
\[ \| S_g \nabla \partial^J L^J u \|_{L^2(\Sigma_t)} \]
\[ = \| \bar{s}^{-1} \partial_\alpha (\bar{m}^{ab} h^{ab}) + 2s^{-1} \partial_\alpha (\bar{s} h^{ab}) \|_{L^\infty(\Sigma_t)} \| s \partial_\alpha \partial^J L^J u \|_{L^2(\Sigma_t)} \]
\[ \leq CC_1 \varepsilon s^{-2} \text{Econ}(s, \partial^J L^J u)^{1/2} \]

Then we see that
\[ \| (\mathcal{K}_g + N_g) \partial^J L^J u \cdot S_g \nabla \partial^J L^J u \|_{L^1(\Sigma_t)} \leq CC_1 \varepsilon s^{-2} \text{Econ}(s, \partial^J L^J u). \]

For the term \(T_0[\partial^J L^J u]\), we see that
\[ T_0[\partial^J L^J u] = - (t/s) \partial_\alpha N_g \cdot (s/t) \partial^J L^J u \cdot (\mathcal{K}_g + N_g) \partial^J L^J u \]
\[ = (t/s) \bar{s}^{ab} \partial_\alpha N_g \cdot s \bar{s} \partial^J L^J u \cdot (s/t) \partial^J L^J u \]

thus we see that
\[ \| T_0[\partial^J L^J u] \|_{L^1(\Sigma_t)} \leq C \| (t/s) \partial_\alpha N_g \|_{L^\infty(\Sigma_t)} \cdot \text{Econ}(s, \partial^J L^J u). \]

Then we apply lemma 9.3 and (9.3), we see that
\[ \| (t/s) \partial_\alpha N_g \|_{L^\infty(\Sigma_t)} \leq CC_1 \varepsilon s^{-2}. \]

Thus we see that the bound on \(M_0\) is bounded by \(CC_1 \varepsilon s^{-2} \text{Econ}(s, \partial^J L^J u)^{1/2}\). Then we apply the bootstrap bound (7.2) and the desired result is established.

Now we are ready to establish the improved energy bound.

**Proof of proposition 7.4** This is by applying the energy estimate (3.11) on the following equation:

\[ \Box \partial^J L^J u + Q^{a\beta\gamma} \partial_\alpha u \partial_\beta \partial_\gamma \partial^J L^J u = - [\partial^J L^J, Q^{a\beta\gamma} \partial_\alpha \partial_\beta \partial_\gamma u] \]

We see that for \(s \in [2, s_1]\),

\[ E_{\text{con}}(s, \partial^J L^J u)^{1/2} \leq CE_{\text{con}}(2, \partial^J L^J u) + C \int_2^s \tau \tau^{-2} \| \partial^J L^J, Q^{a\beta\gamma} \partial_\alpha \partial_\beta \partial_\gamma u \|_{L^2(\Sigma_t)} + M_g(\tau) \) \]
\[ \int_2^s \tau^{-2} d\tau \leq CC_0 \varepsilon + (C_1 \varepsilon)^2. \]

Recall that the initial energy is determined by the initial data and thus can be bounded by \(C_0 \varepsilon\).

Then we substitute the bounds 8.9 and lemma 9.3, we see that
\[ E_{\text{con}}(s, \partial^J L^J u)^{1/2} \leq CC_0 \varepsilon + C(C_1 \varepsilon)^2. \]

Then we see that we chose \(C_1 > 2CC_0\) and \(\varepsilon_0 \leq \frac{C_1 - 2CC_0}{CC_1^2}\). With this choice, we obtain that
\[ E_{\text{con}}(s, \partial^J L^J u)^{1/2} \leq \frac{1}{2} C_1 \varepsilon \]

and this concludes the desired result.
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