Chiral Fermions and Anomalies on a Finite Lattice

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Abstract

Recently Kaplan proposed a new method to simulate chiral fermions on the lattice by introducing a space dependent mass term that looks like a domain wall. He showed that if one starts with an odd dimensional lattice theory the lower (even) dimensional world on the domain wall will have a chiral zero mode and the corresponding anomaly. I test this proposal by computing the gauge and Goldstone-Wilczek currents and their derivatives on a 3-dimensional finite lattice. By determining the spectrum of the finite lattice Hamiltonian I demonstrate that the theory one obtains is chiral below a critical momentum. Furthermore I show that one can see the anomaly on the finite lattice and that in the case of the 3-4-5 model the anomalies in the gauge currents cancel as in the continuum.
1 Introduction

The Standard Model and several of its extensions are chiral theories and consequently chiral
gauge theories play an important role in our theoretical concepts. In principle one would like
to understand these theories not only perturbatively but also at strong coupling. However,
it has turned out that a non-perturbative regularization of chiral gauge theories runs into
problems. If one wants to use the lattice as a regulator one faces the appearance of extra
fermion species, the doubler fermions. Although it is a general belief that the lattice can be
used for regulating every quantum field theory, it seems to fail in the case of chiral gauge
theories [1]. This failure is summarized in the Nielsen-Ninomiya theorem [2] which leaves
us with the choice of either having the unwanted doubler fermions on the lattice or no
doubler modes but also the lack of chiral symmetry. Of course, the problem with the lattice
as a regulator for chiral gauge theories raises the question, of whether this failure can be
attributed only to the lattice or if it is a generic problem also with other non-perturbative
regulators for chiral gauge theories.

The last years have seen numerous attempts to latticize the Standard Model [1]. Some of
them like the Smit-Swift proposal have been carefully analyzed and tested, both in numerical
and analytical computations. However, it is generally accepted by now [3, 4] that the result
of these investigations is negative. So far a successful method to put chiral fermions on the
lattice is missing.

Here I want to start an investigation of a recent proposal by D. Kaplan [5]. He started
with an odd-dimensional vector gauge theory, avoiding in this way problems with the Nielsen-
Ninomiya theorem. In this odd dimensional theory the fermion mass is taken to depend on
one of the coordinates with the structure of a domain wall. Following Kaplan, on this domain
wall –which is a lower (even) dimensional world– one should find a chiral fermion, the correct
anomaly structure and the decoupling of the doubler modes. Therefore for a (lattice) person
who lives on the domain wall and does not know about the extra dimension with the defect,
it would seem as if the Nielsen-Ninomiya theorem is violated whereas a person living in the
higher dimensional world would interprete this result as a consequence of the existence of
the domain wall.

In this letter I provide a first test of Kaplan’s proposal. By using numerical methods I
calculate the spectrum and –in the presence of external gauge fields– the currents on the
finite lattice. In particular I demonstrate the existence of the anomaly on a finite lattice.
2 The model

To be specific I will restrict myself to a 3-dimensional U(1) gauge theory on a lattice $L^2 L_s$. The action is given by

$$S = \frac{1}{2} \sum_{z,\mu} \left[ \bar{\psi}_z \gamma_\mu U_{z,\mu} \psi_{z+\mu} - \bar{\psi}_{z+\mu} \gamma_\mu U_{z,\mu}^* \psi_z \right] + m_0(s) \sum_z \bar{\psi}_z \psi_z + w \sum_{z,\mu} \left[ -2 \bar{\psi}_z \psi_z + \bar{\psi}_x U_{z,\mu} \psi_{z+\mu} + \bar{\psi}_{z+\mu} U_{z,\mu}^* \psi_z \right].$$

(1)

Here a lattice point is denoted by $z = (t, x, s)$ and $\mu$ is a unit vector pointing in one of the three directions. The lattice spacing $a$ is set to one throughout the paper. The fermion fields are 2-component complex spinors and the gauge fields $U_{z,\mu} \in U(1)$. They are related to the vector potential by $U_{z,\mu} = e^{iqA_\mu(z)}$ with $q$ the electric charge. I leave out the gauge field self interaction (i.e. the plaquette term) as I am only interested in external gauge fields at this point. The gamma matrices in euclidean space are given by the Pauli-matrices. Note that in odd dimensions there is no analogue of $\gamma_5$ so that the theory is automatically vector-like.

The action (1) has a mass term depending only on one of the lattice directions, $s$:

$$m_0(s) \equiv \sinh(\mu_0) \theta(s); \quad \theta(s) = \begin{cases} -1 & 2 \leq s \leq \frac{L_s}{2} \\ +1 & \frac{L_s}{2} + 2 \leq s \leq L_s \\ 0 & s = 1, \frac{L_s}{2} + 1 \end{cases}.$$  

(2)

This mass has the form of a kink-antikink pair and generates two domain walls, one located at $s = 1$ and the other at $s = \frac{L_s}{2} + 1$. The existence of the antikink is necessitated by periodic boundary conditions. When one increases the lattice size the distance between these domain walls which is $L_s/2$, increases and for $L_s = \infty$ we are left with only one domain wall.

To get some insight into the spectrum of the theory, let me turn off the gauge fields for a moment and set the Wilson coupling $w$ to be zero (For a discussion of zero modes on domain walls see also [6, 7]). Then the lattice Hamiltonian of the problem is

$$H = -\sigma_1 \left[ \sigma_2 \partial_x + \sigma_3 \partial_s + m_0(s) \right],$$

(3)

where $\partial$ denotes the lattice derivative $\partial_z = \frac{1}{2} [\delta_{z,z+\mu} - \delta_{z,z-\mu}]$.

It is easy to see that

$$\psi^+ = e^{ikx} e^{\mu_0 |s-(\frac{L_s}{2}+1)|} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(4)

is an (unnormalized) eigenstate of the Hamiltonian, which is a massless fermion with chirality $\sigma_3 \psi^+ = + \psi^+$. This solution is centered on the domain wall at $s = 1$ and falls off exponentially as one goes away from the wall. There is a similar solution $\psi^-$

$$\psi^- = e^{-ikx} e^{-\mu_0 |s-(\frac{L_s}{2}+1)|} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(5)
which lives on the domain wall at \( s = \frac{L}{s} + 1 \) with negative chirality. Without the Wilson-term one has also the doubler modes at the corners of the Brillouin zones. They remain massless and appear with the opposite chirality. Therefore the fermion and the doubler modes pair up, the resulting theory is not chiral and so far we have gained nothing.

To decouple the doublers Kaplan added the usual Wilson-term, which in this case is not problematic as we started with a vectorlike theory. Introducing the Wilson-term leads to

\[
H = -\sigma_1 [\sigma_2 \partial_x + \sigma_3 \partial_s + m_0(s) + w(\Delta_x + \Delta_y)] .
\]

(6)

Here \( \Delta \) denotes the second derivative on the lattice. The Hamiltonian (6) can be reduced to only depend on \( s \) by imposing plane wave solutions in the \( x \)-direction. One obtains

\[
H = -\sigma_1 [\sigma_2 \sin(k) + \sigma_3 \partial_s + m_0(s) + 2w(\cos(k) - 1) + w\Delta_s)] .
\]

(7)

In [5] it was shown that for the infinite system which has a single domain wall, the Wilson term gives a large mass to the would-be chiral modes when their momenta exceed a critical value \( k_c = O(1) \). Because the aim at the end is a finite lattice simulation one wants to keep the same scenario at finite volume. Of course, on the finite lattice one has two domain walls and expects two chiral modes, one located at the domain wall at \( s = 1 \) with positive chirality and the other located at \( Ls/2 + 1 \) with negative chirality. The effect of the Wilson term is to delocalize and pair up high momentum bound states of opposite chirality on the two domain walls. For low momentum, the states remain chiral and massless and their overlap is exponentially small \( \approx e^{-\mu_0 Ls/2} \). If we look at only one domain wall we are left with only a single chiral mode in the low energy regime.

To see whether this scenario is true I calculated the eigenvalues and eigenfunctions of the Hamiltonian (7) numerically on a system size of \( L = L_s = 100 \). The lattice momenta \( k \) are given by \( k = \sin((\pi(n + \frac{1}{2}))/L) \) where \( n = 0, ..., L - 1 \). These momenta correspond to antiperiodic boundary conditions in the \( x \)-direction as will be used for the calculation of the anomaly in the next section. Let me discuss the results of the numerical evaluation of the eigenvalues and the eigenfunctions. The eigenvalues occur in \( \pm \lambda(k) \) pairs corresponding to a particle and an anti-particle solution. I plot the lowest momentum, \( k = \sin(\pi/2L) \), eigenfunctions which correspond to the lowest eigenvalues \( \pm \lambda_0 \) in fig.1. The figure shows that the \( +\lambda_0 \) state is located at \( s = 1 \), while the \( -\lambda_0 \) state is localized at \( s = \frac{Ls}{2} + 1 \). They are well separated in agreement with the expected overlap of the wavefunctions which is \( \approx e^{-50} (\mu_0 \text{ was chosen to be } 0.81) \).

In Fig.2a I show the lowest two eigenvalues \( \lambda_0(k) \) and \( \lambda_1(k) \) of the Hamiltonian (7). For \( k < k_c \approx 0.9 \), \( \lambda_0 \) exhibits the dispersion relation of a massless fermion while \( \lambda_1 \) corresponds to a state with a value of the mass at the cut-off and the system has a mass gap. Note that \( k_c \) is already at the order of the cut-off. Increasing the momenta \( k \) above \( k_c \) the eigenvalues degenerate and the energies are above the cut-off. The doubler modes at the corner of the Brillouin zone are decoupled and there is only one zero mode in the spectrum.

Fig.2b shows that this zero mode is a chiral fermion. I plot the ratio

\[
R = \frac{\langle \bar{\psi}\psi \rangle}{\langle \bar{\psi}\sigma_1\psi \rangle} .
\]

(8)
This ratio measures the “violation of chirality”. It is zero if \( \psi \) is chiral. \( R \) being non-zero indicates that the state is no longer a chiral eigenstate. The figure clearly demonstrates that for small momenta the wavefunction on the lattice is chiral. For large values of \( k \approx 0.9 \) the chirality is lost, the large momentum (at the order of the cut-off) modes pair up.

I conclude that if I am restricted to the domain wall at \( s = 1 \) I find a chiral fermion at low energies on the finite lattice. As a consequence I should see the corresponding anomaly. That this is indeed the case will be shown in the next section.

3 The anomaly

3.1 In the continuum

It is well known that the existence of a zero mode on a mass defect produces an anomaly [3, 10]. This anomaly should appear in the 2-dimensional world on the domain wall and is given by

\[
\partial_i j_i(z) = \pm q^2 \frac{E(t)}{2\pi} .
\]

Here \( j_i \) is the current \( j_i = \langle \bar{\psi} \gamma_i \psi \rangle \), \( (i = t, x) \), \( q \) is the charge of the fermion, \( E(t) \) is an applied external \( E \)-field and the \( \pm \) signs stand for the chirality of the zero mode on the domain wall. (The factor of \( i \) appearing in the Euclidean anomaly has been absorbed into the definition of the current \( j_i \).) The existence of this anomaly may seem surprising because we started from a 3-dimensional vector gauge theory and no anomaly should be present. There must be another current with nonzero divergence that cancels the zero mode anomaly (9).

This extra contribution is found in the Goldstone-Wilczek current [9]. It is well known that charge can flow due to the adiabatic change of external fields. In the present example this effect creates the so-called Goldstone-Wilczek current [3, 10], along the \( s \)-direction,

\[
J_{GW}^s(z) = -\frac{q^2}{2\pi} \frac{m(s)}{|m(s)|} E(t) .
\]

This current is perpendicular to the applied \( E \)-field (resembling in this respect a Hall current) and to the domain wall. As the mass \( m_0(s) \) is different on the two sides of the domain wall, the divergence of this current is not zero and exactly opposite to the divergence of the current (9). The 3-dimensional vector theory is anomaly free. The cancellation of the currents (9) and (10) is an example of the general relation of anomalies in odd dimensions to the ones in even dimensions as discussed in [10].

If the picture described above is correct, applying an external \( E \)-field one has to see the anomaly (9) even on a finite lattice. Moreover, one should have no anomaly if one starts with an anomaly free theory like the 3-4-5 model as discussed in [3]. In this model one chooses
as the mass term
\[ m_0(s)\bar{\psi}_z \psi_z \to m_0(s) \left[ (\bar{\psi}_z \psi_z)_3 + (\bar{\psi}_z \psi_z)_4 - (\bar{\psi}_z \psi_z)_5 \right] . \] 

(11)

The indices here mean that the fermions have charge \( q = 3, 4, 5 \), respectively. Because of the minus sign of the \( q = 5 \) fermion mass the \( q = 5 \) fermion and the \( q = 3, 4 \) fermions have opposite chirality. Furthermore, because the anomaly is proportional to the charge squared, the sum of the individual divergencies should vanish. This would not only test that one sees the anomaly on the lattice but also that it is proportional to \( q^2 \) and has therefore the right amplitude.

3.2 On the lattice

To test the above picture I have calculated the gauge current and its derivative numerically. The current on the finite lattice is given by
\[ j^\mu_z = \frac{1}{2} \left[ \bar{\psi}_z \gamma_\mu U_{z,\mu} \psi_z + \psi_{z+\mu} \gamma_\mu U^*_{z,\mu} \bar{\psi}_z \right] + w \left[ \bar{\psi}_z U_{z,\mu} \psi_{z+\mu} - \bar{\psi}_{z+\mu} U^*_{z,\mu} \psi_z \right] \] 

(12)

where \( \mu = 1, 2, 3 \). Note that I leave out the \( i \) in front of the current, because it will drop out in the anomaly equation (9). The total divergence of this current is locally conserved \( \partial^\mu j_\mu = 0 \) because the theory (1) is vectorlike. But if we pretend to live in the 2-dimensional world on one of the domain walls, where we have a chiral fermion, we would see the anomaly (9) realized.

The current (12) can be calculated numerically on a finite lattice from the inverse fermion matrix. I used the conjugate gradient method to invert the matrix in the presence of the external gauge fields. Note that this step is a numerical computation and that no simulation is involved. I have chosen antiperiodic boundary conditions in the \( x \) direction to avoid problems with zero modes that would render the matrix not invertible. To implement an external \( E \)-field along the domain wall, the gauge fields were chosen to be
\[ U_{x,\mu=2} = \exp\left\{ -iq \left[ \frac{L}{2\pi} E_0 \cos\left( \frac{2\pi}{L} (t - 1) \right) \right] \right\} \] 

(13)

The \( U \)’s in the other directions were set to 1.

As is shown in fig.1, the wavefunctions on the lattice have a finite width. The 1 + 1-dimensional world is not localized at only \( s = 1 \) but extends over several \( s \)-slices. The charge is built up in this finite region in \( s \) and the anomaly equation (9) has therefore to be modified on the lattice. Let us define
\[ < \partial_i j_i > \equiv \sum_{s \in \Lambda_\psi} \partial_i j_i(t, x, s) , \quad i = (t, x). \] 

(14)

Here the sum in \( s \) is taken over the support \( \Lambda_\psi \) of the wavefunction (see fig.1) – the “width” of the 1 + 1-dimensional world. By varying the number of \( s \)-slices one can determine how
many slices one has to take in the summation so that the result for $<\partial_i j_i>$ does not change. The anomaly equation on the lattice reads now

$$<\partial_i j_i> = \pm \frac{q^2}{2\pi} E_{eff}(t)$$

(15)

where the effective electric field $E_{eff}$ for small $E_0$ is given by

$$E_{eff} = \frac{\sin(\frac{2\pi}{L})}{\frac{2\pi}{L}} E_0 \sin(t - 1).$$

(16)

One can perform several checks on the program. First, using the wave function (4) as an input and acting on it with the fermion matrix it should reproduce the zero eigenvalue if the gauge fields are switched off. This is an excellent test to see whether the correct matrix is in the program. Second, because the 3-dimensional gauge current is conserved, the total divergence of the current should cancel locally in the presence of the external E-field. Also this I could see clearly by calculating the local currents. It turned out that $\partial_x j_x$ was exactly zero (as one would expect) and that the cancellation only occurred with $\partial_t j_t$ (the zero mode current (9)) and $\partial_s j_s$ (the Goldstone-Wilczek current (10)) at the same lattice point, i.e. $\partial_x j_x = 0$, $\partial_t j_t = -\partial_s j_s$.

I have chosen a $16^3$ lattice with $\mu_0 = 0.81$ (see (2)) and the strength of the E-field $E_0 = 0.001$. The outcome of the numerical evaluation of the current in the presence of the E-field was as follows. Setting the Wilson coupling to zero $<\partial_i j_i>$ was exactly zero. This shows that the doublers did not decouple. They cancel the anomaly and we are left with a vectorlike anomaly free theory.

Next I included a Wilson coupling $w = 0.9$. To see, whether for the $16^3$ lattice and the choice of $\mu_0$ and $w$ used here, the situation is similar to the case of the $L = 100$ system used in section 2, I calculated the lowest eigenvalue $\lambda_0$ and the corresponding eigenfunction from the Hamiltonian (7). I find that the wavefunctions have only a tiny overlap. The support of the eigenfunctions extends over about 7 $s$-slices. For small momenta the $16^3$ system still has a chiral fermion and is appropriate to test the anomaly equation (15).

I calculated $<\partial_i j_i>$ from the inverse fermion matrix. The summation over $s$ (see eq.(14)) was taken over three $s$-slices on each side of the $s = 1$ slice. Increasing the number of $s$-slices did not change the numerical value of $<\partial_i j_i>$. This fits into the picture I have obtained for the wavefunction from the Hamiltonian.

To test the anomaly equation (13) I constructed the ratio

$$R_{anomaly} = \frac{q^2 E_{eff}(t)}{2\pi} / <\partial_i j_i>.$$  

(17)

In the 3-4-5 model I find $R_{anomaly}$ to be independent of $t$ for each fermion with charge $q = 3, 4, 5$, respectively. This shows that the divergencies of the individual flavor currents

\footnote{The choice of $E_0$ was motivated to stay in the low energy regime of the dispersion relation (see fig.2a).}
are anomalous! Moreover, if the anomaly equation (15) is realized \(R_{anomaly}\) should equal one. I find from the numerical computation of \(\langle \partial_i j_i \rangle\), \(R_{anomaly} \approx 0.98\) in very good agreement with the theoretical expectations.

In Fig. 3 I plot \(q \langle \partial_i j_i \rangle\) for the 3-4-5 model. The individual \(\langle \partial_i j_i \rangle\) correspond to the different charges \(q\) in the 3-4-5 model as indicated in the figure. They individually satisfy the relation \(R_a = 1\) within 2%. Notice that in the case of \(q = 5\) the sign of \(\langle \partial_i j_i \rangle\) is reversed showing that the chirality of the zero mode is flipped. Moreover, the figure also shows the sum of all \(\langle \partial_i j_i \rangle\) as the straight line at zero! The anomalies cancel, we end up with a anomaly free gauge current in spite of the fact that the individual flavor currents are seen to be anomalous.

4 Conclusion

In this work I have shown for the first time that it is possible in practice to see anomalous behaviour of currents on the lattice. This was possible by using a recent proposal by Kaplan which circumvents the Nielson-Ninomyia theorem by starting with a three dimensional vectorlike theory with a mass defect and a Wilson term. This mass defect or domain wall guarantees the existence of a zero mode on the domain wall. Kaplan made use of this zero mode to construct a chiral gauge theory in the lower 2-dimensional target gauge theory. By introducing a Wilson-term it is possible to get rid of the unwanted doubler modes and to end up with only one chiral mode.

By solving the Hamiltonian problem on the finite lattice numerically I demonstrated that this scenario is true also for the finite lattice system. I find a chiral fermion for small momenta located at one of the domain walls and that the lattice system has an energy gap. I showed by numerically calculating the gauge current on finite lattices that as a consequence of the existence of the chiral fermion the anomaly can be seen on the lattice and that the divergence of the gauge current \(\langle \partial_i j_i \rangle\), eq. (12), satisfies the anomaly equation eq. (15). In addition I demonstrated that in the case of an anomaly free theory, which was taken to be the 3-4-5 model, the anomalies on the lattice cancel for the gauge current.

Although these results are certainly only a first step, the results are promising. They clearly indicate that the Kaplan proposal for chiral lattice fermions (or regulated chiral fermions in general) has a chance to solve the old puzzle of chiral fermions on the lattice. The next step is to render the gauge fields dynamical and to see whether one can get out a 2-dimensional chiral gauge theory - the chiral Schwinger model- in the continuum limit. Then one might proceed to four dimensions. Work in this direction is in progress.
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**Figure Caption**

**Fig.1** The eigenfunctions corresponding to the lowest eigenvalues $\pm \lambda_0$ for the smallest lattice momentum, $k = \sin(\pi/2L)$, are plotted. The system size is $L = L_s = 100$, the domain wall mass is $\mu_0 = 0.81$ and the Wilson coupling $w = 0.9$. The wavefunction for $+\lambda_0$ is located at $s = 1$, while the wavefunction for $-\lambda_0$ is at $L_s/2 + 1$. They show no overlap.

**Fig.2a** The two lowest positive eigenvalues $\lambda_0$ and $\lambda_1$ from the Hamiltonian (7) as a function of the lattice momenta $k = \sin(\pi(n + 1/2)/L)$ where $n = 0, \ldots, L - 1$. The parameters are as in fig.1. For small $k < k_c \approx 0.9$, the system has a mass gap, the lowest eigenvalue exhibits the dispersion relation of a massless fermion and $\lambda_1$ is at the order of the cut-off.

**Fig.2b** The ratio $R = \langle \bar{\psi}\psi \rangle / \langle \bar{\psi}\sigma_1\psi \rangle$ is shown. $R$ measures the “violation of chirality”. It is zero when the fermion is a chiral eigenstate and the discrepancy from zero indicates that the state is no longer chiral. The figure clearly shows that the lowest eigenvalue $\lambda_0$ in fig.2a (which corresponds also to the wavefunction centered at $s = 1$ in fig.1) belongs to a chiral fermion at low energies.

**Fig.3** The divergence of the 1 + 1-dimensional gauge current in an external $E$-field. The system size here is $16^3$. The parameters are otherwise like in figs.1,2, i.e. $\mu = 0.81$ and $w = 0.9$. The strength of the $E$-field is $E_0 = 0.001$. The different curves correspond to the contributions to $q < \partial_i j_i > , i = x, t$, (see eq.(12)) from fermions of charge $q = 3, 4, 5$ and chirality $+1, +1, -1$, respectively. The straight line at zero is the result of the sum of the three individual curves, showing that the anomalies cancel in the gauge current, even though the individual flavor currents are anomalous. Note that the results shown in the figure do not involve a *simulation* but stem from a direct numerical *computation* of the gauge current. Accordingly they do not have errorbars.