Fuzzy Maximal Sub-Modules

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Abstract
In this paper, we introduce and study the notions of fuzzy quotient module, fuzzy (simple, semisimple) module and fuzzy maximal submodule. Also, we give many basic properties about these notions.

Keywords: (Simple, Semisimple) Fuzzy Modules, Quotient Fuzzy Modules And Maximal Fuzzy Submodule.

Introduction
Recall that an R-module M is called simple if and only if it has no proper non trivial submodules [1]. M is called semisimple if and only if it is a sum of simple submodules of M. While, a submodule N of an R-module M is called maximal if and only if there is no proper submodule M that is different from N containing N properly [1]. Also, a semi − T- maximal submodule introduced as follows [2]: A submodule K of R-module M is called semi − T- maximal if T + K/K is a semisimple R-module M (where T is a submodule of M).

Hamel [3] introduced the definition of fuzzy simple and fuzzy semisimple modules. Some properties of these concepts which are useful in the next sections are given, while we add many other results.

A fuzzy module X is called simple if X has no nontrivial fuzzy submodules .

In other words , X is simple if whenever \( A \leq X \), either \( A = X \) or \( A = 0 \). Moreover , let \( A \leq X \), A is a fuzzy simple submodule of X if A is a fuzzy simple module .

Also, we define and introduce quotient fuzzy modules and fuzzy maximal submodules as a generalization of maximal submodules and give some properties of these concepts.

This paper consists of three sections . In section 1, we recall many definitions and properties which are needed in our work. In section 2, various basic properties about fuzzy simple (semisimple) modules are discussed. In section 3, we study the behavior of fuzzy maximal submodules.

1. Preliminaries
This section contains some definitions and properties of fuzzy sets, fuzzy modules, and quotient fuzzy modules.
**Definition 1.1** [4]

Let $S$ be a non-empty set and $I$ be the closed interval $[0,1]$ of the real line (real numbers). A fuzzy set $A$ in $S$ (a fuzzy subset of $S$) is a function from $S$ into $I$.

**Definition 1.2** [4]

A fuzzy set $A$ of a set $S$ is called a fuzzy constant if $A(x) = t$, for all $x \in S$, where $t \in [0,1]$.

**Definition 1.3** [5]

Let $x_t : S \to [0,1]$ be a fuzzy set in $S$, where $x \in S$, $t \in [0,1]$ defined by:

$$x_t(y) = \begin{cases} t & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

for all $y \in S$. $x_t$ is called a fuzzy singleton or fuzzy point in $S$.

**Definition 1.4** [5]

Let $A$ and $B$ be two fuzzy sets in $S$, then:

1. $A = B$ if and only if $A(x) = B(x)$, for all $x \in S$.
2. $A \leq B$ if and only if $A(x) \leq B(x)$, for all $x \in S$.

If $A < B$ and there exists $x \in S$ such that $A(x) < B(x)$, then $A$ is a proper fuzzy subset of $B$ and written as $A < B$.

By part (2), we can deduce that $x_t \leq A$ if and only if $A(x) \geq t$.

**Definition 1.5** [5],[6]

Let $M$ be an $R$-module. A fuzzy set $X$ of $M$ is called fuzzy module of an $R$-module $M$ if:

1. $X(x-y) \geq \min\{X(x), X, x, y \in M \}$
2. $X(rx) = X(x)$ for all $x \in M$ and $r \in R$.
3. $X(0) = 1$

**Definition 1.6** [6]

Let $X$ and $A$ be two fuzzy modules of $R$-module $M$. $A$ is called a fuzzy submodule of $X$ if $A \leq X$.

**Definition 1.7** [6]

If $A$ is a fuzzy module of an $R$-module $M$, then the submodule $A_t$ of $M$ is called the level submodule of $M$ where $t \in [0,1]$.

**Proposition 1.8** [6]

Let $A$ be a fuzzy set of an $R$-module $M$. Then the level subset $A_t$, $t \in [0,1]$ is a submodule of $M$ if and only if $A$ is a fuzzy submodule of $X$ where $X$ is a fuzzy module of an $R$-module $M$.

Now, we shall give some properties of fuzzy submodules, which are used in the next section.

**Proposition 1.9** [7]

Let $A$ be a fuzzy module in $M$, then we define $A_* = A_{A(0)} = \{x \in M, A(x) = 1\} = A(0_M) = 1$.

**Proposition 1.10** [8]

Let $A$ be a fuzzy module of an $R$-module $M$, then $A_*$ is a submodule of $M$.

We add the following results.

**Proposition 1.11**

Let $X$ be a fuzzy module of an $R$-module $M$ and $A \leq X$ such that $X(0) = 1$, then $A(0) = 1$.

**Proof:** It is clear by the definition of fuzzy modules.

**Remark 1.12**

If $A$ and $B$ are fuzzy modules of a fuzzy module $X$ such that $A \leq B$, then $A_* \leq B_*$. 

**Proof:**

Let $x \in A_*$, then $A(x) = A(0)$. But $B(x) \geq A(x)$, $\forall x \in M$, hence $B(x) \geq A(x) = A(0) = B(0)$. Thus $x \in B_{B(0)} = B_*$. 

**Remark 1.13**

The converse of the above Remark is not true in general as the following example shows:

Let $X : Z \to [0,1]$ defined by: $X(x) = 1, \forall x \in Z$.

Let $A(x) = \begin{cases} 1 & \text{if } x \in 2Z, \\ 0.9 & \text{otherwise} \end{cases}$

Let $B(x) = \begin{cases} 1 & \text{if } x \in 4Z, \\ 1/2 & \text{otherwise} \end{cases}$
It is clear that A and B are fuzzy submodules of X and that $A_x = 2Z = B_x = 4Z$.
Hence $A_y \leq B_y$. But $A \neq B$, since $A(3) = 0.9$, $B(3) = 1/2 = 0.5$.

**Remark**
We assume that if $A_x = B_x$, then $A = B$ is called a **Condition** (*).

**Lemma 1.14** [9]
Let $A$ be a fuzzy submodule of fuzzy module X, $(A:X)_t \leq (A_t:X_t)$ for all $t \in (0,1]$. Also, we can prove that by Lemma 2.3.3.[8].

It follows that if $X = \mathbb{A} \oplus \mathbb{B}$ for $A \oplus B \leq X$ , then $X_r = (A \oplus B)_r = A \oplus B_r$.

**Remark 1.15[9]**
It is not necessarily that $(A:X)_t \leq (A_t:X_t)$ for any fuzzy submodule A of a fuzzy module X.

**Proposition 1.16** [9]
If $A$ is a prime fuzzy submodule of fuzzy X, then $(A:X)_t = (A_t:X_t)$ for all $t \in (0,1]$ where $A_t \neq X_t$.

Recall that a submodule N of R-module M is called an essential if $N \cap H = 0$.

For nontrivial submodule $H$ of $M,[10],[11],[12]$.

Rabi [9] introduced the following definition: Let X be a fuzzy module of an R-module M. A fuzzy submodule A of X is called an essential if $A \cap B = 0_1$ for nontrival fuzzy submodule B of X.

Now we add the following definition.

**Definition 1.17**
Let X be a fuzzy module of an R-module M and let A be fuzzy submodule of X. Define $X/A: M/A \rightarrow [0,1]$ by:

$$X/A(a + A) = \{ \sup \{X(a + b) \} : b \in A_x, a \notin A_x \}$$

for all coset $a + A \in M/A_x$.

**Proposition 1.18**
If X is a fuzzy module of an R-module M and A is a fuzzy submodule of X, then $X/A$ is a fuzzy module of $M/A_x$.

**Proof:**
By definition 1.17, X is a nonempty fuzzy subset of $M/A_x$.

Firstly, we must prove that $X/A$ is well defined.

Let $a_1 + A, a_2 + A \in X/A$, if $a_1 + A = a_2 + A$, implies that $a_1 - a_2 \in A_x$.

Therefore $a_1 - a_2 = b_1$, $b_1 \in A_x$, implies that $a_1 = a_2 + b_1$.

Hence $X/A$ is well defined.

Now, we must prove that $X/A$ is a fuzzy module of $M/A_x$.

1) If $a_1, a_2 \notin A_x$, then either $a_1 - a_2 \in A_x$ or $a_1 - a_2 \notin A_x$, and either $a_1, a_2 \in A$ or $a_1, a_2 \notin A_x$.

Thus, if $a_1 - a_2 \in A_x$, then $X/A(a_1 - a_2 + A_x) = 1 \geq \min X/A(\{ a_1 - a_2 + A_x \})$.

If $a_1 - a_2 \notin A_x$, then

$$X/A(a_1 - a_2 + A_x) = \sup \{X(m), m \in a_1 - a_2 \in A_x \} \geq \sup X(m_1), m_1 \in a_1 + A_x, m_2 \in a_2 + A_x \}$$

$$\geq \sup \{X(m_1), X(m_2) \} \geq \sup \{X(m_1), m_1 \in a_1 + A_x, m_2 \in a_2 + A_x \}$$

$$\geq \sup \{X(m_2), m_2 \in a_2 + A_x \} \sup \{X(m_1), m_1 \in a_1 + A_x \}$$

$$= \min \{X/a_1 + A_x, X/A(a_2 + A_x) \}$$

Hence, $X/A(a_1 - a_2 + A_x) \geq \min X/A(a_1 - a_2 + A_x), X/A(a_2 + A_x)$.

2) To prove that $X/A(rx + A_x) \geq X/A(x + A_x)$ for $r \in R$ and $x \in M$.

If $rx \in A_x$, then $X/A(rx + A_x) = 1$, which implies that $X/A(rx + A_x, X/A(x + A_x))$.

If $rx \notin A_x$, then $x \notin A_x$. Hence, $X/A(rx + A_x) = \sup \{X/A(rx + b) : b \in A_x \}$ and $X/A(x + A_x) = \sup \{X/A(x + b) : b \in A_x \}$.
But, \( \text{sup}\{X(rx + b) : b \in A_\cdot\} \geq \text{sup}\{X(rx + rb) : b \in A_\cdot\} \)
\[ = \sup\{X[r(x + b) : b \in A_\cdot]\} \]
\[ \geq \sup\{X[(x + b) : b \in A_\cdot]\} \]
\[ = X[A(x + A_\cdot)] \]

Thus, \( \{\text{sup}\{X(rx + b) : b \in A_\cdot\}\} \geq X[A(x + A_\cdot)] \)

That is \( X/A(rx + A_\cdot) \geq X/A(x + A_\cdot) \)

3) To prove that \( (0 + \ A_\cdot) = 1, [0 + A_\cdot] = A_\cdot, \)
\[ X/A(a + A_\cdot) = \text{sup}\{X(a + b) : a \in A_\cdot \} \]
\[ \text{if } a \in A_\cdot \]
\[ = \text{sup}\{X(a + b) : a \in A_\cdot \} \]
\[ \text{if } a \in A_\cdot \]

Hence \( X/A(a + A_\cdot) = 1 \), but \( 0 \in A_\cdot \), then \( X/A(0 + A_\cdot) = 1 \).

Thus \( X/A \) is called a quotient fuzzy module.

However, in a previous work [10] there exists a definition of quotient fuzzy module as follows, which is an equivalent to Definition 3.1 where \( X(0) = 1 \).

**Proposition 1.20[10]**

Let \( X \) be a fuzzy module of an \( R \)-module \( M \) and \( A \) be a fuzzy submodule of \( X \). Define \( X/A : M/A_\cdot \rightarrow [0,1] \), such that
\[ X/A(a + A_\cdot) = \text{sup}\{X(a + b) : a \in M, b \in A_\cdot \} \]

**Lemma 1.21**

Let \( A \) be fuzzy submodule of fuzzy module \( X \), then \( X_\cdot/A_\cdot \leq (X/A)_\cdot \).

Proof:

Let \( y' \in X_\cdot/A_\cdot \), hence \( y' = x + A_\cdot \), for \( x \in X_\cdot \), which implies that \( X(x) = 1 \).

Now
\[ X/A(y') = X/A(x + A_\cdot) = \text{sup}\{X(x + a) : a \in A_\cdot \} \]
\[ \geq \text{sup}\{\text{min}\{X(x), A(a)\}\} \]
\[ = \text{sup}\{\text{min}\{1,1\}\} = 1 = \text{sup}\{1\} = 1 \]

Hence \( X/A(y') = 1 \), so \( y' \in (X/A)_\cdot \). Hence \( X_\cdot/A_\cdot \leq (X/A)_\cdot \).

**Proposition 1.22**

Let \( X \) be a fuzzy module of an \( R \)-module \( M \), such that \( X(x) = 1, \forall x \in M \). Then,
\[ (X/A)_\cdot = X_\cdot/A_\cdot \], for each \( A \subseteq X \)

Proof:

\( X_\cdot \leq \left( X/A_\cdot \right) \), by Lemma 1.21.

Now \( X_\cdot = M \) and \( X/A : M/A_\cdot \rightarrow [0,1] \), so for each \( a \in A_\cdot \), \( M + A_\cdot \).

\[ X/A_\cdot(a + A_\cdot) = \text{sup}\{B(a + b) : a \in A_\cdot, b \in A_\cdot \} \]

Since \( X(a + b) \geq \text{min}\{X(x), X(b)\} = 1 \), hence \( X/A(a + A_\cdot) = 1 \).

**Proposition 1.23**

Let \( A_\cdot, B_\cdot \) be fuzzy submodules of fuzzy module \( X \). Then \( X/A_\cdot \cap B_\cdot \approx \text{Fuzzy submodules of } (X/A_\cdot \oplus X/B) \).

**Proof:**

Define \( f : X \rightarrow X[A_\cdot \oplus X/B] \) by \( f(x_\cdot + A_\cdot, x_\cdot + B) \), for each \( x_\cdot \subseteq X \). Then \( f \) is a homomorphism.

\[
\text{ker } f = \{ \{ x_\cdot : x_\cdot \subseteq X \text{ and } f(x) = 0_\cdot \} \} = \{ x_\cdot : x_\cdot \subseteq X \text{ and } f(x_\cdot + A_\cdot, x_\cdot + B) = (0_\cdot, 0_\cdot) \}
\]
\[ = \{ x_\cdot : x_\cdot \subseteq X \text{ and } x_\cdot \in A \cap B \} \]

Thus by First Fundamental theorem
\[
X/\text{ker } f \approx \text{Im } f \approx \text{fuzzy submodules of } (X[A_\cdot \oplus X/B])
\]

**S.2 Fuzzy simple (semisimple) modules**

Recall that an \( R \)-module \( M \) is called simple if and only if it has no proper non trivial submodules and that \( M \) is called semisimple if and only if \( M \) is a sum of simple submodules of \( M \) [1].

Hamel [3] introduced the definition of fuzzy simple...
modules and fuzzy semisimple modules. Some properties of these concepts which are useful in next the sections are given, while we add many other results.

**Definition 2.1**
A fuzzy module $X$ is called simple if $X$ has no nontrivial fuzzy submodules.

In other words, $X$ is simple if whenever $A \leq X$, either $A = X$ or $A = 0$.

Moreover, let $A \leq X$, $A$ is a fuzzy simple submodule of $X$ if $A$ is a fuzzy simple module.

**Remarks 2.2 [3]**
Let $X$ be a fuzzy module, then the followings hold:

1. Every simple fuzzy module is $F$-regular fuzzy module, where $F$-regular is every fuzzy submodule of $X$ is pure.

2. If $X$ is a simple fuzzy module, then $X_t$ is a simple module, $\forall t \in (0,1]$.

3. If $X_t$ is a simple module, $\forall t \in (0,1]$, then is not necessarily that $X$ is a simple fuzzy module.

**Proposition 2.3**
Let $X$ be a fuzzy module of an $R$-module $M$ and $A$ be a fuzzy submodule of $X$. If $A$ is simple, then $A_*$ is a simple submodule in $X$.

Proof:
Suppose that $A_*$ is not simple, then there exists submodule $N < A_*.$

Define $B : M \to [0,1]$ defined by $B(x) = \begin{cases} A(x), & x \in N \\ 0, & \text{otherwise} \end{cases}$

It is clear that $B < A_*$, but $A$ is a simple fuzzy submodule, which is a contradiction. Thus $A_*$ is a simple submodule.

**Remark 2.4**
The converse of proposition 2.3 is not true in general, as the following example shows:

Let $M = Z_6$ as $Z$-module. Define $X : M \to [0,1]$ such that

$X(x) = \begin{cases} 1, & \text{if } x \in \frac{1}{2} > N \\ 1, & \text{otherwise} \end{cases}$

It is clear that $X$ is fuzzy module. Also, let

$A(x) = \begin{cases} 1, & \text{if } x \in \frac{1}{2} > \\ (1/2), & \text{otherwise} \end{cases}$

where $A$ is a fuzzy submodule of $X$ and $A_* = \frac{1}{2} > \frac{1}{2} >$ is a simple submodule in $Z_6$. But $A$ is not simple since there exists fuzzy submodule $B$ of $A$ such that:

$B : M \to [0,1]$, defined by $B(x) = \begin{cases} 1, & \text{if } x \in \frac{1}{2} > \\ 0, & \text{otherwise} \end{cases}$

Thus, $A$ is not a simple fuzzy submodule of $X$.

**Definition 2.5 [3]**
A fuzzy module $X$ is called semisimple if $X$ is the sum of simple fuzzy submodules of $X$.

Next, we need the following lemma.

**Lemma 2.6**
Let $X$ be a fuzzy module of an $R$-module $M$ and let $A$ be a fuzzy direct summand in $X$, then $A_*$ is a direct summand in $X_*$.

Proof:
As $A$ is a fuzzy direct summand of $X$, then $A \bigoplus B = X$, for some $B \leq X$.

Then $A_* \bigoplus B_* = X_*$. And so $A_* \bigoplus B_* = X_*$ by Lemma 1.14. Therefore, $A_*$ is a direct summand of $X_*$.

**Proposition 2.7**
Let $X$ be a fuzzy module of an $R$-module $M$. If $X$ is a fuzzy semisimple module, then $X_*$ is a semisimple module.

Proof:
Let $N \leq X_*$, and let $A : M \to [0,1]$, defined by:

$A(x) = \begin{cases} 1, & \text{if } x \in N \\ 0, & \text{otherwise} \end{cases}$

Hence $N = A_*$. As $A < X$, hence $A$ is a direct summand of $X$. 

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Then $A_\ast$ is a direct summand of $X_\ast$.

So $N$ is a direct summand of $X_\ast$. Thus, $X_\ast$ is semisimple.

**Proposition 2.8**

Let $X$ be a fuzzy module of an $R$-module $M$, if $X_\ast$ is semisimple, then $X$ is semisimple when Condition (*) holds.

Proof:

Let $A \leq X$, then $A_\ast \leq X_\ast$. But $X_\ast$ is semisimple, so $A_\ast \oplus K = X_\ast$, for some $K \leq X_\ast$. By putting:

$$B(x) = \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{otherwise} \end{cases}$$

then $B_\ast = K$. Thus, $A_\ast \oplus B_\ast = X_\ast$, so $A \oplus B = X$ by Condition (*) and $A$ is a direct summand of $X$.

Therefore, $X$ is a semisimple fuzzy module.

**Corollary 2.9**

Any fuzzy submodule of semisimple fuzzy module is semisimple.

Proof:

Let $X$ be a fuzzy semisimple module and let $A \leq X$. Then $X_\ast$ is semisimple module by Proposition 2.8. But $A_\ast \leq X_\ast$, so $A_\ast$ is semisimple submodule of $X$ by Prop 2.8.

**Proposition 2.10**

Let $X$ be a fuzzy module of an $R$-module $M$, where Condition (*) holds. Then the following statements are equivalent:

1. $X$ is semisimple
2. $X$ has no proper essential fuzzy submodule.
3. Every fuzzy submodule of $X$ is a direct summand of $X$.

Proof: $(1) \implies (2)$

Since $X$ is semisimple, then $X_\ast$ is semisimple by prop 2.7, and $X_\ast$ has no proper essential submodule [11]. Now suppose that $X$ has a fuzzy proper essential submodule, say $A$. Then $A_\ast$ is essential submodule in $X_\ast$. But $X_\ast$ has no proper essential submodule, so that $A_\ast = X_\ast$. Then, by codition (*), $A = X$, which is a contradiction.

$(2) \implies (3)$

Let $A \leq X$ then $A_\ast \subseteq X_\ast$, but $X$ is a fuzzy semisimple module, which implies that $X_\ast$ is a semisimple module, so that $A \oplus N = X_\ast$, for some $N \leq X_\ast$.

Let $B : M \to [0,1]$ defined by: $B(x) = \begin{cases} 1 & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases}$

Then $B_\ast = N$ and $B \leq X$. Hence $A_\ast \oplus B_\ast = X_\ast$, which implies that $A \oplus B = X$, for some $B < X$. Then, $A \cap B = 0$, and hence $A$ is not essential in $X$.

$(3) \implies (1)$

If every fuzzy submodule in $X$ is a fuzzy direct summand, then every submodule of $X_\ast$ is a direct summand, by Lemma 2.7. Hence $X_\ast$ is semisimple and, so, $X$ is semisimple.

$(3) \implies (1)$

If every fuzzy submodule in $X$ is a fuzzy direct summand, then every submodule of $X_\ast$ is a direct summand, by prop 2.6. Hence $X_\ast$ is semisimple and, so, $X$ is semisimple by prop 2.7.

**Proposition 2.11**

Let $X$ be a fuzzy module of an $R$-module $M$. Then the followings are equivalent:

1. Every fuzzy submodule of $X$ is a sum of fuzzy simple submodules.
2. $X$ is a direct sum of fuzzy simple submodules of $X$.
3. Every fuzzy submodule of $X$ is a direct summand of $X$.

Proof: It is easy.
**Proposition 2.1.2**

The following are equivalent:
1. Every fuzzy submodule of semisimple fuzzy module is semisimple.
2. Every epimorphic image of semisimple fuzzy module is semisimple.
3. Every sum of semisimple fuzzy modules is semisimple.

Proof: It is easy.

**S.3 Fuzzy maximal submodules**

In this section, we introduce the concept of a fuzzy maximal submodule as a generalization of ordinary concept maximal submodules. Some characterizations of fuzzy maximal submodules are introduced.

**Definition 3.1**

Let A be a fuzzy submodule of fuzzy module X. A is called fuzzy maximal if X/A is a simple fuzzy module.

**Proposition 3.2**

Let A be a fuzzy submodule of fuzzy module X. A is a fuzzy maximal submodule if and only if whenever A \( \leq B < X \), then A = B.

Proof:

Assume that A \( \leq B < X \), then B/A \( \leq X/A \). But, A is maximal, so X/A is simple. Hence, B/A = 0. That is A = B.

To prove that A is fuzzy maximal, we must show that X/A is simple. Suppose there exists B/A \( \langle X/A \rangle \). Then A \( \leq B < X \).

Hence, by assumption, A = B. That is B/A = 0.

Therefore, X/A is simple.

**Proposition 3.3**

Let X be fuzzy module of an R-module M, and let A be a fuzzy submodule of X. If A is a fuzzy maximal submodule of X, then A is a maximal submodule of X.

Proof:

If A is a fuzzy maximal submodule of X, then X/A is simple and hence \( (X/A)_* \) is simple by Prop 1.3. But \( X/A_* \leq X/A \) by Lemma 1.21.

That is \( X/A_* = (X/A)_* \). Hence, \( X/A_* \) is simple and, so, A is a maximal submodule of X.

**Proposition 3.4**

Let X be fuzzy module of an R-module M, and let A be a fuzzy submodule of X. If A is a maximal submodule of X, then A is a fuzzy maximal submodule of X, provided that Condition (*) holds.

Proof:

Let A be a maximal submodule in X. To prove that A is a fuzzy maximal submodule in X, if A \( \leq B < X \), where B is a fuzzy submodule of X, then A \( \leq B_* < X \).

So that A \( = B_* \) and, so, by condition(*) A = B.

**Proposition 3.5**

Let A be fuzzy submodule of fuzzy module X. If A is fuzzy maximal, then \( A < x_t > = X \) for all \( x_t \notin A \), provided that Condition (*) holds.

Proof:

If \( a_1 \notin A \), then \( A(a) < 1 \). Therefore, \( a \notin A_* \). But A is a fuzzy maximal submodule of X. Hence, A is maximal submodule of X by prop. 2.3. Then, \( A_* = X = \langle X \rangle \).

On the other hand,

\[< a_1 > (x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \]

Hence \( (a_1) = \langle a \rangle \).

Hence \( (A < a_1 >)_* = X_* \). Thus \( (A + < a_1 >) = X \)

Since \( a_1 \notin A \), then \( (A + < a_1 >) \geq A \) and A is a fuzzy maximal submodule of X, so \( (A + < a_1 >) = X \).
Corollary 3.6
Let $X$ be a fuzzy module of an $R$–module, and let $X(x) = 1, \forall x \in M$. Let $A$ be a fuzzy submodule of $X$. If $K$ is a maximal submodule in $X$, then there exists a fuzzy submodule $A$ in $X$ such that $A = K$.

Proof:
Let $K$ be a maximal submodule in $X$. Let $A: M \to [0,1]$, defined by:
$$A(x) = \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{otherwise} \end{cases}$$

It is clear that $A$ is a fuzzy submodule of $X$ and $A = K$. To prove that $A$ is a maximal fuzzy submodule in $X$.

Let $B \leq X$, and $A \leq B \leq X$. Then $A \leq B \leq X$, which implies that $K \leq B \leq M$.

Hence $= B$, since $K$ is a maximal in $X$. Thus $A = B$ and, so, $A = B$.

Thus $A$ is a fuzzy maximal submodule of $X$.

Proposition 3.7
If $A$ and $B$ are fuzzy maximal submodules of fuzzy module $X$, then $A + B = X$.

Proof:
Since $A + B > A$ and $A$ is a fuzzy maximal submodule and since $A + B > A$ and $A$ is a fuzzy maximal submodule of $X$, then $A + B = X$.

Proposition 3.8
Let $f: X \to Y$ be fuzzy epimorphism and $\ker f$ is a fuzzy maximal submodule in $X$, then $Y$ is simple.

Proof:
Since $f$ is epimorphism and $X/\ker f \cong Y$, by the first fundamental theorem, thus $Y$ is simple.

Proposition 3.9
If $A$ is fuzzy maximal submodule of fuzzy $X$, then $A$ is a prime fuzzy submodule of $X$.

Proof:
Let $r_t$ be a fuzzy singleton of $R$ and $x_s$ be a fuzzy singleton of $X$ such that $r_t x_s \in A$, and assume that $x_s \in A$.

Then $< x_s + A > \geq A$. But $A$ is a fuzzy maximal submodule.

So $X/A$ is simple.

On the other hand, $< x_s + A > \leq X/A$, so that $< x_s + A > \geq X/A$.

Since $r_t < x_s + A \geq r_t X/A$.

hence $r_t x_s + A \geq r_t X/A$, which implies that $r_t X + A/A = 0_{X/A}$. Then $X + A = A$, so $r_t X \leq A$. Therefore, $r_t \in (A:X)$.

Remark 3.10
The converse of proposition 3.8 is not true in general. The following example shows that:

Example
Let $X: Z \to [0,1]$ defined by $X(x) = 1, \forall x \in Z$. Then $0_1(x) =$

$$0_1(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

is a prime fuzzy submodule of $X$, but is not a fuzzy maximal submodule since there exists a fuzzy submodule $A$ of $X$, such that:

$$A(x) = \begin{cases} 1 & \text{if } x \in 2Z \\ 0 & \text{otherwise} \end{cases}$$

Remark 3.11
The intersection of two fuzzy maximal submodules is not necessarily fuzzy maximal, as shown in the following example.

Example:
Let $M = Z$ as $Z$–module, and define $X : M \to [0,1]$ by $X(x) = 1, \forall x \in M$.

Let $A(x) =$

$$A(x) = \begin{cases} 1 & \text{if } x \in < 2 > \\ 0 & \text{otherwise} \end{cases}$$

Let $B(x) =$

$$B(x) = \begin{cases} 1 & \text{if } x \in < 3 > \\ 0 & \text{otherwise} \end{cases}$$

$A$ and $B$ are fuzzy submodules of $X$ and $A$ and $B$ are maximal of $X$ since $A = < 2 >$ and $B = < 3 >$.
which are maximal of $Z = X$, 
However, $A \cap B_* = (A \cap B)_* < 6$ which is not maximal submodule of $X_*$. 
Thus $A \cap B = \begin{cases} 1 & \text{if } x \in <6> \\ 0 & \text{otherwise} \end{cases}$

$A \cap B$ is not maximal fuzzy submodule of $X$.

**Proposition 3.12**

If $X$ is a finitely generated fuzzy module of an $R$-module $M$, then every proper fuzzy submodule of $X$ is contained in a maximal fuzzy submodule of $X$.

**Proof:**

If $X$ is a finitely generated fuzzy module, then $X_*$ is finitely generated as explained in a previous work [13]. So, $X_*$ is a finitely generated module. Hence every proper submodule is contained in a maximal submodule of $X_*$ [14].

Let $A$ be a fuzzy submodule of $X$. Then, $A_* \leq X_*$. Hence, $A_* \leq N < X_*$, where $N$ is a maximal in $X_*$. Define $B^\ast < X^\ast$ such that $B^\ast:M \to [0,1]$ by:

$$B(x) = \begin{cases} 1 & \text{if } x \notin N \\ 0 & \text{otherwise} \end{cases}$$

Then $B_* = N$ and, so, $A_* \leq B_* < X_*$. But by condition (*), $A < B \leq X$. Since $B_* = N$ is a maximal submodule and Condition (*) holds, then $B$ is a fuzzy maximal submodule of $X$. Thus $A$ is contained in a fuzzy maximal submodule of $X$.

**References**

1. Kasch, F. 1982. *Modules and Rings*, Academic press.
2. Inaam M.A.Hahi, A. and Eelwi., A. 2019. Semi - maximal sumodules. *Iraqi Journal of science*, 60(12): 2725-2731.
3. Maysoun A. 2002. F-regular fuzzy modules , M. Sc. Thesis, University of Baghdad .
4. Zadeh L. A. 1965. *Fuzzy Sets, Information and control*, 8: 338-353, 1965.
5. Zahedi M. M. 1992. On L-Fuzzy Residual Quotient Modules and P. Primary Submodules , *Fuzzy Sets and Systems*, 51: 33-344.
6. Zahedi M. M. 1991. A characterization of L-Fuzzy Prime Ideals, *Fuzzy Sets and Systems*, 44: 147-160.
7. Martinez L. 1996. Fuzzy Modules Over Fuzzy Rings in Connection with Fuzzy Ideal of Ring, *J. Fuzzy Math.*, 4: 843-857.
8. Abd alguad Q. 1999. Some Results On Fuzzy Modules .M. Sc Thesis University of Baghdad.
9. Rabi H. J. 2000. Prime Fuzzy Submodule and Prime Fuzzy Modules , M. Sc. Thesis, University of Baghdad,)
10. Pan F. 1992. "The Various Structures of Fuzzy Quotient Modules, *Fuzzy Sets and Systems*, 50: 187-192.
11. Goodreal K.R. 1976. *Ring Theory –Non Singular Rings and modules*, Marci-Dekker,New York and Basel.
12. Inaam, M.A., Hahi,F., Shyaa, D. and Shukur N. 2019. Essentially Second modules, *Iraqi Journal of science*, 2019, 60: 1374-1380.
13. Inaam. M. Hadi, maysoun, A. and Hamil, A. 2011. Cancellation and Weakly Cancellation Fuzzy Modules, *Journal of Basrah Reserchs*(sciences)), 37(4.D).
14. Burton D.M. 1971. *Abstract and Linear Algebra*, University of Hampshire