ON SINGULAR MODULI FOR LEVEL 2 AND 3

HANS ROSKAM

ABSTRACT. Gross and Zagier proved a formula for the absolute norm \( N(j(\alpha_1) - j(\alpha_2)) \) of a difference of singular values of the modular function \( j \). We formulate and prove the analogues of their result for a number of functions of level 2 and 3.

1. INTRODUCTION

In a famous paper [5], Gross and Zagier established an explicit formula for the expression

\[
J(d_1, d_2) = \left( \prod_{\tau_1, \tau_2} \frac{4}{w_1 w_2} \left( j(\tau_1) - j(\tau_2) \right) \right)^{\frac{1}{4w_1 w_2}}.
\]

Here \( j \) is the elliptic modular function on the complex upper half plane \( \mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \), \( d_1 \) and \( d_2 \) are negative coprime fundamental quadratic discriminants, \( w_1 \) and \( w_2 \) are the number of roots of unity in the corresponding imaginary quadratic orders, and \([\tau_i]\) denotes the equivalence class of \( \tau_i \in \mathbb{H} \) under the natural action of \( \text{SL}_2(\mathbb{Z}) \). By \( \text{disc} \tau_i = d_i \) we mean that \( \tau_i \) is imaginary quadratic, and that its irreducible polynomial over \( \mathbb{Z} \) has discriminant \( d_i \). We know by the theory of complex multiplication that \( J(d_1, d_2) \) is the \( \frac{4}{w_1 w_2} \)-power of the norm of the algebraic integer \( j(\tau_1) - j(\tau_2) \). In particular if \( d_1 \) and \( d_2 \) are both less than \(-4\) then \( J(d_1, d_2) \) is an integer.

In order to give the Gross-Zagier formula, we define for primes \( p \) satisfying \( (d_1 d_2, p) \neq 1 \)

\[
\varepsilon(p) = \begin{cases} 
(d_1, p) & \text{if } \gcd(p, d_1) = 1; \\
(d_2, p) & \text{if } \gcd(p, d_2) = 1.
\end{cases}
\]

We extend \( \varepsilon \) multiplicatively to products \( m = \prod_i p_i^{a_i} \) of such prime numbers, and put

\[
F(m) = \prod_{\substack{nn' = m \\ n, n' > 0}} n^{\varepsilon(n')}
\]

for such a product. Furthermore we set \( F(m) = 1 \) for all \( m \in \mathbb{Q} \) that are not of the above form.

**Theorem 1.** Using the above notation and setting \( D = d_1 d_2 \) the following formula holds:

\[
J(d_1, d_2)^2 = \pm \prod_{\substack{x \in \mathbb{Z} \\ x^2 < D \\ x \neq 0}} F\left( \frac{D - x^2}{4} \right).
\]
The above product can be restricted to those \( x \in \mathbb{Z} \) that satisfy \( x^2 \equiv D \mod 4 \). For these integers one proves the equality \( \varepsilon \left( \frac{D-x^2}{4} \right) = -1 \) using quadratic reciprocity. Furthermore one can prove that for any positive integer \( m \) with \( \varepsilon(m) = -1 \) the value \( F(m) \) is a prime power [1, page 306–307]. More precisely we have

\[
F(m) = \begin{cases} 
  p^{(a+1)} \Pi_{j=1}^{r} (b_j+1) & \text{if } m = p^{2a+1} \cdot \Pi_{i=1}^{r} p_i^{2a_i} \cdot \Pi_{j=1}^{s} q_j^{b_j}, \text{ where } p, p_i \\
  1 & \text{otherwise.}
\end{cases}
\]

Using this formula the following corollary is immediate.

**Corollary 2.** If \( p \) is a prime dividing \( J(d_1, d_2)^2 \) then \( \left( \frac{d_1}{p} \right) \neq 1, \left( \frac{d_2}{p} \right) \neq 1 \) and \( p \) divides a positive integer of the form \( \frac{D-x^2}{4} \). In particular, \( p \leq \frac{D}{4} \).

For algebraic numbers \( \beta_1, \beta_2 \) we set

\[
N(\beta_1, \beta_2) = |N_{\mathbb{Q}(\beta_1, \beta_2)/\mathbb{Q}(\beta_1 - \beta_2)}|.
\]

Hence the above theorem gives an expression for \( N(j(\alpha_1), j(\alpha_2)) \) \( \equiv \frac{8}{\Delta(z)} \), where \( \alpha_i \in \mathbb{H} \) are arbitrarily under the condition \( \text{disc} \alpha_i = d_i \). We will prove similar expressions for \( N(f(\alpha_1), f(\alpha_2)) \) for a number of modular functions \( f \) of level 2 and 3. Here \( \alpha_i \in \mathbb{H} \) of discriminant \( d_i \) is chosen such that \( f(\alpha_i) \) is a class invariant, i.e. such that \( j(\alpha_i) \) and \( f(\alpha_i) \) generate the same field over \( \mathbb{Q}(\sqrt{d_i}) \).

Recall the definition of the \( j \)-function

\[
j(z) = 12^3 g_2(z)^3 \Delta(z) = 12^3 + 6^6 g_3(z)^2 \Delta(z)
\]

where \( g_2(z), g_3(z) \) and \( \Delta(z) \) are the modular forms on \( \text{SL}_2(\mathbb{Z}) \) of weight 4, 6 and 12 respectively defined by

\[
g_2(z) = 60 \sum_{m,n \in \mathbb{Z}, (m,n)\neq(0,0)} \frac{1}{(mz+n)^4}, \quad g_3(z) = 140 \sum_{m,n \in \mathbb{Z}, (m,n)\neq(0,0)} \frac{1}{(mz+n)^6}
\]

and \( \Delta(z) = g_2^3(z) - 27 g_3^2(z) = (2\pi)^{12} \eta(z)^{24} \). Here \( \eta(z) \) is the Dedekind eta function

\[
\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)
\]

with \( q = q(z) = e^{2\pi i z} \).

The first two functions that we will study are the Weber functions

\[
\gamma_3(z) = 6^3 \frac{g_3(z)}{(2\pi)^6 \eta(z)^{12}}, \quad \gamma_2(z) = 12 \frac{g_2(z)}{(2\pi)^4 \eta(z)^8},
\]

which are modular functions of level 2 and 3 respectively. The relations

(1) \quad \gamma_3(z)^2 = j(z) - 12^3, \quad \gamma_2(z)^3 = j(z)
imply that for \( f \in \{ \gamma_3, \gamma_2 \} \) any prime divisor of \( N(f(\alpha_1), f(\alpha_2)) \) also divides \( N(j(\alpha_1), j(\alpha_2)) \). In fact for \( f = \gamma_3 \) the converse also holds. Before we prove this we fix the following notation which we use throughout this paper:

The discriminants \( d_1 \) and \( d_2 \) of the imaginary quadratic fields \( K_1 \) and \( K_2 \) are relatively prime and \( w_1, w_2 \) and \( h_1, h_2 \) denote the number of roots of unity and the class numbers of their ring of integers. Furthermore we set \( D = d_1 d_2 \) and \( h'_i = \frac{2}{w_i} h_i \) for \( i = 1, 2 \).

**Theorem 3.** For \( \alpha_1, \alpha_2 \in \mathcal{H} \) of discriminant \( d_1 \) and \( d_2 \) respectively, the following formula holds:

\[
N(\gamma_3(\alpha_1), \gamma_3(\alpha_2)) = N(j(\alpha_1), j(\alpha_2))^e
\]

with \( e = 2 \) if neither \( d_1 \) nor \( d_2 \) is equal to \(-4\) and \( e = 1 \) otherwise.

**Proof.** First assume that neither \( d_1 \) nor \( d_2 \) is equal to \(-4\). We claim that in this case \( \gamma_3(\alpha_i) \) generates a quadratic extension of \( \mathbb{Q}(j(\alpha_i)) \). Namely, if \( d_i \) is odd then \( \gamma_3(\alpha_i) \) is conjugate over \( \mathbb{Q} \) to one of the numbers \( \pm \gamma_3(\frac{-1+\sqrt{d_i}}{2}) \). As \( j(\frac{-1+\sqrt{d_i}}{2}) \) is real and less then \( 12^2 \) the claim follows from the first equality in (1). If \( d_i \) is even and different from \(-4\), then \( K_i(\gamma_3(\alpha_i)) \) is the ray class field of conductor \( 2 \) of \( K_i \) which is quadratic over \( \mathbb{Q} \) and hence has \( \{ \gamma_3(\tau_i), -\gamma_3(\tau_i) \} \) as a complete set of conjugates over \( \mathbb{Q} \).

As \( K_i(\gamma_3(\alpha_i)) \) is the Hilbert class field or the ray class field of conductor \( 2 \) of \( K_i \), depending on whether \( d_i \) is odd or even, its subfield \( \mathbb{Q}(\gamma_3(\alpha_i)) \) is unramified at the primes not dividing \( d_i \). By assumption \( d_1 \) and \( d_2 \) are relatively prime. The fields \( \mathbb{Q}(\gamma_3(\alpha_1)) \) and \( \mathbb{Q}(\gamma_3(\alpha_2)) \) are therefore linearly disjoint over \( \mathbb{Q} \). Systematically using

\[
(x^2 - y^2)^2 = (x - y)(x + y)(-x + y)(-x - y)
\]

we find \( N(\gamma_3(\alpha_1), \gamma_3(\alpha_2)) = N(j(\alpha_1), j(\alpha_2))^2 \).

If \( \alpha_i \) is of discriminant \(-4\) then \( \gamma_3(\alpha_i) = 0 \) and the formula for the absolute norm of \( \gamma_3(\alpha_1) - \gamma_3(\alpha_2) \) follows easily. \( \square \)

Our result for the function \( f = \gamma_2 \) is less complete. As mentioned above we normalise \( \alpha_i \) such that \( \gamma_2(\alpha_i) \) is a class invariant. Although this normalisation works for all discriminants \( d_i \) that are coprime to \( 3 \), we only obtained a formula in the case \( d_1 \equiv d_2 \equiv 2 \mod 3 \).

**Theorem 4.** Assume that \( d_1 \) and \( d_2 \) are both congruent to \( 2 \) modulo \( 3 \). For \( i = 1, 2 \) define \( \alpha_i \in \mathcal{H} \) by

\[
\alpha_i = \begin{cases} \frac{-3+\sqrt{d_i}}{2} & \text{if } d_i \equiv 1 \mod 4; \\ \frac{\sqrt{d_i}}{2} & \text{if } d_i \equiv 0 \mod 4. \end{cases}
\]

Then the following formula holds:

\[
N(\gamma_2(\alpha_1), \gamma_2(\alpha_2))^{\frac{8}{w_1 w_2}} = 3^{6h_1 h_2} \prod_{x \in \mathbb{Z}, x^2 < D} F\left(\frac{D - x^2}{36}\right).
\]
As a corollary we find that the primes $p \neq 3$ dividing $N(\gamma_2(\alpha_1), \gamma_2(\alpha_2))$ satisfy $p < \frac{D}{90}$. If both $d_1$ and $d_2$ are not equal to $-4$, the exponent $\frac{8}{w_1w_2}$ is equal to 2. In this case one obtains a formula for $N(\gamma_2(\alpha_1), \gamma_2(\alpha_2))$ by restricting the above product to positive $x$ and by replacing the exponent $6h_1h_2$ by $3h_1h_2$.

Next we consider the 24-th powers of the classical Weber $f$-functions:

$$\omega(z) = \frac{\Delta(z^{14})}{\Delta(z)} = -q^{-\frac{1}{2}} \prod_{n=1}^{\infty} (1 + q^n)^{12},$$

$$\omega_1(z) = \frac{\Delta(z)}{\Delta(z)} = q^{-\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^n)^{12},$$

$$\omega_2(z) = 2^{12} \frac{\Delta(2z)}{\Delta(z)} = 2^{12} q \prod_{n=1}^{\infty} (1 + q^n)^{24},$$

where for $z \in H$ and $a \in \mathbb{Q}$ we set $q^a = e^{2\pi i a}$. These functions are modular of level 2 and satisfy the polynomial equation [1, theorem 12.17]

$$(X - \omega)(X - \omega_1)(X - \omega_2) = (X + 16)^3 - jX.$$

**Theorem 5.** Assume that $d_1$ and $d_2$ are both congruent to 1 modulo 8. For $i = 1, 2$ define $\alpha_i \in H$ by $\alpha_i = \frac{-1 + \sqrt{d_i}}{2}$. Then the following formulas hold:

$$N(\omega(\alpha_1), \omega(\alpha_2)) = N(\omega_1(\alpha_1), \omega_1(\alpha_2)) = 2^{12h_1h_2} \cdot \prod_{x \in \mathbb{Z}} F\left(\frac{D - x^2}{8}\right),$$

$$N(\omega_2(\alpha_1), \omega_2(\alpha_2)) = \prod_{0 < x < \sqrt{D}} F\left(\frac{D - x^2}{16}\right).$$

The above formula for $\omega_2$ was conjectured by Yui and Zagier in [16, formula 57].

In the table below we have listed the prime factorization of $N(f(\alpha_1) - f(\alpha_2))$ for each of the functions $f \in \{j, \gamma_2, \omega, \omega_2\}$ and for two choices of the pair $(\alpha_1, \alpha_2)$. Here $\alpha_i \in H$ is of discriminant $d_i$ and normalised depending on $f$ as in the theorems above. The class numbers of the quadratic fields in the first column are equal to $h_1 = 1$ and $h_2 = 4$, in the second column they equal $h_1 = 3$ and $h_2 = 7$.

| $j$       | $3^{26}5^{6}19^{4}47^{2}$ | $3^{140}13^{21}23^{11}53^{6}61^{2}73^{5}79^{4}83^{4}89^{2}179^{2}449^{2}557^{2}$ |
|-----------|---------------------------|---------------------------------------------------------------------------------|
| $\gamma_2$| $3^{14}5^{2}$             | $3^{77}13^{7}23^{5}61^{2}$                                                     |
| $\omega$  | $2^{48}3^{34}5^{8}19^{4}47^{2}$ | $2^{25}3^{390}13^{28}23^{12}53^{8}61^{2}73^{8}79^{4}83^{6}89^{2}179^{2}449^{2}557^{2}$ |
| $\omega_2$| $3^{8}5^{2}19$            | $3^{50}13^{7}23^{3}53^{2}73^{3}83^{2}$                                         |

Gross and Zagier gave two proofs of theorem 1, one algebraic and one analytic. Their algebraic proof uses the reduction theory of elliptic curves and was restricted to the case of prime discriminants. Later Dorman extended it to the general case [2]. Our proof of theorems 4 and 5 is an adaptation of the analytic proof of Gross
and Zagier. It consists of three steps, sections 2, 3 and 4, which are more or less independent. In the outline below we concentrate on the function $\gamma_2$.

In section 2 we use Shimura’s reciprocity law to compute the conjugates of $\gamma_2(\alpha)$ where $\alpha \in \mathbb{H}$ is of discriminant $d \equiv 2 \mod 3$ and normalised as in theorem 4. These conjugates can be written in the form $\gamma_2(\tau)$ with $\tau$ in some finite set $S_d$ of elements $\tau \in \mathbb{H}$ with disc $\tau = d$ and $\text{Tr}_{\mathbb{Q}(\sqrt{d})/\mathbb{Q}}(\tau) \equiv 0 \mod 3$. Next in section 3 we characterise $h(z_1, z_2) = \log |\gamma_2(z_1) - \gamma_2(z_2)|$ as the unique symmetric and harmonic function on $\mathbb{H} \times \mathbb{H}$ with certain invariance and growth conditions. Using Legendre functions and Eisenstein series we then build a function satisfying the same properties. By summing this function over $(z_1, z_2) \in S_{d_1} \times S_{d_2}$ we arrive at a complicated looking expression for $\log N(\gamma_2(\alpha_1), \gamma_2(\alpha_2))$ (theorem 13 below). These kinds of expressions were recognised by Gross and Zagier as being related to the Fourier coefficients of certain holomorphic modular forms of weight 2 on congruence subgroups of SL$_2(\mathbb{Z})$. To make this precise we study in section 4 a family of non-holomorphic Hilbert modular forms and, via restriction to the diagonal and holomorphic projection, the corresponding family of holomorphic modular forms. Finally in section 5 we conclude that $\log N(\gamma_2(\alpha_1), \gamma_2(\alpha_2))$ is up to a simple expression equal to the first Fourier coefficient of one of these holomorphic modular forms. As the corresponding congruence subgroup has genus zero, this Fourier coefficient vanishes and the proof of theorem 4 is complete.

2. Computing conjugates

An element $M \in \text{PSL}_2(\mathbb{Z})$ represented by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ acts on the upper half plane $\mathbb{H}$ by the linear fractional transformation

$$z \mapsto Mz = \frac{az + b}{cz + d}.$$

In the sequel we will identify the transformation $M$ with the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and write $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ instead of $M \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mod \{\pm \text{id}\}$. The left PSL$_2(\mathbb{Z})$-action on $\mathbb{H}$ induces a right action on functions $f$ on $\mathbb{H}$ by $(f \circ M)(z) = f(Mz)$. Fix a positive integer $N$. The (projective) principal congruence modular group $\Gamma(N)$ is defined as the kernel of the map PSL$_2(\mathbb{Z}) \rightarrow \text{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm \text{id}\}$ induced by the natural map $\mathbb{Z} \rightarrow \mathbb{Z}/N\mathbb{Z}$. A modular function of level $N$ is a meromorphic function $f$ on $\mathbb{H}$ that is invariant under $\Gamma(N)$, i.e. $f \circ M = f$ for all $M \in \Gamma(N)$, and that is ‘meromorphic at the cusps’. Recall that the cusps are the points $x \in \mathbb{P}^1(\mathbb{Q})$ which are on the boundary of $\mathbb{H}$. To clarify the condition ‘meromorphic at the cusps’ we need the $N$-th root $q_N^\pm = e^{\frac{2\pi i}{N}}$ of the function $q = q(z) = e^{2\pi iz}$. A $\Gamma(N)$-invariant function $f$ satisfies $f(z + N) = f(z)$ because $\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma(N)$. Hence there is a meromorphic function $f^*$ on the punctured disk $\{q_N^\pm : 0 < |q_N^\pm| < 1\}$ such that $f(z) = f^*(q_N^\pm)$. If $f^*$ is meromorphic at $q_N^\pm = 0$ we say that $f$ is meromorphic at the cusp $\infty$. The $q$-expansion of $f$ is by definition the Laurent expansion of $f^*$ at $q_N^\pm = 0$. Finally we say that $f$ is meromorphic at the cusps if $f \circ M$ is meromorphic at $\infty$ for all $M \in \text{PSL}_2(\mathbb{Z})$. Note that the behaviour of $f$ near the cusp $x \in \mathbb{P}^1(\mathbb{Q})$ is reflected by the behaviour of $f \circ M$ near $\infty$ if $Mx = \infty$.

An equivalent way of introducing modular functions of level $N$ is the following. Let $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ be the extended upper half plane and extend the PSL$_2(\mathbb{Z})$-action on $\mathbb{H}$ in the obvious way to $\overline{\mathbb{H}}$. The orbit space $\overline{\Gamma(N)} \backslash \overline{\mathbb{H}}$ can be given a
complex structure [12, §1.8]. The addition of the set of cusps $\mathbb{P}^1(\mathbb{Q})$ to $\mathbb{H}$ causes the resulting Riemann surface to be compact. The meromorphic functions on this Riemann surface correspond to the modular functions of level $N$.

Let $F_N$ be the field of modular functions of level $N$ for which the $q$-expansion is rational over $\mathbb{Q}(\zeta_N)$, with $\zeta_N = e^{2\pi i/N}$ a $N$-th root of unity. Fix an imaginary quadratic field $K \subset \mathbb{C}$ with ring of integers $\mathcal{O}$. The evaluation of $f \in F_N$ at $\theta \in \mathbb{H} \cap K$ is called a singular modulus. By the theory of complex multiplication a singular modulus generates an abelian extension of the quadratic number field $K$. More precisely the field generated over $K$ by $f(\theta)$ with fixed $\theta$ as above and $f$ ranging over those functions in $F_N$ that are defined at $\theta$, is equal to the ray class field of conductor $N$ [11, page 128].

For example in the case $N = 1$ we have $F_1 = \mathbb{Q}(j)$. If $\theta \in \mathbb{H}$ generates the ring of integers $\mathcal{O}$ the field $K(j(\theta))$ is equal to the Hilbert class field $H$ of $K$, the maximal abelian unramified extension of $K$. By class field theory the Artin map supplies an isomorphism between the ideal class group $C$ of $K$ and the Galois group $\text{Gal}(H/K)$. To describe the action of this group on $j(\theta)$ we represent elements of $C$ by $\text{PSL}_2(\mathbb{Z})$-equivalence classes of primitive positive definite quadratic forms of discriminant equal to the discriminant of $K$. For $d \equiv 0, 1 \mod 4$ a negative integer let

$$Q_d = \{[a, b, c] : a, b, c \in \mathbb{Z}, a > 0, b^2 - 4ac = d\}$$

be the set of positive definite quadratic forms of discriminant $d$. If $d$ is the discriminant of $K$ and $[a, -b, c] \in Q_d$ we have

$$j(\theta)^{[a,-b,c]} = j(\frac{-b + \sqrt{d}}{2a}).$$

Because the function $j$ is $\text{PSL}_2(\mathbb{Z})$-invariant, the above value only depends on the $\text{PSL}_2(\mathbb{Z})$-equivalence class of the quadratic form $[a, -b, c]$. Formula (3) is a reformulation of the well known formula $(b, K/\mathbb{Q})j(\mathcal{O}) = j(b^{-1})$, where $(b, K/\mathbb{Q})$ denotes the Artin automorphism of the $K$-ideal $b$.

More generally for $f \in F_N$ and $\theta$ a generator of $\mathcal{O}$ the singular modulus $f(\theta)$ lies in the ray class field of conductor $N$ of $K$, a field containing the Hilbert class field $H$ of $K$. In the examples treated in this paper $f(\theta)$ does in fact generate $H$ over $K$. In this case $f(\theta)$ is called a class invariant and we will restrict ourselves to this case in the discussion below.

To obtain the generalization of (3) to class invariants we use Shimura’s reciprocity law. First we need to describe the Galois theory of the modular function fields $F_N$ [11, chapter 6]. The field $F_N$ is Galois over $F_1$ with group isomorphic to $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$. For $M \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$ we write $M = \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}$ with $\delta = \text{det} M \in (\mathbb{Z}/N\mathbb{Z})^\times$ and $\bar{M} \in \text{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$. Lift $\bar{M}$ to an element $\bar{\bar{M}} \in \text{PSL}_2(\mathbb{Z})$ and let $\sigma_{\bar{M}}$ be the automorphism of $\mathbb{Q}(\zeta_N)$ induced by $\zeta_N \mapsto \zeta_{\bar{M}}^N$. For $f \in F_N$ with $q$-expansion $\sum_k a_k q^{k/2} \in \mathbb{Q}(\zeta_N)((q^{1/2}))$ the Galois action of $M$ is given by

$$f^M = \bar{f} \circ \bar{\bar{M}},$$

where $\bar{f} \in F_N$ has $q$-expansion $\sum_k \sigma_{\bar{M}}(a_k) q^{k/2}$.

Fix the following generator of $\mathcal{O}$:

$$\theta = \begin{cases} \frac{\sqrt{d}}{2} & \text{if } 2 \mid d; \\ \frac{-1 + \sqrt{d}}{2} & \text{if } 2 \nmid d. \end{cases}$$
To describe the Galois action on a class invariant $f(\theta)$ for some $f \in F_N$ we restrict to the case $N = p$ is prime. Shimura’s reciprocity law [3, theorem 20] states that for $f \in F_p$ such that $f(\theta) \in K(j(\theta))$ and $[a, -b, c] \in C$ there exists an element $M = M(a, b, c) \in \text{GL}_2(\mathbb{Z}/p\mathbb{Z})/\{\pm \text{id}\}$ such that

$$f(\theta)[a, -b, c] = f^M(-\frac{b+\sqrt{d}}{2a}).$$

In case $a$ is prime to $p$ the element $M$ is represented by the following matrix

$$M = \begin{cases} \left( \begin{array}{cc} a & b \\ 0 & 1 \end{array} \right) & \text{if } 2 \mid d; \\ \left( \begin{array}{cc} a & b-1 \\ 0 & 1 \end{array} \right) & \text{if } 2 \nmid d. \end{cases}$$

As each quadratic form is $\text{PSL}_2(\mathbb{Z})$-equivalent to a form $[a, -b, c]$ with $p \nmid a$ [1, lemma 2.3 and 2.25], the above restriction on $a$ is not a serious one.

Before we apply Shimura’s reciprocity law to the function $\gamma_2$ we first determine its stabilizer inside $\text{PSL}_2(\mathbb{Z})$, which is classically denoted by $\Gamma_3$. Recall that $S = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)$ and $T = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)$ generate the group $\text{PSL}_2(\mathbb{Z})$. Their action on the Dedekind eta function is given by

$$\eta(-\frac{1}{z}) = \sqrt{-iz} \eta(z), \quad \eta(z+1) = \zeta_{24} \eta(z)$$

where the square root is positive on the positive real axis [11, page 253]. Using the definition of $\gamma_2$ in the introduction we find

$$\gamma_2 \circ S = \gamma_2, \quad \gamma_2 \circ T = \zeta_3^{-1} \gamma_2.$$ 

Hence $\Gamma_3$ is the kernel of the character $\text{PSL}_2(\mathbb{Z}) \to \langle \zeta_3 \rangle$ sending the transformation $M \to \frac{\gamma_2^N}{\gamma_2^2}$, and therefore normal of index 3 inside $\text{PSL}_2(\mathbb{Z})$. This characterization can be used to show the following equalities [14, page 15-19]

$$\Gamma_3 = \langle \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right), \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} -1 & -1 \\ 2 & 1 \end{array} \right) \rangle = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{PSL}_2(\mathbb{Z}) : ab + cd \equiv 0 \mod 3 \right\}.$$

It follows from the second description that $\Gamma(3)$ is contained in $\Gamma_3$, hence $\gamma_2$ is a modular function of level 3. By applying the Hurwitz formula [12, theorem 4.2.11] we find that the Riemann surface $\Gamma_3 \backslash \mathbb{H}$ is of genus zero.

For a negative integer $d \equiv 0, 1 \mod 4$ we define the $\text{PSL}_2(\mathbb{Z})$-set

$$\mathcal{P}_d = \{ \tau \in \mathbb{H} : a\tau^2 + b\tau + c = 0 \text{ for some } [a, b, c] \in \mathbb{Q}_d \}.$$ 

The map sending $[a, b, c]$ to $-\frac{b+\sqrt{d}}{2a} \in \mathbb{H}$ is a bijection from $\mathbb{Q}_d$ to $\mathcal{P}_d$ and we denote the quadratic form corresponding to $\tau \in \mathcal{P}_d$ by $[a_\tau, b_\tau, c_\tau]$. In case $d$ is a negative fundamental discriminant it follows from the discussion above that $\{ j(\tau) : \tau \in \text{PSL}_2(\mathbb{Z}) \backslash \mathcal{P}_d \}$ is a transitive $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-set. We have the following analogous result for $\gamma_2$. 
Proposition 6. Let \( d \equiv 2 \mod 3 \) be a negative fundamental discriminant and define \( \alpha \in H \) by
\[
\alpha = \begin{cases} 
-\frac{3+\sqrt{d}}{2} & \text{if } d \equiv 1 \mod 4; \\
\frac{\sqrt{d}}{2} & \text{if } d \equiv 0 \mod 4.
\end{cases}
\]
a. The algebraic integer \( \gamma_2(\alpha) \) is of degree \( h \) over \( \mathbb{Q} \).
b. The action of \( \Gamma^3 \) on \( H \) stabilizes the set
\[
P_d^{\gamma_2} = \{ \tau \in H : a\tau^2 + b\tau + c = 0 \text{ for some } [a, b, c] \in \mathbb{Q}_d \text{ with } 3 \mid b \}.
\]
The orbit set \( \Gamma^3 \backslash P_d^{\gamma_2} \) has cardinality \( h \) and \( \{ \gamma_2(\tau) : \tau \in \Gamma^3 \backslash P_d^{\gamma_2} \} \) is a complete set of conjugates of \( \gamma_2(\alpha) \) over \( \mathbb{Q} \).

In the proof of proposition 6 we need the following lemma.

Lemma 7. Let \( Y \subset X \) be an inclusion of sets and \( H \subset G \) be an inclusion of groups. Assume \( G \) acts on \( X \) in such a way that \( Y \) becomes a \( H \)-set. Let \( I \) be a complete set of left coset representatives of \( H \subset G \) and assume
\[
X = \bigsqcup_{g \in I} gY. \quad \text{(disjoint union)}
\]
Then the inclusion \( Y \subset X \) induces a bijection
\[
H \backslash Y \overset{\sim}{\longrightarrow} G \backslash X.
\]

Proof. Let \( y_1, y_2 \in Y \) be \( G \)-equivalent, say \( y_1 = ghy_2 \) with \( g \in I \) and \( h \in H \). Then \( gh y_2 \) is an element of \( gY \cap Y \). As this intersection is non-empty if and only if \( g \in H \), we find that \( y_1 \) and \( y_2 \) are \( H \)-equivalent and the map (9) is injective. The fact that (9) is surjective is immediate from (8).

Proof of proposition 6. As \( \gamma_2 \) is the cube root of \( j \) that is real valued on the positive imaginary axis its \( q \)-expansion lies in \( \mathbb{Q}(q^{1/3}) \). With \( \alpha \) as in the proposition we find that \( \gamma_2(\alpha) \) is real. Because \( \gamma_2(\alpha) \) generates the Hilbert class field of \( K \) over \( K \) [1, theorem 12.2] we conclude that \( \gamma_2(\alpha) \) is of degree \( h \) over \( \mathbb{Q} \). Hence the conjugates of \( \gamma_2(\alpha) \) over \( \mathbb{Q} \) coincide with those over \( K \).

Let \([a, -b, c]\) be a quadratic form of discriminant \( d \). Because \( d \) is congruent to 2 modulo 3, the integer \( a \) is prime to 3 and we can use (4) and (5) to compute the conjugate \( \gamma_2(\alpha)[a, -b, c] \). For odd discriminants we find:
\[
\gamma_2(\alpha)[a, -b, c] = \gamma_2 \circ T^{-1} \left( \frac{\sqrt{d}}{2} \right)[a, -b, c] = \gamma_2^{M} \left( \frac{\sqrt{d}}{2a} \right),
\]
with \( M = T^{-1} \left( \begin{smallmatrix} a & b-1 \\ 0 & 1 \end{smallmatrix} \right) = \left( \begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix} \right) \left( \begin{smallmatrix} b-3/2 \\ a-1/2 \end{smallmatrix} \right) \in \text{GL}_2(\mathbb{Z}/3\mathbb{Z}) \). As the Fourier expansion of \( \gamma_2 \) has rational coefficients, the matrix \( \left( \begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix} \right) \) acts trivially on \( \gamma_2 \). To calculate the action of \( \left( \begin{smallmatrix} a & b-3/2 \\ 0 & a-1/2 \end{smallmatrix} \right) \), we use the decomposition
\[
\left( \begin{smallmatrix} a & b-3/2 \\ 0 & a-1/2 \end{smallmatrix} \right) = ST^{-a}ST^{-a}ST^{-ab-a} \mod 3
\]
from [3, lemma 6] and obtain

$$\gamma_2(\alpha)^{[a,-b,c]} = \gamma_2\left(\frac{-b(1+2a^2)^\frac{1}{2}}{2a}\right).$$

A similar calculation shows that this formula is also valid for even discriminants. Because $a$ is prime to 3, we have $1 + 2a^2 \equiv 0 \mod 3$, hence the conjugates of $\gamma_2(\alpha)$ can be written in the form $\gamma_2(\tau)$ for some $\tau \in \mathcal{P}^\gamma_d$.

To conclude the proof, we use lemma 7 to show that the inclusion $\mathcal{P}^\gamma_d \subset \mathcal{P}_d$ induces a bijection

$$\Gamma^3 \backslash \mathcal{P}^\gamma_d \sim \text{PSL}_2(\mathbb{Z}) \backslash \mathcal{P}_d.$$

For a quadratic form $[a, b, c]$ of discriminant $b^2 - 4ac \equiv 2 \mod 3$ with $3\mid b$, the congruence $a \equiv c \mod 3$ holds. Using this congruence one checks that $\Gamma^3$ acts on $\mathcal{P}^\gamma_d$. The set $I = \{T^k : k = 0, 1, 2\}$ is a complete set of left coset representatives of $\Gamma^3$ in $\text{PSL}_2(\mathbb{Z})$. For $\tau \in \mathcal{P}_d$ and $\tau' = T\tau$ we have $b_{\tau'} = b_\tau - 2a_\tau$. As $a_\tau$ is prime to 3 we find that $T$ changes the residue class of $b_\tau$ modulo 3 and hence $\mathcal{P}_d$ is the disjoint union of $\mathcal{P}^\gamma_d$, $T\mathcal{P}^\gamma_d$ and $T^2\mathcal{P}^\gamma_d$. According to lemma 7, the map above is a bijection.

Using the notation of theorem 4, we saw in the proof above that $\gamma_2(\alpha_1)$ and $\gamma_2(\alpha_2)$ are class invariants. As in the proof of theorem 3 we conclude that $\mathbb{Q}(\gamma_2(\alpha_1))$ and $\mathbb{Q}(\gamma_2(\alpha_2))$ are linearly disjoint over $\mathbb{Q}$. Hence we find the following corollary of proposition 6 which we need in section 3:

$$\log N(\gamma_2(\alpha_1), \gamma_2(\alpha_2)) = \sum_{i=1,2,\tau_i \in \Gamma^3 \backslash \mathcal{P}^\gamma_d} \log |\gamma_2(\tau_i) - \gamma_2(\tau_2)|.$$

The transformations (6) of the Dedekind eta function imply that the functions $\omega, \omega_1$ and $\omega_2$ are permuted under the action of $\text{PSL}_2(\mathbb{Z})$. More precisely we have

$$\omega, \omega_1, \omega_2 \circ S = (\omega, \omega_2, \omega_1), \quad (\omega, \omega_1, \omega_2) \circ T = (\omega_1, \omega, \omega_2).$$

We conclude that $\omega, \omega_1$ and $\omega_2$ are invariant under $T^2$ and $ST^2S$. These transformations generate the congruence subgroup $\Gamma(2)$. Recall that the modular function field $F_2$ of level 2 is a Galois extension of $F_1 = \mathbb{Q}(j)$ with Galois group $\text{GL}_2(\mathbb{Z}/2\mathbb{Z})/\{\pm \text{id}\} \cong S_3$, the permutation group on 3 symbols. As each of the functions $\omega, \omega_1$ and $\omega_2$ generates a different cubic extension of $\mathbb{Q}(j)$ we find that $F_2$ is generated over $\mathbb{Q}$ by these functions. The stabilizer of $\omega_2$ inside $\text{PSL}_2(\mathbb{Z})$ contains $T$ and $\Gamma(2)$ and is therefore equal to

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{Z}) : c \equiv 0 \mod 2 \right\},$$

an index 3 subgroup of $\text{PSL}_2(\mathbb{Z})$. The set of cusp classes $\overline{\Gamma_0(2)} \backslash \mathbb{P}^1(\mathbb{Q})$ is represented by 0 and $\infty$. It follows from the $q$-expansions of $\omega_2$ and $\omega_2 \circ S = \omega_1$ that $\omega_2$ has a single simple zero at $\infty \in \overline{\Gamma_0(2)} \backslash \mathbb{H}$. Consequently the compact Riemann surface $\overline{\Gamma_0(2)} \backslash \mathbb{H}$ is of genus zero and has $\mathbb{C}(\omega_2)$ as its function field.

**Proposition 8.** Let $d \equiv 1 \mod 8$ be a negative fundamental discriminant.
a. The algebraic integers $\omega(-\frac{1+i\sqrt{3}}{2})$ and $\omega_1(-\frac{1+i\sqrt{3}}{2})$ are conjugate and of degree $2h$ over $\mathbb{Q}$. The algebraic integer $\omega_2(-\frac{1+i\sqrt{3}}{2})$ is of degree $h$ over $\mathbb{Q}$.

b. The action of $\Gamma_0(2)$ on $H$ stabilizes the sets

$$\mathcal{P}_d' = \mathcal{P}_d' = \{ \tau \in H : a\tau^2 + b\tau + c = 0 \text{ for some } [a, b, c] \in \mathbb{Q}_d \text{ with } 2 \mid a \}$$

and

$$\mathcal{P}_d' = \{ \tau \in H : a\tau^2 + b\tau + c = 0 \text{ for some } [a, b, c] \in \mathbb{Q}_d \text{ with } 2 \nmid a \}.$$  

The orbit sets $\Gamma_0(2)\backslash \mathcal{P}_d'$ and $\Gamma_0(2)\backslash \mathcal{P}_d'$ have cardinality $2h$ and $h$ respectively.

For $f = \omega$ or $\omega_2$, the set $\{ \omega(\tau) : \tau \in \Gamma_0(2)\backslash \mathcal{P}_d' \}$ is a complete set of conjugates of $f(-\frac{1+i\sqrt{3}}{2})$ over $\mathbb{Q}$.

**Proof.** First we study the action of $\Gamma_0(2)$ on $\mathcal{P}_d$. Let $\tau \in H$ satisfy the quadratic equation $aX^2 + bX + c = 0$. For $(x, y, w) \in \text{PSL}_2(\mathbb{Z})$, the element $(x, y, w)$ $\tau$ satisfies the equation $a'X^2 + b'X + c' = 0$ with

$$a' = aw^2 - bwz + cz^2$$

$$b' = b - 2(awy - byz + cxz)$$

$$c' = ay^2 - byx + cx^2.$$  

We conclude that the greatest common divisor of $a$ and 2 is invariant under the action of $\Gamma_0(2)$, hence $\Gamma_0(2)$ acts on $\mathcal{P}_d'$ and $\mathcal{P}_d'$. If $a$ is even, equation (13) shows that the congruence class of $b$ modulo 4 is also invariant under the action of $\Gamma_0(2)$. Hence for $k \in \{\pm 1\}$ the group $\Gamma_0(2)$ acts on the subset $\mathcal{P}_{d,k}' = \{ \tau \in \mathcal{P}_d : 2 \mid a_\tau \text{ and } b_\tau \equiv k \text{ mod } 4 \}$ of $\mathcal{P}_d'$. We will use lemma 7 to prove the following three bijections:

$$\Gamma_0(2)\backslash \mathcal{P}_d' \sim \text{PSL}_2(\mathbb{Z})\backslash \mathcal{P}_d'$$

$$\Gamma_0(2)\backslash \mathcal{P}_{d,k}' \sim \text{PSL}_2(\mathbb{Z})\backslash \mathcal{P}_d' \quad \text{for } k \in \{\pm 1\}.$$  

In particular $\Gamma_0(2)\backslash \mathcal{P}_d'$ has cardinality $h$ and, because $\mathcal{P}_d' = \mathcal{P}_{d,-1}' \cup \mathcal{P}_{d,1}'$, the set $\Gamma_0(2)\backslash \mathcal{P}_d'$ has cardinality $2h$. The set $I = \{(1 \ 0 \ 0 \ 1), (0 \ -1 \ 1 \ 0), (1 \ 0 \ 1 \ 1)\}$ is a complete set of left coset representatives of $\Gamma_0(2)$ in $\text{PSL}_2(\mathbb{Z})$. As $d = b^2 - 4ac$ is congruent to 1 modulo 8, we find for $\tau \in \mathcal{P}_d$ that $2|a_\tau c_\tau$ and $2 \nmid b_\tau$. In particular we have $\mathcal{P}_{d,2}' = \{ \tau \in \mathcal{P}_d : 2 \mid a_\tau, 2 \mid c_\tau \}$. An easy calculation using (12) and (14) shows the equalities $\mathcal{P}_{d,2}' = \{ \tau \in \mathcal{P}_d : 2 \mid a_\tau, 2 \mid c_\tau \}$ and $\mathcal{P}_{d,2}' = \{ \tau \in \mathcal{P}_d : 2 \mid a_\tau, 2 \mid c_\tau \}$. Hence $\mathcal{P}_d$ is the disjoint union of $\{ M\mathcal{P}_{d,2}' : M \in I \}$ and (15) follows from lemma 7. In a similar way, the bijections (16) follow from the descriptions

$$\mathcal{P}_{d,k}' = \{ \tau \in \mathcal{P}_d : 2 \mid a_\tau \text{ and } b_\tau \equiv k \text{ mod } 4 \},$$

$$\mathcal{P}_{d,k}' = \{ \tau \in \mathcal{P}_d : 2 \mid c_\tau \text{ and } b_\tau \equiv k \text{ mod } 4 \},$$

$$\mathcal{P}_{d,k}' = \{ \tau \in \mathcal{P}_d : (2 \mid a_\tau \text{ and } b_\tau \equiv k \text{ mod } 4) \text{ or } (2 \nmid c_\tau \text{ and } b_\tau \equiv k \text{ mod } 4) \}.$$  

As $d$ is congruent to 1 modulo 8 the Hilbert class field of $K$ coincides with the ray class field of conductor 2 of $K$. Using (2) we find that each of the numbers $f(-\frac{1+i\sqrt{3}}{2})$
with \( f \in \{ \omega, \omega_1, \omega_2 \} \) is an algebraic integer and generates the same field over \( K \) as \( j(-1+\sqrt{d}) \). In particular they are of degree \( h \) over \( K \).

It follows from the \( q \)-expansion of \( \omega_2 \) that \( \omega_2(-1+\sqrt{d}) \) is real, hence of degree \( h \) over \( \mathbb{Q} \). Therefore its conjugates over \( K \) coincide with those over \( \mathbb{Q} \) and we can use (4) and (5) to compute them. Let \([a, -b, c]\) be a quadratic form of discriminant \( d \equiv 1 \) mod 8 with \( a \) odd. We find

\[
\omega_2\left(-\frac{1+\sqrt{d}}{2}\right)_{[a,-b,c]} = \omega_2^{1 - (b-1)/2} \left(\frac{-b+\sqrt{d}}{2a}\right) = \omega_2\left(-\frac{b+\sqrt{d}}{2a}\right),
\]

where the last equality follows from the fact that \( \omega_2 \) is invariant under \( \Gamma_0(2) \). As \( -\frac{b+\sqrt{d}}{2a} \in \mathcal{P}_d^{\omega_2} \) and \( \Gamma_0(2) \mathcal{P}_d^{\omega_2} \) has cardinality \( h \), we conclude that \( \{\omega_2(\tau) : \tau \in \Gamma_0(2) \mathcal{P}_d^{\omega_2}\} \) is a complete set of conjugates of \( \omega_2\left(-\frac{1+\sqrt{d}}{2}\right) \) over \( \mathbb{Q} \).

The conjugates of \( \omega\left(-\frac{1+\sqrt{d}}{2}\right) \) and \( \omega_1\left(-\frac{1+\sqrt{d}}{2}\right) \) over \( K \) are computed similarly using the transformations (11). For \([a, -b, c] \in \mathbb{Q}_d \) with \( a \) odd we find

\[
\omega\left(-\frac{1+\sqrt{d}}{2}\right)_{[a,-b,c]} = \begin{cases} 
\omega_2\left(\frac{b-2a+\sqrt{d}}{2(c-b+a)}\right) & \text{if } b \equiv 1 \text{ mod } 4 \\
\omega_2\left(\frac{b+\sqrt{d}}{2c}\right) & \text{if } b \equiv 3 \text{ mod } 4 
\end{cases}
\]

\[
\omega_1\left(-\frac{1+\sqrt{d}}{2}\right)_{[a,-b,c]} = \begin{cases} 
\omega_2\left(\frac{b+\sqrt{d}}{2c}\right) & \text{if } b \equiv 1 \text{ mod } 4 \\
\omega_2\left(\frac{b-2a+\sqrt{d}}{2(c-b+a)}\right) & \text{if } b \equiv 3 \text{ mod } 4.
\end{cases}
\]

Hence \( \{\omega_2(\tau) : \tau \in \Gamma_0(2) \mathcal{P}_d^{\omega_2}\} \) is a complete set of conjugates of \( \omega\left(-\frac{1+\sqrt{d}}{2}\right) \) over \( K \) and \( \{\omega_2(\tau) : \tau \in \Gamma_0(2) \mathcal{P}_d^{\omega_1}\} \) is a complete set of conjugates of \( \omega_1\left(-\frac{1+\sqrt{d}}{2}\right) \) over \( K \).

Recall that \( \omega_2 \) maps \( \Gamma_0(2) \mathbb{H} \) bijectively to \( \mathbb{P}^1(\mathbb{C}) \). In order words for \( \tau, \tau' \in \mathbb{H} \) the equation \( \omega_2(\tau) = \omega_2(\tau') \) holds if and only if \( \tau = \gamma \tau' \) for some \( \gamma \in \Gamma_0(2) \). Therefore \( \omega\left(-\frac{1+\sqrt{d}}{2}\right) \) and \( \omega_1\left(-\frac{1+\sqrt{d}}{2}\right) \) are not conjugate over \( K \). From the \( q \)-expansion we see that \( \omega\left(-\frac{1+\sqrt{d}}{2}\right) \) and \( \omega_1\left(-\frac{1+\sqrt{d}}{2}\right) \) are complex conjugates and hence conjugate over \( \mathbb{Q} \). We conclude that \( \{\omega_2(\tau) : \tau \in \Gamma_0(2) \mathcal{P}_d^{\omega_2}\} \) is a complete set of \( 2h \) conjugates over \( \mathbb{Q} \) of both \( \omega\left(-\frac{1+\sqrt{d}}{2}\right) \) and \( \omega_1\left(-\frac{1+\sqrt{d}}{2}\right) \). \( \square \)

Similar to (10) we find for \( f \in \{ \omega, \omega_1, \omega_2 \} \) the following corollary which we will use in the next section:

\[
\log N\left(f\left(-\frac{1+\sqrt{d}}{2}\right), f\left(-\frac{1+\sqrt{d}}{2}\right)\right) = \sum_{i=1,2, \tau_i \in \Gamma_0(2) \mathcal{P}_d^{f}} \log |\omega_2(\tau_1) - \omega_2(\tau_2)|
\]

\[
= \sum_{i=1,2, \tau_i \in \Gamma_0(2) \mathcal{P}_d^{f}} \log \left|\frac{\omega_2^{12}}{\omega_2(\tau_1)^{12}} - \frac{\omega_2^{12}}{\omega_2(\tau_2)^{12}}\right| - \begin{cases} 
12h_1h_2 \log 2 & \text{if } f = \omega_2 \\
0 & \text{otherwise}.
\end{cases}
\]

To obtain the last line we used that the absolute value of the norm of \( \omega\left(-\frac{1+\sqrt{d}}{2}\right) \) and \( \omega_2\left(-\frac{1+\sqrt{d}}{2}\right) \) is equal to \( 2^{12h_1} \) and 1 respectively [11, chapter 12 §2].

3. A limit formula for differences of Hauptmodul values

To describe the results of this section, we concentrate for the moment on the function \( \gamma_2 \); similar remarks hold for the other functions. The main result for \( \gamma_2 \) is
be the set of cusps of $\Gamma$. Recall that the cusps of $\Gamma$ are those bounded finite sum over matrices $\gamma \in \Gamma^3$. The second step is more algebraic. We substitute $z_i = \alpha_i$ ($i = 1, 2$) and sum over the conjugates of $\gamma_2(\alpha_1)$ and $\gamma_2(\alpha_2)$. The description of these conjugates in section 2 enables us to transform the infinite sum over matrices into an ‘easier’ infinite sum over integers.

The group of complex analytic automorphisms of $\mathbf{H}$ is isomorphic to $\text{PSL}_2(\mathbb{R})$. Its elements, which we represent by matrices, act on $\mathbf{H}$ by linear fractional transformations. Although we are mainly interested in the subgroups $\Gamma^3$ and $\Gamma_0(2)$ of $\text{PSL}_2(\mathbb{R})$, our setup will be more general. Let $\Gamma \subset \text{PSL}_2(\mathbb{R})$ be a discrete subgroup and let $C_\Gamma$ be the set of cusps of $\Gamma$. Recall that the cusps of $\Gamma$ are those $z \in \mathbb{R} \cup \{\infty\}$ that are the unique fixed point of some $\gamma \in \Gamma$. The action of $\Gamma$ on $\mathbf{H}$ extends to an action on $\overline{\mathbf{H}} = \mathbf{H} \cup C_\Gamma$ and we endow $\Gamma \backslash \overline{\mathbf{H}}$ with the usual Riemann surface structure [12, §1.8]. The $\text{PSL}_2(\mathbb{R})$-invariant measure $\frac{dx\,dy}{y^2}$ on $\mathbf{H}$ gives rise to a measure $\mu$ on $\Gamma \backslash \overline{\mathbf{H}}$. As an example we consider the group $\Gamma = \text{PSL}_2(\mathbb{Z})$. The cusps are the elements of $\mathbb{Q} \cup \{\infty\}$ and $\text{PSL}_2(\mathbb{Z})$ acts transitively on this set. Moreover $\text{PSL}_2(\mathbb{Z}) \backslash \overline{\mathbf{H}}$ has genus zero and $\mu$-measure $\frac{\pi}{3}$ [12, §4.1].

We make the following three assumptions on the group $\Gamma$. First of all we assume that $\Gamma \backslash \overline{\mathbf{H}}$ is compact. By a theorem of Siegel [12, page 32] this is equivalent to the assumption that $\Gamma \backslash \overline{\mathbf{H}}$ has finite volume. This implies in particular that $\Gamma$ is finitely generated [10, page 41] and that the set $C_\Gamma$ has finitely many $\Gamma$-orbits [12, page 27]. Furthermore we want $\Gamma$ to have at least one cusp. To fix notation we take $\infty \in C_\Gamma$. Finally we suppose that the Riemann surface $\Gamma \backslash \overline{\mathbf{H}}$ has genus zero. There are infinitely many conjugacy classes of groups satisfying these three conditions. In this paper we will focus on the groups $\text{PSL}_2(\mathbb{Z})$, $\Gamma^3$ and $\Gamma_0(2)$. In the previous paragraph we noted that $\text{PSL}_2(\mathbb{Z})$ does indeed satisfy the three conditions. The other two groups are of finite index in $\text{PSL}_2(\mathbb{Z})$ and therefore have finite co-volume and $\infty$ as a cusp. The fact that $\Gamma^3 \backslash \overline{\mathbf{H}}$ and $\Gamma_0(2) \backslash \overline{\mathbf{H}}$ are of genus zero was already established in section 2.

As $\Gamma \backslash \overline{\mathbf{H}}$ has genus zero by assumption, its function field is generated by one element. To single out a special kind of generator we first recall that the stabilizer $\Gamma_\infty = \{\gamma \in \Gamma : \gamma \infty = \infty\}$ of the cusp $\infty$ is an infinite cyclic group generated by $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ for some positive $h$. The scaling transformation $\sigma_\infty = \begin{pmatrix} \sqrt{h} & 0 \\ 0 & \frac{1}{\sqrt{h}} \end{pmatrix}$ fixes $\infty$ and $\sigma_\infty^{-1} \Gamma_\infty \sigma_\infty$ is generated by the translation $z \mapsto z+1$. Hence for any generator $f$ of the above function field the meromorphic function $f(\sigma_\infty z)$ has a Laurent expansion in $q = e^{2\pi i z}$. Such a generator $f$ is called a normalised principal modulus (or normalised Hauptmodul) for $\Gamma$ if this expansion has the form

\begin{equation}
(18) \quad f(\sigma_\infty z) = q^{-1} + \sum_{n \geq 0} a_n q^n,
\end{equation}

with $a_n \in \mathbb{C}$. In particular a normalised principal modulus has its pole at $\infty$ and is unique up to an additive constant. We will consider $f$ both as a function on
the Riemann surface $\Gamma \setminus \mathbb{H}$ and as a $\Gamma$-invariant function on $\mathbb{H}$. For example $\gamma_2$ is a normalised principal modulus for the group $\Gamma^3$ and $-2^{12}/\omega_2$ is a normalised principal modulus for $\overline{\Gamma}_0(2)$.

Theorem 10 below states an equality between $\log |f(z_1) - f(z_2)|$ and the limit value of a certain meromorphic function. This equality will follow from the following characterization of the function $\log |f(z_1) - f(z_2)|$.

**Proposition 9.** Let $\Gamma \subset \text{PSL}_2(\mathbb{R})$ be as above and let $f$ be a normalised principal modulus for $\Gamma$. The function

$$h(z_1, z_2) = \log |f(z_1) - f(z_2)|^2$$

is the unique symmetric function on $H \times H - \{(z_1, z_2) : z_1 \not\in \Gamma z_2\}$ that for fixed $z_2 \in H$ satisfies

1. $h(\gamma z_1, z_2) = h(z_1, z_2)$ for all $\gamma \in \Gamma$;
2. $(\frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2})h(z_1, z_2) = 0$, where $x_1 = \text{Re}(z_1)$ and $y_1 = \text{Im}(z_1)$;
3. $h(z_1, z_2) = e_{z_2} \log |z_1 - z_2|^2 + O(1)$ for $z_1 \rightarrow z_2$, where $e_{z_2}$ denotes the order of the stabilizer of $z_2$ in $\Gamma$;
4. $h(\sigma_\infty z_1, z_2) = 4\pi \text{Im}(z_1) + o(1)$ for $z_1 \rightarrow \infty$;
5. $h(z_1, z_2) = O(1)$ for $z_1 \rightarrow a$ with $a \in \mathcal{C}_\Gamma$ not $\Gamma$-equivalent to $\infty$.

**Proof.** The first two properties are obvious. To prove property 3 we use the following equality:

$$f(z_1) - f(z_2) = (z_1 - z_2)^e \cdot \frac{f(z_1) - f(z_2)}{(z_1 - z_2)^e}.$$

with $e = e_{z_2}$. For fixed $z_2 \in H$ the function $f(z_1) - f(z_2)$ has a zero of order $e_{z_2}$ in $z_1 = z_2$. Therefore, the limit for $z_1 \rightarrow z_2$ of the second factor on the right converges to a non-zero value and the third property follows. Property 4 follows by using the Fourier expansion at infinity (18) of the function $f$. For property 5 we use that $f$ is a bijection between $\Gamma \setminus \mathbb{H}$ and the projective line over $\mathbb{C}$. If $z_2 \in H$ is fixed, we find that $f(z_1) \neq f(z_2)$ for $z_1$ close to a cusp $a$. By definition $f$ has its pole at $\infty$, hence for every cusp $a$ not $\Gamma$-equivalent to $\infty$ the value $|f(z_1) - f(z_2)|$ is nonzero and bounded from above for $z_1 \rightarrow a$ and property 5 follows.

To prove uniqueness, we let $k(z_1, z_2)$ be the difference of two functions on $H \times H - \{(z_1, z_2) : z_1 \not\in \Gamma z_2\}$ that satisfy the above five properties. Fix $z_2 \in H$ and consider $k(z_1, z_2)$ as a function on $H - \{z \not\in \Gamma z_2\}$. On this domain the function is $\Gamma$-invariant and harmonic by the first two properties. According to 3,4 and 5 the function $k(z_1, z_2)$ is bounded if $z_1$ approaches an element of $\Gamma z_2$ or a cusp of $\Gamma$. Hence $k(z_1, z_2)$ extends to a harmonic function on the Riemann surface $\Gamma \setminus \mathbb{H}$. As $\Gamma \setminus \mathbb{H}$ is compact by assumption we find that $k(z_1, z_2)$ is constant. Using the fourth property we conclude that the function $k$ is identically zero, which proves the uniqueness of $h$. \[\square\]

We will now `build` a function from scratch which satisfies the five properties above. To motivate the definitions below, we follow the arguments in [7, II §2]. First we make a bi-$\Gamma$-invariant function $G(z_1, z_2)$ on $H \times H$. Let $g(z_1, z_2)$ be a function on $H \times H$ satisfying $g(\gamma z_1, \gamma z_2) = g(z_1, z_2)$ for all $\gamma \in \text{PSL}(\mathbb{R})$. This is equivalent with $g$
being a function of the hyperbolic distance \(d(z_1, z_2)\) between \(z_1\) and \(z_2\). In particular \(g\) is symmetric. Ignoring convergence, the function \(G(z_1, z_2) = \sum_{\gamma \in \Gamma} g(z_1, \gamma z_2)\) is then clearly bi-\(\Gamma\)-invariant. If we want \(G\) to satisfy properties 2 and 3 it seems natural to select a \(g\) that is harmonic in both variables and satisfies \(g(z_1, z_2) = \log|z_1 - z_2|^2 + O(1)\) for \(z_1 \to z_2\). Unfortunately this leads to convergence problems in the definition of \(G\). To resolve this difficulty we weaken the condition that \(g\) should be harmonic. For a complex variable \(z = x + iy \in \mathbb{H}\), let \(\Delta = \Delta_z = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)\) denote the hyperbolic Laplacian in \(x, y\). Recall that \(\Delta\) is an invariant operator: \(\Delta(f(\gamma z)) = (\Delta f)(\gamma z)\) for all sufficiently smooth functions \(f\) on \(\mathbb{H}\) and \(\gamma \in \text{PSL}(\mathbb{R})\). We choose \(g\) to be ‘almost’ harmonic, in the sense that \(g\) satisfies \(\Delta g = \epsilon g\) for some small \(\epsilon > 0\), and with a logarithmic singularity along the diagonal as above.

Note that as \(g\) is symmetric, we can take either \(\Delta = \Delta_{z_1}\) or \(\Delta = \Delta_{z_2}\). Because \(g(z_1, z_2)\) is a function of the hyperbolic distance \(d(z_1, z_2)\), we can write \(g(z_1, z_2) = Q(1 + \frac{|z_1 - z_2|^2}{2 \text{Im}(z_1) \text{Im}(z_2)})\) for some function \(Q\), the argument of \(Q\) being \(\cosh(d(z_1, z_2))\). The partial differential equation \(\Delta g = \epsilon g\) translates into the ordinary differential equation

\[(1 - t^2) \frac{d^2}{dt^2} - 2t \frac{d}{dt} + \epsilon)Q(t) = 0\]

for the function \(Q\). This is the Legendre differential equation of index \(s - 1\), where \(\epsilon = s(s - 1)\) with \(s > 1\). Up to a scalar it has a unique solution that is small at infinity, the Legendre function of the second kind \(Q_{s-1}(t)\). This function, real analytic in \(t \in \mathbb{R}_{>1}\) and holomorphic in \(s \in \mathbb{H}_1 = \{s \in \mathbb{C} : \text{Re}(s) > 1\}\), is given by

\[
Q_{s-1}(t) = \frac{\Gamma(s)^2}{2 \Gamma(2s)} \left( \frac{2}{1 + t} \right)^s F(s, s; 2s; \frac{2}{1 + t}),
\]

where \(F(a, b; c; z)\) is Gauss’s hypergeometric function and \(\Gamma(s)\) is the gamma function (in the sequel it will be clear from the context whether \(\Gamma\) denotes a group or the gamma function). The Legendre function of the second kind has the following asymptotic behaviour [10, lemma 1.7]

\[
Q_{s-1}(t) = -\frac{1}{2} \log(t - 1) + O(1) \quad \text{for} \quad t \downarrow 1,\]

\[
Q_{s-1}(t) = O(t^{-s}) \quad \text{for} \quad t \to \infty.
\]

The above discussion leads to the following definitions. For \((z_1, z_2, s) \in \mathbb{H} \times \mathbb{H} \times \mathbb{H}_1\) with \(z_1 \neq z_2\) the function

\[
g_s(z_1, z_2) = -2Q_{s-1}(1 + \frac{|z_1 - z_2|^2}{2 \text{Im}(z_1) \text{Im}(z_2)})
\]

satisfies

\[
g_s(\gamma z_1, \gamma z_2) = g_s(z_1, z_2) \quad \text{for all} \quad \gamma \in \text{PSL}(\mathbb{R})
\]

\[
\Delta_{z_i}g_s(z_1, z_2) = s(s - 1)g_s(z_1, z_2) \quad (i = 1, 2).
\]

For \((z_1, z_2, s) \in \mathbb{H} \times \mathbb{H} \times \mathbb{H}_1\) with \(z_1 \notin \Gamma z_2\) we define the automorphic Green function, or resolvent kernel function, for \(\Gamma\) by

\[
G_{\Gamma, s}(z_1, z_2) = \sum_{\gamma \in \Gamma} g_s(\gamma z_1, z_2),
\]
which is $4\pi$ times the function studied in [9] and [10]. Using (21) one shows that this sum is uniformly and absolutely convergent on compact subsets of its domain of definition [9, page 31]. Consequently $G_{\Gamma, s}(z_1, z_2)$ is holomorphic as a function of $s \in \mathbb{H}_1$ and, because of (20) and (23), satisfies property 1 and 3 of proposition 9:

\begin{align}
(26) & \quad G_{\Gamma, s}(\gamma z_1, z_2) = G_{\Gamma, s}(z_1, z_2) \quad \text{for } \gamma \in \Gamma \\
(27) & \quad G_{\Gamma, s}(z_1, z_2) = e_{z_2} \log |z_1 - z_2|^2 + O(1) \quad \text{for fixed } z_2 \in \mathbb{H} \text{ and } z_1 \to z_2
\end{align}

Using that (21) is also valid for derivatives of every order of $Q_{s-1}(t)$ one can show that $G_{\Gamma, s}(z_1, z_2)$ is infinitely differentiable as a function of the real variables $x_1, y_1, x_2$ and $y_2$, provided that $z_1$ and $z_2$ are not equivalent modulo $\Gamma$. Furthermore, by equation (24) and the fact that the Laplacian is an invariant operator we have

\begin{equation}
\Delta_z G_{\Gamma, s}(z_1, z_2) = s(s-1)G_{\Gamma, s}(z_1, z_2) \quad (i = 1, 2).
\end{equation}

By taking the limit $s \to 1$ of $G_{\Gamma, s}(z_1, z_2)$ one might hope to obtain a harmonic function. Unfortunately this limit does not exist. The function $G_{\Gamma, s}(z_1, z_2)$ does have a meromorphic continuation to the entire $s$-plane which satisfies (26), (27) and (28). At $s = 1$ it has a simple pole with residue $-4\pi \mu(\Gamma \setminus \mathbb{H})^{-1}$ independent of $z_1$ and $z_2$ [9, chapter 8.6; 10, chapter 7.4]. Therefore we first subtract the ‘singular part of $G_{\Gamma, s}(z_1, z_2)$ at $s = 1′$ and then take the limit $s \to 1$. As we will see below this results in a function which is not only harmonic but satisfies all five properties of proposition 9.

To obtain this ‘singular part’ we fix $(z_2, s) \in \mathbb{H} \times \mathbb{H}_1$, and consider $G_{\Gamma, s}(z_1, z_2)$ as an infinitely differentiable function of $x_1$ and $y_1$ with $y_1 > \sup \{\Im(\gamma z_2) : \gamma \in \Gamma\}$. The function $G_{\Gamma, s}(\sigma_\infty z_1, z_2)$ is invariant under $x_1 \mapsto x_1 + 1$, hence it has a Fourier expansion in $e^{2\pi i x_1}$. One can show [9, (3.3) on page 274] that its zeroth Fourier coefficient is equal to $\frac{4\pi}{1-2s} E_\Gamma(\sigma_\infty z_2, s) y_1^{1-s}$, with

\begin{equation}
E_\Gamma(z, s) = \sum_{\gamma \in \Gamma \setminus \Gamma_\infty} \Im(\sigma^{-1}_\infty \gamma z)^s
\end{equation}

the Eisenstein series for $\Gamma$ and the cusp $\infty$. For the modular group $\text{PSL}_2(\mathbb{Z})$ this function can be written in the more familiar form

\[ E_{\text{PSL}_2}(z, s) = \frac{1}{2} \sum_{(c, d) = 1} y^{s} \frac{y^s}{|cz + d|^{2s}}. \]

The sum (29) converges absolutely and uniformly on compact subsets of $\mathbb{H} \times \mathbb{H}_1$, is real analytic as a function of $z \in \mathbb{H}$ and is holomorphic as a function of $s \in \mathbb{H}_1$. In fact one can prove that $E_\Gamma(z, s)$ admits a meromorphic continuation to the whole $s$-plane with a simple pole at $s = 1$ with residue $\mu(\Gamma \setminus \mathbb{H})^{-1}$ independent of $z$ [10, chapter 6]. At regular points $s \in \mathbb{C}$ the Eisenstein series satisfies the following properties:

\begin{align}
(30) & \quad E_\Gamma(\gamma z, s) = E_\Gamma(z, s) \quad \text{for all } \gamma \in \Gamma \\
(31) & \quad \Delta E_\Gamma(z, s) = s(s-1)E_\Gamma(z, s).
\end{align}
Apart from the zeroth one the Fourier coefficients of \( G_{\Gamma,s}(\sigma_\infty z_1, z_2) \) are all regular at \( s = 1 \) [9, chapter 8.6]. For fixed \( z_2 \in \mathbf{H} \) and \( y_1 \to \infty \) we have

\[
\lim_{s \to 1} \left[ G_{\Gamma,s}(\sigma_\infty z_1, z_2) - \frac{4\pi}{1-2s} E_{\Gamma}(z_2, s)y_1^{1-s} \right] = O(e^{-2\pi y_1}),
\]

which will be used in the proof of theorem 11 below.

If we consider \( E_{\Gamma}(\sigma_\infty z, s) \) as function of \( z = x + iy \in \mathbf{H} \), it is invariant under the translation \( x \mapsto x + 1 \). Its zeroth Fourier coefficient is equal to \( y^s + \phi_{\Gamma}(s)y^{1-s} \) [10, page 66] with

\[
\phi_{\Gamma}(s) = \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \sum_{c>0} c^{-2s} \# \{ d \mod c : \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_\infty^{-1}\Gamma\sigma_\infty \}
\]

for \( s \in \mathbf{H}_1 \). For the group \( \text{PSL}_2(\mathbf{Z}) \) the above sum is over positive integers \( c \) and \( \sigma_\infty \) is the identity transformation. For \( c \in \mathbf{Z}_{>0} \) the cardinality of the set in (33) is equal to the number of positive integers less than and coprime to \( c \). We obtain

\[
\phi_{\text{PSL}_2(\mathbf{Z})}(s) = \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \frac{\zeta(2s - 1)}{\zeta(2s)}
\]

with \( \zeta(s) \) the Riemann zeta function.

The function \( \phi_{\Gamma}(s) \) has a meromorphic continuation to \( \mathbf{C} \) with the same behaviour at \( s = 1 \) as \( E_{\Gamma}(\sigma_\infty z, s) \). More precisely, for \( y \to \infty \)

\[
\lim_{s \to 1} \left[ E_{\Gamma}(\sigma_\infty z, s) - (y^s + \phi_{\Gamma}(s)y^{1-s}) \right] = O(e^{-2\pi y}).
\]

The following result is a slight generalization of proposition 5.1 in [5]. For a normalised principal modulus \( f \) and \( z_1, z_2 \in \mathbf{H} \) it gives a formula for the archimedean local height of the degree zero divisors \( (f(z_1)) - (\infty) \) and \( (f(z_2)) - (\infty) \) on \( \mathbf{P}^1 \) [4; 6, IV §3; 7, II §2].

**Theorem 10.** Let \( \Gamma \subset \text{PSL}_2(\mathbf{R}) \) be a discrete subgroup for which \( \infty \) is a cusp and for which the Riemann surface \( \Gamma \setminus \mathbf{H} \) is compact of genus zero. Furthermore let \( f \) be a normalised principal modulus for \( \Gamma \). For \( z_1, z_2 \in \mathbf{H} \) with \( z_1 \not\in \Gamma \cdot z_2 \) we have the following equality:

\[
\log |f(z_1) - f(z_2)|^2 = \lim_{s \to 1} \left[ G_{\Gamma,s}(z_1, z_2) - \frac{4\pi}{1-2s}(E_{\Gamma}(z_1, s) + E_{\Gamma}(z_2, s) - \phi_{\Gamma}(s)) \right]
\]

**Proof.** It follows from the above discussion that the function between square brackets on the right has at most a simple pole at \( s = 1 \). As the residues add up to zero, we conclude that the limit exists.

Let \( h(z_1, z_2) \) denote the function on the right of the equality sign above. We show that this symmetric function satisfies the five properties of proposition 9, from which the theorem follows. Fix \( z_2 \in \mathbf{H} \) and consider \( h(z_1, z_2) \) as a function of \( z_1 \). The fact that \( h(z_1, z_2) \) is \( \Gamma \)-invariant is obvious from (26) and (30). For \( s \) in some punctured neighbourhood of \( s = 1 \)

\[
G_{\Gamma,s}(z_1, z_2) - \frac{4\pi}{1-2s}E_{\Gamma}(z_1, s)
\]
is a $\Delta_{z_1}$-eigenfunction with eigenvalue $s(s-1)$, because of (28) and (31). In particular its limit $s \to 1$ is harmonic. The remaining terms of $h$ are independent of $z_1$, so we conclude that $h$ is harmonic. Using (27), we find
\[ h(z_1, z_2) = e_{z_2} \log |z_1 - z_2|^2 + O(1) \quad \text{for } z_1 \to z_2, \]
with $e_{z_2}$ the order of the stabilizer of $z_2$ in $\Gamma$.

To study $h(z_1, z_2)$ at the cusp $\infty$ we use the equations (32) and (34) (with $z = z_1$), which are equivalent to
\[ \lim_{s \to 1} \left[ G_{\Gamma, s}(\sigma_{\infty} z_1, z_2) - \frac{4\pi}{1-2s} E_{\Gamma}(z_2, s) \right] = \frac{4\pi}{\mu(\Gamma \backslash \mathbb{H})} \log y_1 + O(e^{-2\pi y_1}) \]
and
\[ \lim_{s \to 1} \frac{4\pi}{1-2s} \left[ E_{\Gamma}(\sigma_{\infty} z_1, s) - \phi_{\Gamma}(s) \right] = -4\pi y_1 + \frac{4\pi}{\mu(\Gamma \backslash \mathbb{H})} \log y_1 + O(e^{-2\pi y_1}), \]
respectively. By subtracting the last equation from the first we conclude that $h(z_1, z_2)$ satisfies the fourth property of proposition 9.

In order to examine $h(z_1, z_2)$ for $z_1$ near a finite cusp, we need to generalize the estimates (32) and (34) to all cusps. Fix a cusp $a \in \mathcal{C}_\Gamma$. Choose a scaling transformation $\sigma_a \in \text{PSL}_2(\mathbb{R})$ such that
\[ \sigma_{a\infty} = a \quad \text{and} \quad \sigma_a^{-1} \Gamma a \sigma_a = \langle \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \rangle. \]
This transformation is unique up to composition on the right with a translation. We need to show that the limit of $h(\sigma_a z_1, z_2)$ for $y_1 \to \infty$ exists if $a$ is not $\Gamma$-equivalent to $\infty$. By comparing $G_{\Gamma, s}(\sigma_a z_1, z_2)$ with its zeroth Fourier coefficient we find for $y_1 \to \infty$
\[ \lim_{s \to 1} \left[ G_{\Gamma, s}(\sigma_a z_1, z_2) - \frac{4\pi}{1-2s} E_{\Gamma, a}(z_2, s) y_1^{1-s} \right] = O(e^{-2\pi y_1}). \]
The Eisenstein series $E_{\Gamma, a}(z, s)$ for $\Gamma$ and the cusp $a$ is for $s \in \mathbb{H}_1$ defined by
\[ E_{\Gamma, a}(z, s) = \sum_{\gamma \in \Gamma} \text{Im}(\sigma_a^{-1} \gamma z)^s. \]
As in the case $a = \infty$ (see (29)) this function has a meromorphic continuation to the whole $s$-plane with a simple pole at $s = 1$ with residue $\mu(\Gamma \backslash \mathbb{H})^{-1}$.

The Fourier expansion of $E_{\Gamma}(\sigma_a z_1, s) = E_{\Gamma, \infty}(\sigma_a z_1, s)$ leads for $y_1 \to \infty$ to
\[ \lim_{s \to 1} \left[ E_{\Gamma}(\sigma_a z_1, s) - (\delta_{a\infty} y_1^s + \phi_{\Gamma, a}(s)y_1^{1-s}) \right] = O(e^{-2\pi y_1}). \]
Here $\delta_{a\infty}$ is 1 if $a$ and $\infty$ are $\Gamma$-equivalent, and 0 otherwise. Furthermore $\phi_{\Gamma, a}(s)$ is a meromorphic function in $s$ with a simple pole at $s = 1$ with the same residue as the Eisenstein series. Note that $\phi_{\Gamma, \infty}(s) = \phi_{\Gamma}(s)$ is defined by (33). Using (35) and (36) and the fact that both $E_{\Gamma, a}(z, s) - E_\Gamma(z, s)$ and $\phi_{\Gamma, a}(s) - \phi_\Gamma(s)$ are bounded for $s \to 1$, we can compute the limit of $h(\sigma_a z_1, z_2)$ for $y_1 \to \infty$. This limit is finite if $a$ is not $\Gamma$-equivalent to $\infty$ because of the $\delta_{a\infty}$ in (36).

Let $d_1$ and $d_2$ be negative fundamental discriminants that are relatively prime and both congruent to 2 modulo 3. For $i = 1, 2$ let $\alpha_i \in \mathbb{H}$ be of discriminant $d_i$ and
normalised as in theorem 4. We will apply the previous theorem for the group \( \Gamma^3 \) and its normalised principal modulus \( \gamma_2 \) together with (10) to obtain a formula for \( \log N(\gamma_2(\alpha_1), \gamma_2(\alpha_2)) \). Therefore we sum each of the four terms on the right hand side of the equation in theorem 10 over \( z_i \in \Gamma^3 \setminus \mathcal{P}^\gamma_{d_i}, i = 1, 2 \).

We start with the automorphic Green function. Replace the summation variable \( \gamma \in \Gamma^3 \) in the definition (25) of \( G_{\Gamma^3,s}(z_1, z_2) \) by \( \kappa_1^{-1} \kappa_2 \) with \( \kappa_1, \kappa_2 \in \Gamma^3 \). The elements \( \kappa_1, \kappa_2 \) are well defined up to right multiplication by elements of the stabilisers \( \Gamma^3_{z_1}, \Gamma^3_{z_2} \) and up to simultaneous left multiplication by an element of \( \Gamma^3 \). Using that the stabilisers \( \Gamma^3_{z_i} \) for \( z_i \in \mathcal{P}^\gamma_{d_i} \) have order \( \frac{w_i}{2} \) and the invariance (23) of the function \( g_s(z_1, z_2) \) we find similarly to [5, page 209] the following:

\[
\frac{4}{w_1 w_2} \sum_{i=1, 2, \tau_i \in \Gamma^3 \setminus \mathcal{P}^\gamma_{d_i}} G_{\Gamma^3,s}(\tau_1, \tau_2) = \sum_{(\tau_1, \tau_2) \in \Gamma^3 \setminus \mathcal{P}^\gamma_{d_1} \times \mathcal{P}^\gamma_{d_2}} g_s(\tau_1, \tau_2),
\]

where \( \Gamma^3 \) acts diagonally on \( \mathcal{P}^\gamma_{d_1} \times \mathcal{P}^\gamma_{d_2} \). By definition (22) we have for \( \tau_i = \frac{-b_i + \sqrt{d_i}}{2a_i} \) the equality

\[
g_s(\tau_1, \tau_2) = -2Q_{s-1}(\frac{B(\tau_1, \tau_2)}{\sqrt{D}})
\]

with \( D = d_1 d_2 \) and where

\[
B(\tau_1, \tau_2) = 2a_1 c_2 + 2a_2 c_1 - b_1 b_2
\]

is an integer which is larger than \( \sqrt{D} \) by (22). Fix an integer \( n > \sqrt{D} \). Because of the absolute convergence of (37), the number of \( (\tau_1, \tau_2) \) on the right hand side of (37) for which the argument of \( Q_{s-1} \) equals \( \frac{n}{\sqrt{D}} \) is finite. We obtain

\[
\frac{4}{w_1 w_2} \sum_{i=1, 2, \tau_i \in \Gamma^3 \setminus \mathcal{P}^\gamma_{d_i}} G_{\Gamma^3,s}(\tau_1, \tau_2) = -2 \sum_{n > \sqrt{D}} \rho^\gamma(n) Q_{s-1}(\frac{n}{\sqrt{D}})
\]

where \( \rho^\gamma(n) \) is the cardinality of the set

\[
S^\gamma_{d_1, d_2, n} = \Gamma^3 \setminus \{(\tau_1, \tau_2) \in \mathcal{P}^\gamma_{d_1} \times \mathcal{P}^\gamma_{d_2} : B(\tau_1, \tau_2) = n\}.
\]

For integers \( n \) with \( n^2 \neq D \mod 36 \) this set is empty. Namely for \( \tau_i = \frac{-b_i + \sqrt{d_i}}{2a_i} \) we have

\[
B(\tau_1, \tau_2)^2 - D = 4(a_1 c_2 - a_2 c_1)^2 + 4(a_1 b_2 - a_2 b_1)(c_1 b_2 - c_2 b_1).
\]

If \( \tau_i \in \mathcal{P}^\gamma_{d_i} \) then \( b_i \) is divisible by 3 and \( a_i \equiv c_i \mod 3 \), hence (39) is divisible by 36.

To determine \( \rho^\gamma(n) \) for integers \( n \) that satisfy \( n^2 \equiv D \mod 36 \) we use the following proposition from [6].

**Proposition 11.** Let \( N \) be a positive integer, let \( d_1 \) and \( d_2 \) be negative relatively prime integers such that \( d_1 \) and \( d_2 \) are squares modulo \( 4N \) and set \( D = d_1 d_2 \). For positive integers \( n \) define the set

\[
S^N_{d_1, d_2, n} = \Gamma_0(N) \setminus \{(\tau_1, \tau_2) \in \mathcal{P}^N_{d_1} \times \mathcal{P}^N_{d_2} : B(\tau_1, \tau_2) = n\}
\]
with $P_d = \{\tau \in P_d : N | a_\tau\}$ for $i = 1, 2$. The set $S_{d_1, d_2, n}^N$ has cardinality

$$\# S_{d_1, d_2, n}^N = \begin{cases} 2t \sum d | n^2 - D \varepsilon(d) & \text{if } n^2 \equiv D \mod 4N; \\ 0 & \text{otherwise}, \end{cases}$$

where $t$ is the number of primes dividing $N$ and $\varepsilon$ is defined as in the introduction.

**Proof.** The case $N = 1$ is proved in proposition 6.1 of [5] using class field theory. The general case is proved using the theory of quaternion algebras on page 516 of [6].

We return to the situation before proposition (39), i.e. $d_1$ and $d_2$ are coprime negative fundamental discriminants both congruent to 2 modulo 3 and $n$ is an integer satisfying $n^2 \equiv D \mod 36$. The inclusions $P_{d_i} \subset P_d$ for $i = 1, 2$ induce a well-defined map

$$(40) \quad S_{d_1, d_2, n}^{\gamma_2} \longrightarrow S_{d_1, d_2, n},$$

where $S_{d_1, d_2, n}^{\gamma_2} = S_{d_1, d_2, n}^1$ is defined as in proposition (39) above. To prove that this map is a bijection we apply lemma 7 to the $\Gamma^3$-set $Y = \{(\tau_1, \tau_2) \in P_{d_1} \times P_{d_2} : B(\tau_1, \tau_2) = n\}$ and the $\text{PSL}_2(\mathbb{Z})$-set $X = \{(\tau_1, \tau_2) \in P_{d_1} \times P_{d_2} : B(\tau_1, \tau_2) = n\}$. Let $I = \{T^k : k = 0, 1, 2\}$ be a complete set of left coset representatives of $\Gamma^3$ in $\text{PSL}_2(\mathbb{Z})$. As we saw in the proof of proposition 6, the translation $T$ changes the residue class of $b_{\tau_1}$ modulo 3. Therefore the sets $MY$ with $M \in I$ are disjoint, and for each $(\tau_1, \tau_2) \in X$ there exists $M \in I$ such that $b_{M^2\tau_1}$ is divisible by 3. Using the assumption on $n$ and the fact that $d_1$ and $d_2$ are congruent to 2 modulo 3 we find by (39) that $b_{M^2\tau_2}$ is also divisible by 3. Hence $M(\tau_1, \tau_2) \in Y$ and $X$ is the disjoint union of $MY$ with $M \in I$. It now follows from lemma 7 that (40) is a bijection. Proposition 11 yields the equality $\rho^{\gamma_2}(n) = \sum d \varepsilon(d)$ where $d$ ranges over the positive divisors of $\frac{1}{4}(n^2 - D)$. If we sort these divisors by their 3-adic valuation and use the equality $\varepsilon(3) = -1$ we obtain

$$(41) \quad \rho^{\gamma_2}(n) = \begin{cases} \sum d | n^2 - D \varepsilon(d) & \text{if } n^2 \equiv D \mod 36; \\ 0 & \text{otherwise}, \end{cases}$$

Next we sum the Eisenstein series $E_{\Gamma^3}(z, s)$ over $z \in \Gamma^3 \setminus P_{d_1}^{\gamma_2}$, where $d$ is equal to $d_1$ or $d_2$. The stabilizer $\Gamma_\infty^3$ is generated by $\left( \begin{smallmatrix} 1 & 3 \\ 0 & 1 \end{smallmatrix} \right)$ so the scaling transformation at $\infty$ is equal to $\sigma_\infty = \left( \begin{smallmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{3} \end{smallmatrix} \right)$. With $\Gamma = \text{PSL}_2(\mathbb{Z})$ the inclusion $\Gamma^3 \subset \Gamma$ induces a bijection $\Gamma_\infty^3 \setminus \Gamma_\infty^3 \rightarrow \Gamma_\infty \setminus \Gamma$ by lemma 7 and we find the equality $E_{\Gamma^3}(z, s) = 3^{-s}E_{\Gamma}(z, s)$. The function that we obtain by substituting $z = \tau$ for some $\tau \in P_d$ in the Eisenstein series $E_{\Gamma}(z, s)$ is essentially the partial zeta function for the ideal class of $\mathbb{Q}(\sqrt{d})$ corresponding to $\tau$. Summing over all $\tau \in \Gamma \setminus P_d$ yields the zeta function of $\mathbb{Q}(\sqrt{d})$. More precisely we have the following result [17, proposition 3iii].

**Proposition 12.** Let $\Delta \equiv 0, 1 \mod 4$ be a negative integer and write $\Delta = df^2$ with $d$ the discriminant of $\mathbb{Q}(\sqrt{\Delta})$. With $\Gamma = \text{PSL}_2(\mathbb{Z})$ and $P_\Delta$ as in (7) we have

$$\sum_{\tau \in \Gamma \setminus P_\Delta} E_{\Gamma}(\tau, s) = \frac{w}{2} \left( \begin{smallmatrix} \Delta \\ 4 \end{smallmatrix} \right)^{s/2} \zeta(s) \zeta(2s) L(s, d) \sum_{m \mid f} \mu(m) \left( \frac{d}{m} \right) m^{-s} \sigma_1 - 2s \left( \frac{f}{m} \right),$$
where \( w \) is the number of roots of unity in \( Q(\sqrt{d}) \), \( \zeta(s) \) denotes the Riemann zeta function, \( L(s, d) = \sum_{n=1}^{\infty} \left( \frac{d}{n} \right) n^{-s} \) and \( \sigma_n(n) = \sum_{m|n} m^n \).

If we apply this proposition with fundamental discriminant \( \Delta = d \equiv 2 \mod 3 \) and use the bijection \( \Gamma^3 \setminus \mathcal{P}_d^3 \to \Gamma \setminus \mathcal{P}_d \), which follows from lemma 7, we obtain

\[
E_{\Gamma}(\tau, s) = 3^{-s} \sum_{\tau \in \Gamma \setminus \mathcal{P}_d} E_{\tau}(\tau, s) = 3^{-s} \frac{w}{2} \left( \frac{|d|}{4} \right)^{s/2} \frac{\zeta(s)}{\zeta(2s)} L(s, d).
\]

Finally we have to calculate the function \( \phi_{\Gamma^3}(s) \). Fix a positive real number \( c \) and let \( \sigma_\infty = \left( \frac{\sqrt{3}}{0} \right) \) be the scaling transformation at \( \infty \). According to definition (33), we need to compute the number \( \# \left\{ d \mod c : \left( \frac{\tau}{d} \right) \in \Gamma^3 \right\} \). For this number to be nonzero \( c \) should be an integer multiple of 3. It follows from the description of \( \Gamma^3 \) in section 2 that if \( \frac{\tau}{d} \) and \( d \) are two relatively prime integers then \( \Gamma^3 \) contains an element of the form \( \left( \frac{\tau}{d} \right) \). With \( \varphi \) the Euler \( \varphi \)-function we find

\[
\phi_{\Gamma^3}(s) = \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \sum_{c \in \mathbb{Z}^3_{>0}} 3\varphi\left( \frac{c}{3} \right) c^{-2s} = 3^{1 - 2s} \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \frac{\zeta(2s - 1)}{\zeta(2s)}.
\]

Theorem 13. For \( i = 1, 2 \) let \( d_i \) and \( \alpha_i \) be as in theorem 4, set \( h_i' = w_i h_i \) and define for \( n \in \mathbb{Z}_{>0} \) the integer \( \rho^{\gamma_2}(n) \) by (41). Then the following formula holds

\[
\log N(\gamma_2(\alpha_1), \gamma_2(\alpha_2)) = \frac{1}{\gamma_2} \lim_{s \to 1} \left[ \frac{4h_1'h_2'}{s-1} - 2 \sum_{n>\sqrt{D}} \rho^{\gamma_2}(n) Q_{s-1} \left( \frac{n}{\sqrt{D}} \right) + 4h_1'h_2'C \right]
\]

with \( C = \frac{1}{\gamma_2} \log D + \frac{L'}{L}(1, d_1) + \frac{L'}{L}(1, d_2) - 2 \frac{\gamma'}{\gamma}(2) - 2 \).

Proof. We need the following Laurent expansions at \( s = 1 \):

\[
\zeta(2s)^{-1} = \frac{4}{\pi^2} - \frac{72\varphi'(2)}{4\pi^4} (s - 1) + O((s - 1)^2)
\]

\[
\zeta(s) L(s, d) = \frac{L(1, d)}{s-1} + \gamma L(1, d) + L'(1, d) + O(s-1)
\]

\[
\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} = \sqrt{\pi} - \sqrt{\pi} \log 4 (s - 1) + O((s - 1)^2)
\]

\[
\zeta(2s-1) = \frac{1}{2(s-1)} + \gamma + O(s-1)
\]

where \( \gamma \) denotes Euler’s constant and \( d \) is a negative fundamental quadratic discriminant. For a derivation of the third expansion above we refer to [11, page 272].

According to the analytic class number formula we have \( L(1, d) = \frac{2}{w} \pi h^2 \log |d|^{-\frac{1}{2}} \) with \( h \) and \( w \) the class number and the number of roots of unity of the quadratic fields of discriminant \( d \). The theorem follows from theorem 10, equations (10), (38), (41), (42), (43) and the expansions above. \( \square \)

Next we fix two negative fundamental discriminants \( d_1 \) and \( d_2 \) that are relatively prime and congruent to 1 modulo 8. In particular both \( w_1 \) and \( w_2 \) are equal to 2. We
want to obtain similar results for \( \log N(f(-\frac{1+\sqrt{d}}{2}), f(-\frac{1+\sqrt{d}}{2})) \), where \( f \) denotes one of the functions \( \omega \) or \( \omega_2 \). According to formula (17) we need to apply theorem 10 for the group \( \Gamma_0(2) \) and its normalised principal modulus \(-2^{12}/\omega_2\) and sum over \( z_i \in \Gamma_0(2) \backslash P_{d_i}^f (i = 1, 2) \).

For \( f \in \{\omega, \omega_2\} \) the summation of the Green function results in a formula analogous to (38):

\[
\sum_{i=1,2, \tau_i \in \Gamma_0(2) \backslash P_{d_i}^f} G_{\Gamma_0(2), s}(\tau_1, \tau_2) = -2 \sum_{n > \sqrt{D}} \rho^f(n)Q_{s-1}(\frac{n}{\sqrt{D}})
\]

where \( \rho^f(n) \) is the cardinality of the set

\[
S_{d_1, d_2, n}^f = \Gamma_0(2) \backslash \{(\tau_1, \tau_2) \in P_{d_1}^f \times P_{d_2}^f : B(\tau_1, \tau_2) = n\}.
\]

Because of the equalities \( P_{d_1}^f = P_{d_2}^f (i = 1, 2) \) which we proved in proposition 8, we can apply proposition 11 with \( N = 2 \) to find

\[
\rho^{\omega_2}(n) = \begin{cases} 
2 \sum_{d \mid n^2 - D} \varepsilon(d) & \text{if } n^2 \equiv D \mod 8; \\
0 & \text{otherwise.}
\end{cases}
\]

The computation of \( \rho^{\omega_2}(n) \) is more involved. For \( \tau \in P_{d_1}^{\omega_2} \) the coefficient \( a_\tau \) is odd by the definition in proposition 8. If the discriminant \( d \) is congruent to 1 modulo 8, this implies that \( b_\tau \) is odd and that \( c_\tau \) is even. Using (39) we find that if \( \rho^{\omega_2}(n) \) is nonzero, then \( n \) satisfies \( n^2 \equiv D \mod 16 \). Therefore we fix a positive integer \( n \) satisfying this congruence. In the following we suppress the dependence on \( d_1, d_2, n \) from the notation. For positive integers \( N, k_1, k_2 \) and \( m \) a positive divisor of \( N \) we define

\[
A_N(k_1, k_2; m) = \Gamma_0(N) \backslash \{(\tau_1, \tau_2) \in P_{d_1} \times P_{d_2} : B(\tau_1, \tau_2) = n, a_1 \equiv m, a_2 \equiv k_2 \}
\]

where \( m \equiv m \) denotes congruence modulo \( m \). This set is well defined because the greatest common divisor of \( N \) and \( a_\tau \) is invariant under the \( \Gamma_0(N) \)-action on \( \tau \in P_d \). To calculate \( \rho^{\omega_2}(n) = \#A_2(1, 1; 2) \) we need the following lemma.

**Lemma 14.** Assume that \( d_1 \) and \( d_2 \) are squares modulo \( 4N \).

a. \( \#A_N(0, 0; N) = 2^t \sum_{d \mid n^2 - D} \varepsilon(d) \), with \( t \) the number of prime divisors of \( N \);

b. \( \#A_N(k_1, k_2; m) = \frac{N}{m} \prod_{p \mid N, p \mid m} (1 + 1/p) \#A_m(k_1, k_2; m) \) for \( m \) dividing \( N \);

c. \( \#A_4(0, 2; 4) = \#A_2(0, 1; 2) \) and \( \#A_4(2, 0; 4) = \#A_2(1, 0; 2) \);

d. \( \#A_4(2, 2; 4) = 2 \#A_2(1, 1; 2) \).

**Proof.** The first formula is the content of proposition 11. Each of the \( \Gamma_0(m) \)-orbits in \( A_m(k_1, k_2; m) \) is the union of \( [\Gamma_0(m) : \Gamma_0(N)] \) \( \Gamma_0(N) \)-orbits. As \( \Gamma_0(N) \) is of index \( N \prod_{p \mid N} (1 + 1/p) \) in \( \text{PSL}_2(\mathbb{Z}) \), lemma 14b follows.

For \( k_1, k_2 \in \{0, 1\} \) define the set

\[
Y(k_1, k_2) = \{(\tau_1, \tau_2) \in P_{d_1} \times P_{d_2} : B(\tau_1, \tau_2) = n, a_1 \equiv k_1, a_2 \equiv k_2, c_1 \equiv c_2 \equiv 0\}.
\]
The greatest common divisors $\gcd(a_\tau, 2)$ and $\gcd(c_\tau, 2)$ are invariant under the $\Gamma(2)$-action on $\tau \in \mathcal{P}_d$. Hence $\Gamma(2)$ acts diagonally on $Y(k_1, k_2)$. For $i = 1, 2$ the maps $\tau_i \mapsto 2\tau_i$, or on coefficients $[a_i, b_i, c_i] \mapsto [\frac{a_i}{4}, b_i, 2c_i]$, induce a map

$$\mathcal{A}_4(2k_1, 2k_2; 4) \to \Gamma(2) \backslash Y(k_1, k_2).$$

It follows from the equality $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right) \Gamma_0(4) = \Gamma(2) \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ that this map is a bijection. If $\{k_1, k_2\} = \{0, 1\}$ we can apply lemma 7 to find that $\Gamma(2) \backslash Y(k_1, k_2)$ has the same cardinality as $\mathcal{A}_2(k_1, k_2; 2)$ and lemma 14c follows. As the discriminants are congruent to 1 modulo 8 by assumption, the conditions on $c_1, c_2$ in the definition of $Y(1, 1)$ follow from those on $a_1, a_2$. In particular we have the equality $\mathcal{A}_2(1, 1; 2) = \Gamma_0(2) \backslash Y(1, 1)$. Each of the $\Gamma_0(2)$-orbits in this set is the union of two $\Gamma(2)$-orbits and the last formula of the lemma follows.

By counting the complement of $\mathcal{A}_2(1, 1; 2)$ inside $\mathcal{A}_2(0, 0; 1)$ we find the equality

$$(45) \quad \# \mathcal{A}_2(1, 1; 2) = \# \mathcal{A}_2(0, 0; 1) - \# \mathcal{A}_2(0, 0; 2) - \# \mathcal{A}_2(1, 1; 2) - \# \mathcal{A}_2(1, 0; 2).$$

To compute the first two terms on the right hand side we apply lemma 14a,b and obtain $\mathcal{A}_2(0, 0; 1) = 3\mathcal{A}_1(0, 0; 1) = 3 \sum \{d \mid \frac{n}{d} \} \varepsilon(d)$ and $\mathcal{A}_2(0, 0; 2) = 2 \sum \{d \mid \frac{n}{d} \} \varepsilon(d)$ with $\bar{n} = n^2 - D$. For the last two terms we first use lemma 14c and then count the complement of $\mathcal{A}_4(0, 2; 4) \cup \mathcal{A}_4(2, 0; 4)$ inside $\mathcal{A}_4(0, 0; 2)$:

$$\# \mathcal{A}_2(0, 1; 2) + \# \mathcal{A}_2(1, 0; 2) = \# \mathcal{A}_4(0, 0; 2) - \# \mathcal{A}_4(0, 0; 4) - \# \mathcal{A}_4(2, 2; 4).$$

As above we apply lemma 14a,b to calculate the first two terms on the right of this equation. By lemma 14d the last term is equal to $2 \# \mathcal{A}_4(1, 1; 2)$, twice the number we are trying to calculate. Substituting all these terms into equation (45) yields

$$\# \mathcal{A}_2(1, 1; 2) = -3 \sum \{d \mid \frac{n^2}{16} \} \varepsilon(d) + 2 \sum \{d \mid \frac{n}{4} \} \varepsilon(d) + 4 \sum \{d \mid \frac{n}{4} \} \varepsilon(d) - 2 \sum \{d \mid \frac{n}{4} \} \varepsilon(d).$$

Using that $\varepsilon$ is multiplicative and $\varepsilon(2) = 1$ we conclude

$$(46) \quad \rho^{\omega_2}(n) = \begin{cases} \sum \{d \mid \frac{n^2 - D}{16} \} \varepsilon(d) & \text{if } n^2 \equiv D \mod 16; \\ 0 & \text{otherwise}. \end{cases}$$

Let $d$ denote one of the discriminants $d_1$ for $d_2$ and set $\Gamma = \text{PSL}_2(\mathbb{Z})$. To calculate the summation of $E_{\Gamma_0(2)}(z, s)$ over $z \in \Gamma_0(2) \backslash \mathcal{P}_d^\omega$ we use proposition 12 and the relation

$$E_{\Gamma_0(2)}(z, s) = \frac{2^s}{2^{2s} - 1} \left( E_{\Gamma}(2z, s) - 2^{-s} E_{\Gamma}(z, s) \right)$$

between the Eisenstein series for $\Gamma$ and $\Gamma_0(2)$ [7, §2 (2.16)]. In the proof of proposition 8 we showed that the map $\Gamma_0(2) \backslash \mathcal{P}_d^\omega \to \Gamma \backslash \mathcal{P}_d$ induced by the inclusion $\mathcal{P}_d^\omega \subset \mathcal{P}_d$ is 2 to 1. In a similar way one proves that the map $\Gamma_0(2) \backslash \mathcal{P}_d^\omega \to \Gamma \backslash \mathcal{P}_d$ sending $\tau$ to $2\tau$ is also 2 to 1 and we obtain

$$\sum_{\tau \in \Gamma_0(2) \backslash \mathcal{P}_d^\omega} E_{\Gamma_0(2)}(\tau, s) = \frac{2^s}{2^{2s} - 1} \sum_{\tau \in \Gamma \backslash \mathcal{P}_d} \left( 2E_{\Gamma}(\tau, s) - 2^{-s} E_{\Gamma}(\tau, s) \right)$$

$$= \frac{2}{2^s + 1} \left( \left\lfloor \frac{|d|}{4} \right\rfloor \right)^{s/2} \zeta(s) \zeta(2s) L(s, d)$$
To obtain a similar result for \( \omega_2 \) let \( Y = \{ \tau \in X : 2 \mid a_\tau, 2 \mid b_\tau \} \) be the image of \( \mathcal{P}_{2d}^\omega \) in \( X = \{ \tau \in \mathcal{P}_{4d} : \gcd(a_\tau, b_\tau, c_\tau) = 1 \} \) under the map \( \tau \mapsto 2\tau \). If we define \( \bar{\Gamma}^0(2) = (\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}) \bar{\Gamma}_0(2)(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix})^{-1} \) we conclude that the first map below is a bijection.

\[
\bar{\Gamma}_0(2) \backslash \mathcal{P}_{2d}^\omega \xrightarrow{\tau \mapsto 2\tau} \bar{\Gamma}^0(2) \backslash Y \xrightarrow{} \Gamma \backslash X
\]

By lemma 7 the second map, induced by the inclusion \( Y \subseteq X \), is also a bijection. Using that the complement of \( X \) in \( \mathcal{P}_{4d} \) is equal to \( \mathcal{P}_d \) and the bijection (15) between \( \bar{\Gamma}_0(2) \backslash \mathcal{P}_{2d}^\omega \) and \( \Gamma \backslash \mathcal{P}_d \) we find

\[
\sum_{\tau \in \bar{\Gamma}_0(2) \backslash \mathcal{P}_{2d}^\omega} E_{\bar{\Gamma}_0(2)}(\tau, s) = \frac{2^s}{2^{2s} - 1} \left[ \sum_{\tau \in \Gamma \backslash \mathcal{P}_d} E_{\Gamma}(\tau, s) - \sum_{\tau \in \Gamma \backslash \mathcal{P}_d} E_{\Gamma}(\tau, s) - 2^{-s} \sum_{\tau \in \Gamma \backslash \mathcal{P}_d} E_{\Gamma}(\tau, s) \right]
\]

\[
= \frac{2^s - 1}{2^s + 1} \left( \frac{|d|}{4} \right)^{s/2} \frac{\zeta(s)}{\zeta(2s)} L(s, d).
\]

The scaling transformation \( \sigma_\infty \) for the group \( \bar{\Gamma}_0(2) \) is the identity, hence definition (33) yields

\[
\phi_{\bar{\Gamma}_0(2)}(s) = \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \sum_{s \in 2\mathbb{Z}_{>0}} \frac{\varphi(c)}{c^{2s}} = \frac{\sqrt{\pi}}{4^s - 1} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \frac{\zeta(2s - 1)}{\zeta(2s)}.
\]

Combining the above formulas as in the proof of theorem 13 we find the following result.

**Theorem 15.** For \( i = 1, 2 \) let \( d_i \) and \( \alpha_i \) be as in theorem 5. Define for \( n \in \mathbb{Z}_{>0} \) the integers \( \rho_\omega(n) \) and \( \rho_\omega^2(n) \) by (44) and (46) respectively. Then the following formulas hold:

\[
\log N(\omega(\alpha_2), \omega(\alpha_2)) = \lim_{s \to 1} \left[ \frac{8h_1h_2}{s^2} - \sum_{n > \sqrt{D}} \rho_\omega(n) Q_{s-1} \left( \frac{n}{\sqrt{D}} \right) \right] + 8h_1h_2 \left( C + \frac{4\log 2}{3} \right)
\]

\[
\log N(\omega_2(\alpha_2), \omega_2(\alpha_2)) = \lim_{s \to 1} \left[ \frac{2h_1h_2}{s-1} - \sum_{n > \sqrt{D}} \rho_\omega^2(n) Q_{s-1} \left( \frac{n}{\sqrt{D}} \right) \right] + 2h_1h_2 \left( C - \frac{2\log 2}{3} \right)
\]

with \( C \) as in theorem 13.

### 4. A Family of Non-holomorphic Hilbert Modular Forms and Holomorphic Projection

In order to finish the proof of the theorems in the introduction we need to compute the limit

\[
\lim_{s \to 1} \left[ \frac{4h_1' h_2'}{s-1} - 2 \sum_{n > \sqrt{D}} \sum_{d \mid n} \varepsilon(d) Q_{s-1} \left( \frac{n}{\sqrt{D}} \right) \right]
\]

which occurs in theorem 13 and the limits occurring in theorem 15. Gross, Kohnen and Zagier proved [6, §3] that limits similar to the one above are ‘almost’ equal to the first Fourier coefficient of certain cusp forms. As their proof does not cover the
above limit we extend their result to a slightly larger class of limits. In this section,
which is independent of the previous ones, we follow Gross and Zagier and study a
family of non-holomorphic Hilbert modular forms. Via holomorphic projection we
obtain a family of holomorphic cusp forms of weight 2 on congruence subgroups
of \( \text{SL}_2(\mathbb{Z}) \). The main result, and the only one which we will use in the sequel, is
theorem 20 below which states that the first Fourier coefficient of each of these cusp
forms is up to a ‘simple expression’ equal to a limit like (47). In the next section we
concentrate on those cusp forms in the family for which the involved congruence
subgroup has genus zero. The corresponding cusp form is then identically zero and
hence (47) is equal to the ‘simple expression’, which completes the proof of the main
theorems.

To indicate the relation between our main theorems and the Fourier coefficients
of modular forms we recall the motivation given by Gross and Zagier on page 214
of [5]. Let \( K \) be a real quadratic field and denote its Dedekind zeta function by \( \zeta_K \).
For \( k \) a positive even integer Siegel expressed \( \zeta_K(1-k) \) as a finite sum of norms of
certain ideals [15]. For \( k = 2, 4 \) these expressions are given by the first equality in
the following formula:

\[
30k\zeta_K(-k+1) = \sum_{\nu \in \mathfrak{d}^{-1}} \sum_{\nu \equiv 0 \mod \text{Tr}(\nu)=1} N_{K/\mathcal{O}}(n)^{k-1} = \sum_{x^2 < D} \sum_{x^2 \equiv D \mod 4} n^{k-1}
\]

where \( \mathfrak{d} = (\sqrt{D}) \) is the different of \( K \) and where we use \( \nu \gg 0 \) to denote that
\( \nu \) is totally positive. The second equality above follows by noting that the totally
positive \( \nu \in \mathfrak{d}^{-1} \) of trace 1 are of the form \( \nu = \frac{x+\sqrt{D}}{2\sqrt{D}} \) for some integer \( x \) which
satisfies \( x \equiv D \mod 2 \) and \( x^2 < D \). Siegel’s proof of the first equality uses the
Hecke-Eisenstein series \( E_{K,k}(z_1, z_2) \), a holomorphic Hilbert modular form of weight
\( k \) on \( \text{SL}_2(\mathcal{O}) \) with \( \mathcal{O} \) the ring of integers of \( K \). To define this series we fix and
embedding \( K \rightarrow \mathbb{C} \), denote the conjugate of \( x \in K \) by \( \bar{x} \) and let \( C \) be the ideal
class group of \( K \). For \( k \in 2\mathbb{Z}_{>0} \) and \((z_1, z_2) \in \mathbb{H} \times \mathbb{H} \) we define

\[
E_{K,k}(z_1, z_2) = \sum_{[a] \in C} \sum_{(m,n) \in (a \times a)/\mathcal{O}} \frac{N_{K/\mathcal{O}}(a)^k}{(mz_1 + n)^k(mz_2 + \bar{n})^k}
\]

where in the case \( k = 2 \) the sum is computed by Hecke summation [15, page 93].
The restriction \( F_{K,k}(z) = E_{K,k}(z, z) \) to the diagonal is an ordinary modular form of
weight \( 2k \) on \( \text{SL}_2(\mathbb{Z}) \). Its zeroth and first Fourier coefficient are equal to \( \zeta_K(k) \) and
\( (\frac{2\pi}{k-1})^{2k} \mathcal{D} \mathcal{H}^{-k} \) times the middle term of (48), respectively. In case \( k \) is equal to 2
or 4 the vector space of modular forms of weight \( 2k \) on \( \text{SL}_2(\mathbb{Z}) \) is one-dimensional
and hence \( F_{K,k}(z) \) is a multiple of the ordinary Eisenstein series of weight \( 2k \). Using
the functional equation of \( \zeta_K \) we obtain equation (48).

The right hand side of (48) resembles the right hand side of the formula for
\( J(d_1, d_2) \) (see theorem 1):

\[
-\frac{1}{2} \log |J(d_1, d_2)|^{\frac{n}{w_1 w_2}} = \sum_{x^2 \equiv D \mod 4} \sum_{D+x^2 \mod 4} \varepsilon(n) \log n.
\]
In order to prove this formula Gross and Zagier adapt the definition of $F_{K,k}(z)$ guided by the distinction between (49) and (48) and study its Fourier coefficients. As our theorems are similar to (49) we will do the same.

For the rest of this section we assume that the discriminant $D$ of $K$ can be written in the form $D = d_1 d_2$ with $d_1$ and $d_2$ two fixed relatively prime negative fundamental discriminants. To account for $\varepsilon$ in (49) we introduce the genus character $\chi : C^+ \to \{\pm 1\}$ of the strict ideal class group $C^+$ of $K$ corresponding to the decomposition $D = d_1 \cdot d_2$. If we identify $C^+$ via the Artin map with the Galois group of the strict Hilbert class field $H^+$ over $K$, the character $\chi$ has kernel $\text{Gal}(H^+/\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}))$. We extend $\chi$ to the idealgroup of $K$ and find $\chi(p) = 1$ for inert primes $p$ and $\chi(p) = \varepsilon(N_{K}/\mathbb{Q}(p))$ for the other primes $p$. As $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})/K$ is ramified at infinity, the character $\chi$ does not factor through the ideal class group $C$ and hence $\chi((\lambda)) = \text{sign}(N_{K}/\mathbb{Q}(\lambda))$ for $\lambda \in K^*$.

To transform the term $n^{k-1}$ of (48) into the term $\log n$ of (49) it seems natural ‘to differentiate $F_{K,k}(z)$ with respect to $k$ and substitute $k = 1’. However the holomorphic Hecke-Eisenstein series is only defined for positive even integers $k$. To resolve this difficulty we introduce for complex $s$ the factor \( \frac{y_1 y_2}{|mz_1 + n|^2|\bar{m}z_2 + \bar{n}|^{2s}} \) into the non-convergent series $E_{K,1}(z_1, z_2)$ (‘Hecke’s trick’), restrict to the diagonal $z_1 = z_2$ and take the derivative with respect to $s$ at $s = 0$.

Fix a primitive ideal $\mathfrak{n} \subset \mathcal{O}$ of norm $N$ such that $\chi(\mathfrak{n}) = \varepsilon(N) = 1$. (We do not assume $\chi$ to be trivial on all divisors of $\mathfrak{n}$ as in [6].) For complex $s$ with $\text{Re}(s) > \frac{1}{2}$ we define the following non-holomorphic Hecke-Eisenstein series on $H \times H$:

\[
E_{n,s}(z_1, z_2) = \sum_{[a] \in C} \sum_{(m,n) \in (\mathfrak{n} \times a)/\mathcal{O}^* \atop (m,n) \neq (0,0)} \chi(a)N_{K}/\mathbb{Q}(a)^{1+2s} (y_1 y_2)^s \frac{1}{(mz_1 + n)(\bar{m}z_2 + \bar{n})|mz_1 + n|^{2s}|\bar{m}z_2 + \bar{n}|^{2s}},
\]

with $y_1 = \text{Im}(z_1)$ and $y_2 = \text{Im}(z_2)$. Although $\chi$ is a character of the strict ideal class group the summation over $[a] \in C$ is well defined, if we replace $a$ by $\lambda a$ for some $\lambda \in K^*$ the summand does not change because of the equality $\chi((\lambda)) = \text{sign}(N_{K}/\mathbb{Q}(\lambda))$. As $D$ is the product of two negative fundamental discriminants the elements of the unit group $\mathcal{O}^*$ have norm 1 and the inner summation over the $\mathcal{O}^*$-orbits is also well defined. For $z_1, z_2$ in some fixed compact subset of $H \times H$ the estimates $|mz_1 + n| = O(\sqrt{m^2 + n^2})$ for $i = 1, 2$ lead to $E_{n,s}(z_1, z_2) = O(\zeta_K(\sqrt{-1})(s + \frac{1}{2}))$. The above sum is therefore absolutely and locally uniformly convergent for $(z_1, z_2, s) \in H \times H \times \{s \in C : \text{Re}(s) > \frac{1}{2}\}$. In particular $E_{n,s}(z_1, z_2)$ is analytic as function of $s$ for $\text{Re}(s) > \frac{1}{2}$. In fact it has a meromorphic continuation to all of $C$ as we will now show.

Fix $s \in C$ with $\text{Re}(s) > \frac{1}{2}$. Using the absolute convergence one easily finds that $E_{n,s}(z_1, z_2)$ transforms like a Hilbert modular form of weight 1:

\[
E_{n,s}(az_1 + b, \bar{a}z_2 + \bar{b}) = (cz_1 + d)(\bar{c}z_2 + \bar{d})E_{n,s}(z_1, z_2)
\]

for all \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(\mathcal{O}, \mathfrak{n}) \) with

\[
\text{PSL}(\mathcal{O}, \mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(K) : a, d \in \mathcal{O}, c \in \mathfrak{n} \text{ and } b \in n^{-1} \right\}.
\]
In particular we find \( E_{n,s}(z_1 + \lambda, z_2 + \tilde{\lambda}) = E_{n,s}(z_1, z_2) \) for each \( \lambda \in n^{-1} \) and the Hilbert modular form can be viewed as a function of \((y_1, y_2, (x_1, x_2)) \in \mathbb{R}_{>0} \times \mathbb{R}^2/n^{-1} \). Here and below we identify any fractional \( K \)-ideal \( \mathfrak{a} \) with the lattice \( \{ (\lambda, \tilde{\lambda}) : \lambda \in \mathfrak{a} \} \subset \mathbb{R}^2 \). The compact group \( \mathbb{R}^2/n^{-1} \) has Lebesque measure \( \sqrt{D}N^{-1} \) and its characters are of the form \( \chi = e^{2\pi i (\nu x_1 + \bar{\nu} x_2)} \) with \( \nu \) in the dual lattice
\[
\{ \nu \in K^* : \text{Tr}_{K/\mathbb{Q}}(\nu n^{-1}) \subset \mathbb{Z} \} = n\mathfrak{d}^{-1}.
\]
Hence \( E_{n,s}(z_1, z_2) \) has a Fourier expansion of the form
\[
E_{n,s}(z_1, z_2) = \sum_{\nu \in n\mathfrak{d}^{-1}} c_{\nu,s}(y_1, y_2)e^{2\pi i (\nu x_1 + \bar{\nu} x_2)}
\]
with
\[
c_{\nu,s}(y_1, y_2) = \frac{N}{\sqrt{D}} \int_{\mathbb{R}^2/n^{-1}} E_{n,s}(z_1, z_2)e^{-2\pi i (\nu x_1 + \bar{\nu} x_2)} dx_1 dx_2
\]
where \( x_i = \text{Re}(z_i) \) and \( y_i = \text{Im}(z_i) \) for \( i = 1, 2 \) and \( dx_1 dx_2 \) denotes the standard Lebesque measure. In the proposition below we will calculate these Fourier coefficients. As it will turn out they can be meromorphically continued to the whole \( s \)-plane which gives us the continuation of \( E_{n,s}(z_1, z_2) \) via (51).

We need to introduce the following functions. For \( \mathfrak{a} \) a nonzero integral ideal of \( \mathcal{O} \) and \( s \in \mathbb{C} \) define the function
\[
\sigma_{s,\chi}(\mathfrak{a}) = \sum_{\mathfrak{b}|\mathfrak{a}} \chi(\mathfrak{b}) N_{K/\mathbb{Q}}(\mathfrak{b})^s
\]
which is clearly analytic in \( s \). This definition also makes sense in case \( \mathfrak{a} \) is the zero ideal if we let \( \mathfrak{b} \) range over the nonzero integral ideals of \( \mathcal{O} \) and restrict \( s \) to the half plane \( \text{Re}(s) < -1 \). For \( \text{Re}(s) > 1 \) we then have the equality \( \sigma_{-s,\chi}(0) = L_K(s, \chi) \) with
\[
L_K(s, \chi) = \sum_{0 \neq \mathfrak{a} \subset \mathcal{O}} \chi(\mathfrak{a}) N_{K/\mathbb{Q}}(\mathfrak{a})^{-s}
\]
the Hecke \( L \)-function associated to the character \( \chi \). Finally for \( s \in \mathbb{C}, \text{Re}(s) > 0 \) and \( t \in \mathbb{R} \) we define the function
\[
\Phi_{s}(t) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i xt}}{(x+i)(x^2+1)^s} dx.
\]

**Proposition 16.** For \( \text{Re}(s) > \frac{1}{2} \) and \( \nu \in n\mathfrak{d}^{-1} \) the Fourier coefficient (52) is given by
\[
c_{\nu,s}(y_1, y_2) = \begin{cases} 
L_K(1 + 2s, \chi)(y_1 y_2)^s + \frac{\Phi_s(0)^2 L_K(2s, \chi)}{\sqrt{D}N^{2s}(y_1 y_2)^s} & \text{if } \nu = 0; \\
\frac{\Phi_s(\nu y_1)\Phi_{\bar{\nu}}(\bar{\nu} y_2)\sigma_{-2s,\chi}((\nu)\mathfrak{d}n^{-1})}{\sqrt{D}N^{2s}(y_1 y_2)^s} & \text{if } \nu \neq 0.
\end{cases}
\]

**Proof.** The following computation is a variation of the methods of Hecke [8, §3]. See [5, page 214] for a discussion on an error of sign made by Hecke.
We split the summation \((m, n) \in (an \times a)/\mathcal{O}^*\) in the definition of \(E_{n,s}(z_1, z_2)\) into two: one sum over \(m = 0\) and \(n \in (a - \{0\})/\mathcal{O}^*\) and one sum over \(m \in (an - \{0\})/\mathcal{O}^*\) and \(n \in a\). The contribution of the first summation to the integral in (52) is

\[
\sum_{[a] \in C} \sum_{n \in a/\mathcal{O}^*} \frac{\chi(a)N(a)^{1+2s}(y_1y_2)^s}{N(n)|N(n)|^{2s}} \int_{\mathbb{R}^2/n-1} e^{2\pi i(\nu x_1 + \bar{\nu} x_2)}dx_1dx_2.
\]

Here and below a prime above a summation sign indicates that the summation variable ranges over nonzero elements. Furthermore \(N(a)\) and \(N(n)\) are abbreviations of the norms \(N_{K/Q}(a)\) and \(N_{K/Q}(n)\). The integral above is zero unless \(\nu = 0\) in which case it equals \(|DN|^{-1}\). The map \(n \mapsto na^{-1}\) is a bijection between \((a - \{0\})/\mathcal{O}^*\) and the set of integral ideals in the ideal class of \(a^{-1}\). Using this bijection we conclude that the first summation contributes nothing to \(c_{\nu, s}\) for \(\nu \neq 0\) and

\[
L_K(1 + 2s, \chi)(y_1y_2)^s
\]
to \(c_{0, s}\).

Fix a complete set of representatives \(S\) for \((an - \{0\})/\mathcal{O}^*\). The summation over \(m \in (an - \{0\})/\mathcal{O}^*\) and \(n \in a\) yields the following contribution to the integral in (52):

\[
\sum_{[a] \in C} \sum_{m \in S} \frac{\chi(a)N(a)^{1+2s}}{N(m)|N(m)|^{2s}} \int_{\mathbb{R}^2/n-1} \sum_{n \in a} \frac{(y_1y_2)^s e^{-2\pi i(\nu x_1 + \bar{\nu} x_2)}}{(z_1 + \frac{n}{m})(z_2 + \frac{n}{m})|z_1 + \frac{n}{m}|^{2s}|z_2 + \frac{n}{m}|^{2s}} dx_1dx_2.
\]

Fix an element \(m \in S\). For \(n \in a\) we write \(n = n_1 + mn_2\) with \(n_1 \in a/mn^{-1}\) and \(n_2 \in n^{-1}\). Summing over \(n_2\) and writing \(n\) for \(n_1\), the above integral is equal to

\[
\int_{\mathbb{R}^2} \sum_{n \in a/mn^{-1}} \frac{(y_1y_2)^s e^{-2\pi i(\nu x_1 + \bar{\nu} x_2)}}{(z_1 + \frac{n}{m})(z_2 + \frac{n}{m})|z_1 + \frac{n}{m}|^{2s}|z_2 + \frac{n}{m}|^{2s}} dx_1dx_2
\]

where we used the equality \(e^{-2\pi i(\nu x_1 + \bar{\nu} x_2)} = 1\) following from the assumption \(\nu \in an^{-1}\). The change of variables \(x_1 = y_1 \tilde{x}_1 - \frac{n}{m}\) and \(x_2 = y_2 \tilde{x}_2 - \frac{n}{m}\) yields

\[
(y_1y_2)^{-s} \Phi_s(\nu y_1) \Phi_s(\bar{\nu} y_2) \sum_{n \in a/mn^{-1}} e^{2\pi i(\frac{mn + \bar{m} n}{m})}.
\]

The finite sum above is zero unless \(\frac{\nu}{m} \in (a\mathcal{O})^{-1}\) in which case it equals \(|N(m)|^{-1}\). Hence the summation over \(m \in (an - \{0\})/\mathcal{O}^*\) and \(n \in a\) yields the following contribution to \(c_{\nu, s}(y_1, y_2)\):

\[
\frac{\Phi_s(\nu y_1) \Phi_s(\bar{\nu} y_2)}{\sqrt{D(y_1y_2)^s}} \sum_{[a] \in C} \sum_{\nu \in m(a\mathcal{O})^{-1}} \frac{\chi(a)N(a)^{2s}}{N(m)|N(m)|^{2s-1}}.
\]

The elements \(m\) in the inner summation are in bijection with the integral ideals in the ideal class of \((an)^{-1}\) that contain \((\nu)\mathcal{O}^{-1}\). By changing the outer summation \([a] \in C\) into \([(an)^{-1}] \in C\) we find that the above double sum is equal to

\[
\sum_{[a] \in C} \sum_{\nu \in m(a\mathcal{O})^{-1}} \frac{\chi(a)N(a)^{2s}}{N(m)|N(m)|^{2s-1}}.
\]
$N^{-2s} \sigma_{-2s, \chi}((\nu) \mathfrak{d} n^{-1})$. Using the equality $\sigma_{-2s, \chi}((0)) = L_K(2s, \chi)$ this concludes the proof of the proposition. \hfill \Box

**Corollary 17.** The function $E_{n,s}(z_1, z_2)$ has a meromorphic continuation to the entire $s$-plane which is regular at $s = 0$. Moreover we have $E_{n,0}(z_1, z_2) = 0$.

**Proof.** First we recall some properties of the function $\Phi_s(t)$ defined by (54), proofs can be found in [7, IV §3]. For $t = 0$ and $\text{Re}(s) > 0$ we have the equality

$$\Phi_s(0) = -\sqrt{\pi i} \frac{\Gamma(s + \frac{1}{2})}{\Gamma(s + 1)}.$$ (56)

Hence $\Phi_s(0)$ admits a meromorphic continuation to the whole $s$-plane which is regular at $s = 0$. By changing the path of integration one can show that for all nonzero $t \in \mathbb{R}$ the function $\Phi_s(t)$ has an analytic continuation to $s \in \mathbb{C}$. For each compact $V \subset \mathbb{C}$ there exist positive constants $c_1, c_2$ such that

$$|\Phi_s(t)| \leq \frac{c_1}{|t|^{c_2}} e^{-\pi|t|} \quad \text{for all } s \in V, t \neq 0,$$

and at $s = 0$ the function is equal to

$$\Phi_0(t) = \begin{cases} -2\pi i e^{-2\pi t} & \text{if } t > 0; \\ -\pi i & \text{if } t = 0; \\ 0 & \text{if } t < 0. \end{cases}$$ (58)

Recall that for a nonzero integral ideal the function $\sigma_{s, \chi}(a)$ defined by (53) is analytic on $\mathbb{C}$. As is well known [13, VII §8] the Hecke $L$-series $L_K(s, \chi)$ has an analytic continuation to $s \in \mathbb{C}$ and satisfies the functional equation

$$D^{s/2} \pi^{-(s+1)} \Gamma(\frac{s+1}{2})^2 L_K(s, \chi) = D^{(1-s)/2} \pi^{s-2} \Gamma(\frac{2-s}{2})^2 L_K(1 - s, \chi).$$

Hence all the functions occuring on the right hand side of (55) have meromorphic continuations to the whole $s$-plane which are regular at $s = 0$, and we conclude that the same holds for $c_{\nu, s}(y_1, y_2)$. Moreover for nonzero $\nu$ this continuation is analytic in $s$.

Because of (57) the Fourier series (51) of $E_{n,s}(z_1, z_2)$ converges absolutely and uniformly on compact subsets of $\mathbb{C}$ that do not contain the poles of $\Phi_s(0)$. This proves the continuation as claimed in the corollary.

To prove that $E_{n,s}(z_1, z_2)$ vanishes at $s = 0$ we show that its Fourier coefficients (55) are all zero. The equality $c_{0,0}(y_1, y_2) = 0$ follows from (56) and the functional equation (59) for $L_K(s, \chi)$. If either $\nu$ or $\bar{\nu}$ is negative, $c_{\nu, s}(y_1, y_2)$ is zero because of (58). Now assume $\nu$ is totally positive. As by assumption $\chi$ is trivial on $\mathfrak{n}$ we find $\chi((\nu) \mathfrak{d} n^{-1}) = \chi((\frac{\nu}{\sqrt{\mathfrak{m}}})) = -\text{sign}(N(\nu)) < 0$. Hence the contribution of $b(\nu) \mathfrak{d} n^{-1}$ to $\sigma_{0,\chi}((\nu) \mathfrak{d} n^{-1})$ is cancelled by the contribution of the complementary divisor $(\nu) \mathfrak{d} (\mathfrak{n} b)^{-1}$. We find $\sigma_{0,\chi}((\nu) \mathfrak{d} n^{-1}) = 0$ so that $c_{\nu,0}(y_1, y_2)$ vanishes for totally positive $\nu$. \hfill \Box

As we argued in the beginning of this section we are interested in the derivative of the Hecke-Eisenstein series at $s = 0$ restricted to the diagonal:

$$F_n(z) = \frac{\sqrt{D}}{8\pi^2} \frac{\partial}{\partial s} E_{n,s}(z, z) \big|_{s=0}.$$
According to (50) this is a non-holomorphic modular form of weight 2 on $\Gamma_0(N)$. The Fourier expansion of $E_{n,s}(z_1, z_2)$ is locally uniformly convergent in $s$ because of (57). By differentiating (51) termwise and using (55), (56), (58) and (59) we obtain the Fourier expansion of $F_n(z)$:

$$F_n(z) = \sum_{n=-\infty}^{\infty} \left( \sum_{\nu \in \mathbb{N}^{-1}_{\mathbb{R}}} c_{\nu}(y) \right) e^{2\pi i n z}$$

where

$$c_{\nu}(y) = \begin{cases} \frac{\sqrt{D}}{2\pi} (L_K(1, \chi) \log y + \kappa) & \text{if } \nu = 0; \\ \sigma'_s((\nu) \mathfrak{d} \mathfrak{n}^{-1}) & \text{if } \nu \gg 0; \\ -\frac{1}{2} \sigma_{0, \chi}((\nu) \mathfrak{d} \mathfrak{n}^{-1}) \Phi(|\nu| y) & \text{if } \nu > 0 > \bar{\nu}; \\ -\frac{1}{2} \sigma_{0, \chi}((\nu) \mathfrak{d} \mathfrak{n}^{-1}) \Phi(|\nu| y) & \text{if } \bar{\nu} > 0 > \nu; \\ 0 & \text{if } \nu << 0, \end{cases}$$

with

$$\kappa = L'_K(1, \chi) + \left( \frac{1}{2} \log(DN) - \log \pi - \gamma \right) L_K(1, \chi),$$

$$\sigma'_s(a) = \frac{\partial}{\partial s} \sigma(s, \chi(a))\big|_{s=0} = \sum_{m|a} \chi(m) \log N_{K/Q}(m)$$

and

$$\Phi(t) = \frac{i}{2\pi} e^{-2\pi t \partial_s \Phi_s(-t)}\big|_{s=0} \text{ for } t \in \mathbb{R}_{>0}.$$

By differentiating under the integral sign one proves the equality

$$\Phi(t) = \int_{1}^{\infty} e^{-4\pi t u} \frac{du}{u} \text{ for } t > 0,$$

[7, IV §3] from which we obtain

$$\Phi(t) = O(t^{-1} e^{-4\pi t}) \text{ for } t > 0,$$

which guarantees the convergence of (60).

To see why these Fourier coefficients can be used to compute limits like (47), we have to introduce the notion of holomorphic projection. Let $S_2(N) = S_2(\Gamma_0(N))$ be the finite dimensional complex vector space of holomorphic cusp forms of weight 2 on $\Gamma_0(N)$. This vector space is equipped with a Hermitian inner product which for $f, g \in S_2(N)$ is defined by

$$\langle f, g \rangle = \int_{\Gamma_0(N) \backslash \mathbb{H}} f(z) \overline{g(z)} y^2 d\mu,$$

where the integral is taken over a fundamental domain for the action of $\Gamma_0(N)$ on $\mathbb{H}$ and where $d\mu$ denotes the $\text{PSL}_2(\mathbb{R})$-invariant measure $\frac{dx dy}{y^2}$ on $\mathbb{H}$ with $z = x + iy$. This inner product supplies an isomorphism between $S_2(N)$ and its dual: for each linear map $L : S_2(N) \to \mathbb{C}$ there is a unique $\phi_L \in S_2(N)$ such that $L(f) = \langle f, \phi_L \rangle$ for all $f \in S_2(N)$. Now let $F$ be a smooth function on $\mathbb{H}$, not necessarily
Theorem 18. Let \( N \) be a positive integer and let \( F(z) = \sum_{m=1}^{\infty} a_m(y)e^{2\pi i mz} \) be a smooth function on \( \mathbf{H} \) that transforms like a modular form of weight 2 on \( \Gamma_0(N) \). Suppose that for every positive divisor \( M \) of \( N \) there exist \( \epsilon > 0 \) and complex numbers \( A(M), B(M) \) such that

\[
(cz+d)^{-2} F\left( \frac{az+b}{cz+d} \right) = A(M) \log y + B(M) + O(y^{-\epsilon}) \quad \text{as} \quad y = \text{Im}(z) \to \infty
\]

for all \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) \( \in \text{SL}_2(\mathbb{Z}) \) with \( \gcd(c,N) = M \). Let \( \tilde{F}(z) = \sum_{m=1}^{\infty} a_m e^{2\pi i mz} \) be the holomorphic projection of \( F \). Then

\[
a_1 = \lim_{s \to 1+} \left[ 4\pi \int_0^{\infty} a_1(y)e^{-4\pi y}y^{s-1} \, dy + \frac{24\alpha}{s-1} \right] - 48\alpha \left[ \sum_{p|N} \frac{\log p}{p^2-1} + \log 2 + \frac{1}{2} + \frac{\zeta'(2)}{\zeta(2)} \right] + 24\beta
\]

with \( \mathbf{H}_1 = \{ s \in \mathbb{C} : \text{Re}(s) > 1 \} \) and

\[
\alpha = \prod_{p|N} (1-p^{-2})^{-1} \sum_{M|N} \frac{\mu(M)A(M)}{M^2},
\]

\[
\beta = \prod_{p|N} (1-p^{-2})^{-1} \sum_{M|N} \frac{\mu(M)[B(M) - 2A(M) \log M]}{M^2}.
\]

Proof. This is the holomorphic projection lemma for the first Fourier coefficient on page 534 of [6] (we corrected two typos). Recall that a cusp form \( f \in S_2(N) \) is exponentially small at the cusps. Using (65), which states that the growth rate of \( F \) at the cusp \( \frac{a}{c} \in \text{P}^1(\mathbb{Q}) \) only depends on \( \gcd(c,N) \), we find that the \( \Gamma_0(N) \)-invariant function \( f(z)\tilde{F}(z)\text{Im}(z)^2 \) is bounded on \( \mathbf{H} \). Therefore the inner product \( \langle f, F \rangle \) converges and hence the holomorphic projection \( \tilde{F} \) of \( F \) is well defined.

In order to sketch the proof we need the Poincaré series

\[
P(z) = \lim_{s \to 1+} \sum_{\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_\infty \backslash \Gamma_0(N)} \frac{1}{(cz+d)^2} \frac{y^{s-1}}{|cz+d|^{2s-2}} e^{2\pi i \frac{az+b}{cz+d}}
\]

where \( \Gamma_\infty \) denotes the stabalizer of \( \infty \) in \( \Gamma_0(N) \). This function is a holomorphic cusp form of weight 2 on \( \Gamma_0(N) \). For each \( f \in S_2(N) \) the inner product \( \langle f, P \rangle \) is equal to \( \frac{1}{4\pi} \) times the first Fourier coefficient of \( f \). In particular we have \( \langle \tilde{F}, P \rangle = \frac{a_1}{4\pi} \). If \( F \) satisfies (65) with \( A(M) = B(M) = 0 \) for all \( M \) then \( \langle F, P \rangle \) is convergent and equal to \( \lim_{s \to 1+} \int_0^{\infty} a_1(y)e^{-4\pi y}y^{s-1} \, dy \). In this special case both \( \alpha \) and \( \beta \) are zero and the theorem follows from the equality \( \langle \tilde{F}, P \rangle = \langle F, P \rangle \).
In general we subtract from $F$ a linear combination of non-holomorphic Eisenstein series of weight 2 to obtain a function $F^*$ which satisfies the asymptotic relation (65) with $A(M) = B(M) = 0$. As these Eisenstein series are perpendicular to the elements of $S_2(N)$, the functions $F$ and $F^*$ have the same holomorphic projection. The theorem follows by applying the arguments of the previous paragraph to the function $F^*$. 

To apply this theorem to $F(z) = F_n(z)$ we have to check condition (65) for this function.

**Proposition 19.** Let $n$ be a primitive ideal of norm $N$ such that $\chi(n) = 1$ and let $M$ be a positive divisor of $N$. For all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}_2(\mathbb{Z})$ with $(c, N) = M$ we have

$$(cz + d)^{-2}F_n\left(\frac{az + b}{cz + d}\right) = A(M) \log y + B(M) + O(y^{-1}) \quad \text{as } y = \text{Im}(z) \to \infty$$

with

$$A(M) = \varepsilon(M) M \frac{A(N)}{N}, \quad B(M) = \varepsilon(M) M \left(\frac{B(N)}{N} + \frac{A(N)}{N} \log \frac{M}{N}\right)$$

and

$$A(N) = \frac{h_1 h'_1}{2}, \quad B(N) = \frac{h_1 h'_1}{2} \left(\frac{1}{2} \log(DN) - \log \gamma + \frac{L'}{L}(1, d_1) + \frac{L'}{L}(1, d_2)\right).$$

**Proof.** The proposition follows from the Fourier expansions of $F_n(z)$ at the various cusps of the group $\Gamma_0(N)$. Fix a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}_2(\mathbb{Z})$ and denote the greatest common divisor of $c$ and $N$ by $M$. It follows from the inclusion $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix} \right) \subset A^{-1}\text{PSL}_2(\mathbb{O}, n)A$ that the function

$$E_{n,s,A}(z_1, z_2) = (cz_1 + d)^{-1}(cz_2 + d)^{-1}E_{n,s}(Az_1, Az_2)$$

satisfies $E_{n,s,A}(z_1 + \lambda, z_2 + \bar{\lambda}) = E_{n,s,A}(z_1, z_2)$ for all $\lambda \in n$. It therefore has a Fourier expansion

$$(66) \quad E_{n,s,A}(z_1, z_2) = \sum_{\nu \in (n\mathbb{O})^{-1}} c_{\nu,s,A}(y_1, y_2) e^{2\pi i (\nu x_1 + \bar{\nu} x_2)}$$

with

$$(67) \quad c_{\nu,s,A}(y_1, y_2) = \frac{1}{N \sqrt{D}} \int_{\mathbb{R}^2/n} E_{n,s,A}(z_1, z_2) e^{-2\pi i (\nu x_1 + \bar{\nu} x_2)} dx_1 dx_2$$

for $\nu \in (n\mathbb{O})^{-1}$. The computation of this integral is very similar to the computation of (52) in the proof of proposition 16. For $\text{Re}(s) > 1$ the function $E_{n,s,A}(z_1, z_2)$ has the same double sum representation as $E_{n,s}(z_1, z_2)$ but now the inner sum ranges over $(m, n) \in (a \times a)/\mathbb{O}^*$ not both zero such that $dm - cn \in \mathfrak{a}$. In order to deal with this last condition we define $n_0$ as the greatest common divisor of the integral ideals $c\mathbb{O}$ and $n$ and let $n_1 = nm_0^{-1}$. Because $n$ is primitive we have $N_{K/Q}(n_0) = M$ and $\chi(n_0) = \varepsilon(M)$. As one easily proves for fixed $m_0 \in n_0 a/\mathfrak{a}$ there is an unique $n_0 \in a/n_1 a$ such that for $m \equiv m_0$ mod $\mathfrak{a}$ the congruence $dm \equiv cn$ mod $\mathfrak{a}$ is
equivalent to $n \equiv n_0 \mod n_1a$. Using this equivalence and the techniques used in the proof of proposition 16 we find for $c_{\nu,s,A}(y_1, y_2)\$

$$
\varepsilon(M) \left(\frac{M}{N}\right)^{1+2s} L_K(1 + 2s, \chi)(y_1y_2)^s + \frac{\varepsilon(M) \Phi_s(0)^2 L_K(2s, \chi)}{NM^{2s-1} \sqrt{D(y_1y_2)^s}} \quad \text{if } \nu \neq 0;
$$

and

$$
\frac{M \Phi_s(\nu y_1) \Phi_s(\nu y_2)}{N \sqrt{D(y_1y_2)^s}} \sum_{[a] \in C} \sum_{m_0 \in n_0//n} \sum_{m \in S} \sum_{m \mod m_0(n)} \frac{\chi(a)N(a)^{2s}e^{2\pi i Tr(\frac{a}{m}n_0)}}{N(m)|N(m)|^{2s-1}} \quad \text{if } \nu \neq 0,
$$

where $S$ is a complete set of representatives for $(a - \{0\})/\mathcal{O}^*$. Note that for $\nu \neq 0$ the triple sum is a finite sum; the elements $m \in S$ such that $\nu \in m(an_1\mathfrak{d})^{-1}$ correspond bijectively to the integral ideals in ideal class of $a^{-1}$ that contain $\nu n_1\mathfrak{d}$.

To find the Fourier expansion of $F_{n,A}(z) = \frac{\sqrt{D}}{8\pi^2} \frac{\partial}{\partial s} E_{n,s,A}(z, z)|_{s=0}$ we substitute $z_1 = z_2$ in (66) and differentiate termwise. It turns out that the growth rate of the function $F_{n,A}(z)$ for $y \to \infty$ is dominated by its zeroth Fourier coefficient. An elementary calculation using (56) and (59) yields

$$
\frac{\sqrt{D}}{8\pi^2} \frac{\partial}{\partial s} c_{0,s,A}(y, y)|_{s=0} = A(M) \log y + B(M)
$$

with

$$
A(M) = \frac{\sqrt{D}M\varepsilon(M)}{2\pi^2 N} L_K(1, \chi)
$$

and

$$
B(M) = \frac{\sqrt{D}M\varepsilon(M)}{2\pi^2 N} \left( L_K'(1, \chi) + \left( \frac{1}{2} \log(DN) - \log \left( \frac{N}{M} \right) - \log \pi - \gamma \right) L_K(1, \chi) \right).
$$

By comparing the Euler product expansions we find the equality $L(s, d_1)L(s, d_2) = L_K(s, \chi)$, where the Dirichlet series $L(s, d_i)$ are defined as in proposition 12, and the class number formula yields $L_K(1, \chi) = \frac{\pi^2}{\sqrt{D}} h_1'h_2'$.

To complete the proof it suffices to show

$$
\sum'_{\nu \in (n\mathfrak{d})^{-1}} \frac{\partial}{\partial s} c_{\nu,s,A}|_{s=0}(y, y)e^{2\pi i x(\nu + \bar{\nu})} = O(y^{-1}) \quad \text{for } y \to \infty.
$$

By changing the path of integration and differentiating under the integral sign in (54) we obtain

$$
\frac{\partial}{\partial s} \Phi_s(t)|_{s=0} = O(|t|^{-1}e^{-\pi|t|}) \quad \text{for all nonzero } t.
$$

(In fact one can replace $\pi$ by any positive number smaller then $2\pi$.) Using (58) we find for all $y > 1$

$$
\frac{\partial}{\partial s} c_{\nu,s,A}|_{s=0}(y, y) = O\left( P(|\nu|, |\bar{\nu}|)e^{-y(|\nu|+|\bar{\nu}|)} \right)
$$

for some polynomial $P \in \mathbb{C}[X, Y]$. For $\nu = \frac{a+by\sqrt{D}}{2\sqrt{D}} \in (n\mathfrak{d})^{-1}$ we have $|\nu| + |\bar{\nu}| = \max(\frac{|a|}{\sqrt{D}}, |b|)$. We conclude that $\#\{\nu \in (n\mathfrak{d})^{-1} : |\nu| + |\bar{\nu}| = x\} = O(x)$ for all $x \in \mathbb{R}_{>0}$ and (68) follows easily. \qed
Before we state the main result of this section we introduce the following function. For an ideal \(a\) and \(s \in H_1 = \{ s \in \mathbb{C} : \text{Re}(s) > 1 \}\) define

\[
T_a(s) = \sum_{\nu \in a\mathfrak{o}^{-1}, \nu > 0 \atop \nu \not\in \mathfrak{o}, \text{Tr}(\nu) = 1} \sigma_0,\chi((\nu)\mathfrak{o}^{-1})Q_{s-1}(1 + 2|\nu|)
\]

where \(Q_{s-1}(t)\) is the Legendre function of the second kind defined by (19) in section 3. For \(\nu\) as in the above sum, the pair \((\nu, \bar{\nu})\) ranges over those elements in the lattice \(a\mathfrak{o}^{-1} \subset \mathbb{R}^2\) that are on the half line \(\{ \text{Tr}(\nu) = 1 : \nu > 0 \atop \nu \not\in \mathfrak{o} \}\subset \mathbb{R}^2\). Because \(\sigma_{0,\chi}(a) = O(N(a)^{\delta})\) for any \(\delta > 0\) and \(Q_{s-1}(t) = O(t^{-s})\) for \(t \to \infty\), the above sum converges absolutely and locally uniformly on \(H_1\).

**Theorem 20.** Let \(n\) be a primitive ideal prime to \(\mathfrak{o}\) of norm \(N\) such that \(\chi(n) = 1\). The first Fourier coefficient \(a_1\) of the holomorphic projection of \(F_n(z)\) is equal to:

\[
a_1 = S_n - \lim_{s \to 1} \left[ T_n(s) + T_{\overline{n}}(s) - \frac{2\alpha}{s-1} \right] + 12\alpha \left( 2C + 2 \sum_{p \mid N} \frac{\log p}{\varepsilon(p)p + 1} - \log \right)
\]

with \(C\) as in theorem 13 and

\[
\alpha = \frac{h'_1h'_2}{2N} \prod_{p\mid N} \frac{p}{p + \varepsilon(p)} \quad \text{and} \quad S_n = \sum_{\nu \in a\mathfrak{o}^{-1}, \nu > 0 \atop \nu \not\in \mathfrak{o}, \text{Tr}(\nu) = 1} \sigma'_\chi((\nu)\mathfrak{o}^{-1}).
\]

**Proof.** First we use proposition 19 to compute \(\alpha\) and \(\beta\) defined in theorem 18 for the function \(F(z) = F_n(z)\). An easy computation yields the following formulas:

\[
\alpha = \prod_{p \mid N} (1 - p^{-2})^{-1} \cdot \frac{A(N)}{N} \sum_{M \mid N} \frac{\mu(M)\varepsilon(M)}{M} = \frac{h'_1h'_2}{2N} \prod_{p \mid N} \frac{1 - \varepsilon(p)p^{-1}}{1 - p^{-2}}
\]

\[
\beta = \alpha \left[ \frac{1}{2} \log\left( \frac{D}{N} \right) - \log \pi - \gamma + \frac{L'}{L}(1, d_1) + \frac{L'}{L}(1, d_2) + \sum_{p \mid N} \frac{\varepsilon(p)\log p}{p - \varepsilon(p)} \right].
\]

As \(n\) is prime to \(\mathfrak{o}\) we have \(\varepsilon(p)^2 = 1\) for all primes \(p\mid N\) and the above expression for \(\alpha\) is equal to the one quoted in the theorem.

By equations (60) and (61) we find that the first Fourier coefficient of \(F_n(z)\) is equal to

\[
a_1(y) = S_n - \frac{1}{2} \sum_{\nu \in a\mathfrak{o}^{-1}, \nu > 0 \atop \nu \not\in \mathfrak{o}, \text{Tr}(\nu) = 1} \sigma_{0,\chi}((\nu)\mathfrak{o}^{-1})\Phi(|\nu|y) - \frac{1}{2} \sum_{\nu \in \mathfrak{o}^{-1}, \nu > 0 \atop \nu \not\in \mathfrak{o}, \text{Tr}(\nu) = 1} \sigma_{0,\chi}((\nu)\mathfrak{o}^{-1})\Phi(|\nu|y),
\]

with \(S_n\) as in the statement of the theorem and where \(\Phi(t)\) is defined by (62). The two sums on the right are infinite sums and converges because of (63). For \(s \in H_1\) we use the integral representation \(\Gamma(s) = \int_0^\infty e^{-y}y^{s-1}dy\) and obtain

\[
4\pi \int_0^\infty a_1(y)e^{-4\pi y}y^{s-1}dy = \frac{\Gamma(s)}{(4\pi)^{s-1}}S_n - \frac{1}{2} \sum_{\nu \in a\mathfrak{o}^{-1}, \nu > 0 \atop \nu \not\in \mathfrak{o}, \text{Tr}(\nu) = 1} \sigma_{0,\chi}((\nu)\mathfrak{o}^{-1})\Psi_{s-1}(|\nu|)
\]

\[
- \frac{1}{2} \sum_{\nu \in \mathfrak{o}^{-1}, \nu > 0 \atop \nu \not\in \mathfrak{o}, \text{Tr}(\nu) = 1} \sigma_{0,\chi}((\nu)\mathfrak{o}^{-1})\Psi_{s-1}(|\nu|),
\]
with
\[ \Psi_{s-1}(\lambda) = 4\pi \int_0^\infty \Phi(\lambda y) e^{-4\pi y y^{s-1}} dy \quad \text{for } \lambda > 0, \Re(s) > 0. \]

For large \( \lambda \) and \( s \) close to 1 this function differs little from the Legendre function of the second kind \( Q_{s-1}(1 + 2\lambda) \). More precisely, using Taylor expansions one can prove [5, page 218]

\[
\Psi_{s-1}(\lambda) - \frac{2\Gamma(2s)}{(4\pi)^s \Gamma(s+1)} Q_{s-1}(1 + 2\lambda) = \begin{cases} 
O(\lambda^{-s-1}) & \text{for } \Re(s) > 0 \text{ and } \lambda \to \infty; \\
0 & \text{for } s = 1 \text{ and } \lambda > 0,
\end{cases}
\]

where the implied constant is independent of \( s \). This estimate allows us to replace \( \Phi_{s-1} \) by \( Q_{s-1} \):

\[
\lim_{s \to 1} \left[ 4\pi \int_0^\infty a_1(y) e^{-4\pi y y^{s-1}} dy + \frac{24\alpha}{s-1} \right] = S_n - \lim_{s \to 1} \left[ T_n(s) + T_\Phi(s) - \frac{24\alpha}{s} \right]
\]

If we substitute this limit into the formula for the first Fourier coefficients of the holomorphic projection of \( F_n(z) \) given in theorem 18 and use the Taylor expansion

\[
\frac{(4\pi)^{s-1} \Gamma(s+1)}{\Gamma(2s)} = 1 + (\log(4\pi) + \gamma - 1)(s-1) + O((s-1)^2)
\]

we obtain

\[
a_1 = S_n - \lim_{s \to 1} \left[ T_n(s) + T_\Phi(s) - \frac{24\alpha}{s-1} \right] = -24\alpha \cdot \left[ 2 + 2 \frac{\zeta'}{\zeta}(2) - \frac{1}{2} \log \left( \frac{D}{N} \right) - \frac{L'}{L}(1, d_1) \right]
\]

\[
- \frac{L'}{L}(1, d_2) + 2 \sum_{p | N} \frac{\log p}{p^2 - 1} - \sum_{p | N} \frac{\varepsilon(p) \log p}{p - \varepsilon(p)}
\]

which is equivalent to the statement in the theorem. \( \square \)

### 5. Conclusion of the Proofs

Let \( d_1 \) and \( d_2 \) the two negative fundamental discriminants that are relatively prime and let \( K \) be the real quadratic field of discriminant \( D = d_1 d_2 \). By making the appropriate choice for the \( K \)-ideal \( n \) we can use theorem 20 to calculate the limits occurring on the right hand side of the equations in the theorems 13 and 15, which will complete the proofs of theorems 4 and 5 in the introduction.

Let \( p \) be a rational prime and let \( a \) be a positive integer that satisfy the following properties:

1. \( p \) splits completely in \( K/\mathbb{Q} \); Denote the primes above \( p \) by \( p \) and \( q \);
2. \( a \) is even in case \( \varepsilon(p) = -1 \);
3. the vector space \( S_n(p^a) \) of holomorphic modular forms of weight 2 on \( \Gamma_0(p^a) \) is zero dimensional.

Recall that property 3 is equivalent with the condition that the compact Riemann surface \( \Gamma_0(p^a) \backslash \mathfrak{H} \) has genus zero and that there are only finitely many pairs \( (p, a) \) with this property. Because of the third condition on \( p^a \), the first Fourier coefficient of the holomorphic projection of \( F_{p^a} \) and \( F_{q^a} \) are both zero. If we add the result of theorem 20 for the ideals \( p^a \) and \( q^a \) we obtain

\[
2 \lim_{s \to 1} \left[ T_{p^a}(s) + T_{q^a}(s) - \frac{24\alpha}{s-1} \right] = S_{p^a} + S_{q^a} + 24\alpha \left( 2C + 2 \frac{\log p}{\varepsilon(p)p + 1} - \log p^a \right)
\]
with $C$ as in theorem 13 and $\alpha = \frac{b_1' b_2'}{2^{p^a - 1} (p + \varepsilon(p))}$,

$$T_{p^a}(s) + T_{q^a}(s) = \sum_{\nu \in \mathcal{P}^{q^{-a}} - 1, \nu \not> 0, \nu \not> 0} \sigma_{0, \chi}(\nu) \mathfrak{d} q^{-a} Q_{s-1}(1 + 2|\nu|) + \sum_{\nu \in \mathcal{P}^{p^{-a}} - 1, \nu \not> 0, \nu \not> 0} \sigma_{0, \chi}(\nu) \mathfrak{d} p^{-a} Q_{s-1}(1 + 2|\nu|)$$

and

$$S_{p^a} + S_{q^a} = \sum_{\nu \in \mathcal{P}^{q^{-a}} - 1, \nu \not> 0, \nu \not> 0} \sigma^\prime_{\chi}(\nu) \mathfrak{d} p^{-a} + \sum_{\nu \in \mathcal{P}^{p^{-a}} - 1, \nu \not> 0, \nu \not> 0} \sigma^\prime_{\chi}(\nu) \mathfrak{d} q^{-a}.$$

The totally positive elements $\nu \in \mathfrak{d}^{-1}$ of trace 1 are of the form $\nu = \frac{n + \sqrt{D}}{2\sqrt{D}}$ with $n$ a rational integer satisfying $|n| < \sqrt{D}$ and $n \equiv D \mod 2$. Such an element $\nu$ is not divisible by an integer, hence its norm is divisible by $p^a$ if and only if either $\nu \in \mathfrak{p}^a \mathfrak{d}^{-1}$ or $\nu \in \mathfrak{q}^a \mathfrak{d}^{-1}$. For $\nu = \frac{n + \sqrt{D}}{2\sqrt{D}} \in \mathfrak{p}^a \mathfrak{d}^{-1}$ the norm maps the ideals dividing the primitive integral ideal $(\nu) \mathfrak{d}^{-a}$ bijectively to the positive divisors of $N_{K/Q}((\nu) \mathfrak{d}^{-a}) = \frac{D - n^2}{4p^a}$. Hence we find for these $\nu$

$$\sigma^\prime_{\chi}(\nu) \mathfrak{d}^{-a} = \sum_{a|\nu \mathfrak{d}^{-a}} \chi(a) \log N_{K/Q}(a) = \sum_{d \mid \frac{D - n^2}{4p^a}} \varepsilon(d) \log d.$$  

The same formula holds for $\sigma^\prime_{\chi}(\nu) \mathfrak{d}^{-a}$ in case $\nu \in \mathfrak{q}^a \mathfrak{d}^{-1}$ and we obtain

$$S_{p^a} + S_{q^a} = \sum_{|n| < \sqrt{D}} \sum_{d \mid \frac{D - n^2}{4p^a}} \varepsilon(d) \log d.$$  

In a similar way we find

$$T_{p^a}(s) + T_{q^a}(s) = \sum_{n > \sqrt{D}} \sum_{d \mid \frac{D - n^2}{4p^a}} \varepsilon(d) Q_{s-1}\left(\frac{n}{\sqrt{D}}\right)$$

and the above limit can be rewritten as

$$\lim_{s \to 1} \left[ \frac{48\alpha}{s - 1} - 2 \sum_{n > \sqrt{D}} \rho_{p^a}(n) Q_{s-1}\left(\frac{n}{\sqrt{D}}\right) \right] =$$

$$24\alpha \left( \log p^a - \frac{2 \log p}{\varepsilon(p)p + 1} - 2C \right) - \sum_{|n| < \sqrt{D}} \sum_{d \mid \frac{D - n^2}{4p^a}} \varepsilon(d) \log d$$

with

$$\rho_{p^a}(n) = \begin{cases} \sum_{d \mid \frac{D - n^2}{4p^a}} \varepsilon(d) & \text{if } n^2 \equiv D \mod 4p^a \\ 0 & \text{otherwise}. \end{cases}$$

Now assume that both $d_1$ and $d_2$ are congruent to 2 modulo 3. In this case the rational prime 3 splits completely in $K/Q$ and $\varepsilon(3) = -1$. As $\Gamma_0(9) \backslash \mathbf{H}$ has genus 0 [12, §4.2] the pair $(p, a) = (3, 2)$ satisfies the three properties that we stated in the
Substituting this limit into the formula of theorem 13 yields
\[\lim_{s \to 1} \left[ \frac{4h'_1h'_2}{s-1} - 2 \sum_{n > \sqrt{D}}^{} \rho^{\gamma_2}(n)Q_{s-1}(\frac{n}{\sqrt{D}}) \right] = 2h'_1h'_2(3 \log 3 - 2C) - \sum_{|n| < \sqrt{D} \, d|} \sum_{D-n^2 \equiv 36} \varepsilon(d) \log d.\]

Substituting this limit into the formula of theorem 13 yields
\[\log N(\gamma_2(\alpha_1), \gamma_2(\alpha_2)) = \frac{8h'_1h'_2}{n_1n_2} = 6h'_1h'_2 \log 3 - \sum_{|n| < \sqrt{D} \, d|} \sum_{D-n^2 \equiv 36} \varepsilon(d) \log d \]
\[= 6h'_1h'_2 \log 3 + \sum_{n^2 < D} \log F \left( \frac{D-n^2}{36} \right) \]

where the last line follows from the equality \(\varepsilon\left(\frac{D-n^2}{36}\right) = -1\) [1, page 306]. This concludes the proof of theorem 4.

Next we assume that both \(d_1\) and \(d_2\) are congruent to 1 modulo 8, in particular \(h'_1 = h_1\). Both pairs \((p, a) = (2, 1)\) and \((p, a) = (2, 2)\) satisfy the desired properties. For \((p, a) = (2, 1)\) we find \(\alpha = \frac{h_1 h_2}{6}\) and because of the equality \(2\rho_2(n) = \rho^{\omega_2}(n)\) (see (44)) equation (69) now reads
\[\lim_{s \to 1} \left[ \frac{8h_1h_2}{s-1} - \sum_{n > \sqrt{D}}^{} \rho^{\omega_2}(n)Q_{s-1}(\frac{n}{\sqrt{D}}) \right] = 4h_1h_2 \left( \frac{1}{3} \log 2 - 2C \right) - \sum_{|n| < \sqrt{D} \, d|} \sum_{D-n^2 \equiv 16} \varepsilon(d) \log d.\]

Applying (69) for \((p, a) = (2, 2), \alpha = \frac{h_1 h_2}{12}\) and \(\rho_4(n) = \rho^{\omega_2}(n)\) (see (46)) and dividing by 2 yields
\[\lim_{s \to 1} \left[ \frac{2h_1h_2}{s-1} - \sum_{n > \sqrt{D}}^{} \rho^{\omega_2}(n)Q_{s-1}(\frac{n}{\sqrt{D}}) \right] = 2h_1h_2 \left( \frac{2}{3} \log 2 - C \right) - \sum_{0 < n < \sqrt{D} \, d|} \sum_{D-n^2 \equiv 16} \varepsilon(d) \log d.\]

If we substitute these limits in the formulas of theorem 15 we find
\[\log N(\omega(\alpha_2), \omega(\alpha_2)) = 12h_1h_2 \log 2 - \sum_{|n| < \sqrt{D} \, d|} \sum_{D-n^2 \equiv 16} \varepsilon(d) \log d\]

and
\[\log N(\omega_2(\alpha_2), \omega_2(\alpha_2)) = - \sum_{0 < n < \sqrt{D} \, d|} \sum_{D-n^2 \equiv 16} \varepsilon(d) \log d,\]

from which theorem 5 follows.

REFERENCES
1. D.A. Cox, Primes of the form \(x^2 + ny^2\), Wiley-Interscience, 1989.
2. D. Dorman, Special values of the elliptic modular function and factorization formulæ, J. Reine Angew. Math. 383 (1988), 207–220.
3. A.C.P. Gee, Class invariant’s by Shimura’s reciprocity law, J. Théor. Nombres Bordeaux 11 (1999), 45–72.
4. D. Gross, Local heights on curves, Arithmetic geometry (G. Cornell, J. Silverman, eds.), Springer-Verlag, 1986, pp. 327–339.
5. B. Gross, D. Zagier, On singular moduli, J. Reine Angew. Math. 355 (1985), 191–220.
6. B. Gross, W. Kohnen, D. Zagier, *Heegner points and derivatives of L-series II*, Math. Ann. **278** (1987), 497–562.

7. B. Gross, D. Zagier, *Heegner points and derivatives of L-series*, Invent. Math. **84** (1986), 225–320.

8. E. Hecke, *Analytische Functionen und algebraische Zahlen, zweiter Teil*, Abh. Math. Sem. Hamburg **3** (1924), 231–236; Mathematische Werke, Göttingen 1970, pp. 381–404.

9. D. A. Hejhal, *The Selberg Trace Formula for PSL_2(R), volume 2*, vol. 1001, Springer Lecture notes in Math., 1983.

10. H. Iwaniec, *Introduction to the Spectral Theory of Automorphic Forms*, Revista Matemática Iberoamericana, 1995.

11. S. Lang, *Elliptic functions, second edition*, vol. GTM 112, Springer-Verlag, 1987.

12. T. Miyake, *Modular Forms*, Springer-Verlag, 1989.

13. J. Neukirch, *Algebraische Zahlentheorie*, Springer-Verlag, 1992.

14. R.A. Rankin, *Modular forms and functions*, Cambridge University Press, 1977.

15. C.L. Siegel, *Berechnung von Zetafunktionen an ganzzahligen Stellen*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II (1969), 87–102.

16. N. Yui, D. Zagier, *On the singular values of Weber modular functions*, Math. Comp. **66** (1997), 1645–1662.

17. D. Zagier, *Modular functions whose Fourier coefficients involve zeta-functions of quadratic fields*, Modular functions of one variable VI (J.P. Serre, D. Zagier, eds.), Lect. Notes 627, Springer-Verlag, 1977, pp. 105–169.