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Elementary properties of triangle in normed spaces

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Abstract. Based on concepts of trigonometric on plane, in this paper we generalized those concept in normed spaces. About the orthogonality concept between two vectors already well known, we are interested to develop elementary properties of triangle, especially the properties of its angle. We propose a non-linear (Wilson) functional to define an angle and explore its properties.

1. Introduction
Before we discuss about triangle in normed spaces, we introduce definition Wilson functional, then we will use it to define an angle as a tool to explore elementary properties of triangle in normed spaces.

Suppose $V$ is a real linear space, and in arbitrary normed spaces $V := (V, \| \cdot \|)$, following [8, 3], we define (Wilson) non-linear functional $\langle \cdot, \cdot \rangle_{\bullet} : V \times V \rightarrow \mathbb{R}$, by

$$\langle x,y \rangle_{\bullet} := \frac{\|x\|^2 + \|y\|^2 - \|x-y\|^2}{2},$$

for every $x,y \in V$.

For some simple properties of our functional above, see [3, p. 26], and our first result is Cauchy-Schwartz inequality as follow.

\textbf{Theorem 1.1} For every $x, y \in V$, we have the following

$$|\langle x, y \rangle_{\bullet}| \leq \|x\| \cdot \|y\|.$$

\textbf{Proof.} Suppose $x, y \in V$. Since $\|x\| - \|y\| \leq \|x-y\|$, then

$$\|x\| - \|y\|^2 \leq \|x-y\|^2 \iff \|x\|^2 - 2\|x\| \cdot \|y\| + \|y\|^2 \leq \|x-y\|^2 \iff \langle x, y \rangle_{\bullet} \leq \|x\| \cdot \|y\|.$$

On the other hand, since $\|x-y\| \leq \|x\| + \|y\|$, then we will obtain

$$\|x-y\|^2 \leq (\|x\| + \|y\|)^2 \iff \|x-y\|^2 - \|x\|^2 - \|y\|^2 \leq 2\|x\| \cdot \|y\| \iff -\langle x, y \rangle_{\bullet} \leq \|x\| \cdot \|y\|,$$

and this complete the proof. \hfill \blacksquare

Note that, our functional $\langle \cdot, \cdot \rangle_{\bullet}$ is continuous. That is, for every $x, y, z \in V$, we have

$$|\langle x, y \rangle_{\bullet} - \langle x, z \rangle_{\bullet}| \leq K_1 \|y-z\|,$$

where $K_1 = K_1(x, y, z) := \|x\| + \|y\| + \|z\|$. Moreover, we always have
Theorem 1.2
For every \( x, y, w, z \in V \), we have the following
\[
|\langle x, y \rangle - \langle w, z \rangle| \leq K_2 \|x - w\| + \|y - z\|, 
\]
with \( K_2 = K_2(x, y, w, z) := \|x\| + \|y\| + \|w\| + \|z\| \).

2. Elementary properties of \( V_\star \)
From the above observations, we define an angle between \( x, y \in V \), denote by \( \angle_\star(x, y) \) (see [3, 4, 8], for more informations about this notions, and their elementary properties), by
\[
\cos \angle_\star(x, y) = \frac{\langle x, y \rangle}{\|x\|\|y\|}.
\]
In this case we only consider \( 0 \leq \angle_\star(x, y) \leq \pi \).

For any normed spaces \( V := (V, \| \cdot \|) \), we will introduce the notations
\[
V_\star := (V, \| \cdot \|, \angle_\star).
\]
In \( V_\star \), we define the orthogonality between two nonzero vectors as follows. Two nonzero vectors \( x, y \in V_\star \) are orthogonal, written \( x \perp y \), if \( \langle x, y \rangle = 0 \). The following is easy, and we omit the detailed proof.

Theorem 2.1
For every \( x, y \in V_\star \), we have the following
(i). \( x \perp y \iff x \perp (-y) \iff y \perp x \iff (-x) \perp y \iff (-x) \perp (-y) \),
(ii). If \( \{x, y\} \) are linearly dependent, then \( \angle_\star(x, y) = 0 \), or \( \angle_\star(x, y) = \pi \).

Proof
(i). The first equivalences is easy, since for every \( x, y \in V_\star \), we have
\[
x \perp y \iff \|x - y\|^2 = \|x\|^2 + \|y\|^2.
\]
(ii). Suppose \( y = kx \), for some \( k \in \mathbb{R} \setminus \{0\} \). Then
\[
\cos \angle_\star(x, y) = \frac{\langle x, y \rangle}{\|x\|\|y\|} = \frac{\langle x, kx \rangle}{\|x\|\|kx\|} = \frac{k}{|k|}.
\]

The converse of statement (ii) in Theorem 2.1 is not true, when we consider a sequence spaces \( \ell^1(\mathbb{R}) \), with its members \( x := \langle 1, 1, 2, 0, 0, \ldots \rangle \), and \( y := \langle 1, 1, 0, 0, \ldots \rangle \), for \( x, y \in \ell^1(\mathbb{R}) \).

The following result is a geometrical interpretation of “parallelogram law” in any normed spaces \( V_\star \).

Theorem 2.2
For every \( a, b \in V_\star \), we have the following
\[
\angle_\star(a, b) + \angle_\star(a, -b) = \pi \iff \|a + b\|^2 + \|a - b\|^2 = 2(\|a\|^2 + \|b\|^2).
\]

Proof. Its proof based on the following two keys observations. That is,
\[
\angle_\star(a, b) + \angle_\star(a, -b) = \pi \iff \cos \angle_\star(a, b) + \cos \angle_\star(a, -b) = 0, \quad \text{and}
\]
\[
\cos \angle_\star(a, b) + \cos \angle_\star(a, -b) = \frac{2\|a\|^2 + 2\|b\|^2 - \|a - b\|^2 - \|a + b\|^2}{2\|a\|\|b\|},
\]
and this complete the proof.
3. Triangle in $V_\bullet$

For every $a, b, c \in V_\bullet \setminus \{0\}$, we introduce notation $\Delta [a, b, c]$ as a set $\{a, b, c\}$, such that $a + c = b$, and for $\Delta [a, b, c]$ we also consider the following $\angle (a, b), \angle (c, -a),$ and $\angle (b, c)$.

From the above informations, we will have the following

\[
\begin{align*}
\|a\|^2 &= \|b\|^2 + \|c\|^2 - 2\|b\| \cdot \|c\| \cos \angle (b, c), \\
\|b\|^2 &= \|a\|^2 + \|c\|^2 - 2\|a\| \cdot \|c\| \cos \angle (c, -a), \\
\|c\|^2 &= \|a\|^2 + \|b\|^2 - 2\|a\| \cdot \|b\| \cos \angle (a, b),
\end{align*}
\]

which we called Cosine Rule for $\Delta [a, b, c]$.

With $K := 2\sqrt{s(s - \|a\|)(s - \|b\|)(s - \|c\|)}$, and $2s := \|a\| + \|b\| + \|c\|$, then we will have Sine Rule as follows.

\[\|b\|\|c\| \cdot \sin \angle (b, c) = \|a\|\|c\| \cdot \sin \angle (c, -a) = \|a\|\|b\| \cdot \sin \angle (a, b) = K.\]

In $\Delta [a, b, c]$, it is easy to see that our cosine rule is equivalent with the following Triangle Sides Formula.

\[
\begin{align*}
\|a\| &= \|b\| \cos \angle (a, b) + \|c\| \cos \angle (c, -a), \\
\|b\| &= \|a\| \cos \angle (a, b) + \|c\| \cos \angle (b, c), \\
\|c\| &= \|a\| \cos \angle (c, -a) + \|b\| \cos \angle (b, c).
\end{align*}
\]

**Theorem 3.1** In every $\Delta [a, b, c]$, cosine rule is equivalent with triangle sides formula, and we can derive sine rule from cosine rule.

4. Elementary properties of angle in $V_\bullet$

In this section we will discuss about the sum of angles in triangle. We start with some special triangle, that is, equilateral triangle.

Suppose $\Delta [a, b, c]$ we have $\|a\| = \|b\| = \|c\|$, its mean that

\[
\cos \angle (b, c) := \frac{\|b\|^2 + \|c\|^2 - \|a\|^2}{2\|b\|\|c\|} = \frac{1}{2}.
\]

So we will have $\angle (b, c) = \angle (c, -a) = \angle (a, b) = \frac{\pi}{3}$, and for equilateral $\Delta [a, b, c]$, we have

$$\angle (a, b) + \angle (b, c) + \angle (c, -a) = \pi.\]

Now we move to right triangle.

Suppose $\cos \angle (b, c) = 0$ or $\angle (b, c) = \frac{\pi}{2}$, then $\|a\|^2 = \|b\|^2 + \|c\|^2$. It is easy to see that

\[
\cos \angle (c, -a) := \frac{\|a\|^2 + \|c\|^2 - \|b\|^2}{2\|a\|\|c\|} = \frac{2\|c\|^2}{2\|a\|\|c\|} = \frac{\|c\|}{\|a\|},
\]

and

\[
\cos \angle (a, b) := \frac{\|a\|^2 + \|b\|^2 - \|c\|^2}{2\|a\|\|b\|} = \frac{2\|b\|^2}{2\|a\|\|b\|} = \frac{\|b\|}{\|a\|}.
\]

Since $\|a\|^2 = \|b\|^2 + \|c\|^2$, then $\cos^2 \angle (c, -a) + \cos^2 \angle (a, b) = 1$ or $\cos^2 \angle (c, -a) = 1 - \cos^2 \angle (a, b) = \sin^2 \angle (a, b).$
Since \( \cos \angle (c, -a) = -\sin \angle (a, b) \) will arrive to a contradiction, then we will conclude that
\[
\cos \angle (c, -a) = \sin \angle (a, b), \quad \text{and} \quad \angle (c, -a) = \pi/2 - \angle (a, b).
\]
For right \( \Delta [a, b, c] \), we will have
\[
\angle (b, c) + \angle (a, b) - \angle (c, -a) = \frac{\pi}{2} + \frac{\pi}{2} = \pi.
\]

Our result below is important for our \( \Delta [a, b, c] \), who serve as a generalization of the above results.

**Theorem 4.1** In every \( \Delta [a, b, c] \), we will have \( \angle (a, b) + \angle (b, c) + \angle (c, -a) = \pi \).

**Proof.** We see that our conclusion is equivalent with the following
\[
\angle (a, b) + \angle (b, c) + \angle (c, -a) = \pi \iff \cos (\angle (a, b) + \angle (b, c)) = \cos (\pi - \angle (c, -a))
\]
\[
\iff \cos \angle (a, b) \cos \angle (b, c) - \sin \angle (a, b) \sin \angle (b, c) = -\cos \angle (c, -a).
\]

Next, we choose \( x, y > 0 \), such that \( \|a\| = x\|c\| \), and \( \|b\| = y\|c\| \).

It means that,
\[
\begin{align*}
x^2 &= y^2 + 1 - 2y \cos \angle (b, c), \\
y^2 &= x^2 + 1 - 2x \cos \angle (c, -a), \\
1 &= x^2 + y^2 - 2xy \cos \angle (a, b),
\end{align*}
\]
that serve as a cosine rule in its simpler form.

After some several simplifications,
\[
\begin{align*}
\cos \angle (a, b) \cos \angle (b, c) - \sin \angle (a, b) \sin \angle (b, c) + \cos \angle (c, -a) & = \frac{x^2 + y^2 - 1}{2xy} \cdot \frac{y^2 + 1 - x^2}{2y} + \frac{x^2 + 1 - y^2}{2x} - \frac{K^2}{\|a\| \|b\| \|c\|} \\
& = \frac{y^4 - (1 - x^2)^2}{4xy^2} + \frac{x^2 + 1 - y^2}{2x} + \frac{[1 - (x + y)^2][1 - (x - y)^2]}{4xy^2} \\
& = \frac{[1 - (x + y)^2][1 - (x - y)^2] + y^4 - (1 - x^2)^2 + 2y^2(x^2 + 1 - y^2)}{4xy^2} \\
& = 0,
\end{align*}
\]
and this completes the proof. ■

From the above elementary result, we have the following. That is, we prove that cosine rule is equivalent with sine rule.

**Theorem 4.2** We can derive cosine rule from sine rule.

**Proof.** From \( \angle (a, b) + \angle (b, c) + \angle (c, -a) = \pi \), we have
\[
\sin \angle (b, c) = \sin (\angle (a, b) + \angle (c, -a)), \quad \cos \angle (b, c) = -\cos (\angle (a, b) + \angle (c, -a)).
\]
And so,
\[
\begin{align*}
K^2(\|b\|^2 + \|c\|^2 - \|a\|^2) & = \|a\|^2 \|b\|^2 \|c\|^2 [\sin^2 \angle (c, -a) + \sin^2 \angle (a, b) - \sin^2 \angle (b, c)] \\
& = -2\|a\|^2 \|b\|^2 \|c\|^2 \sin \angle (a, b) \cdot \sin \angle (c, -a) \cdot \cos \angle (b, c) \\
& = -2K^2 \|b\| \|c\| \cos \angle (b, c),
\end{align*}
\]
and this completes the proof. ■

From Theorem 3.1 and Theorem 4.1, we have the result as follow.
Theorem 4.3 In every $\Delta [a, b, c]$, the following statements are equivalent
(a). Cosine rule
(b). Sine Rule
(c). Triangle sides formula

5. Further results
Following [7, 8], we define, for any $a, b \in V_*$,
$$
S_\ast (a, b) := \{ \lambda a + (1 - \lambda) b : 0 \leq \lambda \leq 1 \};
$$
the line segment joining $a$ and $b$. We also define its length by $|S_\ast (a, b)| := \|b - a\|$.

We denote $[b; a; c]$, the angle between $S_\ast (a, b)$ and $S_\ast (a, c)$, which defined by
$$
\cos [b; a; c] := \frac{\|b - a\|^2 + \|c - a\|^2 - \|b - c\|^2}{2 \|b - a\| \|c - a\|}.
$$

Of course, we only consider that $0 \leq [b; a; c] \leq \pi$.

If we consider the notion of betweenness coined by Pasch in 1882 (see [2] for details), then for every $a, b, c \in V_*$, the following result say that $a$ is between $b$, and $c$, if and only if $[b; a; c] = \pi$.

In particular, 0 is always between $b$, and $-b$, for every $b \in V_* \setminus \{0\}$.

Theorem 5.1 If $b \in S_\ast (a, c)$, or $c \in S_\ast (a, b)$, then $[b; a; c] = 0$. Moreover, we also have $[b; a; c] = \pi$, if and only if $\|b - a\| + \|c - a\| = \|b - c\|$. In particular, we always have $[b; 0; -b] = \pi$.

Proof. Suppose $b := \lambda a + (1 - \lambda) c$, for some $0 < \lambda < 1$. Then
$$
\frac{\langle b - a, c - a \rangle}{\|b - a\| \|c - a\|} = (1 - \lambda) \cdot \frac{\|c - a\|}{\|b - a\|} = 1.
$$

The second part of the theorem follow from the equivalence below,
$$
\|a - c\| + \|a - b\| = \|c - b\| \iff -1 = \frac{\|b - a\|^2 + \|c - a\|^2 - \|b - c\|^2}{2 \|b - a\| \|c - a\|} \iff [b; a; c] = \pi,
$$
and this complete the proof. ■

6. Concluding remark
From the result of this study, we still have another opportunity to extend the above results, but now with the different definition of angle.

Even in any normed spaces $V := (V, \|\cdot\|)$, we still can define another form of angle between $a, b \in V$, by the definition
$$
\cos \angle_I (a, b) := \frac{\|a + b\|^2 - \|a - b\|^2}{4 \|a\| \|b\|}.
$$

From the above definition of angle, we will have another form of cosine rule for $\angle_I$,
$$
\begin{align*}
\|a\|^2 &= \|b + c\|^2 - 4 \|b\| \|c\| \cos \angle_I (b, c), \\
\|b\|^2 &= \|c - a\|^2 - 4 \|c\| \|a\| \cos \angle_I (c, -a), \\
\|c\|^2 &= \|a + b\|^2 - 4 \|a\| \|b\| \cos \angle_I (a, b).
\end{align*}
$$
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