Construction of Diffusion Algebras

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Abstract

In \cite{1} Diffusion algebras have been introduced in the context of one-dimensional stochastic processes with exclusion in statistical mechanics. While this reference is focused on the needs of the physicist reader and thus states results without proofs and focuses on the discussion of lower-dimensional examples, it is the purpose of this paper to present a construction formalism for Diffusion algebras and to use the latter to prove the results in that reference.

1 Introduction

Diffusion algebras play a key role in the understanding of one-dimensional stochastic processes. In the case of \(N\) species of particles with only nearest-neighbour interactions with exclusion on a one-dimensional lattice, Diffusion algebras are useful tools in finding expressions for the probability distribution of the stationary state of these processes. Following the idea of matrix product states \cite{2,3}, the latter are given in terms of monomials built from the generators of a quadratic algebra. Depending on whether the system is closed, i.e. the stochastic process is defined on a ring, or open, in which case boundary conditions at the end of the lattice come into play, this expression varies; \cite{1} presents an exposition of these facts and the reader is referred to this reference and references within for more details about the application of Diffusion algebras in physics.

It is the purpose of this work to treat Diffusion algebras from the mathematician’s point of view and to prove a construction theorem for Diffusion algebras. We consider the following setting:

Let \(\alpha < \beta \in I_N := \{1, \ldots, N\}\) and consider quadratic relations of the form

\[g_{\alpha\beta} D_\alpha D_\beta - g_{\beta\alpha} D_\beta D_\alpha = x_\beta D_\alpha - x_\alpha D_\beta\] (1)

with \(g_{\alpha\beta} \in \mathbb{R} \setminus \{0\}, g_{\beta\alpha} \in \mathbb{R}\) and \(x_\alpha \in \mathbb{C}\). Then on has

\begin{definition}
An algebra with generators \(\{D_\alpha | \alpha \in I_N\}\) and relations of type (1) is called Diffusion algebra, if it admits a linear PBW-basis of ordered monomials of the form

\[D_{\alpha_1}^{k_1} D_{\alpha_2}^{k_2} \ldots D_{\alpha_n}^{k_n}, \quad k_j \in \mathbb{N}^0\] (2)
\end{definition}

\footnote{Note that dependence on all the nonvanishing coefficients \(x_\alpha\) in (1) is easily suppressed by rescaling the elements \(D_\alpha\) as \(x_\alpha D_\alpha\). We choose to display the dependence on these coefficients here, because they are important in applications to stochastic models, which is the physical motivation for the study of this type of algebras.}
with \( \alpha_1 > \alpha_2 > \ldots > \alpha_n \).

We remark that although we formulate the mathematical setting for coefficients in \( \mathbb{R} \) in the relations (1), for physical reasons only relations with positive coefficients \( g_{\alpha\beta} \in \mathbb{R}^{\geq 0} \) and \( g_{\beta\alpha} \in \mathbb{R}^{\geq 0} \) (\( \alpha < \beta \)) are relevant, because they are interpreted as hopping rates in stochastic models. Since we are treating Diffusion algebras from the mathematical point of view here, we will not impose this restriction, but comment on the implications of this restriction on our results after the main theorem.

The requirement of having a PBW basis (2) implies conditions on the coefficients \( g_{\alpha\beta} \) and \( x_\alpha \) in (1) according to the Diamond Lemma in Ring Theory [4]. In particular, the latter gives a criterion to check under which conditions the relations in (1) are of PBW type: it is the case if each subset of three generators \( \{D_\alpha, D_\beta, D_\gamma\} \) with ordering \( \alpha < \beta < \gamma \) is reduction unique with respect to the ordering, that is if the two ways of reducing the monomial \( D_\alpha D_\beta D_\gamma \) to the monomial \( D_\gamma D_\beta D_\alpha \) lead to the same result when expressed in the PBW basis (2).

The task of deriving all Diffusion algebras with \( N \) generators thus reduces to the following two steps:

1. Find all Diffusion algebras with three generators.
2. Find all algebras with \( N \) generators such that each subset of three generators coincides with one of the cases listed before.

The first is a trivial exercise, which amounts to finding those coefficients \( g_{\alpha\beta} \) and \( x_\alpha \) in (1) for which a set \( \{D_\gamma, D_\beta, D_\alpha\} \) of three generators is reduction unique in the above sense. The corresponding list of algebras is given in [1], and we review it here in order to set up notation and render this paper self-contained. The second is a combinatorial problem, and requires to combine in a consistent way the three generator algebras listed before to algebras with \( N \) generators for general \( N > 3 \).

A construction method for Diffusion algebras, and thus a constructive method to approach the second point, is the so-called blending procedure in the above reference, which is an inductive procedure for the construction of Diffusion algebras. It uses the three generator cases and augments them to larger units by attaching further generators in accordance with the requirements of the Diamond Lemma, then giving a prescription how these larger building blocks may be glued (or in the terminology of this reference “blended”) together in order to obtain a general Diffusion algebra of \( N \) generators. The advantage of the inductive procedure is that it facilitates the construction of representations, which are crucial for applications in physics. The main purpose of this paper is to provide a different construction method, which is more suitable for mathematical purposes, in particular, to deliver a proof for the fact that the set of algebras obtained via the blending procedure corresponds exactly to the set of Diffusion algebras in Definition 1.1.

After recalling the three generator case in Section 2, we present in Section 3 a derivation of Diffusion algebras from first principles. Furthermore, we present in this section a compact formulation for the blending procedure, and obtain the exhaustiveness of the inductive approach in [1] as a corollary to our main theorem.

## 2 Review of Diffusion algebras with 3 generators

As mentioned in the introduction, the three generator case provides the building blocks for the derivation of Diffusion algebras according to the Diamond Lemma and we therefore briefly recall the results of [1] for this case.

Consider a set \( \{D_\alpha, D_\beta, D_\gamma\} \) of three generators with an ordering induced by the ordering of the index set \( \alpha < \beta < \gamma \) and relations as in (1). Since \( g_{\alpha\beta} \neq 0 \) for
all $\alpha < \beta \in I_N$ by assumption, we can cast the relations into the following form

$$D_\alpha D_\beta = q_{\beta \alpha} D_\beta D_\alpha + x_{\beta \alpha}^\beta D_\alpha - x_{\alpha \beta}^\gamma D_\beta$$
$$D_\alpha D_\gamma = q_{\gamma \alpha} D_\gamma D_\alpha + x_{\gamma \alpha}^\gamma D_\alpha - x_{\alpha \gamma}^\alpha D_\gamma$$
$$D_\beta D_\gamma = q_{\gamma \beta} D_\gamma D_\beta + x_{\gamma \beta}^\gamma D_\beta - x_{\beta \gamma}^\alpha D_\gamma,$$

where $q_{ji} := \frac{g_{ij}}{g_{ji}}$, $x_{ij}^k := \frac{g_{ik} g_{ij}}{g_{ji}}$ for $k, i < j \in \{\alpha, \beta, \gamma\}$. Then, using (3), any monomial can be expressed in terms of the PBW basis (2). This leads to restrictions on the coefficients $g_{ij}$ and $x_{ij}$ in (3). In particular, they are constrained by a set of 6 equations (see (2.5) – (2.10) in (1)) and their solutions determine all Diffusion algebras of three generators. The latter are listed here for future convenience and in order to set up notations. Throughout this section, we assume $\alpha < \beta < \gamma$ and $x_j \neq 0$ for $j \in \{\alpha, \beta, \gamma\}$.

1. The case of $A_1$:

$$g[D_\alpha, D_\beta] = x_{\beta \alpha} D_\alpha - x_{\alpha \beta} D_\beta$$
$$g[D_\alpha, D_\gamma] = x_{\gamma \alpha} D_\alpha - x_{\alpha \gamma} D_\gamma$$
$$g[D_\beta, D_\gamma] = x_{\gamma \beta} D_\beta - x_{\beta \gamma} D_\gamma$$

where $g \neq 0$.

2. The case of $A_2$:

$$g_{\alpha \beta} D_\alpha D_\beta = x_{\beta \alpha} D_\alpha - x_{\alpha \beta} D_\beta$$
$$g_{\alpha \gamma} D_\alpha D_\gamma = x_{\gamma \alpha} D_\alpha - x_{\alpha \gamma} D_\gamma$$
$$g_{\beta \gamma} D_\beta D_\gamma = x_{\gamma \beta} D_\beta - x_{\beta \gamma} D_\gamma$$

where $g_{ij} := g_i - g_j$ with $g_i \neq g_j$ for all $i < j \in \{\alpha, \beta, \gamma\}$.

3. The case of $B^{(1)}$:

$$g_{\beta} D_\alpha D_\beta - (g_{\beta} - \Lambda) D_\beta D_\alpha = -x_{\alpha \beta} D_\beta$$
$$g_{\delta} D_\alpha D_\gamma - (g_{\delta} - \Lambda) D_\gamma D_\alpha = x_{\gamma} D_\alpha - x_{\alpha \gamma} D_\gamma$$
$$g_{\beta} D_\beta D_\gamma - (g_{\beta} - \Lambda) D_\beta D_\beta = x_{\gamma} D_\beta$$

where $g \neq 0$ and $g_{\beta} \neq 0$. For the same ordering, we also find relations of type $B^{(1)}$ which are relations (5) with an exchange $\alpha \leftrightarrow \beta$ or $\gamma \leftrightarrow \beta$ and restrictions $g \neq 0$ and $g_{\alpha} \notin \{0, \Lambda\}$ or, respectively, $g \neq 0$ and $g_{\gamma} \notin \{0, \Lambda\}$ on the parameters.

4. The case of $B^{(2)}$:

$$g_{\alpha \beta} D_\alpha D_\beta = -x_{\alpha \beta} D_\beta$$
$$g_{\alpha \gamma} D_\alpha D_\gamma - g_{\alpha \gamma} D_\gamma D_\alpha = x_{\gamma} D_\alpha - x_{\alpha \gamma} D_\gamma$$
$$g_{\beta \gamma} D_\beta D_\gamma = x_{\gamma} D_\beta$$

where $g_{\alpha \beta}$, $g_{\alpha \gamma}$ and $g_{\beta \gamma}$ $\neq 0$.

5. The case of $B^{(3)}$:

$$g D_\alpha D_\beta - (g - \Lambda) D_\beta D_\alpha = x_{\beta \alpha} D_\alpha - x_{\alpha \beta} D_\beta$$
$$g D_\alpha D_\gamma = -x_{\alpha \beta} D_\gamma$$
$$g D_\beta D_\gamma = -x_{\beta \gamma} D_\gamma$$

where $g \neq 0$ and $g_{\gamma} \notin \{0, \Lambda\}$.
6. The case of $B^{(4)}$:

\begin{align}
(g_\alpha - \Lambda)D_\alpha D_\beta &= x_\beta D_\alpha \\
g_\alpha D_\alpha D_\gamma &= x_\gamma D_\alpha \\
gD_\beta D_\gamma - (g - \Lambda)D_\gamma D_\beta &= x_\gamma D_\beta - x_\beta D_\gamma
\end{align}

(11)

where $g \neq 0$ and $g_\alpha \notin \{0, \Lambda\}$.

7. The case of $C^{(1)}$:

\begin{align}
 g_\beta D_\alpha D_\beta - (g_\beta - \Lambda) D_\beta D_\alpha &= -x_\alpha D_\beta \\
g_\gamma D_\alpha D_\gamma - (g_\gamma - \Lambda) D_\gamma D_\alpha &= -x_\alpha D_\gamma \\
g_{\beta\gamma} D_\beta D_\gamma - g_{\gamma\beta} D_\gamma D_\beta &= 0
\end{align}

(12)

where $g_\beta, g_\gamma$ and $g_{\beta\gamma} \neq 0$. For the same ordering, we also find relations of type $C^{(1)}$ which are relations (12) with an exchange $\alpha \leftrightarrow \beta$ or $\alpha \rightarrow \gamma \rightarrow \beta \rightarrow \alpha$ and restrictions $g_\alpha \neq \Lambda$ and $g_\gamma, g_{\alpha\gamma} \neq 0$ or, respectively, $g_\alpha, g_\beta \neq \Lambda$ and $g_{\alpha\beta} \neq 0$ on the parameters.

8. The case of $C^{(2)}$:

\begin{align}
g_{\alpha\beta} D_\alpha D_\beta - g_{\beta\alpha} D_\beta D_\alpha &= -x_\alpha D_\beta \\
g_{\alpha\gamma} D_\alpha D_\gamma - g_{\gamma\alpha} D_\gamma D_\alpha &= -x_\alpha D_\gamma \\
D_\beta D_\gamma &= 0
\end{align}

(13)

where $g_{\alpha\beta}$ and $g_{\alpha\gamma} \neq 0$. For the same ordering, we also find relations of type $C^{(2)}$ which are relations (13) with an exchange $\alpha \leftrightarrow \beta$ or $\alpha \rightarrow \gamma \rightarrow \beta \rightarrow \alpha$ and restrictions $g_{\alpha\beta}, g_{\beta\gamma} \neq 0$ or, respectively, $g_{\alpha\gamma}, g_{\beta\gamma} \neq 0$ on the parameters.

9. The case of $D$: With $g_{ij} := \frac{g_i}{g_j}$, $i, j \in \{\alpha, \beta, \gamma\}$ (recall that $g_{ij} \neq 0$ for $i < j$) we have

\begin{align}
 D_\alpha D_\beta - g_{\beta\alpha} D_\beta D_\alpha &= 0 \\
D_\alpha D_\gamma - g_{\gamma\alpha} D_\gamma D_\alpha &= 0 \\
D_\beta D_\gamma - g_{\gamma\beta} D_\gamma D_\beta &= 0
\end{align}

(14)

We remark that the division into algebras of type $A$, $B$, $C$ and $D$ reflects the number of coefficients $x_{ij}$, $j \in \{\alpha, \beta, \gamma\}$, being zero in comparison with the general form (1): for algebras of type $A$, $B$, $C$ and $D$ none respectively one, two, or all three of the coefficients $x_{ij}$ vanish. The subdivision for each type then corresponds to the different choices for the coefficients $g_{i\alpha}$ which are compatible with the Diamond Lemma.

3 The case of general $N$

This section consists of four parts: we start by providing a decomposition of the index set which later facilitates the presentation of the algebras. In other words, we decompose the whole family of algebras, which depends on the ordered set of parameters $\{g_{\alpha\beta}, x_{\alpha}[\alpha, \beta \in I]\}$ in the relations (1), into several subfamilies. Each subfamily is determined by a specific subset of the parameters $x_{\alpha}$ and $g_{\alpha\beta}$, which are subject to a set of conditions formulated below (see conditions (16), (21), (23), (24) and (26) below).

As a next step, we list some general properties specific to Diffusion algebras in each of the subfamilies. They are later used in the proof of the main result. This is followed by the list of Diffusion algebras and a theorem which proves the exhaustiveness of the approach. We finally comment on the counting of Diffusion algebras.
3.1 Decomposition of the index set

The structure of the algebras in (6) – (14) suggests the following decomposition of the index set $I_N = \{1, \ldots, N\}$:

$$I_N = I \cup R$$  \hspace{1cm} (15)

where

$$I := \{\alpha \in I_N | x_\alpha \neq 0\}$$

$$R := \{\alpha \in I_N | x_\alpha = 0\}.$$  \hspace{1cm} (16)

We will use in the following the notation $N_I := |I|$ and $N_R := |R|$ for the cardinalities of these sets.

We introduce the following terminology and notations:

**Definition 3.1** Normal ordering of two generators $D_\alpha$ and $D_\beta$ is defined as

$$: D_\alpha D_\beta : = \begin{cases} D_\alpha D_\beta & \text{if } \alpha < \beta \\ D_\beta D_\alpha & \text{if } \beta < \alpha \end{cases}$$  \hspace{1cm} (17)

**Definition 3.2** For $\alpha < \beta$ we introduce the following short-hand notation:

$$[D_\alpha, D_\beta]_{q_{\beta\alpha}} := D_\alpha D_\beta - q_{\beta\alpha} D_\beta D_\alpha$$  \hspace{1cm} (18)

where the index at the commutator is referring to the coefficients $q_{\beta\alpha}$ in terms of which the commutator is defined.

Using these notation, we subdivide the set $R$ into nonintersecting and non-empty subsets

$$R := R_1 \cup R_2 \cup \ldots \cup R_M,$$  \hspace{1cm} (19)

according to the following requirements:

- Relations between generators from the sets $R_a$ and $R_b$ for $a \neq b$ are given by:

$$: D_{r_1} D_{r_2} : = 0 \quad \forall r_1 \in R_a \text{ and } \forall r_2 \in R_b.$$

- Relations within a set $R_a$ such that $|R_a| \geq 2$ are given by:

$$[D_{r_1}, D_{r_2}]_{q_{r_2r_1}} = 0 \quad \forall r_1 < r_2 \in R_a,$$  \hspace{1cm} (21)

where the coefficients in (21) are subject to the condition opposite to (20), that is: for any subdivision $R_a = R' \cup R''$ into two nonintersecting and non-empty parts $R'$ and $R''$

$$\exists r_1 \in R' \text{ and } r_2 \in R'' : g_{r_1 r_2} r_{r_1 r_2} \neq 0.$$  \hspace{1cm} (22)

In other words, this means that for any pair of indices $r, s \in R_a$ there exists a finite sequence $\{r_k \in R_a | k = 1, \ldots, n\}$ such that $r_1 = r, r_n = s$ and

$$\prod_{k=1}^{n-1} g_{r_k r_{k+1}} g_{r_{k+1} r_k} \neq 0.$$  \hspace{1cm} (23)

Thus, the relations (22) and (23) may be represented graphically via a connectivity condition on an ordered graph the vertices of which are labelled by the indices $r \in R_a$ and the edges connect only those vertices $r_1 < r_2$ for which the condition $q_{r_2 r_1} \neq 0$ is satisfied.
Furthermore, for $N_I \geq 2$ we split the set $R$ into two sets $S$ and $T$ as follows:

For any $R_a \subset R$ we define

$$R_a := \begin{cases} S_a & \text{if } \exists r \in R_a \text{ and } i \in I : g_{ir}g_{ri} \neq 0, \\ T_a & \text{otherwise .} \end{cases} \quad (24)$$

Suppose that the $M_R$ sets $R_a$ in (19) split into $M_S$ sets $S_a$ and $M_T$ sets $T_a$ in this way, thus $M_R = M_S + M_T$. We number these sets as $S_a, a = 1, \ldots, M_S$, and $T_a, a = 1, \ldots, M_T$, and introduce

$$S := \bigcup_{a=1}^{M_S} S_a, \quad T := \bigcup_{a=1}^{M_T} T_a. \quad (25)$$

Although the decomposition of the set $S$ into subsets $S_a$ has been used in the definition of the set $S$, it will not be of practical importance in what follows. Contrary to that, the structure of the set $T$ is crucial and needs further refinement.

For any $T_a \subset T$ define

$$T_a := \begin{cases} T_a^* & \text{if } \exists i < j \in I : T_a \subset \{i + 1, i + 2, \ldots, j - 1\} \text{ and } \\ I \cap \{i + 1, i + 2, \ldots, j - 1\} = \emptyset, \\ T_a^o & \text{otherwise .} \end{cases} \quad (26)$$

Thus in short hand notation $T = \{T_a^*|a = 1, \ldots, M_T^*\} \cup \{T_a^{|o}|a = 1, \ldots, M_T^{|o}\}$ with $M_T = M_T^* + M_T^{|o}$.

### 3.2 General structural remarks about $N$-generator Diffusion algebras

Until now we have primarily discussed index sets. By an abuse of terminology, we will from now on also refer to “generators of a set $I$, $S$, $T$, or $R$” meaning the generators indexed by elements from the corresponding set.

**Definition 3.3** A set of three generators $\{D_x, D_y, D_z\}$ with $x, y$ and $z \in X, Y, Z$, respectively, where $X, Y$ and $Z$ are any of the sets $I, R, S$ and $T$ or any set in their decomposition will be called a triplet (of type) $\{X, Y, Z\}$.

Note that any triplet of type $\{I, I, I\}$ in a Diffusion algebra of $N \geq 3$ generators gives rise to a Diffusion algebra of type $A_I$ or $A_{II}$, any triplet of type $\{I, I, R\}$ to a Diffusion algebra of type $B^{(1)}$, $B^{(2)}$, $B^{(3)}$, or $B^{(4)}$, any triplet of type $\{I, R, R\}$ to a Diffusion algebra of type $C^{(1)}$ or $C^{(2)}$ and any triplet of type $\{R, R, R\}$ to a Diffusion algebra of type $D$.

Then we have:

**Lemma 3.4** For any Diffusion algebra $[4]$ with $N \geq 3$ generators the following statements hold

1. If $N_I \geq 3$, then all subalgebras corresponding to triplets of type $\{I, I, I\}$ are of the same type, which is either $A_I$ (that is, $g_{ij} = g \ \forall i, j \in I$) or $A_{II}$ (that is, $g_{ji} = g, g_{ij} = g_i - g_j, g_i \neq g_j, \forall i < j \in I$).

2. If $N_I \geq 3$ and all subalgebras corresponding to triplets $\{I, I, I\}$ are of type $A_I$ then for any $s \in S$ and for all $i \in I$ one has

$$g_{is} = g_{si} = g_s. \quad (27)$$

3. If $N_I \geq 3$ and all subalgebras corresponding to triplets $\{I, I, I\}$ are of type $A_{II}$ then $S = \emptyset$. 

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4. Let \( N_I \geq 2 \).

For any \( i \in I \) and for all \( t, t' \in T_a \) (here \( T_a \) means both \( T_a^0 \) and \( T_a^* \)) with \( t < i \) and \( t' > i \) the coefficients \( g_{it} \) and \( g_{it'} \) depend only on the index \( a \) of the set \( T_a \) and not on the individual indices \( t \) or \( t' \). If \( t, t' \in T_a \) one furthermore has \( g_{ii} = -g_{it'} \).

For any \( i < j \in I \) and for all \( t, t' \in T_a \): \( t < i \) and \( t' > j \)

\[
\begin{align*}
    g_{it} + \Lambda_{ij} &= g_{ij}, \\
    g_{it'} &= g_{ij} + \Lambda_{ij},
\end{align*}
\]

where \( \Lambda_{ij} := g_{ij} - g_{ji} \).

5. Let \( N_I = 1 \). Denote the only index in \( I \) as \( \mathbf{i} \) in order to stress that it is not a running index. For all \( r \in R_a \) one has

\[
g_{ir} - g_{ri} = \Lambda_a.
\]

Note that both the left and the right hand sides of relation (30) depend only on the index \( a \) of the set \( R_a \) and not on the individual index \( r \).

Proof. 1. It follows from the fact that in each set \( \{D_i, D_j, D_k \mid i, j, k \in I\} \) one has either \( g_{\beta \alpha} \neq 0 \) for all \( \alpha < \beta \in \{i, j, k\} \), or, \( g_{\beta \alpha} = 0 \), for all \( \alpha < \beta \in \{i, j, k\} \), but no mixture thereof, which contradicts a mixing of \( A_I \) and \( A_{II} \) type algebras.

2. Let \( N_I \geq 3 \) and \( g_{ij} = g \forall i, j \in I \). By (25) it is enough to check (27) for any \( S_a \subset S \) which satisfies the condition \( g_{ij} g_{jr} \neq 0 \) for some \( j \in I \). Then, for any \( i \in I \), the triplets \( \{D_r, D_i, D_j\} \) are all of type \( B(1) \) with \( \Lambda = 0 \) and, hence, \( g_{ir} = g_{ri} = g_r \).

Next, take any \( s \in S_a \). By definition, there exists a sequence \( r_k \in S_a \), \( k = 1, \ldots, n \), such that \( r_1 = r \), \( r_n = s \) and such that the connectivity condition (23) is satisfied. Then, for any \( i \in I \), starting with the triplet \( \{D_i, D_j, D_k\} \) one inductively proves that all the triplets \( \{D_{r_k}, D_{r_{k+1}}, D_i\} \) are of type \( C(1) \) with \( \Lambda = 0 \), and hence (27) follows.

3. Let \( N_I \geq 3 \) and \( g_{ij} = 0 \), \( g_{ij} = g_i - g_j \), \( g_i \neq g_j \), \( \forall i < j \in I \) and suppose \( S \neq \emptyset \). Consider some \( S_a \subset S \) and take those indices \( r \in S_a \) and \( i_0 \in I \) for which condition \( g_{ir} g_{r_0} \neq 0 \) is satisfied. For any \( j, k \in I \) the triplets \( \{D_{i_0}, D_j, D_k\} \) and \( \{D_{i_0}, D_k, D_r\} \) are both of type \( B(1) \), which also implies that the triplets \( \{D_j, D_k, D_r\} \), \( \forall j, k \in I \), are all of type \( B(1) \). Now, there is no mutual ordering of any arbitrarily chosen indices \( i < j < k \in I \) and the index \( r \in S_a \) for which the existence of any \( B(1) \)-type triplet \( \{D_i, D_j, D_k\} \), \( \{D_i, D_k, D_r\} \) and \( \{D_j, D_k, D_r\} \) is compatible with the condition that \( g_{ik} = g_{ij} + g_{jk} \), a contradiction.

4. Let \( N_I \geq 2 \) and consider any three indices \( i \in I \) and \( t < t' \in T_a \). Exploiting the connectivity property (23) of the set \( T_a \) one can find a sequence \( \{t_k \mid k = 1, \ldots, n\} \) such that \( t_1 = t \), \( t_n = t' \) and such that all the \( C \)-type triplets \( \{D_{t_k}, D_{t_{k+1}}, D_{t_{k+1}}\} \) are not of type \( C(2) \). Hence, their corresponding nonzero coefficients \( g_{it_k} \) (for \( i < t_k \)) or \( g_{it_k} \) (for \( t_k < i \)) are subject to the conditions for triplets of type \( C(1) \) (see (23)) which together with the definition (23) of the set \( T_a \) implies \( g_{it} = g_{it'} \) in the case \( i < t < t' \), \( g_{ti} = g_{ti} \) in the case \( t < t' < i \) and \( g_{ti} = -g_{it'} \) in the case \( t < i < t' \), thus proving the first part of the fourth statement in the Lemma.

To prove the second part notice that for any four indices \( i, j, \in I \) and \( t, t' \in T_a \) which are ordered as \( t < i < j < t' \) their corresponding triplets \( \{D_i, D_j, D_k\} \) and \( \{D_i, D_j, D_r\} \) are of type \( B(4) \) and, respectively, \( B(3) \). Conditions (28) and (29) then reproduce the relations between the coefficients in those triplets (see eqs. (10) and (11)).
5. Let \( N_I = 1 \) and consider any pair of indices \( r, r' \in R_a \). As before, for every connective set \( R_a \) there exists a chain of \( C \)-type triplets \( \{ D_i, D_{r_k}, D_{r_{k+1}} \} \), \( k = 1, \ldots, n-1 \), with \( r_1 = r \) and \( r_n = r' \) which are not of type \( C^{(2)} \). Hence one obtains \( C^{(1)} \) with one and the same coefficient \( A_\alpha \) for all \( C^{(1)} \) type triplets \( \{ I, R_a, R_a \} \). \( \Box \)

Lemma 3.4 suggests to list Diffusion algebras in families according to the number of generators in the set \( I \) and provides information about the structure of relations among generators from the sets \( I, S, T \) and \( R \) in each case.

3.3 List of Diffusion algebras with \( N \) generators

In this subsection, we list all \( N \)-generator Diffusion algebras and provide a theorem which proves the exhaustiveness of the formalism.

Diffusion algebras with \( N \) generators are listed as five families of algebras: \( A_I, A_{II}, B, C \) and \( D \). As in the case of \( N = 3 \) the number of nonzero coefficients \( x_\alpha \) or, in other words, the cardinality of the set \( I \) is used as a criterion for separating Diffusion algebras into families of the types \( A(N_I \geq 3), B(N_I = 2), C(N_I = 1) \) or \( D(N_I = 0) \). Type \( A \) algebras are separated further into two families \( A_I \) and \( A_{II} \) depending on the number of nonzero coefficients \( g_{ij} \) with indices \( i, j \) in the set \( I \).

Different algebras in the families are obtained in dependence on the choice of the decomposition of the set \( I_N = \{1, 2, \ldots, N\} \) into ordered subsets \( I, S, T_a^0, a = 1, \ldots, M^a \), \( T_b^* \), \( b = 1, \ldots, M^b \) (or \( R_a, a = 1, \ldots, M_R \) for \( N_I = 1 \)) as well as on the choice of coefficients in their defining relations. Below we adopt a notation for Diffusion algebras where the corresponding decomposition of the set \( I_N \) is given explicitly as argument to the family symbol. To avoid any confusion let us stress that subscript indices \( a \) and \( b \) in our notation are treated as running ones so that, e.g.,

\[
A_I(I, S, T_a^0, T_b^*) = A_I(I, S, T_1^0, \ldots, T_M^0, T_1^*, \ldots, T_M^*)
\]

where we imply \( I_N = I \cup S \cup (\cup_{a=1}^{M^a} T_a^0) \cup (\cup_{b=1}^{M^b} T_b^*) \), and \( I, S, T_a^0 \) and \( T_b^* \) are mutually nonintersecting ordered subsets in \( I_N \). The values of the coefficients \( g_{\alpha\beta} \) are not shown explicitly in these notations so that in fact our notation displays connective components in a variety of Diffusion algebras rather than the particular algebras.

All relations in (31) – (35) below are to be complemented by relations (20), (21) for the elements of the subset \( R \) together with the conditions (22) or (23) on the coefficients involved.

1. Diffusion algebras of type \( A_I(I, S, T_a^0, T_b^*) \), \( N_I \geq 3 \):

\[
\begin{align*}
g[D_i, D_j] &= x_j D_i - x_i D_j, \quad \forall i, j \in I, \\
g_s[D_s, D_i] &= x_i D_s, \quad \forall s \in S, i \in I, \\
g_\alpha^a : D_i D_t : &= -x_i D_t, \quad \forall \alpha, t \in T_a^0, i \in I, \\
g_b^\pm D_i D_t &= -x_i D_t, \quad \forall b, t \in T_b^*, i \in I : i < t, \\
g_b^- D_i D_t &= x_i D_t, \quad \forall b, t \in T_b^*, i \in I : i > t,
\end{align*}
\]

where \( g, g_s, g_\alpha^a, g_b^\pm \neq 0 \).

2. Diffusion algebras of type \( A_{II}(I, T_a^0, T_b^*), N_I \geq 3 \):

\[
\begin{align*}
(g_i - g_j) D_i D_j &= x_j D_i - x_i D_j, \quad \forall i < j \in I, \\
(g_i + g_\alpha^a) : D_i D_t : &= -x_i D_t, \quad \forall \alpha, t \in T_a^0, i \in I, \\
(g_i + g_b^-) D_i D_t &= -x_i D_t, \quad \forall b, t \in T_b^*, i \in I : i < t, \\
(g_b^+ - g_b) D_i D_t &= x_i D_t, \quad \forall b, t \in T_b^*, i \in I : i > t,
\end{align*}
\]
where \( g_i \neq g_j \) for \( i \neq j \) and \( g_i \notin \{ g_a, g_b^\pm \} \).

3. Diffusion algebras of type \( B(I = \{ i, j \}, S, T_a, T_b^* \)):

We use the notation \( i \) and \( j \) with \( i < j \) for the two elements of the set \( I \) to emphasize that they are not running indices. Note also that \( i < t < j \) for all \( t \in T_b^* \) in this case.

\[
\begin{align*}
g D_i D_j - (g - \Lambda) D_i D_j &= x_j D_i - x_i D_j, \\
g_s D_i D_s - (g_s - \Lambda) D_s D_i &= -x_i D_s, \quad \forall s \in S, \\
g_s D_s D_j - (g_s - \Lambda) D_j D_s &= x_j D_s, \quad \forall s \in S, \\
g_a^o : D_i D_t : &= -x_i D_t, \quad \forall t \in T_a, \\
(g_a^o - \Lambda) : D_j D_t : &= -x_j D_t, \quad \forall t \in T_a, \\
g_b^o D_i D_t &= -x_i D_t, \quad \forall t \in T_b^* , \\
g_b^o D_j D_t &= x_j D_t, \quad \forall t \in T_b^* , \\
\end{align*}
\]

where \( g \neq 0, g_s \neq 0 \) for all \( s \) and \( g_s \neq \Lambda \) for \( s \) such that either \( s < i \) or \( s > j \), \( g_a^o \notin \{ 0, \Lambda \} \) and \( g_b^\pm \neq 0 \).

4. Diffusion algebras of type \( C(I = \{ i \}, R_a) \):

As in Lemma 3.4 the only element of \( I \) is denoted here as \( i \).

\[
g_r D_i D_r - (g_r - \Lambda_a) D_r D_i = -x_i D_r, \quad \forall r \in R_a, \quad (34)
\]

where \( g_r \neq 0 \) for \( r < i \) and \( g_r \neq \Lambda_a \) for \( r > i \).

5. Diffusion algebras of type \( D(R) \):

\[
D_r D_s - g_{rs} D_s D_r = 0, \quad \forall r < s \in R. \quad (35)
\]

**Theorem 3.5** The list of algebras given above is exhaustive and contains all possible Diffusion algebras with \( N \) generators.

**Proof.** According to the Diamond Lemma, an algebra of \( N \) generators with relations of type \( \{ \} \) is a Diffusion algebra if each of its triplets \( \{ D_a, D_b, D_c \} \) generates a subalgebra coinciding with one of the cases listed in Section 2. Lemma 3.4 provides information about possible consistent combinations of several such triplets and we thus have to demonstrate that the families of algebras \( \{ 31 \} - \{ 33 \} \) exhaust the list of Diffusion algebras which are allowed by this lemma.

Let us start with the case \( N_I \geq 3 \). According to the first statement of Lemma 3.4 there are two possible types of relations between generators from the set \( I \). This gives rise to two families of Diffusion algebras — \( A_I \) and \( A_{II} \). Statement 2 of Lemma 3.4 describes the relations between the generators from the sets \( I \) and \( S \) in the case of the family \( A_I \), and the third statement of Lemma 3.4 excludes the presence of a nonempty set \( S \) in the case of the family \( A_{II} \). The coefficients in the relations between the generators from the set \( I \) and the sets \( T_a \) and \( T_b^* \) are subject to the conditions given in the fourth statement of Lemma 3.4, where \( \Lambda_{ij} = 0 \) for the \( A_I \) family and \( \Lambda_{ij} = (g_i - g_j) \) for the \( A_{II} \) family. These conditions fix the relations between the generators in the families of type \( A_I \) and \( A_{II} \) to the expressions in \( \{ 31 \} \) and \( \{ 32 \} \). Since the triplets of the form \( \{ I, S, T \} \) (occurring only for the \( A_I \) family)
and \( \{I, T_a, T_b\} \) for \( a \neq b \) are of type \( C^{(2)} \), no further conditions arise from these relations. This exhausts all possibilities, and thus no further conditions occur.

Let \( N_I = 2 \). For each subset \( S_a \subset S \) the connectivity property \( 23 \) implies relations of type \( B^{(1)} \) for all triplets \( \{I, I, S_a\} \), and thus for any triplet \( \{I, I, S\} \). The corresponding relations are listed in the first three lines of \( 33 \). The compatibility conditions within the triplets \( \{I, T_a, T_b\} \) and \( \{I, I, T_a\} \) are given in the fourth statement of Lemma \( 3.4 \) where we now have \( A_{ij} \equiv \Lambda \). These conditions fix the form of the last four lines in \( 33 \). Since \( C^{(2)} \) type relations for \( \{I, S, T\} \) and \( \{I, T_a, T_b\} \) \( (a \neq b) \) triplets do not imply further restrictions, no further constraints arise.

In the case \( N_I = 1 \) a decomposition of the set \( R \) into \( S \) and \( T \) is not necessary, and we thus work with the whole set \( R \). Then the form of the relations \( 24 \) is implied by the fifth statement of Lemma \( 3.4 \) which describes the compatibility conditions for the \( \{I, R_a, R_b\} \) triplets. The relations in the \( C^{(2)} \) triplets \( \{I, R_a, R_b\} \) for \( a \neq b \) give no further constraints.

In the case \( N_I = 0 \) all the triplets are of type \( D \), which are compatible without any restrictions on the coefficients. \( \Box \)

Note that while mathematically possible, not all algebras in the families are relevant from the physicist’s point of view. Due to the fact that the structure constants of Diffusion algebras are interpreted as hopping rates, that is probabilities, in the framework of stochastic processes on linear lattices, only non-negative structure constants are relevant. This not only implies restrictions on the structure constants themselves, but also on the decompositions of the set \( I_N \), because some configurations are not compatible with non-negative structure constants. In particular, due to Lemma \( 3.4 \) statement 4 part 1, non-negative structure constants throughout are possible only if the subsets \( T^a \) fulfill one of the following two requirements:

1. \( \forall t \in T^a \) and \( \forall i \in I : t < i \) or 2. \( \forall t \in T^a \) and \( \forall i \in I : t > i \). \( 36 \)

We conclude this section with some comments on the classification problem for Diffusion algebras. To deal with the problem one should first establish criteria of equivalence, and we discuss two natural ones here:

- One can consider linear transformations on the set of generators \( \{D_\alpha | \alpha \in I_N\} \). However, there is the difficulty that not all linear transformations respect the ansatz \( 1 \). There are two special cases: rescaling transformations \( D_\alpha \rightarrow \kappa_\alpha D_\alpha \) and substitution transformations \( D_\alpha \rightarrow D_{\sigma(\alpha)} \), where \( \sigma \) is an element of the symmetric group \( S_N \).

As has already been mentioned in the introduction, rescalings may be used to fix (depending on the context of the physical application) some special values for the nonzero coefficients \( x_\alpha \). In particular, this implies that the values of the nonzero coefficients \( x_\alpha \) are not relevant.

The substitution transformations clearly respect the form of the relations \( 1 \), but may contradict the requirement on the mutual ordering of the generators, that is \( g_{\alpha \beta} \neq 0 \) for \( \alpha < \beta \). In particular, a permutation of the elements from different subsets \( T_\alpha \) and between the subsets \( T \) and \( S \), or \( T \) and \( I \) is strictly forbidden. In addition, one cannot permute two elements \( r < s \) in the same subset \( R_a \) unless \( q_{as} \neq 0 \). On the other hand, permutations inside the subset \( I \) and (in most cases) between the subsets \( I \) and \( S \) are allowed unless they contradict the requirements described above. Thus, substitution transformations establish certain equivalence classes inside each of the families \( A_I(I, S, T^a_\alpha, T^*_b) \), \( A_H(I, T^a_\alpha, T^*_b) \), \( B(\{i, j\}, S, T^a_\alpha, T^*_b) \), \( C(\{i\}, R_a) \) and \( D(R) \).
These equivalence classes can be calculated in concrete cases, but one hardly expects their complete description in the case of general $N$.

Note that besides the rescalings and the substitutions which do always exist there may occur other types of linear transformations which relate different types of Diffusion algebras. For instance, in the case of $N = 3$ the $C^{(1)}$ type algebras in (12) with $\Lambda \neq 0$ can be reduced to (a subclass of) $D$ type algebras by the transformation $D_\alpha \to D_\alpha - x_\alpha / \Lambda$. For general $N$, such transformations allow to reduce the number of nonzero parameters $\Lambda_a$ in the family of $C$ type Diffusion algebras in (34) by 1.

- One can use the algebra antihomomorphism which inverts simultaneously multiplication in the algebra, that is $D_\alpha D_\beta \to D_\beta D_\alpha$, and the order of indices, that is $\alpha < \beta \to \alpha > \beta$. This transformation amounts to a mirror reflection of the corresponding stochastic processes. For example, in the list of Diffusion algebras with $N = 3$ the families $B^{(3)}$ and $B^{(4)}$ are mirror symmetric. Further examples of mirror symmetry for the case $N = 4$ can be found in [1] in Appendix B.

3.4 Description of the blending procedure

The blending procedure is a constructive method to generate Diffusion algebras. The corresponding Construction Theorem states that any Diffusion algebra can be obtained from a set of building blocks (equations (4.1)–(4.7) in [1]) via blending. In the table below we describe the correspondence between the building blocks from [1] (left column) and the specific subclasses of the families in Theorem 3.5 (right column):

\begin{align*}
A^{(1)}_I & : A_I(I, S), \quad T = \emptyset, \\
A^{(2)}_I & : A_I(I, T^o), \quad A_{II}(I, T^\ast), \quad S = \emptyset, \\
A_{II} & : A_{II}(I, T^o), \quad A_{II}(I, T^\ast), \\
B^{(1)} & : B(I = \{i, j\}, S), \quad T = \emptyset, \\
B^{(2)} & : B(I = \{i, j\}, T^o), \quad B(I = \{i, j\}, T^\ast), \quad S = \emptyset, \\
C & : C(I = \{i\}, R), \\
D & : D(R).
\end{align*}

Here it is understood that the sets $T^o$ and $T^\ast$ whenever they appear in the right column of the table are the only connective components in the decomposition (22) of the subset $T \in I_N$. The connectivity condition may be also imposed on the subsets $S$ and $R$ in the right column of (37). We remark that the mathematical setting adopted in the present paper allow us to extract elementary building blocks for the blending procedure. They are shown in the right column of the table (37) and the blocks listed in the left column and used in [1] can be constructed by blending of an arbitrary number of the corresponding blocks from the right column. Note that in the settings of Ref.[1] extracting the elementary blocks would only amount to imposing additional connectivity conditions (23) on the coefficients $g_{\alpha\beta}$ in the relations (4.1)–(4.7) there and so would not suit the purposes of [1].

Furthermore we remark that in contrast to [1] we do not fix the order between $D_i$, $D_j$ and $D_s$ to $i < s < j$ for all $s \in S$ in $B(I = \{i, j\}, S)$, because the other orders are needed when blending with $B(I = \{i, j\}, T^o)$ and $B(I = \{i, j\}, T^\ast)$ in order to obtain all Diffusion algebras. This is an inaccuracy in the formulation of the Construction Theorem in [1]. Despite that in the list of $N = 4$ Diffusion algebras given in Appendix B of [1] the blending of such blocks is treated correctly (see example 13 there).
Let \( X_l(I, U_l) \), \( l = 1, \ldots, K \) denote \( K \) building blocks in the list (37) above, where \( U_l \) refers to the set \( R, S, T^\circ \) or \( T^* \) corresponding to the building block, and which are such that they have the same number of elements \( N_l \) in \( I \) with generators \( D_i \), \( i \in I \) satisfying in all blocks \( X_l \) the same relations among themselves.

Consider an ordered set \( I \) whose elements are labelled by the indices from the sets \( I, U_1, \ldots, U_K \) and such that for any \( l = 1, \ldots, K \) the order of the elements of \( I \) with their labels form \( I \) and \( U_l \) is the same as the order of the indices in the block \( X_l(I, U_l) \). In this situation we say that the order on \( I \) is compatible with the orders in the blocks \( X_l \).

Let us denote as \( X_I(I, U_1, \ldots, U_K) \) the algebra with generators labelled by the elements in the set \( I \) and which satisfies the following conditions:

- For any \( l = 1, \ldots, K \) the generators of \( X_I \) with indices from the subsets \( I, U_l \subseteq I \) satisfy the same relations as their corresponding generators in the blocks \( X_l(I, U_l) \).
- For any \( l_1 \neq l_2 \in \{1, \ldots, K\} \) and for all \( a \in U_{l_1} \subseteq I \) and \( b \in U_{l_2} \subseteq I \) the corresponding generators \( D_a, D_b \in X_I \) satisfy the relation

\[
: D_a D_b := 0 .
\]

The procedure of constructing the algebras \( X_I \) from their building blocks \( X_l(I, U_l) \) is called blending. Clearly the number of different algebras \( X_I \) which are associated with the set of building blocks \( \{X_l(I, U_l)\}_{l=1,\ldots,K} \), coincides with the number of different ordered sets \( I \) whose order is compatible with the orders in all blocks \( X_l \).

The following statement made in [1] is a corollary to the Construction Theorem 3.5:

**Theorem 3.6** Every Diffusion algebra can be obtained via a blending of building blocks in (37).

### 4 Conclusion

We have presented a derivation of Diffusion algebras, which has led to five different families of algebras: \( A_I, A_{II}, B, C \) and \( D \) and it has been shown that the approach is exhaustive. Since these families of algebras correspond to the algebras obtained via the bending procedure in [1], this also proves the Construction Theorem in this reference.

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