On some Chebyshev type inequalities for the complex integral

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Abstract. Assume that \(f\) and \(g\) are continuous on \(\gamma\), \(\gamma \subset \mathbb{C}\) is a piecewise smooth path parametrized by \(z(t), t \in [a, b]\) from \(z(a) = u\) to \(z(b) = w\) with \(w \neq u\), and the complex Chebyshev functional is defined by

\[
D_\gamma(f, g) := \frac{1}{w - u} \int_\gamma f(z) g(z) \, dz - \frac{1}{w - u} \int_\gamma f(z) \, dz \frac{1}{w - u} \int_\gamma g(z) \, dz.
\]

In this paper we establish some bounds for the magnitude of the functional \(D_\gamma(f, g)\) under Lipschitzian assumptions for the functions \(f\) and \(g\), and provide a complex version for the well known Chebyshev inequality.

Keywords: Complex integral, Continuous functions, Holomorphic functions, Chebyshev inequality.

MSC2010: 26D15, 26D10, 30A10, 30A86.

Sobre algunas desigualdades tipo Chebyshev para la integral compleja

Resumen. Sean \(f\) y \(g\) funciones continuas sobre \(\gamma\), siendo \(\gamma \subset \mathbb{C}\) un camino suave por partes parametrizado por \(z(t), t \in [a, b]\) con \(z(a) = u\) y \(z(b) = w\), \(w \neq u\), y el funcional de Chebyshev complejo definido por

\[
D_\gamma(f, g) := \frac{1}{w - u} \int_\gamma f(z) g(z) \, dz - \frac{1}{w - u} \int_\gamma f(z) \, dz \frac{1}{w - u} \int_\gamma g(z) \, dz.
\]

En este artículo establecemos algunas cotas para la magnitud del funcional \(D_\gamma(f, g)\) bajo condiciones de lipschitzianidad para las funciones \(f\) y \(g\), y damos una versión compleja para la conocida desigualdad de Chebyshev.

Palabras clave: Integral compleja, funciones continuas, funciones holomórficas, desigualdad de Chebyshev.
1. Introduction

For two Lebesgue integrable functions \( f, g : [a, b] \to \mathbb{C} \), in order to compare the integral mean of the product with the product of the integral means, we consider the Chebyshev functional defined by

\[
C(f, g) := \frac{1}{b-a} \int_a^b f(t)g(t) \, dt - \frac{1}{b-a} \int_a^b f(t) \, dt \frac{1}{b-a} \int_a^b g(t) \, dt.
\]

In 1934, G. Grüss [17] showed that

\[
|C(f, g)| \leq \frac{1}{4} (M-m)(N-n), \tag{1}
\]

provided \( m, M, n, N \) are real numbers with the property that

\[-\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \text{ a.e. on } [a, b]. \tag{2}\]

The constant \( \frac{1}{4} \) in (1) is sharp.

Another, however less known result, even though it was obtained by Chebyshev in 1882, [8], states that

\[
|C(f, g)| \leq \frac{1}{12} \|f'\|_{\infty} \|g'\|_{\infty} (b-a)^2, \tag{3}
\]

provided that \( f', g' \) exist and are continuous on \([a, b]\) and \( \|f'\|_{\infty} = \sup_{t \in [a, b]} |f'(t)| \). The constant \( \frac{1}{12} \) cannot be improved in the general case.

The Chebyshev inequality (3) also holds if \( f, g : [a, b] \to \mathbb{R} \) are assumed to be absolutely continuous and \( f', g' \in L_{\infty} [a, b] \), while \( \|f'\|_{\infty} = \text{ess sup}_{t \in [a, b]} |f'(t)| \).

For other inequality of Grüss’ type see [1]-[16] and [18]-[28].

In order to extend Grüss’ inequality to complex integral we need the following preparations.

Suppose \( \gamma \) is a smooth path parametrized by \( z(t), t \in [a, b] \) and \( f \) is a complex valued function which is continuous on \( \gamma \). Put \( z(a) = u \) and \( z(b) = w \) with \( u, w \in \mathbb{C} \). We define the integral of \( f \) on \( \gamma_{u,w} = \gamma \) as

\[
\int_{\gamma} f(z) \, dz = \int_{\gamma_{u,w}} f(z) \, dz := \int_a^b f(z(t)) z'(t) \, dt.
\]

We observe that the actual choice of parametrization of \( \gamma \) does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose \( \gamma \) is parametrized by \( z(t), t \in [a, b] \), which is differentiable on the intervals \([a, c]\) and \([c, b]\); then, assuming that \( f \) is continuous on \( \gamma \), we define

\[
\int_{\gamma_{u,v}} f(z) \, dz := \int_{\gamma_{u,v}} f(z) \, dz + \int_{\gamma_{v,w}} f(z) \, dz,
\]

where \( v := z(c) \). This can be extended for a finite number of intervals.
We also define the integral with respect to arc-length:

\[ \int_{\gamma_{u,w}} f(z) |dz| := \int_{a}^{b} f(z(t)) |z'(t)| \, dt, \]

and the length of the curve \( \gamma \) is then

\[ \ell (\gamma) = \int_{\gamma_{u,w}} |dz| = \int_{a}^{b} |z'(t)| \, dt. \]

Let \( f \) and \( g \) be holomorphic in \( G \), an open domain, and suppose \( \gamma \subset G \) is a piecewise smooth path from \( z(a) = u \) to \( z(b) = w \). Then we have the integration by parts formula

\[
\int_{\gamma_{u,w}} f(z) g'(z) \, dz = f(w) g(w) - f(u) g(u) - \int_{\gamma_{u,w}} f'(z) g(z) \, dz. \tag{4}
\]

We recall also the triangle inequality for the complex integral, namely,

\[
\left| \int_{\gamma} f(z) \, dz \right| \leq \int_{\gamma} |f(z)| \, |dz| \leq \|f\|_{\gamma,\infty} \ell (\gamma), \tag{5}
\]

where \( \|f\|_{\gamma,\infty} := \sup_{z \in \gamma} |f(z)| \).

We also define the \( p \)-norm with \( p \geq 1 \) by

\[
\|f\|_{\gamma,p} := \left( \int_{\gamma} |f(z)|^p \, |dz| \right)^{1/p}.
\]

For \( p = 1 \) we have

\[
\|f\|_{\gamma,1} := \int_{\gamma} |f(z)| \, |dz|.
\]

If \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then, by Hölder’s inequality, we have

\[
\|f\|_{\gamma,1} \leq [\ell (\gamma)]^{1/q} \|f\|_{\gamma,p}.
\]

Suppose \( \gamma \subset \mathbb{C} \) is a piecewise smooth path parametrized by \( z(t), t \in [a, b] \) from \( z(a) = u \) to \( z(b) = w \) with \( w \neq u \). If \( f \) and \( g \) are continuous on \( \gamma \), we consider the complex Chebyshev functional defined by

\[
\mathcal{D}_{\gamma}(f,g) := \frac{1}{w-u} \int_{\gamma} f(z) g(z) \, dz - \frac{1}{w-u} \int_{\gamma} f(z) \, dz \frac{1}{w-u} \int_{\gamma} g(z) \, dz.
\]

In this paper we establish some bounds for the magnitude of the functional \( \mathcal{D}_{\gamma}(f,g) \) under various assumptions for the functions \( f \) and \( g \), and provide a complex version for the Chebyshev inequality (3).
2. Chebyshev type results

We start with the following identity of interest:

**Lemma 2.1.** Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in [a, b]$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If $f$ and $g$ are continuous on $\gamma$, then

$$D_\gamma (f, g) = \frac{1}{2(w - u)^2} \int_\gamma \left( \int_\gamma (f(z) - f(w))(g(z) - g(w)) \, dw \right) \, dz$$  \hspace{1cm} (6)

**Proof.** For any $z \in \gamma$ the integral $\int_\gamma (f(z) - f(w))(g(z) - g(w)) \, dw$ exists and

$$I(z) := \int_\gamma (f(z) - f(w))(g(z) - g(w)) \, dw$$

$$= \int_\gamma (f(z)g(z) + f(w)g(w) - g(z)f(w) - f(z)g(w)) \, dw$$

$$= f(z)g(z)\int_\gamma dw + \int_\gamma f(w)g(w) \, dw - g(z)\int_\gamma f(w) \, dw - f(z)\int_\gamma g(w) \, dw$$

$$= (w - u)f(z)g(z) + \int_\gamma f(w)g(w) \, dw - g(z)\int_\gamma f(w) \, dw - f(z)\int_\gamma g(w) \, dw.$$  

The function $I(z)$ is also continuous on $\gamma$, then the integral $\int_\gamma I(z) \, dz$ exists and

$$\int_\gamma I(z) \, dz = \int_\gamma \left[ (w - u)f(z)g(z) + \int_\gamma f(w)g(w) \, dw ight.$$

$$\left. - g(z)\int_\gamma f(w) \, dw - f(z)\int_\gamma g(w) \, dw \right] \, dz$$

$$= (w - u)\int_\gamma f(z)g(z) \, dz + (w - u)\int_\gamma f(w)g(w) \, dw$$

$$- \int_\gamma f(w) \, dw \int_\gamma g(z) \, dz - \int_\gamma g(w) \, dw \int_\gamma f(z) \, dz$$

$$= 2(w - u)\int_\gamma f(z)g(z) \, dz - 2\int_\gamma f(z) \, dz \int_\gamma g(z) \, dz$$

$$= 2(w - u)^2D_\gamma (f, g),$$

which proves the first equality in (6).

The rest follows in a similar manner and we omit the details.

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $h : \gamma \to \mathbb{C}$ a continuous function on $\gamma$. Define the quantity:

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\[ P_\gamma (h, \overline{h}) = \frac{1}{\ell(\gamma)} \int_\gamma |h(z)|^2 |dz| - \frac{1}{\ell(\gamma)} \int_\gamma |h(z)| |dz|^2 \]  
\[ = \frac{1}{\ell(\gamma)} \int_\gamma |h(v) - \frac{1}{\ell(\gamma)} \int_\gamma h(z) |dz|| |dv| \geq 0. \]  

(7)

We say that the function \( f : G \subset \mathbb{C} \to \mathbb{C} \) is \( L-h \)-Lipschitzian on the subset \( G \) if

\[ |f(z) - f(w)| \leq L |h(z) - h(w)| \]

for any \( z, w \in G \). If \( h(z) = z \), we recapture the usual concept of \( L \)-Lipschitzian functions on \( G \).

**Theorem 2.2.** Suppose \( \gamma \subset \mathbb{C} \) is a piecewise smooth path parametrized by \( z(t), t \in [a, b] \) from \( z(a) = u \) to \( z(b) = w \) with \( w \neq u \), \( h : \gamma \to \mathbb{C} \) is continuous, \( f \) and \( g \) are \( L_1, L_2-h \)-Lipschitzian functions on \( \gamma \); then

\[ |D_\gamma (f, g)| \leq L_1 L_2 \frac{\ell^2(\gamma)}{|w - u|^2} P_\gamma (h, \overline{h}). \]  

(8)

**Proof.** Taking the modulus in the first equality in (6), we get

\[ |D_\gamma (f, g)| = \frac{1}{2 |w - u|^2} \left| \int_\gamma \left( \int_\gamma (f(z) - f(w)) (g(z) - g(w)) |dw| \right) |dz| \right| \]
\[ \leq \frac{1}{2 |w - u|^2} \int_\gamma \left| \int_\gamma (f(z) - f(w)) (g(z) - g(w)) |dw| \right| |dz| \]
\[ \leq \frac{1}{2 |w - u|^2} \int_\gamma \left( \int_\gamma |(f(z) - f(w)) (g(z) - g(w))| |dw| \right) |dz| \]
\[ \leq \frac{L_1 L_2}{2 |w - u|^2} \int_\gamma \left( \int_\gamma |h(z) - h(w)|^2 |dw| \right) |dz| =: A. \]
Now, observe that
\[
\int_{\gamma} \left( \int_{\gamma} |h(z) - h(w)|^2 |dw| \right) \, |dz| = \int_{\gamma} \left( \int_{\gamma} |h(z)|^2 - 2 Re \left( h(z) \overline{h(w)} \right) + |h(w)|^2 \right) \, |dw| \, |dz|
\]
\[
= \int_{\gamma} \left( |h(z)|^2 - 2 Re \left( h(z) \int_{\gamma} \overline{h(w)} \, |dw| \right) + \int_{\gamma} |h(w)|^2 \, |dw| \right) \, |dz|
\]
\[
= \int_{\gamma} \left( |h(z)|^2 - 2 Re \left( \int_{\gamma} h(z) \, |dz| \int_{\gamma} \overline{h(w)} \, |dw| \right) \right) \, |dz|
\]
\[
+ \int_{\gamma} \left( |h(w)|^2 \, |dw| \right) \, |dz| - 2 Re \left( \int_{\gamma} h(z) \, |dz| \left( \int_{\gamma} \overline{h(w)} \, |dw| \right) \right)
\]
\[
= 2 \left[ \int_{\gamma} \left| h(z) \right|^2 \, |dz| - \left| \int_{\gamma} h(z) \, |dz| \right|^2 \right] = 2 \ell^2(\gamma) \mathcal{P}_\gamma(h, \overline{h}).
\]

Therefore, by (10) we get
\[
A = L_1 L_2 \frac{\ell^2(\gamma)}{|w - u|^2} \mathcal{P}_\gamma(h, \overline{h}),
\]
and by (9) we get the desired result (8).

Further, for \( \gamma \subset \mathbb{C} \) a piecewise smooth path parametrized by \( z(t) \), and by taking \( h(z) = z \) in (7), we can consider the quantity
\[
\mathcal{P}_\gamma := \frac{1}{\ell(\gamma)} \int_{\Gamma} |z|^2 \, |dz| - \left| \frac{1}{\ell(\gamma)} \int_{\Gamma} z \, |dz| \right|^2.
\]
\[
= \frac{1}{\ell(\gamma)} \int_{\Gamma} \left| v - \frac{1}{\ell(\gamma)} \int_{\Gamma} z \, |dz| \right|^2 \, |dv|
\]
\[
= \frac{1}{2 \ell^2(\gamma)} \int_{\Gamma} \left( \int_{\Gamma} |z - w|^2 \, |dw| \right) \, |dz| \geq 0.
\]

**Corollary 2.3.** Suppose \( \gamma \subset \mathbb{C} \) is a piecewise smooth path parametrized by \( z(t), \ t \in [a, b] \) from \( z(a) = u \) to \( z(b) = w \) with \( w \neq u \) and \( f \) and \( g \) are \( L_1, L_2 \)-Lipschitzian functions on \( \gamma \); then
\[
|\mathcal{D}_\gamma(f, g)| \leq L_1 L_2 \frac{\ell^2(\gamma)}{|w - u|^2} \mathcal{P}_\gamma.
\]

**Remark 2.4.** Assume that \( f \) is \( L \)-Lipschitzian on \( \gamma \). For \( g = f \) we have
\[
\mathcal{D}_\gamma(f, f) = \frac{1}{w - u} \int_{\Gamma} f^2(z) \, dz - \left( \frac{1}{w - u} \int_{\Gamma} f(z) \, dz \right)^2,
\]

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and by (8) we get
\[ |D_{\gamma} (f, f)| \leq L^2 \frac{\ell^2 (\gamma)}{|w - u|^2} \mathcal{P}_\gamma (h, \overline{h}). \tag{14} \]

For \( g = \tilde{f} \) we have
\[ D_{\gamma} (f, \tilde{f}) = \frac{1}{w - u} \int_{\gamma} |f(z)|^2 \, dz - \frac{1}{w - u} \int_{\gamma} f(z) \, dz \cdot \frac{1}{w - u} \int_{\gamma} f(z) \, dz, \tag{15} \]
and by (8) we get
\[ |D_{\gamma} (f, \tilde{f})| \leq L^2 \frac{\ell^2 (\gamma)}{|w - u|^2} \mathcal{P}_\gamma (h, \overline{h}). \tag{16} \]

If \( f \) is \( L \)-Lipschitzian on \( \gamma \), then
\[ |D_{\gamma} (f, f)| \leq L^2 \frac{\ell^2 (\gamma)}{|w - u|^2} \mathcal{P}_\gamma \tag{17} \]
and
\[ |D_{\gamma} (f, \tilde{f})| \leq L^2 \frac{\ell^2 (\gamma)}{|w - u|^2} \mathcal{P}_\gamma. \tag{18} \]

If the path \( \gamma \) is a segment \([u, w]\) connecting two distinct points \( u \) and \( w \) in \( \mathbb{C} \), then we write \( \int_{u}^{w} f(z) \, dz \) as \( \int_{u}^{w} f(z) \, dz \)

Now, if \( f \) and \( g \) are \( L_1, L_2 \)-Lipschitzian functions on \([u, w] := \{(1 - t) u + tw, \ t \in [0, 1]\}\), then by (12) we have
\[ |D_{\gamma} (f, g)| \leq L_1 L_2 \mathcal{P}_{[u, w]}, \]
where
\[ \mathcal{P}_{[u, w]} = \frac{|w - u|^2}{2 |w - u|^2} \int_{0}^{1} \left( \int_{0}^{1} \left| (1 - t) u + tw - (1 - s) u - sw \right|^2 \, dt \right) \, ds \]
\[ = \frac{1}{2} |w - u|^2 \int_{0}^{1} \left( \int_{0}^{1} (t - s)^2 \, dt \right) \, ds = \frac{1}{12} |w - u|^2. \]
Therefore,
\[ \left| \frac{1}{w - u} \int_{\gamma} f(z) g(z) \, dz - \frac{1}{w - u} \int_{\gamma} f(z) \, dz \frac{1}{w - u} \int_{\gamma} g(z) \, dz \right| \leq \frac{1}{12} |w - u|^2 L_1 L_2, \tag{19} \]
if \( f \) and \( g \) are \( L_1, L_2 \)-Lipschitzian functions on \([u, w] \).

If \( f \) is \( L \)-Lipschitzian on \([u, w]\), then
\[ \left| \frac{1}{w - u} \int_{\gamma} f^2(z) \, dz - \left( \frac{1}{w - u} \int_{\gamma} f(z) \, dz \right)^2 \right| \leq \frac{1}{12} |w - u|^2 L^2 \tag{20} \]
and
\[ \left| \frac{1}{w - u} \int_{\gamma} |f(z)|^2 \, dz - \frac{1}{w - u} \int_{\gamma} f(z) \, dz \frac{1}{w - u} \int_{\gamma} |f(z)| \, dz \right| \leq \frac{1}{12} |w - u|^2 L^2. \tag{21} \]
3. Examples for circular paths

Let \([a, b] \subseteq [0, 2\pi]\) and the circular path \(\gamma_{[a, b], R}\) centered in 0 and with radius \(R > 0\):

\[
z(t) = R \exp(it) = R \cos t + i \sin t, \quad t \in [a, b].
\]

If \([a, b] = [0, \pi]\), then we get a half circle, while for \([a, b] = [0, 2\pi]\) we get the full circle.

Since

\[
|e^{is} - e^{it}|^2 = |e^{is}|^2 - 2Re\left(e^{i(s-t)}\right) + |e^{it}|^2
\]

for any \(t, s \in \mathbb{R}\), then

\[
|e^{is} - e^{it}|^r = 2^r \left|\sin \left(\frac{s-t}{2}\right)\right|^r
\]

(22)

for any \(t, s \in \mathbb{R}\) and \(r > 0\). In particular,

\[
|e^{is} - e^{it}| = 2 \left|\sin \left(\frac{s-t}{2}\right)\right|
\]

for any \(t, s \in \mathbb{R}\).

If \(u = R \exp(ia)\) and \(w = R \exp(ib)\), then

\[
w - u = R \left[\exp(ib) - \exp(ia)\right] = R \left[\cos b + i \sin b - \cos a - i \sin a\right]
\]

\[
= R \left[\cos b - \cos a + i \left(\sin b - \sin a\right)\right].
\]

Since

\[
\cos b - \cos a = -2 \sin \left(\frac{a+b}{2}\right) \sin \left(\frac{b-a}{2}\right)
\]

and

\[
\sin b - \sin a = 2 \sin \left(\frac{b-a}{2}\right) \cos \left(\frac{a+b}{2}\right)
\]

hence

\[
w - u = R \left[-2 \sin \left(\frac{a+b}{2}\right) \sin \left(\frac{b-a}{2}\right) + 2i \sin \left(\frac{b-a}{2}\right) \cos \left(\frac{a+b}{2}\right)\right]
\]

\[
= 2R \sin \left(\frac{b-a}{2}\right) \left[-\sin \left(\frac{a+b}{2}\right) + i \cos \left(\frac{a+b}{2}\right)\right]
\]

\[
= 2R i \sin \left(\frac{b-a}{2}\right) \cos \left(\frac{a+b}{2}\right) + i \sin \left(\frac{a+b}{2}\right)
\]

\[
= 2R i \sin \left(\frac{b-a}{2}\right) \exp \left(\frac{a+b}{2}\right) i.
\]
If \( \gamma = \gamma_{[a,b], R} \), then the circular complex Chebyshev functional is defined by

\[
C_{[a,b], R}(f, g) = \mathcal{D}_{[a,b], R}(f, g) := \frac{1}{2 \sin \left( \frac{b-a}{2} \right) \exp \left[ i \frac{a+b}{2} \right]} \int_a^b f(R \exp(it)) g(R \exp(it)) \exp(it) \, dt
\]

If \( \gamma = \gamma_{[a,b], R} \), then

\[
P_{\gamma} := \frac{1}{2 \ell^2(\gamma)} \int_{\gamma} \left( \int_{\gamma} |z-w|^2 \, |dw| \right) \, |dz| \leq \frac{R^2}{4 \sin^2 \left( \frac{b-a}{2} \right)} \left( \frac{(b-a)^2}{2} - \sin^2 \left( \frac{b-a}{2} \right) \right)
\]

We have the following result:

**Proposition 3.1.** Let \( \gamma_{[a,b], R} \) be a circular path centered in \( 0 \), with radius \( R > 0 \) and \( [a, b] \subset [0, 2\pi] \). If \( f \) and \( g \) are \( L_1 \), \( L_2 \)-Lipschitzian functions on \( \gamma_{[a,b], R} \), then

\[
|C_{[a,b], R}(f, g)| \leq \frac{R^2}{\sin^2 \left( \frac{b-a}{2} \right)} \left( \frac{(b-a)^2}{2} - \sin^2 \left( \frac{b-a}{2} \right) \right) L_1 L_2.
\]

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