MINIMAL SURFACES IN THE HEISENBERG GROUP

SCOTT D. PAULS

ABSTRACT. We investigate the minimal surface problem in the three dimensional Heisenberg group, \( H \), equipped with its standard Carnot-Carathéodory metric. Using a particular surface measure, we characterize minimal surfaces in terms of a sub-elliptic partial differential equation and prove an existence result for the Plateau problem in this setting. Further, we provide a link between our minimal surfaces and Riemannian constant mean curvature surfaces in \( H \) equipped with different Riemannian metrics approximating the Carnot-Carathéodory metric. We generate a large library of examples of minimal surfaces and use these to show that the solution to the Dirichlet problem need not be unique. Moreover, we show that the minimal surfaces we construct are in fact \( X \)-minimal surfaces in the sense of Garofalo and Nhieu.

1. INTRODUCTION

The examination of minimal surfaces in various settings has a long and rich history. An exploration of minimal surfaces in \( \mathbb{R}^3 \), \( \mathbb{R}^n \), and Riemannian manifolds led to beautiful and amazing connections to complex analysis, harmonic mappings, and many other diverse settings. These connections led to the solution of many problems in minimal surface theory, providing many different methods for the construction of examples and different proofs of existence and uniqueness results. For example, variational techniques were used to examine minimal surfaces in terms of a partial differential equation and to address existence and uniqueness of solutions with prescribed boundary data via an examination of the partial differential equation. These techniques led to very general existence and uniqueness results for minimal surfaces in many different settings. However, even with these decades of investigation and exploration, the so-called Plateau problem remains compelling and not yet completely understood in many settings:

Plateau Problem: For a given curve, \( \gamma \), can one find a surface of least area spanning \( \gamma \)?

The purpose of this paper is to continue an investigation of this basic question in the setting of the three dimensional Heisenberg group, \( H \), equipped with a Carnot-Carathéodory (CC) metric. To define such a metric, we first must describe some of the features of \( H \). \( H \) is a Lie group defined by a Lie algebra \( \mathfrak{h} \) generated by three vector fields, \( \{X, Y, Z\} \) with one nontrivial bracket relation, namely \([X, Y] = Z\). Moreover, we use this to describe the grading of \( \mathfrak{h} \), namely that

\[ \mathfrak{h} = V \oplus V_2 \]

where \( V = \text{span}\{X, Y\} \) and \( V_2 = \text{span}\{Z\} \). While there are many presentations of the Heisenberg group, for the purposes of this paper, we use an identification of \( H \) with \( \mathbb{R}^3 \) where the vector fields

\textbf{Key words and phrases.} Minimal surfaces, Carnot groups, Heisenberg group, 1-Laplacian.

The author is partially support by NSF grant DMS-9971563.
\{X, Y, Z\} are given by
\[ \begin{aligned}
X &= \partial_x - y\partial_z \\
Y &= \partial_y + x\partial_z \\
Z &= 2\partial_z
\end{aligned} \]

where \{\partial_x, \partial_y, \partial_z\} are the standard basis vector fields in \(\mathbb{R}^3\). Using left translation by group elements, we can think of \(\mathcal{V}\) and \(\mathcal{V}_2\) as left invariant subbundles of the tangent bundle of \(H\). Abusing notation, we will refer to both the vector space and subbundle by the same symbol. Throughout this paper, we often call \(\mathcal{V}\) the bottom of the grading of \(\mathfrak{h}\) and, if a path \(p \subset H\) has tangent vector in \(\mathcal{V}\) at every point, we call \(p\) a horizontal curve. Now, fixing a left invariant inner product, \(\langle \cdot, \cdot \rangle\), on \(\mathcal{V}\) making \(\{X, Y\}\) an orthonormal basis, we define the CC metric as a path metric on \(H\) as follows:
\[
d_{cc}(g, h) = \inf \left\{ \int_I \langle p'(t), p'(t) \rangle^{\frac{1}{2}} \ dt \left| p(0) = g, \ p(1) = h, \text{ and } p \text{ is horizontal} \right. \right\}
\]

It follows from the fact that \(\mathcal{V}\) is bracket generating that \(d_{cc}(g, h) < \infty\) for any \(g, h \in H\). Notice that, by definition, the CC metric is left invariant. Further the CC metric is fractal in the sense that, while the topological dimension of \(H\) is three, the Hausdorff dimension of \(H\), calculated with respect to the CC distance function, is four. In this setting, we investigate a basic version of the question above:

**Plateau Problem in \(H\):** Given a closed curve \(\gamma \in H\), can we find a topologically two dimensional surface \(S \subset H\) spanning \(\gamma\) which minimizes an appropriate surface measure?

The singularity of the CC metric, as illustrated by the disconnect between the topological and Hausdorff dimension mentioned above, makes the notion of “appropriate surface measure” somewhat hard to specify - there are several “natural” candidates for such a measure. For example, in [Gro96], among many other investigations, Gromov addresses such so-called filling problems, although focusing more on isoperimetric type inequalities and questions (see, for example, section 0.7B). He shows that one natural measure for the generic two dimensional surface in \(H\) is the three dimensional Hausdorff measure associated to \(d_{cc}\) as every \(C^1\) surface which has topological dimension two has Hausdorff dimension three (this is an application of results in section 0.6A). Also, the reader should see chapter 2 section 1 for a more general discussion. In a different line of investigation, using a measure theoretic perimeter measure, \(\mathcal{P}\), Garofalo and Nhieu ([GN96]) gave a beautiful and delicate existence result for minimal surfaces in a much more general setting than the Heisenberg group. For the sake of simplicity, we state their theorem only in the case of \(H\) and precise definitions for the necessary objects appear in section 2. To differentiate between their result and others, we follow their convention and call their minimal surfaces \(X\)-minimal surfaces.

**Theorem 1.1.** ([GN96]) Given a bounded open set \(O \subset H\) and an \(X\)-Caccioppoli set \(L \subset H\), there exists an \(X\)-minimal surface in \(O\), i.e. an \(X\)-Caccioppoli set \(E \subset H\) such that \(\mathcal{P}(E) \leq \mathcal{P}(F)\) for every set \(F\) which coincides with \(L\) outside of \(\Omega\).

To interpret this theorem in the language of the question posed above, we must indicate where the curve \(\gamma\) and the two dimensional surface appear. Garofalo and Nhieu’s perimeter measure is an analogue of DeGiorgi’s perimeter measure in Euclidean space and it is a type of area measure on the measure theoretic boundaries of open sets. Thus, to recover a “curve”, we simply intersect the measure theoretic boundaries of \(L\) and \(O\) in the theorem above. The surface spanning this curve is
then the boundary of $E$ inside $O$. Notice that Garofalo and Nhieu’s work addresses only existence and not regularity hence the measure theoretic nature of the result.

We begin our investigation from an a priori different direction. We will consider surfaces bounding a specific curve which minimize an energy based on the three dimensional spherical Hausdorff measure. We use standard variational techniques to find a partial differential equation characterizing these minimal surfaces in the Heisenberg group. If the surface, $S$, is given by $F(x, y, z) = f(x, y) - z = 0$ for $(x, y) \in \Omega$, a region in the $xy$-plane (here we are using the identification of $H$ and $\mathbb{R}^3$, then it is known (see [Pan89] or [Hei95]) that, up to a normalization, the spherical Hausdorff measure takes the form:

$$\mathcal{H}_cc^3(S) = \int_{\Omega} |\nabla_0 F| dA$$

where $\nabla_0$ is the so-called horizontal gradient operator given by

$$\nabla_0 F = (XF, YF)$$

Using this as the basis for our variational setup, we define our energy function to be

$$E(\cdot) = \int_{\Omega} |\nabla_0 \cdot| dA$$

Under these assumption, we can characterize nonparametric minimal surfaces as follows: if the minimal surface is given as the level set $F(x, y, z) = f(x, y) - z = 0$, then it satisfies the partial differential equation:

$$\nabla_0 \cdot \frac{\nabla_0 F}{|\nabla_0 F|} = 0 \quad (1)$$

This “minimal surface equation” is a subelliptic partial differential equation and one cannot immediately conclude existence or uniqueness results from an examination of the defining energy functional. To investigate the standard questions of existence and uniqueness, and to relate these minimal surfaces to the X-minimal surfaces of Garofalo and Nhieu, we rely on approximation of $(H, d_{cc})$ by Riemannian manifolds.

It is well know (see, for example [Gro81a], [Gro81b], and [Pan89]) that $(H, d_{cc})$ can be realized as a limit of dilated Riemannian manifolds. Specifically, we first fix a left invariant Riemannian metric, $g_1$, on $H$ matching $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}$ and making $\{X, Y, Z\}$ an orthonormal basis for $\mathfrak{h}$. Then, we define a dilation map, $h_\lambda : H \to H$ by

$$h_\lambda(e^{aX+bY+cZ}) = e^{a\lambda X+b\lambda Y+c\lambda^2 Z}$$

we define a new Riemannian metric on $H$ for every $\lambda$ by

$$g_\lambda = \frac{1}{\lambda^3} h_\lambda^* g_1$$

Then, if we denote the distance function associated to $g_\lambda$ by $d_\lambda$, the sequence of metric spaces $(H, g_\lambda)$ converges to $(H, d_{cc})$ in the Gromov-Hausdorff topology as $\lambda \to \infty$.

Using this convergence, we attack the minimal surface problem in $(H, d_{cc})$ by considering sequences of surfaces, $\{S_\lambda\}$ where $S_\lambda \subset (H, g_\lambda)$. In particular, we can use sequences of Riemannian minimal surfaces and sequences of Riemannian constant curvature surfaces to construct solutions to (1), i.e. minimizers of our energy functional. This allows us to guarantee existence, at least weakly, for a special class of curves:

**Theorem A.** Let $\Gamma$ be a closed curve in $H$ satisfying the bounded slope condition and which is the graph of a function $\varphi \in C^{2,\alpha}(\mathbb{R})$ over a curve $\gamma \in C^{2,\alpha}(\mathbb{R}^2)$ which bounds a region $\Omega$. Then, there exists $u \in W^{1,p}(\Omega) \cap C^0(\Omega)$ so that $u|_\gamma = \varphi$, $u$ is a weak solution to (1) on $\Omega$ and $u$ minimizes $E(\cdot)$.
on $\Omega$. Moreover, there exists a sequence of functions with the same regularity as $\Gamma$, $\{u_{\lambda_n}\}$, such that $H_{\lambda_n}(u_{\lambda_n}) = 0$ and $u_{\lambda_n} \rightarrow u$ in $W^{1,p}(\Omega)$.

This theorem is proved by showing that the sequence of minimal surfaces with boundary $\Gamma \subset (H, g_{\lambda})$ have a convergent subsequence as $\lambda \rightarrow \infty$ and then showing that the limit is a graph satisfying (1). Basically, this allows us to say that, upon fixing boundary conditions, Riemannian minimal surfaces subconverge to minimal surfaces in the Carnot setting. We note that the bounded slope condition is not optimal - any condition on the curve $\Gamma$ which ensures existence of minimal surfaces in $(H, g_{\lambda})$ is sufficient. While this allows us to conclude an existence result, it says nothing about uniqueness. In fact, the next theorem gives strong evidence for nonuniqueness, by showing that limits of families of constant mean curvature surfaces (if such a limit exists) are minimal surfaces in $(H, d_{cc})$ as well.

**Theorem B.** Let $\Gamma$ be a closed curve with the bounded slope condition in $H$ which is the graph of a function $\varphi \in C^{2,\alpha}(\mathbb{R})$ over a curve $\gamma \in C^{2,\alpha}(\mathbb{R}^2)$ which bounds a region $\Omega$. Suppose $u_n$ are graphs in $(H, d_{\lambda_n})$ spanning $\Gamma$ which have mean curvature given by the function $\kappa_n$. If $\kappa_n \rightarrow 0$ as $n \rightarrow \infty$ and $u_n$ converge to a graph $u$ then $u$ satisfies (1) and is a minimizer of $E(\cdot)$.

As with the previous theorem, this theorem requires the bounded slope condition to ensure existence of minimal surfaces with the same boundary data in $(H, g_{\lambda})$. With regard to the uniqueness question, one now suspects that there may be many different solutions to the Dirichlet problem in this setting generated by taking different convergent subsequences. This suspicion is confirmed in section 5.2 where we construct an explicit example where there are at least two minimal surfaces spanning the same curve. Hence we conclude:

**Theorem C.** Let $\Gamma$ be a closed $C^2$ curve in $H$ which is the graph of a function $\varphi$ over a curve $\gamma \in \mathbb{R}^2$ which bounds a region $\Omega$. Then the solution to the Dirichlet problem for $\Gamma$ need not be unique.

This theorem is a consequence of the work in section 4, where we construct a huge number of solutions to equation 1. We note that, in addition to being examples of the minimal surfaces described above, these are solutions to the 1-subLaplacian equation 1 on $H$ as well as solutions to the least horizontal gradient problem on $\mathbb{R}^3$. Next, we give a compilation of all of the examples, shown via various techniques, in section 4.

**Theorem D.** Given $a, b, c \in \mathbb{R}$, $\alpha \in [-1, 1]$ and $g \in W^{1,p}(\mathbb{R})$, the following graphs satisfy (1) weakly:

(i) $z = ax + by + c$

(ii) $z = a\theta$

(using cylindrical coordinates on $\mathbb{R}^3$)

(iii) $z = \pm \left(\frac{\sqrt{br^2 - 1}}{b} - a \tan^{-1} \left(\frac{1}{\sqrt{br^2 - 1}}\right)\right) + c + a\theta$

Again, these use cylindrical coordinates.

(iv) $z = xy + g(y)$
Although these results are promising, we began with a different measure than in Garofalo and Nhieu’s on X-minimal surfaces and, a priori, our minimal surfaces have no relation to X-minimal surfaces. However, as mentioned above, the Riemannian approximation spaces allow us to demonstrate a link between the two:

**Theorem E.** The minimal surfaces described in theorems A and B are X-minimal as well.

The proof of this theorem rests heavily on standard elliptic theory in Riemannian manifolds coupled with Garofalo and Nhieu’s results and results on X-Caccioppoli sets due to Franchi, Serapioni and Serra Cassano in [FSSC95] and [FSSC99]. In particular, the results in [FSSC99] include a form of the implicit function theorem which allows us to compare the graphs of minimal surfaces with the graphs of X-Caccioppoli sets.

After the completion of this work, the author learned through a personal communication from N. Garofalo that he, D. Danielli, and D.M. Nhieu ([DGN01]) had explored the theory of X-minimal surfaces further and had independently arrived at similar results using somewhat different techniques. In particular, they find the same characterization of X-minimal surfaces given by equation 1 and also address aspects of the Bernstein problem and the construction of examples. In addition, they provide numerous results in other directions.

To conclude the introduction, we summarize the contents of each section. Section 2 reviews many of the definitions above as well as other necessary definitions and results from various sources. Section 3 provides the variational analysis of the energy functional described above and proves theorems A, B and E. In section 4, we construct, through a variety of methods, the examples of minimal surfaces in $(H, d_{cc})$ stated in theorem D. Section 5 points out several consequences of the results in section 3 and the examples in section 4. In particular, it describes the example proving theorem C and discusses the implications of the examples with reference to the classical Bernstein problem. The end of this section discusses some conclusions and avenues of continued exploration.

We wish to thank the referee for many helpful comments and suggestions.

### 2. Further definitions and a recounting of known results

For the convenience of the reader, we begin this section by collecting the definitions embedded in the introduction:

- $H$ denotes the three dimensional Heisenberg group. It is associated to the Lie algebra $\mathfrak{h} = \text{span}\{X, Y, Z\}$ with one nontrivial bracket, $[X, Y] = Z$.
- Using the exponential map, we will often identify $H$ with $\mathbb{R}^3$ using $\{X, Y, Z\}$ as the standard coordinates. We will often describe graphs in $H$ and will use the convention that the graph of a function $g : \mathbb{R}^2 \to \mathbb{R}$ is given by the set $(x, y, g(x, y))$ unless otherwise specified.
• \( h \) has a grading given by
  \[ h = V \oplus V_2 \]
  where \( V = \text{span}\{X, Y\} \) and \( V_2 = \text{span}\{Z\} \).
• \( V \) is thought of both as a subspace of \( h \) and as a left invariant vector bundle on \( H \) via left translation. We will refer to \( V \) as the bottom of the grading or as the horizontal bundle on \( H \).
• Paths whose tangent vectors are always horizontal are called horizontal paths.
• \( \langle \cdot, \cdot \rangle \) denotes the standard inner product on \( V \), i.e. the inner product which makes \( \{X, Y\} \) an orthonormal basis.
• \( \nabla_0 \) denotes the horizontal gradient operator and is defined by
  \[ \nabla_0 f = (Xf, Yf) \]
• Given a \( C^1 \) surface, \( S \), in \( H \), we denote by \( N_0 \) the horizontal normal vector given by projecting the usual normal vector to the horizontal bundle at each point. If we wish to emphasize the dependence on \( S \), we will write \( N_0(S) \). Similarly, to emphasize the dependence of the normal on the base point, we will write \( N_0(p) \) where \( p \) is a point on the surface.
• A point \( p \) on a surface \( S \) is called a characteristic point if \( N_0 \) vanishes at \( p \).
• \( N_0 \) is the unit horizontal normal. Given a surface \( S \), we often think of \( n_0 \) as a map from \( S \) to the circle. In this case, we call \( n_0 \) the horizontal Gauss map.
• The standard Carnot-Carathéodory metric on \( H \) is given by
  \[ d_{cc}(g, h) = \inf \left\{ \int_I \langle p'(t), p'(t) \rangle^{\frac{1}{2}} \, dt \bigg| p(0) = g, \, p(1) = h, \, \text{and } p \text{ is horizontal} \right\} \]
  for \( g, h \in H \).
• \( \mathcal{H}^k_{cc} \) denotes the \( k \)-dimensional spherical Hausdorff measure constructed with respect to the Carnot-Carathéodory metric.
• We denote the open ball of radius \( r \) around a point \( p \in H \) with respect to \( d_{cc} \) by \( B_{cc}(p, r) \).
• \( g_1 \) is the left invariant Riemannian metric on \( H \) which makes \( \{X, Y, Z\} \) an orthonormal basis at each point.
• \( h_\lambda : H \to H \) is a dilation map defined by
  \[ h_\lambda(e^{aX+bY+cZ}) = e^{a\lambda X+b\lambda Y+c\lambda^2 Z} \]
• \( g_\lambda \) is a dilated Riemannian metric on \( H \) given by
  \[ g_\lambda = \frac{1}{\lambda} h_\lambda^* g_1 \]
  Roughly, \( g_\lambda \) measures \( X \) and \( Y \) directions the same way as \( g_1 \) but changes the length of \( Z \) by a factor of \( \lambda \).
• For a given surface, \( N_\lambda \) denotes the normal to the surface computed with respect to \( g_\lambda \).
• If \( \Omega \subset \mathbb{R}^2 \), we denote by \( W^{1,p}(\Omega) \) and \( W^{1,p}_0(\Omega) \) the usual Sobolev spaces.
• Throughout the paper, we will use the “big O” and “little o” notation to describe the rates of decay of various functions. However, we will use the notation to emphasize the dependence on certain variables. For example, \( h(\varepsilon, x, y, z) = o_\varepsilon(1) \) means that
  \[ \lim_{\varepsilon \to 0} h(\varepsilon, x, y, z) = 0 \]
2.1. Surface measures. Next, we review in more detail many of the constructions and theorems mentioned in the introduction. We begin with a description of \( \mathcal{H}^3 \) for smooth surfaces in \( H \). As we will be focusing on graphs over portions of the \( xy \)-plane, we will state the theorems in this setting.

The next theorem was first shown for \( H \) in [Pan89] and was later extended to all Carnot groups by J. Heinonen in [Hei95]. Again, for the purposes of this paper, we will state it only in \( H \).

**Proposition 2.1.** ([Pan89],[Hei95]) Let \( S \) be a smooth surface. Then,

(i) \[ H^3(S) = \int_S \frac{N_0}{N_1} |dA| \]

where \( N_0 \) is the horizontal normal described above, \( N_1 \) is the normal to \( S \) computed with respect to \( g_1 \) and \( dA \) is the Riemannian area element induced by \( g_1 \).

(ii) In particular, if \( S \) be a smooth surface given as the graph of \( f: \Omega \subset \mathbb{R}^2 \to \mathbb{R} \) in \( H \). Then,

\[ H^3(S) = \int_\Omega |\nabla_0 (f(x,y) - z)| \, dx dy \]

In particular, this formula shows that \( H^3 \) is a natural measure to use on topologically two-dimensional surfaces in \( H \) as it is locally finite. We will use this measure as the starting point for our variational analysis. One reason for this choice is that, as demonstrated above, this measure has an extremely nice presentation when the surface is smooth. Moreover, as we will see next, for smooth surfaces, this coincides with the perimeter measure used by Garofalo and Nhieu. To introduce the perimeter measure, we first make several definitions following the notation of [GN96] very closely.

Consider an open set \( O \subset H \) where we think of \( H \) as identified with \( \mathbb{R}^3 \). First, we define the weak horizontal Sobolev space of first order. For \( 1 \leq p < \infty \),

\[ L^{1,p}(O) = \{ f \in L^p(O) | Xf, Yf \in L^p(O) \} \]

In this case, if \( f \) is not smooth, we understand \( Xf \) and \( Yf \) to be distributions. To continue, we next define

\[ \mathcal{F}(O) = \{ \varphi = (\varphi_1, \varphi_2) \in C^1_0(O; \mathbb{R}^2) | ||\varphi||_\infty \leq 1 \} \]

This allows us to define the **X-variation** of a function \( u \in L^1_{loc}(O) \) by

\[ Var_X(u; O) = \sup_{\varphi \in \mathcal{F}(O)} \int_O u(h)(X^* \varphi_1(h) + Y^* \varphi_2(h)) \, dV(h) \]

where \( X^* \) and \( Y^* \) are the formal adjoints of \( X \) and \( Y \) and \( dV \) is Haar measure on \( H \). Notice that if \( u \) is smooth and in \( L^{1,1}(O) \) then an application of the divergence theorem yields that

\[ Var_X(u) = \int_O |\nabla_0(u)| \, dV \]

Finally, we can define the **X-perimeter** of an open set \( E \subset H \) relative to the open set \( O \) by

\[ \mathcal{P}(E; O) = Var_X(\chi_E; O) \]

where \( \chi_E \) is the characteristic function of \( E \). Naturally, we say that a set has finite perimeter (with respect to \( O \)) if \( \mathcal{P}(E; O) < \infty \). Moreover, a set \( E \) is called an **X-Caccioppoli set** if \( \mathcal{P}(E; O) < \infty \) for all open sets \( O \subset H \).

The measure \( \mathcal{P} \) was first introduced in [CDG94a] and is the direct analogue of DeGiorgi’s perimeter measure for \( \mathbb{R}^n \) introduced in [DeG54]. Using this notation, we can define an **X-minimal surface**.
**Definition 2.2.** Let $L \subset H$ be an $X$-Caccioppoli set and $O$ be an open set in $H$. Then, a set $M$ is called an $X$-minimal surface with respect to $O$ and $L$ if, for every set $S$ which coincides with $L$ outside of $O$, we have

$$\mathcal{P}(M;O) \leq \mathcal{P}(S;O)$$

One key component of the proof of theorem 1.1 is the lower semicontinuity of the perimeter measure.

**Lemma 2.3.** Fix an open set $O$ in $H$ and let $\{L_k\}$ be a sequence of $X$-Caccioppoli sets in $H$ converging to the set $L$ in the sense that $\chi(L_k) \to \chi(L)$ in $L^1_{\text{loc}}$. Then,

$$\mathcal{P}(L;O) \leq \liminf_{k \to \infty} \mathcal{P}(L_k;O)$$

A useful technical tool is to approximate sets of finite perimeter by smooth sets. This can be done by adapting the standard mollification process. We state this result (a version of results in [FSSC96] and [GN96]) for the convenience of the reader:

**Proposition 2.4.** If $E$ is a bounded subset of $H$ with finite $X$-perimeter, then there exists a sequence, $E_n$, of smooth sets so that

$$\lim_{n \to \infty} \mathcal{P}(E_n) = \mathcal{P}(E)$$

For the purposes of this paper, we will be interested in surfaces given as graphs over sets in the $xy$-plane (i.e. nonparametric minimal surfaces). To rectify this with Garofalo and Nhieu’s notation, we make the following conventions. Given an open subset $\Omega \subset \mathbb{R}^2$ and a curve $\Gamma \subset H$ given as the graph of a function $\varphi : \partial \Omega \to \mathbb{R}$, we define $O$ to be an open cylinder over $\Omega$,

$$O = \{(x, y, z) \in H | (x, y) \in \Omega\}$$

Further, we will define $L$ in the definition above by fixing a function $f : \mathbb{R}^2 \to \mathbb{R}$ whose graph spans $\Gamma$ and let

$$L = \{(x, y, z) \in H | z < f(x, y)\}$$

Similarly, when considering candidates for $X$-minimal surfaces using this setup, we will specify the graph of a function $u : \Omega \to \mathbb{R}$, extend it to match $f$ outside $\Omega$ and define the open set $M$ by

$$M = \{(x, y, z) | z < u(x, y)\}$$

To summarize, we make the following definition:

**Definition 2.5.** When considering nonparametric $X$-minimal surfaces, given a set $\Omega \subset \mathbb{R}^2$, $\varphi : \partial \Omega \to \mathbb{R}$, $f : \mathbb{R}^2 \to \mathbb{R}$ and $u : \Omega \to \mathbb{R}$ as above. We define the perimeter of $u$ as

$$\mathcal{P}(u) = \mathcal{P}(M;O)$$

and say that the graph of $u$ defines an $X$-minimal surface spanning $\Gamma$ if $M$ is an $X$-minimal surface with respect to $O$ and $L$.

Similarly, we will use a similar convention when denoting the Hausdorff measure of the graph of $u$ over $\Omega$,

$$\mathcal{H}^3_{cc}(u) \equiv \mathcal{H}^3_{cc}(u(\Omega))$$

Next, we state some results that make working with the perimeter measure more tractable. These results are due either to B. Franchi, R. Serapioni and F. Serra Cassano ([FSSC99], see proposition 2.14) or L. Capogna, D. Daneilli and N. Garofalo ([CDG94b] p. 211). The reader should also consult Z. Balough’s paper [Bal] concerning various aspects of the perimeter measure for surfaces as well as a study of different surfaces measure in CC spaces by R. Monti and F. Serra Cassano ([MSC99]).
Proposition 2.6. ([CDG94b],[FSSC99]) Let $S$ be a $C^1$ surface bounding an open set $O$, then

(i) $\mathcal{P}(O) = \int_S \frac{|N_0|}{|N_E|} d\mathcal{H}^2_E$

where $N_0$ is the horizontal normal, $N_E$ is the Euclidean normal (identifying $H$ with $\mathbb{R}^3$) and $\mathcal{H}^2_E$ is the 2-dimensional Hausdorff measure in $\mathbb{R}^3$.

(ii) In particular, if $S$ is given as a graph over a region $\Omega$ of $u \in C^1(\Omega)$ and $\partial \Omega$ is also of class $C^1$ then

$\mathcal{P}(u) = \int_{\Omega} |N_0| \, dx dy$

Franchi, Serapioni and Serra Cassano also prove an extension of the implicit function theorem in the CC setting and use it to show a beautiful structure theorem for $X$-Caccioppoli sets in $H$.

Again, for the purposes of this paper, we do not need to full power of their result and will state only what is necessary. To state the theorem, we first recall some of the definitions in [FSSC99]. The reader should be aware that we choose slightly different notation than that in [FSSC99] to maintain consistency within this paper.

Definition 2.7. We call $S \subset H$ an $H$-regular hypersurface if, for every $p \in S$, there exists an open ball, $B_{cc}(p,r)$ and a function $f : B_{cc}(p,r) \to \mathbb{R}$ such that

(i) Both $f$ and $\nabla_0 f$ are continuous functions.

(ii) $S$ is a level set of $f$,

$S \cap B_c(p,r) = \{ q \in B_{cc}(p,r) | f(q) = 0 \}$

(iii) $\nabla_0 f(p) \neq 0$

Roughly, $H$-regular surfaces are regular in the usual sense in the distributional directions, but are allowed to be nondifferentiable in the $Z$ direction.

To define the reduced boundary, we need to define a generalized unit normal to a surface base on the perimeter measure. Given $E$, an $X$-Caccioppoli set, consider the functional on $C^0_0$ given by

$\varphi \to -\int_E \text{div}_0 \varphi \, dV$

where $\text{div}_0$ is the horizontal divergence operator, $(X,Y)$. The Riesz representation theorem implies that there exists a section of $\mathcal{V}$, $\nu_E$ so that

$-\int_E \text{div}_0 \varphi \, dV = \int_H <\nu_E, \varphi> \, d\mathcal{P}_E$

$\nu_E$ is the generalized unit horizontal normal to $\partial E$. In the case where $\partial E$ is smooth, $\nu_E$ coincides with $n_0$.

Definition 2.8. The reduced boundary of an $X$-Caccioppoli set $E \subset H$, $\partial_{cc} E$ is the set of points $p \in E$ such that

(i) $\mathcal{P}(E \cap B_{cc}(p,r)) > 0$

for all $r > 0$.

(ii) $\nu_0(p) = \lim_{r \to 0} \int_{B_{cc}(p,r)} \nu_0 \, d\mathcal{P}_E$

$\mathcal{P}_E(\cdot) = \mathcal{P}(E \cap \cdot)$.
Note that the reduced boundary does not include any characteristic points of the boundary.
Now we state the pieces of the main theorems that we need for our purposes.

**Theorem 2.9.** ([FSSC99], see theorems 6.4 and 7.1) If $E \subset H$ is an $X$-Caccioppoli set then,

(i) $\mathcal{P}_E = k \mathcal{H}^3_{cc} | \partial^*_{cc} E$

where $k$ is a constant.

(ii) $\partial^*_{cc} E$ is $H$-rectifiable, i.e.

$$\partial^*_{cc} E = N \cup \bigcup_{k=1}^{\infty} C_k$$

where $\mathcal{H}^3_{cc}(N) = 0$ and $C_k$ is a compact subset on an $H$-regular hypersurface $S_k$.

### 3. A Characterization of Minimal Surfaces

For the rest of the paper, we will restrict ourselves to considering the three dimensional Heisenberg group equipped with the Carnot-Carathéodory metric described in section 2, $(H, d_{cc})$. Moreover, we will consider only nonparametric surfaces in $H$, i.e. graphs of functions $u : \mathbb{R}^2 \to \mathbb{R}$ over the $xy$-plane in $H$ (using the identification of $H$ with $\mathbb{R}^3$).

#### 3.1. The Variational Setup.

To characterize minimal surfaces in $H$, we must first specify which surface measure we wish to use. As discussed in section 2, the perimeter measure, $\mathcal{P}$, is a natural measure to use in this setting and, for sufficiently regular surfaces, it coincides with another natural surface measure, the three dimensional spherical Hausdorff measure restricted to the surface. Proposition 2.6 yields that if $S$ is the graph of $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ where $u \in C^1(\Omega)$,

$$\mathcal{P}(u) = \int_{\Omega} |N_0| dxdy$$

where $N_0$ is the horizontal normal of $S$. Taking this as our starting point for a variational analysis, we define our energy Lagrangian function as

$$L(\xi, z, p) = ((\xi_1 - y)^2 + (\xi_2 + x)^2)^{1/2}$$

Thus, for $u \in C^1(\Omega)$,

$$L(\nabla u, u, \bar{x}) = ((u_x - y)^2 + (u_y + x)^2)^{1/2}$$

$$= ((XF)^2 + (YF)^2)^{1/2} = |N_0|$$

where $F(x, y, z) = u(x, y) - z$. Our energy function based on this Lagrangian is

$$E(u) = \int_{\Omega} L(\nabla u, u, \bar{x}) \ dxdy$$

for $u \in W^{1,1}$. Note that if $u \in C^1(\Omega)$ then $E(u) = \mathcal{P}(u)$. For this energy function, the associated Euler-Lagrange equation is

$$-L_{\xi_1}(\nabla u, u, \bar{x})_x - L_{\xi_2}(\nabla u, u, \bar{x})_y = 0$$

Rewriting this in terms of the horizontal gradient operator, $\nabla_0$, this yields:

$$-\nabla_0 \cdot \frac{\nabla_0 F}{|\nabla_0 F|} = 0$$

\[(MSE)\]
Note that this is simply the horizontal 1-Laplacian on the Heisenberg group. In keeping with classical notation, we define a partial differential operator:

\[ H_{cc}(u) = \nabla_0 \cdot \frac{\nabla_0 F}{|\nabla_0 F|} \]

**Lemma 3.1.** \( L \) is convex in \( \xi \).

**Proof:** The Hessian of \( L \) is positive semidefinite with eigenvalues \( \{0, ((\xi_1 - y)^2 + (\xi_2 + x)^2)^{-\frac{1}{2}}\} \).

By the standard variational theory we have that \( E(\cdot) \) is weakly lower semicontinuous on \( W^{1,p} \). Moreover, the convexity implies that any weak solution to \((MSE)\) is a minimizer of the energy function \( E(\cdot) \). A priori, these “minimal surfaces” may not be the \( X \)-minimal surfaces of Garofalo and Nhieu (theorem 1.1) described in section 2. However, we shall see that the solutions to the Plateau problem in the next section are indeed \( X \)-minimal surfaces. We note that it is possible, albeit cumbersome, to give a classical derivation of the same characterization of minimal surfaces assuming that the surfaces are at least \( C^2 \) and are critical with respect to compactly supported \( C^2 \) variations.

We point out that because the equation \((MSE)\) is not strictly elliptic, we do not automatically get uniqueness of solutions with prescribed boundary data. Indeed, as we shall see in the subsequent sections, solutions are not necessarily unique.

### 3.2. Minimal surfaces as the limit of approximating minimal surfaces

In this section, we examine the connection between minimal surfaces in the approximating spaces \((H, d_\lambda)\) and the minimal surfaces in \((H, d_{cc})\). First, we illustrate that some minimal surfaces in \((H, d_{cc})\) arise as limits of sequences of minimal surfaces, each of which is in an \((H, d_\lambda)\). Second, we use this description to give another (more geometric) proof of the existence of solutions to the Plateau problem in \((H, d_{cc})\) by constructing nonparametric solutions via limits of solutions to Plateau problems in the approximating spaces.

In \((H, d_\lambda)\), we use the standard Lagrangian and energy functions used to investigate minimal surfaces:

\[ L_\lambda(\xi, z, p) = \left( (\xi_1 - y)^2 + (\xi_2 + x)^2 + \frac{1}{\lambda^2} \right)^{\frac{1}{2}} \]

\[ E_\lambda(u) = \int_{\Omega} L_\lambda(\nabla u, u, \bar{x}) dxdy \]

Later, we will also compare this functional with the Riemannian area functional on surfaces in \((H, g_\lambda)\). To this end, we define, for a smooth surface \( S \) in \((H, g_\lambda)\),

\[ A_\lambda(S) = \int_S dA_\lambda \]

where \( dA_\lambda \) is the Riemannian area element. Note that if \( S \) is given by the graph of a function \( u : \Omega \to \mathbb{R} \) then \( A_\lambda(u) = E_\lambda(u) \).

**Lemma 3.2.** Given a function \( F(x, y, z) = u(x, y) - z \) on \( H \) where \( u : \Omega \subset \mathbb{R}^2 \to \mathbb{R} \),

\[ E(u) \leq E_\lambda(u) \leq E(u) + \frac{1}{\lambda} \int_{\Omega} dxdy \]

**Proof:** This follows from the definition of the energy functions since \( L_\lambda = ((\xi_1 - y)^2 + (\xi_2 + x)^2 + \frac{1}{\lambda^2})^{\frac{1}{2}} \) and \( L = ((\xi_1 - y)^2 + (\xi_2 + x)^2)^{\frac{1}{2}} \). \( \Box \)
Proposition 3.3. Consider the nonparametric minimal surface equation in \((H,d_\lambda)\), \(H_\lambda = 0\). Suppose for each \(\lambda > 1\), there exists \(u_\lambda : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}\) so that \(u_\lambda\) minimizes \(E_\lambda(\cdot)\). Further suppose that \(\{u_\lambda\}\) converges weakly to some function \(u\) in \(W^{1,p}(\Omega)\). Then \(u\) is a minimizer of \(E(\cdot)\) and \(F = u - z\) is a weak solution to \((MSE)\).

Proof: Let \(m = \inf_\Omega E(g)\) and \(m_\lambda = \inf_\Omega E_\lambda(g)\). Since each \(u_\lambda\) is a minimizing \(E_\lambda\), by the previous lemma, we have
\[
E(u_\lambda) \leq m_\lambda \leq E(u_\lambda) + \frac{1}{\lambda} \int_\Omega dx dy
\]
Since \(m \leq E(u_\lambda)\) and, if \(u\) is a minimizing \(E(\cdot)\) with the same domain,
\[
m \leq E_\lambda(u) \leq m + \frac{1}{\lambda} \int_\Omega dx dy
\]
Combining these, we have
\[
m \leq m_\lambda \leq m + \frac{C}{\lambda}
\]
where \(C\) is a constant depending on \(\Omega\). Thus, as \(\lambda \rightarrow \infty\), we have
\[
\lim_{\lambda \rightarrow \infty} E_\lambda(u_\lambda) = m
\]
In other words, using this and lower semicontinuity, \(\{u_\lambda\}\) converges to a minimizing \(E(\cdot)\). Hence, \(F = u - z\) satisfies \((MSE)\) weakly. \(\square\)

Using standard elliptic estimates, we can produce a weak solution to the Plateau problem in \((H,d_{cc})\) for curves satisfying the bounded slope condition. We first recall the bounded slope condition in this setting:

Definition 3.4. Let \(\Gamma\) be a closed curve in \(H\) given as the graph of a function \(\varphi : \partial \Omega \rightarrow \mathbb{R}\) where \(\Omega\) is a domain in \(\mathbb{R}^2\). Then \(\Gamma\) has the bounded slope condition if, for every point \(p \in B = \{(x,y,z) \in H | (x,y) \in \partial \Omega, z = \varphi(x,y)\}\), there exist two planes \(P_-(p)\) and \(P_+(p)\) passing through \(p\) such that
(i) \(P_-(p) \leq \varphi(x,y) \leq P_+(p)\)
(ii) Let \(s(P)\) denote the slope of a plane. Then, the collection
\[
\{s(P_-(p)) | p \in B\} \cup \{s(P_+(p)) | p \in B\}
\]
is uniformly bounded by some constant \(K\).

Theorem 3.5. Let \(\Gamma\) be a closed curve in \(H\) satisfying the bounded slope condition and which is the graph of a function \(\varphi \in C^{2,\alpha}(\mathbb{R})\) over a curve \(\gamma \in C^{2,\alpha}(\mathbb{R}^2)\) which bounds a region \(\Omega\). Then, there exists \(u \in W^{1,p}(\Omega) \cap C^0(\Omega)\) so that \(u|_\gamma = \varphi\). \(u\) is a weak solution to \((MSE)\) on \(\Omega\) and \(u\) minimizes \(E(\cdot)\) on \(\Omega\). Moreover, there exists a sequence of functions with the same regularity as \(\Gamma\), \(\{u_{\lambda_n}\}\), such that \(H_{\lambda_n}(u_{\lambda_n}) = 0\) and \(u_{\lambda_n} \rightarrow u\) in \(W^{1,p}(\Omega)\).

Proof: For a fixed \(\lambda \geq 1\), \(H_\lambda = 0\) is a uniformly elliptic equation and so, by the standard theory (see, for example [GT01], theorem 11.5), since \(\Gamma\) is a \(C^2\) curve satisfying the bounded slope condition, there exists a solution \(u_\lambda\) to the Plateau problem in \((H,d_\lambda)\). In other words, \(H_\lambda(u_\lambda) = 0\) on \(\Omega\), \(u_\lambda|_\gamma = \Gamma\) and \(u_\lambda\) minimizes \(E_\lambda(\cdot)\). Moreover, standard applications of the quasilinear elliptic maximum principle yield:
\[
\sup_\Omega |u_\lambda| \leq \sup_\gamma |u_\lambda| = \sup_\gamma |\varphi|,
\]
\[
\sup_\Omega |\nabla u_\lambda| \leq \sup_\gamma |u_\lambda| = \sup_\gamma |\varphi|.
\]
Thus, each $u_\lambda$ is bounded in $W^{1,p}$ by a constant $C$ depending only on $\varphi$. In particular, $C$ does not depend on $\lambda$. Thus, \{ $u_\lambda$ \} is uniformly bounded in $W^{1,p}(\Omega)$ and hence, we can extract a subsequence converging weakly to a limit function, $u_\infty \in W^{1,p}_0$. In fact, since the $u_\lambda$ are all $C^2$, the same boundedness yields that the family is uniformly bounded and equicontinuous and hence we may assure that the convergence is uniform. Thus, $u_\infty$ is at least continuous, $u_\infty|_\gamma = \varphi$ and $u_\infty$ is a graph over $\Omega$. By the previous proposition, $F = u_\infty(x, y) - z$ is a weak solution to the Plateau problem in $(H, d_{cc})$ with the specified boundary data. \(\square\)

The previous theorem yields a nice characterization of some solutions to the Plateau problem - they are limits of solutions to the Plateau problem in Riemannian approximates to $(H, d_{cc})$. However, we now show a modification that shows that one can generate solutions with much more flexibility.

**Theorem 3.6.** Let $\Gamma$ be a closed curve with a bounded slope condition in $H$ which is the graph of a function $\varphi \in C^{2, \alpha}(\mathbb{R})$ over a curve $\gamma \in C^{2, \alpha}(\mathbb{R}^2)$ which bounds a region $\Omega$. Suppose $u_n$ are graphs in $(H, d_{cc})$ spanning $\Gamma$ which have mean curvature given by the function $\kappa_n$. If $\kappa_n \to 0$ as $n \to \infty$ and $u_n$ converge to a graph, $u$, then $u$ satisfies (MSE) and is a minimizer of $E(\cdot)$.

**Proof:** First, since $u_n$ is a solution to a prescribed mean curvature equation with boundary values given by $\Gamma$, its energy $E_{\lambda_n}(u_n)$ is close to $E(u_n)$ because the Lagrangian formulation of the prescribed mean curvature problem uses $E_{\lambda}(\cdot)$ as the energy coupled with an additional integral constraint. For a fixed $\lambda_n$, if a convergent sequence of surfaces have mean curvatures converging to zero (uniformly), then they converge to a minimal surface. Moreover, since solutions to the minimal surface Dirichlet problem in $(H, g_{\lambda_n})$ exist (again due to the bounded slope condition of $\Gamma$) and are unique (see theorems 10.1 and 10.2 in [GT01]), they must converge to the unique energy minimizer. Thus,

$$|E_{\lambda_n}(u_n) - E(u_n)| = o_{n-1}(1)$$

Second, we follow the argument in proposition 3.3. Letting $u_{\lambda_n}$ be the minimal surface spanning $\Gamma$ in $(H, d_{cc})$, $u$ a minimizer of $E(\cdot)$ spanning $\Gamma$, $m_n = E_{\lambda_n}(u_{\lambda_n})$ and $m = \inf_g E(g)$, we see that by lemma 3.2

$$E_{\lambda_n}(u_n) = m_n + o_{n-1}(1)$$

$$\geq E(u_{\lambda_n}) + o_{n-1}(1)$$

$$\geq m + o_{n-1}(1)$$

Moreover, again using lemma 3.2 and the argument in proposition 3.3, we have

$$E_{\lambda_n}(u_n) = m_n + o_{n-1}(1)$$

$$\leq E_{\lambda_n}(u) + o_{n-1}(1)$$

$$\leq m + \frac{C}{\lambda_n} + o_{n-1}(1)$$

Taking these together, we have

$$m + o_{n-1}(1) \leq E_{\lambda_n}(u_n) \leq m + \frac{C}{\lambda_n} + o_{n-1}(1)$$

Hence, as $n \to \infty$, $\lambda_n \to \infty$, and $E_{\lambda}(u_n)$ converges to $m = E(u)$. Thus, by lower semicontinuity, we have that $E(u_n) = m$ and hence is a minimizer of $E(\cdot)$ as well. \(\square\)
Remark 1. We point out one feature of the partial differential operator $H_{cc}$, namely that if $H_{cc}(u) = 0$, then

$$H_{\lambda}(u) = \frac{1}{\lambda^2} \frac{\Delta u}{\left((u_x - y)^2 + (u_y + x)^2 + \frac{1}{\lambda^2}\right)^{\frac{3}{2}}}$$

where $\Delta$ is the usual Laplacian on $\mathbb{R}^3$. Therefore, any solution to $H_{cc} = 0$ with boundary data satisfying the bounded slope condition satisfies the hypotheses of theorem 3.6. As we generate examples of solutions to $H_{cc} = 0$ in section 4, this observation provides a link to the energy minimizers in this section.

Next, we provide a link to the $X$-minimal surfaces given in [GN96].

Theorem 3.7. The minimizers of $E(\cdot)$ constructed in theorems 3.5 and 3.6 are also $X$-minimal surfaces. In other words, they are also minimizers for the perimeter measure.

Proof: By the lower semicontinuity of the perimeter measure and the characterization of the perimeter measure for smooth functions in lemma 2.3, we see that the perimeter of an open set, $O$, with boundary given by $u$ (over $\Omega$) is larger or equal to the perimeter of one of Garofalo and Nhieu’s $X$-minimal surfaces with the same boundary conditions. Conversely, given $W$ an $X$-minimal open set with boundary conditions matching that of $u$, we can use the work of Franchi, Serapioni and Serra Cassano ([FSSC95]) as follows. By theorem 2.9, we can realize the reduced boundary of $W$ by level sets of $H$-regular functions up to a set of $H_{cc}$ measure zero. Without loss of generality, we can focus only on one of the level sets given by a function $\psi_0: \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ - we may have to piece together many of these patches, but the result will be the same up to a set of measure zero. Moreover, using proposition 2.4, we can produce a smooth approximate $\psi_\varepsilon$ of $\psi$ which converges in $L^1$ to $\psi$ as $\varepsilon \to 0$ with the property that

$$\lim_{\varepsilon \to 0} P(\psi_\varepsilon) = P(\psi)$$

Unfortunately, $\psi_\varepsilon$ may no longer have the same boundary data as $\psi$. To adjust this, let $T_\varepsilon$ be an $\varepsilon$ neighborhood of $\partial \Omega$ and construct a smooth function $h_\varepsilon$ with support in $T_\varepsilon$ and so that $\psi_\varepsilon - (h_\varepsilon - z) = 0$ restricted to $\partial \Omega$ coincides with the graph of $\varphi$. Note that we may arrange that $h_\varepsilon - z$ has bounded gradient with a bound not depending on $\varepsilon$ and that $h_\varepsilon \to 0$ as $\varepsilon \to 0$. Notice that the perimeter of $\psi_\varepsilon$ and $\psi_\varepsilon - (h_\varepsilon - z)$ can be computed using $P$ by proposition 2.6 as both functions are smooth. Further, letting $W_\varepsilon$ be the open set $\psi_\varepsilon < 0$ and $W_{h,\varepsilon}$ be the open set $\psi_\varepsilon - (h_\varepsilon - z) < 0$,

$$|P(W_{h,\varepsilon}) - P(W_\varepsilon)| \leq \int_{T_\varepsilon \cap \Omega} \left|\nabla_0(\psi_\varepsilon - (h_\varepsilon - z))\right| - \left|\nabla_0(\psi_\varepsilon)\right| \, dx \, dy$$

$$\leq \int_{T_\varepsilon \cap \Omega} \left|\nabla_0(h_\varepsilon - z)\right| \, dx \, dy$$

$$\leq C \int_{T_\varepsilon \cap \Omega} \, dx \, dy$$

Thus,

$$|P(W_{h,\varepsilon}) - P(W_\varepsilon)| \to 0 \quad \text{as} \quad \varepsilon \to 0$$

(2)

Lastly, the relation between $A_\lambda$ and $E_\lambda$ coupled with the representation of $P$ for smooth functions given in proposition 2.6 yields:

$$|A_\lambda(S) - P(O)| = o_{\lambda^{-1}}(1)$$

(3)

for any smooth surface $S$ bounding an open set $O$. 
Now, since $\partial W_{h,\varepsilon}$ is smooth, using proposition 2.6, we see that

$$P(W_{\varepsilon}) = P(W_{h,\varepsilon}) + o_{\varepsilon}(1) \quad \text{by (2)}$$

$$= A_\lambda(\psi_{\varepsilon} - h_{\varepsilon}) + o_{\lambda^{-1}}(1) + o_{\varepsilon}(1) \quad \text{by (3)}$$

$$\geq A_\lambda(u_\lambda) + o_{\lambda^{-1}}(1) + o_{\varepsilon}(1)$$

where $u_\lambda$ is a Riemannian minimal graph with the same boundary data minimizing $E_\lambda(\cdot)$. By the argument in the proof of proposition 3.3 and proposition 2.6, we have that

$$E_\lambda(u_\lambda) \geq E(u_\lambda) = P(u_\lambda)$$

By the lower semicontinuity of $P$ (lemma 2.3), we have

$$\liminf_{\lambda \to \infty} P(u_\lambda) \geq P(u)$$

Putting this together with the previous calculation we have, finally,

$$P(W) = \lim_{\varepsilon \to 0} P(W_{\varepsilon}) \geq P(u)$$

Thus, $u$ and $W$ describe sets of minimal perimeter. \(\square\)

Remarks:

(i) These minimal surfaces in $(H, d_{cc})$ are geometrically realizable as limits and can often be explicitly constructed. In section 4, many of the examples can be realized in this way.

(ii) Theorem 3.6 shows that examples of minimal surfaces in $(H, d_{cc})$ can arise in a wide variety of limits. This, along with the subellipticity of $H_{cc} = 0$ is strong evidence that the solution to the Plateau problem is not necessarily unique.

(iii) Of course, one would like the solutions to have higher regularity than given in the theorem. However, once again the examples in section 4 (particularly those in section 4.2) show that, without imposing boundary conditions, regularity cannot be expected. The question of higher regularity of solution to the Plateau problem with $C^k$ boundary data is open.

4. Examples

4.1. Invariant Solutions. In [Tom93], P. Tomter began a program of finding minimal surfaces in $(H, g_1)$ which are invariant under isometric group actions based on the general technique of Hsiang and Lawson ([HL71]), later expanded by Hsiang and Hsiang ([HH82]). This program is completed in [FMP99] by Figuero, et al. where a number of additional cases are studied. The method rests on reducing the minimal surface partial differential equation to an ordinary differential equation (or system of ODEs) by considering how the PDE descends to the quotient of $H$ by a closed subgroup of isometries. Via the Riemannian submersion given by the process of quotienting by a group of isometries, the solutions of the ODE(s) can be lifted to minimal surfaces in $(H, g_1)$. For complete details on the method in the Heisenberg group, see [Tom93] and [FMP99].

More specifically, the authors mentioned above consider subgroups of rotations, group translations and combinations of these, which we will refer to as “corkscrew” motions. In light of proposition 3.3 and the fact that the subgroups of rotations, translations and corkscrew motions are also subgroups of the isometry group of $(H, d_{cc})$, one expects that there exist group invariant minimal surfaces in $(H, d_{cc})$. In this section, we identify, by brute force, several families of such minimal surfaces. We should remark that our technique is not a generalization of the quotienting technique alluded to above - the quotient mapping is a type of submersion, but it does not behave nearly as...
well as the Riemannian version. Thus, for the purposes of this paper, we rely on the following two methods:

(i) Search for examples as limits of Riemannian examples using proposition 3.3 or theorem 3.6.

(ii) Search for invariant examples directly in \((H, d_{cc})\).

While the first method does reveal specific initial examples, the second is much more efficient in finding the most examples. We will first give some simple isolated examples of the first technique and then explore the second.

4.1.1. **Invariant solutions as limits. Planes and Helicoids.** Consider the nonparametric minimal surface equation in \((H, g_\lambda), H_\lambda(u) = 0\) or:

\[
\begin{aligned}
    u_{xx} \left( \frac{1}{\lambda^2} + (u_y + x)^2 \right) - 2u_{xy}(u_y + x)(u_x - y) + u_{yy} \left( \frac{1}{\lambda^2} + (u_x - y)^2 \right) &= 0
\end{aligned}
\]

By inspection, we see that any plane \(z = ax + by + c\) is a solution to (4) for any \(\lambda > 0\). Thus, by proposition 3.3, all planes are solutions of (MSE).

Further, considering \(R^3\) using cylindrical coordinates and defining the helicoids by \(u(r, \theta) = a\theta\) for \(a \in \mathbb{R}\), then it is a quick computation to show that all of the helicoids satisfy (4).

4.1.2. **Direct methods.** Under the assumption that invariant minimal surfaces exist, in this section we produce smooth nonparametric examples.

**Rotationally and corkscrew invariant examples:** Again viewing \(R^3\) in cylindrical coordinates, we look for solutions of the form \(u(r, \theta) = f(r)\). Under the transformation to cylindrical coordinates, (MSE) becomes:

\[
\begin{aligned}
    u_{rr}(r^5 + ru_\theta^2 + 2r^3u_\theta) - u_{r\theta}(2ru_\theta u_r + 2r^3u_r) + u_{\theta\theta}u_r^2 + 2u_ru_\theta^2 + 2r^2u_\theta u_r + r^2u_r^3 = 0
\end{aligned}
\]

We are looking for all graphs invariant under a combination of rotations and vertical translations (the corkscrew motions), namely

\[
\begin{aligned}
    u(r, \theta) &= v(r) + a\theta \\
    a \in \mathbb{R}
\end{aligned}
\]

Under this assumption, (5) becomes:

\[
\begin{aligned}
    v''(r)(ra^2 + 2r^3a + r^5) + v'(r)(2a^2 + 2r^2a + r^3v'(r)^3) = 0
\end{aligned}
\]

or, under the assumption that the denominator does not vanish,

\[
\begin{aligned}
    v''(r) &= -\frac{v'(r)(2a^2 + 2r^2a + r^3v'(r)^3)}{r(a^2 + 2r^2a + r^4)}
\end{aligned}
\]

This ordinary differential equation can be solved in closed form yielding

\[
\begin{aligned}
    v(r) &= \pm \left( \frac{\sqrt{br^2 - 1}}{b} - a\tan^{-1} \left( \frac{1}{\sqrt{br^2 - 1}} \right) \right) + c \\
    &\quad \text{for } b, c, \in \mathbb{R}
\end{aligned}
\]

Writing the graphs implicitly yields:

\[
\begin{aligned}
    z^2 + 2a\theta z &= \frac{v^2}{b} - \frac{1}{b^2} - \frac{2a}{b} \tan^{-1} \left( \frac{1}{\sqrt{br^2 - 1}} \right) \sqrt{br^2 - 1} + a^2 \left( \tan^{-1} \left( \frac{1}{\sqrt{br^2 - 1}} \right) \right)^2 - a^2\theta^2
\end{aligned}
\]
Figure 1 shows two examples of this type. Notice that when $a \neq 0$, these surfaces are not embedded, but immersed. The shading in the figures shows the change of the horizontal normal - the point is darker the closer $<n_0, X>$ is to 1.

**Translationally invariant examples:**

We now consider graphs which are invariant under left translations by one parameter subgroups of the Heisenberg group. There are two cases, when we translate in the vertical direction and when we translate in a horizontal direction. Since a surface invariant under a vertical translation would not be a graph, we will focus on the second case. Due to the rotational invariance of the metric, it suffices to consider only graphs invariant under the subgroup of isometries generated by translation in the $X$ direction. Thus, we look for graphs $z = u(x, y)$ so that $(x + t, y, u(x + t, y)) = (t, 0, 0) \cdot (x, y, u(x, y))$. This amounts to solving the equation:

$$u(x + t, y) = u(x, y) + ty$$

One quickly sees that any smooth $u$ is of the form $u(x, y) = xy + f(y)$ where $f$ is any smooth function. Under this assumption, the minimal surface equation (MSE) is identically zero. Thus, any graph of the form

$$z = xy + f(y)$$

satisfies the minimal surface equation.

**Remarks:**

(i) Note that so long as $f$ has no singularities, all of these examples are *complete* minimal graphs.

(ii) The horizontal unit normal of these surfaces is a piecewise constant vector. Indeed, $X(xy + f(y) - z) = 0$ and $Y(xy + f(y) - z) = 2x + f'(y)$. So, the horizontal unit normal,

$$n_0 = \left(0, \frac{2x + f'(y)}{|2x + f'(y)|}\right)$$

Notice that the vector field is discontinuous along the characteristic locus over which it changes sign.
4.2. **Prescribed Gauss maps.** In classical minimal surface theory, the Gauss map plays a crucial role in recognizing and constructing examples of minimal surfaces. In this section, we analyze the horizontal Gauss map to help produce families of examples of minimal surfaces derived from the initial examples generated in the previous section.

Recall the definition of the horizontal Gauss map for a nonparametric surface, \( F(x,y,z) = u(x,y) - z = 0 \):

\[
\mathbf{n}_0 = \frac{\nabla_0 F}{|\nabla_0 F|} = \left( \frac{p}{\sqrt{p^2 + q^2}}, \frac{q}{\sqrt{p^2 + q^2}} \right)
\]

Where \( p = u_x - y \) and \( q = u_y + x \). Using this notation, \((MSE)\) becomes

\[
\text{div } \mathbf{n}_0 = 0
\]

where \(\text{div}\) is the standard divergence operator in \(\mathbb{R}^2\). Observing this, one strategy for finding examples is to search for divergence free unit vector fields on \(\mathbb{R}^2\) and associate to them solutions of \((10)\). If \( V = (v_1, v_2) \) is such a vector field, one can use the following procedure to generate examples:

(i) From \((9)\) and \((10)\), we see that we must have

\[
v_1 = \frac{p}{\sqrt{p^2 + q^2}} \quad \text{and} \quad v_2 = \frac{q}{\sqrt{p^2 + q^2}}
\]

(ii) Attempting to solve the equations above for \( u(x,y) \), one possible solution is for \( p \) and \( q \) to satisfy \( v_2 p = v_1 q \).

(iii) This gives rise to a first order partial differential equation:

\[
v_2(u_x - y) - v_1(u_y + x) = 0
\]

Thus, any solution to \((11)\) yields a solution to \((10)\) at points where \(\nabla_0 F \neq 0\). At these, the characteristic points of the graph, the Gauss map is discontinuous. Indeed, a quick calculation shows that if \( v_2 p = v_1 q \),

\[
\mathbf{n}_0 = \left( \frac{p}{|p|} v_1, \frac{q}{|q|} v_2 \right)
\]

Thus, depending on the values of \( p \) and \( q \) when crossing over the characteristic locus, the value of \( \mathbf{n}_0 \) may flip, for example, from \( V \) to \(-V\).

In summary, we have,

**Theorem 4.1.** If \( V = (v_1, v_2) \) is a divergence free unit vector field on \(\mathbb{R}^2\) and \( u(x,y) \) is a solution to

\[
v_2(u_x - y) - v_1(u_y + x) = 0
\]

Then \( u(x,y) \) satisfies \((MSE)\) and its Gauss map, \(\mathbf{n}_0\), matches \( V \) up to sign except at characteristic points, where it is discontinuous.

This theorem allows us to construct many surprising examples.

4.2.1. **Constant Gauss map.** In the initial set of examples generated above, we noticed that of the translationally invariant examples, the surfaces \( z = xy + f(y) \) had piecewise constant horizontal Gauss map. We will now apply the theorem above to find more examples with piecewise constant Gauss map. As we will discover, all of these examples are complete, in other words, they are solutions to the Bernstein problem - complete minimal graphs. To construct them, we simply prescribe

\[
V = (\alpha, \pm \sqrt{1 - \alpha^2}), \ \alpha \in [-1, 1]
\]
Taking first the positive sign on the square root, (11) becomes

\[ \sqrt{1 - \alpha^2}(u_x - y) - \alpha(u_y + x) = 0 \]

Using the method of characteristics and computing the envelope of the resulting solution, we find the following solutions to the above linear partial differential equation:

\[
 u(x, y) = \frac{\alpha}{\sqrt{1 - \alpha^2}} x^2 + xy + g \left( y + \frac{\alpha}{\sqrt{1 - \alpha^2}} x \right)
\]

where \( g \) is any function in \( W^{1,p}(\mathbb{R}) \). In this special case, we have

\[
 p = 2\alpha x + 2\alpha \sqrt{1-\alpha^2} Dg \left( y + \frac{\alpha}{\sqrt{1-\alpha^2}} x \right)
\]

and

\[
 q = 2x + Dg \left( y + \frac{\alpha}{\sqrt{1-\alpha^2}} x \right)
\]

and so

\[
 n_0 = \frac{2x + Dg \left( y + \frac{\alpha}{\sqrt{1-\alpha^2}} x \right)}{|2x + Dg \left( y + \frac{\alpha}{\sqrt{1-\alpha^2}} x \right)|} (\alpha, \sqrt{1 - \alpha^2})
\]

Figure 2 shows the graphs of several examples with different choices of \( g \) and with \( \alpha = \frac{\sqrt{2}}{2} \). The shading in the figures represents the different patches of constant Gauss map which differ from one another in sign. The interface of the darker and lighter shades is precisely the characteristic locus.

There is one degenerate case not covered above, we \( \alpha = \pm 1 \). In the case \( \alpha = 1 \), (11) becomes \( q = u_y + x = 0 \) which implies \( u(x, y) = -xy + g(x) \). So, \( p = -2y + Dg(x) \) and

\[
 n_0 = \left( \frac{-2y + Dg(x)}{|-2y + Dg(x)|}, 0 \right)
\]

The case \( \alpha = -1 \) yields the same solutions.

If one takes \( V = (\alpha, -\sqrt{1 + \alpha^2}) \), the procedure above yields the solutions:

\[
 u(x, y) = \frac{-\sqrt{1 - \alpha^2}}{\alpha} x^2 + xy + g \left( y - \frac{\sqrt{1 - \alpha^2}}{\alpha} x \right)
\]

4.2.2. Helicoidal examples. As we saw in section 4.1, both the plane \( z = 0 \) and the helicoid \( z = \theta \) are solutions to \((MSE)\). Just as in the previous section, we will recover these solutions and many more from considering the divergence-free vector field associated to these solution. For \( z = 0 \) or \( z = \theta \), we have

\[
 n_0 = \left( \frac{-y}{\sqrt{x^2 + y^2}}, \frac{x}{\sqrt{x^2 + y^2}} \right)
\]

Taking this vector field as \( V \), (11) becomes

\[
 \frac{x}{\sqrt{x^2 + y^2}}(u_x - y) + \frac{y}{\sqrt{x^2 + y^2}}(u_y + x) = 0
\]

or

\[
 (12) \quad xu_x + yu_y = 0
\]
Again, solving (12) yields a family of solutions:

\[
\boxed{u(x, y) = g \left( \frac{y}{x} \right)}
\]

where \( g \in W^{1,p}(\mathbb{R}) \). Note that with \( g(s) = 0 \) and \( g(s) = \tan^{-1}(s) \), we recover \( z = 0 \) and \( z = \theta \) respectively. Figure 3a,b give two examples of this type with specific choices of \( g \). As in figure 2

4.2.3. Solutions associated to planes. Expanding on the results of the previous section, we consider the plane \( z = ax + by + c \) which has horizontal Gauss map

\[
N_0 = \left( \frac{a - y}{\sqrt{(a - y)^2 + (b + x)^2}}, \frac{b + x}{\sqrt{(a - y)^2 + (b + x)^2}} \right)
\]

Again using this as \( V \), (11) becomes

\[
(b + x)(u_x - y) - (a - y)(u_y + x) = 0
\]
(a) Example 4.2.2, \( g(s) = \tan^{-1}(s) \)

(b) Example 4.2.2, \( g(s) = \frac{s}{\sqrt{1+s^2}} \)

(c) Example 4.2.3, \( \alpha = \beta = 1 \), \( g(s) = 0 \)

(d) Example 4.2.3, \( \alpha = \beta = 1 \), \( g(s) = s^2 \)

**Figure 3.** Minimal surfaces with Gauss map the same as some plane

Solving this first order partial differential equation yields solutions:

\[
 u(x, y) = \frac{a}{b + x}x^2 + \frac{b^2}{b + x}xy + g \left( \frac{y - a}{b + x} \right)
\]

where, again, \( g \in W^{1, p}(\mathbb{R}) \). Figures 3c,d show some examples with different choices of \( g \) and \( a = b = 1 \). In these figures, the shading is not connected to the horizontal Gauss map. Notice that if we take \( g(s) = s \), we recover the plane \( z = x + y - 1 \).

4.3. **Summary.** Briefly, we revisit the examples generated above, listing them together:

**Theorem 4.2.** Given \( a, b, c \in \mathbb{R} \), \( \alpha \in [-1, 1] \) and \( g \in W^{1, p}(\mathbb{R}) \), the following graphs satisfy (MSE) weakly:

(i) \( z = ax + by + c \)
(ii) \[ z = a \theta \]

(using cylindrical coordinates)

(iii) \[ z = \pm \left( \frac{\sqrt{br^2 - 1}}{b} - a \tan^{-1} \left( \frac{1}{\sqrt{br^2 - 1}} \right) \right) + c + a \theta \]

Again, these use cylindrical coordinates.

(iv) \[ z = xy + g(y) \]

(v) \[ z = \frac{\alpha}{\sqrt{1 - \alpha^2}} x^2 + xy + g \left( y + \frac{\alpha}{\sqrt{1 - \alpha^2}} x \right) \]

(vi) \[ z = -\frac{\sqrt{1 - \alpha^2}}{\alpha} x^2 + xy + g \left( y - \frac{\sqrt{1 - \alpha^2}}{\alpha} x \right) \]

(vii) \[ z = g \left( \frac{y}{x} \right) \]

(viii) \[ z = \frac{a}{b + x} x^2 + \frac{b^2}{b + x} xy + g \left( \frac{y - a}{b + x} \right) \]

5. CONSEQUENCES AND DISCUSSION

The examples of the last sections show that there are a surprising number of minimal surfaces in the setting of the Carnot Heisenberg group. In particular, the examples with prescribed Gauss map are particularly enlightening. These show that even when restricting the horizontal Gauss map to give a specific vector field (up to the ambiguity of sign), one can produce an envelope of infinitely many solutions. This is in stark contrast to the Euclidean and Riemannian cases where, under suitable conditions, one expects reasonable rigidity of minimal surfaces. In the next two sections, we explore the implications of the examples above in the context of two classical problems. The first, the so-called Bernstein problem, asks for the classification of complete minimal graphs. The second deals with the uniqueness of solutions to the Dirichlet problem in this setting.

5.1. The Bernstein Problem. In the theory of minimal surfaces in Euclidean space, Bernstein characterized complete minimal graphs:

**Theorem 5.1** (Bernstein [Ber15]). *Planes are the only complete minimal graphs in \( \mathbb{R}^3 \).*

One can prove this beautiful theorem by showing that the Gauss map can be viewed as a bounded holomorphic function on \( \mathbb{C} \) and thus must be constant. Therefore, the normal to the complete minimal graph is constant, i.e. the graph is a plane. To mark the contrast between minimal surface theory in \( \mathbb{R}^3 \) and in \( (H, d_{cc}) \), we point out that many of the examples listed in section 4 are complete minimal graphs. In particular we’ve seen that we can even find many complete minimal graphs with piecewise constant Gauss map. Theorem 4.2 (iv) (v) and (vi) are examples of this form. We emphasize that in most of these examples, one has a huge amount of flexibility - examples of the form (v) and (vi) depend on a real number as well as an element of \( W^{1,p}(\mathbb{R}) \). This, of course, shows that nothing like Bernstein’s theorem can possibly be true in this setting.
However, if one does not allow discontinuities in the horizontal Gauss map, the question remains: do there exist minimal graphs with constant horizontal Gauss map? If so, how many are there? Having continuous constant Gauss map implies that the surface may not have any characteristic points. Since the submission of this article the author and N. Garofalo have shown ([GP02]) a version of the Bernstein property in the Heisenberg group.

5.2. **Uniqueness for the Dirichlet problem.** In this section, we investigate the uniqueness of solutions to the Dirichlet problem. To be precise, we will investigate the following problem:

**Problem 5.2.** Given a set $\Omega \subset \mathbb{R}^2$ with a smooth closed boundary and a smooth closed curve $\Gamma : \partial \Omega \to H$, does there exist $u \in W^{1,p}(\Omega)$ so that the graph $z = u(x,y)$ defines a surface satisfying the minimal surface equation bounded by $\Gamma$.

In deriving the minimal surface equation ($MSE$) in the third section, we noted that the equation is sub-elliptic and hence, we cannot immediately guarantee uniqueness of solutions to the Dirichlet problem. Indeed, this observation coupled with the wealth of examples produced in section 4 leads one to suspect that solutions may not be unique. In this section we produce two distinct solutions to the Dirichlet problem for a given curve by exploiting the connection between surfaces satisfying $(MSE)$ and sets of minimal perimeter given in theorem 3.7. We first define the minimal surfaces themselves:

\[(S_1) \quad u_1(x,y) = x^2 + xy\]

Note that this is an example of the form produced in section 4.2.1 or theorem 4.2 (v) with $\delta = \sqrt{2}$ and $f(s) = 0$.

\[(S_2) \quad u_2(x,y) = xy + 1 - y^2\]

Note that this is an example of the form produced in section 4.2.1 or theorem 4.2 (v) with $\delta = 0$ and $f(s) = 1 - s^2$.

Observe next that both surfaces contain the same smooth closed curve, $\Gamma$, which is the graph over the unit circle in the plane given parametrically as
\[
\Gamma = \{(\cos(\theta), \sin(\theta), \cos^2(\theta) + \sin(\theta) \cos(\theta)) \mid \theta \in [0, 2\pi]\}
\]

Defining $\varphi(\theta) = \cos^2(\theta) + \sin(\theta) \cos(\theta)$, $\Gamma$ is the graph of $\varphi$ over $S^1$. Figure 5.2 shows both curves bounded by $\Gamma$.

We summarize this observation as follows:

**Theorem 5.3.** The graphs given by $z = u_1(x,y)$ and $z = u_2(x,y)$ over the closed unit ball in $\mathbb{R}^2$ are solutions to the Dirichlet problem:

$$H_{cc}(u) = 0$$

$$u|_{S^1} = \varphi$$

**Remark:** Notice that a direct calculation verifies that the area of the two surfaces is the same:

$$E(u_1) = \int_{\Omega} 8x^2 \, dx \, dy = 2\pi$$

and

$$E(u_2) = \int_{\Omega} 4(x - y)^2 \, dx \, dy = 2\pi$$
5.3. **Further investigations.** The wealth of examples of minimal surfaces described above, while answering several questions concerning the geometry of the Carnot Heisenberg group, leads us to many unsolved problems. In this section we outline a few of these questions:

1. In section 5.1, we mentioned the question of the existence of examples of graphs with constant Gauss map. More generally, we can pose the following question:

**Question 1.** Can one completely classify all examples of prescribed piecewise constant Gauss map?

2. While the examples show that without any extra conditions, one cannot hope for the higher regularity of solutions, the question of higher regularity for solutions of the Dirichlet problem is not ruled out by the examples.

**Question 2.** Given a closed curve, $\Gamma$, in $H$ of class $C^k$ given as a graph over a closed curve in $\mathbb{R}^2$, what is the regularity of a minimal surface spanning $\Gamma$?

3. This paper focused on nonparametric minimal surfaces, mostly because the calculations were not nearly as cumbersome as in the general case. However, the case explored here, that of graphs over the $xy$-plane is set up precisely to match the nonisotropic character of the Carnot-Carathéodory metric. It is not hard to write down an analogous set of equations for graphs over the $yz$-plane, but the resulting partial differential equation is much less tractable and it is not clear that one recovers $X$-minimal surfaces from the solutions.

**Question 3.** Can one extend these techniques to find $X$-minimal surfaces which are not necessarily graphs over domains in the $xy$-plane?

4. Understanding the case of $H_{cc} = 0$ is a first step in the exploration of the analogue of constant mean curvature surfaces in the Carnot-Carathéodory setting. Naively, one can simply find examples, using the same techniques as above, to solutions of $H_{cc} = const.$ For example, one can find a closed solution to $H_{cc} = 1$, the top half of which is given by
the function
\[ f(r) = -\frac{1}{2}r\sqrt{4 - r^2} + 2 \arctan \left( \frac{r}{\sqrt{4 - r^2}} \right) \]
where, like the rotationally invariant solutions detailed above, we use polar coordinates to describe the graph. Figure 5 shows a picture of the whole surface.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig5.png}
\caption{A closed surface satisfying \( H_{cc} = 1 \)}
\end{figure}

The solutions of this equation do indeed minimize area among candidates of fixed volume and so are an appropriate extension of the notion of constant mean curvature surfaces in the Euclidean setting.

In [DGN01], Danielli, Garofalo and Nhieu address the notion of “constant X-mean curvature surfaces” and use them to investigate the isoperimetric profile of \((H, d_{cc})\). Specifically, they find that the surfaces giving the best isoperimetric constant among rotationally symmetric surfaces are indeed those satisfying the equation \( H_{cc} = \text{const} \). This is a tantalizing result which provides a window into the possibilities for understanding completely the connection between the isoperimetric profile and this class of surfaces. In particular, it gives hope that, as in many Riemannian settings, the constant mean curvature surfaces reflect the isoperimetric profile exactly.

\begin{thebibliography}{9}

[Bal] Z. Balogh. Size of characteristic sets and functions with prescribed gradient. To appear in J. für die Reine und Angewandte Mathematik.
[Ber15] S. Bernstein. Sur un théorème de géométrie et ses applications aux équations aux dérivées partielles du type elliptique. Comm. de la Soc. Math. de Kharkov (2-ème série), 15:38-45, 1915.
[CDG94a] Luca Capogna, Donatella Danielli, and Nicola Garofalo. An isoperimetric inequality and the geometric Sobolev embedding for vector fields. Math. Res. Lett., 1(2):263-268, 1994.
[CDG94b] Luca Capogna, Donatella Danielli, and Nicola Garofalo. The geometric Sobolev embedding for vector fields and the isoperimetric inequality. Comm. Anal. Geom., 2(2):203-215, 1994.
[DeG54] E. DeGiorgi. Su una teoria generale della misura \((r - 1)\) dimensionale in uno spazio ad \( r \) dimensioni. Ann. Mat. Pura Appl., 4(36):191-213, 1954.
\end{thebibliography}
[DGN01] D. Danielli, N. Garofalo, and D.M. Nhieu. Minimal surfaces, surfaces of constant mean curvature and isoperimetry in Carnot groups. Preprint, 2001.

[FMP99] Christian B. Figueroa, Francesco Mercuri, and Renato H. L. Pedrosa. Invariant surfaces of the Heisenberg groups. Ann. Mat. Pura Appl. (4), 177:173–194, 1999.

[FSSC95] Bruno Franchi, Raul Serapioni, and Francesco Serra Cassano. Champs de vecteurs, théorème d’approximation de Meyers-Serrin et phénomène de Lavrentev pour des fonctionnelles dégénérées. C. R. Acad. Sci. Paris Sér. I Math., 320(6):695–698, 1995.

[FSSC96] Bruno Franchi, Raul Serapioni, and Francesco Serra Cassano. Meyers-Serrin type theorems and relaxation of variational integrals depending on vector fields. Houston J. Math., 22(4):859–890, 1996.

[FSSC99] Bruno Franchi, Raul Serapioni, and Francesco Serra Cassano. Rectifiability and Perimeter in the Heisenberg group. Math. Ann., 321(3):479–531, 2001.

[GN96] Nicola Garofalo and Duy-Minh Nhieu. Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces. Comm. Pure Appl. Math., 49(10):1081–1144, 1996.

[GP02] Nicola Garofalo and Scott D. Pauls. The Bernstein Problem in the Heisenberg group. Preprint, 2002.

[Gro81a] Mikhail Gromov. Structures métriques pour les variétés riemannienes, volume 1 of Textes Mathématiques [Mathematical Texts]. CEDIC, Paris, 1981. Edited by J. Lafontaine and P. Pansu.

[Gro81b] Mikhail Gromov. Groups of polynomial growth and expanding maps. Intitute Hautes Études Scientifique Publications Mathématique, (53):53–73, 1981.

[Gro96] Mikhail Gromov. Carnot-Carathéodory spaces seen from within. In Sub-Riemannian geometry, volume 144 of Progr. Math., pages 79–323. Birkhäuser, Basel, 1996.

[GT01] David Gilbarg and Neil S. Trudinger. Elliptic partial differential equations of second order. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[Hei95] Juha Heinonen. Calculus on Carnot groups. In Full School in Analysis (Jyväskylä, 1994), pages 1–31. Univ. Jyväskylä, Jyväskylä, 1995.

[HH82] Wu-teh Hsiang and Wu-yi Hsiang. On the existence of codimension-one minimal spheres in compact symmetric spaces of rank 2. II. J. Differential Geom., 17(4):583–594 (1983), 1982.

[HL71] Wu-yi Hsiang and H. Blaine Lawson, Jr. Minimal submanifolds of low cohomogeneity. J. Differential Geometry, 5:1–38, 1971.

[MSC99] Roberto Monti and Francesco Serra Cassano. Surface measures in Carnot-Carathéodory spaces. 1999. Preprint.

[Pan89] Pierre Pansu. Métriques de Carnot-Carathéodory et quasisométries des espaces symétriques de rang un. Annals of Mathematics (2), 129(1):1–60, 1989.

[Tom93] Per Tomter. Constant mean curvature surfaces in the Heisenberg group. In Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990), pages 485–495. Amer. Math. Soc., Providence, RI, 1993.

DARTMOUTH COLLEGE, HANOVER, NH, 03755
E-mail address: scott.pauls@dartmouth.edu