RECONSTRUCTION OF ROUGH CONDUCTIVITIES FROM BOUNDARY MEASUREMENTS

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Abstract. We show the validity of Nachman’s procedure (Ann. Math. 128(3):531–576, 1988) for reconstructing a conductivity \( \gamma \) from its Dirichlet-to-Neumann map \( \Lambda_\gamma \) for less regular conductivities, specifically \( \gamma \in W^{3/2,2n}(\Omega) \) such that \( \gamma \equiv 1 \) near \( \partial \Omega \). We also obtain a log-type stability estimate for the inverse problem when in addition, \( \gamma \in W^{2-s,n/s}(\Omega) \) for \( 0 < s < 1/2 \).

1. Introduction

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) (\( n \geq 2 \)) with sufficiently smooth boundary, and let \( \gamma \) be a positive real-valued function in \( \Omega \) satisfying the uniform ellipticity condition

\[
0 < c < \gamma(x) < c^{-1} \quad \text{for a.e. } x \in \Omega.
\]

Given \( f \in H^{1/2}(\partial \Omega) \), let \( u_f \in H^1(\Omega) \) denote the unique solution to the following Dirichlet boundary value problem:

\[
\begin{cases}
-\nabla \cdot (\gamma \nabla u_f) = 0 \quad \text{in } \Omega \\
u_f = f \quad \text{on } \partial \Omega.
\end{cases}
\]

The Dirichlet-to-Neumann map of \( \gamma \), \( \Lambda_\gamma \), is defined as the map that sends

\[
f \in H^{1/2}(\partial \Omega) \mapsto \gamma \frac{\partial u_f}{\partial \nu} \big|_{\partial \Omega} \in H^{-1/2}(\partial \Omega)
\]

where \( \partial/\partial \nu \) is the outward pointing unit normal vector field on \( \partial \Omega \). Here, \( \gamma \frac{\partial u_f}{\partial \nu} \big|_{\partial \Omega} \) is interpreted in the weak sense as follows: Given \( g \in H^{1/2}(\partial \Omega) \), let \( v_g \in H^1(\Omega) \) be any function such that \( v_g|_{\partial \Omega} = g \). Then

\[
\langle \Lambda_\gamma(f), g \rangle = \left\langle \gamma \frac{\partial u_f}{\partial \nu}, g \right\rangle := \int_\Omega \gamma \nabla u_f \cdot \nabla v_g \, dx.
\]

Physically, if \( \gamma(x) \) represents the electrical conductivity at a point \( x \) inside an object \( \Omega \) and \( f \) is the voltage applied on its boundary \( \partial \Omega \), then the solution \( u_f \) of (1) is precisely the induced electric potential inside \( \Omega \). In this case, \( \gamma \frac{\partial u_f}{\partial \nu} \big|_{\partial \Omega} \) is the induced current flux density at the boundary and therefore, the map \( \Lambda_\gamma \) encodes the set of all possible voltage and current measurements that can be made on the boundary.

The inverse conductivity problem, first proposed by Alberto Calderón in 1980 ([14]), asks whether we can determine the conductivity \( \gamma \) from measurements on the boundary, encoded by \( \Lambda_\gamma \). For there to be any hope of reconstruction, we first need the map \( \gamma \mapsto \Lambda_\gamma \) to be injective. Calderón proved injectivity for a linearized version of the problem where \( \gamma \) was assumed to be a small isotropic perturbation of the identity. For the full nonlinear problem,
injectivity was first proved for \( n \geq 3, \gamma \in C^2 \) by Sylvester and Uhlmann in [45]. Their approach was to reduce the problem to a similar problem for the Schrödinger equation at 0 energy: let \( q \) be a complex valued function in \( \Omega \) such that 0 is not a Dirichlet eigenvalue for \((-\Delta + q)\) on \( \Omega \). Given \( f \in H^{1/2}(\partial \Omega) \), let \( u_f \) denote the unique solution to the following boundary value problem:

\[
\begin{align*}
(-\Delta + q)u_f &= 0 \text{ in } \Omega \\
u_f &= f \text{ on } \partial \Omega.
\end{align*}
\]

The Dirichlet-to-Neumann map for \( q \) is defined as the map \( \Lambda_q : f \mapsto \partial u_f / \partial \nu \big|_{\partial \Omega} \). The corresponding inverse problem is to determine \( q \) from \( \Lambda_q \). Sylvester and Uhlmann showed that the inverse problem for the conductivity equation can be reduced to the inverse problem for the Schrödinger equation with \( q = \gamma^{-1/2} \Delta \gamma^{1/2} \). Next, the authors proved the injectivity of \( q \mapsto \Lambda_q \) using the so-called Complex Geometrical Optics (CGO) solutions to \((-\Delta + q)u = 0\), defined globally in \( \mathbb{R}^n \). These are solutions of the form \( e^{\pm \zeta}(1 + r_\zeta(x)) \), where \( \zeta \in \mathbb{C}^n \) is such that \( \zeta \cdot \zeta = 0 \) and \( r_\zeta \) has certain decay properties as \( |\zeta| \to \infty \).

Once we know that \( \gamma \mapsto \Lambda_\gamma \) is injective, we may try to find a constructive procedure for computing \( \gamma \) from \( \Lambda_\gamma \). In [40], Nachman provided such a constructive procedure for computing \( \gamma \) (resp., \( q \)) from \( \Lambda_\gamma \) (resp., \( \Lambda_q \)) when \( \gamma \in C^{1,1} \) (resp., \( q \in L^\infty \)). The procedure is based on the observation that CGO solutions satisfying certain decay conditions are uniquely determined by their restrictions to \( \partial \Omega \). In turn, these restrictions can be characterized as the unique solutions of certain boundary integral equations on \( \partial \Omega \).

An interesting problem that has received considerable interest is of finding the minimum regularity assumptions on \( \gamma \) (or \( q \)) under which injectivity and the reconstruction procedure hold. This question is also of practical importance. For example, it was pointed out in [17] that if \( q \) arises from a Gaussian random field satisfying certain conditions, almost every instantiation of \( q \) belongs to a Sobolev space of fixed negative order. For \( n \geq 3 \), the regularity assumption for uniqueness was relaxed to \( \gamma \in C^{3/2+} \) in [9], to \( C^{3/2} \) in [43], to \( W^{3/2,2n+} \) in [10], to \( W^{3/2,2+} \) in [42] and to \( \gamma \in C^1 \) or \( \gamma \in C^{0,1} \) with \( \|\nabla \log \gamma\|_{L^\infty} \) small in [23]. The smallness condition was removed in [18]. Note that for \( \gamma \equiv 1 \) in a neighborhood of \( \partial \Omega \) (as we assume in this paper), the regularity assumption in [10] is reduced to \( \gamma \in W^{3/2,2n} \). It was also conjectured by Brown in [10] that uniqueness holds for \( \gamma \in W^{1,n} \) for all \( n \geq 3 \). This was proved for \( n = 3, 4 \) in [22].

For the problem of reconstruction, Nachman’s procedure in [40] was adapted to the case of \( \gamma \in C^1 \) or \( \gamma \in C^{0,1} \) with \( \|\nabla \log \gamma\|_{L^\infty} \) sufficiently small in [20]. The approaches in [42] and [22] are not suitable for reconstruction because they do not guarantee the existence of CGO solutions for \( |\zeta| \) uniformly large, and the approach in [18] produces only local solutions. In this paper, we prove the validity of Nachman’s reconstruction procedure for \( \gamma \in W^{3/2,2n} \) with \( \gamma \equiv 1 \) near \( \partial \Omega \) as in [10]. The key step is establishing bounds on the multiplication operator \( \varphi \mapsto q_\varphi \) in the weighted Sobolev spaces of Sylvester and Uhlmann [45], similar to the bounds in [17]. Note that functions in \( W^{3/2,2n} \) need not be Lipschitz. However, \( W^{3/2,2n} \) is contained in the Zygmund space \( C^1_\ast \) of continuous functions \( f \) such that

\[
\|f\|_{C^1_\ast} = \sup_{x \in \mathbb{R}^n} |f(x)| + \sup_{x,h \in \mathbb{R}^n, h \neq 0} \left| \frac{f(x + h) + f(x - h) - 2f(x)}{h} \right| < \infty.
\]
We refer the reader to the monograph [46] for more on Zygmund spaces.

Another question of interest is of stability of the map $\gamma \mapsto Λ_γ$. It was shown by Alessandrini in [1] that under the a-priori assumption
\[
\|γ_j\|_{H^s(Ω)} \leq M, \quad s > n/2 + 2, \ j = 1, 2,
\]
we have a stability estimate of the form
\[
\|γ_1 - γ_2\|_{L^∞(Ω)} \leq C \left\{ \log \|Λ_γ_1 - Λ_γ_2\|_{H^{1/2} \to H^{-1/2}}^{−\sigma} + \|Λ_γ_1 - Λ_γ_2\|_{H^{1/2} \to H^{-1/2}} \right\}
\]
where $σ = σ(n, s) \in (0, 1)$. Subsequently, the a-priori assumptions were relaxed to $\|γ_j\|_{W^{2,∞}} \leq M$ in [2, 3]. Such logarithmic estimates were shown to be optimal up to the value of the exponent by Mandache in [38] via explicit examples. Later, stability was proved for conductivities bounded a-priori in $C^{1, 1+ε} \cap H^{n/2+ε}$ with $∂Ω$ smooth by Heck in [24] and for a-priori bounds in $C^{1,ε}(Ω)$ with $∂Ω$ Lipschitz by Caro, García and Reyes in [16]. We are able to obtain a similar stability estimate with $\|γ_j\|_{W^{2-s, n/s}(Ω)} \leq M$ for some $0 < s < 1/2$. The main results of this paper are summarized in the following theorem:

**Theorem 1.1.** Let $Ω$ be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$. Let $γ \in W^{3/2, 2n}(Ω)$ be a positive real valued function satisfying
\[
0 < c < γ(x) < c^{-1} \quad \text{for a.e. } x \in Ω
\]
and $γ ≡ 1$ in a neighborhood of $∂Ω$. Then,
(a) One can determine $γ$ from the knowledge of the map $Λ_γ : H^{1/2}(∂Ω) \to H^{-1/2}(∂Ω)$ in a constructive way. Moreover,
(b) We have the following stability estimate: Let $γ_j \in W^{3/2, 2n}(Ω)$, $j = 1, 2$, be such that $γ_j \equiv 1$ near $∂Ω$ and satisfy the ellipticity bound [3]. Suppose in addition that $\|γ_j\|_{W^{2-s, n/s}(Ω)} \leq M$ for some $0 < s < 1/2$, and let $0 ≤ α < 1$. Then there exist $C = C(Ω, n, c, M, s, α) > 0$ and $0 < σ = σ(n, s, α) < 1$ such that
\[
\|γ_1 - γ_2\|_{C^∞(Ω)} \leq C \left\{ \log \|Λ_γ_1 - Λ_γ_2\|_{H^{1/2} \to H^{-1/2}}^{−σ} + \|Λ_γ_1 - Λ_γ_2\|_{H^{1/2} \to H^{-1/2}} \right\}.
\]

As usual, this result will be obtained as a consequence of the corresponding result for the Schrödinger equation:

**Theorem 1.2.** Let $Ω$ be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$. Let $q \in W^{−1/2, 2n}_{comp}(Ω)$ be such that $0$ is not a Dirichlet eigenvalue of the boundary value problem [2]. Then,
(a) One can determine $q$ from the knowledge of the map $Λ_q : H^{1/2}(∂Ω) \to H^{-1/2}(∂Ω)$ in a constructive way. Moreover,
(b) We have the following stability estimate: Let $q_j \in W^{−1/2, 2n}_{comp}(Ω)$, $j = 1, 2$. Suppose in addition that $\|q_j\|_{W^{−s, n/s}} \leq M$ for some $0 < s < 1/2$. Then there exist $C = C(Ω, n, c, M, s) > 0$ and $0 < σ = σ(n, s) < 1$ such that
\[
\|q_1 - q_2\|_{H^{-1}} \leq C \left\{ \log \|Λ_q_1 - Λ_q_2\|_{H^{1/2} \to H^{-1/2}}^{−σ} + \|Λ_q_1 - Λ_q_2\|_{H^{1/2} \to H^{-1/2}} \right\}.
\]

While this paper deals only with the full data Calderón problem, we note here that the problem of partial data, where measurements are made on only a part of the boundary is also of significant interest. Several results have been obtained on uniqueness ([113, 33, 29, 31]), minimum regularity ([34, 47, 44, 33]), reconstruction ([39, 4, 5]) and stability ([25, 36, 15]). We refer the reader to [32] for a survey on the Calderón problem with partial data. The
problem for \( n = 2 \) is also by now well understood. Uniqueness was first proved for \( C^2 \) conductivities in \([11]\). The regularity assumptions were later relaxed to \( W^{1,2} \) in \([11]\), to \( L^\infty \) in \([6]\) and to \( L^{2+} \) in \([12]\). Nachman’s reconstruction procedure has also been extended to \( L^\infty \) conductivities in the plane that are 1 near the boundary in \([37]\). Stability estimates \([7]\) and various partial data results \([27,28,26]\) are also known.

Here is a short outline of the paper: We begin by showing that the problem of reconstructing \( \gamma \in W^{3/2,2n}(\Omega) \) (with \( \gamma \equiv 1 \) near \( \partial \Omega \)) from \( \Lambda_q \) reduces to the problem of reconstructing \( q \in W^{-1/2,2n}_{\text{comp}}(\Omega) \) from \( \Lambda_q \) in Section 2. In section 3, we introduce the necessary function spaces and construct Complex Geometrical Optics (CGO) solutions to \((\Delta + q)u = 0\) in \( \mathbb{R}^n \). These solutions are then used to show uniqueness and reconstruction of \( q \) from \( \Lambda_q \) in Section 4. We conclude by proving the stability estimates \([4]\) and \([5]\) in Section 5.

2. Reduction to the Schrodinger Equation

As in the smooth conductivity case, our first step will be to reduce the conductivity equation \([1]\) to the Schrodinger equation \([2]\) with \( q = \Delta \sqrt{\gamma}/\sqrt{\gamma} \). Recall the class of Bessel potential spaces \( W^{s,p}(\mathbb{R}^n) \), defined by the norms

\[
\|f\|_{W^{s,p}} = \|(I - \Delta)^{s/2}f\|_{L^p}, \quad s \in \mathbb{R}, \ p \geq 1.
\]

For a bounded Lipschitz domain \( \Omega \subset \mathbb{R}^n \), \( W^{s,p}(\Omega) \) is defined as the space of \( W^{s,p}(\mathbb{R}^n) \) functions restricted to \( \Omega \), i.e.,

\[
W^{s,p}(\Omega) := \{u|_{\Omega} : u \in W^{s,p}(\mathbb{R}^n)\}
\]

with the norm

\[
\|f\|_{W^{s,p}(\Omega)} = \inf\{\|u\|_{W^{s,p}(\mathbb{R}^n)} : u|_{\Omega} = f\}.
\]

We will show that when \( \gamma \) is as in Theorem \([11]\) \( q = \gamma^{-1/2}\Delta \gamma^{1/2} \in W^{-1/2,2n}_{\text{comp}}(\Omega) \), where

\[
W^{-1/2,2n}_{\text{comp}}(\Omega) := \{u \in W^{s,p}(\mathbb{R}^n) : \text{supp } u \subset \Omega\}.
\]

Let us note some properties of the Dirichlet boundary value problem \([2]\) when \( q \in W^{-1/2,2n}_{\text{comp}}(\Omega) \).

Proposition 2.1. Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain and \( q \in W^{-1/2,2n}_{\text{comp}}(\Omega) \).

(a) The multiplication operator \( m_q : C^\infty(\Omega) \to \mathcal{D}'(\Omega) \) defined by \( \langle m_q(\varphi), \psi \rangle = \langle q, \varphi \psi \rangle \) extends to a continuous map \( H^1(\Omega) \to H^{-1}(\Omega) \) and is compact.

(b) (The Fredholm Alternative) Exactly one of the following must be true:

(i) For any \( f \in H^{1/2}(\partial \Omega) \) and \( F \in H^{-1}(\Omega) \), there exists a unique \( u \in H^1(\Omega) \) such that

\[
\begin{cases}
-\Delta + m_q u = F & \text{in } \Omega \\
u = f & \text{on } \partial \Omega.
\end{cases}
\]

Moreover, there exists \( C = C(q,\Omega) > 0 \) such that

\[
\|u\|_{H^1(\Omega)} \leq C(\|f\|_{H^{1/2}(\partial \Omega)} + \|F\|_{H^{-1}(\Omega)}).
\]

(ii) There exists \( u \in H^1(\Omega) \), \( u \neq 0 \) such that

\[
\begin{cases}
-\Delta + m_q u = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

That is, 0 is a Dirichlet eigenvalue of \((-\Delta + m_q)\) on \( \Omega \).
Proof. It follows from Theorem 3.8 that $m_q$ maps $H^1(\Omega) \to H_{\text{comp}}^{-1/2}(\Omega)$. The compactness of $m_q : H^1(\Omega) \to H^{-1}(\Omega)$ follows from the compactness of the inclusion $H_{\text{comp}}^{-1/2}(\Omega) \hookrightarrow H^{-1}(\Omega)$.

Next, we note that $(-\Delta + m_q) : H_0^1(\Omega) \to H^{-1}(\Omega)$ is Fredholm, since $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is invertible and $m_q$ is compact. Therefore, (b) follows from standard Fredholm theory. □

As usual, if 0 is not a Dirichlet eigenvalue of $(-\Delta + q)$, we define the Dirichlet-to-Neumann map $\Lambda_q : H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega)$ by duality: Given $f \in H^{1/2}(\partial \Omega)$, let $u \in H^1(\Omega)$ be the unique solution of (2). Then

$$\langle \Lambda_q f, g \rangle = \int_{\partial \Omega} \Lambda_q(f) g \, d\sigma = \int_{\Omega} \nabla u \cdot \nabla v \, dx + \langle m_q u, v \rangle_{L^2(\Omega)}, \quad g \in H^{1/2}(\partial \Omega)$$

where $d\sigma$ is the surface measure on $\partial \Omega$ and $v \in H^1(\Omega)$ is any function such that $v|_{\partial \Omega} = g$. We also get the following integral identity as a consequence of the symmetry of the multiplication operator $m_q$:

**Proposition 2.2.** Let $q_1, q_2 \in W^{-1/2,2n}_{\text{comp}}(\Omega)$ be such that 0 is not a Dirichlet eigenvalue of $(-\Delta + m_{q_j})$ on $\Omega$, $j = 1, 2$. Let $u_1, u_2 \in H^1(\Omega)$ be solutions of $(-\Delta + m_{q_j})u_j = 0$, $j = 1, 2$.

Then

$$\int_{\partial \Omega} (\Lambda_{q_1} - \Lambda_{q_2}) u_1 \cdot u_2 \, d\sigma = \int_{\Omega} (m_{q_1} - m_{q_2}) u_1 \cdot u_2 \, dx$$

where $d\sigma$ is the surface measure on $\partial \Omega$.

Remark on Notation. Henceforth, we will use $qu$ and $m_q(u)$ interchangeably. We will also write $A \lesssim B$ to indicate that there exists $C > 0$ such that $A \leq CB$.

Let us now show how (1) reduces to (2).

**Proposition 2.3.** Let $\gamma \in W^{3/2,2n}(\Omega)$ be such that

$$0 < c < \gamma(x) < c^{-1} \quad \text{a.e. on } \Omega$$

and $\gamma \equiv 1$ on a neighborhood of $\partial \Omega$. Extend $\gamma$ to all of $\mathbb{R}^n$ by defining $\gamma \equiv 1$ on $\mathbb{R}^n \setminus \Omega$ and define $q = \Delta \sqrt{\gamma}/\sqrt{\gamma}$.

(a) $q \in W^{-1/2,2n}_{\text{comp}}(\Omega)$.

(b) $u \in H^1(\Omega)$ solves

$$\begin{cases} -\nabla \cdot (\gamma \nabla u) = 0 & \text{in } \Omega \\ u = f & \text{in } H^{1/2}(\partial \Omega) \end{cases}$$

if and only if $w = \gamma^{1/2}u \in H^1(\Omega)$ solves

$$\begin{cases} (-\Delta + q)w = 0 & \text{in } \Omega \\ w = f & \text{on } \partial \Omega. \end{cases}$$

(c) 0 is not a Dirichlet eigenvalue of $(-\Delta + m_q)$ on $\Omega$ and $\Lambda_q = \Lambda_{\gamma}$.
Proof. (a) That $q$ is compactly supported in $\Omega$ follows from the fact that $\gamma \equiv 1$ outside a compact subset of $\Omega$. Next consider the identity

$$\frac{\Delta \sqrt{\gamma}}{\sqrt{\gamma}} = \frac{1}{2} \Delta \log \gamma + \frac{1}{4} |\nabla \log \gamma|^2$$

$$\Rightarrow \|q\|_{W^{-1/2,2n}} \lesssim \|\Delta \log \gamma\|_{W^{-1/2,2n}} + \|\nabla \log \gamma\|_{W^{-1/2,2n}}^2$$

$$\lesssim \|\log \gamma\|_{W^{3/2,2n}} + \|\nabla \log \gamma\|_{L^n}^2 \quad \text{(as $L^n(\mathbb{R}^n) \hookrightarrow W^{-1/2,2n}(\mathbb{R}^n)$)}$$

$$= \|\log \gamma\|_{W^{3/2,2n}} + \|\nabla \log \gamma\|_{L^n}^2$$

$$\lesssim \|\log \gamma\|_{W^{3/2,2n}} + \|\log \gamma\|_{W^{1,2n}}^2$$

$$\lesssim \|\log \gamma\|_{W^{3/2,2n}} + \|\log \gamma\|_{W^{1,2n}}^2 \quad \text{(as $W^{3/2,2n}(\mathbb{R}^n) \hookrightarrow W^{1,2n}(\mathbb{R}^n)$)}$$

by the monotonicity and Sobolev embedding properties of $W^{s,p}$ spaces (ref. [46]). Next, choose a bounded function $F : \mathbb{R} \rightarrow \mathbb{R}$ that has bounded continuous derivatives up to order 2. We will use the fact that for any $s \geq 1$, $1 < p < \infty$ and $f \in C^{s+1}(\mathbb{R})$ that has bounded derivatives up to order $[s]+1$, the composition map $u \mapsto f \circ u$ maps $W^{s,p}(\Omega') \cap W^{1,s}(\Omega')$ continuously into $W^{s,p}(\Omega')$. Notice that $W^{3/2,2n}(\Omega') \hookrightarrow W^{1,3n}(\Omega')$ by Sobolev embedding (ref. [46], Theorem 3.3.1(ii)). Therefore,

$$\|\log \gamma\|_{W^{3/2,2n}(\Omega')} = \|F \circ \gamma\|_{W^{3/2,2n}(\Omega')} < \infty.$$ 

Finally, observe that $\log \gamma \in W^{3/2,2n}_0(\Omega')$ (i.e., the closure of $C_c^\infty(\Omega')$ in $W^{3/2,2n}(\Omega')$) and extension by 0 is a continuous map from $W^{3/2,2n}_0(\Omega') \to W^{3/2,2n}(\mathbb{R}^n)$ (ref. [46], Section 3.4.3, Corollary and Remark 2). Therefore, we get

$$\|q\|_{W^{-1/2,2n}} \lesssim \|\log \gamma\|_{W^{3/2,2n}(\Omega')} + \|\log \gamma\|_{W^{3/2,2n}(\Omega')}^2 < \infty.$$

(b) We claim that for all $w \in H^1(\Omega)$,

$$\nabla \cdot (\gamma \nabla (\gamma^{-1/2}w)) = \gamma^{1/2} (\Delta w - qw).$$

Indeed, since $\gamma \in [c, c^{-1}]$ a.e.,

$$\gamma \nabla (\gamma^{-1/2}w) = \gamma^{1/2} \nabla w - (\nabla \gamma^{1/2}) w \quad \text{in $H^{-1}(\Omega)$}$$

$$\Rightarrow \nabla \cdot \gamma \nabla (\gamma^{-1/2}w) = \gamma^{1/2} \Delta w + \nabla \gamma^{1/2} \cdot \nabla w - (\Delta \gamma^{1/2}) w - \nabla \gamma^{1/2} \cdot \nabla w$$

$$= \gamma^{1/2}(\Delta w - qw) \quad \text{in $H^{-2}(\Omega)$}$$

and hence also in the sense of distributions. This along with the fact that $\gamma \equiv 1$ on $\partial \Omega$ implies that $w$ solves $[3]$ iff $u = \gamma^{-1/2}w$ solves $[7]$.

(c) 0 is not a Dirichlet eigenvalue as $[4]$ and hence $[6]$ are uniquely solvable. Now suppose $f \in H^{1/2}(\partial \Omega)$ and $u$ and $w$ are as in $[7]$ and $[6]$. Let $\partial \nu$ be the outward pointing unit normal vector field on $\partial \Omega$. Since $\gamma \equiv 1$ near $\partial \Omega$, 

$$\Lambda_q(f) = \frac{\partial w}{\partial \nu} \bigg|_{\partial \Omega} = \gamma \frac{\partial u}{\partial \nu} \bigg|_{\partial \Omega} = \Lambda_\gamma(f).$$

$$\square$$

Now, if we can reconstruct $q = \Delta \sqrt{\gamma}/\sqrt{\gamma}$ from $\Lambda_q = \Lambda_\gamma$, we can reconstruct $\sqrt{\gamma}$ from $q$ as the unique solution of the following boundary value problem:
Proposition 2.4. Let $\gamma$ be as in Theorem 1.1 and $q = \Delta \sqrt{\gamma}/\sqrt{\gamma}$. Then $\sqrt{\gamma}$ is the unique solution in $H^1(\Omega)$ of

\[
\begin{cases}
(-\Delta + q)u = 0 \text{ in } \Omega \\
u \equiv 1 \text{ on } \partial \Omega.
\end{cases}
\]

Proof. $u = \sqrt{\gamma}$ is clearly a solution. Moreover, the solution is unique by Proposition 2.3 (c) and Proposition 2.1(b). $\square$

In the next two sections, we show how to reconstruct $q$ from $\Lambda_q$.

3. Complex Geometrical Optics Solutions

In this section, we will construct CGO solutions to the Schrodinger equation $(-\Delta + q)u = 0$ in $\mathbb{R}^n$. Observe that if $\zeta \in \mathbb{C}^n$ is such that $\zeta \cdot \zeta = \sum_{j=1}^n \zeta_j^2 = 0$, we have $\Delta e^{x \cdot \zeta} = 0$. Viewing $(-\Delta + q)$ as a perturbation of the Laplacian, we look for solutions to $(-\Delta + q)u = 0$ of the form

$$u(x) = e^{x \cdot \zeta}(1 + r_\zeta(x)).$$

Such solutions are called Complex Geometrical Optics (CGO) solutions. We will show the existence of CGO solutions for $|\zeta|$ large enough and establish certain asymptotic bounds on $r_\zeta$ as $|\zeta| \to \infty$. First of all, note that $u = e^{x \cdot \zeta}(1 + r_\zeta)$ solves $(-\Delta + q)u = 0$ iff

\[
\begin{align*}
-\Delta(e^{x \cdot \zeta}r_\zeta) + e^{x \cdot \zeta}qr_\zeta &= -q \\
\Leftrightarrow (-\Delta_\zeta + m_q)r_\zeta &= -q
\end{align*}
\]

where $\Delta_\zeta v := e^{-x \cdot \zeta}\Delta(e^{x \cdot \zeta}v) = (\Delta + 2\zeta \cdot \nabla)v$. There exists a right inverse $G_\zeta$ of $\Delta_\zeta$ given by

$$G_\zeta f = \left(\frac{\hat{f}(\xi)}{-|\xi|^2 + 2i\zeta \cdot \xi}\right)^\vee.$$

Since the denominator $-|\xi|^2 + 2i\zeta \cdot \xi$ vanishes only on a co-dimension 2 sphere in $\mathbb{R}^n$, the right hand side of the above equation is well defined as a tempered distribution whenever $f$ is a Schwartz function. Looking for solutions of the form $r_\zeta = G_\zeta s_\zeta$ to (10), we see that such an $s_\zeta$ should satisfy

$$(I - m_q G_\zeta)s_\zeta = q$$

where $I$ denotes the identity operator. Our goal is to establish bounds on the operators $m_q$ and $G_\zeta$ between appropriate function spaces such that the operator norm $\|m_q G_\zeta\| < 1$ for $|\zeta|$ large enough. If that is the case, the above equation has a unique solution given by the Neumann series

$$s_\zeta = \sum_{j=0}^{\infty} (m_q G_\zeta)^j q.$$

3.1. Function Spaces. We begin by introducing certain weighted $L^2$ spaces necessary for constructing the CGO solutions.

Definition 3.1. Let $\delta \in \mathbb{R}$. We define the weighted $L^2$ space $L^2_\delta(\mathbb{R}^n)$ by the norm

$$\|u\|_{L^2_\delta} = \left(\int_{\mathbb{R}^n} (1 + |x|^2)^\delta |u(x)|^2 \, dx\right)^{1/2}.$$
For any \( m \in \mathbb{N} \), we define the corresponding weighted Sobolev spaces \( H^m_\delta(\mathbb{R}^n) \) through the norms
\[
\|u\|_{H^m_\delta} = \sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^2_\delta}.
\]
Finally, notice that \( L^2_\delta \) and \( L^2_{-\delta} \) are duals of each other with respect to \( \langle \cdot, \cdot \rangle_{L^2} \). Motivated by this, we define the negative order spaces \( H^{-m}_\delta(\mathbb{R}^n) \) for \( m \in \mathbb{N} \) as duals of \( H^m_{-\delta}(\mathbb{R}^n) \).

We will also need the following scaled Sobolev norms.

**Definition 3.2.** Let \( s \in \mathbb{R}, k \geq 1 \). We define \( H^{s,k}(\mathbb{R}^n) \) through the norms
\[
\|u\|_{H^{s,k}} = \|(k^2 - \Delta)^{s/2} u\|_{L^2} = \frac{1}{(2\pi)^{n/2}} \left( \int (k^2 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \right)^{1/2}.
\]
Note that \( H^{s,k}(\mathbb{R}^n) \) and \( H^{-s,k}(\mathbb{R}^n) \) are dual to each other with respect to \( \langle \cdot, \cdot \rangle_{L^2} \). If \( s \in \mathbb{N} \), then for \( \delta \in \mathbb{R} \), we define \( H^{s,k}_\delta(\mathbb{R}^n) \) through the norms
\[
\|u\|_{H^{s,k}_\delta} = \sum_{|\alpha| \leq s} k^{-|\alpha|} \|\partial^\alpha u\|_{L^2_\delta}.
\]
Finally, for negative integers \( s \), we define \( H^{s,k}_\delta(\mathbb{R}^n) \) as the dual of \( H^{-s,k}_\delta(\mathbb{R}^n) \).

Just as in the case of the usual negative order Sobolev spaces, we have the following characterization of \( H^{s,m,k}_\delta \), \( m \in \mathbb{N} \). The proof is similar to the usual \( H^{-m} \) case and therefore is omitted.

**Proposition 3.3.** Let \( m \in \mathbb{N}, \delta \in \mathbb{R}, k \geq 1 \). For every \( u \in H^{-m,k}_\delta(\mathbb{R}^n) \), there exist \( \{u_\alpha \in L^2_\delta(\mathbb{R}^n) : |\alpha| \leq m\} \) such that
\[
\langle u, v \rangle = \sum_{|\alpha| \leq m} \langle \partial^\alpha u_\alpha, v \rangle = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \langle u_\alpha, \partial^\alpha v \rangle_{L^2} \quad \forall v \in H^{m,k}_\delta(\mathbb{R}^n).
\]
Moreover, \( u_\alpha \) can be chosen to satisfy
\[
\sum_{|\alpha| \leq m} k^{-m-|\alpha|} \|u_\alpha\|_{L^2_\delta} = \|u\|_{H^{-m,k}_\delta}.
\]

We record the following simple inequality for future use.

**Lemma 3.4.** Let \( m \in \mathbb{Z}, k \geq 1 \) and \( \delta, \eta \in \mathbb{R} \). Fix \( \varphi \in C_c^\infty(\mathbb{R}^n) \). Then
\[
\|\varphi u\|_{H^{m,k}_\delta} \lesssim_{\varphi, m, \delta, \eta} \|u\|_{H^{m,k}_\eta}, \quad u \in H^{m,k}_\eta(\mathbb{R}^n).
\]

**Proof.** For \( m \geq 0 \), this follows from the fact that for any multi-index \( \alpha \), \( \partial^\alpha \varphi(x)(1 + |x|^2)^{(\delta-n)/2} \) is bounded above. Now suppose \( m < 0 \). Let \( v \in H^{-m,k}_{-\delta}(\mathbb{R}^n) \). Then
\[
\|\langle \varphi u, v \rangle_{L^2} \| = |\langle u, \varphi v \rangle| \leq \|u\|_{H^{m,k}_\eta} \|\varphi v\|_{H^{-m,k}_\eta} \leq \|u\|_{H^{m,k}_\eta} \|v\|_{H^{-m,k}_{-\delta}}.
\]
Taking the supremum of the left hand side over all \( v \) with \( \|v\|_{H^{-m,k}_{-\delta}} \leq 1 \) gives us the desired result. \( \square \)

Now, let us recall the bounds on \( G_\zeta \) proved in \cite{45}.
Proposition 3.5 (Sylvester-Uhlmann). Let $\zeta \in \mathbb{C}^n$ be such that $|\zeta| \geq 1$ and $\zeta \cdot \zeta = 0$, and let $0 < \delta < 1/2$. Then $G_\zeta$ maps $L^2_\delta \to H^2_{-\delta}$ and satisfies the following norm bounds
\[
\|G_\zeta u\|_{L^2_{-\delta}} \lesssim |\zeta|^{-1}\|u\|_{L^3_\delta}
\]
\[
\|G_\zeta u\|_{H^1_{-\delta}} \lesssim \|u\|_{L^3_\delta}
\]
\[
\|G_\zeta u\|_{H^2_{-\delta}} \lesssim |\zeta|\|u\|_{L^3_\delta}
\]

In particular, for $k = |\zeta|$, we have the following scaled estimate:
\[
\|G_\zeta u\|_{H^{2,k}_{-\delta}} \lesssim k\|u\|_{L^2_{-\delta}}.
\]

As an easy corollary, we obtain the following estimate for $G_\zeta$ on negative-order Sobolev spaces:

Corollary 3.6. Let $\zeta \in \mathbb{C}^n$ be such that $\zeta \cdot \zeta = 0$ and $k = |\zeta| \geq 1$, and let $0 < \delta < 1/2$. Then $G_\zeta$ maps $H^{-1,k}_\delta \to H^{-1,k}_\delta(\mathbb{R}^n)$ and satisfies the bound
\[
\|G_\zeta u\|_{H^{-1,k}_\delta} \lesssim k\|u\|_{H^{-1,k}_\delta}, \quad u \in H^{-1,k}_\delta(\mathbb{R}^n).
\]

Proof. Let $u \in H^{-1,k}_\delta(\mathbb{R}^n)$. Then by Proposition 3.3 there exist $u_0, u_1, \ldots, u_n \in L^2_\delta(\mathbb{R}^n)$ such that $u = u_0 + \sum_{j=1}^n \partial_j u_j$ and
\[
k^{-1}\|u_0\|_{L^3_\delta} + \sum_{j=1}^n \|u_j\|_{L^3_\delta} \lesssim \|u\|_{H^{-1,k}_\delta}.
\]

Now, by Proposition 3.5 and the fact that $G_\zeta$ commutes with $\partial_j$, $j = 1, \ldots, n$
\[
\|G_\zeta u_0\|_{L^2_\delta} \lesssim k^{-1}\|u_0\|_{L^2_\delta} \lesssim \|u\|_{H^{-1,k}_\delta},
\]
\[
\|G_\zeta \partial_j u_j\|_{L^2_\delta} \lesssim \|G_\zeta u_j\|_{H^{-1,k}_\delta} \lesssim \|u_j\|_{L^3_\delta} \lesssim \|u\|_{H^{-1,k}_\delta},
\]
\[
\|\nabla G_\zeta u_0\|_{L^2_\delta} \lesssim \|G_\zeta u_0\|_{L^3_\delta} \lesssim \|u_0\|_{L^3_\delta} \lesssim k\|u\|_{H^{-1,k}_\delta},
\]
\[
\|\nabla G_\zeta \partial_j u_j\|_{L^2_\delta} \lesssim \|G_\zeta u_j\|_{H^2_{-\delta}} \lesssim k\|u_j\|_{L^3_\delta} \lesssim k\|u\|_{H^{-1,k}_\delta}.
\]

Combining all the above inequalities, we get (11). ∎

3.2. Bounds on the multiplication operator. Next, we establish $H^{s,k}$ bounds on the multiplication operator $m_q : f \mapsto qf$ when $q$ is of negative Sobolev regularity. We closely follow the proof of Proposition 3.2 in [17]. We begin with the following important lemma.

Lemma 3.7. Let $s > 0$ and $p \in (2, \infty)$ be such that $p \geq n/s$. Then for all $f, g \in \mathcal{S}(\mathbb{R}^n)$, we have
\[
\|fg\|_{W^{s,p'}} \lesssim \|f\|_{H^s}\|g\|_{H^s},
\]
where $1/p + 1/p' = 1$.

Proof. The Kato-Ponce inequality (ref. [21, 30]) implies that
\[
\|fg\|_{W^{s,p'}} \lesssim \|f\|_{H^r}\|g\|_{L^r} + \|f\|_{L^r}\|g\|_{H^s},
\]
where $1/r = 1/p' - 1/2 = 1/2 - 1/p$. Now applying the Sobolev embedding theorem, we get
\[
\|fg\|_{W^{s,p'}} \lesssim \|f\|_{H^s}\|g\|_{H^s} + \|f\|_{H^s}\|g\|_{H^s},
\]
where $1/2 - t/n = 1/r$, or equivalently, $t = n/2 - n/r = n/p$. Now the estimate follows from the fact that $t \leq s$. ∎
Theorem 3.8. Let $s > 0$ and $p \in (2, \infty)$ be such that $p \geq n/s$. Suppose $V \in W^{-s,p}(\mathbb{R}^n)$. Then for $k \geq 1$,
\begin{align}
\|Vf\|_{H^{-s,k}} &\lesssim \omega(k)\|f\|_{H^s} \lesssim \omega(k)\|f\|_{H^{s,k}}, \quad \forall f \in H^t(\mathbb{R}^n),
\end{align}
where $\omega$ is a positive function on $[1, \infty)$ such that $\omega(k) \to 0$ as $k \to \infty$. If in addition we have $0 < s \leq 1$, then
\begin{align}
\|Vf\|_{H^{-1,k}} &\lesssim k^{-(1-s)}\omega(k)\|f\|_{H^1}, \quad \text{and}
\|Vf\|_{H^{-1,k}} &\lesssim k^{-2(1-s)}\omega(k)\|f\|_{H^{1,k}}.
\end{align}

Proof. By duality, it suffices to prove that
\[ |\langle Vf, g \rangle_{L^2} | = |\langle V, fg \rangle_{L^2} | \lesssim \omega(k)\|f\|_{H^s} \|g\|_{H^*_k} \]
for all $f, g \in \mathcal{S}(\mathbb{R}^n)$. Let $W \in L^p(\mathbb{R}^n)$ be such that $V = (I - \Delta)^{s/2}W$. Then we have,
\[ \langle V, fg \rangle = \langle (I - \Delta)^{s/2}W, fg \rangle = \langle W, (I - \Delta)^{s/2}(fg) \rangle. \]
Let $\varphi \in C_c^\infty(\mathbb{R}^n; [0, 1])$ be such that $\int_{\mathbb{R}^n} \varphi(x)dx = 1$. We consider the sequence of mollifiers $\varphi_\epsilon(x) := \epsilon^{-n}\varphi(x/\epsilon)$ and define $W_\epsilon := \varphi_\epsilon * W$. Choosing $t \in (s - n/p, s)$, we may write
\[ \langle V, fg \rangle = \langle W_\epsilon, (I - \Delta)^{s/2}(fg) \rangle + \langle W - W_\epsilon, (I - \Delta)^{s/2}(fg) \rangle \]
\[ = \langle (I - \Delta)^{t/2}W_\epsilon, (I - \Delta)^{(s-t)/2}(fg) \rangle + \langle W - W_\epsilon, (I - \Delta)^{s/2}(fg) \rangle. \]
Now, by Hölder’s inequality,
\begin{align}
|\langle V, fg \rangle | &\leq \| (I - \Delta)^{t/2}W_\epsilon \|_{L^q} \| (I - \Delta)^{(s-t)/2}(fg) \|_{L^{q'}} + \| W - W_\epsilon \|_{L^p} \| (I - \Delta)^{s/2}(fg) \|_{L^{q'}},
\end{align}
where $q = n/(s-t)$ and $p', q'$ are conjugate exponents of $p, q$ respectively. Since $t > s - n/p$, we have $q > p$ and therefore by Young’s convolution inequality,
\[ \| (I - \Delta)^{t/2}W_\epsilon \|_{L^q} = \| ((I - \Delta)^{t/2} \varphi_\epsilon) \ast W \|_{L^q} \]
\[ \leq \| (I - \Delta)^{t/2} \varphi_\epsilon \|_{L^r} \| W \|_{L^p} \]
\[ = \| \varphi_\epsilon \|_{W^{r,t}} \| W \|_{L^p}, \]
where $1/p + 1/r = 1 + 1/q$. Now by Sobolev embedding, $W^{r,t} \hookrightarrow L^u$ where $1/u = 1/r - t/n$. Moreover, it can be easily verified that $\| \varphi_\epsilon \|_{L^u} = \epsilon^{n(1-u)/u} \| \varphi \|_{L^u}$. Therefore,
\[ \| (I - \Delta)^{t/2}W_\epsilon \|_{L^q} \leq \| \varphi_\epsilon \|_{L^\infty} \| W \|_{L^p} \]
\[ \lesssim \epsilon^{-(t+n/q-n/p)} \| W \|_{L^p}. \]
Also, by Lemma 3.7,
\[ \| (I - \Delta)^{(s-t)/2}(fg) \|_{L^{q'}} \lesssim \| f \|_{H^{s-t}} \| g \|_{H^{s-t}}, \quad \text{and} \]
\[ \| (I - \Delta)^{s/2}(fg) \|_{L^{q'}} \lesssim \| f \|_{H^s} \| g \|_{H^s}. \]
Therefore, from (15), we get
\[ |\langle V, fg \rangle | \lesssim \epsilon^{-(t+n/q-n/p)} \| W \|_{L^p} \| f \|_{H^{s-t}} \| g \|_{H^{s-t}} + \| W - W_\epsilon \|_{L^p} \| f \|_{H^s} \| g \|_{H^s} \]
\[ \lesssim \epsilon^{-(t+n/q-n/p)} \| W \|_{L^p} \| f \|_{H^{s-t}} \| g \|_{H^{s-t,k}} + \| W - W_\epsilon \|_{L^p} \| f \|_{H^s} \| g \|_{H^{s,k}} \]
\[ \lesssim (\epsilon^{-(t+n/q-n/p)}k^{-t}) \| W \|_{L^p} + \| W - W_\epsilon \|_{L^p} \| f \|_{H^{s}} \| g \|_{H^{s,k}} \]
Here we have used the easy estimate $\| h \|_{H^{s-t,k}} \lesssim k^{-t} \| h \|_{H^{s,k}}$ for any $h \in \mathcal{S}(\mathbb{R}^n)$. Note that $n/q - n/p \geq (s-t) - s = -t$. Now choose $\epsilon = k^{-1/4}$. Then we get
\begin{align}
|\langle V, fg \rangle | &\lesssim \omega(k)\|f\|_{H^s} \|g\|_{H^{s,k}} \lesssim \omega(k)\|f\|_{H^{s,k}} \|g\|_{H^{s,k}}
\end{align}
where $\omega(k) = k^{-l/2} \|W\|_{L^p} + \|W - W_{k^{-1/4}}\|_{L^p} \to 0$ as $k \to \infty$. This proves \((12)\). Now \((13)\) and \((14)\) follow from the fact that if $0 < s \leq 1$,

\begin{align}
|V, fg| &\lesssim \omega(k) \|f\|_{H^s,k} \|g\|_{H^s,k} \lesssim \omega(k) k^{-(1-s)} \|f\|_{H^s} \|g\|_{H^1,k}, \\
|\langle V, fg \rangle| &\lesssim \omega(k) \|f\|_{H^s,k} \|g\|_{H^s,k} \lesssim \omega(k) k^{-2(1-s)} \|f\|_{H^1,k} \|g\|_{H^1,k}.
\end{align}

\square

If in addition, $V$ is compactly supported, the multiplication operator $m_V$ can be extended to $H^{s,k}_\delta$ spaces.

**Corollary 3.9.** Let $0 < s < 1$ and $q \in W^{-s,n/s}(\mathbb{R}^n)$ be such that supp $q$ is compact. Suppose $\delta, \eta \in \mathbb{R}$. Then $m_q : f \mapsto q f$ satisfies the norm bounds

\begin{align}
\|m_q f\|_{H^{-1,k}_\delta} &\lesssim k^{-(1-s)} \omega(k) \|f\|_{H^s}, \\
\|m_q f\|_{H^{-1,k}_\delta} &\lesssim k^{-2(1-s)} \omega(k) \|f\|_{H^s},
\end{align}

where $\omega$ is a positive function on $[1, \infty)$ that satisfies $\omega(k) \to 0$ as $k \to \infty$.

**Proof.** Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be such that $\varphi \equiv 1$ on supp $q$. Then by \((17)\), for all $f, g \in S(\mathbb{R}^n)$,

\[ |\langle q f, g \rangle_{L^2} - |\langle q, f g \rangle| = |\langle q, (\varphi f)(\varphi g) \rangle| \lesssim \omega(k) k^{-(1-s)} \|\varphi f\|_{H^s} \|\varphi g\|_{H^1,k} \lesssim \omega(k) k^{-(1-s)} \|f\|_{H^s} \|g\|_{H^1,k} \quad \text{by Lemma 3.4} \]

Now \((19)\) follows by density and duality. \((20)\) similarly follows from \((18)\).

\section{3.3. Construction of CGO solutions.}

With the bounds on $m_q$ and $G_\zeta$ in hand, we are now ready to prove the existence of CGO solutions.

**Theorem 3.10.** Let $q \in W^{-s,n/s}(\mathbb{R}^n)$, $0 < s \leq 1/2$ be such that supp $q$ is compact. Fix $\delta \in (0, 1/2)$. Then there exists $M > 0$ such that for all $\zeta \in \mathbb{C}^n$ satisfying $\zeta \cdot \zeta = 0$, $|\zeta| \geq M$,

there exists a unique solution to

\[ (-\Delta + m_q) u = 0 \quad \text{in } \mathbb{R}^n \]

of the form

\[ u = u_\zeta(x) = e^{x \cdot \zeta}(1 + r_\zeta(x)) \]

where $r_\zeta \in H^1_{-\delta}(\mathbb{R}^n)$. Moreover,

\[ \|r_\zeta\|_{H^1_{-\delta}} \lesssim |\zeta|^s, \]

**Proof.** As seen before, $u_\zeta = e^{x \cdot \zeta}(1 + r_\zeta(x))$ satisfies $(-\Delta + q) u = 0$ if and only if

\[ (-\Delta_\zeta + q) r_\zeta = -q \]

where $\Delta_\zeta = e^{-\zeta \cdot x} \Delta e^{\zeta \cdot x}$. We will look for solutions of the form $r_\zeta = G_\zeta s_\zeta$. Such an $s_\zeta$ should satisfy

\[ (I - m_q \circ G_\zeta) s_\zeta = q. \]

\[ I - m_q \circ G_\zeta |_{H^1_{-\delta}} s_\zeta = q. \]
Let $k = |\zeta|$. It follows from Corollary 3.6 and (20) from Corollary 3.9 that
\[
\|G_\zeta\|_{H^{-1,k}_\delta \to H^{1,k}_\delta} \lesssim k,
\]
\[
\|m_q\|_{H^{1,k}_\delta \to H^{-1,k}_\delta} \lesssim k^{-2(1-s)}\omega(k)
\]
where $\omega(k) \to 0$ as $k \to \infty$. Therefore, $\|m_q \circ G_\zeta\| \lesssim k^{-1}2^s\omega(k) \to 0$ as $k \to \infty$ and there exists $M > 0$ such that for $k = |\zeta| \geq M$,
\[
\|m_q \circ G_\zeta\|_{H^{-1,k}_\delta \to H^{1,k}_\delta} \leq \frac{1}{2}.
\]
Moreover, $q \in H^{-1,k}_\delta(\mathbb{R}^n)$. Indeed, suppose $\varphi \in C_c^\infty(\mathbb{R}^n)$ is such that $\varphi \equiv 1$ on supp $q$. Clearly $q = q\varphi = m_q(\varphi)$. Applying Theorem 3.8 with $k = 1$, we get
\[
\|q\|_{H^{-s}} = \|\varphi q\|_{H^{-s}} \lesssim \|\varphi\|_{H^s} \lesssim \|q\|_{H^s}.
\]
Therefore,
\[
\|q\|_{H^{-1,k}_\delta} = \|\varphi q\|_{H^{1,k}_\delta} \lesssim \|q\|_{H^{-1,k}} \quad \text{by Lemma 3.11}
\]
\[
\lesssim k^{-2(1-s)}\|q\|_{H^{-s,k}}
\]
\[
\lesssim k^{-2(1-s)}\|q\|_{H^{-s}}
\]
\[
\lesssim k^{-2(1-s)}\|\varphi\|_{H^s}.
\]
Thus, for all $|\zeta| = k \geq M$, (21) has a unique solution given by the Neumann series
\[
s_\zeta = \sum_{j=0}^{\infty} (m_q \circ G_\zeta)^j q
\]
and we have the estimates
\[
(22) \quad \|s_\zeta\|_{H^{-1,k}_\delta} \lesssim \|q\|_{H^{1,k}_\delta} \lesssim k^{-2(1-s)};
\]
\[
(23) \quad \|r_\zeta\|_{H^{1,k}_\delta} = \|G_\zeta s_\zeta\|_{H^{1,k}_\delta} \lesssim k^s.
\]
This completes the proof. \qed

4. Uniqueness and Reconstruction

Using the integral identity from Proposition 2.2 and appropriate CGO solutions, we will be able to reconstruct the Fourier transform of $q$.

**Theorem 4.1.** Let $\Omega$ be a smooth bounded domain and $q \in W^{1/2,2n}_{\text{comp}}(\Omega)$ be such that 0 is not a Dirichlet eigenvalue of $(-\Delta + q)$ in $\Omega$. Let $\zeta \in \mathbb{R}^n$ be such that $\zeta \neq 0$. Then for $k > 0$ sufficiently large, there exist $\zeta_1, \zeta_2 \in \mathbb{C}^n$ with $\zeta_j \cdot \zeta_j = 0$ and $|\zeta_j| = k$, $j = 1, 2$, such that
\[
\lim_{k \to \infty} \langle (\Lambda_q - \Lambda_0)(u_{\zeta_1}|_{\partial\Omega}), e^{x \cdot \zeta_1} \rangle = \langle q, e^{-ix \cdot \zeta} \rangle = \int_{\Omega} qe^{-ix \cdot \zeta} \, dx
\]
where $\Lambda_0$ denotes the Dirichlet-to Neumann map for $(-\Delta + q)$ and $u_{\zeta_1}$ is the unique solution to $(-\Delta + q)u = 0$ of the form
\[
u_\zeta = e^{x \cdot \zeta}(1 + r_\zeta), \quad r_\zeta \in H^{1,k}_{\delta}
\]
constructed in Theorem 3.10.
Proof. Let $\alpha, \beta$ be unit vectors in $\mathbb{R}^n$ such that $\{\xi/|\xi|, \alpha, \beta\}$ form an orthonormal set. Define $\zeta_1, \zeta_2 \in \mathbb{C}^n$ by

$$\zeta_1 = \frac{k}{\sqrt{2}} \alpha + i\left( -\frac{\xi}{2} + \sqrt{\frac{k^2}{2} - \frac{|\xi|^2}{4}} \beta \right),$$

$$\zeta_2 = -\frac{k}{\sqrt{2}} \alpha + i\left( -\frac{\xi}{2} - \sqrt{\frac{k^2}{2} - \frac{|\xi|^2}{4}} \beta \right).$$

It is easy to check that $k = |\zeta_1| = |\zeta_2|$ and $\zeta_1 \cdot \zeta_1 = \zeta_2 \cdot \zeta_2 = 0$. Therefore, by Theorem 3.10 for $k$ large enough, there exists a solution $u_{\zeta_1} = e^{\zeta_1 \cdot x}(1 + r_{\zeta_1}(x))$ of $(-\Delta + q)u = 0$ such that $\|r_{\zeta_1}\|_{H^{1,k}} \lesssim k^{1/2}$. Moreover, the fact that $\zeta_2 \cdot \zeta_2 = 0$ implies $\Delta e^{x \cdot \zeta_2} = 0$. Therefore, by Proposition 2.2

$$\langle (\Lambda_q - \Lambda_0)(u_{\zeta_1}|_{\partial \Omega}), e^{-x \cdot \zeta_2} \rangle = \langle q, u_{\zeta_1} e^{-x \cdot \zeta_2} \rangle = \langle q, e^{x \cdot (\zeta_1 + \zeta_2)}(1 + r_{\zeta_1}) \rangle = \langle q, e^{-ix \cdot \xi} \rangle + \langle q, e^{-ix \cdot \xi} r_{\zeta_1} \rangle.$$ 

Now, let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be such that $\varphi \equiv 1$ on $\overline{\Omega} \supset \text{supp } q$. By (19) in Corollary 3.9

$$|\langle q, e^{-ix \cdot \xi} r_{\zeta_1} \rangle| \leq k^{-1/2} \omega(k) \|e^{-ix \cdot \xi} \varphi\|_{H^1} \|r_{\zeta_1}\|_{H^{1,k}} \lesssim k^{-1/2} \omega(k) = \omega(k)$$

(Theorem 3.10),

where $\omega(k) \to 0$ as $k \to \infty$. Therefore, it follows that

$$\lim_{k \to \infty} \langle (\Lambda_q - \Lambda_0)(u_{\zeta_1}|_{\partial \Omega}), e^{x \cdot \zeta_2} \rangle = \langle q, e^{-ix \cdot \xi} \rangle = \widehat{q}(\xi).$$

Thus, we see that if $u_{\zeta_1}|_{\partial \Omega}$ can somehow be determined, we can recover $\widehat{q}(\xi)$ for $\xi \neq 0$ from the knowledge of $\Lambda_q$. Since $q$ is compactly supported, $\widehat{q}$ is continuous and thus, $\widehat{q}(0)$ can also be determined by continuity. Therefore, the goal now is to find a procedure to determine $u_{\zeta_1}|_{\partial \Omega}$. We will characterize $u_{\zeta_1}|_{\partial \Omega}$ as the unique solution of a certain boundary integral equation of Fredholm type. The method is due to Nachman [40]. We will mostly follow the presentation and notation in [19].

Let us begin by fixing some notation. We will use $\Omega_+$ to denote the exterior domain $\mathbb{R}^n \setminus \overline{\Omega}$. Let $\gamma : H^1_{\text{loc}}(\mathbb{R}^n) \to H^{1/2}(\partial \Omega)$ denote the usual trace operator $\gamma(u) = u|_{\partial \Omega}$. Similarly, we let $\gamma_+ : H^1(\Omega_+) \to H^{1/2}(\partial \Omega)$ and $\gamma_- : H^1(\Omega) \to H^{1/2}$ denote the trace operators in the exterior and interior domains respectively.

Let $K_0(x, y) = c_n|x - y|^{2-n}$ be the standard Green’s function for the Laplacian. We know that the operator with Schwartz kernel $K_0$ (also denoted by $K_0$) maps $H^{-1}_{\text{comp}}(\mathbb{R}^n) \to H^1_{\text{loc}}(\mathbb{R}^n)$ and satisfies

$$\Delta K_0 f = f , \quad f \in H^{-1}_{\text{comp}}(\mathbb{R}^n).$$

Now let $\zeta \in \mathbb{C}^n$ be such that $\zeta \cdot \zeta = 0$ and $|\zeta| \geq 1$. We define an analogous operator $K_\zeta$ by

$$K_\zeta(f) = e^{x \cdot \zeta} G_\zeta(e^{-x \cdot \zeta} f).$$
Proposition 4.2. The operator $K_\zeta$ maps $H^{-1}_{\text{comp}}(\mathbb{R}^n) \to H^1_{\text{loc}}(\mathbb{R}^n)$ and satisfies the following properties:

(a) $\Delta K_\zeta f = f$ for all $f \in H^{-1}_{\text{comp}}(\mathbb{R}^n)$.

(b) There exists $R_\zeta \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that $K_\zeta = K_0 + R_\zeta$. The operator with Schwartz kernel $R_\zeta$ maps $H^{-k}_{\text{comp}}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ for all $k \in \mathbb{N}$.

Proof. Let $0 < \delta < 1/2$ be arbitrary. Clearly, $f \mapsto e^{-x \cdot \zeta} f$ maps $H^{-1}_{\text{comp}}$ to $H^{-1}_{\text{comp}} \hookrightarrow H^{-1}_{\delta}(\mathbb{R}^n)$. Then by Proposition 3.5, $f \mapsto G_\zeta(e^{-x \cdot \zeta} f)$ takes $H^{-1}_{\text{comp}}$ into $H^{-1}_{\delta}(\mathbb{R}^n)$. Finally, multiplication by $e^{x \cdot \zeta}$ takes $H^{-1}_{\delta}(\mathbb{R}^n) \to H^1_{\text{loc}}(\mathbb{R}^n)$, which proves that $K_\zeta : H^{-1}_{\text{comp}}(\mathbb{R}^n) \to H^1_{\text{loc}}(\mathbb{R}^n)$.

Now, by definition of $K_\zeta$,

$$\Delta K_\zeta f = e^{x \cdot \zeta} \Delta G_\zeta(e^{-x \cdot \zeta} f) = f, \quad \forall f \in H^{-1}_{\text{comp}}(\mathbb{R}^n)$$

since $G_\zeta$ is a right inverse of $\Delta_\zeta$. This proves (a). Next, define $R_\zeta = K_\zeta - K_0$. Then for any $H^{-1}_{\text{comp}}(\mathbb{R}^n)$,

$$\Delta R_\zeta f = \Delta K_\zeta f - \Delta K_0 f = 0.$$

Therefore, (b) follows from the Elliptic Regularity theorem. \hfill \Box

Definition 4.3. The standard Single layer potential is defined as the operator

$$S_0 = K_0 \gamma^* : H^{-1/2}(\partial \Omega) \to H^1_{\text{loc}}(\mathbb{R}^n).$$

Analogously, we define the modified (or Fadeev-type) Single layer potential $S_\zeta$ for $\partial \Omega$ by

$$S_\zeta = K_\zeta \gamma^* : H^{-1/2}(\partial \Omega) \to H^1_{\text{loc}}(\mathbb{R}^n).$$

We will show that $u_\zeta|_{\partial \Omega}$ can be characterized as the unique solution $f \in H^{1/2}(\partial \Omega)$ of the following Boundary Integral Equation:

$$\left(\text{Id} + \gamma S_\zeta(\Lambda_q - \Lambda_0)\right)f = e^{x \cdot \zeta} \quad \text{on } \partial \Omega.$$  \hfill (26)

Theorem 4.4. Let $q \in W^{-1/2,2n}_{\text{comp}}(\Omega)$ be such that $0$ is not a Dirichlet eigenvalue of $(\Delta + q)$ in $\Omega$. Let $\zeta \in \mathbb{C}^n$ be such that $\zeta \cdot \zeta = 0$ and $|\zeta|$ is sufficiently large, and let $0 < \delta < 1/2$. Consider the following problems:

$$\begin{aligned}
&\text{(DE)} \quad \begin{cases}
(\Delta + q)u = 0 & \text{in } \mathbb{R}^n, \\
e^{-x \cdot \zeta}u - 1 \in H^1_{-\delta}(\mathbb{R}^n).
\end{cases} \\
&\text{(EP)} \quad \begin{cases}
(i) \quad \Delta \tilde{u} = 0 & \text{in } \Omega_+, \\
(ii) \quad \tilde{u} = u|_{\Omega_+} \quad \text{for some } u \in H^1_{\text{loc}}(\mathbb{R}^n), \\
(iii) \quad e^{-x \cdot \zeta}\tilde{u} - 1 = r|_{\Omega_+} \quad \text{for some } r \in H^1_{-\delta}(\mathbb{R}^n), \\
(iv) \quad (\partial_\nu u)_+ = \Lambda_q(\gamma u) \quad \text{on } \partial \Omega.
\end{cases} \\
&\text{(BIE)} \quad \begin{cases}
(\text{Id} + \gamma S_\zeta(\Lambda_q - \Lambda_0))f = e^{x \cdot \zeta} & \text{on } \partial \Omega, \\
f \in H^{1/2}(\partial \Omega).
\end{cases}
\end{aligned}$$

Each of these problems has a unique solution. Furthermore, they are equivalent in the following sense: If $u$ solves (DE), $\tilde{u} = u|_{\Omega_+}$ solves (EP) and conversely, if $\tilde{u}$ solves (EP), there exists a solution $u$ of (DE) such that $u = u|_{\Omega_+}$. Also, if $u$ solves (DE), $f := u|_{\partial \Omega}$ solves (BIE) and conversely, if $f$ solves (BIE), there exists a solution $u$ of (DE) such that $f = u|_{\partial \Omega}$.

Proof. (DE) can be rephrased as the problem of finding solutions of the form $u = e^{x \cdot \zeta}(1 + r)$ to the equation

$$(\Delta + q)u = 0 \quad \text{in } \mathbb{R}^n,$$
where \( r \in H^{1,\delta}_0(\mathbb{R}^n) \). Therefore, (DE) has a unique solution by Theorem 3.10 for \(|\zeta|\) sufficiently large. Now we show that (DE) is equivalent to (EP) and (BIE).

\((DE) \Rightarrow (BIE)\): Let \( u \) be the solution of (DE) and let \( f = u|_{\partial \Omega} \). Clearly, \( u \in H^{1}_0(\mathbb{R}^n) \) and hence \( f = \gamma(u) \in H^{1/2}(\partial \Omega) \). Now, fix \( x \in \Omega \) and define the function \( v \) on \( \Omega \) by \( v(y) = K_\zeta(x,y), \ y \in \Omega \). Since \( \Delta v = 0 \) in \( \Omega \), \( v \) is smooth by elliptic regularity. Now, by Green’s theorem,

\[
\int_{\partial \Omega} (u \partial_v v - v \partial_v u) d\sigma = \int_{\Omega} (u \Delta v - v \Delta u).
\]

We know that \( \Delta v = 0 \) and \( \Delta u = qu \). Moreover, since \( u \) and \( v \) satisfy \((-\Delta + q)u = 0 \) and \( \Delta v = 0 \) in \( \Omega \) respectively, \( \partial_\nu u = \Lambda_q(u|_{\partial \Omega}) \) and \( \partial_\nu v = \Lambda_0(v|_{\partial \Omega}) \). Substituting these into the above identity, we get

\[
\int_{\partial \Omega} u \Lambda_0(v|_{\partial \Omega}) d\sigma - \int_{\partial \Omega} K_\zeta(x,y) \Lambda_q(f)(y) d\sigma(y) = - \int_{\Omega} K_\zeta(x,y)(qu(y)) dy
\]

\[
\implies \int_{\partial \Omega} u \Lambda_0(v|_{\partial \Omega}) d\sigma - S_\zeta \Lambda_q f(x) = -K_\zeta(qu)(x).
\]

Next, by symmetry of \( \Lambda_0 \), \( \int_{\partial \Omega} u \Lambda_0(v|_{\partial \Omega}) d\sigma = \int_{\partial \Omega} v \Lambda_0(f) d\sigma = S_\zeta \Lambda_0(f) \). Therefore, the above equation becomes

\[(27) \quad S_\zeta(\Lambda_0 - \Lambda_q)f(x) = -K_\zeta(qu)(x), \quad x \in \Omega_+.
\]

Now, we simplify the right hand side. By definition,

\[
K_\zeta(qu) = e^{x \zeta} G_\zeta(e^{-x \zeta} qu) = e^{x \zeta} G_\zeta(e^{-x \zeta} \Delta u)
\]

\[
= e^{x \zeta} G_\zeta \circ \Delta_\zeta(e^{-x \zeta} u) = e^{x \zeta} G_\zeta \circ \Delta_\zeta(e^{-x \zeta} u - 1).
\]

But we know that \( e^{-x \zeta} u - 1 \in H^{1,\delta}_0(\mathbb{R}^n) \) and \( G_\zeta \) is a right inverse of \( \Delta_\zeta \) on \( H^{1,\delta}_0(\mathbb{R}^n) \). Therefore we get \( K_\zeta(qu) = e^{x \zeta}(e^{-x \zeta} u - 1) = u - e^{x \zeta} \) and

\[
u(x) + S_\zeta(\Lambda_q - \Lambda_0) f(x) = e^{x \zeta}, \quad x \in \Omega_+.
\]

Taking traces along \( \partial \Omega \) on both sides, we get \( (\text{Id} + \gamma S_\zeta(\Lambda_q - \Lambda_0)) f = e^{x \zeta} \) on \( \partial \Omega \), as desired.

\((BIE) \Rightarrow (EP)\): Suppose \( f \) solves (BIE). Define \( \tilde{u} := e^{x \zeta} - S_\zeta(\Lambda_q - \Lambda_0) f \).

Clearly, \( \tilde{u}|_{\partial \Omega} = f \) and \( \Delta \tilde{u} = 0 \) on \( \mathbb{R}^n \setminus \partial \Omega \). Moreover, \((\text{ii})\) follows from the mapping properties of \( S_\zeta \). Next, from the jump properties of single layer potentials, we get

\[
(\partial_\nu \tilde{u})_- - (\partial_\nu \tilde{u})_+ = -(\Lambda_q - \Lambda_0) f.
\]

Since \( \Delta \tilde{u} = 0 \) in \( \Omega \), \( (\partial_\nu \tilde{u})_- = \Lambda_0(\tilde{u}|_{\partial \Omega}) = \Lambda_0 f \). Therefore, \( (\partial_\nu \tilde{u})_+ = \Lambda_q f \) and we have verified \((\text{iv})\). Finally, we note that

\[
e^{-x \zeta} \tilde{u} - 1 = -e^{-x \zeta} S_\zeta(\Lambda_q - \Lambda_0) f = G_\zeta e^{-x \zeta} \gamma^* h,
\]

where \( h = (\Lambda_0 - \Lambda_q) f \in H^{-1/2}(\partial \Omega) \). Since \( e^{-x \zeta} \gamma^* h \in H^{-1}(\mathbb{R}^n) \) is compactly supported, \( e^{-x \zeta} \gamma^* h \in H^{-1}_\delta(\mathbb{R}^n) \) by the usual arguments. Finally, since \( G_\zeta : H^{-1}_\delta(\mathbb{R}^n) \to H^{1,\delta}_0(\mathbb{R}^n) \), we conclude that \( e^{-x \zeta} \tilde{u} - 1 \in H^{1,\delta}_\delta(\mathbb{R}^n) \).
Proof. Let \( \tilde{u} \) solve (EP) and let \( v \in H^1(\Omega) \) be the solution of
\[
\begin{aligned}
\left\{ \begin{array}{l}
(-\Delta + q)v = 0, \\
v|_{\partial\Omega} = \gamma_+ \tilde{u}.
\end{array} \right.
\end{aligned}
\]
Define \( u \) on \( \mathbb{R}^n \) by
\[
u(x) = \begin{cases} v(x) & \text{in } \Omega, \\ \tilde{u}(x) & \text{in } \Omega^+. \end{cases}
\]
We have \( \gamma_-(u) = \gamma_+(u) \) by construction and \( (\partial_\nu u)_- = \Lambda_q(\gamma_+ \tilde{u}) = (\partial_\nu u)_+ \) by EP (iv). Therefore, it follows that \( u \in H^1_{\text{loc}}(\mathbb{R}^n) \) and \( (-\Delta + q)u = 0 \) in \( \mathbb{R}^n \). Finally, \( e^{-x \cdot \zeta} u - 1 \in H^1_{\text{loc}}(\mathbb{R}^n) \) because of EP (iii) and the fact that \( u = \tilde{u} \) on \( \Omega^+ \).

Let us conclude by showing that the Boundary Integral Equation (26) is indeed Fredholm.

**Proposition 4.5.** Let \( q \in W^{-1/2,2n}_{\text{comp}}(\Omega) \) be such that \( 0 \) is not a Dirichlet eigenvalue of \((-\Delta + m_q)\) on \( \Omega \). Then the operator
\[\gamma S_\zeta(\Lambda_q - \Lambda_0) : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)\]
is compact.

**Proof.** Let \( P_q : H^{1/2}(\partial\Omega) \to H^1(\Omega) \) be the solution operator that maps \( f \in H^{1/2}(\partial\Omega) \) to the unique solution \( u \in H^1(\Omega) \) of
\[
\begin{aligned}
\left\{ \begin{array}{l}
(-\Delta + q)u = 0, \\
u|_{\partial\Omega} = f.
\end{array} \right.
\end{aligned}
\]
By the same argument as the one leading to (27), we have
\[\gamma S_\zeta(\Lambda_q - \Lambda_0) f = -\gamma K_\zeta \circ m_q \circ P_q(f), \quad f \in H^{1/2}(\partial\Omega).
\]
But the right hand side is compact since \( m_q : H^1(\Omega) \to H^{-1}_{\text{comp}}(\Omega) \) is compact by Proposition 2.1(a). This proves the result. \( \Box \)

5. **Stability**

In this final section, we will prove the stability estimates (4) and (5). Let us start with the stability estimate for the Schrödinger equation. Given \( q \in W^{-1/2,2n}_{\text{comp}}(\Omega) \), we define the set of Cauchy data for \( q \) as
\[\mathcal{C}_q = \left\{ u|_{\partial\Omega}, \frac{\partial u}{\partial \nu}|_{\partial\Omega} \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega) : (-\Delta + q)u = 0 \right\}.
\]
If \( 0 \) is not a Dirichlet eigenvalue of \((-\Delta + q)\) on \( \Omega \), then \( \mathcal{C}_q \) is precisely the graph of the Dirichlet-to-Neumann map \( \Lambda_q \). Consider the norm on \( H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega) \) given by
\[\|(f, g)\|_{H^{1/2} \oplus H^{-1/2}} = (\|f\|_{H^{1/2} \oplus H^{-1/2}}^2 + \|g\|_{H^{-1/2} \oplus H^{1/2}}^2)^{1/2}.
\]
Given \( q_1, q_2 \in W^{-1/2,2n}_{\text{comp}}(\Omega) \), we define the distance between their Cauchy data sets by
\[\text{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) = \max \left\{ \sup_{(f_1, g_1) \in \mathcal{C}_{q_1}} \inf_{(f_2, g_2) \in \mathcal{C}_{q_2}} \frac{\|(f_1 - f_2, g_1 - g_2)\|_{H^{1/2} \oplus H^{-1/2}}}{\|(f_1, g_1)\|_{H^{1/2} \oplus H^{-1/2}}}, \right. \]
\[\left. \sup_{(f_2, g_2) \in \mathcal{C}_{q_2}} \inf_{(f_1, g_1) \in \mathcal{C}_{q_1}} \frac{\|(f_1 - f_2, g_1 - g_2)\|_{H^{1/2} \oplus H^{-1/2}}}{\|(f_2, g_2)\|_{H^{1/2} \oplus H^{-1/2}}} \right\}.
\]
It can be verified that if $C_{q_j}$ are in fact the graphs of the Dirichlet-to-Neumann maps $\Lambda_{q_j}$, (28)

$$\frac{\|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2} \to H^{-1/2}}}{\sqrt{1 + \|\Lambda_{q_1}\|_{H^{1/2} \to H^{-1/2}}^2 \sqrt{1 + \|\Lambda_{q_2}\|_{H^{1/2} \to H^{-1/2}}^2}}} \leq \text{dist}(C_{q_1}, C_{q_2}) \leq \|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2} \to H^{-1/2}}.$$

We will establish bounds on $\|q_1 - q_2\|_{H^{-1}}$ in terms of $\text{dist}(C_{q_1}, C_{q_2})$, thus including the cases where 0 is a Dirichlet eigenvalue of one of $(-\Delta + q_j)|_\Omega$. The estimate (29) follows from the theorem below:

**Theorem 5.1.** Let $0 < s < 1/2$ and $q_1, q_2 \in W^{s,n/s}_{\text{comp}}(\Omega)$ satisfy the a-priori estimate

$$\|q_j\|_{W^{-s,n/s}} \leq M, \quad j = 1, 2.$$

Then there exists $C > 0$ and $\sigma = \sigma(n, s) \in (0, 1)$ such that

$$\|q_1 - q_2\|_{H^{-1}} \leq C(\log\{\text{dist}(C_{q_1}, C_{q_2})\})^{-\sigma} + \text{dist}(C_{q_1}, C_{q_2}).$$  

**Proof.** Let $u_1, u_2 \in H^1(\Omega)$ satisfy $(-\Delta + q_j)u_j = 0$ in $\Omega, j = 1, 2$. By the weak definition of normal trace, we have

$$\int_{\partial\Omega} \left( u_2 \frac{\partial u_1}{\partial \nu} - u_1 \frac{\partial u_2}{\partial \nu} \right) \, d\sigma = \int_{\Omega} (\nabla u_1 \cdot \nabla u_1 + q_1 u_1 u_2) \, dx - \int_{\Omega} (\nabla u_2 \cdot \nabla u_1 + q_2 u_1 u_2) \, dx = \int_{\Omega} (q_1 - q_2) u_1 u_2 \, dx.$$

Suppose $(f, g) \in C_{q_1}$. Then there exists $v \in H^1(\Omega)$ such that $(-\Delta + q_1)v = 0$, and $v|_{\partial\Omega} = f$, $\frac{\partial v}{\partial \nu}|_{\partial\Omega} = g$.

By the same argument as above,

$$0 = \int_{\partial\Omega} (q_1 - q_2) u_1 v \, dx = \int_{\partial\Omega} \left( f \frac{\partial u_1}{\partial \nu} - u_1 g \right) \, d\sigma$$

and therefore,

$$\int_{\Omega} (q_1 - q_2) u_1 u_2 \, dx = \int_{\partial\Omega} \left( (u_2 - f) \frac{\partial u_1}{\partial \nu} - u_1 \left( \frac{\partial u_2}{\partial \nu} - g \right) \right) \, d\sigma.$$

This implies

$$\left| \int_{\Omega} (q_1 - q_2) u_1 u_2 \, dx \right| \leq \|u_2 - f\|_{H^{1/2}(\Omega)} \left\| \frac{\partial u_1}{\partial \nu} \right\|_{H^{-1/2}(\Omega)} + \|u_1\|_{H^{1/2}(\Omega)} \left\| \frac{\partial u_2}{\partial \nu} - g \right\|_{H^{-1/2}(\Omega)}$$

$$\leq \left\| \left( u_1, \frac{\partial u_1}{\partial \nu} \right) \right\|_{H^{1/2} \oplus H^{-1/2}} \cdot \left\| \left( u_2 - f, \frac{\partial u_2}{\partial \nu} - g \right) \right\|_{H^{1/2} \oplus H^{-1/2}}.$$

Taking supremum over all $(f, g) \in C_{q_1}$,

$$\int_{\Omega} (q_1 - q_2) u_1 u_2 \, dx \leq \left\| \left( u_1, \frac{\partial u_1}{\partial \nu} \right) \right\| \cdot \text{dist}(C_{q_1}, C_{q_2}) \cdot \left\| \left( u_2, \frac{\partial u_2}{\partial \nu} \right) \right\|.$$
Now, we let \( u_1, u_2 \) be the CGO solutions constructed in Theorem 3.10. Choose \( k > 0, \xi \in \mathbb{R}^n \setminus \{0\} \) and let \( \alpha, \beta \) be unit vectors in \( \mathbb{R}^n \) such that \( \{\alpha, \beta, \xi/|\xi|\} \) forms an orthonormal set. Define \( \zeta_1, \zeta_2 \in \mathbb{C}^n \) as in (24)-(25) and let
\[
\begin{align*}
  u_1(x) &= u_{\zeta_1}(x) = e^{x \cdot \zeta_1}(1 + r_1(x)), \\
  u_2(x) &= u_{\zeta_2}(x) = e^{x \cdot \zeta_2}(1 + r_2(x)).
\end{align*}
\]
where \( r_j, j = 1, 2, \) satisfy (23). It follows that
\[
\left\| \left( u_j, \frac{\partial u_j}{\partial v} \right) \right\|_{H^{1/2} \oplus H^{-1/2}} \lesssim \left\| u_j \right\|_{H^1(\Omega)} \lesssim \left\| e^{x \cdot \zeta_j} \right\|_{C^1(\Omega)} \| 1 + r_j \|_{H^1(\Omega)} \lesssim ke^{Rk}(1 + k^s) \quad \text{where } R = \sup_{x \in \Omega} |x| \lesssim e^{Sk}, \quad \text{for some } S > R.
\]
Substituting in (31), we get
\[
\left| \int_{\Omega} (q_1 - q_2)u_1u_2 \, dx \right| \lesssim e^{2Sk} \text{dist}(C_{q_1}, C_{q_2}).
\]
Now consider
\[
(q_1 - \tilde{q}_2)(\xi) = \int_{\Omega} (q_1 - q_2)e^{-ix \cdot \xi} \, dx = \int_{\Omega} (q_1 - q_2)(u_1u_2 - e^{-ix \cdot \xi}(r_1 + r_2 + r_1r_2)) \, dx.
\]
This implies
\[
|\tilde{q}_1 - \tilde{q}_2| \lesssim \int_{\Omega} (q_1 - q_2)u_1u_2 \, dx \left| + |\langle q_1 - q_2, e^{-ix \cdot \xi}(r_1 + r_2) \rangle| \right.
\]
\[
(31) \quad |\langle q_1 - q_2, e^{-ix \cdot \xi}r_1 \rangle, r_2 \rangle| \lesssim \omega(k)\parallel e^{-ix \cdot \xi}r_1\parallel_{H^{s,k}} \parallel \varphi r_2\parallel_{H^{s,k}}
\]
where \( \omega(k) \to 0 \) as \( k \to \infty \). It is obvious from the proof of Theorem 3.8 that
\[
\omega(k) \leq \max_{j=1,2} \parallel q_j\parallel_{W^{-s,n/s}} \leq M \quad \text{for all } k \geq 1.
\]
Also, for any \( f \in H^{s,k}(\mathbb{R}^n) \),
\[
\left\| e^{-ix \cdot \xi} f \right\|_{H^{s,k}}^2 = \frac{1}{(2\pi)^n} \int |\hat{e^{-ix \cdot \xi}} f(\eta)|^2 (k^2 + |\eta|^2)^s \, d\eta
\]
\[
= \frac{1}{(2\pi)^n} \int |\hat{f}(\eta + \xi)|^2 (k^2 + |\eta|^2)^s \, d\eta
\]
\[
= \frac{1}{(2\pi)^n} \int |\hat{f}(\eta)|^2 k^{2s} \left( 1 + \frac{|\eta - \xi|^2}{k^2} \right)^s \, d\eta
\]
\[
\lesssim \int |\hat{f}(\eta)|^2 k^{2s} \left( 1 + \frac{|\eta|^2}{k^2} \right)^s \left( 1 + \frac{|\xi|^2}{k^2} \right)^s \, d\eta \quad \text{(Peetre’s inequality)}
\]
\[
\lesssim (1 + |\xi|^2)^s \| f \|_{H^{s,k}}^2
\]
\[
\lesssim (1 + |\xi|^2)^s \| f \|_{H^{s,k}}^2.
\]
Therefore,

\[
|\langle m_{q_l}(e^{-ix\xi}r_1, r_2)\rangle| \lesssim M(1 + |\xi|^2)^{s/2}\|\varphi r_1\|_{H^{s,k}}\|\varphi r_2\|_{H^{s,k}} \\
\lesssim k^{-2(1-s)}M(1 + |\xi|^2)^{s/2}\|r_1\|_{H^{1,s}}\|r_2\|_{H^{1,s}} \\
\lesssim k^{-4s}M(1 + |\xi|^2)^{s/2} \quad \text{by (23)}.
\]

Next, again by (12), for \( j, l = 1, 2, \)

\[
|\langle q_j, e^{-ix\xi}r_1 \rangle | = |\langle m_{q_l}(\varphi), e^{-ix\xi}\varphi r_1 \rangle | \\
\lesssim \omega(k)\|\varphi\|_{H^{s,k}}\|e^{-ix\xi}\varphi r_1\|_{H^{s,k}} \lesssim M(1 + |\xi|^2)^{s/2}\|\varphi r_1\|_{H^{s,k}} \\
\lesssim M(1 + |\xi|^2)^{s/2}k^{-1+s}\|r_1\|_{H^{1,s}} \\
\lesssim M(1 + |\xi|^2)^{s/2}k^{-2s} \quad \text{by (23)}.
\]

Substituting all these bounds into (31), we get

\[
|\hat{q}_1(\xi) - \hat{q}_2(\xi)| \lesssim e^{2Sk}\text{dist}(C_{q_1}, C_{q_2}) + k^{-2s}M(1 + |\xi|^2)^{s/2}.
\]

We therefore have

\[
\|q_1 - q_2\|_{H^{-1}}^2 = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (1 + |\xi|^2)^{-1}|\hat{q}_1(\xi) - \hat{q}_2(\xi)|^2 d\xi \\
\lesssim \int_{|\xi|\leq \rho} (1 + |\xi|^2)^{-1}|\hat{q}_1(\xi) - \hat{q}_2(\xi)|^2 d\xi + \int_{|\xi| > \rho} (1 + |\xi|^2)^{-1}|\hat{q}_1(\xi) - \hat{q}_2(\xi)|^2 d\xi \\
\lesssim \rho^n e^{4Sk}\text{dist}^2(C_{q_1}, C_{q_2}) + k^{-4s}M^2\rho^{n+2s-2} \\
+ \frac{1}{(1 + \rho^2)^{1-s}} \int (1 + |\xi|^2)^{-s}(\hat{q}_1^2(\xi) + \hat{q}_2^2(\xi)) d\xi \\
\lesssim \rho^n e^{4Sk}\text{dist}^2(C_{q_1}, C_{q_2}) + M^2k^{-4s}\rho^{n-2s-1} + M^2\rho^{-1-2s}.
\]

In order to make the last two terms small and of the same order in \( \rho \), we choose

\[ k = \rho^{\frac{n}{1-s}}, \]

which gives us

\[
\|q_1 - q_2\|_{H^{-1}}^2 \lesssim \rho^n e^{4s\rho^{\frac{n}{1-s}}}\text{dist}^2(C_{q_1}, C_{q_2}) + \rho^{-1-2s} \quad \text{by (32)}.
\]

\[
\|q_1 - q_2\|_{H^{-1}}^2 \lesssim e^{T\rho^{\frac{n}{2}}}\text{dist}^2(C_{q_1}, C_{q_2}) + \rho^{-1-2s} \quad \text{by (33)}.
\]

for fixed \( T > 4S \). Now choose

\[
\rho = \left(\frac{1}{T}\log\{|\text{dist}(C_{q_1}, C_{q_2})|\}\right)^{\frac{1}{1-s}}
\]

so that when \( \text{dist}(C_{q_1}, C_{q_2}) < 1 \),

\[
e^{T\rho^{\frac{n}{2}}}\text{dist}^2(C_{q_1}, C_{q_2}) = \text{dist}(C_{q_1}, C_{q_2}).
\]

Combining this with (33), we see that when \( \text{dist}(C_{q_1}, C_{q_2}) < 1 \),

\[
\|q_1 - q_2\|_{H^{-1}}^2 \lesssim \text{dist}(C_{q_1}, C_{q_2}) + |\log\{|\text{dist}(C_{q_1}, C_{q_2})|\}|^{-\frac{4(1+2s)}{n}} \\
\lesssim |\log\{|\text{dist}(C_{q_1}, C_{q_2})|\}|^{-\frac{4(1+2s)}{n}}.
\]
This gives us \((29)\) when \(\text{dist}(C_{q_1}, C_{q_2}) < 1\) for \(\sigma = 4e(1+2e)/n = 4(1-s)(1-2s)/n\). Moreover, \((29)\) is trivially true when \(\text{dist}(C_{q_1}, C_{q_2}) > 1\) since \(\|q_j\|_{H^{-1}} \lesssim \|q_j\|_{W^{-s,n/s}} \leq M\) for \(j = 1, 2\). Therefore, the proof is complete. \(\Box\)

We can now prove the stability estimate for the conductivity equation. We will use the fact that \(W^{s,p}\) embeds into the Zygmund space \(C_{e}^{q}\) for \(t = s - n/p\).

**Theorem 5.2.** Let \(0 < s < 1/2\) and \(\gamma_1, \gamma_2 \in W^{2-s,n/s}(\Omega)\) be such that \(\gamma_j \equiv 1\) in a neighborhood of \(\partial\Omega\) and

\[
0 < c < \gamma_j(x) < c^{-1}, \quad \text{for a.e. } x \in \Omega, \ j = 1, 2.
\]

Given any \(\alpha \in (0, 1)\), there exists \(C > 0\) and \(\sigma = \sigma(n, s, \alpha) \in (0, 1)\) such that

\[
\|\gamma_1 - \gamma_2\|_{C^{\alpha}(\Omega)} \leq C(\|\log \Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2} \to H^{-1/2}})^{-\sigma} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2} \to H^{-1/2}}).
\]

**Proof.** As in Proposition \((2.3)\) let us extend \(\gamma_j\) to all of \(\mathbb{R}^n\) by defining \(\gamma_j \equiv 1\) on \(\mathbb{R}^n \setminus \Omega\), so that \(\gamma_j - 1 \in W^{2-s,n/s}_{\text{comp}}(\Omega)\). Note that this implies \(\gamma_j \in C_{*}^{1+\epsilon} = C^{1,\epsilon}\) for \(\epsilon = 1 - 2s\). Define \(q_j = \gamma_j^{-1/2} \Delta \gamma_j^{1/2}\). Also choose a bounded domain \(U\) such that \(\partial U \subset U\) and \(\partial U\) is smooth. We observe that the function \(v = \log \gamma_1 - \log \gamma_2\) solves the following elliptic boundary value problem:

\[
\begin{cases}
\nabla \cdot ((\gamma_1 \gamma_2)^{1/2} \nabla v) = 2(\gamma_1 \gamma_2)^{1/2} (q_2 - q_1) & \text{in } U \\
v = 0 & \text{on } \partial U.
\end{cases}
\]

Therefore, we have the estimate

\[
\|\log \gamma_1 - \log \gamma_2\|_{H^1(U)} \lesssim \|q_1 - q_2\|_{H^{-1}(U)} \lesssim \|q_1 - q_2\|_{H^{-1}}.
\]

Now consider the identities

\[
\gamma_1 - \gamma_2 = \left(\int_0^1 e^{t \log \gamma_1 + (1-t) \log \gamma_2} dt\right) \cdot (\log \gamma_1 - \log \gamma_2),
\]

\[
\nabla \gamma_1 - \nabla \gamma_2 = \gamma_1 \nabla \log \gamma_1 - \gamma_2 \nabla \log \gamma_2 = \gamma_1 (\nabla \log \gamma_1 - \log \gamma_2) + \frac{\gamma_1 - \gamma_2}{\gamma_2} \nabla \gamma_2.
\]

Together with the fact that \(\gamma_j \in C^{1,\epsilon}\), these identities imply that

\[
(35) \quad \|\gamma_1 - \gamma_2\|_{H^1(U)} \lesssim \|\log \gamma_1 - \log \gamma_2\|_{H^1(U)} \lesssim \|q_1 - q_2\|_{H^{-1}}.
\]

Next, recall from Proposition \((2.3)\) (c) that \(\Lambda_{\gamma_j} = \Lambda_{q_j}\). By \((29)\), for \(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2} \to H^{-1/2}} = \|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2} \to H^{-1/2}} < 1/2\),

\[
\|q_1 - q_2\|_{H^{-1}} \lesssim \|\log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2} \to H^{-1/2}}\|^{-\sigma}
\]

which along with \((35)\) implies

\[
\|\gamma_1 - \gamma_2\|_{H^1(U)} \lesssim \|\log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2} \to H^{-1/2}}\|^{-\sigma}.
\]

Now, given \(\alpha \in (0, 1)\), define \(p = n/(1 - \alpha)\). By Hölder’s inequality and the fact that \(\gamma_j, \nabla \gamma_j\) are bounded,

\[
\|\gamma_1 - \gamma_2\|_{W^{1,p}(U)} \lesssim \|\gamma_1 - \gamma_2\|_{H^1(U)}^{2/p}.
\]

Therefore, whenever \(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2} \to H^{-1/2}} < 1/2\),

\[
\|\gamma_1 - \gamma_2\|_{W^{1,p}(U)} \lesssim \|\log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2} \to H^{-1/2}}\|^{-\sigma'} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2} \to H^{-1/2}}^{-\sigma'}.
\]
for $\sigma' = \frac{2\alpha}{p} = \frac{8(1-s)(1-2s)(1-\alpha)}{n}$. On the other hand, the above estimate is clearly true when $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2} \rightarrow H^{-1/2}} \geq 1/2$ due to the fact that $\gamma_j \in W^{1,\infty}$. Therefore, in all cases, we have

$$\|\gamma_1 - \gamma_2\|_{W^{1,p}(U)} \lesssim \log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2} \rightarrow H^{-1/2}} \sim \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2} \rightarrow H^{-1/2}}.$$ 

Finally, (34) follows from the fact that $W^{1,p}(U) \hookrightarrow C_0^\alpha(U) = C_0^\alpha(\Omega)$.

□

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