Abstract

In clustering, the choice of initial centers is crucial for the convergence speed of the algorithms. We propose a new initialization scheme for the $k$-median problem in the general metric space (e.g., discrete space induced by graphs), based on the construction of metric embedding tree structure of the data. We propose a novel and efficient search algorithm which finds initial centers that can be used subsequently for the local search algorithm. The so-called HST initialization method can produce initial centers achieving lower error than those from another popular method $k$-median++, also with higher efficiency when $k$ is not too small. Our HST initialization are then extended to the setting of differential privacy (DP) to generate private initial centers. We show that the error of applying DP local search followed by our private HST initialization improves prior results on the approximation error, and approaches the lower bound within a small factor. Experiments demonstrate the effectiveness of our proposed methods.

1 Introduction

Clustering is an important classic problem in unsupervised learning that has been widely studied in statistics, data mining, machine learning, network analysis, etc. (Punj and Stewart, 1983; Dhillon and Modha, 2001; Banerjee et al., 2005; Berkhin, 2006; Abbasi and Younis, 2007). The objective of clustering is to divide a set of data points into clusters, such that items within the same cluster exhibit similarities, while those in different clusters distinctly differ. This is concretely measured by the sum of distances (or squared distances) between each point to its nearest cluster center. One conventional notion to evaluate a clustering algorithms is: with high probability, $cost(C, D) \leq \gamma OPT_k(D) + \xi$, where $C$ is the centers output by the algorithm and $cost(C, D)$ is a cost function defined for $C$ on dataset $D$. $OPT_k(D)$ is the cost of optimal clustering solution on $D$. When everything is clear from context, we will use $OPT$ for short. Here, $\gamma$ is called multiplicative error and $\xi$ is called additive error. Alternatively, we may also use the notion of expected cost.

Two popularly studied clustering problems are 1) the $k$-median problem, and 2) the $k$-means problem. The origin of $k$-median dates back to the 1970’s (e.g., Kaufman et al. (1977)), where one tries to find the best location of facilities that minimizes the cost measured by the distance between clients and facilities. Formally, given a set of points $D$ and a distance measure, the goal is to find $k$ center points minimizing the sum of absolute distances of each sample point to its nearest center. In $k$-means, the objective is to minimize the sum of squared distances instead. There are two general frameworks for clustering. One heuristic is the Lloyd’s algorithm (Lloyd, 1982), which is built upon an iterative distortion minimization approach. In most cases, this method can only be applied to numerical data.
typically in the (continuous) Euclidean space. Clustering in general metric spaces (discrete spaces) is also important and useful when dealing with, for example, the graph data, where Lloyd’s method is no longer applicable. A more generally applicable approach, the local search method (Kanungo et al., 2002; Arya et al., 2004), has also been widely studied. It iteratively finds the optimal swap between the center set and non-center data points to keep lowering the cost. Local search can achieve a constant approximation (i.e., $\gamma = O(1)$) to the optimal solution (Arya et al., 2004). For general metric spaces, the state of the art approximation ratio is 2.675 for $k$-median Byrka et al. (2015) and 6.357 for $k$-means Ahmadian et al. (2017).

**Initialization of cluster centers.** It is well-known that the performance of clustering can be highly sensitive to initialization. If clustering starts with good initial centers with small approximation error, the algorithm may use fewer iterations to find a better solution. The $k$-median++ algorithm (Arthur and Vassilvitskii, 2007) iteratively selects $k$ data points as initial centers, favoring distant points in a probabilistic way, such that the initial centers tend to be well spread over the data points (i.e., over different clusters). The produced initial centers are proved to have $O(\log k)$ multiplicative error. Follow-up works further improved its efficiency and scalability, e.g., Bahmani et al. (2012); Bachem et al. (2016); Lattanzi and Sohler (2019); Choo et al. (2020); Cohen-Addad et al. (2021); Grunau et al. (2023); Fan et al. (2023). In this work, we propose a new initialization framework, called HST initialization, which is built upon a novel search algorithm on metric embedding trees constructed from the data. Our method achieves improved approximation error compared with $k$-median++. Moreover, importantly, our initialization scheme can be conveniently combined with the notion of differential privacy (DP) to protect the data privacy.

**Clustering with Differential Privacy.** The concept of differential privacy (Dwork, 2006; McSherry and Talwar, 2007) has been popular to rigorously define and resolve the problem of keeping useful information for machine learning models, while protecting privacy for each individual. DP has been adopted to a variety of algorithms and tasks, such as regression, classification, principle component analysis, graph distance release, matrix completion, optimization, and deep learning (Chaudhuri and Monteleoni, 2008; Chaudhuri et al., 2011; Abadi et al., 2016; Ge et al., 2018; Wei et al., 2020; Dong et al., 2022; Fan and Li, 2022; Fan et al., 2022; Fang et al., 2023; Li and Li, 2023a,b). Private $k$-means clustering has also been widely studied, e.g., Feldman et al. (2009); Nock et al. (2016); Feldman et al. (2017), mostly in the continuous Euclidean space. Balcan et al. (2017) considered identifying a good candidate set (in a private manner) of centers before applying private local search, which yields $O(\log^3 n)$ multiplicative error and $O((k^2 + d) \log^5 n)$ additive error. Later on, the private Euclidean $k$-means error is further improved by Stemmer and Kaplan (2018), with more advanced candidate set selection. Huang and Liu (2018) gave an optimal algorithm in terms of minimizing Wasserstein distance under some data separability condition.

For private $k$-median clustering, Feldman et al. (2009); Ghazi et al. (2020) considered the problem in high dimensional Euclidean space. However, it is rather difficult to extend their analysis to more general metrics in discrete spaces (e.g., on graphs). The strategy of Balcan et al. (2017) to form a candidate center set could as well be adopted to $k$-median, which leads to $O(\log^{3/2} n)$ multiplicative error and $O((k^2 + d) \log^3 n)$ additive error in the Euclidean space where $n$ is the sample size. In discrete space, Gupta et al. (2010) proposed a private method for the classical local search heuristic, which applies to both $k$-medians and $k$-means. To cast privacy on each swapping step, the authors applied the exponential mechanism of McSherry and Talwar (2007). Their method produced an $\epsilon$-differentially private solution with cost $6OPT + O(\Delta k^2 \log^2 n / \epsilon)$, where $\Delta$ is the diameter of the point set. In this work, we will show that our proposed HST initialization can improve the DP local search for $k$-median of Gupta et al. (2010) in terms of both approximation error and efficiency. Stemmer and Kaplan (2018); Jones et al. (2021) proposed $(\epsilon, \delta)$-differentially private solution also with constant multiplicative error but smaller additive error.

**The main contributions** of this work include the following:

- We introduce the Hierarchically Well-Separated Tree (HST) as an initialization tool for the $k$-median clustering problem. We design an efficient sampling strategy to select the initial center set from the tree, with an approximation factor $O(\log \min\{k, \Delta\})$ in the non-private setting, which is $O(\log \min\{k, d\})$ when $\log \Delta = O(\log d)$. This improves the $O(\log k)$ error of $k$-median++. Moreover, the complexity of our HST based method can be smaller than that of $k$-median++ when the number of clusters $k$ is not too small ($k \geq \log n$), which is a common scenario in practical applications.
• We propose a differentially private version of HST initialization under the setting of Gupta et al. (2010) in discrete metric space. The so-called DP-HST algorithm finds initial centers with $O(\log n)$ multiplicative error and $O(\varepsilon^{-1} \Delta k^2 \log^2 n)$ additive error. Moreover, running DP-local search starting from this initialization gives $O(1)$ multiplicative error and $O(\varepsilon^{-1} \Delta k^2 (\log \log n) \log n)$ additive error, which improves previous results towards the well-known lower bound $O(\varepsilon^{-1} \Delta k \log(n/k))$ on the additive error of DP $k$-median (Gupta et al., 2010) within a small $O(k \log \log n)$ factor. This is the first clustering initialization method with $\varepsilon$-differential privacy guarantee and improved error rate in general metric space.

• We conduct experiments on simulated and real-world datasets to demonstrate the effectiveness of our methods. In both non-private and private settings, our proposed HST-based approach achieves smaller cost at initialization than $k$-median++, which may also lead to improvements in the final clustering quality.

2 Background and Setup

The definition of differential privacy (DP) is as follows.

Definition 2.1 (Differential Privacy (DP) (Dwork, 2006)). If for any two adjacent datasets $D$ and $D'$ with symmetric difference of size one and any $O \subset \text{Range}(A)$, an algorithm $A$ with map $f$ satisfies

$$Pr[A(D) \in O] \leq e^\epsilon Pr[A(D') \in O],$$

then algorithm $A$ is said to be $\epsilon$-differentially private ($\epsilon$-DP).

Intuitively, DP requires that after removing any data point from $D$ (e.g., a node in a graph), the output of $D'$ should not be too different from that of the original dataset $D$. The Laplace mechanism adds $\text{Laplace}(\eta(f)/\epsilon)$ noise to the output where $\eta(f) = \sup_{D \neq D'} |f(D) - f(D')|$ is the sensitivity of $f$, which is known to achieve $\epsilon$-DP. The exponential mechanism is also a tool for many DP algorithms with discrete outputs. Let $O$ be the output domain. The utility function $q : D \times O \rightarrow \mathbb{R}$ is what we aim to maximize. The exponential mechanism outputs an element $o \in O$ with probability $P[A(D) = o] \propto \exp\left(\frac{q(D,o)}{2\eta^2}\right)$. Both mechanisms will be used in our paper.

2.1 $k$-Median Clustering and Local Search

In this paper, we follow the classic problem setting in the metric clustering literature, e.g. Arya et al. (2004); Gupta et al. (2010). Specifically, the definitions of metric $k$-median clustering problem (DP and non-DP) are stated as follow.

Definition 2.2 ($k$-median). Given a universe point set $U$ and a metric $\rho : U \times U \rightarrow \mathbb{R}$, the goal of $k$-median to pick $F \subseteq U$ with $|F| = k$ to minimize

$$k\text{-median: } cost_k(F,U) = \sum_{v \in U} \min_{f \in F} \rho(v,f). \quad (1)$$

Let $D \subseteq U$ be a set of “demand points”. The goal of DP $k$-median is to minimize

$$\text{DP } k\text{-median: } cost_k(F,D) = \sum_{v \in D} \min_{f \in F} \rho(v,f), \quad (2)$$

and the output $F$ is required to be $\epsilon$-differentially private with respect to $D$. We may drop “F” and use “$cost_k(U)$” or “$cost_k(D)$” if there is no risk of ambiguity.

Note that in Definition 2.2, our aim is to protect the privacy of a subset $D \subset U$. To better understand the motivation and application scenario, we provide a real-world example as below.

Example 2.3. Consider $U$ to be the universe of all users in a social network (e.g., Facebook, LinkedIn, etc.). Each user (account) has some public information (e.g., name, gender, interests, etc.), but also has some private personal data that can only be seen by the data server. Let $D$ be a set of users grouped by some feature that might be set as private. Suppose a third party plans to collaborate with the most influential users in $D$ for e.g., commercial purposes, thus requesting the cluster centers of $D$. In this case, we need a differentially private algorithm to safely release the centers, while protecting the individuals in $D$ from being identified (since the membership of $D$ is private).
Well-Separated Tree (HST) is based on the padded decompositions of a general metric space in a hierarchical manner (Fakcharoenphol et al., 2004). Let $U$, $\rho$ be a metric space with $|U| = n$, and we will refer to this metric space without specific clarification. A $eta$-padded decomposition of $U$ achieves $5$ approximation error to the optimal solution, i.e., $\text{cost}(F) \leq 5\text{OPT}$.

### 2.2 $k$-median++ Initialization

Although local search is able to find a solution with constant error, it takes $O(n^2)$ per iteration (Resende and Werneck, 2007) in expected $O(kn \log n)$ steps (which gives total complexity $O(kn^2 \log n)$) when started from a random center set, which would be slow for large datasets. Indeed, we do not need such complicated/meticulous algorithm to reduce the cost at the beginning, i.e., when the cost is large. To accelerate the process, efficient initialization methods find a “roughly” good center set as the starting point for local search. In the paper, we compare our new initialization scheme mainly with a popular (and perhaps the most well-known) initialization method, the $k$-median++ (Arthur and Vassilvitskii, 2007)\(^1\) as presented in Algorithm 2. The output centers $C$ by $k$-median++ achieve $O(\log k)$ approximation error with time complexity $O(nk)$. Starting from the initialization, we only need to run $O(k \log \log k)$ steps of the computationally heavy local search to reach a constant error solution. Thus, initialization may greatly improve the clustering efficiency.

#### Algorithm 2: $k$-median++ initialization (Arthur and Vassilvitskii, 2007)

**Input:** Data points $U$, number of centers $k$  
**Initialization:** Randomly pick a point $c_1 \in U$ and set $F = \{c_1\}$  
for $i = 2, \ldots, k$ do  
    Select $c_i = u \in U$ with probability $\frac{\rho(u,F)}{\sum_{v \in U} \rho(v,F)}$  
    $F = F \cup \{c_i\}$  
**Output:** $k$-median++ initial center set $F$

### 3 Initialization via Hierarchical Well-Separated Tree (HST)

In this section, we propose our new initialization scheme for $k$-median clustering, and provide our analysis in the non-private case solving (1). The idea is based on the metric embedding theory. We will start with an introduction to the main tool used in our approach.

#### 3.1 Hierarchically Well-Separated Tree (HST)

In this paper, for an $L$-level tree, we will count levels in a descending order down the tree. We use $h_v$ to denote the level of $v$, and let $n_i$ be the number of nodes at level $i$. The Hierarchically Well-Separated Tree (HST) is based on the padded decompositions of a general metric space in a hierarchical manner (Fakcharoenphol et al., 2004). Let $(U, \rho)$ be a metric space with $|U| = n$, and we will refer to this metric space without specific clarification. A $\beta$-padded decomposition of $U$ achieves $5$ approximation error to the optimal solution, i.e., $\text{cost}(F) \leq 5\text{OPT}$.

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\(^1\)Arthur et al. (2004) chooses the first swap that improves the cost by $(1 - \alpha/k)$, instead of picking the smallest cost among all such swaps as in Algorithm 1. Both methods are valid and yield the same approximation error.

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\(^2\)In the original paper of Arthur and Vassilvitskii (2007), the $k$-median++ algorithm was called $D^1$-sampling.
Algorithm 3: Build 2-HST($U, L$)

Input: Data points $U$ with diameter $\Delta$, $L$
Randomly pick a point in $U$ as the root node of $T$
Let $r = \Delta/2$
Apply a permutation $\pi$ on $U$ // so points will be chosen in a random sequence
for each $v \in U$ do
  Set $C_v = \{v\}$
  for each $u \in U$ do
    Add $u \in U$ to $C_v$ if $d(v, u) \leq r$ and $u \notin \bigcup_{v' \neq v} C_{v'}$
  Set the non-empty clusters $C_v$ as the children nodes of $T$
for each non-empty cluster $C_v$ do
  Run 2-HST($C_v, L - 1$) to extend the tree $T$; stop until $L$ levels or reaching a leaf node

Output: 2-HST $T$
3.2 HST Initialization Algorithm

Let \( L = \log \Delta \) and suppose \( T \) is a level-\( L \) 2-HST in \((U, \rho)\), where we assume \( L \) is an integer. For a node \( v \) at level \( i \), we use \( T(v) \) to denote the subtree rooted at \( v \). Let \( N_v = |T(v)| \) be the number of data points in \( T(v) \). The search strategy for the initial centers, NDP-HST initialization ("NDP" stands for "Non-Differentially Private"), is presented in Algorithm 4 with two phases.

**Subtree search.** The first step is to identify the subtrees that contain the \( k \) centers. To begin with, \( k \) initial centers \( C_1 \) are picked from \( T \) who have the largest score \( \text{score}(v) = N(v) \cdot 2^{h_v} \). This is intuitive, since to get a good clustering, we typically want the ball surrounding each center to include more data points. Next, we do a screening over \( C_1 \): if there is any ancestor-descendant pair of nodes, we remove the ancestor from \( C_1 \). If the current size of \( C_1 \) is smaller than \( k \), we repeat the process until \( k \) centers are chosen (we do not re-select nodes in \( C_1 \) and their ancestors). This way, \( C_1 \) contains \( k \) root nodes of \( k \) disjoint subtrees.

**Algorithm 4:** NDP-HST initialization

**Input:** \( U, \Delta, k \)

**Initialization:** \( L = \log \Delta, C_0 = \emptyset, C_1 = \emptyset \)

Call Algorithm 3 to build a level-\( L \) 2-HST \( T \) using \( U \)

for each node \( v \) in \( T \) do

\[ N_v \leftarrow |U \cap T(v)|, \text{score}(v) \leftarrow N_v \cdot 2^{h_v} \]

while \( |C_1| < k \) do

Add top \((k - |C_1|)\) nodes with highest score to \( C_1 \)

for each \( v \in C_1 \) do

\[ C_1 = C_1 \setminus \{v\}, \text{if } \exists \ v' \in C_1 \text{ such that } v' \text{ is a descendant of } v \]

\( C_0 = \text{FIND-LEAF}(T, C_1) \)

**Output:** Initial center set \( C_0 \subseteq U \)

**Leaf search.** After we find \( C_1 \) the set of \( k \) subtrees, the next step is to find the center in each subtree using Algorithm 5 ("FIND-LEAF"). We employ a greedy search strategy, by finding the child node with largest score level by level, until a leaf is found. This approach is intuitive since the diameter of the partition ball exponentially decays with the level. Therefore, we are in a sense focusing more and more on the region with higher density (i.e., with more data points).

The complexity of our search algorithm is given as follows. All proofs are placed in Appendix B.

**Proposition 3.2 (Complexity).** Algorithm 4 takes \( O(dn \log n) \) time in the Euclidean space.

**Remark 3.3 (Comparison with \( k \)-median++).** The complexity of of \( k \)-median++ is \( O(dnk) \) in the Euclidean space (Arthur and Vassilvitskii, 2007). Our algorithm would be faster when \( k > \log n \), which is a common scenario. Similar comparison also holds for general metrics.

3.3 Approximation Error of HST Initialization

We provide the error analysis of our algorithm. Firstly, we show that the initial center set produced by NDP-HST is already a good approximation to the optimal \( k \)-median solution. Let \( \rho^T(x, y) = d_T(x, y) \) denote the "2-HST metric" between \( x \) and \( y \) in the 2-HST \( T \), where \( d_T(x, y) \) is the tree distance between nodes \( x \) and \( y \) in \( T \). By Definition 3.1 and since \( \Delta = 2^L \), in the analysis we assume

**Algorithm 5:** FIND-LEAF \((T, C_1)\)

**Input:** \( T, C_1 \)

**Initialization:** \( C_0 = \emptyset \)

for each node \( v \) in \( C_1 \) do

while \( v \) is not a leaf node do

\[ v \leftarrow \arg_{w \in \text{ch}(v)} \max\{N_w, w \in \text{ch}(v)\} \]

Add \( v \) to \( C_0 \)

**Output:** Initial center set \( C_0 \subseteq U \)
We remark that the initial centers found by NDP-HST can be used for k-means clustering with privacy. Since in this problem, \( U \) is the universe of data points, and \( D \subseteq U \) is a demand set that needs to be clustered with privacy. Running Algorithm 1, when the initial centers are produced by our NDP-HST.

Theorem 3.10. When initialized by NDP-HST, local search achieves O(1) approximation error in expected \( O(k \log \log \min\{n, \Delta\}) \) number of iterations for input in general metric space.

We remark that the initial centers found by NDP-HST can be used for k-means clustering analogously. For general metrics, \( E[\text{cost}_{km}(U)] = O(\min\{\log n, \log \Delta\})^2 \text{OPT}_{km}(U) \) where \( \text{cost}_{km}(U) \) is the optimal k-means cost. See Appendix C for more details.

4 HST Initialization with Differential Privacy

In this section, we consider initialization and clustering with differential privacy (DP). Recall (2) that in this problem, \( U \) is the universe of data points, and \( D \subseteq U \) is a demand set that needs to be clustered with privacy. Since \( U \) is public, simply running initialization algorithms on \( U \) would
Algorithm 6: DP-HST initialization

**Input:** $U, D, \Delta, k, \epsilon$

Build a level-$L$ 2-HST $T$ based on input $U$

for each node $v$ in $T$ do

$\hat{N}_v \leftarrow |D \cap T(v)|$, $\tilde{N}_v \leftarrow N_v + \text{Lap}(2^{(L-h_v)}/\epsilon)$, $\text{score}(v) \leftarrow \hat{N}_v \cdot 2^{h_v}$

Based on $\hat{N}_v$, apply the same strategy as Algorithm 4: find $C_1$; $C_0 = \text{FIND-LEAF}(C_1)$

**Output:** Private initial center set $C_0 \subseteq U$

Algorithm 7: DP-HST local search

**Input:** $U$, demand points $D \subseteq U$, parameter $k, \epsilon, T$

**Initialization:** $F_1$ the private initial centers generated by Algorithm 6 with privacy $\epsilon/2$

Set parameter $\epsilon' = \frac{\epsilon}{4\Delta(n+1)}$

for $i = 1$ to $T$ do

Select $(x, y) \in F_i \times (V \setminus F_i)$ with prob. proportional to $\exp(-\epsilon' \times (\text{cost}(F_i - \{x\} + \{y\}))$

Let $F_{i+1} \leftarrow F_i - \{x\} + \{y\}$

Select $j$ from $\{1, 2, \ldots, T + 1\}$ with probability proportional to $\exp(-\epsilon' \times \text{cost}(F_j))$

**Output:** $F = F_j$ the private center set

preserve the privacy of $D$. However, 1) this might be too expensive; 2) in many cases one would probably want to incorporate some information about $D$ in the initialization, since $D$ could be a very imbalanced subset of $U$. For example, $D$ may only contain data points from one cluster, out of tens of clusters in $U$. In this case, initialization on $U$ is likely to pick initial centers in multiple clusters, which would not be helpful for clustering on $D$.

Next, we show how our proposed HST initialization can be easily combined with differential privacy and at the same time contains useful information about the demand set $D$, leading to improved approximation error (Theorem 4.3). Again, suppose $T$ is an $L = \log \Delta$-level 2-HST of universe $U$ in a general metric space. Denote $N_v = |T(v) \cap D|$ for a node point $v$. Our private HST initialization (DP-HST) is similar to the non-private Algorithm 4. To gain privacy, we perturb $N_v$ by adding i.i.d. Laplace noise: $\hat{N}_v = N_v + \text{Lap}(2^{(L-h_v)}/\epsilon)$, where $\text{Lap}(2^{(L-h_v)}/\epsilon)$ is a Laplace random number with rate $2^{(L-h_v)}/\epsilon$. We will use the perturbed $\hat{N}_v$ for node sampling instead of the true value $N_v$, as described in Algorithm 6. The DP guarantee of this initialization scheme is straightforward by the composition theory (Dwork, 2006).

**Theorem 4.1.** Algorithm 6 is $\epsilon$-differentially private.

**Proof.** For each level $i$, the subtrees $T(v, i)$ are disjoint to each other. The privacy budget used in $i$-th level is $\epsilon/2^{(L-i)}$, so by composition the total privacy budget is $\sum_i \epsilon/2^{(L-i)} < \epsilon$. \qed

**Theorem 4.2.** Algorithm 6 finds initial centers such that $E[\text{cost}_k(D)] = O(\log n)(OPT_k(D) + k \epsilon^{-1} \triangle \log n)$.

**DP-HST Local Search.** Similarly, we can use private HST initialization to improve the performance of private $k$-median local search, which is presented in Algorithm 7. After DP-HST initialization, the DP local search procedure follows Gupta et al. (2010) using the exponential mechanism.

**Theorem 4.3.** Algorithm 7 achieves $\epsilon$-differential privacy. The output centers achieve $\text{cost}_k(D) \leq 6OPT_k(D) + O(\epsilon^{-1}k^2 \triangle \log n \log n)$ in $O(k \log \log n)$ iterations with probability $(1 - \frac{1}{poly(n)})$.

In prior literature, the DP local search with random initialization (Gupta et al., 2010) has $6$ multiplicative error and $O(\epsilon^{-1} \triangle^2 \log n)$ additive error. Our result improves the log $n$ term to log log $n$ in the additive error. Meanwhile, the number of iterations needed is improved from $T = O(k \log n)$ to $O(k \log \log n)$ (see Appendix A for an empirical justification). Notably, it has been shown in Gupta et al. (2010) that for $k$-median problem, the lower bounds on the multiplicative and additive error of any $\epsilon$-DP algorithm are $O(1)$ and $O(\epsilon^{-1} \triangle k \log(n/k))$, respectively. Our result matches the lower bound on the multiplicative error, and the additive error is only worse than the bound by a factor of $O(k \log \log n)$. To our knowledge, Theorem 4.3 is the first result in the literature to improve the error of DP local search in general metric space.
5 Numerical Results

5.1 Datasets and Algorithms

Discrete Euclidean space. Following previous work, we test $k$-median clustering on the MNIST hand-written digit dataset (LeCun et al., 1998) with 10 natural clusters (digit 0 to 9). We set $U$ as 10000 randomly chosen data points. We choose the demand set $D$ using two strategies: 1) “balance”, where we randomly choose 500 samples from $U$; 2) “imbalance”, where $D$ contains 500 random samples from $U$ only from digit “0” and “8” (two clusters). We note that, the imbalanced $D$ is a very practical setting in real-world scenarios, where data are typically not uniformly distributed. On this dataset, we test clustering with both $l_2$ and $l_1$ distance as the underlying metric.

Metric space induced by graph. Random graphs have been widely considered in testing $k$-median methods (Balcan et al., 2013; Todo et al., 2019). Our construction of graphs follows a similar approach as the synthetic pmedinfo graphs provided by the popular OR-Library (Beasley, 1990). The metric $\rho$ for this experiment is the (weighted) shortest path distance. To generate a size-$n$ graph, we first randomly split the nodes into 10 clusters. Within each cluster, each pair of nodes is connected with probability 0.2, and with weight drawn from uniform $[0, 1]$. For every pair of clusters, we randomly connect some nodes from each cluster, with weights following uniform $[0.5, r]$. A larger $r$ makes the graph more separable, i.e., clusters are farther from each other (see Appendix A for example graphs). For this task, $U$ has 3000 nodes, and the private set $D$ (500 nodes) is chosen using the similar “balanced” and “imbalance” approaches as described above. In the imbalanced case, we choose the demand set $D$ randomly from only two clusters.

Algorithms. We compare the following clustering algorithms in both non-DP and DP setting: (1) NDP-rand: Local search with random initialization; (2) NDP-kmedian++: Local search with $k$-median++ initialization (Algorithm 2); (3) NDP-HST: Local search with NDP-HST initialization (Algorithm 4), as described in Section 3; (4) DP-rand: Standard DP local search algorithm (Gupta et al., 2010), which is Algorithm 7 with initial centers randomly chosen from $U$; (5) DP-kmedian++: DP local search with $k$-median++ initialization run on $U$; (6) DP-HST: DP local search with HST-initialization (Algorithm 7). For non-DP tasks, we set $L = 6$. For DP clustering, we use $L = 8$.

For non-DP methods, we set $\alpha = 10^{-3}$ in Algorithm 1 and the maximum number of iterations as 20. To examine the quality of initialization as well as the final centers, We report both the cost at initialization and the cost of the final output. For DP methods, we run the algorithms for $T = 20$ steps and report the results with $\epsilon = 1$ (comparisons/results with other $T$ and $\epsilon$ are similar). We test $k \in \{2, 5, 10, 15, 20\}$. The average cost over $T$ iterations is reported for robustness. All the results are averaged over 10 independent repetitions.

5.2 Results

The results on MNIST and graph data are given in Figure 2. Here we present the $l_2$-clustering on MNIST, and the simulated graph with $r = 1$ (clusters are less separable). The comparisons are similar for both $l_1$ metric on MNIST and $r = 100$ graph (see Figure 4 in Appendix A):

- From the left column, the initial centers found by HST has lower cost than $k$-median++ and random initialization, for both non-DP and DP setting, and for both balanced and imbalanced demand set $D$. This confirms that the proposed HST initialization is more powerful than $k$-median++ in finding good initial centers.

- From the right column, we also observe lower final cost of HST followed by local search in DP clustering. In the non-DP case, the final cost curves overlap, which means that despite HST offers better initial centers, local search can always find a good solution eventually.

- The advantage of DP-HST, in terms of both the initial and the final cost, is more significant when $D$ is an imbalanced subset of $U$. As mentioned before, this is because our DP-HST initialization approach also privately incorporates the information of $D$.

To sum up, the proposed HST initialization scheme could perform better with various metrics and data patterns, in both non-private and private setting—in all cases, HST finds better initial centers with smaller cost than $k$-median++. HST considerably outperforms $k$-median++ in the private and imbalanced $D$ setting, for MNIST with both $l_2$ and $l_1$ metric, and for graph with both $r = 100$ (highly separable) and $r = 1$ (less separable).
6 Conclusion

We develop a new initialization framework for the \( k \)-median problem in the general metric space. Our approach, called HST initialization, is built upon the HST structure from metric embedding theory. We propose a novel and efficient tree search approach which provably improves the approximation error of the \( k \)-median++ method, and has lower complexity (higher efficiency) than \( k \)-median++ when \( k \) is not too small, which is a common practice. Moreover, we propose differentially private (DP) HST initialization algorithm, which adapts to the private demand point set, leading to better clustering performance. When combined with subsequent DP local search heuristic, our algorithm is able to improve the additive error of DP local search, which is close to the theoretical lower bound within a small factor. Experiments with Euclidean metrics and graph metrics verify the effectiveness of our methods, which improve the cost of both the initial centers and the final \( k \)-median output.
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A More Details on Experiments

A.1 Examples of Graph Data

In Figure 3, we plot two example graphs (subgraphs of 50 nodes) with $r = 100$ and $r = 1$. When $r = 100$, the graph is highly separable (i.e., clusters are far from each other). When $r = 1$, the clusters are harder to be distinguished from each other.

Figure 3: Example of synthetic graphs: subgraph of 50 nodes. **Upper** $r = 1$. **Bottom**: $r = 100$. Darker and thicker edged have smaller distance. When $r = 100$, the graph is more separable.
A.2 More Experiments

A.3 Improved Iteration Cost of DP-HST

In Theorem 4.3, we show that under differential privacy constraints, the proposed DP-HST (Algorithm 7) improves both the approximation error and the number of iterations required to find a good solution of classical DP local search (Gupta et al., 2010). In this section, we provide some numerical results to justify the theory.

First, we need to properly measure the iteration cost of DP local search. This is because, unlike the non-private clustering, the $k$-median cost after each iteration in DP local search is not decreasing monotonically, due to the probabilistic exponential mechanism. To this end, for the cost sequence with length $T = 20$, we compute its moving average sequence with window size 5. Attaining the
minimal value of the moving average indicates that the algorithm has found a “local optimum”, i.e., it has reached a “neighborhood” of solutions with small clustering cost. Thus, we use the number of iterations to reach such local optimum as the measure of iteration cost. The results are provided in Figure 5. We see that on all the tasks (MNIST with $l_1$ and $l_2$ distance, and graph dataset with $r = 1$ and $r = 100$), DP-HST has significantly smaller iterations cost. In Figure 6, we further report the $k$-median cost of the best solution in $T$ iterations found by each DP algorithm. We see that DP-HST again provide the smallest cost. This additional set of experiments again validates the claims of Theorem 4.3, that DP-HST is able to found better initial centers in fewer iterations.

Figure 5: Iteration cost to reach a locally optimal solution, on MNIST and graph datasets with different $k$. The demand set is an imbalanced subset of the universe.

Figure 6: The $k$-median cost of the best solution found by each differentially private algorithm. The demand set is an imbalanced subset of the universe. Same comparison holds on graph data.

B Technical Proofs

The following composition result of differential privacy will be used in our proof.

**Theorem B.1** (Composition Theorem (Dwork, 2006)). If Algorithms $A_1, A_2, \ldots, A_m$ are $\epsilon_1, \epsilon_2, \ldots, \epsilon_m$ differentially private respectively, then the union $(A_1(D), A_2(D), \ldots, A_m(D))$ is $\sum_{i=1}^{m} \epsilon_i$-DP.
B.1 Proof of Lemma 3.5

Proof. Consider the intermediate output of Algorithm 4, $C_1 = \{v_1, v_2, ..., v_k\}$, which is the set of roots of the minimal subtrees each containing exactly one output center $C_0$. Suppose one of the optimal “root set” that minimizes (4) is $C_1^* = \{v_1', v_2', ..., v_k'\}$. If $C_1 = C_1^*$, the proof is done. Thus, we prove the case for $C_1 \neq C_1^*$. Note that $T(v), v \in C_1$ are disjoint subtrees. We have the following reasoning.

- Case 1: for some $i, j, v_i$ is a descendant node of $v_j'$. Since the optimal center point $f^*$ is a leaf node by the definition of (4), we know that there must exist one child node of $v_j'$ that expands a subtree which contains $f^*$. Therefore, we can always replace $v_j'$ by one of its child nodes. Hence, we can assume that $v_i$ is not a descendant of $v_j'$.

Note that, we have $\text{score}(v_j') \leq \text{score}(v_j)$ if $v_j' \notin C_1^* \cap C_1$. Algorithm 4 sorts all the nodes based on cost value, and it would have more priority to pick $v_j'$ than $v_i$ if $\text{score}(v_j') > \text{score}(v_i)$ and $v_i$ is not a child node of $v_j'$.

- Case 2: for some $i, j$, $v_j'$ is a descendant of $v_i$. In this case, optimal center point $f^*$, which is a leaf of $T(v_i)$, must also be a leaf node of $T(v_j')$. We can simply replace $C_1$ with the swap $C_1 \setminus \{v_i\} + \{v_j'\}$ which does not change $\text{cost}_k^T(U)$. Hence, we can assume that $v_j'$ is not a descendant of $v_i$.

- Case 3: Otherwise. By the construction of $C_1$, we know that $\text{score}(v_j') \leq \min\{\text{score}(v_i), i = 1, ..., k\}$ when $v_j' \in C_1^* \setminus C_1$. Consider the swap between $C_1$ and $C_1^*$. By the definition of tree distance, we have $\text{OPT}_k^T(U) \geq \sum_{v_i \in C_1 \setminus C_1^*} N_{v_i} 2^{h_{v_i}}$, since $\{T(v_i), v_i \in C_1 \setminus C_1^*\}$ does not contain any center of the optimal solution determined by $C_1^*$ (which is also the optimal “root set” for $\text{OPT}_k^T(U)$).

Thus, we only need to consider Case 3. Let us consider the optimal clustering with center set be $C_1^* = \{c_1^*, c_2^*, ..., c_k^*\}$ (each center $c_j^*$ is a leaf of subtree whose root be $c_j''$), and $S_j'$ be the leaves assigned to $c_j'$. Let $S_j$ denote the set of leaves in $S_j'$ whose distance to $c_j''$ is strictly smaller than its distance to any centers in $C_1$. Let $P_j$ denote the union of paths between leaves of $S_j$ to its closest center in $C_1$. Let $v_j'$ be the nodes in $P_j$ with highest level satisfying $T(v_j') \cap C_1 = \emptyset$. The score of $v_j'$ is $2^{h_{v_j'}} N(v_j')$. That means the swap with a center $v_j'$ into $C_1$ can only reduce $4 \cdot 2^{h_{v_j'}} N(v_j')$ to $\text{cost}_k^T(U)$ (the tree distance between any leaf in $S_j$ and its closest center in $C_1$ is at most $4 \cdot 2^{h_{v_j'}}$).

We just use $v_j'$ to represent $v_j''$ for later part of this proof for simplicity. By our reasoning, summing all the swaps over $C_1^* \setminus C_1$ gives

$$\text{cost}_k^T(U) - \text{OPT}_k^T(U) \leq 4 \sum_{v_j' \in C_1^* \setminus C_1} N_{v_j'} 2^{h_{v_j'}}$$

Also, based on our discussion on Case 1, it holds that

$$N_{v_j'} 2^{h_{v_j'}} - N_{v_i} 2^{h_{v_i}} \leq 0.$$ 

Summing them together, we have $\text{cost}_k^T(U) \leq 5\text{OPT}_k^T(U).$ \hfill $\square$

B.2 Proof of Lemma 3.6

Proof. Since the subtrees in $C_1$ are disjoint, it suffices to consider one subtree with root $v$. With a little abuse of notation, let $\text{cost}_k^T(v, U)$ denote the optimal $k$-median cost within the point set $T(v)$ with one center in 2-HST:

$$\text{cost}_k^T(v, U) = \min_{x \in T(v)} \sum_{y \in T(v)} \rho^T(x, y),$$

(7)
which is the optimal cost within the subtree. Suppose \( v \) has more than one children \( u, w, \ldots \), otherwise the optimal center is clear. Suppose the optimal solution of \( \text{cost}^T_k(v, U) \) chooses a leaf node in \( T(u) \), and our HST initialization algorithm picks a leaf of \( T(w) \). If \( u = w \), then HST chooses the optimal one where the argument holds trivially. Thus, we consider \( u \neq w \). We have the following two observations:

- Since one needs to pick a leaf of \( T(u) \) to minimize \( \text{cost}^T_k(v, U) \), we have \( \text{cost}^T_k(v, U) \geq \sum_{x \in \text{ch}(v), x \neq u} N_x \cdot 2^{h_x} \) where \( \text{ch}(u) \) denotes the children nodes of \( u \).
- By our greedy strategy, \( \text{cost}^T_k(v, U) = \sum_{x \in \text{ch}(u)} N_x \cdot 2^{h_x} \leq \text{cost}^T_k(v, U) + N_u \cdot 2^{h_u} \).

Since \( h_u = h_w \), we have

\[
2^{h_u} \cdot (N_u - N_w) \leq 0,
\]

since our algorithm picks subtree roots with highest scores. Then we have \( \text{cost}^T_k(v, U) \leq \text{cost}^T_k(v, U) + N_u \cdot 2^{h_u} \leq 2 \text{cost}^T_k(v, U) \). Since the subtrees in \( C_1 \) are disjoint, the union of centers for \( \text{OPT}^T_k(v, U) \), \( v \in C_1 \) forms the optimal centers with size \( k \). Note that, for any data point \( p \in U \setminus C_1 \), the tree distance \( \rho^T(p, f) \) for \( \forall f \) that is a leaf node of \( T(v) \), \( v \in C_1 \) is the same. That is, the choice of leaf in \( T(v) \) as the center does not affect the \( k \)-median cost under 2-HST metric. Therefore, union bound over \( k \) subtree costs completes the proof.

### B.3 Proof of Proposition 3.2

**Proof.** It is known that the 2-HST can be constructed in \( O(dn \log n) \) (Bartal, 1996). The subtree search in Algorithm 4 involves at most sorting all the nodes in the HST based on the score, which takes \( O(n \log n) \). We use a priority queue to store the nodes in \( C_1 \). When we insert a new node \( v \) into queue, its parent node (if existing in the queue) would be removed from the queue. The number of nodes is \( O(n) \) and each operation (insertion, deletion) in a priority queue based on score has \( O(\log n) \) complexity. Lastly, the total time to obtain \( C_0 \) is \( O(n) \), as the FIND-LEAF only requires a top down scan in \( k \) disjoint subtrees of \( T \). Summing parts together proves the claim.

### B.4 Proof of Theorem 4.2

Similarly, we prove the error in general metric by first analyzing the error in 2-HST metric. Then the result follows from Lemma 3.4. Let \( \text{cost}^T_k(D) \), \( \text{cost}^{T'}_k(D) \) and \( \text{OPT}^T_k(D) \) be defined analogously to (3), (4) and (5), where “\( y \in U \)” in the summation is changed into “\( y \in D \)” since \( D \) is the demand set. That is,

\[
\text{cost}^T_k(D) = \sum_{y \in D} \min_{x \in C_0} \rho^T(x, y),
\]

\[
\text{cost}^{T'}_k(D, C_1) = \min_{|F \cap T(v)| = 1, v \in C_1} \sum_{y \in D} \min_{x \in F} \rho^T(x, y),
\]

\[
\text{OPT}^T_k(D) = \min_{F \subset D, |F| = k} \sum_{y \in D} \min_{x \in F} \rho^T(x, y) \equiv \min_{C_1'} \text{cost}^{T'}_k(D, C_1').
\]

We have the following.

**Lemma B.2.** \( \text{cost}^T_k(D) \leq 10 \text{OPT}^T_k(D) + 10cke^{-1} \Delta \log n \) with probability \( 1 - 4k/n^c \).

**Proof.** The result follows by combining the following Lemma B.4, Lemma B.5, and applying union bound.

**Lemma B.3.** For any node \( v \) in \( T \), with probability \( 1 - 1/n^c \), \( |\hat{N}_v : 2^{h_v} - N_v : 2^{h_v}| \leq cke^{-1} \Delta \log n \).

**Proof.** Since \( \hat{N}_v = N_v + \text{Lap}(2^{(L-h_v)/2}/\epsilon) \), we have

\[
Pr(|\hat{N}_v - N_v| \geq x/\epsilon) = \exp(-x/2^{(L-h_v)}).
\]
As \( L = \log \triangle \), we have
\[
Pr[|\hat{N}_v - N_v| \geq x\triangle/(2^{h_v}c)] \leq \exp(-x).
\]
Hence, for some constant \( c > 0 \),
\[
Pr[|\hat{N}_v - 2^{h_v} - N_v \cdot 2^{h_v}| \leq c\epsilon^{-1}\triangle \log n] \geq 1 - \exp(-c\log n) = 1 - 1/n^c.
\]

Lemma B.4 (DP Subtree Search). With probability \( 1 - 2k/n^c \), \( \text{cost}_k^{T'}(D) \leq 5\text{OPT}_k^{T}(D) + 4ck\epsilon^{-1}\triangle \log n \).

**Proof.** The proof is similar to that of Lemma 3.5. Consider the intermediate output of Algorithm 4, \( C_1 = \{v_1, v_2, \ldots, v_k\} \), which is the set of roots of the minimal disjoint subtrees each containing exactly one output center \( C_0 \). Suppose one of the optimal “root set” that minimizes (4) is \( C_1' = \{v'_1, v'_2, \ldots, v'_k\} \). Assume \( C_1 \neq C_1' \). By the same argument as the proof of Lemma 3.5, we consider for some \( i, j \) such that \( v_i \neq v'_j \), and \( v_i \) is not a descendent of \( v'_j \) and \( v'_j \) is either a descendent of \( v_i \). By the construction of \( C_1 \), we know that \( \text{score}(v'_j) = \min \{ \text{score}(v_i), i = 1, \ldots, k \} \) when \( v'_j \in C_1' \setminus C_1 \). Consider the swap between \( C_1 \) and \( C_1' \). By the definition of tree distance, we have
\[
\text{OPT}_k^{T'}(U) \geq \sum_{v_i \in C_1 \setminus C_1'} N_v^2 \cdot 2^{h_v}, \quad \text{since } \{ T(v_i), v_i \in C_1 \setminus C_1' \} \text{ does not contain any center of the optimal solution determined by } C_1' \text{ (which is also the optimal “root set” for } \text{OPT}_k^{T'}).\]

Let us consider the optimal clustering with center set be \( C^* = \{c^*_1, c^*_2, \ldots, c^*_k\} \) (each center \( c^*_j \) is a leaf of subtree whose root be \( c^*_j \)), and \( S'_j \) be the leaves assigned to \( c^*_j \). Let \( S_j \) denote the set of leaves whose distance to \( c^*_j \) is strictly smaller than its distance to any centers in \( C_1 \). Let \( P_j \) denote the union of paths between leaves of \( S_j \) to its closest center in \( C_1 \). Let \( v'_j \) be the nodes in \( P_j \) with highest level satisfying \( T(v'_j) \cap C_1 = \emptyset \). The score of \( v'_j \) is \( 2^{h_v}N(v'_j) \). That means the swap with a center \( v'_j \) into \( C_1 \) can only reduce \( 4 \cdot 2^{h_v}N(v'_j) \) to \( \text{cost}_k^{T'}(U) \) (the tree distance between any leaf in \( S_j \) and its closest center in \( C_1 \) is at most \( 4 \cdot 2^{h_v} \)). We just use \( v'_j \) to represent \( v''_j \) for later part of this proof for simplicity. Summing all the swaps over \( C_1' \setminus C_1 \), we obtain
\[
\text{cost}_k^{T'}(U) - \text{OPT}_k^{T}(U) \leq 4 \sum_{v'_j \in C_1' \setminus C_1} N_v^2 \cdot 2^{h_v},
\]
\[
\text{OPT}_k^{T}(U) \geq \sum_{v_i \in C_1 \setminus C_1'} N_v^2 \cdot 2^{h_v}.
\]

Applying union bound with Lemma B.3, with probability \( 1 - 2/n^c \), we have
\[
N_v^2 \cdot 2^{h_v} - N_v^2 \cdot 2^{h_v} \leq 2ck\epsilon^{-1}\triangle \log n.
\]

Consequently, we have with probability, \( 1 - 2k/n^c \),
\[
\text{cost}_k^{T'}(D) \leq 5\text{OPT}_k^{T}(D) + 4ck|C_1 \setminus C_1'|\epsilon^{-1}\triangle \log n \leq 5\text{OPT}_k^{T}(D) + 4ck\epsilon^{-1}\triangle \log n,
\]
which proves the claim. \( \square \)

Lemma B.5 (DP Leaf Search). With probability \( 1 - 2k/n^c \), Algorithm 6 produces initial centers with \( \text{cost}_k^{T}(D) \leq 2\text{cost}_k^{T'}(D) + 2ck\epsilon^{-1}\triangle \log n \).

**Proof.** The proof strategy follows Lemma 3.6. We first consider one subtree with root \( v \). Let \( \text{cost}_k^{T'}(v, U) \) denote the optimal \( k \)-median cost within the point set \( T(v) \) with one center in 2-HST:
\[
\text{cost}_k^{T'}(v, D) = \min_{x \in T(v)} \sum_{y \in T(v) \cap D} \rho^T(x, y).
\]

Suppose \( v \) has more than one children \( u, w, \ldots \), and the optimal solution of \( \text{cost}_k^{T'}(v, U) \) chooses a leaf node in \( T(u) \), and our HST initialization algorithm picks a leaf of \( T(w) \). If \( u = w \), then HST chooses the optimal one where the argument holds trivially. Thus, we consider \( u \neq w \). We have the following two observations:
• Since one needs to pick a leaf of $T(u)$ to minimize $cost^T_1(v, U)$, we have $cost^T_1(v, U) \geq \sum_{x \in ch(u), x \neq u} N_x \cdot 2^{h_x}$ where $ch(u)$ denotes the children nodes of $u$.

• By our greedy strategy, $cost^T_1(v, U) \leq \sum_{x \in ch(u)} N_x \cdot 2^{h_x} \leq cost^T_1(v, U) + N_u \cdot 2^{h_u}$.

As $h_u = h_w$, leveraging Lemma B.3, with probability $1 - 2/n^c$, 
$$2^{h_u} \cdot (N_u - N_w) \leq 2^{h_u}(\hat{N}_u - \hat{N}_w) + 2c\epsilon^{-1}\triangle \log n \leq 2c\epsilon^{-1}\triangle \log n,$$

since our algorithm picks subtree roots with highest scores. Then we have $cost^T_1(v, D) \leq cost^T_1(v, D) + N_u \cdot 2^{h_u} + 2c\epsilon^{-1}\triangle \log n \leq 2cost^T_1(v, D) + 2c\epsilon^{-1}\triangle \log n$ with high probability. Lastly, applying union bound over the disjoint $k$ subtrees gives the desired result.

B.5 Proof of Theorem 4.3

Proof. The privacy analysis is straightforward, by using the composition theorem (Theorem B.1). Since the sensitivity of $cost(\cdot)$ is $\triangle$, in each swap iteration the privacy budget is $\epsilon/(T+1)$. Also, we spend another $\epsilon/(T+1)$ privacy for picking a output. Hence, the total privacy is $\epsilon/2$ for local search. Algorithm 6 takes $\epsilon/2$ DP budget for initialization, so the total privacy is $\epsilon$.

The analysis of the approximation error follows from Gupta et al. (2010), where the initial cost is reduced by our private HST method. We need the following two lemmas.

Lemma B.6 (Gupta et al. (2010)). Assume the solution to the optimal utility is unique. For any output $o \in O$ of $2\epsilon$-DP exponential mechanism on dataset $D$, it holds for $\forall t > 0$ that
$$Pr[q(D, o) \leq \max_{o \in O} q(D, o) - (\ln |O| + t)/\epsilon] \leq e^{-t},$$
where $|O|$ is the size of the output set.

Lemma B.7 (Arya et al. (2004)). For any set $F \subseteq D$ with $|F| = k$, there exists some swap $(x, y)$ such that the local search method admits
$$cost_k(F, D) - cost_k(F - \{x\} + \{y\}, D) \geq \frac{cost_k(F, D) - 5OPT(D)}{k}.$$

From Lemma B.7, we know that when $cost_k(F_1, D) > 6OPT(D)$, there exists a swap $(x, y)$ s.t.
$$cost_k(F_1 - \{x\} + \{y\}, D) \leq (1 - \frac{1}{6k})cost_k(F_1, D).$$

At each iteration, there are at most $n^2$ possible outputs (i.e., possible swaps), i.e., $|O| = n^2$. Using Lemma B.6 with $t = 2\log n$, for $\forall i$,
$$Pr[\text{cost}_k(F_{i+1}, D) \geq \text{cost}_k(F_{i+1}^*, D) + \frac{\log n}{\epsilon}] \geq 1 - 1/n^2,$$
where $\text{cost}_k(F_{i+1}^*, D)$ is the minimum cost among iteration $1, 2, \ldots, t + 1$. Hence, we have that as long as $cost(F_1, D) > 6OPT(D) + \frac{24k\log n}{\epsilon}$, the improvement in cost is at least by a factor of $(1 - \frac{1}{6k})$. By Theorem 4.2, we have $cost_k(F_1, D) \leq C(\log n)(6OPT(D) + 6k\triangle \log n/\epsilon)$ for some constant $C > 0$. Let $T = 6Ck\log n$. We have that
$$E[\text{cost}(F_1, D)] \leq (6OPT(D) + 6k\epsilon^{-1}\triangle \log n)C(\log n)(1 - 1/6k)^{6Ck\log n}$$
$$\leq 6OPT(D) + 6k\epsilon^{-1}\triangle \log n \leq 6OPT(D) + \frac{24k\log n}{\epsilon}.$$ 

Therefore, with probability at least $(1 - T/n^2)$, there exists an $i \leq T$ s.t. $cost(F_i, D) \leq 6OPT(D) + \frac{24k\log n}{\epsilon}$. Then by using the Lemma B.7, one will pick an $F_j$ with additional additive error $4\ln n/\epsilon'$ to the min$\{\text{cost}(F_j, D), j = 1, 2, \ldots, T\}$ with probability $1 - 1/n^2$. Consequently, we know that the expected additive error is
$$24k\triangle \log n/\epsilon' + 4\log n/\epsilon' = O(\epsilon^{-1}k^2\triangle(\log \log n)\log n),$$
with probability $1 - 1/poly(n)$.

\qed
C  Extend HST Initialization to $k$-Means

Naturally, our HST method can also be applied to $k$-means clustering problem. In this section, we extend the HST to $k$-means and provide some brief analysis similar to $k$-median. We present the analysis in the non-private case, which can then be easily adapted to the private case. Define the following costs for $k$-means.

$$cost^{T}_{km}(U) = \sum_{y \in U} \min_{x \in C} \rho^{T}(x, y)^{2}, \tag{12}$$

$$cost^{T}_{km}(U, C_{1}) = \min_{F: |F| = 1, N_{v} \in C_{1}} \sum_{y \in U} \min_{x \in F} \rho^{T}(x, y)^{2}, \tag{13}$$

$$OPT^{T}_{km}(U) = \min_{F: |F| = k} \sum_{y \in U} \min_{x \in F} \rho^{T}(x, y)^{2} \equiv \min_{C_{1}} cost^{T}_{km}(U, C_{1}). \tag{14}$$

For simplicity, we will use $cost^{T}_{km}(U)$ to denote $cost^{T}_{km}(U, C_{1})$ if everything is clear from context. Here, $OPT^{T}_{km}$ (14) is the cost of the global optimal solution with 2-HST metric.

**Lemma C.1** (Subtree search). $cost^{T}_{km}(U) \leq 17OPT^{T}_{km}(U)$.

**Proof.** The analysis is similar with the proof of Lemma 3.5. Thus, we mainly highlight the difference. Let us just use some notations the same as in Lemma 3.5 here. Let us consider the clustering with center set be $C^{*} = \{c_{1}^{*}, c_{2}^{*}, ..., c_{k}^{*}\}$ (each center $c_{j}^{*}$ is a leaf of subtree whose root be $c_{j}'$), and $S_{j}'$ be the leaves assigned to $c_{j}^{*}$ in optimal $k$-means clustering in tree metric. Let $S_{j}$ denote the set of leaves in $S_{j}'$ whose distance to $c_{j}^{*}$ is strictly smaller than its distance to any centers in $C_{1}$. Let $P_{j}$ denote the union of paths between leaves of $S_{j}$ to its closest center in $C_{1}$. Let $v''_{j}$ be the nodes in $P_{j}$ with highest level satisfying $T(v''_{j}) \cap C_{1} = \emptyset$. The score of $v''_{j}$ is $2^{h_{v''_{j}}}N(v''_{j})$. That means the swap with a center $v''_{j}$ into $C_{1}$ can only reduce $(4 \cdot 2^{h_{v''_{j}}})^{2}N(v''_{j})$ to $cost^{T}_{km}(U)$. We just use $v'_{j}$ to represent $v''_{j}$ for later part of this proof for simplicity. By our reasoning, summing all the swaps over $C_{1} \setminus C_{1}$ gives

$$cost^{T}_{km}(U) - OPT^{T}_{km}(U) \leq \sum_{v'_{j} \in C_{1} \setminus C_{1}} N_{v'_{j}} \cdot (4 \cdot 2^{h_{v'_{j}}})^{2},$$

$$OPT^{T}_{km}(U) \geq \sum_{v_{j} \in C_{1} \setminus C_{1}} N_{v_{j}} (2^{h_{v_{j}}})^{2}.$$  

Also, based on our discussion on Case 1, it holds that

$$N_{v'_{j}} 2^{h_{v'_{j}}} - N_{v_{j}} 2^{h_{v_{j}}} \leq 0.$$  

Summing them together, we have $cost^{T}_{km}(U) \leq 17OPT^{T}_{km}(U)$.  

Next, we show that the greedy leaf search strategy (Algorithm 5) only leads to an extra multiplicative error of 2.

**Lemma C.2** (Leaf search). $cost^{T}_{km}(U) \leq 2cost^{T}_{km}(U)$.

**Proof.** Since the subtrees in $C_{1}$ are disjoint, it suffices to consider one subtree with root $v$. With a little abuse of notation, let $cost^{T}_{1}(v, U)$ denote the optimal $k$-means cost within the point set $T(v)$ with one center in 2-HST:

$$cost^{T}_{1}(v, U) = \min_{x \in T(v)} \sum_{y \in T(v)} \rho^{T}(x, y)^{2}, \tag{15}$$

which is the optimal cost within the subtree. Suppose $v$ has more than one children $u, w, ...$, otherwise the optimal center is clear. Suppose the optimal solution of $cost^{T}_{1}(v, U)$ chooses a leaf node in $T(u)$, and our HST initialization algorithm picks a leaf of $T(w)$. If $u = w$, then HST chooses the optimal one where the argument holds trivially. Thus, we consider $u \neq w$. We have the following two observations:
• Since one needs to pick a leaf of $T(u)$ to minimize $\text{cost}_{T'}^1(v, U)$, we have $\text{cost}_{T'}^1(v, U) \geq \sum_{x \in \text{ch}(v), x \neq u} N_x \cdot (2^{h_x})^2$ where $\text{ch}(u)$ denotes the children nodes of $u$.

• By our greedy strategy, $\text{cost}_{T}^1(v, U) \leq \sum_{x \in \text{ch}(u)} N_x \cdot (2^{h_x})^2 \leq \text{cost}_{T'}^1(v, U) + N_u \cdot (2^{h_u})^2$.

Since $h_u = h_w$, we have

$$2^{h_u} \cdot (N_u - N_w) \leq 0,$$

since our algorithm picks subtree roots with highest scores. Then we have $\text{cost}_{T}^1(v, U) \leq \text{cost}_{T'}^1(v, U) + N_w \cdot (2^{h_w})^2 \leq 2 \text{cost}_{T'}^1(v, U)$. Since the subtrees in $C_1$ are disjoint, the union of centers for $\text{OPT}_{T}^1(v, U)$, $v \in C_1$ forms the optimal centers with size $k$. Note that, for any data point $p \in U \setminus C_1$, the tree distance $\rho_T^T(p, f)$ for $\forall f$ that is a leaf node of $T(v)$, $v \in C_1$ is the same. That is, the choice of leaf in $T(v)$ as the center does not affect the $k$-median cost under 2-HST metric. Therefore, union bound over $k$ subtree costs completes the proof.

We are ready to state the error bound for our proposed HST initialization (Algorithm 4), which is a natural combination of Lemma C.1 and Lemma C.2.

**Theorem C.3** (HST initialization). $\text{cost}_{km}^T(U) \leq 3 \text{OPT}_{km}^T(U)$.

We have the following result based on Lemma 3.4.

**Theorem C.4.** In a general metric space,

$$E[\text{cost}_{km}(U)] = O(\min\{\log n, \log \triangle\})^2 \text{OPT}_{km}(U).$$