A REMARK ON THE HALF WAVE SCHRÖDINGER EQUATION IN THE ENERGY SPACE

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Abstract. We investigate the Cauchy problem for the half wave Schrödinger equation in the energy space. We derive the local well-posedness in the energy space for the odd power type nonlinearities under certain additional assumption for the initial data, namely $\hat{u}_0 \in L^1_{\xi, \eta}(\mathbb{R}^2)$.

1. Introduction

We study the Cauchy problem for the following equation:

$$i\partial_t u + \partial_x^2 u - |D_y|u = \mu|u|^{p-1}u, \quad (t, x, y) \in [-T, T] \times \mathbb{R}^2,$$

$$u(0, x, y) = u_0(x, y) \in H^{s_1, s_2}(\mathbb{R}^2),$$

where $|D_y| = (-\partial_y^2)^{\frac{1}{2}}, \mu = \pm1, p > 1, T > 0$ and $s_1, s_2 \in \mathbb{R}$. Also, we define the anisotropic Sobolev spaces $H^{s_1, s_2}(\mathbb{R}^2)$ as

$$H^{s_1, s_2}(\mathbb{R}^2) = \{f \in S'(\mathbb{R}^2) : \|f\|_{H^{s_1, s_2}} < \infty\},$$

$$\|f\|_{H^{s_1, s_2}} := \left(\int_{\mathbb{R}^2} \langle \xi \rangle^{2s_1} \langle \eta \rangle^{2s_2} |\hat{f}(\xi, \eta)|^2 \, d\xi \, d\eta\right)^\frac{1}{2},$$

with $\mu = 1, p = 3$ is firstly considered by Xu [5] in the analysis of large time behavior of the solution for smooth data. After [5], Bahri, Ibrahim and Kikuchi [1] obtained the local well-posedness for rough data, namely $s_1 = 0, (1 > s_2 > \frac{1}{2}$ and $1 < p \leq 5$ by the fixed point argument. [1] has the following conservation laws (the mass $M(u)$ and the energy $E(u)$):

$$M(u) = \int_{\mathbb{R}^2} |u|^2 \, dx \, dy,$$

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^2} (|\partial_x u|^2 + |D_y u \cdot \bar{u}|) \, dx \, dy - \frac{\mu}{p + 1} \int_{\mathbb{R}^2} |u|^{p+1} \, dx \, dy.$$

Hence the energy space $E$ for (1.1) lies in $H^{1,0}(\mathbb{R}^2) \cap H^{0,\frac{1}{2}}(\mathbb{R}^2)$ equipped with the norm

$$\|u\|_E := \left(\|\partial_x u\|_{L^2}^2 + \|D_y \frac{1}{2} u\|_{L^2}^2 + \|u\|_{L^2}^2\right)^{\frac{1}{2}}.$$

The well-posedness in $E$ is unknown. In Proposition 5.4 [1], the Strichartz estimates require $s_2 > \frac{1}{2}$, hence we cannot apply it for the problem. Moreover, we cannot apply the Yudovich argument directly to prove uniqueness for (1.1). If we apply the argument, we need to show $\|u\|_{L^q_{t,x,y}} < \infty$ for $q$ large enough. However, the Gagliardo-Nirenberg inequality shows that it only holds for $(2 < q < 6$. These are the main obstacles to obtain well-posedness in $E$.

In this paper, we verify the following local well-posedness result in $E = H^{1,0}(\mathbb{R}^2) \cap H^{0,\frac{1}{2}}(\mathbb{R}^2)$ under additional assumption on the initial data.

Key words and phrases. Cauchy problem, well-posedness, energy space.
The Duhamel term is estimated as follows. Since

\[ u \in C([-T,T]; E) \cap L^\infty_{T,x,y} \quad \text{and} \quad \dot{u} \in L^\infty_T L^1_{\xi,\eta}, \]

\( \dot{u}_0 \in L^1_{\xi,\eta}(\mathbb{R}^2) \) in Theorem 1.1 seems to be somewhat extra assumption. However in order to control \( \|u\|_{L^r_{T,x,y}} \), we need this assumption. Also, we suppose the power \( p \) is odd to estimate the Duhamel term, see section 2 for details. We remark that in Theorem 1.1 \( \|u\|_{L^r_{T,x,y}} \), the norm inflation (ill-posedness) holds in \( E \) for \( p > 5 \), however in our Theorem 1.1, if we additionally suppose \( \dot{u}_0 \in L^1_{\xi,\eta}(\mathbb{R}^2) \), then (local) well-posedness in \( E \) holds even if \( p > 5 \) provided that \( p \) is odd.

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. For a Banach space \( X \) and \( r > 0 \), we define \( B_r(X) := \{ f \in X ; \| f \|_X \leq r \} \). Throughout the paper, \( \hat{\cdot} \) denotes the Fourier transform with respect to spatial variables \( x \) and \( y \).

**Proof of Theorem 1.1** We prove the local well-posedness for (1.1) with initial data \( u_0 \in E \) and \( \dot{u}_0 \in L^1_{\xi,\eta}(\mathbb{R}^2) \) by the fixed point argument. By the Duhamel formula,

\[
\Phi(u) = S(t)u_0 - i\mu \int_0^t S(t-\tau)(|u|^{p-1}u)(\tau) \, d\tau,
\]

where \( S(t) := \exp \left\{ it(\partial_x^2 - |D_y|^2) \right\} \) be the \( L^2_{x,y} \) unitary operator for (1.1). From (2.1),

\[
\widehat{\Phi(u)}(\xi,\eta) = e^{-it(\xi^2 + |\eta|^2)}\hat{u}_0 - i\mu \int_0^t e^{-(t-\tau)(\xi^2 + |\eta|^2)}|\hat{u}|^{p-1}\hat{u}(\tau) \, d\tau.
\]

Let \( Y := \{ u \in C([-T,T]; E) \cap L^\infty_{T,x,y} ; \hat{\dot{u}} \in L^\infty_T L^1_{\xi,\eta} \text{ and } \| u \|_Y < \infty \} \) endowed with the norm

\[ \| u \|_Y := \| u \|_{L^\infty_T E} + \| u \|_{L^\infty_{T,x,y}} + \| \hat{\dot{u}} \|_{L^\infty_T L^1_{\xi,\eta}}. \]

Let us verify \( \Phi \) is a contraction map in \( Y \). Firstly, we show \( \Phi \) is a map in \( Y \). Suppose that \( u_0 \in B_\delta(E), \dot{u}_0 \in B_\delta(L^1_{\xi,\eta}) \) and \( u \in B_r(Y) \). Then by \( S(t)u_0 = e^{-it(\xi^2 + |\eta|^2)}\hat{u}_0 \), it is clear that

\[
\| S(t)u_0 \|_Y = \| S(t)u_0 \|_{L^\infty_T E} + \| S(t)u_0 \|_{L^\infty_{T,x,y}} + \| S(t)u_0 \|_{L^\infty_T L^1_{\xi,\eta}} \leq \| u_0 \|_E + 2\| \dot{u}_0 \|_{L^1_{\xi,\eta}} \leq 3\delta.
\]

The Duhamel term is estimated as follows. Since \( S \) is the unitary operator in \( L^2_{x,y}(\mathbb{R}^2) \) and \( p = 2k + 1, k \in \mathbb{N} \), we obtain

\[
\left\| \int_0^t S(t-\tau)(|u|^{p-1}u)(\tau) \, d\tau \right\|_{L^\infty_T E} \leq \int_0^T \left\| S(t-\tau)(|u|^{p-1}u)(\tau) \right\|_{L^\infty_T E} \, d\tau \leq CT\| u \|^{p-1}_E \leq CT\| u \|^{p-1}_{L^\infty_{T,x,y}} \leq CT\| u \|^{p-1}_{L^\infty_T L^1_{\xi,\eta}} \leq CT\| u \|^{p-1}_{L^\infty_T L^1_{\xi,\eta}} \leq CTp.
\]

By \( p = 2k + 1, k \in \mathbb{N} \) and the Young inequality, we have

\[
\left\| \int_0^t S(t-\tau)(|u|^{p-1}u)(\tau) \, d\tau \right\|_{L^\infty_{T,x,y}} \leq CT\| \hat{u} \|^{p-1}_{L^\infty_T L^1_{\xi,\eta}} \leq CT\| \hat{u} \|^{p-1}_{L^\infty_T L^1_{\xi,\eta}} \leq CTp.
\]
Again by $p = 2k + 1, k \in \mathbb{N}$ and the Young inequality lead
\[
\left\| \int_0^t e^{-i(t-\tau)(\xi^2+|\eta|^2)} |u|^p-1 u(\tau) \, d\tau \right\|_{L_x^p L_t^\infty} \leq CT \|u|^{p-1}u\|_{L_x^p L_t^1} \leq CTr^p. \tag{2.5}
\]

From (2.2)–(2.5), if we take $\delta, T > 0$ such that $3\delta \leq \frac{1}{2}r$ and $3CTr^p \leq \frac{1}{2}r$, then $\Phi$ is a map in $Y$.

Next, we show the contraction of $\Phi$. Set $u, v \in B_r(Y)$. Then from $p = 2k + 1, k \in \mathbb{N}$, we have
\[
\left\| \int_0^t S(t-\tau)(|u|^{p-1}u - |v|^{p-1}v)(\tau) \, d\tau \right\|_{L_x^p L_t^\infty} \leq CT\|u|^{2k}u - |v|^{2k}v\|_{L_x^p L_t^1}.
\]
By induction, we easily check
\[
\|u|^{2k}u - |v|^{2k}v\|_{L_x^p L_t^\infty} \leq (2k + 1)^2 r^{2k}\|u - v\|_{L_x^p L_t^\infty} \cap L_x^p L_t^{1, p}.
\]
Hence we obtain
\[
\left\| \int_0^t S(t-\tau)(|u|^{p-1}u - |v|^{p-1}v)(\tau) \, d\tau \right\|_{L_x^p L_t^\infty} \leq CT(2k + 1)^2 r^{2k}\|u - v\|_{L_x^p L_t^\infty} \cap L_x^p L_t^{1, p} \leq CT(2k + 1)^2 r^{2k}\|u - v\|_Y. \tag{2.6}
\]

From $p = 2k + 1, k \in \mathbb{N}$, we have
\[
\left\| \int_0^t S(t-\tau)(|u|^{p-1}u - |v|^{p-1}v)(\tau) \, d\tau \right\|_{L_x^p L_t^{1, p}} \leq CT\|F_{x,y}\| u|^{2k}u - |v|^{2k}v\|_{L_x^p L_t^{1, p}}. \tag{2.7}
\]
By induction and the Young inequality, we see
\[
\|F_{x,y}\| u|^{2k}u - |v|^{2k}v\|_{L_x^p L_t^{1, p}} \leq (2k + 1)^2 r^{2k}\|\hat{u} - \hat{v}\|_{L_x^p L_t^{1, p}}. \tag{2.8}
\]
From (2.7) and (2.8), we have
\[
\left\| \int_0^t S(t-\tau)(|u|^{p-1}u - |v|^{p-1}v)(\tau) \, d\tau \right\|_{L_x^p L_t^{1, p}} \leq CT(2k + 1)^2 r^{2k}\|u - v\|_Y. \tag{2.9}
\]
From (2.7) and (2.8), we also obtain
\[
\left\| \int_0^t e^{-i(t-\tau)(\xi^2+|\eta|^2)} F_{x,y}[u|^{p-1}u - |v|^{p-1}v](\tau) \, d\tau \right\|_{L_x^p L_t^{1, p}} \leq CT\|F_{x,y}\| u|^{p-1}u - |v|^{p-1}v\|_{L_x^p L_t^{1, p}} \leq CT(2k + 1)^2 r^{2k}\|u - v\|_Y. \tag{2.10}
\]
Collecting (2.10), (2.8), (2.10) and taking $T > 0$ such that $3CT(2k + 1)^2 r^{2k} = \frac{1}{2}$, then
\[
\left\| \int_0^t S(t-\tau)(|u|^{p-1}u - |v|^{p-1}v)(\tau) \, d\tau \right\|_Y \leq \frac{1}{2}\|u - v\|_Y.
\]
Therefore $\Phi : Y \to Y$ is a contraction map. Thus by the fixed point argument, we have the desired result.

\[\square\]

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