Comparing the Selmer group of a $p$-adic representation and the Selmer group of the Tate dual of the representation

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Abstract
For a given “ordinary” $p$-adic representation, we compare its Selmer group with the Selmer group of its Tate dual over an admissible $p$-adic Lie extension. Namely, we show that the generalized Iwasawa $\mu$-invariants associated to the Pontryagin dual of the two said Selmer groups are the same.

Keywords and Phrases: Strict Selmer groups, Greenberg Selmer groups, generalized $\mu$-invariant, admissible $p$-adic Lie extensions, $\mathfrak{M}_H(G)$.

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1 Introduction
The main conjecture of Iwasawa theory is a conjecture on the relation between a Selmer group, which is a module over an (not necessarily commutative) Iwasawa algebra, and a conjectural $p$-adic $L$-function. This $p$-adic $L$-function is expected to satisfy a conjectural functional equation in a certain sense. In [FK], Fukaya and Kato were able to construct such an $p$-adic $L$-function and established a corresponding functional equation for the said $p$-adic $L$-function assuming the (local and global) noncommutative Tamagawa number conjecture. In view of the main conjecture and this functional equation, one would expect to have certain algebraic relationship between the Selmer group attached to a Galois representation and the Selmer group attached to the Tate twist of the dual of the Galois representation which can be thought as an algebraic manifestation of the functional equation. It is precisely a component of this algebraic relationship that this paper aims to investigate. For a module over an Iwasawa algebra, Howson and Venjakob independently developed the notion of a generalized $\mu$-variant which extends the classical $\mu$-invariant. The main conjecture predicts that this invariant attached to a suitable Selmer group should contribute a certain power of $p$ occurring in the leading term of the conjectural $p$-adic $L$-function. In this article, we will show that the Selmer group attached to a Galois representation and the Selmer group attached to the Tate twist of the dual representation have the same generalized $\mu$-variant.

In the case of a cyclotomic $\mathbb{Z}_p$-extension, this study on the $\mu$-invariants has been undertaken in [Gr, Mat]. (Actually, in [Gr], Greenberg also established the full “algebraic” functional equation of the

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Selmer groups, which we will not treat in this article.) There he obtained the equality of the \( \mu \)-invariants by combining a local-global Euler characteristic argument with the asymptotic formulas arising from the structure theory of \( \mathbb{Z}_p[\Gamma] \)-modules, where \( \Gamma \) is the Galois group of the cyclotomic \( \mathbb{Z}_p \)-extension. When the \( p \)-adic Lie extension is a multiple \( \mathbb{Z}_p \)-extension, we adopt the approach of Greenberg, and in our situation, we will need to combine the local-global Euler characteristic argument with the asymptotic formula of Cucuo and Monsky \cite{CM,Mon} which allow us to deal with \( \mathbb{Z}_p[G] \)-modules for \( G \cong \mathbb{Z}_p^r \). We also mention that since there are infinite decomposition of primes in a \( \mathbb{Z}_p^r \)-extension when \( r \geq 2 \), we will need a slightly more careful argument.

When the \( p \)-adic Lie extension is not commutative, the above approach breaks down, as one does not have an asymptotic formula as in the commutative situation. However, if we assume further that our Selmer group satisfies a certain torsion property, then we can establish the equality of the \( \mu \)-invariants via a different (and a rather indirect) approach. Namely, under the assumption of the above said torsion property (with some other assumptions, see Theorem \ref{thm:main} for details), we show that the \( \mu \)-invariant of the Selmer group over the \( p \)-adic extension coincides with \( \mu \)-invariant of the said Selmer group over the cyclotomic \( \mathbb{Z}_p \)-extension. This in turn allows us to deduce the equality of the \( \mu \)-invariants in the noncommutative setting from the cyclotomic \( \mathbb{Z}_p \)-extension case.

We like to emphasize that although this paper is highly motivated by the main conjecture of Iwasawa and the functional equation of the conjectural \( p \)-adic \( L \)-function, we do not assume these conjectures (other than the torsion property on the Selmer groups in the noncommutative \( p \)-adic Lie extension situation) in all our argument.

We now give a brief description of the layout of the paper. In Section 2 we introduce the generalized \( \mu \)-invariant of Howson and Venjakob, and record certain estimates on the order of certain cohomology groups. In Section 3 we recalled a ratio formula of Greenberg for the strict Selmer group of the dual of a finite module and the strict Selmer group of the Tate twist of the Pontryagin dual of the finite module. We will combine this ratio formula with the asymptotic formulas of Cucuo-Monsky to equate the \( \mu \)-invariant of the strict Selmer group of a torsion Galois module and the \( \mu \)-invariant of the strict Selmer group of the Tate twist of its dual over a \( \mathbb{Z}_p^r \)-extension in Section 4. In Section 5 we carry out the study over a general noncommutative \( p \)-adic Lie extension. In Section 6 we compare the strict Selmer groups of the Artin twists of the representations. In Section 7 we compare the strict Selmer groups with another Selmer groups of Greenberg and the Selmer complexes. Via these comparison, we see that the conclusion in the Greenberg strict Selmer groups can be carried over to these groups and complexes. In Section 8 we finally discuss some examples of Galois representations where our results can be applied.

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2 Algebraic Preliminaries

In this section, we recall some algebraic preliminaries that will be required in the later part of the paper.

Fix a prime $p$. Let $O$ be the ring of integers of some finite extension $K$ of $\mathbb{Q}_p$. We fix a local parameter $\pi$ for $O$ and denote the residue field of $O$ by $k$. Let $G$ be a compact pro-$p$ $p$-adic Lie group without $p$-torsion. It is well known that $O[G]$ is an Auslander regular ring (cf. [V, Theorems 3.26]). Furthermore, the ring $O[G]$ has no zero divisors (cf. [Neu]), and therefore, admits a skew field $Q(G)$ which is flat over $O[G]$ (see [GW, Chapters 6 and 10] or [Lam, Chapter 4, §9 and §10]). If $M$ is a finitely generated $O[G]$-module, we define the $O[G]$-rank of $M$ to be

$$\text{rank}_{O[G]} M = \dim_{Q(G)} Q(G) \otimes_{O[G]} M.$$ 

For a general compact $p$-adic Lie group $G$, we follow [CH] and extend the definition of the $O[G]$-rank by the formula

$$\text{rank}_{O[G]} N = \frac{\text{rank}_{O[G_0]} N}{[G : G_0]},$$

where $G_0$ is an open normal uniform pro-$p$ subgroup of $G$. 

Lemma 2.1. The above definition for $O[G]$-rank is independent of the choice of $G_0$.

Proof. Let $G_1$ be another open normal uniform pro-$p$ subgroup of $G$. Since $G_0 \cap G_1$ is also an open normal uniform pro-$p$ subgroup of $G$, one is therefore reduced to proving the equality

$$\text{rank}_{O[G]} M = [G : G_1] \text{rank}_{O[G_0]} M$$

whenever $G_1 \subseteq G_0$. Fix a finite free $O[G_0]$-resolution

$$0 \rightarrow O[G_0]^{n_d} \rightarrow \cdots \rightarrow O[G_0]^{n_0} \rightarrow M \rightarrow 0$$

of $M$. Then the groups $H_i(G_0, M)$ can be computed by the homology of the complex

$$O^{n_d} \rightarrow \cdots \rightarrow O^{n_0},$$

and consequently, we obtain

$$\sum_{i=0}^{d} (-1)^i \text{rank}_O H_i(G_0, M) = \sum_{i=0}^{d} (-1)^i n_i.$$ 

On the other hand, the above $O[G_0]$-free resolution is also a $O[G_1]$-free resolution for $M$. Therefore, the groups $H_i(G_1, M)$ can be computed by the homology of the complex

$$O[G_0 : G_1]^{n_d} \rightarrow \cdots \rightarrow O[G_0 : G_1]^{n_0}$$

which gives

$$\sum_{i=0}^{d} (-1)^i \text{rank}_R H_i(G_1, M) = [G_0 : G_1] \sum_{i=0}^{d} (-1)^i n_i$$

$$= [G_0 : G_1] \sum_{i=0}^{d} (-1)^i \text{rank}_O H_i(G_0, M)$$
Applying a formula of Howson (cf. [Ho, Theorem 1.1]), we have
\[ \operatorname{rank}_{\mathcal{O}[G]} M = [G_0 : G_1] \operatorname{rank}_{\mathcal{O}[G_0]} M \]
as required.

Note that the \( \mathcal{O}[G] \)-rank needs not be an integer in general. We will say that a \( \mathcal{O}[G] \)-module \( M \) is torsion if \( \operatorname{rank}_{\mathcal{O}[G]} M = 0 \).

Now suppose that \( N \) is a \( k[G] \)-module. We then define its \( k[G] \)-rank by
\[ \operatorname{rank}_{k[G]} N = \frac{\operatorname{rank}_{k[G_0]} N}{[G : G_0]}, \]
where \( G_0 \) is an open normal uniform pro-\( p \) subgroup of \( G \). By a similar argument as above, one can show that this definition is independent of the choice of \( G_0 \) (see also [Ho, Proposition 1.6]). Similarly, we will say that that the module \( N \) is a torsion \( k[G] \)-module if \( \operatorname{rank}_{k[G]} N = 0 \).

For a given finitely generated \( \mathcal{O}[G] \)-module \( M \), we denote \( M(\pi) \) to be the \( \mathcal{O}[G] \)-submodule of \( M \) which consists of elements of \( M \) that are annihilated by some power of \( \pi \). Since the ring \( \mathcal{O}[G] \) is Noetherian, the module \( M(\pi) \) is finitely generated over \( \mathcal{O}[G] \). Therefore, one can find an integer \( r \geq 0 \) such that \( \pi^r \) annihilates \( M(\pi) \). Following [Ho, Formula (33)], we define
\[ \mu_{\mathcal{O}[G]}(M) = \sum_{i \geq 0} \operatorname{rank}_{k[G]} (\pi^i M(\pi)/\pi^{i+1}). \]
(For another alternative, but equivalent, definition, see [V, Definition 3.32].) By the above discussion and our definition of \( k[G] \)-rank, the sum on the right is a finite one. It is clear from the definition that \( \mu_{\mathcal{O}[G]}(M) = \mu_{\mathcal{O}[G]}(M(\pi)) \).

Also, it is not difficult to see that this definition coincides with the classical notion of the \( \mu \)-invariant for \( \Gamma \)-modules when \( G = \Gamma \cong \mathbb{Z}_p \). We now record certain properties of this invariant which will be required in the subsequent of the paper.

**Lemma 2.2.** Let \( G \) be a compact \( p \)-adic Lie group and let \( M \) be a finitely generated \( \mathcal{O}[G] \)-module. Then we have the following statements.

(a) If \( G_0 \) is an open subgroup of \( G \), then
\[ [G : G_0] \mu_{\mathcal{O}[G]}(M) = \mu_{\mathcal{O}[G_0]}(M). \]

(b) Let \( \mathcal{O}' \) denote the ring of integers of a finite extension \( K' \) of \( K \). Then we have
\[ \mu_{\mathcal{O}'[G]}(M \otimes_{\mathcal{O}} \mathcal{O}') = \mu_{\mathcal{O}[G]}(M). \]

(c) Viewing \( M \) as a \( \mathbb{Z}_p[G] \)-module, we have
\[ [K : \mathbb{Q}_p] \operatorname{rank}_{\mathcal{O}[G]}(M) = \operatorname{rank}_{\mathbb{Z}_p[G]}(M). \]

(d) Viewing \( M \) as a \( \mathbb{Z}_p[G] \)-module, we have
\[ [k : \mathbb{F}_p] \mu_{\mathcal{O}[G]}(M) = \mu_{\mathbb{Z}_p[G]}(M). \]
(e) Suppose further that $G \cong \mathbb{Z}_p^r$ for $r \geq 1$. Then one has
\[
\mu_{\mathcal{O}[G]}(M/\pi^n) = n \text{rank}_{\mathcal{O}[G]}(M) + \mu_{\mathcal{O}[G]}(M) \quad \text{for } n \gg 0.
\]

Proof. (a), (b) and (c) are immediate from the definition. To prove (d), it suffices, by (a), to consider the case when $G$ is a uniform pro-$p$ group. By [Ho, Corollary 1.7], one has
\[
\mu_{\mathcal{O}[G]}(M) = \sum_{i \geq 0} (-1)^i \text{ord}_q \left(H_i(G, M(\pi))\right),
\]
where $q$ is the order of $k$. (d) is now immediate from this formula and the facts that $M(\pi) = M(p)$ and that $[k : \mathbb{F}_p] \text{ord}_q N = \text{ord}_p N$.

To prove (e), we first consider the case when $M$ is a torsion $\mathcal{O}[G]$-module. By the structure theory of torsion $\mathcal{O}[G]$-module, we have that $M$ is pseudo-isomorphic to
\[
\bigoplus_{i=1}^s \mathcal{O}[G]/\pi^{\alpha_i} \oplus \bigoplus_{j=1}^t \mathcal{O}[G]/f_j
\]
for some nonzero $f_j$ coprime to $\pi$. For $n \geq \max\{\alpha_1, \ldots, \alpha_s\}$, the $\mathcal{O}[G]$-module $M/\pi^n$ is pseudo-isomorphic to
\[
\bigoplus_{i=1}^s \mathcal{O}[G]/\pi^{\alpha_i}.
\]
The equality of (e) in this case is immediate. Now we consider the case when $M$ is a finitely generated $\mathcal{O}[G]$-module with $\mathcal{O}[G]$-rank $r > 0$. Denote $M_t$ to be the maximal torsion $\mathcal{O}[G]$-submodule of $M$ and write $M_{tf} = M/M_t$. Since $M_{tf}$ is torsionfree, we can find an injection $\mathcal{O}[G]^r \hookrightarrow M_{tf}$ with a $\mathcal{O}[G]$-torsion cokernel $N$. By the above argument, for sufficiently large $n$, one has
\[
\mu_{\mathcal{O}[G]}(M_t) = \mu_{\mathcal{O}[G]}(M_{tf}/\pi^n).
\]
As the $\mu_{\mathcal{O}[G]}$-invariant is additive on exact sequences of torsion $\mathcal{O}[G]$-modules, the exact sequence
\[
0 \rightarrow N[\pi^n] \rightarrow N \xrightarrow{\pi^n} N \rightarrow N/\pi^n \rightarrow 0
\]
shows that
\[
\mu_{\mathcal{O}[G]}(N[\pi^n]) = \mu_{\mathcal{O}[G]}(N/\pi^n).
\]
Combining the above equalities with the following two exact sequences
\[
0 \rightarrow M_t/\pi^n \rightarrow M/\pi^n \rightarrow M_{tf}/\pi^n \rightarrow 0
\]
\[
0 \rightarrow N[\pi^n] \rightarrow (\mathcal{O}[G]/\pi^n)^r \rightarrow M_{tf}/\pi^n \rightarrow N/\pi^n \rightarrow 0,
\]
we obtain the required equality, noting that $\mu_{\mathcal{O}[G]}(M) = \mu_{\mathcal{O}[G]}(M_t)$. \(\square\)
Before stating the next lemma, we recall some terminology and notation. Let $\rho : G \rightarrow GL_d(O')$ be a continuous group homomorphism with $O \subseteq O'$. Denote $W_\rho$ to be a free $O'$-module of rank $d$ realizing $\rho$. If $M$ is an $O[G]$-module, we define $\text{tw}_\rho(M)$ to be the $O'$-module $W_\rho \otimes_{\mathbb{Z}_p} M$ with $G$ acting diagonally. We shall say that $\rho$ is an Artin representation if $\rho$ has finite image.

**Lemma 2.3.** Let $G$ be a compact $p$-adic Lie group and let $M$ be a finitely generated $O[G]$-module. Let $\rho : G \rightarrow GL_d(O')$ be an Artin representation with $O \subseteq O'$. Then

$$\mu_{O'[G]}(\text{tw}_\rho(M)) = \mu_{O[G]}(M)d.$$ 

**Proof.** Let $G_0$ be a normal uniform pro-$p$ subgroup of $G$ such that $G_0 \subseteq \ker \rho$. By Lemma 2.2(a), it suffices to show that $\mu_{O'[G_0]}(\text{tw}_\rho(M)) = \mu_{O[G_0]}(M)d$. By our choice of $G_0$, we have $\text{tw}_\rho(M) = (M \otimes O O')^d$ as $O'[G_0]$-modules. Thus, we have

$$\mu_{O'[G_0]}(\text{tw}_\rho(M)) = \mu_{O'[G_0]}(M \otimes O' O')^d = \mu_{O[G_0]}(M)d,$$

where the last equality follows from Lemma 2.2(b). \qed

We now quote a result of Cucuo and Monsky [CM, Theorem 4.13] (see also [Mon, Theorem 3.12] for a finer statement) which extends the classical asymptotic formula to the case when $G \cong \mathbb{Z}_p^r$, $r \geq 2$.

**Theorem 2.4** (Cucuo-Monsky). Suppose that $G \cong \mathbb{Z}_p^r$, where $r \geq 2$. Denote $G_m$ to be $G^p^m$. Let $M$ be a finitely generated torsion $\mathbb{Z}_p[G]$-module which satisfies the property that $\text{rank}_{\mathbb{Z}_p} M_{G_m} = O(p^{(r-2)m})$. Then we have

$$\text{ord}_p(M_{G_m}) = \mu_G(M)p^{rm} + l_0(M)mp^{(r-1)m} + O(p^{(r-1)m}) \text{ for } m \gg 0,$$

where $l_0(M)$ is defined as in [CM, Definition 1.2].

We will require a few more lemmas which estimate the order of certain cohomology groups. For an abelian group $N$, we define its $p$-rank to be the $\mathbb{F}_p$-dimension of $N[p]$ which we denote by $r_p(N)$. If $G$ is a pro-$p$ group, we write $h_1(G) = r_p(H^1(G, \mathbb{Z}/p\mathbb{Z}))$ and $h_2(G) = r_p(H^2(G, \mathbb{Z}/p\mathbb{Z}))$. We now state and prove the following lemma which gives an estimate of the $p$-rank of the first and second cohomology groups.

**Lemma 2.5.** Let $G$ be a pro-$p$ group, and let $M$ be a discrete $G$-module which is cofinitely generated over $\mathbb{Z}_p$. If $h_1(G)$ is finite, then $r_p(H^1(G, M))$ is finite, and we have the following estimate

$$r_p(H^1(G, M)) \leq 2h_1(G) + \text{corank}_{\mathbb{Z}_p}(M) + \text{ord}_p(M/M_{\text{div}}).$$

If $h_2(G)$ is finite, then $r_p(H^2(G, M))$ is finite, and we have the following estimate

$$r_p(H^2(G, M)) \leq 2h_2(G) + \text{corank}_{\mathbb{Z}_p}(M) + \text{ord}_p(M/M_{\text{div}}).$$

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Proof. We shall only give a proof of the upper bound for $r_p(H^1(G, M))$, the proof of the upper bound of $r_p(H^2(G, M))$ being similar. By [NSW, Corollary 1.6.13], the only simple $G$-module is $\mathbb{Z}/p\mathbb{Z}$ with a trivial $G$-action. If $M$ is finite, it follows from a standard d\'evissage argument that $$|H^1(G, M)| \leq |H^1(G, \mathbb{Z}/p\mathbb{Z})| \text{ord}_p(M).$$

This in turn implies that $$r_p(H^1(G, M)) \leq \text{ord}_p(H^1(G, M)) \leq h_1(G) + \text{ord}_p(M).$$

For a general $M$, we denote $M_{\text{div}}$ to be the maximal $p$-divisible subgroup of $M$. Note that $M_{\text{div}}$ is a $G$-submodule of $M$. Then we have a short exact sequence $$0 \rightarrow M_{\text{div}} \rightarrow M \rightarrow M/M_{\text{div}} \rightarrow 0$$

which induces the following exact sequence $$H^1(G, M_{\text{div}}) \rightarrow H^1(G, M) \rightarrow H^1(G, M/M_{\text{div}}).$$

Therefore, we are reduced to showing that $r_p(H^1(G, M_{\text{div}}))$ and $r_p(H^1(G, M/M_{\text{div}}))$ are finite, and that the following inequalities

$$r_p(H^1(G, M_{\text{div}})) \leq h_1(G) + \text{corank}_{\mathbb{Z}_p}(M),$$

$$r_p(H^1(G, M/M_{\text{div}})) \leq h_1(G) + \text{ord}_p(M/M_{\text{div}})$$

hold. Since $M$ is cofinitely generated over $\mathbb{Z}_p$, we have that $M/M_{\text{div}}$ is finite. The finiteness of $r_p(H^1(G, M/M_{\text{div}}))$ and the validity of the second inequality then follow from the above discussion. To see that the first inequality holds, we first note that the short exact sequence

$$0 \rightarrow M_{\text{div}}[p] \rightarrow M_{\text{div}} \rightarrow M_{\text{div}}[p] \rightarrow 0$$

of discrete $G$-modules induces a surjection $H^1(G, M_{\text{div}}[p]) \rightarrow H^1(G, M_{\text{div}})[p]$ and, consequently, the inequality

$$r_p(H^1(G, M_{\text{div}})) \leq r_p(H^1(G, M_{\text{div}}[p])).$$

Again by the above discussion, the latter is less than or equal to $h_1(G) + \text{ord}_p(M_{\text{div}}[p])$, and the required inequality follows from the observation that $\text{ord}_p(M_{\text{div}}[p]) = \text{corank}_{\mathbb{Z}_p}(M)$. \hfill \Box

As an immediate consequence of the preceding lemma, we have the following.

Lemma 2.6. Let $G$ be a pro-$p$ group, and let $M$ be a discrete $G$-module which is cofinitely generated over $\mathbb{Z}_p$. If $h_1(G)$ is finite, then we have the following estimate

$$\text{ord}_p(H^1(G, M)[p^n]) \leq n(2h_1(G) + \text{corank}_{\mathbb{Z}_p}(M) + \text{ord}_p(M/M_{\text{div}})).$$

If $h_2(G)$ is finite, then we have the following inequality

$$\text{ord}_p(H^2(G, M)[p^n]) \leq n(2h_2(G) + \text{corank}_{\mathbb{Z}_p}(M) + \text{ord}_p(M/M_{\text{div}})).$$
3 A ratio formula

As before, let \( p \) be a prime, and let \( F \) be a number field. If \( p = 2 \), we assume further that \( F \) has no real primes. Suppose that we are given the following data:

(a) \( M \) is a finite \( \text{Gal}(\bar{F}/F) \)-module of \( p \)-power order which is unramified outside a finite set of primes of \( F \).

(b) For each prime \( v \) of \( F \) above \( p \), \( M_v \) is a \( \text{Gal}(\bar{F}_v/F_v) \)-submodule of \( M \).

(c) For each real prime \( v \) of \( F \), we write \( M_v^+ = M^{\text{Gal}(\bar{F}_v/F_v)} \).

(d) The following equality
\[
|M| r_2(F) \prod_{v \text{ real}} |M/M_v^+| = \prod_{v \not| p} |M/M_v|
\] (1)
holds. Here \( r_2(F) \) denotes the number of complex primes of \( F \).

We will denote the data by \( (M, \{M_v\}_{v|p}, \{M_v^+\}_{v|\mathbb{R}}) \). Let \( S \) denote a finite set of primes of \( F \) which contains all the primes above \( p \), the ramified primes of \( M \) and all infinite primes. Denote \( F_S \) to be the maximal algebraic extension of \( F \) unramified outside \( S \). For each algebraic extension \( \mathcal{L} \) of \( F \) contained in \( F_S \), we write \( G_S(\mathcal{L}) = \text{Gal}(F_S/F) \).

Lemma 3.1. For a given data \( (M, \{M_v\}_{v|p}, \{M_v^+\}_{v|\mathbb{R}}) \), we have the following equality
\[
\frac{|H^0(G_S(F), M)||H^2(G_S(F), M)|}{|H^1(G_S(F), M)|} = \prod_{v \not| p} \frac{|H^0(F_v, M/M_v)||H^2(F_v, M/M_v)|}{|H^1(F_v, M/M_v)|}.
\]

Proof. This follows from a straightforward application of the local and global Euler characteristic formulas. \( \square \)

Now set
\[
H^1_{\text{str}}(F_v, M) = \begin{cases} 
\ker \left( H^1(F_v, M) \to H^1(F_v, M/M_v) \right) & \text{if } v|p, \\
\ker \left( H^1(F_v, M) \to H^1(F_v^{ur}, M) \right) & \text{if } v \not| p,
\end{cases}
\]
where \( F_v^{ur} \) is the maximal unramified extension of \( F_v \). The Greenberg strict Selmer group (see [Gr, P. 116]) attached to \( (M, \{M_v\}_{v|p}, \{M_v^+\}_{v|\mathbb{R}}) \) is defined by
\[
S(M/F) := \text{Sel}^{\text{str}}(M/F) := \ker \left( H^1(G_S(F), M) \to \bigoplus_{v \in S} H^1(F_v, M)/H^1_{\text{str}}(F_v, M) \right).
\]

For the remainder of the paper, we shall refer to the Greenberg strict Selmer group as the Selmer group. For a data \( (M, \{M_v\}_{v|p}, \{M_v^+\}_{v|\mathbb{R}}) \), we say that \( (M^*, \{M_v^*\}_{v|p}, \{(M^*)_v^+\}_{v|\mathbb{R}}) \) is the dual data given by \( M^* = \text{Hom}_{cts}(M, \mu_p^\infty) \), \( M_v^* = \text{Hom}_{cts}(M/M_v, \mu_p^\infty) \) and \( (M^*)_v^+ = \text{Hom}_{cts}(M/M_v^+, \mu_p^\infty) \). It is a straightforward exercise to verify that the dual data satisfies [1]. The Selmer group \( S(M^*/F) \) for \( (M^*, \{M_v^*\}_{v|p}, \{(M^*)_v^+\}_{v|\mathbb{R}}) \) is defined similarly. Then one has the following proposition.
Proposition 3.2. For a data \((M, \{M_v\}_v, \{M^+_v\}_v)\), we have the following equality

\[
\frac{|S(M/F)|}{|H^0(G^F, M)|} \prod_{v|p} |H^0(F_v, M/F_v)| = \frac{|S(M^*/F)|}{|H^0(G^{F^*}, M^*)|} \prod_{v|p} |H^0(F_v, M^*/F_v^*)|.
\]

Proof. This is proven in the same way as [Gr, Formula (53)] by combining a Poitou-Tate sequence argument with Lemma 3.1. \(\square\)

4 Comparing Selmer groups over multiple \(\mathbb{Z}_p\)-extensions

As before, let \(p\) be a prime. We let \(F\) be a number field. If \(p = 2\), we assume further that \(F\) has no real primes. Denote \(O\) to be the ring of integers of some finite extension \(K\) of \(\mathbb{Q}_p\). Fix a local parameter \(\pi\) for \(O\). Suppose that we are given the following data:

(a) \(A\) is a cofinitely generated cofree \(O\)-module of \(O\)-corank \(d\) with a continuous, \(O\)-linear \(Gal(\overline{F}/F)\)-action which is unramified outside a finite set of primes of \(F\).

(b) For each prime \(v\) of \(F\) above \(p\), \(A_v\) is a \(Gal(\overline{F}/F_v)\)-submodule of \(A\) which is cofree of \(O\)-corank \(d_v\).

(c) For each real prime \(v\) of \(F\), we write \(A^+_v = A^{Gal(F_v/F_v)}\).

(d) The following equality

\[
\sum_{v|p} (d - d_v)[F_v : \mathbb{Q}_p] = dr_2(F) + \sum_{v \text{ real}} (d - d^+_v)
\]

holds. Here \(r_2(F)\) denotes the number of complex primes of \(F\).

We denote the above data as \((A, \{A_v\}_v, \{A^+_v\}_v)\). From these data, we define its dual data as follows. Set \(A^* = \text{Hom}_{cts}(T_\pi(A), \mu_{p^\infty}), A_v^* = \text{Hom}_{cts}(T_\pi(A/A_v), \mu_{p^\infty})\) and \((A^*_v)^+ = \text{Hom}_{cts}(T_\pi(A/A^+_v), \mu_{p^\infty})\). Here \(T_\pi(N)\) denotes the \(\pi\)-adic Tate module of a \(O\)-module \(N\). It is an easy exercise to verify that \((A, \{A_v\}_v, \{(A^*_v)^+\}_v)\) satisfies equality (2). For each \(n\) and an \(O\)-module \(N\), we denote \(N[\pi^n] \to\) to be the kernel of \(\pi^n : N \to N\). One can check easily that the induced data \((A[\pi^n], \{A_v[\pi^n]\}_v, \{A^+_v[\pi^n]\}_v)\) satisfies equality (1).

We now describe briefly the general arithmetic situation, where we can obtain the above data from. (See Section 8 for more explicit examples.) Let \(V\) be a \(d\)-dimensional \(K\)-vector space with a continuous \(G_K(F)\)-action. Suppose that for each prime \(v\) of \(F\) above \(p\), there is a \(d_v\)-dimensional \(K\)-subspace \(V_v\) of \(V\) which is invariant under the action of \(Gal(\overline{F}/F_v)\), and for each real prime \(v\) of \(F\), \(V^{Gal(F_v/F_v)}\) has dimension \(d^+_v\). Choose a \(G_K(F)\)-stable \(O\)-lattice \(T\) of \(V\) (Such a lattice exists by compactness). We can obtain a data from \(V\) by setting \(A = V/T\) and \(A_v = V_v/T \cap V_v\). Note that \(A\) and \(A_v\) depends on the choice of the lattice \(T\).

We now consider the base change property of our data. Let \(L\) be a finite extension of \(F\). We obtain another data \((A, \{A_w\}_w, \{A^+_w\}_w)\) over \(L\) as follows: we consider \(A\) as a \(Gal(\overline{F}/L)\)-module, and for
each prime $w$ of $L$ above $p$, we set $A_w = A_v$, where $v$ is a prime of $F$ below $w$, and view it as a $\text{Gal}(\overline{F}_v/L_w)$-module. Then $d_w = d_v$. For each real prime $w$ of $L$, one sets $A_{\text{Gal} \left( \overline{L}_w/L_w \right)} = A_{\text{Gal} \left( \overline{F}_v/F_v \right)}$ and writes $d_w^+ = d_v^+$, where $v$ is a real prime of $F$ below $w$. In general, the $d_w$'s and $d_w^+$ need not satisfy equality (2). We now record the following lemma which give sufficient condition for equality (2) to hold for the data $(A, \{A_w\}_{w|p}, \{A_w^+\}_{w|R})$ over $L$.

**Lemma 4.1.** Suppose that $(A, \{A_v\}_{v|p}, \{A_v^+\}_{v|R})$ is a data defined over $F$. Suppose further that at least one of the following statements holds.

(i) All the archimedean primes of $F$ are unramified in $L$.

(ii) $[L : F]$ is odd.

(iii) $F$ is totally imaginary.

(iv) $F$ is totally real, $L$ is totally imaginary and

$$\sum_{v \text{ real}} d_v^+ = d[F : Q]/2.$$ 

Then we have the equality

$$\sum_{w|p} (d - d_w)[L_w : Q_p] = dr_2(L) + \sum_{w \text{ real}} (d - d_w^+).$$

**Proof.** Note that if either of the assertions in (ii) or (iii) holds, then the assertion in (i) holds. Therefore, to prove the lemma in these cases, it suffices to prove it under the assumption of (i). We have the following calculation

$$\sum_{w|p} (d - d_w)[L_w : Q_p] = \sum_{v|p} \sum_{w|v} (d - d_w)[L_w : F_v][F_v : Q_p]$$

$$= \sum_{v|p} (d - d_v)[F_v : Q_p] \sum_{w|v} [L_w : F_v]$$

$$= [L : F] \sum_{v|p} (d - d_v)[F_v : Q_p]$$

$$= d[L : F]r_2(F) + [L : F] \sum_{v \text{ real}} (d - d_v^+).$$

Since (i) holds, every prime of $L$ above a real prime (resp., complex prime) of $F$ is a real prime (resp., complex prime). Therefore, one has $[L : F]r_2(F) = r_2(L)$ and

$$[L : F] \sum_{v \text{ real}} (d - d_v^+) = \sum_{w \text{ real}} (d - d_w^+).$$

The required conclusion then follows.

Now suppose that (iv) holds. Then $r_2(F) = 0$ and the above sum is

$$[L : F] \sum_{v \text{ real}} (d - d_v^+) = [L : F] \sum_{v \text{ real}} d - [L : F] \sum_{v \text{ real}} d_v^+$$

$$= [L : Q]d - [L : F]d[F : Q]/2$$

$$= d[L : Q]/2 = dr_2(L).$$

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Let $S$ be a finite set of primes of $F$ which contains all the primes above $p$, the ramified primes of $A$ and all infinite primes. Denote $F_S$ to be the maximal algebraic extension of $F$ unramified outside $S$ and write $G_S(L) = \text{Gal}(F_S/L)$ for every algebraic extension $L$ of $F$ which is contained in $F_S$. Let $L$ be a finite extension of $F$ contained in $F_S$ such that the data $(A, \{A_w\}_{w|p}, \{A_w^+\}_{w|\mathbb{R}})$ satisfies (2). For a prime $w$ of $L$ lying over $S$, set

$$H^1_{str}(L_w, A) = \begin{cases} \ker \left( H^1(L_w, A) \rightarrow H^1(L_w, A/A) \right) & \text{if } w \text{ divides } p, \\ \ker \left( H^1(L_w, A) \rightarrow H^1(L_w^{ur}, A) \right) & \text{if } w \text{ does not divide } p, \end{cases}$$

where $L_w^{ur}$ is the maximal unramified extension of $L_w$. The (Greenberg strict) Selmer group attached to the data is then defined by

$$S(A/L) := \text{Sel}^{str}(A/L) := \ker \left( H^1(G_S(L), A) \rightarrow \bigoplus_{w \in S_L} H^1_s(L_w, A) \right),$$

where we write $H^1_s(L_w, A) = H^1(L_w, A)/H^1_{str}(L_w, A)$ and $S_L$ denotes the set of primes of $L$ above $S$.

It is straightforward to verify that $S(A/L) = \lim_{\rightarrow} \text{Sel}(A[\pi^n]/L)$, where $S(A[\pi^n]/L)$ the Selmer group of $(A[\pi^n], \{A_w[\pi^n]\}_{w|p}, \{A_w^+[\pi^n]\}_{w|\mathbb{R}})$ defined in Section 3 and the direct limit is taken over the maps $S(A[\pi^n]/L) \rightarrow S(A[\pi^n+1]/L)$ induced by the natural injections $A[\pi^n] \hookrightarrow A[\pi^n+1]$ and $A_w[\pi^n] \hookrightarrow A_w[\pi^n+1]$. We will write $X(A/L)$ for its Pontryagin dual. The $S(A^*/L)$ are defined analogously and one also has the identification $S(A^*/L) = \lim_{\rightarrow} S(A^*[\pi^n]/L)$.

For an infinite algebraic extension $L$ of $F$ contained in $F_S$, we define $S(A/L) = \lim_L \text{Sel}(A/L)$, where the limit runs over all finite extensions $L$ of $F$ contained in $L$. We will write $X(A/L)$ for the Pontryagin dual of $S(A/L)$.

Let $F_\infty$ be a $\mathbb{Z}_p$-extension of $F$ which contains the cyclotomic $\mathbb{Z}_p$-extension $F_{cyc}$ of $F$. We denote $G$ to be the Galois group. Let $F_m$ be the unique subextension of $F_\infty$ over $F$ with $\text{Gal}(F_m/F) \cong (\mathbb{Z}/p^m)^r$ and write $G_m = \text{Gal}(F_\infty/F_m)$. We can now prove the following theorem which is the main result of this section.

**Theorem 4.2.** Retain the above assumptions. $X(A/F_\infty)$ and $X(A^*/F_\infty)$ have the same $O[G]$-rank and the same $\mu_O[G]^r$-invariant.

**Proof.** The proof follows the argument in [Gr]. As the case of $r = 1$ is essentially dealt there, we will concentrate on the case when $r \geq 2$. By Lemma 3.2(c)(d), it suffices to show the theorem under the assumption that $O = \mathbb{Z}_p$. Fix an arbitrary positive integer $n$. By Lemma 4.1(ii) and (iii), for each $m$, we may apply Proposition 3.2 to obtain the equality

$$\frac{|S(A[p^n]/F_m)| \prod_{v_m|p} |H^0(F_{m,v_m}, A[p^n]/A_{v_m}[p^n])|}{|H^0(G_S(F_m), A[p^n])|} = \frac{|S(A^*[p^n]/F_m)| \prod_{v_m|p} |H^0(F_{m,v_m}, A^*[p^n]/A_{v_m}^+[p^n])|}{|H^0(G_S(F_m), A^*[p^n])|}.$$
where \( v_m \) runs over all the primes of \( F_m \) above \( p \). Clearly, \( |H^0(G_S(F_m), A[p^n])| \) and \( |H^0(G_S(F_m), A^*[p^n])| \) are bounded independent of \( m \) (for a fixed \( n \)). Since there are only finitely many primes of \( F_{\text{cyc}} \) above \( p \), the decomposition group of \( v \) in \( G \) has at most dimension \( r - 1 \). Therefore, it follows that 
\[
\prod_{v_m \mid p} |H^0(F_{m,v_m}, A[p^n]/A_v[p^n])| \quad \text{and} \quad \prod_{v_m \mid p} |H^0(F_{m,v_m}, A^*[p^n]/A_v^*[p^n])| \quad \text{are both} \quad p^{O(p^{r-1}m)}.
\]
Thus, we have
\[
\text{ord}_p(S(A[p^n]/F_m)) = \text{ord}_p(S(A^*[p^n]/F_m)) + O(p^{r-1}m). \tag{3}
\]
Now we need to estimate the order of the kernels and cokernels of the maps
\[
S(A[p^n]/F_m) \xrightarrow{r_m} S(A/F_m)[p^n] \xrightarrow{s_m} (S(A/F_{\infty})[p^n])^G_m.
\]
One sees easily that \( \ker r_m \subseteq A(F_m)/p^n \) and \( \ker s_m \subseteq H^1(G_m, A(F_{\infty}))[p^n] \). It is clear that one has \( \text{ord}_p(\ker r_m) = O(1) \). On the other hand, it follows from Lemma 2.6 that \( \text{ord}_p(H^1(G_m, A(F_{\infty}))[p^n]) = O(1) \) (noting that \( h_1(G_m) \) is a constant function in \( m \)). Thus, one has \( \text{ord}_p(\ker s_m) = O(1) \).

To estimate \( \text{coker } r_m \) and \( \text{coker } s_m \), one first observes that \( \text{ord}_p(\text{coker } r_m) \leq \text{ord}_p(\ker r_m) \) and that \( \text{ord}_p(\text{coker } s_m) \leq \text{ord}_p(\ker s_m') + \text{ord}_p(H^2(G_m, A(F_{\infty}))[p^n]) \), where \( r_m' \) and \( s_m' \) are given by
\[
r_m' = \left( r_m'_{m,v_m} : \bigoplus_{v_m \in S(F_m)} H^1_{s}(F_{m,v_m}, A[p^n]) \rightarrow \bigoplus_{v_m \in S(F_m)} H^1_{s}(F_{m,v_m}, A)[p^n] \right);
\]
\[
s_m' = \left( s_m'_{m,v_m} : \bigoplus_{v_m \in S(F_m)} H^1_{s}(F_{m,v_m}, A)[p^n] \rightarrow \left( \lim_{\substack{m \rightarrow \infty \\forall v_m \in S(F_m)}} \bigoplus_{v_m \in S(F_m)} H^1_{s}(F_{m,v_m}, A)[p^n] \right)^{G_m} \right).
\]
By Lemma 2.6, one has that \( \text{ord}_p(H^2(G_m, A(F_{\infty}))[p^n]) = O(1) \) (noting that \( h_2(G_m) \) is a constant function in \( m \)). To estimate \( \text{coker } r_m' \), we first observe that
\[
\ker r_m'_{m,v_m} \subseteq \begin{cases} 
\ker (H^1(F_{m,v_m}, A/A_{v_m}[p^n]) \rightarrow H^1(F_{m,v_m}, A/A_{v_m})[p^n]) & \text{if } v_m | p, \\
\ker (H^1(F^r_{m,v_m}, A[p^n]) \rightarrow H^1(F^r_{m,v_m}, A)[p^n]) & \text{if } v_m \not| p,
\end{cases}
\]
\[
= \begin{cases} 
A/A_{v_m}(F_{m,v_m})[p^n] & \text{if } v_m | p, \\
A(F^r_{m,v_m})[p^n] & \text{if } v_m \not| p,
\end{cases}
\]
It is now clear from the above that \( \text{ord}_p(\ker r_m'_{m,v_m}) \) is bounded independent of \( m \) and \( v_m \) (for a fixed \( n \)). Combining these estimates with the fact that the decomposition group of \( v \) in \( G \) has dimension at most \( r - 1 \) for every \( v \in S \), one then has the estimate \( \text{ord}_p(\ker r_m') = O(p^{r-1}m) \).

To estimate \( \text{coker } s_m' \), we now observe that
\[
\ker s_m'_{m,v_m} \subseteq \begin{cases} 
H^1(G_{m,v_m}, A/A_{v_m}(F_{m,v_m}))[p^n] & \text{if } v_m | p, \\
H^1(\text{Gal}(F_{\infty,v_m}/F^r_{m,v_m}), A(F_{m,v_m}))[p^n] & \text{if } v_m \not| p.
\end{cases}
\]
By Lemma 2.6, one can verify that \( \ker s_m'_{m,v_m} \) is bounded independent of \( m \) and \( v_m \) (for a fixed \( n \)). As before, combining these estimates with the fact that the decomposition group of \( v \) in \( G \) has dimension at most \( r - 1 \) for every \( v \in S \), we obtain \( \text{ord}_p(\ker s_m') = O(p^{r-1}m) \). In conclusion, we have
\[
\text{ord}_p(S(A[p^n]/F_m)) = \text{ord}_p(S(A/F_{\infty})[p^n])^{G_m} + O(p^{r-1}m). \tag{4}
\]

Similarly, one also has

$$\text{ord}_p \left( S(A^*/[p^n]/F_m) \right) = \text{ord}_p \left( S(A^*/F_{\infty})[p^n]^{G_m} \right) + O(p^{(r-1)m}).$$  \hspace{1cm} (5)$$

Combining the estimates in (3), (4) and (5), we obtain

$$\text{ord}_p \left( \left( X(A/F_{\infty})/p^n \right)_{G_m} \right) = \text{ord}_p \left( \left( X(A^*/F_{\infty})/p^n \right)_{G_m} \right) + O(p^{(r-1)m}).$$  \hspace{1cm} (6)$$

Since $X(A/F_{\infty})/p^n$ and $X(A^*/F_{\infty})/p^n$ are $p$-torsion modules, we have

$$\text{rank}_{Z_p} \left( X(A/F_{\infty})/p^n \right)_{G_m} = \text{rank}_{Z_p} \left( X(A^*/F_{\infty})/p^n \right)_{G_m} = 0,$$

and therefore, the hypothesis of Theorem 2.3 is satisfied. Hence we may combine the said theorem with the estimate in (6) to conclude that

$$\mu_{Z_p[G]} \left( X(A/F_{\infty})/p^n \right)^{prm} = \mu_{Z_p[G]} \left( X(A^*/F_{\infty})/p^n \right)^{prm} + O(p^{(r-1)m})$$

(note that since $X(A/F_{\infty})/p^n$ and $X(A^*/F_{\infty})/p^n$ are $p$-torsion modules, the $l_0$ quantity attached to these modules vanishes) which in turn implies the equality

$$\mu_{Z_p[G]} \left( X(A/F_{\infty})/p^n \right) = \mu_{Z_p[G]} \left( X(A^*/F_{\infty})/p^n \right).$$

By Lemma 2.2(e), this in turn implies that

$$n \text{rank}_{Z_p[G]} \left( X(A/F_{\infty}) \right) + \mu_{Z_p[G]} \left( X(A/F_{\infty}) \right) = n \text{rank}_{Z_p[G]} \left( X(A^*/F_{\infty}) \right) + \mu_{Z_p[G]} \left( X(A^*/F_{\infty}) \right)$$

for $n \gg 0$. Therefore, we have the equalities $\text{rank}_{Z_p[G]} \left( X(A/F_{\infty}) \right) = \text{rank}_{Z_p[G]} \left( X(A^*/F_{\infty}) \right)$ and $\mu_{Z_p[G]} \left( X(A/F_{\infty}) \right) = \mu_{Z_p[G]} \left( X(A^*/F_{\infty}) \right)$. \hfill \qed

We record several immediate corollaries.

**Corollary 4.3.** $X(A/F_{\infty})$ is a $O[G]$-torsion module if and only if $X(A^*/F_{\infty})$ is a $O[G]$-torsion module.

**Corollary 4.4.** Suppose that $G = \Gamma \cong \mathbb{Z}_p$. Then $X(A/F_{\text{cyc}})$ is a finitely generated $O$ module if and only if $X(A^*/F_{\text{cyc}})$ is a finitely generated $O$ module.

**Remark 4.5.** For most data coming from (nearly) ordinary representations, it is expected that $X(A/F_{\text{cyc}})$ is a torsion $O[\Gamma]$-module (see [Gr] Conjecture 1 or [We] Conjecture 1.7). When the data $(A, \{A_v\}_{v|p})$ comes from the Galois representation attached to a primitive Hecke eigenform for $GL_2$ over $\mathbb{Q}$ and when the base field $F$ is abelian over $\mathbb{Q}$, the torsionness condition (over $F_{\text{cyc}}$) is a deep theorem of Kato [Ka].

**Remark 4.6.** If $F_\infty$ is a general $\mathbb{Z}_p^r$-extension of $F$ (that does not contain $F_{\text{cyc}}$) which has the property such that for each prime $v \in S$, the decomposition group of $\text{Gal}(F_{\infty}/F)$ at $v$ has dimension $\leq r - 1$, then the argument of Theorem 4.2 carries over to yield the same conclusion.
5 Comparing Selmer groups over noncommutative $p$-adic Lie extensions

In this section, we will compare the Selmer groups of $A$ and $A^*$ over a noncommutative $p$-adic Lie extensions. We shall say that $F_\infty$ is an admissible $p$-adic Lie extension of $F$ if (i) $\text{Gal}(F_\infty/F)$ is compact $p$-adic Lie group, (ii) $F_\infty$ contains the cyclotomic $\mathbb{Z}_p$ extension $F^{\text{cyc}}$ of $F$ and (iii) $F_\infty$ is unramified outside a set of finite primes. Write $G = \text{Gal}(F_\infty/F)$, $H = \text{Gal}(F_\infty/F_{\text{cyc}})$ and $\Gamma = \text{Gal}(F_{\text{cyc}}/F)$. Let $S$ denote a finite set of primes of $F$ which contains all the primes above $p$, the ramified primes of $A$, the infinite primes and the primes that are ramified in $F_\infty/F$.

Let $(A, \{A_v\}_{v|p}, \{A_v^+\}_{v|\mathbb{R}})$ denote the data defined in Section 4 and $(A^*, \{A^*_v\}_{v|p}, \{(A^*)_v^+\}_{v|\mathbb{R}})$ the dual data. The Selmer group of $A$ over $F_\infty$ is defined to be $S(A/F_\infty) = \lim_{L} S(A/L)$, where $L$ runs through all finite extensions of $F$ contained in $F_\infty$. As before, we denote the Pontryagin dual of $S(A/F_\infty)$ by $X(A/F_\infty)$. We have similar definitions for $S(A^*/F_\infty)$ and $X(A^*/F_\infty)$.

We shall impose the following hypothesis on our data $(A, \{A_v\}_{v|p}, \{A_v^+\}_{v|\mathbb{R}})$ and $p$-adic Lie extension $F_\infty$, for the rest of this section: for every finite extension $L$ of $F$ contained in $F_\infty$, the induced data $(A, \{A_w\}_{w|p}, \{A_w^+\}_{w|\mathbb{R}})$ over $L$ also satisfies equation (2).

As a start, we record the following special case.

**Proposition 5.1.** $X(A/F_\infty)$ is a finitely generated $\mathcal{O}[H]$-module if and only if $X(A^*/F_\infty)$ is a finitely generated $\mathcal{O}[H]$-module.

**Proof.** Let $L$ be a finite extension of $F$ contained in $F_\infty$ such that $\text{Gal}(F_\infty/L)$ is pro-$p$. Since $H_L = \text{Gal}(F_\infty/L_{\text{cyc}})$ is a subgroup of $H$ of finite index, a $\mathcal{O}[G]$-module is finitely generated over $\mathcal{O}[H]$ if and only if it is finitely generated $\mathcal{O}[H_L]$-module. Now by a standard argument (for instance, see Lemma [CS, Lemma 2.4]), the natural map $X(A/F_\infty)_{H_L} \rightarrow X(A/L_{\text{cyc}})$ has kernel and cokernel that are finitely generated over $\mathcal{O}$. Since $H_L$ is pro-$p$, one can apply a Nakayama lemma argument to conclude that $X(A/F_\infty)$ is finitely generated over $\mathcal{O}[H_L]$ if and only if $X(A/L_{\text{cyc}})$ is finitely generated over $\mathcal{O}$. The conclusion of the proposition now follows from the above discussion and Corollary 4.2. \qed

We say that the admissible extension $F_\infty$ of $F$ is almost abelian if there exists a finite extension $L$ of $F$ contained in $F_\infty$ such that $\text{Gal}(F_\infty/L) \cong \mathbb{Z}_p^r$ for some $r \geq 1$. We then have the following theorem which will follow from Theorem 4.2 and our extended definition of rank and $\mu$-variant.

**Theorem 5.2.** Suppose that $F_\infty$ is an almost abelian $S$-admissible extension of $F$. Then $X(A/F_\infty)$ and $X(A^*/F_\infty)$ have the same $\mathcal{O}[G]$-rank and the same $\mu_{\mathcal{O}[G]}$-invariant. In particular, $X(A/F_\infty)$ is a $\mathcal{O}[G]$-torsion module if and only if $X(A^*/F_\infty)$ is a $\mathcal{O}[G]$-torsion module.

To obtain a similar conclusion for the $\mu$-invariant over a general noncommutative $p$-adic Lie extension, we need to assume a stronger condition which was first introduced in [CF] and was crucial in the formulation of the main conjectures of non-commutative Iwasawa theory (see [CF, FK]). For a finitely
generated torsion $O[G]$-module $M$, we say that $M$ belongs to $\mathfrak{M}_H(G)$ if $M/M(\pi)$ is finitely generated over $O[H]$. Here $M(\pi)$ is the submodule of $M$ consisting of elements of $M$ annihilated by a power of $\pi$.

By the definition of the Selmer group, we have an exact sequence

$$0 \longrightarrow S(A/F_\infty) \longrightarrow H^1(G_S(F_\infty), A) \xrightarrow{\lambda_{A/F_\infty}} \bigoplus_{v \in S} J_v(A/F_\infty),$$

where $J_v(A/F_\infty) = \lim L \xrightarrow{\lambda_v} H^1_v(L, A)$.

**Theorem 5.3.** Let $F_\infty$ be an $S$-admissible $p$-adic Lie extension. Assume that both $X(A/F_\infty)$ and $X(A^*/F_\infty)$ belong to $\mathfrak{M}_H(G)$. Furthermore, suppose that $A(L_{\text{cyc}})$ and $A^*(L_{\text{cyc}})$ are finite for every finite extension $L$ of $F$ contained in $F_\infty$.

Then $X(A/F_\infty)$ and $X(A^*/F_\infty)$ have the same $\mu_{O[G]}$-invariant.

To prove Theorem 5.3, we require two lemmas.

**Lemma 5.4.** Let $F_\infty/F$ be an admissible $p$-adic Lie extension. Assume that $X(A/F_\infty)$ belongs to $\mathfrak{M}_H(G)$. Furthermore, suppose that $A^*(L_{\text{cyc}})$ is finite for every finite extension $L$ of $F$ contained in $F_\infty$. Then $H^2(G_S(F_\infty), A) = 0$ and $\lambda_{A/F_\infty}$ is surjective.

**Proof.** Since $X(A/F_\infty)$ belongs to $\mathfrak{M}_H(G)$, it follows from the argument of [CS Proposition 2.5] that $X(A/L_{\text{cyc}})$ is a torsion $O[\Gamma_L]$-module for every finite extension $L$ of $F$ contained in $F_\infty$, where $\Gamma_L = \text{Gal}(L_{\text{cyc}}/L)$. The Cassel-Poitou-Tate sequence gives an exact sequence

$$0 \longrightarrow S(A/L_{\text{cyc}}) \longrightarrow H^1(G_S(L_{\text{cyc}}), A) \xrightarrow{\lambda_{A/L_{\text{cyc}}}} \bigoplus_{w \in S(L_{\text{cyc}})} H^1_w(L_w, A) \longrightarrow (\hat{S}(A^*/L_{\text{cyc}}))^\vee \longrightarrow H^2(G_S(L_{\text{cyc}}), A) \longrightarrow \bigoplus_{w \in S(L_{\text{cyc}})} H^2(L_w, A).$$

Here $\hat{S}(A^*/L_{\text{cyc}})$ is defined as the kernel of the map

$$\lim_{E} H^1(G_S(L), T_\pi A^*) \longrightarrow \lim_{E} \bigoplus_{w | S} T_\pi H^1(L_w, A^*),$$

where the inverse limit is taken over all finite extensions $E$ of $L$ contained in $L_{\text{cyc}}$. Noting that $\bigoplus_{w \in S(L_{\text{cyc}})} H^2(L_w, A)$ is cofinitely generated over $O$, it then follows from a straightforward $O[\Gamma_L]$-rank calculation that $\hat{S}(A^*/L_{\text{cyc}})$ has zero $O[\Gamma_L]$-rank. On the other hand, by a similar argument to that in [HV] Proposition 7.1, one has an injection

$$\hat{S}(A^*/L_{\text{cyc}}) \longrightarrow \text{Hom}_{O[\text{Gal}(L_{\text{cyc}}/L)]} \left(S(A/L_{\text{cyc}}), O[\text{Gal}(L_{\text{cyc}}/L)]\right)$$

which in turn implies that $\hat{S}(A^*/L_{\text{cyc}})$ is torsionfree over $O[\Gamma_L]$. Hence we must have $\hat{S}(A^*/L_{\text{cyc}}) = 0$ and this in turn implies that $\lambda_{A/L_{\text{cyc}}}$ is surjective. Since $\bigoplus_{w \in S(L_{\text{cyc}})} H^2(L_w, A)$ is cofinitely generated over
O, it follows from the surjectivity of \( \lambda_{A/L_{\text{cyc}}} \) and the above exact sequence that \( H^2(G_S(L_{\text{cyc}}), A) \) is cotorsion over \( \mathcal{O}[\Gamma_L] \). On the other hand, by [Gr] Proposition 4, \( H^2(G_S(L_{\text{cyc}}), A) \) is a cofree \( \mathcal{O}[\Gamma_L] \)-module. Therefore, this will force \( H^2(G_S(L_{\text{cyc}}), A) = 0 \). Since \( H^2(G_S(F_\infty), A) = \lim_{L \to L / \ell} H^2(G_S(L_{\text{cyc}}), A) \) and \( \lambda_{A/F_\infty} = \lim_{L \to L / \ell} \lambda_{A/L_{\text{cyc}}} \), it follows that \( H^2(G_S(F_\infty), A) = 0 \) and \( \lambda_{A/F_\infty} \) is surjective.

\[ \square \]

**Lemma 5.5.** Suppose that \( G \) is a compact pro-\( p \) \( p \)-adic Lie group with no \( p \)-torsion. Assume that \( X(A/F_\infty) \) belongs to \( \mathcal{M}_H(G) \). Suppose also that \( H^2(G_S(F_{\text{cyc}}), A) = 0 \), \( H^2(G_S(F_\infty), A) = 0 \), and \( \lambda_{A/F_{\text{cyc}}} \) and \( \lambda_{A/F_\infty} \) are surjective.

Then \( \mu_{\mathcal{O}[G]}(X(A/F_\infty)) = \mu_{\mathcal{O}[G]}(X(A/F_{\text{cyc}})) \).

**Proof.** Under the assumptions of the lemma, one can apply a similar argument to that of [CSS] Proposition 2.13 (see also [Lan] Proposition 4.7) to conclude that

\[
\mu_{\mathcal{O}[G]}(X(A/F_\infty)) = \mu_{\mathcal{O}[G]}(X(A/F_{\text{cyc}})) + \sum_{n \geq 0}(-1)^{n+1}\mu_{\mathcal{O}[\Gamma]}(H_n(H, X_f(A/F_\infty))),
\]

where \( X_f(A/F_\infty) = X(A/F_\infty)/X(A/F_\infty)(\pi) \). By the hypothesis that \( X(A/F_\infty) \) belongs to \( \mathcal{M}_H(G) \), one has that \( H_n(H, X_f(A/F_\infty)) \) is finitely generated over \( \mathcal{O} \) for every \( n \geq 0 \). This in turn implies that \( \mu_{\mathcal{O}[\Gamma]}(H_n(H, X_f(A/F_\infty))) = 0 \) for every \( n \geq 0 \), and therefore, it follows that \( \mu_{\mathcal{O}[G]}(X(A/F_\infty)) = \mu_{\mathcal{O}[G]}(X(A/F_{\text{cyc}})) \).

\[ \square \]

We can now prove Theorem 5.3.

**Proof of Theorem 5.3.** Let \( L \) be a finite extension of \( F \) contained in \( F_\infty \) such that \( G_L = \text{Gal}(F_\infty/L) \) is pro-\( p \) and has no \( p \)-torsion. Since \( H_L = \text{Gal}(F_\infty/L_{\text{cyc}}) \) is a subgroup of \( H \) of finite index, a \( \mathcal{O}[G] \)-module is finitely generated over \( \mathcal{O}[H] \) if and only if it is finitely generated \( \mathcal{O}[H_L] \)-module. Therefore, we have that \( X(A/F_\infty) \) belongs to \( \mathcal{M}_{H_L}(G_L) \). The hypothesis of the theorem and Lemma 5.4 allows us to apply Lemma 5.5 to conclude that \( \mu_{\mathcal{O}[G_L]}(X(A/F_\infty)) = \mu_{\mathcal{O}[G_L]}(X(A/F_{\text{cyc}})) \) and \( \mu_{\mathcal{O}[G_L]}(X(A^*/F_{\text{cyc}})) = \mu_{\mathcal{O}[G_L]}(X(A^*/F_{\text{cyc}})) \). The required conclusion is now an immediate consequence of Theorem 4.2.

\[ \square \]

We end the section stating some further auxiliary results on the structure of the Selmer group. First, we recall that it is a well-known observation from the structure theory of a finitely generated \( \mathcal{O}[\Gamma] \)-module that a finitely generated \( \mathcal{O}[\Gamma] \)-module \( M \) is finitely generated over \( \mathcal{O} \) if and only if \( M / \mathcal{O} \mathcal{O}[\Gamma] \) has \( \mu_{\mathcal{O}[\Gamma]}(M) = 0 \). Such a statement is false when one replaces \( \Gamma \) by a general compact \( p \)-adic Lie group \( G \) of dimension > 1. For instance, if \( G \) has a quotient \( \Gamma \cong \mathbb{Z}_p \), the module \( \mathcal{O}[\Gamma]/p \) is clearly a torsion \( \mathcal{O}[G] \)-module with \( \mu_{\mathcal{O}[G]}(M) = 0 \), but it is not finitely generated over \( \mathcal{O}[H] \), where \( H \) is the subgroup of \( G \) such that \( G/H = \Gamma \). In fact, the module \( \mathcal{O}[\Gamma]/p \) also belongs to \( \mathcal{M}_H(G) \). Therefore, this example also shows that a \( \mathcal{O}[G] \)-module belonging to \( \mathcal{M}_H(G) \) with \( \mu_{\mathcal{O}[G]}(M) = 0 \) needs not be finitely generated over \( \mathcal{O}[H] \). However, for the dual Selmer group, we have the following interesting observation.

**Proposition 5.6.** Suppose that \( G \) is a compact pro-\( p \) \( p \)-adic Lie group with no \( p \)-torsion. Assume that \( X(A/F_\infty) \) belongs to \( \mathcal{M}_H(G) \). Suppose also that \( H^2(G_S(F_{\text{cyc}}), A) = 0 \), \( H^2(G_S(F_\infty), A) = 0 \), and \( \lambda_{A/F_{\text{cyc}}} \).
and $\lambda_{A/F_\infty}$ are surjective. Then $X(A/F_\infty)$ is finitely generated over $O[H]$ if and only if it belongs to $\mathfrak{M}_H(G)$ with $\mu_{O[\Gamma]}(X(A/F_\infty)) = 0$.

**Proof.** As the “only if” direction is well-known, we will only prove the “if” direction. Suppose that $X(A/F_\infty)$ belongs to $\mathfrak{M}_H(G)$ and $\mu_{O[\Gamma]}(X(A/F_\infty)) = 0$. It then follows from Lemma 5.5 that $\mu_{O[\Gamma]}(X(A/F_{cyc})) = 0$. By [CS] Proposition 2.5, we have that $X(A/F_{cyc})$ is $O[\Gamma]$-torsion. Thus, one can apply the structure theory of $O[\Gamma]$-module to see that $X(A/F_{cyc})$ is finitely generated over $O$. Now appealing to the argument in Proposition 5.1 we have that $X(A/F_\infty)$ is finitely generated over $O[H]$. The proof of the proposition is now completed. \[\square\]

The next result concerns with comparing the structural properties of the dual Selmer groups $X(A/F_\infty)$ and $X(A^*/F_\infty)$. We have seen in Proposition 5.2 that if $F_\infty$ is an almost abelian admissible extension, then $X(A/F_\infty)$ is $O[\Gamma]$-torsion if and only if $X(A^*/F_\infty)$ is $O[\Gamma]$-torsion. Naturally, one may ask if one can establish an analogous result for the property of belonging to $\mathfrak{M}_H(G)$. At this point of writing, we are only able to establish such result in the following special case.

**Proposition 5.7.** Assume that $G = \mathbb{Z}_p^2$. Furthermore, suppose that $A(L_{cyc})$ and $A^*(L_{cyc})$ are finite for every finite extension $L$ of $F$ contained in $F_\infty$.

Then $X(A/F_\infty)$ belongs to $\mathfrak{M}_H(G)$ if and only if $X(A^*/F_\infty)$ belongs to $\mathfrak{M}_H(G)$.

**Proof.** It suffices to show that if $X(A/F_\infty)$ belongs to $\mathfrak{M}_H(G)$, then $X(A^*/F_\infty)$ also belongs to $\mathfrak{M}_H(G)$. Fix a lifting of $\Gamma$ to $G$ so that $G = H \times \Gamma$. Let $L_\infty$ be the fixed field of the subgroup $\Gamma$. Thus, $L_\infty$ is a $\mathbb{Z}_p$-extension of $F$ and set $L_n$ to be its unique subextension of degree $p^n$ over $F$. Now suppose that $X(A/F_\infty)$ belongs to $\mathfrak{M}_H(G)$. By [CS] Proposition 2.5, we have that $X(A/L_{n,cyc})$ is $O[\Gamma]$-torsion for every $n$. Then the proof of [CS] Theorem 3.8] carries over yielding

$$\mu_{O[\Gamma]}(X(A/L_{n,cyc})) = p^n \mu_{O[\Gamma]}(X(A/F_{cyc})).$$

By Theorem 4.2 this in turns implies that $X(A^*/L_{n,cyc})$ is $O[\Gamma]$-torsion for every $n$ and

$$\mu_{O[\Gamma]}(X(A^*/L_{n,cyc})) = p^n \mu_{O[\Gamma]}(X(A^*/F_{cyc})).$$

By the reverse direction of [CS Theorem 3.8], this in turn implies that $X(A^*/F_\infty)$ belongs to $\mathfrak{M}_H(G)$. \[\square\]

6 Artin twists of Selmer groups

We retain the notation of the previous section. Let $\rho : G \to GL_d(O')$ be an Artin representation, where $O'$ is the ring of integers of a finite extension $K'$ of $K$. Denote $W_{\rho}$ to be a free $O'$-module of rank $d$ realizing $\rho$. The (Greenberg strict) Selmer group attached to the Artin twist of the data $(A, \{A_v\}_{v|p}, \{A_v^+\}_{v|R})$ is defined by replacing $A$ by $A \otimes_O W_{\rho}$ and $A_v$ by $A_v \otimes_O W_{\rho}$ in the definition of the Selmer groups. We denote this twisted Selmer group by $S(A \otimes_O W_{\rho}/L)$ and its Pontryagin dual by $X(A \otimes_O W_{\rho}/L)$.

We should mention that the twisted data $(A \otimes_O W_{\rho}, \{A_v \otimes_O W_{\rho}\}_{v|p}, \{(A \otimes_O W_{\rho})^+\}_{v|R})$ need not satisfy equality (2). Therefore, the next two theorems do not follow immediately from the results before.
Theorem 6.1. Suppose that \( F_\infty \) is an almost abelian admissible \( p \)-adic Lie extension of \( F \), and suppose that \( \rho : G \rightarrow GL_d(O') \) is an Artin representation with \( O \subseteq O' \). Then the dual Selmer groups \( X(A \otimes O W_p/F_\infty) \) and \( X(A^* \otimes O W_\hat{\rho}^\prime/F_\infty) \) have the same \( O'[G] \)-rank and the same \( \mu_{O'[G]} \)-invariant.

Theorem 6.2. Let \( F_\infty \) be an \( S \)-admissible \( p \)-adic Lie extension. Assume that both \( X(A/F_\infty) \) and \( X(A^*/F_\infty) \) belong to \( \mathcal{M}_H(G) \). Furthermore, suppose that \( A(L_{\text{cyc}}) \) and \( A^*(L_{\text{cyc}}) \) are finite for every finite extension \( L \) of \( F \).

For each Artin representation \( \rho : G \rightarrow GL_d(O') \) with \( O \subseteq O' \), we have

\[
\mu_{O'[G]}(X(A \otimes O W_p/F_\infty)) = \mu_{O'[G]}(X(A \otimes O W_\hat{\rho}/F_\infty)).
\]

We record the following lemma.

Lemma 6.3. For each Artin representation \( \rho : G \rightarrow GL_d(O') \), we have

\[
tw_\hat{\rho}(X(A/F_\infty)) = X(A \otimes O W_\rho/F_\infty),
\]

where \( \hat{\rho} \) is the contragredient representation of \( \rho \).

Proof. For any \( O \)-module \( M \), we write \( M_{O'} = M \otimes O' \). Since \( \rho \) factors through a finite quotient of \( G \), we have \( S(A \otimes O W_\rho/F_\infty) = S(A/F_\infty)_{O'} \otimes O W_\rho \). Hence

\[
X(A \otimes O W_\rho) = \Hom_{O'}(W_\rho, X(A/F_\infty)_{O'}) = \Hom_{O'}(\Hom_{O'}(O', W_\hat{\rho}), X(A/F_\infty)_{O'}).
\]

Since \( W_\hat{\rho} \) is a free \( O' \)-module, it follows from [CE, Chapter VI, Proposition 5.2] that the latter module is isomorphic to

\[
\Hom_{O'}(O', X(A/F_\infty)_{O'}) \otimes_{O'} W_\rho = tw_\hat{\rho}(X(A/F_\infty)).
\]

This proves the lemma.

The conclusion of Theorem 6.1 is now plain from Lemmas 2.3 and 6.3 and Theorem 5.2. The conclusion of Theorem 6.2 will follow from Lemmas 2.3 and 6.3 and Theorem 5.3.

7 Comparing Greenberg Selmer groups and Selmer complexes

As before, let \( p \) be a fixed prime. Furthermore, we assume that \( F \) has no real primes if \( p = 2 \). Let \( (A, \{A_v\}_{v | p}, \{A^{*v}_v\}_{v | \overline{p}}) \) denote the data defined in Section 4 and \( (A^*, \{A^{*v}_v\}_{v | p}, \{(A^{*v})^*_v\}_{v | \overline{p}}) \) the dual data. Fix a finite set \( S \) of primes of \( F \) which contains all the primes above \( p \), the ramified primes of \( A \) and the infinite primes. Now set

\[
H^1_{\text{Gr}}(F_v, A) = \begin{cases} 
\ker (H^1(F_v, A) \rightarrow H^1(F_v, A/A_v)) & \text{if } v | p, \\
\ker (H^1(F_v, A) \rightarrow H^1(F^{wr}_v, A)) & \text{if } v \nmid p.
\end{cases}
\]
The Greenberg Selmer group attached to these data is then defined by 

\[ \text{Sel}^{Gr}(A/F) = \ker \left( H^1(G_S(F), A) \rightarrow \bigoplus_{v \in S} H^1_{\mathbb{G}}(F_v, A) \right), \]

where we write \( H^1_{\mathbb{G}}(F_v, A) = H^1(F_v, A)/H^1_{\mathbb{G}}(F_v, A) \). For an \( S \)-admissible \( p \)-adic Lie extension \( F_{\infty} \), we define \( \text{Sel}^{Gr}(A/F_{\infty}) = \varprojlim \text{Sel}^{Gr}(A/L) \) and denote \( X^{Gr}(A/F_{\infty}) \) to be the Pontryagin dual of \( \text{Sel}^{Gr}(A/F_{\infty}) \).

The following lemma compares the two Greenberg Selmer groups.

**Lemma 7.1.** We have an exact sequence

\[ 0 \rightarrow S(A/F_{\infty}) \rightarrow \text{Sel}^{Gr}(A/F_{\infty}) \rightarrow N \rightarrow 0, \]

where \( N \) is a cofinitely generated \( \mathcal{O}[H] \)-module.

**Proof.** Now consider the following commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & S(A/F_{\infty}) \rightarrow H^1(G_S(F_{\infty}), A) \rightarrow \bigoplus_{v \in S} J_v(A/F_{\infty}) \\
\downarrow & & \downarrow & & \downarrow \alpha \\
0 & \rightarrow & \text{Sel}^{Gr}(A/F_{\infty}) \rightarrow H^1(G_S(F_{\infty}), A) \rightarrow \bigoplus_{v \in S} J_v^{Gr}(A/F_{\infty})
\end{array}
\]

with exact rows, where \( J_v^{Gr}(A/F_{\infty}) = \varprojlim_{L \supseteq F_{\infty}} H^1_{\mathbb{G}}(L_w, A) \). It therefore remains to show that \( \ker \alpha \) is cofinitely generated over \( \mathcal{O}[H] \). Clearly, \( J_v(A/F_{\infty}) = J_v^{Gr}(A/F_{\infty}) \) for \( v \mid p \). For \( v \nmid p \), choose a prime \( w \) of \( F_{\infty} \) above \( v \). Write \( I_{\infty,w} \) for the inertia subgroup of \( \text{Gal}(\overline{F}_{\infty,w}/F_{\infty,w}) \) and \( U_w = \text{Gal}((\overline{F}_{\infty,w}/F_{\infty,w})/I_{\infty,w}) \).

It then follows from the Hochschild-Serre spectral sequence that we have

\[ 0 \rightarrow H^1(U_w, (A/A_v)^{I_{\infty,w}}) \rightarrow H^1(H_w, A/A_v) \rightarrow H^1(I_{\infty,w}, A/A_v)^{U_w}. \]

Since \( U_w \) is topologically cyclic, \( H^1(U_w, (A/A_v)^{I_{\infty,w}}) \cong (A/A_v)^{I_{\infty,w}} \) \( U_w \) and so is cofinitely generated over \( \mathcal{O} \). Now since the decomposition group of \( G \) at \( v \) has at least dimension one for each \( v \nmid p \), it follows that \( \ker \alpha \) is cofinitely generated over \( \mathcal{O}[H] \), as required.

**Lemma 7.2.** One has

\[ \text{rank}_{\mathcal{O}[G]}(X(A/F_{\infty})) = \text{rank}_{\mathcal{O}[G]}(X^{Gr}(A/F_{\infty})) \]

and

\[ \mu_{\mathcal{O}[G]}(X(A/F_{\infty})) = \mu_{\mathcal{O}[G]}(X^{Gr}(A/F_{\infty})). \]

Furthermore, \( X(A/F_{\infty}) \) belongs to \( \mathfrak{M}_H(G) \) if and only if \( X^{Gr}(A/F_{\infty}) \) belongs to \( \mathfrak{M}_H(G) \).

**Proof.** By the preceding lemma, one has an exact sequence

\[ 0 \rightarrow N' \rightarrow X^{Gr}(A/F_{\infty}) \rightarrow X(A/F_{\infty}) \rightarrow 0 \]

for some finitely generated \( \mathcal{O}[H] \)-module \( N' \). The first equality and the final assertion are then immediate, and the second equality follows from \[ \text{Lim} \] Lemma 2.1(c)(iii)].

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Remark 7.3. Note that Lemma 7.2 does not require any torsion assumptions on either of the Selmer groups.

The next two theorems are immediate from combining the results in Sections 4 and 5 with Lemma 7.2

Theorem 7.4. Suppose that $F_\infty$ is an almost abelian admissible $p$-adic Lie extension of $F$. Then $X^{Gr}(A/F_\infty)$ and $X^{Gr}(A^*/F_\infty)$ have the same $O[G]$-rank and the same $\mu_{O[G]}$-invariant.

Theorem 7.5. Let $F_\infty$ be an $S$-admissible $p$-adic Lie extension. Assume that both $X^{Gr}(A/F_\infty)$ and $X^{Gr}(A^*/F_\infty)$ belong to $\mathfrak{M}_H(G)$. Furthermore, suppose that $A(L_{cyc})$ and $A^*(L_{cyc})$ are finite for every finite extension $L$ of $F$ contained in $F_\infty$.

Then we have

$$\mu_{O[G]}(X^{Gr}(A/F_\infty)) = \mu_{O[G]}(X^{Gr}(A^*/F_\infty)).$$

We now consider the Selmer complex of $\{A_i \{A_{ij}\}_{v|\mathbb{P}}, \{A_{ij}^+\}_{v|R}\}$. The notion of a Selmer complex was first conceived and introduced in [Nek]. In our discussion, we will consider a modified version of Selmer complexes as given in [FK, 4.2.11]. Write $T^* = \text{Hom}_{cts}(A,\mathbb{Q}_p/\mathbb{Z}_p)$ and $T^*_v = \text{Hom}_{cts}(A/A_v,\mathbb{Q}_p/\mathbb{Z}_p)$. For every finite extension $L$ of $F$ and $w$ a prime of $L$ above $p$, write $T^*_w = T^*_v$, where $v$ is the prime of $F$ below $w$. For any profinite group $G$ and a topological abelian group $M$ with a continuous $G$-action, we denote by $C(G,M)$ the complex of continuous cochains of $G$ with values in $M$. Let $F_\infty$ be an $S$-admissible extension of $F$ with Galois group $G$. We define a $(O[G])[G_2(F)]$-module $\mathcal{F}_G(T^*)$ as follows: as an $O$-module, $\mathcal{F}_G(T^*) = O[G] \otimes_O T^*$, and the action of $G_2(F)$ is given by the formula $\sigma(x \otimes t) = x\sigma^{-1} \otimes \sigma t$, where $\sigma$ is the canonical image of $\sigma$ in $G \subseteq O[G]$. We define the $(O[G])[\text{Gal}(\overline{F}_v/F_v)]$-module $\mathcal{F}_G(T^*_v)$ in a similar fashion.

For every prime $v$ of $F$, we write $C(F_v,\mathcal{F}_G(T^*)) = C(\text{Gal}(\overline{F}_v/F_v),\mathcal{F}_G(T^*))$. For each prime $v$ not dividing $p$, let $C_f(F_v,\mathcal{F}_G(T^*))$ be the subcomplex of $C(F_v,\mathcal{F}_G(T^*))$, whose degree $m$-component is $0$ unless $m \neq 0,1$, whose degree $0$-component is $C^0(F_v,\mathcal{F}_G(T^*))$, and whose degree $1$-component is

$$\ker \left( C^1(F_v,\mathcal{F}_G(T^*))_{d=0} \rightarrow H^1(F_v,\mathcal{F}_G(T^*)) \right).$$

The Selmer complex $SC(T^*,T^*_v)$ is defined to be

$$\text{Cone} \left( C(G_2(F),\mathcal{F}_G(T^*)) \rightarrow \bigoplus_{v|p} C(F_v,\mathcal{F}_G(T^+)/\mathcal{F}_G(T^*_v)) \oplus \bigoplus_{v|p} C(F_v,\mathcal{F}_G(T^*))/C_f(F_v,\mathcal{F}_G(T^*)) \right)[-1].$$

Here $[-1]$ is the translation by $-1$ of the complex. We now state the following proposition which is given in [FK] Proposition 4.2.35.

Proposition 7.6. Let $G$ be the kernel of $\text{Gal}(\overline{F}/F) \rightarrow G$. For a place $v$ of $F$, fixing an embedding $F \hookrightarrow F_v$, let $G(v)$ be the kernel of $\text{Gal}(\overline{F}_v/F_v) \rightarrow G$ and let $G_v \subseteq G$ be the image. Then the following statements hold.
(a) $H^i(SC(T^*, T^*_v)) = 0$ for $i \neq 1, 2, 3$.

(b) We have an exact sequence

$$0 \rightarrow X(A/F_\infty) \rightarrow H^2(SC(T^*, T^*_v)) \rightarrow \bigoplus_{v \nmid p} O[G] \otimes_{O[G_\infty]} (T^*_v(-1))_{\mathcal{G}(v)}$$

$$\rightarrow (T^*(-1))_{\mathcal{G}} \rightarrow H^3(SC(T^*, T^*_v)) \rightarrow 0.$$ 

Denote $SC(T, T_v)$ to be the Selmer complex of the dual data $(A^*, \{A_v^*\}_{v \nmid p}, \{(A^*_v)^{\pm}\}_{v \mid p})$. We can now state the following analogous results for the second cohomology groups of the Selmer complexes.

**Theorem 7.7.** Suppose that $F_\infty$ is an almost abelian admissible $p$-adic Lie extension of $F$. Then $H^2(SC(T, T_v))$ and $H^2(SC(T^*, T^*_v))$ have the same $O[G]$-rank and the same $\mu_1$-invariant.

**Theorem 7.8.** Let $F_\infty$ be an $S$-admissible $p$-adic Lie extension of dimension $> 1$. Assume that both $X(A/F_\infty)$ and $X(A^*/F_\infty)$ belong to $\mathfrak{M}_H(G)$. Furthermore, suppose that $A(L_{cyc})$ and $A^*(L_{cyc})$ are finite for every finite extension $L$ of $F$ contained in $F_\infty$.

Then we have

$$\mu_1(O[G]|H^2(SC(T, T_v))) = \mu_1(O[G]|H^2(SC(T^*, T^*_v))).$$

## 8 Examples

We discuss some arithmetic examples of our main results. For brevity, we will only consider the strict Selmer groups. Our choices of examples are motivated by the possible application of Theorem [5.3]. Of course, our Theorem [4.2] is unconditional and can be applied to many other examples that are not discussed here. We refer readers to [We, Section 1.2] for some of these examples.

### 8.1 Abelian varieties

Let $B$ be an abelian variety of dimension $g$ defined over a number field $F$. For simplicity, we will assume that $F$ is totally imaginary and that $B$ has semistable reduction over $F$. We define a data $(A, \{A_v\})$ by first setting $A = B[p^\infty]$. For each prime $v$ of $F$ above $p$, let $\mathcal{F}_v$ be the formal group attached to the Neron model for $B$ over the ring of integers $O_F$ of $F$, and we assume that $\mathcal{F}_v$ is a formal group of height $g$ for all $v|p$. For instance, this is satisfied if $B$ has good ordinary reduction at all $v|p$. We then set $A_v = \mathcal{F}_v[\mathfrak{m}][p^\infty]$, where $\mathfrak{m}$ is the maximal ideal of the rings of integers of $\mathcal{F}_v$. Note that $A_v \cong (Q_p/Z_p)^g$ as an abelian group by our height assumption.

Then the dual data $(A^*, \{A^*_v\})$ is given by $A^* = B^*_{p^\infty}$ and $A_v^* = \mathcal{F}_v[\mathfrak{m}]$ over $B^t$ is the dual abelian variety of $B$ and $\mathcal{F}_v^t$ is the formal group attached to the Neron model for $B^t$ over the ring of integers $O_F$ of $F$. Clearly, $\mathcal{F}_v^t$ has height $g$ and $A_v^* \cong (Q_p/Z_p)^g$ as an abelian group. It is worthwhile to mention that the Greenberg strict Selmer groups attached to these data coincide with the classical Selmer
groups of the abelian variety (see [CG]), when the Selmer groups are considered over an admissible $p$-adic Lie extension. Now if $F_{\infty}$ is an almost abelian $S$-admissible extension of $F$, it follows from Theorem 5.2 that $X(A/F_{\infty})$ and $X(A^*/F_{\infty})$ have the same $\mathbb{Z}_p[G]$-rank and the same $\mu_{\mathbb{Z}_p[G]}$-invariant. In particular, this applies when $B$ is an abelian variety with complex multiplication and $F_{\infty} = F(B[p^\infty])$.

We now consider the situation of a general noncommutative $p$-adic Lie extension. The conditions on the finiteness of $A(L^{\text{cyc}})$ and $A^*(L^{\text{cyc}})$ are consequences of [Win, Theorem 4.3]. Therefore, to apply Theorem 5.3, it remains to require that $X(A/F_{\infty})$ and $X(A^*/F_{\infty})$ both belong to $\mathcal{M}_H(G)$. When $B$ is an elliptic curve, this is conjectured in [CF]. In view of their conjecture, it seems quite reasonable to expect that $X(A/F_{\infty})$ and $X(A^*/F_{\infty})$ both belong to $\mathcal{M}_H(G)$ for a general abelian variety $B$ which has good ordinary reduction at all primes above $p$. However, we should mention that there is currently no known method to prove that the dual Selmer group belongs to $\mathcal{M}_H(G)$ in general.

We like to mention that in a preprint of Bhave [Bh], she was able to establish the equality of $\mu$-invariant for the case that $B$ is an abelian variety without complex multiplication under a weaker torsion condition on the Selmer groups than our Theorem 5.3. Therefore, our results in the abelian varieties cases complement the results there.

We can also apply our results (Theorems 6.1 and 6.2) to the Artin twists of the abelian variety $B$.

### 8.2 Hilbert modular forms

Let $F$ be totally real number field and let $f$ be a primitive Hilbert modular form of parallel weight $(k, k, \ldots, k)$ with $k \geq 2$ which is $p$-ordinary. Denote $K_f$ to be the field generated by the coefficient of $f$. We will write $K$ for the localization of $K_f$ at some fixed prime of $K_f$ above $p$. Let $V$ be the two-dimensional representation over $K$ associated to $f$. Since $f$ is assumed to be $p$-ordinary, it follows that for every prime $v$ of $F$ above $p$, we have a one-dimensional subspace $V_v$ of $V$ which is $\text{Gal}(\overline{F}_v/F_v)$-invariant and has the property that the inertia subgroup acts by a power of the cyclotomic character on this subspace and trivially on the one dimensional quotient. Fix a Galois lattice $T$ of $V$, and for each prime $v$ of $F$ above $p$, we set $T_v = T \cap V_v$. Then in this situation, the data $(A, \{A_v\}_{v|p})$ is given by $A = V/T$ and $A_v = V_v/T_v$. The dual data is then the above definition for the dual modular form $\bar{f}$. In view of Lemma 4.3, we can apply our results to an admissible $p$-adic Lie extension $F_{\infty}$ of $F$ which is either totally real or totally imaginary. Again, if $F_{\infty}$ is an almost abelian $S$-admissible extension of $F$, Theorem 5.2 applies.

We now consider the case of a general noncommutative $p$-adic Lie extension and suppose that $f$ is a primitive cuspidal modular form of positive weight $k \geq 2$ with $K_f = \mathbb{Q}$. The finiteness of $A(L^{\text{cyc}})$ and $A^*(L^{\text{cyc}})$ are established in the proof of [Sn Lemma 2.2]. In order to apply Theorem 5.3, one will require that $X(A/F_{\infty})$ and $X(A^*/F_{\infty})$ both belong to $\mathcal{M}_H(G)$. This latter condition will follow from the $\mathcal{M}_H(G)$ conjecture for Hida families (see [CS]). However, it does not seem to be known how one can verify the $\mathcal{M}_H(G)$-condition in either situations.

Finally, we mention that we can apply our results to Artin twists of the two-dimensional representation of $f$ in a similar fashion as in the previous subsection.
References

[Bh] A. Bhave, Comparison of the $\mu$-invariants of an abelian variety and its dual abelian variety, [arXiv:1305.3444v1 [math.NT]].

[CE] H. Cartan and S. Eilenberg, Homological algebra, Princeton Univ. Press, 1956.

[CF+] J. Coates, T. Fukaya, K. Kato, R. Sujatha and O. Venjakob, The $GL_2$ main conjecture for elliptic curves without complex multiplication, *Publ. Math. IHES* 101 (2005), 163–208.

[CG] J. Coates and R. Greenberg, Kummer theory for abelian varieties over local fields, *Invent. Math.* 124 (1996), 129-174.

[CH] J. Coates and S. Howson, Euler characteristics and elliptic curves II, *J. Math. Soc. Japan* 53 (2001), no. 1, 175–235.

[CSS] J. Coates, P. Schneider and R. Sujatha, Links between cyclotomic and $GL_2$ Iwasawa theory, *Doc. Math.* 2003 (2003) 187–215, Extra Volume: Kazuya Kato’s fiftieth birthday.

[CS] J. Coates and R. Sujatha, On the $\mathcal{M}_H(G)$-conjecture, in: *Non-abelian fundamental groups and Iwasawa theory*, ed. J. Coates, M. Kim, F. Pop, M. Saidi and P. Schneider, London Math. Soc. Lecture Note Ser. 393, Cambridge Univ. Press, 2012, pp. 132–161.

[CM] A. Cucuo and P. Monsky, Class numbers in $\mathbb{Z}_d^p$-extensions, *Math. Ann.* 255 (1981), no. 2, 235–258.

[FK] T. Fukaya and K. Kato, A formulation of conjectures on $p$-adic zeta functions in noncommutative Iwasawa theory, *Amer. Math. Soc. Transl. Ser. 2* 219, 2006, 1–85.

[GW] K. R. Goodearl and R. B. Warfield, *An introduction to non-commutative Noetherian rings*, London Math. Soc. Stud. Texts 61, Cambridge University Press, 2004.

[Gr] R. Greenberg, Iwasawa theory for $p$-adic representations, in *Algebraic Number Theory–in honor of K. Iwasawa*, ed. J. Coates, R. Greenberg, B. Mazur and I. Satake, Adv. Std. in Pure Math. 17, 1989, pp. 97–137.

[HV] Y. Hachimori and O. Venjakob, Completely faithful Selmer groups over Kummer extensions, *Doc. Math.* 2003 (2003) 443–478, Extra Volume: Kazuya Kato’s fiftieth birthday.

[Ho] S. Howson, Euler characteristic as invariants of Iwasawa modules, *Proc. London Math. Soc.* 85 (2002), no. 3, 634–668.

[Ka] K. Kato, $p$-adic Hodge theory and values of zeta functions of modular forms, in: *Cohomologies $p$-adiques et applications arithmétiques. III.*, Astérisque 295, 2004, ix, pp. 117–290.

[Lam] T. Y. Lam, *Lectures on Modules and Rings*, Grad. Texts in Math. 189, Springer, 1999.

[Lim] M. F. Lim, A remark on the $\mathcal{M}_H(G)$-conjecture and Akashi series, [arXiv:1309.4174v1 [math.NT]].

[Mat] K. Matsuno, Finite $\Lambda$-submodules of Selmer groups of abelian varieties over cyclotomic $\mathbb{Z}_p$-extensions, *J. Number Theory* 99 (2003), no. 2, 415–443.

[Mon] P. Monsky, Fine estimate for the growth of $e_n$ in $\mathbb{Z}_d^p$-extensions, in *Algebraic Number Theory–in honor of K. Iwasawa*, ed. J. Coates, R. Greenberg, B. Mazur and I. Satake, Adv. Std. in Pure Math. 17, 1989, pp. 309–330.

[NSW] J. Neukirch, A. Schmidt and K. Wingberg, *Cohomology of Number Fields*, 2nd Ed., Grundlehren Math. Wiss. 323, Springer 2008.

[Nek] J. Nekovář, *Selmer Complexes*, Astérisque 310, 2006.

[Neu] A. Neumann, Completed group algebras without zero divisors, *Arch. Math.* 51 (1988), no. 6, 496–499.

[Su] R. Sujatha, Iwasawa theory and modular forms, *Pure Appl. Math. Q.* 2 (2006), No. 2, 519–538.

[V] O. Venjakob, On the structure theory of the Iwasawa algebra of a $p$-adic Lie group, *J. Eur. Math. Soc.* 4 (2002), no. 3, 271–311.

[We] T. Weston, Iwasawa invariants of galois deformations, *Manuscripta math.* 118 (2005), no. 2, 161–180.

[Win] K. Wingberg, On the rational points of abelian varieties over $\mathbb{Z}_p$-extensions of number fields, *Math. Ann.* 279 (1987), no. 1, 9–24.