Abstract

We establish the well-posedness of an initial-boundary value problem for a general class of linear time-fractional, advection-diffusion-reaction equations, allowing space- and time-dependent coefficients as well as initial data that may have low regularity. Our analysis relies on novel energy methods in combination with a fractional Gronwall inequality and properties of fractional integrals.

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1. Introduction

The main scope of this paper is to investigate the existence and uniqueness of the weak solution of a linear, time-fractional problem of the form

$$\partial_t u - \nabla \cdot (\kappa \nabla \partial_t^{1-\alpha} u - \vec{F} \partial_t^{1-\alpha} u - \vec{G} u) + a \partial_t^{1-\alpha} u + bu = g$$  \hspace{1cm} (1.1)

for $x \in \Omega$ and $0 < t \leq T$. The parameter $\alpha$ in the fractional derivative lies in the range $0 < \alpha < 1$, and the spatial domain $\Omega \subseteq \mathbb{R}^d$ ($d \geq 1$) is bounded and Lipschitz. The transport coefficients $\vec{F}$ and $\vec{G}$, the reaction coefficients $a$ and $b$, as well as the source term $g$, are assumed to be known functions of $x$ and $t$, whereas the generalized diffusivity $\kappa = \kappa(x)$ may depend only on $x$ but is permitted to be a real, symmetric positive-definite
matrix. In \((1.1)\), \(-\nabla \cdot (\kappa \nabla \partial_t^{1-\alpha} u)\) is the non-local diffusion term, whereas \(\nabla \cdot (\vec{F} \partial_t^{1-\alpha} u + \vec{G} u)\) is the non-local/local advection term, and \(a \partial_t^{1-\alpha} u + bu\) is the non-local/local reaction term.

We impose homogeneous Dirichlet boundary conditions,

\[ u(x, t) = 0 \quad \text{for} \quad x \in \partial \Omega \quad \text{and} \quad 0 \leq t \leq T, \quad (1.2) \]

and the initial condition

\[ u(x, 0) = u_0(x) \quad \text{for} \quad x \in \Omega. \quad (1.3) \]

The Riemann–Liouville fractional derivative \([33]\) of order \(1 - \alpha\) is defined via the fractional integral of order \(\alpha\): with \(\omega_{1-\alpha}(t) = \frac{t^{1-\alpha}}{\Gamma(\alpha)}\) we have

\[ \partial_t^{1-\alpha} v(x, t) = \partial_t \mathcal{I}^\alpha v(x, t) \quad \text{where} \quad \mathcal{I}^\alpha v(x, t) = \int_0^t \omega_{1-\alpha}(t-s)v(x, s) \, ds. \]

We denote by \(W^k_p(\Omega)\) the usual Sobolev space of functions whose partial derivatives of order \(k\) or less belong to \(L^p(\Omega)\). The following regularity assumptions on the coefficients will be used:

- \(\kappa \in L_\infty(\Omega)^{d \times d}\), \(\vec{F}, \vec{G} \in C^2([0, T]; W^1_\infty(\Omega)^d)\), \(a, b \in C^1([0, T]; L_\infty(\Omega))\).

In addition, to ensure that the spatial operator \(v \mapsto -\nabla \cdot (\kappa \nabla v)\) is uniformly elliptic on \(\Omega\), we assume that the minimal eigenvalue of \(\kappa(x)\) is bounded away from zero, uniformly for \(x \in \Omega\).

Based on physical models of various subdiffusive transport processes, different classes of time-fractional PDEs arise as special cases of (1.1), including

- fractional Fokker–Planck equations \([4, 10, 16, 30]\), when \(\vec{G} = 0\), \(a = b = 0\) and \(g = 0\);
- fractional reaction-diffusion equations \([11, 12]\), when \(\vec{F} = \vec{G} = 0\);
- fractional cable equations \([19]\), when \(\vec{F} = \vec{G} = 0\);
- fractional advection-dispersion (or fractional convection-diffusion) equations \([25]\), when \(\vec{F} = \vec{F}(x), \vec{G} = 0\) and \(a = b = 0\).

Consider the simplest non-trivial case, when \(\kappa\) is the identity matrix with \(\vec{F} = \vec{G} = 0\), \(a = b = 0\) and \(g = 0\), so that (1.1) reduces to the fractional subdiffusion equation: \(\partial_t u - \nabla^2 \partial_t^{1-\alpha} u = 0\). Let \(\varphi\) denote a Dirichlet eigenfunction of the Laplacian on \(\Omega\), with corresponding eigenvalue \(\lambda > 0\), that is, \(-\nabla^2 \varphi = \lambda \varphi\) in \(\Omega\) with \(\varphi|_{\partial \Omega} = 0\). For the special choice of initial data \(u_0 = \varphi(x)\), the solution of the initial-boundary value problem (1.1)–(1.3) has the separable form \(u(x, t) = E_\alpha(-\lambda t^\alpha)\varphi(x)\), where \(E_\alpha(z) = \sum_{n=0}^\infty z^n / \Gamma(1+n\alpha)\) is the Mittag–Leffler function \([33]\). Notice that \(\partial_t^m u = O(t^{\alpha-m})\) as \(t \to 0\). Moreover, we can extend the classical method of separation of variables for the heat equation to construct a series solution
for arbitrary initial data $u_0 \in L_2(\Omega)$, and the regularity properties of the solution $u$ follow from this representation [28].

Such an explicit construction is no longer possible for the solution of the general equation (1.1). Instead, we proceed by formally integrating (1.1) in time, multiplying both sides by a test function $v$, and applying the first Green identity over $\Omega$ to arrive at the weak formulation

$$\langle u(t), v \rangle + \int_0^t \langle \kappa \nabla \partial_1^{1-\alpha} u(s) - \vec{F}(s) \partial_1^{1-\alpha} u(s) - \vec{G}(s) u(s), \nabla v \rangle \, ds$$

$$+ \int_0^t \langle a(s) \partial_1^{1-\alpha} u(s) + b(s) u(s), v \rangle \, ds = \langle u_0, v \rangle + \int_0^t \langle g(s), v \rangle \, ds$$  \hspace{1cm} (1.5)

for all $v \in H_0^1(\Omega)$, where we have suppressed the dependence of the functions on $x$, and where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L_2(\Omega)$ or $L_2(\Omega)^d$.

Numerical methods for particular cases of (1.1) were extensively studied over the last two decades, see for example [1, 18, 23, 36, 38] for finite differences, [14, 20, 31] for continuous and discontinuous finite elements, and also see [8, 13] for more references. However, due to various types of mathematical difficulties, proof of the well-posedness of the continuous problem is almost missing despite its importance, apart from the case [28] when $\vec{F} = \vec{G} = \vec{0}$ and $a = b = 0$. In this paper, we address these fundamental questions. A related paper [21] treats the fractional Fokker–Planck equation (that is, the case $\vec{G} = \vec{0}$ and $a = b = 0$) via a different, and somewhat simpler, chain of estimates that, for instance, does not use the quadratic operator $Q_1^\mu$ defined below in Section 2.

If the coefficients $\vec{F}$ and $a$ are independent of $t$, and if $\vec{G} = \vec{0}$ and $b = 0$, then by applying the fractional integration operator $I^{1-\alpha}$ to both sides of (1.1) we obtain

$$C \partial_t^{\alpha} u - \nabla \cdot (\kappa \nabla u - \vec{F} u) + a u = \tilde{g},$$  \hspace{1cm} (1.6)

where $C \partial_t^{\alpha} u = I^{1-\alpha} \partial_t u$ denotes the Caputo fractional derivative [33] and where $\tilde{g} = I^{1-\alpha} g$. Existence and uniqueness results for (1.6) were studied by several authors, including Zacher [39], Alikhanov [2], Sakamoto and Yamamoto [34] and Kubica and Yamamoto [17]. Further, the reader can refer to [15, 22, 27, 35]. Some of these papers include results for time-dependent coefficients, but in that case (1.6) is no longer equivalent to (1.1).

To recast the weak formulation (1.5) as a Volterra integral equation, we introduce two bounded linear operators, firstly $K_1(t) : H_0^1(\Omega) \to H^{-1}(\Omega)$ defined by

$$\langle K_1(t)v, w \rangle = \langle \kappa \nabla v, \nabla w \rangle - \langle \vec{F}(t)v, \nabla w \rangle + \langle a(t)v, w \rangle \quad \text{for } v, w \in H_0^1(\Omega),$$

and secondly $K_2(t) : L_2(\Omega) \to H^{-1}(\Omega)$ by
\[ \langle K_2(t)v, w \rangle = \langle b(t)v, w \rangle - \langle G(t)v, \nabla w \rangle \quad \text{for } v \in L_2(\Omega) \text{ and } w \in H_0^1(\Omega). \]

The variational problem (1.5), subject to the initial condition (1.3), can then be written more succinctly as

\[ u(t) + \int_0^t \left[ K_1(s)\partial_s^{1-\alpha}u(s) + K_2(s)u(s) \right] ds = f(t) \equiv u_0 + \int_0^t g(s) ds. \]  

Assuming \( u \) is sufficiently regular that \( (I_\alpha u)^{(0)} = 0 \), and using a dash to indicate a derivative in time, integration by parts leads to

\[ \int_0^t K_1(s)\partial_s^{1-\alpha}u(s) ds = K_1(t)I_\alpha u(t) - \int_0^t K_1'(s)I_\alpha u(s) ds \]

with \( K_1'(t) : H_0^1(\Omega) \to H^{-1}(\Omega) \) given by

\[ \langle K_1'(t)v, w \rangle = -\langle F(t)v, \nabla w \rangle + \langle a'(t)v, w \rangle. \]

Thus, \( u \) satisfies

\[ u(t) + \int_0^t K(t,s)u(s) ds = f(t) \quad \text{for } 0 \leq t \leq T, \]  

where \( K(t,s) : H_0^1(\Omega) \to H^{-1}(\Omega) \) is the weakly-singular, operator-valued kernel

\[ K(t,s) = \omega_\alpha(t-s)K_1(t) + K_2(s) - \int_s^t \omega_\alpha(z-s)K_1'(z)dz. \]  

Following some technical preliminaries in Section 2, we apply the Galerkin method in Section 3 to project the problem (1.8) to a finite dimensional subspace \( X \subseteq H_0^1(\Omega) \), thereby obtaining an approximate solution \( u_X : [0,T] \to X \). Using delicate energy arguments and a fractional Gronwall inequality, we prove a priori estimates for \( u_X \) that are uniform with respect to the dimension of \( X \), allowing us in Section 4 (Theorems 4.1 and 4.2) to establish the existence and uniqueness of a weak solution \( u \) to the original problem (1.1)–(1.3), provided (1.4) holds.

The regularity of the weak solution \( u \) will be studied in a companion paper [29].

2. Preliminaries and notations

Our subsequent analysis makes frequent use of two quadratic operators defined, for \( \mu \geq 0 \) and \( 0 \leq t \leq T \), by

\[ Q_1^\mu(\phi, t) = \int_0^t \langle \phi, I^\mu \phi \rangle ds \quad \text{and} \quad Q_2^\mu(\phi, t) = \int_0^t \| I^\mu \phi \|^2 ds. \]
Since it follows by the Plancherel Theorem that \( \omega \) assuming that (2.5). Mustapha [20, Lemma 3.2]. The fourth inequality follows from (2.3) and Q.

This operators coincide when \( \mu = 0 \) because \( T^0 \phi = \phi \), and so we write \( Q^0 = Q_1^0 = Q_2^0 \). If we put \( \phi(t) = 0 \) for \( t > T \), then the Laplace transform \( \hat{\phi}(z) = \int_T^\infty e^{-zt} \phi(t) \, dt \) is an entire function and \( \hat{\phi}(z) = z^{-\mu} \hat{\phi}(z) \), so it follows by the Plancherel Theorem that

\[
Q_1^\mu(\phi, T) = \frac{\cos(\pi \mu/2)}{\pi} \int_0^\infty y^{-\mu} \|\hat{\phi}(iy)\|^2 \, dy \geq 0,
\]

assuming that \( \phi \) is real-valued; see also [32 Theorem 2]. Note that because \( \omega_\mu \in L_1(0, T) \), the fractional integral defines a bounded linear operator

\[
I^\mu : L_p((0, T); L_2(\Omega)) \to L_p((0, T); L_2(\Omega)) \quad \text{for} \quad 1 \leq p \leq \infty.
\]

Also, \( I^{\mu+\nu} = I^\mu I^\nu \) because \( \omega_\mu * \omega_\nu = \omega_{\mu+\nu} \) for \( \mu > 0 \) and \( \nu > 0 \); here, \( * \) denotes the Laplace convolution.

The next four lemmas establish key inequalities satisfied by \( Q_1^\mu \) and \( Q_2^\mu \).

**Lemma 2.1.** If \( 0 < \alpha < 1 \) and \( \epsilon > 0 \), then

\[
\left| \int_0^t \langle \phi, I^\alpha \psi \rangle \, ds \right| \leq \frac{Q_1^\alpha(\phi, t)}{4\epsilon(1-\alpha)^2} + \epsilon Q_1^\alpha(\psi, t),
\]

(2.3)

\[Q_2^\alpha(\phi, t) \leq \frac{2t^{\alpha}}{1-\alpha} Q_1^\alpha(\phi, t),\]

(2.4)

\[Q_1^\alpha(\phi, t) \leq 2t^{\alpha} Q_0^\alpha(\phi, t),\]

(2.5)

\[
\left| \int_0^t \langle \phi, I^\alpha \psi \rangle \, ds \right| \leq \frac{t^\alpha Q_0^\alpha(\phi, t)}{2\epsilon(1-\alpha)^2} + \epsilon Q_1^\alpha(\psi, t).
\]

(2.6)

**Proof.** The first three inequalities are proved by Le, McLean and Mustapha [20, Lemma 3.2]. The fourth inequality follows from (2.3) and (2.5).

For the next result, note that if \( \phi \in W_1^1((0, T); X) \) for a normed space \( X \), then \( \phi : [0, T] \to X \) is absolutely continuous and

\[
(\partial_1 I^\alpha \phi - I^\alpha \partial_1 \phi)(t) = \phi(0) \omega_\alpha(t) \quad \text{for} \quad 0 < t \leq T.
\]

(2.7)

**Lemma 2.2.** If \( 0 < \alpha \leq 1 \), then for \( \phi \in L_2((0, t), L_2(\Omega)) \),

\[Q_2^\alpha(\phi, t) \leq 2 \int_0^t \omega_\alpha(t-s) Q_1^\alpha(\phi, s) \, ds.
\]

**Proof.** Assume first that \( \phi \in W_1^1((0, T), L_2(\Omega)) \) and let \( \psi = I^\alpha \phi \). Since \( \psi(0) = 0 \), the Caputo fractional derivative of \( \psi \) is

\[
C\partial_1^\alpha \psi = I^{1-\alpha}(\psi') = (I^{1-\alpha} \psi)' - \psi(0) \omega_{1-\alpha} = (I^1 \phi)' = \phi.
\]
Recalling an identity of Alikhanov [3, Corollary 1],

\[ 2 \langle \psi(t), C \partial_t^\alpha \psi(t) \rangle = C \partial_t^\alpha (\| \psi \|^2)(t) + \frac{\alpha}{2 \Gamma(1 - \alpha)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} \left( \int_s^t \psi'(q) \, dq \right)^2 \, ds, \]

we see that

\[ 2 \langle \phi, I^\alpha \phi \rangle = 2 \langle C \partial_t^\alpha \psi, \psi \rangle \geq C \partial_t^\alpha (\| \psi \|^2) = I^{1-\alpha} (\| I^\alpha \phi \|^2)', \]

and thus

\[ I^1 (\| I^\alpha \phi \|^2) = I^2 (\| I^\alpha \phi \|^2)' = I^{1+\alpha} I^{1-\alpha} (\| I^\alpha \phi \|^2)' \leq 2 I^{1+\alpha} (\langle \phi, I^\alpha \phi \rangle) = 2 I^\alpha I^1 (\langle \phi, I^\alpha \phi \rangle), \]

which is equivalent to the desired inequality.

Now let \( \phi \in L_2((0,T), L_2(\Omega)) \), and choose \( \phi_n \in W^1_2((0,T), L_2(\Omega)) \) such that \( \int_0^T \| \phi_n(t) - \phi(t) \|^2 \, dt \to 0 \) as \( n \to \infty \). Using (2.2) with \( \mu = \alpha \) and \( p = 2 \), it follows that \( Q^\alpha_1(\phi_n, t) \to Q^\alpha_1(\phi, t) \) and \( Q^\alpha_2(\phi_n, t) \to Q^\alpha_2(\phi, t) \), uniformly for \( t \in [0, T] \), which implies the result in the general case. \( \square \)

The next lemma will eventually enable us to establish pointwise (in time) estimates for \( u(t) \).

**Lemma 2.3.** Let \( 0 \leq \mu < \alpha \leq 1 \). If the function \( \phi : [0, T] \to L_2(\Omega) \) is continuous with \( \phi(0) = 0 \), and if its restriction to \( (0, T] \) is differentiable with \( \| \phi'(t) \| \leq C t^{-\mu} \) for \( 0 < t \leq T \), then \( \| \phi(t) \|^2 \leq 2 \omega_{2-\alpha}(t) Q^\alpha_1(\phi', t) \).

**Proof.** For \( \alpha = 1 \), equality holds:

\[ 2 \omega_1(t) Q^1_1(\phi', t) = 2 \int_0^t \langle \phi', \phi \rangle \, ds = \| \phi(t) \|^2. \]

For \( 0 < \alpha < 1 \), put \( \psi(t) = I^\alpha \phi' \) and note that \( \| \psi(t) \| \leq C t^{\alpha-\mu} \). By following similar arguments, one can show that (2.8) holds with \( \phi' \) in place of \( \phi \), that is \( 2 \langle \phi', I^\alpha \phi' \rangle = I^{1-\alpha} (\| I^\alpha \phi' \|^2)' \), for almost all \( t > 0 \). Now, applying the operator \( I^1 \) to both sides, and using \( I^\alpha \phi'(0) = \psi(0) = 0 \), we observe that

\[ I^{1-\alpha} (\| I^\alpha \phi' \|^2) (t) = 2 Q^\alpha_1(\phi', t) \quad \text{for } t > 0. \]

Since \( \phi = I^1 \phi' = I^{1-\alpha} \psi \),
\[
\phi(t) \leq \left( \int_0^t \omega_{1-\alpha}(t-s) \|\psi(s)\| ds \right)^2
\]
\[
\leq \int_0^t \omega_{1-\alpha}(t-s) ds \int_0^t \omega_{1-\alpha}(t-s) \|\psi(s)\|^2 ds
\]
\[
= \omega_{2-\alpha}(t) \mathcal{I}^{1-\alpha} \left( \|\mathcal{I}^\alpha \phi'\|^2 \right)(t),
\]
and hence the desired result follows immediately after using (2.9).

**Lemma 2.4.** If \(0 \leq \mu \leq \nu \leq 1\), then \(Q^\nu_2(\phi, t) \leq 2t^{2(\nu-\mu)} Q^\mu_2(\phi, t)\).

**Proof.** See Le, McLean and Mustapha [20, Lemma 3.1].

We will make essential use of the following fractional Gronwall inequality.

**Lemma 2.5.** Let \(\beta > 0\) and \(T > 0\). Assume that \(a\) and \(b\) are non-negative, non-decreasing functions on the interval \([0, T]\). If \(q : [0, T] \to \mathbb{R}\) is an integrable function satisfying
\[
0 \leq q(t) \leq a(t) + b(t) \int_0^t \omega_{\beta}(t-s) q(s) ds
\]
for \(0 \leq t \leq T\), then
\[
q(t) \leq a(t) E_\beta \left( b(t) t^\beta \right)
\]
for \(0 \leq t \leq T\).

**Proof.** See Dixon and McKee [9, Theorem 3.1].

Let \(\mathcal{M}\) denote the operator of pointwise multiplication by \(t\), that is, \((\mathcal{M} \phi)(t) = t \phi(t)\), and note the commutator property
\[
\mathcal{M} \mathcal{I}^\mu - \mathcal{I}^\mu \mathcal{M} = \mu \mathcal{I}^{\mu+1},
\]
for any real \(\mu \geq 0\). We will need the following estimates involving the linear operator \(B^\mu_{\psi}\) defined (for suitable \(\psi\) and \(\phi\)) by
\[
(B^\mu_{\psi} \phi)(t) = \psi(t) \mathcal{I}^\mu \phi(t) - \int_0^t \psi'(s) \mathcal{I}^\mu \phi(s) ds.
\]

**Lemma 2.6.** If \(\psi \in W^1_\infty((0, T); L_\infty(\Omega)^d)\) and \(\phi \in W^1_1((0, T); L_2(\Omega))\), then there is a constant \(C\) (depending only on \(\psi\), \(\mu\) and \(T\)) such that for \(0 \leq t \leq T\),
Thus, noting that \( Q_0^\phi (\phi, t) \leq C Q_2^\mu (\phi, t) \), \( L^0 (MB_1^\mu \phi, t) + L^0 (I^1 B_1^\mu \phi, t) \leq C t^2 Q_2^\mu (\phi, t) \), \( Q_0^\mu ((MB_1^\mu \phi)'_t \leq C Q_2^\mu ((M \phi)', t) + C Q_2^\mu (M \phi, t) + C Q_2^\mu (\phi, t) \). (2.14)

Proof. The assumption on \( \psi \) implies that
\[
\| (B_\psi^\mu \phi) (t) \|^2 \leq C \| (I^\mu \phi) (t) \|^2 + C \int_0^t \| (I^\mu \phi) (s) \|^2 ds,
\]
and (2.12) follows after integrating in time. By the Cauchy–Schwarz inequality,
\[
\| (MB_\psi^\mu \phi) (t) \|^2 + \| (I^1 B_\psi^\mu \phi) (t) \|^2 \leq t^2 \| (B_\psi^\mu \phi) (t) \|^2 + t \int_0^t \| (B_\psi^\mu \phi) (s) \|^2 ds,
\]
and (2.13) follows after integrating in time. The third identity in (2.10) implies that
\[
MB_\psi^\mu \phi = \psi (I^\mu M \phi + \mu I^{\mu+1} \phi) - M I^1 (\psi' I^\mu \phi)
\]
and therefore, differentiating with respect to \( t \),
\[
(MB_\psi^\mu \phi)' = \psi' (I^\mu M \phi + \mu I^{\mu+1} \phi) + \psi ((I^\mu M \phi)' + \mu I^\mu \phi) - (I^1 + M) (\psi' I^\mu \phi).
\]
Thus, noting that \( (I^\mu M \phi)' = I^\mu (M \phi)' \) by (2.7), with
\[
\| I^{\mu+1} \phi (t) \|^2 = \| I^1 (I^\mu \phi) (t) \|^2 \leq t Q_2^\mu (\phi, t)
\]
and \( \| I^1 (\psi' I^\mu \phi) (t) \|^2 \leq C t Q_2^\mu (\phi, t) \), we have
\[
\| (MB_\psi^\mu \phi)' (t) \|^2 \leq C \| I^\mu (M \phi) (t) \|^2 + C \| I^\mu (M \phi)' (t) \|^2 + C \| (I^\mu \phi) (t) \|^2 + C t Q_2^\mu (\phi, t),
\]
so (2.14) follows after integrating in time. \( \square \)

3. The projected equation
Suppose that \( X \) is a finite-dimensional subspace of \( H_0^1 (\Omega) \), equipped with the induced norm: \( \| v \|_X = \| v \|_{H_0^1 (\Omega)} \). We define a bounded linear operator \( K_X (t, s) : X \to X \) in terms of \( K (t, s) \) in (1.9) by
\[
\langle K_X (t, s) v, w \rangle = \langle K (t, s) v, w \rangle \quad \text{for} \ v, w \in X \text{ and } 0 \leq s \leq t \leq T,
\]
and let \( f_X (t) \) denote the \( L_2 \)-projection onto \( X \) of \( f (t) \) from (1.7), that is,
\[
\langle f_X (t), w \rangle = \langle f (t), w \rangle \quad \text{for} \ w \in X \text{ and } 0 \leq t \leq T.
\]
In this way, we arrive at a finite dimensional reduction of the Volterra equation (1.8),
\[ u_X(t) + \int_0^t K_X(t, s)u_X(s) \, ds = f_X(t) \quad \text{for } 0 \leq t \leq T. \] (3.1)

In the next theorem, we outline a self-contained proof of existence and uniqueness under relaxed assumptions on the coefficients in the fractional PDE (1.1). Similar results for scalar-valued kernels are shown by Linz [24, §3.4], Becker [5], and Brunner [6].

Henceforth, \( C \) will denote a generic constant that may depend on the coefficients in (1.1), the spatial domain \( \Omega \), the time interval \( [0, T] \), the fractional exponent \( \alpha \), the parameter \( \eta \), and the integer \( m \) in (1.4). However, any dependence on the subspace \( X \) is indicated explicitly by writing \( C_X \).

We let \( Y = C([0, T]; X) \) with the norm
\[ \|v\|_Y = \max_{0 \leq t \leq T} \|v(t)\|_X. \]

**Theorem 3.1.** Assume that the coefficients in (1.1) satisfy
\[ \kappa \in L_\infty(\Omega)^{d \times d}, \quad \bar{F} \in W^1_\infty((0, T); L_\infty(\Omega)^d), \quad \bar{G} \in L_\infty((0, T); L_\infty(\Omega)^d), \]
\[ a \in W^1_\infty((0, T); L_\infty(\Omega)), \quad b \in L_\infty((0, T); L_\infty(\Omega)). \]
Assume, in addition, that the source term \( g : (0, T] \to L_2(\Omega) \) is a measurable function satisfying
\[ \|g(t)\| \leq Mt^{\eta - 1} \quad \text{for } 0 < t \leq T, \] (3.2)
where \( M \) and \( \eta \) are positive constants, and that the initial data \( u_0 \in L_2(\Omega) \). Then, the weakly-singular Volterra integral equation (3.1) has a unique solution \( u_X \in Y \), and moreover \( \|u_X\|_Y \leq C_X \|f_X\|_Y \leq C_X(\|u_0\| + M) \).

**Proof.** Our assumptions on \( u_0 \) and \( g \) ensure that \( f_X \in Y \). The kernel (1.9) has the form
\[ K(t, s) = \omega_\alpha(t - s)G(t, s) + H(t, s), \]
where
\[ G(t, s) = K_1(t) - \Gamma(\alpha)(t - s) \int_0^1 \omega_\alpha(y)K'_1(s + (t - s)y) \, dy \]
and \( H(t, s) = K_2(s) \) for \( 0 \leq s \leq t \leq T \). Our assumptions on the coefficients of the fractional PDE (1.1) ensure that \( G \) and \( H \) are continuous mappings from the closed triangle \( \Delta = \{(t, s) : 0 \leq s \leq t \leq T\} \) into the space of bounded linear operators \( H^1_0(\Omega) \to H^{-1}(\Omega) \). Likewise,
\[ K_X(t, s) = \omega_\alpha(t - s)G_X(t, s) + H_X(t, s), \]
where \( G_X(t, s) : X \to X \) and \( H_X(t, s) : X \to X \) are defined by
\[ \langle G_X(t, s)v, w \rangle = \langle G(t, s)v, w \rangle \quad \text{and} \quad \langle H_X(t, s)v, w \rangle = \langle H(t, s)v, w \rangle \]
for \((t, s) \in \Delta\) and \(v, w \in X\). Since \(X\) is finite dimensional, \(G_X\) and \(H_X\) are continuous functions from \(\Delta\) into the space of bounded linear operators \(X \to X\). Hence, there is a positive constant \(\gamma_X\) such that 
\[
\|K_X(t, s)v\|_X \leq \gamma_X \omega_\alpha(t-s)\|v\|_X \quad \text{for } (t, s) \in \Delta \text{ and } v \in X,
\]
so we can define the Volterra operator \(K_X : Y \to Y\) by
\[
K_X v(t) = \int_0^t K_X(t, s)v(s) \, ds \quad \text{for } 0 \leq t \leq T \text{ and } v \in Y.
\]
We see that \(\|K_Xv\|_Y \leq \gamma_X \omega_{1+\alpha}(T)\|v\|_Y\). In fact, using the semigroup property,
\[
\int_0^t \omega_\alpha(t-s)\omega_\beta(s) \, ds = \omega_{\alpha+\beta}(t),
\]
we obtain the following estimate for the operator norm of the \(n\)th power of \(K_X\),
\[
\|K_X^n\|_{Y \to Y} \leq \gamma_X^n \max_{0 \leq t \leq T} \int_0^t \omega_{n\alpha}(t-s) \, ds = \gamma_X^n \omega_{1+n\alpha}(T) \quad \text{for } n \geq 1.
\]
It follows that the sum \(R_X = \sum_{n=1}^{\infty} (-1)^{n+1} K_X^n\) defines a bounded linear operator with
\[
\|R_X\|_{Y \to Y} \leq \sum_{n=1}^{\infty} \omega_{1+n\alpha}(T) \gamma_X^n = E_\alpha(\gamma_X T^\alpha) - 1.
\]
The existence and uniqueness of \(u_X \in Y\) is seen by noting
\[
u_X + K_X u_X = f_X \quad \text{if and only if} \quad u_X = f_X - R_X f_X,
\]
from which we also deduce the \textit{a priori} estimate claimed in the theorem.

\[\square\]

For a scalar, weakly-singular, second-kind Volterra equation, it is known that if \(f_X\) admits an expansion in powers of \(t\) and \(t^\alpha\), then so does the solution \(u_X\); see Lubich \[26\] Corollary 3, and also Brunner, Pedas and Vainikko \[7\] Theorem 2.1 (with \(\nu = 1-\alpha\)). To outline a proof that a similar result holds for systems of Volterra equations, let \(C^m_\alpha = C^m_\alpha([0, T]; X)\) denote the space of continuous functions \(v : [0, T] \to X\) that are \(C^m\) on the half-open interval \((0, T]\) and for which the seminorm
\[
|v|_{j, \alpha} = \sup_{0 < t \leq T} t^{j-\alpha} \|v^{(j)}(t)\|_X \quad \text{is finite for } 1 \leq j \leq m.
\]
We make \(C^m_\alpha\) into a Banach space by defining the obvious norm:
\[
\|v\|_{m, \alpha} = \|v\|_Y + \sum_{j=1}^m |v|_{j, \alpha}.
\]
Theorem 3.2. Let \( m \geq 1 \), and strengthen the assumptions (1.1) by requiring
\[
\bar{F}, \bar{G} \in C^{m+1}([0, T]; W^1_{\infty}(\Omega)^d) \quad \text{and} \quad a, b \in C^m([0, T]; L_{\infty}(\Omega)).
\]
If \( u_0 \in L_2(\Omega) \) and \( g : (0, T) \rightarrow X \) is \( C^m \) with \( \|g^{(i-1)}(t)\| \leq Mt^{\alpha-i} \) for \( 1 \leq i \leq m \), then \( u_X \in C^m_\alpha \) and \( \|u_X\|_{m, \alpha} \leq C_X \|f_X\|_{m, \alpha} \leq C_X (\|u_0\| + M) \).

Proof. Our assumptions on \( u_0 \) and \( g \) imply that \( f_X \in C^m_\alpha \). Using the substitution \( z = s + (t-s)y \) in (1.9), we find that if \( j + k \leq m \) and \( 0 \leq s < t \leq T \), then
\[
\| \partial_t^j (\partial_t + \partial_s)^k K(t, s)v \|_{H^{1-1}(\Omega)} \leq C_X (t-s)^{\alpha-1-k} \|v\|_{H^1_0(\Omega)} \quad \text{for} \quad v \in H^1_0(\Omega),
\]
and, since \( X \) is finite dimensional,
\[
\| \partial_t^j (\partial_t + \partial_s)^k K_X(t, s)v \|_{X} \leq C_X (t-s)^{\alpha-1-k} \|v\|_{X} \quad \text{for} \quad v \in X.
\]

Hence, the Volterra operator \( \mathcal{K}_X : C^m_\alpha \rightarrow C^m_\alpha \) is compact [37, Theorem 6.1]. Theorem 3.1 implies that the homogeneous equation, \( u_X + \mathcal{K}_X u_X = 0 \), has only the trivial solution \( u_X = 0 \), and therefore the inhomogeneous equation \( u_X + \mathcal{K}_X u_X = f_X \) is well-posed not only in \( Y \) but also in \( C^m_\alpha \). \( \Box \)

Our goal in the remainder of this section is to obtain bounds for \( \|u_X(t)\| \) and \( \|\nabla u_X(t)\| \) with constants that are independent of \( X \). Our proof relies on a sequence of technical lemmas. To simplify our estimates, we rescale the time variable, if necessary, so that the minimal eigenvalue of \( \kappa \) is bounded below by unity:
\[
\lambda_{\min}(\kappa(x)) \geq 1 \quad \text{for} \quad x \in \Omega. \tag{3.3}
\]
In this way, \( \langle \kappa \nabla v, \nabla v \rangle \geq \|\nabla v\|^2 \) for \( v \in H^1_0(\Omega) \), and we see from (2.1) that for (real-valued) \( \phi \in C([0, T]; H^1_0(\Omega)) \),
\[
\int_0^t \langle \kappa \mathcal{I}^t \nabla \phi, \nabla \phi \rangle \, ds = \frac{\cos(\pi \mu/2)}{\pi} \int_0^\infty y^{-\mu} \langle \kappa \nabla \phi(iy), \nabla \phi(iy) \rangle \, dy \geq \frac{\cos(\pi \mu/2)}{\pi} \int_0^\infty y^{-\mu} \|\nabla \phi(iy)\|^2 \, dy,
\]
so
\[
\int_0^t \langle \kappa \mathcal{I}^t \nabla \phi, \nabla \phi \rangle \, ds \geq \int_0^t \langle \mathcal{I}^t \nabla \phi, \nabla \phi \rangle \, ds = \mathcal{Q}_t^a(\nabla \phi, t). \tag{3.4}
\]
Since (1.7) is equivalent to (1.8), if \( v \in X \) then
Choosing where in time, we see that

Thus, the solution of (3.1) satisfies

and

(3.1)
tion

(2.11) to write

because of (3.3). Thus, after canceling the term

which yields the following estimates (with

Assuming \( \phi \in C_0^1([0, T]; X) \), we may integrate by parts and use the notation (2.11) to write

\[
\tilde{B}_1 \phi(t) = \int_0^t \left( \tilde{F}(s)\partial_s^{1-\alpha} \phi(s) + \tilde{G}(s)\phi(s) \right) ds,
\]

\[
B_2 \phi(t) = \int_0^t \left( a(s)\partial_s^{1-\alpha} \phi(s) + b(s)\phi(s) \right) ds.
\]

Assuming \( \phi \in C_0^1([0, T]; X) \), we may integrate by parts and use the notation (2.11) to write

\[
\tilde{B}_1 = B_1 + B_1^1 \quad \text{and} \quad B_2 = B_a + B_b.
\]

Thus, the solution of (3.1) satisfies

\[
\langle u_X(t), v \rangle + \langle \kappa \nabla T^\alpha u_X(t), \nabla v \rangle - \langle (\tilde{B}_1 u_X)(t), \nabla v \rangle - \langle (B_2 u_X)(t), v \rangle = \langle f_X(t), v \rangle \quad \text{for } v \in X,
\]

(3.7)

which yields the following estimates (with \( C \) independent of \( X \)).

**Lemma 3.1.** For \( 0 \leq t \leq T \), the solution \( u_X \) of the Volterra equation (3.1) satisfies the a priori estimates

\[
Q_1^\alpha(u_X, t) + Q_2^\alpha(\nabla u_X, t) \leq C t^\alpha Q_0^0(f_X, t)
\]

and

\[
Q_0^0(u_X, t) + Q_1^\alpha(\nabla u_X, t) \leq C Q_0^0(f_X, t).
\]

**Proof.** From (3.7),

\[
\langle u_X(t), v \rangle + \langle \kappa \nabla T^\alpha u_X(t), \nabla v \rangle \leq \frac{1}{2} \| \nabla v \|^2 + \frac{1}{2} \| \tilde{B}_1 u_X(t) \|^2 + \frac{1}{2} \| B_2 u_X(t) \|^2 + \frac{1}{2} \| f_X(t) \|^2 + \langle f_X(t), v \rangle.
\]

Choosing \( v = T^\alpha u_X(t) \) we have

\[
\langle \kappa \nabla T^\alpha u_X(t), \nabla v \rangle = \langle \kappa \nabla v, \nabla v \rangle \geq \| \nabla v \|^2
\]

because of (3.3). Thus, after canceling the term \( \frac{1}{2} \| \nabla v \|^2 \) and integrating in time, we see that

\[
Q_1^\alpha(u_X, t) + \frac{1}{2} Q_2^\alpha(\nabla u_X, t) \leq \frac{1}{2} Q_0^0(\tilde{B}_1 u_X, t) + \frac{1}{2} Q_0^0(B_2 u_X, t) + \frac{1}{2} Q_2^\alpha(u_X, t) + \int_0^t \langle f_X(s), T^\alpha u_X(s) \rangle ds.
\]

(3.8)
Using the representation (3.6) and the achieved estimate (2.12),
\[ Q^0(B_1u_X, t) \leq 2Q^0(B_F^0u_X, t) + 2Q^0(B_G^1u_X, t) \]
\[ \leq CQ_2^2(u_X, t) + CQ_2^1(u_X, t) \leq CQ_2^1(u_X, t), \]
where, in the final step, we used Lemma 2.4. In the same way,
\[ Q^0(B_2u_X, t) \leq CQ_2^1(u_X, t). \]
Using (2.5) with \( \phi = f_X, \psi = u_X \) and \( \epsilon = 1/2 \), we deduce that
\[ Q_1^0(u_X, t) + \frac{1}{2}Q_2^0(\nabla u_X, t) \leq CQ_2^2(u_X, t) + C\|u_X\|^2_{L^2} + \frac{1}{2}Q_1^0(u_X, t). \]
Hence, applying Lemma 2.2 with \( \phi = u_X \), we can show that the function \( q(t) = Q_1^0(u_X, t) + Q_2^0(\nabla u_X, t) \) satisfies
\[ q(t) \leq Ct^\alpha Q^0(f_X, t) + C \int_0^t \omega_3(t-s)Q_1^0(u_X, s) ds. \]
Since \( Q_1^0(u_X, s) \leq q(s) \), Lemma 2.5 implies the first estimate.

To show the second estimate, use \(-\langle (B_1u_X)(t), \nabla v \rangle = \langle \nabla \cdot B_1u_X(t), v \rangle \) in (3.7) to obtain
\[ \langle u_X(t), v \rangle + \langle \kappa \nabla I^\alpha u_X(t), \nabla v \rangle \leq \frac{1}{2}\|v\|^2 + \frac{3}{2}\|\nabla \cdot (B_1u_X)(t)\|^2 + \frac{3}{2}\|B_2u_X(t)\|^2 + \frac{3}{2}\|f_X(t)\|^2. \]
Choosing \( v = u_X(t) \), integrating in time, and using (3.4), we have
\[ \frac{1}{2}Q^0(u_X, t) + Q_1^0(\nabla u_X, t) \leq CQ^0(\nabla \cdot B_1u_X, t) + CQ^0(B_2u_X, t) + CQ^0(f_X, t). \]
Since
\[ \nabla \cdot (B_F^0u_X)(t) = (\nabla \cdot \tilde{F}(t))I^\alpha u_X(t) + \tilde{F}(t) \cdot I^\alpha \nabla u_X(t) \]
\[ - \int_0^t \left( (\nabla \cdot \tilde{F}'(s))I^\alpha u_X(s) + \tilde{F}'(s) \cdot I^\alpha \nabla u_X(s) \right) ds \quad (3.9) \]
it follows that
\[ \|\nabla \cdot (B_F^0u_X)(t)\|^2 \leq C\|I^\alpha u_X(t)\|^2 + C\|I^\alpha \nabla u_X(t)\|^2 \]
\[ + C \int_0^t \left( \|I^\alpha u_X(s)\|^2 + \|I^\alpha \nabla u_X(s)\|^2 \right) ds, \]
implicating that \( Q^0(\nabla \cdot B_F^0u_X, t) \leq CQ_2^2(u_X, t) + C\|u_X\|^2_{L^2} \). In the same way, \( Q^0(\nabla \cdot B_G^1u_X, t) \leq CQ_2^1(u_X, t) + C\|u_X\|^2_{L^2} \) and therefore, by Lemma 2.4
\[ Q^0(\nabla \cdot B_1u_X, t) \leq CQ_2^0(u_X, t) + CQ_2^1(\nabla u_X, t). \]
Recall $Q^0(B_{2u_X}, t) \leq C Q^0_2(u_X, t)$ and let $q(t) = Q^0_1(u_X, t) + Q^0_1(\nabla u_X, t)$.

It follows using Lemma 2.22 and 2.25 that

\[
q(t) \leq C Q^0_2(u_X, t) + C Q^0_2(\nabla u_X, t) + C Q^0(f_X, t)
\leq C Q^0_1(f_X, t) + C \int_0^t \omega_\alpha(t-s) \left( Q^0_1(u_X, s) + Q^0_1(\nabla u_X, s) \right) ds
\leq C Q^0_1(f_X, t) + C t^\alpha \int_0^t \omega_\alpha(t-s) q(s) ds.
\]

We may now apply Lemma 2.25 to complete the proof. \hfill \Box

The function $M u_X(t) = tu_X(t)$ satisfies a similar estimate to the first one in Lemma 3.1, but with an additional factor $t^2$ on the right-hand side.

**Lemma 3.2.** The solution $u_X$ of (3.1) satisfies

\[
Q^0_1(M u_X, t) + Q^0_2(M \nabla u_X, t) \leq C t^{2+\alpha} Q^0(f_X, t) \quad \text{for } 0 \leq t \leq T.
\]

**Proof.** Multiplying both sides of (3.7) by $t$, and applying the third identity in (2.10), we find that (since $\kappa$ is independent of $t$)

\[
\langle M u_X, v \rangle + \langle \kappa(\mathcal{I}_1 \mathcal{M} + \alpha \mathcal{I}_1^{\alpha+1}) \nabla u_X, \nabla v \rangle
= \langle M \vec{B}_1 u_X, \nabla v \rangle + \langle M(f_X - B_{2u_X}), v \rangle,
\]

whereas integrating (3.7) in time gives

\[
\langle \kappa \mathcal{I}_1^{\alpha+1} \nabla u_X, \nabla v \rangle = \langle \mathcal{I}_1 \vec{B}_1 u_X, \nabla v \rangle + \langle \mathcal{I}_1(f_X - u_X - B_{2u_X}), v \rangle,
\]

so, after eliminating $\langle \kappa \mathcal{I}_1^{\alpha+1} \nabla u_X, \nabla v \rangle$,

\[
\langle M u_X, v \rangle + \langle \kappa \mathcal{I}_1 \mathcal{M} \nabla u_X, \nabla v \rangle
= \langle (M - \alpha \mathcal{I}_1) \vec{B}_1 u_X, \nabla v \rangle
+ \langle (M - \alpha \mathcal{I}_1)(f_X - B_{2u_X}) + \alpha \mathcal{I}_1 u_X, v \rangle
\leq \frac{1}{2} \| \nabla v \|^2 + \frac{1}{2} \| \vec{B}_3 u_X \|^2 + \frac{1}{2} \| B_{4u_X} \|^2 + \frac{1}{2} \| v \|^2 + \langle (M - \alpha \mathcal{I}_1) f_X + \alpha \mathcal{I}_1 u_X, v \rangle,
\]

where $\vec{B}_3 \phi = (M - \alpha \mathcal{I}_1) \vec{B}_1 \phi$ and $B_{4} \phi = (M - \alpha \mathcal{I}_1) B_2$. By choosing $v = \mathcal{I}^\alpha M u_X$, we have $\langle \kappa \mathcal{I}^\alpha \mathcal{M} \nabla u_X, \nabla v \rangle = \langle \kappa \nabla v, \nabla v \rangle \geq \| \nabla v \|^2$ so, after canceling the term $\frac{1}{2} \| \nabla v \|^2$ and integrating in time,

\[
Q^0_1(M u_X, t) + \frac{1}{2} Q^0_2(M \nabla u_X, t)
\leq \frac{1}{2} Q^0_1(B_{3u_X}, t) + \frac{1}{2} Q^0_1(B_{4u_X}, t) + \frac{1}{2} Q^0_2(M u_X, t)
+ \int_0^t \langle (M - \alpha \mathcal{I}_1) f_X, \mathcal{I}^\alpha M u_X \rangle ds + \alpha \int_0^t \langle \mathcal{I}^1 u_X, \mathcal{I}^\alpha M u_X \rangle ds.
\]
Using (2.6), we find that
\[
\int_0^t \langle (\mathbf{M} - \alpha \mathbf{I}) f_X, \mathcal{I}^a \mathbf{M} u_X \rangle \, ds \leq Ct^\alpha Q^0((\mathbf{M} - \alpha \mathbf{I}) f_X, t) + \frac{1}{4} Q^0_1(\mathbf{M} u_X, t)
\]
and
\[
\int_0^t \langle \mathcal{I}^1 u_X, \mathcal{I}^a \mathbf{M} u_X \rangle \, ds \leq Ct^\alpha Q^0(\mathcal{I}^1 u_X, t) + \frac{1}{4} Q^0_1(\mathbf{M} u_X, t),
\]
so
\[
Q^0_1(\mathbf{M} u_X, t) + Q^0_2(\mathbf{M} \nabla u_X, t) \leq Q^0(B_3 u_X, t) + Q^0(B_4 u_X, t) + 2Q^0_2(\mathbf{M} u_X, t) + Ct^\alpha Q^0((\mathbf{M} - \alpha \mathbf{I}) f_X, t) + Ct^\alpha Q^0(\mathcal{I}^1 u_X, t).
\]
Since
\[
B_3 = (\mathbf{M} - \alpha \mathbf{I}) B_F^\alpha + (\mathbf{M} - \alpha \mathbf{I}) B_G^\alpha
\]
and
\[
B_4 = (\mathbf{M} - \alpha \mathbf{I}) B_0^\alpha + (\mathbf{M} - \alpha \mathbf{I}) B_1^\alpha,
\]
the estimate (2.13) gives
\[
Q^0(\mathcal{I}^3 u_X, t) + Q^0(B_4 u_X, t) \leq Ct^2 Q^0_2(\mathbf{M} u_X, t) + Ct^2 Q^0_2(u_X, t)
\]
\[
\leq Ct^2 Q^0_2(u_X, t),
\]
where, in the last step, we used Lemma 2.4 with \(\mu = \alpha\) and \(\nu = 1\). We easily verify that
\[
Q^0((\mathbf{M} - \alpha \mathbf{I}) f_X, t) \leq Ct^2 Q^0(f_X, t),
\]
and by Lemma 2.4 with \(\mu = 0\) and \(\nu = 1\),
\[
Q^0(\mathcal{I}^1 u_X, t) = Q^0_1(u_X, t) \leq t^2 Q^0_1(u_X, t).
\]
Thus, the function \(q(t) = Q^0_1(\mathbf{M} u_X, t) + Q^0_2(\mathbf{M} \nabla u_X, t)\) satisfies
\[
q(t) \leq Ct^2 Q^0_2(u_X, t) + 2Q^0_2(\mathbf{M} u_X, t) + Ct^{2+\alpha} Q^0(f_X, t) + Ct^{2+\alpha} Q^0(\mathcal{I}^1 u_X, t).
\]
By (2.4) and Lemma 3.1
\[
t^2 Q^0_2(u_X, t) + t^{2+\alpha} Q^0(\mathcal{I}^1 u_X, t) \leq Ct^{2+\alpha} Q^0(u_X, t) \leq Ct^{2+\alpha} Q(\mathcal{I}^1 u_X, t),
\]
and therefore, using Lemma 2.2 with \(\phi = \mathbf{M} u_X\),
\[
q(t) \leq Ct^{2+\alpha} Q^0(f_X, t) + C \int_0^t \omega_\alpha(t-s) q(s) \, ds,
\]
The result now follows by applying Lemma 2.5 \(\square\)

**Lemma 3.3.** The solution \(u_X\) of (3.1) satisfies, for \(0 \leq t \leq T\),
\[
Q^0_1((\mathbf{M} u_X)'(t) + Q^0_2((\mathbf{M} \nabla u_X)'(t) \leq Ct^\alpha Q^0(f_X, t) + Ct^\alpha Q^0((\mathbf{M} f_X)', t).
\]
Proof. By differentiating (3.10) with respect to \( t \), we have
\[
\langle (M u_X)', v \rangle + \langle \kappa \nabla (I^\alpha M u_X)', \nabla v \rangle = \langle \tilde{B}_5 u_X - \alpha \kappa I^\alpha \nabla u_X, \nabla v \rangle + \langle (M f_X)', B_6 u_X, v \rangle, \tag{3.11}
\]
where \( \tilde{B}_5 \phi = (M \tilde{B}_1 \phi)' \) and \( B_6 \phi = (MB_2 \phi)' \). Hence,
\[
\langle (M u_X)', v \rangle + \langle \kappa \nabla (I^\alpha M u_X)', \nabla v \rangle \leq \frac{1}{2} \| \nabla v \|^2 + \| \tilde{B}_5 u_X \|^2 + \frac{1}{2} \| B_6 u_X \|^2 + \frac{1}{2} \| v \|^2 + C \| I^\alpha \nabla u_X \|^2 + \langle (M f_X)', v \rangle.
\]

Putting \( v = I^\alpha (M u_X)' \), we can cancel \( \frac{1}{2} \| \nabla v \|^2 \) because \( v = (I^\alpha M u_X)' \) by (2.7). Thus, by integrating in time and using (2.6) to show
\[
\int_0^t \langle (M f_X)', I^\alpha (M u_X)' \rangle \ ds \leq Ct^\alpha Q^0((M f_X)', t) + \frac{1}{2} Q^0((M u_X)', t),
\]
and using (3.4), we arrive at the estimate
\[
Q^1_1((M u_X)', t) + Q^2_2((M \nabla u_X)', t) \leq 2Q^0(\tilde{B}_5 u_X, t) + Q^0(B_6 u_X, t) + Q^2_2((M u_X)', t) + C Q^0((M f_X)', t).
\]

Since
\[
\tilde{B}_5 u_X = (MB_3^1 u_X)' + (MB_5^1 u_X)',
\]
and
\[
B_6 u_X = (MB_2^1 u_X)' + (MB_6^1 u_X)',
\]
it follows from (2.11) that
\[
Q^0(\tilde{B}_5 u_X, t) + Q^0(B_6 u_X, t) \leq C Q^3_2((M u_X)', t) + C Q^2_2((M u_X)', t) + C Q^2_2(u_X, t).
\]

By Lemmas 2.4, 3.1, and 3.2
\[
Q^2_3(M u_X, t) + Q^3_2(u_X, t) \leq Ct^\alpha Q^1_1(M u_X, t) + Ct^\alpha Q^2_2(u_X, t) \leq C(t^{2+2\alpha} + t^{2\alpha}) Q^0(f_X, t)
\]
and \( Q^2_3(\nabla u_X, t) \leq Ct^\alpha Q^0(f_X, t) \). Hence, the function
\[
q(t) = Q^1_1((M u_X)', t) + Q^2_2((M \nabla u_X)', t)
\]
satisfies
\[
q(t) \leq Ct^\alpha Q^0(f_X, t) + Ct^\alpha Q^0((M f_X)', t) + C Q^2_2((M u_X)', t).
\]

Finally, by Lemma 2.2
\[
Q^2_2((M u_X)', t) \leq C \int_0^t \omega_3(t-s) Q^1_1((M u_X)', s) \ ds \leq C \int_0^t \omega_3(t-s) q(s) \ ds,
\]
and the desired estimate follows by Lemma 2.5. \( \square \)
Lemma 3.4. The solution $u_X$ of (3.1) satisfies, for $0 \leq t \leq T$,

$$Q^0((M_0u_X)'(t)) + Q^0((M\nabla u_X)'(t)) \leq CQ^0(f_X(t)) + CQ^0((M f_X)'(t))$$

Proof. Using $-\langle \bar{B}_5 u_X, \nabla v \rangle = \langle \nabla \cdot \bar{B}_5 u_X(t), v \rangle$ in (3.11), we obtain

$$\langle (M_0u_X)', v \rangle + \langle \kappa T^\alpha (M\nabla u_X)', \nabla v \rangle \leq \frac{1}{2} \|v\|^2 + 2\|\nabla \cdot \bar{B}_5 u_X\|^2 + 2\|B_6 u_X\|^2 + \|((M f_X)'\|^2 - \alpha \langle \kappa T^\alpha \nabla u_X, \nabla v \rangle.$$  

Choosing $v = (M_0u_X)'$, integrating in time, and using (3.4) yields

$$\frac{1}{2}Q^0((M_0u_X)', t) + Q^0((M\nabla u_X)', t) \leq 2Q^0(\nabla \cdot \bar{B}_5 u_X, t) + 2Q^0(B_6 u_X, t)$$

$$+ Q^0((M f_X)', t) - \alpha \int_0^t \langle (M\nabla u_X)'(s), \kappa T^\alpha \nabla u_X(s) \rangle ds.$$  

Recall from (3.9) that $\nabla \cdot B_5^a \phi = B_5^{a, F} \phi + B_5^{a, \nabla \phi}$, where we have used the notation

$$B_5^a \nabla \phi = \bar{F}(t) \cdot T^a \nabla \phi - \int_0^t \bar{F}(s) \cdot T^a \nabla \phi(s) ds.$$  

Thus,

$$\nabla \cdot \bar{B}_5 u_X = \nabla \cdot (M\bar{B}_1 u_X)' = (M\nabla \cdot \bar{B}_1 u_X)'$$

$$= (M\nabla \cdot B_5^a u_X)' + (M\nabla \cdot B_5^a u_X)'$$

$$= (MB_5^a \nabla \phi)' + (MB_5^a \nabla \phi)'$$

$$+ (MB_5^a \nabla \phi)' + (MB_5^a \nabla \phi)'$$

and so, by (2.14),

$$Q^0(\nabla \cdot \bar{B}_5 u_X, t) + Q^0(\nabla \cdot \bar{B}_5 u_X, t) \leq CQ^0((M\nabla u_X)', t) + CQ^0(M u_X, t)$$

$$+ CQ^0((M\nabla u_X)', t) + CQ^0(M\nabla u_X, t) + CQ^0(M\nabla u_X, t)$$

By (2.3),

$$\int_0^t \langle (M\nabla u_X)'(s), \kappa T^\alpha \nabla u_X(s) \rangle ds \leq \frac{1}{2}Q^0((M\nabla u_X)', t) + CQ^0((M\nabla u_X)', t),$$

and thus the function $q(t) = Q^0((M u_X)', t) + Q^0((M\nabla u_X)', t)$ satisfies

$$q(t) \leq CQ^0((M u_X)', t) + CQ^0(M u_X, t) + CQ^0((M\nabla u_X)', t)$$

$$+ CQ^0((M\nabla u_X)', t) + CQ^0((M\nabla u_X)', t) + CQ^0((M\nabla u_X, t)$$

$$+ CQ^0((M f_X)', t) + CQ^0((M f_X)', t)$$

$$\leq CQ^0((M u_X)', t) + CQ^0((M u_X, t) + CQ^0((M f_X)', t)$$

$$+ CQ^0((M\nabla u_X)', t) + CQ^2 \phi(\nabla u_X, t) + CQ^2 \phi(\nabla u_X, t)$$

$$+ CQ^0((M f_X)', t) + CQ^0((M f_X)', t)$$

$$+ CQ^0((M f_X)', t) + CQ^0((M f_X)', t).$$
where, in the second step, we used Lemmas 2.2, 3.1 and 3.2. A further application of Lemmas 3.1 and 3.2 yields
\[ q(t) \leq C Q^0((M f_X)' , t) + C Q^0(f_X , t) + C Q^0_2((M u_X)' , t) + C Q^0_2((M \nabla u_X)' , t). \]
Lemma 2.2 implies that \( Q^0_2((M u_X)' , t) + Q^0_2((M \nabla u_X)' , t) \) is bounded by
\[ C \int_0^t \omega_\alpha(t - s) \left( Q^0_1((M u_X)' , s) + Q^0_1((M \nabla u_X)' , s) \right) ds \leq C \int_0^t \omega_\alpha(t - s) q(s) ds, \]
where we used \( Q^0_1((M u_X)' , s) \leq C t^\alpha Q^0((M u_X)' , s) \), which follows by Lemma 2.4. Finally, Lemma 2.3 implies the desired estimate. \( \square \)

The preceding lemmas yield the main result for this section.

**Theorem 3.3.** Assume that the coefficients satisfy (1.1), that the initial data \( u_0 \in L^2(\Omega) \) and that the source term satisfies (3.2). Then, the solution \( u_X \) of the projected Volterra equation (3.1) satisfies (with \( C \) independent of \( X \))
\[ \| u_X(t) \|^2 + t^\alpha \| \nabla u_X(t) \|^2 \leq C \left( \| u_0 \|^2 + M^2 t^{2n} \right) \text{ for } 0 \leq t \leq T. \]

**Proof.** The function \( \phi = M u_X \) satisfies \( \| \phi'(t) \| \leq C t^\alpha \) by Theorem 3.2 so, applying Lemma 2.3 with \( \mu = 0 \), we see that Lemma 3.3 gives
\[ t^2 \| u_X(t) \|^2 = \| M u_X(t) \|^2 \leq C t^{1-\alpha} Q^0_1((M u_X)' , t) \leq C t^\alpha Q^0(f_X , t) + C t Q^0((M f_X)' , t). \]
Define \( g_X : [0, T] \to X \) by \( g_X(t) = \langle g(t), v \rangle \) for \( v \in X \), and observe that \( f_X = u_0 + \mathcal{L}^1 g_X \) and \( (M f_X)' = f_X + M f_X = f_X + M g_X \). We find using (3.2) that
\[ Q^0(f_X , t) + Q^0((M f_X)' , t) \leq C \int_0^t \left( \| u_0 \|^2 + \| \mathcal{L}^1 g \|^2 + \| M g \|^2 \right) ds \leq C t (\| u_0 \|^2 + M^2 t^{2n}), \]
so the estimate for the first term \( \| u_X(t) \|^2 \) follows at once. Similarly, applying Lemma 2.3 with \( \phi = M \nabla u_X \) followed by Lemma 3.4, we have
\[ t^{2+\alpha} \| \nabla u_X(t) \| = t^{\alpha} \| M \nabla u_X(t) \|^2 \leq C t Q^0_1((M \nabla u_X)' , t) \leq C t Q^0(f_X , t) + C t Q^0((M f_X)' , t), \]
implying the estimate for the second term \( t^\alpha \| \nabla u_X(t) \|^2 \). \( \square \)
4. The weak solution

We will now establish that the weak formulation (1.5) of the initial-boundary value problem (1.1)–(1.3) is well-posed. The proof relies on our estimates from Section 3 and also the following local Hölder continuity properties of \( u_X \).

**Lemma 4.1.** If \( 0 < \delta \leq t_1 < t_2 \leq T \), then
\[
\|u_X(t_2) - u_X(t_1)\|^2 \leq C\delta^{-2}t_2(\|u_0\|^2 + M^2t_2^2)(t_2 - t_1)
\]
and
\[
\|\mathcal{I}^\alpha \nabla u_X(t_2) - \mathcal{I}^\alpha \nabla u_X(t_1)\| \leq C(\|u_0\| + Mt_2^\alpha)\left[\delta^{-2}(t_2 - t_1) + \delta^{-\alpha/2}(t_2 - t_1)^\alpha\right].
\]

**Proof.** The Cauchy–Schwarz inequality implies that
\[
\|u_X(t_2) - u_X(t_1)\|^2 = \left\| \int_{t_1}^{t_2} u_X'(s) \, ds \right\|^2 \leq (t_2 - t_1) \int_{t_1}^{t_2} \|u_X'(s)\|^2 \, ds,
\]
and by the second inequality of Lemma 3.1, together with Lemma 3.4,
\[
\int_{t_1}^{t_2} \|u_X'(s)\|^2 \, ds = \int_{t_1}^{t_2} s^{-\delta/2} \left\| (\mathcal{M}u_X)'(s) - u_X(s) \right\|^2 \, ds
\]
\[
\leq 2\delta^{-2} \int_0^{t_2} \left( \| (\mathcal{M}u_X)' \|^2 + \|u_X\|^2 \right) \, ds
\]
\[
= 2\delta^{-2} \left[ Q^0(\mathcal{M}u_X', t_2) + Q^0(u_X, t_2) \right]
\]
\[
C\delta^{-2} \left[ Q^0(Mf_X', t_2) + Q^0(f_X, t_2) \right].
\]
The first result now follows from (3.12). To prove the second, we write
\[
\mathcal{I}^\alpha \nabla u_X(t_2) - \mathcal{I}^\alpha \nabla u_X(t_1) = \int_{0}^{t_1-\delta/2} \left[ \omega_\alpha(t_2 - s) - \omega_\alpha(t_1 - s) \right] \nabla u_X(s) \, ds
\]
\[
+ \int_{t_1-\delta/2}^{t_1} \left[ \omega_\alpha(t_2 - s) - \omega_\alpha(t_1 - s) \right] \nabla u_X(s) \, ds + \int_{t_1}^{t_2} \omega_\alpha(t_2 - s) \nabla u_X(s) \, ds,
\]
and deduce from Theorem 3.3 that
\[
\|\mathcal{I}^\alpha \nabla u_X(t_2) - \mathcal{I}^\alpha \nabla u_X(t_1)\| \leq C(\|u_0\| + Mt_2^\alpha)(I_1 + I_2 + I_3),
\]
where
\[ I_1 = \int_0^{t_1 - \delta/2} \left[ \omega_\alpha(t_1 - s) - \omega_\alpha(t_2 - s) \right] s^{-\alpha/2} ds, \]

\[ I_2 = \int_{t_1 - \delta/2}^{t_1} \left[ \omega_\alpha(t_1 - s) - \omega_\alpha(t_2 - s) \right] s^{-\alpha/2} ds, \]

\[ I_3 = \int_{t_1}^{t_2} \omega_\alpha(t_2 - s) s^{-\alpha/2} ds. \]

By the mean value theorem,

\[ \omega_\alpha(t_1 - s) - \omega_\alpha(t_2 - s) = (t_2 - t_1)|\omega_\alpha-1(\xi)| \quad \text{with} \quad t_1 - s < \xi < t_2 - s, \]

and if \( 0 < s < t_1 - \delta/2 \) then \( t_1 - s > \delta/2 \) so

\[ I_1 \leq (t_2 - t_1)|\omega_\alpha-1(\delta/2)| \int_0^{t_1 - \delta/2} \frac{ds}{s^{\alpha/2}} \]

\[ \leq \left( \frac{2}{\delta} \right)^{2-\alpha} \frac{1-\alpha}{1-\alpha/2} \frac{(t_1 - \delta/2)^{1-\alpha/2}}{\Gamma(\alpha)}(t_2 - t_1). \]

Moreover,

\[ I_2 \leq (\delta/2)^{-\alpha/2} \int_{t_1 - \delta/2}^{t_1} \left[ \omega_\alpha(t_1 - s) - \omega_\alpha(t_2 - s) \right] ds \]

\[ = (2/\delta)^{\alpha/2} \left[ \omega_{\alpha+1}(t_2 - t_1) + \omega_{\alpha+1}(\delta/2) - \omega_{\alpha+1}(t_2 - t_1 + \delta/2) \right] \]

\[ \leq (2/\delta)^{\alpha/2} \omega_{\alpha+1}(t_2 - t_1) \]

and \( I_3 \leq \delta^{-\alpha/2} \int_{t_1}^{t_2} \omega_\alpha(t_2 - s) ds = \delta^{-\alpha/2} \omega_{\alpha+1}(t_2 - t_1). \)

Our existence theorem is stated as follows. Note the weak continuity at \( t = 0 \) asserted in part 5; we show in the companion paper \[29\] that the solution \( u \) is continuous on the closed interval \([0, T]\) provided \( u_0 \in \dot{H}^\mu(\Omega) \) for some \( \mu > 0. \)

**Theorem 4.1.** Assume that the coefficients satisfy \[1.4\], that the source term satisfies \[3.2\], and that the initial data \( u_0 \in L_2(\Omega) \). Then, the initial-boundary value problem \[1.1\]–\[1.3\] has a weak solution \( u : [0, T] \rightarrow L_2(\Omega) \) with the following properties:

1. The restriction \( u : (0, T] \rightarrow L_2(\Omega) \) is continuous.
2. If \( 0 < t \leq T \), then \( u(t) \in H_0^1(\Omega) \) with

\[ \|u(t)\| + t^{\alpha/2}\|\nabla u(t)\| \leq C\left(\|u_0\| + M^{\eta}\right). \]
(3) The functions \( T^\alpha u \) and \( B_2 u \) are continuous from the closed interval \([0, T]\) to \( L_2(\Omega)\). Likewise, \( T^\alpha \nabla u \) and \( B_1 u \) are continuous from \([0, T]\) to \( L_2(\Omega)^d\).

(4) At \( t = 0 \) we have \( T^\alpha u = B_2 u = 0 \), \( T^\alpha \nabla u = B_1 u = 0 \) and \( u(0) = u_0 \).

(5) If \( t \to 0 \), then \( \langle u(t), v \rangle \to \langle u(0), v \rangle \) for each \( v \in L_2(\Omega) \).

\[
\langle u(t), v \rangle + \langle \kappa(T^\alpha \nabla u)(t), \nabla v \rangle - \langle (B_1 u_n)(t), \nabla v \rangle + \langle (B_2 u_n)(t), v \rangle = \langle f(t), v \rangle.
\]

We may therefore define

\[
u(t) = \lim_{n \to \infty} u_n(t) \quad \text{for} \quad 0 < t \leq T,
\]

and this function satisfies Property 1 because, given any fixed \( \delta \in (0, T) \), the limit is uniform for \( t \in [\delta, T] \). Similarly, the functions \( T^\alpha \nabla u_n \) are bounded and equicontinuous in \( C([\delta, T]; L_2(\Omega)^d) \) so \( T^\alpha \nabla u : [0, T] \to L_2(\Omega)^d \) is continuous. In fact, it will follow from (4.1) below that \( \| T^\alpha \nabla u(t) \| \to 0 \) as \( t \to 0 \), so \( T^\alpha \nabla u : [0, T] \to L_2(\Omega)^d \) is continuous.

By Theorem 3.3

\[
\| u_n(t) \| \leq C(\| u_0 \| + Mt^n) \quad \text{for} \quad 0 < t \leq T,
\]

so by sending \( n \to \infty \) we conclude that \( \| u(t) \| \leq C(\| u_0 \| + Mt^n) \). Also, for \( 0 < t \leq T \),

\[
\| u_n(t) \|_{H^1_0(\Omega)} \| v \|_{H^{-1}(\Omega)} \leq C t^{-\alpha/2}(\| u_0 \| + Mt^n) \| v \|_{H^{-1}(\Omega)}
\]

and sending \( n \to \infty \) it follows that

\[
\| u(t) \| \leq C t^{-\alpha/2}(\| u_0 \| + Mt^n) \| v \|_{H^{-1}(\Omega)} \quad \text{for all} \quad v \in L_2(\Omega),
\]

so \( u(t) \in H^1_0(\Omega) \) with \( \| u(t) \|_{H^1_0(\Omega)} \leq C t^{-\alpha/2}(\| u_0 \| + Mt^n) \), establishing Property 2.

Since \( \| u(t) \| \) is bounded, \( T^\alpha u \) is continuous on \([0, T]\) with
\begin{equation}
\|I^a u(t)\| \leq \int_0^t \omega_\alpha(t-s) \|u(s)\| \, ds 
\leq C \int_0^t (t-s)^{\alpha-1}(\|u_0\| + Ms^n) \, ds \leq C(\|u_0\| + Mt^n) t^\alpha,
\end{equation}

and similarly
\begin{equation}
\|I^a \nabla u(t)\| \leq C \int_0^t (t-s)^{\alpha-1} s^{-\alpha/2}(\|u_0\| + Ms^n) \, ds \leq C(\|u_0\| + Mt^n) t^{\alpha/2}.
\end{equation}

Likewise, for \( n \geq 1 \),
\begin{equation}
\|I^a u_n(t)\| \leq C(\|u_0\| + Mt^n) t^\alpha \quad \text{and} \quad \|I^a \nabla u_n(t)\| \leq C(\|u_0\| + Mt^n) t^{\alpha/2}.
\end{equation}

Continuity of \( \bar{B}_1 u \) and \( B_2 u \) follow from (2.11) and (3.6), completing the proof of property 3, with
\begin{equation}
\|(\bar{B}_1 u)(t)\| + \|(B_2 u)(t)\| \leq C\|[I^a u](t)\| + C \int_0^t \|(I^a u)(s)\| + \|u(s)\| \, ds 
\leq C(\|u_0\| + M) t^\alpha.
\end{equation}

Property 4 follows from the estimates (4.3), (4.4) and (4.6).

If \( 0 \leq \delta < t \leq T \), then
\begin{equation}
\|(I^a u_n)(t) - (I^a u)(t)\| \leq \int_0^t \omega_\alpha(t-s) \|u_n(s) - u(s)\| \, ds 
\leq C \int_0^\delta (t-s)^{\alpha-1}(\|u_0\| + Ms^n) \, ds + \int_0^t (t-s)^{\alpha-1}\|u_n(s) - u(s)\| \, ds 
\leq C\delta^\alpha(\|u_0\| + M\delta^n) + \alpha^{-1}(t-\delta)^\alpha \max_{\delta \leq s \leq t} \|u_n(s) - u(s)\|,
\end{equation}

showing that \( I^a u_n(t) \to I^a u(t) \) in \( L^2(\Omega) \), uniformly for \( t \in [\delta, T] \). In fact, the convergence is uniform for \( t \in [0, T] \), owing to the estimates (4.3) and (4.5). Therefore, we see using (2.11) and (3.6) that, for \( v \in H_0^1(\Omega) \),
\begin{equation}
\langle (\bar{B}_1 u_n)(t), \nabla v \rangle \to \langle (\bar{B}_1 u)(t), \nabla v \rangle \quad \text{and} \quad \langle (B_2 u_n)(t), v \rangle \to \langle (B_2 u)(t), v \rangle.
\end{equation}

Since \( \langle f_n, \psi_j \rangle = \langle f, \psi_j \rangle \) for \( j \leq n \), we have
\begin{equation}
\lim_{n \to \infty} \langle f_n(t), \psi_j \rangle = \langle f(t), \psi_j \rangle \quad \text{for all } j \geq 1 \text{ and } 0 \leq t \leq T,
\end{equation}

and therefore \( \langle f_n(t), v \rangle \to \langle f(t), v \rangle \) for all \( v \in L^2(\Omega) \). Thus, by sending \( n \to \infty \) in (4.2), it follows that (4.1) holds for \( v \in H_0^1(\Omega) \) and \( 0 < t \leq T \). In light of (4.6) and (4.4), the variational equation (4.1) is satisfied when \( t = 0 \) if and only if \( \langle u(0), \psi \rangle = \langle u_0, \psi \rangle \) for all \( \psi \in H_0^1(\Omega), \) which is the case if and only if we define \( u(0) = u_0 \). Moreover, if \( t \to 0 \) then \( \langle u(t), v \rangle \to \langle f(0), v \rangle =
\[ \langle u_0, v \rangle, \text{ for each } v \in H^1_0(\Omega), \text{ and hence by density for each } v \in L^2(\Omega), \text{ establishing Properties 5 and 6.} \]

**Remark 4.1.** Since our estimates rely on Lemma 2.1 the constant \( C \) in part 2 of Theorem 4.1 becomes unbounded as \( \alpha \to 1 \). However, this behavior appears to be an artifact of our method of proof. In the limiting case when \( \alpha = 1 \) and (1.1) reduces to a parabolic PDE, a simple energy argument combined with the classical Gronwall inequality yields the *a priori* estimate

\[ \| u(t) \| \leq C \left( \| u_0 \| + \int_0^t \| g(s) \| \, ds \right) \text{ for } 0 \leq t \leq T; \]

see also the alternative analysis [21] of the fractional Fokker–Planck equation.

**Theorem 4.2.** The weak solution of the initial-boundary value problem (1.1)–(1.3) is unique. More precisely, under the same assumptions as Theorem 4.1 there is at most one function \( u \) that satisfies (4.1) and is such that \( u \) and \( I_\alpha^\nu u \) belong to \( L^2(0,T;L^2(\Omega)) \), and \( I_\alpha^\nu \nabla u \) belongs to \( L^2(0,T;L^2(\Omega)) \).

**Proof.** The problem is linear, so it suffices to show that if \( u_0 = 0 \) and \( g(t) \equiv 0 \) then \( u(t) \equiv 0 \). Thus, suppose that

\[ \langle u(t), v \rangle + \langle \kappa(I_\alpha^\nu u(t)), \nabla v \rangle - \langle (\bar{B}_1 u)(t), \nabla v \rangle + \langle (B_2 u)(t), v \rangle = 0 \]

for \( 0 < t \leq T \) and \( v \in H^1_0(\Omega) \). Proceeding as in the proof of (3.8), we have

\[ Q_1^\alpha(u,t) + \frac{1}{2} Q_2^\alpha(\nabla u,t) \leq \frac{1}{2} Q_0^\alpha(\bar{B}_1 u,t) + \frac{1}{2} Q_0^\alpha(B_2 u,t) + \frac{1}{2} Q_2^\alpha(u,t) \leq C Q_2^\alpha(u,t), \]

where the final step used (2.11), (2.12) and Lemma 2.4. Thus, applying Lemma 2.2 the function \( q(t) = Q_1^\alpha(u,t) + Q_2^\alpha(\nabla u,t) \) satisfies

\[ q(t) \leq C Q_2^\alpha(\nabla u,t) \leq C \int_0^t \omega_\alpha(t-s)q(s) \, ds, \]

and therefore \( q(t) = 0 \) for \( 0 \leq t \leq T \) by Lemma 2.5. In particular, \( Q_1^\alpha(u,T) = 0 \), so if we put \( u(t) = 0 \) for \( t > T \) then the Laplace transform of \( u \) satisfies \( \hat{u}(iy) = 0 \) for \( -\infty < y < \infty \) by (2.1), implying that \( u(t) = 0 \) for \( 0 \leq t \leq T \).
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