1. Introduction

The most advanced optical frequency standards have now reached levels of stability and accuracy [1–11] that significantly surpass the performance of the best caesium primary standards [12–16], raising the possibility of a future redefinition of the SI second [17]. As a first step towards preparing for a redefinition the International Committee for Weights and Measures (CIPM), following a recommendation of the Consultative Committee for Time and Frequency (CCTF), introduced the concept of secondary representations of the second that may be used to realize the second in parallel to the caesium primary standard. Seven optical frequency standards (and one microwave frequency standard) may currently be used as secondary representations of the second, with recommended frequencies and uncertainties being assigned by the Frequency Standards Working Group (WGFS) of the CCTF and Consultative Committee for Length (CCL). These recommended frequency values are periodically updated and published on the website of the International Bureau of Weights and Measures [18].

With a single exception, the data considered so far by the WGFS comes from absolute frequency measurements made relative to caesium fountain primary frequency standards. However future information about the reproducibility of the optical standards will come mainly from direct optical frequency ratio measurements [8, 19–21]. For example, within the EMRP-funded International Timescales with Optical Clocks (ITOC) project [22], a coordinated comparison programme will lead to a set of frequency ratio measurements between high accuracy optical frequency standards being developed within European NMIs, as well as a comprehensive set of absolute frequency measurements with uncertainties at the limit set by caesium primary standards. This set of measurements will be over-determined, in the sense that it will be possible to deduce some of the frequency ratios from several different experiments. For example, a particular frequency ratio $\nu_a/\nu_b$ could be measured directly or it could be determined indirectly by combining two or more other frequency ratio measurements (e.g. $\nu_a/\nu_b = (\nu_a/\nu_c)(\nu_c/\nu_b)$ or $\nu_a/\nu_b = (\nu_a/\nu_d)(\nu_d/\nu_c)(\nu_c/\nu_b)$). If the indirect determinations of this frequency ratio have comparable uncertainties to the
direct determination then all of them must be considered in
deriving a ‘best’ value for the frequency ratio. These multiple
routes to deriving each frequency ratio value will complicate
the task of the WGFS because it will no longer be possible to
treat each optical frequency standard in isolation when con-
sidering the available data.

Here we describe a possible approach to analyzing over-
determined sets of frequency comparison data to deduce
optimized values for the frequency ratios between each con-
tributing standard. The paper is organised as follows. In sec-
tion 2 we describe the analysis procedure, which follows a
least-squares approach. The tests we have carried out to
ensure that the algorithms have been correctly implemented
in our software are described in section 3. In section 4 we
consider the body of frequency comparison data presently
available in the published literature, and discuss how the
CIPM recommended frequency values might change if recent
measurements were included in the evaluation. However this
analysis neglects correlations between the individual meas-
urements, and in section 5 we discuss how this affects the
results obtained. Finally, section 6 contains some conclusions
and perspectives that may be relevant for future discussions
within the WGFS.

2. Analysis procedure

To derive a self-consistent set of optimised frequency ratio
values from a set of clock frequency comparison experiments
we use a least-squares adjustment procedure. This is based
on the approach used by the Committee on Data for Science
and Technology (CODATA) to provide a self-consistent set
of internationally recommended values of the fundamental
physical constants [23]. Our method is illustrated in figure 1.

Suppose that the frequency standards involved in the com-
parison experiments are based on \(N_0\) different reference tran-
sitions, with frequencies \(v_k\) \((k = 1, 2, \ldots, N_0)\). For example \(v_1\)
could be the \(5s^23p \rightarrow 5s5p\) \(\rho_0\) transition in \(87\)Sr at 698 nm, \(v_2\)
the \(6s^23s \rightarrow 4f^36s^2 \rho_2\) transition in \(171\)Yb\(^{+}\) at 467 nm, \(v_3\)
the 9.2 GHz \(6s^23s_{1/2} \approx F = 3\)–\(6s^23s_{1/2} \approx F = 4\) ground-state
hyperfine transition in \(133\)Cs, and so on. The set of frequency
comparison experiments yields a set of \(N\) measured values \(q_i\),
of various quantities, which may be optical frequency ratios,
microwave frequency ratios, or optical-microwave frequency
ratios. These measured values \(q_i\), together with their standard
uncertainties \(u_i\), variances \(u_i^2 = u_i\) and covariances \(u_{ij}\) (where
\(u_{ij} = u_{ji}\)) form the input data to the least-squares adjustment.

As the first step, we choose a set of \(M = N_0 - 1\) adjusted
frequency ratios \(z_j\) (where \(M < N\)) given by

\[ z_j = \frac{v_j}{u_j}, \]  

(1)

where \(j = 1, 2, \ldots, N_0 - 1\). These adjusted frequency ratios are
the variables in the least-squares adjustment and are equiva-
lent to the adjusted constants in the CODATA analysis of the
fundamental physical constants. Our choice of \(z_j\) meets the
necessary condition that no adjusted frequency ratio may be
expressed as a function of the others.

The quantities \(q_i\) that are measured in the frequency com-
parison experiments are next expressed as a function \(f_i\) of one
or more of these adjusted frequency ratios \(z_j\) by the set of \(N\)
equations

\[ q_i \pm f_i(z_1, z_2, \ldots, z_M) \]  

(2)

where \(i = 1, 2, \ldots, N\). Here the symbol \(\pm\) is used to indicate
that in general the left and right hand sides of the equa-
tions are not exactly equal, because the set of equations is
overdetermined. For example, the first observed quantity \(q_1\)
might be \(v_2/v_5\), which can be expressed as \(z_2z_5z_4\), while the
second observed quantity \(q_2\) might be either another mea-
urement of \(v_2/v_5\) or a measurement of a different ratio such as
\(v_1/v_6\).

The equations (2) are, in most cases, nonlinear. Prior to the
least-squares adjustment, they are therefore linearized using a
Taylor expansion around starting values \(s_j\) (initial estimates of
the adjusted frequency ratios):

\[ q_i \pm f_i(s_1, s_2, \ldots, s_M) + \sum_{j=1}^{M} \frac{\partial f_i}{\partial s_j} (z_j - s_j) + \ldots. \]  

(3)

This enables linear matrix methods to be applied. Defining
new variables

\[ y_i \equiv q_i - f_i(s_1, s_2, \ldots, s_M) \]  

(4)

and

\[ x_i \equiv z_j - s_j, \]  

(5)

the linearized equations (3) can be rewritten in the form

\[ y_i \pm a_{ij} x_j, \]  

(6)

where

\[ a_{ij} \equiv \frac{\partial f_i}{\partial s_j}. \]  

(7)

In matrix notation these equations become

\[ Y \pm AX, \]  

(8)

where \(Y\) is a column matrix with \(N\) elements, \(A\) is a rectan-
gular matrix with \(N\) rows and \(M\) columns, and \(X\) is a column
matrix with \(M\) elements. In the same way, we consider the
measured frequency ratios \(q_i\), the functions \(f_i\) and the adjusted
frequency ratios \(z_j\) and their initial estimates \(s_j\) to be elements of
matrices \(Q, F, Z\) and \(S\) respectively.

To obtain the best estimate of \(X\), and hence of the adjusted
frequency ratios \(Z\), a least-squares adjustment is performed,
minimizing the product

\[ S = (Y - AX)^T V^{-1} (Y - AX) \]  

(9)

with respect to \(X\). Here \(V = \text{cov}(Y)\) is the \(N \times N\) covariance
matrix of the input data, with elements \(v_{ij}\). The solution \(X\) is
used to calculate the best estimate \(Z\) of the adjusted frequency
ratios according to

\[ \hat{Z} = S + \hat{X}. \]  

(10)
Although the solution of the linear approximation (3) does not provide an exact solution of the nonlinear equations (2), the values of the adjusted frequency ratios \( \hat{z}_i \) obtained from the least-squares adjustment will be an improvement over the starting values \( S_j \). To obtain more precise values, these improved values of the adjusted frequency ratios are used as starting values for a new linear approximation and a second least-squares adjustment is performed. This procedure is repeated until the new values of the adjusted frequency ratios obtained from the least-squares adjustment differ from the starting values for that iteration by a sufficiently small fraction of their uncertainties. The number of iterations required to satisfy this condition will depend on how close the starting values \( S_j \) are to the values of \( \hat{z}_j \) calculated in the final iteration. Once convergence has been achieved, the best estimate \( \hat{Q} \) of the measured quantities \( Q \) can be calculated from the final solution \( \hat{X} \):

\[
\hat{Q} = F + A\hat{X}.
\]

In general, the values of the adjusted frequency ratios obtained from the least-squares adjustment will be correlated. The covariance matrix \( \text{cov}(\hat{Z}) = \text{cov}(\hat{X}) \), whose elements \( u(\hat{z}_i, \hat{z}_j) \) are the covariances of the adjusted frequency ratios, can be used to evaluate the uncertainty in other frequency ratios calculated from two or more of the adjusted frequency ratios. In general, there will be a total of \( K \) possible frequency ratios \( \hat{r}_i \) which may be expressed in terms of the adjusted frequency ratios \( \hat{z}_j \):

\[
r_i(\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_M)
\]

where \( i = 1, 2, \ldots, K \), including expressions of the form \( \hat{r}_i = \hat{z}_j \). According to the standard formula for the propagation of uncertainty, the covariances of these optimized frequency ratios are given by

\[
u(\hat{r}_k, \hat{r}_l) = \sum_{j=1}^{M} \frac{\partial \hat{r}_k}{\partial \hat{z}_j} \frac{\partial \hat{r}_l}{\partial \hat{z}_j} u(\hat{z}_j, \hat{z}_j).
\]

If \( l = k \), then equation (13) gives the variances \( u^2(\hat{r}_k) = u(\hat{r}_k, \hat{r}_k) \).

Self-consistency checks on the body of data provide verification of the uncertainty evaluations for each individual frequency standard and enable any issues with individual frequency standards to be identified. To obtain a measure of the consistency of the input data, the Birge ratio

\[
R_B = \left( \frac{\chi^2}{N - M} \right)^{1/2}
\]

is computed. Here

\[
\chi^2 = (Q - \hat{Q})^T V^{-1}(Q - \hat{Q}),
\]

is the minimum value of \( S \) as given by equation (9), evaluated in the final iteration of the least-squares adjustment. A Birge ratio significantly larger than one suggests that the input data are inconsistent. Similarly, a normalized residual

![Figure 1. Analysis procedure used to derive a self-consistent set of optimized frequency ratio values from a set of frequency comparison experiments with \( N \) measured frequency ratios involving standards based on \( N_S \) different reference transitions. So that absolute frequencies can be derived, the ground-state hyperfine transition in \(^{133}\text{Cs} \) is included as one of these reference transitions.](image-url)
Table 1. Frequency standards considered in the tests of the least-squares adjustment procedure.

| Atom/ion  | Reference transition |
|----------|----------------------|
| $^{115}$In$^+$ | $^5S_{0}^0$–$^5S_{0}^1P_0$ |
| $^1$H    | $^1S_{1/2}$–$^1S_{1/2}$ |
| $^{199}$Hg | $^6S_{1/2}$–$^6S_{1/2}$ |
| $^{27}$Al$^+$ | $^3P_0$–$^3P_1$ |
| $^{109}$Hg$^+$ | $^5D_{3/2}$–$^5D_{3/2}$ |
| $^{171}$Yb$^+$ | $^6S_{1/2}$–$^6S_{1/2}$ |
| $^{171}$Yb | $^6S_{1/2}$–$^6S_{1/2}$ |
| $^{40}$Ca | $^4S_{1/2}$–$^4S_{1/2}$ |
| $^{88}$Sr$^+$ | $^5S_{1/2}$–$^5S_{1/2}$ |
| $^{87}$Sr | $^5S_{1/2}$–$^5S_{1/2}$ |
| $^{40}$Ca$^+$ | $^4S_{1/2}$–$^4S_{1/2}$ |
| $^{87}$Rb | $^5S_{1/2}$, $^5P_{1/2}$, $^5P_{3/2}$ |
| $^{133}$Cs | $^6S_{1/2}$, $^6S_{1/2}$, $^6S_{1/2}$ |

\[
\rho_i = \frac{q_i - \bar{q}}{u_i} \tag{16}
\]

significantly larger than one for a particular measured frequency ratio $q_i$ suggests that the measurement is inconsistent with the other data.

These algorithms have been implemented in Matlab, with the least-squares solution to equation (8) being found using the routine `lscov()`. This routine provides the option of using two different algorithms for computing the least-squares solution to equation (9). If the matrix $V$ is well-conditioned, it uses the Cholesky decomposition of $V$; however if $V$ is poorly conditioned, for example as a consequence of strong correlation effects, it uses an orthogonal decomposition algorithm that is more stable but more computationally expensive. Due to the extremely high accuracy with which frequency comparisons can be performed, numerical calculations must be performed with a precision of more than 18 significant figures. This is achieved using routines designed for high precision floating point arithmetic [24].

3. Tests of the software algorithms

Several simulated test data sets were generated for testing the analysis software. These were generated in such a way that the software, if functioning correctly, would generate certain known values for the absolute frequencies of each reference transition. Since the software does not calculate these frequencies or their uncertainties directly, but instead calculates them from combinations of the adjusted frequency ratios and the covariance matrix determined from the least-squares adjustment, this approach is considered to provide a good check that the adjusted frequency ratios are also being calculated correctly.

One key test of the software algorithms employed in the analysis procedure is whether they are able to reproduce the CIPM recommended frequency values, when using the same input data employed by the CCL-CCTF WGFS. This means comparing the ratios between the reference transitions in $^{40}$Ca and $^{87}$Sr and the original input data given in [18]. To check this, the set of $N_S = 15$ frequency standards listed in table 1 was considered.

The results obtained for the seven optical secondary representations of the second are shown in figure 2. All these frequency values agree with the CIPM values, but the uncertainties determined from the least-squares adjustment procedure are smaller than the uncertainties assigned by the CIPM, sometimes by a factor of two or three. The explanation for this is that the WGFS takes a conservative approach to estimating uncertainties, typically multiplying the relative standard uncertainty on the weighted mean of a set of frequency values by a factor of two or three to reflect the fact that the measurements originate from only a few independent research groups (or in some cases a single research group). Our analysis procedures yield uncertainties equivalent to the relative standard uncertainty.

The frequency values we obtain for this test are in similar good agreement with the CIPM values, with the sole exception of the $^{40}$Ca optical frequency standard. The reason for this discrepancy is that in this case the WGFS departed from its normal procedure of taking a weighted mean of the available frequency measurements, and instead used an unweighted mean of the two values. Our analysis procedures, on the other hand, are equivalent to taking a weighted mean of the two values.

4. Inclusion of new frequency comparison data

Since the last review of available frequency data by the WGFS, which was performed in September 2012, a number of new frequency comparison results have been reported. These include ten new absolute frequency measurements (one of the 1S–2S transition in $^1$H [25], two each of the E2 [8, 26] and E3 [8, 9] reference transitions in $^{171}$Yb$^+$, one each of the reference transitions in $^{171}$Yb [27] and $^{88}$Sr$^+$ [6] and three of the reference transition in $^{87}$Sr [3, 7, 28]). However three optical frequency ratios have also been measured directly. These are the ratios between the frequency transitions in $^{40}$Ca$^+$ and $^{87}$Sr.
5. Importance of correlations

The above analysis neglects any correlations between the measured frequency ratio values used as input to the least-squares adjustment. In reality, there will be correlations among the input data that should be accounted for. For example, at NPL, the absolute frequencies of the E2 and E3 reference transitions in $^{171}$Yb$^+$ and the direct optical frequency ratio between them were recently all determined during a single measurement campaign, with substantial periods of overlap in the data-taking periods [8]. This means that there are significant correlations between these three values. As frequency comparison experiments involving multiple optical frequency standards and even multiple laboratories become more frequent, such correlations will become more frequent and more significant. However the covariances and corresponding correlation coefficients between different frequency ratio measurements are not normally reported in the literature, even for values obtained in the same laboratory. To evaluate these covariances additional information would typically need to be obtained from each research group involved, just as in the CODATA least-squares adjustment of the fundamental physical constants [23].

To illustrate the importance of properly accounting for correlations, we consider the hypothetical 10 d measurement campaign illustrated in figure 3. This involves a caesium primary frequency standard, which we assume operates 100% of the time, and three optical frequency standards, which we assume each run for 60% of the time, with some periods of overlap. Six different frequency ratios can be determined, each from different periods of the campaign. For these six frequency ratios, there are twelve non-zero correlation coefficients. Correlations arise from both statistical and systematic uncertainties. For example, the absolute frequency measurement of the E2 transition in $^{171}$Yb$^+$ and the absolute frequency measurement of the reference transition in $^{88}$Sr$^+$ are correlated because part of the caesium primary frequency standard data is common to the two. Assuming initially that all other sources of uncertainty are negligible compared to the statistical uncertainty associated with the caesium standard, then since the $^{171}$Yb$^+$ E2 standard runs for a total period $T_A = 6$ d and the $^{88}$Sr$^+$ standard runs for a total period $T_B = 6$ d, with a period of overlap $T_{overlap} = 3$ d, we calculate a correlation coefficient of $\sqrt{T_{overlap}^2/(T_AT_B)} = 0.5$. However in practice for a measurement of this duration, our initial assumption is not a good one and other contributions to the uncertainty must be considered. For this particular example, there is an additional source of correlation because the systematic uncertainty of the caesium fountain is also common to the two measurements. In a similar way, correlations between the
absolute frequency measurement of the E3 transition in $^{171}\text{Yb}^+$ and the $^{171}\text{Yb}^+\text{E2}/^{171}\text{Yb}^+\text{E3}$ frequency ratio measurement arise from both the statistical and the systematic uncertainty of the $^{171}\text{Yb}^+$ E3 standard. If we use the present stabilities and systematic uncertainties of NPL’s frequency standards [6, 8, 15] to estimate the correlation coefficients for this hypothetical measurement campaign, we find that the values of the twelve correlation coefficients range from $-0.102$ to $0.95$. The largest correlation coefficient is for the Yb E3 / Yb E3 and Yb E2 / $^{88}\text{Sr}^+$ frequency ratios, since the uncertainties of both measurements would be dominated by the systematic uncertainty of the Yb E2 standard at its current state of development.

For arbitrarily-selected values of the measured frequency ratios resulting from this hypothetical 10 d measurement campaign, the effect of correlations on the optimized frequency ratios and absolute frequencies can be determined. As illustrated in figure 4, neglecting correlations leads to too much weight being given to these measurements in the least-squares adjustment, resulting in biased frequency values and underestimated uncertainties. For some planned measurement campaigns, even stronger correlations could potentially arise, resulting in more significant biases if these correlations were neglected in the analysis.

Correlations may also arise when comparing frequency standards in different laboratories. For example, when comparing frequency standards based on the same atomic species the uncertainties due to the blackbody Stark shift coefficients will be correlated. As a second example, if a particular frequency standard is compared with two other frequency standards in two separate experiments, careful consideration should be given to the question of whether or not the systematic uncertainty of the first standard is correlated for the two experiments.

The above discussion demonstrates the importance of gathering information about the correlations between the input data, both for intra-laboratory frequency comparisons and for inter-laboratory frequency comparisons.

6. Conclusion

In summary, we have described an analysis procedure that can be applied to determine a self-consistent set of frequency ratios between high accuracy frequency standards (both optical and microwave) based on all experimental data available at any particular time, and including correlations among the data. Currently the matrix of frequency comparison data is rather sparsely populated, but as the number of frequency ratio measurements made without reference to the caesium primary standard increases, our methods and software could be used to provide valuable information about the relative performance of different candidates for an optical redefinition of the SI second. They can also be used to determine optimized values and uncertainties for the absolute frequencies of each optical standard relative to the current definition of the SI second, since these are simply special cases of frequency ratios involving the caesium primary standard. Such absolute frequency values will be required to maximize the potential contribution of optical clocks to international timescales prior to any redefinition.

Our work also identifies the key issues likely to be encountered in assessing an over-determined set of frequency comparison data. Firstly, it will be necessary to identify and critically review all possible input data, with particular attention being paid to the standard uncertainty associated with each measurement. Secondly, the correlations between the input data must be considered, since these can have a significant effect on the frequency ratio values and uncertainties obtained from the least-squares adjustment. However the information reported in the literature is in many cases insufficient to calculate the correlation coefficients, meaning that it will be necessary to seek additional information from the groups that carried out the measurements in order that each input datum is given the appropriate weight in the least-squares adjustment. Finally, it will be necessary to investigate the extent to which each input datum contributes to the determination of the adjusted frequency ratios as well as the effects of omitting inconsistent or inconsequential data as may be deemed appropriate. None of these issues are unique to the problem discussed here; indeed they are common to those that have been faced by the CODATA Task Group on Fundamental Constants for many years. As such they are likely to be highly relevant to future discussions within the CCL-CCTF Frequency Standards Working Group.

Acknowledgments

This work was performed within the ITOC project as part of the European Metrology Research Programme (EMRP). The EMRP is jointly funded by the EMRP participating countries within EURAMET and the European Union. The authors gratefully acknowledge helpful discussions with partners in the ITOC project consortium and with members of the CCL-CCTF WGFS, in particular those who participated in a working group on this topic led by Chris Oates from NIST prior to the 2012 meeting of the WGFS.

References

[1] Chou C W, Hume D B, Koelmeij J C J, Wineland D J and Rosenband T 2010 Frequency comparison of two high-accuracy Al$^+$ optical clocks Phys. Rev. Lett. 104 070802
[2] Madej A A, Dubé P, Zhou Z, Bernard J E and Gertsvald M 2012 $^{88}\text{Sr}^+$ 445-THz single-ion reference at the $10^{-17}$ level via control and cancellation of systematic uncertainties and its measurement against the SI second Phys. Rev. Lett. 109 203002
[3] Le Targat R et al 2013 Experimental realization of an optical second with strontium lattice clocks Nat. Commun. 4 2109
[4] Hinkley N, Sherman J A, Phillips N B, Schioppo M, Lemke N D, Beloy K, Pizzocaro M, Oates C W and Ludlow A D 2013 An atomic clock with $10^{-18}$ instability Science 341 1215–8
[5] Bloom B J, Nicholson T L, Williams J R, Campbell S L, Bishof M, Zhang X, Zhang W, Bromley S L and Ye J 2014...
An optical lattice clock with accuracy and stability at the $10^{-13}$ level Nature 506 71–5

[6] Barwood G P, Huang G, Klein H A, Johnson L A M, King S A, Margolis H S, Szymaniec K and Gill P 2014 Agreement between two $^{88}$Sr optical clocks to 4 parts in 10$^{13}$ Phys. Rev. A 89 050501

[7] Falke S et al 2014 A strontium lattice clock with $3 \times 10^{-17}$ inaccuracy and its frequency New. J. Phys. 16 073023

[8] Godun R M, Nisbet P B R, Jones J M, King S A, Johnson L A M, Margolis H S, Szymaniec K, Lea S N, Bongs K and Gill P 2014 Frequency ratio of two optical clock transitions in $^{171}$Yb$^+$ and constraints on the time-variation of fundamental constants Phys. Rev. Lett. 113 210801

[9] Huntemann N, Lipphardt B, Tamm C, Gerginov V, Weyers S and Peik E 2014 Improved limit on a temporal variation of $\mu m/\mu s$ from comparisons of Yb$^+$ and Cs atomic clocks Phys. Rev. Lett. 113 210802

[10] Ushijima I, Takamoto M, Das M, Ohkubo T and Katori H 2015 Cryogenic optical lattice clocks Nat. Photonics 9 185–9

[11] Nicholson T L et al 2015 Systematic evaluation of an atomic clock at $2 \times 10^{-16}$ total uncertainty Nat. Commun. 6 6896

[12] Levi F, Calonico D, Lorini L and Godone A 2006 IEN-CsF1 primary frequency standard at INRIM: accuracy evaluation and TAI calibrations Metrologia 43 545–55

[13] Weyers S, Gerginov V, Nemitz N, Li R and Gibble K 2012 Distributed cavity phase frequency shifts of the caesium fountain PTB-CsF2 Metrologia 49 82–7

[14] Guéna J et al 2012 Progress in atomic fountains at LNE-SYRTE IEEE Trans. Ultrason. Ferroelectr. Freq. Control 59 391–410

[15] Szymaniec K, Lea S and Liu K 2014 An evaluation of the frequency shift caused by collisions with background gas in the primary frequency standard NPL-CsF2 IEEE Trans. Ultrason. Ferroelectr. Freq. Control 61 203–6

[16] Heavner T P, Donley E A, Levi F, Costanzo G, Parker T E, Shirley J H, Ashby N, Barlow S and Jefferts S R 2014 First accuracy evaluation of NIST-F2 Metrologia 51 174–82

[17] Gill P 2011 When should we change the definition of the second? Proc. R. Soc. A 369 4109–30

[18] BIPM Recommended values of standard frequencies http://www.bipm.org/en/publications/mises-en-pratique/standard-frequencies.html

[19] Rosenband T et al 2008 Frequency ratio of Al$^+$ and Hg$^+$ single-ion optical clocks; metrology at the 17th decimal place Science 319 1808–11

[20] Matsubara K, Hachisu H, Li Y, Nagano S, Locke C, Nogami A, Kajita M, Hayasaka K, Ido T and Hosokawa M 2012 Direct comparison of a Ca$^+$ single-ion clock against a Sr lattice clock to verify the absolute frequency measurement Opt. Express 20 22034–41

[21] Akamatsu D, Yasuda M, Inaba H, Hosaka K, Tanabe T, Onae A and Hong F-L 2014 Frequency ratio measurement of $^{171}$Yb and $^{87}$Sr optical lattice clocks Opt. Express 22 7898–905

[22] Margolis H S et al 2013 International timescales with optical clocks (ITOC) IEEE Proc. of the 2013 Joint European Frequency and Time Forum and Int. Frequency Control Symp., pp 908–11

[23] Mohr P J and Taylor B N 2000 CODATA recommended values of the fundamental physical constants 1998 Rev. Mod. Phys. 72 351–495

[24] D’Errico J R 2012 Available from the Matlab central file exchange http://www.mathworks.com/matlabcentral/fileexchange/

[25] Matveev A et al 2013 Precise measurement of the hydrogen $1S$–$2S$ frequency via a 920 km fiber link Phys. Rev. Lett. 110 230801

[26] Tamm C, Huntemann N, Lipphardt B, Gerginov V, Nemitz N, Kazda M, Weyers S and Peik E 2014 Cs-based optical frequency measurement using cross-linked optical and microwave oscillators Phys. Rev. A 89 052380

[27] Park C Y et al 2013 Absolute frequency measurement of $^3$So ($F = 1/2$–$^3$P$_0$ ($F = 1/2$) transition of $^{171}$Yb atoms in a one-dimensional optical lattice at KRISS Metrologia 50 119–28

[28] Akamatsu D, Inaba H, Hosaka K, Yasuda M, Onae A, Suzuyama T, Amemiya M and Hong F-L 2014 Spectroscopy and frequency measurement of the $^{88}$Sr clock transition by laser linewidth transfer using an optical frequency comb Appl. Phys. Express 7 012401