General framework of higher order gauge invariant perturbation theory

Kouji Nakamura

Department of Astronomical Science, the Graduate University for Advanced Studies, Osawa, Mitaka, Tokyo 181-8588, Japan.

Abstract

Based on the gauge invariant variables proposed in [K. Nakamura, Prog. Theor. Phys. 110 (2003), 723.], general framework of the second order gauge invariant perturbation theory on arbitrary background spacetime is considered. We derived formulae of the perturbative Einstein tensor of each order, which have the similar form to the definitions of gauge invariant variables for arbitrary perturbative fields. As a result, each order Einstein equation is necessarily given in terms of gauge invariant variables.

The perturbative approach is one of the popular techniques to investigate physical systems. In particular, this approach is powerful when the construction of exactly soluble models is difficult. In general relativity, there are many exact solutions to the Einstein equation but these are often too idealized to properly represent natural phenomena. In this situation, the perturbations around appropriate exact solutions are useful to investigate realistic situations. Cosmological perturbation theory is now the most commonly used technique, and perturbations of black holes and stars have been widely studied to obtain descriptions of the gravitational radiation emitted from them.

In general relativistic perturbations, gauge freedom, which is unphysical degree of freedom, arises due to general covariance. To obtain physically meaningful results, we have to fix these gauge freedom or to extract gauge invariant part of perturbations. These situations are also seen in the recent investigations of the oscillatory behavior of gravitating Nambu-Goto membrane, which are concerning about the dynamical degree of freedom of gravitating extended objects. In these works, it was necessary to distinguish true dynamical degree of freedom from gauge freedom in perturbations and to develop a gauge invariant treatment of general relativistic perturbations.

In particular, we discussed comparison of the oscillatory behavior of a gravitating string with that of a test string, in Ref. [5]. To do this, we have developed two-parameter general relativistic gauge invariant perturbation on the Minkowski spacetime, in which one of the perturbation parameter is the string energy density and the other is the string oscillation amplitude. Such multi-parameter perturbations have many other physical situations to be applied. The perturbation of spherical stars is one of them, in which we choose the gravitational field of a spherical star as the background spacetime for the perturbations, one of the parameters for the perturbations corresponds to the rotation of the star and another is the pulsation amplitude of it. The effects due to the rotation-pulsation coupling are described in the higher order. Even in one-parameter case, it is interest to consider higher order perturbations. In particular, Gleiser et al. reported that the second order perturbations predict accurate wave form of gravitational waves. Thus, there are many physical situations to which higher order gauge invariant perturbations with multi-parameter should be applied, and it is worthwhile to discuss them from general point of view.

In this article, we show our treatments of two-parameter higher order gauge invariant perturbations. In our treatments, we do not specify the physical meanings of parameters for perturbation nor background spacetime, though it is necessary to specify both of them when we apply them to some physical situations.

Though the “general covariance” is mathematically formulated in the concept of spacetime manifolds, it intuitively states that there is no preferred coordinate system in nature. Due to this general covariance, “gauge freedom” in general relativistic perturbations arises. To explain this “gauge freedom”, we must remind what we are doing in perturbation theories.

In any perturbation theory, we always treat two spacetime: one is the physical spacetime which should be described by perturbations; another is the background which is prepared.
for calculations. Keeping these two manifolds in our mind, we always write the equation
\[ Q(p') = Q_0(p) + \delta Q(p). \]  
(1)

where \(Q\) is any physical field on \(M\). Through Eq. (1), we are implicitly assuming that there exists a map \(X : M_0 \to M : p \in M_0 \to p' = M,\) which is usually called a “gauge choice” in perturbation theory\[9\]. Namely, \(Q(p')\) in Eq. (1) is a field on \(M\) and \(p' = M\). On the other hand, we should regard that the background value \(Q_0(p)\) of \(Q(p')\) and its deviation \(\delta Q(p)\) from \(Q_0(p)\) in Eq. (1) are fields on \(M_0\) and \(p \in M_0\). Since Eq. (1) is for fields, it implicitly states that the points \(p' \in M\) and \(p \in M_0\) are same.

Note that the gauge choice \(X\) is not unique when we consider theories in which general covariance is imposed and this degree of freedom of \(X\) is “gauge freedom” of perturbations. If there is a preferred coordinate system on both \(M_0\) and \(M\), we can choose \(X\) using this coordinate system. However, there is no such coordinate system due to general covariance and we have no guiding principle to choose \(X\).

Based on this understanding of “gauge”, the gauge transformation is simply the change of the map \(X\). To see this, we only consider the one-parameter perturbation case, because the essence of discussion for the \(m = n\) is no such coordinate system. Based on this understanding of “gauge”, the gauge transformation is simply the change of the map \(X\) which is a point identification map from \(M_0\) to \(M\). The map \(X\) is a gauge choice discussed above. We also consider another gauge choice \(Y\). The pull-back of each gauge choice maps any field \(Q\) on \(M\), to \(XQ := X^*Q\), and \(YQ := Y^*Q\) on \(M_0\), respectively.

The gauge transformation is induced by the map \(\Phi = X^{-1} \circ Y\),
\[ YQ = Y^*Q|_{M_0} = (X^*X^*Q)|_{M_0} = \Phi^*X^*Q_{\lambda, \epsilon}. \]  
(2)

The substitution of expansions \(XQ = Q + \epsilon L_\alpha Q_\alpha + (\epsilon^2/2) L^2_{\alpha \beta} Q_{\alpha \beta} + O(\epsilon^3)\), \(XQ = Q + \epsilon L_\alpha Q_\alpha + (\epsilon^2/2) L^2_{\alpha \beta} Q_{\alpha \beta} + O(\epsilon^3)\), and \(Q = Q_0 + \epsilon Q_1 + (\epsilon^2/2) Q_2 + O(\epsilon^3)\) into Eq. (2) leads each order gauge transformation rules,
\[ YQ_1 - XQ_1 = \epsilon L_\xi Q_0, \quad YQ_2 - XQ_2 = 2 \epsilon L_\xi Q_0 + (\epsilon L_\xi + L^2_{\xi, \xi}) XQ_0, \]  
(3)

where \(\xi^\mu = u^\alpha - \nu^\alpha, \xi^\mu_0 = [u, v]^\alpha,\) and \(u^\mu (v^\mu)\) is the generator of \(X(e) (Y(e))\).

The gauge transformation \(\Phi = X^{-1} \circ Y\), also induces the coordinate transformation on \(M\). A chart \((U, X)\) on \(M_0\) with a gauge choice \(X\), becomes a chart \((X(U), X \circ X^{-1})\) on \(M\) \(\{(x^\mu)\}\). Another gauge choice \(Y\) induces another chart \((Y(U), X \circ Y^{-1})\) on \(M\) \(\{(y^\mu)\}\). In the passive point of view, we obtain
\[ y^\mu (q) := x^\mu (p) = \left( (\Phi^{-1})^* x^\mu \right) (q) = x^\mu (q) - \epsilon x^\mu \xi^\mu_0 (q) + \frac{\epsilon^2}{2} \{ -\xi^\mu_0 (q) + \xi^\mu_1 (q) \partial_\nu \xi^\mu_1 (q) \} + O(\epsilon^3). \]  
(4)

This includes the additional degree of freedom \(\xi^\mu_0\) and this does show that the gauge freedom in perturbations is more than the usual assignment of coordinate labels.

Similarly, the gauge transformation rules in two-parameter perturbations are given by
\[ XQ_{\epsilon, \lambda} := X^*Q_{\epsilon, \lambda} = \sum_{k, k'} \frac{\lambda^k k'}{kk'} \delta^{(k, k')} X Q_{\epsilon, \lambda} = Q_0, \]  
(5)

\[ \delta^{(p, q)}_Y Q - \delta^{(p, q)}_X Q = \epsilon L_{\xi, \epsilon} Q_0, \quad (p, q) = (0, 1), (1, 0), \]  
(6)

\[ \delta^{(p, q)}_Y Q - \delta^{(p, q)}_X Q = 2 \epsilon L_{\xi, \epsilon} Q_0 + \left\{ L_{\xi, \epsilon} Q_{\mu} + L^2_{\xi, \xi} Q_{\mu} \right\} Q_0, \quad (p, q) = (0, 2), (2, 0), \]  
(7)

\[ \delta^{(1, 1)}_Y Q - \delta^{(1, 1)}_X Q = \epsilon L_{\xi, \epsilon} Q_{\lambda} + \left\{ L_{\xi, \epsilon} Q_{\lambda} + \frac{1}{2} \epsilon L_{\xi, \epsilon} L_{\xi, \epsilon} Q_{\lambda} \right\} Q_0. \]  
(8)

Now, we define gauge invariant variables. Our starting point to construct gauge invariant variables is the assumption which states that we have already known the procedure to find gauge invariant variables for the linear metric perturbations. Then, linear metric perturbations \((1, 0) h_{ab} \) \((0, 1) h_{ab}\) decomposed as
\[ (p, q) h_{ab} := (p, q) H_{ab} + 2 \nabla_{(a} (p, q) X_{b)}, \quad (p, q) = (1, 0), (0, 1), \]  
(9)
where \(^{(p,q)}H_{ab}\) \((^{(p,q)}X_a)\) is gauge invariant (variant). (Henceforth, we omit the gauge index \(\lambda\)).

This assumption is non-trivial. However, once we accept this assumption, we can always find gauge invariant variables for higher order perturbations.\(^7\) In the second order, the metric perturbations are decomposed as

\[
\begin{align*}
^{(p,q)}h_{ab} &= \left(^{(p,q)}h_{ab} + 2\mathcal{L}(\tilde{\nabla} \tilde{x})X_{\tilde{\nabla}}^{(p,q)}h_{ab} + \left(\mathcal{L}^{(p,q)}X - \mathcal{L}^{(0,0)}X\right)^{(0)}g_{ab}\right), \quad (p, q) = (2, 0), (0, 2); \quad (10) \\
^{(1,1)}h_{ab} &= \left(1^{(1,1)}h_{ab} + \mathcal{L}(0)X^{(1,0)}h_{ab} + \mathcal{L}(1,0)X^{(0,1)}h_{ab}
+ \left\{\mathcal{L}(1,0)X - \frac{1}{2}\mathcal{L}(1,0)X\mathcal{L}(0,1)x - \frac{1}{2}\mathcal{L}(0,1)X\mathcal{L}(1,0)X\right\}^{(0)}g_{ab}\right), \quad (11)
\end{align*}
\]

where \(^{(p,q)}h_{ab}\) \((^{(p,q)}X_a)\) is gauge invariant (variant).

Next, we show the expression of perturbative Einstein equations of each order using these gauge invariant variables defined above. We consider the pull-back \(\mathcal{X}_a^\lambda \tilde{G}_a^b\) on \(\mathcal{M}_0\) of the Einstein tensor \(\tilde{G}_a^b\) and expand \(\mathcal{X}_a^\lambda \tilde{G}_a^b\) as Eq. \([19]\). In terms of gauge invariant and variant variables of metric perturbations defined above, the perturbative Einstein tensors are given by

\[
\begin{align*}
^{(p,q)}G_a^b &= \left(1^{(p,q)}G_a^b \left[\begin{array}{c}
^{(p,q)}h_{c}^d
\end{array}\right] + \mathcal{L}^{(p,q)}X G_a^b
\right), \quad \text{for} \quad (p, q) = (0, 1), (1, 0), \quad (15) \\
^{(p,q)}G_a^b &= \left(1^{(p,q)}G_a^b \left[\begin{array}{c}
^{(p,q)}h_{c}^d
\end{array}\right] + \left(2^{(p,q)}G_a^b \left[\begin{array}{c}
^{(p,q)}h_{c}^d
\end{array}\right] + \mathcal{L}^{(p,q)}X G_a^b
\right) + 2\mathcal{L}(\tilde{\nabla} \tilde{x})X^{(p,q)}G_a^b
\right), \quad \text{for} \quad (p, q) = (2, 0), (0, 2), \quad (16) \\
^{(1,1)}G_a^b &= \left(1^{(1,1)}G_a^b \left[\begin{array}{c}
^{(1,1)}h_{c}^d
\end{array}\right] + \left(2^{(1,1)}G_a^b \left[\begin{array}{c}
^{(1,0)}h_{c}^d
\end{array}\right] + \mathcal{L}(1,0)X^{(0,1)}G_a^b
\right) + \mathcal{L}(1,0)X^{(0,1)}G_a^b
\right) + \left\{\mathcal{L}(1,0)X - \frac{1}{2}\mathcal{L}(0,1)X\mathcal{L}(1,0)X - \frac{1}{2}\mathcal{L}(0,1)X\mathcal{L}(1,0)X\right\}^{(0)}G_a^b, \quad (17) \\
^{(1)}G_a^b \left[A_c^d\right] &= \left[-2\nabla_{[a}A_{b]d} - A_{[a}^{d}bR_{ab} + \frac{1}{2}\delta_{[a}^{d}b \left(2\nabla_{[e}A_{d]e} + R_{cd}A_{e}^{cd}\right)\right], \quad (18) \\
^{(2)}G_a^b \left[A_c^d, B_c^d\right] &= \left[2R_{a[d}B_{c]}^{d}A^{d}c + 2A_{[a}^{d}B_{c]}^{d}B_{[c]}^{e} + 2B_{[a}^{d}A_{c]}^{d} + 2A_{[a}^{d}B_{c]}^{d}A_{c]}^{d} + 2B_{[a}^{d}B_{c]}^{d} + 2A_{[a}^{d}B_{c]}^{d}A_{c]}^{d}
\right.
\end{align*}
\]

where \(A_{abc} := \nabla_{(a}A_{b)c} - (1/2)\nabla_{c}A_{ab}\) and \(B_{abc}\) and \(^{(p,q)}H_{abc}\) follow the same definition. We note that \(^{(1)}G_a^b \left[A_c^d\right]\) \(^{(2)}G_a^b \left[A_c^d, B_c^d\right]\) and \(^{(0,0)}H_{abc}\) have the similar form to Eqs. \([12]-[15]\), respectively.

Next, to consider the Einstein equation, we expand the energy momentum tensor as Eq. \([5]\) and we impose the perturbed Einstein equation of each order \(^{(p,q)}G_a^b = 8\pi G \,(^{(p,q)}T_a^b)\). We defining the each order gauge invariant variable \(^{(p,q)}T_a^b\) for the perturbative energy momentum tensor by Eqs. \([12]-[14]\).
Then, perturbative Einstein equation of each order is given by

\[
8\pi G \left(\begin{array}{c} p, q \\
\end{array}\right) T^b_a = (1) g^b_a \left[\begin{array}{c} p, q \\
\end{array}\right] \mathcal{H}_c^d,
\]

for \((p, q) = (0, 1), (1, 0)\), \(20\)

\[
8\pi G \left(\begin{array}{c} p, q \\
\end{array}\right) T^b_a = (1) g^b_a \left[\begin{array}{c} p, q \\
\end{array}\right] \mathcal{H}_c^d + (2) g^b_a \left[\begin{array}{c} p, q \end{array}\right] \mathcal{H}_{c'}^{d'} \mathcal{H}_{c''}^d,
\]

for \((p, q) = (0, 2), (2, 0)\), \(21\)

\[
8\pi G \left(\begin{array}{c} 1, 1 \\
\end{array}\right) T^b_a = (1) g^b_a \left[\begin{array}{c} 1, 1 \\
\end{array}\right] \mathcal{H}_c^d + (2) g^b_a \left[\begin{array}{c} 1, 0 \\
\end{array}\right] \mathcal{H}_{c'}^d \mathcal{H}_{c''}^d,
\]

for \((p, q) = (0, 2), (2, 0)\), \(22\)

Thus, order by order Einstein equations are necessarily given in terms of gauge invariant variables only.

In summary, we showed the general framework of higher order gauge invariant perturbations of Einstein equation. We have confirm two facts First, *if the linear order gauge invariant perturbation theory is well established, its extension to higher order and multi-parameter perturbation is straightforward*. Second, *perturbative Einstein equations of each order are necessarily given in gauge invariant form*. The second result is trivial because any equation can be written in the form that the right hand side is equal to “zero” in any gauge. This “zero” is gauge invariant. Then the left hand side of this equation should be gauge invariant. In this sense, the above second result is trivial. However, we have to note that this trivial result implies that our framework is mathematically correct at this level.

Further, we also note that in our framework, we do not specify anything about the background spacetime and physical meaning of the parameters for the perturbations. Our framework is based only on general covariance. Hence this framework is applicable to any theory in which general covariance is imposed and it has very many applications. Though this framework is not complete, yet, we are planning to apply this to some physical problems. We leave these applications as our future works.

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