A Constrained Tropical Optimization Problem: Complete Solution and Application Example

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Abstract

A multidimensional optimization problem is considered, which is formulated in terms of tropical mathematics as to minimize a nonlinear objective function subject to linear inequality constraints. The optimization problem is motivated by a problem in project scheduling when an optimal schedule is given by minimizing the flow time of activities in a project under various activity precedence constraints. To solve the problem, we follow the approach based on the introduction of an additional unknown variable to reduce the problem to solving a linear inequality, where the variable plays the role of a parameter. A necessary and sufficient condition for the inequality to hold is used to evaluate the parameter, whereas the general solution of the inequality is taken as a complete solution of the original problem. Under fairly general conditions, a complete direct solution to the problem is obtained in a compact vector form. The obtained result is applied to solve the motivating problem in project scheduling and illustrated with a numerical example.

Key-Words: idempotent semifield, finite-dimensional semimodule, tropical optimization problem, nonlinear objective function, linear inequality constraint, project scheduling.

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1 Introduction

Tropical optimization problems form a rapidly evolving research domain in the area of tropical (idempotent) mathematics. Multidimensional optimization problems formulated and solved in the framework of tropical mathematics were apparently first considered in [1, 2] at the same time and shortly after the pioneering works in the area have made their appearance, including [3, 4, 5, 6, 7].

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The tropical optimization problems arise in real-world applications in various fields, among them are project scheduling \[2, 8, 9, 10, 11, 12, 13\] and location analysis \[14, 15, 16, 17\]. Further examples include solutions to problems in transportation networks \[8, 10\], decision making \[18\] and discrete event systems \[19\].

The problems are formulated in the tropical mathematics setting as to minimize a linear or nonlinear objective function defined on vectors of a finite-dimensional semimodule over an idempotent semifield. Both unconstrained and constrained problems are under consideration, where the constraints have the form of linear vector equations and inequalities in the semimodule.

There are tropical optimization problems that are examined in the literature in terms of particular idempotent semifields, whereas some other problems are solved in a more general context, which includes such semifields as a particular case. Proposed solutions often take the form of an iterative numerical procedure that produces a solution if any, or indicates that no solution exists, otherwise. In other cases, explicit direct solutions are obtained in a closed vector form. Many existing approaches, however, are only able to offer particular solutions rather than give comprehensive solutions to the problems.

Among the long-known and extensively studied optimization problems is a direct tropical analog of linear programming problems, which has a linear objective function and linear inequality constraints. Complete direct solutions and related duality results are obtained for the problem under various algebraic assumptions. In the case of general semirings, the problem is examined in \[1\], where a solution based on an abstract extension of the conventional linear programming duality is given, and in \[8\], which offers a residual-based solution technique. Solutions for particular idempotent semifields are proposed by \[20\] based on further development of results in \[1\], and by \[21\] in the context of the theory of max-separable functions.

An extended problem with more constraints is considered in \[22, 23, 24, 10\] within the framework of max-separable functions. Explicit solutions for the problem are given basically in conventional terms rather than in terms of tropical vector algebra.

A problem with a linear objective function and two-sided equality constraints is examined in \[25, 26, 11, 13\]. Specifically, \[11\] presents a pseudopolynomial algorithm that produces a solution if any or indicates that no solution exists, otherwise. A heuristic approach is suggested by \[13\] to get an approximate solution of the problem by use of local search techniques.

A tropical optimization problem with a nonlinear objective function, which arises in the best underestimating approximation in the Chebyshev norm, is investigated in \[2\]. A complete explicit solution is given as an application of an abstract theory of linear operators on an idempotent semifield developed there. A similar solution to the problem is suggested by \[8\].
A problem of minimizing a Chebyshev-like distance function under some constraints is solved with a polynomial-time threshold-type algorithm by [9]. An explicit solution to a problem of minimizing the range norm is given in [12]. Another problem with a nonlinear objective function and two-sided equality constraints is solved in [27] by use of an iterative computational scheme.

Finally, both unconstrained and constrained problems with nonlinear objective functions formulated in terms of a general idempotent semifield are investigated in [28, 29, 30, 17, 31, 32, 33]. A solution technique is implemented based on new results in tropical spectral theory and solutions of linear tropical equations and inequalities developed in [28, 29, 30]. With this technique, direct explicit solutions to the problems are obtained in a compact vector form.

In this paper, we examine a multidimensional optimization problem that extends problems in [19, 28, 30, 15, 32, 33] by eliminating restrictions on matrices involved as well as by introducing additional inequality constraints. The problem is drawn from project scheduling when an optimal schedule is given by minimizing the flow time of activities in a project under various activity precedence constraints.

We formulate the problem in terms of a general idempotent semifield. We follow the approach proposed in [32, 33] and based on the introduction of an additional unknown variable to reduce the problem to solving a linear inequality, where the variable plays the role of a parameter. A necessary and sufficient condition for the inequality to hold is used to evaluate the parameter, whereas the general solution of the inequality is taken as a complete solution of the original problem. Under fairly general conditions, a complete direct solution to the problem is obtained in a compact vector form suitable for both further analysis and applications.

The rest of the paper is as follows. We start with an example problem in Section 2 which is taken from project scheduling and intended to motivate and illustrate the main result of the paper. Section 3 offers a short concise introduction to tropical algebra to provide a formal basis for subsequent solution of optimization problems. In Section 4, we examine a system of simultaneous linear inequalities and offer a complete direct solution to the system as a key element for solving optimization problems that follows. We formulate a general tropical optimization problem, give complete direct solution to the problem, and consider some particular cases in Section 5. Finally, Section 6 is concerned with application of the results to solve the motivating scheduling problem.
2 A motivating example

We start with a real-world problem taken from project scheduling and intended to both motivate and illustrate subsequent results. The problem is described in the common setting of project scheduling as having a set of activities operating under various precedence relations between their initiation and completion times, and formulated as to design a schedule that minimize the maximum flow time over all activities in a project, subject to various activity precedence constraints. For further details and references on project scheduling, one can consult [34, 35].

Consider a project with $n$ activities (jobs, tasks) constrained by precedence relations, including start-start, start-finish, early-start, and late-finish time constraints. For any two activities, the start-start constraints define a minimum allowed time interval between their initiations. The start-finish constraints place a lower bound on the time lag between the initiation of one activity and the completion of another. The activities are assumed to complete as early as possible within the constraints. For each activity, the early-start and late-finish constraints respectively specify the earliest possible time of initiation and the latest possible time of completion.

Every activity in the project involves its associated flow (turnaround, processing) time defined as the time interval between its initiation and completion. The optimal scheduling problem is to find an initiation time for each activity to minimize the maximum flow time over all activities, subject to the above constraints.

For each activity $i = 1, \ldots, n$, we denote by $x_i$ the initiation time to be scheduled. Let $g_i$ be a lower bound on the initiation time, and $b_{ij}$ be a minimum possible time lag between the initiation of activity $j = 1, \ldots, n$ and the initiation of $i$.

The start-start constraints imply that, given the time lags $b_{ij}$, the initiation times are to satisfy the relations

$$x_j + b_{ij} \leq x_i, \quad j = 1, \ldots, n.$$

Note that if a time lag is not actually fixed, we set it to be equal to $-\infty$.

These relations taken together lead to one inequality of the form

$$\max(x_1 + b_{i1}, \ldots, x_n + b_{in}) \leq x_i.$$

Since, according to the early-start constraints, activity $i$ cannot start earlier than at a predefined time $g_i$, we arrive at the inequalities

$$\max(\max_{1 \leq j \leq n} (b_{ij} + x_j), g_i) \leq x_i, \quad i = 1, \ldots, n.$$

Furthermore, for each activity $i$, let $y_i$ be the completion time. We denote by $a_{ij}$ a given minimum possible time lag between the initiation of
activity \( j \) and the completion of \( i \), and by \( h_i \) a given upper bound on the completion time for \( i \). As before, if a time lag \( a_{ij} \) appears to be undefined, we put \( a_{ij} = -\infty \).

The start-finish constraints require that the completion time \( y_i \) be subject to the relations
\[
x_j + a_{ij} \leq y_i, \quad j = 1, \ldots, n,
\]
with at least one inequality among them holding as an equality.

By combining the inequalities and adding the upper bound for the completion time, we get the relations
\[
\max_{1 \leq j \leq n} (a_{ij} + x_j) = y_i, \quad h_i \geq y_i, \quad i = 1, \ldots, n.
\]

Now we formulate a scheduling problem as to minimize the maximum flow time over all activities. With an objective function that is readily given by
\[
\max(y_1 - x_1, \ldots, y_n - x_n),
\]
we arrive at a constrained optimization problem to find \( x_i \) for all \( i = 1, \ldots, n \) to
\[
\text{minimize} \quad \max_{1 \leq i \leq n} (y_i - x_i),
\]
subject to
\[
\max_{1 \leq j \leq n} (b_{ij} + x_j, g_i) \leq x_i,
\]
\[
\max_{1 \leq j \leq n} (a_{ij} + x_j) = y_i, \quad h_i \geq y_i, \quad i = 1, \ldots, n.
\]

3 Preliminary definitions and results

In this section we present a short overview of basic definitions, notation and preliminary results of tropical (idempotent) mathematics to provide a formal framework for the analysis and solution of optimization problems in the rest of the paper.

Concise introductions to and comprehensive presentations of tropical mathematics are given in different forms in a range of published works, including recent publications [36, 37, 38, 39, 40, 26]. In the overview below, we mainly follow [29, 28, 30], which offer the prospect of complete direct solution of the problems of interest in a compact vector form. For additional details on and deep insight into the theory and methods of tropical mathematics one can consult the works listed before.

3.1 Idempotent semifield

We consider a commutative idempotent semifield \( \langle \mathcal{X}, 0, 1, \oplus, \otimes \rangle \) over a set \( \mathcal{X} \), which is closed under addition \( \oplus \) and multiplication \( \otimes \), and has zero
0 and identity 1. Both addition and multiplication are associative and commutative operations, and multiplication is distributive over addition.

Addition is idempotent, which implies that \( x \oplus x = x \) for any \( x \in \mathbb{X} \). The addition induces on \( \mathbb{X} \) a partial order such that the relation \( x \leq y \) holds for \( x, y \in \mathbb{X} \) if and only if \( x \oplus y = y \). With respect to the order, the addition is isitone in each argument and has an extremal property that \( x \leq x \oplus y \) and \( y \leq x \oplus y \).

The partial order is considered as extendable to a total order, and so we assume the semifield to be linearly ordered. In what follows, the relation symbols and minimization problems are thought in terms of this order. Note that, according to the order, \( x \geq 0 \) for all \( x \in \mathbb{X} \). We also assume that the set \( \mathbb{X} \) includes (or can be enlarged to include) a maximal element \( \infty \) such that \( x \leq \infty \) for any \( x \in \mathbb{X} \).

For each \( x \in \mathbb{X}^+ \), where \( \mathbb{X}^+ = \mathbb{X} \setminus \{0\} \), there exists an inverse \( x^{-1} \) that yields \( x^{-1} \otimes x = 1 \). We take 0 and \( \infty \) as reciprocal to write \( 0 \otimes \infty = 1 \).

The integer power is routinely defined in the semifield. For any \( x \in \mathbb{X}^+ \) and integer \( p \geq 1 \), we have \( x^0 = 1 \), \( x^p = x^{p-1} \otimes x \), \( x^{-p} = (x^{-1})^p \), \( 0^p = 0 \), and \( \infty^0 = 1 \). We further assume that the power notation can be extended to the rational exponents, and so treat the semiring as radicable. Below we omit the multiplication symbol for the sake of brevity and employ the power notation only in the sense defined.

Examples of the radicable linearly ordered idempotent semifield under consideration include

\[
\mathbb{R}_{\text{max},+} = \langle \mathbb{R} \cup \{-\infty\}, -\infty, 0, \max, + \rangle, \quad \mathbb{R}_{\text{min},+} = \langle \mathbb{R} \cup \{+\infty\}, +\infty, 0, \min, + \rangle,
\]

\[
\mathbb{R}_{\text{max},\times} = \langle \mathbb{R} \cup \{0\}, 0, 1, \max, \times \rangle, \quad \mathbb{R}_{\text{min},\times} = \langle \mathbb{R} \cup \{+\infty\}, +\infty, 1, \min, \times \rangle,
\]

where \( \mathbb{R} \) denotes the set of real numbers, and \( \mathbb{R}^+ = \{x \in \mathbb{R} | x > 0\} \).

Specifically, the semifield \( \mathbb{R}_{\text{max},+} \) has addition and multiplication defined, respectively, as maximum and arithmetic addition. It is equipped with the zero \( 0 = -\infty \) and the identity \( 1 = 0 \). Each \( x \in \mathbb{R} \) is endowed with an inverse \( x^{-1} \) that is equal to \(- x\) in the ordinary notation. The power \( x^y \) is actually defined for any \( x, y \in \mathbb{R} \) and coincides with the arithmetic product \( xy \). The order induced by idempotent addition corresponds to the natural linear order on \( \mathbb{R} \). The number \(+\infty\) can be taken and added to \( \mathbb{R} \) to play the role of the maximal element \( \infty \).

In the semifield \( \mathbb{R}_{\text{min},\times} \), we have \( \oplus = \min, \otimes = \times, \quad 0 = +\infty, \) and \( 1 = 1 \). The symbols of inverse end exponent have ordinary meaning. Idempotent addition produces a reverse order to the natural order on \( \mathbb{R} \). The maximal element is \( \infty = 0 \).

### 3.2 Matrix and vector algebra

Now we concern with matrices with entries in \( \mathbb{X} \). We denote by \( \mathbb{X}^{m \times n} \) the set of matrices having \( m \) rows and \( n \) columns. A matrix with all entries
equal to $\emptyset$ is the zero matrix denoted by $\emptyset$.

Addition and multiplication of conforming matrices, as well as multiplication by scalars are defined in the regular way through the scalar operations on $\mathbb{X}$. Specifically, for matrices $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, and a scalar $x$, we have

\[
\begin{align*}
\{A \oplus B\}_{ij} &= a_{ij} \oplus b_{ij}, \\
\{AC\}_{ij} &= \bigoplus_k a_{ik}c_{kj}, \\
\{xA\}_{ij} &= xa_{ij}.
\end{align*}
\]

Based on properties of scalar addition and multiplication, the matrix operations are element-wise isotone in each argument. For any matrices $A$ and $B$ of the same size, the element-wise inequalities $A \preceq A \oplus B$ and $B \preceq A \oplus B$ are valid as well.

Consider square matrices of order $n$ in the set $\mathbb{X}^{n \times n}$. Any matrix with the off-diagonal entries equal to $\emptyset$ is a diagonal matrix. A diagonal matrix that has all diagonal entries equal to 1 presents the identity matrix denoted by $I$.

The matrix power with non-negative integer exponents is given in the usual way. For any square matrix $A$ and integer $p \geq 1$, we have $A^0 = I$, $A^p = A^{p-1}A$.

The trace of a matrix $A = (a_{ij})$ is conventionally defined as

\[
\text{tr} \ A = \bigoplus_{i=1}^{n} a_{ii}.
\]

It is easy to verify that, for any matrices $A$ and $B$, and scalar $x$, the trace exhibits the standard properties in the form of equalities

\[
\begin{align*}
\text{tr}(A \oplus B) &= \text{tr} A \oplus \text{tr} B, \\
\text{tr}(AB) &= \text{tr}(BA), \\
\text{tr}(xA) &= x \text{tr}(A).
\end{align*}
\]

As usual, a matrix that has only one column (row) is considered as a column (row) vector. The set of column vectors of order $n$ is denoted by $\mathbb{X}^n$. A vector that has all components equal to $\emptyset$ is the zero vector. A vector with nonzero components is called regular. The set of regular vectors in $\mathbb{X}^n$ is denoted by $\mathbb{X}^n_\neq$.

For any column vector $x = (x_i) \in \mathbb{X}^n$, we define a multiplicative conjugate transpose to be a row vector $x^- = (x_i^-)$ with entries $x_i^- = x_i^{-1}$.

Below we will use some properties of multiplicative conjugate transposition, which are easy to verify. Specifically, for any vectors $x, y \in \mathbb{X}^n$, the component-wise inequality $x \preceq y$ implies that $x^- \succeq y^-$ and vice versa.

Moreover, for any vector $x \in \mathbb{X}^n$, it holds that $x^- x = 1$ and $xx^- \succeq I$.

### 3.3 Spectral radius

Every square matrix $A \in \mathbb{X}^{n \times n}$ defines a linear operator on $\mathbb{X}^n$ with certain spectral properties. As usual, a scalar $\lambda$ is an eigenvalue of $A$, if there
exists a nonzero vector $x$ such that

$$Ax = \lambda x.$$  

Any nonzero vector that satisfies the equality is an eigenvector related to $\lambda$.

The maximum eigenvalue (in the sense of the order on $\mathbb{X}$) is called the spectral radius of the matrix $A$ and given by

$$\lambda = \bigoplus_{m=1}^{n} \text{tr}^{1/m}(A^m).$$

The spectral radius $\lambda$ of any matrix $A \in \mathbb{X}^{n \times n}$ possesses a useful extremal property, which says that

$$\min x^{-1}Ax = \lambda,$$

where the minimum is over all regular vectors $x \in \mathbb{X}^n$.

## 4 Linear inequalities

Solution to optimization problems in the subsequent sections make well use of complete direct solutions of linear tropical inequalities. This section begins with the presentation of results based on solutions given in [29, 30, 33] for linear vector inequalities of two types. Furthermore, a problem of simultaneous solution of a system of linear inequalities is considered, which is of independent interest.

### 4.1 Preliminary results

Suppose that there is a square matrix $A \in \mathbb{X}^{n \times n}$ and a vector $b \in \mathbb{X}^n$. Consider a problem of finding all regular solutions $x \in \mathbb{X}^n$ of the inequality

$$Ax \oplus b \leq x.$$  

(2)

To describe a general solution, we introduce a function that takes the matrix $A$ to a scalar

$$\text{Tr}(A) = \bigoplus_{m=1}^{n} \text{tr} A^m.$$  

Provided that $\text{Tr}(A) \leq 1$, we define a matrix

$$A^* = I \oplus A \oplus \cdots \oplus A^{n-1}.$$  

With this notation, we offer a new form for a result, which is apparently first suggested by [41] and is referred to below as the Carré inequality. The
result is as follows. Under the condition that $\text{Tr}(A) \leq 1$, for all integer $k \geq 0$, it holds that

$$A^k \leq A^*.$$

The next assertion provides a general solution to inequality (2).

**Theorem 1.** Let $x$ be the general regular solution of inequality (2). Then the following statements are valid:

1. If $\text{Tr}(A) \leq 1$, then $x = A^*u$ for all regular $u$ such that $u \geq b$.

2. If $\text{Tr}(A) > 1$, then there is no regular solution.

Now we consider another problem. Given a matrix $C \in \mathbb{R}^{m \times n}$ and a vector $d \in \mathbb{R}^m$, find all regular vectors $x \in \mathbb{R}^n$ to satisfy the inequality

$$Cx \leq d.$$  \hspace{1cm} (3)

**Lemma 2.** A vector $x$ is a solution of inequality (3) with a regular vector $d$ if and only if

$$x \leq (d - CA^*)^-.$$

### 4.2 A system of inequalities

Consider a problem of simultaneous solution of inequalities (2) and (3) combined into a system

$$Ax + b \leq x,$$

$$Cx \leq d.$$  \hspace{1cm} (4)

A general solution of the system is given by the following statement.

**Lemma 3.** Let $x$ be the general regular solution of system (4) with a regular vector $d$. Denote $\Delta = \text{Tr}(A) + d^-CA^*b$. Then the following statements hold:

1. If $\Delta \leq 1$, then $x = A^*u$, where $b \leq u \leq (d^-CA^*)^-$. 

2. If $\Delta > 1$, then there is no regular solution.

**Proof.** It follows from Theorem 1 that the first inequality has regular solutions if and only if the condition $\text{Tr}(A) \leq 1$ holds true and that all solutions take a general form $x = A^*u$ for any regular vector $u \geq b$.

Assume the above condition is satisfied and take the general solution of the first inequality. Substitution of the solution into the second inequality leads to a new system of inequalities with respect to $u$, which is given by

$$CA^*u \leq d,$$

$$u \geq b.$$
Application of Lemma 2 to the first inequality gives a general solution in the form \( u \leq (d^{-1}CA^*)^- \), which, combined with the second inequality, results in two-sided boundary conditions \( b \leq u \leq (d^{-1}CA^*)^- \).

The conditions specify a nonempty set only when \( b \leq (d^{-1}CA^*)^- \). It is not difficult to verify that the inequality is equivalent to \( d^{-1}CA^*b \leq 1 \). Indeed, multiplying the first inequality on the left by \( d^{-1}CA^* \) directly produces the second. Now we take the second inequality, multiply it from the left by \( (d^{-1}CA^*)^- \), and then note that \( b \leq (d^{-1}CA^*)^- d^{-1}CA^*b \leq (d^{-1}CA^*)^- \), which yields the first inequality.

Both conditions \( \text{Tr}(A) \leq 1 \) and \( d^{-1}CA^*b \leq 1 \) are combined into one equivalent condition \( \Delta = \text{Tr}(A) \oplus d^{-1}CA^*b \leq 1 \), which completes the proof.

Remark 1. It is possible to represent \( \Delta = \text{Tr}(A) \oplus d^{-1}CA^*b \), provided that \( \Delta \leq 1 \), in another form to be exploited below. In fact, in this case it holds that

\[
\text{Tr}(A) \oplus d^{-1}CA^*b = d^{-1}Cb \oplus \bigoplus_{m=1}^{n} \text{tr}(A^m(I \oplus bCd^-C)).
\] (5)

To verify the equality, first note that the condition \( \Delta \leq 1 \) involves \( \text{Tr}(A) \leq 1 \). It follows from the Carré inequality that \( A^n \leq A^* \), and thus \( I \oplus A \oplus \cdots \oplus A^n = A^* \). The left side is now represented as

\[
\text{Tr}(A) \oplus d^{-1}CA^*b = \bigoplus_{m=1}^{n} \text{tr} A^m \oplus d^{-1}Cb \oplus \bigoplus_{m=0}^{n} d^{-1}CA^m b.
\]

Inserting the trace operator into the last term and combining both terms involving the trace together lead to the desired result.

5 An optimization problem

In this section we examine an optimization problem with nonlinear objective function and linear inequality constraints. A complete direct solution to the problem is given in a compact vector form.

The problem extends those in [19, 23, 30, 15, 32, 33] by eliminating restrictions on matrices involved as well as by introducing additional inequality constraints.

5.1 Problem formulation

Suppose \( \mathcal{X} \) is a linearly ordered radicable idempotent semifield. Given matrices \( A, B \in \mathcal{X}^{n \times n}, \ C \in \mathcal{X}^{m \times n} \) and vectors \( g \in \mathcal{X}^n, \ h \in \mathcal{X}^m \), the problem
is to find all regular vectors $x \in X^n$ that

$$\begin{align*}
\text{minimize} \quad & x - Ax, \\
\text{subject to} \quad & Bx \oplus g \leq x, \\
& Cx \leq h.
\end{align*}$$

(6)

Consider the inequality constraints. It follows from Lemma 3 that the constraints may have no common regular solution, and so make the entire problem unsolvable. The lemma gives necessary and sufficient conditions for the inequality to define a nonempty feasible set in the form of the inequality

$$\text{Tr}(B) \oplus h^\ast CB^\ast g \leq 1$$

Below we derive a solution to the problem under fairly general assumptions. Some particular cases of the problem are also discussed.

5.2 The main result

Now we give a complete direct solution to problem (6), which is based on the approach suggested and developed in [32, 33]. We introduce an auxiliary variable to represent the minimum value of the objective function, and then reduce the problem to solution of a system of linear inequalities, where the variable has a role of a parameter. Necessary and sufficient conditions for the system to have regular solutions are used to evaluate the parameter. Finally, a general solution to the system is exploited as a general solution of the problem.

**Theorem 4.** Suppose $A$ is a matrix with spectral radius $\lambda > 0$. Let $B$ and $C$ be matrices, $g$ and $h$ be vectors such that $\text{Tr}(B) \oplus h^\ast CB^\ast g \leq 1$. Define a scalar

$$\theta = \bigoplus_{k=1}^{n} \bigoplus_{0 \leq i_0 + i_1 + \cdots + i_k \leq n-k} \text{tr}^{1/k}(B^{i_0}(AB^{i_1} \cdots AB^{i_k})(I \oplus gh^\ast C)).$$

(7)

Then the minimum in (6) is equal to $\theta$ and attained if and only if

$$x = (\theta^{-1}A \oplus B)^\ast u$$

(8)

for all regular vectors $u$ such that

$$g \leq u \leq (h^\ast C(\theta^{-1}A \oplus B)^\ast)^\ast.$$

(9)

**Proof.** Since the inequality $\text{Tr}(B) \oplus h^\ast CB^\ast g \leq 1$ is valid by the conditions of the theorem, the feasible set of regular vectors in the problem is not empty. Note that the condition implies both inequalities $\text{Tr}(B) \leq 1$ and $h^\ast CB^\ast g \leq 1$. 


Denote by $\theta$ the minimum value of the objective function on the feasible set and take note that $\theta \geq \lambda > 0$. Any regular vector $x$ that yields the minimum must satisfy the system

$$x^{-}Ax = \theta,$$
$$Bx \oplus g \leq x,$$
$$Cx \leq h.$$ 

Since for all $x$ it holds that $x^{-}Ax \geq \theta$, the solution set for the system remains the same if we replace the first equality by an inequality $x^{-}Ax \leq \theta$. Moreover, it is easy to verify that for all regular $x$ the new inequality is equivalent to an inequality $\theta^{-1}Ax \leq x$. Indeed, after left multiplication of the former inequality by $\theta^{-1}x$, we have $\theta^{-1}Ax \leq \theta^{-1}xx^{-}Ax \leq x$, which yields the latter inequality. At the same time, left multiplication of the inequality $\theta^{-1}Ax \leq x$ by $\theta x^{-}$ leads to the inequality $x^{-}Ax \leq \theta x^{-}x = \theta$, and thus both inequalities are equivalent.

The above system now takes the form

$$\theta^{-1}Ax \leq x,$$
$$Bx \oplus g \leq x,$$
$$Cx \leq h.$$ 

By combining the first two inequalities into one, we arrive at a system in the form of (11),

$$(\theta^{-1}A \oplus B)x \oplus g \leq x,$$ 
$$Cx \leq h.$$ 

By Lemma 3, the system has regular solutions if and only if

$$\text{Tr}(\theta^{-1}A \oplus B) \oplus h^{-}C(\theta^{-1}A \oplus B)^{\top}g \leq 1.$$ 

With (4), the left side in inequality (11) can be written in another form

$$h^{-}Cg \oplus \bigoplus_{m=1}^{n} \text{tr}((\theta^{-1}A \oplus B)^{m}(I \oplus gh^{-}C)) \leq 1.$$ 

To further rearrange the inequality, we write a binomial identity

$$(\theta^{-1}A \oplus B)^{m} = B^{m} \oplus \bigoplus_{k=1}^{m} \bigoplus_{i_{0}+i_{1}+\cdots+i_{k}=m-k} \theta^{-k}B^{i_{0}}(AB^{i_{1}}\cdots AB^{i_{k}}).$$ 

Substitution of the identity together with some algebra result in

$$\bigoplus_{m=1}^{n} \text{tr}B^{m} \oplus \bigoplus_{m=0}^{n} h^{-}CB^{m}g$$
$$\oplus \bigoplus_{m=1}^{n} \bigoplus_{k=1}^{m} \theta^{-k} \text{tr}(B^{i_{0}}(AB^{i_{1}}\cdots AB^{i_{k}})(I \oplus gh^{-}C)) \leq 1.$$ 

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Consider the first two terms on the left. Note that $\text{Tr}(B) \leq 1$, and so $B^n \leq B^*$. Therefore, we have
\[
\bigoplus_{m=1}^{n} \text{tr} B^m \oplus \bigoplus_{m=0}^{n} \text{tr} h^{-1} C B^m g = \text{Tr}(B) \oplus h^{-1} C B^* g.
\]

Since the inequality $\text{Tr}(B) \oplus h^{-1} C B^* g \leq 1$ is provided by the conditions of the theorem, these terms can be eliminated to write inequality (11) in the form
\[
\bigoplus_{m=1}^{n} \bigoplus_{k=1}^{m} \bigoplus_{i_0+i_1+\cdots+i_k=m-k} \theta^{-k} \text{tr}(B^{i_0}(AB^{i_1} \cdots AB^{i_k})(I \oplus gh^{-1} C)) \leq 1.
\]

After rearrangement of terms we get an inequality
\[
\bigoplus_{k=1}^{n} \bigoplus_{0 \leq i_0+i_1+\cdots+i_k \leq n-k} \theta^{-k} \text{tr}(B^{i_0}(AB^{i_1} \cdots AB^{i_k})(I \oplus gh^{-1} C)) \leq 1,
\]
which is equivalent to a system of inequalities
\[
\bigoplus_{0 \leq i_0+i_1+\cdots+i_k \leq n-k} \theta^{-k} \text{tr}(B^{i_0}(AB^{i_1} \cdots AB^{i_k})(I \oplus gh^{-1} C)) \leq 1, \quad k = 1, \ldots, n.
\]

By solving each inequality in the system, we arrive at inequalities
\[
\theta \geq \bigoplus_{0 \leq i_0+i_1+\cdots+i_k \leq n-k} \text{tr}^{1/k}(B^{i_0}(AB^{i_1} \cdots AB^{i_k})(I \oplus gh^{-1} C)), \quad k = 1, \ldots, n,
\]
which are further combined to get a lower bound for $\theta$, which is given by
\[
\theta \geq \bigoplus_{k=1}^{n} \bigoplus_{0 \leq i_0+i_1+\cdots+i_k \leq n-k} \text{tr}^{1/k}(B^{i_0}(AB^{i_1} \cdots AB^{i_k})(I \oplus gh^{-1} C)).
\]

Since $\theta$ is assumed to be the minimum of the objective function in the problem, the last inequality must be satisfied as an equality, which leads to (12).

Application of Lemma 3 to system (10) gives the solution vectors $x$ defined by (8) and (9).

5.3 Particular cases

In this section we discuss problems that present noteworthy particular cases of the general problem examined above. Another particular case is examined in the next section in the context of solving scheduling problems.
First we assume \( C = I \) and consider a problem given by

\[
\begin{align*}
\text{minimize} & \quad x^T Ax, \\
\text{subject to} & \quad Bx \oplus g \leq x, \\
& \quad x \leq h.
\end{align*}
\] (12)

A direct application of the Theorem 4 immediately gives the next solution.

**Corollary 5.** Suppose \( A \) is a matrix with a nonzero spectral radius, \( B \) is a matrix, \( g \) and \( h \) are vectors such that \( \text{Tr}(B) \oplus h^* Bg \leq 1 \). Define a scalar

\[
\theta = \bigoplus_{k=1}^{n} \bigoplus_{1 \leq i_0 + i_1 + \cdots + i_k \leq n-k} \text{tr}^{1/k}(B^{i_0}(AB^{i_1} \cdots AB^{i_k})(I \oplus gh^-)).
\]

Then the minimum in (12) is equal to \( \theta \) and attained if and only if

\[
x = (\theta^{-1}A \oplus B)^* u, \quad g \leq u \leq (h^*(-1A \oplus B)^*)^-.\]

Furthermore, we suppose that \( C = 0 \). Under this assumption, problem (6) takes the form

\[
\begin{align*}
\text{minimize} & \quad x^T Ax, \\
\text{subject to} & \quad Bx \oplus g \leq x.
\end{align*}
\] (13)

Substitution of \( C = 0 \) into the statement of Theorem 4 leads to a following solution with a simplified expression for \( \theta \) instead of that of (7).

**Corollary 6.** Suppose \( A \) is a matrix with spectral radius \( \lambda > 0 \), \( B \) is a matrix with \( \text{Tr}(B) \leq 1 \), and \( g \) is a vector. Define a scalar

\[
\theta = \lambda \bigoplus_{k=1}^{n-1} \bigoplus_{1 \leq i_1 + \cdots + i_k \leq n-k} \text{tr}^{1/k}(AB^{i_1} \cdots AB^{i_k}).
\]

Then the minimum in (13) is equal to \( \theta \) and attained if and only if

\[
x = (\theta^{-1}A \oplus B)^* u, \quad u \geq g.
\]

Note that this result is coincides with that in (33).

Finally, suppose that \( B = 0 \) and \( C = I \). Problem (6) takes the form

\[
\begin{align*}
\text{minimize} & \quad x^T Ax, \\
\text{subject to} & \quad g \leq x \leq h.
\end{align*}
\] (14)

In this case, Theorem 4 reduces to the next statement.
Corollary 7. Suppose $A$ is a matrix with spectral radius $\lambda > 0$, $g$ and $h$ are vectors such that $h - g \leq 1$. Define a scalar

$$\theta = \lambda \oplus \bigoplus_{k=1}^{n} (h - A^k g)^{1/k}. $$

Then the minimum in (14) is equal to $\theta$ and attained if and only if

$$x = (\theta^{-1} A)^* u, \quad g \leq u \leq (h - (\theta^{-1} A)^*)^-. $$

The last result extends the solution obtained in [19, 28, 30] for problem (14) without constraints.

6 Applications to optimal scheduling

Now we turn back to the scheduling problem described in Section 2. In this section we offer a compact vector representation of the problem in terms of tropical mathematics and then give a complete direct solution to the problem.

6.1 Representation of scheduling problem

We start with the representation of the problem given by (1) in ordinary notation. Since it involves only usual operations max, addition, and additive inversion, we translate the representation into the language of the semifield $\mathbb{R}_{\text{max},+}$.

First we replace the standard operations at (1) by their tropical counterparts to write the problem in a scalar form

- minimize $\bigoplus_{j=1}^{n} x_i^{-1} y_i,$
- subject to $\bigoplus_{j=1}^{n} b_{ij} x_j \oplus y_i \leq x_i,$
  $\bigoplus_{j=1}^{n} a_{ij} x_j = y_i, \quad h_i \geq y_i, \quad i = 1, \ldots, n.$

Furthermore, we introduce matrices and vectors

$A = (a_{ij}), \quad B = (b_{ij}), \quad g = (g_i), \quad h = (h_i), \quad x = (x_i), \quad y = (y_i),$

and then rewrite the scalar representation in matrix-vector notation as to

- minimize $x \odot y,$
- subject to $Bx \oplus g \leq x,$
  $Ax = y, \quad h \geq y.$

(15)
6.2 Solution of scheduling problem

A complete direct solution to the scheduling problem is given in terms of the semifield $\mathbb{F}_{\text{max,+}}$ by the next result.

**Theorem 8.** Let $x$ and $y$ be the general regular solution of problem (15) with a matrix $A$ having a nonzero spectral radius, $\Delta = \text{Tr}(B) \oplus h^{-AB}g$ and

$$
\theta = \bigoplus_{k=1}^{n} \bigoplus_{0 \leq i_0 + i_1 + \cdots + i_k \leq n-k} \text{tr}^{1/k}(B^{i_0}(AB)^{i_1} \cdots (AB)^{i_k})(I \oplus gh^{-A}).
$$

(16)

Then the following statements are valid:

1. If $\Delta \leq 1$, then $\theta$ is the minimum in (15), attained at

$$
x = S^*u, \quad y = AS^*u,
$$

where $S = \theta^{-1}A \oplus B$, and $u$ is any regular vector such that

$$
g \leq u \leq (h^{-AS^*})^{-};
$$

(18)

2. If $\Delta > 1$, then there is no regular solution.

**Proof.** To solve problem (15), we first eliminate the unknown vector $y$ by the substitution $y = Ax$ wherever it appears. By this means, we arrive at a problem with respect to the vector $x$, which takes the form of (6) with $C = A$. Application of Theorem 4 to the last problem gives a solution in terms of $x$.

Back substitution of $x$ into the equality $y = Ax$ completes the proof. 

6.3 Numerical example

To illustrate the above result, we take an example project of three activities under constraints given by

$$
A = \begin{pmatrix}
4 & 0 & 0 \\
2 & 3 & 1 \\
1 & 1 & 3
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & -2 & 1 \\
0 & 0 & 2 \\
-1 & 0 & 0
\end{pmatrix}, \quad g = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}, \quad h = \begin{pmatrix}
5 \\
5 \\
5
\end{pmatrix},
$$

where the notation $\emptyset = -\infty$ is used to save writing.

We start with verification of the existence conditions of a regular solution in Theorem 8. We take the matrix $B$ and calculate

$$
B^2 = \begin{pmatrix}
0 & \emptyset & 0 \\
1 & -2 & 1 \\
\emptyset & -3 & 0
\end{pmatrix}, \quad B^3 = \begin{pmatrix}
-1 & -2 & 1 \\
0 & -1 & 2 \\
-1 & 0 & -1
\end{pmatrix}, \quad \text{Tr}(B) = 0.
$$
Furthermore, we obtain
\[ B^* = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 0 & 2 \\ -1 & -3 & 0 \end{pmatrix}, \quad AB^* = \begin{pmatrix} 4 & 2 & 5 \\ 4 & 3 & 5 \\ 2 & 1 & 3 \end{pmatrix}, \]
and then find
\[ h^- AB^* = \begin{pmatrix} -1 & -2 & 0 \end{pmatrix}, \quad h^- AB^* g = 0. \]

Since it holds that \( \text{Tr}(B) \oplus h^- AB^* g = 0 = 1 \), we conclude that the problem under study has regular solutions.

To get the solutions, we need to evaluate \( \theta \) which is given by (16).

Considering that \( n = 3 \), we represent \( \theta \) with three terms as follows
\[ \theta = \text{tr}(C_1 D) \oplus \text{tr}^{1/2}(C_2 D) \oplus \text{tr}^{1/3}(C_3 D), \]
where
\[ C_1 = A \oplus BA \oplus AB \oplus B^2A \oplus BAB \oplus AB^2, \]
\[ C_2 = A^2 \oplus B A^2 \oplus ABA \oplus A^2 B, \]
\[ C_3 = A^3, \]
\[ D = I \oplus gh^- A. \]

First we calculate the matrices
\[ A^2 = \begin{pmatrix} 8 & 4 & 1 \\ 6 & 6 & 4 \\ 5 & 4 & 6 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 12 & 8 & 5 \\ 10 & 9 & 7 \\ 9 & 7 & 9 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & -2 & -2 \\ -1 & 0 & -2 \\ -1 & -2 & 0 \end{pmatrix}. \]

To obtain the first term in the representation of \( \theta \), we successively find
\[ BA = \begin{pmatrix} 2 & 2 & 4 \\ 4 & 3 & 5 \\ 3 & -1 & 0 \end{pmatrix}, \quad AB = \begin{pmatrix} 0 & 2 & 5 \\ 3 & 0 & 5 \\ 2 & -1 & 3 \end{pmatrix}, \quad B^2A = \begin{pmatrix} 4 & 1 & 3 \\ 5 & 2 & 4 \\ 1 & 1 & 3 \end{pmatrix}, \]
\[ BAB = \begin{pmatrix} 3 & 0 & 4 \\ 4 & 2 & 5 \\ -1 & 1 & 4 \end{pmatrix}, \quad AB^2 = \begin{pmatrix} 4 & -2 & 4 \\ 4 & 1 & 4 \\ 2 & 0 & 3 \end{pmatrix}. \]

Now we have
\[ C_1 = \begin{pmatrix} 4 & 2 & 5 \\ 5 & 3 & 5 \\ 3 & 1 & 4 \end{pmatrix}, \quad C_1 D = \begin{pmatrix} 4 & 3 & 5 \\ 5 & 3 & 5 \\ 3 & 2 & 4 \end{pmatrix}, \quad \text{tr}(C_1 D) = 4. \]

Furthermore, we compute the matrices
\[ BA^2 = \begin{pmatrix} 6 & 5 & 7 \\ 8 & 6 & 8 \\ 7 & 3 & 0 \end{pmatrix}, \quad ABA = \begin{pmatrix} 6 & 6 & 8 \\ 7 & 6 & 8 \\ 6 & 4 & 6 \end{pmatrix}, \quad A^2 B = \begin{pmatrix} 4 & 6 & 9 \\ 6 & 4 & 8 \\ 5 & 3 & 6 \end{pmatrix}, \]
and then get the second term
\[ C_2 = \begin{pmatrix} 8 & 6 & 9 \\ 8 & 6 & 8 \\ 7 & 4 & 6 \end{pmatrix}, \quad C_2D = \begin{pmatrix} 8 & 7 & 9 \\ 8 & 6 & 8 \\ 7 & 5 & 6 \end{pmatrix}, \quad \text{tr}(C_2D) = 8. \]

After evaluation of the third term,
\[ C_3D = \begin{pmatrix} 12 & 10 & 10 \\ 10 & 9 & 8 \\ 9 & 7 & 9 \end{pmatrix}, \quad \text{tr}(C_3D) = 12, \]
we arrive at the minimum in the problem, which is given by \( \theta = 4 \).

Now we derive the solution vectors \( x \) and \( y \) according to (17) and (18). First we compute the matrices
\[ S = \begin{pmatrix} 0 & -2 & 1 \\ 0 & -1 & 2 \\ -1 & -3 & -1 \end{pmatrix}, \quad S^2 = \begin{pmatrix} 0 & -2 & 1 \\ 1 & -1 & 1 \\ -1 & -3 & 0 \end{pmatrix}, \quad S^* = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 0 & 2 \\ -1 & -3 & 0 \end{pmatrix}. \]

Furthermore, we get
\[ AS^* = \begin{pmatrix} 4 & 2 & 5 \\ 4 & 3 & 5 \\ 2 & 1 & 3 \end{pmatrix}, \quad \hat{h}^{-1}AS^* = \begin{pmatrix} -1 & -2 & 0 \end{pmatrix}, \quad (\hat{h}^{-1}AS^*)^{-} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}. \]

Denote by \( u_1 \) and \( u_2 \) the lower and upper bounds for the vector \( u \), which are defined by (18), and write
\[ u_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}. \]

The bounds on vector \( u \) produce corresponding bounds \( x_1 \) and \( x_2 \) on the vector \( x \). Evaluation of the bounds on \( x \) gives
\[ x_1 = S^*u_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad x_2 = S^*u_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}. \]

Since the bounds actually define a single vector, we arrive at a unique solution to the problem
\[ x = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad y = Ax = \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix}. \]
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