Photon Orbits and Thermodynamic Phase Transition of Regular AdS Black Holes

Naveena Kumara A.,* Ahmed Rizwan C.L.,† and Ajith K.M.‡

Department of Physics, National Institute of Technology Karnataka, Surathkal 575 025, India

Md Sabir Ali§

Department of Physics, Indian Institute of Technology, Ropar, Rupnagar, Punjab 140 001, India

Abstract

We probe the phase structure of regular AdS black holes using the null geodesics. The radius of photon orbit and minimum impact parameter shows a nonmonotonous behaviour below critical pressure, corresponds to the phase transition in extended phase space. The difference of the radius of unstable circular orbit and the minimum impact parameter can be seen as the order parameter for the small-large black hole phase transition. Our study shows that there exists a close relationship between the gravity and thermodynamics for the regular AdS black holes.

PACS numbers:

Keywords: Photon orbits, Black hole thermodynamics, Phase transition, Regular black holes.

*Electronic address: naviphysics@gmail.com
†Electronic address: ahmedrizwancl@gmail.com
‡Electronic address: ajithkm@gmail.com
§Electronic address: alimd.sabir3@gmail.com
I. INTRODUCTION

The importance of black hole thermodynamics is undeniable due to its intriguing applications, since the seminal work of Hawking and Bekenstein [1, 2]. The identification of temperature and entropy from the surface gravity and area of the event horizon, respectively, enable one to view the black hole as a thermodynamic system. Interestingly, as in the case of an ordinary thermodynamic system, black holes undergo phase transitions in a quite lot of ways. However, the phase transitions of black holes in AdS space has gained wide attention due to thermal stability. In the pioneering work, Hawking and Page [3] had shown the possibility of a phase transition between the thermal radiation and large black hole in the AdS cavity. Since the introduction of proper pressure term for black hole using the cosmological constant in AdS space by Kubiznak and Mann [4], it was observed that a van der Waals (vdW) like phase transition is exhibited by the charged AdS black holes. The phase transition in this scenario is between the small and large black holes (SBH-LBH) analogous to the liquid-gas transition in a vdW fluid.

The characteristic features of the material particles in the very vicinity of the event horizon can be utilized to unveil the information encoded in the concerned black hole. This motivation leads us to directly link the analysis of the particle motions that are affected by the strong gravity near the compact objects such as a black hole, neutron stars etc. The study of geodesics of a test particle plays an important role in the understanding of some observational effects such as the strong gravitational lensing and black hole silhouette, as well as quasinormal modes [5, 6]. Attempts to unravel the vdW phase transition of a black hole through astrophysical observations has its roots in quasinormal mode (QNM) studies [7]. It is reported that during the SBH-LBH phase transition, the slope of the quasinormal mode changes drastically.

Prompted by the study relating the dynamics and thermodynamics, in the context of AdS black holes, recently there were attempts to establish a relationship between the gravity and thermodynamics [8, 9]. The correlation between the gravity and the critical behaviour is seen through the unstable null geodesic which is encoded with the phase transition details. The radius of the photon sphere and the minimum impact parameter of that exhibits an oscillatory behaviour during the vdW phase transition. Above the critical point of phase transition, the behaviour becomes monotonous one. Another important result is that the
differences in the radius and the minimum impact parameters act as an order parameter for SBH-LBH phase transition with the critical exponent of $1/2$. The phase transition is scrutinised using photon orbit method for several black holes in different contexts [10–16].

The Penrose censorship conjecture states the existence of singularity dressed by an event horizon [17, 18]. Therefore all the electrovacuum solution of Einstein general relativity are in accordance with such point of view. However such conjectures do not forbid us to consider the regular black hole spacetimes freed from the singularity. Regular black holes were proposed to overcome such singular points, where the central singularity is replaced by a repulsive de-Sitter core. In this regard Bardeen, motivated by the ideas of Sakharov [19] and Gliner [20] proposed first regular black hole solution [21]. The subsequent study of all the regular black holes was inspired by Bardeen’s idea [22–24]. Later Ayon-Beato-Garcia found the first exact regular black hole solution of Einstein field equations coupled to nonlinear electrodynamic source. Various properties such as the black hole thermodynamics [25, 26], the rotating black hole shadows [27, 28], quasinormal modes [29] as well as the strong gravitational lensing [30] have been investigated in the background of regular black hole spacetimes. Some regular black hole solutions were also considered in alternative theories of gravity such as Lovelock gravity [31], massive gravity theories [32]. Regular black holes have been also extended to higher dimensions to study its horizon structure and thermodynamical properties [33, 34]. In the present work, we study the phase transition of regular black holes in AdS spacetime by considering the correspondence between photon orbits and the extended phase space thermodynamics. We show the parametric effect induced in the black hole solution due to the presence of nonlinear charge.

The organisation of the paper is as follows. In the next section, we discuss the phase transition of regular Hayward AdS black hole. In section III we carry out a similar investigation for the regular Bardeen AdS black hole. Finally, we conclude the paper in section IV.
II. REGULAR HAYWARD BLACK HOLE

A. Thermodynamics of Regular Hayward Black Hole

The metric of regular Hayward black hole is given by \[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega^2 \] (1)

where,

\[ f(r) = 1 - \frac{2Mr^2}{g^3 + r^3} + \frac{8}{3}\pi Pr^2. \] (2)

The pressure \( P \) is related to the cosmological constant \( \Lambda \) as \( P = -\Lambda/8\pi \). The Hawking temperature of the black hole is obtained as,

\[ T = \frac{f'(r_+)}{4\pi} = \frac{2Pr^4}{g^3 + r^3} - \frac{g^3}{2\pi r(g^3 + r^3)} + \frac{r^2}{4\pi (g^3 + r^3)}. \] (3)

The entropy of the black hole is taken as

\[ S = \pi r_+^2 \] (4)

where the first law will be modified with a redefined mass. However the modified mass has no role in our calculation. The equation of state reads as,

\[ P = \frac{g^3}{4\pi r^5} + \frac{g^3T}{2r^4} - \frac{1}{8\pi r^2} + \frac{T}{2r}. \] (5)

The regular black hole exhibits a vdW like critical behaviour, which has been studied extensively in the literature \[36\]. The critical point of this phase transition is given by,

\[ T_{cH} = \left( \frac{5\sqrt{2} - 4\sqrt{3}}{4\sqrt{22/9}} \right)^{2/3} \] (6)

\[ P_{cH} = \frac{3(\sqrt{6} + 3)}{16^{2/3}(\sqrt{6} + 7)^{5/3}\pi g^2} \] (7)

\[ S_{cH} = \left( 6\sqrt{6} + 14 \right)^{2/3} \pi g^2. \] (8)

B. Geodesic equations of motion

To obtain the relationship between the null geodesics and the phase transition of the black hole we consider a free photon orbiting around the black hole on the equitorial plane, i.e.,
\[ \theta = \pi/2. \] Then the lagrangian is,

\[ 2\mathcal{L} = -f(r)i^2 + \frac{\dot{r}^2}{f(r)} + r^2\dot{\phi}^2. \] (9)

The dots over variables stands for the differentiation with respect to an affine parameter. The generalised momentum corresponding to this Lagrangian can easily be obtained as,

\[ p_t = -f(r)i \equiv E \] (10)
\[ p_\phi = r^2\dot{\phi} \equiv L \] (11)
\[ p_r = \dot{r}/f(r). \] (12)

In the above, \( E \) and \( L \) are the energy and orbital angular momentum of the photon, respectively, which are the constants of motion. The \( t \) motion and \( r \) motion can be written as,

\[ \dot{i} = \frac{E}{f(r)} \] (13)
\[ \dot{\phi} = \frac{L}{r^2 \sin^2 \theta}. \] (14)

The Hamiltonian for the system is,

\[ 2\mathcal{H} = -E\dot{i} + L\dot{\phi} + \dot{r}^2/f(r) = 0. \] (15)

The expression for the radial \( r \) motion is rewritten as

\[ \dot{r}^2 + V_{\text{eff}} = 0 \] (16)

where \( V_{\text{eff}} \) is the effective potential, which has the following explicit form,

\[ V_{\text{eff}} = \frac{L^2}{r^2} f(r) - E^2. \] (17)

The behaviour of \( \bar{V}_{\text{eff}} = V_{\text{eff}}/E \) is shown in fig. 1.
Figure 1: The effective potential for the regular Hayward black hole. Here we take $S = 5$, $g = 0.8$ and $P = 0.003$. The thick red line corresponds to the critical angular momentum.

The accessible region for the photon is $V_{\text{eff}} < 0$, since $\dot{r}^2 > 0$. From the figure 1 it is clear that, the photon fall into the black hole for small values of $L$, whereas it is reflected for large values of $L$, as it approaches the black hole. Between these two conditions there is an unstable circular photon orbit which corresponds to the critical angular momentum (red thick line in figure 1). At the peak of that particular effective potential the radial velocity of the photon is zero. The corresponding value of $r$ at the peak is the radius of the photon sphere. The unstable circular orbit is characterised by,

$$V_{\text{eff}} = 0 \quad , \quad V'_{\text{eff}} = 0 \quad , \quad V''_{\text{eff}} < 0,$$

(18)

where the prime denotes the differentiation with respect to $r$. Expanding the second equation,

$$2f(r_{\text{ps}}) - r_{\text{ps}} \partial_r f(r_{\text{ps}}) = 0.$$

(19)

The solution of this gives the radius of photon sphere $r_{\text{ps}},$

$$r_{\text{ps}} = \frac{1}{4} \left( 2 \sqrt{\frac{8 g^3 M}{\sqrt{3} Y}} + \frac{9 M^2}{2} + \frac{27 M^3}{4 \sqrt{\frac{9 M^2}{4} + X + \sqrt{Y}}} - \sqrt{Y} + \sqrt{9 M^2 + 4 \left( X + \sqrt{Y} \right) + 3 M} \right),$$

(20)

where

$$X = \frac{8 g^3 M}{\sqrt[3]{27 g^3 M^3 + \sqrt{729 g^6 M^6 - 512 g^9 M^3}}},$$

(21)
and

\[ Y = 27g^3M^3 + \sqrt{729g^6M^6 - 512g^9M^3} \]  

(22)

The solution of the first equation, \( V_{\text{eff}} = 0 \), gives the minimum impact parameter of the photon,

\[ u_{ps} = \frac{L_c}{E} = \frac{r}{\sqrt{f(r)}} \bigg|_{r_{ps}} \]  

(23)

The explicit form of this can be obtained by using equation 20. write about BH lensing.

We can relate these key quantities, \( r_{ps} \) and \( u_{ps} \), to the thermodynamic variables \( P \) and \( S \) by using the expression for mass of the black hole \( M \). \( r_{ps}(P, S) \) is a complicated expression which we have not written here. The behaviour \( r_{ps} \) and \( u_{ps} \) against the temperature is studied in reduced parameter space (figure 2(a) and 2(b)). The similar study is carried out for \( \tilde{P} - \tilde{r}_{ps} \) and \( \tilde{P} - \tilde{u}_{ps} \) plots for a fixed value of reduced temperature.

![Figure 2](image)

(a) \( \tilde{T} \) vs. \( \tilde{r}_{ps} \)

(b) \( \tilde{T} \) vs. \( \tilde{u}_{ps} \)

Figure 2: The behaviour of photon sphere radius and minimum impact parameter with temperature in reduced space. These plots are for a fixed value of \( g = 0.8 \) and the reduced pressure \( \tilde{P} = 0.7, 0.8, 1, 1.2 \).
Figure 3: The behaviour of photon sphere radius and minimum impact parameter with pressure in reduced space. These plots are for a fixed value of $g = 0.8$ and the reduced temperature $\tilde{T} = 0.8, 0.9, 1, 1.2$.

Both the photon sphere radius and the critical impact parameter shows nonmonotonic behaviour below the critical pressure. All the isobars for $\tilde{P} < 1$ have one minimum and a maximum. There is an inflection point for the isobar $\tilde{P} = 1$. For all the values above $\tilde{P} = 1$ the oscillating behaviour disappears. This behaviour of $\tilde{T} - \tilde{r}_{ps}$ and $\tilde{T} - \tilde{u}_{ps}$ is quite similar to the isobars in $\tilde{T} - \tilde{S}$ plane for the vdW fluid. Similar inference can be drawn from $\tilde{P} - \tilde{r}_{ps}$ and $\tilde{P} - \tilde{u}_{ps}$ plots figure 3. If not for the exact depiction of the SBH-LBH phase transition, this could be seen as the indication of the phase structure of the black hole.

C. Critical behaviour from unstable photon orbits

As in $\tilde{T} - \tilde{S}_{ps}$ plane we can construct the equal area law for these curves for the coexistence curve. The sudden change in $\Delta \tilde{r}_{ps}$ and $\Delta \tilde{u}_{ps}$ exists in the regions corresponding to the first order phase transitions. The difference becomes zero at the the critical value of $\tilde{P}$ is approached, where the second order phase transition is observed. The differences $\Delta \tilde{r}_{ps}$ and $\Delta \tilde{u}$ are plotted against the reduced temperature $\tilde{T}$, which is shown in figure 4 and 5. The $\tilde{r}_{ps}$ and $\tilde{u}_{ps}$ has two branches corresponding to SBH and LBH phases of the black hole. The behaviour of $\Delta \tilde{r}_{ps}$ and $\Delta \tilde{u}_{ps}$ is observed near the critical point in the inlets of figure 4(b) and 5(b). It is clearly seen that their pattern near that point is similar to that of the order parameters for the black hole phase transition. i.e. $\Delta \tilde{r}_{ps} \approx (1 - \tilde{T})^{1/2}$ and
\[ \Delta \tilde{u}_{ps} \approx (1 - \tilde{T})^{1/2}. \] This, once again confirms our earlier observation on the connection between the photon orbits and thermodynamic phase transitions.

\[ \Delta \tilde{r}_{ps} \approx (1 - \tilde{T})^{1/2}. \] This, once again confirms our earlier observation on the connection between the photon orbits and thermodynamic phase transitions.

Figure 4: The behaviour of photon sphere radius and its difference along the coexistence curve.

\[ \Delta \tilde{u}_{ps} \approx (1 - \tilde{T})^{1/2}. \] This, once again confirms our earlier observation on the connection between the photon orbits and thermodynamic phase transitions.

Figure 5: The behaviour of critical impact parameter and its difference along the coexistence curve.

III. REGULAR BARDEEN BLACK HOLE

The Bardeen solution of the black hole in AdS spacetime has the following form \[35],

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega^2 \]  \hspace{1cm} (24)
with
\[ f(r) = 1 - \frac{2Mr^2}{(g^2 + r^2)^{3/2}} + \frac{8}{3} \pi Pr^2. \] (25)

As earlier the pressure \( P \) is related to the cosmological constant \( \Lambda \) as \( P = -\Lambda/8\pi \). The Hawking temperature can be easily obtained as,
\[ T = \frac{f'(r_+)}{4\pi} = \frac{2P r^3}{g^2 + r^2} + \frac{r}{4\pi (g^2 + r^2)} - \frac{g^2}{2\pi r (g^2 + r^2)}. \] (26)

The entropy of the black hole is taken in the usual form,
\[ S = \pi r_+^2. \] (27)

With these inputs the equation of state can be written as,
\[ P = \frac{g^2}{4\pi r^4} + \frac{g^2 T}{2r^3} - \frac{1}{8\pi r^2} + \frac{T}{2r} \] (28)

from which an oscillatory behaviour of isotherms below critical temperature and hence the phase transition is evident, which is well studied [37]. The critical values of the thermodynamic for this phase transition is determined by the usual methods, which are,
\[ T_{cB} = -\left(\sqrt{273} - 17\right) \sqrt{\frac{1}{2}} \left(\sqrt{273} + 15\right) \] (29)
\[ P_{cB} = \frac{\sqrt{273} + 27}{12 (\sqrt{273} + 15)^2 \pi g^2} \] (30)
\[ S_{cB} = \frac{1}{2} \left(\sqrt{273} + 15\right) \pi g^2. \] (31)

A. Photon Orbit and Phase transition

The geodesic for the photon moving in the equatorial plane of regular Bardeen AdS black hole is analysed in the same line as that of Hayward case. The behaviour of \( \tilde{V}_{eff} = V_{eff}/E \) is shown in fig. 6. The interpretation of this in the Hayward case is also applicable here. However, there is a distinct difference in the critical effective potential, which is clear from fig 6 compared to fig 1.
Figure 6: The effective potential for the regular Bardeen black hole. Here we take $S = 5$, $g = 0.8$ and $P = 0.001$. The thick red line corresponds to the critical angular momentum.

The photon orbit expression is obtained by solving the second relation in equation 19 for the Bardeen background, which has a relatively simple form,

$$r_{ps} = \frac{2^{2/3}M^2}{\sqrt{Z}} + M + \frac{3\sqrt{Z}}{2^{2/3}}$$  \hfill (32)

where

$$Z = -15g^2M + \sqrt{15}\sqrt{15g^4M^2 - 8g^2M^4 + 4M^3}.$$  \hfill (33)

With the use of this and equation 23 we obtain the minimum impact parameter $u_{ps}$ for the Bardeen case. As argued earlier, the photon sphere radius and minimum impact parameter are the key quantities in probing the phase transitions of the black hole. The isobars in $\tilde{T}$ vs. $\tilde{r}_{ps}$ and $\tilde{T}$ vs. $\tilde{u}_{ps}$ and isotherms in $\tilde{P}$ vs. $\tilde{r}_{ps}$ and $\tilde{P}$ vs. $\tilde{u}_{ps}$ planes shows the corresponding vdW like phase transition in regular Bardeen black hole (figure 7 and 8). The difficulty in solving the equal area law analytically forbids us in investigating the behaviour of differences in the photon sphere radius and minimum impact parameter. However based on the similarities in the isobar and isotherm studies of Bardeen and Hayward cases, we claim that the result is evident in any regular spacetime background.
Figure 7: The behaviour of photon sphere radius and minimum impact parameter with temperature in reduced space. These plots are for a fixed value of $g = 0.8$ and the reduced pressure $\tilde{P} = 0.7, 0.8, 1, 1.2$.

Figure 8: The behaviour of photon sphere radius and minimum impact parameter with pressure in reduced space. These plots are for a fixed value of $g = 0.8$ and the reduced temperature $\tilde{T} = 0.8, 0.9, 1, 1.2$.

IV. CONCLUDING REMARKS

In this article, using the formalism of unstable circular null geodesics for a class of regular black holes including Hayward-AdS and Bardeen-AdS spacetimes, we find a close connection between the gravity and thermodynamics in the extended phase space. The well-known van der Waals-like phase structure is probed via the photon orbit radius $r_{ps}$ and minimum
critical impact parameter $u_{ps}$. The isobars and isotherms in each of these key parameters show oscillatory behaviour below the critical values of the temperature $\tilde{T}$ and the pressure $\tilde{P}$, respectively, in the reduced parameter space. Such behaviours are in accordance with the van der Waals-like phase transition of the black holes. This first-order phase transition disappears above the critical in the respective plots of $r_{ps}$ and $u_{ps}$. Moreover, the differences $\Delta r_{ps}$ and $\Delta u_{ps}$ serve as order parameters for the critical behaviour. Furthermore, near the second-order phase transition points, these differences exhibit a change concavity with critical exponents $\delta = 1/2$. Our results show that regular black holes are in close proximity with charged AdS black holes, in phase transition perspectives. Thus a regular modification to the electrovacuum solutions of Einstein field equations may be a possible candidate for probing its thermal properties. Our results may also be constrained to the acceptance or discarding of the existence of regular black holes, as it is an effort to connect the thermodynamic phase transition to observational aspects of the concerned compact objects.

Acknowledgments

Author N.K.A. would like to thank U.G.C. Govt. of India for financial assistance under UGC-NET-SRF scheme.

[1] S. W. Hawking, Particle Creation by Black Holes, Commun. Math. Phys. 43 (1975) 199–220, doi:10.1007/BF02345020, 10.1007/BF01608497.
[2] J. D. Bekenstein, Black holes and entropy, Phys. Rev. D7 (1973) 2333–2346. doi:10.1103/PhysRevD.7.2333.
[3] S. W. Hawking, D. N. Page, Thermodynamics of Black Holes in anti-De Sitter Space, Commun. Math. Phys. 87 (1983) 577. doi:10.1007/BF01208266.
[4] D. Kubiznak, R. B. Mann, P-V criticality of charged AdS black holes, JHEP 07 (2012) 033. arXiv:1205.0559, doi:10.1007/JHEP07(2012)033.
[5] V. Cardoso, A. S. Miranda, E. Berti, H. Witek, V. T. Zanchin, Geodesic stability, Lyapunov exponents and quasinormal modes, Phys. Rev. D79 (2009) 064016. arXiv:0812.1806, doi:10.1103/PhysRevD.79.064016.
[6] I. Z. Stefanov, S. S. Yazadjiev, G. G. Gyulchev, Connection between Black-Hole Quasinormal Modes and Lensing in the Strong Deflection Limit, Phys. Rev. Lett. 104 (2010) 251103. arXiv:1003.1609, doi:10.1103/PhysRevLett.104.251103.

[7] Y. Liu, D.-C. Zou, B. Wang, Signature of the Van der Waals like small-large charged AdS black hole phase transition in quasinormal modes, JHEP 09 (2014) 179. arXiv:1405.2644, doi:10.1007/JHEP09(2014)179.

[8] S.-W. Wei, Y.-X. Liu, Photon orbits and thermodynamic phase transition of $d$-dimensional charged AdS black holes, Phys. Rev. D97 (10) (2018) 104027. arXiv:1711.01522, doi:10.1103/PhysRevD.97.104027.

[9] S.-W. Wei, Y.-X. Liu, Y.-Q. Wang, Probing the relationship between the null geodesics and thermodynamic phase transition for rotating Kerr-AdS black holes, Phys. Rev. D99 (4) (2019) 044013. arXiv:1807.03455, doi:10.1103/PhysRevD.99.044013.

[10] Y.-M. Xu, H.-M. Wang, Y.-X. Liu, S.-W. Wei, Photon sphere and reentrant phase transition of charged Born-Infeld-AdS black holes, Phys. Rev. D100 (10) (2019) 104044. arXiv:1906.03334, doi:10.1103/PhysRevD.100.104044.

[11] M. Zhang, S.-Z. Han, J. Jiang, W.-B. Liu, Circular orbit of a test particle and phase transition of a black hole, Phys. Rev. D99 (6) (2019) 065016. arXiv:1903.08293, doi:10.1103/PhysRevD.99.065016.

[12] B. Chandrasekhar, S. Mohapatra, A Note on Circular Geodesics and Phase Transitions of Black Holes, Phys. Lett. B791 (2019) 367–374. arXiv:1805.05088, doi:10.1016/j.physletb.2019.02.042.

[13] M. Chabab, H. El Moumni, S. Iraoui, K. Masmar, Probing correlation between photon orbits and phase structure of charged AdS black hole in massive gravity background (2019). arXiv:1902.00557.

[14] H. Li, Y. Chen, S.-J. Zhang, Photon orbits and phase transitions in Born-Infeld-dilaton black holes (2019). arXiv:1908.09570.

[15] S.-W. Wei, Y.-X. Liu, Null Geodesics, Quasinormal Modes, and Thermodynamic Phase Transition for Charged Black Holes in Asymptotically Flat and dS Spacetimes (2019). arXiv:1909.11911.

[16] S.-Z. Han, J. Jiang, M. Zhang, W.-B. Liu, Photon orbits and thermodynamic phase transition in Gauss-Bonnet AdS black holes (2018). arXiv:1812.11862.
[17] S. W. Hawking, R. Penrose, The Singularities of gravitational collapse and cosmology, Proc. Roy. Soc. Lond. A314 (1970) 529–548. doi:10.1098/rspa.1970.0021.

[18] S. W. Hawking, G. F. R. Ellis, The Large Scale Structure of Space-Time, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 2011. doi:10.1017/CBO9780511524646.

[19] A. D. Sakharov, The Initial Stage of an Expanding Universe and the Appearance of a Nonuniform Distribution of Matter, Soviet Journal of Experimental and Theoretical Physics 22 (1966) 241.

[20] E. B. Gliner, Algebraic Properties of the Energy-momentum Tensor and Vacuum-like States + Matter, Soviet Journal of Experimental and Theoretical Physics 22 (1966) 378.

[21] J. Bardeen, Non-singular general-relativistic gravitational collapse, in proceedings of the international conference gr5, Tbilisi, USSR (1968).

[22] S. A. Hayward, Formation and evaporation of regular black holes, Phys. Rev. Lett. 96 (2006) 031103. arXiv:gr-qc/0506126, doi:10.1103/PhysRevLett.96.031103.

[23] E. Ayon-Beato, A. Garcia, Regular black hole in general relativity coupled to nonlinear electrodynamics, Phys. Rev. Lett. 80 (1998) 5056–5059. arXiv:gr-qc/9911046, doi:10.1103/PhysRevLett.80.5056.

[24] E. Ayon-Beato, A. Garcia, The Bardeen model as a nonlinear magnetic monopole, Phys. Lett. B493 (2000) 149–152. arXiv:gr-qc/0009077, doi:10.1016/S0370-2693(00)01125-4.

[25] J. Man, H. Cheng, The calculation of the thermodynamic quantities of the Bardeen black hole, Gen. Rel. Grav. 46 (2014) 1660. arXiv:1304.5686, doi:10.1007/s10714-013-1660-4.

[26] J. Man, H. Cheng, The description of phase transition of Bardeen black hole in the Ehrenfest scheme (2013). arXiv:1312.6566.

[27] A. Abdujabbarov, M. Amir, B. Ahmedov, S. G. Ghosh, Shadow of rotating regular black holes, Phys. Rev. D93 (10) (2016) 104004. arXiv:1604.03809, doi:10.1103/PhysRevD.93.104004.

[28] M. Amir, S. G. Ghosh, Shapes of rotating nonsingular black hole shadows, Phys. Rev. D94 (2) (2016) 024054. arXiv:1603.06382, doi:10.1103/PhysRevD.94.024054.

[29] A. Flachi, J. P. S. Lemos, Quasinormal modes of regular black holes, Phys. Rev. D87 (2) (2013) 024034. arXiv:1211.6212, doi:10.1103/PhysRevD.87.024034.

[30] E. F. Eiroa, C. M. Sendra, Gravitational lensing by a regular black hole, Class. Quant. Grav. 28 (2011) 085008. arXiv:1011.2455, doi:10.1088/0264-9381/28/8/085008.
[31] R. Aros, M. Estrada, Regular black holes and its thermodynamics in Lovelock gravity, Eur. Phys. J. C79 (3) (2019) 259. arXiv:1901.08724, doi:10.1140/epjc/s10052-019-6783-7.

[32] C. H. Nam, Non-linear charged AdS black hole in massive gravity, Eur. Phys. J. C78 (12) (2018) 1016. doi:10.1140/epjc/s10052-018-6498-1.

[33] M. S. Ali, S. G. Ghosh, Exact $d$-dimensional bardeen-de sitter black holes and thermodynamics, Phys. Rev. D 98 (2018) 084025. doi:10.1103/PhysRevD.98.084025. URL https://link.aps.org/doi/10.1103/PhysRevD.98.084025

[34] A. Kumar, D. Veer Singh, S. G. Ghosh, $D$-dimensional Bardeen-AdS black holes in Einstein-Gauss-Bonnet theory, Eur. Phys. J. C79 (3) (2019) 275. arXiv:1808.06498, doi:10.1140/epjc/s10052-019-6773-9.

[35] Z.-Y. Fan, X. Wang, Construction of Regular Black Holes in General Relativity, Phys. Rev. D94 (12) (2016) 124027. arXiv:1610.02636, doi:10.1103/PhysRevD.94.124027.

[36] Z.-Y. Fan, Critical phenomena of regular black holes in anti-de Sitter space-time, Eur. Phys. J. C77 (4) (2017) 266. arXiv:1609.04489, doi:10.1140/epjc/s10052-017-4830-9.

[37] A. G. Tzikas, Bardeen black hole chemistry, Phys. Lett. B788 (2019) 219–224. arXiv:1811.01104, doi:10.1016/j.physletb.2018.11.036.