Activated dynamic scaling in the random-field Ising model: a nonperturbative functional renormalization group approach

Ivan Balog
Institute of Physics, P.O.Box 304, Bjoenichica cesta 46, HR-10001 Zagreb, Croatia and LPTMC, CNRS-UMR 7600, Université Pierre et Marie Curie, boîte 121, 4 Pl. Jussieu, 75252 Paris cédex 05, France

Gilles Tarjus
LPTMC, CNRS-UMR 7600, Université Pierre et Marie Curie, boîte 121, 4 Pl. Jussieu, 75252 Paris cédex 05, France

(Dated: January 26, 2015)

The random-field Ising model shows extreme critical slowdown that has been described by activated dynamic scaling: the characteristic time for the relaxation to equilibrium diverges exponentially with the correlation length, \( \ln \tau \sim \xi^\psi / T \), with \( \psi \) an \textit{a priori} unknown barrier exponent. Through a nonperturbative functional renormalization group approach, we show that for spatial dimensions \( d \) less than a critical value \( d_{DR} \approx 5.1 \), also associated with dimensional-reduction breakdown, \( \psi = \theta \) with \( \theta \) the temperature exponent near the zero-temperature fixed point that controls the critical behavior. For \( d > d_{DR} \) on the other hand, \( \psi = \theta - 2 \lambda \) where \( \theta = 2 \) and \( \lambda > 0 \) a new exponent. At the upper critical dimension \( d = 6 \), \( \lambda = 1 \) so that \( \psi = 0 \), and activated scaling gives way to conventional scaling. We give a physical interpretation of the results in terms of collective events in real space, avalanches and droplets. We also propose a way to check the two regimes by computer simulations of long-range \( 1-d \) systems.

PACS numbers: 11.10.Hi, 75.40.Cx

Activated dynamic scaling is a phenomenological description of the extreme slowdown of dynamics observed in some disordered or glassy systems: systems in the presence of a quenched random field. An ordered system, an elastic manifold pinned in a random environment is a phenomenological description of the extreme slowdown of dynamics observed in some disordered or glassy systems: systems in the presence of a quenched random field. It is characterized by the presence of a quenched random field, which is controlled in the renormalization group sense, by a unknown barrier exponent. Activated scaling leads to a broad distribution of relaxation times, which shows up in the time or frequency dependence of the response and correlation functions, and has also consequences for the nonequilibrium dynamics.

The random field Ising model is one system whose dramatic critical slowing down is expected to be described by activated dynamic scaling. Its critical point is controlled, in the renormalization group sense, by a zero-temperature fixed point at which the “dangerously irrelevant” renormalized temperature is characterized by an exponent \( \theta > 0 \). The dangerous irrelevancy leads to a breakdown of the hyperscaling relation between critical exponents and anomalous thermal fluctuations, all controlled by the exponent \( \theta \) and further rationalized at a physical level by the “droplet scenario”. The simplest droplet assumption would be to set \( \psi = \theta \). Actually, this equality has been found in the dynamics of a simpler disordered system, an elastic manifold pinned in a random potential, but in the case of the RFIM there has been no attempt to compute the barrier exponent \( \psi \).

The functional renormalization group (FRG) is a tool of choice to provide a theoretical treatment beyond phenomenology and compute the barrier exponent \( \psi \). In its perturbative form, it has been successfully applied to the dynamics of the pinned elastic manifolds. For the RFIM, as was shown for the (static) equilibrium behavior, the FRG must be nonperturbative (NP-FRG). We therefore generalize the NP-FRG approach to describe the critical slowing down of the RFIM.

As we are interested in the long-time, collective behavior of the RFIM, a coarse-grained field theory provides an appropriate starting point. The relaxation dynamics of the scalar field \( \psi_{xt} \) is thus described by a Langevin equation (for simplicity we consider the case of a nonconserved order parameter, known as model \( \Delta^2 \)).

\[
\partial_t \psi_{xt} = -\Omega_B \frac{\delta S[\psi]}{\delta \psi_{xt}} + \eta_{xt},
\]

where \( \eta_{xt} \) is a Gaussian random noise term with zero mean and variance \( \langle \eta_{xt} \eta_{x't'} \rangle = 2 T \Omega_B \delta^{(d)}(x-x') \delta(t-t') \). The “action” or effective Hamiltonian \( S[\psi] \) is given by

\[
S[\psi; h + J] = S_B[\psi] - \int_x [h(x) + J_x] \psi_x,
\]

\[
S_B[\psi] = \int_x \left\{ \frac{1}{2} [\partial_x \psi_x]^2 + \frac{r}{2} \psi_x^2 + \frac{u}{4!} \psi_x^4 \right\},
\]

where \( \int_x \equiv \int d^dx \), \( J_x \) is an external source, \( h_x \) is a random “source” (a random magnetic field) taken with a Gaussian distribution characterized by a zero mean and a variance \( \langle h_x h_{x'} \rangle = \Delta_B \delta^{(d)}(x-x') \).

The generating functional of the multi-point and multi-time correlation and response functions can be built as...
usual by following the MSR formalism. After introducing an auxiliary “response” field \( \hat{\varphi}_{xt} \) and taking into account the fact that the solution of Eq. (1) is unique, one obtains the “partition function”

\[
Z_{\hbar,\eta}[\hat{J}, J] = \int D\varphi D\hat{\varphi} \exp \left\{ - \int_{xt} \hat{\varphi}_{xt} \left[ \partial_t \varphi_{xt} + \Omega_B \frac{\delta S_B[\varphi]}{\delta \varphi_{xt}} - \eta_{xt} - h_x \right] + \int_{xt} \left( J_{xt} \varphi_{xt} + J_{xt} \hat{\varphi}_{xt} \right) \right\}
\]

(3)

where we have used the Itô prescription (which amounts to setting to 1 the Jacobian of the transformation between the thermal noise and the field). It turns out to be convenient for describing the cumulants of the renormalized disorder to introduce copies or replicas of the system. The copies have the same disorder \( h \) but are coupled to distinct sources and are characterized by independent thermal noises. After averaging over the thermal noises and the disorder, one obtains

\[
Z[\hat{J}_a, J_a] = \int \prod_a D\varphi_a D\hat{\varphi}_a e^{-S_{dyn}[(\varphi_a, \hat{\varphi}_a)] + \sum_a \int_{xt} (J_{xt} \varphi_{xt} + J_{xt} \hat{\phi}_{xt} \varphi_{xt})} \]

(4)

where the (bare) dynamical action is

\[
S_{dyn}[(\varphi_a, \hat{\varphi}_a)] = \sum_a \int_{xt} \hat{\varphi}_{xt} \left[ \delta_t \varphi_{xt} - T \hat{\varphi}_{xt} \right] + \frac{\delta S_B[\varphi_a]}{\delta \varphi_{xt}} - \frac{\Delta_B}{2} \sum_{ab} \int_{t,t'} \hat{\varphi}_{xt} \varphi_{bt} \]

(5)

and where we have set \( \Omega_B = 1 \); \( \ln Z \) is the sought generating functional of the response and correlation functions. In the long-time limit, the relaxation toward equilibrium satisfies, in addition to the causality requirement, an invariance under time translation (TTI) and a time-reversal symmetry (TRS). The latter in turn implies the fluctuation-dissipation theorem. The TRS corresponds to an invariance of the theory under the simultaneous transformations \( t \to -t, \varphi_a \to \varphi_a, \) and \( \hat{\varphi}_a \to \hat{\varphi}_a - (1/T) \partial_t \phi_a. \)

Next, we apply the NP-FRG formalism to this dynamical field theory. To do this we introduce an infrared (IR) regulator \( \Delta S_k \) to the action \( \delta \Gamma \), whose role is to suppress the integration over slow modes associated with momenta \( |q| \lesssim k \) in the functional integral. The latter choice

\[
\Delta S_k[(\varphi_a)] = \frac{1}{2} \int_{xx',tt'} \text{tr} \left[ \sum_a \varphi_{a,xt} \hat{R}_k(x-x', t-t') \Phi_{a,xt}^{\top} \right] + \frac{1}{2} \sum_{ab} \varphi_{a,xt} \hat{R}_k(x-x', t-t') \Phi_{b,xt}^{\top},
\]

(6)

where \( \Phi_a \equiv (\varphi_a, \hat{\varphi}_a), \Phi_{a,xt}^{\top} \) its transpose, and \( \hat{R}_k \) and \( \hat{R}_k \) are symmetric \( 2 \times 2 \) matrices of mass-like IR cutoff functions that enforce the decoupling between fast (high-momentum) and slow (low-momentum) modes in the partition function. Following Ref. \( \delta \Gamma \) it proves sufficient to control the contribution of the fluctuations through their momentum dependence and take \( \hat{R}_{k,11} = \hat{R}_{k,22} = 0, \hat{R}_{k,12} = \hat{R}_{k,21} = 0, \hat{R}_{k,02} = \hat{R}_{k,01}(x-x'), \) and \( \hat{R}_{k,11} = \hat{R}_{k,12} = \hat{R}_{k,21} = 0, \hat{R}_{k,22} = \hat{R}_{k}(x-x') \) where \( \hat{R}_k(q^2) \) and \( \hat{R}_k(q^2) \) are chosen (in Fourier space) such that the integration over modes with momentum \( |q| \lesssim k \) is suppressed. To avoid an explicit breaking of the underlying super-rotations of the theory \( \hat{R}_k(q^2) \propto \hat{R}_k(q^2) / q^2 \). Note that this choice of IR regulator satisfies the TRS, a crucial property.

Through this addition \( Z[J_a, J_a] \) is replaced by a \( k \)-dependent quantity, \( Z_k[(J_a, J_a)] \), where \( J_a \) denotes \( (\hat{J}_a, J_a) \). The central quantity of the NP-FRG is the “effective average action” \( \Gamma_k \) which is the generating functional of the 1-particle irreducible correlation functions at the scale \( k \). It is defined from \( \ln Z_k[(J_a, J_a)] \) via a Legendre transform:

\[
\Gamma_k[(\varphi_a)] + \ln Z_k[(J_a, J_a)] = \sum_a \int_{xt} \text{tr} J_{a,xt} \Phi_{a,xt}^{\top} \delta S_k[(\varphi_a)],
\]

(7)

where \( \varphi_a \equiv (\varphi_a, \hat{\varphi}_a) \) now denotes the “classical” (or average) fields with \( \delta \ln Z_k / \delta J_{a,xt} = (\varphi_a, \hat{\varphi}_a) \) and \( \hat{\varphi}_{a,xt} = \delta \ln Z_k / \delta J_{a,xt} = (\varphi_a, \hat{\varphi}_a) \); the trace operation is over the 2 components of \( \Phi_a \) and \( J_a \).

Expansions in generalized cumulants are then generated by expanding the functional in increasing number of unrestricted sums over copies,

\[
\Gamma_k[(\varphi_a)] = \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{p!} \sum_{\delta_0, \ldots, \delta_{ap}} \Gamma_{kp}[(\Phi_a, \ldots, \Phi_a)],
\]

(8)

where \( \Gamma_{kp} \) can be formally expressed as

\[
\Gamma_{kp} = \int_{x_1 t_1 \ldots x_p t_p} \hat{\phi}_{a_1, x_t} \hat{\phi}_{a_p, x_t} \gamma_{kp}(x_{t_1} \ldots x_{t_p}),
\]

(9)

with \( \gamma_{kp} \) a functional of the fields \( \phi_{a_1, t_1}, \ldots, \phi_{a_p, t_p} \) and of their time derivatives, \( \partial_t \phi_{a_1, t_1}, \ldots, \partial_t \phi_{a_p, t_p} \), \( q > 1 \). When the fields are chosen uniform in time with \( \phi_{a,xt} = \phi_{a,xt} = 0 \), the \( \gamma_{kp} \)'s reduce to the cumulants of the renormalized random field at equilibrium already studied in Refs. \( \delta \Gamma \). For generic fields, the additional contributions then represent kinetic terms.

\( \Gamma_k \) satisfies an exact RG flow equation that describes its evolution with the IR cutoff \( k \).

\[
\partial_k \Gamma_k[(\varphi_a)] = \frac{1}{2} \text{Tr} \left\{ (\partial_k \varphi_a)(\varphi_a)^2 \right\} + (\varphi_a)^{-1},
\]

(10)

where the trace is over space-time coordinates, copy indices, and components, and \( \Gamma^{(2)} \) is the matrix formed by the second functional derivatives of the effective average action. (In what follows, superscripts within a parenthesis are used to indicate derivatives with respect to the appropriate arguments.) By inserting the expansion in increasing number of sums over copies and proceeding to the associated algebraic manipulations, one then derives an infinite hierarchy of ERGE’s for the generalized cumulants \( \Gamma_{kp} \) or, alternatively, for the functional \( \gamma_{kp}. \)
One cannot hope to solve this infinite hierarchy of functional flow equations exactly, but one can describe the long-distance physics of the problem by means of a nonperturbative approximation scheme. We combine the minimal truncation of the effective average action already shown to successfully describe the equilibrium critical behavior of the RFIM with an account of the dynamics through a truncation of the expansion in kinetic coefficients allowing us to describe the characteristic relaxation time (but not its distribution: to do this the next orders of the truncation would be required, as discussed in Refs. [3,4]). By taking into account the TRS, we arrive at the following ansatz:

\[
\gamma_{k_1; x_1}[\Phi] = \frac{\delta}{\phi_{x_1}} \left[ U_k(\phi_{x_1}) + \frac{1}{2} Z_k(\phi_{x_1})(\partial_x \phi_{x_1})^2 \right] + X_k(\phi_{x_1})(\partial_x \phi_{x_1} - T \hat{\phi}_{x_1})
\]

\[
\gamma_{k_2; x_1, x_2}[\Phi_1, \Phi_2] = \delta^{(d)}(x_1 - x_2) \Delta_k(\phi_{1, x_1, t_1}, \phi_{2, x_2, t_2})
\]

while the \(\gamma_{kp}\)'s with \(p \geq 3\) are set to zero.

The next step is to derive the RG flow equations for the functions contained in the ansatz. As already mentioned, we work in the Itô discretization scheme and the corresponding prescription can be systematically implemented in the NP-FRG equations by following the simple procedure developed in Ref. [4]. The derivation is tedious but straightforward and will be presented elsewhere [see also the Supplemental Material(SM)]. The output is a set of coupled flow equations for 3 functions of one field, \(U_k(\phi), Z_k(\phi), X_k(\phi)\), and one function of 2 fields \(\Delta_k(\phi_{1, x_1, t_1}, \phi_{2, x_2, t_2})\). As a result of the TRS, the renormalized kinetic function \(X_k\) does not enter the flow of the static ones \((U_k', Z_k, \Delta_k)\).

The flow equations involve the renormalized propagators at scale \(k\) evaluated for fields that are uniform in space and time, \(\Phi_{a,x,t} \equiv (\phi_a, 0)\), and for the lowest order of the expansions is sums over copies. In Fourier (momentum) space, these propagators are expressed as \(\mathbf{P}_k(q; t', t) = \mathbf{P}_k(q; \phi_a; t', t)\delta_{ab} + \mathbf{P}_k(q; \phi_a, \phi_b; t; t')\) where the \(2 \times 2\) matrix \(\mathbf{P}_k\) has a structure following from causality, TTI and fluctuation-dissipation theorem, with

\[
\mathbf{P}_{12}^{11}(q; \phi; t' - t) = \Theta(t' - t) X_k(\phi)^{-1} e^{-\frac{|q - t|}{\tau_{k}(\phi)}}
\]

the response function, \(\mathbf{P}_{22}^{21}(t' - t) = \mathbf{P}_{12}^{12}(t' - t)\), \(\mathbf{P}_{11}\) the 2-time disorder-connected correlation function given by

\[
\mathbf{P}_{11}^{11}(q; \phi; t' - t) = T[Z_k(\phi)q^2 + \tilde{R}_k(q^2) + U_k''(\phi)]^{-1} e^{-\frac{|q - t|}{\tau_{k}(\phi)}}
\]

and \(\mathbf{P}_{22} = 0\); the characteristic relaxation time is defined as \(\tau_{k}(q; \phi) = X_k(\phi)/[Z_k(\phi)q^2 + \tilde{R}_k(q^2) + U_k''(\phi)]\). In addition, the only nonzero component of \(\mathbf{P}_k\) is

\[
\mathbf{P}_{11}^{11}(q; \phi_1; t) \int \mathbf{P}_{22}^{21}(q_2; t') [\Delta_k(\phi_1, \phi_2) - \tilde{R}_k(q_2)]
\]

which, in this truncation, is simply the static (equilibrium) disorder-disconnected correlation function.

Finally, to study the vicinity of the relevant zero-temperature critical fixed point the NP-FRG equations are cast in a dimensionless form by introducing appropriate scaling dimensions: \(\phi \sim k^{(d-4+\eta)/2}, Z_k \sim k^{-\eta}, U_k'' \sim k^{(d-2n+\eta)/2}, \Delta_k \sim k^{-2(\eta-n)}\) and the renormalized temperature \(T_k \sim T_k^0\), where \(\theta = 2 - \eta - \bar{\eta} > 0\). We express the results in terms of the dimensionless fields \(\phi = \frac{\gamma_{k_1}}{\eta_{k_1}}\) and \(\delta \phi = \frac{\Delta_k}{T_k}\). With lowercase letters denoting dimensionless quantities, one can formally write the flow equations for the static quantities as

\[
k \partial_k u_k(\phi) = \beta_{u_0}(\phi) + T_k \beta_{u_1}(\phi),
\]

\[
k \partial_k z_k(\phi) = \beta_{z_0}(\phi) + T_k \beta_{z_1}(\phi),
\]

\[
k \partial_k \delta_k(\phi, \delta \phi) = \beta_{\delta0}(\phi, \delta \phi) + T_k \beta_{\delta1}(\phi, \delta \phi),
\]

(16)

where the beta functions depend on \(u_k, z_k, \delta_k\), their derivatives, and on the (dimensionless) cutoff functions.

These equations generalize to nonzero temperature those given in Ref. [3]. In the latter, i.e., when \(T = 0\), it was previously found that the fixed-point solution displays two regimes: (1) for \(d < d_{DR} \approx 5.1\), a “cusp” in \(|\delta \phi|\) present in the fixed-point function \(\delta_k\) when \(\delta \phi \rightarrow 0\). This cusp is associated with the presence of avalanches on all scales at the critical point \(\Delta\) (2) For \(d > d_{DR}\) the fixed-point function \(\delta_k\) is “cuspless”, which ensures that the \(d \rightarrow d - 2\) dimensional-reduction property of the critical exponents \(\gamma_{2,3}\) is valid (and the super-reaction is not spontaneously broken along the RG flow). Avalanches are still present but only lead to a subdominant cusp, \(\delta_k(\delta \phi, \phi) = \delta_k(\phi, 0) - \delta_{k,a}(\phi)|\delta \phi| + O(\delta \phi^2)\) where \(\delta_{k,a}(\phi) \sim k^h\) when \(k \rightarrow 0\) with \(h > 0\) characterizing the (diverging) number of spanning avalanches \(\Delta_{1,2,3}\).

When \(T > 0\), the beta function of \(\delta_k\) in the limit \(\delta \phi \rightarrow 0\) and \(T_k \rightarrow 0\) can be written as

\[
k \partial_k \delta_k(\phi, \delta \phi) \approx \beta_{\delta,reg}(\phi) + \frac{a_{1k}(\phi)}{2} \frac{\delta_k(\phi, \delta \phi) - \delta_k(\phi, 0)^2}{\delta(\delta \phi)^2} - T_k a_{2k}(\phi)\delta_{k}^{(02)}(\phi, \delta \phi)
\]

(17)

where \(\beta_{\delta,reg}\) is the contribution that is independent of the derivatives of \(\delta_k\) with respect to \(\delta \phi, a_{1k}\) and \(a_{2k}\) are regular functions obtained from the static equations (see SM). The cusp in \(|\delta \phi|\) present in \(T = 0\) is rounded at finite temperature because of the last term in Eq. (17). Instead, \(\delta_k\) develops a “thermal boundary layer”,

\[
\delta_k(\phi, \delta \phi) = \delta_k(\phi, 0) + T_k b_k(\phi, y = \frac{\delta \phi}{T_k}) + O(T_k^2, \delta \phi^2)
\]

(18)

when \(T_k, \delta \phi \rightarrow 0\). It is easy to derive that the solution has the explicit form

\[
b_k(\phi, y) = \frac{a_{2k}(\phi)}{a_{1k}(\phi)} \left(1 - \sqrt{1 + \frac{a_{1k}(\phi)^2 \delta_{k,a}(\phi)^2}{a_{2k}(\phi)^2 y^2}}\right),
\]

(19)

where \(a_{ps}\) are the (nonzero) fixed-point functions and \(\delta_{k,a}(\phi)\) behaves differently when \(k \rightarrow 0\) for \(d < d_{DR}\) and \(d > d_{DR}\) (see above).
Acknowledgments

We acknowledge support from the ERC grant NPRGGLASS (G. Biroli).

* Electronic address: balog@ifs.hr

† Electronic address: tarjus@lptmc.jussieu.fr
I. SUPPLEMENTAL MATERIAL

The theoretical formalism we use to describe the long-time, long-distance physics of the RFIM near its critical point is the nonperturbative functional renormalization group (NP-FRG). We give here some technical details of our implementation. We have generalized the formalism developed for the (static) equilibrium properties of the RFIM\textsuperscript{1,2} by combining it with the approach put forward by Canet et al\textsuperscript{3} for the critical dynamics of the Ising model in the absence of quenched disorder.

With the chosen ansatz [Eqs. (11,12) of the main text], the derivation of the NP-FRG equations for the static quantities, the derivative of the effective average potential $U'_N(\phi)$, the field renormalization function $Z_k(\phi)$, and the second cumulant of the renormalized random field $\Delta_k(\phi_1, \phi_2)$ is very similar to that already obtained from the superfield formalism\textsuperscript{1} and the replica formalism\textsuperscript{1,2}, since one can show that the time-reversal symmetry guarantees that there is no feedback from the kinetic term $X_k(\phi)$ to these equations. In the formal notation of Eq. (16) of the main text, $\beta_{\alpha_0}(\varphi)$ and $\beta_{\alpha_0}(\varphi)$ coincide with the zero-temperature beta functions explicitly given in Ref. 3, and $\beta_{\alpha_1}(\varphi)$ and $\beta_{\alpha_1}(\varphi)$ are regular functions whose expression is unilluminating and is not given here. The beta function for $\delta_k$ is more tricky as it shows a nonuniform convergence when $k \to 0$ and $\delta \varphi \to 0$; $\beta_{\alpha}(\varphi, \delta \varphi)$ coincides with the $T = 0$ beta function of Ref. 3 and $\beta_{\delta}(\varphi, \delta \varphi)$ is given by

$$
\beta_{\delta}(\varphi, \delta \varphi) = -\frac{1}{8} \int \frac{d^d q}{(2\pi)^d} \left[ \delta_k(\hat{q}^2 + 2\varphi + \delta \varphi)^2 + \text{sym} \right] \times

\left[ \delta_k^{(02)}(\varphi, \delta \varphi) + \delta_k^{(20)}(\varphi, \delta \varphi) + 2\delta_k(\hat{q}^2 + 2\varphi + \delta \varphi)^2 + \text{sym} \right] \times

\delta_k^{(11)}(\varphi, \delta \varphi) + 2\delta_k^{(01)}(\varphi, \delta \varphi) \frac{\partial}{\partial \delta \varphi} \left[ \delta_k(\hat{q}^2 + 2\varphi + \delta \varphi)^2 + \text{sym} \right] + 2\delta_k^{(10)}(\varphi, \delta \varphi) \frac{\partial}{\partial \delta \varphi} \left[ \delta_k(\hat{q}^2 + 2\varphi + \delta \varphi)^2 + \text{sym} \right]
$$

(22)

where $q = q/k$, $\int_q \equiv \int d^d q/(2\pi)^d$, the dimensionless cut-off function $\hat{r}(\hat{q}^2)$ is defined through $\hat{R}_k(q^2) = Z_k q^2 \hat{r}(\hat{q}^2)$ and $\hat{\partial}_\varphi \hat{r}(\hat{q}^2) \equiv [-q^2 \hat{r}(\hat{q}^2) + 2q^4 \hat{r}'(\hat{q}^2)]$ is a symbolic notation for the term obtained from $k \partial_k \hat{R}_k(q^2)$. (Similarly, one defines $\hat{r}(\hat{q}^2)$ from $\hat{R}_k(q^2) = \Delta_k \hat{r}(\hat{q}^2)$ but it is simply related to $\hat{r}$ via $\hat{r}(\hat{q}^2) = -\hat{\partial}_\varphi [\hat{r}(\hat{q}^2)]$ from the relation between $\hat{R}_k$ and $\hat{R}_k$ given in the main text.) The dimensionless hat propagator is given by $\hat{p}_k(\hat{q}, \varphi) = (\hat{q}^2 [z_k(\varphi) + \hat{r}(\hat{q}^2)] + u_k^2(\varphi))^{-1}$. Finally, sym denotes a term obtained by changing $\delta \varphi$ in $-\delta \varphi$.

The appearance of a cusp in $|\delta \varphi|$ when $\delta \varphi \to 0$ is associated with the divergence of $\delta_k^{(02)}(\varphi, \delta \varphi \to 0)$. It was shown in Refs. 1,2,3 that two different cases had to be considered depending on the spatial dimension. For $d$ less than a critical dimension $d_{DR} \approx 5.1$, the amplitude of the cusp stays finite (nonzero) at the fixed point whereas for $d > d_{DR}$, the cusp is subdominant and its amplitude goes to zero at the fixed point as $k^\lambda$ with $\lambda > 0$. This
can be described in the limit $\delta \varphi \to 0$ by

$$\delta_k(\varphi, \delta \varphi) = \delta_k(\varphi, 0) - \delta_{k,a}(\varphi)\delta \varphi + O(\delta \varphi^2)$$  \hspace{1cm} (23)

with $\delta_{k,a}(\varphi) \to \delta_{s,a}(\varphi) > 0$ for $d < d_{DR}$ and $\delta_{k,a}(\varphi) \sim k^\lambda \to 0$ for $d > d_{DR}$. In the latter case, the usual static critical exponents describing scaling around the critical point are exactly given by the dimensional reduction (they are equal to those of the pure Ising model in dimension $d-2$) and the temperature exponent $\theta = 2$.

On the other hand, describing the critical slowing down boundary layer of width $\delta \varphi$ and $\delta \varphi$ then takes the form given in Eq. (17) of the main text where $a_{1k}(\varphi)$ is the prefactor of the anomalous contribution in $\beta_{0\varphi}(\varphi, \delta \varphi)$ whose limit when $\delta \varphi \to 0$ is nonzero only in the presence of a cusp,

$$a_{1k}(\varphi) = \frac{1}{2} \int q \partial_q \delta(q^2)\hat{p}_k(q; \varphi)^3$$  \hspace{1cm} (24)

and $a_{2k}(\varphi)$ is the prefactor of the potentially singular piece in $\beta_{0\varphi}(\varphi, \delta \varphi)$,

$$a_{2k}(\varphi) = \frac{1}{4} \int q \partial_q \delta(q^2)\hat{p}_k(q; \varphi)^2.$$  \hspace{1cm} (25)

We now turn to the NP-DRG equation for the kinetic term $X_k(\varphi)$. It is obtained from the renormalization prescription

$$X_k(\varphi) = -\frac{1}{4} \frac{\partial}{\partial \phi} \gamma_{1:xt}(\phi, \tilde{\phi}) \bigg|_{\tilde{\phi}=0}. \hspace{1cm} (26)$$

In graphical terms the flow equation reads

$$\partial_x X_k(\varphi) = \frac{1}{4} \delta_x \int_q \left( \begin{array}{c} 2 \\ -4 \\ \vdots \\ -8 \end{array} - \begin{array}{c} 2 \\ -8 \\ \vdots \\ -4 \end{array} + \begin{array}{c} 4 \\ +8 \\ \vdots \\ +4 \end{array} \right)$$

where a cross in the circle denotes a vertex $X_k(\varphi)$, the lines denote static propagators $\hat{p}_k$ and the dotted lines the disorder vertex $\Delta_k(\phi_1, \phi_2)$ (after having taken the needed derivatives, one sets $\phi_1 = \phi_2 = \phi$); $\delta_x$ is a shorthand notation to indicate a derivative acting only on the cutoff functions, i.e., $\delta_x \equiv \partial_x \hat{R}_k \delta / \delta \hat{R}_k + \partial_x \hat{R}_k \delta / \delta \hat{R}_k$. For implementing Itô prescription we have followed the trick devised by Canet et al.\(\underline{2}\), which amounts to shifting the time dependence of the response field by an infinitesimal amount in the renormalized response functions.

After having introduced the dimensionless quantities, Eq. (27) can be rewritten as

$$k \partial_k X_k(\varphi) = \beta_{X0}(\varphi) + T_k \beta_{X1}(\varphi) \hspace{1cm} (28)$$

where

$$\beta_{X0}(\varphi) = \frac{d - 4 + \sqrt{2}}{2} \varphi X_k'(\varphi) + \frac{1}{4} \left\{ -2 \partial_q \delta(q^2)\hat{p}_k(q; \varphi)^2 \times \right.$$  \hspace{1cm}

$$\left. \times X_k''(\varphi) - 4 \hat{p}_k(q; \varphi)^3[\hat{q}^2 z_k'(\varphi) + u_k^{(3)}(\varphi)] - 2 \partial_q \delta(q^2) \times \right.$$  \hspace{1cm}

$$\left. + 3 \partial_q \delta(q^2)\hat{p}_k(q; \varphi)(\delta_k(\varphi, 0) - \hat{r}(q^2)) \times \right.$$  \hspace{1cm}

$$\left. \times \hat{p}_k(q; \varphi)^4 X_k(\varphi) - 2 \delta_k(q^2) \times \right.$$  \hspace{1cm}

$$\left. + 4 \partial_q \delta(q^2)(\delta_k(\varphi, 0) - \hat{r}(q^2))\hat{p}_k(q; \varphi)[\hat{q}^2 z_k'(\varphi) + u_k^{(3)}(\varphi)]^2 \right.$$  \hspace{1cm}

$$\left. - 2 \partial_q \delta(q^2)\hat{p}_k(q; \varphi)^3 \left[ 2(\delta(\varphi, 0) - \hat{r}(q^2)) X_k''(\varphi) \right] \right\} X_k(\varphi) + \left[ -\delta_k^{(0)}(\varphi, 0) + \delta_k^{(2)}(\varphi, 0) X_k(\varphi) \right] \right\}$$

where $\phi \equiv \partial_q \delta(q^2)$ is a symbolic notation for $[2\eta - \tilde{\eta}][\hat{r}(q^2) + \hat{q}^2 \hat{p}(q^2)] + 2[2\eta^{2} \hat{r}(q^2) + \hat{q}^2 \hat{p}(q^2)]$. The term $\beta_{X1}(\varphi)$ is regular and leads only to subdominant terms near the fixed point; it is not given here. Note that we have kept $X_k$ in a dimensionful form. In the case of a conventional critical slowing down one introduces a dynamical exponent $z$ such that the characteristic relaxation time scales as $\tau_k \sim k^{-z}$ near the fixed point. The kinetic term then has dimension $X_k \sim k^{-z}$. However, in the presence of a nonzero random-field strength, where one anticipates an unconventional activated dynamic scaling and $\tau_k$ of the form given in Eq. (21) of the main text, one should rather focus on $F_k = \ln X_k(\varphi)$ where if needed $X_k$ can be made dimensionless inside the logarithm by dividing by a $k$-independent factor).

By inserting the boundary layer solution in Eq. (29) and working at the dominant orders when $T_k \to 0$ (and $k \to 0$), it is easy to derive the flow of $F_k(\varphi) = \ln X_k(\varphi)$ in the form of Eq. (20) of the main text with the various prefactors explicitly given at the relevant order in $T_k$ by

$$\hat{c}_k(\varphi) = \frac{1}{2} \int q \left[ -2 \partial_q \delta(q^2)\hat{p}_k(q; \varphi)^4[\hat{q}^2 z_k'(\varphi) + u_k^{(3)}(\varphi)]^2 \right.$$  \hspace{1cm}

$$+ \partial_q \delta(q^2)\hat{p}_k(q; \varphi)^3 \left( -\delta_k^{(2)}(\varphi, 0) + 2 \hat{p}_k(q; \varphi)[\hat{q}^2 z_k'(\varphi) + u_k^{(3)}(\varphi)] \right) +$$  \hspace{1cm}

$$\times \left( \hat{q}^2 z_k'(\varphi) + u_k^{(3)}(\varphi) \right) \right] + O(T_k), \hspace{1cm} (30)$$
\[
\tilde{c}_{3k} = -\frac{1}{2} \int \left( \hat{\partial}_s \hat{r}(\hat{q}^2) + 2 \hat{\partial}_s \hat{r}(\hat{q}^2) [\delta_k(\varphi, 0) - \hat{r}(\hat{q}^2)] \hat{p}_k(\hat{q}; \varphi) \right) \\
\times \hat{p}_k(\hat{q}; \varphi)^2 - T_k \int \hat{\partial}_s \hat{r}(\hat{q}^2) \hat{p}_k(\hat{q}; \varphi)^2.
\]

(32)

As stated in the main text, the asymptotic solution in the vicinity of the fixed point \( k \rightarrow 0 \) is of the form

\[
F_k(\varphi) = \frac{1}{T_k} e_k + \frac{1}{\sqrt{T_k}} g_k(\varphi) + O(1)
\]

(33)

with \( e_k \) independent of \( \varphi \) but behaving as \( k \rightarrow 0 \) differently for \( d < d_{DR} \) and \( d > d_{DR} \), as discussed in the main text. (Note that the asymptotic slopes \( e_k \), as illustrated in Fig. 1 of the main text, do not depend on the chosen value of the bare temperature \( T \), as required.)

We illustrate here the flow of \( \sqrt{T_k [F_k(\varphi) - F_k(0)]} \) which should asymptotically converge to \( g_k(\varphi) - g_k(0) \). We can see from Fig. A1 that the fixed-point function is indeed well-behaved.

We have numerically solved the NP-FRG equations for a wide range of dimensions between 3 and 6. To do so, we have discretized the fields on a grid and used a variation of the Newton-Raphson method. For the cutoff function \( \hat{r}_k(\hat{q}^2) \) we have used the same form as in previous work and optimized the parameters by stability considerations.

\[^{1}\text{Electronic address: balog@ifs.hr}\]
\[^{2}\text{Electronic address: tarjus@lptmc.jussieu.fr}\]
\[^{3}\text{G. Tarjus and M. Tissier, Phys. Rev. Lett 93, 267008 (2004); Phys. Rev. B 78, 024203 (2008).}\]
\[^{4}\text{M. Tissier and G. Tarjus, Phys. Rev. Lett 96, 087202 (2006); Phys. Rev. B 78, 024204 (2008).}\]
\[^{5}\text{M. Tissier and G. Tarjus, Phys. Rev. Lett. 107, 041601 (2011); Phys. Rev. B 107, 104202 (2012); ibid 85, 104203 (2012).}\]
\[^{6}\text{L. Canet and H. Chaté, J. Phys. A: Math. Theor. 40, 1937 (2007). L. Canet and H. Chaté and B. Delamotte, J. Phys. A: Math. Theor. 44, 459001 (2011).}\]
\[^{7}\text{G. Tarjus, M. Baczek, and M. Tissier, Phys. Rev. Lett. 110, 135703 (2013).}\]
\[^{8}\text{See also P. Chauve et al., Phys. Rev. B 62, 6241 (2000), L. Balents and P. Le Doussal, Phys. Rev. E 69, 061107 (2004).}\]
\[^{9}\text{P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. 49, 435 (1977).}\]