THE $\Lambda$-ADIC SHINTANI-WALDSPURGER CORRESPONDENCE

MATTEO LONGO, MARC-HUBERT NICOLE

Abstract. We generalize the $\Lambda$-adic Shintani lifting for $GL_2(\mathbb{Q})$ to indefinite quaternion algebras over $\mathbb{Q}$.

1. INTRODUCTION

Langlands’s principle of functoriality predicts the existence of a staggering wealth of transfers (or lifts) between automorphic forms for different reductive groups. In recent years, attempts at the formulation of $p$-adic variants of Langlands’s functoriality have been articulated in various special cases. We prove the existence of the Shimura-Shintani-Waldspurger lift for $p$-adic families.

More precisely, Stevens, building on the work of Hida and Greenberg-Stevens, showed in [21] the existence of a $\Lambda$-adic variant of the classical Shintani lifting of [20] for $GL_2(\mathbb{Q})$. This $\Lambda$-adic lifting can be seen as a formal power series with coefficients in a finite extension of the Iwasawa algebra $\Lambda := \mathbb{Z}_p[X]$ equipped with specialization maps interpolating classical Shintani lifts of classical modular forms appearing in a given Hida family.

Shimura in [19], resp. Waldspurger in [22] generalized the classical Shimura-Shintani correspondence to quaternion algebras over $\mathbb{Q}$, resp. over any number field. In the $p$-adic realm, Hida ([7]) constructed a $\Lambda$-adic Shimura lifting, while Ramsey ([17]) (resp. Park [12]) extended the Shimura (resp. Shintani) lifting to the overconvergent setting.

In this paper, motivated by ulterior applications to Shimura curves over $\mathbb{Q}$, we generalize Stevens’s result to any non-split rational indefinite quaternion algebra $B$, building on work of Shimura [19] and combining this with a result of Longo-Vigni [9]. Our main result, for which the reader is referred to Theorem 3.8 below, states the existence of a formal power series and specialization maps interpolating Shimura-Shintani-Waldspurger lifts of classical forms in a given $p$-adic family of automorphic forms on the quaternion algebra $B$. The $\Lambda$-adic variant of Waldspurger’s result appears computationally challenging (see remark in [15, Intro.]), but it seems within reach for real quadratic fields (cf. [13]).

As an example of our main result, we consider the case of families with trivial character. Fix a prime number $p$ and a positive integer $N$ such that $p \nmid N$. Embed the set $\mathbb{Z}^{\geq 2}$ of integers greater or equal to 2 in $\text{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$ by sending $k \in \mathbb{Z}^{\geq 2}$ to the character $x \mapsto x^{k-2}$. Let $f_\infty$ be an Hida family of tame level $N$ passing through a form $f_0$ of level $\Gamma_0(Np)$ and weight $k_0$. There is a neighborhood $U$ of $k_0$ in $\text{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$ such that, for any $k \in \mathbb{Z}^{\geq 2} \cap U$, the weight $k$ specialization of $f_\infty$
gives rise to an element $f_k \in S_k(\Gamma_0(Np))$. Fix a factorization $N = MD$ with $D > 1$ a square-free product of an even number of primes and $(M, D) = 1$ (we assume that such a factorization exists). Applying the Jacquet-Langlands correspondence we get for any $k \in \mathbb{Z}_{>2} \cap U$ a modular form $f_k^{JL}$ on $\Gamma$, which is the group of norm-one elements in an Eichler order $R$ of level $Mp$ contained in the indefinite rational quaternion algebra $B$ of discriminant $D$. One can show that these modular forms can be $p$-adically interpolated, up to scaling, in a neighborhood of $k_0$. More precisely, let $\mathcal{O}$ be the ring of integers of a finite extension $F$ of $\mathbb{Q}_p$ and let $\mathbb{D}$ denote the $\mathcal{O}$-module of $\mathcal{O}$-valued measures on $\mathbb{Z}_p^2$ which are supported on the set of primitive elements in $\mathbb{Z}_p^2$. Let $\Gamma_0$ be the group of norm-one elements in an Eichler order $R_0 \subseteq B$ containing $R$. There is a canonical action of $\Gamma_0$ on $\mathbb{D}$ (see [9, §2.4] for its description). Denote by $F_k$ the extension of $F$ generated by the Fourier coefficients of $f_k$. Then there is an element $\Phi \in H^1(\Gamma_0, \mathbb{D})$ and maps $\rho_k : H^1(\Gamma_0, \mathbb{D}) \rightarrow H^1(\Gamma, F_k)$ such that $\rho(k)(\Phi) = \phi_k$, the cohomology class associated to $f_k^{JL}$, with $k$ in a neighborhood of $k_0$ (for this we need a suitable normalization of the cohomology class associated to $f_k^{JL}$, which we do not touch for simplicity in this introduction). We view $\Phi$ as a quaternionic family of modular forms. To each $\phi_k$ we may apply the Shimura-Shintani-Waldspurger lifting ([19]) and obtain a modular form $h_k$ of weight $k + 1/2$, level $4Np$ and trivial character. We show that this collection of forms can be $p$-adically interpolated. For clarity’s sake, we present the liftings and their $\Lambda$-adic variants in a diagram, in which the horizontal maps are specialization maps of the $p$-adic family to weight $k$; JL stands for the Jacquet-Langlands correspondence; SSW stands for the Shimura-Shintani-Waldspurger lift; and the dotted arrows are constructed in this paper:

More precisely, as a particular case of our main result, Theorem 3.8, we get the following

**Theorem 1.1.** There exists a $p$-adic neighborhood $U_0$ of $k_0$ in $\text{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$, $p$-adic periods $\Omega_k$ for $k \in U_0 \cap \mathbb{Z}_{\geq 2}$ and a formal expansion

$$\Theta = \sum_{\xi \geq 1} a_\xi q^\xi$$

with coefficients $a_\xi$ in the ring of $\mathbb{C}_p$-valued functions on $U_0$, such that for all $k \in U_0 \cap \mathbb{Z}_{\geq 2}$ we have

$$\Theta(k) = \Omega_k \cdot h_k.$$ 

Further, $\Omega_{k_0} \neq 0$. 
2. Shintani integrals and Fourier coefficients of half-integral weight modular forms

We express the Fourier coefficients of half-integral weight modular forms in terms of period integrals, thus allowing a cohomological interpretation which is key to the production of the $\Lambda$-adic version of the Shimura-Shintani-Waldspurger correspondence. For the quaternionic Shimura-Shintani-Waldspurger correspondence of interest to us (see \cite{15}, \cite{22}), the period integrals expressing the values of the Fourier coefficients have been computed generally by Prasanna in \cite{16}.

2.1. The Shimura-Shintani-Waldspurger lifting. Let $4M$ be a positive integer, $2k$ an even non-negative integer and $\chi$ a Dirichlet character modulo $4M$ such that $\chi(-1) = 1$. Recall that the space of half-integral weight modular forms $S_{k+1/2}(4M,\chi)$ consists of holomorphic cuspidal functions $h$ on the upper-half place $\mathbb{H}$ such that $h(\gamma(z)) = j_1^{1/2}(\gamma,z)^{2k+1}\chi(d)h(z)$, for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4M)$, where $j_1^{1/2}(\gamma,z)$ is the standard square root of the usual automorphy factor $j_1(\gamma,z)$ (cf. \cite[2.3]{15}).

To any quaternionic integral weight modular form we may associate a half-integral weight modular form following Shimura’s work \cite{19}, as we will describe below.

Fix an odd square free integer $N$ and a factorization $N = M \cdot D$ into coprime integers such that $D > 1$ is a product of an even number of distinct primes. Fix a Dirichlet character $\psi$ modulo $M$ and a positive even integer $2k$. Suppose that $\psi(-1) = (-1)^k$. Define the Dirichlet character $\chi$ modulo $4N$ by

$$\chi(x) := \psi(x) \left(\frac{-1}{x}\right)^k.$$  

Let $B$ be an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $D$. Fix a maximal order $O_B$ of $B$. For every prime $\ell|M$, choose an isomorphism $i_\ell : B \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \cong M_2(\mathbb{Q}_\ell)$ such that $i_\ell(O_B \otimes_{\mathbb{Z}} \mathbb{Z}_\ell) = M_2(\mathbb{Z}_\ell)$. Let $R \subseteq O_B$ be the Eichler order of $B$ of level $M$ defined by requiring that $i_\ell(R \otimes_{\mathbb{Z}} \mathbb{Z}_\ell)$ is the suborder of $M_2(\mathbb{Z}_\ell)$ of upper triangular matrices modulo $\ell$ for all $\ell|M$. Let $\Gamma$ denote the subgroup of the group $R^\times$ of norm 1 elements in $R^\times$ consisting of those $\gamma$ such that $i_\ell(\gamma) \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ mod $\ell$ for all $\ell|M$. We denote by $S_{2k}(\Gamma)$ the $\mathbb{C}$-vector space of weight $2k$ modular forms on $\Gamma$, and by $S_{2k}(\Gamma,\psi^2)$ the subspace of $S_{2k}(\Gamma)$ consisting of forms having character $\psi^2$ under the action of $R^\times$. Fix a Hecke eigenform $f \in S_{2k}(\Gamma,\psi^2)$ as in \cite[Section 3]{19}.

Let $V$ denote the $\mathbb{Q}$-subspace of $B$ consisting of elements with trace equal to zero. For any $v \in V$, which we view as a trace zero matrix in $M_2(\mathbb{R})$ (after fixing an isomorphism $i_\infty : B \otimes \mathbb{R} \cong M_2(\mathbb{R})$), set

$$G_v := \{ \gamma \in SL_2(\mathbb{R}) | \gamma^{-1}v\gamma = v \}$$

as in \cite[Section 3]{19}.
and put $\Gamma_v := G_v \cap \Gamma$. One can show that there exists an isomorphism $\omega: \mathbb{R}^\times \xrightarrow{\sim} G_v$ defined by $\omega(s) = \beta^{-1}(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \beta$, for some $\beta \in \text{SL}_2(\mathbb{R})$. Let $t_\ell$ be the order of $\Gamma_v \cap \{\pm 1\}$ and let $\gamma_v$ be an element of $\Gamma_v$ which generates $\Gamma_v \langle \pm 1 \rangle / \{\pm 1\}$. Changing $\gamma_v$ to $\gamma_v^{-1}$ if necessary, we may assume $\gamma_v = \omega(t)$ with $t > 0$. Define $V^*$ to be the $\mathbb{Q}$-subspace of $V$ consisting of elements with strictly negative norm. For any $\alpha = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in V^*$ and $z \in \mathcal{H}$, define the quadratic form

$$Q_\alpha(z) := cz^2 - 2az - b.$$ 

Fix $\tau \in \mathcal{H}$ and set

$$P(f, \alpha, \Gamma) := -\left(2(-\text{nr}(\alpha))^{1/2}/t_\alpha\right) \int_{\tau}^{\gamma_\alpha(\tau)} Q_\alpha(z)^{k-1}f(z)dz$$

where $\text{nr}: \mathbb{B} \to \mathbb{Q}$ is the norm map. By [11] Lemma 2.1, the integral is independent on the choice $\tau$, which justifies the notation.

**Remark 2.1.** The definition of $P(f, \alpha, \Gamma)$ given in [11] (2.5)] looks different: the above expression can be derived as in [11] page 629 by means of [11] (2.20) and (2.22)].

Let $R(\Gamma)$ denote the set of equivalence classes of $V^*$ under the action of $\Gamma$ by conjugation. By [11] (2.6)], $P(f, \alpha, \Gamma)$ only depends on the conjugacy class of $\alpha$, and thus, for $C \in R(\Gamma)$, we may define $P(f, C, \Gamma) := P(f, \alpha, \Gamma)$ for any choice of $\alpha \in C$. Also, $q(C) := -\text{nr}(\alpha)$ for any $\alpha \in C$.

Define $O'_B$ to be the maximal order in $B$ such that $O'_B \otimes \mathbb{Z}_\ell \cong O_B \otimes \mathbb{Z}_\ell$ for all $\ell \nmid M$ and $O'_B \otimes \mathbb{Z}_\ell$ is equal to the local order of $B \otimes \mathbb{Q}_\ell$ consisting of elements $\gamma$ such that $i_\ell(\gamma) = \left( \begin{smallmatrix} a & b/M \\ c & d \end{smallmatrix} \right)$ with $a, b, c, d \in \mathbb{Z}_\ell$, for all $\ell | M$. Given $\alpha \in O'_B$, we can find an integer $b_\alpha$ such that

$$i_\ell(\alpha) \equiv \left( \begin{smallmatrix} * & b_\alpha/M \\ * & * \end{smallmatrix} \right) \mod i_\ell(\mathbb{Z}_\ell), \quad \forall \ell | M.$$ 

Define a locally constant function $\eta_\psi$ on $V$ by $\eta_\psi(\alpha) = \psi(b_\alpha)$ if $\alpha \in O'_B \cap V$ and $\eta_\psi(\alpha) = 0$ otherwise, with $\psi(a) = 0$ if $(a, M) \neq 1$ (for the definition of locally constant functions on $V$ in this context, we refer to [11] p. 611]).

For any $C \in R(\Gamma)$, fix $a_C \in C$. For any integer $\xi \geq 1$, define

$$a_\xi(\tilde{h}) := (2\mu(\Gamma \setminus \mathcal{G}))^{-1} \cdot \sum_{C \in R(\Gamma), q(C) = \xi} \eta_\psi(a_C)\xi^{-1/2}P(f, C, \Gamma).$$

Then, by [11] Theorem 3.1$^*$, $\tilde{h} := \sum_{\xi \geq 1} a_\xi(\tilde{h})q^\xi \in S_{k+1/2}(4N, \chi)$ is called the Shimura-Shintani-Waldspurger lifting of $f$.

### 2.2. Cohomological interpretation

We introduce necessary notation to define the action of the Hecke action on cohomology groups; for details, see [9] §2.1. If $G$ is a subgroup of $B^s$ and $S$ a subsemigroup of $B^s$ such that $(G, S)$ is an Hecke pair, we let $\mathcal{H}(G, S)$ denote the Hecke algebra corresponding to $(G, S)$, whose elements are written as $T(s) = GsG = \bigsqcup_s Gs_t$ for $s, s_t \in S$ (finite disjoint union). For any $s \in S$, let $s^* := \text{norm}(s)s^{-1}$ and denote by $S^*$ the set of elements of the form $s^*$ for $s \in S$. For any $\mathbb{Z}[S^*]$-module $M$ we let $T(s)$ act on $H^1(G, M)$ at the level of cochains $c \in Z^1(G, M)$ by the formula $(c|T(s))(\gamma) = \sum_t s_t^*c(t_\gamma)$, where $t_\gamma$ are
defined by the equations $G_{s_1}\gamma = G_{s_j}$ and $s_1\gamma = t_i(\gamma)s_j$. In the following, we will consider the case of $G = \Gamma$ and
\[ S = \{ s \in B^x | i_\ell(s) \text{ is congruent to } (1^\_0 \_1^\ell) \mod \ell \text{ for all } \ell | M \}. \]

For any field $L$ and any integer $n \geq 0$, let $V_n(L)$ denote the $L$-dual of the $L$-vector space $P_n(L)$ of homogeneous polynomials in 2 variables of degree $n$. We let $M_2(L)$ act from the right on $P(x, y)$ as $P|\gamma(x, y) := P(\gamma(x, y))$, where for $\gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$ we put
\[ (ax + by, cx + dy). \]
This also equips $V_n(L)$ with a left action by $\gamma \cdot \varphi(P) := \varphi(P|\gamma)$. To simplify the notation, we will write $P(z)$ for $P(z, 1)$.

Let $F$ denote the finite extension of $\mathbb{Q}$ generated by the eigenvalues of the Hecke action on $f$. For any field $K$ containing $F$, set
\[ \mathbb{W}_f(K) := H^1(\Gamma, V_{k-2}(K))^I \]
where the superscript $f$ denotes the subspace on which the Hecke algebra acts via the character associated with $f$. Also, for any sign $\pm$, let $\mathbb{W}_f^\pm(K)$ denote the $\pm$-eigenspace for the action of the archimedean involution $\iota$. Remember that $\iota$ is defined by choosing an element $\omega_\infty$ of norm $-1$ in $R^x$ such that $i_\ell(\omega_\infty) \equiv (1^\_0 \_0^{-1}) \mod M$ for all primes $\ell | M$ and then setting $\iota := T(\omega_\infty)$ (see [9, §2.1]). Then $\mathbb{W}_f^\pm(K)$ is one dimensional (see, e.g., [9, Proposition 2.2]); fix a generator $\phi_f^\pm$ of $\mathbb{W}_f^\pm(F)$.

To explicitly describe $\phi_f^\pm$, let us introduce some more notation. Define
\[ f|\omega_\infty(z) := (Cz + D)^{-k/2}f(\omega_\infty(z)) \]
where $i_\infty(\omega_\infty) = (A \begin{smallmatrix} B \\ D \end{smallmatrix})$. Then $f|\omega_\infty \in S_{2k}(\Gamma)$ as well. If the eigenvalues of the Hecke action on $f$ are real, then we may assume, after multiplying $f$ by a scalar, that $f|\omega_\infty = f$ (see [19, p. 627] or [10, Lemma 4.15]). In general, let $I(f)$ denote the class in $H^1(\Gamma, V_{k-2}(\mathbb{C}))$ represented by the cocycle
\[ \gamma \mapsto I(\gamma)(f)(P) := \int P \mapsto I(\gamma)(f)(P) := \int f(z)P(z)dz \]
for any $\gamma \in \mathcal{H}$ (the corresponding class is independent on the choice of $\gamma$). With this notation,
\[ P(f, \alpha, \Gamma) = -\left(2(-nr(\alpha))^{1/2}/t_\alpha\right) \cdot I_{\gamma_\alpha}(f)(Q_{\alpha}(z)^{k-1}). \]

Denote by $I^\pm(f) := (1/2) \cdot I(f) \pm (1/2) \cdot I(f)|_{\omega_\infty}$, the projection of $I(f)$ to the eigenspaces for the action of $\omega_\infty$. Then $I(f) = I^+(f) + I^-(f)$ and $I_f^\pm = \Omega_f^\pm \cdot \phi_f^\pm$, for some $\Omega_f^\pm \in C^\times$.

Given $\alpha \in V^*$ of norm $-\xi$, put $\alpha' := \omega_\infty^{-1} \alpha \omega_\infty$. By [19, 4.19], we have
\[ \eta(\alpha)\xi^{-1/2}P(f, \alpha, \Gamma) + \eta(\alpha')\xi^{-1/2}P(f, \alpha', \Gamma) = -\eta(\alpha) \cdot t_\alpha^{-1} \cdot I_{\gamma_\alpha}^+(Q_{\alpha}(z)^{k-1}). \]
We then have
\[ a_\xi(\tilde{h}) = \sum_{c \in R_{2}(\Gamma, \mathcal{O}(\mathcal{C})) = \xi} -\eta(\alpha)\xi^{-1/2}P(f, \alpha, \Gamma) + \eta(\alpha')\xi^{-1/2}P(f, \alpha', \Gamma) = -\eta(\alpha) \cdot t_\alpha^{-1} \cdot I_{\gamma_\alpha}^+(Q_{\alpha}(z)^{k-1}). \]
We close this section by choosing a suitable multiple of $h$ which will be the object of the next section. Given $Q_\alpha(z) = cz^2 - 2az - b$ as above, with $\alpha$ in
V^*,$ define $\tilde{Q}_\alpha(z) := M \cdot Q_\alpha(z)$. Then, clearly, $I^\pm(f)(\tilde{Q}_\alpha(z)^{k-1})$ is equal to $M^{k-1}I^\pm(f)(Q_\alpha(z)^{k-1})$. We thus normalize the Fourier coefficients by setting

$$a_\xi(h) := \frac{a_\xi(h) \cdot M^{k-1} \cdot 2\mu(\Gamma \setminus H)}{\Omega_f} = \sum_{\mathcal{C} \in R(\Gamma, \mathcal{O}(\mathcal{C}))} \frac{\eta_\psi(\alpha_\mathcal{C})}{\mathcal{C} \alpha_\mathcal{C}} \cdot \phi_f^+(\tilde{Q}_\alpha(z)^{k-1}).$$

So

$$h := \sum_{\xi \geq 1} a_\xi(h) q^\xi$$

belongs to $S_{k+1/2}(4N, \chi)$ and is a non-zero multiple of $\tilde{h}$.

3. The $\Lambda$-adic Shimura-Shintani-Waldspurger Correspondence

At the heart of Stevens’s proof lies the control theorem of Greenberg-Stevens, which has been worked out in the quaternionic setting by Longo–Vigni [9].

Recall that $N \geq 1$ is a square free integer and fix a decomposition $N = M \cdot D$ where $D$ is a square free product of an even number of primes and $M$ is coprime to $D$. Let $p \nmid N$ be a prime number and fix an embedding $\bar{Q} \hookrightarrow \bar{Q}_p$.

3.1. The Hida Hecke algebra. Fix an ordinary $p$-stabilized newform

$$f_0 \in S_{k_0}(\Gamma_1(Mp^n) \cap \Gamma_0(D), \epsilon_0)$$

of level $\Gamma_1(Mp^n) \cap \Gamma_0(D)$, Dirichlet character $\epsilon_0$ and weight $k_0$, and write $\mathcal{O}$ for the ring of integers of the field generated over $\mathbb{Q}_p$ by the Fourier coefficients of $f_0$.

Let $\Lambda$ (respectively, $\mathcal{O}[\mathbb{Z}_p^\times]$) denote the Iwasawa algebra of $W := 1 + p\mathbb{Z}_p$ (respectively, $\mathbb{Z}_p^\times$) with coefficients in $\mathcal{O}$. We denote group-like elements in $\Lambda$ and $\mathcal{O}[\mathbb{Z}_p^\times]$ as $[\ell]$. Let $\mathcal{H}_\text{cont}^{\text{alg}}$ denote the $p$-ordinary Hida Hecke algebra with coefficients in $\mathcal{O}$ of tame level $\Gamma_1(N)$. Denote by $\mathcal{L} := \text{Frac}(\Lambda)$ the fraction field of $\Lambda$. Let $\mathcal{R}$ denote the integral closure of $\Lambda$ in the primitive component $\mathcal{K}$ of $\mathcal{H}_\text{cont}^{\text{alg}} \otimes_{\Lambda} \mathcal{L}$ corresponding to $f_0$. It is well known that the $\Lambda$-algebra $\mathcal{R}$ is finitely generated as $\Lambda$-module.

Denote by $\mathcal{X}$ the $\mathcal{O}$-module $\text{Hom}_{\mathcal{O}_{\text{alg}}(\mathcal{R}, \bar{Q}_p)}$ of continuous homomorphisms of $\mathcal{O}$-algebras. Let $\mathcal{X}^\text{arith}$ be the set of arithmetic homomorphisms in $\mathcal{X}$, defined in [9 §2.2] by requiring that the composition

$$W \longrightarrow \Lambda \longrightarrow \bar{Q}_p$$

has the form $\gamma \mapsto \psi_\gamma(\gamma)^{n_\gamma}$ with $n_\gamma = k_\gamma - 2$ for an integer $k_\gamma \geq 2$ (called the weight of $\gamma$) and a finite order character $\psi_\gamma : W \rightarrow \bar{Q}_p$ (called the wild character of $\gamma$). Denote by $r_\gamma$ the smallest among the positive integers $t$ such that $1 + p^t\mathbb{Z}_p \subseteq \ker(\psi_\gamma)$. For any $\kappa \in \mathcal{X}^\text{arith}$, let $P_\kappa$ denote the kernel of $\kappa$ and $\mathcal{R}_\kappa$ the localization of $\mathcal{R}$ at $\kappa$. The field $F_\kappa := \mathcal{R}_\kappa / P_\kappa \mathcal{R}_\kappa$ is a finite extension of $\text{Frac}(\mathcal{O})$. Further, by duality, $\kappa$ corresponds to a normalized eigenform

$$f_\kappa \in S_{k_\kappa}(\Gamma_0(Np^n), \epsilon_\kappa)$$

for a Dirichlet character $\epsilon_\kappa : (\mathbb{Z}/Np^n \mathbb{Z})^\times \rightarrow \bar{Q}_p$. More precisely, if we write $\psi_\mathcal{R}$ for the character of $\mathcal{R}$, defined as in [4] p. 555, and we let $\omega$ denote the Teichmüller character, we have $\epsilon_\kappa := \psi_\kappa \cdot \psi_\mathcal{R} \cdot \omega^{-n_\kappa}$ (see [6] Cor. 1.6). We call $(\epsilon_\kappa, k_\kappa)$ the signature of $\kappa$. We let $\kappa_0$ denote the arithmetic character associated with $f_0$, so $f_0 = f_{\kappa_0}$, $k_0 = k_{\kappa_0}$, $\epsilon_0 = \epsilon_{\kappa_0}$, and $r_0 = r_{\kappa_0}$. The eigenvalues of $f_\kappa$ under the action of the Hecke operators $T_n$ ($n \geq 1$ an integer) belong to $F_\kappa$. Actually, one can show that $f_\kappa$ is a $p$-stabilized newform on $\Gamma_1(Mp^n) \cap \Gamma_0(D)$. 


Let \( \Lambda_N \) denote the Iwasawa algebra of \( \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times \) with coefficients in \( \mathcal{O} \). To simplify the notation, define \( \Delta := (\mathbb{Z}/NP\mathbb{Z})^\times \). We have a canonical isomorphism of rings \( \Lambda_N \simeq \Lambda[\Delta] \), which makes \( \Lambda_N \) a \( \Lambda \)-algebra, finitely generated as \( \Lambda \)-module.

Define the tensor product of \( \Lambda \)-algebras

\[
\mathcal{R}_N := \mathcal{R} \otimes_\Lambda \Lambda_N,
\]

which is again a \( \Lambda \)-algebra (resp. \( \Lambda_N \)-algebra) finitely generated as a \( \Lambda \)-module, (resp. as a \( \Lambda_N \)-module). One easily checks that there is a canonical isomorphism of \( \Lambda \)-algebras

\[
\mathcal{R}_N \simeq \mathcal{R}[\Delta]
\]

(where \( \Lambda \) acts on \( \mathcal{R} \)): this is also an isomorphism of \( \Lambda_N \)-algebras, when we let \( \Lambda_N \simeq \Lambda[\Delta] \) act on \( \mathcal{R}[\Delta] \) in the obvious way.

We can extend any \( \kappa \in \mathcal{X}^{\text{arith}} \) to a continuous \( \mathcal{O} \)-algebra morphism

\[
\kappa_N : \mathcal{R}_N \rightarrow \bar{\mathbb{Q}}_p
\]

setting

\[
\kappa_N \left( \sum_{i=1}^n r_i \cdot \delta_i \right) := \sum_{i=1}^n \kappa(r_i) \cdot \psi_R(\delta_i)
\]

for \( r_i \in \mathcal{R} \) and \( \delta_i \in \Delta \). Therefore, \( \kappa_N \) restricted to \( \mathbb{Z}_p^\times \) is the character \( t \mapsto \epsilon(t)t^n \).

If we denote by \( \mathcal{X}_N \) the \( \mathcal{O} \)-module of continuous \( \mathcal{O} \)-algebra homomorphisms from \( \mathcal{R}_N \) to \( \bar{\mathbb{Q}}_p \), the above correspondence sets up an injective map \( \mathcal{X}^{\text{arith}} \hookrightarrow \mathcal{X}_N \). Let \( \mathcal{X}_N^{\text{arith}} \) denote the image of \( \mathcal{X}^{\text{arith}} \) under this map. For \( \kappa_N \in \mathcal{X}_N^{\text{arith}} \), we define the signature of \( \kappa_N \) to be that of the corresponding \( \kappa \).

### 3.2. The control theorem in the quaternionic setting

Recall that \( B/\mathbb{Q} \) is a quaternion algebra of discriminant \( D \). Fix an auxiliary real quadratic field \( F \) such that all primes dividing \( D \) are inert in \( F \) and all primes dividing \( Mp \) are split in \( F \), and an isomorphism \( i_F : B \otimes \mathbb{Q} F \simeq M_2(F) \). Let \( \mathcal{O}_B \) denote the maximal order of \( B \) obtained by taking the intersection of \( B \) with \( M_2(\mathcal{O}_F) \), where \( \mathcal{O}_F \) is the ring of integers of \( F \). More precisely, define

\[
\mathcal{O}_B := \iota^{-1}(i_F^{-1}(i_F(B \otimes 1) \cap M_2(\mathcal{O}_F)))
\]

where \( \iota : B \hookrightarrow B \otimes \mathbb{Q} F \) is the inclusion defined by \( b \mapsto b \otimes 1 \). This is a maximal order in \( B \) because \( i_F(B \otimes 1) \cap M_2(\mathcal{O}_F) \) is a maximal order in \( i_F(B \otimes 1) \). In particular, \( i_F \) and our fixed embedding of \( \mathbb{Q} \) into \( \bar{\mathbb{Q}}_p \) induce an isomorphism

\[
i_p : B \otimes \mathbb{Q} \mathbb{Q}_p \simeq M_2(\mathbb{Q}_p)
\]

such that \( i_p(\mathcal{O}_B \otimes \mathbb{Z} \mathbb{Z}_p) = M_2(\mathbb{Z}_p) \). For any prime \( \ell | M \), also choose an embedding \( \mathbb{Q} \hookrightarrow \mathbb{Q}_\ell \) which, composed with \( i_F \), yields isomorphisms

\[
i_\ell : B \otimes \mathbb{Q} \mathbb{Q}_\ell \simeq M_2(\mathbb{Q}_\ell)
\]

such that \( i_p(\mathcal{O}_B \otimes \mathbb{Z} \mathbb{Z}_\ell) = M_2(\mathbb{Z}_\ell) \). Define an Eichler order \( R \subseteq \mathcal{O}_B \) of level \( M \) by requiring that for all primes \( \ell | M \) the image of \( R \otimes \mathbb{Z} \mathbb{Z}_\ell \) via \( i_\ell \) consists of upper triangular matrices modulo \( \ell \). For any \( r \geq 0 \), let \( \Gamma_r \) denote the subgroup of the group \( R^r \) of norm-one elements in \( R \) consisting of those \( \gamma \) such that \( i_{\ell}(\gamma) = \left( \begin{smallmatrix} a & b \\ c \cdot d \end{smallmatrix} \right) \) with \( c \equiv 0 \mod Mp \) and \( a \equiv d \equiv 1 \mod Mp \), for all primes \( \ell | Mp \). To conclude this list of notation and definitions, fix an embedding \( F \hookrightarrow \mathbb{R} \) and let

\[
i_\infty : B \otimes \mathbb{Q} \mathbb{R} \simeq M_2(\mathbb{R})
\]

be the induced isomorphism.
Let \( Y := \mathbb{Z}_p^2 \) and denote by \( X \) the set of primitive vectors in \( Y \). Let \( D \) denote the \( \mathcal{O} \)-module of \( \mathcal{O} \)-valued measures on \( Y \) which are supported on \( X \). Note that \( \mathbb{M}_2(\mathbb{Z}_p) \) acts on \( Y \) by left multiplication; this induces an action of \( \mathbb{M}_2(\mathbb{Z}_p) \) on the \( \mathcal{O} \)-module of \( \mathcal{O} \)-valued measures on \( Y \), which induces an action on \( D \). The group \( R^x \) acts on \( D \) via \( t_p \). In particular, we may define the group:

\[
\mathcal{W} := H^1(G_0, D).
\]

Then \( D \) has a canonical structure of \( \mathcal{O}[\mathbb{Z}_p^2] \)-module, as well as \( \mathfrak{h}_\text{ord}^\ast \)-action, as described in [9, §2.4]. In particular, let us recall that, for any \( [t] \in \mathcal{O}[\mathbb{Z}_p^2] \), we have

\[
\int_X \varphi(x, y) d([t] \cdot \nu) = \int_X \varphi(tx, ty) d\nu,
\]

for any locally constant function \( \varphi \) on \( X \).

For any \( \kappa \in \mathcal{X}_\text{arith} \) and any sign \( \pm \in \{-, +\} \), set

\[
\mathcal{W}_\kappa^\pm := \mathcal{W}_2^\pm(F_\kappa) = H^1(\Gamma_{r_\kappa}, V_{n_\kappa}(F_\kappa))^{f_\kappa, \pm}
\]

where \( f_\kappa \) is any Jacquet-Langlands lift of \( f_\kappa \) to \( \Gamma_{r_\kappa} \); recall that the superscript \( f_\kappa \) denotes the subspace on which the Hecke algebra acts via the character associated with \( f_\kappa \), and the superscript \( \pm \) denotes the \( \pm \)-eigenspace for the action of the archimedean involution \( \iota \). Also, recall that \( \mathcal{W}_\kappa^\pm \) is one dimensional and fix a generator \( \phi_\kappa^\pm \) of it.

We may define specialization maps

\[
\rho_\kappa : D \rightarrow V_{n_\kappa}(F_\kappa)
\]

by the formula

\[
\rho_\kappa(\nu)(P) := \int_{\mathbb{Z}_p^2} \epsilon_\kappa(y) P(x, y) d\nu
\]

which induces (see [9, §2.5]) a map:

\[
\rho_\kappa : \mathcal{W}_\kappa^{\text{ord}} \rightarrow \mathcal{W}_\kappa^\pm.
\]

Here \( \mathcal{W}_\kappa^{\text{ord}} \) and \( \mathcal{W}_\kappa^{\text{ord}} \) denote the ordinary submodules of \( \mathcal{W} \) and \( \mathcal{W}_\kappa \), respectively, defined as in [9, Definition 2.2] (see also [9, §3.5]). We also let \( \mathcal{W}_\mathcal{X} := \mathcal{W} \otimes \mathcal{X} \mathcal{R} \), and extend the above map \( \rho_\kappa \) to a map

\[
\rho_\kappa : \mathcal{W}_\kappa^{\text{ord}} \rightarrow \mathcal{W}_\kappa^\pm
\]

by setting \( \rho_\kappa(x \otimes r) := \rho_\kappa(x) \cdot \kappa(r) \).

**Theorem 3.1.** There exists a \( p \)-adic neighborhood \( U_0 \) of \( \kappa_0 \) in \( \mathcal{X} \), elements \( \Phi^\pm \) in \( \mathcal{W}_\mathcal{X}^{\text{ord}} \) and choices of \( p \)-adic periods \( \Omega_\kappa^\pm \) in \( F_\kappa \) for \( \kappa \in U_0 \cap \mathcal{X}_\text{arith} \) such that, for all \( \kappa \in U_0 \cap \mathcal{X}_\text{arith} \), we have

\[
\rho_\kappa(\Phi^\pm) = \Omega_\kappa^\pm \cdot \phi_\kappa^\pm
\]

and \( \Omega_{\kappa_0}^\pm \neq 0 \).

**Proof.** This is an easy consequence of [9, Theorem 2.18] and follows along the lines of the proof of [21, Theorem 5.5], cf. [10, Proposition 3.2]. \( \square \)

We now normalize our choices as follows. With \( U_0 \) as above, define

\[
U_0^{\text{arith}} := U_0 \cap \mathcal{X}_\text{arith}.
\]

Fix \( \kappa \in U_0^{\text{arith}} \) and an embedding \( \mathbb{Q}_p \hookrightarrow \mathbb{C} \). Let \( f_\kappa \) denote a modular form on \( \Gamma_{r_\kappa} \) corresponding to \( f_\kappa \) by the Jacquet-Langlands correspondence, which is well
defined up to elements in $\mathbb{C}^\times$. View $\phi^\pm$ as an element in $H^1(\Gamma_{r_n}, V_n(\mathbb{C}))^\pm$. Choose a representative $\Phi^\pm_\gamma$ of $\Phi^\pm$, by which we mean that if $\Phi^\pm = \sum_i \Phi^\pm_i \otimes r_i$, then we choose a representative $\Phi^\pm_i$ for all $i$. Also, we will write $\rho_\kappa(\Phi)(P)$ as

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p^2} \epsilon_\kappa(y) P(x, y) d\Phi^\pm_\gamma := \sum_i \kappa(r_i) \cdot \int_{\mathbb{Z}_p \times \mathbb{Z}_p^2} \epsilon_\kappa(y) P(x, y) d\Phi^\pm_i.$$ 

With this notation, we see that the two cohomology classes

$$\gamma \mapsto \int_{\mathbb{Z}_p \times \mathbb{Z}_p^2} \epsilon_\kappa(y) P(x, y) d\Phi^\pm_\gamma(x, y)$$

and

$$\gamma \mapsto \Omega^\pm_\kappa \cdot \int_{\tau} \gamma(z) P(z, 1) f_\kappa^\text{HL}(z) dz$$

are cohomologous in $H^1(\Gamma_{r_n}, V_n(\mathbb{C}))$, for any choice of $\tau \in \mathcal{H}$.

3.3. Metaplectic Hida Hecke algebras. Let $\sigma : \Lambda_N \to \Lambda_N$ be the ring homomorphism associated to the group homomorphism $t \mapsto t^2$ on $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$, and denote by the same symbol its restriction to $\Lambda$ and $\mathcal{O}[\mathbb{Z}_p^\times]$. We let $\Lambda_\sigma$, $\mathcal{O}[\mathbb{Z}_p^\times]_\sigma$ and $\Lambda_{N, \sigma}$ denote, respectively, $\Lambda$, $\mathcal{O}[\mathbb{Z}_p^\times]$ and $\Lambda_N$ viewed as algebras over themselves via $\sigma$. The ordinary metaplectic $p$-adic Hida Hecke algebra we will consider is the $\Lambda$-algebra

$$\tilde{\mathcal{R}} := \mathcal{R} \otimes \Lambda \Lambda_\sigma.$$ 

Define as above

$$\tilde{\mathcal{X}} := \text{Hom}_{\mathcal{O}\text{-alg}}(\tilde{\mathcal{R}}, \mathcal{O}_p)$$

and let the set $\tilde{\mathcal{X}}_{\text{arith}}$ of arithmetic points in $\tilde{\mathcal{X}}$ to consist of those $\tilde{\kappa}$ such that the composition

$$W \xrightarrow{\lambda} \Lambda \xrightarrow{1 \otimes \lambda} \tilde{\mathcal{R}} \xrightarrow{\tilde{\kappa}} \mathcal{O}_p$$

has the form $\gamma \mapsto \psi_\kappa(\gamma)^{n_{\kappa}}$ with $n_{\kappa} := k_{\kappa} - 2$ for an integer $k_{\kappa} \geq 2$ (called the weight of $\kappa$) and a finite order character $\psi_\kappa : W \to \mathcal{O}_p$ (called the wild character of $\kappa$). Let $r_{\kappa}$ the smallest among the positive integers $t$ such that $1 + p^t \mathbb{Z}_p \subseteq \ker(\psi_\kappa)$.

We have a map $p : \tilde{\mathcal{X}} \to \mathcal{X}$ induced by pull-back from the canonical map $\mathcal{R} \to \tilde{\mathcal{R}}$. The map $p$ restricts to arithmetic points.

As above, define the $\Lambda$-algebra (or $\Lambda_N$-algebra)

$$\tilde{\mathcal{R}}_N := \mathcal{R} \otimes \Lambda \Lambda_{N, \sigma}$$

via $\lambda \mapsto 1 \otimes \lambda$.

We easily see that

$$\tilde{\mathcal{R}}_N \simeq \tilde{\mathcal{R}}[\Delta]$$

as $\Lambda_N$-algebras, where we enhance $\tilde{\mathcal{R}}[\Delta]$ with the following structure of $\Lambda_N \simeq \Lambda[\Delta]$-algebra: for $\sum_i \lambda_i \cdot \delta_i \in \Lambda[\Delta]$ (with $\lambda_i \in \Lambda$ and $\delta_i \in \Delta$) and $\sum_j r_j \cdot \delta'_j \in \tilde{\mathcal{R}}[\Delta]$ (with $r_j = \sum_h r_{j,h} \otimes \lambda_{j,h} \in \tilde{\mathcal{R}}, r_{j,h} \in \mathcal{R}, \lambda_{j,h} \in \Lambda_\sigma$, and $\delta'_j \in \Delta$), we set

$$\left( \sum_i \lambda_i \cdot \delta_i \right) \cdot \left( \sum_j r_j \cdot \delta'_j \right) := \sum_{i,j,h} \left( r_{j,h} \otimes (\lambda_i \lambda_{j,h}) \right) \cdot (\delta_i \delta'_j).$$
As above, extend \( \tilde{\kappa} \in \tilde{X}_{\text{arith}} \) to a continuous \( \mathcal{O} \)-algebra morphism \( \tilde{\kappa}_N : \mathcal{R}_N \to \bar{\mathbb{Q}}_p \) by setting

\[
\tilde{\kappa}_N \left( \sum_{i=1}^{n} x_i \cdot \delta_i \right) := \sum_{i=1}^{n} \tilde{\kappa}(x_i) \cdot \psi_R(\delta_i)
\]

for \( x_i \in \mathcal{R} \) and \( \delta_i \in \Delta \), where \( \psi_R \) is the character of \( R \). If we denote by \( \tilde{X}_N \) the \( \mathcal{O} \)-module of continuous \( \mathcal{O} \)-linear homomorphisms from \( \mathcal{R}_N \) to \( \bar{\mathbb{Q}}_p \), the above correspondence sets up an injective map \( \tilde{X}_{\text{arith}} \hookrightarrow \tilde{X}_N \) and we let \( \bar{X}_{N, \text{arith}} \) denote the image of \( \tilde{X}_{\text{arith}} \). We also have a map \( \iota : \bar{X}_{N, \text{arith}} \rightarrow \bar{X}_N \) induced from the map \( \mathcal{R}_N \rightarrow \mathcal{R}_N \) taking \( r \mapsto r \otimes 1 \) by pull-back. The map \( \iota \) also restricts to arithmetic points. The maps \( p \) and \( p_N \) make the following diagram commute:

\[
\begin{array}{ccc}
\tilde{X}^{\text{arith}} & \xrightarrow{p} & \tilde{X}^{\text{arith}} \\
P & & \downarrow \iota \\
X^{\text{arith}} & \xrightarrow{p_N} & X^{\text{arith}}
\end{array}
\]

where the projections take a signature \( (\epsilon, k) \) to \( (\epsilon^2, 2k) \).

### 3.4. The \( \Lambda \)-adic correspondence

In this part, we combine the explicit integral formula of Shimura and the fact that the toric integrals can be \( p \)-adically interpolated to show the existence of a \( \Lambda \)-adic Shimura-Shintani-Waldspurger correspondence with the expected interpolation property. This follows very closely [21, §6].

Let \( \tilde{\kappa}_N \in \tilde{X}^{\text{arith}}_N \) of signature \( (\epsilon_\kappa, k_\kappa) \). Let \( L_r \) denote the order of \( \mathcal{M}_2(F) \) consisting of matrices \( \left( \begin{array}{cc} a & b/Mp^r \\ Mp^r c & d \end{array} \right) \) with \( a, b, c, d \in \mathcal{O}_F \). Define

\[
\mathcal{O}_{B,r} := \epsilon_\kappa^{-1}(i_F^{-1}(i_F(B \otimes 1) \cap L_r))
\]

Then \( \mathcal{O}_{B,r} \) is the maximal order introduced in [21] (and denoted \( \mathcal{O}' \) there) defined in terms of the maximal order \( \mathcal{O}_B \) and the integer \( Mp^r \). Also, \( S := \mathcal{O}_B \cap \mathcal{O}_{B,r} \) is an Eichler order of \( B \) of level \( Mp \) containing the fixed Eichler order \( R \) of level \( M \).

With \( \alpha \in V^* \cap \mathcal{O}_{B,1} \), we have

\[
i_F(\alpha) = \left( \begin{array}{cc} a & b/(Mp) \\ c & -a \end{array} \right)
\]

in \( \mathcal{M}_2(F) \) with \( a, b, c \in \mathcal{O}_F \) and we can consider the quadratic forms

\[
Q_\alpha(x, y) := cx^2 - 2axy - (b/(Mp))y^2,
\]

and

\[
\hat{Q}_\alpha(x, y) := Mp \cdot Q_\alpha(x, y) = Mpcx^2 - 2Mpa xy - by^2.
\]

Then \( \hat{Q}_\alpha(x, y) \) has coefficients in \( \mathcal{O}_F \) and, composing with \( F \hookrightarrow \mathbb{R} \) and letting \( x = z, y = 1 \), we recover \( Q_\alpha(z) \) and \( \hat{Q}_\alpha(z) \) of [21] (defined by means of the isomorphism \( i_\infty \)).

Since each prime \( \ell | Mp \) is split in \( F \), the elements \( a, b, c \) can be viewed as elements in \( \mathbb{Z}_\ell \) via our fixed embedding \( \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_\ell \), for any prime \( \ell | Mp \) (we will continue writing \( a, b, c \) for these elements, with a slight abuse of notation). So,
letting \( b_\alpha \in \mathbb{Z} \) such that \( i_\ell(\alpha) = (^{*\ell}_{b_\alpha/(\mathbb{M}_p)}) \) modulo \( i_\ell(\mathbb{S} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell) \), for all \( \ell | M_p \), we have \( b \equiv b_\alpha \) modulo \( \mathbb{M}_p \mathbb{Z}_\ell \) as elements in \( \mathbb{Z}_\ell \), for all \( \ell | M_p \), and thus we get
\[
(9) \quad \eta_\alpha(\alpha) = \epsilon_\ell(b_\alpha) = \epsilon_\ell(b)
\]
for \( b \) as in (7).

For any \( \nu \in \mathbb{D} \), we may define an \( \mathcal{O} \)-valued measure \( j_\alpha(\nu) \) on \( \mathbb{Z}_p^\times \) by the formula:
\[
\int_{\mathbb{Z}_p^\times} f(t) d(j_\alpha(\nu))(t) := \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} f(Q_\alpha(x, y)) dv(x, y).
\]
for any continuous function \( f : \mathbb{Z}_p^\times \to \mathbb{C}_p \). Recall that the group of \( \mathcal{O} \)-valued measures on \( \mathbb{Z}_p^\times \) is isomorphic to the Iwasawa algebra \( \mathcal{O}[\mathbb{Z}_p^\times] \), and thus we may view \( j_\alpha(\nu) \) as an element in \( \mathcal{O}[\mathbb{Z}_p^\times] \) (see, for example, \( \mathcal{R} \) \S 3.2). In particular, for any group-like element \( [\lambda] \in \mathcal{O}[\mathbb{Z}_p^\times] \) we have:
\[
\int_{\mathbb{Z}_p^\times} f(t) d([\lambda] \cdot j_\alpha(\nu))(t) = \int_{\mathbb{Z}_p^\times} \left( \int_{\mathbb{Z}_p^\times} f(ts) d([\lambda](s)) \right) dj_\alpha(\nu)(t) = \int_{\mathbb{Z}_p^\times} f(\lambda t) dj_\alpha(\nu)(t).
\]
On the other hand,
\[
\int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} f(Q_\alpha(x, y)) d(\lambda \cdot \nu) = \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} f(Q_\alpha(\lambda x, \lambda y)) d\nu = \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} f(\lambda^2 Q_\alpha(x, y)) d\nu
\]
and we conclude that \( j_\alpha(\lambda \cdot \nu) = [\lambda^2] \cdot j_\alpha(\nu) \). In other words, \( j_\alpha \) is a \( \mathcal{O}[\mathbb{Z}_p^\times] \)-linear map
\[
j_\alpha : \mathbb{D} \to \mathcal{O}[\mathbb{Z}_p^\times]_\sigma.
\]

Before going ahead, let us introduce some notation. Let \( \chi \) be a Dirichlet character modulo \( \mathbb{M}_p^r \), for a positive integer \( r \), which we decompose accordingly with the isomorphism \( (\mathbb{Z}/\mathbb{N}p^r\mathbb{Z})^\times \simeq (\mathbb{Z}/\mathbb{N}\mathbb{Z})^\times \times (\mathbb{Z}/p^r\mathbb{Z})^\times \) into the product \( \chi = \chi_N \cdot \chi_p \) with \( \chi_N : (\mathbb{Z}/\mathbb{N}\mathbb{Z})^\times \to \mathbb{C}^\times \) and \( \chi_p : (\mathbb{Z}/p^r\mathbb{Z})^\times \to \mathbb{C}^\times \). Thus, we will write \( \chi(x) = \chi_N(x,N) \cdot \chi_p(x_p) \), where \( x_N \) and \( x_p \) are the projections of \( x \in (\mathbb{Z}/\mathbb{N}p^r\mathbb{Z})^\times \) to \( (\mathbb{Z}/\mathbb{N}\mathbb{Z})^\times \) and \( (\mathbb{Z}/p^r\mathbb{Z})^\times \), respectively. To simplify the notation, we will suppress the \( N \) and \( p \) from the notation for \( x_N \) and \( x_p \), thus simply writing \( x \) for any of the two. Using the isomorphism \( (\mathbb{Z}/\mathbb{M}\mathbb{Z})^\times \simeq (\mathbb{Z}/M\mathbb{Z})^\times \times (\mathbb{Z}/D\mathbb{Z})^\times \) decompose \( \chi_N = \chi_M \cdot \chi_D \) with \( \chi_M \) and \( \chi_D \) characters on \( (\mathbb{Z}/M\mathbb{Z})^\times \) and \( (\mathbb{Z}/D\mathbb{Z})^\times \), respectively. In the following, we only need the case when \( \chi_D = 1 \).

Using the above notation, we may define a \( \mathcal{O}[\mathbb{Z}_p^\times] \)-linear map \( J_\alpha : \mathbb{D} \to \mathcal{O}[\mathbb{Z}_p^\times] \) by
\[
J_\alpha(\nu) = \epsilon_\ell,M(b) \cdot \epsilon_\ell,M(-1) \cdot j_\alpha(\nu)
\]
with \( b \) as in (7). Set \( \mathbb{D}_N := \mathbb{D} \otimes_{\mathcal{O}[\mathbb{Z}_p^\times]} \Lambda_N \), where the map \( \mathcal{O}[\mathbb{Z}_p^\times] \to \Lambda_N \) is induced from the map \( \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times \times (\mathbb{Z}/\mathbb{N}\mathbb{Z})^\times \) on group-like elements given by \( x \mapsto x \otimes 1 \). Then \( J_\alpha \) can be extended to a \( \Lambda_N \)-linear map \( J_\alpha : \mathbb{D}_N \to \Lambda_N, \sigma \). Setting \( \mathbb{D}_{\mathcal{R}_N} := \mathcal{R}_N \otimes_{\Lambda_N} \mathbb{D}_N \) and extending by \( \mathcal{R}_N \)-linearity over \( \Lambda_N \) we finally obtain a \( \mathcal{R}_N \)-linear map, again denoted by the same symbol,
\[
J_\alpha : \mathbb{D}_{\mathcal{R}_N} \to \mathcal{R}_N.
\]
For \( \nu \in \mathbb{D}_N \) and \( r \in \mathcal{R}_N \) we thus have
\[
J_\alpha(r \otimes \nu) = \epsilon_\ell,M(b) \cdot \epsilon_\ell,M(-1) \cdot r \otimes j_\alpha(\nu).
\]
For the next result, for any arithmetic point \( \kappa_N \in \mathcal{X}_N^{\text{arith}} \) coming from \( \kappa \in \mathcal{X}^{\text{arith}} \), extend \( \rho_\kappa \) in (5) by \( \mathcal{R}_N \)-linearity over \( \mathcal{O}[\mathbb{Z}_p^\times] \), to get a map

\[
\rho_{\kappa_N} : \mathbb{D}_{\mathcal{R}_N} \rightarrow V_{\kappa_N}
\]
defined by \( \rho_{\kappa_N}(r \otimes \nu) := \rho_\kappa(\nu) \cdot \kappa_N(r) \), for \( \nu \in \mathbb{D} \) and \( r \in \mathcal{R}_N \). To simplify the notation, set

(10) \[
\langle \nu, \alpha \rangle_{\kappa_N} := \rho_{\kappa_N}(\nu)(\tilde{Q}_\alpha^{n_\kappa}/2).
\]
The following is essentially [21 Lemma (6.1)].

**Lemma 3.2.** Let \( \tilde{\kappa}_N \in \tilde{\mathcal{X}}_N^{\text{arith}} \) with signature \( (\epsilon_{\tilde{\kappa}}, k_{\tilde{\kappa}}) \) and define \( \kappa_N := p_N(\tilde{\kappa}_N) \). Then for any \( \nu \in \mathbb{D}_{\mathcal{R}_N} \) we have:

\[
\tilde{\kappa}_N(J_\alpha(\nu)) = \eta_{k_\kappa}(\alpha) \cdot \langle \nu, \alpha \rangle_{\kappa_N}.
\]

**Proof.** For \( \nu \in \mathbb{D}_N \) and \( r \in \mathcal{R}_N \) we have

\[
\tilde{\kappa}_N(J_\alpha(\nu \otimes r)) = \tilde{\kappa}_N(\epsilon_{\tilde{\kappa},M}(b) \cdot \epsilon_{\tilde{\kappa},p}(-1) \cdot r \otimes j_\alpha(\nu))
\]

\[= \epsilon_{\tilde{\kappa},M}(b) \cdot \epsilon_{\tilde{\kappa},p}(-1) \cdot \tilde{\kappa}_N(r \otimes 1) \cdot \tilde{\kappa}_N(1 \otimes j_\alpha(\nu))
\]

\[= \epsilon_{\tilde{\kappa},M}(b) \cdot \epsilon_{\tilde{\kappa},p}(-1) \cdot \kappa_N(r) \cdot \int_{\mathbb{Z}_p^\times} \tilde{\kappa}_N(t) d\alpha(\nu)
\]

and thus, noticing that \( \tilde{\kappa}_N \) restricted to \( \mathbb{Z}_p^\times \) is \( \tilde{\kappa}_N(t) = \epsilon_{\tilde{\kappa},p}(t)t^{n_\kappa} \), we have

\[
\tilde{\kappa}_N(J_\alpha(\nu \otimes r)) = \epsilon_{\tilde{\kappa},M}(b) \cdot \epsilon_{\tilde{\kappa},p}(-1) \cdot \kappa_N(r) \int_{\mathbb{Z}_p^\times} \epsilon_{\tilde{\kappa},p}(\tilde{Q}_\alpha(x,y)) \tilde{Q}_\alpha(x,y)^{n_\kappa}/2 d\nu.
\]

Recalling (5), and viewing \( a, b, c \) as elements in \( \mathbb{Z}_p \), we have, for \( (x, y) \in \mathbb{Z}_p \times \mathbb{Z}_p^\times \),

\[
\epsilon_{\tilde{\kappa},p}(\tilde{Q}_\alpha(x,y)) = \epsilon_{\tilde{\kappa},p}(-by^2) = \epsilon_{\tilde{\kappa},p}(-b)\epsilon_{\tilde{\kappa},p}(y^2) = \epsilon_{\tilde{\kappa},p}(-b)\epsilon_{\tilde{\kappa},p}(y^2) = \epsilon_{\tilde{\kappa},p}(-b)\epsilon_{\tilde{\kappa},p}(y).
\]

Thus, since \( \epsilon_{\tilde{\kappa}}(-1)^2 = 1 \), we get:

\[
\tilde{\kappa}_N(J_\alpha(\nu \otimes r)) = \kappa_N(r) \cdot \epsilon_{\tilde{\kappa},M}(b) \cdot \epsilon_{\tilde{\kappa},p}(b) \cdot \rho_\kappa(\nu)(\tilde{Q}_\alpha^{n_\kappa}/2) = \eta_{k_\kappa}(\alpha) \cdot \langle \nu, \alpha \rangle_{\kappa_N}
\]

where for the last equality use (4) and (10). \( \square \)

Define

\[\mathbb{W}_{\mathcal{R}_N} := \mathbb{W} \otimes_{\mathcal{O}[\mathbb{Z}_p^\times]} \mathcal{R}_N,\]

the structure of \( \mathcal{O}[\mathbb{Z}_p^\times] \)-module of \( \mathcal{R}_N \) being that induced by the composition of the two maps \( \mathcal{O}[\mathbb{Z}_p^\times] \rightarrow \Lambda_N \rightarrow \mathcal{R}_N \) described above. There is a canonical map

\[\vartheta : \mathbb{W}_{\mathcal{R}_N} \rightarrow H^1(\Gamma_0, \mathbb{D}_{\mathcal{R}_N})\]

described as follows: if \( \nu_\gamma \) is a representative of an element \( \nu \) in \( \mathbb{W} \) and \( r \in \mathcal{R}_N \), then \( \vartheta(\nu \otimes r) \) is represented by the cocycle \( \nu_\gamma \otimes r \).

For \( \nu \in \mathbb{W}_{\mathcal{R}_N} \) represented by \( \nu_\gamma \) and \( \xi \geq 1 \) an integer, define

\[\theta_\xi(\nu) := \sum_{C \in \mathcal{R}(\Gamma_0), q(C) = \xi} \frac{J_\alpha(\nu_{\gamma_{\alpha c}})}{t_{\alpha c}}.\]
Definition 3.3. For $\nu \in \mathcal{W}_{\mathcal{R}_N}$, the formal Fourier expansion
\[
\Theta(\nu) := \sum_{\xi \geq 1} \theta_\xi(\nu) q^\xi
\]
in $\mathcal{R}_N[\mathbb{g}]$ is called the $\Lambda$-adic Shimura-Shintani-Waldspurger lift of $\nu$. For any $\tilde{\kappa} \in \tilde{X}^{\text{arith}}$, the formal power series expansion
\[
\Theta(\nu)(\tilde{\kappa}_N) := \sum_{\xi \geq 1} \tilde{\kappa}_N(\theta_\xi(\nu)) q^\xi
\]
is called the $\tilde{\kappa}$-specialization of $\Theta(\nu)$.

There is a natural map
\[
\mathcal{W}_\mathcal{R} \rightarrow \mathcal{W}_{\mathcal{R}_N}
\]
taking $\nu \otimes r$ to itself (use that $\mathcal{R}$ has a canonical map to $\mathcal{R}_N \simeq \mathcal{R}[\Delta]$, as described above). So, for any choice of sign, $\Phi^+ \in \mathcal{W}_{\mathcal{R}}$ will be viewed as an element in $\mathcal{W}_{\mathcal{R}_N}$.

From now on we will use the following notation. Fix $\tilde{\kappa}_0 \in \tilde{X}^{\text{arith}}$ and put $\kappa_0 := p(\tilde{\kappa}_0) \in X^{\text{arith}}$. Recall the neighborhood $\mathcal{U}_0$ of $\kappa_0$ in Theorem 3.1. Define
\[
\tilde{\mathcal{U}}_0 := p^{-1}(\mathcal{U}_0) \text{ and } \tilde{\mathcal{U}}_0^{\text{arith}} := \mathcal{U}_0 \cap \tilde{X}^{\text{arith}}.
\]
For each $\tilde{\kappa} \in \tilde{\mathcal{U}}_0^{\text{arith}}$ put $\kappa = p(\tilde{\kappa}) \in \mathcal{U}_0^{\text{arith}}$. Recall that if $(\epsilon^\kappa, k^\kappa)$ is the signature of $\tilde{\kappa}$, then $(\epsilon^\kappa, k^\kappa) := (\epsilon^\tilde{\kappa}, 2k^\tilde{\kappa})$ is that of $\kappa_0$. For any $\kappa := p(\tilde{\kappa})$ as above, we may consider the modular form
\[
f_{\kappa}^{\text{JL}} \in S_{k^\kappa}(\Gamma_1^r, \epsilon^\kappa)
\]
and its Shimura-Shintani-Waldspurger lift
\[
h_{\kappa} = \sum_{\xi} a_\xi(h_{\kappa}) q^\xi \in S_{k^\kappa + 1/2}(4Np^{r_{\kappa}}, \chi_{\kappa}), \text{ where } \chi_{\kappa}(x) := \epsilon_{\tilde{\kappa}}(x) \left(\frac{-1}{x}\right)^{k^\kappa},
\]
normalized as in [2] and [3]. For our fixed $\kappa_0$, recall the elements $\Phi := \Phi^+$ chosen as in Theorem 3.1 and define $\phi^\kappa := \phi_{\kappa}^+$ and $\Omega_{\kappa} := \Omega_{\kappa}^+$ for $\kappa \in \mathcal{U}_0^{\text{arith}}$.

Proposition 3.4. For all $\tilde{\kappa} \in \tilde{\mathcal{U}}_0^{\text{arith}}$ such that $r_{\kappa} = 1$ we have
\[
\tilde{\kappa}_N(\theta_\xi(\Phi)) = \Omega_{\kappa} \cdot a_\xi(h_{\kappa}) \text{ and } \Theta(\Phi)(\tilde{\kappa}_N) = \Omega_{\kappa} \cdot h_{\kappa}.
\]
Proof. By Lemma 3.2 we have
\[
\tilde{\kappa}_N(\theta_\xi(\Phi)) = \sum_{\xi \in R(\Gamma_1; q(C))=\xi} \eta_\xi(\alpha_C) \rho_{\kappa_N}(\Phi)(\tilde{Q}_{\alpha_C}^{n_{\kappa}/2}).
\]
Using Theorem 3.1 we get
\[
\tilde{\kappa}_N(\theta_\xi(\Phi)) = \sum_{\xi \in R(\Gamma_1; q(C))=\xi} \eta_\xi(\alpha_C) \cdot \Omega_{\kappa} \cdot \alpha_C(\tilde{Q}_{\alpha_C}^{n_{\kappa}/2}).
\]
Now [2] shows the statement on $\tilde{\kappa}_N(\theta_\xi(\Phi))$, while that on $\Theta(\Phi)(\tilde{\kappa}_N)$ is a formal consequence of the previous one.

Corollary 3.5. Let $a_p$ denote the image of the Hecke operator $T_p$ in $\mathcal{R}$. Then
\[
\Theta(\Phi)T_p^2 = a_p \cdot \Theta(\Phi).
\]
Proof. For any \( \kappa \in \mathcal{X}_{\text{arith}} \), let \( a_p(\kappa) := \kappa(T_p) \), which is a \( p \)-adic unit by the ordinary assumption. For all \( \tilde{\kappa} \in \mathcal{U}_{\text{arith}} \) with \( r_\kappa = 1 \), we have
\[
\Theta(\Phi)(\tilde{\kappa}_N)|T_p^2 = \Omega_\kappa \cdot h_\kappa|T_p^2 = a_p(\kappa) \cdot \Omega_\kappa \cdot h_\kappa = a_p(\kappa) \cdot \Theta(\Phi)(\tilde{\kappa}_N).
\]
Consequently,
\[
\tilde{\kappa}_N(\theta_{\text{ad}}(\Phi)) = a_p(\kappa) \cdot \tilde{\kappa}_N(\theta_{\text{ad}}(\Phi))
\]
for all \( \tilde{\kappa} \) such that \( r_\kappa = 1 \). Since this subset is dense in \( \tilde{X}_N \), we conclude that \( \theta_{\text{ad}}(\Phi) = a_p \cdot \theta_{\text{ad}}(\Phi) \) and so \( \Theta(\Phi)|T_p^2 = a_p \cdot \Theta(\Phi) \). \( \square \)

For any integer \( n \geq 1 \) and any quadratic form \( Q \) with coefficients in \( F \), write \([Q]_n\) for the class of \( Q \) modulo the action of \( i_F(\Gamma_n) \). Define \( \mathcal{F}_{n,\xi} \) to be the subset of the \( F \)-vector space of quadratic forms with coefficients in \( F \) consisting of quadratic forms \( \tilde{Q}_\alpha \) such that \( \alpha \in V^* \cap \mathcal{O}_{B,n} \) and \(-\text{nr}(\alpha) = \xi \). Writing \( \delta_{Q,\alpha} \) for the discriminant of \( Q, \alpha \), the above set can be equivalently described as
\[
\mathcal{F}_{n,\xi} := \{ \tilde{Q}_\alpha \mid \alpha \in V^* \cap \mathcal{O}_{B,n}, \delta_{Q,\alpha} = Np^n \xi \}.
\]
Define \( \mathcal{F}_{n,\xi}/\Gamma_n \) to be the set \( \{ [\tilde{Q}_\alpha]_n \mid \tilde{Q}_\alpha \in \mathcal{F}_{n,\xi} \} \) of equivalence classes of \( \mathcal{F}_{n,\xi} \) under the action of \( i_F(\Gamma_n) \). A simple computation shows that \( Q_{g^{-1}ag} = Q_{\alpha} \) for all \( \alpha \in V^* \) and all \( g \in \Gamma_n \), and thus we find
\[
\mathcal{F}_{n,\xi}/\Gamma_n = \{ [\tilde{Q}_\alpha]_n \mid C \in R(\Gamma_n), \delta_{Q,\alpha} = Np^n \xi \}.
\]
We also note that, in the notation of [24], if \( f \) has weight character \( \psi \), defined modulo \( Np^n \), and level \( \Gamma_n \), the Fourier coefficients \( a_\xi(h) \) of the Shimura-Shintani-Waldspurger lift \( h \) of \( f \) are given by
\[
a_\xi(h) = \sum_{[Q] \in \mathcal{F}_{n,\xi}/\Gamma_n} \frac{\psi(Q)}{t_Q} \phi_j^+(Q(z)^{k-1})
\]
and, if \( Q = \tilde{Q}_\alpha \), we put \( \psi(Q) := \eta_\psi(b_\alpha) \) and \( t_Q := t_\alpha \). Also, if we let
\[
\mathcal{F}_{n}/\Gamma_n := \prod_{\xi} \mathcal{F}_{n,\xi}/\Gamma_n
\]
we can write
\[
h = \sum_{[Q] \in \mathcal{F}_{n}/\Gamma_n} \frac{\psi(Q)}{t_Q} \phi_j^+(Q(z)^{k-1}) q^{\delta_{Q}/(Np^n)}.
\]
Fix now an integer \( m \geq 1 \) and let \( n \in \{1, m\} \). For any \( t \in (\mathbb{Z}/p^n\mathbb{Z})^\times \) and any integer \( \xi \geq 1 \), define \( \mathcal{F}_{n,\xi,t} \) to be the subset of \( \mathcal{F}_{n,\xi} \) consisting of forms such that \( Np^n b_\alpha \equiv t \mod Np^m \). Also, define \( \mathcal{F}_{n,\xi,t}/\Gamma_n \) to be the set of equivalence classes of \( \mathcal{F}_{n,\xi,t} \) under the action of \( i_F(\Gamma_n) \). If \( \alpha \in V^* \cap \mathcal{O}_{B,m} \) and \( i_F(\alpha) = (a b \ c \ d) \), then
\[
\tilde{Q}_\alpha(x, y) = Np^n cx^2 - 2Np^n axy - Np^n by^2
\]
from which we see that there is an inclusion \( \mathcal{F}_{m,\xi,t} \subset \mathcal{F}_{1,\xi,\rho^m-1,t} \). If \( \tilde{Q}_\alpha \) and \( \tilde{Q}_{\alpha'} \) belong to \( \mathcal{F}_{m,\xi,t} \), and \( \alpha' = gag^{-1} \) for some \( g \in \Gamma_m \), then, since \( \Gamma_m \subseteq \Gamma_1 \), we see that \( \tilde{Q}_\alpha \) and \( \tilde{Q}_{\alpha'} \) represent the same class in \( \mathcal{F}_{1,\xi,\rho^m-1,t}/\Gamma_1 \). This shows that \( [\tilde{Q}_\alpha]_m \mapsto [\tilde{Q}_\alpha]_1 \) gives a well-defined map
\[
\pi_{m,\xi,t} : \mathcal{F}_{m,\xi,t}/\Gamma_m \longrightarrow \mathcal{F}_{1,\xi,\rho^m-1,t}/\Gamma_1.
\]
Lemma 3.6. The map \( \pi_{m,\xi,t} \) is bijective.
Proof. We first show the injectivity. For this, suppose \( \tilde{Q}_\alpha \) and \( \tilde{Q}_{\alpha'} \) are in \( F_{m,\ell,t} \) and 
\[ [\tilde{Q}_\alpha]_1 = [\tilde{Q}_{\alpha'}]_1. \]
So there exists \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) in \( i_F(\Gamma_1) \) such that such that \( \tilde{Q}_\alpha = \tilde{Q}_{\alpha'} g \).
If \( \tilde{Q}_\alpha = cx^2 - 2axy - by^2 \), and easy computation shows that \( \tilde{Q}_{\alpha'} = c'x^2 - 2a'xy - b'y^2 \) with
\[
c' = ca^2 - 2a\alpha\gamma - b\gamma^2
\]
\[
a' = -ca\beta + a\alpha\gamma + a\alpha\delta + b\gamma\delta
\]
\[
b' = -c\beta^2 + 2a\beta\delta + b\delta^2.
\]
The first condition shows that \( \gamma \equiv 0 \mod Np^m \). We have \( b \equiv b' \equiv t \mod Np^m \),
so \( \delta^2 \equiv 1 \mod Np^m \). Since \( \delta \equiv 1 \mod Np \), we see that \( \delta \equiv 1 \mod Np^m \) too.
We now show the surjectivity. For this, fix \( [\tilde{Q}_{\alpha C}]_1 \) in the target of \( \pi \), and choose a representative
\[ \tilde{Q}_{\alpha C} = cx^2 - 2axy - by^2 \]
(recall \( Mp^m \xi|\tilde{Q}_{\alpha C} \), \( Np|c, Np|a \), and \( b \in O_F^\times \)), the last condition due to \( \eta_F(\alpha C) \neq 0 \).
By the Strong Approximation Theorem, we can find \( \tilde{g} \in \Gamma_1 \) such that
\[ i_F(\tilde{g}) \equiv \begin{pmatrix} 1 \\ ab^{-1} \\ 0 \\ 1 \end{pmatrix} \mod Np^m \]
for all \( \ell|Np \). Take \( g := i_F(\tilde{g}) \), and put \( \alpha := g^{-1}\alpha C g \). An easy computation, using
the expressions for \( a', b', c' \) in terms of \( a, b, c \) and \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) as above, shows that
\( \alpha \in V^* \cap O_{B,m}, \eta_F(\alpha) = t \) and \( \delta_{\tilde{Q}_\alpha} = Np^m \xi \), and it follows that \( \tilde{Q}_\alpha \in F_{m,\ell,t} \).
Now
\[ \pi ([\tilde{Q}_\alpha]_m) = [\tilde{Q}_\alpha]_1 = [\tilde{Q}_{g^{-1}\alpha C g}]_1 = [\tilde{Q}_{\alpha C}]_1 \]
where the last equality follows because \( g \in \Gamma_1 \).

Proposition 3.7. For all \( \tilde{\kappa} \in \tilde{U}^\overline{0}_{\theta} \) we have
\[ \Theta(\Phi)(\tilde{\kappa}_N)|T_p^{r_{\kappa} - 1} = \Omega_\kappa \cdot h_\kappa. \]

Proof. For \( r_\kappa = 1 \), this is Proposition 3.4 above, so we may assume \( r_\kappa \geq 2 \). As in
the proof of Proposition 3.4 above, we may assume \( r_\kappa \geq 2 \). As in
the proof of Proposition 3.4 above, we may assume \( r_\kappa \geq 2 \). As in
the proof of Proposition 3.4 above, we may assume \( r_\kappa \geq 2 \). As in
Therefore, splitting the above sum over \( t \in (\mathbb{Z}/Np^r\mathbb{Z})^\times \), we get

\[
\Theta(\Phi)(\tilde{\kappa}_N)[T_p^{r-1}] = \sum_{t \in (\mathbb{Z}/p^r\mathbb{Z})^\times} \sum_{[Q] \in \mathcal{F}_{t,t}} \frac{\epsilon_{\tilde{\kappa}(Q) \cdot \Omega_{\kappa}(Q^{k_{\kappa}})} \cdot \phi_{\kappa}(Q^{k_{\kappa}-1})q^{\delta_{Q}/(Np^r)}}{tQ} = \sum_{t \in (\mathbb{Z}/p^r\mathbb{Z})^\times} \sum_{[Q] \in \mathcal{F}_{m,t,T_m}} \frac{\epsilon_{\tilde{\kappa}(Q) \cdot \Omega_{\kappa}(Q^{k_{\kappa}})} \cdot \phi_{\kappa}(Q^{k_{\kappa}-1})q^{\delta_{Q}/(Np^r)}}{tQ} = \sum_{[Q] \in \mathcal{F}_{m,T_m}} \frac{\epsilon_{\tilde{\kappa}(Q) \cdot \Omega_{\kappa}(Q^{k_{\kappa}})} \cdot \phi_{\kappa}(Q^{k_{\kappa}-1})q^{\delta_{Q}/(Np^r)}}{tQ}.
\]

Comparing this expression with (12) gives the result.

We are now ready to state the analogue of [21, Thm. 3.3], which is our main result. For the reader’s convenience, we briefly recall the notation appearing below. We denote by \( X \) the points of the ordinary Hida Hecke algebra, and by \( X_{\text{arith}} \) its arithmetic points. For \( \kappa_0 \in X_{\text{arith}} \), we denote by \( U_0 \) the \( p \)-adic neighborhood of \( \kappa_0 \) appearing in the statement of Theorem 3.8 and put \( U_{\text{arith}}^0 := U_0 \cap X_{\text{arith}} \). We also denote by \( \Phi = \Phi^+ \in \mathcal{W}_{\text{ord}} \) the cohomology class appearing in Theorem 3.1. The points \( X \) of the metaplectic Hida Hecke algebra defined in [3,3] are equipped with a canonical map \( p : X_{\text{arith}} \to X_{\text{arith}} \) on arithmetic points. Let \( U_{\text{arith}}^0 := U_0 \cap X_{\text{arith}} \).

For each \( \tilde{\kappa} \in U_{\text{arith}}^0 \) put \( \kappa = p(\tilde{\kappa}) \in U_{\text{arith}}^0 \). Recall that if \( (\epsilon_{\tilde{\kappa}}, k_{\tilde{\kappa}}) \) is the signature of \( \tilde{\kappa} \), then \( (\epsilon_{\kappa}, k_{\kappa}) := (\epsilon_{\tilde{\kappa}}, 2k_{\tilde{\kappa}}) \) is that of \( \kappa_0 \). For any \( \kappa := p(\tilde{\kappa}) \) as above, we may consider the modular form

\[
f_{\kappa}^{\text{HL}} \in S_{\kappa}(\Gamma_{1,\kappa}, \epsilon_{\kappa})
\]

and its Shimura-Shintani-Waldspurger lift

\[
h_{\kappa} = \sum_{\xi} a_{\xi}(h_{\kappa}) q^{\xi} \in S_{\kappa+1/2}(4Np^r, \chi_{\kappa}), \quad \text{where} \quad \chi_{\kappa}(x) := \epsilon_{\kappa}(x) \left( \frac{-1}{x} \right)^{k_{\kappa}},
\]

normalized as in [2] and [3]. Finally, for \( \tilde{\kappa} \in X_{\text{arith}} \), we denote by \( \tilde{\kappa}_N \) its extension to the metaplectic Hecke algebra \( \mathcal{R}_N \) defined in [3,3].

**Theorem 3.8.** Let \( \kappa_0 \in X_{\text{arith}} \). Then there exists a choice of \( p \)-adic periods \( \Omega_{\kappa} \) for \( \kappa \in U_0 \) such that the \( \Lambda \)-adic Shimura-Shintani-Waldspurger lift of \( \Phi \)

\[
\Theta(\Phi) := \sum_{\xi \geq 1} \theta_{\xi}(\Phi)q^{\xi}
\]

in \( \mathcal{R}_N \llbracket q \rrbracket \) has the following properties:

1. \( \Omega_{\kappa_0} \neq 0 \).
2. For any \( \tilde{\kappa} \in U_{\text{arith}}^0 \), the \( \tilde{\kappa} \)-specialization of \( \Theta(\Phi) \)

\[
\Theta(\nu)(\tilde{\kappa}_N) := \sum_{\xi \geq 1} \tilde{\kappa}(\theta_{\xi}(\Phi)) q^{\xi} \text{ belongs to } S_{\kappa+1/2}(4Np^r, \chi_{\kappa}'),
\]

where \( \chi_{\kappa}'(x) := \chi_{\kappa}(x) \cdot \left( \frac{N}{x} \right)^{k_{\kappa}-1} \), and satisfies

\[
\Theta(\Phi)(\tilde{\kappa}_N) = \Omega_{\kappa} \cdot h_{\kappa} |T_p^{1-r_{\kappa}}.
\]
Proof. The elements $\Omega_k$ are those $\Omega^+_k$ appearing in Theorem 3.1 which we used in Propositions 3.4 and 3.7 above, so (1) is clear. Applying $T_p^\infty \cdot$ to the formula of Proposition 3.7 using Corollary 3.5 and applying $a_p(\kappa) t^{-r_j}$ on both sides gives

$$\Theta(\Phi)(\kappa_N) = a_p(\kappa) t^{-r_j} \Omega_k \cdot h_0|T_p^\infty \cdot .$$

By [18] Prop. 1.9, each application of $T_p$ has the effect of multiplying the character by $(\,\,)$, hence

$$h'_k := h_k|T_p^{-r_j} \in S_{k,+1/2}(4Np^{r_j}, \chi'_k)$$

with $\chi'_k$ as in the statement. This gives the first part of (2), while the last formula follows immediately from Proposition 3.7. \qed

Remark 3.9. Theorem 1.1 is a direct consequence of Theorem 3.8 as we briefly show below.

Recall the embedding $\mathbb{Z}^{\geq 2} \hookrightarrow \text{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$ which sends $k \in \mathbb{Z}^{\geq 2}$ to the character $x \mapsto x^{k-2}$. Extending characters by $\mathcal{O}$-linearity gives a map

$$\mathbb{Z}^{\geq 2} \hookrightarrow \mathcal{X}(\Lambda) := \text{Hom}(\mathcal{O}_{\text{alg}}(\Lambda, \mathbb{Q}_p)).$$

We denote by $k(\Lambda)$ the image of $k \in \mathbb{Z}^{\geq 2}$ in $\mathcal{X}(\Lambda)$ via this embedding. We also denote by $\varpi : \mathcal{X} \rightarrow \mathcal{X}(\Lambda)$ the finite-to-one map obtained by restriction of homomorphisms to $\Lambda$. Let $k^{(\mathcal{R})}$ be a point in $\mathcal{X}$ of signature $(k, 1)$ such that $\varpi(k^{(\mathcal{R})}) = k(\Lambda)$. A well-known result by Hida (see [6] Cor. 1.4]) shows that $\mathcal{R}/\Lambda$ is unramified at $k^{(\mathcal{R})}$. As shown in [21] §1, this implies that there is a section $s_{\mathcal{R}, \Lambda}$ of $\varpi$ which is defined on a neighborhood $\mathcal{U}_{k^{(\mathcal{R})}}$ of $k^{(\mathcal{R})}$ in $\mathcal{X}(\Lambda)$ and sends $k^{(\mathcal{R})}$ to $k^{(\mathcal{R})}$.

Fix now $k_0$ as in the statement of Theorem 1.1 corresponding to a cuspform $f_0$ of weight $k_0$ with trivial character. The form $f_0$ corresponds to an arithmetic character $k_0^{(\mathcal{R})}$ of signature $(1, k_0)$ belonging to $\mathcal{X}$. Let $\mathcal{U}'_0$ be the inverse image of $\mathcal{U}_0$ under the section $s_{\mathcal{R}, \Lambda}^{(-1)}$, where $\mathcal{U}_0 \subseteq \mathcal{X}$ is the neighborhood of $k_0^{(\mathcal{R})}$ in Theorem 3.8. Extending scalars by $\mathcal{O}$ gives, as above, an injective continuous map $\text{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times) \hookrightarrow \mathcal{X}(\Lambda)$, and we let $\mathcal{U}'_0$ be any neighborhood of the character $x \mapsto x^{k_0-2}$ which maps to $\mathcal{U}'_0$ and is contained in the residue class of $k_0$ modulo $p - 1$. Composing this map with the section $\mathcal{U}'_0 \hookrightarrow \mathcal{U}_0$ gives a continuous injective map

$$\zeta : \mathcal{U}_0 \longrightarrow \mathcal{U}'_0 \longrightarrow \mathcal{U}_0$$

which takes $k_0$ to $k_0^{(\mathcal{R})}$, since by construction the image of $k_0$ by the first map is $k_0^{(\mathcal{R})}$. We also note that, more generally, $\zeta(k) = k^{(\mathcal{R})}$ because by construction $\zeta(k)$ restricts to $k^{(\Lambda)}$ and its signature is $(1, k)$, since the character of $\zeta(k)$ is trivial. To show the last assertion, recall that the character of $\zeta(k)$ is $\psi_k \cdot \psi_{\mathcal{R}} \cdot \omega^{-k}$, and note that $\psi_k$ is trivial because $k^{(\Lambda)}(x) = x^{k-1}$, and $\psi_{\mathcal{R}} \cdot \omega^{-k}$ is trivial because the same is true for $k_0$ and $k \equiv k_0$ modulo $p - 1$. In other words, arithmetic points in $\zeta(U_0)$ correspond to cuspforms with trivial character. This is the Hida family of forms with trivial character that we considered in the Introduction.

We can now prove Theorem 1.1. For all $k \in U_0 \cap \mathbb{Z}^{\geq 2}$, put $\Omega_k := \Omega_{k^{(\Lambda)}}$ and define $\Theta := \Theta(\Phi) \circ \zeta$ with $\Phi$ as in Theorem 3.8 for $k_0 = k_0^{(\mathcal{R})}$. Applying Theorem 3.8 to $k_0^{(\mathcal{R})}$, and restricting to $\zeta(U_0)$, shows that $U_0$, $\Omega_k$, and $\Theta$ satisfy the conclusion of Theorem 1.1.
Remark 3.10. For $\tilde{\kappa} \in \tilde{\mathcal{U}}_{\operatorname{arith}}^0$ of signature $(\epsilon_{\tilde{\kappa}}, k_{\tilde{\kappa}})$ with $r_{\tilde{\kappa}} = 1$ as in the above theorem, $h_{\tilde{\kappa}}$ is trivial if $(-1)^{k_{\tilde{\kappa}}} = 1$. However, since $\phi_{\kappa_0} \neq 0$, it follows that $h_{\kappa_0}$ is not trivial as long as the necessary condition $(-1)^{k_0} = 1$ is verified.

Remark 3.11. This result can be used to produce a quaternionic $\Lambda$-adic version of the Saito-Kurokawa lifting, following closely the arguments in [8, Cor. 1].

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Dipartimento di Matematica, Università di Padova, Via Trieste 63, 35121 Padova, Italy

E-mail address: mlongo@math.unipd.it
Institut de mathématiques de Luminy, Université d’Aix-Marseille, campus de Luminy, case 907, 13288 Marseille cedex 9, France

E-mail address: nicole@iml.univ-mrs.fr