On the spectral stability of soliton-like solutions to a non-local hydrodynamic-type model

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Abstract

A model of nonlinear elastic medium with internal structure is considered. The medium is assumed to contain cavities, microcracks or inclusions consisting of substances that differ sharply in physical properties from the base material. To describe the wave processes in such a medium, the averaged values of physical fields are used. This leads to nonlinear evolutionary PDEs, differing from the classical balance equations. Using some transformations, these equations can be presented in the Hamiltonian form. It is shown that the system under investigation has a family of invariant soliton-like solutions. The present work is devoted to study of the spectral stability of the solitary wave solutions.

Keywords: nonlocal hydrodynamic-type model, Hamiltonian formulation, multisymplectic formulation, stability test, spectral stability of soliton-like solutions

1. Introduction

Basing on the paper [1], it was proposed in [2] the following model describing a nonlinear elastic medium with internal inclusions:

\[ u_t + \frac{1}{\nu + 2} \partial_x \left( \beta + \sigma \partial_x^2 \right) \rho^{\nu+2} = 0, \]
\[ \rho_t + \rho^2 u_x = 0, \]

where \( \beta > 0, \sigma \neq 0, \nu > -1 \). In the paper [2] traveling wave (TW) solutions satisfying [1]-[2] are investigated and conditions are formulated under which soliton-type solutions exist in this family. Depending on sign of the parameter \( \sigma \), the solitary wave solutions describe the waves of compression (when \( \sigma > 0 \)) or the waves of rarefaction (corresponding to \( \sigma < 0 \)). In the same paper, the stability of soliton-like solutions are investigated using a numerical analysis of the Evans function [3, 4, 5]. Unfortunately, with the help of numerical studies it is impossible to obtain stability criteria and their dependence on the model parameters. The possibility of rigorous studies of the stability of solutions to this system appears in connection with the presence of a Hamiltonian representation. The rigorous studies
of stability properties of TW solutions to various nonlinear models have been carried out in papers [6, 7, 8, 9, 10, 11] in which some results are formulated concerning the general properties of spectral operators that make it possible to estimate the number of eigenvalues responsible for the instability. In some cases, it is possible to completely eliminate the presence of eigenvalues associated with unstable modes, by investigating function of the spectral parameter put forward by Evans [3, 4] and bearing his name. In general case, this function can only be obtained numerically, but for our purposes it is sufficient to study its asymptotic properties, as well as the behavior at the origin, which can be done analytically. Within the framework of this approach, we succeeded in obtaining rigorous restrictions on parameters the fulfillment of which guarantees the spectral stability of soliton-like TW solutions. The structure of this work is following. In section 2 we pass from the system (1)-(2) to another equivalent system having nice Hamiltonian representation and state the conditions assuring the existence of soliton-like TW solutions. In the first part of the section 3 we perform the stability test and specify the range of parameters for which the soliton-like solutions rather cannot be stable. In further parts of this section we concentrate on the spectral stability which is studied by means of consideration of the operator of linearization about the traveling wave solution. Using the approach based on the Sturm-like theorems, we first estimate the maximal number of unstable modes and next, in the final part, pose the conditions, which assure their absence. In this part we use the arguments based on the analysis of the multisymplectic representation of Hamiltonian system, which is rather cumbersome. So in order not to clutter the main text, we include some technical statements in Appendices A and B.

2. Hamiltonian representation and soliton-like solutions

Let us consider the following substitution

\[ u = (\gamma - \kappa \partial_x^2) w, \quad \eta = \frac{1}{\rho}, \]

where \( \gamma = \beta/(\nu + 2) > 0, \kappa = -\sigma/(\nu + 2) > 0 \). Inserting it into (1)-(2), we get the system

\[
\begin{align*}
-\frac{1}{\eta^2} \{ \eta_t - (\gamma - \kappa \partial_x^2) w_x \} &= 0, \\
(\gamma - \kappa \partial_x^2) \{ w_t + \partial_x \eta^{-(\nu+2)} \} &= 0.
\end{align*}
\]

Under the above assumption the operator \( \mathcal{T} = \gamma - \kappa \partial_x^2 \) is invertible, so we can rewrite the system in the following equivalent form

\[
\begin{align*}
w_t &= -\partial_x \eta^{-(\nu+2)}, \quad (3) \\
\eta_t &= (\gamma - \kappa \partial_x^2) w_x. \quad (4)
\end{align*}
\]

A direct verification shows that the following assertion holds:

**Lemma 1.** The system (3)-(4) allows the Hamiltonian representation

\[ U_t = \partial_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \delta H = J \cdot \delta H, \]

(5)
where \( U = (w, \eta)^{tr} \),
\[
H = \int_{-\infty}^{+\infty} \left\{ \left[ \gamma w^2 + \kappa w_x^2 \right] + \int_{\eta}^{\eta_\infty} [p(\xi) - p(\eta_\infty)] \, d\xi \right\} \, dx,
\]
p(\xi) = 1/\xi^{\nu+2}, \eta_\infty = \lim_{|x| \to \infty} \eta(t, x).

In the sequel we will be interested in a family of the TW solutions \( w = w_s(z), \eta = \eta_s(z) \), where \( z = x - st \), so it is instructive to rewrite the system (5) in the traveling wave variables \( \bar{t} = t, \bar{z} = x - st \), in which it will take the following form:
\[
U_{\bar{t}} = \partial_{\bar{z}} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \delta(H + s Q),
\]
where
\[
Q = \int_{-\infty}^{+\infty} w(\eta - \eta_\infty) \, dz
\]
is the generalized momentum (we omit bars over the independent variables in what follows).

**Corollary 2.** The TW solutions \( U^s(z) = (w_s(z), \eta_s(z))^{tr} \) satisfy the variational equation
\[
\partial_{\bar{z}} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \delta(H + s Q)|_{w_s(z), \eta_s(z)} = 0.
\]

Now we are going to formulate the conditions which guarantee the existence of homoclinic solutions representing the solitary waves. The variational equation (7) can be rewritten as follows:
\[
\partial_{\bar{z}} \left\{ \left( \gamma - \kappa \partial_{zz} \right) w_s + \eta_s - \eta_\infty \right\} = 0,
\]
\[
\partial_{\bar{z}} \left\{ s w_s - \eta_s^{-(\nu+2)} + \eta_\infty^{-(\nu+2)} \right\} = 0.
\]

Integrating both of the equations from \( -\infty \) to \( z \) and taking into account the asymptotics
\[
\lim_{|z| \to \infty} \eta_s(z) = \eta_\infty, \quad \lim_{|z| \to \infty} w_s(z) = 0,
\]
we get the system
\[
s w_s + \eta_\infty^{-(\nu+2)} - \eta_s^{-(\nu+2)} = 0,
\]
\[
(\gamma - \kappa \partial_{zz}) w_s + s (\eta_s - \eta_\infty) = 0.
\]

Next, excluding the variable \( w_s \) from the second equation, introducing the function \( \theta = \eta_s' \), and using the integrating factor \( \varphi = \eta_s^{-(\nu+3)} \) we get the Hamiltonian system
\[
\begin{align*}
\frac{d}{dT} \eta_s &= \theta \kappa (\nu + 2) \varphi^2 = \mathcal{H}_\theta, \\
\frac{d}{dT} \theta &= \varphi \left\{ \kappa (\nu + 2)(\nu + 3)\theta^2 \eta_s^{-(\nu+4)} - \\
&\quad - \left[ s^2 (\eta_s - \eta_\infty) + \gamma \left( \frac{1}{\eta_s^{\nu+2}} - \frac{1}{\eta_\infty^{\nu+2}} \right) \right] \right\} = -\mathcal{H}_{\eta_s},
\end{align*}
\]
where \( \frac{d}{dT} = \kappa (\nu + 2) \varphi^2 \frac{d}{dz} \), \( \mathcal{H} = E_k(\eta_s, \theta) + V(\eta_s) \), the term
\[
E_k(\eta_s, \theta) = \frac{\kappa}{2} (\nu + 2) \eta^{-2(\nu+3)} \theta^2,
\]
and
is associated with the kinetic energy, while the term

\[ V(\eta_s) = s^2 \left[ \frac{\eta_\infty}{(\nu + 2)\eta_s^{\nu+2}} - \frac{1}{(\nu + 1)\eta_s^{\nu+1}} \right] + \gamma \left[ \frac{1}{(\nu + 2)(\eta_s\eta_\infty)^{\nu+2}} - \frac{1}{2(\nu + 2)\eta_s^{2(\nu+2)}} \right] \]

with the potential energy.

Using the well-known properties of two-dimensional Hamiltonian systems [12], we can perform exhaustible qualitative analysis of the system [13]. It is seen that all the stationary points of this system are placed on the horizontal axis. Non-zero components of coordinates of the stationary points satisfy the equation

\[ s^2 (\eta_\infty - \eta) = \gamma \left( \frac{1}{\eta^{\nu+2}} - \frac{1}{\eta_\infty^{\nu+2}} \right). \]  

Eq. (14) is evidently satisfied by \( \eta = \eta_\infty \). We are looking for a soliton solution which corresponds to the trajectory of a dynamical system that is doubly asymptotic to a saddle point. On virtue of conditions [10], the required stationary point is just the point \((\eta_\infty, 0)\). In order that this point be a saddle, it is necessary that the eigenvalues of the Jacobi matrix

\[ \mathcal{P}|_{\eta=\eta_\infty, \theta=0} = \begin{pmatrix} 0 & \varphi(\eta_\infty) \kappa(\nu + 2) \eta_\infty^{-(\nu+3)} \\ -\varphi(\eta_\infty) \left[ s^2 - \gamma (\nu + 2) \eta_\infty^{-(\nu+3)} \right] & 0 \end{pmatrix}, \]

be real numbers of different signs. This condition will be fulfilled if

\[ s^2 < \gamma (\nu + 2) \eta_\infty^{-(\nu+3)}. \]  

![Graphical solution of the equation (14).](image)

When (16) takes place, the configuration shown in Fig. 1 arises. It is also seen that another solution \( \eta_1 > \eta_\infty \) exists, for which the inequality

\[ s^2 > \gamma (\nu + 2) \eta_1^{-(\nu+3)} \]
takes place. The eigenvalues of the Jacobi matrix \( P \) in the stationary point \((\eta_1, 0)\) are pure imaginary, so it is a center.

An extra condition, which, together with \( (16) \), guarantees the existence of the homoclinic loop is connected with the general features of two-dimensional Hamiltonian systems. As is well-known \(^\text{[12]}\), the Hamiltonian \( H(\eta, \theta) \) retains a constant value on the phase trajectories of the system \( (13) \). The potential energy of the system \( (13) \) has exactly two local extrema, namely, the local maximum at the point \( \eta_{\infty} \) and the local minimum at the point \( \eta_1 \). It is easy to check that \( \lim_{\eta \to +\infty} V(\eta) = 0 \) and, depending on the values of the parameters, two distinct configurations occur. If \( V(\eta_{\infty}) \geq 0 \), then the level line passing through the point of local maximum unlimitedly extends to the right without intersecting the graph of the function \( V(\eta) \) (see Fig. 2, left panel). In this case the saddle separatrices do not overlap, and the region of the phase plane \((\eta, \theta)\) bounded by these separatrices is filled with the periodic trajectories spreading up to infinity. If \( V(\eta_{\infty}) < 0 \), then the level line passing through the point of local maximum intersects the graph of the function \( V(\eta) \) at a point \( \eta_s, \eta_1 < \eta_s < \infty \) (see Fig. 3).

![Figure 2](image1.png)

Figure 2: Graph of potential energy \( V(\eta) \), case \( V(\eta_{\infty}) > 0 \) (left panel) and the corresponding phase portrait (right panel). All the trajectories shown represent the periodic solutions.

![Figure 3](image2.png)

Figure 3: Graph of potential energy \( V(\eta) \), case \( V(\eta_{\infty}) < 0 \) (left panel) and the corresponding phase portrait (right panel). Dashed line corresponds to the homoclinic loop; solid lines represent the periodic solutions.

The phase trajectory cannot have a coordinate \( \eta \) greater than \( \eta_s \), so at the point \((\eta_s, 0)\) the trajectory incoming from the saddle point \((\eta_{\infty}, 0)\) is reflected. Since the Hamiltonian
function is not changed under the replacement \( \theta \) by \( -\theta \), the straight and reflected trajectories are symmetric with respect to the horizontal axis and form a single homoclinic trajectory bi-asymptotic to the saddle. Condition \( V(\eta_{\infty}) < 0 \), together with the condition which guarantees that the stationary point \((\eta_{\infty}, 0)\) is a saddle, forms the pair of inequalities

\[
\frac{\beta(\nu + 1)}{2(\nu + 2)\eta_{\infty}^{\nu+3}} < s^2 < \frac{\beta}{\eta_{\infty}^{\nu+3}} \tag{17}
\]

assuring the presence of soliton-like regimes in the set of TW solutions.

**Remark 1.** If we introduce the parameter \( \eta_{0,\infty} = s^{2/(\nu+3)} \eta_{\infty} \), then the velocity \( s \) will be eliminated from (17) which acquires the following form

\[
\frac{\beta(\nu + 1)}{2(\nu + 2)} < \eta_{0,\infty}^{\nu+3} < \beta. \tag{18}
\]

In what follows we treat the parameter \( \eta_{0,\infty} \) as independent of \( s \).

To conclude, let us note that the conditions presented in (17) coincide with those obtained in paper \[2\] after the performance of substitution \( R_1 = \eta_{\infty}^{-1} \). Let us note, that exhaustive qualitative analysis of the system equivalent to (13) over a wide range of parameters values is done in the paper \[13\].

3. **Spectral stability of the soliton-like solutions**

3.1. **Stability test**

Now that we have defined the conditions of existence of homoclinic loops among the set of TW solutions, it is natural to ask the question of the stability of these solutions corresponding to the solitary waves. Here it is appropriate to note, that the system (11)-(12) describing the soliton-like TW solutions \( w_s(z), \eta_s(z) \) can be represented in the variational form

\[
\delta(H + sQ)|_{w_s(z),\eta_s(z)} = 0. \tag{19}
\]

Among the many formulations of the stability conditions for solutions of PDEs, there is also a formulation concerning the solutions of variational problems \[5\], to which, as we have seen, belong the soliton solutions \( w_s(z), \eta_s(z) \). While the necessary condition for \( \Lambda = H + sQ \) to attain the minimum on the soliton solutions takes the form of equation (19), the sufficient condition is formulated in terms of the positiveness of the second variation of the corresponding functional, which, in turn, guarantees the so called *orbital stability* of the TW solutions \[3\]. Here we do not touch upon the problem of strict estimating the signs of the second variation of \( \Lambda \). Instead of this, we follow the approach suggested in \[14, 15, 16\], which enables to test a mere possibility of the local minimum appearance on a selected sets of perturbations of TW solutions. Based on this test, the implementation of which, as a rule, is simple, one cannot, of course, make statements about the stability of the solution being studied. However, as it was shown on the set of examples \[14, 15, 16\], with its help it is possible to specify ranges of values of parameters for which achievement of the minimum of the functional is impossible.
Let us consider the following family of perturbations

\[ w_s(z) \rightarrow \lambda^\alpha w_s(z), \quad \eta_s(z) \rightarrow \lambda^\theta \eta_s(z), \quad \eta_\infty \rightarrow \lambda^\theta \eta_\infty. \tag{20} \]

Choosing \( \theta = -\alpha \), we get

\[ Q[\lambda] = \lambda^{\alpha + \theta} \int_{-\infty}^{+\infty} [w_s(z) (\eta_s(z) - \eta_\infty)] \, dz = Q[1]. \tag{21} \]

Thus, under this choice \( Q[\lambda] \) keeps its unperturbed value. Imposing this condition, we reject "fake" perturbations, associated with the translational symmetry \( T_\delta U_s(z) = U_s(z)(z + \delta) \), where \( U_s(z) = (w_s(z), \eta_s(z)) \).

Thus, we get the following functions to be tested:

\[ \Lambda[\lambda] = (H + c Q)[\lambda] = \lambda^{2\alpha} J + \lambda^{\alpha(\nu+1)} J + s Q[1], \tag{22} \]

where

\[ J = \frac{1}{2} \int_{-\infty}^{+\infty} \left\{ \gamma w_s^2(z) + \kappa [w_s'(z)]^2 \right\} \, dz, \]

\[ J = \int_{-\infty}^{+\infty} \left\{ \frac{1}{\nu + 1} \eta_s(z) \left[ \eta_s^{-\nu+2}(z) + (\nu + 1) \eta_\infty^{-\nu+2} \right] - \frac{\nu + 2}{\nu + 1} \eta_s^{-\nu+1}(z) \right\} \, dz. \]

Let us note, that for \( \nu > -2 \) the functional \( J \) is positive. If the functional \( \Lambda = H + c Q \) attains the extremal value on the soliton solution, then the function \( \Lambda[\lambda] \) has the corresponding extremum in the point \( \lambda = 1 \). The verification of this property is used as a test.

A necessary condition of the extremum

\[ \frac{d}{d\lambda} \Lambda[\lambda] \bigg|_{\lambda=1} = 0 \]

gives us the equality

\[ J = -\frac{2}{\nu + 1} J. \tag{23} \]

The sufficient condition

\[ \frac{d^2}{d\lambda^2} \Lambda[\lambda] \bigg|_{\lambda=1} > 0, \]

together with the equation (23), gives us the inequality

\[ 2 \alpha^2 (1 - \nu) J > 0, \tag{24} \]

which is fulfilled providing that \(-2 < \nu < 1\). It follows from the results of the test, that the stability of the solitary wave arising in the model considered can take place at moderate values of the non-linearity of the equation of state [2] connecting the pressure with the average density of the medium.
3.2. Spectral stability: obtaining restrictions on the number of unstable modes

In order to study the stability of solitary wave solutions, the following set of perturbations is considered:

\[
\begin{pmatrix}
  w(t, z) \\
  \eta(t, z)
\end{pmatrix} = \begin{pmatrix}
  w_s(z) \\
  \eta_s(z)
\end{pmatrix} + \varepsilon e^{\lambda t} \begin{pmatrix}
  M(z) \\
  N(z)
\end{pmatrix}.
\]

(25)

Inserting (25) into (6), we get, up to \(O(\varepsilon^2)\) the eigenvalue problem

\[
\lambda \begin{pmatrix}
  M(z) \\
  N(z)
\end{pmatrix} = J \mathcal{L}^s \begin{pmatrix}
  M(z) \\
  N(z)
\end{pmatrix} =: \mathcal{L} \begin{pmatrix}
  M(z) \\
  N(z)
\end{pmatrix},
\]

(26)

where

\[
\mathcal{L}^s = \delta^2 (H + s Q) \big|_{w_s, \eta_s} = \begin{pmatrix}
  \gamma - \kappa \partial_z^2 + \frac{s}{s + \nu + 3} \\
  \nu + 2
\end{pmatrix}.
\]

(27)

We denote the spectrum of the operator \(\mathcal{L}\) by \(\sigma(\mathcal{L})\) and accept the following definition:

**Definition 1.** The soliton-like solution \(U_s(z) = (w_s(z), \eta_s(z))^{tr}\) is said to be spectrally stable if the intersection of \(\sigma(\mathcal{L})\) with the positive half-plane \(\mathbb{C}^+\) of the complex plane is empty.

In this section the following assertion will be proven:

**Theorem 3.** The set \(\sigma(\mathcal{L}) \cap \mathbb{C}^+\) consists at most of one isolated point \(\lambda_0\). If \(\sigma(\mathcal{L}) \cap \mathbb{C}^+\) is nonempty, then \(\lambda_0\) is a real positive number.

The proof of this theorem is based on a number of auxiliary statements. Some of them are sufficiently general and applicable to a wide class of spectral problems. To begin with, let us localize the essential spectrum \(\sigma_{ess}(\mathcal{L})\). In the case under consideration it coincides with the spectrum of the limiting operator \[17, 5\]

\[
\mathcal{L}_\infty = \mathcal{L}_{\pm \infty} = \lim_{|z| \to \infty} J \cdot \mathcal{L}^s = J \cdot \begin{pmatrix}
  \gamma - \kappa \partial_z^2 + \frac{s}{s + \nu + 3} \\
  \nu + 2
\end{pmatrix}.
\]

The spectrum of the differential operator \(\mathcal{L}_\infty\) having the constant coefficients is determined with the help of the Fourier transformation. It coincides with the set

\[
\sigma_{ess}(\mathcal{L}) = \left\{ \lambda \in \mathbb{C} : \det \begin{pmatrix}
  -i \xi s - \lambda & -i \xi (\nu + 2) / \eta^3 \\
  -i \xi (\gamma + \kappa \xi^2) & -i \xi s - \lambda
\end{pmatrix} = 0, \xi \in \mathbb{R}\right\}.
\]

The set of possible values of the spectral parameter \(\lambda\) is given by the formula

\[
\lambda = -i \xi s \pm i \sqrt{\xi^2 (\nu + 2) (\gamma + \kappa \xi^2) / \eta^3}, \quad \xi \in \mathbb{R}.
\]

It coincides with the imaginary axis. Next, the following general statement is applied to our problem (cf with \[5\]).

**Lemma 4.** The point spectrum \(\sigma_{pt}(\mathcal{L})\) is symmetric with respect to the coordinate axes, that is, if \(\lambda \in \sigma_{pt}(\mathcal{L})\), then simultaneously \(-\lambda,\) and \(\pm \lambda\) belong to the point spectrum of the operator \(\mathcal{L}\).
Proof. Suppose that $\lambda \in \sigma_{pt}(L)$, $\psi \in L^2(\mathbb{R})$ is the eigenvector corresponding to $\lambda$. Then

$$\bar{L}\bar{\psi} = L\bar{\psi} = \bar{\lambda}\bar{\psi},$$

hence $\lambda \in \sigma_{pt}(L)$ implies $\bar{\lambda} \in \sigma_{pt}(L)$. Next, if $L\psi = J \cdot L^s \psi = \lambda \psi$, then

$$(\psi|J \cdot L^s \psi) = (\psi|\lambda \psi) = \lambda (\psi|\psi) = (\bar{\lambda} \psi|\psi).$$

On the other hand,

$$(\psi|J \cdot L^s \psi) = -(J \psi|L^s \psi) = -(L^s \cdot J \psi|\psi),$$

hence $L^s \cdot J \psi = -\bar{\lambda} \psi$, which implies the equality $L^s (J \psi) = -\bar{\lambda} (J \psi)$. Thus, $\lambda \in \sigma_{pt}(L)$ implies $-\bar{\lambda} \in \sigma_{pt}(L)$ and similarly $\bar{\lambda} \in \sigma_{pt}(L)$ implies $-\lambda \in \sigma_{pt}(L)$.

Next statement, borrowed from the paper \[7\] is the following.

**Theorem 5.** Suppose that $J$ is a skew-symmetric operator while $L^s$ is self-adjoint. Suppose in addition that $L^s$ has exactly $k$ strictly negative eigenvalues, counting multiplicities and $k < \infty$. Then $L = J \cdot L^s$ has at most $k$ eigenvalues in the right half-plane of the complex plane.

So, all the auxiliary assertions needed have been formulated, and we can now concentrate on estimating the number of discrete eigenvalues of the operator $L^s$ lying on the negative semiaxis $\mathbb{R}^-$. Thus, we consider the spectral problem $L^s (M, N)^{tr} = \mu (M, N)^{tr}$, which can be presented as follows:

$$\begin{cases}
(\gamma - \kappa \partial_z^2) M + s N = \mu M, \\
 s M + \frac{\nu + 2}{\eta^2} N = \mu N.
\end{cases} \quad (28)$$

If we put in the above equations $\mu = 0$ and take the derivative with respect to $z$, then we obtain, up to the notation, the system \[8\]-\[9\]. Hence the following statement is true:

**Lemma 6.** $U'_s = (w'_s, \eta'_s)^{tr}$ is the eigenvector of the operator $L^s$, corresponding to the eigenvalue $\mu = 0$.

Now, using the second equation of the system \[28\], we can express the function $N$ as follows:

$$N = s M \left( \mu - \frac{\nu + 2}{\eta^2 + 3} \right)^{-1}. \quad (29)$$

Inserting \[29\] in the first equation of the system \[28\], we get the following generalized eigenvalue problem:

$$\kappa \frac{d^2}{dz^2} M = \left[ \gamma - \mu + \frac{s^2}{\mu - \frac{\nu + 2}{\eta^2 + 3}} \right]. \quad (30)$$

From the lemma \[6\] we immediately obtain the following:

**Corollary 7.** Function $M(z) = w'_s(z)$ is the eigenvector of the generalized spectral problem \[30\] corresponding to the eigenvalue $\mu = 0$. 

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Now, let us consider the wronskian

$$W(z) = M'_1(z) M_2(z) - M'_2(z) M_1(z),$$  
(31)

where $$\{M_i\}_{i=1}^2$$ are solutions of the equation (30) corresponding to the eigenvalues $$\mu_i, i=1,2.$$ on the connected set $$I \in \mathbb{R}$$ (finite or infinite). Taking the derivative of (31) with respect to $$z$$ and then integrating the expression obtained we get

$$W(\xi) |_{\xi=a} = \int_a^z W'(\xi) d\xi,$$

which after some manipulation attains the form

$$W(z) - W(a) = \frac{\mu_2 - \mu_1}{\kappa} \int_a^z M_1(\xi) M_2(\xi) \Phi(\xi) d\xi,$$

where

$$\Phi(\xi) = 1 + \frac{s^2 \eta_\kappa^{2(\nu+3)}(\xi)}{[(\nu+2) - \mu_1 \eta_\kappa^{\nu+3}(\xi)] \cdot [(\nu+2) - \mu_2 \eta_\kappa^{\nu+3}(\xi)].$$

Let us note, that $$\Phi(\xi) > 0$$ when $$\mu_i$$ are non-positive.

**Lemma 8.** Let us assume that $$\mu_1 < \mu_2 \leq 0$$ are eigenvalues while $$M_1(z), M_2(z)$$ are the corresponding eigenfunctions of the generalized spectral problem (30) and the following conditions hold:

- $$\lim_{z \to a+0} M_1(z) = \lim_{z \to a+0} M_2(z) = 0;$$
- $$M_2|_{(a,b)} > 0; \quad \exists \epsilon > 0 : M_1|_{(a,a+\epsilon)} > 0.$$  

Then $$M_1 |_{(a,b)} > 0.$$ If in addition the conditions

- $$M_2(b) = 0, \quad M'_2(b) < 0$$

are fulfilled, then $$M_1(b) > 0.$$  

**Proof.** The proof of the first part: let there exists $$c \in (a,b)$$ such that $$M_1(c) = 0$$ and $$M'_1(c) < 0$$ (we’ll assume that $$c$$ is the first point at which $$M_1$$ intersects the horizontal axis). Then we can conclude that the function (32) is growing and non-negative on $$(a,c).$$ On the other hand, under the above assumption, $$W(c) = M'_1(c) M_2(c) < 0.$$ The contradiction obtained proves the first statement. Further, using the additional assumptions we conclude from (32) that $$W(b) > 0.$$ Simultaneously, from the formula (31) appears that $$W(b) = -M'_2(b) M_1(b).$$ But the r.h.s of this formula is positive if and only if $$M_1(b) > 0.$$

**Lemma 9.** We use the same assumptions as in the lemma 8. In addition, we assume that

- $$M_2(b) = 0; \quad M'_2(b) < 0; \quad \exists d > b : M_2|_{(b,d)} < 0;$$
- $$\lim_{z \to d-0} M_1(z) = \lim_{z \to d-0} M_2(z) = 0.$$  

Then $$M_1 |_{(b,d)} > 0.$$

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Proof. The lemma is proved by contradiction. Assume that \( M_1(z) \) intersects the horizontal axis \( OZ \) for the first time at some point \( c \in (b, d) \). Then \( M_1(c) = 0, \ M_1'(c) < 0 \), hence \( W(c) = M_1'(c)M_2(c) - M_2'(c)M_1(c) > 0 \). Let us make at first an additional assumption that \( M_1 \) does not have intersections with the horizontal axis on the segment \((c, d)\). Then we get

\[
W(d) = W(c) + \int_c^d M_1(\xi) M_2(\xi) \Phi(\xi) \, d\xi > 0.
\]

On the other hand, \( M_1'(d)M_2(d) - M_2'(d)M_1(d) = 0 \), so we get the contradiction. Now let us assume that there exists \( f \in (c, d) \) such that \( M_1(f) = 0 \) and \( M_1'(f) > 0 \). Then

\[
W(f) = W(c) + \int_c^f M_1(\xi) M_2(\xi) \Phi(\xi) \, d\xi > 0.
\]

On the other hand, \( W(f) = M_1'(f)M_2(f) - M_2'(f)M_1(f) < 0 \). The contradiction obtained ends the proof.

Lemma 10. Suppose that the spectral problem (30) has three discrete eigenvalues \( \mu_0 < \mu_1 < \mu_2 \leq 0 \), and the corresponding eigenfunctions \( M_0(z), \ M_1(z), \ M_2(z) \) are defined on \((a, d)\). We assume in addition that

- \( \lim_{z \to a+0} M_i(z) = 0, \ i = 0, 1, 2; \)
- there exists \( b \in (a, d) \) such that \( M_2(b) = 0, \ M_2'(b) < 0 \), and \( M_2(z) \) does not have another points of intersection with the horizontal axis on the segment \((a, d)\).

Then there does not exist the eigenfunction \( M_0(z) \), not identically equal to zero corresponding to the eigenvalue \( \mu_0 \).

Proof. Without the loss of generality, we can assume that \( M_1|_{(a, d)} > 0 \) and \( M_2|_{(a, b)} > 0 \) (on virtue of lemmas 8, 9 \( M_1(z) \) does not intersect the horizontal axis on \((a, d)\)). Now assume that \( M_0(z) \) is not identically zero on \((a, d)\). Then, in accordance with the lemma 8 \( M_0(z) \) does not intersect the horizontal axis on this segment (we compare \( M_0 \) with the function \( M_2 \)) and we can assume in addition that \( M_0|_{(a,d)} > 0 \). On virtue of the above assumptions, the function

\[
W(z) = \frac{\mu_1 - \mu_0}{\kappa} \int_a^z M_0(\xi) M_1(\xi) \Phi(\xi) \, d\xi
\]

is growing and non-negative on the segment \((a, d)\), hence \( W(d) > 0 \). But on the other hand, \( W(d) = M_0'(d)M_1(d) - M_1'(d)M_0(f) = 0 \), so we have a contradiction.

Corollary 11. The following assertions are true:

- The eigenvalue problem (30) has at most one discrete eigenvalue \( \mu < 0 \), corresponding to the nonzero eigenfunction \( M(z) \).
- If such an eigenvalue does exist, then it is simultaneously the discrete eigenvalue of the operator \( \mathcal{L}^s \), corresponding to the eigenfunction

\[
\left\{ M(z), \ s \, M(z) \left( \mu - \frac{\nu + 2}{\eta(z)\nu+3} \right)^{-1} \right\}^{tr}.
\]

- The operator \( \mathcal{L} = J \cdot \mathcal{L}^s \) has at most one discrete eigenvalue lying in \( \mathbb{C}^+ \).
- If such an eigenvalue does exist, then it belongs to the positive semiaxis \( \mathbb{R}^+ \).
3.3. The Evans function and spectral stability

3.3.1. Introductory remarks

We proceed to the formulation of conditions that make it possible to exclude the existence of the discrete eigenvalues of the operator $L$ belonging to $\mathbb{C}^+$. For this purpose, we use a technique based on some properties of the Evans function \[3, 4, 5\] - an analytic function of the spectral parameter $\lambda \in \mathbb{C}^+$ that nullifies on those values of the parameter $\lambda$ which belong to the set $\sigma_{pt}(L) \cap \mathbb{C}^+$. Usually, $E(\lambda)$ is defined as the determinant of the Wronskian constructed on the solutions of a dynamical system equivalent to the corresponding spectral problem. The Evans function most often is found numerically, but some of its asymptotic properties (essentially used in this paper) can be obtained analytically. To begin with, let us note that, since $\lambda = 0$ is an eigenvalue of the operator $L$, then $E(0) = 0$. Next, differentiating the equation

$$J \cdot \delta (H + s Q) \ U^s = 0,$$

where $U_s = (w_s, \eta_s)^{tr}$, with respect to $s$, we obtain

$$J \cdot \delta^2 (H + s Q) \frac{\partial U_s}{\partial s} + J \cdot \delta Q|_{w_s, \eta_s} = L(\frac{\partial U_s}{\partial s}) + U'_s.$$

From this we conclude that $\frac{\partial U_s}{\partial s}$ is a generalized eigenfunction of $L$ corresponding to zero eigenvalue, which, in turn, implies the equality $E'(0) = 0$. If we were able to estimate the sign of $E''(0) \neq 0$, and also the sign of $E(+\infty)$ (the latter can be done by several standard methods), then from the equality

$$\text{sign}E(+\infty) \cdot \text{sign}E''(0) = +1, \tag{33}$$

it would follow that the number of intersections of the graph of the function $E(\lambda), \ \lambda \in \mathbb{R}^+$ of the horizontal axis $\text{Re}(\lambda)$ should be even. However, since this contradicts the results obtained above (see the corollary at the end of the previous subsection), then there would be no intersections in this case at all. On the contrary, the negativity of the product appearing in the formula (33) indicates the existence of an unstable mode.

3.3.2. Multi-symplectic representation and evaluation of the sign $E''(0)$

In the evaluation of the sign of $E''(0)$ we follow the papers \[8, 9\]. From there, most of the designations are borrowed. The main formula is based on the theory of multi-symplectic systems. We will not here state this rather cumbersome theory completely, but only concentrate on those fragments that are necessary for deriving the basic formula. Thus, first of all, the Hamiltonian system must be written in the equivalent multi-symplectic form

$$\tilde{M} Z_t + \tilde{K} Z_x = \nabla S(Z), \tag{34}$$

where $Z \in \mathbb{R}^{2n}$, $\tilde{M}, \tilde{K}$ are $2n \times 2n$ skew-symmetric constant matrices, $S(Z)$ is a smooth function and $\nabla$ is the gradient in $\mathbb{R}^{2n}$. The matrices $\tilde{M}, \tilde{K}$ generate in $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ two-forms

$$\omega(\zeta_1, \zeta_2) = (\tilde{M}\zeta_1, \zeta_2), \quad k(\zeta_1, \zeta_2) = (\tilde{K}\zeta_1, \zeta_2)$$

and

$$\Omega(\zeta_1, \zeta_2) = (\tilde{J}_s\zeta_1, \zeta_2),$$
where $\hat{J}_s = \hat{K} - s \hat{M}$. It is assumed that $\det \hat{J}_s \neq 0$ and hence the form $\Omega$ is not degenerate. In the multi-symplectic approach the function $\tilde{Z}(z; a, b)$ is considered, describing the shape of multiparameter family of solitary waves and satisfying the dynamical system

$$\hat{J}_s \tilde{Z}' = \nabla V(\tilde{Z}),$$

where $V(\cdot)$ is $S(\cdot)$ plus additional addends arising from symmetry (we'll give the precise definition of them when addressing the system (3)-(4)). The linearization $U(z)$ about the solitary wave solutions satisfies the dynamical system

$$U'(z) = A(z, \lambda, a, b, s)U(z), \quad U \in C^2$$

where $\lambda \in \mathbb{C}$ is the spectral parameter,

$$A(z, \lambda, a, b, s) = \hat{J}_s^{-1} \left\{ D^2V \left( \tilde{Z}(x; a, b, s) - \lambda \hat{M} \right) - \lambda \hat{M} \right\}.$$ 

It can be shown that the shape function $\tilde{Z}(z; a, b, s)$ satisfied the variational equation

$$\frac{\delta}{\delta \tilde{Z}} \left( H(\tilde{Z}) - s I(\tilde{Z}) \right) = 0,$$

where

$$H(\tilde{Z}) = \frac{1}{2} \int_{-\infty}^{+\infty} \left[ k(\tilde{Z}, \tilde{Z}') + 2 V(\tilde{Z}) \right] dz$$

is the Hamiltonian function, while

$$I(\tilde{Z}) = \frac{1}{2} \int_{-\infty}^{+\infty} \omega(\tilde{Z}, \tilde{Z}') dz$$

is the generalized momentum. In this notation the sign of $E''(0)$ is expressed as follows:

$$\text{sgn} \ E''(0) = \zeta_{60} \left[ \frac{dI}{ds} - B(s) \right],$$

where $\zeta_{60}$ and $B(s)$ are expressed in terms of the combination of the boundary vectors $Z_0^{-}(a, b) \lim_{z \to -\infty} \tilde{Z}(z; a, b, s)$ and $Z_0^{+}(a, b, s) \lim_{z \to +\infty} \tilde{Z}(z; a, b, s)$. The multi-symplectic formalism is described below with reference to the system under study.

3.3.3. Multi-symplectic representation of the system (3)-(4) and evaluation of the sign $E''(0)$

In order to take advantage of the formalism proposed in [8, 9], we should write down the initial system in the multi-symplectic form. Introducing new functions

$$q = \eta^{-\nu+2}, \quad \Phi_x = q^{-\nu+2}, \quad v = w_x, \quad r_x = w - C_0, \quad p = -\Phi_t + \gamma w - \kappa v_x,$$

we can rewrite (3)-(4) as the first-order system

$$-\Phi_t - \kappa v_x = p - \gamma w,$$

$$\Phi_x = q^{-\nu+2},$$

(37)
\[
\begin{align*}
  w_x &= v, \\
  w_t + q_x &= 0, \\
  p_x &= 0, \\
  r_x &= w - C_0.
\end{align*}
\] (39) (40) (41) (42)

The multi-symplectic form of the system \((37)-(42)\) is as follows
\[
\dot{M} Z_t + \dot{K} Z_x = \nabla S,
\] (43)

where \(Z = (w, q, v, \Phi, r, p)^{tr}\),
\[
\dot{M} = \begin{pmatrix}
  0 & 0 & 0 & -1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\quad \dot{K} = \begin{pmatrix}
  0 & 0 & -\kappa & 0 & 0 & 0 \\
  0 & 0 & 0 & -1 & 0 & 0 \\
  \kappa & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & -1 \\
  0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix},
\]
\[
S = p(w - C_0) - \frac{\gamma}{2} w^2 - \frac{1}{\alpha} q^\alpha + \frac{\kappa}{2} v^2, \quad \alpha = \frac{\nu + 1}{\nu + 2}.
\]

The next step will be the use of symmetry properties for the purpose of constructing a manifold at infinity \(\mathcal{M}(a, b)\). The system \((43)\) is evidently invariant with respect to the translation group \(Z \to Z + \epsilon(0, 0, 0, 1, 0)^{tr}, \epsilon \in \mathbb{R}\) having the generator \(\dot{X} = \partial/\partial \Phi\).

To this symmetry corresponds a pair of functions \([8, 9]\) \(P = -w, Q = -q\) with the properties
\[
\dot{M} \dot{X}(Z) = \nabla P(Z), \quad \dot{K} \dot{X}(Z) = \nabla Q(Z).
\]

In addition, the symmetry of the initial system will be used, which allows one to extend the homoclinic solution to a two-parameter family of analogous solutions. A direct verification shows that the following assertion holds

**Lemma 12.** The system \((3)-(4)\) is invariant with respect to the family of transformations:
\[
\bar{t} = e^\mu t, \quad \bar{x} = x, \quad \bar{w} = e^{-\frac{\mu + 1}{\nu + 3}} w + A, \quad \bar{\eta} = e^{\frac{2}{\nu + 3}} \eta,
\] (44)

where \(\mu, A \in \mathbb{R}\) are arbitrary parameters.

The above symmetry induces the following group of invariance of the system \((8)-(9)\) describing the TW solutions:
\[
\bar{w}(z) = e^{-\frac{\mu + 1}{\nu + 3}} w_s(z) + A, \quad \bar{\eta}(z) = e^{\frac{2}{\nu + 3}} \eta_s(z), \quad \bar{\eta}_\infty = e^{\frac{2}{\nu + 3}} \eta_\infty, \quad \bar{s} = e^{-\mu} s.
\] (45)

Combining the translational symmetry of \((43)\) with the symmetry of the system \((8)-(9)\) which makes it possible to extend the set of homoclinic solutions to a multiparametric family, we can eventually construct a non-degenerate manifold \(\mathcal{M}(a, b)\), which is necessary for analyzing formula \((36)\) in our particular case. The vector-function \(\bar{Z} = (\bar{w}, \bar{q}, \bar{v}, \bar{\Phi}, \bar{r}, \bar{p})^{tr}\) satisfies the following variational equation (cf with \([9]\)):
\[
\left(\dot{K} - \bar{s} \dot{M}\right) \bar{Z}' = \nabla S(\bar{Z}) - a \nabla P(\bar{Z}) - b \nabla Q(\bar{Z}),
\] (46)
which, when written out componentwise, looks as follows

\[ \tilde{s} \tilde{\Phi} \tilde{z} - \kappa \tilde{v} \tilde{z} = \tilde{p} + a - \gamma \tilde{w}, \quad (47) \]

\[ - \tilde{\Phi} \tilde{z} = b - \tilde{q} \tilde{w}^{\frac{1}{2}}, \quad (48) \]

\[ \kappa \tilde{w} \tilde{z} = \kappa \tilde{v}, \quad (49) \]

\[ - \tilde{s} \tilde{w} \tilde{z} + \tilde{q} \tilde{z} = 0, \quad (50) \]

\[ - \tilde{p} \tilde{z} = 0, \quad (51) \]

\[ \tilde{r} \tilde{z} = \tilde{w} - C_0. \quad (52) \]

Integrating equation (48) within the limits \((-\infty, z)\) and using the requirement that the function \(\Phi(z)\) should be bounded on \(\mathbb{R}\), we immediately obtain the condition

\[ e^{\mu} = \left( \frac{b}{\eta_{\infty}} \right)^{\frac{\nu+3}{2}}. \]

Taking this condition into account, we can express \(\Phi\) in the form

\[ \tilde{\Phi}(z) = \frac{b}{\eta_{\infty}} \int_{-\infty}^{z} [\eta_s(\xi) - \eta_{\infty}] \, d\xi + C_1. \quad (53) \]

Similarly, the requirement that the function \(\tilde{r}(z)\) be bounded leads to the condition \(C_0 = A\), under which we get the expression

\[ \tilde{r}(z) = \left( \frac{\eta_{\infty}}{b} \right)^{\frac{\nu+1}{2}} \int_{-\infty}^{z} w_s(\xi) \, d\xi + C_2. \quad (54) \]

The remaining functions are expressed as follows:

\[
\begin{align*}
\tilde{w} &= \left( \frac{\eta_{\infty}}{b} \right)^{\frac{\nu+1}{2}} w_s(z) + a(1 + \gamma^{-1}), \\
\tilde{q} &= \left( \frac{b \eta_s(z)}{\eta_{\infty}} \right)^{-\nu+2}, \\
\tilde{v} &= \left( \frac{\eta_{\infty}}{b} \right)^{\nu+1} w_s'(z), \\
\tilde{p} &= \gamma a,
\end{align*}
\]

and, thus, the vector-valued function \(\tilde{Z}\) is represented in the form

\[
\begin{align*}
\tilde{Z} &= \left( \left( \frac{\eta_{\infty}}{b} \right)^{\frac{\nu+1}{2}} w_s(z) + a(1 + \gamma^{-1}), \left( \frac{b \eta_s(z)}{\eta_{\infty}} \right)^{-\nu+2}, \left( \frac{\eta_{\infty}}{b} \right)^{\frac{\nu+1}{2}} w_s(z)', \theta(z) + C_1, \varphi(z) + C_2, \gamma a \right)^{tr}, \\
\tilde{Z}' &= \left( \left( \frac{\eta_{\infty}}{b} \right)^{\frac{\nu+1}{2}} w_s(z)', -(\nu + 2) \eta_{\infty}^{\nu+2} \eta_s(z)', \left( \frac{\eta_{\infty}}{b} \right)^{\frac{\nu+1}{2}} \eta_s(z)'' \right)^{tr} \\
&\quad \left( \frac{b}{\eta_{\infty}} [\eta_s(z) - \eta_{\infty}], \left( \frac{\eta_{\infty}}{b} \right)^{\frac{\nu+1}{2}} w_s(z), 0 \right)^{tr},
\end{align*}
\]
where
\[
\theta(z) = \frac{b}{\eta_\infty} \int_{-\infty}^{z} \left[ \eta(\xi) - \eta_\infty \right] d\xi, \quad \varphi(z) = \left( \frac{b}{\eta_\infty} \right)^{-\frac{\nu+1}{\nu}} \int_{-\infty}^{z} w_s(\xi) d\xi.
\]  

From the formula (56) there follow such conditions at infinity:
\[
Z^{-}_0 = \lim_{z \to -\infty} \tilde{Z}(z) = (a(1 + \gamma^{-1}), b^{-(\nu+2)}, 0, C_1, C_2, \gamma a)^{tr},
\]
\[
Z^{+}_0 = \lim_{z \to +\infty} \tilde{Z}(z) = (a(1 + \gamma^{-1}), b^{-(\nu+2)}, 0, \theta_\infty + C_1, \varphi_\infty + C_2, \gamma a)^{tr},
\]
where \( \theta_\infty = \lim_{z \to +\infty} \theta(z) \), \( \varphi_\infty = \lim_{z \to +\infty} \varphi(z) \).

Now that almost all the necessary tools have been laid out, we can proceed to an analysis of the sign of the second derivative of the Evans function, which, according to [9], is expressed by the relation
\[
E''(0) = \chi_{00}^- \left( \frac{d}{ds} I(\tilde{Z}) - \omega(Z^{+}_0, \partial_s Z^{+}_0) \right).
\]  

The coefficient \( \chi_{00}^- \) is obtained from the condition for normalizing the eigenvectors of the matrix \( A_\infty = \lim_{z \to +\infty} A(z; a, b, s) \) and the eigenvectors of the adjoining matrix \( A^{*}_\infty \). The computations of this quantity are rather cumbersome, so they are taken to Appendix A, in which the following formula is derived:
\[
\chi_{00}^- = \frac{(s \eta_\infty^{\nu+3})^2}{2 \tilde{C}^2 D^{3/2} (\nu + 2)^2}, \quad D = \frac{\beta - s^2 \eta_\infty^{\nu+3}}{\kappa (\nu + 2)} > 0,
\]  

\( \tilde{C} \) is a positive constant.

Thus, we proceed to calculate the remaining terms appearing in the formula (61). Since we are not interested in the whole extended family (44), but only in the special case of solutions of system (8)-(9) satisfying the asymptotic conditions (10), we carry out calculations for \( a = 0 \) and \( b = \eta_\infty \). The generalized impulse is calculated on the basis of the formula (35):
\[
I(\tilde{Z})|_{b=\eta_\infty} = \int_{-\infty}^{+\infty} w_s(z) [\eta_s(z) - \eta_\infty] \, dz.
\]  

In order to calculate the derivative of the functional \( I(\tilde{Z}) \) with respect to the variable \( s \), we need to obtain the explicit dependence of \( w_s(z) \) and \( \eta_z \) on the velocity. This can be done if we exclude the speed from the system (11)-(12) using the scaling \( w_s(z) = s^\alpha w_0(z) \), \( \eta_s(z) = s^\delta \eta_0(z) \). Indeed, if we put \( \alpha = (\nu + 1)/\nu + 3 \), \( \delta = -2/(\nu + 3) \), then we obtain the system
\[
w_0 - \eta_0^{-(\nu+2)} + \eta_{0,\infty}^{-(\nu+2)} = 0,
\]
\[
(\gamma - \kappa \partial_z^2) w_0 + \eta_0 - \eta_{0,\infty} = 0,
\]
which does not contain the parameter \( s \). Thus we have:
\[
\frac{d}{ds} I(\tilde{Z})|_{b=\eta_\infty} = \frac{\nu - 1}{\nu + 3} s^{-4/(\nu+3)} \int_{-\infty}^{+\infty} w_0(z) [\eta_0(z) - \eta_{0,\infty}] \, dz.
\]  

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Since the homoclinic loop representing the solitary wave solution lies to the right from the saddle point \((\eta_\infty, 0)\) then \(\eta_t(z) - \eta_\infty > 0\), and this induces the inequality \(\eta_0(z) - \eta_{0,\infty} > 0\). The inequality \(w_0(z) < 0\), in turn, appears directly from the equation (64). So the integral in the formula (66) is negative and the whole expression is positive if \(\nu \in (-3, 1)\).

We must also calculate the expression \(\omega(Z_0^+, \partial_s Z_0^+)\), which appears in formula (61). Taking the derivative of (60) with respect to \(s\), we get:

\[
\partial_s Z_0^+ = \left(0, 0, 0, -\frac{2}{\nu + 3} s^{-\frac{\nu + 3}{\nu + 1}} \int_{-\infty}^{+\infty} [\eta_0(z) - \eta_{0,\infty}] dz, \right.
\]

\[
\left. \frac{\nu + 1}{\nu + 3} s^{-\frac{2}{\nu + 1}} \int_{-\infty}^{+\infty} w_0(z) dz, 0 \right). \]

Thus,

\[
\omega(Z_0^+, \partial_s Z_0^+) = -\frac{2}{\nu + 3} a \left(1 + \gamma^{-1}\right) s^{-\frac{\nu + 3}{\nu + 1}} \int_{-\infty}^{+\infty} [\eta_0(z) - \eta_{0,\infty}] dz,
\]

which is zero when \(a = 0\). Combining (66) with (18) and taking into account that the homoclinic loop exists when \(\nu > -1\), we can formulate the following assertion:

**Theorem 13.** The solitary wave solution of the system (3)-(4) moving with velocity \(s > 0\) and having the asymptotics \(\lim_{|x| \to +\infty} w(t, x) = 0\), \(\lim_{|x| \to +\infty} \eta(t, x) = s^{-\frac{2}{\nu + 1}} \eta_{0,\infty} > 0\) is spectrally stable if \(\nu \in (-1, 1)\), and the following inequality holds:

\[
\frac{\beta(\nu + 1)}{2(\nu + 2)} < \eta_{0,\infty}^{\nu + 3} < \beta.
\]

Thus, we established that under the conditions listed above the operator of the spectral problem (26) does not have discrete eigenvalues belonging to \(\mathbb{C}^+\). In connection with this, the question arises: can the result obtained be used to formulate the conditions for the stability of soliton-like TW solutions of the system (1)-(2)? As it was mentioned before, the stability of soliton-like solutions supported by this system was investigated in [2] using the numerical methods which do not give the possibility to obtain any qualitative result. In order to formulate the spectral problem, let us consider the system (1)-(2) written in TW coordinates \(t, z = x - st\):

\[
\begin{align*}
  u_t &= s u_z - \partial_z \left(\gamma - \kappa \partial_z^2\right) \rho^{\nu + 2}, \\
  \rho_t &= s \rho_z - \rho^2 u_z,
\end{align*}
\]

(67)

(68)

where \(\gamma = \beta/(\nu + 2) > 0\), \(\kappa = -\sigma/(\nu + 2) > 0\). In accordance with the work [2], we denote the soliton solutions of the system (4) by the symbols \(u = u_s(z), \rho = R_s(z)\). Inserting the perturbations of the form

\[
u(t, z) = u_s(z) + e^{\lambda t} \tilde{U}(z), \quad \rho(t, z) = R_s(z) + e^{\lambda t} \tilde{\rho}(z)
\]

into (67)-(68) and dropping out the terms of the order \(O(\epsilon^2)\), we get the following spectral problem:

\[
\begin{align*}
  \lambda \tilde{U} &= \partial_z \left[s \tilde{U} - (\gamma - \kappa \partial_z^2) (\nu + 2) R_s^{\nu + 1} \tilde{\rho}\right], \\
  \lambda \tilde{\rho} &= s \partial_z \tilde{\rho} - 2 R_s u_s' \tilde{\rho} - R_s^2 \partial \tilde{U}.
\end{align*}
\]

(69)

Below we’ll show that the following assertion holds:
**Theorem 14.** The point spectra of the problems (69) and (26) are identical.

**Proof.** In analysing the connection between the two spectral problems, we will use the following easily verifiable identities:

\[
\begin{align*}
R_s(z) &= \eta_s^{-1}(z), & u_s(z) &= (\gamma - \kappa \partial_z^2) w_s(z), \\
\hat{\rho}(z) &= -\frac{1}{\eta_s^2} N(z), & U(z) &= (\gamma - \kappa \partial_z^2) M(z).
\end{align*}
\]

(70)

Taking into account the invertibility of the operator \((\gamma - \kappa \partial_z^2)\) and using the relations (70), we can rewrite the first equation of the system (26) in the form

\[
\lambda (\gamma - \kappa \partial_z^2)^{-1} \hat{U} = (\gamma - \kappa \partial_z^2)^{-1} \partial_z \left[ s U - (\gamma - \kappa \partial_z^2) (\nu + 2) R_s^{\nu+1} \hat{\rho} \right],
\]

which is identical with the first equation of the system (69). The second equation of the system (26) can be converted in the following way:

\[
\lambda M = \partial_z \left[ (\gamma - \kappa \partial_z^2) M + s N \right]
\]

implies

\[
-\lambda \eta_s^2 \hat{\rho} = \partial_z \left[ \hat{U} - s \eta_s^2 \hat{\rho} \right],
\]

or

\[
\lambda \hat{\rho} = R_s^2 \partial_z \left[ s \frac{1}{R_s^2} \hat{\rho} - \hat{U} \right],
\]

or

\[
\lambda \hat{\rho} = R_s^2 \left[ s \left( \frac{1}{R_s^2} \partial_z \hat{\rho} - \frac{2}{R_s^2} R_s' \hat{\rho} \right) - \partial_z \hat{U} \right] = s \partial_z \hat{\rho} - R_s^2 \partial_z \hat{U} - 2 s \frac{R_s'}{R_s} \hat{\rho},
\]

which is identical with the second equation of the system (69).

Moving in the opposite direction, we can rewrite the equation

\[
\lambda \hat{U} = \partial_z \left[ s \hat{U} - (\gamma - \kappa \partial_z^2) (\nu + 2) R_s^{\nu+1} \hat{\rho} \right],
\]

as

\[
\lambda (\gamma - \kappa \partial_z^2) M = \partial_z \left[ (\gamma - \kappa \partial_z^2) \left[ s M + \frac{\nu + 2}{\eta_s^{\nu+3}} N \right] \right],
\]

which is equivalent to the first equation of the system (26).

The equation

\[
\lambda \hat{\rho} = s \partial_z \hat{\rho} - 2 R_s u_s' \hat{\rho} - R_s^2 \partial_z \hat{U}
\]

is equivalent to

\[
-\lambda \frac{N}{\eta_s^2} = -s \partial_z \left( \frac{N}{\eta_s^2} \right) + 2 R_s u_s' \left( \frac{N}{\eta_s^2} \right) - R_s^2 \partial_z \left( \gamma - \kappa \partial_z^2 \right) M,
\]

or

\[
\lambda \frac{N}{\eta_s^2} = s \partial_z \left( \frac{N}{\eta_s^2} \right) - 2 R_s u_s' \left( \frac{N}{\eta_s^2} \right) - \frac{1}{\eta_s^2} \partial_z \left( \gamma - \kappa \partial_z^2 \right) M,
\]

or

\[
\lambda N = \eta_s^2 \left\{ s \left[ \frac{1}{\eta_s^2} \partial_z N - \frac{2 N}{\eta_s^2} \partial_z \eta_s \right] + 2 s N \frac{\eta_s'}{\eta_s^2} + \frac{1}{\eta_s^2} \left( \gamma - \kappa \partial_z^2 \right) M \right\},
\]

which is equivalent to the second equation of the system (26). To obtain the last equality, we took advantage of the identities \(u_s' = -s R_s'/R_s^2\) and \(R_s' = -\eta_s'/\eta_s^2\).
Since earlier in [2] it was shown that the essential spectrum of the operator appearing in formula (69) coincides with the imaginary axis, then on the basis of the results obtained above it is possible to formulate

**Corollary 15.** Under the assumptions of the theorem [13] the soliton-like TW solutions of the system (1)-(2) are spectrally stable.

**Appendix A**

In order to trace the behavior of the vectors \( \tilde{Z}, \tilde{Z}' \) for large values of the arguments, we consider the linearization of the dynamical system

\[
\begin{aligned}
\frac{d}{dz} \eta_s &= \tilde{J}_s, \\
\frac{d}{dz} \tilde{J}_s &= \frac{\eta_s^{\nu+3}}{\kappa(\nu+2)} \left[ \kappa(\nu+2)(\nu+3)\tilde{J}_s^2/\eta_s - s^2(\eta_s - \eta_\infty) + \gamma \eta_s^{\nu+2} - \eta_\infty^{\nu+2} \right],
\end{aligned}
\] (71)

which is equivalent to (13). The linear part of the system (71) in variables \( x = \eta_s - \eta_\infty \), \( y = \tilde{J}_s \) will have the following form:

\[
\begin{pmatrix}
x \\
y
\end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \mathcal{D} & 0 \end{pmatrix} \begin{pmatrix} x \\
y \end{pmatrix},
\] (72)

where \( \mathcal{D} = \frac{1}{\kappa(\nu+2)} [\beta - s^2 \eta_\infty^{\nu+3}] > 0 \). Thus, for \( |z| \gg 1 \) we obtain the asymptotics

\[
x = \eta_s - \eta_\infty \approx \begin{cases} 
\tilde{C} e^{\sqrt{\mathcal{D}} z}, & z \ll -1, \\
\tilde{C} e^{-\sqrt{\mathcal{D}} z}, & z \gg 1.
\end{cases}
\]

Constants standing at the exponential functions are the same for large and small \( z \) because of the symmetry of the homoclinic trajectory with respect to the horizontal axis. Using the first equation of the system (11), we get the asymptotics for another component of the homoclinic solution:

\[
w_s \approx -\frac{\nu + 2}{s \eta_\infty^{\nu+3}} \begin{cases} 
\tilde{C} e^{\sqrt{\mathcal{D}} z}, & z \ll -1, \\
\tilde{C} e^{-\sqrt{\mathcal{D}} z}, & z \gg 1.
\end{cases}
\]

From this we get the asymptotics (cf with [9]):

\[
\Psi^\pm = \lim_{z \to \pm \infty} e^{\pm \sqrt{\mathcal{D}} z} \tilde{Z}' = \tilde{C} \sqrt{\mathcal{D}} \begin{pmatrix} \pm (\eta_\infty/b)^{\nu+1} & \nu + 2 \eta_\infty^{\nu+3} & \nu^2 + 2 \nu + 2 \eta_\infty^{\nu+3} \\
\pm (\eta_\infty/b)^{\nu+1} & \nu + 2 \eta_\infty^{\nu+3} & \nu^2 + 2 \nu + 2 \eta_\infty^{\nu+3} \\
(\eta_\infty/b)^{\nu+1} & \nu + 2 \eta_\infty^{\nu+3} & \nu + 2 \eta_\infty^{\nu+3} \\
\end{pmatrix} \begin{pmatrix} 0 \\
1 \\
0 \\
\end{pmatrix}.
\]

The coefficient \( \chi_{00}^- \) is obtained from the normalization condition

\[
1 = \left( \hat{J}_s \eta_1^-, \Psi^+ \right),
\]

where \( \eta_1^- = \chi_{00}^- \Psi^- \), \( \hat{K} = \hat{K} - s \hat{M} \) (see [9], section 3). In the general case, the above formula is very cumbersome, but in the case we are interested in, i.e., when \( b = \eta_\infty \), a more straightforward expression emerges:

\[
1 = \chi_{00}^- \left( \hat{J}_s \Psi^- \right)^{tr} \Psi^+ =
\]
\[ C^2 D \chi_0 - \kappa (\nu + 2) \frac{D + s^2 \eta_\infty^{\nu+3}}{s \eta_\infty^{\nu+3} \sqrt{D}}; - \frac{1}{\sqrt{D}}; - \frac{\kappa (\nu + 2)}{s \eta_\infty^{\nu+3}}; 0; 0; - \frac{\nu + 2}{s \sqrt{\Omega} \eta_\infty^{\nu+3}} \times \]
\[ \left[ \begin{array}{c} \nu + 2 \\ \eta_\infty^{\nu+3} \end{array} \right] ; - \frac{\sqrt{D} (\nu + 2)}{s \eta_\infty^{\nu+3}}; \frac{1}{\sqrt{D}}; - \frac{\nu + 2}{s \eta_\infty^{\nu+3} \sqrt{D}}; 0 \right]^{tr} = 2 \chi_0 \tilde{C}^2 D^{3/2} (\nu + 2)^2 (s \eta_\infty^{\nu+3})^2.

Hence
\[ \chi_0 = \frac{(s \eta_\infty^{\nu+3})^2}{2 \tilde{C}^2 D^{3/2} (\nu + 2)^2} > 0. \]

Appendix B

Here we analyze the fulfillment of the hypotheses from [9], which guarantee the existence of the normalization of the Evans function, under which \( \lim_{\lambda \to +\infty} E(\lambda) = 1 \). Substituting into Eq. (43) a perturbation of the form \( Z(t, z) = Z(z) + \varepsilon e^{\lambda t} U(z) \), performing elementary algebraic transformations, and dropping the higher-order terms in \( \varepsilon \), we get the linear dynamical system:
\[ U' = \hat{A}(z; \lambda) U(z), \quad U \in \mathbb{R}^n, \quad (73) \]
where
\[ \hat{A}(z; \lambda) = \hat{J}^{-1} \left( \hat{B}(z, \lambda) - \lambda \hat{M} \right) = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ -\lambda & 0 & s & 0 & 0 & 0 \\ \gamma / \kappa & -s R / \kappa & 0 & \lambda / \kappa & 0 & -1 / \kappa \\ 0 & R & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \]
\[ \hat{B}(z, \lambda) = D^2 S(Z), \quad R = \eta_\infty^{\nu+3} / (\nu + 2). \]
The matrix \( \hat{A}(z; \lambda) \) is related to a pair of constant matrices
\[ \hat{A}^\pm(\lambda) = \lim_{z \to \pm \infty} \hat{A}(z; \lambda). \]

In our case
\[ A^+(\lambda) = A^-(\lambda) = A_\infty(\lambda) = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ -\lambda & 0 & s & 0 & 0 & 0 \\ \gamma / \kappa & -s R_\infty / \kappa & 0 & \lambda / \kappa & 0 & -1 / \kappa \\ 0 & R_\infty & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \]
where \( R_\infty = \eta_\infty^{\nu+3} / (\nu + 2). \) The spectral problem for the matrix \( A_\infty(\lambda) \) can be written as follows:
\[ \det [A_\infty(\lambda) - \mu I] = \mu^2 [\kappa \mu^4 - \gamma \mu^2 + R_\infty(\lambda - s \mu)^2] = 0. \quad (74) \]
We prove the following assertions necessary for applying the results of [9] to the investigation of the asymptotics of the Evans function at infinity.

**Lemma 16.** 1. The spectrum of the matrix \( A_\infty(0) \) contains a pair of real eigenvalues \( \pm \hat{\beta} \neq 0 \); whereas the remaining eigenvalues nullify.
2. If \( \lambda \in \mathbb{C}^+ \), then the spectrum of \( A_\infty(\lambda) \) contains a pair of eigenvalues \( \mu_1, \mu_2 \) with \( \text{Re}(\mu_1) < \text{Re}(\mu_2) < 0 \), and the remaining eigenvalues have non-negative real parts.

**Proof.** The first assertion is obvious, since for \( \lambda = 0 \) it is not difficult to calculate the eigenvalues from the formula (74):

\[
\mu_{1,2} = \pm \sqrt{\frac{\eta_\infty^{\nu+3} \left( \beta \eta_\infty^{-(\nu+3)} - s^2 \right)}{\kappa}},
\]

while \( \mu_{3,\ldots,6} = 0 \). To prove the second assertion, we apply the method of asymptotic expansions, representing the eigenvalues in the form of a series \( \mu = a_0 + a_1 \lambda + \ldots \). Substituting this expression in (74) and equating the coefficients of the corresponding powers of \( \lambda \) to zero, we obtain a system of algebraic equations. In view of the awkwardness of the computations, we used the Mathematica package for deriving these equations. Thus, nullifying zero-order coefficient, we get the equation

\[
a_0^2 \left[ \kappa + a_0^2 \left( s^2 R_\infty - \gamma \right) \right] = 0. \tag{75}
\]

Eq. (75) has a pair of nonzero solutions

\[
a_0^\pm = \pm \frac{\kappa (\nu + 2)}{\eta_\infty^{\nu+3} \left( \beta \eta_\infty^{-(\nu+3)} - s^2 \right)}
\]

and a pair of zero solutions.

Equating to zero the coefficient of \( \lambda^1 \), we get the equation

\[
2 a_0 \left[ a_1 \left( s^2 R_\infty - \gamma \right) + 2 a_0^2 a_1 \kappa - s R_\infty \right] = 0. \tag{76}
\]

For \( a_0 \neq 0 \) Eq. (76) gives the expression

\[
a_1 \bigg|_{a_0^\pm} = \frac{s \eta_\infty^{\nu+3}}{\beta - s^2 \eta_\infty^{\nu+3}}. \]

So for \( 0 < \lambda << 1 \), we have the following pair of the roots:

\[
\mu_{1-} = -\frac{\sqrt{\Delta}}{\kappa} + a_1 \lambda + O(\lambda^2) < 0
\]

and

\[
\mu_{3+} = \frac{\sqrt{\Delta}}{\kappa} + a_1 \lambda + O(\lambda^2) > 0.
\]

The second pair of roots corresponding to \( a_0 = 0 \) is obtained from the following approximation. Setting under this condition the coefficient of \( \lambda^2 \) to be equal to zero, we obtain the equation

\[
R_\infty (s a_1 - 1)^2 - a_1^2 = 0, \tag{77}
\]

whose solutions are expressed as follows:

\[
a_{11} = \frac{s R_\infty - \sqrt{\gamma R_\infty}}{s^2 R_\infty - \gamma} > 0,
\]
\[
a_{12} = \frac{sR_\infty + \sqrt{\gamma R_\infty}}{s^2 R_\infty - \gamma} < 0.
\]

Using these solutions, we obtain the second pair of the roots:

\[
\mu_2^- = a_{12} \lambda + O(\lambda^2) < 0,
\]

\[
\mu_2^+ = a_{11} \lambda + O(\lambda^2) > 0.
\]

Since the characteristic equation always has a pair of zero solutions, the above construction exhausts all possible cases corresponding to small values of the parameter \(\lambda > 0\). And this is quite enough to complete the proof of the second point because of the fact that, as can easily be shown, \(Re(\mu_i)\) change signs only when the parameter \(\lambda\) belongs to the imaginary axis. Thus, equation (74) will have exactly two solutions with negative real part for any \(\lambda\) with positive real part.

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