ALGEBRAIC STABILITY THEOREM FOR DERIVED CATEGORIES OF ZIGZAG PERSISTENCE MODULES

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Abstract. The interleaving and bottleneck distances between ordinary persistence modules can be extended to the derived setting. Using these distances, we prove an algebraic stability theorem in the derived category of ordinary persistence modules. It is well known that the derived categories of ordinary and arbitrary zigzag persistence modules are equivalent. Through this derived equivalence, these distances can also be defined on the derived category of arbitrary zigzag persistence modules, and the algebraic stability theorem holds even in this setting. As a consequence, an algebraic stability theorem for arbitrary zigzag persistence modules is proved.

1. Introduction

Topological data analysis has recently become popular for studying the shape of data in various research areas (Hiraoka et al. 2016; Lee et al. 2017; Oyama et al. 2019; Saadatfar et al. 2017). In topological data analysis, one of the standard tools is persistent homology, the original concept for which was introduced by Edelsbrunner, Letscher, and Zomorodian (2000). For a filtration

$$\mathcal{X} : X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_n$$

of topological spaces, the $q$-th persistent homology is defined by

$$H_q(\mathcal{X}) : H_q(X_1) \rightarrow H_q(X_2) \rightarrow \cdots \rightarrow H_q(X_n),$$

where $H_q(-)$ is the $q$-th homology functor with a field coefficient. Persistent homology is utilized to study the persistence of topological features in the filtration $\mathcal{X}$ such as connected components, loops, voids, and so on, for each dimension $q$. The algebraic structure of persistent homology is expressed using the notion of $(1D)$ persistence modules, which are representations of an equioriented $A_n$-type quiver. This was pointed out by Carlsson and de Silva (2010).

From Gabriel (1972) and the Krull-Schmidt Theorem, any persistence module can be uniquely decomposed into interval representations, which are exactly indecomposable representations in this setting. The endpoints of these interval representations define the birth-death parameters of the topological features, and those topological features are summarized in a barcode (or a persistence diagram). Then, the persistence of a topological feature is expressed by the lifetime defined as the difference between its death and birth parameters.

Here, the Krull-Schmidt Theorem reduces the description of the category of representations of quivers into that of the full subcategory consisting of indecomposable representations. To explicitly compute indecomposable representations, the Auslander-Reiten (AR) quiver was introduced (see Auslander et al. 1997) and has been studied in representation theory of finite-dimensional algebras since the 1970s.

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For details on the AR quiver, refer to Schiffler (2014, Section 3.1) (see Assem et al. 2006, Chapter IV for a more general setting).

From the viewpoint of AR theory, the barcode of a persistence module can be defined as a map from the set of vertices in the AR quiver of the equioriented $A_n$-type quiver to the integers, sending an interval representation to its multiplicity. In this sense, the AR quiver is hidden behind the barcode.

Unlike ordinary homology, it is significant that a stability theorem holds for persistent homology, which was first proved by Cohen-Steiner, Edelsbrunner, and Harer (2007) for the persistent homology of the sublevel set filtration induced by an $\mathbb{R}$-valued function. The theorem guarantees that the barcode is stable (precisely, 1-Lipschitz) with respect to small changes in a given data set.

The algebraic perspective on persistent homology allows for a generalization of the stability theorem, the so-called algebraic stability theorem (AST). Chazal et al. (2009) introduced the interleaving distance between persistence modules to weaken the assumptions needed for the stability theorem, and then proved the AST by using that distance. The AST guarantees that the barcode is stable with respect to small changes in the given persistence module. Following this algebraic generalization, Bauer and Lesnick (2014) provided a simpler proof of the AST via the induced matching theorem (IMT) (see Theorem 2.8). It should be noted that the converse of the AST also holds (Lesnick 2015), hence giving the isometry theorem for persistence modules.

Here, the representation of an $A_n$-type quiver with alternating (resp. arbitrary) orientation is called a purely zigzag (resp. zigzag) persistence module herein, while 1D persistence modules are said to be ordinary. Zigzag persistence modules can also be applied to address characteristic topological features not captured by the theory of ordinary persistence modules (Carlsson and de Silva 2010). For example, let us study time-series data given by a sequence $X_1, \cdots, X_t, \cdots, X_T$ of topological spaces $X_t$ for each time $t$. In general, this sequence is not a filtration with respect to $t$, but we can consider the following zigzag diagram:

$$X_1 \hookrightarrow X_1 \cup X_2 \hookleftarrow X_2 \hookrightarrow \cdots \hookleftarrow X_{T-1} \cup X_T \hookrightarrow X_T.$$

By applying a homology functor $H_q(-)$ to this diagram, we obtain a purely zigzag persistence module

$$H_q(X_1) \rightarrow H_q(X_1 \cup X_2) \leftarrow H_q(X_2) \rightarrow \cdots \leftarrow H_q(X_{T-1} \cup X_T) \leftarrow H_q(X_T).$$

Recall that a purely zigzag persistence module can also be decomposed into interval representations. Hence, it has a well-defined barcode and the persistence of topological features in the time-series data $X_1, \cdots, X_T$ is encoded in the barcode. This generalization is enabled by the algebraic viewpoint of persistent homology as a representation of an $A_n$-type quiver.

It was proved in Botnan and Lesnick (2018) that an AST also holds for purely zigzag persistence modules. Bjerkevik (2016) improved the theorem with a tight bound and provided an isometry theorem for purely zigzag persistence modules. Note that zigzag persistence modules in Botnan and Lesnick (2018); Bjerkevik (2016) are purely zigzag ones in our convention.

In this paper, we first generalize the AST of the equioriented $A_n$-type quiver into the derived category and then show that this generalization naturally provides a proof of the AST for zigzag persistence modules. Botnan and Lesnick (2018) proved the stability theorem for a class of modules, called block-decomposable 2D persistence modules, into which purely zigzag persistence modules can be embedded. In contrast, our strategy focuses on the equivalence of derived categories of ordinary and zigzag persistence modules (see Happel 1988). This enables us to obtain an AST for the wider class (i.e., arbitrary orientations) in a unified manner than the
result of Botnan and Lesnick. In particular, through the derived equivalence, we show the following statements:

(a) The interleaving and bottleneck distances on the ordinary persistence modules can be extended into its derived category (Definition 3.7 and Definition 3.10).
(b) By using these distances, the AST is generalized on the derived category of ordinary persistence modules (Theorem 4.1).
(c) The derived equivalence naturally defines the distances on the derived category of zigzag persistence modules, and induces the AST on it (Definition 4.2, Definition 4.5, and Theorem 4.6 with Proposition 4.12).
(d) As a corollary, the AST on the zigzag persistence modules holds (Theorem 4.14).

In the above, the isometry theorems also hold as in previous studies (Theorem 5.2, Corollary 5.3, and Theorem 5.4).

Let us briefly address prominent issues in order to derive the above statements.

(a) Recall that the derived category is defined by the Verdier localization of the homotopy category of cochain complexes with quasi-isomorphisms as denominators. Then, cochain complexes of ordinary persistence modules can be uniquely decomposed by their cohomologies in the derived category. This fact provides natural extensions of the interleaving and bottleneck distances in the derived setting.
(b) The derived interleaving and the derived bottleneck distances defined in (a) enable us to generalize the AST to the derived category of ordinary persistence modules since these distances on cochain complexes are determined by their cohomologies.
(c) For a category \( D \) equivalent to the derived category of ordinary persistence modules, the distances on \( D \) can be canonically obtained from the known ones on the latter derived category. In fact, equivalences preserve isomorphisms and the indecomposability of objects, and these properties are utilized to define the distance. As mentioned above, the derived categories of ordinary and zigzag persistence modules are equivalent. Under the derived equivalence, the distances on the derived category of zigzag persistence modules can be induced from the derived interleaving and the derived bottleneck distances defined in (a). Then, an AST for the derived category of zigzag persistence modules follows from (b).
(d) As a consequence of (a), (b), and (c), we obtain an AST for zigzag persistence modules. Indeed, the category of zigzag persistence modules can be regarded as a full subcategory of its derived category.

Finally, let us also note relationships between our induced distance and the distance used in Botnan and Lesnick (2018). For direct calculation of our induced distance on the category of zigzag persistence modules, we need to fix a derived equivalence between derived categories of ordinary and zigzag persistence modules. In this paper, we consider the derived equivalence given by a classical tilting module (see Assem et al. 2006; Brenner and Butler 1980; Bongartz 1981; Happel and Ringel 1982). Indeed, for a classical tilting module \( T \), the right derived functor of the functor \( \text{Hom}(T, -) \) gives a derived equivalence (see Happel 1988). By using the derived equivalence, we can calculate our induced distance and will then show that this is incomparable with the distance in Botnan and Lesnick (2018).

From the perspective of the sheaf theory, Kashiwara and Schapira (2018) first defined the derived interleaving distance in the setting of constructible sheaves. Berkouk and Ginot (2018) proved the isometry theorem by using that distance. Indeed, they introduced the bottleneck distance between graded barcodes and then
proved that this is equal to the derived interleaving distance in the sense of Kashiwara and Schapira. Our future work is to investigate a relationship between their distance and our derived distance.

The remainder of the paper is organized as follows. Section 2 reviews the basic concepts of persistent homology from the viewpoint of representation theory and recalls the AST in the ordinary setting. Section 3 reviews the basics of the derived category and refines it for the category of persistence modules. Then we introduce the interleaving distance and the bottleneck distance in the derived setting. Section 4 proves the main results: the AST for the derived category of ordinary persistence modules and that for zigzag persistence modules. Section 5 extends the result of Section 4 to isometry theorems. In Section 6, we explicitly calculate our induced distance on zigzag persistence modules. Finally, Section 7 confirms that the distance of Botnan and Lesnick and our induced distance are incomparable in the purely zigzag setting.

2. Preliminaries

2.1. Quiver representations. Throughout, $k$ denotes an algebraically closed field, and all vector spaces, algebras, and linear maps are assumed to be finite-dimensional $k$-vector spaces, finite-dimensional $k$-algebras, and $k$-linear maps, respectively. Furthermore, all categories and functors are assumed to be additive.

A **quiver** $Q$ is a directed graph. Formally, a quiver $Q$ is a quadruple $Q = (Q_0, Q_1, s, t)$ of sets $Q_0$ of vertices and $Q_1$ of arrows, and maps $s, t : Q_1 \to Q_0$. We draw an arrow $\alpha \in Q_1$ as $\alpha : 1 \to 2$ if $s(\alpha) = 1, t(\alpha) = 2 \in Q_0$. The **opposite quiver** $Q^{\text{op}}$ of a quiver $Q = (Q_0, Q_1, s, t)$ is $Q^{\text{op}} = (Q_0, Q_1, t, s)$. For example, the opposite quiver of $1 \to 2$ is $1 \leftarrow 2$. A quiver $Q$ is **finite** if $Q_0$ and $Q_1$ are finite. Herein, only finite quivers are considered, otherwise stated.

A **quiver morphism** $f$ from a quiver $Q = (Q_0, Q_1, s, t)$ to a quiver $Q' = (Q'_0, Q'_1, s', t')$ is a pair $f = (f_0, f_1)$ of maps $f_i : Q_i \to Q'_i$ for $i = 0, 1$ such that $s' \circ f_1 = f_0 \circ s$ and $t' \circ f_1 = f_0 \circ t$. For example, $1_Q = (1_{Q_0}, 1_{Q_1})$ is a quiver morphism $Q \to Q$, which is called the identity morphism. A quiver morphism $f : Q \to Q'$ is called an **isomorphism** if there is a quiver morphism $g : Q' \to Q$ such that $f \circ g = 1_Q$ and $g \circ f = 1_{Q'}$. A quiver $Q$ is isomorphic to a quiver $Q'$, denoted by $Q \cong Q'$, if there is an isomorphism from $Q$ to $Q'$. For example, a quiver of the form $1 \xrightarrow{\alpha} 2$ is isomorphic to a quiver of the form $X \xrightarrow{\beta} Y$. Indeed, we have an isomorphism $f = (f_0, f_1)$ defined by $f_0(1) = X, f_0(2) = Y$, and $f_1(\alpha) = \beta$.

Here, we introduce the $A_n$-type quiver $A_n(a)$ with orientation $a$, whose underlying graph is the Dynkin diagram of type $A : 1 \leftarrow 2 \leftarrow \cdots \leftarrow n$ for $n \in \mathbb{N}$. Then $A_n(a)$ is the quiver

$$1 \leftrightarrow 2 \leftrightarrow \cdots \leftrightarrow n,$$

where $\leftrightarrow$ means $\to$ or $\leftarrow$ assigned by the orientation $a$. In this paper, the following $A_n$-type quivers with certain orientations are frequently used. The $A_n$-type quiver with equi-orientation

$$1 \to 2 \to \cdots \to n$$

is called the **equioriented** $A_n$-type quiver, which is denoted by $A_n(= A_n(\epsilon))$, and an $A_n$-type quiver with alternating orientation is called a **purely zigzag** $A_n$-type quiver, which is denoted by $A_n(z)$. Moreover, if the vertex 1 of a purely zigzag $A_n$-type quiver $Q$ is a sink vertex, $Q$ is denoted by $A_n(z_1)$. Otherwise, it is denoted by $A_n(z_2)$. Namely, $A_n(z_1)$ is the following quiver:

$$1 \leftarrow 2 \to 3 \leftarrow \cdots \leftarrow n$$

if $n$ is odd, $1 \leftarrow 2 \to 3 \leftarrow \cdots \leftarrow n$ if $n$ is even,
Moreover, a non-zero representation \( M \) implies \( N \) of the category of ordinary persistence modules in order to define the interleaving morphisms of an ordinary persistence module and an endofunctor \( M \): 

\[
A_n : 1 \xrightarrow{\alpha_{1,2}} 2 \xrightarrow{\alpha_{2,3}} \cdots \xrightarrow{\alpha_{n-1,n}} n
\]

has the following form:

\[
M_1 \xrightarrow{M_{n1,2}} M_2 \xrightarrow{M_{n2,3}} \cdots \xrightarrow{M_{nn-1,n}} M_n.
\]

A subrepresentation \( N \) of \( M \) is defined as a representation of \( Q \) such that \( N_x \subseteq M_x \) for each vertex \( x \in Q_0 \) and \( N_\alpha = M_\alpha|_{N_x} \) for each arrow \( \alpha : x \to y \in Q_1 \). The direct sum \( M \oplus N \) of representations \( M \) and \( N \) is defined by \( (M \oplus N)_x = M_x \oplus N_x \) for each vertex \( x \in Q_0 \) and \( (M \oplus N)_\alpha = M_\alpha \oplus N_\alpha \) for each arrow \( \alpha \in Q_1 \). The dimension of \( M \) is defined by \( \dim M := \sum_{x \in Q_0} \dim M_x \). All representations are assumed to be finite-dimensional, namely \( \dim M < \infty \).

Let \( M, N \) be representations of \( Q \). Then a morphism \( f : M \to N \) is a family of linear maps \( f_x : M_x \to N_x \) on each vertex \( x \in Q_0 \) such that the following diagram commutes for any arrow \( \alpha : x \to y \in Q_1 \):

\[
\begin{array}{ccc}
M_x & \xrightarrow{f_x} & N_x \\
\downarrow{M_\alpha} & & \downarrow{N_\alpha} \\
M_y & \xrightarrow{f_y} & N_y
\end{array}
\]

For example, \( 1_M = (1_{M_x})_{x \in Q_0} \) is a morphism \( M \to M \), which is called the identity morphism. A morphism \( f : M \to N \) is called an isomorphism if there is a morphism \( g : N \to M \) such that \( f \circ g = 1_N \) and \( g \circ f = 1_M \). A representation \( M \) is isomorphic to a representation \( N \), denoted by \( M \cong N \), if there is an isomorphism from \( M \) to \( N \). Moreover, a non-zero representation \( M \) is said to be indecomposable if \( M \cong N \oplus L \) implies \( N = 0 \) or \( L = 0 \).

The abelian category of representations of \( Q \) is denoted by \( \text{rep}_Q \). Note that \( \text{rep}_Q \) is a Krull-Schmidt category (see Schiffler 2014, p.11, Theorem 1.2 for example). Indeed, for any \( M \in \text{rep}_Q \), we have unique decomposition

\[
M \cong M^1 \oplus \cdots \oplus M^s
\]

up to permutations and isomorphisms, where each \( M^i \) is indecomposable.

### 2.2. Persistence modules

We call each \( M \in \text{rep}_Q A_n \), each \( N \in \text{rep}_Q A_n(z) \), and each \( L \in \text{rep}_Q A_n(a) \) a (ordinary) persistence module, a purely zigzag persistence module, and a zigzag persistence module, respectively. In this subsection, we will define the internal morphisms of an ordinary persistence module and an endofunctor of the category of ordinary persistence modules in order to define the interleaving distance.

For any \( A_n \)-type quiver \( A_n(a) \), \( \alpha_{x,y} \) denotes the arrow between \( x \) and \( y \) with \( 1 \leq x < y \leq n \). Then the equioriented \( A_n \)-type quiver \( A_n \) is

\[
A_n : 1 \xrightarrow{\alpha_{1,2}} 2 \xrightarrow{\alpha_{2,3}} \cdots \xrightarrow{\alpha_{n-1,n}} n
\]

and a persistence module \( M \) has the form:

\[
M_1 \xrightarrow{M_{n1,2}} M_2 \xrightarrow{M_{n2,3}} \cdots \xrightarrow{M_{nn-1,n}} M_n.
\]
Moreover, when \( n \) is odd, the purely zigzag \( A_n \)-type quiver \( A_n(z_1) \) is
\[
1 \xleftarrow{\alpha_{1,2}} 2 \xrightarrow{\alpha_{2,3}} \cdots \xrightarrow{\alpha_{n-1,n}} n
\]
and a purely zigzag persistence module \( M \in \text{rep}_k A_n(z_1) \) has the form:
\[
M_1 \leftarrow M_{n,1,2} \cdots M_{n,n-1} \rightarrow M_n.
\]
In other cases, we can similarly describe the zigzag \( A_n \)-type quivers and the zigzag persistence modules.

**Definition 2.1.** Let \( M, N \) be persistence modules and \( \delta \) an integer.

(1) For \( 1 \leq s \leq t \leq n \), the linear map \( \phi_M(s, t) : M_s \rightarrow M_t \) is defined by
\[
\phi_M(s, t) = \begin{cases} 
1 & \text{if } s = t \\
M \alpha_{t-1} \circ \cdots \circ M \alpha_{s+1} & \text{otherwise}
\end{cases}
\]
By definition, we have \( \phi_M(s, t) = \phi_M(t-1, t) \circ \cdots \circ \phi_M(s, s+1) \).

(2) The \( \delta \)-shift \( M(\delta) \) of \( M \) is defined by
\[
(M(\delta))_x = \begin{cases} 
M_{x+\delta} & 1 \leq x+\delta \leq n \\
0 & \text{otherwise}
\end{cases}
\]
and
\[
(M(\delta))_{\alpha_{x,x+1}+\delta} = \begin{cases} 
M_{\alpha_{x+\delta,x+1+\delta}} & 1 \leq x+\delta \leq x+1+\delta \leq n \\
0 & \text{otherwise}
\end{cases}
\]
for each vertex \( x \) of \( A_n \). For a morphism \( f : M \rightarrow N \) in \( \text{rep}_k A_n \), the \( \delta \)-shift \( f(\delta) \) of \( f \) is defined by
\[
(f(\delta))_x = \begin{cases} 
f_{x+\delta} & 1 \leq x+\delta \leq n \\
0 & \text{otherwise}
\end{cases}
\]
for each vertex \( x \) of \( A_n \). This defines the \( \delta \)-shift functor \( (\delta) : \text{rep}_k A_n \rightarrow \text{rep}_k A_n \).

(3) The transition morphism \( \phi_M^{\delta} : M \rightarrow M(\delta) \) in \( \text{rep}_k A_n \) is defined by \( (\phi_M^{\delta})_x = \phi_M(x, x+\delta) \) for each vertex \( x \) of \( A_n \). For any morphism \( f : M \rightarrow N \), we have the following commutative diagram:
\[
\begin{array}{ccc}
M & \xrightarrow{f^\delta} & M(\delta) \\
\downarrow{\phi_M} & & \downarrow{\phi_N} \\
N & \xrightarrow{f(\delta)} & N(\delta).
\end{array}
\]
This defines a natural transformation \( \phi^{\delta} : I \rightarrow (\delta) \) from the identity functor \( I \) to the \( \delta \)-shift functor \( (\delta) \).

(4) A persistence module \( M \) is \( \delta \)-trivial if the transition morphism \( \phi_M^{\delta} : M \rightarrow M(\delta) \) is zero.

In our setting, the functor \( (\delta) \) is not an equivalence but an exact functor. Indeed, let \( M, N, L \) be persistence modules. A sequence
\[
0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0
\]
is exact if and only if the sequence
\[
0 \rightarrow M_x \rightarrow N_x \rightarrow L_x \rightarrow 0
\]
is exact for each vertex \( x \) of \( A_n \). This means that the sequence
\[
0 \rightarrow M(\delta) \rightarrow N(\delta) \rightarrow L(\delta) \rightarrow 0
\]
is exact.
2.3. Interleaving distance. In this paper, a distance on a set $X$ means an extended pseudometric. Specifically, it is a function $d : X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that, for every $x, y, z \in X$,

1. $d(x,x) = 0$,
2. $d(x, y) = d(y, x)$, and
3. $d(x, z) \leq d(x, y) + d(y, z)$ if $d(x, y), d(y, z) < \infty$.

Let us recall the interleaving distance between persistence modules.

**Definition 2.2.** Let $\delta$ be a non-negative integer. Two persistence modules $M$ and $N$ are said to be $\delta$-interleaved if there exist morphisms $f : M \to N(\delta)$ and $g : N \to M(\delta)$ such that the following diagrams commute:

$$
\begin{array}{ccc}
M & \xrightarrow{\phi^\delta_M} & M(2\delta), \\
\downarrow f & & \downarrow g(\delta) \\
N(\delta) & & M(\delta)
\end{array}
$$

In this case, we call the pair of $f : M \to N(\delta)$ and $g : N \to M(\delta)$ a $\delta$-interleaving pair. Moreover, we call a morphism $f : M \to N(\delta)$ a $\delta$-interleaving morphism if there is a morphism $g : N \to M(\delta)$ such that the pair $(f, g)$ is a $\delta$-interleaving pair.

For persistence modules $M, N$, the **interleaving distance** is defined as

$$d_I(M, N) := \inf\{\delta \in \mathbb{Z}_{\geq 0} \mid M \text{ and } N \text{ are } \delta\text{-interleaved}\}.$$

We remark that in our setting, $d_I(M, N) = 0$ if and only if $M$ and $N$ are isomorphic. Thus, the interleaving distance can measure how far these modules are from being isomorphic. We will extend this concept to the derived setting later (see Definition 3.7).

2.4. Intervals and barcodes. We recall that the category $\text{rep}_k A_n(a)$ of zigzag persistence modules is a Krull-Schmidt category, i.e., a representation of $A_n(a)$ is isomorphic to a direct sum of indecomposable representations. In this subsection, we discuss all indecomposable representations of $A_n(a)$.

**Definition 2.3.** For $1 \leq b \leq d \leq n$, the **interval representation** $\mathbb{I}[b, d] \in \text{rep}_k A_n(a)$ is defined by

$$(\mathbb{I}[b, d])_x := \begin{cases} 
\mathbb{I}_k, & b \leq x \leq d \\
0, & \text{otherwise}
\end{cases}$$

and

$$(\mathbb{I}[b, d])_{a_{x,y}} := \begin{cases} 
\mathbb{I}_k, & b \leq x < y \leq d \\
0, & \text{otherwise}
\end{cases}.$$ 

Any interval representation is indecomposable. The converse also holds as follows.

**Theorem 2.4 (Gabriel 1972).** Any indecomposable representation of $A_n(a)$ is isomorphic to an interval representation $\mathbb{I}[b, d]$ for some $1 \leq b \leq d \leq n$.

Thus, for a representation $M$ of $A_n(a)$, we obtain the unique interval decomposition

$$M \cong \bigoplus_{1 \leq b \leq d \leq n} \mathbb{I}[b, d]^{m(b,d)},$$

leading to the definition of the **barcode** (or the persistence diagram) $B(M)$ of $M$ by

$$\{ (b, d, m) \mid 1 \leq b \leq d \leq n, 1 \leq m \leq m(b,d) \text{ such that } m(b,d) \neq 0 \}.$$
Below, we will use the same notation as in Bauer and Lesnick (2014). Recall that a multiset is a pair \((S, m)\) of a set \(S\) and a map \(m : S \to \mathbb{Z}_{>0}\) and that a representation \(\text{Rep}(S, m)\) of a multiset \((S, m)\) is
\[
\text{Rep}(S, m) = \{(s, n) \in S \times \mathbb{Z}_{>0} \mid n \leq m(s)\}.
\]
If \(S\) is a totally ordered set, then a representation \(\text{Rep}(S, m)\) is also a totally ordered set with order obtained by restricting the lexicographic order on \(S \times \mathbb{Z}_{>0}\) to \(\text{Rep}(S, m)\).

For a persistence module \(M\), the barcode \(\mathcal{B}(M)\) is regarded as a representation of a multiset \((\mathcal{I}_M, m)\) of the set \(\mathcal{I}_M := \{(b, d) \mid m(b, d) \neq 0\}\) and the map \(m : \mathcal{I}_M \to \mathbb{Z}_{>0}\) sending \((b, d)\) to \(m(b, d)\). For simplicity, write an element \((b, d, m)\) of \(\mathcal{B}(M)\) as \((b, d)\), which is called an interval. For \(1 \leq b \leq n\), \(\mathcal{B}(M)_{(b, \cdot)}\) denotes the subset of \(\mathcal{B}(M)\) consisting of the intervals \((b, c)\) for some \(b \leq c \leq n\), and \(\mathcal{B}(M)_{(\cdot, d)}\) denotes the subset of \(\mathcal{B}(M)\) consisting of the intervals \((c, d)\) for some \(1 \leq c \leq d\). Note that \(\mathcal{B}(M)_{(b, \cdot)}, \mathcal{B}(M)_{(\cdot, d)}\) are regarded as totally ordered sets with the total order induced by the reverse inclusion relation on intervals. Indeed, if \(c < c'\), then \((b, c) \prec (b, c')\) in \(\mathcal{B}(M)_{(b, \cdot)}\) and \((c, d) \prec (c', d)\) in \(\mathcal{B}(M)_{(\cdot, d)}\).

From the perspective of AR theory, the barcode of a representation \(M\) of \(A_n(a)\) can be defined as a map \(\Gamma_0 \to \mathbb{Z}\) sending an interval \([b, d]\) to its multiplicity \(m(b, d)\) in the decomposition of \(M\), where \(\Gamma_0\) is the set of all interval representations. Note that \(\Gamma_0\) is the set of vertices of the AR quiver of \(A_n(a)\) (for details, see Schiffler 2014, Section 1.5 and 3.1, Chapter 7), and in this sense, AR quivers are hidden behind the barcodes. The AR quivers are important tools in the representation theory of quivers. Indeed, under a certain assumption, the AR quiver can recover the category of representations.

**Example 2.5.** The AR quiver \(\Gamma(A_3)\) of \(A_3\) is
\[
\Gamma(A_3) = \begin{array}{c}
\text{[1, 3]} \\
\text{[2, 3]} \\
\text{[3, 3]} \\
\end{array}
\begin{array}{c}
\text{[2, 2]} \\
\text{[1, 2]} \\
\text{[1, 1]} \\
\end{array}
\begin{array}{c}
\text{[1, 3]} \\
\text{[2, 3]} \\
\text{[3, 3]} \\
\end{array}
\begin{array}{c}
\text{[1, 2]} \\
\text{[1, 1]} \\
\end{array}
\]
while the AR quiver \(\Gamma(A_3(z_1))\) of \(A_3(z_1) : 1 \leftarrow 2 \to 3\) is
\[
\Gamma(A_3(z_1)) = \begin{array}{c}
\text{[1, 1]} \\
\text{[2, 3]} \\
\text{[3, 3]} \\
\end{array}
\begin{array}{c}
\text{[2, 2]} \\
\text{[1, 2]} \\
\end{array}
\begin{array}{c}
\text{[1, 3]} \\
\end{array}
\]

### 2.5. Matching and the bottleneck distance.
A matching from a set \(S\) to a set \(T\) (written as \(\sigma : S \rightarrow T\)) is a bijection \(\sigma : S' \rightarrow T'\) for some subset \(S'\) of \(S\) and some subset \(T'\) of \(T\). For a matching \(\sigma : S \rightarrow T\), we write \(S'\) as \(\text{Coim}\sigma\) and \(T'\) as \(\text{Im}\sigma\).

For totally ordered sets, a matching can be defined canonically as follows: let \(S = \{S_i \mid i = 1, \cdots, s\}\) and \(T = \{T_i \mid i = 1, \cdots, t\}\) be finite totally ordered sets such that for \(a \leq b\), \(S_a \leq S_b\) and \(T_a \leq T_b\). Then a canonical matching \(\sigma : S \rightarrow T\) is a matching given by \(\sigma(S_i) = T_i\) for \(i = 1, \cdots, \min\{s, t\}\). In this case, either \(\text{Im}\sigma = S\) or \(\text{Coim}\sigma = T\) is satisfied.

We next define a \(\delta\)-matching between barcodes.

**Definition 2.6.** Let \(\delta\) be a non-negative integer. For a barcode \(\mathcal{B}\), let \(\mathcal{B}_\delta\) be the subset of \(\mathcal{B}\) consisting of intervals \([b, d]\) such that \(d - b \geq \delta\). A \(\delta\)-matching between
barcodes $\mathcal{B}$ and $\mathcal{B}'$ is defined by a matching $\sigma : \mathcal{B} \to \mathcal{B}'$ such that

$$\mathcal{B}_{2\delta} \subseteq \text{Coim} \, \sigma, \mathcal{B}'_{2\delta} \subseteq \text{Im} \, \sigma,$$

for all $\sigma(b, d) = (b', d')$.

Two barcodes $\mathcal{B}$ and $\mathcal{B}'$ are said to be $\delta$-matched if there is a $\delta$-matching between $\mathcal{B}$ and $\mathcal{B}'$. Then the bottleneck distance is defined as

$$d_B(\mathcal{B}, \mathcal{B}') := \inf \{ \delta \in \mathbb{Z}_{\geq 0} | \mathcal{B} \text{ and } \mathcal{B}' \text{ are } \delta \text{-matched} \}.$$

Note that equation (1) implies that the interval representations associated with $(b, d), (b', d')$ are $\delta$-interleaved.

### 2.6. Algebraic stability theorem

In this subsection, we will explain the proof of an AST for $\text{rep}_k A_n$ following Bauer and Lesnick (2014). Their strategy utilizes the IMT.

**Definition 2.7.** Let $f : M \to N$ be a morphism in $\text{rep}_k A_n$. Then the induced matching $\mathcal{B}(f) : \mathcal{B}(M) \to \mathcal{B}(N)$ is defined as follows:

1. when $f$ is injective, $\mathcal{B}(f)$ is defined via the family of canonical matchings from $\mathcal{B}(M)_{(\cdot, \cdot)}$ to $\mathcal{B}(N)_{(\cdot, \cdot)}$.
2. when $f$ is surjective, $\mathcal{B}(f)$ is defined via the family of canonical matchings from $\mathcal{B}(M)_{(b, \cdot)}$ to $\mathcal{B}(N)_{(b, \cdot)}$.
3. For any morphism $f$, $\mathcal{B}(f)$ can be decomposed into the surjective morphism $\pi : M \to \text{Im} f$ and the injective morphism $\mu : \text{Im} f \to N$. Then $\mathcal{B}(f) := \mathcal{B}(\mu) \circ \mathcal{B}(\pi)$ by (1) and (2).

This matching is what yields the IMT (see Bauer and Lesnick 2014, Theorem 4.2). To state the IMT in our setting, we extend representations $M$ in $\text{rep}_k A_n$ to those in $\text{rep}_k A_{\ell}$ for $\ell \geq n$ as

$$0 \to \cdots \to 0 \to M_1 \to \cdots \to M_n \to 0 \to \cdots \to 0 \in \text{rep}_k A_{\ell}.$$

Moreover, for a given representation $M \in \text{rep}_k A_n$ and non-negative integer $\delta$, the map $r_M^\delta : \mathcal{B}(M(\delta)) \to \mathcal{B}(M)$ is defined by $r_M^\delta(b, d) := (b + \delta, d + \delta)$. In general, the map $r_M^\delta$ is not injective. However, we can take an integer $\ell \geq n$ large enough such that $M$ and $M(\delta)(-\delta)$ are isomorphic as representations of $A_{\ell}$. In this case, the map $r_M^\delta$ is injective.

Then, the IMT is stated as follows:

**Theorem 2.8 (IMT).** Let $f : M \to N$ be a morphism in $\text{rep}_k A_n$. Assume that $\ker f$ and $\text{Coker} f$ are $2\delta$-trivial. Moreover, taking an integer $\ell \geq n$ large enough such that $r_M^\delta$ is injective, $M, N$ are regarded as representations of $A_{\ell}$. Then $\mathcal{B}(f) \circ r_M^\delta$ is a $\delta$-matching $\mathcal{B}(M(\delta)) \to \mathcal{B}(N)$.

Let $f : M \to N(\delta)$ be a $\delta$-interleaving morphism. It is easily seen that $\ker f$ and $\text{Coker} f$ are $2\delta$-trivial. Thus, Theorem 2.8 induces the following theorem (see Bauer and Lesnick 2014, Theorem 4.5):

**Theorem 2.9 (AST).** Let $M, N$ be persistence modules in $\text{rep}_k A_n$. Then

$$d_B(\mathcal{B}(M), \mathcal{B}(N)) \leq d_I(M, N).$$

**Proof.** Let $f : M \to N(\delta)$ be a $\delta$-interleaving morphism in $\text{rep}_k A_n$ and $\ell \geq n$ an integer large enough such that $r_M^\delta$ and $r_N^\delta$ are injective. Then $M$ and $N$ are regarded as representations of $A_{\ell}$. Since $\ker f$ and $\text{Coker} f$ are $2\delta$-trivial,

$$r_N^\delta \circ \mathcal{B}(f) = r_N^\delta \circ (\mathcal{B}(f) \circ r_M^\delta) \circ (r_M^\delta)^{-1} : \mathcal{B}(M) \to \mathcal{B}(N(\delta)) \to \mathcal{B}(N)$$

is a $\delta$-matching by Theorem 2.8, as desired. □
3. Derived categories

3.1. Definition and basic properties. Let \( \mathcal{A} \) be an abelian category. We start this section with the definition of its derived category (see Happel 1988, Chapter I, 3).

**Definition 3.1.** (1) A cochain complex \( X^\bullet \) over \( \mathcal{A} \) is a family \( X^i = (X^i, d_X^i)_{i \in \mathbb{Z}} \) of objects \( X^i \) of \( \mathcal{A} \) and morphisms \( d_X^i : X^i \to X^{i+1} \) in \( \mathcal{A} \) satisfying \( d_X^{i+1} \circ d_X^i = 0 \).

A cochain complex \( X^\bullet \) is said to be bounded if \( X^i = 0 \) for \( |i| \gg 0 \). If \( X^i = 0 \) for \( i \neq l \), then it is called a stalk complex concentrated at the \( l \)-th term. For each cochain complex \( X^\bullet \) and each \( i \in \mathbb{Z} \), we have the \( i \)-th cohomology functor \( H^i(X^\bullet) := \text{Ker} d_X^i / \text{Im} d_X^{i-1} \).

Let \( X^\bullet, Y^\bullet \) be cochain complexes over \( \mathcal{A} \). Then a cochain map \( f^\bullet : X^\bullet \to Y^\bullet \) is a family \( f^i = (f^i)_i \in \mathbb{Z} \) of morphisms \( f^i : X^i \to Y^i \) in \( \mathcal{A} \) satisfying \( f^{i+1} \circ d_X^i = d_Y^i \circ f^i \).

We use \( \mathcal{C}^b(\mathcal{A}) \) to denote the category of bounded cochain complexes and cochain maps over \( \mathcal{A} \). Then the \( l \)-translation functor \( [l] : \mathcal{C}^b(\mathcal{A}) \to \mathcal{C}^b(\mathcal{A}) \) is defined by \( X^\bullet[l] := ((X^i[l])^i, d_X^i[l])_{i \in \mathbb{Z}} \) with \( (X^i[l])^i = X^{i+l} \), \( d_X^i[d_X^i] = (-1)^l d_X^{i+l} \) and \( (f^\bullet[l])^i := f^{i+l} \) for a cochain complex \( X^\bullet = (X^i, d_X^i)_{i \in \mathbb{Z}} \) and a cochain map \( f^\bullet \). The 0-translation functor \([0] \) is exactly the identity functor.

(2) A cochain map \( f^\bullet \) naturally induces the morphism \( H^i(f^\bullet) : H^i(X^\bullet) \to H^i(Y^\bullet) \) for each \( i \). Then we have the \( i \)-th cohomology functor \( H^i(-) : \mathcal{C}^b(\mathcal{A}) \to \mathcal{A} \).

A cochain map \( f^\bullet \) is called a quasi-isomorphism if the induced morphisms \( H^i(f^\bullet) \) are isomorphisms for all \( i \).

(3) A cochain map \( f^\bullet : X^\bullet \to Y^\bullet \) is said to be null-homotopic if there exists a family \( (h^i)_{i \in \mathbb{Z}} \) of morphisms \( h^i : X^i \to Y^{i-1} \) such that \( f^i = h^{i+1} \circ d_X^i + d_Y^{i-1} \circ h^i \) for each \( i \).

Let \( I \) be the ideal of \( \mathcal{C}^b(\mathcal{A}) \) consisting of null-homotopic cochain maps. Then the bounded homotopy category \( K^b(\mathcal{A}) \) of \( \mathcal{A} \) is defined as the quotient category of \( \mathcal{C}^b(\mathcal{A}) \) by the ideal \( I \). Since a cochain map \( f^\bullet \) is null-homotopic if and only if \( f^\bullet[I] \) is so, we can extend the \( l \)-translation functor \([l] \) to the setting of the bounded homotopy category such that the following diagram commutes, where \( \pi : \mathcal{C}^b(\mathcal{A}) \to K^b(\mathcal{A}) \) is the canonical quotient functor.

\[
\begin{array}{ccc}
\mathcal{C}^b(\mathcal{A}) & \xrightarrow{\pi} & K^b(\mathcal{A}) \\
[l] \downarrow & & \downarrow l \\
\mathcal{C}^b(\mathcal{A}) & \xrightarrow{\pi} & K^b(\mathcal{A})
\end{array}
\]

It is well-known that the homotopy category \( K^b(\mathcal{A}) \) with the 1-translation functor \([1] \) forms a triangulated category (see Happel 1988).

Moreover, if a cochain map \( f^\bullet \) is null-homotopic, then \( H^i(f^\bullet) = 0 \). Thus, we obtain the \( i \)-th cohomology functor \( H^i(-) : K^b(\mathcal{A}) \to \mathcal{A} \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{C}^b(\mathcal{A}) & \xrightarrow{\pi} & K^b(\mathcal{A}) \\
H^i \downarrow & & \downarrow H^i \\
\mathcal{A} & \xrightarrow{H^i} & K^b(\mathcal{A})
\end{array}
\]

A quasi-isomorphism in \( K^b(\mathcal{A}) \) is a morphism \( f^\bullet \) such that \( H^i(f^\bullet) \) are isomorphisms for all \( i \).

(4) The bounded derived category \( D^b(\mathcal{A}) \) of \( \mathcal{A} \) is the triangulated category given by the Verdier localization of the bounded homotopy category \( K^b(\mathcal{A}) \) with respect to the collection of all quasi-isomorphisms. Thus, a morphism in \( D^b(\mathcal{A}) \) is an isomorphism if and only if it is a quasi-isomorphism in \( K^b(\mathcal{A}) \). The construction is
analogous to that of the localization of a ring. Indeed, a morphism from $X^\bullet$ to $Y^\bullet$ in the derived category is represented by a pair $(f^\bullet, s^\bullet)$ of a morphism $f^\bullet : X^\bullet \to Z^\bullet$ and a quasi-isomorphism $Y^\bullet \to Z^\bullet$ with some cochain complex $Z^\bullet$.

By the universal property of the localization, we obtain the $i$-th cohomology functor $H^i(\cdot) : \mathcal{D}^b(\mathcal{A}) \to \mathcal{A}$ such that the following diagram commutes, where $\iota : K^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{A})$ is the canonical localization functor.

$$
\begin{array}{c}
\mathcal{D}^b(\mathcal{A}) \\
\downarrow H^i \\
\mathcal{A}
\end{array}
\begin{array}{c}
\mathcal{D}^b(\mathcal{A}) \\
\downarrow H^i \\
\mathcal{A}
\end{array}
$$

The abelian category $\mathcal{A}$ can be regarded as a full subcategory of its bounded derived category $\mathcal{D}^b(\mathcal{A})$. Indeed, we have the fully faithful functor $\xi$ from $\mathcal{A}$ to $\mathcal{D}^b(\mathcal{A})$ sending $X$ to the stalk complex

$$
\cdots \to 0 \to X \to 0 \to \cdots
$$

concentrated at the 0-th term. We denote this stalk complex as $X[0]$, using the 0-translation functor $[0]$. The stalk complex concentrated at the $l$-th term is written as $X[-l]$.

We use $\text{proj}\, \mathcal{A}$ to denote the full subcategory of $\mathcal{A}$ consisting of projective objects. An abelian category $\mathcal{A}$ is said to have enough projectives if for each object $M \in \mathcal{A}$, there exists an epimorphism $P \to M$ with $P \in \text{proj}\, \mathcal{A}$. In this case, a projective resolution of $M$ can be defined as a cochain complex

$$
P^\bullet : \cdots \to P_1 \to P_0
$$

over $\text{proj}\, \mathcal{A}$ satisfying

$$
H^i(P^\bullet) \cong \begin{cases} 
M, & i = 0 \\
0, & \text{otherwise}
\end{cases}
$$

In other words, we have an exact sequence

$$
\cdots \to P_1 \to P_0 \to M \to 0.
$$

The projective dimension of $M$ can be defined as the infimum of the lengths of projective resolutions of $M$, and the global dimension of $\mathcal{A}$ can be defined as the supremum of all projective dimensions.

Similarly, we use $\text{inj}\, \mathcal{A}$ to denote the full subcategory of $\mathcal{A}$ consisting of injective objects, and we can dually consider the concept of having enough injectives and the injective and global dimensions. Note that the global dimensions defined by projective and injective dimensions coincide if the abelian category $\mathcal{A}$ has enough projectives and injectives.

Then the following lemma is important for understanding the bounded derived category (see Happel 1988, Chapter I, 3.3). Note that the categories $\mathcal{C}^b(\mathcal{A})$ and $K^b(\mathcal{A})$ can be defined if $\mathcal{A}$ is an additive category, e.g., $K^b(\text{proj}\, \mathcal{A})$ and $K^b(\text{inj}\, \mathcal{A})$.

**Lemma 3.2.** Assume that $\mathcal{A}$ has enough projectives (resp. injectives) and has a finite global dimension. Then $K^b(\text{proj}\, \mathcal{A})(\text{resp. } K^b(\text{inj}\, \mathcal{A})) \hookrightarrow K^b(\mathcal{A}) \xrightarrow{\sim} \mathcal{D}^b(\mathcal{A})$ is an equivalence as a triangulated category.

This lemma states that an object of a derived category $\mathcal{D}^b(\mathcal{A})$ is assumed to be a cochain complex over $\text{proj}\, \mathcal{A}$ (or $\text{inj}\, \mathcal{A}$), and a morphism in $\mathcal{D}^b(\mathcal{A})$ can be written as a morphism in $K^b(\text{proj}\, \mathcal{A})$ (or $K^b(\text{inj}\, \mathcal{A})$). All representatives in a morphism in $\mathcal{D}^b(\mathcal{A})$ from $X^\bullet$ to $Y^\bullet$ need some unknown cochain complex $Z^\bullet$ (see Definition 3.1 (4)). In contrast, a morphism in $K^b(\text{proj}\, \mathcal{A})$ (or $K^b(\text{inj}\, \mathcal{A})$) is concretely written as the residue class of a cochain map. Thus, under the equivalence in Lemma 3.2, we can well understand the bounded derived category $\mathcal{D}^b(\mathcal{A})$. 

Let \( \mathcal{B} \) be another abelian category. \( \mathcal{A} \) is said to be derived equivalent to \( \mathcal{B} \) if \( D^b(\mathcal{A}) \) and \( D^b(\mathcal{B}) \) are equivalent as triangulated categories.

Let \( F : \mathcal{A} \to \mathcal{B} \) be a functor. Any such functor yields the canonical functor \( C(F) : C^b(\mathcal{A}) \to C^b(\mathcal{B}) \) given by \( C(F)(X^\bullet) := (FX^i, Fd_X^i)_{i \in \mathbb{Z}} \) for each cochain complex \( X^\bullet = (X^i, d_X^i)_{i \in \mathbb{Z}} \). Then \( C(F) \) naturally extends to the functor \( K(F) : K^b(\mathcal{A}) \to K^b(\mathcal{B}) \). Moreover, we assume that \( F \) is exact.

For any cochain complex \( \mathcal{D} \), there is a canonical functor \( \delta : D^b(\mathcal{D}) \to D^b(\mathcal{D}) \) given by \( \delta : (X^i, d_X^i)_{i \in \mathbb{Z}} \mapsto (X^i, d_X^i)_{i \in \mathbb{Z}} \).

\[ \delta \] induces a functor \( \delta : D^b(\text{rep}_k A_n) \to D^b(\text{rep}_k A_n) \) via \( X^\bullet(\delta) = (X^i(\delta), d_X^i(\delta))_{i \in \mathbb{Z}} \). Then \( H^t \circ (\delta) \) is identified with \( (\delta) \circ H^t \).

If \( \mathcal{A} \) has enough injectives and \( F \) is left exact, then we can define the right derived functor \( RF : D^b(\mathcal{A}) \to D^b(\mathcal{B}) \) as \( RF(X^\bullet) := (FX^i, Fd_X^i)_{i \in \mathbb{Z}} \) for each cochain complex \( X^\bullet = (X^i, d_X^i)_{i \in \mathbb{Z}} \). We often write \( C(F) \), \( K(F) \), \( D(F) \) as \( F \) in their respective contents and identify \( H^t \circ F \) with \( F \circ H^t \). For example, since the \( \delta \)-shift functor \( \delta : \text{rep}_k A_n \to \text{rep}_k A_n \) is exact, it induces a functor

\[ \delta : D^b(\text{rep}_k A_n) \to D^b(\text{rep}_k A_n) \]

3.2. Derived category of \( \text{rep}_k A_n(a) \). We discuss some specific properties of \( D^b(\text{rep}_k A_n(a)) \) for an \( A_n \)-type quiver \( A_n(a) \) with arbitrary orientation \( a \). Set \( \text{proj}_A(a) := \text{proj}(\text{rep}_k A_n(a)) \). Note that \( \text{rep}_k A_n(a) \) has enough projectives and injectives, and has global dimension 0 for \( n = 1 \) and 1 for \( n > 1 \). Then we always have a projective resolution \( 0 \to P_1 \to P_0 \to M \to 0 \) of length at most 1 for any representation \( M \in \text{rep}_k A_n(a) \). In particular, any subrepresentation of a projective representation is also projective in this setting. In addition, by Lemma 3.2, we obtain an equivalence between \( D^b(\text{rep}_k A_n(a)) \) and \( K^b(\text{proj}_A(a)) \).

In the case of \( \text{rep}_k A_n(a) \), we have the following strong characterization of a cochain complex in \( D^b(\text{rep}_k A_n(a)) \) by its cohomologies.

**Lemma 3.3.** For any cochain complex \( X^\bullet \in D^b(\text{rep}_k A_n(a)) \),

\[ X^\bullet \cong \bigoplus_{i \in \mathbb{Z}} H^i(X^\bullet)[-i] \]

in \( D^b(\text{rep}_k A_n(a)) \). More generally,

\[ X^\bullet(\delta) \cong \bigoplus_{i \in \mathbb{Z}} H^i(X^\bullet(\delta))[-i]. \]

**Proof.** We may assume that \( X^i \) is projective for all \( i \in \mathbb{Z} \), since \( D^b(\text{rep}_k A_n(a)) \cong K^b(\text{proj}_A(a)) \) by Lemma 3.2. The statement will be proved by induction on the length of its non-zero terms. From the boundedness of \( X^\bullet \), let us set \( s := \min \{ i \in \mathbb{Z} \mid X^i \neq 0 \} \) and \( t := \max \{ i \in \mathbb{Z} \mid X^i \neq 0 \} \). Then the length \( \ell(X^\bullet) \) of non-zero terms of \( X^\bullet \) is defined by \( t - s + 1 \). Since the global dimension of \( \text{rep}_k A_n(a) \) is at most 1, \( \text{Im} \ d_{X}^{t-1} \) is projective. Hence the exact sequence

\[ 0 \to \text{Ker} \ d_{X}^{t-1} \to X^{t-1} \to \text{Im} \ d_{X}^{t-1} \to 0 \]

is split, implying that \( \text{Ker} \ d_{X}^{t-1} \) is also projective. Thus, we have \( X^\bullet \cong Y^\bullet \oplus Z^\bullet \), where the complexes \( Y^\bullet, Z^\bullet \in K^b(\text{proj}_A(a)) \) are given by

\[ Y^\bullet = \cdots \to X^{t-2} \to \text{Ker} \ d_{X}^{t-1} \to 0 \] \[ Z^\bullet = \cdots \to 0 \to \text{Im} \ d_{X}^{t-1} \to X^t. \]
Here, $Z^\bullet \cong H^i(X^\bullet)[-t]$ and $H^i(Y^\bullet) \cong \begin{cases} H^i(X^\bullet), & \text{if } i < t \\ 0, & \text{if } i \geq t \end{cases}$. The length $\ell(Y^\bullet)$ is less than the length $\ell(X^\bullet)$, so by induction

$$Y^\bullet \cong \bigoplus_{i < t} H^i(Y^\bullet)[-i] \cong \bigoplus_{i < t} H^i(X^\bullet)[-i].$$

Therefore, we obtain $X^\bullet \cong Y^\bullet \oplus Z^\bullet \cong \bigoplus_{i \in \mathbb{Z}} H^i(X^\bullet)[-i]$ in $\mathbb{D}^b(\text{rep}_k A_n(a))$.

The second statement follows from the first statement and the fact that the functor $\delta$ is exact. $\square$

The fact that the global dimension of $\text{rep}_k A_n(a)$ is at most 1 is essential to the proof of the foregoing lemma. As a consequence of Lemma 3.3, we can characterize all indecomposable objects of $\mathbb{D}^b(\text{rep}_k A_n(a))$.

**Corollary 3.4.** A cochain complex $X^\bullet \in \mathbb{D}^b(\text{rep}_k A_n(a))$ is indecomposable if and only if $X^\bullet$ is isomorphic to a stalk complex

$$[b, d][−i]: \cdots \to 0 \to [b, d] \to 0 \to \cdots$$

concentrated at the $i$-th term in $\mathbb{D}^b(\text{rep}_k A_n(a))$ for some $1 \leq b \leq d \leq n$ and some $i \in \mathbb{Z}$. Thus, any cochain complex $X^\bullet$ is isomorphic to

$$\bigoplus_{b \leq d, i} ([b, d][−i])^{m(b, d, i)},$$

where the non-negative integer $m(b, d, i)$ is the multiplicity of $[b, d][−i]$.

Since $\mathbb{D}^b(\text{rep}_k A_n(a))$ is a Krull-Schmidt category (see Chen et al. 2008), the interval decomposition in the corollary above is unique. By using this result, we propose the ‘derived’ barcode.

**Definition 3.5.** Let $X^\bullet, Y^\bullet$ be cochain complexes of $\mathbb{D}^b(\text{rep}_k A_n(a))$. Then the derived barcode $B^D(X^\bullet)$ is defined as

$$B^D(X^\bullet) := \bigsqcup_{i \text{ with } H^i(X^\bullet) \neq 0} B(H^i(X^\bullet))$$

where $B(H^i(X^\bullet))$ is the ordinary barcode of $H^i(X^\bullet)$ (see the paragraph following Theorem 2.4).

As in Section 2.4, the AR quiver of $\mathbb{D}^b(\text{rep}_k A_n(a))$ with arbitrary orientations can be defined (see Happel 1988, Chapter I, 4 and 5). Similar to the case of $\text{rep}_k A_n(a)$, the derived barcode of $X^\bullet$ can be defined as a map $\Gamma_0 \to \mathbb{Z}$ sending $[b, d][−i]$ to the multiplicity $m(b, d, i)$, where $\Gamma_0$ is the set of vertices of the AR quiver of $\mathbb{D}^b(\text{rep}_k A_n(a))$. Thus, AR quivers are hidden behind the barcodes in this setting, too. Moreover, the AR quiver of $\mathbb{D}^b(\text{rep}_k A_n(a))$ consists of all shifted copies of the AR quiver $\Gamma(A_n(a))$ of $A_n(a)$.

**Example 3.6.** The AR quiver $\Gamma(\mathbb{D}^b(\text{rep}_k A_3))$ of $\mathbb{D}^b(\text{rep}_k A_3)$ is

$$\Gamma(\mathbb{D}^b(\text{rep}_k A_3)) = \begin{array}{c}
\cdots \\
[1, 2][−1] & [1, 3] & [3, 3][1] \\
\cdots \\
[3, 3] & [2, 2] & [1, 1] \\
\cdots \\
[1, 1] 
\end{array}$$

More generally, the AR quiver $\Gamma(\mathbb{D}^b(\text{rep}_k A_n))$ of $\mathbb{D}^b(\text{rep}_k A_n)$ is as described in Figure 1.
Moreover, the AR quiver \( \Gamma(\mathbb{D}^b(\text{rep}_k A_n)) \) of \( \mathbb{D}^b(\text{rep}_k A_3(z_1)) \), where \( A_3(z_1) : 1 \leftarrow 2 \rightarrow 3 \) is

\[
\Gamma(\mathbb{D}^b(\text{rep}_k A_3(z_1))) = \begin{array}{c}
\cdots I[1, 1] \cdots \\
I[1, 2] I[1, 3] I[2, 2] I[2, 3] I[3, 3][1] \\
\cdots I[3, 3] I[1, 2] I[1, 1][1]
\end{array}
\]

Similarly to Figure 1, the AR quiver \( \Gamma(\mathbb{D}^b(\text{rep}_k A_n(z_1))) \) of \( \mathbb{D}^b(\text{rep}_k A_n(z_1)) \) is given by all shifted copies of the AR quiver of \( A_n(z_1) \).

3.3. Derived interleaving and bottleneck distances. In this subsection, we propose distances on the derived category of persistence modules by extending the original interleaving and bottleneck distances.

Recall that the \( \delta \)-shift functor \( (\delta) : \text{rep}_k A_n \rightarrow \text{rep}_k A_n \) induces a functor

\[
(\delta) : \mathbb{D}^b(\text{rep}_k A_n) \rightarrow \mathbb{D}^b(\text{rep}_k A_n)
\]

via \( X^\bullet(\delta) = (X^i(\delta), d_X^i(\delta)) \) is exact since the functor \((\delta)\) is exact.

**Definition 3.7.** Let \( X^\bullet, Y^\bullet \) be cochain complexes in \( \mathbb{D}^b(\text{rep}_k A_n) \) and \( \delta \) a non-negative integer. Then \( X^\bullet \) and \( Y^\bullet \) are said to be derived \( \delta \)-interleaved if there exist morphisms \( f^\bullet : X^\bullet \rightarrow Y^\bullet(\delta) \) and \( g^\bullet : Y^\bullet \rightarrow X^\bullet(\delta) \) such that for each \( i \in \mathbb{Z} \), \((H^i(f^\bullet), H^i(g^\bullet))\) is a \( \delta \)-interleaving pair between \( H^i(X^\bullet) \) and \( H^i(Y^\bullet) \) in the sense of Definition 2.2. Namely, the following diagrams commute for each \( i \in \mathbb{Z} \):

\[
\begin{array}{ccc}
H^i(X^\bullet) & \xrightarrow{g^i_{H^i(X^\bullet)}} & H^i(X^\bullet)(2\delta) \\
H^i(f^\bullet) & & H^i(g^\bullet)(\delta)
\end{array}, \quad \begin{array}{ccc}
H^i(Y^\bullet) & \xrightarrow{f^i_{H^i(Y^\bullet)}} & H^i(Y^\bullet)(2\delta) \\
H^i(g^\bullet) & & H^i(f^\bullet)(\delta)
\end{array}
\]

In this case we also call the pair of \( f^\bullet : X^\bullet \rightarrow Y^\bullet(\delta) \) and \( g^\bullet : Y^\bullet \rightarrow X^\bullet(\delta) \) a derived \( \delta \)-interleaving pair. Moreover, we call a morphism \( f^\bullet : X^\bullet \rightarrow Y^\bullet(\delta) \) a derived \( \delta \)-interleaving morphism if there is a morphism \( g^\bullet : Y^\bullet \rightarrow X^\bullet(\delta) \) such that the pair \((f^\bullet, g^\bullet)\) is a derived \( \delta \)-interleaving pair.

For cochain complexes \( X^\bullet, Y^\bullet \in \mathbb{D}^b(\text{rep}_k A_n) \), the derived interleaving distance is defined as

\[
d^I_D(X^\bullet, Y^\bullet) := \inf \{ \delta \in \mathbb{Z}_{\geq 0} \mid X^\bullet \text{ and } Y^\bullet \text{ are derived } \delta \text{-interleaved} \}.
\]

**Remark 3.8.** (1) Similarly to the original setting, \( d^I_D(X^\bullet, Y^\bullet) = 0 \) for two cochain complexes \( X^\bullet, Y^\bullet \in \mathbb{D}^b(\text{rep}_k A_n) \) if and only if \( X^\bullet \) and \( Y^\bullet \) are isomorphic in \( \mathbb{D}^b(\text{rep}_k A_n) \). Thus, the derived interleaving distance can also measure how far these complexes are from being isomorphic.
(2) We define the derived interleaving distance independently of Berkouk’s (Berkouk 2019). The derived interleaving distance of Berkouk is defined as a generalization of the distance on the base abelian category. In contrast, ours is defined via the characterization of an object of the derived category by its cohomologies. It is also obvious that our definition can be generalized to arbitrary abelian categories $\mathcal{A}$ with some natural transformation from the identity functor $\mathbf{1}$ to an exact endofunctor. Although these two ideas are different, it can be proved that the Berkouk interleaving implies our interleaving. The converse does not hold in general (e.g., for abelian categories having a higher global dimension). For $\text{rep}_k A_n$, these concepts coincide since $\text{rep}_k A_n$ has global dimension at most 1.

Note that if $X^\bullet$ and $Y^\bullet$ are derived $\delta$-interleaved, then $H^i(X^\bullet)$ and $H^i(Y^\bullet)$ are $\delta$-interleaved for all $i$. It follows from Lemma 3.3 that the converse also holds.

**Corollary 3.9.** Let $X^\bullet, Y^\bullet$ be cochain complexes in $\mathbf{D}^b(\text{rep}_k A_n)$. Then $X^\bullet$ and $Y^\bullet$ are derived $\delta$-interleaved if and only if $H^i(X^\bullet)$ and $H^i(Y^\bullet)$ are $\delta$-interleaved for all $i$.

**Proof.** For each $i$, let $(f_i, g_i)$ be a $\delta$-interleaved pair between $H^i(X^\bullet)$ and $H^i(Y^\bullet)$. Then the pair of

$$
(f_i[-i]): \bigoplus_i H^i(X^\bullet)[-i] \to \bigoplus_i H^i(Y^\bullet)(\delta)[-i]
$$

and

$$
(g_i[-i]): \bigoplus_i H^i(Y^\bullet)[-i] \to \bigoplus_i H^i(X^\bullet)(\delta)[-i]
$$

is a derived $\delta$-interleaved pair. By Lemma 3.3, $X^\bullet$ and $Y^\bullet$ are derived $\delta$-interleaved. Indeed, using the isomorphisms in Lemma 3.3, we can construct a derived $\delta$-interleaving pair of morphisms

$$
f^\bullet: X^\bullet \to \bigoplus_i H^i(X^\bullet)[-i] \xrightarrow{(f_i[-i])} \bigoplus_i H^i(Y^\bullet)(\delta)[-i] \to Y^\bullet(\delta)
$$

and

$$
g^\bullet: Y^\bullet \to \bigoplus_i H^i(Y^\bullet)[-i] \xrightarrow{(g_i[-i])} \bigoplus_i H^i(X^\bullet)(\delta)[-i] \to X^\bullet(\delta)
$$

such that $H^i(f^\bullet) = f_i$ and $H^i(g^\bullet) = g_i$ for any $i$. \hfill \Box

Finally, we propose the ‘derived’ bottleneck distance between derived barcodes in the sense of Definition 3.5 in this setting.

**Definition 3.10.** Let $X^\bullet, Y^\bullet$ be cochain complexes of $\mathbf{D}^b(\text{rep}_k A_n)$. Two derived barcodes $B^D(X^\bullet)$ and $B^D(Y^\bullet)$ are said to be $\delta$-matched if $B(H^i(X^\bullet))$ and $B(H^i(Y^\bullet))$ are $\delta$-matched in the sense of Definition 2.6 for all $i \in \mathbb{Z}$.

For derived barcodes $B^D(X^\bullet), B^D(Y^\bullet)$, the derived bottleneck distance is defined as

$$
d^D_B(B^D(X^\bullet), B^D(Y^\bullet)) := \inf\{\delta \in \mathbb{Z}_{\geq 0} \mid B^D(X^\bullet) \text{ and } B^D(Y^\bullet) \text{ are } \delta\text{-matched}\}.
$$

4. **Main results**

In this section, we derive an AST for zigzag persistence modules from an AST for ordinary ones by using the derived category. We adopt a different approach from that of Botnan and Lesnick (2018) by considering that the distances on zigzag persistence modules may be naturally induced by the known interleaving and bottleneck distances on ordinary ones using derived categories. This enables us to obtain an AST for a wider class compared to that of Botnan and Lesnick.
4.1. AST for derived categories. We first prove an AST for derived categories of ordinary persistence modules.

**Theorem 4.1** (AST for derived categories). Let $X^\bullet, Y^\bullet$ be cochain complexes of $\text{D}^b(\text{rep}_k A_n)$. Then

$$d_B^D(\mathcal{B}^D(X^\bullet), \mathcal{B}^D(Y^\bullet)) \leq d_B^D(X^\bullet, Y^\bullet).$$

**Proof.** Assume that $X^\bullet$ and $Y^\bullet$ are derived $\delta$-interleaved. Then for all $i \in \mathbb{Z}$, $H^i(X^\bullet)$ and $H^i(Y^\bullet)$ are $\delta$-interleaved, and hence $\mathcal{B}(H^i(X^\bullet))$ and $\mathcal{B}(H^i(Y^\bullet))$ are $\delta$-matched by Theorem 2.9. Thus, by definition, the inequality

$$d_B^D(\mathcal{B}^D(X^\bullet), \mathcal{B}^D(Y^\bullet)) \leq d_B^D(X^\bullet, Y^\bullet)$$

holds.

Next, we consider a derived equivalence. Let $\mathcal{A}$ be an abelian category. Assume that there exists a derived equivalence $E$ from $\text{D}^b(\mathcal{A})$ to $\text{D}^b(\text{rep}_k A_n)$.

**Definition 4.2.** Two objects $X$ and $Y$ of $\text{D}^b(\mathcal{A})$ are said to be $\delta$-interleaved with respect to $E$ if $E(X)$ and $E(Y)$ are derived $\delta$-interleaved in the sense of Definition 3.7. The interleaving distance $d_I^{E,\mathcal{A}}(X, Y)$ with respect to $E$ is defined as

$$d_I^{E,\mathcal{A}}(X, Y) := \inf \{ \delta \in \mathbb{Z}_{\geq 0} \mid X \text{ and } Y \text{ are } \delta\text{-interleaved with respect to } E \}. $$

Namely, $d_I^{E,\mathcal{A}}(X, Y) = d_B^D(E(X), E(Y))$ holds.

**Remark 4.3.** By Remark 3.8 (1), $d_I^{E,\mathcal{A}}(X, Y) = 0$ if and only if $E(X)$ and $E(Y)$ are isomorphic in $\text{D}^b(\text{rep}_k A_n)$. Since $E$ is an equivalence, this means that $X$ and $Y$ are isomorphic in $\text{D}^b(\mathcal{A})$. Thus, the interleaving distance defined as above can also measure how far these objects are from being isomorphic. This justifies calling the distance an interleaving distance.

**Remark 4.4.** The $\delta$-shift functor cannot be defined in the zigzag setting, so neither is the original interleaving distance. One of the advantages of our approach is that we can define the interleaving distance even in the zigzag setting through the derived equivalence.

Since $E$ is an equivalence, in particular, a fully faithful functor, $X \in \text{D}^b(\mathcal{A})$ is indecomposable if and only if $E(X) \in \text{D}^b(\text{rep}_k A_n)$ is indecomposable. Hence, since $\text{D}^b(\text{rep}_k A_n)$ is a Krull-Schmidt category, so is $\text{D}^b(\mathcal{A})$. Consequently, the derived equivalence $E$ induces a bijection between

$$\mathcal{B}^D_{\mathcal{A}}(X) := \{ Z \in \text{D}^b(\mathcal{A}) \mid Z \text{ is indecomposable and a direct summand of } X \}$$

and $\mathcal{B}^D(E(X))$ (see Definition 3.5). Then the following distance between $\mathcal{B}^D_{\mathcal{A}}(X)$ and $\mathcal{B}^D_{\mathcal{A}}(Y)$ is naturally derived by passing through the derived equivalence $E$.

**Definition 4.5.** For two objects $X, Y$ of $\text{D}^b(\mathcal{A})$, $\mathcal{B}^D_{\mathcal{A}}(X)$ and $\mathcal{B}^D_{\mathcal{A}}(Y)$ are said to be $\delta$-matched with respect to $E$ if $\mathcal{B}^D(E(X))$ and $\mathcal{B}^D(E(Y))$ are $\delta$-matched in the sense of Definition 3.10. The bottleneck distance $d_B^{E,\mathcal{A}}(\mathcal{B}^D_{\mathcal{A}}(X), \mathcal{B}^D_{\mathcal{A}}(Y))$ with respect to $E$ is defined as

$$d_B^{E,\mathcal{A}}(\mathcal{B}^D_{\mathcal{A}}(X), \mathcal{B}^D_{\mathcal{A}}(Y)) := \inf \left\{ \delta \in \mathbb{Z}_{\geq 0} \mid \mathcal{B}^D_{\mathcal{A}}(X) \text{ and } \mathcal{B}^D_{\mathcal{A}}(Y) \text{ are } \delta\text{-matched with respect to } E \right\}.$$

Namely, $d_B^{E,\mathcal{A}}(\mathcal{B}^D_{\mathcal{A}}(X), \mathcal{B}^D_{\mathcal{A}}(Y)) = d_B^D(E(X), E(Y))$ holds.

In our convention, an AST states that the interleaving distance between objects $X$ and $Y$ gives an upper bound for the bottleneck distance between their barcodes. Thus, as a consequence of Theorem 4.1, Definition 4.2, and Definition 4.5, we have the following AST for the derived category $\text{D}^b(\mathcal{A})$. 

Theorem 4.6. Let $\mathcal{A}$ be an abelian category and $X, Y$ objects in $D^b(\mathcal{A})$. Assume that there exists a derived equivalence $E$ from $D^b(\mathcal{A})$ to $D^b(rep_k A_n)$. Then
\[
d_B^{\mathcal{A}}(B^2_d(X), B^2_d(Y)) \leq d_B^{\mathcal{A}}(X, Y).
\]

4.2. AST for zigzag persistence modules. We first discuss an AST for an abelian category $\mathcal{A}$ which is derived equivalent to $rep_k A_n$. Recall that $\mathcal{A}$ can be regarded as a full subcategory of $D^b(\mathcal{A})$. As a consequence of Theorem 4.6, we have the following result.

Corollary 4.7. Let $\mathcal{A}$ be an abelian category. Assume that $\mathcal{A}$ is derived equivalent to $rep_k A_n$. Then an AST also holds for $\mathcal{A}$.

Next, we will discuss an AST for zigzag persistence modules through a derived equivalence given by a classical tilting module between the categories $D^b(rep_k A_n(a))$ and $D^b(rep_k A_n)$ for an $A_n$-type quiver $A_n(a)$ with arbitrary orientation $a$.

Definition 4.8 (Assem et al. 2006; Brenner and Butler 1980; Bongartz 1981; Happel and Ringel 1982). Let $T$ be a persistence module. Then $T$ is called a classical tilting module if it satisfies the following three conditions:

1. the projective dimension of $T$ is at most 1,
2. $\text{Ext}^i(T, T) = 0$ for all $i > 0$, and
3. $T$ has exactly $n$ non-isomorphic indecomposable direct summands.

Bongartz (1981) proved that our classical tilting modules are equivalent to the original ones (see Bongartz 1981 or Happel and Ringel 1982 for the original definition), which are exactly tilting modules with projective dimension at most 1 in the sense of Miyashita (1986) (see also Happel 1988, p.118). Here, we recall that $\text{Hom}(T, -)$ is a functor from $rep_k A_n$ to the module category mod $\text{End}(T)^{\text{op}}$ of the endomorphism algebra on $T$. Then we have the following property of classical tilting modules.

Lemma 4.9 (Happel 1988, Chapter III). Let $T$ be a classical tilting module. Assume that the endomorphism algebra $B = \text{End}(T)^{\text{op}}$ is presented by the quiver $QB$ with no relations. Then the functor $\text{RHom}(T, -)$ is a derived equivalence from $D^b(rep_k A_n)$ to $D^b(rep_k QB)$.

Now, we construct a classical tilting module whose endomorphism algebra is presented by the quiver $A_n(a)$ (with no relations).

Let $\tau, \tau^{-1}$ be the AR translations in $rep_k A_n$ (see Assem et al. 2006, Chapter IV.2, 2.3 Definition or Schiffler 2014, 2.3.3 for definition). For an indecomposable non-projective (resp. non-injective) representation $M$, $\tau(M)$ is indecomposable non-injective and $\tau^{-1}(M) \cong M$ (resp. $\tau^{-1}(M)$ is indecomposable non-projective and $\tau \tau^{-1}(M) \cong M$). The $\tau$-orbit of $M$ is the set of indecomposable representations of the form $\tau^m(M)$ or $\tau^{-m}(M)$ for some non-negative integer $m$, where $\tau^{-m} := (\tau^{-1})^m$.

Since the AR quiver of $A_n$ is finite and connected, there are finitely many $\tau$-orbits. Moreover, each $\tau$-orbit contains exactly one indecomposable projective representation (see Schiffler 2014, 3.1.2).

Let $O(P_i)$ be the $\tau$-orbit of the indecomposable projective representation $P_i$ corresponding to the vertex $i$ of $A_n$. Note that there are $n$ $\tau$-orbits $O(P_1), \cdots, O(P_n)$, and that $O(P_i) = \{P_i\}$ since $P_i$ is projective-injective. Then, the set $\{O(P_i) \mid i = 1, \cdots, n\}$ is just the set of all $\tau$-orbits, the $\tau$-orbit $O(P_i)$ is finite for any $i$, and any indecomposable representation of $A_n$ belongs to the $\tau$-orbit $O(P_i)$ for some $i$. A section of the AR quiver $\Gamma(A_n)$ is a connected full subquiver formed by representatives of all $\tau$-orbits $O(P_i)$. 
Example 4.10. In the AR quiver $\Gamma(A_3)$

\[
\begin{array}{c}
\bullet \[1,3] \\
\bullet \[2,3] \\
\bullet [3,3] \\
\bullet [2,2] \\
\end{array}
\]

\[
\begin{array}{c}
\bullet [1,2] \\
\bullet [1,1] \\
\end{array}
\]

the actions of $\tau$ are denoted by dashed arrows, meaning that there are three $\tau$-orbits $O([3,3]) = \{[3,3], [2,2], [1,1]\}$, $O([2,3]) = \{[2,3], [1,2]\}$, and $O([1,3]) = \{[1,3]\}$, of projective representations $[3,3]$, $[2,3]$, and $[1,3]$, respectively. In this case, for example, $\Sigma = \{[1,3], [2,3], [2,2]\}$ is a section. These representations are written as rectangle-surrounded vertices in the foregoing AR quiver.

More generally, it is easily understood that a section $\Sigma$ in the AR quiver $\Gamma(A_n)$ of $A_n$ is described like the red line in Figure 2.

![Figure 2. Example of a section $\Sigma$ in the AR quiver $\Gamma(A_n)$ of $A_n$](image)

Fix a section $\Sigma$ with vertices $\Sigma_0 = \{X_1 = P_1, \ldots, X_n\}$, where $X_i \in O(P_i)$ for each $i$. Note that $P_1$ is the top vertex of the AR quiver $\Gamma(A_n)$ of $A_n$. Then we define the module $T(\Sigma)$ as follows:

\[
T(\Sigma) = \bigoplus_{i=1}^{n} X_i.
\]

Lemma 4.11 (Bongartz 1981, 2.6 Corollary). For any section $\Sigma$, $T(\Sigma)$ is a classical tilting module in $\text{rep}_k A_n$.

In this setting, $P_1$ is a direct summand of any classical tilting module. More generally, any projective-injective representation is a direct summand of any classical tilting module.

The endomorphism algebra $\text{End}(T(\Sigma))^{op}$ is presented by the quiver $\Sigma^{op}$. By definition, every section is an $A_n(a)$-type quiver with some orientation $a$ and any $A_n(a)$-type quiver appears as a section. Then, the following result is obtained as a consequence of Lemma 4.9 and Lemma 4.11.

Proposition 4.12. If a section $\Sigma$ is isomorphic to $A_n(a)^{op}$, then the functor $R\text{Hom}(T(\Sigma), -)$ is a derived equivalence from $\text{D}^b(\text{rep}_k A_n)$ to $\text{D}^b(\text{rep}_k A_n(a))$.

Remark 4.13. Under the derived equivalence $R\text{Hom}(T(\Sigma), -)$, the AR quiver $\Gamma(A_n(a))$ of $A_n(a)$ can be embedded into the AR quiver of $\text{D}^b(\text{rep}_k A_n)$ as in Figure 3.
Figure 3. The red-bordered polygon except for the broken polygonal line is the area of the AR quiver $\Gamma(A_n(a))$ of $A_n(a)$. For example, the AR quiver $\Gamma(A_3(z_1))$ of $A_3(z_1)$ was described as in Example 2.5

In the case that $\mathcal{A} = \text{rep}_k A_n(a)$ and $E$ is the quasi-inverse of $\text{RHom}(T(\Sigma), -)$ with $\Sigma^{op} \cong A_n(a)$, we put $d_t^{E,a} := d_t^{E,A}$ and $d_B^{E,a} := d_B^{E,A}$. Thus, we conclude the following AST for $\text{rep}_k A_n(a)$ with arbitrary orientations $a$. Note that $\mathcal{B}_D^a(X)$ in Definition 4.5 and the ordinary barcode $\mathcal{B}(X)$ coincide for any $X \in \text{rep}_k A_n(a)$.

Theorem 4.14 (AST for zigzag). Let $X, Y$ be zigzag persistence modules in the category $\text{rep}_k A_n(a)$ with arbitrary orientation $a$. Then

$$d_B^{E,a}(B(X), B(Y)) \leq d_t^{E,a}(X, Y).$$

Proof. Since we have a derived equivalence $\text{RHom}(T(\Sigma), -)$ from $D^b(\text{rep}_k A_n)$ to $D^b(\text{rep}_k A_n)$ by Proposition 4.12, the statement follows from Corollary 4.7. □

5. Isometry theorem

In this section, we will prove an isometry theorem for the category $\text{rep}_k A_n(a)$ of zigzag persistence modules. Theorem 2.9 gives the inequality $d_B \leq d_t$, which is a part of the following isometry theorem (see Bauer and Lesnick 2014, Theorem 3.1 and Section B.1).

Theorem 5.1 (Isometry theorem). Let $M, N$ be persistence modules. Then

$$d_B(\mathcal{B}(M), \mathcal{B}(N)) = d_t(M, N).$$

From Theorem 5.1, we obtain the isometry theorem for the derived category of persistence modules.

Theorem 5.2 (Isometry theorem for derived categories). Let $X^\bullet, Y^\bullet$ be cochain complexes in $D^b(\text{rep}_k A_n)$. Then

$$d_B^{D}(B^D(X^\bullet), B^D(Y^\bullet)) = d_t^{D}(X^\bullet, Y^\bullet).$$

Proof. From Theorem 4.1, we have only to show

$$d_B^{D}(B^D(X^\bullet), B^D(Y^\bullet)) \geq d_t^{D}(X^\bullet, Y^\bullet).$$

If $B^D(X^\bullet)$ and $B^D(Y^\bullet)$ are $\delta$-matched, then there exists a $\delta$-matching between $\mathcal{B}(H^i(X^\bullet))$ and $\mathcal{B}(H^i(Y^\bullet))$ for each $i$ by definition. Then, by Theorem 5.1, there exists a $\delta$-interleaving pair $(f_i, g_i)$ between $H^i(X^\bullet)$ and $H^i(Y^\bullet)$ for each $i$. By Corollary 3.9, $X^\bullet$ and $Y^\bullet$ are derived $\delta$-interleaved in $D^b(\text{rep}_k A_n)$. □
As a consequence of Theorem 5.2, we can extend Theorem 4.6 and Corollary 4.7 by Definition 4.2 and Definition 4.5.

**Corollary 5.3.** Let \( A \) be an abelian category and \( X, Y \) objects in \( A \) or \( \mathbf{D}^b(A) \). Assume that there exists a derived equivalence \( E \) from \( \mathbf{D}^b(A) \) to \( \mathbf{D}^b(\text{rep}_k A_n) \). Then

\[
d_{B}^{E,A}(B_{A}^{D}(X), B_{A}^{D}(Y)) = d_{E,A}^{D}(X, Y).
\]

Finally, we can extend Theorem 4.14 by Corollary 5.3 as follows.

**Theorem 5.4** (Isometry theorem for zigzag). Let \( X, Y \) be zigzag persistence modules in the category \( \text{rep}_k A_n(a) \) with arbitrary orientation \( a \). Then

\[
d_{T,a}^{B}(B(X), B(Y)) = d_{I,a}^{T}(X, Y).
\]

**Proof.** Since we have a derived equivalence \( \text{RHom}(T(\Sigma),-) \) from \( \mathbf{D}^b(\text{rep}_k A_n) \) to \( \mathbf{D}^b(\text{rep}_k A_n(a)) \) by Proposition 4.12, the statement follows from Corollary 5.3. \( \square \)

The special case of Theorem 5.4 is exactly an isometry theorem for purely zigzag persistence modules.

6. **DIRECT CALCULATION OF THE INDUCED DISTANCE ON \( \text{rep}_k A_n(a) \)**

We start this section with the following remark.

**Remark 6.1.** There are actually multiple derived equivalences from \( \mathbf{D}^b(\text{rep}_k A_n) \) to \( \mathbf{D}^b(\text{rep}_k A_n(a)) \). For example, all translations of a classical tilting module are two-sided tilting complexes, which give the derived equivalences (see Rickard 1991). The induced distance on \( \text{rep}_k A_n(a) \) by \( \text{rep}_k A_n \) depends on the choice of derived equivalences. However, in all cases, the isometry theorem holds for \( \text{rep}_k A_n(a) \) by Corollary 5.3.

The purpose of this section is to provide a direct calculation of the induced distance on \( \text{rep}_k A_n(a) \). Let us fix a classical tilting module \( T := T(\Sigma) \) given by a section \( \Sigma \) such that \( \Sigma^{op} \cong A_n(a) \) (see Section 4).

Let \( X \) be a representation of \( A_n(a) \). By Proposition 4.12, there exists a unique cochain complex \( M^* \in \mathbf{D}^b(\text{rep}_k A_n) \) up to isomorphism such that \( \text{RHom}(T,M^*) \cong X \). This complex \( M^* \) is called the corresponding complex of \( X \).

It follows from Theorem 5.4 that the interleaving \( d_{T,a}^{d} \) and bottleneck \( d_{B}^{d} \) coincide, so we put \( d^a := d_{T,a}^{I} = d_{B}^{I} \). By definition, we have

\[
d^a(X, Y) = d_{I}^{T}(M^*, N^*),
\]

where \( X, Y \in \text{rep}_k A_n(a) \) and \( M^*, N^* \) are the corresponding complexes of \( X, Y \), respectively (see Definition 3.7 and Definition 4.2). Thus, we need to deal with the corresponding complexes to calculate \( d^a \) on \( \text{rep}_k A_n(a) \). For this purpose, the classical tilting torsion theory discussed below will be useful (for details, see Assem et al. 2006, Chapter VI).

Let \( \mathcal{T} \) be the full subcategory of \( \text{rep}_k A_n \) consisting of representations \( V \) satisfying \( \text{Ext}^1(T, V) = 0 \) and \( \mathcal{F} \) the full subcategory of \( \text{rep}_k A_n \) consisting of representations \( V \) satisfying \( \text{Hom}(T, V) = 0 \). Moreover, let \( \mathcal{X} \) be the full subcategory of \( \text{rep}_k A_n(a) \) consisting of representations \( V \) satisfying \( T \otimes V = 0 \) and \( \mathcal{Y} \) the full subcategory of \( \text{rep}_k A_n(a) \) consisting of representations \( V \) satisfying \( \text{Tor}_1(T, V) = 0 \). It is known that \( \mathcal{T}, \mathcal{F}, \mathcal{X} \), and \( \mathcal{Y} \) are closed under taking extensions. Indeed, for a short exact sequence

\[
0 \to M \to N \to L \to 0,
\]

\( M, L \in \mathcal{T} \) (resp. \( \mathcal{F}, \mathcal{X} \), or \( \mathcal{Y} \)) implies \( N \in \mathcal{T} \) (resp. \( \mathcal{F}, \mathcal{X} \), or \( \mathcal{Y} \)).
Remark 6.2. The pairs \((T, F)\) and \((X, Y)\) are the so-called torsion pairs (for definition, see Assem et al. 2006, Chapter VI.1, 1.1 Definition). Moreover, these pairs are splitting, namely, for each indecomposable representation \(M \in \text{rep}_k A_n\), we have \(M \in T\) or \(M \in F\), and for each indecomposable representation \(N \in \text{rep}_k A_n(a)\), we have \(N \in X\) or \(N \in Y\) (see also Assem et al. 2006, Chapter VI.5).

The functor \(\text{Hom}(T, -)\) gives an equivalence from \(T\) to \(Y\) and the functor \(\text{Ext}^1(T, -)\) gives an equivalence from \(F\) to \(X\) (see Assem et al. 2006, Chapter VI.3, 3.8 Theorem). From the perspective of the right derived functor \(\text{RHom}(T, -)\), these results can be combined as follows.

**Proposition 6.3.** The derived equivalence \(\text{RHom}(T, -)\) induces the equivalences from \(T[0]\) to \(Y[0]\) and from \(F[1]\) to \(X[0]\).

**Proof.** Note that we have \(H^0(\text{RHom}(T, -)) \cong \text{Hom}(T, -)\) and \(H^1(\text{RHom}(T, -)) \cong H^0(\text{RHom}(T[-1]) \cong \text{Ext}^1(T, -)\).

Then, the claim follows from Assem et al. (2006, Chapter VI.3, 3.8 Theorem). \(\square\)

Proposition 6.3 states that for a given indecomposable representation \(X \in \text{rep}_k A_n(a)\), the corresponding complex \(M^\bullet\) is a stalk complex \(M^\bullet = L[0]\) or \(L[1]\) for some \(L \in T\) or \(F\), respectively. The representation \(L\) is called the corresponding representation of \(X\).

Furthermore, by using the AR quiver, Proposition 6.3 can be described as in Figure 4, where \(F = \text{Hom}(T, -), G = \text{Ext}^1(T, -),\) and \(RF = \text{RHom}(T, -)\).

![Figure 4. Correspondence between the AR quivers of \(\text{rep}_k A_n\) and \(\text{rep}_k A_n(a)\) in the derived category](image)

We will calculate the derived interleaving distance between two stalk complexes.

**Proposition 6.4.** Let \(M, N\) be representations of \(A_n\). For each pair \(i, j\) of integers,

\[
d^D_i(M[i], N[j]) = \begin{cases} 
  d_i(M, N), & i = j \\
  \max\{d_i(M, 0), d_i(N, 0)\}, & i \neq j
\end{cases}
\]
Proof. Note that
\[ H^i(M[i]) = \begin{cases} M, & i = l \\ 0, & i \neq l \end{cases}, \quad H^i(N[j]) = \begin{cases} N, & j = -l \\ 0, & j \neq -l \end{cases}. \]
Thus, in the case that \( i = j \), \( M[i] \) and \( N[j] \) are derived \( \delta \)-interleaved if and only if \( M \) and \( N \) are \( \delta \)-interleaved by Corollary 3.9. Otherwise, \( M[i] \) and \( N[j] \) are derived \( \delta \)-interleaved if and only if \( M \) and 0 are \( \delta \)-interleaved and 0 and \( N \) are \( \delta \)-interleaved by Corollary 3.9.

As a consequence, for indecomposable representations, we have the following calculation of the derived interleaving distance by definition.

**Corollary 6.5.** Let \( M = \llbracket x, y \rrbracket, N = \llbracket s, t \rrbracket \) be indecomposable representations of \( A_n \). For each pair \( i, j \) of integers,
\[ d^P_i(M[i], N[j]) = \begin{cases} \min \left\{ \max\{x - s, y - t\}, \max\left\{ \left\lfloor \frac{1}{2} y - x + 1 \right\rfloor, \left\lfloor \frac{1}{2} l - s + 1 \right\rfloor \right\} \right\}, & i = j \\ \max\{x - s, y - t\}, & i \neq j \end{cases}, \]
where \( \lfloor \cdot \rfloor \) is the ceiling function.

**Proof.** Since
\[ d_I(M, N) = \begin{cases} \min \left\{ \max\{x - s, y - t\}, \max\left\{ \left\lfloor \frac{1}{2} y - x + 1 \right\rfloor, \left\lfloor \frac{1}{2} l - s + 1 \right\rfloor \right\} \right\}, & M \neq 0, N \neq 0 \\ \left\lfloor \frac{1}{2} y - x + 1 \right\rfloor, & N = 0 \\ \left\lfloor \frac{1}{2} l - s + 1 \right\rfloor, & M = 0 \end{cases}, \]
we obtain the desired statement by Proposition 6.4. \( \square \)

By combining Proposition 6.3 and Proposition 6.4, we can calculate the distance \( d^a \) on \( \text{rep}_b A_n(a) \) by the interleaving distance on \( \text{rep}_b A_n \).

**Corollary 6.6.** Let \( X, Y \) be indecomposable representations of \( A_n(a) \) and \( M, N \) the corresponding representations of \( X, Y \) respectively. Then
\[ d^a(X, Y) = \begin{cases} d_I(M, N), & \text{if } X, Y \in \mathcal{X} \text{ or } X, Y \in \mathcal{Y} \\ \max\{d_I(M, 0), d_I(N, 0)\}, & \text{otherwise}. \end{cases} \]

By Corollary 6.5 and Corollary 6.6, we can calculate the value \( d^a(X, Y) \) concretely when we fix the orientation \( a \).

### 7. Comparison between the block distance and the induced distance

Botnan and Lesnick (2018) proved an AST for purely zigzag persistence modules. In that paper, they introduced the interleaving and bottleneck distances on purely zigzag persistence modules, and Bjerkevik (2016) proved that those distances actually coincide. Here, we call those distances the *block distance*, denoted by \( d_{BL} \), following the paper Meehan and Meyer (2020).

In this section, we will compare the distance \( d_{BL} \) with our induced distance in the purely zigzag setting.

#### 7.1. General correspondence

Let \( Z \) be the poset of integers with usual order and \( Z^{op} \) its opposite poset. As per Botnan and Lesnick (2018), let \( ZZ \) be the subposet of the poset \( Z^{op} \times Z \) given by
\[ ZZ := \{(i, j) \mid i \in Z, j \in \{i, i - 1\}\}. \]
Note that this can be expressed by the infinite purely zigzag quiver

\[
\begin{align*}
Q = \begin{tikzpicture}
\node (i+1-i+1) at (0,0) {\cdots};
\node (i-i) at (1,0) {\cdots};
\node (i+1-i) at (2,0) {\cdots};
\node (i-i-1) at (3,0) {\cdots};
\node (i+1-i+1) at (0,-1) {\cdots};
\node (i-i) at (1,-1) {\cdots};
\node (i+1-i) at (2,-1) {\cdots};
\node (i-i-1) at (3,-1) {\cdots};
\end{tikzpicture}
\end{align*}
\]

so that a (locally finite-dimensional) representation of \( \text{ZZ} \) is just that of the quiver \( Q \). Then there are injections \( \mu_i : A(z_i) \to \text{ZZ} \) \((i = 1, 2)\) defined by

\[
\begin{align*}
\mu_1(i) &= \begin{cases}
(m, m), & i = 2m - 1 \\
(m + 1, m), & i = 2m
\end{cases}, \\
\mu_2(i) &= \begin{cases}
(m, m), & i = 2m \\
(m + 1, m), & i = 2m - 1
\end{cases}.
\end{align*}
\]

Moreover, in Botnan and Lesnick (2018), the intervals \((b, d)_{\text{ZZ}}\) \((b \leq d)\) of \( \text{ZZ} \) are divided into the following 4 types:

- closed interval \([b, d]_{\text{ZZ}} := \{(i, j) \in \text{ZZ} \mid (b, b) \leq (i, j) \leq (d, d)\},\)
- right-open interval \([b, d]_{\text{ZZ}} := \{(i, j) \in \text{ZZ} \mid (b, b) \leq (i, j) < (d, d)\},\)
- left-open interval \((b, d)_{\text{ZZ}} := \{(i, j) \in \text{ZZ} \mid (b, b) < (i, j) \leq (d, d)\},\)
- open interval \((b, d)_{\text{ZZ}} := \{(i, j) \in \text{ZZ} \mid (b, b) < (i, j) < (d, d)\}.\)

We use \( \mathbb{I}^{(b, d)_{\text{ZZ}}} \) to denote the interval representation of \( \text{ZZ} \) associated with the interval \((b, d)_{\text{ZZ}}\). Note that the interval representation \( \mathbb{I}^{(b, d)_{\text{ZZ}}} \) of \( \text{ZZ} \) is uniquely determined by the interval \((b, d)_{\text{ZZ}}\). Indeed, \( \mathbb{I}^{(b, d)_{\text{ZZ}}} \) is the representation given by

\[
\mathbb{I}^{(b, d)_{\text{ZZ}}} = \begin{cases}
\mathbb{k}, & (i, j) \in (b, d)_{\text{ZZ}} \\
0, & \text{otherwise}
\end{cases}
\]

and is called a closed (resp. right-open, left-open, and open) interval representation if \((b, d)_{\text{ZZ}}\) is closed (resp. right-open, left-open, and open). Then for \(i = 1, 2\), the map \( \mu_i \) induces the correspondence \( \tilde{\mu}_i \) from the set of interval representations of \( A_n(z_i) \) to the subset of interval representations of \( \text{ZZ} \). More precisely, we have the following result.

**Proposition 7.1.**

1. For any interval representation \( \mathbb{I}[s, t] \) of \( A_n(z_1) \),
   - \( \mathbb{I}[s, t] \in \text{Y} \) if and only if \( \tilde{\mu}_1(\mathbb{I}[s, t]) \) is closed or right-open.
   - \( \mathbb{I}[s, t] \in \text{X} \) if and only if \( \tilde{\mu}_1(\mathbb{I}[s, t]) \) is open or left-open.
2. For any interval representation \( \mathbb{I}[s, t] \) of \( A_n(z_2) \),
   - \( \mathbb{I}[s, t] \in \text{Y} \) if and only if \( \tilde{\mu}_1(\mathbb{I}[s, t]) = \mathbb{I}^{(b, d)_{\text{ZZ}}} \) is closed or right-open, or \((b, d)_{\text{ZZ}}\) is of the form \((1, \cdot)_{\text{ZZ}}\).
   - \( \mathbb{I}[s, t] \in \text{X} \) if and only if \( \tilde{\mu}_1(\mathbb{I}[s, t]) = \mathbb{I}^{(b, d)_{\text{ZZ}}} \) is open or left-open except for when \((b, d)_{\text{ZZ}}\) is of the form \((1, \cdot)_{\text{ZZ}}\).

To prove this proposition, we need the following lemma.

**Lemma 7.2.**

1. For any interval representation \( \mathbb{I}[s, t] \) of \( A_n(z_1) \),
   - \( \mathbb{I}[s, t] \in \text{Y} \) if \( s \) is odd,
   - \( \mathbb{I}[s, t] \in \text{X} \) otherwise.
2. For any interval representation \( \mathbb{I}[s, t] \) of \( A_n(z_2) \),
   - \( \mathbb{I}[s, t] \in \text{Y} \) if \( s \) is 1 or even,
is immediately obtained.

Let \( Q(s) = \text{Hom}(T, X_s) \in \mathcal{Y} \) be the indecomposable projective representation of \( A_n(z_1) \) corresponding to the vertex \( 1 \leq s \leq n \). Note that \( Q(s) \cong \mathbb{I}[s, s] \) if \( s \) is odd, \( Q(n) \cong \mathbb{I}[n-1, n] \) if \( n \) is even, and \( Q(s) \cong \mathbb{I}[s-1, s+1] \) otherwise. Then we have \( \text{Hom}(X_1, Y_{1,t}) \cong \text{Hom}(Q(1), \mathbb{I}[1,t]) \neq 0 \) for any \( t \). Since \( X_1 = P(1) \) is projective-injective and a direct summand of \( T \), \( \text{Hom}(X_1, Y_{1,t}) \neq 0 \) implies that \( Y_{1,t} \in \mathcal{T}[0] \) and hence \( \mathbb{I}[1,t] \in \mathcal{Y} \) by Proposition 6.3. For any odd integer \( s > 1 \), we have an exact sequence

\[
0 \to \mathbb{I}[s, s] \to \mathbb{I}[s, t] \oplus \mathbb{I}[1, s] \to \mathbb{I}[1, t] \to 0.
\]

Since \( \mathcal{Y} \) is closed under taking extensions, \( \mathbb{I}[s, s], \mathbb{I}[1, t] \in \mathcal{Y} \) implies \( \mathbb{I}[s, t] \in \mathcal{Y} \).

Statement (b) then follows from (a) and the fact that the torsion pair \((\mathcal{X}, \mathcal{Y})\) is splitting.

(2). First, we have \( \mathbb{I}[s, s] \in \mathcal{Y} \) when \( s \) is even and \( \mathbb{I}[1, t] \in \mathcal{Y} \) as in the proof of (1). Next, for any even integer \( s > 1 \), we similarly have an exact sequence

\[
0 \to \mathbb{I}[s, s] \to \mathbb{I}[s, t] \oplus \mathbb{I}[1, s] \to \mathbb{I}[1, t] \to 0.
\]

Since \( \mathcal{Y} \) is closed under taking extensions, \( \mathbb{I}[s, s], \mathbb{I}[1, t] \in \mathcal{Y} \) implies \( \mathbb{I}[s, t] \in \mathcal{Y} \).

Statement (b) then follows from (a) and the fact that the torsion pair \((\mathcal{X}, \mathcal{Y})\) is splitting.

As a consequence of Lemma 7.2, Proposition 7.1 is immediately obtained.

Proof of Proposition 7.1. (1). \( \mathbb{I}[s, t] \in \mathcal{Y} \) if and only if \( s \) is odd. In this case, \( \mu_1(s) = (m, m) \) when we write \( s = 2m - 1 \). Then, \( s \) is odd if and only if \( \tilde{\mu}_1(\mathbb{I}[s, t]) \) is closed or right-open, as desired.

(2). Similar to the proof of (1).

Remark 7.3. Proposition 7.1 tells us that the AR quiver of \( A_n(z_1) \) \((l = 1, 2)\) can be divided into 2 areas consisting of 4 kinds of intervals in the sense of Botnan and Lesnick (2018) with respect to classical tilting torsion theory.

7.2. Comparison. Using the results in Subsection 7.1, we will directly compare the block distance \( d_{BL} \) with our induced distance. For this purpose, we consider the following orientation:

\[
A_n(z_1) : 1 \leftrightarrow 2 \to \cdots \to n,
\]

where \( n \) is odd. In this case, we denote the induced distance by \( d^{zzl} \) instead of \( d^a \) (see Section 6). By Proposition 7.1, the interval representations \( \mathbb{I}[s, t] \) of \( A_n(z_1) \) can be divided into 4 kinds of representations \( \mathbb{I}^{(b, d)_{zz}} \). More precisely, we have the following correspondence between \((s, t)\) and \((b, d)\):

- closed interval \([b, d]_{zz} \) \((s = 2b - 1, t = 2d - 1)\),
- right-open interval \([b, d]_{zz} \) \((s = 2b - 1, t = 2d - 2)\),
- left-open interval \([b, d]_{zz} \) \((s = 2b, t = 2d - 1)\),
- open interval \([b, d]_{zz} \) \((s = 2b, t = 2d - 2)\).

Since \( 1 \leq s \leq t \leq n \), we have \( 1 \leq b \leq d \leq \lfloor \frac{n}{2} \rfloor \). In this setting, by Proposition 7.1, \( \mathbb{I}[s, t] \in \mathcal{Y} \) if and only if \( \tilde{\mu}_1(\mathbb{I}[s, t]) \) is closed or right-open. We use \( \mathcal{Y}_c, \mathcal{Y}_{ro} \) to denote the sets of interval representations \( \mathbb{I}[s, t] \in \mathcal{Y} \) which correspond to closed or right-open interval representations of \( ZZ \), respectively. Similarly, we use \( \mathcal{X}_o, \mathcal{X}_{lo} \) to denote the sets of interval representations \( \mathbb{I}[s, t] \in \mathcal{X} \) which correspond to open or left-open interval representations of \( ZZ \), respectively.
From the proof of Proposition 7.1, we recall that $s$ is odd if and only if $\mu_1([s, t])$ is closed or right-open, and that $t$ is odd if and only if $\mu_1([s, t])$ is closed or left-open. Let $I$ be an interval representation of $A_n(z_1)$, and let us set

$$S_I := \{s = 1, \ldots, n \mid \text{Hom}([s, s], I) \neq 0 \text{ or } \text{Hom}(I, [s, s]) \neq 0\}.$$

In this case, $I[s, s]$ is simple projective if $s$ is odd, and is simple injective otherwise. Consequently, $\text{Hom}(I, I[s, s]) = 0$ if $s$ is odd, and $\text{Hom}(I, [s, s]) = 0$ otherwise. Thus, we have $I = [s, t]$ with $s := \min S_I$ and $t := \max S_I$. Note that simple projective representations are source vertices and simple injective representations are sink vertices in the AR quiver (see Assem et al. 2006, Chapter IV.3, 3.6 Corollary).

Then $Y_c, Y_{co}, X_s$, and $X_{oc}$ can be described in the AR quiver $\Gamma(A_n(z_1))$ of $A_n(z_1)$ as in Figure 5:

\[\Gamma(A_n(z_1))\]

Figure 5. Division of the AR quiver $\Gamma(A_n(z_1))$ of $A_n(z_1)$

It is noteworthy that Meehan and Meyer (2020) give the same division of the AR quiver of purely zigzag persistence modules as our model.

The division gives us the following correspondence between the interval representation of $A_n$ and $\mathbb{Z}Z$.

**Lemma 7.4.** Let $[s, t]$ be an interval representation of $A_n(z_1)$. Then for the interval representation $I^{(b, d)}_{\mathbb{Z}Z} := \mu_1([s, t])$ of $\mathbb{Z}Z$, we have the corresponding representation $\mathbb{Z}Z[x, y] \in \text{rep}_{\mathbb{Z}Z} A_n$ of $[s, t]$, where $(x, y)$ is given by the following:

\[
\begin{align*}
(x, y) &= (b, n - d + 1) \quad \langle b, d \rangle_{\mathbb{Z}Z} = [b, d]_{\mathbb{Z}Z} \quad (s = 2b - 1, t = 2d - 1), \\
(x, y) &= (b, d - 1) \quad \langle b, d \rangle_{\mathbb{Z}Z} = [b, d]_{\mathbb{Z}Z} \quad (s = 2b - 1, t = 2d - 2), \\
(x, y) &= (n - d + 2, n - b + 1) \quad \langle b, d \rangle_{\mathbb{Z}Z} = [b, d]_{\mathbb{Z}Z} \quad (s = 2b, t = 2d - 1), \\
(x, y) &= (d, n - b + 1) \quad \langle b, d \rangle_{\mathbb{Z}Z} = [b, d]_{\mathbb{Z}Z} \quad (s = 2b, t = 2d - 2).
\end{align*}
\]

**Proof.** The endpoint formulas can be easily calculated by Proposition 6.3 and the above discussion. \(\square\)

By Botnan and Lesnick (2018, Lemma 3.1, Lemma 4.1), we have the following calculation of $d_{BL}$ for the 4 kinds of representations above.

**Proposition 7.5** (Botnan and Lesnick 2018, Lemma 3.1, Lemma 4.1). Let $\langle b, d \rangle_{\mathbb{Z}Z}, \langle e, f \rangle_{\mathbb{Z}Z}$ be intervals of $\mathbb{Z}Z$. Then the following holds.

\[
d_{BL}([b, d]_{\mathbb{Z}Z}, 0) = \begin{cases} 
\infty, & \text{if } [b, d]_{\mathbb{Z}Z} \text{ is closed} \\
\frac{1}{2}|d - b|, & \text{if } [b, d]_{\mathbb{Z}Z} \text{ is half-open} \\
\frac{1}{2}|d - b|, & \text{if } [b, d]_{\mathbb{Z}Z} \text{ is open}
\end{cases}
\]
Moreover, if \((b,d)_\mathbb{Z}, (e,f)_\mathbb{Z}\) have the same type, then
\[
d_{BL}(I^{(b,d)}_{\mathbb{Z}}, I^{(e,f)}_{\mathbb{Z}}) = \min \left\{ \max\{|b-e|, |d-f|\}, \max\{d_{BL}(I^{(b,d)}_{\mathbb{Z}}, 0), d_{BL}(I^{(e,f)}_{\mathbb{Z}}, 0)\} \right\}.
\]
Otherwise,
\[
d_{BL}(I^{(b,d)}_{\mathbb{Z}}, I^{(e,f)}_{\mathbb{Z}}) = \max\{d_{BL}(I^{(b,d)}_{\mathbb{Z}}, 0), d_{BL}(I^{(e,f)}_{\mathbb{Z}}, 0)\}.
\]

Then, Proposition 7.1 and Proposition 7.5 lead to the following.

**Proposition 7.6.** Let \(I[s,t], I[u,v]\) be interval representations of \(A_n(z_1)\). For interval representations \(I^{(b,d)}_{\mathbb{Z}} := \mu(I[s,t]), I^{(e,f)}_{\mathbb{Z}} := \mu(I[u,v])\) of \(\mathbb{Z}\), the following inequalities hold:

1. \(d_{BL}(I^{(b,d)}_{\mathbb{Z}}, 0) \leq d^{\ast}((I[s,t], 0)\) if \(I[s,t] \in \mathcal{Y}_c, \mathcal{X}_o, \mathcal{X}_{oc}\),
2. \(d^{\ast}(I[s,t], 0) < d_{BL}(I^{(b,d)}_{\mathbb{Z}}, 0)\) is \(\infty\) if \(I[s,t] \in \mathcal{Y}_c\),
3. \(d_{BL}(I^{(b,d)}_{\mathbb{Z}}, I^{(e,f)}_{\mathbb{Z}}) \leq d^{\ast}(I[s,t], I[u,v])\) if \(I[s,t], I[u,v] \in \mathcal{Y}_c, \mathcal{X}_o\) or \(\mathcal{X}_{oc}\),
4. \(d^{\ast}(I[s,t], I[u,v]) \leq d_{BL}(I^{(b,d)}_{\mathbb{Z}}, I^{(e,f)}_{\mathbb{Z}})\) if \(I[s,t] \in \mathcal{Y}_c\).

**Proof.** In each case, the value of \(d_{BL}\) can be calculated by Proposition 7.5. On the other hand, the value of \(d^{\ast}\) in each case can be calculated by Lemma 7.4, Corollary 6.5, and Corollary 6.6.

Then we have
\[
d^{\ast}(I[s,t], 0) = \begin{cases} \left\lfloor \frac{1}{2}n - (b + d) \right\rfloor, & I[s,t] \in \mathcal{Y}_c, \\
\left\lceil \frac{1}{2}n - (b + d) + 2\right\rceil, & I[s,t] \in \mathcal{Y}_o, \mathcal{Y}_{oc}\end{cases}
\]
When \(1 \leq b \leq d \leq \left\lfloor \frac{n}{2} \right\rfloor\), it is easy to check the inequality \(|d-b| < |n-(b+d)+2|\). Thus, we obtain inequality (1). Since \(d^{\ast}(I[s,t], 0) < \infty\) always holds, we obtain inequality (2).

Moreover, by the symmetry of the distance, \(d := d^{\ast}(I[s,t], I[u,v])\) can be calculated as follows.

(a) if \(I[s,t], I[u,v] \in \mathcal{Y}_c\), then
\[
d = \min\{\max\{|b-e|, |d-f|\}, \max\{\left\lfloor \frac{1}{2}n - (b + d) + 2\right\rfloor, \left\lceil \frac{1}{2}n - (e + f) + 2\right\rceil\}\},
\]
(b) if \(I[s,t], I[u,v] \in \mathcal{Y}_o\) or \(I[s,t], I[u,v] \in \mathcal{X}_{oc}\), then
\[
d = \min\{\max\{|b-e|, |d-f|\}, \max\{\left\lfloor \frac{1}{2}n - (b + d) + 2\right\rfloor, \left\lfloor \frac{1}{2}n - (e + f) + 2\right\rfloor\}\},
\]
(c) if \(I[s,t], I[u,v] \in \mathcal{X}_o\), then
\[
d = \min\{\max\{|b-e|, |d-f|\}, \max\{\left\lfloor \frac{1}{2}n - (b + d) + 2\right\rfloor, \left\lfloor \frac{1}{2}n - (e + f) + 2\right\rfloor\}\},
\]
(d) if \(I[s,t] \in \mathcal{Y}_c, I[u,v] \in \mathcal{Y}_o\), then
\[
d = \min\{\max\{|b-e|, |n-(d+f)+2|\}, \max\{\left\lfloor \frac{1}{2}n - (b + d) + 2\right\rfloor, \left\lceil \frac{1}{2}f - e\right\rceil\}\},
\]
(e) if \(I[s,t] \in \mathcal{X}_o, I[u,v] \in \mathcal{X}_c\), then
\[
d = \min\{\max\{|b-e|, |n-(d+f)+2|\}, \max\{\left\lfloor \frac{1}{2}d - b\right\rfloor, \left\lfloor \frac{1}{2}n - (e + f) + 2\right\rfloor\}\},
\]
and
(f) if \(I[s,t] \in \mathcal{Y}, I[u,v] \in \mathcal{X}\), then
\[
d = \max\{d^2(I[s,t], 0), d^2(I[u,v], 0)\}.
\]
Since the inequality $|b - d| < |n - (b + d)| + 2$ (1 ≤ b ≤ d ≤ ⌈n/2⌉) holds, by inequality (1), inequality
\[
d_{BL}(\|b,d\|^x, \|e,f\|^z) \leq d_1(\|s,t\|, \|u,v\|)
\]
holds in cases (b), (c), and (f) except for when \(\|s,t\| \in \mathcal{Y}_c\) in case (f). Thus, we obtain inequality (3).

In case (f), if \(\|s,t\| \in \mathcal{Y}_c\), then \(d_{BL}(\|b,d\|^z, 0) = \infty\), and hence it is obvious that
\[
d_1(\|s,t\|, \|u,v\|) < d_{BL}(\|b,d\|^z, \|e,f\|^z).
\]

In case (a), since \(\|s,t\|, \|u,v\| \in \mathcal{Y}_c\), \(d_{BL}(\|b,d\|^z, 0) = d_{BL}(\|e,f\|^z, 0) = \infty\). Then by definition, the inequality
\[
d_1(\|s,t\|, \|u,v\|) \leq d_{BL}(\|b,d\|^x, \|e,f\|^z) = \max\{|b - e|, |d - f|\}
\]
holds.

In case (d), since \(\|s,t\| \in \mathcal{Y}_c\), \(\|u,v\| \in \mathcal{Y}_ca\),
\[
d_{BL}(\|b,d\|^z, \|e,f\|^z) = \max\{d_{BL}(\|b,d\|^z, 0), d_{BL}(\|e,f\|^z, 0)\} = \infty.
\]

Then it is obvious that \(d_1(\|s,t\|, \|u,v\|) < d_{BL}(\|b,d\|^z, \|e,f\|^z)\).

Thus, we obtain inequality (4).

\[\square\]

Remark 7.7. In Proposition 7.6, the case in which \(\|s,t\| \in \mathcal{X}_{oc}\), \(\|u,v\| \in \mathcal{X}_o\) remains. In this case, we have
\[
d_{BL}(\|b,d\|^z, \|e,f\|^z) = \max\{d_{BL}(\|b,d\|^z, 0), d_{BL}(\|e,f\|^z, 0)\} = \max\{\frac{1}{2}|b - d|, \frac{1}{4}|f - e|\},
\]
and \(d_{BL}\) and \(d_1\) are incomparable for large \(n\). For example, in the case that \(n = 7\), we consider representations \(\|2,7\|, \|2,6\|,\) and \(\|1,2\|\) of \(A_n(z_1)\). Then we have \(\bar{\mu}_1(\|2,7\|) = \|1,4\|^z, \bar{\mu}_1(\|2,6\|) = \|1,4\|^z,\) and \(\bar{\mu}_1(\|1,2\|) = \|1,2\|^z\). By Proposition 7.5, Lemma 7.4, and Corollary 6.5, and Corollary 6.6, the inequalities
\[
d_{BL}(\|1,4\|^z, \|1,4\|^z) \geq d_1(\|2,7\|, \|2,6\|)
\]
and
\[
d_{BL}(\|1,4\|^z, \|1,2\|^z) \leq d_1(\|2,7\|, \|2,2\|)
\]
hold, as desired.

We conclude that the block distance \(d_{BL}\) in Botnan and Lesnick (2018) and our induced distance \(d_1\) are incomparable. Indeed, Proposition 7.6 (3) and (4) inform us that the inequality for comparing the distances is dependent on interval type.

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Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.
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