Rigorous description of holograms of particles illuminated by an astigmatic elliptical Gaussian beam

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Abstract The digital holography is a non-intrusive optical metrology and well adapted for the measurement of the size and velocity field of particles in the spray of a fluid. The simplified model of an opaque disk is often used in the treatment of the diagrams and therefore the refraction and the third dimension diffraction of the particle are not taken into account. We present in this paper a rigorous description of the holographic diagrams and evaluate the effects of the refraction and the third dimension diffraction by comparison to the opaque disk model. It is found that the effects are important when the real part of the refractive index is near unity or the imaginary part is non zero but small.

1. INTRODUCTION

It is essential for the laboratory research to study the combustion engine in an automobile or aircraft and industrial processes in order to improve the efficiency and reduce the pollutant emission. The digital holography is a non-intrusive optical method well adapted for the measurement of the size and 3D velocity fields of particles in a spray of fluid.

The scalar theory of diffraction is often used to describe the recording and reconstruction of particle holograms. The mathematical tools like wavelet transformation or fractional Fourier transformation give a useful tool to perform the reconstruction. However, the particle is often considered as an opaque disk and the laser beam used is non-polarized (Nicolas et al, 2005; Nicolas, 2007). As a result, the effects of the third dimension diffraction and the polarization of the incident beam are not taken into account.

A study based on a rigorous theoretical tool – Generalized Lorenz-Mie Theory (GLMT) (Goesbet et al 1988; Ren 1995) has been conducted to predict the holographic image of a spherical particle illuminated by different forms of laser beam – plane wave, circular Gaussian beam and astigmatic elliptical Gaussian beam (AEG beam). The influences of the third dimension diffraction, the complex refractive index, the position of the particle, and the polarization and the nature of the incident beam are evaluated by comparison between the holograms obtained by this method and the images of an opaque disk predicted by diffraction theory.

The paper is organized as follows. Section 2 recalls the principles of the two models used in the paper to describe the hologram – opaque disk and rigorous electromagnetic scattering of a
homogeneous sphere. The simulated results by the two models are presented and compared in section 3 to evaluate the effects of the refraction and the third dimension diffraction. Section 4 is devoted to conclusions.

2. PRINCIPLES OF HOLOGRAM SIMULATION

2.1 Experimental setup of digital in-line holography

The system of digital holography under study is composed of a laser, a beam collimator and a camera, as shown in Fig. 1. A particle located at a distance \( z \) from the camera is illuminated by the laser beam and the hologram is recorded by the camera. To obtain a collimated beam, a lens of focal length \( f = 200 \) mm is added between the laser and the particle. A astigmatic elliptical Gaussian beam is obtained by a cylindrical lens of focal length \( f_e \) and \( f_h \). The obtained beam has a diameter of \( 2w_0 = 14 \) mm.

![Experimental setup for in-line hologram recording.](image)

2.2 Model of opaque disk - scalar theory of diffraction

Consider an opaque disk of diameter \( d \) located at the center of the plane \((\xi, \eta)\) with a distance \( z \) from the CCD camera as shown in Fig. 1. According to the Huygens-Fresnel integral, the amplitude of the electromagnetic field in the \((x,y)\) plane is given by:

\[
A_i(x, y) = \exp\left(\frac{i 2\pi}{\lambda z} z\right) \int \int E(\xi, \eta) \left[1 - \text{circ}(\xi, \eta)\right] \exp\left\{\frac{i \pi}{\lambda z} \left[(\xi - x)^2 + (\eta - y)^2\right]\right\} d\xi d\eta
\]  

where \( E(\xi, \eta) \) is the scalar field of the incident beam in the \((\xi, \eta)\) plane, \( \lambda \) is the wavelength and \( \text{circ}(\xi, \eta) \) the transmittance function in the object plane defined by:

\[
\text{circ}(\xi, \eta) = \begin{cases} 
1 & \text{if } \sqrt{\xi^2 + \eta^2} < d/2, \\
0 & \text{else}.
\end{cases}
\]

In the cases of circular Gaussian beam and astigmatic elliptical Gaussian beam on-axis illumination, the integration (1) can be carried out analytically (see part 4.2 in Nicolas 2007). For example, when the disk is illuminated by a plane wave, the amplitude in the image plane is given by:

\[
A_i = \exp\left(\frac{i 2\pi}{\lambda z} z\right) \int \int \left[1 - \text{circ}(\xi, \eta)\right] \exp\left\{\frac{i \pi}{\lambda z} \left[(\xi - x)^2 + (\eta - y)^2\right]\right\} d\xi d\eta
\]  

In far-field case \((\lambda z \gg 1)\), the intensity distribution on the image plane can be calculated by:
\[ I(x, y) = A(x, y) A^*(x, y) = 1 - \frac{2}{\lambda z} \sin \left( \frac{\pi}{\lambda z} (x^2 + y^2) \right) F(x, y) \]  \hspace{1cm} (4)

where

\[ F(x, y) = \frac{J_1}{2} \left( \frac{\pi d}{\lambda z} \sqrt{x^2 + y^2} \right) \]  \hspace{1cm} (5)

\( J_1 \) is the first kind Bessel function of the first order. In the more general cases the analytical expressions given by F. Nicolas (Nicolas, 2007) are used to predict the holographical diagrams of an opaque disk.

### 2.3 Model of sphere – Generalized Lorenz-Mie theory

The generalized Lorenz-Mie theory (GLMT) is rigorous theory to describe the scattering of shaped beam by a spherical, homogeneous and isotropic particle (Gouesbet et al. 1988; Ren, 1995). All the particle parameters and the incident beam characteristics, such as refractive index, position of particle and polarization of incident beam, are taken into account. The particle can be located at any point in the beam (Ren, 1995; Ren et al., 1998).

In the framework of GLMT, the holographic diagram of a sphere is obtained by the superposition of the indent field and the scattered one (Ren, 1995; Girasole et al., 2000). The total intensity on the image plane is given by

\[ I_{\text{total}} = \left| E_{\text{total}} \right|^2 \]  \hspace{1cm} (6)

where the total electric field is the summation of the incident beam and the scattered wave: \( E_{\text{total}} = E_{\text{incident}} + E_{\text{scattered}} \).

![Fig. 2. Geometry of scattering problem by GLMT](image)

### 3. SIMULATION DIAGRAMS AND DISCUSSION

We present in this section the holographic diagrams predicted by the two methods. By comparison of the diagrams we investigate the effects of the refraction and the third dimension diffraction of the particle according to its optical properties such as the refractive index of the particle, the polarization and the form of the beam.

#### 3.1 Holograms of different beam

#### 3.1.1 Case of a plane wave

Suppose an opaque disk of diameter \( d = 100\mu\text{m} \) located at a distance \( z = 80 \text{ mm} \) from the CCD camera is illuminated by a plane wave of wavelength \( \lambda = 632.8\text{nm} \). The intensity distribution
predicted by the opaque disk model is illustrated in Fig. 3. The images simulated by the GLMT as function of complex refractive index are presented in Fig. 4-5.

Fig. 3. Intensity distribution observed at a distance $z = 80\text{mm}$ of an opaque disk of diameter $d = 100\mu\text{m}$ illuminated by a plane wave of wavelength $\lambda = 632.8\text{nm}$.

Fig. 4. Holographic images of a sphere simulated by the GLMT with the refractive index $m = (1.33, 0)$ or $m = (1.33, 0.1)$, in the same configuration as Fig. 3.

Fig. 5. The same case as Fig. 4. except $m = (1.33, 0.01)$. 
By comparison of the Figs. 4 and 5 to Fig. 3, we find that the form of holographic images simulated by the GLMT is similar to that of the opaque disk. But in the case \( m = (1.33, 0.01) \), an obvious asymmetry is observed. This asymmetry is due to the polarization of incident beam. To confirm this hypothesis, a study on the polarization influence has been carried out and will be presented in section 3.3.

To quantify the difference between the two models, we compared the intensity along \( x \) axis in the image plane in Fig. 6 for different refractive index \( m = (1.33, 0.1) \), \( m = (1.33, 0.005) \) and \( m = (1.33, 0.01) \). We find that when \( m = (1.33, 0.1) \), the results of two models are in good agreement. In other words, in this case, a spherical particle can be considered as an opaque disk. In the contrary, when \( m = (1.33, 0.005) \) and \( m = (1.33, 0.01) \), the difference between the two models is evident. That is to say, in these two cases, we can not simplify a spherical particle as an opaque disk. The refraction and diffraction of third dimension must be taken into account.

![Fig. 6](image)

Fig. 6. Comparison of intensity distribution along \( x \) axis obtained by the two models with different refractive index in the same configuration as Fig. 3.

3.1.2 Case of a circular Gaussian beam

The diffraction pattern predicted by the scalar theory of diffraction is shown in Fig. 7 for an opaque disk of a diameter \( d = 150\mu m \) illuminated by a circular Gaussian beam and located at \( z = 120mm \) from the CCD camera. The local waist radii and curvatures of the Gaussian beam in the object plane are \( w_e = w_h = 1.75mm \) and \( R_e = R_h = 50mm \). The holographic diagram simulated by the GLMT in the same case is shown in Fig. 8. We find that the form of holographic images simulated by the GLMT is similar to that obtained by the model of an opaque disk. Similar as in the plane wave case, the difference of diagrams obtained by the two models is also observable.

3.1.3 Case of an astigmatic elliptical Gaussian beam

In the case of a sphere illuminated by an AEG beam, the diffraction patterns are no longer symmetry since the beam is asymmetry. The diffraction pattern obtained by opaque disk model in the same case as Fig. 7 is shown as Fig. 9, but disk is located at \( z = 107mm \) after the beam waist. Conversely, Fig. 10. presents the diffraction pattern observed at \( z = 154mm \) where the disk is before the beam waist. By comparison to the diagrams simulated with GLMT given in Figs. 11 and 12 for the same cases, similar comments to the circular Gaussian beam illumination case can be made for the effects of the refractive index on the holographic digrams.
Fig. 7. Diffraction pattern of an opaque disk $d=150\mu m$ illuminated by a circular Gaussian beam $w_c = w_\theta = 1.75 mm$, $R_e = R_\theta = 50 mm$ and $z = 120 mm$.

Fig. 8. Holographic images simulated by the GLMT for a sphere of diameter $d=150\mu m$ and refractive index $n_\perp = (1.33, 0)$ or $n_\perp = (1.33, 0.1)$ located at the distance $z=120 mm$ illuminated by a circular Gaussian beam of wavelength $\lambda = 632.8 nm$.

Fig. 9. Diffraction pattern of an opaque disk illuminated by an AEG beam $d=150\mu m$, $w_c = 7 mm$, $w_\theta = 1.75 mm$, $R_e = -2.36 \times 10^5 m$, $R_\theta = -50 mm$ et $z = 107 mm$. 
3.2 Effect of refractive index

In the previous section, we have chosen certain values of refractive index for the simulation and compared the holographic diagrams of two models. In this section, the effects of refractive index will be assessed systematically. Fig. 13, displays the intensity at the centre of image as function of imaginary part of refractive index with two values of real part of refractive index.
Fig. 13. Intensity at the centre of image as function of the imaginary part of refractive index.

We find that the intensity is very sensitive to the imaginary part of refractive index, especially when the real part of index equals to 1.087. That is to say, when the real part of refractive index is near unity, a spherical particle cannot be considered as an opaque disk.

We have also examined the effects of real part of refractive index and we can conclude that if the imaginary part of refractive index is null or relatively important (0.1 for example), the image remains the same whatever the real part of refractive index between 1 and 1.8. But if the imaginary part is relatively small but not null (0.01 or 0.005 for example), the real part of refractive index plays an important role in the form of the diagram (see Fig. 14).

Fig. 14. Images simulated by the GLMT with imaginary part of refractive index 0.01 in the same configuration Fig. 3.
Fig. 15. represents the intensity at the centre of the image as function of real part of refractive index with four values of imaginary part. We find that when the imaginary part is 0 and 0.1, the effects of the real part of refractive index on holograms are not obvious at all. But when the imaginary part equals to 0.005 and 0.01, the real part of refractive index plays a role in holograms, especially when the real part of index is close to 1.

![Fig. 15. Intensity evolution at centre of image in function of different real part of refractive index.](image)

3.3 Effect of light polarization

Fig. 16-18. illustrate the simulated holographic images of a sphere illuminated by a plane wave in three polarization states. The intensity distributions along x axis for this three cases are compared in Fig. 19. Naturally, the image of polarization in y corresponds to the image of polarization in x but turned by 90 degrees, and the image in non-polarized case is perfectly axisymmetric.

In the cases of a sphere illuminated by a circular Gaussian beam and an AEG beam, there is also the polarization influence for absorbing particles (Re(m)=0.003), but the influence is less obvious relative to the case of a plane wave.

![Fig. 16. Image simulated by the GLMT at z=80mm for a sphere of diameter d=100µm and refractive index m= (1.33, 0.01) illuminated by a plane wave polarized in the x direction of wavelength λ=632.8nm.](image)
Fig. 17. The same case as Fig. 16. but the incident wave is polarized in the y direction.

Fig. 18. The same case as Fig. 17. but the incident wave is non-polarized.

Fig. 19. Intensity distributions along the axis x with the three polarization states.
3.4 Effect of particle position

Since the AEG beam used in the simulation is much more focused in $y$ direction than in $x$ direction, we just move the particle along $y$ axis. We find that the figure structure remains the same but the images moved in the opposite direction (Fig. 20).

Fig. 20. Images simulated by GLMT for a sphere of $d=150\mu$m and $m= (1.33, 0.003)$ located off optical axis ($x_0=0$, $y_0=100\mu$m) and illuminated by a AEG beam of $\lambda=632.8$nm.

4. CONCLUSIONS

In this paper, the holograms of a homogeneous sphere are simulated by a rigorous model based on the generalized Lorenz-Mie theory and compared with those obtained by a simplified model of an opaque disk in order to evaluate the various effects of the refraction and third dimension diffraction of a particle, such as the refractive and the position of particle, the form and the polarization of the incident beam. It is found that the effects of the third dimension (diffraction and refraction ) is obvious when the imaginary part of refractive index is moderate (not zero nor too large) or the real part is near unity. We find also that the polarization effects is more important in the plane wave case than in the circular or astigmatic elliptical Gaussian beam illumination cases.

NOMENCLATURE

| Symbol | Description                                    | Unit       |
|--------|------------------------------------------------|------------|
| $f$    | focal length of lens                           | [mm]       |
| $A$    | complex amplitude of EM field                  | [V/m]      |
| $E$    | electrical field                               | [V/m]      |
| $d$    | particle diameter                              | [mm]       |
| $I$    | intensity                                      | [V^2/m^2]  |
| $m$    | complex refractive index of particle           |            |
| $Im(m)$| imaginary part of refractive index             |            |
| $Re(m)$| real part of refractive index                  |            |
\[ R \quad \text{curvatures of the wave front} \quad [\text{mm}] \]
\[ w \quad \text{waist radius} \quad [\text{mm}] \]

**Greek Letters**

\[ \lambda \quad \text{wavelength} \quad [\text{nm}] \]

**Subscripts**

\[ \xi \quad \text{coordinate in the image plane} \]
\[ \eta \quad \text{coordinate in the image plane} \]

**REFERENCES**

Girasole, T. et al (2000). "Particle imaging : GLMT simulation, J. of Visualization, 3(2), pp195-202.

Gouesbet, G. et al (1988). "Light scattering from a sphere arbitrarily located in a Gaussian beam, using a Bromwich formulation," J. Opt. Soc. Am. A 5, pp1427-1443.

Nicolas, F. et al (2005). "Application of the fractional Fourier transformation to digital holography recorded by an elliptical, astigmate Gaussian beam," J.Opt. Soc. Am. A. 22(11), pp2569-2577

Nicolas, F. (2007). "Holographie numérique femtoseconde : Modélisation et restitution par transformation de Fourier fractionnaire," PhD thesis, University of Rouen, France.

Ren, K. F. (1995). "Diffusion des Faisceaux Feuille Laser par une Particule Sphérique et Application aux Ecoulements Diphasiques," PhD Thesis, University of Rouen, France.

Ren, K. F. et al (1998). "The integral localized approximation in generalized Lorenz-Mie theory," Appl. Opt., 37(19), pp.4218-4225.