ON RICCI COEFFICIENTS OF NULL HYPERSURFACES WITH TIME FOLIATION IN EINSTEIN VACUUM SPACE-TIME

Qian Wang

Abstract. The main objective of this paper is to control the geometry of null cones with time foliation in Einstein vacuum spacetime under the assumptions of small curvature flux and a weaker condition on the deformation tensor for T. We establish a series of estimates on Ricci coefficients, which plays a crucial role to prove the improved breakdown criterion in [12].

1. Introduction

Consider a (3+1)-dimensional Einstein vacuum spacetime \((M, g)\) foliated by \(\Sigma_t\) which are level hypersurfaces of a time function \(t\) monotonically increasing towards the future. Let \(\mathbf{D}\) and \(\nabla\) denote the covariant differentiations with respect to \(g\) and the induced metric \(g\) on \(\Sigma_t\) respectively. We define on each \(\Sigma_t\) the lapse function \(n\) and the second fundamental form \(k\) by

\[ n := (-g(\mathbf{D}t, \mathbf{D}t))^{1/2} \quad \text{and} \quad k(X, Y) := -g(\mathbf{D}X T, Y), \]

where \(T\) denotes the future directed unit normal to \(\Sigma_t\) and \(X, Y \in T\Sigma_t\). For any coordinate chart \(O \subset \Sigma_{t_0}\) with coordinates \(x = (x^1, x^2, x^3)\), let \(x^0 = t, x^1, x^2, x^3\) be the transported coordinates obtained by following the integral curves of \(T\). Under these coordinates the metric \(g\) takes the form

\[ g = -n^2 dt^2 + g_{ij} dx^i dx^j, \quad \partial_t g_{ij} = -2nk_{ij}. \]

1.1. Main result. Consider an outgoing null cone contained in \((M, g)\), whose vertex is denoted by \(p\) and intersections with \(\Sigma_t\) are denoted by \(S_t\). The null vector \(l_\omega\) in \(T_p M\) parametrized with \(\omega \in S^2\), is normalized by \(g(l_\omega, T_p) = -1\). We denote by \(\Gamma_\omega(s)\) the outgoing null geodesic from \(p\) with \(\Gamma_\omega(0) = p, \quad \frac{d}{ds}\Gamma_\omega(0) = l_\omega\) and define the null vector field \(L\) by

\[ L(\Gamma_\omega(s)) = \frac{d}{ds}\Gamma_\omega(s). \]

Then \(\mathbf{D}_t L = 0\). The affine parameter \(s\) of null geodesic is chosen such that \(s(p) = 0\) and \(L(s) = 1\). Let \(\mathcal{H} = \cup_{0 < t \leq 1} S_t\), \(t(p) = 1\) and suppose the exponential map \(G_t : \omega \rightarrow \Gamma_\omega(s(t))\) is a global diffeomorphism from \(S^2\) to \(S_t\) for any \(t \in (0, 1]\). We now define a conjugate null vector \(\mathbf{L}\) on \(\mathcal{H}\) with \(g(\mathbf{L}, \mathbf{L}) = -2\) and such that

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\[ ^{1}\text{we can always suppose } t(p) = 0 \text{ and } 0 \leq t \leq 1 \text{ on a null cone by a standard rescaling of the coordinates } (t, x) \text{ of } (M, g). \text{ The normalized time function, for simplicity, is still denoted by } t. \]
$\mathbf{l}$ is orthogonal to the leaves $S_t$. In addition we can choose an orthonormal frame $(e_A)_{A=1,2}$ tangent to $S_t$ such that $(e_A)_{A=1,2}$, $e_3 = L$, $e_4 = \mathbf{l}$ form a null frame, i.e.

$$g(L, \mathbf{l}) = -2, \ g(L, L) = g(\mathbf{l}, \mathbf{l}) = g(L, e_A) = g(\mathbf{l}, e_A) = 0, \ g(e_A, e_B) = \delta_{AB}. $$

Let $a^{-1} = -\langle L, T \rangle$ with $a(p) = 1$. It follows that along any null geodesic $\Gamma_\omega$, there holds

$$\frac{dt}{ds} = n^{-1}a^{-1}, \ t(p) = 0. $$

Let $N$ be the outward unit normal of $S_t$ on $\Sigma_t$. Then

$$T = \frac{1}{2}(aL + a^{-1}L), \ N = \frac{1}{2}(aL - a^{-1}L). $$

We define the Ricci coefficients $\chi, \check{\chi}, \zeta, \check{\zeta}, \varpi$ via the frame equations

$$D_A L = \check{\chi}_{AB} e_B - \zeta_A L, \quad D_A \mathbf{l} = \check{\chi}_{AB} e_B + \zeta_A \mathbf{l}, \quad D_L e_A = 2\check{\zeta}_A e_A, \quad D_L L = 2\check{\zeta}_A e_A - 2\varpi \mathbf{l}. $$

Thus we also have

$$D_L e_A = \nabla_L e_A + \check{\zeta}_A L, \quad D_B e_A = \nabla_B e_A + \frac{1}{2}\check{\chi}_{AB} e_3 + \frac{1}{2}\check{\zeta}_{AB} e_4 $$

where $\nabla$ denotes the covariant derivative restricted on $S_t$.

Let $\lambda = -\frac{1}{3}\text{Tr}k$, where $\text{Tr}k = g^{ij}k_{ij}$. We decompose $\check{k}_{ij} := k_{ij} + \lambda g_{ij}$, the traceless part of $k$, relative to the orthonormal frame $\{N, e_A, A = 1, 2, \}$ along null cone $\mathcal{H}$ by introducing the following components

$$\eta_{AB} = \check{k}_{AB}, \quad \epsilon_A = \check{k}_{AN}, \quad \delta = \check{k}_{NN}. $$

Denote by $\dot{\eta}_{AB}$ the traceless part of $\eta$. Since $\delta^{AB}\eta_{AB} = -\delta$, it is easy to see

$$\dot{\eta}_{AB} = \eta_{AB} + \frac{1}{2}\delta_{AB}\delta. $$

We denote by $\nabla$ one of the following $S_t$ tangent tensors $\{\nabla, \partial, \nabla, \log n, -\nabla N, \log n \}$. It is easy to check by definition that the Ricci coefficients $\check{\zeta}, \check{\chi}, \nu$ verify

$$\nu := -L(a) = -\nabla_N \log n + \delta - \lambda, $$

$$\zeta_A = \nabla_A \log a + \epsilon_A, \quad \check{\zeta}_A = \nabla_A \log n - \epsilon_A. $$

Let us define $\theta_{AB} := (D_A N, e_B)$. By definition of $\chi, \check{\chi}$ and (1.3), it follows that

$$ax_{AB} = \theta_{AB} - k_{AB}, \quad a^{-1}\check{x}_{AB} = -\theta_{AB} - k_{AB}, \quad a^{-1}\check{r}_A = -\text{tr}\theta + \delta + 2\lambda. $$

We define the null components of Riemannian curvature tensor relative to $t$-foliation,

$$\alpha_{AB} = R(L, e_A, L, e_B), \quad \beta_A = \frac{1}{2}R(e_A, L, \mathbf{l}, L), $$

$$\rho = \frac{1}{4}R(L, L, \mathbf{l}, L), \quad \sigma = \frac{1}{4}R(L, L, L, L), $$

$$\beta_A = \frac{1}{2}R(e_A, L, \mathbf{l}, L), \quad \alpha_{AB} = R(L, e_A, L, e_B). $$
We define also the mass aspect functions \( \mu \) and \( \mu^\ast \) as follows

\begin{align}
(1.10) \quad \mu &= -\frac{1}{2} D_3 \text{tr} \chi + \frac{a^2}{4} (\text{tr} \chi)^2 - \omega \text{tr} \chi, \\
(1.11) \quad \mu^\ast &= D_3 \text{tr} \chi + \frac{1}{2} \text{tr} \chi \cdot \text{tr} \chi.
\end{align}

Denote \( \gamma_i := \gamma(t, \omega) \) the induced metric of \( g \) on \( S_t \), relative to normal coordinates \( \omega = (\omega_1, \omega_2) \) in the tangent space at \( p \). Define the radius function of \( S_t \) to be \( r(t) = \sqrt{(4\pi)^{-1}|S_t|} \) and define the metric \( \tilde{\gamma} \) by \( \tilde{\gamma} = r^{-2} \gamma \). We denote by \( \gamma^{(0)} \) the canonical metric on \( S^2 \). On each \( S_t \) we introduce the ratio of area elements

\begin{equation}
(1.12) \quad v_t(\omega) := \frac{\sqrt{\gamma^{(0)}}}{\sqrt{\gamma}}, \quad \omega \in S^2.
\end{equation}

For smooth scalar functions \( f \), the average of \( f \) on \( S_t \) is defined by \( \bar{f} := \frac{1}{|S_t|} \int_{S_t} f \, d\mu_\gamma \).

For any scalar functions \( f \),

\[ \int_{\mathcal{H}} f := \int_0^1 \int_{S_t} f \, n a d\mu_\gamma \, dt = \int_0^1 \int_{|\omega| = 1} naf v_t d\mu_{S^2} \, dt. \]

We define \( L^p_{\mathcal{H}} \) norm for smooth functions \( f \) on \( S_t \) by \( \|f\|_{L^p_{\mathcal{H}}}^p = \int_{S_t} |f|^p \, d\mu_{S^2} \), and define \( L^2 \) norm on null cone \( \mathcal{H} \) for any smooth function \( f \) by

\[ \|f\|^2_{L^2(\mathcal{H})} = \int_0^1 \int_{S_t} |f|^2 n a d\mu_\gamma \, dt. \]

For simplicity, we will suppress the \( \mathcal{H} \) in the definition of norms on \( \mathcal{H} \) whenever there occurs no confusion. Define for any \( S_t \) tangent tensor \( F \) the norm on \( \mathcal{H} \)

\[ \mathcal{N}_1(F) = \|\nabla_L F\|_{L^2} + \|\nabla F\|_{L^2} + \|r^{-1} F\|_{L^2}. \]

Define \( \mathcal{R}(\mathcal{H}) \), curvature flux on \( \mathcal{H} \) relative to \( t \)-foliation, by

\[ \mathcal{R}(\mathcal{H})^2 = \int_0^1 \int_{S_t} an(|\alpha|^2 + |\beta|^2 + |\rho|^2 + |\sigma|^2 + |\beta|)^2 d\mu_\gamma \, dt. \]

For any \( S_t \)-tangent tensor field \( F \) we define the norm \( \|F\|_{L^\infty_{\mathcal{H}} L^2_{\mathcal{H}}} \) by

\[ \|F\|^2_{L^\infty_{\mathcal{H}} L^2_{\mathcal{H}}} := \sup_{\omega \in S^2} \int_0^1 |F|^2 n a d\gamma := \sup_{\omega \in S^2} \int_{\Gamma_\omega} \int |F|^2 n a d\tau. \]

The main result of this paper is the following

**Theorem 1.1 (Main Theorem).** Let \((M, g)\) be a smooth 3+1 Einstein vacuum spacetime foliated by \( \Sigma_t \), the level hypersurfaces of a time function \( t \) with lapse function \( n \). Consider an outgoing null hypersurface \( \mathcal{H} = \cup_{0 < t < 1} S_t \) in \((M, g)\) initiating from a point \( p \), whose leaves are \( S_t = \Sigma_t \cap \mathcal{H} \) and \( t(p) = 0 \). Assume \( C^{-1} < n < C \) on \( \mathcal{H} \) with certain positive constant \( C \) and assume that

\begin{align}
(1.13) \quad \mathcal{R}(\mathcal{H}) + \mathcal{N}_1(\chi) &\leq \mathcal{R}_0, \text{ on } \mathcal{H}
\end{align}

with \( \mathcal{R}_0 \) sufficiently small. Then the following estimates hold true

\begin{align}
(1.14) \quad \left\| \text{tr} \chi - \frac{2}{s} \right\|_{L^\infty(\mathcal{H})} &\lesssim \mathcal{R}_0^2, \\
(1.15) \quad |a - 1| &\leq \frac{1}{2},
\end{align}
Throughout this paper we will use the notation $A \lesssim B$ to mean $A \leq C \cdot B$ for some appropriate constant $C$.

It is worthy to remark that (1.14) is the most important estimate in the main theorem and none of the estimates among (1.14)-(1.20) can be proved independent of others. Therefore we have to prove all of them simultaneously with a delicate bootstrap argument.

1.2. Application. By a standard rescaling argument, see [7, page 363], the estimates (1.14)-(1.21) in Theorem 1.1 can be rephrased as [12, Theorem 7, Proposition 14], which are the crucial components in the proof of an improved breakdown criterion for Einstein vacuum spacetime in CMC gauge, stated as follows

**Theorem 1.2.** [12, Theorem 1] Let $(\mathcal{M}, g)$ be a globally hyperbolic development of $\Sigma_{t_0}$ foliated by the CMC level hypersurfaces of a time function $t < 0$. Then the space-time together with the foliation $\Sigma_t$ can be extended beyond any value $t_* < 0$ for which,

$$\int_{t_0}^{t_*} \left( \|k\|_{L^\infty(\Sigma_t)} + \|\nabla \log n\|_{L^\infty(\Sigma_t)} \right) dt = K_0 < \infty. \quad (1.22)$$

Under the assumption (1.22) only, we have proved in [12, Section 3] that $C^{-1} < n < C$ with $C$ depending only on $t^*$, $K_0$ and the Bel-Robinson energy on the initial slice $\Sigma_{t_0}$. The condition (1.13) has also been achieved in [12, Theorems 5,6] under the assumption (1.22) with the help of a bootstrap argument (see [12, BA1–BA3]), in particular involved with energy estimate for the geometric wave equation of the second fundamental form $k$. Therefore we can apply Theorem (1.14) to close the proof for Theorem 1.2 ([12, Theorem 1]).

We recall that in [6, 7] the estimates in Theorem 1.1 were obtained under the assumption

$$\sup (\|k\|_{L^\infty(\Sigma_t)} + \|\nabla \log n\|_{L^\infty(\Sigma_t)}) = \Lambda_0 < \infty \quad (1.23)$$

combined with the same assumptions on $\mathcal{R}(\mathcal{H})$ and $n$, both of which can be obtained under (1.23) or the weaker assumption (1.22). The key improvement in Theorem 1.1 lies in that it relies on the weaker assumption

$$\mathcal{N}_1(\xi) \leq R_0, \quad (1.24)$$
which comes naturally as a consequence of (1.22).

Due to the much weaker assumption (1.24), we no longer can adopt the approach contained in [5, 6, 7] to control Ricci coefficients and $|a - 1|$. For instance, let us consider the estimate on $|a - 1|$. As required in [7] and [12], we need to show that

$$|a - 1| \leq \frac{1}{2}. \tag{1.25}$$

For $q > 1$, an assumption of

$$\left( \int_{t_0}^{t_*} \left( \|k\|_{L^\infty(\Sigma)}^q + \|\nabla \log n\|_{L^\infty(\Sigma)}^q \right) \right)^{\frac{1}{q}} < \Lambda_0 < \infty \tag{1.26}$$

by rescaling takes the following form on null cone $\mathcal{H}$

$$\|k\|_{L^q_t L^\infty_x(\mathcal{H})} + \|\nabla \log n\|_{L^q_t L^\infty_x(\mathcal{H})} < \mathcal{R}_0,$$

which by (1.5) and (1.6) immediately gives

$$\|\nu\|_{L^q_t L^\infty_x(\mathcal{H})} + \|\zeta\|_{L^q_t L^\infty_x(\mathcal{H})} \lesssim \mathcal{R}_0. \tag{1.27}$$

In view of the definition $\nu := -\nabla_L a$ and $a(p) = 1$, we can obtain (1.25) by integrating along any null geodesic $\Gamma_\omega$ as long as $\mathcal{R}_0$ is sufficiently small.

If $q = 1$, the above simple argument fails due to the fact that by rescaling, there holds

$$\|k\|_{L^1_t L^\infty_x(\mathcal{H})} + \|\nabla \log n\|_{L^1_t L^\infty_x(\mathcal{H})} < K_0$$

which fails to be small. Thus, it is impossible to derive (1.27) immediately from assumption. Theorem 1.1 however provides the trace estimate for $\|\nu\|_{L^\infty_x L^2_t(\mathcal{H})}$ in (1.17) which is strong enough to guarantee $|a - 1| \leq \frac{1}{2}$.

As remarked after the statement of Theorem 1.1, estimates such as $\|\zeta\|_{L^\infty_x L^2_t(\mathcal{H})}$ are indispensable for establishing (1.14), (1.16) and estimates on $\mu$ and $\nabla \operatorname{tr} \chi$, all of which were employed to prove breakdown criterion in [7] and [12]. The above estimate for $\zeta$ can not directly follow from the assumption (1.24) with $q < 2$.

It is well known that the embedding $H^1(S_i) \hookrightarrow L^\infty(S_i)$ fails. Therefore the assumption (1.24), which is a consequence of (1.22) by energy estimate, can neither control $\|\zeta, \nu\|_{L^2_t L^\infty_x}$ immediately nor by Sobolev embedding. This forces us to estimate the weaker norm $\|\cdot\|_{L^\infty_x L^2_t(\mathcal{H})}$, which, according to our experience, would succeed only when special structures for $\nabla \zeta$ and $\nabla \nu$ can be found.

1.3. Comparison to geodesic foliation. Let us draw comparisons between geodesic foliation and time foliation on null hypersurfaces as follows.

(1) In the case of geodesic foliation ([2, 11]), the complete set of estimates in the main theorem can be obtained under the small curvature flux only. However in time foliation, (1.13) contains one more assumption (1.24). In order to understand the reason for assuming (1.24), let us sketch the approach to derive (1.17). We will prove and employ the sharp trace inequality (see Theorem 5.1) on null cones with time foliation, which lied in the heart of [2, 4, 3] for null hypersurfaces with geodesic foliation. To implement this idea, it is a must to control $\mathcal{N}_1(\nu, \zeta)$. In view of (1.3) and (1.6), $\nu$ and $\zeta$ are combinations of elements of $\mathcal{F}$, therefore the assumption (1.24) guarantees the necessary control on $\nu, \zeta$.\]
The identity \( \zeta + \zeta = 0 \) only holds on null hypersurface \( \mathcal{H} \) with geodesic foliation. Therefore, in the case of time foliation, the estimates for \( \zeta \) are no longer identical to those for \( \zeta \). Note that \( \zeta \) and \( \zeta \) are on an equal footing in structure equations (2.4) and (2.15), we need \( \mathcal{N}_1(\cdot) \) and \( \| \cdot \|_{L^\infty L^2(\mathcal{H})} \) estimates for both of them. \( \zeta \) will be treated in the same fashion as in geodesic foliation, while the estimate for \( \| \zeta \|_{L^\infty L^2(\mathcal{H})} \) requires further study on structure equations. Similar to (2.8) and (2.9), the quantity \( \mu \) is connected to \( \zeta \) by the Hodge system, (2.8) and (2.7).

\[
\begin{aligned}
\text{div} \zeta &= -\bar{\rho} - \mu + \cdots \\
\text{curl} \zeta &= -\bar{\sigma} + \bar{\mu} + \cdots \\
\text{curl} \zeta &= -\bar{\sigma}.
\end{aligned}
\]

However \( \mu \) fails to satisfy a similar transport equation as (2.15) for \( \mu \), consequently does not verify \( \| \cdot \|_{p_0} \) estimate as \( \mu \), the treatment of \( \zeta \) is therefore different from \( \zeta \). Note that if there holds the following decomposition for \( \nabla \zeta \).

\[
(1.28) \quad \nabla \zeta = \nabla L P + E
\]

with \( P \) and \( E \) satisfying appropriate estimates, we may rely on the sharp trace inequality to estimate \( \| \zeta \|_{L^\infty L^2} \). To obtain the important structure (1.28), we first derive a refined Hodge system, (2.16) and (2.7).

\[
(1.29) \quad \begin{aligned}
\text{div} \zeta &= -\bar{\rho} + L(a\delta + 2a\lambda) + \cdots \\
\text{curl} \zeta &= -\bar{\sigma}.
\end{aligned}
\]

The pair of quantities \((\bar{\rho}, \bar{\sigma})\) can be decomposed in the same way as contained in (2.11). We set \( D_1 : F \to (\text{div} F, \text{curl} F) \) for any smooth \( S_t \) tangent 1-form 1. \( L \), then will be obtained from (1.29) by commuting \( \nabla L \) with \( \nabla D_1^{-1} \). The new commutator \([\nabla L, \nabla D_1^{-1}]\)(a\delta + 2a\lambda) will be decomposed in Proposition 6.1 together with other commutators arising in control of \( \| \chi, \zeta \|_{L^\infty L^2(\mathcal{H})} \).

Recall that in order to control \( |a - 1| < 1/2 \) in (1.25), we need to estimate \( \| \nu \|_{L^\infty L^2} \), which is a quantity that does not arise in the case of geodesic foliation. We again rely on sharp trace inequality to derive \( \| \nu \|_{L^\infty L^2} \), which requires another remarkable structure of the form,

\[
\nabla \nu = \nabla L P + E.
\]

This will be done by deriving the transport equation for \( \nabla a \), i.e. (2.17),

\[
\nabla \nu = -\nabla L(\nabla a) - \frac{1}{2}\text{tr} a\nabla a + \cdots .
\]

To prove sharp trace inequality and to control \( P \) and \( E \) actually dominate the article. Further comparison on technical details will be made in Section 2.

1.4. Organization of the paper. This paper is organized as follows. In Section 2 we start with providing all the structure equations and making bootstrap assumptions. Then by using structure equations (2.1)-(2.18) and Sobolev embedding, we establish a series of preliminary estimates including weakly spherical property for the metric \( \bar{\gamma} \). In Section 3 we prove \( \| L^{-a} K \|_{L^\infty L^2} \lesssim \Delta^2_0 + R_0 \) with \( a \geq 1/2 \) and establish a series of elliptic estimates. In Section 4 we briefly review the theory of geometric Littlewood Paley decomposition (GLP) and define Besov norms. We give the equivalence relation on Besov norms in Proposition 4.2 and the reduction argument in Lemma 4.1 based on the weakly spherical property for \( \bar{\gamma} \). With the help of these two arguments, in Section 5 we prove the sharp trace theorem, i.e.
Theorem 5.1. In order to obtain the structure required in Theorem 5.1, the commutators involved are decomposed in Proposition 6.1, which is the main purpose of Section 6. In Section 7, we estimate $\|\hat{\chi}, \zeta, \zeta, \nu\|_{L^\infty_t L^2_x}$ by using Theorem 5.1 and Proposition 6.1. In Section 8, we prove dyadic Sobolev inequalities and (6.76) in Theorem 6.1.

2. Preliminary estimates

2.1. Structure equations. The proof of Main Theorem relies crucially on the following set of structure equations. We will prove (2.16) and (2.17) and refer the reader to [1, Chapter 11] and [2, Section 2] for the derivation of all other formulae.

\begin{align}
(2.1) & \quad \frac{d}{ds} \text{tr}(\chi) + \frac{1}{2} (\text{tr}(\chi))^2 = -|\hat{\chi}|^2, \\
(2.2) & \quad \frac{d}{ds} \hat{\chi}_{AB} + \text{tr}(\chi) \hat{\chi}_{AB} = -\alpha_{AB}, \\
(2.3) & \quad \frac{d}{ds} \zeta_A = -\chi_{AB} \zeta_B + \chi_{AB} \zeta_B - \beta_A, \\
(2.4) & \quad \frac{d}{ds} \nabla \text{tr}(\chi) + \frac{3}{2} \text{tr}(\chi) \nabla \text{tr}(\chi) = -\hat{\chi} \cdot \nabla \text{tr}(\chi) - 2\hat{\chi} \cdot \nabla \hat{\chi} - (\zeta + \zeta)(|\hat{\chi}|^2 + \frac{1}{2} (\text{tr}(\chi))^2), \\
(2.5) & \quad \frac{d}{ds} \text{tr}(\chi) + \frac{1}{2} \text{tr}(\chi) \hat{\chi} = 2 \text{div}\zeta - \hat{\chi} \cdot \hat{\chi} + 2|\zeta|^2 + 2\rho, \\
(2.6) & \quad \text{div}\hat{\chi} = \frac{1}{2} \nabla \text{tr}(\chi) + \frac{1}{2} \text{tr}(\chi) \cdot \zeta - \hat{\chi} \cdot \zeta - \beta, \\
(2.7) & \quad \text{curl}\zeta = \frac{1}{2} \hat{\chi} \wedge \hat{\chi} - \sigma, \\
(2.8) & \quad \text{div}\zeta = -\mu - \hat{\rho} - |\zeta|^2 + \frac{1}{2} a \delta \text{tr}(\chi) + a \lambda \text{tr}(\chi), \\
(2.9) & \quad \text{curl}\zeta = \hat{\sigma}
\end{align}

In what follows, we record null Bianchi equations

\begin{align}
(2.10) & \quad \nabla L^\beta_A = \text{div}\alpha - 2\text{tr}(\chi) \beta_A + (2\zeta_A + \zeta_A) \alpha_{AB} \\
(2.11) & \quad L\hat{\rho} + \frac{3}{2} \text{tr}(\chi) \cdot \hat{\rho} = \text{div}\beta + (\zeta + 2\zeta) \cdot \beta - \frac{1}{2} \hat{\chi} \cdot (\nabla \hat{\zeta} - \frac{1}{2} \text{tr}(\chi) \cdot \hat{\chi} + \zeta \hat{\zeta}), \\
(2.12) & \quad L\hat{\sigma} + \frac{3}{2} \text{tr}(\chi) \cdot \hat{\sigma} = -\text{curl}\beta - (\zeta + 2\zeta) \wedge \beta - \frac{1}{2} \hat{\chi} \wedge (\nabla \hat{\zeta} + \zeta \hat{\zeta}), \\
(2.13) & \quad \nabla L^\beta_A + \text{tr}(\chi) \beta_A = -\nabla \rho + (\nabla \sigma)^* + 2\hat{\chi} \cdot \beta - 3(\zeta \rho - \zeta^* \sigma)
\end{align}

where

\begin{align}
(2.14) & \quad \hat{\rho} = \rho - \frac{1}{2} \hat{\chi} \cdot \hat{\chi}, \quad \hat{\sigma} = \sigma - \frac{1}{2} \hat{\chi} \wedge \hat{\chi}.
\end{align}
Moreover, there hold
\[
L(\mu) + \text{tr} \chi \mu = 2 \hat{\chi} \cdot \nabla \chi + (\zeta - \zeta) \cdot (\nabla A \text{tr} \chi + \text{tr} \chi A) - \frac{1}{2} \text{tr} \chi (\hat{\chi} \cdot \hat{\chi} - 2 \rho + 2 \zeta \cdot \zeta)
\]
(2.15)
\[
+ 2 \zeta \cdot \hat{\chi} \cdot \zeta + \left( \frac{1}{4} a^2 \text{tr} \chi - a(\delta + 2 \lambda) \right) |\hat{\chi}|^2 - \frac{1}{2} a \nu (\text{tr} \chi)^2;
\]
(2.16)
\[
L(a \delta + 2a \lambda) + \frac{3}{2} a\delta \text{tr} \chi = \text{div} \zeta + \hat{\rho} + |\zeta|^2 + a \text{tr} \chi \nabla_N \log \, n,
\]
(2.17)
\[
\nabla_L (\nabla a) + \frac{1}{2} \text{tr} \chi \nabla a = - \nabla \nu - \hat{\chi} \cdot \nabla a - (\zeta + \zeta) \cdot \nu.
\]

The Gauss curvature $K$ on each $S_t$ verifies
\[
K = - \frac{1}{4} \text{tr} \chi \text{tr} \chi + \frac{1}{2} \hat{\chi} \cdot \hat{\chi} - \rho.
\]

The following two commutation formulas hold:

(i) For any scalar functions $U$,
\[
\frac{d}{ds} \nabla_A U + \chi_{AB} \nabla_B U = \nabla_A F + (\zeta + \zeta) F;
\]
(2.19)

where $F = \frac{d}{ds} U$.

(ii) For $S_t$ tangent 1-form $U_A$ satisfying $\frac{dU_A}{ds} = F_A$, there holds
\[
\frac{d}{ds} \text{div} U + \chi_{AB} \nabla_A U_B = \text{div} F + (\zeta + \zeta) \cdot F + \left( \frac{1}{2} \text{tr} \chi \zeta_A - \hat{\chi} \zeta_B + \beta_A \right) U_A.
\]
(2.20)

Proof of (2.16). In view of (2.4) and (2.1),
\[
L(a^{-1} \text{tr} \chi) = L(\text{tr} \chi) a^{-1} + \text{tr} \chi L(a^{-1})
\]
\[
= a^{-1}(2 \text{div} \zeta + 2 \hat{\rho} + 2 |\zeta|^2 - \frac{1}{2} \text{tr} \chi \cdot \text{tr} \chi) - a^{-2} L(a) \text{tr} \chi,
\]
\[
L(\text{attr} \chi) = L(\text{tr} \chi) a + \text{tr} \chi L(a) = - a(|\hat{\chi}|^2 + \frac{1}{2} (\text{tr} \chi)^2) + L(a) \text{tr} \chi.
\]

By using (1.8),
\[
L(2 \delta + 4 \lambda) = a^{-1}(2 \text{div} \zeta + 2 \hat{\rho} + 2 |\zeta|^2) - \frac{1}{2} \text{tr} \chi (a^{-1} \text{tr} \chi + \text{attr} \chi) + L \log a(\text{attr} \chi - a^{-1} \text{tr} \chi)
\]
\[
= a^{-1}2(\text{div} \zeta + \hat{\rho} + |\zeta|^2) - \text{tr} \chi (\delta + 2 \lambda) + 2L(\log a) \text{tr} \theta,
\]
we can obtain with the help of (1.8) and (1.9),
\[
L(a \delta) = a L(\delta) + \delta L(a)
\]
\[
= (\text{div} \zeta + \hat{\rho} + |\zeta|^2) - \frac{1}{2} \text{attr} \chi (\delta + 2 \lambda) + L(a) (\text{tr} \theta + \delta) - 2a \lambda L(\lambda)
\]
\[
= \text{div} \zeta + \hat{\rho} + |\zeta|^2 + \text{attr} \chi (\frac{3}{2} \delta + \nabla_N \log \, n) - L(2a \lambda)
\]
which gives the desired formula. \qed

Proof of (2.17). Recall that by $L(a) = - \nu$, combined with (2.19)
\[
[\nabla_L, \nabla ] a = - \chi \cdot \nabla a - (\zeta + \zeta) \nu
\]
we may obtain relative to orthornomal frame on $S_t$,
\[
\nabla_L \nabla_B a + \frac{1}{2} \text{tr} \chi \nabla_B a = - \nabla_B \nu - \hat{\chi} BC \cdot \nabla_C a - (\zeta_B + \zeta_B) \nu
\]
This completes the proof.

Let $\mathcal{D}_t$ denote $\frac{d}{dt}$ along a null geodesics initiating from $p$. In view of (2.21), $\mathcal{D}_t = an \frac{d}{ds}$. In comparison with geodesic foliation, the term $(\zeta + \xi)\nabla U$ in (2.20) is no longer trivial due to $\zeta + \xi \neq 0$. This term however can be avoided if we consider the commutator $[\mathcal{D}_t, \nabla]U$ instead. Similarly, in view of [1] Lemma 13.1.2], it is simpler to consider $[\mathcal{D}_t, \nabla]F$ than $[\nabla L, \nabla]F$ for $S$-tangent tensor fields $F$.

**Proposition 2.1.** For any smooth scalar function $f$,

\[(2.21) \quad [\mathcal{D}_t, \nabla]f = -an\chi \cdot \nabla f\]

In view of (2.21), (2.20) and (2.6), there holds

\[
[D_t, \Delta]f = -an\chi \cdot \nabla^2 f + 2an\zeta \cdot \nabla f - 2an\zeta \cdot \nabla f
\
- an\zeta tr \nabla f - an\nabla tr \nabla f.
\]

Combining [11] P.288 with the comparison formulas in [2] Section 2, we have

**Lemma 2.1.**

$V, \nabla a, r \nabla tr \chi, r^2 \mu \to 0$ as $t \to 0$, $\lim_{t \to 0} \|\chi, \zeta, \nu\|_{L^\infty(S_t)} < \infty$

For $S$ tangent tensor fields $F$ on $\mathcal{H}$, we introduce the following norms. For $1 \leq p, q \leq \infty$ we define the $L^p_xL^q_t$ norm on $\mathcal{H}$

\[
\|F\|_{L^p_xL^q_t} := \left( \int_0^1 \left( \int_{|\omega| = 1} |F(t, \omega)|^p n\nu_2 d\mu_2 \right)^{\frac{1}{p}} dt \right)^{\frac{1}{q}}
\]

and the $L^\infty_xL^\infty_t$ norm

\[
\|F\|_{L^\infty_xL^\infty_t} := \int_{S^2} \sup_{t \in \Gamma \omega} (\nu_1 |F|^p) d\mu_2.
\]

**2.2. Notations and Bootstrap assumptions.** We fix the following conventions

- $\hat{\xi}$ denotes the collection of $\hat{\eta}, \epsilon, \delta, \nabla N \log n, \nabla \log n, \lambda$,
- $t := \text{tr} \chi - \frac{2}{n}, V := \text{tr} \chi - \frac{2}{n}, \kappa := \text{tr} \chi - (an)^{-1} \text{tr} \chi$,
- $\hat{A}$ denotes the collection of $A$ and $\hat{\chi}$, $\nabla \log a, \hat{\xi}$,
- The pair of quantities $(M, D_0 M)$ denotes either $(\nabla \text{tr} \chi, \nabla \hat{\chi})$, or $(\mu, \nabla \zeta)$,
- $R_0$ denotes the collection of $\alpha, \beta, \rho, \sigma, \beta$.
- $\hat{R}$ denotes the collection of $R_0, \text{tr} \chi \hat{A}, \hat{A}$.
- $\mathcal{H}_t := \cup_{t \in [0, t]} S\nu$, with $0 < t \leq 1$,
- $S := S_t, \gamma := r^{-2} \gamma, \hat{\gamma} := \gamma \hat{a}, \hat{K} := K - \frac{1}{r}$.

**Assumption 2.1.** We make the following bootstrap assumption:

\[(BA1) \quad \|V\|_{L^\infty(S_t)} \leq \Delta_0, \quad \|\hat{\chi}, \nu, \zeta, \nu\|_{L^\infty_xL^2_t(S_t)} \leq \Delta_0, \quad |a - 1| \leq \frac{1}{2},
\]

where we can assume that $0 < R_0 < \Delta_0 < 1/2$. 
2.3. Estimates for \(N_1(A)\), \(\|r^{1/2}M\|_{L^1_tL^\infty_x}\) and \(\|M\|_{L^2}\). Recall a few results that have been proved in \([12]\) and \([2, 10]\).

**Proposition 2.2.** \([12]\) Under the assumption BA1, there hold

\[
C^{-1} \leq \frac{v_l}{s^2} \leq C, \quad C^{-1} < \frac{r}{s} < C,
\]

where \(C\) is a positive constant.

It is easy to derive from \(|V| \leq \Delta_0\) in BA1 and Proposition 2.2 that

\[
\text{str}_\chi + |\text{tr}_\chi| \leq C
\]

and from \(|a - 1| \leq \frac{1}{2}\) in BA1 and \(C^{-1} < n < C\) that

\[
C^{-1} < an < C,
\]

with \(C\) positive constants.

With the help of Proposition 2.2, there hold the following simple inequalities by Sobolev embedding in 2-D slices \(S = S_t\).

- Let \(\text{Osc}(f) := f - \bar{f}\) for any smooth function \(f\) on \(S\) where \(\bar{f} = \frac{1}{|S|} \int_S f d\mu\), there holds the Poincare inequality

\[
\|r^{-1} \text{Osc}(f)\|_{L^2(S)} \lesssim \|\nabla f\|_{L^2(S)}.
\]

- For a smooth function \(\Omega\) on \(S\) with vanishing mean, there holds the following Sobolev inequality (see \([3]\))

\[
\|\Omega\|_{L^\infty(S)} \lesssim \|\nabla^2 \Omega\|_{L^1(S)} + \|\nabla \Omega\|_{L^2(S)},
\]

which implies

\[
\begin{align*}
|\Omega| & \lesssim \|
abla^2 \Omega\|_{L^2(S)} + |\text{tr}_\chi| + |\text{tr}_\chi|, \\
|\nabla \Omega| & \lesssim \|
abla^2 \Omega\|_{L^2(S)} + |\nabla \Omega| + |\text{tr}_\chi|.
\end{align*}
\]

- Let \(F\) be a tangent tensor field, (see \([3]\))

\[
\|F\|_{L^p(S)} \lesssim \|\nabla F\|_{L^2(S)}^{1 - \frac{p}{2}} \|F\|_{L^2(S)}^{\frac{p}{2}} + \|r^{-1} \nabla^2 F\|_{L^2(S)}, \quad \text{with } 2 < p < \infty.
\]

- Let \(F\) be a tangent tensor field, there hold (see \([2, 11]\))

\[
\begin{align*}
\|r^{-1/2} F\|_{L^2_tL^\infty_x} + \|F\|_{L^1_tL^\infty_x} + \|F\|_{L^6} & \lesssim N_1(F), \\
\|r^{-\frac{1}{2}} F\|_{L^\infty} + \|r^{-1} F\|_{L^1_tL^\infty_x} & \lesssim N_2(F),
\end{align*}
\]

where

\[
\begin{align*}
N_2(F) := \|r^{-2} F\|_{L^2} + \|r^{-1} \nabla^2 F\|_{L^2} + \|r^{-1} \nabla F\|_{L^2} + \|\nabla \partial_t F\|_{L^2} + \|\nabla^3 F\|_{L^2}.
\end{align*}
\]

By interpolation,

\[
\|r^{-\frac{1}{2}} F\|_{L^1_tL^4_x} + \|r^{-\frac{1}{4} - \frac{1}{2}} F\|_{L^1_tL^2_x} \lesssim N_1(F), \quad \text{with } b \geq 4, q \geq 2.
\]
Lemma 2.2.

\[ \|V\|_{L^\infty} \lesssim \Delta_0^2 \]

**Proof.** This can be obtained by integrating along \( \Gamma_\omega \) the equation (2.1), i.e.

\[ \frac{d}{ds} V + \frac{2}{s} V = -\frac{1}{2} V^2 - |\hat{\chi}|^2 \]

with the help of Proposition 2.2, Lemma 2.1 and \( \|V\|_{L^\infty} + \|\hat{\chi}\|_{L^\infty_t L^2} \leq \Delta_0 \) in BA1.

\[ \square \]

Lemma 2.3. For a \( S \) tangent tensor field \( F \) verifying

\[ \nabla_L F + \frac{p}{2} \text{tr}_F = G \cdot F + H \]

with \( p \geq 1 \) certain integer, if \( \lim_{t \to 0} r(t)^p F = 0 \) and \( \|G\|_{L^\infty_t L^2} \lesssim \Delta_0 \), then the following estimate holds

\[ |F| \lesssim v_0^{-\frac{p}{2}} \int_0^t v \nu \frac{\partial |H|}{\partial t}. \]

We will constantly use the Hardy-Littlewood inequality for scalar \( f \) on \( H \),

\[ \left\| \frac{1}{s} \int_0^s |f| \right\|_{L^2} \lesssim \|f\|_{L^2} \]

With the help of Lemmas 2.1 and 2.2, 2.3, 2.4, BA1, 2.32 and (1.13) we obtain

Lemma 2.4.

\[ \|r^{-1}\hat{\chi}, r^{-1/2}\zeta\|_{L^2} + \|r^{-1/2}\hat{\chi}, r^{-1/2}\zeta\|_{L^2 L^\infty} \lesssim R_0, \]

(2.34)

\[ \|\nabla_L \hat{\chi}, \nabla_L \zeta\|_{L^2} \lesssim R_0 + \Delta_0^2. \]

In view of Lemma 2.2, 1.13, and (SobM1), by definition of elements of \( A \), we can summarize the estimates for \( A \).

Proposition 2.3.

\[ \|r^{-1}A\|_{L^2} + \|r^{-1/2}A\|_{L^2 L^\infty} + \|\nabla_L A\|_{L^2} \lesssim \Delta_0^2 + R_0. \]

For the proofs of Lemmas 2.3 and 2.4 and Proposition 2.3 see [12].

With the help of Proposition 2.3 and Lemma 2.3 we prove the following result under the assumption of BA1.

Lemma 2.5.

\[ \|\nabla \log s\|_{L^\infty_t L^2} \lesssim \Delta_0, \]

(2.36)

\[ \|\nabla \log s\|_{L^2_t L^2} + \|s^{1/2} \nabla \log s\|_{L^\infty_t L^2} \lesssim \Delta_0^2 + R_0, \]

(2.37)

\[ \|Osc(\frac{1}{s})\|_{L^2_t L^2} + \|s^{1/2} Osc(\frac{1}{s})\|_{L^\infty_t L^2} \lesssim \Delta_0^2 + R_0, \]

(2.38)

\[ \|\kappa\|_{L^2_t L^2} + \|r^{1/2}\kappa\|_{L^\infty_t L^2} \lesssim \Delta_0^2 + R_0. \]
Proof. Apply (2.19) to \(U = s\), we can derive the transport equation
\[
\frac{d}{ds}\nabla_A(s) + \frac{1}{2} \text{tr}_\chi \nabla_A(s) = -\chi_{AB} \nabla_B(s) + \zeta_A + \zeta_A.
\]
Note that \(c_A(s) \to 0\), as \(t \to 0\)\(^3\) in view of Lemma (2.23) with \(G = \hat{\chi}\) and BA1, we can derive by integrating along a null geodesic \(\Gamma_{\omega}\) initiating from vertex,
\[
|s^{-1} \nabla(s)(t)| \lesssim \frac{1}{s} v^\frac{1}{2} \int_0^t v^\frac{1}{2} |\zeta| \text{d}t'.
\]
Taking \(L^2_t\) norm first then \(L^\infty_s(\mathbb{S}^2)\), with the help of BA1,
\[
\|s^{-1} \nabla(s)\|_{L^\infty_s L^2_t} \lesssim \|\zeta + \zeta\|_{L^2_s L^2_t} \lesssim \Delta_0
\]
which gives (2.35).

By taking \(L^2_t\) norm first then \(L^2_s(\mathbb{S}^2)\), we can obtain from (2.40) by using (2.32) that
\[
\|s^{-1} \nabla(s)\|_{L^2_s L^2_t} \lesssim \|r^{-1}(\zeta + \zeta)\|_{L^2_s} \lesssim \Delta^2_0 + \mathcal{R}_0,
\]
where for the last inequality we employed \(\|r^{-1}\Delta\|_{L^2_s} \lesssim \Delta^2_0 + \mathcal{R}_0\) in Proposition (2.20).
Similarly
\[
\|s^{-1/2} \nabla\|_{L^2_s L^\infty_t} \lesssim \Delta^2_0 + \mathcal{R}_0.
\]
Hence (2.39) is proved.

Applying (Poin) to \(f = \frac{1}{s}\), (2.37) follows as a consequence of (2.36).

According to definition, we can derive
\[
\kappa = \text{tr}_\chi - \frac{2}{s} - (an)^{-1} \text{an} \text{tr}_\chi - \frac{2}{s} + \frac{2}{s} (1 - (an)^{-1} \text{an}) \]
\[
+ 2 \text{Osc} \left(\frac{1}{s}\right) (an)^{-1} \text{an} - 2 (an)^{-1} s^{-1} \text{Osc}(an).
\]
By (Poin), (2.21) and also in view of Propositions (2.24) and (2.25), we obtain
\[
\|s^{-1} \text{Osc}(an)\|_{L^s L^2_t} \lesssim \|\nabla \log(an)\|_{L^s L^2_t} \lesssim \|r^{-1}(\zeta + \zeta)\|_{L^2_s} \lesssim \Delta^2_0 + \mathcal{R}_0,
\]
and similarly
\[
\|s^{-1} \text{Osc}(an)\|_{L^s L^2_t} \lesssim \|r^{-1}(\zeta + \zeta)\|_{L^2_s} \lesssim \Delta^2_0 + \mathcal{R}_0.
\]
Using (2.21) and (2.42), the last term in (2.41) can be estimated as follows
\[
\|s^{-1} \text{Osc}(an)\|_{L^s L^2_t} \lesssim \|s^{-1} \text{Osc}(an)\|_{L^s L^2_t} \lesssim \Delta^2_0 + \mathcal{R}_0.
\]
Combining (2.42), (2.44) and (2.29), we obtain
\[
\|r^{-1} \kappa\|_{L^2_t} \lesssim \|r^{-1} \text{Osc}(an)\|_{L^s L^2_t} + \Delta^2_0 + \mathcal{R}_0 \lesssim \Delta^2_0 + \mathcal{R}_0.
\]
where, for the last inequality, we employed (2.37).

By (2.37), (2.43) and (2.29), we can get
\[
\|r^{-1} \kappa\|_{L^s L^2_t} \lesssim \|r^{-1} \text{Osc}(an)\|_{L^s L^2_t} + \|r^{-1} \text{Osc}(an)\|_{L^s L^2_t} + |V|
+ \|s^{-1/2}((an)^{-1} an - 1)\|_{L^s L^2_t} \lesssim \Delta^2_0 + \mathcal{R}_0.
\]
The proof is complete.

\(^2\)This initial condition can be easily checked by using the comparison formulas in [2] Section 2.
Lemma 2.6.

\begin{align}
| \text{tr} \chi - \frac{2}{r} |_{L_t^2} & \lesssim \Delta_0^2 + R_0, \\
| \text{tr} \chi - \frac{2}{r} |_{L_t^3 L_{-3}} & \lesssim \Delta_0^2 + R_0.
\end{align}

Proof. We can derive the transport equation
\begin{equation}
\frac{d}{ds} (r (\text{tr} \chi - \frac{2}{r})) = (an)^{-1} \frac{r}{2} \text{antr} \chi \kappa - r (an)^{-1} an | \chi |^2.
\end{equation}
by combining (2.54) with the help of (2.29), (2.37) and (2.45).

\text{Proof.}

Denote by $\text{tr} \chi_A$ the transport equation
\begin{equation}
\frac{d}{ds} r (\text{tr} \chi - \frac{2}{r}) = (an)^{-1} \frac{r}{2} \text{antr} \chi.
\end{equation}
with
\begin{equation}
\frac{d}{ds} \text{tr} \chi = -(an)^{-1} \text{antr} \chi \cdot \text{tr} \chi + (an)^{-1} an \left( \frac{1}{2} (\text{tr} \chi)^2 - | \chi |^2 \right)
\end{equation}
which can be checked in view of the definition of $\text{tr} \chi$ and (2.1).

Integrate (2.47) in $t$ in view of $r \text{tr} \chi - 2 \to 0$ as $t \to 0$,

\begin{equation}
| \text{tr} \chi - \frac{2}{r} |_{L_t^2} \lesssim \frac{1}{r} \int_0^{s(t)} \left\{ (an)^{-1} \frac{r}{2} \text{antr} \chi \kappa + r (an)^{-1} an | \chi |^2 \right\} ds(t').
\end{equation}
In view of (2.24), taking $L_t^2$ with the help of (2.32) yields
\begin{equation}
| \text{tr} \chi - \frac{2}{r} |_{L_t^2} \lesssim \| r \text{tr} \chi \|_{L_t^2 L_{-3}} + \| V | \chi |^2 \|_{L_t^2 L_{-3}}.
\end{equation}
By (2.29) and (2.38), we can obtain
\begin{equation}
| r \text{tr} \chi \|_{L_t^2 L_{-3}} \lesssim \| V \kappa \|_{L_t^2 L_{-3}} + \| \frac{r}{s} \|_{L_t^2 L_{-3}} \lesssim (\Delta_0^2 + 1) \| \kappa \|_{L_t^2 L_{-3}} \lesssim \Delta_0^2 + R_0.
\end{equation}
By Proposition 2.3, we have
\begin{equation}
\| V | \chi |^2 \|_{L_t^2 L_{-3}} \lesssim \| r^{1/2} \chi \|_{L_t^2 L_{-3}} \| r^{1/2} \chi \|_{L_t^2 L_{-3}} \lesssim \Delta_0^2 + R_0.
\end{equation}
(2.45) then follows by connecting (2.49), (2.50) and (2.51).

Note that it is straightforward to have
\begin{equation}
\text{tr} \chi - \frac{2}{r} = V - \nabla + 2 \text{OSc} \left( \frac{1}{s} \right) + \text{tr} \chi - \frac{2}{r},
\end{equation}
hence
\begin{equation}
| \text{tr} \chi - \frac{2}{r} |_{L_t^2 L_{-3}} \lesssim \| V \|_{L_{\infty}} + \| \text{OSc} \left( \frac{1}{s} \right) \|_{L_t^2 L_{-3}} + \| \text{tr} \chi - \frac{2}{r} \|_{L_t^2}
\end{equation}
which implies (2.46) with the help of (2.29), (2.37) and (2.45).

Lemma 2.7. Denote by $\bar{R}$ one of the quantities, $R_0$, $\text{tr} \chi_A$, $A : \cdot A$, there holds
\begin{equation}
| \bar{R} |_{L_t^2 (H_4)} \lesssim \Delta_0^2 + R_0.
\end{equation}
Proof. The estimate about $R_0$ can be obtained directly from (1.13). With the help of Proposition 2.3 and BA1
\begin{equation}
\| A : \cdot A \|_{L_t^2 (H_4)} \lesssim \| A \|_{L_t^2 L_{-3}} \| A \|_{L_t^2 L_{-3}} \lesssim \Delta_0^2 + R_0.
\end{equation}
By BA1 and Proposition 2.3, we have
\begin{equation}
\| \text{tr} \chi_A \|_{L_t^2 (H_4)} \lesssim \| V : \cdot A \|_{L_t^2 (H_4)} + \| s^{-1} A \|_{L_t^2 (H_4)} \lesssim \| r^{-1} A \|_{L_t^2} \lesssim \Delta_0^2 + R_0.
\end{equation}
The estimate thus follows.
Lemma 2.8. Let $\overline{K} = K - \frac{1}{r^2}$, then
\[
\|\overline{K}\|_{L^2(\mathcal{H}_t)} \lesssim \Delta_0^2 + R_0.
\]

Proof. In view of (2.18) and (1.8),
\[
(2.55) \quad K - \frac{1}{r^2} = a^2 - 1 \frac{a}{r^2} + \frac{\nu(V + \frac{2}{3})a^2}{4} - at\chi(\lambda + \frac{1}{2}r) - \rho
\]
By $\nabla La = -\nu$, (2.32) and (1.13)
\[
(2.56) \quad \frac{a^2 - 1}{r^2} \lesssim \frac{1}{r^2} \int_0^r 2a\nu \int_0^r t \lesssim \frac{1}{r^2} \int_0^r t \lesssim \Delta_0^2 + R_0.
\]
In view of (2.55), by (2.29) and $r \approx s$ in (2.22), also using (2.56), (2.46), (2.53),
\[
\left\|K - \frac{1}{r^2}\right\|_{L^2(\mathcal{H}_t)} \lesssim \frac{a^2 - 1}{r^2} \|F\|_{L^2} + \|\nabla F\|_{L^2} \lesssim \Delta_0^2 + R_0
\]
which is the desired estimate. \hfill \Box

With Lemma 2.8, we can prove the following estimates.

Lemma 2.9. Let $D_1$ be the operator that takes any $S$ tangent 1-form $F$ to $(\text{div}F, \text{curl}F)$. Let $D_2$ be the operator that takes any $S$-tangent symmetric, traceless, 2-tensor fields $F$ to $\text{div}F$. Denote by $D$ one of the operators $D_1, D_2$. For any appropriate $S$ tangent tensor fields $F$ in the domain of $D$, if $\|r^{-\frac{1}{2}} F\|_{L^2 L^2} \lesssim \Delta_0$, there holds
\[
(2.57) \quad \|\nabla F\|_{L^2(\mathcal{H}_t)} + \|r^{-\frac{1}{2}} F\|_{L^2(\mathcal{H}_t)} \lesssim \|DF\|_{L^2(\mathcal{H}_t)} + \Delta_0^2 + R_0.
\]
For smooth scalar functions $\Omega$, if $\|r^{-\frac{1}{2}} \nabla \Omega\|_{L^2 L^2} \lesssim \Delta_0$, then
\[
(2.58) \quad \|\nabla^2 \Omega\|_{L^2(\mathcal{H}_t)} + \|r^{-1} \nabla \Omega\|_{L^2(\mathcal{H}_t)} \lesssim \|\nabla \Omega\|_{L^2(\mathcal{H}_t)} + \Delta_0^2 + R_0.
\]

Proof. Let us prove (2.58) first. In view of Böchner identity on $S$,
\[
(2.59) \quad \int_S |\nabla^2 \Omega|^2 + K|\nabla \Omega|^2 = \int_S |\Delta \Omega|^2,
\]
we obtain
\[
(2.60) \quad \int_{S_{t'}} |\nabla^2 \Omega|^2 + r^{-2}|\nabla \Omega|^2 = \int_{S_{t'}} |\Delta \Omega|^2 - \int_{S_{t'}} K|\nabla \Omega|^2.
\]
Noticing that on $S_{t'}$, there holds by (SoB) and $\|r^{-\frac{1}{2}} \nabla \Omega\|_{L^2 L^2} \lesssim \Delta_0$,
\[
(2.61) \quad \|\nabla \Omega\|_{L^2_{t'}(S_{t'})} \lesssim \|\nabla^2 \Omega\|_{L^2_{t'}(S_{t'})} \|\nabla \Omega\|_{L^2_{t'}(S_{t'})} + r^{-1} \|\nabla \Omega\|_{L^2_{t'}(S_{t'})} \lesssim \Delta_0 \|\nabla^2 \Omega\|_{L^2_{t'}(S_{t'})} + \Delta_0^2.
\]
Integrating (2.60) on $0 < t' < t$, in view of (2.61) and Lemma 2.8 (2.58) follows by using Young's inequality.

For $D_1 : F \to (\text{div}F, \text{curl}F)$ and $D_2 : F \to \text{div}F$ we recall the identities (see [1, Proposition 2.2.1])
\[
\int_S |\nabla F|^2 + K|F|^2 = \int_S |D_1 F|^2, \quad \int_S |\nabla F|^2 + 2K|F|^2 = 2 \int_S |D_2 F|^2.
\]
Then (2.57) follows in the same way as (2.58). \hfill \Box

By Lemma 2.9 and (2.53), we can derive the following
Lemma 2.10. For $M = \nabla \text{tr} \chi, \mu$, there holds

\begin{equation}
\|D_0 M\|_{L^2(H_t)} \lesssim \Delta_0^2 + \mathcal{R}_0 + \|M\|_{L^2(H_t)}.
\end{equation}

More precisely,

\begin{align*}
\|\nabla \hat{\chi}\|_{L^2(H_t)} & \lesssim \Delta_0^2 + \mathcal{R}_0 + \|\nabla \text{tr} \chi\|_{L^2(H_t)}, \\
\|\nabla \zeta\|_{L^2(H_t)} & \lesssim \Delta_0^2 + \mathcal{R}_0 + \|\mu\|_{L^2(H_t)}.
\end{align*}

Proof. By using $\|r^{-\frac{3}{2}} A\|_{L^2 L^\infty} \lesssim \Delta_0^2 + \mathcal{R}_0$ in Proposition 2.3 and applying Lemma 2.9 to $F = \hat{\chi}, \zeta$, we can derive that

\begin{align*}
\|\nabla \hat{\chi}\|_{L^2(H_t)} & \lesssim \Delta_0^2 + \mathcal{R}_0 + \|D_2 \hat{\chi}\|_{L^2(H_t)} + \Delta_0^2 + \mathcal{R}_0, \\
\|\nabla \zeta\|_{L^2(H_t)} & \lesssim \Delta_0^2 + \mathcal{R}_0 + \|D_1 \zeta\|_{L^2(H_t)} + \Delta_0^2 + \mathcal{R}_0.
\end{align*}

In view of (2.6), (2.8) and (2.9),

\begin{align*}
D_2 \hat{\chi} = \nabla \text{tr} \chi + \bar{R}, \\
D_1 \zeta = (\mu, 0) + \bar{R}.
\end{align*}

By using (2.53), (2.62) can be proved. □

Recall that $M = \nabla \text{tr} \chi$ or $\mu$. (2.4) and (2.15) can be symbolically recast as

\begin{equation}
\hat{\nabla} L M + \frac{p}{2} \text{tr} \chi M = \hat{\chi} \cdot M + H_1 + H_2 + H_3
\end{equation}

where

\begin{align*}
H_1 = A \cdot (D_0 M + F), \\
H_2 = A \cdot A \cdot A, \\
H_3 = r^{-1} \bar{R} \cdot \bar{R},
\end{align*}

with

\begin{equation}
(p, F) = \begin{cases} 
(2, \nabla \text{tr} \chi) & \text{if } M = \mu \\
(3, 0) & \text{if } M = \nabla \text{tr} \chi.
\end{cases}
\end{equation}

We will establish the following estimates

**Proposition 2.4.** Let $M$ denote either $\nabla \text{tr} \chi$ or $\mu$, there hold

\begin{equation}
\|r^{1/2} M\|_{L^2 L^\infty} + \|M\|_{L^2} \lesssim \mathcal{R}_0 + \Delta_0^2,
\end{equation}

\begin{equation}
\|\nabla \zeta, \nabla \hat{\chi}\|_{L^2} \lesssim \mathcal{R}_0 + \Delta_0^2.
\end{equation}

Proof. Noticing that (2.66) can be obtained immediately by combining the second estimate in (2.65) with Lemma 2.10, we first consider the second norm in (2.65). By (2.31), (2.63) and Lemma 2.1 integrating along the null geodesic $\Gamma_\omega$ initiating from vertex, we obtain

\begin{equation}
|M| \leq \sum_{i=1}^{3} \left| v_i^{-\frac{n}{2}} \int_{0}^{t} v_i^{-\frac{n}{2}} |H_i| nadj \right| = I_1(t) + I_2(t) + I_3(t).
\end{equation}

\(^3\)This means signs and coefficients on the right side of the expression can be ignored.

\(^4\)In view of $|a - 1| \leq 1/2$ in BA1, the factors $a^m, m \in \mathbb{N}$ can be ignored in $H_i$ when we employ (2.63) to prove Proposition 2.4.
Consequently, in view of (2.69), (2.70) and (2.72), if
\begin{equation}
(2.72)
\end{equation}
Hence in view of (2.62) and BA1
\begin{equation}
(2.70)
\end{equation}
Take \( L = \parallel \), we obtain in view of BA1 and Proposition 2.3 that
\begin{equation}
(2.71)
\end{equation}
\begin{equation}
(2.72)
\end{equation}
By taking \( L \), we obtain in view of BA1 and Proposition 2.3 that
\begin{equation}
(2.70)
\end{equation}
Now consider \( H := A \cdot (D_0 M + F) \). We have
\begin{equation}
(2.71)
\end{equation}
Take \( L = \parallel \),
\begin{equation}
(2.72)
\end{equation}
Consequently, in view of (2.69), (2.70) and (2.72), if \( M = \nabla \text{tr} \chi \), we have
\begin{equation}
(2.73)
\end{equation}
and if \( M = \mu 
\end{equation}
(2.74)
From (2.73), noticing that \( 0 < \Delta_0 < 1/2 \), we conclude
\begin{equation}
(2.75)
\end{equation}
\begin{equation}
(2.76)
\end{equation}

Now we consider the first estimate in (2.65).
\[
\frac{4}{3} I_3(t) \leq \frac{4}{3} - \frac{2}{3} \int_0^{s(t)} v_i^{\frac{2}{3}} s^{\frac{2}{3}} |\tilde{R}| ds(t') \lesssim s^{\frac{2}{3} - p} \int_0^{s(t)} s^{p-2} |\tilde{R}| ds(t')
\]
\[
\lesssim \left( \int_0^{s(t)} |\tilde{R}|^2 ds(t') \right)^{\frac{1}{2}}
\]
where we used \(s^2 \approx v_i, r \approx s\) in Proposition 2.22 to derive the second inequality. Thus
\[
(2.76) \quad \left\| \sup_{t \in (0,1)} |v_i^{\frac{4}{3}} I_3(t)| \right\|_{L^2_x} \lesssim \|\tilde{R}\|_{L^2(H)} \lesssim \Delta_0^2 + R_0.
\]
We can proceed in a similar fashion for the other two terms,
\[
v_i^{\frac{2}{3}} I_2(t) \leq \frac{2}{3} - \frac{2}{3} \int_0^{s(t)} v_i^{\frac{2}{3}} A^2 |\hat{A}| ds(t') \lesssim s^{\frac{2}{3} - p} \int_0^{s(t)} s^{p}|A|^2 |\hat{A}| ds(t')
\]
\[
\lesssim r \|A\|_{L^2_t}^2 \|r^{1/2} \hat{A}\|_{L^\infty},
\]
then
\[
(2.77) \quad \left\| \sup_{0 < t \leq 1} |v_i^{\frac{2}{3}} I_2(t)| \right\|_{L^2_x} \lesssim \|A\|^2_{L^2_x} \|r^{1/2} \hat{A}\|_{L^2_x \ L^\infty} \lesssim \Delta_0^2 (\Delta_0^2 + R_0).
\]
Similarly,
\[
v_i^{\frac{2}{3}} I_1(t) \lesssim \|A\|_{L^2_t} \|r (|\mathcal{D} M| + |\mathcal{F}|)|_{L^2_t} r^{1/2}.
\]
Taking \(L^2_w\), using (2.66) for \(\mathcal{D} M\) and (2.75) for \(\mathcal{F}\), also in view of BA1
\[
(2.78) \quad \left\| \sup_{0 < t \leq 1} |v_i^{\frac{2}{3}} I_1(t)| \right\|_{L^2_x} \lesssim \|\mathcal{D} M\|_{L^2_x} + \|\mathcal{F}\|_{L^2_x} \|A\|_{L^2_x} \lesssim (\Delta_0^2 + R_0) \Delta_0.
\]
Combine (2.60), (2.77), (2.78) we conclude that \(\|r^{1/2} M\|_{L^2_x L^\infty} \lesssim \Delta_0^2 + R_0\). □

Let us summarize the major estimates that have been obtained so far.

**Proposition 2.5.** Let \(M\) denote either \(\mu\) or \(\nabla \text{tr} \chi\). There holds
\[
(2.79) \quad \mathcal{N}_{1}(A) + \|r^{1/2} M\|_{L^2_x L^\infty} + \|M\|_{L^2} \lesssim \Delta_0^2 + R_0
\]

**Remark 2.1.** By SobM1, Sob and (1.13), it is easy to check \((a^m A)\) and \((a^m A)\) with \(m \in \mathbb{N}\) verify the same estimates as \(A\) and \(\hat{A}\) respectively. Consequently they can also be regarded as elements of \(A\) and \(\hat{A}\) respectively.

2.4. **More Estimates for \(\kappa\) and \(\iota\).** The main purpose of this subsection is to provide estimates for \(\mathcal{N}_1(\kappa, \iota)\) and \(\|\kappa, \iota\|_{L^2_x L^\infty}\). We first derive a simple consequence from (2.79) with the help of (2.20).

**Lemma 2.11.**
\[
(2.80) \quad \|r^{-1} \text{Osc}(an)\|_{L^2_x L^\infty} \lesssim \Delta_0^2 + R_0
\]
Proof. Apply (2.20) to \( \Omega = an - \pi n \),

\[
(2.81) \quad \|r^{-1} \Omega \|_{L^2_{t}L^{\infty}_{x}} \lesssim \|\nabla^2 \Omega \|_{L^2} + \|r^{-1} \nabla \Omega \|_{L^2}.
\]

Note that with the help of \( \zeta + \hat{\zeta} = \nabla \log(an) \) and (2.24), there holds

\[
\|r^{-\frac{1}{2}} \nabla \Omega \|_{L^2_{t}L^{\infty}_{x}} \lesssim \|r^{-\frac{1}{2}} \nabla \log(an)\|_{L^2_{t}L^{\infty}_{x}} + \|\zeta + \hat{\zeta}\|_{L^1_{t}L^2_{x}} \lesssim \Delta_0^2 + R_0.
\]

in view of (2.88), we deduce

\[
\|r^{-1} \Omega \|_{L^2_{t}L^{\infty}_{x}} \lesssim \|\Delta \Omega \|_{L^2} + \Delta_0^2 + R_0.
\]

We obtain in view of \( \Delta(an) = an(|\zeta + \hat{\zeta}|^2 + \Delta \log(an)) \), (2.24)

\[
\|r^{-1} \Omega \|_{L^2_{t}L^{\infty}_{x}} \lesssim \|\Delta \log(an)\|_{L^2} + \|\zeta + \hat{\zeta}\|_{L^1_{t}L^2_{x}} + \Delta_0^2 + R_0
\]

\[
(2.82) \quad \lesssim \mathcal{N}_1(\zeta + \hat{\zeta})(1 + \mathcal{N}_1(\zeta + \hat{\zeta})) + \Delta_0^2 + R_0
\]

\[
\lesssim \Delta_0^2 + R_0
\]

where we employed (SobM1) and (2.79) for the last two inequalities. \( \square \)

Proposition 2.6.

\[
(2.83) \quad \|r\nabla^2 \left( \frac{1}{s} \right) \|_{L^2} \lesssim \Delta_0^2 + R_0,
\]

\[
(2.84) \quad \|\text{Osc}(\frac{1}{s}), \text{Osc}(tr\chi), \kappa \|_{L^2_{t}L^{\infty}_{x}} \lesssim \Delta_0^2 + R_0.
\]

Proof. Using (2.39), in view of the commutation formula in \([11\) Lemma 13.1.2], symbolically, we obtain

\[
\frac{d}{ds} \nabla^2 s + tr\chi \nabla^2 s = -\frac{1}{2} \nabla tr\chi \nabla s + \hat{\chi} \cdot \nabla^2 s - \frac{1}{2} tr\chi(\zeta + \hat{\zeta}) \nabla s - \nabla \hat{\chi} \cdot \nabla s - (\zeta + \hat{\zeta}) \nabla^2 s - \hat{\chi} \cdot \nabla s + \chi \cdot \zeta + \zeta \cdot \chi.
\]

We then rewrite it as

\[
\frac{d}{ds} \nabla^2 s + tr\chi \nabla^2 s = \hat{\chi} \cdot \nabla^2 s + (\hat{R} + M + \nabla \hat{\chi}) \cdot \nabla s + A \cdot A + \nabla (\zeta + \hat{\zeta}).
\]

Apply Lemma 2.23 to the above equation with the help of \( \|\hat{\chi}\|_{L^{\infty}_{t}L^2_{x}(\mathcal{U})} \leq \Delta_0 \) in BA1 and \( \lim_{s \to 0} r^2 \nabla^2 s = 0 \),

\[
(2.85) \quad \|\nabla^2 s\| \lesssim e^{-1} \int_0^{s(t)} e^{\nu}(\hat{R} + M + \nabla \hat{\chi}) \cdot \nabla s + A \cdot A + \nabla (\zeta + \hat{\zeta})|ds(t').
\]

Hence, by Hölder inequality and (2.32), we have

\[
\|\nabla^2 s\|_{L^2_{t}L^{\infty}_{x}} \lesssim \|s^{-1} \nabla s\|_{L^2_{t}L^1_{x}} (\|\hat{R}\|_{L^2} + \|M\|_{L^2} + \|\nabla A\|_{L^2})
\]

\[
(2.86) \quad + \|r(A \cdot A + \nabla A)\|_{L^2_{t}L^2_{x}},
\]

By (2.53), (2.65), (2.35), (2.54) and (2.79), we obtain

\[
(2.87) \quad \|\nabla^2 s\|_{L^2_{t}L^2_{x}} \lesssim \Delta_0^2 + R_0.
\]

By a straightforward calculation,

\[
-s^2 \nabla^2 \left( \frac{1}{s} \right) = s^2 \nabla (s^{-2} \nabla s) = \nabla^2 s - 2s^{-1} \nabla s \cdot \nabla s
\]

we deduce

\[
\|r \nabla^2 \left( \text{Osc}(1/s) \right)\|_{L^2} \lesssim \|\nabla^2 s\|_{L^2_{t}L^2_{x}} + \|\nabla \log s \cdot \nabla s\|_{L^2_{t}L^2_{x}}.
\]
By (2.30) and (2.35)

\begin{align*}
|\nabla \log s \cdot \nabla s|_{L^2 L^\infty} & \lesssim |\nabla \log s|_{L^\infty L^2} |s\nabla \log s|_{L^2 L^\infty} \\ & \lesssim \Delta_0 (\Delta_0^2 + R_0).
\end{align*}

Combined with (2.87),

\begin{equation}
|\nabla \nabla (\text{Osc}(1/s))|_{L^2} \lesssim \Delta_0^2 + R_0.
\end{equation}

Now we apply (2.26) to \( \Omega = r \text{Osc}(\frac{1}{s}) \). By (2.30), \( |\nabla (\text{Osc}(\frac{1}{s}))|_{L^2} \lesssim \Delta_0^2 + R_0 \).

Also in view of (2.89), we conclude that

\begin{equation}
|\text{Osc}(\frac{1}{s})|_{L^2 L^\infty} \lesssim \Delta_0^2 + R_0.
\end{equation}

By \( \text{Osc}(\text{tr} \chi) = \text{Osc}(V) + 2 \text{Osc}(\frac{1}{s}) \) and (2.29), it follows from (2.90) that

\begin{equation}
|\text{Osc}(\text{tr} \chi)|_{L^2 L^\infty} \lesssim \Delta_0^2 + R_0.
\end{equation}

In view of \( t = \text{Osc}(\text{tr} \chi) + \text{tr} \chi - \frac{2}{r} \), (2.91) together with (2.45) implies

\[ \| t \|_{L^2 L^\infty} \lesssim \Delta_0^2 + R_0. \]

In view of (2.41), symbolically,

\begin{equation}
\kappa = V - (an)^{-1} \text{tr} V + \text{Osc}(\frac{1}{s}) \text{tr} \chi - (an)^{-1} \text{Osc}(an) + (an)^{-1} s^{-1} \text{Osc}(an).
\end{equation}

Taking \( L^2 L^\infty \) of \( \kappa \) in view of (2.92), by (2.80) and (2.24), the last two terms are bounded by \( \Delta_0^2 + R_0 \). Due to (2.90) and (2.29), \( L^2 L^\infty \) of the remaining three terms are bounded by \( \Delta_0^2 + R_0 \). Thus we conclude \( \| \kappa \|_{L^2 L^\infty} \lesssim \Delta_0^2 + R_0 \).

\begin{proposition}
N_1(t) + N_1(\kappa) \lesssim \Delta_0^2 + R_0.
\end{proposition}

\begin{proof}
In view of (2.1) and (2.38), we have

\[ \nabla \nabla (\text{tr} \chi - \frac{2}{r}) = -\frac{1}{2} (\text{tr} \chi - \frac{2}{r}) \text{tr} \chi - |\chi|^2 - \frac{1}{r} \kappa. \]

By BA1 and (2.40), (2.38) and (2.51), we have \( \| \nabla \nabla (t) \|_{L^2} \lesssim \Delta_0^2 + R_0 \). Together with (2.30) and (2.79), we conclude \( N_1(t) \lesssim \Delta_0^2 + R_0 \).

Now we take \( L^2 (H) \) norm of \( \frac{d}{ds} \kappa \) with the help of

\begin{equation}
\frac{d}{ds} \kappa + \text{tr} \chi \cdot \kappa = -|\chi|^2 + (an)^{-2} |an \chi|^2 + \frac{1}{2} \kappa^2 - \frac{1}{2} (an)^{-2} (an)^2 |\kappa|^2
\end{equation}

\[ + (an)^{-2} \left( \frac{\text{Osc}(\nabla L(an))}{\text{Osc}(\nabla L(an))} - \frac{\text{Osc}(\nabla L(an))}{\text{Osc}(\nabla L(an))} \right). \]

By (2.38) and (2.23), \( \| \text{tr} \chi \kappa \|_{L^2} \lesssim \Delta_0^2 + R_0 \). For the terms on the right of (2.93), we first claim

\begin{equation}
\| r^{-1} \text{Osc}(\nabla L(an)) \|_{L^2} \lesssim \Delta_0^2 + R_0
\end{equation}

Indeed,

\[ \nabla \nabla (an) = \nabla (an \nabla \log (an)) = \nabla D_t \log (an) \]

by (2.21),

\begin{equation}
\nabla \nabla (an) = D_t \nabla \log (an) + an \chi \cdot \nabla \log (an) = an \{ \nabla L(\zeta + \zeta) + \chi \cdot (\zeta + \zeta) \}.
\end{equation}

By (2.90) and (2.24),

\[ \| r^{-1} \text{Osc}(\nabla L(an)) \|_{L^2} \lesssim \| \nabla \nabla L(an) \|_{L^2} \]

\begin{equation}
\lesssim \| \text{tr} \chi \|_{L^2} + \| \nabla L \|_{L^2} + \| \chi \cdot A \|_{L^2}
\end{equation}

\[ \lesssim \Delta_0^2 + R_0. \]
\[ (2.91) \] follows by using \[ (2.53) \] and Proposition \[ 2.3 \].
Using \[ (2.91), (2.24) \] and \[ (2.23) \],
\[ \|(an)^{-1} \nabla \log(\nabla L(\gamma))\|_{L^2} \lesssim \|r^{-1} \nabla L^2(\gamma(\nabla \log(\nabla L(\gamma))))\|_{L^2} \lesssim \Delta_0^2 + R_0. \]

Similarly,
\[ \|(an)^{-2} \nabla \log(\nabla L(\gamma))\|_{L^2} \lesssim \|\nabla L^2(\gamma(\nabla \log(\nabla L(\gamma))))\|_{L^2} \lesssim \Delta_0^2 + R_0. \]

Now consider other terms on the right of \[ (2.93) \], by \[ (2.38) \] and \[ (2.84) \],
\[ \|\kappa^2\|_{L^2} \lesssim \|r\kappa\|_{L^2 L^2} \|\kappa\|_{L^2 L^2} \lesssim \Delta_0^2 + R_0, \]
by \[ (\text{SobM1}) \] and \[ (2.79) \],
\[ \|\hat{\chi} \cdot \hat{\chi}\|_{L^2} \lesssim \|\hat{\chi}\|^2_{L^2} \lesssim \Delta_0^2 + R_0 \]
and the other two terms can be estimated similarly in view of \[ (2.24) \]. Hence\[ (2.97) \]
\[ \|\nabla L^2(\gamma)\|_{L^2} \lesssim \Delta_0^2 + R_0. \]

At last, we obtain by definition
\[ \nabla \kappa = \nabla \log(\nabla L) \cdot (an)^{-1} \nabla \log(\nabla L). \]
By using \[ (2.23), (2.24) \] and \[ (2.79) \],
\[ \|\nabla \kappa\|_{L^2(\mathcal{H})} \lesssim \|\nabla \log(\nabla L)\|_{L^2(\mathcal{H})} + \|r^{-1}(\nabla \log(\nabla L)^2)\|_{L^2(\mathcal{H})} \lesssim \Delta_0^2 + R_0. \]
Combined with \[ (2.38) \] and \[ (2.77) \], we can conclude \[ N_{1}(\kappa) \lesssim \Delta_0^2 + R_0. \]

\[ \text{Remark 2.2.} \] By Proposition \[ 2.7 \] and \[ (2.84) \], we can regard \( \kappa \) and \( \iota \) as elements of \( A \).

Without making the strong assumption \[ (1.23) \] (see \[ 7 \]), under the weaker condition \[ (1.13) \] only, \( \text{tr} \gamma - \frac{2}{\varepsilon} \) no longer satisfies the \( L^\infty(\mathcal{H}) \) estimate as \[ (1.14) \] for \( \text{tr} \gamma - \frac{2}{\varepsilon} \). Since \( S_t \) is not a level set of affine parameter \( s \), obviously, \( \nabla r = 0 \) while \( \nabla s \neq 0 \).
We will check the weakly spherical property for \( \gamma = r^{-2} \gamma \). Integral operators \( \Lambda^{-\alpha} \), geometric Littlewood Paley decompositions \( P_k \) and Besov norms will then be defined by heat flow \( U(t) \) with respect to \( \gamma \) instead of \( s^{-2} \gamma \). Due to the factor \( "an\) in \[ (2.48) \], the nontrivial evolution of \( r \) adds technical complexity. This issue can be settled by proving \( \text{tr} \gamma - (an)^{-1} \nabla \log(\nabla L) \) and \( \text{tr} \gamma - \gamma \) verify stronger estimates than \( \nabla \gamma \).
Examples of the application of \[ (2.84) \] and Proposition \[ 2.7 \] can be seen in the proof of \[ (2.100) \], Proposition \[ 3.3 \] Theorem \[ 5.1 \] etc.

2.5. **Weakly spherical surfaces.** Let \( \gamma \) be the restriction metric on \( S_t \), and define the rescaled metric \( \gamma \) on \( S_t \) by \( \gamma = r^{-2} \gamma \). Let \( \gamma_{ij}^{(0)} \) denote the canonical metric on \( \mathbb{S}^2 \). Note that
\[ (2.98) \]
\[ \lim_{t \to 0} \gamma_{ij}^{(0)} = \gamma_{ij}, \quad \lim_{t \to 0} \partial_k \gamma_{ij}^{(0)} = \partial_k \gamma_{ij} \]
where \( i, j, k = 1, 2 \). With its aid, using the bootstrap assumption BA1, \[ (2.79) \], we will prove that for each \( 0 < t \leq 1 \) the leave \( (S_t, \gamma) \) is a weakly spherical surface.
Proposition 2.8. For the transport local coordinates \((t, \omega)\), the following properties hold true for all surfaces \(S_t\) of the time foliation on the null cone \(\mathcal{H}\): the metric \(\gamma_{ij}(t)\) on each \(S_t\) verifies weakly spherical conditions i.e.

\[
\| \gamma_{ij}(t) - \gamma^{(0)}_{ij} \|_{L^\infty_t} \lesssim \Delta_0
\]

(2.99)

\[
\| \partial_k \gamma_{ij}(t) - \partial_k \gamma^{(0)}_{ij} \|_{L^2_t L^\infty_x} \lesssim \Delta_0
\]

(2.100)

Proof. Since relative to the transport coordinate on \(\mathcal{H}\), \(\frac{d}{ds} \gamma_{ij} = 2 \chi_{ij}\),

(2.101)

\[
\frac{d}{dt} \gamma_{ij} = an(\kappa \cdot \gamma_{ij} + 2 \hat{\chi}_{ij} \gamma_{ij})
\]

where \(i, j, k = 1, 2\), with the initial condition given by (2.98), \(\| \kappa \|_{L^2_t L^\infty_x} \lesssim \Delta_0^2 + \mathcal{R}_0\) in (2.84) and a similar argument to Lemma 2.3 we can obtain

\[
\left| \partial_k \gamma_{ij}(t) - \partial_k \gamma^{(0)}_{ij} \right| \lesssim \int_0^{s(t)} \left( \partial_k \text{tr} \gamma_{ij} \gamma_{ij} + r^{-2}(\nabla_k \hat{\chi}_{ij} - \Gamma \cdot \hat{\chi}) \right.
\]

\[
+ \partial_k \kappa \gamma_{ij} + 2 \partial_k \log(\kappa) \hat{\chi}_{ij} r^{-2} + \kappa \partial_k \gamma^{(0)}_{ij} \right) ds(t'),
\]

where \(\Gamma\) represents Christoffel symbols, and \(\Gamma \cdot \hat{\chi}\) stands for the terms \(\sum_{l=1}^2 \Gamma_{ki}^l \hat{\chi}_{lj}\) with \(l = 1, 2\). Then with the help of (2.99),

\[
\left\| \sup_{0 < t \leq 1} \left| \partial_k \gamma_{ij}(t) - \partial_k \gamma^{(0)}_{ij} \right| \right\|_{L^2_t}
\]

\[
\lesssim \| \Gamma \|_{L^2_t L^\infty_x} \| \hat{\chi} \|_{L^\infty_t L^2_x} + \| \nabla \text{tr} \gamma_{ij} \|_{L^2_t L^1_x} + \| \nabla \hat{\chi} \|_{L^2_t L^2_x}
\]

(2.102)

By BA1, (2.33), (2.51), we obtain the terms in the line of

\[
\| \Gamma \|_{L^2_t L^\infty_x} (\| A \|_{L^\infty_t L^2_x} + 1) + \| A \cdot A \|_{L^2(\mathcal{H})} \lesssim \Delta_0^2 + \mathcal{R}_0.
\]

Using Proposition 2.4

\[
\left\| \sup_{0 < t \leq 1} \left| \partial_k \gamma_{ij}(t) - \partial_k \gamma^{(0)}_{ij} \right| \right\|_{L^2_t}
\]

\[
\lesssim \| \Gamma \|_{L^2_t L^\infty_x} + \| \hat{\chi} \|_{L^\infty_t L^2_x} \| \Gamma \|_{L^2_t L^2_x}.
\]

Sum over all \(i, j, k = 1, 2\), also using (2.99)

\[
\| \Gamma \|_{L^2_t} \lesssim \sum_{i, j, k=1, 2} \| \partial_k (\gamma_{ij} - \gamma^{(0)}_{ij}) \|_{L^2_t} + C,
\]

where \(C\) is the constant such that the Christoffel symbol of \(\gamma^{(0)}\) satisfies \(|\partial \gamma^{(0)}| \leq C\), (2.100) then follows by using \(\| \hat{\chi} \|_{L^\infty_t L^2_x} \leq \Delta_0 < 1/2\) in BA1. \(\square\)
3. \( \|\Lambda^{-\alpha}K\|_{L^p L^2} \) and elliptic estimates

Define the operator \( \Lambda^a \) with \( a \leq 0 \) such that for any \( S \)-tangent tensor fields \( F \)
\[
\Lambda^a F := \frac{r^{-a}}{\Gamma(-a/2)} \int_0^\infty \tau^{-\frac{a}{2}-1} e^{-\tau} U(\tau) F d\tau,
\]
where \( \Gamma \) denotes Gamma function and \( U(\tau) F \) is defined on \((S, \gamma)\) by
\[
\frac{\partial}{\partial \tau} U(\tau) F - \Delta_\gamma U(\tau) F = 0, \quad U(0) F = F.
\]
The definition of \( \Lambda^a \) extends to the range \( a > 0 \) by defining for \( 0 < a < 2m \) that
\[
\Lambda^a F = \Lambda^{a-2m} \cdot (r^{-2} Id - \Delta_\gamma)^m F.
\]
We record the basic properties of \( \Lambda^a \) in the following result (see [3]).

**Proposition 3.1.**
(i) \( \Lambda^0 = Id \) and \( \Lambda^a \cdot \Lambda^b = \Lambda^{a+b} \) for any \( a, b \in \mathbb{R} \).
(ii) For any \( S \)-tangent tensor field \( F \) and any \( a \leq 0 \)
\[
r^a \|\Lambda^a F\|_{L^2(S)} \leq \|F\|_{L^2(S)}.
\]
(iii) For any \( S \)-tangent tensor field \( F \) and any \( b \geq a \geq 0 \)
\[
r^a \|\Lambda^a F\|_{L^2(S)} \leq r^b \|\Lambda^b F\|_{L^2(S)} \quad \text{and} \quad \|\Lambda^a F\|_{L^2(S)} \leq \|\Lambda^b F\|_{L^2(S)} \|F\|_{L^2(S)}^{1-\frac{a}{b}}.
\]
(iv) For any \( S \)-tangent tensor fields \( F \) and \( G \) and any \( 0 \leq a < 1 \)
\[
\|\Lambda^a(F \cdot G)\|_{L^2(S)} \leq \|\Lambda F\|_{L^2(S)} \|\Lambda G\|_{L^2(S)} + \|\Lambda^a F\|_{L^2(S)} \|\Lambda G\|_{L^2(S)}.
\]
(v) For any \( S \)-tangent tensor field \( F \) there holds with \( 2 < p < \infty \) and \( a > 1 - \frac{2}{p} \)
\[
\|F\|_{L^p(S)} \leq \|\Lambda^a F\|_{L^2(S)}.
\]
(vi) For any \( a \in \mathbb{R} \) and any \( S \)-tangent tensor field \( F \)
\[
\|F\|_{H^a(S)} := \|\Lambda^a F\|_{L^2(S)}
\]

**Proposition 3.2.** Under the assumption of BA1 , if \( R_0 > 0 \) is sufficiently small, then for all \( 1/2 \leq \alpha < 1 \) there hold
\[
\mathcal{K}_\alpha := \|\Lambda^{-\alpha}(K - r^{-2})\|_{L^p L^2} \lesssim \Delta_0^2 + \mathcal{R}_0,
\]
\[
\|\Lambda^{-\alpha} \tilde{b}\|_{L^p L^2} \lesssim \Delta_0^2 + \mathcal{R}_0.
\]

For smooth scalar functions \( f \) on \( \mathcal{H} \), with \( p > 2, \alpha \geq 1/2 \), define the good part of the commutator \([\Lambda^{-\alpha}, \mathcal{D}_t]^g \] by
\[
[\Lambda^{-\alpha}, \mathcal{D}_t]^g f := r^a C_\alpha \int_0^\infty \tau^{-\frac{a}{2}-1} e^{-\tau} [U(\tau), \mathcal{D}_t] f d\tau, \quad \text{with} \quad C_\alpha = \frac{1}{\Gamma(\frac{1}{2})}.
\]

We first prove Proposition 3.2 by assuming the following result.

**Proposition 3.3.** Let \( f \) be a smooth scalar function \( \mathcal{H} \), there holds
\[
\|r^{-\alpha} [\Lambda^{-\alpha}, \mathcal{D}_t]^g f\|_{L^1 L^2} \lesssim (1 + I_\alpha^{1-\frac{2}{a}})(\Delta_0^2 + \mathcal{R}_0) \|f\|_{L^2},
\]
where \( p > 2 \) and \( 1/2 \leq \alpha < 1 \), and \( I_\alpha := 1 + \mathcal{K}_\alpha^{-\alpha} + \mathcal{K}_\alpha^{\alpha} \).
Proof of Proposition 3.2. It suffices to prove the following estimates

\begin{align}
(3.5) & \quad \|\Lambda^{-\alpha}(K + \tilde{\rho})\|_{L^\infty_t L^2_x} \lesssim \Delta_0^2 + R_0, \\
(3.6) & \quad \|\Lambda^{-\alpha}\tilde{\rho}\|_{L^p_t L^2_x} \lesssim (\Delta_0^2 + R_0)(1 + L_0^{1-\frac{2}{p}}), \quad p > 2.
\end{align}

By (3.5) and (3.6), according to the definition of \(I_\alpha\), with \(\frac{1}{L_0} \cdot (1 - \frac{2}{p}) < 1\), we can obtain (3.3).

Let us first prove (3.6). By (3.5),

\begin{align}
(3.7) & \quad K + \tilde{\rho} = \frac{a^2 - 1}{r^2} + \frac{a^2 \vartheta}{2r} + \frac{\vartheta a^2}{4} + \vartheta \Lambda^2.
\end{align}

By Proposition 3.1 (ii), BA1, (2.22) Proposition 2.3 and (2.23), for \(\alpha \geq 1/2\),

\[ \|\Lambda^{-\alpha}(\vartheta \Lambda^2)\|_{L^\infty_t L^2_x} \lesssim \|r^{\alpha+1} \vartheta \Lambda^2\|_{L^\infty_t L^2_x} \lesssim \Delta_0^2 + R_0. \]

By Proposition 3.1 (ii) and (1.13), we have for \(\alpha \geq \frac{1}{2}\),

\[ \left\| \Lambda^{-\alpha} \left( \frac{a^2 - 1}{r^2} \right) \right\|_{L^\infty_t L^2_x} \lesssim \|r^{-1} \Lambda^{-\alpha}(a^2 - 1)\|_{L^\infty_t L^2_x} \lesssim \|r^{-1}(a^2 - 1)\|_{L^\infty_t L^2_x} \lesssim \|\vartheta\|_{L^\infty_t L^2_x} \lesssim \|\vartheta\|_{L^\infty_t L^2_x} \lesssim \|\vartheta\|_{L^\infty_t L^2_x} \lesssim \Delta_0^2 + R_0. \]

By Proposition 3.1 (ii), (SobM1) and Proposition 2.4, we have for \(\alpha \geq \frac{1}{2}\),

\[ \left\| \Lambda^{-\alpha} \left( \frac{a^2 \vartheta}{r^2} \right) \right\|_{L^\infty_t L^2_x} \lesssim \|a^2 \vartheta\|_{L^\infty_t L^2_x} \lesssim \|\vartheta\|_{L^\infty_t L^2_x} \lesssim \Delta_0^2 + R_0. \]

Similarly, by (2.23),

\[ \|\Lambda^{-\alpha}(a^2 \vartheta \vartheta)\|_{L^\infty_t L^2_x} \lesssim \|r^{1+\alpha}(a^2 \vartheta \vartheta)\|_{L^\infty_t L^2_x} \lesssim \|r^{\alpha}\|_{L^\infty_t L^2_x} \lesssim \Delta_0^2 + R_0. \]

This finishes the proof of (3.5).

Next we prove (3.6). Let \(W(t) = (\Lambda^{-\alpha}\tilde{\rho})^2(t) - (\Lambda^{-\alpha}\tilde{\rho})^2(0)\). Then

\begin{align}
W(t) & \quad = \int_0^t \int_{S_r'} \left( \frac{d}{dt}(\Lambda^{-\alpha}\tilde{\rho})^2 + \vartheta \Lambda^{-\alpha}(\Lambda^{-\alpha}\tilde{\rho})^2 \right) \, d\mu, dt' \\
& \quad = \int_0^t \int_{S_r'} \left( 2[D_{\tilde{\rho}}, \Lambda^{-\alpha}]\tilde{\rho} \cdot \Lambda^{-\alpha}\tilde{\rho} + (\vartheta \Lambda^{-\alpha} + \vartheta \vartheta \Lambda^{-\alpha})(\Lambda^{-\alpha}\tilde{\rho})^2 \right) \\
& \quad \quad + 2\Lambda^{-\alpha}D_{\tilde{\rho}} \cdot \Lambda^{-\alpha}\tilde{\rho} \right) \, d\mu, dt'.
\end{align}

(3.8)

Let \(\vartheta(t)\) be a smooth cut-off function with \(\vartheta(0) = 1\) and supported in \([0, \frac{1}{2}]\), \(|\vartheta| \leq 1\),

\begin{align}
(3.9) & \quad (\Lambda^{-\alpha}\tilde{\rho}(0))^2 = (\vartheta \Lambda^{-\alpha}\tilde{\rho})(0)^2 - (\vartheta \Lambda^{-\alpha}\tilde{\rho})(1)^2
\end{align}
can be treated similar to $W(t)$, then
\[
\|\Lambda^{-\alpha}\tilde{\rho}\|_{L^2_t L^2_x}^2
\leq \|\mathcal{D}_t\Lambda^{-\alpha}\tilde{\rho}\|_{L^2_t L^2_x} \cdot \|\Lambda^{-\alpha}\tilde{\rho}\|_{L^\infty_t L^2_x}^2 + \|(\text{anttr}\chi + \alpha\text{anttr}\chi)(\Lambda^{-\alpha}\tilde{\rho})^2\|_{L^2_t L^2_x}^2
\]  
(3.10)
\[+ \int_0^1 \left| \int_S \Lambda^{-\alpha}\mathcal{D}_t\rho \cdot \Lambda^{-\alpha}\tilde{\rho} \mu_{\nu} \right| dt' + \int_0^1 \left| \int_S \partial^2\Lambda^{-\alpha}\mathcal{D}_t\rho \cdot \Lambda^{-\alpha}\tilde{\rho} \mu_{\nu} \right| dt'
\]  
(3.11)
\[+ \int_0^1 \int_{S_{t'}} \left| \frac{d}{dt} \partial(\Lambda^{-\alpha}\tilde{\rho})^2 \right| d\mu_{\nu}, dt'.
\]  
(3.12)
In view of (2.23) and Proposition 3.1 (ii),
\[\int_0^1 \int_{S_{t'}} (\text{anttr}\chi + \text{anttr}\chi)(\Lambda^{-\alpha}\tilde{\rho})^2 d\mu_{\nu}, dt' \lesssim \|r^{-\frac{\alpha}{2}}\Lambda^{-\alpha}\tilde{\rho}\|_{L^2}^2 \lesssim \|\tilde{\rho}\|_{L^2}^2.
\]
By Proposition 3.1 (ii), for $\alpha \geq \frac{1}{2}$,
\[\|\Lambda^{-\alpha}\tilde{\rho}\|_{L^2}^2 \lesssim \|\tilde{\rho}\|_{L^2}^2.
\]
We only need to estimate the first term in (3.11), and the second one will follow similarly.
\[\int_0^1 \int_{S_{t'}} \Lambda^{-\alpha}\mathcal{D}_t\rho \cdot \Lambda^{-\alpha}\tilde{\rho} \mu_{\nu} \right| dt' \lesssim \|\Lambda^{-2\alpha}\mathcal{D}_t\rho\|_{L^2_t} \|\tilde{\rho}\|_{L^2_t}.
\]
Assuming
\[\|\Lambda^{-2\alpha}\mathcal{D}_t\rho\|_{L^2_t} \lesssim \Delta_0^2 + \mathcal{R}_0,
\]
then (3.13) \(\lesssim (\Delta_0^2 + \mathcal{R}_0)^2\), and it follows that
\[\|\Lambda^{-\alpha}\tilde{\rho}\|_{L^2_t L^2_x}^2 \lesssim (\Delta_0^2 + \mathcal{R}_0)^2(M^1 + M^2)\|\Lambda^{-\alpha}\tilde{\rho}\|_{L^\infty_t L^2_x}^2 + (\Delta_0^2 + \mathcal{R}_0)^2.
\]
Thus (3.1) is proved. (3.4) follows as an immediate consequence.

To prove (3.14), we rely on the transport equation derived by (2.21),
\[\mathcal{D}_t\tilde{\rho} + \frac{3}{2}\text{anttr}\chi\tilde{\rho} = \text{div}(an\beta) - \nabla(an)\beta + anA \cdot \tilde{\mathcal{R}} = \text{div}(an\beta) + anA \cdot \tilde{\mathcal{R}}.
\]
By Proposition 3.1 (2.23) and (1.13), with $\alpha \geq \frac{1}{2}$,
\[\|\Lambda^{-2\alpha}(\text{anttr}\chi\tilde{\rho})\|_{L^2_t} \lesssim \|r^{2\alpha}\text{tr}\chi\tilde{\rho}\|_{L^2_t} \lesssim \|\tilde{\rho}\|_{L^2_t} \lesssim \Delta_0^2 + \mathcal{R}_0.
\]
By Proposition 3.1 and (1.13), for $\alpha \geq \frac{1}{2}$,
\[\|\Lambda^{-2\alpha}\text{div}(an\beta)\|_{L^2_t} \lesssim \|r^{2\alpha-1}\text{div}(an\beta)\|_{L^2_t} \lesssim \Delta_0^2 + \mathcal{R}_0.
\]
By Proposition 3.1 (v) and Hölder inequality, we obtain
\[\int_0^1 \|\Lambda^{-2\alpha}(anA \cdot \tilde{\mathcal{R}})\|_{L^2_t(S_{t'})}^2 dt' = \int_0^1 \int_{S_{t'}} \Lambda^{-4\alpha}(anA \cdot \tilde{\mathcal{R}}) \cdot (anA \cdot \tilde{\mathcal{R}}) d\mu_{\nu}, dt'
\]  
\[\leq \|\Lambda^{-4\alpha}(anA \cdot \tilde{\mathcal{R}})\|_{L^2_t L^2_x} \|anA \cdot \tilde{\mathcal{R}}\|_{L^2_t L^2_x} \lesssim \|\Lambda^{-4\alpha+\frac{\alpha}{2}}(anA \cdot \tilde{\mathcal{R}})\|_{L^2_t L^2_x} \|anA \cdot \tilde{\mathcal{R}}\|_{L^2_t L^2_x} \|\tilde{\mathcal{R}}\|_{L^2}.
\]
Consequently, by Sobolev inequalities, (2.79), (2.53) and Proposition 3.1 (ii),
\[\int_0^t \|\Lambda^{-2\alpha}(anA \cdot \tilde{\mathcal{R}})\|_{L^2_t(S_{t'})}^2 dt' \lesssim \|r^{2\alpha-\frac{\alpha}{2}}\Lambda^{-2\alpha}(anA \cdot \tilde{\mathcal{R}})\|_{L^2(U)}(\Delta_0^2 + \mathcal{R}_0),
\]
which implies
\[ \|A^{-2\alpha}(anA \cdot \hat{R})\|_{L^2(H)} \lesssim \Delta_0^2 + \mathcal{R}_0. \]

Thus we complete the proof of (3.14). □

Proof of Proposition 3.4. By definition (3.1) and Proposition 2.1,
\[ r^{-\alpha}[\Lambda^{-\alpha}, D_t]_g f \]
\[ = C_\alpha \int_0^\infty d\tau \tau^{\frac{4}{2} - 1} e^{-\tau} \int_0^\tau r^2 U(\tau - \tau')(\Lambda, D_t)\Lambda U(\tau')f - \frac{an\hat{\chi}}{\tau} \Delta U(\tau')f d\tau' \]
(3.17) \[ = C_\alpha \int_0^\infty d\tau \tau^{\frac{4}{2} - 1} e^{-\tau} \int_0^\tau r^2 U(\tau - \tau')(\nabla \phi_1(\tau') + \phi_2(\tau')), \]
where
\[ \phi_1(\tau') = an(\hat{\chi} + \kappa) \cdot \nabla U(\tau')f, \]
\[ \phi_2(\tau') = \{an(\beta + \nabla A + A \cdot A + r^{-1} A) + \nabla (an\kappa)\} \nabla U(\tau')f. \]

Noticing that \( \kappa \) can be regarded as an element of \( A \), thus we have \( \nabla (an\kappa) = an\cdot A + an\nabla A \), and
\[ \phi_1(\tau') = anA \cdot \nabla U(\tau')f, \quad \phi_2(\tau') = an(\beta + \nabla A + A \cdot A + r^{-1} A) \nabla U(\tau')f. \]

Let us set
\[ \Phi_1 = C_\alpha \int_0^\infty d\tau \tau^{\frac{4}{2} - 1} e^{-\tau} \int_0^\tau r^2 U(\tau - \tau')\nabla \phi_1(\tau')d\tau' \]
\[ \Phi_2 = C_\alpha \int_0^\infty d\tau \tau^{\frac{4}{2} - 1} e^{-\tau} \int_0^\tau r^2 U(\tau - \tau')\phi_2(\tau')d\tau'. \]

The difference between \( \Phi_1 \) and those in [10, Page 41] is the extra factor “an” in (3.18), which can be easily treated by using the estimates \( \|\nabla (an)\|_{L^\infty L^1} \lesssim \Delta_0^2 + \mathcal{R}_0 \) and (2.24). Based on the estimate \( \mathcal{N}(\mathcal{A}) \lesssim \Delta_0^2 + \mathcal{R}_0 \) which has been proved in (2.79) and Proposition 2.7, also using (2.24), we can proceed exactly as in Appendix or [10, pages 41–43] to obtain for \( \frac{1}{2} \leq \alpha < 1 \) and \( p > 2 \),
\[ \|\Phi_1\|_{L^1 L^2} + \|\Phi_2\|_{L^1 L^2} \lesssim \|f\|_{L^2}(1 + I_{\alpha - \frac{3}{2}})(\Delta_0^2 + \mathcal{R}_0). \]

The proof is therefore complete. □

3.1. \( L^2 \) estimates for Hodge operators. Consider the following Hodge operators on 2-surfaces \( S := S_t \)
\begin{itemize}
  \item The operator \( D_1 \) takes any 1-forms \( F \) into the pairs of functions (div \( F \), curl \( F \)).
  \item The operator \( D_2 \) takes any symmetric traceless 2-tensors \( F \) on \( S \) into the 1-forms div \( F \).
  \item The operator \( \ast D_1 \) takes the pairs of scalar functions \( (\rho, \sigma) \) into the 1-forms \( -\nabla \rho + (\nabla \sigma)^\ast \) on \( S \).
  \item The operator \( \ast D_2 \) takes 1-forms \( F \) on \( S \) into the 2-covariant, symmetric, traceless tensors \( \frac{1}{2} \mathcal{L}_F \gamma \), where
    \[ (\mathcal{L}_F \gamma)_{ab} = \nabla_b F_a + \nabla_a F_b - (\text{div} F) \gamma_{ab}. \]
\end{itemize}

Using Proposition 3.2 and Bôchner identity, we can follow the same way as [2] to obtain elliptic estimates for Hodge operators.

\[ ^5 \text{For various properties of these operators please refer to [1] Page 38 and [2] Section 4.} \]
Proposition 3.4. The following estimates hold on $H$,

(i) Let $\mathcal{D}$ denote either $\mathcal{D}_1$ or $\mathcal{D}_2$. The operator $\mathcal{D}$ is invertible on its range, for $S$ tangent tensor $F$ in the range of $\mathcal{D}$,

\[ \|\nabla \mathcal{D}^{-1}F\|_{L^2(S)} + \|r^{-1} \mathcal{D}^{-1}F\|_{L^2(S)} \lesssim \|F\|_{L^2(S)}.\]

(ii) The operator $(-\Delta)$ is invertible on its range and its inverse $(-\Delta)^{-1}$ verifies the estimate

\[ \|\nabla^2(-\Delta)^{-1}f\|_{L^2(S)} + \|r^{-1} \nabla(-\Delta)^{-1}f\|_{L^2(S)} \lesssim \|f\|_{L^2(S)}.\]

(iii) The operator $*\mathcal{D}_1$ is invertible as an operator defined for pairs of $H^1$ functions with mean zero (i.e. the quotient of $H^1$ by the kernel of $*\mathcal{D}_1$) and its inverse $*\mathcal{D}_1^{-1}$ takes $S$-tangent $L^2$ 1-forms $F$ (i.e. the full range of $*\mathcal{D}_1$) into pair of functions $(\rho, \sigma)$ with mean zero, such that $-\nabla \rho + (\nabla \sigma)^* = F$, verifies the estimate

\[ \|\nabla^* \mathcal{D}_1^{-1}F\|_{L^2(S)} \lesssim \|F\|_{L^2(S)},\]

and by (i) and duality argument

\[ \|r^{-1} \mathcal{D}_1^{-1}F\|_{L^2(S)} \lesssim \|F\|_{L^2(S)}.\]

(iv) The operator $*\mathcal{D}_2$ is invertible as an operator defined on the quotient of $H^1$-vector fields by the kernel of $*\mathcal{D}_2$. Its inverse $*\mathcal{D}_2^{-1}$ takes $S$-tangent 2-forms $Z$ which is in $L^2$ space into $S$ tangent 1-forms $F$ (orthogonal to the kernel of $\mathcal{D}_2$), such that $*\mathcal{D}_2F = Z$, verifies the estimate

\[ \|\nabla^* \mathcal{D}_2^{-1}Z\|_{L^2(S)} \lesssim \|Z\|_{L^2(S)}.\]

As a consequence of (i)-(iv), let $\mathcal{D}^{-1}$ be one of the operators $\mathcal{D}_1^{-1}$, $\mathcal{D}_2^{-1}$, $*\mathcal{D}_1^{-1}$ or $*\mathcal{D}_2^{-1}$. By duality argument, we have the following estimate for appropriate tensor fields $F$,

\[ \|\mathcal{D}^{-1} \text{div} F\|_{L^2(S)} \lesssim \|F\|_{L^2(S)}.\]

Using Proposition 3.2 and Lemma 2.8 and following the similar argument in [2] Proposition 4.24 and Lemma 6.14] we can obtain

Lemma 3.1. Let $\mathcal{D}$ one of the operators $\mathcal{D}_1$, $\mathcal{D}_2$ and $*\mathcal{D}_1$. For $F$ pairs of scalar functions in the first case, $S$-tangent one form for the second and third case, there hold

\[ \mathcal{N}_2(\mathcal{D}^{-1}F) \lesssim \mathcal{N}_1(F) \quad \text{and} \quad \mathcal{N}_1(\nabla \mathcal{D}^{-1}F) \lesssim \mathcal{N}_1(F).\]

4. A brief review of theory of geometric Littlewood Paley

Consider $S$ the collection of smooth functions on $[0, \infty)$ vanishing sufficiently fast at $\infty$ and verifying the vanishing moment property

\[ \int_0^\infty \tau^{k_1} \partial^{k_2} m(\tau) d\tau = 0, \quad k_1 + k_2 \leq N.\]

We set $m_k(\tau) := 2^{k}m(2^k \tau)$ for some smooth function $m \in S$. Recall from [3] the geometric Littlewood-Paley (GLP) projections $P_k$ associated to $m$ which take the form

\[ P_k F := \int_0^\infty m_k(\tau) U(\tau) F d\tau.\]

\[ \text{By “appropriate” , we mean the tensor } F \text{ such that } \text{div} F \text{ is in the space where } \mathcal{D}^{-1} \text{ is well-defined.}\]
for any $S_t$ tangent tensor field $F$, where $U(\tau)F$ is defined by the heat flow $\Box$ on $(S_t, \gamma)$.

**Proposition 4.1.** There exists $m \in S$ such that the GLP projections $P_k$ associated to $m$ verify $U(\infty) + \sum_{k \in \mathbb{Z}} P_k^2 = \text{Id}$. By $f$ we denote a scalar function and $F$ a $S$-tangent tensor field on $\mathcal{H}$, the GLP projections $P_k$ associated to arbitrary induced function $m$ verify the following properties:

(i) **($L^p$-boundedness)** For any $1 \leq p \leq \infty$, and any interval $I \subset \mathbb{Z}$,

\[ \|P_I F\|_{L^p(S)} \lesssim \|F\|_{L^p(S)} \]

(ii) **(Bessel inequality)** For any tensorfield $F$ on $S$,

\[ \sum_k \|P_k F\|_{L^2(S)}^2 \lesssim \|F\|_{L^2(S)}^2, \quad \sum_k 2^{2k} r^{-2} \|P_k F\|_{L^2(S)}^2 \lesssim \|\nabla F\|_{L^2(S)}^2 \]

(iii) **(Finite band property)** For any $1 \leq p \leq \infty$, $k \geq 0$,

\[ \|\Delta P_k F\|_{L^p(S)} \lesssim 2^{2k} r^{-2} \|F\|_{L^p(S)}. \]

Moreover given $m \in S$ we can find $\tilde{m} \in S$ such that $2^{2k} P_k = \Delta \tilde{P}_k$, with $\tilde{P}_k$ the geometric Littlewood Paley projections associated to $\tilde{m}$, then

\[ P_k F = 2^{-2k} \tilde{P}_k \Delta F, \quad \|P_k F\|_{L^p(S)} \lesssim 2^{-2k} r^{-2} \|\Delta F\|_{L^p(S)}. \]

In addition, there hold $L^2$ estimates

\[ \left\{ \begin{array}{l}
\|\nabla P_k F\|_{L^2(S)} \lesssim 2^k r^{-1} \|F\|_{L^2(S)} \\
\|P_k \nabla F\|_{L^2(S)} \lesssim 2^k r^{-1} \|F\|_{L^2(S)} \\
\|\nabla P_{\leq 0} F\|_{L^2(S)} \lesssim r^{-1} \|F\|_{L^2(S)}
\end{array} \right. \]

and

\[ \|P_k F\|_{L^2(S)} \lesssim 2^{-k} r \|\nabla F\|_{L^2(S)} \]

(iv) **(Bernstein Inequality)** For appropriate tensor fields $F$ on $S$, we have weak Bernstein inequalities

\[ \left\{ \begin{array}{l}
\|P_k F\|_{L^p(S)} \lesssim r^{\frac{1}{p}-1} \left(2^{\left(1-\frac{1}{p}\right) k} + 1 \right) \|F\|_{L^2(S)}, \\
\|P_{\leq 0} F\|_{L^p(S)} \lesssim r^{\frac{1}{p}-1} \|F\|_{L^2(S)} \\
\|P_k F\|_{L^2(S)} \lesssim r^{\frac{1}{p}-1} \left(2^{\left(1-\frac{1}{p}\right) k} + 1 \right) \|F\|_{L^p(S)} \\
\|P_{\leq 0} F\|_{L^2(S)} \lesssim r^{\frac{1}{p}-1} \|F\|_{L^p(S)}
\end{array} \right. \]

where $2 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, $k \geq 0$.

(v) **With the help of Proposition 3.2**, we have the sharp Bernstein inequalities

\[ \|P_k f\|_{L^\infty(S)} \lesssim 2^k r^{-1} \|f\|_{L^2(S)}, \quad \|P_k f\|_{L^1(S)} \lesssim 2^k r^{-1} \|f\|_{L^1(S)}. \]

We will also use the notations for any $S$-tangent tensor field $F$,

\[ F_n := P_n^2 F, \quad F_{\leq 0} := \sum_{k \leq 0} P_k^2 F. \]

\[ \text{[Footnote 7]: For more properties, one can refer to [3, 4] and [10].} \]
Now we define for $0 \leq \theta \leq 1$ the Besov $B^\theta$, $P^\theta$ norms for $S$-tangent tensor fields $F$ on $\mathcal{H}$ and $B^\theta_{2,1}$ norm on $S$ as follows:

\begin{align*}
\|F\|_{B^\theta} &= \sum_{k>0} \|(2^k r^{-1})^\theta P_k F\|_{L^r_1 L^2} + \|r^{-\theta} F\|_{L^r_1 L^2}, \\
\|F\|_{P^\theta} &= \sum_{k>0} \|(2^k r^{-1})^\theta P_k F\|_{L^r_2 L^2} + \|r^{-\theta} F\|_{L^r_2 L^2}, \\
\|F\|_{B^\theta_{2,1}(S)} &= \sum_{k>0} \|(2^k r^{-1})^\theta P_k F\|_{L^2} + \|r^{-\theta} F\|_{L^2}.
\end{align*}

Remark 4.1. For any $a \in \mathbb{R}$ and any $S$-tangent tensor field $F$ 

$$
\|F\|^2_{H^a(S)} := \|\Lambda^a F\|^2_{L^2(S)} \approx \sum_{k>0} 2^{2k a} r^{-2a} \|P_k F\|^2_{L^2(S)} + r^{-2a} \|P_{\leq 0} F\|^2_{L^2}.
$$

In certain situations, it is more convenient to work with the Besov norms defined by the classical Littlewood-Paley (LP) projections $E_k$. Recall that (see $[8,9]$) for any scalar function $f$ on $\mathbb{R}^2$ we can define

$$
E_k f = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \varphi(\xi/2^k) \hat{f}(\xi) e^{i\xi \cdot \xi} d\xi,
$$

where $\varphi$ is a smooth function support in the dyadic shell $\{ \frac{1}{2} \leq |\xi| \leq 2 \}$ and satisfying $\sum_{k \in \mathbb{Z}} \varphi(2^{-k} \xi) = 1$ when $\xi \neq 0$.

Define for any $0 \leq \theta < 1$ the $B^\theta$ and $P^\theta$ norms of any scalar function $f$ on $\mathcal{H}$ by

\begin{align*}
\|f\|_{B^\theta} &= \sum_{k>0} \|(2^k r^{-1})^\theta E_k f\|_{L^r_1 L^2} + \|r^{-\theta} f\|_{L^r_1 L^2}, \\
\|f\|_{P^\theta} &= \sum_{k>0} \|(2^k r^{-1})^\theta E_k f\|_{L^r_2 L^2} + \|r^{-\theta} f\|_{L^r_2 L^2}.
\end{align*}

Using Proposition 2.8 and BA1 we can adapt $[7, \text{Proposition 3.28}]$ to obtain the following lemma.

**Lemma 4.1.** Under the bootstrap assumptions (BA1), there exists a finite number of vector fields $\{X_i\}_{i=1}^l$ verifying the conditions

\[
\begin{cases}
\|X_i r \nabla_0 X_i\|_{L^\infty L^2} \lesssim 1, \\
\|r \nabla_0 (\nabla_0 X_i)\|_{L^1 L^\infty} \lesssim 1, \\
\|\nabla_0 - \nabla_0 X_i\|_{L^1 L^\infty} \lesssim \Delta_0, \quad \nabla_0 X_i = 0,
\end{cases}
\]

where $\nabla_0$ represents the covariant derivative induced by the metric $r^2 \gamma^{(0)}$. For appropriate $S$-tangent tensor $F \in L^\infty L^2$, $F \in B^\theta$ if and only if $F \cdot X_i \in B^\theta$, and

$$
C^{-1} \sum_i \|F \cdot X_i\|_{B^\theta} \leq \|F\|_{B^\theta} \leq C \sum_i \|F \cdot X_i\|_{B^\theta}, \quad \text{with } 0 \leq \theta < 1,
$$

where $C$ is a positive constant. The same results hold for the spaces $P^\theta$. Moreover $\mathcal{N}_1(F \otimes X) + \|F \otimes X\|_{L^\infty L^2} \lesssim \mathcal{N}_1(F) + \|F\|_{L^\infty L^2}$, where $\otimes$ stands for either a tensor product or a contraction.

**Lemma 4.1** allows us to define Besov norms for arbitrary $S$-tangent tensor fields $F$ on $\mathcal{H}$ by the classical LP projections.  

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8See $[3, \text{Page 2}]$ for the finite band and sharp Berstein inequalities of $E_k$. 
**Definition 4.2.** Let $F$ be an $(m,n)$ $S$-tangent tensor field on $\mathcal{H}$ and let $F_{i_{1}j_{2}...j_{m}}^{j_{1}j_{2}...j_{m}}$ be the local components of $F$ relative to $\{X_{i}\}_{i=1}^{l}$. We define the $\mathcal{B}^{\theta}$ and $\mathcal{P}^{\theta}$ norms of $F$ by

$$
\|F\|_{\mathcal{B}^{\theta}} = \sum \|F_{i_{1}j_{2}...j_{m}}^{j_{1}j_{2}...j_{m}}\|_{\mathcal{B}^{\theta}} \quad \text{and} \quad \|F\|_{\mathcal{P}^{\theta}} = \sum \|F_{i_{1}j_{2}...j_{m}}^{j_{1}j_{2}...j_{m}}\|_{\mathcal{P}^{\theta}},
$$

where the summation is taken over all possible $(i_{1}\cdots i_{n};j_{1}\cdots j_{m})$.

The equivalence between $\mathcal{B}^{\theta}$, $\mathcal{P}^{\theta}$ norms and $\mathcal{B}^{\theta}$, $\mathcal{P}^{\theta}$ norms is given in the following result whose proof can be found in [10].

**Proposition 4.2.** Under the bootstrap assumptions BAI for arbitrary $S$-tangent tensor fields $F$ on $\mathcal{H}$ there hold for $0 \leq \theta < 1$,

$$
\|F\|_{\mathcal{B}^{\theta}} \approx \|F\|_{\mathcal{B}^{\theta}} \quad \text{and} \quad \|F\|_{\mathcal{P}^{\theta}} \approx \|F\|_{\mathcal{P}^{\theta}}.
$$

**Lemma 4.2.** Let $H$ be any $S_{l}$ tangent tensor and let $f$ be a smooth function,

$$
\|fH\|_{p_{0}} \lesssim \|H\|_{p_{0}} \|f\|_{L^{\infty}} + \|H\|_{L^{2}(\mathcal{H})} \|f^{1/2} \nabla f\|_{L_{x}^{\infty}L_{t}^{4}}.
$$

and the similar estimates hold for $\mathcal{B}^{0}$ and $\mathcal{B}_{2,1}^{0}(S)$.

**Proof.** By GLP decomposition, $H = \tilde{H} + \sum_{n \in \mathbb{N}} P_{n}^{2}H + P_{0}H$, where $\tilde{H} = U(\infty)H$.

$$
\|fH\|_{p_{0}} \leq \sum_{k>0} \|P_{k}(fH_{<k})\|_{L^{2}} + \sum_{k>0} \|P_{k}(fH_{\geq k})\|_{L^{2}} + \sum_{k>0} \|P_{k}(\tilde{H})\|_{L^{2}} + \|fH\|_{L^{2}}.
$$

Let $H_{n} := P_{n}^{2}H$. Consider the first term in (4.14). By (4.1), (4.5) and (4.3),

$$
\sum_{k>0, k>n>0} \|P_{k}(fH_{n})\|_{L^{2}} \lesssim \sum_{k>0, k>n>0} 2^{-k} \|rP_{k}(\nabla fH_{n})\|_{L^{2}}
$$

$$
\lesssim \sum_{k>0, k>n>0} 2^{-k} \left( \|rP_{k}(\nabla f \cdot H_{n})\|_{L^{2}} + \|rP_{k}(f \cdot \nabla H_{n})\|_{L^{2}} \right)
$$

$$
\lesssim \sum_{k>0, k>n>0} \left( 2^{-k+\frac{\theta}{2}} \|r^{\frac{1}{2}} \nabla f \cdot H_{n}\|_{L_{x}^{2}L_{t}^{4/3}} + 2^{-k+n} \|f\|_{L^{\infty}} \|H_{n}\|_{L^{2}} \right)
$$

$$
\lesssim \sum_{k>0, k>n>0} \left( 2^{-\frac{\theta}{2}} \|r^{\frac{1}{2}} \nabla f\|_{L_{x}^{4}L_{t}^{4}} \|H_{n}\|_{L^{2}} + 2^{-k+n} \|f\|_{L^{\infty}} \|P_{n}H\|_{L^{2}} \right)
$$

$$
\lesssim \|H\|_{L^{2}} \|r^{1/2} \nabla f\|_{L_{x}^{4}L_{t}^{4}} + \|f\|_{L^{\infty}} \|H\|_{p_{0}},
$$

and similarly,

$$
\sum_{k>0} \|P_{k}(fH_{\leq 0})\|_{L^{2}} \lesssim \|H\|_{L^{2}} \|r^{1/2} \nabla f\|_{L_{x}^{4}L_{t}^{4}} + \|f\|_{L^{\infty}} \|H\|_{p_{0}}.
$$

Now consider the second term. By (4.2), (4.5) and (4.5), we obtain

$$
\sum_{k>0, n \geq k} \|P_{k}(fH_{n})\|_{L^{2}} \lesssim \sum_{k>0, n \geq k} 2^{-2n} \|r^{2}P_{k}(f \Delta H_{n})\|_{L^{2}}
$$

$$
\lesssim \sum_{k>0, n \geq k} 2^{-2n} \left( \|r^{2}P_{k}(\nabla \nabla fH_{n})\|_{L^{2}} + \|r^{2}P_{k}(\nabla f \cdot \nabla H_{n})\|_{L^{2}} \right)
$$

$$
\lesssim \sum_{k>0, n \geq k} \left( 2^{-2n+k+n} \|f\|_{L^{\infty}} \|P_{n}H\|_{L^{2}} + 2^{-2n+n} \|r^{\frac{1}{2}} \nabla f \cdot \nabla H_{n}\|_{L_{x}^{2}L_{t}^{4/3}} \right).
$$
By (4.3), we have
\[2^{-2n+k/2} \|r^{p/2} \nabla f \cdot \nabla H_n\|_{L^2 L^{1/3}} \lesssim 2^{-2n+k/2} \|r^{1/2} \nabla f\|_{L^p L^1} \|r \nabla H_n\|_{L^2}\]
\[\lesssim 2^{k-n} \|r^{1/2} \nabla f\|_{L^p L^1} \|H\|_{L^2},\]
thus
\[
\sum_{k>0, n \geq k} \|P_k(f H_n)\|_{L^2} \lesssim \|H\|_{L^2} \|r^{1/2} \nabla f\|_{L^p L^1} + \|f\|_{L^p} \|H\|_{L^2}.
\]
At last, due to \(\nabla H = 0\) and (SobM1), we can deduce
\[
\sum_{k>0} \|P_k(f H_n)\|_{L^2} \lesssim \sum_{k>0} 2^{-k} \|r^{1/2} \nabla f\|_{L^p L^1} \|H\|_{L^2} \lesssim \|r^{1/2} \nabla f\|_{L^p L^1} \|H\|_{L^2}.
\]
The proof is complete. \(\square\)

4.1. Product estimates and Intertwine estimates.

**Proposition 4.3.** Let \(D\) be one of the operators \(D_1, D_2\) and \(*D_1\). Then for \(1 < p \leq 2\) and any \(S\)-tangent tensor \(F\) on \(\mathcal{H}\) there holds
\[
\|D^{-1}F\|_{L^2(S)} \lesssim \|r^{2-\frac{2}{p}}F\|_{L^p(S)}.
\]

**Proof.** For \(p = 2\), the inequality follows immediately from Proposition 3.4. For \(p > 2\), from (Sob) and Proposition 3.3 we infer for \(p' > 2\) satisfying \(\frac{1}{p} + \frac{1}{p'} = 1\) that
\[
\|r^{\frac{2}{2} - 2\cdot D^{-1}F}\|_{L^{p'}(S)} \lesssim \|r^{*D^{-1}F}\|_{L^{2}(S)}^{(1/2)} \|r^{-1 \cdot D^{-1}F}\|_{L^{2}(S)}^{(1/2)} + \|r^{-1 \cdot D^{-1}F}\|_{L^{2}(S)}^{(1/2)} \lesssim \|F\|_{L^{2}(S)}.
\]
Thus, by duality, we complete the proof. \(\square\)

**Lemma 4.3.** Let \(D\) denote one of the Hodge operators \(D_1, D_2, *D_1\) and \(*D_2\), let \(D^{-1}\) denote the inverse of \(D\). For \(P_k F\) with \(P_k\) the GLP projections associated to the heat equation \((3.2)\), there hold for \(k > 0\), \(1 < p \leq 2\),
\[
\|D^{-1}P_k F\|_{L^2} \lesssim 2^{-k/2} \|F\|_{L^2} \quad \text{and} \quad \|P_k D^{-1} F\|_{L^2} \lesssim 2^{-(2-\frac{2}{p})k - \frac{2}{p}} \|F\|_{L^2}.
\]

**Proof.** The first inequality can be proved by using (3.1) in Proposition 4.1 and Proposition 3.4. The second can be proved by duality with the help of the first inequality and (SobM1). \(\square\)

The following result follows from the second estimate in Lemma 4.3 immediately.

**Proposition 4.4.** Let \(D^{-1}\) denote either \(D_1^{-1}, *D_1^{-1}, D_2^{-1}\), then for appropriate \(S\)-tangent tensor fields \(F\) on \(\mathcal{H}\) and any \(1 < p \leq 2\),
\[
(4.15) \quad \|D^{-1} F\|_{P^0} \lesssim \|r^{2-\frac{2}{p}} F\|_{L^p L^p}.
\]

Since the proof of the Hodge-elliptic estimate for geodesic foliation contained in [11] pages 295–301) only relied on
\[
\|K\|_{L^2} + \|\Lambda^{-\alpha_0} K\|_{L^p L^1} \lesssim \Delta^2_0 + R_0, \quad \text{with} \quad \alpha_0 \geq 1/2.
\]
Hence, based on Lemma 2.8 and Proposition 3.2, the same proof also applies to the case of time foliation. We can obtain the result on the Hodge-elliptic \(P^0\) estimates.
Theorem 4.3 (Hodge-elliptic $\mathcal{P}^s$-estimate). Let $\mathcal{D}$ denote either $\mathcal{D}_1$, $\mathcal{D}_2$ or their adjoint operators $^*\mathcal{D}_1$ and $^*\mathcal{D}_2$. Then for any $S$-tangent tensor fields $\xi$ and $F$ satisfying $\mathcal{D}\xi = F$ and any $\frac{1}{2} > \sigma \geq 0$,
\begin{equation}
\|\nabla \xi\|_{p^*} \lesssim \|F\|_{p^*} + \Delta_0 \|D^{-1}F\|_{L^q_L H^1} \|F\|_{L^q_L H^1}^{1-q},
\end{equation}
where $1/2 < a < q < 1 - \sigma$ and $b > 4$.

We now give a series of product estimates for Besov norms. Proofs can be seen in [11] pages 302–304.

Lemma 4.4. For any $S$-tangent tensor fields $F$ and $G$,
\begin{align}
\|F \cdot G\|_{p_0} &\lesssim N_1(F)(\|r^{-\frac{1}{2}}G\|_{L^q_L H^2} + \|r^{-\frac{1}{2}}\nabla G\|_{L^p_L H^2}) \text{ with } b > 4, \\
\|F \cdot G\|_{p_0} &\lesssim N_2(r^{1/2}F)\|G\|_{p_0}, \\
\|F \cdot G\|_{p_0} &\lesssim N_4(\|\nabla F\|_{L^2_L H^1} + \|G\|_{L^2_L H^1}).
\end{align}

Corollary 1. Regard $\kappa, \iota$ also as elements of $A$, there hold
\begin{align}
\|A \cdot F\|_{p_0} &\lesssim (\Delta_0^2 + \mathcal{R}_0)N_1(r^{\frac{1}{2}}F), \\
\|(\text{tr} \chi, r^{-1})F\|_{p_0} \lesssim N_1(F),
\end{align}
\begin{align}
\|\text{an} A \cdot F\|_{p_0} &\lesssim (\Delta_0^2 + \mathcal{R}_0)N_1(r^{\frac{1}{2}}F), \\
\|\text{an}(\text{tr} \chi, r^{-1})F\|_{p_0} \lesssim N_1(F).
\end{align}

Proof. Let us prove (4.20) first. Using (4.17) for $b > 4$, we have
\begin{equation}
\|A \cdot F\|_{p_0} \lesssim N_1(A)(\|r^{\frac{1}{2}}\nabla F\|_{L^2_L H^1} + \|r^{-\frac{1}{2}}F\|_{L^q_L H^1}) \lesssim N_1(A) \cdot N_1(r^{\frac{1}{2}}F)
\end{equation}
where the last inequality follows by using (4.28).

By finite band property of GLP in Proposition 4.1, it is straightforward to obtain
\begin{equation}
\|r^{-1}F\|_{p_0} \lesssim \|\nabla F\|_{L^2} + \|r^{-1}F\|_{L^2}.
\end{equation}

Noticing that $\text{tr} \chi \cdot F = \iota \cdot F + 2r^{-1} \cdot F$ and $N_1(\iota) \lesssim \Delta_0^2 + \mathcal{R}_0$ in Proposition 2.7.

The other inequality in (4.20) follows by (4.22) and (4.23) with $A$ replaced by $\iota$.

Applying (4.13) to $f = \text{an}$ and $H = A \cdot F$, $\text{tr} \chi F$, $r^{-1}F$, (4.21) can be derived by using (4.20) for $H$ and the following inequalities for $f$
\begin{equation}
\|\nabla (\text{an})\|_{L^p_L H^1} \approx \|\zeta\|_{L^p_L H^1} \lesssim \Delta_0^2 + \mathcal{R}_0, \quad \|\text{an}\|_{L^\infty} \lesssim 1.
\end{equation}

5. Sharp trace theorem

The purpose of the section is to prove

Theorem 5.1 (Sharp trace theorem). Let $F$ be an $S$-tangent tensor which admits a decomposition of the form $\nabla (\text{an} F) = \mathcal{D}_1 P + E$ with tensors $P$ and $E$ of the same type as $F$, suppose $\lim_{t \to 0} \|F\|_{L^\infty} < \infty$ and $\lim_{t \to 0} r|\nabla F| < \infty$, there holds the following sharp trace inequality,
\begin{equation}
\|F\|_{L^\infty_L H^1} \lesssim N_1(F) + N_1(P) + \|E\|_{p_0}.
\end{equation}

Remark 5.2. We will employ this theorem to estimate $\|F\|_{L^\infty_L L^2}$ for $F = \zeta, \nu, \hat{\chi}, \zeta$. By local analysis, the two initial assumptions can be checked for the four quantities.

The following result gives the important inequalities to prove Theorem 5.1
Proposition 5.1. Let $p \geq 1$ be any integer, for any $S$-tangent tensor fields $F$, $H$ and $G$ of the same type. There holds

\begin{equation}
(5.2) \quad \left\| r^{-p} \int_0^t r^p H \cdot G dt' \right\|_{\mathcal{B}^0} \lesssim \left\| H \right\|_{p_0} (\mathcal{N}_1(G) + \left\| G \right\|_{L^\infty L^2}).
\end{equation}

Let us further assume

\begin{equation}
(5.3) \quad \lim_{t \to 0} r^p \left\| F \right\|_{L^\infty} = 0, \quad \lim_{t \to 0} \left\| G \right\|_{L^\infty} < \infty, \text{ when } p \geq 2,
\end{equation}

then there holds for $p \geq 1$,

\begin{equation}
(5.4) \quad \left\| r^{-p} \int_0^t r^p D_t F \cdot G dt' \right\|_{\mathcal{B}^0} \lesssim \mathcal{N}_1(F) \mathcal{N}_1(G).
\end{equation}

Using a modified version of [4] Lemma 5.3, (see in Lemma 8.1 in Appendix), also using Proposition 4.2 Proposition 5.1 follows by repeating the procedure in [4] and Lemma 4.1. We omit the detail of the proof of Proposition 5.1.

**Proof of Theorem 5.1.** We set $\varphi(t) = \int_0^t an|F|^2 dt'$, then $D_t \varphi = an|F|^2$. Due to [2] Proposition 5.1] we have

\begin{equation}
(5.5) \quad \left\| \varphi \right\|_{L^\infty(H)} \lesssim \left\| \nabla \varphi \right\|_{\mathcal{B}^0} + \left\| r^{-1} \varphi \right\|_{L^\infty L^2}.
\end{equation}

It is easy to see

\begin{equation}
(5.6) \quad \left\| r^{-1} \varphi \right\|_{L^\infty L^2} \lesssim \left\| F \right\|_{L^\infty L^2} \cdot \left\| r^{-1} F \right\|_{L^2}.
\end{equation}

We now estimate $\left\| \nabla \varphi \right\|_{\mathcal{B}^0}$. In view of (2.19), we obtain

\begin{equation}
(5.7) \quad \nabla_L \nabla \varphi + \frac{1}{2} \text{tr} \chi \nabla \varphi = 2(an)^{-1} \nabla (an \cdot F) \cdot F - \chi \cdot \nabla \varphi - (\zeta + \underline{\zeta})|F|^2.
\end{equation}

Using the decomposition $\nabla(an \cdot F) = D_t P + E$ and Lemma 4.1] we pair $\nabla \varphi$ with vector field $X = X_t$ to obtain

\begin{equation}
(5.8) \quad \nabla_L (r \nabla \varphi \cdot X) = -\frac{1}{2} r^2 \kappa \nabla \varphi \cdot X - r \nabla \varphi \otimes X
\end{equation}

\[+ \left\{ 2 \nabla_L (P \cdot F) \otimes X + 2(an)^{-1} E \cdot F \otimes X - |F|^2 (\zeta + \underline{\zeta}) \cdot X \right\}.
\]

We will not distinguish "\otimes" with ":", and also suppress $X$ whenever there occurs no confusion. Note that under the transport coordinate $(s, \omega_1, \omega_2)$, we have

\begin{equation}
\frac{\partial \varphi}{\partial \omega_i} = \int_0^t \left\{ 2(\partial_{\omega_i} F, F) + |F|^2 \partial_{\omega_i} \log(an) \right\} nadt'
\end{equation}

by assumptions on initial condition, $\lim_{t \to 0} \frac{\partial \varphi}{\partial \omega_i} = 0$. It then follows by integrating (5.8) along a null geodesic $\Gamma_{\omega}$ from 0 to $s(t)$ that, symbolically,

\begin{equation}
(5.9) \quad \nabla \varphi(t) = r^{-1} \int_0^t \left\{ r' an(\kappa + \underline{\kappa}) \nabla \varphi + r' E \cdot F \right\} dt'
\end{equation}

\[+ r^{-1} \int_0^t r' D_t P \cdot F dt' + r^{-1} \int_0^t anr'|F|^2 (\zeta + \underline{\zeta}) dt'.
\]

Since by Proposition 4.2

\begin{equation}
(5.10) \quad \left\| \nabla \varphi \right\|_{\mathcal{B}^0} \approx \sum_{k > 0} \left\| E_k (\nabla \varphi) \right\|_{L^\infty L^2} + \left\| \nabla \varphi \right\|_{L^\infty L^2},
\end{equation}

where
and the estimate of $\|\nabla \varphi\|_{L^2 L^\infty}$ can be obtained in view of (5.9) and (SobM1).

(5.11) \[ \|\nabla \varphi\|_{L^2 L^\infty} \lesssim (\|\kappa\|_{L^\infty L^2} + \|\hat{\chi}\|_{L^\infty L^2}) \|\nabla \varphi\|_{L^2} + (\|\varphi\|_{L^2} + \|\nabla L P\|_{L^2} + \mathcal{N}_1(A)\mathcal{N}_1(F)) r^{-1} F \|_{L^2} \]

where the term on the right of (5.11) can be absorbed in view of $\|\kappa, \hat{\chi}\|_{L^\infty L^2} \lesssim \Delta_0$, obtained from (2.84) and BA1. It remains to estimate the first term in (5.10).

Applying Littlewood Paley decomposition $E_k$ to $\nabla \varphi \otimes X$ via (5.9), we only need to estimate the following terms

\[ I_1 = \sum_{k>0} \left\| E_k \int_0^t r' a_n(\kappa + \hat{\chi}) \nabla \varphi \right\|_{L^\infty L^2} , \quad I_2 = \sum_{k>0} \left\| E_k \int_0^t r' E \cdot F dt' \right\|_{L^\infty L^2} , \]
\[ I_3 = \sum_{k>0} \left\| E_k \int_0^t r' D_t P \cdot F dt' \right\|_{L^\infty L^2} , \quad I_4 = \sum_{k>0} \left\| E_k \int_0^t r' \|F\|^2(\zeta + \zeta') dt' \right\|_{L^\infty L^2} . \]

Use (5.2) and Proposition 4.2

\[ I_1 \lesssim (\mathcal{N}_1(\hat{\chi}) + \|\hat{\chi}\|_{L^\infty L^2} + \mathcal{N}_1(\kappa) + \|\kappa\|_{L^\infty L^2}) \|\nabla \varphi\|_{\mathcal{P}_0} . \]

Consequently, by (2.84), Proposition 2.7 (2.79) and $\|\hat{\chi}\|_{L^\infty L^2} \leq \Delta_0$ in BA1,

\[ I_1 \lesssim \Delta_0 \|\nabla \varphi\|_{\mathcal{P}_0} . \]

Apply (5.2) to $I_2$,

\[ I_2 \lesssim \|\varphi\|_{\mathcal{P}_0} (\mathcal{N}_1(F) + \|F\|_{L^\infty L^2}) . \]

By (5.4) and Lemma 4.1

\[ I_3 \lesssim \mathcal{N}_1(P) (\mathcal{N}_1(F) + \|F\|_{L^\infty L^2}) . \]

By (5.2), (4.19), BA1 and (2.79), we have

\[ I_4 \lesssim (\mathcal{N}_1(\zeta + \zeta') + \|\zeta + \zeta\|_{L^\infty L^2}) \|F\|_{\mathcal{P}_0} \lesssim \Delta_0 \mathcal{N}_1(F) (\|\nabla F\|_{L^2} + \|F\|_{L^\infty L^2}) . \]

Thus we conclude

\[ \|\nabla \varphi\|_{\mathcal{P}_0} \lesssim (\mathcal{N}_1(P) + \|\varphi\|_{\mathcal{P}_0} \left( \mathcal{N}_1(F) + \|F\|_{L^\infty L^2} \right) + \Delta_0 \|\nabla \varphi\|_{\mathcal{P}_0} + \Delta_0 \mathcal{N}_1(F) (\|\nabla F\|_{L^2} + \|F\|_{L^\infty L^2}) . \]

This inequality, together with the fact that $\|\nabla \varphi\|_{\mathcal{P}_0} \lesssim \|\nabla \varphi\|_{\mathcal{E}_0}$, yields

\[ \|\nabla \varphi\|_{\mathcal{E}_0} \lesssim (\mathcal{N}_1(F) + \|F\|_{L^\infty L^2}) (\mathcal{N}_1(P) + \|E\|_{\mathcal{P}_0} + \Delta_0 \mathcal{N}_1(F)) . \]

Combine the above inequality with (5.3) and (5.6), we get

\[ \|F\|_{L^\infty L^2}^2 \lesssim (\mathcal{N}_1(F) + \|F\|_{L^\infty L^2}) (\mathcal{N}_1(P) + \|E\|_{\mathcal{P}_0} + \mathcal{N}_1(F) \Delta_0) + \|F\|_{L^\infty L^2} \cdot r^{-1} F \|_{L^2} \]

which implies (5.1) by Young’s inequality. □
6. Error estimates

We will employ the following conventions:

- \( \mathcal{R} \) denotes either the pair \((\mathcal{R}, -\mathcal{R})\) or \(\mathcal{R}\)
- \(\mathcal{D}^{-1}\mathcal{R}\) denotes either \(\mathcal{D}_1^{-1}(\mathcal{R}, -\mathcal{R})\) or \(\mathcal{D}_1^{-1}\mathcal{R}\)
- \(\mathcal{D}^{-2}\mathcal{R}\) denotes either \(\mathcal{D}_2^{-1}\mathcal{D}_1^{-1}(\mathcal{R}, -\mathcal{R})\) or \(\mathcal{D}_1^{-1}\mathcal{D}_2^{-1}\mathcal{R}\)
- \(\mathcal{D}^{-1}\mathcal{D}_1\mathcal{R}\) denotes either \(\mathcal{D}_1^{-1}\mathcal{D}_1\mathcal{R}\) or \(\mathcal{D}_1^{-1}\mathcal{D}_2(\mathcal{R}, -\mathcal{R})\)
- \(\mathcal{C}(\mathcal{R})\) denotes \([\mathcal{D}_t, \mathcal{D}_1^{-1}]\mathcal{R}\) or \([\mathcal{D}_t, \mathcal{D}_1^{-1}]\mathcal{R}\)
- \(\mathcal{D}^{-2}\mathcal{D}_1\mathcal{R}\) denotes \(\mathcal{D}_2^{-1}\mathcal{D}_1^{-1}\mathcal{D}_1(\mathcal{R}, -\mathcal{R})\) or \(\mathcal{D}_1^{-1}\mathcal{D}_2^{-1}\mathcal{D}_1\mathcal{R}\)
- \(\mathcal{F}\) denotes \(\mathcal{D}^{-1}\mathcal{R}\) or \((a\lambda + 2a\lambda, 0)\)
- \(\mathcal{D}^{-1}\mathcal{F}\) denotes either \(\mathcal{D}^{-2}\mathcal{R}\) or \(\mathcal{D}_1^{-1}(a\lambda + 2a\lambda)\).

6.1. Commutation formula. We will study error terms which arise from commuting \(\mathcal{D}_t\) with Hodge operators. Regard \(\mathcal{R}\) also an element of \(\mathcal{A}\), symbolically, the commutation formula and its good part can be written as follows

(6.1) \[ [\mathcal{D}_t, \nabla] F = an \left( (A + \frac{1}{r})\nabla F + (A + \frac{1}{r}) \cdot A \cdot F + \beta \cdot F \right), \]

(6.2) \[ [\mathcal{D}_t, \nabla_g] F := an \left( (A + \frac{1}{r})\nabla F + (A + \frac{1}{r}) \cdot A \cdot F \right). \]

Due to the nontrivial factor \(\text{“} an \text{”} \) in (6.1), the treatment in [11] Section 6 has to be modified. We rewrite equations (2.11), (2.12) and (2.13) as

(6.3) \[ L(\mathcal{R}, -\mathcal{R}) = \mathcal{D}_1\beta + r^{-1}\mathcal{R} + A \cdot \mathcal{R}, \]

(6.4) \[ \nabla_{\mathcal{L}} = \mathcal{D}_1(\mathcal{R}, \sigma) + r^{-1}\mathcal{R} + A \cdot \mathcal{R}, \]

where \(\mathcal{R} := R_0 + \nabla A + A \cdot A + r^{-1}A\).

We will consider the commutators

(6.5) \[ \mathcal{C}(\mathcal{R}) = (\mathcal{C}_1(\mathcal{R}), \mathcal{C}_2(\mathcal{R}), \mathcal{C}_3(\mathcal{R})) \]

given in [2] Definition 6.3] which, by using the above conventions, can be written symbolically as

\[
\begin{align*}
\mathcal{C}_1(\mathcal{R}) &= \nabla\mathcal{D}^{-1}[\mathcal{D}_t, \mathcal{D}^{-1}]\mathcal{R}, \\
\mathcal{C}_2(\mathcal{R}) &= \nabla[\mathcal{D}_t, \mathcal{D}^{-1}]\mathcal{D}^{-1}\mathcal{R}, \\
\mathcal{C}_3(\mathcal{R}) &= [\mathcal{D}_t, \nabla]\mathcal{D}^{-2}\mathcal{R}.
\end{align*}
\]

Corresponding to (6.3) and (6.4), we introduce the error terms

(6.6) \[ Err := \mathcal{D}_1^{-1}\mathcal{D}_1(\mathcal{R}, -\mathcal{R}) - an\beta \quad \text{and} \quad \mathcal{E}rr := \mathcal{D}_1^{-1}\mathcal{D}_1\beta - an(\mathcal{R}, \sigma). \]

Denote by \(\mathfrak{g}\) either \(Err\) or \(\mathcal{E}rr\). Symbolically, \(\mathfrak{g}\) has the form

\[ \mathfrak{g} = \mathcal{D}^{-1}\{an(r^{-1}\mathcal{R} + A \cdot \mathcal{R})\}. \]

We then infer from (6.3) and (6.4) the symbolic expression

(6.7) \[ \mathcal{D}^{-1}\mathcal{D}_1\mathcal{R} = anR_0 + \mathfrak{g}. \]

---

9 For simplicity, we use \((a\lambda + 2a\lambda)\) to denote the pair of quantities \((a\lambda + 2a\lambda, 0)\).
By using (1.15) with $\theta = 0$ and $p = \frac{1}{3}$, Proposition 3.4 and the Hölder inequality we infer that
\[
\|\tilde{f}\|_{P^0} \lesssim \|r^{\frac{2}{3}} A \cdot \hat{R}\|_{L^\infty_t L^\frac{6}{5}_x} + \|\hat{R}\|_{L^2} \lesssim \Delta_0^2 + R_0. \tag{6.8}
\]

The purpose of this section is to prove

**Proposition 6.1.** There hold the following decomposition for commutators,
\[
C(\hat{R}) = D_t P + E,
\]
where $P, P'$ and $E, E'$ are $S$ tangent tensors verifying
\[
N_1(P) + N_1(P') + \|E\|_{P^0} + \|E'\|_{P^0} \lesssim \Delta_0^2 + R_0.
\]
\[
\lim_{t \to 0} (r\|P\|_{L^\infty(S)} + r\|P'\|_{L^\infty(S)}) = 0.
\]

**Proposition 6.2.** For the error terms $C_0(\hat{R}), C_1(\hat{R}), C_2(\hat{R}), C_3(\hat{R})$, etc, there hold
\[
\|C_0(\hat{R})\|_{P^0} \lesssim \Delta_0^2 + R_0, \tag{6.11}
\]
\[
\|C_1(\hat{R})\|_{P^0} \lesssim \Delta_0^2 + R_0, \tag{6.12}
\]
\[
C_2(\hat{R}) = \nabla D^{-1}(an\beta \cdot D^{-1} \hat{R}) + err, \tag{6.13}
\]
\[
C_3(\hat{R}) = an\beta \cdot D^{-2} \hat{R} + err \tag{6.14}
\]
\[
[\nabla D_t, D^{-1}](\alpha\delta + 2\alpha\lambda) = \nabla D^{-1}(an\beta \cdot D^{-1} \hat{R}) + err \tag{6.15}
\]
\[
[\nabla D_t, D^{-1}](\alpha\delta + 2\alpha\lambda) = an\beta \cdot D^{-1}(\alpha\delta + 2\alpha\lambda) + err \tag{6.16}
\]
with
\[
\|err\|_{P^0} \lesssim \Delta_0^2 + R_0.
\]

We will rely on (2.79), (2.80), (1.13), Proposition 2.7 and Remark 2.2, i.e.
\[
\|A\|_{L^\infty_t L^\frac{6}{5}_x} + \|A\|_{L^6} + N_1(A) + \|r^{\frac{2}{3}} \nabla \text{tr} \chi\|_{L^\infty_t L^\frac{6}{5}_x} + \|\nabla \text{tr} \chi\|_{L^2} + \|R_0\|_{L^2} \lesssim \Delta_0^2 + R_0,
\]
where $\iota, \kappa$ are regarded as elements of $A$.

**Step 1.** We first prove (6.11). In view of (6.1),
\[
\|C_0(\hat{R})\|_{P^0} \lesssim \Delta_0^2 + R_0.
\]

The proof of Proposition 6.2: Part I. In order to prove Proposition 6.1, let us consider the structure of commutators. We first use (6.1) to write
\[
\|C_0(\hat{R})\|_{P^0} \lesssim \Delta_0^2 + R_0,
\]
\[
C_1(\hat{R}) = \nabla D^{-1}(an\beta \cdot D^{-1} \hat{R}) + err, \tag{6.13}
\]
\[
C_2(\hat{R}) = \nabla D^{-1}(an\beta \cdot D^{-1} \hat{R}) + err, \tag{6.14}
\]
\[
[\nabla D_t, D^{-1}](\alpha\delta + 2\alpha\lambda) = \nabla D^{-1}(an\beta \cdot D^{-1} \hat{R}) + err \tag{6.15}
\]
\[
[\nabla D_t, D^{-1}](\alpha\delta + 2\alpha\lambda) = an\beta \cdot D^{-1}(\alpha\delta + 2\alpha\lambda) + err \tag{6.16}
\]
with
\[
\|err\|_{P^0} \lesssim \Delta_0^2 + R_0.
\]

We will rely on (2.79), (2.80), (1.13), Proposition 2.7 and Remark 2.2, i.e.
\[
\|A\|_{L^\infty_t L^\frac{6}{5}_x} + \|A\|_{L^6} + N_1(A) + \|r^{\frac{2}{3}} \nabla \text{tr} \chi\|_{L^\infty_t L^\frac{6}{5}_x} + \|\nabla \text{tr} \chi\|_{L^2} + \|R_0\|_{L^2} \lesssim \Delta_0^2 + R_0,
\]
where $\iota, \kappa$ are regarded as elements of $A$. 

**Step 1.** We first prove (6.11). In view of (6.1),
\[
C_0(\hat{R}) = D^{-1}(an((A + r^{-1})(\nabla D^{-1} \hat{R}) + (A + r^{-1}) \cdot D^{-1} \hat{R} + \beta \cdot D^{-1} \hat{R})).
\]
Using (4.13) with \( \theta = 0 \) and \( p = \frac{4}{3} \), Proposition 3.4 also with the help of (6.17) and Hölder inequality, we can estimate the various terms in (6.18) to get
\[
\|C_0(\tilde{R})\|_{p_0} \lesssim \Delta_0^2 + \mathcal{R}_0 + \Delta_0 \cdot \mathcal{N}_1(D^{-1}\tilde{R}).
\]
By the definition of \( \mathcal{N}_1(D^{-1}\tilde{R}) \) and Proposition 3.4 it follows that
\[
\mathcal{N}_1(D^{-1}\tilde{R}) \lesssim \mathcal{R}_0 + \Delta_0^2 + \|D^{-1}D_t\tilde{R}\|_{L_t^2 L_x^2} + \|C_0(\tilde{R})\|_{L_t^2 L_x^2}.
\]

While it follows from (6.7), (6.8) and (1.13) that
\[
\|D^{-1}D_t\tilde{R}\|_{L_x^2} \lesssim \|\tilde{R}\|_{p_0} + \|anR_0\|_{L_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0.
\]
Combining the above three inequalities and using the smallness of \( \Delta_0 \) we obtain (6.11).

In the above proof, together with Lemma 3.1 (2.79) and Remark 2.1 we have also verified the following

**Proposition 6.3.**

\[
\begin{align*}
(6.20) & \quad \|D^{-1}D_t\tilde{R}\|_{L_x^2} \lesssim \mathcal{R}_0 + \Delta_0^2, \\
(6.21) & \quad \|D_t, D^{-1}\tilde{R}\|_{L_x^2} \lesssim \mathcal{R}_0 + \Delta_0^2, \\
(6.22) & \quad \mathcal{N}_1(F) \lesssim \mathcal{R}_0 + \Delta_0^2, \\
(6.23) & \quad \mathcal{N}_1(D^{-1}F) \lesssim \mathcal{R}_0 + \Delta_0^2, \quad \mathcal{N}_2(D^{-1}F) \lesssim \mathcal{R}_0 + \Delta_0^2,
\end{align*}
\]

where \( F \) denotes either \( D^{-1}\tilde{R} \) or \((a\delta + 2a\lambda)\).

**Step 2.** We will prove (6.13)-(6.16). Let us first establish the following

**Lemma 6.1.** Denote by \( D^{-1} \) one of the operators among \( D_1^{-1}, D_2^{-1} \) and \( *D_1^{-1} \). For any \( S_t \) tangent tensor field \( H \), \( b > 4 \)

\[
\begin{align*}
(6.24) & \quad \|r^{-\frac{1}{2}}D^{-1}(an\nabla D^{-1}H)\|_{L_t^4 L_x^2} \lesssim \mathcal{N}_1(D^{-1}H)\|\zeta + \zeta\|_{L_r^\infty L_x^1} + \mathcal{N}_1(r^{-\frac{1}{2}}D^{-1}H).
\end{align*}
\]

**Proof.** Using Propositions 3.4 and 4.3

\[
\begin{align*}
\|r^{-\frac{1}{2}}D^{-1}(an\nabla D^{-1}H)\|_{L_t^4 L_x^2} & \lesssim \|r^{-\frac{1}{2}}(D^{-1}(\nabla(an)D^{-1}H) + \mathcal{D}^{-1}(\nabla(an)D^{-1}H))\|_{L_t^4 L_x^2} \\
& \lesssim \|r^{-\frac{1}{2}}\nabla(an)D^{-1}H\|_{L_t^4 L_x^{4/3}} + \|r^{-\frac{1}{2}}D^{-1}H\|_{L_t^4 L_x^2}.
\end{align*}
\]

Since by using (2.28), we obtain

\[
\begin{align*}
\|r^{-\frac{1}{2}}\nabla(an)D^{-1}H\|_{L_t^4 L_x^{4/3}} & \lesssim \|r^{-\frac{3}{4}}D^{-1}H\|_{L_t^4 L_x^2} \|\nabla(an)\|_{L_r^\infty L_x^1} L_t^4 L_x^2 \\
& \lesssim \mathcal{N}_1(D^{-1}H)\|\zeta + \zeta\|_{L_r^\infty L_x^1},
\end{align*}
\]

and \( \|r^{-\frac{1}{2}}D^{-1}H\|_{L_t^4 L_x^2} \lesssim \mathcal{N}_1(r^{-1/2}D^{-1}H) \). Then (6.24) follows. \[\Box\]

The proof of (6.13)-(6.16) can be completed by using (6.22) combined with the following result.

**Lemma 6.2.** Denote by \( D^{-1} \) either \( D_1^{-1} \) or \( D_2^{-1} \). For appropriate \( S \)-tangent tensor field \( F \), there hold

\[
\begin{align*}
(6.25) & \quad \|[D_t, \nabla]gD^{-1}F\|_{p_0} + \|\nabla[D^{-1}, D_t]gF\|_{p_0} \lesssim \mathcal{N}_1(F), \\
(6.26) & \quad \|[D_t, \nabla D^{-1}]gF\|_{p_0} \lesssim \mathcal{N}_1(F).
\end{align*}
\]
Proof. Observe that
\[ \| [D_t, \nabla^{-1}] g F \|_{p_0} \lesssim \| [D_t, \nabla] g D^{-1} F \|_{p_0} + \| \nabla [D^{-1}, D_t] g F \|_{p_0}, \]
it suffices to prove (6.25) only.

We first derive in view of (6.1) by using Lemma 4.4 with \(4 < b < \infty\),
\begin{equation}
\| [D_t, \nabla] g D^{-1} F \|_{p_0} \lesssim N_1 (\nabla D^{-1} F) \left( \| r^{1/2} \nabla (anA) \|_{L_t^\infty L_x^2} + \| r^{-1/2} anA \|_{L_t^\infty L_x^2} \right) + N_2 (D^{-1} F) (|anA \cdot A|_{p_0} + \| r^{-1/2} anA \|_{p_0}) + \| r^{-1} an \nabla D^{-1} F \|_{p_0}.
\end{equation}
(6.27)

By (6.24) and (6.17), we obtain
\[ \| r^{-1} anA \|_{p_0} \lesssim \Delta_0^2 + R_0, \quad \| anA \cdot A \|_{p_0} \lesssim \Delta_0 N_1 (A) \lesssim \Delta_0^2 + R_0. \]

Then the term in (6.27) can be bounded by \((\Delta_0^2 + R_0) N_2 (D^{-1} F)\).
We then consider the term in (6.28). In view of (4.43) and Theorem 1.3
\[ \| r^{-1} an \nabla D^{-1} F \|_{p_0} \lesssim \| r^{-1} F \|_{p_0} + \Delta_0 \| r^{-1} D^{-1} F \|_{p_0}^q + \| r^{-1} F \|_{L_t^2 L_x^q} \| r^{-1} F \|_{L_t^2 (H)}. \]

Since by Proposition 4.3 and (SobM1), we obtain,
\[ \| r^{-1} D^{-1} F \|_{L_t^2 L_x^2} \lesssim \| F \|_{L_t^2 L_x^2} \lesssim N_1 (F), \]
also by using (4.23), we deduce
\[ \| r^{-1} an \nabla D^{-1} F \|_{p_0} \lesssim N_1 (F) + \Delta_0 N_1 (F). \]

By (6.17), it is easy to check
\[ \| r^{1/2} \nabla (anA) \|_{L_t^\infty L_x^2} + \| r^{-1/2} anA \|_{L_t^\infty L_x^2} \lesssim \Delta_0^2 + R_0. \]

Consequently, we conclude that
\begin{equation}
\| [D_t, \nabla] g D^{-1} F \|_{p_0} \lesssim (\Delta_0^2 + R_0) (N_1 (\nabla D^{-1} F) + N_2 (D^{-1} F)) + N_1 (F).
\end{equation}
(6.29)

We then infer from (6.29) and Lemma 3.1 that
\begin{equation}
\| [D_t, \nabla] g D^{-1} F \|_{p_0} \lesssim N_1 (F).
\end{equation}
(6.30)

Next, we prove for S-tangent field tensor fields \(F\) on \(H\) the following inequality hold\(^{10}\)
\begin{equation}
\| [D_t, D^{-1}] g F \|_{L_t^p L_x^2} \lesssim N_1 (F) \quad \text{with} \quad 4 < b < \infty.
\end{equation}
(6.31)

Indeed, by using Proposition 4.3 with \(p = 4/3\), Proposition 3.4, (SobM1), (6.17), (6.24) and Lemma 3.1 we then derive that
\begin{align*}
\| [D_t, D^{-1}] g F \|_{L_t^p L_x^2} & \lesssim \| r^{1/2} A \cdot \nabla D^{-1} F \|_{L_t^{5/4} L_x^{4/3}} + \| r^{1/2} A \cdot D^{-1} F \|_{L_t^{5/4} L_x^{4/3}} \\
& \quad + \| r^{-1} D^{-1} (an \nabla D^{-1} F) \|_{L_t^{5/4} L_x^{4/3}} + \| r^{-1} A \cdot D^{-1} F \|_{L_t^{5/4} L_x^{4/3}} \\
& \lesssim \| A \|_{L_t^{5/4} L_x^2} \| \nabla D^{-1} F \|_{L_t^{5/2} L_x^2} \\
& \quad + \| D^{-1} F \|_{L_t^{5/2} L_x^2} \times (\| A \cdot A \|_{L_t^{5/4} L_x^2} + \| r^{-1/2} A \|_{L_t^{5/4} L_x^2}) \\
& \quad + N_1 (r^{-1/2} D^{-1} F) + \Delta_0 N_1 (D^{-1} F) \\
& \lesssim N_1 (\nabla D^{-1} F) + N_2 (D^{-1} F) \lesssim N_1 (F).
\end{align*}

\(^{10}\) We will improve the right hand side of (6.31) to be \(N_1 (D^{-1} F)\) in the next section.
The combination of (6.31), (6.30) and (4.10) gives
\[ ||\nabla[D^{-1}, D_t]gF||_{L^p_0} \lesssim ||[D_t, \nabla]D^{-1}F||_{L^p_0} + \Delta_0 ||[D_t, D^{-1}]gF||_{L^q}\cdot ||[D_t, \nabla]D^{-1}F||_{L^q}^{-q} \lesssim N_1(F), \]
where in the above inequalities, \( b > 4 \) and \( \alpha_0 < q < 1 \).

\[ \square \]

6.3. Proof of Proposition 6.2: Part II. We record the following result, which, for completeness, will be proved in Appendix by using Lemma 5.1.

Lemma 6.3. For any smooth S-tangent tensor field F and for exponent \( 2 \leq q \leq \infty \), we have the following inequality for \( k > 0 \)
\begin{align*}
(6.32) & \quad ||r^{-\frac{1}{2}} \frac{1}{k} F||_{L^q_0} \lesssim 2^{-\frac{1}{2} - \frac{1}{4}} N_1(F), \\
(6.33) & \quad ||r^{-\frac{1}{2}} F||_{L^q_0} \lesssim 2^{-\frac{1}{2} - \frac{1}{4}} N_1(F).
\end{align*}

We will complete the proof of Proposition 6.2 by studying error type terms in Proposition 6.4.

Let us first establish the following result with the help of Lemma 6.3:

Proposition 6.4. Let \( D^{-1} \) denote either \( D_1^{-1} \) or \( \star D_1^{-1} \). For any S-tangent tensor fields \( F \) and \( G \) on \( \mathcal{H} \), there holds
\[ ||r^{-\frac{1}{2}} D^{-1}(an F \cdot \nabla G)||_{L^q_0} \lesssim N_1(F)N_1(G), \quad \text{with} \quad 4 < b < \infty. \]

Proof. Note that based on Lemma 5.1, (6.25), (6.26) still hold true. We only need to modify the case \( l < n < m \) of the proof in [11], P.310-311.

When \( l < n < m \),
\[ anF_n \cdot \nabla G_m = \nabla(anF_n \cdot G_m) - \nabla(an)F_n \cdot G_m - an \nabla F_n \cdot G_m, \]
thus we need to consider the three terms
\begin{align*}
\mathcal{I}^{1}_{mn} & = ||r^{-\frac{1}{2}} P_l D^{-1}(an \nabla F_n \cdot G_m)||_{L^q_0} \\
\mathcal{I}^{2}_{mn} & = ||r^{-\frac{1}{2}} P_l D^{-1}(an \nabla F_n \cdot G_m)||_{L^q_0} \\
\mathcal{I}^{3}_{mn} & = ||r^{-\frac{1}{2}} P_l D^{-1}(\nabla(an)F_n \cdot G_m)||_{L^q_0}.
\end{align*}

Using \( C^{-1} < an < C \), following the same procedure in [11], we can get
\[ \sum_{0 < l < n < m} (\mathcal{I}^{1}_{mn} + \mathcal{I}^{2}_{mn}) \lesssim N_1(F)N_1(G). \]

Now consider \( \mathcal{I}^{3}_{mn} \).

By Lemma 1.3 with \( p = \frac{1}{2} \), Proposition 1.1 (i) followed with \( \text{(SobM1)} \), and (6.33),
\begin{align*}
\mathcal{I}^{3}_{mn} & = ||r^{-\frac{1}{2}} P_l D^{-1}(an(\zeta + \zeta)F_n \cdot G_m)||_{L^q_0} \\
& \lesssim 2^{-\frac{1}{2}} ||r^{-\frac{1}{2}} r^{-\frac{1}{2}} (\zeta + \zeta)F_n \cdot G_m||_{L^q_0} \\
& \lesssim 2^{-\frac{1}{2}} ||r^{-\frac{1}{2}} r^{-\frac{1}{2}} F_n \cdot G_m||_{L^q_0} A_{L^q_0} \\
& \lesssim 2^{-\frac{1}{2}} ||r^{-\frac{1}{2}} r^{-\frac{1}{2}} G_m||_{L^q_0} F_n_{L^q_0} \\
& \lesssim 2^{-\frac{1}{2}} ||r^{-\frac{1}{2}} r^{-\frac{1}{2}} N_1(G) . N_1(F), \]

The combination of (6.31), (6.30) and (4.10) gives
\[ ||\nabla[D^{-1}, D_t]gF||_{L^p_0} \lesssim ||[D_t, \nabla]D^{-1}F||_{L^p_0} + \Delta_0 ||[D_t, D^{-1}]gF||_{L^q_0} \cdot ||[D_t, \nabla]D^{-1}F||_{L^q_0}^{-q} \lesssim N_1(F), \]
we obtain
\[ \sum_{0 \leq j < n < m} I_{lmn}^3 \leq N_1(G)N_1(F). \]
The proof is therefore complete.

Let \( \mathcal{D} \) be one of the operators \( \mathcal{D}_1, \mathcal{D}_1^* \) or \( \mathcal{D}_2 \). In the following result, we use Proposition 6.4 to estimate the error type terms.

**Proposition 6.5.** For \( S \)-tangent tensors \( G \) on \( \mathcal{H} \) verifying \( N_1(G) < \infty \), set
\[ \mathcal{E}_1(G) := r^{-\frac{2}{3}} \mathcal{D}^{-1}(anA \cdot G) \text{ or } \mathcal{D}^{-1}(anA \cdot A \cdot G), \]
\[ \mathcal{E}_2(G) := \mathcal{D}^{-1}(an\nabla A \cdot G) \text{ or } \mathcal{D}^{-1}(an\nabla G \cdot A), \]
The following estimates hold
\[ \|r^{-\frac{2}{3}}\mathcal{E}_1(G)\|_{L^1_tL^2_x} + \|r^{-\frac{2}{3}}\mathcal{E}_2(G)\|_{L^1_tL^2_x} \lesssim (\Delta_0^2 + R_0)N_1(G) \]
where \( 4 < b < \infty \).

**Proof.** Using Proposition 4.3 with \( p = \frac{4}{3} \), (SobM1) and (6.17), we get
\[ \|r^{-\frac{2}{3}}\mathcal{D}^{-1}(anA \cdot A \cdot G)\|_{L^1_tL^2_x} \lesssim \|r^{-\frac{2}{3}}A \cdot A \cdot G\|_{L^\infty_tL^2_x} \lesssim \|A\|_{L^\infty_tL^2_x}^2 \mathcal{D}^{-1}(G)_{L^1_tL^2_x} \]
by using Proposition 3.3 (2.28), (6.17) and (SobM1), we have
\[ \|r^{-\frac{2}{3}}\mathcal{D}^{-1}(anA \cdot G)\|_{L^1_tL^2_x} \lesssim \|r^{-\frac{2}{3}}A \cdot G\|_{L^1_tL^2_x} \lesssim \|r^{-\frac{2}{3}}A\|_{L^1_tL^2_x} \mathcal{D}^{-1}(G)_{L^1_tL^2_x} \]
We thus obtain the desired estimates.

By analyzing the expression of \( \beta \) and \( C_0(F) := [\mathcal{D}, \mathcal{D}^{-1}]F \), we have

**Corollary 2.** The following inequalities hold for any \( S \)-tangent tensor \( F \),
\[ \|r^{-\frac{2}{3}}\mathcal{D}^{-1}(an\beta \cdot F)\|_{L^1_tL^2_x} \lesssim N_1(F)(\Delta_0^2 + R_0) \]
\[ \|r^{-\frac{2}{3}}C_0(F)\|_{L^1_tL^2_x} \lesssim N_1(r^{-\frac{2}{3}}\mathcal{D}^{-1}F) \]
where \( 4 < b < \infty \).

**Proof.** Using Codazzi equation (2.6), i.e.
\[ an\beta = an\nabla A + an(A \cdot A + r^{-1}A), \]
we infer
\[ \mathcal{D}^{-1}(an\beta \cdot F) = \mathcal{E}_1(F) + \mathcal{E}_2(F). \]
Whence (6.35) follows from Proposition 6.5.

Similarly, using (6.11) we can write
\[ C_0(F) = \mathcal{E}_1(\mathcal{D}^{-1}F) + \mathcal{E}_2(\mathcal{D}^{-1}F) + r^{-1}\mathcal{D}^{-1}(an\nabla \mathcal{D}^{-1}F). \]
For the last term, using (6.22) with \( H = F \), we infer
\[ \|r^{-\frac{2}{3}}\mathcal{D}^{-1}(an\nabla \mathcal{D}^{-1}F)\|_{L^1_tL^2_x} \lesssim N_1(r^{-\frac{2}{3}}\mathcal{D}^{-1}F)(\Delta_0^2 + R_0 + 1) \]
The desired estimate then follows from Proposition 6.5.
Proof of (6.12) in Proposition 6.3. Combining Proposition 4.3, (6.36) and (6.22), we derive for $D^{-1}$ either $D_2^{-1}$ or $D_1^{-1}$, $4 < b < \infty$

\begin{equation} \label{eq:6.37}
\| r^{\frac{1}{b}} D^{-1} C_0(\hat{R}) \|_{L^4 L^2} \lesssim \| r^{\frac{1}{b}} C_0(\hat{R}) \|_{L^4 L^2} \lesssim \mathcal{N}_1(r^{\frac{1}{b}} D^{-1} \hat{R}) \lesssim \Delta_0^2 + R_0.
\end{equation}

Observe that $C_1(\hat{R})$ can be written symbolically in the form

\[ C_1(\hat{R}) = \nabla D^{-1} C_0(\hat{R}). \]

By Hodge-elliptic estimate (4.11) with $\frac{1}{2} < q < 1$ and $4 < b < \infty$, (6.11), (6.21) and (6.37)

\[ \| C_1(\hat{R}) \|_{\mathcal{P}^0} \lesssim \| C_0(\hat{R}) \|_{\mathcal{P}^0} + \Delta_0 \| D^{-1} C_0(\hat{R}) \|_{L^4 L^2} \| C_0(\hat{R}) \|_{L^2(\mathcal{H})}^{1-q} \lesssim \Delta_0^2 + R_0 + \Delta_0 \| D^{-1} C_0(\hat{R}) \|_{L^4 L^2} (\Delta_0^2 + R_0)^{1-q} \lesssim \Delta_0^2 + R_0. \]

This is the desired estimate. \hfill \Box

6.4. A preliminary estimate for $(\rho, \sigma)$. Define

\[ \bar{\rho} = \frac{1}{|S_t|} \int_{S_t} \rho \, d\mu_\gamma \quad \text{and} \quad \bar{\sigma} = \frac{1}{|S_t|} \int_{S_t} \sigma \, d\mu_\gamma. \]

We have

**Lemma 6.4.**

\begin{align*}
\| r^{\frac{1}{2}} \bar{\rho} \| + \| r^{\frac{1}{2}} \bar{\sigma} \| & \lesssim \Delta_0^2 + R_0, \quad (6.38) \\
\| r^{\frac{3}{2}} \bar{\rho} \| + \| r^{\frac{3}{2}} \bar{\sigma} \| & \lesssim \Delta_0^2 + R_0, \quad (6.39)
\end{align*}

**Proof.** By [2] Eq. (41), i.e.

\begin{equation} \label{eq:6.40}
\frac{d}{ds} \rho + \frac{3}{2} \text{tr} \chi \rho = F
\end{equation}

where $F = \text{div} \beta - \frac{3}{2} \bar{\rho} \cdot \alpha + (\zeta + 2\chi) \cdot \beta$, the transport equation for $\bar{\rho}$ can be obtained as follows

\begin{align*}
\frac{d}{ds} (\bar{\rho}) &= \nabla L(r^{-2} \int_S \rho) \\
&= -\text{antr} \chi (an)^{-1} \bar{\rho} + r^{-2} (an)^{-1} \int_S (\nabla L \rho + \text{tr} \chi \rho) \, d\mu_\gamma \\
&= -(an)^{-1} \text{antr} \chi \bar{\rho} + (an)^{-1} \left( \frac{1}{2} \text{antr} \chi \rho + anF \right),
\end{align*}

and

\begin{align*}
\frac{d}{ds} (r^{\frac{3}{2}} \bar{\rho}) &= \frac{3r^3}{2} \text{antr} \chi (an)^{-1} \bar{\rho} \\
&+ r^3 \left\{ -(an)^{-1} \text{antr} \chi \bar{\rho} + (an)^{-1} \left( \frac{1}{2} \text{antr} \chi \rho + anF \right) \right\} \\
&= -\frac{1}{2} r^3 (an)^{-1} \text{antr} \chi (\rho - \bar{\rho}) + r^3 (an)^{-1} \text{antr} \chi F.
\end{align*}

Integrating the above identity in $t$, due to $\lim_{t \to 0} r^\theta \rho = 0$ for any $\theta > 0$, we can obtain

\[ r^{\frac{3}{2}} \bar{\rho}(t) = r^{-\frac{3}{2}} \int_0^t r^{3} \left( \frac{1}{2} \text{antr} \chi \cdot \text{Osc}(\rho) + anF \right) \, dt'. \]
In view of (2.23), (1.13) and Proposition 2.2, we obtain
\[
r^{-\frac{1}{2}} \int_0^t \left| r^3 \text{trn} \chi \cdot \text{Osc}(\rho) \right| \leq \| r \text{Osc}(\rho) \|_{L^2_t L^2_x} \| r' \|_{L^2(0,t)} r^{-\frac{1}{2}} \lesssim \| \rho \|_{L^2(\mathcal{H})} \lesssim R_0.
\]

By integration by part on \( S = S_t \), we can obtain
\[
r^2 \text{an} F = \int_S \left( \text{an} (\text{div} \beta - \frac{1}{2} \chi \cdot \alpha) + \text{an} (\chi + 2 \zeta) \cdot \beta \right) \mu_\gamma.
\]
\[
= \int_S \text{an} (\chi \beta - \frac{1}{2} \chi \cdot \alpha) \mu_\gamma = \int_S \text{an} \frac{\partial \mu_\gamma}{\partial t}.
\]
Hence by (1.13) and Proposition 2.3,
\[
\left| r^{-\frac{1}{2}} \int_0^t r^3 \text{an} F dt \right| \leq r^{-\frac{1}{2}} \| r' \|_{L^2(0,t)} \| (r')^2 \frac{\partial \mu_\gamma}{\partial t} \|_{L^2_t L^2_x}
\]
\[
\leq \| R_0 \|_{L^2} \| \frac{\partial \mu_\gamma}{\partial t} \|_{L^2} \lesssim \Delta_0^2 + R_0.
\]

Following the same procedure as above, we can obtain the same estimate for \( \bar{\sigma} \) in view of [2 Eq. (42)].

Note that
\[
| r (\bar{\sigma} \cdot \bar{\chi}) \bar{\chi} \cdot \bar{\chi} | \lesssim \| r \bar{\sigma} \bar{\chi} \|_{L^r_t L^r_x} \lesssim (\Delta_0^2 + R_0)^2.
\]

(6.41) follows by connecting (6.38) with (6.39).

\[ \square \]

**Proposition 6.6.** For \( 4 < b < \infty \), there hold
\[
\| r^{-\frac{1}{2} - \frac{1}{2}} D_1^{-1} (an(\bar{\rho}, -\bar{\sigma})) \|_{L^4_t L^2_x} \lesssim \Delta_0^2 + R_0.
\]
\[
\| r^{-\frac{1}{2} - \frac{1}{2}} D_1^{-1} (an(\rho, -\sigma)) \|_{L^4_t L^2_x} \lesssim \Delta_0^2 + R_0.
\]

**Proof.** Let \( H = (\bar{\rho} - \bar{\rho}, -\bar{\sigma} + \bar{\sigma}) \). In view of \( D_1 D_1^{-1} H = H \),
\[
D_1^{-1} (an(\bar{\rho}, -\bar{\sigma})) = D_1^{-1} (an D_1 D_1^{-1} H) + D_1^{-1} (an(\bar{\rho}, -\bar{\sigma})).
\]

By Proposition 1.3,
\[
\| r^{-\frac{1}{2} - \frac{1}{2}} D_1^{-1} (an(\bar{\rho}, -\bar{\sigma})) \|_{L^4_t L^2_x} \lesssim \| r^{-\frac{1}{2} - \frac{1}{2}} an(\bar{\rho}, -\bar{\sigma}) \|_{L^4_t L^2_x}.
\]

Hence, in view of (6.39) and (1.13),
\[
\| r^{-\frac{1}{4} - \frac{1}{4}} (an(\bar{\rho}, -\bar{\sigma})) \|_{L^4_t L^2_x} \lesssim \| r^{-\frac{1}{4} - \frac{1}{4}} an(\bar{\rho}, -\bar{\sigma}) \|_{L^4_t L^2_x} \lesssim \Delta_0^2 + R_0.
\]

By Leibnitz rule, Proposition 4.3 and (2.28), we have
\[
\| r^{-\frac{1}{2} - \frac{1}{2}} D_1^{-1} (an D_1 D_1^{-1} H) \|_{L^4_t L^2_x} \lesssim \| r^{-\frac{1}{2} - \frac{1}{2}} D_1^{-1} (an D_1^{-1} H) \|_{L^4_t L^2_x} + \| r^{-\frac{1}{2} - \frac{1}{2}} D_1^{-1} (\nabla (an) D_1^{-1} H) \|_{L^4_t L^2_x} \lesssim \| r^{-\frac{1}{2} - \frac{1}{2}} D_1^{-1} H \|_{L^4_t L^2_x} + \| r^{-\frac{1}{2} - \frac{1}{2}} D_1^{-1} H \|_{L^4_t L^2_x} \lesssim N_1 (D_1^{-1} H). \]

For the last two inequalities, we employed (6.17) and (6.22).
Thus, in order to prove (6.46), in view of (6.48), it remains to show

\[
\| r^{-b} D^{-1} E_1^G \|_{L^1_t L^2_x} \lesssim (\Delta_0^2 + R_0) N_1(F).
\]

More precisely,

\[
\| r^{-b} D^{-1} (E_1^G) \|_{L^1_t L^2_x} \lesssim (\Delta_0^2 + R_0) N_1(F)
\]

\[
\| r^{-b} D^{-1} (E_1^G) \|_{L^1_t L^2_x} \lesssim (\Delta_0^2 + R_0) N_1(F).
\]

where \( Err \) is defined in (6.6) and \( 4 < b < \infty \).

In order to prove Proposition 6.7, we may use the error type terms introduced in Proposition 6.5 to rewrite (6.6) in view of (2.11) and (2.12) as

\[
Err = D_1^{-1}(\text{antr} (\rho, -\sigma)) + E_1(A) + E_2(A).
\]

Proof of Proposition 6.7. (6.47) can be obtained by using Proposition 4.3, (6.36), (6.22) and \( \text{SobM1} \) as follows,

\[
\| r^{-b} D^{-1} (C_0(\tilde{R}) \cdot F) \|_{L^1_t L^2_x} \lesssim \| r^{-b} D^{-1} F \|_{L^2_t L^2_x} \lesssim (\Delta_0^2 + R_0) N_1(F),
\]

and similarly, by using Proposition 6.5 for \( i = 1, 2 \)

\[
\| r^{-b} D^{-1} (E_i(A) \cdot F) \|_{L^1_t L^2_x} \lesssim \| r^{-b} D^{-1} F \|_{L^2_t L^2_x} \lesssim (\Delta_0^2 + R_0) N_1(F).
\]

Thus, in order to prove (6.48), in view of (6.43), it remains to show

\[
\| r^{-b} D^{-1} (\text{antr} (\rho, -\sigma)) \cdot F \|_{L^1_t L^2_x} \lesssim N_1(F) \Delta_0.
\]

For this estimate, we proceed as follows. Let \( H = (\dot{\rho} - \tilde{\rho}, -\dot{\sigma} + \tilde{\sigma}) \), then \( H = D_1 D_1^{-1} H \).

\[
\| r^{-b} D^{-1} (\text{antr} (\rho, -\sigma)) \cdot F \|_{L^1_t L^2_x} \lesssim \| r^{-b} D^{-1} (\text{antr} \chi D_1 D_1^{-1} H) \cdot F \|_{L^1_t L^2_x} + \| r^{-b} D^{-1} (\text{antr} (\rho, -\sigma)) \cdot F \|_{L^1_t L^2_x}
\]

By \( I_1 \) and \( I_2 \), we denote the two terms on the right of the inequality. Using Proposition 1.3, \( \text{SobM1} \), (2.28) and (6.22),

\[
I_1 \lesssim \| r^{-b} D^{-1} (\text{antr} \chi D_1^{-1} H) \cdot F \|_{L^1_t L^2_x} + \| r^{-b} D^{-1} (\text{antr} \chi D_1^{-1} H) \cdot F \|_{L^1_t L^2_x}
\]

\[
\lesssim \| r^{-b} D^{-1} H \cdot F \|_{L^1_t L^2_x} + \| r^{-b} D^{-1} H \|_{L^1_t L^2_x} \| F \|_{L^2_t L^2_x}
\]

\[
\lesssim \| F \|_{L^2_t L^2_x} \| r^{-b} D^{-1} H \|_{L^1_t L^2_x} + \| r^{-b} D^{-1} H \|_{L^1_t L^2_x} \| F \|_{L^2_t L^2_x}
\]

\[
\lesssim N_1(F) N_1(D^{-1} H) \lesssim (\Delta_0^2 + R_0) N_1(F)
\]

(6.43) can be obtained by combining (6.42) with the estimate for \( r^{-1} D^{-1} (anA \cdot A) \) in view of the estimate for \( E_1(A) \) in Proposition 6.3. 

6.5. \( L^1_t L^2_x \) estimates for \( D^{-1} E_1^G \). For arbitrary \( S \)-tangent tensor field \( F \), we denote by \( E_1^G \) either \( [D_1 D_1^{-1}] (\dot{\rho}, -\dot{\sigma}) \cdot F \) or \( \text{Err} \cdot F \). In what follows, we establish estimates for \( \| D^{-1} E_1^G \|_{L^1_t L^2_x} \) with \( 4 < b < \infty \), which will be employed for the Hodge-elliptic \( P^0 \) estimates of error terms arising in the decomposition procedure in Section 6.7.

Proposition 6.7. Denote by \( D \) either \( D_1 \) or \( D_2 \), for appropriate \( S \)-tangent tensor fields \( F \), the following estimates hold

\[
\| r^{-b} D^{-1} E_1^G \|_{L^1_t L^2_x} \lesssim (\Delta_0^2 + R_0) N_1(F).
\]

More precisely,

\[
\| r^{-b} D^{-1} (E_1^G) \|_{L^1_t L^2_x} \lesssim (\Delta_0^2 + R_0) N_1(F)
\]

\[
\| r^{-b} D^{-1} (D_1^{-1}(\text{antr} (\rho, -\sigma)) \cdot F) \|_{L^1_t L^2_x} \lesssim (\Delta_0^2 + R_0) N_1(F).
\]

where \( Err \) is defined in (6.6) and \( 4 < b < \infty \).

In order to prove Proposition 6.7, we may use the error type terms introduced in Proposition 6.5 to rewrite (6.6) in view of (2.11) and (2.12) as

\[
Err = D_1^{-1}(\text{antr} (\rho, -\sigma)) + E_1(A) + E_2(A).
\]

Proof of Proposition 6.7. (6.47) can be obtained by using Proposition 4.3, (6.36), (6.22) and \( \text{SobM1} \) as follows,
where we employed
\[ \| r^{1/2} \bar{\nabla} (\bar{\nabla} \chi) \|_{L^1_t L^2_x} \lesssim \| r^{1/2} \bar{\nabla} (\bar{\nabla} \chi) \|_{L^1_t L^2_x} \| \bar{\nabla} \chi \|_{L^1_t L^2_x} \lesssim \Delta_0^2 + R_0. \]

By Proposition 6.3 [6.34] and (SobM1),
\[ I_2 \lesssim \| r^{-1/2} D^{-1} (\bar{\nabla} \chi) \|_{L^1_t L^2_x} \| F \|_{L^1_t L^1_x} \]
\[ \lesssim \| r^{-1/2} \bar{\nabla} \chi (\bar{\rho}, -\bar{\sigma}) \|_{L^1_t L^2_x} \| F \|_{L^1_t L^1_x} \]
\[ \lesssim \| r^{-1/2} \bar{\nabla} (\bar{\rho}, -\bar{\sigma}) \|_{L^1_t L^2_x} \mathcal{N}_1 (F) \]
\[ \lesssim (\Delta_0^2 + R_0) \mathcal{N}_1 (F). \]

The proof is complete. \( \square \)

6.6. \( L^1_t L^2_x \) estimates for \( \bar{\nabla} L^{-1} F \). We will establish the following

**Proposition 6.8.** Denote by \( D^{-1} F \) either \( D^{-2} \bar{R} \) or \( D^{-1} (a \delta + 2a \lambda) \). There holds
\[ \| r^{-1/2} D_t D^{-1} F \|_{L^1_t L^2_x} \lesssim R_0 + \Delta_0^2, \quad 4 < b < \infty. \]

**Case 1:** \( F = D^{-1} \bar{R} \).

We denote by \( D^{-1} \bar{F} \) either \( D^{-1} \bar{E} \bar{r} \) or \( D^{-1} \bar{E} \bar{r} \). To prove Proposition [6.8] we will rely on (6.49) in the following result.

**Proposition 6.9.** For \( \bar{F} = (\bar{E} \bar{r}, \bar{E} \bar{r}, \bar{E} \bar{r}) \) with \( \bar{E} \bar{r} \) and \( \bar{E} \bar{r} \) given by (6.44), there hold
\[ \| r^{-1/2} D^{-1} \bar{F} \|_{L^1_t L^2_x} \lesssim \Delta_0^2 + R_0, \quad 4 < b < \infty, \]
\[ \| \bar{\nabla} D^{-1} \bar{F} \|_{L^0_t L^0_x} \lesssim \Delta_0^2 + R_0. \]

Assuming (6.49), now we prove Proposition 6.8.

**Proof of Proposition 6.8 for Case 1.** In view of the formula
\[ D_t D^{-2} \bar{R} = [D_t, D^{-1}] D^{-1} \bar{R} + D^{-1} [D_t, D^{-1}] \bar{R} + D^{-2} D_t \bar{R}, \]
we only need to show for \( 4 < b < \infty \), there hold
\[ \| r^{-1/2} [D_t, D^{-1}] D^{-1} \bar{R} \|_{L^1_t L^2_x} \lesssim \Delta_0^2 + R_0 \]
\[ \| r^{-1/2} D^{-1} [D_t, D^{-1}] \bar{R} \|_{L^1_t L^2_x} \lesssim \Delta_0^2 + R_0 \]
\[ \| r^{-1/2} D^{-2} D_t \bar{R} \|_{L^1_t L^2_x} \lesssim \Delta_0^2 + R_0. \]

(6.51) follows from (6.39) with \( F = D^{-1} \bar{R} \), by using the fact that \( \mathcal{N}_1 (r^{-1/2} D^{-2} \bar{R}) \lesssim \mathcal{N}_2 (D^{-2} \bar{R}) \lesssim \Delta_0^2 + R_0 \). (6.52) was proved in (6.37).

It only remains to prove (6.53). Consider first the case \( D^{-2} D_t \bar{R} = D^{-1} D_t \bar{R} (\bar{\rho}, -\bar{\sigma}) \). By \( an \beta = \bar{\nabla} (an A) + an (A \cdot A + r^{-1} A) \), (SobM1) and (6.17),
\[ \| r^{-1/2} D^{-1} (an \beta) \|_{L^1_t L^2_x} \lesssim \| r^{-1/2} D^{-1} (\bar{\nabla} (an A) + an (A \cdot A + r^{-1} A)) \|_{L^1_t L^2_x} \]
\[ \lesssim \| r^{-1/2} A \|_{L^1_t L^2_x} + \| r^{-1/2} + 1 A \cdot A \|_{L^1_t L^2_x} \]
\[ \lesssim \mathcal{N}_1 (A) + \mathcal{N}_1 (A)^2 \lesssim \Delta_0^2 + R_0. \]

Then by (6.49), we obtain
\[ \| r^{-1/2} D^{-1} (Err + an \beta) \|_{L^1_t L^2_x} \lesssim \Delta_0^2 + R_0. \]
In view of the definition of $Err$ in (6.6), we have

\[(6.54) \quad \|r^{-\frac{4}{5}}D^{-2}D_t(\check{p}, \check{\sigma})\|_{L_t^2 L_x^2} \lesssim \Delta_0^2 + R_0.\]

Using $D_1^{-1}D_t\beta = an(p, \sigma) + \widetilde{Err}$, Proposition 6.6 and (6.54),

\[(6.55) \quad \|r^{-\frac{4}{5}}D^{-2}D_t\beta\|_{L_t^2 L_x^2} \lesssim \|r^{-\frac{4}{5}}D_1^{-1}(an(p, \sigma))\|_{L_t^2 L_x^2} + \|r^{-\frac{4}{5}}D^{-1}\check{\gamma}\|_{L_t^2 L_x^2} \lesssim \Delta_0^2 + R_0\]

In view of (6.54) and (6.55), (6.53) is proved.

To prove Proposition 6.9, we will rely on the following result.

**Lemma 6.5.** Let $D^{-1}$ denote one of the operators $D_1^{-1}$, $D_2^{-1}$ or $*D_1^{-1}$. For any appropriate $S$-tangent tensor field $G$, there holds

\[(6.56) \quad \|D^{-1}(an\hat{p} \cdot G)\|_{L_t^2 L_x^2} \lesssim \|\Lambda^{-\alpha_0}\check{p}\|_{L_t^\infty L_x^2 N_1(G)}\]

where $\alpha_0 \geq \frac{1}{2}$ and $4 < b < \infty$.

**Proof.** We can adapt the proof for [11] Lemma 4.4 in view of $\|\nabla(an)\|_{L_t^4 L_x^\infty} \lesssim \Delta_0^2 + R_0$ and $an < C$, to derive the following estimates for $S$ tangent tensor fields $F$,

\[(6.57) \quad \|\Lambda^{-\alpha} (an\hat{p} \cdot F_m)\|_{L_t^2(S)} \lesssim \|\Lambda^{-\alpha_0}\check{p}\|_{L_t^\infty L_x^2} 2^m r^{-1} \|P_m F\|_{L_t^2(S)},\]

\[(6.58) \quad \|P_m(an\hat{p} \cdot D^{-1} P_l F)\|_{L_t^2(S)} \lesssim \|\Lambda^{-\alpha_0}\check{p}\|_{L_t^\infty L_x^2} 2^m r^{-\alpha_0} \|P_l F\|_{L_t^2(S)},\]

where $\alpha > \alpha_0 \geq 1/2$ and $D^{-1}$ denotes either $D_1^{-1}$ or $*D_1^{-1}$.

Set $\Omega_{nl} := D^{-1} P_l^2 (an\hat{p} \cdot P_n^2 G)$, with $l, n \in \mathbb{N}$. We now prove

\[(6.59) \quad \sum_{l,n>0} \|\Omega_{nl}\|_{L_t^2 L_x^2} \lesssim \|\Lambda^{-\alpha_0}\check{p}\|_{L_t^\infty L_x^2 N_1(G)},\]

and lower frequency terms can be treated similarly. By duality argument, Proposition 3.1 (iii) and Lemma 4.3,

\[(6.60) \quad \|D^{-1} \Lambda^\alpha P_l F\|_{L_t^2(S)} \lesssim 2^{(-1+\alpha)l} r^{1-\alpha} \|F\|_{L_t^2(S)}.\]

We first prove (6.59) for the case $0 < n < l$. With the help of (6.60) and (6.57),

\[\|\Omega_{nl}\|_{L_x^2} \lesssim 2^{(1-\alpha)l} 2^\alpha r^{-\alpha} \|\Lambda^{-\alpha_0}\check{p}\|_{L_t^\infty L_x^2} \|P_n G\|_{L_t^2}.\]

Take $L_t^b$ norm for $4 < b < \infty$, and (6.32) in Lemma 6.3,

\[\|\Omega_{nl}\|_{L_t^b L_x^2} \lesssim 2^{(1-\alpha)l} 2^\alpha r^{-\alpha} \|\Lambda^{-\alpha_0}\check{p}\|_{L_t^\infty L_x^2 N_1(G)}.\]

Since we can choose $\alpha_0 < \alpha < \frac{1}{3} + \frac{1}{b}$, we deduce

\[\sum_{0<n<l} \|\Omega_{nl}\|_{L_t^2 L_x^2} \lesssim \|\Lambda^{-\alpha_0}\check{p}\|_{L_t^\infty L_x^2 N_1(G)}.\]

For the case $0 < l < n$, let us pair $\Omega_{nl}$ with any $S$ tangent tensor $F$ with $\|F\|_{L_t^2(S)} \leq 1$. By (6.58),

\[\langle \Omega_{nl}, F \rangle = \langle P_l (an\hat{p} P_n^2 G), P_l^* D^{-1} F \rangle \lesssim 2^\alpha r^{-\alpha \alpha_0} \|\Lambda^{-\alpha_0}\check{p}\|_{L_t^\infty L_x^2} \|P_n G\|_{L_t^2}.\]
Hence, by (6.32) in Lemma 6.3
\[ \| \Omega_{nl} \|_{L^2_t L^4_x} \lesssim 2^{\alpha_n - (\frac{1}{2} + \frac{1}{2}) \alpha} \rho^{-\alpha} + \frac{1}{2} + \frac{1}{2} \| \Lambda^{-\alpha} \hat{\rho} \|_{L^p_t L^2_x} N_1(G). \]
Consequently,
\[ \sum_{0 < l < n} \| \Omega_{nl} \|_{L^2_t L^4_x} \lesssim \| \Lambda^{-\alpha} \hat{\rho} \|_{L^p_t L^2_x} N_1(G). \]
(6.50) follows. \[\square\]

We are ready to prove Proposition 6.9

**Proof of Proposition 6.9.** (6.50) can be derived by using (6.49), Theorem 4.3 and (6.56).

Now we consider (6.49). By letting \( F = 1 \) in (6.46), we obtain for \( 4 < b < \infty \) that
\[ \| r^{-\frac{1}{2}} D^{-1}_{t} \tilde{E} r \|_{L^2_t L^2_x} \lesssim \Delta_0^2 + R_0. \]
Thus we only need to consider \( D^{-1}_{t} \tilde{E} r \).

By definition of \( \tilde{E} r \) in (6.6), in view of (2.6), we rewrite \( \tilde{E} r \) symbolically as follows
\[ \tilde{E} r = * D^{-1}_{t} (an \chi \beta + an \chi \sigma - \nabla A + \nabla A + r^{-1} A) = * D^{-1}_{t} (an \chi \cdot \rho - \zeta^* \sigma). \]
By Propositions 4.3 and 6.5
\[ \| r^{-\frac{1}{2}} D^{-2}(an \chi \cdot \rho) \|_{L^2_t L^2_x} \lesssim \Delta_0^2 + R_0. \]
According to (6.61), we consider \( W = \| r^{-\frac{1}{2}} D^{-2}(an \chi \cdot \rho) \|_{L^2_t L^2_x} \), and
\[ U = \| r^{-\frac{1}{2}} D^{-2}(an \chi \cdot \rho) \|_{L^2_t L^2_x}, \quad V = \| r^{-\frac{1}{2}} D^{-2}(an \chi \cdot \rho) \|_{L^2_t L^2_x}. \]
By \( \beta = * D^{-1}_{t} \chi \beta \), also using Propositions 3.4 and 4.3, (2.28), (6.17) and (6.22)
\[ W \lesssim \| r^{-\frac{1}{2}} D^{-2}(\chi \beta) \|_{L^2_t L^2_x} \lesssim \| \chi \beta \|_{L^2_t L^2_x} \lesssim \| \chi \|_{L^2_t L^2_x} \| \beta \|_{L^2_t L^2_x} \]
By (2.7), clearly \( \hat{\sigma} = \text{curl} \hat{\zeta} \). Thus by Propositions 4.3 and 6.5 we obtain
\[ V = \| r^{-\frac{1}{2}} D^{-1} \chi \beta \|_{L^2_t L^2_x} \lesssim N_1(\zeta) N_1(\zeta) \lesssim (\Delta_0^2 + R_0)^2. \]
By Proposition 3.4 and (6.56), we have
\[ U \lesssim \| r^{-\frac{1}{2}} D^{-2}(an \rho \cdot \zeta) \|_{L^2_t L^2_x} \lesssim \| r^{-\frac{1}{2}} D^{-1}(an \rho \cdot \zeta) \|_{L^2_t L^2_x} \]
where we employed (2.7) and (3.4) to obtain the last inequality. \[\square\]

**Case 2:** \( F = (a \delta + 2a \lambda) \). We first give a slightly stronger result than Proposition 6.8 for Case 2.

**Proposition 6.10.** There holds for \( 4 < b < \infty \),
\[ \| r^{-\frac{1}{2}} D^{-1}_{t} (a \delta + 2a \lambda) \|_{L^2_t L^2_x} \lesssim \Delta_0^2 + R_0. \]
Thus, we proved
\[
\|r^{-\frac{1}{2} - \frac{1}{D}} D_t (a \delta + 2a \lambda)\|_{L_t^1 L_x^2} \lesssim \|r^{-\frac{1}{2} - \frac{1}{D}} D_t (a \delta + 2a \lambda)\|_{L_t^1 L_x^2} + \|r^{-\frac{1}{2} - \frac{1}{D}} [D_t, D_t^{-1}] (a \delta + 2a \lambda)\|_{L_t^1 L_x^2}.
\]
(6.63)

By (2.16) and (2.7),
\[
\|r^{-\frac{1}{2} - \frac{1}{D}} D_t (a \delta + 2a \lambda)\|_{L_t^1 L_x^2} \lesssim \|r^{-\frac{1}{2} - \frac{1}{D}} (a \lambda)^2\|_{L_t^1 L_x^2} + \|r^{-\frac{1}{2} - \frac{1}{D}} D_t^{-1} (a \lambda)\|_{L_t^1 L_x^2}
\]
(6.64)
\[
\|r^{-\frac{1}{2} - \frac{1}{D}} D_t (a \delta + 2a \lambda)\|_{L_t^1 L_x^2} \lesssim \|r^{-\frac{1}{2} - \frac{1}{D}} [D_t, D_t^{-1}] (a \delta + 2a \lambda)\|_{L_t^1 L_x^2}.
\]
(6.65)

By (2.28) and Proposition 6.6, the two terms on the right of (6.65) can be bounded by
\[
N_1^1 (\zeta) + \Delta_0^2 + R_0 \lesssim \Delta_0^2 + R_0.
\]
For (6.66), by Proposition 5.4 and (6.17), also in view of (2.23) and (2.28), we deduce
\[
\|r^{-\frac{1}{2} - \frac{1}{D}} D_t^{-1} \text{err}\|_{L_t^1 L_x^2} \lesssim \|r^{-\frac{1}{2} - \frac{1}{D}} \text{err}\|_{L_t^1 L_x^2}
\]
(6.67)
\[
\lesssim \|r^{-\frac{1}{2} - \frac{1}{D}} a \text{tr} \chi\|_{L_t^1 L_x^2} + \|r^{-\frac{1}{2} - \frac{1}{D}} A \cdot A\|_{L_t^1 L_x^2}
\]
\[
\lesssim \|r^{-\frac{1}{2} - \frac{1}{D}} \|_{L_t^1 L_x^2} + \|A\|_{L_t^1 L_x^2} \lesssim \Delta_0^2 + R_0.
\]
Thus, we proved
\[
\|r^{-\frac{1}{2} - \frac{1}{D}} D_t^{-1} D_t (a \delta + 2a \lambda)\|_{L_t^1 L_x^2} \lesssim \Delta_0^2 + R_0.
\]
(6.68)

Repeat the derivation for (6.69), also using Lemma 3.1
\[
\|r^{-\frac{1}{2} - \frac{1}{D}} [D_t^{-1}, D_t] (a \delta + 2a \lambda)\|_{L_t^1 L_x^2} \lesssim (\Delta_0^2 + R_0) N_1 (r^{-1} D_t^{-1} (a \delta + 2a \lambda))
\]
\[
\lesssim (\Delta_0^2 + R_0) N_1 (a \text{tr} \chi) \lesssim (\Delta_0^2 + R_0)^2.
\]
Hence (6.62) is proved. \qed

**Lemma 6.6.**
\[
\|\text{err}_t\|_{p_0} + \|\nabla D^{-1} \text{err}_t\|_{p_0} \lesssim \Delta_0^2 + R_0.
\]
(6.69)

**Proof.** By (4.21), we have
\[
\|an \cdot A\|_{p_0} \lesssim \Delta_0 N_1 (A) \lesssim \Delta_0^2 + R_0
\]
(6.70)
Using (4.21) and (4.13), we obtain
\[
\|a^2 \text{tr} \chi\|_{p_0} \lesssim N_1 (a \text{tr} \chi) \lesssim \Delta_0^2 + R_0.
\]
(6.71)
Thus the first inequality of (6.69) is proved. The second one can be proved by combining (6.70), (6.71), (6.67) and Theorem 4.3. \qed
6.7. Decomposition for commutators. In order to prove Proposition [6.1] it remains to decompose the “bad” terms which have not been treated in Proposition 6.2, i.e.

\[ an_\beta \cdot \mathcal{D}^{-1} F, \nabla \mathcal{D}^{-1}(an_\beta \cdot \mathcal{D}^{-1} F), \]

with \( \mathcal{F} \) either \( \mathcal{D}^{-1} \mathcal{R} \) or \((a \delta + 2a \lambda)\). In view of (6.23) and Proposition 6.8, the assumptions in the following theorem are satisfied with \( \mathcal{F} = \mathcal{D}^{-1} \mathcal{F} \). Then the proof of Proposition 6.1 is complete by using the following

**Theorem 6.1.** Assume that \( F \) is an \( S \)-tangent tensor field of appropriate order on \( \mathcal{H} \) verifying \( \mathcal{N}_2(F) < \infty \) and \( \| r^{-\frac{1}{2}} \mathcal{L}_F \|_{L^1 L^2} < \infty \) with \( 4 < b < \infty \). Then we have

(i) There exists a 1-form \( E_0 \) such that\(^{11}\)

\[ \text{an}_\beta = \mathcal{D}_1 \mathcal{D}_1^{-1} \mathcal{R} + E_0 \quad \text{with} \quad \| E_0 \|_{p_0} \lesssim \Delta_0^2 + R_0 \]

(ii) There exists a decomposition \( \text{an}_\beta \cdot F = \mathcal{D}_1 P + E \), where \( P \) and \( E \) are tensor fields of the same type as \( \text{an}_\beta \cdot F \) with

\[ \lim_{t \to 0} \| P \|_{L^\infty} = 0 \]

and the estimates

\[ \mathcal{N}_1(P) \lesssim \Delta_0 \mathcal{N}_2(F), \| E \|_{p_0} \lesssim \Delta_0 \cdot (\mathcal{N}_2(F) + \| r^{-\frac{1}{2}} \mathcal{L}_F \|_{L^1 L^2}) . \]

(iii) There exist tensors \( \mathcal{P} \) and \( \mathcal{E} \) verifying (6.71) so that

\[ \nabla \mathcal{D}^{-1}(\text{an}_\beta \cdot F) = \mathcal{D}_1 P + \mathcal{E}, \]

where \( \mathcal{D} \) denote either \( \mathcal{D}_1 \) or \( \mathcal{D}_2 \), and

\[ \lim_{t \to 0} \| \mathcal{P} \|_{L^\infty} < \infty \]

**Proof.** In view of (6.6), we have

\[ \text{an}_\beta = \mathcal{D}_1 \mathcal{D}_1^{-1} \mathcal{R} + C_0(\mathcal{R}) + \text{Err} \]

This proves (i) by noting that \( E_0 := \text{Err} + C_0(\mathcal{R}) \) satisfies \( \| E_0 \|_{p_0} \lesssim \Delta_0^2 + R_0 \) in view of (6.3) and (6.11).

Now we prove (ii). We have from (6.77) that

\[ \text{an}_\beta \cdot F = (\mathcal{D}_1 \mathcal{D}_1^{-1} \mathcal{R} + \text{Err} + C_0(\mathcal{R})) \cdot F = \mathcal{D}_1(\mathcal{D}_1^{-1} \mathcal{R} \cdot F) + E_1^B + E_1^G, \]

where

\[ E_1^B := -\mathcal{D}_1^{-1} \mathcal{R} \cdot \mathcal{D}_1 F \quad \text{and} \quad E_1^G := (\text{Err} + C_0(\mathcal{R})) \cdot F. \]

By (4.18), (6.11) and (6.8) we obtain

\[ \| E_1^B \|_{p_0} \lesssim \mathcal{N}_1(\mathcal{D}^{-1} \mathcal{R})(\| r^{-\frac{1}{2}} \mathcal{L}_F \|_{L^1 L^2} + \| r^{-\frac{1}{2}} \nabla \mathcal{D}_1 F \|_{L^1 L^2}) \]

\[ \lesssim (R_0 + \Delta_0^2)(\mathcal{N}_2(F) + \| r^{-\frac{1}{2}} \mathcal{L}_F \|_{L^1 L^2}). \]

Now we set

\[ P_1 := \mathcal{D}_1^{-1} \mathcal{R} \cdot F \quad \text{and} \quad E_1 := E_1^B + E_1^G. \]

\(^{11}\)In Theorem 6.1 and the following proofs, \( \mathcal{R} = (\dot{\rho}, -\ddot{\sigma}) \) and \( C_0(\mathcal{R}) = [\mathcal{D}_1, \mathcal{D}_1^{-1}](\dot{\rho}, -\ddot{\sigma}) \), since the other case that \( \mathcal{R} = \dot{\sigma} \) will not come up here.
from the above estimates we have
\[ \|E_1\|_{p_0} \lesssim (\Delta_0^2 + R_0)(\mathcal{N}_2(F) + \|r^{-\frac{2}{3}}\nabla_L F\|_{L_t^1 L_x^2}). \]

In order to estimate \( \mathcal{N}_1(P_1) \), let us estimate \( \|E_1\|_{L^2} \) first. By using Hölder’s inequality and Sobolev embedding, we can obtain
\[ \|E_1^B\|_{L^2} = \|D^{-1}\dot{R} \cdot \nabla_L F\|_{L^2} \lesssim \|D^{-1}\dot{R}\|_{L_x^\infty} \|\nabla_L F\|_{L_t^1 L_x^2} \lesssim \mathcal{N}_1(D^{-1}\dot{R})(\|\nabla D_t F\|_{L^2} + \|r^{-\frac{2}{3}}\nabla_L F\|_{L^2}), \]
and by using \( \|E_1^G\|_{L^2(\mathcal{H})} \lesssim \|E_1^G\|_{p_0} \) and (6.78) we can obtain
\[ \|E_1^G\|_{L^2} \lesssim (\Delta_0^2 + R_0)\mathcal{N}_2(F). \]
Therefore
\[ (6.80) \quad \|E_1\|_{L^2} \lesssim (\Delta_0^2 + R_0)\mathcal{N}_2(F). \]
Now we show
\[ (6.81) \quad \mathcal{N}_1(P_1) \lesssim \mathcal{N}_2(F)(\Delta_0^2 + R_0). \]
With the help of \( D_t P_1 = an\beta \cdot F - E_1 \) and (6.80) we can estimate \( \|\nabla_L P_1\|_{L^2} \) as follows
\[ \|\nabla_L P_1\|_{L^2} \lesssim \|\beta \cdot F\|_{L_t^2 L_x^2} + \|E_1\|_{L^2} \lesssim (\Delta_0^2 + R_0)\mathcal{N}_2(F). \]
Similar to [2] Section 6.12, we get \( \|\nabla P_1\|_{L_t^2 L_x^2} \lesssim (\Delta_0^2 + R_0)\mathcal{N}_2(F) \). We complete the proof of (6.74).

By (6.79)
\[ (6.82) \quad \|P_1\|_{L_t^\infty} \leq \|F\|_{L_t^\infty} \|D_t^{-1}\dot{R}\|_{L_t^\infty} \lesssim \frac{1}{\dot{R}} \|D_t^{-1}\dot{R}\|_{L_t^\infty} \mathcal{N}_2(F) \]
Since \( \mathcal{N}_2(F) < \infty \) and \( \lim_{t \to 0} \|D_t^{-1}\dot{R}\|_{L_t^\infty} < \infty \), (6.73) follows by letting \( t \to 0 \) in (6.82). Therefore (ii) is proved.

Finally we prove (iii) by using the iteration procedure in [2] Section 6.12. (6.76) will be proved in Section 8. Let \( P_0 := D F \), then we can apply (ii) to construct recurrently two sequences of \( S \)-tangent tensor fields \( \{P_i\} \) and \( \{E_i\} \) such that
\[ (6.83) \quad an\beta \cdot D_t^{-1} P_{i-1} = D_t P_i + E_i \]
where \( D_t^{-1} \) denote either \( D_t^{-1} \) or \( D_t^{-2} \) and
\[ (6.84) \quad \mathcal{N}_1(P_1) \leq C(\Delta_0^2 + R_0)\mathcal{N}_2(D_t^{-1} P_{i-1}), \]
\[ (6.85) \quad \|E_i\|_{p_0} \leq C(\Delta_0^2 + R_0) \left( \mathcal{N}_2(D_t^{-1} P_{i-1}) + \|r^{-\frac{2}{3}}\nabla_L D_t^{-1} P_{i-1}\|_{L_t^1 L_x^2} \right). \]
Such \( P_i \) and \( E_i = E_i^B + E_i^G \) can be constructed as in the proof of (ii). Then for \( i = 1, 2, \ldots \)
\[ (6.86) \quad P_i = D_t^{-1}\dot{R} \cdot D_t^{-1} P_{i-1}, \quad P_0 = D F \]
\[ (6.87) \quad E_i^B := -D_t^{-1}\dot{R} \cdot D_t D_t^{-1} P_{i-1}, \quad E_i^G := (Err + C_0(\dot{R})) \cdot D_t^{-1} P_{i-1}. \]
In particular, \( P_1 \) and \( E_1 \) have been given by (6.79).
With the above definition of \( P_0 \) and \( E_k \), using (6.83)
\[ \nabla D_t^{-1}(an\beta \cdot F) = D_t \tilde{P}_k + \nabla D_t^{-1}(D_t P_k) + \tilde{E}_k, \]
where
\begin{align*}
\tilde{P}_k &= \nabla D^{-1}(P_1 + \ldots + P_{k-1}) + P_2 + \ldots + P_k \\
E_k &= [\nabla D^{-1}, D_t]_0(P_1 + \ldots + P_{k-1}) + \nabla D^{-1}(E_1 + \ldots + E_k) \\
&\quad + E_2 + \ldots + E_k.
\end{align*}

By using Lemma 3.1, it is easy to see from (6.84) that
\begin{equation}
\mathcal{N}_1(P_k) \leq (C(\Delta_0^2 + R_0))^k \mathcal{N}_2(F).
\end{equation}

Moreover we have

**Proposition 6.11.** For \( \{P_k\}_{k=1}^\infty \) and \( \{E_k\}_{k=1}^\infty \) there hold
\begin{align*}
(6.89) \quad &\|r^{-\frac{1}{2}} \nabla L D^{-1} P_k\|_{L^1_t L^2_x} \lesssim \left( \Delta_0^2 + R_0 \right) (N_2(D^{-1} P_{k-1}) + \|\nabla L D^{-1} P_{k-1}\|_{L^1_t L^2_x}), \\
(6.90) \quad &\|\nabla D^{-1} E_k\|_{\mathcal{P}_0} \lesssim \|E_k\|_{\mathcal{P}_0} + (\Delta_0^2 + R_0) (N_2(D^{-1} P_{k-1}) + \|\nabla L D^{-1} P_{k-1}\|_{L^1_t L^2_x}).
\end{align*}

We will prove this result at the end of this section. We observe that Lemma 3.1, (6.89), (6.91) and (6.85) clearly imply
\begin{equation}
\mathcal{N}_1(\bar{P}_k - \bar{P}_j) \leq N_2(F) \sum_{j \leq m \leq k-1} (C(\Delta_0^2 + R_0))^m \lesssim (C(\Delta_0^2 + R_0))^2 N_2(F),
\end{equation}

and
\begin{align*}
\|\bar{E}_k - \bar{E}_j\|_{\mathcal{P}_0} &\leq (N_2(F) + \|r^{-\frac{1}{2}} \nabla L F\|_{L^1_t L^2_x}) \sum_{j \leq m \leq k-1} (C(\Delta_0^2 + R_0))^m \\
&\lesssim (C(\Delta_0^2 + R_0))^2 (N_2(F) + \|r^{-\frac{1}{2}} \nabla L F\|_{L^1_t L^2_x}).
\end{align*}

Therefore \( \{\bar{P}_k\} \) forms a Cauchy sequence relative to the norm \( \mathcal{N}_1(\cdot) \), while \( \{\bar{E}_k\} \) forms a Cauchy sequence relative to the \( \mathcal{P}_0 \) norm. Denote by \( \bar{P} \) and \( \bar{E} \) their corresponding limits, we have
\begin{align*}
\mathcal{N}_1(\bar{P}) \lesssim (\Delta_0^2 + R_0) N_2(F) \quad \text{and} \quad \|\bar{E}\|_{\mathcal{P}_0} \lesssim (\Delta_0^2 + R_0) (N_2(F) + \|r^{-\frac{1}{2}} \nabla L F\|_{L^1_t L^2_x}).
\end{align*}

We also observe that for sufficiently small \( \Delta_0 \),
\begin{equation}
\|\nabla D^{-1}(an \beta \cdot F) - D_t \bar{P} - \bar{E}\|_{L^2} = \|\nabla D^{-1}(D_t P_k)\|_{L^2} \lesssim \mathcal{N}_1(P_k).
\end{equation}

Letting \( k \to +\infty \), we get
\begin{equation}
\|\nabla D^{-1}(an \beta \cdot F) - D_t \bar{P} - \bar{E}\|_{L^2} = 0.
\end{equation}

Hence \( \nabla D^{-1}(an \beta \cdot F) = D_t \bar{P} + \bar{E} \). This completes the proof of (6.75) in (iii). (6.76) will be proved in Appendix.

Now we conclude this section by proving Proposition 6.11. We first prove (6.91). By using (4.16) we have
\begin{equation}
\|\nabla D^{-1} E_k\|_{\mathcal{P}_0} \lesssim \|E_k\|_{\mathcal{P}_0} + (\Delta_0^2 + R_0) \|D^{-1} E_k\|_{L^1_t L^2_x} \|E_k\|_{L^2}^{1-q},
\end{equation}
where \( 4 < b < \infty \) and \( 1/2 < q < 1 \).

Thus it suffices to show for \( 4 < b < \infty \) that
\begin{equation}
\|r^{-\frac{1}{2}} \nabla D^{-1} E_k\|_{L^1_t L^2_x} \lesssim (\Delta_0^2 + R_0) \left( \mathcal{N}_2(D^{-1} P_{k-1}) + \|\nabla L D^{-1} P_{k-1}\|_{L^1_t L^2_x} \right).
\end{equation}
By the construction of $P_k$ and $E_k$, it suffices to show it for $k = 1$. To this end, in view of $E_1 = E_1^R + E_1^L$, we can complete the proof by using Proposition 6.7 for $\|r^{-\frac{1}{2}}D^{-1} E_1^L\|_{L_t^\infty L_x^2}$ and the estimate

$$\|r^{-\frac{1}{2}}D^{-1} E_1^L\|_{L_t^\infty L_x^2} \lesssim \|r^{-\frac{1}{2}}E_1^B\|_{L_t^\infty L_x^{1.5}} \lesssim \|D_1^{-1} R\|_{L_t^\infty L_x^2} \|D_1 F\|_{L_t^\infty L_x^2}$$

$$\lesssim \mathcal{N}_1(D_1^{-1} R) \|D_1 F\|_{L_t^\infty L_x^2} \lesssim (\Delta^2_0 + R_0) \|\nabla L F\|_{L_t^\infty L_x^2},$$

which follows from Proposition 4.3 Hölder inequality, SobM1 and (6.22).

In order to prove (6.90), we first note that

$$\|r^{-\frac{1}{2}} D^{-1} P_k\|_{L_t^\infty L_x^2} \lesssim \|r^{-\frac{1}{2}} [D_1, D^{-1}] P_k\|_{L_t^\infty L_x^2} + \|r^{-\frac{1}{2}} D^{-1} D_1 P_k\|_{L_t^\infty L_x^2}.$$ 

By using (6.36), the first term on the right hand side of (6.94) can be estimated as

$$\|r^{-\frac{1}{2}} D^{-1} D_1 P_k\|_{L_t^\infty L_x^2} \lesssim \mathcal{N}_1(r^{-\frac{1}{2}} D^{-1} P_k) \lesssim \mathcal{N}_2(r^{-\frac{1}{2}} D^{-1} P_k) \lesssim \mathcal{N}_1(P_k),$$

while by using (6.83), (6.93) and (6.35), the second term can be estimated as

$$\|r^{-\frac{1}{2}} D^{-1} D_1 P_k\|_{L_t^\infty L_x^2} \lesssim \|r^{-\frac{1}{2}} D^{-1} (an\beta \cdot D^{-1} P_k - E_k)\|_{L_t^\infty L_x^2}$$

$$\lesssim \|r^{-\frac{1}{2}} D^{-1} (an\beta \cdot D^{-1} P_k - E_k)\|_{L_t^\infty L_x^2} + \|r^{-\frac{1}{2}} D^{-1} E_k\|_{L_t^\infty L_x^2} \lesssim (\Delta^2_0 + R_0) (\mathcal{N}_2(D^{-1} P_k - E_k) + \|\nabla L D^{-1} P_k - E_k\|_{L_t^\infty L_x^2}).$$

Therefore (6.90) is proved. □

7. Main estimates

Lemma 7.1. Let $F = \nabla tr\chi, \mu, A \cdot a, r^{-1} A$, there holds

$$\|\nabla D^{-1}(anF)\|_{p_0} \lesssim \Delta^2_0 + R_0 + \|F\|_{p_0}$$

where $D^{-1}$ is one of the operators $D_1^{-1}, D_2^{-1}, D_1^{-1}$. 

Proof. By Proposition 4.13 and (2.79), we have

$$\|D^{-1}(anF)\|_{L_t^\infty L_x^2} \lesssim \|an F\|_{L_t^\infty L_x^2} \lesssim \|r F\|_{L_t^\infty L_x^2} \lesssim \Delta^2_0 + R_0.$$ 

By (2.79) and (4.13), we have

$$\|F\|_{L_t^\infty L_x^2} \lesssim \Delta^2_0 + R_0.$$ 

We can infer from Theorem 4.13 and (4.13) that

$$\|\nabla D^{-1}(anF)\|_{p_0} \lesssim \|an F\|_{p_0} + \Delta^2_0 + R_0 \lesssim \|F\|_{p_0} + \Delta^2_0 + R_0,$$

as desired. □

Now we improve BA1 with the help of Theorem 5.1

7.1. Estimates for $\nu$ and $|a - 1|.$

Proposition 7.1.

$$\|\nu\|_{L_t^\infty L_x^2} \lesssim \Delta^2_0 + R_0, \quad |a - 1| \leq \frac{1}{4}.$$ 

Proof. We rewrite (2.17) as follows

$$-\nabla \nu = \nabla \nabla \chi + \text{err}_2,$$

and $\text{err}_2 = \frac{1}{2} \text{tr} \chi \nabla a + \nabla \chi \cdot \nu a + A \cdot \nu.$

Let us denote symbolically $\text{err}_2 = \text{tr} \chi \cdot A + A \cdot A$, hence

$$-\nabla (an\nu) = D_1 \nabla a + an\text{err}_2$$

with $\text{err}_2 = -\nabla \log(an) \nu + \text{err}_2 = A \cdot A + r^{-1} A$. By (4.21) we can obtain

$$\|an\text{err}_2\|_{p_0} \lesssim \Delta^2_0 + R_0.$$
Applying Theorem 5.1 to $P = \nabla a$ and $E = an \cdot \tilde{e}_{r_2}$, we have
\[
\|\nu\|_{L^\infty L^t_1} \lesssim N_1(\nu) + N_1(P) + \|an\tilde{e}_{r_2}\|_{p_0} \lesssim \Delta_0^2 + R_0,
\]
where in view of (2.77), $N_1(\nu) + N_1(P) \lesssim \Delta_0^2 + R_0$.

In view of $\nu := -\frac{d}{ds}a$ and $a(p) = 1$,
\[
|a - 1| \leq \int_0^t |\nu|nadt' \lesssim \|\nu\|_{L^\infty L^t_1} \lesssim \Delta_0^2 + R_0.
\]
With $\Delta_0^2 + R_0$ being sufficiently small, $|a - 1| \leq \frac{1}{4}$ can be achieved. Then Proposition 7.1 follows. \hfill $\square$

7.2. Estimate for $\zeta$.

Proposition 7.2.
\[
\|\zeta\|_{L^\infty L^t_1} \lesssim \Delta_0^2 + R_0.
\]

Proof. By (6.8)
\[
\nabla D^{-1}_1 (an(\rho, \sigma)) = \nabla D^{-1}_1 D_1 D_2 \beta - \nabla D^{-1}_1 \tilde{E} + \nabla D^{-1}_1 \tilde{\delta}.
\]

By Proposition 6.1 there exists $P$ and $E$ such that $C(\tilde{R}) = D_1 P + E$.

Let $\tilde{P} = P + \nabla D^{-2} \tilde{R}$ and $\tilde{E} = \nabla D^{-1} \tilde{\delta} + E$. Then by (6.23) and (6.50)
\[
\nabla D^{-1}_1 (an(\rho, \sigma)) = D_1 \tilde{P} + \tilde{E}, \quad N_1(\tilde{P}) + \|\tilde{E}\|_{p_0} \lesssim \Delta_0^2 + R_0.
\]

In view of (6.94),
\[
\nabla (an\zeta) = \nabla D^{-1}_1 (D_1(\alpha \delta + 2a\lambda)) + \nabla D^{-1}_1 (an(\rho, \sigma)) + \nabla D^{-1}_1 \text{err}_1.
\]
(7.2)
\[
= D_1 \nabla D^{-1}_1 (\alpha \delta + 2a\lambda) + [\nabla D^{-1}_1, D_1](\alpha \delta + 2a\lambda) + \nabla D^{-1}_1 \text{err}_1 + D_1 \tilde{P} + \tilde{E}.
\]

In view of Proposition 6.1 there exists $P'$ an $E'$ such that
\[
[\nabla D^{-1}_1, D_1](\alpha \delta + 2a\lambda) = D_1 P' + E', \quad \text{with } N_1(P') + \|E'\|_{p_0} \lesssim \Delta_0^2 + R_0.
\]

Let $P'' = P' + \nabla D^{-1}_1 (\alpha \delta + 2a\lambda)$, we conclude that
\[
D_1 \nabla D^{-1}_1 (\alpha \delta + 2a\lambda) + [\nabla D^{-1}_1, D_1](\alpha \delta + 2a\lambda) = D_1 P'' + E'.
\]

By (6.23), $N_1(P'') \lesssim \Delta_0^2 + R_0$. Hence, we obtain
\[
\nabla (an\zeta) = D_1 P_3 + E_3,
\]
where $E_3 = E' + \nabla D^{-1}_1 \text{err}_1 + \tilde{E}$ and $P_3 = P'' + \tilde{P}$. Also using Lemma 6.6 for $\|\nabla D^{-1}_1 \text{err}_1\|_{p_0}$, we can conclude that
\[
\|E_3\|_{p_0} \lesssim \Delta_0^2 + R_0, \quad N_1(P_3) \lesssim \Delta_0^2 + R_0.
\]
Proposition 7.2 then follows by using Theorem 5.1 and $N_1(\zeta) \lesssim \Delta_0^2 + R_0$. \hfill $\square$
7.3. Estimate for $\hat{\chi}$.

**Proposition 7.3.**

$$\|\hat{\chi}\|_{L^2_t L^2_x} \lesssim \Delta_0^2 + R_0$$

**Proof.** First, by (2.6),

$$\text{div}(a n \hat{\chi}) = \frac{1}{2} a n \nabla \text{tr} \chi + \frac{1}{2} a n \text{tr} \chi - a n \beta + a n \xi \hat{\chi}$$

$$= a n M + a n A \cdot A + r^{-1} a n \zeta - a n \beta$$

from (6.6),

$$a n \beta = D^{-1}_t D_t (\hat{\rho}, -\hat{\sigma}) - \hat{\delta} \quad \text{with} \quad \hat{\delta} = \text{Err.}$$

Hence

$$\text{div}(a n \hat{\chi}) = a n M + a n A \cdot A + r^{-1} a n \zeta - D^{-1}_t D_t (\hat{\rho}, -\hat{\sigma}) + \hat{\delta}$$

This gives

$$a n \hat{\chi} = -D^{-1}_t D^{-1}_t D_t (\hat{\rho}, -\hat{\sigma}) + D^{-1}_t (\hat{\delta} + a n M + a n A \cdot A + r^{-1} a n \zeta).$$

Set $D^{-2} = D^{-1}_2 D^{-1}_t$ and $D^{-1} = D^{-1}_t$, we obtain after taking covariant derivatives

(7.3) \[ \nabla (a n \hat{\chi}) = -\nabla D^{-2}_t \nabla \hat{R} + F + \nabla D^{-1} (a n M), \]

where $F = \nabla D^{-1} (\hat{\delta} + a n A \cdot A + r^{-1} a n \zeta)$ and $M = \nabla \text{tr} \chi$.

By (6.5) and (7.1),

(7.4) \[ \|F\|_{p_0} \lesssim \Delta_0^2 + R_0. \]

Consider the first term on the right of (7.3). By using the notations in (6.5), we can write

$$\nabla D^{-2}_t \nabla (\hat{R}) = D_t (\nabla D^{-2}_t \hat{R}) + C(\hat{R}).$$

where, by Proposition 6.1, there exist tensors $P'$ and $E'$ so that $C(\hat{R}) = D_t (P' + E')$ and

(7.5) \[ \mathcal{N}_1 (P') + \|E'\|_{p_0} \lesssim \Delta_0^2 + R_0, \quad \lim_{t \to 0} r^2 \|P'\|_{L^2_x} = 0. \]

Thus (7.3) becomes

(7.6) \[ \nabla (a n \hat{\chi}) = D_t P + \nabla D^{-1} (a n M) + E \]

where $P = \nabla D^{-2} \hat{R} + P'$ and $E = F + E'$. By using (6.28), (7.4) and (7.5)

(7.7) \[ \mathcal{N}_1 (P) + \|E\|_{p_0} \lesssim \Delta_0^2 + R_0, \quad \lim_{t \to 0} r^2 \|P\|_{L^2_x} = 0 \]

By combining (7.6) with (2.4) we obtain

$$\frac{d}{ds} M + \frac{3}{2} \text{tr} \chi M = -\hat{\chi} \cdot M - 2 (a n)^{-1} \hat{\chi} (D_t P + E + \nabla D^{-1} (a n M)) - \frac{1}{2} (\text{tr} \chi)^2 (\hat{\chi} + \hat{\zeta}),$$

regarding $\epsilon$ as an element of $A$, symbolically,

$$\nabla L M + \frac{3}{2} \text{tr} \chi M = A \cdot M + (a n)^{-1} \hat{\chi} (D_t P + E + \nabla D^{-1} (a n M))$$

$$+ (r^{-1} A + A \cdot A) \cdot A + r^{-2} A.$$

Then

$$D_t (r^3 M) = -3 \frac{3}{2} r^3 \text{an} M + r^3 \text{an} \{ A \cdot M + (a n)^{-1} \hat{\chi} (D_t P + E + \nabla D^{-1} (a n M))$$

$$+ r^3 \text{an} A (A \cdot A + r^{-1} A) + \text{ran} A.$$
Let us pair $\nabla \text{tr} \chi$ with vector fields $X_i$ in Lemma 4.11, which is still denoted by $M$. Regarding $\kappa$ also as an element of $A$, integrating in $t$, in view of $\lim_{t \to 0} r \nabla \text{tr} \chi = 0$,

$$M = r^{-3} \int_0^t r^3 \text{an} A \cdot M + r^3 \tilde{\kappa} \cdot D_t P$$

$$+ r^{-3} \int_0^t r^3 A \cdot (E + \nabla D^{-1}(anM) + anA \cdot A + r^{-1} \text{an} A) + an' Adt'.$$

Using Lemma 4.11, 5.2 and 5.4 in view of $\lim |\hat{\chi}| L^\infty < \infty$, we can obtain

$$\|r^{-3} \int_0^t r^3 A \cdot (an(M + A \cdot A) + E + \nabla D^{-1}(anM)) + r^3 \tilde{\kappa} \cdot D_t P\|_{g_0}$$

$$\lesssim (N_1(A) + |\text{an}|_{L^\infty L^2})(N_1(P) + \|E\|_{p_0} + \|\nabla D^{-1}(anM)\|_{p_0}$$

$$+ |anM|_{p_0} + |r^{-1} \text{an} A|_{p_0} + |\text{an} A \cdot A|_{p_0}).$$

By Proposition 4.12, 2.82, 4.21, and 2.79, we obtain

$$\|r^{-3} \int_0^t \text{an}' A\|_{p_0} \lesssim \|r^{-1} \text{an} A\|_{p_0} \lesssim N_1(A) \lesssim \Delta_0^2 + R_0.$$ 

Hence, in view of 4.13, BA1, 4.21, and 7.1, we can obtain

$$\|M\|_{p_0} \lesssim (N_1(P) + \|M\|_{p_0} + \|E\|_{p_0} + \Delta_0^2 + R_0) \left(\frac{N_1(A) + |\text{an}|_{L^\infty L^2}}{p_0}\right) + \Delta_0^2 + R_0$$

(7.8)

$$\lesssim \Delta_0 \left(\|M\|_{p_0} + N_1(P) + \|E\|_{p_0} + \Delta_0^2 + R_0\right) + \Delta_0^2 + R_0.$$ 

Since $0 < \Delta_0 < 1/2$ can be chosen to be sufficiently small such that the first term of (7.8) can be absorbed, in view of (7.7) we then obtain that

(7.9) $\|\nabla \text{tr} \chi\|_{p_0} \lesssim \Delta_0^2 + R_0$.

Thus, by setting $\tilde{E} = E + \nabla D^{-1}(anM)$ we obtain from (7.6), (7.1), and (7.9) the decomposition

(7.10) $\nabla (an\hat{\chi}) = D_t P + \tilde{E}$ and $\mathcal{N}_1(P) + \|\tilde{E}\|_{p_0} \lesssim \Delta_0^2 + R_0$.

By Theorem 6.1 and (2.79), we conclude

$$\|\hat{\chi}\|_{L^\infty L^2} \lesssim \mathcal{N}_1(\hat{\chi}) + \mathcal{N}_1(P) + \|\tilde{E}\|_{p_0} \lesssim \Delta_0^2 + R_0.$$ 

as expected.

Similar to the derivation of (7.3), we can get

(7.11) $\|r^2 \nabla \text{tr} \chi\|_{g_0} \lesssim \Delta_0^2 + R_0$.

7.4. Estimate for $\zeta$.

Proposition 7.4.

$$\|\zeta\|_{L^\infty L^2} \lesssim \Delta_0^2 + R_0.$$ 

Proof. By using 2.8 and 2.9,

$$\text{div} (an\zeta) = an\zeta \cdot \hat{\zeta} - an\mu - an\rho + \frac{1}{2} \mu n \delta \text{tr} \chi,$$

$$\text{curl} (an\zeta) = an\sigma + an(\zeta + \hat{\zeta} \wedge \zeta).$$
Symbolically, $\mathcal{D}_1(an\zeta) = anA \cdot A - an(\mu, 0) - an(\rho, -\sigma) + an(a\delta\text{tr}A, 0)$. Hence

$$an\zeta = -\mathcal{D}^{-1}_1(an(\rho, -\sigma)) + \mathcal{D}^{-1}_1(anA \cdot A) - \mathcal{D}^{-1}_1(an(\mu, 0)) + \mathcal{D}^{-1}_1(an(r^{-1}A, 0)).$$

Let $J$ be the involution $(\rho, \sigma) \to (-\rho, \sigma)$, $\bar{\mathcal{F}} = \mathcal{E}rr$ is given by (6.6),

$$\nabla(an\zeta) = \nabla \mathcal{D}^{-1}_1 \cdot J \cdot \mathcal{D}^{-1}_1D_1 \beta + \nabla \mathcal{D}^{-1}_1 \cdot J \cdot \bar{\mathcal{F}} - \nabla \mathcal{D}^{-1}_1(an\mu, 0)
+ \nabla \mathcal{D}^{-1}_1(an \cdot A \cdot A) + \nabla \mathcal{D}^{-1}_1(r^{-1}anA, 0).$$

Set $\mathcal{D}^{-2} = \mathcal{D}^{-1}_1 \cdot J \cdot \mathcal{D}^{-1}_1$ and $\mathcal{D}^{-1} = \mathcal{D}^{-1}_1$. By using (6.5), we get

$$\nabla(an\zeta) = \mathcal{D}_1 \nabla \mathcal{D}^{-2}_1 + C(\bar{\mathcal{R}}) + \nabla \mathcal{D}^{-1}_1(anM) + F$$

where $M = (\mu, 0)$ and $F = \nabla \mathcal{D}^{-1}_1(\bar{\mathcal{F}} + an(A \cdot A + r^{-1}A))$.

By (6.50) and (7.1), we derive $\|F\|_{p^0} \lesssim \Delta_0^2 + R_0$.

In view of Proposition 5.1, for some tensors $\bar{P}$ and $\bar{E}$ such that $C(\bar{R}) = \mathcal{D}_1 \bar{P} + \bar{E}$. With $E = \bar{E} + F$, $P = \bar{P} + \nabla \mathcal{D}^{-2}_1$, we can write

$$\nabla(an\zeta) = \mathcal{D}_1 P + \nabla \mathcal{D}^{-1}_1(anM) + E,$$

and

$$(7.12) \quad \mathcal{N}_1(P) + \|E\|_{p^0} \leq \Delta_0^2 + R_0, \quad \lim_{t \to 0} \|P\|_{L^\infty} = 0$$

Let $M = (\mu, 0)$, (2.13) can be written as

$$\frac{d}{ds} M + \text{tr} \chi M = 2(an)^{-1}\dot{\chi} \left( \mathcal{D}_1 P + E + \nabla \mathcal{D}^{-1}_1(anM) \right) - 2\dot{\chi} \cdot \dot{\zeta} + (\zeta - 2\zeta)\text{tr} \chi \zeta - \nabla \text{tr} \chi (\zeta - \dot{\zeta}) + \text{tr} \chi \bar{\rho} + \left( \frac{1}{4} a^2 \text{tr} \chi + A \right) |\dot{\chi}|^2 - \frac{1}{2} an\nu(\text{tr} \chi)^2.$$

Symbolically,

$$\frac{d}{ds} M + \text{tr} \chi M = (an)^{-1}\dot{\chi} \cdot (\mathcal{D}_1 P + \nabla \mathcal{D}^{-1}_1(anM) + E) + \text{tr} \chi \bar{\rho}
+ A \cdot (A \cdot A + r^{-1}A + \nabla \text{tr} \chi) + r^{-1} an\text{tr} \chi A.$$

In view of $\lim_{t \to 0} r^{-2} \mu = 0$ in Lemma 2.1, we deduce

$$(7.14) \quad M = r^{-2} \int_0^t r^2 \left( ank \cdot M + \dot{\chi} \cdot (\mathcal{D}_1 P + A \cdot F + an\text{tr} \chi \bar{\rho} + r^{-1} a^2 an\text{tr} \chi A) \right) dt'$$

with $F = an(A \cdot A + \nabla \text{tr} \chi + r^{-1}A) + E + \nabla \mathcal{D}^{-1}_1(anM)$.

In view of Proposition 5.1 regarding $\kappa$ as an element of $A$,

$$(7.15) \quad \left\| r^{-2} \int_0^t r^2 \left( ank \cdot M + \dot{\chi} \cdot (\mathcal{D}_1 P + A \cdot F) \right) dt' \right\|_{p^0}
\lesssim \left( \mathcal{N}_1(A) + \|A\|_{L^\infty L^2} \right) \left( \|anM\|_{p^0} + \mathcal{N}_1(P) + \|\bar{F}\|_{p^0} \right).$$

Note that by 4.21, 4.13, 7.1 and 6.17, we deduce

$$\|\bar{F}\|_{p^0} \lesssim \|an(A \cdot A + r^{-1}A)\|_{p^0} + \|an\nabla \text{tr} \chi\|_{p^0} + \|E\|_{p^0} + \|\nabla \mathcal{D}^{-1}_1(anM)\|_{p^0}
\lesssim \|\nabla \text{tr} \chi\|_{p^0} + \|E\|_{p^0} + \|M\|_{p^0} + \Delta_0^2 + R_0.$$

By (7.9) and (7.12),

$$\|\bar{F}\|_{p^0} \lesssim \Delta_0^2 + R_0 + \|M\|_{p^0}.\]
Hence, by (6.17), BA1, (4.13) and (7.12)

\[ (7.15) \lesssim \Delta_0(\|M\|p_0 + \Delta_0^2 + \mathcal{R}_0). \]

Assuming the following estimate for \( \mathcal{P}^0 \) norm of the last two terms in (7.24)

\[ (7.16) \left\| r^{-2} \int_0^t r'^2 \left( \text{antr} \hat{\rho} + r'^{-1} a^2 \text{antr} \chi A \right) dt' \right\| p_0 \lesssim \Delta_0^2 + \mathcal{R}_0, \]

since \( 0 < \Delta_0 < 1/2 \) can be chosen sufficiently small, we conclude that

\[ (7.17) \|M\|p_0 \lesssim \Delta_0^2 + \mathcal{R}_0. \]

By (7.1) and (2.79), \( \|\nabla \mathcal{D}^{-1}(anM)\|p_0 \lesssim \Delta_0^2 + \mathcal{R}_0. \)

In view of (7.12), we have \( \nabla (an\zeta) = D_t P + E''', \) with \( E''' = E + \nabla \mathcal{D}^{-1}(anM). \)

By Theorem 5.1

\[ \|\zeta\|_{L^\infty L^2} \lesssim \mathcal{N}_1(P) + \|E'''\|p_0 + \mathcal{N}_1(\zeta) \lesssim \Delta_0^2 + \mathcal{R}_0. \]

We now prove (7.16). With the help of (6.4), by letting

\[ p' := *\mathcal{D}^{-1}_1 \beta, \quad e' = *\mathcal{D}^{-1}_1 \beta - \text{Err} + anA \cdot A, \]

also noting that (6.11) gives \( \|D_t \cdot \mathcal{D}^{-1}_1 \beta\|p_0 \lesssim \Delta_0^2 + \mathcal{R}_0, \) combined with (6.8), (4.21) and (6.17), we can get the following decomposition

\[ (7.18) \quad an(\hat{\rho}, \hat{\sigma}) = D_t p' + e' \quad \text{with} \quad \mathcal{N}_1(p') + \|e'\|p_0 \lesssim \Delta_0^2 + \mathcal{R}_0. \]

(7.16) can be derived by establishing the following inequalities

\[ (7.19) \left\| r^{-2} \int_0^t r'^2 \text{tr} D_t p' dt' \right\| p_0 \lesssim \Delta_0^2 + \mathcal{R}_0, \]

\[ (7.20) \left\| r^{-2} \int_0^t r'^2 \text{tr} \left( e' + r'^{-1} a^2 \right) dt' \right\| p_0 \lesssim \Delta_0^2 + \mathcal{R}_0. \]

To prove (7.20), by Proposition 4.24, (2.29), also in view of (4.21) and (4.15), we can obtain

\[ \left\| r^{-2} \int_0^t r' \left( e' + r'^{-1} a^2 \right) dt' \right\| p_0 \lesssim \|e'\|p_0 + \|r^{-1} a^2\|p_0 \lesssim \Delta_0^2 + \mathcal{R}_0. \]

By (6.2) and (7.18), we get

\[ \left\| r^{-2} \int_0^t r'^2 \cdot e' dt' \right\| \lesssim \mathcal{N}_1(\zeta) + \|\zeta\|_{L^\infty L^2} \|e'\|p_0 \lesssim \Delta_0^2 + \mathcal{R}_0, \]

and similarly

\[ \left\| r^{-2} \int_0^t r' \cdot a^2 dt' \right\| \lesssim \mathcal{N}_1(\zeta) + \|\zeta\|_{L^\infty L^2} \|r^{-1} a^2\|p_0 \lesssim \Delta_0^3 + \mathcal{R}_0. \]

The proof of (7.20) is complete.

Now we prove (7.19). Recall that \( p' = *\mathcal{D}^{-1}_1 \beta, \) then

\[ (7.21) \lim_{s \to 0} r p' = 0, \quad \|\text{tr} \chi p'\|p_0 \lesssim \mathcal{N}_1(p') \lesssim \mathcal{R}_0 + \Delta_0^2. \]
Using Proposition 4.2 and (7.21), also in view of (7.21), we derive
\[
\left\| \frac{1}{r^2} \int_0^t r^2 \text{tr} \chi D_\xi p' dt' \right\|_{L^2_t L^2_x}
\leq \sum_{k>0} \left\| E_k \frac{1}{r} \int_0^t r^2 \text{tr} \chi D_\xi p' dt' \right\|_{L^2_t L^2_x} + \left\| \frac{1}{r} \int_0^t r^2 \text{tr} \chi D_\xi p' dt' \right\|_{L^2_t L^2_x}
\leq \sum_{k>0} \left( \| E_k (r^2 \text{tr} \chi ) \|_{L^2_t L^2_x} + \| E_k r^{-1} \int_0^t D_t (r^2 \text{tr} \chi p') dt' \|_{L^2_t L^2_x} \right) + \| \nabla L^p \|_{L^2_t L^2_x}
\leq \| \text{tr} \chi p' \|_{L^p(\mathbb{R})} + \| \text{tr} \chi p' \|_{L^2(\mathbb{R})} + \| \nabla L^p \|_{L^2_t L^2_x}
+ \left( \sum_{k>0} \left( \left\| E_k r^{-1} \int_0^t r^2 \text{antr} \chi \text{tr} \chi p' dt' \right\|_{L^2_t L^2_x} \right) \right)
\]
where for the last inequality, we employed (7.18).
It is easy to see by (4.20)
\[
\sum_{k>0} \left\| E_k r^{-1} \int_0^t r^2 \text{antr} \chi \text{tr} \chi p' dt' \right\|_{L^2_t L^2_x} \lesssim \| \text{tr} \chi p' \|_{L^2_t L^2_x} \lesssim \mathcal{N}_1(p')
\]
and
\[
\sum_{k>0} \left\| E_k r^{-1} \int_0^t r^2 \text{an} \chi \text{tr} \chi p' dt' \right\|_{L^2_t L^2_x} \lesssim (\Delta_0^2 + R_0) \mathcal{N}_1(p') \lesssim \Delta_0^2 + R_0.
\]
where we employed (7.20) and
\[
(7.23) \quad \| \text{an} \chi \cdot \text{tr} \chi p' \|_{L^p(\mathbb{R})} \lesssim \mathcal{N}_1(p')(\Delta_0^2 + R_0).
\]
To see (7.23), we first deduce with the help of (4.13) that
\[
\| \text{an} \chi \cdot \text{tr} \chi p' \|_{L^p(\mathbb{R})} \lesssim \| \chi \cdot \text{tr} \chi p' \|_{L^p(\mathbb{R})}.
\]
By (4.17) with \( G = r \text{tr} \chi \)
\[
\| \chi \cdot \text{tr} \chi p' \|_{L^p(\mathbb{R})} \lesssim \mathcal{N}_1(p') \left( \| r^\frac{5}{6} \chi (r \text{tr} \chi \kappa) \|_{L^2(\mathbb{R})} + \| r^{1-\frac{5}{6}} \chi \text{tr} \chi \kappa \|_{L^2_t L^2_x} \right)
\]
where $b > 4$. Since $\|r^\kappa\|_{L^\infty} \lesssim C$ and

$$\|r^\frac{1}{2} \nabla (r^\kappa)\|_{L^2(\mathcal{H})} \lesssim \|r^\frac{3}{2} \nabla r^\kappa\|_{L^2(\mathcal{H})} + \|r^\frac{7}{2} \nabla r^\kappa\|_{L^2(\mathcal{H})} \lesssim \|\nabla r^\kappa\|_{L^2(\mathcal{H})} + \|\nabla \kappa\|_{L^2(\mathcal{H})} \lesssim \Delta_0^2 + R_0,$$

where for the last inequality, we employed (2.79) and Proposition 2.7. And by (2.38), we have

(8.1)

Proof of Lemma 6.3.

The result is trivial when $2.8$ and (1.13), more precisely

where we employed Lemma 8.1 to derive the last inequality.

Thus in view of (7.25) and (7.24), (7.23) is proved. In view of (7.21), (7.19) is proved.

Similar to (7.17), we can obtain

$$\|r^\frac{7}{2} \mu\|_{B^0} \lesssim \Delta_0^2 + R_0.$$  

8. Appendix

8.1. Dyadic Sobolev inequalities. We start with proving Lemma 6.3 and a few useful consequences, which will be used to prove (6.76) in the second subsection. Let us still regard $\kappa$ and $\iota$ as elements of $A$. Using Propositions 2.5, 2.7, Lemma 2.8, and (1.13), more precisely

$$\|K\|_{L^2_1 L^2_2} + \|\beta\|_{L^2_1 L^2_2} + N_1(A) \lesssim \Delta_0^2 + R_0,$$

we can adapt the approach in [4, Lemma 5.3] and [10, Chapter 9] to derive

Lemma 8.1. For any smooth $S_t$ tangent tensor fields $F$ and all $q < 2$ sufficiently close to $q = 2$,

$$(8.1) \quad \|r^{-\frac{3}{2} - \frac{1}{q}} [P_k, D_t] F\|_{L^1_t L^2_x} + 2^{-k} \|r^{-\frac{3}{2} + \frac{1}{q}} \nabla [P_k, D_t] F\|_{L^1_t L^2_x} \lesssim 2^{-\frac{k}{q} + 1} N_1(F),$$

$$(8.2) \quad \|r^{-\frac{3}{2}} [P_k, D_t] F\|_{L^1_t L^2_x} + 2^{-k} \|r^\frac{1}{2} \nabla [P_k, D_t] F\|_{L^1_t L^2_x} \lesssim 2^{-\frac{k}{q} + 1} N_1(F).$$

Now we are ready to prove Lemma 6.3.

Proof of Lemma 6.3. The result is trivial when $q = 2$. So we only need to consider the case $q > 2$. It is easy to get the following estimate

$$\|r^{-\frac{3}{2} - \frac{1}{q}} P_k F\|_{L^1_t L^2_x} \lesssim \|r^{-1} P_k F\|_{L^1_t L^2_x} \|r^{-\frac{3}{2}} P_k F\|_{L^\infty_t L^2_x}.$$

Moreover we have by integrating along an arbitrary null geodesic,

$$\|r^{-\frac{3}{2}} P_k F\|_{L^\infty_t L^2_x} \lesssim \|P_k (D_t F)\|_{L^2_t L^2_x} \|r^{-1} P_k F\|_{L^1_t L^2_x} + \|r^{-1} [P_k, D_t] F \cdot P_k F\|_{L^1_t L^2_x}$$

$$(8.3) \quad + \|r^{-1} P_k F\|_{L^2_t L^2_x}.$$

We can obtain the following estimate with $\frac{1}{q} + \frac{1}{q} = 1$ and $2 < q < \infty$

$$\|r^{-1} [P_k, D_t] F \cdot P_k F\|_{L^1_t L^2_x} \lesssim \|r^{-\frac{3}{2}} [P_k, D_t] F\|_{L^1_t L^2_x} \|r^{-\frac{3}{2} - \frac{1}{q}} P_k F\|_{L^1_t L^2_x}$$

$$(8.4) \quad \lesssim 2^{-\frac{k}{q} + 1} N_1(F) \|r^{-\frac{3}{2} - \frac{1}{q}} P_k F\|_{L^1_t L^2_x}$$

where we employed Lemma 8.1 to derive the last inequality.
Combining the above estimates we obtain

\[
\| r^{-\frac{1}{2}+\frac{1}{k}}P_kF \|_{L^q_tL^2_x}^{p} \lesssim \| r^{-1}P_kF \|_{L^q_tL^2_x}^{p} \left( \| P_k(D_tF) \|_{L^q_tL^2_x} + \| r^{-\frac{1}{2}}P_kF \|_{L^q_tL^2_x} + \| r^{-1}P_kF \|_{L^q_tL^2_x} \right)^{\frac{q}{2}-1}.
\]

(8.5)

Using (4.4), we get for any \(0 \leq \alpha < \frac{\frac{1}{q} - \frac{1}{2}}{1\frac{1}{p} + \frac{1}{q}}\),

\[
\| r^{-\frac{1}{2}+\frac{1}{k}}P_kF \|_{L^q_tL^2_x} \lesssim 2^{-k(\frac{1}{q} + \frac{1}{2})}(1 + 2^{-\alpha k})N_1(F).
\]

Using (8.3) and (6.32) with \(2 < q \leq \infty \), by using \(4.4\) we obtain

\[
\| r^{-\frac{1}{2}}P_kF \|_{L^q_tL^2_x} \lesssim 2^{-\frac{1}{2}k}N_1(F).
\]

To see (6.33), we derive by Sob and (4.3)

\[
\| r^{-\frac{1}{2}}F_k \|_{L^q_tL^2_x} \lesssim \| r^{-\frac{1}{2}}r^\frac{1}{q}\nabla F_k \|_{L^q_tL^2_x} + \| r^{-\frac{1}{2}}P_kF \|_{L^q_tL^2_x} \lesssim (2^q + 1)\| r^{-\frac{1}{2}}P_kF \|_{L^q_tL^2_x}.
\]

Combined with (6.32), (6.33) follows.

\[\Box\]

**Lemma 8.2.** Let \(D^{-1}\) denote one of the operators \(D_1^{-1}, D_2^{-1}\) and \(\star D_1^{-1}\). There hold the following estimates for appropriate \(S\) tangent tensor \(G\),

\[
\| D^{-1}P_k^2G \|_{L^p_tL^2_x} \lesssim r^{-\frac{1}{2}}2^{-\frac{k}{2}}N_1(G), \quad k > 0,
\]

(8.6)

\[
\| r^{-1}P_k^2G \|_{L^q_tL^2_x} \lesssim N_1(G) \left( 1 + 2^{-\frac{1}{2}k}r^{-\frac{1}{2}}\| K \|_{L^2_x} \right), \quad k > 0.
\]

(8.7)

**Proof.** (8.6) can be obtained by using Lemma 4.3 and (6.32). Now consider (8.7). For any \(S\) tangent tensor fields \(F\), define

\[
\| F \|_{H^s} = \| \nabla F \|_{L^2(S)} + \| r^{-1}F \|_{L^2(S)}.
\]

Recall by Böchner identity contained in [3], there holds

\[
\| \nabla^2 F \|_{L^2_x} \lesssim \| \Delta F \|_{L^2_x} + \| \mathbf{K} \cdot F \|_{L^2_x} + \| \mathbf{K} \|_{L^2_x}^{\frac{3}{2}}\| \nabla F \|_{L^4_x} + r^{-1}\| F \|_{H^2_x},
\]

(8.8)

and by Sobolev embedding

\[
\| F \|_{L^\infty_x} \lesssim r^\frac{1}{p}\| \nabla^2 F \|_{L^2_x}^{\frac{1}{2}}\| F \|_{H^\frac{1}{p}_x}^{\frac{1}{2}} + \| F \|_{H^2_x}, \quad 2 < p < \infty.
\]

(8.9)

It is easy to observe from [11] P.38 that symbolically

\[
\Delta = \star \mathcal{D} \pm (\mathbf{K} + r^{-2}Id).
\]

Hence with \(2 < p < \infty\),

\[
\| \nabla^2 F \|_{L^2_x} \lesssim \| \star \mathcal{D} F \|_{L^2_x} + \| \mathbf{K} \|_{L^2_x}^{\frac{1}{2}}\| \mathcal{D}^{-1}F \|_{H^0_x} + \| \mathbf{K} \|_{L^2_x}^{\frac{1}{2}}\| \nabla F \|_{L^2_x} + r^{-1}\| F \|_{H^2_x}.
\]

(8.10)

For \(F = \mathcal{D}^{-1}H\), using Proposition 3.4

\[
\| \nabla^2 \mathcal{D}^{-1}H \|_{L^2_x} \lesssim \| \star \mathcal{D} H \|_{L^2_x} + \| \mathbf{K} \|_{L^2_x}^{\frac{1}{2}}\| \mathcal{D}^{-1}H \|_{H^0_x} + \| \mathbf{K} \|_{L^2_x}^{\frac{1}{2}}\| \nabla \mathcal{D}^{-1}H \|_{L^2_x} + r^{-1}\| \mathcal{D}^{-1}H \|_{H^0_x}.
\]

\[
\lesssim \| \star \mathcal{D} H \|_{L^2_x} + \| \mathbf{K} \|_{L^2_x}^{\frac{1}{2}}\| H \|_{L^2_x} + \| \mathbf{K} \|_{L^2_x}^{\frac{1}{2}}\| \mathcal{D}^{-1}H \|_{L^2_x} + \| \mathbf{K} \|_{L^2_x}^{\frac{1}{2}}\| \nabla \mathcal{D}^{-1}H \|_{L^2_x} + r^{-1}\| \mathcal{D}^{-1}H \|_{H^0_x}.
\]

(8.11)
Let \( H = P_k^2 G \), by \((4.3)\), Proposition 4.3, Lemma 4.3, we obtain,
\[
\| \nabla^2 D^{-1} P_k^2 G \|_{L^2} \lesssim \| \ast D P_k^2 G \|_{L^2} + (\| K \|_{L^2} + r^{-1}) \| P_k^2 G \|_{L^2} \\
\lesssim \left( (2^k + 1) r^{-1} + \| \tilde{K} \|_{L^2} \right) \| P_k G \|_{L^2}.
\]

By Sob, Proposition 3.4, \((6.32)\),
\[
\| \nabla D^{-1} P_k^2 G \|_{L^2} \lesssim \| \nabla^2 D^{-1} P_k^2 G \|_{L^2} \| \nabla D^{-1} P_k^2 G \|_{L^2} \| + r^{-\frac{3}{2}} \| \nabla D^{-1} P_k^2 G \|_{L^2} \\
\lesssim (1 + 2^{-\frac{3}{2}} r \| \tilde{K} \|_{L^2}) N_1(G).
\]

\(\square\)

8.2. Proof of \((6.76)\). We first prove \((6.76)\) by assuming the following results.

**Lemma 8.3.** Denote by \( \mathcal{D}^{-1} G \) one of the terms \( \mathcal{D}_1^{-1} G, \mathcal{D}_2^{-1} G \) and \( \ast \mathcal{D}_1^{-1} G \) for appropriate \( S \) tangent tensor \( G \). Let \( F = \mathcal{D}^{-1} \hat{R} \cdot \mathcal{D}^{-1} G \), we have
\[
\| \nabla F \|_{B^2_{1,1}} \lesssim N_1(G) \left( \Delta_0^2 + \mathcal{R}_0 + c_0 r^{\frac{3}{2}} \left( \| \mathcal{D}^{-1} \hat{R} \|_{L^\infty} + \| \hat{R} \|_{L^2} + r^2 \| \mathcal{D} \hat{R} \|_{L^\infty} \right) \right)
\]
where \( c_0 \) depends on \( \| r \tilde{K} \|_{L^2} + K_{0,0} \).

**Lemma 8.4.** Let \( \mathcal{D}^{-1} \) denotes one of the operators \( \mathcal{D}_1^{-1}, \mathcal{D}_2^{-1} \) and \( \ast \mathcal{D}_1^{-1} \). For \( S \) tangent tensor \( H \), there hold
\[
\| \nabla \mathcal{D}^{-1} H \|_{B^2_{1,1}} \lesssim \| \ast D H \|_{B^2_{1,1}} + \| H \|_{L^2} + c_0 r^{\frac{3}{2}} N_1(H),
\]
\[
(8.12)
\]
\[
\| \nabla \mathcal{D}^{-1} H \|_{L^2} \lesssim \| \ast D H \|_{B^2_{1,1}} + (c_0 r^2 + 1) (\| \nabla H \|_{L^2} + \| H \|_{L^2} + c_0 r^{\frac{3}{2}} N_1(H)),
\]
\[
(8.13)
\]
where \( c_0 \) is depending on the quantity \( r \| \tilde{K} \|_{L^2} + K_{0,0} + r \| \nabla \tilde{K} \|_{L^2} \), and \( \theta > 0 \) is very close to 0.

Let us set \( P = \sum_{i=1}^\infty P_i \) and \( \hat{P} = \nabla \mathcal{D}^{-1} P \). Since \( \hat{P} = \hat{P} + \hat{P} - P_1 \) and in view of \((6.73)\) that \( \lim_{i \to 0} \| P_1 \|_{L^2} = 0 \), \((6.74)\) can be proved by establishing
\[
\lim_{i \to 0} \| \hat{P} \|_{L^2} = 0, \quad \text{and} \quad \lim_{i \to 0} \| \hat{P} \|_{L^2} < \infty.
\]
\((8.14)\)

In view of \((6.80)\), we have by SobM2, Lemma 3.1 and \((6.89)\),
\[
\| P \|_{L^\infty} \lesssim \| D_1^{-1} \hat{R} \|_{L^\infty} \| D^{-1} P_{i-1} \|_{L^2} \lesssim r^{\frac{3}{2}} \| D_1^{-1} \hat{R} \|_{L^\infty} N_2(D^{-1} P_{i-1}) \\
\lesssim r^{\frac{3}{2}} N_1(P_{i-1}) \| D_1^{-1} \hat{R} \|_{L^2} \lesssim r^{\frac{3}{2}} (C(\Delta_0^2 + \mathcal{R}_0))^{-1} N_2(F) \| D_1^{-1} \hat{R} \|_{L^\infty}.
\]
Summing over \( i \geq 1 \), with \( (\Delta_0^2 + \mathcal{R}_0) \) sufficiently small,
\[
\| P \|_{L^\infty} \lesssim r^{\frac{3}{2}} N_2(F) \| D_1^{-1} \hat{R} \|_{L^\infty}.
\]

Noting that \( \lim_{i \to 0} \| D_1^{-1} \hat{R} \|_{L^\infty} < \infty \) and \( N_2(F) < \infty \),
\[
\lim_{i \to 0} \| P \|_{L^\infty} = 0.
\]

It remains to prove the second part of \((8.14)\). By \((8.13)\), there holds
\[
\| \hat{P} \|_{L^\infty} \lesssim \| \nabla P \|_{B^2_{1,1}} + (c_0 r^2 + 1) \left( \| \nabla P \|_{L^2} + \| P \|_{L^2} + c_0 r^{\frac{3}{2}} N_1(P) \right),
\]
\((8.15)\)
Recall the definition of $P_t$ in (6.86). For each $P_t = D^{-1} \tilde{R} \cdot D^{-1} P_{t-1}$, by Lemma 8.3 there holds

\[
\|\nabla P_t\|_{B^0_{2,1}} \lesssim c_0 N_1(P_{t-1}) r^{\frac{1}{2}} \left( \|D^{-1} \tilde{R}\|_{L^\infty_{\nu}} + \|\tilde{R}\|_{L^2_{\nu}} + r^2 \|D \tilde{R}\|_{L^\infty_{\nu}} \right) + N_1(P_{t-1})(\Delta_0^2 + R_0).
\]

In view of (6.89), summing over $i \geq 1$ gives

\[
\|\nabla P\|_{B^0_{2,1}} \lesssim \sum_{i \geq 1} \|\nabla P_i\|_{B^0_{2,1}} \lesssim c_0 \sum_{i \geq 1} (C(\Delta_0^2 + R_0))^i N_2(F) r^{\frac{i}{2}} \left( \|D^{-1} \tilde{R}\|_{L^\infty_{\nu}} + \|\tilde{R}\|_{L^2_{\nu}} + r^2 \|D \tilde{R}\|_{L^\infty_{\nu}} \right) + \sum_{i \geq 1} (C(\Delta_0^2 + R_0))^i N_2(F)(\Delta_0^2 + R_0).
\]

Hence

\[
\|\nabla P\|_{B^0_{2,1}} \lesssim c_0 N_2(F) r^{\frac{1}{2}} \left( \|D^{-1} \tilde{R}\|_{L^\infty_{\nu}} + \|\tilde{R}\|_{L^2_{\nu}} + r^2 \|D \tilde{R}\|_{L^\infty_{\nu}} \right) + N_2(F)(\Delta_0^2 + R_0).
\]

(8.16)

By (6.86), (SobM1), Lemma 3.1, (6.89) and (6.22),

\[
\|P\|_{L^2_{\nu}} \lesssim r^{-1} \|D^{-1} \tilde{R}\|_{L^\infty_{\nu}} \|D^{-1} P_{t-1}\|_{L^2_{\nu}} \lesssim N_1(D^{-1} \tilde{R}) r^{-1} N_1(D^{-1} P_{t-1}) \lesssim N_1(P_{t-1})(\Delta_0^2 + R_0) \lesssim (C(\Delta_0^2 + R_0))^i(\Delta_0^2 + R_0) N_2(F).
\]

Therefore

(8.17)

\[
\|P\|_{L^2_{\nu}} \lesssim (\Delta_0^2 + R_0) N_2(F).
\]

It is easy to derive from $P = \sum_{i \geq 1} P_i$ and (6.89) that

(8.18)

\[
N_1(P) \lesssim (\Delta_0^2 + R_0) N_2(F).
\]

Thus in view of (8.15), the combination of (8.16), (8.17) and (8.18) implies

\[
\lim_{t \to 0} \|\tilde{P}\|_{L^\infty_{\nu}} \lesssim N_2(F)(\Delta_0^2 + R_0) < \infty,
\]

as desired.

\textit{Proof of Lemma 8.3} Let us compute $\|\nabla F\|_{L^2_{\nu}}$ first.

\[
\|\nabla F\|_{L^2_{\nu}} = \|\nabla(D^{-1} \tilde{R} \cdot D^{-1} G)\|_{L^2_{\nu}} \lesssim \|\nabla D^{-1} \tilde{R}\|_{L^2_{\nu}} \|D^{-1} G\|_{L^\infty_{\nu}} + \|D^{-1} \tilde{R}\|_{L^2_{\nu}} \|\nabla D^{-1} G\|_{L^2_{\nu}}.
\]

By (SobM2), (SobM1), Proposition 3.4, Lemma 3.1 and (6.22),

\[
\|\nabla F\|_{L^2_{\nu}} \lesssim r^{\frac{1}{2}} \|\tilde{R}\|_{L^2_{\nu}} N_2(D^{-1} G) + N_1(D^{-1} \tilde{R}) N_1(\nabla D^{-1} G) \lesssim \left( r^{\frac{1}{2}} \|\tilde{R}\|_{L^2_{\nu}} + N_1(D^{-1} \tilde{R}) \right) N_1(G) \lesssim \left( r^{\frac{1}{2}} \|\tilde{R}\|_{L^2_{\nu}} + \Delta_0^2 + R_0 \right) N_1(G).
\]

Now we prove

(8.19)

\[
\sum_{k>0} \|P_k \nabla(D^{-1} \tilde{R} \cdot D^{-1} G)\|_{L^2_{\nu}} \lesssim c_0 N_1(G) r^{\frac{1}{2}} \left( \|D^{-1} \tilde{R}\|_{L^\infty_{\nu}} + \|\tilde{R}\|_{L^2_{\nu}} + r^2 \|\nabla \tilde{R}\|_{L^\infty_{\nu}} \right).
\]
Indeed, we will employ GLP decompositions to write
\[ G = \sum_{m > 0} P_m G + P_{\leq 0} G + U(\infty) G. \]

For simplicity, we consider the high frequency terms \( \sum_{m > 0} P_m G. \) The other two terms can be treated similar to Case 2.

**Case 1**: \( k < m. \) By (4.3) and (8.6),
\[ \| P_k \nabla(D^{-1} \hat{R} \cdot D^{-1} G_m) \|_{L^2_x} \lesssim 2^{k - \frac{2n}{p}} r^\frac{k}{2} N_1(G) \| D^{-1} \hat{R} \|_{L^\infty_x}. \]

Thus we obtain
\[ \sum_{k > 0} \sum_{m > k} \| P_k \nabla(D^{-1} \hat{R} \cdot D^{-1} G_m) \|_{L^2_x} \lesssim r^\frac{k}{2} N_1(G) \| D^{-1} \hat{R} \|_{L^\infty_x}. \]

**Case 2**: \( k > m. \) We decompose further such that
\[ P_k \nabla(D^{-1} \hat{R} \cdot D^{-1} G_m) = P_k \nabla(\sum_{n > 0} P_n^2 + P_{\leq 0})(D^{-1} \hat{R} \cdot D^{-1} G_m). \]

For simplicity we consider the high frequency terms, and the low frequency terms can be treated similarly. We can adapt the proof for [11, Proposition 4.5] to obtain the following inequality for \( S \) tangent tensor field \( F \) and \( 1 > \alpha > \alpha_0 \geq \frac{1}{2} \)
\[ \| P_k \nabla P_n^2 F \|_{L^2_x} \lesssim \left(2^{\min(k,n)} 2^{-2|n-k|} r^{-1} + 2^{\min(k,n)} 2^{-(1-\alpha) \max(k,n)} K_{\alpha_0} r^{-\alpha}\right) \|P_n F\|_{L^2_x}. \]

Let \( T_{nm} = \| P_n(D^{-1} \hat{R} \cdot D^{-1} G_m) \|_{L^2_x}, \) we have
\[ \| P_k \nabla P_n^2(D^{-1} \hat{R} \cdot D^{-1} G_m) \|_{L^2_x} \lesssim \left(2^{\min(k,n)} 2^{-2|n-k|} r^{-1} + 2^{\min(k,n)} 2^{-(1-\alpha) \max(k,n)} K_{\alpha_0} r^{-\alpha}\right) \|P_n F\|_{L^2_x}. \]

Now we estimate \( T_{nm}. \) Let us first consider the case that \( n > m > 0. \) By Proposition 4.1 (iii), we have
\[ T_{nm} \lesssim r^2 2^{-2n} \| P_n \Delta(D^{-1} \hat{R} \cdot D^{-1} G_m) \|_{L^2_x} \]
\[ \lesssim r^2 2^{-2n} \left(\| \Delta D^{-1} \hat{R} \cdot D^{-1} G_m \|_{L^2_x} + \| P_n(\nabla D^{-1} \hat{R} \cdot \nabla D^{-1} G_m) \|_{L^2_x} + \| \Delta D^{-1} G_m \cdot D^{-1} \hat{R} \|_{L^2_x}\right). \]

By (8.6) and (8.10), we have
\[ \| \Delta D^{-1} \hat{R} \cdot D^{-1} G_m \|_{L^2_x} \lesssim \left(\| \nabla \hat{R} \|_{L^\infty} + \| K \|_{L^\infty} \| D^{-1} \hat{R} \|_{L^\infty_x} + r^{-2} \| D^{-1} \hat{R} \|_{L^\infty_x}\right) 2^{-\frac{2n}{p}} r^\frac{k}{2} N_1(G). \]

By (4.3), Propositions 3.3 and (8.7),
\[ \| P_n(\nabla D^{-1} \hat{R} \cdot \nabla D^{-1} G_m) \|_{L^2_x} \lesssim 2^\frac{n}{2} r^{-\frac{k}{2}} \| \nabla D^{-1} \hat{R} \|_{L^2_x} \| \nabla D^{-1} G_m \|_{L^2_x} \]
\[ \lesssim 2^\frac{n}{2} r^{-\frac{k}{2}} \| \hat{R} \|_{L^2_x} N_1(G) \left(1 + 2^{-\frac{2n}{p}} r^\frac{k}{2} \| K \|_{L^2_x}\right). \]
By (8.10), (8.13) and (6.32), (8.6), we obtain
\[
\|\Delta D^{-1}G_m \cdot D^{-1}\tilde{R}\|_{L^2_\infty} \\
\lesssim \|D^{-1}\tilde{R}\|_{L^p_\infty} \left( \|D\Delta G_m\|_{L^2_0} + \|\mathbf{K}\|_{L^p_\infty} \|D^{-1}G_m\|_{L^2_0} + r^{-2}\|D^{-1}G_m\|_{L^2_0} \right) \\
\lesssim \|D^{-1}\tilde{R}\|_{L^p_\infty} \left( 2^{\frac{\theta}{2}}r^{-\frac{\theta}{2}} + \|\mathbf{K}\|_{L^p_\infty} 2^{-\frac{3m}{2}}r^\frac{\theta}{2} + r^{-\frac{\theta}{2}}2^{-\frac{3m}{2}} \right) N_1(G).
\]
Hence
\[
I_{nm} \lesssim r^{2}2^{2n}N_1(G) \left( 2^{\frac{\theta}{2}}r^{-\frac{\theta}{2}}\|D^{-1}\tilde{R}\|_{L^p_\infty} + 2^{\frac{\theta}{2}}r^{-\frac{\theta}{2}}\|\tilde{R}\|_{L^p_\infty} (1 + 2^{-\frac{m}{2}}r^\frac{\theta}{2}\|\mathbf{K}\|_{L^p_\infty}) \right)
+ \|\nabla\tilde{R}\|_{L^p_\infty} 2^{-\frac{3m}{2}}r^\frac{\theta}{2} + \|\mathbf{K}\|_{L^p_\infty} \|D^{-1}\tilde{R}\|_{L^p_\infty} 2^{-\frac{3m}{2}}r^\frac{\theta}{2}.
\]
Combined with (8.22), summing over \(k,n,m > 0\) for the cases \(k > n > m\) and \(n > k > m\), we summarize the results as follows
\[
\sum_{k,n,m > 0, k > m, m > n} \|P_k \nabla P_m^2 (D^{-1}\tilde{R} \cdot D^{-1}G_m)\|_{L^2_\infty} \\
\lesssim c_0 N_1(G) r^\frac{\theta}{2} \left( \|D^{-1}\tilde{R}\|_{L^p_\infty} + \|\tilde{R}\|_{L^p_\infty} + r^2\|\nabla\tilde{R}\|_{L^p_\infty} \right)
\]
and \(c_0\) depends on \(\|r^2\mathbf{K}\|_{L^p_\infty} + K_{x0} + \|\mathbf{K}\|_{L^2_0}\).

It remains to estimate \(I_{nm}\) when \(k > m > n\). By (8.6),
\[
I_{nm} \lesssim \|D^{-1}\tilde{R} \cdot D^{-1}G_m\|_{L^2_\infty} \lesssim 2^{-\frac{3m}{2}}r^\frac{\theta}{2} N_1(G) \|D^{-1}\tilde{R}\|_{L^p_\infty}.
\]
Combined with (8.21)
\[
\sum_{k,n,m > 0, k > m, m > n} \|P_k \nabla P_m^2 (D^{-1}\tilde{R} \cdot D^{-1}G_m)\|_{L^2_\infty} \leq c_0 N_1(G) r^\frac{\theta}{2} \|D^{-1}\tilde{R}\|_{L^p_\infty}
\]

Recall the following expression holds symbolically for any \(S\) tangent tensor \(F\), (see [3, page 300]),
\[
[\nabla, \Delta]F = \nabla (K \cdot F) + K \cdot \nabla F.
\]

**Proof of Lemma** (8.4). Assuming (8.12), we first prove (8.13). For simplicity let us set \(H = \nabla D^{-1}H\). We have by [3, Proposition 3.20 (x)],
\[
\|H\|_{L^p_\infty} \lesssim \sum_{k > 0} 2^{k}r^{-1}\|P_k H\|_{L^2_\infty} + r^\theta c_0 (\|\nabla\tilde{H}\|_{L^2_\infty} + \|r^{-1}\tilde{H}\|_{L^2_\infty}),
\]
where \(c_0\) depends on \(\|K\|_{L^2_0}\), and \(\theta > 0\) is very close to 0.

The first term on the right of (8.27) can be bounded in view of (8.12),
\[
\|\tilde{H}\|_{B^\theta_{p,1}_1} \lesssim \|D^\theta H\|_{B^\theta_{p,1}_1} + \|H\|_{L^2_0} + r^\theta c_0 N_1(H).
\]
We then estimate \(\|\nabla\tilde{H}\|_{L^2_\infty}\), by applying (8.11) to \(F = D^{-1}H\). By using Proposition 8.4 and Sobolev, we can obtain
\[
\|\nabla\tilde{H}\|_{L^2_\infty} \lesssim \|D^\theta H\|_{L^2_\infty} + r^{-1}\|H\|_{L^2_\infty} + r^\theta c_0 N_1(H),
\]
with \(c_0\) depending on \(\|K\|_{L^2_0}\).

By combining (8.29), (8.28) and (8.27) and Proposition 3.3, (8.13) follows.
Now consider (8.12). Using GLP projections, we need to prove
\begin{align}
\sum_{k,m>0} 2^{k}r^{-1} \| P_k \nabla P_m D^{-1} H \|_{L^2} & \lesssim \| D H \|_{B^{0}_{2,1}} + c_0 r \frac{1}{2} N_1(H) \\
\sum_{k,m>0} 2^{k}r^{-1} \| P_k \nabla P_{\leq 0} D^{-1} H \|_{L^2} & \lesssim c_0 r \frac{1}{2} N_1(H), \tag{8.31}
\end{align}
where $c_0$ depends on $\| K \|_{L^2} + r \| \nabla K \|_{L^2} + r \| K \|_{L^\infty}$.

The proof of (8.31) is similar to the following Case 1 of the treatment for $T_{km} := 2^{k}r^{-1} \| P_k \nabla P_m D^{-1} H \|_{L^2}$, thus we will give the proof of (8.30) only.

**Case 1:** $k > m$. By (4.2),
\begin{align}
T_{km} & \leq 2^{-k} r \left( \| P_k \nabla \Delta P^2_m D^{-1} H \|_{L^2} + \| P_k [\Delta, \nabla] P^2_m D^{-1} H \|_{L^2} \right) \tag{8.32}
\end{align}
Let us denote the two terms on the right by $T^1_{km}$ and $T^2_{km}$ respectively. In view of (8.10), (4.3) and Lemma 4.3 we have
\begin{align}
T^1_{km} & \lesssim 2^{m-k} \| P_m D H \|_{L^2} + 2^{-k} r \| H \|_{L^2} + 2^{-k} r \| P_k \nabla P_m^2 (K \cdot D^{-1} H) \|_{L^2} \tag{8.33}
\end{align}
We only need to employ (8.21) to estimate the last term of (8.33).
\begin{align}
2^{-k} r \| P_k \nabla P_m^2 (K \cdot D^{-1} H) \|_{L^2} & \lesssim \left( 2^{-3|m-k|} + 2^{-|m-k|} r |K|_{L^2} \right) \| P_m (K D^{-1} H) \|_{L^2} \tag{8.34}
\end{align}
For the last two terms, in view of $k > m$, (SobM2) and Lemma 3.1
\begin{align}
\sum_{k > m} \left( 2^{-|m-k|} r |K|_{L^2} \right) \| P_m (K D^{-1} H) \|_{L^2} & \lesssim \left( r^{1-\alpha} + r |K|_{L^2}^{1-\alpha} \right) \| K \|_{L^2} r^{\frac{1}{2}} N_1(H). \tag{8.35}
\end{align}
Let us decompose $K = \sum_{n} P_n^2 K + \bar{K}$ and consider the high frequency term for the purpose of simplicity. With the help of Proposition 3.3 the proof contained in [11] pages 299–300 implies for $m, n > 0$
\begin{align}
\| P_m (K_n D^{-1} H) \|_{L^2} & \lesssim 2^{-\frac{3}{2} m-n} \| P_n K \|_{L^2} \left( \| D^{-1} H \|_{L^\infty} + \| H \|_{L^2} \right), \tag{8.36}
\end{align}
Therefore the first term on the right of (8.34) can be estimated as follows,
\begin{align}
\sum_{k > m} \sum_{n > 0} 2^{-3|m-k|} \| P_m (K_n D^{-1} H) \|_{L^2} & \lesssim \sum_{k > m} \sum_{n > 0} 2^{-3|m-k|} 2^{-\frac{3}{2} m-n} \| P_n K \|_{L^2} \left( \| D^{-1} H \|_{L^\infty} + \| H \|_{L^2} \right) \\
& \lesssim \| K \|_{B^{0}_{2,1}} r^{\frac{1}{2}} N_1(H)
\end{align}
where we employed (SobM2), Lemma 3.1 and (SobM1) to obtain the last inequality. It is easy to check by (4.4) that $\| K \|_{B^{0}_{2,1}} \lesssim \| K \|_{L^2} + r \| \nabla K \|_{L^2}$. Consequently,
\begin{align}
\sum_{k > m} T^1_{km} & \lesssim \| D H \|_{B^{0}_{2,1}} + c_0 r \frac{1}{2} N_1(H).
\end{align}
Now we consider $I_{k,m}$ with the help of (8.20) and (SobM2), also using (4.3) and Lemma 3.1.

$$
I_{k,m} \lesssim 2^{-k} \left( \| P_k \nabla (K \cdot P_m^2 D^{-1} H) \|_{L^2} + \| P_k (K \cdot \nabla P_m^2 D^{-1} H) \|_{L^2} \right)
$$

$$
\lesssim 2^{-k} \left( \| \nabla K \|_{L^2} \| P_m^2 D^{-1} H \|_{L^2} + \| K \|_{L^\infty} \| \nabla P_m^2 D^{-1} H \|_{L^2} \right)
$$

$$
+ r^{-2} \| P_k (\nabla P_m^2 D^{-1} H) \|_{L^2}
$$

$$
\lesssim 2^{-k} \left( r \| \nabla K \|_{L^2} + r \| K \|_{L^\infty} \right) r^{-1} N_1(H) + r^{-1} \| H \|_{L^2}.
$$

Also using Lemma 3.1 and (SobM1)

$$
\sum_{k, m > 0} I_{2,k,m} \lesssim (r \| \nabla K \|_{L^2} + r \| K \|_{L^\infty}) r^{-1} N_1(H) + r^{-1} \| H \|_{L^2}.
$$

Case 2: $k < m$. Consider $I_{k,m}$ in this case by (4.2) and (4.1).

$$
I_{k,m} \leq 2^{k-2m} \| P_k \nabla \Delta P_m^2 D^{-1} H \|_{L^2}
$$

$$
\leq 2^{k-2m} \left( \| P_k \nabla \Delta P_m^2 D^{-1} H \|_{L^2} + \| P_k \nabla \Delta P_m^2 D^{-1} H \|_{L^2} \right)
$$

$$
\leq 2^{k-2m} \left( \| P_m^2 \Delta H \|_{L^2} + \| P_m^2 (K \Delta H) \|_{L^2} + r^{-2} \| P_m^2 D^{-1} H \|_{L^2} \right)
$$

Let us denote by $I_{1,k,m}$ the first term in the line of (8.37) and by $I_{2,k,m}$ the second term. Consider $I_{1,k,m}$ first. By (4.3), (4.2) and (8.10)

$$
I_{1,k,m} \lesssim 2^{k-4m} \| P_m^2 D^{-1} H \|_{L^2}
$$

$$
\lesssim 2^{k-4m} \left( \| \nabla K \|_{L^2} \| D^{-1} H \|_{L^2} + \| K \|_{L^\infty} \| H \|_{L^2} \right)
$$

By (4.3), Proposition 3.1, (SobM2) and Lemma 3.1

$$
\| P_m^2 (K \Delta H) \|_{L^2} \lesssim 2^{-m} (r \| \nabla K \|_{L^2} \| D^{-1} H \|_{L^2} + \| K \|_{L^\infty} \| H \|_{L^2})
$$

$$
\lesssim 2^{-m} r \| \nabla K \|_{L^2} + \| K \|_{L^\infty} \| H \|_{L^2}.
$$

Also using Lemma 4.3

$$
I_{1,k,m} \lesssim 2^{k-4m} \left( \| \nabla P_m^2 \Delta H \|_{L^2} + r^{-1} \| H \|_{L^2} + \| \nabla K \|_{L^2} + \| K \|_{L^\infty} \right).
$$

Hence, we obtain

$$
\sum_{k, m > 0, k < m} I_{1,k,m} \lesssim \| \nabla^2 H \|_{L^2} + r^{-1} \| H \|_{L^2} + \| \nabla K \|_{L^2} + \| K \|_{L^\infty} \right).
$$

Now consider $I_{2,k,m}$ in view of (8.20).

$$
I_{2,k,m} \lesssim 2^{k-2m} \| P_k \nabla (K \cdot P_m^2 D^{-1} H) \|_{L^2} + \| P_k (K \cdot \nabla P_m^2 D^{-1} H) \|_{L^2}
$$

$$
+ r^{-2} \| P_k \nabla P_m^2 D^{-1} H \|_{L^2}
$$

By (4.3) and Lemma 4.3

$$
I_{2,k,m} \lesssim 2^{k-2m} \left( \| K \|_{L^2} \| H \|_{L^2} + 2^{k-2m} \| H \|_{L^2} \right).
$$

Also using (SobM1),

$$
\sum_{k, m > 0, m > k} I_{2,k,m} \lesssim r^{-1} \| H \|_{L^2} + \| \nabla K \|_{L^\infty} \cdot r \| K \|_{L^\infty}.
$$
Thus
\[ \sum_{k,m>0,m>k} \mathcal{I}_{km} \lesssim r^{-1} \|H\|_{L_x^2} + c_0 r^\frac{1}{2} \mathcal{N}_1(H) \]
where \( c_0 \) depends on \( r(\|K\|_{L_x^\infty} + \|\nabla K\|_{L_x^2}) \).

\[ \square \]

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Department of Mathematics, Stony Brook University, Stony Brook, NY 11794

E-mail address: qwang@math.sunysb.edu