THE CONFORMAL GROUP SU(2,2)
AND INTEGRABLE SYSTEMS
ON A LORENTZIAN HYPERBOLOID

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Abstract. Eleven different types of “maximally superintegrable” Hamiltonian systems on the real hyperboloid $(s^0)^2 - (s^1)^2 + (s^2)^2 - (s^3)^2 = 1$ are obtained. All of them correspond to a free Hamiltonian system on the homogeneous space $SU(2, 2)/U(2, 1)$, but to reductions by different maximal abelian subgroups of $SU(2, 2)$. Each of the obtained systems allows 5 functionally independent integrals of motion, from which it is possible to form two or more triplets in involution (each of them includes the hamiltonian). The corresponding classical and quantum equations of motion can be solved by separation of variables on the $O(2, 2)$ space.
1.- INTRODUCTION.

In a recent article [1] we constructed a family of completely integrable finite-dimensional Hamiltonian systems on a real $O(p, q)$ hyperboloid

$$g_{\mu\nu} s^\mu s^\nu = 1, \quad g = g^T \in \mathbb{R}^{(p+q) \times (p+q)},$$  \hfill (1.1)

$$\text{sign}(g) = (p, q), \quad p \geq q \geq 0, \quad s \in \mathbb{R}^{p+q}.$$  

The classical Hamiltonian had the form

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu + V(s),$$ \hfill (1.2a)

where $p_\mu$ are the momenta canonically conjugate to the coordinates $s^\mu$ and the corresponding quantum mechanical one is

$$H = -\frac{1}{2} \sum_{\mu, \nu=1}^{n} \frac{1}{\sqrt{g_0}} \frac{\partial}{\partial s^\mu} \sqrt{g_0} g^{\mu\nu} \frac{\partial}{\partial s^\nu} + V(s),$$ \hfill (1.2b)

with $g_0 = \det g$. The momenta $p_\mu$ satisfy

$$p_\mu s^\mu = 0.$$ \hfill (1.3)

The potential $V(s)$ was obtained by projecting free motion on a projective complex hyperboloid

$$\mathcal{P}H^{n+1} \sim SU(p, q)/U(p-1, q), \quad p + q = n + 1,$$ \hfill (1.4)

onto the real $O(p, q)$ hyperboloid.

The starting point is thus a free Hamiltonian

$$H_F = \frac{1}{2} g^{\mu\nu} \bar{p}_\mu p_\nu,$$ \hfill (1.5)

where $p_\mu$ are the complex momenta canonically conjugate to the complex coordinates $y^\mu$, satisfying

$$g_{\mu\nu} \bar{y}^\mu y^\nu = 1.$$ \hfill (1.6)

The projection was performed by introducing $n$ ignorable variables on $\mathcal{P}H^{n+1}$, corresponding to the diagonalization of all elements of a Cartan subalgebra of $u(p, q)$. We recall that a variable, in a certain coordinate system, is called ignorable if the corresponding metric tensor $g$ does not depend on it. Since $u(p, q)$ (for $p \geq q$) has $q + 1$ different Cartan subalgebras $M_r$, $r = 0, 1, ..., q$, we obtained $q + 1$ different integrable Hamiltonian systems.
In addition to Cartan subalgebras, that are maximal abelian and selfnormalizing subalgebras [2], the \( su(p, q) \) algebras have maximal abelian subalgebras (MASAs) that contain nilpotent elements. All MASAs of \( su(p, q) \) have recently been classified [3] and in particular \( su(2, 2) \) has 12 \( SU(2, 2) \)-conjugacy of MASAs, 3 of them Cartan subalgebras.

The aim of this article is to show that non Cartan MASAs can also be used to introduce ignorable variables and to generate integrable systems of the type (1.2).

We shall restrict our study to the algebra \( su(2, 2) \) and to the homogeneous (and symmetric) space \( SU(2, 2)/U(2, 1) \). This is actually a case of particular physical interest, since \( SU(2, 2) \) is locally isomorphic to the conformal group \( C(3, 1) \) of Minkowski spacetime. We are thus studying conformally invariant integrable systems on the space (1.4). The integrability of the systems with Hamiltonian (1.2) is a consequence of the original conformal invariance of the free Hamiltonian system on \( PH^4 \).

The motivation for generating new integrable systems was discussed in our previous article [1]. We mention here that the Hamiltonian systems obtained in this article are not just integrable, but rather “maximally superintegrable” [4,5]. This means that instead of allowing \( n \) integrals of motion in involution (for \( n \) degrees of freedom), such systems allow \( 2n - 1 \) functionally independent integrals of motion that are well defined functions on phase space. Amongst these, different sets of \( n \) integrals in involution can be chosen. Such superintegrable systems are rare. They include the Coulomb system, the harmonic oscillator and a few others [5, . . . ,12]. The classical trajectories in such systems are periodic, if they are finite. The energy levels of the corresponding quantum systems are degenerate. The integrals of motion generate a Lie algebra, the representation theory of which explains the “accidental” degeneracy [5, . . . ,12]. The underlying group structure makes it possible to solve the equations of motion analytically: a very useful feature in applications.

Completely integrable potentials such as the Pöschl-Teller potential [13], have found interesting applications in molecular, atomic, nuclear and particle physics [14, . . . ,17].

2.- THE GROUP \( SU(2,2) \) AND ITS LIE ALGEBRA.

A short review of some pertinent information on the group \( SU(2, 2) \), its Lie algebra \( su(2, 2) \) and the homogeneous space \( PH^4 \) will be presented in this section. Most of its contents are classical results on complex geometry and Lie theory.

2.1.- The \( su(2,2) \) Lie Algebra.

We shall realize the Lie algebra \( su(2, 2) \) and group \( SU(2, 2) \) by matrices \( X \) and \( G \) satisfying

\[
\begin{align*}
XK + KX^\dagger &= 0, \\
\text{Tr}X &= 0, \\
K, X, G &\in \mathbb{C}^{4\times 4}
\end{align*}
\]

\[
GKG^\dagger = K, \\
\det G &= 1, \\
K &= K^\dagger,
\]

(2.1)
where $K$ is a hermitian matrix of signature $(++-)$.

A convenient basis for $su(2,2)$ is provided by 15 matrices $X_k, k = 1, \ldots, 15$. Their specific form depends on the choice of the matrix $K$.

We shall actually need 6 different bases, corresponding to 6 realizations of $su(2,2)$ (with different matrices $K$). The transformation from one realization to another is given by

\[
gKg^+ = K', \quad gXg^{-1} = X', \quad g \in SL(4, \mathbb{C}),
\]

(2.2)

\[
X(1) = \begin{bmatrix}
iw_1 & w_4 + iw_5 & w_6 + iw_7 & w_8 + iw_9 \\
w_4 - iw_5 & -iw_1 + iw_2 & w_{10} + iw_{11} & w_{12} + iw_{13} \\
-w_6 + iw_7 & w_{10} - iw_{11} & -iw_2 + iw_3 & w_{14} + iw_{15} \\
w_8 - iw_9 & -w_{12} + iw_{13} & w_{14} - iw_{15} & -iw_3
\end{bmatrix},
\]

$K_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \end{bmatrix}$.

(2.3a)

\[
X(2) = \begin{bmatrix}
iw_1 + iw_2 & w_4 + iw_5 & w_6 + iw_7 & w_8 + iw_9 \\
w_4 - iw_5 & -iw_1 + iw_2 & w_{10} + iw_{11} & w_{12} + iw_{13} \\
-w_8 + iw_9 & w_{12} - iw_{13} & -iw_2 + w_3 & iw_{14} \\
-w_6 + iw_7 & w_{10} - iw_{11} & iw_{15} & -iw_2 - w_3
\end{bmatrix},
\]

$K_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{bmatrix}$.

(2.3b)

\[
X(3) = \begin{bmatrix}
w_1 + iw_3 & iw_4 & w_6 + iw_7 & w_8 + iw_9 \\
-w_1 + iw_3 & -w_4 + iw_7 & w_{10} + iw_{11} & w_{12} + iw_{13} \\
-w_{12} + iw_{13} & -w_8 + iw_9 & -w_2 - iw_3 & iw_{14} \\
-w_{10} + iw_{11} & -w_6 + iw_7 & iw_{15} & -w_2 - iw_3
\end{bmatrix},
\]

$K_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{bmatrix}$.

(2.3c)

\[
X(4) = \begin{bmatrix}
w_1 + iw_2 & w_4 + iw_5 & iw_7 & w_{10} + iw_{11} \\
w_8 + iw_9 & w_6 - iw_2 + iw_3 & -w_6 + iw_7 & w_{12} + iw_{13} \\
iw_5 & -w_8 + iw_9 & -w_1 + iw_2 & w_{14} + iw_{15} \\
w_{14} - iw_{15} & w_{12} - iw_{13} & w_{10} - iw_{11} & -iw_2 - iw_3
\end{bmatrix},
\]

$K_4 = \varepsilon = \begin{bmatrix} 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 \end{bmatrix}$.

(2.3d)

\[
X(5) = \begin{bmatrix}
w_1 + iw_2 & w_4 + iw_5 & iw_8 & w_{10} + iw_{11} \\
w_6 + iw_7 & w_3 - iw_2 & -w_{10} + iw_11 & iw_9 \\
-iw_{12} & w_{14} + iw_{15} & -w_1 - iw_2 & -w_6 + iw_7 \\
-w_{14} + iw_{15} & iw_{13} & -w_4 + iw_5 & -w_3 - iw_2
\end{bmatrix},
\]

$K_5 = \begin{bmatrix} 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \end{bmatrix}$.

(2.3e)

\[
X(6) = \begin{bmatrix}
w_1 + iw_2 & w_4 + iw_5 & w_6 + iw_7 & iw_{12} \\
w_8 + iw_9 & w_3 - iw_2 & iw_{13} & -w_6 + iw_7 \\
w_{10} + iw_{11} & iw_{14} & -w_3 - iw_2 & -w_4 + iw_5 \\
-iw_{15} & -w_{10} + iw_{11} & -w_8 + iw_9 & -w_1 + iw_2
\end{bmatrix},
\]

$K_6 = \begin{bmatrix} 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \end{bmatrix}$.

(2.3f)

where $\varepsilon = \pm 1$.

We denote the matrix elements of $X(a), (a = 1, \ldots, 6)$, $w_i (i = 1, \ldots, 15)$ in all cases, though a notation $\omega_{(a)i}$ would be more consistent. Notice that a basis element $X_i$, obtained by setting $\omega_i = 1, \omega_j = 0, j \neq i$, in one realization is not necessarily conjugate to $X_i$ in another realization. For instance $X_3$ in the realization of $K_1$ generates a
rotation (compact), in the realization of $K_2$ the element $X_3$ corresponds to a hyperbolic rotation (non compact). Similarly, $X_8$ in the case of $K_1$ corresponds to a hyperbolic rotation, whereas $X_8$ in the case of $K_5$ is nilpotent.

The form of the second order Casimir operator $C_2$ of $su(2,2)$ is basis dependent. For instance in the first case, with $K_1$ adapted to the Cartan subalgebras, we have

$$C = 3X_1^2 + 4X_2^2 + 3X_3^2 + 2\{X_1, X_2\} + 2\{X_2, X_3\} + \{X_1, X_3\} + 2(-X_4^2 - X_5^2 + X_6^2 + X_7^2 - X_8^2 - X_9^2 - X_{10}^2 - X_{11}^2 + X_{12}^2 + X_{13}^2 - X_{14}^2 - X_{15}^2).$$

(2.4)

2.2.- The Homogeneous Space $SU(2,2)/U(1,2)$.

In Ref.1 we have constructed the space $SU(p, q)/U(p-1, q)$ in general, so we shall not present any details here. The group $SU(2,2)$ acts transitively on the complex hyperboloid $H^4 \sim SU(2,2)/SU(1,2)$,

$$y^+K y = 1.$$

(2.5)

If we identify any two points $y$ and $y'$ on $H^4$, satisfying

$$y' = ye^{i\phi}, \quad 0 \leq \phi < 2\pi,$$

(2.6)

we obtain the projective hyperboloid

$$P \mathcal{H}^4 \sim SU(2,2)/U(1,2).$$

(2.7)

Its real dimension is 6. The identification (2.6) can be realized by introducing affine coordinates

$$z^j = \frac{y^j}{y^0}, \quad j = 1, 2, 3.$$

(2.8)

The flat Hermitian metric

$$h(y, y') = g_{\mu \nu}y^\mu \bar{y}^\nu$$

then reduces to a noncompact version [1] of the Fubini-Study metric [18].

Real vector fields in the tangent space $T\mathcal{H}^4$ to the hyperboloid $\mathcal{H}^4$, realizing the Lie algebra $u(2,2)$, acting on differentiable functions on $\mathcal{H}^4$, are given by

$$\hat{Z} = -y^\mu (Z)_\mu^\nu \partial y_\nu + \text{c.c.}, \quad Z \in u(2,2),$$

(2.10)

where c.c. denotes complex conjugation. We include the $u(1)$ basis element for which the vector field is

$$\hat{Y}_0 = -i(y^0 \partial y^0 + y^1 \partial y^1 + y^2 \partial y^2 + y^3 \partial y^3).$$

(2.11)
Functions $F(y)$ on $\mathcal{H}^4$ that project properly onto $P\mathcal{H}^4$ must be homogeneous, i.e. satisfy
\[ F(y^0, y^1, y^2, y^3) = F(1, y^1/y^0, y^2/y^0, y^3/y^0). \tag{2.12} \]
On these functions we have
\[ \dot{Y}_0 F = 0, \tag{2.13} \]
and for the corresponding constant of motion we get
\[ y^0 p_0 + y^1 p_1 + y^2 p_2 + y^3 p_3 = 0. \]

2.3.- Maximal Abelian Subalgebras of $su(2,2)$.

Below in Section 3 we shall separate variables in the Hamilton–Jacobi, or the Laplace–Beltrami equation on the homogeneous space $P\mathcal{H}^4 \sim SU(2,2)/U(1,2)$. To do this we need to construct complete sets of commuting first and second order operators in the enveloping algebra of $su(2,2)$. The coordinate systems relevant for us will involve three ignorable variables, i.e. coordinates that do not occur in the corresponding metric tensor [19]. Such coordinates are associated with MASAs of the Lie algebra of the isometry group of the considered space [20], in our case $SU(2,2)$. The ignorable coordinates are obtained by simultaneously diagonalizing all elements of a MASA. This is possible, if the vector fields representing the action of the MASA on differentiable functions over the considered space, are linearly independent at generic points of the space.

The MASAs of $su(2,2)$ were recently classified as an application of a general study of MASAs of $su(p,q)$ for $p \geq q \geq 1$ [3]. We reproduce the results in Table 1. There are $12\, SU(2,2)$ conjugacy classes of MASAs of $su(2,2)$. Their type is given in column 2. The first 7 MASAs are orthogonally decomposable and we give the decomposition pattern. The first 3 of them are Cartan subalgebras, containing maximal compact subalgebras of dimension $3$, $2$ and $1$, respectively. The MASA $M_9$ is decomposable, but not orthogonally. The MASAs $M_{10}$, $M_{11}$ and $M_{12}$ are indecomposable and they are maximal abelian nilpotent algebras (the matrices representing them are nilpotent). In Column 3 we give a basis for each MASA. The corresponding $su(2,2)$ matrices can be read off from one of equations (2.3). Which one is to be used is indicated by the form of the metric given in Column 4.

The choice of representative MASAs in Table 1 is not the same as in Ref. 3. The reason is that it is convenient, as will become clear below, to choose representatives that are pure imaginary matrices $X = -\bar{X}$. That is possible for $M_1, \ldots, M_{11}$, not however for $M_{12}$.

In all cases, if we add the basis element $iI$ to $M_a$, $1 \leq a \leq 12$, we obtain a MASA of $u(2,2)$.
3. IGNORABLE COORDINATES ON $PH^{n+1}$ AND THE REDUCTION TO AN $O(p,q)$ HYPERBOLOID.

While this article is essentially devoted to the six-dimensional space $SU(2,2)/U(1,2)$ we shall present results in this section for arbitrary $p$ and $q$ satisfying $p \geq q \geq 0$, $p+q = n+1$.

3.1.- The Ignorable Variables.

In Ref.1 we developed a general method for introducing coordinates on the space $P^H^{n+1}$ containing $n+1$ ignorable variables, corresponding to the elements of a Cartan subalgebra. Here we shall generalize these results to a larger class of abelian subalgebras of $u(p,q)$.

**Theorem 1.** Let $M$ be an abelian subalgebra of $u(p,q)$, $p \geq q \geq 0$, $p+q = n+1$, satisfying the following conditions:

1. $\dim M = p+q = n+1$.
2. The $n+1$ vectors fields
$$\hat{Y}_a = -y^\mu(Y_a)_\mu \partial y^\nu, \quad a = 0, 1, \ldots, n, \quad (3.1)$$

(where $Y_a$ are the $u(p,q)$ matrices representing the abelian algebra $M$) are linearly independent at a generic point of $PH^{n+1}$.
3. All elements $Y_a$ are represented by pure imaginary matrices in the considered realization in $\mathbb{C}^{(p+q) \times (p+q)}$.

Then the vector fields $\hat{Y}_a + \bar{\hat{Y}}_a$, $a = 0, 1, \ldots, n$ representing the algebra $M$ in the tangent space $T^H^{n+1}$ can be simultaneously straightened out to

$$\hat{Y}_\mu + \bar{\hat{Y}}_\mu = -\partial x^\mu, \quad \mu = 0, 1, \ldots, n. \quad (3.2)$$

This is achieved by a transformation of coordinates $(y^\mu, \bar{y}^\mu) \rightarrow (s^\mu, x^\mu)$, where $s^\mu$ and $x^\mu$ are real and satisfy

$$y^\mu = B(x)^\mu_\nu s^\nu, \quad B(x) = \exp(x^\rho Y_\rho). \quad (3.3)$$

The matrices $Y_\rho$ span the considered MASA, the variables $x^\rho$ are the ignorable ones, while $s$ satisfies

$$s^T K s = 1, \quad (3.4)$$

when $y$ satisfies eq.(2.5). The vector fields $\hat{Z}$ of eq.(2.10) representing the algebra $u(p,q)$, reduce to

$$\hat{Z} = -\frac{1}{2} B^\mu_\nu s^\nu (Z)_\mu^\alpha [(A^{-1})^\beta_\alpha \partial x^\beta + (B^{-1})^\beta_\alpha \partial s^\beta], \quad (3.5)$$
with
\[ A^\mu_\nu = \frac{\partial y^\mu}{\partial x^\nu} = (Y_\nu)_\mu^\rho y^\rho. \] (3.6)

Proof. The proof coincides with the proof of Theorem 1 of Ref.1, though the Theorem presented in this paper is more general. We shall not repeat the proof here. Useful elements of it are that the Jacobian of the transformation (3.3) and its inverse are

\[ j = \frac{\partial (y, \bar{y})}{\partial (x, s)} = \begin{pmatrix} A & B \\ \bar{A} & \bar{B} \end{pmatrix}, \quad j^{-1} = \frac{1}{2} \begin{pmatrix} A^{-1} & \bar{A}^{-1} \\ B^{-1} & \bar{B}^{-1} \end{pmatrix}, \] (3.7)

with \( A \) as in (3.6) and \( B \) as in (3.3). We hence have

\[ \frac{\partial}{\partial y^\mu} = \frac{1}{2} \left[ (A^{-1})^\kappa_\mu \frac{\partial}{\partial x^\nu} + (B^{-1})^\kappa_\mu \frac{\partial}{\partial s^\nu} \right]. \] (3.8)

3.2.- Reduction of the Elements of the Lie Algebra.

Let us consider a basis for \( su(p, q) \) in which all basis elements are either real, or pure imaginary (all bases in (2.3) satisfy this requirement). For such a basis eq.(3.5) simplifies. Indeed, let \( X_{Re} \) and \( X_{Im} \) denote real and imaginary \( su(p, q) \) matrices, respectively. The corresponding vector fields are expressed in terms of the coordinates \((s, x)\) and conjugate momenta \((p_s, p_x)\) as

\[ \hat{X}_{Re} = -p_s^T (X_{Re}) s, \] \[ \hat{X}_{Im} = -p_x^T A^{-1} (X_{Im}) s. \] (3.9)

In order to restrict to the real hyperboloid of eq.(3.4), we set \( x = 0 \) and \( p_x \) = constant. The vector fields \( \hat{X}_{Re} \) then generate the algebra \( o(p, q) \) and lie in the tangent space to the \( O(p, q) \) hyperboloid. Their form depends only on that of the matrices \( X_{Re} \), i.e. of the chosen metric. The vector fields \( \hat{X}_{Im} \), on the other hand, reduce to functions of \( s \), depending on some constants \( k = (k_0, k_1, \ldots, k_n) \). Their form depends on \( A \) and is hence different for each MASA.

3.3.- Reduction of the Hamiltonian.

The free Hamiltonian on the space \( PH^4 \), corresponding to geodesic motion on this space, is

\[ H = \frac{c}{2} g^{\mu\nu} p_\mu p_\nu, \] (3.11)

where \( p_\mu \) are the complex momenta canonically conjugate to \( y^\mu \). In the coordinates \((s, x)\) the Hamiltonian reduces to

\[ H = \frac{c}{8} [ p_s^T g p_s + p_x^T (A^T g A)^{-1} p_x ] . \] (3.12)
Restricting to the $O(p, q)$ hyperboloid we obtain the Hamiltonian (1.2) (for $c = 4$) with the potential

$$V(s) = \frac{c}{8} p_x^T (A^T g A)^{-1} p_x,$$

where $p_x \in \mathbb{R}^{n+1}$ is a constant vector.

Thus the “kinetic energy” part in eq.(3.12) is always the same, but the potential $V(s)$ depends on the matrix $A$ and is hence different for each specific abelian subalgebra of $su(p, q)$ (satisfying the conditions of Theorem 1).

### 4.- THE COORDINATE SYSTEMS AND REDUCED HAMILTONIANS FOR SU(2,2).

We now return to the case $p = q = 2$ and make the results of Section 3 concrete for each of the MASAs of Table 1. The MASAs $M_1, M_2, \ldots, M_{11}$ all satisfy the conditions of Theorem 1, however the four-dimensional MANS $M_{12}$ does not. Indeed the corresponding vector fields of eq.(3.1) for it are

$$\hat{Y}_1 \equiv \hat{X}_6 = -(y^2 \partial_{y^0} - y^3 \partial_{y^1}), \quad \hat{Y}_2 \equiv \hat{X}_7 = -i(y^2 \partial_{y^0} + y^3 \partial_{y^1}),$$

$$\hat{Y}_3 \equiv \hat{X}_{12} = -iy^3 \partial_{y^0}, \quad \hat{Y}_4 \equiv \hat{X}_{13} = -iy^2 \partial_{y^1},$$

$$\hat{Y}_0 \equiv \hat{X}_0 = -i(y^0 \partial_{y^0} + y^1 \partial_{y^1} + y^2 \partial_{y^2} + y^3 \partial_{y^3}).$$

They satisfy

$$y^2 y^3 \hat{Y}_1 + i(y^2)^2 \hat{Y}_3 - i(y^3)^2 \hat{Y}_4 = 0,$$

$$y^2 y^3 \hat{Y}_2 - (y^2)^2 \hat{Y}_3 - (y^3)^2 \hat{Y}_4 = 0,$$

so only 3 of them are linearly independent at a generic point of $\mathcal{H}^4$ (and we need 4).

We shall now run through the other MASAs, $M_1, M_2, \ldots, M_{11}$ and in each case specialize eq.(3.3) and (3.13) to obtain the coordinates on the complex hyperboloid of eq.(2.5) and the real potential, figuring in eq.(1.2). The potential $V(s)$ will in each case be given in coordinates adapted to the specific metric $K_a, (a = 1, \ldots, 6)$ that we use. Note that using formula (3.3), the expression of the coordinates would be $y^\mu = (\exp(c^\rho Y_\rho)_{\nu}) s^\nu$, where $c^\mu$ are the ignorable variables. However, in order to simplify notation we will write $x^\mu$ for the ignorable variables, which will be either the same as $c^\mu$ or linear combinations of them (this last case can be seen as a basis change for the MASA). The momenta $p_x$ conjugate to $x^\mu$ or linear combinations of them will be denoted by $k$. 
1.- The Compact Cartan Subalgebra $M_1$; Metric $K_1$

Coordinates:
\[ y^\mu = e^{ix^\mu} s^\mu, \quad \mu = 0, 1, 2, 3. \] (4.3)

Hamiltonian:
\[
H_1 = \frac{1}{2} \left( p_0^2 - p_1^2 + p_2^2 - p_3^2 + \frac{k_0^2}{(s^0)^2} - \frac{k_1^2}{(s^1)^2} + \frac{k_2^2}{(s^2)^2} - \frac{k_3^2}{(s^3)^2} \right). \] (4.4)

The potential is singular along each of the surfaces $s^\mu = 0$, i.e. on two–sheeted two–dimensional hyperboloids for $s^0 = 0$, or $s^2 = 0$ and one–sheeted hyperboloids for $s^1 = 0$, or $s^3 = 0$.

2.- The Noncompact Cartan Subalgebra $M_2$; Metric $K_1$.

Coordinates:
\[
\begin{align*}
y^0 &= e^{ix^0} s^0, \\
y^1 &= e^{ix^1} s^1, \\
y^2 &= e^{ix^2}(s^2 \cosh x^3 + is^3 \sinh x^3), \\
y^3 &= e^{ix^3}(-is^2 \sinh x^3 + s^3 \cosh x^3).
\end{align*}
\] (4.5)

Hamiltonian:
\[
H_2 = \frac{1}{2} \left( p_0^2 - p_1^2 + p_2^2 - p_3^2 + \frac{k_0^2}{(s^0)^2} - \frac{k_1^2}{(s^1)^2} + \frac{((s^2)^2 - (s^3)^2)(k_2 - k_3)^2 + 4s^2 s^3 k_2 k_3}{((s^2)^2 + (s^3)^2)^2} \right). \] (4.6)

The potential $V_2(s)$ is singular along the hyperboloids $s^0 = 0$ and $s^1 = 0$ and also along the hyperbola $s^2 = s^3 = 0$, $(s^0)^2 - (s^1)^2 = 1$.

3.- The Noncompact Cartan Subalgebra $M_3$; Metric $K_1$.

Coordinates:
\[
\begin{align*}
y^0 &= e^{ix^0} (s^0 \cosh x^1 + is^1 \sinh x^1), \\
y^1 &= e^{ix^0}(-is^0 \sinh x^1 + s^1 \cosh x^1), \\
y^2 &= e^{ix^2}(s^2 \cosh x^3 + is^3 \sinh x^3), \\
y^3 &= e^{ix^3}(-is^2 \sinh x^3 + s^3 \cosh x^3).
\end{align*}
\] (4.7)

Hamiltonian:
\[
\begin{align*}
H_3 &= \frac{1}{2} \left( p_0^2 - p_1^2 + p_2^2 - p_3^2 + \frac{1}{((s^0)^2 + (s^1)^2)^2} \right) \left[ (s^0)^2 - (s^1)^2 \right] (k_2 - k_3)^2 + 4s^0 s^1 k_0 k_1 \\
&\quad + \frac{1}{((s^2)^2 + (s^3)^2)^2} \left[ (s^2)^2 - (s^3)^2 \right] (k_2^2 - k_3^2) + 4s^2 s^3 k_2 k_3 \right]. \] (4.8)

The potential $V_3(s)$ is singular along two hyperbolas, $s^0 = s^1 = 0$ and $s^2 = s^3 = 0$. 
4.- The Orthogonally Decomposable MASA $M_4$; Metric $K_2$.

Coordinates:

$$y^0 = e^{ix^0} s^0, \quad y^1 = e^{ix^1} s^1, \quad y^2 = e^{ix^2} (s^2 + ix^3 s^3), \quad y^3 = e^{ix^2} s^3. \quad (4.9)$$

Hamiltonian:

$$H_4 = \frac{1}{2} \left( p_0^2 - p_1^2 + 2p_2p_3 + \frac{k_0^2}{(s^0)^2} - \frac{k_1^2}{(s^1)^2} + 2k_3(k_2s^3 - k_3s^2) \right). \quad (4.10)$$

The singularity surfaces $s^0 = 0$ and $s^1 = 0$ are 2–dimensional hyperboloids; $s^3 = 0$ is a 2–dimensional hyperbolic cylinder $(s^0)^2 - (s^1)^2 = 1$, $s^3 = 0$, $s^2 \in \mathbb{R}$.

5.- The Orthogonally Decomposable MASA $M_5$; Metric $K_2$.

Coordinates:

$$y^0 = e^{ix^0} (s^0 \cosh x^1 + is^1 \sinh x^1), \quad y^1 = e^{ix^0} (-is^0 \sinh x^1 + s^1 \cosh x^1),$$

$$y^2 = e^{ix^2} (s^2 + ix^3 s^3), \quad y^3 = e^{ix^2} s^3. \quad (4.11)$$

Hamiltonian:

$$H_5 = \frac{1}{2} \left( p_0^2 - p_1^2 + 2p_2p_3 + \frac{((s^0)^2 - (s^1)^2)(k_0^2 - k_1^2) + 4s^0 s^1 k_0 k_1}{((s^0)^2 + (s^1)^2)^2} + 2k_3(k_2s^3 - k_3s^2) \right). \quad (4.12)$$

The potential $V_5(s)$ is singular on a one–dimensional hyperbola $s^0 = s^1 = 0$ and on a 2–dimensional hyperbolic cylinder $s^3 = 0$, $(s^0)^2 - (s^1)^2 = 1$, $s^2 \in \mathbb{R}$.

6.- The OD MASA $M_6$; Metric $K_3$.

Coordinates:

$$y^0 = e^{ix^0} (s^0 + ix^1 s^1), \quad y^1 = e^{ix^0} s^1,$$

$$y^2 = e^{ix^2} (s^2 + ix^3 s^3), \quad y^3 = e^{ix^2} s^3. \quad (4.13)$$

Hamiltonian:

$$H_6 = \left( p_0p_1 + p_2p_3 + \frac{k_1(k_0 s^1 - k_1 s^0)}{(s^1)^3} + \frac{k_3(k_2 s^3 - k_3 s^2)}{(s^3)^3} \right). \quad (4.14)$$

The potential is singular along the two hyperbolic cylinders $s^1 = 0$ and $s^3 = 0$. 

7.- The OD MASAs $M_7$ and $M_8$; Metric $K_{4,\epsilon}$.

Coordinates:
\[
y^0 = e^{i\alpha_0}(s^0 + ix^1s^1 + (ix^2 - \frac{1}{2}(x^1)^2)s^2), \quad y^1 = e^{i\alpha_0}(s^1 + ix^1s^2),
\]
\[
y^2 = e^{i\alpha_0}s^2, \quad y^3 = e^{i\alpha_0}s^3.
\]

Hamiltonian:
\[
H_{7,8} = \frac{\epsilon}{2} \left( 2p_0p_2 + p_1^2 - p_3^2 + \frac{k_1^2 + 2k_0k_2}{(s^2)^2} - \frac{4k_1k_2s^1}{(s^2)^3} + \frac{k_2^2(s^1)^2 - 2s^0s^2}{(s^2)^4} - \frac{k_3^2}{(s^3)^2} \right).
\]

The potential is singular along the hyperboloid $s^3 = 0$ and the hyperbolic cylinder $s^2 = 0$.

8.- The OID and D MASA $M_9$; metric $K_5$.

Coordinates:
\[
y^0 = e^{i\alpha_0}([s^0 + i(s^2x^2 + s^3x^3)]\cosh x^1 + i[s^1 + i(s^2x^3 - s^3x^2)]\sinh x^1),
\]
\[
y^1 = e^{i\alpha_0}(-i[s^0 + i(s^2x^2 + s^3x^3)]\sinh x^1 + i[s^1 + i(s^2x^3 - s^3x^2)]\cosh x^1),
\]
\[
y^2 = e^{i\alpha_0}(s^2 \cosh x^1 - is^3 \sinh x^1), \quad y^3 = e^{i\alpha_0}(is^2 \sinh x^1 + s^3 \cosh x^1).
\]

Hamiltonian:
\[
H_9 = (p_0p_2 + p_1p_3) + \frac{(k_0k_2 + k_1k_3)((s^2)^2 - (s^3)^2) + 2(k_0k_3 - k_1k_2)s^2s^3}{((s^2)^2 + (s^3)^2)^2}
\]
\[
+ \frac{1}{((s^2)^2 + (s^3)^2)^2}\left\{((k_2^2 - k_3^2)[s^1 s^3(3(s^2)^2 - (s^3)^2) - s^0 s^2((s^2)^2 - 3(s^3)^2)]
\right.
\]
\[
- 2k_2k_3[s^1 s^2((s^2)^2 - 3(s^3)^2) + s^0 s^3(3(s^2)^2 - (s^3)^2)]\right\}.
\]

This Hamiltonian is actually nonsingular, since the point $s^2 = s^3 = 0$ does not lie on the hyperboloid $2(s^0s^2 + s^1s^3) = 1$.

9.- The MANS $M_{10}$; metric $K_6$.

Coordinates:
\[
y^0 = e^{i\alpha_0}(s^0 + is^1x^1 + is^2x^2 + (ix^3 - x^1x^2)s^3), \quad y^1 = e^{i\alpha_0}(s^1 + ix^1s^3),
\]
\[
y^2 = e^{i\alpha_0}(s^2 + ix^1s^3), \quad y^3 = e^{i\alpha_0}s^3.
\]
Hamiltonian:

\[ H_{10} = (p_0 p_3 + p_1 p_2) + \frac{k_0 k_3 + k_1 k_2}{(s^3)^2} - \frac{2k_3}{(s^3)^3} (k_1 s^2 + k_2 s^1) + \frac{k_3^2}{(s^3)^4} (-s^0 s^3 + 3 s^1 s^2). \] (4.20)

The singularity surface of the potential is the hyperbolic cylinder \( s^3 = 0 \), \( 2 s^1 s^2 = 1 \), \( s^0 \in \mathbb{R} \).

10.- The MANS \( M_{11} \); Metric \( K_6 \).

Coordinates:

\[
\begin{align*}
y^0 &= e^{ix_0} (s^0 + is^1 x^1 + (ix^2 - \frac{(x^1)^2}{2}) s^2 + [i(x^3 - \frac{(x^1)^2}{6}) - x^1 x^2] s^3), \\
y^1 &= e^{ix_0} (s^1 + ix^1 s^2 + (ix^2 - \frac{(x^1)^2}{2}) s^3), \\
y^2 &= e^{ix_0} (s^2 + ix^1 s^3), \\
y^3 &= e^{ix_0} s^3.
\end{align*}
\] (4.21)

Hamiltonian:

\[
H_{11} = (p_0 p_3 + p_1 p_2) + \frac{k_0 k_3 + k_1 k_2}{(s^3)^2} - \frac{(k_3)^2 s^0 + k_3^2 s^2 + 2k_3(k_1 s^2 + k_2 s^1)}{(s^3)^3} \\
+ \frac{3k_3 s^2 (k_3 s^1 + k_2 s^2)}{(s^3)^4} - \frac{2k_3^2 (s^2)^3}{(s^3)^5}.
\] (4.22)

The singularity surface of the potential is the hyperbolic cylinder \( s^3 = 0 \), \( 2 s^1 s^2 = 1 \), \( s^0 \in \mathbb{R} \).

5.- SUPERINTEGRABILITY

OF THE REDUCED HAMILTONIAN SYSTEMS.

5.1.- Complete Sets of Commuting Operators.

In order to show that the Hamiltonian \( H \) of eq. (1.2) is completely integrable, we need to present two integrals of motion, say \( Q_1 \) and \( Q_2 \), that are in involution and are well defined on the phase space \( \{s, p\} \). Indeed, in view of the constraint (1.1), we have just three degrees of freedom, hence the set

\[ \{H, Q_1, Q_2\} \] (5.1)

guarantees completely integrability. Maximal superintegrability, as defined in the introduction, requires the existence of \( 2n - 1 = 5 \) functionally independent integrals of
motion, pairwise in involution. We shall show that each of the Hamiltonians $H_1, \ldots, H_{11}$ constructed in Section 4 is maximally superintegrable.

To do this we return to the free Hamiltonian (1.5) on the complex hyperboloid (1.6). The underlying configuration space is seven–dimensional. The Hamiltonian system is hence completely integrable, if there exist seven functionally independent integrals of motion in involution. Five of them we already know, namely the Hamiltonian $H$ proportional to the $SU(2, 2)$ second order Casimir operator and four basis elements of the considered MASA of $u(2, 2)$. Thus, to the MASA $\{Y_1, Y_2, Y_3\}$ of $su(2, 2)$ we add the diagonal element

$$Y_0 = iI_4,$$  \hspace{1cm} (5.2)

represented by the vector field

$$\hat{Y}_0 = -i(y^0 \partial y^0 + y^1 \partial y^1 + y^2 \partial y^2 + y^3 \partial y^3).$$ \hspace{1cm} (5.3)

Functions $f(y)$ that project properly onto $P\mathcal{H}^4$ satisfy

$$\hat{Y}_0 f(y) = 0.$$ \hspace{1cm} (5.4)

We hence need to find two more integrals of motion $T_1$ and $T_2$ to form the complete set

$$\{H, Y_0, Y_1, Y_2, Y_3, T_1, T_2\}.$$ \hspace{1cm} (5.5)

Since we wish to solve the Hamilton–Jacobi, or Schrödinger equation for the reduced Hamiltonian system by separation of variables, we shall require that $T_1$ and $T_2$ be second order operators in the enveloping algebra [20–23] of $su(2, 2)$:

$$T_\alpha = \sum_{j=1}^{15} \sum_{i \leq j} a_{ij}^\alpha (X^i X^j + X^j X^i), \quad \alpha = 1, 2, \quad a_{ij}^\alpha \in \mathbb{R}.$$ \hspace{1cm} (5.6)

The first order operator $Y_0$ is fixed once and for all. The first order operators $\{Y_1, Y_2, Y_3\}$ form a basis for the MASA $M_a$, $a = 1, 2, \ldots, 11$ and are given for each MASA in column 3 of Table 1. Constructing the operators $T_\alpha$, satisfying

$$[Y_i, T_\alpha] = 0, \quad i = 1, 2, 3, \quad \alpha = 1, 2,$$ \hspace{1cm} (5.7)

is then a problem of linear algebra ($[Y_0, T_\alpha] = 0$ is satisfied automatically).

The construction of the operators $T_\alpha$ can also be viewed as a problem of group representation theory. Indeed, what is needed, is to find the invariants of the action of the abelian group $G_a = \exp M_a$ on the enveloping algebra of $su(2, 2)$. They can be obtained by considering the coadjoint action of $G_a$ on the dual $su(2, 2)^*$ of the Lie algebra $su(2, 2)$:

$$V' = e^Y V e^{-Y}, \quad Y \in M_a, \quad V \in su(2, 2)^*.$$ \hspace{1cm} (5.8)
This was done for all Cartan subalgebras of $su(p, q)$ in Ref. 1.

Here we skip all details and simply present, in Appendix 1, the sets of second order invariants for each of the MASAs $M_a$. To each set given in Appendix 1 we must add the six operators $Y_i Y_k$, $(i, k = 1, 2, 3)$ in the enveloping algebra of the considered MASA, in order to obtain a complete set of second order invariants. In each case, one linear combination of the invariants is equal to the second order Casimir operator of $su(2, 2)$.

The notation in Appendix 1 is adapted to the metric used for the MASA in Table 1. Thus, say for $M_6$, the metric is $K_3$ and the elements $X_i$, $(i = 1, \ldots, 15)$, are to be read off from eq. (2.3c).

The considered MASAs are all three–dimensional. The corresponding abelian subgroups act on the 15 dimensional space $su(2, 2)^*$. We hence expect to have 12 functionally independent invariants in each case, though they do not, a priori, have to be second order polynomials. In all cases, except $M_{10}$, we do get 12 second order invariants. For $M_{10}$ we get 13 of them; however a polynomial relation between them exists (a “syzygy”), making it possible to express a power of one of them as a polynomial in the others.

Obviously the second order operators $T_\alpha$, obtained for a given MASA, do not all commute amongst each other. Their commutation relations are given in Appendix 2. The letters $A, \ldots, D$ denote certain third order polynomials in the enveloping algebra of $su(2, 2)$. Their actual form is immaterial, except for the fact that they are always linearly independent.

From Appendix 2 we see that it is possible to choose commuting pairs $\{T_1, T_2\}$ in many different ways and thus to obtain different complete sets of commuting operators (5.5). The operators $T_1$ and $T_2$ will restrict to the operators $Q_1$ and $Q_2$ of eq. (5.1) upon reduction to the $O(2, 2)$ hyperboloid. Maximal superintegrability is assured by having (at least) two different pairs $\{T_1, T_2\}$ and $\{T_3, T_4\}$ for a given MASA, such that the four are linearly independent and satisfy

$$[T_1, T_2] = 0, \quad [T_3, T_4] = 0.$$  \hspace{1cm} (5.9)

A systematic search for commuting pairs of operators $[T_a, T_b] = 0$ is related to the problem of separating variables in Hamilton–Jacobi and Laplace–Beltrami equations on the $O(2, 2)$ hyperboloid $\sim O(2, 2)/O(2, 1)$. This problem was solved by Kalnins and Miller [24] (see also Ref.[20, 25–27]). They obtained 74 families of separable coordinates (3 of them nonorthogonal) and classified the corresponding second order operators. The number 74 is actually somewhat ambiguous, since sometimes several systems complement each other to cover the entire $O(2, 2)$ hyperboloid and some systems are related to others by an outer automorphism of $O(2, 2)$.

In order to classify commuting pairs of operators $\{Q_1, Q_2\}$ obtained by projecting the commuting pairs $\{T_1, T_2\}$, we shall need a classification of $O(2, 2)$ pairs of operators and separable coordinates [20, 24].
We recall that an “ignorable variable” is one that does not figure in the metric tensor \(g_{ik}(s)\) written in the corresponding separable coordinate system.

For \(O(2, 2)\) we shall now use the metric \(g = K_1\). A basis for the \(O(2, 2)\) algebra is given by

\[
\begin{align*}
K_{01} &= s^0 \partial_{s^1} + s^1 \partial_{s^0}, & K_{23} &= s^2 \partial_{s^3} + s^3 \partial_{s^2}, \\
K_{03} &= s^0 \partial_{s^3} + s^3 \partial_{s^0}, & L_{02} &= s^0 \partial_{s^2} - s^2 \partial_{s^0}, \\
K_{12} &= s^1 \partial_{s^2} + s^2 \partial_{s^1}, & L_{13} &= s^1 \partial_{s^3} - s^3 \partial_{s^1}.
\end{align*}
\]

The types of coordinates and commuting operators that occur for \(O(2, 2)/O(2, 1)\) are the following:

I.- **Two ignorable variables.**

The operators \(Q_1\) and \(Q_2\) are squares of elements of a MASA of \(o(2, 2)\). Six inequivalent MASAs exist, yielding the pairs

\[
\begin{align*}
1. &\quad (L_{02}^2, L_{13}^2), \\
2. &\quad (K_{01}^2, K_{23}^2), \\
3. &\quad ((L_{02} + K_{03})^2, (L_{13} + K_{12})^2), \\
4. &\quad ((L_{02} - L_{13})^2, (K_{12} + K_{03})^2), \\
5. &\quad ((L_{02} - L_{13})^2, (L_{02} + L_{13} + K_{12} + K_{03})^2), \\
6. &\quad ((K_{12} - K_{03})^2, (L_{02} + L_{13} + K_{12} + K_{03})^2).
\end{align*}
\]

The first three types correspond to the group reductions \(O(2, 2) \supset O(2) \times O(2),\) \(O(2, 2) \supset O(1, 1) \times O(1, 1)\) and \(O(2, 2) \supset E(1) \times E(1),\) respectively. The separable coordinates are orthogonal (\(E(1)\) is the one-dimensional Euclidean group, i.e., translations generated e.g. by \((L_{02} + K_{03})\)). The last three types correspond to the reductions \(O(2, 2) \supset O(2) \times O(1, 1),\) \(O(2, 2) \supset O(2) \times T(1)\) and \(O(2, 2) \supset O(1, 1) \times T(1),\) respectively; the coordinates are nonorthogonal.

II.- **One ignorable variable.**

A). Subgroup type.

The operator \(Q_1\) is the square of an element of the Lie algebra \(o(2, 2),\) \(Q_2\) is the Casimir operator of a subalgebra of \(o(2, 2),\) i.e. either \(e(1, 1)\) (the Poincaré algebra in one space dimension), or \(o(2, 1).\)

The corresponding pairs are:

\[
\begin{align*}
7. &\quad (K_{01}^2, (L_{02} + K_{03})^2 - (K_{12} + L_{13})^2), & O(2, 2) \supset E(1, 1) \supset O(1, 1), \\
8. &\quad (L_{02}^2, K_{01}^2 + K_{12}^2 - L_{02}^2), & O(2, 2) \supset O(2, 1) \supset O(2), \\
9. &\quad (K_{01}^2, K_{01}^2 + K_{12}^2 - L_{02}^2), & O(2, 2) \supset O(2, 1) \supset O(1, 1), \\
10. &\quad ((K_{01} + L_{02})^2, K_{01}^2 + K_{12}^2 - L_{02}^2), & O(2, 2) \supset O(2, 1) \supset E(1).
\end{align*}
\]
We note here that the algebra $o(2, 2)$ allows an outer automorphism realized, e.g., by the permutation of indices
\[ 0 \leftrightarrow 1, \quad 2 \leftrightarrow 3 \] (5.13)
in the coordinates $s^\mu$ and vector fields (5.10). This permutation may provide new coordinate systems and new commuting pairs of operators. This is the case of eq.(5.12), when the $O(2,1)$ subgroup is replaced by $O(1,2)$. In the case of relation (5.12) the permutation (5.13) provides an equivalent system (i.e., the new one can be rotated back into the old one). We shall not analyze this question any further. Below it is to be understood that the permutation (5.13) should be applied whenever it yields new systems (they are always very similar to those listed).

B). Generic Type.

The operator $Q_1$ is the square of an element of $o(2, 2)$, whereas $Q_2$ is generic, i.e., not a Casimir operator of any Lie subgroup.

B$_1$). The $O(2)$ element $L_{02}$ yields
\[ 11. - (L_{02}^2, K_{01}^2 + K_{12}^2 + a(K_{03}^2 + K_{23}^2) + bL_{13}^2 + c\{K_{01}, K_{23}\} - \{K_{12}, K_{03}\}) + d(\{K_{01}, K_{03}\} + \{K_{12}, K_{23}\})]. \] (5.14)
The curly brackets $\{ , \}$ denote anticommutators. Separable coordinates [24] are obtained for
\[ b = c = d = 0, \quad \text{or} \quad a = c = d = 0. \] (5.15)
The permutation (5.13) provides further systems, but we shall not discuss them here (nor below).

B$_2$). The $o(1, 1)$ element $K_{01}$ yields
\[ 12. - (K_{01}^2, a(K_{12}^2 - L_{02}^2) + bK_{23}^2 + c(K_{03}^2 - L_{13}^2) + d(\{L_{13}, K_{12}\} - \{L_{02}, K_{03}\}) + e((L_{02} - K_{03})^2 - (L_{13} - K_{12})^2)). \] (5.16)
Different separable coordinates are obtained for
\[ a = 1, \quad c = d = e = 0, \] (5.17a)
\[ a = 1, \quad b = d = e = 0, \] (5.17b)
\[ d = 1, \quad a = c, \quad b = e = 0, \] (5.17c)
\[ b = 1, \quad e = 1, \quad a = c = d = 0. \] (5.17d)

B$_3$). The $e(1)$ element $L_{02} + K_{12}$ yields:
\[ 13. - ((L_{02} + K_{12})^2, a(K_{12}^2 + K_{01}^2 - L_{02}^2) + b(L_{13} + K_{03})^2 + c(\{K_{23}, L_{02} + K_{12}\} + \{K_{01}, L_{13} + K_{03}\})). \] (5.18)
Different separable coordinates are obtained for:

\begin{align*}
a &= 1, & b &= \pm 1, & c &= 0, & \quad (5.19a) \\
a &= b = 0, & c &= 1. & \quad (5.19b)
\end{align*}

Other pairs of commuting operators satisfying \( Q_1 = X^2, \ X \in o(2,2) \), exist, but they do not correspond to separable coordinates on the \( O(2,2) \) hyperboloid.

III. **No ignorable variables.**

**A). Subgroup type.**

The operator \( Q_1 \) is a Casimir operator of a subalgebra of \( o(2,2) \), i.e. \( e(1,1) \) or \( o(2,1) \). The operator \( Q_2 \) lies in the enveloping algebra of the corresponding subalgebra and is not the square of an element of \( o(2,2) \).

\( A_1). \quad e(1,1) = (K_{01}, L_{02} + K_{03}, L_{13} + K_{12}). \)

Notice that the \( o(2,2) \) element \( K_{23} \) acts like a dilation on the translations \( L_{02} + K_{03} \equiv P_0 \) and \( L_{13} + K_{12} \equiv P_1 \). We have

\[ Q_1 = (L_{02} + K_{03})^2 - (L_{13} + K_{12})^2 = P_0^2 - P_1^2, \quad (5.20) \]

and \( Q_2 \) runs through 8 possibilities:

\[ K_{01}^2 \pm (P_0^2 + P_1^2), \ K_{01}^2 + 2P_0P_1, \ K_{01}^2 \pm (P_0 + P_1)^2, \quad (5.21) \]

\[ \{K_{01}, P_0\}, \ \{K_{01}, P_1\}, \ \{K_{01}, P_0 - P_1\} + (P_0 + P_1)^2. \]

The ninth possibility \( Q_2 = \{K_{01}, P_0 - P_1\} \) does not correspond to separable coordinates [24–27].

\( A_2). \quad o(2,1) = (K_{01}, K_{12}, L_{02}). \quad (5.22) \)

We have

\[ Q_1 = K_{01}^2 + K_{12}^2 - L_{02}^2, \quad (5.23) \]

and \( Q_2 \) runs through the following cases [22]:

\[ K_{01}^2 + aK_{12}^2, \ \{L_{02}, K_{01}\} + a(L_{02}^2 - K_{01}^2), \ K_{01}^2 \pm (K_{12} + L_{02})^2, \ \{K_{01}, K_{12} + L_{02}\}. \quad (5.24) \]

**B). Generic Type.**

The operators \( Q_1 \) and \( Q_2 \) are neither perfect squares, nor Casimir operators of subalgebras of \( o(2,2) \).
Essentially 9 different classes of such commuting pairs exist. For details we refer to Kalnins and Miller [24].

5.2.- Restriction of the su(2,2) Lie Algebra and of the Integrals of Motion to the O(2,2) Hyperboloid.

In Section 3 we presented general formulas for reducing the elements of the Lie algebra of su(2,2) to the real O(2,2) hyperboloid, namely eq.(3.9) and (3.10). In Section 4 we introduced the real coordinates (s, x) for each MASA of su(2,2) and used them to reduce the SU(2,2) free Hamiltonian to 11 different O(2,2) Hamiltonians with nontrivial potentials.

Let us now perform a similar reduction for the generators \( \hat{X}_i \) of su(2,2) and for the integrals of motion \( T_a \) of Appendix 1. Generally speaking the integral \( T_a \) after reduction will have the form

\[
\tilde{T}_a = \tilde{T}_a^{\text{kin}} + \tilde{T}_a^{\text{pot}},
\]

where the “kinetic” part comes from products of terms like (3.9), the “potential” part from terms like (3.10). We will be mainly interested in \( \tilde{T}_a^{\text{kin}} \). It lies in the enveloping algebra of \( \mathfrak{o}(2,2) \) and determines the separable coordinate systems.

Note that if two operators \( (T_a, T_b) \) in the enveloping algebra of su(2,2) commute, then so do the corresponding reduced operators \( (\tilde{T}_a, \tilde{T}_b) \). In this case, if the “kinetic” parts \( (\tilde{T}_a^{\text{kin}}, \tilde{T}_b^{\text{kin}}) \) correspond to a certain coordinate system for which the free O(2,2) Hamiltonian allows the separation of variables, then the reduced Hamiltonian with the induced potential also allows the separation of variables in the same system.

Let us now run through the individual MASAs. We shall use the notations (5.10) for the \( \mathfrak{o}(2,2) \) basis elements, i.e., all the second order integrals of motion are given in the diagonal metric \( g = K_1 = \text{diag}(1, -1, 1, -1) \). We shall use the same notation for the reduced invariants, as for the original ones, i.e., drop the tildes of eq. (5.25).

1.- The Compact Cartan Subalgebra \( M_1 \).

The invariants \( T_1, \ldots, T_6 \) of Appendix 1 restricted to \( o(2,2) \) are

\[
T_1 = K_{01}^2 + (k_0 s_1^1 s_0^0 - k_1 s_1^0)^2, \quad T_2 = L_{02}^2 + (k_0 s_2^2 s_0^0 + k_2 s_2^0 s_1^0)^2,
\]

\[
T_3 = K_{03}^2 + (k_0 s_3^3 s_0^0 - k_3 s_3^0)^2, \quad T_4 = K_{12}^2 + (k_1 s_1^2 s_2^0 - k_2 s_1^0 s_2^0)^2,
\]

\[
T_5 = L_{13}^2 + (k_1 s_3^3 s_1^1 + k_3 s_3^1 s_1^1)^2, \quad T_6 = K_{23}^2 + (k_2 s_3^3 s_2^2 - k_3 s_3^2 s_2^2)^2.
\]

The Hamiltonian (4.4) is recovered as

\[
H = \frac{1}{2}(-T_1 + T_2 - T_3 - T_4 + T_5 - T_6) + \sum \frac{k_{\mu}^2}{2} - 2 \sum_{\mu < \nu} k_{\mu} k_{\nu}.
\]
The following pairs of commuting operators and separable $O(2, 2)$ coordinates can be constructed:

I. 2 ignorable variables.

1. $- (T_2, T_5)$,
2. $- (T_1, T_6)$.

The coordinates are orthogonal and correspond to the reductions $O(2, 2) \supset O(2) \times O(2)$ and $O(2, 2) \supset O(1, 1) \times O(1, 1)$, respectively.

II. 1 ignorable variable.

A. Subgroup type.

3. $- (T_1 + T_4 - T_2, T_2)$, $O(2, 2) \supset O(2, 1) \supset O(2)$,
4. $- (T_1 + T_4 - T_2, T_1)$, $O(2, 2) \supset O(2, 1) \supset O(1, 1)$.

B. Generic.

5. $- (T_2, T_1 + T_4 + a(T_3 + T_6) + bT_5)$, $a \neq 0$
6. $- (T_1, T_4 - T_2 + a(T_3 - T_5) + bT_6)$, $a \neq 0$.

III. No ignorable variables.

A. Subgroup type.

7. $- (T_1 + T_4 - T_2, T_1 + aT_4)$, $a \neq 0$.

B. Generic type

8. $- (T_2 + aT_1 + bT_3, T_5 + \frac{a - b}{b}T_1 + \frac{a - b}{(1 + a)b}T_4)$, $b \neq 0$, $a \neq -1$.

2. The Noncompact Cartan Subalgebra $M_2$.

The invariants restricted to $O(2, 2)$ are

\[ T_1 = K^2_{01} + (k_0^s s^1 + k_1^s s^0)^2, \]
\[ T_2 = -K^2_{23} + \frac{1}{((s^2)^2 + (s^3)^2)^2} \left[ s^2(s^2 k_2 + s^3 k_3) + s^3(s^2 k_3 - s^3 k_2) \right]^2, \]
\[ T_3 = L^2_{02} - K^2_{03} + \left[ k_0^s s^2 \frac{s^0}{(s^2)^2 + (s^3)^2} (s^2 k_2 + s^3 k_3) \right]^2 - \left[ k_0^s s^3 \frac{s^0}{(s^2)^2 + (s^3)^2} (s^2 k_3 - s^3 k_2) \right]^2, \]
\[ T_4 = K^2_{12} - L^2_{13} + \left[ k_1^s s^2 \frac{s^1}{(s^2)^2 + (s^3)^2} (s^2 k_2 + s^3 k_3) \right]^2 - \left[ k_1^s s^3 \frac{s^1}{(s^2)^2 + (s^3)^2} (s^2 k_3 - s^3 k_2) \right]^2, \]
\[ T_5 = -\{L_{02}, K_{03}\} - 2[k_0^s s^2 \frac{s^0}{(s^2)^2 + (s^3)^2} (s^2 k_2 + s^3 k_3)] [k_0^s s^3 \frac{s^0}{(s^2)^2 + (s^3)^2} (s^2 k_3 - s^3 k_2)], \]
\[ T_6 = -\{K_{12}, L_{13}\} + 2[k_1^s s^2 \frac{s^1}{((s^2)^2 + (s^3)^2)} (s^2 k_2 + s^3 k_3)] [k_1^s s^3 \frac{s^1}{((s^2)^2 + (s^3)^2)} (s^2 k_3 - s^3 k_2)]. \]
The Hamiltonian, up to additive and multiplicative constants, is given by $-T_1 + T_2 + T_3 - T_4$.

The commuting pairs of second order integrals of motion, and the coordinates are as follows.

I. **2 ignorable variables**

1. $- (T_1, T_2), \quad O(1, 1) \times O(1, 1)$. \hspace{1cm} (5.33)

II. **1 ignorable variable**

A). Subgroup type

2. $- (T_2 + T_3, T_2), \quad O(2, 2) \supset O(2, 1) \supset O(1, 1)$. \hspace{1cm} (5.34)

B). Generic.

3. $- (T_1, a(T_3 - T_4) + b(T_5 - T_6))$, \hspace{1cm} (5.35)

4. $- (T_2, aT_3 + bT_4), \quad ab \neq 0$.

III. **No ignorable variables**

A). Subgroup type.

5. $- (T_2 + T_3, T_5 + aT_2)$. \hspace{1cm} (5.36)

B). Generic.

6. $- (T_1 + aT_3 + bT_5, T_2 + \frac{b^2}{a(a + 1) + b^2}T_3 - \frac{(a + 1)b}{a(a + 1) + b^2}T_5 + \frac{b}{a(a + 1) + b^2}T_6), \quad a^2 + b^2 + a \neq 0$, \hspace{1cm} (5.37)

7. $- (T_1 - T_3, T_2 + aT_6), \quad a \neq 0$.

8. $- (T_1 + aT_3 + bT_5 + cT_6, (b + c + ac)T_3 + (bc - a - 1)T_5 + (1 + c^2)T_6)$, with $a + a^2 + b^2 + bc = 0$.

3.- **The Noncompact Cartan Subalgebra** $M_3$.

From here on we shall spell out the “kinetic” parts of the invariants only; the remaining “potential” parts are easy to calculate (and are available from the authors upon request).

\[ T_1 = -K_{01}^2 + f_1(s), \quad T_2 = -K_{23}^2 + f_2(s), \quad (5.38) \]
\[ T_3 = L_{02}^2 - K_{03}^2 - K_{12}^2 + L_{13}^2 + f_3(s), \quad T_4 = -\{L_{02}, K_{03}\} + \{K_{12}, L_{13}\} + f_4(s), \]
\[ T_5 = -\{L_{02}, K_{12}\} + \{K_{03}, L_{13}\} + f_5(s), \quad T_6 = \{L_{02}, L_{13}\} + \{K_{03}, K_{12}\} + f_6(s). \]
The Hamiltonian satisfies $H \sim T_1 + T_2 + T_3$. 

Commuting pairs and coordinates:

I. **2 ignorable variables.**

1. $- (T_1, T_2)$, $O(1, 1) \times O(1, 1)$. **(5.39)**

II. **1 ignorable variable.**

A). **Subgroup type:** None

B). **Generic:**

2. $- (T_1, T_4 + aT_2)$. **(5.40)**

III. **No ignorable variables.**

A). **Subgroup type:** None.

B). **Generic:**

3. $Q_1 = T_1 + aT_4 + bT_5 + cT_6$, $Q_2 = T_2 - T_1 + \frac{bc - a}{c^2}T_4 - \frac{a}{c}T_5$,

with $c \neq 0$ and $c(a^2 + b^2 + c^2) - ab = 0$,

4. $Q_1 = T_1 + a^2T_2 + aT_6$, $Q_2 = T_4 + aT_5$, $a \neq 0$. **(5.41)**

4. **The OD MASA M_4.**

The reduced integrals of motion are

\[
T_1 = K_{01}^2 + f_1(s), \quad T_2 = \frac{1}{2}(L_{02} + K_{03})^2 + f_2(s), \\
T_3 = \frac{1}{2}(L_{13} + K_{12})^2 + f_3(s), \quad T_4 = K_{23}^2 + f_4(s), \\
T_5 = L_{02}^2 - K_{03}^2 + f_5(s), \quad T_6 = -L_{13}^2 + K_{12}^2 + f_6(s).
\] **(5.42)**

We have $H \sim T_1 + T_4 - T_5 + T_6$.

The commuting pairs are:

I. **2 ignorable variables.**

1. $- (T_1, T_4)$, $O(1, 1) \times O(1, 1)$,

2. $- (T_2, T_3)$, $E(1) \times E(1)$. **(5.43)**
II. 1 ignorable variable.

A. Subgroup type.

3. \((T_5 - T_4, T_4)\), \(O(2, 2) \supset O(2, 1) \supset O(1, 1)\),

4. \((T_5 - T_4, T_2)\), \(O(2, 2) \supset O(2, 1) \supset E(1)\),

5. \((T_2 - T_3, T_1)\), \(O(2, 2) \supset E(1, 1) \supset O(1, 1)\).

B. Generic.

6. \((T_1, T_4 + a(T_2 - T_3))\), \(a \neq 0\),

7. \((T_4, T_1 + aT_5)\), \(a \neq 0\),

8. \((T_2, T_4 - T_5 + aT_3)\), \(a \neq 0\).

III. No ignorable variables.

A. Subgroup type.

9. \((T_2 - T_3, T_1 + aT_2)\), \(a \neq 0\),

10. \((T_5 - T_4, T_4 + aT_2)\), \(a \neq 0\).

B. Generic.

11. \((T_1 + aT_3 + b(T_4 - T_5), a(b - 1)T_2 + aT_3 + bT_4)\), \(a(b - 1) \neq 0\).

5. The OD MASA \(M_5\).

The integrals of motion now are:

\[
T_1 = -K_{01}^2 + f_1(s), \quad T_2 = K_{23}^2 + f_2(s),
\]

\[
T_3 = \frac{1}{2}[(L_{02} - K_{03})^2 - (L_{13} - K_{12})^2] + f_3(s), \quad T_4 = \frac{1}{2}[L_{02} - K_{03}, -L_{13} + K_{12}] + f_4(s),
\]

\[
T_5 = L_{02}^2 - K_{03}^2 + L_{13}^2 - K_{12}^2 + f_5(s), \quad T_6 = \{L_{02}, K_{12}\} - \{K_{03}, L_{13}\} + f_6(s).
\]

We have \(H \sim T_1 - T_2 + T_5\).

The commuting pairs are:

I. 2 ignorable variables.

1. \((T_1, T_2)\), \(O(1, 1) \times O(1, 1)\),

2. \((T_3, T_4)\), \(E(1) \times E(1)\).
Note that while $T_3$ and $T_4$ are not directly squares of elements of a MASA, diagonalizing them is equivalent to diagonalizing the commuting pair $(L_{02} - K_{03}, L_{13} - K_{12})$, a MASA of $o(2, 2)$. Similar situations with two ignorable variables occur below.

II. 1 ignorable variable.
A). Subgroup type.

3. $- (T_3, T_1), \quad O(2, 2) \supset E(1, 1) \supset O(1, 1)$.  \hspace{1cm} (5.50)

B). Generic.

4. $- (T_1, T_2 + aT_3), \quad a \neq 0,

5. $- (T_2, T_6 + aT_1)$.  \hspace{1cm} (5.51)

III. No ignorable variables.
A). Subgroup type.

6. $- (T_3, T_1 + aT_4), \quad a \neq 0$.  \hspace{1cm} (5.52)

B). Generic.

7. $- (T_2 + aT_3 + bT_4, aT_1 + b^2T_3 - abT_4 - bT_6), \quad a \neq 0,  \hspace{1cm} (5.53)

8. $- (T_2 + aT_4, T_6 - aT_3), \quad a \neq 0.$

6. The OD MASA $M_6$.

The integrals of motion:

\begin{align*}
T_1 &= \frac{1}{4}(L_{02} - K_{03} + K_{12} - L_{13})^2 + f_1(s), \quad T_2 = K_{01}^2 + f_2(s),  \hspace{1cm} (5.54) \\
T_3 &= K_{23}^2 + f_3(s), \quad T_4 = \frac{1}{2}((L_{02} + K_{12})^2 - (K_{03} + L_{13})^2) + f_4(s), \\
T_5 &= \frac{1}{2}((L_{02} - K_{03})^2 - (K_{12} - L_{13})^2) + f_5(s), \quad T_6 = L_{02}^2 + L_{13}^2 - K_{03}^2 - K_{12}^2 + f_6(s).
\end{align*}

We have $H \sim T_6 - T_2 - T_3$.

The commuting pairs are:

I. Two ignorable variables.

1. $- (T_2, T_3), \quad O(2, 2) \supset O(1, 1) \times O(1, 1),$

2. $- (T_1, T_4), \quad O(2, 2) \supset E(1) \times E(1)$.  \hspace{1cm} (5.55)
II. One ignorable variable.
A). Subgroup type.

3. \(- (T_4, T_3), \quad O(2, 2) \supset O(2, 1) \supset O(1, 1). \) \hspace{1cm} (5.56)

B). Generic.

4. \(- (T_2, T_3 + aT_5), \quad a \neq 0, \)
5. \(- (T_1, T_4 + aT_5), \quad a \neq 0. \) \hspace{1cm} (5.57)

III. No ignorable variables.
A). Subgroup type.

6. \(- (T_4, T_3 + aT_1). \) \hspace{1cm} (5.58)

B). Generic.

7. \(- (T_2 + abT_1 + aT_4, T_3 + abT_1 + bT_5), \quad ab \neq 0. \) \hspace{1cm} (5.59)

7. The OD MASAs \(M_7\) and \(M_8\).

The integrals of motion are:

\[
T_1 = \frac{1}{2} (L_{13} - K_{23})^2 + f_1(s), \quad T_2 = \frac{1}{2} (K_{01} + L_{02})^2 + f_2(s),
\]
\[
T_3 = K_{03}^2 + K_{23}^2 - L_{13}^2 + f_3(s), \quad T_4 = 4(K_{12}^2 + K_{01}^2 - L_{02}^2) + f_4(s), \quad (5.60)
\]
\[
T_5 = \frac{1}{\sqrt{2}} \{K_{03}, -L_{13} + K_{23}\} + f_5(s), \quad T_6 = \sqrt{2}\{K_{12}, K_{01} + L_{02}\} + f_6(s).
\]

We have \(H \sim T_3 + \frac{1}{4} T_4.\)

The commuting pairs are:
I. Two ignorable variables.

1. \(- (T_1, T_2), \quad O(2, 2) \supset E(1) \times E(1). \) \hspace{1cm} (5.61)

II. One ignorable variable.
A). Subgroup type.

2. \(- (T_4, T_2), \quad O(2, 2) \supset O(2, 1) \supset E(1). \) \hspace{1cm} (5.62)

B). Generic.

3. \(- (T_1, 2T_5 + T_6 + aT_2), \)
4. \(- (T_2, T_4 + aT_1), \quad a \neq 0. \) \hspace{1cm} (5.63)
III. No ignorable variables.
A). Subgroup type.

5. \(- (T_1 - T_2, T_5 + aT_1),\)
6. \(- (T_4, T_6 + aT_2), \ a \neq 0. \) (5.64)

B). Generic.

7. \(- (T_1 + aT_2 + bT_6, b^2T_4 + a(a + 1)T_2 + 2bT_5 + (a + 1)bT_6), \ b \neq 0. \) (5.65)

8. The OID & D MASA \( M_9 \)

The invariants are

\[
\begin{align*}
T_1 &= \frac{1}{4}(L_{02} - K_{03} + K_{12} - L_{13})^2 + f_1(s), \\
T_2 &= (K_{01} + K_{23})^2 - (L_{02} + L_{13})^2 + f_2(s), \\
T_3 &= (K_{01} - K_{23})^2 + 2(K_{03}^2 + K_{12}^2) - (L_{02} - L_{13})^2 + f_3(s), \\
T_4 &= \frac{1}{2}\{K_{01} - K_{23}, L_{02} - K_{03} + K_{12} - L_{13}\} + f_4(s), \\
T_5 &= \frac{1}{2}\{K_{03} + K_{12}, L_{02} - K_{03} + K_{12} - L_{13}\} + f_5(s), \\
T_6 &= \{K_{01} + K_{23}, L_{02} + L_{13}\} + f_6(s).
\end{align*}
\] (5.66)

We have \( H \sim T_2 + T_3. \)

The commuting pairs are:
I. Two ignorable variables.

1. \(- (T_2, T_6), \ O(2, 2) \supset O(1, 1) \times O(2),\)
2. \(- (T_1, T_4), \ O(2, 2) \supset O(1, 1) \times E(1). \) (5.67)

II. One ignorable variable.
A). Subgroup type: None.
B). Generic

3. \(- (T_1, T_4 + aT_5), \ a \neq 0. \) (5.68)

III. No ignorable variables.
A). Subgroup type: None.
B). Generic

4. \(- (-a^2 + b^2)T_1 + T_2 + aT_4 + bT_5, -bT_4 + aT_5 + T_6), \ a^2 + b^2 \neq 0. \) (5.69)
The first two pairs above correspond to nonorthogonal coordinates on the \( O(2, 2) \) hyperboloid [24]. The third does not correspond to any separable system.

9. The MANS \( M_{10} \)

The situation is different in this case than for all other MASAs of \( su(2, 2) \). First of all, there are 7 second order integrals in the enveloping algebra of \( su(2, 2) \), rather than 6 as in all other cases. Secondly, one of them, \( T_7 \) in Appendix I, does not reduce to the \( O(2, 2) \) manifold as in eq. (5.25). Instead, it reduces to a first order operator, once the momenta, conjugate to ignorable variables are set equal to the constants \( k_\mu \). Indeed, after the reduction the 7 invariants are:

\[
\begin{align*}
T_1 &= \frac{1}{4}(K_{01} - L_{13} + L_{02} + K_{23})^2 + f_1(s), \\
T_2 &= \frac{1}{4}(-K_{01} + L_{13} + L_{02} + K_{23})^2 + f_2(s), \\
T_3 &= \frac{1}{4}((L_{02} + K_{23})^2 - (K_{01} - L_{13})^2) + f_3(s), \\
T_4 &= \frac{1}{2}\{K_{03} + K_{12}, \ K_{01} - L_{13} + L_{02} + K_{23}\} + f_4(s), \\
T_5 &= \frac{1}{2}\{K_{03} - K_{12}, \ -K_{01} + L_{13} + L_{02} + K_{23}\} + f_5(s), \\
T_6 &= 2(K_{03}^2 + K_{12}^2 + K_{01}^2 + K_{23}^2 - L_{02}^2 - L_{13}^2) + f_6(s), \\
T_7 &= -4k_3K_{12} + (k_1 + k_2)(K_{01} - L_{13}) + (k_2 - k_1)(L_{02} + K_{23}).
\end{align*}
\]

For the Hamiltonian we have \( H \sim T_6 \).

The commuting pairs of integrals can be organized as follows:

I. Two ignorable variables.

1. \(- (T_1, T_3), \quad O(2, 2) \supset E(1) \times E(1).\) \( (5.71) \)

II. One ignorable variable.

A). Subgroup type.

2. \(- (T_3, T_7^2), \quad O(2, 2) \supset E(1, 1) \supset O(1, 1).\) \( (5.72) \)

B). Generic.

3. \(- (T_1, T_4 + aT_3).\) \( (5.73) \)

III. No ignorable variables.

A). Subgroup type.

4. \(- (T_3, T_7^2 + aT_1 + bT_2), \quad (a, b) \neq (0, 0).\) \( (5.74) \)
B). Generic.

\[ 5. - (T_4 + bT_3 + aT_5, T_1 + a^2T_2 - aT_3), \quad a \neq 0. \quad (5.75) \]

10. The MANS \( M_{11} \).

The reduced integrals are:

\[
T_1 = \frac{1}{4}(-K_{01} + L_{13} + L_{02} + K_{23})^2 + f_1(s),
\]
\[
T_2 = \frac{1}{2}(K_{01} - L_{13} + L_{02} + K_{23})^2 + \{K_{12}, -K_{01} + L_{13} + L_{02} + K_{23}\} + f_2(s),
\]
\[
T_3 = 2(K_{03}^2 + K_{12}^2 + K_{01}^2 + K_{23}^2 - L_{13}^2 - L_{02}^2) + f_3(s),
\]
\[
T_4 = \frac{1}{2}((L_{02} + K_{23})^2 - (K_{01} - L_{13})^2) + f_4(s),
\]
\[
T_5 = \frac{1}{2}\{K_{03} - K_{12}, -K_{01} + L_{13} + L_{02} + K_{23}\} + f_5(s),
\]
\[
T_6 = \frac{1}{2}((K_{23} + L_{13})^2 - (K_{01} - L_{02})^2 - \{K_{03} + K_{12}, K_{01} - L_{13} + L_{02} + K_{23}\}) + f_6(s).
\]

The integral \( T_3 \) is directly related to the Hamiltonian.

The commuting pairs of integrals are:

II. One ignorable variable.

A). Subgroup type:

\[
1. - (T_1, T_4), \quad O(2, 2) \supset E(1, 1). \quad (5.77)
\]

B). Generic.

\[
2. - (T_1, T_5 + aT_4). \quad (5.78)
\]

III. No ignorable variables.

A). Subgroup type

\[
3. - (T_4, T_2). \quad (5.79)
\]

B). Generic.

\[
4. - (2abT_1 + aT_2 + bT_4 + T_6, 2bT_1 + T_2 + 4aT_4 + 2T_5). \quad (5.80)
\]
6.- CONCLUSIONS.

The results contained in this article can be situated in three complementary research programs. They are: 1) The classification of maximal abelian subalgebras of the classical Lie algebras and their applications in physics. 2) A systematic search for integrable and “superintegrable” Hamiltonian systems in various homogeneous spaces. 3) The theory of the separation of variables in partial differential equations, invariant under some Lie group of local point transformations.

Physical applications that we have in mind can be either direct or indirect ones. The integrability of the considered systems makes it possible to calculate eigenvalues and eigenfunctions of the quantum systems, as well as trajectories in the classical ones. The usefulness of the above facts hinges on the question whether the deduced potentials can be associated with realistic physical phenomena, as was the case for $\tilde{O}(2,1)$ potentials [13, . . . , 17]. We mention that integrable systems in spaces with indefinite metric were studied by Kibler [28].

A different and less direct application occurs in soliton theory [29]. Indeed, finite dimensional integrable systems can sometimes be associated with infinite dimensional ones. Solutions of the finite dimensional systems then provide interesting special solutions of the infinite dimensional systems [30–32].

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References.

[1]. del Olmo, M.A., Rodríguez, M.A. and Winternitz, P., J. Math. Phys. 34, 5118, (1993).
[2]. Jacobson, N., Lie Algebras , Wiley, New York, (1962).
[3]. del Olmo, M.A., Rodríguez, M.A., Winternitz, P. and Zassenhaus H., Lin. Alg. and Applic. 135, 79 (1990).
[4]. Wojciechowski, S., Phys. Lett. 95A, 279 (1983); 100A, 471 (1984).
[5]. Evans, N.W., Phys. Rev. 41, 5666 (1990), Phys. Lett. 147 A, 483 (1990); J. Math. Phys. 32, 3369 (1991).
[6]. Fock, V.A., Z. Phys. 98, 145 (1935).
[7]. Bargman, V., Z. Phys. 99, 576 (1936).
[8]. Friš I., Mandrasov V., Smorodinsky, Ya. A., Uhlíř, M. and Winternitz, P., Phys. Lett. 16, 354 (1965).
[9]. Makarov, A.A., Smorodinsky, Ya. A., Valiev, Kh., and Winternitz, P., Nuovo Cim. A52, 1061 (1967).
[10]. Kibler, M. and Winternitz, P., J. Phys. A 20, 4097 (1987); Phys. Lett. 147A, 338 (1990).
[11]. Kibler, M., Lamot, G.H., Winternitz, P., Int. J. Quantum Chem. 43, 625 (1992).
[12]. Moshinsky, M., The Harmonic Oscillator in Modern Physics: From Atoms to Quarks, Gordon and Breach, New York, 1969.
[13]. Pöschl, G. and Teller, E., Z. Phys. 83, 143 (1933).
[14]. Alhassid, Y., Gursey, F. and Iachello, F., Phys. Rev. Lett. 50, 873 (1983); Chem. Phys. Lett. 99, 27 (1983).
[15]. Frank, A. and Wolf, K.B., Phys. Rev. Lett. 52, 1737 (1984).
[16]. Wulfman, C. and Levine, R.D., Chem. Phys. Lett., 60, 372 (1979).
[17]. Wehrhahn, R.F., Smirnov, Yu.F., and Shirokov, A.M., J. Math. Phys. 33, 2384 (1992).
[18]. Kobayashi, S. and Nomizu, K., Foundations of Differential Geometry I, II, Wiley, New York, 1963 and 1969.
[19]. Eisenhart, L.P., Ann. Math. 35, 284 (1934).
[20]. Miller, W. Jr., Patera, J. and Winternitz, P., J. Math. Phys. 22, 251 (1981).
[21]. Winternitz, P. and Friš I., Sov. J. Nucl. Phys. 1, 636 (1965).
[22]. Winternitz, P., Lukač, I. and Smorodinsky, Ya. A., Sov. J. Nucl. Phys. 7, 139 (1968).
[23]. Miller, W. Jr., Symmetry and Separation of Variables, Addison Wesley, New York, (1977).
[24] Kalnins, E.G. and Miller, W. Jr., Proc. Roy. Soc. Edinb. 79A, 227 (1977).
[25]. Kalnins, E.G. and Miller, W. Jr., SIAM J. Math. Anal. 9, 12 (1978).
[26]. Kalnins, E.G. and Miller, W. Jr., J. Diff. Geom. 14, 221 (1979).
[27]. Kalnins, E.G., SIAM J. Math. Anal. 6, 340 (1975).
[28]. Kibler, M., in Group Theoretical Methods in Physics, Lecture Notes in Phys. 318, p. 238, Springer–Verlag, Berlin (1988).
[29]. Ablowitz, M.J. and Clarkson, P.A., Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge (1991).
[30]. Previato, E., Physica D18, 312 (1986), Duke Math. J. 52, 329 (1985).
[31]. Adams, M.R., Harnad, J. and Previato, E., Commun. Math. Phys. 117, 451 (1988).
[32]. Adams, M.R., Harnad, J. and Hurtubise, J., Commun. Math. Phys. 134, 555 (1990).
Table 1. MASAs of $su(2,2)$. The basis elements $X_i$ are to be identified with the matrices in eq. (2.3a–f). Which realization to take is indicated by the subscript of the metric $K_i$. The abbreviations used are OD for an orthogonally decomposable MASA; OID means orthogonally indecomposable, and D means decomposable, but not orthogonally.

| $N_0$ | Type | Basis: $Y_1, Y_2, Y_3$ | Realization |
|-------|------|------------------------|-------------|
| $M_1$ | Cartan compact | $X_1, X_2, X_3$ | $K_1$ |
|       | OD (1, 0) + (0, 1) + (1, 0) + (0, 1) | | |
| $M_2$ | Cartan noncompact | $X_1, X_1 + 2X_2 + X_3, X_{15}$ | $K_1$ |
|       | OD (1, 0) + (0, 1) + (1, 1) | | |
| $M_3$ | Cartan noncompact | $X_1 + 2X_2 + X_3, X_5, X_{15}$ | $K_1$ |
|       | OD (1, 1) + (1, 1) | | |
| $M_4$ | OD (1, 0) + (0, 1) + (1, 1) | $X_1, X_2, X_{14}$ | $K_2$ |
| $M_5$ | OD (1, 1) + (1, 1) | $X_2, X_5, X_{14}$ | $K_2$ |
| $M_6$ | OD (1, 1) + (1, 1) | $X_3, X_4, X_{14}$ | $K_3$ |
| $M_{7,8}$ | OD (2, 1) + (0, 1) | $X_2 + 2X_3, X_4, X_7$ | $K_{4,\varepsilon}$ |
| $M_9$ | OID & D | $X_5 - X_7, X_8 - X_9, X_{11}$ | $K_5$ |
| $M_{10}$ | MANS (121) | $X_5, X_7, X_{12}$ | $K_6$ |
| $M_{11}$ | MANS (121) | $X_5 + X_{13}, X_7, X_{12}$ | $K_6$ |
| $M_{12}$ | MANS (202) | $X_6, X_7, X_{12}, X_{13}$ | $K_6$ |
APPENDIX 1. Second order operators in the enveloping algebra commuting with MASAs. The operators are written in the presentation corresponding to the metric $K_1$. A basis of each MASA in this metric is also given. The notations coincide with of those of Table I only for the Cartan subalgebras $M_1$, $M_2$ and $M_3$.

**MASA 1**: $\{X_1, X_2, X_3\}$

\[
\begin{align*}
T_1 &= X_4^2 + X_5^2 \\
T_2 &= X_6^2 + X_7^2 \\
T_3 &= X_8^2 + X_9^2 \\
T_4 &= X_{10}^2 + X_{11}^2 \\
T_5 &= X_{12}^2 + X_{13}^2 \\
T_6 &= X_{14}^2 + X_{15}^2
\end{align*}
\]

**MASA 2**: $\{X_1, X_1 + 2X_2 + X_3, X_{15}\}$

\[
\begin{align*}
T_1 &= X_4^2 + X_5^2 \\
T_2 &= -X_{14}^2 + X_3^2 \\
T_3 &= X_6^2 - X_8^2 + X_7^2 - X_9^2 \\
T_4 &= X_{10}^2 - X_{12}^2 + X_{11}^2 - X_{13}^2 \\
T_5 &= \{X_6, X_8\} + \{X_7, X_9\} \\
T_6 &= \{X_{10}, X_{12}\} + \{X_{11}, X_{13}\}
\end{align*}
\]

**MASA 3**: $\{X_1 + 2X_2 + X_3, X_5, X_{15}\}$

\[
\begin{align*}
T_1 &= -X_4^2 + X_1^2 \\
T_2 &= -X_{14}^2 + X_3^2 \\
T_3 &= X_6^2 - X_8^2 - X_{10}^2 + X_{12}^2 + X_7^2 - X_9^2 - X_{11}^2 + X_{13}^2 \\
T_4 &= \{X_6, X_8\} - \{X_{10}, X_{12}\} + \{X_7, X_9\} - \{X_{11}, X_{13}\} \\
T_5 &= \{X_6, X_{10}\} - \{X_8, X_{12}\} + \{X_7, X_{11}\} - \{X_9, X_{13}\} \\
T_6 &= \{X_6, X_{12}\} + \{X_8, X_{10}\} + \{X_7, X_{13}\} + \{X_9, X_{11}\}
\end{align*}
\]
MASA 4: \( \{X_1, 2X_2 + X_3, X_3 - X_{15}\} \)

\[ T_1 = X_4^2 + X_5^2 \]
\[ T_2 = \frac{1}{2}(X_6 - X_8)^2 + \frac{1}{2}(X_7 - X_9)^2 \]
\[ T_3 = \frac{1}{2}(X_{10} - X_{12})^2 + \frac{1}{2}(X_{11} - X_{13})^2 \]
\[ T_4 = X_{14}^2 - X_3^2 + X_{15}^2 \]
\[ T_5 = X_6^2 - X_8^2 + X_7^2 - X_9^2 \]
\[ T_6 = X_{10}^2 - X_{12}^2 + X_{11}^2 - X_{13}^2 \]

MASA 5: \( \{X_1 + 2X_2 + X_3, X_5, X_3 + X_{15}\} \)

\[ T_1 = -X_4^2 + X_1^2 \]
\[ T_2 = X_{14}^2 - X_3^2 + X_{15}^2 \]
\[ T_3 = \frac{1}{2}(X_6 + X_8)^2 - \frac{1}{2}(X_{10} + X_{12})^2 + \frac{1}{2}(X_7 + X_9)^2 - \frac{1}{2}(X_{11} + X_{13})^2 \]
\[ T_4 = -\frac{1}{2}\{X_6 + X_8, X_{10} + X_{12}\} - \frac{1}{2}\{X_7 + X_9, X_{11} + X_{13}\} \]
\[ T_5 = X_6^2 - X_8^2 - X_{10}^2 + X_{12}^2 + X_7^2 - X_9^2 - X_{11}^2 + X_{13}^2 \]
\[ T_6 = -\{X_6, X_{10}\} + \{X_8, X_{12}\} - \{X_7, X_{11}\} + \{X_9, X_{13}\} \]

MASA 6: \( \{X_1 + 2X_2 + X_3, X_1 + X_5, X_3 + X_{15}\} \)

\[ T_1 = \frac{1}{4}(X_6 + X_8 - X_{10} - X_{12})^2 + \frac{1}{4}(X_7 + X_9 - X_{11} - X_{13})^2 \]
\[ T_2 = X_4^2 - X_1^2 + X_5^2 \]
\[ T_3 = X_{14}^2 - X_3^2 + X_{15}^2 \]
\[ T_4 = \frac{1}{2}(X_6 - X_{10})^2 - \frac{1}{2}(X_8 - X_{12})^2 + \frac{1}{2}(X_7 - X_{11})^2 - \frac{1}{2}(X_9 - X_{13})^2 \]
\[ T_5 = \frac{1}{2}(X_6 + X_8)^2 - \frac{1}{2}(X_{10} + X_{12})^2 + \frac{1}{2}(X_7 + X_9)^2 - \frac{1}{2}(X_{11} + X_{13})^2 \]
\[ T_6 = X_6^2 - X_8^2 - X_{10}^2 + X_{12}^2 + X_7^2 - X_9^2 - X_{11}^2 + X_{13}^2 \]
MASA 7: \( \{X_1 + 2X_2 + 3X_3, X_2 - X_{11}, X_5 - X_7\} \)

\[
T_1 = \frac{1}{2}(X_{12} + X_{14})^2 + \frac{1}{2}(X_{13} + X_{15})^2
\]
\[
T_2 = \frac{1}{2}(X_4 - X_6)^2 + \frac{1}{2}\{X_1 - X_3, X_2 - X_{11}\}
\]
\[
T_3 = X_8^2 - X_{12}^2 + X_{14}^2 + X_9^2 - X_{13}^2 + X_{15}^2
\]
\[
T_4 = 4(X_9^2 - X_6^2 + X_{10}^2) + 4(X_5^2 - X_7^2 + X_{11}^2) - 5X_1^2 - 4X_2^2 + 3X_3^2 - 2\{X_1, X_2\} + \{X_1, X_3\} + 2\{X_2, X_3\}
\]
\[
T_5 = \frac{1}{\sqrt{2}}\{X_8, X_{12} + X_{14}\} + \frac{1}{\sqrt{2}}\{X_9, X_{13} + X_{15}\}
\]
\[
T_6 = \sqrt{2}\{X_{10}, X_4 - X_6\} + \frac{1}{\sqrt{2}}\{X_1 - X_3, X_5 - X_7\} + \sqrt{2}(X_2 - X_{11}, X_5 + X_7)
\]

MASA 8: \( \{X_1 + 2X_2 - X_3, X_1 + X_2 + X_3 - X_9, X_5 + X_{13}\} \)

\[
T_1 = \frac{1}{2}(X_6 + X_{14})^2 + \frac{1}{2}(X_7 - X_{15})^2
\]
\[
T_2 = \frac{1}{2}(X_4 - X_{12})^2 - \frac{1}{2}\{X_1 - X_3, X_1 + X_2 + X_3 - X_9\}
\]
\[
T_3 = -X_6^2 + X_{10}^2 + X_{14}^2 - X_7^2 + X_{11}^2 + X_{15}^2
\]
\[
T_4 = 4(X_4^2 + X_8^2 - X_{12}^2) + 4(X_5^2 + X_9^2 - X_{13}^2) - 5X_1^2 - 4X_2^2 + 5X_3^2 - 2\{X_1, X_2\} - 3\{X_1, X_3\} - 6\{X_2, X_3\}
\]
\[
T_5 = \frac{1}{\sqrt{2}}\{X_{10}, X_6 + X_{14}\} + \frac{1}{\sqrt{2}}\{X_{11}, X_7 - X_{15}\}
\]
\[
T_6 = \sqrt{2}\{X_8, X_4 - X_{12}\} + \frac{1}{\sqrt{2}}\{X_1 - X_3, X_5 + X_{13}\} - \sqrt{2}(X_2 + X_3 - X_9, X_5 - X_{13})
\]

MASA 9: \( \{X_9 + X_{11}, X_1 - X_3 + X_5 - X_{15}, X_7 + X_9 - X_{11} - X_{13}\} \)

\[
T_1 = \frac{1}{4}(X_6 + X_8 - X_{10} - X_{12})^2 - \frac{1}{2}\{X_1 + X_5, X_3 + X_{15}\}
\]
\[
T_2 = (X_4 + X_{14})^2 - (X_6 + X_{12})^2 - (X_1 - X_3)^2 + (X_5 - X_{15})^2 - (X_7 - X_{13})^2 + (X_9 - X_{11})^2
\]
$T_3 = (X_4 - X_{14})^2 - (X_5 - X_{12})^2 + 2(X_8^2 + X_{10}^2) - 2(X_1 + X_2)^2 - 2(X_3 + X_3)^2 - 2\{X_1, X_3\} + (X_5 + X_{15})^2 - (X_7 + X_{13})^2 + (X_9 + X_{11})^2$

$T_4 = \frac{1}{2}\{X_4 - X_{14}, X_6 + X_8 - X_{10} - X_{12}\} +$ \\ $\frac{1}{2}\{X_1 + X_5, X_7 - X_9 - X_{11} + X_{13}\} +$ \\ $\frac{1}{2}\{X_3 + X_{15}, X_7 + X_9 + X_{11} + X_{13}\}$

$T_5 = X_{10}^2 - X_8^2 - \frac{1}{2}\{X_6 - X_{12}, X_8 + X_{10}\} + (X_1 + X_3)^2 +$ \\ $\{X_1 + X_3, X_2\} + \frac{1}{2}\{X_1 + 2X_2 + X_3, X_5 + X_{15}\}$

$T_6 = -\{X_4 + X_{14}, X_6 + X_{12}\} - \{X_5 - X_{15}, X_7 - X_{13}\} +$ $\{X_1 - X_3, X_9 - X_{11}\}$

**MASA 10:** $\{X_1 + X_2 + X_3 + X_9, X_5 - X_{13}, X_7 + X_{15}\}$

$T_1 = \frac{1}{4}(X_4 - X_6 + X_{12} + X_{14})^2 + \frac{1}{2}\{X_1 + X_2 + X_3 + X_9, X_2 - X_{11}\}$

$T_2 = \frac{1}{4}(X_4 + X_6 + X_{12} - X_{14})^2 + \frac{1}{2}\{X_1 + X_2 + X_3 + X_9, X_2 + X_{11}\}$

$T_3 = -\frac{1}{2}(X_4 + X_{12})^2 + \frac{1}{2}(X_6 - X_{14})^2$ \\ $+ X_1^2 - X_3^2 + \frac{1}{2}\{X_1 - X_3, X_2 + X_9\}$

$T_4 = -\frac{1}{2}\{X_4 - X_6 + X_{12} + X_{14}, X_8 + X_{10}\}$ \\ $- \frac{1}{2}\{X_1 + 2X_2 + X_3 + X_9 - X_{11}, X_5 + X_{15}\}$ \\ $+ \frac{1}{2}\{X_1 + X_3 + X_9 + X_{11}, X_7 - X_{13}\}$

$T_5 = \frac{1}{2}\{X_4 + X_6 + X_{12} - X_{14}, X_8 - X_{10}\}$ \\ $+ \frac{1}{2}\{X_1 + 2X_2 + X_3 + X_9 + X_{11}, X_5 - X_{15}\}$ \\ $+ \frac{1}{2}\{X_1 + X_3 + X_9 - X_{11}, X_7 + X_{13}\}$

$T_6 = 2(X_4^2 + X_8^2 + X_{10}^2 + X_{14}^2 - X_6^2 - X_{12}^2) + 3X_1^2 - 4X_2^2 - 3X_3^2 - 2\{X_1, X_2\} - \{X_1, X_3\} - 2\{X_2, X_3\} +$ \\ $2(X_8^2 - X_7^2 + X_9^2 + X_{11}^2 - X_{13}^2 + X_{15}^2)$
MASA 11: \( \{X_1 + X_2 + X_3 + X_9, X_5 + X_7 - X_{13} + X_{15}, 2X_5 - 2X_{13} - X_2 - X_{11}\} \)

\[
T_1 = \frac{1}{4}(X_4 + X_6 + X_{12} - X_{14})^2 + \\
\frac{1}{2}\{X_1 + X_2 + X_3 + X_9, X_5 - X_7 - X_{13} - X_{15}\}
\]

\[
T_2 = \frac{1}{2}(X_4 - X_6 + X_{12} + X_{14})^2 + \{X_{10}, X_4 + X_6 + X_{12} - X_{14}\} + \\
\frac{1}{2}(X_5 - X_7 - X_{13} - X_{15})^2 + \{X_2 - X_{11}, X_1 + X_2 + X_3 + X_9\} + \\
\frac{1}{2}\{X_7 + X_{15}, -X_1 + 2X_2 + X_3 + 2X_{11}\} + \\
\frac{1}{2}\{X_5 - X_{13}, -X_1 - 2X_2 + X_3 - 2X_{11}\}
\]

\[
T_3 = 2(X_4^2 + X_8^2 + X_{10}^2 + X_{14}^2 - X_6^2 - X_{12}^2) - 3X_1^2 - 4X_2^2 - 3X_3^2 - \\
2\{X_1, X_2\} - \{X_1, X_3\} - 2\{X_2, X_3\} + \\
2(X_6^2 - X_7^2 + X_9^2 + X_{11}^2 - X_{13}^2 + X_{15}^2)
\]

\[
T_4 = -\frac{1}{2}(X_4 + X_{12})^2 + \frac{1}{2}(X_6 - X_{14})^2 + X_1^2 - X_3^2 + \frac{1}{2}\{X_1, X_2\} - \\
\frac{1}{2}\{X_2, X_3\} + \frac{1}{2}\{X_9, X_1 - X_3\} - \frac{1}{2}(X_5 - X_{13})^2 + \frac{1}{2}(X_7 + X_{15})^2
\]

\[
T_5 = \frac{1}{2}\{X_4 + X_6 + X_{12} - X_{14}, X_8 - X_{10}\} + \\
\frac{1}{2}\{X_1 + 2X_2 + X_3 + X_9 + X_{11}, X_5 - X_{15}\} + \\
\frac{1}{2}\{X_1 + X_3 + X_9 - X_{11}, X_7 + X_{13}\}
\]

\[
T_6 = -\frac{1}{2}(X_4 + X_6)^2 + \frac{1}{2}(X_{12} - X_{14})^2 - \\
\frac{1}{2}\{X_4 - X_6 + X_{12} + X_{14}, X_8 + X_{10}\} - \\
\frac{1}{2}\{X_1 + 2X_2 + X_3 + X_9 - X_{11}, X_5 + X_{15}\} + \\
\frac{1}{2}\{X_1 + X_3 + X_9 + X_{11}, X_7 - X_{13}\} - \\
\frac{1}{2}\{X_1, X_2\} + \frac{1}{2}\{X_2, X_3\} - \frac{1}{2}(X_5 + X_7)^2 - \frac{1}{2}\{X_{11}, X_1 - X_3\} + \\
\frac{1}{2}(X_{13} - X_{15})^2
\]
**Appendix 2.** Commutation tables of second order elements commuting with each MASA. The operators $A, B, C, D$ are linearly independent third order elements in the enveloping algebra and are different in each case.

**MASA 1:**

|   | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ |
|---|---|---|---|---|---|---|
| $T_1$ | $A$ | $B$ | $A$ | $B$ | $0$ |   |
| $T_2$ | $C$ | $A$ | $0$ | $-C$ |   |   |
| $T_3$ | $0$ | $-B$ | $-C$ |   |   |   |
| $T_4$ | $D$ | $D$ |   |   |   |   |
| $T_5$ |   |   |   |   |   |   |
| $T_6$ |   |   |   |   |   |   |

**MASA 2:**

|   | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ |
|---|---|---|---|---|---|---|
| $T_1$ | $0$ | $A$ | $A$ | $B$ | $B$ |   |
| $T_2$ | $0$ | $0$ | $C$ | $D$ |   |   |
| $T_3$ | $A$ | $-C$ | $B$ |   |   |   |
| $T_4$ | $-B$ | $D$ |   |   |   |   |
| $T_5$ |   |   |   |   |   |   |
| $T_6$ |   |   |   |   |   |   |

**MASA 3:**

|   | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ |
|---|---|---|---|---|---|---|
| $T_1$ | $0$ | $0$ | $A$ | $B$ |   |   |
| $T_2$ | $0$ | $C$ | $D$ | $0$ |   |   |
| $T_3$ | $-C$ | $-A$ | $-B$ | $-D$ |   |   |
| $T_4$ | $-B$ | $D$ | $A$ |   |   |   |
| $T_5$ | $C$ |   |   |   |   |   |
| $T_6$ |   |   |   |   |   |   |

**MASA 4:**

|   | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ |
|---|---|---|---|---|---|---|
| $T_1$ | $A$ | $A$ | $0$ | $B$ | $B$ |   |
| $T_2$ | $0$ | $C$ | $D$ | $0$ |   |   |
| $T_3$ | $D$ | $-A$ | $-D$ |   |   |   |
| $T_4$ | $0$ | $0$ | $B$ |   |   |   |
| $T_5$ |   |   |   |   |   |   |
| $T_6$ |   |   |   |   |   |   |

**MASA 5:**

|   | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ |
|---|---|---|---|---|---|---|
| $T_1$ | $0$ | $0$ | $A$ | $0$ | $B$ |   |
| $T_2$ | $C$ | $D$ | $0$ | $0$ |   |   |
| $T_3$ | $0$ | $-C$ | $-A$ | $-D$ |   |   |
| $T_4$ | $A$ | $D$ | $C$ |   |   |   |
| $T_5$ | $-B$ |   |   |   |   |   |
| $T_6$ |   |   |   |   |   |   |

**MASA 6:**

|   | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ |
|---|---|---|---|---|---|---|
| $T_1$ | $A$ | $B$ | $0$ | $0$ | $A$ | $+B$ |
| $T_2$ | $0$ | $C$ | $D$ | $0$ |   |   |
| $T_3$ | $D$ | $-A$ | $-D$ |   |   |   |
| $T_4$ | $0$ | $0$ | $B$ |   |   |   |
| $T_5$ |   |   |   |   |   |   |
| $T_6$ |   |   |   |   |   |   |

**MASA 7,8:**

|   | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ |
|---|---|---|---|---|---|---|
| $T_1$ | $0$ | $A$ | $-4A$ | $B$ | $-2B$ |   |
| $T_2$ | $0$ | $0$ | $B$ | $C$ |   |   |
| $T_3$ | $0$ | $D$ | $0$ |   |   |   |
| $T_4$ | $-4D$ | $0$ |   |   |   |   |
| $T_5$ | $-2A$ |   |   |   |   |   |
| $T_6$ |   |   |   |   |   |   |

**MASA 9:**

|   | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ |
|---|---|---|---|---|---|---|
| $T_1$ | $A$ | $-A$ | $0$ | $0$ | $B$ |   |
| $T_2$ | $0$ | $C$ | $D$ | $0$ |   |   |
| $T_3$ | $-C$ | $-D$ | $0$ |   |   |   |
| $T_4$ | $B$ | $-D$ |   |   |   |   |
| $T_5$ | $C$ |   |   |   |   |   |
| $T_6$ |   |   |   |   |   |   |

**MASA 10:**

|   | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ | $T_7$ |
|---|---|---|---|---|---|---|---|
| $T_1$ | $A$ | $0$ | $0$ | $B$ | $0$ | $E$ |   |
| $T_2$ | $0$ | $C$ | $0$ | $0$ | $F$ |   |   |
| $T_3$ | $B$ | $C$ | $0$ | $0$ |   |   |   |
| $T_4$ | $D$ | $0$ | $G$ |   |   |   |   |
| $T_5$ | $H$ |   |   |   |   |   |   |
| $T_6$ | $0$ |   |   |   |   |   |   |
| $T_7$ |   |   |   |   |   |   |   |

**MASA 11:**

|   | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ |
|---|---|---|---|---|---|---|
| $T_1$ | $A$ | $0$ | $0$ | $0$ | $B$ |   |
| $T_2$ | $0$ | $C$ | $D$ | $0$ |   |   |
| $T_3$ | $0$ | $0$ | $0$ |   |   |   |
| $T_4$ | $B$ | $C$ | $/2$ |   |   |   |
| $T_5$ | $D$ | $/2$ |   |   |   |   |
| $T_6$ |   |   |   |   |   |   |