Thermalization in Backgrounds with Hyperscaling Violating Factor

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Abstract

We present an analytic solution of a Vaidya-charged black hole with hyperscaling violating factor in an Einstein-Maxwell-Dilaton model, where the scalar potential is key to the existence of a solution. By making use of this result we will study the process of thermalization after a global quench in a theory whose gravitational description is provided by the resultant solution. In particular we shall probe the system by entanglement entropy and show that it exhibits certain scaling behaviours during the process.
1 Introduction

One of the interesting phenomena which appears in different areas of physics is the process of thermalization of a non-equilibrium state. To understand the process of thermalization there are several questions one may ask. These questions include: how fast a process may occur and what are the useful quantities which could probe it? A subtlety to address these questions is the fact that the standard thermodynamics which usually provides a useful tool to study long range physics is not applicable in this case where the system is out of equilibrium.

Evolution of a system after a global quantum quench [1] is an example of the thermalization. It is known that such an evolution cannot be studied within the context of the standard thermodynamics. Indeed, being out of equilibrium the thermodynamical quantities such as temperature, entropy, pressure, etc. may not be well defined during the process of thermalization. Nevertheless the process may be probed by entanglement entropy which, indeed, exhibits different scaling behaviours in different regions as the system evolves with time.

Of course for a generic quantum system it is difficult to compute entanglement entropy, though for those systems (typically strongly coupled) which have gravitational description [2] one may utilize the holographic description of the entanglement entropy [3,4] to study its behaviour. We note, however, that since the system is time-dependent one should use its covariant proposal [5].

Actually in the context of gauge/gravity duality the thermalization process after a global quantum quench may be map to a black hole formation due to a gravitational collapse. More precisely quantum quench in the field theory may occurs due to a sudden change in the system which might be caused by turning on the source of an operator in an interval $\delta t \to 0$. This change can excite the system to an excited state with non-zero energy density that could eventually thermalize to an equilibrium state. From gravity point of view this might be described by a gravitational collapse of a thin shell of matter which can be modelled by an AdS-Vaidya metric. The corresponding metric for a collapse of a neutral matter in $d + 1$ dimensions is

$$ds^2 = \frac{1}{\rho^2} \left( f(\rho, v) dv^2 - 2d\rho dv + d\vec{x}^2 \right), \quad f(\rho, v) = 1 - m \theta(v) \rho^d,$$

where $\theta(v)$ is the step function. Therefore for $v < 0$ the geometry is an AdS metric and for $v > 0$ it is a Schwarzschild black hole.

It is then natural to compute the holographic entanglement entropy in this time-dependent background using its covariant description. Indeed, by making use of this duality, entanglement entropy after a global quench for strongly coupled systems has been studied in several papers, including [6–16]. Other quantities such as Wilson loop and two point function of an operator in the dual theory have been also studied in most of these papers (see also [17]). General features of these quantities during the evolution of the system consist of a period of growth with time and then saturation phase when the corresponding quantities saturate to their equilibrium values.
Holographically, these quantities can be computed by extremizing certain hypersurfaces with different dimensions in the above geometry. For example in the case of entanglement entropy it is a co-dimension two hypersurface, while for the Wilson loop it is a two dimensional surface and for two point function it is just a geodesic.

Typically the model has two time scales set by the radius of the horizon, $\rho_H$, and the size of the subsystem we are considering. For example in the case of the entanglement entropy it is the size of entangling region $\ell$. Indeed the theory would reach local equilibrium at $t \sim \rho_H$ when the system stop producing thermodynamic entropy, while at $t \sim \frac{\ell}{2}$ the entanglement entropy saturates to its equilibrium value. When $\frac{\ell}{2} \lesssim \rho_H$ by the time the system ceased production the thermodynamic entropy, it had already passed the saturation time when the entanglement entropy was saturated. On the other hand for $\frac{\ell}{2} \gg \rho_H$ the entanglement entropy still increases even though the system is locally equilibrated.

From gravity point of view when $\frac{\ell}{2} \lesssim \rho_H$ the corresponding co-dimension two hypersurface all the time remains outside the horizon, though for $\frac{\ell}{2} \gg \rho_H$ and for $t \gtrsim \rho_H$ the co-dimension two hypersurface may probe inside the horizon. Actually in this case the main contribution is controlled by the geometry around and inside the horizon [18,19].

The aim of this paper is to extend the consideration of [18,19] for those strongly coupled theories whose gravitational descriptions are provided by hyperscaling violating metrics [21–23]. To do so, we first need to find a Vaidya metric with a hyperscaling violating factor. This is, indeed, what we will do in the next section. Then we will study time-dependent behaviours of certain quantities in this background. Although one could explore all of these quantities within a uniform formalism, we will mainly consider entanglement entropy in more details.

The paper is organized as follows. In the next section we will find an analytic solution of an Einstein-Maxwell-Dilaton model representing of a Vaidya metric with hyperscaling violating factor. In section three we will consider holographic entanglement entropy where we will also set up a formalization to study other quantities. In section four we will explore general behaviours of the quantities as the system evolves with time. In particular we will consider entanglement entropy in more details for the case where the entangling region is much more bigger than the radius of horizon. The last section is devoted to conclusions. In an appendix we will review holographic entanglement entropy in a static black hole with a hyperscaling violating factor.

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1In this paper we will only consider an entangling region in the shape of a strip with a width $\ell$, though its generalization to a sphere is straightforward.

2Actually it was observed in [20] that the extremal surfaces in the bulk cannot penetrate through the horizon of a static black hole.

3Entanglement entropy for Vaidya-Lifshitz geometry has been studied in [24].
2 Infalling shell solutions

Hyperscaling violating geometries with non-zero charged has been studied in [25] where it was shown that an Einstein-Maxwell-Dilaton model with a particular potential admits such solutions. The corresponding action is

\[ S = -\frac{1}{16\pi G_N} \int d^{D+2}x \sqrt{-g} \left[ R - \frac{1}{2}(\partial \phi)^2 + V(\phi) - \frac{1}{4} \sum_{i=1}^{N_g} e^{\lambda_i \phi} F_i^2 \right], \]  

(2.1)

where \( V(\phi) = V_0 e^{\gamma \phi} \), \( G_N \) is the Newton constant and \( \lambda_1, \lambda_2, \gamma \) and \( V_0 \) are free parameters of the model. One of the gauge field is required to produce an anisotropy while the above particular form of the potential is needed to get hyperscaling violating factor. The other gauge fields make the background charged. In what follows we will only consider \( N_g = 2 \) though its generalization to other \( N_g \) is obvious. In this case the corresponding solution, with a proper normalization of \( \phi \), is [25]

\[ ds^2 = r^{-2}\beta^2 \left( -r^{2z} f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 d\vec{x}^2 \right), \quad \phi = \beta \ln r, \]

\[ A_i^{(1)} = \sqrt{\frac{2(z-1)}{D-\theta+z}} r^{D-\theta+z}, \quad A_i^{(2)} = \sqrt{\frac{2(D-\theta)}{D-\theta+z-2}} r^{D-\theta+z-2}, \]  

(2.2)

with \( \beta = \sqrt{2(D-\theta)(z-1-\theta/D)} \) and

\[ f(r) = 1 - \frac{m}{r^{D-\theta+z}} + \frac{Q^2}{r^{2(D-\theta+z-1)}}. \]  

(2.3)

The parameters of the action are also found

\[ \lambda_1 = \frac{2\theta(D-1) - 2D^2}{D\beta}, \quad \lambda_2 = \frac{\beta}{D-\theta}, \quad \gamma = \frac{2\theta}{D\beta}, \quad V_0 = (D-\theta+z-1)(D-\theta+z) \]  

(2.4)

This is indeed a charged black brane solution whose radius of horizon, \( r_H \), is obtained by setting \( f(r_H) = 0 \). It is useful to define an effective dimension \( d = D - \theta + 1 \) by which the function \( f \) reads

\[ f(r) = 1 - \frac{m}{r^{d-1+z}} + \frac{Q^2}{r^{2(d-\theta+z)}}. \]  

(2.5)

Therefore the radius of horizon can be obtained from following algebraic equation for \( r_H \):

\[ r_H^{2(d-\theta+z)} - mr_H^{d-3+z} + Q^2 = 0. \]  

(2.6)
The Hawking temperature and the thermal entropy of the solution are
\[
T = \frac{(d - 1 + z)r_H^2}{4\pi} \left( 1 - \frac{(d - 3 + z)Q^2}{d - 1 + z} r_H^{2(d-2+z)} \right), \quad S_{BH} = \frac{V_D r_H^{d-1}}{4G_N} \equiv V_D S_{BH},
\]
where \( V_D \) is the volume of the spatial directions, \( x_i, i = 1, \cdots, D \), and \( S_{BH} \) is the entropy density.

In what follows we would like to find a background representing an infalling shell of massless and pressureless charged matter in a hyperscaling violating geometry. The corresponding geometry may be thought of as a Vaidya metric with a hyperscaling violating factor. The Vaidya-Lifshitz geometry has been studied in [24].

To proceed, it is useful to use an Eddington-Finkelstein like coordinate system as follows
\[
dv = dt + \frac{dr}{f(r)r^{z+1}}.
\]

In this coordinate the above metric reads
\[
ds^2 = r^{-2\phi \frac{\phi}{D}} \left( -r^{2z} f(r)dv^2 + 2r^{z-1} drdv + r^2 d\vec{x}^2 \right).
\]

Moreover in the gauge of \( A_r^{(i)} = 0 \) the non-zero component of the gauge field is \( A_v^{(i)} \). The dilaton remains unchanged. The Vaidya space-time is sourced by an energy momentum tensor and a current density of a massless null charged matter. Therefore in order to get such a solution one needs to add a proper extra matter field to the action. Doing so, the equations of motion of the action (2.1) should be modified as follows
\[
R_{\mu\nu} + \frac{V(\phi)}{D} g_{\mu\nu} = \frac{1}{2} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} \sum_{i=1}^{2} e^{\lambda_i \phi} \left( F^{(i)}_{\mu \rho} F^{(i)}_{\rho \nu} - \frac{g_{\mu\nu}}{2D} F^{(i)}_{\rho \sigma} F^{(i)}_{\rho \sigma} \right) + T_{\mu\nu},
\]
\[
\nabla^2 \phi = -\frac{dV(\phi)}{d\phi} + \frac{1}{4} \sum_{i=1}^{2} \lambda_i e^{\lambda_i \phi} F^{(i)}_{\mu \nu} F^{(i)}_{\mu \nu}, \quad \nabla_\mu \left( \sqrt{-g} e^{\lambda_1 \phi} F^{(i)}_{\mu \nu} \right) = J^{(i)}_{\nu},
\]
where \( T_{\mu\nu} \) and \( J^{(i)}_{\nu} \) are the energy momentum and current density of the charged matter field, respectively. As we will see for the model we are considering the corresponding non-zero components of the energy momentum and current density are \( T_{vv} \) and \( J^{(2)}_v \).

Motivated by the above solution, let us consider the following ansatz for the metric, scalar and gauge field
\[
ds^2 = r^{-2\phi \frac{\phi}{D}} \left( -r^{2z} f(r,v) dv^2 + 2r^{z-1} drdv + r^2 d\vec{x}^2 \right), \quad \phi = \phi(r), \quad F^{(1)}_{rv}(r) \neq 0, \quad F^{(2)}_{rv}(r, v) \neq 0,
\]
all the other components of gauge fields are set to zero. Note that in the ansatz the scalar field
and the first gauge field which are needed to support an anisotropic solution with hyperscaling violating metric are independent of the null coordinate $v$, while the metric component $f$ and the second gauge field are functions of both $r$ and $v$ coordinates.

It is straightforward to plug this ansatz into the equations of motion to find the unknown functions. Actually from the $rr$ component of the Einstein equations one finds

$$\phi = \sqrt{2(D - \theta)(z - 1 - \theta/D)} \ln r = \beta \ln r.$$  \hspace{1cm} (2.12)

Note that the null energy condition requires $(D - \theta)(z - 1 - \theta/D) \geq 0$. From the $ii$'s components of the Einstein equations, taking into account the identifications of (2.4), one can fix the function $f$ as follows (see [25] for more details)

$$f(r, v) = 1 - \frac{m(v)}{r^{D-\theta+z}} + \frac{Q(v)^2}{r^{2(D-\theta+z-1)}},$$  \hspace{1cm} (2.13)

where $m(v)$ and $Q(v)$ are arbitrary functions of $v$. The gauge fields can be also obtained from the $v$ component of the Maxwell equations. With these equations one can fix the solution completely. Actually the solution is

$$ds^2 = r^{-2} \left( -r^{2z} f(r, v) dv^2 + 2r^{z-1} dr dv + r^2 d\vec{x}^2 \right), \quad \phi = \beta \ln r,$$

$$A_v^{(1)} = \sqrt{\frac{2(z - 1)}{D - \theta + z}} r^{D-\theta+z}, \quad A_v^{(2)} = \sqrt{\frac{2(D - \theta)}{D - \theta + z - 2}} \frac{Q(v)}{r^{D-\theta+z-2}}.$$  \hspace{1cm} (2.14)

Of course so far we have not considered all equations. Indeed from $vv$ component of the Einstein equations and $r$ component of the Maxwell equations one may read the energy momentum and the current density needed to support an infalling shell solution. More precisely from the corresponding components of the equations of motion one finds that the energy momentum and current density of the charged infalling pressureless matter are given by $T_{\mu\nu} = \rho U_{\mu} U_{\nu}$ and $J^{(2)}_{\mu} = \varrho e U_{\mu}$ with $U_{\mu} = \delta_{\mu v}$, and

$$\rho = \frac{\theta - D \partial f(r, v)}{2} r^z, \quad \varrho e = \frac{\partial Q(v)}{\partial v} \sqrt{2(D - \theta)(D - \theta + z - 2)} r^{\theta-D}.$$  \hspace{1cm} (2.15)

Note that the null energy condition requires $\varrho > 0$. Finally the last non-trivial equation needs to be checked is the $rv$ component of the Einstein equations. Actually it is easy to see that this equation is also satisfied without imposing any further constraints.

The solution (2.14) can be thought of as a gravity solution which describes a gravitational collapse of a charged matter in a model that has an anisotropy with a hyperscaling violating factor. Using the gauge/gravity duality it is natural to conjecture that this solution provides a

\footnote{In this paper we will consider the case of $z > 1$ and $D > \theta$.}
gravity description for the process of thermalization after a global quantum quench in a strongly coupled field theory with an anisotropy and violation of scaling. In what follows we would like to probe this process by certain quantities including entanglement entropy. To do so, we will utilize the holographic description of the covariant entanglement entropy to compute it in the above background.

3 Entanglement entropy

Entanglement entropy may be considered as a useful parameter which could probe the system when it undergoes a rapid change. A global quantum quench is a prototype example of a rapid change after which the system is out of equilibrium and it takes time for the system to approach an equilibrium state when it is thermalized. In this context it is important to see when the system is thermalized and how fast the thermalization occurs. Entanglement entropy may be useful to address these questions. Therefore in this section utilizing the holographic description of entanglement entropy we will study the entanglement entropy for a strip in a theory undergoing a rapid change whose gravity dual is given by the solution (2.14).

To proceed for simplicity we set the charge to zero, \( Q(v) = 0 \), so that the background is neutral. In this case setting \( r = \rho^{-1} \) one has

\[
ds^2 = \rho^{2(D-\theta)} \left( -\rho^{2-2z} f(\rho, v) dv^2 - 2\rho^{1-z} d\rho d\bar{x}^2 \right), \quad \text{with} \quad f = 1 - m(v) \rho^{d-1+z}. \quad (3.1)
\]

Here, as before, \( d = D - \theta + 1 \) is the effective dimension.

To compute the entanglement entropy for a strip with width \( \ell \), let us consider the following strip

\[-\ell/2 \leq x_1 = x \leq \ell/2, \quad 0 \leq x_a \leq L, \quad \text{for} \quad a = 2, \cdots, D. \quad (3.2)\]

Of course since the metric is not static one needs to use the covariant proposal for the holographic entanglement entropy \[5\]. Therefore the corresponding co-dimension two hypersurface in the bulk may be parametrized by \( v(x) \) and \( \rho(x) \). Then the induced metric on the hypersurface is

\[
ds_{\text{ind}}^2 = \rho^{2(D-\theta)} \left[ \left( 1 - \rho^{2-2z} f(\rho, v) v'^2 - 2\rho^{1-z} \rho' \right) dx^2 + dx_a^2 \right]. \quad (3.3)
\]

where “prime” represents derivative with respect to \( x \). Using the above induced metric the area of the hypersurface reads

\[
A = \frac{L^{D-1}}{2} \int_{-\ell/2}^{\ell/2} dx \frac{\sqrt{1 - 2\rho^{1-z} v' \rho' - \rho^{2-2z} v'^2 f}}{\rho^{d-1}}. \quad (3.4)
\]

Although our aim was to explore certain properties of the entanglement entropy during a global
quench after which the system undergoes a thermalization process, with a small change one may do even more. Indeed beside the entanglement entropy we could also study Wilson loop and geodesics. To do so, let us define the quantity $A$ as follows

$$A = \frac{L_D^{D+1}}{2} \int_{-\ell/2}^{\ell/2} dx \sqrt{1 - 2p^{1-z}v'p' - p^{2-2z}v'^2 f} = \frac{L_D^{D+1}}{2} \int_{-\ell/2}^{\ell/2} dx \frac{L_D}{\rho^n}. \quad (3.5)$$

Note that for entanglement entropy one should set $n = d - 1$, while for the Wilson loop $n = \frac{2(d-1)}{D}$ and for geodesics $n = d - 1$. Having found $A$ the entanglement entropy, Wilson loop and geodesics can be found respectively as follows:

$$S = \frac{A}{4G_N}, \quad \langle W(t) \rangle \sim e^{-\frac{A}{\pi\alpha'}} \quad G(\ell, t) \sim e^{-MA}, \quad \#(3.6)$$

where $2\pi\alpha'$ is the string tension and $M$ is the mass of the dual operator.

In order to extremize $A$ we note that the above expression may be thought of as a one dimensional action for a dynamical system for the fields $v(x)$ and $\rho(x)$. We note, however, that since the action is independent of $x$ the corresponding Hamiltonian is a constant of motion

$$\rho^n L = H = \text{constant}. \quad (3.7)$$

Moreover we have two equations of motion for $v$ and $\rho$. Indeed, by making use of the above conservation law the corresponding equations of motion read

$$\partial_x P_v = \frac{P_v^2}{2} \frac{\partial f}{\partial v'}, \quad \partial_x P_\rho = \frac{P_\rho^2}{2} \frac{\partial f}{\partial \rho} + \frac{n}{\rho^{2n+1}} H^2 + \frac{1 - z}{\rho^{2-z}} P_\rho P_v, \quad \#(3.8)$$

where

$$P_v = \rho^{1-z}(\rho' + \rho^{1-z}v' f), \quad P_\rho = \rho^{1-z}v', \quad (3.9)$$

are the momenta conjugate to $v$ and $\rho$ up to a factor of $H^{-1}$, respectively. These equations have to be supplemented by the following boundary conditions

$$\rho(\frac{\ell}{2}) = 0, \quad v(\frac{\ell}{2}) = t, \quad \rho'(0) = 0, \quad v'(0) = 0, \quad \#(3.10)$$

and

$$\rho(0) = \rho_t, \quad v(0) = v_t, \quad \#(3.11)$$

where $(\rho_t, v_t)$ is the coordinate of the turning point. With this boundary condition one has $H = \rho^n_t$.

Given a particular form for $m(v)$ one may solve the above equations of motion to find the extremal surface, Then the corresponding quantity can be found from the equation (3.6) evaluated at the

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Note that in the case of the Wilson loop, $\ell$ is the size of loop and for geodesics it is the distance between two operators in the boundary.
extremal surface. Of course in general it is not possible to solve the equations analytically, though one may use a numerical method.

As far as the entanglement entropy is concerned it is worth mentioning that although the model is fixed by three free parameters given by the dimension of the space time, \( D \), the scaling violating parameter, \( \theta \), and the dynamical exponent \( z \), the entanglement entropy is just sensitive to the effective dimension \( d \) and the dynamical exponent \( z \). Therefore even though we are dealing with the hyperscaling violating geometry, the resultant entanglement entropy is the same as that of a Lifshitz geometry in \( d + 1 \) dimensions. Of course this is not the case for Wilson loop and geodesics where they are sensitive to both \( D \) and \( \theta \).

Actually the holographic entanglement entropy for a \( d + 1 \) dimensional Vaidya-Lifshitz solution has been studied in \[24\]. Thus for given effective dimension \( d \) (to be identified with the dimension of the Lifshitz geometry) one can read the results from that in \[24\]. In particular by making use of a numerical method it was shown that the entanglement entropy has a linear growth with time before it saturates at a saturation time given by \( t_s \sim \ell/2 \) after which the entanglement entropy is equal to that of the corresponding black hole solution. In the next section we shall further explore the behaviour of entanglement entropy (and other quantities as \( (3.6) \)) for a global quantum quench holographically modelled by the time-dependent geometry \( (2.14) \) for the case of large entangling region.

\section{4 General consideration for large entangling region}

In this section following the recent papers \[18\][19] we will further study different aspects of quantities given in the equation \( (3.6) \) during a global quantum quench in a theory whose gravity description is given by the solution \( (2.14) \). Since we are interested in a quench where there is a rapid change in the theory, one may assume that \( f(\rho, v) = 1 - \theta(v)g(\rho) \), where \( \theta(v) \) is the step function. In what follows we will consider the case where \( \ell \gg r_H \). Therefore the hypersurface on the bulk may probe near and behind the horizon.

Since the function \( f \) contains a step function, most of time it does not depend on \( v \). In other words mostly one has \( \frac{\partial f(\rho, v)}{\partial v} = 0 \) which in turn means that the momentum conjugate of \( v \) is also a constant of motion

\[ P_v = \rho^{1-z}(\rho' + \rho^{1-z}v'\tilde{f}(\rho)) = \text{constant}, \quad \text{with} \quad \tilde{f}(\rho) = 1 - g(\rho). \quad (4.1) \]

In the present case one has \( g(\rho) = \left(\frac{\rho}{\rho_H}\right)^{d-1+z} \) with \( m = \frac{1}{\rho_H^{1+z}} \).

The process we will be considering for the thermalization after a global quantum quench consists of three phases: initial phase, intermediate phase and final phase. In the initial phase where the system is in a vacuum state the gravity dual will be given by the hyperscaling violating solution \( (2.14) \) with \( \tilde{f}(\rho) = 1 \). On the other hand in the final phase where the system is thermalized, the
corresponding gravity solution is given by a static black hole with a hyperscaling violating factor as in the equation (2.14). In the intermediate phase where the global quench occurs one has a discontinuity due to injection of the infalling shell matter. Therefore it is important to match the initial and final solutions with proper matching conditions in the intermediate region.

Before going into details it is useful to fix our notation. During the course of our study we will be dealing with three distinguished points: turning point denoted by \((\rho_t, v_t)\), crossing point, \((\rho_c, v_c)\), where the hypersurface intersects the null shell and position of the horizon \(\rho_H\) where \(\tilde{f}(\rho_H) = 0\).

In the initial phase one has

\[ P_{(i)v} = \rho' + \rho^{1-z} v' = 0, \] (4.2)

which together with the conservation law (3.7) yields to (see also the appendix A for entanglement entropy)

\[ v(\rho) = v_t + \frac{1}{z}(\rho_t^z - \rho^z), \quad x(\rho) = \int_{\rho}^{\rho_t} \frac{d\xi \xi^n}{\sqrt{\rho_t^{2n} - \xi^{2n}}}. \] (4.3)

On the other hand for the final phase using the equations (4.1) and (3.7) one arrives at

\[ \rho'^2 = \frac{P_{(f)v}^2}{\rho^{2-2z}} + \left( \left( \frac{\rho_t}{\rho} \right)^{2n} - 1 \right) \tilde{f}(\rho) \equiv V_{eff}(\rho), \] (4.4)

which can also be used to find

\[ \frac{dv}{d\rho} = -\frac{1}{\rho^{2(1-z)}\tilde{f}(\rho)} \left( \rho^{1-z} + \frac{P_{(f)v}}{\sqrt{V_{eff}(\rho)}} \right). \] (4.5)

Here \(V_{eff}(\rho)\) might be thought of as an effective potential for a one dimensional dynamical system whose dynamical variable is \(\rho\). In particular the turning point of the potential can be found by setting \(V_{eff}'(\rho) = 0\). As we will see such an interpretation has a physical impact in the exploring of the behaviour of quantities (3.6).

It is now important to match two phases at the null shell. Of course since \(\rho\) and \(v\) are the coordinates of the space time they should be continuous across the null shell. Note also that at the null shell where \(v = 0\), from the equation (4.3), one gets

\[ \rho_c^z = \rho_t^z + zv_t \] (4.6)

which, indeed, gives the point where the hypersurface intersects the null shell. Moreover, by making use of the equation (3.7) in the initial phase one finds

\[ \rho_{(i)}' = -\rho_{(i)}^{1-z} v_{(i)}' = -\sqrt{\left( \frac{\rho_t}{\rho_c} \right)^{2n} - 1} \] (4.7)
Since one is injecting the matter in \(v\) direction, one would expect that its corresponding momentum conjugate jumps once one moves from the initial phase to the final phase. While the momentum conjugate of \(\rho\) must be continuous. Therefore one gets \(v'_{(f)} = v'_{(i)}\). On the other hand by integrating the equations of motion across the null shell one arrives at

\[
\rho'_{(f)} = \left(1 - \frac{1}{2}g(\rho_c)\right)\rho'_{(i)}, \quad \mathcal{L}_{(f)} = \mathcal{L}_{(i)}. \tag{4.8}
\]

It is, then, easy to read the momentum conjugate of \(v\) in the final phase

\[
P_{(f)v} = \frac{1}{2}\rho_c^{1-z}g(\rho_c)\rho'_c = -\frac{1}{2}\rho_c^{1-z}g(\rho_c)\sqrt{\left(\frac{\rho_c}{\rho_c}\right)^{2n} - 1}. \tag{4.9}
\]

Now we have all ingredients to find the area of the corresponding extremal hypersurface in the bulk. In general the hypersurface could extend in both black hole and horizonless regions in the bulk. Therefore the width \(\ell\) and the boundary time are found

\[
\frac{\ell}{2} = \rho_t \left(\int_{\frac{\rho_t}{\rho_H}}^{1} \frac{d\xi}{\sqrt{1 - \xi^{2n}}} + \int_{0}^{\frac{\rho_t}{\rho_H}} \frac{d\xi}{\sqrt{R(\xi)}}\right), \quad t = \rho_t \int_{0}^{\frac{\rho_H}{\rho_H}} \frac{d\xi}{h(\xi)} \left(1 + \frac{\xi^{z-1}E}{\sqrt{R(\xi)}}\right), \tag{4.10}
\]

where \(E = P_{(f)v}\rho_{t}^{z-1}\) and

\[
h(\xi) = \tilde{f}(\rho_t \xi) = 1 - \left(\frac{\rho_t}{\rho_H}\right)^{d-1+z} \xi^{d-1+z},
\]

\[
R(\xi) = V_{eff}(\rho_t \xi) = E^2 \xi^{2(z-1)} + \left(\frac{1}{\xi^{2n}} - 1\right) h(\xi). \tag{4.11}
\]

Finally the quantity \(A\) reads (see also the appendix A for the entanglement entropy)

\[
A = \frac{L^{\frac{d}{n}+1}}{\rho_t^{n-1}} \left(\int_{\frac{\rho_t}{\rho_H}}^{1} \frac{d\xi}{\xi^{n}\sqrt{1 - \xi^{2n}}} + \int_{0}^{\frac{\rho_H}{\rho_H}} \frac{d\xi}{\xi^{2n}\sqrt{R(\xi)}}\right). \tag{4.12}
\]

In the rest of this section using this general consideration we will study the behaviour of the quantities \(3.6\) in different limits.

### 4.1 Early times growth

In this subsection we will study behaviours of the quantities given in \(3.6\) at the early times where \(t \ll \rho_t^{z}\). In this case the crossing point is very close to the boundary, so that \(\tilde{\omega}_{ph} \ll 1\). Therefore one can expand the expressions of \(t, \ell\) and \(A\) in this limit to find the corresponding quantities.
More precisely, from the equation (4.10) at leading order one finds

\[ t \approx \rho^z \int_0 \frac{\xi^{z-1}}{h(\xi)} \frac{d\xi}{\rho c} \left[ 1 + \frac{z}{d-1+2z} \left( \frac{\rho_c}{\rho_H} \right)^{d-1+z} + \cdots \right], \]  

(4.13)

while from the equation (4.12) and for \( d + z - n \neq 0 \) at leading order one gets

\[ \Delta A = A - A_{\text{vac}} \approx \frac{L^{\rho_{d+n-1}}}{2(d-n+z)} m \rho_c^{d-n+z}. \]  

(4.14)

Therefore at leading order one arrives at

\[ \Delta A \approx \frac{L^{\rho_{d+n-1}} m}{2(d-n+z)} (zt)^{1 + \frac{d-n}{z}}, \]  

(4.15)

which reduces to that of Vaidya-AdS considered in [18, 19] for \( z = 1 \). Note that for entanglement entropy when \( n = d - 1 \) even though we are dealing with a hyperscaling violating geometry, the early time growth depends only on the dynamical exponent \( z \) as if we had considered Lifshitz geometry. It is also interesting to note that for sufficiently large \( z \) one finds a linear growth at the earlier times

\[ \Delta A \approx \frac{L^{\rho_{d+n-1}} m}{2} t, \]  

(4.16)

which has a universal form for all quantities in (3.6).

On the other hand for \( d + z - n = 0 \) one gets a logarithmic behaviour as follows\(^6\)

\[ \Delta A = A - A_{\text{vac}} \approx \frac{L^{\rho_{d+n-1}} m}{2} \ln \frac{\rho_c}{\rho_t}, \]  

(4.17)

so that

\[ \Delta A \approx \frac{L^{\rho_{d+n-1}} m}{2z} \ln \frac{zt}{\rho_t}. \]  

(4.18)

It is important to note that for the case of entanglement entropy where \( n = d - 1 \) the logarithmic behaviour occurs for \( z = -1 \) which using the null energy condition it requires to assume \( \theta > D \). We note however that in this case the corresponding solution might be unstable [23].

---

\(^6\)In fact in [26] it was shown that in many body strongly interacting disordered systems the entanglement entropy presents this universal slow growth behaviour at early times. We would like to thank Juan F. Pedraza for a comment on this point.
4.2 Late time saturation

In general the late time behaviours of the quantities (3.6) after a global quench where \( t \sim \frac{\ell}{2} \) depends on the details of the system as well as the shape of the entangling region or Wilson loop. Nevertheless one would expect that if one waits enough these quantities saturate to their equilibrium values which are essentially that of a thermal state. Actually for large \( \ell \) it is proportional to the corresponding thermal quantities of the final state. In fact this behaviour can be readily seen from the equation (4.12). More precisely from holographic description point of view at the late time when the system is thermalized the hypersurface in the bulk remain entirely outside the horizon and therefore the equation (4.12) reduces to that of a static black hole.

Indeed, it is possible to make a rough estimation of the quantities at the saturation time using the general formula (4.10). Actually the onset of the saturation may be given by a point where \( \rho_c \approx \rho_t \to \rho_H \). In this limit \( P_{(f)u} \approx 0 \), and therefore from the equations (4.10) and (4.12) one finds

\[
\frac{\ell}{2} \approx \rho_t \int_{1-\epsilon}^{1} \frac{d\xi}{\sqrt{R(\xi)}} \approx \rho_t \int_{1-\epsilon}^{1} \frac{d\xi \xi^n}{\sqrt{(1 - \xi^{2n})h(\xi)}} = -\frac{\rho_H}{\sqrt{-2n h'(1)}} \ln \epsilon + \cdots ,
\]

\[
\Delta A \approx \frac{L \pi^{n-1}}{\rho_H^{n-1}} \int_{1-\epsilon}^{1} \frac{d\xi}{\xi^{2n} \sqrt{R(\xi)}} = -\frac{L \pi^{n-1}}{\rho_H^{n-1}} \frac{1}{\sqrt{-2n h'(1)}} \ln \epsilon + \cdots ,
\]

(4.19)

where \( h'(1) = h'(\xi)|_{\xi=1} \) and \( \epsilon \) is a small number showing how close \( \rho_c \) to \( \rho_t \) is. Therefore one arrives at

\[
\Delta A = \frac{L \pi^{n-1} \ell}{2 \rho_H^n} + \cdots ,
\]

(4.20)

which for the entanglement entropy when \( n = d - 1 \) it reduces to (A.9). It is also possible to estimate the saturation time. Indeed from the equation (4.10) one gets

\[
t_s \approx \rho_t^z \int_{1-\epsilon}^{1} d\xi \frac{\xi^{z-1}}{h(\xi)} = \rho_H^z \frac{h'}{h'(1)} \ln \epsilon + \cdots ,
\]

(4.21)

leading to

\[
t_s \approx \rho_H^{-z-1} \sqrt{\frac{2n}{d - 1 + z}} \frac{\ell}{2} .
\]

(4.22)

Normalizing the radius of horizon to one, \( \rho_H = 1 \), we observe that the saturation time decreases as we increase the dynamical exponent \( z \). It is, however, important to note that when the saturation is not continuous (which is the case for the entanglement entropy in the present example), this estimation of the saturation time might not be trusted.

4.3 Growth in the intermediate region

In this subsection we will consider the intermediate region where \( \frac{\ell}{2} \gg t \gg \rho_H \). In this case the hypersurface stays mostly behind the horizon and typically intersects the null shell at \( \rho_c > \rho_H \) and
reaches the turning point at \( \rho_t > \rho_c \).

A crucial observation which has been made in \[18, 19\] is as follows (see also \[27\] for further discussions). Actually there is a critical extremal hypersurface which intersects the null shell behind the horizon at a critical point \( \rho_c^* \). Those hypersurfaces which intersect the null shell at \( \rho_c < \rho_c^* \) will reach the boundary, while those with \( \rho_c > \rho_c^* \) never reach the boundary and indeed fall into the singularity.

To study the critical extremal hypersurface we note that the equation (4.4) may be considered as the energy conservation law for a one dimensional dynamical system whose effective potential is given by \( V_{\text{eff}}(\rho) \). Stable trajectories may occur around the minimum of the potential. Indeed one may consider a special case where at the minimum both velocity and acceleration are zero. In this case the particle remains fixed at this point. Of course it is not obvious whether such a point is exist in general.

Actually for a fixed turning point, \( \rho_t \), there is a free parameter in the effective potential given by \( \rho_c \) which may be tuned to a particular value \( \rho_c = \rho_c^* \) such that the minimum of the effective potential becomes zero. In other words, one has

\[
\left. \frac{\partial V_{\text{eff}}(\rho)}{\partial \rho} \right|_{\rho_m, \rho_c^*} = 0, \quad V_{\text{eff}}(\rho)|_{\rho_m, \rho_c^*} = 0. \tag{4.23}
\]

Therefore if the hypersurface intersects the null shell at the critical point it remains fixed at \( \rho_m \). This is indeed the critical hypersurface which is responsible for the linear growth in the intermediate region.

To compute the quantities given in the equations (4.10) and (4.12) around the critical extremal hypersurface we will consider \( \rho_c = \rho_c^*(1 - \epsilon) \) for \( \epsilon \ll 1 \). In this limit the main contributions to the integrals in (4.10) and (4.12) come from \( \rho \sim \rho_m \) points where we are close to the minimum of the effective potential. Indeed in this limit the dominant term in the equations (4.10) and (4.12) is \( \frac{1}{\sqrt{R(\xi)}} \). More precisely, for \( \rho_c = \rho_c^*(1 - \epsilon) \) and near \( \xi \sim \xi_m \) one gets\footnote{For details of computations and different possibilities for relative positions of different distinguished points see \[19\].}

\[
t \approx -\rho_t^* \frac{\xi_m^{2(z-1)} E^*}{h(\xi_m) \sqrt{\frac{1}{2} R''(\xi_m)}} \ln \epsilon, \quad \frac{\ell}{2} \approx c_n \rho_t - \frac{\rho_t}{\sqrt{\frac{1}{2} R''(\xi_m)}} \ln \epsilon, \quad \tag{4.24}
\]

where \( c_n = \frac{\sqrt{\pi} \Gamma\left(\frac{1}{2} + \frac{1}{2n}\right)}{\Gamma\left(\frac{1}{2}\right)} \), \( \xi_m = \frac{\rho_m}{\rho_t} \) and

\[
R''(\xi_m) = \left. \frac{\partial^2 R(\xi)}{\partial \xi^2} \right|_{\xi_m, \rho_c^*} = \rho_t^* \left. \frac{\partial^2 R(\rho)}{\partial \rho^2} \right|_{\rho_m, \rho_c^*}, \quad E^* = -\left( \frac{\rho_t}{\rho_m} \right)^{z-1} \sqrt{-\tilde{f}(\rho_m) \left( \left( \frac{\rho_t}{\rho_m} \right)^{2n} - 1 \right)}.
\]
On the other hand from (4.12) one finds
\[
\Delta A \approx -\frac{L^D n^{-1}}{\rho_t^{n-1}} \frac{1}{\xi_m 2^n \sqrt{\frac{1}{2} R^\prime}(\xi_m)} \ln \epsilon.
\] (4.26)

Therefore we find the following linear growth
\[
\Delta A = \frac{L^D n^{-1}}{\rho_t^{n+z-1}} \frac{h(\xi_m)}{\xi_m^{2(n+z-1)}} \ln t,
\] (4.27)

which in large \( \rho_t \) limit it reads
\[
\Delta A = L^D n^{-1} \sqrt{-\tilde{f}(\rho_m)} \rho_m^{n+z-1} t,
\] (4.28)

Denoting values of quantities (3.6) at the final phase by \( A_f \) (where the system is thermalized and may be described by a static black hole) the above equation may be recast into the following form
\[
\Delta A = L^D n^{-1} A_f v_n \rho_H^{1-z} t,
\] (4.29)

where
\[
v_n = \sqrt{-\tilde{f}(\rho_m)} \left( \frac{\rho_H}{\rho_m} \right)^{n+z-1}
\] (4.30)

might be thought of the velocity of propagation of the quantities which is an indicator to show how fast the saturation could occur. Indeed the saturation occurs when \( v_n t \sim \ell \) by which the corresponding quantities reach their equilibrium values. Therefore the saturation time is
\[
t_s \sim \ell \frac{\rho_H^{z-1}}{v_n} \frac{2 \left( \frac{\rho_m}{\rho_H} \right)^{n+z-1}}{\sqrt{\left( \frac{\rho_m}{\rho_H} \right)^{d+z-1} - 1}} \frac{\ell}{2}.
\] (4.31)

### 4.4 More details for entanglement entropy

In the previous subsections we have studied general behaviour of the quantities (3.6) in the different regions. Here we would like to further explore the results by making use of an explicit example. To do so, we will consider the case of the entanglement entropy where \( n = d - 1 \).

In this case the effective potential is minimized at \( \rho_m \) which can be obtained from the following equation
\[
\rho_t^{2(d-1)} = \rho_m^{2(d-1)} \frac{2 \rho_m \tilde{f}^\prime(\rho_m) + (z - 1) g^2(\rho_c) \left( \frac{\rho_m}{\rho_c} \right)^{2(z-1)}}{2 \rho_m \tilde{f}^\prime(\rho_m) - 4(d - 1) \tilde{f}(\rho_m) + (z - 1) g^2(\rho_c) \left( \frac{\rho_m}{\rho_c} \right)^{2(d-2+z)}}.
\] (4.32)
Note that, unlike $z = 1$ case, for fixed radius of horizon the value of radial coordinate at which the effective potential is minimized, $\rho_m$, is not only a function of the turning point, but also it depends on the point where the hypersurface intersects the null shell. Therefore given the turning point would not completely fix $\rho_m$.

We note, however, for the critical extremal hypersurface where the hypersurface intersects the null shell at the critical point $\rho^*_c$ it is possible to fixed both $\rho_m$ and $\rho^*_c$. Indeed for the critical point we require that the effective potential is also zero at the minimum point. So, one finds

$$\rho_t^{2(d-1)} = \rho_m^{2(d-1)} \frac{4\tilde{f}(\rho_m) + g^2(\rho^*_c)(\frac{\rho_m}{\rho^*_c})^{2(z-1)}}{4\tilde{f}(\rho_m) + g^2(\rho^*_c)(\frac{\rho_m}{\rho^*_c})^{2(d-2+z)}}. \tag{4.33}$$

This may also be considered as another relation which fixes $\rho_m$ as a function of $\rho_t$ at the critical point. Therefore solving the equations \((4.32)\) and \((4.33)\) together one can find $\rho^*_c$ and the corresponding $\rho_m$ of the critical extremal hypersurface. Indeed as far as our considerations in the previous section are concerned, these are what we need to proceed exploring the behaviour of the entanglement entropy in the intermediate region (the situation for other quantities is the same). In particular in large $\rho_t$ limit, assuming $\rho_m$ and $\rho^*_c$ remain finite one gets

$$\frac{\rho_m}{\rho_H} = \left(\frac{2(d+z-2)}{d+z-3}\right)^{\frac{1}{d+z-1}}, \quad \frac{\rho^*_c}{\rho_H} = 2\sqrt{\frac{d+z-1}{d+z-3}} \left(\frac{d+z-3}{2(d+z-2)}\right)^{\frac{d+z-2}{d+z-1}}, \tag{4.34}$$

which are well defined for $d+z > 3$. In this case, the entanglement entropy of the system exhibits a linear growth in the intermediate region with the velocity

$$v_E = \left(\frac{d+z-3}{2(d+z-2)}\right)^{\frac{d+z-2}{d+z-1}} \sqrt{\frac{d+z-1}{d+z-3}}. \tag{4.35}$$

On the other hand for $d+z = 3$ for large large $\rho_t$ the critical point $\rho^*_c$ remains finite while $\rho_m$ becomes large

$$\rho_m = \sqrt{\rho_H \rho_t}, \quad \rho^*_c = 2\rho_H. \tag{4.36}$$

In this case even though one might be in an arbitrary dimension, the entanglement entropy has a linear growth in the intermediate region with $v_E = 1$. The situation is very similar to that of two dimensional quantum quench where the entanglement entropy for the vacuum solution has a logarithmic behaviour and has a linear growth with velocity $v_E = 1$ until it saturates at $t_s \sim \frac{\ell}{2}$.

\[8\] For $d+z < 3$ both $\rho_m$ and $\rho^*_c$ are large for large $\rho_t$. In this case the corresponding quantities do not have a linear growth in the intermediate region.
5 Conclusions

In this paper we have considered an Einstein-Maxwell-Dilaton theory with a non-trivial potential for the dilaton. We have obtained an analytic solution with a form of Vaidya-charged black hole solution with a hyperscaling violating factor. This solution may be thought of as a model describing gravitational collapse of a charged matter to make a charged black hole with hyperscaling violating factor.

From gauge/gravity point of view this geometry may provide a holographic description for a global quantum quench for a theory with hyperscaling violation and a non-trivial dynamical exponent. The quantum quench which might be caused by a rapid change in the theory would correspond to an instant injection of a matter in a time interval $\delta t \to 0$.

This system may also be used to examine the process of the thermalization in the model after a global quantum quench. Therefore in order to probe the thermalization caused by a global quantum quench we have studied time dependence behaviour of certain quantities (3.6). In particular we have considered behaviours of entanglement entropy during the process of the thermalization.

Holographically these quantities can be computed by extremizing certain hypersurfaces in the bulk of the geometry (2.14). For example in the entanglement entropy it is a co-dimension two hypersurface, while for the Wilson loop it is a two dimensional surface. In this paper we have only considered an entangling region or the Wilson loop in the shape of a strip with a width $\ell$.

In this paper following [18, 19] we have considered the case where $\frac{\ell}{2} \gg \rho_H$ and therefore the evaluation of the corresponding hypersurface is controlled by regions inside and around the horizon. We have found that at early times the entropy growth depends on the dynamical exponent which in turns indicates that entanglement entropy at the early times is sensitive to the state, while in the intermediate region it always grows linearly. We have, however, observed that for large $z$ limit the early times behaviour is universal and it is a linear growth too.

For the interesting case of $d = 2$ where the dual theory exhibits a Fermi surface [28], and for an arbitrary $z > 1$ the velocity reads

$$\frac{1}{2} \leq v_E = \left( \frac{z-1}{2z} \right)^{\frac{z}{z+1}} \sqrt{\frac{z+1}{z-1}} < 1.$$ (5.1)

Here one may reach $v_E = \frac{1}{2}$ in the large $z$ limit where we have linear growth all the way from initial phase up to the saturation phase.

As one observes the velocity is sensitive to the dynamical exponent which in turns shows that the velocity (the growth) depends on the initial state. Note also that for $d \geq 3$ and arbitrary $z$ the velocity is always less than one. It would be interesting to see whether this behaviour of velocity may be understood by a free streaming model [1].
Following [18] one may define a dimensionless rate of growth as follows

$$\mathcal{R}(t) = \frac{1}{L \frac{D}{d^{n-1}} A f^{1-z} \frac{\partial S}{\partial t}},$$

which in the intermediate region where one has a linear growth it is equal to the velocity of evolution growth that is always less than one.

Although in this paper we have only considered the strip case, its generalization to a sphere is straightforward. Actually based on our results one would expect that for the case of sphere the results should be similar to that of AdS-Vaidya case [18, 19]. In particular there would be a forth phase which is the late time memory loss where the entanglement entropy depends on the distance to the saturation point.

Moreover we have only considered the gravitational collapse of a neutral matter, though one could also consider a charged matter so that the end point of the thermalization is a charged black hole. Of course the only change is to replace $f$ with that of the charged black hole. In this case we would still have the same phases, though the velocity $v_E$ should be less than that in the neutral case [18, 19].

### Acknowledgements

We would like to thank Mohammad Reza Tanhayi for related discussions. M.R.M.M. would also like to thank Ali Mollabashi for useful discussions. We would also like to thank J. F. Pedraza for his useful comments.

### Appendix

#### A Entanglement entropy for static solution

In this appendix we will review certain properties of the entanglement entropy of a strongly coupled field theory whose gravitational description is given by a gravity on the background \([2,2]\) (for details see for example \([23,25,29]\)). To compute the entanglement entropy via AdS/CFT correspondence one needs to minimize a surface in the bulk gravity. More precisely, given a gravitational theory with the bulk Newton’s constant $G_N$, the holographic entanglement entropy is given by \([3,4]\)

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N},$$

(A.1)

where $\gamma_A$ is the minimal surface in the bulk whose boundary coincides with the boundary of the entangling region.
To proceed let us consider a long strip in the dual theory given by

\[ t = \text{fixed}, \quad \frac{\ell}{2} \leq x_1 \leq \frac{\ell}{2}, \quad 0 \leq x_i \leq L \quad \text{for} \quad i = 2, \cdots, D. \]  

(A.2)

The co-dimension two hypersurface \( \gamma_A \) in the bulk may be parametrized by \( x_1 = x(r) \), so that the induced metric on this hypersurface, setting \( r = \frac{1}{\rho} \), is

\[ ds^2 = \rho^2 \frac{\varphi^2}{\rho^{d-2}} \left[ \left( \frac{1}{f(\rho)} + x'^2 \right) d\rho^2 + d\vec{x}^2 \right]. \]  

(A.3)

Therefore the area \( A \) reads

\[ A = \frac{L^{D-1}}{2} \int d\rho \frac{\sqrt{f^{-1} + x'^2}}{\rho^{d-1}}, \quad \text{with} \quad f(\rho) = 1 - m\rho^{d-1+z} + Q^2 \rho^{2(d-2+z)}. \]  

(A.4)

where “prime” represents derivative with respect to \( \rho \). It is then straightforward to minimize the above area to arrive at

\[ \frac{\ell}{2} = \int_0^{\rho_t} \frac{d\rho}{\sqrt{f(\rho) \left( 1 - \left( \frac{\rho}{\rho_t} \right)^{2(d-1)} \right)}} \]  

\[ 1 \quad S = \frac{L^{D-1}}{4G_N} \int_{\epsilon}^{\rho_t} \frac{1}{\rho^{d-1} \sqrt{f(\rho) \left( 1 - \left( \frac{\rho}{\rho_t} \right)^{2(d-1)} \right)}} \]  

(A.5)

where \( \rho_t \) is a turning point and \( \epsilon \) is a UV cut-off.

For \( f = 1 \) which corresponds to a vacuum solution one finds \([23]\)

\[ S_{\text{vac}} = \begin{cases} \frac{L^{D-1}}{4G_N} \left( -\frac{1}{(d-1)\epsilon^{d-2}} + \frac{c_0}{\epsilon^{d-2}} \right) & \text{for} \quad d \neq 2, \\ \frac{L^{D-1}}{4G_N} \ln \frac{\ell}{\epsilon} & \text{for} \quad d = 2, \end{cases} \]  

(A.6)

with \( c_0 \) being a numerical factor.

On the other hand for an excited state whose gravitational dual is provided by the black brane solution \([22]\) the corresponding entanglement entropy may be found by minimizing the area when \( f \neq 1 \). In this case, in general, it is not possible to find an explicit expression for the entanglement entropy, though in certain limits one may extract the general behaviour of the entanglement entropy.

Actually for sufficiently small entangling region it is possible to expand the area expression \([A.4]\) to find the change of area due to the change of the geometry. More precisely for \( m\ell \ll 1 \) one has

\[ \delta A = \frac{L^{D-1}}{2} \int d\rho \ \delta f \left( \frac{\sqrt{f^{-1} + x'^2}}{\rho^{d-1}} \right) \bigg|_{f=1} \delta f, \]  

(A.7)
which leads to the following expression for the change of the entropy

\[ \Delta S = S - S_{\text{vac}} = \frac{L^{D-1} \ell z}{16G_N} \left( c_m m \ell - c_Q Q^2 \ell^{d-2+z} \right), \]

(A.8)

where \( c_m \) and \( c_Q \) are numerical factors. Note that upon the identification of entanglement temperature with \( \ell \), as \( T_E \sim \ell^{-z} \) the above expression may be identified as the first law of the entanglement entropy \([30][33]\), see also \([34]\) for \( z = 1 \) case.

For sufficiently large entangling region main contributions come from the limit where the minimal surface is extended all the way to the horizon so that \( \rho_t \sim \rho_H \) and then

\[ S = \frac{L^{D-1} \ell}{8G_N \rho_H^{d-1}} = \frac{L^{D-1} \ell}{2} S_{BH}, \]

(A.9)

where \( S_{BH} \) is the density of the thermal entropy given in the equation (2.7).

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