Antagonistic interactions can stabilise fixed points in randomly coupled, linear dynamical systems with inhomogeneous growth rates

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Abstract

We analyse the stability of large, linear dynamical systems of degrees of freedoms with inhomogeneous growth rates that interact through a fully connected random matrix. We show that in the absence of correlations between the coupling strengths a system with interactions is always less stable than a system without interactions. On the other hand, interactions that are antagonistic, i.e., characterised by negative correlations, can stabilise linear dynamical systems. In particular, we show that systems that have a finite fraction of the degrees of freedom that are unstable in isolation can be stabilised when introducing antagonistic interactions that are neither too weak nor too strong. On contrary, antagonistic interactions that are too strong destabilise further random, linear systems and thus do not help in stabilising the system. These results are obtained with an exact theory for the spectral properties of fully connected random matrices with diagonal disorder.

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1 Introduction

We consider a dynamical system described by \(n\) variables \(x_j(t) \in \mathbb{R}\) that are labeled by indices \(j = \{1, 2, \ldots, n\} = [n]\) and where \(t \in \mathbb{R}^+\) is the time index. The evolution in time of the variables \(x_j(t)\) is described by a set of randomly coupled, linear differential equations of the form

\[
\partial_t x_j(t) = \sum_{k=1}^{n} A_{jk} x_k(t)
\]

where the \(A_{jk}\) are the entries of a random matrix \(A\) of dimension \(n \times n\). The fixed point \(\vec{x} = 0\) of the set of Eqs. \((1)\) is stable when all eigenvalues of \(A\) have negative real parts. On the other hand, if there exists at least one eigenvalue with a positive real part, then the fixed point is unstable.

Differential equations of the form Eq. \((1)\) appear in linear stability analyses of complex systems described by nonlinear differential equations of the form \(\partial_t \vec{y}(t) = \vec{f}(\vec{y}(t))\). The matrix \(A\) is then the Jacobian of the function \(\vec{f}\) evaluated at a fixed point \(\vec{y}^*\) of the nonlinear dynamics, defined as \(\vec{f}(\vec{y}^*) = 0\), and \(\vec{x} = \vec{y} - \vec{y}^*\) is a vector that denotes the deviation of \(\vec{y}\) from the fixed point; note that we used the vector notation \(\vec{x} = (x_1, x_2, \ldots, x_n)\). The linear stability of a complex system that settles in a fixed point state is thus determined by the real part of the leading eigenvalue \(\lambda_1\), which is defined as an eigenvalue of the Jacobian matrix \(A\) that has the largest real part.

For complex systems the matrix \(A\) can be random, and the task at hand is then to determine the real part of the leading eigenvalue as a function of the parameters that define the random matrix ensemble \(A\), see e.g. Refs. \([1, 7]\). Although one should be careful in drawing conclusions about the dynamics of nonlinear systems from the study of randomly coupled linear differential equations, random matrix theory has the advantage of providing generic analytical insights about the influence of interactions on linear stability. In fact, linear
stability analyses with random matrix theory have been used to study the onset of chaos in random neural networks [8, 9], the stability of ecosystems [1, 3–5, 10, 11], economies [12], or gene regulatory networks [13], and recently exact results for more realistic models based on complex networks have been derived [6, 7, 14].

Following May’s seminal work [1], so far most studies have concentrated on the case where the diagonal entries of $A$, which we also call the growth rates, are fixed to a constant value $d$, i.e.,

$$A_{jk} = \frac{J_{jk}}{\sqrt{n}} (1 - \delta_{j,k}) + d \delta_{j,k},$$

where $\delta_{j,k}$ is the Kronecker delta function and the coupling strengths $J_{jk}$ are random variables drawn from a certain distribution. In this paper, following Refs. [3], we consider the case where the pairs of random variables $(J_{jk}, J_{kj})$ are independent and identically (i.i.d.) distributed random variables drawn from a distribution with

$$\langle J_{ij} \rangle = 0, \quad \langle J_{ij}^2 \rangle = \sigma^2, \quad \text{and} \quad \langle J_{ij} J_{ji} \rangle = \tau \sigma^2,$$

where the variance $\sigma^2$ of the entries $J_{ij}$ quantifies the strength of the interactions, and $\tau \in [-1, 1]$ is the Pearson correlation coefficient between the variables $J_{jk}$ and $J_{kj}$. Of particular interest for theoretical ecology is the case when $\tau$ is negative, as this represents antagonistic interactions, which are also known as predator-prey interactions [3–5]. The leading eigenvalue of random matrices of the form (2) is given by

$$\text{Re}(\lambda_1) = \sigma (1 + \tau) + d.$$  

It follows from Eq. (4) that in the case of homogeneous relaxation rates $d < 0$ is required for a linear system to be stable. Hence, when the diagonal entries of $A$ are fixed to a constant value $d$, then interactions $J_{jk}$ always destabilise fixed points in large dynamical systems.

In the model given by Eq. (2) it holds that either all degrees of freedom are stable in isolation ($d < 0$) or all degrees of freedom are unstable ($d > 0$). In this paper, we relax this condition and consider random matrix models with growth rates $A_{jj} = D_j$ that fluctuate from one variable to the other. In the symmetric case ($\tau = 1$), such random matrices are called deformed Wigner matrices [15–17] and in this case a functional equation that determines the spectral distribution in the limit of large $n$ has been derived by Pastur in Ref. [15]. Another case that has been studied in the literature is when $A$ is the adjacency matrix of a random directed graph with diagonal disorder [6, 14, 18], which corresponds in the dense limit with $\tau = 0$ [19], and for which a simple equation for the boundary of the spectrum as a function of the distribution of diagonal matrix entries has been derived. On the other hand, in the present paper we focus on the case of negative $\tau$ that is of particular interest for ecology. Interestingly, we will find that for negative $\tau$ the leading eigenvalue can be negative, even if a finite fraction of the relaxation rates $D_j$ are positive. In particular, we find that antagonistic interactions can stabilise linear systems when the interactions are neither too weak nor too strong, even if a finite fraction of degrees of freedom are unstable in isolation.

The paper is organised as follows. In Sec. 2 we define the model that we study, which is a fully connected random matrix with diagonal disorder. In Sec. 3 we discuss the cavity method, which is the mathematical method we use to study the model in the limit of infinitely large random matrices. In Sec. 4 we present the main results for the boundary for the spectrum of fully connected matrices with diagonal disorder, and in Sec. 5 we use these results to derive phase diagrams for the linear stability of fixed points. We end the paper with a discussion
in Sec. The paper also contains a few appendices where we present details about the mathematical derivations.

2 Fully connected random matrices with diagonal disorder

We consider the random matrix model

\[ A_{jk} = \frac{J_{jk}}{\sqrt{n}}(1 - \delta_{j,k}) + D_j \delta_{j,k}, \tag{5} \]

with the off-diagonal pairs \((J_{ij}, J_{ji})\) i.i.d. random variables drawn from a joint distribution \(p_{J_1, J_2}\) with moments as specified in the Eqs. \([3]\), and with diagonal elements \(D_j\) that are i.i.d. random variables drawn from a distribution \(p_D\).

As will become clear later, in the limit of \(n \gg 1\) the leading eigenvalue \(\lambda_1\) is a deterministic variable that only depends on the moments of the distribution of \((J_{ij}, J_{ji})\) given in Eq. \([3]\), and hence we will not need to specify \(p_{J_1, J_2}\). On the other hand, the leading eigenvalue \(\lambda_1\) depends in a nontrivial way on the distribution \(p_D\), and therefore it will be interesting to study the effect that the shape of \(p_D\) has on the leading eigenvalue.

In the special case when \(p_D(x) = \delta(x - d)\) we recover the model given by Eq. \([4]\). A more interesting case is when the growth rates \(D_j\) are heterogeneous, and arguably the most simple model for heterogeneous growth rates considers that the \(D_j\) can take two possible values, yielding a bimodal distribution

\[ p_D(x) = p \delta(x - d_-) + (1 - p) \delta(x - d_+), \tag{6} \]

with \(d_- < 0, d_+ > 0, \) and \(p \in [0, 1]\), and where \(\delta(x - d)\) denotes the Dirac delta distribution. In this example, a fraction \((1 - p)\) of variables \(x_j\) are unstable in the absence of interactions \((\sigma^2 = 0)\). We also consider cases where \(p_D\) is a continuous distribution. One example of a continuous distribution is the uniform distribution defined on an interval \([d_-, d_+]\), i.e.,

\[ p_D(x) = \begin{cases} 
0 & \text{if } x \notin [d_-, d_+], \\
\frac{1}{d_+ - d_-} & \text{if } x \in [d_-, d_+]. 
\end{cases} \tag{7} \]

Since the uniform distribution is supported on a finite set, we will also consider an example for which \(p_D\) has unbounded support, namely, we will consider the Gaussian distribution

\[ p_D(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{8} \]

with zero mean and unit variance.

The main question we address in this paper is whether the interaction variables \(J_{ij}\) can stabilise a linear dynamical system even when a finite fraction of variables are unstable in the absence of interactions, i.e., a finite fraction of species \(i \in [n]\) have a positive growth rate \(D_i\). In other words, we ask whether it is possible to have \(\text{Re} \lambda_1 < 0\) even when \(p_D\) has finite support on positive values \(d > 0\).
3 Cavity method for the empirical spectral distribution of infinitely large matrices

To determine the leading eigenvalue of the adjacency matrices $A$, we first determine the empirical spectral distribution $\rho$ of the eigenvalues $\lambda_j$ of $A$, defined by

$$
\rho(z) = \lim_{n \to \infty} \frac{1}{n} \left( \sum_{j=1}^{n} \delta(x - \text{Re}(\lambda_j))\delta(y - \text{Im}(\lambda_j)) \right)
$$

for all $z = x + iy \in \mathbb{C}$. The spectral distribution determines the leading eigenvalue through

$$
\lambda_1 = \text{argmax}_{\{z \in \mathbb{C} : \rho(z) > 0\}} \text{Re}(z).
$$

Equation (10) holds as long as the spectrum of $A$ does not have eigenvalue outliers [6, 18], which is the case for the model defined in Sec. 2 as $\langle J_{ij} \rangle \neq 0$ [6, 18]. The convergence in Eq. (9) should be understood as weak convergence [20], which implies that the average of any bounded and continuous function $f(z)$ defined on the complex plane converges in the limit of large $n$ to $\int_{\mathbb{C}} dz \rho(z)f(z)$. Also, we can drop the average in the right-hand side of Eq. (9) as the spectral distribution converges almost surely and weakly to $\rho$ [20], and hence also the leading eigenvalue $\lambda_1$ as defined in Eq. (10) is a deterministic variable for large values of $n$.

The limiting distribution $\rho$ of random matrix models as defined in Sec. 2 have been studied before in several special cases. Notably, for the symmetric case with $\tau = 1$ Pastur derived a functional equation that determines $\rho$ [15]. Recently, the symmetric case was revisited in [17], and in that reference also the large deviations of $\lambda_1$ were computed in the case when the matrix entries $J_{ij}$ are drawn from a Gaussian distribution; note that large deviations are not universal and depend on the statistics of $(J_{ij}, J_{ji})$ as determined by the distribution $p_{D_{1,2}}$. In the case when $\tau = 0$ and $p_D$ is a bimodal distribution the spectral distribution $\rho$ has been determined in Refs. 21, 22 and the $\tau = 0$ case for general $p_D$ has been considered in [19]. Lastly, let us mention that for random directed graphs with a prescribed distribution of indegrees and outdegrees, which corresponds with the case $\tau = 0$ in the limit of large mean degrees, a simple equation was derived for the boundary of the spectrum [6, 14, 18].

We determine the spectral density $\rho(z)$ from the resolvent of the matrix $A$, which can be determined with the cavity method [23, 24]. The resolvent is defined as

$$
G = (z1_n - A)^{-1}, \quad z \notin \{\lambda_1, \lambda_2, \ldots, \lambda_n\},
$$

where $1_n$ is the identity matrix of size $n$. The spectral distribution can be expressed in terms of the diagonal elements of the resolvent by [21]

$$
\rho(z) = \lim_{n \to \infty} \frac{1}{\pi n} \partial^* \text{Tr} G(z), \quad \text{where} \quad \partial^* = \frac{1}{2} \frac{\partial}{\partial x} + \frac{i}{2} \frac{\partial}{\partial y}.
$$

For non-Hermitian matrices, the eigenvalues are in general complex-valued, and therefore in the limit of $n \to \infty$ we cannot get $\rho(z)$ from $\text{Tr} G(z)$ [2]. To overcome this, we use the Hermitization method [21] that considers the enlarged $2n \times 2n$ matrix

$$
H = \begin{pmatrix} \eta 1_n & z 1_n - A \\ z^* 1_n - A^T & \eta 1_n \end{pmatrix},
$$

where $\eta$ is a normalizing factor.
where we have introduced a regulator $\eta$ that keeps all quantities well-defined in the limit of large $n$, where $A^T$ is the transpose of the matrix $A$, and where $z^*$ is the complex conjugate of $z$. Defining the $jk$-th block of the generalized resolvent as

$$G_{jk} = \left( \begin{array}{cc} [H^{-1}]_{j,k} & [H^{-1}]_{j,k+n} \\ [H^{-1}]_{j+n,k} & [H^{-1}]_{j+n,k+n} \end{array} \right),$$

(14)

the spectral distribution can be written as \cite{24}

$$\rho(x, y) = \lim_{n \to \infty} \lim_{\eta \to 0} \frac{1}{\pi} \partial z^* g_{21},$$

(15)

where

$$g = \left( \begin{array}{cc} g_{11} & g_{12} \\ g_{21} & g_{22} \end{array} \right) = \frac{1}{n} \sum_{j=1}^{n} G_{jj}.$$  

(16)

In Appendix A we use the Schur formula for the inverse of a block matrix to determine a selfconsistent equation for the matrix $g$ at fixed $\eta$ in the limit of $n \gg 1$, viz.,

$$\left( \begin{array}{cc} g_{11} & g_{12} \\ g_{21} & g_{22} \end{array} \right) = \left( \begin{array}{cc} \eta - \sigma^2 g_{22} & z - D - \tau \sigma^2 g_{21} \\ z^* - D - \tau \sigma^2 g_{12} & \eta - \sigma^2 g_{11} \end{array} \right)^{-1} D,$$

(17)

where $\langle \ldots \rangle_D$ denotes the average over the he distribution $p_D$.

Note that in the present approach we have determined $g$ at finite values of $\eta$ in the limit of large $n$, and afterwards we take the limit of $\eta \to 0$. Hence, we interchange the two limits in Eq. (15), which is not evident as the leading order correction terms to the Eq. (15), at large values of $n$ and small values of $\eta$, intertwine the two limits. Demonstrating that these two limits can be interchanged constitutes the main challenge in rigorous approaches to nonHermitian random matrix theory, see e.g. Refs. \cite{20,25,26}, and it involves bounding the rate at which the least singular value of $z1_n^* - A$ converges to zero for large values of $n$, as this quantity determines the leading order correction to Eq. (15). In what follows, we use the theoretical physics approach, i.e., we exchange the two limits in good faith and then corroborate theoretical results with direct diagonalisation results. In the next section, we use the Eq. (17) together with Eq. (15) to determine the boundary of the support set of $\rho$ in the complex plane.

### 4 Boundary of the spectrum

The support set of $\rho(z)$ is defined as

$$S = \{ z \in \mathbb{C} : \rho(z) > 0 \}$$

(18)

where $\overline{\cdot}$ denotes the closure of a set. From Eq. (10) it follows that the support set determines the leading eigenvalue whenever the spectrum does not contain eigenvalue outliers \cite{18}.

The support set $S$ follows from the solutions to the Eqs. (15)-(17). The Eq. (17) admits two type of solutions \cite{6,7}. First, there is the trivial solution for which $g_{11} = g_{22} = 0$ and $\partial z^* g_{21} = 0$, yielding a distribution $\rho = 0$ for $z \notin S$. Second, there is the nontrivial solution
Figure 1: Spectra of two random matrices $A$ as defined in Eq. (5) for the uncorrelated case $\tau = 0$ and with diagonal elements that are independently drawn from a uniform distribution [Panel (a)] or a Gaussian distribution [Panel (b)]. Markers denote the eigenvalues of a random matrix of size $n = 3000$ and with off-diagonal elements $J_{ij}$ that are drawn independently from a Gaussian distribution with zero mean and unit variance. The red solid line denotes the solution to Eq. (23), which provides boundary of the support set $S$ in the limit of infinitely large $n$. Panel (a) shows the analytical solution Eq. (25) and Panel (b) is obtained by numerically solving Eq. (23).
$g_{11} > 0$ and $g_{22} > 0$, we obtain that for all values of $z \in \mathcal{S}$ it holds that
\[
\left\langle \frac{\sigma^2}{(D - z^* + \tau \sigma^2 g_{12})(D - z + \tau \sigma^2 g_{21})} \right\rangle_D \geq 1 \tag{19}
\]
and the boundary of the support set is given by
\[
\left\langle \frac{\sigma^2}{(D - z^* + \tau \sigma^2 g_{12})(D - z + \tau \sigma^2 g_{21})} \right\rangle_D = 1. \tag{20}
\]
Note that, in general, Eq. (20) is coupled with the Eq. (17) and therefore these equations have to be solved together.

In what follows, we first analyse the Eqs. (17) and (20) in two limiting cases, and then we discuss the general case.

4.1 Symmetric matrices with $J_{ij} = J_{ji}$ ($\tau = 1$)

For symmetric random matrices the Eq. (17) reduces to a functional equation for the resolvent of a Wigner matrix with diagonal disorder derived originally by Pastur in Ref. [15]. Indeed, in this case $g_{22} = g_{11} = 0$ for all $z$ with nonzero imaginary part so that
\[
g_{21} = \int_{\mathbb{R}} dx \, p_D(x) \frac{1}{z - x - \sigma^2 g_{21}} \tag{21}
\]
for all $z \notin \mathbb{R}$, which is identical to Equation (1.6) in Ref. [15]. Since $g_{21}$ is the Stieltjes transform of the spectral distribution defined on the real line, we can use the Sokhotski-Plemelj inversion formula [27]
\[
\rho(x + iy) = \frac{1}{\pi} \delta(y) \lim_{\epsilon \to 0^+} \text{Im} \left( g_{21}(x - i\epsilon) \right) \tag{22}
\]
to obtain the spectral distribution.

4.2 Uncorrelated interaction variables $J_{ij}$ and $J_{ji}$ ($\tau = 0$)

In the absence of correlations between $J_{ij}$ and $J_{ji}$, the Eq. (20) decouples from the Eq. (17). Therefore, the $\tau = 0$-case is mathematically simpler to solve than the $\tau \neq 0$ case.

The boundary of the support set $\mathcal{S}$ is determined by the values of $\lambda \in \mathbb{C}$ that solve the equation
\[
1 = \sigma^2 \int_{\mathbb{R}} dx \, p_D(x) \frac{1}{|\lambda - x|^2}, \tag{23}
\]
which is closely related to the results obtained for the boundary of spectra of random directed graphs in Refs. [6,14,18] and to those of perturbed random matrices with uncorrelated matrix entries [19].

Equation (23) implies that for $\tau = 0$ the leading eigenvalue satisfies
\[
\text{Re} \left( \lambda_1 \right) \geq d_+ = \max \{ x \in \mathbb{R} : p_D(x) > 0 \}. \tag{24}
\]
In other words, in the absence of correlations between the interaction variables $J_{ij}$ and $J_{ji}$, interactions always increase the real part of the leading eigenvalue and have thus a destabilising effect on system stability.
Let us analyse the boundary of the spectrum and the leading eigenvalue for a couple of examples. As shown in Appendix B, when \( p_D(x) \) is the uniform distribution supported on the interval \([d_-, d_+]\), then the boundary of the support set \( S \) is given by values of \((x, y)\) that solve

\[
\left( (d_- - x) (d_+ - x) + y^2 \right) = \frac{y (d_+ - d_-)}{\tan\left( \frac{\pi \sigma^2}{d_+ - d_-} \right)}, \quad y \in \left( -\frac{\pi \sigma^2}{d_+ - d_-}, \frac{\pi \sigma^2}{d_+ - d_-} \right) \setminus \{0\}. \tag{25}
\]

For \((d_+ - d_-)/\sigma^2 \ll 1\), we recover the celebrated circular law \([20, 28]\), while for \((d_+ - d_-)/\sigma^2 \approx 1\) the formula Eq. (25) expresses a deformed circular law replacing the constant radius \( \sigma \) by \( y (d_+ - d_-) / \tan\left( \frac{\pi \sigma^2}{d_+ - d_-} \right) \). In Fig. 1(a) we have plotted the curve Eq. (25) for the case \( d_+ = 1 \) and \( d_- = -1 \) and we show that this theoretical results is well corroborated by the spectrum obtained from numerically diagonalising a matrix. From Eq. (25) it follows that the leading eigenvalue is given by

\[
\text{Re}(\lambda_1) = \frac{1}{2} \left( \sqrt{(d_- - d_+)^2 + 4\sigma^2} + d_- + d_+ \right). \tag{26}
\]

Eq. (26) reveals that \( \text{Re}(\lambda_1) > d_+ \) for any value of \( \sigma \), and hence the interactions make the system less stable. For \( d_+ = d_- = d \) we recover the formula Eq. (4), and in the limit of large \( d_- \) we get \( \lim_{d_- \to -\infty} \text{Re}(\lambda_1) = d_+ \).

![Figure 2](image)

**Figure 2:** Comparison between the spectra of two random matrices \( A \) with two different values of \( \tau \). Eigenvalues plotted are for two matrices of size \( n = 3000 \) whose diagonal elements are drawn from the bimodal distribution Eq. (6) with \( d_- = -1, p = 0.9 \) and \( d_+ = 0.1 \), and whose off-diagonal entries are drawn from a normal distribution with zero mean, variance \( \sigma^2/n = 1/n \), and \( \tau = 0 \) [Panel (a)] or \( \tau = -0.7 \) [Panel (b)]. The red line denotes the solution to the Eqs. (28) and (29).

As a second example we consider the case when \( p_D \) is a Gaussian distribution with zero mean and unit variance. In Fig. 1(b), we compare the solution to Eq. (23) with the spectrum of a random matrix drawn from the ensemble defined in Sec. 2. In this case the spectrum \( S \) in the limit \( n \to \infty \) contains the whole real axis, contrarily to the case where \( p_D \) is a uniform distribution (compare Fig. 1(a) with Fig. 1(b)). The distinction between the two cases follows from the fact that \( p_D \) is supported on a compact interval in the uniform case, while it is supported on the whole real axis in the Gaussian case. Indeed, Eq. (23) implies
that in the former case the spectrum $S$ is a finite subset of the complex plane, while in the latter case it contains the real axis. Consequently, for a compactly supported distribution $p_D$ the leading eigenvalue converges to a finite value as a function of $n$, while for a distribution $p_D$ that is supported on the real axis the leading eigenvalue diverges. The rate of divergence as a function of $n$ of the average of the leading eigenvalue, $\langle \lambda_1 \rangle$, is determined by the scaling of the maximum value of the diagonal entries $D_i$ as a function of $n$. Since the maximum of $n$ i.i.d. random variables drawn from a Gaussian distribution with zero mean and unit variance scales as $\sqrt{\log(n)}$ (see Theorem 1.5.3 in Ref. [29]), it holds that

$$\langle \lambda_1 \rangle = O_n((D_{\text{max}})) = O(\sqrt{\log(n)})$$

when $p_D$ is Gaussian, where $D_{\text{max}} = \max\{D_1, D_2, \ldots, D_n\}$ and where $O(\cdot)$ is the big O notation.

### 4.3 The case of generic correlations between $J_{ij}$ and $J_{ji}$ ($\tau \in [-1, 1]$)

We consider now the case of nonzero correlations between the interaction variables $J_{ij}$ and $J_{ji}$. In this case, it is more difficult to find the values of $z$ that solve the Eq. (20), as contrarily to the $\tau = 0$ case Eq. (20) is coupled with Eq. (17). Nevertheless, we can simplify the Eqs. (20) and (17) by using generic properties of $H$ and $A$.

Using that $H$ is Hermitian, which is implied by the definition Eq. (14), we obtain that $g_{12} = g_{21}^*, \text{Im}(g_{11}) = 0$ and $\text{Im}(g_{22}) = 0$. In addition, since $A$ and $A^T$ have the the same statistical properties, we can set $g_{11} = g_{22}$. Also, since we are interested in the boundary of the continuous part of the spectrum, which is located at the edge between the trivial and the nontrivial solutions, we can set $g_{11} = g_{22} = 0$, as this is satisfied for the trivial solution. Furthermore, we make the ansatz that $\text{Im}(g_{12})$ is independent of the distribution $p_D$, and therefore $\text{Im}(g_{12}) = y/\sigma^2(\tau - 1)$, which is the solution when $p_D(x) = \delta(x)$. In addition, using that $g_{11}g_{22} = 0$, we can express the Eq. (17) as

$$\text{Re}(g_{12}) = \left< \frac{D - x + \text{Re}(g_{12})\tau \sigma^2}{- \left((D - x) + \text{Re}(g_{12})\tau \sigma^2\right)^2 - \frac{y^2}{(1-\tau)^2}} \right>_D$$

and Eq. (20) reads

$$1 = \left< \frac{\sigma^2}{\left((D - x) + \text{Re}(g_{12})\tau \sigma^2\right)^2 + \frac{y^2}{(1-\tau)^2}} \right>_D.$$  

We could not simplify these equations further, and hence we will obtain the boundary of the spectrum by solving the Eqs. (28-29).

In Figs. 2 and 3, we corroborate the the boundary of the spectrum, obtained from solving the Eqs. (28-29), with numerical results for the eigenvalues of matrices of finite size, obtained with numerical diagonalisation routines. We show the boundary of the spectrum for the case of the bimodal distribution $p_D$ given by Eq. (6). Figure 2 compares two spectra with the same $\sigma$ but different values of $\tau$, whereas Fig. 3 considers one negative value of $\tau$ and observes how the spectrum changes as a function of $\sigma$. Note that the the real part of the leading eigenvalue $\text{Re}(\lambda_1)$ decreases as a function of $\tau$.

The leading eigenvalue is obtained by solving Eqs. (28-29) at $y = 0$. For bimodal $p_D$ we obtain a quartic equation in $x$ and we identify the largest real-valued solution of this quartic
Figure 3: Comparison between the spectra of random matrices $\mathbf{A}$ with different values of the interaction strength $\sigma$. Eigenvalues plotted are for three matrices of size $n = 3000$ whose off-diagonal elements $(J_{ij}, J_{ji})$ are drawn from a joint Gaussian distribution with zero mean, a Pearson correlation coefficient $\tau = -0.7$, and a variance $\sigma^2/n$ as indicated. The diagonal elements follow a bimodal distribution with parameters $p = 0.9$, $d_- = -1$, $d_+ = 0.1$. The red line denotes the solution to the Eqs. (28) and (29).
equation with $\text{Re}(\lambda_1)$. We have obtained an analytical expression for $\text{Re}(\lambda_1)$ as a function of the system parameters, which we omit here as it is a very long mathematical formula without clear interpretation. In Fig. 4 we compare this formula with numerical results of the leading eigenvalue obtained through the direct diagonalisation of matrices of finite size $n$. The numerics corroborate well the analytical results that are valid for infinitely large $n$.

For uniform $p_D$ the Eqs. (28)-(29) can be solved explicitly as shown in Appendix C. Remarkably, in this case we obtain a simple, analytical expression for the leading eigenvalue, viz.,

$$
\text{Re}(\lambda_1) = \frac{1}{2} \left( \sqrt{(d_+ - d_-)^2 + 4\sigma^2 + d_+ + d_-} \right) + \frac{\sigma^2}{d_+ - d_-} \log \left( \frac{\sqrt{(d_+ - d_-)^2 + 4\sigma^2 + d_+ - d_-}}{\sqrt{(d_+ - d_-)^2 + 4\sigma^2 - d_+ + d_-}} \right),
$$

(30)

where $d_+ > d_- \in \mathbb{R}$. One readily verifies that for $\tau = 0$ Eq. (30) reduces to Eq. (26) and for $\tau = 1$ Eq. (30) recovers the result in Ref. [17] for the case of symmetric matrices with entries drawn from a Gaussian distribution. Since the sign of the second term of Eq. (30) is equal to the sign of $\tau$, the leading eigenvalue $\lambda_1$ decreases as a function of negative values of $\tau$.

![Figure 4](image-url)

**Figure 4:** Effect of the interaction strength $\sigma$ on the real part of the leading eigenvalue $\lambda_1$ for $\tau = 0$ (triangle, dotted), $\tau = -0.8$ (circle, solid) and $\tau = 1$ (diamond, dashed). Lines show the Eq. (30). Markers are numerical results obtained for random matrices $A$ with diagonal elements $D_j$ that are drawn independently from a uniform $p_D$ supported on the interval $[d_-, d_+] = [-1, 0.1]$ and with pairs of off-diagonal elements $(J_{ij}, J_{ji})$ that are drawn independently from a normal distribution with mean 0, variance $\sigma^2/n$, and Pearson correlation coefficient $\tau$ as provided. Each marker represents the largest eigenvalue of one matrix realisation of size $n = 7000$. 
5 Stability of linear dynamical systems

We discuss the implications of the spectral results obtained for the stability of linear systems of the form given by Eq. (1).

5.1 Uncorrelated interactions destabilise dynamical systems

For $\tau = 0$ it holds that $\text{Re}(\lambda_1) \geq d_+ + d_-$ for all values of $\sigma$ [see Eq. (24)], which has a couple of interesting implications for the stability of linear dynamical systems. First, a linear dynamical systems with $\tau = 0$ cannot be stable if the support of $p_D$ covers the positive axis. Second, interactions $J_{ij}$ destabilise linear dynamical systems as $\lambda_1$ is an increasing function of $\sigma$ (see also Fig. 3). Third, if the support of $p_D$ covers the whole real line, then the leading eigenvalue $\lambda_1$ diverges as a function of $n$. In the latter case we obtain a tradeoff between diversity, as measured by $n$, and stability, as measured by $\text{Re}(\lambda_1)$ [1,30]. Indeed, when $p_D$ has unbounded support, then for any realisation of the system parameters $\sigma, \tau$, and $p_D$, there will exist a value $n^*$ so that with large probability $\text{Re}(\lambda_1) > 0$ when $n > n^*$. In Ref. [7], the latter scenario is referred to as size-dependent stability, as the system size $n$ is an important parameter in determining system stability.

5.2 Antagonistic interactions can render dynamical systems stable

In the case of negative $\tau$ values the interactions $J_{ij}$ can stabilise linear dynamical systems when they are neither too strong nor too weak. To understand how this works, consider linear systems $A$ for which there exist values $x \in \mathbb{R}^+$ with $p_D(x) > 0$, such that the system is unstable in the absence of interactions. As illustrated in Fig. 3 adding antagonistic interactions to a linear system can retract the real part $\text{Re}(\lambda_1)$ of the leading eigenvalue and make it negative. This example demonstrates that unlike the uncorrelated case with $\tau = 0$, interactions can contribute to the stability of a system when $\tau < 0$. However, as shown in Fig. 4 for large values of the interaction strength $\sigma$ the leading eigenvalue increases as a function of $\sigma$, and hence antagonistic interactions stabilise linear dynamical systems as long as they are neither too strong nor too weak.

Fig. 5 draws the lines of marginal stability, corresponding with $\text{Re}(\lambda_1) = 0$, in the $(\sigma, \tau)$ plane for the cases with homogeneous growth rates $p_D(x) = \delta(x - D)$ (dotted line), a bimodal distributions $p_D$ (dashed line), and a uniform distribution $p_D$ (solid line). For all cases we have set $\langle D \rangle = -1$ so that we can analyse the effect of fluctuations in $D$ on system stability. Note that for the dotted line $d_+ = -1$, whereas for the dashed and solid lines $d_+ = 0.1$. As a consequence, for the dotted line a stable region exists when $\tau = 0$ and $\sigma$ is small enough, while for the other cases there is no stable region when $\tau = 0$. Interestingly, for negative values of $\tau$ and for interaction strengths $\sigma$ that are neither too weak nor too strong, there exists a stable region with $\text{Re}(\lambda_1) < 0$. This region exists even though $d_+ > 0$ (solid and dashed lines). On the other hand, for $\tau = 0$ a stable region can only exist when $d_+ < 0$, which is the case of the dotted line with homogeneous rates.

Fig. 5 shows that $\text{Re}(\lambda_1)$ is for fixed $\langle D \rangle$ and large $\sigma$ independent of $p_D$. We explore this universal behaviour in more depth. Expanding the expression of $\text{Re}(\lambda_1)$ for Eq. (30) in large values of $\sigma$ we obtain

$$\text{Re}(\lambda_1) = (1 + \tau)\sigma + \frac{1}{2}(d_- + d_+) + (3 + \tau)\frac{(d_- - d_+)^2}{24\sigma} + O(1/\sigma^2).$$

(31)
Figure 5: Phase diagram for the stability of a linear dynamical system with antagonistic interactions. Lines $\text{Re}(\lambda_1) = 0$ of marginal stability of that separate a stable region with $\text{Re}(\lambda_1) < 0$ (below the lines) from an unstable region $\text{Re}(\lambda_1) > 0$ (above the lines). Results are for the random matrix model defined in Sec. 2 for various distributions $p_D$ with the same mean $\langle D \rangle = -1$. The solid line represents a uniform disorder on the interval $I = [-2.1, 0.1]$; the dashed line represents a bimodal disorder with parameters $p = 0.5, d_- = -2.1, d_+ = 0.1$; and the dotted line represents the case where all diagonal elements take the value $-1$ with no disorder.

Identifying the mean and variance of the uniform distribution $p_D$ in Eq. (31), we can write

$$\text{Re}(\lambda_1) = (1 + \tau)\sigma + \langle D \rangle - \langle D^2 \rangle (\tau - 3)^2 + O(1/\sigma^2),$$

where $\langle D^2 \rangle$ represents the variance of the diagonal elements. If $D$ is a deterministic variable with zero variance, then we recover the Eq. (4). This suggests that when the interactions are strong enough, only the first moment of the diagonal elements is important, rather than the distribution of their elements. Although the relation Eq. (32) is derived for the uniform case, numerical evidence shows that it also holds for the bimodal case, and therefore we conjecture that it holds for arbitrary $p_D$ distributions. Demonstrating the validity of the Eq. (32) beyond the uniform case would be an interesting extension of the present work.

6 Discussion

We have obtained exact results for the boundary of the spectrum of random matrices of the form Eq. (5), where the pairs $(J_{ij}, J_{ji})$ are i.i.d. random variables drawn from a joint distribution with moments as given in Eq. (3), and where the diagonal elements $D_i$ are i.i.d. random variables drawn from a distribution $p_D$.

If the Pearson correlation coefficient $\tau = 0$, then the boundary of the spectrum solves the Eq. (23), which implies that $\lambda_1 \geq d_+$ [see Eq. (24)]. Hence, in this case interactions
render dynamical systems less stable, irrespective of the form of $p_{J_1,J_2}$ and $p_D$. On the other hand, if the Pearson correlation coefficient $\tau$ between the pairs $(J_{ij}, J_{ji})$ is negative and the variance of the distribution $p_D$ is nonzero, then $\lambda_1$ can exhibit a nonmonotonic behaviour as a function of the strength $\sigma$ of the off-diagonal matrix elements, as illustrated in Fig. 4. As a consequence, antagonistic interactions that are neither too strong nor too weak can stabilise linear dynamical systems when the diagonal entries $D_i$ are heterogeneous, see Fig. 5.

The results in Fig. 4 can also be understood perturbatively. Indeed, a perturbative expansion of the eigenvalues of $A$ in the parameter $\sigma$ starting from the diagonal case with $\sigma = 0$ leads to the expression (see App. D)

$$\lambda_j(\sigma) = D_j + \sum_{i=1(i\neq j)}^{n} \frac{J_{ij} J_{ji}}{(D_j - D_i)} + O(\sigma^3).$$

(33)

For the leading eigenvalue $j = 1$ it holds that $D_1 - D_i \geq 0$ for all values $i \in [n]$, and hence the second term is negative whenever $J_{ij} J_{ji} < 0$, leading to the initial decrease of the leading eigenvalue $\lambda_1$ in Fig. 4. For larger values of $\sigma$ we need to consider the higher order terms in the perturbative expansion of $\lambda_1$, which are in general positive leading to the nonmonotonic behaviour of $\lambda_1$ in Fig. 4.

For the case of a uniform distribution $p_D$ of the diagonal elements we have obtained an analytical expression for $\lambda_1$, which is given by Eq. (30), and in the case of $\tau = 0$ we have obtained a closed form expression for the boundary of the support of the spectral density $\rho$, which is given by Eq. (25) and which also holds for uniform $p_D$. The peculiar solvability of the uniform disorder case is consistent with results obtained recently in Ref. [17] for symmetric matrices ($\tau = 1$). Reference [17] shows that when $D_j = a + bj/n$, with $a$ and $b$ arbitrary constants, and when the entries $J_{ij}$ are complex-valued and Gaussian distributed, then an explicit expression for the joint distribution of eigenvalues can be obtained. Based on the results in the present paper, one may speculate that these results are extendable to the case of $\tau = 0$, which will be interesting to explore in future work.

Another interesting extension of the present work is to consider models of the form

$$A_{ij} = J_{ij} C_{ij} + \delta_{i,j} D_i$$

(34)

where $C_{ij}$ is now the adjacency matrix of a random graph. The case of random directed graphs with a prescribed degree distribution $p_{K^{in},K^{out}}$ of indegrees and outdegrees has been solved in Refs. [6,14,18]. This is the sparse equivalent of the $\tau = 0$ case, and in fact the oriented and locally tree-like structure of random directed graphs leads to a decoupling similar to those of Eqs. (17) and (20) in the $\tau = 0$ case. For this reason, random directed graphs are analytically tractable, and Refs. [6,14,18] derived for the boundary of the spectrum an equation similar to Eq. (23), but with a prefactor that is given by $\sigma^2 \langle K^{in} K^{out} \rangle c$, i.e.,

$$1 = \sigma^2 \langle K^{in} K^{out} \rangle c \int_{\mathbb{R}} dx \ p_D(x) \frac{1}{|\lambda - x|^2}. \quad (35)$$

The case of antagonistic interactions ($\tau < 0$) is considerably more challenging as one needs to know the distribution of the diagonal entries $[G]_{ii}$ of the resolvent, which is not self-averaging in the sparse case, see Ref. [7]. Nevertheless, Ref. [7] analysed the antagonistic case without diagonal disorder and found that systems with antagonistic interactions are significantly more stable than systems with mutualistic and competitive interactions (in fact, in the limit $n \to \infty$...
they are infinitely more stable). Ref. [7] did however not study the effect of diagonal disorder on system stability.

Let us end the paper with a word of caution when using the present results to understand the dynamics of nonlinear systems. In a linear system there exist one fixed point, i.e., \( \vec{x} = 0 \), and the system parameters will not affect the existence and uniqueness of this fixed point. However, in nonlinear systems this is in general not the case, see e.g. [31,32], and therefore system stability can also be affected by bifurcations that eliminate fixed points. Another issue is that \( \mathbf{A} \) is the Jacobian matrix, which is in general different from the interaction matrix. Nevertheless, studies in, among others, ecology [10,11] and neuroscience [8,9], show that nonlinear systems do exhibit regimes with one unique stationary fixed point and random matrix theory can provide insights on system stability in this regime.

When preparing the manuscript, we became aware of the preprint [33] that also studies the spectral properties of matrices of the type defined in Sec. 2. However, the paper [33] discusses the case of \( \tau > 0 \) for which interactions further destabilise fixed points, whereas we were interested in the potentially stabilising effect of interactions for \( \tau < 0 \).

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A Derivation of the generalised resolvent equation (17) using the Schur complement formula

We derive the Eq. (17) using two useful properties.

First, we use that permutation of a matrix and matrix inversion are two commutable operations. Indeed, let \( \mathbf{P} \) be the orthogonal matrix that represents an arbitrary permutation of the integers \( \{1,2,\ldots,n\} \), then

\[
\mathbf{P} \mathbf{H}^{-1} \mathbf{P}^{-1} = (\mathbf{P} \mathbf{H} \mathbf{P}^{-1})^{-1}.
\]

(36)

We use this property to perform the permutation [14,24]

\[
[\mathbf{H}]_{j,k} \rightarrow \begin{cases} 
[\bar{\mathbf{H}}]_{j,2j-1,2k-1} & \text{if } 1 \leq j, k \leq n, \\
[\bar{\mathbf{H}}]_{j,2j-n,2k-1} & \text{if } n+1 \leq j \leq 2n, 1 \leq k \leq n, \\
[\bar{\mathbf{H}}]_{j,2j-1,2k-n} & \text{if } 1 \leq j \leq n, n+1 \leq k \leq 2n, \\
[\bar{\mathbf{H}}]_{j,2j-n,2k-n} & \text{if } n+1 \leq j, k \leq 2n,
\end{cases}
\]

(37)

where \( \bar{\mathbf{H}} \) is the matrix of permuted entries of \( \mathbf{H} \). The effect of this permutation is to bundle together the elements of \( \bar{\mathbf{H}} \) that depend on pairs of entries \( (A_{ij}, A_{ji}) \).

Second, we use the Schur inversion formula for the inverse of a \( 2 \times 2 \) block matrix [24,34],

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} s_d & -s_d b d^{-1} \\ d^{-1} c s_d & s_a \end{pmatrix},
\]

(38)
where \( s_d = (a - bd^{-1}c)^{-1} \) and \( s_a = (d - ca^{-1}b)^{-1} \) are the inverse of the Schur complements of the blocks \( d \) and \( a \), respectively. If we choose for \( a \) the upper diagonal \( 2 \times 2 \) block of \( \tilde{H} \), then we obtain

\[
G_{11} = \left( \begin{pmatrix} \eta & z - D_1 \\ z^* - D_1 & \eta \end{pmatrix} - \sum_{\ell,j=1,\ell,j\neq i}^n \begin{pmatrix} 0 & J_{ij} \\ J_{ji} & 0 \end{pmatrix} G_j^{(i)} \begin{pmatrix} 0 & J_{i\ell} \\ J_{i\ell} & 0 \end{pmatrix} \right)^{-1}. \tag{39}
\]

Permuting the entries of the matrix, we obtain the analogous formula

\[
G_{ii} = \left( \begin{pmatrix} \eta & z - D_i \\ z^* - D_i & \eta \end{pmatrix} - \sum_{\ell,j=1,\ell,j\neq i}^n \begin{pmatrix} 0 & J_{ij} \\ J_{ji} & 0 \end{pmatrix} G_j^{(i)} \begin{pmatrix} 0 & J_{i\ell} \\ J_{i\ell} & 0 \end{pmatrix} \right)^{-1}. \tag{40}
\]

Using the law of large numbers, it holds in the limit of \( n \gg 1 \) that

\[
\sum_{\ell,j=1,\ell,j\neq i}^n \begin{pmatrix} 0 & J_{ij} \\ J_{ji} & 0 \end{pmatrix} G_j^{(i)} \begin{pmatrix} 0 & J_{i\ell} \\ J_{i\ell} & 0 \end{pmatrix} = (n - 1) \begin{pmatrix} \langle J_{ik}^2 \rangle \langle [G_{kk}]^{(i)}_{22} \rangle & \langle J_{ik} J_{ki} \rangle \langle [G_{kk}]^{(i)}_{21} \rangle \\ \langle J_{ik} J_{ki} \rangle \langle [G_{kk}]^{(i)}_{12} \rangle & \langle J_{ik}^2 \rangle \langle [G_{kk}]^{(i)}_{11} \rangle \end{pmatrix} + (n - 1)(n - 2) \begin{pmatrix} \langle J_{ik} J_{i\ell} \rangle \langle [G_{kk}]^{(i)}_{22} \rangle & \langle J_{ik} J_{i\ell} \rangle \langle [G_{kk}]^{(i)}_{21} \rangle \\ \langle J_{i\ell} J_{ki} \rangle \langle [G_{kk}]^{(i)}_{12} \rangle & \langle J_{i\ell} J_{ki} \rangle \langle [G_{kk}]^{(i)}_{11} \rangle \end{pmatrix}, \tag{41}
\]

where we have also used that \( G_j^{(i)} \) is statistically independent of \( J_{ij} \) and \( J_{ji} \).

Defining \( g = \langle G_{ii} \rangle \) and using that \( \sigma^2 = \langle J_{ik}^2 \rangle, \tau \sigma^2 = \langle J_{ik} J_{ki} \rangle, \) and \( \langle J_{i\ell} J_{ki} \rangle = 0 \) when \( k \neq \ell \), we readily obtain Eq. (17).

### B Derivation of Eq. (25) for the boundary of the spectrum when \( \tau = 0 \) and \( p_D \) is uniform

Equation (23) for the uniform distribution Eq. (7) gives

\[
\int_{d_-}^{d_+} \frac{1}{(x - u)^2 + y^2} du = \frac{d_+ - d_-}{\sigma^2}, \tag{42}
\]

where we have used \( z = x + iy \). Using the formula

\[
\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \arctan \frac{x}{a} + \text{constant} \tag{43}
\]

for the indefinite integral of \( 1/(x^2 + a^2) \) when \( a \neq 0 \), we obtain that for \( y \neq 0 \)

\[
\arctan \left( \frac{d_+ - x}{y} \right) - \arctan \left( \frac{d_- - x}{y} \right) = \frac{(d_+ - d_-)}{\sigma^2} y. \tag{44}
\]

Notice that since \( \arctan(a) \in (-\pi/2, \pi/2) \), we have the condition \( y \in (-\pi \sigma^2/(d_+ - d_-), \pi \sigma^2/(d_+ - d_-)) \). Subsequently, using

\[
\arctan a - \arctan b = \arctan \left( \frac{a - b}{1 + ab} \right) \tag{45}
\]

in Eq. (44), we obtain Eq. (25).
C Derivation of Eq. (30) for the leading eigenvalue $\lambda_1$ in the case of uniformly distributed diagonal elements

We derive Eq. (30) for the leading eigenvalue $\lambda_1$ when $p_D$ is the uniform distribution given by Eq. (7). Using the assumption that the leading eigenvalue $\lambda_1 \in \mathbb{R}$, we set $y = 0$ in equations Eqs. (28-29) yielding

$$\text{Re}(g_{12}) = -\frac{1}{(D - x) + \text{Re}(g_{12})\tau\sigma^2}$$

(46)

and

$$1 = \frac{\sigma^2}{[(D - x) + \text{Re}(g_{12})\tau\sigma^2]^2}$$

(47)

Integrating the equations Eq. (46-47) over the uniform distribution $p_D$ supported on the interval $[d_-, d_+]$, we obtain

$$\text{Re}(g_{12}) = \frac{\log(x - d_- - \text{Re}(g_{12})\sigma^2\tau) - \log(x - d_+ - \text{Re}(g_{12})\sigma^2\tau)}{d_+ - d_-}$$

(48)

and

$$1 = \frac{\sigma^2}{(d_+ - x + \text{Re}(g_{12})\tau\sigma^2)(d_- - x + \text{Re}(g_{12})\tau\sigma^2)}.$$ 

(49)

We first solve Eq. (49) towards $\text{Re}(g_{12})$ with solutions

$$\text{Re}(g_{12}) = \frac{x}{\sigma^2\tau} - \frac{(d_- + d_+)}{2\sigma^2\tau} + s\sqrt{(d_+ - d_-)^2 + 4\sigma^2}$$

(50)

where $s = \pm 1$. Replacing $\text{Re}(g_{12})$ in Eq. (48) by this solution gives a linear equation in $x$. The solutions of this linear equation provide the intersection points of the boundary of the support of the spectral distribution with the real axis, viz.,

$$x = \frac{1}{2} \left( -s\sqrt{(d_- - d_+)^2 + 4\sigma^2} + d_+ + d_- \right) + \tau\frac{\sigma^2}{d_+ - d_-} \log \left( \frac{-s\sqrt{(d_- - d_+)^2 + 4\sigma^2} + d_+ - d_-}{-s\sqrt{(d_- - d_+)^2 + 4\sigma^2} - d_+ + d_-} \right).$$

(51)

For $s = 1$ we obtain the leading eigenvalue given by Eq. (30).

D Perturbation theory for the leading eigenvalue

We use perturbation theory to understand the functional behaviour of $\text{Re}(\lambda_1)$ as a function of $\sigma$ in Fig. 4.

Let $D + \sigma J$, where $D$ is a diagonal matrix, $\sigma$ a small parameter, and $J$ an arbitrary $\sigma$-independent matrix. Let $\lambda^{(0)}_j$, $r^{(0)}_j$, and $l^{(0)}_j$ denote the eigenvalues, right eigenvectors, and left eigenvectors of $D$, respectively.
Let $\lambda_j(\sigma)$ denote the eigenvalues of $D + \sigma J$. An expansion around $\sigma \approx 0$ gives
\begin{equation}
\lambda_j(\sigma) = \lambda_j^{(0)} + \lambda_j^{(1)} \sigma + \lambda_j^{(2)} \sigma^2 + O(\sigma^3)
\end{equation}
with $\lambda_j^{(0)} = D_j$. Note that in this paper we use the convention that $\lambda_1^{(0)} \geq \lambda_2^{(0)} \geq \ldots \lambda_n^{(0)}$ and thus $D_1 \geq D_2 \geq \ldots D_n$.

It then holds
\begin{equation}
\lambda_j^{(1)} = \frac{\bar{l}_j^{(0)} \cdot J \bar{r}_j^{(0)}}{\bar{l}_j^{(0)} \cdot \bar{r}_j^{(0)}}
\end{equation}
and
\begin{equation}
\lambda_j^{(2)} = \frac{1}{\bar{l}_j^{(0)} \cdot \bar{r}_j^{(0)}} \sum_{i=1; i \neq j}^n \frac{[\bar{l}_j^{(0)} \cdot J \bar{r}_i^{(0)}] [\bar{r}_i^{(0)} \cdot J \bar{l}_j^{(0)}]}{(\bar{l}_j^{(0)} \cdot \bar{r}_i^{(0)}) (\lambda_j^{(0)} - \lambda_i^{(0)})}.
\end{equation}

Since $D$ is a diagonal matrix, we can set $\bar{l}_i^{(0)} \cdot \bar{r}_i^{(0)} = \delta_{i,j}$ and $\bar{l}_j^{(0)} \cdot J \bar{l}_i^{(0)} = J_{ji}(1 - \delta_{i,j})$. In this case, it holds that
\begin{equation}
\lambda_j(\sigma) = D_j + \sum_{i=1; i \neq j}^n \frac{J_{ij} J_{ji}}{(D_j - D_i)} \sigma^2 + O(\sigma^3).
\end{equation}

For $\sigma = 0$, it holds that $D_1 = D_{\text{max}}$ and thus the denominator in Eq. (55) is positive. From this it follows that when $J_{ij} J_{ji} < 0$, $\lambda_1$ initially decreases as a function of $\sigma$. However, when $\sigma$ is larger, then the $O(\sigma^3)$ becomes relevant, which provides the nonmonotonic behaviour in Fig. 5.

**References**

[1] R. M. May, *Will a large complex system be stable?*, Nature 238(5364), 413 (1972), doi:10.1038/238413a0.

[2] H. J. Sommers, A. Crisanti, H. Sompolinsky and Y. Stein, *Spectrum of large random asymmetric matrices*, Phys. Rev. Lett. 60, 1895 (1988), doi:10.1103/PhysRevLett.60.1895.

[3] S. Allesina and S. Tang, *Stability criteria for complex ecosystems*, Nature 483(7388), 205 (2012), doi:10.1038/nature10832.

[4] A. Mougi and M. Kondoh, *Diversity of interaction types and ecological community stability*, Science 337(6092), 349 (2012), doi:10.1126/science.1220529.

[5] A. Mougi and M. Kondoh, *Stability of competition–antagonism–mutualism hybrid community and the role of community network structure*, Journal of theoretical biology 360, 54 (2014), doi:10.1016/j.jtbi.2014.06.030.

[6] I. Neri and F. L. Metz, *Linear stability analysis of large dynamical systems on random directed graphs*, Phys. Rev. Research 2, 033313 (2020), doi:10.1103/PhysRevResearch.2.033313.
[7] A. M. Mambuca, C. Cammarota and I. Neri, *Dynamical systems on large networks with predator-prey interactions are stable and exhibit oscillations*, arXiv preprint arXiv:2009.11211 (2020).

[8] H. Sompolinsky, A. Crisanti and H. J. Sommers, *Chaos in random neural networks*, Physical Review Letters 61, 259 (1988), doi:10.1103/PhysRevLett.61.259.

[9] J. Kadmon and H. Sompolinsky, *Transition to chaos in random neuronal networks*, Phys. Rev. X 5, 041030 (2015), doi:10.1103/PhysRevX.5.041030.

[10] G. Biroli, G. Bunin and C. Cammarota, *Marginally stable equilibria in critical ecosystems*, New Journal of Physics 20(8), 083051 (2018), doi:10.1088/1367-2630/aada58.

[11] F. Roy, G. Biroli, G. Bunin and C. Cammarota, *Numerical implementation of dynamical mean field theory for disordered systems: Application to the lotka–volterra model of ecosystems*, Journal of Physics A: Mathematical and Theoretical 52(48), 484001 (2019), doi:10.1088/1751-8121/ab1f32.

[12] J. Moran and J.-P. Bouchaud, *May’s instability in large economies*, Phys. Rev. E 100, 032307 (2019), doi:10.1103/PhysRevE.100.032307.

[13] Y. Guo and A. Amir, *Exploring the effect of network topology, mrna and protein dynamics on gene regulatory network stability*, Nature communications 12(1), 1 (2021), doi:10.1038/s41467-020-20472-x.

[14] W. Tarnowski, I. Neri and P. Vivo, *Universal transient behavior in large dynamical systems on networks*, Phys. Rev. Research 2, 023333 (2020), doi:10.1103/PhysRevResearch.2.023333.

[15] L. A. Pastur, *On the spectrum of random matrices*, Theoretical and Mathematical Physics 10(1), 67 (1972), doi:10.1007/BF01035768.

[16] J. O. Lee and K. Schnelli, *Extremal eigenvalues and eigenvectors of deformed wigner matrices*, Probability Theory and Related Fields 164(1-2), 165 (2016), doi:10.1007/s00440-014-0610-8.

[17] P. Mergny and S. N. Majumdar, *Stability of large complex systems with heterogeneous relaxation dynamics*, Journal of Statistical Mechanics: Theory and Experiment 2021(12), 123301 (2021), doi:10.1088/1742-5468/ac3b47.

[18] I. Neri and F. L. Metz, *Eigenvalue outliers of non-hermitian random matrices with a local tree structure*, Phys. Rev. Lett. 117, 224101 (2016), doi:10.1103/PhysRevLett.117.224101.

[19] B. Khoruzhenko, *Large-n eigenvalue distribution of randomly perturbed asymmetric matrices*, Journal of Physics A: Mathematical and General 29(7), L165 (1996), doi:10.1088/0305-4470/29/7/003.

[20] T. Tao, *Topics in random matrix theory*, vol. 132, American Mathematical Soc., doi:10.1090/gsm/132 (2012).
[21] J. Feinberg and A. Zee, *Non-hermitian random matrix theory: Method of hermitian reduction*, Nuclear Physics B **504**(3), 579 (1997), doi:10.1016/S0550-3213(97)00502-6.

[22] S. Hikami and R. Pnini, *Density of state in a complex random matrix theory with external source*, Journal of Physics A: Mathematical and General **31**(35), L587 (1998), doi:10.1088/0305-4470/31/35/001.

[23] T. Rogers and I. P. Castillo, *Cavity approach to the spectral density of non-hermitian sparse matrices*, Physical Review E **79**(1), 012101 (2009), doi:10.1103/physreve.79.012101.

[24] F. L. Metz, I. Neri and T. Rogers, *Spectral theory of sparse non-hermitian random matrices*, Journal of Physics A: Mathematical and Theoretical (2019), doi:10.1088/1751-8121/ab1ce0.

[25] Z. Bai and J. W. Silverstein, *Spectral analysis of large dimensional random matrices*, vol. 20, Springer, doi:10.1007/978-1-4419-0661-8 (2010).

[26] C. Bordenave and D. Chafaï, *Around the circular law*, Probability surveys **9**, 1 (2012), doi:10.1214/11-PS183.

[27] G. Livan, M. Novaes and P. Vivo, *Introduction to random matrices: theory and practice*, vol. 26, Springer, doi:10.1007/978-3-319-70885-0 (2018).

[28] H. J. Sommers, A. Crisanti, H. Sompolinsky and Y. Stein, *Spectrum of large random asymmetric matrices*, Phys. Rev. Lett. **60**, 1895 (1988), doi:10.1103/PhysRevLett.60.1895.

[29] M R Leadbetter, Georg Lindgren, and Holger Rootzén, *Extremes and related properties of random sequences and processes*, Springer-Verlag, 1st edn., doi:10.1007/978-1-4612-5449-2 (1983).

[30] M. R. Gardner and W. R. Ashby, *Connectance of large dynamic (cybernetic) systems: critical values for stability*, Nature **228**(5273), 784 (1970), doi:10.1038/228784a0.

[31] Y. V. Fyodorov and B. A. Khoruzhenko, *Nonlinear analogue of the may-wigner instability transition*, Proceedings of the National Academy of Sciences **113**(25), 6827 (2016), doi:10.3410/f.726406145.793528093.

[32] S. Belga Fedeli, Y. V. Fyodorov and J. R. Ipsen, *Nonlinearity-generated resilience in large complex systems*, Phys. Rev. E **103**, 022201 (2021), doi:10.1103/PhysRevE.103.022201.

[33] B. Lacroix-A-Chez-Toine and Y. Fyodorov, *Counting equilibria in a random non-gradient dynamics with heterogeneous relaxation rates*, arXiv preprint arXiv:2112.11250 (2021).

[34] J. Bun, J.-P. Bouchaud and M. Potters, *Cleaning large correlation matrices: tools from random matrix theory*, Physics Reports **666**, 1 (2017), doi:10.1016/j.physrep.2016.10.005.

[35] J. H. Wilkinson, *The algebraic eigenvalue problem*, Oxford University Press, Inc., doi:10.1017/S0013091500012104 (1988).