Quantum Kolmogorov Complexity and Quantum Key Distribution

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We discuss the Bennett-Brassard 1984 (BB84) quantum key distribution protocol in the light of quantum algorithmic information. While Shannon’s information theory needs a probability to define a notion of information, algorithmic information theory does not need it and can assign a notion of information to an individual object. The program length necessary to describe an object, Kolmogorov complexity, plays the most fundamental role in the theory. In the context of algorithmic information theory, we formulate a security criterion for the quantum key distribution by using the quantum Kolmogorov complexity that was recently defined by Vitányi. We show that a simple BB84 protocol indeed distribute a binary sequence between Alice and Bob that looks almost random for Eve with a probability exponentially close to 1.

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I. INTRODUCTION

Cryptography is one of the most important arts in modern society. It enables us to communicate securely with our friends who live far away. In 1984, Bennett and Brassard [1] proposed a simple but also an astonishing protocol which is called the BB84 protocol. The protocol uses quantum theory as its essential part in achieving unconditionally secure key distribution [2, 3, 4, 5]. The security notion of the BB84 protocol is based on Shannon’s information theory [6]. Roughly speaking, the security criterion demands that a random variable representing a final key and another random variable representing Eve’s guess are almost independent. That is, the Shannon entropy of the final key from Eve’s viewpoint should attain a value sufficiently close to its maximum value. In this paper, we give an alternative point of view on this problem. We reconsider the protocol in algorithmic information theory. In the middle of 1960’s, Kolmogorov [7] and independently Chaitin [8] described an innovative idea that makes a bridge between information theory and computation theory. While Shannon’s conventional information theory treats probability distributions and needs them to define a notion of information, their theory, algorithmic information theory, takes randomness with respect to the algorithm as the heart of the information. Their formalism thus does not need a probability to define information, and can assign a notion of information to each individual object such as a binary sequence. The theory has been applied to problems in various fields including physics [9]. As entropy does in Shannon’s information theory, in algorithmic information theory a quantity called the Kolmogorov complexity plays the most fundamental role. The Kolmogorov complexity is defined as the length of the shortest description of an object. Kolmogorov complexity has some good properties and behaves rather rationally, as does entropy in Shannon’s information theory. Thus, the security criterion that we are to consider in this paper should not be based on Shannon’s entropy, but on Kolmogorov complexity instead. Moreover, since Eve has a quantum state, the Kolmogorov complexity has to be extended to be able to treat quantum states as its inputs. That is, a secure final key should have sufficiently large quantum Kolmogorov complexity for Eve.

Recently, some versions of quantum Kolmogorov complexity have been proposed. We employ one of them which was defined by Vitányi [10]. It has a natural interpretation in terms of classical programs for quantum Turing machines. In Sec. II we give a brief review of Vitányi’s definition. Its two properties that play important roles in our paper are explained. In Sec. IIIA we discuss the security that can be attained in a classical communication using a shared random binary sequence. We investigate a one-time pad and show that it provides a secure communication also in the context of algorithmic information. In Sec. IIIC the main part of the present paper, the security proof of the BB84 protocol is discussed. We introduce a simple BB84 quantum key distribution protocol and show that it enables Alice and Bob to share a binary sequence that looks almost random to Eve with probability exponentially close to 1. In Sec. IIIV we give some discussion of our results and future problems.

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II. QUANTUM KOLMOGOROV COMPLEXITY BASED ON CLASSICAL DESCRIPTION

Recently some quantum versions of Kolmogorov complexity were proposed by a several researchers. Svozil \[11\], in his pioneering work, defined the quantum Kolmogorov complexity as the minimum classical description length of a quantum state through a quantum Turing machine \[12, 13\]. As is easily seen by comparing the cardinality of a set of all the programs with that of a set of all the quantum states, the value often becomes infinity. Vitányi’s definition \[10\], while similar to Svozil’s, does not have this disadvantage. He added a term that compensates for the difference between a target state and an output state. Berthiaume, van Dam, and Laplante \[14\] defined their quantum Kolmogorov complexity as the length of the shortest quantum program that outputs a target state. The definition was settled and its properties were extensively investigated by Müller \[15, 16\]. Gacs \[17\] employed a different starting point related to the algorithmic probability to define his quantum Kolmogorov complexity.

In this paper we employ the definition given by Vitányi \[10\]. The use of Vitányi’s definition is justified for the following reason. Since, as will be seen in the next section, we are interested in the randomness of a classical final key for Eve, to consider its classical description is sufficient even if Eve has a quantum state. This way of thinking is natural in quantum-information theory. That is, when one is interested in classical outputs, the inputs to be considered are also classical. Vitányi gave a description of a one-way quantum Turing machine and utilized it to define a prefix quantum Kolmogorov complexity. A one-way quantum Turing machine consists of four tapes and an internal control. (See \[10\] for more details.) Each tape is a one-way infinite qubit chain and has a corresponding head on it. One of the tapes works as the input tape and is read-only from left-to-right. A program is given on this tape as an initial condition. The second tape works as the work tape. The work tape is initially set to 0 for all the cells. The head on it can read and write a cell and can move in both directions. The third tape is called an auxiliary tape. One can put an additional input on this tape. The additional input is written to the leftmost qubits and can be a quantum state or a classical state. This input is needed when one treats conditional Kolmogorov complexity. The fourth tape works as the output tape. It is assumed that after halting the state of this tape will not be changed. The internal control is a quantum system described by a finite-dimensional Hilbert space which has two special orthogonal vectors $|q_0⟩$ (initial state) and $|q_f⟩$ (halting state). After each step one makes a measurement of a coarse-grained observable $\{|q_f⟩⟨q_f|, 1 − |q_f⟩⟨q_f|\}$ on the internal control to know if the computation halts. Although there are subtle problems \[18, 19, 20, 21\] in the halting process of the quantum Turing machine, we do not get into this problem and employ a simple definition of the halting. A computation halts at time $t$ if and only if the probability to observe $q_f$ at time $t$ is one, and at any time $t' < t$ a probability to observe $q_f$ is zero. By using this one-way quantum Turing machine, Vitányi defined the quantum Kolmogorov complexity as follows. He treated the length of the shortest classical description of a quantum state. That is, the programs of the quantum Turing machine are restricted to classical ones. While the programs must be classical, the auxiliary inputs can be quantum states. We write $U(p, y) = |x⟩$ if and only if a quantum Turing machine $U$ with a classical program $p$ and an auxiliary (classical or quantum) input $y$ halts and outputs $|x⟩$. The following is the precise description of Vitányi’s definition.

Definition 1 \[10\] The (self-delimiting) quantum Kolmogorov complexity of a pure state $|x⟩$ with respect to a one-way quantum Turing machine $U$ with $y$ (possibly a quantum state) as conditional input given for free is

$$K_U(|x⟩, | y⟩) := \min_{p, z} \{l(p) + \lceil -\log_2 (|z| |x⟩^2) \rceil : U(p, y) = |z⟩\},$$

where $l(p)$ is the length of a classical program $p$, and $[a]$ is the smallest integer larger than $a$.

The one-way quality of the quantum Turing machine ensures that the halting programs compose a prefix-free set. Because of this, the length $l(p)$ is defined consistently. The term $\lceil -\log_2 (|z| |x⟩^2) \rceil$ represents how insufficiently an output $|z⟩$ approximates the desired output $|x⟩$. This additional term has a natural interpretation using the Shannon-Fano code. Vitányi has shown the following invariance theorem, which is very important.

Theorem 1 \[10\] There is a universal quantum Turing machine $U$, such that for all machines $Q$ there is a constant $c_Q$, such that for all quantum states $|x⟩$ and all auxiliary inputs $y$ we have

$$K_U(|x⟩, | y⟩) \leq K_Q(|x⟩, | y⟩) + c_Q.$$ 

Thus the value of the quantum Kolmogorov complexity does not depend on the choice of the quantum Turing machine if one neglects the unimportant constant term $c_Q$. Thanks to this theorem, one often writes $K$ instead of $K_U$. Moreover, the following theorem is crucial for our discussion.

Theorem 2 \[10\] On classical objects (that is, finite binary strings that are all directly computable) the quantum Kolmogorov complexity coincides up to a fixed additional constant with the self-delimiting Kolmogorov complexity. That is, there exists a constant $c$ such that for any classical binary sequence $|x⟩$,

$$\min_{q} \{l(q) : U(q, y) = |x⟩\} \geq K(|x⟩, | y⟩) \geq \min_{q} \{l(q) : U(q, y) = |x⟩\} − c$$

where $l(q)$ is the shortest classical description length of a quantum state. That is, the programs of the quantum Turing machine are restricted to classical ones. While the programs must be classical, the auxiliary inputs can be quantum states. We write $U(p, y) = |x⟩$ if and only if a quantum Turing machine $U$ with a classical program $p$ and an auxiliary (classical or quantum) input $y$ halts and outputs $|x⟩$. The following is the precise description of Vitányi’s definition.
holds.

According to this theorem, for classical objects it essentially suffices to treat only programs that exactly output the object.

### III. SECURITY PROOF OF QUANTUM KEY DISTRIBUTION IN THE LIGHT OF QUANTUM ALGORITHMIC INFORMATION

#### A. Security of one-time pad

The goal of the quantum key distribution is to distribute a secret key only between legitimate users. In the context of algorithmic information, a secret key is nothing but a binary sequence that looks random to Eve. We first show one-time pad. That is, Alice sends a binary sequence and uses a public linear code for privacy amplification. Although there are more sophisticated or realistic ones, we introduced. We consider a quantum key distribution protocol that uses a preshared secret key for error correction and uses a public linear code for privacy amplification. Although there are more sophisticated or realistic ones, we treat one of the simplest protocols since our aim is to present a different viewpoint from the algorithmic information.

Theorem 3 There exists a constant \( c \) (that depends only on a choice of a quantum Turing machine) such that the following statement holds. Let \( M \) be an arbitrary positive integer and let \( k \in \{0,1\}^M \) be a binary sequence. For any \( \delta > 0 \), we define a set \( B_\delta \subseteq \{0,1\}^M \) as

\[
B_\delta := \{ x | K(x|x \oplus k, M) \leq K(k|M) - \delta M - c \}.
\]

The size of this \( B_\delta \) is bounded by

\[
|B_\delta| \leq 2^{(1-\delta)M}.
\]

**Proof:** See Appendix B.

The following corollary is obvious.

**Corollary 1** There exists a constant \( c \) such that the following statement holds. Let \( M \) be an arbitrary positive integer and let \( k \in \{0,1\}^M \) be a binary sequence that looks random to Eve, who knows its length only. That is, \( K(k|M) \geq M \) holds. For any \( \delta > 0 \), we define a set \( B_\delta \subseteq \{0,1\}^M \) as

\[
B_\delta := \{ x | K(x|x \oplus k, M) \leq (1-\delta)M - c \}.
\]

The size of this \( B_\delta \) is bounded by

\[
|B_\delta| \leq 2^{(1-\delta)M}.
\]

The size \( |B_\delta| \) in this corollary is thus much smaller than \( |\{0,1\}^M| = 2^M \). This corollary shows that, if Alice and Bob share a random binary sequence only between them, they can achieve a secret communication by one-time pad.

Let us note a remark. One may wonder whether one can show that the size of a set \( \{ x | K(x|x \oplus k, M) \leq M - c \} \) is exponentially small compared with \( |\{0,1\}^M| = 2^M \). It is not possible because many \( x \)'s have a small Kolmogorov complexity even if Eve does not know \( x \oplus k \). For instance, the Kolmogorov complexity of \( x = 00\ldots00 \in \{0,1\}^M \) is almost vanishing. Thus even \( |\{ x | K(x|M) \leq M - c \}| \) can be comparable with \( 2^M \), while \( |\{ x | K(x|M) \leq (1-\delta)M - c \}| \leq 2^{(1-\delta)M} \) holds for \( c \geq 0 \).

#### B. BB84 protocol

As was discussed in the last section, if Alice and Bob share a binary sequence that is random for Eve, they can communicate securely by using the sequence. Our goal in the following is to show that a quantum key distribution indeed achieves this distribution of a random binary sequence. In this section, a concrete protocol to be analyzed is introduced. We consider a quantum key distribution protocol that uses a preshared secret key for error correction and uses a public linear code for privacy amplification. Although there are more sophisticated or realistic ones, we treat one of the simplest protocols since our aim is to present a different viewpoint from the algorithmic information. Let us introduce the protocol.
We denote the a posteriori state on Bob’s information bit and Eve’s apparatus as obtaining a sifted key. For any
Theorem 4
its restriction on Eve’s apparatus as
(iv) Alice and Bob check the error rate in the test bits by public discussions. If the error rate is larger than a
preagreed threshold \( p \), they abort the protocol.
(v) Alice and Bob perform an error correction by the one-time pad using a preshared secret key. They consume \( Nh(p) + \text{const} \) secret bits for this procedure.
(vi) Alice and Bob perform a privacy amplification. (See below for the details.)

After error correction, Alice and Bob have a common sifted key \( x \in \{0,1\}^N \) (information bits). On the other hand, Eve has a quantum state that may be correlated with \( x \). Due to this correlation, Eve may have a part of the information on \( x \). Alice and Bob, therefore, cannot use \( x \) itself as the final key. Privacy amplification is a protocol that extracts a shorter final key which cannot be guessed by Eve at all. The privacy amplification in our protocol is performed by use of a linear code. All players including Eve know a set of linear independent vectors \( \{v_1, v_2, \ldots, v_M\} \subset \{0,1\}^N \) which span a linear code \( C \). The vectors could be announced before the whole protocol. Its Hamming distance \( d(C) = \min\{|v| : v \neq 0, v \in C\} \) is assumed to satisfy \( d(C) > 2N(p + \epsilon) \), where \( p \) is the allowed error rate in test bits and \( \epsilon > 0 \) is a small security parameter.

The final key is obtained from the sifted key by a function \( f : \{0,1\}^N \rightarrow \{0,1\}^M \) which is defined as
\[
 f(x) = x \cdot v := (x \cdot v_1, x \cdot v_2, \ldots, x \cdot v_M).
\]

Eve’s purpose is to obtain knowledge of \( f(x) \).

C. Security proof

Suppose that Alice has chosen a basis \( b \in \{0,1\}^{2N} \), test bits \( T \), a value of the test bits \( z_T \in \{0,1\}^N \), and Bob has obtained \( z'_T \in \{0,1\}^N \) as the value of the test bits. After (v) in the above protocol Eve also knows all of them. As is well-known, one can view the protocol also from an Ekert 1991 protocol (E91) like setting. In the E91 like setting, after the error correction there is an entangled state over Alice’s information bits, Bob’s information bits and Eve’s apparatus. We denote the state as \( \rho_{b,T,z_T,z'_T} \). Alice makes a measurement \( X_A = \{|x\rangle\langle x|\} \) on her information bits to obtain a sifted key \( x \in \{0,1\}^N \). This measurement changes the state on Bob’s information bits and Eve’s apparatus \( 24 \). We denote the a posteriori state on Bob’s information bit and Eve’s apparatus as \( \rho_{x,b,T,z_T,z'_T} \). We further write its restriction on Eve’s apparatus as \( \rho^E_{x,b,T,z_T,z'_T} \). Eve’s purpose is to extract information on the final key \( f(x) \) from this quantum state and her knowledge, \( b, T, z_T, z'_T \), and \( f \). Therefore in the context of quantum Kolmogorov complexity, Eve’s uncertainty on the final key is written as \( K(f(x)|\rho^E_{x,b,T,z_T,z'_T}, f, b, T, z_T, z'_T) \) \( 22 \). We prove the following theorem.

**Theorem 4** There exists a constant \( c \) (that depends only on the choice of the quantum Turing machine) such that the following statement holds. For any \( N \), any \( p \), any \( \epsilon > 0 \), any independent vectors \( \{v_1, v_2, \ldots, v_M\} \) whose span \( C \) satisfies \( d(C) > 2N(p + \epsilon) \), and any \( \delta > 0 \),
\[
 Pr\left( K(f(x)|\rho^E_{x,b,T,z_T,z'_T}, f, b, T, z_T, z'_T) \leq M - \delta N - c \wedge |z_T \oplus z'_T| < Np \right) \leq 2^{-\delta N} + 3e^{-\frac{c^2}{4N}}
\]
holds.

**Proof:**
We fix a universal quantum Turing machine \( U \) and discuss the values of the quantum Kolmogorov complexity with respect to it. Since \( f(x) \) is classical, to discuss the quantum Kolmogorov complexity of \( f(x) \) it essentially suffices to consider programs that exactly output \( f(x) \) thanks to Theorem 2. For each output \( x \in \{0,1\}^M \), there is a shortest program \( t_{x,b,T,z_T,z'_T} \) (take an arbitrary one if it is not unique) that produces \( f(x) \) exactly as its output with auxiliary
inputs $\rho_{x,b,T,x',z'}^E$ and $f,b,T,z_T,z_T'$. Although the $t_{x,b,T,x',z'}^E$’s may have different halting times, thanks to a lemma proved by Müller (Lemma 2.3.4. in [10]), there exists a completely positive map (CP map) $\Gamma_U : \Sigma(H_A \otimes H_I) \rightarrow \Sigma(H_O)$ satisfying

$$\Gamma_U(\rho_{x,b,T,x',z'}^E \otimes |t_{x,b,T,x',z'}\rangle\langle t_{x,b,T,x',z'}|) = |f(x)\rangle\langle f(x)|,$$

where $H_A$ is the Hilbert space for the auxiliary input and $H_I$ is the Hilbert space for programs and $H_O = \otimes^M C_2$ is the Hilbert space for outputs, and $\Sigma(H)$ denotes the set of all the density operators on $H$.

For a while we proceed with our analysis for fixed $b,T,z_T,z_T'$, for each $t \in \{0,1\}^*$ (a set of all the finite length binary sequences), let us define a set $\mathcal{E}_t^{b,T,x',z'} \subset \{0,1\}^N$ as $\mathcal{E}_t^{b,T,x',z'} = \{x | t_{x,b,T,z_T,z_T'} = t\}$. That is, for each $x \in \mathcal{E}_t^{b,T,x',z'}$ the program $t$ with auxiliary inputs $\rho_{x,b,T,x',z'}^E$ and $f,b,T,z_T,z_T'$ produces exactly $f(x)$. The set is further decomposed with respect to their outputs as $\mathcal{E}_t^{b,T,z_T,z_T'} = \bigcup_{y} \mathcal{E}_t^{b,T,x',z'}(y)$, where $\mathcal{E}_t^{b,T,z_T,z_T'}(y) := \{x | t_{x,b,T,z_T,z_T'} = t, f(x) = y\}$. That is, for each $x \in \mathcal{E}_t^{b,T,z_T,z_T'}(y)$ the program $t$ with an auxiliary input $\rho_{x,b,T,z_T,z_T'}^E$, $f,b,T,z_T,z_T'$ produces $y$. Since the CP map $\Gamma_U$ does not increase distinguishability among states, for any $x \in \mathcal{E}_t^{b,T,z_T,z_T'}(y)$ and $x' \in \mathcal{E}_t^{b,T,x',z'}(y')$ with $y \neq y'$, $\rho_{x,b,T,z_T,z_T'}^E$ and $\rho_{x,b,T,z_T,z_T'}^{E'}$ must be completely distinguishable. We denote by $E_t^{b,T,z_T,z_T'} := \{E_t^{b,T,x',z'}(y)\}_y$ a projection valued measure (PVM) that perfectly distinguishes states which belong to different $y$. That is,

$$\text{tr}(E_t^{b,T,z_T,z_T'}(y)\rho_{x,b,T,z_T,z_T'}^E) = \delta_{f(x)y}$$

holds for each $x$ and $y$.

Let us consider the problem in an E91 like setting. Now Alice, Bob, and Eve have a state $\rho_{b,T,z_T,z_T'}$ over their systems. For an arbitrary fixed finite $L \subset \{0,1\}^*$, let us consider an observable over Alice’s information bits and Eve’s apparatus;

$$Q_L^{b,T,z_T,z_T'} := \sum_{t \in L} \sum_{y} A_t^{b,T,z_T,z_T'}(y) \otimes E_t^{b,T,z_T,z_T'}(y),$$

where $A_t^{b,T,z_T,z_T'}(y)$ is defined as

$$A_t^{b,T,z_T,z_T'}(y) := \sum_{x \in \mathcal{E}_t^{b,T,z_T,z_T'}(y)} |x\rangle\langle x|.$$

One can easily show that this $Q_L^{b,T,z_T,z_T'}$ is a projection operator. We hereafter consider an expectation value of this projection operator with respect to the state $\rho_{b,T,z_T,z_T'}$. ($Q_L^{b,T,z_T,z_T'}$ is naturally identified with an operator $Q_L^{b,T,z_T,z_T'} \otimes 1_B$ on Alice, Bob, and Eve’s tripartite system.) One can write it as follows:

$$\text{tr}(\rho_{b,T,z_T,z_T'}Q_L^{b,T,z_T,z_T'}) = \langle Q_L^{b,T,z_T,z_T'} \rangle_{b,T,z_T,z_T'} = \sum_{t \in L} \sum_{y} \langle A_t^{b,T,z_T,z_T'}(y) \otimes E_t^{b,T,z_T,z_T'}(y) \rangle_{b,T,z_T,z_T'},$$

where we put $\langle \cdot \rangle_{b,T,z_T,z_T'} = \text{tr}(\rho_{b,T,z_T,z_T'} \cdot )$. If we consider Alice’s measurement $X_A = \{|x\rangle\langle x|\}$ on her information bits and denote by $p(x|b,T,z_T,z_T')$ the probability to obtain $x$, it is represented as

$$\sum_{t \in L} \sum_{y} \sum_{x \in \mathcal{E}_t^{b,T,z_T,z_T'}(y)} \langle x| \otimes E_t^{b,T,z_T,z_T'}(y) \rangle_{b,T,z_T,z_T'} p(x|b,T,z_T,z_T') \text{tr}(\rho_{x,b,T,z_T,z_T'}^E E_t^{b,T,z_T,z_T'}(y))$$

$$= \sum_{t \in L} \sum_{y} \sum_{x \in \mathcal{E}_t^{b,T,z_T,z_T'}(y)} p(x|b,T,z_T,z_T') \text{Pr}(x \in \bigcup_{t \in L} \mathcal{E}_t^{b,T,z_T,z_T'}(y) | b,T,z_T,z_T'),$$

which is the expression in the text.
where we have used the condition (1).

In addition, this quantity can be represented in a different form (see Appendix C for its proof).

Lemma 1 Suppose that Alice virtually makes a measurement on her information bits with a PVM \( Z_A := \{|z\rangle\langle z|\} \) which is conjugate to \( X_A = \{|x\rangle\langle x|\} \) that is actually measured to obtain a sifted key, and Bob virtually makes a measurement on his information bits with \( Z_B := \{|\bar{z}\rangle\langle \bar{z}|\} \) which is conjugate to \( X_B \) that is actually measured. We denote their outcomes \( z_I \) and \( z'_I \). It holds that

\[
\text{tr}(\rho_{b,T,z_I,z'_I} Q^b_{L,T,z_I,z'_I}) \leq |L|^2 - 3 + 3 \sqrt{\text{Pr}(|z_I \oplus z'_I| > N(p + \epsilon)| b, T, z_I, z'_T)},
\]

where the second term in the right hand side is the square root of the probability to obtain distant \( z_I \) and \( z'_I \) with respect to a state \( \rho_{b,T,z_I,z'_I} \).

Combining these different expressions (2) and (3), we obtain

\[
\text{Pr} \left( x \in \bigcup_{t \in L} \mathcal{E}^b_{t,T,z_I,z'_I} \bigm| b, T, z_T, z'_T \right) \leq |L|^2 - 3 + 3 \sqrt{\text{Pr}(|z_I \oplus z'_I| > N(p + \epsilon)| b, T, z_I, z'_T)}.
\]

Now if \( L \) is taken as \( L := \{t | l(t) \leq M - \delta N\} \), since \( |L| \leq 2^M - \delta N \) holds the above inequality can be rewritten as

\[
\text{Pr} \left( x \in \bigcup_{t:l(t) \leq M - \delta N} \mathcal{E}^b_{t,T,z_I,z'_I} \bigm| b, T, z_T, z'_T \right) \leq 2^{-\delta N} + 3 \sqrt{\text{Pr}(|z_I \oplus z'_I| > N(p + \epsilon)| b, T, z_I, z'_T)}.
\]

Thanks to Theorem 2 there exists a constant \( c \) such that, if \( x \) satisfies \( K(f(x)|\rho_{x,b,T,x_I,x'_I}^E, b, T, z_T, z'_T, f) \leq M - \delta N - c \), then \( l(t,b,T,z_T,z'_T) \leq M - \delta N \) follows. That is, we obtain

\[
\text{Pr} \left( K(f(x)|\rho_{x,b,T,x_I,x'_I}^E, b, T, z_T, z'_T, f) \leq M - \delta N - c \bigm| b, T, z_I, z'_I \right) \leq 2^{-\delta N} + 3 \sqrt{\text{Pr}(|z_I \oplus z'_I| > N(p + \epsilon)| b, T, z_I, z'_T)}.
\]

We multiply both sides of this inequality by \( p(b, T, z_T, z'_T) \) which is defined as the probability to obtain \( b, T, z_T, z'_T \) and take a summation with respect to \( b, T, z_T, z'_T \) for all \( b, T, z_T, z'_T \) with \( |z_T \oplus z'_T| \leq Np \), and use Jensen’s inequality. We finally derive

\[
\text{Pr} \left( K(f(x)|\rho_{x,b,T,z_T,z'_T}^E, b, T, z_T, z'_T, f) \leq M - \delta N - c \bigwedge |z_T \oplus z'_T| < Np \right) \leq \text{Pr} \left( |z_I \oplus z'_I| < Np \bigm| z_T \oplus z'_T \right) 2^{-\delta N} + 3 \sqrt{\text{Pr} \left( |z_I \oplus z'_I| \leq Np \right) \text{Pr} \left( |z_I \oplus z'_I| > N(p + \epsilon) | z_T \oplus z'_T \right) Np}.
\]

The second term of the right hand side is bounded by Hoeffding’s lemma as \( \text{Pr}(|z_I \oplus z'_I| > N(p + \epsilon), |z_T \oplus z'_T| \leq Np) \leq e^{-\frac{2}{N}} \) (see e.g. [5]). We thus obtain

\[
\text{Pr} \left( K(f(x)|\rho_{x,b,T,z_T,z'_T}^E, b, T, z_T, z'_T, f) \leq M - \delta N - c \bigwedge |z_T \oplus z'_T| < Np \right) \leq 2^{-\delta N} + 3e^{-\frac{2}{N}}.
\]

This ends the proof.

Q.E.D.

IV. DISCUSSIONS

In this paper, we considered the security of the quantum key distribution protocol in the light of quantum algorithmic information. We employed the quantum Kolmogorov complexity defined by Vitányi as the fundamental quantity, discussed a possible security criterion, and showed that the simple BB84 protocol satisfies it. According to the main theorem, a probability for Eve to obtain an almost random final key is exponentially close to 1. The length of the final keys \( M \) is determined by a condition for the Hamming distance. One can take it as \( M \simeq N(1 - h(2(p + \epsilon))) \). Since the legitimate users have consumed \( Nh(p + \epsilon) \) bits for the error correction, the length of the key produced amounts to \( N(1 - h(2(p + \epsilon)) - h(p + \epsilon)) \). It coincides with the rate obtained in [3] where the security criterion was based on Shannon’s information theory.
Although we hope that the present work can be a first step toward the study of quantum cryptography from the viewpoint of quantum algorithmic information, there still remain a lot of things to be investigated. The security criterion employed in this paper utilizes the quantum Kolmogorov complexity, but it still needs the probability. Therefore, the original motivation of the algorithmic information theory, in some sense, has not been perfectly accomplished. Comparison between security notions based on algorithmic information and Shannon’s information is an important future problem to be considered. While the simple BB84 protocol satisfies both criteria, it is not clear whether one can be derived from another in some sense. The relation between these criteria will become more subtle if we will deepen our algorithmic information theoretical discussion so as to avoid an appearance of probability completely. For instance, as was shown, in the one-time pad protocol, while an individual secret key cannot be discussed in the conventional Shannon’s information theory, it can be treated in the algorithmic information theory. In addition, as we noted in Sec. III there are some other definitions of quantum Kolmogorov complexity. It is interesting to investigate whether one can apply them to the security problem. Application of our argument to other protocols will be another interesting problem.

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APPENDIX A: TECHNICAL LEMMAS

Lemma 2. For any state ρ and any projection operators Q and P, it holds that

$$|\text{tr}(\rho Q) - \text{tr}(PQ P)| \leq 3\text{tr}(\rho (1 - P))^{1/2}.$$

Proof: Since $1 = P + (1 - P)$ holds, Q can be decomposed as

$$Q = 1Q1 = PQP + PQ(1 - P) + (1 - P)QP + (1 - P)Q(1 - P).$$

Thus we obtain

$$|\text{tr}(\rho Q) - \text{tr}(\rho PQP)| \leq |\text{tr}(\rho PQ(1 - P))| + |\text{tr}(\rho (1 - P)QP)| + |\text{tr}(\rho (1 - P)Q(1 - P))|.$$ 

The Cauchy-Schwarz inequality bounds the first term in the right-hand side as

$$|\text{tr}(\rho PQ(1 - P))| = |\text{tr}(\rho PQP)|^{1/2}|\text{tr}(\rho (1 - P))|^{1/2} \leq \text{tr}(\rho (1 - P))^{1/2}.$$ 

Other terms can be bounded in a similar manner. This ends the proof. \( Q.E.D. \)

Lemma 3. For given linearly independent vectors \( \{v_1, v_2, \ldots, v_M\} \subset \{0, 1\}^N \), we define \( f : \{0, 1\}^N \rightarrow \{0, 1\}^M \) as \( f(x) = (x \cdot v_1, x \cdot v_2, \ldots, x \cdot v_M) \). Let \( C \) be a code generated by \( \{v_1, v_2, \ldots, v_M\} \) and \( d(C) \) be its Hamming distance. For \( s, t \in \{0, 1\}^N \) satisfying \( |s|, |t| < \frac{d(C)}{2} \) and for any \( y \in \{0, 1\}^M \),

$$\sum_{x : f(x) = y} (-1)^{x \cdot (s \oplus t)} = \delta_{st}2^{N - M}$$

holds, where \( \delta_{st} \) is Kronecker’s delta.

Proof: If we fix an element \( w_y \in \{0, 1\}^N \) satisfying \( f(w_y) = y \), \( \{x \mid f(x) = y\} \) is represented as \( w_y \oplus C^\perp \). Thus we obtain

$$\sum_{x : f(x) = y} (-1)^{x \cdot (s \oplus t)} = (-1)^{w_y \cdot (s \oplus t)} \sum_{x \in C^\perp} (-1)^{x \cdot (s \oplus t)}.$$ 

For \( s \oplus t \in C \), it gives \( 2^{N - M}(-1)^{w_y \cdot (s \oplus t)} \). Since \( |s \oplus t| \leq |s| + |t| < d(C) \) holds, \( s \oplus t \in C \) means \( s = t \). For \( s \oplus t \notin C \), thanks to Lemma D.1 in [3],

$$\sum_{x \in C^\perp} (-1)^{x \cdot (s \oplus t)} = 0$$

holds. This ends the proof. \( Q.E.D. \)
Appendix B: Proof of Theorem 3

Proof of Theorem 3: According to the fundamental properties of Kolmogorov complexity it is known that

\[ |K(x, k|M) - (K(k|M) + K(x|k, K(k), M))| \leq c_1 \]

holds for some constant \( c_1 \). (The proof also holds for the quantum Kolmogorov complexity thanks to Theorem 2.)

For a fixed \( \delta > 0 \), we define a set \( D_\delta \subset \{0, 1\}^M \) as

\[ D_\delta := \{x | K(x, k, K(k), M) \leq (1 - \delta)M\}. \]

It can be easily shown that \( |D_\delta| \leq 2^{(1 - \delta)M} \) holds. Now let us consider its complement \( D_\delta^c = \{x \in \{0, 1\}^M | K(x, k, K(k), M) > (1 - \delta)M\} \). For \( x \in D_\delta^c \), \( K(x, k|M) > K(k|M) + (1 - \delta)M - c_1 \) holds. By the way we have, in general,

\[ K(x, k|M) = K(x \oplus k, k|M) + c_2 \leq K(x \oplus k|M) + K(x|x \oplus k, M) + c_3 \]

for some \( c_2, c_3 \). Thus, for \( x \in D_\delta^c \), we have

\[ K(x \oplus k|M) + K(x|x \oplus k, M) + c_3 > K(k|M) + (1 - \delta)M - c_1. \]

Since \( K(x \oplus k|M) \leq M + c_4 \) holds for some \( c_4 \), if we put \( c = c_1 + c_3 + c_4 \) we obtain

\[ K(x|x \oplus k, M) > K(k|M) - \delta M - c. \]

for \( x \in D_\delta^c \). Thus \( D_\delta^c \subset B_\delta \) and \( B_\delta \subset D_\delta \) holds. Thanks to \( |D_\delta| \leq 2^{(1 - \delta)M} \), this ends the proof.

Q.E.D.

Appendix C: Proof of Lemma 1

Proof of Lemma 1: Let \( \rho_{b,T,z,T,z}_r \) be a state over Alice’s information bits, Bob’s information bits, and Eve’s apparatus. Suppose that Bob virtually makes a measurement of \( Z_B = \{\Xi | \Xi\} \) on his system (information bits). This observable is conjugate with \( X_B \), which is actually measured by Bob to obtain a sifted key. Suppose that Bob obtains an outcome \( z'_f \). We denote by \( p(z'_f | b, T, z_T, z'_T) \) a probability to obtain \( z'_f \). The a posteriori state on Alice’s information bits and Eve’s apparatus is denoted as \( \rho_{z'_f} \).

Define a projection operator \( P_{z'_f}^{b,T} \) on Alice’s information bits as

\[ P_{z'_f}^{b,T} := |s\rangle \langle s| \]

Applying Lemma 2 with \( P_{z'_f}^{b,T} = P \), \( Q_{L}^{b,T,z_T,z'_T} = Q \) and \( b_{T,z_T,z'_T} = \rho \), we obtain

\[ |(Q_{L}^{b,T,z_T,z'_T})_{b,T,z_T,z'_T} - (P_{z'_f}^{b,T} Q_{L}^{b,T,z_T,z'_T} P_{z'_f}^{b,T})_{b,T,z_T,z'_T} z'_T)| \leq 3(1 - P_{z'_f}^{b,T})^{1/2}, \]

where we put \( (\cdot)_{b,T,z_T,z'_T,z'_T} = \text{tr}(b_{T,z_T,z'_T}(\cdot)) \). In addition, if we introduce \( A^{b,T}(y) := \sum_{x} f(x) = y |x\rangle \langle x| \), it satisfies \( A^{b,T}(y) \leq A^{b,T}(y) \). Thus one can easily show that

\[ Q_{L}^{b,T,z_T,z'_T} \leq \sum_{y} A^{b,T}(y) \otimes E_{z'_f}^{b,T,z_T,z'_T}(y) \]

holds. It follows that

\[ P_{z'_f}^{b,T} Q_{L}^{b,T,z_T,z'_T} P_{z'_f}^{b,T} \leq \sum_{y} A^{b,T}(y) \otimes E_{z'_f}^{b,T,z_T,z'_T}(y). \]
Combining Eqs. (C1) and (C2), we obtain the inequality

\[
\sum_{t \in L} \sum_y \langle P_{z_i}^{b,T} A^{b,T}(y) P_{z_i}^{b,T} \otimes E_t^{b,T,z,t,z'_t} (y) \rangle_{b,T,z,t,z'_t,z'_j} \leq \sum_{t \in L} \sum_y \langle P_{z_i}^{b,T} A^{b,T}(y) P_{z_i}^{b,T} \otimes E_t^{b,T,z,t,z'_t} (y) \rangle_{b,T,z,t,z'_t,z'_j} + 3\langle 1 - P_{z_i}^{b,T} \rangle_{b,T,z,t,z'_t,z'_j}.
\]

(C3)

\[
\langle P_{z_i}^{b,T} A^{b,T}(y) P_{z_i}^{b,T} \otimes E_t^{b,T,z,t,z'_t} (y) \rangle_{b,T,z,t,z'_t,z'_j} \text{ in the first term on the right-hand side of Eq. (C3) is estimated as follows.}
\]

Suppose that, with respect to \(\rho_{z_i}^{b,T,z,t,z'_t} \), Eve made a measurement of the PVM \(E_t^{b,T,z,t,z'_t} \) and obtained \(y\). The probability to obtain \(y\) is denoted as \(p(y|b,T,z,t,z'_t)\). The a posteriori state over Alice’s information bits is denoted as \(\rho_{z_i}^{b,T,y,z,t,z'_t} \). We write its diagonalization as $|\phi_\nu\rangle = \sum s c_s^n |z'_j + s\rangle$. Now \(\langle \phi_\nu | P_{z_i}^{b,T} A^{b,T}(y) P_{z_i}^{b,T} | \phi_\nu \rangle\) is calculated as

\[
\langle \phi_\nu | P_{z_i}^{b,T} A^{b,T}(y) P_{z_i}^{b,T} | \phi_\nu \rangle = \sum_s \sum_t f(x-y) \sum_x (-1)^{x(s+t)} \leq 2^{-M},
\]

where we have used lemma 3 and \(\sum_s |c_s^n|^2 = 1\) to obtain the last inequality. We thus obtain for each \(y\) and \(z'_t\)

\[
\langle P_{z_i}^{b,T} A^{b,T}(y) P_{z_i}^{b,T} \otimes E_t^{b,T,z,t,z'_t} (y) \rangle_{b,T,z,t,z'_t,z'_j} \leq \langle P_{z_i}^{b,T} A^{b,T}(y) P_{z_i}^{b,T} \otimes E_t^{b,T,z,t,z'_t} (y) \rangle_{b,T,z,t,z'_t,z'_j} \leq 2^{-M}.
\]

Multiplying both sides of this inequality with \(p(y|b,T,z,t,z'_t)\) and summing it up with respect to \(y\), we obtain

\[
\sum_{y} \sum_{z'_t} \sum_{t \in L} \sum_y \langle P_{z_i}^{b,T} A^{b,T}(y) P_{z_i}^{b,T} \otimes E_t^{b,T,z,t,z'_t} (y) \rangle_{b,T,z,t,z'_t,z'_j} \leq |L| 2^{-M}.
\]

Summation of this inequality over \(t \in L\) further gives

\[
\sum_{y} \sum_{z'_t} \sum_{t \in L} \sum_y \langle P_{z_i}^{b,T} A^{b,T}(y) P_{z_i}^{b,T} \otimes E_t^{b,T,z,t,z'_t} (y) \rangle_{b,T,z,t,z'_t,z'_j} \leq |L| 2^{-M}.
\]

We next estimate the second term \(3\langle 1 - P_{z_i}^{b,T} \rangle_{b,T,z,t,z'_t,z'_j}^{1/2}\) in Eq. (C3). This term can be represented in a simple form by considering Alice’s measurement on her information bits with \(Z_A := \{|x\} \langle x|\} \) which is conjugate to \(X_A = \{|x| x\}\) that is actually measured to obtain a sifted key in the E91 like picture. One can show

\[
\langle 1 - P_{z_i}^{b,T} \rangle_{b,T,z,t,z'_t,z'_j} = \Pr (|z_t \oplus z'_t| \not> N(p + \epsilon) | b,T,z_t,z'_t,z'_j) ,
\]

where the right-hand side is the probability for Alice to obtain a distant \(z_t\) from Bob’s \(z'_t\). Combining the above estimates, we obtain

\[
\langle Q_{L}^{b,T,z,t,z'_t} \rangle_{b,T,z,t,z'_t,z'_j} \leq |L| 2^{-M} + 3 \sqrt{\Pr (|z_t \oplus z'_t| > N(p + \epsilon) | b,T,z_t,z'_t,z'_j)}.
\]

We multiply both sides of this inequality by \(p(z'_t|b,T,z_t,z'_t)\) and take a summation over \(z'_t\) to obtain

\[
\langle Q_{L}^{b,T,z,t,z'_t} \rangle_{b,T,z,t,z'_t,z'_j} \leq |L| 2^{-M} + 3 \sqrt{\Pr (|z_t \oplus z'_t| > N(p + \epsilon) | b,T,z_t,z'_t,z'_j)} ,
\]

where we have used Jensen’s inequality once. Q.E.D.

[1] C. H. Bennett and G. Brassard, In Proc. of IEEE Int. Conf. on Computers, Systems and Signal Processing, 175 (1984).
2. D. Mayers, in Advances in Cryptology - CRYPTO’96, LNCS 1109, 343 (1996).
3. H-K. Lo and H-F. Chau, Science, 283, 2050 (1999).
4. P. W. Shor and J. Preskill, Phys. Rev. Lett., 85, 441 (2000).
5. E. Biham, M. Boyer, P. O. Boykin, T. Mor, and V. Roychowdhury, J. of Cryptology 19, 381 (2006).
6. Not all the security notions use the Shannon entropy explicitly. They can be interpreted in any case in terms of probability theory.
7. A. N. Kolmogorov, Probl. Inform. Transm. 1, 1, 1 (1965).
8. G. Chaitin, J. Assoc. Comput. Mach., 13, 547 (1966).
9. M. Li and P. Vitányi, An Introduction to Kolmogorov Complexity and Its Applications, New York: Springer-Verlag (1997).
10. P. Vitányi, IEEE Trans. Inform. Theory, 47, 6, 2464 (2001).
11. K. Svozil, J. of Universal Comput. Sci., 2, 311 (1996).
12. D. Deutsch, Proc. Roy. Soc. London A, 400, 96 (1985).
13. A. Bernstein and U. Vazirani, SIAM J. Comput., 26, 1411 (1997).
14. A. Berthiaume, W. van Dam, and S. Laplante, J. Comput. System. Sci., 63, 201 (2001).
15. M. Müller, IEEE Trans. Inform. Theory, 54: 2, 763 (2008).
16. M. Müller, Ph.D. thesis, Technical University of Berlin, 2007.
17. P. Gacs, J. Phys. A: Math. Gen., 34, 1 (2001).
18. J. M. Myers, Phys. Rev. Lett., 78, 1823 (1997).
19. N. Linden and S. Popescu, quant-ph/9806054.
20. M. Ozawa, Phys. Rev. Lett., 80, 631 (1998).
21. T. Miyadera and M. Ohya, Open Sys. Info. Dyn., 12, 261 (2005).

The word “probabilistically” in this section is used to mean “randomly in a probabilistic sense.” That is, we use an unbiased probability $1/|\Omega|$ to choose a sample from a sample space $\Omega$ (say $\Omega = \{0,1\}^N$). (To avoid a possible confusion of this with randomness in the algorithmic sense, we just write “probabilistically.”)

To treat $\rho^E_{x,T,x',T}$ as an auxiliary input for a quantum Turing machine, Eve’s apparatus is identified with a system consisting of qubits. Our discussion does not depend on this identification.

In general, the $a$ posteriori state after a measurement is determined as follows. Suppose that there exist two system $A$ and $B$ that are described by Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Let us consider a state $\rho$ over the bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$. Suppose that on $A$ one made a measurement described by a positive-operator-valued measure $F = \{F_x\}$ and obtained an outcome $x$. The $a$ posteriori state on the system $B$ conditioned with this $x$ becomes

$$\rho_x = \frac{\text{tr}_A(\rho F_x \otimes 1)}{\text{tr}_\rho F_x \otimes 1}.$$

That is, it is a unique state that satisfies $\text{tr}(\rho G)\text{tr}(\rho(F_x \otimes 1)) = \text{tr}(\rho(F_x \otimes G))$ for any operator $G$ on $\mathcal{H}_B$. 