A note on the enclosure method for an inverse obstacle scattering problem with a single point source

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Abstract
This paper gives a note on an application of the enclosure method to an inverse obstacle scattering problem governed by the Helmholtz equation in two dimensions. It is shown that one can uniquely determine the convex hull of an unknown sound-hard polygonal obstacle from the trace of the total wave that was exerted by a single point source onto a known circle surrounding the obstacle provided the source is sufficiently far from the obstacle. The result contains a formula that extracts the value of the support function of the obstacle at a generic direction. Some other applications to thin obstacles, obstacles in a layered medium and the far-field equation in the linear sampling method are also included.

1. Introduction

The enclosure method was introduced in [12] for inverse boundary value problems for elliptic equations which are motivated by the possibility of applications to electrical impedance tomography, diffraction tomography, etc. Therein the observation data are formulated by using the Dirichlet-to-Neumann map (or Neumann-to-Dirichlet map) associated with the governing equation of a ‘signal’ propagating inside the medium. It aims at extracting information about the location and shape of unknown discontinuity embedded in a known reference medium that gives an effect on the propagation of the signal, such as an obstacle, inclusion, crack, etc from data observed on the boundary of the medium. Now we have many applications of this method, see, e.g., [11, 17, 23–25].

In [10] it was shown that in a simplified situation a single set of the Dirichlet and Neumann data gives information about the convex hull of unknown discontinuity. It was the starting point of the single measurement version of the enclosure method and we have already many applications, e.g. [13–16, 18–22].

This paper is closely related to [16]. Therein we considered an inverse obstacle scattering problem of acoustic wave in two dimensions. The problem is to reconstruct a two-dimensional
obstacle from the Cauchy data on a circle surrounding the obstacle of the total wave field generated by a single incident plane wave with a fixed wave number. Let us make a review of one of the results therein.

We consider a polygonal obstacle denoted by $D$, that is, $D \subset \mathbb{R}^2$ takes the form $D_1 \cup \cdots \cup D_m$ with $1 \leq m < \infty$ where each $D_j$ is open and a polygon; $\overline{D}_j \cap \overline{D}_{j'} = \emptyset$ if $j \neq j'$.

The total wave field $u$ outside the obstacle $D$ takes the form $u(x; d, k) = e^{ikx \cdot d} + w(x)$ with $k > 0$, $d \in S^1$ and satisfies

$$
\begin{align*}
\Delta u + k^2 u &= 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D}, \\
\frac{\partial u}{\partial v} &= 0 \quad \text{on } \partial D, \\
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial w}{\partial r} - ikw \right) &= 0,
\end{align*}
$$

where $r = |x|$ and $v$ denotes the unit outward normal relative to $D$. The last condition above is called the Sommerfeld radiation condition.

Let $B_R$ be an open disc with radius $R$ centred at a fixed point satisfying $\overline{D} \subset B_R$. We assume that $B_R$ is known. Our data are $u = u(\cdot; d, k)$ and $\partial u / \partial v$ on $\partial B_R$ for a fixed $d$ and $k$, where $v$ is the unit outward normal relative to $B_R$. Let $\omega$ and $\omega^\perp$ be two unit vectors perpendicular to each other. We always choose the orientation of $\omega^\perp$ and $\omega$ coincides with that of $e_1$ and $e_2$ and thus $\omega^\perp$ is unique.

We make use of the special complex exponential solution of the Helmholtz equation $(\Delta + k^2)v = 0$ in $\mathbb{R}^2$:

$$
v_\tau(x; \omega) = e^{\tau(x \cdot \tau \omega + \sqrt{\tau^2 + k^2} \omega^\perp)}, \quad x \in \mathbb{R}^2,
$$

where $\tau > 0$ is a parameter.

Recall the support function of $D$: $h_D(\omega) = \sup_{x \in D} x \cdot \omega$. We say that $\omega$ is regular with respect to $D$ if the set $\partial D \cap \{x \in \mathbb{R}^2 | x \cdot \omega = h_D(\omega)\}$ consists of only one point.

Define

$$
I(\tau; \omega, d, k) = \int_{\partial B_R} \left( \frac{\partial u}{\partial v} v_\tau - \frac{\partial v_\tau}{\partial v} u \right) dS.
$$

**Theorem 1.1** ([16]). Assume that $\omega$ is regular with respect to $D$. Then the formula

$$
\lim_{\tau \to \infty} \frac{1}{\tau} \log |I(\tau; \omega, d, k)| = h_D(\omega)
$$

is valid. Moreover, we have the following:

- if $t \geq h_D(\omega)$, then $\lim_{\tau \to \infty} e^{-\tau t} |I(\tau; \omega, d, k)| = 0$;
- if $t < h_D(\omega)$, then $\lim_{\tau \to \infty} e^{-\tau t} |I(\tau; \omega, d, k)| = \infty$.

In [19] a similar formula has been established by using the far-field pattern $F_D(\varphi, d, k)$, $\varphi \in S^1$, of the scattered wave $w = u - e^{ikx \cdot d}$ for fixed $d$ and $k$ which determines the leading term of the asymptotic expansion of $w$ as $r \to \infty$ in the following sense:

$$
w(r\varphi) \sim \frac{e^{ikr}}{\sqrt{r}} F_D(\varphi, d; k).
$$

Moreover, therein, instead of volumetric obstacle, similar formulae for thin sound-hard obstacle (or screen) have also been established with two incident plane waves.

In this section, we describe another inverse obstacle scattering problem in which a point source located within a finite distance from an unknown obstacle generates a scattered wave
and one measures the total wave on a known circle surrounding an unknown obstacle. One can see this type of problem in, e.g., a mathematical formulation of microwave tomography [31], subsurface radar [5], etc.

Let \( y \in \mathbb{R}^2 \setminus \overline{D} \). Let \( E = E_D(x, y) \) be the unique solution of the scattering problem:

\[
(\Delta + k^2) E = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{D},
\]

\[
\frac{\partial}{\partial \nu} E = -\frac{\partial}{\partial \nu} \Phi_0(., y) \quad \text{on} \quad \partial D,
\]

\[
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial E}{\partial r} - ikE \right) = 0,
\]

where

\[
\Phi_0(x, y) = \frac{i}{4} H_0^{(1)}(k|x - y|)
\]

and \( H_0^{(1)} \) denotes the Hankel function of the first kind [27].

The total wave outside \( D \) exerted by the point source located at \( y \) is given by the formula

\[
\Phi_D(x, y) = \Phi_0(x, y) + E_D(x, y), \quad x \in \mathbb{R}^2 \setminus \overline{D}.
\]

**Inverse problem.** Let \( R_1 > R \). Fix \( k > 0 \) and \( y \in \partial B_{R_1} \). Extract information about the location and shape of \( D \) from \( \Phi_D(x, y) \) given at all \( x \in \partial B_R \).

The aim of this paper is to show that the single measurement version of the enclosure method still works for this problem.

Define

\[
J(\tau; \omega, y, k) = \int_{\partial B_R} \left( \frac{\partial}{\partial \nu} \Phi_D(x, y) \cdot v_\tau(x; \omega) - \frac{\partial}{\partial \nu} v_\tau(x; \omega) \cdot \Phi_D(x, y) \right) dS(x).
\]

The first result of this paper is as follows.

**Theorem 1.2.** Assume that \( \omega \) is regular with respect to \( D \) and that

\[
diam D < \text{dist}(D, \partial B_{R_1}). \tag{1.1}
\]

It holds that

\[
\lim_{\tau \to \infty} -\frac{1}{\tau} \log |J(\tau; \omega, y, k)| = h_D(\omega).
\]

Moreover, we have the following:

if \( t \geq h_D(\omega) \), then \( \lim_{\tau \to \infty} e^{-\tau t} |J(\tau; \omega, y, k)| = 0 \);

if \( t < h_D(\omega) \), then \( \lim_{\tau \to \infty} e^{-\tau t} |J(\tau; \omega, y, k)| = \infty \).

It should be pointed out that \( \frac{\partial}{\partial \nu} \Phi_D(x, y) \) for \( x \in \partial B_R \) can be computed from \( \Phi_D(x, y) \) for \( x \in \partial B_R \) by solving the exterior Dirichlet problem for the Helmholtz equation:

\[
(\Delta + k^2) \tilde{E} = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{B_R},
\]

\[
\tilde{E} = \Phi_D(., y) - \Phi_0(., y) \quad \text{on} \quad \partial B_R,
\]

\[
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial \tilde{E}}{\partial r} - ik\tilde{E} \right) = 0.
\]

The computation formula is

\[
\frac{\partial}{\partial \nu} \Phi_D(x, y) = \frac{\partial}{\partial \nu} \Phi_0(x, y) + \frac{\partial}{\partial \nu} \tilde{E}(x) \quad \text{on} \quad \partial B_R.
\]

Needless to say, this kind of remark works also for \( \partial u / \partial \nu \) in theorem 1.1.
Condition (1.1) can be satisfied if $R_1$ is sufficiently large compared with $R$. It is not known whether condition (1.1) can be dropped completely. To suggest a possibility next we present a partial result which does not employ (1.1).

For the description of the second result we introduce special scattered and total fields. Given $d \in S^1$ choose $\vartheta \in S^1$ in such a way that $\vartheta^\perp = \vartheta$. Let $x_0 \in \partial D$ and $w = w(x; -d, k, x_0)$ be the unique solutions of the scattering problem:

$$
(\Delta + k^2)w = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \mathcal{T},
$$

$$
\frac{\partial w}{\partial \nu} = -\frac{\partial}{\partial \nu}((x_0 - x) \cdot \vartheta e^{-ikx \cdot d}) \quad \text{on} \quad \partial D,
$$

$$(1.2)
$$

$$
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial w}{\partial r} - ikw \right) = 0.
$$

Define

$$
\Phi_D(rd, y) = (x_0 - x) \cdot \vartheta e^{-ikx \cdot d} + w(x; -d, k, x_0).
$$

Note that the function $x \mapsto (x_0 - x) \cdot \vartheta e^{-ikx \cdot d}$ satisfies the Helmholtz equation in the whole plane and the radiation condition for $w(x; -d, k, x_0)$ yields

$$
\Phi_D(rd, y) = (x_0 - x) \cdot \vartheta e^{-ikx \cdot d} + O(r^{-1/2})
$$

$$(1.4)
$$
as $r \to \infty$. The second result of this paper is as follows.

**Theorem 1.3.** Let $\omega$ be regular with respect to $D$ and $x_0 \in \partial D$ be the point satisfying $x_0 \cdot \omega = h_D(\omega)$. Assume that $y \in \partial B_{R_1}$ satisfies $u(y; -d_1, k, x_0) \neq 0$ or $u(y; -d_2, k, x_0) \neq 0$, where $d_1, d_2 \in S^1$ are directions of two sides of $D$ that meet at $x_0$. Then all the conclusions in theorem 1.2 are valid.

Note that in this theorem (1.1) is not assumed at the price of introducing another implicit restriction on the location of $y$ relative to $D$. From (1.4) we see that given $\delta > 0$ and $\vartheta \in S^1$ if $R_0 > 0$ is sufficiently large, for all $R_1 \geq R_0$ and $y \in \partial B_{R_1}$ with $|(x_0 - y) \cdot \vartheta| \geq \delta$ it holds that $u(y; -d, k, x_0) \neq 0$ with $d = \vartheta^\perp$. We leave the problem of removing the condition completely as an open problem. See also remark 2.2 in section 2.

Since the set of all $\omega$ which is not regular with respect to given $D$ is finite and the support function of $D$ is continuous, as a corollary of theorem 1.2, we have uniqueness of determining the convex hull of $D$ from $\Phi_D(x, y)$ given at all $x \in \partial B_R$ for a single $y \in \partial B_{R_1}$ and $k > 0$ provided (1.1). It seems that this type of uniqueness with a single point source had not appeared in the previous study. See [7] and references therein for uniqueness of a polygonal obstacle with the far-field pattern of the scattered wave exerted by a single plane wave. It seems that their argument heavily depends on the fact that the total wave approaches the plane wave at infinity. This is not true for the total wave exerted by a point source since we have, as $r \to \infty$,

$$
\Phi_D(rd, y) \sim \frac{\hat{u}(\vartheta, r)}{\sqrt{8\pi k r}} u(y; -d, k).
$$

$$(1.5)
$$
See [9, 28, 30] for the derivation for $D$ with a smooth boundary. In our case $\partial D$ is not smooth; however, a minor modification of the proof still works. Equation (1.5) means that the far-field pattern of $\Phi_D(x, y)$ as a function of $x$ is given by $u(y; -d, k)$ multiplied by a known constant. This formula has been used in the probe method [9] and the singular sources method [29].

**Theorem 1.2** together with (1.5) yields

**Corollary 1.1.** Let $R_1 > R$ and $\partial D \subset B_{R_1}$. Assume that $D$ satisfies (1.1). Fix $k > 0$ and $y \in \partial B_{R_1}$. One can uniquely determine the convex hull of $D$ from the data $u(y; -d, k)$ given at all $d \in S^1$. 


Proof. The proof is divided into three steps.

(i) Use (1.5) to compute the far-field pattern of \( \Phi_d(x, y) \) from \( u(y; -d, k) \) given at all \( d \in S^1 \).

(ii) Use, e.g., the point source method \([28]\) to compute \( \Phi_d(x, y) \) together with its normal derivative for \( x \in \partial B_R \) from the far-field pattern of \( \Phi_d(x, y) \).

(iii) Use theorem 1.2 to compute \( h_d(\omega) \) for a generic \( \omega \) from \( \Phi_d(x, y) \) together with its normal derivative for \( x \in \partial B_R \).

\[ \square \]

Summing up, we obtained two procedures for estimating the convex hull of an unknown sound-hard polygonal obstacle by using two types of data.

The first type of data are given by the following process:

(A) produce the total wave by a fixed point source located outside a known circle surrounding an unknown obstacle and observe the wave at all points on the circle.

The second is as follows:

(B) produce the total waves by incident plane waves for all directions and observe the waves at a fixed point outside a known circle surrounding an unknown obstacle.

Note that these are different from the reciprocity principle \([14]\) which is the identity \( F_D(\varphi, d; k) = F_D(-d, -\varphi; k) \) since in this identity the incident wave is always a plane wave and one observes the scattered wave at infinity.

A brief outline of this paper is as follows. Theorems 1.2 and 1.3 are proved in section 2. Both proofs have a common starting point with the proof of theorem 1.1 which we recall before describing subsections 2.1 and 2.2. In those subsections we complete the proof of theorems 1.2 and 1.3. In the last section three other applications are given. Two of them are concerned with some extensions to thin obstacles and obstacles in a layered medium. In the last of the applications we consider the far-field equation which plays the central role in the linear sampling method \([3]\). We show that a modification of the argument for the proof of theorem 1.2 gives unsolvabilty of the far-field equation for polygonal obstacles.

2. Proof of theorems 1.2 and 1.3

First we follow the argument for the proof of theorem 1.1 (see also \([18]\)). For simplicity of notation we set \( u(x) = \Phi_D(x, y) \).

Let \( x_0 \) denote the single point of the set \( \{x \mid x \cdot \omega = h_D(\omega)\} \cap \partial D \). \( x_0 \) has to be a vertex of \( D_j \) for some \( j \). In what follows we denote by \( B_R(x_0) \) the open disc with radius \( R \) centred at \( x_0 \). Let \( \Theta \) denote the outside angle of \( D \) at \( x_0 \). \( \Theta \) satisfies \( \pi < \Theta < 2\pi \) since \( \omega \) is regular with respect to \( D \).

If one chooses a sufficiently small \( \eta > 0 \), then one can write

\[ B_{2\eta}(x_0) \cap (B_R \setminus \overline{D}) = \{x_0 + r(\cos \theta \alpha + \sin \theta \alpha^\perp)|0 < r < 2\eta, 0 < \theta < \Theta\}, \]

\[ B_{\eta}(x_0) \cap \partial D = \Gamma_p \cup \Gamma_q \cup \{x_0\}, \]

where \( \alpha = \cos p \omega^\perp + \sin p \omega, \alpha^\perp = - \sin p \omega^\perp + \cos p \omega; -\pi < p < 0; \Gamma_p = \{x_0 + r \alpha|0 < r < \eta\}, \Gamma_q = \{x_0 + r(\cos \Theta \alpha + \sin \Theta \alpha^\perp)|0 < r < \eta\}. \) Note that the orientation of \( \alpha, \alpha^\perp \) coincides with that of \( e_1, e_2 \). See also figure 1 of \([10]\).

The quantity \( -p \) means the angle between two vectors \( \omega^\perp \) and \( \alpha \). \( p \) satisfies \( \Theta > \pi + (-p) \).

Set \( q = \Theta - 2\pi + p \). Then we have \( -\pi < q < p < 0 \) and the expression

\[ \Gamma_p = \{x_0 + r(\cos p \omega^\perp + \sin p \omega)|0 < r < \eta\}, \]

\[ \Gamma_q = \{x_0 + r(\cos q \omega^\perp + \sin q \omega)|0 < r < \eta\}. \]

\[ \]
This is the meaning of \( p \) and \( q \).

We set

\[
   u(r, \theta) = u(x), \quad x = x_0 + r(\cos \theta a + \sin \theta a^\perp).
\]

The \( u \) can be expanded as

\[
   u(r, \theta) = \alpha_1 J_0(kr) + \sum_{n=2}^{\infty} \alpha_n J_{\lambda_n}(kr) \cos \lambda_n \theta, \quad 0 < r < \eta, \quad 0 < \theta < \Theta,
\]

where the \( \lambda_n \) describes the singularity of \( u \) as \( r \to 0 \) and in this case explicitly given by the formula

\[
   \lambda_n = (n - 1) \pi / \Theta, \quad J_{\lambda_n} \text{ stands for the Bessel function of order } \lambda_n.
\]

One of the key points is introducing a new parameter \( s \) instead of \( \tau \) by the equation

\[
   s = \sqrt{\tau^2 + k^2 + \tau},
\]

we obtain, as \( s \to \infty \), the complete asymptotic expansion

\[
   J(\tau; \omega, y, k) e^{-i \sqrt{\tau^2 + k^2} x_0 \cdot \omega} e^{-\tau h D(\omega)} \sim \sum_{n=2}^{\infty} e^{i \pi \lambda_n} \frac{\alpha_n K_n}{s^{\lambda_n}}, \quad (2.1)
\]

where \( K_n \) are constants given by the formula

\[
   K_n = e^{i \pi \Theta / \pi + (1)^n} e^{i \pi \pi / \pi}.
\]

For the derivation of this expansion see [16]. Note that constants \( K_n \) are exactly the same as the corresponding ones in [10, 16].

Now all the statements in theorem 1.2 follow from (2.1) and another key point: \( \exists n \geq 2 \) \( \alpha_n K_n \neq 0 \). This is due to a contradiction argument. Assume that the assertion is not true, that is,

\[
   \forall n \geq 2 \alpha_n K_n = 0.
\]

**Case A.** First we consider the case when \( \Theta / \pi \) is irrational. It is easy to see that \( K_n \neq 0 \) for all \( n \geq 2 \). Thus, \( \alpha_n = 0 \) and this yields \( u(r, \theta) = \alpha_1 J_0(kr) \) for \( 0 < r < \eta \) and \( 0 < \theta < \Theta \). Since this right-hand side is an entire solution of the Helmholtz equation, the unique continuation property of the solution of the Helmholtz equation yields \( u(x) = \alpha_1 J_0(k|x - x_0|) \) in \( \mathbb{R}^2 \setminus \overline{D} \).

This implies that \( u \) has to be bounded in a neighbourhood of \( y \). However, since

\[
   \Phi_0(x, y) \sim \frac{1}{2\pi} \log \frac{1}{|x - y|},
\]

as \( x \to y \) and \( E_D(\cdot, y) \) is smooth in a neighbourhood of \( y \), one knows that \( u(x) = \Phi_0(x, y) + E_D(x, y) \) is not bounded in any neighbourhood of \( y \). Contradiction.

**Case B.** Next consider the case when \( \Theta / \pi \) is rational. One can write

\[
   \frac{\Theta}{\pi} = 1 + \frac{b}{a},
\]

where \( a(\geq 2) \) and \( b(\geq 1) \) are integers and **mutually prime**. Then we have

\[
   \{n \geq 2 | K_n = 0\} = \{1 + l(a + b) | l = 1, 2, \ldots\}. \quad (2.2)
\]

Note also that \( \lambda_{1+l(a+b)} = al, l = 1, 2, \ldots \). From the assumption of the contradiction argument one knows that if \( n \) satisfies \( K_n \neq 0 \), then \( \alpha_n = 0 \). From this together with (2.2) we have

\[
   u(r, \theta) = \sum_{l=0}^{\infty} \alpha_{1+l(a+b)} J_{al}(kr) \cos al \theta. \quad (2.3)
\]

Hereafter we take **two courses** corresponding to theorems 1.2 and 1.3.
2.1. Completion of the proof of theorem 1.2

Since each $a_l$ in (2.3) is an integer, the right-hand side of (2.3) gives a continuation of $u$ onto $\mathbb{R}^2 \setminus \{(y)\}$ as a solution of the Helmholtz equation and the continuation which we denote by $\tilde{u}$ satisfies the rotation invariance in $B_0(x_0)$:

$$\tilde{u}(r, \theta + \frac{2\pi}{a}) = \tilde{u}(r, \theta).$$  \hspace{1cm} (2.4)

Now having (1.1) and (2.4), one can apply Friedman–Isakov’s extension argument [6] to $\tilde{u}$. See also [18] for the detail of the argument applied to a penetrable obstacle case. As a result one gets a continuation of $\tilde{u}$ onto $\mathbb{R}^2 \setminus \{(y)\}$ as a solution of the Helmholtz equation. Since $\tilde{u}(x) = u(x) = \Phi_0(x, y) + E_D(x, y)$ in $\mathbb{R}^2 \setminus \overline{D}$ and $E_D(x, y)$ is smooth in a neighbourhood of $y$, one concludes that $E_D(\cdot, y)$ can be continued as a solution of the Helmholtz equation in $\mathbb{R}^2$. The continuation satisfies the Sommerfeld radiation condition and therefore has to be identically zero. This gives, in particular, $\tilde{u}(x) = \Phi_0(x, y)$ in $B_0(x_0)$ and from (2.4) one gets

$$\Phi_0(x_0 + rz(\theta), y) = \Phi_0 \left( x_0 + rz \left( \theta + \frac{2\pi}{a} \right), y \right), \hspace{1cm} 0 < r < \eta, \hspace{0.5cm} \theta \in \mathbb{R},$$  \hspace{1cm} (2.5)

where $z(\theta) = \cos \theta a + \sin \theta a^\perp$.

Since both sides of (2.5) satisfy the same Helmholtz equation in $|x - x_0| < |y - x_0|$, the unique continuation property of the solution of the Helmholtz equation yields: (2.5) is valid for all $r$ with $r < |y - x_0|$. Choose a $\theta_0$ in such a way that $y = x_0 + |y - x_0| z(\theta_0)$. Since $2\pi/a \leq \pi$, we have $y \neq x_0 + |y - x_0| z(\theta_0 + 2\pi/a)$. Then letting $\theta = \theta_0$ and $r \rightarrow |y - x_0|$ in (2.5), we have a contradiction because of the singularity of $\Phi_0(x, y)$ as $x \rightarrow y$.

This completes the proof of theorem 1.2. \hspace{1cm} \Box

Remark 2.1. In the proof of theorem 1.1, we never make use of (2.4) after having (2.3) and instead take another course for the total field $u = u(\cdot; d, k)$ in theorem 1.1.

The argument is as follows. Set

$$d_1 = \cos \theta a + \sin \theta a^\perp|_{\theta = \theta_0 - \pi}, \hspace{1cm} \theta_1 = \cos \theta a + \sin \theta a^\perp|_{\theta = \theta_0 - \pi + \pi/2},$$

$$d_2 = \cos \theta a + \sin \theta a^\perp|_{\theta = \theta_0}, \hspace{1cm} \theta_2 = \cos \theta a + \sin \theta a^\perp|_{\theta = \theta_0 + \pi/2}.$$  \hspace{1cm} (2.6)

Note that $d_1$ and $d_2$ are directed along the two sides that meet at $x_0$.

From the right-hand side of (2.3) one gets for all $r$ with $0 < r \ll 1$, $\nabla u(x_0 + rd_1) \cdot \theta_1 = 0$ and $\nabla u(x_0 + rd_2) \cdot \theta_2 = 0$. Then a reflection argument in [1] yields that this is true for all $r > 0$. However, from this together with the asymptotic behaviour of $\nabla u \sim \nabla e^{ik \cdot d}$ as $r \rightarrow \infty$ one gets $d \cdot \theta_1 = d \cdot \theta_2 = 0$. Contradiction.

The advantage of this argument is that one does not need to use (1.1). In the following subsection we employ this argument after (2.3).

2.2. Completion of the proof of theorem 1.3

We use the same notation as (2.6). First we claim that, as $r \rightarrow \infty$,

$$\nabla_x \Phi_D(x_j, y) \cdot \theta_j = \frac{ik^{1/2}}{4} \left[ \frac{2}{\pi} e^{ik/2} e^{ik(x_j - y)} d \frac{e^{ikr}}{r^{3/2}} u(y; -d_j, k, x_0) + O \left( \frac{1}{r^{3/2}} \right) \right],$$  \hspace{1cm} (2.7)

where $x_j = x_0 + rd_j$.

This is proved as follows. The total field $\Phi_D(\cdot, y)$ has the expression

$$\Phi_D(x, y) = \Phi_0(x, y) + \int_{\partial D} \frac{\partial}{\partial \nu(z)} \Phi_0(z, x) \Phi_D(z, y) dS(z), \hspace{1cm} x \in \mathbb{R}^2 \setminus \overline{D}. \hspace{1cm} (2.8)$$
Since
\[ \nabla_x \Phi_0(x, y) = \frac{ik}{4} \frac{x - y}{|x - y|} (H_0^{(1)})'(k|x - y|), \]
we have
\[ \nabla_x \Phi_0(x, y) \cdot \partial_j = \frac{ik}{4} \frac{(x_j - y) \cdot \partial_j}{|x_j - y|} (H_0^{(1)})'(k|x_j - y|). \]
Here we note that \((x_j - y) \cdot \partial_j = (x_0 - y) \cdot \partial_j + x_j \cdot \partial_j = (x_0 - y) \cdot \partial_j \) since \(d_j \cdot \partial_j = 0\). This gives
\[ \nabla_x \Phi_0(x, y) \cdot \partial_j = \frac{ik}{4} \frac{(x_0 - y) \cdot \partial_j}{|x_0 - y|} (H_0^{(1)})'(k|x_0 - y|). \]

By (4.03) on p 238 in [27], we know that as \(r \to \infty\), \(H_0^{(1)}(r)\) and its derivatives satisfy
\[ H_0^{(1)}(r) = \sqrt{\frac{2}{\pi}} e^{-i\pi/4} r^{-1/2} e^{i \phi} (1 + O(r^{-1})), \]
(2.9)
and
\[ (H_0^{(1)})'(r) = \sqrt{\frac{2}{\pi}} e^{i\pi/4} r^{-1/2} e^{i \phi} (1 + O(r^{-1})) \]
and
\[ (H_0^{(1)})''(r) = i \sqrt{\frac{2}{\pi}} e^{i\pi/4} r^{-1/2} e^{i \phi} (1 + O(r^{-1})). \]
(2.10)
Using
\[ |x_j - y| = r + (x_0 - y) \cdot d_j + O(r^{-1}) \]
and (2.9), we have
\[ (H_0^{(1)})'(k|x_j - y|) = \sqrt{\frac{2}{\pi}} e^{i\pi/4} k^{-1/2} r^{1/2} e^{ik(x_0 - y) \cdot d_j} (1 + O(r^{-1})) \]
and thus
\[ \nabla_x \Phi_0(x, y) \cdot \partial_j = \frac{ik^{1/2}}{4} \frac{(x_0 - y) \cdot \partial_j}{r^{3/2}} \sqrt{\frac{2}{\pi}} e^{i\pi/4} e^{ik(x_0 - y) \cdot d_j} + O(r^{-5/2}). \]
(2.11)

Let \(z \in \partial D\). We have
\[ \frac{\partial}{\partial v(z)} \Phi_0(z, x) = -\frac{ik}{4} \frac{v(z)}{|x - z|} (H_0^{(1)})'(k|x - z|) \]
and thus
\[ \nabla_x \left( \frac{\partial}{\partial v(z)} \Phi_0(z, x) \right) = \frac{ik}{4} \frac{v(z)}{|x - z|} (H_0^{(1)})'(k|x - z|) + \frac{ik}{4} \frac{(x - z) \cdot v(z)(x - z)}{|x - z|^3} \]
\[ \times (H_0^{(1)})'(k|x - y|) - \frac{ik^2}{4} \frac{(x - z) \cdot v(z)(x - z)}{|x - z|^2} (H_0^{(1)})''(k|x - z|). \]

This together with \((x - z) \cdot \partial_j = (x_0 - z) \cdot \partial_j \) yields
\[ \nabla_x \left( \frac{\partial}{\partial v(z)} \Phi_0(z, x) \right) \bigg|_{z=x_j} \cdot \partial_j = \frac{ik}{4} \frac{v(z)}{|x_j - z|} (H_0^{(1)})'(k|x_j - z|) \]
\[ + \frac{ik}{4} \frac{(x_j - z) \cdot v(z)(x_0 - z)}{|x_j - z|^3} (H_0^{(1)})'(k|x_j - z|) \]
\[ - \frac{ik^2}{4} \frac{(x_j - z) \cdot v(z)(x_0 - z)}{|x_j - z|^2} (H_0^{(1)})''(k|x_j - z|). \]
Note that the second term of this right-hand side is estimated by $O(r^{-5/2})$. It follows from these and (2.10) that
\[
\nabla_x \left( \frac{\partial}{\partial v(z)} \cdot \Phi_D(z, x) \right) \Big|_{x=x_j} \cdot \frac{\partial j}{\rho} = - \frac{ik^{1/2}}{4} v(z) \cdot \frac{\partial j}{\rho} + \frac{ikd_j \cdot v(z)(x_0 - z) \cdot \frac{\partial j}{\rho}}{r^{3/2}} 
\]
\[
\times \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} e^{ikr} e^{i(k(x_0 - z) \cdot d_j)} + O(r^{-5/2}).
\] (2.12)

Now from (2.7), (2.8), (2.11) and (2.12), we obtain
\[
\nabla_x \Phi_D(x_j, y) \cdot \frac{\partial j}{\rho} = \frac{ik^{1/2}}{4} \sum_{k=0}^{\infty} e^{ikr} e^{i(k(x_0 - y) \cdot d_j)} \frac{e^{i(k(x_0 - z - y) \cdot d_j)} U(y; d, k, x_0) + O\left(1 \frac{1}{r^{3/2}}\right)}{r^{3/2}},
\]
where
\[
U(y; d, k, x_0) = (x_0 - y) \cdot \frac{\partial j}{\rho} e^{i(k(x_0 - y) \cdot d_j)}
\]
\[
- \int_{\partial D} (v(z) \cdot \frac{\partial j}{\rho} + ikd_j \cdot v(z)(x_0 - z) \cdot \frac{\partial j}{\rho}) e^{i(k(x_0 - z) \cdot d_j)} \Phi_D(z, y) dS(z).
\] (2.13)
Define
\[
\Psi_j(x) = (x_0 - x) \cdot \frac{\partial j}{\rho} e^{-ikx \cdot d_j}.
\]
Since
\[
\frac{\partial}{\partial v(z)}((x_0 - z) \cdot \frac{\partial j}{\rho} e^{-ikz \cdot d_j}) = -(v(z) \cdot \frac{\partial j}{\rho} + ikd_j \cdot v(z)(x_0 - z) \cdot \frac{\partial j}{\rho}) e^{-ikz \cdot d_j} u,
\]
one can rewrite (2.13) as
\[
U(y; d, k, x_0) = \Psi_j(y) + \int_{\partial D} \frac{\partial}{\partial v(z)} \Psi_j(z) \cdot \Phi_D(z, y) dS(z).
\] (2.14)

On the other hand, a combination of the Green’s identity, the Sommerfeld radiation condition for $w_j(z) = w(z; -d_j, k, x_0)$ and $\Phi_D(z, y)$ and the boundary condition in (2.2) gives
\[
w_j(y) = \int_{\partial D} \left( w_j(z) \frac{\partial}{\partial v(z)} \Phi_D(z, y) - \Phi_D(z, y) \frac{\partial}{\partial v(z)} w_j(z) \right) dS(z)
\]
\[
= - \int_{\partial D} \Phi_D(z, y) \frac{\partial}{\partial v(z)} w_j(z) dS(z)
\]
\[
+ \int_{\partial D} \frac{\partial}{\partial v(z)}((x_0 - z) \cdot \frac{\partial j}{\rho} e^{-ikz \cdot d_j}) \Phi_D(z, y) dS(z).
\]

Therefore, we see that the left-hand side of (2.14) coincides with $u(y; -d_j, k, x_0)$. This completes the proof of (2.7).

Now the proof of theorem 1.3 starts with having (2.3). By the same reason described in remark 2.1, from (2.3) we have
\[
\nabla_x \Phi_D(x_j, y) \cdot \frac{\partial j}{\rho} = 0,
\] (2.15)
where $x_j = x_0 + r d_j$ and $0 < r \ll 1$. First consider the case when $y \neq x_0 + rd_j$ for all $r > 0$. In this case a reflection argument in [1] ensures that (2.15) is valid for all $r > 0$ and thus (2.7) yields
\[
u(y; -d_j, k, x_0) = 0.
\] (2.16)
If $y = x_0 + |y - x_0| d_j$, then $\nabla_x \Phi_D(x, y) \cdot \frac{\partial j}{\rho} = 0$ for $x = x_0 + r d_j$ with $0 < r < |y - x_0|$ and $r > |y - x_0|$. Then form (2.15) we have $\nabla_x E(x, y) \cdot \frac{\partial j}{\rho} = 0$ for $x = x_0 + r d_j$ with $0 < r \ll 1$ and this is true for all $r > 0$ by a reflection argument in [1]. Therefore, we again have (2.15) for all $r > |y - x_0|$ and thus also (2.16).
Summing up, in any case, we obtain equation (2.16) for \( j = 1, 2 \). This is a contradiction. This completes the proof of theorem 1.3

**Remark 2.2.** However, (2.16) is coming from only the leading term of the asymptotic expansion (2.7). Thus, our next problems in this direction are as follows:

(i) determine the complete asymptotic expansion of \( e^{-ikr} \nabla_x \Phi_D(x_j, y) \cdot \vartheta_j \) as \( r \to \infty \):

\[
e^{-ikr} \nabla_x \Phi_D(x_j, y) \cdot \vartheta_j \sim \sum_{m=0}^{\infty} A_m(y, x_0, d_j, k) r^{-(3/2+2m)}.
\]

(ii) If \( A_m(y, x_0, d_j, k) = 0 \) for all \( m \), then what happens?

The main obstruction in this approach is the complexity of computing the asymptotic expansion in (i) as can be seen in the proof of theorem 1.3

3. Other applications

In this last section, instead we give three applications of the argument done in theorems 1.2 and 1.3.

3.1. Thin obstacle

It should be pointed out that the advantage of the assumption in theorem 1.3 is that the result can be extended to a thin obstacle case.

First we review a result in [16] which employs a single plane wave as an incident wave and corresponds to theorem 1.1.

Let \( \Sigma \) be the union of finitely many disjoint closed piecewise linear segments denoted by \( \Sigma_1, \Sigma_2, \ldots, \Sigma_m \). Assume that there exists a simply connected open set \( D \) such that \( D \) is a polygon and each \( \Sigma_j \) consists of sides of \( D \).

We assume that \( \overline{D} \subset B_R \) with a \( R > 0 \). We denote by \( \nu \) the unit outward normal on \( \partial D \) relative to \( B_R \setminus D \) and set \( \nu^+ = \nu \) and \( \nu^- = -\nu \) on \( \Sigma \). Given \( k > 0 \) and \( d \in S^1 \) let \( u = u(x), x \in \mathbb{R}^2 \setminus \Sigma, \) be the solution of the scattering problem

\[
(\Delta + k^2) u = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \Sigma,
\]

\[
\frac{\partial u}{\partial \nu}^+ = 0 \quad \text{on} \quad \Sigma,
\]

\[
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial u}{\partial r} - ik w \right) = 0,
\]

where \( w = u - e^{i k x \cdot d}, u^+ = u|_{\mathbb{R}^2 \setminus \overline{D}} \) and \( u^- = u|_D \). Note that this is a brief description of the problem and for the exact one see [16]. Define

\[
I_\Sigma(\tau; \omega, d, k) = \int_{\partial B_R} \left( \frac{\partial u}{\partial \nu} v_t - \frac{\partial v}{\partial \nu} u \right) \text{d}S.
\]

In [16] we have established the following result.

**Theorem 3.1 ([16]).** Let \( \omega \) be regular with respect to \( \Sigma \). If every end point of \( \Sigma_1, \Sigma_2, \ldots, \Sigma_m \) satisfies \( x \cdot \omega < h_\Sigma(\omega) \), then the formula

\[
\lim_{\tau \to \infty} \frac{1}{\tau} \log |I_\Sigma(\tau; \omega, d, k)| = h_\Sigma(\omega)
\]

is valid. Moreover, we have the following:
if \( t \geq h/\Sigma(\omega) \), then \( \lim_{\tau \to \infty} e^{-\tau t} |J_{\Sigma}(\tau; \omega, d, k)| = 0; \)

if \( t < h/\Sigma(\omega) \), then \( \lim_{\tau \to \infty} e^{-\tau t} |J_{\Sigma}(\tau; \omega, d, k)| = \infty. \)

If there is an end point \( x_0 \) of some \( \Sigma_j \) such that \( x_0 \cdot \omega = h/\Sigma(\omega) \), then, for \( d \) that is not perpendicular to \( v \) on \( \Sigma_j \) near the point, the same conclusions as above are valid.

Note that \( v \) on \( \Sigma_j \cap B_\eta(x_0) \) for sufficiently small \( \eta > 0 \) becomes a constant vector if \( x_0 \) is an end point of \( \Sigma_j \).

Here we present a result in which, instead of a single plane wave, we make use of a single point source as an incident wave.

Let \( y \in \mathbb{R}^2 \setminus D \). Let \( E = E_{\Sigma}(x, y) \) be the unique solution of the scattering problem:

\[
(\Delta + k^2)E = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \Sigma, \\
\frac{\partial E}{\partial v} = -\frac{\partial \Phi_0}{\partial v} \quad \text{on} \quad \Sigma, \\
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial E}{\partial r} - ikE \right) = 0.
\]

The total wave outside \( \Sigma \) exerted by the point source located at \( y \) is given by the formula:

\[
\Phi_{\Sigma}(x, y) = \Phi_0(x, y) + E_{\Sigma}(x, y), \quad x \in \mathbb{R}^2 \setminus \Sigma.
\]

Given \( d \in S^1 \) choose \( \vartheta \in S^1 \) in such a way that \( \vartheta \perp d \). Let \( x_0 \in \Sigma \) and \( w = w_{\Sigma}(x; -d, k, x_0) \) be the unique solution of the scattering problem:

\[
(\Delta + k^2)w = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \Sigma, \\
\frac{\partial w}{\partial v} = -\frac{\partial \Phi_0}{\partial v} \quad \text{on} \quad \Sigma, \\
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial w}{\partial r} - ikw \right) = 0.
\]

Define

\[
u_{\Sigma}(x; -d, k, x_0) = (x_0 - x) \cdot \vartheta e^{-ikx \cdot d} + w_{\Sigma}(x; -d, k, x_0).
\]

Let \( R_1 > R \) and \( y \in \partial B_{R_1} \). Define

\[
J_{\Sigma}(\tau; \omega, y, k) = \int_{\partial B_R} \left( \frac{\partial}{\partial v} \Phi_{\Sigma}(x, y) \cdot v_\tau(x; \omega) - \frac{\partial}{\partial v} v_\tau(x; \omega) \cdot \Phi_{\Sigma}(x, y) \right) dS.
\]

The following theorem is what we call an extension of theorem 1.3 to thin obstacles.

**Theorem 3.2.** Let \( \omega \) be regular with respect to \( \Sigma \) and let \( x_0 \in \Sigma \) be the point with \( x_0 \cdot \omega = h/\Sigma(\omega) \). Assume that \( y \in \partial B_{R_1} \) satisfies \( u_{\Sigma}(y; -d, k, x_0) \neq 0 \) for a direction \( d \in S^1 \) that meets at \( x_0 \) along \( \Sigma_j \). Then the formula

\[
\lim_{\tau \to \infty} \frac{1}{\tau} \log |J_{\Sigma}(\tau; \omega, y, k)| = h/\Sigma(\omega)
\]

is valid. Moreover, we have the following:

if \( t \geq h/\Sigma(\omega) \), then \( \lim_{\tau \to \infty} e^{-\tau t} |J_{\Sigma}(\tau; \omega, y, k)| = 0; \)

if \( t < h/\Sigma(\omega) \), then \( \lim_{\tau \to \infty} e^{-\tau t} |J_{\Sigma}(\tau; \omega, y, k)| = \infty. \)

The proof of theorem 3.2 is based on the convergent series expansion of \( \Phi_{\Sigma}(\cdot, y) \) at a corner or end point of \( \Sigma \). See propositions 4.4 and 4.5 in [16]. Since the proof of theorem 3.2 can be done along the same line with that of theorem 1.3, we omit the description.
3.2. Obstacle in a layered medium

We consider a medium that consists of two parts. One is given by $\mathbb{R}^2 \setminus B_R$ and another is $B_R$. We assume that the propagation speeds of waves in two parts can be different from each other. An obstacle $D$ is embedded in $B_R$ as before. We assume that $D$ is polygonal. Let us describe a mathematical formulation of the problem.

Define
\[
\gamma(x) = \begin{cases} 
\gamma_+, & x \in \mathbb{R}^2 \setminus B_R, \\
\gamma_-, & x \in B_R.
\end{cases}
\]
where $\gamma_\pm$ are known positive constants.

Fix $y \in \mathbb{R}^2 \setminus B_R$. Set $k_+ = k/\sqrt{\gamma_+}$. Define
\[
\Phi_1^\gamma (x, y) = i \frac{4}{\sqrt{\gamma_+}} H_0^1(k_+|x - y|),
\]
where
\[
\gamma ^2 \Phi_1^\gamma (x, y) = - \nabla \cdot (\gamma_+ \nabla \Phi_1^\gamma (x, y)) \quad \text{in} \quad \mathbb{R}^2,
\]
\[
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial \gamma}{\partial r} - i k_+ \gamma \right) = 0.
\]

Let $E = E_{D,\gamma}(x, y), x \in \mathbb{R}^2 \setminus D$ solve
\[
(\nabla \cdot \gamma \nabla + k_+^2)E = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus D,
\]
\[
\gamma \frac{\partial E}{\partial v} = - \gamma \frac{\partial \Phi_1^\gamma (x, y)}{\partial v} \quad \text{on} \quad \partial D,
\]
\[
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial E}{\partial r} - i k_+ E \right) = 0.
\]

Define
\[
\Phi_{D,\gamma}(x, y) = \Phi_\gamma(x, y) + E_{D,\gamma}(x, y).
\]

Note that the existence and uniqueness of the solutions $\epsilon$ and $E$ can be established by using a variational formulation, for example, see [8].

Define
\[
K(\tau; \omega, y, k) = \int_{\partial B_R} \left( \gamma_+ \frac{\partial}{\partial v} \Phi_{D,\gamma}(x, y) \cdot v^-_\tau(x; \omega) - \gamma_+ \frac{\partial}{\partial v} v^-_\tau(x; \omega) \cdot \Phi^-_{D,\gamma}(x, y) \right) d\mathcal{S}(x),
\]
(3.1)

where
\[
v^-_\tau(x; \omega) = e^{i(\tau + i\sqrt{\tau^2 + k^2} \omega)},
\]
\[
\Phi^-_{D,\gamma}(x, y) = \Phi_{D,\gamma}(x, y), \quad x \in \overline{B_R \setminus D}
\]
and $k_- = k/\sqrt{\gamma_-}$.

Theorem 3.3. Assume that $\omega$ is regular with respect to $D$ and that
\[
\text{diam } D < \text{dist } (D, \partial B_R).
\]
(3.2)
Moreover assume that there exists a $j \in \{1, \ldots, m\}$ such that $k_j^2$ is not a Neumann eigenvalue for $-\Delta$ in $D_j$. It holds that
\[
\lim_{\tau \to \infty} \frac{1}{\tau} \log |K(\tau; \omega, y, k)| = h_D(\omega).
\]
Moreover, we have the following:

if \( t \geq h_D(\omega) \), then \( \lim_{\tau \to \infty} e^{-\tau t} |K(\tau; \omega, y, k)| = 0 \);

if \( t < h_D(\omega) \), then \( \lim_{\tau \to \infty} e^{-\tau t} |K(\tau; \omega, y, k)| = \infty \).

**Proof.** Instead of \( u(x) = \Phi_D(x, y) \) in the proof of theorem 1.2 set \( u(x) = \Phi_{D,\gamma}(x, y) \). For this case the same argument with (3.2) instead of (1.1) as described in Cases A and B in the proof of theorem 1.2 works. Thus we have a continuation \( \tilde{u} \) of \( u \) in \( B_R \backslash D \) onto \( B_R \) as a solution of the Helmholtz equation \( \Delta \tilde{u} + k^2 \tilde{u} = 0 \) in \( B_R \). Since \( u \) satisfies the Neumann boundary condition \( \partial u / \partial v = 0 \) on \( \partial D \), \( \tilde{u} \) also satisfies the condition on \( \partial D \) and by the assumption \( k^2 \), it must hold that \( \tilde{u} = 0 \) in \( D_j \) for some \( j \). Then, the unique continuation theorem for the Helmholtz equation yields \( \tilde{u} = 0 \) in \( B_R \) for some \( j \). Then, the unique continuation theorem for the Helmholtz equation yields \( u = 0 \) in \( B_R \). This yields that the Cauchy data of \( \Phi_{D,\gamma}(x, y) \) on \( \partial B_R \) vanish and thus \( \Phi_{D,\gamma}(x, y) = 0 \) for \( x \in R^2 \backslash (\overline{B_R} \cup \{y\}) \) by the uniqueness of the Cauchy problem for the Helmholtz equation with wave number \( k^* \). Since \( \Phi_{D,\gamma}(x, y) \) is singular as \( x \to y \), this is a contradiction. \( \Box \)

In this theorem the data are given by the Cauchy data of the total wave field on \( \partial B_R \). It is an interesting open problem when the receivers are located on \( \partial B_R + \epsilon \) with \( \epsilon > 0 \) how one can apply the enclosure method in an explicit form.

Here we propose one heuristic approach based on theorem 3.3 in the case when \( \epsilon \) is sufficiently small.

Assume that we have \( \Phi_{D,\gamma}(x, y) \) for all \( x \in \partial B_{R+\epsilon} \) exactly. Solve the exterior problem in \( R^2 \backslash B_{R+\epsilon} \):

\[
\Delta \Psi + k^2 \Psi = 0 \quad \text{in} \quad R^2 \backslash B_{R+\epsilon},
\]

\[
\Psi(x) = \Phi_{D,\gamma}(x, y) - \Phi^+(x, y) \quad \text{on} \quad \partial B_{R+\epsilon},
\]

\[
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial \Psi}{\partial r} - ik \Psi \right) = 0.
\]

Then we have \( \Psi(x) = \Phi_{D,\gamma}(x, y) - \Phi^+(x, y) \) for \( x \in R^2 \backslash B_{R+\epsilon} \). This gives

\[
\frac{\partial \Phi_{D,\gamma}}{\partial v}(x + \epsilon v(x), y) = \frac{\partial \Phi^+}{\partial v}(x + \epsilon v(x), y) + \frac{\partial \Psi}{\partial v}(x + \epsilon v(x)), \quad x \in \partial B_R.
\]

We use for the computation of the Cauchy data of \( \Phi_{D,\gamma}(x, y) \) on \( \partial B_R \) from outside \( B_R \) the approximation

\[
\Phi_{D,\gamma}(x, y) \approx \Phi_{D,\gamma}(x + \epsilon v(x), y),
\]

\[
\frac{\partial \Phi_{D,\gamma}}{\partial v}(x, y) \approx \frac{\partial \Phi^+}{\partial v}(x + \epsilon v(x), y) + \frac{\partial \Psi}{\partial v}(x + \epsilon v(x)).
\]

Using these computed Cauchy data from outside \( B_R \) and the transmission condition

\[
\Phi_{D,\gamma}(x, y) = \Phi_{D,\gamma}(x, y), \quad \gamma_+ \frac{\partial \Phi_{D,\gamma}}{\partial v}(x, y) = \gamma_- \frac{\partial \Phi_{D,\gamma}}{\partial v}(x, y), \quad x \in \partial B_R
\]

which is implicitly included in the governing equation, we compute \( K(\tau; \omega, y, k) \) by replacing \( \Phi_{D,\gamma}(x, y) \) and \( \gamma_- (\partial \Phi_{D,\gamma} / \partial v)(x, y) \) on the right-hand side of (3.1) with \( \Phi_{D,\gamma}(x + \epsilon v(x), y) \) and \( \gamma_+ (\partial \Phi^+ / \partial v)(x + \epsilon v(x), y) + (\partial \Psi / \partial v)(x + \epsilon v(x)) \), respectively. Clearly, the effective range of \( \tau \) shall depend on the size of \( \epsilon \).

It would be interesting to test this approach numerically and check its performance. This belongs to a next research plan.
3.3. Unsolvability of the far-field equation for polygonal obstacles

Let \( k > 0 \) and \( d \in S^1 \). Let \( F_D(\varphi; d, k) \) denote the far-field pattern of the scattered wave \( w(x) = u(x; d, k) - e^{ikx \cdot d} \).

Given \( y \in \mathbb{R}^2 \) the far-field equation for unknown \( g \in L^2(S^1) \)

\[
\int_{S^1} F_D(\varphi; d, k)g(d) \, dS(d) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\varphi y}, \quad \varphi \in S^1,
\]  

(3.3)

plays the central role in the linear sampling method [3].

Note that the right-hand side of (3.3) coincides with the far-field pattern of the field \( \Phi_0(x, y) \) with \( x = r\varphi \) as \( r \to \infty \); the left-hand side of (3.3) coincides with the far-field pattern of the scattered field \( w = w_g \) which is the unique solution of the scattering problem:

\[
(\triangle + k^2)w = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus D,
\]

\[
\frac{\partial w}{\partial v} = -\frac{\partial v_g}{\partial v} \quad \text{on} \quad \partial D,
\]

\[
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial w}{\partial r} - ik w \right) = 0,
\]

where \( v_g \) denotes the Herglotz wavefunction with density \( g \):

\[
v_g(x) = \int_{S^1} e^{ikx \cdot d} g(d) \, dS(d), \quad x \in \mathbb{R}^2.
\]

Note that \( w_g \) satisfies \( w_g|_{B_R} \in H^1(B_R \setminus D) \) for a sufficiently large \( R \) and the inhomogeneous Neumann boundary condition on \( \partial D \) should be considered in a weak sense.

In this section, using the idea of the proof of theorem 1.2, we give a proof of unsolvability of equation (3.3) for any \( k > 0 \) and \( y \in \mathbb{R}^2 \) provided \( D \) is polygonal.

**Theorem 3.4.** For any \( k > 0 \) and \( y \in \mathbb{R}^2 \) there exists no solution \( g \) of equation (3.3).

**Proof.** We employ a contradiction argument. Assume that equation (3.3) admits a solution \( g \). Then the coincidence of both far-field patterns of \( w_g \) and \( \Phi_0(\cdot, x) \) yields \( w_g(x) = \Phi_0(x, y) \) for \( x \in \mathbb{R}^2 \setminus B_R \) with a sufficiently large \( R \). From the unique continuation property for the Helmholtz equation this coincidence gives

\[
w_g(x) = \Phi_0(x, y), \quad \forall x \in (\mathbb{R}^2 \setminus D) \setminus \{y\}.
\]  

(3.4)

Since \( w_g|_{B_R} \in H^1(B_R \setminus D) \) and \( \Phi_0(\cdot, y')|_{B_R} \) does not belong to \( H^1(B_R \setminus D) \) for all \( y' \in \mathbb{R}^2 \setminus D \) from (3.4) one gets \( y \in D \). Note that this part or this type of argument is well known in the linear sampling method. It shows that if (3.3) is solvable, then \( y \in D \). The problem is the next to the intermediate conclusion \( y \in D \). Now we have

\[
w_g(x) = \Phi_0(x, y), \quad \forall x \in \mathbb{R}^2 \setminus D.
\]  

(3.5)

Define

\[
u_g(x) = v_g(x) + w_g(x), \quad x \in \mathbb{R}^2 \setminus D.
\]

Note that \( u = u_g \) satisfies the Helmholtz equation in \( \mathbb{R}^2 \setminus D \) and the homogeneous Neumann boundary condition \( \partial u / \partial v = 0 \) on \( \partial D \). From (3.5) we have \( u_g(x) = v_g(x) + \Phi_0(x, y), \quad x \in \mathbb{R}^2 \setminus D \), and this right-hand side gives a continuation \( \tilde{u}_g \) of \( u_g \) onto \( \mathbb{R}^2 \setminus \{y\} \) as a solution of the Helmholtz equation.

Choose a \( \omega \in S^1 \) that is regular with respect to \( D \) and define

\[
I(\tau) = \int_{\partial B_R} \left( \frac{\partial u_g}{\partial v} v_\tau - \frac{\partial v_g}{\partial v} u_g \right) \, dS, \quad \tau > 0,
\]  

(3.6)

where \( v_\tau(x) = e^{ik(\tau_0 + \tau + 2\tau R^2 \omega \cdot x)} \).
Let $x_0 \in \partial D$ with $x_0 \cdot \omega = h_D(\omega)$. One may assume that $y \in D_1$, where $D_1$ is a connected component of $D$. Then one can choose a small $\delta > 0$ such that if $|x - y| \leq \delta$, then $x \in D_1$ and $x \cdot \omega < h_D(\omega) - \delta$. Replacing $u_\omega$ in (3.6) with $\tilde{u}_\omega$ and applying integration by parts, we obtain, as $\tau \to \infty$,

$$
e^{-\tau h_D(\omega)} I(\tau) = e^{-\tau h_D(\omega)} \int_{\partial D} \left( \frac{\partial \tilde{u}_\omega}{\partial \nu} v_\tau - \frac{\partial v_\tau}{\partial \nu} \tilde{u}_\omega \right) dS$$

$$= e^{-\tau h_D(\omega)} \int_{|x-y|=\delta} \left( \frac{\partial \tilde{u}_\omega}{\partial \nu} v_\tau - \frac{\partial v_\tau}{\partial \nu} \tilde{u}_\omega \right) dS = O(\tau e^{-\delta \tau}). \quad (3.7)$$

Hereafter we make use of the same notation as those of the proof of theorem 1.2.

Recalling the boundary condition $\partial u_\omega / \partial \nu = 0$ on $\partial D$, one has the expansion

$$u_\omega (r, \theta) = \alpha_1 J_0(kr) + \sum_{n=2}^{\infty} \alpha_n J_n(kr) \cos \lambda_n \theta, \quad 0 < r < \eta, \quad 0 < \theta < \Theta$$

and applying the argument for deriving (2.1), we obtain

$$I(\tau) e^{-i\nu + k^2 \omega^2 \omega^2} e^{-\tau h_D(\omega)} \sim -i \sum_{n=2}^{\infty} \frac{e^{i\frac{\tau}{2} + k^2 s^2 s^2} \alpha_n K_n}{s^{2s}}, \quad (3.8)$$

where $K_n$ are constants given by the formula $K_n = e^{i\rho k_n} + (-1)^n e^{i\rho k_n}$ and $s = \sqrt{r^2 + k^2 + \tau}$. Since (3.7) implies that $e^{-\tau h_D(\omega)} I(\tau)$ is rapidly decreasing as $\tau \to \infty$, all the coefficients of the right-hand side of (3.8) have to vanish, that is,

$$\alpha_n K_n = 0, \quad \forall \ n \geq 2. \quad (3.9)$$

First consider the case when $\Theta / \pi$ is irrational. It is easy to see that $K_n \neq 0$ for all $n \geq 2$. Thus, from (3.9) one gets $\alpha_n = 0$ and this yields $u_\omega(r, \theta) = \alpha_1 J_0(kr)$ for $0 < r < \eta$ and $0 < \theta < \Theta$. Since this right-hand side is an entire solution of the Helmholtz equation, the unique continuation property of the solution of the Helmholtz equation yields $\tilde{u}_\omega(x) = \alpha_1 J_0(k|x - x_0|)$ in $\mathbb{R}^3 \setminus \{y\}$ and thus one gets

$$\Phi_0(x, y) = \alpha_1 J_0(k|x - x_0|) - v_\omega(x), \quad x \neq y.$$

Comparing the behaviour as $x \to y$ on both sides, we obtain a contradiction.

Next consider the case when $\Theta / \pi$ is rational. Applying the same argument for the derivation of (2.4), we have a continuation of $u_\omega$ onto $(\mathbb{R}^3 \setminus \overline{D}) \cup B_\eta(x_0)$ as a solution of the Helmholtz equation and its continuation which we denote by $\tilde{u}'$ satisfies the rotation invariance

$$\tilde{u}'(r, \theta + \frac{2\pi}{a}) = \tilde{u}'(r, \theta), \quad 0 < r < \eta, \quad \theta \in \mathbb{R},$$

where $a \geq 2$ is an integer. Since the unique continuation property gives $\tilde{u}_\omega(r, \theta) = \tilde{u}'(r, \theta)$ for $0 < r < \eta$ and thus one gets

$$\tilde{u}_\omega(r, \theta + \frac{2\pi}{a}) = \tilde{u}_\omega(r, \theta), \quad 0 < r < \eta, \quad \theta \in \mathbb{R}. \quad (3.10)$$

Since $\tilde{u}_\omega$ satisfies the Helmholtz equation for $|x - x_0| < |x_0 - y|$, it follows from the unique continuation property and the rotation invariance of the Helmholtz equation that $\eta$ in (3.10) can be replaced with $|x_0 - y|$:

$$\tilde{u}_\omega(x_0 + rz(\theta)) = \tilde{u}_\omega \left( x_0 + rz \left( \theta + \frac{2\pi}{a} \right) \right), \quad 0 < r < |x_0 - y|, \quad \theta \in \mathbb{R}. \quad (3.11)$$
where \( z(\theta) = \cos \theta a + \sin \theta a^\perp \). Now choose a \( \theta_0 \) in such a way that \( y = x_0 + |y-x_0| z(\theta_0) \).

Since \( 2\pi/a \leq \pi \), we have \( y \neq x_0 + |y-x_0| z(\theta_0 + 2\pi/a) \). Then letting \( \theta = \theta_0 \) and \( r \uparrow |y-x_0| \) in (3.11), we have a contradiction since \( \tilde{u}_g(x) = v_g(x) + \Phi_0(x, y) \) for \( x \neq y \) and

\[
\Phi_0(x, y) \sim \frac{1}{2\pi} \log \frac{1}{|x-y|}
\]

as \( x \to y \). \( \square \)

Using a variational formulation in, e.g., [8], one can formulate and establish the unique solvability of the scattering problem of the acoustic wave by a sound-hard obstacle \( D \) with a Lipschitz boundary. We use the same notation as those in the case when \( D \) is polygonal. Having the far-field pattern for \( D \) with a Lipschitz boundary, one can extend theorem 3.4 to a slightly general case. We say that \( D \) with a Lipschitz boundary has a \textit{horn}, if there exist a \( \omega \in S^1 \) that is regular with respect to \( D \) and \( \delta > 0 \) such that the set \( V \equiv \{ x \in D \mid x \cdot \omega > h_D(\omega) - \delta \} \) becomes a finite cone with the vertex at the point in \( \{ x \mid x \cdot \omega = h_D(\omega) \} \cap \partial D \) and the base on \( x \cdot \omega = h_D(\omega) - \delta \). We say that \( V \) is a \textit{horn}.

The conclusion of this section is the following statement and since the proof is really a minor modification of that of theorem 3.4, we omit the description of the proof.

**Corollary 3.1.** If \( D \) with a Lipschitz boundary has a horn, then for any \( k > 0 \) and \( y \in \mathbb{R}^2 \) there exists no solution \( g \) of equation (3.3).

Note that, in [26], the far-field equation for a single circular obstacle with an arbitrary radius has been considered and it is shown that the equation is not solvable except for its centre point. Corollary 3.1 means that the existence of a horn \( V \) even it is small prevents the existence of the solution of (3.3) for any \( k > 0 \) and \( y \in \mathbb{R}^2 \).

It should be pointed out that the linear sampling method is not based on the solvability of the far-field equation. Instead a family of approximate solutions of the far-field equation is taken. See [2] for an interesting study of the method itself.

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