Effect of temperature gradient on heavy quark anti-quark potential using gravity dual model

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The quark-gluon plasma (QGP) is an expanding fireball, with finite dimensions. Given the finite dimensions, the temperature would be highest at the center, and close to the critical temperature, $T_c$, at the boundary, giving rise to a temperature gradient inside the QGP. A heavy quark anti-quark pair immersed in the QGP medium would see this temperature gradient. The effect of a static temperature gradient on the quark anti-quark potential is analyzed using a gravity dual model. A non-uniform black string metric is developed, by perturbing the Schwarzchild metric, which allows to incorporate the temperature gradient in the dual AdS space. Finally, an expression for the quark antiquark potential, in the presence of a temperature gradient, is derived.

Keywords: QGP, potential model, AdS-CFT, gravity dual model, Wilson loop, black string, temperature gradient.

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I. INTRODUCTION

The suppression of $J/\psi$ and $\Upsilon$ are two prominent signatures to detect the presence of QGP in heavy-ion collision experiments, as well as to study its properties [1][2]. In [10, 11], it has been shown that at high temperatures, the heavy quark anti-quark separation increases, which leads to an increase in formation time for both $J/\psi$ and $\Upsilon$. This in turn results in a drastic reduction in suppression due to the color screening mechanism [10, 11]. The temperature inside the QGP medium would be non-uniform spatially. As the heavy quark anti-quark separation increases, the heavy quark and anti-quark in the QGP medium will be subjected to two different temperatures. The effect of this temperature gradient on quark anti-quark potential in the presence of QGP has not been formulated in the existing literature. Accurate determination of heavy quark anti-quark potential is essential to determine the $J/\psi$ and $\Upsilon$ suppression precisely. In this work, we attempt to explore the effect of this temperature gradient in the QGP medium on the potential between the heavy quark and anti-quark using a gravity dual model.

Any system with a temperature gradient is expected to evolve in time till the gradient decays to zero. If $\Theta(z, t)$ is the temperature of the system, at time $t$ and location $z$, then $\Theta(z, t)$ can be decomposed into a time independent $\Theta_0(z)$ and time dependent $\Delta\Theta(z, t)$, where $\Theta_0(z)$ is the time averaged value of $\Theta(z, t)$, i.e.,

$$\Theta(z, t) = \Theta_0(z) + \Delta\Theta(z, t). \quad (1)$$

It is the static temperature gradient $\Theta_0(z)$, that we attempt to model in this paper.

The gravity dual model in the form of AdS-CFT correspondence was first proposed by Maldacena in his seminal work [12, 13]. AdS-CFT has been an area of current research for understanding QGP properties [14, 15] and other areas [17, 18]. It has been shown that the strong coupling in the framework of Quantum Chromodynamics (QCD), corresponds to a weak coupling in the gravity dual domain, and hence calculations can be more easily be performed in the gravity dual domain. With the increase in the separation between the quark and the anti-quark, due to higher temperature, the coupling constant would become larger, and hence, the potential cannot be calculated accurately using perturbative QCD. The gravity dual model provides an ideal mathematical framework to determine the effect of a temperature gradient on the heavy quark
anti-quark potential, when the separation between them is large. However, the gravity model in $AdS_5$ space is dual to $\mathcal{N} = 4$ supersymmetric Yang-Mills Lagrangian, and hence our calculation would be for a $\mathcal{N} = 4$ supersymmetric Yang-Mills Lagrangian instead of the QCD Yang-Mills Lagrangian.

The organization of the rest of the paper is as follows. Section II frames the problem in both the Wilson loop domain and in the gravity dual domain. Section III dwells into the determination of the metric due to the insertion of a non-uniform black string, which is required for creating a temperature gradient in the dual domain. The calculation of the string action in the dual gravity domain, and hence the quark-antiquark potential, in the presence of a temperature gradient, is treated in Sec. IV. The conclusion is finally drawn in Sec. V.

II. WILSON LOOP AND DUAL DOMAIN FORMULATION

A. Wilson Loop

The Wilson loop that has been used to model the scenario of the temperature gradient is depicted in Fig. 1. The relation of the Wilson loop to the heavy quark anti-quark potential could be modeled as [21]:

$$\langle W(C) \rangle = e^{-V_{QQ} \beta}, \quad (2)$$

where, $\beta = \frac{1}{\Theta}$ and $\Theta$ is the temperature of the system. $V_{QQ}$ is the potential between the heavy quark and anti-quark. From Maldacena’s conjecture, the vacuum expectation value of the Wilson loop for a $\mathcal{N} = 4$ supersymmetric Yang-Mills Lagrangian, could be equated to the string action $S = \frac{1}{2\pi\alpha'} \int_{\Sigma} d\sigma d\tau \sqrt{G_{MN} \partial_\alpha X^M \partial_\beta X^N}$, where $\Sigma$ is the domain covered by the Wilson loop, which lives at the Minkowski boundary of the $AdS_5$ space. In the integral, $\tau = \beta$, and $\sigma$ takes the value from 0 to $L$.

For the Wilson loop shown in Fig. 1, one can break the corresponding string action into 2 parts:

$$S = \frac{1}{2\pi\alpha'} \int_{\Sigma_1} d\sigma d\tau \sqrt{G_{MN} \partial_\alpha X^M \partial_\beta X^N} + \frac{1}{2\pi\alpha'} \int_{\Sigma_2} d\sigma d\tau \sqrt{G_{MN} \partial_\alpha X^M \partial_\beta X^N} = S_1 + S_2. \quad (3)$$

The two string actions $S_1$ and $S_2$ correspond to the two Wilson loops $L_1$ and $L_2$ (Fig. 2), which make up the original Wilson loop $L$. Varying $S$, we get

$$\delta S = \delta S_1 + \delta S_2 = 0.$$

A sufficient, albeit stricter condition for $\delta S = 0$ is that the individual variation in action be zero, i.e. $\delta S_1 = \delta S_2 = 0$. This allows the two loops $L_1$ and $L_2$ to be treated independently.

We give the intuition behind treating the two actions $S_1$ and $S_2$ separately, by analyzing the equivalent Wilson loop domain. Let us consider the

FIG. 1: Wilson loop to model the potential of a heavy quark anti-quark pair immersed in a QGP medium with a temperature gradient.

FIG. 2: The two Wilson loops $L_1$ and $L_2$, with the original loop $L_0$. 

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zero temperature case, where the ”temporal” axis is time. As the time interval ”$T$" becomes large, only the smallest eigenvalue (the ground state) survives in the Wilson loop. To see why, consider the Wilson loop, $W$, for a rectangular loop $L \times T$. We treat the Wilson loop in axial gauge, with $A$ along the time axis equal to zero. This gives \cite{20},

$$W = \langle tr \Psi(0) \Psi^\dagger(T) \rangle,$$

where,

$$\Psi_{ij}(t) = \left[ \exp(-i \int_0^L A_\mu(z, t) dz) \right]_{ij}.$$ 

Inserting a complete set of intermediate eigenstates $\langle n \rangle$,

$$W = \sum_n \langle \Psi_{ij}(0) \langle n \mid \Psi^\dagger_{ji}(T) \rangle = \sum_n |\langle \Psi_{ij}(0) \langle n \rangle|^2 e^{-E_n T},$$

where $E_n$ is the energy of the state $\langle n \rangle$. As $T \to \infty$, only $E_0$ survives.

We now decompose the rectangular region $L \times T$ into $L \times T_1$ and $L \times T_2$, with $T_1 + T_2 = T$. The domains, $\Omega_1$ and $\Omega_2$ denote the two rectangular regions $L \times T_1$ and $L \times T_2$. Under the conditions when only one eigenvalue survives, or is the dominant contributor,

$$tr \left( \exp \left\{ (-i \int_{\Omega_1} A_\mu dx^\mu) + (-i \int_{\Omega_2} A_\mu dx^\mu) \right\} \right) \approx$$

$$tr \left( \exp(-i \int_{\Omega_1} A_\mu dx^\mu) \right) tr \left( \exp(-i \int_{\Omega_2} A_\mu dx^\mu) \right)$$

where, $A_\mu = A^a_{\mu a}$. This implies that the two actions corresponding to the two Wilson loops can be treated separately as an approximation. In Malda- cenna’s derivation \cite{12}, it can explicitly be seen that, if the action is broken into two actions corresponding to two loops $L \times T_1$ and $L \times T_2$, the final result is the same as the single loop $L \times T$. Extending to the finite temperature case, if only the ground state contributes towards the action $S$, i.e., when $\beta$ is large, then $S_1$ and $S_2$ may be treated independently as an approximation.

### B. Finite Temperature in Dual Domain

Temperature is introduced in the AdS space, by introducing a black hole. The black hole emits Hawking radiation, and creates a temperature, $\Theta$, in the system. The resulting $AdS_5 \times S_5$ near-horizon metric, of near extremal D3-branes in Type IIB string theory at finite temperature is given by \cite{13, 21}:

$$ds^2 = \alpha' \left[ \frac{u^2}{R^2} \left( -V dt^2 + dx_\|^2 \right) + \frac{R^2}{u^2} \left( \frac{1}{V} du^2 + u^2 d\Omega_5^2 \right) \right].$$

Here,

$$V = 1 - \frac{a^4}{u^4},$$

$$R^4 = 4\pi gN, \text{ with } N \text{ large.}$$

Imposing a periodicity condition on $\tau(=it)$ to prevent a conical singularity near the horizon, determines the temperature $\Theta$.

As discussed, when the QGP medium is in local equilibrium, the quark and anti-quark can be at slightly different temperatures, $\Theta$ and $\Theta + \Delta \Theta$. Thus the two particles exist at two different points in space, with slightly different metric, due to different temperatures. In the next section, Sec. \textsc{iii} we show that it is possible to achieve a temperature gradient by inserting a non-uniform black string. The non-uniform black string metric can be obtained by perturbing the Schwarzchild black hole metric, namely,

$$ds^2 = \alpha' \left[ \frac{u^2}{R^2} \left( -V + \Delta H_u(t, u, x_\|) \right) dt^2 + dx_\|^2 \right)$$

$$+ \frac{R^2}{u^2} \left( \frac{1}{V} + \Delta H_{uu}(t, u, x_\|) \right) du^2$$

$$+ \left( u^2 + \Delta H_{\Omega} \right) d\Omega_5^2. \tag{6}$$

The values of $\Delta H(t, u, x_\|)$ would be calculated in Sec. \textsc{iii}. It is shown how these perturbations lead to a temperature gradient. It is also shown that, after imposing periodicity conditions, under appropriate conditions, one can have $\beta$ varying linearly with $x_\|$. 

### III. Non Uniform Black String

#### A. Perturbation of Black String

In this section, we explore the incorporation of a temperature gradient in the AdS-CFT framework...
by inserting a non-uniform black string. A non-uniform black string, in itself, need not necessarily result in a non-uniform temperature $^{22}$. We arrive upon a non-uniform black string, which gives rise to a temperature gradient, by perturbing a uniform black string. This section discusses the perturbation mechanism and the resultant temperature gradient.

The $qq$ pair is taken to lie in the spatial $z$-direction. This allows the 10 dimensional space to be first reduced to 8 dimensions by representing the spatial $x_{||}$ space defined by the three $x$, $y$ and $z$ dimensions, by the single $z$ dimension, for the purpose of analyzing black strings. A uniform black string can be modeled in an 8 dimensional space, by using the Schwarzschild metric, as:

$$ds^2 = -V dt^2 + \frac{du^2}{V} + u^2 d\Omega_5^2 + dz^2,$$

where $V = 1 - \frac{a^2}{u^2}$. We now apply a perturbation $\Delta H_{bc}$ to the above metric, $g_{bc}$, in order to model a non-uniform black string. Motivated by the work done in $^{23}$, we take the general form of $\Delta H_{bc}$ to be:

$$\epsilon \Omega e^{-mz}$$

$$\begin{bmatrix}
  h_{tt} & h_{tu} & 0 & 0 & \ldots & h_{tz} \\
  h_{ut} & h_{uu} & 0 & 0 & \ldots & h_{uz} \\
  0 & 0 & 0 & W & \ldots & 0 \\
  0 & 0 & 0 & W \sin^2(\phi_1) & \ldots & 0 \\
  \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
  h_{zt} & h_{zu} & 0 & 0 & \ldots & h_{zz}
\end{bmatrix}.$$

Here, $h_{tt}$, $h_{uu}$, $h_{ut}$, $h_{tu}$ and $W$ are some functions of $u$. The perturbation $\Delta H_{bc}$ satisfies the equation $\Delta_L(\Delta H_{bc}) = 0$, where $\Delta_L$ is the Lichnerowicz operator. Taking a transverse tracefree gauge, $h^b_b = 0$ and $\nabla_b h^b_c = 0$, the equation, $\Delta_L(\Delta H_{bc}) = 0$, reduces to $^{23}$:

$$\nabla^2(\Delta H_{bc}) + 2R_{bdc}^e(\Delta H_{de}^d) = 0,$$  \hspace{1cm} (7)

with $R_{bdc}^e$, being the Riemann curvature tensor. In $^{23}$, the $z$ dependence has been taken as $e^{imz}$. It can be seen that, with $e^{imz}$, the value of $m$ which would solve the Lichnerowicz equation for our chosen value of $\Omega$, is imaginary. Hence from hindsight, we take the $z$ dependence as $e^{-mz}$ with real and positive $m$. A real physical black string is more likely to be of the form $e^{-m|z|}$. But such a function is not well behaved at $z = 0$. Since we are mainly interested in the region, $z > 0$, we have preferred to proceed with the more well behaved $e^{-mz}$ instead of $e^{-m|z|}$. The two functions have the same value in the region of interest $0 < z < L$.

As we are interested in a particular solution, and not in the most general solution, we take, $h_{zz} = h_{tz} = h_{zt} = 0$. From Eq. (7) one can then get the following set of differential equations:

$$\left[ V h''_{tt} + \left( \frac{5V}{u} - V' \right) h'_t - V'' h_{tt} + \left( \frac{h_{tt}}{2V} - \frac{V}{2} \frac{h_{uu}}{u} \right) V'^2 \right]$$

$$+ \left( \Omega h_{ut} + \Omega h_{tu} - \frac{5h_{tt}}{u} \right) V' - \left( -m^2 + \frac{\Omega^2}{V} \right) h_{tt} - V \left( V h_{uu} V'' + \frac{5WV'}{u^3} \right) = 0, \hspace{1cm} (8)$$

$$\left[ V h''_{uu} + \left( 3V' + \frac{5V}{u} \right) h_{uu} + h_{uu} V'' + \left( \frac{h_{uu}}{2V} - \frac{h_u}{u^2} V' \right) V'^2 \right]$$

$$+ \left( \frac{5h_{uu}}{u} + \frac{\Omega}{V^2} (h_{tu} + h_{ut}) \right) V'$$

$$+ \left( -\frac{10V}{u^2} - ( -m^2 + \frac{\Omega^2}{V} ) \right) h_{uu}$$

$$+ \frac{10W}{u^4} - \frac{V'' h_{tt}}{V} + \frac{5WV'}{u^3} = 0, \hspace{1cm} (9)$$

The gauge condition, $\nabla_b h^b_c = 0$, leads to:

$$\left( \frac{W' - 2W}{u} \right) V' + WV'' + \frac{V}{u} W'$$

$$+ 2V^2 h_{uu} - \frac{6WV}{u^2} - \left( -m^2 W + \frac{\Omega^2 W}{V} \right)$$

$$+ u(V h_{uu} - \frac{h_{tt}}{V}) V' + \frac{8WV}{u^2} - \frac{8W}{u} = 0. \hspace{1cm} (10)$$

The gauge condition, $\nabla_b h^b_c = 0$, gives:

$$V h'_{uu} + V' h_{uu} + \frac{5V}{u} h_{uu} - \frac{\Omega}{V} h_{tt}$$

$$- \frac{1}{2} h_{uu} V' + \frac{1}{2} h_{tt} V' = 0, \hspace{1cm} (11)$$

and

$$V h'_{uu} + V h_{uu} + \frac{5V}{u} h_{uu} - \frac{\Omega}{V} h_{tt}$$

$$+ \frac{1}{2} h_{uu} V' + \frac{h_{uu} V'}{2V^2} - \frac{5W}{u^3} = 0. \hspace{1cm} (12)$$

The final gauge condition, $h^b_b = 0$, gives:

$$\left( -\frac{h_{tt}}{V} + h_{uu} V \right) + 5W = 0 \hspace{1cm} (13)$$
The superscript ‘t’ refers to partial derivative w.r.t \(u\). The region of interest is the near horizon region, as \(u \to a\). Let \(u_+ = u - a\), and in the region of interest, \(u_+ \ll u\). In these limits, the lowest power of \(u_+\) becomes the most significant. We further assume, that, the solution in the lowest power of \(u_+\), to be of the form, \(h_{tt}(u) = A(m, \Omega, \alpha)u_+^n\), \(h_{uu}(u) = B(m, \Omega, \alpha)u_+^l\), \(h_{tt}(u) = h_{tt}(u) = C(m, \Omega, \alpha)u_+^k\) and \(W = W_0(m, \Omega, \alpha)u_+^k\).

From the gauge condition, \(h_0^b = 0\), i.e., Eq. [13], we get \(n = l + 2\) and \(k = n - 1\). Given that \(h_{tt}\) and \(h_{uu}\) vary as \(u_+^{n-2}\) and \(u_+^n\), we go with the ansatz that \(h_{tt}\) varies as \(u_+^{n-1}\), i.e., \(n = q - 1\). Since the lowest power of \(u_+\) becomes dominant as \(u_+ \to 0\), by equating the coefficients of the lowest power of \(u_+\) to 0, the equations [8], [9], [10] and [12] give respectively,

\[
\begin{align*}
4A \frac{(n^2 - 2n + 0.5)}{a} + \frac{32B}{a^3} + \frac{8\Omega C}{a} - \frac{\Omega^2 A a}{4} &= 0, \\
4B \frac{(l^2 + 2l + 0.5)}{a} - \frac{8\Omega A c}{8} + \frac{4\Omega a C}{2} - \frac{2\Omega a A}{4} &= 0, \\
4C \frac{(q + 1)}{a} - \frac{\Omega a A}{4} &= 0, \\
6B + 4Bl - \frac{8\Omega a C}{a} + \frac{4A}{8} &= 0.
\end{align*}
\]

(14)

The above equations are satisfied for \(n = 1\), \(l = -1\) and \(q = 0\), with the constraint that \(\frac{\Omega a}{4} = \pm 1\). In [23], the authors have solved the equations for arbitrary values of \(\Omega\), but in this work, we have restricted ourselves to certain values of \(\Omega\), since it gives rise to a form of metric perturbation, which is more suitable for temperature gradient calculations. It is seen that our solution corroborates with [23] for the specific values of \(\Omega = \frac{4}{a}\) when \(h_{tt} \propto u_+^{-1}\), \(\Omega = -\frac{4}{a}\) when \(h_{tt} \propto u_+^{-1}\). Further, we get \(C = \frac{\Omega a^2 A}{16}\), \(B = \frac{a^2 A}{16}\).

(15)

Inserting the value of \(B\) in the gauge condition, \(h_0^b = 0\), we find that \(W\) vanishes. Clearly, the non zero contribution to \(W\) comes from equating with higher powers of \(u_+\) of \(h_{tt}\) and \(h_{uu}\). To determine \(W\), we then solve Eq. [14] to obtain,

\[
(4k^2W_0 + \frac{\Omega^2 a W_0}{4}) u_+^{k-1} + \text{higher powers of } u_+ = 0.
\]

In the limit \(u_+ \to 0\), we get, \(k = \pm \frac{\Omega a}{4}\). Thus,

\[
5W = -(\frac{h_{tt}}{V} + Vh_{uu}) \approx 5W_0 u_+^{k-1}, \quad (17)
\]

which then corroborates with the solution of \(-\frac{h_{tt}}{V} + Vh_{uu}\) in [24].

However, for the purpose of determining the temperature of the perturbed black string, near the horizon, we are mainly interested in the dominant, lowest power of \(u_+\) behavior of \(h_{tt}\) and \(h_{uu}\). Thus, with \(n = 1\), \(l = -1\), we get, \(h_{tt} = Au_+, \) and \(h_{uu} = \frac{Aa^2}{16u_+}\), along with the value of \(\Omega = \pm \frac{4}{a}\). For large \(a\), \(\Omega\) is small, indicating a slow time varying system. For clarity, we summarize the final form of \(\Delta H_{tt}, \Delta H_{u_+u_+}\) and \(\Delta H_{tu_+}\) as:

\[
\begin{align*}
\Delta H_{tt} &= e^{\Omega t} e^{-mz} Au_+, \\
\Delta H_{u_+u_+} &= e^{\Omega t} e^{-mz} B/u_+, \\
\Delta H_{tu_+} &= e^{\Omega t} e^{-mz} C.
\end{align*}
\]

The perturbed black string metric now becomes:

\[
\begin{align*}
ds^2 &= (-V + e^{\Omega t} e^{-mz} Au_+) dt^2 \\
&\quad + \left(\frac{1}{V} + \frac{e^{\Omega t} e^{-mz} B}{u_+}\right) du_+^2 \\
&\quad + (e^{\Omega t} e^{-mz} C) dt du_+ \\
&\quad + (...)d\Omega_5^2 + dz^2. \quad (18)
\end{align*}
\]

where we have replaced \(du\) by \(du_+\).

In order to eliminate the \(dt du_+\) term, we apply a linear transformation \(t' \leftarrow t + \alpha u_+\), and \(u' \leftarrow u_+\). We obtain the metric,

\[
\begin{align*}
ds^2 &= (-V + e^{\Omega t'} e^{-mz} A u_') dt'^2 \\
&\quad + \left(\frac{1}{V} + \frac{e^{\Omega t'} e^{-mz} B}{u_'}\right) du'^2 \\
&\quad - \frac{e^{2\Omega t'} e^{-2mz} C a}{4u_+ (4 + e^{\Omega t'} A u e^{-mz})} du'^2 \\
&\quad + (...)d\Omega_5^2 + dz^2. \quad (19)
\end{align*}
\]

In the above equation, we have used the fact that, as \(u_+ \to 0\), \(u'_+ \to 0\) and \(e^{\Omega t' - \alpha u_+} \to e^{\Omega t'}\). We take the interquark distance \(L \ll 1/m\) along the \(z\)-axis. In Sec. [11] we shall see that, this would be true if \(a\) is very large, leading to a small \(m\). In this region of the black string, i.e., \(0 < z < L\), we can approximate \(e^{-mz} \approx (1 - mz)\). In the dual domain,
these assumptions become equivalent to the slope of the temperature gradient being very small. With these approximations, and with $C^2 < A,B$, and dropping the primes from $t'$ and $u'_+$, we get the first two terms of the metric as,

$$\begin{align*}
(-V + e^{\Omega t}(1-mz)Aa_{u_+})\,dt^2 \\
+ \left(\frac{1}{V} + \frac{e^{\Omega t}B(1-mz)}{u_+} \right)\,du_+^2.
\end{align*}$$

(20)

It would be tempting to relate the evolution, $e^{\Omega t}$, with the evolution of the QGP temperature. If $\Omega = -\frac{4}{a}$, it indicates that after infinite time, the perturbation to the metric vanishes. This qualitatively agrees with the fact that the temperature gradient in the QGP system should also vanish after sufficiently long time. Its also possible that a different function $f(t)$ (say, for example $f(t) = \sum_i c_i e^{\Omega_i t}$) may more correctly determine the QGP evolution.

We, however, in this work, do not probe this further. Relating the evolution $e^{\Omega t}$, with the evolution of the QGP, however has a drawback that the QGP lifetime is finite, while the metric evolution $e^{\Omega t}$ decays over infinite time. We premise that the metric is the dual to the QGP only till QGP lifetime $t_{QGP}$. We split the $e^{\Omega t}$ into a time averaged $F_1$ (averaged over $t_{QGP}$), and a time dependent $F_2(t)$ part. The first two terms of the metric in (20) then become

$$\begin{align*}
(-V + (F_1 + F_2(t))(1-mz)Aa_{u_+})\,dt^2 \\
+ \left(\frac{1}{V} + \frac{(F_1 + F_2(t))B(1-mz)}{u_+} \right)\,du_+^2,
\end{align*}$$

(21)

where

- $F_1 = (e^{\Omega_{QGP}} - e^{\Omega_{t_1}})/(\Omega(t_{QGP} - t_1)).$

and $t_1$ corresponding to the beginning of the QGP phase.

As mentioned in Sec. I, we are attempting to model $\Theta_0(z)$ in Eq. [1]. Its logical to expect the time independent part of the metric to be the dual of the time independent QGP system, and the time dependent part of the metric to model the duality of the time dependent QGP system. Thus the first two terms of the metric corresponding to the time independent QGP system would be

$$\begin{align*}
(-V + F_1(1-mz)Aa_{u_+})\,dt^2 \\
+ \left(\frac{1}{V} + \frac{F_1 B(1-mz)}{u_+} \right)\,du_+^2.
\end{align*}$$

(22)

Near horizon, as $u \to a$, $V = 1 - \frac{a^4}{4} \approx 4a_+ / a$. Setting $\rho = \sqrt{u_+}$, the first two terms become:

$$\begin{align*}
&\left(\frac{a + 4Be^{\Omega t}mz}{\rho^2 a^2 (1 + 4BF_1(1-mz)/a)}\right)dt^2 + \rho^2 (23)
\end{align*}$$

We are now at a point where we can define the temperature.

With $\beta = it$, and taking the periodicity condition to avoid conical singularity, we get,

$$\begin{align*}
\sqrt{-\frac{4 + aF_1(1-mz)A}{a^2 (1 + 4BF_1(1-mz)/a)}} \beta = 2\pi, \text{ giving}
\end{align*}$$

$$\begin{align*}
\beta = \frac{1}{\Theta} = 2\pi a \sqrt{\frac{(1 + 4BF_1(1-mz)/a)}{-4 + aF_1(1-mz)A}}.
\end{align*}$$

(24)

Thus we have obtained $\beta = 1/\Theta$ as a function of $z$.

If, we consider very small perturbations, i.e., $B$ and $A$ are small, $\beta$ can be seen to be approximately a linear function of $z$. If $\frac{4}{a} \gg F_1e^{-mz}A \approx F_1 (1-mz)A$, then we get:

$$\begin{align*}
ds^2 \approx (a + 4BF_1(1-mz))
\times \left[\frac{\rho^2}{a^2 (1 + (4BF_1/a + AF_1/a/4)(1-mz))}dt^2 + \rho^2 \right] + (...)d\Omega_5^2 + dz^2. (25)
\end{align*}$$

We rewrite the metric as:

$$\begin{align*}
ds^2 = (a + 4BF_1(1-mz))
\times \left[\frac{\rho^2 (4/a^2)}{(1 + E mz)}d\beta^2 + \rho\beta^2 \right] + (...)d\Omega_5^2 + dz^2. (26)
\end{align*}$$

with $E = -(4BF_1/a + AF_1/a/4) = -AF_1/a/2$, and since the perturbations are small, $|E| \ll 1$. Taking the periodicity of $\frac{2\beta}{a\sqrt{1 + Emz}} = 2\pi$, we get,

$$\begin{align*}
\frac{1}{\Theta} = \pi a \sqrt{1 + Emz} \approx \pi a (1 + \frac{Em}{2}z). (27)
\end{align*}$$

This gives temperature $= \Theta = 1/\beta$ as a function of $z$. Since the quark and antiquark lie along the $z$ axis, the temperature can be seen varying along the quark-antiquark distance, leading to a temperature gradient. The inverse temperature gradient $= \frac{d\Theta}{dz} = \pi aEm/2$. The linear gradient is a valid assumption only when the perturbation $E$ and $m$, are small. For larger perturbation, there would still be a temperature gradient along the quark-antiquark axis, but the gradient would not be linear.
It can be seen that a large $a$ corresponds to a large $\beta$ and a small $\Omega$, justifying both the assumptions of a large $\beta$ and a pseudo static system. In Sec. III B, we will see that $m = 12/a$, leads to a solution of Lichnerowicz equation for $\Omega = -4/a$. Hence, a large value of $a$ also leads to a small value of $m$, and thus consistent with the assumptions used in our derivation.

B. Numerical Simulation

We now fully solve the Lichnerowicz equation and show that the solution is bounded for the value of $\Omega = -4/a$. The equation for $h_{tu}$, with all the other variables eliminated, has been provided in [23]. As $u_+ \to 0$, taking the lowest power of $u_+$, the equation reduces to

$$
\left( -\Omega^2 + \frac{(D-3)^2}{4a^2} \right) h_{tu}'' - 3(D-3) \left( \frac{\Omega^2}{aV} + \frac{(D-3)^2}{4a^3V} \right) h_{tu}' + \left[ \left( \frac{\Omega^2}{V} \right)^2 + \frac{-(5\Omega^2)}{4a^2V^2} + \frac{(D-3)^2}{4a^4V^2} \right] h_{tu} = 0,
$$

(28)

where, $D + 1 = 8$ = black string dimensions. It is easily seen that $h_{tu} = Cu_+^4$, with $g = 0$, solves the above equation for $D=7$ and $\Omega = \pm 4/a$.

We now numerically solve for $h_{tu}$, $h_{uu}$ and $h_{tt}$, for all $u_+$, using Eqs. 8, 9 and the equation of $h_{tu}$ in [23]. They have been solved for $\Omega = -4/a$ and $m = 12/a$, using matlab’s ode15s function. The values of "RelTol" and "AbsTol" are $1e^{-6}$. For the initial conditions, i.e. values as $u_+ \to 0$, we have taken the values accurate till the next higher order in $u_+$. Explicitly,

$$
h_{tt}(u_+ \to 0) = Au_+ + A_2 u_+^2,
$$

$$
h_{uu}(u_+ \to 0) = B/u_+ + B_2,
$$

$$
h_{tu}(u_+ \to 0) = C + C_2 u_+,
$$

(29)

where $C_2 = \frac{3a^2}{2} A$, $B_2 = \frac{a^2}{16} A$, and $A_2 = \frac{10}{A}$. From Eq. 15 $A = \frac{16c^2}{10a}$, and we take $C = 1$ as reference. The differential equations have been solved from $u_+ = 1e^{-5}$ to $3a$ (i.e. to $u$ varying from $(a + 1e^{-5})/a$ to $4a$), along a logarithmically spaced grid with $10^6$ points. Figure 3 depicts the solution, with $\Omega = -4/a$ and $m = 12/a$. It is seen that $h_{tt}/V$, $Vh_{tu}(u_+)$ and $h_{tu}(u_+)$ are bounded for all values of $u_+$. Evidently, $e^{\Omega u} e^{-mz}h_{tt}/V$, $e^{\Omega u} e^{-mz}Vh_{uu}$ and $e^{\Omega u} e^{-mz}h_{tu}$ would be bounded for all values of $t$ and $u$, and for $z > 0$. Hence, we have arrived at a solution of the Lichnerowicz equation, with value of $\Omega = -4/a$, and $m = 12/a$, which is never going to blow up for positive $z$.

C. Magnitude of Perturbation

This section explores further the magnitude of perturbation for all $u$ and $t$. Large perturbations can lead to divergence of the integrals in the calculation of the action. Near horizon, $g_{tt} + \Delta H_{tt} = V + e^{\Omega u} e^{-mz}Au_+ \approx \left( \frac{1}{a} + e^{\Omega u} e^{-mz} A \right) u_+$. In Sec. III A for linearity to be valid, we had assumed that $A$ is small enough, such that $A \ll 4/a$. Thus, the perturbation is small near the horizon for $g_{tt}$. Again, near horizon, $g_{uu} + \Delta H_{uu} = \frac{1}{V} + e^{\Omega u} e^{-mz}B\frac{u_+}{u_+} \approx \left( \frac{1}{4} + e^{\Omega u} e^{-mz} B \right) \frac{1}{u_+}$. As $u_+ \to 0$, the perturbation and the original metric, $g_{uu}$, approach infinity. Here, we need to define ”a small perturbation”. When we say that the perturbation is small, we do not mean $\Delta H_{uu}$ itself, but the coefficient of $1/u_+$ in $\Delta H_{uu}$, compared to the coefficient of $1/u_+$ in $g_{uu}$. From Eq 15 $B = \frac{a^2}{16} A$. Thus, $g_{uu} + \Delta H_{uu} = \frac{a^2}{16} \left( \frac{1}{a} + e^{\Omega u} e^{-mz} A \right) \frac{1}{u_+}$.

We now show that the perturbation remains
small for all \( u \) and any \( t = t_0 \). From Sec. III B and Fig. 3 it can be seen that the solution for the black string perturbation, namely, \( h_{tt}/V \) and \( Vh_{uu} \) are bounded for all values of \( u \), and in fact, roughly decaying with \( u \). Comparing with the original metric, where, \(-g_{tt}/V = Vg_{uu} = 1\), if the initial values of \( h_{tt}/V \) and \( Vh_{uu} \) are much smaller than \( g_{tt}/V \) and \( Vg_{uu} \) respectively, they will remain so for all \( u \). We identify the \( z \) variable in this section with \( x_{||} \) in Eq. 6 and understand that it lies in the range \( 0 < x_{||} < L \). The negative values of \( \Omega \), namely, \( \Omega = -4/a \), indicates that if the perturbation is small at \( t = 0 \), it would be small at any \( t = t_0 \). Summarizing, the perturbation to the metric remains smaller than the original metric in the region of interest \( 0 < x_{||} < L \), for all \( u \), and for all \( t_0 \).

For a positive value of \( \Omega = 4/a \), the existence of Lichnerowicz solution leads to an unstable black string, as the solution blows up \[23\]. In a real astrophysical black string, both positive and negative values of \( \Omega \) would be possible, leading to the instability. However, in the QCD domain, the temperature or the temperature gradient of the QGP is not going to blow up, but rather decay with time. Hence we identify the negative value of \( \Omega \), namely, \( \Omega = -4/a \), to model the appropriate mathematical dual.

### IV. QQ POTENTIAL UNDER A TEMPERATURE GRADIENT

#### A. Formulation

We have so far derived the perturbation functions in the metric in Eq. 6 which can be used to calculate the quark anti-quark potential, in the presence of a temperature gradient. We have seen that \( \beta \) can vary linearly with \( z \) (or equivalently \( x_{||} \)). The Wilson loop, corresponding to a linear variation of \( \beta \) is shown in Fig. 4 which is what we require to compute. One can identify \( \beta \) and \( \Delta \beta \) in Fig. 4 with black string metric parameters in Eq. 27 at \( z = L \), as:

\[
\Delta \beta = \frac{\pi a E mL}{2}, \quad \beta = \frac{\pi a}{2}
\]

The problem has now been reduced to mathematical computation of the string action for the Wilson loop in Fig. 4 using the metric in Eq. 6. The general approach would require a numerical approach. However, most parts of the solution can be analyzed analytically.

For convenience we state below, the static part of the perturbed metric:

\[
ds^2 = \alpha' \left[ \frac{u^2}{R^2} \left( (-V + F_0 h_{tt}) dt^2 + dx_{||}^2 \right) \right. \\
\left. + \frac{R^2}{u^2} \left( \frac{1}{V} + F_0 h_{uu} \right) du^2 + (u^2 + ... ) d\Omega^2 \right],
\]

with \( F_0 = F_1 e^{-mz} \approx F_1 (1 - mz) \). The values of \( g_{\alpha \beta} = G_{\alpha MN} \partial_\alpha X^M \partial_\beta X^N \), with \( \beta = i t \), are:

\[
g_{00} = \frac{u^2}{R^2} V - \frac{u^2 F_0 h_{tt}}{R^2}; \quad g_{11} = \frac{u^2}{R^2} + \frac{R^2}{u^2} (1/V + F_0 h_{uu}) u^2; \quad \text{and} \quad g_{01} = g_{10} = 0;
\]

After ignoring the second order term \( h_{tt}h_{uu} \), this gives

\[
S = \frac{1}{2\pi} \int d\sigma dt \\
\times \sqrt{\frac{u^4 V}{R^4} - \frac{u^4 F_0 h_{tt}}{R^4} + (1 - \frac{F_0 h_{tt}}{V} + F_0 V h_{uu}) u^2} \tag{31}
\]

The Hamiltonian \( H \) is then

\[
H = \frac{\frac{u^4 V}{R^4} - \frac{u^4 F_0 h_{tt}}{R^4}}{\sqrt{\frac{u^4 V}{R^4} + u^2 - \left\{ \frac{u^4}{R^4} F_0 h_{tt} + 5F_0 W u^2 \right\}}}, \tag{32}
\]
with $5W$ determined from Eq. [17] At minima, $u' = 0$ and $u = u_0$, giving

$$H = \frac{u_0^4 V - u_0^4 h_{tt}}{\sqrt{u_0^4 V - u_0^4 h_{tt}}}$$

(33)

The perturbation being small is unlikely to modify the string $u$ significantly. For the purpose of determination of the string equation, we ignore the perturbation to get,

$$\frac{u'}{u_0} = \frac{u_0}{R^2} \frac{u^4}{u_0^4} - \frac{u^4}{u_0^4} - 1.$$

Again, defining $y = u/u_0$, and $\omega = a/u_0$,

$$y' = \frac{u_0}{R^2} \sqrt{y^4 - \omega^4 \sqrt{y^4 - 1}}.$$

This gives:

$$\int_{L/2}^L dr = L/2 = \frac{R^2}{u_0} \int_0^\infty \frac{dy}{\sqrt{y^4 - \omega^4 \sqrt{y^4 - 1}}},$$

(34)

or

$$u_0 = \frac{2R^2}{L} \int_0^\infty \frac{dy}{\sqrt{y^4 - \omega^4 \sqrt{y^4 - 1}}} = I(R, y, \omega)/L,$$

(35)

where (the integral $I(R, y, \omega)$ has been evaluated in [21]):

$$I(R, y, \omega) = 2R^2 \int_0^\infty \frac{dy}{\sqrt{y^4 - \omega^4 \sqrt{y^4 - 1}}}$$

$$= \frac{2R^2}{4(\omega^2)} \left[ K(\sqrt{\omega/2+1/(2\omega)+1/(\omega+1/\omega)}) - K(\sqrt{\omega/2+1/(2\omega)-1/(\omega+1/\omega)}) \right],$$

with $K$ being the complete elliptic integral of the first kind;

$y = u/u_0$, with $u_0$ being the minimum of $u$; and $\omega = a/u_0$.

The action in Eq. [31] can be written as the sum of two actions $S_1 + S_2$

$$S_2 = \frac{1}{2\pi} \int_{\Sigma_2} dtd\sigma \sqrt{u'^2 + \frac{u^4 V}{R^4}}$$

$$\times \left[ 1 + \frac{1}{2} \left( -\frac{u^4}{R^4} h_{tt} + 5Wu'^2 \right) F_0 \right]$$

(37)

In $S_2$, perturbation terms in the integrand lead to second order terms which involve the product of perturbations (the domain $\Sigma_2$ already contains the perturbation to the metric). Thus ignoring the perturbation terms in the metric for $S_2$, This leads to

$$S_2 = \frac{1}{2\pi} \int_{\Sigma_2} dtd\sigma \sqrt{u'^2 + \frac{u^4 V}{R^4}}$$

(38)

The action $S_1$ is further split into two:

$$S_{1a} = \frac{1}{2\pi} \int dtd\sigma \sqrt{u'^2 + \frac{u^4 V}{R^4}}$$

(39)

$$S_{1b} = \frac{1}{2\pi} \int dtd\sigma \left[ \frac{1}{2} \left( -\frac{u^4}{R^4} h_{tt} + 5Wu'^2 \right) F_0 \right]$$

(40)

The action $S_{1a}$, can be evaluated using standard techniques. The action $S_{1b}$ has to be evaluated numerically.

**B. Calculation of $S_{1a}$**

Using standard techniques [12, 13], and after subtracting the contribution of the self energy of the heavy quark and anti-quark system to the action, it can be seen that

$$S_{1a} = 2\beta I(R, y, \omega) \left( \int_1^\infty dy \left( \frac{y^2}{\sqrt{y^4 - 1}} - 1 \right) - 1 \right)$$

$$= \beta I(R, y, \omega) \left( \frac{(2\pi)^{3/2}}{\Gamma(1/2^2)} \right).$$

(41)

**C. Convergence of $S_{1b}$**

While this has to be evaluated numerically, it is possible to confirm that it converges.
\[ S_{1b} = \frac{1}{2\pi} \int dt d\sigma \sqrt{\frac{u^4 V}{R^4} + u'^2} \]
\[ \times \frac{1}{2} \left( \left[ \frac{-u'^2 h_{tt}}{R^4 + u'^2} \right] + \left[ \frac{5Wu'^2}{R^4 + u'^2} \right] \right) F_0. \] (42)

Since the denominator terms in the integrand are always positive, \( \frac{-u'^2 h_{tt}}{R^4 + u'^2} < \frac{1}{|u'|} \) and \( \frac{5Wu'^2}{R^4 + u'^2} < 5|W| \).

Hence,
\[ S_{1b} < \frac{1}{2\pi} \int dt d\sigma \frac{F_0}{2} \left( -\frac{h_{tt}}{V} + 5W \right) \sqrt{\frac{u^4 V}{R^4} + u'^2}. \]

From Sec. III B and III C we know that \( h_{tt} \) and \( W \) are bounded. Thus,
\[ S_{1b} < (|h_{tt}|_{\text{max}} + |5W|_{\text{max}}) \frac{F_0}{2} \]
\[ \times \frac{1}{2\pi} \int dt d\sigma \sqrt{\frac{u^4 V}{R^4} + u'^2} \] (43)

After subtracting the self-energy of the quarks from \( S_{1b} \), it evaluates to:
\[ S_{1b} < (|h_{tt}|_{\text{max}} + |5W|_{\text{max}}) \frac{F_0}{2} \]
\[ \times \beta \frac{I(R, y, \omega)}{2\pi L} \left( \frac{(-2\pi)^{3/2}}{1/(1/4)^2} \right), \] (44)

Hence \( S_{1b} \) converges.

**D. Calculation of Action \( S_2 \)**

As discussed, for calculation of the action \( S_2 \), we continue with the same metric in Eq. 5 for both the particles. The string action for the triangular region, with \( \tau = \beta \) and \( \sigma = x_\parallel \), would be given by,
\[ S_2 = \frac{1}{2\pi} \int_{\Sigma_2} d\sigma d\tau \sqrt{u'^2 + \frac{u^4 V}{R^4}}, \]
where \( \Sigma_2 \) represents the triangular region shown in Fig. 5. As can be seen in Fig. 5, the string contours in triangular region are most naturally represented in polar coordinates. The representation of the triangular region with a sector, and using the polar coordinates is the second approximation we make. Again, this approximation would give accurate results only if the temperature gradient is small. Converting to polar coordinates,
\[ S_2 = \frac{1}{2\pi} \int d\theta dr r \sqrt{u'^2 + \frac{u^4 V}{R^4}}, \] (45)

**FIG. 5:** Dashed lines represent the string contours in top view in the triangular region. The triangular region is approximated as a sector of a circle.

where, \( u \) and \( V \) are now expressed in polar coordinates. From the symmetry of the problem, \( u \) would depend on \( r \) alone. The Lagrangian explicitly depends on \( r \), and hence the Hamiltonian would also explicitly depend on \( r \). The Hamiltonian, \( H^a = \frac{rV u^4 / R^4}{\sqrt{u'^2 + V u'^4 / R^4}} \).

We express the above Hamiltonian as \( H^a = rH_0^a \). We have separated out the constant \( H_0^a \) from the \( r \) dependent Hamiltonian. If \( u_0 \) is the value of \( u \) at the minima, then \( H_0^a = \frac{u_0^5 \sqrt{V}}{R^2} \). This gives
\[ rH_0^a = \frac{rV u^4 / R^4}{\sqrt{u'^2 + V u'^4 / R^4}}. \]

Substituting \( H_0^a \) and solving for \( u \),
\[ \frac{u'}{u_0} = \frac{u_0}{R^2} \sqrt{\frac{u'^4}{u_0^4} - \frac{a^4}{u_0^4}} \sqrt{\frac{u'^4}{u_0^4} - 1}. \]

It is seen that the explicit \( r \) dependence is eliminated. Again, defining \( y = u / u_0 \), and \( \omega = a / u_0 \),
\[ y' = \frac{u_0}{R^2} \sqrt{\frac{y'^4}{\omega^4} - \omega^4} \sqrt{\frac{y'^4}{\omega^4} - 1}. \]

This gives:
\[ \int_{L/2}^{L} dr = \frac{R^2}{u_0} \int_\omega^\infty \frac{dy}{\sqrt{y'^4 - \omega^4} \sqrt{y'^4 - 1}}. \] (46)
or

\[
u_0 = \frac{2R^2}{L} \int_\omega^\infty \frac{dy}{\sqrt{y^2 - \omega^4 y^4 - 1}} = I(R, y, \omega)/L,
\]

(47)

with \(I(R, y, \omega)\) as defined earlier.

Substituting the expression for \(y'\) and \(dr\) in the action in Eq. 45, and noting that the integrand is independent of \(\theta\), we get the result,

\[
S_2 = \frac{u_0}{2\pi} \theta_0 \left( \int_1^{u/u_0} dy r(y) \frac{y^2}{\sqrt{y^4 - 1}} \right.
+ \left. \int_0^1 dy r(y) \frac{y^2}{\sqrt{y^4 - 1}} \right).
\]

The string extends from \(\infty\) to \(u_0\) and then from \(u_0\) to \(\infty\). Hence, the integration is divided into two limits, \(1\) to \(u/u_0\) and \(u_0/u\) to \(1\). The heavy quark antiquark system self energy needs to be subtracted.

The self energy contribution to the action (and remembering that \(u\) is now in polar coordinates) =

\[
S_{SE} = \frac{1}{2\pi} \int_0^{\theta_0} r\theta_r \text{d}u + \frac{1}{2\pi} \int_0^{\theta_0} r\theta_r \text{d}u + L\theta_0 \frac{u_0}{2\pi} = u_0 \theta_0 \int_1^{u/u_0} dy + u_0 \theta_0 \int_0^1 dy + u_0 L \theta_0 \frac{u_0}{2\pi}.
\]

After subtracting \(S_{SE}\) from \(S_2\), \(S_2\) becomes,

\[
S_2 = \frac{u_0}{2\pi} \theta_0 \left( \int_1^{u/u_0} dy r(y) \left( \frac{y^2}{\sqrt{y^4 - 1}} - 1 \right) \right.
+ \left. \int_0^1 dy r(y) \left( \frac{y^2}{\sqrt{y^4 - 1}} - 1 \right) - L \frac{u_0}{2\pi} \right).
\]

As \(y\) ranges from \(u/u_0\) to \(1\), \(r\) ranges from \(0\) to \(L/2\), and as \(y\) ranges from \(1\) to \(u/u_0\), \(r\) ranges from \(L/2\) to \(L\). Substituting the expression for \(u_0\) and recognizing that for small \(\theta_0\), \(\theta_0 = \frac{\Delta \beta}{\sqrt{r^2}}\),

Integrating by parts,

\[
S_2 = \frac{I(R, y, \omega) \Delta \beta}{2\pi L^2} \left\{ \left[ \int_1^{u/u_0} dy r(y) \left( \frac{y^2}{\sqrt{y^4 - 1}} - 1 \right) \right]_{y=1}^{y=\infty}
+ \left[ \int_0^1 dy r(y) \left( \frac{y^2}{\sqrt{y^4 - 1}} - 1 \right) \right]_{y=\infty}^{y=L} \right. \]

\[
- \frac{I(R, y, \omega) \Delta \beta}{2\pi L^2} \int_0^\infty dy \int_0^1 dy \left( \frac{y^2}{\sqrt{y^4 - 1}} - 1 \right).
\]

Solving the integrals,

\[
S_2 = \frac{I(R, y, \omega) \Delta \beta}{2\pi L^2} \left\{ \left[ \int_1^{u/u_0} dy r(y) \left( \frac{y^2}{\sqrt{y^4 - 1}} - 1 \right) \right]_{y=1}^{y=\infty}
+ \left[ \int_0^1 dy r(y) \left( \frac{y^2}{\sqrt{y^4 - 1}} - 1 \right) \right]_{y=\infty}^{y=L} \right. \]

\[
- \frac{I(R, y, \omega) \Delta \beta}{2\pi L^2} \int_0^\infty dy \int_0^1 dy \left( \frac{y^2}{\sqrt{y^4 - 1}} - 1 \right).
\]

with \(C_0\) being a positive constant and is equal to

\[
2 \int_1^\infty dy \int_0^1 dy \left( \frac{y^2}{\sqrt{y^4 - 1}} - 1 \right).
\]

The \(L^{-1}\) term is conformally invariant. The \(L^{-2}\) term is however not conformally invariant. The \(L^{-2}\) term is intriguing, and one needs to determine if its a consequence of mathematical approximations, or if it indeed breaks conformal invariance. A more detailed probe is therefore required.

From Eqs. 41, 42 and 48 and taking \(\beta \sim 1/\Theta\), the final potential between the heavy quark and anti-quark is:

\[
<V_{QQ}(L, T) >= S/\beta \sim S\Theta
\]

\[
= I(R, y, \omega) \left\{ \int_1^{u/u_0} dy r(y) \left( \frac{y^2}{\sqrt{y^4 - 1}} - 1 \right) \right. \]

\[
+ \left. \int_0^1 dy r(y) \left( \frac{y^2}{\sqrt{y^4 - 1}} - 1 \right) - L \frac{u_0}{2\pi} \right\}.
\]

In terms of the linear perturbation of the black string metric, from Eq. 40 it becomes,

\[
<V_{QQ}(L, T) >=
\]

\[
= I(R, y, \omega) \left\{ \int_1^{u/u_0} dy r(y) \left( \frac{y^2}{\sqrt{y^4 - 1}} - 1 \right) \right. \]

\[
+ \left. \int_0^1 dy r(y) \left( \frac{y^2}{\sqrt{y^4 - 1}} - 1 \right) - L \frac{u_0}{2\pi} \right\}.
\]

where it is understood that heavy quark self energy is subtracted from \(S_{1b}\).
V. CONCLUSIONS

The effect of a temperature gradient on the heavy quark, anti-quark potential has been analyzed and calculated. The temperature gradient in the dual AdS space is obtained by inserting a non-uniform black string. The non-uniform black string metric is developed by perturbing the Schwarzchild solution. The modification in the analytical part of the potential due to the temperature gradient is seen to be proportional to ∆β. The term contains a conformally invariant $L^{-1}$ term and a non-invariant $L^{-2}$ term. The $L^{-2}$ term needs to be probed further, which we plan to do in future. This calculation, provides the heavy quark anti-quark potential, for a $\mathcal{N}=4$ supersymmetric Yang-Mills Lagrangian, instead of the QCD Lagrangian.

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