Mathematical modelling of two layer shallow water flow with incline and uneven bottom

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Abstract. This study aims to construct mathematical model of two layer shallow water flow based on physical phenomenon. The proposed model is with attention to incline and uneven bottom, usually is chosen as function of space \( b(x) \). The result is the governing equation of each layer consist of mass continuity and momentum equations. The layers of the model are determined by the density of the fluid. The first layer is water exposed to sunshine which the temperature is higher than the second layer. The difference of density between the first and the second layer results the difference of gravity force which in the second layer we use reduced gravity force (gravity force decreases due to density ratio).

1. Introduction

The definition of shallow water is a fluid layer with constant density where the horizontal scale of the flow is greater than the layer depth. The shallow water describes a thin layer of constant density fluid in hydrostatic balance that is bounded from below by a rigid surface and from above by a free surface, which is assumed to contact with another fluid of negligible inertia. The single layer model is one of the simplest useful model in geophysical fluid dynamics, because it allows to consider the effect of rotation in a simple framework without considering the complicated effect of stratification. By adding layers, we can subsequently study the effects of stratification. The two layer model not just a simple model of a stratified fluid, it is a surprisingly good model of many phenomena in the ocean and atmosphere [7].

Fluid flow can be described as a multi layer of fluid flow which one layer flows over another layer to form a different layer. This layer difference can be caused by temperature difference resulting different density. For example, water in the ocean can be considered as a two layer fluid. The first layer is water that is exposed to the sunshine such that it has higher temperature than the second layer water below the first layer [7].

The two layer fluid flow model over the incline plane was first studied by Pascal [4], which was applied to a flat bottom. The non-Newtonian fluid is used in the model that is not too thick and applied to the mud flow in water. Then the fluid flow model with inclination was developed by Yadav and Usha [8] for uneven bottom. In this model it is assumed that the fluid consists of only single layer, and it flows in two directions, namely horizontal direction and vertical direction.

On the other hand, a shallow water flow model for flat bottom in the strait of Gilbraltar was derived by Chakir, et al [1] by considering wind stresses. In this model it is assumed that the flow is quasi-horizontal neglecting the \( y \) component, and the density is uniform in each layer. Chiapolino and Saurel [2] also derived two layer shallow water flow model with some limitations, namely by neglecting the vertical velocity component and by assuming that the velocity is uniform in cross sections of each layer.
Different from the previous researcher, we propose a mathematical model for two layer shallow water flow with inclination and uneven bottom. Boundary condition is applied by referring Chiapolino and Saurel [2]. The model shows a variety of features, because it has many conservation laws. Based on the physical phenomenon, the governing equation is derived by apply conservation laws of mass and momentum as presented in Section 2. This model can be applied to river flow as explained in Section 3.

2. Method
The proposed method in this study is to construct model based on physical phenomenon. The governing equation of each layer that consists of mass continuity, momentum equations, and the boundary conditions. The governing equation is integrated from the bottom of domain to the free surface. The steps this study are shown in Figure 1.

2.1. Physical phenomenon: two layer shallow water flows with incline and uneven bottom
The problem of two layer shallow water flow with an inclined and uneven bottom is illustrated in Figure 2. The pressure of the upper layer is lower than the under layer. The domain of the first layer is \( \eta_2(x,t) \leq z \leq \eta_1(x,t) \) and the second layer is \( b(x) \leq z \leq \eta_2(x,t) \), where \( \eta_1(x,t) \) is the vertical displacement of free surface and \( \eta_2(x,t) \) is the height of the interface. \( h_1(x,t) \) and \( h_2(x,t) \) is the thickness of the liquid column. In a flat bottom \( \eta_2(x,t) = h_2(x,t) \), whereas in general \( h_2(x,t) = \eta_2(x,t) - b(x) \), where \( b(x) \) describes the uneven bottom topography. The inclination of the bottom is represented by the angle \( \theta \).

The fluid motion is fully determined by the mass continuity and the momentum equations. In this model, the fluid flows in \( x \) and \( z \) directions respectively, hence we have mass continuity of each layer as follows

\[
\frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z} = 0, \tag{1}
\]

\[
\frac{\partial u_2}{\partial x} + \frac{\partial w_2}{\partial z} = 0, \tag{2}
\]
Figure 2. A sketch of the two-layer model

where \( u_i, w_i \) represent the components of the velocity along the \( x \) and \( z \) directions respectively, \( i = 1, 2 \).

In the momentum equation, the hydrostatics approximation is well satisfied since the vertical velocity is small enough. Let the forces acting on the fluid volume element are only the surface force and body force. In actual circumstances, body force can be in the form of gravity force. Furthermore, for incline case, we can see in Figure 3 which the gravity force \( f_g \) in \( x \) and \( z \) direction are

\[
\begin{align*}
    f_{gx} &= w \sin \theta, \\
    f_{gz} &= -w \cos \theta,
\end{align*}
\]

where \( w = mg \) is weight of an object, \( m = \rho \Delta x \Delta y \Delta z \) denotes mass of object, and \( g \) is gravity.

Figure 3. Gravity force on the incline bottom

Therefore the body force in \( x \) and \( z \) directions respectively are given as follows

\[
\begin{align*}
    f_{gx} &= \rho g \sin \theta \Delta x \Delta y \Delta z, \\
    f_{gz} &= -\rho g \cos \theta \Delta x \Delta y \Delta z,
\end{align*}
\]
where \( \rho \) is mass density. The momentum equation are

\[
\rho_1 \left( \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + w_1 \frac{\partial u_1}{\partial z} \right) = -\frac{\partial p_1}{\partial x} + \rho_1 g \sin \theta, \quad (3)
\]

\[
\rho_1 \left( \frac{\partial w_1}{\partial t} + u_1 \frac{\partial w_1}{\partial x} + w_1 \frac{\partial w_1}{\partial z} \right) = -\frac{\partial p_1}{\partial z} - \rho_1 g \cos \theta, \quad (4)
\]

\[
\rho_2 \left( \frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + w_2 \frac{\partial u_2}{\partial z} \right) = -\frac{\partial p_2}{\partial x} + \rho_2 g \sin \theta, \quad (5)
\]

\[
\rho_2 \left( \frac{\partial w_2}{\partial t} + u_2 \frac{\partial w_2}{\partial x} + w_2 \frac{\partial w_2}{\partial z} \right) = -\frac{\partial p_2}{\partial z} - \rho_2 g \cos \theta, \quad (6)
\]

where \( \rho \) denotes the pressure with \( p_1 < p_2 \) and \((f_{gx}, f_{gz})\) are the gravity force component of \( x \) and \( z \). Since at the top of layer is the ambient fluid, the pressure on the fluid at the top is zero

\[
p_1 = 0, \quad \frac{\partial u_1}{\partial z} = 0, \quad \text{at} \quad z = \eta_1(x,t), \quad (7)
\]

where \( z = \eta_1(x,t) \) is the elevation of the free surface. At the interface between the two layers we have

\[
p_1 = p_2, \quad \frac{\partial u_2}{\partial z} = 0, \quad \text{at} \quad z = \eta_2(x,t), \quad (8)
\]

where \( z = \eta_2(x,t) \) is the elevation of the interface. We also have the condition of the velocity

\[
u_1 = u_2, \quad w_1 = w_2 \quad \text{at} \quad z = \eta_2(x,t). \quad (9)
\]

The kinematic conditions at the surface of the ambient fluid and at the interface between the two fluids can be expressed as

\[
\frac{\partial \eta_1}{\partial t} + u_1 \frac{\partial \eta_1}{\partial x} = w_1, \quad \text{at} \quad z_1 = \eta_1(x,t), \quad (10)
\]

\[
\frac{\partial \eta_2}{\partial t} + u_1 \frac{\partial \eta_2}{\partial x} = w_1, \quad \text{at} \quad z_1 = \eta_2(x,t), \quad (11)
\]

\[
\frac{\partial \eta_2}{\partial t} + u_2 \frac{\partial \eta_2}{\partial x} = w_2, \quad \text{at} \quad z_2 = \eta_2(x,t), \quad (12)
\]

\[
u_2 \frac{\partial b}{\partial x} = w_2, \quad \text{at} \quad z_2 = b(x). \quad (13)
\]

### 2.2. Construction mathematical model of two layer shallow water flows with incline and uneven bottom

In the shallow waters, the vertical velocity in the \( z \) direction is very small \( \left( \frac{Dw}{Dt} \ll 0 \right) \), such that equation (4) and (6) can be written as

\[
0 = \frac{\partial p_1}{\partial z} - \rho_1 g \cos \theta, \quad (14)
\]

\[
0 = -\frac{\partial p_2}{\partial z} - \rho_2 g \cos \theta. \quad (15)
\]

In the each layer, pressure is given by the hydrostatic approximation. To get the pressure equations, we integrate equation (14) using the boundary conditions (7). The first layer equation is

\[
\int \frac{\partial p_1}{\partial z} dz = - \int \rho_1 g \cos \theta dz, \quad p_1 = -\rho_1 g \cos \theta + C, \quad p_1 = \rho_1 g (\eta_1 - z) \cos \theta. \quad (16)
\]
In the same way, we can find the pressure equation for the second layer by integrating equations (15) and using the boundary conditions (8)

\[
    p_2 = \rho_1 g \cos \theta \eta_1 - \rho_2 g \cos \theta \eta_z + g \cos \theta (\rho_2 - \rho_1) \eta_2.
\]  

(17)

The term involving \( z \) is irrelevant for the dynamics, because only the horizontal derivative is in the equation of motion. We yield new equation by inserting these expressions into the \( x \)-momentum equations (3) and (5).

\[
\begin{align*}
\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + w_1 \frac{\partial u_1}{\partial z} &= -\rho_1 g \cos \theta \frac{1}{\rho_1} \frac{\partial \eta_1}{\partial x} + g \sin \theta, \\
\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + w_2 \frac{\partial u_2}{\partial z} &= (g' - g) \cos \theta \frac{\partial \eta_1}{\partial x} - g' \cos \theta \frac{\partial \eta_2}{\partial x} + g \sin \theta.
\end{align*}
\]  

(18)

(19)

where \( g' = \frac{g(\rho_2 - \rho_1)}{\rho_2} \) is the reduced gravity (gravity force decreases due to density ratio).

The continuity equations (1) and (2), and the momentum equations (18) and (19) are integrated with depth. The equations are integrated using the Leibniz rule [3], as follows:

\[
\frac{\partial}{\partial t} \int_{\alpha(t)}^{\beta(t)} f(z,t) dz = \int_{\alpha(t)}^{\beta(t)} \frac{\partial f(z,t)}{\partial t} dz + f(\beta(t),t) \frac{\partial \beta}{\partial t} - f(\alpha(t),t) \frac{\partial \alpha}{\partial t}.
\]

Therefore, the integral equations of (1), (2), (18), and (19) are written as

\[
\frac{\partial}{\partial t} \left( \eta_1(x,t) - \eta_2(x,t) \right) + \frac{\partial}{\partial x} \int_{\eta_1(x,t)}^{\eta_2(x,t)} u_1(x,z,t) dz = 0,
\]

(20)

\[
\frac{\partial}{\partial t} \left( \eta_2(x,t) - \eta_1(x,t) \right) + \frac{\partial}{\partial x} \int_{\eta_1(x,t)}^{\eta_2(x,t)} u_2(x,z,t) dz = 0,
\]

(21)

\[
\frac{\partial}{\partial t} \int_{\eta_1(x,t)}^{\eta_2(x,t)} u_1(x,z,t) dz + \frac{\partial}{\partial x} \left( \int_{\eta_1(x,t)}^{\eta_2(x,t)} u_1^2(x,z,t) dz + \frac{1}{2} \rho_1 \cos \theta \eta_1^2 \right)
\]

\[
= g \sin \theta (\eta_1(x,t) - \eta_2(x,t)) + g \cos \theta \eta_2 \frac{\partial \eta_1}{\partial x},
\]

(22)

\[
\frac{\partial}{\partial t} \int_{\eta_1(x,t)}^{\eta_2(x,t)} u_2(x,z,t) dz + \frac{\partial}{\partial x} \left( \int_{\eta_1(x,t)}^{\eta_2(x,t)} u_2^2(x,z,t) dz + \frac{1}{2} \rho_2 \cos \theta \eta_2^2 \right)
\]

\[
= (g' - g) \cos \theta \frac{\partial \eta_1}{\partial x} + g' \cos \theta \eta_2 \frac{\partial \eta_2}{\partial x} + g \sin \theta (\eta_2 - \eta_1).
\]

(23)

If we define

\[
\begin{align*}
\bar{u}_1 &= \frac{1}{\eta_1 - \eta_2} \int_{\eta_1}^{\eta_2} u_1 dz, \\
\bar{u}_2 &= \frac{1}{\eta_2 - \eta_1} \int_{\eta_2}^{\eta_1} u_2 dz,
\end{align*}
\]

\[
\begin{align*}
q_1 &= (\eta_1 - \eta_2) \bar{u}_1, \\
q_2 &= (\eta_2 - \eta_1) \bar{u}_2.
\end{align*}
\]

where \( \bar{u}_1 \) and \( \bar{u}_2 \) are the depth-averaged velocities and \( q_1 \) and \( q_2 \) are water debit in the first and second
layer respectively. The equations (20)-(23) can be expressed as

\[ \frac{\partial}{\partial t} (\eta_1 - \eta_2) + \frac{\partial q_1}{\partial x} = 0, \]  
\[ \frac{\partial \eta_1}{\partial t} + \frac{\partial q_2}{\partial x} = 0, \]  
\[ \frac{\partial q_1}{\partial t} + \frac{\partial}{\partial x} \left( \frac{q_1^2}{(\eta_1 - \eta_2)} + \frac{1}{2} g \cos \theta \eta_1^2 \right) = \] 
\[ g \sin \theta (\eta_1 - \eta_2) + g \cos \theta \eta_2 \frac{\partial \eta_1}{\partial x}, \]  
\[ \frac{\partial q_2}{\partial t} + \frac{\partial}{\partial x} \left( \frac{q_2^2}{(\eta_2 - b)} + \frac{1}{2} g' \cos \theta \eta_2^2 \right) = \] 
\[ (g' - g) \cos \theta \frac{\partial \eta_1}{\partial x} (\eta_2 - b) + g' \cos \theta b \frac{\partial \eta_2}{\partial x} + g \sin \theta (\eta_2 - b). \]  

Equations (24)-(27) are called a two layer shallow water model with incline and uneven bottom in dimensional equation.

2.3. Non-dimensional variable
It is easier to express the equation of motion in a non-dimensional variable where we express every variable (such as velocity) as the ratio of its value to some reference value. The observed surface waves have small \( H \) amplitudes and the long \( L \) wave lengths, so it is needed to find shallow water equations on a scale variable. Scale is used to compare between real and actual circumstances with a model or picture. Now, the equations of system (24)-(27) are non-dimensionalized. The \((x, z)\) coordinates will be non-dimensionalized differently, recognizing the fact that there will be larger gradients in the vertical direction than the horizontal. Finally, we introduce the non-dimensional quantities

\[ x = L \tilde{x}, \quad z = H \tilde{z}, \quad t = \frac{L}{U} \tilde{t}, \quad \eta_1 = H \tilde{\eta}_1, \]  
\[ \eta_2 = H \tilde{\eta}_2, \quad b = H \tilde{b}, \quad q_1 = U H \tilde{q}_1, \quad q_2 = U H \tilde{q}_2, \]

where

\[ U = \left( H g' \right)^{\frac{1}{2}}, \quad L = \frac{U^2}{g}, \quad \gamma = \frac{g'}{g}. \]

3. Result and discussion
Consider the variety-density momentum equation in Cartesian coordinates. If a typical velocity is \( U \), a typical length is \( L \), and a typical time scale is \( T \), by using the non-dimensional quantities into equations (24)-(27) with chain rules and removing the tildes as non-dimensional variables, we obtain

\[ \frac{\partial \eta_1}{\partial t} - \frac{\partial \eta_2}{\partial t} + \frac{\partial q_1}{\partial x} = 0, \]  
\[ \frac{\partial \eta_1}{\partial t} + \frac{\partial q_2}{\partial x} = 0, \]  
\[ \frac{\partial q_1}{\partial t} + \frac{\partial}{\partial x} \left( \frac{q_1^2}{(\eta_1 - \eta_2)} + \frac{1}{2} \gamma \cos \theta \eta_1^2 \right) = \] 
\[ \sin \theta (\eta_1 - \eta_2) + \frac{\cos \theta}{\gamma} \eta_2 \frac{\partial \eta_1}{\partial x}, \]  
\[ \frac{\partial q_2}{\partial t} + \frac{\partial}{\partial x} \left( \frac{q_2^2}{(\eta_2 - b)} + \frac{1}{2} \gamma \cos \theta \eta_2^2 \right) = \] 
\[ \left( 1 - \frac{1}{\gamma} \right) \cos \theta (\eta_2 - b) \frac{\partial \eta_1}{\partial x} + \cos \theta b \frac{\partial \eta_2}{\partial x} + \sin \theta (\eta_2 - b). \]
Equations (28) - (31) is non-dimensional from of two layers shallow water model with incline and uneven bottom. For the first layer (upper layer), the shallow water flow is expressed by equations (28) and (30) that shows continuity and momentum equations. This equations shows fluid flow at free surface. For second layer, the layer between upper and bottom layer, the mathematical model of the shallow water is expressed by equations (29) and (31). The continuity equation at the first layer, the depth of water is affected by the height of the first and second layers, whereas for the bottom layer is only affected by the height of the second layer. The affected on second layer because the bottom surface is steady state. In the momentum equation, the pressure of the first layer is influenced by the height of the first and second layers. Furthermore, the pressure force of the second layer is influenced by the height of the free surface (first layer), the second layer, and the bottom surface. Because of this reason, the pressure at the first layers is smaller than the pressure at the second layer ($p_1 < p_2$). A fluid of density $\rho_1$ lies over a denser fluid of density $\rho_2$. In the reduced gravity case, the lower layer may be arbitrarily thick and is assumed stationary so that it does not have a horizontal pressure gradient.

The pressure is continuous across the interface, but the density jumps discontinuously. This allows the horizontal velocity to have a corresponding discontinuity. Thus in this research, shallow water is assumed steady state. Shallow water in steady state is if the mass flux over the depth and the mechanical energy at every point on the water surface is constant [6]. The condition following in this paper is assumed the water debit and height are constan with respect to time [5]. Equations (28) - (31) becomes

\[
\begin{align*}
\frac{\partial q_1}{\partial x} &= 0, \\
\frac{\partial q_2}{\partial x} &= 0, \\
\frac{\partial}{\partial x} \left( \frac{q_1^2}{\eta_1 - \eta_2} + \frac{1}{2\gamma} \cos \theta \eta_1^2 \right) &= \sin \theta (\eta_1 - \eta_2) + \frac{\cos \theta}{\gamma} \eta_2 \frac{\partial \eta_1}{\partial x}, \\
\frac{\partial}{\partial x} \left( \frac{q_2^2}{(\eta_2 - b)} + \frac{1}{2} \cos \theta \eta_2^2 \right) &= \left( 1 - \frac{1}{\gamma} \right) \cos \theta (\eta_2 - b) \frac{\partial \eta_1}{\partial x} + \cos \theta b \frac{\partial \eta_2}{\partial x} + \sin \theta (\eta_2 - b).
\end{align*}
\]

The steady state of equation (32) - (35) can be applied to river at rest conditions. The river can be seen as two layers. When exposed to sunshine, the temperature in the first layer is smaller than the temperature in the second layer. This causes the pressure in the first layer to be smaller than the pressure in the second layer. When the interface between two layers varies with position (when it is wavy), the layers exert a pressure force on each other. Likewise when the uneven bottom, the topography and the bottom layer will in general exert forces on each other. due to steady state, so changes in the first and second layers follow an uneven bottom shape.

4. Conclusion
In this paper, we have constructed the model of shallow water with two layer. We derived the mathematical model based on its physical phenomenon that is with incline and uneven bottom. The governing equation of each layer consists of mass continuity and momentum equations. The layers of the model are determined by the density of the fluid. The difference of density between the first and the second layer results in differences of gravity force which in the second layer we use reduced gravity force. The pressure at the first layer is smaller than the pressure at the second layer ($p_1 < p_2$). The two layer shallow water equation is obtained by changing the variable of dimensional to non-dimensional.

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