ZERMELO NAVIGATION ON RIEMANNIAN MANIFOLDS

DAVID BAO*, COLLEEN ROBLES** AND Z. SHEN

Abstract. In this paper, we study Zermelo navigation on Riemannian manifolds and use that to solve a long standing problem in Finsler geometry. Namely, the complete classification of strongly convex Randers metrics of constant flag curvature.

1. Introduction

1.1. Purpose. We have three goals in this paper.

The first is to describe Zermelo’s problem of navigation on Riemannian manifolds. Zermelo aims to find the paths of shortest travel time in a Riemannian manifold \((M, h)\), under the influence of a wind or a current which is represented by a vector field \(W\). We point out that the solutions are the geodesics of a strongly convex Finsler metric, which is of Randers type and is necessarily non-Riemannian unless \(W\) is zero. Conversely, we show constructively that every strongly convex Randers metric arises as the solution to Zermelo’s navigational problem on some Riemannian landscape \((M, h)\), under the influence of an appropriate wind \(W\). This is the content of Proposition 1 in §2.3.

Randers metrics are interesting not only as solutions to Zermelo’s problem of navigation. They form a ubiquitous class of metrics with a strong presence in both the theory and applications of Finsler geometry. Of particular interest are Randers metrics of constant flag curvature, the latter being the Finslerian analog of the Riemannian sectional curvature.

It is the second goal of this paper to describe strongly convex Randers metrics of constant flag curvature via Zermelo navigation. Unlike previous characterization results [BR03, MS02], the navigation description has the advantage of clearly illuminating the underlying geometry. More precisely, suppose \((h, W)\) is the navigation data of a strongly convex Randers metric \(F\). Then: \(F\) has constant flag curvature \(K\) if and only if there exists a constant \(\sigma\) such that, \(h\) has constant sectional curvature \(K + \frac{1}{16}\sigma^2\) and \(W\) satisfies the equation \(L_W h = -\sigma h\) (namely, \(W\) is an infinitesimal homothety of \(h\)). This is Theorem 3 in §4.4 of the paper.

Our third goal has two components. (1) Use the navigation description to classify Randers metrics of constant flag curvature. This problem was proposed by Ingarden

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about half a century ago. Until 2002, it was erroneously thought to have been solved by Yasuda–Shimada in 1977; see [BR03, MS02] for references therein. Our classification lists, explicitly, all the vector fields $W$ that will perturb a Riemannian space form $h$ into a strongly convex Randers metric of constant flag curvature. This is done in Theorem 7 of §6.1. (2) Parametrize the moduli space for the equivalence classes of locally isometric $n$-dimensional Randers metrics with constant flag curvature $K$. In striking contrast with the Riemannian setting – where the moduli space consists of a single point for each value of $K$ – the Randers moduli space is of dimension $n/2$ when $n$ is even, and either $(n+1)/2$ or $(n-1)/2$ when $n$ is odd. The specifics are detailed in Propositions 8 (§7.2.5), 9 (§7.3.1), 10 (§7.4.4), respectively for $K$ positive, zero, or negative.

Finally, we test the usefulness of the classification by applying it to two special cases.

- First, those $W$ which effect projectively flat strongly convex Randers metrics of constant flag curvature $K$ are singled out. We find that up to local isometry, the non-Riemannian ones consist of a 1-parameter family of locally Minkowskian metrics when $K = 0$, and a single variant of the Funk metric for each $K < 0$. In particular, every projectively flat strongly convex Randers metric of constant positive flag curvature must be locally isometric to a standard (Riemannian) sphere. This discussion constitutes §8.3. Our conclusions are also compared with the main result of Shen [S02a].

- Next, the general classification is specialized to the case in which the tensor $\theta_i := b^s \text{curl}_s$ vanishes. This enables us to list explicitly all the Randers metrics addressed by systems of nonlinear partial differential equations in the corrected Yasuda–Shimada theorem [BR03, MS02]. Such is the thesis of §9.2. We find that strongly convex non-Riemannian Randers metrics of constant flag curvature $K$ and $\theta = 0$ comprise, up to local isometry, three small but distinguished camps.
  - $K < 0$: there is just a single variant of the Funk metric for each $K$.
  - $K = 0$: the Yasuda–Shimada theorem tells us that only locally Minkowski metrics belong to this camp; we show that, in fact, there is simply a one parameter family of them.
  - $K > 0$: this is the most enigmatic case. There is exactly a 1-parameter family of the $\theta = 0$ metrics on the odd dimensional spheres, and none on the even dimensional spheres. Furthermore, such conclusion holds whether the metrics being sought are locally or globally defined.

The classification of the $K > 0$ metrics within the $\theta = 0$ family has previously been done by Bejancu–Farran [BF02, BF03]. However, our description of the local isometry classes offers a perspective which is totally different from theirs.

1.2. Summary of contents. Section 2 presents Zermelo’s problem of navigation on Riemannian manifolds, and its solution.

We specialize to concrete 3-dimensional Riemannian space forms in §3. These examples are categorized into three subsections, dealing with spheres, Euclidean space, and the Klein model of hyperbolic geometry, respectively. For each model we present examples of Zermelo’s navigation which produce Randers metrics of constant flag curvature. In §3.4, we review the definition of Finsler metrics of constant flag curvature.
Section 4 begins by recalling a previous characterization result. It also includes a Matsumoto identity which exhibits the interplay between the constant $\sigma$ (in the equation $\mathcal{L}_W h = -\sigma h$) and the constant flag curvature $K$. This is followed by the navigation description of strongly convex Randers metrics of constant flag curvature $K$.

Before presenting the classification theorem we pause in §5 to derive a complete list of allowable vector fields for each of the three standard models of Riemannian space forms. With the list in hand, §6 gives the classification of strongly convex Randers metrics of constant flag curvature; both local and global aspects are treated. The isometry classes of constant flag curvature Randers metrics comprise the focus of §7. We make explicit the requisite Lie theory (mostly for a non-compact subgroup of the Lorentz group) in the Appendix (§10), and (then) give concrete descriptions of the moduli space and its dimension in §7. Section 8 contains a discussion of projectively flat Randers metrics of constant flag curvature. Finally, in §9, we specialize our classification to the $\theta = 0$ case.

The dimension counts established in sections 7 through 9 can in essence be summarized by the following table:

| Moduli space’s dimension | CFC metrics | dim $M$ | $K > 0$ | $K = 0$ | $K < 0$ |
|--------------------------|-------------|---------|---------|---------|---------|
| Riemannian $b$ equiv. $W = 0$ | $n \geq 2$ | 0 | empty |
| Projectively flat | $n \geq 2$ | $0^*$ | $1$ | $0^*$ | $0^!$ |
| Yasuda–Shimada $\theta = 0$ | even $n$ | $0^*$ | $1$ | $0^*$ | $0^!$ |
| odd $n$ | 1 | | |
| Unrestricted Randers | even $n$ | $n/2$ | |
| odd $n$ | $(n + 1)/2$ | $(n - 1)/2$ | |

* The moduli spaces of dimension $\theta$ consist of a single point.
† The single isometry class is Riemannian.
‡ The single isometry class is non-Riemannian, of Funk type.

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2. **Zermelo navigation**

2.1. **Perturbing Riemannian metrics by vector fields.**

2.1.1. **Background Riemannian metric and perturbing vector field.** Given any Riemannian metric $h$ on a differentiable manifold $M$, denote the corresponding norm-squared of tangent vectors $y \in T_x M$ by

$$|y|^2 := h_{ij} y^i y^j = h(y, y).$$
Think of $|y|$ as the time it takes, using an engine with a fixed power output, to travel from the base(point) of the vector $y$ to its tip. Note the symmetry property $|−y| = |y|$.

The unit tangent sphere in each $T_yM$ consists of all those tangent vectors $u$ such that $|u| = 1$. Now introduce a vector field $W$ such that $|W| < 1$, thought of as the spatial velocity vector of a mild wind on the Riemannian landscape $(M, h)$. Before $W$ sets in, a journey from the base to the tip of any $u$ would take 1 unit of time, say, 1 second. The effect of the wind is to cause the journey to veer off course (or merely off target if $u$ is collinear with $W$). Within the same 1 second, we traverse not $u$ but the resultant $v = u + W$ instead.

As an example, suppose $|W| = \frac{1}{2}$. If $u$ points along $W$ (that is, $u = 2W$), then $v = \frac{3}{2}u$. Alternatively, if $u$ points opposite to $W$ (namely, $u = −2W$), then $v = \frac{5}{2}u$. In these two scenarios, $|v|$ equals $\frac{3}{2}$ and $\frac{5}{2}$ instead of 1. So, with the wind present, our Riemannian metric $h$ no longer gives the travel time along vectors. This prompts the introduction of a function $F$ on the tangent bundle $TM$, in order to keep track of the travel time needed to traverse tangent vectors $y$ under windy conditions. For all those resultants $v = u + W$ mentioned above, we have $F(v) = 1$. In other words, within each tangent space $T_xM$, the unit sphere of $F$ is simply the $W$-translate of the unit sphere of $h$. Since this $W$-translate is no longer centrally symmetric, $F$ cannot possibly be Riemannian.

2.1.2. Formula for the new norm $F$. Start with the fact $|u| = 1$; equivalently, $h(u, u) = 1$. Into this, we substitute $u = v − W$ and then $h(v, W) = |v||W|\cos θ$. After using the abbreviation $λ := 1 − |W|^2$ to reduce clutter, we have $|v|^2 − (2|W|\cos θ)|v| − λ = 0$. Since $|W| < 1$, the resultant $v$ is never zero, hence $|v| > 0$. This leads to $|v| = |W|\cos θ + \sqrt{|W|^2\cos^2 θ + λ}$, which we abbreviate as $p + q$. Since $F(v) = 1$, we see that

$$F(v) = 1 = |v|\frac{1}{q + p} = |v|\frac{q - p}{q^2 − p^2} = \frac{\sqrt{[h(W, v)]^2 + |v|^2λ}}{λ} − \frac{h(W, v)}{λ}.$$ 

It remains to deduce $F(y)$ for an arbitrary $y \in TM$. Note that every nonzero $y$ is expressible as a positive multiple $c$ of some $v$ with $F(v) = 1$. For $c > 0$, traversing $cv$ under the windy conditions should take $c$ seconds. Consequently, $F$ is positively homogeneous. Using this homogeneity and the formula derived for $F(v)$, we find that:

$$F(y) = \frac{\sqrt{[h(W, y)]^2 + |y|^2λ}}{λ} − \frac{h(W, y)}{λ}.$$ 

It is now manifest that $F(−y) ≠ F(y)$. By hypothesis, $|W| < 1$, hence $λ > 0$. We see from the formula for $F(y)$ that it is positive whenever $y ≠ 0$. Also, $F(0) = 0$ as expected.

2.1.3. New Riemannian metric and 1-form. Our formula for $F$ has two parts.

- The first term is the norm of $y$ with respect to a new Riemannian metric

$$a_{ij} = \frac{h_{ij}}{λ} + \frac{W_i W_j}{λ},$$

where $W_i := h_{ij} W^j$ and $λ = 1 − W^i W_i$. 

• The second term is the value on $y$ of a differential 1-form

$$b_i = \frac{-W_i}{\lambda}.$$ 

Under the influence of $W$, the most efficient navigational paths are no longer the geodesics of the Riemannian metric $h$; instead, they are the geodesics of the Finsler metric $F$. For $\mathbb{R}^2$, this phenomenon is treated by Carathéodory [C99] as Zermelo’s navigation problem [Z31]. Shen [S02] showed that the same phenomenon holds for arbitrary Riemannian backgrounds in all dimensions.

2.2. Ubiquitous class of Finsler metrics. The Finsler metric $F$ derived from the perturbation has the simple form

$$F = \alpha + \beta,$$

where

$$\alpha(x, y) := \sqrt{a_{ij}(x) y^i y^j}, \quad \beta(x, y) := b_i(x) y^i.$$ 

This is the defining feature of Randers metrics, which were introduced by Randers in 1941 [Ra41] in the context of general relativity, and later named by Ingarden [I57].

The function $F$ is positive on the manifold $TM \setminus 0$, whose points are of the form $(x, y)$, with $0 \neq y \in T_x M$. Over each point $(x, y)$ of $TM \setminus 0$ (treated as a parameter space), we designate the vector space $T_x M$ as a fiber, and name the resulting vector bundle $\pi^* TM$.

There is a canonical symmetric bilinear form $g_{ij} dx^i \otimes dx^j$ on the fibers of $\pi^* TM$, with

$$g_{ij} := \frac{1}{2} (F^2)_{y^i y^j}.$$ 

The subscripts $y^i, y^j$ signify partial differentiation, and the matrix $(g_{ij})$ is known as the fundamental tensor. A Finsler metric $F$ is said to be strongly convex if the said bilinear form is positive definite, in which case it defines an inner product on each fiber of $\pi^* TM$.

For a Randers metric to be strongly convex, it is necessary and sufficient to have

$$\|b\| := \sqrt{b_i b^i} < 1, \quad \text{where} \quad b^i := a^{ij} b_j.$$ 

See [BCS00] or [AIM93] for the proof of this fact. In our case, using $a^{ij} = \lambda(h^{ij} - W^i W^j)$ and $b^i = -\lambda W^i$, we find that

$$\|b\|^2 := a^{ij} b_i b_j = h_{ij} W^i W^j =: |W|^2,$$

which is less than 1 by hypothesis. Therefore the described perturbation of Riemannian metrics $h$ by vector fields $W$ with $|W| < 1$ always generates strongly convex Randers metrics.

2.3. An inverse problem. A question naturally arises: can every strongly convex Randers metric be realized through the perturbation of some Riemannian metric $h$ by some vector field $W$ satisfying $|W| < 1$?

Happily, the answer to this question is yes. Indeed, let us be given an arbitrary Randers metric $F$ with data $a$ and $b$, respectively a Riemannian metric and a differential 1-form, such that $\|b\|^2 := a^{ij} b_i b_j < 1$. Set $b^i := a^{ij} b_j$, and $\varepsilon := 1 - \|b\|^2$. Construct $h$ and $W$ as follows:

$$h_{ij} := \varepsilon (a_{ij} - b_i b_j), \quad W^i := -b^i/\varepsilon.$$
Note that $F$ is Riemannian if and only if $W = 0$, in which case $h = a$. Also, we have $W_i := h_{ij} W^j = -\varepsilon b_i$. Using this, it can be directly checked that perturbing the above $h$ by the stipulated $W$ gives back the Randers metric we started with. Furthermore,

$$|W|^2 := h_{ij} W^i W^j = a^{ij} b_i b_j := \|b\|^2 < 1.$$ 

Let us summarize:

**Proposition 1.** A strongly convex Finsler metric $F$ is of Randers type if and only if it solves the Zermelo navigation problem on some Riemannian manifold $(M, h)$, under the influence of a wind $W$ with $h(W, W) < 1$. Also, $F$ is Riemannian if and only if $W = 0$.

Incidentally, the inverse of $h_{ij}$ is $h^{ij} = \varepsilon^{-1} a^{ij} + \varepsilon^{-2} b^i b^j$. This $h^{ij}$, together with $W^i$, defines a Cartan metric $F^*$ of Randers type on the cotangent bundle $T^* M$. A comparison with [HS96] shows that $F^*$ is the Legendre dual of the Finsler-Randers metric $F$ on $TM$. It is remarkable that the Zermelo navigation data of any strongly convex Randers metric $F$ is so simply related to its Legendre dual. See also [Zi82] and [S02b].

**2.4. Remark about isometries.** Two Finsler spaces $(M_1, F_1)$ and $(M_2, F_2)$ are said to be isometric if there exists a diffeomorphism $\phi : M_1 \to M_2$ which, when lifted to a map between $TM_1$ and $TM_2$, satisfies $\phi^* F_2 = F_1$.

Now consider two strongly convex Randers metrics $F_1$ and $F_2$, where $F_i$ has Riemannian data $(a_i, b_i)$. By the above proposition, they arise as solutions to Zermelo’s navigation problem with $(h_1, W_1)$ and $(h_2, W_2)$, respectively. A moment’s thought gives the lemma below.

**Lemma 2.** Let $\phi : M_1 \to M_2$ be a diffeomorphism. The following three statements are equivalent:

- $\phi$ lifts to an isometry between $F_1$ and $F_2$.
- $\phi^* a_2 = a_1$ and $\phi^* b_2 = b_1$.
- $\phi^* h_2 = h_1$ and $\phi^* W_1 = W_2$.

**3. Zermelo navigation on Riemannian space forms**

This section illustrates a variety of perturbations on 3-dimensional Riemannian space forms. In each example, with the exception of the radial perturbation on the Euclidean metric (§3.2.2), $W$ is an infinitesimal isometry of $h$. It happens that all the resulting strongly convex Randers metrics are of constant flag curvature. The concept of flag curvature is a natural extension of Riemannian sectional curvatures to the Finslerian realm; see §3.4 for a review.

Since all our examples are in three dimensions, we let $(x, y, z)$ denote position coordinates, and expand arbitrary tangent vectors as $u \partial_x + v \partial_y + w \partial_z$. We give expressions for the norm $a := \sqrt{a(y, y)}$ instead of $a_{ij}$ because the former are more compact. The Riemannian metric $a$ (defined in §2.2) can be recovered via $a_{ij} = \left(\frac{1}{2} a^2\right)_{y^i y^j}$.

**3.1. Spheres.**
3.1.1. Rotational perturbation. Let $S^3$ denote the standard unit sphere in $\mathbb{R}^4$. Using its tangent spaces at the east and west poles, we may parametrize the sphere by

$$(x, y, z) \mapsto \frac{1}{\sqrt{1 + x^2 + y^2 + z^2}} (s, x, y, z);$$

here, $s = \pm 1$, respectively, for the eastern and western hemispheres. Note that the equator corresponds to asymptotic infinity on the above tangent spaces. Fix any constant $0 < \tau < 1$ and perturb via the infinitesimal rotation $W = \tau (y, -x, 0)$, with $|W| = \tau \sqrt{\frac{x^2 + y^2}{1 + x^2 + y^2 + z^2}} < 1$.

The bound on $\tau$ is needed to maintain $|W| < 1$ globally on $S^3$. The resulting Randers metric $F = \alpha + \beta$ has constant flag curvature $K = 1$. Explicitly,

$$\alpha = \sqrt{\frac{\rho^2(u^2 + v^2) - (\rho + \tau^2 \varphi)(xu + yv)^2 + \eta \{(\rho - z^2)w^2 - 2zw(xu + yv)\}}{\rho \eta^2}},$$

$$\beta = \frac{\tau (-yu + xv)}{\eta},$$

where $\varphi := 1 + z^2$, $\rho := 1 + x^2 + y^2 + z^2$, and $\eta := 1 + (1 - \tau^2)(x^2 + y^2) + z^2$.

3.1.2. Perturbing by a privileged Killing field. Again, start with the unit sphere $S^3$ in $\mathbb{R}^4$, parametrized as above. For each constant $K > 1$, let $h$ be $\frac{1}{K}$ times the standard Riemannian metric induced on $S^3$. The re-scaled metric has sectional curvature $K$.

Perturb $h$ by the Killing vector field

$$W = \sqrt{K-1} (s(1 + x^2), z - sxy, -y - sxz),$$

with $|W| = \sqrt{\frac{K-1}{K}}$.

This $W$ is tangent to the $S^1$ fibers in the Hopf fibration of $S^3$. The resulting Randers metric $F$ has constant flag curvature $K$. Moreover, it is not projectively flat [BS02]. This is in stark contrast with the Riemannian case because, according to Beltrami’s theorem, a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. Explicitly, $F = \alpha + \beta$, where

$$\alpha = \sqrt{\frac{K(su - zw + yw)^2 + (zu + sv - xw)^2 + (-yu + xv + sw)^2}{1 + x^2 + y^2 + z^2}},$$

$$\beta = \frac{\sqrt{K-1}(su - zw + yw)}{1 + x^2 + y^2 + z^2}.$$

3.2. Euclidean space.
3.2.1. **Rotational perturbation.** The Riemannian metric to be perturbed is the flat metric on $\mathbb{R}^3$. The perturbing vector field is the infinitesimal rotation $W := y \partial_x - x \partial_y + 0 \partial_z$. The resulting Randers metric $F = \alpha + \beta$ solves the least time problem for fish that are surface-feeding in a cylindrical tank with a rotating current. $F$ is defined on the open cylinder $x^2 + y^2 < 1$ in $\mathbb{R}^3$, and has constant flag curvature $K = 0$. Explicitly,

$$\alpha = \frac{\sqrt{(-yu + xv)^2 + (u^2 + v^2 + w^2)(1 - x^2 - y^2)}}{1 - x^2 - y^2},$$

$$\beta = \frac{-yu + xv}{1 - x^2 - y^2}, \quad \text{with} \quad |W|^2 = x^2 + y^2.$$

3.2.2. **Radial perturbation.** Again, we perturb the Euclidean metric, but this time $M$ is the open ball of radius $R$ in $\mathbb{R}^3$, centered at the origin. The perturbing vector field is the radial $W = \tau(x \partial_x + y \partial_y + z \partial_z)$, where $\tau$ is a constant. Impose the constraint $|\tau| \leq \frac{1}{R}$ to ensure that $|W| < 1$ on $M$. The resulting Randers metric $F = \alpha + \beta$ is of constant flag curvature $K = -\frac{1}{4} r^2$, and is given by

$$\alpha = \frac{\sqrt{\tau^2(xu + yv + zw)^2 + (u^2 + v^2 + w^2)(1 - \tau^2(x^2 + y^2 + z^2))}}{1 - \tau^2(x^2 + y^2 + z^2)},$$

$$\beta = \frac{-\tau(xu + yv + zw)}{1 - \tau^2(x^2 + y^2 + z^2)}, \quad \text{with} \quad |W| = \sqrt{\tau^2(x^2 + y^2 + z^2)}.$$

When $R = 1$ and $\tau = -1$, the perturbation generates the Funk metric $\mathbb{F}$ on the unit ball in $\mathbb{R}^3$. See also [OS3] [SO1]. The Funk metric is isometric to the so-called Finslerian Poincaré ball. A 2-dimensional version of the latter is analyzed in [BCS00].

3.2.3. **Perturbing by a translation.** As above, $h$ is the Euclidean metric $\delta_{ij}$. Choose any three constants $p, q, r$ which satisfy $p^2 + q^2 + r^2 < 1$. We perturb $h$ by the vector field

$$W = (p, q, r), \quad \text{with} \quad |W| = \sqrt{p^2 + q^2 + r^2}.$$

The resulting Randers metric $F = \alpha + \beta$ has the form

$$\alpha = \frac{\sqrt{(pu + qv + rw)^2 + (u^2 + v^2 + w^2)(1 - (p^2 + q^2 + r^2))}}{1 - (p^2 + q^2 + r^2)},$$

$$\beta = \frac{-(pu + qv + rw)}{1 - (p^2 + q^2 + r^2)}.$$

This $F$ has constant flag curvature $K = 0$, and is a (locally) Minkowski metric.

3.3. **Hyperbolic space.**

3.3.1. **Rotational perturbation.** Consider the Klein metric

$$h_{ij} = \frac{(1 - x^2 - y^2 - z^2)\delta_{ij} + x_i x_j}{(1 - x^2 - y^2 - z^2)^2}$$
on the unit ball $\mathbb{B}^3 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\}$. Here $x_i := \delta_{ix} x^i$. We perturb by the infinitesimal rotation

$$W = (y, -x, 0), \quad \text{with} \quad |W| = \sqrt{\frac{x^2 + y^2}{1 - x^2 - y^2 - z^2}}.$$ 

In order that $|W| < 1$, we restrict to the domain $\{2x^2 + 2y^2 + z^2 < 1\}$. Define $\varphi = 1 - 2x^2 - 2y^2 - z^2$. Perturbing $h$ by $W$ produces a Randers metric $F = \alpha + \beta$, with

$$\alpha = \sqrt{\varphi [(1 - z^2)(u^2 + v^2) + (1 - x^2 - y^2)u^2 + 2zw(xu + yv)] + (x^2 + y^2)(yu - xv)^2},$$

$$\beta = \frac{-yu + xv}{\varphi}.$$ 

It is of constant flag curvature $K = -1$.

3.3.2. Perturbing by a type $(S)$($\S$7.4.2) Killing field. In this section we will consider a more complicated perturbation of the Klein metric. The perturbing vector field is

$$W = \tau (1 - x^2, z - xy, -y - xz).$$ 

In order to effect $|W| < 1$, we restrict to the domain $\{(1 - \tau^2)x^2 + (1 + \tau^2)(y^2 + z^2) < 1 - \tau^2\}$. The resulting Randers metric $F = \alpha + \beta$ is also of constant flag curvature $K = -1$. The Riemannian portion $\alpha$ of $F$ is substantial. Before writing it down we introduce some abbreviations: $\psi_1 = 1 - x^2 - y^2 - z^2$, $\psi_2 = 1 - x^2 + y^2 + z^2$, $\psi_3 = \psi_1^2 + \tau^2(-1 - x^2 + y^2 + z^2)(y^2 + z^2)$, $\psi_4 = \psi_1^2 + \tau^2(-1 + x^2 - y^2 + z^2)(1 - x^2)$, $\psi_5 = \psi_1^2 + \tau^2(-1 + x^2 + y^2 - z^2)(1 - x^2)$, $\psi_6 = 2\tau^2(z\psi_1 - xy\psi_2)$, $\psi_7 = -2\tau^2(1 - x^2)yz$, $\psi_8 = -\tau^2(y\psi_1 + xz\psi_2)$, $\psi_9 = (1 - \tau^2)(1 - x^2) - (1 + \tau^2)(y^2 + z^2)$. Then $\alpha$ is given implicitly by

$$\psi_1^2 \alpha^2 = \psi_1(xu + yv + zw)^2 + \psi_3 u^2 + \psi_4 v^2 + \psi_5 w^2 + \psi_6 uv + \psi_7 vw + \psi_8 uw.$$ 

The linear term is $\beta = \{\tau(-u - vz + yw)\}/\psi_9$.

3.3.3. Perturbing by a type $(T)$($\S$7.4.3) Killing field. In our final perturbation of the Klein metric we consider the infinitesimal isometry

$$W = (1 - y - x^2, x - xy, -xz).$$ 

The condition $|W| < 1$ holds if we restrict to the domain $(1 - y)^2 < 1 - x^2 - y^2 - z^2$. On this domain the perturbation produces a strongly convex Randers metric $F = \alpha + \beta$ of constant flag curvature $K = -1$. For convenience, define $\varphi_1 = 1 - x^2 + y^2 - z^2$, $\varphi_2 = (1 + y)^2$, $\varphi_3 = (1 - y^2 - z^2)\varphi_1 - x^2\varphi_2$, $\varphi_4 = (1 - z^2)\varphi_1 - (1 - x^2 - z^2)\varphi_2$, $\varphi_5 = (1 - x^2 - y^2)\varphi_3$, $\varphi_6 = -2\varphi(\varphi_1 + y\varphi_2)$, $\varphi_7 = 2yz\varphi_9$, $\varphi_8 = 2xz\varphi_9$, $\varphi_9 = \varphi_1 - \varphi_2$. We have

$$\varphi_1\varphi_9^2 \alpha^2 = \varphi_3 u^2 + \varphi_4 v^2 + \varphi_5 w^2 + \varphi_6 uv + \varphi_7 vw + \varphi_8 uw,$$

and $\beta = \{(xy - (y + 1)u)/\varphi_9\}$. 

ZERMULO NAVIGATION ON RIEMANNIAN MANIFOLDS 9
3.4. Finsler metrics of constant flag curvature. Given any Finsler metric $F$, the Chern connection on the pulled-back tangent bundle $\pi^*TM$ gives rise to two curvature tensors, one of which, $R_{ijkl}$, is analogous to the curvature tensor in Riemannian geometry. Indices on $R$ are raised and lowered by the fundamental tensor $g_{ij}$ and its inverse $g^{ij}$.

At any point $x$ on $M$, a flag consists of a flagpole $0 \neq y \in T_xM$ and a transverse edge $V \in T_xM$. The corresponding flag curvature is defined as

$$K(x, y, V) := \frac{V^i (g(y,y) R_{jikl} y^j y^l) V^k}{g(y,y) g(V,V) - [g(y,V)]^2}.$$ 

In the generic Finslerian setting, both the Chern $hh$-curvature $R$ and the inner product $g$ (given by the fundamental tensor $g_{ij}$) depend on the flagpole $y$. This dependence is absent whenever we specialize to the Riemannian realm, in which case the flag curvature becomes the familiar sectional curvature. For details and conventions, see [BCS00]. A Finsler metric is said to have constant flag curvature $K$ if $K(x, y, V)$ has the constant value $K$ for all locations $x \in M$, flagpoles $y$, and transverse edges $V$.

We note an interesting phenomenon shared by all our examples. In each case, the constant flag curvature of the resulting Randers metric $F$ does not exceed the constant sectional curvature of the original Riemannian metric $h$.

4. Navigation description of Randers metrics of constant flag curvature

4.1. Characterization. Let $F = \alpha + \beta$, with $\alpha^2 := a_{ij} y^i y^j$ and $\beta := b_i y^i$, be a Randers metric. Using $a_{ij}$ to raise the index on the components $b_j$ of the 1-form $b$, we get a vector field $b^i = b^i \partial_i$. Let us introduce the abbreviations

$$\text{curl}_{ij} := \partial_{x^j} b_i - \partial_{x^i} b_j \quad \text{and} \quad \theta_j := b^i \text{curl}_{ij}.$$ 

Note that curl is the 2-form $-db$, and interior multiplication of curl by the vector field $b^i$ gives the 1-form $\theta$.

Define the geometric quantity

$$\sigma := \frac{2 \text{div } b^i}{n - \|b\|^2}.$$ 

A theorem in [BR03] states that the Randers metric $F$ has constant flag curvature $K$ if and only if $\sigma$ is constant,

$$\mathcal{L}_{b^k} a = \sigma (a - b \otimes b) - (b \otimes \theta + \theta \otimes b)$$

(where $\mathcal{L}_{b^k} a = b^k \partial_{x^k} a_{ij} + a_{kj} \partial_{x^j} b^k + a_{ik} \partial_{x^i} b^k$ is a Lie derivative), and the Riemann tensor of $a$ has the form

$$^aR_{hijk} = \xi (a_{ij} a_{hk} - a_{ik} a_{hj})$$

$$- \frac{1}{4} a_{ij} \text{curl}^t_h \text{curl}^t_{hk} + \frac{1}{4} a_{ik} \text{curl}^t_h \text{curl}_{ij}$$

$$+ \frac{1}{4} a_{hj} \text{curl}^t_i \text{curl}_{ik} - \frac{1}{4} a_{hk} \text{curl}^t_i \text{curl}_{ij}$$

$$- \frac{1}{4} \text{curl}_{ij} \text{curl}_{hk} + \frac{1}{4} \text{curl}_{ik} \text{curl}_{hj} + \frac{1}{4} \text{curl}_{hi} \text{curl}_{jk},$$

with $\xi := (K - \frac{1}{16} \sigma^2) + (K + \frac{1}{16} \sigma^2) \|b\|^2 - \frac{1}{2} \theta^i \theta_i$. 
In these equations, all tensor indices are raised and lowered by $a$. For later purposes, let us refer to the above as the Basic equation and the Curvature equation, respectively.

The Basic equation alone is equivalent to the statement that the S-curvature (divided by $F$) has the constant value $\frac{1}{4}\sigma(n + 1)$; see [CS03]. While the Basic equation only makes sense for Randers metrics, its characterization in terms of the S-curvature gives a well-defined criterion which can be imposed on Finsler metrics in general.

4.2. Matsumoto identity. In the original statement of the characterization above, there is a third equation, named CC(23), that $a$ and $b$ must satisfy. As such, the said theorem is equivalent in content to one in [MS02]. Recent work shows that the CC(23) equation is derivable from the Basic and Curvature equations with $\sigma$ constant. Hence it is omitted here. See [BR04] for more discussions.

The omitted CC(23) equation is geometrically significant because, in the presence of a preliminary form of the Basic and Curvature equations, it is equivalent to the constancy of $\sigma$ (or the S-curvature). The CC(23) equation is also useful. For example, it leads to the following Matsumoto identity, which describes the interplay between $\sigma$ and $K$:

$$\sigma(K + \frac{1}{16}\sigma^2) = 0 \quad \text{for constant } K \text{ and } n \geq 2.$$  

4.3. Navigation description. According to Proposition 1, our strongly convex Randers metric $F$ can be realized as the perturbation of a Riemannian metric $h$ by a vector field $W$ which satisfies $h(W, W) < 1$. Using this fact and §2.1.3, the tensors $a$ and $b$ that comprise $F$ are expressible as

$$a_{ij} = \frac{h_{ij}}{\lambda} + \frac{W_i W_j}{\lambda}, \quad b_i = -\frac{W_i}{\lambda},$$

where $W_i := h_{ij} W^j$ and $\lambda := 1 - h(W, W) > 0$. For $a^{ij}$ and $b^i$, see §2.2.

4.3.1. Navigation version of the Basic equation. The Basic equation in the stated characterization involves $a$, $b$, $\mathcal{L}_b a$, and $\theta$. Substituting the above formulae for $a$, $b$ and computing the requisite partial derivatives in the remaining two tensors, we obtain an equivalent $\mathcal{L}_W$ equation:

$$\mathcal{L}_W h = -\sigma h.$$  

The left-hand side can be rewritten in terms of the covariant derivative operator “$;$” associated to $h$, and the $\mathcal{L}_W$ equation becomes

$$W_{ij} + W_{ji} = -\sigma h_{ij}.$$  

In this equation,

“$\sigma$ must vanish whenever $h$ is not flat.”

Indeed, let $\varphi_t$ denote the time $t$ flow of the vector field $W$. The $\mathcal{L}_W$ equation tells us that $\varphi_t^* h = e^{-\sigma t} h$. Since $\varphi_t$ is a diffeomorphism, $e^{-\sigma t} h$ and $h$ must be isometric; therefore they have the same sectional curvatures. If $h$ is not flat, this condition on sectional curvatures mandates that $e^{-\sigma t} = 1$, hence $\sigma = 0$. The above argument was pointed out to us by Bryant.
4.3.2. Riemannian connections of $a$ and $h$. To minimize some anticipated clutter, let us introduce the abbreviations

$$C_{ij} := \partial_x W_i - \partial_x W_j = W_{i;j} - W_{j;i}, \quad T_j := W^i C_{ij},$$

and agree to let the subscript 0 denote contraction of any index with $y^i$. Indices on $C$, $T$ are to be manipulated by the Riemannian metric $h$ only.

Let $\alpha^j_{jk}$ and $\alpha^G_i := \frac{1}{2} \alpha^j_{00}$ be, respectively, the Christoffel symbols and geodesic spray coefficients of the Riemannian metric $a$. Likewise, let $bG^i := \frac{1}{2} \alpha^j_{00}$ be the geodesic spray coefficients of $h$. (The factor of $\frac{1}{2}$ here is absent in some references such as [BCS00].) A straight-forward computation, or an application of Rapcsák’s identity [Rap61], together with the $L_W$ equation, shows [BR14] that

$$\alpha^G_i = bG^i + \frac{y^i}{2\lambda} (T_0 - \sigma W_0) - \frac{y^i}{2\lambda} \left( \frac{h_{00}}{4\lambda} + \frac{W_0 W_0}{2\lambda^2} \right) + \frac{C_{ij} W_0}{2\lambda}.$$  

4.3.3. Navigation version of the Curvature equation. Abbreviate the above formula as $\alpha^G_i = bG^i + \zeta^i$. We now use it to relate the curvature tensor $^aR$ of $a$ to the curvature tensor $^bR$ of $h$. To this end, consider the spray curvature [B47a,b] tensors $^aK^i_j = ^aR^i_{j0}$ and $^bK^i_j = ^bR^i_{j0}$. The Riemann tensor can be recovered from the spray curvature through $^aR_{hijk} = \frac{1}{4} \{ (^aK^i_j) y^i y^k - (^aK^i_k) y^i y^j \}$, where the up index on $^aK$ has been lowered by $a$. A similar formula holds for $^bR_{hijk}$ and $^bK_{ij}$, with the index on $^bK$ lowered by $h$. The advantage of working with the spray curvature is that it has less indices than the full Riemann tensor.

The Curvature equation of §4.1 can be recast into the form

$$^aK^i_j = \xi (\alpha^2 \delta^i_j - y^k \zeta^j_k) + \frac{1}{2} \text{curl}^a \left( \alpha^j_k \zeta^i_j + y^i \text{curl} a_j - \text{curl} a_0 \delta^i_j \right) - \frac{1}{2} \alpha^2 \text{curl}^a \zeta^i_j \zeta^j_k - \frac{3}{2} \text{curl}^a \left( \alpha^j_k \zeta^i_j \right),$$

where $\xi$ is as defined in §4.1 and $\zeta^i_j := \alpha^j_k y^k$. Into (the left-hand side of) this we substitute one version of the split covariantized Berwald formula (see [BR04, S01] for expositions and references therein), which says that

$$^aK^i_j = bK^i_j + (2 \zeta^i_{0j})_j - (\zeta^i_{0j}) y^k_0 (\zeta^i_{0k}) y^j_0 - y^i_0 (\zeta^i_{0k}) y^j_0 + 2 \zeta^i (\zeta^i_{0j}) y^j_0.$$  

Here, the subscripts “$y^k_0” mean $\partial_{y^k}$. This is followed by a tedious calculation, in which all quantities are rewritten in terms of the navigation variables $h$, $W$, and the $L_W$ equation is used prodigiously. A formula for $^bK^i_j$ then results, from which we compute the Riemann tensor $^bR_{hijk}$.

The outcome of that calculation is remarkable. It says that given the $L_W$ equation, the said Curvature equation is equivalent to the statement that $h$ is a Riemannian space form of constant sectional curvature $K + \frac{1}{16} \sigma^2$. Namely,

$$^bR_{hijk} = (K + \frac{1}{16} \sigma^2) (h_{ij} h_{hk} - h_{ik} h_{hj}).$$
4.4. Summary.

Theorem 3. A strongly convex Randers metric $F$ has constant flag curvature $K$ if and only if:

- $F$ solves Zermelo’s navigation problem on a Riemannian space form $(M, h)$ of sectional curvature $K + \frac{1}{16} \sigma^2$ for some constant $\sigma$, under the influence of a vector field (“wind”) $W$.
- The wind $W$ satisfies $h(W, W) < 1$, and is coupled to $h$ and $\sigma$ in such a way that $\mathcal{L}_W h = -\sigma h$, where $\mathcal{L}$ denotes Lie differentiation.

For non-flat $h$, $\sigma$ must vanish, in which case $W$ must be a Killing vector field of $h$.

The last statement has already been observed in §4.3.1. Alternatively, since the sectional curvature of $h$ is $K + \frac{1}{16} \sigma^2$, that statement also follows from Matsumoto’s identity (§4.2).

Note that $K$, the flag curvature of $F$, is bounded above by the sectional curvature $K + \frac{1}{16} \sigma^2$ of $h$. This explains the phenomenon we noted at the end of §3.4. Since $\sigma$ must vanish whenever $K + \frac{1}{16} \sigma^2 \neq 0$, we have the following trichotomy.

(+) For $K > 0$: The quantity $K + \frac{1}{16} \sigma^2$ is positive, hence $\sigma = 0$. Consequently the sectional curvature of $h$ must equal $K$, the flag curvature of $F$.

(0) For $K = 0$: The sectional curvature of $h$ reduces to $\frac{1}{16} \sigma^2$. If $\sigma$ were nonzero, $h$ would have to be flat according to the last part of Theorem 3; but that would be incompatible with having sectional curvature $\frac{1}{16} \sigma^2$. So $\sigma$ must vanish, whence $h$ is flat.

(−) For $K < 0$: There are two viable scenarios. The first is $\sigma = \pm 4\sqrt{|K|}$, in which case $h$ is flat. For the second scenario, $K + \frac{1}{16} \sigma^2 \neq 0$; hence $\sigma = 0$ and $h$ must have negative sectional curvature $K$.

5. Complete list of allowable vector fields

Our goal here is towards a classification of Randers metrics of constant flag curvature. By the navigation description, these metrics arise as perturbations of Riemannian space forms $h$ by vector fields $W$ satisfying $W_{ij} + W_{ji} = -\sigma h_{ij}$. For each of the three standard models (Euclidean, spherical and hyperbolic) of Riemannian space forms we derive a formula for $W$.

5.1. Setting some notation with a basic lemma.

Lemma 4. Let $P_i = P_i(x)$ be solutions of the following system

$$\frac{\partial P_i}{\partial x^j} + \frac{\partial P_j}{\partial x^i} = 0.$$ 

Then

$$P_i = Q_{ij} x^j + C_i,$$

where $(C_i)$ is an arbitrary constant row vector and $Q = (Q_{ij})$ is an arbitrary constant skew-symmetric matrix $(Q_{ji} = -Q_{ij})$. 
Proof: Using the defining differential equation three times, we have
\[ \frac{\partial^2 P_i}{\partial x^k \partial x^j} = - \frac{\partial^2 P_j}{\partial x^k \partial x^i} = \frac{\partial^2 P_k}{\partial x^i \partial x^j} = - \frac{\partial^2 P_i}{\partial x^j \partial x^k}. \]
This shows that all second order partial derivatives of \( P_i \) must vanish. Hence \( P_i \) must be linear; that is, it has the form \( P_i = Q_{ij} x^j + C_i \), with constants \( Q_{ij} \) and \( C_i \). Inserting this expression into the defining PDE shows that \( Q_{ij} + Q_{ji} = 0 \). \( \square \)

For the rest of the paper: “·” refers to the standard dot product on \( \mathbb{R}^n \); indices on \( Q \) and \( C \) are raised and lowered by the Kronecker delta \( \delta_{ij} \); and \( Qx + C \) means \((Q^i_j x^j + C^i)\). We regard \((C_i)\) as a row vector and \((Q^i)\) as a column vector.

5.2. The Euclidean case. The first Riemannian space form we consider is the flat Euclidean metric. The admissible perturbing vector fields \( W \) are described in the following proposition.

Proposition 5. Let \( F = \alpha + \beta \) be a strongly convex Randers metric which results from perturbing the flat metric \( h_{ij} = \delta_{ij} \) on \( \mathbb{R}^n \) by a vector field \( W = (W_i) \). Then \( F \) is of constant flag curvature \( K \) if and only if \( W \) has the form
\[ W_i = -\frac{1}{2} \sigma \delta_{ij} x^j + Q^i_j x^j + C^i, \]
where \((Q^i_j)\) is a constant skew-symmetric matrix, \((C^i)\) is a constant column vector, \( \sigma \) is a constant such that \( \sigma^2 = -16K \), and
\[ (Qx + C) \cdot (Qx + C) + \sigma x \cdot \left(\frac{1}{4} \sigma x - C\right) < 1. \]

Remark: Note that by virtue of \( \sigma^2 = -16K \), we see that \( K \) must be \( \leq 0 \).

Proof: Being flat, \( h \) satisfies the space form criterion of the navigation description, with \( K + \frac{1}{16} \sigma^2 = 0 \). The rest of the proof studies the second criterion, which is the equation \( \mathcal{L}_W h = -\sigma h \).

\((\Leftarrow)\) Suppose \( W \), with its index lowered by \( h_{ij} = \delta_{ij} \), is of the form
\[ W_i = -\frac{1}{2} \sigma \delta_{ij} x^j + Q^i_j x^j + C^i. \]
Keeping in mind that the covariant derivative “·” associated with the Euclidean \( h \) is simply partial differentiation, together with the skew-symmetry of \( Q \), we immediately obtain
\[ \mathcal{L}_W h_{ij} = -\sigma \delta_{ij}. \]
Thus the \( \mathcal{L}_W \) equation in the navigation description is satisfied, and \( F \) has constant flag curvature \( K \).

\((\Rightarrow)\) Conversely, suppose \( F \) has constant flag curvature \( K \). By the navigation description, \( W \) must be a solution of \((\ast)\). Note that
\[ W_i = -\frac{1}{2} \sigma \delta_{ij} x^j \]
is a particular solution. Adding to it the solutions of the homogeneous system \( \frac{\partial P_i}{\partial x^j} + \frac{\partial P_j}{\partial x^i} = 0 \) gives the general solution. According to Lemma 10 the latter have the form \( P_i = Q_{ij} x^j + C_i \), where each \( C_i \) is constant and \((Q_{ij})\) is a constant
skew-symmetric matrix. Using \( h^{ij} = \delta^{ij} \), we raise the index on \( W_i \) to effect the \( W^i \) as claimed.

The inequality satisfied by \( Q \), \( C \), and \( \sigma \) comes from the requirement \(|W| < 1\). \(\square\)

5.3. The spherical and hyperbolic cases. We now perturb standard models of Riemannian metrics with constant sectional curvature \( \kappa \neq 0 \). The list of allowable \( W \) is given in the following proposition.

**Proposition 6.** Let \( F = \alpha + \beta \) be a strongly convex Randers metric which results from perturbing the standard, complete, simply connected, \( n \)-dimensional Riemannian space \((M, h)\) of constant sectional curvature \( \kappa \neq 0 \) by a vector field \( W \). Then \( F \) is of constant flag curvature \( K \) if and only if \( K = \kappa \) and \( W \) is Killing, with the following description in terms of a constant vector \( (C^i) \) and a constant skew-symmetric matrix \( (Q^i) \).

(a) \( K = \kappa > 0 \). Employ a projective coordinate system on the unit \( n \)-sphere, one which comes from parametrizing each hemisphere using the tangent space at the pole. Multiply the standard Riemannian metric by \( \frac{1}{\kappa} \) to effect constant sectional curvature \( K \). The \( h \)-norm of any tangent vector \( y \) is given by

\[
|y| := \sqrt{h(y, y)} = \frac{1}{\sqrt{\kappa}} \sqrt{(y \cdot y)(1 + x \cdot x) - (x \cdot y)^2}, \quad y \in T_x \mathbb{R}^n \simeq \mathbb{R}^n.
\]

With respect to this coordinate system,

\[
W^i(x) = Q^i_j x^j + C^i + (x \cdot C) x^i.
\]

(b) \( K = \kappa < 0 \). Let \( h \) be the Klein model of constant sectional curvature \( K \) on the unit ball \( B^n \), with the Cartesian coordinates of \( \mathbb{R}^n \). The \( h \)-norm of any tangent vector \( y \) is given by

\[
|y| := \sqrt{h(y, y)} = \frac{1}{\sqrt{|K|}} \sqrt{(y \cdot y)(1 - x \cdot x) + (x \cdot y)^2}, \quad y \in T_x \mathbb{R}^n \simeq \mathbb{R}^n.
\]

With respect to this coordinate system,

\[
W^i(x) = Q^i_j x^j + C^i - (x \cdot C) x^i.
\]

In each case, \( W \) is subject to the constraint

\[
\frac{1}{1 + \psi(x \cdot x)} \left\{ (Qx + C) \cdot (Qx + C) + \psi(x \cdot C)^2 \right\} < |K|, \quad \text{where } \psi := \frac{K}{|K|}.
\]

**Proof:** Our Riemannian metric \( h \) has constant sectional curvature \( \kappa \neq 0 \). Therefore it satisfies the space form criterion of the navigation description, with \( K + \frac{\sigma^2}{16} = \kappa \).

In particular, \( K + \frac{\sigma^2}{16} \neq 0 \). The Matsumoto identity then implies that \( \sigma \) must vanish. Consequently, \( K = \kappa \).

According to our navigation description, perturbing the above \( h \) by a vector field \( W \) (with \(|W| < 1\)) generates a Randers metric of constant flag curvature \( K \) if and only if the equation \( \mathcal{L}_W h = -\sigma h \) is satisfied. Since \( \sigma = 0 \) here, that equation reduces to the
The statement that $W$ is a Killing vector field of $h$. The proof of this proposition therefore concerns the classification of solutions of the Killing field equation:

$$W_{i,j} + W_{j,i} = 0.$$  

- To minimize notational clutter, let us introduce the abbreviations

$$x_i := \delta_{ij} x^j, \quad \rho := 1 + \psi (x \cdot x).$$

Then

$$h_{ij} = \frac{1}{|K|} \left( \frac{\delta_{ij}}{\rho} - \psi \frac{x_i x_j}{\rho^2} \right), \quad h^{ij} = \rho |K| \{ \delta^{ij} + \psi x^i x^j \}.$$

The Christoffel symbols of $h$ are given by

$$\gamma_{ij}^{k} \ = \ -\psi \delta_{kj} x^i - \delta_{ki} x^j \rho.$$  

Hence

$$W_{i,j} = \frac{\partial W_i}{\partial x^j} + \psi \frac{x_i W_j + x_j W_i}{\rho}.$$

The Killing field equation now reads

$$\frac{\partial W_i}{\partial x^j} + \frac{\partial W_j}{\partial x^i} + \frac{2 \psi}{\rho} (x_i W_j + x_j W_i) = 0.$$

- To solve it, let us replace the dependent variables $W_i$ by new ones that are named $P_i$, as follows:

$$W_i = \frac{1}{\rho |K|} P_i.$$

(The division by $|K|$ effects a simplification later, when we use $h^{ij}$ to raise the index on $W_i$.) Computations give:

$$\frac{\partial W_i}{\partial x^j} + \frac{\partial W_j}{\partial x^i} = \frac{1}{\rho |K|} \left( \frac{\partial P_i}{\partial x^j} + \frac{\partial P_j}{\partial x^i} \right) - \frac{2 \psi}{\rho^2 |K|} (x_i P_j + x_j P_i),$$

$$\frac{2 \psi}{\rho} (x_i W_j + x_j W_i) = \frac{2 \psi}{\rho^2 |K|} (x_i P_j + x_j P_i).$$

This change of dependent variables transforms the above equation into

$$\frac{\partial P_i}{\partial x^j} + \frac{\partial P_j}{\partial x^i} = 0.$$  

By Lemma 4, the solutions $P_i$ have the form

$$P_i = Q_{ij} x^j + C_i,$$

where $(Q_{ij})$ is a constant skew-symmetric matrix, and the $C_i$ are constants. Thus the covariant form (that is, with index down) of the Killing field $W$ is

$$W_i = \frac{Q_{ij} x^j + C_i}{\rho |K|}.$$
To obtain the contravariant form (namely, with index up) of $W$, we raise its index using $h_{ij} = \rho|K|\{\delta_{ij} + \psi x^ix^j\}$. The result reads:

$W^i := h^{ij}W_j = Q^i_jx^j + C^i + \psi(x \cdot C)x^i,$

where $Q^i_j := \delta^i_sQ_{sj}$ and $C^i := \delta^i_sC_s$.

Finally, the constraint on $Q$ and $C$ comes from the requirement $|W| < 1$. □

5.4. **Remarks.** Note that: in the case of flat $h$, both $W_i$ and $W^i$ are polynomials of degree 1 in the position variables $x$; for non-flat $h$, $W_i$ is a rational function in $x$ of degree -1, while $W^i$ is a polynomial of degree 2 in $x$ whenever $C \neq 0$.

We tabulate below the constant skew-symmetric matrix $Q$, the constant vector $C$, and the value of the constant $\sigma$, for all the examples of §3. To reduce clutter, let $0_{3 \times 3}$ denote the 3-by-3 zero matrix, and

$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

| Example | $(Q_{ij})$ | $(C_i)$ | $\sigma$ |
|---------|------------|---------|---------|
| 3.1.1   | $\tau J \oplus 0$ | $(0,0,0)$ | 0 |
| 3.1.2   | $0 \oplus \sqrt{K - 1} J$ | $(-s\sqrt{K - 1},0,0)$ | 0 |
| 3.2.1   | $J \oplus 0$ | $(0,0,0)$ | 0 |
| 3.2.2   | $0_{3 \times 3}$ | $(0,0,0)$ | $-2\tau$ |
| 3.2.3   | $0_{3 \times 3}$ | $(p,q,r)$ | 0 |
| 3.3.1   | $J \oplus 0$ | $(0,0,0)$ | 0 |
| 3.3.2   | $0 \oplus \tau J$ | $(\tau,0,0)$ | 0 |
| 3.3.3   | $J \oplus 0$ | $(1,0,0)$ | 0 |

6. **Classification of Randers metrics with constant flag curvature**

6.1. **The main theorem.** We now combine the navigation description (see §4.4) and the work of §5 to classify Randers metrics of constant flag curvature. Before stating the theorem, we recall that:

- the skew-symmetric matrix $Q = (Q^i_j)$ and the vector $C = (C^i)$ are constant;
- $Qx$ denotes $(Q^i_jx^j)$, and $x := (x^i)$;
- all indices on $Q$, $C$, $x$ are manipulated by the Kronecker deltas $\delta_{ij}$ and $\delta^ij$;
- “.$"$ is the standard Euclidean dot product.
Theorem 7 (Classification). Let $F(x,y) = \sqrt{a_{ij}(x)y^iy^j} + b_i(x)y^i$ be a strongly convex Randers metric on a smooth manifold $M$ of dimension $n \geq 2$. Then $F$ is of constant flag curvature $K$ if and only if the following conditions are satisfied.

(1) The Riemannian metric $a$ and 1-form $b$ have the representation

$$a_{ij} = \frac{h_{ij}}{\lambda} + \frac{W_i W_j}{\lambda}, \quad b_i = -\frac{W_i}{\lambda},$$

where $h$ is a Riemannian space form and $W = W^i \partial_{x^i}$ is an infinitesimal homothety (of $h$), both globally defined on $M$. Here, $W_i := h_{ij}W_j$ and $\lambda := 1 - h(W,W) > 0$.

(2) Up to local isometry, the Riemannian space form $h$ and the vector field $W$ must belong to one of the following four families.

(+) When $K > 0$: $h$ is $\frac{1}{K}$ times the standard metric on the unit $n$-sphere, and $W = Qx + C + (x \cdot C)x$, with

$$\frac{1}{1 + (x \cdot x)} \{(Qx + C) \cdot (Qx + C) + (x \cdot C)^2\} < K.$$

(0) When $K = 0$: $h$ is the Euclidean metric on $\mathbb{R}^n$ and $W = Qx + C$, with

$$(Qx + C) \cdot (Qx + C) < 1.$$

(−) When $K < 0$:

(−)$_e$ either $h$ is the Euclidean metric on $\mathbb{R}^n$, and $W = -\frac{1}{2}\sigma x + Qx + C$ satisfies

$$(Qx + C) \cdot (Qx + C) + \sigma x \cdot (\frac{1}{4}\sigma x - C) < 1$$

with $\sigma = \pm 4\sqrt{|K|}$;

(−)$_k$ or $h$ is the Klein model of sectional curvature $K$ on the unit ball in $\mathbb{R}^n$, and $W = Qx + C - (x \cdot C)x$ satisfies

$$\frac{1}{1 - (x \cdot x)} \{(Qx + C) \cdot (Qx + C) - (x \cdot C)^2\} < |K|.$$

Furthermore, if $M$ is simply-connected and $h$ is complete, then the said local isometry is in fact a global isometry.

Proof:

• By Proposition 1, every strongly convex Randers metric has the representation, stipulated in (1), in terms of the Zermelo navigation variables $(h,W)$.

• Theorem 3 tells us that $h$ must be a Riemannian space form. The discussion after the statement of Theorem 3 reduces the landscape to only four families, in keeping with (2). They are as follows.

(+) For $K > 0$: $h$ must have sectional curvature $K$ and $W$ is Killing.

(0) For $K = 0$: $h$ must be flat and $W$ is Killing.

(−) For $K < 0$: there are two scenarios,

(−)$_e$ either $h$ is flat, $\sigma = \pm 4\sqrt{|K|}$, and $\mathcal{L}_W h = -\sigma h$ (in which case $W$ turns out to be $-\frac{1}{2}\sigma$ times the radial vector $x = (x^i)$, plus an arbitrary Killing field);
$(-)_k$ or $h$ has sectional curvature $K$ and $W$ is Killing.

- Up to (Riemannian) isometry, there are only three standard models for Riemannian metrics $h$ of constant sectional curvature $K$. They are: \( \frac{1}{K} \times \) the standard metric on the unit $n$-sphere, Euclidean $\mathbb{R}^n$, and the Klein metric with sectional curvature $K$ on the unit ball in $\mathbb{R}^n$. In view of Lemma 2, when classifying $F$ up to Finslerian isometry, it suffices to list the allowable vector fields $W$ with respect to each of the three specific models. (A more leisurely discussion of this point is given at the beginning of §7.) For the families $(+)$ and $(-)_k$, this has been done by Proposition 6. Families $(0)$ and $(-)_e$ are handled by Proposition 5, with $\sigma = 0$ and $\sigma = \pm 4 \sqrt{|K|}$, respectively.

- In each of the four families, the constraint that must be satisfied by $Q$, $C$ and $x$ is equivalent to $|W| < 1$, which characterizes the strong convexity of the Randers metric in question. The table in §5.4 shows that this constraint admits non-trivial solutions for all four families. In §7.2–7.4 we enumerate, with the help of normal forms, all the $Q$, $C$ for which there exists an open domain of $x$ on which $|W| < 1$ holds.

- Finally, if $M$ is simply-connected and $h$ is complete, Hopf’s classification theorem assures us that the Riemannian space form $(M, h)$ must be globally isometric to one of the three standard models. \( \square \)

### 6.2. Globally defined solutions on the standard $S^n$. We see in the previous section that all strongly convex Randers metrics of constant flag curvature $K > 0$ arise locally as solutions to Zermelo’s problem of navigation on the unit sphere $S^n$, under the influence of a Killing field (an infinitesimal isometry) of $\frac{1}{K}$ times the standard metric on $S^n$. Let us show that each strongly convex solution on any closed hemisphere has a unique smooth extension to a globally defined strongly convex solution on $S^n$. There is no restriction on the dimension $n$.

#### 6.2.1. An extension. Without loss of generality, let us assume that the hemisphere in question is the closed eastern hemisphere. Parametrize the eastern ($s = +1$) and western ($s = -1$) open hemispheres, as submanifolds of the ambient $\mathbb{R}^{n+1}$, by the maps

$$x \mapsto \psi^\pm(x) := \frac{1}{\sqrt{1 + x \cdot x}}(s, x), \text{ with } x \in \mathbb{R}^n.$$ 

Geometrically, the tangent space at the east pole (resp. west pole) is identified with $\mathbb{R}^n$. Each point $q$ on an open hemisphere lies on a unique ray which emanates from the center of the sphere. This ray intersects the copy of $\mathbb{R}^n$ tangent to the pole, at a point $x$. The above parametrization expresses $q$ in terms of $x$.

According to Theorem 7, on the open eastern hemisphere, the given Randers metric has navigation data $(h, W)$, where $h$ is $\frac{1}{K}$ times the standard Riemannian metric of $S^n$, and $W(x) = Qx + C + (x \cdot C)x$. We find that it is easier to visualize $W(x)$ by considering its image under $\psi^+_x$. Motivated by a Lie-theoretic reason that will be pointed out in §7.2, we convert the image point $p := \psi^+_x(x)$ into a position row vector $p^t$ of $\mathbb{R}^{n+1}$. A
computation gives

\[ [\psi^+ W(x)]^t = p^t \Omega, \quad \text{where} \quad \Omega = \begin{pmatrix} 0 & C^t \\ -C & -Q \end{pmatrix} \]

is an \((n + 1) \times (n + 1)\) skew-symmetric constant matrix, and \(^t\) means transpose. The continuity of \(W\) on the closed hemisphere implies that its value at any point \(p\) on the equator is also the matrix product \(p^t \Omega\).

Extend \(W\) to the open western hemisphere by insisting that \([\psi^- W(x)]^t = [\psi^-(x)]^t \Omega\). The result is \(W(x) = qx + sC + (x \cdot sC)x\), with \(s = -1\).

It is an artifact of local coordinates that \(W\) is constructed from the data \((Q, C)\) on the eastern hemisphere, but from \((Q, -C)\) on the western hemisphere. The actual Killing field on the embedded unit sphere in \(\mathbb{R}^{n+1}\) has the value \(p^t \Omega\) at any point \(p\), including the equator. Since the matrix \(\Omega\) is constant, there is no question that the constructed \(W\) is globally defined and smooth.

6.2.2. Uniqueness of the extension. Let \(W\) be any global extension of the given Killing field. The isometries of \((S^n, h)\) consist of rigid rotations, implemented by constant \((n + 1) \times (n + 1)\) orthogonal matrices right multiplying the row vectors of \(\mathbb{R}^{n+1}\). Since \(W\) is an infinitesimal isometry, it is the initial tangent to a curve of isometries. Thus it also corresponds to a constant matrix which right multiplies all row vectors. For points \(p\) of the eastern hemisphere, we have determined the matrix in question to be the above \(\Omega\). Constancy dictates that the same \(\Omega\) must be used for the western hemisphere as well. This proves that every global extension agrees with the one we presented. In particular, any global \(W\) with data \((Q, C)\) on some hemisphere must have data \((Q, -C)\) on the complement.

6.2.3. Strong convexity. The strong convexity criterion reads \(|W| < 1\). On the two open hemispheres, Proposition 6 helps us deduce that

\[ |W(x)|^2 = \frac{1}{K\{1 + (x \cdot x)\}} \{(Qx + sC) \cdot (Qx + sC) + (x \cdot sC)^2\}. \]

Using this formula, it is straight forward to check that \(|W(x)|^2 = (p^t \Omega) \cdot (p^t \Omega)\), where \(p = \psi^+(x)\). Before the extension, our Randers metric is strongly convex on the closed eastern hemisphere. In particular, \((p^t \Omega) \cdot (p^t \Omega) < 1\) for all points \(p\) of the open eastern hemisphere. Replacing \(p\) by \(-p\) generates all the points of the open western hemisphere, but does not alter \((p^t \Omega) \cdot (p^t \Omega)\). Therefore the extended metric is also strongly convex on the open western hemisphere and hence on all of \(S^n\).

6.2.4. Discussion. The examples of §3.1.1 and §3.1.2 determine globally defined Randers metrics of constant positive flag curvature on \(S^3\). The first example illustrates the necessity of assuming strong convexity on a closed hemisphere. Had we permitted \(\tau = 1\), the norm of \(W\) would have been less than 1 on the open (eastern and western) hemispheres; but strong convexity would fail at the points \((0, p_1, p_2, p_3)\) on the equator.
6.3. **Globally defined solutions on Euclidean** $\mathbb{R}^n$. Because Euclidean $\mathbb{R}^n$ is covered by a single coordinate chart, globality is relatively easy to address. According to scenarios (0) and $(-)_e$ of Theorem 7, navigation on $\mathbb{R}^n$ under an infinitesimal homothety $W$ produces a strongly convex Randers metric of constant flag curvature $K \leq 0$ wherever $|W| < 1$. In particular, the Randers metric is defined globally if and only if

$$|W(x)|^2 = (Qx + C) \cdot (Qx + C) + \sigma x \cdot (\frac{1}{2}\sigma x - C) < 1 \quad \text{for all } x \in \mathbb{R}^n.$$  

Here, $\sigma$ is zero if $K = 0$, and has the values $\pm 4\sqrt{|K|}$ if $K < 0$. Since $|W(x)|^2$ is polynomial in $x$, the displayed criterion is possible if and only if both $\sigma$ and $Q$ vanish, in which case $W = C$, with $C \cdot C < 1$. The resulting Randers metric is locally Minkowski.

This conclusion is consistent with §3.2, where the only globally defined example is that of §3.2.3.

6.4. **Globally defined solutions on the Klein model.** It remains to discuss global solutions to Zermelo’s problem of navigation on the Klein model with constant sectional curvature $K < 0$, under the influence of a Killing vector field $W$. Theorem 7 says that the resulting Randers metric has constant negative flag curvature $K$. Strong convexity of the Randers metric is equivalent to $|W| < 1$. In this subsection we will show that requiring strong convexity on the entire open unit ball forces $W = 0$, whence the negatively curved Randers metric is simply the Klein model itself.

Suppose $|W| < 1$ holds on the entire open unit ball. It is implicit in Proposition 6 that

$$|W(x)|^2 = \frac{(Qx + C) \cdot (Qx + C) - (x \cdot C)^2}{|K| (1 - x \cdot x)}.$$  

Note that $|K|(1 - x \cdot x) > 0$ because $K$ is negative and $x$ is confined to the unit ball. Multiplying the inequality $0 \leq |W|^2 < 1$ by this positive denominator yields

$$0 \leq (Qx + C) \cdot (Qx + C) - (x \cdot C)^2 < |K| (1 - x \cdot x).$$

Letting $x \cdot x \to 1$ leads to $(Qx + C) \cdot (Qx + C) - (x \cdot C)^2 = 0$ for all unit $x$. In particular, $(Qx + C) \cdot (Qx + C) = (-Qx + C) \cdot (-Qx + C)$, which is equivalent to $Qx \cdot C = 0$. The equality above then simplifies to $Qx \cdot (Qx + C) \cdot C - (x \cdot C)^2 = 0$, again for all unit $x$.

Since we are in dimension at least two, there exists a unit $x_0$ such that $x_0 \cdot C = 0$. The ensuing equation $Qx_0 \cdot Qx_0 + C \cdot C = 0$ tells us that $C$ must have been zero to begin with. This reduces our original equality to $Qx = 0$ for all unit $x$, implying that $Q = 0$. Thus $W$ is identically zero, and our assertion follows.

7. The moduli space

7.1. **A strategy.** Theorem 3 of Section 4.4 characterizes the navigation data $(h, W)$ of strongly convex Randers metrics with constant flag curvature $K$. Namely, $h$ must be a Riemannian metric with constant sectional curvature $K + \frac{1}{16} \sigma^2$, and $W$ must be an infinitesimal homothety of $h$. Also, we observed that $\sigma$ can be nonzero only when $h$ is flat.
Consider any Randers metric \((M, F)\) of constant flag curvature \(K\), with navigation data \((h, W)\). There exists a local isometry \(\varphi\) between \((M, h)\) and one of the three standard models:

- the sphere \((S^n, h_+)\) of constant curvature \(K\) when \(K > 0\);
- Euclidean space \((\mathbb{R}^n, h_0)\) when \(K = 0\), or when \(K < 0\) and \(\sigma = \pm 4\sqrt{|K|}\);
- the Klein model \((\mathbb{B}^n, h_-)\) of constant curvature \(K\) when \(K < 0\) and \(\sigma = 0\).

By using this \(\varphi\) to transform the navigation data \((h, W)\) if necessary, we may assume without loss of generality that \(h\) is already one of the standard models. For each such \(h\), Theorem 7 of Section 6.1 lists its infinitesimal homotheties \(W\).

That list contains a good amount of redundancy because it includes Randers metrics that are locally isometric. The redundancy comes from the symmetry/isometry group \(G\) of \(h\), consisting of diffeomorphisms \(\phi\) that leave \(h\) invariant. Since \(\phi^* h = h\), the action of the Lie group \(G\) on the navigation data is \((h, W) \mapsto (h, \phi_* W)\). According to Lemma 2 of Section 2.4, sets of navigation data which lie on the same \(G\)-orbit correspond to locally isometric Randers metrics. The redundancy we described can therefore be eliminated by collapsing each \(G\)-orbit to a point. These “points” constitute the elements of our moduli space \(M_K\) of Randers metrics with constant flag curvature \(K\). It is the goal of §7 to parametrize \(M_K\) and thereby count its dimension.

To this end, we begin with a standard model \(h = h_+\), or \(h_0\), or \(h_-\)) of a Riemannian space form. Identify the isometry group \(G\) of \(h\) with a matrix subgroup of \(GL_{n+1}\mathbb{R}\). The infinitesimal homotheties \(W\) of \(h\) comprise a representation of some matrix Lie subalgebra \(\mathfrak{h}\) of \(gl_{n+1}\mathbb{R}\). The push-forward action \(W \mapsto \phi_* W := \phi_* \circ W \circ \phi^{-1}\) then corresponds to the “adjoint action”

\[ \Omega \mapsto Ad_\Omega := g \Omega g^{-1} \]

of \(G\) on \(\mathfrak{h}\). Here: \((1)\) \(g \in GL_{n+1}\mathbb{R}\) is the matrix which corresponds to the isometry map \(\phi\), and \(\Omega \in \mathfrak{h}\) is the matrix analog of the infinitesimal homothety \(W\) (which is a vector field). \((2)\) \(Ad\) is well defined because the equation \(\mathcal{L}_W h = -\sigma h\), being tensorial, becomes \(\mathcal{L}_{\phi_* W} h = -\sigma h\). Thus, \(\phi_* W\) is an infinitesimal homothety of \(h\) whenever \(W\) is, and the value of \(\sigma\) is invariant under isometries. \((3)\) According to Theorem 3, when \(h\) is not flat, its infinitesimal homotheties are simply its Killing vector fields. In that case, \(\mathfrak{h}\) equals the Lie algebra \(\mathfrak{g}\) of \(G\), and \(Ad\) is the standard adjoint action of a Lie group on its Lie algebra.

The adjoint action \(Ad\) described above partitions \(\mathfrak{h}\) into orbits. These orbits correspond to distinct local isometry classes of Randers metrics with constant flag curvature \(K\). For each orbit, matrix theory singles out a privileged representative \(\bar{\Omega}\), to be referred to as a normal form. These normal forms provide a concrete parametrization of the points in the moduli space \(M_K\), and the number of parameters constitutes its dimension. The linear algebra behind the construction of \(M_K\) depends on the sign of \(K\). Here is an overview.

- For \(K > 0\), \(h = h_+\) is \(\frac{1}{K}\) times the standard metric on the unit \(n\)-sphere. The orbits are those which result from the adjoint action of the orthogonal group \(O(n+1)\) on its Lie algebra \(\mathfrak{o}(n+1)\).
• For $K = 0$, we have $h = h_0$, the standard flat metric on $\mathbb{R}^n$. The orbits come from the adjoint action of the Euclidean group $E(n)$ on its Lie algebra $\mathfrak{e}(n)$. Here, $E(n)$ is the semi-direct product of $O(n)$ with the additive group $\mathbb{R}^n$ of translations.

• For $K < 0$, the orbits consist of two camps. (1) $h = h_-$ is the Klein model, and the $Ad$ orbits arise from a subgroup of the Lorentz group $O(1, n)$, acting on the Lie algebra $\mathfrak{o}(1, n)$. (2) $h = h_0$ is the flat Euclidean metric, and the $Ad$ orbits are those of $E(n)$ acting on a matrix description of the infinitesimal homotheties, with $\sigma = \pm 4\sqrt{|K|}$.

The Lie theory necessary for determining the normal form $\tilde{\Omega}$ is relegated to the Appendix (§10). The material there will be called upon frequently in the following three subsections as we determine the local isometry classes of strongly convex Randers metrics of constant flag curvature.

7.2. The $n$-sphere. The isometry group $G$ of $(S^n, h_+)$ is $O(n + 1)$, whose elements are orthogonal matrices which implement rigid rotations by right multiplying the row vectors of $\mathbb{R}^{n+1}$. As explained in §6.2, each Killing vector field $W$ of $(S^n, h_+)$ also corresponds to a constant matrix which right multiplies those row vectors, and we have identified that skew-symmetric $(n + 1) \times (n + 1)$ matrix to be

$$\Omega := \begin{pmatrix} 0 & C^t \\ -C & -Q \end{pmatrix},$$

an element of the Lie algebra $\mathfrak{o}(n + 1)$. This correspondence between the Killing fields of $(S^n, h_+)$ and $\mathfrak{o}(n + 1)$ is a Lie algebra isomorphism. (Incidentally, if we had let the group $O(n + 1)$ act on column vectors instead, then the matrix $-\Omega$ would correspond to $W$, while the negative of the commutator $[-\Omega_1, -\Omega_2]$ would represent the Lie bracket $[W_1, W_2]$, rendering the correspondence a Lie algebra anti-isomorphism.)

Applying §10.2 (with $\ell := n + 1$) to $\Omega$, we see that there exists a $g \in O(n + 1)$ so that $\tilde{\Omega} = g\Omega g^{-1}$ is in normal form. Explicitly:

- when $n$ is even, $\tilde{\Omega} = a_1 J \oplus \cdots \oplus a_m J \oplus 0$ with $m = n/2$;
- when $n$ is odd, $\tilde{\Omega} = a_1 J \oplus \cdots \oplus a_m J$ with $m = (n + 1)/2$.

Here, $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The matrix $\tilde{\Omega}$ represents the Killing field $\tilde{W} = \phi_\ast W$, where $\phi$ is the map which corresponds to the orthogonal matrix $g$ (§7.1). According to Theorem 7, $\tilde{W}$ has the form $\tilde{Q}x + \tilde{C} + (x \cdot \tilde{C})x$ with respect to the projective coordinates $x$ which parametrize the eastern hemisphere. Comparing the matrix analog

$$\begin{pmatrix} 0 & \tilde{C}^t \\ -\tilde{C} & -\tilde{Q} \end{pmatrix}$$

of $\tilde{W}$ with $\tilde{\Omega}$, we conclude that $\tilde{C}^t = (a_1, 0, \ldots, 0)$ and $-\tilde{Q} = 0 \oplus a_2 J \oplus \cdots \oplus a_m J \oplus 0$ when $n$ is even, $-\tilde{Q} = 0 \oplus a_2 J \oplus \cdots \oplus a_m J$ when $n$ is odd.
The Randers metric which solves Zermelo’s problem of navigation on \((S^n, h_+)\) under the influence of \(\tilde{W}\) must satisfy the strong convexity criterion \(|\tilde{W}| < 1\). In terms of the data \((\tilde{Q}, \tilde{C})\) for \(\tilde{W}\), inequality \((2,+)\) of Theorem 7 expresses this criterion as:

\[
\begin{align*}
& a_1^2 (1 + x_1^2) + a_2^2 (x_2^2 + x_3^2) + \cdots + a_m^2 (x_{n-2}^2 + x_{n-1}^2) < K (1 + x \cdot x), \quad n \text{ even;} \\
& a_1^2 (1 + x_1^2) + a_2^2 (x_2^2 + x_3^2) + \cdots + a_m^2 (x_{n-1}^2 + x_n^2) < K (1 + x \cdot x), \quad n \text{ odd.}
\end{align*}
\]

We wish to demarcate those \(a_i\) that satisfy the above inequalities on an open set.

7.2.1. **Locally defined metrics when \(n\) is even.** Consider the point \(x = (0, \ldots, 0, x_n)\). Here the condition \(|\tilde{W}(x)| < 1\) simplifies to \(a_1^2 < K(1 + x_n^2)\), which can be made to hold for arbitrary but fixed \(a_1\) by choosing \(|x_n|\) large enough. Once we have \(|\tilde{W}(x)| < 1\), the continuity of \(\tilde{W}\) effects \(|\tilde{W}| < 1\) on a neighborhood about this \(x\). Thus, for even \(n\), the moduli space is parametrized by

\[
a_1 \geq \ldots \geq a_m \geq 0
\]

without any upper bound on \(a_1\), and hence none on the \(a_i\).

7.2.2. **Locally defined metrics when \(n\) is odd.** Suppose \(|\tilde{W}| < 1\) holds at some point \(x\). Then \(0 \leq a_m \leq a_1\) implies that

\[
a_m^2 (1 + x \cdot x) \leq a_1^2 (1 + x_1^2) + a_2^2 (x_2^2 + x_3^2) + \cdots + a_m^2 (x_{n-1}^2 + x_n^2) < K (1 + x \cdot x).
\]

In particular, we obtain the necessary condition \(a_m < \sqrt{K}\). Conversely, given \(a_m < \sqrt{K}\), let us consider a point \(x\) of the form \((0, \ldots, 0, x_n)\). At this \(x\), the desired condition \(|\tilde{W}(x)| < 1\) simplifies and can be rearranged to read \(a_1^2 < K + (K - a_m^2) x_n^2\). Since \(a_m^2 < K\), the inequality can be made to hold by choosing \(|x_n|\) large enough. Continuity then extends \(|\tilde{W}| < 1\) from this \(x\) to a neighborhood containing it. Therefore the isometry classes of locally defined Randers metrics on the odd dimensional spheres are parametrized by

\[
a_1 \geq \ldots \geq a_m \geq 0, \quad \text{with} \quad a_m < \sqrt{K}.
\]

7.2.3. **Globally defined metrics.** Here, the criterion \(|\tilde{W}(x)| < 1\) must hold on the entire sphere. In particular, it must hold for all \(x \in \mathbb{R}^n\) parametrizing the open eastern hemisphere. Setting \(x = 0\) in the inequalities immediately before \S 7.2.1 gives \(a_1 < \sqrt{K}\). Conversely, if \(a_1 < \sqrt{K}\), then those inequalities are satisfied for all \(x\) because \(a_1 \geq a_i \geq 0\). Hence the constraint \(a_1 < \sqrt{K}\) is both necessary and sufficient for strong convexity on the open eastern hemisphere. By virtue of \S 6.2.3, the same bound on \(a_1\) effects \(|\tilde{W}| < 1\) on the open western hemisphere. Thus strong convexity holds on the open hemispheres if and only if the condition \(a_1 < \sqrt{K}\) is met.

It turns out that \(a_1 < \sqrt{K}\) ensures strong convexity on the equator as well. To see this, let \(u\) be any unit vector in the copy of \(\mathbb{R}^n\) tangent to the poles. Our parametrization (see \S 6.2.1) of the open hemispheres says that \(\lim_{t \to \infty} tu\) corresponds asymptotically to some point \(p\) on the equator. In fact, \(p = \lim_{t \to \infty} (1 + tu \cdot tu)^{-1/2} (s, tu) = (0, u)\). Calculating with the norm \(|y|^2 := h(y, y)\) given in part (a) of Proposition 6, we find that

\[
|\tilde{W}(p)| = \lim_{t \to \infty} |\tilde{W}(tu)| = \frac{1}{\sqrt{K}} \sqrt{(u \cdot s\tilde{C})^2 + |\tilde{Q}u|^2},
\]
Proposition 8. The local isometry moduli space of globally defined constant flag curvature $K > 0$ Randers metrics on $S^n$ is given by the polytope
\[ \sqrt{K} > a_1 \geq \cdots \geq a_m \geq 0. \]

7.2.4. Global versus local. For the locally defined metrics, the upper bound $a_1 < \sqrt{K}$ is not necessary because the strong convexity criterion $|W| < 1$ only has to hold on some open subset of $S^n$. However, when $n$ is odd, all local solutions have to satisfy $a_m < \sqrt{K}$.

The metric of §3.1.1 illustrates these nuances well. The table in §5.4 tells us that $C^t = (0, 0, 0)$ and $Q = \tau J \oplus 0$. Using the data $(Q, C)$, construct $\Omega$ as in §7.2. Almost by inspection, the normal form is $\hat{\Omega} = \tau J \oplus 0J$, thus $a_1 = \tau$ and $a_m \equiv a_2 = 0$. Since $K$ here is 1, the theory assures us that a locally defined strongly convex solution exists for any $\tau$, while strongly convex global solutions are characterized by $\tau < 1$.

Indeed, §6.2.3 tells us that $W(p) = p^t \hat{\Omega}$, and $|W(p)|^2 = (p^t \hat{\Omega}) \cdot (p^t \hat{\Omega}) = \tau^2 (p_0^2 + p_1^2)$, where $p^t = (p_0, p_1, p_2, p_3)$ gives the coordinates of an arbitrary point on the embedded $S^3$ in $\mathbb{R}^4$. So $|W| < 1$ globally, as long as $\tau < 1$. On the other hand, if $\tau \geq 1$, then $|\hat{W}(p)| < 1$ holds only at those points $p$ on $S^3$ where $p_0^2 + p_1^2 < 1/\tau^2$.

7.2.5. The moduli space for $K$ positive.

Proposition 8. The local isometry moduli space of $n$-dimensional strongly convex Randers spaces of constant flag curvature $K > 0$ is parametrized by $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$ as follows.

- When $n$ is even, $m = n/2$ and the parameter space is given by
  \[ a_1 \geq \cdots \geq a_m \geq 0. \]

- When $n$ is odd, $m = (n + 1)/2$ and the parameter space is given by
  \[ a_1 \geq \cdots \geq a_m \geq 0, \text{ with } \sqrt{K} > a_m. \]

- The globally defined metrics on $S^n$ are parametrized by the polytope
  \[ \sqrt{K} > a_1 \geq \cdots \geq a_m \geq 0. \]

7.3. Euclidean space. The isometry group of $(\mathbb{R}^n, h_0)$ consists of rotations, reflections, and translations; it is the Euclidean group $E(n)$. Though the action of $E(n)$ on $\mathbb{R}^n$ is affine, it can be implemented by matrix multiplication. To this end, we first represent elements $\phi$ of $E(n)$ by matrices $g \in GL_n \mathbb{R}$ of the form
  \[ g = \begin{pmatrix} A & 0 \\ b & 1 \end{pmatrix}, \text{ where } A \in O(n) \text{ and } b \in \mathbb{R}^n. \]

Next we embed Euclidean $n$-space into $\mathbb{R}^{n+1}$ by assigning to each point $x$ the column position vector $\psi(x) = (x, 1) = p$. The matrix action we have in mind is then
  \[ p^t \mapsto p^tg = (x^tA + b, 1). \]

Here, $p^t$ and the output $p^tg$ are both row vectors.
The image of an infinitesimal homothety \( W = -\frac{1}{2} \sigma x + Q x + C \) under the described representation is \( \left[ \phi^*, W(x) \right]^t = p^t \Omega \), where

\[
\Omega := \begin{pmatrix}
-\frac{1}{2} \sigma I_n - Q & 0 \\
C^t & 0 \\
0 & 0
\end{pmatrix}.
\]

Such matrices, with \( \sigma \in \mathbb{R}, \ C \in \mathbb{R}^n \) and \( Q \in \mathfrak{o}(n) \), form a Lie subalgebra \( \mathfrak{h} \) of \( \mathfrak{gl}_{n+1} \). The correspondence between the infinitesimal homotheties \( W \) of \( (\mathbb{R}^n, h_0) \) and the subalgebra \( \mathfrak{h} \) is a Lie algebra isomorphism. When \( \sigma = 0 \), \( \mathfrak{h} \) is the Lie algebra \( \mathfrak{e}(n) \) of \( E(n) \).

The vector field \( \tilde{W} = -\frac{1}{2} \tilde{\sigma} x + \tilde{Q} x + \tilde{C} \) is the push forward of \( W \) under an isometry if and only if its matrix representative \( \tilde{\Omega} \) is given by \( g \Omega g^{-1} \). Since

\[
g^{-1} = \begin{pmatrix}
A^t & 0 \\
-b A^t & 1
\end{pmatrix},
\]
we have

\[
\left( \begin{array}{cc}
-\frac{1}{2} \tilde{\sigma} I_n - \tilde{Q} & 0 \\
\tilde{C}^t & 0
\end{array} \right) = \tilde{\Omega} = g \Omega g^{-1} = \left( \begin{array}{cc}
-\frac{1}{2} \sigma I_n - AQ A^t & 0 \\
[AW(b)]^t & 0
\end{array} \right),
\]

where \( W(b) = -\frac{1}{2} \sigma b + Q b + C \). Thus \( \tilde{\sigma} = \sigma, \tilde{Q} = AQ A^t \), and \( \tilde{C} = AW(b) \); in particular, the value of \( \sigma \) remains unchanged under any isometry, a general fact we pointed out in §7.1. Our objective is to find \( A \) and \( b \), equivalently \( g \in E(n) \), so that \( \tilde{\Omega} \) takes on a simplest form.

7.3.1. The case of \( \sigma = 0 \) and the moduli space for \( K = 0 \). The Randers metrics of constant flag curvature zero arise as perturbation of the Euclidean metric under an infinitesimal isometry. This corresponds to the \( \sigma = 0 \) case in the above discussion.

For ease of exposition, let us abbreviate group elements \( g \in E(n) \) as \( \{A, b\} \) and Lie algebra elements \( \Omega \in \mathfrak{e}(n) \) as \( [-Q, C]^t \).

1. By §10.2, we can find an \( R \in O(n) \) which puts \( -Q \) into the normal form \( -\tilde{Q} = \rho_1 J \oplus \cdots \oplus \rho_h J \oplus 0_{n-2h} \), with \( \rho_1 \geq \cdots \geq \rho_h > 0 \). Thus \( g_1 := \{R, 0\} \) conjugates \( \Omega \) into \( \hat{\Omega}_1 := [-\tilde{Q}, (RC)^t] \).

2. Choose \( r \in O(n-2h) \) to transform the last \( n-2h \) components of \( RC \) into \( (0, \ldots, 0, \xi \geq 0) \), without affecting its first \( 2h \) components \( D := (D_1, \ldots, D_h) \), listed pairwise for convenience as \( D_i = [C_{2i-1}, C_{2i}] \). The corresponding group element \( g_2 := \{I_{2h} \oplus r, 0\} \) conjugates \( \hat{\Omega}_1 \) into \( \hat{\Omega}_2 := [-\tilde{Q}, (D, 0, \ldots, 0, \xi)^t] \).

3. Set \( b := (-\frac{r D_1}{\rho_1}, \ldots, -\frac{r D_h}{\rho_h}, 0, \ldots, 0) \) and note that \( -\tilde{Q} b = (D, 0, \ldots, 0) \). Then \( g_3 := \{I_n, b\} \) conjugates \( \hat{\Omega}_2 \) into \( \hat{\Omega}_3 := [-\tilde{Q}, (0, \ldots, 0, \xi)^t] \).

In short, using \( g := g_3 g_2 g_1 \in E(n) \), we get

\[
\hat{\Omega} := g \Omega g^{-1} = \begin{pmatrix}
\rho_1 J \oplus \cdots \oplus \rho_h J & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0, \ldots, 0, \xi
\end{pmatrix}.
\]
A moment’s thought tells us that $\xi = 0$ whenever $C \in \text{Range} \, Q$, and $\xi > 0$ otherwise. The strong convexity condition $|W| < 1$ restricts our domain to those $x$ which satisfy

$$|\tilde{W}(x)|^2 = (\tilde{Q}x + \tilde{C}) \cdot (\tilde{Q}x + \tilde{C}) = \xi^2 + \sum_{i=1}^{h} \rho_i^2(x_{2i-1}^2 + x_{2i}^2) < 1.$$ 

In particular, we must have $\xi < 1$. As long as this condition is met, the inequality above will always be satisfied on some neighborhood of the origin in $\mathbb{R}^n$.

Suppressing the rank of $Q$ by augmenting the parameters, followed by some appropriate relabeling, simplifies the normal form $\tilde{\Omega}$ to

$$\left( \begin{array}{ccc} a_1 J \oplus \cdots \oplus a_m J & 0 \\ 0, \ldots, 0, a_0 & 0 \end{array} \right)_{\text{for even } n}, \quad \left( \begin{array}{ccc} a_2 J \oplus \cdots \oplus a_m J & 0 & 0 \\ 0 & 0 & a_1 \end{array} \right)_{\text{for odd } n}.$$ 

Here, a priori we have

$$1 > a_0 \geq 0, \quad a_1 \geq \cdots \geq a_m \geq 0, \quad \text{and } m = n/2 \quad \text{for even } n;$$
$$1 > a_1 \geq 0, \quad a_2 \geq \cdots \geq a_m \geq 0, \quad \text{and } m = (n+1)/2 \quad \text{for odd } n.$$ 

However:

- When $n$ is even, $a_0$ and $a_m$ cannot both be nonzero for any fixed $\Omega$. Indeed, if $a_0 > 0$, then $C$ is not in Range $Q$ and we must at least have $a_m = 0$. On the other hand, if $a_m \neq 0$, then $Q$ is surjective and thus $a_0$ must vanish.
- When $n$ is odd, the displayed normal form precludes any sort of rigid coupling between $a_1$ and $a_m$.

For the even $n$ case, whenever $a_0 > 0$ (so that $a_m = 0$), let us agree to relabel the remaining parameters $a_0, a_1, \ldots, a_{m-1}$ as $a_1, a_2, \ldots, a_m$.

**Proposition 9.** The local isometry moduli space of $n$-dimensional strongly convex Randers spaces of constant flag curvature $K = 0$ is parametrized by $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$ as follows.

- When $n$ is even, $m = n/2$ and the parameter space is the disjoint union of
  $$a_1 \geq \cdots \geq a_m \geq 0 \quad \text{and} \quad 1 > a_1 > 0, \quad a_2 \geq \cdots \geq a_m \geq 0.$$ 

- When $n$ is odd, $m = (n+1)/2$ and the parameter space is given by
  $$1 > a_1 \geq 0, \quad a_2 \geq \cdots \geq a_m \geq 0.$$ 

- The globally defined metrics are parametrized by
  $$1 > a_1 \geq 0, \quad a_2 = \cdots = a_m = 0.$$ 

### 7.3.2. When $\sigma$ is nonzero.

Refer to the general discussion at the beginning of §7.3, and the abbreviation introduced in §7.3.1. We see that conjugating $\Omega = [-\frac{1}{2}\sigma I_n - Q, C^t]$ by any $g := \{A, b\} \in E(n)$ converts it to $[-\frac{1}{2}\sigma I_n - AQA^t_t, (AW(b))^t]$. Select $A \in O(n)$ to cast $-Q$ into the following normal form:

- when $n$ is even, $-\tilde{Q} = -AQA^t_t = a_1 J \oplus \cdots \oplus a_m J$, with $m = n/2$;
- when $n$ is odd, $-\tilde{Q} = -AQA^t_t = a_1 J \oplus \cdots \oplus a_m J \oplus 0$, with $m = (n-1)/2$. 

Here, $a_1 \geq \cdots \geq a_m \geq 0$. Note that $W(b) = (Q - \frac{1}{2} \sigma I_n)b + C$. The linear operator $Q - \frac{1}{2} \sigma I_n$ is invertible because the spectrum of $Q$ is pure imaginary (§10.2) whereas $\sigma$ is real and nonzero. Therefore we may select $b$ so that $W(b) = 0$. With this choice of $A$ and $b$, $g := \{A, b\}$ conjugates $\Omega$ into the normal form

$$\tilde{\Omega} = g\Omega g^{-1} = \begin{pmatrix} -\frac{1}{2} \sigma I_n - \tilde{Q} & 0 \\ 0 & 0 \end{pmatrix}.$$ 

The corresponding infinitesimal homothety has $\tilde{C} = 0$ and its formula is $\tilde{W}(x) = -\frac{1}{2} \sigma x + \tilde{Q} x$. Navigating on Euclidean $\mathbb{R}^n$ subject to the wind $\tilde{W}$ generates a Randers metric of negative flag curvature $K = -\frac{1}{4}\sigma^2$. This metric is strongly convex wherever

$$|\tilde{W}(x)|^2 = \tilde{Q} x \cdot \tilde{Q} x + \frac{1}{4} \sigma^2 x \cdot x = \frac{1}{4} \sigma^2 x \cdot x + \sum_{i=1}^m a_i^2 (x_{2i-1}^2 + x_{2i}^2) < 1.$$

- For any choice of $a_i$ and $\sigma \neq 0$, this condition will be satisfied on some neighborhood of the origin in $\mathbb{R}^n$.
- The left-hand side is a nonzero polynomial in $x$. Therefore strong convexity will never hold globally on $\mathbb{R}^n$.
- The space of local isometry equivalence classes is parametrized by

$$a_1 \geq \cdots \geq a_m \geq 0,$$

with $m = n/2$ when $n$ is even, and $m = (n - 1)/2$ when $n$ is odd.

In order to complete our parametrization of the local isometry classes of constant negative flag curvature Randers spaces, it remains to consider perturbations of the Klein model.

### 7.4. Hyperbolic space

In analogy with the spherical (§6.2, §7.2) and Euclidean (§7.3) cases, we embed the Klein model of hyperbolic geometry into an ambient $(n + 1)$-dimensional space. To that end, consider $\mathbb{R}^{n+1}$ equipped with the scalar product $\langle v, w \rangle := v^t E w$, where $E = -1 \oplus I_n$. The isometry group of this space is the Lorentz group $O(1, n)$.

For $K < 0$, define the subspace $H_K := \{x \in \mathbb{R}^{n+1} \mid \langle x, x \rangle = \frac{1}{K} \}$. We make three observations [ON83]:

- $H_K$ consists of two components, each diffeomorphic to $\mathbb{R}^n$.
- $\langle \cdot, \cdot \rangle$ restricts to a Riemannian metric of constant sectional curvature $K$ on $H_K$.
- $O(1, n)$ preserves $H_K$.

Let $h_K$ denote that component which passes through $(1/\sqrt{|K|}, 0, \ldots, 0)$. Then $h_K$ is a complete, simply connected model of hyperbolic space. The isometry group $G$ of $h_K$ consists of those matrices $g \in O(1, n)$ such that $g(h_K) = h_K$. This identifies $G$ as the orthochronous subgroup $O_+(1, n)$. Its Lie algebra is $\mathfrak{o}(1, n)$.

Let us determine the relationship between Killing vector fields on the Klein model and the Lie algebra $\mathfrak{o}(1, n)$. Introduce the diffeomorphism

$$\psi(x) = \frac{1}{|K| \sqrt{1 - x \cdot x}}(1, x)$$

which maps the unit ball in $\mathbb{R}^n$ onto $h_K$. The map $\psi$ is an isometry between the Klein model and $h_K$. Let $p := \psi(x)$ abbreviate the position column vector of the image point.
Then Killing vector fields $W(x) = Qx + C - (x \cdot C)x$ of the Klein model are associated with elements

$$\Omega := \begin{pmatrix} 0 & C^t \\ C & -Q \end{pmatrix} \in \mathfrak{o}(1, n)$$

via $[\psi, W(x)]^t = p^t \Omega$. This correspondence is a Lie algebra isomorphism.

In §10.3 of the Appendix, we show that there exists a $g \in O_+(1, n)$ so that $\tilde{\Omega} = g\Omega g^{-1}$ assumes one of three possible block diagonal forms, as follows.

- **When $i \Omega$ has a timelike eigenvector:**
  
  $\tilde{\Omega}$ has block diagonal form $\mathbb{O}(1) \oplus a_1 J \oplus \cdots \oplus a_m J$, with $m = n/2$; when $n$ is odd, $\tilde{\Omega}$ has block diagonal form $\mathbb{O}(1) \oplus a_1 J \oplus \cdots \oplus a_m J \oplus 0$, with $m = (n-1)/2$.

  Here, $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$. See §3.3.1 for an example of such a normal form.

- **When $i \Omega$ has a null eigenvector with nonzero eigenvalue:**
  
  $\tilde{\Omega}$ has block diagonal form $a_1 S \oplus a_2 J \oplus \cdots \oplus a_m J \oplus 0$, with $m = n/2$; when $n$ is odd, $\tilde{\Omega}$ has block diagonal form $a_1 S \oplus a_2 J \oplus \cdots \oplus a_m J$, with $m = (n+1)/2$.

  Here, $a_1 > 0$ and $a_2 \geq \cdots \geq a_m \geq 0$. See §3.3.2 for an example.

- **When $i \Omega$ has a null eigenvector with zero eigenvalue but no timelike eigenvector:**
  
  $\tilde{\Omega}$ has block diagonal form $a_1 T \oplus a_2 J \oplus \cdots \oplus a_m J$, with $m = n/2$; when $n$ is odd, $\tilde{\Omega}$ has block diagonal form $a_1 T \oplus a_2 J \oplus \cdots \oplus a_m J \oplus 0$, with $m = (n-1)/2$.

  Here, $a_1 > 0$ and $a_2 \geq \cdots \geq a_m \geq 0$. §3.3.3 exemplifies this normal form.

In the above description, $J$, $S$ and $T$ denote the matrices

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

We declare this $\tilde{\Omega}$ to be the normal form of $\Omega$. It remains to determine how the strong convexity criterion $|\tilde{W}| < 1$ constrains the parameters that describe these normal forms. This is where inequality $(2, -)_k$ of Theorem 7 comes into play. It reads: $(\tilde{Q}x + \tilde{C}) \cdot (\tilde{Q}x + \tilde{C}) - (x \cdot \tilde{C})^2 < |K|(1 - x \cdot x)$.

**7.4.1. When $i \Omega$ has a timelike eigenvector.** The type $(J)$ normal form $\tilde{\Omega}$ is derived in §10.3.4. The corresponding Killing field is given by $\tilde{C} = 0$ and

$$\tilde{Q} = a_1 J \oplus \cdots \oplus a_m J,$$

when $n$ is even, $\tilde{Q} = a_1 J \oplus \cdots \oplus a_m J \oplus 0$, with $m = (n-1)/2$.

Here, $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$. Because $\tilde{W}(0) = 0$, the criterion $|\tilde{W}| < 1$ will always be satisfied in some neighborhood of the origin. Therefore the moduli space is parametrized by

$$a_1 \geq \cdots \geq a_m \geq 0.$$
7.4.2. When $i\Omega$ has a null eigenvector with nonzero eigenvalue. The type $(S)$ normal form $\Omega$ is given in §10.3.5. The associated Killing field has data $\tilde{C} = (a_1, 0, \ldots, 0)$ and when $n$ is even, $-\tilde{Q} = 0 \oplus a_2 J \oplus \cdots \oplus a_m J \oplus 0$, with $m = n/2$; when $n$ is odd, $-\tilde{Q} = 0 \oplus a_2 J \oplus \cdots \oplus a_m J$, with $m = (n + 1)/2$.

Here, $a_1 > 0$ and $a_2 \geq \cdots \geq a_m \geq 0$. The condition $|\tilde{W}| < 1$ is equivalent to

$$a_1^2 (1 - x_1^2) + \sum_{j=2}^m a_j^2 (x_{2j-2}^2 + x_{2j-1}^2) < |K| (1 - x \cdot x).$$

In particular, we must have $a_1^2 (1 - x_1^2) < |K| (1 - x \cdot x)$. This forces $a_1 < \sqrt{|K|}$. Conversely, as long as $a_1$ satisfies this bound, we shall have $|\tilde{W}| < 1$ on a neighborhood of the origin. Hence the moduli space is parametrized by

$$\sqrt{|K|} > a_1 > 0, \quad a_2 \geq \cdots \geq a_m \geq 0.$$

7.4.3. When $i\Omega$ has a null eigenvector with zero eigenvalue but no timelike eigenvector. For this case, the normal form $\Omega$ is of type $(T)$ and is determined in §10.3.6. The corresponding Killing field $\tilde{W}$ is specified by $\tilde{C} = (a_1, 0, \ldots, 0)$ and when $n$ is even, $-\tilde{Q} = a_1 J \oplus \cdots \oplus a_m J$, with $m = n/2$; when $n$ is odd, $-\tilde{Q} = a_1 J \oplus \cdots \oplus a_m J \oplus 0$, with $m = (n - 1)/2$.

Here, $a_1 > 0$ and $a_2 \geq \cdots \geq a_m \geq 0$. Given this data, $|\tilde{W}| < 1$ precisely when

$$a_1^2 (1 - x_2^2)^2 + \sum_{j=2}^m a_j^2 (x_{2j-2}^2 + x_{2j-1}^2)^2 < |K| (1 - x \cdot x).$$

Consider a point $x$ of the type $(0, x_2, 0, \ldots, 0)$. For this $x$, the inequality takes the form $a_1^2 (1 - x_2^2) < |K| (1 + x_2)$, which always holds provided that $x_2$ is sufficiently close to 1. Continuity then extends the inequality to a neighborhood of that $x$. Thus strong convexity does not impose any constraint on the $a_i$. We conclude that the moduli space is parametrized by

$$a_1 > 0, \quad a_2 \geq \cdots \geq a_m \geq 0.$$

7.4.4. The moduli space for $K < 0$. Unlike those of positive and zero flag curvature, Randers spaces of negative constant flag curvature may arise in two different fashions, corresponding to the cases $\sigma \neq 0$ and $\sigma = 0$. Since $\sigma$ is invariant under isometries (§7.1), it makes sense to talk about the local isometry classes, and hence the moduli spaces, for these two families.

- Zermelo navigation on Euclidean space under an infinitesimal homothety with $\sigma \neq 0$ produces a metric with flag curvature $K = -\frac{4}{15} \sigma^2$. The local moduli space of these metrics is parametrized in §7.3.2.

- For $\sigma = 0$, the perturbation of a Riemannian space form of negative sectional curvature $K$ by an infinitesimal isometry generates a metric with flag curvature $K$. These spaces are parametrized, up to local isometry, in §7.4.1–7.4.3.

Together the Euclidean and hyperbolic parametrizations provide a complete description of the local isometry classes.
Proposition 10. The local isometry moduli space of $n$-dimensional strongly convex Randers spaces of constant flag curvature $K < 0$ is parametrized by $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$ as follows.

(e) $h$ is the Euclidean metric and $\sigma = \pm 4\sqrt{|K|}$. The parameter space is
\[ a_1 \geq \cdots \geq a_m \geq 0, \]
where $m = n/2$ when $n$ is even, and $m = (n-1)/2$ when $n$ is odd. These metrics may not be extended to all of $\mathbb{R}^n$.

(k) $h$ is the Klein metric. The parameter space is the disjoint union of three sets.
- When $n$ is even, $m = n/2$ and the three sets are
  \[ a_1 \geq \cdots \geq a_m \geq 0, \]
  \[ \sqrt{|K|} > a_1 > 0, \quad a_2 \geq \cdots a_m \geq 0, \]
  and \[ a_1 > 0, \quad a_2 \geq \cdots \geq a_m \geq 0. \]
- When $n$ is odd, $m = (n+1)/2$ and the three sets are
  \[ a_1 \geq \cdots \geq a_m = 0, \]
  \[ \sqrt{|K|} > a_1 > 0, \quad a_2 \geq \cdots a_m \geq 0, \]
  and \[ a_1 > 0, \quad a_2 \geq \cdots \geq a_m = 0. \]

The globally defined metrics are parametrized by $a = 0$.

8. Discussion of projective flatness

Let $M$ be an $n$-dimensional differentiable manifold. A metric on $M$ is said to be projectively flat if $M$ can be covered by coordinate charts in which the geodesics of the metric are straight lines. For Riemannian metrics, Beltrami’s theorem says that the only projectively flat ones are those with constant sectional curvature. There are Finsler metrics of constant flag curvature which are not projectively flat; see for example [Br96, Br02] and [BS02]. Thus Beltrami’s theorem does not extend to the Finslerian setting.

8.1. Douglas’ theorem. A theorem due to Douglas [D28] states that a Finsler metric $F$ is projectively flat if and only if two special curvature tensors are zero. The first is the Douglas tensor. The second is the projective Weyl tensor for $n \geq 3$, and the Berwald–Weyl tensor [B47] for $n = 2$. (The projective Weyl tensor automatically vanishes when $n = 2$, thereby predicated the need for a different invariant in that dimension.) A complete statement of Douglas’ theorem can be found on p.144 of [Ri59].

The projective Weyl tensor vanishes if and only if the flag curvatures of $F$ have no dependence on the transverse edges (but can possibly depend on the position $x$ and the flagpole $y$); see [S77, M80]. The Berwald–Weyl tensor is defined for all $n$, though only relevant in Douglas’ theorem when $n = 2$. Shen has shown that, at least for the $n = 2$ case, this tensor is zero whenever the Ricci scalar and the S-curvature (divided by $F$) are both constant. (The Ricci scalar is the sum of $n-1$ appropriately chosen flag curvatures.)
8.2. Specializing to Randers metrics. For Randers metrics of constant flag curvature, we see that the projective Weyl tensor vanishes. Since the flag curvature is constant, so is the Ricci scalar. Moreover, for such metrics the S-curvature (divided by $F$) is $\frac{1}{4}(n+1)\sigma$, where $\sigma$ is the constant we encountered in §4.1; see [CS03]. Thus the Berwald–Weyl tensor in two dimensions is zero as well.

According to [BaM97], a Randers metric $F$ has vanishing Douglas tensor if and only if the 1-form $b := b_i dx^i$ is closed. Let $W^\flat$ denote the 1-form $W_i dx^i$, where $(h,W)$ is the Zermelo navigation data of $F$. Using the equation $L_W h = -\sigma h$ with constant $\sigma$, it can be checked that the 2-forms curl $:= -db$ (§4.1) and $C := -dW^\flat$ (§4.3.2) are related through curl$^\flat = -\lambda C^\flat$, where $\lambda := 1 - |W|^2$ is positive because of strong convexity (§2.2). In particular, $db = 0 \iff dW^\flat = 0$, whenever the above $L_W$ equation holds. If the Randers metric $F$ has constant flag curvature, then Theorem 3 (§4.4) avails us of this $L_W$ equation; in that case, the vanishing of the Douglas tensor is equivalent to the condition $dW^\flat = 0$.

8.3. Projectively flat Randers metrics of constant flag curvature. By virtue of Douglas’ theorem, we see that a Randers metric $F$ of constant flag curvature is projectively flat if and only if the 1-form $W^\flat$ is closed, namely, $\partial_x W_i - \partial_x W_j = 0$.

- Suppose $F$ is obtained by perturbing the Euclidean metric. Using the formula for $W_i$ given in the proof of Proposition 5, we see that $W^\flat$ is closed if and only if $(Q_{ij})$ is the zero matrix.
- Suppose $F$ is obtained by perturbing the standard sphere or the Klein model. Using the formula for $W_i$ given in the proof of Proposition 6, we find that $W^\flat$ is closed if and only if $W$ is identically zero. The absence of non-trivial closed Killing fields (namely parallel fields) on non-flat Riemannian space forms is consistent with deRham’s decomposition theorem.

The above information, together with our classification (Theorem 7), tells us the following. Up to local isometry, projectively flat strongly convex non-Riemannian Randers metrics of constant flag curvature $K$ comprise exactly two camps.

1. $K = 0$: Zermelo navigation on Euclidean space with a constant vector field $W = C$ satisfying $0 < |C| < 1$. These are the (locally) Minkowski spaces; see §3.2.3. A rotation can be used to transform $W$ into $(0,\ldots,0,|C|)$ without causing the Minkowski metric in question to leave its local isometry class. Thus $|C|$ parametrizes the 1-dimensional moduli space, which is the open unit interval. Excluding Randers metrics which are Riemannian from Proposition 9 gives the same conclusion.

2. $K < 0$: Zermelo navigation on Euclidean space with $W = -\frac{1}{2}\sigma x + C$, $\sigma = \pm 4\sqrt{|K|}$, and $C \cdot C + \sigma x \cdot \left(\frac{1}{2}\sigma x - C\right) < 1$. This camp includes the Funk metric of §3.2.2. A translation transforms $W$ into $\tilde{W} = -\frac{1}{2}\sigma x$. By §7.3.2 and §7.1, the corresponding Randers metrics $F$ and $\tilde{F}$ share the same local isometry class. Closer examination of $\tilde{F}$ reveals that it is a $\tilde{x}$-scaled variant of the Funk metric, one which lives on the open ball of radius $1/(2\sqrt{|K|})$ centered at the origin of
As a corollary of this itemization, every projectively flat strongly convex Randers metric of constant positive flag curvature must be locally isometric to a Riemannian standard sphere.

We see from the table in §5.4 that among the examples in §3, only 3.2.2 and 3.2.3 are projectively flat.

8.4. Comments. The above conclusions about projectively flat Randers metrics \( F \) of constant flag curvature is consistent with the main result of [S02a]. However, other than the fact that the two papers use totally different methods, there are further distinctions. Here, the \( K < 0 \) camp has simple navigation data \((h, W)\), where \( h \) is the Kronecker delta; but the resulting \( F \), when generated with §2.1.3, shows a certain amount of complexity. In [S02a], a simple expression is derived for \( F \) in the \( K < 0 \) camp; but, upon the use of §2.3 to recover the navigation data \((h, W)\), we find that \( h \), though isometric to the Euclidean metric, takes on a somewhat complicated form. Also, the fact that the moduli space for \( K < 0 \) consists of a single point is not manifest in [S02a].

9. Restricting to the \( \theta = 0 \) family

Recall the tensor \( \theta_i := b^s \text{curl}_{si} \) encountered in §4.1. Strongly convex Randers metrics of constant flag curvature and satisfying the additional condition \( \theta = 0 \) have previously been characterized by the corrected Yasuda–Shimada theorem. See [BR03, MS02] for details and references therein.

9.1. Necessary and Sufficient conditions for \( \theta = 0 \). It can be shown (using the machinery in [BR04]) that the tensor \( \theta \) for Randers metrics of constant flag curvature has the navigation description \((1 - |W|^2)\theta_j = (|W|^2)_{ij} + \sigma W_j\). Since our Randers metrics are always presumed to be strongly convex (\(|W| < 1\)), we see that

\[
\theta = 0 \iff (|W|^2)_{ij} + \sigma W_j = 0 .
\]

9.1.1. The Euclidean case. When \( h \) is the flat Euclidean metric, \( W = -\frac{1}{2} \sigma x + Qx + C \) according to Proposition 5. The equation \((|W|^2)_{ij} + \sigma W_j = 0\) is polynomial in the local coordinates \((x^i)\). By considering the coefficients of this polynomial, one can establish that \( \theta = 0 \) if and only if

- \( Q = 0 \) when \( \sigma \neq 0 \);
- \( Q^2 = 0 \) and \( QC = 0 \) when \( \sigma = 0 \).

It is clear, from the normal form \( \tilde{Q} \) (§10.2) of \( Q \), that \( Q^2 = 0 \) if and only if \( Q = 0 \). Hence the two cases can be unified into a single criterion \( Q = 0 \), which is in turn equivalent to the 1-form \( W^\flat := W_j dx^j \) being closed (§8.3). We conclude that, for strongly convex constant flag curvature Randers metrics which are generated by navigating on Euclidean \( \mathbb{R}^n \) under the influence of an infinitesimal homothety \( W \),

\[
\theta = 0 \quad \text{if and only if} \quad dW^\flat = 0.
\]
Such metrics are precisely the projectively flat ones enumerated in §8.3. It is worth recollecting (§8.2) that in the present context, $dW^2 = 0$ is equivalent to $db = 0$.

9.1.2. The spherical and Klein models. When $h$ is either the spherical or hyperbolic metric, $\sigma$ must vanish (§4.4), and we see that

$$\theta = 0 \iff (|W|^2)_{i,j} = 0 \iff |W| \text{ is constant}.$$ 

Proposition 6 says that $W_i = (Q_{ij}x^j + C_i)/(|K|(1 + \psi x \cdot x))$, where $\psi := K/|K|$. Consequently the expression $(|W|^2)_{i,j}$ is rational in the local coordinates $(x^i)$. Hence $\theta = 0$ if and only if the polynomial numerator does, which ultimately leads to the following necessary and sufficient conditions:

$$QC = 0 \quad \text{and} \quad Q^2 = \psi (CC^t - |C|^2 I_n).$$

Here, $C$ is a column and $C^t$ is a row.

The above equations are invariant in form under any orthogonal transformation $R \in O(n)$. Indeed, multiplying each term by $R$ on the left and $R^t$ on the right, those equations become $\tilde{Q}\tilde{C} = 0$ and $\tilde{Q}^2 = \psi (\tilde{C}\tilde{C}^t - |\tilde{C}|^2 I_n)$, where $\tilde{Q} = RQR^t$, $\tilde{C} = RC$.

- Therefore, without any loss of generality, we may assume that $Q$ is already in the normal form derived in §10.2. Namely,

$$Q = q_1J \oplus \cdots \oplus q_kJ \oplus 0_{n-2k}, \quad \text{with} \quad q_1 \geq \cdots \geq q_k > 0.$$

- With this $Q$, the equation $QC = 0$ can be solved immediately to find that the first $2k$ components of $C$ are zero. Its remaining components can be transformed by any $r \in O(n-2k)$ without altering $Q$. Thus we may assume that the column vector $C$ which solves $QC = 0$ has the simplified form

$$C = (0, \ldots, 0, |C|).$$

We now substitute the displayed $Q$ and $C$ into the equation $Q^2 = \psi (CC^t - |C|^2 I_n)$. The outcome reads

$$-(\ast) \quad -q_1^2I_2 \oplus \cdots \oplus -q_k^2I_2 \oplus 0_{n-2k} = -\psi |C|^2 I_{n-1} \oplus 0,$$

where $I_j$ denotes the $j \times j$ identity matrix.

- By inspection, all the $q_i$ are zero if and only if $|C| = 0$. In other words, $Q = 0 \iff C = 0$. The Killing field corresponding to $Q = 0$, $C = 0$ is $W = 0$. In that case the associated Randers metric is simply the original Riemannian space form $h$.

- It remains to examine the scenario in which neither $Q$ nor $C$ is identically zero. Since all the $q_i$, as well as $|C|$, are nonzero, equation $(\ast)$ forces three restrictions.

1. $q_1 = 1$, hence $K = 1$ and $h$ must be the spherical metric.
2. $q_1 \cdots = q_k = |C|$.  
3. $2k = n - 1$; equivalently, $n = 2k + 1$ is odd.

Up to local isometry, the strongly convex Randers metric in question must have arisen from navigation on an odd dimensional sphere, under the influence of a one parameter family of winds $W$.

We hasten to reiterate that these restrictions are obtained from local considerations only; global conditions are not needed in their derivation.
9.2. Refined conclusions of the corrected Yasuda–Shimada theorem. Taken together, the previous subsections and §8.3 allow us to enumerate all strongly convex Randers metrics with constant flag curvature $K$ and $\theta = 0$. They are obtained by Zermelo navigation on Riemannian space forms $h$, subject to the influence of appropriate winds $W$ which satisfy $|W| < 1$. The non-Riemannian ones are as follows.

- When $K < 0$: $h$ is the flat metric on Euclidean $\mathbb{R}^n$, and $W = -\frac{1}{2}\sigma x + C$, with $\sigma = \pm 4\sqrt{|K|}$. As explained in §8.3, the resulting Randers metric is locally isometric to a position-scaled variant of the Funk metric, one which is generated by $\tilde{W} = -\frac{1}{2}\sigma \tilde{x}$ and lives on the open ball of radius $1/(2\sqrt{|K|})$.

- When $K > 0$: $h$ is the flat metric on Euclidean $\mathbb{R}^n$, and $W = C \neq 0$, living on the domain where $|C| < 1$. We saw in §8.3 that up to local isometry, this family, which consists of locally Minkowski metrics, is parametrized by a single parameter $|C|$.

- When $K > 0$: $h$ is the standard metric on the unit sphere $S^n$, with $n = 2k + 1$ odd. The wind $W$ is given in projective coordinates (§5.3, §6.2.1) as $Qx + C + (x \cdot C)x$, where $Q$ and $C$ are specially related on account of $\theta = 0$. In fact (§9.2.1), there is an $R \in O(n)$ such that $\tilde{C} := RC = (0, \ldots, 0, |C|)$ and $\tilde{Q} := RQR^t = |C|(I \oplus \cdots \oplus I) \oplus 0$, respectively. This is equivalent to conjugating the matrix representative (§7.2) of $W$ by the element $1 \oplus R$ in the isometry group of $h$. Thus (§7.1) the Randers metric generated by $\tilde{W} := Qx + \tilde{C} + (x \cdot \tilde{C})x$ lies in the same isometry class as that from $W$. Applying the analysis in §7.2.2 to $\tilde{W}$, we see that strong convexity mandates $|C| < \sqrt{K}$, which as a bonus (§7.2.3) ensures that the metric is global on $S^n$. Thus, up to isometry, there is only a one parameter family (indexed by $|C|$) of non-Riemannian strongly convex Randers metrics with constant flag curvature $K$ and $\theta = 0$ on the odd dimensional spheres. By contrast, no such metric exists on the even dimensional spheres, regardless of whether it is locally or globally defined.

Strongly convex non-Riemannian Randers metrics with constant flag curvature $K$ and $\theta = 0$ are characterized by the corrected Yasuda–Shimada theorem [BR03, MS02]. The conclusion for the $K = 0$ case is as described above. For nonzero $K$, the characterization is in terms of coupled systems of nonlinear partial differential equations. Our discussion above may be viewed as a complete list of solutions to those partial differential equations.

Bejancu–Farran [BF02, BF03], with the help of the corrected Yasuda–Shimada theorem, have recently established a bijection between Sasakian space forms of constant $\phi$-sectional curvature $c \in (-3, 1)$, and Randers metrics of constant flag curvature $K = 1$ with $\theta = 0$. In the course of their study they showed that the underlying manifold $M$ must be of odd dimension, and is necessarily diffeomorphic to a sphere when it is simply connected and complete with respect to $a$. These results can be made equivalent to what we have described for the $K > 0$ case. Our $\theta$ is denoted by $\beta$ in the Bejancu–Farran papers, and their $c$ is $1 - 4\|b\|^2$ in our notation.

It is worth mentioning here that all spheres, of both odd and even dimensions, admit a wealth of non-Riemannian globally defined Randers metrics of constant positive flag curvature, provided that the restriction $\theta = 0$ is lifted.
Here is a straightforward example on $S^4$. Following the treatment of §6.2 we let $p = (p^0, p^1, p^2, p^3, p^4)$ denote the canonical coordinates on $\mathbb{R}^5$. The infinitesimal rotation

$$W(p) = \tau(-p^2 \partial_{p^1} + p^1 \partial_{p^2}), \quad \tau \text{ constant},$$

restricts to a globally defined Killing field on the standard unit sphere $S^4$. As long as $|\tau| < 1$, we have $|W| < 1$ on the entire sphere. Hence $W$ induces a globally defined, strongly convex Randers metric with constant flag curvature $+1$ on $S^4$.

Notice, however, that $\theta \neq 0$. This is immediate from the equation displayed at the beginning of §9.1.2. It says that $\theta$ vanishes if and only if $|W|$ is constant. The norm of our $W$ is certainly not constant. Hence $\theta$ is nonzero.

10. Appendix: Some Lie Theory

Recall from §7.1 that the symmetry/isometry groups $G$ (of the Riemannian space forms) act on the Lie algebras of infinitesimal homotheties, via the adjoint action $Ad$. Our analysis of the moduli space (§7) of constant flag curvature Randers metrics requires detailed knowledge of each $Ad$ orbit, in order to pinpoint a specific representative in the fundamental Weyl chamber.

Though the Lie theory for the orthogonal group is well known, it is invoked so many times in the paper, and in several different contexts, that we feel obligated to at least set the notation and state the facts (§10.2). In the non-compact case $G = O_+(1, n)$, the orthochronous Lorentz group, the information we need is less standard, and is typically not in a form that we could use without substantial modification or synthesis. Since this information plays such a pivotal role in our geometrical conclusions, we are compelled to sketch a cohesive account (§10.3), which takes up the bulk of the Appendix.

Finally, we have chosen to present this material in matrix language for the sake of concreteness.

10.1. Scalar products and the “perp argument”. By a scalar product on any complex vector space $V$, we mean a pairing $\langle , \rangle$ which is $\mathbb{C}$-linear in the first factor, satisfies $\langle u, v \rangle = \overline{\langle v, u \rangle}$, and is non-degenerate (namely, if $\langle u, v \rangle = 0$ for all $v \in V$, then $u$ must vanish). Inner products are simply positive definite scalar products. For example, if $E$ is the diagonal matrix $-1 \otimes I_n$, then $\langle u, v \rangle := u^t E v$ is a scalar product on $\mathbb{C}^{1+n}$, whereas replacing that $-1$ by $+1$ gives the canonical inner product $u^t \theta$ on $\mathbb{C}^{1+n}$. A vector $v$ is said to be spacelike, null, or timelike, respectively, if $\langle v, v \rangle$ is positive, zero, or negative.

Let $W$ be any subspace of a scalar product space $V$. Its perp $W^\perp$ is $\{ v \in V : \langle v, w \rangle = 0 \text{ for all } w \in W \}$. The restriction of $\langle , \rangle$ to $W^\perp$ may fail to be non-degenerate when $W$ contains a null vector. For instance, in $\mathbb{C}^{1+2}$ with $E = \text{diag}(-1, 1, 1)$, if $W = \text{span}\{(1,1,0)\}$, then $\langle , \rangle$ is degenerate on $W^\perp = \text{span}\{(1,1,0),(0,0,1)\}$. On the other hand, if $W = \text{span}\{(1,1,0),(1,-1,0)\}$, then non-degeneracy holds on $W^\perp = \text{span}\{(0,0,1)\}$. These examples illustrate the following useful fact implicit in [ONS3]: $W$ admits a $\langle , \rangle$ orthonormal basis $\iff W \cap W^\perp = \{0\} \iff$ the restriction of $\langle , \rangle$ to $W^\perp$ is non-degenerate, in which case it defines a scalar product there.

Let $A$ be a self-adjoint linear operator on the scalar product space $V$. Suppose the subspace $W$ is invariant under $A$. Then so is $W^\perp$, because $\langle Av, w \rangle = \langle v, Aw \rangle$. Hence the
Suppressing the rank of \( \Omega \), we see that the restricted \( A \) is again operating on a scalar product space, albeit a smaller one. We shall repeatedly invoke this “perp argument”.

10.2. A compact case: normal form for skew-symmetric real matrices. Let \( \Omega \) be any real \( \ell \times \ell \) skew-symmetric matrix. Then \( A := i\Omega \) is a self-adjoint linear operator on the inner product space \( \mathbb{C}^\ell \), with \( \langle u, v \rangle := u^\top \overline{v} \).

- For eigenvectors \( z_1, z_2 \) with eigenvalues \( \lambda_1, \lambda_2 \), respectively, self-adjointness leads to \( \lambda_1 \langle z_1, z_2 \rangle = \overline{\lambda_2} \langle z_1, z_2 \rangle \). Thus eigenvectors corresponding to distinct eigenvalues are \( \langle , \rangle \) orthogonal, and all eigenvalues of \( A \) are real.
- Since \( A = i\Omega \) where \( \Omega \) is real, we have \( Az = \lambda z \) if and only if \( A^\top = -\overline{\lambda} \). Hence the nonzero eigenvalues of \( A \) must occur in pairs \( \pm a \) \( (a > 0) \), with \( \langle , \rangle \) orthogonal eigenvectors \( z \) and \( \overline{z} \).
- For each nonzero pair \( \pm a \), the real vectors \( v := (z + \overline{z})/2 \) and \( u := (z - \overline{z})/(2i) \) satisfy \( Au = -ia^\top v \) and \( Av = iau \). The \( \langle , \rangle \) orthogonality between \( z \) and \( \overline{z} \) gives \( \langle u, u \rangle = \langle v, v \rangle = 0 \). Thus, the normalized versions \( \hat{u}, \hat{v} \) are orthogonal real unit vectors that still satisfy \( A\hat{u} = -ia\hat{v} \) and \( A\hat{v} = ia\hat{u} \).

Enumerate the nonzero eigenvalues of \( A \), counted with multiplicity, as \( \pm a_1, \ldots, \pm a_k \), where \( a_1 \geq \cdots \geq a_k > 0 \). Associated to \( \pm a_1 \) is the subspace \( W_1 := \text{span}\{\hat{u}_1, \hat{v}_1\} \) which is invariant under the self-adjoint \( A \). Since \( \hat{u}_1, \hat{v}_1 \) are orthonormal, the perp argument (§10.1) says that \( A \) restricts to a self-adjoint linear operator on the inner product space \( W_1^\perp \), and its largest eigenvalue pair on \( W_1^\perp \) is \( \pm a_2 \). Repeating this perp argument \( k \) times, we obtain a special orthonormal set of real vectors \( \{\hat{u}_1, \hat{v}_1, \ldots, \hat{u}_k, \hat{v}_k\} \) which are \( \langle , \rangle \) orthogonal to the \((\ell - 2k)\)-dimensional generalized null space of \( A \).

If \( 2k < \ell \), then \( A \) admits 0 as an eigenvalue of algebraic multiplicity \( \ell - 2k \), with at least one eigenvalue \( \xi_{2k+1} \), say, of unit length. This \( \xi_{2k+1} \) also belongs to the null space of \( \Omega \), and hence can be chosen to be real. Successive applications of the perp argument generates an orthonormal set of real eigenvectors \( \{\xi_{2k+1}, \ldots, \xi_\ell\} \) for the eigenvalue 0.

Consequently, for every eigenvalue of \( A \), the algebraic and geometric multiplicities agree. With respect to the the real orthonormal basis \( \{\hat{u}_1, \hat{v}_1, \ldots, \hat{u}_k, \hat{v}_k, \xi_{2k+1}, \ldots, \xi_\ell\} \), the matrix representation of \( A \) is \( ia_1 J \oplus \cdots \oplus ia_k J \oplus 0_{\ell-2k} \), where

\[
J = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
\]

Correspondingly, that of \( \Omega \) is \( \tilde{\Omega} := a_1 J \oplus \cdots \oplus a_k J \oplus 0_{\ell-2k} \), with \( 2k \) being its rank. Suppressing the rank of \( \Omega \), we see that

- when \( \ell \) is even, \( \tilde{\Omega} = a_1 J \oplus \cdots \oplus a_m J \oplus 0_{\ell-2m} \), with \( m = \ell/2 \);
- when \( \ell \) is odd, \( \tilde{\Omega} = a_1 J \oplus \cdots \oplus a_m J \oplus 0 \), with \( m = (\ell - 1)/2 \);

where \( a_1 \geq a_2 \geq \cdots \geq a_m \geq 0 \). This is the desired normal form of \( \Omega \). Note that \( \tilde{\Omega} = B^{-1}\Omega B \), where \( B \) is the orthogonal matrix whose columns are given by the vectors in our real orthonormal basis.

In terms of Lie theory, a skew-symmetric matrix \( \Omega \) is an element in the Lie algebra \( \mathfrak{o}(\ell) \) of the orthogonal group \( O(\ell) \). The fact that \( \exp(a_i J) \) is the \( 2 \times 2 \) rotation matrix
with angle \( a_i \) tells us that \( \exp(\tilde{\Omega}) \) lands in a maximal torus of \( O(\ell) \), and \( \tilde{\Omega} \) itself belongs to a Cartan subalgebra \( \mathcal{H} \) of \( \mathfrak{o}(\ell) \). The condition \( a_1 \geq \cdots \geq a_m \geq 0 \) singles out the fundamental closed Weyl chamber of \( \mathcal{H} \). Setting \( g = B^{-1} \), our arguments show that every real skew-symmetric \( \Omega \) can be \( O(\ell) \)-conjugated into this closed Weyl chamber.

10.3. A non-compact case. We now use the tools and notation set up in §10.2 to derive a normal form for elements of the Lie algebra \( \mathfrak{o}(1,n) \).

Let \( E \) denote the diagonal matrix \(-1 \oplus I_n\). The elements of \( \mathfrak{o}(1,n) \) are real \( (n+1) \times (n+1) \) matrices \( \Omega \) which satisfy the condition \( \Omega^t = -E \Omega E \); equivalently, \( \Omega \) has the defining form

\[
\Omega = \begin{pmatrix} 0 & C^t \\ C & -Q \end{pmatrix},
\]

where \( Q, C \) are real, and \( Q \) is \( n \times n \) skew-symmetric. The Lorentz group \( O(1,n) \) is a non-compact Lie group of which \( \mathfrak{o}(1,n) \) is the Lie algebra. Elements of \( O(1,n) \) are real \( (n+1) \times (n+1) \) matrices \( g \) such that \( g^{-1} = Eg' E \). Thus the first column of \( g \) is a time-like unit vector and the remaining columns are space-like unit vectors. In particular, the top left entry of \( g \) satisfies \((g^0_0)^2 \geq 1\). For reasons that will be made clear later, our interest is in the orthochronous subgroup \( G := O_+(1,n) \), for which \( g^0_0 \geq 1 \).

10.3.1. An available simplification. Our goal here is to select a simplest representative along the \( G \) adjoint orbit of \( \Omega \). To that end, we first invoke §10.2 to find an element \( R \in O(n) \) such that \( RQR^{-1} = q_1 J \oplus \cdots \oplus q_h J \oplus 0_{n-2h} \), where \( q_1 \geq \cdots \geq q_h > 0 \). This has the effect of changing \( C \) to \( RC \). Next, we use an element \( r \in O(n-2h) \) to transform the last \( n-2h \) components of \( RC \) into \((0,\ldots,0,\xi)\) without affecting its first \( 2h \) components. In terms of matrix conjugation, set \( g_1 := 1 \oplus R \) and \( g_2 := 1 \oplus I_{2h} \oplus r \), then \((g_2 g_1) \Omega (g_2 g_1)^{-1}\) has the simplified form

\[
\begin{pmatrix} 0 & D^t & 0 & \xi \\ D & -(q_1 J \oplus \cdots \oplus q_h J) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \xi & 0 & 0 & 0 \end{pmatrix}.
\]

Here, \( D \) is a column of \( 2h \) entries listed pairwise; namely, \( D = (D_1,\ldots,D_h) \), with \( D_j := [(RC)_{1j},(RC)_{j+1}] \). Since \( g_2 g_1 \in G \), the above matrix lies on the same \( Ad \) orbit as \( \Omega \). When necessary, we can use this simplified form for \( \Omega \) with no loss of generality.

10.3.2. Preliminaries about eigenvalues and eigenvectors. Given any \( \Omega \in \mathfrak{o}(1,n) \), the matrix \( A := i\Omega \) is a self-adjoint linear operator on the scalar product space \( C^{1+n} \), with \( \langle U,V \rangle := U^t \bar{E} \nabla \bar{V} \). Let \( V = (v_0,v) \) be any (possibly complex) eigenvector of \( A \) with eigenvalue \( \lambda \). Direct verification tells us that:

1. both \( AV = \lambda V \) and \( A\nabla = -\nabla V \) hold;
2. we have \( \lambda v_0 = iC^t v \) and \( \lambda v = iv_0 C - iQv \);
3. either \( \lambda = 0 \) or \( v_0^2 = v^t v \). This is equivalent to \( \lambda(v_0^2 - v^t v) = 0 \), which follows from (2) and the skew-symmetry of \( Q \).

Suppose \( \lambda \neq 0 \), so that \( v_0^2 = v^t v \) by (3) above.
Then $V$ is either space-like or null. (Equivalently, all time-like eigenvectors must have zero eigenvalue.) This comes about because $\langle V, V \rangle = -|v_0|^2 + |v|^2$ and $|v_0|^2 = |v^T v| = |(v, \overline{v})| \leq |v||\overline{v}| = |v|^2$, where the Cauchy–Schwarz inequality is being applied to the canonical inner product $(\cdot, \cdot)$ on $C^n$.

The space-like eigenvectors have real eigenvalues, which must occur in pairs $\pm a$ ($a > 0$), with corresponding $(\cdot, \cdot)$ orthogonal eigenvectors $V, \overline{V}$. Note that $A$ being self-adjoint implies that $\lambda(V, V) = \overline{\lambda}(V, V)$, hence $\lambda$ is real whenever $V$ is not null. The rest follows from $\lambda = a > 0$, item (1), and $a(V, \overline{V}) = -a(V, \overline{V})$.

The null eigenvectors have pure imaginary eigenvalues, and can always be standardized into the form $V = (1, v)$ with $v$ real. Indeed, $V = (v_0, \bar{v})$ being nonzero and null means that $|v_0|^2 = |\bar{v}|^2$ with $v_0 \neq 0$: dividing by $v_0$ gives $(1, v)$, where $v^T \overline{v} = |v|^2 = 1$. Yet, (3) says that $v^T \overline{v} = 1$. Substituting $v = \text{Re}v + i\text{Im}v$ into these two equations gives $\text{Im}v = 0$. Then (1) tells us that $\lambda = -\overline{\lambda}$.

10.3.3. Categorizing the normal forms of $A = i\Omega$. We first note that
“if $A$ has no timelike eigenvector, then it must admit a null eigenvector.”

Given the absence of timelike eigenvectors, suppose there were no null eigenvectors either. Then all eigenvectors of $A$ would have to be spacelike. Applying the perp argument (§10.1) $n$ times would produce a $(\cdot, \cdot)$ orthonormal basis $B$ which is entirely spacelike (and which diagonalizes $A$). With respect to $B$, the matrix of $(\cdot, \cdot)$ would be $I_{n+1}$ instead of $E = -1 \oplus I_n$, contradicting the invariance of the index of $(\cdot, \cdot)$.

Thus it is reasonable to split our derivation of the normal forms of $A$ into three camps.

- When $A$ has a timelike eigenvector, the normal form is of type $(J)$.
- In the absence of timelike eigenvectors:
  * $A$ has a null eigenvector with nonzero eigenvalue, in which case its normal form is of type $(S)$.
  * $A$ has a null eigenvector with eigenvalue zero; then its normal form is said to be of type $(T)$.

These are discussed separately in §10.3.4, §10.3.5 and §10.3.6. After those discussions, the following will be apparent: (a) The three types of normal forms are mutually exclusive. (b) The absence of timelike eigenvectors is essential for the type $(T)$ normal form to surface. (c) Having a null eigenvector with nonzero eigenvalue automatically rules out timelike eigenvectors; hence the assumption about timelike eigenvectors being absent is not needed in the type $(S)$ case.

10.3.4. In the presence of a timelike eigenvector for $A$. Call this eigenvector $U$; by item (4) of §10.3.2, its eigenvalue must be 0. This puts $U$ in the null space of $A$ and hence that of $\Omega$. Since the latter is real, $U$ can be chosen real. Being timelike, the first component $u_0$ of $U$ cannot vanish. Replace $U$ by $-U$ if necessary to effect $u_0 > 0$, and scale $U$ to unit length.

Set $U := \text{span}\{U\}$. Since $U$ is timelike, the restriction of $(\cdot, \cdot)$ to $U^\perp$ is positive definite. Hence the analysis of $A_{|U^\perp}$ reduces to the compact case considered in §10.2. So there is a real orthonormal basis $B$ for $U^\perp$, with respect to which $A_{|U^\perp}$ has the normal form $i a_1 J \oplus \cdots \oplus i a_k J \oplus 0_{n-2k}$.
The collection $B := \{U\} \cup B$ is a real $\langle \cdot, \cdot \rangle$ orthonormal basis which puts $\Omega$ into the normal form $\Omega := 0 \oplus a_1 J \oplus \cdots \oplus a_k J \oplus 0_{n-2k}$, with $a_1 \geq \cdots \geq a_k > 0$. Since $u_0 > 0$, the corresponding matrix $g := B^{-1}$ belongs to $O_+(1,n)$. Suppressing the rank of $\Omega$ gives the following:

when $n$ is even, $\tilde{\Omega} = 0 \oplus a_1 J \oplus \cdots \oplus a_m J$, with $m = n/2$;
when $n$ is odd, $\tilde{\Omega} = 0 \oplus a_1 J \oplus \cdots \oplus a_m J \oplus 0$, with $m = (n-1)/2$.

Here, $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$.

10.3.5. When $A$ has a null eigenvector with nonzero eigenvalue. Take any such null eigenvector and call it $X$. According to item (6) of §10.3.2, the eigenvalue in question has the form $ia$ with $0 \neq a \in \mathbb{R}$, and $X$ can be chosen as $(1,x)$, where $x$ is real and $|x|^2 = 1$. Incidentally, item (2) of §10.3.2 characterizes $x$ by the equations $a = C^t x$ and $ax = C - Q x$.

There is in fact a companion real null eigenvector $Y$ with the standardized form $(1,y)$, and which has eigenvalue $-ia$. To see this, it suffices to solve $-a = C^t y$ and $-ay = C - Q y$ for a real $y$. The condition $|y|^2 = y^t y = 1$ then follows from these two equations and the fact that $a$ is nonzero.

Since $Q^t = -Q$, we can rewrite the second equation as $y^t (Q + a I) = -C^t$. Also, $Q + aI$ is invertible because the spectrum of $Q$ is pure imaginary (§10.2). Thus $y^t = -C^t (Q + aI)^{-1}$, which is real because $Q$ and $C$ are. Finally, with the help of the hypothesized $x$, we have $C^t y = y^t C = y^t (Q + aI)x = -C^t x = -a$. This proves that the asserted $Y$ exists. Further analysis, based on the self-adjointness of $A$, shows that any standardized null eigenvector with nonzero eigenvalue must be either $X$ or $Y$.

By interchanging $X$ with $Y$ if necessary, we may assume that $a > 0$. For later purposes, relabel it as $a_1$. Define $U := X + Y = (2, x + y)^t$ and $V := X - Y = (0, x - y)^t$. Observe that:

* $\langle U, U \rangle = 2(-1 + x \cdot y) < 0$ and $\langle V, V \rangle = 2(1 - x \cdot y) > 0$;
* $U$ and $V$ are $\langle \cdot, \cdot \rangle$ orthogonal;
* $AU = i a_1 V$ and $AV = i a_1 U$. Since $|\langle U, U \rangle| = \langle V, V \rangle$, that pair of equations remains valid for the normalized vectors $\tilde{U}$ and $\tilde{V}$.

Set $W := \text{span} \{ \tilde{U}, \tilde{V} \}$. Since $\tilde{U}$ is timelike, the scalar product $\langle \cdot, \cdot \rangle$ becomes positive definite on the $(n-1)$-dimensional $W^\perp$, which is invariant under the self-adjoint $A$. In view of §10.2, there is a real orthonormal basis $B$ for $W^\perp$, with respect to which the restricted $A$ has the normal form $ia_2 J \oplus \cdots \oplus ia_k J \oplus 0_{n-1-2(k-1)}$.

The collection $B := \{ \tilde{U}, \tilde{V} \} \cup B$ is a real $\langle \cdot, \cdot \rangle$ orthonormal basis which puts $\Omega$ into the normal form $\bar{\Omega} := a_1 S \oplus a_2 J \oplus \cdots \oplus a_k J \oplus 0_{n+1-2k}$, where

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $a_1 > 0$, $a_2 \geq \cdots \geq a_k > 0$. Since the first component of $\tilde{U}$ is positive, the corresponding matrix $g := B^{-1}$ belongs to $O_+(1,n)$. Suppressing the rank of $\bar{\Omega}$ gives the following:

when $n$ is even, $\bar{\Omega} = a_1 S \oplus a_2 J \oplus \cdots \oplus a_m J \oplus 0$, with $m = n/2$;
when $n$ is odd, $\bar{\Omega} = a_1 S \oplus a_2 J \oplus \cdots \oplus a_m J$, with $m = (n+1)/2$. 
Here, \( a_1 > 0 \) and \( a_2 \geq \cdots \geq a_m \geq 0 \).

10.3.6. When \( A \) has a null eigenvector with zero eigenvalue but no timelike eigenvector. Items (6) and (2) of §10.3.2 tell us that such a null eigenvector \( V \) can always be standardized into the form \( (1, v) \), where \( v \) is real, \( v \cdot v = 1 \), \( Qv = C \), and \( C \cdot v = 0 \). §10.3.1 says there is no loss of generality in assuming that \( Q \) and \( C \) have already been simplified to \( q_1 J \oplus \cdots \oplus q_h J \oplus 0_{n-2h} \) and \( (D_1, \ldots, D_h, 0, \ldots, 0, \xi) \), respectively. Here, \( q_1 \geq \cdots \geq q_h > 0 \) and \( D_j = [C_j, C_{j+1}] \). The hypothesized existence of \( V \) implies that \( Qv = C \) admits a solution. Hence \( C \) is in the range of \( Q \) and \( \xi \) must vanish. The use of \( J^2 = -I \) solves the equation \( Qv = C \) to give

\[
v = \left( \frac{-JD_1}{q_1}, \ldots, \frac{-JD_h}{q_h}, v_{2h+1}, \ldots, v_n \right).
\]

This \( v \) automatically satisfies \( C \cdot v = 0 \) because of the skew-symmetry of \( J \), and its last \( n - 2h \) components are constrained by the requirement \( v \cdot v = 1 \).

For further discussions, set

\[
z := \left( \frac{-JD_1}{q_1}, \ldots, \frac{-JD_h}{q_h}, 0, \ldots, 0 \right).
\]

The null space \( \mathcal{N}_1 \) of \( A = i\Omega \) consists of eigenvectors \( U = \{u, u\} \) with eigenvalue 0, which are characterized by \( Qu = u_0 C \) and \( C^t u = 0 \). Since \( \Omega \) is real, \( U \) may be chosen to be real. A calculation like the one above tells us that \( \mathcal{N}_1 \) admits a basis \( \{1, z\}, \{0, e_j\}, j = 2h + 1, \ldots, n \}, \) where \( e_j \) has a 1 in the \( j \)th entry, and 0 elsewhere. In particular, \( (1, z) \) is an eigenvector of \( A \) with eigenvalue 0.

If \( (1, z) \) are not null, then the components \( v_{2h+1}, \ldots, v_n \) of the hypothesized null eigenvector \( (1, v) \) could not all be zero, whence \( |z|^2 < |v|^2 = 1 \). This would force the eigenvector \( (1, z) \) to be timelike, a scenario forbidden by our hypothesis. Thus \( (1, z) \) has to be null.

Since \( |JD_i| = |D_i| \), the condition \( z \cdot z = 1 \) is equivalent to

\[
(*) \quad \frac{|D_1|^2}{q_1^2} + \cdots + \frac{|D_h|^2}{q_h^2} = 1.
\]

Introduce the column vectors

\[
z_1 := \left( \frac{D_1}{q_1}, \ldots, \frac{D_h}{q_h}, 0, \ldots, 0 \right), \quad z_2 := \left( \frac{JD_1}{q_1}, \ldots, \frac{JD_h}{q_h}, 0, \ldots, 0 \right).
\]

Let \( \mathcal{N}_i \) be the null space of \( A^i \) and abbreviate the vectors \( (1, z), (0, e_j), j = 2h + 1, \ldots, n \) collectively as \( B_0 \). Then

\[
\mathcal{N}_1 = \text{span}\{B_0\}, \quad \mathcal{N}_2 = \text{span}\{0, z_1\}, B_0\}, \quad \mathcal{N}_3 = \text{span}\{0, z_2\}, (0, z_1), B_0\};
\]

\( \mathcal{N}_p = \mathcal{N}_3 \) for any \( p \geq 3 \).

The first three follow from \( Qz = C, Qz_1 = -z, Qz_2 = -z_1 \), and \( C \cdot z = 0, C \cdot z_1 = 1, C \cdot z_2 = 0 \). The fourth is essentially due to the fact that, while certainly there is a \( z_3 \) such that \( Qz_3 = -z_2 \), it is unable to satisfy \( C \cdot z_3 \), because of \( (*) \) above. The union of all the \( \mathcal{N}_i \) is the generalized null space \( \mathcal{N} \) of \( A \). It is invariant under \( A \).
Normalize \((0, z_1), (0, z_2)\) to yield two real \(\langle \cdot, \cdot \rangle\) orthonormal spacelike vectors \(X_1, X_2\). A routine calculation produces the unit timelike real vector

\[
X_0 := \frac{|z_2|}{|z \cdot z_2|}(1, z) + X_2,
\]

which is \(\langle \cdot, \cdot \rangle\) orthogonal to \(X_1, X_2\). Also, with \(a_1 := |z_1|/|z_2|\), we have \(AX_0 = ia_1X_1\), \(AX_1 = ia_1(X_0 - X_2)\), and \(AX_2 = ia_1X_1\). Let \(B_1\) be the real \(\langle \cdot, \cdot \rangle\) orthonormal basis \(\{X_0, X_1, X_2, (0, e_j), j = 2h + 1, \ldots, n\}\) for the generalized null space \(\mathcal{N}\). With respect to \(B_1\), the matrix of \(A_{\mathcal{N}}\) has the form \(ia_1T \oplus 0_{n-2h}\), where

\[
T = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix}.
\]

Correspondingly, the matrix of \(\Omega_{\mathcal{N}}\) is \(a_1T \oplus 0_{n-2h}\), with \(a_1 > 0\).

Since \(X_0\) is timelike, the scalar product becomes positive definite on \(\mathcal{N}^\perp\), which is invariant under the self-adjoint \(A\). By \S 10.2, there is a real orthonormal basis \(B_2\) for \(\mathcal{N}^\perp\) which puts \(A_{\mathcal{N}^\perp}\), and hence \(\Omega_{\mathcal{N}^\perp}\), into normal form. Incidentally, this normal form must look like \(a_2J \oplus \cdots \oplus a_mJ\), where \(a_2 \geq \cdots \geq a_m > 0\), because the kernel of \(\Omega\) has already been accounted for in \(\mathcal{N}\).

Since \(X_0\) is future-pointing, the real \(\langle \cdot, \cdot \rangle\) orthonormal basis \(B := B_1 \cup B_2\) gives an element \(g := B^{-1} \in O_+(1, n)\). The normal form \(\Omega := g\Omega g^{-1}\) is as follows:

- when \(n\) is even, \(\Omega = a_1T \oplus a_2J \oplus \cdots \oplus a_mJ\), with \(m = n/2\);
- when \(n\) is odd, \(\Omega = a_1T \oplus a_2J \oplus \cdots \oplus a_mJ \oplus 0\), with \(m = (n-1)/2\).

Here, \(a_1 > 0\) and \(a_2 \geq \cdots \geq a_m \geq 0\).

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ZERMELO NAVIGATION ON RIEMANNIAN MANIFOLDS

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TX 77204-3008, USA

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROCHESTER, ROCHESTER, NY 14627-0001, USA

DEPARTMENT OF MATHEMATICAL SCIENCES, IUPUI, INDIANAPOLIS, IN 46202-3216, USA

E-mail address: bao@math.uh.edu
E-mail address: robles@math.rochester.edu
E-mail address: zshen@math.iupui.edu