Phase separating solutions for two component systems in general planar domains

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Abstract
In this paper we consider a two component system of coupled nonlinear Schrödinger equations modeling the phase separation in the binary mixture of Bose–Einstein condensates and other related problems. Assuming the existence of solutions in the limit of large interspecies scattering length $\beta$ the system reduces to a couple of scalar problems on subdomains of pure phases (Noris et al. in Commun Pure Appl Math 63:267–302, 2010). Here we show that given a solution to the limiting problem under some additional non degeneracy assumptions there exists a family of solutions parametrized by $\beta \gg 1$.

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1 Introduction

1.1 Motivation

The Gross–Pitaevskii system [9, 15] consisting of two coupled nonlinear Schrödinger equations,
\[
-\imath \frac{\partial}{\partial t} \Phi_j = \Delta \Phi_j - V_j(x)\Phi_j - \mu_j |\Phi_j|^2 \Phi_j - \sum_{i \neq j} \beta_{ij} |\Phi_i|^2 \Phi_j, \quad x \in \Omega, \ t > 0,
\]
\[
\Phi_j = \Phi_j(x, t) \in \mathbb{C}, \ \Phi_j(x, t) = 0, \ x \in \partial \Omega, \ j = 1, 2
\]  
(1.1)
is a mathematical model for the binary Bose–Einstein condensate for the unknown condensate wave functions \( \Phi_j, \ j = 1, 2 \). Here \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^2 \), and the nonnegative constants \( \mu_j \)'s and \( \beta_{ij} \)'s are the intraspecies and interspecies scattering lengths which represent the interactions between like and unlike particles, respectively. It is natural to assume that \( \beta_{ij} \)'s are symmetric, i.e. \( \beta_{ij} = \beta_{ji} \) if \( j \neq i \). The functions \( V_j(x), \ j = 1, 2 \), represent the magnetic trapping potentials. To find solitary wave solutions of the system (1.1), we set \( \Phi_j(x, t) = e^{-i\beta_j t} u_j(x) \), \( \lambda_j \in \mathbb{R} \) and \( u_j \in \mathbb{R} \). Then we may transform the system (1.1) into a system of semilinear elliptic equations given by
\[
- \Delta u_j + (V_j(x) + \lambda_j) u_j = \mu_j u_j^3 + \sum_{i \neq j} \beta_{ij} u_i^2 u_j, \quad x \in \Omega, \ u_j(x, t) = 0, \ x \in \partial \Omega, \ j = 1, 2,
\]  
(1.2)
which are time independent vector Gross–Pitaevskii/Hartree–Fock equations [7, 8] for the condensate wave functions \( u_j \). It was shown in [22] that there are two distinct scenarios of spatial separation: (i) potential separation, caused by the external trapping potentials in much the same way that gravity can separate fluids of different specific weight; (ii) phase separation, which persists in the absence of external potentials. In the fluid analogy, phase separated condensates can be compared to a system of two immiscible fluids, such as oil and water.

Actually, in a binary mixture of Bose–Einstein condensates in the absence of external potentials, i.e. \( V_i = 0 \), a segregation phenomena occurs when intra species scattering lengths \( \mu_j \) are constants and the parameter \( \beta := \beta_{12} \) is large. In this case the two states repel each other and form segregated domains like the mixture of oil and water. Such a phenomenon is called phase separation of a binary mixture of Bose–Einstein condensates and has been investigated extensively by experimental and theoretical physicists [9, 13, 22]. From a mathematical point of view, a lot of work has been done to study the segregation phenomena [3, 5, 18–21]. In particular, in [14] the authors find the governing equations of the limiting functions of the bound state solutions of the system (1.2) as \( \beta \to \infty \): if \( u_{1,\beta} \) and \( u_{2,\beta} \) are \( L^\infty(\Omega) \)-uniformly bounded solutions of
\[
\begin{align*}
-\Delta u_{1,\beta} + \lambda_1 u_{1,\beta} &= \mu_1 u_{1,\beta}^3 - \beta u_{1,\beta} u_{2,\beta}^2, \quad &\text{in } \Omega, \\
-\Delta u_{2,\beta} + \lambda_2 u_{2,\beta} &= \mu_2 u_{2,\beta}^3 - \beta u_{2,\beta} u_{1,\beta}^2, \quad &\text{in } \Omega, \\
u_{1,\beta} = u_{2,\beta} &= 0, \quad &\text{on } \partial \Omega.
\end{align*}
\]  
(1.3)
then, up to a subsequence, as \( \beta \) approaches \( +\infty \) they converge in \( C^{0,\alpha}(\overline{\Omega}) \cap H^1(\Omega) \) to a pair of functions \( u_1 \) and \( u_2 \) having compact disjoint supports (namely \( u_1 u_2 = 0 \) in \( \Omega \)) which
solve
\[
\begin{align*}
-\Delta u_1 + \lambda_1 u_1 &= \mu_1 u_1^3 \quad \text{in } \Omega \cap \{u_1 > 0\}, \\
-\Delta u_2 + \lambda_2 u_2 &= \mu_2 u_2^3 \quad \text{in } \Omega \cap \{u_2 > 0\}, \\
u_1 &= 0 \text{ on } \partial (\Omega \cap \{u_1 > 0\}), \quad u_2 = 0 \text{ on } \partial (\Omega \cap \{u_2 > 0\}).
\end{align*}
\] (1.4)

It is quite natural to ask if any solutions to the limiting equation (1.4) can be seen as the limiting functions of a bound state solutions of the system (1.3). In the present paper we address this question and give a positive answer.

1.2 Statement of the result

Our objective is to construct phase separating solutions of the more general (than (1.3)) system
\[
\begin{align*}
-\Delta u_1 &= f(u_1, x) - \beta u_1 u_2^2 \quad \text{in } \Omega_1, \\
-\Delta u_2 &= f(u_2, x) - \beta u_2 u_1^2 \quad \text{in } \Omega_2, \\
u_1 &= u_2 = 0 \quad \text{on } \partial \Omega.
\end{align*}
\] (1.5)
as the parameter \(\beta\) is large enough. Here \(\Omega\) is a bounded, open domain in \(\mathbb{R}^2\) with smooth boundary, \(f : \mathbb{R}^2 \times \Omega \to \mathbb{R}\) is sufficiently smooth and odd in the first variable
\[
f(u, x) = -f(-u, x).
\] (1.6)
and \(f\) separates phases in the following sense:

**Definition 1.1** We say that the function \(f \in C^1(\mathbb{R} \times \Omega)\) separates phases in \(\Omega\) if the problem
\[
-\Delta w = f(w, x) \quad \text{in } \Omega, \quad w = 0 \text{ on } \partial \Omega
\] (1.7)
has a solution \(w\) such that \(\Gamma = \{x \mid w(x) = 0\} \subset \Omega\) is a regular, simple closed curve dividing \(\Omega\) into disjoint, open components \(\Omega_i, i = 1, 2\) with \(\partial \Omega_i \cap \Omega = \Gamma, \Omega = \Gamma \cup \Omega_1 \cup \Omega_2\) and
\[
\partial_n w(x) := \omega(x) > 0, \quad \text{on } \Gamma,
\] (1.8)
where \(\nu\) denotes the choice of the unit normal to \(\Gamma\) exterior to the fixed component of \(\Omega\).

To explain the definition let us suppose that \(\nu\) is the exterior to \(\Omega_2\) so that \(w > 0\) in \(\Omega_1\) and \(w < 0\) in \(\Omega_2\). Since \(f\) is odd in the first variable it holds
\[
-\Delta w = f(w, x), \quad \text{in } \Omega_1 \quad \text{and} \quad -\Delta(-w) = -f(-w, x), \quad \text{in } \Omega_2.
\]
The vector function
\[
w^0 = (w_1, w_2), \quad w_1(x) = w(x) \mathbb{1}_{\Omega_1}(x) \quad \text{and} \quad w_2(x) = -w(x) \mathbb{1}_{\Omega_2}(x),
\] (1.9)
would be a smooth solution to (1.5) if not for the jump of the derivative along \(\Gamma\). As we will show in the rest of the paper modifying \(w^0\) suitably near \(\Gamma\) the non-continuity of its derivative can be remedied by adding to it a very small function. The solution of (1.5) obtained this way represents a two component system whose phases are separated along \(\Gamma\). To carry out the construction we also need the following non-degeneracy condition.

**Definition 1.2** We say that the phase separating solution to the problem (1.7) is non-degenerate if:
(a) Each of the following linear problems

\[ - \Delta \psi = f_u(w_i, x) \text{ in } \Omega_i, \]
\[ \psi = 0 \text{ on } \partial \Omega_i, \]

\[ i = 1, 2, \] has only the trivial solution.

(b) The problem

\[ - \Delta \psi = \left[ f_u(w_1, x) \chi_{\Omega_1} + f_u(w_2, x) \chi_{\Omega_2} \right] \psi \text{ in } \Omega, \]
\[ \psi = 0 \text{ on } \partial \Omega, \]

has only the trivial solution.

With all the above taken into account the main result of this paper is:

**Theorem 1.1** Suppose that the function \( f \in C^{3, \gamma} (\mathbb{R} \times \Omega) \), with some \( 0 < \gamma < 1 \), and that \( w \in H^s(\Omega), s > 11/2 \) is a non-degenerate solution to the problem (1.7). Then the system (1.5) has a solution \((u_1, u_2)\) such that as \( \beta \to \infty \)

\[ \| w_i - u_i \|_{C^\alpha(\Omega)} = O(\beta^{-(1-\alpha)/4}), \quad 0 \leq \alpha < 1, \]

and

\[ \| w_i - u_i \|_{C^{2,\alpha}(K)} = O(\beta^{-1/4}), \]

over the compacts \( K \subset \Omega \setminus \Gamma \) with the additional restriction \( 0 < \alpha < \frac{1}{2} \).

**Remark 1.1** We believe that the non-degeneracy condition of the phase separating solution as stated in Definition 1.2 is true for generic functions \( f \) or for generic domains \( \Omega \). The proof should rely on some classical transversality argument as in [16, 17, 23].

**Remark 1.2** Our theorem can be extended to the case when \( \Gamma \) consists of two or more disjoint, simple closed curves as long as it divides \( \Omega \) into two disjoint components. To avoid complicated notations and minor technical difficulties we will carry out the proof for a simple, closed curve.

**Remark 1.3** We conjecture that our result also holds in a \( d \)-dimensional domain, once the curve \( \Gamma \) (1.1) is replaced by a \((d - 1)\)-dimensional regular manifold. The construction of the solution close to the \( \Gamma \) is obtained more or less in the same way. What is much more difficult is the gluing of the one dimensional profile around the manifold with the phase separating solution. The radial case, i.e. \( \Omega \) is the unit ball in \( \mathbb{R}^d \), \( d \geq 1 \), when \( f \) as in the Example 1.3 does not depend on \( x \), has been solved by Casteras and Sourdis [4].

**Remark 1.4** In principle our result can be extended to systems with \( m \) equations at least in a two-dimensional setting. In this case the function \( f \) separates \( m \) phases in \( \Omega \) according to (1.1), i.e. the set \( \Gamma = \{ x \mid w(x) = 0 \} \subset \Omega \) is a nonempty, one dimensional boundary set dividing \( \Omega \) into \( m \) disjoint, open components \( \Omega_i, i = 1, \ldots, m \) such that \( \Omega = \bigcup_{i=1}^m \Omega_i \cup \left( \bigcup_{i=1}^{m-1} \Gamma_i \right) \). \( \Gamma_i := \partial \Omega_i \cap \partial \Omega_{i+1} \cap \Omega \) and \( \partial_{v} w(x) = \omega_i(x) > 0 \) along each curve \( \Gamma_i \). The profile of the solution close to each curve \( \Gamma_i \) is the usual one-dimensional profile and far away it is nothing but the multiphase separating solution.
Before discussing several examples where our Theorem applies some comments are in place. As for the smoothness assumption on $f$ we require that $f \in C^{3,\gamma}(\mathbb{R} \times \Omega)$ (these assumptions can be somewhat weakened, see Example 1.3). As a consequence the function $w_1(x) + w_2(x)$ has Lipschitz continuous extension to the whole $\Omega$ and so the function

$$f_u(w_1, x)1_{\Omega_1} + f_u(w_2, x)1_{\Omega_2}.$$ 

In what follows we will need to calculate various functions defined on the curve $\Gamma$ and their derivatives, for instance the function $\omega$, the curvature $\kappa$ of $\Gamma$ etc. We will assume, without being specific, that they are as regular as needed. Also, it will be evident from the discussion that $w \in H^{9/2}(\Omega)$ is sufficient to guarantee enough smoothness for our calculations. Note that a priori we have $\omega \in H^{9/2}(\Gamma) \cap C^3, \alpha(\Gamma), \alpha \in (0, \frac{1}{2})$.

In principle it is possible and physically justified to allow $\Gamma \cap \partial\Omega$ to consist for example of a discrete set of points. This leads to serious, but not unsurmountable technical difficulties however by definition $\Gamma \cap \partial\Omega = \emptyset$. To keep the paper at reasonable length we chose not to deal with the case $\Gamma \cap \partial\Omega \neq \emptyset$ here.

The curve $\Gamma$ divides $\Omega$ into two disjoint components $\Omega_1$ and $\Omega_2$. We agree that the choice of $\nu$ in (1.8) is made in such a way that it is outward to the fixed set $\Omega_i$ at all points of $\Gamma$.

As stated for the purpose of the definition the problem (1.7) is supplied with the homogeneous Dirichlet boundary conditions on $\partial\Omega$ in order to determine the function $w$. In general the type of boundary conditions should depend on the particular physical context. The most common choice would be to impose either the Dirichlet or the Neumann boundary conditions or some combination of the two. Our method allows for some flexibility in the choice of the boundary conditions and can be easily adopted to deal with non homogeneous Dirichlet boundary conditions

$$w = g \text{ on } \partial\Omega. \quad (1.14)$$

We will now discuss some examples.

**Example 1.1** Suppose that $f(u, x) = f(u)$ where $f$ is odd and $|f(u) - f'(0)u| \leq c|u|^p$, $p \geq 2$. Let $w$ be a sign-changing solution to

$$-\Delta w = f(w) \text{ in } \Omega,$$

$$w = 0 \text{ on } \partial\Omega. \quad (1.15)$$

Assume that $\Gamma := \{x \in \Omega \mid w(x) = 0\}$ is a regular, simple closed curve not intersecting the boundary $\partial\Omega$. Then we may set $\Omega_1 := \{x \in \Omega \mid w(x) > 0\}$, $\Omega_2 = \text{int}(\Omega \setminus \Omega_1)$.

**Example 1.2** Suppose that $f \equiv 0$ and that the domain $\Omega$ is at least doubly connected and that its boundary consists of smooth, simple closed curves $G_j$, $j = 1, \ldots, k$, $k \geq 1$ and $\partial\Omega$. Let us fix one of the components of the boundary, say $G_k$ and consider the following problem

$$-\Delta w = 0, \text{ in } \Omega,$$

$$w = -1, \text{ on } \partial\Omega \cup G_1 \cup \cdots \cup G_{k-1},$$

$$w = 1, \text{ on } G_k.$$ 

By the choice of the boundary conditions $\Gamma := \{x \in \Omega \mid w(x) = 0\}$ is non empty and it is reasonable to assume that at least generically it will be a smooth, simple, close curve.
Example 1.3 Another possible situation of the binary phase separation is given by the solution of the following system

\[-\Delta w_1 = g(w_1, x), \text{ in } \Omega_1\]
\[-\Delta w_2 = h(w_2, x), \text{ in } \Omega_2\]

such that

\[w_1 = 0 = w_2, \text{ on } \partial \Omega, \quad \partial_n w_1 = \partial_n w_2 > 0, \text{ on } \Gamma\]

In this case the function \( w = 1_{\Omega_1} w_1 + 1_{\Omega_2} w_2 \) is a weak solution of (1.7) with

\[f(u, x) = 1_{\Omega_1} g(u, x) + 1_{\Omega_2} h(u, x).\]

Strictly speaking the regularity hypothesis in the Definition 1.1 may not be satisfied in this case. This can be remedied by assuming additional smoothness of the function \( \omega \) and the curve \( \Gamma \). If for instance \( \nu \) is exterior to \( \Omega_2 \) assuming that \( h(u, x) \) is odd in \( u \) we can define

\[u^0 = (1_{\Omega_1} w_1, -1_{\Omega_2} w_2)\]

as the model of the phase separating solution of (1.5). Our result generalizes straightforwardly to this case at the expense of some additional technical hypothesis.

The method of the proof of Theorem 1.1 is in part motivated by the approach in [11] and relies on careful separation of the problem into the outer equation whose solution is approximately \( u^0 \), given in (1.9), and the inner equation whose one dimensional, leading order solution is given by a suitable scaled solution of the ODE system (2.3) below. The challenge is to combine them locally, near \( \Gamma \) in a smooth way. To this end we will introduce lower order corrections to the inner and the outer solution as well as some modulations functions. The latter are needed to deal with the problem of small divisors of the linearized inner equation.

This paper is organized as follows: in Sect. 2.1 we introduce the Dirichlet-to-Neumann map needed to improve the initial approximation and guarantee the smallness of its error. In Sects. 2.2–2.4 we study the one dimensional, inner solution. Section 3 is devoted to the construction of the approximate solution. The proof of the main result is carried out in Sect. 4. Lemma 4.7 is proven in Sect. 5. Finally in Sect. 6 we prove an auxiliary result needed in the proof of Proposition 2.1.

In this paper we will use \( c \) to denote a positive constant whose value may change from line to line. The symbol \( g_1 \lesssim g_2 \) will be used to mean \( g_1 \leq c g_2 \) when comparing two quantities. By \( 1 > \alpha \geq 0 \) we will denote the exponent in the Hölder norms \( C^k,\alpha \). We will not specify the precise value of \( \alpha \) unless necessary and it may vary from line to line.

2 The linear theory for the outer and the inner problem

2.1 The Dirichlet-to-Neumann map and the nondegeneracy condition

We will show that the non-degeneracy condition (1.11) implies invertibility of certain operator defined in terms of the Dirichlet-to-Neumann maps associated with the linearization of (1.7) around \( w_i, i = 1, 2 \). To define this map we introduce two symmetric, bilinear forms \( a_i \) in \( H^{1/2}(\Gamma) \) as follows

\[a_i(g_1, g_2) = \int_{\Omega_i} \nabla \varphi_1 \cdot \nabla \varphi_2 - f_u(w_i, x)\varphi_1 \varphi_2,\]
where \( \varphi_{k|R} = g_k \), and
\[
-\Delta \varphi_k = f_u(w_i, x)\varphi_k \quad \text{in } \Omega_i,
\]
\[
\varphi_k = 0 \quad \text{on } \partial \Omega \cap \partial \Omega_i.
\]
(2.1)

By the nondegeneracy assumption (Definition 1.2 (a)) we see that \( \varphi_k \) satisfying the above conditions is unique, or in other words \( g_k = 0 \) implies \( \varphi_k = 0 \). Then, by definition, \( b = D_i(g) \) is the value of the Dirichlet-to-Neumann map of \( g \) on \( \Gamma \) if
\[
a_i(g, h) = \int_{\Gamma} bh, \quad \forall h \in H^{1/2}(\Gamma).
\]

Using the bilinear form \( a_i \) and the standard functional analytic argument for \( i = 1, 2 \) we can obtain \( D_i \) as densely defined, self-adjoint operator in \( L^2(\Gamma) \). The operator we are interested in here is actually the sum of the two \( D = D_1 + D_2 \). Our condition is:
\[
\text{the operator } D : \text{Dom}(D) \to L^2(\Gamma) \text{ has bounded inverse.}
\]
(2.2)

We claim that (1.11) implies (2.2). Indeed it suffices to show that
\[
D(g) = 0 \implies g \equiv 0.
\]

By the definition of the maps \( D_i \) there exist functions \( \varphi_i \in H^2(\Omega_i) \), \( \varphi_{i|R} = g \) such that
\[
\int_{\Gamma} (\partial_{n_1} \varphi_1 + \partial_{n_2} \varphi_2) h = 0, \quad \forall h \in H^{1/2}(\Gamma).
\]

It follows that
\[
\varphi = \varphi_1 \mathbb{1}_{\Omega_1} + \varphi_2 \mathbb{1}_{\Omega_2}
\]
is a weak solution of (1.11) and thus \( \varphi \equiv 0 \) (condition (b) in Definition 1.2) which proves the claim.

2.2 The ODE system

In this section we describe some basic properties of the system
\[
\begin{cases}
-V_1'' + V_1 V_2^2 = 0 \quad \text{in } \mathbb{R}, \\
-V_2'' + V_2 V_1^2 = 0 \quad \text{in } \mathbb{R}.
\end{cases}
\]
(2.3)

In [2] it has been proved that there exists a unique solution to this system such that \( V_1, V_2 > 0 \), in \( \mathbb{R} \) and \( V_1(0) = V_2(0) = 1 \), \( V_1(t) = V_2(-t) \), \( V_1'(t) > 0 \),
\[
V_1(t) = At + B + \mathcal{O}(e^{-ct^2}) \quad \text{as } t \to +\infty,
\]
\[
V_1(t) = \mathcal{O}(e^{-ct^2}) \quad \text{as } t \to -\infty,
\]
(2.4)

for some \( A > 0, B \in \mathbb{R} \) and \( c > 0 \).

Let \( L_0 \) be the linearization of (2.3) around \( V = (V_1, V_2) \):
\[
L_0 = \begin{pmatrix}
-d^2/dx^2 + V_2^2 & 2V_1 V_2 \\
2V_1 V_2 & -d^2/dx^2 + V_1^2
\end{pmatrix}.
\]
Note that (2.3) is invariant under translation \( \tau \mapsto V(\cdot - \tau) \) and scaling \( \mu \mapsto \mu V(\mu \cdot) \). This has a simple consequence for the homogeneous system

\[
L_0 Z = 0, \quad \text{in } \mathbb{R}, \quad Z = (Z_1, Z_2).
\]

Indeed

\[
X = V' \quad \text{and} \quad Y = t V' + V,
\]

which are the derivatives of the solution with respect to the invariance parameters, belong to the fundamental set of the linearized operator. More precise information about this operator, which was proven in [2] (Proposition 5.2) is summarized below.

**Lemma 2.1**

1. The only bounded solution of the problem \( L_0 Z = 0 \) in \( \mathbb{R} \) is \( X = V' \).

2. The solution \( V \) is stable, in other words for any \( f \in C^\infty_0(\mathbb{R}) \) we have

\[
\langle L_0 f, f \rangle \geq 0.
\]

For future use we point out that

\[
V'_1(t) = -V'_2(-t) \quad \text{and} \quad V'_1(t) = A + O\left(e^{-ct^2}\right) \quad \text{as} \quad t \to +\infty,
\]

\[
V'_1(t) = O\left(e^{-ct^2}\right) \quad \text{as} \quad t \to -\infty,
\]

and

\[
tV'_1(t) + V_1(t) = 2At + B + O\left(e^{-ct^2}\right) \quad \text{as} \quad t \to +\infty,
\]

\[
tV'_1(t) + V_1(t) = O\left(e^{-ct^2}\right) \quad \text{as} \quad t \to -\infty.
\]

Our aim is to build a positive solution \((u_1, u_2)\) to (1.5) which looks like \((w_1, w_2)\) far away from \(\Gamma\) (the outer solution) and a suitable scaling of \((V_1, V_2)\) close to \(\Gamma\) (the inner solution).

### 2.3 The inner linear operator

We recall that \(\Gamma\) is a regular, simple closed curve. Denote its length by \(|\Gamma|\). By \(s \in [0, |\Gamma|]\) we will denote the arc length parameter on \(\Gamma\). Let \(b_0: [0, |\Gamma|] \to \mathbb{R}\) be defined by

\[
b_0(s) = \sqrt{\frac{\omega(s)}{A}},
\]

see (1.8) for the definition of the function \(\omega\) and (2.4) for the definition of the constant \(A\). For notational reasons it is convenient to introduce a small parameter

\[
\epsilon = \frac{1}{\sqrt{\beta}}.
\]

Additionally we introduce two more periodic functions \(b_1, b_2 \in C^{3,\alpha}([0, |\Gamma|])\) and define

\[
b(s) = b_0(s) + \epsilon b_1(s) + \epsilon^2 b_2(s).
\]

The results of the present section do not depend on the specific properties of the function \(b_1, b_2\) and we postpone their precise definitions until Sect. 3.4.1. Let \(C\) be the right cylinder \(\mathbb{R} \times [0, |\Gamma|]\). Denote

\[
\mathcal{L} = \begin{pmatrix}
-\frac{\partial^2}{\partial x^2} - \epsilon^2 b^{-2}(y) \frac{\partial^2}{\partial y^2} & V_2^2 \\
2V_1 V_2 & -\frac{\partial^2}{\partial x^2} - \epsilon^2 b^{-2}(y) \frac{\partial^2}{\partial y^2} + V_1^2
\end{pmatrix}
\]

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In this section we will study the equation
\[ \mathcal{L}\varphi = g, \quad \text{in } \mathcal{C}. \] (2.11)
In other words we suppose that the function \( \varphi \) satisfies periodic boundary conditions at \( y = 0 \) and \( y = |\Gamma| \).

Consider the following eigenvalue problem
\[ -\psi'' = \omega^2 b^2(y)\psi, \quad \text{in } (0, |\Gamma|), \]
\[ \psi(0) = \psi(|\Gamma|), \quad \psi'(0) = \psi'(|\Gamma|). \] (2.12)
We denote the eigenvalues and eigenfunctions respectively by \( \omega_k^2 \) and \( \psi_k \), \( k = 0, \ldots \). We note that \( \omega_0 = 0 \) is the eigenvalue corresponding to \( \psi_0 \equiv \frac{1}{\sqrt{|\Gamma|}} \). For large \( k \), by Weyl’s formula \[12\] (Corollary 1.41), we have
\[ \omega_{2k}^2 = \left( \frac{2\pi k}{\ell_0} \right)^2 + O\left( \frac{1}{k^2} \right), \quad \omega_{2k-1}^2 = \left( \frac{2\pi k}{\ell_0} \right)^2 + O\left( \frac{1}{k^2} \right) \quad k \to \infty, \] (2.13)
with
\[ \ell_0 = \int_0^{|\Gamma|} |b(s)| \, ds. \]
It is then convenient to relabel the eigenvalues using the map
\[ 2k \mapsto k, \quad 2k - 1 \mapsto -k. \]
Accordingly, from now on we will denote the eigenvalues by \( \omega_k \) and eigenfunctions by \( \psi_k \) with \( k \in \mathbb{Z} \).

First we will need some preliminary results. Given a function \( g \in L^2(\mathcal{C}) \), we expand it in terms of the eigenfunctions of (2.12):
\[ g(x, y) = \sum_{k = -\infty}^{\infty} g_k(x) \psi_k(y). \] (2.14)
We look for a solution of (2.11) separating variables in the form
\[ \mathcal{L}_{\omega_k} \varphi_k = g_k, \quad \text{in } \mathcal{C}, \] (2.15)
where
\[ \mathcal{L}_{\omega} = \left( -\frac{d^2}{dx^2} + V_2^2 - \frac{2V_1 V_2}{2V_1 V_2} + \frac{d^2}{dx^2} + V_1^2 \right) + \omega^2 \text{Id} \]
and \( \omega \geq 0 \). In what follows we will study the following Dirichlet problem:
\[ \mathcal{L}_{\omega} \varphi = g, \quad \text{in } (-R, R), \]
\[ \varphi(\pm R) = 0, \] (2.16)
where \( R > 0 \). A solution of (2.15) will be obtained taking \( R \to \infty \). Below we will show that \( \mathcal{L}_{\omega} \) satisfies the (ψ) property [6, Definition 1.1] hence the existence and uniqueness of solutions for (2.16). Our goal is to obtain uniform in \( \omega \in [0, \infty) \) estimates for the solution and for this purpose we will study (2.16) in the case when \( g \in C^\alpha((-R, R)) \) is such that its weighted Hölder norm defined by
\[ \| g \|_{C^\alpha((-R, R))} = \| g \|_{C^0((-R, R))} + \sup_{-R < x < R} \cosh(\theta x) \| g \|_{L^\alpha(x, x+1)} \]
is bounded. Obviously if \( g \in C^\alpha_{\partial}((−R, R)) \) then
\[
|g(x)| \leq \|g\|_{C^\alpha_{\partial}((−R, R))} \text{sech}(\theta x).
\]

Now we recall the basic fact about the systems satisfying the \((\psi)\) property [6, Theorem 1.2], which in the case at hand is the comparison principle of the form:
\[
L_\omega u = F \geq 0, \quad u(±R) = 0 \implies u \geq 0.
\]

To check the \((\psi)\) property we define \( \tilde{u}(x) = (\tilde{u}_1(x), \tilde{u}_2(x)) \) by
\[
\tilde{u}_1(x) = K \int_{−\infty}^{x} \int_{−\infty}^{y} \text{sech}(at) \ dt \ dy + M \int_{−\infty}^{x} \text{sech}(ay) \ dy,
\]
\[
\tilde{u}_2(x) = K \int_{x}^{\infty} \int_{y}^{\infty} \text{sech}(at) \ dt \ dy + M \int_{x}^{\infty} \text{sech}(ay) \ dy. \tag{2.17}
\]

Note that \( \tilde{u}(x) > 0 \) if \( K, M \) are positive. Because of the symmetry of the operator \( L_\omega \) due to \( V_1(x) = V_2(−x) \) to check that positive constants \( a, K, M \), can be chosen so that \( \tilde{u} \) is a positive supersolution of our problem it suffices to do this for \( x \geq 0 \). We have
\[
\tilde{u}_1''(x) = K \text{sech}(ax) - Ma \tanh(ax) \text{sech}(ax),
\]
\[
\tilde{u}_2''(x) = K \text{sech}(ax) + Ma \tanh(ax) \text{sech}(ax),
\]
and using
\[
V_1(x) \geq (mx + l), \quad m, l > 0, \quad x > 0,
\]
and
\[
V_2(x) \geq \frac{1}{2} \mathbb{1}_{[−δ, δ]}(x), \quad \text{for some} \quad δ > 0,
\]
we find
\[
L_\omega \tilde{u} \geq \begin{pmatrix}
-K \text{sech}(ax) + Ma \tanh(ax) \text{sech}(ax) + V_2(2) \left( K \int_{−\infty}^{x} \int_{−\infty}^{y} \text{sech}(at) \ dt \ dy + M \int_{−\infty}^{x} \text{sech}(ay) \ dy \right)

-K \text{sech}(ax) - Ma \tanh(ax) \text{sech}(ax) + V_2(1) \left( K \int_{x}^{\infty} \int_{y}^{\infty} \text{sech}(at) \ dt \ dy + M \int_{x}^{\infty} \text{sech}(ay) \ dy \right)
\end{pmatrix}
\]
\[
\geq \begin{pmatrix}
-2K + Ma \tanh(ax) + \mathbb{1}_{[−δ, δ]}(x) \left( Ka^2 + Ma^{-1} - 4K + Ma \tanh(ax) \right)

-2K + Ma \tanh(ax) + (mx + l)^2 \left( Ka^2 + Ma^{-1} \right) e^{-ax}
\end{pmatrix}
\]

Take
\[
M = \frac{8K}{a \tanh(a δ)}, \quad a \leq \min \left\{ 1/4, l \tanh^{1/2}(δa) (2 \tanh(δa) + 8)^{-1/2} \right\} \tag{2.18}
\]
to get
\[
L_\omega \tilde{u} \geq CK e^{-ax} \mathbb{1}, \quad x \geq 0, \tag{2.19}
\]
where \( C \) is a constant depending on \( a \) and \( δ \) and \( \mathbb{1} = (1, 1) \). The constant \( K \) at this point is arbitrary. Similar argument for \( x \leq 0 \) gives
\[
L_\omega \tilde{u} \geq CK \text{sech}(ax) \mathbb{1}, \quad −∞ < x < ∞.
\]

This shows that the \((\psi)\) property holds.
Denote by $\varphi_R$ the solution of (2.16) with the right hand side $g_R \in C_0^0((-R, R))$ such that $\|g\|_{C_0^0((-R, R))} = 1$. Restricting $a$ further if necessary

$$a \leq \min \left\{1/4, l \tanh^{1/2}(\delta a)(2 \tanh(\delta a) + 8)^{-1/2}, \theta \right\},$$

and taking $K$ large we get

$$L_\omega(\tilde{u} - \varphi_R)(x) \geq 0, \quad -R < x < R.$$ 

Using the comparison principle we find

$$\varphi_R(x) \leq C \tilde{u}(x).$$

The lower bound can be obtained similarly. Finally, with $g_R \in C_0^0((-R, R))$ arbitrary we find

$$|\varphi_R(x)| \leq C \|g_R\|_{C_0^0((-R, R))} |\tilde{u}(x)|.$$  \hspace{1cm} (2.20)

In case that the functions $g_R$ converge in $C_0^0(\mathbb{R})$ locally (e.g. $g_R = 1_{[-R, R]}g$ for some $g \in C_0^0(\mathbb{R})$) we can use the estimate (2.20), standard regularity arguments and the Arzela-Ascoli Theorem to pass to the limit locally in $C_0^0(R)$ for $\varphi_R$ and get a weak solution to

$$L_\omega \varphi = g, \quad \text{in} \quad \mathbb{R},$$  \hspace{1cm} (2.21)

such that

$$|\varphi(x)| \leq C \|g\|_{C_0^0(\mathbb{R})} |\tilde{u}(x)|.$$  \hspace{1cm} (2.22)

Again, standard arguments give that $\varphi$ is a classical solution. From (2.22) we see that the function $\varphi$ is locally bounded by $C\|g\|_{C_0^0(\mathbb{R})}$ but in principle $|\varphi_1(x)|$ ($|\varphi_2(x)|$) may grow linearly as $x \to \infty$ ($x \to -\infty$ respectively). We will show that this does not happen when $\omega = 0$ and if we suppose additionally that

$$\int_R g \cdot X \, dx = 0, \quad \int_R g \cdot Y \, dx = 0$$  \hspace{1cm} (2.23)

(Recall $X = V'$, $Y = (x V' + V)$). The following Lemma can be found in [1, Theorem 1.2]. For completeness we include it here together with an alternative demonstration.

**Lemma 2.2** There exists a solution $\varphi$ to (2.21) with $\omega = 0$ satisfying (2.22) such that we have

$$\|\varphi\|_{C_0^0(\mathbb{R})} \leq C \|g\|_{C_0^0(\mathbb{R})},$$  \hspace{1cm} (2.24)

provided that the orthogonality conditions (2.23) hold.

**Proof** We claim that there exists $\bar{R} > 0$ such for all $R > \bar{R}$ if $\varphi_R$ is a solution of (2.16) then

$$\|\varphi_R\|_{C_0^0((-R, R))} \leq C \|g_R\|_{C_0^0((-R, R))}$$  \hspace{1cm} (2.25)

provided that it holds

$$\int_{-R}^R g_R \cdot X \, dx = 0, \quad \int_{-R}^R g_R \cdot Y \, dx = 0.$$  \hspace{1cm} (2.26)
To prove the claim we argue by contradiction. We suppose that there exists a sequence $R_n \to \infty$ and sequences of functions $\varphi_n$ and $g_n$ that satisfy
\[ \|g_n\|_{C^0_0((-R_n,R_n))} \to 0, \quad \|\varphi_n\|_{C^0_0((-R_n,R_n))} = 1. \] (2.27)
By (2.20) we have
\[ |\varphi_n(x)| \leq \min\{\|g_n\|_{C^0_0((-R_n,R_n))}\tilde{u}(x), e^{-\theta|x|} \}, \quad x \in [-R, R]. \] (2.28)
By (2.26) we have
\[
\begin{align*}
\langle L_0\varphi_n, X \rangle &= - (\varphi_n' \cdot X) (x) \bigg|_{-R_n}^{R_n} = 0, \\
\langle L_0\varphi_n, Y \rangle &= - (\varphi_n' \cdot Y) (x) \bigg|_{-R_n}^{R_n} = 0,
\end{align*}
\]
hence using the asymptotic behavior of $V$ and $V'$ we get
\[-A\varphi_n'(-R_n) - A\varphi_n'(-R_n) = \left((\varphi_n'(R_n)| + |\varphi_n'(-R_n)|\right)O \left( e^{-R_n^{2}} \right),
\]
\[-(2AR_n + B)\varphi_{n,1}(R_n) + (-2AR_n + B)\varphi_{n,2}(-R_n) = \left((\varphi_n'(R_n)| + |\varphi_n'(-R_n)|\right)O \left( R_n e^{-cR_n^{2}} \right).\]
From (2.22) using the equation we further conclude
\[ |\varphi_n''(x)| \leq C\|g_n\|_{C^0_0((-R_n,R_n))} \text{sech}(\theta x), \quad |\varphi_n'(x)| \leq C\|g_n\|_{C^0_0((-R_n,R_n))}. \] (2.29)
It follows
\[ |\varphi_n'(R_n)| + |\varphi_n'(R_n)| \leq \|g_n\|_{C^0_0((-R_n,R_n))} O \left( R_n e^{-cR_n^{2}} \right). \] (2.30)
Integrating twice on $(x, R_n)$, $x \geq 0$ the equation
\[-\varphi_n''(x) = -V_2^2\varphi_{n,1} - 2V_1V_2\varphi_{n,2} + g_{n,1},\]
we get
\[ \varphi_n(x) = \varphi_{n,1}(R_n)(R_n - x) + O \left( \|g_n\|_{C^0_0((-R_n,R_n))} \right)e^{-\theta x} + O \left( e^{-cR_n^{2}} \right). \]
It follows that for some large, but independent on $n$ number $\tilde{x} > 0$
\[ |\varphi_n(x)| \leq \frac{e^{-\theta x}}{4}, \quad x \in [\tilde{x}, R_n], \]
for all sufficiently large $n$. Similarly we show
\[ |\varphi_n(x)| \leq \frac{e^{-\theta x}}{4}, \quad x \in [-R_n, -\tilde{x}].\]
These last two estimates and (2.28) show that for $n$ large
\[ |\varphi_n(x)| \leq \frac{1}{2} \text{sech}(\theta x), \quad x \in [-R_n, R_n], \]
which contradicts (2.27).
To finish the proof of the Lemma we use the approximation of the problem on $\mathbb{R}$ by the Dirichlet problem (2.16) replacing $g$ by
\[ g_R = g 1_{[-R,R]}(x) + \lambda_{R,1} e^{-x^2} 1 + \lambda_{R,2} e^{-x^2} 1, \]
with $\lambda_{R,i}, i = 1, 2$ chosen so that $g_R$ satisfies the (2.26). Note that
$$\lambda_{R,i} = O\left(Re^{-\bar{\theta} R}\right).$$
Passing to the limit to obtain $\varphi$ satisfying (2.2) is standard. \hfill \Box

Now we consider the problem (2.21) with $\omega > 0$. Our goal is to find a solution that decays at an exponential rate independent on $\omega$ and for this the right hand side of the equation can not be an arbitrary, exponentially decaying function. Naturally we would like to impose the orthogonality conditions (2.23) but there is an additional complication coming form the fact that the functions $X$ and $Y$ are not in the fundamental set of $L_\omega$. The following a priori estimate settles it.

**Corollary 2.1** There exists $\bar{\omega} > 0$ such that if $\varphi \in C^{2,a}_\bar{\omega}(\mathbb{R})$ is a solution of (2.21) with $\omega \in [0, \bar{\omega}]$ satisfying
$$\int g \cdot X \, dx = \omega \int \varphi \cdot X \, dx, \quad \int g \cdot Y \, dx = \omega \int \varphi \cdot Y \, dx, \quad (2.31)$$
then
$$\|\varphi\|_{C^{2,a}_\bar{\omega}(\mathbb{R})} \leq C \|g\|_{C^\omega(\mathbb{R})}. \quad (2.32)$$

**Proof** The proof is an easy consequence of Lemma 2.2 if we take $\bar{\omega} < \frac{1}{\sqrt{2C}}$, where $C$ is the constant on the right hand side of (2.24). \hfill \Box

We are in position to solve the equation (2.11). We introduce the weighted Hölder spaces $C^\omega_\theta(\hat{C})$ and $C^{2,a}_\theta(\hat{C})$ equipped with the norms
$$\|h\|_{C^\omega_\theta(\hat{C})} = \|h \cosh(\theta x)\|_{C^0(\hat{C})} + \sup_{x \in \mathbb{R}} \cosh(\theta x) [h]_{\alpha, (x, x+1) \times [0, \Gamma]},$$
$$\|h\|_{C^{2,a}_\theta(\hat{C})} = \|h \cosh(\theta x)\|_{C^0(\hat{C})} + \sum_{j=1}^k \sup_{x \in \mathbb{R}} \cosh(\theta x) \|D^j h\|_{\alpha, (x, x+1) \times [0, \Gamma]} + \sup_{x \in \mathbb{R}} \cosh(\theta x) [D^k h]_{\alpha, (x, x+1) \times [0, \Gamma]};$$
and the weighted Sobolev spaces $L^2_\theta(\hat{C})$ and $H^1_\theta(\hat{C})$ equipped, respectively, with the norms
$$\|h\|^2_{L^2_\theta(\hat{C})} = \int_{\hat{C}} \cosh^2(\theta x) |h(x, y)|^2 b_\theta^2(y) \, dx \, dy,$$
$$\|h\|^2_{H^1_\theta(\hat{C})} = \|h\|^2_{L^2_\theta(\hat{C})} + \int_{\hat{C}} \cosh^2(\theta x) \sum_{j=1}^k |D^j h(x, y)|^2 b_\theta^2(y) \, dx \, dy.$$
For any $\omega \leq \frac{1}{\sqrt{2C}}$ (c.f. Corollary 2.1) and for a given $\epsilon$ let $\bar{K}_\epsilon$ be the largest positive integer such that
$$\epsilon \omega_k < \bar{\omega}, \quad \text{for } |k| \leq \bar{K}_\epsilon. \quad (2.33)$$
Note that $\bar{K}_\epsilon = O(\epsilon^{-1})$ by the Weyl theorem. We will suppose that $g \in L^2_\theta(\hat{C})$. For such $g$ we consider the expansion (2.14) and write
$$g = g^\perp + g^\perp,$$
where
\[ g^\parallel = \sum_{k=-\tilde{K}_\epsilon}^{\tilde{K}_\epsilon} g_k \psi_k, \quad g^\perp = g - g^\parallel. \]

When convenient we will write \( P_{\tilde{K}}g \) to be the projection of \( g \) on the first \( 2\tilde{K} \) Fourier modes of \( g \) in the expansion (2.14) so that \( P_{\tilde{K}}g = g^\parallel \) and \( g^\perp = (\text{Id} - P_{\tilde{K}})g \). With these notations we decompose the Eq. (2.11) into the following problems
\[ L_{\epsilon \omega} \varphi_k = g_k, \quad \text{in } \mathbb{R}, \quad |k| = 0, \ldots, \tilde{K}_\epsilon \quad (2.34) \]
and
\[ \mathcal{L} \varphi^\perp = g^\perp, \quad \text{in } \hat{\mathcal{C}}. \quad (2.35) \]

**Proposition 2.1** Suppose that \( \theta < 1 \) and \( g \in L_0^2(\hat{\mathcal{C}}) \cap C_0^\alpha(\hat{\mathcal{C}}) \) is decomposed \( g = g^\parallel + g^\perp \) as above. There exists \( \tilde{\omega} > 0 \), small but \( \epsilon \) independent, such that with \( \tilde{K}_\epsilon \) defined in (2.33) the following hold:

(i) For any \( k \in \{-\tilde{K}_\epsilon, \ldots, \tilde{K}_\epsilon\} \) there exist constants \( \lambda_{X,k}, \lambda_{Y,k} \) and a solution \( \varphi_k \) of
\[ L_{\epsilon \omega} \varphi_k = g_k + \lambda_{X,k} \text{sech} y + \lambda_{Y,k} \text{sech} y, \quad \text{in } \mathbb{R}, \quad |k| = 0, \ldots, \tilde{K}_\epsilon \quad (2.36) \]
such that for any \( \alpha \in (0, 1), k \neq 0 \)
\[ \| \varphi_k \|_{C_{\omega}^{2,\alpha}(\mathbb{R})} \lesssim C|k|^{-\alpha} \| g_k \|_{C_0^\alpha(\mathbb{R})}. \quad (2.37) \]
Moreover, for
\[ \varphi^\parallel = \sum_{k=-\tilde{K}_\epsilon}^{\tilde{K}_\epsilon} \varphi_k \psi_k \]
we have for any \( \alpha \in (0, 1) \)
\[ \| \varphi^\parallel \|_{C_{\omega}^{2,\alpha}(\mathbb{R})} \lesssim \epsilon^{1+\alpha} \| g^\parallel \|_{C_0^\alpha(\hat{\mathcal{C}})}. \quad (2.38) \]

When \( g^\parallel \in C_0^{1,\alpha}(\hat{\mathcal{C}}) \) we have
\[ \| \varphi^\parallel \|_{C_{\omega}^{2,\alpha}(\mathbb{C})} \lesssim |\ln \epsilon| \| g^\parallel \|_{C_0^{1,\alpha}(\hat{\mathcal{C}})} \quad (2.39) \]

(ii) Assuming that \( \theta < \frac{1}{4} \tilde{\omega} \) there exists a unique solution \( \varphi^\perp \) of (2.35) such that
\[ \| \varphi^\perp \|_{C_{\omega}^{2,\alpha}(-\tilde{K}_\epsilon, \tilde{K}_\epsilon)} \lesssim C \| g^\perp \|_{C_\omega^\alpha(\hat{\mathcal{C}})}. \quad (2.40) \]

**Proof** As long as (2.31) is satisfied for each individual mode part (i) follows directly from Corollary 2.1 since, by Proposition 6.1
\[ |g_k(x)| \leq \left| \int_0^{\| \Gamma \|} g(x, y) \psi_k(y) b_0^2(y) dy \right| \lesssim |k|^{-\alpha} \| g^\parallel(x, \cdot ) \|_{C^\alpha(0, \| \Gamma \| )}, \]
hence
\[ \| g_k \|_{C_\omega^\alpha(\mathbb{R})} \lesssim \left| \int_0^{\| \Gamma \|} g(x, y) \psi_k(y) b_0^2(y) dy \right|_{C_\omega^\alpha(\mathbb{R})} \lesssim |k|^{-\alpha} \| g^\parallel \|_{C_\omega^\alpha(\hat{\mathcal{C}})}, \quad -\tilde{K}_\epsilon \leq k \leq \tilde{K}_\epsilon, \quad k \neq 0. \]
However (2.31) does not hold in general and this is why we need to introduce the Lagrange multipliers in (2.36). We fix $k$ and and look for a solution of (2.36) in the form of a fixed point problem as follows

$$L_0\phi_k = -e^2\omega_k^2\bar{\phi}_k + g_k + \lambda_{X,k}X\sech x + \lambda_{Y,k}Y\sech x,$$  \hspace{1cm} (2.41)

with $\bar{\phi}_k \in C^\alpha_g(\mathbb{R})$ given. We chose

$$\lambda_{X,k} = \left(\int_{\mathbb{R}} (e^2\omega_k^2\bar{\phi}_k - g_k) \cdot X \right) \left(\int_{\mathbb{R}} |X|^2\sech x\right)^{-1},$$

$$\lambda_{Y,k} = \left(\int_{\mathbb{R}} (e^2\omega_k\bar{\phi}_k - g_k) \cdot Y \right) \left(\int_{\mathbb{R}} |Y|^2\sech x\right)^{-1}.$$

We have

$$|\lambda_{X,k}| + |\lambda_{Y,k}| \leq C \left(e^2\omega_k^2\|\bar{\phi}_k\|_{C^\alpha_g(\mathbb{R})} + \|g_k\|_{C^\alpha_g(\mathbb{R})}\right).$$

By Lemma 2.2 we know that there exists a solution to (2.41) such that

$$\|\varphi_k\|_{C^\alpha_g(\mathbb{R})} \leq C \left(e\omega_k\|\bar{\phi}_k\|_{C^\alpha_g(\mathbb{R})} + \|g_k\|_{C^\alpha_g(\mathbb{R})}\right).$$

From this and a straightforward implementation of the fixed point argument the first estimate of part (i) follows. To finish the proof we note that as long as $0 \leq k \leq \bar{K}_e \sim 1/\epsilon$ we have (see Lemma 6.1)

$$\|\psi_k\|_{C^0(0,|\Gamma|)} \leq C.$$  

To show (2.38) we first prove that

$$\|\varphi\|_{C^\alpha_g(\bar{C})} \leq C e^{-1+\alpha} \|g\|_{C^\alpha_g(\bar{C})}.$$  \hspace{1cm} (2.42)

Indeed we have

$$\|\varphi\|_{C^\alpha_g(\bar{C})} \leq \sum_{k=-\bar{K}_e}^{\bar{K}_e} \|\varphi_k\|_{C^\alpha_g(\mathbb{R})} \|\psi_k\|_{C^0(0,|\Gamma|)} \lesssim \|g\|_{C^\alpha_g(\bar{C})} \sum_{k=1}^{\bar{K}_e} k^{-\alpha} \lesssim e^{\alpha-1} \|g\|_{C^\alpha_g(\bar{C})}.$$ 

From this we get (2.38) by standard elliptic regularity argument. The second estimate is proven in a similar way using an easy to show fact that the modes of a $C^1(0,|\Gamma|)$ function decay like $k^{-1}$ for $k$ large.

To show (ii) we fix a mode $\omega \geq \tilde{\omega}$ and consider (2.35) projected on this mode

$$L_\omega \varphi^\perp_{\omega} = g^\perp_{\omega}.$$  \hspace{1cm} (2.43)

We claim that assuming that $\theta < 1/4\tilde{\omega}$ there exists a unique solution $\varphi^\perp$ of (2.35) such that

$$\|\varphi^\perp\|_{H^2(\bar{C})} \leq C \|g^\perp\|_{L^2(\bar{C})}.$$  

It is clear that by a standard bootstrap argument we get (2.40) from this.

To show the claim we first show an a priori estimate. Write

$$\tilde{\psi}(x) = \cosh(\theta x)\varphi^\perp_{\omega}(x), \quad \tilde{h} = \cosh(\theta x)g^\perp_{\omega}(x).$$

Then

$$L_0\tilde{\psi} - 2\theta \tanh(\theta x)\tilde{\psi}' + \left[\omega^2 - \theta^2 + 2\theta^2 \tanh^2(\theta x)\right] \tilde{\psi} = \tilde{h}.$$  \hspace{1cm} (2.44)
Multiply this equation by $\tilde{\psi}$ and integrate by parts:

$$\langle L_0 \tilde{\psi}, \tilde{\psi} \rangle + \int_{\mathbb{R}} \left[ \omega^2 - \theta^2 + 2\theta^2 \tanh^2(\theta x) - \theta^2 \text{sech}^2(\theta x) \right] |\tilde{\psi}|^2 \, dx = \langle \tilde{\psi}, \tilde{h} \rangle$$

From Lemma 2.1 (i) we infer

$$\|\tilde{\psi}\|_{H^1(\mathbb{R})} \leq C \|\tilde{h}\|_{L^2(\mathbb{R})},$$

and using the equation we get

$$\|\tilde{\psi}\|_{H^2(\mathbb{R})} \leq C \|\tilde{h}\|_{L^2(\mathbb{R})}. \tag{2.45}$$

In particular the second order differential operator

$$\tilde{L}_{\omega, \theta} = L_0 - 2\theta \tanh(\theta x) \frac{d}{dx} + \left[ \omega^2 - \theta^2 + 2\theta^2 \tanh^2(\theta x) \right]$$

is injective. Considering its adjoint

$$\tilde{L}_{\omega, \theta}^* = L_0 + 2\theta \tanh(\theta x) \frac{d}{dx} + \left[ \omega^2 - \theta^2 + 2\theta^2 \tanh^2(\theta x) + 2\theta^2 \text{sech}^2(\theta x) \right]$$

in a similar way we show that it is also an injective operator and thus $\tilde{L}_{\omega, \theta}$ is surjective. From this we get the existence and uniqueness for the equation (2.44) and from the estimate (2.45) we deduce the existence of solution of (2.43) and also the estimate

$$\|\phi_{\omega}^{-1}\|_{H^2_{\omega}(\mathbb{R})} \leq C \|g_{\omega}^{-1}\|_{L^2_{\omega}(\mathbb{R})}.$$

The proof of (ii) is completed by summing up the Fourier modes and using the Plancherel identity. \qed

### 2.4 An auxiliary Lemma

The result proven below will be needed in Sect. 3.2. We remark that its proof can be found in [4]. For completeness we include it here together with a somewhat different proof.

**Lemma 2.3** There exists a one parameter family of solution of the problem

$$L_0 \psi = V', \quad \text{in} \quad \mathbb{R},$$

of the form

$$\psi = W + CV', \quad W = \frac{1}{2} x^2 V' + O(e^{-c|x|^2}),$$

where C is an arbitrary constant. Moreover $W_1(x) = -W_2(-x)$.

**Proof** To start the proof of the Lemma let $\phi = \frac{1}{2} x^2 V' + \psi$. Replacing this in the equation we get

$$L_0 \psi = x V'',$$ \tag{2.46}

so now the matter is to solve this last equation for $\psi$ with the required properties. To this end we solve first the following problem

$$L_0 \psi_R = x V'', \quad \text{in} \quad (-R, R),$$

$$\psi_R(R) = 0 = \psi_R(-R). \tag{2.47}$$

\[Springer]
We recall that the operator \( L_0 \) satisfies the \((\psi)\) property hence we get the existence and uniqueness for (2.47). More precisely we have

\[
L_0 u = F \geq 0, \quad u(R) = 0, \quad u(-R) = 0 \implies u \geq 0. \tag{2.48}
\]

Next let us consider the function

\[
\tilde{u} = (\tilde{u}_1, \tilde{u}_2), \quad \tilde{u}_1 = K_1 \int_{-\infty}^{x} e^{-a_1y^2} dy, \quad \tilde{u}_2 = K_2 \int_{x}^{\infty} e^{-a_2y^2} dy,
\]

with positive constants \( K_j \) and \( a_j, j = 1, 2 \) are to be chosen. We have

\[
L_0 \tilde{u} = \begin{pmatrix}
2K_1a_1xe^{-a_1x^2} + V_2K_1 \int_{-\infty}^{x} e^{-a_1y^2} dy + 2V_1V_2K_2 \int_{x}^{\infty} e^{-a_2y^2} dy \\
-2K_2a_2xe^{-a_2x^2} + V_1V_2K_2 \int_{x}^{\infty} e^{-a_2y^2} dy + 2V_1V_2K_1 \int_{-\infty}^{x} e^{-a_1y^2} dy
\end{pmatrix}
\]

We know that for \( x \geq 0 \)

\[
V_1(x) \geq mx + l, \quad 0 \leq V_2(x) \leq Ce^{-cx^2}, \quad |V''(x)| \leq Ce^{-cx^2},
\]

with some positive constants \( m, l, C, c \). When \( x \geq 0 \) we get

\[
L_0 \tilde{u} \geq \begin{pmatrix}
2K_1a_1xe^{-a_1x^2} \\
-2K_2a_2xe^{-a_2x^2} + (mx + l)2K_2 \int_{x}^{\infty} e^{-a_2y^2} dy
\end{pmatrix}
\]

It follows that there exist \( K_j > 0 \) large and \( a_j > 0 \) small such that

\[
L_0(\tilde{u} - \psi_R) \geq 0, \quad x \geq 0.
\]

Using the symmetry \( V_1(x) = V_2(-x) \) we get similar estimate for \( x \leq 0 \) and since \( \tilde{u} \geq 0 \) at \( x = R, -R \) by (2.48) we get

\[
\psi_R(x) \leq \tilde{u}(x).
\]

Another comparison argument based on (2.48) gives a lower bound (adjusting \( K_j, a_j \) if necessary) on \( \psi_R \) so we get

\[
|\psi_R(x)| \leq \tilde{u}(x).
\]

Using this estimate and the equation we get

\[
|\psi''_R(x)| \leq Ce^{-cx^2} \implies |\psi''_R(x)| \leq C, \quad -R < x < R
\]

Finally we note that from \( V_1(x) = V_2(-x), V'(x) = -V_2'(-x) \) and the uniqueness of solutions of (2.47) we get

\[
\psi_{R,1}(x) = -\psi_{R,2}(-x). \tag{2.49}
\]

Using the Arzela-Ascoli Theorem we conclude that \( \psi_R \to \psi \) in \( C_{loc}^1(\mathbb{R}) \) where \( \psi \) is a bounded solution of

\[
L_0 \psi = xV'' \quad \text{in} \quad \mathbb{R}. \tag{2.50}
\]

This solution inherits the symmetry (2.49), so that \( \psi_1(x) = -\psi_2(-x) \), and the bounds on \( \psi_R \). Using this and the equation we have with some positive constants \( K, a \)

\[
|\psi_1(x)| \leq K \int_{-\infty}^{x} e^{-ay^2} dy, \quad |\psi_2(x)| \leq K \int_{x}^{\infty} e^{-ay^2} dy, \quad |\psi'(x)| + |\psi''(x)| \leq Ke^{-ax^2}.
\]

\[
(2.51)
\]
Integrating twice the equation we get also that \( \lim_{x \to \infty} \psi_1(x) = c_1 A \) and \( \lim_{x \to -\infty} \psi_2(x) = -c_1 A \) for some constant \( c_1 \) hence
\[
\psi(x) = c_1 V'(x) + O(e^{-ax^2}). \tag{2.52}
\]
Setting \( \varphi = \frac{1}{2} x^2 V' + \psi + (c - c_1) V' \), with an arbitrary constant \( c \in \mathbb{R} \), we obtain the family of solutions in the form claimed. This ends the proof of the Lemma. \( \square \)

3 The approximate solution

3.1 Local coordinates near \( \Gamma \)

The curve \( \Gamma \) is parametrized by the arc length
\[
\gamma(s) := (\gamma_1(s), \gamma_2(s)), \quad s \in [0, \ell],
\]
where \( \ell := |\Gamma| \). The tangent vector and the inner normal at \( \gamma(s) \in \Gamma \) are given respectively by
\[
\tau(s) := (\dot{\gamma}_1(s), \dot{\gamma}_2(s)), \quad \nu(s) := (-\dot{\gamma}_2(s), \dot{\gamma}_1(s)).
\]
Note that the orientation of \( \Gamma \) is compatible with the choice of the unit normal in the Definition 1.1. For \( \delta > 0 \) small enough, let
\[
U_\delta := \{ x \in \Omega \mid \text{dist}(x, \Gamma) \leq \delta \}
\]
be a small neighborhood of the curve \( \Gamma \). Taking \( \delta \) smaller if needed for any \( x \in U_\delta \) there exists the unique \( (t, s) \in (-\delta, \delta) \times [0, \ell) \) such that
\[
x = \gamma(s) + t\nu(s) = (\gamma_1(s) - t\dot{\gamma}_2(s), \gamma_2(s) + t\dot{\gamma}_1(s)). \tag{3.1}
\]
By \( S_\ell \) we denote the interval \([0, \ell]\) with the ends identified and by \( \hat{C}_{\delta, \ell} \) the truncated cylinder \((-\delta, \delta) \times S_\ell \). The map
\[
X: \hat{C}_{\delta, \ell} \to U_\delta,
\]
\[
(t, s) \mapsto x,
\]
is a diffeomorphism between \( U_\delta \) and \( \hat{C}_{\delta, \ell} \). Notice that \( \{ X(0, s), s \in S_\ell \} = \Gamma \). We also have
\[
\Omega_1 \cap U_\delta = \{ x = X(t, s) \in \Omega \mid t > 0 \} \quad \text{and} \quad \Omega_2 \cap U_\delta = \{ x = X(t, s) \in \Omega \mid t < 0 \}.
\]
We will need later the expression of the Laplace operator in local coordinates. If by \( \kappa(s) \) we denote the curvature of \( \Gamma \) and \( A(t, s) = (1 - t\kappa(s))^2 \) then
\[
\Delta(t, s) = \partial_{tt} + \frac{1}{A} \partial_{ss} + \frac{\partial_t A}{2A} \partial_t - \frac{\partial_s A}{2A^2} \partial_s. \tag{3.2}
\]
We recall (c.f (1.9)) the initial outer approximation of the solution
\[
w_1^0(x) = w(x) \circ_{\Omega_1}(x), \quad w_2^0(x) = -w(x) \circ_{\Omega_2}(x).
\]
For future use we will calculate some basic quantities associated with \( w_1^0 \) along \( \Gamma \). We gather the estimates needed later in the following:
Lemma 3.1 Let $x = X(t, s)$ be the local coordinates described above. It holds
\begin{align*}
\partial_t w^0_i \circ X^{-1}(0, s) &= \omega \circ X^{-1}(0, s), \\
\partial_t w^0_2 \circ X^{-1}(0, s) &= -\omega \circ X^{-1}(0, s), \\
\partial_s w^0_i \circ X^{-1}(0, s) &= 0, \\
\partial_s w^0_2 \circ X^{-1}(0, s) &= 0, \\
\partial_{tt} w^0_i \circ X^{-1}(0, s) &= \kappa(s) \partial_t w^0_i \circ X^{-1}(0, s). \tag{3.3}
\end{align*}

Proof We only proof the last assertion. Using (3.2) and the equation satisfied by $w$ we get along $\Gamma = \{ t = 0 \}$
\begin{align*}
\partial_{tt} w(0, s) - \kappa(s) \partial_t w(0, s) = 0.
\end{align*}
We conclude invoking the definition of $w^0_i$. \hfill \Box

In what follows for a function $\varphi$ defined in a small neighborhood of $\Gamma$ we will write
\begin{align*}
\varphi(s, t) = \varphi \circ X^{-1}(s, t),
\end{align*}
and for a function $\varphi$ defined on $\Gamma$ we will write
\begin{align*}
\varphi(s) = \varphi \circ X^{-1}(s, 0),
\end{align*}
as long as it does not cause confusion.

3.2 The ansatz

In the rest of this paper we will denote $\epsilon = \beta^{-1/4}$ and consider the limit $\epsilon \to 0$ instead of $\beta \to \infty$. In this section we will introduce a preliminary candidate for the solution (1.5) which consists of an inner and outer approximation and is of the form
\begin{align*}
U_i(x) = \chi(x) \left( \epsilon v^0_i(x) + \epsilon^2 v^1_i(x) \right) + \chi_i(x) \left( w^0_i(x) + \epsilon w^1_i(x) + \epsilon^2 w^2_i(x) + \sum_{j=3}^{5} \epsilon^j w^j_i(x) \right). \tag{3.4}
\end{align*}
Here $\chi, \chi_1$ and $\chi_2$ are cut-off functions such that
\begin{align*}
supp \chi(x) &\subset \{ |t| \leq 2 \eta \}, \ supp \chi_1(x) \subset \{ t > \eta \}, \ supp \chi_2(x) \subset \{ t < -\eta \}, \\
\chi &= 1 \text{ if } |t| \leq \eta \text{ and } \chi + \chi_1 + \chi_2 = 1, \tag{3.5}
\end{align*}
where
\begin{align*}
\eta = K \epsilon |\ln \epsilon|, \ \epsilon \to 0, \tag{3.6}
\end{align*}
and $K > 0$ is a constant to be chosen later (see Sect. 4.1). In particular
\begin{align*}
\chi_1(x) = 0 \text{ if } x \in \Omega_2 \quad \text{and} \quad \chi_2(x) = 0 \text{ if } x \in \Omega_1.
\end{align*}
The functions $v^j = (v^j_1, v^j_2), j = 0, 1$ expressed in local variables (3.1) of $\Gamma$ have form
\begin{align*}
v^0(t, s) &= bV(\epsilon^{-1}b(t - \epsilon \xi)) \\
v^1(t, s) &= \kappa W(\epsilon^{-1}b(t - \epsilon \xi)). \tag{3.7}
\end{align*}
As for the functions $b$ and $\xi$ we will assume a priori the following expansions
\begin{align*}
b(s) &= b_0(s) + \epsilon b_1(s) + \epsilon^2 b_2(s), \\
\xi(s) &= \xi_1(s) + \epsilon \xi_2(s). \tag{3.8}
\end{align*}
Explicitly the function $b_0$ is given in (2.9) and is assumed to be of class $C^{3,\alpha}(0, |\Gamma|)$. For the purpose of formal calculations below the functions $b_k, \zeta_k, k = 1, 2$ are assumed \textit{a priori} to be $C^{3,\alpha}(0, |\Gamma|)$ as well. The functions $w_i^j, j = 1, \ldots, 5, i = 1, 2$ will be determined below.

### 3.3 The matching error along the interface

Expanding $w_i^0$ as Taylor polynomial of order 3 in $t$ and denoting its remainder by $T_3 w_i^0, i = 1, 2$, from Lemma 3.1 we get

$$w_1^0(t, s) = \omega(s) \left( t + \frac{1}{2} \kappa(s) t^2 \right) + (T_3 w_1^0)(t, s) \quad \text{if } t > 0,$$

$$w_2^0(t, s) = -\omega(s) \left( t + \frac{1}{2} \kappa(s) t^2 \right) + (T_3 w_2^0)(t, s) \quad \text{if } t < 0. \tag{3.9}$$

Denoting

$$v(t, s) = \epsilon v^0(t, s) + \epsilon^2 v^1(t, s)$$

and using (2.4), (2.7) and (2.8), we find that when $0 \geq t$:

$$v_1(t, s) = \epsilon b V_1 \left( \epsilon^{-1} b(t - \epsilon \zeta) \right) + \epsilon^2 b W_1 \left( \epsilon^{-1} b(t - \epsilon \zeta) \right)$$

$$= \epsilon b \left( A \epsilon^{-1} b(t - \epsilon \zeta) + B \right) + \frac{A k}{2} b^2 (t - \epsilon \zeta)^2 + \mathcal{O}_{C^{2,\alpha}} (\epsilon e^{-ct^2 \epsilon^{-2}})$$

$$= Ab^2 \left( t + \frac{1}{2} \kappa t^2 \right) + \epsilon Bb - \epsilon Ab^2 \zeta - \epsilon Ab^2 \zeta t + \epsilon^2 \frac{A}{2} \kappa b^2 \zeta^2 + \mathcal{O}_{C^{2,\alpha}} (\epsilon e^{-ct^2 \epsilon^{-2}}). \tag{3.10}$$

Similarly when $t \leq 0$ changing only $A$ to $-A$

$$v_2(t, s) = -Ab^2 \left( t + \frac{1}{2} \kappa t^2 \right) + \epsilon Bb + \epsilon Ab^2 \zeta + \epsilon Ab^2 \zeta t$$

$$- \epsilon^2 \frac{A}{2} \kappa b^2 \zeta^2 + \mathcal{O}_{C^{2,\alpha}} (\epsilon e^{-ct^2 \epsilon^{-2}}). \tag{3.11}$$

Moreover, by (2.4), (2.7) and (2.8) we also deduce

$$v_2(t, s) = \mathcal{O}_{C^{2,\alpha}} (\epsilon e^{-ct^2 \epsilon^{-2}}), \quad \text{if } 0 \geq t,$$

$$v_1(t, s) = \mathcal{O}_{C^{2,\alpha}} (\epsilon e^{-ct^2 \epsilon^{-2}}), \quad \text{if } t \leq 0. \tag{3.12}$$

To indicate that the expansions (3.10)—(3.12) can be differentiated and are valid in the $C^{2,\alpha}$ sense we have used the symbol $\mathcal{O}_{C^{2,\alpha}} (\cdot)$.
From (3.8), (3.9), (3.10) and (3.11) it follows

\[ v_1 - w_1^0 = \epsilon \left( 2Ab_1b_0t + Bb_0 - Ab_0^2\xi_1 - \kappa b_0^2\xi_1 t \right) + \epsilon^2 \left[ A (b_1 + 2b_2b_0) t + Bb_1 - A \left( 2b_1b_0\xi_1 + b_0^2\xi_2 \right) - A \left( \kappa b_0b_1\xi_1 + \kappa b_0^2\xi_2 \right) t \right] + \dfrac{A}{2} \kappa b_0^2\xi_1^2 + \mathcal{O}_{C^2.\alpha} \left( e^{-ct^2}e^{-s^2} \right) - (T_3 w_1^0)(t, s) \]

\[ v_2 - w_2^0 = \epsilon \left( -2Ab_1b_0t + Bb_0 + Ab_0^2\xi_1 + \kappa b_0^2\xi_1 t \right) + \epsilon^2 \left[ -A (b_1 + 2b_2b_0) t + Bb_1 + A \left( 2b_1b_0\xi_1 + b_0^2\xi_2 \right) + A \left( \kappa b_0b_1\xi_1 + \kappa b_0^2\xi_2 \right) t \right] - \dfrac{A}{2} \kappa b_0^2\xi_1^2 + \mathcal{O}_{C^2.\alpha} \left( e^{-ct^2}e^{-s^2} \right) - (T_3 w_2^0)(t, s) \]  

(3.13)

respectively for \( 0 \leq t \) and \( 0 \leq t \). In the next section we will find a correction to the outer expansion of the form \( w^0 + \epsilon w^1 + \epsilon^2 w^2 \) so that the terms of order \( \mathcal{O}(\epsilon) \) and \( \mathcal{O}(\epsilon^2) \) in (3.13) are removed.

### 3.4 The outer approximation

#### 3.4.1 The matching condition along \( \Gamma \)

We recall that by \( v \) we denoted to unit normal of \( \Gamma \) outer to \( \Omega_2 \). For the purpose of this section we will denote by \( n_i, i = 1, 2 \) the unit normals of \( \Gamma \) outer to \( \Omega_i \). Thus along \( \Gamma \) we have

\[ \partial_{n_1} = -\partial_v = -\partial_t, \quad \partial_{n_2} = \partial_v = \partial_t. \]

Formally, the matching conditions at order \( \mathcal{O}(\epsilon) \) will be satisfied if we require that

\[ w_1^1(t, s) = A \left( 2b_1b_0 - \kappa b_0^2\xi_1 \right) t + (Bb_0 - Ab_0^2\xi_1) + (T_2 w_1^1)(t, s), \]

\[ w_2^1(t, s) = -A \left( 2b_1b_0 - \kappa b_0^2\xi_1 \right) t + (Bb_0 + Ab_0^2\xi_1) + (T_2 w_2^1)(t, s). \]

(3.14)

At order \( \mathcal{O}(\epsilon^2) \) we impose

\[ w_1^2(t, s) = Bb_1 - A(2b_0b_1\xi_1 + b_0^2\xi_2 - \dfrac{\kappa}{2} b_0^2\xi_1) + A \left( 2b_2b_0 + b_1^2 - \kappa b_0^2\xi_2 - 2\kappa b_0b_1\xi_1 \right) t + (T_2 w_1^2)(t, s), \]

\[ w_2^2(t, s) = Bb_1 + A(2b_0b_1\xi_1 + b_0^2\xi_2 - \dfrac{\kappa}{2} b_0^2\xi_1) - A \left( 2b_2b_0 + b_1^2 - \kappa b_0^2\xi_2 - 2\kappa b_0b_1\xi_1 \right) t + (T_2 w_2^2)(t, s). \]

(3.15)

This implies that on \( \Gamma \) the following matching conditions at order \( \mathcal{O}(\epsilon) \) should hold:

\[ w_1^1(0, s) = Bb_0 - Ab_0^2\xi_1, \quad \partial_{n_1} w_1^1(0, s) = -2Ab_0b_1 + A\kappa b_0^2\xi_1, \]

\[ w_2^1(0, s) = Bb_0 + Ab_0^2\xi_1, \quad \partial_{n_2} w_2^1(0, s) = -2Ab_0b_1 + A\kappa b_0^2\xi_1. \]

(3.16)

Likewise at order \( \mathcal{O}(\epsilon^2) \) we should require

\[ w_1^2(0, s) = Bb_1 - A(2b_0b_1\xi_1 + b_0^2\xi_2 - \dfrac{\kappa}{2} b_0^2\xi_1), \]

\[ \partial_{n_1} w_1^2(0, s) = -A \left( 2b_2b_0 + b_1^2 - \kappa b_0^2\xi_2 - 2\kappa b_0b_1\xi_1 \right), \]

\[ w_2^2(0, s) = Bb_1 + A(2b_0b_1\xi_1 + b_0^2\xi_2 - \dfrac{\kappa}{2} b_0^2\xi_1), \]

\[ \partial_{n_2} w_2^2(0, s) = -A \left( 2b_2b_0 + b_1^2 - \kappa b_0^2\xi_2 - 2\kappa b_0b_1\xi_1 \right). \]

(3.17)
Note that conditions (3.16) and (3.17) share the general structure with respect to the unknowns \((b_1, \zeta_1)\) and \((b_2, \zeta_2)\) appearing on the right hand sides. Thus it is convenient to take a more general point of view in which both problems can be treated in a common framework. To this end suppose that we are given functions \(g_i \in H^{s+1/2}(\Gamma), h_i \in H^s(\Gamma)\) with \(s \geq 1\) to be specified later. To solve the matching problem (3.16) or (3.17) we consider the problem of finding functions \(k_1\) and \(k_2\) in such a way that the following linear problems can be solved for \(\psi = (\psi_1, \psi_2)\):

\[
\begin{cases}
- \Delta \psi_1 - f_u(w^0_{1i}, x)\psi_1 = 0 & \text{in } \Omega_1, \\
\psi_1 = 0 & \text{on } \partial \Omega \cap \partial \Omega_1, \\
\psi_1 = -Ab_0^2k_1 + g_1 & \text{on } \Gamma, \\
\partial_{\nu_i} \psi_1 = -2Ab_0 k_2 + A\kappa b_0^2k_1 + h_1 & \text{on } \Gamma, \\
\end{cases}
\quad \begin{cases}
- \Delta \psi_2 - f_u(w^0_{2i}, x)\psi_2 = 0 & \text{in } \Omega_2, \\
\psi_2 = 0 & \text{on } \partial \Omega \cap \partial \Omega_2, \\
\psi_2 = Ab_0^2k_1 + g_2 & \text{on } \Gamma, \\
\partial_{\nu_i} \psi_2 = -2Ab_0 k_2 + A\kappa b_0^2k_1 + h_2 & \text{on } \Gamma.
\end{cases}
\tag{3.18}
\]

This problem can be stated in terms of the Dirichlet-to-Neumann maps \(D_i\) and the map \(D = D_1 + D_2\) defined in Sect. 2.1 as the following system of equations for the unknowns \(k_1, k_2\):

\[
\begin{align*}
D_1(-Ab_0^2k_1 + g_1) &= -2Ab_0 k_2 + A\kappa b_0^2k_1 + h_1 \\
D_2(Ab_0^2k_1 + g_2) &= -2Ab_0 k_2 + A\kappa b_0^2k_1 + h_2
\end{align*}
\]

which, after adding and subtracting becomes

\[
-2Ab_0 k_2 + \frac{1}{2} \left[ D_1 \left( Ab_0^2 k_1 \right) - D_2 \left( Ab_0^2 k_1 \right) \right] + Ab_0^2 k_1
\]

\[
= \frac{1}{2} \left[ \left( -h_1 + h_2 + D_1(g_1) + D_2(g_2) \right) D_1( Ab_0^2 k_1) + D_2( Ab_0^2 k_1) \right] \\
= D_1(g_1) - D_2(g_2) + h_2 - h_1 \\
\tag{3.19}
\]

The second equation is of the form

\[
D( Ab_0^2 k_1) = D_1(g_1) - D_2(g_2) + h_2 - h_1
\]

and can be solved since, by (2.2), the operator \(D\) has bounded inverse in \(L^2(\Gamma)\). Once \(k_1\) is determined, we can determine \(k_2\) from the first relation in (3.19) using the fact that \(b_0(s) \neq 0, s \in [0, |\Gamma|]\). Summarizing we have:

**Lemma 3.2** Given functions \(g_i \in H^{s+1/2}(\Gamma), h_i \in H^s(\Gamma)\) with \(s \geq 1\) the problem (3.18) has a unique solution \(\psi \in H^{s+1}(\Omega_1), \psi \in H^{s+1}(\Omega_2), k_1 \in H^{s+1/2}(\Gamma), k_2 \in H^s(\Gamma)\).

**Proof** The proof follows directly from the properties of the Dirichlet-to-Neumann maps \(D_i\) described in Sect. 2.1. For instance under the hypothesis of the Lemma the right hand side of the second equation in (3.19) belongs to \(H^s(\Gamma)\) hence \(D( Ab_0^2 k_1) \in H^s(\Gamma)\) and then \(Ab_0^2 k_1 \in H^{s+1/2}(\Gamma)\). Since \(b_0 > 0\) is a smooth function we get \(k_1 \in H^{s+1/2}(\Gamma)\). We proceed similarly with the first equation in (3.19) to show \(k_2 \in H^s(\Gamma)\). \(\square\)

### 3.4.2 Improvement of the outer approximation

We will suppose that \(b_0 \in H^{s+1/2}(\Gamma), s > 1\). To improve the outer approximation at order \(O(\epsilon)\) we need to solve

\[
\begin{align*}
- \Delta w^1_i - f_u(w^0_{1i}, x)w^1_i &= 0 & \text{in } \Omega_i \\
w^1_i &= 0 & \text{on } \partial \Omega \cap \partial \Omega_i,
\end{align*}
\tag{3.20}
\]
supposing additionally that along $\Gamma$ the matching conditions (3.16) hold. In this case we have

$$g_i = Bb_0 \in H^{s+1/2}(\Gamma), h_i \equiv 0 \text{ and}$$

$$D(Ab_0^2 \xi_1) = D_1(Bb_0) - D_2(Bb_0)$$

$$-2 Ab_0 b_1 = -\frac{1}{2} \left[ D_1 \left( Ab_0^2 \xi_1 \right) - D_2 \left( Ab_0^2 \xi_1 \right) \right] - Ab_0^2 \kappa \xi_1 + \frac{1}{2} \left[ D_1(Bb_0) + D_2(Bb_0) \right]$$

(3.21)

It follows by Lemma 3.2 that there exists a unique solution of the above system such that $(\xi_1, b_1) \in H^{s+1/2}(\Gamma) \times H^s(\Gamma)$ and $w_1^1 \in H^{s+1}(\Omega_i), i = 1, 2$.

Next, to find the improvement of the outer approximation at order $O(\epsilon^2)$ we need to solve again the system (3.20) with $w_1^1$ replaced by $w_2^1$ together with the boundary conditions (3.17). However we should also take into account the fact that in the formal expansion of the outer problem terms of the order $O(\epsilon^2)$ depending on $w_1^1$ will appear in the right hand side. Thus we look for $w_2^1$ in the form $w_2^1 = \tilde{w}_2^1 + \hat{w}_2^1$ where

$$-\Delta \tilde{w}_2^1 - f_u(w_1^0, x)\tilde{w}_2^1 = \frac{1}{2} f_{uu}(w_1^0, x) \left( w_1^1 \right)^2 \text{ in } \Omega_i$$

$$\tilde{w}_2^1 = 0 \text{ on } \partial \Omega_i \cap \partial \Omega_i,$$

$$\tilde{w}_2^1 = 0, \text{ on } \Gamma,$$

and $\tilde{w}_2^1$ solves (3.19) with $\psi_i = \tilde{w}_2^1, k_1 = \xi_2, k_2 = b_2$ and

$$g_1 = Bb_1 + 2 Ab_0 b_1 \xi_1, \quad g_2 = Bb_1 - 2 Ab_0 b_1 \xi_1$$

$$h_1 = -Ab_1 + Ab_0 b_1 \xi_1 - \partial_n \tilde{w}_2^1, \quad h_2 = -Ab_1 + Ab_0 b_1 \xi_1 - \partial_n \hat{w}_2^2.$$

Similarly as above, by Lemma 3.2, we obtain the unique solution such that $(\xi_2, b_2) \in H^{s}(\Gamma) \times H^{s-1/2}(\Gamma)$. We summarize our discussion in the following:

**Corollary 3.1**: Let $s > 1$ be such that $b_0 \in H^{s+1/2}(\Gamma)$. There exist functions $w_j^i, \xi_j, b_j, j$, satisfying the matching conditions (3.16) and (3.17), respectively for $j = 1$ and $j = 2$ such that $w_j^1 \in H^{s+1}(\Omega_i), (\xi_1, b_1) \in H^{s+1/2}(\Gamma) \times H^s(\Gamma)$ and $w_2^1 \in H^{s+1/2}(\Omega_i) (\xi_2, b_2) \in H^{s}(\Gamma) \times H^{s-1/2}(\Gamma)$.

We see that the approximation at order $O(\epsilon^2)$ entails loss of regularity with respect to the approximation at order $O(\epsilon)$. For this reason we will assume a priori that $b_0$ is as smooth as we need or in other words we can take $s > 1$ in the Corollary 3.1 large enough to ensure that all the calculations below are justified. Since we need $b_j, \xi_j \in C^{3,\alpha}(\Gamma), j = 1, 2$ and the lower regularity a priori is that of $b_2 \in H^{s-1/2}(\Gamma)$ standard embeddings imply that if $s > 4$ the required regularity will hold with some $\alpha \in (0, 1)$. Additionally this implies that $w_i^j \in C^{3,\alpha}(\Omega_i)$.

### 3.5 Further refinement of the outer expansion

In this section we will further refine the outer expansion in $\Omega_i$. At this moment we will not need to satisfy the matching conditions. The outer approximation we have calculated so far is of the form

$$\psi = (\psi_1 I_{\Omega_1}, \psi_2 I_{\Omega_2}), \quad \psi_i = w_i^0 + \epsilon w_i^1 + \epsilon^2 w_i^2.$$

The error of this approximation is

$$\Theta = (\Theta_1 I_{\Omega_1}, \Theta_2 I_{\Omega_2}), \quad \Theta_i = -\Delta \psi_i - f(\psi_i, x), \text{ in } \Omega_i.$$
Assuming enough regularity for \( f(u, x) \) we can expand the error in powers of \( \epsilon \):
\[
\Theta_i = \epsilon^3 \Theta_i^3 + \epsilon^4 \Theta_i^4 + \epsilon^5 \Theta_i^5 + \mathcal{O}(\epsilon^6).
\]

By the non-degeneracy condition (b) in Definition 1.2 we can solve for \( j = 3, 4, 5 \) the following problems
\[
-\Delta w^j - \left[ f_u(w^0_1, x) \mathbb{1}_{\Omega_1} + f_u(w^0_2, x) \mathbb{1}_{\Omega_2} \right] w^j = -\Theta^j_1 \mathbb{1}_{\Omega_1} - \Theta^j_2 \mathbb{1}_{\Omega_2}, \quad \text{in} \quad \Omega,
\]
\[
w^j = 0 \quad \text{on} \quad \partial \Omega,
\]
We set
\[
w^j = w^j \mathbb{1}_{\Omega_i}, \quad j = 3, 4, 5,
\]
and define the the ansatz far away from the curve by setting
\[
w = (\mathbb{1}_{\Omega_1} w_1, \mathbb{1}_{\Omega_2} w_2), \quad w_i := \sum_{j=0}^{5} \epsilon^j w^j_i \quad \text{in} \quad \Omega_i. \quad (3.22)
\]
For convenience we set \( w_i \equiv 0 \) in \( \Omega_{3-i}, i = 1, 2 \) whenever we need to consider these functions in the whole \( \Omega \).

## 4 Proof of Theorem 1.1

### 4.1 Decomposition into the inner and the outer problem

We define smooth cutoff functions \( \chi, \tilde{\chi}, \hat{\chi} \) and \( \chi_i, \tilde{\chi}_i, \hat{\chi}_i, i = 1, 2 \) as follows: let \( \varphi \) be a smooth, even, cutoff function such that \( \mathbb{1}_{[-1,1]} \leq \varphi \leq \mathbb{1}_{[-2,2]} \) and \( \varphi'(x) \leq 0, x \geq 0 \). With \( \eta = K \epsilon |\ln \epsilon|^{-1/4} \), and \( \beta, \tilde{m}, \tilde{m}, \hat{m} \) to be fixed later on we set
\[
\chi(t) = \varphi(t/\eta), \quad \tilde{\chi}(t) = \varphi(\tilde{m}t/\eta), \quad \hat{\chi}(t) = \varphi(\hat{m}t/\eta),
\]
\[
\chi_1(t) = (1 - \chi(t)) \mathbb{1}_{(0,\infty)}, \quad \tilde{\chi}_1(t) = (1 - \varphi(\tilde{m}t/\eta)) \mathbb{1}_{(0,\infty)},
\]
\[
\chi_2(t) = (1 - \chi(t)) \mathbb{1}_{(-\infty,0)}, \quad \tilde{\chi}_2(t) = (1 - \varphi(\tilde{m}t/\eta)) \mathbb{1}_{(-\infty,0)}. \quad (4.1)
\]
We set \( \tilde{\chi} = \tilde{\chi}_1 + \tilde{\chi}_2 \). Furthermore, we chose \( \tilde{m} < 1, \tilde{m} < \hat{m} < 1 \) and \( \hat{m} > 1 \) such that
\[
\tilde{\chi} \equiv 1 \quad \text{in} \quad \text{supp} \nabla \chi \quad \text{and} \quad \text{supp} \nabla \tilde{\chi}_i, \quad \chi \equiv 1 \quad \text{in} \quad \text{supp} \nabla \tilde{\chi}_i,
\]
\[
\tilde{\chi}^2 \chi = \chi, \quad \tilde{\chi}_i \chi_i = \chi_i, \quad \tilde{\chi} \hat{\chi} = \hat{\chi}, \quad (1 - \tilde{\chi}) \hat{\chi} = (1 - \hat{\chi}). \quad (4.2)
\]
Recall the definition of inner approximate solution \( v(s, t) \) in (3.7). This function depends on the functions \( b = b(s) \) and \( \xi = \xi(s) \) defined in (3.8) and determined explicitly up to order \( \mathcal{O}(\epsilon^2) \) in Sect. 3.4.2 through the matching conditions. We introduce new stretched variable \( \tau \) as follows
\[
\tau = \epsilon^{-1} b(s) (t - \epsilon \xi(s)), \quad (4.3)
\]
and we will refer to \((\tau, s)\) as the inner variables. We also modify the initial ansatz by adding two new modulations functions \( \mu = \mu(\tau, s) \) and \( \xi = \xi(\tau, s) \) which are a priori assumed to be of order \( o(\epsilon) \). We will denote
\[
v^0(\tau, s) = b(s) (1 + \mu(\tau, s)) V ((1 + \mu(\tau, s)) \tau + \xi(\tau, s)),
\]
\[
v^1(\tau, s) = \kappa(s) W ((1 + \mu(\tau, s)) \tau + \xi(\tau, s)),
\]
so that
\[ v(\tau, s) = \epsilon v^0(\tau, s) + \epsilon^2 v^1(\tau, s). \]  
(4.4)

With the outer ansatz \( w \) defined in (3.22) we set
\[ U = \chi v + (\chi_1 w_1, \chi_2 w_2). \]
(4.5)

This is the new initial approximation to our problem. Additionally we introduce the additive inner perturbation \( \tilde{v} = (\tilde{v}_1, \tilde{v}_2) \in C^2(\Omega) \) and the outer perturbation \( \tilde{w} = (\tilde{w}_1, \tilde{w}_2) \in C^2(\Omega) \). We look for a solution of (4.7) in the form
\[ u = U + \tilde{\chi} \tilde{v} + \tilde{\chi} \tilde{w}. \]  
(4.6)

Denoting
\[ F(u, x) = \left( \frac{f(u_1, x) - \epsilon^{-4} u_1^2}{2 \epsilon^{-4} U_1 U_2}, \frac{2 \epsilon^{-4} U_1 U_2 - f(u_2, x) + \epsilon^{-4} U_2}{2 \epsilon^{-4} U_1 U_2} \right) : = \left( q_1, q_2 \right) \]
(4.8)

and
\[ q^0 = \left( \frac{2 \epsilon^{-2} v_1^0}{2 \epsilon^{-2} v_1^0}, \frac{2 \epsilon^{-2} v_1^0}{2 \epsilon^{-2} v_1^0} \right) : = \left( q^0_1, q^0_2 \right) \]
(4.9)

and denote
\[ E(U, x) = \Delta U + F(U, x), \]
\[ N(u, x) = F(u, x) - F(U, x) - D_u F(U, x) \cdot (u - U). \]

We need two more cutoff functions: \( \tilde{\rho} \) compactly supported and such that \( \tilde{\rho} \tilde{\chi} = \tilde{\chi} \) and \( \tilde{\rho}_i \) supported in the set \( |i| \geq \frac{m}{2} \) and satisfying \( \tilde{\rho}_i \tilde{\chi}_i = \tilde{\chi}_i \). To achieve this we may take
\[ \tilde{\rho}(t) = \chi \left( \frac{\tilde{\ell} t}{\eta} \right), \quad \tilde{\ell} < \frac{m}{2}, \quad \tilde{\rho}_i(t) = \chi_i \left( \frac{\tilde{\ell} t}{\eta} \right), \quad \tilde{\ell} > 2m. \]

Set \( \tilde{\rho} = \tilde{\rho}_1 + \tilde{\rho}_2 \) and \( \delta_i = v_i - w_i \). With these notations we decompose (4.7):
\[ - \Delta \tilde{v}_i + q_i^0 \cdot \tilde{v} = - \tilde{\rho} (q_i - q_i^0) \cdot \tilde{v} + \chi (\Delta \tilde{v}_i + F_i(U, x) + f_u(w_i, x)(1 - \chi) \delta_i) \]
\[ + \tilde{\chi} N_i + [\Delta, \chi] \delta_i + [\Delta, \tilde{\chi}] \tilde{w}_i, \quad i = 1, 2, \]
\[ - \Delta \tilde{w}_i + q_i \cdot \tilde{w} = (1 - \chi) (\Delta \tilde{w}_i + F_i(U, x) - f_u(w_i, x) \chi \delta_i) + (1 - \tilde{\chi}) N_i \]
\[ + [\Delta, \tilde{\chi}] \tilde{v}_i, \quad \text{in } \Omega, \quad i = 1, 2. \]  
(4.10)

Multiply the first two equations by \( \tilde{\chi} \) and use the relevant relations in (4.2) to get
\[ \tilde{\chi} (- \Delta \tilde{v}_i + q_i \cdot \tilde{v}) = \chi (\Delta \tilde{v}_i + F_i(U, x) + f_u(w_i, x)(1 - \chi) \delta_i) + \tilde{\chi} N_i + [\Delta, \chi] \delta_i + [\Delta, \tilde{\chi}] \tilde{w}_i \]
and multiply the remaining equations by \( \tilde{\chi} \) each to get
\[
\tilde{\chi} \left( -\Delta \tilde{w}_i + q_i \cdot \tilde{w} \right) = (1 - \chi) (\Delta w_i + F_i(U, x) - f_\mu(w_i, x) \chi \delta_i) + (1 - \tilde{\chi}) N_i + [\Delta, \tilde{\chi}] \tilde{v}_i.
\]

Adding the corresponding inner and outer equations we obtain (4.7).

The first equation in (4.10) is the inner problem and the second is the outer problem. Note that at this stage we do not require matching conditions between the inner and the outer part of the solution. This is due to the fact that their respective commutators are small enough thanks to the \( O(\epsilon^3) \) approximation. However we need to guarantee that the inner unknown \( \tilde{v} \) is exponentially decaying and for this reason we introduced the modulation functions \( \mu \) and \( \xi \) in the ansatz.

### 4.2 Norms of the perturbations

In general we will measure the size of the perturbations \( \tilde{w} \) and \( \tilde{v} \), respectively, in Hölder and in weighted Hölder norms. Therefore we will suppose \textit{a priori} that
\[
\| \tilde{w}_i \|_{C^1,\alpha(O_1)} + \| \tilde{w}_i \|_{C^1,\alpha(O_3, \cdot)} \lesssim \epsilon^{4+\tilde{\varsigma}}, \quad i = 1, 2.
\]

The inner problem in (4.10) is more conveniently stated in terms of the stretched variables \((\tau, s) \in \hat{C}\). We will use the weighted Hölder norms for the perturbations and suppose \textit{a priori}
\[
\| \tilde{v} \|_{C^2,\alpha(\hat{C})} \lesssim \epsilon^{2+\tilde{\varsigma}}.
\]

The constant \( \theta \) appearing in the exponential weight will be adjusted \textit{a posteriori}.

We will now discuss the modulation functions. Consistently with the assertions of Lemma 2.1 we suppose
\[
(\text{Id} - P_{\tilde{K}_e}) \mu \equiv 0, \quad (\text{Id} - P_{\tilde{K}_e}) \xi \equiv 0,
\]
and more explicitly we will assume the following. We set
\[
\mu(\tau, s) = \left( \sum_{-\tilde{K}_e}^{\tilde{K}_e} \hat{\mu}_k \psi_k(s) \right) \phi_{\mu}(\tau), \quad \xi(\tau, s) = \left( \sum_{-\tilde{K}_e}^{\tilde{K}_e} \hat{\xi}_k \psi_k(s) \right) \phi_{\xi}(\tau),
\]
where \( \hat{\mu}_k, \hat{\xi}_k \) are unknowns to be determined. At this point we do not need the precise form of the weight functions \( \phi_{\mu} \) and \( \phi_{\xi} \) (they will be introduced later on) but we suppose \textit{a priori} that they are even functions such that
\[
|\phi_{\mu}(\tau)| \lesssim \text{sech}(\hat{\lambda} \tau), \quad |\phi_{\xi}(\tau)| \lesssim \text{sech}(\hat{\lambda} \tau),
\]
where \( \hat{\lambda} > 0 \) is a large constant and with similar estimates for the derivatives. We will denote
\[
\hat{\mu}(s) = \sum_{-\tilde{K}_e}^{\tilde{K}_e} \hat{\mu}_k \psi_k(s), \quad \hat{\xi}(s) = \sum_{-\tilde{K}_e}^{\tilde{K}_e} \hat{\xi}_k \psi_k(s)
\]
and introduce the norms to be used for the modulation functions \( \mu(\tau, s) \) and \( \xi(\tau, s) \)
\[
\| \mu \|_{C^k,\alpha(\Gamma)} = \| \hat{\mu} \|_{C^k,\alpha((0, |\Gamma|))}, \quad \| \xi \|_{C^k,\alpha(\Gamma)} = \| \hat{\xi} \|_{C^k,\alpha((0, |\Gamma|))}.
\]

We will suppose \textit{a priori}
\[
\| \mu \|_{C^2,\alpha(\Gamma)} + \| \xi \|_{C^2,\alpha(\Gamma)} \lesssim \epsilon^{2-\tilde{\varsigma}}.
\]
We have as well (we agree that \( \theta < \lambda \))
\[
\| \mu \chi \|_{C^k_{\theta}}(\hat{\mathcal{C}}) + \| \xi \chi \|_{C^k_{\theta}}(\hat{\mathcal{C}}) \lesssim (\| \mu \|_{C^{k,0}(\Gamma)} + \| \xi \|_{C^{k,0}(\Gamma)}).
\] (4.17)

Above \( \tilde{\zeta}, \overline{\zeta}, \zeta > 0 \) are constants to be determined later.

In what follows it is important to estimate some functions coming from the outer equation and appearing in the inner equation and vice versa. Note that in the nonlinear term \( \chi \) of the inner equation in (4.10) the outer perturbation \( \hat{\mathcal{U}} \) appears multiplied by \( \tilde{\chi} \). Thus, consider a function \( \phi \in C^{1,\delta} (\Omega) \). We express \( \tilde{\chi} \hat{\phi} \) in terms of the stretched variables \((\tau, s)\) to get
\[
\| \tilde{\chi} \hat{\phi} \|_{C^{1,\delta}(\Omega)} \lesssim e^{2\theta\delta |\ln \epsilon|} \| \tilde{\phi} \|_{C^{1,\delta}(\Omega)}.
\] (4.18)

We also have a commutator term, which can be estimated as follows
\[
\| [\Delta, \tilde{\chi}] \hat{\phi} \|_{C^{1,\delta}(\hat{\mathcal{C}})} \lesssim e^{2\theta|\ln \epsilon|/\bar{m}} \left( e^{-2} \| \tilde{\phi} \|_{C^{1,\delta}(\Omega)} + e^{-1} \| \tilde{\phi} \|_{C^{1,\delta}(\hat{\mathcal{C}})} \right).
\] (4.19)

Considering the outer equation in (4.10) we note that the nonlinear function \((1 - \chi)\hat{\mathcal{N}}_i\) contains "inner" functions of the form \( \tilde{\chi} \tilde{\phi} \). We estimate them as follows
\[
\| (1 - \chi)\tilde{\chi} \tilde{\phi} \|_{C^{1,\delta}(\Omega)} \lesssim e^{-\alpha} e^{-K\theta|\ln \epsilon|/\bar{m}} \| \tilde{\phi} \|_{C^{1,\delta}(\hat{\mathcal{C}})}.
\] (4.20)

The commutator term satisfies
\[
\| [\Delta, \tilde{\chi}] \tilde{\phi} \|_{C^{1,\delta}(\hat{\mathcal{C}})} \lesssim e^{-2\alpha} e^{-K\theta|\ln \epsilon|/\bar{m}} \| \tilde{\phi} \|_{C^{1,\delta}(\hat{\mathcal{C}})}.
\] (4.21)

### 4.3 Expansion of the nonlinear term

With the notation as above we set
\[
\rho = u - U.
\]

We will separate the components of the outer equation as follows
\[
\tilde{w}_i = \tilde{w}_{i1} + \tilde{w}_{i2}, \quad \tilde{w}_{ij} = \tilde{w}_{i}(\overline{\Omega}_j), \quad i, j = 1, 2.
\]

Then
\[
\rho_i = \tilde{\chi} \tilde{w}_i + \bar{\chi}_1 \tilde{w}_{i1} + \bar{\chi}_2 \tilde{w}_{i2}.
\]

We also write
\[
N(U + \rho, x) = N_f(U + \rho, x) - \epsilon^{-4} M(U + \rho),
\] (4.22)

where
\[
N_f(U + \rho, x) = \begin{pmatrix} f(U_1 + \rho_1, x) - f(U_1, x) - f_u(U_1, x) \rho_1 \\ f(U_2 + \rho_2, x) - f(U_2, x) - f_u(U_2, x) \rho_2 \end{pmatrix},
\]
\[
M(U + \rho) = \begin{pmatrix} U_1 \rho_1^2 + 2U_2 \rho_1 \rho_2 + \rho_1 \rho_2^2 \\ U_2 \rho_1^2 + 2U_1 \rho_1 \rho_2 + \rho_1 \rho_2^2 \end{pmatrix}.
\]

In what follows we will need to estimate nonlinear functions of the perturbations localized near \( \Gamma \) and also supported in the outer region. These estimates are straightforward for \( N_f \) but the second term \( M(U + \rho) \) is more complicated.
Expanding \( f \) in the Taylor polynomial we get for \( i, j = 1, 2 \):
\[
\chi_j |N_{f,i}(U_i + \rho_i, x)| \lesssim \chi_j |\rho_i|^2 \lesssim \chi_j (\chi^2 |\bar{v}|^2 + |\bar{w}_{i1}|^2 + |\bar{w}_{i3-i}|^2). \tag{4.23}
\]
For later purpose we calculate
\[
\chi_i M_i(U + \rho) = \chi_i U_i \rho_{3-i}^2 + 2 \chi_i U_{3-i} \rho_i \rho_{3-i} + \chi_i \rho_i \rho_{3-i}^2 \tag{4.24}
\]
We have also
\[
\chi_{3-i} M_i(U + \rho) = \chi_{3-i} U_i \rho_{3-i}^2 + 2 \chi_{3-i} U_{3-i} \rho_i \rho_{3-i} + \chi_{3-i} \rho_i \rho_{3-i}^2. \tag{4.25}
\]

### 4.4 The outer problem

Let \( w = (w_1, w_2) \) be the outer approximation defined in (3.22). We set
\[
E_i^{\text{out}} := -\Delta w_i - f(w_i, x).
\]
Let \( v = (v_1, v_2) \) be the inner approximation defined in (4.4). When needed we will denote \( v = v(\tau, s; \mu, \xi) \) to indicate the dependence on the unknown modulation functions. Keep in mind that the function \( v \) depends on the local variables \( (t, s) \) through (4.3).

The error of the outer approximation and the matching error between the inner and the outer approximation is estimated in the following:

**Lemma 4.1** It holds
\[
\|E_i^{\text{out}}\|_{C^\alpha(\Omega_i)} \lesssim \epsilon^\delta, \quad i = 1, 2, \tag{4.26}
\]
with some \( \alpha > 0 \). In addition for any \( C > 1 \) we have
\[
\|w_i - v_i\|_{C^{0,\alpha}(\Omega_i \cap \{C^{-1} < |t|/\epsilon < C\})} \lesssim (\epsilon |\ln \epsilon|)^{3-\alpha} + \epsilon^{1-\alpha} e^{-c|\ln \epsilon|^2} + \epsilon e^{-\lambda C^{-1}|\ln \epsilon|}\left(\|\mu\|_{C^\alpha(\Gamma)} + \|\xi\|_{C^\alpha(\Gamma)}\right).
\]
\[
\|\partial_t (w_i - v_i)\|_{C^{0,\alpha}(\Omega_i \cap \{C^{-1} < |t|/\epsilon < C\})} \lesssim (\epsilon |\ln \epsilon|)^{2-\alpha} + \epsilon^{-\alpha} e^{-c|\ln \epsilon|^2} + \epsilon^{-\alpha} |\ln \epsilon| e^{-\lambda C^{-1}|\ln \epsilon|}\left(\|\mu\|_{C^{1,\alpha}(\Gamma)} + \|\xi\|_{C^{1,\alpha}(\Gamma)}\right). \tag{4.27}
\]

**Proof** The proof of (4.26) follows directly by construction and is omitted.

To prove (4.27) we write
\[
w_i - v_i = w_i - v_i(\cdot; \mu, \xi) = w_i - v_i(\cdot; 0, 0) + v_i(\cdot; 0, 0) - v_i(\cdot; \mu, \xi).
\]
By construction
\[
\|w_i - v_i(\cdot; 0, 0)\|_{C^{0,\alpha}(\Omega_i \cap \{C^{-1} < |t|/\epsilon < C\})} \lesssim (\epsilon |\ln \epsilon|)^{3-\alpha} + \epsilon^{1-\alpha} e^{-c|\ln \epsilon|^2} \tag{4.28}
\]
Similarly as in (4.20), taking into account (4.14), we estimate
\[
\|v_i(\cdot; \mu, \xi) - v_i(\cdot; 0, 0)\|_{C^{0,\alpha}(\Omega_i \cap \{C^{-1} < |t|/\epsilon < C\})} \lesssim \epsilon^{1-\alpha} |\ln \epsilon| e^{-\lambda C^{-1}|\ln \epsilon|}\left(\|\mu\|_{C^\alpha(\Gamma)} + \|\xi\|_{C^\alpha(\Gamma)}\right). \tag{4.29}
\]
Adding (4.28) and (4.29) we obtain the first estimate in (4.27).
To show the second estimate in (4.27) we write
\[ \mathbf{w}^0 + \epsilon \mathbf{w}^1 + \epsilon^2 \mathbf{w}^2 = \mathbf{\bar{w}}, \quad \mathbf{w} - \mathbf{\bar{w}} = \mathbf{r}. \]

By construction
\[ \| \partial_t (\mathbf{\bar{w}}_i - \mathbf{v}_i (\cdot, 0, 0)) \|_{C^{0, \alpha}(\Omega_t \cap \{C^{-1} < |t|/\epsilon |\ln \epsilon| < C\})} \lesssim (\epsilon |\ln \epsilon|)^{2-\alpha} + \epsilon^{-\alpha} e^{-c|\ln \epsilon|^2}, \tag{4.30} \]
and
\[ \| \partial_t \mathbf{r}_i \|_{C^{0, \alpha}(\Omega_t \cap \{C^{-1} < |t|/\epsilon |\ln \epsilon| < C\})} \lesssim \epsilon^3. \tag{4.31} \]

Estimating as in (4.20)
\[ \| \partial_t (\mathbf{v}_i (\cdot, 0, 0) - \mathbf{v}_i (\cdot, \mathbf{\mu}, \xi)) \|_{C^{0, \alpha}(\Omega_t \cap \{C^{-1} < |t|/\epsilon |\ln \epsilon| < C\})} \lesssim \epsilon^{-\alpha} e^{-\lambda C^{-1}|\ln \epsilon|} \| \mathbf{\mu} \|_{C^{1, \alpha}(\Gamma)} + \| \xi \|_{C^{1, \alpha}(\Gamma)} \tag{4.32} \]
The last three estimates give the second bound in (4.27). \hfill \Box

As a byproduct of the proof we observe that the map \((\mathbf{\mu}, \xi) \mapsto \mathbf{v}(\cdot; \mathbf{\mu}, \xi)\) is Lipschitz in the following sense

**Corollary 4.1** The following estimates hold
\[
\begin{align*}
\| \mathbf{v}_i (\cdot; \mathbf{\mu} + \delta \mathbf{\mu}, \xi + \delta \xi) - \mathbf{v}_i (\cdot; \mathbf{\mu}, \xi) \|_{C^{0, \alpha}(\Omega_t \cap \{C^{-1} < |t|/\epsilon |\ln \epsilon| < C\})} \\
\lesssim \epsilon^{-\alpha} |\ln \epsilon| e^{-\lambda C^{-1}|\ln \epsilon|} \left( \| \delta \mathbf{\mu} \|_{C^{1, \alpha}(\Gamma)} + \| \delta \xi \|_{C^{1, \alpha}(\Gamma)} \right), \\
\| \partial_t (\mathbf{v}_i (\cdot; \mathbf{\mu} + \delta \mathbf{\mu}, \xi + \delta \xi) - \mathbf{v}_i (\cdot; \mathbf{\mu}, \xi)) \|_{C^{0, \alpha}(\Omega_t \cap \{C^{-1} < |t|/\epsilon |\ln \epsilon| < C\})} \\
\lesssim \epsilon^{-\alpha} |\ln \epsilon| e^{-\lambda C^{-1}|\ln \epsilon|} \left( \| \delta \mathbf{\mu} \|_{C^{1, \alpha}(\Gamma)} + \| \delta \xi \|_{C^{1, \alpha}(\Gamma)} \right). \tag{4.33} \\
\end{align*}
\]

We will further recast the outer problem to give it a more convenient form. We have
\[
\mathbf{\tilde{\rho}}_i q_{ii} = -\mathbf{\tilde{\rho}} f_u(U_i, x) + O(e^{-c|\ln \epsilon|^2}) \\
= -f_u(w_i^0, x) + (1 - \mathbf{\tilde{\rho}}) f_u(w_i^0, x) - \mathbf{\bar{\rho}} \left( f_u(U_i, x) - f_u(w_i^0, x) \right) + O(e^{-c|\ln \epsilon|^2}) \\
\mathbf{\tilde{\rho}} q_{ij} = O(e^{-c|\ln \epsilon|^2}), \quad i \neq j.
\]
We note that
\[
f_u(w_i^0, x) = f_u(w_i^0, x) \mathbf{1}_{\Omega_i}.
\]
Thus, we have
\[
\mathbf{\tilde{\rho}} q = \begin{pmatrix} -f_u(w_i^0, x) \mathbf{1}_{\Omega_i} & 0 \\ 0 & -f_u(w_j^0, x) \mathbf{1}_{\Omega_j} \end{pmatrix} + \epsilon \mathbf{q}^1,
\]
where \( \mathbf{q}^1 \) is a 2 \times 2 matrix whose entries are Hölder continuous functions.

To solve the nonlinear outer problem we will first solve the following linear system with \( i = 1, 2 \):
\[
-\Delta \psi_i - f_u(w_i^0, x) \mathbf{1}_{\Omega_i} \psi_i + \epsilon (\mathbf{q}^1 \cdot \mathbf{\psi})_i = (1 - \chi) h_i, \quad \text{in} \quad \Omega \setminus \Gamma, \\
\psi_i = 0, \quad \text{on} \quad \partial \Omega. \tag{4.34}
\]
We assume that the right hand sides in (4.34) are Hölder continuous. We look for a solution in the form
\[
\mathbf{\psi} = (\psi_1, \psi_2) = (\mathbf{1}_{\Omega_1}, \mathbf{1}_{\Omega_2}) \hat{\mathbf{\psi}} + (\mathbf{1}_{\Omega_2} \varphi_1, \mathbf{1}_{\Omega_1} \varphi_2) \tag{4.35}
\]
where \( \tilde{\psi} \in C^{2,\alpha} \) and \( \varphi_1 \in C^{2,\alpha}(\Omega_2), \varphi_2 \in C^{2,\alpha}(\Omega_1) \) are determined from the following system

\[
- \Delta \tilde{\psi} - \left\{ \left[ f_u(w_1^0, x) + \epsilon q_{11}^1 \right] \mathbb{1}_{\Omega_1} + \left[ f_u(w_2^0, x) + \epsilon q_{12}^2 \right] \mathbb{1}_{\Omega_2} \right\} \tilde{\psi} \\
+ \epsilon \left[ q_{12}^1 \mathbb{1}_{\Omega_1} \varphi_2 + q_{21}^1 \mathbb{1}_{\Omega_2} \varphi_1 \right] = (1 - \chi) (h_1 \mathbb{1}_{\Omega_1} + h_2 \mathbb{1}_{\Omega_2}), \quad \text{in} \ \Omega,
\]

\( \tilde{\psi} = 0, \quad \text{on} \ \partial \Omega. \) (4.36)

and

\[
- \Delta \varphi_1 + \epsilon q_{11}^1 \mathbb{1}_{\Omega_2} \varphi_1 + \epsilon q_{12}^1 \mathbb{1}_{\Omega_2} \tilde{\psi} = (1 - \chi) \mathbb{1}_{\Omega_2} h_1, \quad \text{in} \ \Omega_2
\]

\[
- \Delta \varphi_2 + \epsilon q_{22}^1 \mathbb{1}_{\Omega_1} \varphi_2 + \epsilon q_{21}^1 \mathbb{1}_{\Omega_1} \tilde{\psi} = (1 - \chi) \mathbb{1}_{\Omega_1} h_2, \quad \text{in} \ \Omega_1
\]

\( \varphi_1 = 0 = \varphi_2, \quad \text{on} \ \Gamma
\]

\( \varphi_1 = 0 = \varphi_2, \quad \text{on} \ \partial \Omega. \) (4.37)

The last boundary condition could be vacuous for one of the functions \( \varphi_i \).

**Lemma 4.2** Assuming that \( (1 - \chi)h_i \in C^\alpha(\Omega), i = 1, 2, \) for each sufficiently small \( \epsilon > 0 \) the system (4.36)–(4.37) has a unique solution \( \tilde{\psi} \in C^{2,\alpha}(\Omega) \) and \( \varphi_1 \in C^{2,\alpha}(\Omega_2), \varphi_2 \in C^{2,\alpha}(\Omega_1) \) such that

\[
\| \tilde{\psi} \|_{C^{2,\alpha}(\Omega)} + \| \varphi_1 \|_{C^{2,\alpha}(\Omega_2)} + \| \varphi_2 \|_{C^{2,\alpha}(\Omega_1)} \leq \|(1 - \chi)h_1 \|_{C^\alpha(\Omega_1)} + \|(1 - \chi)h_2 \|_{C^\alpha(\Omega_2)}.
\]

Moreover the function \( \tilde{\psi} \) defined in (4.35) is a solution of (4.34).

**Proof** By the hypothesis of non degeneracy (1.11) given Hölder continuous functions \( \varphi_i \) we can solve (4.36) uniquely. Standard elliptic theory and non degeneracy imply the following estimate:

\[
\| \tilde{\psi} \|_{C^{2,\alpha}(\Omega)} \leq C \left( \|(1 - \chi)h_1 \|_{C^\alpha(\Omega_1)} + \|(1 - \chi)h_2 \|_{C^\alpha(\Omega_2)} \right)
\]

\[+ O(\epsilon) \left( \| \varphi_1 \|_{C^{\alpha}(\Omega_2)} + \| \varphi_2 \|_{C^{\alpha}(\Omega_1)} \right). \] (4.39)

Since, for a given \( \tilde{\psi} \in C^\alpha(\Omega) \) each of the equations in (4.37) is a perturbation of the Poisson equation in \( \Omega_i \) similarly we get the existence of a unique solutions such that

\[
\| \varphi_1 \|_{C^{2,\alpha}(\Omega_2)} \leq C \left\| (1 - \chi)h_1 \right\|_{C^\alpha(\Omega_2)} + O(\epsilon) \| \tilde{\psi} \|_{C^\alpha(\Omega)}
\]

\[
\| \varphi_2 \|_{C^{2,\alpha}(\Omega_1)} \leq C \left\| (1 - \chi)h_2 \right\|_{C^\alpha(\Omega_1)} + O(\epsilon) \| \tilde{\psi} \|_{C^\alpha(\Omega)} \] (4.40)

Combining (4.39)–(4.40) with a straightforward fixed point argument we get the existence and uniqueness together with the estimate (4.38). Checking directly we verify the rest of the Lemma.

Our next objective is to estimate the right hand side of the outer equation in (4.10). Denote by \( e_1 = (1, 0), e_2 = (0, 1) \) and

\[
h = h_1 e_1 + h_2 e_2
\]

\[
h_i = (\Delta w_i + F_i(U, x) - f_u(w_i, x)\chi(v_i - w_i)) + (1 - \hat{\chi})N_i + [\Delta, \hat{\chi}]\tilde{v}_i
\]

Lemma 4.2 suggests that we should further decompose

\[
h = \mathbb{1}_{\Omega_1} h + \mathbb{1}_{\Omega_2} h = \sum_{i,j} h_{ij} e_i, \quad h_{ij} = h_i \mathbb{1}_{\Omega_j}.
\]
Observe that
\[
(1 - \chi)h = \sum_{i,j} \chi_j h_{ij} e_i.
\]

As we will see the size of the error \(\chi_j h_{ij}\) will depend on whether \(i = j\) or \(i \neq j\).

**Lemma 4.3** The following estimates hold for \(i = 1, 2:\)
\[
\|h_{ii}\|_{C^a(\Omega_i)} \lesssim (\epsilon | \ln \epsilon |)^{5-\alpha} + e^{-c|\ln \epsilon|^2} + \epsilon^{-4-\alpha} \|\bar{w}_{3-i}\|_{C^a(\Omega_i)}^2 \lesssim \epsilon^{1-3\alpha} e^{-c|\ln \epsilon|^2}.
\]

**Proof** We start with the estimate for \(h_{ii}\). We have
\[
h_{ii} = \chi_i (\Delta w_i + F_i(U, x) - f_u(w_i, x) \chi(v_i - w_i)) + \chi_i (1 - \hat{\chi}) N_i + \chi_i [\Delta, \hat{\chi}] \bar{v}_i.
\]

To estimate the first term on the right hand side above we denote \(\delta_i = v_i - w_i\) and write
\[
\chi_i (\Delta w_i + F_i(U, x) - f_u(w_i, x) \chi \delta_i) = \chi_i E_{i}^{out} + \chi_i (f(w_i + \chi \delta_i, x) - f(w_i, x) - f_u(w_i, x) \chi \delta_i) - \epsilon^{-4} \chi_i (w_i + \chi \delta_i) (w_{3-i} + \chi \delta_{3-i})^2.
\]

Let us consider the last term above. By definition of the outer approximation \(w_{3-i} \equiv 0\) in \(\Omega_i\). In addition, by definition of the inner solution \((3.7)\) and a priori assumed size of the modulation functions \(\mu\) and \(\xi\) we see that
\[
\|\chi_i (w_i + \chi \delta_i) (w_{3-i} + \chi \delta_{3-i})^2 \|_{C^a(\Omega_i)} \equiv \|\chi_i \chi^2 \delta_{3-i} (w_i + \chi \delta_i) \|_{C^a(\Omega_i)} \lesssim \epsilon^{1-3\alpha} e^{-c|\ln \epsilon|^2}.
\]

From this and Lemma 4.1 it follows
\[
\|\chi_i (\Delta w_i + F_i(U, x) - f_u(w_i, x) \chi \delta_i) \|_{C^a(\Omega_i)} \lesssim \|\chi_i E_{i}^{out} \|_{C^a(\Omega_i)} + \|\chi_i \chi \delta_i \|_{C^a(\Omega_i)}^2 + \epsilon^{-4} \|\chi_i (w_i + \chi \delta_i) (w_{3-i} + \chi \delta_{3-i})^2 \|_{C^a(\Omega_i)} \lesssim (\epsilon | \ln \epsilon |)^{5-\alpha} + e^{-c|\ln \epsilon|^2}.
\]

To estimate the second term on the right hand side of (4.42) we use the decomposition (4.22). From (4.23) we we have
\[
\|\chi_i (1 - \hat{\chi}) N_{f,i} (U_i + \rho_i, x) \|_{C^a(\Omega_i)} \lesssim \|\chi_i (1 - \hat{\chi}) \chi^2 \delta_i \|_{C^a(\Omega_i)} + \|\chi_i \bar{w}_{ii} \|_{C^a(\Omega_i)} \lesssim \epsilon^{-\alpha} \left( e^{-K|\ln \epsilon|/\tilde{m}} \|\bar{v} \|_{C^a(\tilde{\Omega})}^2 + \|\bar{w}_i \|_{C^a(\Omega_i)}^2 \right) .
\]
Also from (4.22), using (4.24) and \( |\tilde{\chi} U_i| \lesssim \epsilon \ln \epsilon \) we find
\[
e^{-4} \| \chi(1 - \hat{\chi}) M_j(U + \rho) \|_{C^0(\Omega_i)} \lesssim e^{-4-\alpha} \left( e^{-K\theta |\ln \epsilon|/\tilde{m}} \epsilon \ln \epsilon \| \tilde{v}_i \|_{C^0_\alpha(\tilde{C}_i)} \right)
\]
\[
+ e^{-4-\alpha} e^{-c|\ln \epsilon|^2} \left( \| \tilde{w}_1 \|_{C^0(\Omega_i)}^2 + \| \tilde{w}_2 \|_{C^0(\Omega_i)}^2 \right)
\]
\[
+ e^{-4-\alpha} \| \tilde{w}_i \|_{C^0(\Omega_i)} \| \tilde{w}_3-i \|_{C^0(\Omega_i)}^2.
\]

Finally, the commutator term is estimated using (4.21) by
\[
\| [\Delta, \tilde{\chi}] \tilde{v}_i \|_{C^0(\Omega_i)} \lesssim e^{-2-\alpha} e^{-K\theta |\ln \epsilon|/\tilde{m}} \| \tilde{v}_i \|_{C^0_{1,\alpha}(\tilde{C})}
\] (4.43)

Summarizing these estimates (and taking \( \epsilon \) small) we get the first bound in (4.41).

To show the second bound in (4.41) we write for \( i = 1, 2 \):
\[
h_{3-i} = \chi_{3-i} (\Delta w_i + F_j(U, x) - f_{\mu}(w_i, x) \chi \delta_j) + \chi_{3-i} (1 - \tilde{\chi}) N_i + \chi_{3-i} [\Delta, \tilde{\chi}] \tilde{v}_i.
\] (4.44)

In the supp \( \chi_{3-i} \) we have \( w_i \equiv 0 \), and \( v_i = \mathcal{O}(\epsilon^{-c|\ln \epsilon|^2}) \) by construction hence
\[
\| \chi_{3-i} (\Delta w_i + F_j(U, x) - f_{\mu}(w_i, x) \chi \delta_j) \|_{C^0(\Omega_{3-i})} \lesssim e^{-c|\ln \epsilon|^2}. \]

Similarly as above
\[
\| \chi_{3-i} (1 - \tilde{\chi}) N_{f,j}(U_i + \rho_i, x) \|_{C^0(\Omega_{3-i})} \lesssim e^{-2-\alpha} \left( e^{-2K\theta |\ln \epsilon|/\tilde{m}} \| \tilde{v}_i \|_{C^0_{1,\alpha}(\tilde{C})} \right)
\]
from (4.25) we get
\[
e^{-4} \| \chi_{3,i} (1 - \tilde{\chi}) M_j(U + \rho) \|_{C^0(\Omega_{3-i})} \lesssim e^{-4-\alpha} e^{-K\theta |\ln \epsilon|/\tilde{m}} \epsilon \ln \epsilon \| \tilde{v}_i \|_{C^0_{1,\alpha}(\tilde{C})}
\]
\[
+ e^{-4-\alpha} \left( \| \tilde{w}_1 \|_{C^0(\Omega_{3-i})}^2 + \| \tilde{w}_2 \|_{C^0(\Omega_{3-i})}^2 \right)
\]
\[
+ e^{-4-\alpha} \| \tilde{w}_i \|_{C^0(\Omega_{3-i})} \| \tilde{w}_3-i \|_{C^0(\Omega_{3-i})}. \]

The commutator term is estimated in a similar was as in the estimate for \( h_{ij} \). Combing the above bounds we conclude the proof.

As a byproduct of the above we have

**Corollary 4.2** The maps

\[
(\rho, \mu, \xi) \mapsto h_{ij}(\cdot; \rho, \mu, \xi)
\]

are Lipschitz in the following sense
\[
\| h_{ij}(\cdot; \rho, \mu, \xi) - h_{ij}(\cdot; \rho + \delta \rho, \mu + \delta \mu, \xi + \delta \xi) \|_{C^0(\Omega_j)} \lesssim e^{-\epsilon} e^{-K\tilde{\chi}|\ln \epsilon|} \left( \| \delta \mu \|_{C^{1,\alpha}(\Gamma)} + \| \delta \xi \|_{C^{1,\alpha}(\Gamma)} \right)
\]
\[
+ e^{-\epsilon} \left( e^{-2K\theta |\ln \epsilon|/\tilde{m}} + e^{-2-\alpha} e^{-2K\theta |\ln \epsilon|/\tilde{m}} \right) \| \delta \tilde{v}_i \|_{C^0_{1,\alpha}(\tilde{C})}
\]
\[
+ e^{-\epsilon} \sum_{i=1,2} \left( \| \delta \tilde{w}_i \|_{C^0(\Omega_i)} + \| \delta \tilde{w}_{3-i} \|_{C^0(\Omega_{3-i})} \right), \quad (4.45)
\]

where, respectively, \( \tilde{\epsilon}, \tilde{\epsilon} > 0 \) are the constants in (4.11) and (4.12).
4.5 The error of the inner approximation

We denote the right hand side of the first equation in (4.10) by \( g = (g_1, g_2) \):

\[
g_i = -\tilde{\rho}(q_i - q_i^0) \cdot \bar{v} + \chi (\Delta v_i + F_i(U, x) + f_u(w_i, x)(1 - \chi)\delta_i) + \tilde{\chi} N_i + [\Delta, \chi] \delta_i + [\Delta, \tilde{\chi}] \tilde{w}_i. \tag{4.46}
\]

The function \( g \) represents the inner error we will calculate now.

Below we will will use the same symbol for functions of the local, inner variables \((t, s)\) and the stretched variables \((\tau, s)\) defined in (4.3). Using (3.2) it is straightforward to find the expression of the Laplacian in terms of \((\tau, s)\). We denote

\[
A_\epsilon(\tau, s) = (1 - \epsilon \kappa(s)(\tau b^{-1}(s) + \xi(s)))^2,
\]

and

\[
\dot{b}(s) = \frac{d}{ds} b(s), \quad \ddot{b}(s) = \frac{d^2}{ds^2} b(s), \quad \dot{\xi}(s) = \frac{d}{ds} \xi(s), \quad \ddot{\xi}(s) = \frac{d}{ds} \kappa(s),
\]

with similar rule for other functions of \(s\). With this notation

\[
\Delta = \epsilon^{-2} \left( b^2 \partial_{\tau\tau} + \epsilon^2 A_\epsilon^{-1} \partial_{ss} \right) - \epsilon^{-1} \kappa b A_\epsilon^{-1/2} \partial_\tau + a_{11} \partial_\tau \partial_\tau + a_{12} \partial_\tau \partial_s + c_1 \partial_\tau + \epsilon c_2 \partial_s,
\]

where

\[
a_{11} = A_\epsilon^{-1} (b \dot{b} - b \dot{\xi})^2, \quad a_{12} = 2 A_\epsilon^{-1} (b \dot{b} - b \dot{\xi}),
\]

\[
c_1 = \left( \epsilon A_\epsilon^{-3/2} \kappa (\tau b^{-1} + \xi) (b \dot{b} - b \dot{\xi}) + A_\epsilon^{-1} (b \dot{b} - b \dot{\xi}) \right),
\]

\[
c_2 = A_\epsilon^{-3/2} \kappa (b^{-1} \tau + \xi).
\]

For later purpose we set

\[
D_\epsilon^0 = \epsilon^{-2} \left( b^2 \partial_\tau \partial_\tau + \epsilon^2 A_\epsilon^{-1} \partial_\tau \partial_\tau \right) - \epsilon^{-1} \kappa b A_\epsilon^{-1/2} \partial_\tau,
\]

\[
D_\epsilon^1 = a_{11} \partial_\tau \partial_\tau + a_{12} \partial_\tau \partial_s + c_1 \partial_\tau + \epsilon c_2 \partial_s.
\]

Our first task is to calculate

\[
E^{\text{in}}(v) = \begin{pmatrix}
\Delta v_1 - \epsilon^{-4} v_1 v_2 \\
\Delta v_2 - \epsilon^{-4} v_1 v_2
\end{pmatrix},
\]

in the neighborhood \( U_\chi = \{ \chi \neq 0 \} \), the interior of the supp \( \chi \), around \( \Gamma \). Recall that the approximate solution \( v \) depends on the modulation functions \( \mu \) and \( \xi \). For notational convenience we set

\[
m(\tau, s) = b(s)(1 + \mu(\tau, s)), \quad k(\tau, s) = \tau \mu(\tau, s) + \xi(\tau, s),
\]

so that

\[
v(\tau, s) = \epsilon m V(\tau + k) + \epsilon^2 \kappa(s) W(\tau + k).
\]

We will denote

\[
V'(\tau + k) = \left. \frac{d}{dx} V(x) \right|_{x = \tau + k}, \quad V''(\tau + k) = \left. \frac{d^2}{dx^2} V(x) \right|_{x = \tau + k},
\]

\[
W'(\tau + k) = \left. \frac{d}{dx} W(x) \right|_{x = \tau + k}, \quad W''(\tau + k) = \left. \frac{d^2}{dx^2} W(x) \right|_{x = \tau + k}.
\]
We calculate
\[ D_e^0(\epsilon m V) = -\frac{1}{4} \left( b^2 \partial_{tt} + A_e^{-1} \partial_{ss} \right) (mV) - \kappa b A_e^{-1/2} \partial_t (mV) \]
\[ = -b^2 \epsilon m V'' - \kappa b \eta m V' + \Omega_1 + \Theta_1 \]
where
\[ \Omega = -b^2 \epsilon V (b^2 \partial_t m + e^2 A_e^{-1} \partial_s m) + e^2 V' (b^2 (2 \partial_t m + \partial_s m) + e^2 A_e^{-1} m \partial_s m) \]
\[ + 2 \epsilon^2 \eta m V'' \partial_t k, \]
\[ \Theta_1 = -b^2 \epsilon V (2 b^2 \partial_t m \partial_s k + 2 e^2 A_e^{-1} \partial_s m \partial_t k) V' + \epsilon^2 (b^2 \partial_t k) V'' + (e^2 A_e^{-1} (\partial_s k)^2) V'' \]
\[ - \kappa b A_e^{-1/2} \partial_t m V - \kappa b \epsilon^2 (\tau b - 1 + \xi) A_e^{-1/2} m V'. \]

Similarly we have
\[ e^2 D_e^0(\kappa W) = (b^2 \partial_{tt} + e^2 A_e^{-1} \partial_{ss}) (\kappa W) - \kappa b A_e^{-1/2} \partial_t (\kappa W) = bm \kappa W'' + \Theta_2 \]
where
\[ \Theta_2 = b \kappa (2 b \partial_t k + b (\partial_s k)^2 - \mu k) W'' + b^2 \kappa \partial_t k W' + e^2 A_e^{-1} \partial_{ss} (\kappa W) - \kappa b A_e^{-1/2} \partial_t (\kappa W). \]

Note that the terms involved in $D_e^1$ are similar to those of $D_e^0$ but multiplied additionally by $\epsilon$.

The components of the nonlinear term in $E^{in}$ are, with $i = 1, 2$:
\[ -\epsilon^4 v_i V_{3-i} = -\epsilon^4 (m V_i + \epsilon \kappa W_i) (m V_{3-i} + \epsilon \kappa W_{3-i})^2 \]
\[ = -\epsilon^4 b^2 \mu V_i V_{3-i} + \kappa b m (V_{3-i} W_i + 2 V_{3-i} V_i W_{3-i}) + \Theta_3 \]
where $\Theta_3 = (\Theta_3_1, \Theta_3_2)$ is
\[ \Theta_3_1 = -\epsilon^4 m (m^2 - b^2) \mu V_i V_{3-i}^2 + \kappa m \mu (V_{3-i} W_i - 2 V_{3-i} V_i W_{3-i}) \]
\[ - \epsilon^4 b^2 \mu V_i W_{3-i}^2 - 2 V_{3-i} W_i W_{3-i} \]

We will denote
\[ \Theta_4 = D_e^1(\epsilon m V + e^2 \kappa W) = (a_{11} \partial_{tt} + a_{12} \partial_{ts} + c_1 \partial_t + \epsilon c_2 \partial_s) (\epsilon m V + e^2 \kappa W). \]

Adding the above identities and using the definition of $W$ we get
\[ E^{in} = \Omega + \sum_{j=1}^{4} \Theta_j. \]

We will estimate the terms on the right hand side of the above identity and also study its Lipschitz character as functions of the modulations functions $\mu, \xi$ in the weighted Hölder space $C^\alpha_\mu (\tilde{C})$. Before doing this we will derive a more explicit formula for $\Omega = \Omega_0 + \Omega_1$, where
\[ \Omega_0 = \epsilon^4 b (V + \tau V') (b^2 \partial_{tt} \mu + e^2 A_e^{-1} \partial_{ss} \mu) + \epsilon^4 b V' (b^2 \partial_t \mu + e^2 A_e^{-1} \partial_{ss} \xi) \]
\[ + 4 \epsilon^4 b^3 V' \partial_t \mu + 2 \epsilon^4 b^2 V'' \mu + \tau \partial_t \mu + \partial_t \xi, \]
\[ \Omega_1 = V A_e^{-1} (\epsilon b b^{-1} m + 2 \epsilon b \partial_t \mu) + \epsilon^4 V' b^2 \mu \partial_t k + \epsilon A_e^{-1} V' \mu \partial_{ss} k + 2 \epsilon^4 b^2 \mu V'' (\partial_s \alpha \beta) \]
The operator $(\mu, \xi) \mapsto \Omega_0$ will play an important role in the derivation of the modulation equation later on. We recall that $\chi = \chi(t/\eta)$ with $\eta = K \epsilon |\ln \epsilon|$. Using
\[ t = \epsilon (\tau b(s) + \xi(s)) \]
Lemma 4.4 The following estimates hold
\[
\|\chi \Omega_0 \|_{C^0_t(\hat{C})} \lesssim \epsilon^{-1} \left( \| \mu \|_{C^{2,\alpha}(\Gamma)} + \| \xi \|_{C^{2,\alpha}(\Gamma)} \right),
\]
\[
\|\chi \Omega_1 \|_{C^0_t(\hat{C})} \lesssim \epsilon e^{2K\theta|\ln |} \ln | + \| \mu \|_{C^{1,\alpha}(\Gamma)} + \epsilon^{-1} \left( \| \mu \|_{C^{2,\alpha}(\Gamma)} + \| \xi \|_{C^{2,\alpha}(\Gamma)} \right),
\]
and
\[
\|\chi \Theta_1 \|_{C^0_t(\hat{C})} \lesssim \epsilon e^{2K\theta|\ln |} \ln | + \left( \| \mu \|_{C^{2,\alpha}(\Gamma)} + \| \xi \|_{C^{2,\alpha}(\Gamma)} \right) + \\
\epsilon^{-1} \left( \| \mu \|_{C^{2,\alpha}(\Gamma)} + \| \xi \|_{C^{2,\alpha}(\Gamma)} \right),
\]
\[
\|\chi \Theta_2 \|_{C^0_t(\hat{C})} \lesssim \epsilon e^{2K\theta|\ln |} \ln | + \left( \| \mu \|_{C^{2,\alpha}(\Gamma)} + \| \xi \|_{C^{2,\alpha}(\Gamma)} \right),
\]
\[
\|\chi \Theta_3 \|_{C^0_t(\hat{C})} \lesssim \epsilon e^{2K\theta|\ln |} \ln | + \epsilon^{-1} \| \mu \|_{C^{2,\alpha}(\Gamma)},
\]
\[
\|\chi \Theta_4 \|_{C^0_t(\hat{C})} \lesssim \epsilon e^{2K\theta|\ln |} \ln | + \epsilon \left( \| \mu \|_{C^{2,\alpha}(\Gamma)} + \| \xi \|_{C^{2,\alpha}(\Gamma)} \right).
\]
In addition the function $(\mu, \xi) \mapsto \chi E^{in}$ is Lipschitz and satisfies:
\[
\|\chi E^{in} - \chi E^{in} \|_{C^0_t(\hat{C})} \lesssim \epsilon^{-1} \| \delta \mu \|_{C^{0,\alpha}(\Gamma)} + \| \delta \xi \|_{C^{2,\alpha}(\Gamma)}.
\]
Next we will estimate the remaining terms in the error $\tilde{g}$:
\[
j = g - \chi E^{in},
\]
where $j = \sum_{k=1}^5 j_k$,
\[
j_1 = -\tilde{\rho}(q - q^0) \cdot \tilde{v},
\]
\[
j_2 = \chi \left( F_1(U, x) - f(v_1, x) + f_u(w_1, x)(1 - \chi)\delta_1 + e^{-4}v_1v_2^2 \right),
\]
\[
j_3 = \hat{\chi} N(U + \rho, x)
\]
(see (4.22)), and $j_4$, $j_5$ are the vectors of the commutator terms in (4.46):
\[
j_4 = [\Delta, \chi] \delta, \quad j_5 = [\Delta, \hat{\chi}] \tilde{v}.
\]
Lemma 4.5 With the above notation we have
\[
\|j_1\|_{C^0_t(\hat{C})} \lesssim (\epsilon^{-1} + \epsilon^{-2-\alpha} \| \mu \|_{C^{2,\alpha}(\Gamma)} + \| \xi \|_{C^{2,\alpha}(\Gamma)}) \| \ln | + \| \tilde{v} \|_{C^0_t(\hat{C})},
\]
\[
\|j_2\|_{C^0_t(\hat{C})} \lesssim \epsilon e^{2K\theta|\ln |} \ln | + \epsilon^{-1-\alpha} \| \ln | e^{(-\hat{\lambda} + \theta)K|\ln |} \left( \| \mu \|_{C^{2,\alpha}(\Gamma)} + \| \xi \|_{C^{2,\alpha}(\Gamma)} \right)
\]
\[
+ e^{4} e^{-c|\ln |^2},
\]
\[
\|j_3\|_{C^0_t(\hat{C})} \lesssim \epsilon^{-2} \| \tilde{v} \|_{C^0_t(\hat{C})} + e^{2K\theta|\ln |} \ln | \sum \| \tilde{u}_{ij} \|_{C^{2,\alpha}(\Omega_j)},
\]
\[
\|j_4\|_{C^0_t(\hat{C})} \lesssim (\| \ln | e^{3-\alpha} e^{2K\theta|\ln |} + \epsilon^{-1-\alpha} e^{(-\hat{\lambda} + \theta)K|\ln |} \left( \| \mu \|_{C^{1,\alpha}(\Gamma)} + \| \xi \|_{C^{1,\alpha}(\Gamma)} \right)
\]
\[
\|j_5\|_{C^0_t(\hat{C})} \lesssim \epsilon^{-2-\alpha} e^{2K\theta|\ln |} \ln | \sum \| \tilde{u}_{ij} \|_{C^{1,\alpha}(\Omega_j)},
\]
(4.49)
Proof We begin by proving the first bound in (4.49). Denote, as above, \( \delta_i = v_i - w_i \) so that

\[
U = (v_1, v_2) - ((1 - \chi)\delta_1, (1 - \chi)\delta_2).
\]

Directly from the expressions (4.8) and (4.9) and (4.27) we find

\[
|f_1| \lesssim e^4 \tilde{\rho}(|v_0|^2 + |v_1|^2) + e\tilde{\rho}(1 - \chi)(|v_0| + |v_1|)(|\delta_1| + |\delta_2|).
\]

From this the required bound follows.

Using (4.18), (4.27) we find

\[
\|\chi x_i \delta_i \|_{C^0(\bar{\mathcal{C}})} \lesssim (\epsilon |\ln \epsilon|)^{3-\alpha} e^{2b|\ln \epsilon|} + e^{1-\alpha |\ln \epsilon|} e^{(-\tilde{\lambda} + \theta)K|\ln \epsilon|} (\|\mu\|_{C^{1,0}\Gamma} + \|\xi\|_{C^{1,0}\Gamma}).
\]

The first two terms are estimated by

\[
|\chi f(v_i - \chi \delta_i, x) + f_u(w_i, x)\chi(1 - \chi)\delta_i - \chi f(v_i, x)| \lesssim \chi(1 - \chi)\delta_i
\]

and the last term by

\[
e^{-4}\chi |v_i(2(1 - \chi)v_{3-i}\delta_{3-i} - (1 - \chi)^2\delta_{3-i}^2) - (1 - \chi)\delta_i v_{3-i} - (1 - \chi)\delta_{3-i}^2| \lesssim e^{-4}\chi ((1 - \chi)|v_i||v_{3-i}||\delta_{3-i}| + (1 - \chi)^2|v_i||\delta_{3-i}|^2 + (1 - \chi)\delta_i|v_{3-i}|^2).
\]

Note that

\[
\chi(1 - \chi)|v_i||\delta_{3-i}| + \chi(1 - \chi)v_{3-i}||\delta_i| \lesssim e^{-c|\ln \epsilon|^2}, \quad i = 1, 2.
\]

Using the above we obtain the second estimate in (4.49) in a straightforward way.

To show the third estimate in (4.49) we decompose \( N \) as in (4.22). Using (4.23) we find

\[
\|\hat{\chi} N_f(U + \rho, x)\|_{C^0(\bar{\mathcal{C}})} \lesssim \|\hat{\mu}\|_{C^0(\bar{\mathcal{C}})}^2 + e^{2K|\theta| |\ln \epsilon|/\hat{\mu}} \sum \|\tilde{w}_{ij}\|_{C^0(\bar{\Omega}_j)}^2.
\]

From (4.24) replacing \( \chi \) by \( \hat{\chi} \) we get

\[
\hat{\chi} M_i(U + \rho) = \hat{\chi} U_i \rho_{3-i}^2 + 2\hat{\chi} U_{3-i} \rho_i \rho_{3-i} + \hat{\chi} \rho_i \rho_{3-i}^2,
\]

hence

\[
e^{-4}\|\hat{\chi} M_i(U + \rho)\|_{C^0(\bar{\mathcal{C}})} \lesssim e^{-3|\ln \epsilon|} \left( \|\hat{\mu}\|_{C^0(\bar{\mathcal{C}})}^2 + e^{2K|\theta| |\ln \epsilon|/\hat{\mu}} \sum \|\tilde{w}_{ij}\|_{C^0(\bar{\Omega}_j)}^2 \right).
\]

The latter estimate gives the second bound in (4.49).

Next we estimate the first commutator term the definition of \( g \). Slight modification of (4.19) and (4.27) gives

\[
\|[\Delta, \chi] \delta_i\|_{C^0(\bar{\mathcal{C}})} \lesssim \epsilon(|\ln \epsilon|)^{3-\alpha} e^{2K|\theta| |\ln \epsilon|} + e^{1-\alpha |\ln \epsilon|} e^{(-\tilde{\lambda} + \theta)K|\ln \epsilon|} (\|\mu\|_{C^{1,0}\Gamma} + \|\xi\|_{C^{1,0}\Gamma}).
\]

The remaining estimate is straightforward at this point. □

As for the Lipschitz dependence of \( j \) on the various unknown functions involved we state without a proof the following:
Corollary 4.3 The function

\[(\tilde{v}, \tilde{w}, \mu, \xi) \mapsto j(\cdot; \tilde{v}, \tilde{w}, \mu, \xi),\]

is Lipschitz in the following sense

\[
\|j(\cdot; \tilde{v}, \tilde{w}, \mu, \xi) - j(\cdot; \tilde{v} + \delta \tilde{v}, \tilde{w} + \delta \tilde{w}, \mu + \delta \mu, \xi + \delta \xi)\|_{C^0_\sigma(\mathcal{C})} \\
\lesssim \left( \epsilon^{-1} + \epsilon^{-\frac{\lambda}{2}} \right) \| \ln |\epsilon| \|_{C^0_\sigma(\mathcal{C})} \\
+ \epsilon^{-\frac{\lambda}{2}} e^{2K|\theta| \ln |\epsilon|/\tilde{m}} \sum \| \delta \tilde{w}_{ij} \|_{C^0(\Omega_j)} \\
+ \epsilon^{-\frac{\lambda}{2}} e^{-\lambda |\theta|} |\ln |\epsilon|} \left( \| \delta \mu \|_{C^{1,2}(\Gamma)} + \| \delta \xi \|_{C^{1,2}(\Gamma)} \right) \\
+ \epsilon^{-1} |\ln |\epsilon| \left( \| \tilde{v} \|_{C^0_\sigma(\mathcal{C})} \| \tilde{w} \|_{C^0(\mathcal{C})} \right) + e^{2K|\theta| |\ln |\epsilon|\tilde{m}} \sum \| \tilde{w}_{ij} \|_{C^0(\Omega_j)} \| \delta \tilde{w}_{ij} \|_{C^0(\Omega_j)}
\]

After these preparations we will bring the inner equation (4.10) to the form in which the theory described in Sect. 2.3 applies. We use (4.47) to pass to the inner stretched variables. To simplify suitably the inner equation it is convenient to introduce new unknown

\[z = b^{-1} \tilde{v}.\]

The results of Lemma 4.5 and Corollary 4.3 remain unchanged if we replace \(\tilde{v}\) by \(z\) on the right hand sides of the error estimates.

Recall the operator \(\mathcal{L}\) defined in (2.10). Simple manipulations allow us to write the equation for \(z\) in the form

\[
\mathcal{L}z = \epsilon^2 b^{-3}(s) i(z) + \epsilon^2 b(s)^{-3} g, \tag{4.50}
\]

where

\[
i(z) = \tilde{\rho} \left[ \epsilon^{-2} b^3(s) \mathcal{L}(z) + \Delta_{\tau,s}(b(s)z) - q^0 \cdot (b(s)z) \right], \tag{4.51}
\]

\(\Delta_{\tau,s}\) is the operator on the right hand side of (4.47) and \(g\) is defined in (4.46). To control the right hand side of (4.50) we only need to estimate the first term in (4.51). We set

\[i(z) = i_1(z) + i_2(z),\]

where

\[
i_1(z) = \tilde{\rho} \left[ - \epsilon^{-2} b^3 \partial_{\tau \tau} + b(s) \partial_{ss} \right] z + \Delta_{\tau,s}(b(s)z),
\]

\[
i_2(z) = \epsilon^{-2} \tilde{\rho} \left( \frac{b_0^2 V_1^2 - (v_0^0)^2}{2 b_0^2 V_1 V_2 - 2v_0^0 v_0^1} \frac{b_0^2 V_1^2 - (v_0^0)^2}{2 b_0^2 V_1 V_2 - 2v_0^0 v_0^1} \right) (b(s)z).
\]

Note that \(i_2\) depends additionally on the unknown modulation functions \(\mu, \xi\). The estimates gathered below are not hard to prove at this point.

Lemma 4.6 It holds

\[
\|i_1(z)\|_{C^0_\sigma(\mathcal{C})} \lesssim \epsilon^{-1} |\ln |\epsilon| \|z\|_{C^3(\mathcal{C})},
\]

\[
\|i_2(z)\|_{C^0_\sigma(\mathcal{C})} \lesssim \epsilon^{-1} |\ln |\epsilon| \left( 1 + \epsilon^{-1} |\mu|_{C^\sigma(\Gamma)} + \epsilon^{-1} |\xi|_{C^\sigma(\Gamma)} \right) \|z\|_{C^0(\mathcal{C})}. \tag{4.52}
\]
Also, the functions \((z, \mu, \xi) \mapsto i_i(z; \mu, \xi)\) are Lipschitz in the following sense
\[
\|i_1(z) - i_1(z + \delta z)\|_{C^1_b(\hat{\mathcal{C}})} \lesssim \epsilon^{-1} \ln \epsilon \|\delta z\|_{C^2_b(\hat{\mathcal{C}})}
\]
\[
\|i_2(z; \mu, \xi) - i_2(z + \delta z; \mu + \delta \mu, \xi + \delta \xi)\|_{C^1_b(\hat{\mathcal{C}})} \lesssim \epsilon^{-2} \ln \epsilon \|\epsilon + \|\mu\|_{C^4(\Gamma')} + \|\xi\|_{C^4(\Gamma')}\|\delta z\|_{C^1_b(\hat{\mathcal{C}})} + \epsilon^{-2} \ln \epsilon \|\|\delta \mu\|_{C^4(\Gamma')} + \|\delta \xi\|_{C^4(\Gamma')}\|\delta z\|_{C^1_b(\hat{\mathcal{C}})}.
\]

Anticipating later developments we observe that several error terms in Lemma 4.4 and Lemma 4.5 carry terms of the size \(\mathcal{O}(\epsilon e^{-\frac{1}{2}\ln \epsilon} |\ln \epsilon|) \ln |\ln \epsilon|\) with some \(q > 0\). If we rely on (2.38) of Proposition 2.1 to solve (4.50) then it seems hard to close the estimates consistently with (4.12). To overcome this technical difficulty we will use \(C^1,\alpha\) estimates to control the leading order term of the function \(z\) (cf (2.39) of Proposition 2.1). We state now the estimates needed later on.

**Corollary 4.4** It holds
\[
\|\chi \Omega(\cdot; 0)\|_{C^1_b(\hat{\mathcal{C}})} + \|\chi \Theta_l(\cdot; 0)\|_{C^1_b(\hat{\mathcal{C}})} + \|f_i(\cdot; 0)\|_{C^1_b(\hat{\mathcal{C}})} \lesssim \epsilon e^{-2\frac{1}{\ln \epsilon} |\ln \epsilon|} \ln \epsilon \|^5
\]
(4.53)
where \(l = 1, \ldots, 4, i = 2, 4\) and \((\cdot; 0)\) indicates that we take \(\mu = 0, \xi = 0\) in the definition of the corresponding function.

To prove (4.53) we note that all terms appearing on the left hand side depend only on already known functions which have sufficient smoothness to state \(C^1,\alpha(\hat{\mathcal{C}})\) estimate as claimed. We omit standard details similar to those in the proofs of Lemmas 4.4 and 4.5.

### 4.6 The modulation equation

So far the functions \(\mu\) and \(\xi\) were merely parameters. Now we will derive the modulation equation which determines them in terms of the remaining unknowns. This will complete the derivation of the nonlinear problem we need to solve. From the definition of the modulation functions (4.14) we see that in reality we will obtain a system of nonlinear equations for finitely many Fourier coefficients \(\hat{\mu}_{e,k}, \hat{\xi}_{e,k}, |k| \leq \hat{K}_e \sim \epsilon^{-1}\). For brevity we denote
\[
\hat{\mu}(s) = \left(\sum_{-\hat{K}_e}^{\hat{K}_e} \hat{\mu}_k \psi_k(s)\right), \quad \hat{\xi}(s) = \left(\sum_{-\hat{K}_e}^{\hat{K}_e} \hat{\xi}_k \psi_k(s)\right),
\]
so that
\[
\mu(\tau, s) = \hat{\mu}(s)\phi_\mu(\tau), \quad \xi(\tau, s) = \hat{\xi}(s)\phi_\xi(\tau).
\]
We set
\[
\mathcal{D} = -\partial_{\tau \tau} - \epsilon^2 b^{-2}(s) \partial_{ss}.
\]
Our heuristic is based on the fact that
\[
\Omega_0 = -\epsilon^{-1} b^3 \left[X \Omega \xi + Y \mathcal{D} \mu - 4X \partial_{\tau} \mu + 2V''(\mu + \tau \partial_{\tau} \mu + \partial_{\tau} \xi)\right] + \ldots,
\]
where \(\ldots\) indicate terms which should be smaller cf (4.48). We denote
\[
\hat{\Omega}_0(\mu, \xi) = X \mathcal{D} \xi + Y \mathcal{D} \mu - 4X \partial_{\tau} \mu + 2V''(\mu + \tau \partial_{\tau} \mu + \partial_{\tau} \xi).
\]
Lemma 4.7  There exist even functions $\phi_\mu$, $\phi_\xi$ satisfying (4.15) and positive constants $\ell_\mu$ and $\ell_\xi$ such that

\[
\begin{align*}
\langle \hat{\Omega}_0(\mu, \xi), X \rangle &= \ell_\xi \hat{\xi} + \mathcal{O}(\bar{\omega}) \hat{\xi}, \\
\langle \hat{\Omega}_0(\mu, \xi), Y \rangle &= \ell_\mu \hat{\mu} + \mathcal{O}(\bar{\omega}) \hat{\mu}.
\end{align*}
\] (4.54)

We prove this lemma in Sect. 5.

We will first use the above to solve the leading order terms $\mu^0, \xi^0$ (in $\epsilon$) of the modulation functions $\mu, \xi$. They depend only on the already known functions and in particular on $b$ and $\zeta$ which are $C^{3, \alpha}$ regular by assumption. Let us denote (c.f. Corollary 4.4)

\[
\Psi_0 = b^{-3} \left[ \Omega_1(\cdot; 0) \chi + \sum_{l=1}^{4} \Theta_l(\cdot; 0) \chi + \sum_{i=2, 4} j_i(\cdot; 0) \right].
\]

Denote

\[
\Psi_0^\parallel = P_{\bar{K}_\epsilon} \Psi_0, \quad \Psi_0^\perp = \Psi_0 - \Psi_0^\parallel,
\]

c.f notation of Proposition 2.1). Our goal is to solve the system

\[
\begin{align*}
\langle \chi \hat{\Omega}_0(\mu^0, \xi^0), X \rangle &= \epsilon \langle \Psi_0^\parallel, X \rangle, \\
\langle \chi \hat{\Omega}_0(\mu^0, \xi^0), Y \rangle &= \epsilon \langle \Psi_0^\parallel, Y \rangle,
\end{align*}
\] (4.55)

for the unknown functions $\mu^0$ and $\xi^0$ of the form

\[
\mu^0(\tau, s) = \hat{\mu}^0(s) \phi_\mu(\tau), \quad \xi^0(\tau, s) = \hat{\xi}^0(s) \phi_\xi(\tau).
\]

where

\[
\hat{\mu}^0(s) = \left( \sum_{k=-\bar{K}_\epsilon}^{\bar{K}_\epsilon} \hat{\mu}_k^0 \psi_k(s) \right), \quad \hat{\xi}^0(s) = \left( \sum_{k=-\bar{K}_\epsilon}^{\bar{K}_\epsilon} \hat{\xi}_k^0 \psi_k(s) \right),
\]

in the Hölder class $C^{2, \alpha}$. For this purpose and also having in mind later developments we state:

Lemma 4.8  Consider the following equations

\[
\langle \chi \hat{\Omega}_0(a, b), T \rangle = \langle \gamma^\parallel, T \rangle
\]

where $T = X$ or $T = Y$. There exist even functions $\phi_\mu$ and $\phi_\xi$ such that for any $\alpha \in (0, 1)$ we have

\[
\| \hat{a} \|_{C^{2, \alpha}(\Gamma)} + \| \hat{b} \|_{C^{2, \alpha}(\Gamma)} \lesssim \epsilon^{-1+\alpha} \| \gamma^\parallel \|_{C_\theta^\alpha(\hat{\mathcal{C}})}
\]

and this estimate can be improved

\[
\| \hat{a} \|_{C^{2, \alpha}(\Gamma)} + \| \hat{b} \|_{C^{2, \alpha}(\Gamma)} \lesssim | \ln \epsilon \| \| \gamma^\parallel \|_{C_\theta^\alpha(\hat{\mathcal{C}})}
\]

Proof  We have

\[
a = \left( \sum_{k=-\bar{K}_\epsilon}^{\bar{K}_\epsilon} \hat{a}_k \right) \psi_k(s) \phi_\mu, \quad b = \left( \sum_{k=-\bar{K}_\epsilon}^{\bar{K}_\epsilon} \hat{b}_k \psi_k(s) \phi_\xi.
\]
We also decompose the right hand side

\[
\langle \mathbf{y}, \mathbf{T} \rangle = \sum_{k=-K_e}^{K_e} \hat{y}_k \psi_k(s).
\]

By Lemma 4.7 we get

\[
|\hat{a}_k| + |\hat{b}_k| \lesssim |\hat{y}_k| \lesssim k^{-\alpha} \| \mathbf{y} \|_{C^\alpha([0,1])} \lesssim k^{-\alpha} \| \mathbf{y} \|_{C^\alpha_0(\hat{C})}.
\]

Using Lemma 6.1 and summing up the modes we conclude. The second statement is proven similarly.

By the above Lemma, Corollary 4.4 we have:

**Lemma 4.9** The system (4.55) has a solution such that

\[
\| \mu_0 \|_{C^{2,\alpha}(\Gamma)} + \| \xi_0 \|_{C^{2,\alpha}(\Gamma)} \lesssim \varepsilon^2 e^{2K\theta|\ln \varepsilon|} |\ln \varepsilon|^7
\]

(4.56)

and this estimate holds as well for \( C^{3,\alpha}(\Gamma) \) norms by a bootstrap argument.

Note that this is consistent with (4.16) provided that \( K\theta \) is sufficiently small. Next we look for the modulation functions \( \mu \) and \( \xi \)

\[
\mu = \mu^0 + \mu^1, \quad \xi = \xi^0 + \xi^1,
\]

where the new unknowns are of the form

\[
\mu^1(\tau, s) = \hat{\mu}^1(s) \phi_\mu(\tau), \quad \hat{\mu}^1(s) = \left( \sum_{k=-K_e}^{K_e} \hat{\mu}_k \psi_k(s) \right),
\]

\[
\xi^1(\tau, s) = \hat{\xi}^1(s) \phi_\xi(\tau), \quad \hat{\xi}^1(s) = \left( \sum_{k=-K_e}^{K_e} \hat{\xi}_k \psi_k(s) \right).
\]

We will now derive the modulation equation for \( \mu^1, \xi^1 \). In fact it is an equation of the form (4.55) but with the right hand side depending on the unknown functions of the problem coupling it with the inner and the outer problem described above. Recalling the error terms treated in Sect. 4.5 we write

\[
P_{K_e} \left[ b^{-3}(s) g + b^{-3}(s) i(z) \right] = -\varepsilon^{-1} P_{K_e} \hat{\Theta}_0(\mu^1, \xi^1) + \Psi_1^\parallel,
\]

where

\[
\Psi_1^\parallel = P_{K_e} \left[ b^{-3}(s) g + b^{-3}(s) i(z) - \varepsilon^{-1} \chi \hat{\Theta}_0(\mu, \xi) - \chi \Psi_0 \right].
\]

Based on the results of Sect. 4.5 the following is straightforward:

**Lemma 4.10** It holds

\[
\| \Psi_1^\parallel \|_{C^{3,\alpha}_0(\hat{C})} \lesssim \left( 1 + \varepsilon^{-1-\alpha} e^{-\left( \hat{\lambda}_{-\theta} \right) K|\ln \varepsilon|} \left( \| \mu \|_{C^{2,\alpha}(\Gamma)} + \| \xi \|_{C^{2,\alpha}(\Gamma)} \right) + e^{-1} \| \mu \|_{C^{2,\alpha}(\Gamma)} + \| \xi \|_{C^{2,\alpha}(\Gamma)} \right) + e^{-2-4\theta} \| \mu \|_{C^{2,\alpha}(\Gamma)} + \| \xi \|_{C^{2,\alpha}(\Gamma)} \| \ln \varepsilon \| \| z \|_{C^2_{\theta}(\hat{C})} + e^{-3} |\ln \varepsilon| \left( \| z \|_{C^2_{\theta}(\hat{C})} + e^{2K\theta|\ln \varepsilon|/\tilde{m}} \sum \| \tilde{w}_{ij} \|_{C^{1,\alpha}(\Omega_j)} \right). \]

(4.57)
We turn our attention to Lemma 4.5. Again, subtraction of operator \( \hat{\Psi}_1 \) hence only complicated term is the one involving \( \mu \) or \( \xi \) on the right hands are removed from \( \Psi_1 \) by subtracting \( \Psi_0 \). The only complicated term is the one involving \( e^{-1} \|\mu\|_{C^0(\Gamma)} \) in \( \Theta_3 \) but this term is part of the operator \( \hat{\Theta}_0 \). In all, denoting by \( \Psi_{12} \) the terms coming from \( \Theta_1 \) and \( \Theta_j \), \( j = 1, \ldots, 4 \) we get

\[
\|\psi_{12}\|_{C^2(\hat{\mathcal{C}})} \lesssim \left( \|\mu\|_{C^2(\Gamma)} + \|\xi\|_{C^2(\Gamma)} \right) + \epsilon^{-1} \left( \|\mu\|_{C^2(\Gamma)}^2 + \|\xi\|_{C^2(\Gamma)}^2 \right).
\]

We turn our attention to Lemma 4.5. Again, subtraction of \( \Psi_0 \) removes some terms from \( j \) and denoting what is left by \( \psi_{13} \) we have

\[
\|\psi_{13}\|_{C^2(\hat{\mathcal{C}})} \lesssim \epsilon^{-1} \|\mu\|_{C^1(\Gamma)} + \|\xi\|_{C^1(\Gamma)} \left( \|\mu\|_{C^2(\Gamma)} + \|\xi\|_{C^2(\Gamma)} \right) + \epsilon^{2} \|\mu\|_{C^1(\Gamma)} + \|\xi\|_{C^1(\Gamma)} \left( \|\mu\|_{C^2(\Gamma)} + \|\xi\|_{C^2(\Gamma)} \right) + \epsilon^3 \|\mu\|_{C^1(\Gamma)} + \|\xi\|_{C^1(\Gamma)} \left( \|\mu\|_{C^2(\Gamma)} + \|\xi\|_{C^2(\Gamma)} \right).
\]

Finally, denoting by \( \Psi_{14} \) the projection of \( i \) (Lemma 4.6) we get

\[
\|\psi_{14}\|_{C^0(\hat{\mathcal{C}})} \lesssim \epsilon^{-1} \|\mu\|_{C^0(\Gamma)} + \|\xi\|_{C^0(\Gamma)} \left( \|\mu\|_{C^2(\Gamma)} + \|\xi\|_{C^2(\Gamma)} \right) \|\mathbf{z}\|_{C^0(\hat{\mathcal{C}})}.
\]

This ends the proof. \( \square \)

For later purpose we state the following

**Corollary 4.5** Estimate (4.57) holds with \( \Psi_1 \) replaced by

\[
\Psi_1 = b^{-3}(s) \mathbf{g} + b^{-3}(s) \mathbf{i}(\mathbf{z}) + \epsilon^{-1} \hat{\Theta}_0(\mu, \xi) - \hat{\Theta}_0.
\]

We state the nonlinear system for the modulation functions \( \mu^1 \), \( \xi^1 \):

\[
\begin{align*}
\langle \hat{\Theta}_0(\mu^1, \xi^1), X \rangle &= \epsilon \langle \Psi_1, X \rangle \\
\langle \hat{\Theta}_0(\mu^1, \xi^1), Y \rangle &= \epsilon \langle \Psi_1, Y \rangle.
\end{align*}
\]
4.7 Conclusion of the proof

It this section we will parallel the developments of the previous section: we will first improve the inner approximation and then state the nonlinear problem for the correction. We look for the solution of (4.50) in the form \( z = z^0 + z^1 \) where

\[
\mathcal{L}z^0 = \epsilon^2 \left( -\epsilon^{-1} \chi \hat{\Omega}_0(\mu^0, \xi^0) + \Psi_0 \right).
\]  

(4.59)

Observe that the right hand side is of class \( C^{1,\beta} \) hence we can rely on (2.38) and (2.39) of Proposition 2.1 to obtain

\[
\|z^0\|_{\mathcal{C}^2(\hat{\Gamma})} \lesssim \epsilon^2 |\ln \epsilon| \left( \|\chi \hat{\Omega}_0(\mu^0, \xi^0)\|_{\mathcal{C}^1(\hat{\Gamma})} + \|\Psi_0\|_{\mathcal{C}^1(\hat{\Gamma})} \right) \lesssim \epsilon^3 e^{2K\theta} |\ln \epsilon| |\ln \epsilon|^{10}.
\]  

(4.60)

Next we write the nonlinear problem for the correction \( z^1 \):

\[
\mathcal{L}z^1 = \epsilon^2 \left( -\epsilon^{-1} \hat{\Omega}_0(\mu^1, \xi^1) + \Psi_1 \right).
\]  

(4.61)

To complete the nonlinear system we need to solve to finish the proof we restate the outer equation in (4.10) in the form (4.34) (see also Sect.4.3)

\[
- \Delta \tilde{w}_i - f^i(u^0_i, x) \|_{\Omega_0} \tilde{w}_i + \epsilon (q^1 \cdot \tilde{w})_i = (1 - \chi) h_i, \quad \text{in} \quad \Omega \setminus \Gamma,
\]

\[
\tilde{w}_i = 0, \quad \text{on} \quad \partial \Omega.
\]  

(4.62)

The right hand sides of this system are described and estimated in Lemma 4.3. To solve the outer problem we use the results of Lemma 4.2.

Next we solve the system (4.58), (4.61), (4.62) using the Banach Fixed Point Theorem. To this end we suppose that we are given the modulation functions \( \tilde{\mu}, \tilde{\xi} \) satisfying (4.16), the inner function \( \tilde{z} \) satisfying (4.12) and the outer function \( \tilde{w} = (\tilde{w}_1, \tilde{w}_2) \) satisfying (4.11). We replace \( \mu, \xi \) and \( z \) on the right hand sides of the system by \( \mu^0 + \tilde{\mu}, \xi^0 + \tilde{\xi} \) and \( z^0 + \tilde{z} \). Taking into account the \( \mu^0 \) and \( \xi^0 \) satisfy (4.56) and \( z^0 \) satisfies (4.60) and using the theory developed above (Proposition 2.1, Lemma 4.2 and Lemma 4.8) we obtain with \( \alpha \in (0, 1) \)

\[
\|\mu^1\|_{\mathcal{C}^2,\alpha(\Gamma)} + \|\xi^1\|_{\mathcal{C}^2,\alpha(\Gamma)} \lesssim \epsilon^\alpha \left[ e^{-1-\alpha e^{-(\hat{\lambda}-\theta)K}|\ln \epsilon|} (e^{3 e^{2K\theta} |\ln \epsilon|} |\ln \epsilon|^{10} + e^{2-\zeta}) + e^{3 e^{4K\theta} |\ln \epsilon|} |\ln \epsilon|^{10} + e^{2+\xi - \alpha e^{2K\theta} |\ln \epsilon|/\tilde{\eta}} + e^{-1} + e^{-\alpha e^{2K\theta} |\ln \epsilon|} |\ln \epsilon|^{10} + e^{2+\zeta} e^{2K\theta} |\ln \epsilon|/\tilde{\eta} \right].
\]  

(4.63)

The first line above can be made as small as we wish by choosing \( \hat{\lambda} \) large. The second, the third and the fourth lines can be made of size \( o(\epsilon^2) \) if \( \theta \) is chosen small and

\[
\alpha - \zeta > -1, \quad \zeta + \tilde{\zeta} + \alpha > 1, \quad 2\tilde{\zeta} + \alpha > 1.
\]  

(4.64)

Based on Corollary 4.5 we see that under the same restrictions

\[
\|z^1\|_{\mathcal{C}^3(\hat{\Gamma})} \lesssim o(\epsilon^3).
\]  

(4.65)
To estimate the size of the outer solution we rely on Lemmas 4.2 and 4.3:

\[ \|\tilde{u}_i\|_{C^{1,\alpha}(\Omega_i)} + \|\tilde{u}_2\|_{C^{1,\alpha}(\Omega_3+\epsilon)} \lesssim (\epsilon |\ln \epsilon|)^{5-\alpha} + \epsilon^{4-\alpha+2\tilde{\xi}} + (\epsilon^{\tilde{\xi} - \alpha} + \epsilon^{3e^{2K\theta|\ln\epsilon|} |\ln \epsilon|^2}) \times (e^{2K\theta|\ln\epsilon|/\tilde{m}} + e^{-2K\theta|\ln\epsilon|/\tilde{m}}). \]  

(4.66)

for \( i = 1, 2 \). The first two terms are of order \( o(\epsilon^{4+\tilde{\xi}}) \) when

\[ 1 - \alpha > \tilde{\xi}, \quad \tilde{\xi} > \alpha, \]  

(4.67)

and the second line can be made of the same size by choosing \( \tilde{m} < \hat{m} < 1 \) small. It is easily checked that conditions (4.64) and (4.67) are satisfied when

\[ 1 > \alpha + \tilde{\xi}, \quad \tilde{\xi} > \alpha, \quad \tilde{\xi} + \tilde{\xi} > 1 - \alpha, \quad \tilde{\xi} > \frac{1 - \alpha}{2}. \]  

(4.68)

These conditions should be satisfied together with

\[ \epsilon^2 e^{2K\theta|\ln\epsilon| |\ln\epsilon|^{10}} \lesssim o(\epsilon^{2-\tilde{\xi}}), \quad \epsilon^3 e^{2K\theta|\ln\epsilon| |\ln\epsilon|^{10}} \lesssim o(\epsilon^{2+\tilde{\xi}}). \]  

(4.69)

Explicitly we can chose \( \alpha \) smaller than but close to 1/2 and then \( \tilde{\xi} \) larger than, close to 1/2 and \( \tilde{\xi} = 1/4 \) but smaller than 1/2. In fact it can be checked that for any \( \alpha \in (0, 1/2) \) it is possible to chose the other constants so that (4.69) holds. Finally we fix \( K \) and chose \( \theta \) small to satisfy (4.69). Summarizing we see that the nonlinear map

\[ (\tilde{\mu}, \tilde{\xi}, \tilde{\zeta}, \tilde{w}) \mapsto (\mu^1, \xi^1, \zeta^1, \tilde{w}) \]

defined by (4.58), (4.61), (4.62) transforms the set

\[ \mathcal{B}((\tilde{\mu}, \tilde{\xi}, \tilde{\zeta}, \tilde{w})) = \{(\mu, \xi, \zeta, w) \text{ satisfying (4.16), (4.12), (4.11)}\}, \]

into itself. It remains to check that this map is also a contraction. This is standard given the results of Corollaries 4.2, 4.3 and Lemmas 4.4, 4.6. We omit somewhat tedious details. To conclude the proof of the theorem we need to show (1.12) and (2.40). By construction of the solution it is evident that it suffices to show that corresponding facts replacing the function \( u = (u_1, u_2) \) by the approximate solution \( U = (U_1, U_2) \). Straightforward direct argument using the explicit definition of the approximate solution then shows the required estimates. This ends the proof of the theorem.

## 5 Proof of Lemma 4.7

We have

\[ \langle \hat{\Omega}_0(\mu, \xi), X \rangle = \langle X \partial \xi, X \rangle + \langle Y \partial \mu, X \rangle - 4\langle X \partial \mu, X \rangle + 2\langle V'' \mu, X \rangle + 2\langle V'' \partial \mu, X \rangle + 2\langle V'' \partial \xi, X \rangle. \]

We observe that because \( V_1(\tau) = V_2(-\tau) \) the product \( X \cdot Y \) is odd while functions \( \phi_\mu \) and \( \phi_\xi \) are even. From this we see that

\[ \langle \hat{\Omega}_0(\mu, \xi), X \rangle = \langle X \partial \xi, X \rangle + 2\langle V'' \partial \xi, X \rangle. \]

We have

\[ \langle X \partial \xi, X \rangle = -\hat{\xi} \langle X \phi''_\xi, X \rangle + \mathcal{O}(\hat{\omega})\hat{\xi} = -\hat{\xi} \int (|X|^2)_{\tau} \phi_\xi + \mathcal{O}(\hat{\omega})\hat{\xi} \]

\[ = -2\hat{\xi} \int (V''_1 \phi''_1 + V''_2 \phi''_2)_{\tau} \phi_\xi + \mathcal{O}(\hat{\omega})\hat{\xi} = -2\hat{\xi} \int (V'' \cdot X)_{\tau} \phi_\xi + \mathcal{O}(\hat{\omega})\hat{\xi}. \]

\[ \hat{\xi} \] Springer
It follows

\[ \langle \hat{\Omega}_0(\mu, \xi), X \rangle = -4\hat{\xi} \int (V'' \cdot X)_{\tau} \phi_{\xi} + O(\overline{\omega})\hat{\xi}. \]

We set

\[ \phi_{\xi} = -(V'' \cdot X)_{\tau}. \]

We get explicitly

\[ \int (V'' \cdot X)_{\tau} \phi_{\xi} = \int |(V_1 V_1' V_2^2 + V_2 V_2' V_1^2)_{\tau}|^2 d\tau > 0 \]

provided that \( \phi_{\xi} \not\equiv 0 \). To show that \( \phi_{\xi} \not\equiv 0 \) we note that

\[ \phi_{\xi}(0) = -2(V'_{\infty}(0)^2). \]

In [1] it was shown that \( V'_1(\infty)^2 = \frac{1}{2} \). Using the equation it is not hard to show that

\[ V'_1(0)^2 = \frac{1}{2}(1 + V'_1(\infty)^2) = \frac{1}{2} \left(1 + \frac{1}{2}\right) < 1. \]

hence \( \phi_{\xi}(0) < 0 \). Note that

\[ |\phi_{\xi}(\tau)| \lesssim e^{-c|\tau|^2}. \]

This ends the proof of the first identity in (4.54).

Next we will show the second identity in (4.54). Similarly as above

\[ \langle \hat{\Omega}_0(\mu, \xi), Y \rangle = \langle X \partial_\xi, Y \rangle + \langle Y \partial_\mu, Y \rangle - 4\langle X \partial_\tau \mu, Y \rangle \]

\[ + 2\langle V'' \mu, Y \rangle + 2\langle V'' \tau \partial_\tau \mu, Y \rangle + 2\langle V'' \partial_\tau \xi, Y \rangle \]

\[ = \hat{\mu} \left( -\int (|Y|^2)_{\tau \tau} \phi_{\mu} + \int (X \cdot Y)_{\tau} \phi_{\mu} + 2 \int V'' \cdot Y \phi_{\mu} - 2 \int (\tau V'' \cdot Y)_{\tau} \phi_{\mu} \right) \]

\[ + O(\overline{\omega})\hat{\mu}. \]

We set

\[ \phi_{\mu}(\tau) = (-(|Y|^2)_{\tau \tau} + (X \cdot Y)_{\tau} + 2V'' \cdot Y - 2(\tau V'' \cdot Y)_{\tau}) \text{sech}(\hat{\lambda}\tau). \]

With this definition \( \phi_{\mu} \) is even by the symmetry of \( V \). To show that \( \phi_{\mu} \not\equiv 0 \) we note that after some elementary calculations we get

\[ -(|Y|^2)_{\tau \tau} + (X \cdot Y)_{\tau} + 2V'' \cdot Y - 2(\tau V'' \cdot Y)_{\tau} = -4|V'(\tau)|^2 + O(e^{-c|\tau|^2}), \]

hence for \( \tau \to \infty \)

\[ \phi_{\mu}(\tau) = -4|V'(\infty)|^2 \text{sech}(\hat{\lambda}\tau) + O(e^{-c|\tau|^2}) = -2 \text{sech}(\hat{\lambda}\tau) + O(e^{-c|\tau|^2}) \not\equiv 0. \]

This end the proof of the Lemma.
6 The rate of decay of Fourier modes for Hölder continuous functions

Using Liouville Normal Form of the Sturm-Liouville operator

\[ y(s) = \int_0^s b_0(\sigma) \, d\sigma, \quad \tilde{\psi}(y) = \psi(s(y))\sqrt{b_0(s(y))}, \quad \tilde{q} = \frac{d^2}{dy^2} \sqrt{b_0(s(y))}, \]

we can transform (2.12) to

\[ -\delta_{yy} \tilde{\psi} + \tilde{q} \tilde{\psi} = \omega^2 \tilde{\psi}, \quad y \in [0, \tilde{\ell}], \quad \tilde{\ell} = \int_{|\Gamma|} b_0(\sigma) \, d\sigma \]

with periodic boundary conditions. Under this change of variables, for any two functions \( u_1, u_2 \) in \([0, |\Gamma|]\) we have

\[ \int_{|\Gamma|} u_1(s) u_2(s) b_0^2(s) \, ds = \int_0^{\tilde{\ell}} \tilde{u}(y) \tilde{v}(y) \, dy, \quad \tilde{u}_j(y) = u_j(s(y)) \sqrt{b_0(s(y))}. \]

Consider, more generally, the following operator \( L_q = -\partial^2_{ss} + q(s) \) in \( S^1 \), and let us assume that \( q \) is such that \( L_q \geq 0 \). Let \( \omega_k^2 \geq 0, k \in \mathbb{Z} \), be the eigenvalues and \( \psi_k \) the eigenfunctions of \( L_q \). Note that by the normal form transformation and possible further scaling of the independent variables (2.12) becomes an eigenvalue problem for the operator of the form \( L_q \). From now on we will consider the operator \( L_q \) on \( S^1 \). From the preceding discussion the result of Proposition 6.1 (below) applies in the original case.

The following summarizes the contents of Lemma 1.2.1 and Corollary 1.4.1 in [12]

**Lemma 6.1** Let \( \omega_k \) be the eigenvalues and \( \psi_k \) normalized eigenfunctions of \( L_q \).

(i) The following asymptotic formula holds

\[ \omega_k^2 = k^2 + \mathcal{O}(k^{-2}), \quad k \in \mathbb{Z} \]

as \( k \to \infty \).

(ii)

\[ \psi_k = a_k \cos \omega_k x + b_k \sin \omega_k x + \mathcal{O}(|\omega_k|^{-1}), \]

where \( a_k = \mathcal{O}(1) \) and \( b_k = \mathcal{O}(1) \).

For any \( u \in L^2(S^1) \) we set

\[ \hat{u}_k = \int_{S^1} u(x) \psi_k(x) \, dx. \]

Now we prove:

**Proposition 6.1** For any \( u \in C^\alpha(S^1), \alpha \in (0, 1) \) it holds

\[ |u_k| \lesssim |k|^{-\alpha} \|u\|_{C^\alpha(S^1)}. \]

**Proof** By a well known result in harmonic analysis:

\[ \left| \int_{S^1} u(x)e^{-ikx} \, dx \right| \lesssim |k|^{-\alpha} \|u\|_{C^\alpha(S^1)}. \]

From this and Lemma 6.1 our claim follows easily. \( \square \)

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