A note on parameter derivatives of classical orthogonal polynomials

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Abstract

Coefficients in the expansions of the form

\[ \frac{\partial P_n(\lambda; z)}{\partial \lambda} = \sum_{k=0}^{n} a_{nk}(\lambda)P_k(\lambda; z), \]

where \( P_n(\lambda; z) \) is the \( n \)th classical (the generalized Laguerre, Gegenbauer or Jacobi) orthogonal polynomial of variable \( z \) and \( \lambda \) is a parameter, are evaluated. A method we adopt in the present paper differs from that used by Fröhlich [Integral Transforms Spec. Funct. 2 (1994) 253] for the Jacobi polynomials and by Koepf [Integral Transforms Spec. Funct. 5 (1997) 69] for the generalized Laguerre and the Gegenbauer polynomials.

**KEY WORDS:** orthogonal polynomials; generalized Laguerre polynomials; Gegenbauer polynomials; Jacobi polynomials; parameter derivatives

**AMS subject classification:** 42C05, 33C45, 42C10

1 Introduction

The problem of evaluation of coefficients in the expansions of the form

\[ \frac{\partial P_n(\lambda; z)}{\partial \lambda} = \sum_{k=0}^{n} a_{nk}(\lambda)P_k(\lambda; z), \]

where \( P_n(\lambda; z) \) is the \( n \)th classical (the generalized Laguerre, Gegenbauer or Jacobi [1–3]) orthogonal polynomial of variable \( z \), with \( \lambda \) being a parameter, is not new. Fröhlich [4] investigated the case of the Jacobi polynomials, \( P_n(\lambda; z) = P_n^{(\alpha, \beta)}(z) \), with either of the two parameters \( \alpha \) or \( \beta \) playing the role of \( \lambda \). He started with a relationship between the Jacobi polynomials and the Gauss hypergeometric function, found partial derivatives of the latter with respect to the second and the third parameters, and then ingeniously used some combinatorial identities to obtain closed-form expressions for the coefficients in two (one corresponding to the choice \( \lambda = \alpha \), the other to the choice \( \lambda = \beta \)) relevant expansions of the form (1.1). Using a notation more suitable for the present purposes than that used originally in [4], Fröhlich’s results may be written as (cf [5, Eqs. (1.2.23.1)
and (1.2.23.2))

\[
\frac{\partial P_n^{(\alpha,\beta)}(z)}{\partial \alpha} = \left[\psi(2n + \alpha + \beta + 1) - \psi(n + \alpha + \beta + 1)\right]P_n^{(\alpha,\beta)}(z)
\]

\[\quad + \frac{\Gamma(n + \alpha + \beta + 1)}{\Gamma(n + \alpha + \beta + 1)} \sum_{k=0}^{n-1} \frac{2k + \alpha + \beta + 1}{(n-k)(k+n + \alpha + \beta + 1)}
\]

\[\times \frac{\Gamma(k + \alpha + \beta + 1)}{\Gamma(k + \alpha + 1)} P_k^{(\alpha,\beta)}(z)
\]

(1.2)

and

\[
\frac{\partial P_n^{(\alpha,\beta)}(z)}{\partial \beta} = \left[\psi(2n + \alpha + \beta + 1) - \psi(n + \alpha + \beta + 1)\right]P_n^{(\alpha,\beta)}(z)
\]

\[\quad + (-)^n \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + \alpha + \beta + 1)} \sum_{k=0}^{n-1} (-)^k \frac{2k + \alpha + \beta + 1}{(n-k)(k+n + \alpha + \beta + 1)}
\]

\[\times \frac{\Gamma(k + \alpha + \beta + 1)}{\Gamma(k + \alpha + 1)} P_k^{(\alpha,\beta)}(z),
\]

(1.3)

where

\[
\psi(\zeta) = \frac{1}{\Gamma(\zeta)} \frac{d\Gamma(\zeta)}{d\zeta}
\]

(1.4)

is the digamma function. Somewhat later, Koepf [6] pointed out that one might use the Fröhlich’s results [1.2] and [1.3] and the following known relationship between the Gegenbauer and the Jacobi polynomials:

\[
C_n^{(\lambda)}(z) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(2\lambda)} \frac{\Gamma(n + 2\lambda)}{\Gamma(n + \lambda + \frac{1}{2})} P_{n-1/2,\lambda-1/2}(z)
\]

(1.5)

to obtain an expansion of the form [1.11] for \(C_n^{(\lambda)}(z)\). With some simplifying notational changes, Koepf’s result is (cf [5, Eq. (1.22.2.2)])

\[
\frac{\partial C_n^{(\lambda)}(z)}{\partial \lambda} = \left[\psi(n + \lambda) - \psi(\lambda)\right]C_n^{(\lambda)}(z) + 4 \sum_{k=0}^{n-1} \frac{(-)^k + n}{2} \frac{k + \lambda}{(n - k)(k + n + \lambda + 1)} C_k^{(\lambda)}(z),
\]

(1.6)

which may be also rewritten as (cf [5, Eq. (1.22.2.1)])

\[
\frac{\partial C_n^{(\lambda)}(z)}{\partial \lambda} = \left[\psi(n + \lambda) - \psi(\lambda)\right]C_n^{(\lambda)}(z) + \sum_{k=1}^{\text{int}(n/2)} \frac{n - 2k + \lambda}{k(n - k + \lambda)} C_{n-2k}^{(\lambda)}(z).
\]

(1.7)

Moreover, Koepf [6] observed that combining the Fröhlich’s result [1.3] with the known formula

\[
L_n^{(\alpha)}(z) = \lim_{\beta \to \infty} P_n^{(\alpha,\beta)}\left(1 - \frac{2z}{\beta}\right),
\]

(1.8)

relating the generalized Laguerre and the Jacobi polynomials, one obtains the remarkably simple expansion

\[
\frac{\partial L_n^{(\lambda)}(z)}{\partial \lambda} = \sum_{k=0}^{n-1} \frac{1}{n - k} L_k^{(\lambda)}(z).
\]

(1.9)

In [6], Koepf gave also an alternative proof of this formula.

The relationship [1.9] finds applications in relativistic quantum mechanics where it is used to determine corrections of the order \(\alpha^2\) (\(\alpha\) is the fine-structure constant) to wave functions of particles bound in the Coulomb potential.
Our interest in the matter discussed in the present paper arose and evolved in the course of carrying out research on derivatives of the associated Legendre function of the first kind, \( P_\nu^n(z) \), with respect to its parameters \( \mu \) and \( \nu \) \([7, 8]\). In particular, we have found \([8]\) that the derivatives \( [\partial_P^{\pm M}(z)/\partial\nu]_{\nu=N} \), with \( M, N \in \mathbb{N} \), may be expressed in terms of the parameter derivatives of the Jacobi polynomials, \( \partial P_\nu^{(\alpha,\beta)}(z)/\partial\beta \), with suitably chosen \( n, \alpha \) and \( \beta \). Being at the time of doing the aforementioned research unaware of the Fröhlich’s paper \([4]\), we derived the relationships \((1.2)\) and \((1.3)\) independently. Then it appeared to us that our way of reasoning might be applied to derive the relationships \((1.6)\) and \((1.9)\) as well. Since our approach turns out to be entirely different from the methods adopted by Fröhlich \([4]\) and Koepf \([6]\), we believe it is worthy to be presented.

Thus, in the present paper, we re-derive the expansions \((1.2), (1.3), (1.6)\) and \((1.9)\). The path we shall follow for each particular polynomial family comprises the following steps. First, using the lemma proved in the Appendix and explicit representations of the polynomials in terms of powers of \( z - z_0 \), with suitably chosen \( z_0 \), for each family we shall deduce the coefficient \( a_{nk}(\lambda) \) standing in Eq. \((A.1)\) at \( P_n(\lambda; z) \). Then, from the homogeneous differential equation satisfied by \( P_n(\lambda; z) \), in each particular case we shall derive an inhomogeneous differential equation obeyed by \( \partial P_n(\lambda; z)/\partial\lambda \). Next, we shall show how, in each case, after exploiting a relevant Christoffel–Darboux identity, the inhomogeneity in that differential equation may be cast to a form of a linear combination of the polynomials \( P_k(\lambda; z) \) with \( 0 \leq k \leq n - 1 \). In the final step, we shall insert this combination and the expansion \((1.1)\) into the aforementioned inhomogeneous differential equation for \( \partial P_n(\lambda; z)/\partial\lambda \), and then straightforwardly deduce the coefficients \( a_{nk}(\lambda) \) with \( 0 \leq k \leq n - 1 \).

All properties of the classical orthogonal polynomials we have exploited in the present paper may be found in \([3]\).

## 2 The generalized Laguerre polynomials

The explicit formula defining the generalized Laguerre polynomials is

\[
L_n^{(\lambda)}(z) = \sum_{k=0}^{n} \frac{(-)^k}{k!} \binom{n + \lambda}{n - k} z^k.
\]

The coefficient at \( z^n \) is

\[
k_n(\lambda) = \frac{(-)^n}{n!}
\]

and is seen to be independent of \( \lambda \). Hence, using Eq. \((A.1)\), we deduce that the coefficient \( a_{nn}(\lambda) \) in the expansion

\[
\frac{\partial L_n^{(\lambda)}(z)}{\partial\lambda} = \sum_{k=0}^{n} a_{nk}(\lambda) L_k^{(\lambda)}(z)
\]

vanishes:

\[
a_{nn}(\lambda) = 0.
\]

The coefficients \( a_{nk}(\lambda) \) with \( 0 \leq k \leq n - 1 \) may be found through the following reasoning. The differential equation obeyed by \( L_n^{(\lambda)}(z) \) is

\[
\left[ z \frac{d^2}{dz^2} + (\lambda + 1 - z) \frac{d}{dz} + n \right] L_n^{(\lambda)}(z) = 0,
\]

which implies that

\[
\left[ z \frac{d^2}{dz^2} + (\lambda + 1 - z) \frac{d}{dz} + n \right] \frac{\partial L_n^{(\lambda)}(z)}{\partial\lambda} = -\frac{dL_n^{(\lambda)}(z)}{dz}.
\]

We shall express the derivative \( dL_n^{(\lambda)}(z)/dz \) as a linear combination of the polynomials \( L_k^{(\lambda)}(z) \) with \( 0 \leq k \leq n - 1 \). To this end, at first we invoke the known relation

\[
\frac{dL_n^{(\lambda)}(z)}{dz} = \frac{nL_n^{(\lambda)}(z) - (n + \lambda)L_{n-1}^{(\lambda)}(z)}{z}.
\]
Next, we recall that the pertinent Christoffel–Darboux identity is
\[
\sum_{k=0}^{n-1} \frac{k!}{\Gamma(k + \lambda + 1)} f^{(\lambda)}_k(z) L^{(\lambda)}_k(z') = -\frac{n!}{\Gamma(n + \lambda)} \frac{L^{(\lambda)}_n(z)L^{(\lambda)}_{n-1}(z')}{z - z'}.
\] (2.8)

If we set here \( z' = 0 \) and use
\[
L^{(\lambda)}_k(0) = \frac{\Gamma(k + \lambda + 1)}{k! \Gamma(\lambda + 1)} k!
\] (2.9)
Eq. (2.8) becomes
\[
\sum_{k=0}^{n-1} L^{(\lambda)}_k(z) = -\frac{n}{z} L^{(\lambda)}_n(z) + \frac{n(n + \lambda)}{z} L^{(\lambda)}_{n-1}(z).
\] (2.10)

Combining Eqs. (2.7) and (2.10) yields
\[
\frac{d}{dz} L^{(\lambda)}_n(z) = -\sum_{k=0}^{n-1} L^{(\lambda)}_k(z). \tag{2.11}
\]

Next, we insert Eqs. (2.3) and (2.11) into the left- and right-hand sides of Eq. (2.6), respectively, and then simplify the left-hand side of the resulting equation with the aid of Eq. (2.5). In this way, we obtain
\[
\sum_{k=0}^{n-1} (n - k) a_{nk}(\lambda) L^{(\lambda)}_k(z) = \sum_{k=0}^{n-1} L^{(\lambda)}_k(z). \tag{2.12}
\]

Equating coefficients at \( L^{(\lambda)}_k(z) \) on both sides of Eq. (2.12), we arrive at the sought expression for \( a_{nk}(\lambda) \):
\[
a_{nk}(\lambda) = \frac{1}{n - k} \quad (0 \leq k \leq n - 1). \tag{2.13}
\]

From Eqs. (2.3), (2.4) and (2.13) the relationship (1.9) follows.

3 The Gegenbauer polynomials

The Gegenbauer polynomials may be defined through the formula
\[
C^{(\lambda)}_n(z) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(2\lambda)} \sum_{k=0}^{n} \frac{\Gamma(k + n + 2\lambda)}{k!(n - k)!\Gamma(k + \lambda + \frac{1}{2})} \left( \frac{z - 1}{2} \right)^k. \tag{3.1}
\]

To find the coefficient \( a^{(\lambda)}_{nn} \) in the expansion
\[
\frac{\partial C^{(\lambda)}_n(z)}{\partial \lambda} = \sum_{k=0}^{n} a_{nk}(\lambda) C^{(\lambda)}_k(z), \tag{3.2}
\]
we infer from Eq. (3.1) that the coefficient at \( (z - 1)^n \) is
\[
k_n(\lambda) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(2\lambda)} \frac{\Gamma(2n + 2\lambda)}{n!\Gamma(n + \lambda + \frac{1}{2})} = \frac{2^n \Gamma(n + \lambda)}{n!\Gamma(\lambda)}. \tag{3.3}
\]
Hence, application of the lemma from the Appendix gives
\[
a_{nn}(\lambda) = \psi(n + \lambda) - \psi(\lambda). \tag{3.4}
\]

To proceed further, we recall that the Gegenbauer polynomials obey the differential identity
\[
\left[ (1 - z^2) \frac{d^2}{dz^2} - (2\lambda + 1)z \frac{d}{dz} + n(n + 2\lambda) \right] C^{(\lambda)}_n(z) = 0. \tag{3.5}
\]
Differentiating with respect to $\lambda$, we obtain
\[
\left[ (1 - z^2) \frac{d^2}{dz^2} - (2\lambda + 1)z \frac{d}{dz} + n(n + 2\lambda) \right] \frac{\partial C_n^{(\lambda)}(z)}{\partial \lambda} = 2z\frac{dC_n^{(\lambda)}(z)}{dz} - 2nC_n^{(\lambda)}(z).
\]
Eq. (3.6)

The next step is a bit tricky. It consists of writing
\[
2z\frac{dC_n^{(\lambda)}(z)}{dz} - 2nC_n^{(\lambda)}(z) = \left[ (z + 1)\frac{dC_n^{(\lambda)}(z)}{dz} - nC_n^{(\lambda)}(z) \right] + \left[ (z - 1)\frac{dC_n^{(\lambda)}(z)}{dz} - nC_n^{(\lambda)}(z) \right].
\]
Eq. (3.7)

Using the known relation
\[
(z^2 - 1)\frac{dC_n^{(\lambda)}(z)}{dz} = nzC_n^{(\lambda)}(z) - (n + 2\lambda - 1)C_{n-1}^{(\lambda)}(z),
\]
Eq. (3.8)

Eq. (3.7) may be cast to the form
\[
2z\frac{dC_n^{(\lambda)}(z)}{dz} - 2nC_n^{(\lambda)}(z) = \frac{nC_n^{(\lambda)}(z) - (n + 2\lambda - 1)C_{n-1}^{(\lambda)}(z)}{z - 1} - \frac{nC_n^{(\lambda)}(z) + (n + 2\lambda - 1)C_{n-1}^{(\lambda)}(z)}{z + 1}.
\]
Eq. (3.9)

Now, the Christoffel–Darboux identity for the Gegenbauer polynomials is
\[
\sum_{k=0}^{n-1} \frac{k!(k + \lambda)}{\Gamma(k + 2\lambda)} C_k^{(\lambda)}(z)C_k^{(\lambda)}(z') = \frac{n!}{2\Gamma(n + 2\lambda - 1)} C_n^{(\lambda)}(z)C_{n-1}^{(\lambda)}(z') - C_{n-1}^{(\lambda)}(z)C_n^{(\lambda)}(z').
\]
Eq. (3.10)

If we set in this identity $z' = \pm 1$ and use
\[
C_k^{(\lambda)}(\pm 1) = (\pm)^k \frac{\Gamma(k + 2\lambda)}{k!\Gamma(2\lambda)}.
\]
Eq. (3.11)

we obtain
\[
\sum_{k=0}^{n-1} (\pm)^k (k + \lambda)C_k^{(\lambda)}(z) = \frac{(\pm)^n nC_n^{(\lambda)}(z) - (n + 2\lambda - 1)C_{n-1}^{(\lambda)}(z)}{z \mp 1}.
\]
Eq. (3.12)

Combining this result with Eq. (3.9) yields the expansion
\[
2z\frac{dC_n^{(\lambda)}(z)}{dz} - 2nC_n^{(\lambda)}(z) = 4\sum_{k=0}^{n-1} 1 - (-)^{k+n} \frac{k + \lambda}{2}(k + \lambda)C_k^{(\lambda)}(z).
\]
Eq. (3.13)

If we plug the expansions (3.2) and (3.13) into Eq. (3.6), use Eq. (3.5) and equate coefficients at $C_k^{(\lambda)}(z)$ on both sides of the resulting equation, this gives
\[
a_{nk}(\lambda) = 4\frac{1 + (-)^{k+n}}{2} \frac{k + \lambda}{(n - k)(k + n + 2\lambda)} (0 \leq k \leq n - 1).
\]
Eq. (3.14)

Insertion of Eqs. (3.14) and (3.14) into Eq. (3.2) results in the expansion (1.6).

4 The Jacobi polynomials

Finally, we shall determine coefficients in the expansion
\[
\frac{\partial P_n^{(\alpha,\beta)}(z)}{\partial \alpha} = \sum_{k=0}^{n} a_{nk}^{(\beta)}(\alpha)P_k^{(\alpha,\beta)}(z).
\]
Eq. (4.1)

where
\[
P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + \alpha + \beta + 1)} \sum_{k=0}^{n} \frac{\Gamma(k + n + \alpha + \beta + 1)}{k!(n - k)!\Gamma(k + \alpha + 1)} \left( \frac{z - 1}{2} \right)^k
\]
Eq. (4.2)
is the Jacobi polynomial. Observe that once these coefficients are determined, coefficients in the counterpart expansion of $\partial P_n^{(\alpha,\beta)}(z)/\partial \beta$ are also known, because in virtue of the relationship

$$P_n^{(\alpha,\beta)}(z) = (-)^n P_n^{(\beta,\alpha)}(-z) \quad (4.3)$$

one has

$$\frac{\partial P_n^{(\alpha,\beta)}(z)}{\partial \beta} = \sum_{k=0}^{n} (-)^{k+n} a_k^{(\alpha)}(\beta) P_k^{(\alpha,\beta)}(z). \quad (4.4)$$

The coefficient at $(z-1)^{n}$ on the right-hand side of Eq. (4.2) is

$$k_n(\alpha,\beta) = \frac{\Gamma(2n + \alpha + \beta + 1)}{2^n n! \Gamma(n + \alpha + \beta + 1)} \quad (4.5)$$

hence, the coefficient $a_n^{(\beta)}(\alpha)$ in the expansion (4.1) is

$$a_n^{(\beta)}(\alpha) = \psi(2n + \alpha + \beta + 1) - \psi(n + \alpha + \beta + 1). \quad (4.6)$$

To find the remaining coefficients $a_n^{(\beta)}(\alpha)$ with $0 \leq k \leq n - 1$, we differentiate the identity

$$\left\{ (1 - z^2) \frac{d^2}{dz^2} + [\beta - \alpha - (\alpha + \beta + 2)z] \frac{d}{dz} + n(n + \alpha + \beta + 1) \right\} P_n^{(\alpha,\beta)}(z) = 0 \quad (4.7)$$

with respect to $\alpha$, obtaining

$$\left\{ (1 - z^2) \frac{d^2}{dz^2} + [\beta - \alpha - (\alpha + \beta + 2)z] \frac{d}{dz} + n(n + \alpha + \beta + 1) \right\} \frac{\partial P_n^{(\alpha,\beta)}(z)}{\partial \alpha} = (z + 1) \frac{d P_n^{(\alpha,\beta)}(z)}{d z} - n P_n^{(\alpha,\beta)}(z). \quad (4.8)$$

Using the known relationship

$$(2n + \alpha + \beta)(z^2 - 1) \frac{d P_n^{(\alpha,\beta)}(z)}{d z} = n[\beta - \alpha + (2n + \alpha + \beta)z] P_n^{(\alpha,\beta)}(z) - 2(n + \alpha)(n + \beta) P_{n-1}^{(\alpha,\beta)}(z), \quad (4.9)$$

the right-hand side of Eq. (4.8) may be rewritten as

$$(z + 1) \frac{d P_n^{(\alpha,\beta)}(z)}{d z} - n P_n^{(\alpha,\beta)}(z) = \frac{2(n + \beta)}{2n + \alpha + \beta} \frac{n P_n^{(\alpha,\beta)}(z) - (n + \alpha) P_{n-1}^{(\alpha,\beta)}(z)}{z - 1}. \quad (4.10)$$

The Christoffel–Darboux formula for the Jacobi polynomials is

$$\sum_{k=0}^{n-1} \frac{(2k + \alpha + \beta + 1)}{(k + \alpha + \beta + 1) \Gamma(k + \beta + 1)} P_k^{(\alpha,\beta)}(z) P_k^{(\alpha,\beta)}(z') = \frac{2n! \Gamma(n + \alpha + \beta + 1)}{(2n + \alpha + \beta) \Gamma(n + \alpha) \Gamma(n + \beta)} \frac{P_n^{(\alpha,\beta)}(z) P_{n-1}^{(\alpha,\beta)}(z') - P_{n-1}^{(\alpha,\beta)}(z) P_n^{(\alpha,\beta)}(z')}{z - z'}. \quad (4.11)$$

We set in this formula $z' = 1$ and then use

$$P_k^{(\alpha,\beta)}(1) = \frac{\Gamma(k + \alpha + 1)}{k! \Gamma(\alpha + 1)}. \quad (4.12)$$
This yields
\[
\sum_{k=0}^{n-1} (2k + \alpha + \beta + 1) \frac{\Gamma(k + \alpha + \beta + 1)}{\Gamma(k + \beta + 1)} P_k^{(\alpha, \beta)}(z) = \frac{2\Gamma(n + \alpha + \beta + 1)}{(2n + \alpha + \beta) \Gamma(n + \beta)} n P_n^{(\alpha, \beta)}(z) - (n + \alpha) P_{n-1}^{(\alpha, \beta)}(z) \quad z - 1.
\]
Combining this result with Eq. (4.10) gives us the following expansion of the expression standing on the right-hand side of Eq. (4.8):
\[
(z + 1) \frac{d P_n^{(\alpha, \beta)}(z)}{dz} - n P_n^{(\alpha, \beta)}(z) = \frac{\Gamma(n + \beta + 1)}{\Gamma(n + \alpha + \beta + 1)} \sum_{k=0}^{n-1} (2k + \alpha + \beta + 1) \frac{\Gamma(k + \alpha + \beta + 1)}{\Gamma(k + \beta + 1)} P_k^{(\alpha, \beta)}(z).
\]
Substitution of Eq. (4.1) into the left-hand side and of Eq. (4.14) into the right-hand side of Eq. (4.8), followed by the use of Eq. (4.7), after some obvious movements leads to
\[
\alpha_n^{(\beta)}(\alpha) = \frac{2k + \alpha + \beta + 1}{(n - k)(k + n + \alpha + \beta + 1)} \frac{\Gamma(n + \beta + 1)\Gamma(k + \alpha + \beta + 1)}{\Gamma(k + \beta + 1)\Gamma(n + \alpha + \beta + 1)}
\]
\[
(0 \leq k \leq n - 1).
\]
Plugging Eqs. (4.7) and (4.15) into Eq. (4.1) results in the relation (1.2), while Eqs. (4.4), (4.6) and (4.15) imply Eq. (1.3).

A Appendix

In this Appendix we shall prove the following

**Lemma 1.** The coefficient \( a_{n\lambda}(\lambda) \) in the expansion (4.14) is given by
\[
a_{n\lambda}(\lambda) = \frac{\partial \ln k_n(\lambda)}{\partial \lambda},
\]
where \( k_n(\lambda) \) is the coefficient at \((z - z_0)^n\) (with \( z_0 \in \mathbb{C} \) arbitrary) in the expansion of the polynomial \( P_n(\lambda; z) \) in powers of \( z - z_0 \).

**Proof.** We have
\[
P_n(\lambda; z) = k_n(\lambda)(z - z_0)^n + Q_{n-1}(\lambda, z_0; z)
\]
(usually, \( k_n(\lambda) \) is independent of \( z_0 \)), where \( Q_{n-1}(\lambda, z_0; z) \) is a polynomial in \( z - z_0 \) of degree \( n - 1 \). Hence, it follows that
\[
\frac{\partial P_n(\lambda; z)}{\partial \lambda} = \frac{\partial k_n(\lambda)}{\partial \lambda}(z - z_0)^n + \frac{\partial Q_{n-1}(\lambda, z_0; z)}{\partial \lambda}.
\]
On the other side, from Eqs. (1.1) and (A.2) we have
\[
\frac{\partial P_n(\lambda; z)}{\partial \lambda} = a_{n\lambda}(\lambda) P_n(\lambda; z) + \sum_{k=0}^{n-1} a_{nk}(\lambda)P_k(\lambda; z)
\]
\[
= a_{n\lambda}(\lambda)k_n(\lambda)(z - z_0)^n + R_{n-1}(\lambda, z_0; z),
\]
where \( R_{n-1}(\lambda, z_0; z) \) is a polynomial in \( z - z_0 \) of degree \( n - 1 \). Equating coefficients at \((z - z_0)^n\) on the right-hand sides of Eqs. (A.3) and (A.4) yields the expression (A.1). \( \square \)
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