Numerical construction of magnetosphere with relativistic two-fluid plasma flows

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ABSTRACT

We present a numerical model in which a cold pair plasma is ejected with relativistic speed through a polar cap region and flows almost radially outside the light cylinder. Stationary axisymmetric structures of electromagnetic fields and plasma flows are self-consistently calculated. In our model, motions of positively and negatively charged particles are assumed to be determined by electromagnetic forces and inertial terms, without pair creation and annihilation or radiation loss. The global electromagnetic fields are calculated by the Maxwell’s equations for the plasma density and velocity, without using ideal magnetohydrodynamic condition. Numerical result demonstrates the acceleration and deceleration of plasma due to parallel component of the electric fields. Numerical model is successfully constructed for weak magnetic fields or highly relativistic fluid velocity, i.e. kinetic energy dominated outflow. It is found that appropriate choices of boundary conditions and plasma injection model at the polar cap should be explored in order to extend present method to more realistic pulsar magnetosphere, in which the Poynting flux is dominated.

Key words: magnetosphere – MHD – relativity – pulsars: general.

1 INTRODUCTION

A global structure of pulsar magnetosphere is one of the key issues to understanding energy outflow to the exterior. The numerical model has been successfully developed in the past decade, although the basic equation was already derived in the early days of pulsar theory. Extensive reviews are available in some books (e.g. Michel 1991; Beskin, Gurevich & Istomin 1993; Mestel 1999). Contopoulos, Kazanas & Fendt (1999) calculated the stationary axially symmetric magnetosphere based on the force–free approximation. They for the first time showed a solution with dipole magnetic field lines near a neutron star, which smoothly pass through the light cylinder to the wind region at infinity. In the model, there is a current sheet flowing on the separatrix and equator outside the light cylinder. The magnetosphere model is subsequently explored in detail by several authors: some physical properties represented by the solution (Ogura & Kojima 2003), the Y-point singularity between open and closed field lines on the equator (Uzdensky 2003; Timokhin 2006) and the electromagnetic luminosity in high numerical resolution (Gruzinov 2005, 2006). Numerical construction of the magnetosphere around an aligned rotator is also performed using time-dependent codes with the force–free and magnetohydrodynamic (MHD) approximations (Komissarov 2006; McKinney 2006). A stationary state, which is very similar to the solution given by Contopoulos et al. (1999), is obtained with certain initial and boundary conditions. The approach is extended to an oblique rotator by three-dimensional simulation codes (Spitkovsky 2006; Kalapotharakos & Contopoulos 2009).

An ideal MHD condition $E = B \times v / c$ is used to determine the electric field in these calculations irrespective of the numerical methods. Consequently, two electromagnetic field vectors are always orthogonal $B \cdot E = 0$, and the parallel component of $E$ along the plasma motion vanishes everywhere. This condition holds if the plasma density much exceeds the Goldreich–Julian density (Goldreich & Julian 1969). The global structure based on the force–free and MHD approximations is obviously a good step to understanding the whole magnetosphere. However, it is important to study how and where the condition breaks down, and how this changes the electromagnetic field structure and plasma behaviour. An alternative approach, in which the ideal MHD condition is relaxed, is necessary to address these problems. Breakdown of ideal MHD condition in the pulsar magnetosphere is qualitatively pointed out in the literature (e.g. Mestel & Shibata 1994; Goodwin et al. 2004). It is our purpose to further study the problem by actual modelling. It is necessary to determine the electric fields from the distribution of the charge density, since $E = B \times v / c$ is no longer used. The electric acceleration or deceleration of fluids will be allowed elsewhere, since $v \cdot E \neq 0$. The location may be important to the observation.

In this paper, we present an approach based on a two-fluid plasma consisting of positively and negatively charged particles. In this approach, the electromagnetic fields are modelled by Maxwell’s...
2 Assumptions and Equations

2.1 Electromagnetic fields and plasma flows

Axially symmetric electromagnetic fields in the stationary state are expressed by three functions $\Phi(r, \theta)$, $G(r, \theta)$, $S(r, \theta)$ as

$$E = -\nabla \Phi,$$

$$B = \frac{1}{R} \nabla \times e_\phi + \frac{S}{R} \hat{e}_\phi,$$

where we use spherical coordinate $(r, \theta, \phi)$ and $R = r \sin \theta$. It is convenient to use the non-corotational potential $\Psi = \Phi - \Omega G/c$, where $\Omega$ is angular velocity of a central star. Maxwell equations with charge density $\rho_e$ and poloidal and toroidal components of current $(j_r, j_\phi)$ are given by

$$DG = -\frac{4\pi}{c} j_\phi,$$

$$\frac{1}{R} \nabla \times e_\phi = \frac{4\pi}{c} j_\phi,$$

$$\nabla^2 \Psi = -4\pi \left( \rho_e - \frac{\Omega R}{c^2} j_\phi \right),$$

where operators $\nabla^2$ and $\nabla^2$ in spherical coordinate are given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin \theta}{\theta} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right),$$

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right).$$

We adopt a treatment in which the plasma is modelled as a two-component fluid. Each component, consisting of positively or negatively charged particles, is described by a number density $n_\pm$ and velocity $v_\pm = v_{\mp} + e_\mp e_s$. Note that the proper density $n_\pm^2$ is related with the lab-frame density $n_\pm$ by $n_\pm^2 = n_\pm / \gamma_\pm$, where $\gamma_\pm$ is a Lorentz factor $\gamma_\pm = \left[ 1 - (v_\pm/c)^2 \right]^{-1/2}$ (e.g. Goodwin et al. 2004). We assume that the positive particle has mass $m$ and charge $q$, while the negative one has mass $m$ and charge $-q$. The charge density and electric current are given in terms of $n_\pm$ and $v_\pm$ as

$$\rho_e = q(n_+ - n_-),$$

$$j = q(n_+ v_+ - n_- v_-).$$

Continuity equation for each component in the stationary axisymmetric conditions is

$$0 = \nabla \cdot (n_\pm v_\pm) = \nabla \cdot (n_\pm v_{\mp}),$$

The poloidal velocity components $v_{\mp}$ are satisfied by introducing a stream function $F_\pm(r, \theta)$ as

$$n_\pm v_{\pm} = \frac{1}{R} \nabla F_\pm \times e_\phi.$$

From the definition, the number density is given by

$$n_\pm = \frac{|\nabla F_\pm|}{R(v_{\mp}^2 + v_{\mp}^2)^{1/2}}.$$

From equations (9) and (11), the current function $S$ in equation (4) can be solved as

$$S = \frac{4\pi q}{c} (F_+ - F_-).$$

The electromagnetic force is dominant so that collision, thermal pressure and gravity are ignored. The interaction between two-component fluids is assumed only through the global electromagnetic fields. The equation of motion for each component with mass $m$ and charge $\pm q$ in the stationary state is given by

$$(v_\pm \cdot \nabla) v_\pm = \pm \frac{q}{m} \left[ E + \frac{v_\pm}{c} \times B \right].$$

By adding and subtracting equations (14) for two components, we have an equation of one-fluid bulk motion and a generalized Ohm’s law. See, for example, Melatos & Melrose (1996) and Goodwin et al. (2004) for the detailed discussion. We do not follow such a treatment, but rather solve equation (14) for each component. Using the identity $(v \cdot \nabla) v = (\nabla \times E) \times v + \nabla \nu c^2$, we find two conserved quantities along each stream line, corresponding to axially symmetric and stationary conditions. They are generalized angular momentum $J_\pm$ and Bernoulli integral $K_\pm$, which are obtained by the azimuthal component of equation (14) and a scalar product of $v$ and equation (14) (Mestel 1999). Their explicit expressions are given by

$$J_\pm = \gamma_\pm v_\pm c R \pm \frac{q}{mc} G,$$

$$K_\pm = \gamma_\pm \pm \frac{q}{mc^2} \Phi.$$
a cross product of \( \mathbf{v} \) and equation (14). This means a perpendicular component to the stream lines, which is given by

\[
DF_\pm = \nabla \ln \left( \frac{n_\pm}{\gamma_\pm} \right) \cdot \mathbf{v} \pm \frac{e^2}{2} \left( K'_{\pm} - \frac{v_n \pm v_\theta}{\gamma_\pm} J'_{\pm} \right)
\]

where \( J_{\pm} \) and \( K'_{\pm} \) are derivatives of \( J_{\pm} \) and \( K_{\pm} \) with respect to \( F_{\pm} \). Using \( \nabla J_{\pm} = J'_{\pm} \mathbf{v} F_{\pm}, \nabla K_{\pm} = K'_{\pm} \mathbf{v} F_{\pm} \) and equation (12), equation (17) can be written in an alternative form

\[
DF_\pm = \left( \nabla \ln \left( \frac{n_\pm}{\gamma_\pm} \right) \right) \cdot \mathbf{v} F_{\pm} \pm \frac{q}{mc} \frac{n_\pm}{\gamma_\pm} S,
\]

where \( S \) is an equator. The numerical domain in our calculations are given in terms of the corotating condition \( \Phi = \Omega G/c, j_\pm = \rho_\pm \Omega R e_\theta \). From equation (5), the corotating charge density is given by

\[
4\pi \rho_{\pm} = \frac{2\gamma_\pm}{(c^2 - \Omega^2 c^2 R^2)} \left( \sin \theta \frac{\partial G}{\partial r} + \cos \theta \frac{\partial G}{\partial \theta} \right).
\]

Three velocity components, Lorentz factor and number density are determined by equation (12) and two integrals (15) and (16), if four functions \( G, \Psi \) and \( F_{\pm} \) are known. The charge density (8) and current (9) are calculated from these fluid quantities of both species. In the corotating region, they are given by corotating charge density and current. Irrespective of the spatial region, the source terms of partial differential equations for \( G, \Psi \) and \( F_{\pm} \) depend on themselves in a non-linear manner. Some iterative methods are needed to self-consistently solve a set of equations (3), (5) and (18). There is no established method so far to solve non-linearly coupled equations, so that our numerical procedure is rather primitive. Initial guess for these functions is assumed, say \( G^{(0)}, \Psi^{(0)} \) and \( F_{\pm}^{(0)} \). Using these functions, the source terms are calculated, and a new set of functions \( G^{(1)}, \Psi^{(1)} \) and \( F_{\pm}^{(1)} \) are solved from these source terms with appropriate boundary conditions. The procedure is repeated until the convergence, say, \( |G^{(m+1)} - G^{(m)}|, |\Psi^{(m+1)} - \Psi^{(m)}|, |F_{\pm}^{(m+1)} - F_{\pm}^{(m)}| < \varepsilon \), where \( \varepsilon \) is a small number. The iteration scheme may not necessarily lead to a convergent solution, since there is no mathematical proof.

In order to examine our numerical scheme, we have performed a test for the split-monopole case, for which an analytic solution is known (Michel 1973). The non-corotational electric potential in the solution is zero everywhere, so that the condition \( \Psi = 0 \) is used and a reduced system of \( G \) and \( F_{\pm} \) is checked. These functions are numerically solved by a finite difference method with appropriate boundary conditions in the upper half plane. Results for the convergence to the solution are given in Table 1. Two types of initial trial functions and two different grid numbers are used. Deviation from the analytic solution is shown by a norm \( ||f||_n \), which is evaluated at all grid points as

\[
||f||_n = \left[ \sum (f_{n}^{(0)}(r, \theta) - f^{(n)}(r, \theta))^2 \right]^{1/2} / \left[ \sum (f^{(0)}(r, \theta))^2 \right]^{1/2},
\]

where \( f^{*} \) is the analytic solution and \( f^{(n)} \) is numerical result after \( n \) iterations. We have repeated until the relative error \( \varepsilon = 1 \times 10^{-1} \) in this test problem. We have started from \( G^{(0)} = 0 \), so that \( ||\delta G||_0 = 1 \). The initial choice of \( G^{(0)} \) is not so important, since the numerical solution approaches the analytic one at the first step. On the other hand, the choice of initial guess for \( F_{\pm} \) is important. It is not easy to set large deviation at the initial step, since the function \( F_{\pm} \) should be monotonic. If there is a maximum or minimum, where \( \nabla F_{\pm} = 0 \) inside the numerical domain, the flow vanishes. This causes a numerical difficulty at that point. From the monotonic nature consistent with the boundary conditions, the initial norm \( ||\delta G||_0 \) cannot be large. Table 1 shows that the numerical solutions successfully converge on the analytic ones within certain errors. Convergence factor \( \varepsilon \) does not exactly correspond to deviation from true solution, but gives an estimate. The true solution is not known in most problems, and the deviation cannot be calculated. The convergence factor \( \varepsilon \) can be regarded as error estimate.

### 2.2 Boundary conditions

We assume that the axially symmetry around a polar axis and the reflection symmetry across an equator. The numerical domain in the spherical coordinate \((r, \theta)\) is \( r_0 \leq r \leq r_1, 0 \leq \theta \leq \pi/2 \). The inner and outer radii in our calculations are \( r_0 = r_\star/5 \) and \( r_1 = 3r_\star \), where \( r_\star = c/\Omega \) is the distance to the light cylinder. Fig. 1 schematically represents the numerical domain. The functions at the inner boundary \( r_0 \) are closely related with plasma injection model, which is separately discussed in the next subsection. We here discuss the boundary conditions at the axis, equator and outer radius.

We solve the magnetic flux function \( G \) in the upper half plane between \( r_0 \) and \( r_1 \), which is the region enclosed by a curve \( PBRQP \) in Fig. 1. Poloidal magnetic field at the inner boundary \( r_0 \), i.e. on

![Figure 1. Meridian region of numerical calculation. The region enclosed by a curve ABLA is a corotating region. Plasma is injected through the polar region PA and goes out through the outer radius QR. A line OQ is polar axis, and OR is an equator.](https://academic.oup.com/mnras/article-abstract/398/1/271/1095895)
and the current function $S$ is calculated as

$$S = -\frac{8\pi q\lambda\alpha}{c r_0} \sin^2 \theta \left[ 1 - \left( \frac{\sin \theta}{\sin \theta_0} \right)^2 \right]. \quad (21)$$

The constant $\alpha$ determines the deviation from the dipolar field. The poloidal current completely vanishes in the limit of $\alpha = 0$, where the positively charged particles and negatively charged particles move along the common stream lines. The scale factor $\lambda$ is chosen as $4\pi q\lambda\alpha = \mu \Omega$, so that the current function can be written as $S \approx -2Gd\Omega/c$ near the polar region, where $G_d = \mu \sin^2 \theta/r$. The current function corresponds to the split-monopole solution near the polar region (Michel 1973). Our current function $S$ smoothly goes to zero at the edge of pole cap, $\theta = \theta_0$. This property is different from that of the force–free and ideal MHD approximations in which the function has a discontinuity (Contopoulos et al. 1999; Ogura & Kojima 2003; Gruzinov 2005). That is, there is a current sheet.

The current density as a function of angle is calculated as $j_\lambda = -4\mu \cos \theta [1 - 2(\sin^2 \theta) / \sin \theta_0^2 / r_0^2]$ at the polar cap. The electric current is negative for $0 < \sin \theta < \sin \theta_0 / \sqrt{2}$ and positive for $\sin \theta_0 / \sqrt{2} < \sin \theta < \sin \theta_0$. The total current ejected through the polar cap region is zero, since $(\sin \theta_0 \theta_0) = 0$. The positive or negative current flow is produced from the charge-separated plasma, i.e. different number-density distribution between two components in our model. The injection flow at $r_0$ is calculated from equation (20) as

$$n_{\pm} \nu_{\pm} = 2\lambda \cos \theta / c r_0 \left[ 1 + \alpha \left( 1 - 2 \left( \frac{\sin \theta}{\sin \theta_0} \right)^2 \right) \right]. \quad (22)$$

We assume that the flow speed $v_0$ is relativistic, $\gamma_0 = [1 - (v_0/c)^2]^{-1/2} \gg 1$, and is independent of $\theta$. The property is assumed to be the same for each particle type. The flow direction through the small polar region cos $\theta \approx 1$ is almost radial and the velocity is $v_0 \sim c$, so that the number density at $r_0$ is approximately given by

$$n_{\pm} = 2\lambda / c r_0 \left[ 1 + \alpha \left( 1 - 2 \left( \frac{\sin \theta}{\sin \theta_0} \right)^2 \right) \right]. \quad (23)$$

The charge density is given by

$$n_e = - \frac{4q\lambda\alpha}{c r_0} \left[ 1 - 2 \left( \frac{\sin \theta}{\sin \theta_0} \right)^2 \right] = - \frac{\mu \Omega}{\pi c r_0} \left[ 1 - 2 \left( \frac{\sin \theta}{\sin \theta_0} \right)^2 \right]. \quad (24)$$

where our choice of parameter $4q\lambda\alpha = \mu \Omega$ is used. The typical value of parameter (24) $\mu \Omega / (\pi c r_0^2)$ is the Goldreich–Julian charge density for the field strength $B_d$ of magnetic dipole.

Near the polar cap region, the force–free condition is satisfied, so that the current function $S$ and electric potential $\Psi$ depend on the magnetic flux function $G$. We adopt the following forms, $S_p(G_d)$ and $\Phi_p(G_d)$, as a function of dipolar flux function $G_d$ as

$$S_p = - \frac{2\Omega G_d}{c} (1 - b G_d), \quad (25)$$

$$\Phi_p = \frac{\Omega G_d}{c} + \frac{\Omega \Phi_0}{2 \pi c R} \left[ 1 - (b G_d)^2 \right], \quad (26)$$

where $b = r_0 / (\mu \sin^2 \theta_0)$. Equation (25) is reduced to equation (21) at $r_0$, and the Poisson equation is approximately satisfied for $\Phi_p$ (26) and the charge density (24). The electric current in the force–free condition is generally given by

$$j = e c \frac{d S}{d G} B + \rho_R c R \frac{d \Phi}{d G} e_p. \quad (27)$$
By the straightforward calculations, it is found that the poloidal components of equation (27) with the expressions $S_p(G_d)$ and $\Phi_p(G_d)$ are satisfied and toroidal component gives a small value $j_\phi \approx \rho_c c \times (r_0/r_L)^{3/2} \ll \rho_c c$. We regard $j_\phi = 0$ and impose the $\phi$-component of the fluid velocity as

$$v_\phi = -\frac{n_+ - n_-}{2n_+} R_0 \Omega, \quad v_\psi = -\frac{n_+ + n_-}{2n_-} R_0 \Omega,$$  \hspace{1cm} (28)

where $R_0 = r_0 \sin \theta$. For this choice, total angular momentum of plasma flow is $m(n_+ + n_-) \gamma_0 R_0^2 \Omega$ at the inner boundary.

We here summarize the boundary conditions at $r_0$. Equation (20) is used for $F_\pm$ with $4 \pi q_0 \alpha = \mu \Omega$ and equation (26) for $\Phi$. Two integrals $J_{\pm}$ and $K_{\pm}$ are calculated at $r_0$ as a function of polar angle $\theta$ from equations (23), (26), (28), $G_4$ and $\gamma_0$. The relations $J_{\pm}(F_{\pm})$ and $K_{\pm}(F_{\pm})$ are constructed by eliminating $\theta$ in terms of equation (20).

### 3 Numerical Results

We use a finite difference method to solve a set of partial differential equations. The typical grid number in the spherical coordinate $(r, \theta)$ is $300 \times 100$ for $0.2 \leq r/r_L \leq 5$ and $0 \leq \theta \leq \pi/2$. The polar cap region at the inner boundary is covered by approximately 30 grid points. We have obtained the same result by changing the grid numbers as 150 $\times$ 50 or 450 $\times$ 150. The convergent factor is $\varepsilon \approx 10^{-2}$, and cannot be improved so much by the grid refinement.

We demonstrate numerically constructed magnetosphere. Parameters used in the numerical calculation are a deviation parameter $\alpha = 0.2$, Lorentz factor $\gamma_0 = 10^2$ and $q_0/(mc^2 r_L^2) = 10$. The last dimensionless parameter is magnetic gyration frequency to angular velocity $\Omega = c/r_L$ of a star. Numerical results of the plasma flows and electromagnetic fields depend on a single combination, $\eta = q_0/(mc^2 r_L^2)$, as far as $\gamma_0 \gg 1$, since the source terms for equations (3), (5) and (18) are scaled by it. Thus, $\eta$ is a key parameter to determining the global structure. See Appendix for the details.

Fig. 2 shows numerical solution of the magnetic function $G$. We also show that of dipole field $G_d = \mu \sin^2 \theta/r$ for the comparison. The poloidal magnetic field is dipole near the inner radius $r_0/r_L = 0.2$. The field gradually deviates from the dipole, and becomes open outside the light cylinder. The field configuration is eventually radial near outer radius $r_1/r_L = 5$. The numerical result provides

![Figure 2. Magnetic flux functions. Solid curves denote the flux surfaces of numerical solution for $G_d/\mu = 0.1, 0.2, \ldots, 1.4$, and dotted curves those of dipole for $G_d/\mu = 0.2, 0.4, \ldots, 1.4$ starting from the polar axis.](https://academic.oup.com/mnras/article-abstract/398/1/271/1095895)

![Figure 3. Stream functions for negatively and positively charged fluids. Solid curves denote the flux surfaces of $F_-$, and dotted curves those of $F_+$ in intervals of 0.1 ($\lambda \sin^2 \theta_0/r_0$).](https://academic.oup.com/mnras/article-abstract/398/1/271/1095895)

the last open field line as $G_0 \approx 1.15 \mu/r_L$. The critical value is $G_0 = 1.592 \mu/r_L$ in magnetosphere filled with rigidly rotating plasma (Michel 1973, 1991; Mestel & Pryce 1992), and $G_0 = 1.36 \mu/r_L$ (Contopoulos et al. 1999), $G_0 = 1.27 \mu/r_L$ (Gruzinov 2005) in a solution with the force-free approximation, and $G_0 = 1.26 \mu/r_L$ (Kommassov 2006) in MHD simulation. Our value is smaller than that of other models, but is not so different. The total current is different from that of these models, so that there is no reason why the critical value should agree.

Fig. 3 shows the results of the stream functions for both species. The global structure of stream lines is almost the same as that of the magnetic field lines shown in Fig. 2, although the numerical agreement is not so complete. Thus, flows in meridian plane is almost parallel to the magnetic field lines. A difference between $F_-$ and $F_+$ at large radius originates from the inner boundary condition at $r_0$, where the fraction of negatively charged plasma is slightly larger in polar region $\theta \approx 0$, but smaller for $\theta \approx \theta_0$. The property extends to the outer radius. At large radius, the flow becomes radial and the velocity is still relativistic, so that the number density decreases with the radius, $n_\pm \approx (\rho_0 F_\pm/(r^2 \sin \theta)) \propto r^{-2}$. However, the fraction $(n_+ - n_-)/(n_+ + n_-)$ is still finite, and the charge separation remains. Our numerical model shows that the negatively and positively charged regions are separated approximately by a curve with $F_\pm \approx 0.5 (\lambda \sin^2 \theta_0/r_0)$.

We show numerical result of the electric potential. The contour of the non-corotating part $\Psi = \Phi - \Omega G/c$ is shown in Fig. 4. There is a peak on the polar axis at $r \approx 0.4 r_L$. The function $\Psi$ decreases towards the outer boundaries, where $\Psi = 0$ is imposed as the boundary condition at outer radius, and on the last open magnetic field. Numerical result shows the maximum value $\Psi = 0.85 \mu \Omega/(cr_L)$ at $(r, \theta) = (0.4 r_L, 0)$. This value is not small, since the maximum of corotating electric potential is $\Omega G/c \approx 1.15 \mu \Omega/(cr_L)$. Total electric potential $\Phi = \Psi + \Omega G/c$ is shown in Fig. 5. Overall structure is very different from the magnetic flux function $G$ or stream functions $F_\pm$ shown in Figs 2–3. The difference is clear at the polar region, whereas the agreement becomes better at high-latitude region near the equator. Ideal MHD condition $B \cdot E = 0$ is not assumed in our model. The deviation is very large on the polar axis. This feature is closely related with the boundary conditions. As discussed in Section 2, the boundary condition of $\Psi$ is not $E_r = 0$, but $E_\theta = 0$.
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Figure 4. Contour of non-corotating part of electric potential $\Psi = \Phi - \Omega G/c$. Contour levels outwardly decrease in intervals of $0.1 \mu \Omega/(c r_L)$.

Figure 5. Contour of electric potential $\Phi$. Contour levels upwardly decrease in intervals of $0.1 \mu \Omega/(c r_L)$.

on the axis. This mathematical condition may allow the large value of $\Psi$ on the axis. On the other hand, $\Psi$ almost remains zero near the equatorial region from the boundary condition.

We discuss a consequence of non-ideal MHD field $B \cdot E \neq 0$. Fig. 6 shows contour of the Lorentz factor of positively charged particles normalized by initial one $\gamma_0$. At the injection boundary, $\gamma_+ \text{ is fixed as } \gamma_+ = \gamma_0 = 10^4$ for all polar cap angle, but there is a gradual increase towards the outer radius. The increase is remarkable at low $\theta$, but $\gamma_+$ is almost constant for the flow along the equator. The increase of $\gamma_+$ is determined by the Bernoulli integral as $\Delta \gamma_+ = -q \Delta \Phi/(mc^2)$, since $0 = \Delta K_+ = \Delta \gamma_+ + q \Delta \Phi/2mc^2$ along each flow line. Thus, large acceleration of positively charged particles towards the polar region can be understood, since available potential difference $-\Delta \Phi$ is large as inferred from Fig. 5. The Lorentz factor $\gamma_-$ of negatively charged particles is opposite in sign, and is given by $\Delta \gamma_- = q \Delta \Phi/(mc^2)$. They are, therefore, decelerated towards the polar region.

Fig. 7 demonstrates electromagnetic forces acting on a positively charged particle on the stream lines. Two vectors $qE$ and $q(v \times B)/c$ in the meridian plan are shown by arrows. The sum of these two forces causes a net acceleration. It is clear that the vector $E$ is not perpendicular to the flow lines, and hence accelerates outwardly. The acceleration mechanism works better for the flow towards the polar region. This is another explanation for the increase of the Lorentz factor $\gamma_+$ as shown in Fig. 6. This electric effect is opposite for negatively charged particles, which should be decelerated.

In Fig. 8, we show the current function $S$. The poloidal current flows along a curve with a constant value of $S$. Fig. 8 demonstrates a return current. That is, two distinct positions at $r_0$ are connected by a curve, say, the curve with $S = -0.3 \mu r_0^2$. Such a global return current is generally produced due to the inertial term in our model. The integral $J_\pm$ in equation (15) is replaced by $G$ in the limit of $m = 0$. The stream function $F_\pm$ is constant on a constant magnetic surface $G$, and the current function $S = 4 \pi q(F_+ - F_-)/c$ is also constant. The global structure of $S$ should be the same as that of $G$. Therefore, no loop of $S$ is allowed in the limit of $m = 0$, since $G$ is open field outside corotation region. In the context of generalized Ohm’s law, the inertial term is a kind of resistivity. This term causes the dissipation of global current, and return current is produced in our model.
We consider the effect of the current decay on the toroidal magnetic field, which is given by $B_\phi = S/R$. Fig. 9 shows the global structure. The function is zero at the polar axis, on the last open field line of $G$, and has a maximum at $\theta \approx \theta_0/\sqrt{2}$ of the polar cap region. The function decreases outwardly. Ratio to the poloidal component is important since the magnetic field strength both of poloidal and toroidal components decreases with radius. The ratio is small, $|B_\theta/B_\phi| \leq (r_0/r_L)^{3/2}/2 \approx 0.04$ at the inner boundary. We numerically estimated and found that $|B_\theta/B_\phi| \approx 0.5$ at $(r_L, \pi/4)$, and $|B_\theta/B_\phi| \approx 1$ at $(4r_L, \pi/4)$. Outside the light cylinder, the poloidal magnetic field is monopole-like as shown in Fig. 2, so that $|B_\phi| \propto r^{-2}$. On the other hand, $|B_\theta| \propto r^{-1}$ along the stream lines in the limit of $m = 0$. The slope of $|B_\phi|$ slightly becomes steep due to the inertial term, but it is not so steep as $\propto r^{-2}$. In this way, toroidal component of the magnetic field is gradually important with radius, although the resistivity is involved in our model.

The electromagnetic luminosity through a sphere at $r$ is evaluated by radial component of the Poynting flux as

$$L_{\text{em}}(r) = 2 \int_0^{\pi/2} \frac{c}{4\pi} (E \times B)_r 2\pi r^2 \sin \theta d\theta.$$

(29)

Figure 8. Contour of current stream function $S$ in intervals of $0.1\mu/r_L^2$.

Figure 9. Contour of toroidal magnetic field $B_\phi$. Contour levels are outwardly for $-B_\phi r_L^2/\mu = 3.2, 1.6, 0.8, 0.4, 0.2, 0.1, 0.05$.

The luminosity of the plasma flow is a sum of both species as

$$L_{\text{plasma}}(r) = 2 \int_0^{\pi/2} mc^2 (\mu n_+ v_{+r} + \mu n_- v_{-r}) 2\pi r^2 \sin \theta d\theta. \quad (30)$$

The energy conversion between two flows is possible through the Joule heating $j \cdot E$, but the total $L_{\text{em}}(r) + L_{\text{plasma}}(r)$ should be conserved. In Table 2, numerical results are shown for different radii. We can check the conservation within a numerical error. The energy flux by plasma flow is always much larger than the electromagnetic one, and the conversion is very small in our model. The magnitude of the luminosities is almost fixed by the injection condition. We analytically evaluate these luminosities at $r_0$ using the inner boundary conditions, and find that $L_{\text{em}} = (\mu \Omega^2 \sin^2 \theta_0/r_0^2)/(3c) \approx 0.4 \mu \Omega^2 \sin^2 \theta_0/r_0 = 2 \mu \Omega^2 \sin^2 \theta_0/(ar\eta_0) \approx 115 \mu \Omega^2 /c^3$, where the numerical values $\alpha = 0.2, \eta = 0.1$ and $\sin^2 \theta_0 = 1.15 \Omega/c$ are used. In order to simulate the Poynting flux dominated case, it is necessary to increase the parameter $\eta$.

4 CONCLUSION

We have numerically constructed a stationary axisymmetric model of magnetosphere with charge separated plasma outflow. The flow lines of pair plasma are determined by electromagnetic forces and inertial term. The massless limit corresponds to the force–free and ideal MHD approximations. The global structures of electromagnetic fields and plasma flows are calculated by taking into account the inertial term. In particular, the non-ideal MHD effects are studied. The electrical acceleration or deceleration region depending on the charge species appears. Poloidal current slightly dissipates. Numerical results depend on a single parameter $\eta = q\mu/(mc^2 \gamma_0 r_L^2)$ as far as $\gamma_0 \gg 1$. The number of our model demonstrated in Section 4 is $\eta = 0.1$, and is small when applying to the pulsar magnetosphere. Typical number $\eta$ is estimated for electron–positron pair plasma, magnetic field $B_0$ at the surface and spin period $P$ as $\eta = 10^4 (B_0/10^2G)(P/1s)\gamma_0/10^2$. Our present model is only applicable to highly relativistic injection ($\gamma_0 \gg 1$) or weaker magnetic fields ($B_0 \ll 10^2G$). It will be necessary to scale up many orders of magnitude to $\eta \sim 10^4$ in order to apply the present method to more realistic cases. We in fact tried scaling up in our numerical calculations, but found that it is not straightforward.

The difficulty and the limitation to smaller value of $\eta$ are closely related with boundary conditions and involved physics, as explained below. The Bernoulli integral (16) should satisfy a constraint $K_{\pm} + q\Psi/mc^2 = \gamma > 1$. In our model, we specify $K_{\pm}$ by the injection condition which is fixed at the inner boundary. During the numerical iterations, the magnitude of the function $\Psi$ becomes very large in a certain region, where the condition $K_{\pm} + q\Psi/mc^2 > 1$ is no longer satisfied. It is easily understood that this easily happens for large value of $\eta = q\mu/(mc^2 \gamma_0 r_L^2)$, since the typical scale of $\Psi$ is

| $r/r_L$ | $L_{\text{em}} c^3/(\mu^2 \Omega^4)$ | $L_{\text{plasma}} c^3/(\mu^2 \Omega^4)$ |
|--------|---------------------------------|---------------------------------|
| 1.5    | 0.18                            | 115.57                          |
| 2.0    | 0.20                            | 115.55                          |
| 2.5    | 0.21                            | 115.54                          |
| 3.0    | 0.22                            | 115.53                          |
| 3.5    | 0.23                            | 115.53                          |
| 4.0    | 0.23                            | 115.53                          |
| 4.5    | 0.23                            | 115.53                          |
\[ \frac{\mu}{r^2}. \] This gives a certain upper limit to the choice of \( \eta \). The actual estimate of the limit is somewhat complicated, since the potential \( \Psi \) depends on the choice of boundary conditions, especially injection model at inner radius. It is, therefore, necessary to explore consistent boundary conditions or to include some physical process, in order to calculate the models with larger \( \eta \). Adjusting mechanism may be required at the boundary or within the numerical domain. For example, our present inner boundary is one-way, i.e. injection only at fixed rate. During numerical iterations, the charge density might numerically blow up elsewhere due to poor boundary condition if the injection rate is able to be adjusted or plasma is absorbed through the boundary, the increase may be suppressed. However, this is a very difficult back-reaction problem. The boundary conditions are normally used to determine the inner structures. The adjustable boundary conditions should be controlled by the interior. Thus, the boundary conditions and the inner structures should be determined simultaneously. Such numerical scheme is not known and should be developed in future. Otherwise, extensive study to find out consistent boundary conditions for large \( \eta \) is required. Our numerical method will be improved by either approach.

Pulsar magnetosphere is described in most spatial region by the force-free and ideal MHD approximations, which correspond to the limit of \( \eta \gg 1 \). Stationary axisymmetric magnetosphere is constructed so far by a solution of the Grad–Shafranov equation with these approximations (Contopoulos et al. 1999; Ogura & Kojima 2003; Gruzinov 2005). Poloidal magnetic field approaches a quasi-spherical wind at infinity. There is a discontinuity in toroidal magnetic field at the boundary of the corotating region, where the current sheet is formed. Lovelace, Turner & Romanova (2006) have obtained an alternative solution of the same equation, but with a different injection current. Their solution exhibits a jet along polar axis and a disc on the equator. There is no current sheet in their numerical model. Thus, there are at least two models in the strong magnetic field limit, \( \eta \gg 1 \). The poloidal magnetic field at infinity is quasi-spherical and there is no current sheet in our numerical solution. It is interesting to examine the model sequence by increasing the parameter \( \eta \). Some plasma flow should be highly constrained to a thin region, or the jet-disc system should be formed in the large \( \eta \) limit. It is unclear whether or not many solutions exist, depending on physical situations including the plasma state. In order to address these questions, it is necessary to construct the magnetosphere with plasma flow beyond the force–free and ideal MHD approximations. We have here presented a possible approach, although the improvement is needed.

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APPENDIX A: DIMENSIONLESS FORMS

We consider dimensionless forms of equations (3), (5) and (18). The magnetic function \( G \) is normalized in terms of the magnetic dipole moment \( \mu \) and the distance to light cylinder \( r_L \) as \( G = \mu G^\dag / r_L \), where a symbol \( \dag \) denotes a dimensionless quantity. The electric potential \( \Psi \) is expressed as \( \Psi = \mu \Omega \Psi^\dag / (c r_L) = \mu \Psi^\dag / r_L^2 \). Two integrals (15) and (16) are written as \( J^\dag = \gamma_0 c r_L j^\dag_\pm \) and \( K^\dag = \gamma_0 K^\dag_\pm \), where

\[ J^\dag = \left( \frac{\gamma_\pm}{\gamma_0} \right) \left( \frac{\nu_0}{c} \right) \left( \frac{R}{r_L} \right) \pm \eta G^\dag, \tag{A1} \]

\[ K^\dag = \left( \frac{\gamma_\pm}{\gamma_0} \right) \pm \eta \Psi^\dag. \tag{A2} \]

These values depend on a dimensionless parameter \( \eta = q u / (mc^2 \gamma_0 r_L^2) \). As for the stream function \( F_\pm \), the normalization constant is \( \mu \Omega / ( q r_L ) = \mu c / ( q r_L^2 ) \) and \( F_\pm = \mu \Omega F^\dag / (q r_L^2) \). The number density is normalized from equation (12) as \( n_\pm = \mu \Omega n^\dag / ( q c r_L^2 ) \). Electric charge and current densities are normalized as \( \rho_\pm = \mu \Omega \rho^\dag / ( c r_L^2 ) \) and \( j = \mu \Omega j^\dag / r_L^2 \). Using these dimensionless functions, equations (3), (5) and (18) can be written as

\[ D^\dag G^\dag = -4\pi R^1 \hat{J}^\dag, \tag{A3} \]

\[ (\nabla^\dag)^2 \Psi^\dag = -4\pi \left( \rho^\dag_\pm - R^1 j^\dag_\pm \right) - 2 \left( \frac{1}{r_L^2} \right)^2 \left( r^1 \frac{\partial}{\partial r^1} + \cot \theta \frac{\partial}{\partial \theta} \right) G^\dag, \tag{A4} \]
\[ \mathcal{D}^\dagger F^\dagger_\pm = \left[ \nabla^\dagger \ln \left( \frac{n^\dagger_\pm \gamma_0}{\gamma_\pm} \right) + \frac{c^2 \gamma_0}{\gamma_\pm (v^\dagger_\pm + v^\dagger_\parallel)} \left( \nabla^\dagger K^\dagger_\pm - \frac{v^\dagger_\pm}{c R^\dagger} \nabla^\dagger J^\dagger_\perp \right) \right] \cdot \nabla^\dagger F^\dagger_\pm \pm 4\pi \eta \frac{n^\dagger_\pm \gamma_0}{\gamma_\pm} \left( F^\dagger_+ - F^\dagger_- \right), \quad (A5) \]

where \( R^\dagger = R/r_L, \rightarrow r/r_L \) and the differential operators with the symbol \( \dagger \) are defined by \( r^\dagger = r/r_L \). From these expressions, we find that \( \eta \) is an important parameter.

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