Infinite geodesics, competition interfaces and the second class particle in the scaling limit

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Contents

1 Introduction 2
2 Impressions from a landscape 11
3 Geodesics 14
4 Competition interfaces 32
5 A portrait of interfaces 54
6 The 2nd class particle 58

Abstract

We establish fundamental properties of infinite geodesics and competition interfaces in the directed landscape. We construct infinite geodesics in the directed landscape, establish their uniqueness and coalescence, and define Busemann functions. We then define competition interfaces in the directed landscape. We prove the second class particle in tasep converges under KPZ scaling to a competition interface. Under suitable conditions, we show the competition interface has an asymptotic direction, analogous to the speed of a second class particle, and determine its law. Moreover, we prove the competition interface has an absolutely continuous law on compact sets with respect to infinite geodesics.

Keywords: competition interface, KPZ universality, last passage percolation, second class particle, tasep

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1 Introduction

Imagine a random metric over the plane. Given two points, consider the locus of all points that are equidistant from them. In Euclidean geometry this is of course a straight line, the perpendicular bisector between the points, and also an infinite geodesic of the Euclidean metric. What is the shape of this curve in the random metric? Of interest are its local behaviour, its relation to geodesics, and its global nature.

We study geodesics and competition interfaces in the directed landscape, which is expected to be the scaling limit of the most natural planar discrete random metric: first passage percolation. The directed landscape is the scaling limit of all classical last passage percolation models, and carries the metric geometry inherent in models belonging to the KPZ universality class. Its therefore describes universal KPZ geometry.

Several questions we consider have been studied in the literature in the pre-limit, for last passage percolation models, and some of our ideas do go back to the work of other authors. There are, however, crucial differences. One is that in the directed landscape, there are arbitrary small scales. This give rise to complications as well as interesting new questions. The second is that this limit has more symmetries, which allows us to give explicit answers to many questions. Before giving such an example, let us give a definition of the directed landscape.

Assign independent exponential random variables $\omega_p$ to points $p \in \mathbb{Z}_2^2 = \{(x, y) \in \mathbb{Z}^2, x + y \text{ even}\}$. A finite sequence $\pi_0, \ldots, \pi_n$ of points in $\mathbb{Z}_2^2$ is an (upwards) directed path in $\mathbb{Z}_2^2$ if $\pi_{i+1} - \pi_i = (1, 1)$ or $(-1, 1)$ for all $i$. For $a, b \in \mathbb{Z}_2^2$, let

$$L(a; b) = \max_{\pi} \sum_{u \in \pi} \omega_u$$

where the maximum is taken over directed paths beginning at $a$ and ending in $b$; let $L(a; b) = -\infty$ if there is no such path, and extend $L$ to $(\mathbb{R}^2)^2$ by rounding arguments to the nearest point in $\mathbb{Z}_2^2$.

In [12] it is shown that as $\epsilon \to 0$, jointly as functions of $x, t, y, s$

$$\frac{\epsilon}{2} L\left(\frac{2x}{\epsilon^2}, t/\epsilon^3 ; \frac{2y}{\epsilon^2}, s/\epsilon^3 \right) - (t - s)/\epsilon^2 \xrightarrow{\text{law}} \mathcal{L}(x, t; y, s),$$

where the random function $\mathcal{L} : (\mathbb{R}^2)^2 \to \mathbb{R} \cup \{-\infty\}$ is the directed landscape. This function is a directed metric on $\mathbb{R}^2$: it satisfies $\mathcal{L}(p, p) = 0$ and the reverse triangle inequality $\mathcal{L}(p, q) \geq \mathcal{L}(p, r) + \mathcal{L}(r, q)$. For distinct points $p, q$ with $s \geq t$, $\mathcal{L}(p, q) = -\infty$. Otherwise $\mathcal{L}(p, q) + (x - y)^2/(t - s)$ has a GUE Tracy-Widom law scaled by $(t - s)^{1/3}$.

An explicit formula available in the limit is the following. Let $\mathcal{H} = \{(x, t) : x \in \mathbb{R}, t > 0\}$
denote the open upper half plane.

**Theorem 1.1 (Direction of bisectors).** Consider the points \((-1, 0)\) and \((1, 0)\) and the locus of all points \((x, t) \in \mathbb{H}\) that are \(L\)-equidistant from them, namely \(L(-1, 0; x, t) = L(1, 0; x, t)\). This set is the graph \(\{(I(t), t) : t > 0\}\) of a continuous function \(I(t)\), and there is a random variable \(D\) which is Normally distributed with mean 0 and variance \(1/4\) such that, almost surely, 

\[
\lim_{t \to \infty} \frac{I(t)}{t} = D.
\]

Now consider the line segment from \([-1, 1] \times \{0\}\). A point \((x, t) \in \mathbb{H}\) is \(L\)-equidistant from its left half \([-1, 0]\) and its right half \([0, 1]\) if 

\[
\sup_{y \in [-1,0]} L(y, 0; x, t) = \sup_{y \in [0,1]} L(y, 0; x, t).
\]

The \(L\)-equidistant points form the graph \(\{(I(t), t) : t > 0\}\) of a continuous function and, almost surely, 

\[
\lim_{t \to \infty} \frac{I(t)}{t} = D \text{ law} = (\beta_2 - \beta_1)\chi_5 - N.
\]

The pair \((\beta_1, \beta_2)\) are the first two components from a Dirichlet(1,1,2) random vector. The random variable \(\chi_5\) has the chi-five distribution and \(N\) has a standard Normal distribution. All three of \((\beta_1, \beta_2), \chi_5\) and \(N\) are independent.

Bisectors in last passage percolation are called competition interfaces, and their directions are well studied in the literature. In a seminal paper [18] Ferrari and Pimentel derived the direction of the competition interface in exponential last passage percolation with the step initial condition. This was generalized by Ferrari, Martin and Pimentel [19] to initial conditions having different inclinations. Georgiou, Rassoul-Agha and Seppäläinen [26] showed existence of the direction for step initial condition in the last passage percolation model with continuous i.i.d. weights. Cator and Pimentel [7] considered exponential last passage percolation and the Hammersley process with rather general initial conditions, and described the limiting direction of competition interfaces in terms of suprema of certain random walks. We believe the next theorem is the first geometric representation.

An initial condition \(h_0 : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}\) is an upper semicontinuous function that is finite at some point and satisfies \(h_0(x) \leq c(1 + |x|)\) for all \(x\) and some constant \(c\). For \((x, t) \in \mathbb{H}\), the distance from \(h_0\) to \((x, t)\) is 

\[
L(h_0; x, t) = \sup_{y \in \mathbb{R}} \{h_0(y) + L(y, 0; x, t)\}.
\]

The initial condition \(h_0\) is good if (i) there are \(a < 0 < b\) such that \(h_0(a)\) and \(h_0(b)\) are finite, and (ii) for a two-sided Brownian motion \(B(x)\) with diffusivity constant \(\sqrt{2}\), and every compact interval \([-a, a]\), \(h_0(x) + B(x)\) almost surely does not have a maximum over
Let \([-a, a]\) at \(x = 0\). Let \(h_0^-, h_0^+\) be the restrictions of \(h_0\) to the non-positive and non-negative axes, taking value \(-\infty\) elsewhere. If \(h_0\) is good, the set of points \((x, t) \in \mathcal{H}\) that are \(\mathcal{L}\)-equidistant from \(h_0^-\) and \(h_0^+\) forms the graph

\[
\{(I(t), t), t > 0\}
\]

of a continuous function \(I(t)\). The curve \(I(t)\) is the bisector of \(h_0\) from the origin \(x = 0\), which is called a competition interface.

Let \(\text{cmaj}(f)\) denote the concave majorant of the function \(f\): the infimum of all concave functions that are at least \(f\). Let \(B\) be a two-sided Brownian motion with constant diffusivity \(\sqrt{2}\). The following theorem gives the direction of \(I(t)\); see Theorem 4.2 and Corollary 4.3 for more general statements.

**Theorem 1.2** (Direction of interfaces). Suppose \(h_0\) is a good initial condition and there are \(a > 0\) and \(b \in \mathbb{R}\) such that \(h_0(x) \leq -a|x| + b\) for every \(x\). Then there is an almost sure limit:

\[
\lim_{t \to \infty} \frac{I(t)}{t} = D^{\text{law}} = -\frac{1}{2} \text{cmaj}(h_0 + B)'(0)
\]

In words, the vector \((2D, 1)\) is perpendicular to the graph of \(\text{cmaj}(h_0 + B)\) at zero. This is a randomized analogue of Euclidean plane geometry, where the direction is perpendicular to \(2 \text{cmaj}(h_0)\). Theorem 1.2 allows us to compute the direction in several settings, in particular, recent work of Ouaki and Pitman [37] about \(\text{cmaj}(B|_{(0, \infty)})\) allows us to deduce Theorem 1.1 and variants such as when \(h_0\) is itself a Brownian motion with negative drift.

Next, we study the local nature of the curve \(I(t)\). In Euclidean geometry, for some examples, the curve could be a piecewise linear function. So at points where the segments meet, the curve is locally different. In random geometry, the curve \(I(t)\), although random, is more uniform: locally it is indistinguishable from a geodesic of \(\mathcal{L}\).

Specifically, there is almost surely an unique infinite geodesic of \(\mathcal{L}\) from \((0, 0)\) in direction zero, see Theorems 3.2 and 3.3. This is the unique \(\mathcal{L}\)-measurable continuous function \((g, t \geq 0)\) such that (i) \(g(0) = 0\) and \(g(t)/t \to 0\), and (ii) for every \(t_1 < t_2 < t_3\) and \(z_i = (g(t_i), t_i)\), \(\mathcal{L}(z_1, z_3) = \mathcal{L}(z_1, z_2) + \mathcal{L}(z_2, z_3)\).

**Theorem 1.3** (Geodesic-like interfaces). For any compact interval \([a, b] \subset (0, \infty)\), the law of the restricted competition interface \(I|_{[a, b]}\) is absolutely continuous with respect to the law of the restricted geodesic \(g|_{[a, b]}\).

The implications of such absolute continuity are very strong: all local almost sure properties of \(g(t)\) are inherited by \(I(t)\). For example, \(I(t)\) has three-halves variation and is Holder continuous with any exponent less than 2/3, see [14]. Moreover, from a single
sample of $I(t)$ one can recover the law of the entire doubly-infinite geodesic $\gamma_*$ defined in [14]. Specifically, let $A$ be an event defined in terms of $\gamma_*$ restricted to some compact time-interval $[a,b]$. Let $I_{n,k}(t) = n^{2/3}(I(1+t/n+k/n) - I(1+k/n))$ for $t \in [a,b]$ and $1 \leq k \leq n$. Then along a deterministic sequence of natural numbers,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}(I_{n,k} \in A) = P(\gamma_* \in A) \quad \text{a.s.}$$

Finally, we use interfaces to derive the scaling limit of a 2nd class particle in the totally asymmetric simple exclusion process (tasep). A fundamental result of Ferrari, Martin and Pimentel is that competition interfaces are related to the trajectory of a 2nd class particle in tasep via a time-change [18, 19]. We show that, under suitable conditions, the scaling limit of a competition interface in exponential last passage percolation is a corresponding interface $I(t)$. This leads to the scaling limit of a 2nd class particle in tasep. There are no previous functional limit theorems about the 2nd class particle, but the theory built so far allows us to prove such a result. In §1.1 we elaborate on this result with its background.

Apart from these results the paper develops the theory of geodesics and interfaces of the directed landscape, summarized in §1.2 and §1.3 below. Geodesics, interfaces and 2nd class particles are all tied to each other in interesting ways. We hope the reader will find our discussion informative and the connections appealing. Several open problems are mentioned throughout for further thought.

### 1.1 The 2nd class particle in tasep

In the continuous time tasep particles initially occupy sites along the integers, at most one per site, and those sites which are not occupied are called holes. Each particle tries to jump randomly at rate 1 to its neighbouring site to the right, however, a jump is allowed only if a particle is displacing a hole, as per the exclusion rule each site can have at most one particle.

A **2nd class particle** in tasep initially starts at site 0, surrounded by other particles and holes. It too tries to jump right like a regular particle, displacing holes but not particles. But when a regular particle wants to jump to the site of the 2nd class particle it is allowed to do so, moving the 2nd class particle leftward.

See Liggett’s book [31] for both a rigorous construction of tasep and of tasep with 2nd class particles.

Consider tasep with a single 2nd class particle at the origin and initial condition $X_0(\cdot)$, where $X_0(n)$ is the initial location of regular particle number $n$, with particles labelled right to left and particle number 1 being the first one below site 0. Let $X(t)$ be the position
of the 2nd class particle at time t.

Suppose there is a sequence of tasep initial conditions $X^\varepsilon_0$ for $\varepsilon > 0$, and it converges under the re-scaling

$$\varepsilon^{1/2}(X^\varepsilon_0([\varepsilon^{-1}x]) + 2\varepsilon^{-1}x) \to -h_0(-x) \quad \text{as } \varepsilon \to 0,$$

(1.5)

where $h_0$ is an initial condition in the continuum as introduced earlier. Suppose the convergence is uniform on compact subsets of $x \in \mathbb{R}$. Assume $h_0$ is a good initial condition. Then there is a competition interface $I(t)$ associated to $h_0$ as in (1.4).

**Theorem 1.4 (Scaling limit of the 2nd class particle).** Let $X^\varepsilon(t)$ be the position of the 2nd class particle at time t with initial condition $X^\varepsilon_0$. Under the assumptions above, as $\varepsilon \to 0$, the re-scaled trajectory of the 2nd class particle converges to the interface:

$$\frac{(\varepsilon/2)X^\varepsilon(2\varepsilon^{-3/2}t)}{I(t)}$$

in law, under the topology of uniform convergence over compact subsets of $t \in (0, \infty)$.

The mode of convergence (1.5) of initial conditions can be relaxed to include a broader class. We state and prove the more general theorem in §6 as Theorem 6.1.

Theorem 1.4 is new even for the classical flat and stationary initial conditions. The flat initial condition consists of particles at every even integer site, and under the re-scaling (1.5) it converges to $h_0(x) \equiv 0$. The stationary initial condition has a particle at each site independently with probability $1/2$; under diffusive scaling (1.5) it converges in law to a two-sided Brownian motion.

The flat interface $I_{\text{flat}}(t)$ has an invariant law under re-scaling $\sigma^{-2}I_{\text{flat}}(\sigma^3t)$. Its law at time $t = 1$ is

$$I_{\text{flat}}(1)^{\text{law}} = -\frac{1}{2} \text{cmaj}(\mathcal{A})'(0)$$

where $\mathcal{A}(x) = \mathcal{L}(0, 0; x, 1)$ is the parabolic Airy process. Indeed, in Proposition 4.7 we provide a geometric characterization of the single-time law of interfaces.

The stationary interface $I_{\text{stat}}(t)$ has the same law as the aforementioned infinite geodesic of $\mathcal{L}$ from $(0, 0)$ in direction zero (Lemma 4.2). This equality is based on a duality between geodesics and competition interfaces in last passage percolation [1, 47]. The law of the infinite geodesic is in turn described by Corollary 3.4, in terms of geodesics from the stationary (Brownian) initial condition. The law of $I_{\text{stat}}(t)$ is scale invariant like $I_{\text{flat}}(t)$ and

$$I_{\text{stat}}(1)^{\text{law}} = \text{argmax } (\mathcal{A}(x) + B(x))$$

where $B$ is a two-sided Brownian motion with diffusivity constant $\sqrt{2}$ and independent
A. Ferrari and Spohn [23] found the distribution function of $I_{\text{stat}}(1)$ in terms of the KPZ scaling function. Balázs, Cator, and Seppäläinen [1] gave optimal bounds on the fluctuation of the 2nd class particle and competition interface in stationarity.

We now discuss the background behind Theorem 1.4.

### 1.1.1 Background

Tasep is a microscopic model for Burgers equation and the 2nd class particle represents a microscopic characteristic. Suppose the initial condition of tasep is such that there is a macroscopic particle density $u_0(x)$:

$$
\int_0^x u_0(y) \, dy = \lim_{\varepsilon \to 0} \varepsilon \cdot \# \{\text{particles inside } [0, \varepsilon^{-1}x] \text{ at time } 0\}.
$$

Then for further times there is an almost sure macroscopic density $u(x,t)$ [43]:

$$
\int_0^x u(y, t) \, dy = \lim_{\varepsilon \to 0} \varepsilon \cdot \# \{\text{particles inside } [0, \varepsilon^{-1}x] \text{ at time } \varepsilon^{-1}t\}.
$$

The density $u(x,t)$ satisfies Burgers equation (a scalar conservation law) in the form

$$
\partial_t u + \partial_x (u(1-u)) = 0; \quad u(x,0) = u_0(x).
$$

There are 3 classical initial conditions for which the interplay between tasep and Burgers equation has been studied. These are the rarefaction, shock and flat initial conditions. They are characterized by two parameters $\rho_-, \rho_+ \in [0,1]$ for which

$$
u_0(x) = \rho_- 1_{x<0} + \rho_+ 1_{x>0}.
$$

The rarefaction case is when $\rho_- > \rho_+$, shock being $\rho_- < \rho_+$, and flat being $\rho_- = \rho_+$.

In all three cases the Burgers equation is solved by finding characteristics, which are curves $x(t)$ along which $u(x(t),t)$ is constant. Characteristics satisfy the equation $x'(t) = 1 - 2u(x(t),t)$, starting from some $x(0) = x_0$. The characteristics are straight lines.

In the rarefaction case there is a fan of characteristics emanating from the origin, and the solution $u(x,t)$ equals $u(x/t,1)$ where

$$
u(x,1) = \begin{cases} 
\rho_- & \text{if } x \leq 1 - 2\rho_- \\
\rho_+ & \text{if } x \geq 1 - 2\rho_+ \\
\text{linear in between} & \text{if } x \in [1 - 2\rho_-, 1 - 2\rho_+].
\end{cases}
$$
A classical result is that in the rarefaction setting, a 2nd class particle in tasep started from the origin follows one of the characteristics inside the fan at random. If $X(t)$ is the position of the 2nd class particle at time $t$, then $X(t)/t$ converges in probability to a random variable uniformly distributed over the interval $[1 - 2\rho_-, 1 - 2\rho_+]$ [17]. This convergence is later shown to hold almost surely [35].

In the shock case characteristics collide and Burgers equation is not well-posed in the usual sense. There is the entropic solution, which, based on physical grounds, is taken to mean the correct solution in this setting, see [16, 42]. The entropic solution is simply the travelling front $u(x, t) = u_0(x - (1 - \rho_- - \rho_+)t)$. The line $x = (1 - \rho_- - \rho_+)t$ carries the discontinuity of the initial condition. It is also the macroscopic trajectory of a 2nd class particle in that a 2nd class particle from the origin has an asymptotic speed, the limit of $X(t)/t$, which is $1 - \rho_- - \rho_+$ [31].

Fluctuations of the 2nd class particle around its deterministic speed has been investigated in the shock setting. If the shock density profile is modelled by a Bernoulli random initial condition, $X(t)$ has Gaussian fluctuations on the scale of $t^{1/2}$ [31]. When the initial condition is deterministic, say particles are placed periodically in large blocks to achieve the shock density profile, then $X(t)$ has Tracy-Widom fluctuations on the scale of $t^{1/3}$ [21, 41].

Finally, consider the flat case $\rho_- = \rho_+ = \rho$. The characteristics are the lines $x = x_0 + (1 - 2\rho)t$ and $u(x, t) \equiv \rho$. A 2nd class particle from the origin has asymptotic speed $1 - 2\rho$ [16, 42, 46]. It is therefore interesting to investigate the fluctuations of its position $X(t)$ around $(1 - 2\rho)t$, which had remained open. This is the content of Theorem 1.4. We set $\rho = 1/2$. The initial conditions $X^\varepsilon_0$ have a macroscopically flat density as $\varepsilon \to 0$, but can differ from the identically flat density on scales of $\varepsilon^{-1}$ along the $x$-axis and $\varepsilon^{1/2}$ along the $u$-axis. This makes for the scaling (1.5).

1.2 Geodesics of the directed landscape

A geodesic of the directed landscape is a continuous function $g$ from a closed interval $J$ so that for all $t_1 < t_2 < t_3$ in $J$ and $z_i = (g(t_i), t_i)$, the triangle inequality is sharp: $\mathcal{L}(z_1, z_3) = \mathcal{L}(z_1, z_2) + \mathcal{L}(z_2, z_3)$. Section 3 studies geodesics.

First, we consider geodesics from an initial condition $h_0$. Given $(x, t) \in \mathcal{H}$, a geodesic from $h_0$ to $(x, t)$ is a geodesic $g$ of the directed landscape defined on $[0, t]$ with $g(t) = x$ and so that

$$y \mapsto h_0(y) + \mathcal{L}(y, 0; x, t)$$

is maximized at $y = g(0)$. In §3.1, regularity properties of geodesics from an initial condition are established, and the following unique geodesic condition is proved, see
Theorem 3.1. For every initial condition $h_0$, almost surely for every $t > 0$, for all but countably many $x$, there exists a unique geodesic from $h_0$ to $(x, t)$. This is a statement about the forest of geodesics emanating from $h_0$ and allows us to study competition interfaces later.

Infinite geodesics of the directed landscape are constructed and studied in §3.6. These are geodesics defined on time intervals of the form $[t, \infty)$. Such geodesics are parametrized by their starting point and direction. Existence of an infinite geodesic from a given point in a given direction is shown in Theorem 3.2, along with an effective bound on its deviation. Next, the almost sure uniqueness of an infinite geodesic from a given point in a given direction is proven, and it is shown that uniqueness fails only for a countable number of directions, see Theorem 3.3. Finally, the almost sure coalescence of all infinite geodesics in a given direction is established in Theorem 3.4, which shows that the infinite geodesics in a given direction form a one-ended tree.

Busemann functions of the directed landscape are studied in §3.8. They capture the asymptotic geometry of infinite geodesics. We focus on understanding the Busemann function in a fixed direction, whose law is Brownian and evolves according to the KPZ fixed point as described by Corollary 3.3 and Theorem 3.5. This is then used to identify the law of an infinite geodesic in terms of geodesics from a Brownian initial condition seen backwards in time, see Corollary 3.4.

The results on infinite geodesics and Busemann functions are inspired by, and complement, what has been proven about infinite geodesics in last passage percolation. Our results are nevertheless self-contained, a key difference being that in the directed landscape one has to deal with both small and large time scales. Busani, Seppäläinen and Sorensen [5] use these results to study Busemann functions and geodesics of the directed landscape. They characterize the Busemann process simultaneously across all directions and establish geometric properties of infinite geodesics along all directions and starting points, for instance, characterizing the countable dense set of directions with non-unique geodesics. Ganguly and Zhang [24] recently and independently study infinite geodesics and Busemann functions of the directed landscape as well.

1.2.1 Background

The study of geodesics in first and last passage percolation models has a rich history, beginning with the work of Newman and co-authors [36]. Here we survey works in the last passage percolation literature. Ferrari and Pimentel [18] were among the first to study infinite geodesics in exponential last passage percolation, proving several key results, which were extended by Coupier [11]. Seppäläinen [47] provides a recent exposition about geodesics in last passage percolation, including a discussion of history and proofs.
of the main results for exponential last passage percolation. Busemann functions in last
passage percolation are intimately connected to its geometry. Seppäläinen and co-authors
have a large body of work in this direction, establishing fine geometric properties, see
[15, 25, 28, 48]. Cator and Pimentel [6, 7] have also studied similar questions.

1.3 Competition interfaces of the directed landscape

Section 4 discusses competition interfaces of the directed landscape. They are defined
in §4.2, and basic properties such as continuity and distributional invariances are estab-
lished in §4.3 and §4.4. The law of the competition interface at a given time is expressed
geometrically in §4.7, in terms of a concave majorant associated to the Airy process.

The asymptotic direction of competition interfaces is studied in §4.8. Theorem 4.1
proves that when \( h_0 \) is asymptotically flat, its interfaces are indeed asymptotically vertical.
This for instance means that two-sided Brownian motion and \( h_0(x) = |x|^\alpha \) (0 ≤ \( \alpha < 1 \))
has vertical interfaces. However, interfaces may also have random directions depending
on the initial condition. Theorem 4.2 identifies the direction of interfaces associated to a
large class of initial conditions, roughly, initial conditions that are dominated from above
by a downward wedge. The theorem is then used to compute the direction of interfaces
for several families of examples.

The absolute continuity of competition interfaces with respect to geodesics is proven in
§3.9.

Section 5 looks at the family of all competition interfaces associated to an initial condi-
tion \( h_0 \). We call this the interface portrait of \( h_0 \). It is complementary to the geodesic forest
from \( h_0 \) and is itself a forest lying in the upper half plane, as proven in Theorem 5.1. The
portrait has curious topological and geometric properties. For instance, it is quite “thin” –
its horizontal sections are always a discrete set as shown in Proposition 5.2. There should
be an interesting duality between interface portraits and geodesic forests that remains to
be understood.

1.3.1 Background

Ferrari, Martin and Pimentel [18, 19] introduced competition interfaces in last passage
percolation. Competition interfaces are closely related to the trajectory of a 2nd class
particle in tasep, and identify directions with non-unique geodesics in last passage per-
colation. The asymptotic direction of competition interfaces in last passage percolation
has been studied by various authors in [7, 18, 20, 26]. Balázs, Cator, and Seppäläinen [1]
prove an interesting duality relating competition interfaces to geodesics. Ferrai and Nejjar
[22] find single-point limiting fluctuations for the competition interface in the presence of
shocks.
2 Impressions from a landscape

We gather here some preliminaries and basic notions used throughout the article.

2.1 The directed landscape

The objects in our discussion are governed by the underlying randomness of the directed landscape \( L(y, s; x, t) \), defined for spatial coordinates \( x, y \in \mathbb{R} \) and time coordinates \( s, t \in \mathbb{R} \) with \( s < t \). It was introduced by Dauvergne, Ortmann and Virág [13].

The function \( L \) acts like a directed metric on \( \mathbb{R}^2 \): it satisfies \( L(p; p) = 0 \) and the reverse triangle inequality \( L(p; q) \geq L(p; r) + L(r; q) \). In fact, it has the metric composition property

\[
L(y, s; x, t) = \max_{z \in \mathbb{R}} \{L(y, s; z, u) + L(z, u; x, t)\}
\]

for every \( x, y \) and \( s < u < t \). For distinct points \( (y, s) \) and \( (x, t) \), \( L(y, s; x, t) = -\infty \) if and only if \( s \geq t \). Otherwise \( L(y, s; x, t) + (x - y)^2/(t - s) \) has GUE Tracy-Widom law scaled by \( (t - s)^{1/3} \).

\( L \) is a continuous function, stationary separately in the space and time coordinates, and has independent increments in time in the metric composition semigroup. We use the notation

\[
A(y, s; x, t) = L(y, s; x, t) + (x - y)^2 \frac{t - s}{t - s}
\]

for the “centered” version of \( L \). Several properties of \( L \) and \( A \) will be used throughout. A bound that will be used a lot, see [13, Corollary 10.7], is that

\[
|A(y, 0; x, t)| \leq Ct^{1/3} \log^{4/3} \left( \frac{2(||u|| + 2)}{t} \right) \log^{2/3}(||u|| + 2)
\]

for a random constant \( C \), where \( u = (y, 0; x, t) \) and ||u|| is the Euclidean norm. In particular, this bound implies that

\[
|A(y, 0; x, t)| \leq C(t^{1/2} + |x|^{1/2} + |y|^{1/2} + 1)
\]

for a random constant \( C \).

Another property of \( L \) is shear invariance [13, Lemma 10.2], which states that for any \( d \in \mathbb{R} \),

\[
L(x + ds, s; y + dt, t) \overset{law}{=} L(x, s; y, t) + \frac{(x - y)^2 - (x - y - d(t - s))^2}{t - s}
\]

for a random constant \( C \).

Other properties of \( L \) will be discussed as the time comes. The properties mentioned...
here are proved in [13], see also [10].

2.2 The KPZ fixed point

Denote the open upper half plane by \( \mathbb{H} = \{(x, t) : x \in \mathbb{R}, t > 0\} \) and by \( \overline{\mathbb{H}} \) its closure. An initial condition is a function

\[ h_0 : \mathbb{R} \to \mathbb{R} \cup \{-\infty\} \]

which is upper semicontinuous, finite on at least one point, and satisfies \( h_0(x) \leq c(1 + |x|) \) for some constant \( c \). Define the height function with initial condition \( h_0 \) as

\[ h(x, t) = \sup_{y \in \mathbb{R}} \{ h_0(y) + \mathcal{L}(y, 0; x, t) \}, \quad (x, t) \in \mathbb{H}. \quad (2.5) \]

The law of \( h \) is called the KPZ fixed point, first constructed by Matetski, Quastel and Remenik [33] as a Markov process on function space. The height function is continuous in \( x \) and \( t \), and for every \( t \), \( h(\cdot, t) \) also satisfies the definition of an initial condition [33].

The KPZ fixed point is also a stochastic integrable system in that its finite dimensional laws can be expressed in terms of Fredholm determinants; see [29, 30, 32, 33] for such results on the spacetime law of \( h \), and [4, 8, 40] for surveys on the KPZ universality class.

The directed landscape gives a simultaneous coupling of the KPZ fixed point for all initial conditions through the variational formula (2.5). This allows to compare different height functions through a common underlying randomness. We will thus call \( h(x, t) \) the directed landscape height function when it is realized as (2.5).

2.3 Geodesics and their properties

The function \( \mathcal{L}(y, s; x, t) \) can be thought of as giving a random geodesic length from \((y, s)\) to \((x, t)\), and in this sense the equation (2.5) measures the length of a geodesic weighted by an initial condition. These concepts can be formalized by discussing geodesics of \( \mathcal{L} \).

A path \( p \) from \((y, s)\) to \((x, t)\) for \( s < t \) is a continuous function \( p : [s, t] \to \mathbb{R} \) with \( p(s) = y \) and \( p(t) = x \). Paths are considered to be moving upwards, so \( p \) begins at \((y, s)\) and ends at \((x, t)\). A path may be identified with its graph

\[ \{(p(u), u), \ u \in [s, t]\} \]

lying in the spacetime plane \( \mathbb{R}^2 \). The Hausdorff metric on closed subsets of the plane then induces a topology on the space of paths.

Here are some basic notions about paths.
Ordering  A path \( p \) from \((y, s) \to (x, t)\) is to the left of a path \( p' \) from \((y', s) \to (x', t)\) if \( p(u) \leq p'(u) \) for every \( u \in [s, t] \).

Crossing  Paths \( p \) and \( p' \) cross if there are times \( s \neq t \) such that \( p(s) < p'(s) \) and \( p(t) > p'(t) \).

Coalescing  Paths \( p \) and \( p' \) coalesce downward from time \( t \) if \( p(s) = p'(s) \) for every \( s \leq t \), and coalesce upward from time \( t \) if \( p(s) = p'(s) \) for every \( s \geq t \).

The length of a path \( p \) with respect to \( L \) is
\[
\ell(p) = \inf_{\text{all partitions}} \sum_{i=1}^{n} L(p(t_{i-1}), t_{i-1}; p(t_i), t_i)
\] (2.6)
where \( s = t_0 < t_1 < \cdots < t_n = t \) is a partition of \([s, t]\).

A geodesic of \( L \) from \((y, s) \) to \((x, t)\) is a path of maximal length. Recall \( L \) satisfies the reverse triangle inequality. The triangle inequality becomes an equality along points of a geodesic, and geodesics are the paths for which that holds: for a geodesic \( g \) from \((y, s) \to (x, t)\),
\[
\ell(g) = \sum_{i=1}^{n} L(g(t_{i-1}), t_{i-1}; g(t_i), t_i)
\]
for any partition \( \{t_i\} \) of \([s, t]\). The geodesic length from \((y, s) \) to \((x, t)\) is \( L(y, s; x, t) \).

Geodesics of \( L \) have properties that hold almost surely in the underlying randomness, some of which are listed below as they will be used throughout our discussion. These are derived from [3, 13, 14].

Existence  There is a geodesic between every two points \((y, s) \) and \((x, t)\) for \( s < t \). There are also leftmost and rightmost geodesics from \((y, s) \) to \((x, t)\). See [13, Lemma 13.2].

Closure  If a sequence of geodesics converges in terms of the Hausdorff metric on their graphs, then the limit is also a geodesic. The length functional is continuous over geodesics. See [14, Lemma 3.1].

Compactness  The collection of geodesics whose graphs have endpoints lying in a given compact subset of \( \{(y, s; x, t) \in \mathbb{R}^4 : s < t\} \) is itself compact in the Hausdorff topology. See [14, Lemma 3.1].

Nearby geodesics meet  For every compact subset \( K \subset \mathbb{R}^2 \), there is an \( \epsilon > 0 \) such that if \( g \) and \( g' \) are two geodesics whose graphs lie in \( K \) and are within distance \( \epsilon \), then there is a common time \( t \) such that \( g(t) = g'(t) \). See [3, Theorem 1.18].
3 Geodesics

3.1 Geodesics from an initial condition

Fix an initial condition $h_0$ as in §2.2. A path from $h_0$ to $(x, t) \in \mathbb{H}$ is a path $p : [0, t] \to \mathbb{R}$ with $p(t) = x$. The starting point of $p$ is $p(0)$ and we may say that $p$ goes from the axis to $(x, t)$. The length of the path relative to $h_0$ is

$$\ell(p; h_0) = h_0(p(0)) + \ell(p),$$

where $\ell(p)$ is defined by (2.6).

A geodesic from $h_0$ to $(x, t)$ is a path from $h_0$ to $(x, t)$ of maximal relative length, that is, $p \mapsto \ell(p; h_0)$ is maximal among all paths $p$ from $h_0$ with endpoint $(x, t)$. The geodesic length $\mathcal{L}(h_0, x, t)$ from $h_0$ to $(x, t)$ is the value of the height function $h(x, t)$ from (2.5), which then equals $\mathcal{L}(0, g(0); x, t) + h_0(g(0))$ for any geodesic $g$ from $h_0$ to $(x, t)$.

Geodesics from $h_0$ are a sub-collection of geodesics of $\mathcal{L}$. A geodesic $h_0$ to $(x, t)$ is obtained by maximizing over all geodesics of $\mathcal{L}$ from $(y, 0)$ to $(x, t)$, for $y \in \mathbb{R}$, the relative length above. This leads to the variational formula (2.5).

3.2 Basic properties of geodesics from an initial condition

Some properties of geodesics from $h_0$ come from properties of $\mathcal{L}$-geodesics, holding whenever the properties of $\mathcal{L}$-geodesics from §2.3 do. The assertions in the following lemmas hold almost surely in the underlying randomness,

**Lemma 3.1** (Existence). Given an initial condition $h_0$, there is a geodesic from $h_0$ to every $(x, t) \in \mathbb{H}$. There are also leftmost and rightmost geodesics from $h_0$ to every point.

**Proof.** Consider the function $y \mapsto \mathcal{L}(y, 0; x, t) + h_0(y)$. It decays rapidly to $-\infty$ because the $\mathcal{L}$-term decays to $-\infty$ parabolically while $h_0$ is bounded above by a linear term. More concretely, fix an $\alpha \in \mathbb{R}$ for which $h_0(\alpha)$ is finite. From the estimate (2.3) for $\mathcal{L}$ and the bound $h_0(y) \leq B(1 + |y|)$ it follows that there is a compact interval $I$, depending only on $\alpha, B$, the random constant $C$ and the point $(x, t)$, such that the supremum of $\mathcal{L}(y, 0; x, t) + h_0(y)$ over $y$ cannot be attained outside $I$. Inside of $I$ the supremum is achieved because $h_0$ is upper semicontinuous and bounded above (and $\mathcal{L}$ is continuous). Thus, the maximizers of the function lie in the compact interval $I$, and being upper semicontinuous and bounded above, the set of maximizers form a non-empty compact subset $K \subset I$.

A geodesic from $h_0$ to $(x, t)$ is obtained by picking any point $y \in K$ and choosing an $\mathcal{L}$-geodesic from $(y, 0)$ to $(x, t)$. The rightmost geodesic is going to be the rightmost $\mathcal{L}$-geodesic from the rightmost point in $K$ to $(x, t)$; likewise for the leftmost geodesic. ■
Lemma 3.2 (Closure, Continuity, Compactness). For an initial condition \( h_0 \), the following properties hold.

1. The limit of a sequence of geodesics from \( h_0 \) in the Hausdorff metric is also a geodesic from \( h_0 \).

2. The relative length function \( \ell(\cdot; h_0) \) is continuous over geodesics from \( h_0 \).

3. Geodesics from \( h_0 \) that lie in a compact set \( K \subset \mathbb{H} \) are compact in the Hausdorff topology. Consequently, if \( (x_n, t_n) \to (x, t) \) and \( g_n \) is a geodesic from \( h_0 \) with endpoint \( (x_n, t_n) \), then a subsequence of these \( g_n \) converges to a geodesic from \( h_0 \) to \( (x, t) \).

Proof. The assertions are proven one by one.

1. Let \( g_n \) be a sequence of geodesics from \( h_0 \) that are convergent. Suppose \( g_n \) starts at \( (y_n, 0) \) and ends at \( (x_n, t_n) \). The \( g_n \)'s are geodesics of \( \mathcal{L} \), so their limit \( g \) is too, say from \( (y, 0) \) to \( (x, t) \) where \( (x_n, t_n) \to (x, t) \) and \( y_n \to y \). Upper semicontinuity of \( h_0 \) means \( \limsup_n h_0(y_n) \leq h_0(y) \), and since the lengths \( \ell(g_n) \to \ell(g) \), it follows that \( \limsup_n \ell(g_n; h_0) \leq \ell(g; h_0) \). Now \( \ell(g_n; h_0) = h(x_n, t_n) \), which tends to \( h(x, t) \) by continuity; so \( \ell(g; h_0) \geq h(x, t) \). The reverse inequality \( \ell(g; h_0) \leq h(x, t) \) holds because \( g \) is a path from \( h_0 \) to \( (x, t) \). It follows that \( g \) is a geodesic from \( h_0 \).

2. The relative length \( \ell(g; h_0) \) equals \( h(x, t) \) for a geodesic \( g \) from \( h_0 \) to \( (x, t) \). Its continuity is implied by the continuity of \( h \).

3. Compactness follows immediately from the compactness of \( \mathcal{L} \)-geodesics and the closure property in (1). The assertion in the consequence is implied by the compactness property if there is a compact set that contains every \( g_n \). In this regard note that there are \( x_-, x_+ \) and \( T \) such that \( x_- \leq x_n \leq x_+ \) and \( t_n \leq T \) for every \( n \). The graphs of \( g_n \) are constrained by the leftmost and rightmost geodesics to \( [x_-, T] \) and \( [x_+, T] \), respectively.

\[ \square \]

3.3 Unique geodesics and the unique geodesic condition

We want to establish uniqueness of geodesics to an initial condition in a strong enough sense over all points in \( \mathbb{H} \). Such a notion is needed for the discussion in the next section on competition interfaces. Such considerations motivate the following definition.

Definition 3.1. Given a sample from \( \mathcal{L} \) and an initial condition \( h_0 \), a point \( (x, t) \in \mathbb{H} \) is a unique geodesic point, UGP for short, if there is a unique geodesic from \( h_0 \) to \( (x, t) \).

The initial condition \( h_0 \) satisfies the unique geodesic condition if the following holds almost surely in the underlying randomness of \( \mathcal{L} \):
For every $t > 0$, the set of points $x \in \mathbb{R}$ such that $(x, t)$ is not a UGP is countable.

The unique geodesic condition is abbreviated as UGC.

We are going to prove that every initial condition satisfies the UGC. A first step is uniqueness of the geodesic starting point.

**Lemma 3.3.** Let $(x, t) \in \mathbb{H}$ be given. Every geodesic from $h_0$ to $(x, t)$ starts at a unique common point almost surely.

Proof. We have to show the process $y \mapsto \mathcal{L}(y, 0; x, t) + h_0(y)$ has a unique maximizer almost surely. Lemma 3.1 says there is a maximizer, so we must show there cannot be more than one. If there were, then one could find two disjoint intervals with rational endpoints that capture different maximizers. So it is enough to show that for any two disjoint compact intervals $[a, b]$ and $[c, d]$, with $b < c$, the maximum of the process over the intervals are different almost surely. We can assume $h_0$ takes some finite values over both $[a, b]$ and $[c, d]$, for otherwise there would be no maximizers there.

The process $y \mapsto \mathcal{L}(y, 0; x, t) − \mathcal{L}(a, 0; x, t)$ is a re-centred and re-scaled Airy process. It is absolutely continuous with respect to a Brownian motion $B(y)$ on the interval $[a, d]$, started from $B(a) = 0$ and with diffusivity constant $\sqrt{2}$ [9, 45]. So it suffices to prove $B + h_0$ has different maximums over $[a, b]$ and $[c, d]$ almost surely. Observe that

\[
\max_{y \in [a, b]} \{B(y) + h_0(y)\} − \max_{y \in [c, d]} \{B(y) + h_0(y)\} = \\
\max_{y \in [a, b]} \{B(y) − B(b) + h_0(y)\} − \max_{y \in [c, d]} \{B(y) − B(c) + h_0(y)\} + B(b) − B(c).
\]

The three terms on the right hand side above are independent. The first two terms are finite random variables and the third has a Normal distribution. So the right hand side has an absolutely continuous law and avoids the value 0 almost surely. $\blacksquare$

The second step is uniqueness of the geodesic to a given point.

**Lemma 3.4.** Let $(x, t) \in \mathbb{H}$ be given. There is an unique geodesic from $h_0$ to $(x, t)$ almost surely. Moreover, the set of non UGPs from $h_0$ has empty interior almost surely.

Proof. Given a point $(x, t)$, let $g$ be a geodesic from $h_0$ to $(x, t)$. If the values of $g$ are almost surely unique for all rational times $s \in [0, t]$, then $g$ is unique almost surely by continuity. The claim about the rational times follows from a union bound if for any given $s \in [0, t]$, the value of $g(s)$ is unique almost surely.

Let $h_s(y) = \sup_{z \in \mathbb{R}} \{h_0(z) + \mathcal{L}(z, 0; y, s)\}$ be the height function grown to time $s$ from $h_0$. Set $\hat{\mathcal{L}}(y, u; y', u') = \mathcal{L}(y, s + u; y', s + u')$ for $0 \leq u < u'$ and $y, y' \in \mathbb{R}$. So $\hat{\mathcal{L}}$ is $\mathcal{L}$ viewed
from time $s$ onwards. The process $\hat{\mathcal{L}}$ has the same law as $\mathcal{L}$ and is independent of $\mathbf{h}_s$ [13]. The value $g(s)$ is the starting point of a geodesic from $\mathbf{h}_s$ to $(x, t - s)$ according to $\hat{\mathcal{L}}$. It is then unique almost surely by Lemma 3.3.

Now that we know any given point has a unique geodesic from $\mathbf{h}_0$ to it almost surely, it follows that almost surely every rational point has a unique geodesic from $\mathbf{h}_0$ to it. The set of non UGPs from $\mathbf{h}_0$ has an empty interior when it contains no rational point. ■

The third observation is that geodesics from $\mathbf{h}_0$ have a natural ordering.

**Lemma 3.5 (Geodesic ordering).** For an initial condition $\mathbf{h}_0$, the following holds almost surely in the underlying randomness. Suppose $g_1$ and $g_2$ are geodesics from $\mathbf{h}_0$ to $(x, t)$ and $(y, t)$, respectively. If $x < y$ then $g_1(s) \leq g_2(s)$ for every $s \in [0, t]$, that is, $g_1$ stays to the left of $g_2$. If $g_1$ and $g_2$ meet then they coalesce downward from that point. As a result, no two geodesics from $\mathbf{h}_0$ to any points of $\mathcal{H}$ can cross.

**Proof.** Condition on the almost sure event that the non UGPs from $\mathbf{h}_0$ have an empty interior.

If $g_1$ and $g_2$ do not meet, then continuity and $x < y$ implies $g_1(s) < g_2(s)$ for every $s$. Suppose they do meet for the last time at $s \in (0, t)$ but do not coalesce downward. Then $p = (g_1(s), s) = (g_2(s), s)$ is not a UGP. There is an open region of points between the graphs of $g_1$ and $g_2$ from time $s$ to $t$. No point in this region can be a UGP either. Indeed, any point $q$ in this region has two geodesics from $\mathbf{h}_0$: one by following $g_1$ to $p$ and then any geodesic from $p$ to $q$, and another by following $g_2$ to $p$ instead. But this contradicts the set of non UGPs having an empty interior. ■

The three previous lemmas culminate into a proof of the unique geodesic condition.

**Theorem 3.1 (Unique geodesic condition).** For every initial condition $\mathbf{h}_0$, almost surely, the unique geodesic condition holds.

**Proof.** Condition again on the event that the non UGPs from the initial condition $\mathbf{h}_0$ have an empty interior, which implies the geodesic ordering property from Lemma 3.5. Condition also on the almost sure event about existence of geodesics from $\mathbf{h}_0$ asserted by Lemma 3.1.

For a point $p = (x, t)$, let $A_p$ be the interior of the region between the leftmost and rightmost geodesics from $\mathbf{h}_0$ to $(x, t)$. The set $A_p$ is empty if and only if $p$ is a UGP. For distinct points $p = (x, t)$ and $q = (y, t)$ with the same time coordinate, $A_p$ and $A_q$ are disjoint by the geodesic ordering property. The sets $A_p$ as $p$ varies over all non UGPs with time coordinate $t$ are then disjoint, non-empty open sets. There can only be countably
many of them (each contains a different rational point). As $t$ is arbitrary, the UGC is established.

3.4 Good samples assumption

All properties of geodesics and height functions discussed so far hold almost surely in the underlying randomness. Many of them are interrelated, and it is convenient to have a single event over which they are all satisfied. This will be useful in the coming sections as well. Such an event – let us call it the good samples – is described below.

The good samples, relative to an initial condition $h_0$, are samples of $L$ for which all the following properties hold, which they do almost surely.

- The process $A$ is continuous and satisfies the bound from (2.2).
- The properties of geodesics of $L$ described in §2.3 hold.
- The unique geodesic condition for $h_0$ is satisfied.

3.5 Where do geodesics emanate from?

For $(x, t) \in H$, consider the point from where the rightmost geodesic from $h_0$ to $(x, t)$ emanates:

$$e(x, t) = \text{starting point of the rightmost geodesic from } (x, t) \text{ to } h_0$$

(3.2)

It encapsulates many properties of geodesics and will also help to understand interfaces in the next section. Assume good samples of $L$.

**Proposition 3.1.** For every $t > 0$, the mapping $x \mapsto e(x, t)$ is a non-decreasing, right continuous step function with a discrete set of discontinuities. The left limit of $e(\cdot, t)$ at $x$ is the starting point of the leftmost geodesic to $(x, t)$. The image of $e$ is also countable.

**Proof.** Let $e_t(x) = e(x, t)$. Ordering of geodesics from $h_0$ implies $e_t$ is non-decreasing. Right continuity comes from the ordering and compactness properties of geodesics. The left limit is derived by the same reasoning.

In order to show $e_t$ is a step function with a discrete set of discontinuities it is enough to show that for every $x$, there are $w$ and $y$ with $w < x < y$, such that $e_t$ is constant on the interval $[x, y]$ (with common value $e_t(x)$), and also on the interval $[w, x]$ (the common value being the left limit of $e_t(x)$). Now when $y > x$ is sufficiently close to $x$, the rightmost geodesic from $(y, t)$ has to meet the rightmost geodesic from $(x, t)$ by geodesic compactness and the property that nearby geodesics meet. The ordering property implies
that these geodesics coalesce downward, so \( e_t(x) = e_t(y) \). Similarly, for \( w < x \) sufficiently close, the rightmost geodesic from \((w, t)\) has to meet and coalesce downward with the leftmost geodesic from \((x, t)\).

The image of \( e_t \) is countable because \( e_t \) is a step function with a discrete set of discontinuities. Now observe that the image of \( e_t \) is contained in the image of \( e_s \) if \( t > s \), since a rightmost geodesic from time \( t \) contains a rightmost geodesic from time \( s \). So the image of \( e \) is a countable union, \( \text{Im}(e) = \bigcup_{n \geq 1} \text{Im}(e_{1/n}) \), of countable sets. ■

### 3.6 Infinite geodesics

We turn to the construction and properties of infinite geodesics of \( L \). A highlight of this section is Theorem 3.4, which describes the tree of infinite geodesics in a common direction.

**Definition 3.2.** A continuous function \( g : [s, \infty) \to \mathbb{R} \) is an **infinite geodesic** of \( L \) if it is a geodesic of \( L \) when restricted to every interval \([s, T]\). Its starting point is \((g(s), s)\).

The **direction** of an infinite geodesic \( g \) is the limit of \( g(t)/t \) as \( t \) tends to infinity, if it exists.

Infinite geodesics may be identified with their graphs lying in the spacetime plane. When convenient we will call an infinite geodesic in direction \( d \) simply as a geodesic with direction \( d \).

**Theorem 3.2 (Existence of infinite geodesics).** Given a starting point \( p = (x, s) \in \mathbb{R}^2 \) and direction \( d \in \mathbb{R} \), there is almost surely an infinite geodesic \( g \) of \( L \) from \( p \) with direction \( d \). More precisely, there is \( a > 1 \) and a random constant \( C \) with \( E_a C^3 < \infty \) so that

\[
|g(s + t) - x - dt| \leq Ct^{2/3} \left( \log \log t \right)^{1/3} \quad \text{for} \quad t \geq 10. \tag{3.3}
\]

**Proof.** Due to shear and translation invariance of \( L \), see §2.1, it is enough to consider the starting point \( p = (0, 0) \) and direction \( d = 0 \).

For each \( n \), let \( g_n(t) \) be a geodesic of \( L \) from \((0, 0)\) to \((0, n)\). These geodesics are almost surely unique for every \( n \) and the law of \( g_n(t) \) is \( n^{2/3} g_{1/(t/n)} \) [13]. The following bound on geodesics is derived from Lemma 3.6 below by scale invariance. There are constants \( c \) and \( d \) such that for every \( t > 0 \) and \( n \geq t \),

\[
\Pr \left( \sup_{s \in [0,t]} |g_n(s)| > \lambda \right) \leq ce^{-d\lambda^3}. \tag{3.4}
\]

Indeed, \( \sup_{s \in [0,t]} |g_n(s)| \) has the same law as \( t^{2/3} \sup_{s \in [0,1]} |g_n(t/s)| \), and then the conclusion of Lemma 3.6 can be applied.
Now we argue as in the proof of the law of the iterated logarithm for random walks. A union bound with (3.4) over $t$ that are powers of 2 implies there are absolute constants $c'$ and $d'$ for which

$$
\Pr\left( \sup_{t \in [0,n]} \frac{|g_n(t)|}{t^{2/3}(1 \vee \log^{1/3} \log t)} > \lambda \right) \leq c' e^{-d' \lambda^3}
$$

(3.5)

for every $n$. Let $E(n, \lambda)$ be the event in (3.5).

Fatou’s lemma applied to the indicator of the events $E(n, \lambda)$ over $n$ implies

$$
\Pr(\text{E}(n, \lambda) \text{ occurs eventually for all } n) \leq c' e^{-d' \lambda^3}.
$$

Thus, by the Borel-Cantelli lemma applied over integer values of $\lambda$, there is a random $\Lambda$ with tail bounds as in (3.5) such that $E(n, \Lambda)^c$ occurs infinitely often along some random subsequence. By passing to such a subsequence we may assume that for every $n$

$$
\sup_{t \in [0,n]} \frac{|g_n(t)|}{t^{2/3}(1 \vee \log^{1/3} \log t)} \leq \Lambda.
$$

(3.6)

The next step is a sample-wise compactness and diagonalization argument. Assume good samples of $\mathcal{L}$ with respect to the narrow wedge initial condition at zero.

Along the subsequence where (3.6) holds, the graphs of the geodesics $g_n(t)$ for $t \in [0,1]$ have endpoints that remain inside a compact subset of $\overline{\mathbb{H}}$. By the compactness property of geodesics, there is a further subsequence on which $g_n$ restricted to $[0,1]$ converges to a geodesic $\gamma_1$ for times $t \in [0,1]$. After we pass to this subsequence, the same argument implies there is a further subsequence on which $g_n$ restricted to $[0,2]$ converges to a geodesic $\gamma_2$. Continuing like this for integer times and then moving to the diagonal subsequence gives the desired infinite geodesic $g$, and a subsequence along which $g_n$ tends to $g$ on every compact interval of times.

Since $g$ is the limit of geodesics satisfying (3.6), it satisfies the bound (3.3)

Lemma 3.6. There are constants $c, d > 0$ such that for every $T \geq 1$, the geodesic $g_T$ from $(0,0)$ to $(0,T)$ satisfies

$$
\Pr\left( \sup_{t \in [0,1]} |g_T(t)| > \lambda \right) \leq c e^{-d \lambda^3}.
$$

Proof. The proof of the lemma is based on the following two bounds. The first bound is derived from [13, Proposition 12.3] and the second is from [14, Corollary 2.14].
For every $b > 0$, there exists a $a > 1$ and a random variable $C$ with $E[a^{C^3}] < \infty$ such that for any geodesic $g$ from a point in $[-b, b] \times \{0\}$ to a point in $[-b, b] \times \{1\}$,

$$|g(s) - g(t)| < C|s - t|^{2/3} \log^{1/3}(2/|s - t|).$$  \hfill (3.7)

There are also constants $c_1$ and $d_1$ such that for every $t \in (0, 1]$,

$$\Pr\left(\frac{|g_1(t)|}{t^{2/3}} > \lambda\right) \leq c_1 e^{-d_1 \lambda^3}. \hfill (3.8)$$

Since $g_T(t)$ has law $T^{2/3} g_1(t/T)$, it follows from (3.8) that for every $T$ and $t \in (0, T]$,

$$\Pr\left(\frac{|g_T(t)|}{t^{2/3}} > \lambda\right) \leq c_1 e^{-d_1 \lambda^3}. \hfill (3.9)$$

Let $\gamma_x$ be a geodesic from $(0, 0)$ to $(x, 1)$. The bound

$$\Pr\left(\sup_{x \in [0,1], t \in [0,1]} |\gamma_x(t)| > \lambda\right) \leq c_2 e^{-d_2 \lambda^3}$$

follows from (3.7) with $s = 0$. Here $c_2$ and $d_2$ are absolute constants derived from the fact that $E[a^{C^3}] < \infty$.

Now consider geodesics $\gamma_x$ for $x \in [k, k+1]$. By shear invariance (2.4), $\gamma_x$ has law $\gamma_{x-k}(t) + kt$. Therefore, using the bound above,

$$\Pr\left(\sup_{x \in [k,k+1], t \in [0,1]} |\gamma_x(t) - tk| > \lambda\right) \leq c_2 e^{-d_2 \lambda^3}.$$  

This bound implies that if $\lambda \geq |2k|$ then

$$\Pr\left(\sup_{x \in [k,k+1], t \in [0,1]} |\gamma_x(t)| > \lambda\right) \leq c_2 e^{-d_2 (\lambda/2)^3}.$$  

By a union bound for $k = -m, -m + 1, \ldots, m$, it follows that for $|\lambda| > 2m$,

$$\Pr\left(\sup_{x \in [-m,m], t \in [0,1]} |\gamma_x(t)| > \lambda\right) \leq 2mc_2 e^{-d_2 (\lambda/8)^3} \leq c_3 e^{-d_3 \lambda^3} \hfill (3.10)$$

Now

$$\sup_{t \in [0,1]} |g_T(t)| > 2m$$
implies that either $|g_T(1)| > m$ or $g_T(1) \in [-m, m]$ and the segment of $g_T$ over $[0,1]$ violates the event in (3.10) with $\lambda = 2m$. By (3.9) and (3.10), the probabilities of either of these events are bounded above as claimed.

Since there is an infinite geodesic in any given direction, are there infinite geodesics simultaneously in every direction? Are they unique?

**Theorem 3.3 (Geodesics in every direction).** Given $p \in \mathbb{R}^2$, there are infinite geodesics of $L$ starting from $p$ in every direction almost surely. In fact, almost surely, there are leftmost and rightmost geodesics $g_d^-$ and $g_d^+$ from $p$ in direction $d$ for every $d$. Given a specific direction, there is a unique geodesic from $p$ in that direction almost surely. There are in fact at most countably many (random) directions with non-unique geodesics.

**Proof.** We can take $p = (0,0)$ as before. Assume good samples of $L$ with respect to the narrow wedge initial condition at zero, so that all geodesics start from $(0,0)$.

By Theorem 3.2 and a union bound, there are almost surely geodesics from $(0,0)$ along every rational direction. Geodesic ordering (Lemma 3.5) implies that if $d < d'$, any geodesic $g_d$ with direction $d$ stays to the left of any geodesic $g_{d'}$ with direction $d'$, that is, $g_d(t) \leq g_{d'}(t)$ for every $t$. Now geodesics can be built along every direction sample-wise.

Let $d$ be an arbitrary direction. Approximate $d$ by rational directions $d_{2n}$ that are decreasing to $d$ as well as rational directions $d_{2n+1}$ that are increasing to it. Let $g_n$ be a geodesic from $(0,0)$ along direction $d_n$. Geodesic ordering and the compactness property, used as in the proof of Theorem 3.2, imply that a subsequence $g_{n_k}$ converges, sample-wise, to a geodesic $g$ uniformly on compact time intervals. To see that $g$ has direction $d$ note that for every $n$, by geodesic ordering, $g_{2n+1}(t) \leq g(t) \leq g_{2n}(t)$. Dividing by $t$ and taking limits imply that

$$d_{2n+1} \leq \liminf_{t \to \infty} \frac{g(t)}{t} \leq \limsup_{t \to \infty} \frac{g(t)}{t} \leq d_{2n}.$$  

Taking the limit over $n$ shows the direction of $g$ is indeed $d$.

The leftmost and rightmost geodesics in direction $d$ are defined according to

$$g_d^-(t) = \inf_g g(t) \quad \text{and} \quad g_d^+(t) = \sup_g g(t),$$

where $g$ ranges over all geodesics from $(0,0)$ in direction $d$. Geodesic ordering and compactness ensures $g_d^\pm$ are in fact geodesics with direction $d$.

Uniqueness of a geodesic in a fixed direction comes almost for free from the above construction and invariances of $L$.  

22
Suppose there is more than one geodesic from \((0, 0)\) along some direction \(d\). Then there is a non-empty open region \(A_d\) of points between the leftmost and rightmost geodesics in direction \(d\). (Let \(A_d\) be empty if there is only one geodesic with direction \(d\)). If \(d \neq d'\) then the sets \(A_d\) and \(A_{d'}\) are disjoint due to geodesic ordering. So the sets \(A_d\), as the direction varies, are a collection of open, pair-wise disjoint sets. Only a countable number of them can be non-empty. Thus the set \(D\) of directions along which there are multiple infinite geodesics is at most countable.

Shear invariance of \(L\) (2.4) implies the set \(D\) is translation invariant. A countable, translation invariant set has zero probability of containing any given number. Indeed,

\[
\int_{-\infty}^{\infty} 1_{x \in D} \, dx = 0,
\]

and on taking expectation this gives \(\Pr(x \in D) = 0\) for almost every \(x\). But this probability does not depend on \(x\). So any fixed direction has a unique infinite geodesic from \((0, 0)\) almost surely.

Note the set \(D\) is also scale invariant by the KPZ scale invariance of \(L\). As such, although countable, it should be dense over the real line. This has indeed been proven recently in [5].

**Corollary 3.1.** Almost surely, the following holds. Let \(p = (x, s) \in \mathbb{R}^2\), let \(t_n \to \infty\) and let \(x_n/t_n \to d \in \mathbb{R}\). Let \(g_n\) be a geodesic from \(p\) to \((x_n, t_n)\) and let \(g\) be a geodesic from \(p\) in direction \(d\). Assume that \(g\) is unique. Then for every \(r > s\), \(g_n = g\) on \([s, r]\) for all large \(n\).

**Proof.** Let \(\varepsilon > 0\) and let \(g_-\) and \(g_+\) be geodesics from \(p\) in the directions \(d \pm \varepsilon\). Then for all large enough \(n\), \(g_-(t_n) < g_n(t_n) < g_+(t_n)\), and by geodesic ordering, \(g_-(t) \leq g_n(t) \leq g_+(t)\) for \(t \in [s, t_n]\). By the compactness property of geodesics, \(g_n\) is precompact in the Hausdorff topology restricted to compact time intervals, and every limit point \(\gamma\) satisfies

\[
d - \varepsilon \leq \liminf_{t \to \infty} \frac{\gamma(t)}{t} \leq \limsup_{t \to \infty} \frac{\gamma(t)}{t} \leq d + \varepsilon.
\]

Since this holds for all \(\varepsilon > 0\), \(\gamma\) is a geodesic in direction \(d\), and by uniqueness, \(\gamma = g\). So \(g_n \to g\) on compacts.

Observe that if \(g_n(t) = g(t)\) for some time \(t\) then \(g_n = g\) on \([s, t]\). Indeed, \(g\) is the unique geodesic from \(p = (g(s), s)\) to \((g(t), t)\), and if \(g_n\) differs from \(g\) on \([s, t]\) then \(g\) could not be unique. Finally, observe that \(g_n\) meets \(g\) at arbitrarily large times. Indeed, \(g_n\) tends to \(g\) on any compact time interval \([t_1, t_2]\) with \(t_1 \geq s\). So by the property that nearby geodesics meet, \(g_n\) must meet \(g\) at some time in \([t_1, t_2]\). As \(t_1\) may be arbitrarily
large, the claim follows. These two observations imply that for any compact interval \([s, r]\),
\(g_n = g\) on \([s, r]\) for all large \(n\).

### 3.7 The geodesic tree

The family of infinite geodesics in a common direction makes an one-ended tree – the
geodesic tree. This result will be proved in steps, the key observation being that any two
infinite geodesics in a common direction coalesce upward.

By Theorem 3.3, for every \(x \in \mathbb{R}\), there is almost surely a unique geodesic \(g_x\) with
direction 0 started from \((x, 0)\). So the event

\[ E = \{\text{there is a unique geodesic in direction 0 from every rational point } (x, 0)\} \tag{3.11} \]

has probability one. We will condition on the event \(E\) repeatedly in this section.

**Lemma 3.7.** The following ordering of geodesics holds on the event \(E\), and thus almost surely.
If \(g\) and \(g'\) are any two geodesics in direction 0 started from \((y, 0)\) and \((y', 0)\) respectively, then
\(g(t) \leq g'(t)\) for every \(t\) when \(y < y'\).

**Proof.** Condition on the event \(E\) in (3.11) and let \(g_x\) be the unique geodesic in direction 0 stated from the rational point \((x, 0)\). For \(y < y'\) let \(g\) and \(g'\) be geodesics from \((y, 0)\) and \((y', 0)\), respectively, in direction 0. Let \(x\) be a rational number between \(y\) and \(y'\). We
claim that \(g(t) \leq g_x(t) \leq g'(t)\) for every \(t\), which proves the lemma.

If \(g\) does not remain weakly to the left of \(g_x\) then they must cross, meeting for the
first time a some point \(p = (g(t), t)\). However, this means the geodesic from \((x, 0)\) is
not unique; a second geodesic is obtained by going from \((x, 0)\) to \(p\) along \(g_x\) and then
following \(g\) afterwards. For the same reason, \(g'\) stays weakly to the right of \(g_x\). ■

**Lemma 3.8.** The following coalescing of geodesics holds almost surely. If \(g\) and \(g'\) are any two
geodesics in direction 0 started from the real axis, then they coalesce upward, meaning there is a \(t\)
such that \(g(s) \neq g'(s)\) for \(s < t\) and \(g(s) = g'(s)\) for \(s \geq t\).

**Proof.** Condition on the almost sure event \(E\) in (3.11) and also on the properties of \(L\)-
geodesics discussed in §2.3.

Due to the ordering of geodesics property from Lemma 3.7, it is enough to show the
geodesics \(g_x\) and \(g_{x'}\) coalesce for every rational \(x\) and \(x'\). By a union bound it suffices to
show this for every specific pair of rationals \(x < x'\).

Consider the event

\[ C_{[x,y]} = \{g_x \text{ meets } g_y\} \]
If $C_{[x,y]}$ occurs then $g_x$ and $g_y$ must coalesce, or else these geodesics would not be unique. Thus we need to show that $\Pr(C_{[x,y]}) = 1$ for every rational $x < y$. However, due to translation and scale invariance of $L$, $C_{[x,y]}$ has the same probability for every interval $[x,y]$. So the lemma follows from having $\Pr(C_{[0,1]}) = 1$.

Observe that $C_{[x,y]}$ is non-increasing in that $C_{[x,y]} \subset C_{[x',y']}$ if $[x',y'] \subset [x,y]$. This is due to geodesic ordering. As a result,

$$\Pr(C_{[0,1]}) = \lim_{n \to \infty} \Pr(C_{[0,1/n]}) = \Pr(\bigcup_n C_{[0,1/n]}).$$

The event $C_{[0,1/n]}$ occurs for sufficiently large $n$, so the probability above is one. Indeed, consider $g_0$ and $g_{1/n}$ only up to time 1. By geodesic compactness and ordering, $g_{1/n}$ tends to $g_0$ for times $t \in [0,1]$ as $n$ goes to infinity. Then, due to the property that nearby geodesics meet, they must in fact meet by time 1 for all large $n$. ■

**Theorem 3.4 (The geodesic tree).** Consider all infinite geodesics of $L$ in direction zero started from points along the real axis. It has the following almost sure properties.

1. There is a geodesic in direction zero from every point $(x,0)$ for $x \in \mathbb{R}$. There are also leftmost and rightmost geodesics $g_x^-$ and $g_x^+$ in direction zero starting from every $(x,0)$.

2. If $x < y$ then $g_x^+$ stays weakly to the left of $g_y^-$. Thus, geodesics along direction zero are naturally ordered according to their start points and they never cross.

3. There are only countably many $x \in \mathbb{R}$ for which there is more than one geodesic from $(x,0)$ in direction zero (that is, $g_x^- \neq g_x^+$).

4. Every pair of geodesics started from the real axis along direction zero coalesce upward.

**Proof.** Condition on the almost sure event $E$ in (3.11) and on the properties of $L$-geodesics from §2.3. These ensure the conclusion of Lemmas 3.7 and 3.8.

Part (1) is proved in the same way as Theorem 3.3. The unique geodesics $g_x$ for rational $x$ can be used to build geodesics from every starting point, sample-wise, by the ordering and compactness property of geodesics. The leftmost and rightmost geodesics from $(x,0)$ are the pointwise infimum and supremum, respectively, of all geodesics from $(x,0)$ along direction zero.

Part (2) follows from Lemma 3.7. The proof of part (3) is the same as in Theorem 3.3. Part (4) is the conclusion of Lemma 3.8. ■

**Corollary 3.2.** The geodesic tree of $L$ consists of infinite geodesics in direction zero started from points in $\mathbb{R}^2$. It has the following almost sure properties.
1. For every \( p \in \mathbb{R}^2 \), there is a leftmost geodesic \( g_p^- \) and rightmost geodesic \( g_p^+ \) from \( p \) in direction zero.

2. For any given \( p \), the geodesic is unique.

3. Let \( g \) and \( g' \) be geodesics from \((x, t)\) and \((x', t')\) in direction zero with \( t' \leq t \). There is an \( s \geq t \) so that \( g - g' \) is non-zero on \((t, s)\) and zero on \([s, \infty)\). In particular, either \( g \leq g' \) or \( g' \leq g \) on \([t, \infty)\).

**Proof.** By Theorem 3.4 the geodesics \( g_p^\pm \) exist almost surely for every \( p = (x, t) \) with \( x \in \mathbb{R} \) and \( t \) rational. For a generic point \( p \) define

\[
g_p^-(t) = \lim_{\epsilon \to 0} \inf_{g : |g^0 - p| < \epsilon} g(t)
\]

where \( g^0 \) is the starting point of \( g \), and the infimum is over all geodesics \( g \) along direction 0 starting at rational times. The limit exists and is a geodesic by geodesic compactness, and has direction zero due to geodesic ordering.

Part (2) is proved in Theorem 3.3. Part (3) follows from Theorem 3.4 by considering the restrictions of \( g \) and \( g' \) to intervals \([t'', \infty)\) for rational \( t'' > t \). The final claim follows from continuity. ■

### 3.8 Busemann functions and the law of the infinite geodesic

The geodesic tree allows us to define and study the Busemann function of the directed landscape in a given direction. In turn, Busemann functions help to identify the law of the infinite geodesic, see Corollary 3.4. Other notable facts are that the Busemann function in a given direction and at a given time has a Brownian law (Corollary 3.3), and that backwards in time it evolves as a KPZ fixed point (Theorem 3.5).

First we define the Busemann function. For \( p \in \mathbb{R}^2 \) let \( g_p \) be the rightmost geodesic from \( p \) in direction zero, and let \( \kappa(p, q) \) be the point where the geodesics \( g_p \) and \( g_q \) coalesce upward. Define

\[
W(p; q) = \mathcal{L}(p; \kappa(p, q)) - \mathcal{L}(q; \kappa(p, q)).
\]  

(3.12)

Note that if \( \kappa' \) is any point on the geodesics after they coalesce then \( W(p, q) = \mathcal{L}(p, \kappa') - \mathcal{L}(q, \kappa') \) because, by the triangle equality along geodesics, \( \mathcal{L}(p, \kappa') = \mathcal{L}(p, \kappa(p, q)) + \mathcal{L}(\kappa(p, q), \kappa') \) and likewise for \( \mathcal{L}(q, \kappa') \). It follows that \( W(p; q) = -W(q; p) \) and \( W(p; q) + W(q; r) = W(p; r) \). So all values can be expressed from

\[
W(p) := W(p; 0, 0),
\]
which is the Busemann function (in direction 0 with basepoint \((0, 0)\)) of \(L\). The Busemann function \(W_d\) in direction \(d\) may be defined by considering geodesics in direction \(d\) instead.

Clearly \((0, 0)\) is just a reference point where \(W\) is anchored, and for any \(q \in \mathbb{R}^2\), by translation invariance of \(L\), as functions on \(\mathbb{R}^2\),

\[
W(\cdot + q) - W(q) \overset{\text{law}}{=} W(\cdot).
\]  

(3.13)

It also follows from the definition and time stationarity of \(L\) that \(t \mapsto W(\cdot, t) - W(0, t)\) is a stationary process.

The following proposition gives an alternate definition of \(W\) without the use of geodesics.

**Proposition 3.2.** Almost surely, the following holds. Let \(t_n \to \infty, x_n/t_n \to 0\) and set \(q_n = (x_n, t_n)\). Then for every compact interval \(I\) and \(t \in \mathbb{R}\),

\[
W(p) = \mathcal{L}(p; q_n) - \mathcal{L}(0, 0; q_n)
\]

for all \(p \in I \times \{t\}\) for all sufficiently large \(n\).

**Proof.** Let \([y_1, y_2]\) be an interval containing \(I\) so that there are unique geodesics \(g_i\) in direction zero from \((y_i, t)\). There is almost surely a unique geodesic from \((0, 0)\) in direction zero. Let \((z, r)\) be where all three of these geodesics coalesce for the first time.

By Corollary 3.1 there is an \(n_0\) so that for \(n \geq n_0\), the geodesics \(g_{n,i}\) from \((y_i, t)\) to \(q_n\) as well as the geodesic \(g_n\) from \((0, 0)\) to \(q_n\) satisfy \(g_{n,i}(r) = g_n(r) = z\). By geodesic ordering, any geodesic \(g\) from any \(p \in [y_1, y_2] \times \{t\}\) to \(q_n\) satisfies \(g(r) = z\). This implies that for such \(p\) and \(n \geq n_0\),

\[
\mathcal{L}(p; q_n) - \mathcal{L}(0, 0; q_n) = \mathcal{L}(p; z, r) - \mathcal{L}(0, 0; z, r) = W(p).
\]

\[\square\]

**Corollary 3.3.** For every \(t\), the process \(x \mapsto W(x, t) - W(0, t)\) is a two-sided Brownian motion with diffusivity \(\sqrt{2}\).

**Proof.** By stationary, it suffices to show this for \(t = 0\). By Proposition 3.2, \(W(x, 0)\) restricted to compacts is the \(n \to \infty\) limit of

\[
W_n(x) = \mathcal{L}(x, 0; n^3) - \mathcal{L}(0, 0; n^3).
\]

By the definition and scaling invariance of \(\mathcal{L}\), the process \(W_n(x)\) has the law of \(x \mapsto n[A(x/n^2) - A(0)] - x^2/n^3\) where \(A\) is the stationary Airy-two process.
By the locally Brownian property of the Airy-two process \([9]\), \(x \mapsto n[A(x/n^2) - A(0)]\) converges in law to a two-sided Brownian motion with diffusivity \(\sqrt{2}\). The topology of convergence is that of uniformly on compacts. The term \(x^2/n^3\) vanishes uniformly on compacts as \(n \to \infty\). Thus \(x \mapsto W(x, 0)\) is Brownian.

Backwards in time, the Busemann function evolves as the KPZ fixed point.

**Theorem 3.5 (Busemann law).** For any pair of times \(s < t\),

\[
W(x, -t) = \sup_y \{L(x, -t; y, -s) + W(y, -s)\}. \tag{3.14}
\]

The maximum is attained exactly at those \(y\) for which \((y, -s)\) is on a geodesic from \((x, -t)\) in direction 0.

Moreover, the process \((W(x, -t) ; x \in \mathbb{R}, t \geq 0)\) is a KPZ fixed point with initial condition \(W(\cdot, 0)\), a two-sided Brownian motion with diffusivity \(\sqrt{2}\).

**Proof.** Set \(p = (x, -t)\) and \(q = (y, -s)\). Let \(u\) be the maximum of the time coordinates of \(\kappa(p, (0, 0))\), \(\kappa(q, (0, 0))\) and the time when the leftmost geodesic from \(p\) in the direction zero meets \(g(0,0)\). \(\) (The latter is the “left” version of the time of \(\kappa(p, (0, 0))\), needed here since the geodesic from \(p\) may not be unique). Let \(r = (g(0,0)(u), u)\). Since \(g_p, g_q\) and \(g_{(0,0)}\) have coalesced by time \(u\) at point \(r\),

\[
W(p) = L(p; r) - L(0,0; r) \quad \text{and} \quad W(q) = L(q; r) - L(0,0; r).
\]

The reverse triangle inequality applied to \(p, q, r\) implies

\[
W(p) \geq L(p, q) + W(q),
\]

with equality if and only if \(q\) is on a \(p\) to \(r\) geodesic. Applying this for all \(y\) gives (3.14). Now by geodesic ordering, all geodesics from \(p\) meet \(g(0,0)\) by time \(u\), so \(q\) is on a \(p\) to \(r\) geodesic if and only if \(q\) is on an infinite geodesic from \(p\) in direction zero.

By definition, \(W(\cdot, 0)\) is contained in the \(\sigma\)-field \(\mathcal{F}_{\geq 0}\) where

\[
\mathcal{F}_{\geq r} = \sigma(\mathcal{L}(x, s; y, t) : x, y \in \mathbb{R}, s, t \geq r). \tag{3.15}
\]

Since the values of \(\mathcal{L}(p; y, -s)\) are independent of \(\mathcal{F}_{\geq 0}\), \(W(\cdot, -t)\) evolves as a KPZ fixed point with random initial condition \(W(\cdot, 0)\). The law of \(W(\cdot, 0)\) is given by Corollary 3.3.

It follows from Theorem 3.5, Corollary 3.3 and the stationarity of \(t \mapsto W(\cdot, t) - W(0, t)\)
that a two-sided Brownian initial condition with diffusivity $\sqrt{2}$ is, up to an additive
constant, a stationary distribution for the KPZ fixed point. Furthermore, Theorem 3.5 and
the continuity of the KPZ fixed point [33] implies $W$ is almost surely continuous on the
lower half plane. Due to the shift-invariance (3.13), $W$ is then almost surely continuous
everywhere.

When $(x, t) = (0, 0)$, the first part of Theorem 3.5 can be interpreted as follows. For any
time $s \geq 0$, $g_{(0,0)}$ restricted to $[0, s]$ is just the geodesic from $(0, 0)$ to a “final condition”
$W(\cdot, s)$ at time $s$. By reversing time, it implies the following.

**Corollary 3.4** (Law of the infinite geodesic). Let $g_{(0,0)}$ be the infinite geodesic of $L$ from $(0, 0)$
in direction zero. For $s > 0$, let $\gamma_s$ denote the geodesic from the stationary two-sided Brownian
initial condition with diffusivity $\sqrt{2}$ to $(0, s)$. As random continuous functions from $[0, s] \to \mathbb{R}$,

$$g_{(0,0)}(\cdot) \xrightarrow{\text{law}} \gamma_s(s - \cdot).$$

The distribution function of $g_{(0,0)}(1)$ is the well-known KPZ scaling function [39].

**Remark 3.1** (Busemann functions in other directions). For any direction $d$, the Busemann
function $W_d$ in direction $d$ is related in law to $W_0$ through the shear invariance of the
directed landscape. The two-sided Brownian law of $W_d(\cdot, 0)$ picks up a drift of $2d$. In
fact, the reverse quadrangle inequality (see Figure 1) and Proposition 3.2 imply that for
any $t$ and $x < y$, the increment $W_d(y, t) - W_d(x, t)$ is non-decreasing in $d$. This means
for any fixed $t$, the function $(d, x) \mapsto W_d(x, t)$ is the cumulative distribution function of a
random measure, a limiting version of the shock measure introduced in [12].

### 3.9 Absolute continuity of geodesics

There is almost surely a unique geodesic of $L$ from a given point $p = (y, s)$ to another
point $q = (x, t)$ when $s < t$ [13]. In this section we will show that for two unique geodesics
with a common endpoint, the law of the shorter one is absolutely continuous with respect
to the longer geodesic. This technical fact will be utilized later in the proof of Theorem
4.3.

We will need a fact about overlap between geodesics. For two geodesics $\gamma$ and $\pi$, the
overlap is the set of times $t$ where $\gamma(t) = \pi(t)$. For a geodesic $\gamma$ that is the unique geodesic
between its endpoints, its overlap with any other geodesic $\pi$ is an interval. Furthermore,
if a sequence of geodesics $\gamma_n$ converges to $\gamma$, and $\gamma$ is unique, then their overlaps tend to
the full interval of times where $\gamma$ is defined. These facts, which will be used below, are
proved in [14, Lemma 3.3].

**Proposition 3.3.** Let $\gamma$ be the unique geodesic from $q = (0, 0)$ to $r = (x_1, t_1)$ with $t_1 > 0$, and
let $\pi$ be the unique geodesic from $p = (x_0, t_0)$ to $r$ with $t_0 < 0$. Let $s \in (0, t_1)$. Then $\gamma$ restricted to $[s, t_1]$ is absolutely continuous with respect to $\pi$ restricted to $[s, t_1]$.

Proof. Let $\gamma'$ and $\pi'$ be the restrictions of $\gamma$ and $\pi$ to $[s, t_1]$, respectively. Let $\mathcal{F}$ be the sigma algebra generated by $L$ between times $0$ and $t_1$.

We will show that

$$\Pr(\gamma' = \pi' | \mathcal{F}) > 0 \text{ almost surely.} \quad (3.16)$$

This suffices for absolute continuity by the following argument. Let $\pi_1, \pi_2, \ldots$ be conditionally independent samples from the conditional distribution of $\pi$ given $\mathcal{F}$ and let $\pi'_k$ be their restrictions to $[s, t_1]$. There is a random, finite first time $T$ so that $\pi'_T = \gamma'$, as there is a positive probability of this occurring for each sample. Then for any set $A$,

$$\Pr(\gamma' \in A) = \sum_{k=1}^{\infty} \Pr(\pi'_k \in A, T = k).$$

This means that if $A$ is a nullset for $\pi'$, then the right hand side is zero, and $A$ is also a nullset for $\gamma'$ as required.

Now, using the fact about overlaps between geodesics mentioned above, there is an $\varepsilon > 0$ such that all geodesics from $(x, 0)$ to $r$ with $|x| \leqslant \varepsilon$ coincide with $\gamma$ on $[s, t_1]$. Indeed, as $\varepsilon \to 0$ the geodesics from $(x, 0)$ to $r$ tends to $\gamma$ by compactness, and since $\gamma$ is unique, their overlap must eventually occupy $[s, t_1]$. Therefore, to conclude (3.16) it is enough that for any $\varepsilon > 0$ one has $\Pr(\pi(0) \in [-\varepsilon, \varepsilon] | \mathcal{F}) > 0$ almost surely.

In fact $\pi(0)$ is the maximizer of the function

$$x \mapsto L(p; x, 0) + L(x, 0, r). \quad (3.17)$$

The two summands are independent, continuous, and have finite maxima almost surely. The second is $\mathcal{F}$-measurable; we will think of it as a fixed function.

The first summand is a re-scaled and re-centred Airy process, the top line of an Airy line ensemble. We condition on all values of the Airy line ensemble except for the top line in $[-\varepsilon, \varepsilon]$. The Brownian Gibbs property gives that the conditional distribution is a Brownian bridge $B$ conditioned to be greater than the second to top line on $[-\varepsilon, \varepsilon]$ [9]. Let $m$ be the maximum of (3.17) outside $[-\varepsilon, \varepsilon]$, and let $h$ be the minimum of the second summand on $[-\varepsilon, \varepsilon]$. If $\max_{[-\varepsilon, \varepsilon]} B > m - h$, then (3.17) is maximized in $[-\varepsilon, \varepsilon]$. The conditional probability of this is always positive.

There is a parallel result about the absolute continuity of infinite geodesics in a common direction. Recall there is almost surely a unique infinite geodesic from a given point in a
Proposition 3.4. Let $g_0$ be the unique infinite geodesic of $\mathcal{L}$ from $(0, 0)$ in direction zero. Let $g_p$ be the unique infinite geodesic of $\mathcal{L}$ from $p = (x_0, t_0)$ with $t_0 < 0$ also in direction zero. For any $s > 0$, the law of the restriction of $g_0$ to $[s, \infty)$ is absolutely continuous with respect to the restriction of $g_p$ to $[s, \infty)$.

Proof. Let $\mathcal{F} = \mathcal{F}_{\geq 0}$ be the sigma algebra generated by the restriction of $\mathcal{L}$ to times after 0, see (3.15). Let $g'_0$ and $g'_p$ be the restriction of $g_0$ and $g_p$ to times in $[s, \infty)$. It is enough to show that $\Pr(g'_0 = g'_p | \mathcal{F}) > 0$ almost surely.

Using the fact about geodesic overlaps, there is an $\epsilon > 0$ such that any geodesic from $(x, 0)$ in direction zero with $|x| \leq \epsilon$ coincides with $g_0$ on $[s, \infty)$. Indeed, due to uniqueness of $g_0$, the geodesic from $(x, 0)$ coalesces with $g_0$ once they meet, and they must meet before time $s$ if $\epsilon$ is small enough by the overlap condition. We have to prove that for any $\epsilon$, $\Pr(g_p(0) \in [-\epsilon, \epsilon] | \mathcal{F}) > 0$ almost surely.

The location $g_p(0)$ is the maximizer of

$$x \mapsto \mathcal{L}(p; x, 0) + W(x)$$

where $W(x) = \lim_n \mathcal{L}(x, 0; 0, n) - \mathcal{L}(0, 0; 0, n)$. This limit exists by Proposition 3.2, is $\mathcal{F}$-measurable and has the law of a Brownian motion by Corollary 3.3. We think of $W$ as a fixed function. The two summand are independent, continuous and the sum has a finite maximum $M$ outside of $[-\epsilon, \epsilon]$ almost surely.

The first summand is a re-scaled and re-centred Airy process, the top line of an Airy line ensemble. We condition on all values of the Airy line ensemble except for the top line in $[-\epsilon, \epsilon]$. The Brownian Gibbs property gives that the conditional distribution is a Brownian bridge $B$ conditioned to be greater than the second to top line on $[-\epsilon, \epsilon]$. Let $H = \min_{[-\epsilon, \epsilon]} W$. If $\max_{[-\epsilon, \epsilon]} B > M - H$, then (3.18) is maximized in $[-\epsilon, \epsilon]$. The conditional probability of this is always positive.

3.10 Questions

Law of the geodesic starting point Let $h_0$ be an initial condition. The geodesic $g$ from $h_0$ to $(0, 1)$ is unique almost surely. Compute the law of its starting point $g(0)$ (which is also $e(0, 1)$ in the notation of §3.5). In particular, does its distribution function have a determinantal form?

The law of the starting points for the flat and stationary initial conditions have been computed in determinantal form [34, 39].
Law of the coalescence point  Study the law of the random point $\kappa(p, q)$ where the geodesic from $p$ in direction 0 first coalesces with the geodesic from $q$ in direction zero. It would be interesting to compute the law of $\kappa(p, q)$ exactly. See [38] for results on last passage percolation models.

4  Competition interfaces

This section studies competition interfaces of the directed landscape. Let $h_0$ be an initial condition as in §2.2. Throughout this discussion, assume good samples of $\mathcal{L}$ relative to $h_0$ as in §3.4, over which one may work sample-wise.

4.1  The competition function

A point $p \in \mathbb{R}$ is a reference point for an initial condition $h_0$ if there are points $a \leq p \leq b$ such that $h_0(a)$ and $h_0(b)$ are finite. The point $p$ is an interior reference point if this holds for some $a < p < b$. Define the left and right height functions split at a reference point $p$ as

$$h_p^-(x) = \begin{cases} h_0(x) & \text{if } x \leq p \\ -\infty & \text{if } x > p \end{cases} \quad \text{and} \quad h_p^+(x) = \begin{cases} h_0(x) & \text{if } x \geq p \\ -\infty & \text{if } x < p \end{cases}. \quad (4.1)$$

Let $h_p^-(x, t)$ and $h_p^+(x, t)$ be the directed landscape height functions associated to initial conditions $h_p^-$ and $h_p^+$, respectively. In other words,

$$h_p^-(x, t) = \sup_{y \leq p} \{h_0(y) + \mathcal{L}(0, y; x, t)\} \quad \text{and} \quad h_p^+(x, t) = \sup_{y \geq p} \{h_0(y) + \mathcal{L}(0, y; x, t)\}. \quad (4.2)$$

The competition function from $p$ is

$$d_p(x, t) = h_p^+(x, t) - h_p^-(x, t) \quad \text{for } (x, t) \in \mathcal{H}. \quad (4.3)$$

The competition function is continuous on $\mathcal{H}$ owing to the continuity of height functions.

Proposition 4.1 (Monotonicity). For every $t > 0$, the competition function is non-decreasing in the variable $x$.

Proof. Suppose $x < y$ and consider geodesics from $h_p^\pm$ to $(x, t)$ and from $h_p^\pm$ to $(y, t)$. The values of $h_p^\pm(x, t)$ and $h_p^\pm(y, t)$ are the relative lengths of these geodesics. The geodesic from $h_p^-$ to $(x, t)$ has to meet the one from $h_p^-$ to $(y, t)$ because the $(x, t)$-geodesic starts to the right of the $(y, t)$-geodesic and ends to its left. Suppose they meet at a point $a = (z, s)$. See Figure 1 for the setup (and essentially the proof).
Let $g_1$ be the path from $h_p^-$ to $(x,t)$ obtained by going from $h_p^-$ to $a$ along the $h_p^- \rightarrow (y,t)$ geodesic and then from $a$ to $(x,t)$ along the $h_p^+ \rightarrow (x,t)$ geodesic. Similarly, let $g_2$ be the path from $h_p^+$ to $(y,t)$ obtained by switching geodesics at $a$. The sum of the lengths of $g_1$ and $g_2$ is $h_p^+(x,t) + h_p^-(y,t)$, as it is the same as the sum of the lengths of the $h_p^+ \rightarrow (x,t)$ and $h_p^- \rightarrow (y,t)$ geodesics. On the other hand, the length of $g_1$ is at most $h_p^-(x,t)$, which is the geodesic length from $h_p^-$ to $(x,t)$. Similarly, $g_2$ has length at most $h_p^+(y,t)$. So, $h_p^+(x,t) + h_p^-(y,t) \leq h_p^-(x,t) + h_p^+(y,t)$, which says that $d_p(x,t) \leq d_p(y,t)$.

### 4.2 Defining an interface

Given an initial condition $h_0$ and a reference point $p$ for it, the interface with reference point $p$ is roughly the zero set of the competition function $d_p$. It represents points $(x,t) \in \mathbb{H}$ that have geodesics from $h_0$ starting both to the left and right of $p$. It is the boundary of this zero set that encodes information, leading to the following definition of interfaces.

**Definition 4.1.** The left and right interfaces from a reference point $p$ are as follows. For every $t > 0$,

\[
I_p^-(t) = \inf\{x \in \mathbb{R} : d_p(x,t) \geq 0\} \\
I_p^+(t) = \sup\{x \in \mathbb{R} : d_p(x,t) \leq 0\}
\]

In follows that $-\infty \leq I_p^-(t) \leq I_p^+(t) \leq +\infty$ since $d_p$ is continuous and non-decreasing in $x$. With the appropriate conventions, $d_p^{-1}(-,t)(0) = [I_p^-(t), I_p^+(t)]$.

Interfaces can be infinite, for instance $I_p^-(t)$ will be identically $-\infty$ if $h_0(x) = -\infty$ for every $x < p$. The right and left interfaces can also be separated by a region of points...
Definition 4.2. The interface from reference point $p$ is uniquely defined if $I^p_t = I^+_p(t) \in \mathbb{R}$ for every $t > 0$.

To visualize the definitions, note the upper half place $H$ is partitioned into the sets \{d_p < 0\}, \{d_p = 0\} and \{d_p > 0\}. These sets are connected and \{d_p = 0\} is sandwiched between \{d_p < 0\} to the left and \{d_p > 0\} to the right. For every $t > 0$, the intersections of $\mathbb{R} \times \{t\}$ with these sets are intervals (possibly empty or infinite). The left and right boundaries of \{d_p = 0\} make the left and right interfaces, respectively. It is convenient to visualize and identify a finite interface $I_p$ with its graph \{(I_p(t), t), t > 0\} lying in $H$.

4.3 Basic properties of interfaces

The following properties of interfaces hold for good samples of $L$.

Proposition 4.2 (Finiteness criterion). Let $p$ be a reference point for $h_0$. The interface $I^p_t$ is identically $-\infty$ if $h_0(x) \equiv -\infty$ for $x < p$; otherwise it is finite. Similarly, $I^+_p(t)$ is identically $+\infty$ if $h_0(x) \equiv -\infty$ for $x > p$ and otherwise it is finite. So the left and right interfaces from an interior reference point are finite.

Proof. If $h_0(x)$ is identically $-\infty$ for $x < p$ then $d_p(x, t) \geq 0$ because all geodesics start at or to the right of $p$. Thus, $I^p_t(t) \equiv -\infty$. Now suppose there is a point $a < p$ such that $h_0(a)$ is finite. We will prove that for every $t$, $d_p(x, t) < 0$ for sufficiently negative values of $x$. Then the set of points $x$ such that $d_p(x, t) \geq 0$ is bounded from below and non-empty (the latter because $h_0(x)$ is finite for some $x \geq p$). So $I^p_t(t)$ is finite. The same argument applies for the finiteness of $I^+_p(t)$.

In order to see that $d_p(x, t)$ becomes negative, consider $h^+_p(x, t)$ separately. For $x \leq a - 1$,

$$h^-_p(x, t) \geq A(a, 0; x, t) - \frac{(x - a)^2}{t} + h_0(a).$$

This is to be compared to $L(y, 0; x, t) + h_0(y)$ for $y \geq p$. Observe that $(x - a)^2 - (x - y)^2 = -(y - a)(y - x + a - x) \leq -(y - a)(y - x + 1)$ since $x \leq a - 1$ and $y \geq p > a$. So,

$$\frac{(x - a)^2}{t} \geq \frac{(x - y)^2}{t} + \frac{(y - a)(y - x + 1)}{t}.$$

One also has the bound

$$|A(a, 0; x, t)| + |A(y, 0; x, t)| \leq C(t^{1/2} + |y|^{1/2} + |x|^{1/2} + 1)$$
for a random constant $C$ by (2.3). As a result, since $h_0(y) \leq B(1 + |y|)$,

$$
\mathcal{L}(a, 0; x, t) + h_0(a) \geq \mathcal{L}(y, 0; x, t) + h_0(y) + f(y)
$$

where

$$
f(y) = \frac{(y - a)(y - x + 1)}{t} - C|y|^{1/2} - C|x|^{1/2} - B|y| - D_{t, a}
$$

for constants $C$, $B$ and $D_{t, a}$, some of which are random.

The dominating term in $f$ is the quadratic $(y - a)(y - x + 1)/t$, which is positive and tends to $+\infty$ as $x \to -\infty$ faster than the other terms (note $y \geq p > a > x$.) If $x$ is sufficiently negative in terms of the parameters $a, t, C, B, D_{t, a}$, then $f(y)$ will be strictly positive over all $y \geq p$. For such $x$,

$$
\mathcal{L}(a, 0; x, t) + h_0(a) > \sup_{y \geq p} \{\mathcal{L}(y, 0; x, t) + h_0(y)\},
$$

and so $h^{-}_p(x, t) > h^{+}_p(x, t)$ as claimed. \hfill \blacksquare

**Proposition 4.3** (Continuity). *When the interface $I^+_p(t)$ or $I^-_p(t)$ is finite, it is continuous.*

**Proof.** Take for instance the right interface, and a time $t > 0$ at which we want continuity. Fix an $\epsilon > 0$. The point $(I^+_p(t) + \epsilon, t)$ satisfies $d_p(I^+_p(t) + \epsilon, t) > 0$. By continuity of $d_p$, there is a $\delta_1 > 0$ such that $d_p(I^+_p(t) + \epsilon, s) > 0$ for every $s \in (t - \delta_1, t + \delta_1)$. Consequently,

$$
I^+_p(s) \leq I^+_p(t) + \epsilon \quad \text{for } s \in (t - \delta_1, t + \delta_1).
$$

We need to show that there is a $\delta_2 > 0$ such that $I^+_p(t) - \epsilon \leq I^+_p(s)$ for all $s \in (t - \delta_2, t + \delta_2)$. Consider the point $(I^+_p(t) - \epsilon, t)$. If $d_p$ is negative at this point then arguing as above there is a $\delta_2 > 0$ such that $I^+_p(t) - \epsilon \leq I^+_p(s)$ for every $s \in (t - \delta_2, t + \delta_2)$. So we are reduced to considering the case $d_p(I^+_p(t) - \epsilon, t) = 0$.

Suppose that $d_p(I^+_p(t) - \epsilon, t) = 0$. Consider the point $a = (I^+_p(t) - \epsilon, t + \delta)$ for $\delta > 0$. If $d_p(a) \leq 0$ then $I^+_p(t + \delta) \geq I^+_p(t) - \epsilon$. We will argue below that there is a $\delta_3 > 0$ such that $d_p(a) \leq 0$ for all $\delta \leq \delta_3$. Thus, $I^+_p(s) \geq I^+_p(t) - \epsilon$ for all $s \in [t, t + \delta_3]$. Similarly, for points of the form $b = (I^+_p(t) - \epsilon, t - \delta)$ there is a $\delta_4 > 0$ such that $d_p(b) \leq 0$ for all $\delta \leq \delta_4$, which implies $I^+_p(s) \geq I^+_p(t) - \epsilon$ for $s \in [t - \delta_4, t]$. This proves continuity at $t$.

Suppose that $d_p(a) > 0$ with $a$ as above. Consider any geodesic $g$ from $h_0$ to $a$, which then has to start to the right of $p$ since $d_p(a) > 0$. At time $t$ this geodesic must be to the right of $I^+_p(t)$. Otherwise, if $g(t) < I^+_p(t)$, then there is a geodesic to $(g(t), t)$ which starts at a point $\leq p$. Then $g$ can be modified to a geodesic to $a$ that starts at a point $\leq p$ by starting with the former geodesic and switching to $g$ at time $t$. Since this cannot happen
due to $d_p(a) > 0$, it must be that $g(t) \geq I_p^+(t)$. It follows that $|g(t + \delta) - g(t)| \geq \varepsilon$. But by continuity and compactness of geodesics, this cannot happen for sufficiently small $\delta$. Thus, there is a $\delta_3 > 0$ such that if $s \in [t, t + \delta_3]$ then $d_p(I_p^+(t) - \varepsilon, s) \leq 0$.

A similar argument works for points of the form $b = (I_p^+(t) - \varepsilon, t - \delta)$ to show that there is a $\delta_4 > 0$ such that $d_p(b) \leq 0$ for all $\delta \leq \delta_4$. Indeed, let $g$ be a geodesic from $h_0$ to $(I_p^+(t), t)$ that starts at a point $\leq p$. With $b$ as above, suppose $d_p(b) > 0$. Then all geodesics to $b$ start at points $> p$. As the geodesic $g$ starts at a point $\leq p$, geodesic ordering implies $g(t - \delta)$ must be to the left of $I_p^+(t) - \varepsilon$. So $|g(t - \delta) - g(t)| \geq \varepsilon$ if $d_p(b) > 0$. Geodesic continuity implies this won’t happen for all small values of $\delta$, so indeed $d_p(b) \leq 0$ for all $\delta \leq \delta_4$.

What is the location of an interface as time approaches zero? It need not be that an interface from reference point $p$ begins there, as seen in §4.5.

**Proposition 4.4** (Starting location). Suppose for a reference point $p$ there are $a < p < b$ such that $h_0(a)$ and $h_0(b)$ are finite. Then $\liminf_{t \to 0} I_p^-(t) \geq a$ and $\limsup_{t \to 0} I_p^+(t) \leq b$. As a result, if for every $\varepsilon > 0$ there are $a$ and $b$ such that $p - \varepsilon < a < p < b < p + \varepsilon$ and $h_0(a), h_0(b)$ are both finite, then $I_p^+(t) \to p$ as $t \to 0$.

**Proof.** We will show that the liminf of $I_p^-(t)$ is at least $a$ at $t \to 0$; the argument for the limsup being similar. Fix a point $x < a$ and consider points $(x, t) \in \mathbb{H}$ as $t \to 0$. We will argue that $d_p(x, t) < 0$ for all sufficiently small $t$, almost surely in the underlying randomness. This implies that $\liminf_{t \to 0} I_p^-(t) \geq a$.

Recall the functions $h_p^\pm(x, t)$ from (4.2). Clearly,

$$h_p^-(x, t) \geq A(0, a; x, t) - \frac{(x - a)^2}{t} + h_0(a).$$

Let $y_t \geq p$ be a point where the supremum in the definition of $h_p^+(x, t)$ is obtained. So,

$$h_p^+(x, t) = A(0, y_t; x, t) - \frac{(x - y_t)^2}{t} + h_0(y_t).$$

Such a $y_t$ exists by the assumptions of $h_0$, namely that it is upper semicontinuous and bounded form above by a linear function. Moreover, almost surely in the underlying randomness, $y_t$ will be bounded as $t \to 0$. It follows that

$$h_p^-(x, t) - h_p^+(x, t) \geq A(0, a; x, t) - A(0, y_t; x, t) + h_0(a) - h_0(y_t) + \frac{(x - y_t)^2 - (x - a)^2}{t}.$$

There is a random constant $C$ such that $|A(a, 0; x, t)| + |A(y, 0; x, t)| \leq C(t^{1/2} + |x|^{1/2} +
\(|y|^{1/2} + |a|^{1/2} + 1\) by (2.3). This means there is a random constant \(C'\) such that for \(t \leq 1\), 
\(|A(a, 0; x, t)| + |A(y, 0; x, t)| \leq C'\) because \(y_t\) remains bounded while \(a\) and \(x\) are now fixed. There is also a random constant \(C''\) such that \(h_0(a) - h_0(y) \geq C''\) since \(h_0\) is bounded from above by a linear function and \(y_t\) is bounded.

Consider now the term \((x - y_t)^2 - (x - a)^2\), which equals \((y_t - a)(a + y_t - 2x)\). Due to \(y_t \geq p > a\), we find that \(y_t - a \geq p - a\) and \(a + y_t - 2x \geq 2(a - x)\). So,

\[
\frac{(x - y_t)^2}{t} - \frac{(x - a)^2}{t} \geq \frac{2(a - x)(p - a)}{t} = \frac{2|a - x||p - a|}{t}.
\]

Combined with the bounds from above we infer that

\[
h_p(x, t) - h^+_p(x, t) \geq \frac{2|a - x||p - a|}{t} + C'' - C'.
\]

It follows that \(h_p(x, t) > h^+_p(x, t)\) for sufficiently small values of \(t\), and so \(d_p(x, t) < 0\) for such \(t\).

### 4.4 Distributional invariances of interfaces

Distributional invariances of the directed landscape [13] give way to distributional invariances of interfaces. These are recorded here. In order to track dependence of interfaces on the initial condition, write \(I_p^\pm(t; h_0(x))\) for the left or right interface with reference point \(p\) and initial condition \(h_0\).

The following invariances hold jointly in \(p\) and \(t\), and \(h_0\).

**Scaling.** For \(\sigma > 0\),

\[
I_p^{\pm}(\sigma^3; h_0) \overset{\text{law}}{=} \sigma^2 I_p^{\pm}(t; h_0(\sigma^3) / \sigma).
\]

This is due to the scaling invariance of the directed landscape height functions:

\[
h(\sigma^2, \sigma^3; h_0(y)) / \sigma \overset{\text{law}}{=} h(x, t; h_0(\sigma^2) / \sigma),
\]

which in turn implies

\[
d_p(\sigma^2, \sigma^3; h_0(y)) / \sigma \overset{\text{law}}{=} d_p(\sigma^2, \sigma^3; h_0(\sigma^2) / \sigma).
\]

**Translation.** For \(a \in \mathbb{R}\),

\[
I_p^{\pm}(t; h_0(x) + a) \overset{\text{law}}{=} I_p^{\pm}(t; h_0(x)).
\]

Indeed, \(h(x, t; h_0(x) + a)\) has law \(h(x, t; h_0(x)) + a\) and the competition function remains unchanged by such translations. Similarly the law of \(h(x, t; h_0(y + a))\) is
\( h(x + a, t; h_0(y)) \), which implies that
\[
I_p^\pm(t; h_0(x)) \overset{law}{=} I_p^\pm(t; h_0(x - a)) - a.
\]

Also, if \( h_0 \) satisfies \( h_0(x - a) = h_0(x) - h_0(0) + h_0(-a) \) for every \( x \), then \( I_p^\pm(t; h_0(x)) \) has the same law as \( I_p^\pm(t; h_0(x)) - a \). As examples, if \( h_0 \) is flat or a two-sided Brownian motion then \( I_p^\pm(t; h_0) \) has the same law as \( I_p^\pm(t; h_0) - a \) for every \( a \).

**Affine shift.** For \( a \in \mathbb{R} \),
\[
I_p^\pm(t; h_0(x) + ax) \overset{law}{=} I_p^\pm(t; h_0(x)) - \frac{at}{2}.
\]

This comes from the invariance
\[
h(x, t; h_0(y) + ay) \overset{law}{=} h(x + (at)/2, t; h_0(y)),
\]
which implies that \( d_p(x, t; h_0(y) + ay) \) has law \( d_p(x + (at)/2, t; h_0(y)) \).

### 4.5 Example: bisector between two points

Consider interfaces for the initial condition consisting of two narrow wedges:
\[
h_0(x) = \begin{cases} 
0 & \text{if } x = \pm 1 \\
-\infty & \text{otherwise}.
\end{cases}
\]

Its competition function has been studied in [2] and called the Airy difference profile. The finite interfaces are \( I_p^\pm(t) \) for \( p \in (-1, 1) \), which are the same for every \( p \). This interface is uniquely defined: \( I_0^- (t) = I_0^+ (t) \) for every \( t \). Indeed, the zero set of the function \( x \to d_0(x, t) \) is a non-empty interval by Propositions 4.1 and 4.2. Any point in the zero set can not be a UGP as it must have geodesics ending at both \(-1\) and \(1\). Since the initial condition satisfies the UGC, the zero set is countable and so it must be a singleton.

The unique interface \( I(t) = I_0^+ (t) \) consists of points in \( \mathbb{H} \) that are \( \mathcal{L} \)-equidistant from \((-1, 0)\) and \((1, 0)\). It begins at the origin: \( \lim_{t \to 0} I(t) = 0 \). To see this note that
\[
d_0(x, t) = A(1, 0; x, t) - A(-1, 0; x, t) + \frac{4x}{t}.
\]
The bound (2.3) implies \( |A(1, 0; x, t) - A(-1, 0; x, t)| \leq C(|x|^{1/2} + t^{1/2} + 1) \) for a random constant \( C \). Suppose \( t \leq 1 \). So if \( x \) is such that \( d_0(x, t) = 0 \), then
\[
|x| \leq \frac{t}{4} C(|x|^{1/2} + 2),
\]

38
from which it is easy to conclude that $|x| \leq C't$ for a random constant $C'$. Consequently, $|I(t)| \leq C't$ as $t \to 0$.

The following proposition establishes the first half of Theorem 1.1.

**Proposition 4.5.** There is a random variable $D$ having a Normal distribution with mean 0 and variance $1/4$ such that, almost surely,

$$\lim_{t \to \infty} \frac{I(t)}{t} = D.$$ 

**Proof.** For $d \in \mathbb{R}$, almost surely there is an infinite geodesic $g$ from the initial condition $h_0$ with asymptotic direction $d$. Its existence can be proved as in Theorem 3.2. So $g(t)$ for $t \geq 0$ is a path that satisfies $g(t)/t \to d$ as $t \to \infty$, and for which the segment $g(s)$ for $0 \leq s \leq t$ is a geodesic to the initial condition for every $t$.

An infinite geodesic cannot cross the bisector, that is, satisfy $g(t) < I(t)$ and $g(t') > I(t')$ for different times $t$ and $t'$. If it did cross the bisector then it would have to end simultaneously at $-1$ and $1$. So it lies either to the left or to the right of the bisector. If it is to the left ($g(t) \leq I(t)$) then $\liminf_{t \to \infty} I(t)/t \geq d$. While if it stays to the right then $\limsup_{t \to \infty} I(t)/t \leq d$.

So almost surely, for every $d \in \mathbb{Q}$, either $\liminf_{t \to \infty} I(t)/t \geq d$ or $\limsup_{t \to \infty} I(t)/t \leq d$. This means there is a limit of $I(t)/t$ or else there would be a rational $d$ satisfying $\liminf_{t \to \infty} I(t)/t < d < \limsup_{t \to \infty} I(t)/t$.

In order to derive the law of $D$, note $\Pr(D > a)$ is the limit of $\Pr(d_0(at, t) < 0)$ as $t \to \infty$. Now $d_0(at, t) = \mathcal{L}(1, 0; at, t) - \mathcal{L}(-1, 0; at, t)$, which by shear invariance of $\mathcal{L}$ has the law of $\mathcal{L}(1, 0; 0, t) - \mathcal{L}(-1, 0; 0, t) + 4a$. The difference $\mathcal{L}(1, 0; 0, t) - \mathcal{L}(-1, 0; 0, t)$ converges to $W(1, 0) - W(-1, 0)$ by Proposition 3.2, where $W$ is the Busemann function. The law of $W(x, 0)$ is Brownian with diffusivity $\sqrt{2}$ by Corollary 3.3, so $W(1, 0) - W(-1, 0)$ has a Normal distribution with mean 0 and variance 4. Therefore, $\Pr(D > a) = \Pr(2Z + 4a < 0)$, where $Z$ is a standard Normal random variable. This shows $D$ is Normal with mean 0 and variance $1/4$. ■

### 4.6 Criterion for a uniquely defined interface

It is often necessary to know when an interface is uniquely defined, so we present a criterion to check this condition.

Let $h_0(x)$ be an initial condition. Let $[a, b]$ be a compact interval and suppose $h_0$ is not identically $-\infty$ on $[a, b]$.

**Definition 4.3.** The polar measure of $h_0$ over $[a, b]$, denoted $p_{h_0([a, b])}$, is the law of the maximizer of $h_0(x) + B(x)$ for $x \in [a, b]$, where $B$ is a Brownian motion started from
B(a) = 0 and with diffusivity constant $\sqrt{2}$.

The proof of Lemma 3.3 shows the maximizer is unique, so the measure is well defined.

The polar points of $h_0$ are

$$\mathcal{pp}(h_0) = \{ x \in \mathbb{R} : \mu_{h_0,[a,b]}(\{x\}) > 0 \text{ for some } a < x < b \}. \tag{4.4}$$

If $[a, b] \subset [c, d]$ then $\mu_{h_0,[a,b]}$ is absolutely continuous with respect to $\mu_{h_0,[c,d]}$. So the polar points of $h_0$ are the atoms of its polar measures over increasing compact intervals. There are at most a countable number of polar points of any $h_0$.

**Proposition 4.6.** Let $p$ be a non polar point of $h_0$ that is an interior reference point (there are $a < p < b$ such that $h_0(a), h_0(b)$ are both finite). Then the interface of $h_0$ from reference point $p$ is uniquely defined almost surely.

**Proof.** Since $p$ is an interior reference point, $I_p^\pm(t)$ are finite valued curves by Proposition 4.2.

Recall $I_p^\pm$ are the boundaries of the set $\{d_p = 0\}$ and, by continuity of interfaces, they are equal if and only if this set has an empty interior. A point in the interior of $\{d_p = 0\}$ has the property that all its geodesics from $h_0$ start at $p$; this is shown in Lemma 5.3 below. So to show $\{d_p = 0\}$ has empty interior almost surely, it is enough to prove that any given $(x, t) \in \mathcal{H}$ has a geodesic from $h_0$ starting at $p$ with zero probability. For then the interior of $\{d_p = 0\}$ contains no rational points almost surely and so it is empty.

Now if $(x, t) \in \mathcal{H}$ has a geodesic from $h_0$ starting at $p$ then $y = p$ maximizes $y \mapsto \mathcal{L}(0, y; x, t) + h_0(y)$. Let $[c, d]$ be any compact interval of positive length containing $p$. Then $p$ is a maximizer of $y \mapsto \mathcal{L}(0, y; x, t) - \mathcal{L}(0, c; x, t) + h_0(y)$ for $y \in [c, d]$. Let $B(y)$ be a Brownian motion started from $B(c) = 0$. The former process is absolutely continuous with respect to $B(y) + h_0(y)$ over $[c, d]$. As $p$ is not a polar point of $h_0$, it maximizes $B + h_0$ over $[c, d]$ with zero probability. The same therefore holds for the former process, as required. 

A sufficient condition for $p$ being a non polar point of $h_0$ is if $h_0$ the modulus of continuity of $h_0$ at $p$ has Hölder exponent more than $1/2$. Indeed, assuming without loss of generality that $p = 0$ and $h_0(0) = 0$, note that by the law of iterated logarithm there exists arbitrary small $t > 0$ so that $B(t) > \sqrt{T \log \log(1/T)}$, and for those $t$ one has $B(t) + h(t) > 0$.

Another example: given $p \in \mathbb{R}$, when $h_0$ is Brownian motion then $p$ is non-polar almost surely. This follows from the fact that sum of two independent Brownian motions is another Brownian motion. In a similar vein, any given $p$ is almost surely a non-polar
point of the KPZ fixed point $h(x, t)$ at any fixed positive time. This is because $h(x, t)$ is absolutely continuous with respect to Brownian motion on any compact interval [45].

### 4.7 Distribution of interfaces

There is a geometric description of the law of the interface $I_{\pm}^p(t)$ associated to an initial condition $h_0$ in terms of the concave majorant of a function. This also helps to derive the law of the asymptotic direction of interfaces.

The **concave majorant** of a function $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is the smallest concave function $c$ such that $c(x) \geq f(x)$ for every $x$. We denote it by $\text{cmaj}(f)$.

Let $\partial_l c$ and $\partial_r c$ be the left and right derivatives, respectively, of a concave function $c$. Recall these are defined as $\partial_l c(p) = \lim_{\varepsilon \downarrow 0} \frac{c(p) - c(p - \varepsilon)}{\varepsilon}$ and $\partial_r c(p) = \lim_{\varepsilon \downarrow 0} \frac{c(p + \varepsilon) - c(p)}{\varepsilon}$. The concave majorant has the property that all maximizers of $f$ are strictly to the right of $p$ if and only if $\partial_r \text{cmaj}(f)(p) > 0$. Similarly, all maximizers of $f$ are strictly to the left of $p$ if and only if $\partial_l \text{cmaj}(f)(p) < 0$. Indeed, $f$ and $\text{cmaj}(f)$ are equal at the maximizers of $f$ and the derivative of $\text{cmaj}(f)$ is non-increasing.

Denote by $A_t(x) = L(0, 0; x, t)$ the (parabolic) Airy process at time $t$ as a process in $x$.

**Proposition 4.7.** Given an initial condition $h_0$, an interior reference point $p$ and a time $t > 0$,

$$I_{-}^p(t) \overset{\text{law}}{=} -\frac{t}{2} \partial_l \text{cmaj}(h_0 + A_t)(p) \quad \text{and} \quad I_{+}^p(t) \overset{\text{law}}{=} -\frac{t}{2} \partial_r \text{cmaj}(h_0 + A_t)(p).$$

**Proof.** We prove the first claim as the second is symmetric. Note $I_{-}^p(t) > y$ if and only if $L(\cdot, 0; y, t) + h_0(\cdot)$ takes its maximum strictly to the left of $p$. Since $h_0$ is an initial condition, hence does not grow faster than linearly, and $L(\cdot, 0; y, t)$ decays quadratically, the concave majorant of $L(\cdot, 0; y, t) + h_0(\cdot)$ exists, and the above is equivalent to

$$\partial_l \text{cmaj} \left( L(\cdot, 0; y, t) + h_0(\cdot) \right)(p) < 0.$$

By shear invariance, as functions of $x$,

$$L(x, 0; y, t) \overset{\text{law}}{=} L(x, 0; 0, t) + \frac{y(2x - y)}{t}.$$

$L(x, 0; 0, t)$ has the same law as $L(0, 0; x, t)$ as a function of $x$ by the time reversal symmetry of $L$ [12]. Therefore,

$$\partial_l \text{cmaj} \left( L(\cdot, 0; y, t) + h_0(\cdot) \right)(p) \overset{\text{law}}{=} \partial_l \text{cmaj}(A_t + h_0)(p) + \frac{2y}{t}.$$
As a result, \( \Pr(I_p^-(t) > y) = \Pr(\partial_1 \text{cmaj}(h_0 + A_t)(p) < -2y/t) \) as claimed.

Corollary 4.1. For any \( t > 0 \),
\[
\Pr(I_p^-(t) = I_p^+(t)) = 1 \iff \Pr(\text{cmaj}(h_0 + A_t) \text{ is differentiable at } p) = 1.
\]

Proof. Since \( I_p^-(t) \leq I_p^+(t) \) and \( \partial_r \text{cmaj}(h_0 + A_t)(p) \leq \partial_1 \text{cmaj}(h_0 + A_t)(p) \), each of these pairs of random variables are equal with probability one if and only if they have the same law. Therefore, by Proposition 4.7, \( I_p^-(t) \) and \( I_p^+(t) \) have the same law if and only if \( \text{cmaj}(h_0 + A_t) \) is differentiable at \( p \).

4.8 Asymptotic direction of interfaces

Proposition 4.5 showed that the bisector between two points has an asymptotically Normal direction. What about other initial conditions? Intuition from Euclidean geometry suggests asymptotically flat initial conditions should have asymptotically vertical interfaces – a fact confirmed by Theorem 4.1. More generally, in §4.8.1 we compute the direction of interfaces for initial conditions that do not grow too fast.

Theorem 4.1. Suppose an initial condition \( h_0(x) \) is asymptotically flat in that \( |h_0(x)|/|x| \to 0 \) as \( |x| \to \infty \). Then for every \( p \), almost surely, \( \lim_{t \to \infty} I_p^\pm(t)/t = 0 \).

Combining the theorem with the affine shift invariance of interfaces leads to

Corollary 4.2. Suppose \( h_0(x) = \alpha x + h(x) \) where \( h(x) \) is asymptotically flat and \( \alpha \in \mathbb{R} \). Then for every \( p \), almost surely, \( \lim_{t \to \infty} I_p^\pm(t)/t = -\alpha/2 \).

Proof. We may assume that \( p = 0 \) by translation invariance of interfaces. Since \( I_p^-(t) \) and \( I_p^+(t) \) are the first and last zeroes of the monotone function \( d(\cdot, t) \), it is enough to show that for every \( \varepsilon > 0 \) it holds that \( d(\varepsilon t, t) > 0 \) and \( d(-\varepsilon t, t) < 0 \) for all large values of \( t \). We will prove that \( d(\varepsilon t, t) > 0 \) and the argument for the other inequality is entirely similar.

Suppose \( 0 < \varepsilon < 1/2 \) and let \( \delta \in (0, \varepsilon^2/5) \). The bound (2.2) for \( \mathcal{A}(y, 0; x, t) \) implies that there is a random \( t_0 \) such that if \( t \geq t_0 \) then
\[
|\mathcal{A}(y, 0; x, t)| \leq \delta(t + |x| + |y|).
\]

Since \( |h_0(x)|/|x| \) tends to zero, there is a \( t_1 \) such that for \( t \geq t_1 \),
\[
|h_0(x)| \leq \delta(|x| + \frac{t}{2}).
\]

Suppose \( t \geq \max(t_0, t_1) \) so that both the bounds above hold. We will show that \( d(\varepsilon t, t) > 0 \), which is sufficient.
Consider $h_0^-(\epsilon t, t) = \sup_{y \leq 0} \{h_0(y) + \mathcal{L}(y, 0; \epsilon t, t)\}$. The bounds above imply

$$h_0^-(\epsilon t, t) \leq \sup_{y \leq 0} \left\{ \frac{(y - \epsilon t)^2}{t} - 2\delta y + 2\delta t \right\}$$

since $\epsilon < 1/2$. The optimization may be rearranged as

$$t(2\delta - \epsilon^2) + \sup_{y \leq 0} \left\{ \frac{y^2}{t} + 2y(\epsilon - \delta) \right\} = t(2\delta - \epsilon^2),$$

since $\delta < \epsilon$ and so the maximum occurs at $y = 0$.

A similar lower bound holds for $h_0^+(\epsilon t, t) = \sup_{y \geq 0} \{h_0(y) + \mathcal{L}(y, 0; \epsilon t, t)\}$.

$$h_0^+(\epsilon t, t) \geq \sup_{y \geq 0} \left\{ \frac{(y - \epsilon t)^2}{t} - 2\delta y - 2\delta t \right\}$$

$$= -t(2\delta + \epsilon^2) + \sup_{y \geq 0} \left\{ \frac{y^2}{t} + 2y(\epsilon - \delta) \right\}$$

$$= -t(2\delta + \epsilon^2) + t(\epsilon - \delta)^2$$

$$\geq -3\delta t.$$

In the final line we used that $\epsilon < 1/2$.

Combining the upper bound on $h_0^- (\epsilon t, t)$ with the lower bound on $h_0^+ (\epsilon t, t)$ shows that

$$d(\epsilon t, t) \geq (\epsilon^2 - 5\delta)t > 0.$$

\[\Box\]

### 4.8.1 Directions for sub-wedge initial conditions

For an initial condition $h_0$, its left and right upper linear growth rates are

$$a_- = \limsup_{x \to -\infty} \frac{h_0(x)}{x}, \quad a_+ = \limsup_{x \to \infty} \frac{h_0(x)}{x}.$$

Note $a_\pm$ take values in $\mathbb{R} \cup \{-\infty, \infty\}$. An initial condition is called sub-wedge if $a_+ < a_-$. For instance, $h_0(x) = -a|x| + b$ for $a > 0$ is sub-wedge, as is any $h_0$ that is $-\infty$ outside a compact interval or outside a half-line of the form $[p, \infty)$ or $(-\infty, p]$. We determine the asymptotic direction of interfaces to sub-wedge initial conditions in Theorem 4.2 and Corollary 4.3.

If $B(x)$ is two-sided Brownian motion (of diffusivity $\sqrt{2}$) and $h_0$ is a sub-wedge initial condition, then

$$\sup_{y \in \mathbb{R}} \{h_0(y) + B(y) + by\} \quad (4.5)$$
is almost surely finite and achieved for $-a_- < b < -a_+$. For $b < -a_-$ or $b > -a_+$, it is almost surely infinite. Recall the Busemann function $W_d$ in direction $d$, for which $W_d(x, 0)$ has law $B(x) + 2dx$. Consequently, if $d \in (-a_-/2, -a_+/2)$ then, almost surely,

$$W_d(h_0) = \sup_{y \in \mathbb{R}} \{ h_0(y) + W_d(y, 0) \}$$  \hfill (4.6)

is finite and achieved.

A geodesic $\gamma$ from an initial condition $h_0$ in direction $d$ is a geodesic $\gamma : [0, \infty) \to \mathbb{R}$ of $\mathcal{L}$ in direction $d$ such that $\gamma$ restricted to $[0, t]$ is a geodesic from $h_0$ for every $t$.

**Lemma 4.1.** Let $h_0$ be an sub-wedge initial condition with upper linear growth rates $a_\pm$. For every $d \in (-a_-/2, -a_+/2)$, there is almost surely a geodesic $\gamma$ from $h_0$ in direction $d$. Such a geodesic is obtained by choosing a maximizer $z$ of $W_d(h_0)$ from (4.6) and following a geodesic of $\mathcal{L}$ from $(z, 0)$ in direction $d$. With $z = \gamma(0)$, for any point $p$ on $\gamma$,

$$\mathcal{L}(h_0; p) = \mathcal{L}(z, 0; p) + h_0(z)$$ \hfill (4.7)

$$= W_d(z, 0) + h_0(z) - W_d(p)$$ \hfill (4.8)

$$= W_d(h_0) - W_d(p)$$ \hfill (4.9)

**Proof.** Recall the notation $\mathcal{L}(h_0; p) = \sup_y \{ h_0(y) + \mathcal{L}(0, y; p) \}$. By definition of the Busemann function $W_d$, for any two points $p, q$ in increasing time order on a geodesic $\gamma$ of $\mathcal{L}$ in direction $d$,

$$\mathcal{L}(p, q) = W_d(p) - W_d(q).$$ \hfill (4.10)

Let $d \in (-a_-/2, -a_+/2)$ be fixed. By the discussion around (4.6), there is a $z$ such that for every $y \in \mathbb{R}$,

$$W_d(y, 0) + h_0(y) \leq W_d(z, 0) + h_0(z) = W_d(h_0).$$ \hfill (4.11)

Let $\gamma$ be the geodesic of $\mathcal{L}$ from $(z, 0)$ in direction $d$. We will show $\gamma$ is a geodesic from $h_0$.

Let $p$ be any point on $\gamma$ and $y \in \mathbb{R}$. Let $r$ be the point where the geodesic of $\mathcal{L}$ from $(y, 0)$ to $p$ first meets $\gamma$. Since the $(y, 0) \to r$ and $r \to p$ segments are part of a geodesic in direction $d$, the triangle equality along geodesics together with (4.10) implies

$$\mathcal{L}(y, 0; p) = \mathcal{L}(y, 0; r) + \mathcal{L}(r, p) = W_d(y, 0) - W_d(p).$$

Now (4.11) gives

$$\mathcal{L}(y, 0; p) + h_0(y) \leq W_d(z, 0) + h_0(z) - W_d(p) = \mathcal{L}(z, 0; p) + h_0(z)$$

44
by (4.10), showing (4.8). Thus the left hand side is maximized at $z$, showing (4.7). Since this holds for every $p$ on $\gamma$, $\gamma$ is a geodesic from $h_0$. The definition of $z$ implies (4.9).

So far we have considered the geodesic tree and the Busemann function in a fixed direction. For the next result we need to consider them simultaneously in every direction. We have not constructed them like so, but we can get around this difficulty by noting that our construction allows to define geodesic trees and Busemann functions simultaneously in every rational direction. This will suffice. Lemma 4.1 implies that, almost surely, there are geodesics from $h_0$ in every rational direction $d$ for $d \in (-a-/2, -a+/2)$.

Recall the splitting of an initial condition $h_0$ at a reference point $p$, which defines split functions $h_{p}^{\pm}$ according to (4.1). The function $d \mapsto W_d(h_{p}^{+}) - W_d(h_{p}^{-})$ is non-decreasing (restricted to rational $d$ as we do). This may be proved in the same way as Proposition 4.1, using the reverse quadrangle inequality, by interpreting $W_d(h_0)$ as a geodesic length from $h_0$ in direction $d$. In fact, the difference is the limit as $t \to \infty$ of the function $d_p(x, t)$ where $d_p(x, t)$ is the competition function from (4.3). This motivates the following theorem.

**Theorem 4.2 (Asymptotic direction).** Let $h_0$ be a sub-wedge initial condition with upper linear growth rates $a_{\pm}$. Let $p$ be an interior reference point of $h_0$. Then the left and right interfaces $I_{p}^{\pm}(t)$ have almost sure directions: $D_{p}^{-} = \lim_{t \to \infty} I_{p}^{-}(t) / t$ and $D_{p}^{+} = \lim_{t \to \infty} I_{p}^{+}(t) / t$. They are determined by

\[
D_{p}^{-} = \inf\{d \in Q : W_d(h_{p}^{+}) \geq W_d(h_{p}^{-})\}
\]
\[
D_{p}^{+} = \sup\{d \in Q : W_d(h_{p}^{+}) \leq W_d(h_{p}^{-})\}
\]

In particular the limits lie in the interval $[-a-/2, -a+/2]$.

**Proof.** We will prove the proposition for the left interface since the proof for the right interface is symmetric.

If $d > -a+/2$ then $W_d(h_{p}^{+}) = \infty$ and $W_d(h_{p}^{-}) < \infty$; so $D_{p}^{-} \leq -a+/2$. Similarly, $D_{p}^{-} \geq -a-/2$.

Let $d < D_{p}^{-}$ be an arbitrary rational direction with $d \in (-a-/2, -a+/2)$. By Lemma 4.1 there is a geodesic $\gamma$ from $h_{p}^{+}$ in direction $d$. Moreover, for all times $t$,

\[
\mathcal{L}(h_{p}^{+}; (\gamma(t), t)) = W(h_{p}^{+}) - W_d((\gamma(t), t)).
\]

(4.12)

Since $W_d(h_{p}^{-}) > W_d(h_{p}^{+})$, there is a $y$ such that $W_d(y, 0) + h_{p}^{+}(y) > W_d(h_{p}^{+})$. Let $\gamma'$ be
the geodesic of \( \mathcal{L} \) from \((y, 0)\) in the direction \(d\). Then
\[
\mathcal{L}(h_p^-, (\gamma'(t), t)) \geq h_p^-(y) + \mathcal{L}(y, 0; (\gamma'(t), t)) \\
= h_p^-(y) + W_d(y, 0) - W_d((\gamma'(t), t)) \\
> W_d(h_p^+) - W_d((\gamma'(t), t)).
\]

Comparing this with (4.12), it follows that for all times after \(\gamma\) and \(\gamma'\) meet,
\[
\mathcal{L}(h_p^+; (\gamma(t), t)) < \mathcal{L}(h_p^-; (\gamma(t), t)),
\]
which implies \(\gamma(t) \leq l_p^-(t)\). Since \(\gamma\) has a direction \(d\), this means \(d \leq \lim \inf_t l_p^-(t)/t\). As this holds for all rational \(d < D_p^-\), it means \(D_p^- \leq \lim \inf_t l_p^-(t)/t\).

A symmetric argument for the other side gives that for all rational \(d > D_p^-\) inside \((-a_-/2, -a_+/2)\), for the geodesic \(\gamma\) from \(h_p^-\) in direction \(d\),
\[
\mathcal{L}(h_p^+, (\gamma(t), t)) \geq \mathcal{L}(h_p^-, (\gamma(t), t)) \quad \text{for all large enough } t,
\]
which implies \(l_p^-(t) \leq \gamma(t)\). It follows that \(\lim \sup_t l_p^-(t)/t \leq D_p^-\).

**Corollary 4.3.** Let \(h_0\) be a sub-wedge initial conditions with an interior reference point \(p\). Then,
\[
\begin{align*}
D_p^- \overset{\text{law}}{=} & -\frac{1}{2} \partial_1 \text{cma}(h_0 + B)(p) \\
D_p^+ \overset{\text{law}}{=} & -\frac{1}{2} \partial_r \text{cma}(h_0 + B)(p)
\end{align*}
\]
where \(B\) is a two-sided Brownian motion of diffusivity \(\sqrt{2}\).

**Proof.** For rational values of \(d\), it follows that if \(d < D_p^-\) then \(W_d(h_p^+) < W_d(h_p^-)\), which means all geodesics from \(h_0\) start strictly from the left of \(p\). So \(h_0(y) + W_d(y, 0)\) has all maximizers to the left of \(p\), which is equivalent to \(\partial_1 \text{cma}(h_0 + W_d)(p) < 0\). Since \(W_d\) equals \(B(x) + 2dx\) in law, \(\Pr(\partial_1 \text{cma}(h_0 + W_d)(p) < 0) = \Pr(\partial_1 \text{cma}(h_0 + B)(p) < -2d)\). So \(\Pr(D_p^- > d) = \Pr(\partial_1 \text{cma}(h_0 + B)(p) < -2d)\). Taking complements gives
\[
\Pr\left(-\frac{1}{2} \partial_1 \text{cma}(h_0 + B)(p) \leq d\right) \leq \Pr(D_p^- \leq d).
\]

On the other hand, \(D_p^- < d\) implies \(W_d(h_p^+) \geq W_d(h_p^-)\). The latter is equivalent to \(\partial_1 \text{cma}(h_0 + W_d)(p) \geq 0\). The probability of this event is \(\Pr(\partial_1 \text{cma}(h_0 + B)(p) \geq -2d)\). Therefore, \(\Pr(D_p^- < d) \leq \Pr(\partial_1 \text{cma}(h_0 + B)(p) \geq -2d)\). Combined with the bound
above, this gives
\[ \Pr(D_p^- < d) \leq \Pr \left( -\frac{1}{2} \partial_t \text{cmaj}(h_0 + B)(p) \leq d \right) \leq \Pr(D_p^- \leq d). \]

As this holds of all rational values of \(d\), the two distribution functions are equal. The law of \(D_p^+\) is derived in the same way. ■

The asymptotic direction of interfaces can often be computed via Corollary 4.3. This was the case for the two narrow wedges initial condition in §4.5. Three other examples are discussed below, the second of which proves the second half of Theorem 1.1.

**One-sided Brownian initial condition** Suppose \(h_0\) is a one-sided Brownian motion with diffusivity \(\sqrt{2\sigma}\) for \(\sigma \geq 0\) and equals \(-\infty\) for \(x < 0\). Every reference point \(p > 0\) has a uniquely defined interface with a random asymptotic direction \(D_p\). The law of \(D_p\) is
\[ \sqrt{(\sigma^2 + 1)/2} \text{cmaj}(B_+)'(p), \]
where \(B_+\) is a one-sided standard Brownian motion.

It is shown in [27, Corollary 2.2] that the random variable \(K = \text{cmaj}(B_+)'(1)\) exists with probability one and has the probability density
\[ \Pr(K \in dx) = 4\Pr(N \in dx) - 4x\Pr(N > x)dx, \quad x \geq 0, \]
where \(N\) is a standard normal variable. Also, \(K\) has the law of the product of a standard \(\chi_5\) random variable and an independent beta\((1, 3)\) random variable \(\beta_1\); see the example below. By Brownian scaling, for \(p > 0\),
\[ D_p \overset{\text{law}}{=} -\sqrt{\frac{\sigma^2 + 1}{2p}} \chi_5 \beta_1. \]

**Flat initial condition on an interval** When \(h_0\) is 0 on \([0, 1]\) and \(-\infty\) elsewhere, and \(p \in (0, 1)\), the quantity in question is the slope at \(p\) of the convex majorant of Brownian motion on \([0, 1]\). To compute the distribution, we use an argument suggested by Jim Pitman based on the results in [37]. First write Brownian motion on \([0, 1]\) as \(Nt + R(t)\), where \(R(t)\) is a Brownian bridge and \(N\) is an independent standard normal.

Next, Doob’s transformation \(f \mapsto g, g(u) = (1 - u)f(u/(1 - u))\), maps the line with slope \(a\) and intercept \(b\) to the line with slope \(a - b\) and intercept \(b\). Also, it maps the standard Brownian motion to a standard Brownian bridge. As a result,
\[ -\sqrt{2D_p} \overset{\text{law}}{=} N + K_t - I_t \overset{\text{law}}{=} N + K_1/\sqrt{t} - I_1\sqrt{t}, \quad t = p/(1 - p), \]
where \(K_t, I_t\) are the slope and intercept of the tangent line at time \(t\) in the concave majo-
rant of standard Brownian motion. The law of \((K_1, I_1)\) is is given in Ouaki and Pitman [37] (see Proposition 1.2, density \(f_3\)). If the vector \((\beta_1, \beta_2, \beta_3)\) are chosen from Dirichlet(1, 1, 2) distribution, then for an independent \(\chi_5\)-random variable, \((K_1, I_1) \overset{law}{=} (\beta_1, \beta_2) \chi_5\). So

\[
D_p \overset{law}{=} \frac{p\beta_2 - (1 - p)\beta_1}{\sqrt{2(1 - p)p}} \chi_5 - \frac{N}{\sqrt{2}}.
\]

**Brownian minus a wedge initial condition** Take \(h_0(x) = \sigma B(x) - \mu |x|\) where \(\sigma \geq 0, \mu > 0\) and \(B(x)\) is a two-sided standard Brownian motion. Then the interface from \(p = 0\) has direction

\[
D_0 \overset{law}{=} \text{Uniform}[\mu/2, \mu/2].
\]

This is based on the fact that if \(V_{\sigma, \mu}(x) = \sigma B(x) - \mu |x|\), then for every \(\sigma, \mu > 0\),

\[
\text{cmaj}(V_{\sigma, \mu})'(0) \overset{law}{=} \text{Uniform}[-\mu, \mu]. \tag{4.13}
\]

To prove this, note that the law of the derivative does not depend on \(\sigma\). This is because \(\text{cmaj}(f)'(0)\) is unchanged under the transformation \(f \mapsto \lambda f(x/\lambda)\) for any \(\lambda > 0\), and this takes the law of \(V_{\sigma, \mu}\) to the law of \(V_{\sqrt{\lambda} \sigma, \mu}\). Thus we may assume \(\sigma = 1\).

For any function \(V\) with a concave majorant, \(\text{cmaj}(V)'(0) = \lim_{\alpha \to \infty} \text{cmaj}(V|_{[-\alpha, \infty)})'(0)\). Consider the transformation \(V \mapsto \tilde{V}\) where

\[
\tilde{V}(x) = \frac{V((x - 1)\alpha) - V(-\alpha)}{\sqrt{\alpha}} + \mu \sqrt{\alpha}x \quad \text{for} \quad x \geq 0.
\]

This takes the law of \(V_{i, \mu}\) restricted to \([a, \infty)\) to the law of \(B_{2\mu\sqrt{\alpha}}\), where \(B_{\nu}(x) = B_0(x) + \nu(\min\{x, 1\})\) for \(x \geq 0\) and \(B_0(x)\) is a one-sided standard Brownian motion. As \(\text{cmaj}(V)'(0) = \frac{1}{\sqrt{\alpha}} \text{cmaj}(\tilde{V})'(1) - \mu\), it is enough to show that as \(\nu \to \infty\),

\[
\frac{1}{\nu} \text{cmaj}(B_{\nu})'(1) \overset{law}{\to} \text{Uniform}[0, 1]. \tag{4.14}
\]

Let \(K_\nu = \text{cmaj}(B_{\nu})(1)\). Then \(K_\nu - B_{\nu}(1) \geq 0\) and it is non-increasing in \(\nu\). This is because \(K_\nu - B_{\nu}(1) = \text{cmaj}(B_{\nu} - B_{0}(1) - \nu)(1)\) and \(B_{\nu}(\cdot) - B_{0}(1) - \nu\) is non-increasing in \(\nu\). Since \(B_{\nu}(1)/\nu \to 1\) almost surely, it follows that \(K_\nu/\nu \to 1\) as well.

When \(\nu = 0\), the results of Ouaki and Pitman [37] (Proposition 1.2, density \(f_3\)) imply that conditionally on \((K_\nu, B_{\nu}(1))\) the law of \(\text{cmaj}(B_{\nu})'(1)\) is uniform on \([0, K_\nu]\). Since \(B_{\nu}\) has the law of \(B_0\) biased by \(e^{\nu B_0(1) - \nu^2/2}\), the same property holds for all \(\nu\), and (4.14) follows from \(K_\nu/\nu \to 1\).
4.9 Interfaces are locally absolutely continuous with respect to geodesics

This section proves Theorem 1.3. A theorem about exponential last passage percolation states that the infinite geodesic in point-to-point last passage percolation has the same law as the competition interface in stationary last passage percolation [1, 47]. This suggests a likeliness between geodesics and interfaces in the limit, which we formalize in terms of absolute continuity.

**Definition 4.4.** Let I = (I(t), t ≥ s) be a random process. Let g be the (almost surely unique) infinite geodesic of L from (0, s) in direction zero. The process I is **geodesic-like on compacts** if for every compact interval [a, b] ⊂ (s, ∞), the process I|_{[a,b]} is absolutely continuous with respect to g|_{[a,b]}.

We re-state Theorem 1.3 using the terminology established.

**Theorem 4.3.** Let h0 be an initial condition. Suppose p is an interior reference point for h0 which is non-polar, so there is a uniquely defined interface I_p(t) for t ≥ 0 from reference point p. The process I_p(t) − p is geodesic-like on compacts.

The theorem will be proved in several steps through a sequence of lemmas.

**Lemma 4.2.** Let I_0(t) be the uniquely defined interface from reference point 0 associated to the stationary Brownian initial condition. The process I_0 has the same law as the infinite geodesic of L from (0, 0) with direction zero.

**Proof.** In order to prove this result we shall consider exponential last passage percolation, whose setting and terminology are described in the Introduction and in §6.3 below.

Consider point-to-point last passage percolation on \( \mathbb{Z}^2_{even} \) with i.i.d. exponential weights. Let \( L(a, b) \) be the last passage time from point \( a \in \mathbb{Z}^2_{even} \) to \( b \in \mathbb{Z}^2_{even} \) as defined by (1.1). Thus \( L(a, b) \) is the maximal sum of weights over all directed paths from \( a \) to \( b \). A geodesic is a directed path that achieves the last passage time between its endpoints.

In this setting there is an almost surely unique infinite geodesic \( \Gamma(n) \), for \( n = 0, 1, 2, 3, \ldots \), from (0, 0) in direction zero [47]. This means that \( n \mapsto (n, \Gamma(n)) \) is an infinite directed path that is a geodesic when restricted to any finite interval of times, \( \Gamma(0) = 0 \), and \( \Gamma(n)/n \) tends to 0 as \( n \) goes to infinity.

The second process to consider is the competition interface in stationary last passage percolation. Stationary last passage percolation looks at last passage times from an initial condition \( h_0(x) \) that is a two-sided simple symmetric random walk on \( \mathbb{Z} \). See §6.3 for the setup in detail. The stationary competition interface is a process \( \Gamma'(n) \), for \( n = 0, 1, 2, 3, \ldots \), which is the process \( X_n \) from (6.4) for the stationary initial condition. We note that \( \Gamma'(n) \)
records the position of a 2nd class particle starting from the origin in stationary tasep after \( n \) steps.

A surprising fact is that the geodesic \( \Gamma \) and the competition interface \( \Gamma' \) have the same law as random processes \([47, \text{Proposition 5.2}]\).

We may now consider the KPZ scaling limit of these two processes. Under KPZ scaling, \( \Gamma \) and \( \Gamma' \) are re-scaled to \( t \mapsto T^{-2/3} \Gamma(\lfloor Tt \rfloor) \) and \( t \mapsto T^{-2/3} \Gamma'(\lfloor Tt \rfloor) \) for a large parameter \( T \) and \( t \geq 0 \). As proved in Corollary 6.1, the KPZ re-scaled competition interface converges in law to the uniquely defined interface \( I_0(t) \) associated to the stationary Brownian initial condition. The convergence here is under the topology of uniform convergence on compacts. Note further that \( I_0(t) \) has asymptotic direction zero because Brownian motion is almost surely an asymptotically flat initial condition, so Theorem 4.1 applies.

As a result one also concludes that the KPZ re-scaled geodesic has a tight law and admits distributional limits. Any limit point \( g(t) \) of the re-scaled geodesic must be an infinite geodesic of \( L \) because limits of geodesics in last passage percolation are geodesics of \( L \) \([12] \). The process \( g(t) \) also has direction zero because \( I_0(t) \) does so. Since \( L \) almost surely has a unique infinite geodesic from \((0,0)\) with direction zero (Theorem 3.3), it follows that \( \Gamma(n) \) converges in law to said geodesic in the KPZ scaling limit.

From the equality of laws between \( \Gamma \) and \( \Gamma' \), one concludes that \( I_0(t) \) has the same law as the infinite geodesic of \( L \) from \((0,0)\) with direction zero. \( \blacksquare \)

Let \( h_0 \) be a random initial condition independent of \( L \). We say \( h_0 \) is Brownian on compacts if for every interval \([-n,n]\), the law of \( h_0(x) - h_0(0) \) on \([-n,n]\) is absolutely continuous with respect to that of a two-sided Brownian motion with diffusivity \( \sqrt{2} \). Observe in this case every deterministic \( p \in \mathbb{R} \) is a non-polar point of \( h_0 \) and, thus, the interface \( I_p(t) \) of \( h_0 \) is uniquely defined by Proposition 4.6.

**Lemma 4.3.** Let \( h_0 \) be a random initial condition independent of \( L \) that is Brownian on compacts. For any \( p \in \mathbb{R} \), the process \( I_p(t) - p \) is geodesic-like on compacts.

**Proof.** We may assume without loss of generality that \( p = 0 \). The proof is built up in three stages of increasing complexity, which are:

1. When \( h_0 \) is a two-sided Brownian motion.
2. When \( h_0 \) is a two-sided Brownian motion over \([-n,n]\) for some \( n > 0 \) and \(-\infty\) outside the interval.
3. When \( h_0 \) is simply Brownian on compacts.

For (1) the claim follows from Lemma 4.2 with an equality of laws.
For the proofs of (2) and (3) we note that a probability measure \(\mu\) is absolutely continuous with respect to \(\nu\) if there are events \(A_m\) such that \(\mu(A_m) \to 1\) and \(\mu |_{A_m}\) is absolutely continuous with respect to \(\nu\).

Also, for an initial condition \(h\), denote by \(I_0(t; h)\) the interface from reference point 0 associated to \(h\) (provided it is uniquely defined as it will be in the following).

For the proof of (2), let \(f_m\) be the function that is 0 on \([-n, n]\), equal to \(-m\) on the complement of \([-n-1, n+1]\), and linear on \([-n-1, -n]\) and \([n, n+1]\). Let \(f_\infty\) be the pointwise limit of \(f_m\). Let \(B\) denote a two-sided Brownian motion. Observe that \(h_0 = B + f_\infty\). Let \(h_0^m = B + f_m\).

By the Cameron-Martin Theorem, \(h_0^m\) is absolutely continuous with respect to \(B\). So \(I_0(t; h_0^m)\) is geodesic-like on compacts by (1). Let \(t > 0\). We claim the event
\[
A_m = \{I_0(s; h_0^m) = I_0(s; h_0) \text{ for all } s \leq t\}
\]
holds eventually for all large \(m\). In particular, \(\Pr(A_m) \to 1\). This implies that \(I_0(\cdot, h_0)\) is geodesic-like on every interval \([0, t]\), and since \(t\) is arbitrary, (2) follows.

To see that the events \(A_m\) eventually hold, choose \(b \in \mathbb{R}\). Let \(\gamma_m\) be any geodesic to \((-b, t)\) from \((h_0^m)^-\) and \(\gamma_\infty\) a geodesic to \((-b, t)\) from \((h_0)^-\). By definition,
\[
\gamma_m(0) = \arg\max_{y \leq 0} \{h_0^m(y) + \mathcal{L}(y, 0; -b, t)\}.
\]
Let \(Y = B(-n) + \mathcal{L}(-n, 0; -b, t)\). Since \(-n\) is not a polar point for \(B\), the maximum on the interval \([-n, 0]\) is strictly greater than \(Y\). On the other hand, the maximum on the interval \((-\infty, n)\) converges to \(Y\) as \(m \to \infty\). Thus, for large enough \(m\), \(\gamma_m(0) \in (-n, 0)\), and so \(\gamma_m(0) = \gamma_\infty(0)\). Once this happens, \(\gamma_m = \gamma_\infty\) if the geodesics are unique. This will indeed be the case by the unique geodesic condition as \((-b, t)\) may be chosen to be a UGP for every \((h_0^m)^-\) and \((h_0)^-\).

Note that by definition of the interface, when \(-b < I_0(t, h_0) < b\), \(\gamma_m = \gamma_\infty\), and the analogous condition holds for the geodesics from \((h_0^m)^+\) and \((h_0)^+\) to \((b, t)\), then the event \(A_m\) holds. By choosing \(b\) appropriately and sufficiently large, we find that \(A_m\) holds for all large \(m\).

For the proof of (3), let \(t > 0\) be arbitrary. Let \(b > 0\) and consider the event \(A_b\) that \(I_0(t; h_0) \in (-b, b)\). Now let \(A_{b,n}\) be the intersection of \(A_b\) with the event that all geodesics from \(h_0, h_0^+, h_0^-\) to \((-b, t)\) and \((b, t)\) start from \([-n, n]\). On \(A_{b,n}\),
\[
I_0(s; h_0) = I_0(s; h_0[-n,n]) \text{ for all } s \leq t.
\]
Here \(h_0 \mid_{[-n,n]}\) equals \(h_0\) over \([-n, n]\) and is \(-\infty\) outside.
Note that \( I_0(\cdot; h_0 |_{[−n,n]} ) = I_0(\cdot, h_0 |_{[−n,n]} − h_0(0)) \) since the interface does not change when the initial condition is shifted by a constant. Since \( h_0 |_{[−n,n]} − h_0(0) \) is absolutely continuous with respect to \( B |_{[−n,n]} \), part (2) implies that on the event \( A_{b,n} \) the process \( I_0(\cdot; h_0) \) restricted to \([0,t]\) is geodesic-like. Taking \( b \) and then \( n \) large, we conclude (3).

The next part of the argument requires some new notation. For an initial condition \( h_0 \) and \( s \geq 0 \), denote by \( h_s \) the function \( x \mapsto h(x,s) \), which is \( h_0 \) grown to time \( s \). For \( t > s \), let \( t \mapsto I(s,h_s,p,t) \) denote the interface from reference point \( p \) for the initial condition \( h_s \). We imagine that the interface starts from time \( s \), and \( p \) is assumed to be such that the interface is uniquely defined.

The Markovian evolution of \( s \mapsto h_s \) implies a Markovian evolution of interfaces together with its environment: for any \( s \geq 0 \) it holds for every \( t > s \) that

\[
I(0,h_0,p,t) = I(s,h_s,p_s,t) \quad \text{with} \quad p_s = I(0,h_0,p,s).
\]

This follows from the observation that a geodesic to \((x,t)\) from \( h_s \) is the restriction of a geodesic to \((x,t)\) from \( h_0 \) to times \( u \in [s,t] \).

**Lemma 4.4.** Let \( p \) be a non-polar point of \( h_0 \) which is also an interior reference point. So then \( I(0,h_0,p,t) \) is uniquely defined. The following condition holds almost surely. For every \( t > 0 \) there is an \( s_0 \) with \( 0 < s_0 < t \) such that for any \( s \leq s_0 \),

\[
I(0,h_0,p,t') = I(s,h_s,p,t') \quad \text{for all} \quad t' \geq t.
\]

**Proof.** We may assume that \( p = 0 \). By the evolution of interfaces, it suffices to show the identity for \( t' = t \). For all \( s \in (0,t) \),

\[
z := I(0;h,0;t) = I(s;h_s,p_s;t), \quad p_s = I(0;h,0;s)
\]

The right hand side, by the definition of interface, means that

\[
\mathcal{L}(h_s|_{(-\infty,p_s]},s;z,t) = \mathcal{L}(h_s|_{[p_s,\infty)},s;z,t) = \mathcal{L}(h_s|_{\mathbb{R},s};z,t)
\]

where

\[
\mathcal{L}(h|_I,s;z,t) = \sup_{y \in I} \{h(y) + \mathcal{L}(y,s;z,t)\}
\]

Let \((a_s,s)\) be the starting point of the leftmost geodesic to \((z,t)\) from \( h_s \) achieving the distance, or supremum, in (4.16). Similarly, let \((b_s,s)\) be the starting point of the rightmost geodesic achieving the distance. Due to the metric composition property of \( \mathcal{L} \), the map \( s \mapsto a_s \) is the leftmost geodesic from \( h_0 \) to \((z,t)\). Likewise, \( s \mapsto b_s \) is the rightmost such geodesic.
Since 0 is not a polar point of $h_0$, $a_0 < 0 < b_0$ almost surely. Therefore, by continuity of geodesics, there is an $s_0 > 0$ such that $a_s \leq 0 \leq b_s$ for all $s \in [0, s_0]$. Consider (4.16) for $s \in (0, s_0]$.

Suppose $p_s \geq 0$. Then, since $a_s \leq 0$,

$$L(h_s|_{(-\infty, p_s]}, s; z, t) = L(h_s|_{(-\infty, 0]}, s; z, t).$$

On the other hand, since $p_s \geq 0$,

$$L(h_s|_{[p_s, \infty)}, s; z, t) \leq L(h_s|_{[0, \infty)}, s; z, t) \leq L(h_s; z, t) = L(h_s|_{[p_s, \infty)}, s; z, t).$$

So,

$$L(h_s|_{(-\infty, 0]}, s; z, t) = L(h_s|_{[0, \infty)}, s; z, t).$$

which, by definition, means that $z = I(s; h_s, 0; t)$ as required. A symmetric argument applies when $p_s \leq 0$ because $b_s \geq 0$.

We can now complete the proof of Theorem 4.3.

**Proof.** We may assume $p = 0$ without loss of generality. Let $[a, b] \subset (0, \infty)$ be a compact interval. By Lemma 4.4 there is an $s \in (0, a)$ such that $I(0, h_0, 0, t) = I(s, h_s, 0, t)$ for $t \in [a, b]$. By Lemma 4.3, the process $t \mapsto I(s, h_s, 0, t)$ over $[a, b]$ is absolutely continuous with respect to the geodesic of $L$ from $(0, s)$ in direction zero. This geodesic has the law of $t \mapsto g(t - s)$ for $t \geq s$, where $g$ is the geodesic of $L$ from $(0, 0)$ in direction zero. Now $g(t - s)$ is absolutely continuous over $t \in [a, b]$ with respect to $g(t)$ over $[a, b]$ by the absolute continuity of geodesics shown in Proposition 3.4. Consequently, the process $I(0, h_0, 0, t)$ is absolutely continuous over $[a, b]$ with respect to $g$.

**4.10 Questions**

**No polar points for Brownian motion**  Show that Brownian motion almost surely has an empty polar set.

**Law of an interface**  Compute the distribution function of $I_0(1)$ for various initial conditions $h_0$. This is interesting because interfaces are also the scaling limit of 2nd class particles in tasep.

**Likelihood between interfaces and geodesics**  Suppose for an initial condition $h_0$ the interface $I_0(t)$ from reference point 0 is uniquely defined. Is there a distributional identity between $I_0(t)$ and the infinite geodesic of $L$ from $(0, 0)$ in direction zero? Such an identity should be stronger than the absolute continuity result in Theorem 4.3.
5 A portrait of interfaces

The interface portrait of an initial condition $h_0$ is all points lying on the finite interfaces associated to $h_0$:

$$I = \{(x, t) \in H : x = I^-_p(t) \text{ or } I^+_p(t) \text{ for some reference point } p\}.$$ 

Figure 2: An interface portrait.

This is the family of all finite interfaces, from all reference points, associated to $h_0$. It is an intriguing geometric object, see Figure 2. Interfaces lie between geodesics, and the interface portrait is complementary to the forest of geodesics emanating from $h_0$. The topic of this section are the topological and geometric properties of the portrait.

Fix an initial condition $h_0$ and assume good samples of $\mathcal{L}$ as in §3.4.

5.1 Ordering of interfaces

Lemma 5.1. If $p < q$ then $I^+_p(t) \leq I^-_q(t)$ for every $t$. So interfaces to not cross.

Proof. Any point $x$ that satisfies $I^-_q(t) \leq x \leq I^+_p(t)$ for some $t$ has a geodesic from $h_0$ to $(x, t)$ that starts at $\leq p$ and another that starts at $\geq q$. So $(x, t)$ is not a UGP. By the UGC property there can not be an interval of such points $x$, and so $I^+_p(t) \leq I^-_q(t)$. ■

54
Lemma 5.2. Suppose the interface \( I_p^-(t) \) is finite. Then there are reference points \( p_n \to p \) from the left such that \( I_{p_n}^+(t) \to I_p^-(t) \) for every \( t \). Similarly, a finite right interface can be approximated from the right by left interfaces.

Proof. Since \( I_p^- (t) \) is finite there are points \( a < p < b \) such that \( h_0(a) \) and \( h_0(b) \) are finite, according to the finiteness criterion in Proposition 4.2. For points \( q \in (a, p) \) the right interface \( I_q^+(t) \) is finite by the finiteness criterion. Moreover, by the previous lemma, \( I_q^+(t) \) is non-decreasing in \( q \) and bounded by \( I_p^- (t) \); so it has a limit \( t(t) \) as \( q \to p \) from the left. If \( t(t) < I_p^-(t) \) then consider a point \( x \) between them. Any geodesic to \( (x, t) \) must begin strictly to the left of \( p \) and simultaneously to the right of every \( q < p \). Such is impossible, so \( t(t) = I_q^+(t) \) as required. ■

5.2 Topological view of the portrait

For a reference for \( p \), consider the set

\[
G_p = \{(x, t) \in \mathbb{H} : \text{there is a geodesic from } h_0 \text{ to } (x, t) \text{ starting at } p \}.
\]

The set \( G_p \) lies inside the zero set \( \{d_p = 0\} \) of the competition function \( d_p \). Like the zero set, \( G_p \) is connected and for every \( t > 0 \) the intersection of \( G_p \) with \( \mathbb{R} \times \{t\} \) is an interval (possibly empty or infinite).

Lemma 5.3. The set \( G_p \) has the same interior as the zero set \( \{d_p = 0\} \). All geodesics to interior points of \( G_p \) start from \( p \). The interfaces \( I_p^\pm(t) \) are thus given by \( \{d_p = 0\} \setminus \text{Int}(G_p) \). Moreover, if \( p \neq q \) then \( \{d_p = 0\} \) and \( \{d_q = 0\} \) (as well of \( G_p \) and \( G_q \)) have disjoint interiors and can only intersect along their boundaries.

Proof. As \( G_p \) lies inside the zero set \( \{d_p = 0\} \), \( \text{Int}(G_p) \subseteq \text{Int}(\{d_p = 0\}) \). Suppose \( (x, t) \) is an interior point of the zero set and consider any geodesic \( g \) from \( h_0 \) to \( (x, t) \), starting at \( g(0) \). We will prove that \( g(0) = p \), which shows that \( \text{Int}(\{d_p = 0\}) \subseteq \text{Int}(G_p) \) and all geodesics to \( (x, t) \) start at \( p \).

Notice that a UGP in the zero set of \( d_p \) must have its unique geodesic ending at \( p \). By the UGC condition, since \( (x, t) \) is an interior point, there are UGPs \( (x', t) \) and \( (x'', t) \) that lie in the zero set and for which \( x' \leq x \leq x'' \). The unique geodesic from \( (x', t) \) has to stay to the left of \( g \), so that \( p \leq g(0) \). Similarly, the geodesic from \( (x'', t) \) stays to the right of \( g \), so \( g(0) \geq p \) as well.

Finally, if \( p \neq q \) and a point lies in both \( \{d_p = 0\} \) and \( \{d_q = 0\} \) then it cannot be a UGP. Due to the UGC condition, for every \( t > 0 \), the intervals \( \{d_p = 0\} \cap (\mathbb{R} \times \{t\}) \) and \( \{d_q = 0\} \cap (\mathbb{R} \times \{t\}) \) can intersect in at most one point. Therefore the sets \( \{d_p = 0\} \) and \( \{d_q = 0\} \) can only intersect along their boundaries and have disjoint interiors. ■
Proposition 5.1. The interface portrait is a closed subset of $H$. It is also the closure in $H$ of the set

$$\{(x,t) \in H : x = 1_p^+(t) \text{ for some reference point } p\}$$

of points on finite right interfaces or, alternatively, the closure of points on finite left interfaces.

Proof. Let $U$ be the union of the interior of the sets $\{d_p = 0\}$ over all reference points $p$ of $h_0$. This is an open set and we claim that its complement in $H$ is the interface portrait $I$, which is then closed.

If a point $(x,t)$ belongs to $I$ then it is at the boundary of some zero set $\{d_p = 0\}$. So $(x,t)$ cannot lie in the interior of any zero set $\{d_q = 0\}$ by Lemma 5.3. Therefore $I$ lies inside the complement of $U$.

Now suppose a point $(x,t)$ is not in $U$, which by Lemma 5.3 means that it does not lie in the interior of any $G_p$. The union of the sets $G_p$ over all reference points $p$ is the entirety of $H$. So there is some $p$ such that $(x,t) \in G_p \setminus \text{Int}(G_p)$. But then $(x,t)$ lies at the boundary of $\{d_p = 0\}$ because $G_p \subset \{d_p = 0\}$ and they have the same interior. So the complement of $U$ belongs to $I$.

The fact that $I$ is the closure of all the finite right or left interfaces comes from Lemma 5.2. ■

Recall the function $e$ from (3.2) that shows where geodesics emanate. It describes the interface portrait.

Proposition 5.2. Call a point $(x,t)$ a discontinuity of the function $e$ if $x$ is a discontinuity of $e(\cdot,t)$. The interface portrait $I$ is the set of discontinuities of $e$. Thus, $I \cap (\mathbb{R} \times \{t\})$ is a discrete set for every $t > 0$.

Proof. Let $e_t(x) = e(x,t)$ for a given $t$. Recall properties of $e$ from Proposition 3.1.

If $x$ is not a discontinuity of $e_t$ then $e_t$ is constant on $(x-\delta, x+\delta)$ for some $\delta > 0$ because $e_t$ is a step function with a discrete set of discontinuities. Let $p$ denote this constant. So the interval $(x-\delta, x+\delta) \times \{t\}$ lies in the zero set $\{d_p = 0\}$. It is easy to see from this and continuity of the interfaces $I_p^\pm$ that the point $(x,t)$ then lies in the interior of $\{d_p = 0\}$. So it cannot lie on any interface by Lemma 5.3.

Now suppose that $x$ is a discontinuity of $e_t$, meaning that the leftmost geodesic from $(x,t)$ ends at some point $a$ and the rightmost one ends at $b > a$. Let $p$ be a point between $a$ and $b$. It is a reference point because $h_0(a)$ and $h_0(b)$ are finite. Clearly $d_p(x,t) = 0$. By Lemma 5.3 the point $(x,t)$ does not belong to the interior of $\{d_p = 0\}$ because not all geodesics from it ends at $p$. So it belongs to the boundary of $\{d_p = 0\}$ and lies on an interface.
Discreteness of $I \cap (\mathbb{R} \times \{t\})$ comes from discreteness of the discontinuities of $e_t$, see Proposition 3.1.

5.3 Geometric view of the portrait

Geometrically, the interface portrait makes a forest.

Theorem 5.1 (Interface portrait is a forest). The interface portrait $I$ is a forest in $\mathbb{H}$ in the following sense: whenever two different interfaces meet at a positive time, they coalesce upward from then onwards. Moreover, geodesics do not cross interfaces. The only geodesics that meet $I$ are those from points in $I$.

The theorem follows from a combination of the two lemmas below.

Lemma 5.4. If $(x, t) \not\in I$ then every geodesic from it lies outside $I$. If $(x, t) \in I$ then a geodesic from $(x, t)$ can follow an interface on which it belongs for a while and, if the geodesic ventures off, it doesn’t meet $I$ again. Every $(x, t) \in I$ also has geodesics from it that lie (weakly) to the left and to the right of any interface on which it belongs.

Proof. No UGP can lie on $I$ because every point on an interface has a different leftmost and rightmost geodesic by Proposition 5.2. So the geodesic from a UGP won’t meet any interface either. Now suppose $(x, t) \not\in I$. Then there are UGPs $(x_-, t)$ and $(x_+, t)$ that are not in $I$ and for which $x_- \leq x \leq x_+$. This is because of the UGC condition and the complement of $I$ being open by Proposition 5.1. Any geodesic from $(x, t)$ is confined by the geodesics from $(x_-, t)$ and $(x_+, t)$ by the geodesic ordering property. So it cannot meet $I$ either.

If $(x, t)$ lies on an interface $I$ then any geodesic from it can stay on $I$ and if it ever leaves $I$ then it won’t meet $I$ again due to the previous assertion. By approaching $(x, t)$ with UGPs from the left and the right, and using geodesic compactness, it follows that there are geodesics from $(x, t)$ that stay to the left and to the right of $I$.

Lemma 5.5. If two different interfaces meet at a positive time then they coalesce upwards from then onwards.

Proof. Let $I$ and $J$ be finite interfaces (left or right). Suppose they meet at some point $(x, t) \in \mathbb{H}$, that is $I(t) = J(t) = x$. Now if $I$ and $J$ do not coalesce from time $t$ onward then there is a UGP not in $I$ lying between them after time $t$, by the UGC condition and Proposition 5.2. Its unique geodesic cannot meet $I$ or $J$ by Lemma 5.4, but since $I$ and $J$ meet at time $t$ and the geodesic ends between them after time $t$, it must encounter a point on $I$ or $J$ as it goes down to the initial condition. The contradiction means $I$ and $J$ do coalesce.
5.4 Questions

Geometry of interface portraits Find a criterion to determine when an interface portrait is a tree. For instance, prove that the flat (when $h_0 \equiv 0$) and stationary (when $h_0$ is Brownian motion) interface portraits are trees.

Duality between interfaces and geodesics The interface portrait of the stationary initial condition (Brownian motion with diffusivity $\sqrt{2}$) is scale and translation invariant. Study the distributional and geometric properties of the stationary portrait in detail. In particular, show that it has the same law at the geodesic tree from Theorem 3.4.

6 The 2nd class particle

In this section we derive the scaling limit of the 2nd class particle in tasep – Theorem 1.4. As mentioned in the Introduction, we prove a more general result, Theorem 6.1, by relaxing the mode of convergence in (1.5). There are a couple of steps to the proof. First, we model the 2nd class particle in terms of a hole–particle pair in tasep, and map its trajectory to a competition interface in last passage percolation. This mapping is well known and has been utilized for instance in [7, 21, 18, 19]. Next, we establish the scaling limit of the competition interface by using the fact that exponential last passage percolation scales to the directed landscape [12]. Together, this leads to the scaling limit of the 2nd class particle.

6.1 Statement of the limit theorem

Recalling back to the the Introduction, consider tasep with a 2nd class particle at the origin and an initial condition $X_0(\cdot)$ of regular particles. Encode the initial condition as a height function $h_0 : \mathbb{Z} \to \mathbb{Z}$ according to

$$h_0(0) = 0 \quad \text{and} \quad h_0(x + 1) - h_0(x) = \begin{cases} 1 & \text{if initially there is a hole as site } x \\ -1 & \text{if initially there is a particle as site } x \end{cases}$$  (6.1)

The 2nd class particle is treated as a particle in the above.

Let $h_0^\varepsilon(x)$ be a sequence of initial height functions for $\varepsilon > 0$. Extend them to $x \in \mathbb{R}$ by linear interpolation. The sequence converges if

$$-\varepsilon^{1/2}h_0^\varepsilon(2\varepsilon^{-1}x) \to h_0(x) \quad \text{as } \varepsilon \to 0.$$  (6.2)

The sense in which this limit should hold is as follows, called hypograph convergence. Assume $h_0(x)$ is an initial condition for the directed landscape, namely an upper semi-
continuous function with values in $\mathbb{R} \cup \{-\infty\}$ and finite somewhere. The convergence as $\varepsilon \to 0$ should be in the natural sense for upper semicontinuous functions: or every $x \in \mathbb{R}$ and sequence $x^\varepsilon \to x$,

$$\limsup_{\varepsilon \to 0} -\varepsilon^{1/2} h_0^\varepsilon(2\varepsilon^{-1} x^\varepsilon) \leq h_0(x),$$

and there exists some sequence $x^\varepsilon \to x$ such that

$$\liminf_{\varepsilon \to 0} -\varepsilon^{1/2} h_0^\varepsilon(2\varepsilon^{-1} x^\varepsilon) \geq h_0(x).$$

Assume also that there is a constant $c$ such that for every $\varepsilon$ and $x$,

$$-\varepsilon^{1/2} h_0^\varepsilon(2\varepsilon^{-1} x) \leq c(1 + |x|).$$

**Theorem 6.1** (Limit of the 2nd class particle). Consider tasep with a sequence of initial condition $X_0^\varepsilon$ containing a 2nd class particle at the origin. Let $X^\varepsilon(t)$ be the position of the 2nd class particle at time $t$. Let $h_0^\varepsilon$ denote the corresponding initial height functions given by (6.1).

Suppose $h_0^\varepsilon$ converges to $h_0$ as in (6.2). Furthermore, suppose the interface $I_0(t)$ of the directed landscape from reference point $p = 0$ for the initial condition $h_0$ is uniquely defined (see Definition 4.2 and §4.6 for such a criterion).

Then the re-scaled trajectory of the 2nd class particle

$$X^\varepsilon(t) = (\varepsilon/2)X^\varepsilon(\varepsilon^{-3/2} t)$$

(6.3)

converges to $I_0(t/2)$ in law, as $\varepsilon \to 0$, uniformly in $t$ over compact subsets of $(0, \infty)$.

The convergence of $X^\varepsilon(t)$ to $I_0(t/2)$ may not extend to $t = 0$. The 2nd class particle can move a positive KPZ-distance in zero KPZ-time. For example, consider the initial condition $h_0(x)$ with narrow wedges at $x = -1$ and $x = 1$ only. The interface from reference point $p = 1/2$, say, begins at $x = 0$ at time zero. In tasep terms this means that when the initial conditions $h_0^\varepsilon$ approximate $h_0$, a 2nd class particle started from site $x = \varepsilon^{-1}/2$ moves to a site $x = o(\varepsilon^{-1})$ in time scales of order $o(\varepsilon^{-3/2})$.

**6.2 The 2nd class particle as a hole - particle pair**

Consider tasep with some initial configuration of particles but with no 2nd class particle. Let there be a particle at site 0, labelled 0, and a hole at site $-1$ with label 0, too. Then label the particles right to left and the holes left to right, as in Figure 3. Write $X_t(n)$ for the position of particle number $n$ at time $t$; for example, $X_0(1) = -3$ in Figure 3.

Consider the pair hole 0 – particle 0 that are initially adjacent at sites $-1$ and 0. Mark it as a pair like so: $\circ \rightarrow \bullet$. This pair remains intact under the dynamics of tasep and moves
Figure 3: Snapshot of a tasep initial configuration. Particles are labelled left to right with particle 0 at site 0; holes are labelled right to left with hole 0 at site −1.

left or right together as shown in Figure 4. Note although the label of the particle or hole in the pair changes, the pair itself is always intact.

Figure 4: How the \( \circ - \bullet \) pair moves in tasep. In (i) it moves right and in (ii) it moves left. It does not move otherwise.

The \( \circ - \bullet \) pair moves according to the same rules as a 2nd class particle! Let \( X(t) \) be the position of the particle in the \( \circ - \bullet \) pair at time \( t \). Consider, at each time, the configuration obtained by removing the hole in the pair and then sliding all particles and holes to its left by one unit rightward. Mark the surviving particle from the \( \circ - \bullet \) pair. The following observation is due to Ferrari, Martin and Pimentel; the lemma is from [18, Lemma 6].

**Lemma 6.1.** The new system (after removing, sliding and marking) evolves like tasep with a 2nd class particle, the 2nd class particle being the marked one. So \( X(t) \) is the location of the 2nd class particle at time \( t \).

In order to analyse tasep with a 2nd class particle, it is therefore enough to follow the \( \circ - \bullet \) pair in the original system, and this is what we will do. Let \( X_n \) be the position of the particle in the \( \circ - \bullet \) pair after it has taken \( n \) steps:

\[
X_n = \text{position of the particle in the } \circ - \bullet \text{ pair after its taken } n \text{ steps.} \quad (6.4)
\]

Equivalently, by Lemma 6.1, \( X_n \) is the position of the 2nd class particle after \( n \) steps. Clearly, \( X_0 = 0 \) and \( X_{n+1} - X_n \in \{\pm 1\} \). The process \( X(t) \) is a time change of \( X_n \). Let \( \tau_n \) be the time when the \( \circ - \bullet \) pair takes its \( n \)-th step. Then, with \( \tau_0 = 0 \),

\[
X(t) = X_n \quad \text{for } t \in [\tau_n, \tau_{n+1}). \quad (6.5)
\]
We will describe the evolution of $X_n$ in terms of a competition interface in last passage percolation and use (6.5) to get the scaling limit of the 2nd class particle.

### 6.2.1 Height functions

Consider the encoding (6.1) of an initial condition $X_0$ of tasep as a height function $h_0$. When $X_0$ contains a 2nd class particle at the origin, map it to an initial condition $\hat{X}_0$ with a $\circ - \bullet$ pair at $-1 - 0$ as follows. Remove the 2nd class particle, slide all particles and holes of $X_0$ below the origin by one unit leftward, and insert a $\circ - \bullet$ pair at $-1 - 0$. Let $\hat{h}_0$ be the height function associated to $\hat{X}_0$. The relation between $h_0$ and $\hat{h}_0$ is

$$
\hat{h}_0(x) = \begin{cases} 
  h_0(x) & \text{if } x \geq 0 \\
  h_0(x + 1) - 1 & \text{if } x < 0
\end{cases}
$$

When a sequence of initial height functions $h_0^\varepsilon$ of tasep converges to $h_0$ according to (6.2), the corresponding functions $\hat{h}_0^\varepsilon$ converge to $h_0$ under the same scaling. Therefore, in order to prove Theorem 6.1 by way of Lemma 6.1, it is enough to consider tasep with initial conditions that have a $\circ - \bullet$ pair instead of a 2nd class particle, assume the initial height functions converge according to (6.2), and prove that the trajectory of the $\circ - \bullet$ pair converges to the interface under the scaling (6.3).

The initial height function $h_0(x)$ is a simple random walk path. Such a path has a peak at site $x$ if $h_0(x \pm 1) = h_0(x) - 1$, and a valley at $x$ if $h_0(x \pm 1) = h_0(x) + 1$. Having a $\circ - \bullet$ pair at sites $-1 - 0$ corresponds to $h_0$ having a peak at $x = 0$.

The evolution of tasep defines a growing family of height functions $h_t(x)$ for $t \geq 0$ that are simple random walk paths. Starting from $h_0(x)$, valleys simply turn to peaks at exponential rate 1 and $h_t(x)$ is the state of the function at time $t$. The trajectory of the $\circ - \bullet$ pair is that of the initial peak at $x = 0$, so the 2nd class particle has the same evolution as a peak of $h_0$.

### 6.3 Mapping the 2nd class particle to a competition interface

The previous section explained how tasep with a 2nd class particle can be identified as tasep with a marked $\circ - \bullet$ pair. We now explain how the trajectory of the $\circ - \bullet$ pair can be mapped to a competition interface.
6.3.1 Last Passage Percolation

From the evolution of tasep define the function

\[ G(i,j) = \text{time when hole } i \text{ interchanges position with particle } j. \]

Set \( G(i,j) = 0 \) if hole \( i \) is initially left of particle \( j \), with the conventions that position(hole \( i \)) = \(-\infty\) if there is no hole \( i \) and position(particle \( j \)) = \( \infty \) if there is no particle \( j \). Recall particles are labelled right to left and holes left to right, and particle 0 is initially at site 0 and hole 0 is initially at site \(-1\).

Function \( G \) is the model of last passage percolation. This mapping between tasep and last passage percolation goes back to Rost [44]. Consider the set

\[ \Lambda = \{(i,j) \in \mathbb{Z}^2 : G(i,j) = 0 \text{ and } G(i+1,j+1) > 0 \}. \]

\( \Lambda \) is a down/right path in \( \mathbb{Z}^2 \) which is determined by, and in one to one correspondence with, the initial configuration of tasep. Let \( \omega_{i,j} \) be independent exponential random variables with mean 1 for \( i,j \in \mathbb{Z} \). Then \( G(i,j) \) is determined by

\[ G(i,j) = \max\{ G(i-1,j), G(i,j-1) \} + \omega_{i,j} \quad (6.6) \]

with boundary conditions \( G(i,j) = 0 \) on \( \Lambda \). Indeed, in order for particle \( j \) and hole \( i \) to swap positions they must first be adjacent, which happens at time \( \max\{G(i-1,j), G(i,j-1)\} \). Then they wait an independent exponential unit of time before swapping, which accounts for the plus \( \omega_{i,j} \).

Now change coordinates in last passage percolation from \((i,j) \in \mathbb{Z}^2\) to

\[ x = i - j \quad \text{and} \quad y = i + j. \]

This takes \( \mathbb{Z}^2 \) to

\[ \mathbb{Z}^2_{\text{even}} = \{(x,y) \in \mathbb{Z}^2 : x + y \text{ is even}\}. \]

Define

\[ g(x,y) = G((x+y)/2, (y-x)/2) \quad \text{for } (x,y) \in \mathbb{Z}^2_{\text{even}}. \]

(6.7)

The boundary \( \Lambda \) is mapped to the graph of the initial height function \( h_0(x) \) defined by (6.1). Unwrapping the recursion (6.6) shows that, above the graph of \( h_0 \), \( g \) satisfies

\[ g(x,y) = \max_{\pi} \sum_{(x',y') \in \pi} \omega_{x',y'} \quad (6.8) \]

where \( \pi \) is any upward directed path in \( \mathbb{Z}^2_{\text{even}} \) from the graph of \( h_0 \) to \((x,y)\). Here directed
means that at each step the path moves in the direction \((-1, 1)\) or \((1, 1)\).

As the \(\omega_{i,j}\) have a continuous distribution, we may assume that almost surely \(g(x, y)\) takes distinct and positive values for every \((x, y)\) above the graph of \(h_0\). Moreover, the maximizing path \(\pi\) in (6.8) is unique for every such \((x, y)\). Also, the maximizing path always starts from a valley of \(h_0\) because it can be continued down a slope of \(h_0\) until it reaches a valley.

Recall the process \(X(t)\) which is the position of the particle in the \(\circ - \bullet\) pair at time \(t\), and the process \(X_n\) from (6.4) that records the steps of the pair. The following lemma is from [18, Proposition 3]; it describes the evolution of \(X(t)\).

**Lemma 6.2.** Starting with \(X_0 = 0\), for every \(n \geq 0\),

\[
X_{n+1} = \begin{cases} 
X_n + 1 & \text{if } g(X_n + 1, n + 1) < g(X_n - 1, n + 1) \\
X_n - 1 & \text{if } g(X_n - 1, n + 1) < g(X_n + 1, n + 1)
\end{cases}.
\]

In other words, \((X_{n+1}, n+1) = \arg\min\{g(X_n - 1, n + 1), g(X_n + 1, n + 1)\}. Moreover, let

\[
\tau_n = g(X_n, n)
\]

for \(n \geq 1\) with \(\tau_0 = 0\). Then \(X(t) = X_n\) for \(t \in [\tau_n, \tau_{n+1})\).

### 6.3.2 Competition interface

Split the initial height function \(h_0\) at the origin according to

\[
h_0^-(x) = \begin{cases} 
h_0(x) & \text{for } x \leq 0 \\
x & \text{for } x > 0
\end{cases}
\]

and

\[
h_0^+(x) = \begin{cases} 
h_0(x) & \text{for } x > 0 \\
-x & \text{for } x \leq 0
\end{cases}
\]

Let

\[
g^\pm(x, y)
\]

denote the last passage times associated to the initial height functions \(h_0^+\) and \(h_0^-\), respectively. Note the same weights \(\omega_{x,y}\) are being used for \(g^\pm\) as for \(g\), and it is only the initial conditions that differ.

The variational principle (6.8) shows that

\[
g(x, y) = \max\{g^-(x, y), g^+(x, y)\}.
\]

The \(\circ - \bullet\) pair induces a peak of \(h_0\) at \(x = 0\). Therefore, the unique maximizing path \(\pi\) from \(h_0\) to \((x, y)\) starts to the left or right of the origin and there are no ties between
\(g^{-}(x,y)\) and \(g^{+}(x,y)\). The discrete competition function \(d(x,y) = g^{+}(x,y) - g^{-}(x,y)\) is non-decreasing in \(x\) and never vanishes. Define the sets Yellow \((Y)\) and Blue \((B)\) above the graph of \(h_{0}\) by

\[
Y = \{d < 0\} = \{(x, y) \in \mathbb{Z}_{\text{even}}^{2} : \text{maximizing path from } h_{0} \text{ to } (x, y) \text{ starts at } h_{0}^{-}\}
\]
\[
B = \{d > 0\} = \{(x, y) \in \mathbb{Z}_{\text{even}}^{2} : \text{maximizing path from } h_{0} \text{ to } (x, y) \text{ starts at } h_{0}^{+}\}
\]

If \((x, y) \in B\) then both \((x + 1, y + 1) \in B\) as well. Indeed, the maximizing path to \((x + 1, y + 1)\) from \(h_{0}\) lies weakly to the right of the maximizing path to \((x, y)\). This observation implies if \((x, y) \in B\) then \((x + 2, y) \in B\), by the chain of inclusions \((x, y) \rightarrow (x + 1, y + 1) \rightarrow (x + 2, y)\). This shows \(B\) is connected and filled in to the right. Likewise, \(Y\) is connected and filled in to the left because if \((x, y) \in Y\) then \((x - 1, y + 1) \in Y\) too.

We infer that for every \(n \geq 1\) there is an \(I_{n} \in \mathbb{Z}\) such that

\[
Y \cap (\mathbb{Z} \times \{n\}) = \{\cdots, I_{n} - 4, I_{n} - 2, I_{n}\} \text{ and } B \cap (\mathbb{Z} \times \{n\}) = \{I_{n} + 2, I_{n} + 4, I_{n} + 6, \cdots\}.
\]

The discrete competition interface \((I_{n}, n \geq 0)\) is defined by \(I_{0} = 0\) and

\[
I_{n} = \text{rightmost point of } Y \cap (\mathbb{Z} \times \{n\}). \tag{6.11}
\]

**Lemma 6.3.** For every \(n \geq 0\), \(X_{n} \in \{I_{n}, I_{n} + 2\}\) and \((X_{n} - 1, n + 1) \in Y\) while \((X_{n} + 1, n + 1) \in B\).

**Proof.** Clearly, \(X_{0} = I_{0} = 0\). Observe that \((-1, 1)\) is yellow and \((1, 1)\) is blue because \(h_{0}\) has a peak at the origin. So the claims hold for \(n = 0\) and we proceed by induction. Assuming the claims hold for \(n\), consider the colour of site \((X_{n}, n + 2)\). We will prove the claims at time \(n + 1\) when \((X_{n}, n + 2)\) is yellow. The proof when \((X_{n}, n + 2)\) is blue is analogous.

Suppose \((X_{n}, n + 2)\) is yellow. Since \((X_{n} - 1, n + 1) \in Y\) and \((X_{n} + 1, n + 1) \in B\) by hypothesis, the maximizing path to \((X_{n}, n + 2)\) enters from \((X_{n} - 1, n + 1)\) instead of \((X_{n} + 1, n + 1)\). This means \(g(X_{n} + 1, n + 1) < g(X_{n} - 1, n + 1)\), so \(X_{n+1} = X_{n} + 1\) by Lemma 6.2. Now \((X_{n+1} + 1, n + 2)\) is blue since its lower-left site \((X_{n+1}, n + 1) = (X_{n} + 1, n + 1)\) is blue, and \((X_{n+1} - 1, n + 2) = (X_{n}, n + 2)\) is yellow by assumption. Also, \(I_{n+1} = X_{n} - 1\) and so \(X_{n+1} = X_{n} + 1 = I_{n+1} + 2\). Both claims at time \(n + 1\) are established when \((X_{n}, n + 2)\) is yellow.

Lemma 6.3 establishes that the steps of the \(\circ - \bullet\) pair are uniformly close to the discrete competition interface. The next section derives the scaling limit of the competition interface.
6.4 Limit of the competition interface

We first discuss the scaling limit of last passage times followed by the limit of competition interfaces. Assume \( h_0^\varepsilon \) is a sequence of initial height functions that converges to \( h_0 \) as stipulated in (6.2).

6.4.1 Scaling limit of last passage percolation

Let \( g^\varepsilon(x, y) \) be last passage times associated to the initial height function \( h_0^\varepsilon \). Extend them to \( x \in \mathbb{R} \) by linear interpolation.

The KPZ scaling of last passage times re-scales them as

\[
 g^\varepsilon(x, t) = \left( \varepsilon^{1/2}/2 \right) \left( 2\varepsilon^{-1}x, \left\lceil \varepsilon^{-3/2}t \right\rceil \right) - \varepsilon^{-1} t \quad \text{for } (x, t) \in \mathbb{H}. \tag{6.12}
\]

With this re-scaling \( g^\varepsilon(x, t) \) converges to the directed landscape height function \( h(x, t) \) from (2.5) with initial condition \( h_0 \) from (6.2). The sense in which this convergence takes places is explained next.

Recall the directed landscape \( \mathcal{L} \) from §2.1 which governs the underlying randomness. The functions \( g^\varepsilon \) and \( \mathcal{L} \) can be realized on the same probability space. Under this coupling, \( g^\varepsilon(x, t) \) converges almost surely to \( h(x, t) \) as \( \varepsilon \to 0 \), which it does uniformly in \( (x, t) \) over compact subsets of \( \mathbb{H} \). The directed landscape height function \( h(x, t) \) is realized in terms of the variational principle (2.5), the continuum analogue of (6.8).

Recall the last passage times \( g^\pm(x, y) \) from (6.10) obtained by splitting an initial height function \( h_0 \) at the origin. Let \( g^{\pm, \varepsilon}(x, t) \) be the split last passage times associated to \( h_0^\varepsilon \) and re-scaled as in (6.12). The split initial height functions \( h_0^{\pm, \varepsilon} \) converge, after re-scaling as in (6.2), to the respective initial conditions \( h_0^{\pm} \), which is the splitting of \( h_0 \) at the reference point \( p = 0 \) as defined in §4.1. Both \( g^{\pm, \varepsilon}(x, t) \) converge, jointly and in the aforementioned coupled sense, to the directed landscape height functions \( h_0^{\pm}(x, t) \) in (4.2) that are associated to the split initial conditions \( h_0^{\pm} \).

This convergence of last passage times can be proven with the methods of [12]. Indeed, [12, Theorem 1.20] proves convergence of the tasep height functions \( h_1^\varepsilon(x) \) in the aforementioned sense to the directed landscape height functions. Convergence of the last passage times \( g^\varepsilon \) can be shown with the same argument, as explained below.

Define, for \( (a, b) \) and \( (c, d) \) in \( \mathbb{Z}_{\text{even}}^2 \)

\[
 L(a, b; c, d) = \max_{\pi} \sum_{(x, y) \in \pi} \omega_{x, y}
\]

where \( \pi \) ranges over all upward directed paths in \( \mathbb{Z}_{\text{even}}^2 \) from \( (a, b) \) to \( (c, d) \). Such paths
exists if \(|a - c| \leq d - b\), and otherwise set \(L(a, b; c, d) = -\infty\). Then the variational principle (6.8) implies
\[
g(x, y) = \max_{z \in \mathbb{Z}} [L(z, h_0(z); x, y)]. \tag{6.13}
\]

For \((x, t) \in \mathbb{R}^2\), let \((x, t)^\varepsilon\) denote the closest point in \(\mathbb{Z}^2_{\text{even}}\) to \((2\varepsilon^{-1}x, \varepsilon^{-3/2}t)\). Re-scale the function \(L\) according to
\[
L^\varepsilon(y, s; x, t) = (\varepsilon^{1/2}/2) L((y, s)^\varepsilon; (x, t)^\varepsilon) - \varepsilon^{-1}(t - s).
\]
Here \(x, y \in \mathbb{R}\) and \(s, t \in \mathbb{R}\) satisfy \(s < t\). According to [12, Theorem 1.7] there is a coupling of all the \(L^\varepsilon\) with \(L\) such that almost surely \(L^\varepsilon\) converges to \(L\) uniformly on compact subsets of \(x, y \in \mathbb{R}\) and \(s, t \in \mathbb{R}\) with \(s < t\).

Consider last passage time \(g^\varepsilon(x, y)\) associated to initial height functions \(h_0^\varepsilon(x)\). Using (6.13) and re-scaling variables one sees that the re-scaled last passage times \(g^\varepsilon(x, t)\) equal
\[
g^\varepsilon(x, t) = \max_{y \in \mathbb{R}} [L^\varepsilon(y, \varepsilon^{3/2}h_0^\varepsilon(2\varepsilon^{-1}y); x, t) - \varepsilon^{1/2}h_0^\varepsilon(2\varepsilon^{-1}y)]. \tag{6.14}
\]

The functions \(h_0^\varepsilon(y) = -\varepsilon^{1/2}h_0^\varepsilon(2\varepsilon^{-1}y)\) tend to \(h_0\) is the hypograph topology. Since \(h_0^\varepsilon\) is a simple random walk path, \(|h_0^\varepsilon(x)| \leq |x|\) and so \(|\varepsilon^{3/2}h_0^\varepsilon(2\varepsilon^{-1}x)| \leq 2\varepsilon^{1/2}|x|\), which tends to zero uniformly on compact subsets as \(\varepsilon \to 0\). Consequently,
\[
L^\varepsilon(y, \varepsilon^{3/2}h_0^\varepsilon(2\varepsilon^{-1}y); x, t) \to L(y, 0; x, t) \tag{6.15}
\]
almost surely, under the aforementioned coupling, over compact subsets of \((y, x, t) \in \mathbb{R}^2 \times (0, \infty)\). Now suppose \(h_0\) is compactly supported in the sense that it equals \(-\infty\) outside a compact interval. Then hypograph convergence of \(h_0^\varepsilon\) to \(h_0\), the convergence (6.15) and the representation (6.14) imply that
\[
g^\varepsilon(x, t) \to \sup_{y \in \mathbb{R}} [h_0(y) + L(y, 0; x, t)] = h(x, t)
\]
uniformly over compact subsets of \((x, t) \in \mathbb{R} \times (0, \infty)\). This shows convergence of the re-scaled last passage times to the directed landscape height functions for compactly supported initial conditions \(h_0\).

When the initial condition \(h_0\) is not compactly supported, one needs an additional tightness argument to derive the limit of \(g^\varepsilon\) in terms of the variational principle above. This argument is provided in [12, Theorem 15.5]. We omit it here for brevity.
6.4.2 Scaling limit of the competition interface

Let $I^n_\varepsilon$ be the competition interface for last passage percolation with initial condition $h^\varepsilon_0$ as defined in (6.11). Re-scale it according to

$$I^\varepsilon(t) = \frac{\varepsilon}{2} \lfloor I^{\varepsilon-3/2}_\varepsilon(t) \rfloor \text{ for } t > 0. \quad (6.16)$$

Recall the discrete competition function $d(x, y)$ and let $d^\varepsilon(x, t) = g^+, \varepsilon(x, t) - g^-, \varepsilon(x, t)$ be the re-scaled competition function for the initial condition $h^\varepsilon_0$. The function is non-decreasing in $x$ and the interface $I^\varepsilon$ satisfies

$$d^\varepsilon(I^\varepsilon(t), t) < 0 \quad \text{while} \quad d^\varepsilon(I^\varepsilon(t) + \varepsilon, t) > 0. \quad (6.17)$$

Following the earlier discussion, there is a coupling of the $d^\varepsilon(x, t)$ with $\mathcal{L}$ under which $d^\varepsilon(x, t)$ converges, almost surely and uniformly over compact subsets of $\mathbb{H}$, to the limiting competition function

$$d_0(x, t) = h^+_0(x, t) - h^-_0(x, t)$$

from (4.3).

Recall from §4.2 the left and right interfaces $I^\pm_0(t)$ from the reference point $p = 0$ in the directed landscape with initial condition $h_0$, and that the interface is uniquely defined if $I^\pm_0(t)$ are finite and $I^-_0(t) = I^+_0(t)$ for every $t > 0$.

**Proposition 6.1.** Suppose $d^\varepsilon(x, t)$ converges uniformly on compact subsets of $\mathbb{H}$ to $d_0(x, t)$. For every compact interval $[a, b] \subset (0, \infty)$ and $\delta > 0$, there is an $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ then

$$I^-_0(t) - \delta \leq I^\varepsilon(t) \leq I^+_0(t) + \delta \quad \text{for } t \in [a, b].$$

As a consequence, if the interface from reference point 0 for the initial condition $h_0$ is uniquely defined then $I^\varepsilon(t)$ converges to it uniformly in $t$ over compact subsets of $(0, \infty)$.

**Proof.** Consider the set $K^+ = (I^+_0(t) + \delta, t)$ for $t \in [a, b]$. As $I^+_0(t)$ is continuous, this is a compact subset of $\mathbb{H}$. The competition function $d_0$ is positive on $K^+$ by definition of $I^+_0$. Being continuous, $d_0$ has a positive minimum value over $K^+$, say $\eta$.

The function $d^\varepsilon$ converges to $d_0$ uniformly over $K^+$. So there is an $\varepsilon_1$ such that for $\varepsilon < \varepsilon_1$, $d^\varepsilon(x, t) \geq \eta/2$ on $K^+$. Consequently, by monotonicity and (6.17), $I^\varepsilon(t) \leq I^+_0(t) + \delta$ for every $t \in [a, b]$.

By a parallel argument applied to $K^- = (I^-_0(t) - \delta/2, t)$ for $t \in [a, b]$, there is an $\varepsilon_2 > 0$...
such that $I_0^-(t) - (\delta/2) - \epsilon \leq I^\epsilon(t)$ for every $t \in [a, b]$, if $\epsilon < \epsilon_2$.

The proposition follows by taking $\epsilon_0 = \min\{\epsilon_1, \epsilon_2, \delta/2\}$.

The proposition and the aforementioned convergence of last passage times imply

**Corollary 6.1** (Interface limit). Suppose the initial height functions $h_0^\epsilon(x)$ for tasep converge to $h_0(x)$ under the re-scaling (6.2). Suppose also the interface from reference point $p = 0$ and initial condition $h_0$ is uniquely defined in the directed landscape. Call this unique interface $I_0(t)$ for $t > 0$.

The re-scaled competition interface $I^\epsilon(t)$ in (6.16) associated to $h_0^\epsilon$ converges in law, uniformly in $t$ over compact subsets of $(0, \infty)$, to $I_0(t)$.

### 6.5 Scaling limit of the 2nd class particle

We can now prove Theorem 6.1 using Lemma 6.3 and Corollary 6.1. First we must understand the random times when the $\circ - \bullet$ pair jumps. Recall $X_n$ from (6.4), which is the position of the particle in the $\circ - \bullet$ pair after its $n$-th jump, and $\tau_n$ is the time of this jump. Lemma 6.2 describes the evolution of $X_n$ and $\tau_n$. Let $n_t$ be the integer such that $t \in [\tau_{n_t}, \tau_{n_t+1})$, which means that the position $X(t)$ of the particle in the $\circ - \bullet$ pair satisfies

$$X(t) = X_{n_t}. \quad (6.18)$$

**Lemma 6.4.** Assume the conditions of Theorem 6.1: tasep initial conditions $h_0^\epsilon(x)$ converge to $h_0(x)$ under the re-scaling (6.2) and there is a uniquely defined interface from reference point 0 for and initial condition $h_0$.

Denote by $X_n^\epsilon$, $\tau_n^\epsilon$ and $n_t^\epsilon$ the quantities $X_n$, $\tau_n$ and $n_t$ above for the tasep with initial condition $h_0^\epsilon$. Given a compact interval $[a, b] \subset (0, \infty)$, the following holds uniformly for $s \in [a, b]$. There is a random constant $C$ such that

$$\sup_{s \in [a, b]} \left| \frac{n_t^\epsilon - \frac{3}{2}s}{\epsilon^{-3/2}s} - \frac{1}{2} \right| \leq C\epsilon.$$ 

*Proof.* Fix $\epsilon$ and to save notation suppress its dependence on particle locations, jump times, last passage times, etc. Recall from Lemma 6.2 that $\tau_n = g(X_n, n)$ and

$$\tau_{n+1} = g(X_{n+1}, n + 1) = \min(g(X_n + 1, n + 1), g(X_n - 1, n + 1)) \leq g(X_n, n + 2).$$

Since $\tau_{n_t} \leq t \leq \tau_{n_t+1}$, it follows that

$$g(X_{n_t}, n_t) \leq t \leq g(X_{n_t}, n_t + 2). \quad (6.19)$$
Now $X_n \in \{I_n, I_{n+2}\}$ for every $n$ by Lemma 6.3. The process $I_n$ re-scaled according to (6.16) converges to the uniquely defined interface $I_0(t)$ from reference point 0 and initial condition $h_{0r}$, as stated in Corollary 6.1. In fact, there is a coupling of the $I_n$ and $I_0(t)$, over all $\epsilon$, for which the convergence holds almost surely. So there is a random constant $L$ such that $|I_n| \leq Ln^{2/3}$ for every $n = n_t$ with $t = s\epsilon^{-3/2}$ and $s \in [a, b]$. Thus $X_n$ satisfies the same bound, under this coupling, for a slightly larger $L$ and the same values of $n$.

From the convergence of the KPZ re-scaled last passage function $g$ in (6.12) to the directed landscape height function $h$ as discussed in §6.4.1 (the dependence on $\epsilon$ is suppressed here), it is easy to see that if $|x| \leq Ln^{2/3}$ then there is a random constant $C'$ such that
\[
g(x, n) = 2n + C(x, n)n^{1/3} \quad \text{with } |C(x, n)| \leq C'.
\]
Applying with to $x = X_{n_t}$ and then using (6.19) implies that
\[
|t - 2n_t| \leq C'n_t^{1/3} \tag{6.20}
\]
for every $t = s\epsilon^{-3/2}$ with $s \in [a, b]$. Now $t \leq (C' + 2)n_t$, which shows that $n_t \geq c\epsilon^{-3/2}$ for every $t$ as before and a random constant $c$. As a result, one deduces from (6.20) that
\[
\left| \frac{t}{n_t} - 2 \right| \leq C''\epsilon
\]
for a random constant $C''$. The estimate in the lemma follows after taking reciprocals. ■

Now we complete the proof of Theorem 6.1.

**Proof.** Consider $X_{n/t}^\epsilon$, the position of the particle in the $o - \bullet$ pair after $n$ steps for tasep with initial height function $h_{0r}^\epsilon$. Since $|X_{n_t}^\epsilon - X_{m_t}^\epsilon| \leq |n - m|$, $|X_{n_t}^\epsilon - X_{t/2}^\epsilon| \leq |n_t - [t/2]|$.

Plugging in $t = s\epsilon^{-3/2}$ for $s \in [a, b] \subset (0, \infty)$, with $[a, b]$ compact, and using Lemma 6.4,
\[
|X_{n_t}^\epsilon|_{s\epsilon^{-3/2}} - X_{s\epsilon^{-3/2}}^\epsilon| \leq C\epsilon^{-1/2}.
\]
Here $C$ is a random constant from the aforementioned lemma.

Since $X^\epsilon(t) = X_{n_t}^\epsilon$, $X^\epsilon(s) = (\epsilon/2)X_{n_t}^\epsilon|_{s\epsilon^{-3/2}}$. Multiply the display above by $\epsilon/2$ and, writing $\overline{X}^\epsilon(s) = (\epsilon/2)X_{s\epsilon^{-3/2}}^\epsilon$, it becomes
\[
|X^\epsilon(s) - \overline{X}^\epsilon(s/2)| \leq (C/2)\epsilon^{1/2}
\]
for every \( s \in [a, b] \).

Finally, recall \( I^\varepsilon(s) = (\varepsilon/2)I^\varepsilon_{s-3/2} \). Lemma 6.3 implies \( |X^\varepsilon(s/2) - I^\varepsilon(s/2)| \leq \varepsilon \). So,

\[
|X^\varepsilon(s) - I^\varepsilon(s/2)| \leq \varepsilon + (C/2)\varepsilon^{1/2}.
\]

The convergence of \( X^\varepsilon(s) \) to \( I_0(s/2) \) follows from the convergence of \( I^\varepsilon(t) \) to \( I_0(t) \) as stated in Corollary 6.1.

\( \blacksquare \)

### 6.6 Questions

**Multiple 2nd class particles** Consider tasep with more than one 2nd class particle. What is their joint scaling limit? It is no longer possible to model multiple 2nd class particles by using hole–particle pairs. Ideally, there should be a framework to study the joint scaling limits of 1st, 2nd, 3rd and so on class particles over suitable initial conditions.

**Universality of the 2nd class particle** There are several particle systems in the KPZ universality class with a notion of a 2nd class particle, for instance, the general asymmetric simple exclusion process. Is the scaling limit of the 2nd class particle, namely the competition interface, universal for these models?

An intriguing perspective would be to define the competition interface in terms of an SDE, mirroring the notion of characteristics in Burgers equation. It may then be possible to define a 2nd class particle in models like the KPZ equation, and to show that it converges to the competition interface in the KPZ scaling limit.

**2nd class particles and characteristics** Here is another question about where does a 2nd class particle find its characteristic, based on discussions with Jeremy Quastel. Suppose tasep has initial density \( \rho_- = 1 \) and \( \rho_+ = 0 \), so initially there are particles at all the negative sites. Consider a 2nd class particle started from \( X(0) = x_t \). If \( x_t \) is large enough then \( X(t) \) will be outside the rarefaction fan. What is the smallest \( x_t \), as \( t \) tends to infinity, for which the 2nd class particle stays outside the fan? Formally, find the smallest positive sequence \( x_t \) such that for every \( \varepsilon > 0 \),

\[
\lim_{t \to \infty} \Pr(X(t) \leq (1 - \varepsilon)t \mid X(0) = x_t) = 0.
\]

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71
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