Sharp Weak Type Estimates for a Family of Soria Bases

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Abstract
Let \( B \) be a collection of rectangular parallelepipeds in \( \mathbb{R}^3 \) whose sides are parallel to the coordinate axes and such that \( B \) contains parallelepipeds with side lengths of the form \( s, \frac{2^N}{s}, t \), where \( s, t > 0 \) and \( N \) lies in a nonempty subset \( S \) of the natural numbers. We show that if \( S \) is an infinite set, then the associated geometric maximal operator \( M_B \) satisfies the weak type estimate

\[
\left| \left\{ x \in \mathbb{R}^3 : M_B f(x) > \alpha \right\} \right| \leq C \int_{\mathbb{R}^3} \frac{|f|}{\alpha} \left( 1 + \log^+ \frac{|f|}{\alpha} \right)^2,
\]

but does not satisfy an estimate of the form

\[
\left| \left\{ x \in \mathbb{R}^3 : M_B f(x) > \alpha \right\} \right| \leq C \int_{\mathbb{R}^3} \phi \left( \frac{|f|}{\alpha} \right)
\]

for any convex increasing function \( \phi : [0, \infty) \to [0, \infty) \) satisfying the condition

\[
\lim_{x \to \infty} \frac{\phi(x)}{x(\log(1 + x))^2} = 0.
\]

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1 Introduction

This paper is concerned with sharp weak type estimates for a class of maximal operators naturally arising from work surrounding the so-called Zygmund conjecture in multiparameter harmonic analysis. Let us recall that the strong maximal operator $M$ is defined on $L^1_{\text{loc}}(\mathbb{R}^n)$ by

$$Mf(x) = \sup_{x \in R} \frac{1}{|R|} \int_R |f|,$$

where the supremum is over all rectangular parallelepipeds in $\mathbb{R}^n$ containing $x$ whose sides are parallel to the coordinate axes. An important inequality associated with the strong maximal operator is

$$\left| \left\{ x \in \mathbb{R}^n : Mf(x) > \alpha \right\} \right| \leq C_n \int_{\mathbb{R}^n} \frac{|f|}{\alpha} \left( 1 + \log^+ \frac{|f|}{\alpha} \right)^{n-1}.$$

This inequality may be found in de Guzmán [5, 6] (see also the related paper [3] of Córdoba and Fefferman as well as the paper [1] of Capri and Fava) and may be used to provide a proof of the classical Jessen–Marcinkiewicz–Zygmund Theorem [8], which tells us that the integral of any function in $L(\log^+ L)^{n-1}(\mathbb{R}^n)$ is strongly differentiable.

Now, the strong maximal operator in $\mathbb{R}^n$ is associated with an $n$-parameter basis of rectangular parallelepipeds. It is natural to consider weak type estimates for maximal operators in $\mathbb{R}^n$ associated with $k$-parameter bases. The Zygmund Conjecture in this regard is the following:

**Conjecture 1** (Zygmund Conjecture; now disproven) *Let $B$ be a collection of rectangular parallelepipeds in $\mathbb{R}^n$ whose sides are parallel to the coordinate axes and whose sidelengths are of the form

$$\phi_1(t_1, \ldots, t_k), \ldots, \phi_n(t_1, \ldots, t_k),$$

where the functions $\phi_i$ are nonnegative and increasing in each variable separately. Define the associated maximal operator $M_B$ by

$$M_B f(x) = \sup_{x \in R \in B} \frac{1}{|R|} \int_R |f|.$$*
Then $M_B$ satisfies the weak type estimate
\[
\left| \{ x \in \mathbb{R}^n : M_B f(x) > \alpha \} \right| \leq C_n \int_{\mathbb{R}^n} \frac{|f|}{\alpha} \left( 1 + \log^+ \frac{|f|}{\alpha} \right)^{k-1}.
\]

(1.1)

This conjecture was disproven by Soria in [9]. That being said, it does hold in many important cases. For example, A. Córdoba proved in [2] that the Zygmund Conjecture holds in the case that $B$ consists of rectangular parallelepipeds in $\mathbb{R}^3$ with sides parallel to the coordinate axes and whose sidelengths are of the form $s, t, \phi(s, t)$, where $\phi$ is nonnegative and increasing in the variables $s, t$ separately. Of particular interest to us in this paper is the following extension of Córdoba’s result due to Soria in [9]:

**Proposition 1** Let $B$ be a collection of rectangular parallelepipeds in $\mathbb{R}^3$ whose sides are parallel to the coordinate axes. Furthermore, suppose that, given a parallelepiped $R$ in $B$ of sidelengths $r_1, r_2, r_3$ and another parallelepiped $R'$ in $B$ of sidelengths $r'_1, r'_2, r'_3$, if $r_1 > r'_1$, then either $r_2 > r'_2$ or $r_3 > r'_3$. Then
\[
\left| \{ x \in \mathbb{R}^3 : M_B f(x) > \alpha \} \right| \leq C \int_{\mathbb{R}^3} \frac{|f|}{\alpha} \left( 1 + \log^+ \frac{|f|}{\alpha} \right).
\]

Note that this proposition encompasses bases that can be quite different in character than the ones consider by Córdoba. In particular, in [9], Soria mentions as an example the basis of parallelepipeds with sidelengths of the form $s, t, \frac{1}{7}$.

At this point, we introduce another strand of research associated with Zygmund’s Conjecture. It is natural to consider, given a translation invariant basis $B$ of rectangular parallelepipeds, whether or not the *sharp* weak type estimate associated with $M_B$ must be of the form
\[
\left| \{ x \in \mathbb{R}^n : M_B f(x) > \alpha \} \right| \leq C_n \int_{\mathbb{R}^n} \frac{|f|}{\alpha} \left( 1 + \log^+ \frac{|f|}{\alpha} \right)^{k-1}
\]
for some integer $1 \leq k \leq n$. In [10], Stokolos proved the following:

**Proposition 2** Let $B$ be a translation invariant basis of rectangles in $\mathbb{R}^2$ whose sides are parallel to the coordinate axes. If $B$ does not satisfy the weak type $(1, 1)$ estimate
\[
\left| \{ x \in \mathbb{R}^2 : M_B f(x) > \alpha \} \right| \leq C \int_{\mathbb{R}^2} \frac{|f|}{\alpha},
\]
then $M_B$ satisfies the weak type estimate
\[
\left| \{ x \in \mathbb{R}^2 : M_B f(x) > \alpha \} \right| \leq C \int_{\mathbb{R}^2} \frac{|f|}{\alpha} \left( 1 + \log^+ \frac{|f|}{\alpha} \right)
\]
but does not satisfy a weak type estimate of the form
\[
\left| \{ x \in \mathbb{R}^2 : M_B f(x) > \alpha \} \right| \leq C \int_{\mathbb{R}^2} \phi \left( \frac{|f|}{\alpha} \right)
\]
for any nonnegative convex increasing function $\phi$ such that $\phi(x) = o(x \log x)$ as $x$ tends to infinity.

In essence, this proposition tells us that, if $B$ is a translation invariant basis of rectangles in $\mathbb{R}^2$ whose sides are parallel to the coordinate axes, then the optimal weak type estimate for $M_B$ must be inequality 1.1 for $k = 1$ or $k = 2$. Optimal weak type estimates of this form when, say, $k = \frac{3}{2}$ are ruled out. The proof of Stokolos’ result is very delicate and involves the idea of crystallization that we will return to.

It is of interest that Proposition 2 has at the present time never been extended to encompass translation invariant bases consisting of (some, but not all) rectangular parallelepipeds in dimensions 3 or higher. In particular, one might expect that the optimal weak type estimate for the maximal operator associated with such a basis of parallelepipeds in $\mathbb{R}^3$ would be inequality 1.1 when $n = 3$ and $k$ is either 1, 2, or 3.

The purpose of this paper is, motivated by Propositions 1 and 2 above, to consider sharp weak type estimates associated with the translation invariant basis of rectangular parallelepipeds in $\mathbb{R}^3$ whose sides are parallel to the coordinate axes and whose side-lengths are of the form $s, \frac{2N}{s}, t$, where $s, t > 0$ and $N$ lies in a nonempty subset $S$ of the natural numbers. The end result, although not its proof, is strikingly straightforward and is stated as follows:

**Theorem 1** Let $B$ be a collection of rectangular parallelepipeds in $\mathbb{R}^3$ whose sides are parallel to the coordinate axes and such that $B$ consists of all parallelepipeds with side lengths of the form $s, \frac{2N}{s}, t$, where $s, t > 0$ and $N$ lies in a nonempty subset $S$ of the natural numbers.

If $S$ is a finite set, then the associated geometric maximal operator $M_B$ satisfies the weak type estimate

$$\left| \left\{ x \in \mathbb{R}^3 : M_B f(x) > \alpha \right\} \right| \leq C \int_{\mathbb{R}^3} \frac{|f|}{\alpha} \left( 1 + \log^+ \frac{|f|}{\alpha} \right)$$

(1.2)

but does not satisfy an estimate of the form

$$\left| \left\{ x \in \mathbb{R}^3 : M_B f(x) > \alpha \right\} \right| \leq C \int_{\mathbb{R}^3} \phi \left( \frac{|f|}{\alpha} \right)$$

for any convex increasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying the condition

$$\lim_{x \to \infty} \frac{\phi(x)}{x (\log(1 + x))} = 0.$$ 

If $S$ is an infinite set, then the associated geometric maximal operator $M_B$ satisfies the weak type estimate

$$\left| \left\{ x \in \mathbb{R}^3 : M_B f(x) > \alpha \right\} \right| \leq C \int_{\mathbb{R}^3} \frac{|f|}{\alpha} \left( 1 + \log^+ \frac{|f|}{\alpha} \right)^2,$$
but does not satisfy an estimate of the form
\[ \left| \left\{ x \in \mathbb{R}^3 : M_B f(x) > \alpha \right\} \right| \leq C \int_{\mathbb{R}^3} \phi \left( \frac{|f|}{\alpha} \right) \]
for any convex increasing function \( \phi : [0, \infty) \to [0, \infty) \) satisfying the condition
\[ \lim_{x \to \infty} \frac{\phi(x)}{x(\log(1 + x))^2} = 0. \]

The remainder of the paper is devoted to a proof of this theorem. Note that for inequality 1.2, it is easily seen that the constant \( C \) is at most linearly dependent on the number of elements in \( S \), although the sharp dependence of \( C \) on the number of elements of \( S \) is potentially a quite difficult issue that we do not treat here. The primary content of the above theorem is the sharpness of the weak type estimate of \( M_B \) in the case that \( S \) is infinite. In harmonic analysis, we typically show that an optimal weak type estimate on a maximal operator is sharp by testing the operator on a bump function or the characteristic function of a small interval or rectangular parallelepiped. This can be done, for instance, with the Hardy–Littlewood maximal operator, the strong maximal operator, or even the maximal operator associated with rectangles whose sides are parallel to the axes with sidelengths of the form \( t, \frac{1}{t} \) [9]. However, in dealing with maximal operators associated with rare bases of the type featured in Theorem 1, such simple functions do not provide examples illustrating the sharpness of the optimal weak type results, and more delicate constructions such as will be seen here are needed.

We remark that a recent paper of D’Aniello and Moonens [4] also treats the subject of translation invariant rare bases; in particular, they provide sufficient conditions on a rare basis \( B \) for the estimate 1.1 to be sharp when \( k = n \). However, certain bases covered in Theorem 1 (such as when \( S = \{ 2^m : m \in \mathbb{N} \} \)) do not fall into the scope of those considered in their paper, although the interested reader is strongly encouraged to consult it.

2 Crystallization and Preliminary Weak Type Estimates

In this section, we shall introduce a collection of two-dimensional “crystals” that we will use to prove Theorem 1. We remark that similar types of crystalline structures were used by Stokolos in [10–12] as well as by Hagelstein and Stokolos in [7].

Let \( m_1 < m_2 < \cdots \) be an increasing sequence of natural numbers. We may associate to this sequence and any \( k \in \mathbb{N} \) a set in \([0, 2^{mk}]\) denoted by \( Y_{\{m_j\}^k_{j=1}} \) defined by
\[
Y_{\{m_j\}^k_{j=1}} = \left\{ t \in [0, 2^{mk}] : \sum_{j=1}^k r_0 \left( \frac{t}{2^{m_j}} \right) = k \right\}.
\]
Here \( r_0(t) \) denotes the standard Rademacher function defined on \([0, 1)\) by
\[
r_0(t) = \chi_{[0, \frac{1}{2})}(t) - \chi_{(\frac{1}{2}, 1]}(t)
\]
and extended to be 1-periodic on \( \mathbb{R} \).

Note that
\[
\mu_1(Y_{\{m_j\}_{j=1}^k}) = 2^{-k} 2^{m_k}.
\]
Associated with the set \( Y_{\{m_j\}_{j=1}^k} \) is the crystal \( Q_{\{m_j\}_{j=1}^k} \subset [0, 2^{m_k}] \times [0, 2^{m_k}] \) defined by
\[
Q_{\{m_j\}_{j=1}^k} = Y_{\{m_j\}_{j=1}^k} \times Y_{\{m_j\}_{j=1}^k}.
\]
Note
\[
\mu_2(Q_{\{m_j\}_{j=1}^k}) = 2^{-2k} 2^{2m_k}.
\]
Here \( \mu_j \) refers to the Lebesgue measure on \( \mathbb{R}^j \).

We also associate with \( \{m_j\}_{j=1}^k \) the geometric maximal operator \( M_{\{m_j\}_{j=1}^k} \) defined on \( L^1_{loc}(\mathbb{R}^2) \) by
\[
M_{\{m_j\}_{j=1}^k} f(x) = \sup_{x \in \mathbb{R}^2} \frac{1}{|R|} \int_R |f|,
\]
where the supremum is over all rectangles in \( \mathbb{R}^2 \) containing \( x \) whose sides are parallel to the coordinate axes with areas in the set \( \{2^{m_1}, \ldots, 2^{m_k}\} \).

In the case that the context is clear, we may refer to the set \( Y_{\{m_j\}_{j=1}^k} \) simply as \( Y_k \), the set \( Q_{\{m_j\}_{j=1}^k} \) simply as \( Q_k \), and the maximal operator \( M_{\{m_j\}_{j=1}^k} \) simply as \( M_k \).

A few basic observations regarding the sets \( Y_k \) and \( Q_k \) are in order.

First, note that \( Y_{k+1} \) is a disjoint union of \( \frac{2^{m_k+1} - 1}{2^{m_k}} \) copies of \( Y_k \). In fact, defining the translation \( \tau_s E \) of a set \( E \) in \( \mathbb{R}^2 \) by \( \chi_{\tau_s E}(x) = \chi_E(x-s) \), we have
\[
Y_{k+1} = \bigcup_{l=0}^{2^{m_k+1} - 1} \tau_{l2^{m_k}} Y_k.
\]
Furthermore, by induction, we see that if \( 1 \leq r \leq k \), we have \( Y_{k+1} \) is a disjoint union of
\[
\frac{2^{m_k+1} - 1}{2^{m_k}}, \frac{2^{m_k-1} - 1}{2^{m_k-1}}, \ldots, \frac{2^{m_r+1} - 1}{2^{m_r}} = 2^{m_{k+1} - m_r} - k + r - 1
\]
copies of $Y_r$, with

$$Y_{k+1} = \bigcup_{(l_0, \ldots, l_k)} \tau_{l_0} \tau_{l_0+1} \tau_{l_0+2} \cdots \tau_{l_k} 2^{m_k} Y_r.$$ 

We also remark that the average of $\chi_{Y_k}$ over $[0, 2^{m_j}]$ is exactly $2^{-j}$, and moreover, the average of $\chi_{Y_k}$ over any translate $\tau_{l_0} 2^{m_j} \tau_{l_0+1} \tau_{l_0+2} \cdots \tau_{l_k} 2^{m_k} [0, 2^{m_j}]$ with $0 \leq l_i \leq 2^{m_i+1} - m_i - 1$ is also $2^{-j}$. Observe that the number of such translates is

$$2^{m_{j+1} - m_j - 1} \times 2^{m_{j+2} - m_{j+1} - 1} \cdots 2^{m_{k-1} - m_k - 1} = 2^{m_k - m_j - j - k}.$$ 

We now consider how $M_k$ acts on $\chi_{Q_k}$. We will do so in the special case that, for $1 \leq j \leq k$, we have that $m_{k-j} \leq m_{k-j+1} - m_j$. (This will be the case if the $m_j$ increases rapidly in $j$, for example if $m_{j+1} \geq 2m_j$ for all $j$.)

Fix now $1 \leq j \leq \frac{k}{2}$. We are going to show that there exist pairwise a.e. disjoint rectangles with sides parallel to the coordinate axes in $[0, 2^{m_k}]$ whose areas are all $2^{m_{k-j+1}}$ and such that the average of $\chi_{Q_k}$ over these rectangles is $2^{-k}$. Moreover, each of these rectangles will be a translate of $[0, 2^{m_j}] \times [0, 2^{m_{k-j+1} - m_j}]$. Accordingly, the measure of the union of these rectangles will be $2^{m_k-k}$.

We have already indicated above that the average of $\chi_{Y_k}$ over each of $2^{m_k-m_j-k+j}$ pairwise a.e. disjoint translates of $[0, 2^{m_j}]$ is $2^{-j}$. Somewhat more technically, we now need to prove that the average of $\chi_{Y_k}$ over $2^{m_k-m_{k-j+1}}$ pairwise a.e. disjoint intervals of length $2^{m_{k-j+1} - m_j}$ is equal to $2^{-j-k}$.

Note that the average of $\chi_{Y_k}$ over $[0, 2^{m_{k-j}}]$ is $2^{j-k}$ as well as any translate $\tau[0, 2^{m_{k-j}}]$ of this interval where $\tau$ is of the form $l \cdot 2^{m_{k-j}}$ for $0 \leq l \leq 2^{m_{k-j+1} - m_j - m_{k-j}} - 1$. The union of these intervals is the interval $I := [0, 2^{m_{k-j+1} - m_j}]$ over which the average of $\chi_{Y_k}$ is $2^{-j-k}$. It is especially important to recognize here that

$$Y_k \cap [0, 2^{m_k-j+1}] = \bigcup_{i=0}^{2^{m_k-j+1} - m_k-j-1} \tau_{i} 2^{m_k} Y_{k-j},$$

where the latter is a pairwise a.e. disjoint union. It is here that we need the condition that $m_{k-j} \leq m_{k-j+1} - m_j$, so that $[0, 2^{m_{k-j+1} - m_j}]$ can be tiled by pairwise a.e. disjoint intervals of length $2^{m_{k-j}}$ over which the average of $\chi_{Y_{k-j}}$ is $2^{-j-k}$.

Now, $[0, 2^{m_k}]$ contains many pairwise a.e. disjoint translates of $I \cap Y_k$, each of whom being contained in a collection of translates of $I$ that are themselves pairwise a.e. disjoint; we count them here. The number of translates is the number of pairwise a.e. disjoint translates of $I$ whose union is the left half of $[0, 2^{m_{k-j+1}}]$ (which is $2^{m_{k-j+1} - 1 - m_{k-j+1} + m_j} = 2^{-m_j-1}$) times the number of translates of $Y_{k-j+1}$ needed
to form $Y_k$ (which is $2^{m_k - m_{k-j+1}} (k-j+1) = 2^{m_k - m_{k-j+1} - j+1}$.) Hence, the total number of translates is

$$2^{m_{j-1}} 2^{m_k - m_{k-j+1} - j+1} = 2^{m_j + m_k - m_{k-j+1} - j}.$$ 

Hence, $Y_k$ contains $2^{m_{j} + m_k - m_{k-j+1} - j}$ pairwise a.e. disjoint intervals of length $2^{m_k - m_{j+1} - m_{j}}$ over each of which the average of $\chi_{Y_k}$ is $2^{j-k}$. As we have already shown that the average of $\chi_{Y_k}$ over each of $2^{m_{k-j} - k} + j$ pairwise a.e. disjoint translates of $[0, 2^{m_{j}}]$ is $2^{-j}$, we have then that there exist $2^{m_{j} + m_k - m_{k-j+1} - j}$ pairwise a.e. disjoint rectangles in $[0, 2^{m_{j}}] \times [0, 2^{m_k}]$ of size $2^{m_k - m_{j+1} - m_{j}}$.

$m_j = 2^{m_{k-j+1}}$ over each of which the average of $\chi_{Q_k}$ is $2^{-j} 2^{j-k} = 2^{-k}$. Note the measure of the union of these rectangles is

$$2^{m_k - m_{k-j+1} - k} 2^{m_{k-j+1}} = 2^{m_k - k}.$$

We come now to a crucial observation. By the construction of $Y_k$, any dyadic interval of length $2^{m_{j}}$ is at most only half filled by the translates of intervals of length $2^{m_{j-1}}$ such that the union of those translates acting on $Y_{j-1}$ is $Y_j$. Accordingly, the union of the above $2^{m_{k-j} - m_{j+1} - k}$ pairwise a.e. disjoint rectangles in $[0, 2^{m_{k}}] \times [0, 2^{m_{k}}]$ of size $2^{m_k - j+1}$ over each of which the average of $\chi_{Q_k}$ is $2^{-k}$ is at most only half filled by the corresponding set of rectangles of size $2^{m_{k-j} - (j+1)}$. Hence the union of all the rectangles $R$ in $[0, 2^{m_{k}}] \times [0, 2^{m_{k}}]$ whose sides are parallel to the coordinate axes and of area in the set $\{2^{m_{k-j}} : j = 1, \ldots, \left\lceil \frac{k}{2} \right\rceil \}$ and such that the average of $\chi_{Q_k}$ over $R$ is greater than or equal to $2^{-k}$ must exceed $\frac{1}{2} \cdot 2^{2m_k - k} = \frac{k}{8} 2^{m_k - k}$.

This series of observations leads to the proof of the following lemma.

**Lemma 1** Let the geometric maximal operator $M_{\{m_j\}_{j=1}^k}$ and the set $Q_{\{m_j\}_{j=1}^k}$ be defined as above. Suppose for $1 \leq j \leq \frac{k}{2}$ we have that $m_{k-j} \leq m_{k-j+1} - m_j$. Then

$$\mu_2 \left( \left\{ x \in [0, 2^{m_k}] \times [0, 2^{m_{k}}] : M_{\{m_j\}_{j=1}^k} \chi_{Q_{\{m_j\}_{j=1}^k}}(x) \geq 2^{-k} \right\} \right) \geq \frac{k}{8} 2^{m_k - k} = \frac{1}{8} 2^{-k} \mu_2 \left( Q_{\{m_j\}_{j=1}^k} \right).$$

**3 Proof of Theorem 1**

**Proof of Theorem 1** Let $B$ be a collection of rectangular parallelepipeds in $\mathbb{R}^3$ whose sides are parallel to the coordinate axes and such that $B$ contains parallelepipeds with side lengths of the form $s, \frac{2N}{s}, t$, where $t > 0$ and $S$ is a nonempty set consisting of natural numbers.

If $S$ is a finite set, then the associated geometric maximal operator $M_B$ is comparable to the maximal operator averaging over rectangular parallelepipeds with side lengths of the form $s, \frac{1}{s}, t$. In [9], Soria showed that this operator maps $L(1 + \log^+ L)(\mathbb{R}^3)$
continuously into weak $L^1(\mathbb{R}^3)$ but does not map any larger Orlicz class into weak $L^1(\mathbb{R}^3)$. So Theorem 1 holds in this case.

Suppose now $S$ is an infinite set. Note that the maximal operator $M_B$ is dominated by the strong maximal operator in $\mathbb{R}^3$, so the weak type estimate

$$\left| \left\{ x \in \mathbb{R}^3 : M_B f(x) > \alpha \right\} \right| \leq C \int_{\mathbb{R}^3} \frac{|f|}{\alpha} \left( 1 + \log^+ \frac{|f|}{\alpha} \right)^2$$

automatically holds.

Since $S$ is an infinite set, there exists a subset $\{m_j\}_{j=1}^\infty$ of $S$ satisfying the condition that $2m_j \leq m_{j+1}$ for all $j$. So the hypothesis of Lemma 1 holds for $\{m_j\}_{j=1}^k$ for all $k$.

For each natural number $k$, we let $Z_k \subset \{0, 2^{m_k}\} \times \{0, 2^{m_k}\} \times \{0, 2^k\}$ be defined by

$$Z_k = Q_k \times [0, 1].$$

To show the estimate

$$\left| \left\{ x \in \mathbb{R}^3 : M_B f(x) > \alpha \right\} \right| \leq C \int_{\mathbb{R}^3} \phi \left( \frac{|f|}{\alpha} \right)$$

does not hold for any convex increasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying the condition

$$\lim_{x \to \infty} \frac{\phi(x)}{x (\log(1 + x))^2} = 0,$$

it suffices to show that

$$\mu_3 \left( \left\{ x \in [0, m_k] \times [0, m_k] \times [0, 2^k] : M_B \chi_{Z_k}(x) \geq 2^{-k} \right\} \right) \geq \frac{1}{32} \frac{k^2}{2^k} \mu_3(Z_k).$$

Fix $1 \leq r \leq k$. Note that, just as $Y_k$ is a disjoint union of $2^{m_k-m_r-k+r}$ copies of $Y_r$, we have that $Q_k$ is a disjoint union of $2^{2(m_k-m_r-k+r)}$ copies of $Q_r$, with each of these copies being contained in pairwise a.e. disjoint squares of sidelength $2^{m_r}$. By Lemma 1, for each one of these squares $\hat{Q}$,

$$\mu_2 \left( \left\{ x \in \hat{Q} : M_r \chi_{\hat{Q} \cap Q_k}(x) \geq 2^{-r} \right\} \right) \geq \frac{r}{8} 2^{2m_r-r}.$$

Note each of the rectangles associated with the maximal operator $M_r$ has sidelength in the set $\{2^m, \ldots , 2^{m_r}\} \subset \{2^{m_1}, \ldots , 2^{m_k}\}$ and hence for any of these rectangles $R$, the associated parallelepiped $R \times [0, 2^{k-r}]$ lies in the basis $B$. Note that if

$$\frac{1}{\mu_2(R)} \int_R \chi_{\hat{Q} \cap Q_k} \geq 2^{-r},$$
then
\[
\frac{1}{\mu_3(R \times [0, 2^{k-r-1}])} \int_{R \times [0, 2^{k-r-1}]} \chi_{Q_k \times [0, 1]} \geq 2^{-r} 2^{r-k} = 2^{-k}.
\]

Taking into account only the top half of these parallelepipeds, for any one of the above squares \(\tilde{Q}\), we obtain
\[
\mu_3\left(\left\{x \in [0, 2^{m_k}] \times [0, 2^{m_k}] \times [2^{k-r-1}, 2^{k-r}] : M_B \chi_{Z_k}(x) \geq 2^{-k}\right\}\right)
\geq 2^{2(m_k - m_r - k + r)} \mu_2\left(\left\{x \in \tilde{Q} : M_r \chi_{\tilde{Q} \cap Q_k}(x) \geq 2^{-r}\right\}\right) \cdot 2^{k-r-1}
\geq 2^{2(m_k - m_r - k + r)} \frac{r}{8} 2^{m_r - r} \cdot 2^{k-r-1} = \frac{r}{16} 2^{m_k - k}.
\]

We now take advantage of the fact that, for different values of \(r\), the sets \([0, 2^{m_k}] \times [0, 2^{m_k}] \times [2^{k-r-1}, 2^{k-r}]\) are pairwise a.e. disjoint. In particular, we have
\[
\mu_3\left(\left\{x \in [0, 2^{m_k}] \times [0, 2^{m_k}] \times [0, 2^k] : M_B \chi_{Z_k}(x) \geq 2^{-k}\right\}\right)
\geq \sum_{r=1}^{k} \mu_3\left(\left\{x \in [0, 2^{m_k}] \times [0, 2^{m_k}] \times [2^{k-r-1}, 2^{k-r}] : M_B \chi_{Z_k}(x) \geq 2^{-k}\right\}\right)
\geq \sum_{r=1}^{k} \frac{r}{16} 2^{m_k - k} = \frac{1}{32} \frac{k^2}{2^{2m_k}} = \frac{1}{32} \frac{k^2}{2^{2k}} \mu_3(Z_k),
\]
as desired. \(\square\)

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