A NOTE ON CLUSTER AUTOMORPHISM GROUPS

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Abstract. We conjecture a characterization of a cluster automorphism as an algebra homomorphism from the cluster algebra to itself that restricts to a bijection between two clusters. This formulation does not require that the map commutes with mutations as in the original definition of cluster automorphisms.

We prove the conjecture in the case where at least one of the two clusters is bipartite.

1. Introduction

Cluster algebras were introduced in [11] and studied in a multitude of papers since. For introductory texts to cluster algebras we refer to the books [15, 17, 4], and the online notes [10]. A cluster algebra \( A \) of rank \( n \) is a subalgebra of a field \( F \) of rational functions in \( n \) variables. Its generators, the cluster variables, are grouped into overlapping ordered sets, the clusters, all of the same cardinality \( n \). The set of all clusters is constructed recursively by an operation called mutation, which replaces the \( k \)-th cluster variable \( x_k \) of a cluster \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \) by a new cluster variable \( x'_k \) that is defined by a binomial exchange relation of the form

\[
x'_k = \frac{p_k u_k + q_k v_k}{x_k},
\]

where \( u_k, v_k \) are coprime monomials in \( \mathbf{x} \setminus \{x_k\} \) and \( p_k, q_k \) are coprime coefficients.

Cluster automorphisms were introduced in [2] and further studied in [5, 3, 8, 9]. A different definition was given in [18] and recently proved to be equivalent in [6]. Generalizations were studied in [1, 13, 7]. By definition, a cluster automorphism is an algebra automorphism \( f \) such that there exist two clusters \( \mathbf{x} \) and \( \mathbf{x} \) such that \( f \) restricts to a bijection from \( \mathbf{x} \) to \( \mathbf{x} \) that commutes with the mutations at \( \mathbf{x} \) and \( \mathbf{x} \). Thus a cluster automorphism preserves both the algebraic structure and the combinatorial structure of the cluster algebra.

In this note, we consider the question if the defining conditions can be relaxed without changing the defined object. On the algebraic side, we only require \( f \) to be an algebra homomorphism, and, on the combinatorial side, we only require that there exists two clusters \( \mathbf{x} \) and \( \mathbf{x} \) such that \( f \) restricts to a bijection from \( \mathbf{x} \) to \( \mathbf{x} \). Thus, a priori, \( f \) may not be invertible and may not commute with mutations.

It is easy to see that the invertibility of \( f \) follows from it being a bijection between the two clusters. On the other hand, we claim that the requirement that the homomorphism \( f \) maps \( A \) to itself is already enough to guarantee that \( f \) commutes with mutations. We thus make the following conjecture.

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**Conjecture 1.** A \( \mathbb{Z} \)-algebra homomorphism \( f \) from a cluster algebra \( A \) to itself is a cluster automorphism if and only if there exist two clusters \( x \) and \( \overline{x} \) such that \( f \) restricts to a bijection from \( x \) to \( \overline{x} \).

**Remark 1.1.** It is also shown in \([2, 8]\) that a \( \mathbb{Z} \)-algebra homomorphism \( f: A \to A \) is a cluster automorphism if and only if it is a \( \mathbb{Z} \)-algebra automorphism that maps every cluster to a cluster. Thus our conjecture can also be understood as relaxing the condition that \( f \) maps every cluster to a cluster to the condition that \( f \) maps one cluster to a cluster.

Any cluster is a set of algebraic-independent generators of the ambient field \( F \), and thus a bijection between two clusters induces a field automorphism of \( F \). We will show the above conjecture is equivalent to the following.

**Conjecture 2.** A field automorphism \( f \) of the ambient field \( F \) induced by a map between two clusters gives a cluster automorphism of \( A \) if and only if \( f(A) \subseteq A \).

In the course of the proof, we have found that these conjectures are related to following.

**Conjecture 3.** The variable \( x_k' \), \( 1 \leq k \leq n \), is the only element in \( A \) which is of the form \( x_k^{-1}(pu +qv) \), where \( u \) and \( v \) are coprime monomials in \( x - \{x_k\} \) and \( p \) and \( q \) are coprime elements in \( \mathbb{Z}P \).

Our main result is the proof of these conjectures in the case where the cluster is bipartite.

**Theorem 1.2.** Let \( A \) be a cluster algebra.

(a) Let \( f : A \to A \) be a \( \mathbb{Z} \)-algebra homomorphism which maps a cluster \( x \) bijectively to a cluster \( \overline{x} \). If \( x \) or \( \overline{x} \) is bipartite then \( f \) is a cluster automorphism.

(b) Conjecture 3 holds for bipartite seeds.

**Remark 1.3.** The theorem applies in particular for cluster algebras of rank 2, since every seed in rank 2 is bipartite.

We also prove that Conjecture 3 implies Conjecture 1, and hence Conjecture 2.

2. Preliminaries

In this section, we recall definitions and basic properties of cluster algebras from \([11]\) and cluster automorphisms from \([2, 8]\). Throughout the paper, we use the notation \( [x]_+ = \max(x, 0) \) for \( x \in \mathbb{Z} \).

2.1. **Seed.** A seed of rank \( n \) is a triple \( \Sigma = (x, p, B) \), where

- \( x = (x_1, \ldots, x_n) \) is an ordered set with \( n \) elements;
- \( p = (x_{n+1}, \ldots, x_{n+l}) \) is an ordered set with \( l \) elements;
- \( B = (b_{x_i, x_j}) \in M_{(n+l) \times n}(\mathbb{Z}) \) is a matrix labeled by \( (x \cup p) \times x \), and it is extended skew-symmetric; that is, there exists a diagonal matrix \( D \) with positive integer entries such that \( DB \) is skew-symmetric, where \( B \) is a submatrix of \( B \) consisting of the first \( n \) rows.

The set \( x \cup p \) is called the (extended) cluster of \( \Sigma \), where the elements in \( x \) are called the cluster variables, while the elements in \( p \) are called the coefficient variables. We shall write \( b_{ji} \) for the element \( b_{x_j, x_i} \) in \( B \) for brevity. We also assume that \( B \) is connected and \( n > 1 \) for convenience.
Definition 2.1. A skew-symmetrizable matrix is called bipartite if each row is non-negative or non-positive. We say $i$ is a source if the $i$-th row is non-negative, otherwise a sink. A seed is bipartite if its matrix is, and a cluster is bipartite if it lies in a bipartite seed.

2.2. Mutation. Given an cluster variable $x_k$, one may produce a new seed by a seed mutation. The seed $\mu_k(\Sigma) = (\mu_k(x), \mathbf{p}, \mu_k(B))$ obtained by mutation of $\Sigma$ in direction $k$ (or direction $x_k$) is given by:

- $\mu_k(x) = \{\mu_{x_k}(x_j), 1 \leq j \leq n\}$, where $x'_j := \mu_{x_k}(x_j)$ satisfies

\begin{equation}
x'_{j} = x_{j}, j \neq k,
\end{equation}

\begin{equation}
x_kx'_k = p^+_k \prod_{j=1}^{n} x_j^{[b_{jk}]} + p^-_k \prod_{j=1}^{n} x_j^{-[b_{jk}]},
\end{equation}

\begin{equation}
p^+_k = \prod_{j=n+1}^{n+l} x_j^{[b_{jk}]}, \quad p^-_k = \prod_{j=n+1}^{n+l} x_j^{-[b_{jk}]}.
\end{equation}

- $\mu_k(B) = (b'_{ji})_{(n+l) \times n} \in M_{(n+l) \times n}(\mathbb{Z})$ is given by

\begin{equation}
b'_{ji} = \begin{cases} 
-b_{ji} & \text{if } i = k \text{ or } j = k; \\
-b_{ji} + [-b_{jk}] + b_{ki} + b_{jk}[b_{ki}] & \text{otherwise.}
\end{cases}
\end{equation}

2.3. Base ring $\mathbb{ZP}$ and ambient field $\mathcal{F}$. For a finite set $\mathbf{p} = \{x_{n+1}, \ldots, x_{n+l}\}$, let $\mathbb{P}$ be the free abelian group (written multiplicatively) generated by the elements of $\mathbf{p}$ and define an addition $\oplus$ in $\mathbb{P}$ by

\begin{equation}
\prod_{j} x_j^{a_j} \oplus \prod_{j} x_j^{b_j} = \prod_{j} x_j^{\min(a_j, b_j)}.
\end{equation}

Then $(\mathbb{P}, \oplus, \cdot)$ is a semifield called tropical semifield. Let $\mathbb{ZP}$ be its group ring and $\mathbb{QP}$ its field of fractions. Let $\mathcal{F} = \mathbb{QP}(x_1, x_2, \ldots, x_n)$ be the field of rational functions in $n$ independent variables with coefficients in $\mathbb{QP}$. $\mathcal{F}$ is called the ambient field of the cluster algebra.

2.4. Cluster algebra. Denote by $\mathcal{X}$ the union of all clusters obtained from an initial seed $\Sigma = (\mathbf{x}, \mathbf{p}, B)$ by iterated mutations. The cluster algebra $\mathcal{A}$ is the $\mathbb{ZP}$-subalgebra of the ambient field $\mathcal{F}$ generated by cluster variables in $\mathcal{X}$, that is $\mathcal{A} = \mathbb{ZP}[\mathcal{X}]$.

The elements of $\mathcal{X}$ are called cluster variables, while the elements in $\mathbb{ZP}$ are called coefficients. The cluster algebra is said to be of geometric type if $\mathbb{P}$ is the tropical semifield defined above. In this paper, all cluster algebras are of geometric type.

2.5. Cluster automorphism. Cluster automorphisms were introduced in [2] for cluster algebras with trivial coefficients, that is, $\mathbf{p}$ is empty, and in [8] for general case.

Let $\mathcal{A}$ be a cluster algebra. A $\mathbb{Z}$-algebra automorphism $f: \mathcal{A} \rightarrow \mathcal{A}$ is called a cluster automorphism if there exists a seed $(\mathbf{x}, \mathbf{p}, B)$ such that

- $(i)$ $f(\mathbf{p}) = \mathbf{p}$ as unordered sets,
- $(ii)$ $f(\mathbf{x})$ is a cluster,
- $(iii)$ $f$ commutes with mutations, that is, for every $x, x' \in \mathbf{x},$

\begin{equation}
f(\mu_{x,x'}(x')) = \mu_{f(x),f(x)}(f(x')).
\end{equation}
Lemma 3.1. For all \( m \in \mathbb{Z}^n \), the following lemma follows immediately from these assumptions.

(2.7) \[ x_i^{<m_i>} = \begin{cases} x_i^{m_i} & \text{if } m_i \geq 0; \\ x_i^{-m_i} & \text{if } m_i < 0. \end{cases} \]

Then we also have

(2.8) \[ x_i^{<m_i>} = x_i^{m_i} \left( p_i^+ \prod_{j=1}^{n} x_j^{[b_{ji}]} + p_i^- \prod_{j=1}^{n} x_j^{-[b_{ji}]} \right)^{[-m_i]} . \]

Fact 3. We may order both the standard monomial \( x^{<m>} \) and the ordinary Laurent monomial \( x^{m} = x_1^{m_1} \cdots x_n^{m_n} \) as follows:

\( m < m' \) if the first nonzero difference \( m'_j - m_j \) is positive.

3. Main Result

Let \( \Sigma = (x, p, B) \) be a bipartite seed of a cluster algebra \( A \). We may assume that the sinks correspond to the first \( t \) rows and columns of \( B \) and the sources correspond to the last \( n - t \) rows and columns of \( B \). Let \( x_1, \ldots, x_n \) denote the cluster variables in \( x \), and let \( x'_i \) be the cluster variable obtained by mutating the cluster \( x \) in direction \( i \). The following lemma follows immediately from these assumptions.

Lemma 3.1. For all \( i = 1, 2, \ldots, n \), we have

\[ x_i x'_i = \begin{cases} p_i^+ \prod_{j=t+1}^{n} x_j^{b_{ji}} + p_i^- \prod_{j=1}^{n} x_j^{b_{ji}} & \text{if } i \leq t; \\ p_i^+ p_i^- \prod_{j=1}^{t} x_j^{b_{ji}} & \text{if } i > t. \end{cases} \]

3.1. Conjecture 3 holds for bipartite seeds. We are ready to prove our first main result.

Theorem 3.2. The cluster variable \( x'_k \) is the only element in \( A \) which is of the form \( x_k^{-1}(pu + qv) \), where \( p \) and \( q \) are coprime elements of \( \mathbb{ZP} \), and \( u \) and \( v \) are coprime monomials in \( x - \{x_k\} \).
Proof. We shall prove the result only in the case where \( k \) is a sink, since the proof is similar if \( k \) is a source. We may assume without loss of generality that \( k = 1 \). Let 
\[
a = x_1^{-1}(pu + qv) \in \mathcal{A}
\]
as in the statement. Then \( a \) can be written in terms of the standard monomials as follows
\[
(3.1) \quad a = \sum_{\mathbf{m} \in \mathbb{Z}^n} C_\mathbf{m} \mathbf{x}(\mathbf{m}).
\]

Using the definition of the standard monomials as well as Lemma 3.1, we have
\[
\mathbf{x}(\mathbf{m}) = \prod_{i: m_i > 0} x_i^{m_i} \prod_{i: m_i < 0} (x_i')^{-m_i}
\]
\[
= \prod_{i: m_i > 0} x_i^{m_i} \prod_{m_i < 0, i \leq t} x_i^{m_i} \left( p_i^+ \prod_{j=t+1}^{n} x_j^{b_{ij}} + p_i^- \right)^{-m_i}
\]
\[
= \mathbf{x}^{\mathbf{m}} \prod_{m_i < 0, i \leq t} \left( p_i^+ \prod_{j=t+1}^{n} x_j^{b_{ij}} + p_i^- \right)^{-m_i}
\]
\[
\prod_{m_i > 0, i > t} \left( p_i^+ \prod_{j=t+1}^{n} x_j^{b_{ij}} + p_i^- \right)^{-m_i}
\]
(3.2) \quad \mathbf{x}^{\mathbf{m}}(p_m + \text{higher order terms}),
\]
with \( p_m \in \mathbb{Z}_+ \) a nonzero constant term. Now let \( \mathbf{m}' \) be the smallest vector (in the order defined in subsection 2.6) such that \( C_{\mathbf{m}'} \neq 0 \). Let \( \mathbf{m} \neq \mathbf{m}' \) be another vector such that \( C_{\mathbf{m}} \neq 0 \) and let \( j \) be the smallest integer such that \( m_j \neq m'_j \). Since \( \mathbf{m}' < \mathbf{m} \), we have \( m'_j < m_j \). Equation (3.2) implies that in every monomial in \( \mathbf{x}(\mathbf{m}) \) the exponent of \( x_j \) is at least \( m_j > m'_j \). In particular, none of the monomials in \( \mathbf{x}(\mathbf{m}) \) has the same degree as the leading term \( \mathbf{x}^{\mathbf{m}} p_{\mathbf{m}'} \) of \( \mathbf{x}^{\mathbf{m}'} \). Therefore
\[
(3.3) \quad a = C_{\mathbf{m}}' \mathbf{x}^{\mathbf{m}'} p_{\mathbf{m}'} + \text{terms of other degrees}.
\]

It follows that \( C_{\mathbf{m}'} \mathbf{x}^{\mathbf{m}'} p_{\mathbf{m}'} = pu x_1^{-1} \) or \( C_{\mathbf{m}'} \mathbf{x}^{\mathbf{m}'} p_{\mathbf{m}'} = q v x_1^{-1} \), with \( C_{\mathbf{m}'} p_{\mathbf{m}'} q \in \mathbb{Z}_+ \).

Since both \( u \) and \( v \) are ordinary monomials in \( x_2, \ldots, x_n \), we must have \( m'_i = -1 \) and \( m_i' \geq 0 \), for \( i = 2, 3, \ldots, n \). Moreover, since \( \mathbf{m}' < \mathbf{m} \) we must have \( m_i \geq -1 \), whenever \( C_{\mathbf{m}} \neq 0 \). Thus equation (3.1) becomes
\[
(3.4) \quad a = \sum_{\mathbf{m}: m_1 = -1} C_{\mathbf{m}} \mathbf{x}(\mathbf{m}) + \sum_{\mathbf{m}: m_1 \geq 0} C_{\mathbf{m}} \mathbf{x}(\mathbf{m}).
\]

We shall show that the first sum is equal to \( x_1' \) and the second sum is zero.

Multiplying equation (3.4) by \( x_1 \) and replacing \( a \) by \( x_1^{-1}(pu + qv) \), we obtain
\[
pu + qv = x_1 \sum_{\mathbf{m}: m_1 = -1} C_{\mathbf{m}} \mathbf{x}(\mathbf{m}) + x_1 \sum_{\mathbf{m}: m_1 \geq 0} C_{\mathbf{m}} \mathbf{x}(\mathbf{m})
\]
\[
(3.5) \quad = \left( p_i^+ \prod_{j=t+1}^{n} x_j^{b_{ij}} + p_i^- \right) \sum_{(m_2, \ldots, m_n)} C_{-1, m_2, \ldots, m_n} x_2^{(m_2)} \cdots x_n^{(m_n)} + x_1 \sum_{\mathbf{m}: m_1 \geq 0} C_{\mathbf{m}} \mathbf{x}(\mathbf{m}).
\]
where the last equality holds because if \( m_1 = -1 \) then \( x^{(m)} = x_1^{(m_2)} \cdots x_n^{(m_n)} \). Setting \( x_1 = 0 \) and dividing by the binomial, we obtain

\[
\frac{pu + qv}{p_i^+ \prod_{j=t+1}^n x_j^{b_{ji}} + p_i^-} = \sum_{(m_2, \ldots, m_n)} C_{(-1, m_2, \ldots, m_n)} x_2^{(m_2)} \cdots x_n^{(m_n)}.
\]

Since the right hand side is a Laurent polynomial in \( x_1, x_2, \ldots, x_n \), so must be the left hand side. However, since \( u, v \) are coprime and \( p, q \) are coprime, the Laurentness implies that the left hand side is equal to the integer one. Thus

\[
a = \frac{pu + qv}{x_1} = \frac{p_i^+ \prod_{j=t+1}^n x_j^{b_{ji}} + p_i^-}{x_1} = x_1'.
\]

3.2. Conjecture 1 holds for bipartite seeds. We are now ready to prove part (a) of Theorem 1.2.

**Theorem 3.3.** Let \( A \) be a cluster algebra. Let \( f : A \to A \) be a \( \mathbb{Z} \)-algebra homomorphism which maps a cluster \( x \) bijectively to a cluster \( \bar{x} \). If \( x \) or \( \bar{x} \) is bipartite then \( f \) is a cluster automorphism.

**Proof.** Let \( x = (x_1, \ldots, x_n) \) and \( \bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \) and let \( \Sigma = (x, p, B) \) and \( \bar{\Sigma} = (\bar{x}, \bar{p}, \bar{B}) \) be the seeds of \( A \) containing \( x \) and \( \bar{x} \), respectively, where \( B = (b_{ji})_{n+l \times n} \) and \( \bar{B} = (\bar{b}_{ji})_{(n+l) \times n} \). Let \( \mathbb{P} = \text{Trop}(x_{n+1}, \ldots, x_{n+l}) \) be the coefficient semifield of \( A \). Up to relabeling the elements in \( x \sqcup p \) and \( \bar{x} \sqcup \bar{p} \), and simultaneously relabeling the columns and rows of \( B \) and \( \bar{B} \), we may assume that \( f(x_i) = \bar{x}_i \), for \( i = 1, 2, \ldots, n + l \), where \( \bar{x}_i = x_i \) for \( i \geq n + 1 \) and some \( n + 1 \leq j \leq n + l \). According to Remark 2.2, to show \( f : A \to A \) is a cluster automorphism it suffices to show \( B = \bar{B} \) or \( B = -\bar{B} \).

Since \( f \) is a homomorphism, we have

\[
f(x'_i) = f \left( \frac{p_i^+ \prod_{j=1}^n x_j^{b_{ji}} + p_i^- \prod_{j=1}^n x_j^{-b_{ji}}}{x_i} \right) = \frac{f(p_i^+ \prod_{j=1}^n f(x_j)^{b_{ji}} + f(p_i^- \prod_{j=1}^n f(x_j)^{-b_{ji}})}{f(x_i)}
\]

\[
\tag{3.7}
= \frac{p_i^+ \prod_{j=1}^n x_j^{b_{ji}} + p_i^- \prod_{j=1}^n x_j^{-b_{ji}}}{x_i},
\]

for all \( i = 1, 2, \ldots, n \). Thus we see that \( f(x'_i) \) is of the form \((pu + qv)/x_i\), with \( u, v \) coprime as monomials in \( \bar{x} - \{x_i\} \) and \( p, q \) coprime as elements of \( \mathbb{Z} \mathbb{P} \). Therefore when \( \bar{x} \) is bipartite, Theorem 3.2 implies that \( f(x'_i) = \bar{x}_i' \), for \( i = 1, 2, \ldots, n \). Since \( \bar{x}_i' \) is defined by the same formula as in the last row of (3.7) except for replacing the \( b_{ji} \) by \( \bar{b}_{ji} \), we see that \( b_{ji} = \bar{b}_{ji} \) or \( b_{ji} = -\bar{b}_{ji} \), for all \( i, j \). Therefore \( B = \bar{B} \) or \( B = -\bar{B} \), noticing that \( B \) and \( \bar{B} \) is connected, and we are done. When \( x \) is bipartite, we prove the result by a similar argument for the homomorphism \( f^{-1} \) on \( A \), noticing that \( f \) is invertible on \( A \) because it is induced by a bijection between two clusters, which are algebraic-independent generators of the ambient field \( \mathcal{F} \).

From the proof we also obtain the following result which does not require bipartite seeds.

**Corollary 3.4.** Conjecture 3 implies Conjecture 1.
Now we prove the following

**Proposition 3.5.** Conjecture 1 is equivalent to Conjecture 2.

**Proof.** Let $f$ be a field automorphism of $\mathcal{F}$ induced from a map between two clusters. If $f(A) \subseteq A$, then $f$ induces an algebra homomorphism on $A$, which bijectively maps a cluster to a cluster. The validity of Conjecture 1 says that $f$ is a cluster automorphism of $A$. This shows Conjecture 1 implies Conjecture 2.

Let $f : A \to A$ be an algebra homomorphism which restricts as a bijection between two clusters. Since any cluster is a set of algebraic-independent generators of $\mathcal{F}$, the bijection between two clusters induces a field automorphism $f'$ of $\mathcal{F}$, and $f'(A) = f(A) \subseteq A$. By the validity of Conjecture 2, $f'$ induces the cluster automorphism $f$ on $A$. Thus Conjecture 2 implies Conjecture 1. \hfill \Box

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