DECAY OF SOLUTIONS FOR A DISSIPATIVE HIGHER-ORDER BOUSSINESQ SYSTEM ON A PERIODIC DOMAIN

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Abstract. In this paper we are concerned with a Boussinesq system for small-amplitude long waves arising in nonlinear dispersive media. Considerations will be given for the global well-posedness and the time decay rates of solutions when the model is posed on a periodic domain and a general class of damping operator acts in each equation. By means of spectral analysis and Fourier expansion, we prove that the solutions of the linearized system decay uniformly or not to zero, depending on the parameters of the damping operators. In the uniform decay case, the result is extended for the full system.

1. Introduction. In recent years, Boussinesq equations have attracted a great deal of attention from all aspects of wave dynamic researchers due to the wide range of practical applications, specially for simulating waves propagation in coastal zones. Theoretical models of water waves are often derived under application driven assumptions facilitating analysis and numerical computation. The hope is that these models are accurate enough for the intended applications. There are numerous models because no single model can capture all the phenomena associated with shallow water waves. For example, the family of Korteweg-de Vries equations (KdV) describes the uni-directional propagation of shallow water waves, whereas the family of Boussinesq equations describes the bi-directional propagation of such waves. Each model within each family has its own range of applicability.

Considered herein is a variant of the classical Boussinesq system and its higher-order generalizations proposed by J. J. Bona, M. Chen and J.-C. Saut in [3, 4]. Such equations were first derived by Boussinesq to describe the two-way propagation of small-amplitude, long wavelength, gravity waves on the surface of water in a canal, but they arise also when modeling the propagation of long-crested waves on large
lakes or the ocean:

\[
\eta_t + w_x + aw_{xxx} - b \eta_{xx} + a_2 w_{xxxx} + b_2 \eta_{xxxx} = -(\eta w)_x + b(\eta w)_{xxx} - (a + b - \frac{1}{3}) (\eta w)_{xx},
\]

\[w_t + \eta_x + c \eta_{xxx} - d w_{tx} + c_2 \eta_{xxxx} + d_2 w_{txxxx} = -w w_x - c(w w_x)_{xx} - (\eta \eta_{xx})_x + (c + d - 1) w_x w_{xx} + (c + d) \eta \eta_{xxx}.\]  

(1)

In the system above, \(\eta\) is the elevation of the fluid surface from the equilibrium position, \(w = w_0\) is the horizontal velocity in the flow at height \(\theta h\), where \(h\) is the undisturbed depth of the liquid. The parameters \(a, b, c, d, a_2, c_2, b_2, d_2\) are required to fulfill the relations

\[a + b = \frac{1}{2} (\theta^2 - \frac{1}{3}), \quad c + d = \frac{1}{2} (\theta^2 - \frac{1}{2}),\]

\[a_2 - b_2 = -\frac{1}{2} (\theta^2 - \frac{1}{3}) b + \frac{5}{24} (\theta^2 - \frac{1}{3})^2,\]

\[c_2 - d_2 = \frac{1}{2} (1 - \theta^2) c + \frac{5}{24} (1 - \theta^2) (\theta^2 - \frac{1}{5}),\]

where \(\theta \in [0, 1]\). Observe that this class of systems contains some of the well-known systems, such as the classical Boussinesq system \((a = b = c = a_2 = c_2 = b_2 = d_2 = 0, d = \frac{1}{2}) [7]\).

Despite of the developments obtained for Boussinesq systems, there are many issues still open that deserve further attention, specially when dissipative mechanisms are incorporated to the models. In real physical situations, dissipative effects are often as important as nonlinear and dispersive effects (see, for instance, [5, 6, 10]) and this fact has given currency to the study of water wave model in nonlinear dispersive media. Indeed, it was clear from the experimental outcomes that damping effects must be accounted in addition to those of nonlinearity and dispersion for good quantitative agreement with model predictions. When suitable dissipation was added, it appeared that these long-wave models provided an accurate description of reality for a reasonably wide range of amplitudes and frequencies. This in itself presents an interesting challenge.

In this paper, our purpose is to investigate such questions by considering a general class of damping operator with nonnegative symbol. Considerations will be given to the Boussinesq system, posed on a periodic domain, when the parameters in (2) are such that \(a_2 = c_2 = 0\). More precisely, we consider the model

\[
\begin{aligned}
\eta_t + w_x - b \eta_{xx} + a_2 \eta_{xxxx} + c_2 w_{xxxx} + \beta_1 M_{\alpha_1} \eta &= -(\eta w)_x + b(\eta w)_{xxx} - (a + b - \frac{1}{3}) (\eta w)_{xx}, \\
\eta_x - d w_{tx} + c_2 \eta_{xxxx} + \beta_2 M_{\alpha_2} w &= -w w_x - c(w w_x)_{xx} - (\eta \eta_{xx})_x + (c + d - 1) w_x w_{xx} + (c + d) \eta \eta_{xxx}, \\
\eta^r(t, 0) &= \frac{\partial^r \eta}{\partial x^r}(t, 2\pi), \quad \eta^w(t, 0) = \frac{\partial^r w}{\partial x^r}(t, 2\pi), \quad t > 0, \\
\eta(x, 0) &= \eta^0(x), \quad w(x, 0) = w^0(x), \quad x \in (0, 2\pi),
\end{aligned}
\]

(3)

where \(0 \leq r, q \leq 3\),

\[\beta_1, \beta_2 \geq 0, \quad \alpha_1, \alpha_2 \in [0, 4], \quad b, d, b_2, d_2 > 0, \quad a, c < 0 \text{ or } a = c > 0.\]
The number of boundary conditions depends on the values such parameters. For our purpose, we need to work in a rigorous functional framework in which Sobolev spaces of periodic functions play a crucial role, namely $H^s_p(0, 2\pi)$, $s \in \mathbb{R}$. With this notation, we introduce the operators $M_{\alpha_j} : H_p^{s+\frac{1}{2}}(0, 2\pi) \rightarrow H_p^{s}(0, 2\pi)$, which are Fourier multiplier operators defined in terms of their Fourier coefficients as follows

$$M_{\alpha_j} \left( \sum_{k \in \mathbb{Z}} \hat{a}_k e^{ikx} \right) = \sum_{k \in \mathbb{Z}} (1 + k^2)^{\frac{\alpha_j}{2}} \hat{a}_k e^{ikx}, \quad j = 1, 2. \tag{4}$$

The operators $M_{\alpha_j}$ are, in some sense, similar to fractional derivative operators. Indeed, for a periodic function $h(x) = \sum_{k \in \mathbb{Z}} a_k e^{ikx}$, the Weyl fractional derivative operator of order $\alpha \geq 0$ applied to $h$ is defined by (see Samko et al. [17])

$$W^\alpha_x h(x) = \sum_{k \in \mathbb{Z}^*} (ik)^\alpha a_k e^{ikx}. \tag{5}$$

Consequently, the Fourier coefficients of $M_{\alpha_j} h$ and $W^\alpha_t h$ behave in the same manner for large $k$.

A natural question arises as to whether dissipative effects overcome the nonlinear-dispersive interaction and leads to the decay of solutions, and allows to obtain at which rate they decay. It is to this and related questions that the present work is directed. But before to address the issue, a global (in time) existence result is necessary, i.e., we need to show that system (3) has unique solutions corresponding to reasonably smooth initial data. Therefore, in order to make more precise the idea we have in mind, let us define

$$E[\eta, w](t) = \int_0^{2\pi} \left( \left| (I - b_1 \partial_t^2 + b_2 \partial_x^2)^{1/2} \eta(t, x) \right|^2 + \left| (I - b_1 \partial_t^2 + d_2 \partial_x^2)^{1/2} \mathcal{H} w(t, x) \right|^2 \right) dx,$$

with the operator $\mathcal{H}$ given in the following way

$$\mathcal{H} \left( \sum_{k \in \mathbb{Z}} \hat{a}_k e^{ikx} \right) = \sum_{k \in \mathbb{Z}^*} \frac{w_1}{w_2} \hat{a}_k e^{ikx},$$

where $w_1 = \frac{1 - nk^2}{1 + nk^2 + b_2 k^4}$ and $w_2 = \frac{1 - nk^2}{1 + nk^2 + d_2 k^4}$. Then, we obtain a positive constant $C > 0$, depending only on $\beta_1, \beta_2, a$ and $c$, such that (see (61)-(64))

$$\frac{d}{dt} E[\eta, w](t) \leq -C \left( \left\| \eta \right\|^2_{H^{s+\frac{1}{2}}_p(0, 2\pi)} + \left\| w \right\|^2_{H^{s-\frac{1}{2}}_p(0, 2\pi)} \right) + \left( N(\eta, w), (\eta, w) \right)_{(L^2_p(0, 2\pi))^2}, \tag{5}$$

for any $t \geq 0$, where $N(\eta, w)$ denotes the nonlinear terms on the right hand side of the equations in (3). The estimate above indicates that the terms $M_{\alpha_1}$ and $M_{\alpha_2}$ play the role of feedback damping mechanisms, at least for the linearized system, but $E[\eta, w]$ does not have a definite sign. Moreover, it is not easy to ascertain that a solution of (3) defined locally in time has a global extension if it remains bounded in a suitable norm on bounded time intervals. Indeed, the standard energy techniques seem unable to establish the a priori bounds needed to guarantee global existence. Thus, the decay of solutions, as well as, the question of smoothing and an associated well-posedness theory set in certain function classes are interesting issues.

In the present paper, we first study the linearized system. Through a careful spectral analysis of the associated differential operator and an explicit Fourier series
expansion of the solution in terms of the eigenvectors, we show that the solutions of the linearized system goes to zero, as the time $t$ tends to infinity. Moreover, if $\alpha_1 = \alpha_2 = 4$ and $\beta_1, \beta_2 > 0$, it is possible to prove that they decay exponentially to zero in the $H^s$--setting, for any $s \in \mathbb{R}$, whenever $(\eta_0, w_0) \in (H^s_p(0, 2\pi))^2$. On the other hand, if $\min\{\alpha_1, \alpha_2\} \in [0, 4)$, $\beta_1, \beta_2 \geq 0$ and $\beta_1^2 + \beta_2^2 > 0$, we derive a polynomial decay rate, also in the $H^s$--setting, but considering more regular data. From this second case we can conclude that the exponential decay does not holds when damping term acts in only one equation. In both cases, the same spectral approach combined with the semigroup theory allow to prove the well-posedness, also in the $H^s$--setting. However, the linear well-posedness does not by itself imply that the nonlinear problem will be well-posed. The same is true for the stabilization property, i.e., the decay of solutions of linearized system does not assure the decay of the solutions of the full system. However, by combining the well-posedness and the exponential decay estimate obtained for the linear problem we prove the global well-posedness together with the exponential stability of the solutions (3) issued from small initial data in a convenient weighted space.

Our analysis were inspired by an earlier work on the subject [15] and aims to add considerably to the conclusions drawn therein (see also [2]). By using the same approach the authors derived similar results when the parameters given in (2) are such that $a = c = b_2 = d_2 = 0$ with nonlinearities $N(\eta, w) = (-\eta w)_x, -ww_x$. We note that, in the absence of the damping, the model considered in [15] is the so-called Boussinesq system of Benjamin-Bona-Mahony type (BBM-BBM), also derived in [3, 4]. When the model is posed on a bounded interval, the previous work [14] addresses the stabilization problem for the linearized Benjamin-Bona-Mahony system ($N \equiv 0$), with a localized damping term that acts in one equation only. By considering Dirichlet boundary conditions, it was proved that the energy associated to the model converges to zero as time goes to infinity. A similar problem posed on the whole real axis was studied by Chen and Goubet [11], where they prove the exponential decay when the damping is active in both equations. Finally, let us mention that the study of the controllability and stability properties for model (3) was initiated in [13], when $a = c = 0$, $N(\eta, w) = (-\eta w)_x, -ww_x$ and a periodic domain is considered. The space of the controllable data for the associated linear system is determined for each value of the four parameters. Then, as an application of the newly established exact controllability results, some simple feedback controls are constructed for some particular choice of the parameters, such that the resulting closed-loop systems are exponentially stable. Later on, the stabilization problem was studied in [8, 16] for Boussinesq system of KdV-KdV type ($b = d = 0$) posed on a bounded interval. In any case, depending on the values of its parameters, system (3) couples two equations that may be of KdV-KdV or BBM-BBM types. It is therefore interesting to see to which extent the stability properties of each model are maintained and/or improved.

The plan of the present article is as follows: Section 2 is devoted the study of the well-posedness of the linearized problem. In Section 3 we obtain the decay rates for the associated linear semigroup. Finally, in Section 4 we give the asymptotic behavior of the nonlinear system (3) for the case in which the linearized system has an exponential decay rate.

2. The linearized system. The aim of this section is to study the main properties of the linearized model corresponding to (3). Its well-posedness in the $H^s_p$--setting
will be investigated in subsection 2.1. In order to do that, we make a careful spectral analysis of the state operator and Fourier expansion. Then, having the well-posedness result in hands, the stabilization will be established in subsection 2.2.

We consider the following system

\[
\begin{align*}
\eta_t + w_x - b_2 \eta_t x x + b_2 \eta_t x x + a w_x x x + \beta_1 M_{\alpha}, \eta = 0, & \quad x \in (0, 2\pi), \quad t > 0 \\
w_t + \eta_x - d w_t x x + d_2 w_t x x + c \eta x x + \beta_2 M_{\alpha_2} w = 0, & \quad x \in (0, 2\pi), \quad t > 0 \\
\frac{\partial \eta}{\partial x}(t, 0) = \frac{\partial \eta}{\partial x}(t, 2\pi), & \quad \frac{\partial \eta}{\partial x}(t, 0) = \frac{\partial \eta}{\partial x}(t, 2\pi), \quad t > 0, \quad 0 \leq r, q \leq 3, \\
\eta(0, x) = \eta^0(x), & \quad w(0, x) = w^0(x), \quad x \in (0, 2\pi).
\end{align*}
\]

(6)

The next steps are devoted to prove the well-posedness and stabilization results.

2.1. Well-posedness. Given \( s \in \mathbb{R} \), let us introduce the Hilbert space

\[
V^s = H^s_p(0, 2\pi) \times H^s_p(0, 2\pi),
\]

(7)

endowed with the inner product defined by

\[
\langle (f_1, f_2), (g_1, g_2) \rangle = \langle f_1, g_1 \rangle_s + \langle \mathcal{H} f_2, \mathcal{H} g_2 \rangle_s,
\]

(8)

and the operator \( \mathcal{H} \) defined in the following way

\[
\mathcal{H} \left( \sum_{k \in \mathbb{Z}} \tilde{a}_k e^{ikx} \right) = \sum_{k \in \mathbb{Z}} \frac{w_1}{w_2} \tilde{a}_k e^{ikx},
\]

where \( w_1 = \frac{1 - a k^2}{1 + a k^2 - b_2 k^4} \) and \( w_2 = \frac{1 - c k^2}{1 + d k^2 - d_2 k^4} \). Let us remark that system (6) can be written in the following vectorial form

\[
\left( \begin{array}{c} \eta \\ w \end{array} \right)_t + A \left( \begin{array}{c} \eta \\ w \end{array} \right)_x = \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \quad \left( \begin{array}{c} \eta \\ w \end{array} \right)(0) = \left( \begin{array}{c} \eta^0 \\ w^0 \end{array} \right),
\]

(9)

where \( A \) is the linear compact operator in \( V^s \) defined by

\[
A = \left( \begin{array}{cc} \beta_1 (I - b_2 \partial_x^2 + b_2 \partial_x^4)^{-1} M_{\alpha} & (I - b_2 \partial_x^2 + b_2 \partial_x^4)^{-1} (\partial_x + a \partial_x^3) \\ (I - d_2 \partial_x^2 + d_2 \partial_x^4)^{-1} (\partial_x + c \partial_x^3) & \beta_2 (I - d_2 \partial_x^2 + d_2 \partial_x^4)^{-1} M_{\alpha_2} \end{array} \right).
\]

(10)

We pass now to study the existence of solutions to (6). If we assume that the initial data in (6) are given by

\[
(\eta^0, w^0) = \sum_{k \in \mathbb{Z}} (\tilde{\eta}_k^0, \tilde{w}_k^0) e^{ikx},
\]

(11)

then, at least formally, the solution of (6) can be written as

\[
(\eta, w)(t, x) = \sum_{k \in \mathbb{Z}} (\hat{\eta}_k(t), \hat{w}_k(t)) e^{ikx},
\]

(12)

where \( (\hat{\eta}_k(t), \hat{w}_k(t)) \) fulfills

\[
\begin{align*}
(1 + b k^2 + b_2 k^4) \hat{\eta}_k + i k (1 - a k^2) \hat{w}_k + \beta_1 (1 + k^2) \hat{\eta}_k = 0, & \quad t \in (0, T), \\
(1 + d k^2 + d_2 k^4) \hat{w}_k + i k (1 - c k^2) \hat{\eta}_k + \beta_2 (1 + k^2) \hat{w}_k = 0, & \quad t \in (0, T),
\end{align*}
\]

(13)

\[
\hat{\eta}_k(0) = \hat{\eta}_k^0, \quad \hat{w}_k(0) = \hat{w}_k^0.
\]
The next result provides an explicit formula for the eigenvalues of the operator $A$, as well as, for the solution of (13).

**Lemma 2.1.** The eigenvalues of the operator $A$ defined by (10) are given by

$$
\lambda_k^\pm = \frac{1}{2} \left( \beta_1 (1 + k^2) \frac{\omega_2}{\omega_1 + 1} + \beta_2 (1 + k^2) \frac{\omega_3}{\omega_1 + 1} \pm 2 |k| \sqrt{\omega_1 \omega_2} \sqrt{c_k^2 - 1} \right) \quad (k \in \mathbb{Z}^*),
$$

where

$$
e_k = \frac{1}{2k \sqrt{(1 - ak^2)(1 - ck^2)}} \times
\left( \beta_1 (1 + k^2) \frac{\omega_2}{\omega_1 + 1} \sqrt{1 + dk^2 + d_2 k^4} - \beta_2 (1 + k^2) \frac{\omega_3}{\omega_1 + 1} \sqrt{1 + bk^2 + b_2 k^4} \right)
$$

and $c_k = e_k - \sqrt{c_k^2 - 1} \quad (k \in \mathbb{Z}^*)$. The solution $(\tilde{\eta}_k(t), \tilde{w}_k(t))$ of (13) is given by

$$
\begin{align*}
\tilde{\eta}_k(t) &= \frac{1}{1 - c_k^2} \left[ (e^{-\lambda_k^+ t} - \zeta_k^2 e^{-\lambda_k^- t}) \tilde{\eta}_k^0 + i \sqrt{\frac{\omega_1}{\omega_2}} \tilde{w}_k \left( e^{-\lambda_k^+ t} - e^{-\lambda_k^- t} \right) \tilde{\eta}_k^0 \right], \\
\tilde{w}_k(t) &= \frac{1}{1 - c_k^2} \left[ (e^{-\lambda_k^+ t} - \zeta_k^2 e^{-\lambda_k^- t}) \tilde{w}_k^0 + i \sqrt{\frac{\omega_2}{\omega_1}} \tilde{\eta}_k \left( e^{-\lambda_k^+ t} - e^{-\lambda_k^- t} \right) \tilde{w}_k^0 \right],
\end{align*}
$$

if $|e_k| \neq 1$ and $k > 0$,

$$
\begin{align*}
\tilde{\eta}_k(t) &= \frac{1}{1 - c_k^2} \left[ (1 - |k| \sqrt{\omega_1 \omega_2 t}) \tilde{\eta}_k^0 - i k w_1 t \tilde{w}_k^0 \right] e^{-\lambda_k^+ t}, \\
\tilde{w}_k(t) &= \left[ -ik w_2 \tilde{\eta}_k^0 + (1 + |k| \sqrt{\omega_1 \omega_2 t}) \tilde{w}_k^0 \right] e^{-\lambda_k^- t},
\end{align*}
$$

if $|e_k| \neq 1$ and $k < 0$,

$$
\begin{align*}
\tilde{\eta}_k(t) &= \left[ -ik w_2 \tilde{\eta}_k^0 + (1 + |k| \sqrt{\omega_1 \omega_2 t}) \tilde{w}_k^0 \right] e^{-\lambda_k^+ t}, \\
\tilde{w}_k(t) &= \frac{1}{1 - c_k^2} \left[ (1 - |k| \sqrt{\omega_1 \omega_2 t}) \tilde{\eta}_k^0 - i k w_1 t \tilde{w}_k^0 \right] e^{-\lambda_k^- t},
\end{align*}
$$

if $|e_k| = 1$ and $k \neq 0$, and finally,

$$
\begin{align*}
\tilde{\eta}_0(t) &= \tilde{\eta}_0^0 e^{-\beta t}, \\
\tilde{w}_0(t) &= \tilde{w}_0^0 e^{-\beta t}.
\end{align*}
$$

**Proof.** It is easy to see that (13) is equivalent to

$$
\begin{pmatrix}
\tilde{\eta}_k \\
\tilde{w}_k
\end{pmatrix}_t + A(k) \begin{pmatrix}
\tilde{\eta}_k \\
\tilde{w}_k
\end{pmatrix}(t) = \begin{pmatrix}
0 \\
0
\end{pmatrix},
$$

where

$$
A(k) = \begin{pmatrix}
\beta_1 (1 + k^2) \frac{\omega_2}{\omega_1 + 1} & i k (1 - ak^2) \\
\frac{1}{1 + bk^2 + b_2 k^4} & \frac{1}{1 + bk^2 + b_2 k^4} \frac{\omega_3}{\omega_1 + 1} \frac{\beta_2 (1 + k^2) \omega_2}{\omega_1 + 1} \\
\frac{ik (1 - ck^2)}{1 + dk^2 + d_2 k^4} & \frac{1}{1 + dk^2 + d_2 k^4}
\end{pmatrix}.
$$

Hence, the solution of (13) is given by

$$
\begin{pmatrix}
\tilde{\eta}_k \\
\tilde{w}_k
\end{pmatrix}(t) = e^{-A(k) t} \begin{pmatrix}
\tilde{\eta}_k^0 \\
\tilde{w}_k^0
\end{pmatrix}.
$$

(20)
The eigenvalues $\lambda_k^\pm$ of the matrix $A(k)$ are:

$$
\lambda_k^\pm = \frac{1}{2} \left( \frac{\beta_1 (1+k^2)\frac{\alpha_k}{2}}{1+bk^2+b_2k^4} + \frac{\beta_2 (1+k^2)\frac{\alpha_k}{2}}{1+dk^2+d_2k^4} \right) \pm \left\{ \left( \frac{\beta_1 (1+k^2)\frac{\alpha_k}{2}}{1+bk^2+b_2k^4} - \frac{\beta_2 (1+k^2)\frac{\alpha_k}{2}}{1+dk^2+d_2k^4} \right)^2 - \frac{4k^2(1-ak^2)(1-ck^2)}{(1+bk^2+b_2k^4)(1+dk^2+d_2k^4)} \right\}^{\frac{1}{2}}
$$

(21)

$k \in \mathbb{Z}^*$, that can be rewritten as (14).

In order to compute the solutions of (13), we can make use of the following result, whose proof can be found in [1]:

**Proposition 1.** Let $A$ a $2 \times 2$ matrix with eigenvalues $\lambda_1 \neq \lambda_2$. If

$$
Q_1 = \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2}, \quad Q_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1},
$$

then,

1. $A = \lambda_1 Q_1 + \lambda_2 Q_2$;
2. $Q_1^2 = Q_1$ ; $Q_2^2 = Q_2$ ; $Q_2 Q_1 = Q_1 Q_2 = 0$;
3. $A^k = \lambda_1^k Q_1 + \lambda_2^k Q_2$, $\forall k \in \mathbb{N}$;
4. $e^{At} = e^{\lambda_1 t} Q_1 + e^{\lambda_2 t} Q_2$;

Moreover, if $\lambda_1 = \lambda_2 = \lambda_0$ and $Q = A - \lambda_0 I$, then $e^{At} = (I + tQ)e^{\lambda_0 t}$.

Let us analyze the following cases:

1. **Case $|e_k| \neq 1$ and $k \neq 0$.**
   
   We have that, $\lambda_k^+ \neq \lambda_k^-$. Let

$$
Q_1 = \frac{A(k) - \lambda_k^- I}{\lambda_k^+ - \lambda_k^-} \quad \text{and} \quad Q_2 = \frac{A(k) - \lambda_k^+ I}{\lambda_k^- - \lambda_k^+}.
$$

(22)

It is straightforward to verify that

$$
Q_1 = \begin{cases}
\begin{pmatrix}
\frac{1}{2} \left( 1 + \frac{\text{sgn}(k)e_k}{\sqrt{\zeta_k^2 - 1}} \right) & \frac{\text{sgn}(k) \sqrt{w_2/w_1}}{2\sqrt{\zeta_k^2 - 1}} \\
\frac{\text{sgn}(k) \sqrt{w_2/w_1}}{2\sqrt{\zeta_k^2 - 1}} & 1 - \frac{1}{\sqrt{\zeta_k^2 - 1}}
\end{pmatrix} & \text{if } k > 0,
\end{cases}
$$

(23)

$$
= \begin{cases}
\frac{1}{1 - \zeta_k^2} \begin{pmatrix}
1 & i \sqrt{w_2/w_1} \zeta_k \\
i \sqrt{w_2/w_1} \zeta_k & -\zeta_k^2
\end{pmatrix} & \text{if } k > 0,
\end{cases}
$$

$$
\begin{cases}
\frac{1}{1 - \zeta_k^2} \begin{pmatrix}
-\zeta_k^2 & -i \sqrt{w_2/w_1} \zeta_k \\
i \sqrt{w_2/w_1} \zeta_k & 1
\end{pmatrix} & \text{if } k < 0.
\end{cases}
$$
and

\[
Q_2 = \begin{pmatrix}
\frac{1}{2} \left( 1 - \frac{\text{sgn}(k)e_k}{\sqrt{e_k^2 - 1}} \right) & -i \frac{\text{sgn}(k)\sqrt{w_1}}{w_2} \\
\frac{1}{2} \frac{\text{sgn}(k)\sqrt{w_2}}{w_1} & 1 + \frac{\text{sgn}(k)e_k}{\sqrt{e_k^2 - 1}}
\end{pmatrix}
\]  \tag{24}

where \( \zeta_k = e_k - \sqrt{e_k^2 - 1} \). On the other hand, from Proposition 1 we have that

\[ e^{-A(k)t} = e^{-\lambda_k^+ t}Q_1 + e^{-\lambda_k^- t}Q_2. \]  \tag{25}

Thus, from (25) and (20) the solution of (13) is given by (16) and (17) in the respective cases.

2. Case \(|e_k| = 1\) and \(k \neq 0\).

In this case,

\[ \lambda_k^+ = \lambda_k^- = \frac{1}{2} \left( \frac{\beta_1(1 + k^2)^{\frac{n}{2}}}{1 + bk^2 + b_2 k^4} + \frac{\beta_2(1 + k^2)^{\frac{n}{2}}}{1 + dk^2 + d_2 k^4} \right). \]

Observe that, from (21), we have

\[
\frac{1}{2} \left( \frac{\beta_1(1 + k^2)^{\frac{n}{2}}}{1 + bk^2 + b_2 k^4} - \frac{\beta_2(1 + k^2)^{\frac{n}{2}}}{1 + dk^2 + d_2 k^4} \right) = |k|\sqrt{w_1 w_2}.
\]

Let

\[
Q = \lambda_k^+ - A(k)
\]

\[
= \begin{pmatrix}
\frac{1}{2} \left( \frac{\beta_2(1 + k^2)^{\frac{n}{2}}}{1 + dk^2 + d_2 k^4} - \frac{\beta_1(1 + k^2)^{\frac{n}{2}}}{1 + bk^2 + b_2 k^4} \right) & -i k(1 - ak^2) \\
-ik(1 - ck^2) & \frac{1}{2} \left( \frac{\beta_1(1 + k^2)^{\frac{n}{2}}}{1 + bk^2 + b_2 k^4} - \frac{\beta_2(1 + k^2)^{\frac{n}{2}}}{1 + dk^2 + d_2 k^4} \right)
\end{pmatrix}
\]  \tag{26}

\[
= \begin{pmatrix}
-|k|\sqrt{w_1 w_2} - i k w_1 \\
-ik w_2 |k|\sqrt{w_1 w_2}
\end{pmatrix}.
\]

Hence, from (26) and Proposition 1 we infer that

\[ e^{-A(k)t} = e^{-\lambda_k^+ t} (I + tQ) = e^{-\lambda_k^+ t} \begin{pmatrix}
1 - t|k|\sqrt{w_1 w_2} & -ik w_1 \, t \\
-ik w_2 \, t & 1 + |k|\sqrt{w_1 w_2}
\end{pmatrix}. \]  \tag{27}

Thus, (27) and (20) lead to the solution of (13) given by (18).
3. Case $k = 0$.
   It can be deduced from (13).

**Remark 1.** Firstly, we note that $\lambda_k^\pm = \lambda_{-k}^\pm$ and the following holds:

- If $e_k < 1$, then the eigenvalues $\lambda_k^\pm$ are complex numbers.
- If $e_k \geq 1$, then the eigenvalues $\lambda_k^\pm$ are real numbers and $\lambda_k^+ \geq \lambda_k^-.$

The next steps are devoted to analyze the eigenvalues $\lambda_k^\pm$ given by (14). In the sequel $l, M$ and $C$ denote generic positive constants which may change from one row to another.

We have the following result.

**Proposition 2.** Let $\alpha_1 < 4$ or $\alpha_2 < 4$ and $|e_k| \geq 1$. We suppose that, if $\alpha_j = \max\{\alpha_1, \alpha_2\}$, then $\beta_j > 0$. Then, there exists a constant $l_1 > 0$, such that

$$\lambda_k^- \geq \begin{cases} 
\frac{l_1}{|k|^\max\{\alpha_1, \alpha_2\}} & \text{if } \alpha_1 + \alpha_2 \leq 6, \quad \max\{\alpha_1, \alpha_2\} > 3, \\
\frac{l_1}{|k|^4\min\{\alpha_1, \alpha_2\}} & \text{if } \alpha_1 + \alpha_2 > 6, \quad \max\{\alpha_1, \alpha_2\} > 3, \\
\frac{l_1}{|k|^4\max\{\alpha_1, \alpha_2\}} & \text{if } \max\{\alpha_1, \alpha_2\} \leq 3.
\end{cases}$$

(28)

**Proof.** From (21), $\lambda_k^-$ can be written as

$$\lambda_k^- = \frac{1}{2}w_1w_2\left(r - s - \sqrt{(r - s)^2 - 4k^2}\right) = 2\sqrt{w_1w_2}\left(\frac{rs + k^2}{r + s + \sqrt{(r - s)^2 - 4k^2}}\right)$$

(29)

$$\sim \frac{2}{k^2}\sqrt{\frac{ac}{b_2d_2}}\left(\frac{rs + k^2}{r + s + \sqrt{(r - s)^2 - 4k^2}}\right),$$

where

$$r = \frac{1}{\sqrt{(1 - ak^2)(1 - ck^2)}}\beta_1(1 + k^2)^{\alpha_1} \sqrt{\frac{1 + dk^2 + d_2k^4}{1 + bk^2 + b_2k^4}},$$

and

$$s = \frac{1}{\sqrt{(1 - ak^2)(1 - ck^2)}}\beta_2(1 + k^2)^{\alpha_2} \sqrt{\frac{1 + bk^2 + b_2k^4}{1 + dk^2 + d_2k^4}}.$$

From the relations above we obtain that

$$rs \sim \frac{\beta_1\beta_2}{ac}|k|^\max\{\alpha_1, \alpha_2\} - \frac{4}{4}.$$

(30)

Note that $(r + s)$ has order $|k|^\max\{\alpha_1, \alpha_2\} - 2$ and $(r - s)^2$ has order $|k|^2\max\{\alpha_1, \alpha_2\} - 2$. Let us analyze the order of $(r - s)^2 - 4k^2$:

1. If $\max\{\alpha_1, \alpha_2\} > 3$, then $2\max\{\alpha_1, \alpha_2\} - 2 > 2$. Hence, $((r - s)^2 - 4k^2)$ has order $|k|^2\max\{\alpha_1, \alpha_2\} - 2$. Furthermore, $(r + s + \sqrt{(r - s)^2 - 4k^2})$ has order $|k|^\max\{\alpha_1, \alpha_2\} - 2$.
2. If $\max\{\alpha_1, \alpha_2\} \leq 3$, then $2\max\{\alpha_1, \alpha_2\} - 2 \leq 2$. Thus, $((r - s)^2 - 4k^2)$ has order $|k|^2$ and $(r + s + \sqrt{(r - s)^2 - 4k^2})$ has order $|k|$.
Moreover, from (30), if $\alpha_1 + \alpha_2 \leq 6$, we deduce that $(rs + k^2)$ has order $|k|^2$ and, if $\alpha_1 + \alpha_2 > 6$, $(rs + k^2)$ has order $|k|^\alpha_1 + \alpha_2 - 4$. Therefore, from (29), we have the following cases:

- If $\max\{\alpha_1, \alpha_2\} > 3$ and $\alpha_1 + \alpha_2 \leq 6$, then
  \[
  \lambda_k^- \sim \frac{2}{k^2} \sqrt{\frac{ac}{b_2d_2}} \left( \frac{rs + k^2}{l_1} \right) \sim \frac{l_1}{k^2} \left( \frac{k^2}{|k|^\max\{\alpha_1, \alpha_2\} - 2} \right) \tag{31}
  \]
  
- If $\max\{\alpha_1, \alpha_2\} > 3$ and $\alpha_1 + \alpha_2 > 6$, then
  \[
  \lambda_k^- \sim \frac{2}{k^2} \sqrt{\frac{ac}{b_2d_2}} \left( \frac{rs + k^2}{l_1} \right) \sim \frac{l_1}{k^2} \left( \frac{|k|^\alpha_1 + \alpha_2 - 4}{|k|^\max\{\alpha_1, \alpha_2\} - 2} \right) \tag{32}
  \]

- If $\max\{\alpha_1, \alpha_2\} \leq 3$, we obtain that $\alpha_1 + \alpha_2 \leq 6$ and
  \[
  \lambda_k^- \sim \frac{2}{k^2} \sqrt{\frac{ac}{b_2d_2}} \left( \frac{rs + k^2}{l_1} \right) \sim \frac{l_1}{k^2} \left( \frac{k^2}{|k|^\max\{\alpha_1, \alpha_2\}} \right) \tag{33}
  \]

\[ \square \]

**Remark 2.** If $\alpha_1 = \alpha_2 = 4$, then $\lim_{k \to \infty} \lambda_k^- = \min\{\frac{\beta_1}{b_2}, \frac{\beta_2}{d_2}\}$ and $\lim_{k \to \infty} \Re(\lambda_k^+) = \frac{1}{2} \left( \frac{\beta_1}{b_2} + \frac{\beta_2}{d_2} \right)$. In this case,

\[
\lambda_k^- \sim \frac{C}{k^4} \left( \frac{\beta_1 \beta_2}{ac} k^4 + k^2 \right) \geq \begin{cases} 
\frac{l_2}{k^2} & \text{if } \beta_1 \beta_2 = 0, \\
\frac{l_2}{l_2} & \text{if } \beta_1 \beta_2 > 0,
\end{cases}
\]

for some positive constant $l_2$.

**Remark 3.** If $|e_k| < 1$, we have that

\[
\Re(\lambda_k^+) \sim \frac{1}{2} \left( \frac{\beta_1}{b_2} |k|^{\alpha_1 - 4} + \frac{\beta_2}{d_2} |k|^{\alpha_2 - 4} \right).
\]

Hence, we obtain the following cases:

- If $\alpha_1 < 4$ and $\alpha_2 < 4$ (we suppose that, if $\alpha_j = \max\{\alpha_1, \alpha_2\}$, then $\beta_j > 0$), there exists a constant $l_2 > 0$, such that
  \[
  \Re(\lambda_k^+) \geq \frac{l_2}{|k|^\max\{\alpha_1, \alpha_2\}} \left( \frac{\beta_1}{b_2} |k|^{\alpha_1 - \max\{\alpha_1, \alpha_2\}} + \frac{\beta_2}{d_2} |k|^{\alpha_2 - \max\{\alpha_1, \alpha_2\}} \right) \\
  \geq \frac{l_2 \beta_j}{|k|^\max\{\alpha_1, \alpha_2\}}.
\]

- If $\alpha_1 = \alpha_2 = 4$, then there exists a constant $l_2 > 0$, such that
  \[
  \Re(\lambda_k^+) \geq l_2.
\]

Now, we analyze the case of double eigenvalue in details.

**Lemma 2.2.** With the notation introduced in Lemma 2.1, we have that:
1. There exists only a finite number of values \( k \in \mathbb{Z} \) with the property that \( |e_k| = 1 \).

2. There exists a subsequence \((e_{k_m})_{m \geq 1}\) of \((e_k)_{k \geq 1}\), such that \( \lim_{k_m \to \infty} |e_{k_m}| = 1 \) if and only if one of the following cases holds

   (C1) \( \alpha_1 = \alpha_2 = 3 \) and \( \frac{1}{\sqrt{a c}} \left( \beta_1 \sqrt{\frac{q}{a} - \beta_2 \sqrt{\frac{b}{a}} \right) = 2 \),

   (C2) \( 3 > \alpha_1 > \alpha_2 \) and \( \beta_1 = 2 \sqrt{\frac{a c b_2}{d_2}} \),

   (C3) \( 3 > \alpha_2 > \alpha_1 \) and \( \beta_2 = 2 \sqrt{\frac{ac d_2}{b_2}} \).

3. If \( \lim_{k \to \infty} |e_k| = 1 \), there exists a positive constant \( M \), such that \( \frac{1}{|k|^{\alpha_k}} \leq M \).

**Proof.** For the first part of the Lemma, let us suppose that we have an infinite number of different values \((k_m)_{m \geq 1} \subset \mathbb{N}\), such that \( e_{k_m} = 1 \). Without loss of generality, we may assume that \( \lim_{m \to \infty} k_m = \infty \). We have the following cases:

- **If** \( \alpha_1 > \alpha_2 \), then

\[
1 = \lim_{m \to \infty} e_{k_m} = \frac{\beta_1}{2} \sqrt{\frac{d_2}{b_2}} \lim_{m \to \infty} \frac{(1 + k_m^2)^{\frac{3}{2}}}{\sqrt{(1 - a k_m^2)(1 - c k_m^2)k_m}} \]

which implies that \( \alpha_1 = 3 \) and \( \beta_1 = 2 \sqrt{\frac{a c b_2}{d_2}} \). Then,

\[
\frac{\beta_2}{2k_m} \frac{(1 + k_m^2)^{\frac{3}{2}}}{\sqrt{(1 - a k_m^2)(1 - c k_m^2)}} \sqrt{\frac{1 + bk_m^2 + b_2 k_m^4}{1 + dk_m^2 + d_2 k_m^4}} = \frac{l_{k_m} - p_{k_m}}{l_{k_m}} \]

\[
= 1 - \frac{(1 + k_m^2)^{\frac{3}{2}}}{k_m} \sqrt{\frac{a c b_2(1 + dk_m^2 + d_2 k_m^4)}{d_2(1 + bk_m^2 + b_2 k_m^4)}} = \frac{l_{k_m} - p_{k_m}}{l_{k_m}} \frac{1}{1 + \sqrt{\frac{p_{k_m}}{l_{k_m}}}} \tag{34}
\]

where,

\[
l_{k_m} = d_2 k_m^2 (1 - ak_m^2)(1 - ck_m^2)(1 + bk_m^2 + b_2 k_m^4),
\]

\[
p_{k_m} = ac b_2 (1 + k_m^2)^3 (1 + dk_m^2 + d_2 k_m^4).
\]

On the other hand, \( \frac{l_{k_m} - p_{k_m}}{l_{k_m}} = \frac{1}{k_m^2} \sum_{j=0}^{3} \frac{\theta_j^1}{k_m^{\frac{j}{2}}} \sum_{j=0}^{5} \frac{\theta_j^2}{k_m^{\frac{j}{2}}} \), where \( \theta_j^1 \) and \( \theta_j^2 \) are constants that depend only on the parameters \( a, b, d, b_2 \) and \( c \). Moreover, \( \theta_0^2 = ac b_2 d_2 \neq 0 \), \( \lim_{m \to \infty} \frac{p_{k_m}}{l_{k_m}} = \frac{\theta_1^0}{\theta_0^0} \) and \( \lim_{m \to \infty} \sqrt{\frac{p_{k_m}}{l_{k_m}}} = 1 \). Hence, from (34) we obtain

\[
-\frac{\beta_2}{2} \frac{k_m (1 + k_m^2)^{\frac{3}{2}}}{d_2} \frac{(1 + k_m^2)^{\frac{3}{2}}}{\sqrt{(1 - a k_m^2)(1 - c k_m^2)}} \sqrt{\frac{1 + bk_m^2 + b_2 k_m^4}{1 + dk_m^2 + d_2 k_m^4}} = \frac{\sum_{j=0}^{3} \frac{\theta_j^1}{k_m^{\frac{j}{2}}} \sum_{j=0}^{5} \frac{\theta_j^2}{k_m^{\frac{j}{2}}}}{1 + \sqrt{\frac{p_{k_m}}{l_{k_m}}} \frac{p_{k_m}}{l_{k_m}}} \tag{35}
\]
and, consequently,
\[
\lim_{m \to \infty} \left( -\beta_2 \frac{k_m (1 + k_m^2)^{\frac{a_2}{2}}}{2\sqrt{(1 - ak_m^2)(1 - ck_m^2)}} \sqrt{\frac{1 + bk_m^2 + b_2 k_m^4}{1 + dk_m^2 + d_2 k_m^4}} \right) = \frac{\theta_0}{2\theta_0}. \tag{36}
\]
Thus, if \( \alpha_2 \geq 1 \), (36) implies that \( \beta_2 = \theta_0^1 = 0 \). If \( \alpha_2 < 1 \), from (36) we obtain \( \theta_0^1 = 0 \). Then, from (35) we deduce that
\[
\lim_{m \to \infty} \left( -\beta_2 \frac{k_m^3 (1 + k_m^2)^{\frac{a_2}{2}}}{2\sqrt{(1 - ak_m^2)(1 - ck_m^2)}} \sqrt{\frac{1 + bk_m^2 + b_2 k_m^4}{1 + dk_m^2 + d_2 k_m^4}} \right) = \frac{\theta_1}{2\theta_0}. \tag{37}
\]
which implies that \( \beta_2 = \theta_1^1 = 0 \). However, \( e_{k_m} \) can be written as
\[
e_{k_m} = \frac{(1 + k_m^2)^{\frac{a_2}{2}}}{k_m \sqrt{(1 - ak_m^2)(1 - ck_m^2)}} \sqrt{\frac{abcd(1 + dk_m^2 + d_2 k_m^4)}{d_2(1 + bk_m^2 + b_2 k_m^4)}}. \tag{38}
\]
Therefore, \( e_{k_m} = 1 \) is equivalent to an eighth order equation in \( k_m \) which has at most eight solutions. We have obtained a contradiction and, thus, this case is not possible.

- The case \( \alpha_1 < \alpha_2 \) may be treated as before, and we obtain the same conclusion.
- If \( \alpha_1 = \alpha_2 \) we obtain that \( \lim_{m \to \infty} e_{k_m} = 1 \) if and only if \( \alpha_1 = \alpha_2 = 3 \) and the coefficients \( a, b, c, d, \beta_1 \) and \( \beta_2 \) satisfy \( \frac{1}{\sqrt{ac}} \left( \beta_1 \sqrt[4]{\frac{a}{b}} - \beta_2 \sqrt[4]{\frac{b}{a}} \right) = 2 \). However, in this case \( e_{k_m} \) is given by
\[
e_{k_m} = \frac{(1 + k_m^2)^{\frac{a_2}{2}}}{2k_m \sqrt{(1 - ak_m^2)(1 - ck_m^2)}} \left( \beta_1 \sqrt{\frac{1 + dk_m^2 + d_2 k_m^4}{1 + bk_m^2 + b_2 k_m^4}} - \beta_2 \sqrt{\frac{1 + bk_m^2 + b_2 k_m^4}{1 + dk_m^2 + d_2 k_m^4}} \right).
\]
Therefore, \( e_{k_m} = 1 \) is equivalent to a fourteenth order equation in \( k_m \) which has at most fourteen solutions. We have again obtained a contradiction. Hence, there exists only a finite number of values \( k \in \mathbb{Z} \) with the property that \( |e_k| = 1 \).

The second part of the Lemma follows as before, by analyzing the similar three cases.

For the third part of Lemma, we consider the following cases:

- If \( (C_1) \) holds, \( \alpha_1 = \alpha_2 = 3 \) and \( \frac{1}{\sqrt{ac}} \left( \beta_1 \sqrt[4]{\frac{a}{b}} - \beta_2 \sqrt[4]{\frac{b}{a}} \right) = 2 \). Then, from Proposition 2, we obtain a constant \( l_1 > 0 \), such that \( |\lambda_k^-| \geq l_1 |k| \). Thus, there exists \( M = \frac{1}{l_1} \) satisfying \( \frac{|k|}{|\lambda_k^-|} \leq M \).
- For the cases \( (C_2) \) and \( (C_3) \) we proceed in a similar way.

\[ \square \]

Remark 4. When we have complex eigenvalues, if \( \lim_{k \to \infty} |e_k| = 1 \), Remark 3 guarantee the existence of a positive constant \( M \), such that \( \frac{1}{|k||\Re(\lambda_k^\pm)|} \leq M \).

Since \( \Re(\lambda_k^\pm) \geq \lambda_k^- > 0 \), for \( |e_k| \geq 1 \), in the sequel we consider
\[
|\Re(\lambda_k^-)| := |\lambda_k^-|.
\]
We have the following result.
There exists a constant $M > 0$, such that the solution $(\tilde{\eta}_k(t), \tilde{w}_k(t))$ of (13) verifies the following estimate,

$$|\tilde{\eta}_k(t)|^2 + \frac{w_1}{w_2}|\tilde{w}_k(t)|^2 \leq M \left( |\tilde{\eta}_k(0)|^2 + \frac{w_1}{w_2}|\tilde{w}_k(0)|^2 \right) e^{-2t|\Re(\lambda_k^-)|} \quad (t \geq 0, \ k \in \mathbb{Z}). \quad (39)$$

**Proof.** We have to analyze two different cases.

- If there exists no subsequence $(e_{k_m})_{m \geq 1}$ of $(e_k)_{k \geq 1}$, such that $\lim_{k_m \to \infty} |e_{k_m}| = 1$, then

$$\frac{1 + |\zeta_k| + |\zeta_k|^2}{|1 - \zeta_k^2|} = \frac{1 - |\zeta_k||^2}{|1 - \zeta_k|} + \frac{3|\zeta_k|}{|1 - \zeta_k|} = \frac{|1 - |\zeta_k||}{1 + |\zeta_k|} + \frac{3|\zeta_k|}{|1 - \zeta_k|} \leq 1 + \frac{3}{2\sqrt{|e_k|^2 - 1}} \leq M,$$

for some constant $M > 0$. Thus,

$$\limsup_{|k| \to \infty} \frac{1 + |\zeta_k| + |\zeta_k|^2}{|1 - \zeta_k^2|} \leq M. \quad (40)$$

Hence, from (16)-(17) and (40) we have that

$$|\tilde{\eta}_k(t)|^2 \leq e^{-2t|\Re(\lambda_k^-)|} \left( \frac{1 + |\zeta_k|^2}{|1 - \zeta_k^2|} |\tilde{\eta}_k(0)|^2 + 2 \frac{|\zeta_k|}{|1 - \zeta_k^2|} \right)^2 \left( \frac{w_1}{w_2} |\tilde{w}_k(0)|^2 \right) \quad (41)$$

and

$$|\tilde{w}_k(t)|^2 \leq e^{-2t|\Re(\lambda_k^-)|} \left( \frac{1 + |\zeta_k|^2}{|1 - \zeta_k^2|} |\tilde{w}_k(0)|^2 + 2 \frac{|\zeta_k|}{|1 - \zeta_k^2|} \right)^2 \left( \frac{w_2}{w_1} |\tilde{\eta}_k(0)|^2 \right). \quad (42)$$

We multiply (42) by $\frac{w_1}{w_2}$ and add the resulting estimate, hand to hand, to (41) and obtain (39).

- Suppose that exists a subsequence $(e_{k_m})_{m \geq 1}$ of $(e_k)_{k \geq 1}$, such that

$$\lim_{k_m \to \infty} |e_{k_m}| = 1.$$

We claim that there exists a constant $M > 0$, such that

$$\left| e^{-\lambda_k^+ t} - e^{-\lambda_k^- t} \right| \leq \frac{M e^{-2t|\Re(\lambda_k^-)|}}{|1 - \zeta_k^2|} \quad (t \geq 0, \ k \in \mathbb{Z}). \quad (43)$$
Assume that it was proved. Then, from (16)-(17) we obtain
\[ |\tilde{\eta}_m(t)| \]
\[ \leq \left| \frac{1}{1 - \zeta_k^2} \left( (e^{-\lambda_k^+ t} - e^{-\lambda_k^- t}) + (1 - \zeta_k^2) e^{-\lambda_k^- t} \right) \tilde{\eta}_m^0 \right| \]
\[ + \left| \frac{i}{1 - \zeta_k^2} \sqrt{\frac{w_1}{w_2}} \zeta_{k_m} \left( e^{-\lambda_k^+ t} - e^{-\lambda_k^- t} \right) \tilde{w}_m^0 \right| \]
\[ \leq \left( Me^{-t|\Re(\lambda_k^-)|} + e^{-t|\Re(\lambda_k^-)|} \right) |\tilde{\eta}_m^0| + \sqrt{\frac{w_1}{w_2}} |\zeta_{k_m}| Me^{-t|\Re(\lambda_k^-)|} |\tilde{w}_m^0| \]
\[ \leq Me^{-t|\Re(\lambda_k^-)|} \left( |\tilde{\eta}_m^0| + \sqrt{\frac{w_1}{w_2}} |\tilde{w}_m^0| \right). \]

Hence, from (43) it follows that
\[ |\tilde{\eta}_m(t)|^2 \leq M^2 e^{-2t|\Re(\lambda_k^-)|} \left( |\tilde{\eta}_m^0|^2 + \frac{w_1}{w_2} |\tilde{w}_m^0|^2 \right). \]  

Similarly, from (16)-(17) we get
\[ |\tilde{w}_m(t)|^2 \leq M^2 e^{-2t|\Re(\lambda_k^-)|} \left( |\tilde{\eta}_m^0|^2 + \frac{w_1}{w_2} |\tilde{w}_m^0|^2 \right). \]

Combining (45) and (44) we obtain (39). Recall that it remains to prove claim (43). In fact, since \( \lim_{k \to \infty} (\lambda_k^+ - \lambda_k^-) = 0 \), there exists a positive constant \( M \), such that
\[ |e^{-(\lambda_k^+ - \lambda_k^-) t} - 1| \leq M |\lambda_k^+ - \lambda_k^-| t. \]  

Thus, from (46) we obtain the estimate
\[ \left| \frac{e^{-\lambda_k^+ t} - e^{-\lambda_k^- t}}{1 - \zeta_k^2} \right| \leq \frac{Me^{-t|\Re(\lambda_k^-)|} |k_m| \sqrt{w_1w_2} \sqrt{\frac{e_k^2}{e_k^2 - 1} - 1}}{|k_m| (e_k^2 - \frac{e_k^2}{e_k^2 - 1})} t \]
\[ \leq \frac{Me^{-t|\Re(\lambda_k^-)|}}{|k_m| (e_k^2 - \frac{e_k^2}{e_k^2 - 1})} \leq \frac{Me^{-t|\Re(\lambda_k^-)|}}{|k_m| |\Re(\lambda_k^-)|}. \]

Hence, from (47)
\[ |e^{-\lambda_k^+ t} - e^{-\lambda_k^- t}| \leq \frac{Me^{-t|\Re(\lambda_k^-)|}}{|k_m| |\Re(\lambda_k^-)|}. \]  

Since \( \lim_{k_m \to \infty} |e_{k_m}| = 1 \), from Lemma 2.2, we obtain a positive constant \( M \), such that \( \frac{1}{|k_m||\Re(\lambda_k^-)|} \leq M \). Thus, from (48) it follows that the right hand side of (43) can be estimated as follows
\[ \left| \frac{e^{-\lambda_k^+ t} - e^{-\lambda_k^- t}}{1 - \zeta_k^2} \right| \leq Me^{-t|\Re(\lambda_k^-)|}, \]
for some constant $M > 0$.

The following result gives the semigroup associated to our linear problem.

**Theorem 2.4.** The family of linear operators $(S(t))_{t \geq 0}$ defined by

$$S(t)(\eta^0, w^0) = \sum_{k \in \mathbb{Z}} (\hat{\eta}_k(t), \hat{w}_k(t)) e^{ikx}, \quad (\eta^0, w^0) \in V^s, \quad (49)$$

where the coefficients $(\hat{\eta}_k(t), \hat{w}_k(t))$ are given by (16)-(19), is a semigroup in $V^s$ and verifies the following estimate, for each $s \in \mathbb{R}$,

$$\|S(t)(\eta^0, w^0)\|_{V^s} \leq M \| (\eta^0, w^0) \|_{V^s}, \quad (\eta^0, w^0) \in V^s, \quad (50)$$

where $M$ is a positive constant.

**Proof.** From Theorem 2.3, there exists a constant $M > 0$, such that

$$\left\| \sum_{k \in \mathbb{Z}} (\hat{\eta}_k(t), \hat{w}_k(t)) e^{ikx} \right\|_{V^s}^2 \leq \sum_{k \in \mathbb{Z}} \left( |\hat{\eta}_k(t)|^2 + \frac{w_1}{w_2} |\hat{w}_k(t)|^2 \right) (1 + k^2)^s \leq M^2 \sum_{k \in \mathbb{Z}} \left( |\hat{\eta}_k(t)|^2 + \frac{w_1}{w_2} |\hat{w}_k(t)|^2 \right) (1 + k^2)^s = M^2 \left\| (\eta^0, w^0) \right\|_{V^s}^2.$$

Then, $(S(t))_{t \geq 0}$ is a well-defined linear and continuous operator and satisfies (50). It is easy to see that $S(0) = I$, $S(t_1) \circ S(t_2) = S(t_1 + t_2)$ for any $t_1, t_2 \in \mathbb{R}^+$ and, in addition, from (16)-(19) and the analysis developed in Theorem 2.3, we obtain that

$$\|S(t)(\eta^0, w^0) - (\eta^0, w^0)\|_{V^s}^2 \leq C \sum_{k \in \mathbb{Z}} \Psi_k^2(t) \left( |\hat{\eta}_k(t)|^2 + \frac{w_1}{w_2} |\hat{w}_k(t)|^2 \right) (1 + k^2)^s,$$

where

$$\Psi_k(t) = \max \left\{ \left| e^{-\lambda_k t} - \zeta_k^2 e^{-\lambda_k^2 t} \right| / \left| 1 - \zeta_k^2 \right|, \left| e^{-\lambda_k^2 t} - \zeta_k^2 e^{-\lambda_k t} \right| / \left| 1 - \zeta_k^2 \right|, \left| e^{-\lambda_k t} - e^{-\lambda_k^2 t} \right| / \left| 1 - \zeta_k^2 \right| \right\}. \quad$$

Consequently

$$\lim_{t \to 0} S(t)(\eta^0, w^0) = (\eta^0, w^0) \text{ in } V^s \text{ and the proof is complete.} \quad \Box$$

**Theorem 2.5.** The infinitesimal generator of the semigroup $(S(t))_{t \geq 0}$ is a bounded operator $(D(-A), -A)$, where $D(-A) = V^s$ and $A$ is given by (10).

**Proof.** We show that

$$\lim_{t \to 0} \frac{S(t)(\eta^0, w^0) - (\eta^0, w^0)}{t} = -A(\eta^0, w^0), \quad (51)$$

if and only if $(\eta^0, w^0) \in V^s$. This is equivalent to show that the derivative in zero of the series $\sum_{k \in \mathbb{Z}} (\hat{\eta}_k(t), \hat{w}_k(t)) e^{ikx}$, where $(\hat{\eta}_k(t), \hat{w}_k(t))$ is given by (16)-(19), is convergent to $-A(\eta^0, w^0)$ in $V^s$ if and only if $(\eta^0, w^0) \in V^s$.

If we denote by

$$S_N(t) = \sum_{|k| \leq N} (\hat{\eta}_k(t), \hat{w}_k(t)) e^{ikx},$$

then
a partial sum of the series, a straightforward computation which takes into account (13) shows that
\[ [S_N]_t (0) = -A (S_N) (0). \]  
(52)

Let \((D(B), B)\) the infinitesimal generator of the semigroup \((S(t))_{t \geq 0}\). If \((\eta^0, w^0) \in D(B)\), from (52) we obtain that
\[ B(\eta^0, w^0) = \lim_{t \to 0} \frac{S(t)(\eta^0, w^0) - (\eta^0, w^0)}{t} = \left[ \sum_{k \in \mathbb{Z}} (\tilde{\eta}_k(t), \tilde{w}_k(t))e^{ikx} \right]_t (0) \]  
(53)

Hence, \((\eta^0, w^0) \in D(-A) = V^s\) and \(B(\eta^0, w^0) = -A(\eta^0, w^0)\), for any \((\eta^0, w^0) \in D(B)\). On the other hand, let \((\eta^0, w^0) \in D(-A) = V^s\). We have to show that the series
\[ \left[ \sum_{k \in \mathbb{Z}} (\tilde{\eta}_k(t), \tilde{w}_k(t))e^{ikx} \right]_t (0) \]
is convergent. This is equivalent to show that
\[ [S_N]_t (0) = \left[ \sum_{|k| \leq N} (\tilde{\eta}_k(t), \tilde{w}_k(t))e^{ikx} \right]_t (0) \]
is a Cauchy sequence. Indeed,
\[ \| [S_{N+p}]_t (0) - [S_N]_t (0) \|^2 = \sum_{N \leq |k| \leq N+p} \left( |\tilde{\eta}_{k,t}(0)|^2 + \frac{w_1}{w_2} |\tilde{w}_{k,t}(0)|^2 \right) (1 + k^2)^s. \]  
(54)

From (13) we deduce that
\[ |\tilde{\eta}_{k,t}(0)|^2 = \left| \frac{\beta_1(1 + k^2)^{\alpha_1}}{1 + bk^2 + b_2k^4} \right|^2 |\tilde{\eta}(0)|^2 + k^2 w_1^2 |\tilde{w}(0)|^2 \]  
(55)
and
\[ |\tilde{w}_{k,t}(0)|^2 = k^2 w_2^2 |\tilde{\eta}(0)|^2 + \left| \frac{\beta_2(1 + k^2)^{\alpha_2}}{1 + dk^2 + d_2k^4} \right|^2 |\tilde{w}(0)|^2 \]  
(56)
where \(M\) is a positive constant depending only on \(\alpha_1, \alpha_2, \beta_1, \beta_2, b, b_2, d, d_2\). Then, from (55) and (56) we have that
\[ |\tilde{\eta}_{k,t}(0)|^2 + \frac{w_1}{w_2} |\tilde{w}_{k,t}(0)|^2 \]  
\[ \leq k^2 w_1^2 |\tilde{\eta}(0)|^2 + \frac{w_1}{w_2} |\tilde{w}(0)|^2 \]  
\[ + M \left( |\tilde{\eta}(0)|^2 + \frac{w_1}{w_2} |\tilde{w}(0)|^2 \right) \]  
(57)
\[ \leq M \left( |\tilde{\eta}(0)|^2 + \frac{w_1}{w_2} |\tilde{w}(0)|^2 \right) . \]
Therefore, from (54) and (57) we obtain the following estimate,
\[
\| [S_{N+p}]_t (0) - [S_N]_t (0) \|_{V^*} \leq M \sum_{N \leq |k| \leq N+p} \left( |\tilde{\eta}_k(0)|^2 + \frac{w_1}{w_2} |\tilde{\omega}_k(0)|^2 \right) (1 + k^2)^s
\]
(58)
and as \((\eta^0, w^0) \in D(-A) = V^*\),
\[
[S_N]_t (0) = \left[ \sum_{|k| \leq N} (\tilde{\eta}_k(t), \tilde{\omega}_k(t))e^{ikx} \right] (0)
\]
is a Cauchy sequence. Thus,
\[
-A(\eta^0, w^0) = \lim_{N \to \infty} -A(S_N)(0) = \lim_{N \to \infty} [S_N]_t (0) = \left[ \sum_{k \in \mathbb{Z}} (\check{\eta}_k(t), \check{\omega}_k(t))e^{ikx} \right] (0)
\]
(59)
Thus,
\[
-A(\eta^0, w^0) = \lim_{t \to 0} \frac{S(t)\eta^0 - \eta^0}{t} = B(\eta^0, w^0).
\]
Hence, \((\eta^0, w^0) \in D(B)\) and \(-A(\eta^0, w^0) = B(\eta^0, w^0)\), for any \((\eta^0, w^0) \in D(-A) = V^*\).

**Remark 5.** In fact much more can be said about the regularity of solutions of (6). Since (6) is linear and \(-A\) is a bounded operator, we can easily deduce that \((\eta, w) \in C^w([0, \infty); V^*)\), where \(C^w([0, \infty); V^*)\) represents the class of the analytic functions defined in \([0, \infty)\) with values in \(V^*\). Indeed, for \(t_0 \in [0, \infty)\)
\[
\left\| \sum_{n=0}^{\infty} \frac{d^n}{dt^n}(\eta, w)(t_0) \frac{(t - t_0)^n}{n!} \right\|_{V^*} \leq \sum_{n=0}^{\infty} \frac{|t - t_0|^n}{n!} \left\| \frac{d^n}{dt^n}(\eta, w)(t_0) \right\|_{V^*}
\]
\[
\leq \| (\eta, w)(t_0) \|_{V^*} \sum_{n=0}^{\infty} \frac{|t - t_0|^n}{n!} \|A\|_{2(V^*)}^n < \infty.
\]
Hence, the series \(\sum_{n=0}^{\infty} \frac{d^n}{dt^n}(\eta, w)(t_0) \frac{(t - t_0)^n}{n!}\) is (absolutely) convergent and
\[
(\eta, w)(t) = \exp(-A(t - t_0))(\eta, w)(t_0) = \sum_{n=0}^{\infty} \frac{(t - t_0)^n}{n!}(-A)^n(\eta, w)(t_0)
\]
(59)
As a direct consequence of the Theorems 2.4 and 2.5 and the general theory of the evolution equations (see, for instance, [9]), we have the following existence and uniqueness result:

**Theorem 2.6.** Let \(T > 0\) and \(s \in \mathbb{R}\). For any \((\eta^0, w^0) \in V^*\) and \((f, g) \in L^1(0, T; V^*)\), there exists a unique solution \((\eta, w) \in W^{1,1}([0, T]; V^*)\) of the system
\[
\begin{pmatrix}
\eta \\
w
\end{pmatrix}_t + A \begin{pmatrix}
\eta \\
w
\end{pmatrix}(t) = \begin{pmatrix}
f \\
g
\end{pmatrix}, \quad \begin{pmatrix}
\eta \\
w
\end{pmatrix}(0) = \begin{pmatrix}
\eta^0 \\
w^0
\end{pmatrix},
\]
(59)
which verifies the constant variation formula

\[
\left( \begin{array}{c}
\eta \\
\omega
\end{array} \right)(t) = S(t) \left( \begin{array}{c}
\eta^0 \\
\omega^0
\end{array} \right) + \int_0^t S(t-s) \left( \begin{array}{c}
f \\
g
\end{array} \right)(s) \, ds.
\]  

(60)

2.2. Asymptotic behavior. In this section we study the behavior of the solutions of system (6), as the time goes to infinity. In order to have a dissipative system, we assume that

\[
\beta_1 \geq 0, \quad \beta_2 \geq 0, \quad \beta_1^2 + \beta_2^2 > 0.
\]  

(61)

Multiplying both sides of the first equation in (13) by \( \overline{\eta}_k \) and the second equation by \( \overline{\omega}_k \), if \( a = c > 0 \), or by \( \left( \frac{1-ak^2}{1-k^2} \right) \overline{w}_k \) if \( a < 0 \), \( c < 0 \), and then adding the resulting first equation to the conjugate of the resulting second equation, we obtain

\[
\frac{d}{dt}(1 + bk^2 + b_k^2) \left( |\hat{\eta}_k|^2 + \frac{w_1(k)}{w_2(k)} |\hat{\omega}_k|^2 \right) = -2\beta_1(1 + k^2) \overline{\eta}_k^2 - 2\beta_2(1 + k^2) \overline{\omega}_k \left( \frac{1-ak^2}{1-k^2} \right) |\hat{\omega}_k|^2,
\]  

(62)

for all \( k \in \mathbb{Z} \). Thus, if we define

\[
E[\eta, \omega](t) = \int_0^{2\pi} \left( \left| (I - b\partial_x^2 + b_1\partial_x^2)^{1/2} \eta(t,x) \right|^2 + \left| (I - b\partial_x^2 + b_1\partial_x^2)^{1/2} \omega(t,x) \right|^2 \right) \, dx,
\]

then, from (62), we get

\[
\frac{d}{dt} E[\eta, \omega](t) \leq -C \left( \|\eta\|^2_{L^2(0,2\pi)} + \|\omega\|^2_{L^2(0,2\pi)} \right),
\]

(64)

for any \( t \geq 0 \) and some positive constant \( C > 0 \), depending only on \( \beta_1, \beta_2, a \) and \( c \).

Firstly, we analyze the cases in which the solutions of (6) decay exponentially to zero. We recall that the solutions to (6) decay exponentially in \( V^* \) if there exist two positive constants \( M \) and \( \mu \), such that

\[
\|S(t)(\eta^0,\omega^0)\|_{V^*} \leq Me^{-\mu t}\|\eta^0,\omega^0\|_{V^*}, \quad t \geq 0, \quad (\eta^0,\omega^0) \in V^*.
\]  

(65)

We have the following result.

**Theorem 2.7.** The solutions of (6) decay exponentially in \( V^* \) if and only if \( \alpha_1 = \alpha_2 = 4 \) and \( \beta_1, \beta_2 > 0 \). Moreover, \( \mu \) from (65) is given by

\[
\mu = \inf_{k \in \mathbb{Z}} \left\{ |\Re(\lambda_k^-)| \right\},
\]  

(66)

where the eigenvalues \( \lambda_k^- \) are given by (14).

**Proof.** First, let \( \alpha_1 = \alpha_2 = 4 \) and \( \beta_1, \beta_2 > 0 \). In this case, Remarks 2 and 3 ensure that the eigenvalues \( \lambda_k^- \) are uniformly bounded away from the real axis, i.e.,

\[
|\Re(\lambda_k^-)| \geq D > 0, \quad k \in \mathbb{Z},
\]

where \( D \), is a positive number, depending on the parameters \( \beta_1, \beta_2, \alpha_1, \alpha_2, b, d, b_2 \) and \( d_2 \). Then, there exists \( \mu = \inf_{k \in \mathbb{Z}} \left\{ |\Re(\lambda_k^-)| \right\} \), and from the Theorem 2.3, we
obtain that
\[
|\hat{\eta}_k(t)|^2 + \frac{w_1}{w_2}|\hat{w}_k(t)|^2 \leq M \left( |\hat{\eta}_k|^2 + \frac{w_1}{w_2}|\hat{w}_k|^2 \right) e^{-2t|\Re(\lambda_k)|}
\]
\[
\leq Me^{-2t\mu} \left( |\eta_0|^2 + \frac{w_1}{w_2}|w_0|^2 \right),
\]
for some constant \(M > 0\), which implies (65).

On the other hand, if \(\alpha_2 < \alpha_1 < 4\), from Proposition 2 we obtain \(l > 0\) and \(\delta > 0\), such that \(|\Re(\lambda_k^+)| \geq \frac{l}{|\mu_k^+|}\). Thus, from Theorem 2.3 it follows that
\[
|\hat{\eta}_k(t)|^2 + \frac{w_1}{w_2}|\hat{w}_k(t)|^2 \leq Me^{-2t|\Re(\lambda_k^-)|} \left( |\eta_0|^2 + \frac{w_1}{w_2}|w_0|^2 \right)
\]
\[
\leq Me^{-2t\mu} \left( |\eta_0|^2 + \frac{w_1}{w_2}|w_0|^2 \right).
\]
Hence, the decay rate cannot be exponential. If \(\alpha_1 < \alpha_2 < 4\), similar estimates leads to the same result. Therefore, \(\alpha_1 = \alpha_2 = 4\). Now, suppose that \(\beta_1, \beta_2 = 0\). From Remark 2, there exists a constant \(l > 0\), such that \(|\Re(\lambda_k^-)| \geq \frac{l}{|\mu_k^-|}\). Then, we infer that the same conclusion holds. Therefore, \(\beta_1, \beta_2 > 0\).

Now, we analyze the decay rate of solutions in the remaining cases. Since we know from Theorem 2.7 that we do not have an exponential decay, we can only expect a polynomial decay if the initial data have additional smoothness properties. We have the following result:

**Theorem 2.8.** Suppose that (61) holds and \(\alpha_1, \alpha_2 \in [0, 4]\). Let \(\delta > 0\) be defined by
\[
\delta = \begin{cases} 
4 - \max\{\alpha_1, \alpha_2\} & \text{if} \ \max\{\alpha_1, \alpha_2\} \leq 3, \\
\max\{\alpha_1, \alpha_2\} & \text{if} \ \max\{\alpha_1, \alpha_2\} > 3, \ \alpha_1 + \alpha_2 \leq 6 \\
4 - \min\{\alpha_1, \alpha_2\} & \text{if} \ \max\{\alpha_1, \alpha_2\} > 3, \ \alpha_1 + \alpha_2 > 6.
\end{cases}
\tag{67}
\]
Then, there exists \(M > 0\), such that the solutions of (6) satisfy
\[
\|S(t)(\eta^0, w^0)\|_{V_s} \leq \frac{M}{(1 + t)^{\frac{s}{2}}} \|\eta^0 \|_{V_s+q}, \quad t \geq 0, \quad (\eta^0, w^0) \in V_{s+q}, \tag{68}
\]
where \(s \in \mathbb{R}\) and \(q > 0\).

**Proof.** We use an argument developed in [12]. Firstly, we remark that it is sufficient to prove the result for \(t\) sufficiently large. From Proposition 2, there exists a constant \(l > 0\), such that
\[
|\Re(\lambda_k^+)| \geq \frac{l}{|k|}, \quad k \in \mathbb{Z}^*.
\tag{69}
\]
From Theorem 2.3 and (69) we deduce that
\[
\|S(t)(\eta^0, w^0)\|_{V_s}^2 \leq M^2 \sum_{k \in \mathbb{Z}} (1 + k^2)^s e^{-2t \min\{|\Re(\lambda_k^+)|, |\Re(\lambda_k^-)|\}} \left( |\eta_0|^2 + \frac{w_1}{w_2}|w_0|^2 \right)
\]
\[
= M^2 \left( e^{-t s_{\beta_1, \beta_2}} \left( |\eta_0|^2 + \frac{w_1}{w_2}|w_0|^2 \right) \right) + \sum_{k \in \mathbb{Z}^*} \frac{1}{(1 + k^2)^q} e^{\frac{\mu_k}{|\mu_k^-|}} (1 + k^2)^{s+q} \left( |\eta_0|^2 + \frac{w_1}{w_2}|w_0|^2 \right).
\tag{70}
\]
Let us analyze the function $e^{-\frac{2t}{|k|^q}}$. As $\lim_{k \to \infty} e^{-\frac{2t}{|k|^q}} = 1$, there exists a constant $M > 0$, such that $e^{-\frac{2t}{|k|^q}} \geq M$, for all $|k| \geq k_0$ and some $k_0 \in \mathbb{N}$. Moreover, if $1 \leq |k| \leq k_0$, then $e^{-\frac{2t}{|k|^q}} \leq e^{-\frac{2t}{|k_0|^q}}$.

Hence, we obtain that

$$\sum_{k \in \mathbb{Z}} \frac{1}{(1 + k^2)^2} e^{-\frac{2t}{|k|^q}} (1 + k^2)^{s+q} \left( |\eta_k^0|^2 + \frac{u_1}{w_2} |\hat{\omega}_k^0|^2 \right)$$

$$\leq e^{2t} \sum_{1 \leq |k| \leq k_0} \frac{1}{(1 + k^2)^2} e^{-\frac{2t(t+1)}{|k|^q}} (1 + k^2)^{s+q} \left( |\eta_k^0|^2 + \frac{u_1}{w_2} |\hat{\omega}_k^0|^2 \right)$$

$$+ \frac{1}{M} \sum_{k_0 \leq |k|} \frac{1}{(1 + k^2)^2} e^{-\frac{2t(t+1)}{|k|^q}} (1 + k^2)^{s+q} \left( |\eta_k^0|^2 + \frac{u_1}{w_2} |\hat{\omega}_k^0|^2 \right)$$

$$\leq M \sum_{k \in \mathbb{Z}} \frac{1}{(1 + k^2)^2} e^{-\frac{2t(t+1)}{|k|^q}} (1 + k^2)^{s+q} \left( |\eta_k^0|^2 + \frac{u_1}{w_2} |\hat{\omega}_k^0|^2 \right).$$

Let us now study the term $E_k(t) = \frac{1}{(1 + k^2)^2} e^{-\frac{2t(t+1)}{|k|^q}}$, for $k \in \mathbb{Z}^*$. Firstly, we remark that $x \leq e^{x-1}$ for all $x \geq 0$. Then, given $\xi > 0$ the following inequality holds true

$$x^\xi e^{-x} \leq \xi^\xi e^{-\xi}, \quad x \geq 0. \quad (72)$$

By using (72) with $x = \frac{2t(t+1)}{|k|^q}$ and $\xi = \frac{2q}{q}$ we deduce that, there exists a constant $C(q, \delta, \ell) > 0$, such that

$$e^{-\frac{2t(t+1)}{|k|^q}} \leq \left( \frac{2q}{q} \right)^{\frac{2q}{q}} e^{-\frac{2q}{q} |k|^q} \left( \frac{2q}{2q} \right)^{\frac{2q}{q}} \leq C(q, \delta, \ell) \frac{(1 + k^2)^q}{(t+1)^q}.$$

From the last estimate, we obtain for each $k \in \mathbb{Z}^*$

$$|E_k(t)| \leq C(q, \delta, \ell) \frac{(t+1)^q}{(t+1)^q}, \quad t \geq 0. \quad (73)$$

Therefore, from (70), (71) and (73) we have that

$$||S(t)(\eta^0, u^0)||^2 \leq M^2 \left( e^{-\min\{\beta_1, \beta_2\} t} \left( |\eta^0|^2 + \frac{u_1}{w_2} |\hat{\omega}_k^0|^2 \right) + \frac{1}{(t+1)^q} ||(\eta^0, u^0)||^2_{H^{s+q}} \right).$$

3. The nonlinear system. We are now in a position to prove the well-posedness and the stabilization for the solutions of the nonlinear system (3) issued from small initial data, when the linearized system is exponentially stable, i.e., under the hypothesis of Theorem 2.7. The proof will be done by using a fixed point argument. Therefore, the application of the following lemma, proved in [4], will be needed:

**Lemma 3.1.** Let $s \geq -1$. There exists a constant $C > 0$, depending only on $s$, such that

$$||fg||_{H^s_p(0, 2\pi)} \leq C||f||_{H^{s+1}_p(0, 2\pi)} ||g||_{H^{s+1}_p(0, 2\pi)},$$

for any $f, g \in H^{s+1}_p(0, 2\pi)$. 

Remark 6. We write (3) in its integral form
\[
\begin{pmatrix}
\eta \\
w
\end{pmatrix}_t + A \begin{pmatrix}
\eta \\
w
\end{pmatrix}(t) + N \begin{pmatrix}
\eta \\
w
\end{pmatrix}(t) = \begin{pmatrix} 0 \\
0
\end{pmatrix},
\]
(74)
where \(N\) is defined by
\[
N(\eta, w) = \left( \begin{array}{c}
(I - b\partial_x^2 + b_2\partial_x^4)^{-1} \left[ (\eta w)_x - b(\eta w)_{xxx} + (a + b - \frac{3}{2})(\eta w_x)_x \right] \\
(I - b\partial_x^2 + d_2\partial_x^4)^{-1} \left[ w w_x + c(w w_x)_{xx} + (\eta w_x)_x \right] \\
- (c + d - 1)w x w_{xx} - (c + d)w w_{xxx}
\end{array} \right),
\]
(75)
and \(A\) is the compact operator defined by (10). Thus, we obtain that the solution of (74) is given by
\[
(\eta, w)(t) = S(t)(\eta_0, w_0) - \int_0^t S(t - \tau)N(\eta, w)(\tau)\,d\tau,
\]
(76)
where \(\{S(t)\}_{t \geq 0}\) is the semigroup defined in Theorem 2.4.

The main result of this section reads as follows. The proof follows closely the arguments developed in [15].

Theorem 3.2. Let \(s \geq 0\) and suppose that \(\beta_1, \beta_2 > 0\) and \(\alpha_1 = \alpha_2 = 4\). There exists \(r > 0, C > 0\) and \(\mu > 0\), such that, for any \((\eta_0, w_0) \in V^s\), satisfying
\[
||(\eta, w)||_{V^s} \leq r,
\]
the system (3) admits a unique solution \((\eta, w) \in C([0, \infty); V^s)\) which verifies
\[
||(\eta(t), w(t))||_{V^s} \leq Ce^{-\mu t}||(\eta_0, w_0)||_{V^s}, \quad t \geq 0.
\]
(77)
Moreover, \(\mu\) may be taken as in (66).

Proof. We remark that the hypothesis of Theorem 2.7 are verified and there exist \(M, \mu > 0\), such that (65) holds true. In order to use a fixed point argument, we define the space
\[
Y_{s, \mu} = \{(\eta, w) \in C([0, \infty); V^s) : e^{\mu t}(\eta, w) \in C([0, \infty); V^s)\},
\]
with the norm
\[
||(\eta, w)||_{Y_{s, \mu}} := \sup_{0 \leq t < \infty} ||e^{\mu t}(\eta, w)(t)||_{V^s},
\]
and the function \(\Gamma : Y_{s, \mu} \rightarrow Y_{s, \mu}\) by
\[
\Gamma(\eta, w)(t) = S(t)(\eta_0, w_0) - \int_0^t S(t - \tau)N(\eta, w)(\tau)\,d\tau.
\]
From Lemma 3.1, we deduce that
\[
||N(\eta_1, w_1)||_{V^s} \leq C||(\eta_1, w_1)||_{V^s}^2,
\]
and
\[
||N(\eta_1, w_1) - N(\eta_2, w_2)||_{V^s} \\
\leq C(||(\eta_1, w_1)||_{V^s}^2 + ||(\eta_2, w_2)||_{V^s})(||\eta_1, w_1) - (\eta_2, w_2)||_{V^s},
\]
(79)
for any \((\eta_1, w_1), (\eta_2, w_2) \in V^s\) and for some \(C > 0\). Then, combining the estimates above and Theorem 2.7, we obtain
\[
\|\Gamma(\eta, w)(t)\|_{V^s} \leq M e^{-\mu t}\|N(\eta, w)(0)\|_{V^s} + M \int_0^t e^{-\mu (t-\tau)}\|N(\eta, w)(\tau)\|_{V^s} \, d\tau
\]
\[
\leq M e^{-\mu t}\|N(\eta, w)(0)\|_{V^s} + MC e^{-\mu t} \sup_{0 \leq \tau \leq t} \|e^{\mu \tau}(\eta, w)(\tau)\|^2_{V^s},
\]
for any \(t \geq 0\) and some positive constants \(M\) and \(C\). Thus, if we take \((\eta, w) \in B_R(0)\) where
\[
B_R(0) = \{ (\eta, w) \in Y_{s, \mu}; \|N(\eta, w)\|_{Y_{s, \mu}} \leq R \},
\]
from (80) we conclude that
\[
\|\Gamma(\eta, w)\|_{Y_{s, \mu}} \leq M\|N(\eta, w)(0)\|_{V^s} + MC\|N(\eta, w)\|_{Y_{s, \mu}}^2 \leq Mr + MCR^2.
\]
A similar calculations shows that, for any \((\eta_1, w_1), (\eta_2, w_2) \in B_R(0)\), we have that
\[
\| (\Gamma(\eta_1, w_1) - \Gamma(\eta_2, w_2)) (t) \|_{V^s} \leq M e^{-\mu t} \sup_{0 \leq \tau \leq t} \|e^{\mu \tau}(N(\eta_1, w_1) - N(\eta_2, w_2))(\tau)\|_{V^s} \leq MCe^{-\mu t} \sup_{0 \leq \tau \leq t} \{(\|N(\eta_1, w_1)(\tau)\|_{V^s} + \|N(\eta_2, w_2)(\tau)\|_{V^s}) \times \|e^{\mu \tau}((\eta_1, w_1) - (\eta_2, w_2))(\tau)\|_{V^s} \leq 2RMC \sup_{0 \leq \tau \leq t} \|e^{\mu \tau}((\eta_1, w_1) - (\eta_2, w_2))(\tau)\|_{V^s} \)
\]
Therefore,
\[
\|\Gamma(\eta_1, w_1) - \Gamma(\eta_2, w_2)\|_{Y_{s, \mu}} \leq 2RMC\|((\eta_1, w_1) - (\eta_2, w_2))\|_{Y_{s, \mu}}.
\]
By choosing \(R = 2Mr\) and \(r \leq \frac{1}{8CM^2}\), from (81) and (83) we deduce that the map
\[
\Gamma : B_R(0) \subseteq Y_{s, \mu} \rightarrow B_R(0)
\]
is a contraction, hence it admits a unique fixed point \((\eta, w) \in B_R(0)\) which solves the integral equation (74). Moreover,
\[
\|e^{\mu t}(\eta, w)(t)\|_{V^s} \leq R = 2Mr, t \geq 0.
\]
The proof of the Theorem is complete.

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**REFERENCES**

[1] D. K. Arrowsmith and C. M. Place, *Dynamical Systems: Differential Equations, Maps and Chaotic Behaviour*, Chapman and Hall, London, 1992.

[2] G. J. Bautista and A. F. Pazoto, Large-time red behavior of a linear Boussinesq system for the water waves, *J. Dyn. Diff. Equ.*, 31 (2019), 959–978.

[3] J. L. Bona, M. Chen and J.-C. Saut, Boussinesq equations and other systems for smallamplitude long waves in nonlinear dispersive media. I: Derivation and linear theory, *J. Nonlinear Sci.*, 12 (2002), 283–318.

[4] J. L. Bona, M. Chen and J.-C. Saut, Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. II: Nonlinear theory, *Nonlinearity*, 17 (2004), 925–9052.
[5] J. L. Bona, G. W. Pritchard and L. R. Scott, An evaluation of a model equation for water waves, Philos. Trans. Roy. Soc. London Ser. A, 302 (1981), 457–510.
[6] J. L. Bona and J. Wu, Zero-dissipation limit for nonlinear waves, M2AN Math. Model. Numer. Anal., 34 (2000), 275–301.
[7] J. Boussinesq, Théorie de l’intumescence liquide appelée onde solitaire ou de translation se propageant dans un canal rectangulaire, Comptes Rendus de l’Académie de Sciences, 72 (1871), 755–759.
[8] R. A. Capistrano-Filho, A. F. Pazoto and L. Rosier, Control of a Boussinesq system of KdV-KdV type on a bounded interval, ESAIM Control Optim. Calc. Var., DOI: https://doi.org/10.1051/cocv/2018036.
[9] T. Cazenave and A. Haraux, An Introduction to Semilinear Evolution Equations, Oxford Lecture Series in Mathematics and its Applications, 13, The Clarendon Press, Oxford University Press, New York, 1998.
[10] J.-P. Chehab, P. Garnier and Y. Mammeri, Long-time behavior of solutions of a BBM equation with generalized damping, Discrete Contin. Dyn. Syst. Ser. B, 20 (2015), 1897–1915.
[11] M. Chen and O. Goubet, Long-time asymptotic behavior of dissipative Boussinesq systems, Discrete Contin. Dyn. Syst., 17 (2007), 509–528.
[12] W. Littman and L. Markus, Some recent results on control and stabilization of flexible structures, Proc. COMCON on Stabilization of Flexible Structures (Montpellier, France), 1987, 151–161.
[13] S. Micu, J. H. Ortega, L. Rosier and B.-Y. Zhang, Control and stabilization of a family of Boussinesq systems, Discrete Contin. Dyn. Syst., 24 (2009), 273–313.
[14] S. Micu and A. F. Pazoto, Stabilization of a Boussinesq system with localized damping, J. Anal. Math., 137 (2019), 291–337.
[15] S. Micu and A. F. Pazoto, Stabilization of a Boussinesq system with generalized damping, Systems Control Lett., 105 (2017), 62–69.
[16] A. F. Pazoto and L. Rosier, Stabilization of a Boussinesq system of KdV-KdV type, Systems Control Lett., 57 (2008), 595–601.
[17] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach Science, New York, 1987.

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