SPACES OF STABILITY CONDITIONS VIA EXCEPTIONAL COLLECTION: LENGTH 4 CASE

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ABSTRACT. In this paper, we study the space of stability conditions on triangulated categories generated by an exceptional collection. We give an exact description of subspace of stability conditions generated by length 4 complete exceptional collection.

1. Introduction

The space of stability conditions on a triangulated category $\mathcal{T}$ was introduced by Bridgeland [1], with inspiration from work by Douglas [2] in string theory. Roughly speaking, a stability condition $\sigma$ on a triangulated category $\mathcal{T}$ consists of data $(Z, \mathcal{P})$ where $Z$ is called the central charge, and $\mathcal{P}$ is a slicing on $\mathcal{T}$. There are many examples and conjectures about the constructions of the stability conditions, we refer to surveys [3, 4, 5] for more information and applications.

Based on [6, 7, 8], Macrì [9, 10] gave a procedure generating the stability conditions on triangulated categories generated by an exceptional collection which was developed further by Collins and Polishchuk [11]. To a fixed complete exceptional collection $\mathcal{E} = \{E_0, \cdots, E_n\}$ on a triangulated category $\mathcal{T}$, Macrì [9, 10] obtained an subspace of stability conditions, denoted by $\Theta_\mathcal{E}$, generated by $\mathcal{E}$.

Motivated by the work of Macrì [9, 10], Dimitrov and Katzarkov [12, 13] introduced the notation $\sigma$-exceptional collection and reformulated Macrì’s construction of $\Theta_\mathcal{E}$. For length 3 complete exceptional collection $\mathcal{E} = \{E_0, E_1, E_2\}$, Dimitrov and Katzarkov [12] gave an exact description of $\Theta_\mathcal{E}$. For longer length cases, it is difficult to give an exact description of $\Theta_\mathcal{E}$ following the approach in [12]. As explained in [12, Remark 2.5], the main difficulty is that the similarly equality in [12, Lemma 2.4] does not hold for $n \geq 3$.

In this paper we give a description of $\Theta_\mathcal{E}$ for length 4 exceptional collection $\mathcal{E}$ as follows
Theorem 1.1. Let $\mathcal{T}$ be a finite linear triangulated category over $\mathbb{C}$. Let $\mathcal{E} = \{E_0, E_1, E_2, E_3\}$ be a complete exceptional collection, denoted by

$$k_{ij} = \min\{k : \text{Hom}^k(E_i, E_j) \neq 0\} \in \mathbb{Z},$$

$$\mathcal{K}_{02} = \min\{k_{02}, k_{01} + k_{12} - 1\}, \quad \mathcal{K}_{13} = \min\{k_{13}, k_{12} + k_{23} - 1\},$$

$$\mathcal{K}_{03} = \min\{k_{01} + k_{12} + k_{23} - 2, k_{01} + k_{13} - 1, k_{02} + k_{23} - 1, k_{03}\},$$

and

$$\Theta_1 = \left\{(y_0, y_1, y_2, y_3) \in \mathbb{R}^4 \mid \begin{array}{l}
y_0 - y_1 < k_{01}, \quad y_1 - y_2 < k_{12} \\
y_2 - y_3 < k_{23}, \quad y_1 - y_3 < \mathcal{K}_{13} \\
y_0 - y_3 < \mathcal{K}_{03}, \quad y_0 - y_2 < \mathcal{K}_{02}
\end{array} \right\}$$

Then $\Theta_\mathcal{E}$ is homeomorphic with the set

$$\mathbb{R}_{>0}^4 \times (\Theta_1 \setminus \Delta).$$

where $\Delta$ is the union of following five sets

$$\Delta^1 = \left\{(x_0, x_1, x_2, x_3) \in \Theta_1 \mid \begin{array}{l}
\mathcal{R}_{23}^1, \quad \mathcal{R}_{13}^1 \\
x_0 - x_1 = k_{01} - 1, \quad [x_0 - x_2] < \mathcal{K}_{02} - 1, \\
x_2 - x_3 = k_{23} - 1, \quad x_1 - x_2 = [x_1 - x_2]
\end{array} \right\}$$

$$\Delta^2 = \left\{(x_0, x_1, x_2, x_3) \in \Theta_1 \mid \begin{array}{l}
\mathcal{R}_{13}^1, \quad \mathcal{R}_{10}^0 \\
x_0 - x_2 < \mathcal{K}_{02} - 1, \\
x_2 - x_3 = k_{23} - 1, \quad x_1 - x_2 = [x_1 - x_2]
\end{array} \right\}$$

$$\Delta^3 = \left\{(x_0, x_1, x_2, x_3) \in \Theta_1 \mid \begin{array}{l}
\mathcal{R}_{12}^1, \quad \mathcal{R}_{10}^0 \\
x_0 - x_1 < k_{01} - 1, \\
x_1 - x_2 = k_{12} - 1, \quad x_2 - x_3 = [x_2 - x_3]
\end{array} \right\}$$

$$\Delta^4 = \left\{(x_0, x_1, x_2, x_3) \in \Theta_1 \mid \begin{array}{l}
\mathcal{R}_{12}^1, \quad \mathcal{R}_{13}^0 \\
x_0 - x_1 < k_{01} - 1, \\
x_1 - x_2 = k_{12} - 1, \quad [x_0 - x_3] = \mathcal{K}_{03} - 1
\end{array} \right\}$$

$$\Delta^5 = \left\{(x_0, x_1, x_2, x_3) \in \Theta_1 \mid \begin{array}{l}
\mathcal{R}_{12}^0, \quad \mathcal{R}_{13}^1 \\
x_0 - x_1 < k_{01} - 1, \\
x_0 - x_1 = k_{01} - 1, \quad [x_0 - x_2] = \mathcal{K}_{02} - 1
\end{array} \right\}$$

where $\mathcal{R}_{ij}^\alpha, \alpha \in \{0, 1\}, 1 \leq i, j \leq 3$ are the relations defined by (5).

Proof idea. Our method is different from [12]. We refer to Section 3 for complete notations. Our proof based on two ideas. First, we can see that $x_0 = y_0$, then we need to $z_i \in (x_0 - 1, x_0 + 1)$, then the possible choices of $p_i$ are $[x_0 - x_i]$ or $[x_0 - x_i]$, then we reduce to check conditions $(z_0, z_1, z_2, z_3) \in S^3(-1, 1)$ and $(0, p_1, p_2, p_3) \in A_0$, secondly, our
main technical observation is the properties of floor function $\lfloor x \rfloor$ (See Lemma 3.1, Lemma 3.2, Lemma 3.4, Corollary 3.3 and Corollary 3.5) which can classify $\Theta_1$ into different cases, and check them case by case (See Lemma 3.6-3.9). This classification can give all elements of $\Delta$ and avoid the more technical arguments of [12, Lemma 2.4].

The paper is structured as follows. In the next section we review the basic notations and constructions. In Section 3 we give the complete proof. In Section 4 we give the examples and some remarks of author’s original motivation.

2. Review: basic definitions and constructions

In this section we review the basic notations and facts concerning the construction of the stability conditions following Macrì [9, 10], Dimitrov–Katzarkov [12, 13] and Bridgeland [1]. For details the reader is referred to the original papers [9, 10, 12, 13, 1, 6].

Let $\mathcal{T}$ be a linear triangulated category and of finite type over $\mathbb{C}$. We denoted by $\langle E \rangle$ the extension closed subcategory of $\mathcal{T}$ generated by subcategory $E$ of $\mathcal{T}$ and $K(\mathcal{T})$ the Grothendieck group of $\mathcal{T}$. Similarly, the Grothendieck group of an abelian category $\mathcal{A}$ is denoted $K(\mathcal{A})$.

2.1. Stability conditions.

Definition 2.1 ([1]). A stability condition $\sigma = (Z, \mathcal{P})$ on a triangulated category $\mathcal{T}$ consists of a group homomorphism $Z : K(\mathcal{T}) \rightarrow \mathbb{C}$, and full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{T}$ for each $\phi \in \mathbb{R}$, satisfying the following axioms:

(a) if $E \in \mathcal{P}(\phi)$ then $Z(E) = m(E) \exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$,
(b) for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$,
(c) if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$ then $\text{Hom}_\mathcal{T}(A_1, A_2) = 0$,
(d) for each nonzero object $E \in \mathcal{T}$ there are a finite sequence of real numbers

$\phi_1 > \phi_2 > \cdots > \phi_n$

and a collection of triangles

$$
\begin{array}{ccccccc}
0 & \rightarrow & E_0 & \rightarrow & E_1 & \rightarrow & E_2 & \cdots & \rightarrow & E_{n-1} & \rightarrow & E_n = E \\
\downarrow A_1 & & \downarrow A_2 & & \downarrow & & \downarrow & & \downarrow & & \downarrow A_n \\
& & & & & & & & & & & \\
\end{array}
$$

with $A_j \in \mathcal{P}(\phi_j)$ for all $j$. 
Nonzero objects in \( P(\phi) \) are said to be semistable of phase \( \phi \). For interval \( I \subseteq \mathbb{R} \), we denote \( P(I) \) the extension closed subcategory of \( T \) generated by the subcategories \( P(\phi), \phi \in I \). If there exists some \( \varepsilon > 0 \) such that each \( P((\phi - \varepsilon, \phi + \varepsilon)) \) is of finite length, then we call the stability condition is locally finite.

By [1, Theorem 1.2, Corollary 1.3], the space of locally finite stability conditions, denoted by \( \text{Stab}(T) \), is a complex manifold. By [1, Lemma 8.2], \( \text{Stab}(T) \) carries a right action of the group \( \widetilde{\text{GL}}^+(2, \mathbb{R}) \), the universal covering space of \( \text{GL}^+(2, \mathbb{R}) \), and a left action by isometries of the group of exact autoequivalences of \( T \).

For an abelian category \( A \), Bridgeland [1, Definition 2.1] defined the stability function \( Z : K(A) \to \mathbb{C} \), and [1, Definition 2.2] introduced the Harder-Narasimhan property for the stability function. Then Bridgeland [1, Proposition 5.3] showed that to give a stability condition on a triangulated category \( T \) is equivalent to giving a bounded \( t \)-structure on \( T \) and a stability function on its heart with the Harder-Narasimhan property.

2.2. Exceptional objects. The theory of exceptional collections and helix theory developed in the Rudakov seminar [14]. In order to get a heart of a bounded \( t \)-structure on \( T \) generated by the exceptional collection \( E \), Macrì [9, 10] introduced the notation \( \text{Ext} \) exceptional collection.

Now, we recall the basic definitions. Following [6] we denote

\[
\Hom^\bullet(A, B) = \bigoplus_{k \in \mathbb{Z}} \Hom^k(A, B)[-k],
\]

where \( A, B \in T, \Hom^k(A, B) = \Hom(A, B[k]) \). An object \( E \in T \) is called exceptional if it satisfies

\[
\Hom^i(E, E) = \begin{cases} 
\mathbb{C} & \text{if } i = 0, \\
0 & \text{otherwise.}
\end{cases}
\]

An ordered collection of exceptional objects \( \{E_0, \cdots, E_n\} \) is called exceptional in \( T \) if it satisfies

\[
\Hom^\bullet(E_i, E_j) = 0, \text{ for } i > j.
\]

We call \( \{E_0, \cdots, E_n\} \) the length \( n + 1 \) exceptional collection, and we call an exceptional collection of two objects an exceptional pair.
Let \((E, F)\) an exceptional pair. We define objects \(L_E F\) and \(R_F E\) (which we call \textit{left mutation} and \textit{right mutation} respectively) by the distinguished triangles
\[
\begin{align*}
L_E F &\rightarrow \text{Hom}^\bullet(E, F) \otimes E \rightarrow F, \\
E &\rightarrow \text{Hom}^\bullet(E, F)^* \otimes F \rightarrow R_F E,
\end{align*}
\]
where \(V[k] \otimes E\) denotes an object isomorphic to the direct sum of \(\dim V\) copies of the object \(E[k]\). A \textit{mutation} of an exceptional collection \(E = \{E_0, \cdots, E_n\}\) is defined as a mutation of a pair of adjacent objects in this collection:
\[
\begin{align*}
R_i E &= \{E_0, \cdots, E_{i-1}, E_{i+1}, E_i, E_{i+2}, \cdots, E_n\}, \\
L_i E &= \{E_0, \cdots, E_{i-1}, L_{E_i} E_{i+1}, E_i, E_{i+2}, \cdots, E_n\},
\end{align*}
\]
for \(i = 0, \cdots, n - 1\).

**Definition 2.2** ([9]). Let \(E = \{E_0, \cdots, E_n\}\) be an exceptional collection. We call \(E\)
- \textit{strong}, if \(\text{Hom}^k(E_i, E_j) = 0\) for all \(i\) and \(j\), with \(k \neq 0\);
- \textit{Ext}, if \(\text{Hom}^{\leq 0}(E_i, E_j) = 0\) for all \(i \neq j\);
- \textit{complete}, if \(E\) generates \(T\) by shifts and extensions.

**2.3. Basic construction.** Let \(E = \{E_0, \cdots, E_n\}\) be a complete exceptional collection on \(T\). For \(p = (p_0, \cdots, p_n) \in \mathbb{Z}^{n+1}\), we denote by \(E[p] := \{E_0[p_0], \cdots, E_n[p_n]\}\) the shift of the exceptional collection \(E\) and \(A_{E[p]} = \langle E_0[p_0], \cdots, E_n[p_n] \rangle\) the extension closed subcategory generated by \(E[p]\). We write
\[
A_0 = \{p : p = (p_0, \cdots, p_n) \in \mathbb{Z}^{n+1} \text{ such that } E[p] \text{ is Ext}\}.
\]

From now, we assume \(p \in A_0\). By [9, Lemma 3.14], \(A_{E[p]}\) is the heart of a bounded \(t\)-structure on \(T\). Fix \(z_0, \cdots, z_n \in \mathbb{H} := \{z \in \mathbb{C} : z = |z| \exp(i\pi\phi), 0 < \phi \leq 1\}\). Define a stability function \(Z_p : K_0(A_{E[p]}) \rightarrow \mathbb{C}\) by
\[
Z_p(E_i[p_i]) = z_i, \text{ for all } i.
\]

By [9, Lemma 3.16], \(A_{E[p]}\) is an abelian category of finite length and with finitely many simple objects, then by [1, Proposition 2.4] any stability function \(Z_p\) on \(A_{E[p]}\) satisfies the Harder–Narasimhan property. By [1, Proposition 5.3], this extends to a unique stability condition on \(T\) which is locally finite.
Following [13, Defintion 3.13], we denote by $\mathbb{H}^E[p]$ the set of stability conditions on $\mathcal{T}$ constructed above, denote $\Theta'_E[p] = \mathbb{H}^E[p] \cdot \mathcal{GL}^+ (2, \mathbb{R})$, and

$$\Theta_E = \bigcup_{p \in A_0} \Theta'_E[p].$$

For the convenience, Dimitrov–Katzarkov [12, Definition 2.3] introduced the notation $S^n(-1, 1)$, which defined by

$$S^n(-1, +1) = \{(y_0, \cdots, y_n) \in \mathbb{R}_{n+1} : -1 < y_i - y_j < 1, i < j \} \subset \mathbb{R}_{n+1}.$$

By the comment in Dimitrov–Katzarkov [12, p. 833-834] and [12, formula (18)], there exists a homeomorphism which we denote by $f_E$:

$$f_E : \Theta_E \longrightarrow \mathbb{R}^{n+1} \times \left( \bigcup_{p \in A_0} S^n(-1, +1) - p \right).$$

**Remark 2.3.** Dimitrov–Katzarkov [12, p. 833-834] defined the homeomorphism $f_E$ by the restriction. The exact definition of $f_E$ is not necessary for our proof. For the details of $f_E$ the reader is referred to the original paper [12].

For $n = 2$ case, Dimitrov–Katzarkov [12] gave an exact description of $\Theta_E$. In the next section we give a complete description of $\Theta_E$ for $n = 3$ case using different method.

3. **Main Theorem**

In this section we prove the main theorem. For $x \in \mathbb{R}$, let us denote floor function by

$$\lfloor x \rfloor = \max \{ n \in \mathbb{Z} : n \leq x \},$$

ceiling function by $\lceil x \rceil = \lfloor x \rfloor + 1$ and fractional part of $x$ by $\{x\} = x - \lfloor x \rfloor$. It is easy to check that for every $x$ and $y$, the following inequality holds

$$\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1. \quad (4)$$

For $(x_0, x_1, x_2, x_3) \in \mathbb{R}^4, 1 \leq i < j \leq 3$ and $\alpha \in \{0, 1\}$, by the inequality (4), we can define $\mathcal{R}_{ij}^\alpha$ as the relation

$$\lfloor x_0 - x_j \rfloor - \lfloor x_0 - x_i \rfloor = \lfloor x_i - x_j \rfloor + \alpha. \quad (5)$$

By the inequality (4), the following lemma holds.
Lemma 3.1. For fixed $i, j$, $R^0_{ij}$ and $R^1_{ij}$ do not hold at the same time.

Moreover, for triple relations $R^0_{12}$, $R^2_{13}$ and $R^3_{23}$, the following lemma holds.

Lemma 3.2. Let $(x_0, x_1, x_2, x_3) \in \mathbb{R}^4$, $\alpha, \beta, \gamma \in \{0, 1\}$. If relations $R^\alpha_{12}$, $R^\beta_{13}$ and $R^\gamma_{23}$ hold at the same time. Then

$$\beta \leq \alpha + \gamma \leq \beta + 1.$$

In particular, all possibly values of triple $(\alpha, \beta, \gamma)$ are

$$(0, 0, 0), \quad (1, 1, 0), \quad (0, 0, 1), \quad (1, 0, 0), \quad (1, 1, 1), \quad (0, 1, 1).$$

Proof. If relations $R^\alpha_{12}$, $R^\beta_{13}$ and $R^\gamma_{23}$ hold at the same time, i.e.

$$\lfloor x_0 - x_2 \rfloor - \lfloor x_0 - x_1 \rfloor = \lfloor x_1 - x_2 \rfloor + \alpha$$

$$\lfloor x_0 - x_3 \rfloor - \lfloor x_0 - x_1 \rfloor = \lfloor x_1 - x_3 \rfloor + \beta$$

$$\lfloor x_0 - x_3 \rfloor - \lfloor x_0 - x_2 \rfloor = \lfloor x_2 - x_3 \rfloor + \gamma$$

then we have

$$\lfloor x_1 - x_3 \rfloor + \beta = \lfloor x_0 - x_3 \rfloor - \lfloor x_0 - x_1 \rfloor = \lfloor x_2 - x_3 \rfloor + \lfloor x_1 - x_2 \rfloor + \alpha + \gamma$$

By the inequality (4), we have

$$\lfloor x_2 - x_3 \rfloor + \lfloor x_1 - x_2 \rfloor \leq \lfloor x_1 - x_3 \rfloor \leq \lfloor x_2 - x_3 \rfloor + \lfloor x_1 - x_2 \rfloor + 1$$

Hence

$$\beta \leq \alpha + \gamma \leq \beta + 1.$$

Then we can list all possibly values of $(\alpha, \beta, \gamma)$ as follows

$$(0, 0, 0), \quad (1, 1, 0), \quad (1, 0, 0), \quad (0, 0, 1), \quad (1, 1, 1), \quad (0, 1, 1).$$

Then we have following corollary

Corollary 3.3. Relations $R^0_{12}$, $R^1_{13}$ and $R^0_{23}$ do not hold at the same time. Relations $R^0_{12}$, $R^1_{13}$ and $R^1_{23}$ do not hold at the same time.

Before the main theorem, we recall more properties of $R^\alpha_{ij}, \alpha \in \{0, 1\}$.

Lemma 3.4. Let $(x_0, x_1, x_2, x_3) \in \mathbb{R}^4$. If $|x_i - x_j| = x_i - x_j$ hold, then $R^0_{ij}$ hold.
Proof. If $|x_i - x_j| = x_i - x_j$ holds, then $\{x_i\} = \{x_j\}$, we also have $\{x_0 - x_i\} = \{x_0 - x_j\}$. By the definition of $\{x\}$, we have

$$x_0 - x_i - \lfloor x_0 - x_i \rfloor = x_0 - x_j - \lfloor x_0 - x_j \rfloor$$

then $\mathcal{R}_{ij}^0$ hold. \qed

Hence we have

**Corollary 3.5.** If $\mathcal{R}_{ij}^0$ holds, then

$$(x_i + \lfloor x_0 - x_i \rfloor) - (x_j + \lfloor x_0 - x_j \rfloor) = x_i - x_j - \lfloor x_i - x_j \rfloor \in [0, 1).$$

If $\mathcal{R}_{ij}^1$ holds, then $|x_i - x_j| \neq x_i - x_j$, and

$$(x_i + \lfloor x_0 - x_i \rfloor) - (x_j + \lfloor x_0 - x_j \rfloor) = x_i - x_j - \lfloor x_i - x_j \rfloor - 1 \in (-1, 0).$$

Before the main theorem, we prove the following four lemmas which is useful in the proof of the main theorem.

**Lemma 3.6.** If $|x_0 - x_1| = k_{01} - 1$ and $|x_0 - x_2| = k_{02} - 1$, then for any $(x_0, x_1, x_2, x_3) \in \Theta_1$, there exists $(0, p_1, p_2, p_3) \in A_0$ and $z_i \in S^3(-1, +1)$ such that $x_0 = z_0, x_i = z_i - p_i, i = 1, 2, 3$.

Proof. If $|x_0 - x_1| = k_{01} - 1$ and $|x_0 - x_2| = k_{02} - 1$, then for any $(x_0, x_1, x_2, x_3) \in \Theta_1$, we can set

$$p_i = \lfloor x_0 - x_i \rfloor, z_i = x_i + p_i, i = 1, 2, 3.$$ 

Then we have

$$p_2 - p_1 = k_{02} - 1 - k_{01} + 1 \leq k_{12} - 1,$$

$$p_3 - p_1 \leq k_{03} - 1 - k_{01} + 1 \leq k_{13} - 1,$$

$$p_3 - p_2 \leq k_{03} - 1 - k_{02} + 1 \leq k_{23} - 1$$

Hence $(0, p_1, p_2, p_3) \in A_0$. By the Corollary 3.5, we have $z_i \in S^3(-1, +1)$. \qed

**Lemma 3.7.** Let

$$\Delta_{II} = \left\{ (x_0, x_1, x_2, x_3) \in \Theta_1 \middle| \begin{array}{l}
\mathcal{R}_{23}^1, \\
| x_0 - x_1 | = k_{01} - 1, | x_0 - x_2 | < k_{02} - 1, \\
| x_2 - x_3 | = k_{23} - 1, x_1 - x_2 = | x_1 - x_2 |
\end{array} \right\}.$$ 

If $|x_0 - x_1| = k_{01} - 1$ and $|x_0 - x_2| < k_{02} - 1$, then for any $(x_0, x_1, x_2, x_3) \in \Theta_1 \setminus \Delta_{II}$, there exists $(0, p_1, p_2, p_3) \in A_0$ and $z_i \in S^3(-1, +1)$ such that $x_0 = z_0, x_i = z_i - p_i, i = 1, 2, 3$. 


Proof. If \(|x_0 - x_1| = k_{01} - 1\) and \(|x_0 - x_2| < K_{02} - 1\), by the Lemma 3.1, we can consider following cases

(1) If the relation \(R_{23}^0\) holds, then we can set
\[
p_i = |x_0 - x_i|, \quad z_i = x_i + p_i, \quad i = 1, 2, 3
\]
Then we have
\[
p_2 - p_1 \leq K_{02} - 1 - k_{01} + 1 \leq k_{12} - 1,
p_3 - p_1 \leq K_{03} - 1 - k_{01} + 1 \leq K_{13} - 1,
p_3 - p_2 = |x_0 - x_3| - |x_0 - x_2| = |x_2 - x_3| \leq k_{23} - 1
\]
Hence \((0, p_1, p_2, p_3) \in A_0\). By the Corollary 3.5, we have \(z_i \in S^3(-1, +1)\).

(2) If the relation \(R_{12}^1\) holds,

(i) If \(|x_2 - x_3| \leq k_{23} - 2\), we can set
\[
p_i = |x_0 - x_i|, \quad z_i = x_i + p_i, \quad i = 1, 2, 3
\]
Then we have
\[
p_2 - p_1 \leq K_{02} - 1 - k_{01} + 1 \leq k_{12} - 1,
p_3 - p_1 \leq K_{03} - 1 - k_{01} + 1 \leq K_{13} - 1,
p_3 - p_2 = |x_0 - x_3| - |x_0 - x_2| = |x_2 - x_3| + 1 \leq k_{23} - 1
\]
Hence \((0, p_1, p_2, p_3) \in A_0\). By the Corollary 3.5, we have \(z_i \in S^3(-1, +1)\).

(ii) If the relation \(R_{12}^0\) and the condition \(|x_1 - x_2| \neq x_1 - x_2\) hold, we set
\[
p_1 = |x_0 - x_1|, \quad p_2 = |x_0 - x_2|, \quad p_3 = |x_0 - x_3|, \quad z_i = x_i + p_i, \quad i = 1, 2, 3
\]
Then we have
\[
p_2 - p_1 \leq K_{02} - 1 - k_{01} + 1 \leq k_{12} - 1,
p_3 - p_1 \leq K_{03} - 1 - k_{01} + 1 \leq K_{13} - 1,
p_3 - p_2 = |x_0 - x_3| - |x_0 - x_2| = |x_2 - x_3| + 1 \leq k_{23} - 1
\]
Hence \((0, p_1, p_2, p_3) \in A_0\). By the Corollary 3.5, we have \(z_i \in S^3(-1, +1)\).

If \(R_{23}^1\) and \(R_{12}^1\) hold, then only case 2i is possible. Then we can write \(\Delta_{II}\) by

\[
\Delta_{II} = \left\{ (x_0, x_1, x_2, x_3) \in \Theta_1 \left| \begin{array}{c}
R_{23}^1, \\
|x_0 - x_1| = k_{01} - 1, |x_0 - x_2| < K_{02} - 1, \\
|x_2 - x_3| = k_{23} - 1, x_1 - x_2 = |x_1 - x_2| \end{array} \right. \right\}
\]
Lemma 3.8. Let $\Delta_{III} = \Delta^1_{III} \cup \Delta^2_{III} \cup \Delta^3_{III}$ where

\[
\Delta^1_{III} = \left\{ (x_0, x_1, x_2, x_3) \in \Theta_1 \mid \begin{array}{l}
\mathcal{R}^1_{12}, \mathcal{R}^0_{13}, \\
|x_0 - x_1| < k_{01} - 1, [x_0 - x_2] = k_{02} - 1,
\end{array} \right\}
\]

\[
\Delta^2_{III} = \left\{ (x_0, x_1, x_2, x_3) \in \Theta_1 \mid \begin{array}{l}
\mathcal{R}^1_{12}, \mathcal{R}^0_{13}, \\
|x_0 - x_1| < k_{01} - 1, [x_0 - x_2] = k_{02} - 1,
\end{array} \right\}
\]

\[
\Delta^3_{III} = \left\{ (x_0, x_1, x_2, x_3) \in \Theta_1 \mid \begin{array}{l}
\mathcal{R}^0_{12}, \mathcal{R}^1_{13}, [x_1 - x_3] = K_{13} - 1,
\end{array} \right\}
\]

If $|x_0 - x_1| < k_{01} - 1$ and $[x_0 - x_2] = K_{02} - 1$, then for any $(x_0, x_1, x_2, x_3) \in \Theta_1 \setminus \Delta_{III}$, there exists $(0, p_1, p_2, p_3) \in A_0$ and $z_i \in S^3(-1, +1)$ such that $x_0 = z_0, x_i = z_i - p_i, i = 1, 2, 3$.

Proof. If $|x_0 - x_1| < k_{01} - 1$ and $[x_0 - x_2] = K_{02} - 1$, as the similar analysis in Lemma 3.7, by Lemma 3.1, we can consider the following cases:

(1) If the relation $\mathcal{R}^0_{13}$ holds,

(i) If the relation $\mathcal{R}^0_{12}$ holds, then we can find

\[p_i = [x_0 - x_i], z_i = x_i + p_i, i = 1, 2, 3\]

such that $(0, p_1, p_2, p_3) \in A_0$ and $z_i \in S^3(-1, +1)$.

(ii) If the relation $\mathcal{R}^1_{12}$ and $|x_1 - x_2| \leq k_{12} - 2$ hold, then we can find

\[p_i = [x_0 - x_i], z_i = x_i + p_i, i = 1, 2, 3\]

such that $(0, p_1, p_2, p_3) \in A_0$ and $z_i \in S^3(-1, +1)$.

(iii) If relations $\mathcal{R}^1_{12}, \mathcal{R}^0_{23}, [x_2 - x_3] \neq x_2 - x_3$ and $[x_0 - x_3] \leq K_{03} - 1$ hold, then we can find

\[p_1 = [x_0 - x_1], p_2 = [x_0 - x_2], p_3 = [x_0 - x_3], z_i = x_i + p_i, i = 1, 2, 3\]

such that $(0, p_1, p_2, p_3) \in A_0$ and $z_i \in S^3(-1, +1)$.

(2) If the relation $\mathcal{R}^1_{13}$ holds,

(i) If $\mathcal{R}^0_{12}$ and $x_1 - x_3 \neq [x_1 - x_3] \leq K_{13} - 2$ hold, we can find

\[p_i = [x_0 - x_i], z_i = x_i + p_i, i = 1, 2, 3\]
such that $(0, p_1, p_2, p_3) \in A_0$ and $z_i \in S^3(-1, +1)$.

(ii) If $R^1_{12}$ holds, we can find

$$p_1 = [x_0 - x_1], p_2 = [x_0 - x_2], p_3 = [x_0 - x_3], \quad z_i = x_i + p_i, i + 1, 2, 3$$

such that $(0, p_1, p_2, p_3) \in A_0$ and $z_i \in S^3(-1, +1)$.

By Lemma 3.4, if $[x_1 - x_3] = x_1 - x_3$, then the relation $R^0_{13}$ holds. Hence we can write $\Delta_{III}$ as

$$\Delta_{III} = \Delta^1_{III} \cup \Delta^2_{III} \cup \Delta^3_{III}$$

where

$$\Delta^1_{III} = \begin{cases} (x_0, x_1, x_2, x_3) \in \Theta_1 \mid & R^1_{12}, R^0_{13}, \\
[ x_0 - x_1 ] < k_{01} - 1, [ x_0 - x_2 ] = K_{02} - 1, \\
[ x_1 - x_2 ] = k_{12} - 1, x_2 - x_3 = [ x_2 - x_3 ] \end{cases}$$

$$\Delta^2_{III} = \begin{cases} (x_0, x_1, x_2, x_3) \in \Theta_1 \mid & R^1_{12}, R^0_{13}, \\
[ x_0 - x_1 ] < k_{01} - 1, [ x_0 - x_2 ] = K_{02} - 1, \\
[ x_1 - x_2 ] = k_{12} - 1, [ x_0 - x_3 ] = K_{03} - 1 \end{cases}$$

$$\Delta^3_{III} = \begin{cases} (x_0, x_1, x_2, x_3) \in \Theta_1 \mid & R^0_{12}, R^1_{13}, [ x_1 - x_3 ] = K_{13} - 1, \\
[ x_0 - x_1 ] < k_{01} - 1, [ x_0 - x_2 ] = K_{02} - 1 \end{cases}$$

Lemma 3.9. Let $\Delta_{IV} = \Delta^1_{IV} \cup \Delta^2_{IV} \cup \Delta^3_{IV}$ where

$$\Delta^1_{IV} = \begin{cases} (x_0, x_1, x_2, x_3) \in \Theta_1 \mid & R^1_{12}, R^0_{13}, R^0_{23}, \\
[ x_0 - x_1 ] < k_{01} - 1, [ x_0 - x_2 ] < K_{02} - 1, \\
[ x_1 - x_2 ] = k_{12} - 1, [ x_2 - x_3 ] = x_2 - x_3 \end{cases}$$

$$\Delta^2_{IV} = \begin{cases} (x_0, x_1, x_2, x_3) \in \Theta_1 \mid & R^1_{12}, R^0_{13}, R^0_{23}, \\
[ x_0 - x_1 ] < k_{01} - 1, [ x_0 - x_2 ] < K_{02} - 1, \\
[ x_1 - x_2 ] = k_{12} - 1, [ x_0 - x_3 ] = K_{03} - 1 \end{cases}$$

$$\Delta^3_{IV} = \begin{cases} (x_0, x_1, x_2, x_3) \in \Theta_1 \mid & R^0_{12}, R^0_{13}, R^1_{23}, \\
[ x_0 - x_1 ] < k_{01} - 1, [ x_0 - x_2 ] < K_{02} - 1, \\
[ x_1 - x_2 ] = x_1 - x_2, [ x_2 - x_3 ] = k_{23} - 1 \end{cases}$$

If $[ x_0 - x_1 ] < k_{01} - 1$ and $[ x_0 - x_2 ] < K_{02} - 1$, then for any $(x_0, x_1, x_2, x_3) \in \Theta_1 \setminus \Delta_{IV}$, there exists $(0, p_1, p_2, p_3) \in A_0$ and $z_i \in S^3(-1, +1)$ such that $x_0 = z_0, x_i = z_i - p_i, i = 1, 2, 3$.

Proof. If $[ x_0 - x_1 ] < k_{01} - 1$ and $[ x_0 - x_2 ] < K_{02} - 1$, as similar analysis in Lemma 3.7, we can list all possible choices as follows
(A) If the relation $R^0_{12}$ holds, then we have the following possible choices
(1) $p_1 = [x_0 - x_1]$ and $p_2 = [x_0 - x_2]$,
(2) $p_1 = [x_0 - x_1]$ and $p_2 = [x_0 - x_2]$,
(3) If $x_1 - x_2 \neq [x_1 - x_2]$ and $|x_1 - x_2| \leq k_{12} - 2$, then we can choose
\[ p_1 = [x_0 - x_1] \quad \text{and} \quad p_2 = [x_0 - x_2]. \]

(B) If the relation $R^1_{12}$ holds, then we have the following possible choices
(1) $p_1 = [x_0 - x_1]$ and $p_2 = [x_0 - x_2]$,
(2) If $|x_1 - x_2| \leq k_{12} - 2$ then $p_1 = [x_0 - x_1]$ and $p_2 = [x_0 - x_2]$, or
(3) If $|x_1 - x_2| \leq k_{12} - 2$ then $p_1 = [x_0 - x_1]$ and $p_2 = [x_0 - x_2]$.

(C) If the relation $R^0_{13}$ holds, then we have the following possible choices
(1) $p_1 = [x_0 - x_1]$ and $p_3 = [x_0 - x_3]$,
(2) If $[x_0 - x_3] \leq k_{03} - 1$, then we can choose $p_1 = [x_0 - x_1]$ and $p_3 = [x_0 - x_3]$,
(3) If $x_1 - x_3 \neq [x_1 - x_3]$, $|x_1 - x_3| \leq k_{13} - 2$, and $[x_0 - x_3] \leq K_{03} - 1$ then we can choose
\[ p_1 = [x_0 - x_1] \quad \text{and} \quad p_3 = [x_0 - x_3]. \]

(D) If the relation $R^1_{13}$ holds, then we have the following possible choices
(1) $p_1 = [x_0 - x_1]$ and $p_3 = [x_0 - x_3]$,
(2) If $|x_1 - x_3| \leq k_{13} - 2$ then we can choose $p_1 = [x_0 - x_1]$ and $p_3 = [x_0 - x_3]$, or
(3) If $|x_1 - x_3| \leq k_{13} - 2$ and $[x_0 - x_3] \leq K_{03} - 1$ then we can choose $p_1 = [x_0 - x_1]$ and $p_3 = [x_0 - x_3]$.

(E) If the relation $R^0_{23}$ holds, then we have the following possible choices
(1) $p_2 = [x_0 - x_2]$ and $p_3 = [x_0 - x_3]$,
(2) If $[x_0 - x_3] \leq K_{03} - 1$, then we can choose $p_2 = [x_0 - x_2]$ and $p_3 = [x_0 - x_3]$, or
(3) If $x_2 - x_3 \neq [x_2 - x_3]$, $|x_2 - x_3| \leq k_{23} - 2$, and $[x_0 - x_3] \leq K_{03} - 1$ then we can choose
\[ p_2 = [x_0 - x_2] \quad \text{and} \quad p_3 = [x_0 - x_3]. \]

(F) If the relation $R^1_{23}$ holds, then we have the following possible choices
(1) $p_2 = [x_0 - x_2]$ and $p_3 = [x_0 - x_3]$,
(2) If $|x_2 - x_3| \leq k_{23} - 2$, then we can choose $p_2 = [x_0 - x_2]$ and $p_3 = [x_0 - x_3]$, or
(3) If \(|x_2 - x_3| \leq k_{23} - 2\) and \(|x_0 - x_3| \leq K_{03} - 1\), then we can choose \(p_2 = \lfloor x_0 - x_2 \rfloor\) and \(p_3 = \lfloor x_0 - x_3 \rfloor\).

By Lemma 3.2, Corollary 3.3, Corollary 3.5 and above analysis (A)-(B)-(C)-(D)-(E)-(F), for this case, we have the following choices

(a) If \(R^0_{12}, R^0_{13}\) and \(R^0_{23}\) hold, we can choose (A1)-(C1)-(E1) we can find

\[
p_i = \lfloor x_0 - x_i \rfloor, \quad z_i = x_i + p_i, \ i = 1, 2, 3,
\]

such that \((0, p_1, p_2, p_3) \in A_0\) and \(z_i \in S^3(-1, +1)\).

(b) If \(R^0_{12}, R^1_{13}\) and \(R^1_{23}\) hold, we can choose (A2)-(D1)-(F1), we can find

\[
p_1 = \lfloor x_0 - x_1 \rfloor, \quad p_2 = \lfloor x_0 - x_2 \rfloor, \quad p_3 = \lfloor x_0 - x_3 \rfloor, \quad z_i = x_i + p_i, \ i = 1, 2, 3
\]

such that \((0, p_1, p_2, p_3) \in A_0\) and \(z_i \in S^3(-1, +1)\).

(c) If \(R^1_{12}, R^1_{13}\) and \(R_0^{0}\) hold, we can choose (B1)-(D1)-(E1), we can find

\[
p_1 = \lfloor x_0 - x_1 \rfloor, \quad p_2 = \lfloor x_0 - x_2 \rfloor, \quad p_3 = \lfloor x_0 - x_3 \rfloor, \quad z_i = x_i + p_i, \ i = 1, 2, 3
\]

such that \((0, p_1, p_2, p_3) \in A_0\) and \(z_i \in S^3(-1, +1)\).

(d) If \(R^1_{12}, R^0_{13}\) and \(R^0_{23}\) hold,

(i) if \(|x_2 - x_3| \leq k_{23} - 2\), \(x_2 - x_3 \neq |x_2 - x_3|\) and \(|x_0 - x_3| \leq K_{03} - 2\), we can choose (B1)-(C2)-(E3), we can find

\[
p_1 = \lfloor x_0 - x_1 \rfloor, \quad p_2 = \lfloor x_0 - x_2 \rfloor, \quad p_3 = \lfloor x_0 - x_3 \rfloor, \quad z_i = x_i + p_i, \ i = 1, 2, 3
\]

such that \((0, p_1, p_2, p_3) \in A_0\) and \(z_i \in S^3(-1, +1)\),

(ii) if \(|x_1 - x_2| \leq k_{12} - 2\), we can choose (B2)-(C1)-(E1), we can find

\[
p_i = \lfloor x_0 - x_i \rfloor, \quad z_i = x_i + p_i, \ i = 1, 2, 3
\]

or moreover \(|x_0 - x_3| \leq K_{03} - 2\), we can also choose (B3)-(C2)-(E2), we can find

\[
p_i = \lfloor x_0 - x_i \rfloor, \quad z_i = x_i + p_i, \ i = 1, 2, 3
\]

such that \((0, p_1, p_2, p_3) \in A_0\) and \(z_i \in S^3(-1, +1)\),

(e) If \(R^0_{12}, R^0_{13}\) and \(R^1_{23}\) hold,

(i) if \(|x_2 - x_3| \leq k_{23} - 2\), we can choose (A1)-(C1)-(F2), we can find

\[
p_i = \lfloor x_0 - x_i \rfloor, \quad z_i = x_i + p_i, \ i = 1, 2, 3
\]
or moreover if \( |x_0 - x_3| \leq K_{03} - 1 \), we can also choose (A2)-(C2)-(F3), we can find
\[
p_i = [x_0 - x_i], z_i = x_i + p_i, i = 1, 2, 3
\]
such that \((0, p_1, p_2, p_3) \in A_0\) and \(z_i \in S^3(-1, +1)\),
(ii) if \( x_1 - x_2 \neq [x_1 - x_2], |x_1 - x_2| \leq k_{12} - 2 \), we can choose (A3)-(C1)-(F1), we can find
\[
p_i = [x_0 - x_i], p_2 = [x_0 - x_2], p_3 = [x_0 - x_3], z_i = x_i + p_i, i = 1, 2, 3
\]
such that \((0, p_1, p_2, p_3) \in A_0\) and \(z_i \in S^3(-1, +1)\),
(f) If \( R_{12}^1, R_{13}^1 \) and \( R_{23}^1 \) hold,
(i) if \( |x_2 - x_3| \leq k_{23} - 2 \), we can choose (B1)-(D1)-(F2), we can find
\[
p_1 = [x_0 - x_1], p_2 = [x_0 - x_2], p_3 = [x_0 - x_3], z_i = x_i + p_i, i = 1, 2, 3
\]
such that \((0, p_1, p_2, p_3) \in A_0\) and \(z_i \in S^3(-1, +1)\),
(ii) if \( |x_1 - x_2| \leq k_{12} - 2 \), we can choose (B2)-(D1)-(F1), we can find
\[
p_1 = [x_0 - x_1], p_2 = [x_0 - x_2], p_3 = [x_0 - x_3], z_i = x_i + p_i, i = 1, 2, 3
\]
such that \((0, p_1, p_2, p_3) \in A_0\) and \(z_i \in S^3(-1, +1)\).

If \( R_{12}^0, R_{13}^0 \) and \( R_{23}^0 \) hold, i.e.
\[
\begin{align*}
|x_0 - x_2| - |x_0 - x_1| &= |x_1 - x_2| + 1 \\
|x_0 - x_3| - |x_0 - x_1| &= |x_1 - x_3| \\
|x_0 - x_3| - |x_0 - x_2| &= |x_2 - x_3|
\end{align*}
\]
then we have
\[
|x_1 - x_3| = |x_1 - x_2| + |x_2 - x_3| + 1
\]
If moreover,
\[
|x_1 - x_2| = k_{12} - 1, |x_2 - x_3| = k_{23} - 1
\]
then we have
\[
|x_1 - x_3| = |x_1 - x_2| + |x_2 - x_3| + 1 = k_{12} + k_{23} - 1
\]
But
\[
|x_1 - x_3| \leq K_{13} - 1 \leq k_{12} + k_{23} - 2
\]
This is a contraction. Hence, if \(|x_1 - x_2| = k_{12} - 1\) and \(|x_2 - x_3| = k_{23} - 1\) hold then \(R^1_{12}, R^0_{13}\) and \(R^0_{23}\) does not hold at the same time. And the same reason, if \(|x_1 - x_2| = k_{12} - 1\) and \(|x_2 - x_3| = k_{23} - 1\) hold then

- \(R^1_{12}, R^1_{13}\) and \(R^1_{23}\) does not hold at the same time.
- \(R^0_{12}, R^0_{13}\) and \(R^1_{23}\) does not hold at the same time.

By above analysis, we can write \(\Delta_{IV}\) by

\[
\Delta_{IV} = \Delta^1_{IV} \cup \Delta^2_{IV} \cup \Delta^3_{IV}
\]

where

\[
\Delta^1_{IV} = \left\{ (x_0, x_1, x_2, x_3) \in \Theta_1 \mid \begin{array}{l}
R^1_{12}, R^0_{13}, R^0_{23}, \\
|x_0 - x_1| < k_{01} - 1, |x_0 - x_2| < \mathcal{K}_{02} - 1, \\
|x_1 - x_2| = k_{12} - 1, |x_2 - x_3| = x_2 - x_3,
\end{array} \right\}
\]

\[
\Delta^2_{IV} = \left\{ (x_0, x_1, x_2, x_3) \in \Theta_1 \mid \begin{array}{l}
R^1_{12}, R^0_{13}, R^0_{23}, \\
|x_0 - x_1| < k_{01} - 1, |x_0 - x_2| < \mathcal{K}_{02} - 1, \\
|x_1 - x_2| = k_{12} - 1, |x_0 - x_3| = \mathcal{K}_{03} - 1,
\end{array} \right\}
\]

\[
\Delta^3_{IV} = \left\{ (x_0, x_1, x_2, x_3) \in \Theta_1 \mid \begin{array}{l}
R^0_{12}, R^0_{13}, R^1_{23}, \\
|x_0 - x_1| < k_{01} - 1, |x_0 - x_2| < \mathcal{K}_{02} - 1, \\
|x_1 - x_2| = x_1 - x_2, |x_2 - x_3| = k_{23} - 1,
\end{array} \right\}
\]

\[\square\]

Using inequality (4), Lemma 3.2, Lemma 3.6, Lemma 3.7, Lemma 3.8, Lemma 3.9, Corollary 3.3 and Corollary 3.5, we will prove the Theorem 1.1 of this paper:

**Proof.** We write

\[
\Theta_2 = \bigcup_{(0,p_1,p_2,p_3) \in A_0} \left\{ (x_0, x_1, x_2, x_3) \mid \begin{array}{l}
-1 + p_i \quad < x_0 - x_i < \quad 1 + p_i, i = 1, 2, 3 \\
-1 + p_k - p_j \quad < x_j - x_k < \quad 1 + p_k - p_j, 1 \leq j < k \leq 3
\end{array} \right\}.
\]
By the definition of \( A_0 \), the following inequalities hold

\[
\begin{align*}
(7) \quad p_1 & \leq k_{01} - 1, \\
(8) \quad p_2 & \leq K_{02} - 1, \\
(9) \quad p_3 & \leq K_{03} - 1, \\
(10) \quad p_2 - p_1 & \leq k_{12} - 1, \\
(11) \quad p_3 - p_2 & \leq k_{23} - 1, \\
(12) \quad p_3 - p_1 & \leq K_{13} - 1
\end{align*}
\]

By the above inequalities (7)-(12), we have
\[ \Theta_2 \subseteq \Theta_1 \]

By the homeomorphism (3), \( \Theta_\varepsilon \) is homeomorphism to \( \mathbb{R}_{>0}^3 \times \Theta_2 \). To prove the theorem, we need to show that for every \( (x_0, x_1, x_2, x_3) \in \Theta_1 \setminus \Delta \), there exist \( (0, p_1, p_2, p_3) \in A_0 \) and \( z_i \in S^3(-1, +1) \) such that \( (x_0, x_1, x_2, x_3) = (z_0, z_1, z_2, z_3) - (0, p_1, p_2, p_3) \) and \( \Theta_2 \cap \Delta = \emptyset \).

By the definition of \( S^3(-1, +1) \), we need to choose \( z_i \) such that \( z_i \in (x_0 - 1, x_0 + 1) \), hence the possible choices of \( p_i \) are \( p_i = \lfloor x_0 - x_i \rfloor \) or \( p_i = \lceil x_0 - x_i \rceil \) and \( z_i = x_i + p_i \). We need to check the inequalities (7)-(12) for \( p_i \) and \( z_i \in S^3(-1, +1) \).

By relations \( R_{ij}^0, R_{ij}^1 \), the inequalities (7)-(12) and Corollary 3.5, we need to consider the following four cases.

(I) \[ \lfloor x_0 - x_1 \rfloor = k_{01} - 1 \text{ and } \lfloor x_0 - x_2 \rfloor = K_{02} - 1 \]

(II) \[ \lfloor x_0 - x_1 \rfloor = k_{01} - 1 \text{ and } \lfloor x_0 - x_2 \rfloor < K_{02} - 1 \]

(III) \[ \lfloor x_0 - x_1 \rfloor < k_{01} - 1 \text{ and } \lfloor x_0 - x_2 \rfloor = K_{02} - 1 \]

(IV) \[ \lfloor x_0 - x_1 \rfloor < k_{01} - 1 \text{ and } \lfloor x_0 - x_2 \rfloor < K_{02} - 1 \]

By the above analysis, we can describe the set \( \Delta \) as follows.

- For Case I, by Lemma 3.6, there are no elements in \( \Delta \).
- For Case II, by Lemma 3.7, Then the set of elements of \( \Delta \) in this case is \( \Delta_{II} \)
- For Case III, by Lemma 3.8, Then the set of elements of \( \Delta \) in this case is \( \Delta_{III} \)
- For Case IV, by Lemma 3.9 , Then the set of elements of \( \Delta \) in this case is \( \Delta_{IV} \)

By the Corollary 3.3, we can rewrite \( \Delta_{IV}, \Delta_{IV}^2, \Delta_{IV}^3 \) as
\[
\Delta^1_{IV} = \left\{(x_0, x_1, x_2, x_3) \in \Theta_1 \mid \begin{array}{l}
\mathcal{R}^1_{12}, \mathcal{R}^0_{13}, \\
[x_0 - x_1] < k_{01} - 1, \ |x_0 - x_2| < \mathcal{K}_{02} - 1, \\
[x_1 - x_2] = k_{12} - 1, \ |x_2 - x_3| = x_2 - x_3,
\end{array} \right\}
\]
\[
\Delta^2_{IV} = \left\{(x_0, x_1, x_2, x_3) \in \Theta_1 \mid \begin{array}{l}
\mathcal{R}^1_{12}, \mathcal{R}^0_{13}, \\
[x_0 - x_1] < k_{01} - 1, \ |x_0 - x_2| < \mathcal{K}_{02} - 1, \\
[x_1 - x_2] = k_{12} - 1, \ |x_0 - x_3| = \mathcal{K}_{03} - 1,
\end{array} \right\}
\]
\[
\Delta^3_{IV} = \left\{(x_0, x_1, x_2, x_3) \in \Theta_1 \mid \begin{array}{l}
\mathcal{R}^0_{13}, \mathcal{R}^1_{23}, \\
[x_0 - x_1] < k_{01} - 1, \ |x_0 - x_2| < \mathcal{K}_{02} - 1, \\
[x_1 - x_2] = x_1 - x_2, \ |x_2 - x_3| = k_{23} - 1,
\end{array} \right\}
\]

By the Corollary 3.3, we can rewrite \(\Delta_{II}\) as \(\Delta_{II} = \Delta^1 \cup \Delta^2_{II}\), where

\[
\Delta^1 = \left\{(x_0, x_1, x_2, x_3) \in \Theta_1 \mid \begin{array}{l}
\mathcal{R}^1_{23}, \mathcal{R}^1_{13}, \\
[x_0 - x_1] = k_{01} - 1, \ |x_0 - x_2| < \mathcal{K}_{02} - 1, \\
[x_2 - x_3] = k_{23} - 1, \ x_1 - x_2 = |x_1 - x_2|
\end{array} \right\}
\]
\[
\Delta^2_{II} = \left\{(x_0, x_1, x_2, x_3) \in \Theta_1 \mid \begin{array}{l}
\mathcal{R}^1_{23}, \mathcal{R}^0_{13}, \\
[x_0 - x_1] = k_{01} - 1, \ |x_0 - x_2| < \mathcal{K}_{02} - 1, \\
[x_2 - x_3] = k_{23} - 1, \ x_1 - x_2 = |x_1 - x_2|
\end{array} \right\}
\]

Then we hvae

\[
\Delta^2 := \Delta^3_{IV} \cup \Delta^2_{II} = \left\{(x_0, x_1, x_2, x_3) \in \Theta_1 \mid \begin{array}{l}
\mathcal{R}^1_{23}, \mathcal{R}^0_{13}, \\
[x_0 - x_2] < \mathcal{K}_{02} - 1, \\
[x_2 - x_3] = k_{23} - 1, \ x_1 - x_2 = |x_1 - x_2|
\end{array} \right\}
\]
\[
\Delta^3 := \Delta^1_{II} \cup \Delta^1_{IV} = \left\{(x_0, x_1, x_2, x_3) \in \Theta_1 \mid \begin{array}{l}
\mathcal{R}^1_{12}, \mathcal{R}^0_{13}, \\
[x_0 - x_1] < k_{01} - 1, \\
[x_1 - x_2] = k_{12} - 1, \ x_2 - x_3 = |x_2 - x_3|
\end{array} \right\}
\]
\[
\Delta^4 := \Delta^2_{II} \cup \Delta^2_{IV} = \left\{(x_0, x_1, x_2, x_3) \in \Theta_1 \mid \begin{array}{l}
\mathcal{R}^1_{12}, \mathcal{R}^0_{13}, \\
[x_0 - x_1] < k_{01} - 1, \\
[x_1 - x_2] = k_{12} - 1, \ |x_0 - x_3| = \mathcal{K}_{03} - 1
\end{array} \right\}
\]

For the convenience we rewrite \(\Delta^3_{II}\) by \(\Delta^5\).

Hence we prove that

\[
\Delta = \Delta_{II} \cup \Delta_{III} \cup \Delta_{IV} = \bigcup_{i=1}^{5} \Delta^i.
\]

By the construction of \(\Delta\), we have \(\Theta_2 \cap \Delta = \emptyset\). We complete the proof. \(\square\)
4. Examples

In this section we give an application of the Theorem 1.1. Let $X$ be a quadric surface, which isomorphism to $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathcal{D}^b(X)$ is the bounded derived category of coherent sheaves on $X$. We use the notation

$$\mathcal{O}(a, b) = \pi_1^* \mathcal{O}_{\mathbb{P}^1}(a) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(b)$$

where $\pi_1, \pi_2 : X \to \mathbb{P}^1$ are the projections.

Then $\mathcal{E} = (\mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1))$ is a complete strong exceptional collection on $\mathcal{D}^b(X)$. For $\mathcal{E}$, we have

$$k_{ij} = 0, K_{02} = K_{13} = -1, \text{ and } K_{03} = -2.$$  

Then we can rewrite the Theorem 1.1 as followes:

**Proposition 4.1.** Let $X$ be a quadric surface, $\mathcal{E} = (\mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1))$ be a complete strong exceptional collection on $\mathcal{D}^b(X)$.

Then

$$\Theta_1 = \left\{ (y_0, y_1, y_2, y_3) \mid \begin{array}{l} y_0 < y_1 < y_2 < y_3, \quad y_1 - y_3 < -1 \\ y_0 - y_3 < -2, \quad y_0 - y_2 < -1 \end{array} \right\}.$$  

and we can express $\Theta_\mathcal{E}$ as followes:

$$\mathbb{R}^4_{>0} \times (\Theta_1 \setminus \Delta)$$
where $\Delta$ is the union of following five sets

$$
\Delta^1 = \left\{ (x_0, x_1, x_2, x_3) \in \Theta_1 \right\} | \begin{array}{l}
    [x_1 - x_3] = [x_0 - x_3], \\
    [x_0 - x_1] = -1, [x_0 - x_2] = [x_0 - x_3] < -2, \\
    [x_2 - x_3] = -1, x_1 - x_2 = [x_1 - x_2]
\end{array} \}
$$

$$
\Delta^2 = \left\{ (x_0, x_1, x_2, x_3) \in \Theta_1 \right\} | \begin{array}{l}
    [x_0 - x_3] = [x_0 - x_2] < -2, \\
    [x_0 - x_3] - [x_0 - x_1] = [x_1 - x_3], \\
    [x_2 - x_3] = -1, x_1 - x_2 = [x_1 - x_2]
\end{array} \}
$$

$$
\Delta^3 = \left\{ (x_0, x_1, x_2, x_3) \in \Theta_1 \right\} | \begin{array}{l}
    [x_0 - x_1] = [x_0 - x_2] < -1, \\
    [x_0 - x_3] - [x_0 - x_1] = [x_1 - x_3], \\
    [x_1 - x_2] = -1, x_2 - x_3 = [x_2 - x_3]
\end{array} \}
$$

$$
\Delta^4 = \left\{ (x_0, x_1, x_2, x_3) \in \Theta_1 \right\} | \begin{array}{l}
    [x_0 - x_1] = [x_0 - x_2] < -1, \\
    [x_0 - x_1] + [x_1 - x_3] + 3 = 0, \\
    [x_1 - x_2] = -1, [x_0 - x_3] = -3
\end{array} \}
$$

$$
\Delta^5 = \left\{ (x_0, x_1, x_2, x_3) \in \Theta_1 \right\} | \begin{array}{l}
    [x_0 - x_1] < -1, \\
    [x_0 - x_2] = -2, [x_1 - x_3] = -2 \\
    [x_0 - x_1] = - [x_1 - x_2] - 2 = [x_0 - x_3] + 1
\end{array} \}.
$$

**Remark 4.2.** The author’s original motivation was to give a detail description of stability conditions on $X$ and excepted to get a contractible connected component of the spaces of stability conditions on $\mathcal{D}^b(X)$.

Compared to the $\mathbb{P}^2$ case [15, 10], $\mathbb{P}^1 \times \mathbb{P}^1$ case is more difficult. As compute in the Theorem 1.1, $\Delta$ is the union of five sets. We can not give a simple description of $\Delta$ now. We wish to prove that $\Theta_2$ is contractible. Another problem is that the strong exceptional collection on $\mathbb{P}^1 \times \mathbb{P}^1$ is not strong after the mutation. The space of stability condition generated by exceptional collection on $\mathbb{P}^1 \times \mathbb{P}^1$ is still not clear.

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