ON WEAKLY LOCALLY UNIFORMLY ROTUND NORMS WHICH ARE NOT LOCALLY UNIFORMLY ROTUND

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Abstract. We show that every infinite-dimensional Banach space with separable dual admits an equivalent norm which is weakly locally uniformly rotund but not locally uniformly rotund.

1. Introduction

Recall that a norm in a Banach space is called strictly convex (SC) if for arbitrary points $x, y$ from the unit sphere the equality $\|x + y\| = 2$ implies that $x = y$. The norm is called weakly locally uniformly rotund (wLUR) if for arbitrary points $x_n (n = 1, 2, \ldots)$ and $x$ from the unit sphere the equality $\lim_{n \to \infty} \|x_n + x\| = 2$ implies the weak convergence of the sequence $(x_n)_{n=1}^{\infty}$ to $x$; if the last convergence is strong, then the norm is called locally uniformly rotund (LUR). In the preceding definitions it is sufficient to require that $\lim_{n \to \infty} \|x_n\| = \|x\|$ and $\lim_{n \to \infty} \|x_n + x\| = 2 \|x\|$. It is clear that $wLUR \implies SC$ and $LUR \implies wLUR$; it is also well-known that none of these implications reverses. Indeed, the space $\ell_\infty$ can be renormed in a strictly convex manner, but it does not admit an equivalent $wLUR$ norm (cf. [2, §4.5]). M.A. Smith [5, Example 2] gave an example of a $wLUR$ norm on $\ell_2$ which is not $LUR$, however, in the next section we shall present a somewhat simpler example (which is a particular case of our main result, but slightly different).

D. Yost [7, Theorem 2.1] showed that the implication $wLUR \implies SC$ does not reverse in the strong sense, namely, that every infinite-dimensional separable Banach space admits an equivalent strictly convex norm which is not $wLUR$. Of course, the analogous theorem does not hold for the implication $LUR \implies wLUR$, because of the Schur property, e.g., of the space $\ell_1$. However, it is true when assuming that the dual of the underlying space is separable; this is what our main result states:

Theorem 1. Every infinite-dimensional Banach space with separable dual admits an equivalent $wLUR$ norm which is not $LUR$.

Remark 2. It is worth mentioning that the class of Banach spaces having a $wLUR$ renorming coincides with the class of Banach spaces having a $LUR$ renorming [3, Theorem 1.11]. However, Theorem 1 (and, all the more, Corollary 6) suggests that in a large class of Banach spaces with a $wLUR$ renorming not every $wLUR$ norm is automatically $LUR$.

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2. AN EXAMPLE OF A wLUR NORM WHICH IS NOT LUR

The norm

\[
\|x\| = \|x\|_\infty + \left(\sum_{n=1}^{\infty} 2^{-n} |x(n)|^2 \right)^{1/2}
\]

for \( x \in c_0 \),

where \( \|\cdot\|_\infty \) stands for the standard supremum norm, was given in \([3, p. 1]\) as an example of a strictly convex norm which is not LUR. Nonetheless, we shall show that this norm is wLUR.

**Lemma 3.** Suppose that \((x_n)_{n=1}^\infty \subset c_0\) is pointwise convergent to \(\alpha x\), where \(\alpha \in [0, \infty)\) and \(x \in c_0 \setminus \{0\}\). If

\[
\lim_{n \to \infty} (\|x_n + x\|_\infty - \|x_n\|_\infty) = \|x\|_\infty
\]

and the limit \(\lim_{n \to \infty} \|x_n\|_\infty\) exists, then \(\lim_{n \to \infty} \|x_n\|_\infty = \alpha \|x\|_\infty\).

**Proof.** We shall show that the sequence \((\|x_n\|_\infty)_{n=1}^\infty\) has a subsequence which is convergent to \(\alpha \|x\|_\infty\).

Let

\[ K = \{k: |x(k)| = \|x\|_\infty\}; \]

by our assumptions \(K\) is a non-empty finite set. Furthermore, if \(n\) and \(k\) are positive integers such that \(k \notin K\), then

\[
|x_n + x|(k) - \|x_n\|_\infty \leq |x_n(k)| + |x(k)| - \|x_n\|_\infty \leq |x(k)|
\]

\[
\leq \max\{|x(l)|: l \notin K\} < \|x\|_\infty.
\]

It means that there is a \(k_0 \in K\) such that \(|x_n + x|(k_0) = \|x_n + x\|_\infty\) for infinitely many \(n\). Let \((n_l)_{l=1}^\infty\) be a strictly increasing sequence of positive integers such that

\[
|(x_{n_l} + x)(k_0)| = \|x_{n_l} + x\|_\infty \quad \text{for } l = 1, 2, \ldots.
\]

Passing with \(l\) to infinity we obtain

\[
(1 + \alpha|x(k_0)|) = \lim_{l \to \infty} \|x_{n_l} + x\|_\infty = \lim_{l \to \infty} \|x_{n_l}\|_\infty + \|x\|_\infty,
\]

which completes the proof. \(\square\)

**Proposition 4.** The norm given by \((1)\) is wLUR.

**Proof.** Fix a sequence \((x_n)_{n=1}^\infty\) in the unit sphere of \((c_0, \|\cdot\|)\) and a point \(x\) from this sphere such that \(\lim_{n \to \infty} \|x_n + x\| = 2\). We shall show that each subsequence of \((x_n)_{n=1}^\infty\) has a subsequence which is weakly convergent to \(x\). To this end fix an arbitrary subsequence of \((x_n)_{n=1}^\infty\) which for convenience will be still denoted by \((x_n)_{n=1}^\infty\).

Set

\[ y_n = (2^{-k/2}x_n(k))_{k=1}^\infty \quad \text{for } n = 1, 2, \ldots \quad \text{and} \quad y = (2^{-k/2}x(k))_{k=1}^\infty. \]

The equality

\[
2 - \|x_n + x\| = \|x_n\| + \|x\| - \|x_n + x\|
\]

\[
= \|x_n\|_\infty + \|x\|_\infty - \|x_n + x\|_\infty + \|y_n\|_2 + \|y\|_2 - \|y_n + y\|_2,
\]

implies
where \( \| \cdot \|_2 \) stands for the norm in \( \ell_2 \), implies the existence and the equality of the following limits:

\[
\lim_{n \to \infty} (\|x_n\|_\infty + \|x\| - \|x_n + x\|_\infty) = 0
\]

and

\[
\lim_{n \to \infty} (\|y_n\|_2 + \|y\|_2 - \|y_n + y\|_2) = 0.
\]

Passing to a further subsequence of \((x_n)^\infty_{n=1}\) (still denoted by \((x_n)^\infty_{n=1}\)) we may assume that the limits \(\lim_{n \to \infty} \|x_n\|_\infty\) and \(\lim_{n \to \infty} \|y_n\|_2\) exist. Using the equality (3) we obtain

\[
\lim_{n \to \infty} (\|y_n\|_2 + \|y\|_2)^2 = \lim_{n \to \infty} \|y_n + y\|_2^2 = \lim_{n \to \infty} (\|y_n\|_2^2 + 2(y_n|y) + \|y\|_2^2),
\]

where \((\cdot, \cdot)_\ell\) stands for the real inner product. Whence

\[
\lim_{n \to \infty} (y_n|y) = \lim_{n \to \infty} \|y_n\|_2 \cdot \|y\|_2 = \alpha \|y\|_2^2,
\]

where \(\alpha = \lim_{n \to \infty} \|y_n\|_2 / \|y\|_2\). Thus

\[
\lim_{n \to \infty} \|y_n - \alpha y\|_2^2 = \lim_{n \to \infty} (\|y_n\|_2^2 - 2\alpha(y_n|y) + \alpha^2 \|y\|_2^2)
= \alpha^2 \|y\|_2^2 - 2\alpha^2 \|y\|_2^2 + \alpha^2 \|y\|_2^2 = 0,
\]

which means that the sequence \((y_n)^\infty_{n=1}\) is convergent (in the space \(\ell_2\)) to \(\alpha y\). In particular, the sequence \((y_n)^\infty_{n=1}\) is pointwise convergent to \(\alpha y\), therefore the sequence \((x_n)^\infty_{n=1}\) is pointwise convergent to \(\alpha x\). By the equality (2) and Lemma 3 \(\lim_{n \to \infty} \|x_n\|_\infty = \alpha \|x\|_\infty\). Therefore

\[
1 = \lim_{n \to \infty} \|x_n\| = \lim_{n \to \infty} \|x_n\|_\infty + \lim_{n \to \infty} \|y_n\|_2
= \alpha \|x\|_\infty + \alpha \|y\|_2 = \alpha \|x\| = \alpha.
\]

Finally, the sequence \((x_n)^\infty_{n=1}\) is weakly convergent to \(x\) as it is bounded and converges pointwise to this point. \(\square\)

3. The Proof of the Main Result

Throughout this section \(X\) denotes an infinite-dimensional Banach space. We shall need a simple lemma about the weak convergence (a trivial proof will be omitted).

**Lemma 5.** Assume that \((x_n)^\infty_{n=1}\) is a bounded sequence in \(X\), \(\Gamma\) is a set and \(\{x_\gamma^* : \gamma \in \Gamma\} \subset X^*\). If the space \(\text{span}\{x_\gamma^* : \gamma \in \Gamma\}\) is dense in \(X^*\) and

\[
\lim_{n \to \infty} x_\gamma^*(x_n) = 0 \quad \text{for each } \gamma \in \Gamma,
\]

then the sequence \((x_n)^\infty_{n=1}\) is weakly null.

**Proof of Theorem 7** Assume that \(X^*\) is separable. According to the result of A. Pełczyński [4] Remark A] there exists an \(M\)-basis \((e_n, e_n^*)^\infty_{n=1}\) of the space \(X\) which is both bounded and shrinking. It means that

\[
\sup\{\|e_n\| : n = 1, 2, \ldots\} < \infty
\]
and the functionals $e_n^*$ are linearly dense in $X^*$.

Without loss of generality we may assume that $\|e_n\| = 1$ for $n = 1, 2, \ldots$. Define a functional $\|\cdot\|_0 : X \to [0, \infty)$ by

$$\|x\|_0 = \max \left\{ \frac{1}{2} \|x\|, \sup_n |e_n^*(x)| \right\} \text{ for } x \in X.$$ 

One can easily see that $\|\cdot\|_0$ is a norm on $X$ and by the boundedness of the $M$-basis $(e_n, e_n^*)_{n=1}^{\infty}$ this norm is equivalent to the original one.

Define a functional $\|\cdot\| : X \to [0, \infty)$ by

$$\|x\|^2 = \|x\|_0^2 + \sum_{n=1}^{\infty} 4^{-n} |e_n^*(x)|^2 \text{ for } x \in X.$$ 

One can easily observe that $\|\cdot\|$ is an equivalent norm on $X$. We shall show that it is wLUR but not LUR.

For the proof of the first part consider a sequence $(x_n)_{n=1}^{\infty}$ and a point $x$ in the unit sphere of $(X, \|\cdot\|)$ such that $\lim_{n \to \infty} \|x_n + x\| = 2$. Set

$$y_n = (\|x_n\|_0, 2^{-1}e_1^*(x_n), 2^{-2}e_2^*(x_n), \ldots) \text{ for } n = 1, 2, \ldots$$

and

$$y = (\|x\|_0, 2^{-1}e_1^*(x), 2^{-2}e_2^*(x), \ldots).$$

We have

$$\|y_n + y\|^2 = (\|x_n\|_0 + \|x\|_0)^2 + \sum_{m=1}^{\infty} 4^{-m} |e_m^*(x_n + x)|^2 \geq \|x_n + x\|_0^2 + \sum_{m=1}^{\infty} 4^{-m} |e_m^*(x_n + x)|^2 = \|x_n + x\|^2 \xrightarrow{n \to \infty} 4,$$

and by the local uniform rotundity of the norm in the (Hilbert) space $\ell_2$, we obtain $\lim_{n \to \infty} \|y_n - y\|_2 = 0$. In particular,

$$\lim_{n \to \infty} e_m^*(x_n) = e_m^*(x) \quad \text{for } m = 1, 2, \ldots.$$ 

Lemma 5 and the fact that the $M$-basis $(e_n, e_n^*)_{n=1}^{\infty}$ is shrinking give the weak convergence of the sequence $(x_n)_{n=1}^{\infty}$ to $x$.

To see that the norm $\|\cdot\|$ is not LUR consider the sequence $(e_1 + e_n)_{n=1}^{\infty}$ and the point $e_1$. One can easily verify that

$$\lim_{n \to \infty} \|e_1 + e_n\| = \frac{1}{2} \sqrt{5} = \|e_1\|$$

and

$$\lim_{n \to \infty} \|2e_1 + e_n\| = \sqrt{5},$$

while $\|e_n\| \geq 1$ for $n = 1, 2, \ldots$. 

Corollary 6. Every Banach space which admits an equivalent LUR norm, in particular every separable Banach space, and has an infinite-dimensional subspace with separable dual admits an equivalent wLUR norm which is not LUR.

Proof. Suppose that $Y$ is an infinite-dimensional subspace of $X$ with separable dual. By Theorem 1 the space $Y$ admits an equivalent wLUR norm which is not LUR. According to Tang’s Theorem [6, Theorem 1.1] it extends to an equivalent wLUR norm on the whole $X$. Obviously, this extension fails to be LUR. □

Remark 7. The statement of Tang’s Theorem does not include the case of wLUR norm literally, however, the theorem is also valid in this case (cf. [6, Remark 1.1]). Indeed, one can easily verify that the proof works without major changes.

Remark 8. Corollary 6 implies that every Banach space which admits an equivalent LUR norm, in particular every separable Banach space, and enjoys the Schur property has no infinite-dimensional subspace with separable dual. Of course, it is not a new result, as it is well-known that every Banach space having the Schur property is $\ell_1$-saturated. However, this fact follows from Rosenthal’s $\ell_1$-Theorem (cf. [11, §10.2]), so its proof is much less elementary than the one given in this paper.

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