Structures and production cross sections of p-shell
Lambda-hypernuclei calculated with
multi-configuration shell model

Atsushi Umeya¹, Toshio Motoba²,³, Kazunori Itonaga⁴

¹Liberal Arts and Sciences, Nippon Institute of Technology, Miyashiro, Saitama 345-8501,
²Laboratory of Physics, Osaka Electro-Communication University, Osaka 572-8530, Japan,
³Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan,
⁴Faculty of Education, Gifu University, Gifu 501-1193, Japan

E-mail: aumeya@nit.ac.jp

Abstract. The high-resolution data of the $^{10}$B $(e, e' K^+)$ $^{10}$Be experiment done at the Jefferson National Laboratory reported a new bump, which cannot be explained by the DWIA calculation by employing the conventional shell-model wave functions. To describe the new bump, we have extended the model space by introducing a new configuration with $1\hbar$ excitation for both nuclear and hyperon states. We show that the DWIA calculation by using the extended shell-model wave functions successfully explains the new bump for the first time and that the $p$-state in the lower molecular orbit plays an important role in the appearance of the new bump.

1. Introduction

Hypernuclear studies have played an important role to understand hyperon-nucleon fundamental interaction properties and also to disclose characteristic structures of many-body systems with strange particle(s) which are free from the nucleon Pauli principle. Hypernuclear production experiments have been carried out first at CERN [1–3] by making use of the $(K^-, \pi^-)$ reaction with recoiless condition of kaon momentum $p_K \approx 0.8$ GeV/c. The reaction is based on the strangeness-exchange elementary process: $n + K^- \rightarrow \Lambda + \pi^-$. Such experiments have been extended at Brookhaven National Laboratory [3–5] to include also the $(\pi^+, K^+)$ reaction which has been performed also at KEK. The latter reaction experiments have been successfully performed with various targets including very heavy one such as $^{208}$Pb, although the energy resolution is limited to around 1.5 MeV at best. On the other hand, the $(e, e'K^+)$ reaction experiments done recently at Hall-A [6–8] and Hall-C [9–13] of the Jefferson National Laboratory (JLab) have provided us with remarkably high-resolution data showing hypernuclear structure details within the energy resolution of few hundreds of keV. Such reaction spectroscopic data naturally attracts much theoretical attention how to describe low-lying level structure details but also some high-lying interesting excited states of hypernuclei. For convenience one may refer to some Refs. [14–16].

In this paper in particular, we focus our attention to the understanding of the whole spectrum of $^{10}$B $(e, e' K^+)$ $^{10}$Be production experiment [13]. This experiment has confirmed for the first time the major peaks predicted by the DWIA calculation in which the $s$-orbit $\Lambda$ particle is coupled with the nuclear core states confined within the $0\hbar\omega$ $p$-shell configuration [17]. On
the other hand, at the same time, the data disclose sizable extra strengths (bump) at 8.34 MeV excitation that seem difficult to be explained by the conventional shell model. In order to describe the new bump, we have extended the model space by introducing the new configuration with $1\hbar\omega$ excitation for both nuclear and hyperon states [18]. The energy levels for the core nucleus $^9$Be, the target nucleus $^{10}$B, and the hypernucleus $^{10}_Λ$Be are calculated within the multi-configuration model space up to the necessary excitation energies. Then we show that the DWIA calculation by using the extended wave functions successfully explains the new bump in addition to the sharp four peaks predicted before. In order to understand the appearance of such new states, we have also calculated the $(K^{-}, \pi^{-})$ and $(\pi^{+}, K^{+})$ reaction spectra and compare them with the $(\gamma, K^{+})$ spectrum. We will emphasize that $Λ$-$p$-orbital state comes down appreciably due to the deformation effect of nuclear core and hence the coupling between $[^9$Be$(J^-_c) \times \Lambda(p)]$ and $[^9$Be$(J^+_c) \times \Lambda(s)]$ configurations plays an important role in the appearance of the new bump. The nuclear excited states with different parities are both contained in the bump wave functions, which manifests itself as a new characteristic feature mediated by the existence of a $Λ$ hyperon.

2. Multi-configuration shell model framework

In present study, we extend the model space by employing shell-model wave functions with nuclear core parity mixing mediated by the $Λ$-hyperon, in order to express states of the $p$-shell $Λ$-hypernuclei measured with high resolution. In the case of $^{10}_Λ$Be, the wave functions are shown symbolically by

$$
\Psi(^{10}_Λ$Be; $J^-) = \sum_i \alpha_i [\psi_c(^9$Be; $J^-_c) \otimes \Lambda(s)]^{(i)}_J + \sum_j \beta_j [\psi_c(^9$Be; $J^+_c) \otimes \Lambda(p)]^{(j)}_J,
$$
(1)

$$
\Psi(^{10}_Λ$Be; $J^+) = \sum_i \alpha'_i [\psi_c(^9$Be; $J^+_c) \otimes \Lambda(s)]^{(i)}_J + \sum_j \beta'_j [\psi_c(^9$Be; $J^-_c) \otimes \Lambda(p)]^{(j)}_J,
$$
(2)

where the summation stands for all the possible energies and angular momentum couplings. The nuclear core eigenstates $\{\psi_c(^9$Be; $J^-_c)\}$ consist of four inert nucleons in the $0s_{1/2}$ orbit and five valence nucleons in the $p$-shell ($0p_{3/2}$ and $0p_{1/2}$) orbits, that is $0\hbar\omega$ configurations. In the $\{\psi_c(^9$Be; $J^+_c)\}$ eigenstates, one nucleon excited from the $0s_{1/2}$ orbit to the $p$-shell orbits or from the $p$-shell orbits to the $sd$-shell ($0d_{5/2}$, $1s_{1/2}$, and $0d_{3/2}$) orbits, that is $1\hbar\omega$ $1p$-$1h$ configurations. $\Lambda(s)$ and $\Lambda(p)$ denote the state of the $Λ$-hyperon in the $0s_{1/2}$ orbit ($0\hbar\omega$) and the $p$-shell orbits ($1\hbar\omega$), respectively. It is an interesting point to note that the nuclear core eigenstates with different parities are mixed in Eqs. (1) and (2), respectively. In the positive parity states of $^{10}_Λ$Be, which is expressed by Eq. (2), both $[\psi_c(^9$Be; $J^+_c) \otimes \Lambda(s)]^{(i)}_J$ and $[\psi_c(^9$Be; $J^-_c) \otimes \Lambda(p)]^{(j)}_J$ belong to $1\hbar\omega$ in the naive picture, and these two configurations easily mix by the $ΛΣ$ interaction.

We construct the Hamiltonian of the $^9$Be core nucleus by adopting the Cohen-Kurath interaction [19] for the $p$-shell part, the modified Kuo-Brown $G$-matrix [20, 21] for the $sd$-shell part, and the Millener-Kurath interaction [22] for the $p$-$sd$ cross-shell part, respectively, which are reasonable $NN$ effective interaction in the shell model. In order to remove the spurious center-of-mass motion effects, we include the ATB interaction [23]. In addition, in order to obtain wave functions of the $^{10}_Λ$Be hypernucleus, we try to adopt the Nijmegen soft-core model NSC97e for the $ΛN$ interaction [24]. We determine the single-particle energies of the $Λ(s)$ and $Λ(p)$ states to reproduce the sharp $2^-$ and $3^+$ peaks, which are uniquely identified respectively, in the $(e,e'K^+)$ experimental production spectrum of $^{12}$Be [12].

3. DWIA calculations for $(\gamma, K^+)$, $(K^-, \pi^-)$ and $(\pi^+, K^+)$ reaction spectra

Corresponding to the actual kinematics of the $^{10}$B $(e,e'K^+)^{10}_Λ$Be reaction experiment [13], we estimate the $^{10}$B $(\gamma, K^+)^{10}_Λ$Be cross sections at $E_\gamma = 1.5$ GeV and $\theta_{K^{\text{lab}}} = 7^\circ$. In the analyses
of hypernuclear reaction spectroscopy, it is important to make use of the characteristic nature of particular reaction process. In the \((\gamma, K^+)\) or \((e, e'K^+)\) reaction, the spin-flip amplitudes dominate and therefore high-spin unnatural parity states in hypernuclear energy levels are selectively excited due to the sizable momentum transfer \([25]\). On the other hand, the \((K^-, \pi^-)\) reaction at recoiless condition at \(p_{K^0}^{\text{lab}} \approx 0.8\, \text{GeV}/c\) and forward angle excites preferentially the substitutional states in which a surface neutron is simply replaced with \(\Lambda\) in the same orbit because of the \(\Delta J = 0\) \((\Delta L = 0\) and \(\Delta S = 0)\) transition. In the \((\pi^+, K^+)\) reaction which involves the sizable momentum transfer \(q \approx 350\, \text{MeV}/c\) with spin-nonflip dominance, the hypernuclear high-spin states with natural parity are selectively excited.

In order to take account of these selective nature of each reaction process, we start with the microscopic amplitudes that describe the elementary process of hyperon production. For the \((\gamma, K^+)\) reaction, we employ the \(t\)-matrix of the \(\gamma + p \rightarrow \Lambda + K^+\) reaction expressed in the two-body Lab system as

\[
\langle k_{\gamma} - p, p | t | k_\gamma, 0 \rangle^{\text{Lab}}_{(\gamma, K)} = \epsilon_0 (f_0 + g_0 \sigma_0) + \epsilon_x (g_1 \sigma_1 + g_{-1} \sigma_{-1}) .
\]

Here \(k_\gamma\) and \(p\) are the incident photon and out-going kaon momenta, while \(\epsilon_0\) and \(\epsilon_x\) denote the photon polarization in the \(z\) and \(x\) directions, and \(\sigma_i\) is the baryon spin operators, respectively. The spin-nonflip coefficients \((f_0)\) and spin-flip ones \((g_0, g_1\) and \(g_{-1})\) are complex and momentum-dependent. For the definitions one may refer to Ref. \([17]\). The hypernuclear production cross section \(\frac{d\sigma}{d\Omega} (\theta_K)\) for the real photon \((\gamma, K^+)\) reaction is calculated in DWIA on the basis of the many-body transition matrix elements \(R(f_i; M_f)\) by employing the target wave function \(\Phi_i\) and hypernuclear ones \(\Psi_f\) solved within the multi-configuration shell model space, respectively:

\[
R(f_i; M_f) = \frac{1}{2J_i + 1} \sum_{M_f} |\langle \Psi(J_f M_f T_f \tau_f) | \Phi(J_i M_i T_i \tau_i) \rangle|^2 ,
\]

\[
\mathcal{O}_{(\gamma, K)} = \int d^3r \chi^{(--) \ast}_K(p, \xi, r) \chi^{(+)}(k, r) \sum_{\nu=1}^{A} V_{\nu}(\nu) \delta(r - \eta \xi_{\nu}) \langle k_{\gamma} - p, p | t | k_{\gamma}, 0 \rangle ,
\]

where the factors \(\xi = M_A/M_H\) and \(\eta = (M_H - M_A)/M_A\) are introduced so as to take the recoil effect into account. The operator \(V^{(\nu)}\) converts a proton into a \(\Lambda\) hyperon. For incident photon the plane wave is employed, while the \(K^+\) distorted wave \(\chi^{(+)}_K\) is evaluated by solving the Klein-Gordon equation, where the optical potential \(U_K(r)\) is assumed to be proportional to an appropriate nuclear density \([26]\): \(2\omega U_K(r) = -b_0 k^2 L \rho(r)\) with a purely absorptive parameter \(\text{Im} b_0 = (\sigma_{\text{tot}}^{\text{jet}})_{av} / k_L\).

For the meson-induced reaction cases of \((K^-, \pi^-)\) and \((\pi^+, K^+)\) reactions, denoted by \((a, b)\), in which a neutron is converted into \(\Lambda\), their elementary \(t\)-matrices are given respectively in terms of the spin-nonflip and spin-flip components as

\[
\langle k_a - p_b, p_b | t | k_a, 0 \rangle^{\text{Lab}}_{(a, \pi^+), (\pi, K)} = f' + g' (\sigma \cdot n) ,
\]

where \(\sigma\) is the baryon spin and \(n\) is the unit vector perpendicular to the reaction plane. The hypernuclear production cross sections for \(^A Z (K^-, \pi^-) \Lambda^Z\) and \(^A Z (\pi^+, K^+) \Lambda^Z\) reactions are similarly calculated in DWIA with the corresponding meson distorted waves and the elementary \(t\)-matrices given by Eq. \((6)\). For the detailed expressions of the many-body transition matrix elements \(R(f_i; M_f)\), one can refer to Ref. \([27]\).

In the theoretical reaction spectroscopy, it is useful to compare three reaction spectra of different characters as mentioned above. As the major purpose, we show the strength function of the \(^{10}\text{B} (\gamma, K^+) ^{10}\text{Be}\) production reaction in Fig.\,1 (b). Then we make use of the recoiless \((K^-, \pi^-)\) reaction in order to identify the substitutional states among many excited energy
levels obtained in the multi-configuration calculations. It is noted that the p-state single-particle energy of $\Lambda$ is fixed so as to reproduce the unique $[p_{3/2}^1p_{3/2}^\Lambda]$ ($J = 3^+$) state observed at 10.99 MeV excitation energy in the $^{12}$C ($e, e'K^+$) $^{12}_\Lambda$B production experiment [12]. It is interesting to see dynamical change of $p$-state energy position when we go from the spherical $^{12}_\Lambda$B to the $A = 10$ and $9$ hypernuclei under the possible effect of nuclear core deformation. For this purpose we diagonalize the energy levels of $^{10}_\Lambda$Be and $^9$Be, respectively, with the same input of $\Lambda$ $p_{3/2}$-state energy mentioned above. With these obtained wave functions, we use the $(K^-, \pi^-)$ reaction spectra in order to find the energy positions of $p$-orbit $\Lambda$ particle when it is coupled with deformed core nuclei such as $^9$Be and $^8$Be, respectively.

4. Results and discussion
We perform numerical calculations of the $^{10}_\Lambda$Be hypernucleus to evaluate the cross sections of the $^{10}$B ($\gamma, K^+$) $^{10}_\Lambda$Be production reaction, which corresponds to the $^{10}$B ($e, e'K^+$) $^{10}_\Lambda$Be production reaction done at JLab [13]. Figure 1 (a) shows the experimental and calculated energy levels of $^{10}_\Lambda$Be. In the calculated energy levels (right panel), solid lines denote the $J^-$ states, and dashed lines denote the $J^+$ states that can be expressed by the extended shell-model. In the experimental energy levels (left panel) that are reported in the JLab experiment, #1, #2, #3, and #4 levels at 0.00, 2.78, 6.26, and 10.83 MeV have been expressed by the conventional shell-model calculation [17,28,29] and other model calculations [30,31]. However, the bump (a) about 8.34 MeV is not predicted by these calculations. The present calculation by using the extended shell-model, for the first time, can express the bump as the several $J^+$ states, which is clearly shown in Fig. 1 (b). The DWIA cross sections in Fig. 1 (b) are based on the elementary amplitudes Saclay-Lyon model $\Lambda$ [32] and are in satisfactory agreement with the experimental ones [13]. We emphasize that the present calculation by using the extended shell-model can provide us with new $J^+$ states at right position that correspond to the new bump (a) of the JLab experiment, in addition to the major peaks (#1–#4). The numerical values of excitation

---

**Figure 1.** (a) Experimental (left) and calculated (right) energy levels of $^{10}_\Lambda$Be. (b) Calculated spectrum of the $^{10}$B ($\gamma, K^+$) $^{10}_\Lambda$Be reaction. (c) Calculated energy levels of $^{10}_\Lambda$Be (left) and spectroscopic factors of proton pickup reaction from the $^{10}$Be ground state (right).
Figure 2. (a) Calculated spectrum of the $^{10}$B ($K^-,\pi^-$) $^{10}$B reaction (left) and the $^9$Be ($K^-,\pi^-$) $^9$Be reaction (right). (b) Images of the $\Lambda(p_\perp)$ and $\Lambda(p_{\parallel})$ states in the cluster-model picture.

energies and cross sections are listed in Ref. [18].

In order to investigate the structure of the bump, we calculate the spectroscopic factors of proton pickup reaction from the $^{10}$B ground state, which play an important role in production cross-sections of the ($\gamma, K^+$) reaction. Figure 1 (c) shows the spectroscopic factors (right), together with the energy levels of the $^9$Be core nucleus (left). In the proton pickup reaction from the $^{10}$B ground state, the low-lying $J^-_c$ states of $^9$Be are strongly excited, and, by coupling with the $\Lambda(s)$ state, these states express major peaks in the $^{10}$B ($e,e'K^+$) $^{10}$Be reaction. On the other hand, the $J^+_c$ states are hardly excited and then do not contribute to the ($\gamma, K^+$) reaction directly. Thus, the $J^-_c$ core states coupled with the $\Lambda(p)$ state mainly contribute to the bump. The peak broadness of the bump is attributed to a group of several $J^+$ states that are obtained by the configuration mixing due to the $\Lambda N$ interaction in the present extended shell model.

In the cross sections of the $^{10}$B ($e,e'K^+$) $^{10}$Be reaction at JLab, the bump observed about 8.34 MeV seems considerably low in energy when one compares it with the $\Lambda$ single-particle energy difference $\varepsilon^\Lambda_p - \varepsilon^\Lambda_s \approx 11$ MeV underlying the present multi-configuration shell model. In order to understand this low energy $\Lambda(p)$ state, the analysis of the $^9$Be hypernucleus plays an important role. In $^9$Be, it is well known that the $\Lambda(p)$ state splits into two orbital states, $\Lambda(p_\perp)$ and $\Lambda(p_{\parallel})$, which occurs due to the strong coupling with nuclear core deformation having the $\alpha-\alpha$ structure [33], and the excitation energy of the $\Lambda(p_{\parallel})$ state comes down to 7 MeV. Figure 2 shows the DWIA cross sections of the $^{10}$B ($K^-,\pi^-$) $^{10}$B and $^9$Be ($K^-,\pi^-$) $^9$Be production reactions (a) by using the wave functions in the extended shell model, together with the images of the $\Lambda(p_\perp)$ and $\Lambda(p_{\parallel})$ states in the cluster-model picture (b). The present DWIA calculation of the $^9$Be ($K^-,\pi^-$) $^9$Be reaction is good agreement with that in the cluster-model calculation [33]. In the $^9$Be ($K^-,\pi^-$) $^9$Be production reaction, the $\Lambda(p_\perp)$ and $\Lambda(p_{\parallel})$ states are observed as two peaks. The $3/2^-_c$ peak at hypernuclear energy $E^\Lambda \approx 0$ MeV corresponds to the $\Lambda(p_{\parallel})$ state, and the $3/2^-_c$ peak at $E^\Lambda \approx 6$ MeV corresponds to the $\Lambda(p_\perp)$ state. Since the $\Lambda(p_\perp)$ state is the $^9$Be analog state, the big peak of the $\Lambda(p_\perp)$ state appears in the recoilless condition case with the
incident momentum of 0.80 GeV/c. The similar structure is realized also in cross sections of the $^{10}\text{B} (K^-, \pi^-) ^{10}\Lambda\text{B}$ reaction in Fig. 2 (a). The big 3$^+$ peak at $E_A \approx 4$ MeV is the $^{10}\text{Be}$ analog state, and corresponds to the $\Lambda(p_{\perp})$ state in the $^{10}\Lambda\text{B}$ case. Below the big peak, we can see the 3$^+$ peak at $E_A \approx 0$ MeV, which corresponds to the $\Lambda(p_{||})$ state in the $^{10}\Lambda\text{B}$ case and to the bump in the $^{10}\text{B} (e, e'K^+) ^{10}\Lambda\text{Be}$ reaction. It is difficult to obtain such state within the conventional shell model. We can show the structure of the $\Lambda(p)$ state by using the extended shell model. The detailed analysis of the $^{10}\Lambda\text{Be}$ structure with the $\Lambda(p)$ state will be published elsewhere, together with the DWIA results for other reactions.

5. Summary
We have extended the model space by introducing a new configuration with $\hbar\omega$ excitation for both nuclear and hyperon states, in order to explain the new bump observed at $E_A \simeq 0$ MeV in the $^{10}\text{B} (e, e'K^+) ^{10}\Lambda\text{Be}$ reaction done at JLab. Our new calculation has successfully explained the new bump as a sum of cross sections of several $J^+$ states. These $J^+$ states mainly consist of the $^9\text{Be}(J^c)$ nuclear core states coupled with the $\Lambda(p_{||})$ hyperon state, which is one of the splitting $\Lambda(p)$ states occurred due to the strong coupling with nuclear core deformation. It is also interesting to point out that some of these $J^+$ states contain the other component such as $^9\text{Be}(J^c) \times \Lambda(s)$. In other words, the nuclear core states having different parity are mixed due to the mediation by the $\Lambda$ hyperon, which is one of the novel aspects realized in hypernuclei.

References
[1] Bruecker W et al 1978 Phys. Lett. B 79 157; Bertini R et al 1983 Phys. Lett. B 83 306
[2] Povh B 1980 Nucl. Phys. A 335 233, and references therein
[3] Dalitz R H 1980 Nucl. Phys. A 354 110, and references therein
[4] Chrien R E et al 1979 Phys. Lett. B 89 31
[5] May M et al 1981 Phys. Rev. Lett. 47 1106; May M et al 1983 Phys. Rev. Lett. 51 2085
[6] Iodice M et al 2007 Phys. Rev. Lett. 99 052501
[7] Cusanno F et al 2009 Phys. Rev. Lett. 103 202501; Cusanno F et al 2010 Nucl. Phys. A 835 129
[8] Garibaldi F et al 2013 Nucl. Phys. A 914 34
[9] Miyoshi T et al 2003 Phys. Rev. Lett. 90 232502; Yuan L et al 2006 Phys. Rev. C 73 044607
[10] Hashimoto O et al 2010 Nucl. Phys. A 835 121
[11] Nakamura S N et al 2013 Phys. Rev. Lett. 110 012502
[12] Tang L et al 2014 Phys. Rev. C 90 034320
[13] Gogami T et al 2016 Phys. Rev. C 93 034314
[14] Hashimoto O and Tamura H 2006 Prog. Part. Nucl. Phys. 57 564, and references therein
[15] Hiyama E, Motoba T and Yamamoto Y (eds.) 2010 Prog. Theor. Phys. Suppl. 185
[16] Gal A, Hungerford E V and Millener D J 2016 Rev. Mod. Phys. 88 035004
[17] Motoba T, Setona M and Itonaga K 1994 Prog. Theor. Phys. Suppl. 117 123
[18] Umeya A, Motoba T and Itonaga K 2009 JPS Conf. Proc. 26 023016
[19] Cohen S and Kurath D 1965 Nucl. Phys. 73 1
[20] Kuo T S 1967 Nucl. Phys. A 103 71
[21] Ando K, Bandô H and Nagata S 1974 Prog. Theor. Phys. 52 509
[22] Millener D J and Kurath D 1975 Nucl. Phys. A 255 315
[23] Anantaraman N, Toki H and Bertsch G F 1983 Nucl. Phys. A 398 269
[24] Rijken Th A, Stoks V G J and Yamamoto Y 1999 Phys. Rev. C 59 21
[25] Bydžovský P, Setona M, Motoba T, Itonaga K, Ogawa K and Hashimoto O 2012 Nucl. Phys. A 881 199
[26] Itonaga K, Motoba T and Bandô H 1990 Prog. Theor. Phys. 84 291
[27] Itonaga K, Motoba T, Richter O and Setona M 1994 Phys. Rev. C 49 1045
[28] Motoba T, Bydžovský P, Setona M and Itonaga K 2010 Prog. Theor. Phys. Suppl. 185 224
[29] Millener D J 2012 Nucl. Phys. A 881 298
[30] Hiyama E and Yamamoto Y 2012 Prog. Theor. Phys. 128 105
[31] Iwakami H, Homma H, Kinkawa M, Dote A and Ohnishi A 2013 Few-Body Syst. 54 1219
[32] Mizutani T, Fayard C, Lamot G -H and Saghai B 1998 Phys. Rev. C 58 75
[33] Motoba T, Bandô H, Ikeda K and Yamada T 1985 Prog. Theor. Phys. Suppl. 81 42