ON SOME TWO PARAMETER QUANTUM AND JORDANIAN
DEFORMATIONS, AND THEIR COLOURED EXTENSIONS

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Abstract

This paper surveys some recent algebraic developments in two parameter Quantum
defORMations and their Nonstandard (or Jordanian) counterparts. In particular, we discuss the contraction procedure and the quantum group homomorphisms associated to these deformations. The scheme is then set in the wider context of the coloured extensions of these deformations, namely, the so-called Coloured Quantum Groups.

I Introduction

Recent years have witnessed considerable development in the study of multiparameter quantum deformations from both, the algebraic as well the differential geometric point of view. These have also found profound applications in many diverse areas of Mathematical Physics. Despite of the intensive and successful development of the mathematical theory of multiparameter quantum deformations or quantum groups, various important aspects still need thorough investigation. Besides, all quantum groups seem to have a natural coloured extension thereby defining corresponding coloured quantum groups. It is the aim of this paper to address some of the key issues involved.

Two parameter deformations provide an obvious step in constructing generalisations of single parameter deformations. Besides being mathematically interesting in their own right, two parameter quantum groups serve as very good examples in generalising physical theories based on the quantum group symmetry. $GL_{p,q}(2)$ and $GL_{h,h'}(2)$ are well known examples of two parameter Quantum and Jordanian deformations of the space of $2 \times 2$ matrices. Just as both these quantum groups are of great significance in building up various mathematical and physical theories, it is worthwhile to look for other possible examples, including the ‘coloured’ ones, which might play a fundamental role in future.

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researches. We wish to focus our attention on a new two parameter quantum group [1], $G_{r,s}$ which sheds light on some of the above mentioned issues. $G_{r,s}$ is a quasitriangular Hopf algebra generated by five elements, four of which form a Hopf subalgebra isomorphic to $GL_q(2)$, while the fifth generator relates $G_{r,s}$ to $GL_{p,q}(2)$.

The $G_{r,s}$ quantum group, which is the basis of our investigation, is defined in Section II. In Section III, we give a new Jordanian analogue of $G_{r,s}$, denoted $G_{m,k}$ and establish a homomorphism with $GL_{h,h'}(2)$. Both $G_{r,s}$ and $G_{m,k}$ admit a natural coloured extension and this is given in Section IV. Section V generalises the contraction procedure to the case of coloured quantum groups and discusses various homomorphisms. In section VI, we make concluding remarks and give possible physical significance of our results. Throughout this paper, we shall endeavour to refrain from too much of technical details, which can be found in the appropriate references.

II Two parameter $q$- deformations

The quantum group $G_{r,s}$ was defined in [1] as a quasitriangular Hopf algebra with two deformation parameters $r$ and $s$, and generated by five elements $a, b, c, d$ and $f$. The generators $a, b, c, d$ of this Hopf algebra form a subalgebra, in fact a Hopf subalgebra, which coincides exactly with the single parameter dependent $GL_q(2)$ quantum group when $q = r^{-1}$. Moreover, the two parameter dependent $GL_{p,q}(2)$ can also be realised through the generators of this $G_{r,s}$ Hopf algebra, provided the sets of deformation parameters $(p, q)$ and $(r, s)$ are related to each other in a particular fashion. This new algebra can, therefore, be used to realise both $GL_q(2)$ and $GL_{p,q}(2)$ quantum groups. Alternatively, this $G_{r,s}$ structure can be considered as a two parameter quantisation of the classical $GL(2) \otimes GL(1)$ group. The first four generators of $G_{r,s}$, i.e. $a, b, c, d$ correspond to $GL(2)$ group at the classical level and the remaining generator $f$ is related to $GL(1)$. In fact, $G_{r,s}$ can also be interpreted as a quotient of multiparameter $q$- deformation of $GL(3)$.

The elements of $G_{r,s}$ can be conveniently arranged in the matrix $T = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix}$ and the coalgebra and counit are $\Delta(T) = T \otimes T$, $\varepsilon(T) = 1$. It should be mentioned that the quantum determinant $\delta = Df$ (where $D = ad - r^{-1}bc$) is group-like but not central. The above block diagonal form of the $T$- matrix is particularly convenient to understand the related schematics. The $G_{r,s}$ $R$-matrix is given in [1,2], and the most general Hopf algebra generated by this $R$-matrix is multiparameter $GL(3)$ with the $T$-matrix of the form $\begin{pmatrix} a & b & x_1 \\ c & d & x_2 \\ v_1 & v_2 & f \end{pmatrix}$. It can be shown that the two-sided Hopf ideal generated by $x_1, x_2$ when factored out yields the Inhomogenous multiparameter $IGL(2)$. Furthermore, if one factors out yet another two-sided Hopf ideal generated by elements $v_1, v_2$, what one obtains is precisely the $G_{r,s}$ Hopf algebra. The relation of $G_{r,s}$ with various
known q-deformed groups can be exhibited as

\[
\begin{array}{ccc}
GL_Q(3) & \xrightarrow{\mathcal{Q}} & IGL_Q(2) \\
\downarrow & & \downarrow \\
GL(2) \otimes GL(1) & \xleftarrow{\mathcal{L}} & G_{r,s} & \xrightarrow{\mathcal{F}} & GL_{p,q}(2) \\
\downarrow & & \downarrow S & & \downarrow GL_q(2)
\end{array}
\]

where \( \mathcal{Q}, \mathcal{F}, S \) and \( \mathcal{L} \) denote the Quotient, Hopf algebra homomorphism, (Hopf)Subalgebra and (classical) Limit respectively. \( GL_Q(3) \) denotes the multiparameter q-deformed \( GL(3) \) and \( IGL_Q(2) \) is the inhomogenous multiparameter q-deformation of \( GL(2) \). Motivated by the rich structure of \( G_{r,s} \), this quantum group has recently been studied by the authors in detail \([2]\). As an initial step in the further understanding of \( G_{r,s} \), the authors have derived explicitly the dual algebra and showed that it is isomorphic to the single parameter deformation of \( gl(2) \oplus gl(1) \), with the second parameter appearing in the costructure. In \([2]\), the authors have also constructed a differential calculus on \( G_{r,s} \), which in turn provides a realisation of the calculus on \( GL_{p,q}(2) \).

### III Two parameter h-deformations

Jordanian deformations (also known as h-deformations) of Lie groups and Lie algebras have attracted a lot of attention in recent years. A peculiar feature of this deformation is that the corresponding \( R \)-matrix is triangular i.e. \( R_{12}R_{21} = 1 \). These deformations are called ‘Jordanian’ due to the Jordan normal form of the \( R \)-matrix. It was shown in \([3]\) that upto isomorphism, \( GL_q(2) \) and \( GL_h(2) \) are the only possible distinct deformations (with central determinant) of the group \( GL(2) \). In \([4]\), an interesting observation was made that the h-deformations could be obtained by a singular limit of a similarity transformation from the q-deformations, and this was generalised to multiparameter deformations as well as to higher dimensions i.e space of \( n \times n \) quantum matrices \([5]\). For the purpose of current investigation, the authors have applied the contraction procedure to \( G_{r,s} \) to obtain a new Jordanian quantum group \( G_{m,k} \) \([6]\). It turns out that this new structure is also related to other known Jordanian quantum groups.

The \( G_{m,k} \) quantum group can be defined as a triangular Hopf algebra generated by the \( T \)-matrix \( \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix} \). The set of commutation relations consisting of elements \( a,b,c,d \) form a subalgebra that coincides exactly with the single parameter Jordanian \( GL_h(2) \) for \( m = h \). This is exactly analogous to the q-deformed case where the first four elements of \( G_{r,s} \) form the \( GL_q(2) \) Hopf subalgebra. Again, the remaining fifth element \( f \) generates the \( GL(1) \) group, as it did in the q-deformed case, and the second parameter appears only through the cross commutation relations between \( GL_m(2) \) and \( GL(1) \) elements. Therefore, \( G_{m,k} \)
can also be considered as a two parameter Jordanian deformation of classical $GL(2) \otimes GL(1)$ group. Furthermore, $G_{m,k}$ also provides a realisation of the two parameter Jordanian $GL_{h,h'}(2)$. Besides, it may be interpreted as a quotient of the multiparameter Jordanian deformation of $GL(3)$, denoted $GL_J(3)$ as well as that of inhomogenous $IGL(2)$, denoted $IGL_J(2)$ quantum groups. This can be represented as follows

\[
\begin{array}{c}
GL_J(3) \\
\downarrow \mathcal{Q} \\
IGL_J(2) \\
\downarrow \mathcal{Q} \\
GL(2) \otimes GL(1) \\
\mathcal{L} \\
\downarrow \mathcal{S} \\
GL_{h,h'}(2) \\
\end{array}
\]

\[G_{m,k} \xleftarrow{\mathcal{F}} GL_{h,h'}(2) \]

where the maps $\mathcal{Q}$, $\mathcal{F}$, $\mathcal{S}$ and $\mathcal{L}$ are as before.

**IV Coloured Extensions**

The standard quantum group relations can be extended by parametrising the corresponding generators using some continuous ‘colour’ variables and redefining the associated algebra and coalgebra in a way that all Hopf algebraic properties remain preserved [1,7,8]. For the case of a single parameter quantum deformation of $GL(2)$ (with deformation parameter $r$), its ‘coloured’ version [1] is given by the $R$-matrix, denoted $R^\lambda_{\mu}^\nu$ which satisfies

\[R^\lambda_{12}R^\lambda_{13}R^\mu_{23} = R^\mu_{23}R^\lambda_{13}R^\lambda_{12}\]

the so-called ‘Coloured’ Quantum Yang Baxter Equation (CQYBE). It should be stressed at this point that the coloured $R$-matrix provides a nonadditive-type solution $R^\lambda_{\mu} \neq R(\lambda - \mu)$ of the Yang-Baxter equation, which is in general multicomponent and the parameters $\lambda$, $\mu$, $\nu$ are considered as ‘colour’ parameters. Such solutions were first discovered in the study of integrable models [9].

This gives rise to the coloured $RTT$ relations

\[R^\lambda_{\mu}T_1T_2 = T_2R^\lambda_{\mu} \]

(\text{where } T_{1\lambda} = T_1 \otimes 1 \text{ and } T_{2\mu} = 1 \otimes T_\mu) \text{ in which the entries of the } T \text{ matrices carry colour dependence}. \text{ The coproduct and counit for the coalgebra structure are given by } \Delta(T_\lambda) = T_\lambda \otimes T_\lambda, \varepsilon(T_\lambda) = 1 \text{ and depend only on one colour parameter. By contrast, the algebra structure is more complicated with generators of two different colours appearing simultaneously in the algebraic relations. The full Hopf algebraic structure can be constructed and results in a coloured extension of the quantum group. Since $\lambda$ and $\mu$ are continuous variables, this implies the coloured quantum group has an infinite number of generators.}
The above coloured generalisation of the FRT formalism was given by Kundu and Basu-Mallick [1,10] and that of the Drinfeld-Jimbo formulation of quantised universal enveloping algebras has been given by Bonatos, Quesne et al [11]. In the context of knot theory, Ohtsuki [12] introduced some coloured quasitriangular Hopf algebras, which are characterised by the existence of a coloured universal $R$-matrix, and he applied his theory to $U_q\mathfrak{sl}(2)$. Coloured generalisations of quantum groups can also be understood as an application of the twisting procedure, in a manner similar to the multiparameter generalisation of quantum groups. Jordanian deformations also admit coloured extensions [7]. The associated $R$-matrix satisfies the CQYBE and is 'colour' triangular i.e. $R_{12}^{\lambda,\mu} = (R_{21}^{\mu,\lambda})^{-1}$, a coloured extension of the notion of triangularity.

**Coloured Extension of $G_{r,s}$ : $G_{r,s'}^{s,s'}$**

The coloured extension of $G_{r,s}$ proposed in [1] has only one deformation parameter $r$ and two colour parameters $s$ and $s'$. The second deformation parameter of the uncoloured case now plays the role of a colour parameter. In such a coloured extension, the first four generators $a, b, c, d$ are kept independent of the colour parameters while the fifth generator $f$ is now parameterised by $s$ and $s'$. The matrices of generators are

$$T_s = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f_s \end{pmatrix}, \quad T_{s'} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f_{s'} \end{pmatrix}$$

From the $RTT$ relations, one observes that the commutation relations between $a, b, c, d$ remain unchanged whereas $f_s$ and $f_{s'}$ now satisfy two colour copies of the relations satisfied by $f$ of the uncoloured $G_{r,s}$. In addition, the relation $[f_s, f_{s'}] = 0$ holds. The associated coloured $R$-matrix, denoted $R_{r,s'}^{s,s'}$ satisfies the CQYBE

$$R_{12}(r; s, s')R_{13}(r; s, s'')R_{23}(r; s', s'') = R_{23}(r; s', s'')R_{13}(r; s, s'')R_{12}(r; s, s')$$

and the corresponding coloured quantum group is denoted $G_{r,s'}^{s,s'}$.

**Coloured Extension of $G_{m,k}$ : $G_{m}^{k,k'}$**

Similar to the case of $G_{r,s}$, we have proposed [13] a coloured extension of the Jordanian quantum group $G_{m,k}$. The first four generators remain independent of the colour parameters $k$ and $k'$ whereas the generator $f$ is parameterised by $k$ and $k'$. Again, the second deformation parameter $k$ of the uncoloured case now plays the role of a colour parameter and the $T$-matrices are

$$T_k = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f_k \end{pmatrix}, \quad T_{k'} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f_{k'} \end{pmatrix}$$

The commutation relations between $a, b, c, d$ remain unchanged whereas $f_k$ and $f_{k'}$ satisfy two colour copies of the relations satisfied by $f$ of the uncoloured $G_{m,k}$.
In addition, the relation \([f_k, f_{k'}] = 0\) holds. The associated coloured \(R\)-matrix, denoted \(R_{m,k,k'}^k\), is a solution of the CQYBE
\[
R_{12}(m; k, k') R_{13}(m; k, k'') R_{23}(m; k', k'') = R_{23}(m; k', k'') R_{13}(m; k, k'') R_{12}(m; k, k')
\]
and is colour triangular. The corresponding coloured Jordanian quantum group is denoted \(G_{m,k,k'}\).

\[\text{V Constructions and Homomorphisms}\]

The \(R\)-matrix of the Jordanian (or \(h\)-deformation) can be viewed as a singular limit of a similarity transformation on the \(q\)-deformed \(R\)-matrix [4]. Let \(g(\eta)\) be a matrix dependent on a contraction parameter \(\eta\) which is itself a function of one of the deformation parameters of the \(q\)-deformed algebra. This can be used to define a transformed \(q\)-deformed \(R\)-matrix
\[
R_h = (g^{-1} \otimes g^{-1}) R_q (g \otimes g)
\]
The \(R\)-matrix of the Jordanian deformation is then obtained by taking a limiting value of the parameter \(\eta\). Even though the contraction parameter \(\eta\) is undefined in this limit, the new \(R\)-matrix is finite and gives rise to a new quantum group structure through the \(RTT\)-relations. For example, in the contraction process which takes \(GL_q(2)\) to \(GL_h(2)\), the contraction matrix is
\[
g(\eta) = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}
\]
where \(\eta = \frac{h}{h_q}\) with \(h\) a new free parameter. Such transformations have proved to be powerful tools in establishing various connections between the \(q\)- and the \(h\)-deformed quantum groups, which were previously obscure. In the context of the quantum groups under consideration in the present paper, the contraction procedure was successfully applied [6] to the \(G_{r,s}\) quantum group of Section II to obtain the Jordanian \(G_{m,k}\) given in Section III. Furthermore, the multiparameter Jordanian \(GL_J(3)\) and hence the multiparameter Inhomogeneous \(IGL_J(2)\) were also obtained by contracting their respective \(q\)-deformed counterparts [14].

The Hopf algebra homomorphism \(F\) from \(G_{r,s}\) to \(GL_{p,q}(2)\), which provides a realisation of the latter, is given by
\[
F : G_{r,s} \rightarrow GL_{p,q}(2)
\]
\[
F \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \rightarrow \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = f^N \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)
\]
The elements \(a',b',c'\) and \(d'\) are the generators of \(GL_{p,q}(2)\) and \(N\) is a fixed non-zero integer. The relation between the deformation parameters \((p, q)\) and \((r, s)\) is given by
\[
p = r^{-1} s^N, \quad q = r^{-1} s^{-N}
\]
A Hopf algebra homomorphism

$$\mathcal{F} : G_{m,k} \rightarrow GL_{h,h'}(2)$$

of exactly the same form as in the $q$-deformed case, exists between the generators of $G_{m,k}$ and $GL_{h,h'}(2)$ provided that the two sets of deformation parameters $(h, h')$ and $(m, k)$ are related via the equation

$$h = -m + Nk \quad , \quad h' = -m - Nk$$

Note that for vanishing $k$, one gets the one parameter case. In addition, using the above realisation together with the coproduct, counit and antipode axioms for the $G_{m,k}$ algebra and the respective homeomorphism properties, one can easily recover the standard coproduct, counit and antipode for $GL_{h,h'}(2)$. Thus, the Jordanian $GL_{h,h'}(2)$ group can in fact be reproduced from the newly defined Jordanian $G_{m,k}$. It is curious to note that if we write $p = e^h, q = e^{h'}, r = e^m$ and $s = e^k$, then the relations between the parameters in the $q$-deformed case and the $h$-deformed case are identical. The systematics of the uncoloured quantum groups discussed here can be summarised in the following commutative diagram

$$
\begin{array}{cccccc}
GL_Q(3) & \xrightarrow{\mathcal{Q}} & IGL_Q(2) & \xrightarrow{\mathcal{Q}} & G_{r,s} & \xrightarrow{\mathcal{F}} & GL_{p,q}(2) \\
\downarrow{C} & & \downarrow{C} & & \downarrow{C} & & \downarrow{C} \\
GL_J(3) & \xrightarrow{Q} & IGL_J(2) & \xrightarrow{Q} & G_{m,k} & \xrightarrow{\mathcal{F}} & GL_{h,h'}(2)
\end{array}
$$

where $\mathcal{Q}, C$ and $\mathcal{F}$ denote the quotient, contraction and the Hopf algebra homomorphism. The contraction procedure discussed above has been successfully applied [13] to the case of coloured quantum groups yielding new coloured Jordanian deformations. We apply to $R^\lambda_{r}^\mu$, the coloured $R$-matrix for $q$-deformed $GL(2)$, the transformation

$$(g \otimes g)^{-1}R^\lambda_{r}^\mu(g \otimes g)$$

where $g$ is the two dimensional transformation matrix $\begin{pmatrix} \eta \\ 0 \end{pmatrix}$ and $\eta$ is chosen to be $\eta = \frac{m}{r-1}$. In the limit $r \rightarrow 1$, we obtain a new $R$-matrix, $R^\lambda_{m}^\mu$ which is a coloured $R$-matrix for a Jordanian deformation of $GL(2)$. The contraction is then also used to obtain the coloured extension $G^h_{m,k'}$ of $G_{m,k}$, from the coloured extension $G_{r,s}'$ of $G_{r,s}$. The $R$-matrix $R^k_{m,k'}$ is obtained as the contraction limit of the $R$-matrix for the coloured extension of $G_{r,s}$ via the transformation

$$R^k_{m,k'} = \lim_{r \rightarrow 1} (G \otimes G)^{-1}R^r_{s,s'}(G \otimes G)$$

where

$$G = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} : \quad g = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}, \quad \eta = \frac{m}{r-1}$$
The Hopf algebra homomorphism from $G_{s,s'}^r$ to $GL_{r}^{\lambda,\mu}(2)$

$$
\mathcal{F}_N : G_{s,s'}^r \rightarrow GL_{r}^{\lambda,\mu}(2)
$$
is given by

$$
\mathcal{F}_N : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a'_{\lambda} & b'_{\lambda} \\ c'_{\lambda} & d'_{\lambda} \end{pmatrix} = f_s^{N}(a \ b \\ c \ d)
$$

$$
\mathcal{F}_N : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a'_{\mu} & b'_{\mu} \\ c'_{\mu} & d'_{\mu} \end{pmatrix} = f_{s'}^{N}(a \ b \\ c \ d)
$$

where $N$ is a fixed non-zero integer and the sets of colour parameters $(s, s')$ and $(\lambda, \mu)$ are related through quantum deformation parameter $r$ by

$$
s = r^{2N\lambda} , \quad s' = r^{2N\mu}
$$

The primed generators $a'_{\lambda}, b'_{\lambda}, c'_{\lambda}, d'_{\lambda}$ and $a'_{\mu}, b'_{\mu}, c'_{\mu}, d'_{\mu}$ belong to $GL_{r}^{\lambda,\mu}(2)$ whereas the unprimed ones $a, b, c, d, f_s$ and $f_{s'}$ are generators of $G_{s,s'}^r$. If we now denote the generators of $GL_{m}^{\lambda,\mu}(2)$ by $a'_{\lambda}, b'_{\lambda}, c'_{\lambda}, d'_{\lambda}$ and $a'_{\mu}, b'_{\mu}, c'_{\mu}, d'_{\mu}$ and the generators of $G_{m,k}^{k,k'}$ by $a, b, c, d, f_k$ and $f_{k'}$ then a Hopf algebra homomorphism from $G_{m,k}^{k,k'}$ to $GL_{m}^{\lambda,\mu}(2)$

$$
\mathcal{F}_N : G_{m,k}^{k,k'} \rightarrow GL_{m}^{\lambda,\mu}(2)
$$
is of exactly the same form

$$
\mathcal{F}_N : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a'_{\lambda} & b'_{\lambda} \\ c'_{\lambda} & d'_{\lambda} \end{pmatrix} = f_k^{N}(a \ b \\ c \ d)
$$

$$
\mathcal{F}_N : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a'_{\mu} & b'_{\mu} \\ c'_{\mu} & d'_{\mu} \end{pmatrix} = f_{k'}^{N}(a \ b \\ c \ d)
$$

The sets of colour parameters $(k, k')$ and $(\lambda, \mu)$ are related to the Jordanian deformation parameter $m$ by

$$
Nk = -2m\lambda , \quad Nk' = -2m\mu
$$

and $N$, again, is a fixed non-zero integer. The schematics of our analysis for the coloured quantum groups is represented in the diagram

$$
\begin{array}{c}
G_{r,s} \xrightarrow{c} G_{r,s'}^r \xrightarrow{\mathcal{F}} GL_{r}^{\lambda,\mu}(2) \\
\downarrow \quad \downarrow c \\
G_{m,k} \xrightarrow{\mathcal{E}} G_{m}^{k,k'} \xrightarrow{\mathcal{F}} GL_{m}^{\lambda,\mu}(2)
\end{array}
$$

where $\mathcal{C}, \mathcal{F}$ and $\mathcal{E}$ denote the contraction, Hopf algebra homomorphism and coloured extension respectively. In both of the commutative diagrams above, the objects at the top level are the $q$ deformed ones and the corresponding Jordanian counterparts are shown at the bottom level.
VI Conclusions

In the present work, we have obtained a new Jordanian quantum group $G_{m,k}$ by contraction of the $q$-deformed quantum group $G_{r,s}$. We then used this new structure to establish quantum group homomorphisms with other known two parameter quantum groups at the Jordanian level. At the same time we also showed that such homomorphisms commute with the contraction procedure. Our analysis is then set in the wider context of coloured quantum groups. We give a coloured generalisation of the contraction procedure and obtain new coloured Jordanian quantum groups. A careful study of the properties of both $G_{r,s}$ and $G_{m,k}$ lead to their respective coloured extensions. Furthermore, we show that the homomorphisms of the uncoloured case naturally extend to the coloured case.

The physical interest in studying $G_{r,s}$ lies in the observation that when endowed with a $*$-structure, this quantum group specialises to a two parameter quantum deformation of $SU(2) \otimes U(1)$ which is precisely the gauge group for the theory of electroweak interactions. Since gauge theories have an obvious differential geometric description, the study of differential calculus [2] provides insights in constructing a $q$-gauge theory based on $G_{r,s}$. It would also be of significance to generalise the formalism of differential calculus to the case of coloured quantum groups and explore possible physical applications.

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