Lorentzian non-stationary dynamical systems

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Abstract

In this paper, we introduce a Lorentzian Anosov family (LA-family) up to a sequence of distributions of null vectors. We prove for each \( p \in M_i \), where \( M_i \) is a Lorentzian manifold for \( i \in \mathbb{Z} \), the tangent space \( M_i \) at \( p \) has a unique splitting and this splitting varies continuously on a sequence via the distance function created by a unique torsion-free semi-Riemannian connection. We present three examples of LA-families. Also, we define Lorentzian shadowing property of type I and II and prove some results related to this property.

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1. Introduction

There are some fantastic results on Riemannian manifolds which are extendable for Lorentzian manifolds. For instance, in [8] Porwal and Mishra introduced a new class of geodesic local E-convex sets and geodesic local (semilocal) E-convex functions on Riemannian manifolds and studied their features and in [9] pointwise slant submersions from almost product Riemannian manifolds onto Riemannian manifolds were investigated by Sepet and Ergut. Also, Anosov family as a generalization of Anosov diffeomorphisms were introduced in 2005 by P. Arnoux and A. M. Fisher [1] is a non-stationary dynamical system defined on a sequence of compact Riemannian manifolds. Since we are enthusiastic about the dynamical system on Lorentzian manifolds then the concept of Anosov families on Lorentzian manifolds is extended. For this purpose we take a sequence of Hausdorff Lorentzian manifolds of the same dimension with Lorentzian metrics. We recall that a Lorentzian metric for a manifold as a tensor field of type with the diagonal form according to whether or or . We take as a disjoint union of i.e. and its atlas is the set of charts of the form such that and is defined by where is a chart for and gives a chart on . These charts are expanded by the former method to a maximal atlas for . The construction of as a manifold implies that if the image of a curve is in a chart , then there is a unique such that .

In section 3, we extend Anosov family to a Lorentzian Anosov family up to a sequence of distribution , where is null subspace of the tangent space of the ambient Lorentzian manifold that is an extension of hyperbolic autonomous discrete dynamical system on semi-Riemannian manifold [6] to non-autonomous one. In proposition 3.5 we prove the tangent space of at a given has a unique splitting. In theorem 3.6 we determine the form of splitting.

In section 4 we provide two examples of Lorentzian Anosov families. In section 5 by using the results of [10] for non-autonomous discrete dynamical system and induced dynamics in hyperspace of them, we...
define Lorentzian shadowing property of type I and type II on $L$-family $(M,\langle\cdot,\cdot\rangle,F)$ and on the induced dynamic by it. In theorem 5.5 we prove shadowing property of type I is invariant by uniformly conjugacy and the product of two $L$-families has shadowing property of type I if and only if they have shadowing property of type I. In proposition 5.6 we prove if $L$-family $(M,g,F)$ has shadowing property of type II then it has shadowing property of type I.

2. Preliminaries

We begin this section by recalling the concepts of connections and parallel translation in semi-Riemannian geometry which the Lorentzian geometry is a special case of it [2].

Let $\chi(M)$ denote the set of all smooth vector fields defined on $M$ and let $\mathcal{F}(M)$ denote the ring of all smooth real-valued functions on $M$. A connection is a mapping $\nabla: \chi(M) \times \chi(M) \to \chi(M)$ with the following properties

i) $\nabla_v(X + Y) = \nabla_vX + \nabla_vY$;

ii) $\nabla_{fV + hw}(X) = f\nabla_v(X) + h\nabla_v(Y)$;

iii) $\nabla_v(fX) = f\nabla_vX + V(f)X$;

for all $f,h \in \mathcal{F}(M)$ and all $X,Y, V, W \in \chi(M)$.

The vector $\nabla_{X(p)}Y = \nabla_XY|_p$ at a point $p \in M$ depends only on the value $X(p) = X_p$ of $X$ at $p$ and the values of $Y$ along any smooth curve which passes through $p$ which has the velocity $X(p)$ at $p$. Let the connection $\nabla$ on $M$, a curve $\gamma:[-\varepsilon,\varepsilon]\to M$ and a smooth vector field $Y$ along $\gamma$ be given. Then for $t_0 \in [-\varepsilon,\varepsilon]$ we can locally extend $Y$ to a smooth vector field defined on a neighborhood of $\gamma(t_0)$. We denote the covariant derivative of $Y \in \chi(M)$ along $\gamma$ by $\frac{DY}{dt}$ and it is defined by $\nabla_{\gamma(t)}Y$. A vector field $Y$ along $\gamma$ which satisfies $\nabla_{\gamma(t)}Y(t) = 0$ for all $t \in [-\varepsilon,\varepsilon]$ is called a parallel vector field along $\gamma$. If $v \in T_pM$ and $\gamma:(-\varepsilon,\varepsilon)\to M$ is a smooth curve passing through $p$, that is, $\gamma(0) = p$, then it is proved that there is a unique parallel vector field $Y$ along $\gamma$ with $Y_p = v$. The mapping $P_t:T_pM\to T_{\gamma(t)}M, \ v \mapsto Y_{\gamma(t)}$ is called a parallel transition.

The torsion tensor $T$ of $\nabla$ is the mapping $T: \chi(M) \times \chi(M) \to \chi(M)$ defined by $T(X,Y) = \nabla_XY - \nabla_YX - [X,Y]$. A connection $\nabla$ with $T = 0$ is said to be torsion free or symmetric.
Two vectors \( v, w \) in \( T_pM \) are orthogonal if \( g(v, w) = 0 \). A given vector \( v \in T_pM \) is said to be a unit vector if \( |g(v, v)| = 1 \). Thus an orthonormal basis \( \{e_1, e_2, \ldots, e_n\} \) of \( T_pM \) satisfies \( |g(e_i, e_j)| = \delta_{ij} \). Let \((M, g)\) be an \( n \)-dimensional manifold \( M \) with a semi-Riemannian metric \( g \) of arbitrary signature \((-,-,+,-,+)\). There exists a unique connection \( \nabla \) on \( M \) such that

i) \( Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \) and

ii) \( \nabla_X Y - \nabla_Y X = [X, Y] \) for all \( X, Y, Z \in \mathcal{X}(M) \). This connection is called the Levi-Civita connection of \((M, g)\). The condition (i) means that the connection \( \nabla \) is compatible with the metric \( g \) and condition (ii) means that \( \nabla \) is torsion free. By replacing \( Z = \gamma' \) in (i), we find a unique parallel translation of vector fields along a given smooth curve \( \gamma \) of \( M \) which preserves \( g \) \cite{2}.

As stated in \cite{6} we use parallel translation to define a distance function \( d \) on the subspaces of the tangent spaces. Let \( \gamma : (-\varepsilon, \varepsilon) \to M \) be a smooth curve passing through \( p \). Then

\[
d(u, B_E) = \min \{ |g_{\gamma(t)}(P_{\gamma(-t)}(u) - w, P_{\gamma(-t)}(u) - w)| : w \in B_F \},
\]

where \( u \in T_{\gamma(0)}M \), \( E \) is a subspace of \( T_{\gamma(\zeta)}M \) with the basis \( B_E \) and \( t, \zeta \in (-\varepsilon, \varepsilon) \). For two given subspaces \( E \) and \( F \) of \( T_{\gamma(\zeta)}M \) and \( T_{\gamma(t)}M \) with the basis \( B_E \) and \( B_F \) we define the distance function \( d(B_E, B_F) \) by

\[
d(B_E, B_F) = \max \{ \max \{ d(v, B_E) : v \in B_E \}, \max \{ d(u, B_E) : u \in B_F \} \}
\]

(2.1)

3. Lorentzian Anosov family

In this section \( M \) is the disjoint union of the Hausdorff Lorentzian manifolds \( \{M_i\}_{i \in \mathbb{Z}} \), with Lorentzian metrics \( \{g_i\} \). We can define the Lorentzian metric \( g = \langle \cdot, \cdot \rangle \) on \( M \) by \( \langle \cdot, \cdot \rangle|_{M_i} = \langle \cdot, \cdot \rangle_i \) for each \( i \in \mathbb{Z} \).

**Definition 3.1:** A non-stationary dynamical system (or nsds) \((M, \langle \cdot, \cdot \rangle, F)\) is a mapping \( F : M \to M \) such that, for each \( i \in \mathbb{Z} \), \( F|_{M_i} = f_i : M_i \to M_{i+1} \) is a \( C^1 \)-diffeomorphism. We use the notation \( F = (f_i)_{i \in \mathbb{Z}} \). The \( n \)-th composition of \( F \) on \( M_i \) is defined by
\[
F^n_i = \begin{cases} 
 f_{i+n-1} \circ \cdots \circ f_i : M_i \to M_{i+n} & \text{if } n > 0 \\
 f_{i+n}^{-1} \circ \cdots \circ f_i^{-1} : M_i \to M_{i+n} & \text{if } n < 0 \\
 I_i : M_i \to M_i & \text{if } n = 0, 
\end{cases}
\]

where \( I_i : M_i \to M_i \) is the identity on \( M_i \). In this definition \( f_i \) may not be an isometry and the Lorentzian metric vary on each \( M_i \). We also use of the name \( L \)-family for the non-stationary dynamical system (or \( nsds \) \((M,\langle , \rangle, F)\)).

**Definition 3.2 :** An \( L \)-family \((M,\langle , \rangle, F)\) is called a Lorentzian Anosov family (\( LA \)-family) up to a distribution \( p \mapsto E^s(p) \), if there exist constants \( 0 < \lambda < 1 \), \( c > 0 \) and a continuous splitting \( T_pM = E^s(p) \oplus E^n(p) \oplus E^u(p) \) for each \( p \in M \) such that

i) Each vector of \( E^s(p) \) is a null vector and each non-zero vector in \( E^s(p) \) or \( E^u(p) \) is spacelike or timelike:

ii) The splitting \( T_pM = E^s(p) \oplus E^n(p) \oplus E^u(p) \) up to the sequence of distribution \( p \mapsto E^s(p) \) is \( DF \)-invariant, i.e., for each \( p \in M \), \( D_pF(E^s_p) = E^s_{F(p)} \) and \( D_pF(E^u_p) = E^u_{F(p)} \), where \( T_pM \) is the tangent space at \( p \);

iii) For each \( i \in \mathbb{Z} \), \( n \in \mathbb{N} \) and \( p \in M_i \), we have:

\[
| g_{F^n_i(p)} (D_p F_i^n(v), D_p F_i^n(v)) | \leq c \lambda^n | g_p(v,v) | \text{ if } v \in E^s(p)
\]

and

\[
| g_{F^{-n}_i(p)} (D_p F_i^{-n}(v), D_p F_i^{-n}(v)) | \leq c \lambda^n | g_p(v,v) | \text{ if } v \in E^u(p)
\]

iv) For each \( \xi \in E^u_p \) and for each \( \nu \in T_pM \) with the property

\[
| g_{F^{-n}_i(p)} (D_p F_i^{-n}(\nu), D_p F_i^{-n}(\nu)) | \leq c \lambda^n | g_p(\nu,\nu) |
\]

we have \( \lim_{n \to \infty} g_{F^{-n}_i(p)} (D_p F_i^{-n}(\xi), D_p F_i^{-n}(\nu)) = 0 \).

The subspaces \( E^s(p) \) and \( E^u(p) \) are called stable and unstable subspaces respectively.

**Lemma 3.3 :** The statement

\[
| g_{F_i^n(p)} (D_p F_i^n(v), D_p F_i^n(v)) | \leq c \lambda^n | g_p(v,v) | 
\]

for each \( i \in \mathbb{Z} \) and \( n \in \mathbb{N} \) and for \( v \in E^s_p \) is equivalent to

\[
| g_{F_i^{-n}(p)} (D_p F_i^{-n}(v), D_p F_i^{-n}(v)) | \geq c^{-1} \lambda^{-n} | g_p(v,v) |
\]

and the same is true for the condition on the unstable subspace.
Proof: Let \( w = D_p F^u_i(v) \) where \( w \in E^u_{F_p_i} \). Then

\[
|g_{F_i^u} (v,w)| = |g_{F_i^u} (D_p F^u_i(v), D_p F^u_i(v))| \\
\leq c \lambda^n |g_{f_p} (v,v)| \\
= c \lambda^n |g_{F_i^u} (D_p F^u_i(w), D_p F^u_i(w))|
\]

If we put \( F^u_i = q \), then \( |g_{F_i^u} (w,w)| \geq c^{-1} \lambda^{-n} |g_q (w,w)| \). Since \( D_p F_i \) is an isomorphism then the inequality is true for each element of \( E^u_i \).

Remark 3.4: Base on the previous lemma the condition (iii) of definition 3.2 can be replaced by:

\[
\begin{align*}
\text{(iii') } & |g_{F_i^u} (D_p F^u_i(v), D_p F^u_i(v))| \leq c \lambda^n |g_{f_p} (v,v)| \quad \text{if } v \in E^u(p) \quad \text{and} \\
& |g_{F_i^u} (D_p F^u_i(v), D_p F^u_i(v))| \geq c^{-1} \lambda^{-n} |g_{f_p} (v,v)| \quad \text{if } v \in E^u(p)
\end{align*}
\]

\[
\begin{align*}
\text{(iii'') } & |g_{F_i^u} (D_p F^u_i(v), D_p F^u_i(v))| \geq c^{-1} \lambda^{-n} |g_{f_p} (v,v)| \quad \text{if } v \in E^u(p) \quad \text{and} \\
& |g_{F_i^u} (D_p F^u_i(v), D_p F^u_i(v))| \leq c \lambda^n |g_{f_p} (v,v)| \quad \text{if } v \in E^u(p)
\end{align*}
\]

Proposition 3.5: Given an LA-family \((M, F)\) up to a sequence of distributions \( p \mapsto E^u(p) \). Then for each \( p \in M \), the tangent space of \( M \) at \( p \) has a unique splitting.

Proof: Since the splitting is invariant then it is determined by the splitting on each component, so we restrict our proof on \( M_0 \). Let \( p \in M_0 \) be given and let \( T_p M_0 = E^u(p) \oplus E^u(p) \oplus E^u(p) = \tilde{E}^u(p) \oplus \tilde{E}^u(p) \oplus E^u(p) \) up to the distribution \( p \mapsto E^u(p) \). Since \( E^u(p) \oplus E^u(p) \oplus E^u(p) = \tilde{E}^u(p) \oplus \tilde{E}^u(p) \) then it is enough to prove that \( E^u = \tilde{E}^u \). If \( \xi \in E^u(p) \), then \( \xi = \nu + \omega \), where \( \nu \in \tilde{E}^u(p) \) and \( \omega \in \tilde{E}^u(p) \). We show that the vector \( \omega \) is a null vector. By lemma 3.3 we have

\[
c^{-1} \lambda^{-n} |g_{f_p} (\omega, \omega)| \leq |g_{F_i^u} (D_p F^u_i(\omega), D_p F^u_i(\omega))| \\
= |g_{F_i^u} (D_p F^u_i(\xi - \nu), D_p F^u_i(\xi - \nu))| \\
= |g_{F_i^u} (D_p F^u_i(\xi), D_p F^u_i(\xi)) + g_{F_i^u} (D_p F^u_i(\nu), D_p F^u_i(\nu))|
\]
Since $0 < \lambda < 1$, then the above inequality tends to zero as $n$ tends to infinity. Hence the vector $\omega$ is a null vector. So $E^u(p) \subseteq E^\sigma(p)$. We can show by similar calculation $E^\sigma(p) \subseteq E^u(p)$, therefore $E^u(p) = E^\sigma(p)$ and the splitting is unique.

**Theorem 3.6**: Let $(M, F)$ be an LA-family up to a $d$ dimensional distribution $p \mapsto E^u(p)$. Let $\gamma$ be a curve with $\gamma(t_n) \in M$ such that $t_n \to 0$ and $\gamma(t_n) \to p \in M$ when $n \to \infty$, then for a subsequence $\{\zeta_n\}$ of $\{t_n\}$, which we call it again $\{t_n\}$, we have

$$E^u_{\gamma(t_n)} \to E^u(p) \quad \text{when} \quad n \to \infty,$$

and

$$E^u_{\gamma(t_n)} \to E^u(p) \quad \text{when} \quad n \to \infty.$$

**Proof**: We take a point $p = (i, m_i) \in M$ with $m_i \in M_i$ and the map $F : M \to M$ with $F(i, m_i) = (i + 1, f_i(m_i))$, $f_i(m_i) \in M_{i+1}$ and a curve $\gamma(t_n)$ on $M$ such that $t_n \to 0$ when $n \to \infty$. In fact $\gamma(t_n) = (i, \gamma_i(t_n))$ where $\gamma_i(t_n) \in M_i$. First we prove $d(B_{E^u_{\gamma(t_n)}}, B_{E^u(p)}) \to 0$ when $n \to \infty$ where $B_{E^u_{\gamma(t_n)}}$ is a basis for $E^u_{\gamma(t_n)}$ and $B_{E^u(p)}$ is a basis for $E^u(p)$. Second we show the convergence of the vectors in $E^u_{\gamma(t_n)}$ to the vectors in $E^u(p)$.

Let $m$ be the dimensional of $M$. Then $0 \leq \dim(E^u(\gamma(t_n))) \leq m$ for all $n \in \mathbb{N}$. There exist a constant $l$ and a subsequence $\{\zeta_n \in [-\varepsilon / 2, \varepsilon / 2]: n \in \mathbb{N}\}$ such that $\dim(E^u(\gamma(\zeta_n))) = l$ for all $n \in \mathbb{N}$. We take an orthonormal basis $B_{E^u(\gamma(\zeta_n))} = \{v_{11}, \ldots, v_{1l}\}$ and we translate it parallelly via linear isomorphism $P_1$ to $B_{E^u(\gamma(\zeta_n))}$ by $\{v_{n1} = P_{\zeta_n - \zeta_1}(v_{11}), v_{n2} = P_{\zeta_n - \zeta_1}(v_{12}), \ldots, v_{nl} = P_{\zeta_n - \zeta_1}(v_{1l})\}$ that is an orthonormal basis for $E^u(\gamma(\zeta_n))$. The sequence $\{v_{nj}\}$ is a convergent sequence in the tangent bundle $TM$. This limit is $v_j = \lim_{n \to \infty} v_{nj} = \lim_{n \to \infty} P_{\zeta_n - \zeta_1}(v_{1j}) = P_{\lim_{n \to \infty} \zeta_n - \zeta_1}(v_{1j})$. Obviously $v_j \in E^u(p) \oplus E^\sigma(p)$, thus $v_j = u + w$ where $u \in E^e(p)$ and $w \in E^\sigma(p)$. By using of Lemma 3.3 we prove $u = 0$. In fact we have

$$-2g_{F_{\gamma(p)}^{-1}}(D_{i}F^{-n}_{\gamma}(\xi), D_{\gamma}F^{-n}_{\gamma}(\nu))|$$

$$\leq c\lambda^n(|g_{p}(\xi, \xi)| + |g_{p}(\nu, \nu)|)$$

$$+ 2|g_{F_{\gamma(p)}^{-1}}(D_{i}F^{-n}_{\gamma}(\xi), D_{\gamma}F^{-n}_{\gamma}(\nu))|$$
\[ c^{-1} \lambda^{-n} |g_p(u,u)| \]
\[ \leq |g_{F_p^{-1}(p)}(D_p F_i^{-n}(u), D_p F_i^{-n}(u))| = |g_{F_i^{-1}(p)}(D_p F_i^{-n}(u+w-w), D_p F_i^{-n}(u+w-w))| \]
\[ \leq |g_{F_i^{-1}(p)}(D_p F_i^{-n}(v_j), D_p F_i^{-n}(v_j))| + |g_{F_i^{-1}(p)}(D_p F_i^{-n}(w), D_p F_i^{-n}(w))| + 2 |g_{F_i^{-1}(p)}(D_p F_i^{-n}(v_j), D_p F_i^{-n}(w))| \]
\[ \leq \lim_{t_i \to 0} |g_{F_i^{-1}(p)}(D_p F_i^{-n}(v_{\eta_j}), D_p F_i^{-n}(v_{\eta_j}))| + |g_{F_i^{-1}(p)}(D_p F_i^{-n}(w), D_p F_i^{-n}(w))| + 2 \lim_{t_i \to 0} |g_{F_i^{-1}(p)}(D_p F_i^{-n}(v_{\eta_j}), D_p F_i^{-n}(P_t v))| \]
\[ \leq \lim_{t_i \to 0} c \lambda^{-n} |g_{F_i^{-1}(p)}(v_{\eta_j}, v_{\eta_j})| + c \lambda^{-n} |g_{F_i^{-1}(p)}(w, w)| + 2 \lim_{t_i \to 0} |g_{F_i^{-1}(p)}(D_p F_i^{-n}(v_{\eta_j}), D_p F_i^{-n}(P_t v))|. \]

Base on Definition 3.2 \( \lim_{t_i \to 0} |g_{F_i^{-1}(p)}(D_p F_i^{-n}(v_{\eta_j}), D_p F_i^{-n}(P_t v))| = 0. \) Thus \( |g_p(u,u)| = 0. \) Hence \( u = 0 \) or \( u \) is a null vector. Since \( u \in E^s(p) \) then \( u = 0. \) Therefore the set \( \{v_1,...,v_l\} \) is an orthonormal subset of \( E^s(p). \) Thus \( \dim (E^s(p)) \geq l. \) By the same calculations \( \dim (E^s(p)) \geq m - l - d \) where the dimension of \( E^s(p) \) is \( d. \) Hence \( \dim (E^s(p)) = l \) and \( \dim (E^s(p)) = m - l - d. \) Thus with choosing sufficiently large \( n, \) \( \dim (E^s(\gamma(t_n))) = l \) and \( \dim (E^s(\gamma(t_n))) = m - l - d. \) For the second part we consider an arbitrary unit vector \( u_k = \sum_{j=1}^{l} \alpha_j v_{\eta_j} \in E^s(\gamma(t_k)) \), we have \( \sum_{j=1}^{l} \alpha_j^2 = 1. \) Since \( v_{\eta_j} \to v_j \) then \( |g_p(\sum_{j=1}^{l} \alpha_j v_{\eta_j}, \sum_{j=1}^{l} \alpha_j v_j)| = 1 \) and \( \lim_{t_k \to 0} |g_p(P_{t_k} v_{\eta_j} - v_j, P_{t_k} v_{\eta_j} - v_j)| = 0. \) Last equality implies for given \( \delta > 0 \) there is \( M > 0 \) such that for all \( k > M \) and \( j, j' \in [1,...,l] \) we have
\[ |g_p(P_{t_k} v_{\eta_j} - v_j, P_{t_k} v_{\eta_j} - v_{j'})| < \delta / \left( 1 + \sum_{j=1}^{l} \sum_{j'=1}^{l} \alpha_j \alpha_{j'} \right). \]

Therefore
Now we take an arbitrary unit vector \( v = \sum_{j=1}^{l} \alpha_j v_j \in E^u(p) \). Since \( \nu_n \to v_j \) then \( |g_{\gamma(t_n)}(\sum_{j=1}^{l} \alpha_j v_{n_j}, \sum_{j=1}^{l} \alpha_j v_j)| = 1 \). Thus for given \( \delta > 0 \) we have

\[
|g_{\gamma(t_n)}(P_{t_n}(v_j) - v_{n_j}, P_{t_n}(v_j) - v_{n_j})| < \delta \left( 1 + \frac{\sum_{j=1}^{l} \sum_{j=1}^{l} \alpha_j \alpha_j'}{1 + \sum_{j=1}^{l} \sum_{j=1}^{l} \alpha_j \alpha_j'} \right).
\]

Consequently

\[
|g_{\gamma(t_n)}(P_{t_n}(v) - \sum_{j=1}^{l} \alpha_j v_{n_j}, P_{t_n}(v) - \sum_{j=1}^{l} \alpha_j v_j)|
\]

\[
= |g_{\gamma(t_n)}(\sum_{j=1}^{l} \alpha_j P_{t_n}(v_j) - \sum_{j=1}^{l} \alpha_j v_{n_j}, \sum_{j=1}^{l} \alpha_j P_{t_n}(v_j) - \sum_{j=1}^{l} \alpha_j v_{n_j})|
\]

\[
= |\sum_{j=1}^{l} \sum_{j=1}^{l} \alpha_j \alpha_j' g_{\gamma(t_n)}(P_{t_n}(v_j) - v_{n_j}, P_{t_n}(v_j) - v_{n_j})|
\]

\[
\leq \delta \left( \sum_{j=1}^{l} \sum_{j=1}^{l} \alpha_j \alpha_j' \right) \left( 1 + \sum_{j=1}^{l} \sum_{j=1}^{l} \alpha_j \alpha_j' \right) \leq \delta \quad (**)
\]

The inequalities (*) and (**) implies \( E^u(\gamma(t_n)) \) convergence to \( E^u(p) \). The second part of Theorem can prove similarly.

4. Examples of Lorentzian non-stationary dynamical system

Let \( M_0 \) be a smooth Lorentzian manifold with a metric \( g_0 \) which has the splitting \( TM_0 = E_0^+ \oplus E_0^- \oplus E_0^u \) for each \( p \in M_0 \). We make an LA-family \( (M_i, \langle \cdot, \cdot \rangle, F) \) by taking \( M = \bigcup_{i \in \mathbb{Z}} M_i \), where \( M_i = \{ i \} \times M_0 \) and we define a Lorentzian metric on each \( M_i \) by \( g_i |_{e^i} = \alpha^{-1} g_0 |_{e^i} \), \( g_i |_{e^i} = \alpha^{-1} g_0 |_{e^i} \), and \( g_i |_{e^n} = g_0 |_{e^n} \), where \( i \neq 0 \) and \( \alpha > 1 \). In fact we contract \( g_0 \) exponentially along the subspace \( E_0^+ \) and we expand it along the subspace \( E_0^u \). If we take \( f_i : M_i \to M_{i+1} \) with the properties \( Df_i(E^+) = E^+ \), \( Df_i(E^-) = E^u \) and \( Df_i(E^u) = E^u \), then \((M_i, \langle \cdot, \cdot \rangle, F = \{ f_i \})\) is an LA-family.
Now we present another example.

**Example 4.1:** Take the solid torus $N = S^1 \times D^2$ where $D^2$ is the unit disk in $\mathbb{R}^2$. The coordinates of this manifold are of the forms $(\theta, u, v)$ such that $\theta \in S^1$ and $(u, v) \in D^2$, with $u^2 + v^2 \leq 1$. Using these coordinates the dynamical system $f : N \to N$ defined by

$$f(\theta, u, v) = \left(2\theta, \frac{1}{10}u + \frac{1}{10}\cos\theta, \frac{1}{10}v + \frac{1}{10}\sin\theta\right)$$

creates a solenoid or Smale attractor by the iterations of $f$. In fact it is the maximal invariant hyperbolic set $\Lambda := N \cap f(N) \cap f^2(N) = \bigcap_{n=0}^{\infty} f^n(N)$ [4]. If we restrict $f$ on $\Lambda$ then we have an Anosov map in the sense of Riemannian metric. We define a mapping $\tilde{f} : \Lambda \times \mathbb{R} \subset \mathbb{R}^4 \to \Lambda \times \mathbb{R} \subset \mathbb{R}^4$ by $\tilde{f}(\theta, u, v, z) = (f(\theta, u, v), z)$. On $\Lambda \times \mathbb{R}$ we take the induced metric of $\mathbb{R}^4$ defined by $g_p(X, Y) = \theta_1 \theta_2 + u_1 u_2 + v_1 v_2 - z_1 z_2$ where $p = (\theta, u, v) \in T_p \mathbb{R}^4$. If we take $E^u(p) = \{(a, a, \sqrt{5}a) : a \in \mathbb{R}\}$ then $\Lambda \times \mathbb{R}$ is a Lorentzian hyperbolic set for $\tilde{f}$ and it is a Lorentzian Anosov map. We define an LA-family $(M, F)$ by taking the components of $M$ i.e. $M_i$ as distinct copies of $\Lambda \times \mathbb{R}$ and the mapping $\tilde{f}_i : M_i \to M_{i+1}$ by $(i, x) \mapsto (i+1, \tilde{f}_i(x))$, for $i \in \mathbb{Z}$ (see figure 1).

**Example 4.2:** Let $M, g$ be a connected two dimensional $C^k$ Riemannian manifold. We take $M_i = I_i \times M$ where $I_i = (0, 2^i)$ for each $i \in \mathbb{Z}$.

By putting a Lorentzian metric $g_i(v_i \oplus w_1, v_2 \oplus w_2) = -v_i v_2 + d^i g(w_1, w_2)$ where $0 < d < 1$, $v_i \oplus w_1, v_2 \oplus w_2 \in T_{(0,1)}(I \times M)$, each $M_i$ is a warped product manifold. We make an LA-family by defining $f_i : I_i \times M \to I_{i+1} \times M$, $(x, y) \mapsto (2x, h(y))$ where $h$ is a diffeomorphism on $M$. If we take

$$E^u_{i,p} = \langle d[1] \sqrt{g}((0,1),(0,1)), 0, 1 \rangle, \quad E^u_i = \langle 1, 0, 0 \rangle \quad \text{and} \quad E^s_i = \langle 0, 1, 1 \rangle,$$

then $(M, \{f_i\})$ is an LA-family.

![Figure 1](image-url)
5. Shadowing property on Lorentzian non-stationary dynamical system

In this section we define Lorentzian shadowing property of type I and type II on \( L \)-family \((M,\langle ., . \rangle, F)\) and on the induced dynamic by it. We prove shadowing property of type I is invariant by uniformly conjugacy and if \( L \)-family \((M,\langle ., . \rangle, F)\) has shadowing property of type II then it has shadowing property of type I.

**Definition 5.1** : Consider a \( L \)-family \((M,\langle ., . \rangle, F)\) for \( \delta > 0 \), the sequence \( \{v_i\}_{i \in \mathbb{Z}} \) where \( v_i \in T_p M_i \) is said to be a \( \delta \) -pseudo orbit of type I if \( |g_{i+1}(D_f^j(v_i), v_{i+1})| < \delta \), for \( i \in \mathbb{Z} \).

For given \( \epsilon > 0 \) a \( \delta \) -pseudo orbit \( \{v_i\}_{i \in \mathbb{Z}} \) is called to be \( \epsilon \) -traced by \( w \in T_p M_i \) if \( |g_{i+n}(F^n(w), v_{i+n})| < \epsilon \) for \( n \in \mathbb{Z} \).

**Definition 5.2** : A \( L \)-family \((M,\langle ., . \rangle, F)\) is said to have Lorentzian shadowing property of type I if for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that every \( \delta \) -pseudo orbit is \( \epsilon \) -traced by some vector in \( T_p M_i \).

Consider two \( L \)-family \((M,\langle ., . \rangle, F)\) and \((\tilde{M},\tilde{g},\tilde{F})\). We say a homeomorphism \( h : M \rightarrow \tilde{M} \) is uniform continuous if for given \( \epsilon > 0 \) there exists an \( \tilde{\epsilon} > 0 \) such that \( |g(v, w)| < \epsilon \) implies \( |\tilde{g}(h(v), h(w))| < \tilde{\epsilon} \) where \( v, w \in T_p M_i \) and \( i \in \mathbb{Z} \).

**Definition 5.3** : A topological conjugacy between \( L \)-family \((M,\langle ., . \rangle, F)\) and \((\tilde{M},\tilde{g},\tilde{F})\) is a map \( h : M \rightarrow \tilde{M} \) such that for \( i \in \mathbb{Z} \), \( h_{|_{M_i}} = h_i : M_i \rightarrow \tilde{M}_i \) is a homeomorphism and \( h_{i+1} \circ f_i = \tilde{f}_i \circ h_i \).

**Definition 5.4** : A topological conjugacy \( h : M \rightarrow \tilde{M} \) is uniformly conjugate if \( h_i : M_i \rightarrow \tilde{M}_i \) and \( h_i^{-1} : \tilde{M}_i \rightarrow M_i \) are uniformly continuous.

**Theorem 5.5** : Let \( L \)-family \((M,\langle ., . \rangle, F)\) be uniformly conjugate \((\tilde{M},\tilde{g},\tilde{F})\). Then \((M,\langle ., . \rangle, F)\) has shadowing property of type I if and only if \((\tilde{M},\tilde{g},\tilde{F})\) has shadowing property of type I.

**Proof** : Suppose \( L \)-family \((M,\langle ., . \rangle, F)\) has shadowing property of type I. Given \( \epsilon > 0 \), because of uniform continuity of \( h \) there exists \( 0 < \epsilon_0 < \epsilon \) such that \( |g(u_i, v_i)| < \epsilon_0 \) implies \( |\tilde{g}(h_i(u_i), h_i(v_i))| < \epsilon \) for all \( u_i, v_i \in T_p M_i \). Since \( L \)-family \((M,\langle ., . \rangle, F)\) has shadowing property of type I there exists \( 0 < \delta_0 < \epsilon_0 \) such that for every \( \delta \) -pseudo orbit of \( L \)-family \((M,\langle ., . \rangle, F)\) is \( \epsilon_0 \) -traced by \( w \in T_p M_i \). By uniform continuous of \( h^{-1} \), there exists \( 0 < \delta < \delta_0 \) such that \( |\tilde{g}(\tilde{u}_i, \tilde{v}_i)| < \delta \) implies \( g(h^{-1}(\tilde{u}_i), h^{-1}(\tilde{v}_i)) < \delta_0 \) for any vector \( \tilde{u}_i, \tilde{v}_i \in T_p \tilde{M}_i \). Now we prove that every \( \delta \) -pseudo orbit of
L-family \((\tilde{M}, \tilde{g}, \tilde{F})\) is \(\epsilon\)-traced by some vector of \(T_p \tilde{M}\). Suppose \(\{\tilde{v}_i\}\) is a \(\delta\)-pseudo orbit of \(L\)-family \((\tilde{M}, \tilde{g}, \tilde{F})\) i.e \(|\tilde{g}_{i+1}(f_i(\tilde{v}_i), \tilde{v}_{i+1})| < \delta\) so \(|h_{i+1}^{-1}(\tilde{F}_i(\tilde{v}_i), h_{i+1}^{-1}(\tilde{v}_{i+1}))| < \delta_0\) then \(|\tilde{g}_{i+1}(f_i \circ h_{i+1}^{-1}(\tilde{v}_i), h_{i+1}^{-1}(\tilde{v}_{i+1}))| < \delta_0\). Hence \(\{h_{i+1}^{-1}(\tilde{v}_i)\}\) is a \(\delta_0\)-pseudo orbit for \(L\)-family \((M, g, F)\). Thus there exist \(w \in T_p M\) such that \(|\tilde{g}_{i+1}(F_i^n(w), h_{i+1}^{-1}(\tilde{v}_{i+1}))| < \epsilon_0\) for \(n \in \mathbb{Z}\). So by using uniform continuous of \(h\) we have \(|\tilde{g}_{i+1}(F_i^n(\tilde{v}), h_{i+1}^{-1}(\tilde{v}_{i+1}))| < \epsilon\) for \(n \in \mathbb{Z}\) where \(h_i(w) = \tilde{w} \in T_p \tilde{M}\). Hence \(\{\tilde{v}_i\}\) is \(\epsilon\)-traced by \(\tilde{w}\). The converse is proved by similar argument.

For the next theorem we define a metric on all non-empty compact subspaces of \(TM = \bigcup_{i \in \mathbb{Z}} T_i M\). Because \(T_p M_i \cong \mathbb{R}^n\) where \(\dim M_i = n\) we put usual topology on \(T_p M_i\). Again consider \(\gamma : (-\epsilon, \epsilon) \to M_i\) is a smooth curve passing through \(p\). If \(u \in T_{\gamma(t)} M_i\) and \(K\) is a non-empty compact subspace of \(T_{\gamma(t)} M_i\) then \(d(u, K) = \inf \{|g_{\gamma(t)}(P_{\gamma(t)}(u), v)| : v \in K\}\) where \(t, \zeta \in (-\epsilon, \epsilon)\). So for two non-empty compact subspaces \(A\) and \(B\) of \(T_{\gamma(t)} M_i\) and \(T_{\gamma(t)} M_i\) we define \(d(A, B) = \max \{d(u, B), d(v, A) : u \in A\) and \(v \in B\)\). Consider homeomorphism \(D_p : T_{\gamma(t)} M_i \to T_{f(p)} M_{i+1}\) for \(i \in \mathbb{Z}\), it induces a homeomorphism \(\mathcal{F} : K(M_i) \to K(M_{i+1})\) by \(\mathcal{F}(K) = D_p f_i(K)\) for every \(K \in K(M_i)\). Now we extend definitions 5.1 and 5.2 on \(K(M) = \bigcup_{i \in \mathbb{Z}} K(M_i)\).

Consider a \(L\)-family \((M, \langle \cdot, \cdot \rangle, F)\) for \(\delta > 0\), the sequence \(\{V_i\}_{i \in \mathbb{Z}}\) where \(V_i \in K(M_i)\) is said to be a \(\delta\)-pseudo orbit of type \(II\) if \(d(\mathcal{F}(V_i), V_{i+1}) < \delta\) for \(i \in \mathbb{Z}\).

A \(L\)-family \((M, \langle \cdot, \cdot \rangle, F)\) is said to have Lorentzian shadowing property of type \(II\) if for every \(\epsilon > 0\), there exists a \(\delta > 0\) such that every \(\delta\)-pseudo orbit is \(\epsilon\)-traced by some vector in \(K(M_i)\).

**Theorem 5.6**: If \(L\)-family \((M, g, F)\) has shadowing property of type \(II\) then it has shadowing property of type \(I\).

**Proof**: Suppose \(L\)-family \((M, g, F)\) has shadowing property of type \(II\). Let \(\{v_i\}_{i \in \mathbb{Z}}\) be a \(\delta\)-pseudo orbit of type \(I\) then \(\{v_i\}_{i \in \mathbb{Z}}\) is a \(\delta\)-pseudo orbit of type \(II\). For any \(\epsilon > 0\) a \(\delta\)-pseudo orbit \(\{V_i = [v_i]\}_{i \in \mathbb{Z}}\) is \(\epsilon\)-traced by \(W \in K(M_i)\).
Then \(d(F_i^n(W), V_{i+n}) < \varepsilon\) for \(n \in \mathbb{Z}\). We can write \(|g_{i+n}(F_i^n(w), v_{i+n})| < \varepsilon\) where \(w \in W\) and \(v_{i+n} \in V_{i+n}\) for \(n \in \mathbb{Z}\). So \(L\)-family \((M, g, F)\) has shadowing property of type \(I\).

6. Conclusion

We extend the notion of Anosov family on a sequence of compact Riemannian manifolds \(\{M_i\}_{i \in \mathbb{Z}}\) to Lorentzian Anosov family by using of a distributions \(p \mapsto E^a(p)\). We see that when each \(M_i\) is a Lorentzian manifold then the tangent space of \(M_i\) at \(p \in M_i\) has a unique splitting to stable, unstable and a subset of null vectors which is also a vector space. We consider the behavior of this splitting by using of the distance function created by the unique torsion-free connection.

Lorentzian shadowing property of type \(I\) and type \(II\) on \(L\)-family \((M, \langle \cdot, \cdot \rangle, F)\) and on the induced dynamic by it has been introduced. We prove shadowing property of type \(I\) is invariant by uniformly conjugacy. Also we prove if \(L\)-family \((M, g, F)\) has shadowing property of type \(II\) then the induced dynamic of it has shadowing property of type \(I\).

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