We give two formulations of exclusion statistics (ES) using a variable number of bosonic or fermionic single-particle states which depend on the number of particles in the system. Associated bosonic and fermionic ES parameters are introduced and are discussed for FQHE quasiparticles, anyons in the lowest Landau level and for the Calogero-Sutherland model. In the latter case, only one family of solutions is emphasized to be sufficient to recover ES; appropriate families are specified for a number of formulations of the Calogero-Sutherland model. We extend the picture of variable number of single-particle states to generalized ideal gases with statistical interaction between particles of different momenta. Integral equations are derived which determine the momentum distribution for single-particle states and distribution of particles over the single-particle states in the thermal equilibrium.

PACS Numbers: 05.30.-d, 05.70.Ce, 74.20.Kk, 73.40.Hm

The idea of reduction of single-particle Hilbert space upon adding particles into the system introduced by Haldane into definition of statistics for identical particles has recently received great attention. This idea was originally proposed to give an alternative definition of fractional statistics based on a generalized exclusion principle [exclusion statistics (ES)], with a specific law of reduction of the Hilbert space determined by an ES parameter. Associated formula for the dimension of the many-particle Hilbert space was suggested which was then applied to FQHE quasiparticles to evaluate numerically and analytically their ES parameters in simple cases.

The latter formula applied locally in the phase space was also used as a starting point for formulation of the statistical mechanics of anyons (see also Ref. 7). The statistical distribution for ES turned out to coincide with that previously derived for fractional statistics in one dimension [the latter was introduced in the algebraic (Heisenberg) approach to quantization of identical particles and was argued to be modeled by systems with an inverse square interaction (the Calogero-Sutherland model)]. On related studies of ES in terms of the Calogero-Sutherland models, see Refs. 12–15.

Interpretation of anyons confined to the lowest Landau level (LLL) in terms of ES statistics was discovered in various ways (see also Ref. 7). This is consistent with the interpretation of anyons in the LLL, where their dynamics becomes effectively one-dimensional, in terms of fractional statistics in one dimension in the algebraic approach.

The idea of reduction of the space of single-particle states was used to define statistical interaction between particles of different momenta. The latter notion was also discussed in the language of generalized single-particle statistical distributions for free identical particles.

Haldane has defined ES by the condition that upon adding ΔN particles into the system (we restrict ourselves to the case of one species of particles), the number of single-particle states D available for further particles is altered by

\[ \Delta D = -g \Delta N, \]

where g is the ES parameter (with g = 0 and g = 1 for bosons and fermions). He also suggested the formula for the dimension of the space of many-particle states:

\[ W = \frac{(D + N - 1)!}{N!(D - 1)!}. \]

If one denotes by G the number of single-particle states available for the first particle and uses the expression

\[ D = G - g(N - 1), \]

consistent with (1), then (2) interpolates between the numbers of ways of placing N bosons (g = 0) and N fermions (g = 1) over G single-particle states.

We stress that in order to derive the dimension of the space of many-particle states starting from the dimension of the space of single-particle states, it is necessary to know “accommodation” properties of single-particle states. In the usual interpretation of Eqs. (2)–(3) as counting ways of placing N particles into G states, Eq. (2) (which in fact determines implicitly “accommodation” properties of single-particle states) should be viewed as independent of (1).

In this paper we give an alternative interpretation of Eqs. (1), (3), in terms of usual bosonic single-particle states. In this case, (3) is a consequence of (1). We also give a dual description of ES which uses fermionic single-particle states. The two formulations of the ES are somewhat similar to the boson-based and fermion-based descriptions of anyons. We extend these formulations to involve local statistical interaction in the phase space as well as statistical interaction between particles of different energies. Based on this, we reexamine systems of anyons in the LLL, the Calogero-Sutherland systems and interacting systems of spinless particle solvable by the thermodynamic Bethe ansatz (TBA), refining and extending previous results on thermodynamic

Bosonic and Fermionic Single-Particle States in the Haldane Approach to Statistics for Identical Particles

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(Received 12 May 1995)
equivalence of the above systems to systems of free identical particles.

**Bosonic and fermionic single-particle states.** Consider a system that may be described in terms of bosonic (fermionic) single-particle states and assume that the total number of single-particle states depends on the number of particle in the system: \( D^{b,f} = D^{b,f}(N) \). Single-particle states may thus appear or disappear on adding particles into the system. The formula similar to (4) then defines variation the number of single-particle states upon adding \( \Delta N \) particles into the system:

\[
\Delta D^{b,f}(N) = -g^{b,f} \Delta N ,
\]

where we have introduced the *bosonic* and *fermionic* ES parameters \( g^{b} \) and \( g^{f} \). In place of (4), we have

\[
D^{b,f} = G - g^{b,f}(N - 1) ,
\]

where \( G \) is the number of single-particle states in the absence of particles. The dimension of the space of many-particle states follows from (4) to be

\[
W^{b} = \frac{(D^{b} + N - 1)!}{N!(D^{b} - 1)!} , \quad W^{f} = \frac{D^{f}!}{N!(D^{f} - N)!} .
\]

Note that \( W^{b} \) and \( W^{f} \) with \( D^{b} \) and \( D^{f} \) from (4) coincide, resulting in the same statistical properties, if

\[
g^{b} = 1 + g^{f} .
\]

We see that Haldane’s original ES parameter corresponds to the *bosonic* ES parameter in our notation.

As an example, we consider ES parameters for FQHE quasiparticles. The ES parameters for the quasiholes (qh) and quasielectrons (qe) above the Laughlin states with the filling factors \( \nu = \frac{1}{m} \) with \( m \) odd were found by estimations of the dimension of many-particle space for quasiparticles in the spherical geometry

\[
\begin{align*}
\rho &= 1 - \frac{1}{m} \quad \text{for quasihole}, \\
\rho &= 1 - \frac{1}{m} \quad \text{for quasielectron},
\end{align*}
\]

which corresponds to the bosonic ES parameters. The results of Refs. 2–4 then read \( g_{\text{qh}}^{b} = \frac{1}{m} \) and \( g_{\text{qe}}^{f} = 2 - \frac{1}{m} \). According to (7), we may as well say that the fermionic ES parameters for the qh and qe are

\[
g_{\text{qe}}^{f} = -g_{\text{qh}}^{f} = 1 - 1/m ,
\]

possessing the symmetry \( g_{\text{qe}}^{f} = -g_{\text{qh}}^{f} \) which was originally suggested for the bosonic ES parameters.

We thus have the two descriptions of ES, in terms of bosonic and fermionic single-particle states, respectively. This is reminiscent of the two descriptions of anyons in two spatial dimensions: anyons with the same exchange statistics parameter \( \theta \) with \( 0 \leq \theta < 2\pi \) can be described either as bosons carrying the magnetic flux (in units of quantum flux \( \phi_{0} = 2\pi/e, e > 0 \)) \( \phi^{b} = \theta/\pi \) with \( 0 \leq \phi^{b} < 2 \) or as fermions carrying the magnetic flux \( \phi^{f} \) with \(-1 \leq \phi^{f} < 1 \), where the two descriptions are connected by

\[
\phi^{b} = 1 + \phi^{f} .
\]

This relation is quite similar to (7). We show below that for anyons confined to the LLL where the notion of ES applies, this similarity becomes the precise correspondence: the bosonic and fermionic ES parameters for anyons in the LLL exactly coincide with \( \phi^{b} \) and \( \phi^{f} \).

**Particles of the same energy and anyons in the LLL.** Let all the particles have the same energy \( \varepsilon \). It follows from (7) that the entropy \( S = \ln W \) is

\[
S^{b,f} = D^{b,f} \left( \pm (1 \pm n^{b,f}) \ln(1 \pm n^{b,f}) - n^{b,f} \ln n^{b,f} \right) ,
\]

(the upper and lower signs refer to the bosonic-state and fermionic-state descriptions, respectively), where

\[
n^{b,f} \equiv N/D^{b,f} = n/(1 - g^{b,f} n) ,
\]

and \( n = N/G \). We maximize (9) with respect to \( n^{b,f} \) subject to the constraints of fixed total number of particles \( N = D^{b,f} n^{b,f} \) and the total energy \( E = D^{b,f} \varepsilon n^{b,f} \), with the associated Lagrange multipliers \( \beta \mu \) and \(-\beta \) (\( \beta = 1/T, \mu \) is the chemical potential). With (10), we obtain the equation for \( n^{b,f} \) in the thermal equilibrium:

\[
n^{b,f} (1 \pm n^{b,f})^{\pm g^{b,f} - 1} = x \, ,
\]

where \( x = e^{\beta(\mu - \varepsilon)} \).

The equation of state can be derived from (9) and (11) using the relations \( \Omega = E - TS - \mu N \) and \( \Omega = -PA \) (specified for two spatial dimensions with \( A \) the area occupied by the gas) to be

\[
P\beta = \pm(G/A) \ln(1 \pm n^{b,f}) .
\]

Let us compare this equation of state with that obtained for anyons in a strong magnetic field [1] where only particles in the LLL (all having the same energy) contribute into the equation of state. Regarding anyons as bosons carrying the magnetic flux of value \( \phi^{b} \) (\( 0 \leq \phi^{b} < 2 \)) in the direction antiparallel to the external magnetic field \( B \), the equation of state was found to be

\[
P\beta = \rho L \ln[1 + \nu/(1 - \phi^{b} \nu)] ,
\]

where \( \rho L = eB/2\pi \) is the density of states in the LLL, \( \nu = N/\rho L A \) is the filling factor. We may as well rewrite this equation of state regarding anyons as fermions carrying magnetic flux \( \phi^{f} \) (\(-1 \leq \phi^{f} < 1 \)) connected with \( \phi^{b} \) by (4). The two equations of state coincide with (12), if one identifies the density of states \( G/A \) in (12) with \( \rho L \) (and hence, \( n \) with \( \nu \)), and, in addition, makes the identifications

\[
g^{b,f} = \phi^{b,f} .
\]

Let us comment on formula (4) in this context. Consider the mean-field approximation corresponding to the change of a flux carried by a particle with a uniform magnetic field of the same flux. Adding a particle into the system then diminishes the total magnetic flux by \( \phi^{b,f} ,\)
hence the number of states in the LLL, equal to the total magnetic flux of in units of the flux quantum, is

\[ D^{b,f} = eBA/2\pi - \phi^{b,f} N. \]  

(14)

This corresponds to (3) with \( g^{b,f} \) from (13). But we stress that (14) holds only in the mean-field approximation.

**Statistical interaction between particles of the same momentum.** — Let now particles in different single-particle states may have different energies. For definiteness, we assume that single-particle states may be labeled with the momenta which particles have in these states. We apply the construction of the previous section to particles of the same momentum \( k \) (supplying all the extensive quantities of the previous section with the subscript \( k \), which turns all the quantities into densities per unit momentum). We assume that upon adding particles of momentum \( k \) into the system, there may appear or disappear single-particle states of only the same momentum:

\[ D^{b,f}_k = G_k - g^{b,f}_k N_k. \]  

(15)

The entropy for the whole system is

\[ S^{b,f} = \sum_k G_k \ln W^{b,f}_k, \]  

(16)

where \( W^{b,f}_k \) are obtained from (3) by the changes \( N \to N_k \) and \( D^{b,f} \to D^{b,f}_k \). Again, the statistical properties given by the bosonic and fermionic pictures are identical if the condition (7) is fulfilled.

For \( n^{b,f}_k \equiv N_k / D^{b,f}_k \) in equilibrium, in place of (11), we have

\[ n^{b,f}_k (1 \pm n^{b,f}_k \pm g^{b,f}_k - 1 = x_k, \]  

(17)

where \( x_k = e^{\beta(\mu-e_k)} \), and \( e_k \) is the energy of a particle with momentum \( k \). The substitution \( n^{b,f}_k = n_k/(1 - g^{b,f}_k n_k) \), yields the equation for \( n_k \equiv N_k / G_k \):

\[ n_k (1 - g^{b,f}_k n_k)^{g^{b,f}_k - 1}(1 - g^{b,f}_k n_k \pm n_k)^{g^{b,f}_k - 1} = x_k. \]  

(18)

For the bosonic ES parameter, (18) recovers the equation that was derived in the formulation of ES where (3) was viewed as counting the ways of placing \( N \) particles over \( G \) states. That formulation implied an invariable number of single-particle states, and the function \( n_k \) played the part of the distribution of particles over states \( k \) (‘statistical distribution’). Unlike that, in the present formulation of ES, with a variable number of single-particle states, the distribution of particles over states is given by the functions \( n^{b,f}_k \).

Eq. (18) for the bosonic ES parameter was discussed in the context of the Calogero-Sutherland model as a system which reveals ES (11) and (14). Here we extend that analysis with specific emphasis on the fact that only one family of solutions of this model is sufficient to recover ES.

We discuss the Calogero-Sutherland model in a harmonic well governed by the Hamiltonian

\[ H = \frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i<j} \frac{\lambda(\lambda - 1)}{(x_i - x_j)^2} + \frac{1}{2} \omega^2 \sum_i x_i^2. \]  

(19)

Consider the coordinate domain \( x_1 \leq \cdots \leq x_N \). Due to the symmetry of the Hamiltonian (19) under the change \( \lambda \to 1 - \lambda \), the system (19) has two classes of solutions of the form

\[ \psi^b(\lambda) = \Delta^{\lambda} \Phi_+ (\{x_i\}, \lambda), \psi^f(\lambda) = \Delta^{1-\lambda} \Phi_+ (\{x_i\}, 1 - \lambda), \]  

(20)

where \( \Delta = \prod_{i>j} (x_i - x_j), \{x_i\} \) is the set with \( i = 1, \ldots, N \), and \( \Phi_+ (\{x_i\}, \lambda) = P(\{x_i\}, \lambda) \exp(-\frac{1}{2} \sum_i x_i^2) \), where \( P(\{x_i\}, \lambda) \) is a polynomial symmetric in the particle coordinates \( \{x_i\} \) with a non-zero free term. The solutions \( \psi^b(\lambda) \) and \( \psi^f(\lambda) \) are quadratically integrable and non-singular for \( \lambda \geq 0 \) and for \( \lambda \leq 1 \), respectively.

We may rewrite the solutions of class II as \( \psi^f(\lambda) = \Delta^{-\lambda} \Phi_- (\{x_i\}, -\lambda) \), where \( \Phi_- (\{x_i\}, -\lambda) = \Delta \Phi_+ (\{x_i\}, 1 - \lambda) \) is an antiasymmetric function of the coordinates. We can then extend continuously the solutions \( \psi^f \) for \( \lambda \geq 0 \) and \( \psi^f \) for \( \lambda \leq 1 \) to the regions \( xQ_1 \leq \cdots \leq xQ_N \), where \( Q \) is a permutation of \( 1, \ldots, N \), in the symmetric and antsymmetric way, respectively. Making, in addition, the changes \( \lambda \to \lambda_B \) in \( \psi^b \) and \( -\lambda \to -\lambda_F \) in \( \psi^f \), we arrive at symmetric (‘boson’) and antisymmetric (‘fermion’) wave functions \( \psi^B \) and \( \psi^F \) of the form

\[ \psi^{B,F} = |\Delta^{\lambda_B,F} \Phi_+ (\{x_i\}, \lambda_B,F) , \]  

(21)

with \( \lambda_B \geq 0 \) and \( \lambda_F \geq -1 \). These function are solutions of the boson and fermion Hamiltonians \( H^B \) and \( H^F \), which are obtained from the Hamiltonian (13) by the changes of the two-body potential \( V(x) \) with the potentials

\[ V^{B,F}(x) = \lambda_B,F (\lambda_B,F + 1)/x^2, \]  

(22)

where \( \lambda_B \geq 0 \) and \( \lambda_F \geq -1 \).

The energy levels for the solutions \( \psi^b \) and \( \psi^f \), or, equivalently, the solutions \( \psi^B \) and \( \psi^F \), are given by \( E^{B,F} = E_0^{B,F} + \sum_{n_1} \lambda \bar{n}_i \), where \( \{n_i\} \) is a set of non-negative integers obeying \( 0 \leq n_1 \leq \cdots \leq n_N \), and \( E_0^B = \omega[\frac{1}{2} N + \frac{1}{2} \lambda_B(N-1)] \) and \( E_0^F = \omega[\frac{1}{2} N + \frac{1}{2}(1 + \lambda_F)N(N-1)] \) are the ground state energies (see also Ref. 23). The corresponding partition functions are

\[ Z_N^{B,F} = e^{-\beta E_0^{B,F}} \prod_{i=1}^N (1 - q^{-1}). \]  

(23)

The family of solutions \( \psi^B \) (21) for the boson Calogero-Sutherland model (or, equivalently, the family of solutions of class II for (19)) was analyzed in Ref. 8 in the limit \( \omega \to 0 \) in terms of generalized single-state partition functions and statistical distributions in the approach with nonvariable number of single-particle states, resulting in the statistical distributions (18) with \( g^b = \lambda_B \).
Repeating the analysis of Ref. 8 for the family of solutions \( \psi^F \) for the fermion Hamiltonian (or, equivalently, for the solutions of class II in (23)), with the partition functions \( Z_N^F \) (23), we arrive at the statistical distributions \( n_k \) coinciding with those given by (13) with \( g^f = \lambda^F \).

We note that the analysis given in Ref. 13 implies using the family of the fermion solutions \( \psi^F \) of (21) with the parameter \( g \) used in Ref. 13 equal to our \( \lambda^F + 1 \). Refs. 12 and 15 use the family \( \psi^B \) (21).

**Statistical interaction between particles of different momenta.** — We now give a generalization of the considerations of the previous section assuming that on adding particles of momentum \( k \) into the system, there may appear or disappear single-particle states of any other momenta (cf. Eq. (15)):

\[
D^{b,f}_k = G_k - \sum_{k'} G_{k'} g^{b,f}(k,k') n_{k'},
\]

where \( g^{b,f}(k,k') \) may be called the bosonic and fermionic statistical interaction functions.

The entropy (16) is written as

\[
S^{b,f} = \sum_k G_k (\pm (d^{b,f}_k \pm n_k) \ln(d^{b,f}_k \pm n_k) \\
+ d^{b,f}_k \ln d^{b,f}_k - n_k \ln n_k),
\]

where

\[
d^{b,f}_k = D_k^{b,f}/G_k = 1 - \sum_{k'} G_{k'} g^{b,f}(k,k') n_{k'}. \tag{26}
\]

It follows from (23)-(26) that the bosonic and fermionic pictures result in the same statistical properties if

\[
g^{b,f}(k,k') = \frac{2m}{L} \delta(k-k') + g^f(k,k') \tag{27}
\]

generalizing (1). Maximization of the entropy with respect to \( n_k \), subject to the total particle number and total energy constraints, yields the equation for \( n_k \) in equilibrium

\[
\ln \frac{n_k}{x_k} = \sum_{k'} G_{k'} \left\{ \pm g^{b,f}(k',k) \ln(d^{b,f}_{k'} \pm n_{k'}) \\
+ \frac{2m}{L} \delta(k'-k) \mp g^{b,f}(k',k) \ln(d^{b,f}_{k'} \pm n_{k'}) \right\}, \tag{28}
\]

with \( d^{b,f}_k \) of (26). Eq. (28) can be rewritten as an equation for the distribution of particles over single-particle states \( n^{b,f}_k \):

\[
\ln \frac{n^{b,f}_k}{x_k} = \sum_{k'} G_{k'} \left[ \frac{2m}{L} \delta(k'-k) \mp g^{b,f}(k',k) \right] \ln(1 \pm n^{b,f}_{k'}). \tag{29}
\]

Eq. (29) together with the equation for the distribution of single-particle states \( d^{b,f}_k \),

\[
d^{b,f}_k = 1 - \sum_{k'} G_{k'} g^{b,f}(k,k') n^{b,f}_{k'} d^{b,f}_{k'}, \tag{30}
\]

which follows from (26) and the equality \( n_k = d^{b,f}_k n^{b,f}_k \), enable one in principle to determine all the thermodynamic quantities of the ideal gas (e. g. \( E = \sum_k G_k n^{b,f}_k / \epsilon_k n^{b,f}_k \)). For \( g^{b,f}(k,k') \propto \delta(k-k') \), Eqs. (28) and (29) reduce to Eqs. (15) and (17), respectively.

The fermionic state description is convenient to study low-temperature properties since \( n_k \) has a non-singular simple form for \( T = 0 \). Indeed, for \( T = 0 \), all the states with \( |k| \leq k_0 \), where \( k_0 \) is some boundary momentum (the ‘Fermi momentum’), are occupied, the others are empty, that is, \( n^f_k = 1 \) for \( |k| \leq k_0 \), and \( n^f_k = 0 \) for \( |k| > k_0 \). From Eq. (29) we then get for \( T = 0 \) the integral equation for the distribution of the fermionic states

\[
d^f_k = 1 - \sum_{|k'| \leq k_0} G_{k'} g^f(k,k') d^f_{k'}, \tag{31}
\]

as well as the equation for \( n_k \) (\( k_0 \leq k \leq k_0 \))

\[
n_k = 1 - \sum_{|k'| \leq k_0} G_{k'} n^{f}(k,k') n_{k'} \tag{32}
\]

For sufficiently low temperatures, \( n_k \) will slightly differ from \( n_k \) at \( T = 0 \).

Refs. 19 and 14 argued that the TBA equations for spinless particles may be viewed as encoding statistical interaction between particles of different momenta. We now discuss relation of the above single-particle state definition of statistics to state counting in the TBA.

In the description of the TBA we follow Refs. 25 and 26. Let the wave function have the Bethe ansatz form

\[
\Psi(x_1, \ldots, x_N) = \sum_P A(P) \exp(\frac{i}{\hbar} \sum_{j=1}^N k_P j x_j) \tag{33}
\]

in the asymptotic region \( x_1 \ll \cdots \ll x_N \), where \( k_1 > \cdots > k_N \), and \( P \) is a permutation of \( 1, 2, \ldots N \). The coefficients \( A(P) \equiv A(k_{P1}, \ldots, k_{PN}) \) are related by \( A(\ldots k', k, \ldots)/A(\ldots k, k', \ldots) = S(k-k') \), where \( S(k-k') = e^{-i\theta(k-k')} \) is the two-body ‘scattering matrix’, and \( \theta(k-k') \) is the two-body ‘phase shift’, antisymmetric in \( k-k' \) with \( k \) and \( k' \) the particle momenta. Imposing periodic boundary conditions on the wave function (33) yields the Bethe ansatz equations \( e^{ik_j L} \prod_{j \neq j} S(k_j - k_l) = 1 \) for allowed momentum values. Note that the wave function (33), which vanishes if \( j = l \) for any two momenta \( S(0) = -1 \), implies description of the system in terms of fermionic states.

In the thermodynamic limit the Bethe ansatz equations results in the following picture. The density of levels \( r^f \) (being the sum of the particle and hole densities, \( r^f = \rho + \rho_h \)) over which the particles are distributed is determined by the integral relation

\[
\frac{\pi}{L} \delta(k-k') \mp g^{b,f}(k',k) \ln(d^{b,f}_{k'} \pm n_{k'}) \right\}, \tag{28}
\]

\[
\ln \frac{n^{b,f}_k}{x_k} = \sum_{k'} G_{k'} \left[ \frac{2m}{L} \delta(k'-k) \mp g^{b,f}(k',k) \right] \ln(1 \pm n^{b,f}_{k'}). \tag{29}
\]
\[
2\pi r^f(k) = 1 - \int \varphi(k - k')\rho(k')L\,dk',
\]
with \(\varphi(k) = \frac{1}{2}\partial^2 \theta(k)/\partial k\) the derivative of the phase shift, thus depending on the particle density. Fermionic nature of the states is also displayed in the expression for the entropy:

\[
S^f = \int [-r^f - \rho \ln(r^f - \rho) + r^f \ln r^f - \rho \ln \rho ]L\,dk.
\]

We see that the TBA picture exactly corresponds to the above definition of statistics in terms of fermionic states: Eq. (34) and the entropy coincide with (25) and (23) after the identifications \(n_k \leftrightarrow 2\pi\rho, d_k^f \leftrightarrow 2\pi r^f_k\), and

\[
g^f(k, k') = \varphi(k - k').
\]

Ref. 14 used the same fermionic state counting in the TBA as above but the bosonic statistical interaction function. With this remark, the identification (13) agrees with that obtained in Ref. 14 if one takes into account the relation (27). But we stress that interpretation of the formula of reduction of the space of single-particle states (24) in the TBA scheme of Refs. 25 and 26 necessarily implies that single-particle states are to be specified as fermionic. On the other hand, it is natural to expect that in an alternative, bosonic state counting in the TBA, which is discussed in Ref. 28, the reduction formula (24) with the bosonic single-particle states should be used.

As the first example, we consider the system of bosons with the two-body \(\delta\)-function potential \(V(x) = c\delta(x)\). That this system models a possible statistics for identical particles in one dimension was pointed out long ago. The two-particle phase shift reads

\[
\theta(k) = -2\tan^{-1}(k/c).
\]
Eq. (55) then yields \(g^f(k, k') = -2c/L(k_x^2 + (k - k')^2)\). In the limits \(c \to 0\) and \(c \to \infty\) we have \(g^f(k, k') = \frac{2c}{L}\delta(k - k')\) and \(g^f(k, k') = 0\), recovering the thermodynamics for free bosons and free fermions as it should be. Note that after the identifications \(n_k \leftrightarrow 2\pi\rho\) and (22), Eq. (22) coincides with the \(T = 0\) equation for the momentum distribution for bosons with the \(\delta\)-function interaction.

Consider now the Calogero-Sutherland system (19) (without the harmonic potential). The Schrödinger equation for the relative problem of two particles \((-\partial_x^2 + \lambda(\lambda - 1)/x^2)\psi = k^2\psi\), where \(x = x_2 - x_1\) and \(k = \frac{1}{2}(k_1 - k_2)\) are the relative coordinate and momentum, reduces to a Bessel equation so that any solution of the Schrödinger equation can be represented as a linear combination of the two solutions, \(\sqrt{x}J_{\lambda-1/2}(kx)\) and \(\sqrt{x}J_{1/2-\lambda}(kx)\), if \(\lambda - \frac{1}{2}\) is a noninteger number (for \(\lambda - \frac{1}{2}\) integer, the two above solutions are linear dependent, and another fundamental system of solutions, \(\sqrt{x}Y_{\lambda-1/2}(kx)\) and \(\sqrt{x}Y_{1/2-\lambda}(kx)\), involving Bessel functions of the first and second kinds, should be used). These two solutions, for \(\lambda \geq 0\) and \(\lambda \leq 1\), respectively, correspond to the solutions of classes I and II for the Calogero-Sutherland model discussed in the previous section.

Consider the solution \(\sqrt{x}J_{\lambda-1/2}(kx)\) for \(\lambda \geq 0\). For \(kx \gg 1\), it has the asymptotics \(\pi^{-1/2}\cos(kx - \frac{1}{2}\pi\lambda)\). Comparing this with the asymptotic wave function (32) for two particles, \(\Psi \propto e^{-ikx} + S(k)e^{ikx}\), we get \(S(k) = e^{-i\pi\lambda}k\) for \(k > 0\). In order to determine the phase shift \(\theta(k) = i\ln(-S(k))\) for all the values of \(\lambda\) simultaneously, we consider the \(\ln\) on its Riemann surface glued of the complex planes \(D_k(k = 0, \pm 1, \pm 2, \ldots)\) which are cut along their positive semiaxes and are characterized by \(2k\pi < \arg z < 2(k + 1)\pi\); in the \(D_0\), the branch of the logarithm \(\ln z = \ln|z| + i\arg z\) with \(0 < \arg z < 2\pi\) is chosen. Then we obtain \(\theta(k) = \pi(\lambda - 1)\) for \(k > 0\), and accounting for the antisymmetry of the phase shift, finally, \(\theta(k) = \pi(\lambda - 1)\ln(k)\). This recovers the result obtained by Sutherland\(26\) our choice of the unique solution of the scattering problem thus corresponds to that of Ref. 26.

Eq. (33) then yields \(g^f(k, k') = \frac{2\pi}{T}(\lambda_B - 1)\delta(k - k')\). According to (27), the same statistics may be described by the bosonic statistical interaction function \(g^f(k, k') = \frac{2\pi}{T}\lambda_B\delta(k - k')\). Similar considerations show that the fermion Calogero-Sutherland system is equivalent to a free system with \(g^f(k, k') = \frac{2\pi}{T}\lambda_F\delta(k - k')\). These results agree with those obtained in the previous section for the Calogero-Sutherland system in a harmonic well.

In conclusion, we have proposed a formulation of the Haldane approach to statistics for spinless identical particles considering a variable number of bosonic (fermionic) single-particle states which depends on the number of particles in the system. The variation of the number of single-particle states is governed by the bosonic (fermionic) statistical interaction function which reduces to a single parameter for exclusion statistics [bosonic (fermionic) ES parameter]. Thermodynamic quantities for the ideal gas are determined by the momentum distributions for single-particle states together with distributions of particles over single-particle states. The equations for these two distributions in the most generic case are Eqs. (21)–(23). The fermionic state picture seems to be the most convenient for studying the low-temperature properties.

The picture of variable number of single-particle state exactly corresponds to the description of states in the TBA for spinless particles. We also note that the proposed formulation of ES allows one to avoid encountering negative probabilities which arise in the formulation posed formulation of ES allows one to avoid encounter-

I thank J. M. Leinaas, J. Myhrum, and S. Ouvry for
helpful discussions. I also acknowledge NORDITA and Department of Physics of University of Oslo for their hospitality while this work was in progress. The work was supported in part by the Russian Foundation for Fundamental Research, grant No. 95-02-04337.

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