Coadjoint orbits of the odd real symplectic group

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Abstract

We give a representative of every coadjoint orbit of the odd symplectic group. Our argument follows that used for the Poincaré group but the details differ.

Key words: odd symplectic group, coadjoint orbit, cotype, modulus.

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We introduce the notion of cotype and reduce the classification of coadjoint orbits of the odd real symplectic group to the classification of types for the Lie algebra of the real symplectic group, see [2, p.352-3]. For a classification of the adjoint orbits of the odd symplectic group see [3]. We follow the line of argument in [4] for classifying the coadjoint orbits of the Poincaré group, but there are some new features.

1 The odd symplectic group

We begin by defining some basic concepts. Let \((V, \omega)\) be a real symplectic vector space. A real linear map \(P : V \rightarrow V\), which preserves the symplectic form \(\omega\), that is, \(P^* \omega = \omega\), is called a linear symplectic map. The set of all real linear symplectic maps is the Lie group \(\text{Sp}(V, \omega)\). Its Lie algebra \(\text{sp}(V, \omega)\) is the set of all real linear maps \(Y : V \rightarrow V\) such that \(\omega(Yv, w) + \omega(v, Yw) = 0\) for every \(v, w \in V\). Let \(v^0\) be a nonzero vector in \(V\). Let \(\text{Sp}(V, \omega)_{v^0}\) be the Lie group of all real linear symplectic maps of \((V, \omega)\) into itself, which leave \(v^0\) fixed. \(\text{Sp}(V, \omega)_{v^0}\) is called the odd symplectic group, see [1]. Its Lie algebra \(\text{sp}(V, \omega)_{v^0}\) consists of all \(Y \in \text{sp}(V, \omega)\) such that \(Yv^0 = 0\). Let \(Y = S + N\) be the Jordan decomposition of \(Y\) into a sum of a semisimple linear map \(S\) and a commuting nilpotent linear map \(N\), which both lie in \(\text{sp}(V, \omega)\). Because \(S\) and \(N\) are polynomials in \(Y\) with real coefficients and no constant terms. Since \(Yv^0 = 0\), it follows that \(Sv^0 = 0 = Nv^0\). So \(S\) and \(N\) lie in \(\text{sp}(V, \omega)_{v^0}\).

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2 Classification of coadjoint orbits

A tuple \((V, Y, v; \omega)\) is a real symplectic vector space \((V, \omega)\), a vector \(v \in V\), and a real linear map \(Y \in \text{sp}(V, \omega)\). Two tuples \((V, Y, v; \omega)\) and \((V', Y', v'; \omega')\) are equivalent if there is a bijective real linear map \(P : V \to V'\) such that i) \(Pv = v'\), ii) \(P^* \omega' = \omega\), and iii) there is a vector \(w \in V\) so that \(Y' = P(Y + L_v,w)P^{-1}\). Here \(L_{v,w} = v \circ w^* + w \circ v^*\), where \(v^*\) is the linear function on \(V\) which sends \(u\) to \(\omega(v, u)\). Note that \(L_{v,w} = L_{w,v}\) for every \(v, w \in V\).

**Fact 1.** For every \(v, w \in V\), we have \(L_{v,w} \in \text{sp}(V, \omega)\).

**Proof.** For \(x, y \in V\)

\[
\omega((v \circ w^*)x + (w \circ v^*)x, y) + \omega(x, (v \circ w^*)y + (w \circ v^*)y) = \\
= \omega(\omega(w, x)v + \omega(v, x)w, y) + \omega(x, \omega(w, y)v + \omega(v, y)w) \\
= \omega(w, x)\omega(v, y) + \omega(v, x)\omega(w, y) + \omega(x, w)\omega(v, y) + \omega(x, v)\omega(w, y) \\
= 0, \quad \text{since } \omega \text{ is skew symmetric.}
\]

Therefore \(L_{v,w} \in \text{sp}(V, \omega)\). \(\square\)

**Fact 2.** If \(P \in \text{Sp}(V, \omega)\), then \(PL_{v,w}P^{-1} = L_{Pv,Pw}\).

**Proof.** For \(z \in V\) we have

\[
(PL_{v,w}P^{-1})z = PL_{v,w}(P^{-1}z) = P((v \circ w^*)P^{-1}z + (w \circ v^*)P^{-1}z) \\
= w^*(P^{-1}z)Pv + v^*(P^{-1}z)Pw \\
= \omega(w, P^{-1}z)Pv + \omega(v, P^{-1}z)Pw \\
= \omega(Pw, z)Pv + \omega(Pv, z)Pw, \quad \text{since } P \in \text{Sp}(V, \omega) \\
= (Pv \circ (Pw)^*)z + (Pw \circ (Pv)^*)z = L_{Pv,Pw}(z). \quad \square
\]

Being equivalent is an equivalence relation on the collection of all tuples. An equivalence class of tuples is called a cotype, which is denoted by \(\nabla\). Suppose that the tuple \((V, Y, v; \omega)\) represents the cotype \(\nabla\). If \(v \neq 0\), then \(\nabla\) is an affine cotype. Define the dimension of \(\nabla\) to be \(\text{dim } V\). The height of the cotype \(\nabla\) is the largest positive integer \(n\) such that \(N^{n+1}V = \{0\}\), where \(N\) is the nilpotent summand in the Jordan decomposition of \(Y\).

**Lemma 3.** Every affine cotype \(\nabla\) of dimension \(2n + 2\) has a representative tuple \((\mathbb{R}^{2n+2}, Y', e_{2n+1}; J)\).

**Proof.** Let \((V, Y, v; \omega)\) be a tuple which represents the cotype \(\nabla\). Let \(\mathcal{f} = \{v_0, v_1, \ldots, v_{2n+1} = v\}\) be a basis of \(V\) with respect to which the matrix
If $\omega$ is $J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -I_n & 0 \\ -1 & 0 & 0 \end{pmatrix}$, where $\bar{J} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Let $P : V \to \mathbb{R}^{2n+2}$ be the bijective real linear map which sends the basis vector $v_i$ of $V$ to the standard basis vector $e_i$ of $\mathbb{R}^{2n+2}$ for $i = 0, \ldots, 2n+1$. Then $P^*J = \omega$ and $Pv = e_{2n+1}$. The matrix $Y'$ of $Y$ with respect to the standard basis $\epsilon = \{e_0, e_1, \ldots, e_{2n}, e_{2n+1}\}$ of $\mathbb{R}^{2n+2}$ is

$$Y' = PYP^{-1} = P(Y + L_{0,v})P^{-1}.$$}

Thus the tuple $(\mathbb{R}^{2n+2}, Y', e_{2n+1}; J)$ is equivalent to the tuple $(V, Y, v; \omega)$. □

**Proposition 4.** For $Y \in \text{sp}(\mathbb{R}^{2n+2}, J)$ let $\ell_Y$ be the linear function on $\text{sp}(\mathbb{R}^{2n+2}, J)$ which sends $Z$ to $\text{tr}(YZ)$. The map

$$(\mathbb{R}^{2n+2}, Y, e_{2n+1}; J) \mapsto \ell_Y|\text{sp}(\mathbb{R}^{2n+2}, J)_{e_{2n+1}}$$

induces a bijection between affine cotypes and coadjoint orbits of the odd real symplectic group $\text{Sp}(\mathbb{R}^{2n+2}, J)_{e_{2n+1}}$ on $\text{sp}(\mathbb{R}^{2n+2}, J)_{e_{2n+1}}^\epsilon$, the dual of its Lie algebra.

**Proof.** Suppose that the tuples $(\mathbb{R}^{2n+2}, Y, e_{2n+1}; J)$ and $(\mathbb{R}^{2n+2}, Y', e_{2n+1}; J)$ are equivalent. Then there is a real linear map $P \in \text{Sp}(\mathbb{R}^{2n+2}, J)_{e_{2n+1}}$ and a vector $w \in \mathbb{R}^{2n+2}$ such that $Y' = P(Y + L_w, e_{2n+1})P^{-1}$. We begin our argument with three observations.

**Observation 1.** Let $w = w_0e_0 + \bar{w} + w_{2n+1}e_{2n+1} \in \mathbb{R}^{2n+2}$. Then the matrix of $L_w, e_{2n+1}$ with respect to the standard basis $\epsilon$ of $\mathbb{R}^{2n+2}$ is $\begin{pmatrix} w_0 & 0 & 0 \\ \bar{w} & 0 & 0 \\ 2w_{2n+1} & \bar{w} & -w_0 \end{pmatrix}$.

To see this we compute

$$L_{w,e_{2n+1}}(e_0) = (w \otimes e_{2n+1})^*e_0 + e_{2n+1} \otimes w^*e_0 = (e_0^TJe_{2n+1} + e_0^TJw)e_{2n+1} = w + w_{2n+1}e_{2n+1} = w_0e_0 + \bar{w} + 2w_{2n+1}e_{2n+1},$$

$$L_{w,e_{2n+1}}(e_i) = (e_i^TJe_{2n+1} + e_i^TJw)e_{2n+1} = \bar{w}^*e_i^*e_{2n+1},$$

$$L_{w,e_{2n+1}}(e_{2n+1}) = (e_{2n+1}^TJe_{2n+1} + e_{2n+1}^TJw)e_{2n+1} = -w_0e_{2n+1}.$$
**Observation 2.** For \( P \in \text{Sp}(\mathbb{R}^{2n+2}, J) \) and \( Y \in \text{sp}(\mathbb{R}^{2n+2}, J) \) we have \( \ell_{PY^{-1}} = \text{Ad}_{P^{-1}}^T \ell_Y \). To see this, we compute. Let \( Z \in \text{sp}(\mathbb{R}^{2n+2}, J) \). Then

\[
\ell_{PY^{-1}}(Z) = \text{tr}((PY^{-1})Z) = \text{tr}(PY^{-1}Z) = \text{tr}(YP^{-1}Z) = \ell_Y(\text{tr}Z).
\]

\( \ell_Y(\text{tr}Z) = \ell_Y(\text{Ad}_{P^{-1}}Z) = (\text{Ad}_{P^{-1}}^T \ell_Y)Z. \)

**Observation 3.** Let \( \text{sp}(\mathbb{R}^{2n+2}, J)_{\ell_{2n+1}}^0 \) be the set of all \( X \in \text{sp}(\mathbb{R}^{2n+2}, J) \) such that \( \ell_X(Y) = 0 \) for every \( Y \in \text{sp}(\mathbb{R}^{2n+2}, J)_{\ell_{2n+1}} \). Then

\[
\text{sp}(\mathbb{R}^{2n+2}, J)_{\ell_{2n+1}}^0 = \{ \ell_{L_v,\ell_{2n+1}} \in \text{sp}(\mathbb{R}^{2n+2}, J)^0 \mid v \in \mathbb{R}^{2n+2} \}.
\]

**Proof of proposition 4.** Suppose that the tuples \( (\mathbb{R}^{2n+2}, Y, e_{2n+1}; J) \) and \( (\mathbb{R}^{2n+2}, Y', e_{2n+1}; J) \) are equivalent. Then there is \( P \in \text{Sp}(\mathbb{R}^{2n+2}, J)_{\ell_{2n+1}} \) and a vector \( w \in \mathbb{R}^{2n+2} \) such that \( Y' = P(Y + L_w,e_{2n+1})P^{-1} \). For every \( Z \in \text{sp}(\mathbb{R}^{2n+2}, J)_{\ell_{2n+1}} \) we have
\(\ell_p(Y + L_w, e_{2n+1}) P^{-1}(Z) = \ell_p Y P^{-1}(Z) + \ell_p L_w, e_{2n+1} P^{-1}(Z)\)

\(= \ell_p Y P^{-1}(Z) + \ell_p L_w, e_{2n+1} P^{-1}(Z), \) since \(P \in \text{Sp}(\mathbb{R}^{2n+2}, J)_{e_{2n+1}}\)

\(= \ell_p Y P^{-1}(Z), \) since \(Z \in \text{sp}(\mathbb{R}^{2n+2}, J)_{e_{2n+1}}\)

\((\text{Ad}_{P^{-1}} \ell_y) Z = (\text{Ad}_{P^{-1}} (\ell_y | \text{sp}(\mathbb{R}^{2n+2}, J)_{e_{2n+1}})) Z.\)

Thus the affine cotype, represented by \((\mathbb{R}^{2n+2}, Y, e_{2n+1}; J)\), corresponds to the unique coadjoint orbit of \(\text{Sp}(\mathbb{R}^{2n+2}, J)_{e_{2n+1}}\) through \(\ell_Y | \text{sp}(\mathbb{R}^{2n+2}, J)_{e_{2n+1}}\) in \(\text{sp}(\mathbb{R}^{2n+2}, J)^*_{e_{2n+1}}.\)

Suppose that for some \(Y, Y' \in \text{sp}(\mathbb{R}^{2n+2}, J)\) and some \(P \in \text{Sp}(\mathbb{R}^{2n+2}, J)_{e_{2n+1}}\) we have \(\ell_{Y - \text{Ad}_{P^{-1}} (Y')} = 0\) on \(\text{sp}(\mathbb{R}^{2n+2}, J)_{e_{2n+1}}.\) In other words, suppose that \(\ell_Y | \text{sp}(\mathbb{R}^{2n+2}, J)_{e_{2n+1}}\) lies in the \(\text{Sp}(\mathbb{R}^{2n+2}, J)_{e_{2n+1}}\) coadjoint orbit through \(\ell_Y | \text{sp}(\mathbb{R}^{2n+2}, J)_{e_{2n+1}}.\) Then \(\ell_{Y - P(Y') P^{-1}} \in \text{sp}(\mathbb{R}^{2n+2}, J)_{e_{2n+1}}.\) Therefore for some \(v \in \mathbb{R}^{2n+2}\) we have \(\ell_{Y - P(Y') P^{-1}} = \ell_{L_v, e_{2n+1}}.\) So

\[Y = P(Y') P^{-1} + L_v, e_{2n+1} = P(Y' + L_{P^{-1} v, e_{2n+1}}) P^{-1}.\]

Hence the tuples \((\mathbb{R}^{2n+2}, Y, e_{2n+1}; J)\) and \((\mathbb{R}^{2n+2}, Y', e_{2n+1}; J)\) are equivalent. Thus a coadjoint orbit uniquely determines an affine cotype. Since every element of \(\text{sp}(\mathbb{R}^{2n+2}, J)_{e_{2n+1}}^*\) is of the form \(\ell_Y | \text{sp}(\mathbb{R}^{2n+2}, J)_{e_{2n+1}}\) for some \(Y \in \text{sp}(\mathbb{R}^{2n+2}, J)\), the map from the affine cotype, represented by the tuple \((\mathbb{R}^{2n+2}, Y, e_{2n+1}; J)\), to an \(\text{Sp}(\mathbb{R}^{2n+2}, J)_{e_{2n+1}}\) coadjoint orbit through \(\ell_Y | \text{sp}(\mathbb{R}^{2n+2}, J)_{e_{2n+1}}\) is surjective. This proves proposition 4. \(\square\)

Let \((V, Y, v; \omega)\) be a tuple which represents the cotype \(\nabla.\) If \(V = V_1 \oplus V_2,\) where \(v \in V_1, V_2 \neq \{0\},\) and \(V_i\) are \(Y\)-invariant, \(\omega\)-perpendicular, \(\omega\)-non-degenerate subspaces of \((V, \omega),\) then \(\nabla\) is the sum of a cotype \(\nabla,\) represented by the tuple \((V_1, Y|V_1, v; \omega|V_1),\) and a type \(\Delta,\) represented by the pair \((V_2, Y|V_2; \omega|V_2),\) see [2]. We write \(\nabla = \nabla + \Delta.\) We say that the cotype \(\nabla\) is indecomposable if it cannot be written as the sum of a cotype and a type. If \(V_1 = \{0\},\) then \(v = 0\) and the tuple \((\{0\}, 0, 0; 0)\) represents the zero cotype, which we denote by \(0.\) Note that \(\text{dim } 0 = 0.\)

**Proposition 5.** An indecomposable affine cotype \(\nabla\) is represented by the tuple \((V, Y, v^0; \omega),\) where \(Y\) is nilpotent.

**Proof.** Let \(S\) be the semisimple summand in the Jordan decomposition of \(Y.\) The following argument shows that \(v^0 \in \ker S.\) Since \(\text{V} = \ker S \oplus \text{im } S,\)
we may write \( v^0 = v_1 + v_2 \in \ker S \oplus \im S \). So \( 0 = Y v^0 = Y v_1 + Y v_2 \in \ker S \oplus \im S \). Thus \( Y v_2 = 0 \). Since \( Y \) is invertible on \( \im S \), it follows that \( v_2 = 0 \). Thus \( v^0 = v_1 \in \ker S \). Let \( N \) be the nilpotent summand of the Jordan decomposition of \( Y \). By results of [2] the type \( \Delta' \), represented by the tuple \((V' = \ker S, N = Y|\ker S; \omega|\ker S)\) is a sum of indecomposable types, which is unique up to reordering of the summands. Since \( v^0 \in \ker N \), there is exactly one indecomposable summand \( \Delta'' \), represented by the tuple \((V'', N|V''; \omega|V'')\) where \( v^0 \in V'' \). So \( V'' \) is a \( Y \) invariant, \( \omega \) nondegenerate subspace of \((V', \omega|V')\), and hence of \((V, \omega)\), which contains \( v^0 \). On \( V'' \) the linear map \( Y \) is nilpotent. Thus \((V'', Y|V'', v^0; \omega|V'')\) is a tuple, which represents a nilpotent affine cotype \( \nabla'' \). Since the affine cotype \( \nabla \) is indecomposable, it follows that the tuples \((V, Y, v^0; \omega)\) and \((V'', Y''|V'', v^0; \omega|V'')\) are equal. Thus the affine cotype \( \nabla \) is nilpotent.

\[\square\]

### 3 Classification of affine cotypes

In this section we classify indecomposable affine cotypes.

For \( w = w_0 e_0 + \tilde{w} + w_{2n+1} e_{2n+1} \in \mathbb{R}^{2n+2} \) the matrix of \( L_{w, e_{2n+1}} \) with respect to the standard basis \( \varepsilon \) is

\[
\begin{pmatrix}
\tilde{w}_0 & 0 & 0 \\
\tilde{w}_1 & 0 & 0 \\
\vdots & \vdots & \vdots \\
\tilde{w}_{2n+1} & 0 & 0
\end{pmatrix}.
\]

With respect to the basis \( \varepsilon \) the matrix of \( Y \in \sp(\mathbb{R}^{2n+2}, J) \) is

\[
\begin{pmatrix}
a & -\tilde{v}^* & c \\
\tilde{d} & \tilde{Y} & \tilde{v} \\
2f & \tilde{d}^* & -a
\end{pmatrix},
\]

where \( \begin{pmatrix} a & c \\ 2f & -a \end{pmatrix} \in \sp(\mathbb{R}^2, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \), using the basis \( \{e_0, e_{2n+1}\} \) of \( \mathbb{R}^2 \). Also \( \tilde{d}, \tilde{v} \in \mathbb{R}^{2n} \). If \( (\mathbb{R}^{2n+2}, Y, e_{2n+1}; J) \) is a tuple with the matrix of \( Y \) given by (4), then we call the \((0, 2n+1)\)th entry \( c \) of \( Y \), the \textit{parameter of the tuple}.

**Lemma 6.** Let \( (\mathbb{R}^{2n+2}, Y, e_{2n+1}; J) \) be a tuple with parameter \( c \), which represents the affine cotype \( \nabla \). Then \( c \) does not depend on the choice of representative of \( \nabla \).

**Proof.** Let \( (\mathbb{R}^{2n+2}, Y', e_{2n+1}; J) \) be another representative of the cotype \( \nabla \). Then the tuples \( (\mathbb{R}^{2n+2}, Y, e_{2n+1}; J) \) and \( (\mathbb{R}^{2n+2}, Y', e_{2n+1}; J) \) are equivalent. In other words, there is a \( P \in \Sp(\mathbb{R}^{2n+2}, J) \) such that \( Y' = P(Y + L_{w, e_{2n+1}})P^{-1} \).

\[\text{(5)}\]
We need only calculate the right hand side of equation (5). Observe that

$$\text{Sp}(\mathbb{R}^{2n+2}, J)_{e_{2n+1}} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ \bar{u} & \bar{P} & 0 \\ k & \bar{u}^*P & 1 \end{pmatrix} \right| k \in \mathbb{R}, \bar{u} \in \mathbb{R}^{2n}, \bar{P} \in \text{Sp}(\mathbb{R}^{2n}, J) \right\}. $$

Then

$$Y' = \begin{pmatrix} 1 & 0 & 0 \\ \bar{u} & \bar{P} & 0 \\ k & \bar{u}^*P & 1 \end{pmatrix} \begin{pmatrix} a + w_0 & -\bar{y} & c \\ d + \bar{w} & (d + \bar{w})^* & -(a + w_0) \\ Y & \bar{y} & \bar{y}^* \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\bar{P}^{-1}\bar{u} & 0 & 0 \\ -k & -\bar{u}^* & 1 \end{pmatrix} \quad (6)$$

$$= \begin{pmatrix} * & c \\ * & 0 \\ * & 1 \end{pmatrix} \begin{pmatrix} * & c \\ * & 0 \\ * & 1 \end{pmatrix} = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}. $$

Thus the tuple \((\mathbb{R}^{2n+2}, Y', e_{2n+1}; J)\) has parameter \(c\) as well. \(\square\)

From the conclusion of lemma 6 it makes sense to call \(c\) the parameter of the affine cotype \(\nabla\), represented by the tuple \((\mathbb{R}^{2n+2}, Y, e_{2n+1}; J)\).

Calculating the right hand side of equation (6) more explicitly gives

$$Y' = \begin{pmatrix} a' & -(c\bar{u} + \bar{P}\bar{u})^* & c \\ \tilde{d}' & \tilde{P}\tilde{y}\tilde{P}^{-1} - L_{\tilde{u}, \tilde{P}^*} - c\bar{u}^* & \bar{u}^* + \bar{P}\bar{u} \\ 2f' & (\tilde{d}')^* & -a' \end{pmatrix} \quad (7)$$

for some \(a', f' \in \mathbb{R}, \tilde{d}' \in \mathbb{R}^{2n}\), which depend on \(k, \bar{u} \in \mathbb{R}^{2n}\) and \(\bar{P} \in \text{Sp}(\mathbb{R}^{2n}, J)\).

Suppose that the tuple \((\mathbb{R}^{2n+2}, Y, e_{2n+1}; J)\) has parameter \(c\), which is nonzero. Set \(k = 0, \bar{P} = I_{2n}, \) and \(\bar{u} = -\frac{1}{c}\bar{v}\). This determines matrix of \(P \in \text{Sp}(\mathbb{R}^{2n+2}, J)_{e_{2n+1}}\) in equation (6). Then equation (7) reads

$$Y'' = \begin{pmatrix} a'' & 0 & c \\ \tilde{d}'' & \tilde{y} + \frac{1}{c}L_{\bar{e}, \bar{v}} & 0 \\ 2f'' & (\tilde{d}'')^* & -a'' \end{pmatrix} = Y'' + L_{w'', e_{2n+1}}. $$

Here \(Y'' = \begin{pmatrix} 0 & 0 & c \\ 0 & \bar{v} & 0 \\ 0 & 0 & 0 \end{pmatrix}\) with \(\tilde{Y}' = \tilde{Y} + \frac{1}{c}L_{\bar{e}, \bar{v}}\), and \(w'' = a''e_0 + \bar{d}'' + f''e_{2n+2}\). Therefore the tuple \((\mathbb{R}^{2n+2}, Y'', e_{2n+1}; J)\) is equivalent to the tuple \((\mathbb{R}^{2n+2}, Y'', e_{2n+1}; J)\). Thus we have proved

**Proposition 7.** If the affine cotype \(\nabla\), represented by the tuple \((\mathbb{R}^{2n+2}, Y', e_{2n+1}; J)\), has a nonzero parameter \(c\), then it is decomposable into a sum of a two dimensional indecomposable nilpotent affine cotype \(\nabla_1(0), c\), represented by the tuple \((\mathbb{R}^2, \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}, e_{2n+1}; \begin{pmatrix} 0 & 0 \\ a_{-1} & 0 \end{pmatrix})\) with basis \(\{e_0, e_{2n+1}\}\) of \(\mathbb{R}^2\), and a type \(\Delta\), represented by the pair \((\mathbb{R}^{2n}, Y'; J)\).
We can sharpen the conclusion of proposition 7 a bit. By proposition 4 of [2] the height of the type \( \Delta \) in proposition 7 above is strictly less than the height of \( \Delta_1, c \), which is 1. Thus the height of \( \Delta \) is 0, that is, \( \Delta \) is a semisimple type.

Suppose that the affine cotype \( \nabla \), represented by the tuple \( (\mathbb{R}^{2n+2}, Y, e_{2n+1}; J) \), where \( Y \) is given in equation (4), has parameter equal to 0. The tuple \( (\mathbb{R}^2, \bar{Y}, \bar{v}, \bar{J}) \) represents the little cotype \( \nabla_\ell \) associated to \( \nabla \).

**Lemma 8.** The little cotype \( \nabla_\ell \) is uniquely determined by the affine cotype \( \nabla \) with parameter 0.

**Proof.** Let \( (\mathbb{R}^{2n+2}, Y', e_{2n+1}; J) \) be another representation of the affine cotype \( \nabla \) with parameter 0. Here

\[
Y' = \begin{pmatrix} a' & \bar{d} & -(\bar{P}\bar{v})^* & 0 \\
\bar{d} & \bar{P}Y\bar{P}^{-1} - L_{\bar{u}, \bar{v}} & \bar{P}\bar{v} \\
2f' & (\bar{d})^* & -a' \end{pmatrix},
\]

for some \( \bar{u} \in \mathbb{R}^n \). Equation (8) is obtained from (7) by setting \( c = 0 \). The tuple \( (\mathbb{R}^2, \bar{Y}', \bar{P}\bar{v}; \bar{J}) \), where \( Y' = \bar{P}(\bar{Y} - L_{\bar{P}^{-1}\bar{u}, \bar{v}})\bar{P}^{-1} \), is equivalent to the tuple \( (\mathbb{R}^2, \bar{Y}, \bar{v}; \bar{J}) \). Hence it also represents \( \nabla_\ell \).

**Lemma 9.** Let \( \nabla \) be an affine cotype with parameter equal to 0. Then \( \nabla \) is uniquely determined by its little cotype \( \nabla_\ell \).

**Proof.** Suppose that the affine cotypes \( \nabla \) and \( \nabla' \) with parameter zero, represented by the tuples \( (\mathbb{R}^{2n+2}, Y, e_{2n+1}; J) \) and \( (\mathbb{R}^{2n+2}, Y', e_{2n+1}; J) \), respectively, both have the same little cotype \( \nabla_\ell \), represented by \( (\mathbb{R}^2, \bar{Y}, \bar{v}; \bar{J}) \). Say that

\[
Y = \begin{pmatrix} w_0 & -\bar{v}^* & 0 \\
\bar{w} & \bar{Y} & \bar{v} \\
2w_{2n+1} & \bar{w}^* & -w_0 \end{pmatrix} \quad \text{and} \quad Y' = \begin{pmatrix} w_0' & -(\bar{v}')^* & 0 \\
\bar{w}' & \bar{Y}' & \bar{v}' \\
2w_{2n+1}' & (\bar{w}')^* & -w_0' \end{pmatrix},
\]

where \( w = w_0e_0 + \bar{w} + w_{2n+1}e_{2n+1} \) and \( w' = w_0'e_0 + \bar{w}' + w_{2n+1}'e_{2n+1} \) lie in \( \mathbb{R}^n \) and \( \bar{Y}, \bar{Y}' \) lie in \( \text{sp}(\mathbb{R}^{2n}, \bar{J}) \). By hypothesis the tuples \( (\mathbb{R}^{2n}, \bar{Y}, \bar{v}; \bar{J}) \) and \( (\mathbb{R}^{2n}, \bar{Y}', \bar{v}; \bar{J}) \) are equivalent. In other words, there is \( \bar{P} \in \text{Sp}(\mathbb{R}^{2n}, \bar{J}) \) such that \( \bar{P}\bar{v} = \bar{v}' \) and a vector \( \bar{w} \in \mathbb{R}^n \) such that \( \bar{Y}' = \bar{P}(\bar{Y} + L_{\bar{w}, \bar{v}})\bar{P}^{-1} \). Let

\[
P = \begin{pmatrix} 1 & 0 & 0 \\
0 & \bar{P} & 0 \\
0 & 0 & \bar{P}^{-1} \end{pmatrix}
\]

Then \( P \in \text{Sp}(\mathbb{R}^{2n+2}, J)_{e_{2n+1}} \). With \( c = 0, k = 0, \) and \( \bar{u} = \bar{y} \) equation (7) reads

\[
PYP^{-1} = \begin{pmatrix} \bar{a}^T & -(\bar{P}\bar{v})^* & 0 \\
\bar{d}^T & \bar{P}Y\bar{P}^{-1} - L_{\bar{u}, \bar{v}} & \bar{P}\bar{v} \\
2\bar{f}^T & (\bar{d})^* & -\bar{a}^T \end{pmatrix}
\]

(9)
Setting \( \tilde{y} = -\tilde{P}\tilde{w}, Pw = \tilde{a}^\dagger e_0 + \tilde{d}^\dagger + 2\tilde{f}^\dagger e_{2n+1}, \) and using the fact that \( \tilde{Y}' = P(Y + L_{\tilde{w},\tilde{v}})\tilde{P}^{-1} \) by hypothesis, equation (9) reads

\[
PYP^{-1} = \begin{pmatrix}
0 & -\tilde{v}'^* & 0 \\
0 & \tilde{Y}' & \tilde{v}' \\
0 & 0 & 0
\end{pmatrix} + PL_{w, e_{2n+1}}P^{-1}
\]

\[
= \begin{pmatrix}
w_0' & -\tilde{v}'^* & 0 \\
\tilde{w}' & \tilde{Y}' & \tilde{v}' \\
2w_{2n+1}' & -(\tilde{w}')^* & -w_0'
\end{pmatrix} - L_{w', e_{2n+1}} + L_{Pw, e_{2n+1}}
\]

\[
= Y' - P(L_{P^{-1}w', e_{2n+1}})P^{-1}.
\]

Therefore the tuples \((\mathbb{R}^{2n+2}, Y, e_{2n+1}; J)\) and \((\mathbb{R}^{2n+2}, Y', e_{2n+1}; J)\) are equivalent. So the affine cotypes \(\nabla\) and \(\nabla'\) with parameter 0 are equal. \(\square\)

**Lemma 10.** If \(\nabla\) is not an affine cotype, then \(\nabla\) is the sum of the zero cotype 0 and a type.

**Proof.** Suppose that \(\nabla\) is represented by the tuple \((V, Y, v; \omega)\). Since \(\nabla\) is not affine, \(v = 0\). Because \(\{0\}\) and \(V\) are \(Y\)-invariant, \(\omega\)-perpendicular, \(\omega\)-symplectic subspaces of \((V, \omega)\) with \(V = \{0\} \oplus V\) and \(0 \in \{0\}\), we can write the tuple \((V, Y, 0; \omega)\) as the sum of the tuple \((\{0\}, 0, 0; 0)\) and the pair \((V, Y; \omega)\). Therefore \(\nabla\) is the sum of the zero cotype and a type, represented by the pair \((V, Y; \omega)\). \(\square\)

**Proposition 11.** Every affine cotype is either the sum of a nonzero nilpotent indecomposable affine cotype and a sum of indecomposable types or the sum of the zero cotype and a sum of indecomposable types. This decomposition is unique up to reordering of the summands which are types.

**Proof.** Let \(\nabla\) be an affine cotype. Then by lemma 6 it has a parameter, say \(c\). If \(c \neq 0\), then by proposition 7 the cotype \(\nabla\) is the sum of the two dimensional nilpotent indecomposable affine cotype \(\nabla_1(0)\) and a semisimple type \(\Delta\), which may be decomposed into a sum of indecomposable semisimple types that are unique up to reordering of the summands, see \([2\) table 3, p.349]. The argument stops here. If \(c = 0\), then by lemma 8 the affine cotype \(\nabla\) has a unique little cotype \(\nabla_\ell\), which uniquely determines \(\nabla\). This correspondence respects decomposition, that is, if \(\nabla\) is decomposable, then so is \(\nabla_\ell\) and conversely. If \(\nabla_\ell\) is not affine, then by lemma 10 it is the sum of the zero cotype and a type \(\Delta\), which may be written as a sum of indecomposable types. The argument stops here. Otherwise \(\nabla_\ell\) is affine and we may repeat the above argument. Only a finite number of repetitions are needed before the argument comes to a stop, because \(\dim \nabla_\ell < \dim \nabla < \infty\). \(\square\)
Proposition 12. Let $\nabla$ be a nilpotent indecomposable affine cotype of dimension $2r + 2$ with $r \leq n$. Then exactly one of the following possibilities holds.

1. Suppose that $r = 0$ and the parameter $d$ is *not* equal to 0. Then $(\mathbb{R}^2, Y, e_1; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$ is a representative of $\nabla$. Moreover, there is a basis $e' = \{e_0, e_1\}$ of $\mathbb{R}^2$ such that the matrix of $Y$ with respect to the basis $e'$ is $\begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix}$. Note $Y^2 = 0$. So $Y$ has height 1. The matrix of $\omega$ with respect to the basis $e'$ is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We denote this nilpotent indecomposable affine cotype by $\nabla_{1}(0, d)$. Here the $d$ is nonzero and is called a *modulus*.

2. Suppose that $r = 0$ and the parameter $d$ is equal to 0. Then $(\mathbb{R}^2, Y, e_1; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$ is a representative of $\nabla$. Moreover, there is a basis $e' = \{e_0, e_1\}$ of $\mathbb{R}^2$ such that the matrix of $Y$ with respect to the basis $e'$ is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. The matrix of $\omega$ with respect to the basis $e'$ is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We denote this nilpotent indecomposable affine cotype by $\Delta_0(0, 0)$. There is no modulus and the parameter of $\Delta_0(0, 0)$ is 0.

3. $\nabla$ is obtained by taking $r$-iterated little cotypes which ends at the cotype $\nabla_{1}(0)$, $d$ with $d \neq 0$. Let $(\mathbb{R}^{2r+2}, Y, e_{2r+2}; J)$ be a representative of $\nabla$. Then there is a vector $z \in \mathbb{R}^{2r+2}$ and a basis $f = \{Y^{r+1}z; (-1)^rY^{r+1}z, (-1)^{r-1}Y^{r+2}z, \ldots, -Y^{2r}z; (-1)^r dY^r z, (-1)^r dY^{r-1}z, \ldots, (-1)^r dz\}$ of $\mathbb{R}^{2r+2}$ such that $Y^{2r+2} = 0$. The matrix of $Y$ with respect to the basis $f$ is

$$
\begin{pmatrix}
0 & -e_r^T & 0 & 0 \\
0 & -N^T & D & 0 \\
0 & 0 & -N & e_r \\
0 & 0 & 0 & 0
\end{pmatrix},
$$

where $N$ is the $r \times r$ upper Jordan block

$$
\begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

and $D$ is the $r \times r$ antidiagonal matrix $\begin{pmatrix} 0 \\ \vdots \\ d \end{pmatrix}$. $Y$ is nilpotent of height $2r + 1$ and $\mathbb{R}^{2r+2}$ is spanned by one Jordan chain of length $2r + 2$. The
matrix of $\omega$ with respect to the basis $\mathcal{f}$ is

$$J = \begin{pmatrix}
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & 0 & -I_r & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix},$$

where $I_r$ is the $r \times r$ diagonal matrix diag$(1,1,\ldots,1)$. We denote this nilpotent indecomposable affine cotype by $\nabla_{2r+1}(0), d$. Here $d$ is a nonzero modulus and the parameter of $\nabla_{2r+1}(0)$ is zero.

4. $\nabla$ is obtained by taking $r$-iterated little cotype which ends at the cotype $\nabla_0(0,0)$. Let $(\mathbb{R}^{2r+2}, Y, e_{2r+2}; J)$ be a representative of $\nabla$. Then there are vectors $z, w \in \mathbb{R}^{2r+2}$ and a basis

$$\mathcal{f}' = \{(-1)^r Y^r z; z, -Y z, \ldots, (-1)^{r-1} Y^{r-1} z, Y^r w, Y^{r-1} w, \ldots; w\}$$

of $\mathbb{R}^{2r+2}$ such that $Y^{r+1} z = Y^{r+1} w = 0$. The matrix of $Y$ with respect to the basis $\mathcal{f}'$ is

$$J = \begin{pmatrix}
0 & -e_r^T & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -N^T \\
0 & 0 & -N & e_{2r} \\
\end{pmatrix},$$

where $N$ is the $r \times r$ upper Jordan block. $Y$ is nilpotent of height $r$ and $\mathbb{R}^{2r+2}$ is spanned by two Jordan chains of length $r + 1$. The matrix of $\omega$ with respect to the basis $\mathcal{f}'$ is $J$. Denote this nilpotent indecomposable affine cotype by $\nabla_r(0,0)$. There is no modulus and the parameter of $\nabla_r(0,0)$ is 0.

**Proof.**
1. and 2. The cotypes $\nabla_1(0), d$ and $\Delta_0(0,0)$ are clearly indecomposable.

Suppose that $r \geq 1$. Then according to the proof of proposition 11, the indecomposable affine cotype $\nabla$ is obtained by taking the $r$-iterated little cotype which ends in either a) the cotype $\nabla_1(0), d$ with $d \neq 0$ or b) the cotype $\Delta_0(0,0)$.

3. Alternative a) holds. Using the basis $\tilde{\mathcal{f}}$ given by

$$\{Y^{2r+1} z, -Y^{2r} z, Y^{2r-1} z, \ldots, (-1)^r Y^{r+1} z, (-1)^r d Y^r z, (-1)^r d Y^{r-1} z, \ldots, (-1)^r dz\},$$
instead of the basis $\mathfrak{f}$, we see that the matrix of $Y$ and $\omega$ are respectively
\[
\begin{pmatrix}
0 & -1 & -1 \\
0 & 0 & 0 \\
\vdots & -1 & \ddots \\
0 & 0 & 0 \\
-1 & 0 & 0 \\
1 & -1 & 0 \\
\end{pmatrix}
\quad\text{and}\quad
\begin{pmatrix}
0 & 1 & 1 \\
0 & -1 & 1 \\
\vdots & -1 & \ddots \\
0 & 0 & 0 \\
1 & -1 & 0 \\
-1 & 1 & 0 \\
\end{pmatrix}.
\]

From this it is clear that the little cotype of the affine cotype $\nabla_{2r+1}(0), d$ is the affine cotype $\nabla_{2r-1}(0), d$. Because $\nabla_1(0), d$ is an indecomposable cotype, it follows that $\nabla_{2r+1}(0), d$ is also. Note that $Y$ is nilpotent of height $2r + 1$ and that $\mathbb{R}^{2r+2}$ is spanned by one Jordan chain of length $2r + 2$.

4. Alternative b) holds. Using the basis $\tilde{\mathfrak{f}}'$ given by
\[
\{Y^r z, -Y^{r-1} z, Y^{r-2} z, \ldots, (-1)^r z, (-1)^r Y^r w, (-1)^r Y^{r-1} w, \ldots, (-1)^r w\},
\]
instead of the basis $\mathfrak{f}'$, we see that the matrix of $Y$ and $\omega$ are respectively
\[
\begin{pmatrix}
0 & -1 & -1 \\
0 & 0 & 0 \\
\vdots & -1 & \ddots \\
0 & 0 & 0 \\
-1 & 0 & 0 \\
1 & -1 & 0 \\
\end{pmatrix}
\quad\text{and}\quad
\begin{pmatrix}
0 & 1 & 1 \\
0 & -1 & 1 \\
\vdots & -1 & \ddots \\
0 & 0 & 0 \\
1 & -1 & 0 \\
-1 & 1 & 0 \\
\end{pmatrix}.
\]

From this it is clear that the little cotype of the affine cotype $\nabla_r(0, 0)$ is the affine cotype $\nabla_{r-1}(0, 0)$. Because $\nabla_0(0, 0)$ is an indecomposable cotype, it follows that $\nabla_r(0, 0)$ is also. Note that $Y$ is nilpotent of height $r$ and that $\mathbb{R}^{2r+2}$ is spanned by two Jordan chains both of length $r + 1$. □

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