TANGENT BUNDLES WITH COMPLETE LIFT OF THE BASE METRIC AND ALMOST HYPERCOMPLEX HERMITIAN-NORDEN STRUCTURE

MANCHO MANEV

Dedicated to the 75th anniversary of Prof. Kostadin GRIBACHEV

Abstract
The tangent bundle of an almost Norden manifold and the complete lift of the Norden metric is considered as a $4n$-manifold. It is equipped with an almost hypercomplex Hermitian-Norden structure. It is characterized geometrically. The case when the base manifold is an h-sphere is considered.

Key words: tangent bundle, complete lift, almost hypercomplex structure, Hermitian metric, Norden metric.

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1. Introduction
The investigating of the tangent bundle $TM$ of a manifold $M$ help us to study the manifold $M$. Moreover, $TM$ has own structure closely related to the structure of $M$, which implies mutually related geometric properties.

The geometry of the almost hypercomplex manifolds with Hermitian metric is known (e.g. [1]). A parallel direction including indefinite metrics is the developing of the geometry of the almost hypercomplex manifolds with Hermitian-Norden metric structure. It has a natural origination from the geometry of the $n$-dimensional quaternionic Euclidean space.

The beginning was put by our joint works with K. Gribachev and S. Dimiev in [2] and [3]. More precisely we have combined the Hermitian metric with the Norden metric with respect to the almost complex structures of a hypercomplex structure.

The aim of the present work is consideration of an almost Norden manifold as a base manifold and generation of its tangent bundle with a metric, which is a prolongation of the base metric by its complete lift. In that way we get the tangent bundle with almost hypercomplex Hermitian-Norden structure and characterize it.

2. Differentiable manifolds with almost complex structures

2.1. Almost complex manifolds with Hermitian metric or Norden metric.
The notion of the almost complex manifold $(M^{2n}, J)$ is well-known. There exists a possibility it to be equipped with two different kinds of metrics. When $J$ acts as an isometry on each tangent space then the manifold is an almost Hermitian manifold.

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But, in the case when \( J \) acts as an anti-isometry on each tangent space, the notion of the so-called almost Norden manifold (or almost anti-Hermitian manifold) is available. Let us consider more precisely the latter one.

Every \( n \)-dimensional complex Riemannian manifold induces a real \( 2n \)-dimensional manifold \((M^{2n}, J, g, \bar{g})\) with a complex structure \( J \), a metric \( g \) and an associated metric \( \bar{g} = g(\cdot, J\cdot) \). Both metrics are indefinite of signature \((n,n)\). This manifold is called an almost Norden manifold because it is introduced by A.P. Norden in [4]. An almost Norden manifold is of K"ahler type (sometimes it is called briefly a K"ahler-Norden manifold) if \( J \) is parallel with respect to the Levi-Civita connection \( \nabla \) of the metric \( g \). The class of these manifolds is contained in every other class of the almost Norden manifolds. A classification with respect to \( \nabla J \) consisting of three basic classes is given in [5], whereas the corresponding classification of almost Hermitian manifolds is known from [6] and it contains four basic classes.

2.2. Almost hypercomplex manifolds with Hermitian-Norden structure. Let us recall the notion of the almost hypercomplex structure \( H \) on a manifold \( M^{4n} \). It is the triple \( H = (J_\alpha) \ (\alpha = 1, 2, 3) \) of anticommuting almost complex structures satisfying the property \( J_3 = J_1 \circ J_2 \) ([1], [7]). Further, the index \( \alpha \) runs over the range \( \{1, 2, 3\} \) unless otherwise stated.

A pseudo-Riemannian metric \( g \) of signature \((2n,2n)\) (or a neutral semi-Riemannian metric) on \((M^{4n}, H)\) is introduced as follows (2)

\[
g(\cdot, \cdot) = g(J_1\cdot, J_1\cdot) = -g(J_2\cdot, J_2\cdot) = -g(J_3\cdot, J_3\cdot).
\]

We have called such metric a Hermitian-Norden metric. It generates a K"ahler 2-form \( \Phi \) and two Hermitian-Norden metrics \( g_2 \) and \( g_3 \) by the following way

\[
\Phi := g(J_1\cdot, \cdot), \quad g_2 := g(J_2\cdot, \cdot), \quad g_3 := g(J_3\cdot, \cdot).
\]

Let us note that \( g \) (\( g_2, g_3 \), respectively) has a Hermitian compatibility with respect to \( J_1 \) (\( J_3, J_2 \), respectively) and a Norden compatibility with respect to \( J_2 \) and \( J_3 \) (\( J_1 \) and \( J_2, J_1 \) and \( J_3 \), respectively). On the other hand, a quaternionic inner product \( < \cdot, \cdot > \) generates in a natural way the bilinear forms \( g, \Phi, g_2 \) and \( g_3 \) by the following decomposition: \( < \cdot, \cdot > = -g + i\Phi + jg_2 + kg_3 \).

We have called the structure \((H, G) = (J_1, J_2, J_3; g, \Phi, g_2, g_3)\) on \( M^{4n} \) an almost hypercomplex Hermitian-Norden structure or shortly an almost hcH\(N\)-structure. We have called the manifold \((M, H, G)\) an almost hypercomplex Hermitian-Norden manifold or shortly an almost hcH\(N\)-manifold.

It is well known, that the almost hypercomplex structure \( H = (J_\alpha) \) is a hypercomplex structure if the Nijenhuis tensors \( N_\alpha(\cdot, \cdot) = [\cdot, \cdot] + J_\alpha [\cdot, J_\alpha \cdot] + J_\alpha [J_\alpha \cdot, \cdot] - [J_\alpha \cdot, J_\alpha \cdot] \) vanish for each \( \alpha \). Moreover, a structure \( H \) is hypercomplex if and only if two of \( N_\alpha \) vanish.

We have introduced three structure \((0,3)\)-tensors of the almost hcH\(N\)-manifold by \( F_\alpha(x, y, z) = g((\nabla_x J_\alpha) y, z) = (\nabla_x g_\alpha)(y, z) \), where \( \nabla \) is the Levi-Civita connection generated by \( g \) and \( x, y, z \in T_p M \) at any \( p \in M \). Relations between the tensors \( F_\alpha \) are valid, e.g. \( F_1(\cdot, \cdot, \cdot) = F_2(\cdot, J_3\cdot, \cdot) + F_3(\cdot, \cdot, J_2\cdot) \) [2].
3. TANGENT BUNDLE WITH ALMOST \(hcHN\)-STRUCTURE

Our purpose is a determination of an almost \(hcHN\)-structure \((H,G)\) on \(TM\) when the base manifold \(M\) has an almost Norden structure \((J,g,\tilde{g})\).

We will use the horizontal and vertical lifts of the vector fields on \(M\) to get the corresponding components of the considered tensor fields on \(TM\). These components are sufficient to describe the characteristic tensor fields on \(TM\) in general.

3.1. Almost hypercomplex structure on the tangent bundle. As it is known [8], for any affine connection in \(M\), the induced horizontal and vertical distributions in \(TM\) are mutually complementary. Then we define tensor fields \(J_1, J_2\) and \(J_3\) in \(TM\) by their action over the horizontal and vertical lifts of an arbitrary vector field in \(M\):

\[
J_1: \begin{cases} 
X^H \rightarrow -(JX)^H \\
X^V \rightarrow (JX)^V 
\end{cases}, 
J_2: \begin{cases} 
X^H \rightarrow X^V \\
X^V \rightarrow -X^H 
\end{cases}, 
J_3: \begin{cases} 
X^H \rightarrow (JX)^V \\
X^V \rightarrow (JX)^H 
\end{cases},
\]

where \(J\) is the given almost complex structure on \(M\).

By direct computations we get the following

Proposition 3.1. There exists an almost hypercomplex structure \(H\), defined by \((\cdot)^H, (\cdot)^V \in T^0_1(TM)\) in \(TM\) over an almost complex manifold \((M,J)\) with an affine connection \(\nabla\). The constructed 4n-dimensional manifold is an almost hypercomplex manifold \((TM,H)\).

Let \(N_{\alpha}\) denotes the Nijenhuis tensor of \(J_{\alpha}\) for each \(\alpha\) and \(\hat{X}, \hat{Y} \in T^0_1(TM)\), i.e.

\[
N_{\alpha}(\hat{X}, \hat{Y}) = [\hat{X}, \hat{Y}] + J_{\alpha}[J_{\alpha}\hat{X}, \hat{Y}] + J_{\alpha}\hat{X}, J_{\alpha}\hat{Y} - [J_{\alpha}\hat{X}, J_{\alpha}\hat{Y}].
\]

Further, we denote the horizontal and vertical lifts by \((\cdot)^H, (\cdot)^V \in T^0_1(TM)\) of any \(X,Y,Z,W \in T^0_1(M)\) at \(u \in T_p M\).

For the Levi-Civita connection \(\nabla\) of \(M\) and its curvature tensor \(R\), then at \(u \in T_p M\) we have (see also [8])

\[
[X^H,Y^H] = [X,Y]^H - \{R(X,Y)u\}^V, \\
[X^H,Y^V] = (\nabla_X Y)^V, \\
[X^V,Y^V] = 0, \\
[X^V,Y^H] = -(\nabla_X Y)^V.
\]

Using (3.1), (3.2) and (3.3), we get

Proposition 3.2. Let \((M,J)\) be an almost complex manifold. Then the Nijenhuis tensors of the structure \(H\) in \(TM\) for the horizontal and vertical lifts have the form

\[
N_1(X^H, Y^H) = (N(X,Y))^H \\
+ (R(JX,JY)u + J\nabla(X,Y)u + J\nabla(X,JY)u - R(X,Y)u)^V, \\
N_1(X^H, Y^V) = ((\nabla_X J)\{Y\})^V - (\nabla_X J)(JY)^V, \\
N_1(X^V, Y^H) = ((\nabla_Y J)\{X\})^V - (\nabla_Y J)(JX)^V, \\
N_1(X^V, Y^V) = 0;
\]

\[
N_2(X^H, Y^H) = -N_2(X^V, Y^V) = -(R(X,Y)u)^V, \\
N_2(X^H, Y^V) = N_2(X^V, Y^H) = -(R(X,Y)u)^H; \\
N_3(X^H, Y^H) = (J(\nabla_X J)\{Y\})^H - (J(\nabla_Y J)(X))^H - (R(X,Y)u)^V, \\
N_3(X^H, Y^V) = (J(\nabla_X J)\{Y\})^V + (\nabla_Y J)(JX))^V - (J\nabla(X,Y)u)^H,
\]
\[ N_3(X^V,Y^H) = -((\nabla_{JX}J)(Y) + J(\nabla_{JY}J)(X))^V - (JR(JX,Y)u)^H, \]
\[ N_3(X^V,Y^V) = -((\nabla_{JX}J)(Y) - (\nabla_{JY}J)(X))^H + (R(JX,JY)u)^V. \]

The last equalities for \( N_\alpha \) imply the following necessary and sufficient conditions for integrability of \( J_\alpha \) and \( H \).

**Theorem 3.3.** Let \( TM \) be the tangent bundle manifold with an almost hypercomplex structure \( H = (J_1, J_2, J_3) \) defined as in (3.1) and \( M \) be its base manifold with almost complex structure \( J \). Then the following interconnections hold:

(i) \((TM, J_\alpha)\) for \( \alpha = 1 \) or 3 is complex if and only if \( M \) is flat and \( J \) is parallel;

(ii) \((TM, J_2)\) is complex if and only if \( M \) is flat;

(iii) \((TM, H)\) is hypercomplex if and only if \( M \) is flat and \( J \) is parallel.

**Corollary 3.4.**

(i) \((TM, J_1)\) is complex if and only if \((TM, J_3)\) is complex.

(ii) If \((TM, J_1)\) or \((TM, J_3)\) is complex then \((TM, H)\) is hypercomplex.

3.2. **Complete lift of the base metric on the tangent bundle.** Let us introduce a metric \( \tilde{g} \) on \( TM \), which is the complete lift \( g^C \) on \( TM \) of the base metric \( g \), by

\[ \tilde{g}(X^H,Y^H) = \tilde{g}(X^V,Y^V) = 0, \quad \tilde{g}(X^H,Y^V) = \tilde{g}(X^V,Y^H) = g(X,Y). \]

It is known that \( g^C \), associated with a (semi-)Riemannian metric \( g \), is a semi-Riemannian metric on \( TM \) of signature \((m,m)\), where \( m = \dim M \), which coincides with the horizontal lift of \( g \) when this is considered with respect to the Levi-Civita connection of \( g \). This metric is introduced by Yano and Kobayashi as \((TM, g^C)\) has zero scalar curvature and it is an Einstein space if and only if \( M \) is Ricci-flat [8].

As it is known from [9], if \( \nabla \) is the Levi-Civita connection of \( M \) with respect to the pseudo-Riemannian metric \( g \), then \( \nabla^C \) is the Levi-Civita connection of \( TM \) with respect to \( g^C \). Since \( \nabla \) is the Levi-Civita connection of \( \tilde{g} \) on \( TM \) as \( \nabla \) of \( g \) on \( M \), then using the Koszul formula we obtain the covariant derivatives of the horizontal and vertical lifts of vector fields on \( TM \) at \( u \in T_pM \) as follows (see also [8])

\[ \nabla_{X^H}Y^H = (\nabla_XY)^H + (R(u,X)Y)^V, \quad \nabla_{X^H}Y^V = (\nabla_XY)^V, \quad \nabla_{X^V}Y^H = 0, \quad \nabla_{X^V}Y^V = 0. \]

After that, we calculate the components of the curvature tensor \( \tilde{R} \) of \( \nabla \) with respect to the horizontal and vertical lifts of the vector fields on \( M \) and we obtain the following non-zero components for the curvature tensors \( R \) and \( \tilde{R} \) as well as the Ricci tensors \( \rho \) and \( \tilde{\rho} \), corresponding to the metrics \( g \) and \( \tilde{g} \) on \( M \) and \( TM \), respectively (see also [8])

\[ \tilde{R}(X^H,Y^H)Z^H = \{R(X,Y)Z\}^H + \{(\nabla_uR)(X,Y)Z\}^V, \]
\[ \tilde{R}(X^H,Y^H)Z^V = \tilde{R}(X^H,Y^V)Z^H = \{R(X,Y)Z\}^V; \quad \tilde{\rho}(Y^H,Z^H) = 2\rho(Y,Z). \]

Hence, we get

**Corollary 3.5.**

(i) \((TM, \tilde{g})\) is flat if and only if \((M, g)\) is flat.

(ii) \((TM, \tilde{g})\) is Ricci flat if and only if \((M, g)\) is Ricci flat.

(iii) \((TM, \tilde{g})\) is scalar flat.

**Remark.** The results of (3.5), (3.6) and Corollary 3.5 are confirmed also by [8], where \( g \) is a Riemannian metric.
3.3. **Tangent bundle with almost hcHN-structure.** Suppose that \((M, J, g, \tilde{g})\) is an almost complex manifold with the pair of Norden metrics \(\{g, \tilde{g}\}\) and that \((TM, H)\) is its almost hypercomplex tangent bundle with the Hermitian-Norden metric structure \(\tilde{G} = (\tilde{g}, \tilde{\Phi}, \tilde{g}_2, \tilde{g}_3)\) derived (as in (2.2)) from the metric \(\tilde{g}\) on \(TM\) – the complete lift of \(g\). The generated 4n-dimensional manifold we denote by \((TM, H, \tilde{G})\).

Bearing in mind (3.1), we verify immediately that \(\tilde{g}\) satisfies (2.1) and therefore it is valid the following

**Theorem 3.6.** The tangent bundle \(TM\) equipped with the almost hypercomplex structure \(H\) and the metric \(\tilde{g}\), defined by (3.1) and (3.4), respectively, is an almost hypercomplex Hermitian-Norden manifold \((TM, H, G)\).

In order to characterize the structure tensors \(F_\alpha\) with respect to \(\tilde{g}\) and \(\tilde{\nabla}\) at each \(u \in T_p M\) on \((TM, H, \tilde{G})\), we use (3.5) and (3.1), whence we obtain the following

**Proposition 3.7.** The nonzero components of \(F_\alpha\) with respect to the horizontal and vertical lifts of the vector fields depend on structure tensor \(F\) and the curvature tensor \(R\) on \((M, J, g, \tilde{g})\) by the following way:

\[
\begin{align*}
F_1(X^H, Y^H, Z^H) &= -R(u, X, JY, Z) - R(u, X, Y, JZ), \\
F_1(X^H, Y^H, Z^V) &= -F_1(X^H, Y^V, Z^H) = -F(X, Y, Z); \\
F_2(X^H, Y^H, Z^V) &= -F_2(X^H, Y^V, Z^H) = R(u, X, Y, Z); \\
F_3(X^H, Y^H, Z^V) &= F_3(X^H, Y^V, Z^V) = F(X, Y, Z), \\
F_3(X^H, Y^H, Z^V) &= -R(u, X, Y, JZ), \\
F_3(X^H, Y^V, Z^H) &= R(u, X, JY, Z).
\end{align*}
\]

By direct verification of the definition conditions of the classes in the corresponding classifications in [6] and [5], we obtain

**Proposition 3.8.**

(i) The almost Hermitian manifold \((TM, J_1, \tilde{g})\) belongs to the class \(\{AH \setminus \{NK \cup H\}\} \cup K\), where \(AH, NK, H, K\) and \(K\) are stand for the classes of almost Hermitian, nearly Kähler, Hermitian and Kähler manifolds, respectively. For the 4-dimensional case, the class of \((TM, J_1, \tilde{g})\) is restricted to the class \(AK\) of almost Kähler manifolds.

(ii) The almost Norden manifold \((TM, J_\alpha, \tilde{g})\), \((\alpha = 2, 3)\), belongs to the class \((W_1 \oplus W_2 \oplus W_3) \setminus \{(W_1 \oplus W_2) \cup (W_1 \oplus W_3)\} \cup W_6\).

The corresponding Lee forms are determined in an arbitrary basis \(\{e_i\}_{i=1}^{4n}\) by \(\theta_\alpha(\cdot) = g^{ij}F_\alpha(e_i, e_j, \cdot)\). Hence, we compute them with respect to an adapted frame.

Let \(\{\tilde{e}_A\} = \{e_A^H, e_A^V\}\) be the adapted frame at each point of \(TM\) derived by the orthonormal basis \(\{e_i\}\) of signature \((n, n)\) at each point of \(M\). The indices \(i, j, \ldots\) run over the ranges \(\{1, 2, \ldots, 2n\}\), while the indices \(A, B, \ldots\) the range \(\{1, 2, \ldots, 4n\}\). The summation convention is used in relation to this system of indices.

For example, we compute as follows

\[
\theta_3(Z^H) = g^{AB}F_3(\tilde{e}_A, \tilde{e}_B, Z^H) = g^{ij}\{F_3(e_i^H, e_j^V, Z^H) + F_3(e_i^V, e_j^H, Z^H)\} = g^{ij}F_3(e_i^H, e_j^V, Z^H) = g^{ij}R(u, e_i, J e_j, Z) = \rho^*(u, Z).
\]
Analogously, we have \( \theta_3(Z^V) = g^{ij} F_3(e_i^H, e_j^V, Z^V) = g^{ij} F(e_i, e_j, Z) = \theta(Z) \). Thus, we obtain the following nonzero components of the Lee forms \( \theta_\alpha \):

\[
\theta_1(Z^H) = \theta_3(Z^V) = \theta(Z), \quad \theta_2(Z^H) = -\rho(u, Z), \quad \theta_3(Z^H) = \rho^*(u, Z),
\]

where \( \rho \) and \( \rho^* \) are the Ricci tensor and its associated Ricci tensor regarding \( g \) and \( J \). Therefore, we obtain

**Proposition 3.9.** (i) \( \theta_1 = 0 \) if and only if \( \theta = 0 \);
(ii) \( \theta_2 = 0 \) if and only if \( \rho = 0 \);
(iii) \( \theta_3 = 0 \) if and only if \( \theta = 0 \) and \( \rho^* = 0 \).

**Remark.** Let us recall that the condition for vanishing of the corresponding Lee form determines the class SK of the semi-Kähler manifolds among the almost Hermitian manifolds and the class \( W_2 \oplus W_3 \) among the almost Norden manifolds, respectively.

Bearing in mind Proposition 3.7 and Theorem 3.3 it is easy to conclude the following

**Proposition 3.10.** (i) \((TM, J_\alpha, \tilde{g}) \) for \( \alpha = 1 \) or \( 3 \) has a parallel complex structure \( J_\alpha \) if and only if \((M, J, g, \tilde{g}) \) is flat and \( J \) is parallel.
(ii) \((TM, J_2, \tilde{g}) \) has a parallel complex structure \( J_2 \) if and only if \((M, J, g, \tilde{g}) \) is flat.
(iii) \((TM, H, G) \) is a pseudo-hyper-Kähler manifold if and only if \((M, J, g, \tilde{g}) \) is flat and \( J \) is parallel.

**Remark.** We say that a Hermitian-Norden manifold is a pseudo-hyper-Kähler manifold, if \( F_\alpha = 0 \) for each \( \alpha \in \{1, 2, 3\} \), i.e. the manifold is of Kähler type with respect to each \( J_\alpha \). According to [4], each pseudo-hyper-Kähler manifold is a flat pseudo-Riemannian manifold of neutral signature.

**Corollary 3.11.** (i) \( J_1 \) is parallel if and only if \( J_3 \) is parallel.
(ii) If \( J_1 \) or \( J_3 \) is parallel then \((TM, H, \tilde{G}) \) is pseudo-hyper-Kähler.

**Corollary 3.12.** (i) The only complex manifolds \((TM, J_\alpha, \tilde{g}) \) for some \( \alpha \) are the corresponding manifolds of Kähler type with respect to the same \( J_\alpha \).
(ii) The only hypercomplex manifolds \((TM, H, \tilde{G}) \) are the pseudo-hyper-Kähler manifolds.

**Corollary 3.13.** (i) If \((M, J, g, \tilde{g}) \) is flat, then \( TM \) has parallel \( J_2 \).
(ii) If \((M, J, g, \tilde{g}) \) is flat and its Lee form \( \theta \) is zero, i.e. \((M, J, g, \tilde{g}) \in \{W_2 \oplus W_3\} \), then \((TM, H, \tilde{G}) \in SK(J_1) \cup W_0(J_2) \cup \{W_2 \oplus W_3\}(J_3) \); 
(iii) If \((M, J, g, \tilde{g}) \) is a Kähler-Norden manifold, then \((TM, J_1, \tilde{g}) \in AK \).

**Remark.** By comparison with the Riemannian case, in [10] it is shown that \((TM, J_1, \tilde{g}) \) is almost Kähler (i.e. symplectic) for any Riemannian metric \( g \) on the base manifold when the connection used to define the horizontal lifts is the Levi-Civita connection.

### 3.4. Tangent bundle of h-sphere

Let \((M, J, g, \tilde{g}) \) be a Kähler-Norden manifold, \( \dim M = 2n \geq 4 \). Let \( x, y, z, w \) be arbitrary vectors in \( T_p M \), \( p \in M \). The curvature tensor \( R \) of type \((0, 4)\) defined by \( R(x, y, z, w) = g(R(x, y)z, w) \) has the Kähler property \( R(x, y, z, w) = -R(x, y, Jz, Jw) \) in this case. This implies that the associated tensor \( R^* \) of type \((0, 4)\) defined by \( R^*(x, y, z, w) = R(x, y, z, Jw) \) has the property \( R^*(x, y, z, w) = \).
Therefore, \( R^* \) has the properties of a curvature tensor, i.e. it is a curvature-like tensor. The following tensors are essential curvature-like tensors:

\[
\begin{align*}
\pi_1(x, y, z, w) &= g(y, z)g(x, w) - g(x, z)g(y, w), \\
\pi_2(x, y, z, w) &= g(y, z)\tilde{g}(x, w) - \tilde{g}(x, z)g(y, w), \\
\pi_3(x, y, z, w) &= -g(y, z)\tilde{g}(x, w) + g(x, z)\tilde{g}(y, w) - \tilde{g}(y, z)g(x, w) + \tilde{g}(x, z)g(y, w).
\end{align*}
\]

Every non-degenerate 2-plane \( \beta \) with respect to \( g \) in \( T_pM, p \in M \), has the following two sectional curvatures \( k(\beta; p) = cR(x, y, x, y), k^*(\beta; p) = cR^*(x, y, x, y) \), where \( c = \pi_1(x, y, y, x)^{-1} \) and \( \{x, y\} \) is a basis of \( \beta \).

A 2-plane \( \beta \) is said to be holomorphic (resp., totally real) if \( \beta = J\beta \) (resp., \( \beta \perp J\beta \neq \beta \)) with respect to \( g \) and \( J \).

The orthonormal \( J \)-basis \( \{e_i, e_j\} \), where \( i \in \{1, 2, \ldots, n\} \), is an adapted \( H \)-basis.

Thus, the following basic 2-planes in \( T_u(TM) \) are special with respect to \( H \) (\( i \neq j \)):

- \( J_\alpha \)-totally-real 2-planes (\( \alpha = 1, 2, 3 \)): \( \{\xi_i, \xi_j\}, \{\xi_i, \eta_j\}, \{\eta_i, \eta_j\}, \{\eta_i, \xi_j\} \);
- \( J_1 \)-holomorphic and \( J_\alpha \)-totally-real 2-planes (\( \alpha = 2, 3 \)): \( \{\xi_i, \eta_j\}, \{\eta_i, \xi_j\} \);
- \( J_2 \)-holomorphic and \( J_\alpha \)-totally-real 2-planes (\( \alpha = 1, 3 \)): \( \{\eta_i, \xi_j\}, \{\eta_i, \eta_j\} \);
- \( J_3 \)-holomorphic and \( J_\alpha \)-totally-real 2-planes (\( \alpha = 1, 2 \)): \( \{\xi_i, \xi_j\}, \{\eta_i, \eta_j\} \).

The sectional curvatures \( \hat{k} \) of these 2-planes and the sectional curvatures \( k_{ij}, k_{ij}, k_{ij} \) of the special basic 2-planes in \( T_pM - J \)-totally-real 2-planes \( \{e_i, e_j\}, \{e_i, e_j\} \) (\( i \neq j \)) and \( J \)-holomorphic 2-planes \( \{e_i, e_j\} \), respectively – are related as follows:

\[
\begin{align*}
\hat{k}(\xi_i, \xi_j) &= \frac{1}{4} (\nabla u_k)_{ij} + k_{ij}, \\
\hat{k}(\xi_i, \eta_j) &= -\frac{1}{4} (\nabla u_k)_{ij}, \\
\hat{k}(\eta_i, \eta_j) &= \frac{1}{4} (\nabla u_k)_{ij} - k_{ij}.
\end{align*}
\]

Therefore, we obtain

**Proposition 3.14.** The manifold \( (TM, H, \hat{G}) \) for an arbitrary almost Norden manifold \( (M, J, g) \) has equal sectional curvatures of the \( J_1 \)-holomorphic 2-planes and vanishing sectional curvatures of the \( J_2 \)-holomorphic 2-planes.

Identifying the point \( z = (x^1, \ldots, x^{n+1}; y^1, \ldots, y^{n+1}) \) in \( \mathbb{R}^{2n+2} \) with the position vector \( Z \), we consider the holomorphic hypersurface \( S^{2n}(z_0; a, b) \) in the Kähler-Norden manifold \( (\mathbb{R}^{2n+2}, J', g') \) defined by \( g(z - z_0, z - z_0) = a, \bar{g}(z - z_0, z - z_0) = b \), where \( (0, 0) \neq (a, b) \in \mathbb{R}^2 \). Every \( S^{2n} \), \( n \geq 2 \), has vanishing holomorphic sectional curvatures and constant totally real
sectional curvatures $\nu = \frac{a}{a^2 + b^2}$, $\nu^* = \frac{b}{a^2 + b^2}$. Then, according to [12], the curvature tensor of the $h$-sphere is
\begin{equation}
R = \nu(\pi_1 - \pi_2) + \nu^*\pi_3
\end{equation}
and therefore $\nabla R = 0$. Moreover, we have $\rho = 2(n-1)(\nu g - \nu^* \bar{g})$, $\rho^* = 2(n-1)(\nu^* \bar{g} + \nu g)$, $\tau = 4n(n-1)\nu$, $\tau^* = 4n(n-1)\nu^*$, where $\rho = \rho(R^*)$, $\tau = \tau(R^*)$. Because of the form of $\rho$, $S^{2n}$ is called almost Einstein.

Let us consider the tangent bundle with almost $hcHN$-structure $(TS, H, \hat{G})$ with the $h$-sphere $(S, J, g)$ with parameters $(a, b)$ as its base Kähler-Norden manifold. Then, bearing in mind Proposition [3.7], Proposition [3.8] and Corollary [3.13] (iii), we get that $(TS, J_1, \hat{g}) \in \mathcal{AK}$ and $(TS, J_\alpha, \hat{g}) (\alpha = 2, 3)$ belongs to $(W_1 \oplus W_2 \oplus W_3) \setminus (W_i \oplus W_j)$, where $i \neq j \in \{1, 2, 3\}$. Moreover, according to [3.6] and [3.7], we obtain the components of the curvature tensor $\hat{R}$ of $(TS, H, \hat{G})$. Then we have
\begin{align*}
\hat{k}(\xi, \xi_j) &= \hat{k}(\xi, \xi_j) = \hat{k}(\xi, \xi_j) = \hat{k}(\eta, \eta_j) = -\hat{k}(\eta, \eta_j) = -\hat{k}(\eta, \eta_j) = \frac{a}{a^2 + b^2}, \\
\hat{k}(\xi, \eta_j) &= \hat{k}(\xi, \eta_j) = \hat{k}(\xi, \eta_j) = \hat{k}(\xi, \eta_j) = 0.
\end{align*}

**Corollary 3.15.** The manifold $(TS, H, \hat{G})$ for an arbitrary $h$-sphere $(S, J, g)$ has constant sectional curvatures of $J_\alpha$-totally-real 2-planes and vanishing sectional curvatures of $J_\alpha$-holomorphic 2-planes $(\alpha = 1, 2, 3)$.

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Department of Algebra and Geometry
Faculty of Mathematics, Informatics and IT
Paisii Hilendarski University of Plovdiv
236 Bulgaria Blvd
4027 Plovdiv, Bulgaria
e-mail: mmanev@uni-plovdiv.bg