Oscillatory matrix model in Chern-Simons theory and
Jacobi-theta determinantal point process

Yuta Takahashi ∗ and Makoto Katori †
Department of Physics, Faculty of Science and Engineering,
Chuo University, Kasuga, Bunkyo-ku, Tokyo 112-8551, Japan
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Abstract

The partition function of the Chern-Simons theory on the three-sphere with the unitary
group $U(N)$ provides a one-matrix model. The corresponding $N$-particle system can be mapped
to the determinantal point process whose correlation kernel is expressed by using the Stieltjes-
Wigert orthogonal polynomials. The matrix model and the point process are regarded as $q$-
extensions of the random matrix model in the Gaussian unitary ensemble and its eigenvalue
point process, respectively. We prove the convergence of the $N$-particle system to an infinite-
dimensional determinantal point process in $N \to \infty$, in which the correlation kernel is expressed
by Jacobi’s theta functions. We show that the matrix model obtained by this limit realizes the
oscillatory matrix model in Chern-Simons theory discussed by de Haro and Tierz.

1 Introduction

Chern-Simons theory on a three-manifold $M$ with a simply-laced gauge group $G$ is specified by the
action

$$S(A) = \frac{k}{4\pi} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right),$$

(1.1)

where $A$ is a $G$-connection on $M$ and $k$ is an integer. The partition function of the Chern-Simons
theory is then given by

$$Z_k (M, G) = \int \mathcal{D}A e^{iS(A)}$$

(1.2)

with $i = \sqrt{-1}$. Based on [31, 32], Mariño showed in [14] that the partition function of Chern-
Simons theory on Seifert spaces can be calculated in a combinatorial way and expressed by multiple
integrals. In particular, when the gauge group $G$ is chosen as the unitary group $U(N)$,
$N \in \{2, 3, \cdots \}$, the Chern-Simons partition function on the three-sphere $S^3$ is expressed by [14]

$$Z_k \left( S^3, U(N) \right) = \frac{e^{-g_s N(N^2-1)/12}}{N!} \int_{\mathbb{R}^N} \prod_{j=1}^N \frac{d\phi_j}{2\pi} e^{-\phi_j^2/2g_s} \prod_{1 \leq j < k \leq N} \left( 2 \text{sinh} \frac{\phi_k - \phi_j}{2} \right)^2,$$

(1.3)
where the string coupling constant $g_s$ is given by

$$g_s = \frac{2\pi i}{k + N}.$$  \hfill (1.4)

The structure of (1.3) is similar to those of partition functions of one-matrix models [17,6]. Tierz [28] put

$$q = e^{-g_s} \tag{1.5}$$

and regarded (1.3) as the partition function of matrix model associated with the Stieltjes-Wigert polynomials $p_n(\cdot; q)$, $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, which are $q$-extensions of Hermite polynomials (see Section 2.1). He performed the integral (1.3) by using orthonormality of $p_n(\cdot; q)$’s, which is generally valid for $0 < |q| < 1$, and obtained the exact and explicit expression for (1.3)

$$Z_k \left( S^3, U(\mathbb{N}) \right) = e^{i\pi N^2/4} (k + N)^{-N/2} \prod_{j=1}^{N-1} \left( 2 \sin \frac{\pi j}{k + N} \right)^{N-j} \tag{1.6}$$

by setting (1.4) in the result [28].

The above fact leads us to consider the Chern-Simons partition function (1.3) in the way that the constant $g_s$ is a free parameter and is not restricted by (1.4). In the present paper, we consider the case that the parameter $g_s$ is positive and regard (1.3) as the partition function of a statistical mechanics system of $N$ particles. The variables $\phi_j \in \mathbb{R}$, $1 \leq j \leq N$ in the integral (1.3) are considered to be realizations of $N$ random variables on $\mathbb{R}$ whose probability law is given by the probability density function

$$\tilde{P}_N \left( \{ \phi_j \}^N_{j=1} \right) = \tilde{c}_N \prod_{j=1}^{N} e^{-\phi_j^2/2g_s} \prod_{1 \leq j < k \leq N} \left( 2 \sinh \frac{\pi j}{k + N} \right)^2. \tag{1.7}$$

Here $\tilde{c}_N$ is the normalization constant which will be explicitly given in Section 2.1. From it, the partition function (1.3) is obtained by

$$Z_k \left( S^3, U(\mathbb{N}) \right) = \frac{g_s^{N/2} e^{-g_s N(N^2 - 1)/12}}{\tilde{c}_N N!} \tag{1.8}$$

Let $\mathbb{R}_+ = \{ x \in \mathbb{R} : x > 0 \}$. By the mapping

$$f_N : \phi \in \mathbb{R} \mapsto x \in \mathbb{R}_+, \quad x = f_N(\phi) = e^{\phi + Ng_s}, \tag{1.9}$$

(1.7) is transformed to the probability density function

$$P_N \left( \{ x_j \}^N_{j=1} \right) = c_N \prod_{j=1}^{N} e^{-\frac{(\ln x_j)^2}{2g_s}} \prod_{1 \leq j < k \leq N} \left( x_k - x_j \right)^2, \tag{1.10}$$

where $c_N$ is given by

$$c_N = e^{-g_s N^3/2} \tilde{c}_N. \tag{1.11}$$

In the Gaussian unitary ensemble (GUE) with variance $\sigma^2$, the eigenvalues of $N \times N$ random Hermitian matrices obey the probability density

$$P_{N \text{GUE}} \left( \{ x_j \}^N_{j=1} \right) = c_{N \text{GUE}} \prod_{j=1}^{N} e^{-x_j^2/2\sigma^2} \prod_{1 \leq j < k \leq N} \left( x_k - x_j \right)^2, \tag{1.12}$$
where $c_N^{\text{GUE}} = 1/\sigma^{N^2-N} \prod_{j=1}^{N} j!$. In the present ensemble (1.10), individual point $x_j$ follows the log-normal distribution on $\mathbb{R}_+$ instead of the Gaussian distribution on $\mathbb{R}$, while the repulsive interactions represented by $\prod_{1 \leq j < k \leq N} (x_k - x_j)^2$ are common.

Let $x_j$, $1 \leq j \leq N$ be random variables having the probability density function (1.10) on $\mathbb{R}_+$. Since $P_N$ as well as $P_N^{\text{GUE}}$ are symmetric functions of $x_j$, $1 \leq j \leq N$, we shall represent a configuration as unlabeled. Let $\mathfrak{M}$ be the space of nonnegative integer-valued Radon measure on $\mathbb{R}_+$. Any element $\xi \in \mathfrak{M}$ is represented as $\xi(\cdot) = \sum_{j=1}^{N} \delta_{x_j}(\cdot)$ with a countable index set $I$, where $\delta_x(\cdot)$ denotes a point mass (the delta measure) on $x$. There a sequence of points in $\mathbb{R}_+$, $x = (x_j)_{j=1}^\infty$, satisfies $\xi(K) = \sharp \{x_j : x_j \in K\} < \infty$ for any compact subset $K \subset \mathbb{R}_+$. Then we regard the present particle system as $\mathfrak{M}$-valued and write it as $(\xi, P_N)$ with

\begin{equation}
\xi(\cdot) = \sum_{j=1}^{N} \delta_{x_j}(\cdot). \quad (1.13)
\end{equation}

Let $C_0(\mathbb{R}_+)$ be the set of all continuous real-valued function with compact support on $\mathbb{R}_+$. For $f \in C_0(\mathbb{R}_+)$, the moment generating function of the system is given by the following generalized Laplace transform of the distribution (1.10),

\begin{equation}
\mathcal{G}_N[f] = E_N \left[ \exp \left( \int_{\mathbb{R}_+} f(x) \xi(dx) \right) \right] = \int_{\mathbb{R}_+^N} d\mathbf{x} P_N(\{x_j\}_{j=1}^N) e^{\sum_{k=1}^{N} f(x_k)}, \quad (1.14)
\end{equation}

where $d\mathbf{x} = \prod_{j=1}^{N} dx_j$ and $E_N$ denotes the expectation with respect to $P_N$. It is expanded with respect to the ‘test function’ $g(\cdot) = e^{f(\cdot)} - 1$ as

\begin{equation}
\mathcal{G}_N[f] = \int_{\mathbb{R}_+^N} d\mathbf{x} P_N(\{x_j\}_{j=1}^N) \prod_{k=1}^{N} \left( 1 + g(x_k) \right) = 1 + \sum_{N' = 1}^{N} \frac{1}{N'!} \int_{\mathbb{R}_+^{N'}} d\mathbf{x}'(N') \prod_{j=1}^{N'} g \left( x_j^{(N')} \right) \rho_N^{(N')}(\mathbf{x}^{(N')}), \quad (1.15)
\end{equation}

where $\mathbf{x}^{(N')}$ denotes $(x_1^{(N')}, x_2^{(N')}, \ldots, x_{N'}^{(N')})$, $1 \leq N' \leq N$. Here $\rho_N^{(N')}$ gives the $N'$-point correlation function for $(\xi, P_N)$, which is a symmetric function, $1 \leq N' \leq N$. Given an integral kernel $K(x, y)$, $(x, y) \in \mathbb{R}_+^2$, a Fredholm determinant with $g \in C_0(\mathbb{R}_+)$ is defined as

\begin{equation}
\text{Det}_{(x, y) \in \mathbb{R}_+^2} \left[ \delta(x - y) + K(x, y)g(y) \right] = 1 + \sum_{N' = 1}^{N} \frac{1}{N'!} \int_{\mathbb{R}_+^{N'}} d\mathbf{x}'(N') \prod_{j=1}^{N'} g \left( x_j^{(N')} \right) \text{det}_{1 \leq k, \ell \leq N'} \left[ K \left( x_k^{(N')}, x_\ell^{(N')} \right) \right]. \quad (1.16)
\end{equation}

If the system $(\xi, P_N)$ has an integral kernel $K$ such that any moment generating function (1.14) is given by a Fredholm determinant (1.16), $(\xi, P_N)$ is said to be a determinantal point process with
the correlation kernel $K \overset{[22,24]}{=}$. By definition, we have

$$
\rho^{(N')}_{\tilde{P}}(x^{(N')}) = \frac{N!}{(N-N')!} \int_{\mathbb{R}^{N-N'}} dx^{(N-N')} P_{N} \left( \left\{ x^{(N')}, x^{(N-N')} \right\} \right)
$$

(1.17)

$$
= \det_{1 \leq j,k \leq N'} \left[ K \left( x_j^{(N')}, x_k^{(N')} \right) \right],
$$

(1.18)

$x^{(N')} \in \mathbb{R}^{N'}_+$, $1 \leq N' \leq N$. Note that the terminology ‘point process’ does not mean any stochastic process but does a spatial distribution of points as usually used in probability theory (e.g. Poisson processes). Determinantal point process is also called fermion point process $[24,22]$. By definition, we have

$K_N(x,y) = \sum_{n=0}^{N-1} p_n(x;q)p_n(y;q) \sqrt{w(x;q)w(y;q)}$

(1.19)

$$
= \frac{1-q^N}{q^{2N}} \frac{p_N(x;q)p_{N-1}(y;q) - p_N(y;q)p_{N-1}(x;q)}{x-y} \sqrt{w(x;q)w(y;q)},
$$

$(x,y) \in \mathbb{R}^2_+, x \neq y$, (1.19)

$$
K_N(x,x) = \sum_{n=0}^{N-1} p_n(x;q)^2 w(x;q)
$$

(1.20)

where $p_n(\cdot;q)$, $n \in \mathbb{N}_0$, $0 < q < 1$ are the Stieltjes-Wigert polynomials, $p_n'(\cdot;q)$, $n \in \mathbb{N}_0$ are their derivatives, and $w(\cdot;q)$ is their weight function for orthogonality, which will be explicitly given in Section 2.1. The second equality in (1.19) is given by the Christoffel-Darboux formula $[25]$. This fact implies that the original system $(\Xi, P_N)$ is a determinantal point process with the correlation kernel $K_N(x, x)$.

This fact implies that the original system

$$
\Xi(\cdot) = \sum_{j=1}^{N} \delta_{\phi_j}(\cdot)
$$

(1.21)

with the probability density (1.17) associated with the Chern-Simons partition function is also a determinantal point process on $\mathbb{R}$. By (1.19) the correlation kernel of $(\Xi, \tilde{P}_N)$ is given by

$$
\mathcal{K}_N(\phi, \varphi) = e^{(\phi + \varphi)/2 + Nq} K_N \left( e^{\phi + Nq}, e^{\varphi + Nq} \right), \quad (\phi, \varphi) \in \mathbb{R}^2.
$$

(1.22)

These finite point processes $(\Xi, P_N)$ and $(\Xi, \tilde{P}_N)$ are fully studied in $[25,13,26]$. The purpose of the present paper is to consider an $N \to \infty$ limit of the systems $(\Xi, P_N)$ and $(\Xi, \tilde{P}_N)$. For $n \in \mathbb{N}_0$, let

$$
\chi(n) = \begin{cases} 
0 & \text{if } n \text{ is even}, \\
1 & \text{if } n \text{ is odd}, 
\end{cases}
$$

(1.23)
and for \( x \in \mathbb{R} \), let \( [x] \) be the least integer not less than \( x \). Then for \( 0 < q < 1 \) and \( 2/3 < \tau < 2 \), we will prove

\[
q^{-\tau N - x(\tau N)} K_N \left( q^{-\tau N - x(\tau N)} u, q^{-\tau N - x(\tau N)} v \right) \rightarrow K^\Theta(u, v), \quad \text{as } N \rightarrow \infty, \tag{1.24}
\]

(\( u, v \) \in \( \mathbb{R}_+^2 \) (Proposition 3), where

\[
K^\Theta(u, v) = \frac{\Theta(-q^{1/2}u|q)\Theta(-q^{-1/2}v|q) - \Theta(-q^{-1/2}v|q)\Theta(-q^{1/2}u|q)}{u - v} \sqrt{w(u; q)w(v; q)} \prod_{k=1}^\infty (1 - q^k)^3, \quad (u, v) \in \mathbb{R}_+^2, u \neq v, \tag{1.25}
\]

\[
K^\Theta(u, u) = \left\{ \frac{1}{\sqrt{q}} \Theta(-q^{1/2}u|q)\Theta'(-q^{-1/2}u|q) - \sqrt{q}\Theta'(-q^{1/2}u|q)\Theta(-q^{-1/2}u|q) \right\}, \quad u \in \mathbb{R}_+, \tag{1.26}
\]

with the weight function \( w(\cdot; q) \), and the conditional limit depending on \( \tau \), denoted by \( N \rightarrow \infty \), is defined at the beginning of Section 3.1. Here \( \Theta(\cdot|q) \) is a version of Jacobi’s theta function defined by (see, for instance, [30]),

\[
\Theta(z|q) = \sum_{k=-\infty}^{\infty} q^{k^2} z^k \tag{1.27}
\]

\[
= \prod_{k=1}^{\infty} (1 - q^{2k})(1 + zq^{2k-1})(1 + z^{-1}q^{2k-1}), \tag{1.28}
\]

for \( 0 < |q| < 1 \) and \( 0 < |z| < \infty \), and \( \Theta'(\cdot|q) \) is its derivative

\[
\Theta'(z|q) = \frac{d\Theta(z|q)}{dz} = \frac{1}{z} \sum_{k=-\infty}^{\infty} kq^{k^2} z^k. \tag{1.29}
\]

Thus we call the correlation kernel \( K^\Theta \) given by (1.25) and (1.26) the Jacobi-theta kernel. The convergence (1.24) of correlation kernel in \( N \rightarrow \infty \) implies that of moment generating function \( G_N \) to the Fredholm determinant associated with the Jacobi-theta kernel (1.25), (1.26),

\[
\det_{(u, v) \in \mathbb{R}_+^2} \left[ \delta(u - v) + K^\Theta(u, v)g(v) \right], \tag{1.30}
\]

\( g \in C_0(\mathbb{R}_+) \). Then all correlation functions \( \rho_N' \), \( N' \in \mathbb{N} \) are determined and expressed by determinants. In this sense, as the \( N \rightarrow \infty \) limit of \( (\Xi, P_N) \) and \( (\tilde{\Xi}, \tilde{P}_N) \), determinantal point processes with infinite numbers of particles are obtained (Theorem 4 and Corollary 5).

In [4], de Haro and Tierz discussed an oscillatory matrix model, which seems to appear in sufficiently large but finite \( N \) in the Chern-Simons theory with the \( U(N) \) gauge. With the restriction (1.4), the \( N \rightarrow \infty \) limit is identified with the ‘t Hooft limit; \( N \rightarrow \infty \) with \( Ng_s = \text{constant} \). Since it corresponds to \( q \rightarrow 1 \) by (1.5), the oscillatory behavior will vanish and only classical matrix model is obtained in the limit. The oscillatory matrix model of de Haro and Tierz is realized as a crossover phenomenon in [4]. In the present paper, we fix \( g_s \) so that \( 0 < q = e^{-g_s} < 1 \) and
take limit $N \to \infty$. The infinite-dimensional determinantal point process obtained by this limit of $(\tilde{\Xi}, P_N)$ will be a stationary realization of the oscillatory matrix model observed by de Haro and Tierz [4]. The oscillatory behavior will be demonstrated in Section 4.1 with figures. In Section 4.2 we will confirm that if we take the further limit $g_s \to 0$ (i.e. $q \to 1$), the system becomes classical with the sine-kernel as it should. In other words, in the context of random matrix theory [17,6] the present paper reports a $q$-extension of the bulk scaling limit of the Hermite kernel in GUE. The $q$-extension of the edge scaling limit described by the Airy kernel will be reported in a forthcoming paper [27].

The paper is organized as follows. In Section 2, we define the Stieltjes-Wigert polynomials $p_n$ and give their asymptotic expansions as the degree of polynomials $n \to \infty$. In Section 3, we present asymptotic form of the Stieltjes-Wigert kernel described by (1.19) and (1.20) and explain its connection with the oscillatory matrix model. Section 4 is devoted to showing the oscillatory behaviors of the infinite-particle systems. Proofs of Lemma 1 and Proposition 3 are given in Section 5. In Appendix A, we rewrite the correlation kernel of the oscillatory matrix model in standard notations of theta functions [30] and in terms of Gosper’s $q$-trigonometric functions [8]. Appendix B complements the proof of Lemma 1.

2 Preliminaries

2.1 Some $q$-special functions

For $0 < |q| < 1$ and $z \in \mathbb{C}$, we introduce the $q$-Pochhammer symbol

$$(z; q)_0 = 1, \quad (z; q)_n = \prod_{k=0}^{n-1} (1 - z q^k), \quad n \in \mathbb{N},$$

and

$$(z; q)_\infty = \lim_{n \to \infty} (z; q)_n. \quad (2.2)$$

The following identity follows from the $q$-binomial theorem [11],

$$(z; q)_n = \sum_{k=0}^{n} \frac{(q; q)_n q^{k(k-1)/2} (-z)^k}{(q; q)_k (q; q)_{n-k}}, \quad n \geq 0,$$

and then

$$(z; q)_\infty = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} (-z)^k}{(q; q)_k}. \quad (2.4)$$

For $0 < q < 1$ and $n \in \mathbb{N}_0$, the orthonormal Stieltjes-Wigert polynomials are defined by [25]

$$p_n(x; q) = (-1)^n q^{n/2 + 1/4} \sqrt{(q; q)_n} \sum_{k=0}^{n} \frac{q^{k^2} (-q^{1/2} x)^k}{(q; q)_k (q; q)_{n-k}}, \quad x > 0. \quad (2.5)$$

They satisfy the orthonormality relations

$$\int_{0}^{\infty} p_n(x; q) p_m(x; q) w(x; q) dx = \delta_{nm}, \quad n, m \in \mathbb{N}_0, \quad (2.6)$$
with respect to the weight function

\[ w(x; q) = \frac{1}{\sqrt{2\pi \ln q}} \exp \left[ -\frac{(\ln x)^2}{2\ln q} \right], \quad x > 0. \]  

(2.7)

This gives a density for a log-normal distribution and solves the functional equation

\[ w(q^s x; q) = q^{s^2/2} x^s w(x; q), \quad s \in \mathbb{R}. \]  

(2.8)

By using the Stieltjes-Wigert polynomials and their orthonormality (2.6), normalization constant \( c_N \) in (1.10) is determined as

\[ c_N = e^{-g_s N(4N^2-1)/6} \frac{N!}{\prod_{k=1}^{N-1} (e^{-g_s}; e^{-g_s})_k}, \]  

(2.9)

where (1.5) was assumed. Then \( \tilde{c}_N \) is given by (1.11), and through (1.8) we obtain

\[ Z_k \left( S^3, U(N) \right) = \left( \frac{g_s}{2\pi} \right)^{N/2} e^{g_s N(N^2-1)/12} \prod_{j=1}^{N-1} (e^{-g_s}; e^{-g_s})_j. \]  

(2.10)

We have the identity

\[ \prod_{j=1}^{N-1} (e^{-g_s}; e^{-g_s})_j = e^{-g_s N(N^2-1)/12} e^{i\pi N(N-1)/4} \prod_{j=1}^{N-1} \left( 2\sin \frac{jg_s}{2i} \right)^{N-j}, \]  

(2.11)

and obtain the expression (1.6) by substituting (1.4) into (2.10) [28].

The theta function (1.28) can be written as

\[ \Theta(z | q) = (q^2; q^2)_\infty (-zq; q^2)_\infty (-z^{-1}q; q^2)_\infty, \]  

(2.12)

which is called Jacobi’s triple product identity. One can prove the functional equation

\[ \Theta(q^2 z | q) = q^{-1} z^{-1} \Theta(z | q), \]  

(2.13)

directly from the definition (1.27).

A \( q \)-exponential function is defined as

\[ e_q(z) = (z; q)_\infty^{-1}, \quad |z| < 1. \]  

(2.14)

### 2.2 Asymptotic expansions for the Stieltjes-Wigert polynomials

For \( \tau \in (0, 2), n \in \mathbb{N} \), let

\[ m = \lfloor (2 - \tau)n \rfloor, \]  

(2.15)

and

\[ \lambda = (2 - \tau)n - m, \]  

(2.16)

where \( \lfloor x \rfloor \) denotes the integer part of \( x \in \mathbb{R} \). We also introduce the indicator function \( 1_A(\omega) \) of a set \( A \) such that \( 1_A(\omega) = 1 \) if \( \omega \in A \) and \( 1_A(\omega) = 0 \) otherwise.
Lemma 1  Let \( 0 < q < 1 \) and \( 0 < \tau < 2 \). Then the orthonormal Stieltjes-Wigert polynomials (2.3) have the following asymptotic expansions as the degree of polynomials \( n \to \infty \),

\[
p_n(q^{-n\tau} u; q) = (-1)^n q^{n^2/2 + 1/4 + n^2(1-\tau) - \lfloor m/2 \rfloor (|m/2| + \chi(m) + \lambda)} \sqrt{|q; q^n_\infty \Theta \left(-q^{\lambda + \chi(m) + 1/2} u \big| q \right)}
\]

\[
+ q^{\lfloor m/2 \rfloor} \left( q^{1/2 - (1-\tau) m} u \frac{1_{(0,1/3)}(\tau) - q 1_{(2,3/2)}(\tau)}{1_{(0,1/3)}(\tau) - q 1_{(2,3/2)}(\tau)} \right) \Theta \left(-q^{\lambda + \chi(m) - 1/2} u \big| q \right)
\]

\[
+ O \left(q^{\tau n + 2(1-\tau) n 1_{[1,2]}(\tau)}\right). \quad (2.17)
\]

The proof is given in Section 5.1 with Appendix B. Since the Stieltjes-Wigert polynomial is a \( q \)-extension of the Hermite polynomials [11], this result can be regarded as a \( q \)-analogue of the celebrated Plancherel-Rotach asymptotic formula [20]. We note that the leading term given by the first term in the parenthesis in the RHS was given by Ismail and Zhang as equations (2.19) and (2.23) in [9] (and (16) and (19) in [10]). This lemma improves their estimate. In order to obtain the Jacobi-theta kernel given by (12.20) and (12.20) as a limit of the Stieltjes-Wigert kernel expressed by (1,19) and (1,20), the correction term given by the second term in the parenthesis is necessary (Proposition 3). Owing to the factor \( \lfloor m/2 \rfloor \) with (2.14) in the formula, asymptotic behavior in \( n \to \infty \) will depend on whether \( \tau \) is rational or irrational as discussed in [9].

Lemma 1 gives the following asymptotic expansions for \( p_n(q^{-n\tau} u; q) \) multiplied by the weight function (2.7); as \( n \to \infty \),

\[
p_n(q^{-n\tau} u; q) \sqrt{w(q^{-n\tau} u; q)}
\]

\[
= \sqrt{|q; q^n_\infty \Theta \left(-q^{\lambda + \chi(m) + 1/2} u \big| q \right)}
\]

\[
+ q^{\lfloor m/2 \rfloor} \left( q^{1/2 - (1-\tau) m} u \frac{1_{(0,1/3)}(\tau) - q 1_{(2,3/2)}(\tau)}{1_{(0,1/3)}(\tau) - q 1_{(2,3/2)}(\tau)} \right) \Theta \left(-q^{\lambda + \chi(m) - 1/2} u \big| q \right)
\]

\[
+ O \left(q^{\tau n + 2(1-\tau) n 1_{[1,2]}(\tau)}\right). \quad (2.18)
\]

Then, we can find the following.

Lemma 2  For \( 0 < q < 1 \), \( 0 < \tau < 2 \),

\[
p_n(q^{-\tau n} - \chi(\tau n)) u; q) \sqrt{w(q^{-\tau n} - \chi(\tau n)) u; q)}
\]

\[
= \sqrt{|q; q^n_\infty \Theta \left(-q^{1/2} u \big| q \right)}
\]

\[
+ q^{-\tau n} \left( q^{1/2 + n + \tau n} u \frac{1_{(0,1/3)}(\tau) - q 1_{(2,3/2)}(\tau)}{1_{(0,1/3)}(\tau) - q 1_{(2,3/2)}(\tau)} \right) \Theta \left(-q^{1/2} u \big| q \right)
\]

\[
+ O \left(q^{\tau n + 2(1-\tau) n 1_{[1,2]}(\tau)}\right), \quad \text{as} \ n \to \infty. \quad (2.19)
\]
3 Main theorems

3.1 Jacobi-theta determinantal point process

Since we obtained the asymptotic forms of the Stieltjes-Wigert polynomials as Lemmas 1 and 2, we will be able to determine the asymptotics of the Stieltjes-Wigert kernel given by (1.19) and (1.20) in $N \to \infty$. For a technical reason, here we assume $\tau \in (2/3, 2)$. We consider a monotonically increasing series of integers $(N_j)_{j \in \mathbb{N}}$ such that

$$[\tau(N_j - 1)] = [\tau N_j] + [-\tau], \quad j \in \mathbb{N},$$

(3.1)

holds. Note that $[-\tau] = 1_{[1,2]}(\tau)$ for $\tau \in (2/3, 2)$. Then $(N_j^{(0)})_{j \in \mathbb{N}}$ and $(N_j^{(1)})_{j \in \mathbb{N}}$ are defined as subsequences of $(N_j)_{j \in \mathbb{N}}$ such that $[\tau N_j^{(0)}]$ are even and $[\tau N_j^{(1)}]$ are odd, respectively, $j \in \mathbb{N}$. For a given $\tau \in (2/3, 2)$, we take the limit $N \to \infty$ following the subsequences $(N_j^{(0)})_{j \in \mathbb{N}}$ and $(N_j^{(1)})_{j \in \mathbb{N}}$. We write this conditional limit as $N \xrightarrow{\tau} \infty$.

We have the following result.

**Proposition 3** Let $(u, v) \in \mathbb{R}^2_+$ and $2/3 < \tau < 2$. Then (1.24) holds.

We expect that the statement will be extended for $\tau \in (0, 2/3]$, but we need further improvement of Lemmas 1 and 2 to prove it.

Proposition 3 means the convergence of integral operators

$$\hat{K}_N f(\cdot) = \int_{\mathbb{R}_+} dv q^{-[\tau N] - \chi(\lfloor \tau N \rfloor)} K_N \left( q^{-[\tau N] - \chi(\lfloor \tau N \rfloor)} x - q^{-[\tau N] - \chi(\lfloor \tau N \rfloor)} y \right) f(v)$$

$$\to \hat{K}^\Theta f(\cdot) = \int_{\mathbb{R}_+} dv K^\Theta(\cdot, v) f(v), \quad \text{in } N \xrightarrow{\tau} \infty,$$

(3.2)

$f \in C_0(\mathbb{R}_+)$. The convergence of integral operators $\hat{K}_N \to \hat{K}^\Theta$ implies that of Fredholm determinants to (1.30) for $g \in C_0(\mathbb{R}_+)$. Since the Fredholm determinants are identified with the moment generating functions in determinantal point process, we can conclude the following.

**Theorem 4** Let $2/3 < \tau < 2$. The determinantal point process $(\Xi, P_N)$ converges to the determinantal point process in $N \xrightarrow{\tau} \infty$, whose correlation kernel is given by the Jacobi-theta kernel $K^\Theta$ defined by (1.25) and (1.26). In other words, for any $N' \in \mathbb{N}$,

$$\rho^{(N')}_{\mathcal{N}} \left( q^{-[\tau N] - \chi(\lfloor \tau N \rfloor)} x^{(N')} \right) \to \rho^{(N')} \left( x^{(N')} \right) = \frac{\det_{1 \leq j, k \leq N'} \left[ K^\Theta \left( x_j^{(N')}, x_k^{(N')} \right) \right]}{\det_{1 \leq j \leq N'} \left( q^{-[\tau N] - \chi(\lfloor \tau N \rfloor)} x_j^{(N')} \right)^{N'}}$$

(3.3)

where $q^{-[\tau N] - \chi(\lfloor \tau N \rfloor)} x^{(N')} \equiv \left( q^{-[\tau N] - \chi(\lfloor \tau N \rfloor)} x_j^{(N')} \right)_{j=1}^{N'}$.

3.2 Mapping to the matrix model

We obtained an infinite-particle system on $\mathbb{R}_+$ in the previous subsection. Here, we explain that the particle system is then mapped to an infinite-particle system on $\mathbb{R}$, which will be regarded as a stationary realization of the oscillatory matrix model considered in [4].
First, we remind that the $N$-particle systems $(\Xi, \overline{P}_N)$ and $(\Xi, P_N)$ were related by the mapping (1.9). On the other hand, when we take the $N \to \infty$ limit in Theorem 4, we performed the scaling of variables as

$$h_{N,\tau} : x \in \mathbb{R}_+ \mapsto u \in \mathbb{R}_+,$$ 

where $q = e^{-g_s}$. Then, the combination of the mappings (1.9) and (3.4) gives

$$h_{N,\tau} \circ f_N : \phi \in \mathbb{R} \mapsto u \in \mathbb{R}_+,$$

$$u = h_{N,\tau}(\phi) = e^{\phi + g_s(N - \lfloor \tau N \rfloor)}(N - \lfloor \tau N \rfloor).$$

This suggests that, only in the case of $\tau = 1$, the $N$-dependent factor $N - \lfloor \tau N \rfloor$ vanishes. (The value of the factor $\chi \left( \lfloor \tau N \rfloor \right)$ is fixed to be 0 in the series $(N_j^0)_{j \in \mathbb{N}}$ and 1 in $(N_j^1)_{j \in \mathbb{N}}$ by the definition of $\chi$, (1.23), and of $N \to \infty$.) Hence, in the case $\tau = 1$, if we take an infinite-particle limit of the scaled version of $(\Xi, P_N)$ by (3.3), we can obtain an infinite-dimensional model in Chern-Simons theory by simply putting $u = e^{\phi + g_s \chi(N)}$.

Let

$$K_\infty(\phi, \varphi) = e^{(\phi + \varphi)/2}K^\Theta(e^\phi, e^{\varphi}) = e^{-g_s(\phi^2 + \varphi^2)/2g_s}/(e^{-g_s}; e^{-g_s})_\infty \times \Theta(-e^{\phi + g_s^2/2}e^{-g_s}) \Theta(-e^{\varphi + g_s^2}e^{-g_s}) \Theta(-e^{\phi + g_s^2/2}e^{-g_s}) \Theta(-e^{\varphi + g_s^2/2}e^{-g_s}) \frac{2 \sinh \frac{\phi - \varphi}{2}}{2},$$

$(\phi, \varphi) \in \mathbb{R}^2$, $\phi \neq \varphi$, (3.6)

and

$$K_\infty(\phi, \phi) = e^{\phi}K^\Theta(e^{\phi}, e^{\phi}) = e^{-g_s(\phi^2)/2g_s}/(e^{-g_s}; e^{-g_s})_\infty \times \left\{ e^{g_s}\Theta(-e^{\phi + g_s^2/2}e^{-g_s}) \Theta(-e^{\phi + g_s^2/2}e^{-g_s}) \Theta(-e^{g_s^2}e^{-g_s}) \Theta(-e^{g_s^2}e^{-g_s}) \right\},$$

$\phi \in \mathbb{R}$. (3.7)

Then, for $(\Xi, \overline{P}_N)$ with correlation kernel (1.22), we have the following corollary from Proposition 3 and Theorem 4.

**Corollary 5** Set $\tau = 1$. Then

$$K_N(\phi + g_s \chi(N), \varphi + g_s \chi(N)) \to K_\infty(\phi, \varphi), \quad \text{as } N \stackrel{\tau \to 1}{\longrightarrow} \infty.$$ (3.8)

Then, the system $(\Xi, \overline{P}_N)$ converges to an infinite-dimensional determinantal point process on $\mathbb{R}$ with the correlation kernel $K_\infty$ in $N \stackrel{\tau \to 1}{\longrightarrow} \infty$. The correlation functions are given by

$$\tilde{\rho}^{(N')} \left( \phi^{(N')} \right) = \det_{1 \leq j, k \leq N'} K_\infty \left( \phi^{(N')}_j, \phi^{(N')}_k \right), \quad N' \in \mathbb{N},$$ (3.9)

for the limit system, where $\phi^{(N')}$ denotes $(\phi^{(N')}_1, \phi^{(N')}_2, \ldots, \phi^{(N')}_{N'}) \in \mathbb{R}^{N'}$. 

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When \( \tau = 1 \), the condition (3.1) is always satisfied; \( [N_j - 1] = [N_j] - 1 \). Then the limit \( N \to \infty \) is just a conditional limit such that we consider the limit in the even numbers \( N_{j}^{(0)} = 2j \) and in the odd numbers \( N_{j}^{(1)} = 2j + 1, j \in \mathbb{N} \), separately.

We note that in the infinite-particle limit \( N \to \infty \) in Corollary 5, we do not need any scaling, while we do in the bulk scaling limit of the GUE-determinantal point process [17,6]. The situation seems to be quite similar to the \( N \to \infty \) limit of the Ginibre determinantal point process on \( \mathbb{C} \approx \mathbb{R}^2 \) [7].

Other expressions of \( K_{\infty} \) by using Jacobi’s theta functions \( \vartheta_j \), \( 1 \leq j \leq 4 \) [30] and Gosper’s \( q \)-trigonometric functions [8] are given in Appendix A.

In the present paper, we consider the determinantal point processes on \( \mathbb{R} \) and \( \mathbb{R}_+ \), which are mapped by (1.9) to each other. It will be an interesting future problem to study the matrix model and the associated point process on the unit circle, which will be transformed from the present systems as discussed in [18,19,21,23].

4 Oscillatory matrix model in Chern-Simons theory

4.1 Oscillatory behavior

From the pseudo-periodicity of Jacobi’s theta function (2.13) we have

\[
\Theta(-e^{(\phi+2gs)\pm gs/2}|e^{-gs}) = -e^{gs}e^{(\phi\mp gs/2)}\Theta(-e^{\phi\pm gs/2}|e^{-gs})
\]

(4.1)

for theta functions used to express \( K_{\infty} \) in (3.6) and (3.7). Then we can readily prove the periodicity of the correlation kernel \( K_{\infty} \),

\[
K_{\infty}(\phi + 2ng_s, \varphi + 2ng_s) = K_{\infty}(\phi, \varphi), \quad (\phi, \varphi) \in \mathbb{R}^2, n \in \mathbb{Z}.
\]

(4.2)

Note that the Jacobi-theta kernel \( K^{\Theta} \) given by (1.25) and (1.26) does not have such periodicity and only has the quasi-periodicity,

\[
q^{2n}K^{\Theta}(q^{2n}u, q^{2n}v) = K^{\Theta}(u, v), \quad (u, v) \in \mathbb{R}_+^2, 0 < q < 1, n \in \mathbb{Z}.
\]

(4.3)

Then all correlation functions of the infinite-particle system on \( \mathbb{R} \) obtained in Corollary 5 have oscillatory behavior caused by (4.2). In order to demonstrate it, we consider the case \( N' = 1 \) here. In this case, (1.18) gives the density of the number of particles

\[
\tilde{\rho}(\phi) = \tilde{\rho}^{(1)}_{\infty}(\phi) = K_{\infty}(\phi, \phi), \quad \phi \in \mathbb{R},
\]

(4.4)

which is given by (3.7). We can prove

\[
\tilde{\rho}(\phi + 2ng_s) = \tilde{\rho}(\phi), \quad \phi \in \mathbb{R}, n \in \mathbb{Z}.
\]

(4.5)

Figures 1-3 show the density profiles (4.4) for \( g_s = 1, 5, \) and 25, respectively. We can see that as the period \( 2g_s \) increases, both of the mean value of \( \tilde{\rho}(\phi) \) and the amplitude of oscillation decrease. The mean value of \( \tilde{\rho}(\phi) \) can be estimated by a ‘mean-field’ analysis. Mariño computed a density profile of the system \((\Xi, P_N)\) by this approximation [14]. (See also [2,13].) It gives the density profile of the system \((\Xi, P_N)\) as

\[
\tilde{\rho}^{mf}_{N}(\phi) = \frac{1}{\pi g_s} \tan^{-1} \left[ \sqrt{\frac{e^{Ng_s} - (\cosh(\phi/2))}{\cosh(\phi/2)}} \right].
\]

(4.6)
Since \( \lim_{x \to \infty} \tan^{-1} x = \pi/2 \), in the limit \( N \to \infty \) (4.6) becomes

\[
\tilde{\rho}_{mf}(\phi) = \frac{1}{2g_s}.
\] (4.7)

It is the mean value of the present rigorous result \( \tilde{\rho}(\phi) \). As shown in Figure 3, when \( g_s \) is much large the density profile seems to be an equidistant set of peaks, that is, a lattice structure appears. In the vicinity of the origin,

\[
\tilde{\rho}(\phi) \sim \tilde{\rho}(0) \cosh \phi, \quad |\phi| \ll g_s,
\] (4.8)

with

\[
\tilde{\rho}(0) \equiv \frac{2e^{-g_s/2}}{\sqrt{2\pi g_s}} \to 0, \quad \text{as } g_s \to \infty.
\] (4.9)

In [4], de Haro and Tierz reported a numerical observation of \( N \)-dependence of the density profile \( \tilde{\rho}_N^{(1)}(\phi) \) for \((\tilde{\Xi}, \tilde{\Pi}_N)\), in which the relation (1.4) between \( g_s \) and \( N \) was not imposed and \( N \) and \( q = e^{-g_s} \in (0,1) \) were treated as free parameters. They demonstrated that for any finite \( N \) with \( 0 < q < 1 \), oscillatory behaviors are observed. They found that, if \( q \) is fixed to be less than 1, the oscillatory behavior remains even in setting \( N \) large. As claimed by them, however, the relation (1.4) implies that \( N \to \infty \) limit gives \( g_s \to 0 \) and \( q = e^{-g_s} \to 1 \), and hence the remarkable oscillatory behavior will be smoothed out in \( N \to \infty \). In the present paper, however, we take the \( N \to \infty \) limit with keeping \( 0 < g_s < \infty \) i.e. \( 0 < q < 1 \). The obtained determinantal point process with the correlation kernel \( K_\infty \) is a stationary realization of the oscillatory matrix model of de Haro and Tierz which is constructed uniformly on \( \mathbb{R} \). The present limit \( N \to \infty \) with fixed \( 0 < q = e^{-g_s} < 1 \) has not been able to be rigorously studied so far because of the lack of suitable \( q \)-analogous formulas of the Plancherel-Rotach asymptotics for the Stieltjes-Wigert polynomials as mentioned in [4]. This problem was solved by Lemmas 1 and 2 in the present paper. See also [29] and [12] for other study on the asymptotics of the Stieltjes-Wigert polynomials.

![Figure 1](image.png)

Figure 1: The density profile \( \tilde{\rho}(\phi) \) with \( g_s = 1 \). The period is \( 2g_s = 2 \) and the mean value is \( 1/2g_s = 0.5 \).
Figure 2: The density profile $\tilde{\rho}(\phi)$ with $g_s = 5$. The period is $2g_s = 10$ and the mean value is $1/2g_s = 0.1$.

Figure 3: The density profile $\tilde{\rho}(\phi)$ with $g_s = 25$. The period is $2g_s = 50$ and the mean value is $1/2g_s = 0.02$. Equidistant peaks imply a lattice structure on $\mathbb{R}$.
4.2 Reduction to the sine-kernel

For consistency with the consideration and observation by de Haro and Tierz \[4\], the further limit $g_s \to 0$ ($q \to 1$) after $N \to \infty$ should reduce our oscillatory model to be the classical one-matrix model. That is, the determinantal point process with the correlation kernel $K_\infty$ should converge to that with the sine-kernel. We consider the sine-kernel with density 1,

$$K_{\text{sin}}(\phi, \varphi) = \begin{cases} \frac{\sin(\pi(\phi - \varphi))}{\pi(\phi - \varphi)}, & \phi, \varphi \in \mathbb{R}, \ \phi \neq \varphi, \\ 1, & \phi = \varphi \in \mathbb{R}. \end{cases}$$ (4.10)

The following proposition ensures the fact.

**Proposition 6** We have

$$\lim_{g_s \to 0} 2g_s K_\infty(2g_s \phi, 2g_s \varphi) = K_{\text{sin}}(\phi, \varphi).$$ (4.11)

The proof of Proposition 6 is owed to the following asymptotic expansion for the $q$-exponential function, which was given as equation (3.13) in \[3\].

**Lemma 7** \[3\] For $|\arg z| \leq 2\pi$,

$$\frac{1}{e_q(z)} = \frac{2\sin(\pi z/\ln q)}{(z^{-1}; q)_\infty} \exp \left[ \frac{1}{2} \ln z - \frac{1}{\ln q} \left\{ -\frac{\pi^2}{3} + \frac{1}{2} (\ln z)^2 \right\} \frac{1}{12} \ln q \\
+ \sum_{k=1}^{\infty} \frac{\cos(2\pi k \ln z/\ln q) \exp(2\pi^2 k/\ln q)}{k \sinh(2\pi^2 k/\ln q)} \right]. \quad (4.12)$$

As pointed out in \[3\], since $1/e_q(z) = (z; q)_\infty$, Lemma 7 gives an expansion formula for the function $(z; q)_\infty(z^{-1}; q)_\infty$. Hence by Jacobi’s triple product identity \[212\] for $\Theta$, we have

$$\Theta(-z|q) = (q^2; q^2)_\infty 2 \cos \left( \frac{\pi z}{2 \ln q} \right) \exp \left[ -\frac{1}{2} \ln q \left\{ -\frac{\pi^2}{3} + \frac{1}{2} (\ln z)^2 \right\} \frac{1}{12} \ln q \\
+ \sum_{k=1}^{\infty} \frac{\cos(\pi k \ln z/\ln q + \pi k) \exp(\pi^2 k/\ln q)}{k \sinh(\pi^2 k/\ln q)} \right], \quad \text{for } |\arg z| \leq 2\pi. \quad (4.13)$$

Then, for the theta function used to express the kernel $K_\infty$ in (3.6) and (3.7), we obtain

$$\Theta(-e^{i\phi}; e^{-gs}) = (e^{-2gs}, e^{-2gs})_\infty 2 \cos \left( \frac{\pi \phi}{2gs} \pm \frac{\pi}{4} \right)$$

$$\times \exp \left[ -\frac{\pi^2}{6gs} + \frac{\phi^2}{4gs} \pm \frac{\phi}{4gs} + \frac{1}{48} \sum_{k=1}^{\infty} \frac{\cos(\pi k \phi/gs \pm \pi k/2 - \pi k) \exp(-\pi^2 k/gs)}{k \sinh(-\pi^2 k/gs)} \right], \quad \phi \in \mathbb{R}. \quad (4.14)$$

Here, we note the following asymptotic expansion for the $q$-Pochhammer symbol (Theorem 2 in \[16\]). For $q = e^{-\epsilon}$, $\epsilon > 0$,

$$(q; q)_\infty = \sqrt{\frac{2\pi}{\epsilon}} e^{-\pi^2/6\epsilon} (1 + O(\epsilon)), \quad \text{as } \epsilon \to 0. \quad (4.15)$$
Proof of Proposition 6. Asymptotic expansion of $2g_s K_\infty(2g_s\phi, 2g_s\varphi)$ as $g_s \to 0$ is obtained by (4.14) and (4.15). The last term in the exponential in (4.14) given by an infinite sum are irrelevant to other terms in $g_s \to 0$. The contribution to $2g_s K_\infty(2g_s\phi, 2g_s\varphi)$ which comes from $\exp(-\pi^2/6g_s)$ in (4.14) is completely canceled by that from $q$-Pochhammer symbols in (3.6), (3.7), and (4.14) by the formula (4.15). Therefore, in the limit $g_s \to 0$, only the factor $\cos(\pi\phi/2g_s \pm \pi/4)$ in (4.14) is relevant, and the proof is completed.

In order to demonstrate the difference between the kernel $K_\infty(\phi, \varphi)$ of the oscillatory model and $K_{\sin}(\phi, \varphi)$ of the classical model, we plotted them as functions of $(\phi, \varphi)$ in Figures 3 and 4.

![Figure 4: The correlation kernel $K_\infty(\phi, \varphi)$ of the oscillatory matrix model as a function of $(\phi, \varphi)$. The case with $g_s = 1$ ($q = e^{-g_s} \simeq 0.37$) is plotted. The oscillatory behaviors in the diagonal directions show breakdown of the translational invariance.](image-url)
Figure 5: The sine-kernel $K_{\sin}(\phi, \varphi)$ is plotted as a function of $(\phi, \varphi)$. The value depends only on $\phi - \varphi$ and the translational invariance of the system is shown.

5 Proofs of Lemma 1 and Proposition 3

Assume that $0 < q < 1$. The following estimate is obtained from Lemma 3.1 in [9]. Let

$$
\frac{(q; q)_{\infty}}{(q; q)_n} = 1 + \tilde{R}(q; n).
$$

(5.1)

Then

$$
|\tilde{R}(q; n)| \leq \frac{(-q^3; q)_{\infty}}{1 - q} q^{n+1}.
$$

(5.2)

The following lemma improves this estimate.

Lemma 8 Let

$$
\frac{(q; q)_{\infty}}{(q; q)_n} = 1 - \frac{q^{n+1}}{1 - q} + R(q; n).
$$

(5.3)

Then

$$
|R(q; n)| < \frac{(-q; q)_{\infty}}{(1 - q)(1 - q^3)} q^{2n+2}.
$$

(5.4)

Proof From the definition of the $q$-Pochhammer symbol (2.4), we have

$$
\frac{(q; q)_{\infty}}{(q; q)_n} = (q^{n+1}; q)_{\infty} = 1 - \frac{q^{n+1}}{1 - q} + \sum_{k=2}^{\infty} \frac{q^{k(k-1)/2}}{(q; q)_k} (-1)^k q^{k(n+1)}.
$$

(5.5)
Then, by (5.3),

\[
R(q; n) = \sum_{k=2}^{\infty} \frac{q^{k(k-1)/2}}{(q; q)_k} (-1)^k q^{k(n+1)} = q^{2(n+1)} \sum_{k=0}^{\infty} \frac{q^{(k+2)(k+1)/2}}{(q; q)_{k+2}} (-1)^k q^{kn+1}.
\]

(5.6)

By taking the absolute value of (5.6), we have

\[
|R(q; n)| < q^{2(n+1)} \sum_{k=0}^{\infty} \frac{q^{(k+2)(k+1)/2}}{(q; q)_{k+2}} q^{k(n+1)} = q^{2n+2} \sum_{k=0}^{\infty} q^{k(k-1)/2} (q; q)_k q^k \frac{q^{2k+1+kn}}{(1-q^{k+1})(1-q^{k+2})}.
\]

(5.7)

Since for any \(n, k \in \mathbb{N}_0\), \(q^{2k+1+kn} < 1\) and \(1/(1-q^{k+1})(1-q^{k+2}) \leq 1/(1-q)(1-q^2)\), we have

\[
|R(q; n)| < \frac{q^{2n+2}}{(1-q)(1-q^2)} \sum_{k=0}^{\infty} q^{k(k-1)/2} (q; q)_k q^k = \frac{(-q; q)_\infty}{(1-q)(1-q^2)} q^{2n+2},
\]

(5.8)

which proves (5.4).

Let

\[
S_n(x; q) = \sum_{k=0}^{n} \frac{q^{k^2}(-x)^k}{(q; q)_k(q; q)_{n-k}} = q^n (-x)^n \sum_{k=0}^{n} \frac{q^{k^2-2kn}(-x)^{-k}}{(q; q)_k(q; q)_{n-k}}, \quad n \in \mathbb{N}_0.
\]

(5.9)

The orthonormal Stieltjes-Wigert polynomials (2.5) are then given by

\[
p_n(x; q) = (-1)^n q^{n/2+1/4} \sqrt{(q; q)_n} S_n(q^{1/2} x; q), \quad n \in \mathbb{N}_0.
\]

(5.10)

### 5.1 Proof of Lemma 1

We set \(x = q^{-n\tau} u, \quad u > 0, \quad \tau \in (0, 2)\) and assume (1.23), (2.15), and (2.16). By definitions (2.15) and (2.16), \(m + \lambda = (2 - \tau)n\), and we have

\[
S_n \left( q^{-n\tau} u; q \right) = \frac{(-u)n^2(1-\tau)}{(q; q)_{2\infty}} \sum_{k=0}^{n} \frac{(q; q)_k^2 q^{k^2}}{(q; q)_k(q; q)_{n-k}} \left( -q^{-m-\lambda} u^{-1} \right)^k.
\]

(5.11)

Here we split the sum \(S_n \left( q^{-n\tau} u; q \right)\) into two parts as follows,

\[
S_n \left( q^{-n\tau} u; q \right) = S_n^{(1)} + S_n^{(2)},
\]

(5.12)

\[
S_n^{(1)} = \frac{(-u)n^2(1-\tau)}{(q; q)_{2\infty}} \sum_{k=0}^{[m/2]} \frac{(q; q)_k^2 q^{k^2}}{(q; q)_k(q; q)_{n-k}} \left( -q^{-m-\lambda} u^{-1} \right)^k,
\]

(5.13)

\[
S_n^{(2)} = \frac{(-u)n^2(1-\tau)}{(q; q)_{2\infty}} \sum_{k=[m/2]+1}^{n} \frac{(q; q)_k^2 q^{k^2}}{(q; q)_k(q; q)_{n-k}} \left( -q^{-m-\lambda} u^{-1} \right)^k.
\]

(5.14)
We rewrite \( S_n^{(1)} \) as
\[
S_n^{(1)} = \frac{q^{n^2(1-r)-[m/2](\lfloor m/2 \rfloor + \chi(m) + \lambda)}}{(-u)^{[m/2]-n}(q; q)_{\infty}^{2}} \sum_{k=0}^{[m/2]} q^{k^2} \left( -q^{\lambda+\chi(m)} u \right)^k \frac{(q; q)_{\infty}}{(q; q)_{[m/2]-k}} \frac{(q; q)_{n-[m/2]+k}}{(q; q)_{n-[m/2]+k}}. \tag{5.15}
\]

Applying Lemma 8 yields nine terms as follows,
\[
S_n^{(1)} = \frac{q^{n^2(1-r)-[m/2](\lfloor m/2 \rfloor + \chi(m) + \lambda)}}{(-u)^{[m/2]-n}(q; q)_{\infty}^{2}} \sum_{k=0}^{[m/2]} q^{k^2} \left( -q^{\lambda+\chi(m)} u \right)^k \left( 1 - q^{n-[m/2]+k+1} \right) - \frac{q^{[m/2]-k+1}}{1-q} \frac{(q; q)_{n-[m/2]+k}}{(q; q)_{n-[m/2]+k}} R(q; n - \lfloor m/2 \rfloor + k)
\]
\[
+ q^{n+2(1-q^2)} + R(q; n - \lfloor m/2 \rfloor + k) - \frac{q^{[m/2]-k+1}}{1-q} R(q; n - \lfloor m/2 \rfloor + k)
\]
\[
+ R(q; \lfloor m/2 \rfloor - k) - \frac{q^{n-[m/2]+k+1}}{1-q} R(q; \lfloor m/2 \rfloor - k)
\]
\[
+ R(q; \lfloor m/2 \rfloor - k) R(q; n - \lfloor m/2 \rfloor + k) \right). \tag{5.16}
\]

Then we write \( S_n^{(1)} \) as
\[
S_n^{(1)} = \frac{q^{n^2(1-r)-[m/2](\lfloor m/2 \rfloor + \chi(m) + \lambda)}}{(-u)^{[m/2]-n}(q; q)_{\infty}^{2}} \sum_{k=0}^{\infty} q^{k^2} \left( -q^{\lambda+\chi(m)} u \right)^k \frac{q^{1+n-[m/2]}}{1-q} \sum_{k=0}^{\infty} q^{k^2+k} \left( -q^{\lambda+\chi(m)} u \right)^k
\]
\[
- \frac{q^{1+[m/2]}}{1-q} \sum_{k=0}^{\infty} q^{k^2-k} \left( -q^{\lambda+\chi(m)} u \right)^k + r_1(n) \right), \tag{5.17}
\]

where \( r_1(n) \) consists of nine terms shown explicitly as \( B.1 \) in Appendix B. Similarly, we rewrite \( S_n^{(2)} \) as
\[
S_n^{(2)} = \frac{q^{n^2(1-r)-[m/2](\lfloor m/2 \rfloor + \chi(m) + \lambda)}}{(-u)^{[m/2]-n}(q; q)_{\infty}^{2}} \sum_{k=1}^{n-[m/2]} \frac{(q; q)_{\infty}^2 q^{k^2} \left( -q^{\lambda+\chi(m)} u \right)^{-k}}{(q; q)_{[m/2]+k}(q; q)_{n-[m/2]-k}}, \tag{5.18}
\]

and through the Lemma 8, we write
\[
S_n^{(2)} = \frac{q^{n^2(1-r)-[m/2](\lfloor m/2 \rfloor + \chi(m) + \lambda)}}{(-u)^{[m/2]-n}(q; q)_{\infty}^{2}} \sum_{k=1}^{\infty} q^{k^2} \left( -q^{\lambda+\chi(m)} u \right)^{-k} \frac{d^{1+n-[m/2]}}{1-q} \sum_{k=1}^{\infty} q^{k^2-k} \left( -q^{\lambda+\chi(m)} u \right)^{-k}
\]
\[
- \frac{q^{1+[m/2]}}{1-q} \sum_{k=1}^{\infty} q^{k^2+k} \left( -q^{\lambda+\chi(m)} u \right)^{-k} + r_2(n) \right), \tag{5.19}
\]
where \( r_2(n) \) consists of nine terms shown explicitly as (5.22) in Appendix B. Thus we have

\[
S_n^{(1)} + S_n^{(2)} = \frac{q^{n^2(1-\tau)-[m/2]}([m/2]+\chi(m)+\lambda)}{(-u)^{[m/2]-n}(q;q)_\infty^2} \sum_{k=-\infty}^{\infty} q^{k^2} (-q^\lambda+\chi(m)u)^k - \frac{q^{1+n-[m/2]}}{1-q} \sum_{k=-\infty}^{\infty} q^{k^2+k} (-q^\lambda+\chi(m)u)^k
\]

\[
\frac{q^{1+[m/2]}}{1-q} \sum_{k=-\infty}^{\infty} q^{k^2-k} (-q^\lambda+\chi(m)u)^k + r_1(n) + r_2(n). \tag{5.20}
\]

Then infinite sums in the parenthesis in (5.20) can be expressed by using the theta functions (1.27). Therefore, by (5.12), we obtain

\[
S_n(q^{-\tau n}u; q) = \frac{q^{n^2(1-\tau)-[m/2]}([m/2]+\chi(m)+\lambda)}{(-u)^{[m/2]-n}(q;q)_\infty^2} \left\{ \Theta \left( -q^\lambda+\chi(m)u \right) \right\}
\]

\[
- \frac{q^{1+n-[m/2]}}{1-q} \Theta \left( -q^\lambda+\chi(m)+1u \right) - \frac{q^{1+[m/2]}}{1-q} \Theta \left( -q^\lambda+\chi(m)-1u \right) + r_1(n) + r_2(n). \tag{5.21}
\]

It is shown in Appendix B that the terms \( r_1(n) + r_2(n) \) can be evaluated as

\[
r_1(n) + r_2(n) = O \left( q^{\tau n+2(1-\tau)n1_{[1,2]}(\tau)} \right). \tag{5.22}
\]

Since

\[
\frac{\tau n}{2} \leq n - [m/2] < \frac{\tau n}{2} + 1 \tag{5.23}
\]

and

\[
\frac{(2-\tau)n}{2} - 1 < [m/2] \leq \frac{(2-\tau)n}{2}, \tag{5.24}
\]

we have to care about the order of \( \tau n/2, (2-\tau)n/2, \tau n, \) and \( (2-\tau)n \). We can readily see that in the case of \( 2/3 < \tau < 4/3 \), equation (5.21) holds with (5.22) as it stands. In the case of \( 0 < \tau \leq 2/3 \), the last two relevant terms in the parenthesis in (5.21) can be replaced by \( -\frac{q^{1+[m/2]}}{1-q} \Theta \left( -q^\lambda+\chi(m)+1u \right) \) and the estimate \( O(q^{\tau n+2(1-\tau)n1_{[1,2]}(\tau)}) \) for the irrelevant terms by \( O(q^\tau) \). In the case of \( 4/3 \leq \tau < 2 \), the former can be replaced by \( -\frac{q^{1+[m/2]}}{1-q} \Theta \left( -q^\lambda+\chi(m)-1u \right) \) and the latter \( O(q^{\tau n+2(1-\tau)n1_{[1,2]}(\tau)}) \) by \( O(q^{(2-\tau)n}) \). Hence, for the orthonormal Stieltjes-Wigert polynomials (5.10), we have

\[
p_n(q^{-\tau n}u; q) = (-1)^n q^{n^2/2+1/4+n(1-\tau)-[m/2]+\chi(m)+\lambda} \sqrt{(q;q)_n} \left( q^{-\tau n} q^{1/2}u \right) q^{n^2/2+1/4+n^2(1-\tau)-[m/2]+\chi(m)+\lambda} \sqrt{(q;q)_n}
\]

\[
\times \left\{ \Theta \left( -q^\lambda+\chi(m)+1/2u \right) - \frac{q^{1+n-[m/2]}}{1-q} \Theta \left( -q^\lambda+\chi(m)+3/2u \right) 1_{(0,1/3)}(\tau) \right\}
\]

\[
- \frac{q^{1+[m/2]}}{1-q} \Theta \left( -q^\lambda+\chi(m)-1/2u \right) 1_{(2/3,2)}(\tau) + O \left( q^{\tau n+2(1-\tau)n1_{[1,2]}(\tau)} \right) \tag{5.25}
\]

Further, by noting the functional equation (2.13) and \( [m/2] = m/2 - \chi(m)/2 \), (5.21) is obtained. It completes the proof. \( \square \)
5.2 Proof of Proposition 3

By Lemma 2 and the definition of $N \to \infty$, we have

$$
p_{N-1} \left( q^{-[\tau N]} - \chi([\tau N]) u; q \right) \sqrt{w \left( q^{-[\tau N]} - \chi([\tau N]) u; q \right)} = \frac{\sqrt{(q; q)_{N-1} \left( -1 \right)^{N-1} q^{N/2 - 1/4 + [\tau N]/4 + \chi([\tau N])/4}}}{q} \sqrt{w(v; q)} \left\{ \Theta \left( -q^{1/2} u \right) q \right\} 
$$

\[ + q^{N-\tau N}/2 - \chi([\tau N])/2 \left( q^{1/2 - N + [\tau N] + \chi([\tau N])} v \right) 1_{(0, 4/3)}(\tau) - 1_{(2/3, 2)}(\tau) \left\{ \Theta \left( -q^{-1/2} v \right) q \right\} + \mathcal{O} \left( q^{\tau N + 2(1-\tau)N_{1, 1}(\tau)} \right), \] as $N \to \infty$. (5.26)

In the parenthesis in (5.26), there are two terms expressed by using $\Theta$, in which the first term is the leading term and the second one includes indicators $1_{(0, 4/3)}(\tau)$ and $1_{(2/3, 2)}(\tau)$, and an irrelevant term of order $\mathcal{O}(q^{\tau N + 2(1-\tau)N_{1, 1}(\tau)} \tau)$. We put (5.26) and the similar estimate of $p_N \left( q^{-[\tau N]} - \chi([\tau N]) u; q \right)$ into the product

$$
p_N \left( q^{-[\tau N]} - \chi([\tau N]) u; q \right) \sqrt{w \left( q^{-[\tau N]} - \chi([\tau N]) u; q \right)} \times p_{N-1} \left( q^{-[\tau N]} - \chi([\tau N]) u; q \right) \sqrt{w \left( q^{-[\tau N]} - \chi([\tau N]) u; q \right)}. \tag{5.27}
$$

We find the product of the leading terms expressed by $\Theta$ is symmetric in exchanging $u$ and $v$, and the product of the second terms expressed by $\Theta$ with indicators becomes irrelevant since it is in order $q^{\tau N + 2(1-\tau)N_{1, 1}(\tau)}$. Then, when we consider the Christoffel-Darboux kernel (1.19), the leading terms are canceled out and the cross terms of the first terms with $\Theta$ and the second terms with $\Theta$ and indicators become relevant. Moreover, we find that the cross terms which possess the indicator $1_{(0, 4/3)}(\tau)$ are also completely canceled. This is the reason why we assume $2/3 < \tau < 2$ in this proposition. Therefore, if we take the $N \to \infty$ limit, we obtain the Jacobi-theta kernel. Then the proof is completed.

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Appendix A: Expressions of $\mathcal{K}_\infty$ using Jacobi’s theta functions and Gosper’s $q$-trigonometric functions

Let $z = e^{2i\zeta}$, $\zeta \in \mathbb{C}$, and $q = e^{\pi i \omega}$, $\Im \omega > 0$. The theta function defined by (1.27) is written by Jacobi’s theta function $\vartheta_3$ [30] as

$$
\Theta(z|q) = \vartheta_3(\zeta|\omega) = \sum_{k=-\infty}^{\infty} e^{2(k+1)^2 \pi i \omega + 2ki \zeta}. \tag{A.1}
$$
Theta functions used to express $K_\infty$ in (3.6) are written by

$$
\Theta(-e^{\phi \pm g_s/2} e^{-g_s}) = \sum_{k=-\infty}^{\infty} (-1)^k e^{-k^2g_s} e^{k(\phi \pm g_s/2)} = \vartheta_4 \left( \frac{1}{2i} \left( \phi \pm \frac{g_s}{2} \right) \right), \quad \phi \in \mathbb{R},
$$

where $\vartheta_4$ is defined by

$$
\vartheta_4(\zeta|\omega) = \sum_{k=-\infty}^{\infty} (-1)^k e^{k^2 \pi i \omega + 2k i \zeta}.
$$

The quasi-periodicity (4.1) comes from the equality

$$
\vartheta_4(\zeta + \omega \pi|\omega) = -e^{-i \pi \omega - 2i \zeta} \vartheta_4(\zeta|\omega).
$$

Jacobi’s imaginary transformation [30] for $\vartheta_4$ reads

$$
\vartheta_4(\zeta|\omega) = \frac{1}{\sqrt{-i \omega}} e^{\zeta^2 / \pi i \omega} \vartheta_2 \left( \frac{\zeta}{\omega} \right) - \frac{1}{\omega},
$$

where $\vartheta_2$ is defined by

$$
\vartheta_2(\zeta|\omega) = e^{i \zeta + \pi i \omega / 4} \vartheta_4(\zeta + \pi / 2 + \pi \omega / 2|\omega) = \sum_{k=-\infty}^{\infty} e^{(k+1/2)^2 \pi i \omega + (2k+1)i \zeta}.
$$

Theta functions $\vartheta_2$, $\vartheta_3$, and $\vartheta_4$ are all even functions of $\zeta$.

Gosper’s $q$-trigonometric functions are defined as [8]

$$
\sin_q(\pi z) = q^{(z-1/2)^2} (q^{-2z}; q^{-2})_\infty \left( q^{-2z+2}; q^{-2} \right)_\infty (q; q^{-2})^{-2},
$$

$$
\cos_q(\pi z) = \sin_q(\pi (z + 1/2)) = q^{z^2} (q^{2z+1}; q^2)_\infty \left( q^{-2z+1}; q^2 \right)_\infty (q; q^{-2})^{-2}.
$$

It is easy to find the following properties,

$$
\sin_q(-z) = -\sin_q(z), \quad \cos_q(-z) = \cos_q(z),
$$

$$
\sin_q(z + \pi) = -\sin_q(z), \quad \cos_q(z + \pi) = -\cos_q(z).
$$

By using the product form of $\vartheta_4$ (put $q = e^{\pi i \omega}$ and $z = -e^{2i \zeta}$ in (2.12)), we have the equality

$$
\vartheta_4(\zeta|\omega) = e^{\zeta^2 / \pi i \omega} (e^{\pi i \omega}; e^{\pi i \omega})_\infty (e^{\pi i \omega}; e^{2 \pi i \omega})_\infty \cos_{e^{\pi i \omega}} \left( \frac{\zeta}{\omega} \right).
$$

By (A.5) we also find

$$
\frac{1}{\sqrt{-i \omega}} \vartheta_2 \left( \frac{\zeta}{\omega} \right) - \frac{1}{\omega} = (e^{\pi i \omega}; e^{\pi i \omega})_\infty (e^{\pi i \omega}; e^{2 \pi i \omega})_\infty \cos_{e^{\pi i \omega}} \left( \frac{\zeta}{\omega} \right).
$$
Then $K_\infty$ is rewritten in terms of $\vartheta_4$, $\vartheta_2$, and Gosper’s $q$-trigonometric functions as follows,

$$K_\infty(\varphi, \varphi) = \frac{1}{\sqrt{2\pi g_s}} \frac{e^{-(\varphi^2+\varphi^2)/4g_s}}{(e^{-g_s}; e^{-g_s})_\infty^3} \vartheta_4\left(\frac{1}{2i} \left(\varphi - \frac{g_s}{2}\right) \frac{i g_s}{\pi}\right) \vartheta_4\left(\frac{1}{2i} \left(\varphi + \frac{g_s}{2}\right) \frac{i g_s}{\pi}\right) - \vartheta_4\left(\frac{1}{2i} \left(\varphi - \frac{g_s}{2}\right) \frac{i g_s}{\pi}\right) \vartheta_4\left(\frac{1}{2i} \left(\varphi + \frac{g_s}{2}\right) \frac{i g_s}{\pi}\right) \times 2\sinh \frac{\varphi - \varphi}{2}$$

$$= \frac{1}{g_s} \sqrt{\pi} \frac{e^{g_s/8}}{(e^{-g_s}; e^{-g_s})_\infty^3} \frac{e^{-(\varphi^2+\varphi^2)/4g_s}}{(e^{-g_s}; e^{-g_s})_\infty^3} \frac{e^{-(\varphi^2+\varphi^2)/4g_s}}{(e^{-g_s}; e^{-g_s})_\infty^3} \vartheta_2\left(\frac{\pi \varphi}{2g_s} + \frac{\pi}{4} \frac{i g_s}{g_s}\right) \vartheta_2\left(\frac{\pi \varphi}{2g_s} + \frac{\pi}{4} \frac{i g_s}{g_s}\right) - e^{-(\varphi^2+\varphi^2)/4g_s} \vartheta_2\left(\frac{\pi \varphi}{2g_s} + \frac{\pi}{4} \frac{i g_s}{g_s}\right) \vartheta_2\left(\frac{\pi \varphi}{2g_s} + \frac{\pi}{4} \frac{i g_s}{g_s}\right) \times 2\sinh \frac{\varphi - \varphi}{2}$$

$$(\varphi, \varphi) \in \mathbb{R}^2.$$

**Appendix B: Proof of (5.22)**

The last term in (5.17) is given by

$$r_1(n) = \sum_{j=1}^{9} r_{1j}(n), \quad (B.1)$$

where

$$r_{11}(n) = - \sum_{k=\lfloor m/2 \rfloor + 1}^{\infty} q^k \left(-q^{\lambda + \chi(m)}u\right)^k,$$

$$r_{12}(n) = q^{1+n-\lfloor m/2 \rfloor} \sum_{k=\lfloor m/2 \rfloor + 1}^{\infty} q^{k^2+k} \left(-q^{\lambda + \chi(m)}u\right)^k,$$

$$r_{13}(n) = q^{1+\lfloor m/2 \rfloor} \sum_{k=\lfloor m/2 \rfloor + 1}^{\infty} q^{k^2-k} \left(-q^{\lambda + \chi(m)}u\right)^k,$$

$$r_{14}(n) = \frac{q^{2+n}}{1-q} \sum_{k=0}^{\lfloor m/2 \rfloor + 1} q^k \left(-q^{\lambda + \chi(m)}u\right)^k,$$

and

$$r_{15}(n) = \sum_{k=0}^{\lfloor m/2 \rfloor} q^k \left(-q^{\lambda + \chi(m)}u\right)^k R(q; n - \lfloor m/2 \rfloor + k).$$
\[ r_{16}(n) = -\frac{q^{1+\lfloor m/2 \rfloor}}{1 - q} \sum_{k=0}^{\lfloor m/2 \rfloor} q^{k^2-k} \left(-q^{\lambda+\chi(m)}u\right)^k R(q; n - \lfloor m/2 \rfloor + k), \]
\[ r_{17}(n) = \sum_{k=0}^{\lfloor m/2 \rfloor} q^{k^2} \left(-q^{\lambda+\chi(m)}u\right)^k R(q; \lfloor m/2 \rfloor - k), \]
\[ r_{18}(n) = -\frac{q^{1+n-\lfloor m/2 \rfloor}}{1 - q} \sum_{k=0}^{\lfloor m/2 \rfloor} q^{k^2+k} \left(-q^{\lambda+\chi(m)}u\right)^k R(q; \lfloor m/2 \rfloor - k), \]
\[ r_{19}(n) = \sum_{k=0}^{\lfloor m/2 \rfloor} q^{k^2} \left(-q^{\lambda+\chi(m)}u\right)^k R(q; \lfloor m/2 \rfloor - k)R(q; n - \lfloor m/2 \rfloor + k). \]

Noting \( q \in (0, 1) \), we have the following inequalities,

\[ |r_{11}(n)| < \sum_{k=0}^{\infty} q^{k^2} \left(q^{\lambda+\chi(m)}u\right)^k \]
\[ = q^{[m/2]^2+2[m/2]} \left(q^{\lambda+\chi(m)}u\right)^{[m/2]+1} \sum_{k=0}^{\infty} q^{k^2+2k+1} q^{2[m/2]k} \left(q^{\lambda+\chi(m)}u\right)^k \]
\[ < q^{[m/2]^2+2[m/2]} \left(q^{\lambda+\chi(m)}u\right)^{[m/2]+1} \sum_{k=0}^{\infty} q^{k^2-2k} \left(q^{\lambda+\chi(m)}u\right)^k \]
\[ < \frac{q^{[m/2]^2+2[m/2]} \left(q^{\lambda+\chi(m)}u\right)^{[m/2]+1}}{1 - q} \sum_{k=0}^{\infty} q^{k^2-2k} \left(q^{\lambda+\chi(m)}u\right)^k, \]

and

\[ |r_{12}(n)| < \frac{q^{[m/2]^2+2[m/2]+1+n}}{1 - q} \left(q^{\lambda+\chi(m)}u\right)^{[m/2]+1} \sum_{k=0}^{\infty} q^{2k-2k} \left(q^{\lambda+\chi(m)}u\right)^k, \]
\[ |r_{13}(n)| < \frac{q^{[m/2]^2+2[m/2]+1}}{1 - q} \left(q^{\lambda+\chi(m)}u\right)^{[m/2]+1} \sum_{k=0}^{\infty} q^{k^2-2k} \left(q^{\lambda+\chi(m)}u\right)^k, \]
\[ |r_{14}(n)| < \frac{\left(-q; q\right)_{\infty}^2}{(1 - q)^2(1 - q^2)^2} q^n \sum_{k=0}^{\infty} q^{k^2-2k} \left(q^{\lambda+\chi(m)}u\right)^k. \]

For the others, applying Lemma 8 yields

\[ |r_{15}(n)| < \frac{\left(-q; q\right)_{\infty}^{2+n-\lfloor m/2 \rfloor}}{(1 - q)(1 - q^2)} \sum_{k=0}^{\infty} q^{k^2} \left(q^{\lambda+\chi(m)}u\right)^k \]
\[ < \frac{\left(-q; q\right)_{\infty}^2}{(1 - q^2)^2} q^{2+n-\lfloor m/2 \rfloor} \sum_{k=0}^{\infty} q^{k^2-2k} \left(q^{\lambda+\chi(m)}u\right)^k, \]

and

\[ |r_{16}(n)| < \frac{\left(-q; q\right)_{\infty}^2}{(1 - q)^2(1 - q^2)^2} q^{2n-\lfloor m/2 \rfloor} \sum_{k=0}^{\infty} q^{k^2-2k} \left(q^{\lambda+\chi(m)}u\right)^k, \]

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\[ |r_{17}(n)| < \frac{(-q; q)_\infty^3}{(1 - q)^2 (1 - q^2)^3} q^{2[m/2]} \sum_{k=0}^{\infty} q^{k^2 - 2k} (q^{\lambda + \chi(m)} u)^k, \]

\[ |r_{18}(n)| < \frac{(-q; q)_\infty^3}{(1 - q)^2 (1 - q^2)^3} q^{n + [m/2]} \sum_{k=0}^{\infty} q^{k^2 - 2k} (q^{\lambda + \chi(m)} u)^k, \]

\[ |r_{19}(n)| < \frac{(-q; q)_\infty^3}{(1 - q)^2 (1 - q^2)^3} q^{2n} \sum_{k=0}^{\infty} q^{k^2 - 2k} (q^{\lambda + \chi(m)} u)^k. \]

On the other hand, the last term in (5.19) is given by

\[ r_2(n) = \sum_{j=1}^{9} r_{2j}(n), \quad \text{(B.2)} \]

where

\[ r_{21}(n) = - \sum_{k=n - [m/2] + 1}^{\infty} q^{k^2} (-q^{\lambda + \chi(m)} u)^{-k}, \]

\[ r_{22}(n) = \frac{q^{1 + n - [m/2]}}{1 - q} \sum_{k=n - [m/2] + 1}^{\infty} q^{k^2 - k} (-q^{\lambda + \chi(m)} u)^{-k}, \]

\[ r_{23}(n) = \frac{q^{1 + [m/2]}}{1 - q} \sum_{k=n - [m/2] + 1}^{\infty} q^{k^2 + k} (-q^{\lambda + \chi(m)} u)^{-k}, \]

\[ r_{24}(n) = \frac{q^{2n}}{(1 - q)^2} \sum_{k=0}^{\infty} q^{k^2} (-q^{\lambda + \chi(m)} u)^{-k}, \]

and

\[ r_{25}(n) = \sum_{k=1}^{n - [m/2]} q^{k^2} (-q^{\lambda + \chi(m)} u)^{-k} R(q; n - [m/2] - k), \]

\[ r_{26}(n) = -\frac{q^{1 + [m/2]}}{1 - q} \sum_{k=1}^{n - [m/2]} q^{k^2 + k} (-q^{\lambda + \chi(m)} u)^{-k} R(q; n - [m/2] - k), \]

\[ r_{27}(n) = \sum_{k=1}^{n - [m/2]} q^{k^2} (-q^{\lambda + \chi(m)} u)^{-k} R(q; [m/2] + k), \]

\[ r_{28}(n) = -\frac{q^{1 + n - [m/2]}}{1 - q} \sum_{k=1}^{n - [m/2]} q^{k^2 - k} (-q^{\lambda + \chi(m)} u)^{-k} R(q; [m/2] + k), \]

\[ r_{29}(n) = \sum_{k=1}^{n - [m/2]} q^{k^2} (-q^{\lambda + \chi(m)} u)^{-k} R(q; [m/2] + k) R(q; n - [m/2] - k). \]
Similarly, by noting $q \in (0, 1)$, we have

$$|r_{21}(n)| < \frac{q^{(n-\lfloor m/2 \rfloor)/2}}{1 - q} \left( q^{\lambda+\chi(m)u} \right)^{-(n-\lfloor m/2 \rfloor)} \sum_{k=1}^{\infty} q^{k^2-2k} \left( q^{\lambda+\chi(m)u} \right)^{-k},$$

$$|r_{22}(n)| < \frac{q^{(n-\lfloor m/2 \rfloor)/2+1}}{1 - q} \left( q^{\lambda+\chi(m)u} \right)^{-(n-\lfloor m/2 \rfloor)} \sum_{k=1}^{\infty} q^{k^2-2k} \left( q^{\lambda+\chi(m)u} \right)^{-k},$$

$$|r_{23}(n)| < \frac{q^{(n-\lfloor m/2 \rfloor)/2+1+n}}{1 - q} \left( q^{\lambda+\chi(m)u} \right)^{-(n-\lfloor m/2 \rfloor)} \sum_{k=1}^{\infty} q^{k^2-2k} \left( q^{\lambda+\chi(m)u} \right)^{-k},$$

$$|r_{24}(n)| < \frac{q^n}{(1-q)^2(1-q^2)^2} \sum_{k=1}^{\infty} q^{k^2-2k} \left( q^{\lambda+\chi(m)u} \right)^{-k}.$$

For the others, applying Lemma 8 yields

$$|r_{25}(n)| < \frac{(-q; q)_{\infty}^2}{(1-q)^2(1-q^2)^2} 2^{(n-\lfloor m/2 \rfloor)/2} \sum_{k=1}^{\infty} q^{k^2-2k} \left( q^{\lambda+\chi(m)u} \right)^{-k},$$

$$|r_{26}(n)| < \frac{(-q; q)_{\infty}^2}{(1-q)^2(1-q^2)^2} q^{2n-\lfloor m/2 \rfloor} \sum_{k=1}^{\infty} q^{k^2-2k} \left( q^{\lambda+\chi(m)u} \right)^{-k},$$

$$|r_{27}(n)| < \frac{(-q; q)_{\infty}^2}{(1-q)^2(1-q^2)^2} q^{2\lfloor m/2 \rfloor} \sum_{k=1}^{\infty} q^{k^2-2k} \left( q^{\lambda+\chi(m)u} \right)^{-k},$$

$$|r_{28}(n)| < \frac{(-q; q)_{\infty}^2}{(1-q)^2(1-q^2)^2} q^{n+\lfloor m/2 \rfloor} \sum_{k=1}^{\infty} q^{k^2-2k} \left( q^{\lambda+\chi(m)u} \right)^{-k},$$

$$|r_{29}(n)| < \frac{(-q; q)_{\infty}^2}{(1-q)^2(1-q^2)^2} q^{2n} \sum_{k=1}^{\infty} q^{k^2-2k} \left( q^{\lambda+\chi(m)u} \right)^{-k}.$$
Therefore we have

\[ |r_1(n) + r_2(n)| \leq \sum_{j=1}^{9} |r_{1j}(n)| + \sum_{j=1}^{9} |r_{2j}(n)| \]

\[ < (1 + q^{1+n} + q) \frac{q^{\lfloor m/2 \rfloor^2 + 2 \lfloor m/2 \rfloor}}{1 - q} \left( q^{\lambda + \chi(m)} u \right)^{\lfloor m/2 \rfloor + 1} \sum_{k=0}^{\infty} q^{k^2 - 2k} \left( q^{\lambda + \chi(m)} u \right)^k \]

\[ + \left( q^n + q^{2(n - \lfloor m/2 \rfloor)} + q^{2n - \lfloor m/2 \rfloor} + q^{2\lfloor m/2 \rfloor} + q^{n + \lfloor m/2 \rfloor} + q^{2n} \right) \]

\[ \times \frac{(-q; q)_\infty}{(1 - q)^2 (1 - q^2)^2} \sum_{k=0}^{\infty} q^{k^2 - 2k} \left( q^{\lambda + \chi(m)} u \right)^k \]

\[ + (1 + q + q^{1+n}) \frac{q^{(n - \lfloor m/2 \rfloor)^2}}{1 - q} \left( q^{\lambda + \chi(m)} u \right)^{-(n - \lfloor m/2 \rfloor)} \sum_{k=1}^{\infty} q^{k^2 - 2k} \left( q^{\lambda + \chi(m)} u \right)^{-k} \]

\[ + \left( q^n + q^{2(n - \lfloor m/2 \rfloor)} + q^{2n - \lfloor m/2 \rfloor} + q^{2\lfloor m/2 \rfloor} + q^{n + \lfloor m/2 \rfloor} + q^{2n} \right) \]

\[ \times \frac{(-q; q)_\infty}{(1 - q)^2 (1 - q^2)^2} \sum_{k=1}^{\infty} q^{k^2 - 2k} \left( q^{\lambda + \chi(m)} u \right)^{-k} \]. \quad \text{(B.3)}

By (5.24), we obtain

\[ q^n + q^{2(n - \lfloor m/2 \rfloor)} + q^{2n - \lfloor m/2 \rfloor} + q^{2\lfloor m/2 \rfloor} + q^{n + \lfloor m/2 \rfloor} + q^{2n} \]

\[ < q^n + q^{\tau n} + q^{n + \tau n/2} + q^{2(2 - \tau)n - 2} + q^{n + (2 - \tau)n/2 - 1} + q^{2n} \]

\[ < q^{-2} \left( q^n + q^{\tau n} + q^{n + \tau n/2} + q^{2(2 - \tau)n} + q^{n + (2 - \tau)n/2} + q^{2n} \right) \]

\[ < q^{-2} \left( q^{\tau n} + 4q^n + q^{(2 - \tau)n} \right) \]

\[ \leq 6q^{-2}q^{\tau n + 2(1 - \tau)n/1,2(\tau)}. \quad \text{(B.4)}\]
Hence, by $1 + q + q^{1+n} < 3$ and $\lambda + \chi(m) \in [0, 2)$, we have
\[
|r_1(n) + r_2(n)| \leq \frac{3q^{m/2}2q^{m/2}}{1 - q} (q^\lambda + \chi(m)u)^{m/2+1} \sum_{k=0}^{\infty} q^{2k-2k} (q^\lambda + \chi(m)u)^k \\
+ 6q^{-2}q^{n+2(1-\tau)n1_{[1,2]}(\tau)} (q; q)_{\infty}^2 \frac{(1 - q)^2}{1 - q^2(1 - q^2)^2} \sum_{k=0}^{\infty} q^{k^2-2k} (q^\lambda + \chi(m)u)^k \\
+ 3q^{n-[m/2]^2} (q^\lambda + \chi(m)u)^{-(n-[m/2])} \sum_{k=1}^{\infty} q^{k^2-2k} (q^\lambda + \chi(m)u)^{-k} \\
+ 6q^{-2}q^{n+2(1-\tau)n1_{[1,2]}(\tau)} (q; q)_{\infty}^2 \frac{(1 - q)^2}{1 - q^2(1 - q^2)^2} \sum_{k=1}^{\infty} q^{k^2-2k} (q^\lambda + \chi(m)u)^{-k}.
\]
(B.5)

Finally, by (5.21), we have
\[
|r_1(n) + r_2(n)| \leq \frac{3q^{2-\tau}2q^{2n-4n-1}}{1 - q} u^{m/2+1} \sum_{k=0}^{\infty} q^{k^2-2k} u^k \\
+ 6q^{-2}q^{n+2(1-\tau)n1_{[1,2]}(\tau)} (q; q)_{\infty}^2 \frac{(1 - q)^2}{1 - q^2(1 - q^2)^2} \sum_{k=0}^{\infty} q^{k^2-2k} u^k \\
+ 3q^{-2n^2/4-\tau n-2} u^{-(n-[m/2])} \sum_{k=1}^{\infty} q^{k^2-4k} u^{-k} \\
+ 6q^{-2}q^{n+2(1-\tau)n1_{[1,2]}(\tau)} (q; q)_{\infty}^2 \frac{(1 - q)^2}{1 - q^2(1 - q^2)^2} \sum_{k=1}^{\infty} q^{k^2-4k} u^{-k} \\
\equiv M(n).
\]
(B.6)

Since, for any $\alpha > 0$, $\beta > 0$, and $q \in (0, 1)$, $q^{\alpha n^2} \beta^n \to 0$, as $n \to \infty$, we have
\[
\frac{M(n)}{q^{r_n+2(1-\tau)n1_{[1,2]}(\tau)}} \to 6q^{-2} (q; q)_{\infty}^2 \frac{(1 - q)^2}{1 - q^2(1 - q^2)^2} \left[ \sum_{k=0}^{\infty} q^{k^2-2k} u^k + \sum_{k=1}^{\infty} q^{k^2-4k} u^{-k} \right],
\]
as $n \to \infty$. (B.7)

Thus, $|r_1(n) + r_2(n)|/q^{r_n+2(1-\tau)n1_{[1,2]}(\tau)}$ is bounded in $n \to \infty$. Hence the proof of (5.22) is completed. 

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