On the equivalence between the sets of the trigonometric polynomials

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Abstract

In this paper we construct an injection from the linear space of trigonometric polynomials defined on \( T^d \) with bounded degrees with respect to each variable to a suitable linear subspace \( L^1_E \subset L^1(T) \). We give such a quantitative condition on \( L^1_E \) that this injection is a isomorphism of a Banach spaces equipped with \( L^1 \) norm and the norm of the isomorphism is independent on the dimension \( d \).

Introduction

The purpose of this article is to study the equivalence between sets of trigonometric polynomials defined on \( T \) with sets of trigonometric polynomials in higher dimensions. For a given vector of integers \( \tau = (\tau_1, \ldots, \tau_d) \) we want to study operator

\[
 f(x) = \sum_{\lambda} a_\lambda e^{2\pi i \langle \lambda, x \rangle} \rightarrow T f(z) = \sum_{\lambda} a_\lambda e^{2\pi i \langle \lambda, \tau \rangle} z \quad \forall x \in T^d \quad \forall z \in T
\]

with some bounds on vectors \( \lambda = (\lambda_1, \ldots, \lambda_d) \). We want to find a quantitative criterion on \( \tau \) for \( T \) being isomorphism in \( L^1 \) norm. To give a precise formulation we introduce following notation.

Definition 1. As a \( L^p_{\mathcal{A}}(T^k) \) we will denote a subspace of Banach space \( L^p(T^k) \) defined below

\[
 L^p_{\mathcal{A}}(T^k) = \{ f \in L^p(T^k) : \text{supp } \hat{f} \subset \mathcal{A} \}
\]

Definition 2. For a given sequence of natural numbers \( (a_n)_{n \in \mathbb{N}} \) and a sequence of integers \( (\tau_n)_{n \in \mathbb{N}} \) we define sets \( E \subset \mathbb{Z} \) and \( F \subset \mathbb{Z}^N \), where by \( \mathbb{Z}^N \) we denote a dual group to \( T^N \), in the following way:

\[
 F = \{ \lambda \in \mathbb{Z}^N : |\lambda_n| \leq |a_n| \}
 E = \{ \beta \in \mathbb{Z} : \beta = \sum_{k=1}^{\tau_k} \lambda_k \lambda_k \text{ for } \lambda_k \in F \}.
\]

Using that notation we can state the main Theorem of this article:

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Theorem 1. For a given sequence of natural numbers \((a_n)_{n\in\mathbb{N}}\) and a sequence of integers \(\tau_n\) satisfying
\[
\tau_{k+1} \geq 3a_k |\tau_k| \quad \forall k \in \mathbb{N},
\]
\[
\sum_{j=1}^{\infty} \frac{|a_j||\tau_j||a_{j+1}|}{|\tau_{j+1}|} < \infty.
\]
(2)
Then operator \(T : L^1_k(\mathbb{T}^d) \to L^1_k(\mathbb{T})\) given by the formula
\[
Tf(x) = \sum_{\lambda \in F} \hat{f}(\lambda) e^{2\pi i (\lambda, \tau)x}
\]
(3)
is an isomorphism, moreover
\[
K^{-1}\|f\|_{L^1_k(\mathbb{T}^d)} \leq \|Tf\|_{L^1_k(\mathbb{T})} \leq K\|f\|_{L^1_k(\mathbb{T}^d)}
\]
with the constant \(K\) depending only on the value of \(\sum_{j=1}^{\infty} \frac{|a_j||\tau_j||a_{j+1}|}{|\tau_{j+1}|}\) for all proofs seem to be incomplete. We fix this problem with an elementary proof for \(L^1\) norm. For examples of use of such a criterion one can check [2], [4].

Auxiliary lemmas

We start with the estimate on the approximation of trigonometric polynomial by simple functions.

Lemma 1. Let \(s_d, N_d \in \mathbb{N}\) and \(f\) is a trigonometric polynomial. Assume that the degree with respect to the last variable of the polynomial \(f\) is less than or equal to \(s_d\) (\(\deg_z(f) \leq s_d\)). For a function \(\tilde{f}\) given by the formula
\[
\tilde{f}(y', z) = \sum_{j=0}^{N_d-1} \chi_{I_j}(z) f \left( y', \frac{j}{N_d} \right) \quad \forall y' \in \mathbb{T}^{d-1}; \quad z \in \mathbb{T},
\]
where \(I_j = \left[ \frac{j}{N_d}; \frac{j+1}{N_d} \right]\), we have
\[
\|f\|_{L^1(\mathbb{T}^d)} - \|\tilde{f}\|_{L^1(\mathbb{T}^d)} \leq \frac{s_d}{N_d} \|f\|_{L^1(\mathbb{T}^d)}.
\]

Proof. We can estimate the difference of \(L^1\) norms of the functions using the norm of the partial derivative of \(f\).

\[
\|f\|_{L^1(\mathbb{T}^d)} - \|\tilde{f}\|_{L^1(\mathbb{T}^d)} = \left| \int_{\mathbb{T}^d} |f(y', z)| dy'dz - \int_{\mathbb{T}^d} |\tilde{f}(y', z)| dy'dz \right|
\]
\[
= \left| \sum_{j=0}^{N_d-1} \int_{\mathbb{T}^{d-1}} \int_{I_j} |f(y', z)| - |\tilde{f}(y', \frac{j}{N_d})| dz dy' \right|
\]
\[
\leq \sum_{j=0}^{N_d-1} \int_{\mathbb{T}^{d-1}} \int_{I_j} |f(x)| - \left| f \left( \frac{j}{N_d} \right) \right| dz dy'
\]
\[
\leq \sum_{j=0}^{N_d-1} \int_{\mathbb{T}^{d-1}} \int_{I_j} \int_{0}^{N_d} \frac{\partial}{\partial z} (y', z + y) dy dy' dz
\]
\[
\leq \sum_{j=0}^{N_d-1} \int_{I_j} \int_{\mathbb{T}^{d-1}} \int_{0}^{N_d} \frac{\partial}{\partial z} (y', z + y) dy dy' dz 
\leq \frac{s_d}{N_d} \|f\|_{L^1(\mathbb{T}^d)}.\]
Due to Bernstein inequality (see e.g. [5]) we get
\[
\left\| f \right\|_{L^1(\mathbb{T}^d)} - \left\| \tilde{f} \right\|_{L^1(\mathbb{T}^d)} \leq \frac{\| \partial_x f \|_{L^1(\mathbb{T}^d)}}{N_d} \leq \frac{\text{deg}_z(f)}{N} \|f\|_{L^1(\mathbb{T}^d)} \leq \frac{s_d}{N_d} \|f\|_{L^1(\mathbb{T}^d)}.
\]
\[\square\]

**Lemma 2.** For trigonometric polynomials \(f_{l_d}, f_{l_d+1}, \ldots, f_{k_d} \in L^1(\mathbb{T}^d)\), \(-N_d < l_d < k_d \leq N_d\) and
\[
w_{d+1}(y', y, z) = \sum_{j=l_d}^{k_d} e^{2\pi i y j_d} f_j(y', z),
\]

following estimates are satisfied
\[
\|w_{d+1}\|_{L^1(\mathbb{T}^{d+1})} \leq \sum_{j=l_d}^{k_d} \|f_j\|_{L^1(\mathbb{T})} \leq |k - l| \|w_{d+1}\|_{L^1(\mathbb{T}^{d+1})}.
\]

**Proof.** Left hand side of the inequality is just a triangle inequality. To get the right hand side of inequality we just observe that following inequalities are satisfied and add them up.
\[
\int_{\mathbb{T}^d} |f_j| dy' dz = \int_{\mathbb{T}^d} \left| \int_{\mathbb{T}} e^{-2\pi i y j_d} w_{d+1} dy \right| dy' dz \int_{\mathbb{T}^d} \leq \|w_{d+1}\|_{L^1(\mathbb{T}^{d+1})}.
\]
\[\square\]

**Lemma 3.** Assume that trigonometric polynomials \(f_{l_d}, f_{l_d+1}, \ldots, f_{k_d} \in L^1(\mathbb{T}^d)\) satisfy \(\text{deg}_z(f_j) \leq s_d\). We define functions
\[
w_d(y', z) := \sum_{j=l_d}^{k_d} e^{2\pi i y j_d} f_j(y', z),
\]
\[
w_{d+1}(y', y_d, z) := \sum_{j=l_d}^{k_d} e^{2\pi i y j_d} f_j(y', z).
\]

This pair of functions satisfies following estimates
\[
\left(1 - 2|k_d - l_d| \frac{s_d}{N_d}\right) \|w_{d+1}\|_{L^1(\mathbb{T}^{d+1})} \leq \|w_d\|_{L^1(\mathbb{T}^d)} \leq \left(1 + 2|k_d - l_d| \frac{s_d}{N_d}\right) \|w_{d+1}\|_{L^1(\mathbb{T}^{d+1})}.
\]

**Proof.** Let us define functions
\[
\tilde{w}_{d+1}(y', y_d, z) := \sum_{j=l_d}^{k_d} e^{i y_d j_d} \tilde{f}_j(y', z),
\]
\[
\tilde{w}_d(y', z) := \sum_{j=l_d}^{k_d} e^{i y_d j_d} \tilde{f}_j(y', z).
\]

Using triangle inequality we get
\[
\|w_d\|_{L^1(\mathbb{T}^d)} - \|w_{d+1}\|_{L^1(\mathbb{T}^{d+1})} \leq \left| \|w_d\|_{L^1(\mathbb{T}^d)} - \|\tilde{w}_d\|_{L^1(\mathbb{T}^d)} \right| + \left| \|\tilde{w}_{d+1}\|_{L^1(\mathbb{T}^{d+1})} - \|\tilde{w}_d\|_{L^1(\mathbb{T}^d)} \right|
\]
\[
+ \left| \|w_{d+1}\|_{L^1(\mathbb{T}^{d+1})} - \|\tilde{w}_{d+1}\|_{L^1(\mathbb{T}^{d+1})} \right|. \tag{4}
\]
Once again using the triangle inequality we obtain
\[ \|w_d\|_{L^1(\mathbb{T}^d)} - \|\tilde{w}_d\|_{L^1(\mathbb{T}^d)} + \|w_{d+1}\|_{L^1(\mathbb{T}^{d+1})} - \|\tilde{w}_{d+1}\|_{L^1(\mathbb{T}^{d+1})} \leq 2 \sum_{j=0}^{k} \|f_j - \tilde{f}_j\|_{L^1(\mathbb{T}^d)}. \]

Then the definition of the function $\tilde{f}_j$ leads to estimates from Lemma [1] and Lemma [2]
\[ \|w_d\|_{L^1(\mathbb{T}^d)} - \|\tilde{w}_d\|_{L^1(\mathbb{T}^d)} + \|w_{d+1}\|_{L^1(\mathbb{T}^{d+1})} - \|\tilde{w}_{d+1}\|_{L^1(\mathbb{T}^{d+1})} \leq 2k_d - l_d \frac{s_d}{N_d} \|w_{d+1}\|_{L^1(\mathbb{T}^{d+1})}. \] (5)

Now we pass to estimate of the second term of the right hand side of the inequality [4]. We know that the function $f_j(y', \cdot)$ is a constant on a interval $I_k$ for every $j \in \{1, \ldots, N_d - 1\}$ and every $y' \in \mathbb{T}^{d-1}$. We denote this value by $h_j(k, y')$. This property is crucial in the following calculations.

\[ \|\tilde{w}_d\|_{L^1(\mathbb{T}^d)} = \int_{\mathbb{T}^d} |\tilde{w}(y', z)| dy' dz \]
\[ = \int_{\mathbb{T}^{d-1}} \sum_{k=0}^{N-1} \int_{I_k} \sum_{j=1}^{k_d} e^{2\pi i j N z} f_j(y', z) \, dz dy' \]
\[ = \int_{\mathbb{T}^{d-1}} \sum_{k=0}^{N-1} \int_{I_k} \sum_{j=1}^{k_d} e^{2\pi i j N z} h_j(k, y') \, dz dy' \]
\[ = \frac{1}{N} \int_{\mathbb{T}^{d-1}} \sum_{k=0}^{N-1} \int_{T} \sum_{j=1}^{k_d} e^{2\pi i j y_d} h_j(k, y') \, dy_d dy' \]
\[ = \int_{\mathbb{T}^{d+1}} \sum_{j=1}^{k_d} e^{2\pi i j y_d} \tilde{f}_j(y', z) \, dz dy_d dy' \]
\[ = \int_{\mathbb{T}^{d+1}} |\tilde{w}_{d+1}| = \|\tilde{w}_{d+1}\|_{L^1(\mathbb{T}^{d+1})}. \]

We have obtain
\[ \|\tilde{w}_d\|_{L^1(\mathbb{T}^d)} = \|\tilde{w}_{d+1}\|_{L^1(\mathbb{T}^{d+1})} \]

Above equality together with [4] and [5] gives us
\[ \|w_d\|_{L^1(\mathbb{T}^d)} - \|\tilde{w}_d\|_{L^1(\mathbb{T}^d)} + \|w_{d+1}\|_{L^1(\mathbb{T}^{d+1})} - \|\tilde{w}_{d+1}\|_{L^1(\mathbb{T}^{d+1})} \leq 2k_d - l_d \frac{s_d}{N_d} \|w_{d+1}\|_{L^1(\mathbb{T}^{d+1})} \]
which is equivalent to the inequality from the statement of the lemma. \( \square \)

Now using above lemmas we can prove the main theorem.

**Proof of the main Theorem**

*Proof.* Let us take polynomial $f \in L^1_F(\mathbb{T}^n)$ which depends only on first $n$ variables. Then from the definition of operator $T$ we have
\[ Tf(z) = \sum_{(\lambda_1, \ldots, \lambda_n) \in F} \tilde{f}(\lambda_1, \ldots, \lambda_n) e^{2\pi i (\sum_{i=1}^{n} \tau_i \lambda_i) z} \quad \forall z \in \mathbb{T}, \]
which we can rewrite in the form
\[ w_1(z) := Tf(z) = \sum_{j=-a_n}^{a_n} e^{2\pi i j \tau_n z} g_j^1(z), \]
where \( g_j \) are suitable polynomials such that \( \deg_z (g_j^1) \leq \sum_{j=1}^{n-1} |a_j| \tau_j \). By the Lemma we get
\[ \left( 1 - 4a_n \sum_{j=1}^{n-1} a_j |\tau_j| \right) \|w_2\|_{L^1(\mathbb{T})} \leq \|w_1\|_{L^1(\mathbb{T})} \leq \left( 1 + 4a_n \sum_{j=1}^{n-1} a_j |\tau_j| \right) \|w_2\|_{L^1(\mathbb{T})}, \]
where
\[ w_2(y, z) = \sum_{(\lambda_1, \ldots, \lambda_n) \in F} \tilde{f}(\lambda_1, \ldots, \lambda_n) e^{2\pi i (\lambda_1 \cdots \lambda_{n-d+1}) y} e^{2\pi i (\sum_{i=1}^{n-d} \tau_i \lambda_i) z}. \]
Analogously as in the case \( d = 1 \) we proceed with \( d > 1 \). We obtain trigonometric polynomials of the form
\[ w_d(y', z) := \sum_{(\lambda_1, \ldots, \lambda_n) \in F} \tilde{f}(\lambda_1, \ldots, \lambda_n) e^{2\pi i ((\lambda_1 \cdots \lambda_{n-d+1}) y')} e^{2\pi i (\sum_{i=1}^{n-d} \tau_i \lambda_i) z} \quad \forall z \in \mathbb{T} \forall y' \in \mathbb{T}^{d-1}. \]
We can rewrite them in following way
\[ w_d(y', z) = \sum_{j=-a_d}^{a_d} e^{2\pi i j \tau_d z} g_j^d(y', z), \]
with polynomials \( g_j^d \) satisfying \( \deg_z (g_j^d) \leq \sum_{j=1}^{n-d} |a_j| \tau_j \). By the Lemma we have
\[ (1 - K(d)) \|w_{d+1}\|_{L^1(\mathbb{T}^{d+1})} \leq \|w_d\|_{L^1(\mathbb{T}^d)} \leq (1 + K(d)) \|w_{d+1}\|_{L^1(\mathbb{T}^{d+1})}, \]
where constant \( K(d) \) is given by the formula
\[ K(d) = 4a_{n-d+1} \sum_{j=1}^{n-d} a_j |\tau_j| \]
Combining above inequalities for \( d = 1, \ldots, n-1 \) we get
\[ \Pi_{j=1}^{n-1} (1 - K(j)) \|w_n\|_{L^1(\mathbb{T}^n)} \leq \|T f\|_{L^1(\mathbb{T})} \leq \Pi_{j=1}^{n-1} (1 + K(j)) \|w_n\|_{L^1(\mathbb{T}^n)}. \]
Let us observe that the function \( w_n \) is equal to the function \( f \) up to a permutation of the variables. Hence \( \|w_n\|_{L^1(\mathbb{T}^n)} = \|f\|_{L^1(\mathbb{T}^n)} \). Since \( a_j, \tau_j \) satisfy we have
\[ \sum_{d=1}^{\infty} K(d) = \sum_{d=1}^{\infty} 4a_{n-d+1} \sum_{j=1}^{n-d} a_j |\tau_j| \leq C \sum_{d=1}^{\infty} a_d (a_{d-1} + 1) \frac{\tau_{d-1}}{\tau_d} \leq \infty. \]
Hence there exist such a constant \( K \) that following inequality is satisfied
\[ K^{-1} \|f\|_{L^1(\mathbb{T}^n)} \leq \|T f\|_{L^1(\mathbb{T})} \leq K \|f\|_{L^1(\mathbb{T}^n)}, \quad (6) \]
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