Abstract

In this paper, we give some bounds for principal eigenvector and spectral radius of connected uniform hypergraphs in terms of vertex degrees, the diameter, and the number of vertices and edges.

Keywords: Hypergraph, Spectral radius, Principal eigenvector, Principal ratio

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1. Introduction

For a positive integer $n$, let $[n] = \{1, 2, \ldots, n\}$. Let $\mathbb{C}^{[m,n]}$ be the set of order $m$ dimension $n$ tensors over the complex field $\mathbb{C}$. For $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in \mathbb{C}^{[m,n]}$, if all the entries $a_{i_1i_2\ldots i_m} \geq 0$ (or $a_{i_1i_2\ldots i_m} > 0$) of $\mathcal{A}$, then $\mathcal{A}$ is called nonnegative (or positive) tensor. When $m = 2$, $\mathcal{A}$ is a $n \times n$ matrix. Let $\mathcal{I} = (\delta_{i_1i_2\ldots i_m}) \in \mathbb{C}^{[m,n]}$ be the unit tensor, where $\delta_{i_1i_2\ldots i_m}$ is Kronecker function.

In 2005, Qi [1] and Lim [2] defined the eigenvalue of tensors, respectively. For $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in \mathbb{C}^{[m,n]}$ and $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n$, $\mathcal{A}x^{m-1}$ is a dimension $n$ vector whose the $i$-th component is

$$\left(\mathcal{A}x^{m-1}\right)_i = \sum_{i_2,i_3,\ldots,i_m=1}^n a_{i_1i_2\ldots i_m}x_{i_2}x_{i_3}\cdots x_{i_m}.$$  

If there exists a number $\lambda \in \mathbb{C}$, a nonzero vector $x = (x_1, \ldots, x_n)^T \in \mathbb{C}^n$ such that

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

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then $\lambda$ is called an eigenvalue of $A$, $x$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda$, where $x^{[m-1]} = (x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1})^T$. Let $\sigma(A)$ denote the set of all eigenvalues of $A$, the spectral radius $\rho(A) = \max \{|\lambda| | \lambda \in \sigma(A)\}$.

Chang et al. [11], Yang et al. [3, 4], Friedland et al. [12] gave Perron-Frobenius theorem of nonnegative tensors. Let $A = (a_{i_1i_2\ldots i_m})$ be an order $m$ dimension $n$ nonnegative tensor, if for any nonempty proper index subset $\alpha \subset \{1, \ldots, n\}$, there is at least an entry

$$a_{i_1\ldots i_m} > 0, \text{ where } i_1 \in \alpha \text{ and at least an } i_j \notin \alpha, \ j = 2, \ldots, m,$$

then $A$ is called nonnegative weakly irreducible tensor (see[4]).

Let $V(G) = \{1, 2, \ldots, n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$ denote the vertex set and edge set of a hypergraph $G$, respectively. If each edge of $G$ contains exactly $k$ distinct vertices, then $G$ is called a $k$-uniform hypergraph. In particular, 2-uniform hypergraphs are exactly the ordinary graphs. For a connected $k$-uniform hypergraph $G$, $e_i$ denotes an edge that contains vertex $i$, the degree of a vertex $i$ of $G$ is denoted by $d_i$, $\Delta = \max\{d_i\}$, $\delta = \min\{d_i\}$, for $i = 1, \ldots, n$. If all vertices of $G$ have the same degree, then $G$ is regular. A path $P$ of a $k$-uniform hypergraph is defined to be an alternating sequence of vertices and edges $P = v_0e_1v_1e_2\ldots v_{l-1}e_lv_l$, where $v_0, \ldots, v_l$ are distinct vertices of $G$, $e_1, \ldots, e_l$ are distinct edges of $G$ and $v_{i-1}, v_i \in e_i$, for $i = 1, \ldots, l$. The number of edges in $P$ is called the length of $P$. If there exists a path starting at $u$ and terminating at $v$ for all $u, v \in V(G)$, then $G$ is connected. Let $u, v$ be two distinct vertices of $G$, the distance between $u$ and $v$ is defined to be the length of the shortest path connecting them, denoted by $d(u, v)$. The diameter of a connected $k$-uniform hypergraph $G$ is the maximum distance among all vertices of $G$, denoted by $D$. In 2012, Cooper and Dutle [6] gave the concept of adjacency tensor of a $k$-uniform hypergraph. The adjacency tensor of a $k$-uniform hypergraph $G$ denoted by $A_G$, is an order $k$ dimension $n$ nonnegative symmetric tensor with entries

$$a_{i_1i_2\ldots i_k} = \begin{cases} 
\frac{1}{(k-1)!}, & \text{if } \{i_1, i_2, \ldots, i_k\} \in E(G), \\
0, & \text{otherwise}. 
\end{cases}$$

Eigenvalues of $A_G$ are called eigenvalues of $G$, the spectral radius of $A_G$ is called the spectral radius of $G$, denoted by $\rho(G)$.

Let $G$ be a connected $k$-uniform hypergraph, then the adjacency tensor $A_G$ of $G$ is nonnegative weakly irreducible (see[8]), by Perron-Frobenius theorem of
nonnegative tensors (see [4]), $\rho(G)$ is an eigenvalue of $A_G$, there exists a unique positive eigenvector $x = (x_1, \ldots, x_n)^T$ corresponding to $\rho(G)$ and $\sum_{i=1}^{n} x_i = 1$, $x$ is called and the principal eigenvector of $A_G$, the maximum and minimum entries of $x$ are denoted by $x_{\text{max}}$ and $x_{\text{min}}$, respectively. $\gamma = \frac{x_{\text{max}}}{x_{\text{min}}}$ is called the principal ratio of $A_G$ (see [5]). In this paper, the principal eigenvector and the principal ratio of $A_G$ are called the principal eigenvector and the principal ratio of $G$. Let $\rho(G)$ be the spectral radius of a $k$-uniform hypergraph $G$ with eigenvector $x = (x_1, \ldots, x_n)^T$. Since $A_Gx^{k-1} = \rho(G)x^{[k-1]}$, we know that $cx$ is also an eigenvector of $\rho(G)$ for any nonzero constant $c$. When $\sum_{i=1}^{n} |x_i| = 1$, let $x^e = x_{i_1}x_{i_2}\cdots x_{i_k}$, $\{i_1, i_2, \ldots, i_k\} = e$ (see [6, 10]), we have

$$\rho(G) = \frac{x^T(A_Gx^{k-1})}{x^T(Ix^{k-1})} = x^T(A_Gx^{k-1}) = \sum_{i_1, \ldots, i_k=1}^{n} a_{i_1 \cdots i_k}x_{i_1} \cdots x_{i_k} = \sum_{e \in E(G)} k! \frac{1}{(k-1)!}x^e = k \sum_{e \in E(G)} x^e.$$ 

In spectral graph theory, there are some work concern relations among the spectral radius, principal eigenvector and graph parameters [5, 7]. The interest of this paper is to consider similar problems in spectral hypergraph theory. This paper is organized as follows. In Section 2, we give some bounds for the principal ratio and the maximum and minimum entries of principal eigenvector of connected uniform hypergraphs. In Section 3, we show some bounds for the spectral radius of connected uniform hypergraphs via degrees of vertices, the principal ratio and diameter.

2. The principal eigenvector of hypergraphs

Let $G$ be a connected $k$-uniform hypergraph, $G$ is regular if and only if $\gamma = 1$. Thus, $\gamma$ is an index which measure the irregularity of $G$. In 2005, Zhang [7] gave some bounds of the principal ratio of irregular graph $G$, these results were used to obtain a bound of the spectral radius of $G$.

Let $G$ be a connected uniform hypergraph with maximum degree $\Delta$, minimum degree $\delta$. We give the lower bound for the principal ratio $\gamma$ of $G$, which extend the result of Zhang [7, Theorem 2.3] to hypergraphs.
Theorem 2.1. Let $G$ be a connected $k$-uniform hypergraph, then

$$\gamma \geq \left( \frac{\Delta}{\delta} \right)^{\frac{1}{2(k-1)}}. \quad (2.1)$$

If equality in (2.1) holds, then $\rho(G) = \sqrt{\Delta \delta}$.

Proof. Let $G$ be a connected $k$-uniform hypergraph, $A_G$ be the adjacency tensor of $G$, $x = (x_1, \ldots, x_n)^T$ be the principal eigenvector of $G$. Suppose that $d_p = \Delta$, $d_q = \delta$, $(p, q \in V(G))$, since $A_G x^{k-1} = \rho(G) x^{k-1}$, we have

$$\rho(G) x^{k-1}_p = \sum_{e_p \in E(G)} x^{e_p \setminus \{p\}} \geq \Delta x^{k-1}_{\text{min}}, \quad (2.2)$$

$$\rho(G) x^{k-1}_q = \sum_{e_q \in E(G)} x^{e_q \setminus \{q\}} \leq \delta x^{k-1}_{\text{max}}. \quad (2.3)$$

where $x^{e_1 \setminus \{i\}} = x_{i_2} x_{i_3} \cdots x_{i_k}, \{i, i_2, \ldots, i_k\} = e_i$.

By (2.2) and (2.3), we have

$$\frac{\Delta}{\rho(G)} \frac{\rho(G)}{\delta} \leq \left( \frac{x_p}{x_{\text{min}}} \right)^{k-1} \left( \frac{x_{\text{max}}}{x_q} \right)^{k-1} \leq \left( \frac{x_{\text{max}}}{x_{\text{min}}} \right)^{2(k-1)} = \gamma^{2(k-1)}. \quad (2.4)$$

i.e.

$$\gamma \geq \left( \frac{\Delta}{\delta} \right)^{\frac{1}{2(k-1)}}.$$ 

Since equality in (2.1) holds, three equalities in (2.2), (2.3) and (2.4) hold. If equality in (2.4) holds, we have

$$x_p = x_{\text{max}}, \quad x_q = x_{\text{min}}. \quad (2.5)$$

When equalities in (2.2) and (2.3) hold, by (2.5), we obtain

$$\rho(G) = \sqrt{\Delta \delta}.$$ 

Applying the bound of the principal ratio $\gamma$, we obtained the result as follows.
Theorem 2.2. Let $G$ be a connected $k$-uniform hypergraph, $x = (x_1, \ldots, x_n)^T$ be the principal eigenvector of $G$, then

(1) $x_{\text{max}} \geq \left[ \left( \frac{\delta}{\Delta} \right)^{\frac{k}{2(k-1)}} + n - 1 \right]^{-\left( \frac{1}{k} \right)}$;

(2) $x_{\text{min}} \leq \left[ \left( \frac{\delta}{\Delta} \right)^{\frac{k}{2(k-1)}} + n - 1 \right]^{-\left( \frac{1}{k} \right)}$.

Proof. Let $G$ be a connected $k$-uniform hypergraph, $x = (x_1, \ldots, x_n)^T$ be the principal eigenvector of $G$, then

$$1 = \sum_{i=1}^{n} x_i^k \leq x_{\text{min}}^k + (n - 1) x_{\text{max}}^k.$$

Let $\gamma$ be the principal ratio of $G$, we obtain

$$x_{\text{max}}^{-k} \leq \gamma^{-k} + n - 1,$$

$$x_{\text{max}}^k \geq (\gamma^{-k} + n - 1)^{-1}. \quad (2.6)$$

By Theorem 2.1, we know that $\gamma \geq \left( \frac{\Delta}{\delta} \right)^{\frac{1}{2(k-1)}}$, so

$$x_{\text{max}}^k \geq \left[ \left( \frac{\delta}{\Delta} \right)^{\frac{k}{2(k-1)}} + n - 1 \right]^{-1},$$

i.e.

$$x_{\text{max}} \geq \left[ \left( \frac{\delta}{\Delta} \right)^{\frac{k}{2(k-1)}} + n - 1 \right]^{-\left( \frac{1}{k} \right)} \quad (2.7)$$

Since

$$1 = \sum_{i=1}^{n} x_i^k \geq (n - 1) x_{\text{min}}^k + x_{\text{max}}^k,$$

$$x_{\text{min}}^{-k} \geq \gamma^k + n - 1,$$

$$x_{\text{min}}^k \leq (\gamma^k + n - 1)^{-1}.$$
By Theorem 2.1 we know that $\gamma \geq \left( \frac{\Delta}{\delta} \right)^{\frac{k}{2(k-1)}}$, so

$$x_{\min}^k \leq \left[ \left( \frac{\Delta}{\delta} \right)^{\frac{k}{2(k-1)}} + n - 1 \right]^{-1},$$

i.e.

$$x_{\min} \leq \left[ \left( \frac{\Delta}{\delta} \right)^{\frac{k}{2(k-1)}} + n - 1 \right]^{-\frac{1}{k}}.$$ 

Theorem 2.3. Let $G$ be a connected $k$-uniform hypergraph with $n$ vertices and $m$ edges, $x = (x_1, \ldots, x_n)^T$ be the principal eigenvector of $G$, then

$$x_{\max} \geq \left( \frac{\rho(G)}{km} \right)^{\frac{1}{k}},$$

with equality if and only if $G$ is regular.

Proof. Let $G$ be a connected $k$-uniform hypergraph with $n$ vertices and $m$ edges, $A_G = (a_{i_1i_2\ldots i_k})$ be the adjacency tensor of $G$, $x = (x_1, \ldots, x_n)^T$ be the principal eigenvector of $G$, then

$$\rho(G) = x^T (A_G x^{k-1}) = k \sum_{e \in E(G)} x^e \leq km x_{\max}^k.$$ 

(2.8)

$$x_{\max} \geq \left( \frac{\rho(G)}{km} \right)^{\frac{1}{k}}.$$ 

(2.9)

Clearly, equality in (2.9) holds if and only if equality in (2.8) holds, i.e. $x_1 = x_2 = \cdots = x_n$, therefore $G$ is regular.

3. The spectral radius of hypergraphs

Let $G$ be a connected uniform hypergraph with maximum degree $\Delta$, minimum degree $\delta$. We obtain some bounds for the spectral radius of $G$ via degrees of vertices, the principal ratio and diameter.

We give some auxiliary lemmas which will be used in the sequel.
Lemma 3.1. Let \( y_1, \ldots, y_n \) be nonnegative real numbers \((n \geq 2)\), then
\[
\frac{y_1 + \cdots + y_n}{n} - (y_1 \cdots y_n)^{\frac{1}{n}} \geq \frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} (\sqrt{y_i} - \sqrt{y_j})^2.
\]

Lemma 3.2. Let \( a, b, y_1, y_2 \) be positive numbers. Then
\[
a(y_1 - y_2)^2 + by_2^2 \geq \frac{ab}{a+b}y_1^2.
\]

Proof. By computation, we have
\[
a(y_1 - y_2)^2 + by_2^2 = (a + b)(y_2 - \frac{ay_1}{a+b})^2 + \frac{ab}{a+b}y_1^2 \geq \frac{ab}{a+b}y_1^2.
\]

\(\square\)

Theorem 3.3. Let \( G \) be a connected \( k \)-uniform hypergraph, then
\[
\frac{\Delta}{\gamma^{k-1}} \leq \rho(G) \leq \gamma^{k-1} \delta.
\]

Proof. Let \( x = (x_1, \ldots, x_n)^T \) be the principal eigenvector of \( G \), for all \( i \in V(G) \), we have
\[
\rho(G) x_i^{k-1} = \sum_{e_i \in E(G)} x_{e_i \setminus \{i\}} \geq d_i x_i^{k-1} > 0.
\]
Suppose that \( d_{\mu} = \delta, \mu \in V(G) \), we obtain
\[
\rho(G) = \sum_{e_{\mu} \in E(G)} x_{e_{\mu} \setminus \{\mu\}}^{k-1} \leq \gamma^{k-1} \delta.
\]
Similarly, we have
\[
\rho(G) \geq \frac{\Delta}{\gamma^{k-1}}.
\]
Thus,
\[
\frac{\Delta}{\gamma^{k-1}} \leq \rho(G) \leq \gamma^{k-1} \delta.
\]

\(\square\)

Theorem 3.4. Let \( G \) be an irregular connected \( k \)-uniform hypergraph with \( n \) vertices and \( m \) edges, then
\[
\rho(G) < \frac{km\Delta}{km + (n\Delta - km)\gamma^{-k} + \frac{k}{2(k-1)D} \left[1 - \gamma^{-\frac{k}{2}}\right]^2},
\]

\[7\]
where \( D \) is the diameter of \( G \).

**Proof.** Let \( G \) be an irregular connected \( k \)-uniform hypergraph with \( n \) vertices and \( m \) edges, \( x = (x_1, \ldots, x_n)^T \) is the principal eigenvector of \( G \), then

\[
\Delta - \rho(G) = \Delta \sum_{i=1}^{n} x_i^k - k \sum_{e \in E(G)} x^e
\]

\[
= \sum_{i=1}^{n} (\Delta - d_i) x_i^k + \sum_{i=1}^{n} d_i x_i^k - k \sum_{e \in E(G)} x^e
\]

\[
= \sum_{i=1}^{n} (\Delta - d_i) x_i^k + \sum_{\{i_1 \cdots i_k\} \in E(G)} (x_{i_1}^k + \cdots + x_{i_k}^k - kx^e).
\]

Let \( x_u = \max x_i, \ x_v = \min x_i, \) since \( x > 0 \), \( G \) be an irregular connected \( k \)-uniform hypergraph, by Lemma 3.1, it yields that

\[
\Delta - \rho(G) > (n\Delta - km)x_v^k + \frac{1}{k-1} \sum_{i,j \in E(G)} (x_i^k - x_j^k)^2. \quad (3.1)
\]

Let \( P = v_0e_1v_1e_2 \cdots v_{l-1}e_lv_l \) be the shortest path from vertex \( u \) to vertex \( v \), where \( u = v_0, \ v = v_lv_{l-1}, v_i \in e_i, \) for \( i = 1, \ldots, l \), we have

\[
\sum_{i,j \in E(P)} (x_i^k - x_j^k)^2 \geq \sum_{i=0}^{l-1} (x_{v_i}^k - x_{v_{i+1}}^k)^2
\]

\[
+ \sum_{i=0}^{l-1} \sum_{u_i \in e_i \setminus \{v_{i-1}, v_i\}} [(x_{v_i}^k - x_{v_{i+1}}^k)^2 + (x_{u_{i+1}}^k - x_{v_{i+1}}^k)^2]
\]

\[
\geq \sum_{i=0}^{l-1} (x_{v_i}^k - x_{v_{i+1}}^k)^2 + \frac{1}{2} \left( \sum_{i=0}^{l-1} \sum_{u_i \in e_i \setminus \{v_{i-1}, v_i\}} (x_{u_i}^k - x_{v_{i+1}}^k)^2 \right)
\]

\[
= \sum_{i=0}^{l-1} (x_{v_i}^k - x_{v_{i+1}}^k)^2 + \frac{k-2}{2} \left( \sum_{i=0}^{l-1} (x_{v_i}^k - x_{v_{i+1}}^k)^2 \right).
\]
By Cauchy-Schwarz inequality, we obtain
\[
\sum_{i,j \in E(P)} (x_i^k - x_j^k)^2 \geq \frac{1}{l} \left( \sum_{i=0}^{l-1} (x_{u_i}^k - x_{v_{i+1}}^k) \right)^2 + \frac{k-2}{2l} \left( \sum_{i=0}^{l-1} (x_{u_i}^k - x_{v_{i+1}}^k) \right)^2
\]
\[
= \frac{1}{l} (x_u^k - x_v^k)^2 + \frac{k-2}{2l} (x_{u}^k - x_{v})^2
\]
\[
= \frac{k}{2l} (x_u^k - x_v^k)^2.
\]

Let \( D \) is the diameter of \( G \), since \( l = d(u, v) \leq D \), so
\[
\sum_{i,j \in E(P)} (x_i^k - x_j^k)^2 \geq \frac{k}{2D} (x_u^k - x_v^k)^2. \tag{3.2}
\]

By (3.1) and (3.2), it yields that
\[
\Delta - \rho(G) > (n\Delta - km)x_u^k + \frac{k}{2(k-1)D}(x_u^k - x_v^k)^2. \tag{3.3}
\]

Let \( \gamma \) be the principal ratio of \( G \), we have
\[
\frac{\Delta - \rho(G)}{x_u^k} > (n\Delta - km)\gamma^{-k} + \frac{k}{2(k-1)D} \left[ 1 - \gamma^{-\frac{k}{2}} \right]^2. \tag{3.4}
\]

It follows from Theorem 2.3 that \( x_u^k \geq \frac{\rho(G)}{km} \), so
\[
\frac{(\Delta - \rho(G)) km}{\rho(G)} > \frac{\Delta - \rho(G)}{x_u^k} > (n\Delta - km)\gamma^{-k} + \frac{k}{2(k-1)D} \left[ 1 - \gamma^{-\frac{k}{2}} \right]^2,
\]
\[
\rho(G) < \frac{km\Delta}{km + (n\Delta - km)\gamma^{-k} + \frac{k}{2(k-1)D} \left[ 1 - \gamma^{-\frac{k}{2}} \right]^2}.
\]

\[\square\]

**Theorem 3.5.** Let \( G \) be an irregular connected \( k \)-uniform hypergraph with \( n \) vertices and \( m \) edges, \( D \) is the diameter of \( G \), then
\[
\rho(G) < \Delta - \frac{2(k-1)D(n\Delta - km)\gamma^{-k} + k \left( 1 - \gamma^{-\frac{k}{2}} \right)^2}{2 (\gamma^{-k} + n - 1) (k-1)D}.
\]

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**Proof.** Let $x = (x_1, \ldots, x_n)^T$, $\gamma$ be the principal eigenvector and the principal ratio of $G$, respectively, $x_u = \max_i x_i$, by (3.4), we know

$$\Delta - \rho(G) > (n\Delta - km)\gamma^{-k} + \frac{k}{2(k-1)D} \left[1 - \gamma^{-\frac{k}{2}}\right]^2.$$  

By (2.6), we have

$$x_u^k \geq (\gamma^{-k} + n - 1)^{-1}.$$  

Thus

$$\Delta - \rho(G) > \frac{(n\Delta - km)\gamma^{-k} + \frac{k}{2(k-1)D} \left[1 - \gamma^{-\frac{k}{2}}\right]^2}{\gamma^{-k} + n - 1} = \frac{2(k-1)D(n\Delta - km)\gamma^{-k} + k \left(1 - \gamma^{-\frac{k}{2}}\right)^2}{2 (\gamma^{-k} + n - 1) (k-1)D}.$$  

$$\rho(G) < \Delta - \frac{2(k-1)D(n\Delta - km)\gamma^{-k} + k \left(1 - \gamma^{-\frac{k}{2}}\right)^2}{2 (\gamma^{-k} + n - 1) (k-1)D}.$$  

\[\square\]

**Theorem 3.6.** Let $G$ be an irregular connected $k$-uniform hypergraph with $n$ vertices and $m$ edges, $D$ is the diameter of $G$, then

$$\rho(G) < \Delta - \frac{k(n\Delta - km)}{[2(k-1)D(n\Delta - km) + k] \left[(\delta \Delta)^{\frac{k}{2(k-1)}} + n - 1\right]}.$$  

**Proof.** Let $G$ be an irregular connected $k$-uniform hypergraph with $n$ vertices and $m$ edges, Let $x = (x_1, \ldots, x_n)^T$, $\gamma$ be the principal eigenvector and the principal ratio of $G$, respectively, $x_u = \max_i x_i$, $x_v = \min_i x_i$, by (3.4), we know

$$\Delta - \rho(G) > (n\Delta - km)x_v^k + \frac{k}{2(k-1)D} (x_u^k - x_v^k)^2.$$
By Lemma 3.2 we get
\[ \Delta - \rho(G) > \frac{k(n\Delta - km)}{2(k-1)D(n\Delta - km) + k} x_u. \]  
(3.5)

By Theorem 2.2 we have
\[ x_u \geq \left[ \left( \frac{\delta}{\Delta} \right)^{\frac{k}{k-1}} + n - 1 \right]^{-\left( \frac{1}{k} \right)}. \]

Thus,
\[ \Delta - \rho(G) > \frac{k(n\Delta - km)}{[2(k-1)D(n\Delta - km) + k] \left[ \left( \frac{\delta}{\Delta} \right)^{\frac{k}{k-1}} + n - 1 \right]}. \]

i.e.
\[ \rho(G) < \Delta - \frac{k(n\Delta - km)}{[2(k-1)D(n\Delta - km) + k] \left[ \left( \frac{\delta}{\Delta} \right)^{\frac{k}{k-1}} + n - 1 \right]}. \]

**Theorem 3.7.** Let \( G \) be an irregular connected \( k \)-uniform hypergraph with \( n \) vertices and \( m \) edges, then
\[ \rho(G) < \frac{2m\Delta(k-1)D(n\Delta - km) + km\Delta}{2m(k-1)D(n\Delta - km) + n\Delta}, \]
where \( D \) is the diameter of \( G \).

**Proof.** Let \( x = (x_1, \ldots, x_n)^T \), \( \gamma \) be the principal eigenvector and the principal ratio of \( G \), respectively, \( x_u = \max_i x_i \), by (3.5), we know
\[ \Delta - \rho(G) > \frac{k(n\Delta - km)}{2(k-1)D(n\Delta - km) + k} x_u. \]

By Theorem 2.3 we have
\[ x_u \geq \left( \frac{\rho(G)}{km} \right)^{\frac{1}{k}}, \]

Thus
\[ km(\Delta - \rho(G)) > \frac{k(n\Delta - km)\rho(G)}{2(k-1)D(n\Delta - km) + k}. \]
\[ \rho(G) < \frac{km\Delta}{km + \frac{k(n\Delta - km)}{2(k-1)D(n\Delta - km) + km}} = \frac{2m\Delta(k-1)D(n\Delta - km) + km\Delta}{2m(k-1)D(n\Delta - km) + n\Delta}. \]

Let \( G \) be an irregular connected \( k \)-uniform hypergraph with \( n \) vertices and \( m \) edges, when \( k = 2 \), by Theorem 3.6 and Theorem 3.7 we can obtain results as follows.

**Corollary 3.8.** Let \( G \) be an irregular connected graph with \( n \) vertices and \( m \) edges, \( D \) is the diameter of \( G \), then

\[ \Delta - \rho(G) > \frac{n\Delta - 2m}{[D(n\Delta - 2m) + 1] \left[ \frac{2}{D} + n - 1 \right]} > \frac{n\Delta - 2m}{n(D(n\Delta - 2m) + 1)}. \]

**Corollary 3.9.** Let \( G \) be an irregular connected graph with \( n \) vertices and \( m \) edges, \( D \) is the diameter of \( G \), then

\[ \Delta - \rho(G) > \Delta - \frac{2m\Delta D(n\Delta - 2m) + 2m\Delta}{2mD(n\Delta - 2m) + n\Delta} > \frac{n\Delta - 2m}{n(D(n\Delta - 2m) + 1)}. \]

**Remark:** For a connected irregular graph \( G \) with \( n \) vertices and \( m \) edges, Cioabă, Gregory and Nikiforov [13] obtain the following bound

\[ \Delta - \rho(G) > \frac{n\Delta - 2m}{n(D(n\Delta - 2m) + 1)}. \] (3.6)

Clearly, the results of Corollary 3.8 and Corollary 3.9 improve bound (3.6).

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