Harnack inequality and continuity of solutions to quasi-linear degenerate parabolic equations with coefficients from Kato-type classes

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Abstract

For a general class of divergence type quasi-linear degenerate parabolic equations with measurable coefficients and lower order terms from non-linear Kato-type classes, we prove local boundedness and continuity of solutions, and the intrinsic Harnack inequality for positive solutions.

Keywords: quasi-linear degenerate parabolic equations, local boundedness, continuity, Harnack inequality.

1 Introduction and main results

In this paper we are concerned with general divergence type quasi-linear degenerate parabolic equations with measurable coefficients and lower order terms. This class of equations has numerous applications and has been attracting attention for several decades (see, e.g. the monographs [7, 17, 31], survey [8] and references therein).

Let $\Omega$ be a domain in $\mathbb{R}^n$, $T > 0$. Set $\Omega_T = \Omega \times (0, T)$. We study solutions to the equation

$$u_t - \text{div} A(x, t, u, \nabla u) = a_0(x, t, u, \nabla u), \quad (x, t) \in \Omega_T. \quad (1.1)$$

Throughout the paper we suppose that the functions $A : \Omega \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $a_0 : \Omega \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ are such that $A(\cdot, \cdot, u, \zeta)$, $a_0(\cdot, \cdot, u, \zeta)$ are Lebesgue measurable for all $u \in \mathbb{R}, \zeta \in \mathbb{R}^N$, and $A(x, t, \cdot, \cdot)$, $a_0(x, t, \cdot, \cdot)$ are continuous for almost all $(x, t) \in \Omega_T$.

We also assume that the following structure conditions are satisfied:

$$A(x, t, u, \zeta) \zeta \geq c_1 |\zeta|^p, \quad \zeta \in \mathbb{R}^n,$$

$$|A(x, t, u, \zeta)| \leq c_2 |\zeta|^{p-1} + g_1(x) |u|^{p-1} + f_1(x),$$

$$|a_0(x, t, u, \zeta)| \leq h(x) |\zeta|^{p-1} + g_2(x) |u|^{p-1} + f_2(x), \quad \quad (1.2)$$

where $2 < p < n$, $c_1, c_2$ are positive constants and $f_1(x), f_2(x), g_1(x), g_2(x), h(x)$ are nonnegative functions, satisfying conditions which will be specified below. The constants in (1.2), $n$ and $p$ are further referred to as the data. The aim of this paper is to establish basic qualitative properties such as local boundedness of weak solutions, their continuity and the Harnack inequality for positive solutions under minimal possible restrictions on the coefficients in structure conditions (1.2). These properties are indispensable in the qualitative theory of second-order elliptic and parabolic equations. For equation (1.1) with $g_1 = g_2 = h = 0$ and $f_1, f_2$ constants the local boundedness and Hölder continuity of solutions

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was know since mid-1980s (see [7, 8] for the results, references and historical notes), and a recent break through has been made in [10], where the intrinsic Harnack inequality has been proved. Before stating precisely our results we make several remarks related to lower order terms of (1.1) and refer the reader for an extensive survey of the regularity issues to [7, 8, 10].

Local boundedness and Hölder continuity of weak solutions to homogeneous linear divergence type second-order elliptic equations with measurable coefficients without lower order terms is known since the famous results by De Giorgi [6] and Nash [23], and the Harnack inequality since Moser’s celebrated paper [22]. However in presence of lower order term in the equation weak solutions may have singularities and/or internal zeroes, and the Harnack inequality in general may not be valid, as one can easily realize looking at the equation \(-\Delta u + \frac{1}{|x|} u = 0\). It was Serrin [24] who generalized Moser’s result to the case of quasi-linear equations with lower order terms with conditions expressed in terms of \(L^p\)-spaces. Using probabilistic techniques Aizenman and Simon in their famous paper [11] proved the Harnack inequality and continuity of weak solutions to the equation \(-\Delta u + Vu = 0\) under the local Kato class condition on the potential \(V\). Moreover, they showed that the Kato type condition on the potential \(V\) is necessary for the validity of the Harnack inequality. Soon after that Chiarenza, Fabes and Garofalo [5] developed a real variables techniques to prove the Harnack inequality for a linear equation of divergence type with measurable coefficients and the potential from the Kato class, thus extending Aizenman, Simon’s result.

Kurata [15] extended the method of Chiarenza, Fabes and Garofalo and proved the same for the equation \(-\sum_{k,j} \partial_k a_{kj} \partial_j u + \sum_j b_j \partial_j u + Vu = 0\), with \(|b|^2, V\) from the Kato class. Both papers [5] and [15] make a heavy use of Green’s functions which makes this approach inapplicable to quasi-linear equations. To treat the quasi-linear case of \(p\)-Laplacian with a lower order term Biroli [3, 4] introduced the notion of the nonlinear Kato class and gave the Harnack inequality for positive solutions to \(-\Delta_p u + Vu^{p-1} = 0\). This was extended in [28] to the general case of quasi-linear elliptic equations with lower order terms.

For second-order linear parabolic equations with measurable coefficients (without lower order terms) Hölder continuity of solutions was first proved by Nash [23]. Moser [22] proved the validity of the Harnack inequality which was extended to the case of quasi-linear equations with \(p = 2\) in the structure conditions and structure coefficients from \(L^p\)-classes in [2, 29]. The continuity of weak solutions and the Harnack inequality for second-order linear elliptic equations with lower order coefficients from Kato-classes was proved by Zhang [32, 33].

The parabolic theory for degenerate quasi-linear equations differs substantially from the "linear" case \(p = 2\) which can be already realized looking at the Barenblatt solution to the parabolic \(p\)-Laplace equation. DiBenedetto developed an innovative intrinsic scaling method (see [2] and the references to the original papers there; see also a nice exposition in [30] where some recent advances are included) and proved the Hölder continuity of weak solutions to (1.1) for \(p \neq 2\) for the case \(g_1 = g_2 = h = 0\) and \(f_1, f_2\) from \(L^q\)-classes, and the intrinsic Harnack inequality for the parabolic \(p\)-Laplace equations. For the measurable coefficients in the main part of (1.1) the intrinsic Harnack inequality was proved in the recent break-through paper [10]. It is natural to conjecture that the intrinsic Harnack inequality holds for the parabolic \(p\)-Laplace equation perturbed by lower order terms with coefficients from Kato classes. The difficulty is that seemingly neither De Giorgi nor Moser iteration techniques work in this situation.

In this paper following the strategy of [10] but using a different iteration, namely the Kilpeläinen-Malý technique [13] properly adapted to the parabolic equations (cf. [20, 27]), we establish the local boundedness and continuity of solutions to (1.1) and the intrinsic Harnack inequality.

Following Biroli [3, 4] we introduce the non-linear Kato \(K_p\) class by

\[
K_p := \left\{ g \in L^1(\Omega) : \limsup_{R \to 0} \sup_{x \in \Omega} \int_0^R \frac{1}{r^{n-p}} \int_{B_r(x) \cap \Omega} |g(z)| dz \, dr = 0 \right\},
\]

where \(B_r(x) = \{ z \in \Omega : |z - x| < r \}\). As one can easily see, for \(p = 2\), \(K_p\) reduces to the standard definition of the Kato class as defined in [11, 23].

We will also need the class \(\widetilde{K}_p\) of functions \(g \in L^1(\Omega)\) satisfying the condition

\[
\limsup_{R \to 0} \sup_{x \in \Omega} \int_0^R \frac{1}{r^{n-p}} \int_{B_r(x)} |g(z)| dz \, dr = 0.
\]

2
It is easy to see that $\widetilde{K}_p \subset K_p$. We assume that

\begin{align}
F_1 & := (g_1 + f_1)^{\frac{p}{p-1}} \in \widetilde{K}_p, \\
F_2 & := h^p + g_2 + f_2 \in K_p.
\end{align}

In what follows we use the following quantities

\begin{align}
\mathcal{F}_1(R) & = \sup_{x \in \Omega} \int_0^R \left( \frac{1}{r^{n-p}} \int_{B_r(x)} F_1(z) dz \right) \frac{dr}{r}, \\
\mathcal{F}_2(R) & = \sup_{x \in \Omega} \int_0^R \left( \frac{1}{r^{n-p}} \int_{B_r(x)} F_2(z) dz \right) \frac{dr}{r}.
\end{align}

Before formulating the main results, let us remind the reader of the definition of a weak solution to equation (1.1).

We say that $u$ is a weak solution to (1.1) if $u \in V(\Omega_T) := W^{1,p}_{\text{loc}}(\Omega_T) \cap C([0,T]; L^2_{\text{loc}}(\Omega))$ and for any interval $[t_1, t_2] \subset (0,T)$ the integral identity

\begin{align}
\int_{t_1}^{t_2} \int_{\Omega} u \phi dx + \int_{t_1}^{t_2} \int_{\Omega} \{ -u \phi_{\tau} + A(x, \tau, u, \nabla u) \nabla \phi - a_0(x, \tau, u, \nabla u) \phi \} \, dx \, d\tau = 0
\end{align}

for any $\phi \in \overset{\circ}{V}(\Omega_T)$.

The first main result of this paper is the local boundedness of solutions.

**Theorem 1.1.** Let conditions (1.2), (1.5) and (1.6) be fulfilled. Let $u$ be a weak solution to equation (1.1). Then $u$ locally bounded, that is $u \in L^\infty_{\text{loc}}(\Omega_T)$.

The proof of Theorem 1.1 is based on the adaptation of the Kilpeläinen-Malý technique [13] to parabolic equations using ideas from [26, 27]. Having established the local boundedness we proceed with the continuity. At this stage we can assume that the solutions are bounded in $\Omega_T$.

**Theorem 1.2.** Let conditions (1.2), (1.5) and (1.6) be fulfilled and $h(x) = 1$. Let $u$ be a bounded weak solution to equation (1.1). Then $u$ is continuous, that is $u \in C(\Omega_T)$.

Next is the Harnack inequality for positive solutions to (1.1).

Let $u$ be a nonnegative solution to (1.1). Fix a point $(x_0, t_0) \in \Omega_T$ such that $u(x_0, t_0) > 0$. Consider the cylinders

\begin{align}
Q^\theta(x_0, t_0) = B_\rho(x_0) \times (t_0 - \theta \rho^p, t_0 + \theta \rho^p), \quad \theta = \left( \frac{c}{u(x_0, t_0)} \right)^{p-2},
\end{align}

where $c > 0$ is fixed.

**Theorem 1.3.** Let conditions the conditions of Theorem 1.2 be fulfilled and $h(x) = 1$. Let $u$ be a positive solution to (1.1). Then there exist positive constants $c, \gamma$ depending only on the data and $\sup_{\Omega_T} u(x, t)$, such that for all intrinsic cylinders $Q^\theta(x_0, t_0) \subset \Omega_T$ either $u(x_0, t_0) \leq \gamma (\rho + F_1(2\rho) + F_2(2\rho))$ or

\begin{align}
\sup_{Q^\theta(x_0, t_0)} u(x, t) \leq \gamma \inf_{B_\rho(x_0)} u(x, t_0 + \theta \rho^p), \quad \theta = \left( \frac{c}{u(x_0, t_0)} \right)^{p-2}.
\end{align}

Moreover, if $g_1 = g_2 = 0$, the constant $\gamma$ can be chosen independent from $\sup_{\Omega_T} u(x, t)$. 


Remark 1.4. In the linear theory the Kato class is known to be the optimal condition on the zero order term of the equation $\Delta u + Vu = 0$ to imply the continuity of solutions and the Harnack inequality. The same is true for the quasi-linear equations. For the equation $\Delta_p u + Vu|u|^{p-2} = 0$ with $V$ behaving around zero like $c \left( \log \frac{1}{|x|} \right)^{1-p}$, depending on the sign of $V$, one can easily produce a solution with singularity at zero, or with internal zero at zero (see, e.g. [18, 19]). On the other hand, the function $c \left( \log \frac{1}{|x|} \right)^{1-p+\varepsilon}$ is from the Kato class $K_p$ for any $\varepsilon > 0$, and Theorems 1.1, 1.2, 1.3 apply.

Let $u$ be a weak solution to (1.1) in $\Omega_T$. Let $(y, s) \in \Omega_T$ be an arbitrary point. Consider the cylinder $Q^\theta_{4\rho}(y, s) \subset \Omega_T$,

$$Q^\theta_{4\rho}(y, s) = B_\rho(y) \times (s - \theta \rho^p, s), \quad \theta > 0.$$ 

Denote by $\mu_\pm$ and $\omega$ non-negative numbers such that

$$\mu_+ \geq \operatorname{ess sup} u(x, t), \quad \mu_- \leq \operatorname{ess inf} u(x, t), \quad \omega \geq \mu_+ - \mu_-.$$

As was already mentioned, our strategy of the proof of the Harnack inequality is the same as in [10]. Namely, Theorems 1.2, 1.3 will be consequences of the following two theorems.

The next theorem is a De Giorgi-type lemma (cf. [10]), and its formulation is almost the same as in [10]. However, due to the different structure conditions the De Giorgi type iteration cannot be used. Instead, we adapt the Kilpeläinen-Malý iteration [13] combined with ideas from [26, 27], where the Kilpeläinen-Malý technique was adapted to parabolic equations.

**Theorem 1.5.** Let the conditions of Theorem 1.3 be fulfilled. Fix $\xi, a \in (0, 1)$, $(\xi \omega)^{p-2} \geq \frac{1}{\theta}$. There exist numbers $B \geq 1$ and $\nu \in (0, 1)$ depending only on the data and $\theta, \xi, \omega$ and $a$ such that if

$$(1.10) \quad \left| \{(x, t) \in Q^\theta_{2\rho}(y, s) : u(x, t) \leq \mu_- + \xi \omega \} \right| \leq \nu|Q^\theta_{2\rho}(y, s)|,$$

then either $\xi \omega \leq B(\rho + F_1(2\rho) + F_2(2\rho))$, or

$$(1.11) \quad u(x, t) \geq \mu_- + a \xi \omega \quad \text{for almost all (a.a.)} \quad (x, t) \in Q^\theta_{\rho}(y, s).$$

Likewise, if

$$(1.12) \quad \left| \{(x, t) \in Q^\theta_{2\rho}(y, s) : u(x, t) \geq \mu_+ - \xi \omega \} \right| \leq \nu|Q^\theta_{2\rho}(y, s)|,$$

then either $\xi \omega \leq B(\rho + F_1(2\rho) + F_2(2\rho))$, or

$$(1.13) \quad u(x, t) \leq \mu_+ - a \xi \omega \quad \text{for almost all (a.a.)} \quad (x, t) \in Q^\theta_{\rho}(y, s),$$

where $F_1(\rho)$, $F_2(\rho)$ are defined in (1.5), (1.6).

The following theorem is an expansion of positivity result, analogous in formulation as well as in the proof to [10] Lemma 3.1.

**Theorem 1.6.** Let the conditions of Theorem 1.3 be fulfilled. There exist positive numbers $B, b_1 < b_2$ and $\sigma \in (0, 1)$ depending only on the data such that if

$$(1.14) \quad u(x, s) \geq \mu_- + N \quad \text{for} \quad x \in B_\rho(y),$$

then either $N \leq B(\rho + F_1(2\rho) + F_2(2\rho))$, or

$$(1.15) \quad u(x, t) \geq \mu_- + \sigma N \quad \text{for a.a.} \quad x \in B_{2\rho}(y),$$

for all

$$(1.16) \quad s + N^{2-p} b_1 \rho^p \leq t \leq s + N^{2-p} b_2 \rho^p.$$
If on the other hand
\begin{equation}
(1.17)\quad u(x, s) \leq \mu_+ - N \quad \text{for} \quad x \in B_\rho(y),
\end{equation}
then either \( N \leq B(\rho + F_1(2\rho) + F_2(2\rho)) \), or
\begin{equation}
(1.18)\quad u(x, t) \leq \mu_+ - \sigma N \quad \text{for a.a.} \quad x \in B_{2\rho}(y),
\end{equation}
for all \( t \) satisfying \((1.10)\).

The rest of the paper contains the proof of the above theorems. In Section 2 we collect some auxiliary propositions and required integral estimates of solutions. In Section 3 we give a proof of local boundedness of solutions which is based on the parabolic modification of the Kilpeläinen-Malý technique [13]. Section 4 contains the proof of the variant of De Giorgi lemma, Theorem 1.3. Expansion of positivity, Theorem 1.6 is proved in Section 5. In Section 6 we prove continuity of solutions following [7]. Finally, in Section 7 we sketch a proof of the intrinsic Harnack inequality, Theorem 1.3, leaving out details for which we refer to [10].

2 Auxiliary material and integral estimates of solutions

2.1 Local energy estimates

Lemma 2.1. Let \( u \) be a solution to \((1.1)\) in \( \Omega_T \). Then there exists \( \gamma > 0 \) depending only on \( n, p, c_1, c_2 \) such that for every cylinder \( Q^\rho_k(y, s) = B_\rho(y) \times (s - \theta \rho^p, s) \subset \Omega_T \), any \( k \in \mathbb{R}^1 \) and any smooth \( \xi(x, t) \) which is zero for \( (x, t) \in \partial B_\rho(y) \times (s - \theta \rho^p, s) \) one has
\begin{equation}
(2.1)\quad \sup_{s - \theta \rho^p < t < s} \int_{B_\rho(y)} \left( u - k \right)^2 \xi^p dx + c_1 \int_{Q^\rho_k(y, s)} |\nabla (u - k)\xi|^p dx dt \\
\leq \int_{B_\rho(y)} \left( u - k \right)^2 \xi^p dx + \gamma \int_{Q^\rho_k(y, s)} \left( (u - k)^p |\nabla \xi|^p + (u - k)^2(|\xi|) \right) dx dt \\
+ \gamma \int_{A_\rho(y, s)} |f_1(x)| \frac{\rho}{\theta^p} + g_1(x) \frac{\rho}{\theta} |u|^p \xi dx dt + \gamma \int_{Q^\rho_k(y, s)} (u - k)^2 h(x)\xi^p dx dt,
\end{equation}
where \( A_\rho(y, s) = Q^\rho_k(y, s) \cap \{(u - k)_\pm > 0\} \).

Proof. Test \((1.1)\) by \( \varphi = (u - k)_\pm \xi^p \) and use conditions \((1.2)\) and the Hölder and Young inequalities. □

Let
\[ H^\pm_k := \text{ess sup}_{Q^\rho_k(y, s)} |(u - k)\pm|, \quad \Psi^\pm(u) := \left( \ln \frac{H^\pm_k}{H^\pm_k - (u - k)\pm + c} \right)_+, \quad 0 < c < H^\pm_k. \]

Lemma 2.2. Let \( u \) be a solution to \((1.1)\) in \( \Omega_T \). Then there exists \( \gamma > 0 \) depending only on \( n, p, c_1, c_2 \) such that for every cylinder \( Q^\rho_k(y, s) = B_\rho(y) \times (s - \theta \rho^p, s) \subset \Omega_T \), any \( k \in \mathbb{R}^1 \) and any smooth \( \xi(x) \) which is zero for \( |x - y| > \rho \) one has
\begin{equation}
(2.2)\quad \sup_{s - \theta \rho^p < t < s} \int_{B_\rho(y)} \Psi^2_\pm(u)\xi^p dx \leq \int_{B_\rho(y) \times (s - \theta \rho^p, s)} \Psi^2_\pm(u)\xi^p dx + \gamma \int_{Q^\rho_k(y, s)} \Psi^2_\pm(u)|\xi|^{2-p}|\nabla \xi|^p dx dt \\
+ \gamma \int_{A_\rho(y, s)} \Psi_\pm |\Psi_\pm(u)|^{2-p} h(x)\xi^p dx dt + \gamma \int_{A_\rho(y, s)} \Psi_\pm |\Psi_\pm(u)|^2 \left( f_1(x) \frac{\rho}{\theta^p} + g_1(x) \frac{\rho}{\theta} |u|^p \right) dx dt,
\end{equation}
The proof is analogous to that of [7] Proposition 3.2, Chapter II.
2.2 Auxiliary propositions

The following two lemmas will be used in the sequel. The first one is the well known De Giorgi-Poncaré lemma (see [7, Chapter I], [16, Chapter II, Lemma 3.9]).

**Lemma 2.3.** Let $u \in W^{1,1}(B_\rho(y))$ for some $\rho > 0$ and $y \in \mathbb{R}^n$. Let $k$ and $l$ be real numbers such that $k < l$. Then there exists a constant $\gamma$ depending only on $n$ such that

$$
(2.3) \quad (l - k)|A_{k,\rho}|B_\rho \leq \gamma \rho^{n+1} \int_{A_{l,\rho}\setminus A_{k,\rho}} |\nabla u| dx,
$$

where $A_{k,\rho} = \{ x \in B_\rho : u(x) < k \}$.

The next lemma is the time-dependent version of the measure-theoretic lemma from [9], which can be extracted from [10, Section 8].

**Lemma 2.4.** Let $Q_1 = B_1(0) \times (-1,0)$ and $v \in V(Q_1)$. Let $v$ satisfy (2.1). Suppose that there exist constants $\gamma > 0$ and $\nu \in (0,1)$ such that

$$
(2.4) \quad \int_{Q_1} |\nabla v|^p dx \leq \gamma \quad \text{and} \quad |\{ (x,t) \in Q_1 : v(x,t) > 1 \}| > \nu.
$$

Then for any $\lambda \in (0,1)$ and $\nu_0 \in (0,1)$ there exist a point $(y,s) \in Q_1$, a number $\eta_0 \in (0,1)$ and a cylinder $Q_{2\eta_0}(y,s) \subset Q_1$ such that

$$
(2.5) \quad |\{ (x,t) \in Q_{\eta_0}(y,s) : v(x,t) > \lambda \}| \geq (1 - \nu_0)|Q_{\eta_0}(y,s)|,
$$

where $Q_R(y,s) = B_R(y) \times (s - R^p, s)$.

In what follows we will frequently use the following lemma which is due to Biroli [3, 4].

**Lemma 2.5.** Let $1 < q < n$. For any $\varepsilon > 0$ there exist $R_0 < 1$ and $\tau > 0$ such that the inequality

$$
(2.6) \quad \sup_{x \in B_1(0)} \int_0^{R_0} \left( \frac{1}{r^{n-q}} \int_{B_r(x) \cap B_1(0)} H(z) dz \right)^{\frac{1}{1-\frac{q}{n}}} \frac{dr}{r} < \tau
$$

implies that

$$
(2.7) \quad \int_{B_R(x_0)} H(x)|\varphi(x)|^q dx \leq \varepsilon \int_{B_R(x_0)} |\nabla \varphi(x)|^q dx
$$

for any $\varphi \in \dot{W}^{1,q}(B_R(x_0))$ if $R \leq R_0$ and $B_{4R_0}(x_0) \subset B_1(0)$.

2.3 Integral estimates of solutions

Set

$$
G(u) = \begin{cases} 
    u & \text{for } u > 1, \\
    u^{2\lambda} & \text{for } 0 < u \leq 1.
\end{cases}
$$

**Lemma 2.6.** Let the conditions of Theorem 1.1 be fulfilled. Let $u$ be a solution to (1.1). Then there exists a constant $\gamma > 0$ depending only on $n, p, c_1, c_2$ such that for any $\varepsilon \in (0,1)$, $l, \delta > 0$ and any cylinder

$$
Q_{\rho}^{(\delta)}(y,s) = B_\rho(y) \times (s - \delta^2 - p^p, s + \delta^2 - p^p) \subset \Omega_T, \quad \rho \leq R
$$
and any $\xi \in C^{\infty}_c(Q^\delta_p(y,s))$ such that $\xi(x,t) = 1$ for $(x,t) \in Q^\delta_{p/2}(y,s)$
\[
I_1 := \delta^2 \int_{L(t)} u(x,t) G \left( \frac{u(x,t) - l}{\delta} \right) \xi(x,t)^k dx
\]
\[
+ \int \int_L \left\{ \int_{l}^{u} \left( 1 + \frac{s - l}{\delta} \right)^{-1+\lambda} \left( \frac{s - l}{\delta} \right)^{-2\lambda} ds + u \left( 1 + \frac{u - l}{\delta} \right)^{-1+\lambda} \left( \frac{u - l}{\delta} \right)^{-2\lambda} \right\} |\nabla u|^p \xi(x,t) dx dt
\]
\[
\leq \gamma \delta^2 \int \int L \left( \frac{u - l}{\delta} \right)^{1-\lambda \left( p - 1 \right)} \xi^k dx dt + \gamma \delta^p \int \int L u \left( 1 + \frac{u - l}{\delta} \right)^{-1-\lambda} \left( \frac{u - l}{\delta} \right)^{-2\lambda} ds F_1(x) \xi^k dx dt
\]
\[
+ \varepsilon \delta^p \int \int L u \left( 1 + \frac{u - l}{\delta} \right)^{-1-\lambda} \xi^k dx dt + \gamma \int \int L \int L u^p \int_{l}^{u} \left( 1 + \frac{s - l}{\delta} \right)^{-1+\lambda} \left( \frac{s - l}{\delta} \right)^{-2\lambda} ds F_2(x) \xi^k dx dt
\]
\[
(2.8) + \gamma \rho^p \delta^3 \int_{B_\rho(y)} f_2(x) dx := \sum_{i=2}^s I_i,
\]
where $L = Q^\delta_p(y,s) \cap \{ u > l \}$, $L(t) = L \cap \{ t = t \}$ and $\lambda = \min \left\{ \frac{1}{p+1}, \frac{p+2}{p-1} \right\}$, $k = \frac{(p+2)(p-1)(1+\lambda)}{p-1-\lambda} + p$.

**Proof.** First, note that
\[
\int_{l}^{u} \left( 1 + \frac{s - l}{\delta} \right)^{-1+\lambda} \left( \frac{s - l}{\delta} \right)^{-2\lambda} ds \leq \gamma \delta,
\]
and
\[
\int_{l}^{u} w \int_{l}^{u} \left( 1 + \frac{s - l}{\delta} \right)^{-1+\lambda} \left( \frac{s - l}{\delta} \right)^{-2\lambda} ds = \frac{1}{2} \int_{l}^{u} \left( 1 + \frac{s - l}{\delta} \right)^{-1+\lambda} \left( \frac{s - l}{\delta} \right)^{-2\lambda} (u^2 - s^2) ds
\]
\[
\geq \frac{1}{4} u(u - l) \int_{l}^{u+l} \left( 1 + \frac{s - l}{\delta} \right)^{-1+\lambda} \left( \frac{s - l}{\delta} \right)^{-2\lambda} ds = \frac{\delta^2}{4} u \left( \frac{u - l}{\delta} \right) \int_0^{u+l} (1 + z)^{-1+\lambda} z^{-2\lambda} dz
\]
\[
(2.10) \geq \gamma \delta^2 u G \left( \frac{u - l}{\delta} \right).
\]
Test \((1.3)\) by $\varphi$ defined by
\[
\varphi(x,t) = u(x,t) \left[ \int_{l}^{u(x,t)} \left( 1 + \frac{s - l}{\delta} \right)^{-1+\lambda} \left( \frac{s - l}{\delta} \right)^{-2\lambda} ds \right] \xi(x,t)^k,
\]
and $t_1 = s - \delta^{2-p} \rho^p$, $t_2 = t$. Using \((1.2)\), we have for any $t > 0$
\[
\int_{L(t)} \int_{l}^{u} w \int_{l}^{u} \left( 1 + \frac{s - l}{\delta} \right)^{-1+\lambda} \left( \frac{s - l}{\delta} \right)^{-2\lambda} ds \xi^k dx dt
\]
\[
+ \int \int_{L} \left\{ \int_{l}^{u} \left( 1 + \frac{s - l}{\delta} \right)^{-1+\lambda} \left( \frac{s - l}{\delta} \right)^{-2\lambda} ds + u \left( 1 + \frac{u - l}{\delta} \right)^{-1+\lambda} \left( \frac{u - l}{\delta} \right)^{-2\lambda} \right\} |\nabla u|^p \xi^k dx dt
\]
\[
\leq \gamma \int \int_{L} \int_{l}^{u} w \int_{l}^{u} \left( 1 + \frac{s - l}{\delta} \right)^{-1+\lambda} \left( \frac{s - l}{\delta} \right)^{-2\lambda} ds \xi^k \xi^{k-1} dx dt
\]
\[
+ \gamma \int \int_{L} \int_{l}^{u} \left( 1 + \frac{s - l}{\delta} \right)^{-1+\lambda} \left( \frac{s - l}{\delta} \right)^{-2\lambda} ds \left[ |\nabla u|^{p-1} + g_1 u^{p-1} + f_1 \right] |\nabla \xi| \xi^{k-1} dx dt
\]
\[
+ \gamma \int \int_{L} \int_{l}^{u} \left( 1 + \frac{s - l}{\delta} \right)^{-1+\lambda} \left( \frac{s - l}{\delta} \right)^{-2\lambda} ds \left[ h|\nabla u|^{p-1} + g_2 u^{p-1} + f_2 \right] \xi^k dx dt.
\]
From this using \((2.9)\), \((2.10)\) and Young’s inequality we obtain the required \((2.8)\).
Proof. I

Lemma 2.7. Let the conditions of Lemma 2.6 be fulfilled. Then there exist \( \nu_1 \in (0,1) \) depending only on \( n, p, c_1, c_2 \) such that the inequality

\[
\mathcal{F}_1(2R) + \mathcal{F}_2(2R) \leq \nu_1
\]

implies that

\[
\int \left( \frac{u - l}{\delta} \right)^{\frac{\nu_1^\frac{1}{2}}{\rho^p}} \xi^k d\tau + \delta^{p-2} \int L u (1 + \frac{u - l}{\delta})^{(1+\lambda)(p-1)} F - \xi^k - p d\tau \\
+ \gamma \frac{\delta^{p-2}}{\rho^p} \int L u \left( 1 + \frac{u - l}{\delta} \right)^{2\lambda(p-1)} \xi^k - p d\tau \\
+ \gamma \nu_1 \frac{1}{\rho^p} \int L f_1(x) dx + \gamma \frac{1}{\rho^p} \int f_2(x) dx
\]

(2.13)

(2.14)

Proof. In the notation of Lemma 2.6 with \( \varepsilon = \nu_1^{\frac{1}{2}} \), using the Young inequality we have

\[
I_2 + I_3 + I_4 \leq \nu_1^{\frac{1}{2}} \frac{\delta p}{\rho^p} \int L u \left( 1 + \frac{u - l}{\delta} \right)^{(1+\lambda)(p-1)} \xi^k - p d\tau \\
+ \gamma \frac{\delta^{p-2}}{\rho^p} \int L u \left( 1 + \frac{u - l}{\delta} \right)^{2\lambda(p-1)} \xi^k - p d\tau.
\]

(2.15)

Set

\[
I_9 = \gamma \int L (u - l)^p \int L (1 + \frac{s - l}{\delta})^{1+\lambda} \left( \frac{s - l}{\delta} \right)^{-2\lambda} d\tau \\
I_{10} = \gamma \nu_1^{\frac{1}{2}} \int L (u - l)^{p+1} \left( 1 + \frac{u - l}{\delta} \right)^{-1-\lambda} F_1(x) \xi^k d\tau.
\]

First we estimate \( I_{10} \). By the Young inequality and Lemma 2.5 we obtain

\[
I_{10} \leq \gamma \nu_1^{\frac{1}{2}} \int L (u - l)^p \left( 1 + \frac{u - l}{\delta} \right)^{-1-\lambda} |\nabla u|^{\rho^p} \xi^k d\tau \\
+ \gamma \nu_1^{\frac{1}{2}} \delta^{-p} \int L (u - l)^{p+1} \left( 1 + \frac{u - l}{\delta} \right)^{-1-\lambda-p} |\nabla u|^{\rho^p} d\tau \\
+ \gamma \nu_1^{\frac{1}{2}} \rho^p \int L (u - l)^{p+1} \left( 1 + \frac{u - l}{\delta} \right)^{-1-\lambda} \xi^k - p d\tau
\]

(2.16)

To estimate \( I_9 \) we consider the weak solution to the problem

\[-\Delta \varphi = F_2, \quad H \in W^{1,p}(B_\rho(y)),\]

i.e.

\[
\int_{B_\rho(y)} \nabla H \nabla \varphi dx = \int_{B_\rho(y)} F_2 \varphi dx,
\]

(2.17)
for any $\varphi \in W^{1,p}(B_\rho(y))$. By (13) we have
\begin{equation}
\|H\|_{L^\infty(B_\rho(y))} \leq \gamma F_2(2\rho).
\end{equation}

Testing (2.17) by
\[ \varphi = (u - l)^p \int_l^u \left(1 + \frac{s - l}{\delta}\right)^{-1+\lambda} \left(\frac{s - l}{\delta}\right)^{-2\lambda} \, ds \xi^k, \]
and using the Young inequality we have
\begin{align*}
\int_{B_\rho(y)} (u - l)^p \int_l^u \left(1 + \frac{s - l}{\delta}\right)^{-1+\lambda} \left(\frac{s - l}{\delta}\right)^{-2\lambda} \, ds \, F_2(x) \xi^k \, dx
\leq \gamma \int_{B_\rho(y)} \nabla H \, \left| \int_l^u \left(1 + \frac{s - l}{\delta}\right)^{-1+\lambda} \left(\frac{s - l}{\delta}\right)^{-2\lambda} \, ds \xi^k \right| \, dx
\leq \frac{\nu_1^2}{8\gamma} \int_{B_\rho(y)} \int_l^u \left(1 + \frac{s - l}{\delta}\right)^{-1+\lambda} \left(\frac{s - l}{\delta}\right)^{-2\lambda} \, ds \, |\nabla u|^p \xi^k \, dx
\end{align*}
\begin{equation}
\quad + \frac{\nu_1^2}{8\gamma} \int_{B_\rho(y)} \int_l^u (u - l)^p \left(1 + \frac{s - l}{\delta}\right)^{-1+\lambda} \left(\frac{s - l}{\delta}\right)^{-2\lambda} \, ds \, |\nabla H|^p \xi^k \, dx
\end{equation}
\begin{equation}
\quad + \gamma \nu_1 \frac{\nu_1^2}{8\gamma} \int_{B_\rho(y)} \int_l^u (u - l)^p \left(1 + \frac{s - l}{\delta}\right)^{-1+\lambda} \left(\frac{s - l}{\delta}\right)^{-2\lambda} \, ds \, |\nabla H|^p \xi^k \, dx
\end{equation}
\begin{equation}
\quad + \frac{\nu_1^2}{8\gamma} \int_{B_\rho(y)} (u - l)^p \xi^k \, dx.
\end{equation}

Using the definition of the weak solution to (2.17) again, we have
\begin{align*}
\int_{B_\rho(y)} (u - l)^p \int_l^u \left(1 + \frac{s - l}{\delta}\right)^{-1+\lambda} \left(\frac{s - l}{\delta}\right)^{-2\lambda} \, ds \, |\nabla H|^p \xi^k \, dx
= \int_{B_\rho(y)} F_2(x) H(u - l)^p \int_l^u \left(1 + \frac{s - l}{\delta}\right)^{-1+\lambda} \left(\frac{s - l}{\delta}\right)^{-2\lambda} \, ds \xi^k \, dx
\end{align*}
\begin{equation}
\quad + \int_{B_\rho(y)} H |\nabla H|^{p-2} \nabla H \cdot \nabla \left( (u - l)^p \int_l^u \left(1 + \frac{s - l}{\delta}\right)^{-1+\lambda} \left(\frac{s - l}{\delta}\right)^{-2\lambda} \, ds \xi^k \right) \, dx,
\end{equation}
and
\begin{align*}
\int_{B_\rho(y)} (u - l)^{p+1} \left(1 + \frac{u - l}{\delta}\right)^{-1+\lambda} \left(\frac{u - l}{\delta}\right)^{-2\lambda} \, |\nabla H|^p \xi^k \, dx
= \int_{B_\rho(y)} H(u - l)^{p+1} \left(1 + \frac{u - l}{\delta}\right)^{-1+\lambda} \left(\frac{u - l}{\delta}\right)^{-2\lambda} \, ds \, F_2(x) \xi^k \, dx
\end{align*}
\begin{equation}
\quad + \int_{B_\rho(y)} H |\nabla H|^{p-2} \nabla H \cdot \nabla \left( (u - l)^{p+1} \left(1 + \frac{u - l}{\delta}\right)^{-1+\lambda} \left(\frac{u - l}{\delta}\right)^{-2\lambda} \xi^k \right) \, dx.
\end{equation}

The terms in the right hand side of (2.20) have been estimated in (2.19). The right hand side of (2.21) is estimated similarly to (2.16) using the Young inequality. Thus using (2.13) and (2.18) and collecting (2.15), (2.16), (2.19) - (2.21) we arrive at the required (2.14).
3 Local boundedness of solutions. Proof of Theorem 1.1

Let \((x_0, t_0)\) be an arbitrary point in \(\Omega_T\). Let

\[
R \leq \frac{1}{2} \min \left\{ 1, \text{dist} (x_0, \partial \Omega_T), t_0^{1/2}, (T - t_0)^{1/2} \right\}.
\]

Let

\[
Q_R(x_0, t_0) = B_R(x_0) \times (t_0 - R^2, t_0 + R^2).
\]

Fix a point \((y, s) \in Q_{\frac{1}{2}}(x_0, t_0)\). For \(j = 1, 2, \ldots \) set

\[
\rho_j = R^{2^{-j}}, \quad Q_j = B_j \times (s - \delta_j^{2-p} \rho_j^p, s + \delta_j^{2-p} \rho_j^p), \quad B_j = B_{\rho_j}(y), \quad L_j = Q_j \cap \Omega_T \cap \{ u(x, t) > l_j \}.
\]

Let \(\xi_j \in C^\infty_0(Q_j)\) be such that \(\xi_j(x, t) = 1\) for \((x, t) \in B_{j+1} \times (s - \frac{3}{4} \delta_j^{2-p} \rho_j^p, s + \frac{3}{4} \delta_j^{2-p} \rho_j^p), \| \nabla \xi_j \| \leq \gamma \rho_j^{-1}, \| \partial_j \xi_j \| \leq \gamma \delta_j^{p-2} \rho_j^{-p}.
\)

The sequences of positive numbers \((l_j)_{j \in \mathbb{N}}\) and \((\delta_j)_{j \in \mathbb{N}}\) are defined inductively as follows.

Let \(l_0 = 1\) and assume that \(l_1, l_2, \ldots, l_j\) and \(\delta_0, \delta_1, \ldots, \delta_{j-1}\) have been already chosen. Let us show how to chose \(l_{j+1}\) and \(\delta_j\).

Define the sequence \((\alpha_j)_{j \in \mathbb{N}}\) by

\[
\alpha_j = \rho_j + \left( \frac{1}{\rho_j^{n-p}} \int_{B_j} F_1(x) dx \right)^{\frac{1}{n-p}} + \left( \frac{1}{\rho_j^{n-p}} \int_{B_j} F_2(x) dx \right)^{\frac{1}{n-p}}.
\]

For \(l \geq l_j + \alpha_j\) set

\[
A_j(l) = \frac{(l - l_j)^{p-2}}{\rho_j^{n+p}} \int_{L_j} \frac{u}{l - l_j} \left( \frac{u - l_j}{l - l_j} \right)^{(1 + \lambda)(p-1)} \frac{\xi_j^k}{d \tau} + \sup_{\frac{l_j}{l} \rho_j} \int_{L_j(t)} \frac{u}{l - l_j} \xi_j^k dx,
\]

where \(L_j = \bar{Q}_j \cap \Omega_T \cap \{ u(x, t) > l_j \}, \bar{Q}_j = B_j \times (s - (l - l_j)^{2-p} \rho_j^p, s + (l - l_j)^{2-p} \rho_j^p)\).

Fix a positive number \(\varkappa \in (0, 1)\) depending on \(n, p, c_1, c_2\), which will be specified later. If

\[
A_j(l_j + \alpha_j) \leq \varkappa,
\]

we set \(l_{j+1} = l_j + \alpha_j\).

Note that \(A_j(l) \searrow 0\) as \(l \to \infty\). So if

\[
A_j(l_j + \alpha_j) > \varkappa,
\]

there exists \(\bar{l} > l_j + \alpha_j\) such that \(A_j(\bar{l}) = \varkappa\). In this case we set \(l_{j+1} = \bar{l}\).

In both cases we set \(\delta_j = l_{j+1} - l_j\). Note that our choices guarantee that \(\bar{Q}_j \subset Q_R(x_0, t_0)\) and

\[
A_j(l_{j+1}) \leq \varkappa.
\]

Lemma 3.1. Let the conditions of Theorem 1.1 be fulfilled. The for all \(j \geq 1\) there exists \(\gamma > 0\) depending on the data, such that

\[
\delta_j \leq \frac{1}{2} \delta_{j-1} + \alpha_j + \gamma(1 + l_j) \nu_1 \rho_j^{p-n} \int_{B_j} F_1(x) dx + \gamma(1 + l_j) \rho_j^{p-n} \int_{B_j} F_2(x) dx
\]

Proof. Fix \(j \geq 1\). Without loss assume that

\[
\delta_j > \frac{1}{2} \delta_{j-1}, \quad \delta_j > \alpha_j,
\]

since otherwise (3.7) is evident. The second inequality in (3.7) guarantees that \(A_j(l_{j+1}) = \varkappa\) and \(\bar{Q}_j = Q_j\).
Let us estimate the terms in the right hand side of (3.2) with \( l = l_{j+1} \). For this we decompose \( L_j \) as \( L_j = L'_j \cup L''_j \),
\[
L'_j = \left\{ x \in L_j : \frac{u(x) - l_j}{\delta_j} < \varepsilon_1 \right\}, \quad L''_j = L_j \setminus L'_j,
\]
where \( \varepsilon_1 \) depending on \( n_p, c_1, c_2 \) is small enough to be determined later.

We also have
\[
l_j \geq 1.
\]

Recall that \( \xi_{j-1} = 1 \) on \( Q_j \). By (3.5) we have
\[
\frac{\delta_j^{p-2}}{\rho_j^{p+n}} \int_{L'_j} u \left( \frac{u - l_j}{\delta_j} \right)^{(1+\lambda)(p-1)} \xi_j^{k-p} dx d\tau \leq 2 \varepsilon_1^{(1+\lambda)(p-1)} \rho_j^{-n} \sup_t \int_{L'_j(t)} u \left( \frac{u - l_j-1}{\delta_j-1} \right) \xi_j^{k} dx 
\]
\[
\leq 2^{n+1} \varepsilon_1^{(1+\lambda)(p-1)} \rho_j^{-n} \sup_t \int_{L_{j-1}(t)} u \left( \frac{u - l_j-1}{\delta_j-1} \right) \xi_j^{k} dx \leq 2^{n+1} \varepsilon_1^{(1+\lambda)(p-1)} \kappa.
\]

Let
\[
\psi_j(x, t) = \frac{1}{\delta_j} \left( \int_{l_j}^{u(x,t)} s^{\frac{1-\lambda}{p}} \left( \frac{s - l_j}{\delta_j} \right)^{-\frac{\lambda}{p}} ds \right)^{\frac{1}{1-\lambda}}.
\]

Then by the Young inequality
\[
\int_{L''_j} u \left( \frac{u - l_j}{\delta_j} \right)^{(1+\lambda)(p-1)} \xi_j^{k-p} dx d\tau \leq \varepsilon_2 \int_{L''_j} u(x, \tau) dx d\tau
\]
\[
+ \gamma(\varepsilon_2) \int_{L''_j} u \left( \frac{u - l_j}{\delta_j} \right)^{(1+\lambda)(p-1)z} \xi_j^{k-p^*} dx d\tau,
\]
where
\[
z = \frac{n + \rho(\lambda)}{n} \frac{p - 1 - \lambda}{(1 + \lambda)(p - 1)}, \quad \rho(\lambda) = \frac{p}{p - 1 - \lambda}.
\]

Similarly to (3.9) we have
\[
\int_{L'_{j+1}} u(x, \tau) dx d\tau \leq 2^{n+1} \frac{\rho^{p+n}}{\delta_j^{p+n}} l_j \kappa.
\]

Using the evident inequality
\[
c(\varepsilon_1)^{-1} \psi_j(x, t)^{\rho(\lambda)} \leq u(x, t)^{\frac{1-\lambda}{p}} \left( \frac{u(x, t) - l_j}{\delta_j} \right) \leq c(\varepsilon_1) \psi_j(x, t)^{\rho(\lambda)}, \quad (x, t) \in L'_j,
\]
the Sobolev inequality and Lemma [2.7] with \( l = l_j, \delta = \delta_j \), we obtain
\[
\int_{L''_j} u \left( \frac{u - l_j}{\delta_j} \right)^{(1+\lambda)(p-1)z} \xi_j^{(k-p)z} dx d\tau \leq \gamma(\varepsilon_1) \int_{L''_j} u^{-\frac{\rho(\lambda)}{n}} \psi_j^{\frac{n + \rho(\lambda)}{n}} \xi_j^{(k-p)z} dx d\tau
\]
\[
\leq \gamma(\varepsilon_1) l_j^{\frac{\rho(\lambda)}{n}} \int_{L''_j} \psi_j^{\frac{n + \rho(\lambda)}{n}} \xi_j^{(k-p)z} dx d\tau
\]
\[
\leq \gamma(\varepsilon_1) l_j^{\frac{p}{n}} \left( \sup_t \int_{L''_j(t)} u^{-\frac{p-2}{p} \rho(\lambda)} \xi_j^{(k-p)z} dx \right) \cdot \int_{L''_j} \nabla \left( \psi_j \xi_j^{(k-p)z} \right)^{p} dx d\tau
\]

11
\[
\leq \gamma(\varepsilon_1)\delta_j^{2-n} l_{j+1}^{\frac{n}{p}} \left[ \frac{\partial^{p-2}}{\partial_j^p} \int_{L_{j+1}} u \left( 1 + \frac{u - l_j}{\delta_j} \right)^{(1+\lambda)(p-1)} \xi_j d\tau \right]
+ \gamma(\varepsilon_1)\delta_j^{p-2} \rho_j^p \int_{L_j} u \left( 1 + \frac{u - l_j}{\delta_j} \right)^{(1-\lambda)(p-1)} \xi_j d\tau
+ \nu_1^{-\frac{1}{p} + \frac{1}{p} + 1} \left( \frac{\rho_j}{\delta_j} \right)^p \int_{B_j} F_1(x) dx + \left( \frac{\rho_j}{\delta_j} \right)^p \int_{B_j} F_2(x) dx
+ \nu_2^{-\frac{1}{p} + \frac{1}{p} + 1} \left( \frac{\rho_j}{\delta_j} \right)^p \int_{B_j} f_1(x)^\frac{p}{p-1} dx + \delta_j \left( \frac{\rho_j}{\delta_j} \right)^p \int_{B_j} f_2(x) dx \right]^{1+\frac{2}{p}}.
\]

From (3.9) and (3.12) and from the fact that \( \xi_j = 1 \) on \( Q_{j-1} \), we obtain
\[
\begin{align*}
\frac{\delta_j^{p-2}}{\rho_j^{-p}} \int_{L_{j+1}} u \left( 1 + \frac{u - l_j}{\delta_j} \right)^{(1+\lambda)(p-1)} \xi_j d\tau & \leq \left( 2^{n+1} \varepsilon_1^{(1+\lambda)(p-1)} + 2^{n+1} \varepsilon_2 \right) \varkappa \\
\varepsilon & = \nu_1^{-\frac{1}{p} + \frac{1}{p} + 1} \int_{B_j} F_1(x) dx + \left( \frac{\rho_j}{\delta_j} \right)^p \int_{B_j} F_2(x) dx
+ \nu_1^{-\frac{1}{p} + \frac{1}{p} + 1} \left( \frac{\rho_j}{\delta_j} \right)^p \int_{B_j} f_1(x)^\frac{p}{p-1} dx + \left( \frac{\rho_j}{\delta_j} \right)^p \int_{B_j} f_2(x) dx.
\end{align*}
\]

The first two terms of the right hand side of (3.14) were estimated in (3.9) and in (3.12). Therefore we conclude from (3.13), (3.14) that
\[
\begin{align*}
\varkappa & \leq \left( 2^{n+1} \varepsilon_1^{(1+\lambda)(p-1)} + 2^{n+1} \varepsilon_2 + \nu_1 + \varepsilon_1^{2\lambda(p-1)} \right) \varkappa + \gamma(\varepsilon_1, \varepsilon_2) \left[ \nu_1^{-\frac{1}{p} + \frac{1}{p} + 1} \int_{B_j} F_1(x) dx + \left( \frac{\rho_j}{\delta_j} \right)^p \int_{B_j} f_1(x)^\frac{p}{p-1} dx + \left( \frac{\rho_j}{\delta_j} \right)^p \int_{B_j} f_2(x) dx \right]
+ \gamma(\varepsilon_1, \varepsilon_2) \left[ \nu_1^{-\frac{1}{p} + \frac{1}{p} + 1} \int_{B_j} F_1(x) dx + \left( \frac{\rho_j}{\delta_j} \right)^p \int_{B_j} f_1(x)^\frac{p}{p-1} dx + \left( \frac{\rho_j}{\delta_j} \right)^p \int_{B_j} f_2(x) dx \right]^{1+\frac{2}{p}}.
\end{align*}
\]

Choose \( \nu_1 < \frac{\varkappa}{16} \), \( \varepsilon_1, \varepsilon_2 \), such that
\[
2^{n+1} \varepsilon_1^{(1+\lambda)(p-1)} + 2^{n+1} \varepsilon_2 + \varepsilon_1^{2\lambda(p-1)} < \frac{1}{16},
\]
and \( \varkappa \) such that \( \gamma(\varepsilon_1, \varepsilon_2) \varkappa^{\frac{2}{p}} = \frac{\varkappa}{16} \). Hence (3.15) yields (3.6) which completes the proof of the lemma. \( \square \)
In order to complete the proof of Theorem \[14\] we sum up (3.10) with respect to \(j\) from 1 to \(J - 1\)

\[
l_j \leq \gamma \delta_0 + \gamma \sum_{j=1}^{\infty} \alpha_j + \gamma (1 + l_j) \nu_1^{-\frac{p}{p-n}} \sum_{j=1}^{\infty} \left( \rho_j^p f_1(x) \right)^{\frac{1}{p}} + \gamma (1 + l_j) \sum_{j=1}^{\infty} \left( \rho_j^p f_2(x) \right)^{\frac{1}{p}}
\]

\[
\leq \gamma \delta_0 + \gamma R + \gamma (1 + l_j) \left[ \nu_1^{-\frac{p}{p-n}} \int_0^{2R} \left( \frac{1}{r^{n-p}} \int_{B_r(y)} F_1(x) \right) \frac{dr}{r} \right]^{\frac{1}{p}}
\]

(3.16) \[\quad \int_0^{2R} \left( \frac{1}{r^{n-p}} \int_{B_r(y)} F_2(x) \right) \frac{dr}{r} \]

(3.17) \[\delta_0 \leq \gamma \left( \frac{1}{R^{p+n}} \int_{Q_{R/2}(x_0,t_0)} u^{p+\lambda(p-1)} \right)^{\frac{1}{2+\lambda(p-1)}} + \gamma \sup_{t_0 - R^2 < t < t_0 + R^2} \left( \frac{1}{R^n} \int_{B_R(x_0)} u^2 \right)^{\frac{1}{2}}.
\]

Now we choose \(R > 0\) such that \(F_1(R_0) + F_2(R_0) < \nu_1 \leq \frac{1}{2} \gamma^{-1} \nu_1^{\frac{p}{p-1}}\), where \(\gamma\) as in last line of (3.10). Then by (3.10), for \(R \leq R_0\) we obtain

\[
l_j \leq \gamma + \gamma \left( \frac{1}{R^{p+n}} \int_{Q_{R/2}(x_0,t_0)} u^{p+\lambda(p-1)} \right)^{\frac{1}{2+\lambda(p-1)}} + \gamma \sup_{t_0 - R^2 < t < t_0 + R^2} \left( \frac{1}{R^n} \int_{B_R(x_0)} u^2 \right)^{\frac{1}{2}}
\]

(3.18) \[\quad + \gamma R + \gamma \left( \frac{1}{R^{p+n}} \int_{B_{R/2}(y)} F_1(x) \right)^{\frac{1}{p}} + \gamma \int_0^{2R} \left( \frac{1}{r^{n-p}} \int_{B_r(y)} F_2(x) \right) \frac{dr}{r} \]

Hence the sequence \((l_j)_{j \in \mathbb{N}}\) is convergent, and \(\delta_j \to 0 (j \to \infty)\), and we can pass to the limit \(J \to \infty\) in (3.18). Let \(l = \lim_{j \to \infty} l_j\). From (3.10) we conclude that

\[
\frac{1}{\rho_j^{p+n}} \int_{Q_i} u (u - l)^{(1+\lambda)(p-1)} \leq \gamma \varepsilon l_j^{1+\lambda(p-1)} \to 0 \quad (j \to \infty).
\]

Choosing \((y, s)\) as a Lebesgue point of the function \(u (u - l)^{(1+\lambda)(p-1)}\) we conclude that \(u(y, s) \leq l\) and hence \(u(y, s)\) is estimated from above by the right hand side of (3.18). Applicability of the Lebesgue differentiation theorem follows from [12 Chap. II, Sec. 3].

Taking essential supremum over \(Q_{R/2}(x_0,t_0)\) we complete the proof. \(\square\)

### 4 Proof of Theorem [1.5]

In this section we prove the Theorem [1.5] which is a De Giorgi-type lemma [10]. Here we assume the structure conditions

\[
\mathbf{A}(x, t, u, \zeta) \zeta \geq c_1 |\zeta|^p, \quad \zeta \in \mathbb{R}^n,
\]

\[
|\mathbf{A}(x, t, u, \zeta)| \leq c_2 |\zeta|^{p-1} + f_1(x),
\]

\[
|a_0(x, t, u, \zeta)| \leq |\zeta|^{p-1} + f_2(x),
\]

with some positive constants \(c_1, c_2\) and nonnegative functions \(f_1(x), f_2(x)\). These assumptions follow from (1.2) due to the boundedness of \(u\) and \(h = 1\).

We assume that

\[
f_1^{\frac{1}{p-1}} \in \overline{K}_p, \quad f_2 \in K_p.
\]

We provide the proof of (1.11), while the proof of (1.13) is completely similar.

Set \(v = u - \mu, \ M = \text{ess sup}_{Q_{r}} |u(x, t)|\). In the sequel \(\gamma\) will denote a constant depending on the data and \(M\), which, as usual, can vary from line to line.
Lemma 4.1. Let $u$ be a solution to (4.1). Then for any $l, \delta > 0$ and $\varepsilon \in (0, 1)$ and any cylinder
\[ Q_\rho^\delta(y, s) = B_\rho(y) \times (s - \delta^2 - p, s + \delta^2 - p) \subset \Omega_T, \quad \rho \leq R \]
and any $\xi \in C_0^\infty(Q_\rho^\delta(y, s))$ such that $\xi(x, t) = 1$ for $(x, t) \in Q_{\rho/2}^\delta(y, s)$ with $|\nabla \xi| \leq \gamma \frac{1}{p}, |\xi_t| \leq \gamma \frac{\delta^{p-2}}{p}$ we have
\[
\begin{align*}
\delta^2 \int_{L(t)} G \left( \frac{l - v(x, t)}{\delta} \right) \xi(x, t)^k dx & + \int \int_L \left( 1 + \frac{l - v}{\delta} \right)^{-1 + \lambda} \left( \frac{l - v}{\delta} \right)^{-2\lambda} |\nabla v|^p \xi(x, t)^k dx d\tau \\
& \leq \frac{\gamma \delta^p}{p^\rho} \int \int_L \left( 1 + \frac{l - v}{\delta} \right)^{(1-\lambda)(p-1)} \left( \frac{l - v}{\delta} \right)^{2\lambda(p-1)} \xi^{k-p} dx d\tau \\
& + \frac{\varepsilon}{p^\rho} \int \int_L \left( 1 + \frac{l - v}{\delta} \right)^{(1+\lambda)(p-1)} \xi^{k-p} dx d\tau \\
& + \gamma \varepsilon \frac{\rho^2 \delta^2 - p}{p} \int_{B_\rho(y)} F_1(x) dx + \gamma \rho^p \delta^{3-p} \int_{B_\rho(y)} F_2(x) dx,
\end{align*}
\]
(4.2)
where $L = Q_\rho^\delta(y, s) \cap \{v < l\}$, $L(t) = L \cap \{t = t\}$ and $\lambda = \min\left(\frac{1}{p}, \frac{p-2}{2}\right)$, $k = p + \frac{(p+2)(p-1)(1+\lambda)}{p-1-\lambda}$ and $G(v)$ is defined in the previous section.

Proof. The proof is similar to that of Lemma 2.3 with the choice of the test function
\[
\varphi(x, t) = \left[ \int_{v(x, t)}^l \left( 1 + \frac{l - s}{\delta} \right)^{-1+\lambda} \left( \frac{l - s}{\delta} \right)^{-2\lambda} ds \right] + \xi(x, t)^k. \quad \square
\]

Set
\[
w(x, t) = \left( \frac{1}{\delta} \int_{v(x, t)}^l \left( 1 + \frac{l - s}{\delta} \right)^{-\frac{1}{p} + \lambda} \left( \frac{l - s}{\delta} \right)^{-\frac{2\lambda}{p}} ds \right) + \]
(4.3)

Note that
\[
c(\varepsilon)w(x, t)^p(\lambda) \leq \frac{l - v}{\delta} \leq C(\varepsilon)w(x, t)^p(\lambda) \quad \text{if} \quad \frac{l - v}{\delta} \geq \varepsilon,
\]
with
\[
(4.4)
\rho(\lambda) = \frac{p}{p - 1 - \lambda}.
\]

The next lemma follows from Lemma 4.1 with $\varepsilon = \nu_1$ via the arguments similar to (2.16)–(2.21).

Lemma 4.2. Let $u$ be a solution to (4.1). Then for any $l, \delta > 0$ and $\varepsilon \in (0, 1)$ and any cylinder
\[ Q_\rho^\delta(y, s) = B_\rho(y) \times (s - \delta^2 - p, s + \delta^2 - p) \subset \Omega_T, \quad \rho \leq R \]
and any $\xi \in C_0^\infty(Q_\rho^\delta(y, s))$ such that $\xi(x, t) = 1$ for $(x, t) \in Q_{\rho/2}^\delta(y, s)$ with $|\nabla \xi| \leq \gamma \frac{1}{p}, |\xi_t| \leq \gamma \frac{\delta^{p-2}}{p}$ there exists $\nu_1 \in (0, 1)$ depending only on $n, p, c_1, c_2$ such that the inequality
\[
F_1(2R) + F_2(2R) \leq \nu_1
\]
(4.6)
implies that

\[
\sup_t \int_{L(t)} \xi(x,t) \xi(x,t)^k = \frac{\xi(x,t)^k}{\xi(x,t)^k} dx + \delta^p \int_{L} \frac{\xi(x,t)^k}{\xi(x,t)^k} dx d\tau \\
\leq \gamma t_{B} \frac{\delta^p}{\rho^p} \int_{L} \left( 1 + \frac{l-v}{\delta} \right)^{(1+\lambda)(p-1)} \xi^{k-p} dx dt \\
+ \gamma \frac{\delta^p}{\rho^p} \int_{L} \left( 1 + \frac{l-v}{\delta} \right)^{(1-\lambda)(p-1)} \left( \frac{l-v}{\delta} \right)^{2\lambda(p-1)} \xi^{k-p} dx dt \\
(4.7) + \gamma \frac{\rho^p}{\delta^p} \int_{B_r(x,y)} F_1(x) dx + \gamma t_{B} \frac{\delta^p}{\rho^p} \int_{B_r(x,y)} F_2(x) dx,
\]

where \( L, \lambda, k \) and \( G \) are the same as in Lemma 4.1.

Further we assume that

\[
(4.8) \xi \omega \geq B(\rho + F_1(2\rho) + F_2(2\rho)).
\]

Let \((x_1, t_1) \in Q^0_{\rho}(y,s)\). Set

\[
r_j = \frac{\rho_0}{4}, \rho_0 = \frac{\rho}{C}, B_j = B_{r_j}(x_1), Q_j(l) = B_j \times \left( t_1 - \frac{r_j}{(l_j - l)^{p-2}}, t_1 + \frac{r_j}{(l_j - l)^{p-2}} \right), j = 1, 2, \ldots,
\]

\( C \geq 16 \) will be fixed later depending only on the known data.

Let \( 1_{B_j+1} \leq \xi_j(x) \leq 1_{B_j} \),

\[
\frac{1}{1} \left( t_1 - \frac{r_j}{(l_j - l)^{p-2}}, t_1 + \frac{r_j}{(l_j - l)^{p-2}} \right) \leq \theta_j(t) \leq \frac{1}{1} \left( t_1 - \frac{r_j}{(l_j - l)^{p-2}}, t_1 + \frac{r_j}{(l_j - l)^{p-2}} \right), \xi_j(x,t) = \xi_j(x)\theta_j(t).
\]

We start with the choice of the sequences \( l_j, \delta_j, j = 0, 1, 2, \ldots \).

Set

\[
A_j(l) = \frac{(l_j - l)^{p-2}}{r_j^{p-2}} \int_{L_j(t)} \left( \frac{l_j - v}{l_j - l} \right)^{(1+\lambda)(p-1)} \xi_j(x,t)^{k-p} dx dt + \inf_{t} \frac{1}{r_j^{p-2}} \int_{L_j(l,t)} G \left( \frac{l_j - v}{l_j - l} \right) \xi_j(x,t)^{k} dx,
\]

where \( L_j(l) = Q_j(l) \cap \Omega_T \cap \{ v \geq l_j \}, \quad L_j(l,t) = L_j \cap \{ \tau = t \} \). Define the sequence \( \alpha_j \in \mathbb{N} \) by

\[
(4.10) \alpha_j = r_j + \int_0^{r_j} \frac{dr}{r} \left( r^{p-n} \int_{B_r(x_1)} F_1(z)dz \right)^{\frac{1}{p}} + \int_0^{r_j} \frac{dr}{r} \left( r^{p-n} \int_{B_r(x_1)} F_2(z)dz \right)^{\frac{1}{p-1}}, \quad j = -1, 0, 1, 2, \ldots.
\]

By the definition of the Kato class \( \alpha_j \downarrow 0 \) as \( j \to \infty \). Note that

\[
(4.9) \alpha_j - \alpha_j \geq \gamma \left[ \left( r_j^{p-n} \int_{B_j} F_1(z)dz \right)^{\frac{1}{p}} + \left( r_j^{p-n} \int_{B_j} F_2(z)dz \right)^{\frac{1}{p-1}} \right],
\]

and also

\[
(4.10) \alpha_j - \alpha_j \leq 3r_j + \gamma \left[ \left( r_j^{p-n} \int_{B_j} F_1(z)dz \right)^{\frac{1}{p}} + \left( r_j^{p-n} \int_{B_j} F_2(z)dz \right)^{\frac{1}{p-1}} \right].
\]

Set \( l_0 = \xi \omega, \quad \bar{t} = \frac{\xi \omega}{2} + \frac{B_{\alpha_0}}{4} \). Then \( \xi \omega - \bar{t} = \frac{\xi \omega}{2} - \frac{B_{\alpha_0}}{4} \geq \frac{\xi \omega}{4} \geq \frac{B_{\alpha_0}}{4} \) and moreover from (1.10) it...
follows that
\[
\frac{(\xi_\omega - \bar{\gamma})^{p-2}}{r_0^{n+p}} \iint_{L_0(\bar{\gamma})} \left( \frac{\xi_\omega - \nu}{\xi_\omega - \bar{\gamma}} \right)^{(1+\lambda)(p-1)} \xi(x,t)^{k-p} dx dt
\]
\[
\leq \frac{4(1+\lambda)(p-1)}{r_0^{n+p}} \{(x,t) \in Q_0(\bar{\gamma}) : v(x,t) \leq \xi_\omega\}
\]
\[
\leq \frac{4(1+\lambda)(p-1)}{r_0^{n+p}} \{(x,t) \in Q_0^b(y,s) : u(x,t) \leq \mu_+ + \xi_\omega\}
\]
\[
(4.11)
\]
By Lemma 4.2
\[
\text{ess sup}_{t} \frac{1}{r_0^n} \int_{L_0(\bar{\gamma},t)} G \left( \frac{\xi_\omega - v}{\xi_\omega - \bar{\gamma}} \right) \xi(x,t)^k dx
\]
\[
\leq \gamma \frac{(\xi_\omega)^{p-2}}{r_0^{n+p}} \iint_{L_0(\bar{\gamma})} \left( 1 + \frac{\xi_\omega - v}{\xi_\omega - \bar{\gamma}} \right)^{(1-\lambda)(p-1)} \left( \frac{\xi_\omega - v}{\xi_\omega - \bar{\gamma}} \right)^{2\lambda(p-1)} \xi(x,t)^{k-p} dx dt,
\]
\[
+ \gamma \varepsilon \frac{(\xi_\omega)^{p-2}}{r_0^{n+p}} \iint_{L_0(\bar{\gamma})} \left( 1 + \frac{\xi_\omega - v}{\xi_\omega - \bar{\gamma}} \right)^{(1+\lambda)(p-1)} \xi(x,t)^{k-p} dx dt
\]
\[
+ \gamma \varepsilon \frac{r_0^{p-n}}{(\xi_\omega - \bar{\gamma})^p} \int_{B_0} F_1(x) dx + \gamma \frac{r_0^{p-n}}{(\xi_\omega - \bar{\gamma})^{p-1}} \int_{B_0} F_2(x) dx
\]
\[
(4.12)
\]
\[
\leq \gamma \nu \theta (\xi_\omega)^{p-2} C^{n+p} + \gamma \varepsilon \frac{1}{r_0^n} C^{n-p} (B^{1-p} + B^{-p}).
\]

Fix a number \( \varepsilon \in (0, 1) \) depending on the known data. First, choose \( \varepsilon = \nu \), next choose \( \nu \) from the condition \( \gamma \nu \theta (\xi_\omega)^{p-2} C^{n+p} \leq \frac{\varepsilon}{2} \) and \( B \) from the condition \( B^{1-p} \gamma \nu \frac{1}{r_0^n} C^{n-p} \leq \frac{\varepsilon}{2} \). Then we obtain from (4.11), (4.12) that \( A_0(\bar{\gamma}) \leq \frac{\varepsilon}{2} \).

**Lemma 4.3.** Suppose we have chosen \( l_1, \ldots, l_j \) and \( \delta_0, \ldots, \delta_{j-1} \) such that
\[
(4.13) \quad \frac{l_{i-1}}{2} + B \frac{\alpha_{i-1}}{4} < l_i \leq l_{i-1} - \frac{1}{4} (\alpha_{i-2} - \alpha_{i-1}), \quad i = 1, 2, \ldots, j,
\]
\[
(4.14) \quad A_{i-1}(l_i) \leq \varepsilon, \quad i = 1, 2, \ldots, j,
\]
\[
(4.15) \quad l_i > B \frac{\alpha_{i-1}}{2}, \quad i = 1, 2, \ldots, j.
\]

Then
\[
(4.16) \quad A_j(\bar{\gamma}) \leq \frac{\varepsilon}{2}, \quad \bar{l} = \frac{l_j}{2} + B \frac{\alpha_j}{4}.
\]

**Proof.** Let us decompose \( L_j(\bar{\gamma}) \) as
\[
L_j(\bar{\gamma}) = L'_j(\bar{\gamma}) \cup L''_j(\bar{\gamma}), \quad L'_j(\bar{\gamma}) = \left\{ \frac{l_i - v}{l_j - \bar{l}} \leq \varepsilon_1 \right\}, \quad L''_j(\bar{\gamma}) = L_j(\bar{\gamma}) \setminus L'_j(\bar{\gamma}).
\]
Using that $\xi_{j-1}(x, t) = 1$ for $(x, t) \in Q_j(\tilde{I})$ and inequality (4.14) we have

$$
\frac{(l_j - \tilde{l})^{p-2}}{r_j^{n+p}} \int_{L_j^p(\tilde{I})} \left( \frac{l_j - u}{l_j - \tilde{l}} \right)^{(1+\lambda)(p-1)} \xi_j(x, t)^{k-p} dx dt
$$

$$
\leq \frac{(l_j - \tilde{l})^{p-2}}{r_j^{n+p}} e_1^{(1+\lambda)(p-1)} |L_j(\tilde{I})| \leq \frac{e_1^{(1+\lambda)(p-1)}}{r_j^n} \text{ess sup} \int_{L_j^p(\tilde{I}, t)} \xi_j(x, t)^k dx
$$

(4.17)

$$
\leq \frac{e_1^{(1+\lambda)(p-1)}}{r_j^n} \text{ess sup} \int_{L_j^p(\tilde{I}, t)} G \left( \frac{l_j - u}{l_j - \tilde{l}} \right) \xi_j(x, t)^k dx \leq 2^n e_1^{(1+\lambda)(p-1)} \kappa.
$$

Above we also used the following inequality, which follows from (4.13), (4.15).

$$
l_j - \tilde{l} = \frac{l_j}{2} - \frac{B}{4} \alpha_j \geq \frac{l_j}{4} + \frac{B}{8} \alpha_j - \frac{B}{4} \alpha_j
$$

(4.18)

It follows from (4.18) that $Q_j(\tilde{I}) \subset Q_{j-1}(l_j)$.

Using the Young inequality we have

$$
\frac{(l_j - \tilde{l})^{p-2}}{r_j^{n+p}} \int_{L_j^p(\tilde{I})} \left( \frac{l_j - u}{l_j - \tilde{l}} \right)^{(1+\lambda)(p-1)} \xi_j(x, t)^{k-p} dx dt
$$

(4.19)

$$
\leq e_1 \frac{(l_j - \tilde{l})^{p-2}}{r_j^{n+p}} |L_j(\tilde{I})| + \gamma(e_1) \frac{(l_j - \tilde{l})^{p-2}}{r_j^{n+p}} \int_{L_j^p(\tilde{I})} \left( \frac{l_j - u}{l_j - \tilde{l}} \right)^{\rho \frac{p+2(\lambda)}{p(\lambda+n)(\lambda+n)}} \xi_j(x, t)^{(k-p)z} dx dt,
$$

where $\rho(\lambda) = \frac{p}{p-1-\lambda}$, $z(1+\lambda)(p-1) = \frac{p+2(\lambda)}{p(\lambda+n)(\lambda+n)}$ due to our choice of $\lambda$, $z > 1$.

Similarly to (4.17), the first term in the right hand side of (4.19) is estimated as

(4.20)

$$
\frac{(l_j - \tilde{l})^{p-2}}{r_j^{n+p}} |L_j(\tilde{I})| \leq e_1 2^n \kappa.
$$

Define

$$
w_j(x, t) = \frac{1}{l_j - \tilde{l}} \left[ \int_{L_j^p(x, t)} \left( 1 + \frac{l_j - s}{l_j - \tilde{l}} \right)^{-\frac{1}{p} + \frac{1}{\lambda}} \left( \frac{l_j - s}{l_j - \tilde{l}} \right)^{-\frac{2}{p(n+p)}} ds \right].
$$

Using the embedding theorem and Lemma 4.2 we have

$$
\gamma(e_1) \frac{(l_j - \tilde{l})^{p-2}}{r_j^{n+p}} \int_{L_j^p(\tilde{I})} \left( \frac{l_j - u}{l_j - \tilde{l}} \right)^{\rho \frac{p+2(\lambda)}{p(\lambda+n)(\lambda+n)}} \xi_j(x, t)^{(k-p)z} dx dt
$$

$$
\leq \gamma(e_1) \frac{(l_j - \tilde{l})^{p-2}}{r_j^{n+p}} \int_{L_j^p} w_j(x, t)^{\rho \frac{p+2(\lambda)}{p(\lambda+n)(\lambda+n)}} \xi_j(x, t)^{(k-p)z} dx dt
$$

$$
\leq \gamma(e_1) \frac{(l_j - \tilde{l})^{p-2}}{r_j^{n+p}} \left( \text{ess sup} \int_{L_j^p(\tilde{I}, t)} w_j(x, t)^{\rho(\lambda) \xi_j(x, t)^{\frac{p(\lambda+k-p)}{p(\lambda+n)(\lambda+n)}}} dx \right)^{\frac{1}{\rho}}
$$
Let us take $\varepsilon = 1$. Using the inequality $l_j \leq l_{j-1}$ and \(4.12\), \(4.14\), \(4.18\) we have
\[
\gamma \leq \frac{(l_j - \bar{l})^{p-2}}{r_j^{n+p}} \int_{\mathcal{L}_{j}(\bar{l})} \left( 1 + \frac{l_j - u}{l_j - l} \right)^{(1-\lambda)(p-1)} \left( \frac{l_j - u}{l_j - l} \right)^{2\lambda(p-1)} dxdt
\]
\[
+ \gamma \varepsilon \frac{(l_j - \bar{l})^{p-2}}{r_j^{n+p}} \int_{\mathcal{L}_{j}(\bar{l})} \left( 1 + \frac{l_j - u}{l_j - l} \right)^{(1-\lambda)(p-1)} dxdt
\]
\[
(4.21)
\]
\[
+ \gamma \varepsilon \frac{r_j^{p-n}}{(l_j - l)^{p}} \int_{B_j} F_1(x) dx + \gamma \frac{r_j^{p-n}}{(l_j - l)^{p-1}} \int_{B_j} F_2(x) dx \bigg\}
\]

Let us take $\varepsilon = 1$. Using the inequality $l_j \leq l_{j-1}$ and \(4.12\), \(4.14\), \(4.18\) we have
\[
\gamma \leq \frac{(l_j - \bar{l})^{p-2}}{r_j^{n+p}} \int_{\mathcal{L}_{j}(\bar{l})} \left( 1 + \frac{l_j - u}{l_j - l} \right)^{(1-\lambda)(p-1)} \left( \frac{l_j - u}{l_j - l} \right)^{2\lambda(p-1)} dxdt
\]
\[
+ \gamma \varepsilon \frac{(l_j - \bar{l})^{p-2}}{r_j^{n+p}} \int_{\mathcal{L}_{j}(\bar{l})} \left( 1 + \frac{l_j - u}{l_j - l} \right)^{(1-\lambda)(p-1)} dxdt
\]
\[
(4.22)
\]
\[
\leq \lambda \varepsilon + \gamma \varepsilon \delta_{l_j-1}^{p-2} \int_{\mathcal{L}_{j-1}(\bar{l})} \left( \frac{l_j - u}{l_j - l} \right)^{(1-\lambda)(p-1)} \xi_{j-1}(x,t)^{k-p} dxdt \leq \lambda \varepsilon.
\]

Furthermore, \(4.18\) implies that $l_j - \bar{l} \geq \frac{B}{4} (\alpha_{j-1} - \alpha_j)$. Therefore by \(4.3\) we have
\[
(4.23) \quad \gamma \frac{r_j^{p-n}}{(l_j - l)^{p}} \int_{B_j} F_1(x) dx + \gamma \frac{r_j^{p-n}}{(l_j - l)^{p-1}} \int_{B_j} F_2(x) dx \leq \gamma (B^{1-p} + B^{-p}).
\]

Using Lemma \(4.2\) again, we obtain
\[
\frac{1}{r_j^n} \int_{\mathcal{L}_{j}(\bar{l})} G \left( \frac{l_j - u}{l_j - l} \right) \xi_j(x,t)^{k} dx
\]
\[
\leq \gamma \frac{(l_j - \bar{l})^{p-2}}{r_j^{n+p}} \int_{\mathcal{L}_{j}(\bar{l})} \left( 1 + \frac{l_j - u}{l_j - l} \right)^{(1-\lambda)(p-1)} \left( \frac{l_j - u}{l_j - l} \right)^{2\lambda(p-1)} \xi_j(x,t)^{k-p} dxdt
\]
\[
+ \gamma \varepsilon \frac{(l_j - \bar{l})^{p-2}}{r_j^{n+p}} \int_{\mathcal{L}_{j}(\bar{l})} \left( 1 + \frac{l_j - u}{l_j - l} \right)^{(1+\lambda)(p-1)} \xi_j(x,t)^{k-p} dxdt
\]
\[
(4.24)
\]
\[
+ \gamma \varepsilon \frac{r_j^{p-n}}{(l_j - l)^{p}} \int_{B_j} F_1(x) dx + \gamma \frac{r_j^{p-n}}{(l_j - l)^{p-1}} \int_{B_j} F_2(x) dx.
\]
Using the decomposition $L_j(\bar{l}) = L'(\bar{l}) \cup L''(\bar{l})$ we have

$$\frac{\gamma (l_j - \bar{l})^{p-2}}{r_j^{n+p}} \int \int_{L_j(\bar{l})} \left( 1 + \frac{l_j - v}{l_j - t} \right)^{(1-\lambda)(p-1)} \left( \frac{l_j - v}{l_j - t} \right)^{2\lambda(p-1)} \xi_j(x, t)^k_\lambda dx dt$$

$$\leq \frac{\gamma (l_j - \bar{l})^{p-2}}{r_j^{n+p}} \varepsilon_1^{2\lambda(p-1)} (1 + \varepsilon_1)^{(1-\lambda)(p-1)} |L_j'(\bar{l})|$$

(4.25) + $\gamma (\varepsilon_1) \frac{(l_j - \bar{l})^{p-2}}{r_j^{n+p}} \int \int_{L_j'(\bar{l})} \left( \frac{l_j - v}{l_j - t} \right)^{(1+\lambda)(p-1)} \xi_j(x, t)^k_\lambda dx dt$.

Similarly

$$\gamma \varepsilon \frac{(l_j - \bar{l})^{p-2}}{r_j^{n+p}} \int \int_{L_j(\bar{l})} \left( 1 + \frac{l_j - v}{l_j - t} \right)^{(1-\lambda)(p-1)} \xi_j(x, t)^k_\lambda dx dt$$

$$\leq \gamma \varepsilon (1 + \varepsilon_1)^{(1-\lambda)(p-1)} \frac{(l_j - \bar{l})^{p-2}}{r_j^{n+p}} |L_j'(\bar{l})|$$

(4.26) + $\varepsilon \gamma (\varepsilon_1) \frac{(l_j - \bar{l})^{p-2}}{r_j^{n+p}} \int \int_{L_j'(\bar{l})} \left( \frac{l_j - v}{l_j - t} \right)^{(1+\lambda)(p-1)} \xi_j(x, t)^k_\lambda dx dt$.

Combining estimates (4.17)-(4.26) we have

(4.27) $A_j(\bar{l}) \leq \gamma (\varepsilon_1^{1+\lambda)(p-1)} + \varepsilon_1^{2\lambda(p-1)} + \varepsilon (1 + \varepsilon_1)^{(1-\lambda)(p-1)} \varkappa + \gamma (\varepsilon_1, \varepsilon) (B^{1-p} + B^{-p}) + \gamma (\varepsilon_1, \varepsilon) \{ \varkappa + (B^{1-p} + B^{-p}) \}^{1 + \frac{\lambda}{16}}$

First choose $\varepsilon_1$ from the condition

(4.28) $\varepsilon_1^{1+\lambda)(p-1)} + \varepsilon_1^{2\lambda(p-1)} = \frac{1}{16}$

Next we choose $\varepsilon$ from the equality

(4.29) $\varepsilon (1 + \varepsilon_1)^{(1-\lambda)(p-1)} = \frac{1}{16}$

Fix $\varkappa$ by

(4.30) $\gamma (\varepsilon_1, \varepsilon) \frac{\varkappa}{16} = \frac{1}{16}$

and choosing $B$ large enough so that

(4.31) $B^{1-p} + B^{-p} \leq \frac{\varkappa}{16}$

we conclude from (4.27) that $A_j(\bar{l}) \leq \frac{\varkappa}{16}$, which completes the proof of Lemma 4.3.

Further, since $A_j(l)$ is an increasing and continuous function and $A_j(l) \to \infty$ if $l \to l_j$, inequality (4.16) ensures the existence of $\bar{l} \in (l, l_j)$ such that $A_j(\bar{l}) = \varkappa$. If $\bar{l} < l_j - \frac{1}{4}(\alpha_{j-1} - \alpha_j)$ we set $l_{j+1} = \bar{l}$. If $\bar{l} \geq l_j - \frac{1}{4}(\alpha_{j-1} - \alpha_j)$, then we set $l_{j+1} = l_j - \frac{1}{4}(\alpha_{j-1} - \alpha_j)$ and in both cases we set $\delta_j = l_j - l_{j+1}$.

In what follows

$$Q_j = Q_j(l_{j+1}), \quad L_j = L_j(l_{j+1})$$

Lemma 4.4. Let the conditions of Theorem 1.2 be fulfilled. Then for any $j \geq 1$ the following inequality holds

(4.32) $\delta_j \leq \frac{1}{2} \delta_{j-1} + \gamma r_j + \gamma \left( r_j^{p-n} \int_{B_{j-1}} F_1(x) dx \right)^{\frac{\lambda}{p-n}} + \gamma \left( r_j^{p-n} \int_{B_{j-1}} F_2(x) dx \right)^{\frac{\lambda}{p-n}}$. 

19
Obtain \( \delta_j > \frac{1}{2} \delta_{j-1} \), \( \delta_j > \frac{1}{4} (\alpha_{j-1} - \alpha_j) \)
since in the opposite case due to \((4.10)\) inequality \((4.32)\) is obvious. The second inequality in \((4.33)\)
ensures that \( A_j(l_{j+1}) = \infty \). Using the decomposition \( L_j = L_j' \cup L_j'' \) similarly to \((4.17), (4.19) - (4.22)\) we obtain

\[
\frac{\delta_j^{p-2}}{r_j^{n+p}} \left( \int_{L_j} \left( \frac{l_j - v}{\delta_j} \right)^{(p-1)(1+\lambda)} \xi_j(x,t)^{k-p} \right) dx dt \\
\leq \frac{\varepsilon_1^{(1+\lambda)(p-1)}}{r_j^{n+p}} \left| L_j' \right| + \frac{\delta_j^{p-2}}{r_j^{n+p}} \left( \int_{L_j''} \left( \frac{l_j - v}{\delta_j} \right)^{(1+\lambda)(p-1)} \xi_j(x,t)^{k-p} \right) dx dt
\]

\[
\leq \gamma \left( (1+\lambda)^{(p-1)} + \varepsilon_1 \right) + \gamma (\varepsilon_1) \left\{ \lambda (1+\lambda)^{(p-1)} + \varepsilon (1 + \varepsilon_1) \lambda^{p-1} \right\}
\]

Using Lemma \(\ref{lem:4.2}\) in the same way as \((4.24), (4.26)\) we have

\[
\text{ess sup } t \left( \int_{L_j(t)} G \left( \frac{l_j - v}{\delta_j} \right) \xi_j(x,t)^k \right) dx \leq \gamma (\varepsilon_1^{(1+\lambda)(p-1)} + \varepsilon_1^{\lambda(p-1)} + \varepsilon (1 + \varepsilon_1) \lambda^{p-1}) \alpha
\]

\[
+ \gamma (\varepsilon_1, \varepsilon) \left\{ \lambda (1+\lambda)^{(p-1)} + \varepsilon (1 + \varepsilon_1) \lambda^{p-1} \right\}
\]

Choosing \(\varepsilon_1, \varepsilon, \alpha\) from inequalities \((4.28), (4.29), (4.30)\) we conclude that at least one of the two following inequalities holds

\[
\delta_j \leq \gamma \left( \int_{B_j} F_1(x) dx \right)^{\frac{1}{p-1}}, \quad \delta_j \leq \gamma \left( \int_{B_j} F_2(x) dx \right)^{\frac{1}{p-1}},
\]

which proves Lemma \(\ref{lem:4.4}\).

Summing up inequality \((4.32)\) with respect to \(j = 1, \ldots, J - 1\) we obtain

\[
l_j - l_j \leq \delta_0 + \gamma r_0 + \gamma \sum_{j=1}^{\infty} \left( \int_{B_{j-1}} F_1(x) dx \right)^{\frac{1}{p-1}} + \gamma \sum_{j=1}^{\infty} \left( \int_{B_{j-1}} F_2(x) dx \right)^{\frac{1}{p-1}}.
\]

If \(l_1\) is defined by \(l_1 = \xi \omega - \frac{1}{4} (\alpha - \alpha_0)\) then \(\delta_0 = \frac{1}{4} (\alpha - \alpha_0)\). Passing to the limit in \((4.36)\) as \(J \to \infty\) we have

\[
\xi \omega \leq \lim_{j \to \infty} l_j + \gamma r_0 + \gamma \int_0^{p} \left( \frac{1}{r^{n-p}} \int_{B_r(y)} F_1(x) dx \right)^{\frac{1}{p-1}} \frac{dr}{r} + \gamma \int_0^{\rho} \left( \frac{1}{r^{n-p}} \int_{B_r(y)} F_2(x) dx \right)^{\frac{1}{p-1}} \frac{dr}{r}.
\]

If \(l_1 < \xi \omega - \frac{1}{4} (\alpha - \alpha_0)\) then \(A_0(l_1) = \infty\) and at least one of the following inequalities holds

\[
\frac{\delta_0^{p-2}}{r_0^{n+p}} \int_{L_0} \left( \frac{l_0 - v}{\delta_0} \right)^{(1+\lambda)(p-1)} dx dt \geq \frac{\alpha}{2},
\]

20
or

\[ (4.39) \quad \text{ess sup} \frac{1}{r_0^p} \int_{L_0} G \left( \frac{l_0 - v}{\delta_0} \right) dx \geq \frac{\kappa}{2}. \]

Similarly to (4.11), it follows from (4.38) that

\[ (4.40) \quad \frac{\kappa}{2} \leq \text{ess sup} \int_{L_0} G \left( \frac{l_0 - v}{\delta_0} \right) dx \]

\[ \leq \frac{\gamma}{r_0^{n+p}} \int_{L_0} \left( 1 + \frac{l_0 - v}{\delta_0} \right)^{(1+\lambda)(p-1)} dx dt \leq \frac{\gamma}{\delta_0^{1+\lambda(p-1)}} C^{n+p} \nu(\xi \omega)^{(1+\lambda)(p-1)} \theta. \]

Similarly to (4.12) with \( \varepsilon = \nu^{-\frac{1}{2}} \), it follows from (4.39) that

\[ (4.41) \quad \frac{\kappa}{2} \leq \text{ess sup} \int_{L_0} G \left( \frac{l_0 - v}{\delta_0} \right) dx \]

\[ \leq \frac{\gamma}{r_0^{n+p}} \nu^{-\frac{1}{2}} \int_{L_0} \left( 1 + \frac{l_0 - v}{\delta_0} \right)^{(1+\lambda)(p-1)} dx dt + \frac{\gamma}{\delta_0^{1+\lambda(p-1)}} \int_{B_0} F_1(x) dx + \frac{\gamma}{\delta_0^{1+\lambda(p-1)}} \int_{B_0} F_2(x) dx. \]

First we choose \( C > 16 \). Then (4.40), (4.41) imply that

\[ (4.42) \quad \delta_0 \leq \xi \omega (\gamma \frac{1}{C^{n+p}} \nu(\xi \omega)^{p-2} \theta)^{\frac{1}{\lambda(p-1)}} + \frac{\gamma}{\delta_0^{1+\lambda(p-1)}} \frac{1}{\nu^{\frac{1}{p}}} \left( \int_{B_0} F_1(x) dx \right)^{\frac{1}{p}} + \frac{\gamma}{\delta_0^{1+\lambda(p-1)}} \frac{1}{\nu^{\frac{1}{p}}} \left( \int_{B_0} F_2(x) dx \right)^{\frac{1}{p}}. \]

Finally due to the inequality \( \xi \omega \geq \theta \frac{1}{\nu} \) from (4.36), (4.37), (4.42) we have due to \( \nu < 1 \)

\[ (4.43) \quad \xi \omega \leq v(x_1, t_1) + \xi \omega (\gamma \frac{1}{C^{n+p}} \nu(\xi \omega)^{p-2} \theta)^{\frac{1}{\lambda(p-1)}} + \gamma \rho \]

\[ + \int_0^\rho \left( \int_{B_{r}(y)} F_1(x) dx \right)^{\frac{1}{p}} dr + \int_0^\rho \left( \int_{B_{r}(y)} F_2(x) dx \right)^{\frac{1}{p}} dr. \]

Next we fix \( \nu \) from the condition

\[ (4.44) \quad \gamma \frac{1}{C^{n+p}} \nu(\xi \omega)^{p-2} \theta = \left( \frac{1 - a}{2} \right)^{1+\lambda(p-1)} \]

and finally, choosing \( B \) large enough so that

\[ (4.45) \quad B \geq \frac{2\gamma}{1 - a}, \]

we obtain from (4.43)

\[ (4.46) \quad u(x_1, t_1) \geq \mu_\omega + a \xi \omega. \]

Since \( (x_1, t_1) \) is an arbitrary point in \( Q_{\delta}(y, s) \), from (4.40) the required (1.11) follows, which proves Theorem 1.5 \( \square \)
5 Expansion of positivity. Proof of Theorem 1.6

In the proof we closely follow [10], also using the idea of logarithmic estimates from [7]. Our assumption here are again (1.1). In what follows we suppose that

\begin{equation}
N \geq B(\rho + F_1(2\rho) + F_2(2\rho)).
\end{equation}

Let \(0 \leq \tau \leq \frac{1}{2}(p - 2) \ln B\), \(k = \mu_+ + e^{\frac{\tau}{p-2}} N, \theta = e^\tau N^{-(p-2)}, \xi \in C_0^\infty(B_\rho(y)), \xi(x) = 1\) if \(x \in B_{\rho/2}(y), 0 \leq \xi(x) \leq 1, |\nabla \xi(x)| \leq 2\rho^{-1}\). As above, set \(\Psi_-(u) = \ln_{\frac{H_\tau^{-}(k-u)+2^{-s_0}e^{\frac{\tau}{p-2}}N}{H_\tau^{-}}}, s_0\) is a positive number satisfying \(s_0 < \frac{1}{2} \ln B\), which will be determined later depending on the data. Note the evident inequalities

\begin{align*}
(k - u)_+ &\leq e^{-\frac{\tau}{p-2}} N, \quad |\Psi_-(u)| \leq s_0 \ln 2, \\
|\Psi_-(u)| &\leq 2^{s_0} e^{\frac{\tau}{p-2}} N^{-1}, \quad |\Psi_-(u)|^{2-p} \leq e^{-\tau} N^{p-2},
\end{align*}

\[
\int_{B_\rho(y)} F_1(x) dx \leq \gamma F_1(2\rho)^p \rho^{n-p}, \quad \int_{B_\rho(y)} F_2(x) dx \leq \gamma F_2(2\rho)^{p-1} \rho^{n-p}.
\]

Since condition (1.14) guarantees that

\(\Psi_-(u) = 0\) for \(x \in B_\rho(y), t = s\), Lemma 2.2 implies that

\[
\begin{align*}
&\text{ess sup}_{s < t < s + \theta \rho^p} \int_{B_\rho^2(y)} \Psi_-^2(u) dx \leq \gamma \int_{Q_\rho^2(y,s)} \Psi_- |\Psi_-'(u)|^{2-p} |\nabla \xi|^p dx + \gamma \int_{Q_\rho^2(y,s)} \Psi_- |\Psi_-(u)|^{2-p} dx dt \\
&+ \gamma \int_{Q_\rho^2(y,s)} \Psi_- |\Psi_-'(u)|^2 F_1(x) dx dt + \gamma \int_{Q_\rho^2(y,s)} \Psi_- |\Psi_-(u)| F_2(x) dx dt \\
\leq & \quad \gamma s_0 \rho^n + \gamma \frac{s_0}{N} \left( \frac{2^{s_0} e^{\frac{\tau}{p-2}} F_1(2\rho)}{N} \right)^p \rho^n + \gamma \frac{s_0}{N} \left( \frac{2^{s_0} e^{\frac{\tau}{p-2}} F_2(2\rho)}{N} \right)^{p-1} \rho^n \leq \gamma s_0 \rho^n.
\end{align*}
\]

Since

\(\Psi_-(u) \geq (s_0 - 1) \ln 2\) for \(x \in B_{\rho/2}(y) \cap \left\{ u < \mu_+ + \frac{e^{-\frac{\tau}{p-2}} N}{2^{s_0}} \right\}\),

inequality (5.3) yields

\begin{equation}
\left\{ x \in B_{\rho/2}(y) : u(x, t) < \mu_+ + \frac{e^{-\frac{\tau}{p-2}} N}{2^{s_0}} \right\} \leq \gamma \frac{s_0}{(s_0 - 1)^2} |B_{\rho/2}(y)|
\end{equation}

for all \(t \in (s, s + \theta \rho^p), 0 \leq \tau \leq \frac{1}{2}(p - 2) \ln B\). Choosing \(s_0\) from the condition

\begin{equation}
\frac{s_0}{(s_0 - 1)^2} \leq \frac{1}{2},
\end{equation}

we obtain

\begin{equation}
\left\{ x \in B_{\rho/2}(y) : u(x, s + \theta \rho^p) \leq \mu_+ + \frac{e^{-\frac{\tau}{p-2}} N}{2^{s_0}} \right\} \leq \frac{1}{2} |B_{\rho/2}(y)|
\end{equation}

for all \(0 \leq \tau \leq \frac{1}{2}(p - 2) \ln B\).

In the same way as in [10, p. 191] we consider the function

\[
w(x, \tau) = e^{-\frac{\tau}{p-2}} N^{-1} \rho^{\frac{\tau}{p-2}} \left( u(x, s + (e^{-\frac{\tau}{p-2}} N^{-1})^{p-2} \rho^p) - \mu_- \right),
\]

22
and let \( k_0 = 2^{-s_0} \rho^{2/p} \).

Inequality (3.6) translates into \( w \) as \( \{ x \in B_{\rho/2}(y) : w(x, \tau) \leq k_0 \} \leq k_0 \{|B_{\rho/2}(y)|\} \), which yields

\[
(5.7) \quad \{ x \in B_{4\rho}(y) : w(x, \tau) \leq k_0 \} \leq \left(1 - \frac{1}{2 \cdot 8^n}\right) |B_{4\rho}(y)|
\]

for all \( \tau \in (0, 2^{-s_0(p-2)-1}(p-2) \ln B k_0^{2-p} \rho^p) \).

Since \( w \geq 0 \), formal differentiation, which can be justified in a standard way, gives

\[
(5.8) \quad w_\tau = \frac{1}{p-2} w + (e^{-2N^{-1} \rho^{2/p}})^{p-1} u_t \geq \text{div} \tilde{A}(x, t, w, \nabla w) + \tilde{a}_0(x, t, w, \nabla w),
\]

where \( \tilde{A}, \tilde{a}_0 \) satisfy the inequalities

\[
\tilde{A}(x, t, w, \nabla w) \cdot \nabla w \geq c_1 |\nabla w|^p, \quad |\tilde{A}(x, t, w, \nabla w)| \leq c_2 |\nabla w|^{p-1} + (e^{-2N^{-1} \rho^{2/p}})^{p-1} f_1(x), \quad |\tilde{a}_0(x, t, w, \nabla w)| \leq c_2 |\nabla w|^{p-1} + (e^{-2N^{-1} \rho^{2/p}})^{p-1} f_2(x).
\]

**Lemma 5.1.** For every \( \nu \in (0, 1) \) there exists \( s_* > s_0, 2^{s_*} \leq 2^{-s_0(p-2)-1}(p-2) \ln B, \) depending only on the data and \( \nu \) such that

\[
(5.10) \quad \{ |Q^*_\rho : w(x, \tau) < \frac{k_0}{2^{s_*}} \} \leq \nu |Q^*_\rho|,
\]

where \( Q^*_\rho = B_\rho(y) \times (2k_0^{2-p} \rho^p, (2^{s_*} k_0^{-1})^{p-2} \rho^p) \).

**Proof.** Using Lemma 2.3 with \( k = \frac{k_0}{2^{s_*}}, l = \frac{k_0}{2^{s_*}}, s_0 \leq s \leq s_* \), due to (5.7) we obtain the inequality

\[
(5.11) \quad \frac{k_0}{2^{s_*}} |A_{\frac{k_0}{2^{s_*}},4\rho}(\tau)| = \gamma \rho^{\frac{2}{p-2}} \int_{A_{\frac{k_0}{2^{s_*}},4\rho}(\tau)} |\nabla w(x, \tau)| \, dx
\]

for all \( \tau \in (0, 2^{-s_0(p-2)-1}(p-2) \ln B k_0^{2-p} \rho^p) \), where \( A_{k,\rho}(\tau) = \{ x \in B_\rho(y) : w(x, \tau) \leq k \} \).

Integrating the last inequality with respect to \( \tau, \tau \in (2k_0^{2-p} \rho^p, (2^{s_*} k_0^{-1})^{p-2} \rho^p) \), and using the Hölder inequality we obtain

\[
(5.12) \quad \left( \frac{k_0}{2^{s_*}} \right)^{\frac{p}{p-2}} |A_{\frac{k_0}{2^{s_*}},4\rho}|^{\frac{p}{p-2}} \leq \gamma \rho^{\frac{2}{p-2}} \left( \int_{A_{\frac{k_0}{2^{s_*}},4\rho}} |\nabla w(x, \tau)|^p \, dx \, dt \right)^{\frac{1}{p-2}} |A_{\frac{k_0}{2^{s_*}},4\rho} \setminus A_{\frac{k_0}{2^{s_*}},4\rho}|^{\frac{1}{p-2}},
\]

where \( A_{k,\rho} = \int_{2k_0^{2-p} \rho^p}^{(2^{s_*} k_0^{-1})^{p-2} \rho^p} A_{k,\rho}(\tau) \, d\tau \).

To estimate the first factor we use Lemma 2.4 with \( k = \frac{k_0}{2^{s_*}}, \xi \in C_0^\infty(\tilde{Q}^*_\rho), \xi(x, \tau) = 1 \) for \( (x, t) \in \tilde{Q}^*_\rho, 0 \leq \xi(x, \tau) \leq 1, |\nabla \xi| \leq \gamma \rho^{-1}, \left| \frac{\partial \xi}{\partial t} \right| \leq \gamma \left( \frac{k_0}{2^{s_*}} \right)^{p-2} \rho^{-p}, \tilde{Q}^*_\rho = B_\rho(y) \times \left( \frac{k_0^{2-p}}{2^{s_*}} \rho^p, (2^{s_*} k_0^{-1})^{p-2} \rho^p \right) \).

Due to (5.8), (5.9) we obtain

\[
(5.13) \quad \int_{\tilde{Q}^*_\rho} |\nabla w(x, \tau)|^p \, dx \, d\tau \leq \gamma \left( \frac{k_0^{2-p}}{2^{s_*}} - w \right)^p + \frac{k_0^{2-p}}{2^{s_*}} - w \right)^2 \left| \frac{\partial \xi}{\partial t} \right| \, dx \, d\tau
\]

\[
(5.13) \quad + \gamma (e^{-2N^{-1} \rho^{2/p}})^{p-1} \int_{\tilde{Q}^*_\rho} (\frac{k_0^{2-p}}{2^{s_*}} - w) + F_2(x) \, dx \, d\tau + \gamma (e^{-2N^{-1} \rho^{2/p}})^{p} \int_{\tilde{Q}^*_\rho} F_1(x) \, dx \, d\tau = \sum_{i=1}^{3} I_i.
\]
Due to the choice of $\xi(x,t)$ we have
\begin{equation}
I_1 \leq \gamma \left( \frac{k_0}{2^s} \right)^p \rho^{-p} |Q_{4\rho}|.
\end{equation}

Using (5.14) and the definition of the $K_p, \bar{K}_p$-classes we derive
\begin{equation}
I_2 \leq \gamma \left( e^{\frac{\pi}{2s} N^{-1} N^{\frac{p}{2}}} \right)^{p-1} \frac{k_0}{2^s} F_2(2\rho)^{p-1} \rho^{-p} |Q_{4\rho}| = \gamma \left( \frac{e^{\frac{\pi}{2s} N^{\frac{p}{2}}} F_2(2\rho)}{N} \right)^{p-1} \left( \frac{k_0}{2^s} \right)^p \rho^{-p} |Q_{4\rho}| \leq \gamma \left( \frac{k_0}{2^s} \right)^p \rho^{-p} |Q_{4\rho}|.
\end{equation}

Combining estimates (5.12)–(5.16) we obtain
\begin{equation}
\left| A_{\frac{k_0}{2^s+4\rho}} \rho \right| \leq \gamma |Q_{4\rho}| \left| A_{\frac{k_0}{2^s+4\rho}} \rho \right| A_{\frac{k_0}{2^s+4\rho}} \rho \right|.
\end{equation}

Summing up the last inequalities in $s, s_0 < s \leq s_*$, we conclude that
\begin{equation}
(s_* - s_0) |A_{\frac{k_0}{2^s+4\rho}} \rho| \leq \gamma |Q_{4\rho}| \left| A_{\frac{k_0}{2^s+4\rho}} \rho \right|.
\end{equation}

Choosing $s_*$ by the condition
\begin{equation}
(s_* - s_0) \leq \frac{2^s-1}{N} \gamma \leq \nu,
\end{equation}
we obtain inequality (5.10), which proves Lemma 5.1.

Using Theorem 1.5 with $\xi = \frac{1}{2^s}, \omega = k_0, \theta = (2^s k_0)^{-p-2}, a = \frac{1}{2}$ and choosing $\nu$ from condition (4.44) we obtain
\begin{equation}
w(x, \tau) \geq \frac{N}{2^{s+1}} \text{ for } x \in B_2(y)
\end{equation}
and for all $\tau \in \left( k_0^{2-p}(2\rho)^p, (2^s k_0)^{p-2}(2\rho)^p \right)$.

Due to the choice of $k_0$, we have $k_0^{2-p} \rho^p = 2^{o_1(p-2)}$. For $\tau \in \left( k_0^{2-p}(2\rho)^p, (2^s k_0)^{p-2}(2\rho)^p \right)$ there holds
\[ \bar{b}_1 = \exp 2^{(s_0+1)(p-2)+1} e^\tau \leq \exp 2^{(s_*+s_0+1)(p-2)+2} = \bar{b}_2. \]

Inequality (5.20) translates for $u$ into
\begin{equation}
u(x, s + t) \geq 2^{-s-1} b_2^{-1} N = \sigma N \text{ for } x \in B_2(y)
\end{equation}
and for all $b_1 N^{2-p} \rho^p \leq t \leq b_2 N^{2-p} \rho^p$, where
\begin{equation}
b_1 = b_1(s_0) = \bar{b}_1^{-p-2}, b_2 = b_2(s_0, s_*) = \bar{b}_2^{-p-2}
\end{equation}
depend only on the data. This completes the proof of Theorem 1.6.

6 Continuity of solutions. Proof of Theorem 1.2

Here we closely follow [7], Chapter III. Let $(x_0, t_0) \in \Omega_T$ be arbitrary,
\[ Q_R(x_0, t_0) = B_R(x_0) \times (t_0 - R^2, t_0), \quad R < \frac{1}{4} \min\{1, t_0^{1/2}, \mathrm{dist}(x_0, \partial\Omega)\}. \]
Set
\[ \mu_+ = \text{ess sup}_{Q_R(x_0,t_0)} u(x,t), \quad \mu_- = \text{ess inf}_{Q_R(x_0,t_0)} u(x,t), \quad \omega = \mu_+ - \mu_- . \]

Fix a positive number \( s^* \), \( s_1 = \frac{1}{p-2} \log_2 b_1 < s^* < \log_2 b_2 \), which will be determined later depending only on the known data, \( b_1 = b_1(s_0), b_2 = b_2(s_0, s_*) \) are defined in (6.22).

If \( (6.1) \)
\[ \omega \geq b_2(R + F_1(2R) + F_2(2R)), \]
then the cylinder
\[ Q^0_R(x_0,t_0) = B_R(x_0) \times (t_0 - \theta R^p, t_0), \quad \theta = \left( \frac{2s^*}{\omega} \right)^{p-2} \]
is contained in \( Q_R(x_0,t_0) \). In \( Q^0_R(x_0,t_0) \) consider the cylinders
\[ Q^0_R(x_0,\bar{t}) = B_R(x_0) \times (\bar{t} - \eta R^p, \bar{t}), \quad \eta = b_1 \omega^{2-p}, \quad t_0 - \theta R^p \leq \bar{t} - \eta R^p < \bar{t} \leq t_0. \]

Let us fix \( \nu \in (0,1) \) satisfying (4.14) with \( a = \left( \frac{1}{2} \right)^{\frac{1}{p-2}}, \xi = \frac{1}{2} \) and \( \theta = b_1 \omega^{p-2} \).

The following two alternative cases are possible.

**First alternative.** There exists a cylinder \( Q^0_R(x_0,\bar{t}) \subset Q^0_R(x_0,t_0) \) such that
\[ (6.2) \quad \left\| \lbrace (x,t) \in Q^0_R(x_0,\bar{t}) : u(x,t) \leq \mu_- + \frac{\omega}{2} \rbrace \right\| \leq \nu |Q^0_R(x_0,\bar{t})|. \]

**Second alternative.** For all cylinders \( Q^0_R(x_0,\bar{t}) \subset Q^0_R(x_0,t_0) \) the opposite inequality
\[ (6.3) \quad \left\| \lbrace (x,t) \in Q^0_R(x_0,\bar{t}) : u(x,t) \leq \mu_- + \frac{\omega}{2} \rbrace \right\| > \nu |Q^0_R(x_0,\bar{t})|. \]
holds.

### 6.1 Analysis of the first alternative

Here we assume that (6.1) is satisfied. By Theorem 1.5 with \( \xi = \frac{1}{2}, a = \left( \frac{1}{2} \right)^{\frac{1}{p-2}} \) we obtain from (6.2)
\[ (6.4) \quad u(x,\bar{t}) \geq \mu_- + 2^{-1 - \frac{1}{p-2}} \omega \quad \text{for all } x \in B_{\frac{1}{2}}(x_0). \]

Using Theorem 1.6 with \( N = 2^{-1 - \frac{1}{p-2}} \omega \) from (6.4) we conclude that
\[ (6.5) \quad u(x,t) \geq \mu_- + \sigma N \quad \text{for } x \in B_{\frac{1}{2}}(x_0) \]
and for all \( t \in (\bar{t} + \frac{1}{2} b_1 R^p, \bar{t} + \frac{1}{2} b_2 R^p) \), where \( \sigma = \sigma(s_0, s_*), b_1 = b_1(s_0), b_2 = b_2(s_0, s_*) \) are fixed numbers defined in (5.22).

Since \( \bar{t} + \frac{1}{2} b_1 R^p < t_0 < \bar{t} + \frac{1}{2} b_2 R^p \), inequality (6.5) holds for \( (x,t) \in B_{\frac{1}{2}}(x_0) \times (t_0 - \frac{1}{2} b_1 R^p, t_0) \).

Thus we have proved the following

**Proposition 6.1.** Suppose the first alternative holds. Then either \( \omega \leq b_2(R + F_1(2R) + F_2(2R)) \), or
\[ (6.6) \quad \text{ess osc}_{Q^0_R(x_0,t_0)} u \leq (1 - \sigma) \omega. \]

### 6.2 Analysis of the second alternative

This part is almost a literal repetition of the corresponding part from [7] Chapter III and is here for the readers' convenience.

Since (6.3) holds for all cylinders \( Q^0_R(x_0,\bar{t}) \), for the cylinders \( Q^0_R(x_0,\bar{t}) \subset Q^0_R(x_0,t_0) \) we have that
\[ (6.7) \quad \left\| \lbrace (x,t) \in Q^0_R(x_0,\bar{t}) : u(x,t) \geq \mu_+ - \frac{\omega}{2} \rbrace \right\| \leq \left\| \lbrace (x,t) \in Q^0_R(x_0,\bar{t}) : u(x,t) \geq \mu_+ - \frac{\omega}{2} \rbrace \right\| \leq (1 - \nu) |Q^0_R(x_0,\bar{t})|. \]

Further on we assume that (6.1) holds.
Lemma 6.2. Fix a cylinder $Q^0_R(x_0, \bar t)$. Suppose that (6.7) holds. There exists $t_\ast \in (\bar t - \eta R^p, \bar t - \frac{\omega}{2s_1} R^p)$ such that

\begin{equation}
(6.8) \quad \left| \left\{ x \in B_R(x_0) : u(x, t) \geq \mu_+ - \frac{\omega}{2s_1} \right\} \right| \leq \frac{1 - \nu}{1 - \eta} B_R(x_0).
\end{equation}

Proof. Suppose not. Then for all $t \in (\bar t - \eta R^p, \bar t - \frac{\omega}{2s_1} R^p)$ there holds

\[ \left| \left\{ x \in B_R(x_0) : u(x, t) \geq \mu_+ - \frac{\omega}{2s_1} \right\} \right| > \frac{1 - \nu}{1 - \eta} B_R(x_0). \]

Hence

\[ \left| \left\{ (x, t) \in Q^0_R(x_0, \bar t) : u(x, t) \geq \mu_+ - \frac{\omega}{2s_1} \right\} \right| \]
\[ \geq \int_{\bar t - \eta R^p}^{\bar t - \frac{\omega}{2s_1} R^p} \left| \left\{ x \in B_R(x_0) : u(x, t) \geq \mu_+ - \frac{\omega}{2s_1} \right\} \right| dt > (1 - \nu)|Q^0_R(x_0, \bar t)|, \]

which contradicts (6.7).

\[ \square \]

Lemma 6.3. There exists a number $s_1 < s_2 < s^\ast$, which depends only on known data, such that

\begin{equation}
(6.9) \quad \left| \left\{ x \in B_R(x_0) : u(x, t) \geq \mu_+ - \frac{\omega}{2s_1} \right\} \right| \leq \left( 1 - \left( \frac{\nu}{2} \right)^2 \right) |B_R(x_0)|,
\end{equation}

for all $t \in (\bar t - \frac{\omega}{2s_1} R^p, \bar t)$.

Proof. We use Lemma 2.2 in the cylinder $B_R(x_0) \times (t_\ast, \bar t)$ with $k = \mu_+ - \frac{\omega}{2s_1}$, \( \Psi \) defined by

\[ \Psi_+(u) = \ln \left( \frac{H_k^+}{H_k^+ - (\mu_+ - \frac{\omega}{2s_1})^+} + \frac{\omega}{2s_1} \right), \quad H_k^+ = \text{ess sup}_{Q^0_R(x_0, \bar t)} (u - \mu_+ + \frac{\omega}{2s_1})^+ \leq \frac{\omega}{2s_1}, \]

and $\xi$ satisfying $1_{B_R(z_0)} \leq \xi \leq 1_{B_R(x_0)}$, $|\nabla \xi| \leq \frac{2}{\eta R}$. We can assume without loss that $H_k^+ > \frac{\omega}{2s_1} \eta R$, since otherwise the assertion follows. From Lemma 2.2 it follows that

\begin{equation}
(6.10) \quad \int_{B_1(\bar t) \cap B_R(x_0)} \Psi^2_+(u(x, t)) dx \leq \int_{B_R(x_0)} \Psi^2_+(u(x, t)) dx + \gamma (\sigma R)^{-p} \int_{Q^0_R(x_0, \bar t)} \Psi_+ |\Psi'_+(u)|^{2-p} dx dt
\end{equation}

\[ + \gamma \int_{Q^0_R(x_0, \bar t)} \Psi_+ |\Psi'_+(u)|^2 F_1(x) dx dt + \gamma \int_{Q^0_R(x_0, \bar t)} \Psi_+ |\Psi'_+(u)| F_2(x) dx dt. \]

From the definition of $\Psi_+$ it follows that

\[ \Psi_+ \leq s_2 \ln 2, \quad |\Psi'(u)| \leq \frac{2s_2}{\omega}, \quad |\Psi'_+(u)|^{2-p} \leq \gamma \left( \frac{\omega}{2s_1} \right)^{p-2}. \]

On the set $\left\{ x \in B_R(x_0) : u(x, t) \geq \mu_+ - \frac{\omega}{2s_1} \right\}$ we also have $\Psi_+ \geq (s_1 - 1) \ln 2$. Using (6.8) we infer from (6.10) that for all $t \in (t_\ast, \bar t)$

\begin{equation}
(6.11) \quad \left| \left\{ x \in B_R(x_0) : u(x, t) \leq \mu_+ - \frac{\omega}{2s_1} \right\} \right| \]
\[ \leq (s_2 - 1)^2 \ln^{-2} 2 \int_{B_1(\bar t) \cap B_R(x_0)} \Psi^2_+(u(x, t)) dx + n \sigma |B_R(x_0)| \]
\[ \leq \left( \frac{s_2}{s_2 - 1} \right)^2 \frac{1 - \nu}{1 - \eta} |B_R(x_0)| + n \sigma |B_R(x_0)| \]
\[ + \gamma \frac{s_2}{(s_2 - 1)^2} \left\{ \sigma^{-p} + \left( \frac{2s_2 F_1(2R)}{\omega} \right)^p \right\} + \left( \frac{2s_2 F_2(2R)}{\omega} \right)^{p-1} |B_R(x_0)|. \]

26
First choosing $\sigma$ such that $n\sigma \leq \frac{3}{8} = \nu^2$ and then $s_2$ such that
\[
\left( \frac{s_2}{s_2 - 1} \right)^2 \leq (1 - \frac{\nu}{2})(1 + \nu), \quad \gamma \frac{s_2}{(s_2 - 1)^2} (1 + \sigma^{-p}) \leq \frac{3}{8} \nu^2,
\]
due to (6.11) we obtain the required (6.9) from (6.11).

Since inequality (6.9) holds true for all cylinders $Q^\rho_R(x_0, t)$, Lemma 6.2 implies the following assertion.

**Remark 6.4.** For all $t \in (t_0 - \frac{1}{2}\theta R^p, t_0)$ the inequality
\[
(6.12) \quad \left| \{x \in B_R(x_0) : u(x, t) \geq \mu_+ - \frac{\omega}{2s_2} \} \right| \leq \left( 1 - \frac{\nu^2}{2} \right) |B_R(x_0)|
\]
holds.

**Lemma 6.5.** For any $\nu \in (0, 1)$ there exists a number $s^*$, $s_2 < s^* < \log_2 b_2$, depending on the data only, such that
\[
(6.13) \quad \left| \{ (x, t) \in Q^\rho_R(x_0, t_0) : u(x, t) \geq \mu_+ - \frac{\omega}{2s^*} \} \right| \leq \nu |Q^\rho_R(x_0, t_0)|.
\]

The proof of Lemma 6.5 is completely analogous to that of Lemma 5.1.

Using Theorem 1.5 with $\xi = \frac{1}{2} t$, $a = \frac{1}{2}$, $\theta = \left( \frac{2s^*}{\nu} \right)^{p-2}$ and $\nu$ defined by (4.44), from (6.13) we obtain that
\[
(6.14) \quad u(x, t) \leq \mu_+ - \frac{\omega}{2s^*+1} = \mu_+ - \sigma_1 \omega \quad \text{for a.a.} \ (x, t) \in Q^\rho_{2q}(x_0, t_0).
\]

Thus we have proved the following

**Proposition 6.6.** Let the second alternative hold. Then either $\omega \leq b_2(R + \mathcal{F}_1(2R) + \mathcal{F}_2(2R))$, or
\[
(6.15) \quad \text{ess osc}_{Q^\rho_{2q}(x_0, t_0)} u(x, t) \leq (1 - \sigma_1) \omega.
\]

From Propositions 6.1, 6.6 in the same way as in [14, Chapter III, Proposition 3.1] with the help of [11, Lemma 8.23] we obtain:

**Proposition 6.7.** For any $\varepsilon \in (0, 1)$ and for all $\rho \leq R$, there exist $\beta, \gamma > 0$ and $\alpha \in (0, 1)$, depending only on the data, such that
\[
(6.16) \quad \text{ess osc}_{Q(\rho, M)} u(x, t) \leq \gamma \left( \frac{\rho}{R} \right)^{\alpha} \omega(R) + \gamma \mathcal{F}_1(2\rho^p R^{1-\gamma}) + \gamma \mathcal{F}_2(2\rho^p R^{1-\gamma}),
\]
where $Q(\rho, M) = B_\rho(x_0) \times (t_0 - \beta M^{2-p} \rho^p, t_0)$, $M = \text{ess sup}_{\Omega_R} |u(x, t)|$.

This completes the proof of Theorem 1.2.

### 7 Harnack inequality. Sketch of Proof of Theorem 1.3

After we have proved Theorems 1.5, 6.7 the rest of the arguments do not differ from [10]. We give a short sketch here.

Let us consider the cylinder $Q_\tau = B_{\tau R}(x_0) \times \left( t_0 - \frac{\tau^p \rho}{u_0}, t_0 \right)$, $u_0 := u(x_0, t_0)$. Following Krylov-Safonov [14] consider the equation
\[
\max_{Q_\tau} u(x, t) = u_0(1 - \tau)^{-\beta}
\]
Choosing $\beta > 1$ is to be determined only depending on the data. Let $\tau_0$ be the maximal root of the above equation and $u(\bar{x}, \bar{t}) = u_0(1 - \tau)^{-\beta}$. Let $\tilde{Q} = B_{1 - \tau_0}(\bar{x}) \times (\bar{t} - (1 - \tau_0)\rho \frac{\rho^p}{u_0}, \bar{t})$. Since $\tilde{Q} \subset Q_{1 + \tau_0} \subset Q_1$, we have that

$$\max_Q u \leq \max_{Q_{1 + \tau_0}} u \leq 2^\beta(1 - \tau_0)^{-\beta}u_0.$$

Claim 1. There exists a positive number $\nu(\beta)$ such that

$$\{ (x, t) \in \tilde{Q} : u(x, t) \geq \frac{1}{2}(1 - \tau_0)^{-\beta}u_0 \} > \nu(\beta)|\tilde{Q}|.$$

Indeed, in the opposite case we apply Theorem 1.5 with the choices

$$\mu_+ = 2^\beta(1 - \tau_0)^{-\beta}u_0, \xi\omega = (2^\beta - \frac{3}{2})(1 - \tau_0)^{-\beta}u_0, a = \frac{\beta^2 - \frac{3}{2}}{2\beta - \frac{3}{2}}.$$ The condition $u_0 \geq B(\rho + F_1(2\rho) + F_2(2\rho))$ obviously implies that $\xi\omega \geq B(\frac{2\beta^2 - \frac{3}{2}}{2\beta - \frac{3}{2}} \rho + F_1((1 - \tau_0)\rho) + F_2((1 - \tau_0)\rho))$. Therefore we can conclude that $u(\bar{x}, \bar{t}) \leq \frac{3}{4}(1 - \tau_0)^{-\beta}$ reaching a contradiction which proves the claim.

Claim 2. (Analogue of [10 Proposition 8.3]) For every $\nu_0 \in (0, 1)$ there exists a point $(y, s) \in \tilde{Q}$ and $\eta_0 \in (0, 1)$ and a cylinder $Q_* = (y, s) + Q_{\eta_0(1 - \tau_0)}^\beta \subset \tilde{Q}$ such that the inequality $u_0 \geq B(\rho + F_1(2\rho) + F_2(2\rho))$ implies that

$$\{(x, t) \in Q_* : u(x, t) < \frac{1}{4}(1 - \tau_0)^{-\beta}u_0 \leq \nu_0|Q_*|.$$ The proof is the same as in [10]. One writes down the energy inequality (2.1) with $k = \frac{1}{2}(1 - \tau_0)^{-\beta}u_0$ over coaxial cylinders $2\tilde{Q}$ and $Q$ and obtains the inequality

$$\iint_{\tilde{Q}} \nabla (u - k)^- |p| dxdt \leq \gamma \frac{k^p}{R^p} |\tilde{Q}|, \quad R = \frac{1 - \tau_0}{2} \rho.$$ One only needs to note the estimates of the additional terms in the energy inequality (2.1). In the following (1.7), (1.8) are used.

$$\iint_{Q \cap \{u > k\}} f_1^\rho u^p \rho^p dxdt \leq \gamma \left( \frac{1 - \tau_0}{2} \right)^p \rho^p u_0^{2 - p} \left( \frac{1 - \tau_0}{2} \rho \right)^{n - p} F_1(2\rho)^p \leq \gamma \frac{k^p}{R^p} |\tilde{Q}|,$$

$$\iint_{\tilde{Q}} (u - k)^-_f \xi^p dxdt \leq \gamma k \left( \frac{1 - \tau_0}{2} \right)^p \rho^p u_0^{2 - p} \left( \frac{1 - \tau_0}{2} \rho \right)^{n - p} F_2(2\rho)^{p - 1} \leq \gamma \frac{k^p}{R^p} |\tilde{Q}|.$$ The rest of the proof of Claim 2 is the same as in [10 Proposition 8.3] and is based on Lemma 2.4 which in turn relies on [9].

As in [10], by Theorem 1.5 we obtain that there exist $(y, s) \in \tilde{Q}$ and $\eta_0 \in (0, 1)$ such that

$$u(x, s) \geq \frac{u_0}{8}(1 - \tau_0)^{-\beta} \quad \text{for} \quad |x - y| \leq r := \eta_0 \frac{1 - \tau_0}{2} \rho.$$ Then an application of Theorem 1.6 yields that if

$$\frac{u_0}{8}(1 - \tau_0)^{-\beta} \geq B(r + F_1(2r) + F_2(2r))$$

then

$$u(x, t) \geq \sigma \frac{u_0}{8}(1 - \tau_0)^{-\beta} \quad \text{for} \quad |x - y| \leq 2r, \quad s + \left( \frac{u_0}{8}(1 - \tau_0)^{-\beta} \right)^{2 - p} b_1 r^p \leq t \leq s + \left( \frac{u_0}{8}(1 - \tau_0)^{-\beta} \right)^{2 - p} b_2 r^p.$$ After iteration for $j = 1, 2, 3, \ldots$ we have either

$$\sigma^{j - 1} \frac{u_0}{8}(1 - \tau_0)^{-\beta} \leq B(\sigma^j r + F_1(2\sigma^j r) + F_2(2\sigma^j r))$$

or

$$u(x, t) \geq \sigma^j \frac{u_0}{8}(1 - \tau_0)^{-\beta} \quad \text{for} \quad |x - y| \leq 2^j r,$$

$$t_j^{(1)} = t_{j - 1}^{(1)} + \left( \frac{u_0}{8}(1 - \tau_0)^{-\beta} \sigma^j \right)^{2 - p} b_1 r^p \leq t \leq t_{j - 1}^{(2)} + \left( \frac{u_0}{8}(1 - \tau_0)^{-\beta} \sigma^j \right)^{2 - p} b_2 r^p = t_j^{(2)}.$$ Choosing $j$ such that $2^j \eta_0 \frac{1 - \tau_0}{2} = 2$ and $\beta$ by the condition $2^\beta \sigma = 1$ we complete the proof. (see [10] Section 8 for details).
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