Path integrals with discarded degrees of freedom

Luke M. Butcher*
Institute for Astronomy, University of Edinburgh,
Royal Observatory, Edinburgh EH9 3HJ, United Kingdom
(Dated: July 18, 2018)

Whenever variables φ = (φ1, φ2, . . .) are discarded from a system, and the discarded information capacity S(x) depends on the value of an observable x, a quantum correction ∆Veff(x) appears in the effective potential [1]. Here I examine the origins and implications of ∆Veff within the path integral, which I construct using Synge’s world function. I show that the φ variables can be ‘integrated out’ of the path integral, reducing the propagator to a sum of integrals over observable paths x(t) alone. The phase of each path is equal to the semiclassical action (divided by ℏ) including the same correction ∆Veff as previously derived. This generalises the prior results beyond the limits of the Schrödinger equation; in particular, it allows us to consider discarded variables with a history-dependent information capacity S = S(x(t), ∫t f(x(t′))dt′). History dependence does not alter the formula for ∆Veff.

I. INTRODUCTION

In a recent paper [1] I obtained a powerful model for the quantum effect of discarded variables, applicable to any observable x whose classical motion is determined by the nonrelativistic action

\[ I[x(t)] = \int dt \left[ \frac{m}{2} \dot{x}^2 - V_{cl}(x) \right]. \] (1)

In general, discarded variables φ = (φ1, . . . , φd) with information capacity S(x) generate a quantum correction to the effective potential: Vcl → Vcl + ∆Veff, where

\[ \Delta V_{\text{eff}} = \frac{\hbar^2}{8m} \left[ 1 - 4\xi \frac{d+1}{d} \right] (\partial_x S)^2 + 2(1 - 4\xi) \partial_x^2 S, \] (2)

for some ξ ∈ R. The correction (2) appears in the Schrödinger equation for the wavefunction Ψx(x, t) over the observable configuration space:

\[ i\hbar \partial_t \Psi_x = \left[ -\frac{\hbar^2}{2m} \partial_x^2 + V_{cl} + \Delta V_{\text{eff}} \right] \Psi_x. \] (3)

Consequently, ∆Veff directly affects the average motion of the observable:

\[ m\dot{x}_x^2 = -\langle \partial_t V_{cl} + \partial_x \Delta V_{\text{eff}} \rangle. \] (4)

This motivates the use of a semiclassical action

\[ \mathcal{J}[x(t)] = \int dt \left[ \frac{m}{2} \dot{x}^2 - (V_{cl} + \Delta V_{\text{eff}}) \right], \] (5)

which generates trajectories consistent with the mean equation of motion (4).

These results were all obtained within the Schrödinger picture. To complement that work, we now examine the origins and implications of ∆Veff in the path integral formalism. The paper is organised as follows. In section II, we show the quantum propagator can be written as an integral over observable paths x(t) alone, with the phase of each path given by the semiclassical action (5). In section III, this same propagator is expressed as a path integral over the full configuration space, including the discarded variables. In section IV, we reconcile these two viewpoints: by ‘integrating out’ the paths φ(t), we generate the same quantum correction ∆Veff as the Schrödinger approach. Finally, in section V we use the path integral method to extend the formula (2) to cover history-dependent capacities S = S(x(t), ∫t f(x(t′))dt′). For ease of reference, key definitions from the previous paper [1] are included in the appendix.

II. PROPAGATOR OVER OBSERVABLE PATHS

The main goal of this section is to construct the propagator K(xf, φf, x0, φ0; T) of a nonrelativistic particle over the full (d + 1)-dimensional configuration space (A1). We begin by observing that the reduced Schrödinger equation (3) can be solved with an ordinary one-dimensional path integral:

\[ \Psi_x(x_f, T) = \int \! dx_0 \Psi_x(x_0, 0) \int_{x(0)=x_0}^{x(T)=x_f} \! Dx(t) \, e^{i\mathcal{J}[x(t)]/\hbar}, \] (6)

where \( \mathcal{J}[x(t)] \) is the semiclassical action (5).

To understand the implications of equation (6) we recall the definition (A7) of Ψx, and write an arbitrary state as

\[ \Psi = \sum_k \frac{\Phi_k(\phi)}{|\tilde{\phi}(\phi)|^{d/2}} \Psi_x(x, t), \] (7)

where \{Φk\} are ‘energy’ eigenfunctions (A8) with eigenvalues \( E_k^\phi \), forming an orthonormal basis over the discarded configuration space \( M_\phi \):

\[ \int \! d^d\phi \sqrt{\tilde{g}(\phi)} \Phi_k(\phi) \Phi_k^*(\phi) = \delta_{kk'}. \] (8)
As the Schrödinger equation (A6) is linear, each $\Psi_k^b$ will obey its own reduced Schrödinger equation (3) with $E_\phi = E_0^b$ in the classical effective potential (A2). Hence we can use equation (6) to propagate each component of the general state (7):

$$\Psi(x_f, \phi_f, T) = \sum_k \frac{\Phi_k(\phi_f)}{(b_f)^{d/2}} \times \int dx_0 \Psi^k(x_0, 0) \int_{x(0)=x_0}^{x(T)=x_f} Dx(t) e^{i\mathcal{J}_\phi[x(t)]/\hbar}, \quad (9)$$

where $b_f \equiv b(x_f)$ [similarly, we will write $b_0 \equiv b(x_0)$, $\tilde{g}_f \equiv \tilde{g}(\phi_f)$, etc.] and the index on $\mathcal{J}_\phi[x(t)]$ indicates that we set $E_\phi = E_0^b$ in the classical effective potential.

We wish to represent the time evolution (9) as the result of a propagator $K(x_f, \phi_f, x_0, \phi_0; T)$ acting over the entire configuration space:

$$\Psi(x_f, \phi_f, T) = \int dx_0 d^d\phi_0 \sqrt{g_0} K(x_f, \phi_f, x_0, \phi_0; T) \Psi(x_0, \phi_0, 0), \quad (10)$$

where $\sqrt{g} = b^d \sqrt{\tilde{g}}$ is the covariant measure. It is easy to check that

$$K(x_f, \phi_f, x_0, \phi_0; T) = \sum_k \frac{\Phi_k(\phi_f)\Phi_k^*(\phi_0)}{(b_kb_0)^{d/2}} \int_{x(0)=x_0}^{x(T)=x_f} Dx(t) e^{i\mathcal{J}_\phi[x(t)]/\hbar} \quad (11)$$

is the propagator we need – simply substitute (7) and (11) into the right of (10), perform the integral over $\phi_0$, and recover (9) as required. Equation (11) therefore achieves our stated goal: we have found an expression for the propagator as a sum of integrals over observable paths $x(t)$ alone, with the phase of each path given by the semi-classical action (5). The prefactors $\Phi_k(\phi_f)\Phi_k^*(\phi_0)$ simply serve to project the wavefunction onto states of given $E_\phi$, so that the classical effective potential (A2) can be evaluated within $\mathcal{J}[x(t)]$.

Now, it must also be possible to represent the propagator as a path integral over the entire configuration space:

$$K(x_f, \phi_f, x_0, \phi_0; T) = \int_{x(0)=x_0}^{x(T)=x_f} Dx(t) D^d\phi(t) \ldots (12)$$

How is this path integral related to the formula (11) we just derived? To answer this question, we must first construct the path integral (12) appropriate to our curved configuration space (A1); this is covered in section III.

Then, in section IV, we show that (11) follows directly from (12) by ‘integrating out’ the paths $\phi(t)$. This calculation provides an independent derivation of $\mathcal{J}[x(t)]$ and $\Delta V_{\text{eff}}$, establishing their validity beyond the Schrödinger picture.\(^1\) The path integral formalism therefore extends our results to domains where the Schrödinger equation no longer applies – we shall explore this new territory in section V.

### III. PATH INTEGRAL OVER CURVED SPACE

Let us seek a general expression for the propagator of a nonrelativistic particle in a curved space of $D = d + 1$ dimensions:

$$K(q_f, q_0; T) \equiv \langle q_f | e^{-iH T/\hbar} | q_0 \rangle, \quad (13)$$

where $q \equiv (q^1, q^2, \ldots, q^D)$ are arbitrary coordinates, and the Hamiltonian $H$ is the operator in square brackets on the right of the covariant Schrödinger equation (A6). The states $|q\rangle$ form a coordinate-invariant orthonormal basis,

$$\langle q'|q\rangle = \frac{\delta^D(q' - q)}{\sqrt{g(q)}}, \quad \int d^Dq \sqrt{g(q)} |q\rangle \langle q| = 1, \quad (14)$$

and hence the propagator (13) transforms as a scalar at both $q_f$ and $q_0$.

To represent (13) as a path integral, we split the exponential operator into $N$ equal factors, and insert $N - 1$ copies of the identity (14):

$$K(q_f, q_0; T) \equiv \prod_{n=1}^{N-1} \int d^Dq_n \sqrt{g_n} \prod_{n=1}^N K(q_n, q_{n-1}; \epsilon), \quad (15)$$

where $g_n \equiv g(q_n)$, $\epsilon \equiv T/N$, and $q_N \equiv q_f$. The path integral is then obtained as $N \to \infty$ and $\epsilon \to 0$.

#### A. Short-Time Propagator

To evaluate the path integral (15) we require a formula for the short-time propagator $K(q', q; \epsilon)$. To this end, we will prove that

$$K(q', q; \epsilon) = \sqrt{\frac{m}{2\pi i\hbar^2}} \exp\left\{ \frac{i\sigma \epsilon^2}{\hbar} + \frac{R^2 \nabla_i \sigma \nabla_j \sigma}{12} \right\} \times \frac{1}{2m} \left[ \frac{\hbar^2}{\eta} \left( \frac{\xi - 1}{6} R + V_0 \right) + \eta \right], \quad (16)$$

with $\eta = O(\sigma^{3/2}) + O(\epsilon^{1/2}) + O(\epsilon^2)$. Here, $\sigma$ is Synge’s world function [3], so that $\sqrt{2\sigma(q', q)}$ is the geodesic distance from $q$ to $q'$. [Thus $\nabla_i \sigma \approx \text{distance} = O(\sigma^{1/2})$].

\(^1\) Alternatively, we could start with a path integral over the entire phase space: $\int D\phi D\phi_0 D\phi_2 D\phi_3 \ldots$ Storchak has shown that this type of integral can be dealt with by transforming to a new time coordinate [2] with $\Delta V_{\text{eff}}$ then appearing as the Jacobian of the transformation. As this ‘canonical’ approach favours a particular ordering of operators (rather than coordinate invariance) it tacitly sets $\xi = 0$. 
Notice that \( \{g_{ij}, \nabla_i, R_{ij}, V_0\} \) may be defined at either \( q \) or \( q' \): a change in convention generates terms \( O(\sigma^{3/2}) + O(\sigma^{1/2}) \) that can be absorbed into \( \eta \). Thus the short-time propagator is symmetric under \( q' \leftrightarrow q \), modulo terms \( O(\eta) \).

To prove equation (16), we need to check that that this propagator (i) satisfies the initial condition

\[
\lim_{\epsilon \to 0} K(q', q; \epsilon) = \langle q' | q \rangle, \tag{17}
\]

and (ii) solves the covariant Schrödinger equation (A6).

For part (i) we write \( \Delta q \equiv q' - q \), so the squared geodesic distance becomes

\[
2\sigma(q', q) = g_{ij} \Delta q^i \Delta q^j + O(\sigma^{3/2}), \tag{18}
\]

and thus

\[
K(q', q; \epsilon) = \sqrt{\frac{m}{2\pi i \hbar}} \exp\left\{ \frac{i m g_{ij} \Delta q^i \Delta q^j}{2\hbar} + O(\Delta q^3/\epsilon) \right\}
\]

\[
\to \frac{\delta^D(\Delta q)}{\sqrt{\det(g_{ij})}} = \frac{\langle q' | q \rangle}{\epsilon} \quad \text{as} \quad \epsilon \to 0. \tag{19}
\]

For part (ii) we replace \( \Psi \to K(q', q; \epsilon) \), \( \partial_t \to \partial_\epsilon \) in the covariant Schrödinger equation (A6) and evaluate the result by use of the following identities [4]:

\[
\nabla_i \sigma \nabla^i \sigma = 2\sigma,
\]

\[
\nabla^i \nabla_j \sigma = \delta_j^i - \frac{1}{3} R^i_{kj} \nabla^k \sigma \nabla^l \sigma + O(\sigma^{3/2}).
\]

After a great deal of cancelling, we are left with

\[
\partial_\epsilon \eta = O(\epsilon^{-1}) \left[ \nabla^i \sigma \nabla_i \eta + O(\sigma^{3/2}) \right] + O(1) \left[ \nabla^2 \eta + (\nabla \eta)^2 \right] + O(\sigma^{1/2}) + O(\epsilon), \tag{22}
\]

which can then be solved by some \( \eta = \eta_0 + \epsilon \eta_1 + O(\epsilon^2) \), where \( \eta_0 = O(\sigma^{3/2}) \) and \( \eta_1 = O(\sigma^{1/2}) \). This completes the proof.

**B. Continuum Limit**

In general, terms \( O(\epsilon^{3/2}) \) do not contribute to the path integral (15) in the continuum limit: as \( \epsilon \to 0 \), we have

\[
\prod_{n=1}^N \exp\{O(\epsilon^{3/2})\} = 1 + O(\epsilon^{1/2}) \to 1.
\]

Moreover, due to rapid oscillations of the factor \( \exp(i m \sigma / \hbar) \), the short-time propagator (16) gives a negligible contribution to the integrals (15) for \( \sigma \gg \hbar / m \); hence we can treat \( \sigma = O(\epsilon) \), and so \( \eta = O(\epsilon^{3/2}) \) is negligible also. In a similar vein, note that

\[
0 = \left( \frac{2\epsilon \hbar}{im} \right) \partial_{g_{ij}} \sqrt{\frac{2\pi i \hbar}{m}}^D \tag{23}
\]

\[
= \left( \frac{2\epsilon \hbar}{im} \right) \partial_{g_{ij}} \int d^D \Delta q \sqrt{g} e^{i m g_{ij} \Delta q^i \Delta q^j / 2\hbar}
\]

\[
= \int d^D \Delta q \sqrt{g} e^{i m g_{ij} \Delta q^i \Delta q^j / 2\hbar} \left[ \Delta q^i \Delta q^j + \frac{g_{ij}}{2} \left( \frac{2\epsilon \hbar}{im} \right) \right]
\]

\[
= \int d^D \sqrt{g} e^{i m (q', q) / \hbar} \left[ \nabla^i \sigma \nabla_i \sigma - \frac{g_{ij}}{2} \frac{2\epsilon \hbar}{im} + O(\epsilon^{3/2}) \right],
\]

where equation (18) was used in the last line. We can therefore set \( \nabla_i \sigma \nabla_j \sigma = g_{ij} (i \hbar / m) + O(\epsilon^{3/2}) \) inside the path integral (15).

Applying the arguments of the previous paragraph to equation (16) we arrive at a simplified version of the short-time propagator:

\[
K(q', q; \epsilon) = \frac{1}{m} \exp \left\{ \frac{i m \sigma(q', q)}{2} - \frac{\epsilon}{\hbar} \frac{\hbar^2}{2m} \left( \xi - \frac{1}{3} \right) \right\}, \tag{24}
\]

suitable for the path integral (15) in the continuum limit. Comparing this formula with equation (10.153) of Kleinert’s textbook [5], we confirm that our construction generalises the standard result to \( \xi \neq 0 \).

This completes our general treatment of the path integral over curved space. To construct the required path integral (12) we insert the short-time propagator (24) into equation (15), adopt the metric (A1), and take the limit \( \epsilon \to 0 \).

**IV. INTEGRATING OUT DISCARDED PATHS**

Armed with a concrete formulation of the path integral over the full configuration space, we are now in a position to integrate out the discarded degrees of freedom \( \phi \).

**A. General method**

It will be useful to represent propagators in the \( \{ \Phi_b \} \) eigenbasis. Because \( \hat{H} \Phi_b \propto \Phi_b \), the time evolution operator \( \exp(-i \hat{H} t / \hbar) \) will not alter the \( \phi \) dependence of any eigenfunction, so we must have

\[
K(x', \phi', x, \phi; t) = \sum_k \frac{\Phi_k(x') \Phi_k^*(x)}{b_k} K_k(x', x; t), \tag{25}
\]

for some \( K_k(x', x; t) \). [The factors of \( b \equiv \sqrt{b(x') b(x)} \) are a convenient convention.] The components \( K_k(x', x; t) \) are

\[\text{Recall the path integral for a free particle in flat space, \text{composed of integrals over Cartesian coordinates \( \int d^D x_n \) and propagators proportional to \( \exp(im|x - x'|^2/2\hbar) \). How would one generalise that expression to curved space, guided only by Einstein’s equivalence principle and the notion of ‘minimal coupling’? It is natural to add the covariant measure \( \sqrt{g(x_n)} \) to the integrals, and replace \( |x - x'|^2 \to 2\sigma(x, x') \) as the invariant squared distance. However, there would be no reason to introduce \( R \) to the propagator, as in equation (24). In other words, the minimally coupled path integral has \( \xi = 1/3 \), not the value \( \xi = 0 \) of the minimally coupled Schrödinger equation. We see that the meaning of ‘minimal coupling’ depends on ones starting point. In general, this ambiguity justifies an agnostic view of curvature coupling: without a better argument to fix \( \xi \), the parameter must be determined experimentally.} \]
then given by
\[ K_k(x',x;t) = \delta^t \int d^d \phi' d^d \phi \sqrt{g(\phi')g(\phi)} \Phi_k(\phi') \Phi_k(\phi) K(x',\phi',x,\phi;t) , \]
by virtue of orthonormality (8).

To see the value of this representation, let us set the coordinates \((x',\phi',x,\phi;t) \rightarrow (x_f,\phi_f,x_0,\phi_0;T)\) in equation (26), insert the path integral (15) into the right hand side, and use the eigenbasis expansion (25) for the short-time propagators therein:
\[ K_k(x_f,x_0;T) = (b_f b_0)^{d/2} \int d^d \phi_f d^d \phi_0 \sqrt{g_f g_0} \Phi_k(\phi_f) \Phi_k(\phi_0) \times \prod_{n=1}^{N-1} \int dx_n d^d \phi_n b_n^d \sqrt{g_n} \times \prod_{n=1}^{N} \Phi_{k_n}(\phi_n) \Phi_{k_n}(\phi_{n-1}) \frac{1}{(b_n b_{n-1})^{d/2}} K_{k_n}(x_n,x_{n-1};\epsilon). \]

We can now perform the integrals over each \(\phi_n\) in turn. The integral over \(\phi_0\) leaves only the \(k_1 = k\) term in the sum over \(k_1\), then the integral over \(\phi_1\) leaves only the \(k_2 = k\) term in the sum over \(k_2\), and so on. On completing the final integral (over \(\phi_f \equiv \phi_N\)) we have
\[ K_k(x_f,x_0;T) = (b_f b_0)^{d/2} \prod_{n=1}^{N-1} \int dx_n b_n^d \prod_{n=1}^{N} \frac{K_k(x_n,x_{n-1};\epsilon)}{(b_n b_{n-1})^{d/2}} \]
\[ = \prod_{n=1}^{N-1} \int dx_n \prod_{n=1}^{N} K_k(x_n,x_{n-1};\epsilon), \]
which has the form of a path integral over \(x\) alone.

Setting \((x',\phi',x,\phi;t) \rightarrow (x_f,\phi_f,x_0,\phi_0;T)\) in the eigenbasis expansion (25) and inserting the components (28) we arrive at
\[ K(x_f,\phi_f,x_0,\phi_0;T) = \sum_k \Phi_k(\phi_f) \Phi_k(\phi_0) \prod_{n=1}^{N-1} \int dx_n \prod_{n=1}^{N} K_k(x_n,x_{n-1};\epsilon), \]
which is now very close to the formula (11) we wish to rederive. All that remains is to evaluate the short-time components \(K_k(x_n,x_{n-1};\epsilon)\) and take the limit of equation (29) as \(\epsilon \rightarrow 0\).

### B. Calculation for \(d = 1\)

To illustrate the process described above, let us restrict our interest to the configuration space (A1) with \(d = 1\):
\[ ds^2 = dx^2 + [b(x)]^2 d\phi^2, \quad \phi \in [0,2\pi), \]
which was the motivating example in the original paper [1]. In principle, the steps below can also be followed for any configuration space of the form (A1).

The metric (30) fixes \(R = -2(\partial_0^2 b)/b, \tilde{g} = 1, \tilde{R} = 0\), and sets the eigenfunctions (A8) as
\[ \Phi_k = \frac{e^{ik\phi}}{\sqrt{2\pi}}, \quad E_k^0 = \frac{\hbar^2 k^2}{2m}, \quad k \in \mathbb{Z}. \]

Consequently, when we insert the short-time propagator (24) into equation (26) we get
\[ K_k(x',x;\epsilon) = \frac{mb}{2 \hbar \epsilon} \int d\Delta \phi \exp \left\{ \frac{im}{\hbar \epsilon} \sigma(x',x,\Delta \phi) - ik \Delta \phi \right. \]
\[ - \frac{i \epsilon}{\hbar} \left[ \frac{h^2}{m} \left( \frac{1}{3} - \xi \right) \frac{\partial^2 b}{b} + V_0 \right] \right\}, \]
where \(\Delta \phi = \phi' - \phi\). To evaluate \(\sigma(x',x,\Delta \phi)\), we note that (21) implies
\[ \partial^2_\phi \sigma \rightarrow 1, \quad \partial^2_\phi \sigma \rightarrow b^2, \quad \partial^2_\sigma \partial_\phi \partial_\sigma \rightarrow b \partial_\sigma b, \]
\[ \partial^2_\sigma \partial_\phi \partial^2_\phi \sigma \rightarrow 2b \partial_\phi b^3/3, \quad \partial^2_\phi \partial_\phi \sigma \rightarrow (\partial_\phi b)^2, \]
as \(x \rightarrow x'\) and \(\Delta \phi \rightarrow 0\), with all other \(\partial^{(n)} \sigma \rightarrow 0\) for \(n \in \{0,1,2,3,4\}\). We can therefore construct a Taylor expansion in \(\Delta x \equiv x' - x\) and \(\Delta \phi\):
\[ 2\sigma(x + \Delta x, x, \Delta \phi) = \Delta x^2 + \tilde{b}(\Delta \phi)^2 \frac{b \partial^2_\phi b^2}{6} \Delta x^2 \Delta \phi^2 \]
\[ - \frac{(b \partial_\phi b)^2}{12} \Delta \phi^4 + O(\sigma^{5/2}), \]
having used \(\tilde{b}^2 = [b + \Delta x \partial_x b + \Delta x^2 \partial^2_\phi b^2/2 + O(\sigma^{3/2})]/2\).

We now substitute (34) into (32) and perform the integral. In doing so, recall \(\sigma = O(\epsilon)\) and that terms \(O(\epsilon^{1/2})\) can be neglected in the continuum limit; moreover, a previous trick (23) allows us to write \(\Delta x^2 = \left(i\hbar \epsilon/m \right) + O(\epsilon^{1/2})\). We therefore arrive at
\[ K_k(x',x;\epsilon) = \sqrt{\frac{m}{2\pi \hbar \epsilon}} \exp \left\{ \frac{im}{\hbar} \left[ \frac{m \Delta x^2}{2 \epsilon^2} \right. \right. \]
\[ - \left. \left. \frac{\hbar^2 (hk)^2}{2mb^2} + D V_{\text{eff}} \right) \right\}, \]
with \(DV_{\text{eff}}\) given by equation (A11) for \(d = 1\).

Note that \(V_0 + (hk)^2/2mb^2\) is just the classical effective potential (A2) with \(E_0 = E_k^0 = (hk)^2/2m\) as prescribed by the eigenfunctions (31). Thus equation (35) implies
\[ \prod_{n=1}^{N-1} \int dx_n \prod_{n=1}^{N} K_k(x_n,x_{n-1};\epsilon) \]
\[ = \prod_{n=1}^{N-1} \int dx_n \prod_{n=1}^{N} \sqrt{\frac{m}{2\pi \hbar \epsilon}} \exp \left\{ \frac{im}{\hbar} \left[ \frac{m (x_n-x_{n-1})^2}{2 \epsilon^2} \right. \right. \]
\[ - \left. \left. \frac{\hbar^2 (hk)^2}{2mb^2} + D V_{\text{eff}}(x_n) \right) \right\} \}
\[ \int_{x(0)=x_0}^{x(T)=x_f} D x(t) e^{iJ_k[x(t)]/\hbar} \]
as \(\epsilon \rightarrow 0\). Inserting this limit into (29) we finally recover the desired result (11).
C. Remarks

We have shown that the degrees of freedom $\phi$ can be integrated out of the path integral (12) over the curved configuration space (A1). This renders the propagator as a sum of integrals over observable paths alone (11). Crucially, the phase of each observable path $x(t)$ is given by the semiclassical action (5) which includes the same quantum correction (2) as previously derived in the Schrödinger approach [1].

Although the proof was only given explicitly for $d = 1$, the same steps can be followed for any metric of the form (A1). The result of this process must agree with (11) in general, otherwise the path integral would be inconsistent with the covariant Schrödinger equation. Furthermore, when the Schrödinger equation cannot be used, the path integral still provides a route to obtain the quantum correction $\Delta V_{\text{eff}}$ and the semiclassical action $\mathcal{J}[x(t)]$. This is illustrated in the next section.

V. HISTORY-DEPENDENT INFORMATION CAPACITY

Thus far, we have only discussed discarded variables whose information capacity depends on the present value of the observable: $S = S(x(t))$. For such systems, the full configuration space can be described by a curved metric (A1) with a subspace $M_\phi$ that changes size as a function of $x$. Consequently, the covariant Schrödinger equation (A6) naturally determines the dynamics of the quantum state $\Psi(x, \phi, t)$.

But now suppose the discarded information capacity also depends on the history of $x$:

$$S = S(x(t), y(t)), \quad y(t) \equiv \int_{t_0}^{t} dt' f(x(t')),$$  

(37)

for some function $f : \mathbb{R} \to \mathbb{R}$. How should we model this system? Clearly, we cannot continue to use the curved space (A1): this would require the radius $b$ to depend on the history of the particle, which cannot be defined in the Schrödinger approach. Fortunately, the path integral formalism provides a natural environment to quantify particle histories, and will accommodate the new form of information capacity (37) without any great difficulty. To see how this works, we shall first consider the simplest nontrivial case $f = 1$ (where the Schrödinger equation can still be used, with a minor modification) before moving on to the general case (37).

A. Time dependence

For $f = 1$, the information capacity (37) reduces to

$$S = S(x(t), t).$$

(38)

This time-dependent information capacity can be represented in the Schrödinger picture by promoting $b(x) \to b(x, t)$ in the metric (A1). However, because the configuration space is now time-dependent, the Schrödinger equation must become

$$i\hbar \partial_t \left( g^{1/4} \Psi \right) = \left[ \frac{\hbar^2}{2m} (-\nabla^2 + \xi R) + V_0 \right] \left( g^{1/4} \Psi \right),$$

(39)

in order that unitarity be preserved:

$$\partial_t \int d^D q \sqrt{|g|} |\Psi|^2$$

$$= \int d^D q \left[ g^{1/4} \Psi^* \partial_t \left( g^{1/4} \Psi \right) + \partial_t \left( g^{1/4} \Psi^* \right) g^{1/4} \Psi \right]$$

$$= \frac{i\hbar}{2m} \int d^D q \sqrt{|g|} \left[ \Psi^* \nabla^2 \Psi - \Psi \nabla^2 \Psi^* \right]$$

$$= 0.$$  

(40)

For the metric (A1) with $b(x) \to b(x, t)$, the modified Schrödinger equation (39) is simply

$$i\hbar \partial_t \Psi = \left[ \frac{\hbar^2}{2m} (-\nabla^2 + \xi R) + V_0 - \frac{i\hbar d}{2} \partial_t \ln b \right] \Psi.$$  

(41)

With this minor alteration, the derivation of $\Delta V_{\text{eff}}$ then follows the same route as outlined in the appendix. Because the state (A7) contains a factor of $|b(x, t)|^{-d/2}$, the $i\hbar \partial_t$ operator on the left of the modified Schrödinger equation (41) generates a term that exactly cancels the new term $-(i\hbar d/2) \partial_t \ln b$ on the right. Consequently, the formula for the quantum correction (2) is completely unchanged.

Turning our attention to the path integral, we see that time-dependent capacity (38) modifies the formalism of sections III-IV in two ways. First, in order to solve the modified Schrödinger equation (41) the short-time propagator (24) must now be

$$K(q', q; t + \epsilon, t)$$

$$= \sqrt{\frac{m}{2\pi i\epsilon}} \exp \left\{ \frac{im}{\hbar \epsilon} \sigma(q', q) - \frac{i\epsilon C}{\hbar} \left[ \xi - \frac{1}{3} \right] R + V_0 - \frac{i\hbar d}{2} \partial_t \ln b \right\}.$$  

(42)

Here, $\sigma(q', q)$ is evaluated using the metric at $t$; hence when we assemble its Taylor expansion, as in equation (34), the factor of $b^2$ will be replaced by $b(x', t)b(x, t)$.

Second, the measures of the integrals (15) will now be

$\sqrt{b} = (b_n)^d \sqrt{b_n},$ with

$$b_n \equiv b(x_n, t_n) = b(x_n, \tau + n\epsilon).$$  

(43)

In order that all the $b_n$ cancel when assembling the $x(t)$ path integral (28) we must promote $b \to \sqrt{b(x', t + \epsilon)b(x, t)}$ in definitions (25) and (26). Notice
the difference between this replacement and the replacement for $\bar{b}$ in the Taylor expansion of $\sigma(q', q)$; consequently, each integral over $\Delta \phi_n$ leaves a factor

$$\left( \frac{b(x', t + \epsilon)}{b(x', t)} \right)^{d/2} = \exp \left\{ \frac{d}{2} \ln b(x', t + \epsilon) - \ln b(x', t) \right\}$$

$$= \exp \left\{ \frac{d}{2} \epsilon \partial_t \ln b + O(\epsilon^{3/2}) \right\}, \quad (44)$$

by virtue of this mismatch. Fortunately, this factor is cancelled by the new term in the propagator (42). Thus the time-dependence of the information capacity (38) has no effect on the results of our path integral calculation, besides making $\Delta V_{\text{eff}} = \Delta V_{\text{eff}}(x, t)$.

**B. History dependence**

It is now remarkably easy to generalise the path integral formalism to the history-dependent case (37): we simply replace $b(x, t) \rightarrow b(x, y(t))$ as one would expect. Unlike the Schrödinger equation, the short-time propagator (42) remains well-defined, with

$$\partial_t \ln b = \partial_t \ln b(x, y(t)) = f(x) \partial_y \ln b \quad (45)$$

therein. At each time-step $t_n = t_0 + n \epsilon$, the measures (43) are now

$$b_n \equiv b(x_n, y_n), \quad y_n = \sum_{k=1}^{n} \epsilon f(x_k), \quad (46)$$

where the sum represents the integral that appears in equation (37). Thus, exactly as above, each $\Delta \phi_n$ integral leaves us with a factor

$$\left( \frac{b(x_n, y_n)}{b(x_n, y_{n-1})} \right)^{d/2} = \exp \left\{ \frac{d}{2} \ln b(x_n, y_n) - \ln b(x_n, y_{n-1} - \epsilon f(x_n)) \right\}$$

$$= \exp \left\{ \frac{d}{2} \epsilon f(x_n) \partial_y \ln b(x_n, y_n) + O(\epsilon^{3/2}) \right\}, \quad (47)$$

which cancels the new term (45) in the propagator.

We therefore conclude that the formulae for the quantum correction (2), the semiclassical action (5), and the propagator (11) all remain valid when the information capacity depends on the history of the observable (37). Of course, as $\Delta V_{\text{eff}}$ is now a function of the path $x(t)$, extra care must be taken when actually evaluating the path integrals or generating the semiclassical equations of motion.

**VI. CONCLUSION**

We have constructed the path integral (12) for a non-relativistic particle living in the curved configuration space (A1) and developed a general method by which the variables $\phi$ can be integrated out, and hence discarded. This procedure reduces the propagator to a sum of integrals over observable paths $x(t)$ alone (11). As the phase of each path is set by the semiclassical action (5) this provides an independent derivation of the quantum correction $\Delta V_{\text{eff}}$ previously obtained in the Schrödinger picture [1]. Thus, when the Schrödinger equation cannot be used, the path integral grants us a means to calculate $\Delta V_{\text{eff}}$ and hence model the quantum effect of discarded degrees of freedom. As an example of this generalisation, we have demonstrated that the formula for the quantum correction (2) remains valid even when the discarded variables have information capacity that depends on the history of the observable (37).

**ACKNOWLEDGMENTS**

The author is supported by a research fellowship from the Royal Commission for the Exhibition of 1851, and the Institute for Astronomy at the University of Edinburgh.

**Appendix A: Prior Results**

Here, we briefly recap the key definitions and results of the preceding work [1]. For detailed motivation and derivations, the reader should refer to the original paper.

Consider a nonrelativistic particle (of mass $m$) living on a curved $D = d + 1$ dimensional space

$$ds^2 = g_{ij}dq^i dq^j = dx^2 + [b(x)]^2 \tilde{g}_{IJ}(\phi) d\phi^I d\phi^J, \quad (A1)$$

on which there is also a potential $V_0(x)$. We wish to predict the behaviour of the ‘observable’ coordinate $x$, while ignoring the variables $\phi \equiv (\phi^1, \ldots, \phi^d)$ that cover a $d$-dimensional compact manifold $M_\phi$ with metric $\tilde{g}_{IJ}$ and physical volume $\text{Vol}_\phi \propto b^d$. As usual, the metric $g_{ij}$ defines a covariant measure $\sqrt{\tilde{g}} \equiv \sqrt{\text{det}(g_{ij})}$, a derivative operator $\nabla_i$, and curvature tensors $\{R^k_{ijkl}, R_{ij}, R\}$ according to $\{\nabla_i, \nabla_j\}v^k = R^k_{ijl} v^l$, $R_{ij} \equiv R^k_{ikj}, R \equiv R_{ijkl} g^{ij}$. Similarly, $\tilde{g}_{IJ}$ defines $\{\sqrt{\tilde{g}}, \nabla_I, \tilde{g}^{IJ}, R_{IKL}, R_I\}$.

We can predict the $x$ coordinate of a classical particle with the reduced action (1) where the effective potential

$$V_{\text{cl}} = V_0 + E_\phi/b^2 \quad (A2)$$

depends on the conserved ‘energy’

$$E_\phi \equiv \frac{1}{2m} \tilde{g}_{IJ} p_\phi^I p_\phi^J = \frac{mb^2}{2} \tilde{g}_{IJ} \dot{\phi}^I \dot{\phi}^J = \text{const}. \quad (A3)$$

This allows us to treat the particle as though it were living on a reduced configuration space

$$ds^2 = dx^2, \quad x \in \mathbb{R}, \quad (A4)$$

without any reference to the variables $\phi$. $E_\phi$ is now viewed as a parameter of the system.
For a quantum particle, however, we must examine the behaviour of the wavefunction \( \Psi(x, \phi, t) \), which defines probabilities via integrals of the form

\[
P = \int d^Dq \sqrt{g} |\Psi|^2 = \int dx d^d\phi b^d \sqrt{\tilde{g}} |\Psi|^2, \quad (A5)
\]

and obeys the covariant Schrödinger equation

\[
i\hbar \partial_t \Psi = \left[ \frac{\hbar^2}{2m} (-\nabla^2 + \xi R) + V_0 \right] \Psi, \quad (A6)
\]

for some \( \xi \in \mathbb{R} \). (This free parameter reflects a quantisation ambiguity [6–8]. One might appeal to ‘minimal coupling’ and set \( \xi = 0 \), however there is nothing particularly special about this choice – see footnote 2.)

In order to separate the observable \( x \) from the variables \( \phi \), we consider states of the form

\[
\Psi = \frac{\Phi(\phi)}{[b(x)]^{d/2}} \Psi_x(x, t), \quad (A7)
\]

where \( \Phi \) is an eigenfunction over \( \mathcal{M}_\phi \),

\[
\frac{\hbar^2}{2m} \left( -\nabla^2 + \xi \tilde{R} \right) \Phi = E_\phi \Phi, \quad (A8)
\]

with unit norm

\[
\int d^d\phi \sqrt{\tilde{g}} |\Phi|^2 = 1. \quad (A9)
\]

Note that \( \Psi_x \) is normalised such that probabilities \((A5)\) become

\[
P = \int dx |\Psi_x|^2, \quad (A10)
\]

consistent with the metric on the reduced configuration space \((A4)\). Furthermore, the eigenvalue equation \((A8)\) follows the same quantisation rule as the Schrödinger equation \((A6)\): \( g^{ij} p_i p_j \rightarrow \hbar^2 [-\nabla^2 + \xi R] \).

Inserting \((A7)\) into \((A6)\) we derive the reduced Schrödinger equation \((3)\) wherein the effective potential has a quantum correction

\[
\Delta V_{\text{eff}} = \frac{\hbar^2 d}{2m} \left[ \left( \frac{d-2}{4} + \xi(1-d) \right) \left( \frac{\partial_x b}{b} \right)^2 + \frac{1 - 4 \xi}{2} \left( \frac{\partial_x^2 b}{b} \right) \right]. \quad (A11)
\]

To put this result in its final form \((2)\) we imagine dividing the curved space \((A1)\) into a lattice of small cells with spacing \( \ell \ll \min\{b, (b/\partial_x b), \sqrt{b/\partial_x^2 b}\} \). Then, at a given value of \( x \), the information capacity of the \( \phi \) subspace is

\[
S = \ln \Omega = \ln (\text{Vol}_\phi / \ell^d) = d \ln b + \text{const}. \quad (A12)
\]

Hence \((A11)\) is equal to \((2)\) for all \( \ell \), and we are free to return to the continuum limit \( \ell \rightarrow 0 \).

[1] L. M. Butcher, Quantum effective potential from discarded degrees of freedom, Phys. Lett. A (2018). arXiv: 1707.05789.
[2] S. Storchak, Path integrals on warped product manifolds, Physics Letters A 174 (1) (1993) 13 – 18.
[3] J. L. Synge, Relativity: The General Theory, North Holland, 1960.
[4] J. Vines, Geodesic deviation at higher orders via covariant bitensors, General Relativity and Gravitation 47 (5) (2015) 59. arXiv:1407.6992.
[5] H. Kleinert, Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets, 5th Edition, World Scientific, 2009.
[6] B. S. DeWitt, Dynamical theory in curved spaces. i. a review of the classical and quantum action principles, Rev. Mod. Phys. 29 (1957) 377–397.
[7] C. DeWitt-Morette, K. D. Elworthy, B. L. Nelson, G. S. Sammelman, A stochastic scheme for constructing solutions of the Schrödinger equations, Annales de l'I.H.P. Physique theorique 32 (4) (1980) 327–341.
[8] M. Baszak, Z. Domaski, Canonical quantization of classical mechanics in curvilinear coordinates. Invariant quantization procedure, Annals of Physics 339 (2013) 89 – 108.