In this paper, we investigate the Cauchy problem associated to a system of PDE's of Johnson-Segalman type. The considered model describes the evolution of certain viscoelastic fluids within a corotational framework. We show that some widespread results concerning the incompressible Navier-Stokes equations can be extended to the considered system. In particular we show the existence and uniqueness of finite energy solutions for large data in dimension two. This result is supported by suitable condition on the initial data to provide a global-in-time Lipschitz regularity for the flow, which allows to overcome specific challenging due to the non time decay of the main forcing terms.

Secondly, we address the global-in-time well posedness of our model in dimension $d \geq 3$ in suitable critical spaces. We always allow a Lipschitz regularity of the flow and the initial data are just assumed to be small in a critical weak Lebesgue norm.

1. Introduction

The modeling and analysis of the hydrodynamics of viscoelastic fluids has attracted much attention over the last decades [2, 6, 8, 17, 19]. Generally, the physical state of matter of a material can be determined by the degree of freedom of movement about its constitutive molecules. Increasing this degree of freedom, the most of materials evolves from a solid state to a liquid phase and eventually to a gas form. Nevertheless, there exist in nature several materials that present characteristics in the between of an isotropic fluid and a crystallized solid. These materials are usually classified as viscoelastic fluids, since they generally behave as a viscous fluids as well as they share some properties with elastic materials, for instance exhibiting memory effects. We refer the reader to [4, 13, 20, 21] for an overview of the Physics behind the modeling of these complex fluids.
This article is devoted to the analysis of the following evolutionary system of PDE’s, describing the hydrodynamics of specific incompressible viscoelastic fluids:

\[
\begin{aligned}
\frac{\partial \tau}{\partial t} + u \cdot \nabla \tau - \omega \tau + \tau \omega &= 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^d, \\
\frac{\partial u}{\partial t} + u \cdot \nabla u - \nu \Delta u + \nabla p &= \text{div} \tau & \text{in } \mathbb{R}^+ \times \mathbb{R}^d, \\
\text{div} u &= 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^d, \\
(u, \tau)_{|t=0} &= (u_0, \tau_0) & \text{in } \mathbb{R}^d.
\end{aligned}
\]

(1)

Here the constant \( \nu > 0 \) stands for the viscosity of the fluids, while the state variables correspond to \( u = u(t, x) \), the velocity field of a particle \( x \in \mathbb{R}^d \) at a time \( t \in \mathbb{R} \), and \( \tau = (\tau(t,x))_{i,j=1,...,d} \), the conformation tensor in \( \mathbb{R}^{d \times d} \), describing the internal elastic forces that the constitutive molecules exert on each other. To simplify our analysis, we assume our non-Newtonian fluid to occupy the entire whole space \( \mathbb{R}^d \), with dimension \( d \geq 2 \). The evolutionary equation for the conformation tensor \( \tau \) is then driven by the vorticity tensor \( \omega = (\omega(t,x))_{i,j=1,...,d} \), which stands for the skew-adjoint part of the deformation tensor \( \nabla u \):

\[
\omega = \frac{\nabla u - \text{sym} \nabla u}{2}.
\]

The system can be seen as a simplified version of the more general Johnson-Segalman model (cf. [5]):

\[
\begin{aligned}
\frac{\partial \tau}{\partial t} + u \cdot \nabla \tau + a \tau + Q(u, \tau) &= \mu_2 \mathbb{D} & \text{in } \mathbb{R}^+ \times \mathbb{R}^d, \\
\frac{\partial u}{\partial t} + u \cdot \nabla u - \nu \Delta u + \nabla p &= \mu_1 \text{div} \tau & \text{in } \mathbb{R}^+ \times \mathbb{R}^d, \\
\text{div} u &= 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^d, \\
(u, \tau)_{|t=0} &= (u_0, \tau_0) & \text{in } \mathbb{R}^d.
\end{aligned}
\]

(2)

The main parameters \( \nu, a, \mu_1, \mu_2 \) are assumed to be non-negative and they are specific to the characteristic of the considered material. In particular [10] \( \nu, a \) and \( b \) correspond respectively to \( \theta/\text{Re}, 1/\text{We} \) and \( 2(1 - \theta)/(\text{We} \cdot \text{Re}) \), where \( \text{Re} \) is the Reynolds number, \( \theta \) is the ratio between the so called relaxation and retardation times and \( \text{We} \) is the Weissenberg number. The bilinear term \( Q(u, \tau) \) assumes the following form:

\[
Q(u, \tau) = \tau \omega - \omega \tau + b(\mathbb{D} \tau + \tau \mathbb{D}),
\]

where the so-called slip parameter \( b \) is a constant value between \( [-1, 1] \) and \( \mathbb{D} = (\nabla u + \nabla u^T)/2 \) is the adjoint part of the deformation tensor \( \nabla u \). The corotational term \( \tau \omega - \omega \tau \) describes how molecules are twisted by the underlying flow, while the term depending on \( b \) describes how molecules are stretched and deformed by the flow itself.

For the sake of our analysis, in this paper we impose the following restriction on the main parameters of the Johnson-Segalman model:

\[
\mu_1 = 1, \quad \mu_2 = 0, \quad b = 0, \quad a = 0,
\]

from which one obtain our main system (1). The first condition is introduced just for the sake of a clear presentation, while the second and third conditions will play a major role in the analysis techniques we will perform in the forthcoming sections. The last condition \( a = 0 \) increases the challenging of our model, since no damping effect is now assumed on the evolution of the conformation tensor \( \tau \). We hence claim that all our results hold also for the damped case given by \( a > 0 \).

We present an overview of the main results in literature concerning systems (1) and (2). In [10], the authors dealt with the existence and uniqueness of local strong solutions for system (2) in Sobolev space \( H^s(\Omega) \), for a sufficiently-smooth bounded domain \( \Omega \) and a sufficiently large regularity \( s > 0 \). The same authors in [11] showed that these solutions are global if the initial data are sufficiently small as well it is small the coupling between the main terms of the constitutive equations.

Lions and Masmoudi [13] addressed the corotational case of system (2), given by \( b = 0 \) and showed the existence and uniqueness of global-in-time weak solutions in a bidimensional setting.
Lei, Liu and Zhou [16] proved existence and uniqueness of classical solutions near equilibrium of system (2) for small initial data, assuming the domain to be periodic or to be the whole space.

Bresh and Prange [3] analyze the law Weissenberg asymptotic limit of solutions for system (2) in a corotational setting $b = 0$. They focus on the specific formulation of the Johnson-Segalman system in which the main parameter of (2) are explicitly defined in terms of the Weissenberg number $\text{We}$, which compares the viscoelastic relaxation time to a time scale relevant to the fluid flow. The authors study the weak convergence towards the Navier–Stokes system, as $\text{We} \to 0$. Furthermore, they take into account the presence of defect measures in the initial data and show that they do not perturb the Newtonian limit of the corotational system.

Chemin and Masmoudi [5] proved the existence and uniqueness of strong solutions of the Oldroyd model (2), within the framework of homogenous Besov spaces with critical index of regularity. The authors particularly showed local and global-in-time existence of solutions for large and small initial data, respectively, under the assumption of a smallness condition on the coupling parameters of system (2). In [23], Zi, Fang and Zhand extended the mentioned result, relaxing this smallness condition. The main results of this manuscript should be seen as suitable improvements of [5,23], within the setting of system (1). In particular, we relax the assumption of small initial data for global-in-time solution, considering small functions in weak Lebesgue spaces (cf. Theorem 1.4).

In this article, we aim to show that three of the most widespread results about the Navier-Stokes equations can be extended to the Johnson-Segalman model (1), under suitable condition on the initial data. More precisely:

- Existence of global-in-time classical solutions with critical regularity and finite energy in dimension two for large initial data,
- Uniqueness in dimension two of strong solutions with finite energy,
- Existence and uniqueness of global-in-time strong solution in dimension $d \geq 3$, with a Fujita-Kato [9] smallness condition for the initial data.

The first problem we address in this article is the existence and uniqueness of global-in-time solutions when dealing with large initial data in $L^2(\mathbb{R}^2)$. In the case of the classical Navier-Stokes system, existence of this type of solutions for the Navier-Stokes equations was proven by Leray in [14] while the uniqueness was showed by Lions and Prodi in [15]. In this context, specific difficulties arise when dealing with the Johnson-Segalman system. This difficulties should be recognized within the intrinsic structure of the system (1):

- the equation of the conformation tensor $\tau$ is of hyperbolic type, sharing the majority of the difficulties related to a transport equation,
- the fluid equation is driven by a forcing term which behaves as the gradient of the conformation tensor, complicating the behavior of the flow $u$ for large value of time.

In the framework of a Navier-Stokes equation, it is rather common to construct Leray type solutions making use of a compactness method. However, the specific structure of the hyperbolic equation for $\tau$ adds complexity when transposing this technique to our system. For instance, when dealing with nonlinearities such as

$$-\omega \tau + \tau \omega = \frac{\nabla u - \nabla u}{2},$$

in the $\tau$-equation, one should recognize the typical difficulty related to the product of two weakly convergent sequences, when passing to the limit of suitable approximate solutions. In order to overcome this issue, we assume some extra regularity on the initial tensor $\tau_0$ that would to a certain degree allow to achieve suitable strong convergences of solutions. One can evince how the hyperbolic behavior of the conformation equation counteracts against the propagation of this regularity. Typically, this can be overcome when dealing with sufficiently regular flow, such as velocity field $u$ that are Lipschitz in space. This Lipschitz condition, however, is above the properties of Leray type solutions and this leads in considering an additional regularities also for the initial velocity field $u_0$. We can hence summarize our first result in the following statement:

**Theorem 1.1.** Consider system (1) within the bidimensional case $d = 2$. Let the initial data $u_0$ be a free divergence vector field in $L^2(\mathbb{R}^2) \cap \dot{B}^{2/p}_{p,1}$, and $\tau_0$ an element of $L^2(\mathbb{R}^2) \cap \dot{B}^{2/p}_{p,1}$, for an index $p \in [1, \infty)$. Then the system (1)
admits a global-in-time weak solution \((u, \tau)\) within the functional framework

\[
\begin{align*}
    u & \in L^\infty_{\text{loc}}(\mathbb{R}^+; L^2(\mathbb{R}^2)) \cap L^2_{\text{loc}}(\mathbb{R}^+, H^1(\mathbb{R}^2)), \\
    \tau & \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2)), \\
    u & \in L^\infty_{\text{loc}}(\mathbb{R}^+; B^\frac{2}{p-1}_{p,1}) \cap L^1_{\text{loc}}(\mathbb{R}^+, B^\frac{2}{p+1}_{p,1}), \\
    \tau & \in L^\infty_{\text{loc}}(\mathbb{R}^+, B^\frac{2}{p}_{p,1}).
\end{align*}
\]

This solution is unique if \(p \in [1, 4]\). Furthermore, for any time \(t \geq 0\), the \(L^2\)-energy is bounded by

\[
\|\tau(t)\|_{L^2(\mathbb{R}^2)} \leq \|\tau_0\|_{L^2(\mathbb{R}^2)} + \int_0^t \|\nabla u(\tau)\|_{L^2(\mathbb{R}^2)}(\tau) \leq C (\|u_0\|_{L^2(\mathbb{R}^2)} + v^{-1}\|\tau_0\|_{L^2(\mathbb{R}^2)})^2,
\]

for a suitable positive constant \(C\). In addition, the Besov regularities satisfy the following inequalities

\[
\begin{align*}
    &\|\tau(t)\|_{\dot{B}^\frac{2}{p}_{p,1}} \leq \|\tau_0\|_{\dot{B}^\frac{2}{p}_{p,1}} \exp \left\{ C v^{-1} \Upsilon_1(t, u_0, \tau_0) \right\}, \\
    &\|u(t)\|_{\dot{B}^\frac{2}{p-1}_{p,1}} + \int_0^t \|u(s)\|_{\dot{B}^\frac{2}{p+1}_{p,1}} ds \leq \left( \|u_0\|_{\dot{B}^\frac{2}{p-1}_{p,1}} + r \|\tau_0\|_{\dot{B}^\frac{2}{p+1}_{p,1}} \right) \exp \left\{ C v^{-1} \Upsilon_2(t, u_0, \tau_0) (v^{-1} \Upsilon_2(t, u_0, \tau_0) + 1) \right\},
\end{align*}
\]

where \(\Upsilon_1(t, u_0, \tau_0)\) and \(\Upsilon_2(t, u_0, \tau_0)\) are two smooth functions depending on the time \(T\) and on the norms of \((u_0, \tau_0)\) in the functional framework \(L^2(\mathbb{R}^2) \cap \dot{B}^{-1}_{\infty,1} \times L^2(\mathbb{R}^2) \cap \dot{B}^0_{\infty,1}\).

As pointed out, the initial condition \((u_0, \tau_0)\) to belong to \(\dot{B}^{2/p}_{p,1} \times \dot{B}^{2/p}_{p,1}\) (cf. Section 2 for some details about these functional spaces) is the precursor of the Lipschitz behavior of the fluid \(u\). Nevertheless, the real regularity which unlock the Lipschitz condition for \(u\) is somehow hidden in the above statement, although it is reported. We overview some specific about that: when considering the case \(p = 2\), that is \((u_0, \tau_0) \in \dot{B}^1_{2,1} \times \dot{B}^1_{2,1}\), we are dealing with a strict subcase of the framework \((u_0, \tau_0) \in L^2(\mathbb{R}^2) \times \dot{H}^1(\mathbb{R}^2)\), where \(\dot{H}^1(\mathbb{R}^2)\) stands for the homogeneous Sobolev space. It is well known that just considering the simplified case of a transport equation

\[\partial_t \tau + u \cdot \nabla \tau = 0,\]

the Sobolev regularity \(\dot{H}^1(\mathbb{R}^2)\) is propagated by a Lipschitz flow with the following exponential growth:

\[
\|\tau(t)\|_{\dot{H}^1(\mathbb{R}^2)} \leq \|\tau_0\|_{\dot{H}^1(\mathbb{R}^2)} \exp \left\{ \int_0^t \|u\|_{L^p} \right\}.
\]

Coupling this inequality with the structure of system (1) would eventually lead to a bound for the Lipschitz regularity of the flow \(u\) of the following type:

\[
\frac{d}{dt} \|u\|_{L^p} \leq C (\|u_0\|_{L^p}, \|\tau_0\|_{\dot{H}^1(\mathbb{R}^2)}) \exp \left\{ \int_0^t \|u\|_{L^p} \right\},
\]

for which no global-in-time bound is automatically determined. Hence, we will first propagate the norm of the initial data within a largest functional framework than the one specifically stated in Theorem 1.1 namely we will propagate the following regularities:

\[
u_0 \in \dot{B}^{-1}_{\infty,1} \quad \text{and} \quad \tau_0 \in \dot{B}^0_{\infty,1},
\]

in which \(\dot{B}^{2/p-1}_{p,1}\) and \(\dot{B}^{2/p}_{p,1}\) are embedded, respectively. We will show that this particular choice is essential since it can still be propagated by a Lipschitz flow, however just with a linear growth:

\[
\|\tau(t)\|_{\dot{B}^0_{\infty,1}} \leq \|\tau_0\|_{\dot{B}^0_{\infty,1}} \left( 1 + \int_0^t \|u\|_{L^p} \right) \Rightarrow \frac{d}{dt} \|u\|_{L^p} \leq C (\|u_0\|_{L^p}, \|\tau_0\|_{\dot{H}^1(\mathbb{R}^2)}) \left( 1 + \int_0^t \|u\|_{L^p} \right).
\]

Thus a standard Gronwall inequality will unlock the Lipschitz condition on \(u(t)\), globally in time \(t > 0\).
Remark 1.2. In Theorem 1.1 we avoided to explicitly present the form of $\Upsilon^1_v(T, u_0, \tau_0)$ and $\Upsilon^2_v(T, u_0, \tau_0)$ for the sake of a compact formulation. Nevertheless, we can report here their exact expression. We first need to introduce the functions

$$
\Phi_v(T, u_0, \tau_0) = \| u_0 \|_{L^2(\mathbb{R}^2)} + \| u_0 \|^2_{L^2(\mathbb{R}^2)} + v^{-\frac{1}{7}} T^\frac{2}{5} \| \tau_0 \|_{L^2(\mathbb{R}^2)} + v^{-1} T \| \tau_0 \|^2_{L^2(\mathbb{R}^2)},
$$

$$
\Psi_{1,v}(T, u_0, \tau_0) = C \left\{ v^{-\frac{9}{10}} \Phi_v(T, u_0, \tau_0)^2 + v^{-\frac{13}{8}} \Phi_v(T, u_0, \tau_0) \| u_0 \|_{L^2(\mathbb{R}^2)} \right\},
$$

$$
\Psi_{2,v}(T, u_0, \tau_0) = C \left\{ v^{-2} T \| \tau_0 \|_{L^2(\mathbb{R}^2)} + v^{-\frac{13}{8}} \Phi_v(T, u_0, \tau_0) + v^{-1} \| u_0 \|_{L^2(\mathbb{R}^2)} \right\},
$$

for a sufficiently large constant $C$ that will be determined later on. Hence, the exact formulations of $\Upsilon^1_v$ and $\Upsilon^2_v$ is given by

$$
\Upsilon^1_v(T, u_0, \tau_0) := \left\{ \| u_0 \|_{B^{-1}_{\infty,1}} + \| \Psi_{1,v}(T, u_0, \tau_0) + C \| \Psi_{2,v}(T, u_0, \tau_0) + C \| \tau_0 \|_{B_{p,1}^0} T \right\} \times
$$

$$
\exp \left\{ v^{-1} \int_0^T \Psi_{2,v}(t, u_0, \tau_0) dt + CT \right\}.
$$

and also

$$
\Upsilon^2_v(T, u_0, \tau_0) := \| u_0 \|_{B^{-1}_{\infty,1}} + \Psi_{1,v}(T, u_0, \tau_0) + C \left( \Psi_{2,v}(T, u_0, \tau_0) + C \| \tau_0 \|_{B_{p,1}^0} T +
$$

$$
+ C \| \tau_0 \|_{B_{\infty,1}^0} \int_0^T \left( \Psi_{2,v}(t, u_0, \tau_0) + C \right) \Upsilon^1_v(t, u_0, \tau_0) dt \right\}.
$$

The last part of our program concerns to establish a result of global-in-time existence of strong solutions for small initial data in dimension $d \geq 3$, proceeding similarly as in the result of Fujita-Kato [9] for the incompressible Navier-Stokes equations, as well as in the result of Paicu and Danchin in [6] for the so-called Boussinesq system.

We recall that the Fujita-Kato Theorem is proven by reformulating the Navier-Stokes system as a fixed point problem. The last part of our program concerns to establish a result of global-in-time existence of strong solutions for small initial data.

Fujita and Kato showed that the Navier-Stokes system is well posed for small initial data that belong to the homogeneous Sobolev Space $H^{1/2}(\mathbb{R}^3)$.

A trivial computation shows that the Johnson-Segalman system (11) is invariant under the following transformation:

$$
(u(t, x), \tau(t, x)) \rightarrow (\lambda u(\lambda^2 t, \lambda x), \lambda^2 \tau(\lambda^2 t, \lambda x)).
$$

Hence, the critical functional framework for the velocity field $u$ is the same as in the case of the Navier-Stokes equations, while for the conformation tensor $\tau$ we need to impose an additional derivative.

We hence prove the following local result of solutions for system (11) within critical regularities:

Theorem 1.3. Consider a dimension $d \geq 3$, let the initial data $u_0$ be a free divergence vector field in $B^{d-1}_{p,1}$, while $\tau_0$ belongs to $B^d_{p,1}$, for a parameter $p \in [1, 2d)$. Then there exists a time $T^* > 0$ for which system (11) admits a unique local solution $(u, \tau)$ within

$$
u \in C([0, T], B^{d-1}_{p,1}) \cap L^1(0, T, B^{d+1}_{p,1}), \tau \in C([0, T], B^d_{p,1}),
$$

for any $T < T^*$. Furthermore if $u$ belongs to $L^\infty(0, T^*, B^{d-1}_{p,1}) \cap L^1(0, T^*, B^{d+1}_{p,1})$ and $\tau$ belongs to $L^\infty(0, T^*, B^d_{p,1})$, the solution can be extended in time with a life span larger than $T^*$.
The construction of global-in-time classical solution is more delicate than the above local result. Indeed, the lacking of damping term for the conformation tensor $\tau$ does not allow to use the classic fixed point approach, which couples the Picard scheme with standard estimates for the Stokes operator. We hence proceed as follows:

- we determine a suitable functional setting for the initial data for which system (1) preserves the smallness condition of the initial data,
- we hence use this specific small condition, coupled with the Picard fixed point, to show that certain critical regularities are still propagated, globally in time.

We will see that the mentioned functional framework corresponds to the Lorentz spaces $u \in \mathcal{L}^{d,\infty}(\mathbb{R}^d)$ and $\tau_0 \in \mathcal{L}^{d,\infty}(\mathbb{R}^d)$. We mention that these spaces are critical under the considered scaling behavior. We will thus prove the following global result:

**Theorem 1.4.** Let us assume that the dimension $d \geq 3$, and the initial data $u_0$ belongs to $\mathcal{B}^{d/p-1}_{p,1} \cap \mathcal{L}^{d,\infty}(\mathbb{R}^d)$ while $\tau_0$ belongs to $\mathcal{B}^{d/p}_{p,1} \cap \mathcal{L}^{d,\infty}(\mathbb{R}^d)$, with $p \in [1, +\infty)$. Then there exists a small positive constant $\varepsilon$ depending on the dimension $d$ such that, whenever the following smallness condition holds true

$$\| u_0 \|_{\mathcal{L}^{d,\infty}} + \frac{1}{\nu} \| \tau_0 \|_{\mathcal{L}^{d,\infty}} \leq \frac{\varepsilon}{\nu},$$

then the corotational Johnson-Segalman model (1) admits a global-in-time classical solution $(u, \tau)$, satisfying

$$u \in \mathcal{C} \left( \mathbb{R}^+, \mathcal{B}^{d/p}_{p,1} \cap \mathcal{L}^{d,\infty}(\mathbb{R}^d) \right) \cap \mathcal{L}^1_{\text{loc}}(\mathbb{R}^+, \mathcal{B}^{d+1}_{p,1}), \quad \tau \in \mathcal{C} \left( \mathbb{R}^+, \mathcal{B}^{d}_{p,1} \cap \mathcal{L}^{d,\infty}(\mathbb{R}^d) \right).$$

If $p$ belongs to $[1, 2d]$ then the solution is unique. Furthermore, there exists a constant $C_\nu$ depending just on the dimension $d$ such that for any time $T \geq 0$ we have

$$\| \tau(T) \|_{\mathcal{L}^{d,\infty}} \leq \| \tau_0 \|_{\mathcal{L}^{d,\infty}}, \quad \| u(T) \|_{\mathcal{L}^{d,\infty}} \leq C \left( \| u_0 \|_{\mathcal{L}^{d,\infty}} + v^{-1} \| \tau_0 \|_{\mathcal{L}^{d,\infty}} \right);$$

$$\| u \|_{\mathcal{L}^1(0,T;\mathcal{B}^{d+1}_{p,1})} + v \| u \|_{\mathcal{L}^1(0,T;\mathcal{B}^{d}_{p,1})} \leq \left\{ \| u_0 \|_{\mathcal{B}^{d/p}_{p,1}} + CT \| \tau_0 \|_{\mathcal{B}^{d}_{p,1}} \right\} e^{C T \Theta_\nu(u_0, \tau_0, T)};$$

$$\| \tau \|_{\mathcal{L}^\infty(0,T;\mathcal{B}^{d}_{p,1})} \leq \| \tau_0 \|_{\mathcal{B}^{d}_{p,1}} \exp \left\{ CT \Theta_\nu(u_0, \tau_0, T) \right\},$$

where the function $\Theta_\nu(u_0, \tau_0, T)$ is a smooth function depending on the time $T > 0$ and on the norms of the initial data $(u_0, \tau_0)$ in the functional framework $\mathcal{B}^{-1}_{\infty,1} \times \mathcal{B}^0_{\infty,1}$:

$$\Theta_\nu(u_0, \tau_0, T) := C \left\| u_0 \right\|_{\mathcal{B}^{-1}_{\infty,1}} \exp \left\{ CT v^{-1} \| \tau_0 \|_{\mathcal{B}^0_{\infty,1}} \right\} + v \left( \exp \left\{ CT v^{-1} \| \tau_0 \|_{\mathcal{B}^0_{\infty,1}} \right\} - 1 \right).$$

The paper is structured as follows. In Section 2 we present the main functional settings in which we develop our main results. Section 3 is devoted to suitable inequalities related to the tensor equation $\tau$, that will play a major role in the main proofs. Section 4 is devoted to the proof of Theorem 1.3 about the existence and uniqueness of global-in-time strong solutions for large initial data in dimension two. Section 5. Section 6 and Section 7 are devoted to Theorem 1.3 and Theorem 1.4 respectively, namely to the existence and uniqueness of strong solutions in dimension $d \geq 3$.

**2. Functional spaces and toolbox of harmonic analysis**

We begin with recalling the definition of weak Lebesgue spaces $L^{p,\infty}(\mathbb{R}^d)$.

**Definition 2.1.** For any $p \in [1, \infty)$, the functional space $L^{p,\infty}(\mathbb{R}^d)$ is composed by Lebesgue measurable function, for which the following norm is bounded:

$$\| f \|_{L^{p,\infty}} = \sup_{\lambda > 0} \lambda m \left( \left\{ x \in \mathbb{R}^d \text{ such that } |f(x)| > \lambda \right\} \right)^{\frac{1}{p}} < \infty,$$

where $m$ stands for the Lebesgue measure on $\mathbb{R}^d$. 
We can now define the functional set of the homogeneous Besov space as follows:

We state that thanks to (3) the identity \( u = \dot{\Delta}^{\sigma} \).

We then define the homogeneous dyadic block \( \dot{\Delta}^{\sigma} \):

\[
\dot{\Delta}^{\sigma} f = \mathcal{F}^{-1}(\varphi(2^{-\sigma} \xi) \hat{f}(\xi)), \quad \dot{\Delta}^{\sigma} f = \mathcal{F}^{-1}(\chi(2^{-\sigma} \xi) \hat{f}(\xi))
\]

where \( \mathcal{F} \) stands for the standard Fourier transform. We remark that for any tempered distribution \( u \in S'(\mathbb{R}^d) \), the functions \( \dot{\Delta}^{\sigma} u \) and \( \dot{\Delta}^{\sigma} u \) are analytic. Furthermore, if there exists a real \( s \) for which \( u \in H^s(\mathbb{R}^d) \), then both \( \dot{\Delta}^{\sigma} u \) and \( \dot{\Delta}^{\sigma} u \) belong to the space \( H^s(\mathbb{R}^d) = \bigcap_{\sigma \in \mathbb{R}} H^\sigma(\mathbb{R}^d) \).

We state that thanks to (3) the identity \( u = \dot{\Delta}^{\sigma} u + \sum_{q \in \mathbb{N}} \dot{\Delta}^{\sigma} u \) holds in \( S'(\mathbb{R}^d) \) while \( u = \sum_{q \in \mathbb{N}} \dot{\Delta}^{\sigma} u \) for any homogeneous temperate distribution \( u \in S'_h(\mathbb{R}^d) \).

We will frequently use the following orthogonal condition on the dyadic blocks \( \dot{\Delta}^{\sigma} \):

\[
\dot{\Delta}^{\sigma} \dot{\Delta}^{\tau} \equiv 0 \quad \text{if} \quad |\sigma - \tau| \geq 2 \quad \text{and} \quad \dot{\Delta}^{\sigma} \left( \dot{\Delta}^{\tau} u \dot{\Delta}^{\tau} v \right) \equiv 0 \quad \text{if} \quad |\sigma - \tau| \geq 5.
\]

We can now define the functional set of the homogeneous Besov space as follows:

**Definition 2.3.** Let \( s \in \mathbb{R} \), \( (p, r) \in [1, \infty)^2 \) and \( u \in S'(\mathbb{R}^d) \). We denote by

\[
\| u \|_{\dot{B}^{s}_{p, r}} := \left\{ \left( \sum_{q \in \mathbb{Z}} 2^{qs} \| \dot{\Delta}^{\sigma} u \|_{L^p}^r \right)^{\frac{1}{r}} \right\} \quad \text{if} \quad r < +\infty,
\]

\[
\sup_{q \in \mathbb{Z}} 2^{qs} \| \dot{\Delta}^{\sigma} u \|_{L^p} \quad \text{if} \quad r = +\infty.
\]

Thus, we define the homogeneous Besov space \( \dot{B}^{s}_{p, r} = \dot{B}^{s}_{p, r}(\mathbb{R}^d) \) by

\[
\dot{B}^{s}_{p, r} := \left\{ u \in S'(\mathbb{R}^d) \mid \| u \|_{\dot{B}^{s}_{p, r}} < +\infty \right\}
\]

if \( s < d/p \) or \( s = d/p \) with \( r = 1 \), and by

\[
\dot{B}^{s}_{p, r} := \left\{ u \in S'(\mathbb{R}^d) \mid \forall |\alpha| = k + 1 \quad \| \partial^{\alpha} u \|_{\dot{B}^{s-k-1}_{p, r}} < +\infty \right\}
\]

if \( d/p + k \leq s < d/p + k + 1 \) or \( s = d/p + k + 1 \) and \( r = 1 \), for some \( k \in \mathbb{N} \).

**Remark 2.4.** The functional space \( \dot{B}^{s}_{p, r} \) is a Banach space if and only if \( s < d/p \) or \( s = d/p \) and \( r = 1 \). For the sake of completeness we recall also the definition of the non-homogeneous Besov spaces:
\textbf{Definition 2.5.} Let $s \in \mathbb{R}$, $(p, r) \in [1, \infty]^2$ and $u \in S'(\mathbb{R}^d)$. We denote by 
\[ \| u \|_{\dot{B}^s_{p,r}} := \begin{cases} \left( \| \dot{S}_0 u \|_{L^p} + \sum_{q \in \mathbb{N}} 2^{qs} \| \dot{A}_q u \|_{L^p} \right)^{\frac{1}{r}} & \text{if } r < +\infty, \\ \max \{ \| \dot{S}_0 u \|_{L^p}, \sup_{q \in \mathbb{N}} 2^{qs} \| \dot{A}_q u \|_{L^p} \} & \text{if } r = +\infty. \end{cases} \]
The non-homogeneous Besov space $\dot{B}^s_{p,r} = \dot{B}^s_{p,r}(\mathbb{R}^d)$ is the set of temperate distributions for which $\| u \|_{\dot{B}^s_{p,r}}$ is finite.

\textbf{Remark 2.6.} The Besov spaces $\dot{B}^s_{2,2}$ and $\dot{B}^s_{2,2}$ coincide with the Sobolev spaces $\dot{H}^s(\mathbb{R}^d)$ and $H^s(\mathbb{R}^d)$ respectively. Furthermore, if $s \in \mathbb{R}^+ \setminus \mathbb{N}$, the Besov spaces $\dot{B}^s_{\infty,\infty}$ and $\dot{B}^s_{\infty,\infty}$ coincide with the Hölder spaces $C^s$ and $C^s$.

The following estimates are known as Bernstein type inequalities, and they will be frequently used in our proofs.

\textbf{Lemma 2.7.} Let $1 \leq p \leq l \leq \infty$ and $\psi \in C_c^\infty(\mathbb{R}^d)$. We hence have
\[ c 2^{-q \left( \frac{d}{p} - \frac{d}{l} \right)} \| \dot{A}_q u \|_{L^l} \leq \| \dot{A}_q u \|_{L^p} \leq C 2^{q \left( \frac{d}{p} - \frac{d}{l} \right)} \| \dot{A}_q u \|_{L^l} \]
and
\[ \| \dot{S}_q u \|_{L^p} \leq C 2^{q \left( \frac{d}{p} - \frac{d}{l} \right)} \| \dot{S}_q u \|_{L^l}. \]

As a consequence of the Bernstein type inequality and the definition of Besov Spaces $\dot{B}^s_{p,r}$, we have the following proposition:

\textbf{Proposition 2.8.} \( (i) \) There exists a constant $c > 0$ such that
\[ \frac{1}{c} \| u \|_{\dot{B}^s_{p,r}} \leq \| \nabla u \|_{\dot{B}^s_{p,r}} \leq c \| u \|_{\dot{B}^s_{p,r}}. \]
\( (ii) \) For $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$, we gather $\dot{B}^s_{p_1,r_1} \hookrightarrow \dot{B}^{s-d(1/p_1 - 1/p_2)}_{p_2,r_2}$.
\( (iii) \) If $p \in [1, \infty]$ then $\dot{B}^{d/p}_{p,1} \hookrightarrow \dot{B}^{d/p}_{p,\infty} \cap L^\infty$. Furthermore, for any $p \in [1, \infty]$, $\dot{B}^{d/p}_{p,1}$ is an algebra embedded in $L^\infty(\mathbb{R}^d)$.
\( (iv) \) The real interpolation $(\dot{B}^s_{p,r}, \dot{B}^s_{p,r})_{\theta,1}$, for a parameter $\theta \in (0, 1)$, is isomorphic to $\dot{B}^\theta_{p,1} + (1 - \theta)s_2$.

We recall further the following results about inclusions between Lorentz and Besov spaces.

\textbf{Lemma 2.9.} For any $1 < p < q \leq \infty$, we have
\[ L^{p,\infty} \hookrightarrow \dot{B}^{s - \frac{d}{q}}_{q,\infty}. \]

\textbf{Proof.} Denoting by $h_j = 2^{jN} h(2^j \cdot)$ with $h = 2^s \varphi$, we recast that the dyadic block $\dot{A}_j$ as a convolution operator
\[ \dot{A}_j u = h_j \ast u. \]
Hence, making use of the following convolution inequalities between Lorentz spaces
\[ \| \dot{A}_j u \|_{L^q} \leq \| h_j \|_{L^1} \| u \|_{L^{p,\infty}} \quad \text{with} \quad \frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q} \]
and observing that by change of variables
\[ \| h_j \|_{L^{1,1}} = 2^{jd(1 - \frac{1}{q})} \| h \|_{L^q}, \]
we eventually gather that
\[ \sup_{j \in \mathbb{Z}} 2^{j \left( \frac{d}{q} - \frac{d}{r} \right)} \| \dot{A}_j u \|_{L^q} \leq \| h \|_{L^{1,\infty}} \| u \|_{L^{p,\infty}} \]
which concludes the proof of the lemma. \( \square \)
We now consider several a-priori estimates in the functional framework of Besov spaces for the heat semigroup (cf. [1], Lemma 2.4).

**Lemma 2.10.** There exists two constant $c$ and $C$ such that for any $\tau \geq 0$, $q \in \mathbb{Z}$ and $p \in [1, \infty]$, we get
\[
\left\| e^{\tau \Delta} \Delta_q u \right\|_{L^p} \leq C e^{-c \tau 2^q} \left\| \Delta_q u \right\|_{L^p}.
\]

From the above Lemma we can then deduce the following result (cf. [6], Proposition 3.11)

**Proposition 2.11.** Let $s \in \mathbb{R}$, $1 \leq p$, $r$, $\rho_1 \leq \infty$. Let $u_0$ be in $\tilde{B}^s_{p,r}$ and $f$ be in $\tilde{L}^{\rho_1}(0,T;\tilde{B}^{s-2+2/\rho_1}_{p,r})$ for some positive time $T$ (possibly $T = \infty$). Then the heat equation
\[
\begin{aligned}
\partial_t u - \Delta u &= f \\
u_{|t=0} &= u_0
\end{aligned}
\]
for $(0,T) \times \mathbb{R}^d$, $\nu \in L^q(\mathbb{R}^d)$, admits an unique strong solution in $\tilde{L}^{\infty}(0,T;\tilde{B}^s_{p,r}) \cap \tilde{L}^{\rho_1}(0,T;\tilde{B}^{s-2+2/\rho_1}_{p,r})$. Moreover there exist a constant $C$ depending just on the dimension $d$ such that the following estimate holds true for any time $t \in [0,T]$ and $\rho \geq \rho_1$:
\[
\left\| \nu \right\|_{\tilde{L}^{p}(0,t,\tilde{B}^{s+\frac{2}{\rho}}_{p,r})} \leq C \left( \| u_0 \|_{\tilde{B}^s_{p,r}} + \left\| \frac{4p}{2p-1} \left\| f \right\|_{\tilde{L}^{\rho_1}(0,t,\tilde{B}^{s-2+2/\rho_1}_{p,r})} \right\| \right).
\]

In the previous Proposition we introduce the functional space $\tilde{L}^p(0,T;\tilde{B}^s_{p,r})$, which is known as a Chemin-Lerner space. This is defined similarly as in Definition (2.3), imposing
\[
\left\| u \right\|_{\tilde{L}^p(0,T;\tilde{B}^s_{p,r})} := \left( \left\| \tilde{\nu} \right\|_{L^p(0,t,\tilde{B}^{s+\frac{2}{\rho}}_{p,r})} \right)_{\rho \in \mathbb{N}}.
\]

We remark that thanks to the Minkowski inequality
\[
\left\| u \right\|_{\tilde{L}^p(0,T;\tilde{B}^s_{p,r})} \leq C \left\| u \right\|_{L^p(0,T;\tilde{B}^s_{p,r})} \quad \text{for} \quad \rho \geq r,
\]
while the opposite inequality holds when $\rho \leq r$. We hence denote
\[
\tilde{C}([0,T];\tilde{B}^s_{p,r}) := \tilde{L}^\infty(0,T;\tilde{B}^s_{p,r}) \cap C([0,T],\tilde{B}^s_{p,r}) \quad \text{and by} \quad \tilde{L}^p_{\text{loc}}(\mathbb{R}^+;\tilde{B}^s_{p,r}) := \cap_{T>0} \tilde{L}^p(0,T;\tilde{B}^s_{p,r}).
\]

Similarly, one can define the non-homogeneous Chemin-Lerner spaces $\tilde{L}^p(0,T;\tilde{B}^s_{p,r})$. In the particular case of $p = r = 2$ we will use the notation $\tilde{L}^2(0,T;H^s)$ or $\tilde{H}^s(0,T;H^s)$.

**Remark 2.12.** Thanks to Proposition 2.11 and using the fact that the projector $P$ on the free divergence vector fields is an homogeneous Fourier multiplier of degree 0, namely it is continuous from $\tilde{B}^s_{p,r}$ to itself, we can easily solve the non-stationary Stokes problem
\[
\begin{aligned}
\partial_t u - \Delta u + \nabla p &= f \\
\text{div} u &= 0 \\
u_{|t=0} &= u_0
\end{aligned}
\]
with initial data $u_0 \in \tilde{B}^s_{p,r}$ with null divergence and a source term $f$ in $\tilde{L}^1(0,T;\tilde{B}^s_{p,r})$. We hence achieve a unique solution $u$ in the class affinity
\[
u \in \tilde{L}^\infty(0,T;\tilde{B}^s_{p,r}) \cap \tilde{L}^1(0,T;\tilde{B}^{s+2}_{p,r}), \quad \nabla p \in \tilde{L}^1(0,T;\tilde{B}^s_{p,r}),
\]
with $u$ satisfying
\[
\left\| \nu \right\|_{\tilde{L}^p(0,T;\tilde{B}^{s+2}_{p,r})} \leq C \left( \| u_0 \|_{\tilde{B}^s_{p,r}} + \| f \|_{L^1(0,T;\tilde{B}^s_{p,r})} \right).
\]

Furthermore, if $r < \infty$ the solution $u$ belongs to $C([0,T];\tilde{B}^s_{p,r})$. 

---
We first remark that for any integer \( q \) in \( B_{r,s}^p \) and \( f \) in \( \tilde{L}^p(0, T; B_{p,r}^{-2+2/p}) \). The existence and uniqueness of a solution still holds, nevertheless for a constant \( C \) in \( [4] \) that this time depends linearly on the time \( T \).

The proof of a-priori estimates for certain nonlinear terms is mainly handled through the use of the paradifferential calculus, in particular of the so called Bony type decomposition:

\[
fg = \hat{T}fg + \hat{T}g f + \hat{R}(f, g),
\]

where the paraproduct \( \hat{T} \) and the homogenous reminder \( \hat{R} \) are defined by

\[
\hat{T}fg = \sum_{q \in \mathbb{Z}} \hat{S}_{q-1}g \Delta_q f \quad \text{and} \quad \hat{R}(f, g) = \sum_{q \in \mathbb{Z}} \Delta_q f \left( \sum_{|j-q| \leq 1} \Delta_j g \right).
\]

We then state some results of continuity of these operators that we will often use in our proof.

**Proposition 2.14.** Let \( 1 \leq p, p_1, p_2, r, r_1, r_2 \leq \infty \) satisfying \( 1/p = 1/p_1 + 1/p_2 \) and \( 1/r = 1/r_1 + 1/r_2 \). The the homogeneous paraproduct \( \hat{T} \) is continuous

- from \( L^\infty \times B_{p,r}^1 \) into \( B_{p,r}^1 \) for any \( t \in \mathbb{R} \),
- from \( B_{p,r_1}^{-s_1} \times B_{p,r_2}^{-s_2} \) into \( B_{p,r}^{-s} \), for any \( t \in \mathbb{R} \) and \( s > 0 \).

The homogeneous reminder \( \hat{R} \) is continuous

- from \( B_{p_1,r_1}^s \times B_{p_2,r_2}^s \) into \( B_{p,r}^{s+t} \) for any \( (s, t) \in \mathbb{R}^2 \), such that \( s + t > 0 \),
- from \( B_{p_1,r_1}^s \times B_{p_2,r_2}^s \) into \( B_{p_1}^0 \) if \( s \in \mathbb{R} \) and \( 1/r_1 + 1/r_2 \geq 1 \).

The above proposition allows to determine almost any continuity results for the product of two distributions that belong to two Besov spaces. Further extensions of the above result can be achieved assuming some additional regularity of the distributions:

**Lemma 2.15.** Let \( f \) be a function in \( L^2(\mathbb{R}^2) \cap B_{\infty,1}^1 \), then \( f^2 \) belongs to \( B_{\infty,1}^0 \) and satisfies

\[
\| f^2 \|_{B_{\infty,1}^0} \leq C \| f \|_{L^2(\mathbb{R}^2)} \| f \|_{B_{\infty,1}^1}.
\]

**Proof.** We begin with localizing the frequencies of \( f^2 \) through the standard Bony decomposition (5)

\[
f^2 = 2\hat{T}f + \hat{R}(f, f).
\]

Thus, the triangular inequality implies that

\[
\| f^2 \|_{B_{\infty,1}^0} \leq 2\| \hat{T}f \|_{B_{\infty,1}^0} + \| \hat{R}(f, f) \|_{B_{\infty,1}^0},
\]

\[
\leq 2\sum_{q \in \mathbb{Z}} \| \Delta_q \hat{T}f \|_{L^\infty} + \| \Delta_q \hat{R}(f, f) \|_{L^\infty}.
\]

We first remark that for any integer \( q \in \mathbb{Z} \)

\[
\| \Delta_q \hat{T}f \|_{L^\infty} \leq \sum_{|q-j| \leq 5} \| \hat{S}_{j-1}f \|_{L^\infty} \| \Delta_j f \|_{L^\infty}
\]

\[
\leq \sum_{|q-j| \leq 5} \| \hat{S}_{j-1}f \|_{L^2} 2^j \| \Delta_j f \|_{L^\infty} \leq \| f \|_{L^2} \sum_{|q-j| \leq 5} 2^j \| \Delta_j f \|_{L^\infty},
\]

hence, the homogeneous paraproduct is bounded by

\[
\| \hat{T}f \|_{B_{\infty,1}^0} \leq \| f \|_{L^2} \| f \|_{B_{\infty,1}^1}.
\]
We now take into account the homogeneous reminder. By definition, we gather that
\[
\| \tilde{\Delta} q \tilde{R}(f, f) \|_{L^p_t} \leq \sum_{j: q \geq 5} \| \tilde{\Delta} q (\tilde{\Delta} j + v f \tilde{\Delta} j f) \|_{L^p_t} \leq \sum_{j: q \geq 5} 2^q \| \tilde{\Delta} q (\tilde{\Delta} j + v f \tilde{\Delta} j f) \|_{L^2_t}
\]
\[
\leq \sum_{j: q \geq 5} 2^{q-j} \| \tilde{\Delta} j + v f \|_{L^2_t} 2^j \| \tilde{\Delta} j f \|_{L^\infty_t} \leq \| f \|_{L^2_t} \sum_{j \in \mathbb{Z}} 2^{q-j} 1_{(-\infty, 5]}(q-j) 2^j \| \tilde{\Delta} j f \|_{L^\infty_t}.
\]

Defining \( a_j = 2^j 1_{(-\infty, 5]}(j) \) for any \( j \in \mathbb{Z} \), we can recast the last term in convolution form, namely
\[
\sum_{j \in \mathbb{Z}} 2^{q-j} 1_{(-\infty, 5]}(q-j) 2^j \| \tilde{\Delta} j f \|_{L^\infty_t} = \left( (a_j)_{j \in \mathbb{Z}} * (2^j \| \tilde{\Delta} j f \|_{L^\infty_t}) \right)_q,
\]
for any \( q \in \mathbb{Z} \). Hence, applying the Young inequality we deduce that
\[
\left\| (a_j)_{j \in \mathbb{Z}} * (2^j \| \tilde{\Delta} j f \|_{L^\infty_t})_{j \in \mathbb{Z}} \right\|_{L^1_t} \leq \left\| (2^j \| \tilde{\Delta} j f \|_{L^\infty_t})_{j \in \mathbb{Z}} \right\|_{L^1_t} = \| f \|_{\dot{B}^1_{\infty, 1}^1},
\]
from which
\[
\sum_{q \in \mathbb{Z}} \| \tilde{\Delta} q \tilde{R}(f, f) \|_{L^\infty_t} \leq \| f \|_{L^2_t} \| f \|_{\dot{B}^1_{\infty, 1}^1},
\]
which concludes the proof of the Lemma. 

\[\square\]

2.1. Estimates for the conformation tensor

In this section we perform several a priori estimates for the following equation that governs the evolution of the conformation tensor \( \tau(t, x) \):
\[
\begin{aligned}
\tau_{t} + u \cdot \nabla \tau - \omega \tau + \tau \omega &= f \\
\tau_{|t=0} &= \tau_0
\end{aligned}
\quad \mathbb{R}^+ \times \mathbb{R}^d,
\]

We begin with a standard bound for Lebesgue and Lorentz norms.

**Lemma 2.16.** For any \( p \in [1, \infty] \), the following estimate in Lebesgue spaces holds true:
\[
\| \tau(t) \|_{L^p_t} \leq \| \tau_0 \|_{L^p_t} + \int_0^t \| f(s) \|_{L^p_t} \, ds.
\]

More in general, one has
\[
\| \tau(t) \|_{L^{p, \infty}_t} \leq \| \tau_0 \|_{L^{p, \infty}_t} + \int_0^t \| f(s) \|_{L^{p, \infty}_t} \, ds
\]
and the inequality reduces to an equality whenever \( f \) is identically null.

**Proof.** Considering \( p \in [1, \infty) \), we take the matrix inner product between the \( \tau \)-equation and \( \tau | \tau |^{p-2} \). Hence, integrating in spatial domain, we first observe that
\[
- \int_{\mathbb{R}^d} \omega \tau : \tau | \tau |^{p-2} + \int_{\mathbb{R}^d} \tau \omega : \tau | \tau |^{p-2} = 0
\]
from which we deduce the following \( L^p_t \)-bound of \( \tau \)
\[
\frac{1}{p} \frac{d}{dt} \| \tau(t) \|_{L^p_t}^p \leq \| f \|_{L^p} \| \tau \|_{L^{p-1}_t}^{p-1} \Rightarrow \frac{d}{dt} \| \tau(t) \|_{L^p_t} \leq \| f \|_{L^p}.
\]
The case of \( p = +\infty \) can be achieved as the limit case of the previous inequalities. 

\[\square\]
3. Some a-priori estimates for the conformation tensor

In this section we present some a-priori estimates for the $\tau$ equation. We begin with the following lemma about the propagation of Besov regularities.

**Lemma 3.1.** Let $(p, r) \in [1, \infty]^2$ and $s \in \mathbb{R}$. Assume that $u$ is a free divergence vector field whose coefficients belong to $L^1(0, T; \dot{B}^r_{p,1})$, that the source term $f$ belongs to $L^1(0, T; \dot{B}^0_{r,1})$ and that the initial data $\tau_0$ is in $\dot{B}^s_{p,1}$. Then system (6) admits a unique solution $\tau$ in the class affinity

$$\tau \in L^\infty(0, T; \dot{B}^s_{p,1})$$

which fulfills the following estimate for any time $t \in [0, T]$

$$\| \tau \|_{L^\infty(0,t;\dot{B}^s_{p,1})} \leq \left( \| \tau_0 \|_{\dot{B}^s_{p,1}} + \| f(s) \|_{L^1(0,t;\dot{B}^0_{r,1})} \right) \exp \left\{ C \int_0^t \| \nabla u \|_{\dot{B}^0_{\infty,1}} \right\},$$

for a suitable positive constant $C > 0$.

The proof of Lemma 3.1 is equivalent to the one of Proposition 4.7 in [6]. The presence of an exponential term in the a-priori estimate of Lemma 3.2 produces intrinsic difficulties when dealing with the existence of global-time solutions. Nevertheless, we can refine such an inequality taking into account Besov spaces with null index of regularity:

**Lemma 3.2.** Let $\tau$ be a solution of (3) in $L^\infty(0, T; \dot{B}^0_{r,1})$ with $f$ in $L^1(0, T; \dot{B}^0_{r,1})$ and also $\nabla u$ in $L^1(0, T; \dot{B}^0_{\infty,1})$, for some $p \in [1, \infty]$. Then, the following bound holds true for any time $t \in [0, T]$:

$$\| \tau \|_{L^\infty(0,t;\dot{B}^s_{p,1})} \leq C \left( \| \tau_0 \|_{\dot{B}^s_{p,1}} + \int_0^t \| f(s) \|_{\dot{B}^0_{r,1}} \, ds \right) \left( 1 + \int_0^t \| \nabla u \|_{\dot{B}^0_{\infty,1}} \, ds \right)$$

**Proof.** We decompose the solution $\tau = \sum_{q \in \mathbb{Z}} \tau_q$, where $\tau_q$ is solution of the following system of PDE’s:

$$\begin{cases}
\partial_t \tau_q + u \cdot \nabla \tau_q - \omega \tau_q + \tau_q \omega = f_q & \mathbb{R}^+ \times \mathbb{R}^d, \\
\tau_{t=0} = \dot{\Delta}_q \tau_0 & \mathbb{R}^d.
\end{cases}$$

We first remark that, for any fixed positive $N > 0$

$$\| \tau(t) \|_{\dot{B}^s_{p,1}} \leq \sum_{j,q \in \mathbb{Z}} \| \dot{\Delta}_j \tau_q(t) \|_{L^p_t} \leq \sum_{j,q \in \mathbb{Z}, |j-q| \leq N} \| \dot{\Delta}_j \tau_q(t) \|_{L^p_t} + \sum_{j,q \in \mathbb{Z}, |j-q| > N} \| \dot{\Delta}_j \tau_q(t) \|_{L^p_t} =: \mathcal{I}_N + \mathcal{II}_N.$$

Hence, thanks to Lemma 2.16, we gather

$$\mathcal{I}_N = \sum_{j,q \in \mathbb{Z}, |j-q| \leq N} \| \dot{\Delta}_j \tau_q(t) \|_{L^p_t} \lesssim \sum_{j,q \in \mathbb{Z}, |j-q| \leq N} \| \tau_q(t) \|_{L^p_t} \lesssim N \sum_{q \in \mathbb{Z}} \| \tau_q(t) \|_{L^p_t}$$

$$\lesssim N \sum_{q \in \mathbb{Z}} \left( \| \dot{\Delta}_q \tau_0 \|_{L^p_t} + \int_0^t \| f_q(s) \|_{\dot{B}^0_{r,1}} \, ds \right)$$

$$\lesssim N \left( \| \tau_0 \|_{\dot{B}^s_{p,1}} + \int_0^t \| f(s) \|_{\dot{B}^0_{r,1}} \, ds \right).$$

In order to handle $\mathcal{II}_N$, we make use of Lemma 3.1 where the initial data $\dot{\Delta}_q \tau_0$ is assumed in $\dot{B}^{s+\varepsilon}_{p,1}$ for a small parameter $\varepsilon \in (0,1)$ and a source term $f$ in $L^1(0, T; \dot{B}^s_{r,1})$:

$$\| \tau_q(t) \|_{\dot{B}^{s+\varepsilon}_{p,1}} \leq \left( \| \dot{\Delta}_q \tau_0 \|_{\dot{B}^{s+\varepsilon}_{p,1}} + \int_0^t \| \dot{\Delta}_q f \|_{\dot{B}^{s+\varepsilon}_{p,1}} \right) \exp \left\{ C \int_0^t \| \nabla u \|_{\dot{B}^0_{\infty,1}} \right\}.$$
We now prove some a-priori estimates within the functional framework of Lorentz norms, for the non-stationary
Stokes system. The following Lemma allows us to control the term arising from the combination of the conformation
equation within the Navier-Stokes system.

**Lemma 3.3.** For any time \( t \geq 0 \) and viscosity \( \nu > 0 \), the following estimate holds true
\[
\left\| \int_0^t \mathcal{P} e^{(t-s)\Delta} \nabla f(s) ds \right\|_{L^2_t,\infty} \leq \begin{cases} C \| f \|_{L^\infty(0,t;L^4)} & \text{if } d = 2, \\ C \| f \|_{L^\infty(0,t;L^{4,\infty})} & \text{if } d > 2. \end{cases}
\]

**Proof.** Without loss of generality, we recast ourselves to the case of \( \nu = 1 \). We begin remarking that the projector \( \mathcal{P} \) is a bounded operator from \( L^p(\mathbb{R}^d) \) to itself, for any \( 1 > p < \infty \). Hence, by real interpolation we get that \( \mathcal{P} \) is a bounded linear operator within Lorentz spaces
\[
\mathcal{P} \in \mathcal{L}(L^{\frac{4}{d},\infty}(\mathbb{R}^d), L^{\frac{4}{d},\infty}(\mathbb{R}^d)).
\]
This allows us to cancel the presence of the project \( \mathcal{P} \) in any inequality we aim to prove and limit the proof to the case of the heat kernel operator. We hence decouple the integral we aim to bound
\[
\int_0^t e^{(t-s)\Delta} \nabla f(s) ds
\]
by \( J_\varepsilon \) and \( J^\varepsilon \) defined as follows: fixing a small positive parameter \( \varepsilon \)
\[
J_\varepsilon := \int_0^{t-\varepsilon} e^{(t-s)\Delta} \nabla f(s) ds \quad \text{and} \quad J^\varepsilon := \int_{t-\varepsilon}^t e^{(t-s)\Delta} \nabla f(s) ds
\]
For any \( \tau \geq 0 \) and \( 1 \leq p \leq p \leq \infty \) we have
\[
\left\| e^{i\tau \nabla} \right\|_{L^p(L^q)} = \left\| \mathcal{S}^{-1}(e^{-\tau |\xi|^2 i\xi}) \right\|_{L^p} = \frac{1}{\sqrt{\tau}} \left\| \mathcal{S}^{-1}(e^{-\sqrt{\tau} |\xi|^2 i\xi}) \right\|_{L^{\frac{d+1}{2}}} = \left\| \mathcal{S}^{-1}(e^{-\sqrt{\tau} |\xi|^2 i\xi}) \left( \frac{x}{\sqrt{\tau}} \right) \right\|_{L^q}
\]
hence, by a change of variable, we finally get that there exists a constant \( C \) for which
\[
\left\| e^{i\tau \nabla} \right\|_{L^p(L^q)} \leq \frac{C}{\tau^{\frac{1}{r} + \frac{1}{q}}},
\]
where \( \frac{1}{r} + \frac{1}{p} = \frac{1}{q} + 1 \), namely \( \frac{1}{r'} = \frac{1}{p} - \frac{1}{q} \).
We first assume that $d = 2$. We hence get
\[
\| J^e \|_{L^1_t} \leq \int_{t - \varepsilon}^{t} \left\| e^{(t-s)\Delta} \nabla f(s) \right\|_{L^1_t} ds
\]
\[
\leq \int_{t - \varepsilon}^{t} \left\| e^{(t-s)\Delta} \nabla \mathcal{L}(L^1_t, L^1) \right\| f(s) \|_{L^1_t} ds
\]
\[
\leq C \int_{t - \varepsilon}^{t} \frac{\left\| f(s) \right\|_{L^1_t}}{|t - s|^{\frac{1}{2}}} ds
\]
\[
\leq C \| f(s) \|_{L^\infty(0,t;L^1_t)} \int_{0}^{t} \frac{1}{s^{\frac{1}{2}}} ds \leq C \sqrt{\varepsilon} \| f(s) \|_{L^\infty(0,t;L^1_t)},
\]
while similarly
\[
\| J^e \|_{L^\infty} \leq \int_{0}^{t - \varepsilon} \left\| e^{(t-s)\Delta} \nabla f(s) \right\|_{L^\infty} ds
\]
\[
\leq \int_{0}^{t - \varepsilon} \left\| e^{(t-s)\Delta} \nabla \mathcal{L}(L^1_t, L^\infty) \right\| f(s) \|_{L^1_t} ds
\]
\[
\leq C \int_{0}^{t - \varepsilon} \frac{\left\| f(s) \right\|_{L^1_t}}{|t - s|^{\frac{1}{2}}} ds
\]
\[
\leq C \| f(s) \|_{L^\infty(0,t;L^1_t)} \int_{0}^{t} \frac{1}{s^{\frac{1}{2}}} ds \leq C \sqrt{\varepsilon} \| f(s) \|_{L^\infty(0,t;L^1_t)}.
\]

In virtue of Remark 2.2 about the real interpolation $L^2,\infty(\mathbb{R}^2) = (L^1, L^\infty)_{1/2,\infty}$, we hence conclude that in dimension $d = 2$
\[
\left\| \int_{0}^{t} \mathcal{P} e^{(t-s)\Delta} \nabla f(s) ds \right\|_{L^2,\infty} \leq C \| f(s) \|_{L^\infty(0,t;L^1_t)}.
\]

We now address the case of a dimension $d > 2$. With a similar technique as the one used above, we remark that
\[
\| J^e \|_{L^\infty} \leq \int_{t - \varepsilon}^{t} \left\| e^{(t-s)\Delta} \nabla f(s) \right\|_{L^\infty} ds
\]
\[
\leq \int_{t - \varepsilon}^{t} \left\| e^{(t-s)\Delta} \nabla \mathcal{L}(L^d_t, L^\infty) \right\| f(s) \|_{L^\infty} ds
\]
\[
\leq C \int_{t - \varepsilon}^{t} \frac{\left\| f(s) \right\|_{L^\infty}}{|t - s|^{\frac{1}{2}}} ds
\]
\[
\leq C \sqrt{\varepsilon} \| f(s) \|_{L^\infty(0,t;L^\frac{d}{2})},
\]
while
\[
\| J^e \|_{L^\infty} \leq \int_{0}^{t - \varepsilon} \left\| e^{(t-s)\Delta} \nabla f(s) \right\|_{L^\infty} ds
\]
\[
\leq \int_{0}^{t - \varepsilon} \left\| e^{(t-s)\Delta} \nabla \mathcal{L}(L^d_t, L^\infty) \right\| f(s) \|_{L^\infty} ds
\]
\[
\leq C \int_{0}^{t - \varepsilon} \frac{\left\| f(s) \right\|_{L^\infty}}{|t - s|^{\frac{1}{2}}} ds
\]
\[
\leq \frac{C}{\sqrt{\varepsilon}} \| f(s) \|_{L^\infty(0,t;L^\frac{d}{2})}.
\]
We recall again Remark 2.2 concerning this time the real interpolation $L^{d/2, \infty}([R^2]) = (L^{d/2, \infty}, L^{\infty})_{1/2, \infty}$, we hence conclude that in dimension $d \geq 3$
\[
\left\| \int_0^t P e^{(t-s)A} \nabla f(s) \, ds \right\|_{L^{d, \infty}} \leq C \left\| f(s) \right\|_{L^{\infty}(0,T; L_\tau^{d/4, \infty})}.
\]

4. Global-in-time solutions in dimension two

This section is devoted to the proof of Theorem 1.1. We begin with introducing the following Friedrich-type approximation of system (1):
\[
\begin{align*}
\partial_t \tau^n + u^n \cdot \nabla \tau^n - J_n \omega^n \tau^n + \tau^n J_n \omega^n &= 0 \quad \mathbb{R}_+ \times \mathbb{R}^2, \\
\partial_t u^n - \nu \Delta u^n + \nabla p^n &= -J_n (u^n \cdot \nabla u^n) + \text{div} \tau^n \quad \mathbb{R}_+ \times \mathbb{R}^2, \\
\text{div} u^n &= 0 \quad \mathbb{R}_+ \times \mathbb{R}^d, \\
(u^n(\tau), \tau^n)|_{\tau=0} &= (J_n u_0, J_n \tau_0) \quad \mathbb{R}^2.
\end{align*}
\]

Denoting by $1_A$ the characteristic function of a set $A$, for any $n \in \mathbb{N}$ we introduce the regularizing operator $J^n$ by the formula
\[
\mathcal{F}(J^n g)(\xi) := 1_{\mathcal{C}_n}(\xi) \hat{g}(\xi)
\]
which localizes the Fourier transform of a suitable function $g$ into the annulus $\mathcal{C}_n = \{ \xi \in \mathbb{R}^2, |\xi| \in [1/n, n] \}$. Hence, we claim that an approach coupling the Friedrich’s scheme together with the Schaefer fixed point theorem, allows us to construct a sequence of approximate solutions $(u^n(\tau), \tau^n)_{n \in \mathbb{N}}$ satisfying the following class affinity:
\[
\begin{align*}
u^n &\in L^\infty_{\text{loc}}(\mathbb{R}_+, L^2(\mathbb{R}^2) \cap \dot{B}^{2-1}_{p,1}) \cap L^2_{\text{loc}}(\mathbb{R}_+, \dot{H}^{1}(\mathbb{R}^2)) \cap L^1_{\text{loc}}(\mathbb{R}_+, \dot{B}^{2+1}_{p,1}), \\
\tau^n &\in L^\infty_{\text{loc}}(\mathbb{R}_+, L^2(\mathbb{R}^2) \cap \dot{B}^{2}_{p,1}).
\end{align*}
\]

We refer the reader to [7] for some details about this procedure, where the first author showed a similar result for a different system of PDE’s. The purpose of the next sections is to reveal the above regularities of the approximate solutions $(u^n, \tau^n)_{n \in \mathbb{N}}$. This result is achieved into two main steps:

(i) **Propagating the Lipschitz regularity of the velocity field $u^n$:** the initial data $(u_0, \tau_0)$ belongs to $\dot{B}^{2/p-1}_{p,1} \times \dot{B}^{2/p}_{p,1}$ which is embedded into $\dot{B}^{-1}_{\infty,1} \times \dot{B}^{0}_{\infty,1}$. This last regularity will be hence propagated in time, allowing to control $u^n$ into the functional framework given by
\[
\nabla u^n \in L^1_{\text{loc}}(\mathbb{R}_+, \dot{B}^0_{\infty,1}) \hookrightarrow L^1_{\text{loc}}(\mathbb{R}_+, L^\infty(\mathbb{R}^2)),
\]
from which we will deduce that $u^n$ is Lipschitz, globally in time.

(ii) **Propagating higher regularities of solutions:** we will propagate the specific regularity of the initial data $(u_0, \tau_0)$ in $\dot{B}^{2/p-1}_{p,1} \times \dot{B}^{2/p}_{p,1}$, making use of the Lipschitz condition achieved in point (i).

Last, we will estimate the mentioned norms with a bound independent on the index $n \in \mathbb{N}$. This will allow us to pass to the limit and construct a classical solution of system (1) within the functional framework of Theorem 1.1.

4.1. **Lipschitz regularity of the velocity field**

In this section we show some mathematical properties of solutions for the approximate system (7). The main goal is to establish the propagation of Lipschitz regularity for the velocity field $u^n$, namely to show that $\nabla u^n$ belongs to the functional space
\[
L^1_{\text{loc}}(\mathbb{R}_+, \dot{B}^0_{\infty,1}) \hookrightarrow L^1_{\text{loc}}(\mathbb{R}_+, L^\infty(\mathbb{R}^2)).
\]

We also aim in controlling this regularity with a bound which is independent on the index $n \in \mathbb{N}$, in order to keep this property also when passing to the limit.
We collect in the following statement the result we aim to prove.

**Theorem 4.1.** Assume that the initial data \( u_0 \) and \( \tau_0 \) belongs to \( L^2(\mathbb{R}^2) \cap \dot{B}_{\infty,1}^{-1} \) and \( L^2(\mathbb{R}^2) \cap \dot{B}_{\infty,1}^0 \), respectively. Then, the solutions \((u^n, \tau^n)\) of the system (31), belongs to the functional framework determined by

\[
u \parallel u^n \parallel_{L^1(0,T;\dot{B}_{\infty,1}^1)} \leq \Upsilon_1^1(T, u_0, \tau_0) \quad \text{and} \quad \parallel u^n \parallel_{L^\infty(0,T;\dot{B}_{\infty,1}^{-1})} \leq \Upsilon_2^2(T, u_0, \tau_0),\]

Furthermore, there exists two smooth functions \( \Upsilon_{1,v}(T, u_0, \tau_0) \) and \( \Upsilon_{2,v}(T, u_0, \tau_0) \), for which the following inequalities hold true:

\[
u \parallel u^n \parallel_{L^1(0,T;\dot{B}_{\infty,1}^1)} \leq \Upsilon_1^1(T, u_0, \tau_0) \quad \text{and} \quad \parallel u^n \parallel_{L^\infty(0,T;\dot{B}_{\infty,1}^{-1})} \leq \Upsilon_2^2(T, u_0, \tau_0),\]

with also

\[
u \parallel \tau^n \parallel_{L^\infty(0,T;\dot{B}_{\infty,1}^0)} \leq C \parallel \tau_0 \parallel_{\dot{B}_{\infty,1}^0} (1 + \nu^{-1} \Upsilon_1^1(T, u_0, \tau_0)).\]

Both functions \( \Upsilon_{1,v} \) and \( \Upsilon_{2,v} \) vanish when \( T = 0 \), they are increasing in time \( T > 0 \) and they depend uniquely on the norms \( \parallel u_0 \parallel_{L^2(\mathbb{R}^2) \cap \dot{B}_{\infty,1}^{-1}} \) and \( \parallel \tau_0 \parallel_{L^2(\mathbb{R}^2) \cap \dot{B}_{\infty,1}^0} \).

The proof of Theorem 4.1 requires to proceed into several fundamental steps. We will first begin with determining the standard energy inequalities for system (31) (cf. Proposition 4.2). These inequalities will allow us then to unlock some delicate semi-group estimates related in-primis to the mild formulation of the velocity field \( u^n(t) \):

\[
u u^n(t) := e^{\nu\Delta} u_0 + \int_0^t \mathcal{P} e^{(t-s)\Delta} \text{div} (u^n \otimes u^n)(s) \text{ds} + \int_0^t \mathcal{P} e^{(t-s)\Delta} \text{div} \tau^n(s) \text{ds}. \tag{8}\]

Here \( \mathcal{P} \) is the Leray projector into the space of free-divergence vector fields, while the operator \( e^{\nu\Delta} \) stands for the heat semigroup in the whole space. We recognize in the above identity three distinct terms \( u^r, u^r_1, u^r_2 \), the first one related to the linear contribution of system (31), the second one tackling the non-linearity due to the Navier-Stokes contribution to the system and the last one specifically correlated to the evolution of the conformation tensor \( \tau^n \).

Due to the different structures of these terms, each of them will be separately handled, with appropriate estimates in Chemin-Lerner Besov spaces. We will then proceed as follows:

(i) we will first establish some standard energy estimate of our system (cf. Proposition 4.2 and Proposition 4.3).

(ii) we will then propagate suitable regularities in order to control the second term \( u^r_n(t) \) (cf. Lemma 4.4, Remark 4.5, and Proposition 4.7).

(iii) we will hence analyze the remaining term \( u^r_2(t) \) (cf. Lemma 4.6).

(iv) we will summarize our previous estimates at the end of the section, proving Theorem 4.1 and propagating the Lipschitz-in-space regularity of \( u^n \).

Thanks to a classical energy approach we begin with stating the following proposition:

**Proposition 4.2.** For any \( n \in \mathbb{N} \), \((u^n, \tau^n)\) belongs to \( \mathcal{C}_{\text{loc}}(\mathbb{R}_+, L^2(\mathbb{R}^2)) \) and \( \nabla u^n \) belongs to \( L^2(\mathbb{R}_+, L^2(\mathbb{R}^2)) \), satisfying

\[
u \parallel \tau^n \parallel_{L^2(\mathbb{R}^2)} = \parallel \tau_0 \parallel_{L^2(\mathbb{R}^2)}, \quad \parallel u^n \parallel^2_{L^{\infty}(0,T;L^2(\mathbb{R}^2))} + \nu \parallel \nabla u^n \parallel^2_{L^2(0,T;L^2(\mathbb{R}^2))} \leq \parallel u_0 \parallel^2_{L^2(\mathbb{R}^2)} + \nu^{-1} T \parallel \tau_0 \parallel^2_{L^2(\mathbb{R}^2)}, \tag{9}\]

for any time \( T > 0 \).

Furthermore, one can remark that \( u^n \) does belongs to a more refined functional space, namely:

**Proposition 4.3.** For any integer \( n \in \mathbb{N} \) the solution \( u^n \) fulfills

\[
u \parallel u^n \parallel_{L^{\infty}(0,T;L^2(\mathbb{R}^2))} \leq \parallel u_0 \parallel_{L^2(\mathbb{R}^2)} + \left( \parallel u_0 \parallel^2_{L^2(\mathbb{R}^2)} + \nu^{-\frac{1}{2}} T \frac{1}{2} \parallel \tau_0 \parallel_{L^2(\mathbb{R}^2)} + \nu^{-1} T \parallel \tau_0 \parallel^2_{L^2(\mathbb{R}^2)} \right),\]
Proof. Thanks to the mild formulation (3), the velocity field \( u^n \) can be decomposed into three terms \( u^n = u^L + u^n_1 + u^n_2 \). The heat kernel allows to initially estimate the initial term \( u^L \) as follows:

$$
\| u^L \|_{L^\infty(0,T;L^2(\mathbb{R}^2))} \leq \| u_0 \|_{L^2(\mathbb{R}^2)}.
$$

Next, we can estimate \( u^n_1 \) in \( \tilde{L}^\infty(0,T;L^2(\mathbb{R}^2)) \). We apply the dyadic block \( \hat{\Delta}_q \) to \( u^n_1 \), for a fixed integer \( q \in \mathbb{Z} \). We thus gather

$$
\| \hat{\Delta}_q u^n_1 \|_{L^2(\mathbb{R}^2)} \lesssim \int_0^T 2^q e^{-c(t-s)\nu 2^{2q}} \| \hat{\Delta}_q (u^n(s) \otimes u^n(s)) \|_{L^2(\mathbb{R}^2)} \, ds,
$$

hence, the Young inequality applied to the last convolution term leads to

$$
\| \hat{\Delta}_q u^n_1 \|_{L^\infty(0,T;L^2(\mathbb{R}^2))} \leq C \| \hat{\Delta}_q (u^n \otimes u^n) \|_{L^2(0,T;L^2(\mathbb{R}^2))}
\leq C v^{-\frac{1}{2}} \| u^n(t) \|_{L^2(\mathbb{R}^2)} \|_{L^2(0,T)}
\leq C \| u^n \|_{L^\infty(0,T;L^2(\mathbb{R}^2))} v^{-\frac{1}{2}} \| \nabla u^n \|_{L^2(0,T;L^2(\mathbb{R}^2))}
\leq C \left( \| u_0 \|_{L^2(\mathbb{R}^2)}^2 + v^\nu \| \tau_0 \|_{L^2(\mathbb{R}^2)} \right).
$$

Taking the supremum with respect to the parameter \( q \in \mathbb{Z} \), we eventually get that

$$
\| u^n_1 \|_{L^\infty(0,T;L^2(\mathbb{R}^2))} \leq C \left( \| u_0 \|_{L^2(\mathbb{R}^2)}^2 + v^{-\frac{1}{2}} T \| \tau_0 \|_{L^2(\mathbb{R}^2)}^2 \right),
$$

for any integer \( n \in \mathbb{N} \) and for a suitable positive constant \( C \). Next we deal with \( u^n_2 \) and we remark that

$$
\| \hat{\Delta}_q u^n_2 \|_{L^2(\mathbb{R}^2)} \lesssim \int_0^T 2^q e^{-c(t-s)\nu 2^{2q}} \| \hat{\Delta}_q \tau^n(s) \|_{L^2(\mathbb{R}^2)} \, ds,
\lesssim v^{-\frac{1}{2}} \| \hat{\Delta}_q \tau^n \|_{L^2(0,T;L^2(\mathbb{R}^2))} \leq v^{-\frac{1}{2}} \| \tau^n \|_{L^2(0,T;L^2(\mathbb{R}^2))} \leq v^{-\frac{1}{2}} \| \tau_0 \|_{L^2(\mathbb{R}^2)} T^{\frac{1}{2}},
$$

therefore

$$
\| u^n_2 \|_{L^\infty(0,T;L^2(\mathbb{R}^2))} \leq C v^{-1} \| \tau_0 \|_{L^2(\mathbb{R}^2)} T^{\frac{1}{2}},
$$

and this concludes the proof of the proposition. \( \Box \)

**Lemma 4.4.** For any positive integer \( n \in \mathbb{N} \) and for any positive time \( T > 0 \), the velocity field \( u^n \) satisfies the class affinity

$$
u^n \otimes u^n \in \tilde{L}^\infty(0,T;B_{2,1}^\frac{1}{2}).$$

Furthermore, the following bound holds true

$$
\| u^n \otimes u^n \|_{L^\frac{2}{1}(0,T;B_{2,1}^\frac{1}{2})} \leq C v^{-\frac{3}{2}} \left( \| u_0 \|_{L^2(\mathbb{R}^2)} + \| u_0 \|_{L^2(\mathbb{R}^2)} + v^{-\frac{1}{2}} T^{\frac{1}{2}} \| \tau_0 \|_{L^2(\mathbb{R}^2)} + v^{-1} T \| \tau_0 \|_{L^2(\mathbb{R}^2)} \right),
$$

for a positive constant that does not depend on the index \( n \in \mathbb{N} \).

**Proof.** We first claim that for any integer \( n \in \mathbb{N} \) the approximate velocity field \( u^n \) belongs to \( \tilde{L}^\infty(0,T;H^\frac{1}{2}) \), for any time \( T > 0 \). Thanks to Propositions 4.2 and 4.3, \( u^n \in \tilde{L}^\infty(0,T;L^2(\mathbb{R}^2)) \), with \( \nabla u^n \in L^2(0,T;L^2(\mathbb{R}^2)) \). Furthermore, a standard interpolation yields

$$
2^{\frac{3}{4}q} \| \hat{\Delta}_q u^n \|_{L^\frac{8}{5}(0,T;L^2(\mathbb{R}^2))} \lesssim \| \hat{\Delta}_q u^n \|_{L^\infty(0,T;L^2(\mathbb{R}^2))}^{\frac{1}{4}} \left( 2^{\frac{3}{4}q} \| \hat{\Delta}_q u^n \|_{L^2(0,T;L^2(\mathbb{R}^2))} \right)^{\frac{3}{4}},
$$
for any dyadic block of index \( q \in \mathbb{Z} \). Taking the square of the above identity, applying a Cauchy-Schwartz inequality and taking the sum for \( q \in \mathbb{Z} \), we hence deduce that

\[
\| u^n \|_{L^8(0,T;H^{\frac{4}{3}}(\mathbb{R}^2))}^2 = \sum_{q \in \mathbb{Z}} 2^q \| \hat{\Delta}_q u^n \|_{L^8(0,T;L^2(\mathbb{R}^2))}^2 \\
\lesssim v^{-\frac{3}{8}} \sum_{q \in \mathbb{Z}} \| \hat{\Delta}_q u^n \|_{L^8(0,T;L^2(\mathbb{R}^2))}^2 \left( v^{-\frac{1}{2}} 2^q \| \hat{\Delta}_q u^n \|_{L^2(0,T;L^2(\mathbb{R}^2))} \right)^{\frac{3}{8}} \\
\lesssim v^{-\frac{3}{8}} \left( \| u^n \|_{L^\infty(0,T;L^2(\mathbb{R}^2))}^2 + v \| u^n \|_{L^2(0,T;H^1(\mathbb{R}^2))}^2 \right) \\
\lesssim v^{-\frac{3}{8}} \left( \| u_0 \|_{L^2(\mathbb{R}^2)}^2 + \| u_0 \|_{L^2(\mathbb{R}^2)}^2 + v^{-\frac{1}{2}} T^{-\frac{1}{2}} \| \tau_0 \|_{L^2(\mathbb{R}^2)}^2 + v^{-1} T^{-1} \| \tau_0 \|_{L^2(\mathbb{R}^2)}^2 \right).
\]

We hence claim that \( u^n \otimes u^n \) belongs to \( \Hat{L}^{\frac{4}{3}}(0,T;\Hat{B}^{\frac{1}{2},1}_{2,1}) \). Indeed, making use of the Bony decomposition

\[
\| u^n \otimes u^n \|_{L^8(0,T;\Hat{B}^{\frac{1}{2},1}_{2,1})} \leq \sum_{q} 2^q \| \hat{\Delta}_q (u^n \otimes u^n) \|_{L^8(0,T;L^2(\mathbb{R}^2))} + \sum_{q} 2^q \| \hat{\Delta}_q (\hat{\mathcal{R}}(u^n, u^n)) \|_{L^8(0,T;L^2(\mathbb{R}^2))}.
\]

We then control the first term by

\[
2^q A_q \lesssim \sum_{|j-q| \leq 5} 2^q \| \hat{S}_{j-1} u^n \|_{L^8(0,T;L^\infty(\mathbb{R}^2))} \| \hat{\Delta}_j u^n \|_{L^8(0,T;L^2(\mathbb{R}^2))} \\
\lesssim \sum_{|j-q| \leq 5} 2^q \| \hat{S}_{j-1} u^n \|_{L^8(0,T;L^\infty(\mathbb{R}^2))} \| \hat{\Delta}_j u^n \|_{L^8(0,T;L^2(\mathbb{R}^2))} \\
\lesssim \sum_{|j-q| \leq 5} 2^{-\frac{q}{2}} \| \hat{S}_{j-1} u^n \|_{L^8(0,T;L^\infty(\mathbb{R}^2))} 2^\frac{q}{2} \| \hat{\Delta}_j u^n \|_{L^8(0,T;L^2(\mathbb{R}^2))}.
\]

Hence, taking the sum as \( q \in \mathbb{Z} \) we gather that

\[
\sum_{q \in \mathbb{Z}} 2^q A_q \lesssim \left( \sum_{j \in \mathbb{Z}} 2^{-\frac{q}{2}} \| \hat{S}_{j-1} u^n \|_{L^8(0,T;L^\infty(\mathbb{R}^2))}^2 \right)^{\frac{1}{2}} \| u^n \|_{L^8(0,T;\Hat{H}^{\frac{4}{3}}(\mathbb{R}^2))} \\
\lesssim \left( \sum_{j \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} 2^{\frac{k}{2}+q-\frac{k}{2}} \| \hat{\Delta}_k u^n \|_{L^8(0,T;L^2(\mathbb{R}^2))} \right) \right)^{\frac{1}{2}} \| u^n \|_{L^8(0,T;\Hat{H}^{\frac{4}{3}}(\mathbb{R}^2))}.
\]

and the Young inequality eventually leads to

\[
\sum_{q \in \mathbb{Z}} 2^q A_q \lesssim \| u^n \|_{L^8(0,T;\Hat{H}^{\frac{4}{3}}(\mathbb{R}^2))}.
\]
It remains to handle the term $B_q$ of the homogeneous reminder. We proceed as follows:

$$2^\frac{q}{2} B_q \lesssim 2^\frac{q}{2} \sum_{j \geq q-5} 2^j \| \hat{\Delta}_j u^n(\hat{\Delta}_j + \eta) u^n \|_{L^2_T(0,T; L^1(\mathbb{R}^2))}$$

$$\leq \sum_{j \geq q-5} 2^\frac{q}{2} |j| \| \hat{\Delta}_j u^n \|_{L^2_T(0,T; L^2(\mathbb{R}^2))} \mathcal{A}(j+\eta) \| \hat{\Delta}_j + \eta u^n \|_{L^2_T(0,T; L^2(\mathbb{R}^2))}.$$

Applying again the Young inequality we finally deduce that

$$\sum_{q \in \mathbb{Z}} 2^\frac{q}{2} B_q \lesssim \| u^n \|_{L^2_T(0,T; H^\frac{3}{2}(\mathbb{R}^2))}^2. \tag{14}$$

The Lemma is then proven plugging inequalities (14) and (13), into (11) together with (12).

\[\square\]

**Remark 4.5.** Recalling that $u^n_1(t)$ is defined by means of

$$u^n_1(t) := \int_0^t \text{div} \mathcal{P} e^{V(t-s)} \Delta u^n(s) \otimes u^n(s) ds,$$

the previous proposition yields that $u^n_2$ belongs to $\tilde{L}^\frac{3}{2}(0,T; \tilde{B}^\frac{3}{2}_{2,1})$, for any integer $n \in \mathbb{N}$ and the following estimate holds true

$$v^\frac{1}{2} \| u^n_1 \|_{\tilde{L}^\infty(0,T; \tilde{B}^\frac{3}{2}_{2,1})} + v \| u^n_1 \|_{\tilde{L}^2_T(0,T; \tilde{B}^\frac{3}{2}_{2,1})} \lesssim \| \text{div}( u^n \otimes u^n ) \|_{\tilde{L}^2_T(0,T; \tilde{B}^\frac{3}{2}_{2,1})}$$

$$\lesssim v^{-\frac{1}{2}} \left( \| u_0 \|_{L^2(\mathbb{R}^2)} + \| \tau_0 \|_{L^2_T(\mathbb{R}^2)} + v^{-\frac{1}{2}} T^\frac{1}{2} \| \tau_0 \|_{L^2(\mathbb{R}^2)} + v^{-1} T \| \tau_0 \|_{L^2(\mathbb{R}^2)} \right).$$

Similarly, we can also conclude that $u^n_1$ belongs to $\tilde{L}^1(0,T; \tilde{B}^\frac{1}{2}_{2,1}) \cap L^2(0,T; \tilde{B}^\frac{1}{2}_{2,1})$, satisfying

$$v^\frac{1}{2} \| u^n_1 \|_{\tilde{L}^1(0,T; \tilde{B}^\frac{1}{2}_{2,1})} + v^\frac{1}{2} \| u^n_1 \|_{\tilde{L}^2_T(0,T; \tilde{B}^\frac{1}{2}_{2,1})} \lesssim$$

$$\lesssim v^{-\frac{1}{2}} \left( \| u_0 \|_{L^2(\mathbb{R}^2)} + \| \tau_0 \|_{L^2_T(\mathbb{R}^2)} + v^{-\frac{1}{2}} T^\frac{1}{2} \| \tau_0 \|_{L^2(\mathbb{R}^2)} + v^{-1} T \| \tau_0 \|_{L^2(\mathbb{R}^2)} \right). \tag{15}$$

We now perform some suitable bounds for the component of the velocity field $u^n_2(t)$, which is defined by means of

$$u^n_2(t) = \int_0^t \text{div} \mathcal{P} e^{V(t-s)} \Delta J_n \tau^n(s) ds.$$

**Lemma 4.6.** For any positive integer $n \in \mathbb{N}$ the following class affinity holds true

$$u^n_2 \in \tilde{L}^\infty(0,T; \tilde{B}^0_{\infty,2}) \cap \tilde{L}^1(0,T; \tilde{B}^2_{\infty,2}),$$

$$u^n_2 \in \tilde{L}^\infty(0,T; \tilde{B}^0_{2,1}) \cap \tilde{L}^2_T(0,T; \tilde{B}^2_{2,1}).$$

Furthermore

$$\| u^n_2 \|_{\tilde{L}^\infty(0,T; \tilde{B}^0_{\infty,2})} + v \| u^n_2 \|_{\tilde{L}^1(0,T; \tilde{B}^2_{\infty,2})} \leq C \| \tau^n \|_{L^1(0,T; L^2(\mathbb{R}^2))},$$

$$\| u^n_2 \|_{\tilde{L}^\infty(0,T; \tilde{B}^0_{2,1})} + v^\frac{1}{2} \| u^n_2 \|_{\tilde{L}^2_T(0,T; \tilde{B}^2_{2,1})} \leq C \| \tau^n \|_{L^1(0,T; \tilde{B}^0_{\infty,1})},$$

for a suitable positive constant $C$.

**Proof.** We restrict ourselves in proving the first statement, that is $u^n_2$ belongs to $\tilde{L}^\infty(0,T; \tilde{B}^0_{\infty,2}) \cap \tilde{L}^1(0,T; \tilde{B}^2_{\infty,2})$ and satisfies

$$\| u^n_2(t) \|_{\tilde{L}^\infty(0,T; \tilde{B}^0_{\infty,2})} + v \| u^n_2 \|_{\tilde{L}^1(0,T; \tilde{B}^2_{\infty,2})} \lesssim \| \tau^n \|_{L^1(0,T; L^2(\mathbb{R}^2))}.$$
The second part of the Lemma can indeed be achieved with a similar procedure. Applying the dyadic bloc \( \hat{\Delta}_q \) on \( u^n_2(t) \) and taking the \( L^\infty(\mathbb{R}^2) \) norm, one has
\[
\| \hat{\Delta}_q u^n_2 \|_{L^\infty(0,T;L^\infty(\mathbb{R}^2))} \lesssim \int_0^T e^{c(t-s)2^{2q}} \| \hat{\Delta}_q \tau^n(s) \|_{L^\infty(\mathbb{R}^2)} ds \lesssim \int_0^T \| \hat{\Delta}_q \tau^n(s) \|_{L^2(\mathbb{R}^2)} ds
\]
from which
\[
\| u^n_2 \|_{L^\infty(0,T;B_{\infty,2}^0)} = \left( \sum_{q \in \mathbb{Z}} \| \hat{\Delta}_q u^n_2(t) \|_{L^\infty(0,T;L^2(\mathbb{R}^2))}^2 \right)^{\frac{1}{2}} \lesssim \| \tau^n \|_{L^1(0,T;B_{\infty,2}^0)} \lesssim \| \tau^n \|_{L^1(0,T;L^2(\mathbb{R}^2))}.
\]
Furthermore
\[
2^{2q} \| \hat{\Delta}_q u^n_2 \|_{L^\infty(\mathbb{R}^2)} \lesssim \int_0^T 2^{2q} e^{-c(t-s)2^{2q}} \| \hat{\Delta}_q \tau^n(s) \|_{L^\infty(\mathbb{R}^2)} ds \lesssim \int_0^T 2^{2q} e^{-c(t-s)2^{2q}} \| \hat{\Delta}_q \tau^n(s) \|_{L^2(\mathbb{R}^2)} ds,
\]
thus applying the Young inequality we gather that
\[
2^{2q} \| \hat{\Delta}_q u^n_2 \|_{L^1(0,T;L^\infty(\mathbb{R}^2))} \lesssim \nu^{-1} \| \hat{\Delta}_q \tau^n \|_{L^1(0,T;L^2(\mathbb{R}^2))}
\]
and taking the sum as \( q \in \mathbb{Z} \)
\[
\nu \| u^n_2 \|_{L^1(0,T;B_{\infty,2}^0)} \lesssim \| \tau^n \|_{L^1(0,T;L^2(\mathbb{R}^2))}.
\]

**Proposition 4.7.** For any positive integer \( n \in \mathbb{N} \) and for any positive time \( T > 0 \), the velocity field \( u^n \) satisfies the class affinity
\[
div (u^n \otimes u^n) \in L^1(0,T;\dot{B}_{\infty,1}^{-1}).
\]
Furthermore, the following bound holds true
\[
\| \text{div}(u^n \otimes u^n) \|_{L^1(0,T;\dot{B}_{\infty,1}^{-1})} \leq \Psi_{1,\nu}(T, u_0, \tau_0) + \Psi_{2,\nu}(T, u_0, \tau_0) \| \tau^n \|_{L^1(0,T;\dot{B}_{\infty,1}^{-1})},
\]
for a suitable positive constant \( C \), where the smooth-in-time functions \( \Psi_{1,\nu}(T, u_0, \tau_0) \) and \( \Psi_{2,\nu}(T, u_0, \tau_0) \) are defined by
\[
\Psi_{1,\nu}(T, u_0, \tau_0) = \| u_0 \|_{L^2(\mathbb{R}^2)} + \| u_0 \|_{L^2(\mathbb{R}^2)} + \nu^{-\frac{1}{2}} \| \tau_0 \|_{L^2(\mathbb{R}^2)} + \nu^{-1} \| \tau_0 \|_{L^2(\mathbb{R}^2)},
\]
\[
\Psi_{2,\nu}(T, u_0, \tau_0) = C \left\{ \nu^{-\frac{2}{3}} \Phi_{\nu}(T, u_0, \tau_0)^{2} + \nu^{-\frac{1}{3}} \Phi_{\nu}(T, u_0, \tau_0) \| u_0 \|_{L^2(\mathbb{R}^2)} \right\},
\]
for a suitable positive constant \( C \).

**Proof.** We are now in the position to deal with \( \text{div}(u^n \otimes u^n) \) in the functional space \( L^1(0,T;\dot{B}_{\infty,1}^{-1}) \). We keep on using the standard decomposition \( u^n = u^n_1 + u^n_2 + u^n_L \), thus our estimate reduces to
\[
\| \text{div}(u^n \otimes u^n) \|_{L^1(0,T;\dot{B}_{\infty,1}^{-1})} \lesssim \| \text{div}(u^n_1 \otimes u^n_1) \|_{L^1(0,T;\dot{B}_{\infty,1}^{-1})} + \| u^n_1 \cdot \nabla u^n_1 \|_{L^1(0,T;\dot{B}_{\infty,1}^{-1})} + \| u^n_1 \cdot \nabla u^n_L \|_{L^1(0,T;\dot{B}_{\infty,1}^{-1})} + \| u^n_2 \cdot \nabla u^n_2 \|_{L^1(0,T;\dot{B}_{\infty,1}^{-1})} + \| u^n_2 \cdot \nabla u^n_L \|_{L^1(0,T;\dot{B}_{\infty,1}^{-1})} + \| u^n_L \cdot \nabla u^n_1 \|_{L^1(0,T;\dot{B}_{\infty,1}^{-1})} + \| u^n_L \cdot \nabla u^n_L \|_{L^1(0,T;\dot{B}_{\infty,1}^{-1})}.
\]
We hence control any term on the right-hand side of the above inequality. We first remark that
\[
\| \text{div}(u^n_1 \otimes u^n_1) \|_{L^1(0,T;\dot{B}_{\infty,1}^{-1})} \lesssim \| \text{div}(u^n_1 \otimes u^n_1) \|_{L^1(0,T;\dot{B}_{2,1}^{-1})} \lesssim \| u^n_1 \otimes u^n_1 \|_{L^1(0,T;\dot{B}_{2,1}^{-1})} \lesssim \| u^n_1 \|_{L^2(0,T;\dot{B}_{2,1}^{-1})}^2,
\]
hence thanks to the Remark 4.5 together with inequality [15], we obtain
\[
\| \text{div}(u^n_1 \otimes u^n_1) \|_{L^1(0,T;\dot{B}_{\infty,1}^{-1})} \lesssim \nu^{-\frac{2}{3}} \left( \| u_0 \|_{L^2(\mathbb{R}^2)} + \| u_0 \|_{L^2(\mathbb{R}^2)}^2 + \nu^{-\frac{1}{2}} \| \tau_0 \|_{L^2(\mathbb{R}^2)} + \nu^{-1} \| \tau_0 \|_{L^2(\mathbb{R}^2)}^2 \right).
\]
Now, recalling the embedding $\tilde{B}_{2,1}^0 \hookrightarrow \tilde{B}_{\infty,1}^{-1}$ in dimension two, together with the continuity of the product within

$$L^4(0, T; \tilde{B}_{2,1}^\frac{1}{2}) \times L^\infty(0, T; \tilde{B}_{2,2}^\frac{1}{2}) \to L^1(0, T; \tilde{B}_{2,1}^0),$$

we gather

$$\| u_1^T \cdot \nabla u_1^L \|_{L^1(0, T; \tilde{B}_{2,1}^0)} \lesssim \| u_1^T \cdot \nabla u_1^L \|_{L^1(0, T; \tilde{B}_{2,1}^\frac{1}{2})} \lesssim \| u_1^T \|_{L^4(0, T; \tilde{B}_{2,1}^\frac{1}{2})} \| \nabla u_1^L \|_{L^\infty(0, T; \tilde{B}_{2,2}^\frac{1}{2})} \lesssim v^{-\frac{1}{10}} \left( \| u_0 \|_{L^2(\mathbb{R}^2)} + \| u_0 \|_{L^2(\mathbb{R}^2)} + v^{-\frac{1}{2}} T^\frac{1}{2} \right) \| \tau_0 \|_{L^2(\mathbb{R}^2)} + v^{-1} T \| \tau_0 \|_{L^2(\mathbb{R}^2)} \| u_0 \|_{L^2(\mathbb{R}^2)}.

Similarly, the following bound holds true

$$\| u_1^T \cdot \nabla u_1^L \|_{L^1(0, T; \tilde{B}_{2,1}^0)} \lesssim \| u_1^T \cdot \nabla u_1^L \|_{L^1(0, T; \tilde{B}_{2,1}^\frac{1}{2})} \lesssim \| u_1^T \|_{L^4(0, T; \tilde{B}_{2,1}^\frac{1}{2})} \| \nabla u_1^L \|_{L^\infty(0, T; \tilde{B}_{2,2}^\frac{1}{2})} \lesssim v^{-\frac{1}{10}} \| u_0 \|_{L^2(\mathbb{R}^2)} \left( \| u_0 \|_{L^2(\mathbb{R}^2)} + \| u_0 \|_{L^2(\mathbb{R}^2)} + v^{-\frac{1}{2}} T^\frac{1}{2} \right) \| \tau_0 \|_{L^2(\mathbb{R}^2)} + v^{-1} T \| \tau_0 \|_{L^2(\mathbb{R}^2)} \| u_0 \|_{L^2(\mathbb{R}^2)}.

We now take into account

$$\| \text{div}(u_1^T \otimes u_1^L) \|_{L^1(0, T; \tilde{B}_{\infty,1}^{-1})} \lesssim \| u_1^T \otimes u_1^L \|_{L^1(0, T; \tilde{B}_{\infty,1}^0)},$$

hence, applying Proposition 2.15 together with inequality (10), we gather

$$\| \text{div}(u_1^T \otimes u_1^L) \|_{L^1(0, T; \tilde{B}_{\infty,1}^{-1})} \lesssim \| u_1^T \|_{L^\infty(0, T; L^2(\mathbb{R}^2))} \| u_1^L \|_{L^1(0, T; \tilde{B}_{\infty,1}^0)} \lesssim v^{-2} T^\frac{1}{2} \| \tau_0 \|_{L^2(\mathbb{R}^2)} \| \tau_0 \|_{L^1(0, T; \tilde{B}_{\infty,1}^0)}

Next, we estimate the following term

$$\| u_1^T \cdot \nabla u_1^L \|_{L^1(0, T; \tilde{B}_{\infty,1}^{-1})} \lesssim \| u_1^T \cdot \nabla u_1^L \|_{L^1(0, T; \tilde{B}_{2,1}^0)} \lesssim \| u_1^T \|_{L^4(0, T; \tilde{B}_{2,1}^\frac{1}{2})} \| \nabla u_1^L \|_{L^\infty(0, T; \tilde{B}_{2,2}^\frac{1}{2})} \lesssim v^{-\frac{1}{10}} \left( \| u_0 \|_{L^2(\mathbb{R}^2)} + \| u_0 \|_{L^2(\mathbb{R}^2)} + v^{-\frac{1}{2}} T^\frac{1}{2} \right) \| \tau_0 \|_{L^2(\mathbb{R}^2)} + v^{-1} T \| \tau_0 \|_{L^2(\mathbb{R}^2)} \| u_0 \|_{L^2(\mathbb{R}^2)}.

So that, applying Lemma 4.6 and Remark 4.5

$$\| u_1^T \cdot \nabla u_1^L \|_{L^1(0, T; \tilde{B}_{\infty,1}^{-1})} \lesssim \| u_1^T \cdot \nabla u_1^L \|_{L^1(0, T; \tilde{B}_{2,1}^0)} \lesssim v^{-\frac{1}{10}} \left( \| u_0 \|_{L^2(\mathbb{R}^2)} + \| u_0 \|_{L^2(\mathbb{R}^2)} + v^{-\frac{1}{2}} T^\frac{1}{2} \right) \| \tau_0 \|_{L^2(\mathbb{R}^2)} + v^{-1} T \| \tau_0 \|_{L^2(\mathbb{R}^2)} \| u_0 \|_{L^2(\mathbb{R}^2)}.

A similar computation allows us to gather

$$\| u_1^L \cdot \nabla u_1^L \|_{L^1(0, T; \tilde{B}_{\infty,1}^{-1})} \lesssim \| u_1^L \cdot \nabla u_1^L \|_{L^1(0, T; \tilde{B}_{2,1}^0)} \lesssim \| u_1^L \|_{L^4(0, T; \tilde{B}_{2,1}^\frac{1}{2})} \| \nabla u_1^L \|_{L^\infty(0, T; \tilde{B}_{2,2}^\frac{1}{2})} \lesssim v^{-\frac{1}{10}} \left( \| u_0 \|_{L^2(\mathbb{R}^2)} + \| u_0 \|_{L^2(\mathbb{R}^2)} + v^{-\frac{1}{2}} T^\frac{1}{2} \right) \| \tau_0 \|_{L^2(\mathbb{R}^2)} + v^{-1} T \| \tau_0 \|_{L^2(\mathbb{R}^2)} \| u_0 \|_{L^2(\mathbb{R}^2)}.

We now estimate the following term

$$\| u_1^L \cdot \nabla u_1^L \|_{L^1(0, T; \tilde{B}_{\infty,1}^{-1})} \lesssim \| u_1^L \cdot \nabla u_1^L \|_{L^1(0, T; \tilde{B}_{2,1}^0)} \lesssim \| u_1^L \|_{L^4(0, T; \tilde{B}_{2,1}^\frac{1}{2})} \| \nabla u_1^L \|_{L^\infty(0, T; \tilde{B}_{2,2}^\frac{1}{2})} \lesssim v^{-1} \| u_0 \|_{L^2(\mathbb{R}^2)} \| \tau_0 \|_{L^1(0, T; \tilde{B}_{\infty,1}^0)} \| u_0 \|_{L^2(\mathbb{R}^2)}.

Similarly

$$\| u_1^L \cdot \nabla u_1^L \|_{L^1(0, T; \tilde{B}_{\infty,1}^{-1})} \lesssim \| u_1^L \cdot \nabla u_1^L \|_{L^1(0, T; \tilde{B}_{2,1}^0)} \lesssim v^{-1} \| u_0 \|_{L^2(\mathbb{R}^2)} \| \tau_0 \|_{L^1(0, T; \tilde{B}_{\infty,1}^0)} \| u_0 \|_{L^2(\mathbb{R}^2)}.$$
Finally, we have that
\[ \| u^L \cdot \nabla u^L \|_{L^1(0,T;\dot{B}^{-1}_{p,1})} \lesssim \| u^L \|_{L^1(0,T;\dot{B}^{1}_{p,1})}, \] \[ \| \nabla u^L \|_{L^4(0,T;\dot{B}^{-\frac{1}{2}}_{p,2})} \lesssim \nu^{-1} \| u_0 \|_{L^2}. \]

Summarizing the above inequalities, we can then conclude the proof of the proposition.

We are now in the condition to prove that the velocity field is Lipschitz in space.

**Theorem 4.8.** For any index \( n \in \mathbb{N} \) the velocity field \( u^n \) belongs to the functional space \( L^\infty(0,T;\dot{B}^{-1}_{\infty,1}) \cap L^1(0,T;\dot{B}^{1}_{\infty,1}) \), for any time \( T > 0 \). Furthermore the following inequalities hold true
\[ \| u^n \|_{L^1(0,T;\dot{B}^{-1}_{\infty,1})} \leq \Upsilon^1_v(T, u_0, \tau_0), \] \[ \| u^n \|_{L^\infty(0,T;\dot{B}^{-1}_{\infty,1})} \leq \Upsilon^2_v(T, u_0, \tau_0). \]

**Proof.** Recalling the mild formulation of the velocity field \( u^n(t) \)
\[ u^n(t) = \exp \nu \Delta J_p u_0 + \int_0^t \mathcal{P}_\nu \exp \nu(t-s) \Delta \mathcal{D} \left( u^n(s) \otimes u^n(s) \right) \, ds + \int_0^t \mathcal{P}_\nu \exp \nu(t-s) \Delta \mathcal{D} \nu \, ds, \]
hence
\[ \| u^n \|_{L^1(0,T;\dot{B}^{-1}_{\infty,1})} \leq \| u_0 \|_{\dot{B}^{-1}_{\infty,1}} + C \| \mathcal{D} \left( u^n \otimes u^n \right) \|_{L^1(0,T;\dot{B}^{-1}_{\infty,1})} + C \| \mathcal{D} \tau^n \|_{L^1(0,T;\dot{B}^{-1}_{\infty,1})}. \]

Next, applying Proposition 4.7 we deduce that
\[ \| u^n \|_{L^\infty(0,T;\dot{B}^{-1}_{\infty,1})} \leq \| u^n \|_{L^1(0,T;\dot{B}^{-1}_{\infty,1})} + \Psi_1 \nu \left( T, u_0, \tau_0 \right) + C \| \Psi_2 \nu \left( T, u_0, \tau_0 \right) \|_{\dot{B}^{-1}_{\infty,1}}. \]

Hence, Lemma 3.2 allows us to gather
\[ \| u^n \|_{L^\infty(0,T;\dot{B}^{-1}_{\infty,1})} \leq \| u_0 \|_{\dot{B}^{-1}_{\infty,1}} + \Psi_1 \nu \left( T, u_0, \tau_0 \right) + C \| \Psi_2 \nu \left( T, u_0, \tau_0 \right) \|_{\dot{B}^{-1}_{\infty,1}} + C \| \tau^n \|_{\dot{B}^{-1}_{\infty,1}} \int_0^T \left( \Psi_2 \nu \left( t, u_0, \tau_0 \right) + C \right) u^n \|_{L^1(0,T;\dot{B}^{-1}_{\infty,1})} \, dt. \]

Applying the Gronwall inequality we hence deduce that
\[ \nu \| u^n \|_{L^1(0,T;\dot{B}^{-1}_{\infty,1})} \leq \Upsilon^1_v(T, u_0, \tau_0) \]
where the smooth function \( \Upsilon^1_v(T, u_0, \tau_0) \) grows in time \( T \) as in definition 1.2.

\[ \square \]

**4.2. Propagation of \( \dot{B}^{\frac{3}{p}-1}_{\frac{3}{p},1} \times \dot{B}^{\frac{3}{p}-1}_{\frac{3}{p},1} \)-regularity with index \( 1 \leq p < 2 \)**

In the previous section we establish the propagation of a Lipschitz-in-space regularity for any approximate velocity fields \( u^n \). We now use this criterion to propagate higher regularity given by the initial condition
\[ u_0 \in \dot{B}^{\frac{3}{p}-1}_{\frac{3}{p},1} \quad \text{and} \quad \tau_0 \in \dot{B}^{\frac{3}{p}-1}_{\frac{3}{p},1}, \]
where \( p \) is assumed in \( [1, \infty) \).

**Theorem 4.9.** For any index \( n \in \mathbb{N} \) and for any time \( t \in [0,T) \) the following estimate for the conformation tensor \( \tau^n \) holds true
\[ \| \tau^n(t) \|_{\dot{B}^{-\frac{1}{2}}_{\frac{3}{p},1}} \leq \| \tau_0 \|_{\dot{B}^{-\frac{1}{2}}_{\frac{3}{p},1}} \exp \left\{ C \nu^{-1} \Upsilon^1_\nu(T, u_0, \tau_0) \right\}, \]
for a suitable positive constant \( C > 0 \). Furthermore, the approximate velocity field \( u^n \) satisfies
\[ \| u^n(t) \|_{\dot{B}^{\frac{3}{p}-1}_{\frac{3}{p},1}} + \int_0^t \| u^n(s) \|_{\dot{B}^{\frac{3}{p}-1}_{\frac{3}{p},1}} \, ds \leq \| u_0 \|_{\dot{B}^{\frac{3}{p}-1}_{\frac{3}{p},1}} \exp \left\{ C \nu^{-2} \Upsilon^2_\nu(T, u_0, \tau_0) \Upsilon^1_\nu(T, u_0, \tau_0) \right\}. \]
Proof. We recall that \( \tau^n \) satisfies the equation
\[
\partial_t \tau^n + u^n \cdot \nabla \tau^n - \alpha^n \tau^n + \tau^n \omega^n = 0,
\]
hence, applying Lemma 3.1 we deduce that
\[
\| \tau^n(t) \|_{\dot{B}^0_{p,1}} \leq \| \tau_0 \|_{\dot{B}^0_{p,1}} \exp \left\{ C \| u \|_{L^1(0,t;\dot{B}^{-1}_{\infty,1})} \right\}.
\]
We hence apply the first inequality of Theorem 4.3 to eventually gather inequality (15). Similarly, we remark that \( u^n \) satisfies the Stokes equation
\[
\begin{cases}
\partial_t u^n - \nu \Delta u^n + \nabla p^n = -J^n(u^n \cdot \nabla u^n) + \text{div}J^n \tau^n, \\
\text{div} u^n = 0,
\end{cases}
\]
from which we gather that
\[
\| u^n(t) \|_{\dot{B}^0_{p,1}} + \nu \int_0^t \| u^n(s) \|_{\dot{B}^0_{p,1}} \, ds \leq \int_0^t \| u^n(s) \otimes u^n(s) \|_{\dot{B}^1_{p,1}} \, ds + \int_0^t \| \tau^n(s) \|_{\dot{B}^0_{p,1}} \, ds.
\]
Now, applying Proposition 2.14 we remark that \( u^n \otimes u^n \) can be recast as follows
\[
\| u^n \otimes u^n \|_{\dot{B}^1_{p,1}} \leq 2 \| \hat{T} u^n \|_{\dot{B}^1_{p,1}} + \| R(u^n \otimes u^n) \|_{\dot{B}^1_{p,1}} \leq C \| u^n \|_{L^\infty} \| u^n \|_{\dot{B}^0_{p,1}} + \| u^n \|_{\dot{B}^{-1}_{\infty,1}} \| u^n \|_{\dot{B}^0_{p,1}} \leq C \| u^n \|_{\dot{B}^0_{p,1}} \| u^n \|_{\dot{B}^0_{\infty,1}}.
\]
Hence, by interpolation we get
\[
\| u^n \otimes u^n \|_{\dot{B}^1_{p,1}} \leq C \| u^n \|_{\dot{B}^0_{p,1}} \| u^n \|_{\dot{B}^0_{\infty,1}} \leq C \| u^n \|_{\dot{B}^0_{p,1}} \| u^n \|_{\dot{B}^1_{p,1}} \| u^n \|_{\dot{B}^1_{\infty,1}} \| u^n \|_{\dot{B}^1_{p,1}} \| u^n \|_{\dot{B}^1_{\infty,1}} \leq \frac{\nu}{2} \| u^n \|_{\dot{B}^1_{p,1}}^2 + CV^{-1} \| u^n \|_{\dot{B}^0_{\infty,1}} \| u^n \|_{\dot{B}^1_{\infty,1}} \| u^n \|_{\dot{B}^1_{p,1}}.
\]
Replacing the above inequality into (22), we can then apply the Gronwall inequality to eventually gather that
\[
\| u^n(t) \|_{\dot{B}^0_{p,1}} + \nu \int_0^t \| u^n(s) \|_{\dot{B}^0_{p,1}} \, ds \leq \left( \| u_0 \|_{\dot{B}^0_{p,1}} + T \| \tau_0 \|_{\dot{B}^0_{p,1}} \right) \exp \left\{ C(V^{-1}) \| u^n \|_{L^\infty(0,T;\dot{B}^{-1}_{\infty,1})} + 1 \| u^n \|_{L^1(0,T;\dot{B}^{-1}_{\infty,1})} \right\} \leq \left( \| u_0 \|_{\dot{B}^0_{p,1}} + T \| \tau_0 \|_{\dot{B}^0_{p,1}} \right) \exp \left\{ C(V^{-1}) \| u_0 \|_{\dot{B}^0_{p,1}} + T \| \tau_0 \|_{\dot{B}^0_{p,1}} \right\} + 1 \right\}.
\]

4.3. Passage to the limit
To conclude the proof of Theorem 1.1, it remains to pass to the limit in system (7), as \( n \) goes to infinity. We first recall that \( \{u^n\}_n \) is uniformly bounded in
\[
L^\infty_{\text{loc}}(\mathbb{R}^+, \dot{B}^{-1}_{p,1}) \cap L^1_{\text{loc}}(\mathbb{R}^+, \dot{B}^{2}_{p,1}) \quad \text{and also} \quad L^\infty_{\text{loc}}(\mathbb{R}^+, L^2(\mathbb{R}^2)) \cap L^2_{\text{loc}}(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^2)),
\]
while the sequence \( \{\tau^n\}_n \) is uniformly bounded into
\[
L^\infty_{\text{loc}}(\mathbb{R}^+, \dot{B}^{2}_{p,1}) \cap L^\infty_{\text{loc}}(\mathbb{R}^+, L^2(\mathbb{R}^2)).
\]
We hence proceed by extrapolating a convergent subsequence within a suitable functional space. To this end we consider the classical Aubin-Lions Lemma.
The lemma being stated, we can focalize ourselves to the choice of the functional spaces where

\[ q \in B \] is a compact set through the following interpolation:

\[ \text{Denoting by } r = \text{any } 2 \leq s < +\infty \text{ and } q > 2. \]

The compactness embedding \( X_0 \hookrightarrow X \) of Lemma 4.10 is satisfied thanks to the following Proposition (cf. [1], Corollary 2.96).

**Proposition 4.11.** For any \((s', s)\) in \(\mathbb{R}^2\) such that \(s' < s\) and any compact set \(K\) of \(\mathbb{R}^d\), the space \(B^{q'}_{q,\infty}(K)\) is compactly embedded in \(B^q_{q,1}(K)\).

Since \(H^1(\mathbb{R}^2)\) is continuously embedded into \(L^q(\mathbb{R}^2)\), we gather that \((u^n)\) is uniformly bounded into \(L^2_{loc}(\mathbb{R}^+, B^{2/q}_{q,1})\), where \(B^{2/q}_{q,1}\) stands for the non-homogeneous Besov space. Furthermore, since \(B^{2/q}_{q,1}\) is continuously embedded into \(L^\infty(\mathbb{R}^2)\), the sequence \((\tau^n)\) is uniformly bounded into \(L^\infty_{loc}(\mathbb{R}^+, L^q(\mathbb{R}^2))\) and so into \(L^\infty_{loc}(\mathbb{R}^+, B^{2/q}_{q,1})\) which is embedded into \(L^2_{loc}(\mathbb{R}^+, B^{2/q-2}_{q,1})\). This allows us to conclude that

\[ \| \text{div } \tau^n \|_{L^2_{loc}(\mathbb{R}^+, B^{2/q-2}_{q,1})} \leq C, \]

for a suitable constant \(C\) that does not depend on the index \(n \in \mathbb{N}\). Now, we claim that \(\text{div } j^n(u^n \otimes u^n)\) is uniformly bounded in \(L'(0, T; B^{2/q-2}_{q,1}(K))\), for a suitable positive index \(r > 1\). We first remark that

\[ \| \text{div } j^n(u^n \otimes u^n) \|_{B^{2/q-2}_{q,1}(K)} \leq \| j^n(u^n \otimes u^n) \|_{B^{2/q-1}_{q,1}(K)} \leq \| j^n(u^n \otimes u^n) \|_{B^{2/q-1}_{q,1}} \]

\[ \leq \| \hat{S}_{r-1} j^n(u^n \otimes u^n) \|_{L^q(\mathbb{R}^2)} + \| (\text{Id} - \hat{S}_{r-1}) j^n(u^n \otimes u^n) \|_{B^{2/q-1}_{q,1}} \]

\[ \leq \| j^n(u^n \otimes u^n) \|_{L^q(\mathbb{R}^2)} + \| (\text{Id} - \hat{S}_{r-1}) j^n(u^n \otimes u^n) \|_{B^{2/q-1}_{q,1}}, \]

where \(\varepsilon \in (0, 1)\) such that \(2/q - \varepsilon > 0\). Hence we eventually gather that

\[ \| \text{div } j^n(u^n \otimes u^n) \|_{B^{2/q-2}_{q,1}(K)} \leq C \left( \| (u^n \otimes u^n) \|_{L^q(\mathbb{R}^2)} + \| (u^n \otimes u^n) \|_{B^{2/q-1}_{q,1}} \right). \]

Now, making use of the continuity of the product between the functional spaces

\[ L_{loc}^{1/(1-\varepsilon)}(\mathbb{R}^+, B^{2/q-1}_{q,1}) \times L_{loc}^{2}(\mathbb{R}^+, B^{2/q}_{q,1}) \to L_{loc}^{2/q}(\mathbb{R}^+, B^{2/q-\varepsilon}_{q,1}), \]

the term \(u^n \otimes u^n\) is uniformly bounded into \(L_{loc}^{1/(1-\varepsilon)}(\mathbb{R}^+, B^{2/q-1}_{q,1})\). Furthermore, we can bound the \(L^q(\mathbb{R}^2)\) norm through the following interpolation:

\[ \| u^n \otimes u^n \|_{L^q(\mathbb{R}^2)} \leq C \| u^n \|_{L^2(\mathbb{R}^2)}^{2/q} \| \nabla u^n \|_{L^2(\mathbb{R}^2)}^{2/q} \leq C \| u^n \|_{L^2(\mathbb{R}^2)} \| \nabla u^n \|_{L^2(\mathbb{R}^2)}^{2/q} \in L^q_{loc}(\mathbb{R}^+). \]

Denoting by \(r = \min\{2, 1/(1-\varepsilon), q/(q-1)\} > 1\), we gather that the sequence \((\partial_t u^n)\) satisfying

\[ \partial_t u^n = \Delta u^n - J^n(u^n \cdot \nabla u^n) + \nabla p^n + \text{div } \tau^n \]
is uniformly bounded in $L^p_{loc}(\mathbb{R}^+, B^{2/p-2}_{p,1}(K))$. Hence, the Aubin-Lion Lemma [4, 10] and the generality of the compact set $K$ allow us to extract a convergent subsequence $(u^n)_{n} \subset (u^k)_{n}$ such that
\[
    u^n_k \to u \quad \text{in} \quad L^\infty(0, T; (B^{\frac{2}{p}-2}_{p,1})),
\]
Since $(u^n)_{n}$ satisfies estimates (19) and (9), also the limit $u$ belongs to the functional space given by (21).

We now claim that $(\partial_t \tau^n)_{n}$ is uniformly bounded in the non-homogeneous functional space $L^p_{loc}(\mathbb{R}^+, B^{2/p-1}_{p,1}(K))$.

First we recall that the conformation tensor satisfies
\[
    \partial_t \tau^n = -u^n \cdot \nabla \tau^n + \omega^n \tau^n - \tau^n \omega^n,
\]
therefore
\[
    \| \partial_t \tau^n \|_{B^p_{p,1}} \leq \| u^n \|_{B^p_{p,1}} \| \nabla \tau^n \|_{B^p_{p,1}} + \| \nabla u^n \|_{B^p_{p,1}} \| \tau^n \|_{B^p_{p,1}} \in L^2_{loc}(\mathbb{R}^+),
\]
Furthermore
\[
    \| -\omega^n \tau^n + \tau^n \omega^n \|_{L^p(\mathbb{R}^2)} \lesssim \| \nabla u^n \|_{L^{2p}(\mathbb{R}^2)} \| \tau^n \|_{L^{2p}(\mathbb{R}^2)}
\]
\[
    \lesssim \| \nabla u^n \|_{L^{2p}(\mathbb{R}^2)} \| \nabla u^n \|_{L^\infty(\mathbb{R}^2)} \| \tau^n \|_{L^{2p}(\mathbb{R}^2)} \| \tau^n \|_{L^\infty(\mathbb{R}^2)}
\]
\[
    \lesssim \| \nabla u^n \|_{L^{2p}(\mathbb{R}^2)} \| \nabla u^n \|_{L^\infty(\mathbb{R}^2)} \| \tau^n \|_{L^{2p}(\mathbb{R}^2)} \| \tau^n \|_{L^\infty(\mathbb{R}^2)} \in L^2_{loc}(\mathbb{R}^2),
\]
Finally, one has
\[
    \| \Delta_{-1}\text{div}(u^n \otimes \tau^n) \|_{L^p(\mathbb{R}^2)} \lesssim \| u^n \otimes \tau^n \|_{L^p(\mathbb{R}^2)} \lesssim \| u^n \|_{L^{2p}(\mathbb{R}^2)} \| \tau^n \|_{L^{2p}(\mathbb{R}^2)}
\]
\[
    \lesssim \| u^n \|_{L^{2p}(\mathbb{R}^2)} \| u^n \|_{L^\infty(\mathbb{R}^2)} \| \tau^n \|_{L^{2p}(\mathbb{R}^2)} \| \tau^n \|_{L^\infty(\mathbb{R}^2)}
\]
\[
    \lesssim \| u^n \|_{L^{2p}(\mathbb{R}^2)} \| u^n \|_{L^\infty(\mathbb{R}^2)} \| \tau^n \|_{L^{2p}(\mathbb{R}^2)} \| \tau^n \|_{L^\infty(\mathbb{R}^2)} \in L^2_{loc}(\mathbb{R}^2),
\]
which allows us to conclude that $(\partial_t \tau^n)_{n}$ is uniformly bounded in $L^p_{loc}(\mathbb{R}^+, B^{2/p-1}_{p,1})$. Thus, the Aubin-Lion lemma allows us to extract a convergent subsequence $(\tau^n)_{n} \subset (\tau^k)_{n}$ such that
\[
    \tau^n \to \tau \quad \text{in} \quad L^\infty(0, T; (B^{\frac{2}{p}-1}_{p,1})).
\]
Since $(\tau^n)_{n}$ satisfies estimates (18) and (9), also the limit $\tau$ belongs to the functional space given by (21).

These properties allow to pass to the limit to system (31) and thus to show that $(u, \tau)$ is a global-in-time solution of system (1). This concludes the proof of the existence part of Theorem 1.1.

4.4. Uniqueness of classical solutions in dimension two: The case $p \in [2, 4]$.

This section is devoted to the proof of the uniqueness of solutions determined by Theorem 1.1. We consider two solutions $(u_1, \tau_1)$ and $(u_2, \tau_2)$ with same initial data and satisfying the condition of Theorem 1.1
\[
    (u_1, u_2) \in L^\infty(0, T; L^2(\mathbb{R}^2) \cap B^{2/p}_{p,1}) \cap L^2(0, T; H^1(\mathbb{R}^2)) \cap L^1(0, T; B^{2/p+1}_{p,1}),
\]
\[
    (\tau_1, \tau_2) \in L^\infty(0, T; L^2(\mathbb{R}^2) \cap B^{\frac{2}{p}}_{p,1}),
\]
for any time $T > 0$. Defining by $\delta u := u_1 - u_2$ and by $\delta \tau := \tau_1 - \tau_2$, we aim in controlling $(\delta u, \delta \tau)$ in the following Chemin-Lerner spaces:
\[
    \tilde{L}^\infty(0, T; B^{\frac{2}{p}-2}_{p,1}) \cap \tilde{L}^1(0, T; B^{\frac{2}{p}}_{p,1}) \times \tilde{L}^\infty(0, T; B^{\frac{2}{p}-1}_{p,1}).
\]
We begin with introducing the system driving the evolution of \((\delta u, \delta \tau)\):
\[
\begin{align*}
\begin{cases}
\partial_t \delta \tau + u_1 \cdot \nabla \delta \tau - \omega_1 \delta \tau + \delta \tau \omega_1 &= -\delta u \cdot \nabla \tau_2 + \delta \omega \tau_2 - \tau_2 \delta \omega \\
\partial_t \delta u - \nu \Delta \delta u + \nabla \delta p &= -\delta u \cdot \nabla u_1 - u_2 \cdot \nabla \delta u + \text{div} \delta \tau \\
\text{div} \delta u &= 0 \\
(\delta u, \delta \tau)|_{t=0} &= (0, 0)
\end{cases}
\end{align*}
\]
Applying Remark (2.12) to the equation of \(\delta u\), we gather that
\[
\| \delta u \|_{L^\infty(0,T;B_{p,1}^{\frac{2}{p}-2})} + \| \delta u \|_{L^1(0,T;B_{p,1}^{\frac{2}{p}-1})} \lesssim \| \delta u \cdot \nabla u_1 + u_2 \cdot \nabla \delta u \|_{L^1(0,T;B_{p,1}^{\frac{2}{p}-2})} + \| \text{div} \delta \tau \|_{L^1(0,T;B_{p,1}^{\frac{2}{p}-2})}. \tag{22}
\]
Hence, we observe first that thanks to the free-divergence condition on \(\delta u\) and the continuity of the product between
\[
\dot{B}_{p,1}^{\frac{2}{p}-1} \times B_{p,1}^{\frac{2}{p}} \rightarrow \dot{B}_{p,1}^{\frac{2}{p}-1}, \quad p \in [1,4),
\]
the following inequality holds true:
\[
\| \delta u \cdot \nabla u_1 + u_2 \cdot \nabla \delta u \|_{L^1(0,T;B_{p,1}^{\frac{2}{p}-2})} \leq C\| \delta u \otimes (u_1, u_2) \|_{L^1(0,T;B_{p,1}^{\frac{2}{p}-1})} \]
\[
\leq C\| \delta u \|_{L^2(0,T;B_{p,1}^{\frac{2}{p}-1})} \| (u_1, u_2) \|_{L^2(0,T;B_{p,1}^{\frac{2}{p}})} \]
\[
\leq C\| \delta u \|_{L^\infty(0,T;B_{p,1}^{\frac{2}{p}-2})} \| \nabla \delta u \|_{L^1(0,T;B_{p,1}^{\frac{2}{p}-1})} \| (u_1, u_2) \|_{L^2(0,T;B_{p,1}^{\frac{2}{p}})} \]
\[
\leq C\| (u_1, u_2) \|_{L^2(0,T;B_{p,1}^{\frac{2}{p}})} \| \delta u \|_{L^\infty(0,T;B_{p,1}^{\frac{2}{p}-2})} + \frac{\nu}{100} \| \nabla \delta u \|_{L^1(0,T;B_{p,1}^{\frac{2}{p}-1})}
\]
We assume \(T\) sufficiently small satisfying
\[
C\| (u_1, u_2) \|_{L^2(0,T;B_{p,1}^{\frac{2}{p}})} \leq \frac{1}{2}
\]
hence, we can replace the last inequality into (22) and absorb any terms by the left-hand, to gather
\[
\| \delta u \|_{L^\infty(0,T;B_{p,1}^{\frac{2}{p}-2})} + \| \delta u \|_{L^1(0,T;B_{p,1}^{\frac{2}{p}})} \lesssim \| \delta \tau \|_{L^1(0,T;B_{p,1}^{\frac{2}{p}-1})} \tag{23}
\]
We now apply Lemma 3.1 to the \(\delta \tau\)-equation to gather:
\[
\| \delta \tau \|_{L^\infty(0,T;B_{p,1}^{\frac{2}{p}-1})} \leq C \left\{ \| \delta u \cdot \nabla \tau_2 \|_{L^1(0,T;B_{p,1}^{\frac{2}{p}-2})} + \| \delta \omega \tau_2 - \tau_2 \delta \omega \|_{L^1(0,T;B_{p,1}^{\frac{2}{p}-1})} \right\} \exp \left\{ C \int_0^T \| u_1(s) \|_{B_{p,1}^{\frac{2}{p}+1}} \, ds \right\}. \tag{24}
\]
First, we have
\[
\| \delta u \cdot \nabla \tau_2 \|_{L^1(0,T;B_{p,1}^{\frac{2}{p}-2})} \leq C \| \delta u \otimes \tau_2 \|_{L^1(0,T;B_{p,1}^{\frac{2}{p}-1})}
\]
\[
\leq C\| \delta u \|_{L^1(0,T;B_{p,1}^{\frac{2}{p}})} \| \tau_2 \|_{L^\infty(0,T;B_{p,1}^{\frac{2}{p}})}.
\]
Furthermore,
\[
\| \delta \omega \tau_2 - \tau_2 \delta \omega \|_{L^1(0,T;B_{p,1}^{\frac{2}{p}-2})} \leq C \| \delta \omega \|_{L^1(0,T;B_{p,1}^{\frac{2}{p}-1})} \| \tau_2 \|_{L^\infty(0,T;B_{p,1}^{\frac{2}{p}})}
\]
\[
\leq C\| \delta u \|_{L^1(0,T;B_{p,1}^{\frac{2}{p}})} \| \tau_2 \|_{L^\infty(0,T;B_{p,1}^{\frac{2}{p}})}.
\]
Replacing the above inequalities into (24), we eventually that
\[
\| \delta \tau \|_{L^\infty(0,T;B_{p,1}^{\frac{2}{p}-1})} \leq C\| \delta u \|_{L^1(0,T;B_{p,1}^{\frac{2}{p}})}, \tag{25}
\]
for a suitable positive constant $C$ that depends also on the norm of the initial data $(u_0, \tau_0)$ in $L^2(\mathbb{R}^2) \cap \dot{B}^{5/2}_{p,1}$ and $L^2(\mathbb{R}^2) \cap \dot{B}^{2/p}_{p,1}$. Hence, denoting by

$$\delta U(T) := \| \delta u \|_{\dot{L}^\infty(0,T;\dot{B}^{\frac{5}{2}}_{p,1})} + \| \delta u \|_{L^1(0,T;\dot{B}^{\frac{2}{p}}_{p,1})},$$

and replacing the above inequalities into (27), we gather that

$$\delta U(T) \leq C \int_0^T \delta U(t) \, dt,$$

for a sufficiently small $T > 0$. The Gronwall inequality yields that $\delta U(t) = 0$ for any $t \in [0, T]$, for which $u_1(t) \equiv u_2(t)$. Similarly, making use of (25) also the conformations tensors coincide $\dot{\tau}_1(t) \equiv \dot{\tau}_2(t)$ for any time $t \in [0, T]$. Since the time $T$ depends uniquely on the norm of the initial data $u_0$ and $\tau_0$, a standard bootstrap method allows to propagate the uniqueness, globally in time. This concludes the proof of Theorem 1.1 for any $p \in [1, 4]$.

4.5. Uniqueness of classical solutions in dimension two: The case $p = 4$. In order to obtain the uniqueness in the critical case $p = 4$ we have to control the difference between two solutions $(\delta u, \delta \tau)$ in the space

$$\dot{L}^\infty(0,T;\dot{B}^{\frac{5}{2}}_{4,\infty}) \cap \dot{L}^1(0,T;\dot{B}^{\frac{1}{4}}_{4,\infty}) \times \dot{L}^\infty(0,T;\dot{B}^{\frac{1}{2}}_{4,\infty}).$$

Using the system verified by $(\delta u, \delta \tau)$ we get the estimate

$$\| \delta u \|_{\dot{L}^\infty(0,T;\dot{B}^{\frac{5}{2}}_{4,\infty})} + \| \delta u \|_{L^1(0,T;\dot{B}^{\frac{1}{4}}_{4,\infty})} \lesssim \| \delta u \cdot \nabla u_1 + u_2 \cdot \nabla \delta u \|_{L^1(0,T;\dot{B}^{\frac{1}{4}}_{4,\infty})} + \| \text{div} \, \delta \tau \|_{L^1(0,T;\dot{B}^{\frac{1}{2}}_{4,\infty})},$$

and the same type of estimates as previously, the following inequality holds true:

$$\| \delta u \|_{\dot{L}^\infty(0,T;\dot{B}^{\frac{5}{2}}_{4,\infty})} \lesssim \| \delta \tau \|_{L^1(0,T;\dot{B}^{\frac{1}{2}}_{4,\infty})}. \quad (27)$$

Using the equation on $\delta \tau$ we obtain easily

$$\| \delta \tau \|_{\dot{L}^\infty(0,T;\dot{B}^{\frac{1}{4}}_{4,\infty})} \leq C \left\{ \| \delta u \cdot \nabla \tau_2 \|_{L^1(0,T;\dot{B}^{\frac{1}{4}}_{4,\infty})} + \| \text{div} \, \delta \tau_2 \|_{L^1(0,T;\dot{B}^{\frac{1}{2}}_{4,\infty})} \right\} \exp \left\{ C \int_0^T \| u_1(s) \|_{\dot{B}^{\frac{3}{4}}_{4,1}} \, ds \right\}. \quad (28)$$

First, we have

$$\| \delta u \cdot \nabla \tau_2 \|_{L^1(0,T;\dot{B}^{\frac{1}{4}}_{4,\infty})} \leq \| \delta u \|_{L^1(0,T;\dot{B}^{\frac{1}{4}}_{4,\infty})} \| \tau_2 \|_{L^\infty(0,T;\dot{B}^{\frac{1}{4}}_{4,\infty})}.$$\)

Furthermore,

$$\| \text{div} \, \delta \tau_2 \|_{L^1(0,T;\dot{B}^{\frac{1}{2}}_{4,\infty})} \leq \| \delta u \|_{L^1(0,T;\dot{B}^{\frac{1}{4}}_{4,\infty})} \| \tau_2 \|_{L^\infty(0,T;\dot{B}^{\frac{1}{2}}_{4,\infty})}.$$\)

Inserting the above inequalities into (28), we eventually deduce that

$$\| \delta \tau \|_{\dot{L}^\infty(0,T;\dot{B}^{\frac{1}{4}}_{4,\infty})} \lesssim \| \delta u \|_{L^1(0,T;\dot{B}^{\frac{1}{4}}_{4,\infty})}. \quad (29)$$

Now we use the following standard logarithmical estimate

$$\| \delta u \|_{L^1(0,T;\dot{B}^{\frac{1}{4}}_{4,\infty})} \leq \| \delta u \|_{L^1(0,T;\dot{B}^{\frac{1}{4}}_{4,\infty})} \ln \left( e + \frac{\| \text{div} \, \delta \tau \|_{L^1(0,T;\dot{B}^{\frac{1}{2}}_{4,\infty})}}{\| \delta u \|_{L^1(0,T;\dot{B}^{\frac{1}{4}}_{4,\infty})}} \right) \leq \| \delta u \|_{L^1(0,T;\dot{B}^{\frac{1}{4}}_{4,\infty})} \ln \left( e + \frac{\| u^1 + u^2 \|_{L^1(0,T;L^2)} + \| \tau_1 + \tau_2 \|_{L^1(0,T;\dot{B}^{\frac{1}{2}}_{4,\infty})}}{\| \delta u \|_{L^1(0,T;\dot{B}^{\frac{1}{4}}_{4,\infty})}} \right).$$
We now define the functions $\delta u(T) := \|\delta u\|_{L^\infty(0,T;\tilde{B}^d_{p,1})} + \|\delta u\|_{L^1(0,T;\tilde{B}^d_{4,\infty})}$, and replacing the above inequalities into (27), we gather that

$$\delta u(T) \leq C \int_0^T \delta u(t) \ln(e + \frac{C}{\delta u(t)}) dt.$$  

The uniqueness of our solutions is then an application of the classical Osgood lemma.

5. Local-in-time solutions for any initial data

This section is devoted to the proof of the results of local-in-time existence of classical solutions for the Johnson-Segalman model (1) in dimension $d \geq 3$, aiming to prove Theorem 1.3. We adopt a standard strategy, which can be summarized as follows:

- construction of global-in-time approximate solutions,
- uniform estimates on a suitable fixed (small) interval of time,
- convergence of the sequences to a solution in such an interval,
- uniqueness of the solutions.

5.1. Global-in-time approximate solutions

We first introduce the solution $u^L(t) = u^L(t, x)$ of the following linear non-stationary Stokes system:

$$\begin{align*}
\partial_t u_L - \nu \Delta u_L + \nabla p_L &= 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\
\text{div} u_L &= 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\
u u_L |_{t=0} &= u_0 & \mathbb{R}^d.
\end{align*}$$  

(30)

Since $u_0$ belongs to $\tilde{B}^d_{p,1}$, a standard approach allows us to conclude (cf. [6], Remark 3.12) that the solution $u^L(t)$ belongs to the functional framework

$$u_L \in C_b(\mathbb{R}_+, \tilde{B}^d_{p,1}) \cap L^1_{\text{loc}}(\mathbb{R}_+, \tilde{B}^d_{p,1})$$  

and

$$\nabla p_L \in L^1_{\text{loc}}(\mathbb{R}_+, \tilde{B}^d_{p,1}),$$  

satisfying the inequality

$$\|u_L(t)\|_{\tilde{B}^d_{p,1}} + \nu \int_0^t \|u_L(s)\|_{\tilde{B}^d_{p,1}} ds \leq \|u_0\|_{\tilde{B}^d_{p,1}}.$$  

We now define the functions $(\tilde{u}^0(t, x), \tau^0(t, x)) := (0, \tau_0(x))$ and we apply an iterative method to solve the following sequences of linear equations:

$$\begin{align*}
\partial_t \tau^{n+1} + u^n \cdot \nabla \tau^{n+1} - \omega^n \tau^{n+1} + \tau^{n+1} \omega^n &= 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\
\partial_t \tilde{u}^{n+1} - \nu \Delta \tilde{u}^{n+1} + \nabla \tilde{p}^{n+1} &= -u^n \cdot \nabla u^n + \text{div} \tau^{n+1} & \mathbb{R}_+ \times \mathbb{R}^d, \\
\text{div} \tilde{u}^{n+1} &= 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\
(\tilde{u}^{n+1}, \tau) |_{t=0} &= (0, \tau_0) & \mathbb{R}^d.
\end{align*}$$  

(31)

where $u^n$ stands for $u^L + \tilde{u}^n$. Being a linear system of PDE’s, the above equations are globally solvable in time. Furthermore, we claim that an induction method implies the sequence of solutions $(\tilde{u}^n, \tau^n)$ to belong to the functional framework

$$\tilde{u}^n \in C_b(\mathbb{R}_+, \tilde{B}^d_{p,1}) \cap L^1_{\text{loc}}(\mathbb{R}_+, \tilde{B}^d_{p,1}), \quad \tau^n \in C(\mathbb{R}_+, \tilde{B}^d_{p,1}).$$  

(32)
The basic case of $n = 0$ is automatically satisfied by the definition of $(\bar{u}^0, \tau^0)$, thus we focus on the induction step. Thanks to Lemma 3.3, the automatically bound for the conformation tensor $\tau^{n+1}$ holds true

$$
\| \tau^{n+1} \|_{L^\infty([0, T], \dot{B}_{p, 1}^{\alpha})} \leq \| \tau_0 \|_{\dot{B}_{p, 1}^{\alpha}} \exp \left\{ C \| u^n \|_{L^1_{L^2}(\mathbb{R}^+, \dot{B}_{p, 1}^{\alpha+1})} \right\}.
$$

Moreover, applying Remark 2.12 to the mild formulation

$$
\bar{u}^{n+1}(t) = \int_0^t \text{div} \mathcal{P} e^{(t-s)\Delta} \left( - u^n(s) \otimes u^n(s) + \tau^{n+1}(s) \right) ds,
$$

we gather that $u^{n+1}$ satisfies

$$
\| \bar{u}^{n+1} \|_{L^\infty([0, T], \dot{B}_{p, 1}^{\alpha-1})} + \| u^{n+1} \|_{L^1_{L^2}(\mathbb{R}^+, \dot{B}_{p, 1}^{\alpha+1})} \leq C \left\{ \| u^n \|^2_{L^2_{L^2}(\mathbb{R}^+, \dot{B}_{p, 1}^{\alpha})} + \| \tau^{n+1} \|_{L^1_{L^2}(\mathbb{R}^+, \dot{B}_{p, 1}^{\alpha+1})} \right\},
$$

thus the condition (32) is satisfied by induction.

5.2. Uniform estimates on a fixed small interval

In this section we deal with some suitable estimates of our approximate solutions $(u^n, \tau^n)$. We show that for a sufficiently small time $T > 0$, the norms of the solutions $(u^n, \tau^n)$ within the functional framework

$$
\bar{u}^n \in C([0, T], \dot{B}_{p, 1}^{\alpha-1}) \cap L^1(0, T; \dot{B}_{p, 1}^{\alpha+1}), \quad \tau^n \in C([0, T], \dot{B}_{p, 1}^{\alpha})
$$

are bounded uniformly in $n \in \mathbb{N}$.

Thanks to Lemma 3.3 we have that for any time $T \geq 0$ the following inequality holds true:

$$
\| \tau^{n+1} \|_{L^\infty([0, T], \dot{B}_{p, 1}^{\alpha})} \leq \| \tau_0 \|_{\dot{B}_{p, 1}^{\alpha}} \exp \left\{ C \| u^n \|_{L^1_{L^2}(0, T; \dot{B}_{p, 1}^{\alpha+1})} \right\}.
$$

Next, we define the function $\tilde{U}^n(T)$, depending on time $T \geq 0$, by means of

$$
\tilde{U}^n(T) = \| \bar{u}^n \|_{L^\infty(0, T; \dot{B}_{p, 1}^{\alpha-1})} + \| u^n \|_{L^1(0, T; \dot{B}_{p, 1}^{\alpha+1})} + \| \nabla p^n \|_{L^1(0, T; \dot{B}_{p, 1}^{\alpha-1})}.
$$

We thus have

$$
\tilde{U}^{n+1}(T) \lesssim \| \bar{u}^n \|^2_{L^2(0, T; \dot{B}_{p, 1}^{\alpha-1})} + \| u^n \|^2_{L^2(0, T; \dot{B}_{p, 1}^{\alpha+1})} + \| \tau^{n+1} \|_{L^1(0, T; \dot{B}_{p, 1}^{\alpha})}
$$

$$
\leq C \| u^n \|^2_{L^2(0, T; \dot{B}_{p, 1}^{\alpha})} + CV^{-1}\tilde{U}^n(T)^2 + CT \| \tau_0 \|_{\dot{B}_{p, 1}^{\alpha}} \exp \left\{ C \tilde{U}^n(T) \right\}.
$$

Observing that $\tilde{U}^0 \equiv 1$, one can easily show by induction that $\tilde{U}^n(T) \leq \nu \varepsilon / (2C)$, for any integer $n \in \mathbb{N}$ and a small parameter $\varepsilon > 0$, provided that $T > 0$ is sufficiently small, for instance satisfying

$$
C \| u^n \|^2_{L^2(0, T; \dot{B}_{p, 1}^{\alpha})} + CT \| \tau_0 \|_{\dot{B}_{p, 1}^{\alpha}} \exp \left\{ \frac{\nu \varepsilon}{2C} \right\} \leq \frac{\nu \varepsilon}{100C}. \tag{34}
$$

Denoting by $T$ the supremum of the time $t$ satisfying the above inequality, we can finally deduce that the sequence $(u^n, \tau^n, \nabla p^n)$ is uniformly bounded in the functional space given by (33).

5.3. Convergence in small norms

Denoting by $\tilde{\tau}^n := \tau^n - \tau_0$, we claim that the sequence $(\bar{u}^n, \tilde{\tau}^n, \nabla p^n)$ is a Cauchy sequence in the functional setting given by

$$
(u^n, \tau^n, \nabla p^n) \in \mathcal{Y}_T \quad \Rightarrow \quad \begin{cases} 
\bar{u}^n \in \dot{L}^\infty(0, T; \dot{B}_{p, 1}^{\alpha-2}) \cap \dot{L}^1(0, T; \dot{B}_{p, 1}^{\alpha}) , \\
\tilde{\tau}^n \in \dot{L}^\infty(0, T; \dot{B}_{p, 1}^{\alpha-1}) , \\
\nabla p^n \in \dot{L}^1(0, T; \dot{B}_{p, 1}^{\alpha-2}) .
\end{cases} \tag{35}
$$
We will prove that this condition holds true, up to sufficiently decreasing the life span $T > 0$. It is worth to remark that the considered Besov spaces are homogeneous, hence the fact that $(\tilde{u}^n, \tilde{\tau}^n, \nabla \tilde{p}^n)$ belongs to the functional space given by (33) does not automatically imply that the same functions $(\bar{u}^n, \bar{\tau}^n, \nabla \bar{p}^n)$ belong also to $\mathcal{V}_T$. Nevertheless, system (31) together with Proposition 2.14 yields that $\partial_t \tilde{u}^n$ and $\partial_t \tilde{\tau}^n$ belongs to $\bar{L}^{1}(0, T; B^{3/2-p}_{p,1})$ and $\bar{L}^{1}(0, T; B^{3/2-p-1}_{p,1})$, respectively. This remark together with the initial condition $\tilde{u}^n(0) = 0$ and $\tilde{\tau}^n(0) = 0$ justifies the fact that $(\tilde{u}^n, \tilde{\tau}^n, \nabla \tilde{p}^n)$ belongs to the functional space given by $\mathcal{Y}_T$.

We now introduce the notation $\delta \tau^n = \tau^{n+1} - \tau^n$, $\delta u^n = u^{n+1} - u^n$, $\delta \omega^n = \omega^{n+1} - \omega^n$ and $\delta p^n = p^{n+1} - p^n$. Hence, we first remark from the equation of the conformation tensor in (31) that $\delta \tau^n$ satisfies

$$
\delta \partial_t \delta \tau^n + u^n \cdot \nabla \delta \tau^n - \omega^n \delta \tau^n + \delta \tau^n \omega^n = -\text{div} \left( \delta u^{n-1} \otimes \tau^n \right) + \delta \omega^{n-1} \tau^n - \tau^n \delta \omega^{n-1}.
$$

We can then apply Lemma 3.1 about the propagation of Besov regularity of index 0, insuring that

$$
\| \delta \tau^n \|_{L^\infty(0, T; B^{3/2-p}_{p,1})} \leq C \exp \left\{ \int_0^T \| \nabla u^{n-1} \|_{B^{3/2-1}_{p,1}} \right\} \left( \| \delta u^{n-1} \otimes \tau^n \|_{L^1(0, T; B^{3/2-1}_{p,1})} + \| \delta \omega^{n-1} \|_{L^1(0, T; B^{3/2-1}_{p,1})} \right)
$$

for a suitable positive constant $C$. Next, from Proposition 2.14 we gather

$$
\| \delta \tau^n \|_{L^\infty(0, T; B^{3/2-p}_{p,1})} \leq C \exp \left\{ \int_0^T \| \nabla u^{n-1} \|_{B^{3/2-1}_{p,1}} \right\} \| \delta u^{n-1} \|_{L^1(0, T; B^{3/2-1}_{p,1})} \| \tau^n \|_{L^\infty(0, T; B^{3/2-1}_{p,1})}.
$$

Hence, there exists a smooth increasing function $\chi = \chi(T)$ which is null in $T = 0$ and such that the following bound holds true:

$$
\| \delta \tau^n \|_{L^\infty(0, T; B^{3/2-p}_{p,1})} \leq \chi(T) \| \delta u^{n-1} \|_{L^1(0, T; B^{3/2-1}_{p,1})} \leq \chi(T) \delta U^{n-1}(t). (36)
$$

Here, the function $\chi$ depends just on $T$ and not on the index $n$ of the approximate solution.

Next, since $\delta u^n$ is a classical solution of the equation

$$
\partial_t \delta u^n - \nu \Delta \delta u^n + \nabla \delta \tau^n = -\text{div} \left( \delta u^{n-1} \otimes \delta u^{n-1} \right) - \text{div} \left( \delta u^{n-1} \otimes u^n \right) + \text{div} \delta \tau^n,
$$

Then, the Remark 2.12 concerning suitable bounds for the Stokes operator implies that

$$
\delta U^n(t) := \| \delta u^n \|_{L^\infty(0, T; B^{3/2-2}_{p,1})} + \| \delta u^n \|_{L^1(0, T; B^{3/2-1}_{p,1})} \| \nabla \delta \tau^n \|_{L^1(0, T; B^{3/2-1}_{p,1})} \leq C \left( \| u^{n-1} \otimes \delta u^{n-1} \|_{L^1(0, T; B^{3/2-1}_{p,1})} + \| \delta u^{n-1} \otimes u^n \|_{L^1(0, T; B^{3/2-1}_{p,1})} + \| \delta \tau^n \|_{L^1(0, T; B^{3/2-1}_{p,1})} \right). (37)
$$

Hence, recalling that $p \in [1, 2d)$, the product is a continuous function between the functional spaces

$$
\bar{L}^2(0, t; B^{3/2-2}_{p,1}) \times \bar{L}^2(0, t; B^{3/2-1}_{p,1}) \rightarrow \bar{L}^1(0, t; B^{3/2-1}_{p,1}),
$$

we combine (37) together with (36), to get

$$
\delta U^n(t) \leq C \left( \| u^{n-1} \|_{L^2(0, t; B^{3/2-1}_{p,1})} + \| u^n \|_{L^2(0, t; B^{3/2-1}_{p,1})} \right) \nu^{-1/2} \delta U^{n-1}(t) + \nu^{-1} \chi(t) \delta U^{n-1}(t).
$$

From (34), there exists a small parameter $c$, we can assume small enough, such that

$$
\| u^{n-1} \|_{L^2(0, t; B^{3/2-1}_{p,1})} + \| u^n \|_{L^2(0, t; B^{3/2-1}_{p,1})} \leq c \sqrt{\nu}.
$$

We hence conclude that for $T$ sufficiently small, we have

$$
\delta U^n(t) \leq \frac{1}{2} \delta U^{n-1}(t), \quad \forall t \in [0, T] \quad \text{and} \quad \forall n \in \mathbb{N}. (38)
$$
The considered sequence \((\bar{u}^n, \tau^n, \nabla \bar{p}^n)\) is then a Cauchy sequence in \(\mathcal{Y}_T\).

5.4. End of the proof of the local existence

On the one hand, we have achieved that the sequence \((\bar{u}^n, \tau^n, \nabla \bar{p}^n)\) converges towards a function \((\bar{u}, \tau, \nabla \bar{p})\) in the functional space \(\mathcal{Y}_T\). On the other hand the uniform estimates of Section 5.2 allow us to state that \((u^n, \tau^n, \nabla p^n)\) converges to a solution \((u, \tau, \nabla p)\) of the co-rotational Oldroyd model (1)

\[ u = u_L + \bar{u}, \quad \tau = \tau_0 + \tau, \quad \nabla p = \nabla p_L + \nabla \bar{p}. \]

This solution belongs to the functional space given by

\[ L^\infty((0,T); R^n) \times L^1((0,T); R^n) \times L^\infty((0,T); R^n). \]

The uniqueness of Theorem 1.3 can be achieved with similar procedures as the one used in Section 5.3: considering

\[ \begin{cases} u \cdot \nabla \tau - \omega \tau + \tau \omega \end{cases} \]

we proceed similarly as for proving (38). For \(p \cdot \nabla u \cdot \nabla \tau \) we control

\[ \int_0^T \int_\Omega \nabla u \cdot \nabla \tau \, dx \, dt \]

and we use as in the corresponding section in 2D, the following logarithmic estimate

\[ \| \delta u \|_{L^1(0,T; B^\frac{d}{p-1}_{p,1})} \leq \left( \int_{\Omega} \nabla u \cdot \nabla \tau \, dx \right) \ln \left( e + \frac{\| \delta u \|_{L^1(0,T; B^\frac{d}{p-1}_{p,1})}}{\| \delta u \|_{L^1(0,T; B^\frac{d}{p-1}_{p,1})}} \right) \]

and to conclude the uniqueness as an application of the classical Osgood lemma. We hence gather that \(\delta U(T) = 0\), which insures the uniqueness of the solution.

5.5. Uniqueness

The uniqueness of Theorem 1.3 can be achieved with similar procedures as the one used in Section 5.3 considering two solutions \((u^1, \tau^1)\) and \((u^2, \tau^2)\) satisfying the condition of Theorem 1.3, we define the difference \(\delta u := u^1 - u^2\) and \(\delta \tau := \tau^1 - \tau^2\). Hence, for \(p \in [1, 2d)\) we can estimate \((\delta u, \delta \tau)\) in the functional space defined by \(\mathcal{Y}_T\) as in (35): denoting by \(\delta U\) the functional

\[ \delta U(T) := \| \delta u \|_{L^\infty(0,T; B^\frac{d}{p-1}_{p,1})} + \| \delta \tau \|_{L^\infty(0,T; B^\frac{d}{p-1}_{p,1})}, \]

we proceed similarly as for proving (38). For \(p = 2d\) we control

\[ \delta U(T) := \| \delta u \|_{L^\infty(0,T; B^\frac{d}{p-1}_{p,1})} + \| \delta \tau \|_{L^\infty(0,T; B^\frac{d}{p-1}_{p,1})}, \]

and we use as in the corresponding section in 2D, the following logarithmic estimate

\[ \| \delta u \|_{L^1(0,T; B^\frac{d}{p-1}_{p,1})} \leq \left( \int_{\Omega} \nabla u \cdot \nabla \tau \, dx \right) \ln \left( e + \frac{\| \delta u \|_{L^1(0,T; B^\frac{d}{p-1}_{p,1})}}{\| \delta u \|_{L^1(0,T; B^\frac{d}{p-1}_{p,1})}} \right) \]

\[ \leq \| \delta u \|_{L^1(0,T; B^\frac{d}{p-1}_{p,1})} \ln \left( e + \frac{\| u^1 - u^2 \|_{L^1(0,T; L^2)}}{\| \delta u \|_{L^1(0,T; B^\frac{d}{p-1}_{p,1})}} \right), \]

and to conclude the uniqueness as an application of the classical Osgood lemma. We hence gather that \(\delta U(T) = 0\), which insures the uniqueness of the solution.

5.6. Extension criterion

Since the functions \(\tau\) and \(u\) belongs to \(L^\infty(0,T^*; B^\frac{d}{p-1}_{p,1})\) and \(L^\infty(0,T^*; B^\frac{d}{p-1}_{p,1}) \cap L^1(0,T^*; B^\frac{d}{p-1}_{p,1})\). Thanks to (34), for any \(t \in [0, T^*)\) there exists a local solution with initial data \((u(t), \tau(t))\) on a time interval \([t, t + T]\), with \(T > 0\) not depending on \(t\). Combining such a property with the previous uniqueness result, we can extend the solution \((u, \tau)\) for larger time \(t\) than \(T^*\).

This concludes the proof of Theorem 1.3.
6. Global-in-time solutions for small initial data: the case $p \in [1, 2d)$

This section is devoted to the proof of Theorem 1.4 assuming that $p \in [1, 2d)$. The existence of a global in time solution is based on certain suitable estimates. We will begin considering the local solution given by Theorem 1.3 and we will then show that the additional assumptions of Theorem 1.4 allow these solutions to be uniformly-in-time bounded in suitable Lorentz spaces. Hence, under a smallness condition on such a norms, we will propagate the Lipschitz regularity of the velocity field, globally in time. The flow being Lipschitz, we will then be able to propagate the suitable Besov (positive) regularity of Theorem 1.4.

6.1. Propagation of Lorentz regularities

Since the initial data $u_0$ and $\tau_0$ belongs to $\dot{B}^{4d-1}_{p,1}$, for a $p \in [1, 2d)$, thanks to Theorem 1.3 there exists a time $T$ and a unique solution $(u, \tau)$ of the corotational Johnson-Segalman system (11) in the functional space

$$u \in C([0, T], B^{4d-1}_{p,1}) \cap L^1(0, T; B^{4d+1}_{p,1}), \quad \tau \in C([0, T], B^{4d}_{p,1})$$

We first remark that the velocity field $u$ satisfies the mild formulation

$$u(t) = e^{\nu \Delta} u_0 + \int_0^t \text{div} \mathcal{P} e^{\nu (t-s) \Delta} (u(s) \otimes u(s)) ds + \int_0^t \text{div} \mathcal{P} e^{\nu (t-s) \Delta} \tau(s) ds$$

Thus, thanks to Lemma 3.3 and since $\| u \otimes u \|_{L^{d/2, \infty}} \leq \| u \|_{L^{d, \infty}}^2$, we gather that

$$\| u \|_{L^\infty(0, t; L^d; \infty)} \leq C \left( \| u_0 \|_{L^d; \infty} + V^{-1} \| u \|_{L^\infty(0, t; L^d; \infty)}^2 + V^{-1} \| \tau \|_{L^d; \infty} \right),$$

for any time $t \in [0, T]$. Furthermore, thanks to Lemma 2.16 $\tau$ satisfies

$$\| \tau(t) \|_{L^d; \infty} = \tau_0 \|_{L^d; \infty}.$$

We thus conclude that the Lorentz norm of $u$ is uniformly small in time

$$\| u \|_{L^\infty(0, t; L^d; \infty)} \leq \varepsilon V,$$

provided that

$$V^{-1} \| \tau_0 \|_{L^d; \infty} + \| u_0 \|_{L^d; \infty} \leq \varepsilon V$$

for a suitable small parameter $\varepsilon$.

6.2. Propagation of Lipschitz regularities

We now deal with the Lipschitz regularity of the flow $u$ and provide an uniform estimate. We first remark that since $\dot{B}^{4d/p}_{p,1}$, $\dot{B}^{4d/p}_{p,1}$ and $\dot{B}^{4d/p+1}_{p,1}$ are continuously embedded into $\dot{B}^{-1}_{\infty,1}$, $\dot{B}^0_{\infty,1}$ and $\dot{B}^1_{\infty,1}$, respectively, the solution $(u, \tau)$ of Theorem 1.3 belongs to

$$u \in C([0, T], \dot{B}^{-1}_{\infty,1}) \cap L^1(0, T; \dot{B}^1_{\infty,1}), \quad \tau \in C([0, T], \dot{B}^0_{\infty,1}),$$

and we can hence define the continuous functional $U(t)$, $t \in [0, T]$, by means of

$$U(t) := \| u \|_{L^\infty(0, T; \dot{B}^{-1}_{\infty,1})} + V \| u \|_{L^1(0, T; \dot{B}^0_{\infty,1})}.$$

Next, recasting the equation for $u$ into a non stationary linear Stokes problem, we can apply Remark 2.12 to gather

$$U(t) \leq C \left( \| u_0 \|_{\dot{B}^{-1}_{\infty,1}} + \| u \cdot \nabla u \|_{L^1(0, T; \dot{B}^{-1}_{\infty,1})} + \| \tau \|_{L^1(0, T; \dot{B}^0_{\infty,1})} \right).$$

We hence analyze the nonlinear term $u \cdot \nabla u = \text{div}(u \otimes u)$. First we recast it through the Bony decomposition

$$(u \cdot \nabla u)_j = \partial_j \dot{T}_w u_j + \partial_i \dot{T}_{u_i} u_j + \partial_j \dot{R}(u_i, u_j), \quad j = 1, \ldots d.$$
Then, we proceed estimating each term on the right hand side. First, we observe that
\[
\|\mathcal{T}_u u_j\|_{\dot{\mathcal{B}}^0_{\infty,1}} + \|\mathcal{T}_u u_i\|_{\dot{\mathcal{B}}^{1}_{\infty,1}} = \sum_{q \in \mathbb{Z}} \sum_{k \geq q - 5} \|\Delta_q(\mathcal{S}_{k-1} u_i \Delta_k u_j)\|_{L^\infty} + \|\Delta_q(\dot{\mathcal{S}}_{k-1} u_j \dot{\Delta}_k u_i)\|_{L^\infty} \leq \sum_{k \in \mathbb{Z}} 2^{-q} \|\dot{\mathcal{S}}_{k-1} u\|_{L^\infty} 2^q \|\dot{\Delta}_k u\|_{L^\infty} \\
\leq \|u\|_{\dot{\mathcal{B}}^{1}_{\infty,1}} \|u\|_{\dot{\mathcal{B}}^{1}_{\infty,1}} \leq \|u\|_{L^{1,\infty}} \|u\|_{\dot{\mathcal{B}}^{1}_{\infty,1}}.
\]
Moreover
\[
\|\partial_t \mathcal{R}(u, u_j)\|_{\dot{\mathcal{B}}^{0}_{\infty,1}} \leq \sum_{q \in \mathbb{Z}} \sum_{k \geq q - 5} \|\Delta_q(\dot{\Delta}_k u_i \dot{\Delta}_k u_j)\|_{L^\infty} \leq \sum_{q \in \mathbb{Z}} \sum_{k \geq q - 5} 2^q 2^{-q} \|\Delta_q(\dot{\Delta}_k u_i \dot{\Delta}_k u_j)\|_{L^\infty} \leq \sum_{q \in \mathbb{Z}} \sum_{k \geq q - 5} 2^q \|\dot{\Delta}_k u_i \dot{\Delta}_k u_j\|_{\dot{\mathcal{B}}^{1}_{\infty,1}}.
\]
Since \(L^{1,\infty}\) is continuously embedded into \(\dot{\mathcal{B}}^{1}_{\infty,1}\), we hence gather
\[
\|\partial_t \mathcal{R}(u, u_j)\|_{\dot{\mathcal{B}}^{0}_{\infty,1}} \leq C \|u\|_{L^{1,\infty}} \sum_{q \in \mathbb{Z}} \sum_{k \geq q - 5} 1(q - k) 2^q \|\dot{\Delta}_k u\|_{L^\infty} \leq C \|u\|_{L^{1,\infty}} \|u\|_{\dot{\mathcal{B}}^{1}_{\infty,1}}.
\]
Summarizing the previous estimates with (43), we eventually get
\[
\mathcal{U}(t) \leq C \left(\|u_0\|_{\dot{\mathcal{B}}^{1}_{\infty,1}} + \|\tau\|_{L^1(0,T; \dot{\mathcal{B}}^{1}_{\infty,1})}\right) + C \|u\|_{L^{\infty}(0,T; L^{1,\infty})} \mathcal{U}(t),
\]
thus, recalling the smallness condition (41) of the Lorentz norm of \(u\), we deduce that
\[
\mathcal{U}(t) \leq C \left(\|u_0\|_{\dot{\mathcal{B}}^{1}_{\infty,1}} + \|\tau\|_{L^1(0,T; \dot{\mathcal{B}}^{1}_{\infty,1})}\right).
\]
Next, in virtue of Lemma 3.2
\[
\|\tau(t)\|_{\dot{\mathcal{B}}^{0}_{\infty,1}} \leq C \|\tau_0\|_{\dot{\mathcal{B}}^{0}_{\infty,1}} \left(1 + \|u\|_{L^1(0,T; \dot{\mathcal{B}}^{1}_{\infty,1})}\right)
\]
from which we obtain
\[
\mathcal{U}(t) \leq C \|u_0\|_{\dot{\mathcal{B}}^{1}_{\infty,1}} + Ct\|\tau_0\|_{\dot{\mathcal{B}}^{0}_{\infty,1}} + CV^{-1}\|\tau_0\|_{\dot{\mathcal{B}}^{0}_{\infty,1}} \int_0^t \mathcal{U} d\tau,
\]
hence thanks to the Gronwall Lemma we deduce that
\[
\mathcal{U}(t) \leq C \|u_0\|_{\dot{\mathcal{B}}^{1}_{\infty,1}} \exp \left(\frac{Ct}{2} V^{-1}\|\tau_0\|_{\dot{\mathcal{B}}^{0}_{\infty,1}}\right) + V \left(\exp \left(\frac{Ct}{2} V^{-1}\|\tau_0\|_{\dot{\mathcal{B}}^{0}_{\infty,1}}\right) - 1\right),
\]
for any time \(t \in [0, T]\).

6.3. Proof of global existence
Let \(T^*\) be the largest time of existence of the solution \((u, \tau)\) determined by Theorem 1.3. We claim that under the condition of Theorem 1.4 the lifespan \(T^*\) satisfies \(T^* = +\infty\). We proceed by contradiction, assuming \(T^* < +\infty\). For any time \(T \in (0, T^*)\), \((u, \tau)\) belongs to
\[
C([0, T], \dot{\mathcal{B}}^{1}_{p,1}) \cap L^1(0, T; \dot{\mathcal{B}}^{d+1}_{p,1}) \times C([0, T], \dot{\mathcal{B}}^{d}_{p,1}),
\]
which is continuously embedded into
\[
C([0, T], \dot{\mathcal{B}}^{-1}_{\infty,1}) \cap L^1(0, T; \dot{\mathcal{B}}^{-1}_{\infty,1}) \times C([0, T], \dot{\mathcal{B}}^{0}_{\infty,1}).
\]
Furthermore, thanks to (44), the following estimate of the Lipschitz regularity of $u$ holds true for any time $T \in (0, T^*)$:

$$
\|u\|_{L^\infty(0,T; \dot{B}^{-1}_{p,1})} + \|v\|_{L^1(0,T; \dot{B}^{1}_{p,1})} \leq \Theta_v(u_0, \tau_0, T),
$$

where $\Theta_v$ is a growing continuous function depending on time $T$ by

$$
\Theta_v(u_0, \tau_0, T) := C\|u_0\|_{\dot{B}^{-1}_{p,1}} \exp \left\{ C T v^{-1} \|\tau_0\|_{\dot{B}^{0}_{p,1}} \right\} + v \left( \exp \left\{ C T v^{-1} \|\tau_0\|_{\dot{B}^{0}_{p,1}} \right\} - 1 \right) .
$$

Applying Lemma 3.1, we deduce that for any time $T \in (0, T^*)$

$$
\|\tau\|_{L^\infty(0,T; \dot{B}^{-\frac{d}{p}}_{p,1})} \leq \|\tau_0\|_{\dot{B}^{-\frac{d}{p}}_{p,1}} \exp \left\{ C \int_0^T \Theta_v(u_0, \tau_0, t) \, dt \right\}
$$

$$
\leq \|\tau_0\|_{\dot{B}^{-\frac{d}{p}}_{p,1}} \exp \left\{ C T^* \Theta_v(u_0, \tau_0, T^*) \right\} < +\infty.
$$

We hence deduce that $\tau$ belongs to $L^\infty(0,T^*; \dot{B}^{d/p}_{p,1})$.

Next, we take into account the velocity field and we remark that for any $t \in (0, T^*)$

$$
\|u(t)\|_{\dot{B}^{-\frac{d}{p}}_{p,1}} + v \int_0^t \|u(s)\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \, ds \leq \|u_0\|_{\dot{B}^{-\frac{d}{p}}_{p,1}} + \int_0^t \|u(s)\|_{\dot{B}^{0}_{p,1}} \|u(s)\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \, ds + \int_0^t \|\tau(s)\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \, ds
$$

from which we deduce that

$$
\|u(t)\|_{\dot{B}^{-\frac{d}{p}}_{p,1}} + v \int_0^t \|u(s)\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \, ds \leq \|u_0\|_{\dot{B}^{-\frac{d}{p}}_{p,1}} + C v^{-1} \int_0^t \|u(s)\|_{\dot{B}^{0}_{p,1}} \|u(s)\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \, ds +
$$

$$
+ C T^* \|\tau_0\|_{\dot{B}^{-\frac{d}{p}}_{p,1}} \exp \left\{ C T^* \Theta_v(u_0, \tau_0, T^*) \right\}.
$$

Finally, applying the Gronwall inequality we gather that for any time $t \in (0, T^*)$,

$$
\|u(t)\|_{\dot{B}^{\frac{d}{p}}_{p,1}} + v \int_0^t \|u(s)\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \, ds \leq \left\{ \|u_0\|_{\dot{B}^{\frac{d}{p}}_{p,1}} + C T^* \|\tau_0\|_{\dot{B}^{\frac{d}{p}}_{p,1}} e^{C T^* \Theta_v(u_0, \tau_0, T^*)} \right\} e^{C v^{-1} \Theta_v(u_0, \tau_0, T^*)} < +\infty.
$$

The above inequality implies that $u$ belongs to $L^\infty(0,T^*; \dot{B}^{-\frac{d}{p}}_{p,1}) \cap L^1(0,T^*; \dot{B}^{\frac{d}{p}}_{p,1})$. The prolongation criterion of Theorem 1.3 hence allows to extend in time the solution $(u, \tau)$ above the lifespan $T^*$, which contradicts the maximality of $T^*$, itself. Thus $T^* = +\infty$ and this concludes the proof of Theorem 1.4 for $p \in [1, 2d)$.

7. Global-in-time solutions for small initial data: the case $p \in [2d, \infty)$

This section is devoted to conclude the proof of Theorem 1.4 namely showing the existence of global-in-time classical solutions when $p \in [2d, \infty)$. In this setting, the uniqueness of these solutions is not determined as in the case of the previous section, since the regularity of the velocity field is below the critical negative value $d/p - 1 < -1/2$.

We begin with regularizing the initial data $(u_0, \tau_0)$ as follows:

$$
u_0 := J^n u_0, \quad \tau_0 := J^n \tau_0.
$$

The regularized initial data $(u^n_0, \tau^n_0)$ belongs to $\dot{B}^{0}_{d,1} \times \dot{B}^{1}_{d,1}$ as well as the smallness condition

$$
\|u^n_0\|_{L^d1} + \frac{1}{v} \|\tau^n_0\|_{L^\infty} \leq \frac{\varepsilon}{v}.
$$
is still satisfied. Thanks to Theorem 4.1 with \( p = d \in [1, 2d] \), there exists a unique global-in-time solutions \((u^n, \tau^n)\) of the system

\[
\begin{align*}
\begin{cases}
\partial_t \tau^n + u^n \cdot \nabla \tau^n - \omega^n \tau^n + \tau^n \omega^n &= 0 &\quad \mathbb{R}_+ \times \mathbb{R}^2, \\
\partial_t u^n + u^n \cdot \nabla u^n - \nu \Delta u^n + \nabla p^n &= \text{div} \, \tau^n &\quad \mathbb{R}_+ \times \mathbb{R}^d, \\
\text{div} \, u^n &= 0 &\quad \mathbb{R}_+ \times \mathbb{R}^d, \\
(u^n, \tau^n)|_{t=0} &= (u_0^n, \tau_0^n) &\quad \mathbb{R}^2,
\end{cases}
\end{align*}
\]

satisfying

\[u^n \in C\left( \mathbb{R}_+, \dot{B}_{d,1}^{\frac{d}{2}} \cap L^{\infty}(\mathbb{R}^d) \right) \cap L^1_{\text{loc}}(\mathbb{R}_+, \dot{B}_{d,1}^{\frac{d}{2}}), \quad \tau^n \in C\left( \mathbb{R}_+, \dot{B}_{d,1}^{\frac{d}{2}} \cap L^{\infty}(\mathbb{R}^d) \right).
\]

Proceeding as in Section 4.3, we get that \((u^n, \tau^n)\) is uniformly bounded in the functional space

\[u^n \in C\left( \mathbb{R}_+, \dot{B}_{p,1}^{\frac{d}{2}-1} \cap L^{\infty}(\mathbb{R}^d) \right) \cap L^1_{\text{loc}}(\mathbb{R}_+, \dot{B}_{p,1}^{\frac{d}{2}+1}), \quad \tau^n \in C\left( \mathbb{R}_+, \dot{B}_{p,1}^{\frac{d}{2}} \cap L^{2\infty}(\mathbb{R}^d) \right).
\]

In order to conclude, we need to pass to the limit as \( n \) goes to \( \infty \). We proceed similarly as in Section 4.3. We fix a compact set \( K \) in \( \mathbb{R}^2 \) and we introduce the functional spaces

\[X_0 := \dot{B}_{p,1}^{\frac{d}{2}-1} (K), \quad X = X_1 := \dot{B}_{p,1}^{\frac{d}{2}-2} (K),\]

for the Aubin-Lions Lemma 4.10. Since \((u^n)_n\) is uniformly bounded in \( L^2_{\text{loc}}(\mathbb{R}_+, \dot{B}_{d,1}^{\frac{d}{2}}) \) then, by embedding it is uniformly bounded in \( L^2_{\text{loc}}(\mathbb{R}_+, \dot{B}_{p,1}^{\frac{d}{2}}) \). Interpolating this result with the uniform bound of \((u^n)_n\) in \( L^{\infty}_{\text{loc}}(\mathbb{R}_+, L^{d\infty}(\mathbb{R}^d)) \) allows us to conclude that \((u^n)_n\) is uniformly bounded in \( L^{2p/(p-d)}_{\text{loc}}(\mathbb{R}_+, L^p(\mathbb{R}^d)) \) and thus in \( L^2_{\text{loc}}(\mathbb{R}_+, \dot{B}_{p,1}^{\frac{d}{2}}) \), where \( \dot{B}_{p,1}^{\frac{d}{2}} \) is a non-homogeneous Besov space. Furthermore, since \( \dot{B}_{p,1}^{\frac{d}{2}} \) is continuously embedded into \( L^{\infty}(\mathbb{R}^d) \), the sequence \((\tau^n)_n\) is uniformly bounded into \( L^{\infty}_{\text{loc}}(\mathbb{R}_+, L^p(\mathbb{R}^d)) \) and so into \( L^2_{\text{loc}}(\mathbb{R}_+, \dot{B}_{p,1}^{\frac{d}{2}}) \) which is embedded into \( L^2_{\text{loc}}(\mathbb{R}_+, \dot{B}_{p,1}^{2/p-1}) \). This allows us to conclude that

\[\|\text{div} \, \tau^n\|_{L^2_{\text{loc}}(\mathbb{R}_+, \dot{B}_{p,1}^{2/p-2}(K))} \leq C,
\]

for a suitable constant \( C \) that does not depend on the index \( n \in \mathbb{N} \). Now, we claim that \( \text{div} \, j^n(u^n \otimes u^n) \) is uniformly bounded in \( L'(0, T; \dot{B}_{p,1}^{2/p-2}(K)) \), for a suitable positive index \( r > 1 \). We first remark that

\[
\begin{align*}
\|\text{div} \, j^n(u^n \otimes u^n)\|_{\dot{B}_{p,1}^{2/p-2}(K)} &\leq \|j^n(u^n \otimes u^n)\|_{\dot{B}_{p,1}^{2/p-1}(K)} \leq \|j^n(u^n \otimes u^n)\|_{\dot{B}_{p,1}^{2/p-1}} \\
&\lesssim \|\tilde{S}_n j^n(u^n \otimes u^n)\|_{L^p(\mathbb{R}^2)} + \|\text{div} \, (\text{Id} - \tilde{S}_n) j^n(u^n \otimes u^n)\|_{\dot{B}_{p,1}^{2/p-1}} \\
&\lesssim \|j^n(u^n \otimes u^n)\|_{L^p(\mathbb{R}^2)} + \|\text{div} \, (\text{Id} - \tilde{S}_n) j^n(u^n \otimes u^n)\|_{\dot{B}_{p,1}^{2/p-1}},
\end{align*}
\]

where \( \epsilon \in (0, 1) \) such that \( d/p - \epsilon > 0 \). Hence we eventually gather that

\[\|\text{div} \, j^n(u^n \otimes u^n)\|_{\dot{B}_{p,1}^{2/p-2}(K)} \leq C \left( \|u^n \otimes u^n\|_{L^p(\mathbb{R}^2)} + \|u^n \otimes u^n\|_{\dot{B}_{p,1}^{2/p-\epsilon}} \right).
\]

Now, making use of the continuity of the product between the functional spaces

\[L^2_{\text{loc}}(\mathbb{R}_+, \dot{B}_{p,1}^{\frac{d}{2}}) \times L^2_{\text{loc}}(\mathbb{R}_+, \dot{B}_{p,1}^{\frac{d}{2}}) \rightarrow L^\frac{2p}{p-1}_{\text{loc}}(\mathbb{R}_+, \dot{B}_{p,1}^{\frac{d}{2}}),
\]

the term \( u^n \otimes u^n \) is uniformly bounded into \( L^{1/(1-\epsilon)}_{\text{loc}}(\mathbb{R}_+, \dot{B}_{p,1}^{\frac{d}{2}}) \). Furthermore, we can bound the \( L^p(\mathbb{R}^2) \) norm through the following interpolation:

\[\|u^n \otimes u^n\|_{L^p(\mathbb{R}^2)} \leq C \|u^n\|_{L^{2p}(\mathbb{R}^2)} \leq C \|u^n\|_{L^p(\mathbb{R}^2)} \|u^n\|_{L^{\infty}(\mathbb{R}^2)} \in L^\frac{2p}{p-1}_{\text{loc}}(\mathbb{R}_+).
\]
Denoting by $r = \min\{2, 1/(1 - \varepsilon), 2p/(2p - d)\} > 1$, we gather that the sequence $(\partial_t u^n)_N$ satisfying
\[ \partial_t u^n = \Delta u^n - J(u^n) \cdot \nabla u^n + \nabla p^n + \nabla^2 u^n \]
is uniformly bounded in $L_{\text{loc}}^r (\mathbb{R}^+, B^{1/(p-2)}_{p,1}(K))$. Hence, the Aubins-Lion Lemma 4.10 and the generality of the compact set $K$ allow us to extract a convergent subsequence $(u^n)_N \subset (u^n)_N$ such that
\[ u^n \rightarrow u \quad \text{in} \quad L^\infty (0, T; (B^{d-2}_{p,1})_{\text{loc}}). \]

We now claim that $(\partial_t \tau^n)_N$ is uniformly bounded in the non-homogeneous functional space $L_{\text{loc}}^r (\mathbb{R}^+, B^{1/(p-1)}_{p,1}(K))$. First we recall that the conformation tensor satisfies
\[ \partial_t \tau^n = -u^n \cdot \nabla \tau^n + \omega^n \tau^n - \tau^n \omega^n, \]

therefore
\[ \| \partial_t \tau^n \|_{B^{d-1}_{p,1}} \lesssim \| u^n \|_{B^{d-1}_{p,1}} \| \nabla \tau^n \|_{B^{d-1}_{p,1}} + \| \nabla u^n \|_{B^{d-1}_{p,1}} \| \tau^n \|_{B^{d-1}_{p,1}} \in L_{\text{loc}}^2 (\mathbb{R}^+). \]

Furthermore
\[ \| -\omega^n \tau^n + \tau^n \omega^n \|_{L^p(\mathbb{R}^2)} \lesssim \| \nabla u^n \|_{L^p(\mathbb{R}^2)} \| \tau^n \|_{L^\infty (\mathbb{R}^2)} \lesssim \| \nabla u^n \|_{B^{0}_{p,1}} \| \tau^n \|_{B^{d-1}_{p,1}} \]
\[ \lesssim \| u^n \|_{B^{d-1}_{p,1}} \| \tau^n \|_{B^{d-1}_{p,1}} \lesssim \| u^n \|_{B^{d-1}_{p,1}} \| u^n \|_{B^{d-1}_{p,1}} \| \tau^n \|_{B^{d-1}_{p,1}} \in L_{\text{loc}}^2 (\mathbb{R}^+). \]

Finally, one has
\[ \| \Delta_{-1} \text{div}(u^n \otimes \tau^n) \|_{L^p(\mathbb{R}^2)} \lesssim \| u^n \otimes \tau^n \|_{L^p(\mathbb{R}^2)} \lesssim \| u^n \|_{L^p(\mathbb{R}^2)} \| \tau^n \|_{L^\infty (\mathbb{R}^2)} \]
\[ \lesssim \| u^n \|_{L^p(\mathbb{R}^2)} \| \tau^n \|_{B^{d-1}_{p,1}} \in L_{\text{loc}}^2 (\mathbb{R}^2), \]

which allows us to conclude that $(\partial_t \tau^n)_N$ is uniformly bounded in $L_{\text{loc}}^r (\mathbb{R}^+, B^{2/(p-1)}_{p,1}(K))$. Thus, the Aubins-Lion lemma together with the arbitrariness of the compact set $K$ allow us to extract a convergent subsequence $(\tau^n)_N \subset (u^n)_N$ such that
\[ \tau^{n_k} \rightarrow \tau \quad \text{in} \quad L^\infty (0, T; (B^{d-1}_{p,1})_{\text{loc}}). \]

These properties allow to pass to the limit and thus to show that $(u, \tau)$ is a global-in-time solution of system (1). This concludes the proof of Theorem 1.4.

References

[1] H. Bahouri and J.-Y. Chemin, and R. Danchin, “Fourier analysis and nonlinear partial differential equations”, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 343, Springer, Heidelberg, 2011
[2] O. Bejauri and N. Masmoudi, “Global weak solutions for some Oldroyd models”, J. Differential Equations, 254, no. 2, (2013) 660–685
[3] D. Bresch and C. Prange, “Newtonian limit for weakly viscoelastic fluid flows”, SIAM J. Math. Anal., 46, no. 2, (2014) 1116–1159
[4] E. Fernández-Cara, F. Guillén, R. R. Ortega, “Mathematical modeling and analysis of viscoelastic fluids of the Oldroyd kind”, Handb. Numer. Anal., VIII, North-Holland, Amsterdam, (2002) 543–661
[5] J.-Y. Chemini and N. Masmoudi, “About lifespan of regular solutions of equations related to viscoelastic fluids”, SIAM J. Math. Anal., 33, no. 1, (2001) 84–112
[6] R. Danchin and M. Paicu, “Les théorèmes de Leray et de Fujita-Kato pour le système de Boussinesq partiellement visqueux”, Bull. Soc. Math. France, 136, no. 2, (2008) 261–309
[7] F. De Anna, “A global 2D well-posedness result on the order tensor liquid crystal theory ”, J. Differential Equations, 262, no. 7, (2017) 3932–3979
[8] E. Fernández Cara, F. Guillon and R.R. Ortega, “Existence and unicité de solution forte locale en temps pour des fluides non newtoniens de type Oldroyd (version $L^r$–$L^1$)”, C. R. Acad. Sci. Paris Sér. I Math., 319, no. 4, (1994) 411–416
[9] H. Fujita and T. Kato, “On the Navier-Stokes initial value problem. I ”, Arch. Rational Mech. Anal., 16, (1964) 269–315
[10] C. Guillop´ e and J.-C. Saut, “Existence results for the flow of viscoelastic fluids with a differential constitutive law”, Nonlinear Anal., 15, no. 9, (1990) 849–869
[11] C. Guillop´ e and J.-C. Saut, “Global existence and one-dimensional nonlinear stability of shearing motions of viscoelastic fluids of Oldroyd type”, RAIRO Modél. Math. Anal. Numér., 24, no. 3, (1990) 369–401
[12] D. Hu and T. Lelièvre, “New entropy estimates for Oldroyd-B and related models“, Commun. Math. Sci., 5, no. 4, (2007) 909–916
[13] C. Le Bris and T. Lelièvre, “Multiscale modelling of complex fluids: a mathematical initiation”, In Multiscale modeling and simulation in science. Volume 66 of Lect. Notes Comput. Sci. Eng., pages 49–137. Springer, Berlin, 2009
[14] J. Leray, “Sur le mouvement d’un liquide visqueux emplissant l’espace”, Acta Math. 63, (1934) 193–248
[15] J.-L. Lions and G. Prodi, “Un théorème d’existence et unicité dans les équations de Navier-Stokes en dimension 2”, C. R. Acad. Sci. Paris 248 (1959), p. 3519–3521.
[16] Z. Lei, C. Liu and Y. Zhou, “Global solutions for incompressible viscoelastic fluids”, Arch. Ration. Mech. Anal., 188, no. 3, (2008) 371–398
[17] F.H. Lin, C. Liu and P. Zhang, “On hydrodynamics of viscoelastic fluids”, Comm. Pure Appl. Math., 58, no. 11, (2005) 1437–1471
[18] P.L. Lions and N. Masmoudi, “Global solutions for some Oldroyd models of non-Newtonian flows”, Chinese Ann. Math. Ser. B, 21, no. 2, (2000) 131–146
[19] N. Masmoudi, “Global existence of weak solutions to macroscopic models of polymeric flows”, J. Math. Pures Appl. (9), 96, no. 5, (2011) 502–520
[20] P. Oswald, “Rhé ophysique, ou comment coule la matière”, Collection Échelles. Belin, 2005
[21] M. Renardy, “Mathematical analysis of viscoelastic flows”, volume 73 of CBMS-NSF Regional Conference Series in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000.
[22] Y. Zhuan and X. Xu, “Global regularity for the 2D Oldroyd-B model in the corotational case”, Math. Methods Appl. Sci., 39, no. 13, (2016) 3866–3879
[23] R. Zi, Da. Fang and T. Zhang, “Global solution to the incompressible Oldroyd-B model in the critical $L^p$ framework: the case of the non-small coupling parameter”, Arch. Ration. Mech. Anal., 213, no. 2, (2014) 651–687