On the correspondence between the solutions of Dirac equation and electromagnetic 4-potentials

Aristides I. Kechriniotis * Christos Tsonos †
Konstantinos K. Delibasis ‡ Georgios N. Tsigaridas §

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Abstract

In this paper the inverse problem of the correspondence between the solutions of the Dirac equation and the electromagnetic 4-potentials, is fully solved. The Dirac solutions are classified into two classes. The first one consists of degenerate Dirac solutions corresponding to an infinite number of 4-potentials while the second one consists of non-degenerate Dirac solutions corresponding to one and only one electromagnetic 4-potential. Explicit expressions for the electromagnetic 4-potentials are provided in both cases. Further, in the case of the degenerate Dirac solutions, it is proven that at least two 4-potentials are gauge inequivalent, and consequently correspond to different electromagnetic fields. An example is provided to illustrate this case.

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1 Introduction

The Dirac equation has been the first electron equation in quantum mechanics to satisfy the Lorentz covariance [2], initiating the beginning of one of the most powerful theories ever formulated: the quantum electrodynamics. This equation predicted the spin and the magnetic moment of the electrons, the existence of antiparticles and was able to reproduce accurately the spectrum of the hydrogen atom. It has also played an important role in various areas of physics such as high energy physics and nuclear physics, while recently its usefulness has been

*Department of Informatics Engineering, Technological Educational Institute of Central Greece, GR-35100 Lamia, Hellas - Greece. Corresponding author. E-mail: arisk7@gmail.com
†Department of Electronic Engineering, Technological Educational Institute of Central Greece, GR-35100 Lamia, Hellas - Greece
‡Department of Computer Science and Biomedical Informatics, University of Thessaly, GR-35100 Lamia, Hellas - Greece
§Department of Physics, School of Applied Mathematical and Physical Science, National Technical University of Athens, Heroon Polytechniou 9, GR-15780 Zografou, Athens, Hellas - Greece
extended in condensed matter Physics because the electronic band structure in solids sometimes has features similar to Dirac electrons in vacuum [3], [4]. Despite all the work that has been done over the years, Dirac equation is exactly solvable only for very few interactions [5], [6], [7], [8], and the solutions usually come with a strong constraint on the potentials [9], [10], [11].

The majority of the previously reported works focus on the determination of the wave function $\Psi$ when the electromagnetic 4-potential is given. However, the inverse problem, as formulated by Eliezer [12] is also quite interesting: “Given the wave function $\Psi$, what can we say about the electromagnetic potential $A_\mu$, which is connected to $\Psi$ by Dirac’s equation? Is $A_\mu$ uniquely determined, and if not, what is the extent to which it is arbitrary? ”. The expression of $A_\mu$ as a function of the components of $\Psi$ allows the calculation of the components of the electromagnetic tensor, $f_{\mu\nu}$, in the same manner. In his relevant work Eliezer found an expression for the magnetic vector $A_\mu$ and the electric scalar potential $\phi$ as a function of $\Psi$. Eliezer pointed out that the Dirac equation could be written in the matrix form $PX = Q$, where $P$ is $4 \times 4$ matrix containing only Dirac spinor $\Psi$components, $X$ is a column matrix containing only electromagnetic potential components and $Q$ is a column matrix containing expressions of the $\Psi$ components and their derivatives. In this case the four equations of the system are not linearly independent, because $\det P = 0$, and so $P$ is not invertible.

One important work of Radford in this direction is reported in [13]. In this work the Dirac equation is expressed in a 2-spinor form, which allows it to be (covariantly) solved for the electromagnetic 4-potential, in terms of the wave function and its derivatives. This approach subsequently led to some physically interesting results, see also [14], [15], [16] (for a review see [17]). In [18] it was demonstrated that the Dirac equation is indeed algebraically invertible if a real solution for the vector potential is required. Namely, two expressions for the components $A_\mu$ of the electromagnetic 4-potential are presented, equivalent to the one given in [13]. These two expressions, using the standard representation of the Dirac matrices $\gamma_\mu$, $\mu = 1, 2, 3, 4$, can be equivalently written in the following forms $A_\mu = \frac{1}{2q}\Phi_\mu \gamma^\mu \gamma^4 \Psi$ and $A_\mu = \frac{1}{2q}\gamma^\mu \Omega_\mu$, where $\Phi_\mu$ and $\Omega_\mu$ are bilinear forms in terms of $\Psi$ and its derivatives. It is obvious that both expressions do not hold in the case that $\Phi^\mu \gamma_\mu \Psi = \Omega^\mu \gamma_\mu \gamma^4 \Psi = 0$.

However, the above mentioned works, do not study the existence of nonzero Dirac solutions satisfying the aforementioned conditions, and further they do not provide any answer on what happens in this case. Specifically, if it is assumed that there exist solutions of the Dirac equation satisfying the above conditions, it should be investigated if one such solution is connected to a unique real 4-potential or not. Consequently, the problem of the inversion of the Dirac equation remains open.

In this paper we fully solve the inverse problem. Specifically we find the necessary and sufficient condition so that a Dirac spinor corresponds to one and only one 4-potential. The results obtained lead us to classify the set of all Dirac spinor into two classes. The elements of the first class correspond to
one and only one 4-potential, and are called non-degenerate Dirac solutions, while the elements of the second class correspond to an infinite number of 4-potentials, and are called degenerate Dirac solutions. In the first case the 4-potential is fully defined as a function of the Dirac spinor, while in the second one explicit expressions are provided for the infinite number of the 4-potentials connected to the degenerate Dirac spinors. Further, it is proven that at least two of these potentials are Gauge inequivalent and consequently correspond to different electromagnetic fields. From a physical point of view this is quite a surprising result, meaning that for a specific class of spinors it is possible to be in the same state under the influence of different electromagnetic fields. An example is provided to illustrate this fact. It is also proven that every Dirac spinor corresponds to one and only one mass.

2 Preliminaries

In this paper the continuity of the Dirac spinors in the Euclidean space $\mathbb{R}^4$ is used extensively. Therefore, in order to write the Dirac equation in an appropriate form, we employ the fourth coordinate $x_0 = ct$. Then the Dirac equation which includes an electromagnetic 4-potential \cite{19} can be written in the following form:

$$\left[ \sum_{\mu=1}^{3} \gamma_\mu (\partial_\mu - ia_\mu) - i\gamma_4 (\partial_0 + ia_0) + \kappa \right] \Psi = 0, \quad (2.1)$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are the Dirac matrices in the standard representation:

$$\gamma_1 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

$$\gamma_3 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}, \quad \gamma_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$x_{\mu}, \mu = 0, 1, 2, 3$ are real variables, $a_\mu$ are real functions of $x_{\mu}, \kappa = mc/h$, $\partial_0 = \frac{1}{c} \partial_t$ and $\Psi = [\psi_1, \psi_2, \psi_3, \psi_4]^T$ is a 4 component Dirac spinor, which is regarded as a column vector. More specifically, $a_\mu = \frac{c}{e} A_\mu, \mu = 1, 2, 3$ where $A_\mu$ are the magnetic vector potential components and $a_0 = e\Phi/hc$ where $\Phi$ is the electric scalar potential. In the rest of the paper the real constant $\kappa = mc/h$ will be called mass, and this quantity is identical to the inverse of the reduced Compton wavelength of a Dirac particle with mass $m$. The Dirac matrices in the standard representation satisfy $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. 
From now on it will be assumed that both the 4-potential and the mass are real and the Dirac spinor $\Psi$ is continuous, in the sense all of its components $\psi_\mu: \mathbb{R}^4 \to \mathbb{C}$, $\mu = 1, 2, 3, 4$ are continuous wavefunctions.

For convenience some definitions should be introduced.

**Definition 1.** Any solution of equation (2.1) for a 4-potential

$$a := (a_0, a_1, a_2, a_3)$$

and a mass $\kappa$ will be called Dirac solution.

**Definition 2.** A Dirac solution $\Psi$ is said to correspond to a mass $\kappa$, if there exists a 4-potential $a$, such that $\Psi$ is a solution of (2.1) connected with $a$ and $\kappa$.

**Definition 3.** A Dirac solution $\Psi$ is said to correspond to a 4-potential $a$, if there exists a mass $\kappa$, such that $\Psi$ is a solution of (2.1) connected with $a$ and $\kappa$.

We introduce the following notations:

- We denote the following matrices
  $$\gamma_5 := \gamma_1 \gamma_2 \gamma_3 \gamma_4, \quad \delta_\mu := \gamma_\mu + \gamma_5 \gamma_\mu, \quad \mu = 1, 2, 3, 4.$$  

- Let $\Psi$ be a Dirac solution, and $\delta$ is an arbitrary $4 \times 4$ complex or real matrix. If $\partial$ is a differential operator, then,

$$\leftrightarrow \partial \left( \Psi^* \delta \Psi \right) := \Psi^* \delta \partial \Psi - (\partial \Psi^*) \delta \Psi.$$

In [18] two equivalent algebraic expressions for the 4-potential in terms of the Dirac solution are derived. At this point, we reproduce these expressions, using the notation adopted in this work.

The following properties of Dirac matrices will be used: $\gamma_\mu$, $\mu = 1, 2, 3, 4, 5$ are Hermitian, while the matrices $\gamma_\lambda \gamma_\mu$, $1 \leq \lambda \neq \mu \leq 5$ are anti-Hermitian. Eq. (2.1), after some algebraic manipulations, leads to the following sets of relations

$$2a_0 (\Psi^* \gamma_4 \Psi) = - \sum_{\mu=1}^{3} \partial_\mu (\Psi^* \gamma_\mu \Psi) + i \partial_0 (\Psi^* \gamma_4 \Psi) - 2\kappa \Psi^* \Psi. \quad (2.2)$$

$$2ia_1 \Psi^* \gamma_4 \Psi = -2\kappa \Psi^* \gamma_1 \gamma_4 \Psi$$

$$+ \leftrightarrow \partial_1 (\Psi^* \gamma_4 \Psi) - \partial_2 (\Psi^* \gamma_5 \gamma_3 \Psi) + \partial_3 (\Psi^* \gamma_5 \gamma_2 \Psi) + i \partial_0 (\Psi^* \gamma_5 \gamma_1 \Psi), \quad (2.3)$$

$$2ia_2 \Psi^* \gamma_4 \Psi + 2\kappa \Psi^* \gamma_2 \gamma_4 \Psi$$

$$= \partial_1 (\Psi^* \gamma_5 \gamma_3 \Psi) + \leftrightarrow \partial_2 (\Psi^* \gamma_4 \Psi) \quad - \partial_3 (\Psi^* \gamma_5 \gamma_1 \Psi) + i \partial_0 (\Psi^* \gamma_2 \Psi), \quad (2.4)$$
\[ 2\alpha_3 \Psi^* \gamma_4 \Psi + 2\kappa \Psi^* \gamma_3 \gamma_4 \Psi \]

\[ = -\partial_1 (\Psi^* \gamma_5 \gamma_2 \Psi) + \partial_2 (\Psi^* \gamma_5 \gamma_1 \Psi) + \partial_3 (\Psi^* \gamma_4 \Psi) + i\partial_0 (\Psi^* \gamma_3 \Psi), \quad (2.5) \]

and

\[ 2\alpha_0 (\Psi^* \gamma_5 \gamma_4 \Psi) \]

\[ = i \partial_0 (\Psi^* \gamma_5 \gamma_4 \Psi) - \partial_1 (\Psi^* \gamma_5 \gamma_1 \Psi) - \partial_2 (\Psi^* \gamma_5 \gamma_2 \Psi) - \partial_3 (\Psi^* \gamma_5 \gamma_3 \Psi), \quad (2.6) \]

\[ 2\alpha_1 \Psi^* \gamma_5 \gamma_4 \Psi \]

\[ = \partial_1 (\Psi^* \gamma_5 \gamma_4 \Psi) - \partial_2 (\Psi^* \gamma_3 \Psi) + \partial_3 (\Psi^* \gamma_2 \Psi) + i\partial_0 (\Psi^* \gamma_5 \gamma_1 \Psi), \quad (2.7) \]

\[ 2\alpha_2 \Psi^* \gamma_5 \gamma_4 \Psi \]

\[ = \partial_1 (\Psi^* \gamma_3 \Psi) + \partial_2 (\Psi^* \gamma_5 \gamma_4 \Psi) - \partial_3 (\Psi^* \gamma_1 \Psi) + i\partial_0 (\Psi^* \gamma_5 \gamma_2 \Psi), \quad (2.8) \]

\[ 2\alpha_3 \Psi^* \gamma_5 \gamma_4 \Psi \]

\[ = -\partial_1 (\Psi^* \gamma_2 \Psi) + \partial_2 (\Psi^* \gamma_1 \Psi) + \partial_3 (\Psi^* \gamma_5 \gamma_4 \Psi) + i\partial_0 (\Psi^* \gamma_5 \gamma_3 \Psi). \quad (2.9) \]

The two inversion formulas, given by (2.2), (2.3), (2.4), (2.5) and (2.6), (2.17), (2.8), (2.9) respectively, stand only in the case where \( \Psi^* \gamma_4 \Psi \neq 0 \) or \( \Psi^* \gamma_5 \gamma_4 \Psi \neq 0 \). A question arises here: A Dirac solution satisfying the conditions \( \Psi^* \gamma_4 \Psi = \Psi^* \gamma_5 \gamma_4 \Psi = 0 \), corresponds to one and only one 4-potential or not? The main aim in the rest of this paper is to answer this question. The above considerations lead us to introduce the following definition:

First we introduce a notation. Let \( \Psi \) be a non zero Dirac solution, and \( U \) any open subset of \( \text{supp} (\Psi) := \{ x \in \mathbb{R}^4 : \Psi (x) \neq 0 \} = \bigcup_{\mu=1}^{4} \text{supp} (\psi_{\mu}), \)

which is also an open subset of \( \mathbb{R}^4 \), as it follows from the continuity of \( \Psi \) in \( \mathbb{R}^4 \). Then we denote by \( P (\Psi, U) \) the set of the restrictions to \( U \) of all the 4-potentials corresponding to \( \Psi \).

**Definition 4.** A Dirac solution \( \Psi \) is said to be degenerate, if for any open subset \( U \) of \( \text{supp} (\Psi) \), \( P (\Psi, U) \) contains more than one elements. Therefore a Dirac solution \( \Psi \) is said to be non degenerate, if there is a non empty open set \( U \) in \( \text{supp} (\Psi) \), such that \( P (\Psi, U) \) contains one and only one element.

### 3 Uniqueness of mass

In this section we prove that any non zero Dirac solution corresponds to one and only one mass. This result will also be used in the next section.
Definition 5. In the set of all Dirac solutions we define the following relation:

- $\Psi_1 \approx \Psi_2$ if and only if $\Psi_1, \Psi_2$ are gauge equivalent, that is there exists a non zero number $c$ and a differentiable function of the spatial and temporal variables $f : \mathbb{R}^4 \to \mathbb{R}$ such that $\Psi_1 = ce^{if}\Psi_2$. Clearly $\approx$ is an equivalence relation, and by $[\Psi]$ we will denote the equivalence class of $\Psi$.

For the proofs of the next theorems we need the following Lemmas.

Lemma 3.1. Let $\Psi$ be a non zero Dirac solution. Then any element of $[\Psi]$ is a Dirac solution. If $\Psi$ corresponds to a mass $\kappa$, then any element of $[\Psi]$ corresponds to the same mass $\kappa$.

Proof. Since $\Psi$ corresponds to a mass $\kappa$, then there is a 4-potential $(a_0, a_1, a_2, a_3)$ such that $\Psi$ is a solution of (2.1) for this mass and vector field. Let $\Psi_1$ be any element of $[\Psi]$. Then we have $\Psi = c_1 \exp(-i\int x_0 k a_0(s, x_1, x_2, x_3) ds) \Psi_0$, where $k$ is a real constant. Then $\Psi_0 \in [\Psi]$. So from (2.1) we get

$$
\left[ \sum_{\mu=1}^3 \gamma_\mu (\partial_\mu - i(a_\mu - \partial_\mu f)) - i\gamma_4 (\partial_0 - i(a_0 - \partial_0 f)) + \kappa \right] \Psi_1 = 0.
$$

Therefore, $\Psi_1$ corresponds to the mass $\kappa$ and to the 4-potential $(a_0 - \partial_0 f, a_1 - \partial_1 f, a_2 - \partial_2 f, a_3 - \partial_3 f)$.

Lemma 3.2. Let $\Psi$ be any Dirac solution. Then there exists at least one $\Psi_0 \in [\Psi]$, such that

$$
-i\overleftrightarrow{\partial}_0 (\Psi_0^* \gamma_4 \Psi_0) + \sum_{\mu=1}^3 \partial_\mu (\Psi_0^* \gamma_\mu \Psi_0) + 2\kappa \Psi_0^* \Psi_0 = 0.
$$

Proof. We define $\Psi_0$ by

$$
\Psi = \exp \left( -i \int_{x_0} x k a_0(s, x_1, x_2, x_3) ds \right) \Psi_0,
$$

where $k$ is a real constant. Then $\Psi_0 \in [\Psi]$.

Let $\delta$ be any matrix with real or complex entries. Then from (3.2) we easily get,

$$
i\overleftrightarrow{\partial}_0 (\Psi_0^* \delta \Psi) = 2a_0 \Psi_0^* \delta \Psi_0 + i\overleftrightarrow{\partial}_0 (\Psi_0^* \delta \Psi_0),
$$

and

$$
\Psi_0^* \delta \Psi = \Psi_0^* \delta \Psi_0, \partial_\mu (\Psi_0^* \delta \Psi) = \partial_\mu (\Psi_0^* \delta \Psi_0).
$$

Now, if we substitute (3.2) in (2.1), by using (3.3) and (3.4) we get (3.1).
**Theorem 3.3.** Any non zero Dirac solution corresponds to one and only one mass.

**Proof.** Let $\Psi$ be any Dirac solution. Let $B(\Psi)$ be the set of all masses to which $\Psi$ corresponds. Then, according to Lemma 3.1, we have $B(\Psi_0) = B(\Psi)$ for all $\Psi_0 \in [\Psi]$. Therefore it suffices to show that some element of $[\Psi]$ corresponds to exactly one mass: According to Lemma 3.2 there is one $\Psi_0 \in [\Psi]$, which satisfies (3.1). From $\Psi \neq 0$ it follows that there is one $s \in \mathbb{R}^4$ such that $\Psi^*(s) \Psi(s) \neq 0$ and consequently $\Psi_0^*(s) \Psi_0(s) \neq 0$. Therefore, from (3.1) we get the following unique expression of the mass $\kappa$ that corresponds to $\Psi_0$.

$$\kappa = \frac{i \partial_0 (\Psi_0^*(s) \gamma_4 \Psi_0(s)) - \sum_{\mu=1}^{3} \partial_{\mu} (\Psi_0^*(s) \gamma_{\mu} \Psi_0(s))}{2 \Psi_0^*(s) \Psi_0(s)}.$$

Therefore $\Psi$ corresponds to a unique mass. $\square$

**Corollary 3.4.** Let $\Psi$ be a Dirac solution corresponding to the vector fields $A_i = (a_{0i}, a_{1i}, a_{2i}, a_{3i})$, $i = 1, 2$. Then,

$$\left[ (a_{01} - a_{02})^2 - \sum_{\mu=1}^{3} (a_{\mu1} - a_{\mu2})^2 \right]|_{\text{supp}(\Psi)} = 0,$$

where $f|_B$ denotes the restriction of the function $f$ to the set $B$.

**Proof.** Since $\Psi$ corresponds to $A_i$, $i = 1, 2$, according to Theorem 3.3 there is unique mass $\kappa$ such that

$$\left[ \sum_{\mu=1}^{3} \gamma_{\mu} (\partial_{\mu} - ia_{\mu3}) - i \gamma_4 (\partial_0 + ia_{0j}) + \kappa \right] \Psi = 0, \ j = 1, 2.$$

If we subtract the above equations from each other, and then multiply the result by $\sum_{\mu=1}^{3} i \gamma_{\mu} (a_{\mu2} - a_{\mu1}) - \gamma_4 (a_{02} - a_{01})$, we get (3.5). $\square$

### 4 On degenerate and non degenerate Dirac solutions

In this section the Dirac solutions are classified into two classes. Degenerate solutions corresponding to an infinite number of 4-potentials and non-degenerate solutions corresponding to one and only one 4-potential. The 4-potentials are explicitly calculated in both cases. Further, as far as the degenerate solutions are concerned, it is proven that at least two 4-potentials are gauge inequivalent, and consequently correspond to different electromagnetic fields. This results is illustrated by an example where we provide a spinor form corresponding to two Gauge inequivalent 4-potentials. In order to proceed with these results, the following Lemmas are required:
Lemma 4.1. Let $U$ be any non empty subset of $\text{supp} (\Psi)$, where $\text{supp} (f) := \{ x : f(x) \neq 0 \}$ is the support of the function $f$. Then for any $x \in U$ we have,

$$(\Psi^T \gamma_2 \Psi)(\Psi^T \gamma_2 \Psi) + (\Psi^T \gamma_2 \gamma_4 \Psi)(\Psi^T \gamma_2 \Psi) = 0, \quad (4.1)$$

$$(\Psi^T \gamma_2 \Psi)(\Psi^T \gamma_4 \Psi) + (\Psi^T \gamma_4 \Psi)(\Psi^T \gamma_2 \Psi) = 0,$$

$$(\Psi^T \gamma_2 \Psi)(\Psi^T \gamma_2 \gamma_4 \Psi) + (\Psi^T \gamma_2 \gamma_4 \Psi)(\Psi^T \gamma_2 \Psi) = 0.$$

if and only if

$$(\Psi^T \gamma_2 \Psi)(\Psi^* \delta_4 \Psi) = 0, \quad (4.2)$$

where $\Psi = \Psi(x)$.

Proof. Supposing that

$$\Psi = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{bmatrix} \Leftrightarrow \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \Psi,$$

after some algebra, (4.1) can be written as

$$\begin{align*}
(\zeta_1 \zeta_4 - \zeta_2 \zeta_3) (\zeta_1 \zeta_2 - \zeta_3 \zeta_4) &= (\zeta_1 \zeta_4 - \zeta_2 \zeta_3) (\zeta_1 \zeta_2 - \zeta_3 \zeta_4), \\
(\zeta_1 \zeta_4 - \zeta_2 \zeta_3) (\zeta_1 \zeta_2 + \zeta_3 \zeta_4) &= (\zeta_1 \zeta_4 - \zeta_2 \zeta_3) (\zeta_1 \zeta_2 + \zeta_3 \zeta_4), \\
\zeta_1 \zeta_4 \zeta_2 \zeta_3 &= \zeta_1 \zeta_4 \zeta_2 \zeta_3,
\end{align*}$$

which by setting

$$\zeta_1 = \omega_1 \zeta_3 \text{ and } \zeta_4 = \omega_2 \zeta_2 \quad (4.4)$$

takes the following form:

$$\begin{align*}
|\zeta_2 \zeta_3|^2 [(\omega_1 \omega_2 - 1) (\omega_1 - \omega_2)] &= 0, \\
|\zeta_2 \zeta_3|^2 [(\omega_1 \omega_2 - 1) (\omega_1 + \omega_2)] &= 0, \\
|\zeta_2 \zeta_3|^2 (\omega_1 \omega_2 - \omega_1 \omega_2) &= 0.
\end{align*}$$

Consequently

$$\zeta_2 = 0 \text{ or } \zeta_3 = 0 \text{ or } \begin{cases} (\omega_1 \omega_2 - 1) (\omega_1 - \omega_2) = 0, \\
(\omega_1 \omega_2 - 1) (\omega_1 + \omega_2) = 0, \end{cases}$$

which is equivalent to

$$\zeta_2 = 0 \text{ or } \zeta_3 = 0 \text{ or } \omega_1 \omega_2 = 1 \text{ or } \begin{cases} \omega_1 - \omega_2 - \omega_1 + \omega_2 = 0, \\
\omega_1 + \omega_2 = 0, \end{cases}$$

or

$$\zeta_2 = 0 \text{ or } \zeta_3 = 0 \text{ or } \omega_1 \omega_2 - 1 = 0 \text{ or } \omega_1 + \omega_2 = 0,$$
or
\[|\zeta_2|^2|\zeta_3|^2(\omega_1\omega_2 - 1)(\omega_1 + \frac{1}{\omega_2}) = 0,\]

Using (4.4) the above relation takes the form
\[(\zeta_1\zeta_4 - \zeta_2\zeta_3)(\zeta_1\bar{\zeta}_2 + \zeta_3\bar{\zeta}_4) = 0,\]

which through (4.3) can be rewritten as (4.2).

**Lemma 4.2.** Let \(U\) be any non empty open subset of \(\text{supp}(\Psi)\). Then for any \(x \in U\) we have
\[\Psi^T\gamma_2\Psi = \Psi^*\delta_4\Psi = 0,\]
(4.5)

if and only if
\[\Psi = [\psi_1 \psi_2 \psi_1 \psi_2]^T\] or \([\psi_1 \psi_2 - \psi_1 - \psi_2]^T\],
where \(\Psi = \Psi(x)\) and \(\psi_\mu = \psi_\mu(x)\).

**Proof.** Setting (4.3) in (4.5) we get
\[\zeta_1\zeta_4 = \zeta_2\zeta_3, \quad \bar{\zeta}_1\zeta_2 + \bar{\zeta}_3\zeta_4 = 0,\] (4.6)

Suppose that for some \(x \in U\), there holds \(\zeta_1\zeta_2\zeta_4\neq 0\). Then from (4.6) we easily obtain
\[\left|\frac{\zeta_1}{\zeta_3}\right|^2 = -1.\]

Therefore for any \(x \in U\) holds \(\zeta_1\zeta_2\zeta_4 = 0\). Hence, from (4.6) we have that for any \(x \in \text{supp}(\Psi)\)
\[\zeta_1 = \zeta_3 = 0\] or \(\zeta_2 = \zeta_4 = 0\)

which using (4.3) can be rewritten as
\[\psi_4 + \psi_2 = \psi_3 + \psi_1 = 0\] or \(\psi_4 - \psi_2 = \psi_3 - \psi_1 = 0\).

**Lemma 4.3.** Let \(\Psi\) be a Dirac solution. If \(\Psi = [\psi_1 \psi_2 \psi_1 \psi_2]^T\) or \(\Psi = [\psi_1 \psi_2 - \psi_1 - \psi_2]^T\) in an open set \(U\), then for \(\kappa \neq 0\) we have \(\Psi = 0\) in \(U\) while for \(\kappa = 0\) we have either \(\Psi = 0\) or \(\Psi\) is degenerate.

**Proof.** If \(\Psi = [\psi_1 \psi_2 - \psi_1 - \psi_2]^T\), then from (2.1) we obtain the following system of equations (S),
\[
\begin{align*}
i\partial_1\psi_2 + \partial_2\psi_1 + i\partial_3\psi_1 - i\partial_0\psi_1 & = -a_1\psi_2 + ia_2\psi_2 - a_3\psi_1 - a_0\psi_1 - \kappa\psi_1 \\
i\partial_1\psi_1 - \partial_2\psi_1 - i\partial_3\psi_2 - i\partial_0\psi_2 & = -a_1\psi_1 - ia_2\psi_1 + a_3\psi_2 - a_0\psi_2 - \kappa\psi_2 \\
i\partial_1\psi_1 + \partial_2\psi_2 + i\partial_3\psi_1 - i\partial_0\psi_1 & = -a_1\psi_2 + ia_2\psi_2 - a_3\psi_1 - a_0\psi_1 + \kappa\psi_1 \\
i\partial_1\psi_1 - \partial_2\psi_1 - i\partial_3\psi_2 - i\partial_0\psi_2 & = -a_1\psi_1 - ia_2\psi_1 + a_3\psi_2 - a_0\psi_2 + \kappa\psi_2.
\end{align*}
\]

We distinguish the following two cases:
Case 1: \( \kappa \neq 0 \), then subtracting the first equation from the third and the second from the fourth we obtain respectively \( 2 \kappa \psi_1 = 0, 2 \kappa \psi_2 = 0 \). Therefore \( \psi_1 = \psi_2 = 0 \).

Case 2: \( \kappa = 0 \), then (S) obtains the following equivalent form:

\[
\begin{align*}
-a_1 \psi_2 + ia_2 \psi_2 - a_3 \psi_1 - a_0 \psi_1 &= i \partial_1 \psi_2 + \partial_2 \psi_2 + i \partial_3 \psi_1 - i \partial_0 \psi_1 \\
-a_1 \psi_1 - ia_2 \psi_1 + a_3 \psi_2 - a_0 \psi_2 &= i \partial_1 \psi_1 - \partial_2 \psi_1 - i \partial_3 \psi_2 - i \partial_0 \psi_2,
\end{align*}
\]

A trivial solution of (S) is \( \Psi = 0 \). We suppose, (S) has also a solution \( \Psi \neq 0 \).

Setting \( k_i = \text{Re} (\psi_i), l_i = \text{Im} (\psi_i) \), \( \text{Re} (c_i) = n_i, \text{Im} (c_i) = n_i, i = 1, 2 \), where \( c_1 = i \partial_1 \psi_2 + \partial_2 \psi_2 + i \partial_3 \psi_1 - i \partial_0 \psi_1 \) and \( c_2 = i \partial_1 \psi_1 - \partial_2 \psi_1 - i \partial_3 \psi_2 - i \partial_0 \psi_2 \), these two equations can equivalently be written as

\[
\begin{align*}
-k_1 a_0 - k_2 a_1 - l_2 a_2 - k_1 a_3 &= m_1 \\
-l_1 a_0 - l_2 a_1 + k_2 a_2 - l_1 a_3 &= n_1 \\
-k_2 a_0 - k_1 a_1 + l_1 a_2 + k_3 a_3 &= m_2 \\
-l_2 a_0 - l_1 a_1 - k_1 a_2 + a_3 l_2 &= n_2
\end{align*}
\]

At this point we attempt to solve the above real system for the four unknowns \( a_0, a_1, a_2, a_3 \). We may calculate \( \det (P) = 0 \), where

\[
P = \begin{bmatrix}
-k_1 & -k_2 & -l_2 & -k_1 \\
-l_1 & -l_2 & k_2 & -l_1 \\
-k_2 & -k_1 & l_1 & k_2 \\
-l_2 & -l_1 & -k_1 & l_2
\end{bmatrix}.
\]

Therefore the above system of equations has either no solutions or has an infinite number of solutions. Finally, if \( \Psi = [\psi_1 \ \psi_2 \ \psi_1 \ \psi_2]^T \), in a manner similar to the above, we can show that in any case either \( \Psi = 0 \) or \( \Psi \) is degenerate.

In this point we introduce the following notation: Let \( \Psi \) be a Dirac solution. Then

\[ S (\Psi) := \text{supp} (\Psi^* \gamma_4 \Psi) \cup \text{supp} (\Psi^* \gamma_5 \gamma_4 \Psi), \]

**Remark 4.4.** For any Dirac solution \( \Psi \) we have

\[ \text{supp} (\Psi^* \delta_4 \Psi) = S (\Psi). \] \hspace{1cm} (4.7)

**Proof.** For any \( s \in \mathbb{R}^4 \setminus \text{supp} (\Psi^* \delta_4 \Psi) \) we have

\[ \left( \Psi^* \gamma_4 \Psi + \Psi^* \gamma_5 \gamma_4 \Psi \right) \bigg|_{x=s} = 0. \] \hspace{1cm} (4.8)

Since \( \gamma_4 \) is Hermitian and \( \gamma_5 \gamma_4 \) anti-Hermitian, the number \( \left( \Psi^* \gamma_4 \Psi \right) \big|_{x=s} \) is real while \( \Psi^* \gamma_5 \gamma_4 \Psi \big|_{x=s} \) is imaginary. So from (4.8) we get

\[ \left( \Psi^* \gamma_4 \Psi \right) \big|_{x=s} = \Psi^* \gamma_5 \gamma_4 \Psi \big|_{x=s} = 0, \]

that is

\[
\begin{align*}
\mathbb{R}^4 \setminus \text{supp} (\Psi^* \delta_4 \Psi) \\
= (\mathbb{R}^4 \setminus \text{supp} (\Psi^* \gamma_4 \Psi)) \cap (\mathbb{R}^4 \setminus \text{supp} (\Psi^* \gamma_5 \gamma_4 \Psi)).
\end{align*}
\]

Consequently, (4.7) is true. \( \square \)
Theorem 4.5. Let \( \Psi \), with \( \text{supp} (\Psi) \neq \emptyset \), be a Dirac solution corresponding to a mass \( \kappa \) and to a 4-potential \((a_0, a_1, a_2, a_3)\).

1. If \( \Psi^* \delta_4 \Psi = 0 \), then \( \Psi \) is degenerate. Specifically for \( \kappa \neq 0 \) we have:
   i. \( \text{supp} \left( \frac{T \gamma_2 \Psi}{\Psi} \right) = \text{supp} (\Psi) \), \hspace{1cm} (4.9)
   ii. The restrictions, to any non empty open subset \( U \) of \( \text{supp} (\Psi) \), of the functions
      \( \Theta_1 := -i \frac{T \gamma_5 \gamma_3 \Psi}{\Psi}, \Theta_2 := -i \frac{T \gamma_4 \Psi}{\Psi}, \Theta_3 := i \frac{T \gamma_5 \gamma_3 \Psi}{\Psi} \) \hspace{1cm} (4.10)
      are well defined and real,
   iii. for any non empty open subset \( U \) of \( \text{supp} (\Psi) \), we have that \( \text{P} (\Psi, U) \) contains infinitely many elements. More specifically, the set \( \text{P} (\Psi, U) \) is given by
      \[ \text{P} (\Psi, U) = \{ (a_0 + f, a_1 + f \Theta_1, a_2 + f \Theta_2, a_3 + f \Theta_3) / f : \text{supp} (\Psi) \to \mathbb{R} \} \].

2. If \( \Psi^* \delta_4 \Psi \) is not identically zero, that is \( U := S (\Psi) \) is non empty, then \( \Psi \) is non degenerate. More specifically, \( \text{P} (\Psi, U) \) contains exactly one element \((a_0, a_1, a_2, a_3)\) which is given by

   \[
   \begin{bmatrix}
   a_1 \\
   a_2 \\
   a_3 \\
   -ia_0
   \end{bmatrix}
   = \frac{i}{\Psi^* \delta_4 \Psi} \kappa \begin{bmatrix}
   \Psi^* \gamma_1 \gamma_4 \Psi \\
   \Psi^* \gamma_2 \gamma_4 \Psi \\
   \Psi^* \gamma_3 \gamma_4 \Psi \\
   \Psi^* \Psi
   \end{bmatrix}
   \]

   \[
   + \frac{1}{2} \begin{bmatrix}
   i \partial_0 & \partial_3 & -\partial_2 & \partial_1 \\
   -\partial_3 & i \partial_0 & \partial_1 & \partial_2 \\
   \partial_2 & -\partial_1 & i \partial_0 & \partial_3 \\
   -\partial_1 & -\partial_2 & -\partial_3 & i \partial_0
   \end{bmatrix}
   \begin{bmatrix}
   \Psi^* \delta_1 \Psi \\
   \Psi^* \delta_2 \Psi \\
   \Psi^* \delta_3 \Psi \\
   \Psi^* \delta_4 \Psi
   \end{bmatrix}
   \] \hspace{1cm} (4.11)

Proof. 1. First, the case \( \kappa \neq 0 \) is considered.

   i.: From the continuity of \( \Psi \) in its domain, it follows that \( \text{supp} \left( \frac{T \gamma_2 \Psi}{\Psi} \right) \) and \( \text{supp} (\Psi) \) are open sets in \( \mathbb{R}^4 \). Therefore it is sufficient to prove that \( \text{supp} \left( \frac{T \gamma_2 \Psi}{\Psi} \right) \) is dense in \( \text{supp} (\Psi) \): Supposing that \( \text{supp} \left( \frac{T \gamma_2 \Psi}{\Psi} \right) \) is not dense in \( \text{supp} (\Psi) \), then there is a non empty open set \( U \subseteq \text{supp} (\Psi) \) such that

   \[ \left. \frac{T \gamma_2 \Psi}{\Psi} \right|_U = 0. \] \hspace{1cm} (4.12)

   Moreover from the condition \( \Psi^* \delta_4 \Psi = 0 \) it follows

   \[ \left. \Psi^* \delta_4 \Psi \right|_U = 0. \] \hspace{1cm} (4.13)

   Now from (4.12) and (4.13), according to Lemma 4.2 we get that the restriction of \( \Psi \) to \( U \) has the following form: either \( \Psi = [\psi_1 \psi_2 \psi_3 \psi_4]^T \) or
\[ \psi_1 \psi_2 - \psi_1 - \psi_2 \] . Therefore from Lemma 4.3 we get that \( \Psi = 0 \) restricted to \( U \). This is a contradiction since \( \varnothing \neq U \subseteq \text{supp} (\Psi) \).

ii. From (4.9) we have that \( \Theta_1, \Theta_2, \Theta_3 \) are well defined in any \( U \subseteq \text{supp} (\Psi) \), and from Lemma 4.1 that \( \Theta_1, \Theta_2, \Theta_3 \) are all real.

iii. Since \( \Psi \) corresponds to the mass \( \kappa \) and also to the potential \((a_0, a_1, a_2, a_3)\) it follows that \( \Psi \) is a solution of (2.1). Let \( f \) be any function defined in \( \text{supp} (\Psi) \).

Then (2.1) can be rewritten as
\[
\left[ 3 \sum_{\mu=1}^{3} \gamma_{\mu} \left( \partial_{\mu} - i (a_{\mu} + f \Theta_{\mu}) \right) - i \gamma_{4} \left( \partial_{0} + i (a_{0} + f) \right) + \kappa \right] \Psi \\
= -f \left[ i \sum_{\mu=1}^{3} \Theta_{\mu} \gamma_{\mu} - \gamma_{4} \right]
\]

Further after some algebraic calculations we easily get
\[
i \sum_{\mu=1}^{3} \Theta_{\mu} \gamma_{\mu} - \gamma_{4} = 0.
\]

Combining the last two relations we obtain
\[
\left[ 3 \sum_{\mu=1}^{3} \gamma_{\mu} \left( \partial_{\mu} - i (a_{\mu} + f \Theta_{\mu}) \right) - i \gamma_{4} \left( \partial_{0} + i (a_{0} + f) \right) + \kappa \right] \Psi = 0.
\]

Therefore \( \Psi \) corresponds to \((a_0 + f, a_1 + f \Theta_1, a_2 + f \Theta_2, a_2 + f \Theta_3)\) and consequently \( \Psi \) is degenerate.

Conversely: Suppose that \( \Psi \) corresponds also to another 4-potential \((b_0, b_1, b_2, b_3)\).

Then \( \Psi \) satisfies (2.1) for the 4-potential \((b_0, b_1, b_2, b_3)\) and according to Theorem 3.3 for the same mass \( \kappa \). That is
\[
\left[ 3 \sum_{\mu=1}^{3} \gamma_{\mu} \left( \partial_{\mu} - ib_{\mu} \right) - i \gamma_{4} \left( \partial_{0} + ib_{0} \right) + \kappa \right] \Psi = 0. \tag{4.14}
\]

Subtracting (2.1) from (4.14) we obtain
\[
\left[ i \sum_{\mu=1}^{3} (b_{\mu} - a_{\mu}) \gamma_{\mu} - \gamma_{4} (b_{0} - a_{0}) \right] \Psi = 0.
\]

Now, multiplying the last equation with \( \Psi^T \gamma_1 \gamma_2, \Psi^T, \Psi^T \gamma_2 \gamma_3 \) and using the fact that the matrices \( \gamma_1 \gamma_2 \gamma_{\mu}, \mu = 2, 3; \gamma_{\mu} \gamma_\mu, \mu = 1, 3; \gamma_2 \gamma_3 \gamma_{\mu}, \mu = 1, 2 \) are antisymmetric, we get
\[
i (b_1 - a_1) \Psi^T \gamma_2 \Psi = -(b_0 - a_0) \Psi^T \gamma_1 \gamma_2 \gamma_4 \Psi,
\]
\[
i (b_2 - a_2) \Psi^T \gamma_2 \Psi = (b_0 - a_0) \Psi^T \gamma_4 \Psi,
\]
\[
i (b_3 - a_3) \Psi^T \gamma_2 \Psi = (b_0 - a_0) \Psi^T \gamma_2 \gamma_3 \gamma_4 \Psi.
\]
From the last three relations, choosing \( f = b_0 - a_0 \), we immediately get \((b_0, b_1, b_2, b_3) = (a_0 + f, a_1 + f \Theta_1, a_2 + f \Theta_2, a_3 + f \Theta_3)\).

In the case that \( \kappa = 0 \), we distinguish the following two subcases:

a: \( \text{supp}(\Psi^T \gamma_2 \Psi) \neq \emptyset \), then in similar manner as above we can show that
\[
\begin{align*}
P(\Psi, V) &= \{(a_0 + f, a_1 + f \Theta_1, a_2 + f \Theta_2, a_3 + f \Theta_3) \mid f : V \to \mathbb{R}\},
\end{align*}
\]
where \( V = \text{supp}(\Psi^T \gamma_2 \Psi) \).

b: \( \Psi^T \gamma_2 \Psi = 0 \), then according to Lemma 4.3 we have either \( \Psi \) is degenerate or \( \Psi = 0 \). If \( \Psi = 0 \), then \( \Psi \) satisfies the claims 1.i, 1.ii and 1.iii. of the present Theorem. Therefore in any case \( \Psi \) is degenerate.

2. If we add (2.2) with (2.6), (2.3) with (2.7), (2.4) with (2.8), and (2.5) with (2.9) we respectively obtain,
\[
\begin{align*}
2a_0 \Psi^* \delta_4 \Psi + 2 \kappa \Psi^* \Psi
&= -\partial_1 (\Psi^* \delta_1 \Psi) - \partial_2 (\Psi^* \delta_2 \Psi) - \partial_3 (\Psi^* \delta_3 \Psi) + i \partial_0 (\Psi^* \delta_4 \Psi). \\
2ia_1 \Psi^* \delta_4 \Psi &= -2 \kappa \Psi^* \gamma_1 \gamma_4 \Psi + \partial_1 (\Psi^* \delta_4 \Psi) - \partial_2 (\Psi^* \delta_3 \Psi) \quad (4.16) \\
&+ \partial_3 (\Psi^* \delta_2 \Psi) + i \partial_0 (\Psi^* \delta_1 \Psi), \\
2ia_2 \Psi^* \delta_4 \Psi
&= -2 \kappa \Psi^* \gamma_2 \gamma_4 \Psi + \partial_1 (\Psi^* \delta_3 \Psi) - \partial_2 (\Psi^* \delta_4 \Psi) \quad (4.17) \\
&- \partial_3 (\Psi^* \delta_1 \Psi) + i \partial_0 (\Psi^* \delta_2 \Psi), \\
2ia_3 \Psi^* \delta_4 \Psi &= -2 \kappa \Psi^* \gamma_3 \gamma_4 \Psi - \partial_1 (\Psi^* \delta_2 \Psi) + \partial_2 (\Psi^* \delta_1 \Psi) \quad (4.18) \\
&+ \partial_3 (\Psi^* \delta_4 \Psi) + i \partial_0 (\Psi^* \delta_3 \Psi).
\end{align*}
\]
Finally, it is easy to verify, that the system of equations (4.15), (4.16), (4.17), (4.18) can be rewritten as (4.11). The proof is complete.

**Remark 4.6.** It is easy to prove, that \( \Psi^* \delta_4 \Psi = 0 \) if and only if \( \Psi \) has the following form,
\[
\Psi = u \begin{bmatrix} \frac{w}{w} \\ 1 \\ \frac{w}{w} \\ 1 \end{bmatrix} + v \begin{bmatrix} 1 \\ -w \\ -1 \\ w \end{bmatrix}. \quad (4.19)
\]

**Corollary 4.7.** Let \( \Psi \) be any degenerate non zero Dirac solution corresponding to a mass \( \kappa \neq 0 \). Then there are at least two gauge inequivalent 4-potentials connected to \( \Psi \).

**Proof.** According to Theorem 4.4.1.i, \( \Psi^T \gamma_2 \Psi \) is non zero at any point in \( \text{supp}(\Psi) \). If \( \Psi \) corresponds to \( \mathbf{a} = (a_0, a_1, a_2, a_3) \), then from Theorem 4.4.1.iii, choosing
\( f = 1 \) and \( f = x_0 \), we have that the restriction of \( \Psi \) to \( U = \text{supp}(\Psi) \) corresponds also to
\[
a_1 = (a_0 + 1, a_1 + \Theta_1, a_2 + \Theta_2, a_3 + \Theta_3),
\]
and to
\[
a_2 = (a_0 + x_0, a_1 + x_0 \Theta_1, a_2 + x_0 \Theta_2, a_3 + x_0 \Theta_3),
\]
where \( \Theta_1, \Theta_2, \Theta_3 \) are the same as defined in (4.10). We suppose that the 4-potentials \( a_1 \) and \( a_2 \) are gauge equivalent to \( a \). That is the fields \((1, \Theta_1, \Theta_2, \Theta_3), (x_0, x_0 \Theta_1, x_0 \Theta_2, x_0 \Theta_3)\) are conservative. Therefore
\[
\partial_0 \Theta_\mu = 0, \quad \mu = 1, 2, 3, \quad (4.20)
\]
and
\[
x_0 \partial_0 \Theta_\mu + \Theta_\mu = 1, \quad \mu = 1, 2, 3. \quad (4.21)
\]
From (4.20) and (4.21) follows that \( \Theta_\mu = 1, \quad \mu = 1, 2, 3 \), which is equivalent to
\[
i \Psi^T \gamma_1 \gamma_2 \gamma_3 \Psi = -i \Psi^T \gamma_4 \Psi = -i \Psi^T \gamma_2 \gamma_4 \Psi = \Psi^T \gamma_2 \Psi.
\]
Now, after some algebra, we find the set \( L \) of all solutions \( \Psi \) of the above system of bilinear equations, given by
\[
L = \left\{ \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \begin{bmatrix} \psi_1 \\ -\psi_1 \end{bmatrix}, \begin{bmatrix} \psi_1 \\ -\psi_1 \end{bmatrix}, \begin{bmatrix} \psi_2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \psi_2 \end{bmatrix}, \begin{bmatrix} 0 \\ \psi_1 \end{bmatrix} \right\}
\]
Finally it is easy to verify that all elements of \( L \) satisfy identically the equation \( \Psi^T \gamma_2 \Psi = 0 \). Contradiction. Therefore at least one of the 4-potentials \( a_1, a_2 \) is gauge inequivalent to \( a \).

Here, it should be noted that the above corollary is very interesting and to some extend surprising from a physical point of view, because it practically means that for particles described by Dirac spinors satisfying the condition \( \Psi^* \delta_4 \Psi = 0 \), explicitly given by (4.19), it is possible to be in the same state under the influence of different electromagnetic fields. Below, we provide an example to illustrate this argument.

Now we will show the existence of degenerate Dirac solutions: Let \( \Psi \) be given by
\[
\Psi = \begin{bmatrix} e^{i(\phi_1 - \phi_2)} \cos \alpha \\ e^{i(\phi_1 - \phi_2)} \sin \alpha \\ e^{p_1 x_1 + p_2 x_2 + p_3 x_3 + p_0 x_0} \end{bmatrix}, \quad (4.22)
\]
\[ p_0 = -\kappa \tan \phi_2, \]  
\[ p_1 = i\kappa \frac{1 + e^{2i\phi_2} \sin^2 \alpha - e^{2i\phi_1} \cos^2 \alpha}{2e^{i\phi_1} (\cos \alpha) \cos \phi_2}, \]  
\[ p_2 = \kappa \frac{1 + e^{2i\phi_1} \cos^2 \alpha + e^{2i\phi_2} \sin^2 \alpha}{2e^{i\phi_2} (\cos \alpha) \cos \phi_2}, \]  
\[ p_3 = i\kappa \frac{e^{i\phi_2} (\sin \alpha)}{ \cos \phi_2}. \]

\[ \phi_1, \phi_2, \alpha \in \mathbb{R} \text{ with } \alpha, \phi_2 \neq \frac{(2k + 1) \pi}{2}, k \in \mathbb{Z}. \]

Then, after some calculations, it is easy to verify that \( \Psi \) satisfies the force-free Dirac equation, that is
\[ \left[ \sum_{\mu=1}^{3} \gamma_{\mu} \partial_{\mu} - i \gamma_4 \partial_0 + \kappa \right] \Psi = 0, \] as well as the condition \( \Psi^* \delta_4 \Psi = 0 \). Therefore, according to Theorem (4.5), \( \Psi \) is degenerate and corresponds also to the following 4-potentials
\[ (f, -f \cos \alpha \cos (\phi_2 - \phi_1), f \cos \alpha \sin (\phi_2 - \phi_1), -f \sin \alpha) \]
for any function of the spatial and temporal variables \( f : \mathbb{R}^4 \rightarrow \mathbb{R} \). Thus, the spinor given by (4.22) corresponds both to the force-free Dirac equation (zero electromagnetic field) and to the 4-potential given by (4.24) (non-zero electromagnetic field). This is quite a surprising result with many implications, having the potential to revolutionize our view on the interactions of charged particles with electromagnetic fields.

## 5 Summary

In the present study it has been demonstrated that a Dirac solution \( \Psi \) corresponds to one and only one electromagnetic 4-potential, if and only if the quantity \( \Psi^* (\gamma_4 + \gamma_1 \gamma_2 \gamma_3) \Psi \) is not identically equal to zero. Any such Dirac solution is called non-degenerate. When a Dirac solution satisfies the condition \( \Psi^* (\gamma_4 + \gamma_1 \gamma_2 \gamma_3) \Psi = 0 \), it is called degenerate and corresponds to an infinite number of electromagnetic 4-potentials, where it is proven that at least two of them are gauge inequivalent. An example is provided to illustrate this case. Since one Dirac solution describes the state of a Dirac particle, it is strange and surprising the fact that it is possible for a Dirac particle to be in the same state under the presence of different electromagnetic fields. Finally it has been proven that every Dirac solution corresponds to one and only one value of the mass of the Dirac particle. This means that it is impossible for two Dirac particles with different mass to be in the same state.

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