Online inference in Markov modulated nonlinear dynamic systems: a Rao-Blackwellized particle filtering approach

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Abstract
The Markov modulated (switching) state space is an important model paradigm in applied statistics. In this article, we specifically consider Markov modulated nonlinear state-space models and address the online Bayesian inference problem for such models. In particular, we propose a new Rao-Blackwellized particle filter for the inference task which is our main contribution here. The detailed descriptions including an algorithmic summary are subsequently presented.

Keywords:
Rao-Blackwellized particle filter, Markov modulated systems, Markov regime switching, switching nonlinear state-space, Jump Markov nonlinear systems

1. Introduction

In many practical applications of applied science, engineering and econometrics, one often deals with nonlinear dynamic systems involving both a continuous value target state and a discrete value regime variable. Such descriptions imply that the system can switch between different nonlinear dynamic regimes, where the parameters of each regime is governed by the corresponding regime variable. The different regimes can possibly be described in terms of different stochastic processes. The regime variable also
evolves dynamically according to a finite state Markov chain. Both the target state and regime variable are latent and are related to the noisy observations. This model paradigm is often referred to as a Markov regime switching (MRS) state space, sometimes with other monikers like jump Markov, Markov modulated or hybrid dynamic system. Due to its modeling flexibility, MRS is very popular in different disciplines and as such, has been successfully used in diverse areas like econometrics, operations research, control theory, and signal processing, population dynamics, and machine learning among others (Fruhwirth-Schnatter, 2007; Ristic et al., 2004; Luo and Mao, 2007; Mamon and Elliott, 2007; Barber, 2012). However, most of the studies have focused on a special case where each individual regime follows a linear Gaussian state-space model. This special case is known as the jump Markov linear system (JMLS). Nonetheless, for many practical applications of interest, including econometrics (Carvalho and Lopes, 2003), signal processing (Andrieu et al., 2003), target tracking and localization (Driessen and Boers, 2005) are some examples, the individual regimes follows nonlinear dynamics, possibly driven by non-Gaussian processes. Such a system is referred to as a Markov modulated nonlinear dynamic system (MmNDS) or a jump Markov nonlinear system (JMNS). Compared to JMLS, this class of problem is less well studied. Hence we consider the state inference problem for MmNDS.

For certain models, part of the state space may be (conditionally) tractable. It is then sufficient to employ a particle filter (PF) for the remaining intractable part of the state space. By exploiting such analytical substructure, the Monte Carlo based estimation is then confined to a space of lower dimension. Consequently, the estimate obtained is often better and never worse than the estimate provided by the PF targeting the full state space. This efficiency arises due to the implication of the well-known Rao-Blackwell estimator (see the Appendix). For this reason, the resulting method is popularly known as Rao-Blackwellized particle filtering (RBPF) (Chen and Liu, 2000; Doucet et al., 2000; Chopin, 2007; Schönh et al., 2005; Hendeby et al., 2010; Saha et al., 2010).

In this article, we address the online inference problem for MmNDS using PF. Particularly, we propose a new RBPF framework using the conditionally analytical substructure of the regime indicator variable. To the best of our knowledge, this RBPF framework has not yet been exploited.

The organization of the article is as follows. In Section 2, we provide a brief but necessary PF background. This is followed by the problem statement in Section 3, where we first describe the model and then pose the
inference objective. In Section 4, the derivations for the new RBP F scheme are outlined for one complete cycle and an algorithm is also presented. In Section 5, we provide comparisons to other similar models. Finally, we concluded in Section 6.

2. Brief background on Particle filter (PF)

Consider the following discrete time general state-space model relating the latent state $x_k$ to the observation $y_k$ as

$$x_k = f(x_{k-1}, w_{k-1}), \quad (1a)$$
$$y_k = h(x_k, e_k), \quad (1b)$$

where $f(x_{k-1}, w_{k-1})$ describes how the state propagates driven by the process noise $w_{k-1}$, and $h(x_k, e_k)$ describes how the measurements relates to the state and how the measurement is affected by noise, $e_k$. This model can also be expressed with a probabilistic model

$$x_k \sim p(x_k | x_{k-1}), \quad (2a)$$
$$y_k \sim p(y_k | x_k), \quad (2b)$$

where $p(x_k | x_{k-1})$ and $p(y_k | x_k)$ are the corresponding state transition and observation likelihood densities, which are here assumed to be known. Given this model, the density for the initial state (i.e., $p(x_0)$) and the stream of observations $y_{0:k} \equiv \{y_0, y_1, \ldots, y_k\}$ up to time $k$, the inference objective is to optimally estimate the sequence of posterior densities $p(x_{0:k} | y_{1:k})$, and typically their marginals $p(x_k | y_{1:k})$, over time. The above posteriors are in general intractable but can be approximated using PF to arbitrary accuracy. In PF, the posterior distribution associated with the density $p(x_{0:k} | y_{1:k})$ is approximated by an empirical distribution induced by a set of $N$ weighted particles (samples) as

$$\hat{P}_N(dx_{0:k} | y_{1:k}) = \sum_{i=1}^{N} \tilde{w}_k^{(i)} \delta_{x_{0:k}^{(i)}} (dx_{0:k}), \quad (3)$$

where $\delta_{x_{0:k}^{(i)}} (A)$ is a Dirac measure for a given $x_{0:k}^{(i)}$ and a measurable set $A$, and $\tilde{w}_k^{(i)}$ is the associated weight attached to each particle $x_{0:k}^{(i)}$, such that
\[ \sum_{i=1}^{N} \tilde{w}_k^{(i)} = 1. \] Given this PF output, one can approximate the marginal distribution associated with \( p(x_k|y_{1:k}) \) as

\[
\hat{P}_N(dx_k|y_{1:k}) = \sum_{i=1}^{N} \tilde{w}_k^{(i)} \delta_{x_k^{(i)}}(dx_k),
\]

and expectations of the form

\[
I(g_k) = \int g_k(x_{0:k}) p(x_{0:k}|y_{1:k}) \, dx_{0:k}
\]

as

\[
\hat{I}_N(g_k) = \int g_k(x_{0:k}) \hat{P}_N(dx_{0:k}|y_{1:k})
\approx \sum_{i=1}^{N} \tilde{w}_k^{(i)} g_k(x_{0:k}^{(i)}).
\]

Even though the distribution \( \hat{P}_N(dx_{0:k}|y_{1:k}) \) does not admit a well defined density with respect to the Lebesgue measure, the density \( p(x_{0:k}|y_{1:k}) \) is conventionally represented as

\[
\hat{p}_N(x_{0:k}|y_{1:k}) = \sum_{i=1}^{N} \tilde{w}_k^{(i)} \delta(x_{0:k} - x_{0:k}^{(i)}),
\]

where \( \delta(\cdot) \) is the Dirac-delta function. The notation used in (7) is not mathematically rigorous; however, it is intuitively easier to follow than the stringent measure theoretic notations. This is especially useful if we are not concerned with theoretical convergence studies.

Now suppose at time \( k-1 \), we have a weighted particle approximation of the posterior \( p(x_{0:k-1}|y_{1:k-1}) \) as \( \hat{P}_N(dx_{0:k-1}|y_{1:k-1}) = \sum_{i=1}^{N} \tilde{w}_{k-1}^{(i)} \delta_{x_{0:k-1}^{(i)}}(dx_{0:k-1}) \). With the arrival of a new measurement \( y_k \), we wish to approximate \( p(x_{0:k}|y_{1:k}) \) with a new set of samples. The particles are propagated to time \( k \) by sampling a new state \( x_k^{(i)} \) from a proposal kernel \( \pi(x_k|x_{0:k-1}^{(i)}, y_{1:k}) \) and setting \( x_{0:k}^{(i)} \triangleq \left(x_{0:k-1}^{(i)}, x_k^{(i)} \right) \). Since we have

\[
p(x_{0:k}|y_{1:k}) \propto p(y_k|x_{0:k}, y_{1:k-1}) p(x_k|x_{0:k-1}, y_{1:k-1}) p(x_{0:k-1}|y_{1:k-1})
\]
and using the Markovian property (2), the corresponding weights of the particles are obtained as

$$
w_k^{(i)} \propto \tilde{w}_k^{(i)} \frac{p(y_k|x_k^{(i)})p(x_k^{(i)}|x_{k-1}^{(i)})}{\pi(x_k^{(i)}|x_{0:k-1}, y_{1:k})}$$

(9)

$$
\tilde{w}_k^{(i)} = \frac{w_k^{(i)}}{\sum_{j=1}^{N} w_k^{(j)}}.
$$

(10)

To avoid carrying trajectories with small weights and to concentrate upon the ones with large weights, the particles need to be resampled regularly. When resampling, new particles are sampled with replacement from the old ones with the probabilities \(\{\tilde{w}_k^{(i)}\}_{i=1}^{N}\). The effective sample size \(N_{\text{eff}}\), a measure of how many particles that actually contributes to the approximation of the distribution, is often used to decide when to resample. When \(N_{\text{eff}}\) drops below a specified threshold, resampling is performed. For a more general introduction to PF, refer to Doucet and Johansen (2011).

3. Problem Statement

In this section, we first provide a description of the model and subsequently pose the estimation objectives.

3.1. Model description:

Consider the following (hybrid) nonlinear state-space model evolving according to

$$
\Pi(r_k|r_{k-1}),
$$

(11a)

$$
p_{\theta_{r_k}}(x_k|x_{k-1}, r_k),
$$

(11b)

$$
p_{\theta_{r_k}}(y_k|x_k, r_k),
$$

(11c)

where \(r_k \in S \triangleq \{1, 2, \ldots, s\}\), is a (discrete) regime indicator variable with finite number of regimes (i.e., categorical variable), \(x_k \in \mathbb{R}^{n_x}\) is the (continuous) state variable. As the system can switch between different dynamic regimes, for a given regime variable \(l \in S\), the corresponding dynamic regime can be characterized by a set of parameters \(\theta_l\). Both \(x_k\) and \(r_k\) are latent variables, which are related to the measurement \(y_k \in \mathbb{R}^{n_y}\). The time behavior of the regime variable \(r_k\) is commonly modeled by a homogeneous
(time-invariant) first order Markov chain with transition probability matrix (TPM) \( \Pi = [\pi_{ij}]_{ij} \) as

\[
\pi_{ij} \triangleq \mathbb{P}(r_k = j | r_{k-1} = i) \quad (i, j \in S),
\]

(12a)

\[
\pi_{ij} \geq 0; \quad \sum_{j=1}^{s} \pi_{ij} = 1,
\]

(12b)

This model is represented graphically in Figure 1. We also present below the following examples illustrating some real life applications where the above model is used.

**Example 1:** Consider the Markov switching stochastic volatility model (Carvalho and Lopes, 2003), where \( x_k \) is the latent time varying log-volatility, \( y_k \) is the observed value of daily return of stock price or index. The regime variable \( r_k \) is modeled as a \( K \)-state first order Markov process. The model is further specified as

\[
p_{\theta_{rk}}(x_k|x_{k-1}, r_k) = \mathcal{N}(\alpha_{rk} + \phi x_{k-1}, \sigma^2),
\]

(13a)

\[
p(y_k|x_k, r_k) = \mathcal{N}(0, e^{x_k/2}),
\]

(13b)

where the parameter vector is given by \( \theta_{rk} \triangleq \{\alpha_{rk}, \phi, \sigma\} \).

**Example 2:** Consider an altitude based terrain navigation framework (Schön et al., 2005). To keep the description simple, assume that an aircraft is traveling in an one dimensional space (e.g., on a manifold). The aircraft is assumed to follow a constant velocity model. The state-space model is given as

\[
x_{k+1} = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} x_k + \begin{pmatrix} \frac{1}{2} T^2 \\ T \end{pmatrix} w_k
\]

(14a)

\[
y_k = h(x_k) + e_k(r_k),
\]

(14b)
where $T$ is the sampling period, $w_k$ and $e_k(\cdot)$ are the process and the measurement noise, respectively, commonly assumed to be individually independent and also independent of each other. The aircraft latent state $x_k$ consists of position and velocity. The observation $y_k$ denotes the terrain altitude measured by the aircraft at time $k$. This is obtained by deducting the height measurement of the ground looking (on board) radar from the known altitude of the aircraft (obtained using an altimeter). The function $h(x_k)$ relates the terrain altitude to position $x_k$ in the form of a digital terrain database. As the same height can correspond to different locations, $h(\cdot)$ is highly nonlinear.

The distribution of $w_k$ is typically modeled as Gaussian. As radar reflections can come from the ground as well as from the tree canopy, typically the observation noise $e_k$ is modeled as a (bimodal) two component Gaussian mixture. The regime variable $r_k$ indicates the corresponding mixture component. The sufficient statistics (i.e., mean and variance) of each component can be specified by the regime dependent parameters $\theta_{r_k}$. The dynamics of $r_k$ is modeled as two state first order homogeneous Markov process.

3.2. Inference objective:

For the model described by (11)–(12), given the densities for the initial state $\{r_0, x_0\}$ and the measurements up to a time $k$, our interest lies in estimating sequentially the latent states $\{r_k, x_k\}$. More precisely, for the statistical inference purpose, we target the series of filtering distributions $P(r_k|y_{1:k})$ and $p(x_k|y_{1:k})$ recursively over time. However, the above posteriors are in general, computationally intractable. Given this intractability, PF is a suitable candidate for this approximate (real time) inference task. Interestingly, however, we note that conditioned on the sequence $x_{1:k}$, $r_k$ follows a finite state hidden Markov model (HMM), implying that $P(r_k|x_{1:k}, y_{1:k})$ is analytically tractable. Using this analytical substructure, it is possible to implement an efficient RBPF scheme which can reduce the variance of the estimation error. In the sequel, we detail this RBPF framework for the MmNDS.

4. A new RBPF for Markov modulated nonlinear state-space model

In this section we outline a new RBPF framework exploiting the conditionally finite state-space HMM.
4.1. Description of the RBPF approach

At time zero, the initial densities for the state and the regime variables are respectively given by \( p(x_0) \) and \( \mathbb{P}(r_0) \overset{\triangle}{=} \mathbb{P}(r_0|x_0) \), these can be arbitrary but are assumed to be known. We further assume favorable mixing conditions as in Crisan and Doucet (2002).

Suppose that we are at time \( k - 1 \). We consider the extended target density \( p(r_{k-1}, x_{0:k-1}|y_{1:k-1}) \) which can be decomposed as

\[
p(r_{k-1}, x_{0:k-1}|y_{1:k-1}) = p(r_{k-1}|x_{0:k-1}, y_{1:k-1}) p(x_{0:k-1}|y_{1:k-1}). \tag{15}
\]

The posterior propagation of the latent state \( x_{k-1} \) can then be targeted through a PF, where \( p(x_{0:k-1}|y_{1:k-1}) \) is represented by a set of \( N \) weighted random particles as

\[
p(x_{0:k-1}|y_{1:k-1}) \approx \sum_{i=1}^{N} w_{k-1}^{(i)} \delta(x_{0:k-1} - x_{0:k-1}^{(i)}). \tag{16}
\]

Now conditioned on \( \{x_{0:k-1}, y_{1:k-1}\} \), the regime variable \( r_{k-1} \) follows a finite state-space HMM. As a result \( p(r_{k-1}|x_{0:k-1}, y_{1:k-1}) \) is analytically tractable\(^1\), which is represented as

\[
q_{k-1|k-1}^{(i)}(l) \overset{\triangle}{=} \mathbb{P}(r_{k-1} = l|x_{0:k-1}^{(i)}, y_{1:k-1}, \ldots, y_{1:k-1}), \tag{17}
\]

for \( l \in S \) and \( i = 1, \ldots, N \). Now using (16) and (17), the extended target density in (15) can be represented as

\[
\left[ x_{0:k-1}^{(i)}, w_{k-1}^{(i)}, \{q_{k-1|k-1}^{(i)}(l)\}_{l=1}^{S} \right]_{i=1}^{N}. \tag{18}
\]

Now having observed \( y_k \), we want to propagate the extended target density in (15) to time \( k \). This can be achieved in the following steps (a)–(d):

\[^1\]Observe that this distribution depends on the PF path space representation \( x_{0:k-1} \). It is well known that with time, particle filter suffers from a progressively impoverished particle representation. This is caused due to the effect of repeated resampling steps, leading to a path degeneracy problem (Cappé et al., 2007). On the other hand, uniform convergence in time of the particle filter is known under the mixing assumptions as in Crisan and Doucet (2002). This property ensures that any error is forgotten exponentially with time and can explain why the particle filter works for the marginal filter density.
(a) **Prediction step for conditional HMM filter:** this is easily obtained as:

\[
q_{k|k-1}^{(i)}(l) \triangleq \mathbb{P}(r_k = l|x_{0:k-1}^{(i)}, y_{1:k-1}) = \sum_{j=1}^{s} \pi_{jt} q_{k-1|k-1}^{(i)}(j), \quad (l, j) \in S.
\]

(b) **Prediction step for particle filter:** at this stage, generate \(N\) new samples \(x_k^{(i)}\) from an appropriate proposal kernel as

\[
x_k^{(i)} \sim \pi(x_k^{(i)}|x_{0:k-1}^{(i)}, y_{1:k}).
\]

Then set \(x_{0:k}^{(i)} = \{x_{0:k-1}^{(i)}, x_k^{(i)}\}\), for \(i = 1, \ldots, N\), representing the particle trajectories up to time \(k\).

(c) **Update step for conditional HMM filter:** noting that

\[
\mathbb{P}(r_k = l|x_{0:k}, y_{1:k}) \propto p(y_k, x_k|r_k = l, x_{0:k-1}, y_{1:k-1}) \mathbb{P}(r_k = l|x_{0:k-1}, y_{1:k-1}),
\]

we have

\[
q_{k|k}^{(i)}(l) \propto p(y_k, x_k^{(i)}|r_k = l, x_{0:k-1}^{(i)}, y_{1:k-1}) q_{k|k-1}^{(i)}(l)
\]

\[
\propto p_{\theta_l}(y_k|x_k^{(i)}, r_k = l) p_{\theta_l}(x_k^{(i)}|x_{k-1}^{(i)}, r_k = l) q_{k|k-1}^{(i)}(l).
\]

Now defining

\[
\alpha_k^{(i)}(l) \triangleq p_{\theta_l}(y_k|x_k^{(i)}, r_k = l) p_{\theta_l}(x_k^{(i)}|x_{k-1}^{(i)}, r_k = l) q_{k|k-1}^{(i)}(l)
\]

we obtain

\[
q_{k|k}^{(i)}(l) = \frac{\alpha_k^{(i)}(l)}{\sum_{j=1}^{s} \alpha_k^{(i)}(j)},
\]

for \(l \in S\) and \(i = 1, \ldots, N\).

---

2Since each \(q_{k-1|k-1}^{(i)}(\cdot)\) is smaller than 1, and the recursion involves multiplication by the terms, each less than 1, some \(q_{k|k-1}^{(i)}(\cdot)\) can become very small. For this numerical problem, it is better to work with \(\log(q_{k|k-1}^{(i)}(\cdot))\).
(d) **Update step for particle filter:** as the continuous state can be recursively propagated using the following relation:

$$
p(x_{0:k}|y_{1:k}) \propto p(y_k, x_k|x_{0:k-1}, y_{1:k-1}) p(x_{0:k-1}|y_{1:k-1}),
$$

(25)

the corresponding weight update equation for the particle filtering is given by

$$
w_k^{(i)} = \frac{p(x_{k}^{(i)}, y_k|x_{0:k-1}^{(i)}, y_{1:k-1})}{\pi_k(x_{k}^{(i)}|x_{0:k-1}^{(i)}, y_{1:k})} \tilde{w}_{k-1}^{(i)}
$$

(26a)

$$
\tilde{w}_{k}^{(i)} = \frac{w_k^{(i)}}{\sum_{j=1}^{N} w_k^{(j)}},
$$

(26b)

where \( \{\tilde{w}_{k}^{(i)}\}_{i=1}^{N} \) are the normalized weights. The numerator

$$
p(x_{k}^{(i)}, y_k|x_{0:k-1}^{(i)}, y_{1:k-1})
$$

can be obtained as

$$
p(x_{k}^{(i)}, y_k|x_{0:k-1}^{(i)}, y_{1:k-1}) = \sum_{l=1}^{s} p\left(y_k|x_{k}^{(i)}, x_{0:k-1}^{(i)}, y_{1:k-1}\right) \mathbb{P}(r_k = l|x_{0:k-1}^{(i)}, y_{1:k-1}),
$$

(27)

which is basically given by the normalizing constant of (24). Note that the marginal density

$$
p(x_k|y_{1:k})
$$

can be obtained as

$$
p(x_k|y_{1:k}) \approx \sum_{i=1}^{N} \tilde{w}_{k}^{(i)} \delta(x_k - x_k^{(i)}).
$$

(28)

The posterior probability of the regime variable can now be obtained as

$$
\mathbb{P}(r_k = l|y_{0:k}) = \int \mathbb{P}(r_k = l|x_{0:k}, y_{0:k}) p(x_{0:k}|y_{0:k}) dx_{0:k}
$$

(29a)

$$
\approx \sum_{i=1}^{N} q_{l|k}^{(i)}(\tilde{w}_{k}^{(i)}).
$$

(29b)

The mean and variance of the marginal distribution in (28) at time \( k \) can be obtained from the weighted particle representation as

$$
\hat{x}_k = \sum_{i=1}^{N} \tilde{w}_{k}^{(i)} x_k^{(i)},
$$

(30a)

$$
\hat{P}_{k} = \sum_{i=1}^{N} \tilde{w}_{k}^{(i)} (x_k^{(i)} - \hat{x}_k)(x_k^{(i)} - \hat{x}_k)^T,
$$

(30b)
where \((\cdot)^T\) denotes the transpose operation. Let \(\hat{m}^{(i)}_k\) and \(\hat{V}^{(i)}_k\) denote the mean and variance of the conditional HMM filter. They are now obtained as

\[
\hat{m}^{(i)}_k = \sum_{l=1}^{s} (r_k = l) \, q^{(i)}_{k|k}(l),
\]  

\[
\hat{V}^{(i)}_k = \sum_{l=1}^{s} \{(r_k = l) - \hat{m}^{(i)}_k\} \{(r_k = l) - \hat{m}^{(i)}_k\}^T q^{(i)}_{k|k}(l).
\]  

As noted earlier, the posterior of the regime variable is given by \((29b)\). Let \(\hat{m}_k\) and \(\hat{V}_k\) denote the corresponding mean and variance, which can be obtained as

\[
\hat{m}_k = \sum_{i=1}^{N} \hat{w}_k^{(i)} \hat{m}_k^{(i)},
\]  

\[
\hat{V}_k = \sum_{i=1}^{N} \hat{w}_k^{(i)} \left[ \hat{V}_k^{(i)} + (\hat{m}_k^{(i)} - \hat{m}_k)(\hat{m}_k^{(i)} - \hat{m}_k)^T \right].
\]

**Remark 1.** For PF, a popular (but less efficient) choice for the proposal kernel is given by the state transition density \(p(x_k|x_{k-1})\), which in this case can be obtained in the form of a weighted mixture density:

\[
p(x_k|x_{k-1}) = \sum_{l=1}^{s} p_{\theta}(x_k|x_{k-1}^{(i)}, r_k = l) q^{(i)}_{k|k-1}(l),
\]

where \(p_{\theta}(x_k|x_{k-1}^{(i)}, r_k = l)\) is specified in \((11b)\).

### 4.2. Algorithmic summary

The new RBPF for the MmNDS is summarized in Algorithm 1.

### 5. Relation to other similar models

Here we compare our RBPF model to other existing models exploiting similar conditional substructure. Similar conditionally finite state-space HMM have earlier been considered by Doucet et al. (2000) as well as Andrieu and Doucet (2002), although, each framework is fundamentally different. The differences are emphasized below.
Algorithm 1 RBPF for MmNDM

Initialization:
For each particle $i = 1, \ldots, N$ do
- Sample $x_0^{(i)} \sim p(x_0)$,
- Set initial weights $w_0^{(i)} = \frac{1}{N}$,
- Set initial $q_0^{(i)}(l) \triangleq \mathbb{P}(r_0 = l|x_0^{(i)}), \ l = 1, \ldots, s$

Iterations:
Set the resampling threshold $\eta$;
For $k = 1, 2, \ldots$ do
- For each particle $i = 1, \ldots, N$ do
  - Compute $q_{k|k-1}^{(i)}(l)$ using (19b)
  - Sample $x_k^{(i)} \sim \pi(x_k^{(i)})$ using (20)
  - Set $x_{0:k}^{(i)} \triangleq (x_{0:k-1}^{(i)}, x_k^{(i)})$
  - Compute $\alpha_k^{(i)}(l)$ using (23)
  - Compute $d_k^{(i)}(l)$ using (24)
  - Compute $w_k^{(i)}$ using (26a) and (27) as
    \[
    w_k^{(i)} = \frac{\sum_{j=1}^{s} \alpha_k^{(i)}(j) \tilde{w}_{k-1}^{(i)}}{\pi_k(x_k^{(i)} | x_{1:k-1}^{(i)}, y_{1:k})}
    \]
  - Normalize the weights using (26b)
  - Compute $N_{\text{eff}} = \frac{1}{\sum_{i=1}^{N} (\tilde{w}_k^{(i)})^2}$
    - If $N_{\text{eff}} \leq \eta$, resample the particles. Let the resampled particles be \(i^* = 1, \ldots, N\).
    - Copy the corresponding $q_{k|k}^{(i^*)}(l)$ and set $\tilde{w}_k^{(i^*)} = \frac{1}{N}$. 

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In our case \((x_k, y_k)\) follows a nonlinear state-space model, which is modulated by a finite state hidden Markov process \(r_k\). Hierarchically \(r_k\) is at the top level and is not influenced by \(x_k\). This is different from the hierarchical conditionally finite state-space HMM in [Doucet et al. (2000)](#), where \((r_k, y_k)\) follows a finite state-space hidden Markov process, which is modulated by another (hidden) Markov process \(c_k\). Here \(c_k\) is at the top of hierarchy and is not influenced by \(r_k\). In contrast, [Andrieu and Doucet (2002)](#) considered a partially observable finite state-space HMM, where \(r_k\) is a finite state hidden Markov process, \(y_k\) is a latent data process and \(z_k\) is observed data process. Conditioned on the sequence \(z_{1:k}\), here \((r_k, y_k)\) follows a finite state-space HMM.

6. Concluding remarks

Markov modulated nonlinear state-space model, although less well explored, appears quite naturally in many applications of interest. The model implies that the system can switch between different nonlinear dynamic regimes. The regime state is governed by a regime variable, which follows a homogeneous finite state first-order Markov process. In this article, the associated online inference problem for such model is addressed. In particular, a new RBPF is proposed for such inference tasks. This RBPF scheme exploits the analytical marginalization of the regime variable using the conditional HMM structure. This results in improved performance over a standard particle filter in terms of variance of the estimation error. Moreover for a standard particle filter where the regime state is also represented by the particles, degeneracy is commonly observed around regime transition [Driessen and Boers, 2005](#). In our RBPF implementation, as the regime variable follows a conditionally analytical substructure, hence the degeneracy is expected to be less severe.

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Appendix A. Sketch of the variance reduction mechanism through Rao-Blackwellization

Let $\theta$ be an unknown parameter and $Y$ be the random variable corresponding to the observed data. Let $\hat{\theta}(Y)$ be any kind of estimator of $\theta$. Further, if $T$ be the sufficient statistics for $Y$, then the Rao-Blackwell theorem states that the following estimator

$$\hat{\theta}_{RB}(T) = \mathbb{E}[\hat{\theta}(Y)|T]$$  \hspace{1cm} (A.1)

is typically a better estimator of $\theta$, and is never worse. The transformed estimator $\hat{\theta}_{RB}(T)$ using the sufficient statistics is known as the Rao-Blackwell estimator (Lehmann, 1983).

Now suppose $X$ is a random variable admitting a probability density function $p(x)$. Further, let $g(\cdot)$ be a function of $X$ and $\Phi$ be a test function given as the expectation of $g(X)$

$$\Phi = \mathbb{E}[g(X)] = \int g(x)p(x) \, dx.$$ \hspace{1cm} (A.2)

A Monte Carlo based estimator of $\Phi$ can be obtained as

$$\hat{\Phi}_{MC}(X) = \frac{1}{N} \sum_{i=1}^{N} g(x^{(i)}),$$ \hspace{1cm} (A.3)

where $x^{(i)}$, $i = 1, \ldots, N$ are generated according to $p(x)$. The variance of this estimator is

$$\text{Var}(\hat{\Phi}_{MC}(X)) = \frac{\text{Var}[g(X)]}{N},$$ \hspace{1cm} (A.4)

provided that the variance of $g(X)$ is finite.

Now suppose that $X$ is a random vector which can be split into two components as $X = (\Xi, \Lambda)^T$. Using (A.3), we have

$$\hat{\Phi}_{MC}(\Xi, \Lambda) = \frac{1}{N} \sum_{i=1}^{N} g(\xi^{(i)}, \lambda^{(i)}).$$ \hspace{1cm} (A.5)

Using (A.2) and law of iterated expectations, we can write

$$\Phi = \mathbb{E} \left[ \mathbb{E} \{ g(\Xi, \Lambda) | \Xi \} \right].$$ \hspace{1cm} (A.6)
We can subsequently define the following Rao-Blackwell estimator using (A.1) and (A.6) as

$$\hat{\Phi}_{RB}(\Xi) = \mathbb{E}\left[\hat{\Phi}_{MC}(\Xi, \Lambda) \mid \Xi\right]. \quad (A.7)$$

Now using the law of total variance

$$\text{Var}(\Phi) = \text{Var}\left(\mathbb{E}[\Phi \mid \Xi]\right) + \mathbb{E}\left(\text{Var}[\Phi \mid \Xi]\right) \geq 0. \quad (A.8)$$

Consequently, we have

$$\text{Var}\left(\hat{\Phi}_{MC}(\Xi, \lambda)\right) \geq \text{Var}\left(\hat{\Phi}_{RB}(\Xi)\right). \quad (A.9)$$

Rao-Blackwellization is useful when $\mathbb{E}[\Phi \mid \Xi]$ can be computed efficiently. This happens e.g., when part of the integration in (A.2) is analytically tractable.

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