Evaluation of the Mass of an Asymptotically Hyperbolic Manifold

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Abstract
We show that the mass functional of an asymptotically hyperbolic manifold with a noncompact boundary can be evaluated via the Ricci tensor and the second fundamental form by using purely coordinates. The method extends Miao-Tam’s approach to the asymptotically hyperbolic manifold. We also calculate a graphical example.

Keywords Asymptotically hyperbolic · Mass functional · Ricci tensor · Second fundamental form · Noncompact boundary

Mathematics Subject Classification 53C42

1 Introduction

Let $\mathbb{H}^n$ be the standard hyperbolic $n$-space with constant sectional curvature $-1$. Take an arbitrary point $o \in \mathbb{H}^n$ as the origin and let $\rho(x) = \text{dist}_{\mathbb{H}^n}(o, x)$ be the geodesic distance from a point $x$ to the origin and $V_0 = \cosh \rho$. The rotationally symmetric model of the standard metric $b$ on $\mathbb{H}^n$ is then written as $d\rho^2 + \sinh^2 \rho \sigma$ on $(0, \infty) \times S^{n-1}$ where $\sigma$ is the standard metric on the unit $(n - 1)$-sphere.

For another model of the standard $n$-space $\mathbb{H}^n$, we may also take $\mathbb{H}^{n-1} \times \mathbb{R}$ equipped with a warped product metric. Any point $x \in \mathbb{H}^n$ has the coordinate $x = (x', s)$. We assume that $o = (0', 0) \in \mathbb{H}^{n-1} \times \mathbb{R}$ where we take $0'$ as the origin in $\mathbb{H}^{n-1}$. Let $U(x') = \cosh(\text{dist}_{\mathbb{H}^{n-1}}(o', x'))$, now using this model the metric $b$ takes the form

$$b := \tilde{h} + U^2 ds \otimes ds$$

(1.1)
where \( \bar{h} \) is the metric for \( \mathbb{H}^{n-1} \). Note that \( U = V_0 \) when \( s = 0 \). Now define

\[
\mathbb{H}^n_+ = \mathbb{H}^{n-1} \times \{ s \in \mathbb{R} : s \geq 0 \}
\]

to be the \( n \)-dimensional hyperbolic half space and \( B^n_+ = \{ x \in \mathbb{H}^n_+ : \text{dist}_{\mathbb{H}^n}(x, o) \leq 1 \} \) to be the half ball of radius one. Now let \( \bar{\nabla} \) and \( D \) be respectively the standard connection on \( \mathbb{H}^n \) and \( \mathbb{H}^{n-1} = \mathbb{H}^{n-1} \times \{ 0 \} \). Denote the Christoffel symbols of \( \mathbb{H}^{n-1} \) by \( (\Gamma) \).

We let the Latin letters \( i, j, k, l, \ldots \) range from 1 to \( n \) and the Greek letters \( \alpha, \beta, \gamma \ldots \) range from 1 to \( n - 1 \). The letter \( n \) denotes the \( s \) factor of \( (x', s) \in \mathbb{H}^n \).

Motivated by the notion of an asymptotically flat manifold with a noncompact boundary [1], we can formulate a similar notion in the settings of asymptotically hyperbolic manifolds (see [2]).

**Definition 1** A Riemannian manifold \((M^n, g)\) is called asymptotically hyperbolic with a noncompact boundary of decay order \( \tau > \frac{n}{2} \) if there exist a compact set \( K \) and a diffeomorphism \( \Psi : M \setminus K \to \mathbb{H}^n_+ \setminus B^n_+ \) such that \((\Psi^{-1})^* g \) is uniformly equivalent with \( b \) and

\[
\|e\|_b + \|\bar{\nabla} e\|_b + \|\bar{\nabla} \bar{\nabla} e\|_b = O(e^{-\tau \rho})
\]

where \( e := (\Psi^{-1})^* g - b \). For the usual asymptotically hyperbolic manifolds, one could just drop the positive signs in \( \mathbb{H}^n_+ \setminus B^n_+ \). Let \( \theta^j \) be the coordinate of the standard \((n - 1)\)-sphere in \( \mathbb{R}^n \), define \( V_i = \theta^j \sinh \rho \).

The functions \( V_0 \) and \( V_\alpha, \alpha = 1, \ldots, n - 1 \) span a linear space called the space of static potentials denoted by \( \mathcal{N}_b^+ \) in which any \( V \in \mathcal{N}_b^+ \) satisfies the following:

\[
\bar{\nabla}_i \bar{\nabla}_j V = V b_{ij} \text{ in } \mathbb{H}^n_+, \quad \frac{\partial}{\partial \nu} V = 0 \text{ on } \partial \mathbb{H}^n_+.
\]

See [2,(1.2) and (2.1)]. For the definition of the static potential and \( \mathcal{N}_b \) when there is no noncompact boundary present, we refer to [4].

We drop the bar to denote quantities computed with respect to the metric \( g \). Let \( A = \nabla \eta \) be the second fundamental form and \( H \) be the mean curvature of \( \partial M \). We use the Einstein summation convention.

**Definition 2** If \((M^n, g)\) is an asymptotically hyperbolic manifold with decay order \( \tau > \frac{n}{2} \), let \( V \in \mathcal{N}_b^+ \), assume that \( V(R + n(n - 1)) \) is integrable and \( VH \) integrable on \( \partial M \). Let \( \{ D_q \}_{q=1}^\infty \) be a sequence of open sets with Lipschitz boundary \( \partial D_q \). The boundary \( \partial D_q \) are made of two portions, one is \( \Sigma_q = \partial D_q \cap \text{int } M \) and the other is \( \Pi_q = \partial D_q \cap \partial M \). \( \Pi_q \) and \( \Sigma_q \) share the same boundary \( S_q \). Let \( \rho_q = \inf_{x \in \Sigma_q} \text{dist}_{\mathbb{H}^n}(o, x) \) and we assume that

\[
\lim_{q \to \infty} \rho_q = \infty, \quad |S_q| \leq C \sinh^{n-2} \rho_q \text{ and } |\Sigma_q| \leq C \sinh^{n-1} \rho_q,
\]

where \( C \) is a constant independent of \( q \).
Let \( P_{ijkl} = \frac{1}{2} (g_{ik} g_{jl} - g_{il} g_{jk}) \), \( \nu \) be the normal of \( \Sigma_q \) in \( M \) and \( \theta \) be the normal of \( S_q \) in \( \partial M \), then

\[
\mathcal{M}(V) = \lim_{q \to +\infty} \left[ \int_{\Sigma_q} (V \bar{\nabla}_l e_{j k} - e_{j k} \bar{\nabla}_l V) P_{ijkl} \nu_i + \int_{S_q} \frac{V}{V_0} e_{\alpha \theta} \theta^\alpha \right]
\]

exists and is finite.

The quantity \( \mathcal{M}(V_0)^2 - \sum_\alpha \mathcal{M}(V_\alpha)^2 \) is shown to be a geometric invariant by [2, Theorem 3.4].

The definition of \( \mathcal{M}(V) \) extends [4, 17] to the settings of asymptotically hyperbolic manifolds with a noncompact boundary. Let \( G \) denote the Einstein tensor \( R_c - \frac{1}{2} R g \), we define the modified Einstein tensor

\[
\tilde{G} := G - \frac{1}{2} (n - 1)(n - 2) g.
\]

We can evaluate the mass \( \mathcal{M}(V) \) in terms of the Ricci tensor and the second fundamental form.

**Theorem 1.1** Assume that \( (M, g) \) and \( \{D_q\} \) are as Definition 2, then

\[
\mathcal{M}(V) = - \lim_{q \to +\infty} \frac{2}{n - 2} \left[ \int_{\Sigma_q} \tilde{G}(X, \nu) + \int_{S_q} (A - H h)(X, \theta) \right].
\]

**Remark 1.2** The author found independently the definition of \( \mathcal{M}(V_0) \), (1.3) and (1.4) for \( \mathcal{M}(V_0) \). While preparing the article, the author learned the formulas (1.3) and (1.4) for all \( \mathcal{M}(V) \) are found by [2], [8].

It is well known in the community of general relativity that the ADM mass of an asymptotically flat manifold can be evaluating via the Ricci tensor. We refer to the work of Miao and Tam [14] and the references therein for the history. Miao and Tam used purely coordinates, and there is also a work of Herzlich [10] who used instead a coordinate-free approach.

The author in his PhD thesis [5] had proved a similar formula for the ADM type mass defined in [1] using both approaches from Miao-Tam and Herzlich, see also [6] and [8]. In [8], the formula (1.4) is proved based on a method of Herzlich while here in this article is based on purely coordinates.

This type of formula is used to prove the convergence of the Hawking mass and Brown-York mass to the ADM mass in the asymptotically flat manifold case. See [15]. Given a hypersurface \( \Sigma^{n-1} \) with boundary intersecting the ambient boundary orthogonally, with the formula (1.4) and with similar calculations as in [15], one can define a Hawking type mass with boundary,

\[
m_H(\Sigma) := |\Sigma|^{\frac{1}{n-1}} \left\{ \int_\Sigma \left[ R_\Sigma + (n - 1)(n - 2) - \frac{n - 2}{n - 1} H_\Sigma^2 \right] + 2 \int_{\partial \Sigma} H_\partial \Sigma \right\}.
\]
In particular, when $\Sigma$ is a two dimensional surface with boundary, by the Gauss-Bonnet theorem, one has (up to a constant)

$$|\Sigma|^{1/2} \left[8\pi \chi(\Sigma) - \int_{\Sigma} (H^2 - 4)\right]$$

resembling the classical Hawking mass [9]. The Euclidean version of this mass is already proposed by [13] while studying inverse mean curvature flow with boundary.

Motivated by [12] and [7], we want to find a graphical example similar to the asymptotically flat case. We consider the manifold $M$ given by

$$M = \{ x = (x', s) : x \in \mathbb{H}^{n-1} \times U \mathbb{R}, s \geq u(x') \},$$

and $u$ has the decay rate

$$|u| + |\bar{D}u| + |\bar{D}^2 u| = O(e^{-(\tau+1)r}), \tau > \frac{n}{2}. \quad (1.5)$$

over $\mathbb{H}^{n-1}$. Here $U(x') = \cosh r(x')$ as in (1.1) where $r(x') = \text{dist}_{\mathbb{H}^n}(o', x')$. We show that $M$ is asymptotically hyperbolic with a noncompact boundary and the mass component $\mathcal{M}(V_0)$ has a very simple expression given by $u$. We have the following theorem (see also Theorem 3.4).

**Theorem 1.3** The mass component $\mathcal{M}(V_0)$ is simply given by

$$\mathcal{M}(V_0) = \lim_{r \to \infty} 2 \int_{\partial B_r} U^2 \langle \bar{D}u, \partial_r \rangle \sqrt{1 + U^2 |\bar{D}u|^2} d\mathcal{H}^{n-2},$$

where $B_r$ is a geodesic ball of radius $r$ centered at $o'$ in $\mathbb{H}^{n-1}$.

**Remark 1.4** Here, the $V_0$ is computed with a asymptotically hyperbolic coordinate which we assign on $M$ in Lemma 3.2. See (3.3). The expression above is an analogy of the exterior mass in Euclidean case appeared in [16,Definition 2.5] (see also [11]).

The explicit expression of the mean curvature $H = H(u)$ of a graph $M$ in the warped product $\mathbb{H}^{n-1} \times U \mathbb{R}$ was found by [7], however the following divergence form

$$UH = \sum_{i=1}^{n} \bar{D}_i \left( \frac{U^2 \bar{D}_i u}{\sqrt{1 + U^2 |\bar{D}u|^2}} \right),$$

was not observed by the authors. It can be easily obtained through calculation. From integration by parts,

$$\mathcal{M}(V_0) = 2 \int_{\partial M} UH d\mu_b.$$
2 Proof of Theorem 1.1

Recall that \( P^{ijkl} = \frac{1}{2} (g^{ik} g^{jl} - g^{il} g^{jk}) \) and the decay rate (1.2), we have an expansion at infinity,

\[
V(R + n(n - 1)) = 2\tilde{V}_i ((V\tilde{V}_i e_{jk} - e_{jk} \tilde{V}_i V) P^{ijkl}) + O(e^{-2\tau\rho + \rho}),
\]

for any linear combinations of \( V_0 \) and \( \{ V_i \}, i = 1, \ldots, n \). This expansion is used to define the mass functional \( H/\Phi_1(V) \) by Chrusciel and Herzlich [4]. A mass is introduced for an asymptotically flat manifold with a noncompact boundary by Almaraz, Barbosa and de Lima [1], where an expansion of the mean curvature of the noncompact boundary is also used to define an ADM mass. Similarly, in the settings of an asymptotically hyperbolic manifold with a noncompact boundary,

**Lemma 2.1** Let \( V \in \mathcal{N}^+_b \), and \( C^i = (V\tilde{V}_i e_{jk} - e_{jk} \tilde{V}_i V) P^{ijkl}, \) then

\[
2V H = -2\langle C, \eta \rangle + \tilde{h}^{\alpha\beta} \tilde{D}_\alpha \left( \frac{V}{V_0} e_{\beta n} \right) + O(e^{-2\tau\rho + \rho})
\]

along \( \partial M \). Here \( \tilde{D}_\alpha e_{\beta n} \) is understood as the covariant derivative of the tensor field \( e(\partial_n, \cdot)|_{\partial M} \) along \( \partial M \).

**Proof** Since we are using the decomposition \( b = \tilde{h} + U^2 ds \otimes ds \) of the background metric \( b \) on \( M \) and the length of \( \partial_s \) or \( \partial_n \) is of the order \( O(e^\rho) \), the decay of metric \( g \) written in components then has to be handled carefully. We choose the coordinate \( x \) on \( M \) such that the vector fields \( \partial_\alpha \) on the \( \mathbb{H}^{n-1} \) factor have uniformly bounded length (from both below and above) and the \( n \)-th coordinate to be just \( s \). Then we have the decay for the metric \( g \),

\[
g_{\alpha\beta} = b_{\alpha\beta} + O(e^{-\tau\rho}), \quad g_{\alpha n} = O(e^{-\tau\rho + \rho}), \quad g_{nn} = V_0^2 (1 + O(e^{-\tau\rho})).
\]

The we use the adjugate matrix from linear algebra to find the decay for the inverse metric \( g^{-1} \),

\[
g^{\alpha\beta} = b^{\alpha\beta} + O(e^{-\tau\rho}), \quad g^{\alpha n} = O(e^{-\tau\rho - \rho}), \quad g^{nn} = V_0^{-2} (1 + O(e^{-\tau\rho})).
\]

We readily have (see also [1,(2.12)])

\[
\eta = -(g^{nn})^{-1/2} g^{ni} \partial_i, \quad \eta_n = -(g^{nn})^{-1/2}, \quad \eta_\alpha = 0 \quad (2.1)
\]

and the mean curvature

\[
H = -h^{\alpha\beta} \langle \eta, \nabla_\alpha \partial_\beta \rangle = h^{\alpha\beta} (g^{nn})^{-1/2} \Gamma^n_{\alpha\beta}.
\]

Here \( h^{\alpha\beta} \), the induced inverse metric on \( \partial M \), is also the inverse of \( g_{\alpha\beta} \) viewed as an \( (n - 1) \times (n - 1) \) matrix. Indeed, along \( \partial M \), one can check easily from (2.1) that
\[ h^{\alpha \beta} g_{\beta \gamma} = \delta^\alpha_\gamma, \] and the decay of \( h^{\alpha \beta} \) is

\[ h^{\alpha \beta} = g^{\alpha \beta} - \eta^\alpha \eta^\beta = b^{\alpha \beta} + O(e^{-\tau \rho}) = \tilde{h}^{\alpha \beta} + O(e^{-\tau \rho}). \]

The following line from [7, Lemma 3.1] is an easy calculation and often used.

\[ \tilde{\Gamma}^\gamma_{\alpha \beta} = (\Gamma)^\gamma_{\alpha \beta}, \quad \tilde{\Gamma}^n_{\alpha \beta} = \Gamma^n_{\beta \alpha} = 0, \quad \tilde{\Gamma}^n_{\alpha n} = V_0^{-1} \tilde{D}_a V_0 = V_0^{-1} \tilde{\nabla}_a V_0. \]

It is well known that the difference of two Christoffel symbols is a tensor, in fact,

\begin{align*}
\Gamma^n_{\alpha \beta} & = \Gamma^n_{\alpha \beta} - \tilde{\Gamma}^n_{\alpha \beta} \\
& = \frac{1}{2} g^{nl} (\tilde{\nabla}_a g_{\beta l} + \tilde{\nabla}_\beta g_{a l} - \tilde{\nabla}_l g_{a \beta}) \\
& = \frac{1}{2} g^{nl} (\tilde{\nabla}_a e_{\beta l} + \tilde{\nabla}_\beta e_{a l} - \tilde{\nabla}_l e_{a \beta}) \\
& = \frac{1}{2} g^{\alpha \gamma} (\tilde{\nabla}_a e_{\beta \gamma} + \tilde{\nabla}_\beta e_{a \gamma} - \tilde{\nabla}_\gamma e_{a \beta}) + \frac{1}{2} g^{nn} (\tilde{\nabla}_a e_{\beta n} + \tilde{\nabla}_\beta e_{a n} - \tilde{\nabla}_n e_{a \beta}) \\
& = O(e^{-2\tau \rho - \rho}) + \frac{1}{2} V_0^{-2} (1 + O(e^{-\tau \rho}))(\tilde{\nabla}_a e_{\beta n} + \tilde{\nabla}_\beta e_{a n} - \tilde{\nabla}_n e_{a \beta}) \\
& = O(e^{-\tau \rho - \rho}).
\end{align*}

Since \( (g^{nn})^{-1/2} = V_0(1 + O(e^{-\tau \rho})) \),

\[ 2A_{\alpha \beta} = V_0^{-1} (\tilde{\nabla}_a e_{\beta n} + \tilde{\nabla}_\beta e_{a n} - \tilde{\nabla}_n e_{a \beta}) + O(e^{-\tau \rho}). \quad (2.2) \]

Expanding \( 2\mathcal{V}H \) at infinity,

\[ 2\mathcal{V}H = 2V h^{\alpha \beta} (g^{nn})^{-1/2} \Gamma^n_{\alpha \beta} \]

\[ = 2V (b^{\alpha \beta} + O(e^{-\tau \rho})) V_0 (1 + O(e^{-\tau \rho})) \Gamma^n_{\alpha \beta} \]

\[ = b^{\alpha \beta} V V_0^{-1} (\tilde{\nabla}_a e_{\beta n} - \tilde{\nabla}_n e_{a \beta}) + O(e^{-2\tau \rho + \rho}). \quad (2.3) \]

Expanding \( 2\langle \mathcal{C}, \eta \rangle \) at infinity along \( \partial M \),

\[ 2\langle \mathcal{C}, \eta \rangle = \eta_i (V \tilde{\nabla}_l e_{jk} - e_{jk} \tilde{\nabla}_l V)(g^{ik} g^{jl} - g^{il} g^{jk}) \]

\[ = \eta_i (V \tilde{\nabla}_l e_{jk} - e_{jk} \tilde{\nabla}_l V)(b^{ik} b^{jl} - b^{il} b^{jk}) + O(e^{-2\tau \rho + \rho}) \]

\[ = -V_0 (V \tilde{\nabla}_l e_{jk} - e_{jk} \tilde{\nabla}_l V)(b^{nk} b^{jl} - b^{nl} b^{jk}) + O(e^{-2\tau \rho + \rho}) \]

\[ = -(V V_0^{-1} \tilde{\nabla}_l e_{jn} - e_{jn} V_0^{-1} \tilde{\nabla}_l V)b^{jl} \]

\[ + (V V_0^{-1} \tilde{\nabla}_n e_{jk} - e_{jk} V_0^{-1} \tilde{\nabla}_n V)b^{jk} + O(e^{-2\tau \rho + \rho}) \]

\[ = -V V_0^{-1} b^{jl} \tilde{\nabla}_l e_{jn} + e_{an} b^{a\beta} V V_0^{-1} \tilde{\nabla}_\beta V + V V_0^{-1} b^{ji} \tilde{\nabla}_n e_{ij} + O(e^{-2\tau \rho + \rho}), \]
where in the second line we can obtain the expansion by returning temporarily to a coordinate whose coordinate vector fields have uniformly bounded length (from both below and above), and $\bar{\eta}$ is the outward normal to $\partial M$ under the metric $b$, and in the last line we have also used that

$$\bar{\nabla}_n V = \langle \partial_n, \bar{\nabla} V \rangle = \langle \partial_n, \bar{\nabla} r \rangle V' = 0.$$ 

Finally, noting that $b^{\alpha\beta} = \bar{h}^{\alpha\beta}$ and $b_{\alpha\beta} = \bar{h}_{\alpha\beta}$, we have

$$2VH + 2\langle C, \eta \rangle = b^{\alpha\beta} (2\bar{\nabla}_\alpha e_\beta n - \bar{\nabla}_n e_{\alpha\beta}) + O(e^{-2\rho + \rho})$$

$$- b^{ij} \bar{\nabla}_i e_{jn} + e_{an} b^{\alpha\beta} V^{-1} \bar{\nabla}_n V + b^{ij} \bar{\nabla}_n e_{ij}$$

$$= \bar{h}^{\alpha\beta} (V V_0^{-1} \bar{\nabla}_\alpha e_\beta n + e_{\beta n} V_0^{-1} \bar{\nabla}_\alpha V) + O(e^{-2\rho + \rho}).$$

Since we view $e(\partial_n, \cdot)|_{\partial M}$ as a tensor field on $\partial M$, inserting the following relation to the above

$$\bar{\nabla}_\alpha e_{\beta n} = \partial_\alpha e_{\beta n} - \bar{\gamma}^{ji}_{\alpha\beta} e_{in} - \bar{\gamma}^{i}_{an} e_{n\beta}$$

$$= (\partial_\alpha e_{\beta n} - \bar{\gamma}^{\gamma}_{\alpha\beta} e_{\gamma n}) - \bar{\gamma}^{n}_{\alpha\beta} e_{nn} - \bar{\gamma}^{\gamma}_{an} e_{n\beta}$$

$$= \bar{D}_\alpha e_{\beta n} - \bar{\gamma}^{n}_{an} e_{n\beta}$$

$$= \bar{D}_\alpha e_{\beta n} - e_{\beta n} V_0^{-1} \bar{D}_\alpha V_0$$

$$= \bar{D}_\alpha e_{\beta n} - e_{\beta n} V_0^{-1} \bar{\nabla}_\alpha V_0,$$ (2.4)

and we obtain

$$2VH + 2\langle C, \eta \rangle$$

$$= \bar{h}^{\alpha\beta} (V V_0^{-1} \bar{D}_\alpha e_{\beta n} - e_{\beta n} V \bar{\nabla}_\alpha V_0 + e_{\beta n} V_0^{-1} \bar{\nabla}_\alpha V) + O(e^{-2\rho + \rho})$$

$$= \bar{h}^{\alpha\beta} \bar{D}_\alpha \left( \frac{V}{V_0} e_{\beta n} \right) + O(e^{-2\rho + \rho}).$$

We establish the following decay of $\tilde{G}$. The proof is also used later in the paper.

**Lemma 2.2** For an asymptotically hyperbolic manifold $(M, g)$ with decay rate $\tau > n/2$, the modified Einstein tensor expands at infinity as

$$-\tilde{G}_{ik} = 2(1-n)e_{ik} + (\bar{\nabla}_i \bar{\nabla}_k E - \tilde{\nabla}^l \tilde{\nabla}_l e_{kl} - \tilde{\nabla}^l \tilde{\nabla}_k e_{il} + \tilde{\nabla}^l \tilde{\nabla}_l e_{ik})$$

$$- (1-n)b_{ik} E - b_{ik} (\Delta E - \tilde{\nabla}^j \tilde{\nabla}_j e_{ij}) + O(e^{-2\rho}),$$

where we use the shorthand $E := \text{tr}_b e$. 

\[\square\]
Proof Let $\Lambda = \Gamma - \bar{\Gamma}$, $\Lambda$ is the difference of the Christoffel symbols and hence a tensor. More specifically,

$$
\frac{1}{2} g^{kl} (\bar{\nabla}_i e_{jl} + \bar{\nabla}_j e_{il} - \bar{\nabla}_l e_{ij})
= \frac{1}{2} g^{kl} (\bar{\nabla}_i g_{jl} + \bar{\nabla}_j g_{il} - \bar{\nabla}_l g_{ij})
= \frac{1}{2} g^{kl} (\partial_i g_{jl} - \bar{\Gamma}_{il}^s g_{sj} - \bar{\Gamma}_{ij}^s g_{sl}) + \frac{1}{2} g^{kl} (\partial_j g_{il} - \bar{\Gamma}_{jl}^s g_{si} - \bar{\Gamma}_{ij}^s g_{sj})
\quad - \frac{1}{2} g^{kl} (\partial_l g_{ij} - \bar{\Gamma}_{jl}^s g_{si} - \bar{\Gamma}_{ij}^s g_{sj})
= \Gamma_{ij}^k - \bar{\Gamma}_{ij}^k = \Lambda_{ij}^k = O(e^{-\tau \rho}).
$$

Expressing $R_{\ ij\ kl}$ in terms of $\bar{R}_{\ ij\ kl}$ and $\Lambda$, we have

$$
R_{\ ij\ kl} = \partial_i \Gamma_{\ jk}^l - \partial_j \Gamma_{\ ik}^l + \Gamma_{\ ij}^m \Gamma_{\ km}^l - \Gamma_{\ ik}^m \Gamma_{\ jm}^l
= \partial_i (\Lambda_{\ jk}^l + \bar{\Gamma}_{\ jk}^l) - \partial_j (\Lambda_{\ ik}^l + \bar{\Gamma}_{\ ik}^l)
\quad + (\Lambda_{\ jk}^m + \bar{\Gamma}_{\ jk}^m)(\Lambda_{\ jm}^l + \bar{\Gamma}_{\ jm}^l) - (\Lambda_{\ jm}^m + \bar{\Gamma}_{\ jm}^m)(\Lambda_{\ jm}^l + \bar{\Gamma}_{\ jm}^l)
= \partial_i \Lambda_{\ jk}^l - \Lambda_{\ jk}^l \bar{\Gamma}_{\ ij}^s - \Lambda_{\ jk}^l \bar{\Gamma}_{\ is}^l + \Lambda_{\ jk}^l \bar{\Gamma}_{\ is}^l + \bar{R}_{\ ij\ kl}
\quad - \partial_j \Lambda_{\ ik}^l - \Lambda_{\ ik}^l \bar{\Gamma}_{\ ij}^s + \Lambda_{\ ik}^l \bar{\Gamma}_{\ is}^l - \Lambda_{\ ik}^l \bar{\Gamma}_{\ is}^l + \Lambda_{\ ik}^m \Lambda_{\ jm}^l - \Lambda_{\ ik}^m \Lambda_{\ jm}^l
= \bar{R}_{\ ij\ kl} + \bar{\nabla}_i \Lambda_{\ jk}^l - \bar{\nabla}_j \Lambda_{\ ik}^l + \Lambda_{\ jk}^m \Lambda_{\ im}^l - \Lambda_{\ ik}^m \Lambda_{\ jm}^l.
$$

In short, the Riemann curvature tensor has the decay,

$$
R_{\ ij\ kl} = \bar{R}_{\ ij\ kl} + \bar{\nabla}_i \Lambda_{\ jk}^l - \bar{\nabla}_j \Lambda_{\ ik}^l + O(e^{-2\tau \rho}).
$$

We also readily find the decay of $\bar{\nabla}_i g_{jk}$,

$$
\bar{\nabla}_i g_{jk} = \partial_i g_{jk} + \bar{\Gamma}_{\ il}^k g_{jl} + \bar{\Gamma}_{\ il}^j g_{kl}
\quad = (\partial_i g_{jk} + \Gamma_{\ il}^k g_{jl} + \Gamma_{\ il}^j g_{kl}) - (\Lambda_{\ il}^k g_{jl} + \Lambda_{\ il}^j g_{kl})
\quad = -(\Lambda_{\ il}^k g_{jl} + \Lambda_{\ il}^j g_{kl}) = O(e^{-\tau \rho}).
$$

Since $g = b + e$, the inverse metric $g^{-1}$ by the formula of invertible matrices from elementary linear algebra, $g^{ij} = b^{ij} + O(e^{-\tau \rho})$, hence the decay of $\bar{\nabla}_i \Lambda_{\ jk}^l$ is

$$
2 \bar{\nabla}_i \Lambda_{\ jk}^l = \bar{\nabla}_i (g_{ls} (\bar{\nabla}_s e_{kj} + \bar{\nabla}_s e_{jk} - \bar{\nabla}_s e_{jk})))
\quad = b^{\ell l} (\bar{\nabla}_s e_{kj} + \bar{\nabla}_s e_{jk} - \bar{\nabla}_s e_{jk}) + O(e^{-2\tau \rho}).
$$
The decay of the Ricci tensor is
\[-2R_{ik} = 2R_{jk}^i = -2\tilde{R}_{ik} + 2\tilde{\nabla}_i \Lambda_{jk}^j - 2\tilde{\nabla}_j \Lambda_{ik}^j + O(e^{-2\tau\rho})\]
\[= -2\tilde{R}_{ik} + b^{ij}\tilde{\nabla}_j(\tilde{\nabla}_k e_{ik} + \tilde{\nabla}_k e_{il} - \tilde{\nabla}_l e_{ik}) + O(e^{-2\tau\rho})\]
\[= -2(1 - n)b_{ik}\]
\[+ (\tilde{\nabla}_l \tilde{\nabla}_k E - \tilde{\nabla}^l \tilde{\nabla}_i e_{kl} - \tilde{\nabla}^l \tilde{\nabla}_k e_{il} + \tilde{\nabla}^l \tilde{\nabla}_l e_{ik}) + O(e^{-2\tau\rho}).\]

There are two metrics involved, so we should raise and lower the indices explicitly. Also, the scalar curvature $R$ decays as
\[-2R = -2g^{ik}R_{ik}\]
\[= -2g^{ik}b_{ik}(1 - n)\]
\[+ b^{ik}(\tilde{\nabla}_l \tilde{\nabla}_k E - \tilde{\nabla}^l \tilde{\nabla}_i e_{kl} - \tilde{\nabla}^l \tilde{\nabla}_k e_{il} + \tilde{\nabla}^l \tilde{\nabla}_l e_{ik}) + O(e^{-2\tau\rho})\]
\[= -2g^{ik}b_{ik}(1 - n) + 2\Delta E - 2\tilde{\nabla}^k \tilde{\nabla}^l e_{kl} + O(e^{-2\tau\rho}).\]

By a simple Taylor expansion argument, one has the elementary fact from linear algebra: Let $A, B$ be two symmetric $n \times n$ matrices, $A$ is positive definite and all entries of $B$ are sufficiently small such that $C := (A + B)^{-1}$ exists, then
\[\sum_{i,j=1}^{n} C_{ij}A_{ij} = n - \sum_{i,j=1}^{n} A_{ij}B_{ij} + O(|B|^2).\]

So
\[R = (n - E)(1 - n) - (\tilde{\nabla}^k \tilde{\nabla}^l e_{kl}) + O(e^{-2\tau\rho}).\]

Finally, we have the decay of the modified Einstein tensor $\tilde{G}$,
\[-2\tilde{G}_{ik} = -2R_{ik} + Rg_{ik} + (n - 1)(n - 2)g_{ik}\]
\[= -2(1 - n)b_{ik} + (\tilde{\nabla}_l \tilde{\nabla}_k E - \tilde{\nabla}^l \tilde{\nabla}_i e_{kl} - \tilde{\nabla}^l \tilde{\nabla}_k e_{il} + \tilde{\nabla}^l \tilde{\nabla}_l e_{ik})\]
\[+ (n - E)(1 - n)\tilde{g}_{ik} - \tilde{g}_{ik}(\tilde{\nabla}^j \tilde{\nabla}^l e_{jl})\]
\[+ (n - 1)(n - 2)\tilde{g}_{ik} + O(e^{-2\tau\rho})\]
\[= 2(1 - n)e_{ik} + (\tilde{\nabla}_l \tilde{\nabla}_k E - \tilde{\nabla}^l \tilde{\nabla}_i e_{kl} - \tilde{\nabla}^l \tilde{\nabla}_k e_{il} + \tilde{\nabla}^l \tilde{\nabla}_l e_{ik})\]
\[+ (1 - n)b_{ik}E - b_{ik}(\tilde{\nabla}^j \tilde{\nabla}^l e_{jl}) + O(e^{-2\tau\rho}).\]

concluding the proof.  

Now we turn to the proof of Theorem 1.1.
Proof For brevity, we suppress the index $q$ in $D_q$ and similar objects.

Step 1. Term involving the modified Einstein tensor $\tilde{G}$. Set

$$I := \int \tilde{n}^k \tilde{\nabla}^i V[\tilde{\nabla}_i \tilde{\nabla}_k E - \tilde{\nabla}^l \tilde{\nabla}_i e_{kl} - \tilde{\nabla}^l \tilde{\nabla}_k e_{il} + \tilde{\nabla}^l \tilde{\nabla}_l e_{ik}],$$

and

$$J = \int \tilde{n}^k \tilde{\nabla}^j V[\tilde{\nabla}_j \tilde{\nabla}_k E - \tilde{\nabla}^l \tilde{\nabla}_j e_{kl} - \tilde{\nabla}^l \tilde{\nabla}_k e_{jl} + \tilde{\nabla}^l \tilde{\nabla}_j e_{lk}].$$

Note that the integrands of $I$ and $J$ come from $\tilde{G}$.

$$I = \int \tilde{n}^k \tilde{\nabla}^i V[\tilde{\nabla}_i \tilde{\nabla}_k E - \tilde{\nabla}^l \tilde{\nabla}_i e_{kl} - \tilde{\nabla}^l \tilde{\nabla}_k e_{il} + \tilde{\nabla}^l \tilde{\nabla}_l e_{ik}]$$

$$= \left(\int_{\partial D} - \int_{\Pi}\right) \tilde{n}^k \tilde{\nabla}^i V[\tilde{\nabla}_i \tilde{\nabla}_k E - \tilde{\nabla}^l \tilde{\nabla}_i e_{kl} - \tilde{\nabla}^l \tilde{\nabla}_k e_{il} + \tilde{\nabla}^l \tilde{\nabla}_l e_{ik}]$$

$$= \int_D \tilde{\nabla}^k \tilde{\nabla}^i V[\tilde{\nabla}_i \tilde{\nabla}_k E - \tilde{\nabla}^l \tilde{\nabla}_i e_{kl} - \tilde{\nabla}^l \tilde{\nabla}_k e_{il} + \tilde{\nabla}^l \tilde{\nabla}_l e_{ik}]$$

$$+ \int_D \tilde{\nabla}^i V \tilde{n}^k [\tilde{\nabla}_i \tilde{\nabla}_k E - \tilde{\nabla}^l \tilde{\nabla}_i e_{kl} - \tilde{\nabla}^l \tilde{\nabla}_k e_{il} + \tilde{\nabla}^l \tilde{\nabla}_l e_{ik}] - J$$

$$= \int_D \tilde{\nabla}^k \tilde{\nabla}^i V[\tilde{\nabla}_i \tilde{\nabla}_k E - \tilde{\nabla}^l \tilde{\nabla}_i e_{kl} - \tilde{\nabla}^l \tilde{\nabla}_k e_{il} + \tilde{\nabla}^l \tilde{\nabla}_l e_{ik}]$$

$$+ \int_D \tilde{\nabla}^i V[\tilde{\nabla}_k \tilde{\nabla}_i \tilde{\nabla}^l e_{kl} - \tilde{\nabla}^l \tilde{\nabla}_i e_{kl} - \tilde{\nabla}^l \tilde{\nabla}_k e_{jl} - \tilde{\nabla}^l \tilde{\nabla}_l e_{jk}]$$

$$+ \int_D \tilde{\nabla}^i V[\tilde{\nabla}_k \tilde{\nabla}_i \tilde{\nabla}_l \tilde{\nabla}^l e_{kl} - \tilde{\nabla}^l \tilde{\nabla}_i e_{kl} - \tilde{\nabla}^l \tilde{\nabla}_k e_{jl} - \tilde{\nabla}^l \tilde{\nabla}_l e_{jk}]$$

$$= \int_D 2V[\tilde{\nabla}_k \tilde{\nabla}_i \tilde{\nabla}_l \tilde{\nabla}^l e_{kl}] + I_1 + I_2 - J,$$

where we denote

$$I_1 := \int_D \tilde{\nabla}^i V[\tilde{\nabla}_k \tilde{\nabla}_l \tilde{\nabla}_i \tilde{\nabla}^l e_{kl}]$$

and

$$I_2 := \int_D \tilde{\nabla}^i V[\tilde{\nabla}_k \tilde{\nabla}_l \tilde{\nabla}_i \tilde{\nabla}^l e_{kl}].$$

One of the main steps is the calculation of the term $I_1$. First, exchanging the order of indices leads to

$$\tilde{\nabla}^k \tilde{\nabla}_i \tilde{\nabla}_k E = \tilde{\nabla}_i \tilde{\nabla}^k \tilde{\nabla}_k E - \tilde{\nabla}^k \tilde{\nabla}_l \tilde{\nabla}_i \tilde{\nabla}^j e_{lj} = \tilde{\nabla}_i \tilde{\nabla}^k \tilde{\nabla}_k E + (1 - n) \tilde{\nabla}_i E,$$

and similarly,

$$\tilde{\nabla}^k \tilde{\nabla}_l \tilde{\nabla}_i e_{kl} = \tilde{\nabla}^k (\tilde{\nabla}_l \tilde{\nabla}_j e_{kl} - \tilde{\nabla}_k \tilde{\nabla}_i e_{kl} - \tilde{\nabla}_l \tilde{\nabla}_j e_{ik} + \tilde{\nabla}^j \tilde{\nabla}_l e_{ik})$$

$$= (\tilde{\nabla}_l \tilde{\nabla}^k \tilde{\nabla}_i e_{kl} - \tilde{\nabla}^k \tilde{\nabla}_l j \tilde{\nabla}_i e_{kl} - \tilde{\nabla}^k \tilde{\nabla}_l j \tilde{\nabla}_j e_{ik} - \tilde{\nabla}_l \tilde{\nabla}^k j \tilde{\nabla}_j e_{kl}).$$

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\[ - \bar{R}^{l}_{ik} \bar{\nabla}^k e_{jl} - \bar{R}^{l}_{il} \bar{\nabla}^k e_{jk} = \bar{\nabla}_i \bar{\nabla}^i e_{kl} - (2n - 1) \bar{\nabla}^l e_{il} + \bar{\nabla}_i E. \]

Integration by parts gives

\[ I_1 = \int_D \bar{\nabla}^i V \{ \bar{\nabla}^k \bar{\nabla}_i \bar{\nabla}^k E - \bar{\nabla}^k \bar{\nabla}^l e_{kl} \} \]
\[ = \int_D \bar{\nabla}^i V \bar{\nabla}_i \{ \bar{\nabla}^k \bar{\nabla}_k E - \bar{\nabla}^k \bar{\nabla}^l e_{kl} \} \]
\[ + \int_D [(2n - 1) \bar{\nabla}^i V \bar{\nabla}^k e_{ik} - n \bar{\nabla}^i V \bar{\nabla}_i E] \]
\[ = \int_{\partial D} \bar{\nabla}^i V \bar{\nabla}_i \{ \bar{\nabla}^k \bar{\nabla}_k E - \bar{\nabla}^k \bar{\nabla}^l e_{kl} \} - \int_D \bar{\nabla}_i \bar{\nabla}^i V \{ \bar{\nabla}^k \bar{\nabla}_k E - \bar{\nabla}^k \bar{\nabla}^l e_{kl} \} \]
\[ + \int_D [(2n - 1) \bar{\nabla}^i V \bar{\nabla}^k e_{ik} - n \bar{\nabla}^i V \bar{\nabla}_i E]. \]

For the integrand of the term \( I_2 \),

\[ - \bar{\nabla}^k \bar{\nabla}^l \bar{\nabla}^k e_{il} + \bar{\nabla}^l \bar{\nabla}^k \bar{\nabla}^k e_{il} \]
\[ = R^{kl}_{\ j} \bar{\nabla}^j e_{il} + \bar{R}^{kl}_{\ l} \bar{\nabla}^k e_{ij} + \bar{R}^{kl}_{\ i} \bar{\nabla}^k e_{jl} \]
\[ = R^{kl}_{\ j} \bar{\nabla}^j e_{il} \]
\[ = \bar{\nabla}_i E - \bar{\nabla}^l e_{ji}, \]

so

\[ I_2 = \int_D \bar{\nabla}^i V (\bar{\nabla}_i E - \bar{\nabla}^j e_{ji}). \]

Since \( \bar{\nabla}^i \bar{\nabla}^j V = V \delta^i_j, \Delta V = nV, \bar{\eta} \) is orthogonal to \( \bar{\nabla} V \) along \( \partial M \), and a further integration by parts gives

\[ I = (2 - n) \int_D V \{ \bar{\nabla}^k \bar{\nabla}^l e_{kl} \} \]
\[ + \int_{\Sigma} \bar{\eta}^{i} \bar{\nabla}^l V \{ \bar{\nabla}^k \bar{\nabla}_k E - \bar{\nabla}^l \bar{\nabla}^j e_{jk} - (n - 1) \bar{\nabla}^l V \bar{\nabla}_l E \} - J \]
\[ = (2 - n) \int_{\partial D} \bar{\eta}^{k} V \{ \bar{\nabla}_k E - \bar{\nabla}^l e_{kl} \} - J \]
\[ + \int_D n \bar{\nabla}^l V \bar{\nabla}^k e_{jk} - \int_D \bar{\nabla}^l V \bar{\nabla}_l E \]
\[ + \int_{\Sigma} \bar{\eta}^{i} \bar{\nabla}^j V \{ \bar{\nabla}^k \bar{\nabla}_k E - \bar{\nabla}^k \bar{\nabla}^l e_{kl} \}. \]
Combining the above with (2.5), and again from integration by parts,

\[-2 \int_{\Sigma} \tilde{G}_{ik} \eta^k \tilde{\nabla}^i V + o(1)\]

\[= -2 \int_{\Sigma} \tilde{G}_{ik} \eta^k \tilde{\nabla}^i V + o(1)\]

\[= \int_{\Sigma} 2(1 - n) e_{ik} \eta^k \tilde{\nabla}^i V + (2 - n) \int_{\partial D} \eta^k V (\tilde{\nabla}_k E - \tilde{\nabla}^i e_{kl})\]

\[+ \int \int_{\partial D} (n \tilde{\nabla}^j V \tilde{\nabla}^k e_{jk} - \tilde{\nabla}^l V \tilde{\nabla}_l E)\]

\[-\int \int_{\Sigma} (1 - n) E \tilde{\eta}_k \tilde{\nabla}^k V - J + o(1)\]

\[= (2 - n) \int_{\Sigma} [e_{ik} \eta^k \tilde{\nabla}^i V - E \tilde{\eta}_k \tilde{\nabla}^k V + \eta^k V (\tilde{\nabla}_k E - \tilde{\nabla}^l e_{kl})]\]

\[+ (2 - n) \int_{\Pi} \eta^k V (\tilde{\nabla}_k E - \tilde{\nabla}^l e_{kl})\]

\[+ n \int_{\Pi} e_{jk} \eta^k \tilde{\nabla}^j V - \int E \tilde{\eta}_l \tilde{\nabla}^l V - J + o(1).\]

Now we see that the contribution on $\Sigma$ of the mass functional in the above. In fact, where this gives an alternative proof to the case when there is no noncompact boundary present. This extends the proof [14], see also [10].

Writing down the terms that do not appear in the mass expression (1.3) and noting that $\bar{\nabla} V = \bar{\nabla} V$ and $\bar{\eta} \perp \bar{\nabla} V$ along $\partial M$, we have

\[K := (2 - n) \int_{\Pi} \eta^k V (\tilde{\nabla}_k E - \tilde{\nabla}^l e_{kl}) + n \int_{\Pi} e_{jk} \eta^k \tilde{\nabla}^j V - \int_{\Pi} E \tilde{\eta}_l \tilde{\nabla}^l V\]

\[- \int_{\Pi} \eta^k \tilde{\nabla}^i V [\tilde{\nabla}_i \tilde{\nabla}_k E - \tilde{\nabla}^l \tilde{\nabla}_i e_{kl} - \tilde{\nabla}^l \tilde{\nabla}_k e_{il} + \tilde{\nabla}_i \tilde{\nabla}_j e_{ik}]\]

\[= - (2 - n) \int_{\Pi} \frac{\bar{\nabla}_n E - \tilde{\nabla}^l e_{nl}}{V_0} - n \int_{\Pi} \frac{\bar{\nabla}_n e_{an}}{V_0}\]

\[+ \int_{\Pi} \left. \frac{\bar{\nabla}_a \bar{\nabla}_n E - \tilde{\nabla}^l \tilde{\nabla}_a e_{nl} - \tilde{\nabla}^l \tilde{\nabla}_n e_{al} + \tilde{\nabla}_l \tilde{\nabla}_a e_{an}}{V_0} \right|\]

**Step 2. Term involving second fundamental form $A_{\alpha\beta}$.** Similar to (2.4), viewing $\bar{\nabla}_\alpha e_{\beta n}$ as a 2-tensor on $\partial M$, we have

\[\bar{\nabla}_{\gamma} \bar{\nabla}_\alpha e_{\beta n} = \bar{\nabla}_{\gamma} \bar{\nabla}_\alpha e_{\beta n} + V_0^{-1} \bar{\nabla}_{\gamma} V_0 \bar{\nabla}_\alpha e_{\beta n}.\]

We calculate the terms of type $\bar{\nabla}_{\gamma}(V_0^{-1} \bar{\nabla}^\alpha V \bar{\nabla}_\mu e_{\beta n})$ as follows

\[\bar{\nabla}_{\gamma}(V_0^{-1} \bar{\nabla}_\alpha V \bar{\nabla}_\mu e_{\beta n})\]

\[= -V_0^{-2} \bar{\nabla}_{\gamma} V \bar{\nabla}_\alpha V \bar{\nabla}_\mu e_{\beta n} + V_0^{-1} \bar{\nabla}_\gamma \bar{\nabla}_\alpha e_{\beta n} + V_0^{-1} \bar{\nabla}_\gamma \bar{\nabla}_\mu e_{\beta n} + V_0^{-1} \bar{\nabla}_\gamma \bar{\nabla}_\alpha \bar{\nabla}_\mu e_{\beta n}\]
where in the last line we have used the relation \( \bar{D}_\alpha \bar{D}_\beta V = V b_{\alpha \beta} \), and it is important to keep in mind the range of the Greek indices. Switching the roles of the indices \( \mu \) and \( n \), one obtains a similar formula

\[
\bar{D}_\gamma (V_0^{-1} \bar{D}_\alpha V \bar{\nabla}_n e_{\beta \mu}) = V_0^{-1} \bar{D}_\alpha V \bar{\nabla}_n e_{\beta \mu} + b_{\gamma \alpha} \bar{\nabla}_n e_{\beta \mu}.
\]

By recalling from the expansion (2.2) and (2.3),

\[
-2 \int_{\partial M} (A - H h) (\bar{D} V, \theta) + o(1)
\]

\[
= -2 \int_{\partial M} (A - H h) (\bar{D} V, \bar{\theta}) + o(1)
\]

\[
= \int_{\partial M} \bar{D}^\gamma V \bar{\nabla}_\gamma V^{-1} b^{\alpha \beta} (2 \bar{\nabla}_\alpha e_{\beta n} - \bar{\nabla}_n e_{\alpha \beta})
\]

\[
- \bar{D}^\alpha V \bar{\nabla}_\alpha V^{-1} (\bar{\nabla}_\mu e_{\beta n} + \bar{\nabla}_\beta e_{\alpha n} - \bar{\nabla}_n e_{\alpha \beta}) + o(1)
\]

\[
= \int_{\Pi} \bar{D}_\gamma (V^{-1} \bar{D}^\gamma V b^{\alpha \beta} (2 \bar{\nabla}_\alpha e_{\beta n} - \bar{\nabla}_n e_{\alpha \beta}))
\]

\[
- \int_{\Pi} \bar{D}^\beta (V^{-1} \bar{D}_\alpha V (\bar{\nabla}_\alpha e_{\beta n} + \bar{\nabla}_\beta e_{\alpha n} - \bar{\nabla}_n e_{\alpha \beta}))) + o(1)
\]

\[
= (n - 2) \int_{\Pi} b^{\alpha \beta} \frac{V}{V_0} (2 \bar{\nabla}_\alpha e_{\beta n} - \bar{\nabla}_n e_{\alpha \beta})
\]

\[
+ \int_{\Pi} V_0^{-1} \bar{D}^\gamma V b^{\alpha \beta} (2 \bar{\nabla}_\gamma \bar{\nabla}_\alpha e_{\beta n} - \bar{\nabla}_\gamma \bar{\nabla}_n e_{\alpha \beta})
\]

\[
- \int_{\Pi} V_0^{-1} \bar{\nabla}_\gamma V (\bar{\nabla}_\gamma \bar{\nabla}_\alpha e_{\beta n} + \bar{\nabla}_\gamma \bar{\nabla}_\beta e_{\alpha n} - \bar{\nabla}_\gamma \bar{\nabla}_n e_{\alpha \beta}) + o(1)
\]

\[
=: L + o(1).
\]

To finish the proof, it remains to prove that

\[
K + L = (n - 2) \int_{\Pi} \bar{h}^{\alpha \beta} \bar{D}_\alpha \left( \frac{V}{V_0} e_{\beta n} \right).
\]

We collect terms from \( K + L \) involving second covariant derivatives of \( e = g - b \), and also note that there is a common factor \( V_0^{-1} \bar{D}^\alpha V \) shared by these terms after renaming. Not writing down the common factor and using that the \( b = \bar{h} + V_0^2 ds \otimes ds \) along \( \partial M \),

\[
[\bar{\nabla}_\alpha \bar{\nabla}_n E - \bar{\nabla}^l \bar{\nabla}_\alpha e_{nl} - \bar{\nabla}^l \bar{\nabla}_n e_{al} + \bar{\nabla}^l \bar{\nabla}_l e_{an}]
\]
\[-(\bar{\nabla}_\beta \bar{\nabla}_\alpha e_{\beta n} + \bar{\nabla}_\beta e_{\alpha n} - \bar{\nabla}_\alpha e_{\beta \alpha}) + b\gamma^\beta (2\bar{\nabla}_\alpha \bar{\nabla}_\gamma e_{\beta n} - \bar{\nabla}_\alpha \bar{\nabla}_\gamma e_{\gamma \beta})
\]
\[= 2(\bar{\nabla}_\alpha \bar{\nabla}_\beta e_{\beta n} - \bar{\nabla}_\beta \bar{\nabla}_\alpha e_{\beta n}) + (\bar{\nabla}_\alpha \bar{\nabla}_\alpha e_{\alpha n} - \bar{\nabla}_\alpha \bar{\nabla}_\alpha e_{\alpha n})
\]
\[= -2(\bar{\nabla}_\alpha \bar{\nabla}_\beta \gamma \gamma e_{\beta n} - \bar{\nabla}_\alpha \bar{\nabla}_\gamma e_{\gamma \beta}) - 2\bar{\nabla}_\alpha e_{\alpha n}
\]
\[= -2\bar{\nabla}_\alpha \gamma \gamma e_{\gamma n} = -2(1 - n)e_{n\alpha}.
\]

Adding the common factor $V_0^{-1}\bar{\nabla}_\alpha V$ back, we have

\[K + L = (n - 2) \int_\Pi b^{\alpha\beta} \frac{V}{V_0} \bar{\nabla}_\alpha e_{\beta n} - n \int_\Pi e_{\alpha n} \frac{\bar{\nabla}_\alpha V}{V_0}
\]
\[- 2(1 - n) \int_\Pi e_{\alpha n} \frac{\bar{\nabla}_\alpha V}{V_0}
\]
\[= (n - 2) \int_\Pi \bar{\nabla}_\alpha \gamma \gamma D_{\alpha} \left( \frac{V}{V_0} e_{\beta n} \right)
\]

by (2.4) thus finished the proof. \(\square\)

### 3 The Graphical Example

In this section, we do the calculation of the graphical example. Recall the distance formula of the warped product metric on $\mathbb{H}^n = \mathbb{H}^{n-1} \times U \mathbb{R}$.

**Lemma 3.1** Let $(x', s) \in \mathbb{H}^{n-1} \times U \mathbb{R}$, $\rho = \text{dist}_{\mathbb{H}^n}(o, x)$ and $r = \text{dist}_{\mathbb{H}^{n-1}}(o', x')$, then

\[
\cosh \rho = \cosh r \cosh s.
\]

**Proof** It is sufficient to assume that $n = 2$. On $\mathbb{H}^2$, the metric takes the form of a warping product

\[
dr^2 + \cosh^2 r ds^2.
\]

One checks that the map

\[(r, s) \mapsto \phi(r, s) := (\cosh s \cosh r, \sinh s \cosh r, \sinh r)
\]

is an isometry to the hyperboloid model of $\mathbb{H}^2$, where the first component is the time direction. The origin $(r, s) = (0, 0)$ is then sent to $(1, 0, 0)$. The the distance $\rho$ satisfies

\[
\cosh \rho = -\langle o, x \rangle = \cosh s \cosh r
\]

by [3, Section A.5]. Note $\langle \cdot, \cdot \rangle$ is the inner product in three dimensional Minkowski space. \(\square\)

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We need to assign an asymptotically hyperbolic coordinate on $M$. This is done in the next lemma.

**Lemma 3.2** We suppose that $u$ is a function on $\mathbb{H}^{n-1}$ with the decay rate (1.5). Then the manifold

$$M = \{(x', s) \in \mathbb{H}^{n-1} \times U : s \geq u(x')\}$$

with $U H$ integrable on $\partial M$ is an asymptotically hyperbolic manifold with a noncompact boundary.

**Proof** Let $w = w(t)$ be a smooth function with

$$|w| + |w'| + |w''| = O(e^{-\tau t}), \quad (3.1)$$

and $w(0) = 1$. We assign a coordinate $(y', t)$ on $M$ via the map

$$\Psi(y', t) = (y', t + u(y')w(t)) \in \mathbb{H}^{n-1} \times U = \mathbb{H}^n.$$

We use the new coordinate labels $(y', t)$ on $M$ to avoid confusion and we reserve the coordinate labels $x'$ and $s$ for the image of $\Psi$, that is, the map $\Psi$ sends $y'$ to $x'$ and $t$ to $s = t + u(y')w(t)$.

The push-forwards of the coordinate vector fields $\frac{\partial}{\partial y'_i}$ and $\partial_t$ on $M$ are respectively in $(\mathbb{H}^{n-1} \times U, b)$ given by

$$\frac{\partial}{\partial x'_i} = w(t) \frac{\partial u}{\partial y'_i} \partial_s, \quad (1 + w'(y')u(y'))\partial_s.$$

Let $g = \Psi^* g_{\mathbb{H}^n}$, so the metric components are given as

$$g_{ij}(y', t) = b_{ij}(x') + w^2(t)U^2(x') \frac{\partial u}{\partial y'_i} \frac{\partial u}{\partial y'_j},$$

$$g_{tt}(y', t) = w(t) \frac{\partial u}{\partial y'_i} (1 + w'(y')u(y'))U^2(x'),$$

$$g_{ii}(y', t) = (1 + w'(y')u(y'))^2U^2(x').$$

Recall that $y'$ is send to $x'$ under $\Psi$, we replace $x'$ by $y'$ in the above, we obtain the metric $g$ in terms of $y'$ and $t$.

We define $\rho$ on $M$ by the relation

$$\cosh \rho(y', t) := \cosh r(y') \cosh t. \quad (3.2)$$

When $\rho \to +\infty$, then either $t \to +\infty$ or $r(y') \to +\infty$, then the conditions (1.5) and (3.1) ensure that $|g - b|_b$ is of decay rate $O(e^{-\tau \rho(y', t)})$. The asymptotic behavior of $|\bar{\nabla} g|$ and $|\bar{\nabla}^2 g|$ follows similarly. So $(M, g)$ is asymptotically hyperbolic with a noncompact boundary.

**Remark 3.3** Note that the static potential $V_0$ of $(M, g)$ in the definition of its mass functional is

$$V_0(y', t) = \cosh \rho(y', t). \quad (3.3)$$

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And the $V_0$ of $\mathbb{H}^{n-1} \times_U \mathbb{R}$ is given by

$$V_0(x', s) = \cosh \rho(x', s) = \cosh r(y') \cosh(t + u(y') w(t))$$

under the map $\Psi$. They are different.

Obviously $\tilde{G}$ vanishes for $M$, so following from (1.4) we have the following.

**Theorem 3.4** The manifold $(M, g)$ is asymptotically hyperbolic, using notations from Definition 2, the mass component $\mathcal{M}(V_0)$ of $(M, g)$ is given

$$\mathcal{M}(V_0) = - \lim_{q \to +\infty} \frac{2}{n-2} \int_{\partial q} (A - Hg)(X, \theta) \, d\mathcal{H}^{n-2}.$$

Moreover, let $B_r$ be a geodesic ball of radius $r$ centered at $o'$, then

$$\mathcal{M}(V_0) = \lim_{r \to +\infty} 2 \int_{B_r} \frac{U^2 \langle \bar{D}u, \partial_r \rangle}{\sqrt{1 + U^2 |\bar{D}u|^2}} \, d\mathcal{H}^{n-2}.$$

**Proof** Let $W = \sqrt{1 + U^2 |Du|^2}$. Since $\partial M$ is a graph over $\mathbb{H}^{n-1}$ in $\mathbb{H}^{n-1} \times_U \mathbb{R}$, we do the calculations on $\mathbb{H}^{n-1}$. The article [7] computed many quantities for asymptotically hyperbolic graphs, in particular the second fundamental form and mean curvature of the graph $\partial M$, and we use their result. On $\partial M$, $\rho = r$ from (3.2), so

$$A(X, \theta)$$

$$= \sinh r A(\partial_r, \partial_r) + O(e^{-2\tau r + r})$$

$$= \sinh r \frac{V}{W} \left[ (\bar{D}^2u)(\partial_r, \partial_r) + 2 \frac{\bar{D}_{\partial_r} u \bar{D}_{\partial_r} U}{U} + U \langle du, dU \rangle \langle \bar{D}_{\partial_r} u \rangle^2 \right] + O(e^{-2\tau r + r})$$

$$= \sinh r \frac{\cosh r}{W} \left[ (\bar{D}^2u)(\partial_r, \partial_r) + 2 \sinh r \frac{\partial_r u}{\cosh r} + \cosh r \sinh r \langle du, dr \rangle \langle \partial_r u \rangle^2 \right]$$

$$+ O(e^{-2\tau r + r})$$

and

$$Hg(X, \theta)$$

$$= \sinh r H + O(e^{-2\tau r + r})$$

$$= \sinh r \frac{\cosh r}{W} \left[ \Delta_{\mathbb{H}^{n-1}} u - \frac{\cosh^2 r \langle \bar{D}^2 u, du \otimes du \rangle}{W^2} + (1 + W^{-2}) \frac{\sinh r}{\cosh r} \langle du, dr \rangle \right]$$

$$+ O(e^{-2\tau r + r}).$$

So combining the above,

$$(A - Hg)(X, \partial_r)$$

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\[
\begin{align*}
\frac{1}{W} \sinh r \cosh r ((\nabla^2 u)(\partial_r, \partial_r) - \Delta_{\mathbb{H}^{n-1}} u) + O(e^{-2\tau r}) &= -\frac{1}{W} \sinh r \cosh r (\Delta_{\partial B_r} u + H_{\partial B_r} \partial_r u) + O(e^{-2\tau r}),
\end{align*}
\]
where in the last line we have used the decomposition of Laplacian $\Delta_{\mathbb{H}^{n-1}}$ on the geodesic sphere $\partial B_r$. Here $\Delta_{\partial B_r}$ and $H_{\partial B_r}$ are respectively the Laplacian and the mean curvature of $\partial B_r$ in $\mathbb{H}^{n-1}$. Since the mean curvature $H_{\partial B_r}$ in $\mathbb{H}^{n-1}$ of $\partial B_r$ is
\[
H_{\partial B_r} = \sum_{i=1}^{n-2} \langle \nabla e_i, \partial_r \rangle = (n-2) \frac{\partial_r \sinh r}{\sinh r},
\]
by an application of the divergence theorem,
\[
\mathcal{M}(V_0) = -\frac{2}{n-2} \int_{\partial B_r} (A - Hg)(X, \partial_g) d\mathcal{H}^{n-2}_g + o(1)
\]
\[
= \frac{2}{n-2} \int_{\partial B_r} \frac{1}{W} \sinh r \cosh r (\Delta_{\partial B_r} u + H_{\partial B_r} \partial_r u) d\mathcal{H}^{n-2}_g + o(1)
\]
\[
= \frac{2}{n-2} \int_{\partial B_r} \frac{1}{W} \sinh r \cosh r H_{\partial B_r} \partial_r u d\mathcal{H}^{n-2}_g + o(1)
\]
\[
= 2 \int_{\partial B_r} \frac{1}{W} U^2 \partial_r u d\mathcal{H}^{n-2}_g + o(1).
\]
Hence,
\[
\mathcal{M}(V_0) = 2 \lim_{r \to +\infty} \int_{\partial B_r} \frac{U^2 \langle \tilde{\mathcal{D}} u, \partial_r \rangle}{\sqrt{1 + U^2 |\mathcal{D} u|^2}} d\mathcal{H}^{n-2}_b
\]
is the desired result. \hfill \Box

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