Pin^{-}(2)-monopole equations and intersection forms with local coefficients of 4-manifolds

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Pin^{-}(2)-monopole equations

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Proof of Theorem 1
Proof of Theorem 2
Let $X$ be a closed oriented 4-manifold.

**Topological invariants for $X$**
- $\pi_1 X$, cohomology ring, $k$-invariants...

**Intersection form**

$$Q_X : H^2(X; \mathbb{Z})/\text{torsion} \times H^2(X; \mathbb{Z})/\text{torsion} \to \mathbb{Z},$$

$$(a, b) \mapsto \langle a \cup b, [X] \rangle.$$

- $Q_X$ is a symmetric bilinear unimodular form.

[J.H.C. Whitehead '49]

If $\pi_1 X = 1$, the homotopy type of $X$ is determined by the isomorphism class of $Q_X$.

In 4-dim. TOP

$\pi_1 X = 1$

[Freedman '82]

The homeo type of $X$ is determined by
- the iso. class of $Q_X$ if $Q_X$ is even,
- the iso. class of $Q_X$ & $kS(X)$ if $Q_X$ is odd.

$\pi_1 X \neq 1$

If $\pi_1 X$ is “Good” $\Rightarrow$ Freedman theory + Surgery theory.

$\rightarrow$ Difficult.
In 4-dim. DIFF

- Let $X$ be a closed oriented smooth 4-manifold.

[Rohlin] If $X$ is spin $\Rightarrow \text{sign}(X) \equiv 0 \mod 16$.

[Donaldson] If $Q_X$ is definite $\Rightarrow Q_X \sim \text{The diagonal form}$.

[Furuta] If $X$ is spin & $Q_X$ is indefinite, then

$$b_2(X) \geq \frac{10}{8} \left| \text{sign}(X) \right| + 2.$$ 

Refinements, variants

[Furuta-Kametani '05]
The strong 10/8-inequality in the case when $b_1(X) > 0$.

[Froyshov '10]
A local coefficient analogue of Donaldson's theorem.

local coefficients $\leftrightarrow$ double coverings $\leftrightarrow H^1(X;\mathbb{Z}/2)$
Froyshov’s results

4-manifolds and intersection forms with local coefficients, arXiv:1004.0077

- Suppose a double covering \( \tilde{X} \to X \) is given.
- \( l := \tilde{X} \times_{\mathbb{Z}_2} \mathbb{Z} \), a \( \mathbb{Z} \)-bundle over \( X \).
- \( \to H^*(X; l) \): \( l \)-coefficient cohomology.
- Note \( l \otimes l = \mathbb{Z} \). The cup product
  \[ \cup : H^2(X; l) \times H^2(X; l) \to H^4(X; \mathbb{Z}) \cong \mathbb{Z}, \]
  induces the intersection form with local coefficient
  \[ Q_{X,l} : H^2(X; l)/\text{torsion} \times H^2(X; l)/\text{torsion} \to \mathbb{Z}. \]
- \( Q_{X,l} \) is also a symmetric bilinear unimodular form.

A special case of Froyshov’s theorem

- \( X \): a closed connected oriented smooth 4-manifold s.t.
  \[ b^+(X) + \dim_{\mathbb{Z}/2}(\text{tor} H_1(X; \mathbb{Z}) \otimes \mathbb{Z}/2) \leq 2. \] (1)
- \( l \to X \): a nontrivial \( \mathbb{Z} \)-bundle.
  If \( Q_{X,l} \) is definite \( \Rightarrow Q_{X,l} \sim \text{diagonal}. \)

- The original form of Froyshov’s theorem is:
  If \( X \) with \( \partial X = Y : \mathbb{Z} HS^3 \) satisfies (1)
  & \( Q_{X,l} \) is nonstandard definite
  \( \Rightarrow \delta_0 : HF^4(Y; \mathbb{Z}/2) \to \mathbb{Z}/2 \) is non-zero.
- \( Y = S^3 \Rightarrow HF^4(Y; \mathbb{Z}/2) = 0 \Rightarrow \) The above result.
Introduction

Applications

Pin$^-(2)$-monopole equations

Proof of Theorem 1 & 2

Froyshov’s results

Main results

◮ The proof uses the moduli space of $SO(3)$-instantons on a $SO(3)$-bundle $V$.

◮ Twisted reducibles (stabilizer $\cong \mathbb{Z}/2$) play an important role. $V$ is reduced to $\lambda \oplus E$, where $E$ is an $O(2)$-bundle, $\lambda = \det E$: nontrivial.

Cf [Fintushel-Stern’84] gives an alternative proof of Donaldson’s theorem by using $SO(3)$-instantons.

→ Abelian reducibles (stabilizer $\cong U(1)$)

$V$ is reduced to $\mathbb{R} \oplus L$, where $L$ is a $U(1)$-bundle.

- Donaldson’s theorem is proved by Seiberg-Witten theory, too.

Question

Can we prove Froyshov’s result by Seiberg-Witten theory?

→ Our result would be an answer.

Main results

Theorem 1.(N.)

◮ $X$: a closed connected ori. smooth 4-manifold.

◮ $l \to X$: a nontrivial $\mathbb{Z}$-bdl. s.t. $w_1(\lambda)^2 = 0$, where $\lambda = l \otimes \mathbb{R}$.

If $Q_{X,l}$ is definite $\Rightarrow$ $Q_{X,l} \sim$ diagonal.

Cf. Froyshov’s theorem

◮ $X$: s.t. $b^+(X) + \dim_{\mathbb{Z}/2}(\text{tor}H_1(X;\mathbb{Z}) \otimes \mathbb{Z}/2) \leq 2$.

◮ $l \to X$: a nontrivial $\mathbb{Z}$-bundle.

If $Q_{X,l}$ is definite $\Rightarrow$ $Q_{X,l} \sim$ diagonal.
Main results

Theorem 1.(N.)

- $X$: a closed connected ori. smooth 4-manifold.
- $l \to X$: a nontrivial $\mathbb{Z}$-bdl. s.t. $w_1(\lambda)^2 = 0$, where $\lambda = l \otimes \mathbb{R}$.

If $Q_{X,l}$ is definite $\implies Q_{X,l} \sim \text{diagonal}.$

- For the proof, we will introduce a variant of Seiberg-Witten equations
  $\rightarrow$ Pin$^\perp$(2)-monopole equations on Spin$^c$-structures on $X$.
- Spin$^c$-structure is a Pin$^\perp$(2)-variant of Spin$^c$-str. defined by M.Furuta, whose complex structure is “twisted along $l$”.

- The moduli space of Pin$^\perp$(2)-monopoles is compact.
  $\rightarrow$ Bauer-Furuta theory can be developed.

Furuta’s theorem

Let $X$ be a closed ori. smooth spin 4-manifold with indefinite $Q_X$.

$$b_+(X) \geq -\frac{\text{sign}(X)}{8} + 1.$$ 

Theorem 2(N.)

Let $X$ be a closed connected ori. smooth 4-manifold. For any nontrivial $\mathbb{Z}$-bundle $l \to X$ s.t. $w_1(\lambda)^2 = w_2(X)$, where $\lambda = l \otimes \mathbb{R}$,

$$b_+(X; \lambda) \geq -\frac{\text{sign}(X)}{8},$$

where $b_+(X; \lambda) = \text{rank } H^+(X; \lambda)$.
**Applications**

Recall fundamental theorems.

1. [Rohlin] $X^4$: closed spin $\Rightarrow$ $\text{sign}(X) \equiv 0 \mod 16$.
2. [Donaldson] Definite $\Rightarrow$ diagonal.
3. [Furuta] The $10/8$-inequality
4. [Furuta-Kametani] The strong $10/8$-inequality in the case when $b_1 > 0$.

**Corollary 1(N.)**

$\exists$ Nonsmoothable closed indefinite spin $4$-manifolds satisfying

- $\text{sign}(X) \equiv 0 \mod 16$,
- the strong $10/8$-inequality.

**Proof**

- Let $M$ be $T^4$ or $T^2 \times S^2$. $\Rightarrow Q_{T^4} = 3H$, $Q_{T^2 \times S^2} = H$.
- If $l' \to M$ is any nontrivial $\mathbb{Z}$-bundle, $\Rightarrow b_2(M; l') = 0$ \& $w_1(l' \otimes \mathbb{R})^2 = 0$.
- Let $V$ be a topological $4$-manifold s.t. $\pi_1 V = 1$, $Q_V$ is even and definite, $\text{sign}(V) \equiv 0 \mod 16$. ($\Rightarrow V$ is spin.)
- Choose a large $k$ s.t. $X = V \# kM$ satisfies the strong $10/8$-inequality.
- Let $l := \mathbb{Z} \# k l' \to X$. $\Rightarrow Q_{X,l} = Q_V$, $w_1(l \otimes \mathbb{R})^2 = 0$.
- Suppose $X$ is smooth. By Theorem 1, $Q_{X,l} = Q_V \sim \text{diagonal}$. **Contradiction.**

**Remark**

Similar examples can be constructed by using Theorem 2.
Non-spin manifolds

10/8-conjecture

Every non-spin closed smooth 4-manifold $X$ with even form satisfies

$$b_2(X) \geq \frac{10}{8}|\text{sign}(X)|.$$  

[Bohr, '02], [Lee-Li, '00]

If the 2-torsion part of $H_1(X; \mathbb{Z})$ is $\mathbb{Z}/2^i$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$

⇒ the 10/8-conjecture is true.

Corollary 2(N.)

∃ Nonsmoothable non-spin 4-manifolds $X$ with even form s.t.

► the 2-torsion part of $H_1(X; \mathbb{Z}) \cong \mathbb{Z}/2$,

► the 10/8-conjecture is true.

The outline of the proof of Theorem 1

► The proof of Theorem 1 is almost parallel to the SW-proof of Donaldson’s theorem.

► By using $\text{Pin}^-(2)$-monopole moduli, we will prove every characteristic element $w$ of $Q_{X,l}$ satisfies

$$|w^2| \geq \text{rank } H^2(X;l). \iff (\text{The dim. of the moduli}) \leq 0$$

► Then Elkies' theorem implies $Q_{X,l}$ should be standard.

- An element $w$ in a unimodular lattice $L$ is called characteristic if $w \cdot v \equiv v \cdot v \mod 2$ for $\forall v \in L$.

[Elkies '95]

If every characteristic element $w \in L$ satisfies $|w^2| \geq \text{rank } L$, then $L \cong \text{ diagonal.}$
Pin\(^{-}(2)\)-monopole equations

\[ \text{Pin}\(^{-}(2)\) = \langle \text{U}(1), j \rangle = \text{U}(1) \cup j \text{U}(1) \subset \text{Sp}(1) \subset \mathbb{H}. \]

The two-to-one homomorphism Pin\(^{-}(2)\) → O(2) is defined by

\[ z \in \text{U}(1) \subset \text{Pin}\(^{-}(2)\) \mapsto z^2 \in \text{U}(1) \subset \text{O}(2), \]
\[ j \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

**Definition** Spin\(^{c-}\)(n) := Spin(n) × \{±1\} Pin\(^{-}(2)\).

\[ 1 \to \{±1\} \to \text{Spin}^{c-}(n) \to \text{SO}(n) \times \text{O}(2) \to 1. \]

*Cf.* Spin\(^{c}\)(n) = Spin(n) × \{±1\} U(1).

Spin\(^{c-}\)-structures

- Let \( X \) be an oriented \( n \)-manifold.
- Fix a Riemannian metric.
  \[ \to F(X) : \text{The SO}(n)\text{-frame bundle}. \]
- Suppose an O(2)-bundle \( E \) over \( X \) is given.

**Spin\(^{c-}\)-structure**

A **Spin\(^{c-}\)-structure** on \((X, E)\) is given by \((P, \tau)\) s.t.

- \( P \): a Spin\(^{c-}\)(n)-bundle over \( X \),
- \( \tau : P/\{±1\} \xrightarrow{\cong} F(X) \times_X E. \)

**Proposition (Furuta '08)**

\[ \exists \text{Spin}^{c-}\text{-structure on } (X, E) \iff w_2(X) = w_2(E) + w_1(E)^2. \]
The case when $n = 4$

- $\text{Spin}(4) = \text{Sp}(1) \times \text{Sp}(1)$.
- $\text{Spin}^c(4) = (\text{Sp}(1) \times \text{Sp}(1) \times \text{Pin}^-(2))/\{\pm 1\} \ni [q_+, q_-, u]$.

$\text{Spin}^c(4)$-modules $\mathbb{H}_T$, $\mathbb{H}_+$ and $\mathbb{H}_-$

- $\mathbb{H}_T, \mathbb{H}_+, \mathbb{H}_- \cong \mathbb{H}$ as vector spaces.
- The actions of $[q_+, q_-, u] \in \text{Spin}^c(4)$ are given by
  \[
  \mathbb{H}_T \ni v \mapsto q_+vq_-^{-1} \quad \longrightarrow \quad P \times_{\text{Spin}^c(4)} \mathbb{H}_T \cong TX
  \]
  \[
  \mathbb{H}_\pm \ni \phi \mapsto q_\pm \phi u^{-1} \quad \longrightarrow \quad P \times_{\text{Spin}^c(4)} \mathbb{H}_\pm =: S^\pm
  \]

$S^\pm$ are the positive/negative spinor bundles.

The Clifford multiplication
Define the $\text{Spin}^c(4)$-equivariant map

\[
\rho_0 : \mathbb{H}_T \times \mathbb{H}_+ \rightarrow \mathbb{H}_-, (v, \phi) \mapsto \bar{v}\phi.
\]

\[
\longrightarrow \rho : \Omega^1(X) \times \Gamma(S^+) \rightarrow \Gamma(S^-).
\]

Twisted complex version

- $\text{Spin}^c(4) = \text{Spin}(n) \times \{\pm 1\} \text{Pin}^-(2)$ has two components.
- Let $G_0 \subset \text{Spin}^c(4)$ be the identity component.
- Let $\varepsilon : \text{Spin}^c(4) \rightarrow \text{Spin}^c(4)/G_0 \cong \{\pm 1\}$ be the projection.
  
  $\longrightarrow P \times_\varepsilon \mathbb{R} = \det E =: \lambda$

- Let $\text{Spin}^c(4)$ act on $\mathbb{C}$ by complex conjugation via $\varepsilon$.
- Define the $\text{Spin}^c(4)$-equivariant map,

\[
\rho_0 : \mathbb{H}_T \otimes_\mathbb{R} \mathbb{C} \times \mathbb{H}_+ \rightarrow \mathbb{H}_-, (v \otimes a, \phi) \mapsto \bar{v}\phi\bar{a}.
\]

\[
\longrightarrow \rho : \Omega^1(\mathbb{R} \oplus i\lambda) \times \Gamma(S^+) \rightarrow \Gamma(S^-).
\]
Dirac operator
An $O(2)$-connection $A$ on $E +$ Levi-Civita connection
→ A $\text{Spin}^{c-}(4)$-connection $\mathbb{A}$ on $P$
→ Dirac operator

$$D_A: \Gamma(S^+) \to \Gamma(S^-).$$

If $A'$ is another $O(2)$-connection $\Rightarrow a = A - A' \in \Omega^1(i\lambda)$.

$$D_{A+a}\phi = D_A\phi + \rho(a)\phi.$$ 

Quadratic map
Let $x = [q_+, q_-, u] \in \text{Spin}^{c-}(4)$ act on $\text{im } \mathbb{H}$ by

$$\text{im } \mathbb{H} \ni v \mapsto \varepsilon(x)q_+ vq_+^{-1} \to \Gamma(P \times_{\text{Spin}^{c-}(4)} \text{im } \mathbb{H}) \cong \Omega^+(i\lambda).$$

Then $\phi \in \mathbb{H}_+ \mapsto \phi i\bar{\phi} \in \text{im } \mathbb{H}$ is $\text{Spin}^{c-}(4)$-equivariant. We obtain

$$q: \Gamma(S^+) \to \Omega^+(i\lambda).$$

$\text{Pin}^{-}(2)$-monopole equations
Let $\mathcal{A}$ be the space of $O(2)$-connections on $E$.
For $(A, \phi) \in \mathcal{A} \times \Gamma(S^+)$, $\text{Pin}^{-}(2)$-monopole equations are defined by

$$\begin{cases} D_A\phi = 0, \\ F_A^+ = q(\phi). \end{cases}$$
Relation to Seiberg-Witten theory

- $\text{Spin}^c-(4) = \text{Spin}(4) \times \{\pm 1\} \text{Pin}^-(2)$ has two component.
- The identity compo. $G_0 = \text{Spin}(4) \times \{\pm 1\} \text{U}(1) = \text{Spin}^c(4)$.
- $\text{Spin}^c-(4)/G_0 = \mathbb{Z}/2$.
- Let $(P, \tau)$ be a $\text{Spin}^c-$structure on $(X, E)$.
- $\tilde{X} = P/G_0 \to X$ is a double covering s.t.
  \[ \lambda := \tilde{X} \times \{\pm 1\} \mathbb{R} \cong \text{det } E. \]
- $P \to \tilde{X}$ is a $G_0 = \text{Spin}^c(4)$-bundle.

$\ni: \tilde{X} \to \tilde{X}$, the covering transformation.

$J = [1, 1, j] \in (\text{Sp}(1) \times \text{Sp}(1) \times \text{Pin}^-(2))/\{\pm 1\} = \text{Spin}^c-(4)$

The $\text{Spin}^c$-structure $c$ on $\tilde{X}$ is induced from $P \to \tilde{X}$.

The $J$-action induces antilinear involutions $I$ on the spinor bundles and the determinant line bundle of $c$.

$\text{Pin}^-(2)$-monopole theory on $X = I$-invariant SW theory on $\tilde{X}$. 
Gauge transformation group

\[ G := \{ \text{Spin}^c(4)\text{-equiv. diffeos of } P \text{ covering the id. of } P/\text{Pin}^-(2) \} \]
\[ \cong \Gamma(P \times_{\text{ad}} \text{Pin}^-(2)), \]

where “ad” is the adjoint action on \text{Pin}^-(2) by \text{Pin}^-(2)-compo. of \text{Spin}^c(4) = \text{Spin}(4) \times \{ \pm 1 \} \text{Pin}^-(2).

\[ g \in G \text{ acts on } (A, \phi) \in \mathcal{A} \times \Gamma(S^+) \text{ by } g(A, \phi) = (A - 2g^{-1}dg, g\phi). \]

Cf. In the SW-case, \( G_{SW} = \text{Map}(X, S^1). \)

The moduli space \( \mathcal{M} = \{ \text{solutions } \}/G. \)

What is \( G = \Gamma(P \times_{\text{ad}} \text{Pin}^-(2)) ? \)

\begin{itemize}
  \item Pin^-(2) = U(1) \cup j U(1).
  \end{itemize}

For \( u, z \in U(1), \)
\[ \text{ad}_z(u) = zu\bar{z} = u, \]
\[ \text{ad}_{jz}(u) = jzu\bar{z}(-j) = \bar{u}, \]
\[ \text{ad}_z(ju) = z^2ju, \]
\[ \text{ad}_{jz}(ju) = \bar{z}^2j\bar{u}. \]
\[ \Rightarrow G = G_0 \cup G_1, \quad G_0 = \Gamma(P \times_{\text{ad}} U(1)), \]
\[ G_1 = \Gamma(P \times_{\text{ad}} j U(1)). \]

\begin{itemize}
  \item Note \( G_0 \cong \Gamma(\tilde{X} \times \{ \pm 1 \} U(1)), \) where \( \{ \pm 1 \} \) acts on \( U(1) \) by complex conjugation.
\end{itemize}

Define the involution \( I \) on \( G_{SW} = \text{Map}(\tilde{X}; S^1) \) by \( Ig = i^*g, \) where
\( i: \tilde{X} \to \hat{X} \) the covering transformation. \( \Rightarrow \quad G_0 = (G_{SW})^I. \)
Proposition $G_1 = \Gamma(P \times_{\text{ad} j} U(1)) \neq \emptyset \iff \tilde{c}_1(E) = 0$.

- $\tilde{c}_1(E)$ is the Euler class considered in $H^2(X; l)$, where $l$ is the sub-$\mathbb{Z}$-bundle of $\lambda = \det E$.
  Froyshov calls $\tilde{c}_1(E)$ the **twisted 1st Chern class**.
- The iso. classes of $O(2)$-bundle $E$ over $X$ s.t. $\det E \cong \lambda$ are classified by $\tilde{c}_1(E) \in H^2(X; l)$. $\leftarrow$ Proved by Froyshov.
- $\tilde{c}_1(E) = 0 \iff E \cong \mathbb{R} \oplus \lambda$.
- Since $\text{ad}_z(ju) = z^2 ju$ & $\text{ad}_z(j\bar{u}) = \bar{z}^2 j\bar{u}$,
  
  \[ P \times_{\text{ad} j} U(1) \cong S(E) : \text{The bundle of unit vectors of } E. \]

The moduli space

\[ M = \{ \text{solutions} \} / G, \]
\[ M_0 = \{ \text{solutions} \} / G_0. \]

Note $\tilde{c}_1(E) \neq 0 \Rightarrow G = G_0 \Rightarrow M = M_0$.

Proposition

- $M$ is compact.
- The virtual dimension of $M$:
  \[
  d = \frac{1}{4}(\tilde{c}_1(E)^2 - \text{sign}(X)) - (b_0(X; l) - b_1(X; l) + b_+(X; l)).
  \]

If $l$ is nontrivial & $X$ connected $\Rightarrow b_0(X; l) = 0$. 
Reducibles

- Recall $g(A, \phi) = (A - 2g^{-1}dg, g\phi)$.
- If $\phi \neq 0 \Rightarrow G$-action is free.
- The stabilizer of $(A, 0)$ is $\{\pm 1\} \subset G_0 \cong \Gamma(\hat{X} \times \{\pm 1\} \text{U}(1))$, unless $E = \mathbb{R} \oplus \lambda$ and $A$ is flat ($\Rightarrow$ The stabilizer $\cong \mathbb{Z}/4$).
- The elements of the form $(A, 0)$ are called reducibles.

Cf. In the SW-case, the stabilizer of $(A, 0)$ is $S^1 \subset \text{Map}(X, S^1)$.

- In general, $\{\text{reducible solutions}\}/G_0 \cong T^{b_1(X;l)} \subset M_0$.

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Proof of Theorem 1

Theorem 1.(N.)

- $X$: a closed connected ori. smooth 4-manifold.
- $l \rightarrow X$: a nontrivial $\mathbb{Z}$-bdl. s.t. $w_1(\lambda)^2 = 0$, where $\lambda = l \otimes \mathbb{R}$.

If $Q_{X,l}$ is definite $\Rightarrow Q_{X,l} \sim$ diagonal.

Outline of the proof

- We will prove every characteristic element $w$ of $Q_{X,l}$ satisfies $|w^2| \geq \text{rank } H^2(X;l)$,

  by proving for every $E$,

  $d = \dim M_0 \leq 0$.

- Then Elkies’ theorem implies $Q_{X,l}$ should be standard.
The structure of $\mathcal{M}_0$ when $b_+(X; l) = 0$

- Suppose a $\text{Spin}^c$-structure $(P, \tau)$ on $X$ is given.
- For simplicity, assume $b_1(X, l) = 0$.
  $\Rightarrow \exists^1$ reducible class $\rho_0 \in \mathcal{M}_0$.
- Perturb the $\text{Pin}^{-}(2)$-monopole equations by adding $\eta \in \Omega^+(i\lambda)$ to the curvature equation. $\Rightarrow F_A^+ = q(\phi) + \eta$.
- For generic $\eta$, $\mathcal{M}_0 \setminus \{\rho_0\}$ is a $d$-dimensional manifold.
- Fix a small neighborhood $N(\rho_0)$ of $\{\rho_0\}$.
  $\Rightarrow N(\rho_0) \cong \mathbb{R}^d/\{\pm 1\} = \text{a cone of } \mathbb{RP}^{d-1}$

Then $\overline{\mathcal{M}_0} := \overline{\mathcal{M}_0 \setminus N(\rho_0)}$ is a compact $d$-manifold & $\partial \overline{\mathcal{M}_0} = \mathbb{RP}^{d-1}$. 
Let \( B^* = (A \times (\Gamma(S^+) \setminus \{0\})) / G_0 \).

**Proposition** \( B^* \simeq \mathbb{RP}^\infty \times T^{b_1}(X;l) \).

Cf. In the SW-case, \( B^*_{SW} \simeq \mathbb{CP}^\infty \times T^{b_1}(X) \). \( B^* \simeq (B^*_{SW})^I \).

**Lemma**

If \( b_+(X;l) = 0 \) & \( b_1(X;l) = 0 \) \( \Rightarrow \) \( d = \dim \mathcal{M}_0 \leq 0 \).

**Proof**

- Suppose \( d > 0 \).
- Recall \( \overline{\mathcal{M}}_0 \) is a compact \( d \)-manifold s.t. \( \partial \overline{\mathcal{M}}_0 = \mathbb{RP}^{d-1} \).
- \( \exists C \in H^{d-1}(B^*;\mathbb{Z}/2) \cong H^{d-1}(\mathbb{RP}^\infty;\mathbb{Z}/2) \) s.t. 
  \[ \langle C, [\partial \overline{\mathcal{M}}_0] \rangle \neq 0. \Rightarrow \text{Contradiction}. \]

Note \( \text{sign}(X) = b_+(X;l) - b_-(X;l) \) for any \( \mathbb{Z} \)-bundle \( l \).

By Lemma, if \( l \) is nontrivial & \( b_+(X;l) = 0 \) & \( b_1(X;l) = 0 \),

\[
d = \frac{1}{4}(\tilde{c}_1(E)^2 - \text{sign}(X)) - (b_0(X;l) - b_1(X;l) + b_+(X;l))
= \frac{1}{4}(\tilde{c}_1(E)^2 + b_2(X;l)) \leq 0.
\]

Note \( \tilde{c}_1(E)^2 \leq 0 \) if \( b_+(X;l) = 0 \).

Therefore, for any \( E \) which admits a Spin\(^c\)-structure,

\[
b_2(X;l) \leq |\tilde{c}_1(E)|.
\]

By varying \( E \), we can prove every characteristic element \( w \) satisfies

\[
b_2(X;l) \leq |w|^2.
\]
Recall

- $E$ admits a Spin$^c$-structure

$$w_2(X) = w_2(E) + w_1(E)^2 = w_2(E) + w_1(\lambda)^2,$$
where $\lambda = \det E = l \otimes \mathbb{R}$.

- $\tilde{c}_1(E) \in H^2(X; l)$ classifies $E$ s.t. $\det E = l \otimes \mathbb{R}$.

Note that $0 \to l \to l \to \mathbb{Z}/2 \to 0$ induces the mod-2-reduction map $[\cdot]: H^2(X; l) \to H^2(X; \mathbb{Z}/2)$ & $[\tilde{c}_1(E)]_2 = w_2(E)$. We have,

**Theorem**

Suppose $w_1(\lambda)^2 = 0$. For every $C \in H^2(X; l)$ s.t.

$$w_2(X) = [C]_2 + w_1(\lambda)^2 = [C]_2,$$

$$|C^2| \geq b_2(X; l).$$

**Lemma**

For every characteristic element $c$ of $Q_{X, l}$, $\exists$ a torsion $\delta \in H^2(X; l)$ s.t. $[c + \delta]_2 = w_2(X)$.

Then, for $\forall$ characteristic element $c$ of $Q_{X, l}$

$$|c^2| = |(c + \delta)^2| \geq b_2(X; l).$$

By Elkies’ theorem, $Q_{X, l} \sim$-diagonal.
The outline of the proof of Theorem 2

- Suppose \( w_1(\lambda)^2 = w_2(X) \). Let \( E = \mathbb{R} \oplus \lambda \).
  \( \implies \exists \text{Spin}^c\)-structure on \((X, E)\). \( \implies G_1 \neq \emptyset \).
- For simplicity, assume \( b_1(X; l) = 0 \).
- Then, by taking finite dimensional approximation of the monopole map, we obtain a proper \( \mathbb{Z}_4 \)-equivariant map
  \[
  f: \tilde{\mathbb{R}}^m \oplus \mathbb{C}_1^{n+k} \to \tilde{\mathbb{R}}^{m+b} \oplus \mathbb{C}_1^n,
  \]
  where
  - \( \tilde{\mathbb{R}} \) is \( \mathbb{R} \) on which \( \mathbb{Z}_4 \) acts via \( \mathbb{Z}_4 \to \mathbb{Z}_2 = \{\pm 1\} \act \mathbb{R} \),
  - \( \mathbb{C}_1 \) is \( \mathbb{C} \) on which \( \mathbb{Z}_4 \) acts by multiplication of \( i \),
  - \( k = -\text{sign}(X)/8, b = b_+(X; \lambda) \), \( m, n \) are some integers.

Here, \( \mathbb{Z}_4 \) is generated by the constant section
\[
J \in G_1 = \Gamma(\tilde{X} \times \{\pm 1\} \cup (1)).
\]

- By using the techniques of equivariant homotopy theory, e.g., tom Dieck’s character formula, we can see that any proper \( \mathbb{Z}_4 \)-map of the form,
  \[
  f: \tilde{\mathbb{R}}^m \oplus \mathbb{C}_1^{n+k} \to \tilde{\mathbb{R}}^{m+b} \oplus \mathbb{C}_1^n,
  \]
  should satisfy \( b \geq k \).
- That is,
  \[
  b_+(X; \lambda) \geq -\frac{1}{8} \text{sign}(X).
  \]
Finite dimensional approximation

- Take a flat connection $A_0$ on $\mathbb{R} \oplus \lambda$.

**Pin**$^−(2)$-monopole map

$$\mu: \Omega^1(i\lambda) \oplus \Gamma(S^+) \rightarrow (\Omega^0 \oplus \Omega^+)(i\lambda) \oplus \Gamma(S^-) =: \mathcal{W},$$

$$(a, \phi) \mapsto (d^*a, F_{A_0} + d^+a + q(\phi), D_{A_0+a}\phi).$$

- Let $l(a, \phi) := (d^*a, d^+a, D_{A_0}\phi)$ be the linear part of $\mu$.
  $\rightarrow l$ is Fredholm.

- $c = \mu - l$: quadratic, compact.

- Choose a finite dim. subspace $U \subset \mathcal{W}$ s.t. $\dim U \gg 1$,
  $U \supset (\text{im } l)^\perp$

- Let $V := l^{-1}(U) \& p: \mathcal{W} \rightarrow U$ be the $L^2$-projection.

- Define $f: V \rightarrow U$ by $f = l + pc$.
  $\rightarrow f$: proper, $\mathbb{Z}_4$-equiv.

Remarks for future researches

- **Pin**$^−(2)$-monopole invariants
  - Calculation, gluing formula, stable cohomotopy refinements

- Orbifolds with surface singularities
  - Exotic involutions *Cf. [Fintushel-Stern-Snukujian]*
  - Smooth inequivalent but topologically equivalent embedded surfaces *Cf. [H.J.Kim-Ruberman]*

- When $\tilde{X}$: symplectic & $I^*\omega = -\omega$,
  $$\text{Pin}^−(2)\text{-monopole inv.} = \text{real Gromov-Witten inv.}$$
  *Cf. [Tian-Wang]*

- **Pin**$^−(2)$-monopole Floer theory?

- **Pin**$^−(2)$ Heegaard Floer theory?

- “Witten conjecture” for **Pin**$^−(2)$-monopole invariants?
  - *Cf. [Feehan-Leness] SW = Donaldson
    $$\text{Pin}^−(2)\text{-monopole inv.} = ??$$