Abstract

Our daily social and political life is more and more impacted by social networks. The functioning of our living bodies is deeply dependent on biological regulation networks such as neural, genetic, and protein networks. And the physical world in which we evolve, is also structured by systems of interacting particles. Interaction networks can be seen in all spheres of existence that concern us, and yet, our understanding of interaction networks remains severely limited by our present lack of both theoretical and applied insight into their clockworks. In the past, efforts at understanding interaction networks have mostly been directed towards applications. This has happened at the expense of developing understanding of the generic and fundamental aspects of interaction networks (properties and behaviours due primarily to the fact that a system is an interaction network, as opposed to properties and behaviours rather due to the fact a system is a genetic interaction network for instance). Intrinsic properties of interaction networks (e.g., the ways in which they transmit information along entities, their ability to produce this or that kind of global dynamical behaviour depending on local interactions) are thus still not well understood. Lack of fundamental knowledge tends to limit the innovating power of applications. Without more theoretical fundamental knowledge, applications cannot evolve deeply and become more impacting. Hence, it is necessary to better apprehend and comprehend the intrinsic properties of interaction networks, notably the relations between their architecture and their dynamics and how they are affected by and set in time. In this chapter, we use the elementary mathematical model of Boolean automata networks as a formal archetype of interaction networks. We survey results concerning the role of feedback cycles and the role of intersections between feedback cycles, in shaping the asymptotic dynamical behaviours of interaction networks. We pay special attention to the impact of the automata updating modes.

Keywords: Discrete dynamical systems, Boolean networks, Boolean cycles, updating modes, dynamics and combinatorics.
1 Introduction

Interaction networks occupy an important place in our daily life. We see this today in particular with the massive use of social networks, the fundamental implications and mechanisms of which we hardly have any understanding of. And social media is just one example among many other kinds of interaction networks that affect us consequentially. At all levels of our lives there are interaction networks, that can be comprised as sets of entities that interact locally with each other over time.

In an interaction network, local interactions take place. And as a result of these, the network as a whole exhibits global behaviours that generally remain difficult to explain on the sole basis of local processes. Undeniably one of the most telling examples lies at the origin of all living organisms: genes and their regulation, associated with other mechanisms inducing variability (splicing, role of the chromatin, etc.). It is currently accepted in biology and medicine, that a better understanding of genetic regulation is a necessary condition for improving our knowledge of life, which in turn could allow us to achieve a more precise comprehension of pathologic mechanisms, and to access more targeted and thereby more efficient therapies.

Whilst the application aspects of interaction networks are obviously of real and tangible importance, and they have progressed quantitatively at a frantic pace over the last twenty years, the more fundamental aspects aiming at understanding and analysing the intrinsic properties of these networks have so far received less attention from the scientific community. Current applications, in particular those emerging in biology (the privileged domain of application of the present chapter), are awaiting significant theoretical advances to continue their qualitative development. Undoubtedly, such theoretical advances may be the fruit of the combination of computer science and discrete mathematics. Together with bioinformatics, viewed here as “the study of computer processes in biotic systems” [HH70, HH78], they seem particularly suited to meet the needs arising from applications. Indeed, real networks, through the entities and interactions that compose them, can naturally be viewed as discrete objects. They can easily be represented by computer models which capture their essence thanks to a high level of abstraction. Where traditional (continuous) mathematical modelling focuses mainly on the quantitative characteristics of networks, the interest of discrete computer science modelling comes from the qualitative nature of the questions that it raises, which makes it possible to realise that the central elements of networks are not the entities themselves but rather the interactions that link them.

This survey adopts the qualitative point of view of fundamental computer science to examine some properties of interaction networks. This choice of approach is consistent with the origins of modern computing. Indeed, investigations of formal neural networks [MP43] and cellular automata [von66] in the 1940s, both strongly inspired by natural processes, helped establish the first links between data processing and biology. Also, the pioneering work of McCulloch and Pitts introduced automata networks as a fundamental model of interaction networks. For reasons discussed later, we choose to rely on the same model here.

We thus here study automata networks, and more specifically Boolean automata networks, from a fundamental point of view interested in developing a qualitative understanding of networks. We focus on understanding how feedback cycles that are part of network architectures, come to influence the asymptotic dynamical properties of these networks. By way of methods from discrete dynamical systems theory, enumerative combinatorics, algorithms and complexity theory, we explore the behavioural diversity of cycles and their tangential intersections.

Another point of focus in this chapter is how updating modes influence cycles and cycle intersections (and this network behaviours). From both the fundamental and the applied points of view, updating modes are known to be of decisive importance in the shaping of a network’s behaviour. Updating modes define the way in which the states of the automata of a network are updated as a function of a discrete time. The time being unbounded towards the future, there is an infinite number of different possible updating modes which we can come with and update the automata of a network according to. We focus here on the two modes that are the most customary in the literature: the parallel mode and
the asynchronous mode. The parallel mode is a deterministic and periodic mode. At each time step, it updates the states of all the automata in a network. The asynchronous mode is a non-deterministic mode. It allows for any possible series of sequential updates – which update exactly one automaton per time step – to take place. We show how the differences between these two updating modes imply profound differences in the network dynamics.

This chapter is a synthesis of results about interaction cycles and variations of interaction cycles derived since the early 2000s. The list of results presented here is not exhaustive. Details and demonstrations of these results can be found in the literature [DNS12, MNR+15, Nou12b, RMCT03, Sen12].

We will start with a presentation of the history of automata networks in the context of modern fundamental computer science, including a mention of the links that this science has always entertained with biology. Main definitions and notations will be explained after that. Then, we will briefly present seminal results obtained in the 1980s that highlight the crucial importance of the role of feedback cycles on the dynamical expressiveness and behavioural diversity of interaction networks. This will lead us to focus on the dynamical and combinatorial properties of isolated cycles when they are subjected to the parallel and asynchronous updating modes. And finally, before concluding, we will take a step towards contextualising cycles and look into tangential intersections of cycles, called “tangential double-cycles” for short.

2 Automata networks: between fundamental computer science and biology

Automata network research is part of natural computation field in computer science. And the scope of natural computation is twofold. First, it designs and develops models of computations that draw inspiration from natural phenomenology. Second, it manipulates such models so as to build up a firm, albeit necessarily incomplete, grasp of biological reality natural phenomena. As mentioned above, automata networks were initially introduced as a theoretical model of neural networks. And since the late 1960s, the literature evidences that they are also relevant as models of genetic regulation networks. In this section, we provide a comprehensive overview of the interconnections between computer science and biology manifested through automata networks.

2.1 Biology as an inspiration for modern computer science

Generally, when we are interested in the history of the so-called modern computer science, we go back to the 1930s. This period saw the development of classical computing paradigms such as the recursive functions of Herbrand and Gödel, the main works of which are available in [God86], the lambda-calculus developed initially in [Chu32], and also the Turing machines [Tur36]. However, to think of modern computer science only in terms of these paradigms is to forget a whole family of less conventional models that have grounded many developments in computer science of relevance today. This is the family of automata networks that we mainly owe to McCulloch and Pitts, and von Neumann, whose first elements of the theory date from the 1940s. Based on advances at the time, the original works [MP43, von66] highlight a desire to develop the science of computation while inspiring and advancing the modelling of natural biological phenomena. Thus, McCulloch and Pitts introduced the model of formal neural networks which provides an abstraction of neural interactions. They showed in particular that propositional logic can represent neural events and that these networks can be considered, to a certain extent, as a universal model of computation. A little later, in the late 1940s, von Neumann developed cellular automata in order to “compare natural and artificial automata” and “abstract the logical structure of life”. The result of his work was the construction of the first self-reproducing and universal cellular automaton.
Formal neural networks and cellular automata constitute the found base of the theory of automata networks, an automata network being defined “roughly” as a set of entities (automata) which interact with each other. Their interactions happen in a discrete time, according to transition functions which are local to the entities. The two major differences that distinguish these two models relate to the number of interacting entities and the network on which the entities are placed. Indeed, cellular automata are defined by default as having an infinite number of entities (a.k.a. cells). And the entities are placed on a regular and homogeneous network (in general \( \mathbb{Z}^d \), with \( d \geq 0 \)). Formal neural networks (also called threshold Boolean automata networks) have a finite number of entities. And the entities are placed on an irregular and heterogeneous network. These differences give to each of these models its own characteristics that have enabled strong advances in computer science.

By their infinite nature, cellular automata have been mainly studied for their computability properties. Among them, we find the Turing universality of the self-reproducing automaton von66, of the Game of Life of Conway BCG82 and of the elementary cellular automaton 110 Coo04. Also, in 1971, Smith showed that any Turing machine could be simulated by a cellular automaton defined on \( \mathbb{Z} \) Smi71. In addition, cellular automata proved to be good mathematical tools for studying the parallel functioning of computers, of which they were at the origin of systolic architectures KL80. Finally, within the framework of dynamical system theory, the desire to understand their behavioural diversity was at the origin of studies of complexity Wol84 Kur97. Formal neural networks have also brought a lot of progress. In their original article, McCulloch and Pitts showed that they can simulate any Boolean function. Kleene resumed this work. Based on their finite nature, he proved that the languages recognised by these objects are regular, which also allowed him to introduce the concept of finite automata Kle56. Behavioural characterisation conditions were also given and algebraic methods were then developed within this framework Hui59 Els59 Gol67 Cul71. This last reference highlights in particular, strong links between these networks and the Boolean model of genetic regulation networks introduced in Kau69a Kau69b. This latter model is at the origin of numerous works emphasising the interest taken by computer science in the context of research in theoretical biology.

2.2 Computer science as a methodological source for biology

Understanding the mechanisms of biological regulation, in all their diversity, is one of the major current problems in molecular biology. This was notably highlighted by Jacob and Monod in the early 1960s, in particular in JM61 MCJ63. However, from the end of the 1960s, an observation was shared by two biologists, Kauffman (biochemist and biophysicist) and Thomas (biochemist and geneticist). The usual methods of treatment stemming from molecular biology are not adapted to treat as a whole such a problem at the genetic level. According to them, the experimental nature of the methods specific to biology at the time can only provide a piecemeal response to this problem and needs to be supplemented by methodological approaches. On the basis of Delbrück’s work, according to which there are links between differentiated cell types and the attractors of theoretical network models Del49, Kauffman and Thomas proposed to use discrete mathematics to go beyond simple observational knowledge of regulatory systems, advocating in a sense that biology must move towards more general and systematic approaches to living things. This resulted in two articles which organise and federate a whole section of research at the frontier between discrete mathematics and theoretical biology Kau69a Tho73.

Kauffman is thus the first to have proposed a model of genetic regulatory networks, based on formal neural networks. This model is known as Boolean networks. It is a formalisation of regulation where genes are the vertices of a randomly constructed graph. Genes interact over time (discrete). Their interactions are dictated by local Boolean transition functions. They determine whether the genes can be expressed or not, that is to say, transcribed or not Kau71. Originally, this model is based on two strong hypotheses: the interactions are based on an architecture of \( k \)-regular graphs, namely graphs of which all the vertices have the same number of neighbours; the evolution is perfectly synchronous (or parallel). Relaxations of these hypotheses were subsequently carried out Ald03 Ger04. They gave
rise to applications to biological problems such as the analysis of the behaviour of the yeast regulatory network [KPST03], and more generally to the analysis of signalling networks [GBK07, ALB09]. In 1973, Thomas opposed the parallelism hypothesis and developed another method that sought to be closer to “genetic reality” [Tho73]. This method comes with two new ideas. The first proposes to represent the causal dynamics of genetic regulations by means of an asynchronous state transition system. The second idea is to represent the networks themselves by digraphs whose arcs are signed according to the promoting or inhibiting nature of interactions. In [Tho81], Thomas introduces two fundamental conjectures, proven in the discrete framework in [RC07, Ric10]. The first one (resp. the second one) states that the presence of a positive cycle (resp. a negative cycle), composed of an even number (resp. an odd number) of inhibitory edges, in the architecture of the network is necessary for dynamical multi-stationarity (resp. for the existence of an oscillating limit regime). Beyond theoretical research, this method has been widely applied in biology, such as for example the immune response [KT85, Men06, SRSL07] or to infection of Escherichia coli by λ phage [TTK95, TT95, GEHL02].

Although they were initiated by scientists from biology, these two visions emphasise the relevance of automata networks and thus bear the mark of computer science and discrete mathematics. The contribution of mathematician F. Robert played a key role. From the end of the 1960s he pioneered the study of automata networks from a more fundamental and formal point of view. Indeed, Kauffman and Thomas made “arbitrary” choices regarding the ways of updating the automata over time. But Robert was interested in the updating modes as such and their influences on the networks behaviours. He formalised the concepts of block-sequential iterations and chaotic iterations [Rob69, Rob76, Rob80, Rob86, Rob95], which make it possible to obtain updating modes that are partly synchronous and asynchronous. This makes perfect sense in theoretical biology since there is no biological argument today to define the temporal organisation of genetic regulations. In addition, the work carried out by Robert and his collaborators made it possible to establish solid theoretical bases (simple and general) for the behavioural study of automata networks [Gol82, CD85, GFSP85, GM90] while keeping in mind their representational capacities for biology [Dem75, CDLB83, DGTL85]. In this theoretical framework, Robert proved the essential role played by cycles in defining the intrinsic behavioural properties of networks: his theorem stipulates that any acyclic network has a trivial behaviour and admits to the temporal asymptote only a single point fixed. Again, Robert’s work has found many applications in biology, including modelling the genetic control of the flower development of Arabidopsis thaliana [MAB98, DGM10, RGS18] and the study of ventral invagination during gastrointestinal morphogenesis in Drosophila [AGZ06].

2.3 Boolean automata networks, a simple but complex model

From a general point of view, automata networks can be used to model any system which satisfies the following three properties:

- It is made up of distinct entities that interact with each other;
- Each entity is characterised by a variable quantity, which precisely calls to be translated in terms of states of the corresponding automaton in the model;
- The events undergone by the system, like the mechanisms that are at their origin, cannot be observed directly or integrally with certainty. Only their consequences are, that is, changes that are fully accomplished.

These three properties impose very few restrictions on the set of systems that can be abstracted and thus modelled by automata networks. These theoretical objects are therefore generic models of a very wide variety of real systems. It is therefore quite easy to understand the reasons that pushed scientists to use them and to keep studying them in the context of “fundamental bioinformatics”.

5
Let us return to entities’ characteristic "variable quantity" mentioned above. Translating the quantity in terms of automata states, calls for a first exercise of formalisation. This consists in choosing whether what interests us in the variation of the quantity is of a Boolean, discrete or continuous nature. As an illustration, let us take the example of genetic regulation and choose the action of a gene as a variable quantity. If, in the action of this gene, what interests us is its expression (and its non-expression), then the state of the automaton chosen to model the gene should be Boolean. If it is the different ways that this gene has of acting on the other elements of the system that interests us, then we can choose to match an automaton state with each way. This induces a discrete formalism which can obviously be encoded without loss in a Boolean formalism, since an automaton with \( k \) states can be represented by \( \log_2(k) \) Boolean automata. Finally, if we measure the action of the gene by means of the concentration of proteins it produces, continuous formalism turns out to be the most natural. On the other hand, it brings a quantitative character. If this aspect is not desired, the tendency will then be to approximate the protein concentration function at intervals in order to fall back into a discrete, even Boolean framework by considering only extreme concentrations for example.

We can therefore grant different statuses to the Boolean context depending on whether we see it as a direct modelling of reality or as an approximation or encoding of an intrinsically continuous or discrete modelling. Note that direct Boolean modelling is consistent with the choice to focus on the state changes of the automata rather than on their states themselves. By analogy to mechanics, if we see automata as internal combustion engines, the interest relates to the fact that an engine is capable of going from the “off” state to the “on” state (and vice versa) rather than the amount of electricity supplied by the battery to start or on that released by the candles to cause the explosion and initiate the movement. Under this assumption, Boolean abstraction is necessary and sufficient. Furthermore, in order to place ourselves in the context of modelling in biology, it should be emphasised that the discourse of biologists is generally imbued with syntactic elements of propositional logic. It is not uncommon to hear sentences such as: “in the absence of the repressor \( \alpha \), the \( \beta \) gene is expressed” or even “if the products of the \( \alpha \) and \( \beta \) genes form a complex, the latter promotes the expression of the \( \gamma \) gene while these genes tend to inhibit its expression when they are in monomeric form”. This syntax also fits perfectly with a direct modelling of reality in the Boolean formalism.

In addition, Boolean automata networks derive other interesting benefits from their simplicity of definition. In particular, they provide a framework with clearly defined contours, ideal for tackling fundamental problems around the modelling of interacting entity systems. Given the variety of their nature and the current state of our knowledge, the problems in question could not currently benefit from significantly more elaborate frameworks. This would inevitably lead to delaying the initial questions and to destructuring the problem posed by paying attention to additional problems induced by the set of parameters to be considered and not intrinsically included in the initial problem. For these issues, on the contrary, Boolean automata networks offer only what is essential and facilitate the manipulation of a minimal concept of causality, which is rooted in the notion of state changes. Their merit therefore lies in the reliability of the information they potentially provide, delivered by their very high level of abstraction that also makes it possible to obtain general laws that remain valid in more specific contexts. In other words, it is crucial to understand that this simplicity of definition does not necessarily make them “simplistic”, and does not detract from their ability to model complex phenomena that they allow to analyse qualitatively with a surprising subtlety.

3 General definitions and notations

Informally, a Boolean automata network is comprised of abstract entities that interact with each other. The abstract entities are called automata. Automata have states which can take one of two values: either 0 (inactive) or 1 (active). The states of automata can change over the course of a discrete time. They change under the influence of the states of other automata in the network. This section aims to present the formalism of this model, giving the main definitions and useful notations in the rest of the
changes of automata, which leads us to introduce the following notations for all $x$ goes from state $x$ predetermines its evolution from any configuration $p|f$ network $W$ are taken into account, namely a transition for each subset of automata whose states can be updated.

Let $B = \{0,1\}$ and $V = \{0, \ldots , n-1\}$ be a set of $n$ Boolean automata such that $\forall i \in V$, $x_i \in B$ represents the state of automaton $i$. A configuration $x$ of a Boolean automata network $f$ of size $n$ assigns a value of $B$ to each of the automata of $V$ and is classically noted as a vector $x = (x_0, \ldots , x_{n-1})$ that is a vertex of the $n$-cube $B^n$, or as a binary word $x = x_0 \ldots x_{n-1}$. Formally, a Boolean automata network $f$ of size $n$ whose set of automata is $V$ is an ordered set of $n$ Boolean functions, such that $f = (f_i : B^n \rightarrow B \mid i \in V)$. Given $i \in V$, $f_i$ is the local transition function of automaton $i$. It predetermines its evolution from any configuration $x$: if $i$ is updated in configuration $x$ at time $t$, it goes from state $x_i(t)$ to state $f_i(x(t)) = x_i(t+1)$.

Let $s : B \rightarrow \mathbb{I}$, with $\mathbb{I} = \{-1,1\}$, defined such that $s(b) = b - (\neg b)$, the function allowing to convert a Boolean number into a signed integer in $\mathbb{I}$. In this chapter, we pay particular attention to the state changes of automata, which leads us to introduce the following notations for all $x$ in $B^n$:

$$\forall i \in V, \bar{x}^{(i)} = (x_0, \ldots , x_{i-1}, -x_i, x_{i+1}, \ldots , x_{n-1}).$$

$$\forall W \subseteq V, \forall i \notin W, \bar{x}^{W \cup (i)} = \bar{x}^{W \cup \{i\}}.$$ The sign of an interaction from $i$ to $j$ in configuration $x$ is defined by $\text{sign}_x(i,j) = s(x_i) \cdot (f_j(x) - f_j(\bar{x}^{(i)}))$. The effective interactions in $x$ belong to $E(x) = \{(i,j) \in V \times V \mid \text{sign}_x(i,j) \neq 0\}$. From there, we define the interaction graph of $f$ as being the oriented graph $G = (V, E)$, where $E = \bigcup_{x \in B^n} E(x)$ is the set of interactions. In this chapter, the Boolean automata networks discussed (see section 3.3) are special in the sense that their interaction graphs are simple, namely that there can only be one signed interaction $(i,j) \in E$. If it is signed positively (resp. negatively), we say that it is activating (resp. inhibiting) and the state of $j$ tends to mimic (resp. to oppose) that of $i$. In the following, the interaction graphs will be signed for convenience of reading (see Figure 1).

### 3.2 Updating modes and transition graphs

In order to determine the possible behaviours of a Boolean automata network, it is essential to specify the way according to which the states of the automata (or, abusing language, the automata) are updated over time. This specification is what we call an updating mode. The most general point of view is to consider all the possibilities. This amounts to seeing the evolution of a network as a discrete dynamical system associated with a relation so that, for each configuration, $2^n - 1$ outgoing transitions are taken into account, namely a transition for each subset of automata whose states can be updated. More precisely, for all $W \neq \emptyset \subseteq V$, we define the update function $F_W : B^n \rightarrow B^n$ such that:

$$\forall x \in B^n, \forall i \in V, F_W(x)_i = \begin{cases} f_i(x) & \text{if } i \in W, \\ x_i & \text{otherwise.} \end{cases}$$

Figure 1: A Boolean automata network of size 3: its interaction graph (on the left), the ordered set of its local transition functions (on the right).
Thus, for the most general updating mode, called the elementary updating mode, the global behaviour of the network is given by the elementary transition graph $G_e = (B^n, T_e)$, where $T_e = \{ (x, F_W(x)) \mid x \in B^n, W \not= \emptyset \subseteq V \}$, introduced in [Nou12b, Sen12].

The transitions $(x, F_i(x))$ that involve updating a single automaton $i \in V$ are called asynchronous transitions. The transitions $(x, F_W(x))$, with $|W| > 1$, that induce the updating of several automata are called synchronous transitions. The subgraph $G_a = (B^n, T_a)$ of $G_e$ whose set of arcs $T_a = \{ (x, F_{i_1}(x)) \mid x \in B^n, i \in V \}$ equals the set of asynchronous transitions of network is called the asynchronous transition graph. This graph defines the asynchronous dynamics of the network that corresponds to its dynamics when it evolves according to the asynchronous updating mode, i.e. such that in each configuration, only $n$ transitions are considered, one for each automaton. This updating mode has been widely used in the studies of Thomas and his collaborators [RMCT03, RRT08, RCB04, RC07, Ric10, Tho81, Tho91]. As an illustration, the asynchronous transition graph of the network depicted in Figure 1 is presented in Figure 2 (left). However, because it is the most “natural” (mathematically speaking) when only the local transition functions are known, and because that it allows to give relevant insights through transition graphs of smaller sizes, the parallel updating mode occupies a very important place in the literature on discrete dynamical systems in general. When a network evolves in parallel, all of its automata update their states at each time step. In other words, the parallel transition graph of a network is $G_p = (B^n, T_p)$, where $T_p = \{ (x, F_V(x)) \mid x \in B^n \}$ is the subset of the perfectly synchronous transitions of $T_e$. The parallel transition graph of the network given in Figure 1 is presented in Figure 2 (right).

Let us now specify notations and vocabulary relative to dynamical behaviours of networks. Consider an arbitrary Boolean automata network $f$ of size $n$, an updating mode $\mu$, the associated transition graph $G_\mu = (B^n, T_\mu)$, and $x \in B^n$ one of its possible configurations. A trajectory of $x$ is any path in $G_\mu$ that starts from $x$. A strongly connected component of $G_\mu$ which does not admit any outgoing transition is an asymptotic behaviour of $(f, \mu)$, that we classically designate as an attractor of $(f, \mu)$. A configuration of $B^n$ that belongs to an attractor is a recurring configuration. Given an attractor, its length is the number of recurring configurations that compose it. An attractor of length 1 (resp. strictly greater than 1) is a stable configuration, a.k.a. a fixed point (resp. a stable oscillation, a.k.a. a limit cycle) of $(f, \mu)$. If $\mu$ is a deterministic updating mode, such as the parallel mode, attractors are simple cycles and the term period is preferred to refer to their length. Finally, we define the convergence time of a configuration $x$ as the length of its shortest trajectory which makes it reach a recurring configuration. The convergence time of the network is the greatest convergence time of all
its $2^n$ configurations. In the transition graphs presented in this chapter, by convention, we associate the light gray color with stable configurations and the dark gray color with recurrent configurations belonging to stable oscillations. Thus, in Figure 3, it can be seen that configuration $011$ is stable for the asynchronous and parallel updating modes. This is a direct consequence of the fact that a stable configuration that is a fixed point of $f$ (implicitly evolving in parallel) is preserved by every updating mode since it corresponds to the vector of the local fixed points of the local transition functions. On the other hand, we observe that the asynchronous transition graph admits a stable oscillation of size 4 while the parallel update mode admits a stable oscillation of size 3. This illustrates that stable oscillations are generally not conserved when the updating mode is changed. This is a current important field of study in the context of interaction networks and Boolean automata networks.

### 3.3 Isolated cycles and tangential cycles

As mentioned in the introduction, we focus in this chapter on two sorts of Boolean automata networks, namely, **Boolean automata cycles** and **Boolean automata double-cycles**. The first are networks whose interaction graphs are cycles. The second are networks whose interaction graphs are two cycles that intersect tangentially. The founding results that lead us to develop ever more research on these interaction patterns are presented in the following section.

A **Boolean automata cycle** $\mathcal{C}_n$ is a Boolean automata network of size $n$ whose interaction graph $G = (V, E)$ is a cycle, in the sense of graph theory. $V$ is naturally assimilated to $\mathbb{Z}/n\mathbb{Z}$, so that considering two automata $i$ and $j$ of $V$, $i + j$ represents $i + j \mod n$. Thus, a Boolean automata cycle $\mathcal{C}_n$ is defined as an ordered set of local transition functions of arity 1 that are such that: $\forall i \in V$, $f_i : \mathbb{B}^n \to \mathbb{B}$, and either $f_i(x) = x_{i-1}$ or $f_i(x) = \neg x_{i-1}$. Note that there are two types of Boolean automata cycle, positive and negative. A Boolean automata cycle is a **positive cycle** $\mathcal{C}^+$ (resp. a **negative cycle** $\mathcal{C}^-$) if it is composed of an even number (resp. of an odd number) of inhibiting interactions.

A **Boolean automata double-cycle** $\mathcal{D}_n$, with $n = \ell + r - 1$, is a Boolean automata network of size $n$ composed of two Boolean automata cycles $\mathcal{C}_\ell$ and $\mathcal{C}_r$ which tangentially intersect in one automaton, automaton 0. In the following, for reasons of clarity, we prefer the notation $\mathcal{D}_{\ell,r}$. The set of automata of cycle $\mathcal{C}_\ell$ is $V^\ell = \mathbb{Z}/\ell\mathbb{Z} = \{0, \ldots, \ell - 1\}$ and that of cycle $\mathcal{C}_r$ is $V^r = 0 \cup \{\ell - 1 + i \mid i \neq 0 \in \mathbb{Z}/r\mathbb{Z}\}$. In a Boolean automata double-cycle, by definition, all the local transition functions are of arity 1 except that...
of automaton 0 that is of arity 2. In this work, we only consider locally monotonous Boolean automata double-cycle, which induces that function \( f_0 \) is defined as \( f_0(x) = f_0^\ell(x) \circ f_0^\ell(x) = x_{\ell-1} \circ x_{n-1} \), where \( \circ \in \{ \wedge, \vee \} \). Note that the choice of operator \( \circ \) only changes the position of the configurations on the trajectories. In other words, whatever the chosen updating mode, given two Boolean automata double-cycle \( \mathcal{G}_{\ell,r} \) and \( \mathcal{G}'_{\ell,r} \), such that \( f_0(x) = f_{\ell-1}(x) \wedge f_{n-1}(x) \) and \( f'_0(x) = f'_{\ell-1}(x) \lor f'_{n-1}(x) \), their respective transition graphs are identical up to an isomorphism on the configurations. In addition, it is trivial to determine one from the other, replacing the configurations with their opposites. Consequently, in the rest of this chapter, to insist on this property, we will use without loss of generality the above expression to determine one from the other, replacing the configurations with their opposites. Consequently, it is easy to see that there are three distinct types of Boolean automata double-cycles. A positive Boolean automata double-cycle is composed of two positive Boolean automata cycles and is denoted by \( \mathcal{G}_{\ell,r}^+ \); a negative Boolean automata double-cycle is composed of two negative Boolean automata cycles and is denoted by \( \mathcal{G}_{\ell,r}^- \); a mixed Boolean automata double-cycle is composed of a positive Boolean automata cycle \( \mathcal{C}^+ \) tangentially intersected with a negative Boolean automata cycle \( \mathcal{C}^- \), and is denoted by \( \mathcal{G}_{\ell,r}^{+,+} \).

Finally, in [Non12b, Sen12], the authors have shown that the Boolean automata cycles admit canonical representatives, and Boolean automata double-cycles too by induction. The studies presented in the sequel focus on these canonical representatives only. Indeed, canonicity means that two distinct Boolean automata cycles or Boolean automata double-cycles of same sign and same size that evolve following the same updating mode admit the same transition graph up to an isomorphism on the configurations. A positive Boolean automata cycle is said to be canonical when its interaction graph contains only activating interactions. A negative Boolean automata cycle is canonical when its interaction graph admits a single inhibiting interaction, represented by the arc \((n - 1, 0) \in E\). The canonical Boolean automata double-cycles are the canonical Boolean automata cycle compositions, depicted in Figure 3.

4 Seminal results on cycles

The works of Robert [Rob80] and Thomas [Tho81] have highlighted three fundamental results that explain the primordial role that cycles play in the behavioural diversity of interaction networks.

**Theorem 1** ([Rob80] [Rob86] [Rob95]). Let \( f : \mathbb{B}^n \rightarrow \mathbb{B}^n \) be a Boolean automata network of size \( n \) and \( G \) its associated interaction graph. If \( G \) is an acyclic graph, then:

1. \( f \) has a unique attractor that is a stable configuration, let us say \( x \).
2. \( \mathcal{G}_p \) has a path of length at most \( n \) from every configuration \( y \) to \( x \).
3. \( \mathcal{G}_a \) is acyclic and has a a geodesic path from every configuration \( y \) to \( x \).

This theorem is particularly interesting for two reasons. First, it is easy to extend it to any kind of multi-valued automata networks \( g : \prod_{i=1}^{n} X_i \rightarrow \prod_{i=1}^{n} X_i \), where \( X_i \) denotes the set of possible states of automaton \( i \), and to any kind of updating mode such that every automaton is updated an infinite number of times over the course of time (let us call such an updating mode a fair updating mode). Indeed, the general idea of the proof rests on an induction on the depths of the automata of the acyclic interaction graph that admits source automata. Source automata are automata that are governed by constant local transition functions. They inevitably become forever fixed once they are updated for the first time. In an acyclic network, the fixity of source automata propagates. The states of all the other automata progressively become fixed too as a result. The theorem emphasizes that cycles are necessary conditions for interaction networks to admit complex dynamics.
Theorem 2 ([RR10], [RC07], [Tho81]). Let \( g : \prod_{i=1}^{n} X_i \rightarrow \prod_{i=1}^{n} X_i \) be an automata network and \( G \) its associated interaction graph. Under the asynchronous updating mode, the presence of a positive cycle in \( G \) is necessary for the dynamics of \( g \) to admit several stable configurations.

This second theorem sheds light on the role of positive cycles on the ability of interaction networks to stabilise in several ways. Although it was originally stated and demonstrated under the asynchronous updating mode hypothesis, this result has been shown to hold for any fair updating mode ([Nou12], [Sen12]). The general proof starts from the result of [RC07] and is made by contradiction under the assumption of the absence of a positive cycle. In this case, either the interaction graph is acyclic and Theorem 1 applies, or it has at least one negative cycle and it is shown that such cycles, whatever the updating mode, cannot remove the local instabilities on all automata.

Theorem 3 ([Ric10], [Tho81]). Let \( g : \prod_{i=1}^{n} X_i \rightarrow \prod_{i=1}^{n} X_i \) be an automata network and \( G \) its associated interaction graph. Under the asynchronous updating mode, the presence of a negative cycle is necessary for the dynamics of \( g \) to admit a stable oscillation.

As for this third theorem, it turns out not to be general for all updating modes. To be convinced, it suffices to compute the parallel transition graph of an arbitrary positive Boolean automata cycle of size greater than 2 and to note that it admits at least one stable oscillation. Despite this lack of generality, it should be noted that this theorem underlines the singular role of negative cycles in connection with asymptotic dynamic oscillations, as will be explained below.

Taken together, these three theorems, of which we can say that two of them are laws insofar as their generality makes them sorts of meta-theorems of interaction network theory, emphasise that the feedback cycles are the causes of the dynamical complexity of networks. In other words, they effectively constitute the sources of the behavioural diversity of networks and consequently of the computational expressiveness of networks. This naturally brings us to the following parts of this chapter, about major results concerning the dynamical and combinatorial properties of Boolean automata cycles and Boolean automata double-cycles. The results are illustrated with examples. As discussed above, Robert’s result highlighted feedback cycles as kinds of complexity engines. Then, Thomas’ results put forward the necessity to distinguish feedback cycles depending on their (positive or negative) nature to understand their influence. The results presented below about Boolean automata double-cycles are a first step further, towards understanding how cycle combinations operate and what their effects are.

5 Boolean automata cycle dynamics

In this section, we focus on Boolean automata cycles. In a first part, we present in Theorem 4 the main results related to the dynamics of isolated Boolean automata cycles when the latter evolve according to the parallel updating mode. This theorem requires some preliminary definitions and notations of number theory such as the Dirichlet convolution, the Möbius function and the Euler’s totient function. In a second part, we present the results related to isolated Boolean automata cycles when the latter evolve according to the asynchronous updating mode.

5.1 Parallel Boolean automata cycles

The first elements on the dynamics of Boolean automata cycles evolving according to the parallel updating mode were introduced in [RMCT03]. The full characterisation of their dynamics had to wait until [DNS12]. This characterisation could be obtained thanks to an approach combining discrete dynamical systems theory, enumerative and word combinatorics, particularly appropriate to the nature of the mathematical objects in question.
5.1.1 Definitions and notations of pertinent quantities

To describe the results obtained, we give below definitions and notations. First, given an attractor of (minimal) period \( p \), we say that all the multiples of \( p \) are also periods of this attractor. So if \( x \in \mathbb{B}^n \) is a recurring configuration of an attractor of period \( p \), then \( p \) is the period of \( x \) and of any other configuration \( y \) such that there exists \( t \in \mathbb{N} \) such that \( y = F_t(x) \). We denote by \( \mathcal{X}(p) = \{ x \in \mathbb{B}^n \mid x = F_t(x) \} \) the set of recurring configurations of period \( p \) and by \( X(p) = |\mathcal{X}(p)| \) their number. We define the smallest integer \( \omega \) that is a period common to all recurring configurations as the order of the Boolean automata cycle, which is said to be reached when there exists an attractor of minimum period \( \omega \).

Let us consider the function \( \text{one} : n \in \mathbb{N} \mapsto 1 \) as well as the Dirichlet convolution, denoted by \( \star \). Given two functions \( f \) and \( g \), \( \star \) is the binary operator defined such that \( f \star g : n \in \mathbb{N}^* \mapsto \sum_{d|n} f(d) \cdot g(n/d) \). The set of the arithmetic functions with point-to-point addition and Dirichlet convolution is a commutative ring. The identity by the multiplication of this ring is the function \( w \) such that for all \( n \in \mathbb{N}^* \), \( w(n) = 1 \) if \( n > 0 \) has an even number of prime factors, and \( -1 \) if \( n > 0 \) has an odd number of prime factors.

If \( n = \prod_{i=0}^{k} p_i \), where the \( p_i \)'s are the distinct prime numbers taken in increasing order, then \( \mu(n) = (-1)^k \). In our context, this function is of interest through the Möbius inversion formula that is obtained from \( \text{one} \star \mu = \delta \), that is satisfied by all the functions \( f \) and \( g \), and that is such that: \( g = f \star \text{one} \iff f = g \star \mu \). In other words, we have:

\[
\forall n \in \mathbb{N}^*, \quad g(n) = \sum_{d|n} f(d) \iff f(n) = \sum_{d|n} g(d) \cdot \mu(n/d).
\]

Another particularly useful function in the sequel is the Euler’s totient function, denoted by \( \phi \). Given an integer \( n \in \mathbb{N}^* \), it associates the number of strictly positive integers less than or equal to \( n \) that are prime with \( n \), such as: \( \phi(n) = |\{ m \in \mathbb{N}^* \mid m \leq n \text{ and } m \text{ is prime with } n \}| \). Note that there is a relationship between the Möbius function and the Euler’s totient function. Indeed, as \( \phi \) satisfies \( \forall n \in \mathbb{N}^*, \quad n = \phi \star \text{un}(n) \), it respects \( \phi = \mu \star \text{id} \), where \( \text{id} : n \in \mathbb{N}^* \mapsto n \).

In terms of combinatorics, the asymptotic behaviour of an interaction network can be described by means of four quantities [PW01] related to each other and given below. Consider that \( p \) is a divisor of the order \( \omega \) of the Boolean automata cycle studied and the function \( \text{inv} : n \in \mathbb{N}^* \mapsto 1/n \). We then have the following quantities:

- The number \( X(p) \) of configurations of period \( p \) is \( X = \tilde{X} \star \text{one} \);
- The number \( \tilde{X}(p) \) of configurations of minimal period \( p \) is \( \tilde{X} = X \star \mu \);
- The number \( \Lambda(p) = \tilde{X}(p)/p \) of attractors of period \( p \) is \( \Lambda = \text{inv}(X \star \mu) \);
- The total number \( T(\omega) \) of attractors is \( T = \Lambda \star \text{one} = \text{inv}(X \star \phi) \).

The last two quantities correspond to well known formulas in the context of Lyndon words and binary necklaces [BP07 [GKP89 [Rus03]: a Lyndon word \( w \) being such that \( w < v \) for all nonempty words \( v \) such that \( w = uv \) and \( u \) is nonempty; a binary necklace \( w \) of length \( n \) being a circular binary word such that for all \( i \in \mathbb{Z} \), \( w_i = w_{i+n} \mod n \). In particular, the penultimate defining \( \Lambda \) corresponds to the Witt formula counting the number of Lyndon words; the last one defining \( T \) corresponds to the Burnside’s orbit-counting lemma. Note that the last formula is satisfied because \( \text{inv} \) distributes over \( \star \). Finally, note that it suffices to calculate \( X \) to obtain the others.
5.1.2 Results

The qualitative characterisation of the dynamics of Boolean automata cycles can be summarised by the following theorem that presents in the form of a table the set of all the quantities presented above. For reasons of space, the details and the demonstrations related to these results are not presented here. The reader can nevertheless find all the details in [DNS12, Nou12b, Sen12].

**Theorem 4 (DNS12).** The order $\omega$, the numbers $X(p)$ of configurations of (minimal) period $p$, where $X(\omega)$ is the total number of recurring configurations, as well as the number $A(p)$ of attractors of period $p$ and the total number $T(\omega)$ of attractors of positive and negative Boolean automata cycles are:

| Positive cycles $C^+_n$ | Negative cycles $C^-_n$ |
|--------------------------|--------------------------|
| $\omega = n$             | $\omega = 2n$            |
| $X^+(p) = 2^p$           | $X^-_n(p) = -(p|n) \cdot 2^\frac{p}{2}$ |
| $X^+(p) = \sum_{d|p} \mu\left(\frac{n}{d}\right) \cdot 2^d$ | $X^-_n(p) = \sum_{k|p \text{ odd}} \mu(k) \cdot 2^\frac{p}{2}$ |
| $A^+(p) = \frac{T^+(p)}{p}$ | $A^-_n(p) = \frac{T^-_n(p)}{p}$ |
| $A^+(p) = \text{OEIS A27375}(p)$ | $A^-_n(p) = \text{OEIS A1037}(p)$ |
| $T^+(\omega) = \frac{1}{n} \sum_{d|n} \phi\left(\frac{n}{d}\right) \cdot 2^d$ | $T^- (\omega) = \frac{1}{2n} \sum_{k|2n \text{ odd}} \phi(k) \cdot 2^\frac{p}{2}$ |
| $T^+(\omega) = \text{OEIS A31}(n)$ | $T^- (\omega) = \text{OEIS A16}(n)$ |

where $-(p|n)$ is 0 if $p$ divides $n$ and 1 otherwise.

Among the particularly interesting properties of a parallel Boolean automata cycle that emerge from this theorem, it should be noted that all of its configurations are recurring, as illustrated in Figure 4, and that the total number of attractors, for a given period or not, is exponential according to its size $n$. Moreover, as a corollary, for the two types of cycles, the order is reached, and positive cycles admit two stable configurations $x$ and $x^{V}$ (where $x$ is the configuration in which all the automata are at state 0 for canonical Boolean automata cycles). Finally, another important point is that this combinatorial study induces the complete characterisation of the structure of the parallel transition graphs of both positive and negative cycles.
It is also interesting to notice that these results go beyond the parallel updating mode and extend to the block-sequential updating modes. Block-sequential updating modes were introduced by Robert [Rob69]. They are deterministic periodic updating modes defined by ordered partitions of $V$. Given a period $p$, such an updating mode can be defined by a function $\mu : V \rightarrow \mathbb{N}/p\mathbb{N}$. Indeed, in [GN10], the authors showed that the dynamics of a Boolean automata cycle of size $n$ and sign $s \in +, -$ evolving according to a block-sequential updating mode is in essence equivalent to that of a Boolean automata cycle of smaller size and same sign evolving in parallel. The proof rests on substitutions of local transition functions according to their execution over time. Hence, to understand the dynamics of a Boolean automata cycle evolving according to a block-sequential updating mode is a matter of understanding the dynamics in parallel of a smaller Boolean automata cycle.

### 5.2 Asynchronous Boolean automata cycles

In combinatorial terms, the dynamics of asynchronous Boolean automata cycles are much simpler than that of parallel Boolean automata cycles. It is in [RMCT03] that we find the first characterisation of the attractors of asynchronous Boolean automata cycles. The general idea of these results is based on the concept of instability. In a given configuration of the network, an automaton is said to be unstable if the application of its local transition function would then make its state change. A configuration is unstable when it has at least one unstable automaton. In particular, the authors showed that asynchronism makes it possible to reduce the number of instabilities, until there are none (resp. only one) left in the case of positive (resp. negative) Boolean automata cycles. This property implies the existence of at least one stable configuration in the positive case. In the negative case, it implies the absence of stable configurations and thereby the existence of at least one stable oscillation. Theorem 5 derives almost directly from this work. It is illustrated by Figure 5.

**Theorem 5** ([RMCT03]). A positive Boolean automata cycle $\mathcal{C}_n^+$ has two attractors which are two stable configurations $x$ and $x'$. A negative cycle $\mathcal{C}_n^-$ has a single attractor of length $2n$.

It is easy to see that the results stated in this theorem are strongly related to those of Theorem 4. Indeed, first, the two stable configurations of asynchronous positive cycles are identical to those of these same cycles in parallel. And second, the unique stable oscillation of length $2n$ of a negative cycle $\mathcal{C}_n^-$ under the asynchronous mode is identical to that of period $\omega$ of parallel negative cycles. The two configurations $x$ and $x'$ which are stable configurations in the case of positive cycles, belong to the stable oscillations in the case of negative cycles.
Figure 6: Top panel: Interaction graphs of two positive canonical Boolean automata double-cycles. Middle panel: Interaction graphs of two mixed canonical Boolean automata double-cycles. Bottom panel: Interaction graphs of two negative canonical Boolean automata double-cycles. These six networks will serve as examples in the sequel.

Finally, notice that this approach based on instabilities has been generalised in [Nou12b, Sen12] to show the validity of Theorem 2 regardless of the updating mode (see Section 4).

6 Boolean automata double-cycle dynamics

Now that the structural and combinatorial properties of the parallel and asynchronous transition graphs of the Boolean automata cycles are established, we present in this section those related to Boolean automata double-cycles.

6.1 Parallel Boolean automata double-cycles

In this section, we focus on Boolean automata double-cycles evolving according to the parallel updating. The method used to obtain the results presented follows the lines of method used for Boolean automata cycles.

6.1.1 Definitions and notations

Besides the four quantities $X(p)$, $\bar{X}(p)$, $\lambda(p)$ and $T(\omega)$ that we are going to use, we introduce here other definitions and notations to characterise the dynamics of Boolean automata double-cycles that are Boolean automata cycles tangentially interconnected. In this sense, given any Boolean automata double-cycle $D_{\ell, r}$, where $s \in \{(+, +), (-, +), (-, -)\}$, two quantities defined by means of $\ell$ and $r$ will be particularly useful, $\Delta = \gcd(\ell, r)$ and $\Delta_p = \gcd(\Delta, p)$.

Moreover, the results call on combinatorics on words. So we introduce the Lucas’ and Perrin’s sequences. These sequences will allow us to count the number of recurring configurations. The Lucas’ sequence $(L(n))_{n \in \mathbb{N}^*}$ (OEIS A204) is defined by $L(1) = 1$, $L(2) = 3$ and for all $n > 2$, $L(n) = L(n -
1) + L(n - 2), and counts the number of binary necklaces of size \(n\) without the factor 00. The Perrin’s sequence \(\{P(n)\}_{n \in \mathbb{N}}\) (OEIS A1608) is defined by \(P(0) = 3, P(1) = 0, P(2) = 2\) and for all \(n > 2\), \(P(n) = P(n - 2) + P(n - 3)\), and counts the number of binary necklaces of size \(n\) without the factors 00 and 111.

### 6.1.2 Results

On the basis of the previous definitions and notations, Theorem 6 below gives the qualitative characterisation of the dynamics of the different types of Boolean automata double-cycles. It will be illustrated on six distinct Boolean automata double-cycles depicted in Figure 6. The details of the proofs can be found in [Nou12a], [Nou12b], [Sen12].

**Theorem 6 ([Nou12a]).** The order \(\omega\), the numbers \(\mathcal{X}(p)\) and \(\bar{\mathcal{X}}(p)\) of configurations of (minimal) period \(p\), where \(\mathcal{X}(\omega)\) represents the total number of recurring configurations, as well as the number \(A(p)\) of asymptotic behaviours of period \(p\), and the total number \(T(\omega)\) asymptotic behaviours of the positive, mixed and negative Boolean automata double-cycles are:

| Positive double-cycles \(\mathcal{G}_{p,\ell}^{+}\) | Mixed double-cycles \(\mathcal{G}_{p,\ell}^{\ell}\) | Negative double-cycles \(\mathcal{G}_{p,\ell}^{-}\) |
|-------------------------------------------------|-------------------------------------------------|-------------------------------------------------|
| \(\omega = \Delta\) | \(\omega = \ell\) | \(\frac{\ell + 1}{2}\) if \(\frac{\ell + 1}{\Delta} = 4\) \(\ell + r\) (\(\sin\)) |

\[X^+(p) = -(p(\ell) \cdot L(\ell)\Delta_{\ell}), \quad \bar{X}^-(p) = -(p(\Delta) \cdot P(\ell)\Delta_{\ell})\]

\[\bar{X}^{\ell}(p) = \sum_{d \mid p} \phi(Q(p)) \cdot L(\ell)\Delta_{\ell}\]

\[\bar{A}^+(p) = \frac{X^+(p)}{\ell}, \quad \bar{A}^-(p) = \frac{\bar{X}^-(p)}{\ell}\]

\[\bar{T}^+(\omega) = \frac{1}{2} \sum_{d \mid p} \phi(Q(p)) \cdot L(\ell)\Delta_{\ell}, \quad \bar{T}^-(\omega) = \frac{1}{2} \sum_{d \mid p} \phi(Q(p)) \cdot P(\ell)\Delta_{\ell}\]

where \(\neg(p|m)\) equals 0 if \(p\) divides \(m\) and 1 otherwise.

Among other things (see Figure 6), it emerges from these results that (i) positive Boolean automata double-cycles have two stable configurations \(x\) and \(\bar{x}\) (with \(x = (0, \ldots, 0)\) when they are canonical) and that they have an asymptotic behaviour similar to positive Boolean automata cycles of the same order, (ii) mixed Boolean automata double-cycles have a single stable configuration, and (iii) negative Boolean automata double-cycles have no stable configurations.

Also, we remark that, unlike Boolean automata cycles, the order of Boolean automata double-cycles is not necessarily reached. Finally, this theorem highlights again that, like Boolean automata cycles, Boolean automata double-cycles admit an exponential number of attractors according to their size. However, despite its exponential nature, the number of Boolean automata double-cycle attractors is significantly smaller than that of Boolean automata cycles. In other words, the intersections of cycles seem to participate strongly in the reduction of asymptotic degrees of freedom of interaction networks. Based on this idea, studies have been conducted to compare \(T^+(n)\) and \(T^-(2n)\) of Boolean automata cycles with quantities \(\bar{T}^+(\Delta), \bar{T}^{\ell}(r)\) and \(\bar{T}^-(\ell + r)\) of Boolean automata double-cycles.

Consider a network \(f\) of order \(\omega\), so that \(f = X^+\) or \(f = X_{\ell,r}^{s,s'}\), where \(s, s' \in \{+, -, \}\). Let \(p\) be a divisor of \(\omega\). Also note \(Q \in \{\bar{X}, \bar{A}, \bar{T}\}\) one of the four quantities analysed. It has been demonstrated in [Nou12b], [Sen12] that \(Q_{+,s}^{+}(p) = Q^{+}(p), \bar{X}_{+,r}^{+}(p) \leq X^{+}(p), \bar{X}_{+,r}^{-}(p) \leq X^{-}(p)\) and \(Q_{+,s}^{s}(p) = Q_{+,s}^{-}(p)\).

However, the number of recurring configurations of a Boolean automata double-cycle, \(X_{+,s}^{s}(p)\) has been bounded more finely as a function of \(X^+_n(p)\) and \(X^-_n(p)\), based on the previous results as well as on the relation of the Lucas sequence with the golden ratio \(\frac{1 + \sqrt{5}}{2}\).
Figure 7: Parallel transition graphs of the canonical Boolean automata double-cycles depicted in Figure 6, where $\Delta$.

(root of $x^2 - x - 1 = 0$) and that of the Perrin sequence with the plastic number $\xi$, denoted by $\xi = \sqrt{\frac{1}{2} + \frac{1}{6}\sqrt{23}} + \sqrt{\frac{1}{2} - \frac{1}{6}\sqrt{23}}$ (root of $x^3 - x - 1 = 0$). The bounds found led to Theorem 7 that concludes this part by highlighting that a vast majority of recurring configurations have the greatest minimum period possible.

Theorem 7 (Nou12b). Let $f$ be a Boolean automata network of order $\omega$. If $f$ is a Boolean automata cycle or a Boolean automata double-cycle such as $f$ is neither $\mathcal{D}_{2,2}^-$ nor $\mathcal{D}_{1,3}^-$, then its total number of attractors $T(\omega)$ is bounded by its total number of recurring configurations $X(\omega)$ so that:

$$\frac{X(\omega)}{\omega} \leq T(\omega) \leq 2 \cdot \frac{X(\omega)}{\omega}.$$
which means that the attractor periods of $f$ are very large:

$$\sum_{p|\omega} A(p) \frac{A(p)}{\tau(p)} = \frac{X(\omega)}{\tau(\omega)} \geq \frac{\omega}{2}$$

To end this section dedicated to parallel Boolean automata double-cycles, notice that all the results presented here extend naturally to any tangential double-cycles, namely two cycles that admit several automata in common such that these common automata are organised into an isolated path so that each of them have a local transition function of arity 1 except the first one for which the arity is 2. The proof is simple and rests on an induction consisting in transforming each tangential automaton whose local transition function arity is 1 (by following the isolated path in reverse direction) into two copies, one in the left cycle and one in the right cycle. Following this reasoning, it is easy to see that such a tangential double-cycle is equivalent to a Boolean automata double-cycle whose cycles are of bigger sizes and same signs.

Moreover, notice that both Boolean automata cycles and Boolean automata double-cycles admit an exponential number of attractors. This is quite unrealistic if we view these objects as models of genetic regulation networks. Indeed, “real” genetic regulation networks seem to have a number of asymptotic behaviours (cellular types, biological rhythms...) that is polynomial (perhaps linear) according to the number of their genes. Nevertheless, it is important to see that the results obtained show that the number of attractors of Boolean automata double-cycles is drastically smaller than that of Boolean automata cycles. Actually, we think that the polynomial characteristics of the number of attractors of real regulation systems comes from the entanglement of cycles. More precisely, without formalising it, we conjecture that the more there are entangled cycles, the less there are local and global instabilities, the less there are attractors (simply because adding intersections adds dynamical constraints), and thus the less automata networks are sensitive to synchronism.

### 6.2 Asynchronous Boolean automata double-cycles

The last part of this synthesis is devoted to the dynamics of asynchronous Boolean automata double-cycles. It presents the characterisation established in [MNR15]. It is the counterpart of the study of the dynamics of Boolean automata double-cycles under the parallel updating mode. The study of the asynchronous case takes a new approach. It formalises long sequences of updates by way of algorithmic descriptions. This approach allows an elegant and more detailed description of the dynamics of feedbacks in interaction networks than the previous works. In particular it facilitates the study of convergence times.

#### 6.2.1 Definitions and notations

For the sake of clarity, let us first recall that the study surveyed here focuses on canonical Boolean automata double-cycles.

**States and configurations** Here, we will use the classical notation $V = \{0, \ldots, n-1\}$ for representing the automata of a network of size $n$ and its congruence $V \equiv \{c = c_0, c_1, \ldots, c_{n-1}\}$. A configuration $x \in \mathbb{B}^n$ is seen as a vector of two binary words. The first symbol of these two words represents $x_c \equiv x_0$. The null configuration is therefore denoted by $(0^0, 0^0)$. Furthermore, we denote by $x^f$ (resp. $x^r$) the projection of $x$ on the Boolean automata cycle $c^f$ (resp. $c^r$). This way, $x = (x^f, x^r)$, and in configuration $x$, the state of automaton $c^f$ is $x^f_c$. Notice that $x_0 = x_0^f = x_0^r$ since the three notations represent the state of automaton $c$ in $x$. 

18
Expressiveness measure  Let $x$ be a configuration of a Boolean automata cycle $\mathcal{C}_n$. Its expressiveness is the number of factors $01$ that compose it, namely $\{i \mid 0 \leq i \leq n-1, x_i = 0 \text{ and } x_{i+1 \mod n} = 1\}$. The expressiveness of a configuration $x$ of a Boolean automata double-cycle is the sum of the expressiveness of $x'$ and $x''$. From this definition follows that, if $\ell$ and $r$ are even, the least expressive configurations are $(0', 0')$ and $(1', 1')$, and that the most expressive ones are $((01)\hat{x}, (01)\hat{x})$ and $((10)\hat{x}, (10)\hat{x})$.

Elementary instructions  Network trajectories can be very long. To study them, we need a way to efficiently describe the sequence of automata updates that they execute. Our human minds must be able to understand from the description, what is the effect of the trajectory on the network, what changes does the network undergo along the trajectory. To do so, we proposed to view these sequences as instructions that make it easier to capture their effect on configurations. Let us therefore consider:

- a Boolean automata double-cycle $\mathcal{D}_n$,
- one of the Boolean automata cycles of $\mathcal{D}_n$, namely $\mathcal{C}$, whose size is noted $\text{size}(\mathcal{C})$,
- the current configuration $x$ of $\mathcal{C}$, and,
- two automata of $\mathcal{C}$ distinct from $c$, namely $c_i$ and $c_j$, such that $i < j$.

With these notations, the following seven basic instructions are defined:

1. /* update of automaton $c$ */
   \[ \text{sync: } x_c \leftarrow f_c(x) \]
   Instruction \text{sync} is the only instruction that updates automaton $c$ and where both Boolean automata cycles interact with each other. This (key)-instruction will always be called when $c$ can change its state. This instruction can be used either to set $c$ at a desired state or to increase the expressiveness from a configuration. Furthermore, it is the only way to switch a $111$ (resp. $000$) pattern into a $101$ (resp. $010$) pattern and, thus, to increase the expressiveness.

2. /* update of automaton $c_i$ */
   \[ \text{update}(c_i): x_{c_i} \leftarrow f_{c_i}(x) \]
   Instruction \text{update} updates an automaton distinct from $c$.

3. /* incremental updates */
   \[ \text{incUp}(\mathcal{C}, i, j): \text{ for } k = i \text{ upto } j \text{ do update}(c_k) \]
   Instruction \text{incUp} updates consecutive automata in increasing order. In fact, \text{incUp} propagates the state of $c_{i-1}$ along $\mathcal{C}$. Notice that if $j < i$ then no automata are updated. Moreover, since $i \neq 0$ and $j \neq 0$, $c$ cannot be updated with \text{incUp}. This instruction admits the following property. Let $x'$ be the result of the execution of $\text{incUp}(\mathcal{C}, i, j)$ on configuration $x$. Then $\forall k \in \{i, \ldots, j\}, x'_{k} = x_{i-1}$ and $\forall k \notin \{i, \ldots, j\}, x'_{k} = x_{k}$.

4. /* incremental propagation of $x_c$ */
   \[ \text{erase}(\mathcal{C}): \text{incUp}(\mathcal{C}, 1, \text{size}(\mathcal{C}) - 1) \]
   Instruction \text{erase} is a particular case of \text{incUp}. It propagates the state of $c_0$ along $\mathcal{C}$. As a consequence, using \text{erase} on $\mathcal{C}$ decreases its expressiveness to $0$, and thus, is really efficient to reach quickly the least expressive configuration. This instruction admits the following property. Let $x'$ be the result of applying $\text{erase}(\mathcal{C})$ on configuration $x$. Then we have: $\forall k \in \{0, \ldots, \text{size}(\mathcal{C}) - 1\}, x'_{k} = x_{0}$.
5. /* incremental propagation of $x_c$ with no loss of expressiveness */
   \[\text{expand}(\mathcal{C}) : \text{incUp}(\mathcal{C}, 1, \kappa - 1 \in \mathbb{N})\]
   where $\kappa = \min_{1 \leq k \leq \text{size}(\mathcal{C} - 1)} \left\{ k \mid \begin{cases} (x_k = 0) \text{ and } (x_{k+1} \mod \text{size}(\mathcal{C}) = 1) \text{ if } x_c = 1 \\ (x_k = 1) \text{ and } (x_{k+1} \mod \text{size}(\mathcal{C}) = 0) \text{ if } x_c = 0 \end{cases} \right\}.$

   Instruction \text{expand} is another particular case of \text{incUp} that aims at propagating the state of $c_0$ along $\mathcal{C}$ while neither 01 nor 10 patterns are destroyed, which avoids decreasing the expressiveness of $\mathcal{C}$.

6. /* decremental updates */
   \[\text{decUp}(\mathcal{C}, i, j) : \text{for } k = j \text{ downto } i \text{ do update}(c_k)\]

   Instruction \text{decUp} is the converse of instruction \text{incUp}, and updates consecutive automata in decreasing order. Once \text{decUp}(\mathcal{C}, i, j) executed, the information of $c_j$ is lost and that of $c_{i-1}$ is possessed by both $c_{i-1}$ and $c_i$. In fact, \text{decUp} aims at shifting partially a Boolean automata cycle section. As for \text{incUp}, if $j < i$ then no automata are updated and $c$ cannot be updated with \text{decUp}. This instruction admits the following property. Let $x'$ the result of the execution of \text{decUp}(\mathcal{C}, i, j) on $x$. Then $\forall k \in \{i, \ldots, j\}, \ x'_k = x_{k-1}$ and $\forall k \notin \{i, \ldots, j\}, \ x'_k = x_k$.

7. /* complete decremental update (except $c$) */
   \[\text{shift}(\mathcal{C}) : \text{decUp}(\mathcal{C}, 1, \text{size}(\mathcal{C}) - 1)\]

   Instruction \text{shift} is a particular case of instruction \text{decUp}. Once executed, every automaton of $\mathcal{C}$ takes the state of its predecessor, except $c$ whose state does not change. Automaton $c_{\text{size}(\mathcal{C}) - 1}$ excluded, all the information contained along $\mathcal{C}$ is kept safe. This instruction is useful to propagate information along a Boolean automata cycle without losing too much expressiveness (at most one 01 pattern is destroyed).

6.2.2 Results

**More complex instructions** Let $x$ be a configuration of a Boolean automata double-cycle $\mathcal{D}$. Let us consider an algorithm composed of instructions defining an update sequence $\text{sequence}(x)$ from $x$. In every algorithm that follows, Boolean automata double-cycle $\mathcal{D}$ is always considered as a global variable and is not mentioned. Abusing language, $\text{sequence}(x)$ represents the sequence as well as the configuration resulting from its execution.

For the purpose of the study, we introduce three other more complex sequences in Table 1. In addition, Lemma 1 below shows that the \text{copy} instruction allows to transform $x$ into another configuration $x'$ if $x$ is sufficiently expressive.

**Lemma 1 (MNR+15)** Let $\mathcal{D}$ be a Boolean automata double-cycle and let $x$ and $x'$ be two of its configurations such that $x_0 = x'_0$. If, for all $m \in \{\ell, r\}$, one of the following properties holds for $x$:

1. $\forall i \in \{1, \ldots, \text{size}(\mathcal{C}_m) - 1\}, \ x_i^m \neq x_{i-1}^m$,
2. $\forall i \in \{1, \ldots, \text{size}(\mathcal{C}_m) - 2\}, \ x_i^m \neq x_{i-1}^m$ and $x_{\text{size}(\mathcal{C}_m)}^m = x_{\text{size}(\mathcal{C}_m) - 1}^m$,
3. $\forall i \in \{1, \ldots, \text{size}(\mathcal{C}_m) - 2\}, \ x_i^m \neq x_{i-1}^m$ and $\exists p \in \{1, \ldots, \text{size}(\mathcal{C}_m) - 2\}, \ x_p^m \neq x_p^{m'}$,

then \text{copy}(x, x') = x'$ and this sequence executes at most $2(\ell + r - 6)$ updates.

This lemma gives strong insights about the expressive power of instructions and sequences to reveal possible trajectories between configurations. Now let us focus on the dynamical behaviour of Boolean automata double-cycles, from a general point of view.
Theorem 8

Let $D^{+\dagger}$ be a canonical positive Boolean automata double-cycle where $\phi = \land$ and $x$ one of its unstable configurations. If $x$ admits one automaton in state 0, then $fix_0(x) = (0', 0')$. Moreover, if in configuration $x$, there is one automaton in state 1 in each Boolean automata cycle, then $fix_1(x) = (1', 1')$. The convergence time of $D^{+\dagger}$ is at most $2(\ell + r) - 5$.
Mixed Boolean automata double-cycles For mixed asynchronous Boolean automata double-cycles, as for positive ones, asynchronism allows to eliminate all local instabilities. Thus, contrary to parallel Boolean automata double-cycles, asynchronous Boolean automata double-cycles have only one attractor that is a stable configuration. In the canonical case, this is evidenced by the simp sequence given in Table 3 that provides a way to converge towards this stable configuration from any initial configuration \( x \), by reducing progressively its expressiveness. This result is illustrated in Figure 8 (middle) and is formalised by Theorem 9 below.

**Theorem 9** ([MNR+15]). Let \( \mathcal{D}^{\ell \rightarrow r} \) be a canonical mixed Boolean automata double-cycle, where \( \circ = \wedge \).

For any of its configuration \( x \), \( \text{simp}(x) = (\emptyset^\ell, \emptyset^r) \) holds. The convergence time of \( \mathcal{D}^{\ell \rightarrow r} \) is at most \( 2^{\ell + r - 2} \).

Negative Boolean automata double-cycles Here, we distinguish between (1) even negative Boolean automata double-cycles and (2) odd negative Boolean automata double-cycles. The first ones are defined as having two even-sized Boolean automata cycles, the second ones as having at least one odd-sized Boolean automata cycle.

Even negative Boolean automata double-cycles admit a single attractor. This attractor is a stable oscillation of length \( 2^{\ell + r - 1} \). This means that all the configurations are recurring and consequently the convergence time is null. All configurations are reachable. However this result also means that configurations of maximal expressiveness are hard to reach: the number of updates to reach them is quadratic according to the size of the Boolean automata double-cycle. The general idea of the proof follows the following three points:

- Any configuration can reach the less expressive one \((0^\ell, 0^r)\) in linear time. (P1)
- Configuration \((0^\ell, 0^r)\) can reach the highest expressive one \(((10)^\frac{r}{2}, (10)^\frac{\ell}{2})\) in quadratic time. (P2)
- Any configuration can be reached from \(((10)^\frac{r}{2}, (10)^\frac{\ell}{2})\) in linear time. (P3)

Consider P1. It is easy to see that the sequence simp remains effective for reaching \((0^\ell, 0^r)\), which is formalised by Lemma 2 below.

**Lemma 2** ([MNR+15]). For any configuration \( x \) of \( \mathcal{D}^{\ell \rightarrow r} \), \( \text{simp}(x) = (\emptyset^\ell, \emptyset^r) \) and executes at most \( 2^{\ell + r - 2} \) updates.
Figure 8: Asynchronous transition graphs of the canonical Boolean automata double-cycles depicted in Figure 6, where \( \hat{\sigma} \) and where every \( \rightarrow \) (resp. \( \rightarrow \) and \( \rightarrow \)) represents an update of automaton 0 (resp. 1 and 2).

Now, consider P2. Implicitly, P2 requires increasing the expressiveness of \( p^0 \ell, 0^r \) by successive updates, and finding a trajectory that reaches \((10)\hat{\sigma}, (10)\hat{\tau}\). To do so, we proceed in two stages. First, we increase the expressiveness of \( C_{\ell} \) using \( \text{comp}_1 \) (see Lemma 3). Then, we increase the expressiveness of \( C_{r} \). This second stage is carried out without reducing the expressiveness of \( C_{\ell} \) by using \( \text{comp}_2 \) (see Lemma 4). The expected result follows from the composition \( \text{comp} = \text{comp}_2 \circ \text{comp}_1 \). It is formalised in Lemma 5.
Lemma 3 ([MNR+15]). In an even negative Boolean automata double-cycle $D^{-}_{\ell,r}$, sequence
\[
\text{comp}_1(((\ell'), 0'))
\]
leads to configuration $((10)^{\ell'}, 1')$ and executes at most $(\ell-1)(\ell+r-2)$ updates.

Lemma 4 ([MNR+15]). In an even negative Boolean automata double-cycle $D^{-}_{\ell,r}$, sequence
\[
\text{comp}_2(((10)^{\ell'}, 1'))
\]
leads to configuration $((10)^{\ell'}, (10)^{\ell'})$ and executes at most $(r-2)(\ell+r-2) + (2r-1)$ updates.

Lemma 5 ([MNR+15]). In an even negative Boolean automata double-cycle $D^{-}_{\ell,r}$, sequence
\[
\text{comp}((\ell'), 0'))
\]
leads to configuration $((10)^{\ell'}, (10)^{\ell'})$ and executes at most $(\ell+r)^2 - 5(\ell-1) - 3r$ updates.

P3 is developed in Lemma 6 that uses the copy sequence (see Table I).

Lemma 6 ([MNR+15]). In an even negative Boolean automata double-cycle $D^{-}_{\ell,r}$, for any configuration $x'$, sequence
\[
\text{copy}_p(((10)^{\ell'}, (10)^{\ell'}), x')
\]
transforms configuration $((10)^{\ell'}, (10)^{\ell'})$ into $x'$ in at most $3(\ell+r-4) - 1$ updates.

Starting from Lemmas 2 to 6 whatever the configurations $x$ and $x'$, the composition
\[
\text{copy}_p(\text{comp}(\text{simp}(x)), x') = x'
\]
holds, which proves that there is a unique attractor of length $2^{\ell+r-1}$. From this is derived Theorem 10.

It gives some bounds on convergence time.

Theorem 10 ([MNR+15]). Let $D^{-}_{\ell,r}$ be a canonical negative Boolean automata double-cycle, where $\circ = \land$. $D^{-}_{\ell,r}$ admits a unique attractor of length $2^{\ell+r-1}$. In this stable oscillation, any configuration can be reached from any other in $O(\ell^2 + r^2)$ updates. However, configurations $(\ell', 0')$ and $(1', 1')$ can be reached from any other one in $O(\ell + r)$ updates, and the configurations $((11)^{\ell'}, (01)^{\ell'})$ and $((10)^{\ell'}, (10)^{\ell'})$ can reach all the others in $O(\ell + r)$ updates.

Like even negative Boolean automata double-cycles, odd negative Boolean automata double-cycles also admit a single attractor (a stable oscillation). However, all configurations are not necessarily recurring. Indeed, they admit a set $I$ of non-reachable configurations, from which updates are irreversible. The associated result is formalised in Theorem 11 below.

Theorem 11 ([MNR+15]). Let $\rho : \mathbb{N} \to \{0, 1\}$, with $\rho(k) = \begin{cases} 0 & \text{if } k = 0 \text{ or } k \equiv 1 \mod 2 \\ 1 & \text{otherwise} \end{cases}$. Every Boolean automata double-cycle $D^-_{\ell,r}$ admits a unique attractor $\mathbb{B}^{2^{\ell+r-1}} \setminus I$, where $|I| = \alpha(\ell-1) \times 2^{r-1} + \alpha(r-1) \times 2^{\ell-1}$.

This result, whose proof rests on the characterisation of $I$, generalises Theorem 10.
7 Conclusion

In this chapter, we have summarised the major results obtained in recent years on the role of feedbacks in interaction networks. The literature has already established that feedback patterns are “engines of complexity” in the dynamical behaviours of larger networks that contain them. Because of their proven essential character, we have therefore deliberately focused on the dynamics of cycles and double-cycles, by focusing especially on the influence of updating modes. Without going back on the results themselves, this chapter highlights the fundamental differences induced by the “scheduling” of events on the behaviours of the complexity engines of interaction networks: while parallelism tends to render all the asymptotic mathematical diversity of networks, pure asynchronism tends to reduce degrees of freedom of networks, which partly explains why the former is often preferred in theoretical work while the second is often adopted in works that are oriented towards applications in molecular biology. At present, no real knowledge in molecular biology establishes precisely how regulations are implemented over time, even if works state that the chromatin dynamics plays an important role. That is in particular why the type of studies developed in this chapter, focusing on updating modes, remains essential to go further in understanding formal and applied interaction networks. As a consequence, further studies need to be done on the scheduling of updates over time. A first avenue is to focus for instance on more likely updating modes, in agreement with the discussion of what is claimed in [DS20].

Another natural opening highlighted by these works, deeply rooted in fundamental computer science, consists in developing knowledge on the dynamical properties of interaction networks, including relationships between their architecture and structure. But other equally relevant lines of investigation also call for exploration. One of them was opened up by the study of asynchronous double-cycles and deals with the time complexity of networks. A question that remains currently open is the following: does the time complexity of the networks go hand in hand with their behavioural diversity? If yes, can we find a measure of it? Finally, the work carried out on cycles and double-cycles in parallel, developed in [Nou12, Sen12] discusses perspectives around computability, modularity/compositionality, and intrinsic universality of networks. These are certainly, like the ever-growing understanding of the influences of updating modes, among the most promising tracks for future works in the field.

Acknowledgement This work was funded mainly by our salaries as French or German State agents or pensioner (affiliated to Université Grenoble-Alpes (JD), Université d’Évry (TM and DR), Freie Universität Berlin (MN), and Université d’Aix-Marseille (SS)), and secondarily by the ANR-18-CE40-0002 FANs project (SS).

References

[AGZ+06] J. Aracena, M. González, A. Zuñiga, M. A. Mendez, and V. Cambiazo. Regulatory network for cell shape changes during drosophila ventral furrow formation. Journal of Theoretical Biology, 239:49–62, 2006.

[Alb09] R. Albert. Discrete dynamic modeling of cellular signaling networks. Methods in Enzymology, 467:281–306, 2009.

[Ald03] M. Aldana. Boolean dynamics of networks with scale-free topology. Physica D, 185:45–66, 2003.

[Apo76] T. M. Apostol. Introduction to analytic number theory. Springer-Verlag, 1976.

[BCG82] E. R. Berlekamp, J. H. Conway, and R. K. Guy. Winning ways for your mathematical plays. Academic Press, 1982.
[BP07] J. Berstel and D. Perrin. The origins of combinatorics on words. *European Journal of Combinatorics*, 28:996–1022, 2007.

[CD85] M. Cosnard and J. Demongeot. On the definitions of attractors. In *Iteration theory and its functional equations*, volume 1163 of *Lecture Notes in Mathematics*, pages 23–31. Springer, 1985.

[CDLB83] M. Cosnard, J. Demongeot, and A. Le Breton, editors. *Rhythms in biology and other fields of application*, volume 49 of *Lecture Notes in Biomathematics*. Springer, 1983.

[Chu32] A. Church. A set of postulates for the foundation of logic. *Annals of Mathematics*, 33:346–366, 1932.

[Coo04] M. Cook. Universality in elementary cellular automata. *Complex Systems*, 15:1–40, 2004.

[Cul71] P. Cull. Linear analysis of switching nets. *Biological Cybernetics*, 8:31–39, 1971.

[Del49] M. Delbrück. Génétique du bactériophage. In *Unités biologiques douées de continuité génétique*, volume 8 of *Colloques internationaux du CNRS*, pages 91–103, 1949.

[Dem75] J. Demongeot. *Au sujet de quelques modèles stochastiques appliqués à la biologie*. PhD thesis, Université scientifique et médicale de Grenoble, 1975.

[DGM+10] J. Demongeot, E. Goles, M. Morvan, M. Noual, and S. Sené. Attraction basins as gauges of the robustness against boundary conditions in biological complex systems. *PLoS One*, 5:e11793, 2010.

[DGT85] J. Demongeot, E. Goles, and M. Tchuente, editors. *Dynamical systems and cellular automata*. Academic Press, 1985.

[DNS12] J. Demongeot, M. Noual, and S. Sené. Combinatorics of Boolean automata circuits dynamics. *Discrete Applied Mathematics*, 160:398–415, 2012.

[DS20] J. Demongeot and S. Sené. About block-parallel Boolean networks: a position paper. *Natural Computing*, 19:5–13, 2020.

[Els59] B. Elspas. The theory of autonomous linear sequential networks. *IRE Transactions on Circuit Theory*, 6:45–60, 1959.

[GBK+07] S. Gupta, S. S. Bisht, R. Kukreti, S. Jain, and S. K. Brahmachari. Boolean network analysis of a neurotransmitter signaling pathway. *Journal of Theoretical Biology*, 244:463–469, 2007.

[GEHL02] C. C. Guet, M. B. Elowitz, W. Hsing, and S. Leibler. Combinatorial synthesis of genetic networks. *Science*, 296:1466–1470, 2002.

[Ger04] C. Gershenson. Updating schemes in random Boolean networks: do they really matter? In *Proceedings of Artificial Life*, pages 238–243. MIT Press, 2004.

[GFSP85] E. Goles, F. Fogelman-Soulie, and D. Pellegrin. Decreasing energy functions as a tool for studying threshold networks. *Discrete Applied Mathematics*, 12:261–277, 1985.

[GKP89] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete mathematics: a foundation for computer science*. Addison-Wesley, 1989.

[GM90] E. Goles and S. Martínez. *Neural and automata networks: dynamical behavior and applications*, volume 58 of *Mathematics and Its Applications*. Kluwer Academic Publishers, 1990.
E. Goles and M. Noual. Block-sequential update schedules and Boolean automata circuits. In Proceedings of AUTOMATA, pages 41–50. Discrete Mathematics and Theoretical Computer Science, 2010.

K. F. Gödel. Kurt Gödel collected works, volume I – Publications 1929-1936, chapter On undecidable propositions of formal mathematical systems, pages 346–372. Oxford University Press, 1986.

S. W. Golomb. Shift register sequences. Holden-Day Inc., 1967.

E. Goles. Fixed point behavior of threshold functions on a finite set. SIAM Journal on Algebraic and Discrete Methods, 3:529–531, 1982.

B. Hesper and P. Hogeweg. Bioinformatica: een werkconcept. Kameleon, 1:18–29, 1970.

P. Hogeweg and B. Hesper. Interactive instruction on population interactions. Computers in Biology and Medicine, 8:319–327, 1978.

D. A. Huffman. Canonical forms for information-lossless finite-state logical machines. IRE Transactions on Information Theory, 5:41–59, 1959.

F. Jacob and J. Monod. Genetic regulatory mechanisms in the synthesis of proteins. Journal of Molecular Biology, 3:318–356, 1961.

S. A. Kauffman. Homeostasis and differentiation in random genetic control networks. Nature, 224:177–178, 1969.

S. A. Kauffman. Metabolic stability and epigenesis in randomly constructed genetic nets. Journal of Theoretical Biology, 22:437–467, 1969.

S. A. Kauffman. Current topics in developmental biology, volume 6, chapter Gene regulation networks: A theory for their global structures and behaviors, pages 145–181. Elsevier, 1971.

H. T. Kung and C. E. Leiserson. Introduction to VLSI systems, chapter Algorithms for VLSI processor arrays, pages 271–292. Addison-Wesley, 1980.

S. C. Kleene. Automata studies, volume 34 of Annals of Mathematics Studies, chapter Representation of events in nerve nets and finite automata, pages 3–41. Princeton University Press, 1956.

S. A. Kauffman, C. Peterson, B. Samuelsson, and C. Troein. Random Boolean network models and the yeast transcriptional network. Proceedings of the National Academy of Sciences of the USA, 100:14796–14799, 2003.

M. Kaufman and R. Thomas. Towards a logical analysis of the immune response. Journal of Theoretical Biology, 114:527–561, 1985.

P. Kurka. Languages, equicontinuity and attractors in cellular automata. Ergodic Theory and Dynamical Systems, 17:417–433, 1997.

L. Mendoza and E. R. Alvarez-Buylla. Dynamics of the genetic regulatory network for Arabidopsis thaliana flower morphogenesis. Journal of Theoretical Biology, 193:307–319, 1998.

J. Monod, J.-P. Changeux, and F. Jacob. Allosteric proteins and cellular control systems. Journal of Molecular Biology, 6:306–329, 1963.
L. Mendoza. A network model for the control of the differentiation process in Th cells. *Biosystems*, 84:101–114, 2006.

T. Melliti, M. Noual, D. Regnault, S. Sené, and J. Sobieraj. Asynchronous dynamics of Boolean automata double-cycles. In *Proceedings of UCNC*, volume 9252 of *Lecture Notes in Computer Science*, pages 250–262. Springer, 2015.

W. S. McCulloch and W. Pitts. A logical calculus of the ideas immanent in nervous activity. *Journal of Mathematical Biophysics*, 5:115–133, 1943.

M. Noual. Dynamics of circuits and intersection circuits. In *Proceedings of LATA*, volume 7183 of *Lecture Notes in Computer Science*, pages 433–444. Springer, 2012.

M. Noual. *Updating automata networks*. PhD thesis, École normale supérieure de Lyon, 2012. [http://tel.archives-ouvertes.fr/tel-00726560](http://tel.archives-ouvertes.fr/tel-00726560).

Y. Puri and T. Ward. Arithmetic and growth of periodic orbits. *Journal of Integer Sequences*, 4:01.2.1, 2001.

A. Richard and J.-P. Comet. Necessary conditions for multistationarity in discrete dynamical systems. *Discrete Applied Mathematics*, 155:2403–2413, 2007.

A. Richard, J.-P. Comet, and G. Bernot. R. Thomas’ modeling of biological regulatory networks: introduction of singular states in the qualitative dynamics. *Fundamenta Informaticae*, 65:373–392, 2004.

G. A. Ruz, E. Goles, and S. Sené. Reconstruction of Boolean regulatory models of flower development exploiting an evolution strategy. In *Proceedings of CEC*, pages 1–8. IEEE Press, 2018.

P. Ribenboim. *The new book of prime number records*. Springer-Verlag, 1996.

A. Richard. Negative circuits and sustained oscillations in asynchronous automata networks. *Advances in Applied Mathematics*, 44:378–392, 2010.

É. Remy, B. Mossé, C. Chaouiya, and D. Thieffry. A description of dynamical graphs associated to elementary regulatory circuits. *Bioinformatics*, 19:ii172–ii178, 2003.

F. Robert. Blocs-H-matrices et convergence des méthodes itératives classiques par blocs. *Linear Algebra and its Applications*, 2:223–265, 1969.

F. Robert. Contraction en norme vectorielle: convergence d’itérations chaotiques pour des équations non linéaires de point fixe à plusieurs variables. *Linear Algebra and its Applications*, 13:19–35, 1976.

F. Robert. Itérations sur des ensembles finis et automates cellulaires contractants. *Linear Algebra and its Applications*, 29:393–412, 1980.

F. Robert. *Discrete iterations: a metric study*, volume 6 of *Springer Series in Computational Mathematics*. Springer, 1986.

F. Robert. *Les systèmes dynamiques discrets*, volume 19 of *Mathématiques & Applications*. Springer, 1995.

É. Remy, P. Ruet, and D. Thieffry. Graphic requirements for multistability and attractive cycles in a Boolean dynamical framework. *Advances in Applied Mathematics*, 41:335–350, 2008.
[Rus03] F. Ruskey. Combinatorial generation. Book preliminary working draft, 2003.

[Sen12] S. Šené. Sur la bio-informatique des réseaux d’automates. Habilitation thesis, Université d’Évry – Val d’Essonne, 2012. [http://tel.archives-ouvertes.fr/tel-00759287]

[Smi71] A. R. Smith. Simple computation-universal cellular spaces. *Journal of the ACM*, 18:339–353, 1971.

[SRL+07] J. Saez-Rodriguez, L. Simeoni, J. A. Lindquist, R. Hemenway, U. Bommhardt, Arndt B., U.-U. Haus, R. Weismantel, E. D. Gilles, S. Klamt, and B. Schraven. A logical model provides insights into T cell receptor signaling. *PLoS Computational Biology*, 3:e163, 2007.

[Tho73] R. Thomas. Boolean formalization of genetic control circuits. *Journal of Theoretical Biology*, 42:563–585, 1973.

[Tho81] R. Thomas. On the relation between the logical structure of systems and their ability to generate multiple steady states or sustained oscillations. In *Numerical methods in the study of critical phenomena*, volume 9 of *Springer Series in Synergetics*, pages 180–193. Springer, 1981.

[Tho91] R. Thomas. Regulatory networks seen as asynchronous automata: a logical description. *Journal of Theoretical Biology*, 153:1–23, 1991.

[TT95] D. Thieffry and R. Thomas. Dynamical behaviour of biological regulatory networks – II. Immunity control in bacteriophage lambda. *Bulletin of Mathematical Biology*, 57:277–297, 1995.

[TTK95] R. Thomas, D. Thieffry, and M. Kaufman. Dynamical behaviour of biological regulatory networks – I. Biological role of feedback loops and practical use of the concept of the loop-characteristic state. *Bulletin of Mathematical Biology*, 57:247–276, 1995.

[Tur36] A. M. Turing. On computable numbers, with an application to the entscheidungsproblem. *Proceedings of the London Mathematical Society*, 2:230–265, 1936.

[van60] H. van der Laan. *Le nombre plastique, quinze leçons sur l’ordonnance architectonique*. E.J. Brill, 1960.

[von66] J. von Neumann. *Theory of self-reproducing automata*. University of Illinois Press, 1966.

[Wol84] S. Wolfram. Universality and complexity in cellular automata. *Physica D*, 10:1–35, 1984.