Abstract

Motivated by particle physics results, we investigate certain dyonic solutions in arbitrary dimensions. Concretely, we study the stringy constructions of such objects from concrete compactifications. Then, we elaborate their tensor network realizations using multistate particle formalism.

Keywords: String theory; Compactification, Dyonic solutions, Tensor network

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1 Introduction

Since the development of string theory framework and its efficient technique for studying non-perturbative phenomena, a certain number of relevant links with other theories have been comprehended [1,2]. Specially, in such a framework, the formerly familiar group of solitonic p-brane solutions of Type II supergravities in 10 dimensions are known to be characterized in terms of D-branes [2,3]. Non-perturbative properties of supersymmetric Yang-Mills theory, black holes entropy, dyons, magnetic monopoles and a number of appealing issues in various dimensions have been handled in the context of D-brane objects and their dynamics [4,5]. All these relevant connections have promoted such a string theory aspect to one of the most important and promising side to be explored. It is known that in higher dimensions many physical quantities are reinterpreted from the usual four-dimensional spacetime viewpoint as various faces of possible one quantity. In particular, it has been shown that the single electric and magnetic charges in ten dimensions could be viewed from the 4-dimensional non-compact spacetime standpoint as dyonic charges, being particles carrying simultaneous existence of the electric and the magnetic charges [6]. A dyon with a zero electric charge is usually referred to as a magnetic monopole [7,8]. These objects are hypothetical particles predicted in many extended models and grand unified theories [9,10]. Indeed, besides to magnetic monopole solutions, dyonic solutions appear in extended Yang-Mills-Higgs modeling taking places in theories going beyond the ordinary physics associated with material points. It is therefore interesting to see how the D-brane framework automatically encodes this feature by using stringy constructions and computations.

Although there appears to have been no precedent concrete experimental search for dyonic objects, it has been now become possible given the progress made in the direct search of magnetic monopoles at accelerators which has a long history [11]. Actually, seen that dyons possess an electric charge that can, in principle, be significant, with the recent LHC results. For instance ATLAS and MoEDAL results have reported limits on monopole productions as well as on stable objects with electric charge [12,13]. Thus, it turns out that LHC could have a right place for the search of dyons.

The aim of this work is contribute to this field by investigating stringy dyonic objects and the corresponding tensor network realizations. Inspired and motivated by known results obtained from non-trivial theories including string theory, we first provide the origin and certain related concepts associated with such objects. After that, we present a general framework of dyonic solutions in the brane structure by showing how they are naturally built from inspired stringy compactifications on certain complex manifolds. Then, we reveal that such dyonic solutions involve tensor network representations.

The organization of this work is as follows. In section 2, we present shortly the origin and the related concepts of the dyonic objects motivated by known results. In section 3, we elaborate a general framework of dyonic solutions in the brane structure and show how are naturally obtained in various dimensions within stringy scheme compactifications. In section 4, we investigate the dyonic solution constructions using the tensor network formalism and
reveal some dyonic object characteristics. The last section is devoted to some concluding remarks and open questions.

2 Motivations of dyonic objects

Before dealing with the dyonic objects associated with the electromagnetic duality in higher dimensions, we would like to shed light on some related concepts. Indeed, a similar one goes back to the neutrino discovery, where the proton and the neutron can be regarded as two states of a single particle motivated by the observation that they have approximately equal masses $m_p \simeq m_n = m$ [14], which in turn conducted, according to the mass-energy equivalence, to an energy degeneracy of the underlying interaction. This mass degeneracy led then to the existence of a corresponding symmetry whose the interaction obeys. That is to say, these two particles have an identical behavior under the underlying interaction and that their charge content is their solely difference. In such a case, where these two particles are to be viewed as two linearly independent states of the same particle, i.e., nucleus $N$, it is unexceptional to depict them in the form of a two component vector like

$$N : \begin{pmatrix} p \\ n \end{pmatrix}^{m_{p-n}}$$  \hspace{1cm} (2.1)

where $p$ and $n$ refer to the proton and the neutron particles making such a nucleus state. More fundamentally, analogous to the spin-up and the spin-down states of a spin-one half particle, the concept of isospin symmetry is introduced and is governed by an $SU(2)_I$ group rotating doublet components, i.e., proton and neutron, into each other in abstract isospin space. In particular, in the modern formulation, the isospin is defined as a vector quantity in which up and down quarks have a value of $1/2$, with the 3rd-component being $+1/2$ for up quarks, and $-1/2$ for down quarks. In this picture, denoting the total isospin $I$ and its 3rd component $I_3 = \pm 1/2$, an up-down quarks pair can be assembled in a state of the total isospin $1/2$ as

$$\begin{pmatrix} u \\ d \end{pmatrix}_{I=1/2}$$ \hspace{1cm} (2.2)

where, again, $u$ and $d$ refer to the up and down quarks forming such a doublet state. Albeit such a particle state classification is a good approximated one with a symmetry dealing with nuclear interactions. It remains a useful and large concept. In fact, one could go further and extend this view to the electromagnetic charge where each particle $P$ could be represented according to its charge content by the doublet

$$P : \begin{pmatrix} Q_e \\ P_m \end{pmatrix}_{Q_{em}}$$ \hspace{1cm} (2.3)
whose components are the electric and magnetic charges of the involved particle. At first sight, this picture will seem to be strange or even inconsistent with the symmetry of the electromagnetic interaction as dictated by the corresponding $U(1)_{Q_{em}}$ group. However, this apparent inconsistency could be simply alleviated by assuming a hidden symmetry group associated with the magnetic charge. Within the the standard model framework, a simple and economic way to do so is to consider an extended electroweak symmetry $SU(2)_L \times U(1)_e \times U(1)_m$ involving both electric $U(1)_e$ and magnetic $U(1)_m$ symmetries being broken at a certain high energy resulting in the standard low energy electromagnetic symmetry $U(1)_{Q_{em}}$. In this picture, and at a certain energy scale, the particle charge representation in ref gives rise to the following physical objects, for instance

$$
\begin{pmatrix}
0 \\
0
\end{pmatrix}, \begin{pmatrix}
Q_e \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
P_m
\end{pmatrix}, \begin{pmatrix}
Q_e \\
P_m
\end{pmatrix}_{Q_{em}}.
$$

(2.4)

These four objects are nothing but all known possible particles, discovered and hypothetical. In particular, the first neutral doublet is nothing but the neutrino, being neutral. The second doublet corresponds to the ordinary charged particles, the third doublet is that of theoretical magnetic monopoles with a single magnetic charge, while the four doublet is associated with a dyonic particle carrying both electric and magnetic charges. For more clarity, we list them in Tab.(1). We can now see that this particle classification view has given rise to the well motivated hypothetical objects, namely magnetic monocles charged only magnetically, and dyons carrying both the electric and the magnetic charges associated with the electromagnetic duality in higher dimensions. These objects are predicted by most grand unified theories in high energy physics and are now one of the active field of research in many theoretical and experimental works [13]. As we shall see in what follows, such a view can be considered as a part of a larger concept being of great utility to classify physical object families.

| object                               | $Q_e$ | $P_m$ | nature      |
|--------------------------------------|------|------|-------------|
| neutrinos                            | no   | no   | real        |
| ordinary (electrically) charged particles | yes  | no   | real        |
| magnetic monopoles                   | no   | yes  | hypothetical|
| dyons                                | yes  | yes  | hypothetical|

Table 1: The four possible particles, observed and hypothetical, in terms of their electromagnetic charges.

3 Stringy dyonic solutions

In lower dimensional string compactifications, a certain number of dyonic objects can be generated. However, these solutions could be obtained from the geometry of a compact
real manifold $X^n$ on which higher dimensional theories are compactified. The geometric information of such compactifications can be deployed to construct such dyonic objects as doublets carrying both electric and magnetic charges [15]. Indeed, they can be generally arranged as

$$\begin{pmatrix} p \\ q \end{pmatrix}.$$ (3.1)

It is worth noting that now $p$ and $q$ denote the electric and the magnetic solitonic objects, with $p$ and $q$ dimensions respectively, forming the dyonic solutions. To see how this could be constructed, consider a gauge field $C_{p+1}$ coupled to a $p$-brane associated with the field strength $F_{p+2} = dC_{p+1}$. The corresponding electric charge $Q_e$ can be computed using the integration over a $S^{p+2}$ ($(p + 2)$-cycle)

$$Q_e = \int_{(p+2)-\text{cycle}} F_{p+2}. \quad (3.2)$$

However, the magnetic charge, associated with the magnetic object, is calculated via the duality of the field strength $F_{p+2}$ in $d$-dimensional space time over a $S^{d-p-2}$ ($(d - p - 2)$-cycle)

$$P_m = \int_{(d-p-2)-\text{cycle}} \ast dC_{p+1}. \quad (3.2)$$

To build a dyonic object, one put together the electric and the magnetic objects associated with the electromagnetic duality

$$F \longleftrightarrow \ast F. \quad (3.3)$$

In string theory, when a magnetic dual of an electric $p$-brane is a $q$-brane, where $q$ is related to $p$ by the constraint relation

$$p + q = 6 - n. \quad (3.4)$$

This includes two different kinds of the dyonic objects. We refer to them as fundamental dyonic solutions ($p = q$) and non-fundamental dyonic solutions ($p \neq q$), respectively. The first one gives rise to the ordinary dyonic solution required by

$$p = q = 3 - \frac{n}{2}, \quad (3.5)$$

which allows to present any fundamental dyonic state in the form of a doublet of electromagnetic charges with same dimension

$$\begin{pmatrix} p \\ p \end{pmatrix}_{p=3-\frac{n}{2}}. \quad (3.6)$$

This case is analogue of what we call pure state involving only the charged objects of the same spatial dimension appearing in even dimensional space-time being a compact one. It is recalled that one has three families of dyonic objects such as a dyonic particle in a
four dimensional spacetime, a dyonic string in a six dimensional space-time and a dyonic membrane in an eight dimensional space-time given respectively by

\[
\begin{pmatrix}
0 \\
0
\end{pmatrix}, \quad \begin{pmatrix}
1 \\
1
\end{pmatrix}, \quad \begin{pmatrix}
2 \\
2
\end{pmatrix}
\] (3.7)

The second family, defined by \( p \neq q \), are presented by the following constraints

\[ q = 6 - n - p, \quad p \neq 3 - \frac{n}{2}. \]

In this way, these dyonic objects can be formed by two different branes in any compact space. Such dyonic objects are presented by a doublet of the couple \((p, q)\)

\[
\begin{pmatrix}
p \\
q
\end{pmatrix}_{p+q=6-n}
\] (3.8)

The unusual solutions can be built in terms of the ten dimensional space-time theory either from type II superstrings or heterotic brane doublets

\[
\begin{pmatrix}
D_0 \\
D_6
\end{pmatrix}, \quad \begin{pmatrix}
D_2 \\
D_4
\end{pmatrix}, \quad \begin{pmatrix}
D_1 \\
D_5
\end{pmatrix}, \quad \begin{pmatrix}
D_3 \\
D_3
\end{pmatrix}, \quad \begin{pmatrix}
NS_1 \\
NS_5
\end{pmatrix}
\] (3.9)

These configurations of dyonic states represent the maximally number of dyonic objects in ten dimensions [15]. It turns out that the number of such dyonic objects will be decreased by lowering the dimensional space-time obtained from the compactification \( X^n \).

Now, we are in position to discuss explicit models from concrete compactifications. Working with M-theory/superstring inspired models in \( d \) dimensions, we consider the compactification on a \( n \)-dimensional compact manifold product of the identical manifolds of dimensions \( m \) such that

\[ n = km. \] (3.10)

We refer to such manifolds as

\[ X^n = Y^m \times \ldots \times Y^m \] (3.11)

where \( Y^m \) is a \( m \)-dimensional manifold compact space. In connection with such theories in the presence of brane solitonic objects, the \( d \)-dimensional space-time geometry can be split as follows

\[ AdS_{p+2} \times S^{d-p-2-n} \times X^n \] (3.12)

To make contact with dyonic solutions certain conditions should be imposed on \( Y^m \). A close inspection shows that \( Y^m \) should satisfy the following conditions

\[
\begin{cases}
    b_0(Y^m) = 1 \\
    b_i(Y^m) = 0 & 1 < i < m - 1 \\
    b_m(Y^m) = 1
\end{cases}
\] (3.13)
where $b_m(r)$ denote the associated Betti numbers. Performing such compactifications, we can build dyonic solutions from the brane doublets in $d$-dimensional inspired stringy models. Forgetting for while the connection with string theory models, the compactification can produce $2^{k-1}$ dyonic double states carrying both electric and magnetic charges. These states are obtained from branes wrapping $a$-cycles $C_a$ in $X$-manifolds. These cycles are given in terms of the volume forms $\omega_i$ of $Y^m$-manifolds. Indeed, dyonic stringy solutions, involving the electric and the magnetic charges, allow one to consider two indices associated with binary numbers. Using such a property, the cycles can be denoted by

$$C_a \equiv C_{e_1 \ldots e_k}. \quad (3.14)$$

where $e_i$ is a binary number taking either 0 or 1. The associated volume form are given by

$$(\omega_1)^{e_1} \wedge \ldots \wedge (\omega_k)^{e_k} \quad (3.15)$$

such that

$$\int_{C_{e_1 \ldots e_k}} (\omega_1)^{e_1} \wedge \ldots \wedge (\omega_k)^{e_k} = \delta_{e_1}^{e_1'} \ldots \delta_{e_k}^{e_k'}. \quad (3.16)$$

The dyonic doublet states can be obtained from D-branes wrapping cycles dual to the following doublet volume form

$$\left( p F_2, q F_2 \right) \leftrightarrow \left( \omega_1^{e_1} \wedge \ldots \wedge \omega_k^{e_k}, \omega_1^{e_1'} \wedge \ldots \wedge \omega_k^{e_k'} \right). \quad (3.17)$$

These dyonic states correspond to $D$-branes with charges

$$\left( Q^{e_1 \ldots e_k}, P^{e_1' \ldots e_k'} \right) \quad (3.18)$$

which can be obtained by the following integration

$$Q^{e_1 \ldots e_k} = \int_{C_{e_1 \ldots e_k}} F_2, \quad P^{e_1' \ldots e_k'} = \int_{C_{e_1' \ldots e_k'}} * F_2. \quad (3.19)$$

To provide a concrete model, we consider the following compact manifold $X^{2k}$ where $n = 2k$

$$X^{2k} = S^2 \times S^2 \times \ldots \times S^2 \quad (3.20)$$

where $S^2$ is 2-dimensional real sphere. Since $S^2$ is isomorph to $\mathbb{C}P^1$, it is useful to consider the complex geometry. In this way, we take the following factorisation

$$X^{2k} = \underbrace{\mathbb{C}P^1 \times \mathbb{C}P^1 \times \ldots \times \mathbb{C}P^1}_{k} \quad (3.21)$$

According [16,17], a nice way to understand such a geometry is to exploit the $N = 2$ sigma model by embedding the involved compact manifold in local Calabi-Yau manifolds given by

$$\mathcal{O}(-2, \ldots, -2) \to \underbrace{\mathbb{C}P^1 \times \mathbb{C}P^1 \times \ldots \times \mathbb{C}P^1}_{k}. \quad (3.22)$$
To get the associated cycles, one can use the result of the trivial fibration of two spaces \( M \times N \). According to [18], the needed Hodge numbers can be obtained by the following identity

\[
h^{(p,q)}(M \times N) = \sum_{u+v=p \atop r+s=q} h^{(u,v)}(M)h^{(r,s)}(N).
\]

(3.23)

It is recalled that the Hodge diagrams associated with \( k = 1, 2, 3 \) are listed in Tab.2. It has been revealed that the invariant volume forms belong to the cohomology class \( H_{j,j}^{+} \) formed by \( \prod_{\ell=1}^{j} dz_{\ell} \wedge d \bar{z}_{\ell} \) where \( w_{\ell} = dz_{\ell} \wedge d \bar{z}_{\ell} \) denotes the volume form associated with \( \ell \)-th space.

It is easy to calculate that the corresponding Hodge numbers are

\[
\dim H^{j,j}_{+} = h^{j,j}_{+} = \frac{k!}{j!(k-j)!}.
\]

(3.24)

It is clear that one has the following relation

\[
\dim H(X^{n}) = \sum_{j=0}^{k} h^{j,j}_{+} = 2^{k},
\]

(3.25)

associated with the total cohomology class.

### 4 Tensor network representation of dyonic solutions

In this section, we would like to investigate dyonic objects using tensor network formalism [19,20]. In particular, we exploit such a formalism to unveil certain dyonic object properties.
It is recalled that usually tensors are complex having order $k$ and size $L$ \cite{24, 25}. For any tensor, it is interesting to control the associated degrees of freedom as well as its organization. Such data could be used to encode certain physical quantities corresponding to fundamental and non fundamental objects including dyonic ones. For later use, we will consider the real tensors with a fixed size being $L = 2$.

Roughly speaking, the tensors, which will be used here, are defined as a series of real numbers labeled by $k$ indices associated with legs in graph representations. In such a context, a scalar, which is one number and labeled by zero index, is a 0th-order tensor. Graphically, we represent a scalar by dot as illustrated in Fig.1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{tensor_representation.png}
\caption{Tensor representation. (a) scalar, (b) vector, and (c-d-e) tensors associated with polyvalent vertices}
\end{figure}

A 2-component vector consists of 2 real numbers labeled by one index, being a 1st-order tensor. In this way, one can write the state vector of a spin-$1/2$ as follows

$$|\Psi\rangle = C_1|0\rangle + C_2|1\rangle = \sum_{e=0,1} C_e|e\rangle,$$  \hspace{1cm} (4.1)$$

where the coefficients $C_e$ correspond to a two-component vector. It is recalled that $|0\rangle$ and $|1\rangle$ are usually used to represent spin up and down states. Graphically, we use a dot with one open bond to represent such a vector (see (a) in figure 1). However, a matrix is in fact a 2nd-order tensor. Considering two spins as an example, the state vector can be written under an irreducible representation as a four-dimensional vector. Using the local basis of each spin associated with binary notation, the state can be written as follows

$$|\Psi\rangle = C_{00}|00\rangle + C_{01}|01\rangle + C_{10}|10\rangle + C_{11}|11\rangle = \sum_{e_{1}e_{2}=0}^{1} C_{e_{1}e_{2}}|e_{1}e_{2}\rangle,$$  \hspace{1cm} (4.2)$$

where $C_{e_{1}e_{2}}$ represent now a matrix with two indices. It is worth noting that the difference between a $(2 \times 2)$ matrix and a $2^2$-component vector is just the way of labeling the tensor elements. Graphically, we use a dot with two bonds to represent a matrix and its two indices (Figure 1).

It is then natural to define an $k$-th order tensor. Considering $k$ spins, the $2^k$ coefficients can
be written as a $k$-th order tensor $C$, which satisfies

$$|\Psi\rangle = \sum_{e_1...e_k=0}^1 C_{e_1...e_k} |e_1...e_k\rangle. \quad (4.3)$$

Graphically, such a $k$-th order tensor is represented by a dot connected with $k$ open bonds (polyvalent vertex of order $k$, Figure 1). Now we can define Tensor Network (TN), as the contraction of many tensors. A TN is a set of tensors where some, or all, of its indices are contracted according to some pattern. Contracting the indices of a TN is called, for simplicity, contracting the TN. In general, the contraction of a TN with some open indices gives as a result another tensor, and in the case of not having any open indices the result is a scalar.

Having given a concise review on tensors, we move to make contact with dyons. Indeed, a close inspection, in the above concrete stringy models, shows that the dyonic state can be associated with a tensor $T_{e_1...e_k}$ defined by the state

$$|\Psi\rangle = \sum_{e_i=0,1} T_{e_1...e_k} |e_1...e_k\rangle, \quad (4.4)$$

where the vector $|e_1...e_k\rangle$ is given by $|e_1...e_k\rangle = |e_1\rangle \otimes \ldots \otimes |e_k\rangle$.

To give a tensor network realisation of such dyonic objects, we introduce a new tensor defined by

$$T_{e_1...e_k} \rightarrow \bar{T}_{\bar{e}_1...\bar{e}_k} = T_{\bar{e}_1...\bar{e}_k} \quad (4.5)$$

such that $e_i + \bar{e}_i = 1$.

In this way, the above state can be written as

$$|\psi\rangle = \sum_{e_i} T_{e_1...e_k} |e_1...e_k\rangle + \sum_{\bar{e}_i} T_{\bar{e}_1...\bar{e}_k} |\bar{e}_1...\bar{e}_k\rangle \quad (4.6)$$

where the tensor values correspond to the involved charges. Indeed, one proposes

$$T_{e_1...e_k} = Q_{e_1...e_k}$$

$$T_{\bar{e}_1...\bar{e}_k} = P_{\bar{e}_1...\bar{e}_k} \quad (4.7)$$

subject to $e_i + \bar{e}_i = 1$. In this way, the degrees of freedom of $T$ can be split as

$$2^k = 2^{k-1} + 2^{k-1} \quad (4.8)$$

Now, we are in position to build a new tensor carrying information on dyonic objects by combing $T$ and $\bar{T}$ tensor defined previously. The suggested notation is given by

$$X_{e_1...e_k} = \begin{pmatrix} T_{e_1...e_k} \\ T_{\bar{e}_1...\bar{e}_k} \end{pmatrix} \quad (4.9)$$

In this tensor realization, the dyonic state can be expressed as

$$|\psi\rangle = \sum_{e_i+\bar{e}_i=1} X_{e_1...e_k} |e_1\bar{e}_1\rangle \otimes \ldots \otimes |e_k\bar{e}_k\rangle \quad (4.10)$$

$$= \sum_{e_i+\bar{e}_i=1} X_{e_1...e_k} \prod_i \otimes |e_i\bar{e}_i\rangle. \quad (4.11)$$
Graphically, this state can be illustrated by the tensor given in Fig. 2

Roughly, we expect that the theory that supports these objects is a color tensor model with a gauge group $U(2)^{\otimes k}$ which acts on each slot by complex rotation. In fact, multiparticle states are given by linear combinations of the full contraction of $r$ copies of the tensor $T$ with $r$ copies of the tensor $\bar{T}$, providing that the states are gauge invariant. To simplify notation, consider a tensor with three indices, namely $k = 3$. The results can be generalized straightforwardly. For one particle, we have

$$O^{(1)} = T_{ijk} \bar{T}^{ijk},$$

(4.12)

which is the unique scalar that can be constructed (remember that in color tensor models the contractions must always happen on indices of the same slot). For two indices, there are four choices given by

$$O^{(2)}_1 = T_{i_1j_1k_1} T_{i_2j_2k_2} \bar{T}_{i_1j_1k_1} \bar{T}_{i_2j_2k_2}, \quad O^{(2)}_2 = T_{i_1j_1k_1} T_{i_2j_2k_2} \bar{T}_{i_2j_2k_1} \bar{T}_{i_1j_1k_2}, \quad O^{(2)}_3 = T_{i_1j_1k_1} T_{i_2j_2k_2} \bar{T}_{i_1j_2k_1} \bar{T}_{i_2j_1k_2}, \quad O^{(2)}_4 = T_{i_1j_1k_1} T_{i_2j_2k_2} \bar{T}_{i_1j_2k_2} \bar{T}_{i_2j_1k_1}. \quad (4.13)$$

According to [19], $r$ copies we have as many invariants as

$$O_{\alpha\beta\gamma}^{(r)} = T_{i_1j_1k_1} \ldots T_{i_\alpha j_\alpha k_\alpha} \bar{T}_{i_\alpha j_\alpha k_\alpha} \ldots \bar{T}_{i_\beta j_\beta k_\beta} \ldots \bar{T}_{i_\gamma j_\gamma k_\gamma}, \quad \alpha, \beta, \gamma \in S_r, \quad (4.14)$$

subject to the equivalence

$$O_{\alpha\beta\gamma} \sim O_{\alpha'\beta'\gamma'} \quad \text{if} \quad \alpha' = \tau \alpha \sigma, \quad \beta' = \tau \beta \sigma, \quad \gamma' = \tau \gamma \sigma, \quad (4.15)$$

for some $\sigma, \tau \in S_r$. The equivalence takes into account the freedom to shuffle $T$ and $\bar{T}$ slots in (4.14). As a comment, note that the same invariants could have been constructed with $r$ copies of (4.9) and patterns of contraction. We are expressing the same, although we believe that this construction is simpler.

Besides, for $L < r$ (which is going to be the case since $L = 2$), the number of invariants gets reduce in account of redundancies due to the small number of degrees of freedom.
Following [20–23], the exact number of invariants has been computed and is

\[ \# \text{ of invariants} = \sum_{\mu, \nu, \lambda} g_{\mu \nu \lambda} \leq 2 \]

where \( g_{\mu \nu \lambda} \) are the Kronecker coefficients labeled by three Young diagrams or partitions of \( r \), and the fact that \( L = 2 \) translates into the number of the rows of each Young diagram being equal or less than 2, as indicated in the sum (4.16).

It is remarked that Kronecker coefficients are hard to compute in general. However the restriction on the partitions to have at most two rows should make a dramatic simplification. This case could be studied somewhere and some neat formulas found for the counting. It would be interesting to find it.

Now, we could interpret (4.5) as the dyonic condition. It is actually a restriction in the number of degrees of freedom since given \( T_{ijk} \) one immediately knows what is \( \bar{T}_{\bar{i}\bar{j}\bar{k}} \), which is the conjugate number. It is noted that, thanks to (4.5), the invariants are real since

\[ O_{\alpha \beta \gamma} = T_{i_1 j_1 k_1} \cdots T_{i_r j_r k_r} T^\alpha_{(1)J(1)k_{\gamma(1)}} \cdots T^\alpha_{(r)J(\gamma(r))} = \mathcal{O}_{\alpha \beta \gamma}. \]  

After improving the comprehension of the mathematical structure of the matrix product states, for instance the TN, one might wonder why not use such a method to advance in the mathematical foundations of related physics. This is possible thanks to the properties, especially the encoded symmetries, of the TN states in many-body systems. Indeed, the TNs could be employed to explore results in condensed matter physics and quantum information theory. More precisely, this includes the microscopic origin of magnetism with spin systems and quantum computing simulations. For many-body systems, the spectral resolution of the corresponding acting Hamiltonian is always a highly complex task due to the associated large Hilbert space. The latter grows, exponentially, with the number of system particles. Such a fast growth is a relevant characteristic for certain physical behaviors such as phase transitions and quantum computation investigations.

Although it seems that the TNs have not been mastered and exploited well enough, we believe that they are a powerful mathematical framework to deal with certain deepest physical phenomena.

## 5 Conclusion

In this paper, we have investigated stringy dyonic objects and tensor network formalism. Inspired and motivated by known results obtained from non-trivial theories including string theory, we have provided shortly the origin and certain related concepts. After that, we have
elaborated a general framework of dyonic solutions in the brane structure and show how are naturally obtained in various dimensions within stringy compactifications on a concrete complex geometry. Then, we have discussed dyonic solution representations using tensor network formalism by revealing some characteristics associated with extra dimensions.

Assuming the existence of such dimensions, we believe that such a theoretical investigation could shed, somehow, some light on experimental searches for dyons, through either their direct production or indirect detection benchmarks at the current and future accelerators.

This work comes with certain open questions. A naturel question is make contact with MERA being a kind of tensor network states explored in the study of black holes [26]. We try to address such question elsewhere.

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