The unusual normal state properties of high $T_c$ superconductors (SC) have led to many studies exploring possible scenarios for the breakdown of Fermi liquid theory (FLT) [1]. Much of this effort has been in one of two directions: search for genuinely new non-Fermi liquid ground states arising from strong correlations in the proximity of an insulating state, or search for a low energy scale such that for temperatures above it one would see deviations from FLT. In this paper we take a different point of view and ask the question: Is the normal state of a short coherence length SC necessarily a Fermi liquid? More specifically, if the ground state is a condensate of fermion pairs, and the phase transition leads to a degenerate Fermi system above $T_C$, do the correlations in that normal state have to be described by Landau’s FLT? If not, what are the characteristic deviations from FLT?

For a weak coupling SC, with the coherence length $\xi_0 \gg a$, the lattice spacing, the normal state is a FL. As the coherence length decreases, and $\xi_0/a$ is not a small parameter, the $T_c$ at which phase coherence is established and the pair formation scale separate [2]. Certainly in the opposite extreme of tightly bound pairs, the state above $T_c$ is a normal Bose liquid. The question then is whether there is a broad intermediate coupling regime, especially in 2D, where the normal state has a “Fermi surface” and yet exhibits non-Fermi liquid correlations.

The simplest lattice model within which this problem can be studied is the attractive Hubbard model where the pair-size can be tuned by varying the strength of the interaction. In the absence of a small parameter we use quantum Monte Carlo (MC) simulations [3] to gain insight into the intermediate coupling regime above $T_c$. While the $U < 0$ Hubbard model is not a realistic microscopic model for the high $T_c$ materials, it has two merits: its simplicity (fewest parameters) and the reliability of low temperature MC simulations, since it is one of the few interacting fermion model which is free of the “sign problem” at all densities [1].

We begin by summarizing our main results which are obtained in the normal state ($T > T_c$):

1. The momentum distribution $n(k)$ shows structure reminiscent of a Fermi surface, though broadened by both thermal and interaction effects.
2. A pseudogap opens up in the one-particle density of states (DOS) well above $T_c$. The DOS is $T$-dependent – hence the notation $N_T(0)$ – with $dN_T(0)/dT > 0$. We obtain $N_T(0)$ using a new method which avoids numerical analytic continuation.
3. At low $T$, the spin susceptibility $\chi_s(T) = N_T(0)$, while the compressibility $(dn/d\mu)$ is $T$-independent. The spin and charge responses are thus qualitatively different.
4. The low frequency spectral weight $K(q) = \text{Im} \chi(q, \omega)/\omega$ in the spin channel is of the form $K(q) \sim N_T(0)/\Gamma(q)$ where the $T$-dependence is largely in the DOS and $\Gamma(q)$ is essentially the same as in non-interacting system. This naturally explains the spin-gap scaling $1/T_1 \equiv \sum_q K(q) \sim \chi_s(T)$, noted in our previous work [3], and observed in the underdoped high $T_c$ cuprates [4].

These results show that the normal state of a short coherence length 2D SC exhibits marked deviations from usual Fermi liquid behavior. One obtains a kind of “spin-charge separation” with the spin correlations determined by one-particle excitations, while the charge channel is dominated by collective excitations.

Consider the attractive Hubbard Hamiltonian $H = -t \sum_{i,j,\sigma} c_{i\sigma}^\dagger c_{j\sigma} + h.c. - |U| \sum_i n_{i\uparrow} n_{i\downarrow} + \mu \sum_{i,\sigma} n_{i\sigma}$, where the hopping is between nearest neighbor sites. All energies are measured in units of $t$, and the lattice spacing $a = 1$.

We first establish that in the regime $|U| = 4$ and $\langle n \rangle = 0.5$ [4] that we focus on here, our finite system results are indeed characteristic of the normal state. For this we estimate the correlation length $\xi(T)$ from the spatial decay of the SC order parameter correlation function on systems of size up to $L = 16$. We find that for $|U| = 4$
we have $\xi \sim 3 - 4$ at $T = 1/6$; thus $\xi(T) \ll L$ and the system is normal. (Details of this analysis and finite size scaling will be presented elsewhere).

**Single-particle properties:**

To ascertain that the system is degenerate at moderate $|U|$ we study the momentum distribution $n(k) = \sum_\nu \langle c_{k\nu}^\dagger c_{k\nu} \rangle$ on large lattices of size up to $N = L^2 = 16^2$; see Fig. 1. It is clear that $n(k)$ shows a rapid variation with $k$ -- a “Fermi surface” -- even though it is broadened by the temperature and also affected by the interactions. We emphasize that we are not in the large-$|U|$ preformed boson limit, where the constituent fermions (tightly bound into singlet pairs) are non-degenerate and their $n(k)$ is independent of $k$.

We now turn to the single-particle density of states (DOS) $N(\omega)$ with $\omega$ measured from $\mu$. This is given by $N(\omega) = \frac{1}{\pi} \sum_k A(k, \omega)$. Here $A$ is the spectral function, which is related to the imaginary time Green function $G(k, \tau) = -\langle T[c_{k0}(\tau)c_{k0}^\dagger(0)] \rangle$ via

$$G(k, \tau) = -\int_{-\infty}^{\infty} d\omega \frac{\exp(-\omega \tau)}{1 + \exp(-\beta \omega)} A(k, \omega),$$  

for $0 < \tau < \beta = 1/T$. To estimate $A$ and from it the DOS, given MC data for $G(k, \tau)$, involves inverting this Laplace transform. We avoid the numerical complications inherent in such an analytic continuation by deriving a general expression for $N(0)$ in terms of $G(k, \tau)$, which is valid provided there is no low energy scale in the problem. Fourier transforming to real space and looking at the local Green function at $\tau = \beta/2$ we get $G(r = 0, \beta/2) = -\int d\omega \text{sech}(\beta \omega/2) N(\omega)/2$. Let $\Omega$ be the frequency scale on which there is structure in the DOS; for $T \ll \Omega$ we obtain

$$N(0) \simeq -\beta G(r = 0, \tau = \beta/2)/\pi.$$  

The one-particle DOS obtained from (2) is plotted in Fig. 2 as a function of temperature. We see that for $T > T_c$ a pseudo-gap develops at the chemical potential and the DOS is depleted as $T$ is reduced. To emphasize this $T$-dependence of the DOS we use the notation $N_T(0)$. This behavior should be compared with that in weak coupling where the DOS remains featureless above $T_c$, except for a small fluctuation dip. The pseudogap at intermediate coupling may be thought of as the evolution of the weak coupling fluctuation dip into a regime where $a/\xi_0$ is no longer a small parameter.

**Spin and charge correlations:** In Fig. 2 we also plot the uniform, static spin susceptibility $\chi_s$. The $T$-dependence of $\chi_s$ was already noted in ref. [3]: what we see here is that this $T$-dependence comes entirely from that of the one-particle DOS, so that $\chi_s(T) = N_T(0)$ (to within the errors inherent in extracting the latter).

It is interesting to ask whether the charge channel also exhibits the same pseudo-gap. The compressibility $(dn/d\mu)$ was obtained by numerically differentiating [11] the average density $\langle n \rangle$ determined as a function of $\mu$. We found that $(dn/d\mu)$ shows significant system size dependence (much more than, e.g., $\chi_s$); small system data show (finite size) upturns which are pushed down to lower $T$ with increasing $L$. The results on the largest lattice ($L = 16$) plotted in Fig. 3 show that the system becomes more compressible with increasing $|U|$ (see below). More significantly, in sharp contrast to the one-particle DOS, $(dn/d\mu)$ is very weakly $T$-dependent.

Within a simple RPA (p-h bubbles) we obtain $(dn/d\mu)_\text{RPA} = 2N_0(0)/|1 - |U|N_0(0)|$. This is valid only if $|U|N_0(0) \ll 1$, where it correctly explains the trend that attractive interactions increase $(dn/d\mu)$; for large $|U|$ RPA fails in that it predicts an entirely spurious instability (phase separation) when $|U|N_0(0) = 1$. In fact, pairs form at large $|U|$ and their residual interactions are repulsive, so that the compressibility of the system remains finite for all $|U|$. It is simplest to see this in the broken symmetry state at $T = 0$ where we find [12] that, within mean field (MF) theory, $(dn/d\mu)$ decreases monotonically with $|U|$ from $(dn/d\mu)_0 \simeq 2N_0(0)/|1 + |U|N_0(0)|$ for small $|U|$ to $(dn/d\mu) \simeq |U|/4\alpha t^2 - 2/|U|$ for $|U|/t \gg 1$. The $T = 0$ MF result [12] is also shown in Fig. 3 and is found to be of the right order-of-magnitude as the normal state MC result; note that we do not expect $(dn/d\mu)$ to change dramatically as $T$ goes through $T_c$.

The difference between the spin and charge response functions could be characterized as a sort of spin charge separation [13]. In its mildest form this exists even in a Landau Fermi liquid where the $\chi_s$ and $(dn/d\mu)$ are quantitatively different, the two being renormalized by different FL parameters: $F^s_0$ and $F^s_0$. What we see here is much stronger: as a result of strong interactions, $\chi_s$ and $(dn/d\mu)$ acquire qualitatively different $T$-dependences. As argued above, the spin response is dominated by incoherent single-particle excitations, which is $T$-dependent because triplet excitations require breaking up the singlet pair correlations while the charge channel is dominated by collective behavior.

**Spin-gap scaling:** We next use our results for the $T$-dependent DOS $N_T(0)$ to gain insight into the suppression of low frequency spectral weight in the spin channel as probed by $K(q) = \lim_{\omega \to 0} \text{Im} \chi(q, \omega)/\omega$. To contrast with our MC results it may be useful to recall that for a Fermi liquid (all quantities denoted by a subscript 0): $K_0(q) = N_0(0)/\Gamma_0(q)$ is $T$-independent for $T < \epsilon_F$ with $\Gamma_0(q) \sim v_F q$. Further $\sum_q \Gamma_0^{-1}(q) \approx N_0(0)$ leads to the Korringa law $(1/T_1 T) = \sum_q K_0(q) \approx N_0^2(0)$.

In Fig. 4 we plot the MC results for $K(q)$ for $q \neq 0$ and the analytic continuation required to obtain $K$ was done using the method of ref. [13]. From Fig. 4(a) we see that $K(q)$ is more or less uniformly suppressed at all $q$ with decreasing temperature. Further this $T$-dependence is similar to that of $\chi_s(T)$ (or the DOS) shown at $q = 0$ in the Fig. 4(a). For fixed $T$ the $q$-dependence of $K(q)$
The numerical results thus suggest that \( K(q; T) = \alpha \times N_F(0)/T_0(q) \), with \( \alpha \) independent of \( T \) and \( q \), i.e., the \( T \)-dependence comes from a DOS with a pseudo-gap while the \( q \)-dependence is that of the non-interacting system. This form for \( K(q; T) \) leads to \( 1/T_1 T = \sum_q K(q; T) \approx \alpha N_F(0) N_T(0) \), thus providing a natural explanation for the spin gap scaling \( 1/T_1 T \sim \chi_s(T) \) found in our earlier MC studies [8].

We note that \( 1/T_1 T \sim \chi_s(T) \) is observed in the (planar) O and Y NMR in a large number of (usually underdoped) cuprates [9]. What distinguishes the results presented here from spin gap theories [11] based on spin models, is that the anomalies exist in a system with itinerant carriers with a large Fermi surface. Secondly in the present approach these anomalies are directly related to the carrier distribution giving clear evidence for a degenerate 2D short coherence length SC exhibits characteristic deviations from Fermi liquid behavior: while the momentum distribution gives clear evidence for a degenerate Fermi system, the one-particle DOS shows a pseudo-gap [7] and the spin and charge correlations show qualitatively different temperature dependences.

In conclusion we have shown that the normal state of a 2D short coherence length SC exhibits characteristic deviations from Fermi liquid behavior: while the momentum distribution gives clear evidence for a degenerate Fermi system, the one-particle DOS shows a pseudo-gap [7] and the spin and charge correlations show qualitatively different temperature dependences.

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[1] See, e.g., Strongly Correlated Electronic Materials edited by K. Bedell et al., (Adison Wesley, 1994) and High Temperature Superconductivity edited by K. Bedell et al., (Adison Wesley, 1990).

[2] C. Sa de Melo, M. Randeria, and J. R. Engelbrecht, Phys. Rev. Lett. 71, 3202 (1993).

[3] For a detailed review and references on the BCS-Bose crossover, see M. Randeria, in Bose-Einstein Condensation edited by A. Griffin, D. Snoke and S. Stringari (Cambridge University Press, 1994).

[4] (a) R. T. Scalettar et al., Phys. Rev. Lett. 62, 1407 (1989); (b) A. Moreo and D. J. Scalapino, Phys. Rev. Lett. 66, 946 (1991); (c) A. Moreo, D. J. Scalapino and S. R. White, Phys. Rev. B 45, 7544 (1992).

[5] M. Randeria, N. Trivedi, A. Moreo, and R. T. Scalettar, Phys. Rev. Lett. 69, 2001. (1992).

[6] J. E. Hirsch, Phys. Rev. B 28, 4059 (1983).

[7] W. W. Warren et al., Phys. Rev. Lett. 62, 1193 (1989); M. Takigawa et al., Phys. Rev. B 43, 247 (1991); H. Alloul et al., Phys. Rev. Lett. 70, 1171 (1993); R. E. Walstedt et al., Phys. Rev. Lett. 72, 3610 (1994).

[8] We choose \( \langle n \rangle \neq 1 \) since we are not interested in the competition with the CDW state at half-filling. See: R. Micas, J. Ranninger and S. Robaskiewicz, Rev. Mod. Phys. 62, 113 (1990).

[9] For low frequencies \( N(\omega) = N(0) + N'(0) \omega + \ldots \), where \( N'(0) \sim N(0) / \Omega^2 \). Substituting this in the preceding equation we obtain with corrections of order \( (T/\Omega)^2 \).

[10] C. Di Castro, C. Castellani, R. Raimondi and A. Varlamov, Phys. Rev. B 42, 10211 (1990). For the resulting effects on spin correlations, see: M. Randeria and A. Varlamov, Phys. Rev. B 42, 10211 (1994) and D. Rainer (private communication).

[11] We could not use \( \langle n_n \rangle - \langle n \rangle \langle n \rangle \) which is prone to errors due to large cancellations.

[12] From an extension of the \( T = 0 \) analysis by L. Belkhir and M. Randeria, Phys. Rev. B 45, 5087 (1992) and Phys. Rev. B 49, 6829 (1994), we find \( \langle n \rangle / \partial U \rangle = U |(\partial n/\partial U)|/(1 - U |(\partial n/\partial U)|/2) \) with \( \partial n/\partial U = \Delta^2 \sum_q E_p^{-3} + (\sum_q E_p^{-3})^2 / (\sum_q E_p^{-3}) \). Here \( \mu = \mu + \langle n \rangle |U| / 2 \), and the rest of the notation is standard: \( \xi_k \) is the energy from \( \mu \), \( \Delta \) is the gap, and \( E_k \) is the Bogoliubov quasiparticle energy.

[13] The parallels between spin charge separation (SCS) in the resonating valence bond picture and in the BCS ground state have been discussed by S. A. Kivelson and D. S. Rokhsar, Phys. Rev. B 41, 11693 (1990). Here we find that pairing correlations alone can also give rise to SCS without any broken symmetry.

[14] A. J. Millis and H. Monien, Phys. Rev. Lett. 70, 2810 (1993), 71, 210 (1993) (E); A. Sokol and D. Pines, Phys. Rev. Lett. 71, 2813 (1993).

[15] After this work of was completed we learned from R. Micas of self consistent T-matrix calculations at low densities by J. J. Rodriguez-Nunez et al.(unpublished) which also find a pseudo-gap in the DOS.

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FIG. 1. The momentum distribution \( n(k) \) versus \( |k| \) along [1, 0] and [1, 1] for \( |U| = 4 \), \( \langle n \rangle = 0.5 \) and \( T = 0.25 \). The Fermi function \( (U = 0) \) is shown as the long dashed curve and the \( U = -\infty \) Bose limit result is plotted as the short dashed curve. The statistical error bars on all the MC data, unless explicitly shown, are less than the size of the symbols in this and other figures.

FIG. 2. The one-particle density of states \( N_T(0) \) at the chemical potential (full triangles) and the spin susceptibility \( \chi_s \) (open squares) as a function of temperature for \( |U| = 4 \), \( \langle n \rangle = 0.5 \) and \( L^2 = 8^2 \).
FIG. 3. The compressibility \((dn/d\mu)\) for \(\langle n \rangle = 0.5, |U| = 0\) (full line) and \(|U| = 4\) (open circles), as a function of \(T\) obtained on a \(16^2\) lattice. The \(T = 0\) non-interacting and the \(T = 0\) mean field result for \(|U| = 4\) are also shown.

FIG. 4. (a) Low frequency spectral weight in the spin channel \(K(q; T) = \lim_{\omega \to 0} \text{Im} \chi(q, \omega)/\omega\) for \(q \neq 0\) (open symbols) plotted along [1, 0] and [1, 1] for various \(T\). The dashed lines are guides to the eye. The filled symbols plotted at \(q = 0\) show \(2.0 \times \chi_s(T)\) with \(T\) corresponding to that of the open symbols. Note that the \(T\)-dependence of \(K(q)\) is similar to that of the susceptibility \(\chi_s(T)\). All of the data is for \(|U| = 4, \langle n \rangle = 0.5\) and \(L^2 = 8^2\). (b) At a fixed \(T\) the \(q\)-dependence of \(K(q)\) is qualitatively similar to that of the non-interacting case. To show this we compare the \(K(q)\) at \(T = 0.25\) (open squares) with the full curve given by \(2\chi_s(T)K^0(q)\), where \(K^0(q)\) is the essentially \(T\)-independent spectral weight for the non-interacting system.
$U = -4$
$T = 0.25$
$\langle n \rangle = 0.5$

$n(k)$ as a function of $k$ for different lattice sizes: 8x8, 10x10, 12x12, 14x14, 16x16.
$N = 8 \times 8$

$U = -4 \quad \langle n \rangle = 0.5$

$N_T(0), \chi$

$T/t$
