Abstract

We consider three-dimensional domino tilings of cylinders $\mathcal{R}_N = D \times [0, N]$ where $D \subset \mathbb{R}^2$ is a fixed quadriculated disk and $N \in \mathbb{N}$. A domino is a $2 \times 1 \times 1$ brick. A flip is a local move in the space of tilings $\mathcal{T}(\mathcal{R}_N)$: remove two adjacent dominoes and place them back after a rotation. The twist is a flip invariant which associates an integer number to each tiling. For some disks $D$, called regular, two tilings of $\mathcal{R}_N$ with the same twist can be joined by a sequence of flips once we add vertical space to the cylinder. We have that if $D$ is regular then the size of the largest connected component under flips of $\mathcal{T}(\mathcal{R}_N)$ is $\Theta(N^{-\frac{1}{2}}|\mathcal{T}(\mathcal{R}_N)|)$. The domino group $G_D$ captures information of the space of tilings. A disk $D$ is regular if and only if $G_D$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/(2)$; sufficiently large rectangles are regular.

We prove that certain families of disks are irregular. We show that the existence of a bottleneck in a disk $D$ often implies irregularity. In many, but not all, of these cases, we also prove that $D$ is strongly irregular, i.e., that there exists a surjective homomorphism from $G_D^+$ (a subgroup of index two of $G_D$) to the free group of rank two. Moreover, we show that if $D$ is strongly irregular then the cardinality of the largest connected component under flips of $\mathcal{T}(\mathcal{R}_N)$ is $O(c^N|\mathcal{T}(\mathcal{R}_N)|)$ for some $c \in (0, 1)$.
1 Introduction

Domino tilings have been the topic of many questions. A lot of results have been established in dimension two, e.g., [1], [11], [10]. However, generalizations to three dimensions seem to be hard and much less is known. For instance, the counting tilings problem is easy in dimension two [3] and computationally more complex in dimension three [7]. The present paper focuses on the problem of connectivity by local moves.

A flip is a local move that consists of a $90^\circ$ rotation of two adjacent parallel dominoes. Thurston [11] proved that any two domino tilings of a planar simply connected region can be joined by a sequence of flips. For non simply connected regions, the idea of flux of a tiling is presented in [10]. The flux is a flip invariant such that two domino tilings of a planar region can be joined by a sequence of flips if and only if they have the same flux.

The three-dimensional problem of flip connectivity is much more subtle. For instance, the space of tilings of many simply connected regions is no longer connected by flips; see Figure 7 for tilings of $[0, 4]^2 \times [0, 2]$ which admit no flip. Milet and Saldanha [5] introduced the twist of a tiling for several contractible cubiculated regions contained in $\mathbb{R}^3$. In this case, the twist is a flip invariant assuming values in $\mathbb{Z}$: given a tiling $t$ of a suitable region, we have an integer $\text{Tw}(t)$. For a fixed nontrivial quadriculated disk $D \subset \mathbb{R}^2$, as defined in Section 2, the distribution of the twist (according to the number of tilings of $D \times [0, N]$) tends to a Gaussian as $N$ goes to infinity (see [8]). In the same scenario, almost always any two tilings with the same twist can be joined by a sequence of flips. Experimental evidence suggests that similar results hold for large cubical boxes.

The concept of regular disk is defined in [9]. A nontrivial balanced quadriculated disk $D$ is regular if whenever two tilings $t_1$ and $t_2$ of $D \times [0, N] \subset \mathbb{R}^3$ have the same twist then $t_1$ and $t_2$ can be connected by a sequence of flips provided that some vertical space is allowed. We then say that a disk is irregular if it is not regular. Saldanha [9] proved that the rectangle $D = [0, L] \times [0, M]$ with $LM$ even is regular if and only if $\min\{L, M\} \geq 3$; and it was conjectured that “plump” disks are regular. For instance, Figure 1 below exhibits examples of regular disks.
In this paper we prove that other families of disks are irregular. We show that, for some disks $D$, the existence of either a unit square or a domino that disconnects $D$ implies that $D$ is irregular. As a consequence, the disks in Figure 2 are irregular.

We study the irregularity of a disk $D$ by considering its domino group $G_D$. The domino group is the fundamental group of a finite 2-dimensional CW-complex $C_D$, whose construction is described in [9] and in Section 2. Tilings of $D \times [0, N]$ are related to closed paths of length $N$ in $C_D$. Moreover, if two tilings can be joined by a sequence of flips then their corresponding paths are homotopic. The even domino group $G_D^+$ is a normal subgroup of index two of $G_D$. If there exists a domino tiling of $D \times [0, 1]$ then $G_D$ is isomorphic to a semidirect product of $G_D^+$ and $\mathbb{Z}/(2)$. The twist defines a homomorphism from $G_D^+$ to the integers $\mathbb{Z}$; it turns out that $D$ is regular if and only if this homomorphism is an isomorphism. In this case, $G_D = \mathbb{Z} \oplus \mathbb{Z}/(2)$.

We distinguish irregular disks by the behavior of their domino groups. A disk $D$ is called strongly irregular if there exists a surjective homomorphism from the even domino group $G_D^+$ to the free group with two generators $F_2$. The vast majority of the computed examples of irregular disks are strongly irregular. For instance, it follows from our results that the disks shown in the mid and bottom row of Figure 2 are strongly irregular. On the other hand, we have essentially only one example of a irregular disk which is not strongly irregular. Indeed, we show in Example 2.1 that the disks in the top row of Figure 2 are such that their even domino groups are isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. 
The structure of a domino group $G_D$ provides information about $T(R_N)$ (the set of tilings of $D \times [0, N]$) for large values of $N$. Let $\approx$ be the equivalence relation on $T(R_N)$ defined by connectivity via flips. The $\approx$-equivalence classes are called connected components under flips. In the regular case, i.e. $G_D^+ = \mathbb{Z}$, it follows from [8] that the size of the largest component is $\Theta(N^{-\frac{1}{2}}|T(R_N)|)$. We prove that if $D$ is strongly irregular then, for large values of $N$, the connected components under flips consist of exponentially small fractions of $T(R_N)$.

**Theorem 1.** Consider a balanced quadriculated disk $D$. Let $T_1$ and $T_2$ be two independent random tilings of $D \times [0, N]$. If $D$ is strongly irregular then there exists $c \in (0, 1)$ such that $\mathbb{P}(T_1 \approx T_2) = o(c^N)$.

We do not try to obtain sharp estimates of the constant $c$.

**Remark 1.1.** It follows from [8] that if $T_N$ is a random tiling of $D \times [0, N]$ then the twist defines a random variable $Tw(T_N)$ which converges in distribution to a normal distribution. Then, by Theorem 1, $\mathbb{P}(T_1 \approx T_2 | Tw(T_1) = Tw(T_2)) = o(c^N)$.

The following is then an immediate corollary.

**Corollary 1.1.** Consider a strongly irregular balanced quadriculated disk $D$. Then, the cardinality of the largest flip connected component of $T(R_N)$ is $O(c^N|T(R_N)|)$ for some $c \in (0, 1)$.

Normally, it is easier to prove that a disk is strongly irregular than to compute its (even) domino group. We compute the even domino group of very specific disks. Our main example are thin rectangles $D_L = [0, L] \times [0, 2]$ with $L \geq 3$. We now provide a presentation of $G_{D_L}^+$. Let $S_L = \{a_i : i \in \mathbb{Z}_{\neq 0} \text{ and } |i| \leq \left\lfloor \frac{L-1}{2} \right\rfloor \}$ be a set of symbols. Consider the finite set $R_L = \{(m, n) \in \mathbb{Z}^2 : \max\{|m|, |n|, |m-n|\} < \left\lfloor \frac{L}{2} \right\rfloor \}$, see Figure 14. Then, consider the group

$$G_L^+ = \langle S_L | [a_m, a_n] = 1 \text{ for } (m, n) \in R_L \rangle. \quad (1.1)$$

Notice that if $i = \left\lfloor \frac{L-1}{2} \right\rfloor$ then the subgroup $H \leq G_L^+$ generated by $a_{-i}$ and $a_i$ is isomorphic to $F_2$. Let $\phi : G_L^+ \rightarrow H$ be the homomorphism defined by $\phi(a_i) = a_i$, $\phi(a_{-i}) = a_{-i}$ and $\phi(a_j) = e$ for $|j| < i$: $\phi$ is surjective.

**Theorem 2.** Let $L \geq 3$ and consider the disk $D_L = [0, L] \times [0, 2]$. Then, the even domino group $G_{D_L}^+$ is isomorphic to $G_L^+$. In particular, $D_L$ is strongly irregular.
In general, we prove the strong irregularity of a disk $\mathcal{D}$ by constructing a surjective homomorphism from $G^{+}_\mathcal{D}$ to $F_2$. This is the strategy of the proofs of Theorems 3, 4 and 5. These imply that the disks in Figures 3, 4 and 5 are strongly irregular.

**Theorem 3.** Consider a balanced quadriculated disk $\mathcal{D}$. Suppose that $\mathcal{D}$ contains a unit square $s$ such that $\mathcal{D} \setminus s$ has at least three connected components. If the largest component of $\mathcal{D} \setminus s$ has size at most $|\mathcal{D}| - 4$ then $\mathcal{D}$ is strongly irregular.

![Figure 3: Examples of strongly irregular disks; unit squares $s$ as in Theorem 3 are marked by a red line segment.](image)

In contrast with the previous theorem, the next two results are based on the existence of a domino that disconnects $\mathcal{D}$.

**Theorem 4.** Consider a balanced quadriculated disk $\mathcal{D}$ containing a domino $d$ such that $\mathcal{D} \setminus d$ is not connected. Suppose that there exists a $2 \times 2$ square in $\mathcal{D}$ which contains $d$. If every connected component of $\mathcal{D} \setminus d$ which intersects a $2 \times 2$ square contained in $\mathcal{D}$ that contains $d$ has size at most $\frac{|\mathcal{D}| - 2}{2}$ then $\mathcal{D}$ is strongly irregular.

![Figure 4: Examples of strongly irregular disks; dominoes $d$ as in Theorem 4 are marked by a red line segment.](image)

**Theorem 5.** Consider a balanced quadriculated disk $\mathcal{D}$. Suppose there exists a $2 \times 2$ square $s \subset \mathcal{D}$ such that $\mathcal{D} \setminus s$ is the union of two disjoint disks $\mathcal{D}_1$ and $\mathcal{D}_2$ with $|\mathcal{D}_1| = |\mathcal{D}_2|$. Suppose $s$ contains dominoes $d_1$ adjacent to $\mathcal{D}_1$ and $d_2$ adjacent to $\mathcal{D}_2$ such that $\mathcal{D} \setminus d_1$ and $\mathcal{D} \setminus d_2$ are not connected. Then, $\mathcal{D}$ is strongly irregular.

![Figure 5: Examples of strongly irregular disks; dominoes $d_1$ and $d_2$ as in Theorem 5 are marked by a red line segment.](image)
Notice that the hypotheses of Theorems 3, 4 and 5 are not mutually exclusive. For instance, the first and the third disk of Figure 4 are examples of disks for which, besides Theorem 4, Theorem 5 and Theorem 3 also applies. However, in this case, each theorem provides a distinct surjective homomorphism from the even domino group to the free group of rank two.

**Remark 1.2.** Consider an even number $a \geq 4$ and let $b = a^2 - 4$. Then, it follows from [4], that the disk $D$ formed by the union of $[0, a]^2$ and $[a, a + \frac{b}{2}] \times [0, 2]$ is regular. For instance, Figure 1 shows $D$ for $a = 4$. On the other hand, notice that if $d = [a - 1, a] \times [0, 2]$ then $D \setminus d$ is the union of two disjoint disks $D_1$ and $D_2$ such that $|D_1| = a^2 - 2$ and $|D_2| = b = a^2 - 4$. Therefore, in Theorem 4, the hypothesis about the size of the connected components cannot be discarded.

Similarly, the hypothesis in Theorem 3 also cannot be relaxed. Indeed, consider the disks in the first row of Figure 2 and in Example 2.1 whose even domino groups are isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Notice that these disks can be disconnected, by removing a unit square, into three connected components so that the largest component has size $|D| - 3$. ⊥

The text is divided as follows. In Section 2 we briefly review the construction of the domino group (see [9]). In Example 2.1 we show the existence of disks which are irregular but not strongly irregular by explicitly computing their domino groups. The proof of Theorem 1 is presented in Section 3. Section 4 is dedicated to the computation of the domino group of thin rectangles $[0, L] \times [0, 2]$ with $L \geq 3$. We prove Theorems 3, 4 and 5 in Section 5. A rather technical result (Lemma 3.1) concerning random walks on the free group $F_2$ is proved in Section 6, an appendix.

The author thanks Nicolau Saldanha for providing insightful ideas, comments and suggestions. Acknowledgements are also given to Caroline Klivans, Robert Morris, Simon Griffiths and Yoshiharu Kohayakawa who carefully read the author’s Master dissertation [4], which corresponds to part of this paper. The author thanks the referee for the careful reading. The support of CNPq, CAPES, FAPERJ and Projeto Arquimedes (PUC-Rio) are appreciated.
2 Definitions

A quadriculated disk \( \mathcal{D} \subset \mathbb{R}^2 \) is a region, homeomorphic to a closed disk, formed by a finite union of closed unit squares \([a, a+1] \times [b, b+1]\) with \((a, b) \in \mathbb{Z}^2\). The number of unit squares contained in \( \mathcal{D} \) is denoted by \(|\mathcal{D}|\). A unit square \([a, a+1] \times [b, b+1]\) is white (resp. black) if \(a + b\) is even (resp. odd). We say that a quadriculated disk is balanced if it contains the same number of white and black unit squares. Moreover, a disk is trivial if either it is a \(2 \times 2\) square or its unit squares are adjacent to at most other two unit squares. Unless otherwise stated, we always assume that our quadriculated disks are balanced and not trivial. A domino is the union of two adjacent closed unit squares and a domino tiling of a quadriculated disk \( \mathcal{D} \) is a covering of \( \mathcal{D} \) by dominoes with disjoint interiors.

From the graph-theoretical viewpoint, a disk \( \mathcal{D} \) is identified with a bipartite graph \( \mathcal{G}(\mathcal{D}) \) whose vertices are the white and black colored unit squares in \( \mathcal{D} \); two vertices are connected by an edge if and only if their corresponding unit squares are adjacent. Thus, a domino tiling of \( \mathcal{D} \) corresponds to a perfect matching of \( \mathcal{G}(\mathcal{D}) \). We refer to a spanning tree of \( \mathcal{G}(\mathcal{D}) \) as a spanning tree of \( \mathcal{D} \). In that sense, given a disk \( \mathcal{D} \) with a spanning tree \( T \), we say that a domino in \( \mathcal{D} \) is not an edge of \( T \) if its corresponding edge in \( \mathcal{G}(\mathcal{D}) \) is not contained in \( T \).

A cylinder \( \mathcal{R}_N \subset \mathbb{R}^3 \) is a cubiculated region formed by the cartesian product of a balanced quadriculated disk \( \mathcal{D} \) and an interval \([0, N]\) for some \( N \in \mathbb{N} \). Analogously to the two-dimensional case, the unit cubes in \( \mathcal{R}_N \) are colored in black and white alternatively. A domino is a parallelepiped with sides of length 2, 1 and 1. A domino tiling of \( \mathcal{R}_N \) is a covering of \( \mathcal{R}_N \) by dominoes with disjoint interiors. The set of domino tilings of \( \mathcal{R}_N \) is denoted by \( \mathcal{T}(\mathcal{R}_N) \). In particular, for \( N \) even, we have the vertical tiling \( \mathbf{t}_{\text{vert}, N} \in \mathcal{T}(\mathcal{R}_N) \) which consists of only dominoes of the form \([a, a+1] \times [b, b+1] \times [c, c+2] \).

We draw a tiling \( \mathbf{t} \in \mathcal{T}(\mathcal{R}_N) \) by floors, i.e., we describe the behavior of \( \mathbf{t} \) at each floor \( \mathcal{D} \times [K - 1, K] \), as in Figure 6. We fix the \( x \)-axis and the \( y \)-axis, then the floors are exhibited in increasing order from the left to the right. Dominoes which are parallel to either the \( x \)-axis or the \( y \)-axis are represented as planar dominoes. Vertical dominoes, i.e., dominoes parallel to \( z \)-axis, are represented by two unit squares contained in adjacent floors; to avoid confusion the unit square contained in the highest floor, which appears at the right hand side, is left unfilled.
Let $D$ be a quadriculated disk. Consider two tilings $t_1 \in T(R_{N_1})$ and $t_2 \in T(R_{N_2})$. We write $t_1 \approx t_2$ if $N_1 = N_2$ and there exists a sequence of flips joining $t_1$ and $t_2$. The concatenation $t_1 \ast t_2$ is the tiling of $R_{N_1+N_2}$ formed by the union of $t_1$ and the translation of $t_2$ by $(0,0,N_1)$. We say that $t_1 \sim t_2$ if $N_1 \equiv N_2 \pmod{2}$ and there exists $M_1, M_2 \in \mathbb{Z}^N$ such that $N_1 + M_1 = N_2 + M_2$ and $t_1 \ast t_{vert,M_1} \approx t_2 \ast t_{vert,M_2}$.

The Figure 7 shows two tilings $t_1$ and $t_2$ of $[0, 4] \times [0, 2]$ which admit no flip, so $t_1 \not\approx t_2$. However, a computation shows that $t_1 \ast t_{vert,2} \approx t_2 \ast t_{vert,2}$, so $t_1 \sim t_2$.

A plug $p$ is a union of an equal number of white and black unit squares contained in $D$. In particular, we have the empty plug $p_\emptyset = \emptyset$ and the full plug $p_\bullet = D$. The complement $D \setminus \text{int}(p)$ of a plug $p$ is also a plug and is denoted by $p^{-1}$. The number of unit squares in $p$ is denoted by $|p|$. We denote by $\mathcal{P}$ the set of plugs in $D$.

Sometimes, it is useful to consider a region more general than cylinders. Let $p_1, p_2 \in \mathcal{P}$ be two plugs and consider two nonnegative integers $N_1$ and $N_2$ such that $N_2 > N_1 + 2$. The cork $R_{N_1,N_2;p_1,p_2}$ is defined as:

$$R_{N_1,N_2;p_1,p_2} = (D \times [N_1 + 1, N_2 - 1]) \cup (p_1^{-1} \times [N_1, N_1 + 1]) \cup (p_2^{-1} \times [N_2 - 1, N_2]).$$

In other words, the cork $R_{N_1,N_2;p_1,p_2}$ is obtained from $D \times [N_1, N_2]$ by removing the plug $p_1$ from the $(N_1 + 1)$-th floor ($D \times [N_1, N_1 + 1]$) and the plug $p_2$ from the $N_2$-th floor. For instance, notice that $R_{0,N;p_\emptyset,p_\bullet} = R_N$. The inverse of a tiling $t$ of $R_{N_1,N_2;p_1,p_2}$ is defined as the tiling $t^{-1}$ of $R_{N_1,N_2;p_2,p_1}$ obtained by reflecting $t$ on the $xy$ plane. If $t \in T(R_N)$ then $t \ast t^{-1} \approx t_{vert,2N}$ (see Lemma 4.2 of [9]).

Given a quadriculated disk $D$, we construct a 2-complex $C_D$. The domino group $G_D$ will be the fundamental group of $C_D$. The 0-skeleton, i.e. the set of vertices of
\( \mathcal{C}_D \), is the set of plugs \( \mathcal{P} \). The edges of \( \mathcal{C}_D \) represent two-dimensional domino tilings of subregions of \( D \). More precisely, attach an edge between two disjoint plugs \( p_1 \) and \( p_2 \) for each tiling of \( D \setminus \text{int}(p_1 \cup p_2) \). In particular, non disjoint plugs do not share any edge. For the special case \( p_1 = p_2 = p_\emptyset \) (the empty plug), there is a loop based at \( p_\emptyset \) for each tiling of \( D \). This construction defines the 1-skeleton of \( \mathcal{C}_D \).

We now attach the 2-cells. First, attach a disk to each loop by wrapping its boundary twice around the loop, so that the result is a projective plane. For instance, see Figure 8.

![Figure 8: Four loops in \( \mathcal{C}_D \) for \( D = [0, 4]^2 \), the four petals are four projective planes.](image)

The other 2-cells are attached injectively to certain bigons and quadrilaterals. Notice that a bigon consists of two distinct disjoint plugs \( p_1 \) and \( p_2 \) connected by two edges representing distinct tilings of \( D \setminus \text{int}(p_1 \cup p_2) \). We attach a disk to each bigon whose edges correspond to tilings which differ by a single flip, as in Figure 9.

![Figure 9: Two bigons in the complex of \( D = [0, 4]^2 \). In the first row we attach a disk to the bigon. The two tilings in the second row do not differ by a single flip. Therefore, we do not attach a disk to the bigon in the second row.](image)

We attach a 2-cell to each quadrilateral constructed in the following manner. Let \( p_1, p_2, p_3, p_4 \) be four plugs such that \( p_2 \) is disjoint from \( p_1 \) and \( p_4 \). Suppose that \( p_2 \) equals a union of \( p_3 \) and two adjacent unit squares, i.e. a domino \( d \subset D \). Let \( f^*_1 \) be a tiling of \( D \setminus \text{int}(p_1 \cup p_2) \) and \( f^*_2 \) be a tiling of \( D \setminus \text{int}(p_2 \cup p_4) \). Therefore, \( f^*_3 = f^*_1 \cup d \) and \( f^*_4 = f^*_2 \cup d \) are tilings of \( D \setminus \text{int}(p_1 \cup p_3) \) and \( D \setminus \text{int}(p_3 \cup p_4) \),
respectively. Notice that $p_1, p_2, p_3, p_4$ and $f_1^*, f_2^*, f_3^*, f_4^*$ thus define a quadrilateral in the 1-skeleton of $C_D$. Then, attach a 2-cell to this quadrilateral. For instance, see Figure 10.

![Figure 10: Two quadrilaterals in the complex of $D = [0, 4]^2$. In the first row we attach a 2-cell to the quadrilateral. The second row shows an example of a quadrilateral where we do not attach a 2-cell.](image)

This finishes the construction of the complex $C_D$. By definition, $G_D = \pi_1(C_D, p_0)$. In most cases, it is impractical to draw the complex $C_D$. For instance, if $D = [0, 4]^2$ then $C_D$ has 12870 vertices and 36 loops. The calculation of the exact number of 1-cells and 2-cells requires a long computation.

The complex $C_D$ is related to tilings of cylinders of base $D$. Indeed, a tiling $t$ of $D \times [0, N]$ corresponds to an oriented loop of length $N$ based at $p_0$. In order to construct this correspondence we proceed as follows. We describe the behavior of $t$ at each floor $D \times [K-1, K]$ by a triple $f_K = (p_{K-1}, f_K^*, p_K)$. The plug $p_K$ is the union of the unit squares $[a, a+1] \times [b, b+1]$ such that $[a, a+1] \times [b, b+1] \times [K-1, K]$ is contained in $t$. The reduced $K$-th floor $f_K^*$ is formed by the dominoes $d \subset D$ such that $d \times [K-1, K]$ is contained in $t$. Notice that the interiors of $p_{K-1}$ and $p_K$ are disjoint, and $f_K^*$ is a tiling of $D \setminus \text{int}(p_{K-1} \cup p_K)$. Therefore, each triple $f_K$ corresponds to an oriented edge in $C_D$. Thus, by identifying each floor with a triple, the tiling $t$ is described by an oriented loop in $C_D$:

$$t = (p_0, f_1^*, p_1) * (p_1, f_2^*, p_2) * \ldots * (p_{N-1}, f_N^*, p_0).$$

In the complex $C_D$, we specify an orientation each time we move along an edge. Consider two distinct disjoint plugs $p_1$ and $p_2$ and let $f_1^*$ be a tiling of $D \setminus \text{int}(p_1 \cup p_2)$. Thus, $f_1^*$ is an edge of $C_D$. We then denote the two possible orientations of $f_1^*$ by $f = (p_1, f_1^*, p_2)$ and $f^{-1} = (p_2, f_1^*, p_1)$. Due to the connection between tilings and
paths, oriented edges in $C_D$ are also called floors. Notice that, by construction, the two possible orientations of a loop are homotopic.

Under this identification of tilings and paths, concatenation of paths in $G_D$ corresponds to concatenation of tilings. In that sense, flips correspond to homotopies between paths. Then, two tilings are equivalent under $\sim$ if and only if their corresponding paths in $C_D$ are homotopic (see Lemma 5.4 of [9]).

The even domino group $G_D^+$ is a subgroup of $G_D$. More precisely, $G_D^+$ is the kernel of the homomorphism $\psi: G_D \rightarrow \mathbb{Z}/(2)$ which takes a closed path of length $N$ to $N \mod 2$. In other words, $G_D^+$ consists of closed paths of even length. Notice that $G_D^+$ is a normal subgroup of index two of $G_D$.

The even domino group is the fundamental group of a double cover $C_D^+$ of $C_D$, i.e., $\pi_1(C_D^+) = G_D^+$. The set of vertices of $C_D^+$ is the set $\mathcal{P} \times \mathbb{Z}/(2)$, which indicates the plug and the parity of its position. Moreover, if $p_1$ and $p_2$ are two disjoint plugs then each tiling $f_1^*$ of $D \setminus (p_1 \cup p_2)$ corresponds to two edges in $C_D^+$. More specifically, for each $i \in \mathbb{Z}/(2)$, there is an edge $f_{1,i}^*$ between $(p_1, i + 1)$ and $(p_2, i)$. Therefore, $f_i = ((p_1, i + 1), f_{1,i}^*, (p_2, i))$ and $f_i^{-1} = ((p_2, i), f_{1,i}^*, (p_1, i + 1))$ define two orientations of $f_{1,i}^*$. We prefer to describe the orientation of an edge in $C_D^+$ by a pair formed by an oriented edge in $C_D$ and an element $i \in \mathbb{Z}/(2)$. The oriented edge of $C_D$ indicates the initial and the final vertex, the element of $\mathbb{Z}/(2)$ indicates the parity of the final vertex. For instance, an oriented edge $f_i = ((p_1, i + 1), f_{1,i}^*, (p_2, i))$ in $C_D^+$ is described by the pair $(f, i)$ where $f = (p_1, f_{1,i}^*, p_2)$ is an oriented edge in $C_D$. Therefore, oriented edges in $C_D^+$ are also called floors with parity. Notice that if $f_i = (f, i)$ then $f_i^{-1} = (f^{-1}, i + 1)$.

We end this section by computing the even domino group of an important family of disks. The disks in this family are irregular but not strongly irregular.

**Example 2.1.** Consider $L \geq 3$. Let $D_L$ be the quadriculated disk formed by the union of the rectangle $R_L = [0, 2L] \times [0, 1]$ and the two unit squares $s = [1, 2] \times [-1, 0]$ and $s_L = [2L - 2, 2L - 1] \times [-1, 0]$; the disk $D_3$ is shown in the top left corner of Figure 2. We claim that $G_D^+_{D_L}$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

Let $d = [1, 2] \times [-1, 1]$ and $d_L = [2L - 2, 2L - 1] \times [-1, 1]$ be dominoes. We may draw a tiling of $D_L \times [0, 2N]$ in the rectangle $[0, 2L] \times [0, 2N]$, as in Figure 11. Dominoes in $R_L \times [0, 2N]$ are represented by planar dominoes, dominoes in $(d \cup d_L) \times [0, 2N]$ are represented by unit squares, dominoes in $(s \cup s_L) \times [0, 2N]$ are omitted.
Now, with the previous paragraph in mind, it is not difficult to see that $G_{DL}^+$ is generated by tilings containing exactly two dominoes whose projections on $DL$ are equal to either $d$ or $d_L$. It follows that $G_{DL}^+$ is generated by two tilings $t_L$ and $\tilde{t}_L$ of $DL \times [0,6]$. For instance, $G_{D_3}^+$ is generated by the tilings exhibited in Figure 12. In general, the two generators of $G_{DL}^+$ are obtained from $t_3$ and $\tilde{t}_3$ by translating the dominoes contained in $((4,6] \times [0,1] \times [0,6]) \cup ([4,5] \times [-1,0] \times [0,6])$ by $(2L-6, 0, 0)$ and adding horizontal dominoes in $[4,2L-2] \times [0,1] \times [0,6]$. A long computation shows that the tilings $t_L$ and $\tilde{t}_L$ commute, i.e., $t_L \ast \tilde{t}_L \approx \tilde{t}_L \ast t_L$; we omit the details.

![Figure 12: The first row shows the tiling $t_3$ and the second row shows the tiling $\tilde{t}_3$.](image)

Given a tiling of $DL \times [0, 2N]$ we obtain another tiling by reflecting on the plane $x + y = 2L - 1$ the dominoes in $(s_L \cup ([2L-2, 2L] \times [0,1])) \times [0,2N]$; as with the two tilings in Figure 11. This construction defines an automorphism $\psi: G_{DL}^+ \to G_{DL}^+$. Then, the map $\phi: G_{DL}^+ \to Z \oplus Z$ that takes a tiling $t$ to $(Tw(t), Tw \circ \psi(t))$ is a homomorphism. We have that $\phi(t_L) = (−1, 1)$ and $\phi(\tilde{t}_L) = (1, 1)$. Thus, the image of $\phi$ is isomorphic to $Z \oplus Z$. Moreover, since $t_L$ and $\tilde{t}_L$ commute, $\phi$ is injective. Therefore, the even domino group $G_{DL}^+$ is isomorphic to $Z \oplus Z$.

3 Proof of Theorem 1

We need two results to prove Theorem 1. The first result is about random variables which assume values in the free group $F_2 = \langle a, b \rangle$. The second result discusses the properties of tilings of $D \times [0, N]$ for large $N$. 

12
Lemma 3.1. Consider \( s \in (0, \frac{1}{2}) \). Let \( (X_t)_{t \geq 1} \) be a sequence of i.i.d. random variables in \( F_2 \) such that \( \mathbb{P}(X_1 = a) = \mathbb{P}(X_1 = b) = \mathbb{P}(X_1 = a^{-1}) = \mathbb{P}(X_1 = b^{-1}) = s \) and \( \mathbb{P}(X_1 = e) = 1 - 4s \). Let \( (y_t)_{t \geq 0} \) be a sequence of elements in \( F_2 \). Then, \( \mathbb{P}(y_0X_1y_1 \ldots X_ty_t = e) \leq (1 - s(4 - \sqrt{13}))^t \) for all \( t \geq 1 \).

Proof. The result follows from two lemmas proved in detail in the Appendix. Indeed, Lemma 6.8 implies that \( \mathbb{P}(y_0X_1y_1 \ldots X_ty_t = e) \leq \mathbb{P}(X_1 \ldots X_t = e) \). In addition, Lemma 6.7 shows that \( \mathbb{P}(X_1 \ldots X_t = e) \leq (1 - s(4 - \sqrt{13}))^t \).

Let \( N_0 \in \mathbb{N} \) and consider a set of tilings \( \mathcal{B} \subseteq \mathcal{T}(\mathcal{R}_{N_0}) \). We say that a tiling \( t \) of a cork \( \mathcal{R}_{N_1,N_2;p_1,p_2} \) is formed by \((k,M)\)-blocks of \( \mathcal{B} \) if there exist distinct nonnegative integers \( b_1 < b_2 < \ldots < b_k \) and tilings \( b_1, b_2, \ldots, b_k \in \mathcal{B} \) such that, for each \( M_j = b_j(2M + N_0) + M + N_1 \), the restriction of \( t \) to \( D \times [M_j, M_j + N_0) \) equals \( b_j \).

We denote by \( \text{block}_M^k(t) \) the maximum nonnegative integer \( k \) such that \( t \) is formed by \((k,M)\)-blocks of \( \mathcal{B} \). The following lemma shows that \( \text{block}_M^k(t) \) is almost never very small.

Lemma 3.2. Consider a quadriculated disk \( D \), \( N_0 \in \mathbb{N} \) and \( \mathcal{B} \subseteq \mathcal{T}(\mathcal{R}_{N_0}) \). For \( N > N_0 \), let \( T \) be a random tiling of \( D \times [0,N] \). There exist \( M \in 2\mathbb{N} \) and constants \( C, c \in (0, 1) \) such that \( \mathbb{P}(\text{block}_M^k(T) < CN) = o(c^N) \).

Proof. Recall that \( \mathsf{p}_o \) is the empty plug. If follows from from Lemma 13 of [8] that there exist \( \epsilon > 0 \) and an even integer \( N_\epsilon > 2|D| \) such that if \( j \geq N_\epsilon \), \( \tilde{N} \geq j + N_\epsilon \) and \( p_0, p_{\tilde{N}} \in \mathcal{P} \) then for a random tiling \( \tilde{T} \) of \( \mathcal{R}_{0,N_0,p_0,p_{\tilde{N}}} \) we have \( \mathbb{P}(\text{plug}_{j}(\tilde{T}) = \mathsf{p}_o) > \epsilon \).

Let \( m = \max_{p_0,p_{\tilde{N}} \in \mathcal{P}} |\mathcal{T}(\mathcal{R}_{0,N_0,p_0,p_{\tilde{N}}})| \cdot |\mathcal{T}(\mathcal{R}_{0,N_0,N_0,p_0,p_{\tilde{N}}})| \) and \( \delta = \epsilon |\mathcal{B}| m^{-1} \). We claim that if \( \tilde{N} = 2N_\epsilon + N_0 \) then \( \mathbb{P}(\text{block}_M^{N_\epsilon}(\tilde{T}) \geq 1) > \delta \).

Since \( N_\epsilon \geq 2|D| \), by Lemma 4.1 of [9], there exist tilings \( t_1 \) of \( \mathcal{R}_{0,N_0,p_0,p_o} \) and \( t_2 \) of \( \mathcal{R}_{0,N_0,p_0,p_{\tilde{N}}} \). Thus, each tiling \( b \) in \( \mathcal{B} \) defines a tiling \( \tilde{b} = t_1 * b * t_2 \) of the cork \( \mathcal{R}_{0,N_0,p_0,p_{\tilde{N}}} \) with plug_{\tilde{N}}(\tilde{b}) = \mathsf{p}_o \) and block_{\tilde{N}}^{N_\epsilon}(\tilde{b}) = 1. Moreover, notice that the number of tilings of \( \mathcal{R}_{0,N_0,p_0,p_{\tilde{N}}} \) such that the \( N_\epsilon \)-th plug equals \( \mathsf{p}_o \) is smaller than \( m \). Therefore, \( \mathbb{P}(\text{block}_M^{N_\epsilon}(\tilde{T}) \geq 1 | \text{plug}_{\tilde{N}}(\tilde{T}) = \mathsf{p}_o) \geq |\mathcal{B}| m^{-1} \) and the claim above is proved.

Take \( M = N_\epsilon \) and \( C = \frac{\delta}{2(2M + N_0)} \). For each \( i = 1, 2, \ldots, \lfloor \frac{N}{2M + N_0} \rfloor \) let \( A_i \) be the event that the restriction of \( T \) to \( D \times [(i - 1)(2M + N_0), i(2M + N_0)] \) is formed by \((1,M)\)-blocks of \( \mathcal{B} \). Notice that the previous paragraph implies that \( \mathbb{P}(A_i | T \text{ constructed up to floor } (i - 1)(2M + N_0)) > \delta \). Therefore, we have that
\( \mathbb{P}(\text{block}^M(B) < CN) \leq \mathbb{P}(X < CN) \) where \( X \) is a random variable with binomial distribution \( \text{Bin}\left(\frac{N}{2M+N_0}, \delta\right) \). By writing \( X \) as a sum of i.i.d. random variables with Bernoulli distribution \( \text{Bern}(\delta) \), the result follows from Chernoff’s inequality.

**Proof of Theorem 1.** Let \( \phi: G_D^N \to F_2 \) be a surjective homomorphism. Consider \( N_0 \in 2\mathbb{N} \) sufficiently large so that there exist tilings \( b_a \) and \( b_b \) of \( \mathcal{R}_{N_0} \) with \( \phi(b_a) = a \) and \( \phi(b_b) = b \). Let \( b_{e_1} \) be the tiling \( t_{vtx,0} \). We obtain four distinct tilings \( b_{e_2}, b_{e_3}, b_{e_4}, b_{e_5} \) from \( b_{e_1} \) by performing exactly one vertical flip. Define the set \( \mathcal{B} = \{ b_{a}, b_{a}^{-1}, b_{b}, b_{b}^{-1}, b_{e_1}, b_{e_2}, b_{e_3}, b_{e_4}, b_{e_5} \} \subset \mathcal{T}(\mathcal{R}_{N_0}) \). Set \( s = \frac{1}{5} \) (importantly, \( s \in (0, \frac{1}{8}) \) and we will be able to use Lemma 3.1 later). We have \( |\mathcal{B} \cap \phi^{-1}(\{x\})| = s|\mathcal{B}| \) for every \( x \in \{a, a^{-1}, b, b^{-1}\} \) and \( |\mathcal{B} \cap \phi^{-1}(\{e\})| = (1 - 4s)|\mathcal{B}| \). By Lemma 3.2, there exist \( M \in 2\mathbb{N} \) and \( C \in (0, 1) \) such that the probability that \( \text{block}^{1/2}_B(T_i) \) is smaller than \( r = \lfloor CN \rfloor \) goes to zero exponentially. We may therefore assume that \( T_1 \) and \( T_2 \) are formed by at least \((r, M)\)-blocks of \( \mathcal{B} \).

Let \( \mathcal{B}_r(\mathcal{R}_N) \subset \mathcal{T}(\mathcal{R}_N) \) be the set of tilings formed by at least \((r, M)\)-blocks of \( \mathcal{B} \). For each tiling \( t \in \mathcal{B}_r(\mathcal{R}_N) \) let \( b_{1,t} < b_{2,t} < \ldots < b_{r,t} \) be the first \( r \) nonnegative integers such that, for \( j = 1, 2, \ldots, r \) and \( M_{j,t} = b_{j,t}(2M + N_0) + M \), the restriction of \( t \) to \( \mathcal{D} \times [M_{j,t}, M_{j,t} + N_0] \) equals a tiling in \( \mathcal{B} \). We now define an equivalence relation \( \approx \) on \( \mathcal{B}_r(\mathcal{R}_N) \): \( \tilde{t} \approx \hat{t} \) if and only if \( b_{j,t} = b_{j,\hat{t}} \) (for all \( j \)) and \( \tilde{t} \) equals \( \hat{t} \) in the region \( (\mathcal{D} \times [0, N]) \setminus (\bigcup_{j=1}^r \mathcal{D} \times [M_{j,\hat{t}}, M_{j,\hat{t}} + N_0]) \).

Let \( B_1, B_2, \ldots, B_r \) be the \( \approx \)-equivalence classes. Notice that, for each \( i \leq l \), there are fixed tilings \( t_{0,i}, t_{1,i}, \ldots, t_{r,i} \) such that \( B_i \) consists of all tilings of the form \( t_{0,i} \ast b_1 \ast t_{1,i} \ast b_2 \ast t_{2,i} \ast \ldots \ast b_r \ast t_{r,i} \) with \( b_1, b_2, \ldots, b_r \in \mathcal{B} \). Thus, each \( \approx \)-equivalence class has size exactly \(|\mathcal{B}|^r\).

Suppose that \( T_1 \) has been chosen first from \( \mathcal{B}_r(\mathcal{R}_N) \), say \( T_1 = t \). The probability that there exists a sequence of flips joining \( T_2 \) and \( t \) is less than or equal to the probability that \( \phi \) takes \( T_2 \ast t^{-1} \) to the identity. We prove that the later probability decays exponentially with \( N \). To this end, it suffices to show that the conditional probabilities \( \mathbb{P}(\phi(T_2 \ast t^{-1}) = e \mid T_2 \in B_i) \) are uniformly bounded by \((1 - s(4 - \sqrt{13}))^r\).

Consider a sequence \((X_t)_{t \geq 1} \) of i.i.d. random variables which assume values in \( F_2 \) such that \( \mathbb{P}(X_1 = x) = s \) for \( x \in \{a, a^{-1}, b, b^{-1}\} \) and \( \mathbb{P}(X_1 = e) = 1 - 4s \). Then, by construction, the probability \( \mathbb{P}(\phi(T_2 \ast t^{-1}) = e \mid T_2 \in B_i) \) is equal to \( \mathbb{P}(\phi(t_{0,i})X_1\phi(t_{1,i}) \ldots X_r\phi(t_{r,i} \ast t^{-1}) = e) \). The result now follows by Lemma 3.1. \( \square \)
In this section we prove Theorem 2, that is, we compute the even domino group of $\mathcal{D}_L = [0, L] \times [0, 2]$ for $L \geq 3$. The strategy consists in constructing a homomorphism from $G^+_{\mathcal{D}_L}$ to the group $G^+_L$ defined in Equation 1.1. We then prove that this homomorphism is in fact an isomorphism.

The computation of $G^+_{\mathcal{D}_L}$ is inspired by the results of [6], where a flip invariant for tilings of duplex regions is exhibited. By performing a rotation, we think of tilings of $\mathcal{D}_L \times [0, N]$ as tilings of the duplex region $[0, N] \times [0, L] \times [0, 2]$. Therefore, in this section, we say that a flip is horizontal if it is performed in two dominoes contained in one of the two floors of the new rotated tiling; the flip is vertical otherwise.

Let $\mathbf{t}$ be a tiling of $\mathcal{D}_L \times [0, N]$ and orient each domino contained in $\mathbf{t}$ from its white unit cube to its black unit cube. By projecting the two floors of $\mathbf{t}$ on the plane $z = 0$, we obtain a diagram $\mathcal{I}_{\mathbf{t}}$ on $[0, N] \times [0, L]$ containing oriented disjoint cycles and jewels, i.e., unit squares formed by the projections of dominoes parallel to the $z$-axis. A cycle is trivial if it has length two and a jewel is trivial if it is not enclosed by a cycle. The color of a jewel $j$ is defined as the color of its corresponding unit square in the rectangle $[0, N] \times [0, L]$. We write $\text{color}(j) = +1$ if $j$ is white and $\text{color}(j) = -1$ if $j$ is black. The Figure 13 shows an example of a tiling and its associated diagram; we always exhibit trivial cycles in green, counterclockwise cycles in red and clockwise cycles in blue.

![Diagram](image)

Figure 13: The first row shows a tiling $\mathbf{t}$ of $\mathcal{D}_5 \times [0, 6]$. The second row shows $\mathbf{t}$ after a rotation and its diagram $\mathcal{I}_{\mathbf{t}}$.

We order the jewels contained in $\mathcal{I}_{\mathbf{t}}$. Consider two jewels $j_1 = [a, a+1] \times [b, b+1]$ and $j_2 = [c, c+1] \times [d, d+1]$. If $j_1$ and $j_2$ are in different columns, i.e. $a \neq c$, we write $j_1 < j_2$ if $a < c$. If $j_1$ and $j_2$ are in the same column, i.e. $a = c$, we write $j_1 < j_2$ if $b > d$. When this order is used, jewels are called ordered jewels.
For a jewel \( j \) let \( \text{wind}(j) \) be the sum of the winding numbers \( \text{wind}(j, \gamma) \) taken over all the cycles \( \gamma \) in \( I_t \). Notice that \( \text{wind}(j) \) is an integer and \( |\text{wind}(j)| \leq \lfloor \frac{L-1}{2} \rfloor \).

We are especially interested in the winding numbers of jewels contained in the same column. Consider the finite set

\[
R_L = \{(m, n) \in \mathbb{Z}^2 \colon \max\{|m|, |n|, |m - n|\} < \lfloor \frac{L}{2} \rfloor \}.
\]

The Figure 14 below shows the elements of \( R_{10} \) and two tilings of \( D_{10} \times [0, 12] \), notice that in the figure any two jewels \( j_1 \) and \( j_2 \) contained in the same column are such that \( \text{wind}(j_1), \text{wind}(j_2) \in R_{10} \).

![Figure 14: The square lattice with elements of \( R_{10} \) shown in black, and two diagrams of tilings of \( D_{10} \times [0, 12] \).](image)

**Lemma 4.1.** Consider \( L \geq 3 \). Then \((m, n) \in R_L\) if and only if there exist \( N \in \mathbb{N} \) and a tiling \( t \) of \( D_L \times [0, 2N] \) such that \( I_t \) contains two jewels \( j_1 \) and \( j_2 \) in the same column with \( \text{wind}(j_1), \text{wind}(j_2) = (m, n) \).

**Proof.** We first prove the if direction. Suppose that \( j_1 = [a, a + 1] \times [b, b + 1] \) and \( j_2 = [a, a + 1] \times [c, c + 1] \) with \( b > c \). Since \( \text{wind}(j_1) = m \) there exists at least \( |m| \) unit squares in \([a, a + 1] \times [b + 1, L]\). Analogously, there exists at least \( |n| \) unit squares in \([a, a + 1] \times [0, c]\). Moreover, we must have at least \( |m - n| \) cycles not enclosing both jewels. Therefore, there exists at least \( |m - n| \) unit squares in \([a, a + 1] \times [b + 1, c]\). Then, \( |m| + |n| + |m - n| \leq L - 2 \) and we have \( \max\{|m|, |n|, |m - n|\} \leq \lfloor \frac{L}{2} \rfloor \).

For the only if direction let \((m, n) \in R_L\) and take \( N \) sufficiently large. In order to show the existence of a tiling \( t \) of \( D_L \times [0, 2N] \) with the desired properties we proceed backwards. Indeed, since a tiling is entirely determined by its diagram, it suffices to construct \( I_t \).

We first deal with the case in which \( |m| + |n| < \lfloor \frac{L}{2} \rfloor \). Consider two disjoint squares centered in the same column: \( s_m \) of side \( 2|m| + 1 \) and \( s_n \) of side \( 2|n| + 1 \).

16
Let the jewel $j_1$ (resp. $j_2$) be the center of $s_m$ (resp. $s_n$). Construct $|m|$ cycles in $s_m$ and $|n|$ cycles in $s_n$ such that $\text{wind}(j_1) = m$ and $\text{wind}(j_2) = n$. Now, to obtain $\mathcal{I}_k$, fill the rest of $[0, 2N] \times [0, L]$ with trivial cycles and trivial jewels.

We are left with the case $|m| + |n| \geq \lfloor \frac{L}{2} \rfloor$, so that $\text{sign}(m) = \text{sign}(n)$. Suppose that $|m| \geq |n|$ and write $|n| = |m| - r$, where $0 \leq r \leq |m|$. The square $s = [0, 2|m| + 2]^2$ is contained in $[0, 2N] \times [0, L]$, since $|m| < \lfloor \frac{L}{2} \rfloor$. Let $j_1 = [|m|, |m| + 1] \times [|m| + 1, |m| + 2]$ and $j_2 = [|m|, |m| + 1] \times [|m| - r, |m| - r + 1]$. Construct $m$ cycles in $s$ such that $\text{wind}(j_1) = m$. We have $\text{wind}(j_2) = n$, as $\text{sign}(m) = \text{sign}(n)$. The result then follows by proceeding as in the previous paragraph.

Recall that, as in Equation 1.1, $G^+_L = \langle S_L \mid [a_m, a_n] = 1 \text{ for } (m, n) \in R_L \rangle$, where $S_L = \{a_i : i \in \mathbb{Z}_{\neq 0} \text{ and } |i| \leq \lfloor \frac{L-1}{2} \rfloor \}$. We construct a map

$$\Phi: \bigcup_{N \geq 1} \mathcal{T}(\mathcal{D}_L \times [0, 2N]) \rightarrow G^+_L.$$ 

Consider a tiling $\mathbf{t}$ of $\mathcal{D}_L \times [0, 2N]$ and let $j_1 < j_2 < \ldots < j_k$ be the ordered jewels in $\mathcal{I}_k$. Define $\Phi(\mathbf{t}) = b_1 \ldots b_k$ where $b_i = a_{\text{color}(j_i)}^{\text{wind}(j_i)}$ if $\text{wind}(j_i) \neq 0$ and $b_i = e$ if $\text{wind}(j_i) = 0$.

**Lemma 4.2.** The map $\Phi$ induces a homomorphism $\phi: G^+_{\mathcal{D}_L} \rightarrow G^+_L$.

**Proof.** It suffices to check that $\Phi$ is invariant under flips. Let $\mathbf{t}$ be a tiling of $\mathcal{D}_L \times [0, 2N]$. Consider a horizontal flip performed in two dominoes $d_1$ and $d_2$. The horizontal flip either connects two disjoint cycles or disconnects a cycle into two cycles. Suppose the former, the other case is similar. Therefore, $d_1$ and $d_2$ are contained in distinct cycles. If either $d_1$ or $d_2$ is contained in a trivial cycle then it is easy to see that the horizontal flip does not change the winding number of any jewel. Then, suppose that $d_1$ and $d_2$ are contained in nontrivial cycles, Figure 15 below shows an example of the possible cases.

**Figure 15:** Two tilings and the effect of a horizontal flip (highlighted in magenta) on their diagrams.
If $d_1$ and $d_2$ are contained in cycles having the same orientation then the flip connects the two cycles preserving the orientation. If $d_1$ and $d_2$ are contained in cycles having opposite orientations then one cycle must be enclosed by the other. The flip then creates a cycle with the same orientation as the outer cycle. Moreover, the new cycle encloses only jewels enclosed by the outer cycle but not by the inner cycle. Therefore, in any of the possible cases, the flip preserves the winding number of the jewels. Then, $\Phi$ is invariant under horizontal flips.

Consider a vertical flip that takes a trivial cycle to two adjacent jewels (i.e., jewels whose corresponding unit squares are adjacent); a similar analysis holds for the reverse of this flip. The flip creates adjacent jewels $j$ and $j'$ such that $j < j'$, $\text{wind}(j) = \text{wind}(j')$ and $\text{color}(j) \neq \text{color}(j')$. If $j$ and $j'$ are in the same column then the flip clearly preserves the value of $\Phi(t)$. Otherwise, the definition of $G_L^+$ and Lemma 4.1 imply that the contributions of jewels between $j$ and $j'$ commute so that $\Phi$ is invariant under vertical flips.

Our objective is to prove that the homomorphism $\phi$ obtained above is an isomorphism. To achieve this, we now study the even domino group $G_{DL}^+$. We follow [6] to derive a family of generators of $G_{DL}^+$. A tiling $t$ of $D_L \times [0, N]$ is called a boxed tiling if its corresponding diagram $I_t$ is composed of a nontrivial jewel $j$ and trivial jewels outside the square of center $j$ and side $2|\text{wind}(j)| + 1$.

We prefer to work with boxed tilings due to some helpful properties. Notably, we can move via flips the nontrivial jewel of a boxed tiling so that the resulting tiling is a boxed tiling as well. Specifically, consider two boxed tilings $t$ and $\tilde{t}$ of $D_L \times [0, N]$. Let $j$ and $\tilde{j}$ be the nontrivial jewels of $I_t$ and $I_{\tilde{t}}$, respectively. If $\text{color}(j) = \text{color}(\tilde{j})$ and $\text{wind}(j) = \text{wind}(\tilde{j})$ then $t \approx \tilde{t}$. For instance, Figure 16 shows the process of moving a nontrivial jewel with winding number equals 1. The extension to other cases follows inductively, by initially transforming the outer cycle into a rectangle through a sequence of flips.

![Figure 16: The process of moving a nontrivial jewel via a sequence of flips.](image-url)
The family of boxed tilings generates the even domino group $G^+_{DL}$. Indeed, every tiling of a cylinder $D_L \times [0, 2N]$ is ~-equivalent to a concatenation of boxed tilings.

**Lemma 4.3.** Let $t$ be a tiling of $D_L \times [0, 2N]$. Then, there exist $M \in \mathbb{N}$ and boxed tilings $t_1, \ldots, t_k$ of $D_L \times [0, 2M]$ such that $t \sim t_1 \ast \ldots \ast t_k$.

**Proof.** This result is proved, in Lemma 7.4 of [6], for diagrams in $\mathbb{Z}^2$ instead of $[0, 2N] \times [0, L]$. However, the same proof holds in our setting, since the relation $\sim$ allows us to assume that $N$ is arbitrarily large. To avoid repetition, we do not provide further details. \hfill \Box

We now investigate relations between boxed tilings. The lemma below shows that, under specific conditions, two boxed tilings commute with respect to concatenation. Notice that the particular case, where the nontrivial jewels of the two boxed tilings share the same color and winding number, follows from the fact that we can move nontrivial jewels.

**Lemma 4.4.** Consider boxed tilings $t_1$ of $D_L \times [0, 2N_1]$ and $t_2$ of $D_L \times [0, 2N_2]$. Let $j_1$ and $j_2$ be the nontrivial jewels in $I_{t_1}$ and $I_{t_2}$, respectively. If $\text{color}(j_1) = \text{color}(j_2)$ and $(\text{wind}(j_1), \text{wind}(j_2)) \in R_L$ then $t_1 \ast t_2 \approx t_2 \ast t_1$.

**Proof.** Let $(m, n) = (\text{wind}(j_1), \text{wind}(j_2))$, we focus on the diagram $I_{t_1 \ast t_2}$. We consider two cases: $|m| + |n| < \lfloor \frac{L}{2} \rfloor$ and $|m| + |n| \geq \lfloor \frac{L}{2} \rfloor$. First suppose the former. This case is a matter of moving the nontrivial jewels (as in Figure 16), we proceed in three steps. Initially, move $j_1$ to a jewel $\tilde{j}_1$ in $[0, 2N_1] \times [2|n| + 1, L]$ and $j_2$ to a jewel $\tilde{j}_2$ in $[2N_1, 2(N_1 + N_2)] \times [0, 2|n| + 1]$. Since $L \geq 2(|m| + |n| + 1)$, we can then move $\tilde{j}_1$ to a jewel $\overline{j}_1$ in $[2N_2, 2(N_1 + N_2)] \times [2|n| + 1, L]$ and $\tilde{j}_2$ to a jewel $\overline{j}_2$ in $[0, 2N_2] \times [0, 2|n| + 1]$. Finally, move $\overline{j}_1$ (resp. $\overline{j}_2$) in $[2N_2, 2(N_1 + N_2)] \times [0, L]$ (resp. $[0, 2N_2] \times [0, L]$) to obtain copies of $t_1$ and $t_2$. For instance, Figure 17 shows a particular case ($L = 6$ and $(m, n) = (1, -1)$) of the general idea.

![Figure 17: The diagram $I_{t_1 \ast t_2}$ and the effect of three sequences of flips.](image)

We have $R_3 = \{(0, 0)\}$ and $R_4 = R_5 = \{(0, 0), (\pm 1, 0), (0, \pm 1), (1, 1), (-1, -1)\}$. The previous paragraph cover all cases except $(m, n) \in \{(1, 1), (-1, -1)\}$ for $L = 4$.
and $L = 5$. However, in any of these two cases, the nontrivial jewels share the same color and winding number, so that $t_1 \approx t_2$. Therefore, the result holds for $L = 3, 4, 5$.

If $|m| + |n| \geq \left\lceil \frac{L}{2} \right\rceil$ then $\text{sign}(m) = \text{sign}(n)$. The proof follows by induction on $L$. Perform a sequence of flips that takes the largest cycle which encloses $j_1$ to the cycle $\gamma_1$ which encloses the region $[1, 2N_1 - 1] \times [1, L - 1]$. Similarly, enlarge the largest cycle which encloses $j_2$ to obtain the cycle $\gamma_2$ which encloses the region $[2N_1 + 1, 2N_1 + 2N_2 - 1] \times [1, L - 1]$. Since $\text{sign}(m) = \text{sign}(n)$ there exists a flip that connects $\gamma_1$ and $\gamma_2$ into a cycle $\gamma$; as before, enlarge $\gamma$ to obtain a cycle which encloses the region $[1, 2N_1 + 2N_2 - 1] \times [1, L - 1]$.

Now, the winding number of $j_1$ and $j_2$, restricted to $[1, 2N_1 + 2N_2 - 1] \times [1, L - 1]$, is equal to $m - \text{sign}(m)$ and $n - \text{sign}(n)$, respectively. Notice that $(m - \text{sign}(m), n - \text{sign}(n)) \in R_{L-2}$. Then, by the induction hypothesis, there exists a sequence of flips which commutes the nontrivial jewels in $[1, 2N_1 + 2N_2 - 1] \times [1, L - 1]$. Finally, undo the flips of the previous paragraph to conclude that $t_1 \ast t_2 \approx t_2 \ast t_1$. \hfill $\square$

**Proof of Theorem 2.** We prove that the homomorphism $\phi : G_{D_L}^+ \to G_L^+$ of Lemma 4.2 is an isomorphism. Indeed, we obtain a map $\psi : G_L^+ \to G_{D_L}^+$ such that $\psi^{-1} = \phi$. To this end, we first define a homomorphism $\Psi : F(S_L) \to G_{D_L}^+$ from the free group generated by $S_L$ to $G_{D_L}^+$.

Consider a nonzero integer $|i| \leq \left\lfloor \frac{L-1}{2} \right\rfloor$. Let $t_i$ be the boxed tiling, of the cylinder $D_L \times [0, 2(|i| + 1)]$, such that the nontrivial jewel $j_i$ in $I_{t_i}$ is the center of the square $[0, 2|i| + 1]^2$ and $\text{wind}(j_i) = i$. Let $\Psi$ be the homomorphism such that $\Psi(a_i) = t_i$. Then, it follows from Lemma 4.4 that $\Psi(a_m a_n a_m^{-1} a_n^{-1}) = e$ for $(m, n) \in R_L$. Thus, $\Psi$ induces a homomorphism $\psi : G_L^+ \to G_{D_L}^+$.

By definition $\phi(t_i) = a_i$, so that $\psi \circ \phi$ equals the identity map. Now, consider an arbitrary tiling $t$ of $D_L \times [0, 2N]$. By Lemma 4.3, $t$ is $\sim$-equivalent to a concatenation of boxed tilings. Moreover, we know that two boxed tilings whose nontrivial jewels share the same color and winding number are also $\sim$-equivalent. Thus, there exists $i_1, i_2, \ldots, i_k \in \mathbb{Z}$ such that $t \sim t_{i_1}^{e_1} \ast t_{i_2}^{e_2} \ast \ldots \ast t_{i_k}^{e_k}$ for some $e_1, e_2, \ldots, e_k = \pm 1$. Then, $\phi(t) = \phi(t_{i_1}^{e_1} \ast t_{i_2}^{e_2} \ast \ldots \ast t_{i_k}^{e_k}) = a_{i_1}^{e_1} a_{i_2}^{e_2} \ldots a_{i_k}^{e_k}$, and therefore $\psi \circ \phi(t) = t$. \hfill $\square$

**Remark 4.5.** We are able to compute the domino group $G_{D_L}$ once we compute the even domino group $G_{D_L}^+$. Indeed, consider a tiling of $D_L \times [0, 1]$ so that it defines an element of order 2 in $G_{D_L}$ which generates the subgroup $H$. Notice that
every element in $G_{DL}$ is a product of an element in $G^{+}_{DL}$ and an element in $H$. Thus, the domino group is isomorphic to the inner semidirect product of $G^{+}_{DL}$ and $H$. More specifically, let $\psi: \mathbb{Z}/(2) \to \text{Aut}(G^{+}_{L})$ be the homomorphism defined by $\psi(1)(a_i) = a_i^{-1}$ for each $a_i \in S_L$. Therefore, the semidirect product $\mathbb{Z}/(2) \ltimes_\psi G^{+}_{L}$ is isomorphic to the domino group $G_{DL}$. ♦

5 Strongly irregular disks

The proofs of Theorems 3, 4 and 5 are very similar. We construct a surjective homomorphism $\phi: G^{+}_{DL} \to F_2$, where $F_2$ is the free group generated by $a$ and $b$. The map $\phi$ is first defined for oriented edges (floors with parity) in $C^{+}_{DL}$. We then check that the boundary of any 2-cell is mapped to the identity and conclude that $\phi$ extends to a homomorphism from the even domino group $G^{+}_{DL}$ to $F_2$.

In order to fix the ideas, consider the Example 5.1 below of a strongly irregular disk. In this example, we exhibit the only disk that we know of whose strong irregularity does not follow from Theorems 3, 4 and 5.

Example 5.1. Let $D$ be the first disk in the bottom row of Figure 2, i.e., the quadriculated disk formed by the union of the rectangle $[0, 4] \times [0, 1]$ and the two unit squares $[1, 2] \times [-1, 0]$ and $[2, 3] \times [1, 2]$. We construct an isomorphism $\phi: G^{+}_{D} \to F_2$.

Let $d = [1, 3] \times [0, 1]$ be a domino. Consider the plugs $p_0 = [0, 1]^2 \cup ([2, 3] \times [1, 2])$ and $p_1 = p_0^{-1} \setminus d$. Then, $f = (p_0, \{d\}, p_1)$ is an oriented edge of the complex $C_D$. This edge defines four oriented edges in $C^{+}_D$, i.e., floors with parity: $f_0 = (f, 0)$, $f_1 = (f^{-1}, 0)$, $f_0^{-1} = (f^{-1}, 1)$ and $f_1^{-1} = (f, 1)$.

We now define $\phi$ for oriented edges in $C^{+}_D$. Set $\phi(f_0) = a$, $\phi(f_1) = b$, $\phi(f_0^{-1}) = a^{-1}$ and $\phi(f_1^{-1}) = b^{-1}$; all other edges are mapped to the identity. Notice that neither $f_0$ nor $f_1$ is part of the boundary of a 2-cell in $C^{+}_D$. Therefore, $\phi$ takes the boundary of any 2-cell in $C^{+}_D$ to the identity. Thus, $\phi$ extends to a homomorphism from $G^{+}_{D}$ to $F_2$. We obtain the generators $t$ and $\tilde{t}$ of $G^{+}_{D}$, shown in Figure 18, as with Example 2.1. Since $\phi(t) = a$ and $\phi(\tilde{t}) = b$, the map $\phi$ is an isomorphism. ♦
For the proofs of Theorems 3, 4 and 5 we also construct a surjective homomorphism $\phi$. However, in contrast with the example above, the proof of the surjectivity of $\phi$ is based on an algorithm to construct, for each plug $p \in \mathcal{P}$, a tiling $t_p$ of $\mathcal{R}_{0,|p|;p,p}$. We now describe this algorithm (see Lemma 4.3 of [9]).

Let $\mathcal{D}$ be a disk with a spanning tree $T$ and a plug $p \in \mathcal{P}$. For any two unit squares $s$ and $\tilde{s}$ in $\mathcal{D}$ define their distance $d(s, \tilde{s})$ as the length of the path in $T$ connecting them. Consider a sequence of plugs $p_0, p_1, \ldots, p_{|p|}$ such that $p_0 = p$ and $p_{i+1}$ is obtained from $p_i$ by removing two unit squares of opposite colors $s_i$ and $\tilde{s}_i$ at minimal distance. Let $\gamma_i = (s_1, s_{i1}, s_{i2}, \ldots, s_{i|d(s_i, \tilde{s}_i)|}, \tilde{s}_i)$ be the path in $T$ joining $s_i$ and $\tilde{s}_i$. Therefore, each term of $\gamma_i$ corresponds to an unit square in $\mathcal{D}$ and any two consecutive terms correspond to a domino in $\mathcal{D}$. Notice that, since $s_i$ and $\tilde{s}_i$ are of opposite colors, $\gamma_i$ has an even number of vertices.

Define the two reduced floors $f^*_1 = \{s_{i1j-1} \cup s_{i1j} : j = 1, 2, \ldots, \frac{d(s_i, \tilde{s}_i)-1}{2}\}$ and $f^*_{2i+1} = \{s_i \cup s_1 \} \cup \{s_{i2j} \cup s_{i2j+1} : j = 1, 2, \ldots, \frac{d(s_i, \tilde{s}_i)-3}{2}\} \cup \{s_{i\frac{d(s_i, \tilde{s}_i)-1}{2}} \cup \tilde{s}_i\}$. In other words, $f^*_i$ is formed by the consecutive dominoes along the path $\gamma_i \setminus (s_i \cup \tilde{s}_i)$ and $f^*_{2i+1}$ is formed by the consecutive dominoes along $\gamma_i$. Consider the floors $f_2i = (p_i, f^*_1, p_{i-1} \setminus f^*_1)$ and $f_{2i+1} = (p_{i-1} \setminus f^*_1, f^*_2, f^*_{2i+1}, p_{2i+1})$. The tiling $t_p$ is described by the sequence

$$t_p = f_0 \ast f_1 \ast \ldots \ast f_{|p|}.$$ 

The important fact is that the projection on $\mathcal{D}$ of every horizontal domino in $t_p$ is an edge of $T$. The Figure 19 shows an example of the construction of $t_p$ for the disk $\mathcal{D} = [0, 4]^2$.

![Figure 19: The disk $\mathcal{D} = [0, 4]^2$ with a spanning tree, a plug $p$ and $t_p$.](image)

**Proof of Theorem 3.** We first consider the case in which $\mathcal{D} \setminus s$ has exactly three connected components $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$. Suppose $|\mathcal{D}_1| \leq |\mathcal{D}_2| \leq |\mathcal{D}_3|$ so that, by hypothesis, $|\mathcal{D}_1 \cup \mathcal{D}_2| \geq 3$. For $i \in \{1, 2, 3\}$ let $s_i \subset \mathcal{D}_i$ be a unit square adjacent to $s$. Moreover, let $d_1 = s \cup s_1$, $d_2 = s \cup s_2$ and $d_3 = s \cup s_3$ be three dominoes.

We define two classes of floors $\mathcal{F}_0$ and $\mathcal{F}_1$. A floor $(p_0, f^*, p_1)$ belongs to $\mathcal{F}_0$ if and only if:

22
1. \( f^* \) contains the domino \( d_1 \).

2. \( p_0 \) marks all white unit squares in \( D_1 \setminus s_1 \) and all black unit squares in \( D_2 \).

3. \( p_1 \) marks all black unit squares in \( D_1 \setminus s_1 \) and all white unit squares in \( D_2 \).

A floor belongs to \( \mathcal{F}_1 \) if and only if its inverse belongs to \( \mathcal{F}_0 \); Figure 20 shows an example of a disk and its classes \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \). This defines four classes of floors with parity:

\[
\begin{align*}
  f_0 &= (\mathcal{F}_0, 0), \\
  f_1 &= (\mathcal{F}_1, 0), \\
  f_0^{-1} &= (\mathcal{F}_1, 1) \text{ and } f_1^{-1} &= (\mathcal{F}_0, 1).
\end{align*}
\]

Figure 20: A disk and its two classes of floors \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \).

We initially define \( \phi \) for oriented edges in \( C_D^+ \). If \( f \) is a floor with parity not contained in the classes defined above, let \( \phi(f) = e \). Otherwise, set \( \phi(f_0) = a \), \( \phi(f_0^{-1}) = a^{-1} \) and \( \phi(f_1^{-1}) = b^{-1} \).

Consider two adjacent floors with parity in \( C_D^+ \) (i.e. a oriented path of length two) whose reduced floors contain the domino \( d_1 \). We have only two possibilities. First, both floors are neither in \( \mathcal{F}_0 \) nor in \( \mathcal{F}_1 \). Second, either the first floor is in \( \mathcal{F}_0 \) and the second floor is in \( \mathcal{F}_1 \) or vice-versa. Since adjacent edges have opposite parity, in any case we conclude that \( \phi \) maps this path of length two to the identity. With this observation in mind, it is straightforward to check that \( \phi \) maps the boundary of any 2-cell to the identity. Therefore, \( \phi \) extends to a homomorphism from \( G_D^+ \) to \( F_2 \).

We now prove the surjectivity of \( \phi \). Suppose, without loss of generality, that \( s \) is a white unit square. Let \( s_3 \subset D_3 \) be a unit square adjacent to \( s_3 \). Consider a floor \( f = (p_0, \{d_1\}, p_1) \) in \( \mathcal{F}_0 \) such that \( p_1 = p_0^{-1} \setminus d_1 \). We may assume that \( s_3 \not\subset p_0 \) and \( \tilde{s}_3 \subset p_0 \). Notice that the two unit squares of opposite colors in \( p_1 \) at minimal distance are contained in \( D_2 \cup D_3 \). Then, by construction, the floors of \( t_{p_1} \) are contained neither in \( \mathcal{F}_0 \) nor in \( \mathcal{F}_1 \). Let \( t = (p_1 \setminus s_3) \cup s_1 \) be a plug and \( g = (p, \{d_3\}, p_0) \) be a floor. Analogously, the floors of \( t_p \) are contained neither in \( \mathcal{F}_0 \) nor in \( \mathcal{F}_1 \). Therefore, \( t = t_p^{-1} * g * f * t_{p_1} \) is a tiling of \( D \times [0, 2|p_1| + 2] \) such that \( \phi(t) = a \). Similarly, consider the plug \( \tilde{p} = (p_0 \setminus s_2) \cup s_1 \) and the floor \( \tilde{g} = (\tilde{p}, \{d_2\}, p_1) \). Then, the tiling \( \tilde{t} = \tilde{t}_p^{-1} * \tilde{g} * f^{-1} * t_{p_0} \) of \( D \times [0, 2|p_0| + 2] \) is such that \( \phi(\tilde{t}) = b \).

Now, consider the case in which there exists a fourth connected component \( D_4 \). By possibly relabeling the components assume that \(|D_1| \leq |D_2| \leq |D_3| \leq |D_4|\).
If $|D_3| > 1$ proceed as in the previous case: define two classes containing floors $(p_0, f^*, p_1)$ such that $d_1 \subset f^*$ and the plugs $p_0$ and $p_1$ mark alternately the unit squares in $D_1 \cup D_2 \cup D_3$.

Suppose that $|D_1| = |D_2| = |D_3| = 1$. We define two classes of floors $F_0$ and $F_1$. A floor $(p_0, f^*, p_1)$ belongs to $F_0$ if and only if $d_1 \subset f^*$, $s_2 \subset p_0$ and $s_3 \subset p_1$. A floor belongs to $F_1$ if and only if its inverse belongs to $F_0$; see Figure 21. As in the previous case, we have four classes with parity and a homomorphism $\phi: G_D^+ \to F_2$.

![Figure 21: A disk and its two classes of floors $F_0$ and $F_1$.](image)

Let $f = (p_0, \{d_1\}, p_1)$ be a floor in $F_0$ such that $p_1 = p_0^{-1} \setminus d_1$. Let $p = (p_1 \setminus s_3) \cup s_1$ be a plug and $g = (p, \{d_3\}, p_0)$ be a floor. Then, by construction, the floors of $t_p$ and $t_p$ are contained neither in $F_0$ nor in $F_1$. Thus, for $t = t_p^{-1} * g * f * t_{p_1}$ we have $\phi(t) = a$. Analogously, we obtain a tiling $\tilde{t}$ such that $\phi(\tilde{t}) = b$. □

**Proof of Theorem 4.** The proof consists of two cases. We define, in both cases, two classes of floors $F_0$ and $F_1$. The class $F_1$ contains a floor if and only if its inverse belongs to $F_0$, so that it suffices to define $F_0$.

First consider the case in which there exists only one $2 \times 2$ square containing $d$; denote this square by $s$. Let $D_0$ be the connected component of $D \setminus d$ that intersects $s$. In this case, a floor $(p_0, f^*, p_1)$ belongs to $F_0$ if and only if $d \subset f^*$ and $p_0$ (resp. $p_1$) marks all black (resp. white) unit squares in $D_0$; as in Figure 22.

![Figure 22: A disk and its two classes of floors $F_0$ and $F_1$.](image)

The second case is based on the existence of two distinct $2 \times 2$ squares $s_1, s_2 \subset D$ such that $d \subset s_1$ and $d \subset s_2$. Let $D_1$ (resp. $D_2$) be the connected component of $D \setminus s$ that intersects $s_1$ (resp. $s_2$). Suppose that $|D_1| \leq |D_2|$. In this case, a floor $(p_0, f^*, p_1)$ belongs to $F_0$ if and only if:

1. $f^*$ contains the domino $d$. 

24
2. $p_0$ marks all black squares in $D_1$ and all white squares in $D_2$.

3. $p_1$ marks all white squares in $D_1$ and all black squares in $D_2$.

Figure 23: A disk and its two classes of floors $F_0$ and $F_1$.

Notice that the classes of floors are nonempty. Indeed, by hypothesis, in the first case $|D \setminus d| - |D_0| \geq |D_0|$ and in the second case $|D \setminus d| - |D_1| - |D_2| \geq |D_2| - |D_1|$. Therefore, since $D$ is balanced, in both cases there exist plugs satisfying the required properties.

As in the proof of Theorem 3, we have four floors with parity which define $\phi$ for oriented edges in $C^+_D$. By construction, in a floor $(p_0, f^*, p_1)$ of class either $F_0$ or $F_1$, each connected component of $D \setminus d$ that intersects a $2 \times 2$ square which contains $d$ is marked alternately by $p_0$ and $p_1$. Then, again as in the proof of Theorem 3, it is not difficult to check that $\phi$ takes the boundary of any 2-cell to the identity. Thus, $\phi$ extends to a homomorphism from $G^+_D$ to $F_2$.

We now prove that $\phi$ is surjective. Consider a floor $f = (p_0, \{d_1\}, p_1)$ in $F_0$ such that $p_1 = p_0^{-1} \setminus d_1$. Notice that, since $d$ is contained in a $2 \times 2$ square, there exists a spanning tree of $D$ whose set of edges does not contain $d$. Then, by definition, the tilings $\tilde{t} = t^{-1} \ast f \ast (p_1, \emptyset, p_1^{-1}) \ast t^{-1}$ and $\tilde{t} = t^{-1} \ast f^{-1} \ast (p_0, \emptyset, p_0^{-1}) \ast t^{-1}$ are such that $\phi(\tilde{t}) = a$ and $\phi(\tilde{t}) = b$.

**Proof of Theorem 5.** We define four classes of floors: $F_0$, $F_1$, $F_2$, and $F_3$. Let $F_0$ be the class of all floors $(p_0, f^*, p_1)$ such that $d_1 \subseteq f^*$ and $p_0$ (resp. $p_1$) mark all black (resp. white) unit squares in $D_1$. A floor belongs to the class $F_1$ if and only if its inverse belongs to $F_0$, i.e., $F_1 = F_0^{-1}$. Let $F_2$ be the class of all floors $(p_0, f^*, p_1)$ such that $d_2 \subseteq f^*$ and $p_0$ (resp. $p_1$) mark all black (resp. white) unit squares in $D_2$. Finally, let $F_3 = F_2^{-1}$.

The Figure 24 below shows an example of a disk, with $d_1$ and $d_2$ not parallel, and its four classes of floors.
Figure 24: A disk \( \mathcal{D} \), with \( d_1 \) and \( d_2 \) marked by a red line segment, and its 4 classes.

If \( d_1 \) and \( d_2 \) are parallel then the intersections \( \mathcal{F}_0 \cap \mathcal{F}_2 \) and \( \mathcal{F}_1 \cap \mathcal{F}_3 \) are not empty, as shown in Figure 25.

Figure 25: The first row shows a disk \( \mathcal{D} \), with \( d_1 \) and \( d_2 \) marked by a red line segment, and its 4 classes. The second row shows floors which are contained in the intersection of two classes.

We first define \( \phi \) for oriented edges in \( \mathcal{C}_D^+ \). Let \( f = (f, k) \) be a floor with parity. If \( f \) belongs to either two or none of the classes constructed above, set \( \phi(f) = e \). The image of the floors which are not taken to the identity depends on whether \( d_1 \) and \( d_2 \) are parallel. Suppose that \( f \) belongs to the class \( \mathcal{F}_j \) for some \( j = 0, 1, 2, 3 \). If \( d_1 \) and \( d_2 \) are not parallel define \( \phi(f) \) as

\[
\begin{array}{c|cccccccc}
(j,k) & (0,0) & (1,0) & (2,0) & (3,0) & (0,1) & (1,1) & (2,1) & (3,1) \\
\hline
\phi(f) & a & b & b & a & b^{-1} & a^{-1} & b^{-1} & \\
\end{array}
\]

Otherwise, if \( d_1 \) and \( d_2 \) are parallel, define \( \phi(f) \) as

\[
\begin{array}{c|cccccccc}
(j,k) & (0,0) & (1,0) & (2,0) & (3,0) & (0,1) & (1,1) & (2,1) & (3,1) \\
\hline
\phi(f) & a & b & a^{-1} & b^{-1} & b^{-1} & a^{-1} & b & a \\
\end{array}
\]

In both cases, a careful analysis shows that \( \phi \) takes the boundary of any 2-cell to \( e \). Therefore, \( \phi \) extends to a homomorphism from the even domino group \( G_D^+ \) to \( F_2 \). The surjectivity of \( \phi \) follows as in the proof of Theorem 4.

Remark 5.2. In many cases the homomorphisms constructed above are not isomorphisms. For instance, consider the thin rectangles \( \mathcal{D}_L = [0, L] \times [0, 2] \). Then, from the computation of \( G_{\mathcal{D}_L}^+ \) provided in Section 4, it follows that the map \( \phi \) constructed in the proof of Theorem 4 (resp. Theorem 5) is an isomorphism between \( G_{\mathcal{D}_3}^+ \) (resp. \( G_{\mathcal{D}_4}^+ \)) and \( F_2 \). However, for \( L > 4 \) these maps have nontrivial kernel.
More generally, the theorems are based on the existence of either a unit square \( s \) or a domino \( d \) that disconnects \( D \). If one of the connected components of either \( D \setminus s \) or \( D \setminus d \) contains a \( 3 \times 2 \) rectangle then there exists a tiling \( t \) of \( D \times [0, 4] \) such that \( \phi(t) = e \) and \( Tw(t) = 1 \); the kernel of \( \phi \) then contains a nontrivial element. The tiling \( t \) is formed by taking a tiling of the \( 3 \times 2 \times 4 \) box, as in Figure 26, and vertical dominoes outside the box.

\[
\begin{align*}
&\begin{array}{c}
\includegraphics[width=0.2\textwidth]{3x2x4_tiling}
\end{array}
\end{align*}
\]

**Figure 26:** Tiling of a \( 3 \times 2 \times 4 \) box.

### 6 Appendix

This appendix contains two results regarding random walks and random paths on the free group \( F_2 = \langle a, b \rangle \). We first establish a few combinatorial lemmas about finite subsets of \( F_2 \). Henceforth, we speak of a subset in \( F_2 \) and its corresponding forest in the Cayley graph of \( F_2 \) interchangeably. We order the words in \( F_2 \) in increasing order of length by stipulating that \( a < b < a^{-1} < b^{-1} \) and then following the alphabetical order. For each \( n \geq 1 \) let \( v_{n-1} \) denote the \( n \)-th word in \( F_2 \). Then, for instance, the first twelve words are: \( v_0 = e, v_1 = a, v_2 = b, v_3 = a^{-1}, v_4 = b^{-1}, v_5 = aa, v_6 = ab, v_7 = ab^{-1}, v_8 = ba, v_9 = bb, v_{10} = ba^{-1}, v_{11} = a^{-1}b \).

Fix a positive real number \( s < \frac{1}{8} \) and consider a finite subset \( M \subset F_2 \). Let \( 1_M \) be the characteristic function of \( M \). Define the weight function \( w_M : F_2 \to \mathbb{R} \) by

\[
w_M(v) = (1 - 4s)1_M(v) + s(1_M(va) + 1_M(vb) + 1_M(va^{-1}) + 1_M(vb^{-1})).
\]

Notice that \( w_M \) has finite support and \( \sum_{v \in F_2} w_M(v) = |M| \). The \( M \)-weight of a vertex \( v \in F_2 \) is \( w_M(v) \). The interior of \( M \) is the set formed by vertices with \( M \)-weight equals 1; the number of interior points in \( M \) is denoted by \( i_M \). The interior boundary (resp. exterior boundary) of \( M \) is the set formed by vertices with \( M \)-weight equals \( 1 - ks \) (resp. \( ks \)) for some \( 0 < k \leq 4 \). Notice that if \( M \) is a tree then its exterior boundary contains \( 2|M| + 2 \) elements. Moreover, since the sum of all \( M \)-weights equals \( |M| \), the interior boundary of a tree has at least \( \left\lceil \frac{2|M|+2}{3} \right\rceil \) elements and \( i_M \leq \left\lfloor \frac{|M|+2}{3} \right\rfloor \).

We obtain a non-increasing sequence \( x_M \) by ordering the nonzero \( M \)-weights \( w_M(v), v \in F_2 \). Let \( |x_M| \) be the number of terms of \( x_M \). Therefore, if \( M \) is a tree
then \(|x_M| = 3|M| + 2\). Given two finite subsets of the same cardinality \(M_1, M_2 \subset F_2\) we say that \(x_{M_1} > x_{M_2}\) if the sum of the first \(n\) terms of \(x_{M_1}\) is greater than or equal to the sum of the first \(n\) terms of \(x_{M_2}\) for all \(1 \leq n \leq \min\{|x_{M_1}|, |x_{M_2}|\} \).

In order to facilitate the reading we write the repeated terms of \(x_M\) using exponents. For instance, if \(M = \{e, a, a^2, a^3\}\) then \(x_M\) has two terms equal to \(1 - 2s\), two terms equal to \(1 - 3s\) and ten terms equal to \(s\); therefore, we write \(x_M = ((1 - 2s)^2, (1 - 3s)^2, s^{10})\).

**Example 6.1.** Consider \(m \geq 1\) and let \(M_m = \{v_0, v_1, \ldots, v_{m-1}\}\) be the set of the first \(m\) words in \(F_2\). Therefore, \(M\) is a tree and for \(m \leq 5\) we have \(x_{M_1} = (1 - 4s, s^4), x_{M_2} = ((1 - 3s)^2, s^6), x_{M_3} = ((1 - 3s)^2, 1 - 2s, s^8), x_{M_4} = ((1 - 3s)^3, 1 - s, s^{10})\) and \(x_{M_5} = (1, (1 - 3s)^4, s^{12})\).

For \(m > 5\) consider \(l \geq 1\) and \(r \in \{1, 2, \ldots, 43^l\}\) such that \(m = 2.3^l - 1 + r\). Notice that for every \(k \geq 1\) the words of length \(k\) are \(v_{2.3^{k-1}-1}, v_{2.3^{k-1}}, \ldots, v_{2.3^k-2}\). Thus, \(\{v_0, v_1, \ldots, v_{2.3^l-2}\}\) is the set of all words with length at most \(l\). Therefore, an analysis of the three possible values of \(r - 3\lfloor \frac{r}{5} \rfloor\) shows that

\[
x_{M_m} = (1^{2.3^{l-1} - 1 + \lfloor \frac{r}{5} \rfloor}, 1 - s(3 - (r - 3\lfloor \frac{r}{3} \rfloor))(1 - 3s)^{4.3^{l-1} - 1 + r - \lfloor \frac{r}{5} \rfloor}, s^{4.3^{l} + 2r}).
\]

Notice that \(1 - s(3 - (r - 3\lfloor \frac{r}{5} \rfloor)) \in \{1 - s, 1 - 2s, 1 - 3s\}\).

Let \(m \geq 1\). We are interested in obtaining a subset of \(F_2\) of cardinality \(m\) whose corresponding sequence is maximal under the partial order \(>\). The following lemma shows that such subset must be a tree.

**Lemma 6.2.** Let \(m \geq 1\) and consider \(M \subset F_2\) such that \(|M| = m\). Then there exists a tree \(\Bar{M} \subset F_2\) such that \(|\Bar{M}| = m\), \(i_{\Bar{M}} = i_M\) and \(x_{\Bar{M}} > x_M\).

**Proof.** If \(M\) is a tree take \(\Bar{M} = M\). Then, suppose that \(M\) is a forest with \(n\) connected components \(T_1, T_2, \ldots, T_n\). For \(k = 0, 1, 2, 3\) let \(p_k\) (resp. \(q_k\)) be the number of vertices with \(M\)-weight equals \(1 - (4 - k)s\) (resp. \((4 - k)s\)). Therefore, \(x_M = (1^M, (1 - s)^{p_1}, (1 - 2s)^{p_2}, (1 - 3s)^{p_3}, (1 - 4s)^{p_4}, (4s)^{q_1}, (3s)^{q_2}, (2s)^{q_3}, s^{q_4})\).

For \(i = 1, 2, \ldots, n\) denote the exterior boundary of \(T_i\) by \(\partial_i T_i\). Since there exist no cycles in \(F_2\) we may assume, by possibly relabeling the exterior boundaries, that \(\partial_i T_i \cap \bigcup_{i=2}^n \partial_i T_i\) contains at most one element. Let \(w \in M \setminus T_1\) be a word of maximal length. Notice that \(w\) is a leaf of one of the connected components of \(M\).
We consider the two possible cases. First, suppose that $\partial_e T_i \cap (\bigcup_{i=2}^n \partial_e T_i) = \emptyset$. Let $M'$ be the set, with $|M|$ elements, obtained from $M$ by performing a rigid transformation on $T_1$ that takes a leaf to a vertex in $F_2 \setminus M$ adjacent to $w$. Therefore, $x_{M'} = (1^M, (1-s)^{p_3}, (1-2s)^{p_2+2}, (1-3s)^{p_1-2}, (1-4s)^{p_0}, (4s)^{q_0}, (3s)^{q_1}, (2s)^{q_2}, s^q-2)$. 

Now consider the case that $\partial_e T_i \cap (\bigcup_{i=2}^n \partial_e T_i) = \{u\}$ for some $u \in F_2$. Notice that $u$ belongs exactly to $2, 3$ or $4$ exterior boundaries. Suppose that there exist distinct numbers $i, j, k \in \{2, 3, \ldots, n\}$ such that $\partial_e T_i \cap \partial_e T_i \cap \partial_e T_j \cap \partial_e T_k = \{u\}$, the other cases are similar. Construct $M'$ as in the previous paragraph: perform a rigid transformation on $T_1$ that takes a leaf to a vertex in $F_2 \setminus M$ adjacent to $w$. Thus, $x_{M'} = (1^M, (1-s)^{p_3}, (1-2s)^{p_2+2}, (1-3s)^{p_1-2}, (1-4s)^{p_0}, (4s)^{q_0}, (3s)^{q_1}, (2s)^{q_2}, s^q-1)$. 

Then, for any of the two cases above, we construct a set $M'$ such that $|M'| = m$, $i_{M'} = i_M$ and $x_{M'} > x_M$. Moreover, $M'$ has $n-1$ connected components. We then obtain $\tilde{M}$ by repeating the argument above $n-2$ times. 

**Lemma 6.3.** Let $m \geq 5$ and consider $M \subset F_2$ such that $|M| = m$. If $i_M < \lfloor \frac{m-2}{3} \rfloor$ then there exists $\tilde{M} \subset F_2$ such that $|\tilde{M}| = m$, $i_{\tilde{M}} = i_M + 1$ and $x_{\tilde{M}} > x_M$.

**Proof.** We may assume, by Lemma 6.2, that $M$ is a tree. Thus, there exists no $v \in M$ such that $w_M(v) \in \{1 - 4s, 4s, 3s, 2s\}$. Let $p_1, p_2$ and $p_3$ be the number of vertices in $M$ of degree 1, 2 and 3, respectively. Therefore, we have that $x_M = (1^M, (1-s)^{p_3}, (1-2s)^{p_2}, (1-3s)^{p_1}, s^{2m+2})$. Consider $\delta \in \{1, \ldots, \lfloor \frac{m-2}{3} \rfloor \}$ such that $i_M = \lfloor \frac{m-2}{3} \rfloor + \delta$. Since the sum of the $M$-weights equals $m$, it follows that $p_2 + 2p_3 \geq 3\delta$.

We consider three cases: $p_2 = 0, p_3 = 0$ and $p_2, p_3 \neq 0$. First, suppose $p_2 = 0$. Then, $p_3 \geq 2$ and there exist vertices $u, v \in M$ of degree 3. Let $u_1$ (resp. $v_1$) be the vertex adjacent to $u$ (resp. $v$) not contained in $M$.

We may assume that $u$ is adjacent to a leaf. If not, since every vertex adjacent to a leaf has degree at least 3, there exist leaves $l_1, l_2$ adjacent to a vertex of degree 4. Therefore, $(M \setminus \{l_1\}) \cup \{u_1\}$ contains a vertex of degree 3 which is adjacent to a leaf and its corresponding sequence equals $x_M$. 

Let $\tilde{M}$ be the set obtained from $M \cup \{u_1\}$ by removing the leaf contained in $M$ which is adjacent to $u$; Figure 27 shows an example of this construction. Notice that $x_{\tilde{M}} = (1^{M+1}, (1-s)^{p_3-2}, (1-2s)^{p_2+1}, (1-3s)^{p_1}, s^{2m+2})$. 

29
Figure 27: A tree $M$ and the set $\bar{M}$.

Suppose $p_3 = 0$. Then, $p_2 \geq 3$ and therefore there exist vertices $u, v, w \in M$ of degree 2. Let $u_1, u_2$ (resp. $v_1, v_2$) be the vertices in $M$ which are adjacent to $u$ (resp. $v$). Assume, without loss of generality, that $v_2$ belongs to the connected component of $M \setminus \{v\}$ that contains $u$. Analogously, assume that $u_2$ belongs to the connected component of $M \setminus \{u\}$ that contains $v$.

Let $M'$ be the set formed by the union of $M \setminus \{u, v\}$ and the two vertices in $F_2 \setminus M$ which are adjacent to $w$. The tree $\tilde{M}$ with $|M|$ vertices is obtained from $M'$ by performing two rigid transformations; the Figure 28 below shows two examples of this construction. The first transformation is performed on the connected component of $M'$ that contains $u_1$ and takes $u_1$ to $u$. The second transformation is performed on the connected component of $M'$ that contains $v_1$ and takes $v_1$ to $v$. Therefore, $x_{\tilde{M}} = (1^{s+1}, (1 - s)^{p_3}, (1 - 2s)^{p_2 - 3}, (1 - 3s)^{p_1 + 2}, s^{m+2})$.

Figure 28: Two examples of a tree $M$ with $p_3 = 0$ and the construction of $\tilde{M}$.

Suppose $p_3 \neq 0$ and $p_2 \neq 0$. Then there exist vertices $u, v \in M$ of degree 2 and degree 3, respectively. Let $M'$ be the union of $M \setminus \{u\}$ and the vertex in $F_2 \setminus M$ which is adjacent to $v$. Now, as in the previous paragraph, obtain a tree $\tilde{M}$ by...
performing a rigid transformation; the Figure 29 below shows an example of this construction. Then, \( x_M = (1^m + 1, (1 - s)^{p_1} - 1, (1 - 2s)^{p_2} - 1, (1 - 3s)^{p_3+1}, s^{2m+2}). \)

Figure 29: Two examples of a tree \( M \) with \( p_3, p_2 \neq 0 \) and the construction of \( \tilde{M} \).

Therefore, for any of the three possible cases above, we obtain a tree \( \tilde{M} \) with \( m \) vertices. Moreover, notice that \( i_{\tilde{M}} = i_M + 1 \) and \( x_{\tilde{M}} > x_M. \)

**Lemma 6.4.** Consider \( m \geq 1 \) and let \( M_m = \{v_0, v_1, \ldots, v_{m-1}\} \) be the set of the first \( m \) words in \( F_2 \). If \( M \subset F_2 \) is such that \( |M| = m \) then \( x_{M_m} > x_M. \)

**Proof.** We may assume, by Lemma 6.2, that \( M \) is a tree. If \( m \leq 5 \) the result then follows by checking a few cases. If \( m > 5 \) then \( m = 2.3^l - 1 + r \) for some \( l \geq 1 \) and \( r \in \{1, 2, \ldots, 4.3^l\} \). Therefore, as in Example 6.1, we have that \( x_{M_m} = (2.3^l - 1 + \lfloor \frac{r}{3} \rfloor, 1 - s(3 - (r - 3\lfloor \frac{r}{3} \rfloor)), (1 - 3s)^{2.3^l-1 - 1 + r - \lfloor \frac{r}{3} \rfloor}, s^{4.3^l+2r}). \)

By Lemma 6.3 it is sufficient to consider the case \( i_M = \lfloor \frac{m-2}{3} \rfloor \). Let \( p_2 \) (resp. \( p_3 \)) be the number of vertices in \( M \) of degree 2 (resp. 3). Since the sum of the terms of \( x_M \) equals \( m \), it follows that \( \frac{m - 2 - (p_2 + 2p_3)}{3} = \lfloor \frac{m - 2}{3} \rfloor \). Therefore, \( r - 3\lfloor \frac{r}{3} \rfloor = p_2 + 2p_3 \) and \( (p_2, p_3) \in \{(0, 0), (1, 0), (2, 0), (0, 1)\} \). If \( (p_2, p_3) \in \{(0, 0), (1, 0), (0, 1)\} \) then \( x_M \) equals \( x_{M_m} \). If \( (p_2, p_3) = (2, 0) \) then \( x_{M_m} > x_M. \)

Let \( X \) be a random variable which assumes values in \( F_2 \) such that \( \mathbb{P}(X = e) = 1 - 4s \) and \( \mathbb{P}(X = a) = \mathbb{P}(X = b) = \mathbb{P}(X = a^{-1}) = \mathbb{P}(X = b^{-1}) = s. \) Consider a sequence of i.i.d. random variables \( X_1, X_2, \ldots, X_t \) with distribution given by \( X \). We denote by \( P(n, t) \) the probability that \( X_1X_2 \ldots X_t \) equals \( v_n \). For instance, \( P(0, 1) = 1 - 4s, P(0, 2) = (1 - 4s)^2 + 4s^2 \) and \( P(0, 3) = (1 - 4s)((1 - 4s)^2 + 12s^2). \) Notice that \( P(n, t) \geq P(n + 1, t) \) for every \( n \geq 0. \)
Let \( \gamma(n) \) be the number of closed paths of length \( 2n \) contained in \( F_2 \) which start at \( e \); therefore \( \gamma(0) = 1 \). We have, for \( t \geq 1 \), that

\[
P(0, t) = \sum_{n=0}^{\lfloor s \rfloor} \binom{t}{2n} s^{2n}(1 - 4s)^{t-2n} \gamma(n). \tag{5.1}
\]

We compute \( \gamma(n) \) and show that \( P(0, t) \) decays exponentially as \( t \) goes to infinity; we do not try to obtain sharp estimates of the decay rate.

**Lemma 6.5.** For \( n \geq 1 \), \( \gamma(n) = 16^n \left(1 + 2 \sum_{k=1}^{n} (-1)^k \left(\frac{3}{2}\right)^k\right) \) and \( \gamma(n) \leq 13^n \).

**Remark 6.6.** In Lemma 6.5 above the constant 13 is not optimal. However, the important fact is that there exists a constant \( c < 16 \) such that \( \gamma(n) \leq c^n \) for \( n \geq 1 \).

**Proof.** Let \( \kappa(n) \) be the number of closed paths of length \( 2n \) starting at \( a \) in \( F_2 \setminus \{e\} \). Moreover, let \( \tilde{\kappa}(n) \) be the number of closed paths of length \( 2n \) contained in \( F_2 \setminus \{e\} \) that visit \( a \) only at the starting and ending vertices. By convenience, we define \( \tilde{\kappa}(0) = 0 \). Notice that \( \tilde{\kappa}(n) = 3\kappa(n-1) \) and \( \kappa(n) = \sum_{0 < m \leq n} \tilde{\kappa}(m) \kappa(n - m) \).

Define the two series \( k(q) = \sum_{n=0}^{\infty} \kappa(n)q^n \) and \( \tilde{k}(q) = \sum_{n=0}^{\infty} \tilde{\kappa}(n)q^n \). By the previous observations \( k(q) = k(q)\tilde{k}(q) + 1 \) and \( \tilde{k}(q) = 3k(q) \). Thus, \( 3qk(q)^2 - k(q) + 1 = 0 \) so that \( k(q) = \frac{1 - \sqrt{1 - 12q}}{6q} \).

Notice that \( \gamma(n) = \sum_{0 < m \leq n} 4\kappa(m-1)\gamma(n-m) \). Then, the series \( g(q) = \sum_{n=0}^{\infty} \gamma(n)q^n \) is such that \( g(q) = 1 + 4qk(q)g(q) \) and therefore \( g(q) = \frac{1 - 2\sqrt{1 - 12q}}{16q - 1} \). Hence, since \( \sum_{n=0}^{\infty} \gamma(n)(\frac{1}{13})^n = g(\frac{1}{13}) = \frac{13 - 2\sqrt{13}}{3} < 2 \), it follows that \( \gamma(n) \leq 13^n \) for all \( n \geq 1 \).

Moreover, by the binomial theorem \( \sqrt{1 - 12q} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-12)^n q^n \) so that \( g(q) = 1 + \sum_{n=1}^{\infty} (16^n + 2 \sum_{k=1}^{n} (-1)^k \binom{\frac{1}{2}}{k} 12^k 16^{n-k})q^n \). Then, for \( n \geq 1 \), we have that \( \gamma(n) = 16^n + 2 \sum_{k=1}^{n} (-1)^k \binom{\frac{1}{2}}{k} 12^k 16^{n-k} \). \( \square \)

**Lemma 6.7.** For \( t \geq 1 \), \( P(0, t) \leq (1 - s(4 - \sqrt{13}))^t \).

**Proof.** By Lemma 6.5 and equation (5.1) we have \( P(0, t) \leq (1 - 4s)^t \sum_{n=0}^{\lfloor s \rfloor} \binom{t}{2n} (\sqrt{13})^{2n} \). Therefore, \( P(0, t) \leq \frac{1}{2}(1 - 4s)^t((1 + \frac{\sqrt{13}}{1 - 4s})^t + (1 - \frac{\sqrt{13}}{1 - 4s})^t) \leq (1 - s(4 - \sqrt{13}))^t \). \( \square \)
Let \( y_0, y_1, \ldots, y_t \) be fixed but arbitrary elements in \( F_2 \) and consider the random variable \( Z = y_0X_1y_1 \ldots X_ty_t \). We denote by \( Q(n, t) \) the probability that \( Z \) equals \( v_n \). The following lemma compares the probabilities \( P(n, t) \) and \( Q(n, t) \).

**Lemma 6.8.** Let \( t \geq 0 \) and \( m \geq 1 \). If \( n_0, n_1, \ldots, n_{m-1} \in \mathbb{N} \) are distinct then \( Q(n_0, t) + Q(n_1, t) + \ldots + Q(n_{m-1}, t) \leq P(0, t) + P(1, t) + \ldots + P(m-1, t) \).

**Proof.** The proof is by induction on \( t \). Since \( P(0, 0) = 1 \) the case \( t = 0 \) is trivial. The case \( t = 1 \) is also easy. Indeed, notice that \( P(0, 1) = 1 - 4s, P(1, 1) = P(2, 1) = P(3, 1) = P(4, 1) = s \) and \( P(n, 1) = 0 \) for \( n \geq 5 \). Now, consider the distinct numbers \( n_0, n_1, n_2, n_3, n_4 \in \mathbb{N} \) such that \( v_{n_0} = y_0y_1, v_{n_1} = y_0y_1, v_{n_2} = y_0y_1, v_{n_3} = y_0a^{-1}y_1 \) and \( v_{n_4} = y_0b^{-1}y_1 \). Then, \( Q(n_0, 1) = 1 - 4s \) and \( Q(n_1, 1) = Q(n_2, 1) = Q(n_3, 1) = Q(n_4, 1) = s \); at this point it is not difficult to see that the result holds for \( t = 1 \).

It suffices to prove the result for \( Z = X_1y_1 \ldots X_{t-1}y_{t-1}X_t \). Indeed, the general case then follows by considering the distinct numbers \( k_0, k_1, \ldots, k_{m-1} \) such that \( v_{k_i} = y_0^{-1}v_iy_i^{-1} \) for all \( 0 \leq i \leq m - 1 \). Let \( Z' = X_1y_1 \ldots X_{t-1}y_{t-1} \) and define the two finite sets \( M = \{ v_{n_0}, v_{n_1}, \ldots, v_{n_{m-1}} \} \) and \( M_m = \{ v_0, v_1, \ldots, v_{m-1} \} \). Furthermore, let \( j > m \) be such that \( v_{n_m}, v_{n_{m+1}}, \ldots, v_{n_j} \) are the elements in the exterior boundary of \( M \). Suppose, without loss of generality, that \( w_M(v_{n_k}) \geq w_M(v_{n_{k+1}}) \) for all \( k \leq j - 1 \).

Notice that \( Q(n_0, t) + \ldots + Q(n_{m-1}, t) = \mathbb{P}(Z \in M) = \sum_{i=0}^{j} \mathbb{P}(Z' = v_{n_i})w_M(v_{n_i}) \).

Moreover, by the induction hypothesis

\[
\sum_{i=0}^{j} \mathbb{P}(Z' = v_{n_i})w_M(v_{n_i}) = \sum_{i=0}^{j} \left( \sum_{k \leq i} \mathbb{P}(Z' = v_{n_k}) \right) (w_M(v_{n_i}) - w_M(v_{n_{i+1}})) \leq \sum_{i=0}^{j} \left( \sum_{k \leq i} P(k, t - 1) \right) (w_M(v_{n_i}) - w_M(v_{n_{i+1}})) \tag{5.1}
\]

On the other hand, Equation 5.1 equals

\[
\sum_{i=0}^{j} P(i, t - 1)w_M(v_{n_i}) = \sum_{i=0}^{j} \left( \sum_{k \leq i} w_M(v_{n_k}) \right) (P(i, t - 1) - P(i + 1, t - 1))
\]

and therefore, by Lemma 6.4, less than or equal to

\[
\sum_{i=0}^{j} \left( \sum_{k \leq i} w_M(v_{n_k}) \right) (P(i, t - 1) - P(i + 1, t - 1)). \tag{5.2}
\]
The result then follows by noticing that Equation 5.2 equals
\[
\sum_{i=0}^{j} w_{M_m}(v_i) P(i, t - 1) = \mathbb{P}(X_1X_2\ldots X_t \in M_m) = \sum_{i=0}^{m-1} P(i, t).
\]

\[\square\]

References

[1] J. H. Conway and J. C. Lagarias, Tiling with polyominoes and combinatorial group theory, *Journal of combinatorial theory*, Series A, 53:183–208, 1990.

[2] J. Freire, C. J. Klivans, P. H. Milet and N. C. Saldanha, On the connectivity of spaces of three-dimensional tilings, *Transactions of The American Mathematical Society*, 375:1579–1605, 2022.

[3] P. W. Kasteleyn, The statistics of dimers on a lattice: I. the number of dimer arrangements on a quadratic lattice, *Physica*, 27(12):1209–1225, 1961.

[4] R. de Marreiros, Domino tilings of 3D cylinders and regularity of disks, Master dissertation, https://doi.org/10.17771/PUCRio.acad.53188, PUC–Rio, 2021.

[5] P. H. Milet and N. C. Saldanha, Domino tilings of three-dimensional regions: flips and twists, *arxiv:1410.7693*, 2018.

[6] P. H. Milet and N. C. Saldanha, Flip Invariance for Domino Tilings of Three-Dimensional Regions with Two Floors, *Discrete & Computational Geometry*, 53:914–940, 2015.

[7] I. Pak and J. Yang, The Complexity of Generalized Domino Tilings, *The Electronic Journal of Combinatorics*, Vol 20, Issue 4, P12, 2013.

[8] N. C. Saldanha, Domino Tilings of Cylinders: Connected Components under Flips and Normal Distribution of the Twist, *The Electronic Journal of Combinatorics*, Vol 28, Issue 1, P1.28, 2021.

[9] N. C. Saldanha, Domino tilings of cylinders: the domino group and connected components under flips, *Indiana University Mathematics Journal*, Vol 71 No. 3:965–1002, 2022.
[10] N. C. Saldanha, C. Tomei, M. A. Casarin Jr. and D. Romualdo, Spaces of domino tilings, *Discrete & Computational Geometry*, 14(1):207–233, 1995.

[11] W. P. Thurston, Conway’s Tiling Groups, *The American Mathematical Monthly*, 97(8):757–773, 1990.

Raphael de Marreiros  
Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro  
Rua Marquês de São Vicente, 225, Gávea, Rio de Janeiro, RJ 22451-900, Brazil  
raphaeldemarreiros@mat.puc-rio.br