Natural FLRW metrics on the Lie group of nonzero quaternions

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Abstract

It is shown that the Lie group of invertible elements of the quaternion algebra carries a family of natural closed Friedmann-Lemaître-Robertson-Walker metrics.

Introduction.

The quaternion algebra $\mathbb{H}$ is one of the most important and well-studies objects in mathematics (e.g. [Wid02] and references therein) and physics (e.g. [Adl95] and references therein). It has a natural Hermitian form which induces a Euclidean inner product on its additive vector space $\mathbb{H}$. There is also a family of natural Minkowski inner products (signature 2) on $\mathbb{H}$, induced by the structure tensor $\mathbb{H}$ of the quaternion algebra. This result was obtained in [Tri95], where a notion of a natural inner product on a linear algebra over a field $F$ was introduced. The result came out of a study of relationship between natural metric properties of unital algebras and internal logic of topoi they generate. It was shown in [Tri95] that if the logic of a topos is bivalent Boolean then the generating algebra is isomorphic to the quaternion algebra with a family of Minkowski inner products. In this note we show that for a unital algebra the inner products can be naturally extended over the Lie group of its invertible elements, producing a family of principal metrics. In particular, for the quaternion algebra, these metrics are closed Friedmann-Lemaître-Robertson-Walker. These metrics are of interest because they constitute one of the most important classes (as far as our universe is concerned) of solutions of Einstein’s equations, and there are
indications in astrophysics and cosmology that the universe may be spatially closed ([Tr03] and references therein).

Remark 0.1. Some of the notations are slightly nonstandard. Small Greek indices, $\alpha, \beta, \gamma$ and small Latin indices $p, q$ always run 0 to 3 and 1 to 3, respectively. Summation is assumed on repeated indices of different levels. We use the $[m^m]$ device to denote tensor ranks; for example a one-form is a $[0^0]$-tensor. For clarity of the exposition we use $\Box$ at the end of a Proof, and each Remark ends with the sign appearing at the end of this line.

Definition 0.1. An $\mathbb{F}$-algebra, $A$, is an ordered pair $(S_A, A)$, where $S_A$ is a vector space over a field $\mathbb{F}$, and $A$ is a $[1^1]$-tensor on $S_A$, called the structure tensor of $A$. Each vector $a$ of $S_A$ is called an element of $A$, denoted $a \in A$. The dimensionality of $A$ is that of $S_A$.

Remark 0.2. This is an unconventional definition of a linear algebra over $\mathbb{F}$. Indeed, the tensor $A$ induces a binary operation $S_A \times S_A \rightarrow S_A$, called the multiplication of $A$: to each pair of vectors $(a, b)$ the tensor $A$ associates a vector $ab : S_A^* \rightarrow \mathbb{F}$, such that $(ab)(\tilde{\tau}) = A(\tilde{\tau}, a, b), \forall \tilde{\tau} \in S_A^*$. An $\mathbb{F}$-algebra with an associative multiplication is called associative. An element $\mathfrak{i}$, such that $a\mathfrak{i} = \mathfrak{i}a = a, \forall a \in A$ is called an identity of $A$.

Definition 0.2. For an $\mathbb{F}$-algebra $A$ and a nonzero one-form $\tilde{\tau} \in S_A^*$, a principal inner product is a $[2^2]$-tensor, $A[\tilde{\tau}]$, on $S_A$, assigning to each ordered pair $(a, b)$ a number $A[\tilde{\tau}](a, b) := A(\tilde{\tau}, a, b) \in \mathbb{F}$, just in case it is symmetric, $A[\tilde{\tau}](a, b) = A[\tilde{\tau}](b, a), \forall a, b \in A$.

Remark 0.3. In other words, a principal inner product is the contraction of a one-form with the structure tensor.

Definition 0.3. For each $\mathbb{F}$-algebra $A = (S_A, A)$, an $\mathbb{F}$-algebra $[A] = (S_A, [A])$, with the structure tensor defined by

$$[A](\tilde{\tau}, a, b) := A(\tilde{\tau}, a, b) - A(\tilde{\tau}, b, a), \forall \tilde{\tau} \in S_A^*, a, b \in A,$$

is called the commutator algebra of $A$.

Definition 0.4. A finite dimensional associative $\mathbb{R}$-algebra with an identity is called a unital algebra.

Lemma 0.1. The set $A$ of all invertible elements of a unital algebra $A$ is a Lie group with respect to the multiplication of $A$, with $[A]$ as its Lie algebra.
Proof. See, for example, [Pos82] for a proof of this simple lemma.

Remark 0.4. For an \( \mathbb{R} \)-algebra \( A \), its vector space \( S_A \) canonically generates a (linear) manifold \( S_A \) with the same carrier, with a bijection \( j : S_A \to S_A \). We use the normal \((a, u, ...)\) and bold \((\mathbf{a}, \mathbf{u}, ...)\) fonts, respectively, to denote their elements, e.g., \( j(a) = \mathbf{a} \). The tangent space \( T_aS_A \) is identified with \( S_A \) at each point \( a \in S_A \) via an isomorphism \( j^* : T_aS_A \to S_A \) sending a tangent vector to the curve \( \mu : \mathbb{R} \to S_A, \mu(t) = a + tu \), at the point \( \mu(0) = a \in S_A \), to the vector \( \mathbf{u} \in S_A \), with the “total” map \( j^* : T_S A \to S_A \).

A linear map \( F : S_A \to S_A \) induces a vector field \( f : S_A \to T_S A \) on \( S_A \), such that the following diagram commutes,

\[
\begin{array}{ccc}
S_A & \xrightarrow{f} & T_S A \\
\downarrow j & & \downarrow j^* \\
S_A & \xrightarrow{F} & S_A \\
\end{array}
\]

For a unital algebra \( A \) the Lie group \( A \) is a submanifold of \( S_A \), with the inclusion map \( \bar{j} : A \to S_A \), which is the restriction, to \( A \), of the identity map.

Remark 0.5. For each basis \((e_j)\) on the vector space \( S_A \) of a unital algebra, there is a natural basis field on \( A \), namely the basis \((\mathbf{e}_j)\) of left invariant vector fields generated by \((e_j)\). We call \((\mathbf{e}_j)\) a proper frame generated by \((e_j)\). The value, \((\mathbf{e}_j)(a)\), of \((\mathbf{e}_j)\) at \( a \) is basis on the tangent space \( T_aA \); it is referred to as a proper basis (at \( a \)) generated by \((e_j)\). In particular, \((\mathbf{e}_j)(\mathbf{1})\), the proper basis at the identity generated by \((e_j)\), coincides with \((e_j)\).

Definition 0.5. For a unital algebra \( A \), let \((\mathbf{e}_j)\) be a proper frame on \( A \), generated by a basis \((e_j)\) on \( S_A \). The structure field of the Lie group \( A \) is a tensor field \( \mathcal{A} \) on \( A \), assigning to each point \( a \in A \) a \([\frac{1}{2}]\)-tensor \( \mathcal{A}(a) \) on \( T_aA \), with components \( \mathcal{A}^i_{jk}(a) := (\mathcal{A}(a))^i_{jk} \) in the basis \((\mathbf{e}_j)(a)\), defined by

\[
\mathcal{A}^i_{jk}(a) := \mathcal{A}^i_{jk}, \quad \forall a \in A,
\]

where \( \mathcal{A}^i_{jk} \) are the components of the structure tensor \( \mathcal{A} \) in the basis \((e_j)\).

Remark 0.6. Intuitively, the structure field is the constant extension of the structure tensor along the left invariant vector fields.
Definition 0.6. For a unital algebra $A$ and each $a \in A$, an $\mathbb{R}$-algebra $A\{a\} = (S_{A\{a\}}, A\{a\})$, where $S_{A\{a\}} := T_aA$, and $A\{a\} := A(a)$, is called the tangent algebra of the Lie group $A$ at $a$.

Remark 0.7. It is easy to see that for each $a \in A$, the tangent algebra $A\{a\}$ is isomorphic to $A$; in particular, each $A\{a\}$ is unital.

Definition 0.7. For a unital algebra $A$ and a twice differentiable real function $T$ on the Lie group $A$, a principal metric on $A$ is a $[0, 2]$-tensor field $\mathcal{T}$ on $A$, such that that $\mathcal{T}(a) = A\{a\}[\tilde{a}]$, $\forall a \in A$, where $\tilde{a} := d\mathcal{T}(a)$ is the value of the gradient of $\mathcal{T}$ at $a$.

Remark 0.8. In other words, a principal metric is the contraction of a one-form field on $A$ with the structure field of $A$. For each $a \in A$, the value, $\mathcal{T}(a)$, of $\mathcal{T}$ is a principal inner product on the tangent algebra $A\{a\}$.

1 Quaternion algebra.

Definition 1.1. A four dimensional $\mathbb{R}$-algebra, $\mathbb{H} = (S_{\mathbb{H}}, H)$, is called a quaternion algebra (with quaternions as its elements), if there is a basis on $S_{\mathbb{H}}$, in which the components of the structure tensor $H$ are given by the entries of the following matrices,

\[
H_0^{0,\alpha\beta} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad H_0^{1,\alpha\beta} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix},
\]

\[
H_2^{0,\alpha\beta} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}, \quad H_3^{0,\alpha\beta} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\] (2)

We refer to such a basis as canonical.

Remark 1.1. The vectors of the canonical basis are denoted $1, i, j, k$. A quaternion algebra is unital, with the first vector of the canonical basis, $1$, as its identity. Since $(1, i, j, k)$ is a basis on a real vector space, any quaternion $a$ can be presented as $a^01 + a^1i + a^2j + a^3k$, $a^\beta \in \mathbb{R}$. A quaternion $\bar{a} = a^01 - a^1i - a^2j - a^3k$ is called conjugate to $a$. We refer to $a^0$ and $a^\beta i^\gamma$ as
the real and imaginary part of \(a\), respectively. Quaternions of the form \(a^0\mathbf{1}\) are in one-to-one correspondence with real numbers, which is often denoted, with certain notational abuse, as \(\mathbb{R} \subset \mathbb{H}\).

**Remark 1.2.** A linear transformation \(S_{\mathbb{H}} \rightarrow S_{\mathbb{H}}\) with the following components in the canonical basis,

\[
\begin{pmatrix}
1 & 0 \\
0 & \mathbf{B}
\end{pmatrix}, \quad \mathbf{B} \in SO(3),
\]

takes \((1, i, j, k)\) to a basis \((i_\beta)\) in which the components \([2]\) of the structure tensor will not change, and neither will the multiplicative behavior of vectors of \((i_\beta)\). Thus, we have a class of canonical bases parameterized by elements of \(SO(3)\).

**Lemma 1.1.** Every principal inner product on \(\mathbb{H}\) is Minkowski.

**Proof.** For the quaternion algebra the components of the structure tensor \(\mathbf{H}\) in a canonical basis are given by \([2]\). A one-form \(\tilde{\tau}\) on \(S_{\mathbb{H}}\) with components \(\tilde{\tau}_\beta\) in (the dual of) a canonical basis \((i_\beta)\) contracts with the structure tensor into a \([2]\)-tensor on \(S_{\mathbb{H}}\) with the following components in the basis \((i_\beta)\):

\[
\begin{pmatrix}
\tilde{\tau}_0 & \tilde{\tau}_1 & \tilde{\tau}_2 & \tilde{\tau}_3 \\
-\tilde{\tau}_0 & \tilde{\tau}_3 & -\tilde{\tau}_2 & \tilde{\tau}_1 \\
-\tilde{\tau}_3 & -\tilde{\tau}_2 & \tilde{\tau}_1 & \tilde{\tau}_0 \\
\end{pmatrix}.
\]

The only way to make this symmetric is to put \(\tilde{\tau}_1 = -\tilde{\tau}_1, \tilde{\tau}_2 = -\tilde{\tau}_2, \tilde{\tau}_3 = -\tilde{\tau}_3\), which yields \(\tilde{\tau}_1 = \tilde{\tau}_2 = \tilde{\tau}_3 = 0\):

\[
(\mathbf{H}[\tilde{\tau}])_{\alpha\beta} = \begin{pmatrix}
\tilde{\tau}_0 & 0 & 0 & 0 \\
0 & -\tilde{\tau}_0 & 0 & 0 \\
0 & 0 & -\tilde{\tau}_0 & 0 \\
0 & 0 & 0 & -\tilde{\tau}_0
\end{pmatrix}. \tag{3}
\]

\[\quad\]

2 Natural structures on \(\mathcal{H}\).

There is a class of canonical bases on \(S_{\mathbb{H}}\) (see Remark [2]) whose members differ from one another by a rotation in the hyperplane of pure imaginary
quaternions. Each canonical basis \((i_\beta)\) on \(S_\mathbb{H}\) induces a canonical coordinate system \((w, x, y, z)\) on the linear manifold \(S_\mathbb{H}\) canonically generated by \(S_\mathbb{H}\), and therefore also on its submanifold \(\mathcal{H}\) of nonzero quaternions: a point \(a \in \mathcal{H}\) such that \(\mathcal{J}(a) = a = a^0_i \beta\) is assigned coordinates \((w = a^0, x = a^1, y = a^2, z = a^3)\). This coordinate system covers both \(S_\mathbb{H}\) and \(\mathcal{H}\) with a single patch. Since \(0 \notin \mathcal{H}\), at least one of the coordinates is always nonzero for any point \(a \in \mathcal{H}\). For a differentiable function \(R : \mathbb{R} \to \mathbb{R} \setminus \{0\}\) there is a system of natural spherical coordinates \((\eta, \chi, \theta, \varphi)\) on \(\mathcal{H}\), related to the canonical coordinates by

\[
w = R(\eta) \cos \chi, \quad x = R(\eta) \sin \chi \sin \theta \cos \varphi, \quad y = R(\eta) \sin \chi \sin \theta \sin \varphi, \quad z = R(\eta) \sin \chi \cos \theta.
\]

Each canonical basis \((i_\beta)\) can be considered a basis on the vector space of the Lie algebra of \(\mathcal{H}\), i.e., the tangent space \(T_1 \mathcal{H} \cong S_\mathbb{H}\) to \(\mathcal{H}\) at the point \((1, 0, 0, 0)\), the identity of the group \(\mathcal{H}\). There are several natural basis fields on \(\mathcal{H}\) induced by each basis \((i_\beta)\). First of all, we have the proper frame \((\hat{i}_\beta)\), of left invariant vector fields on \(\mathcal{H}\) (see Remark 0.5), which is a noncoordinate basis field. There are also two coordinate basis fields, the canonical frame, \((\partial_w, \partial_x, \partial_y, \partial_z)\) and the corresponding spherical frame \((\partial_\eta, \partial_\chi, \partial_\theta, \partial_\varphi)\). A left invariant vector field \(\hat{u}\) on \(\mathcal{H}\), generated by a vector \(u \in S_\mathbb{H}\) with components \((u^\beta)\) in a canonical basis, associates to each point \(a \in \mathcal{H}\) with coordinates \((w, x, y, z)\) a vector \(\hat{u}(a) \in T_a \mathcal{H}\) with the components \(\hat{u}^\beta(a) = (au)^\beta\) in the basis \((\partial_w, \partial_x, \partial_y, \partial_z)\) of \(T_a \mathcal{H}\):

\[
\hat{u}^0(a) = wu^0 - xu^1 - yu^2 - zu^3, \quad \hat{u}^1(a) = wu^1 + xu^0 + yu^3 - zu^2, \\
\hat{u}^2(a) = wu^2 - xu^3 + yu^0 + zu^1, \quad \hat{u}^3(a) = wu^3 + xu^2 - yu^1 + zu^0.
\]

The system (1) contains sufficient information to compute transformation between the frames. For example, the transformation between the spherical and proper frames is given by

\[
\begin{pmatrix}
R/\dot{R} & 0 & 0 \\
0 & \sin \theta \cos \varphi & \sin \theta \sin \varphi \\
0 & \cos \chi \sin \varphi & \cos \chi \cos \varphi + \sin \chi \sin \varphi \\
0 & \sin \chi \cos \varphi - \cos \chi \sin \varphi & \sin \chi \sin \varphi
\end{pmatrix} = \\
\begin{pmatrix}
0 & \sin \theta \cos \varphi & \sin \theta \sin \varphi \\
\cos \chi \sin \varphi & \cos \chi \cos \varphi + \sin \chi \sin \varphi & \sin \chi \sin \varphi \\
\sin \chi \cos \varphi - \cos \chi \sin \varphi & \sin \chi \sin \varphi & \sin \chi \cos \varphi
\end{pmatrix},
\]

where \(\dot{R} := \frac{dR}{d\eta} : \mathbb{R} \to \mathbb{R} \setminus \{0\}\).
Definition 2.1. A Lorentzian metric on a four dimensional manifold is called closed FLRW (Friedmann-Lemaître-Robertson-Walker) if there is a coordinate system \((x^β)\), such that in the corresponding coordinate frame the components of the metric are given by the entries of the following matrix:

\[
\pm \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -a^2 & 0 & 0 \\
0 & 0 & -a^2 \sin^2(x^1) & 0 \\
0 & 0 & 0 & -a^2 \sin^2(x^1) \sin^2(x^2)
\end{pmatrix},
\]

where \(a : \mathbb{R} \to \mathbb{R}\), referred to as the scale factor, is a function of \(x^0\) only.

3 Principal metrics on \(\mathcal{H}\).

Theorem 3.1. Every principal metric on \(\mathcal{H}\) is closed FLRW.

Proof. Let \(\tilde{T}\) and \((\hat{i}_β)\) be a one-form and a canonical basis on \(S_\mathcal{H}\), respectively. For each point \(a \in \mathcal{H}\) the \(\mathbb{R}\)-algebra \(\mathbb{H}(a)\) is the tangent algebra, at \(a\), of the Lie group \(\mathcal{H}\). For each \(a \in \mathcal{H}\) the components of the structure tensor \(\mathbb{H}\{a\}\) and a principal inner product, \(\mathbb{H}\{a\} [\tilde{T}]\), of \(\mathbb{H}(a)\) in the basis \((\hat{i}_β)(a)\) are given by (2) and (3), respectively. Therefore, the components of a principal metric, \(T\), in the proper frame \((\hat{i}_β)\) must have the form

\[
\pm \begin{pmatrix}
\tilde{T} & 0 & 0 & 0 \\
0 & -\tilde{T} & 0 & 0 \\
0 & 0 & -\tilde{T} & 0 \\
0 & 0 & 0 & -\tilde{T}
\end{pmatrix},
\]

for some function \(\tilde{T} : \mathcal{H} \to \mathbb{R} \setminus \{0\}\). In other words, any principal metric on \(\mathcal{H}\) is obtained by contraction of a one-form field \(\tilde{T}\), whose components in \((\hat{i}_β)\) are \((\tilde{T}, 0, 0, 0)\), with the structure field \(\mathcal{H}\). This one-form is exact, i. e., there exists a twice differentiable function \(T\), such that \(dT = \tilde{T}\). In the spherical frame with \(R(\eta) = \exp(\eta)\) the components of \(\tilde{T}\) are also \((\tilde{T}, 0, 0, 0)\), and,

\[
dT_0 = \frac{\partial T}{\partial \eta} = \tilde{T}, \quad dT_1 = \frac{\partial T}{\partial \chi} = dT_2 = \frac{\partial T}{\partial \theta} = dT_3 = \frac{\partial T}{\partial \phi} = 0.
\]

It follows from (6) that both \(T\) and \(\tilde{T}\) depend on \(\eta\) only. Since \(\frac{\partial T}{\partial \eta}\) is differentiable, \(\tilde{T}\) must be at least continuous. Since \(\tilde{T}(\eta) \neq 0, \forall \eta \in \mathbb{R}\), \(\tilde{T}\) cannot
change sign. Computing the components of the principal metric $\mathcal{T}$ in the spherical frame we get

$$
\mathcal{T}_{\alpha\beta} = \begin{pmatrix}
\tilde{\tau}(\eta)(\frac{k}{R})^2 & 0 & 0 & 0 \\
0 & -\tilde{\tau}(\eta) & 0 & 0 \\
0 & 0 & -\tilde{\tau}(\eta)\sin^2\chi & 0 \\
0 & 0 & 0 & -\tilde{\tau}(\eta)\sin^2\chi\sin^2\theta
\end{pmatrix}.
$$

If $\tilde{\tau}(\eta) > 0$, we take $R(\eta)$ such that $\tilde{\tau}(\eta)(\frac{k}{R})^2 = 1$, which yields

$$
R(\eta) = \exp\int \frac{d\eta}{\pm\sqrt{\tilde{\tau}(\eta)}}.
$$

In other words, with $R(\eta)$ satisfying (7), the components of the principal metric in the spherical frame are

$$
\mathcal{T}_{\alpha\beta} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -a^2 & 0 & 0 \\
0 & 0 & -a^2\sin^2\chi & 0 \\
0 & 0 & 0 & -a^2\sin^2\chi\sin^2\theta
\end{pmatrix},
$$

where the scale factor $a(\eta) := \sqrt{\tilde{\tau}(\eta)}$.

If $\tilde{\tau}(\eta) < 0$, similar considerations show that the metric is also closed FLRW with the scale factor $a(\eta) := \sqrt{-\tilde{\tau}(\eta)}$.

**Corollary 3.1.** $\mathcal{T}$ is a monotonous function of $\eta$ for each principal metric $\mathcal{T}$ of $\mathcal{H}$.

Thus the natural geometry of the Lie group of nonzero quaternions $\mathcal{H}$ is defined by a family of closed Friedmann-Lemaître-Robertson-Walker metrics.

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