Nonlinear Filtering with Optimal MTLL

E. Fischler\textsuperscript{1}, Z. Schuss\textsuperscript{2}

Abstract

We consider the problem of nonlinear filtering of one-dimensional diffusions from noisy measurements. The filter is said to lose lock if the estimation error exits a prescribed region. In the case of phase estimation this region is one period of the phase measurement function, e.g., $[-\pi, \pi]$. We show that in the limit of small noise the causal filter that maximizes the mean time to loose lock is Bellman’s minimum noise energy filter.

1 Introduction

Optimal filtering theory defines different optimality criteria, such as minimizing the conditional mean square estimation error (MSEE), given the measurements \cite{1}, maximizing the a posteriori probability (MAP) density function (pdf) of the signal, given the measurements \cite{2}, Bellman’s criterion of minimum noise energy (MNE) \cite{3}, \cite{4}, \cite{5}, and more. In problems of phase estimation, that lead to loss of lock and cycle slips, an important optimality criterion is maximizing the mean time to lose lock (MTLL) or to exit a given region, which is also a well known control problem \cite{6}, \cite{7}, \cite{8}, \cite{9}. Approximation methods for finding the various optimal filters have been devised for problems with small noise, including large deviations and WKB solutions of Zakai’s equation, the extended Kalman filter (EKF) \cite{10}, \cite{11}, \cite{12}, \cite{13} and others. The EKF and WKB approximations produce explicit suboptimal finite-dimensional filters, which in case of phase estimation are the well known phase trackers, such as the phase locked loop (PLL), delay locked loop (DLL), angle tracking loops, and so on \cite{14}. The MSEE in these phase trackers is asymptotically optimal \cite{10}, \cite{13}.

The suboptimal phase trackers are known to lose lock (or slip cycles) \cite{14}. The MTLL in these filters is simply the mean first passage time (MFPT) of the estimation error to the boundary of the lock region. The MFPT from an attractor of a dynamical system driven by small noise has been calculated by large deviations and singular perturbation methods \cite{15}, \cite{16}, \cite{17}, and in particular, for the PLL \cite{18}. The MTLL in particle filters for phase estimation was found in \cite{19}. It has been found recently that minimizing the MNE leads to a finite, yet much longer MTLL than in the above mentioned phase estimators \cite{20}, \cite{21}. This raises the question of designing a causal (or noncausal) phase estimator with maximal MTLL.

The MTLL is the fundamental performance criterion in phase tracking and synchronization systems. Thus, for example, a phase tracking system is considered locked, as long as the estimation error $e(t) = x(t) - \hat{x}(t)$ is in $(-\pi, \pi)$. When the error exceeds these limits, the estimation is said to be unlocked, and the system relocks on an erroneous equilibrium.

\textsuperscript{1}Department of Systems, School of Electrical Engineering, The Iby and Aladar Fleischman Faculty of Engineering, Tel-Aviv University, Ramat-Aviv Tel-Aviv 69978, Israel.

\textsuperscript{2}Department of Mathematics, Tel-Aviv University, Tel-Aviv 69978, Israel.
point, with a deviation of $2\pi$. Another example is an automatic sight of a cannon. The sight is said to be locked on target if the positioning error is somewhere in between certain limits. Similar problems, in which the maximization of exit time is an optimality criterion, were considered by several authors [7], [8], [9]. In [7], a simpler filtering problem is considered, in which the error $e(t)$ is measured, rather than the state variable $x(t)$. It is solved under the further assumption of a linear measurement inside a domain. In [8], [9] the state process is controlled through its drift, rendering it a control rather than a filtering problem.

In this paper we show that for small noise the maximum MTLL filter is Bellman’s MNE filter [4]. It follows that the result of [21] for the MTLL of the optimal MNE phase filter, is asymptotically an upper bound for any other filtering scheme. In view of the results of [21], the potential gain of the optimal MNE filter over the first order EKF-PLL is 12 dB.

2 Formulation

An important class of filtering problems with small measurements noise can be reduced to the model of a diffusion process

$$dx(t) = m(x, t) \, dt + \varepsilon \sigma \, dw(t),$$

measured in a noisy channel

$$dy(t) = h(x, t) \, dt + \varepsilon \rho \, d\nu(t),$$

where $m(x, t)$ and $h(x, t)$ are possibly nonlinear, continuous functions. The processes $w(t)$ and $\nu(t)$ are independent standard Brownian motions, and $\varepsilon$ is a small parameter. If $m(x, t)$ is a linear function and the noise in (1) is not small, an appropriate scaling of time and dependent variables scales the small measurements noise into the diffusion equation as well, giving the canonic system (1), (2) [20]. The optimal filtering problem is to find a causal estimator $\hat{x}(t)$ of $x(t)$, given the measurements $y_0 = \{y(s) : 0 \leq s \leq t\}$, such that the mean first time the error signal,

$$e(t) = x(t) - \hat{x}(t),$$

leaves a given lock domain $L \subset \mathbb{R}$, is maximal. More specifically, for any adapted function $\hat{x}(t) \in \mathcal{C}(\mathbb{R}^+)$ (measurable with respect to the filtration generated by $y(t)$), we define an error process by (3) and the first time to lose lock by

$$\tau = \inf \{t : e(t) \in \partial L\}.$$  

The optimal filtering problem is to maximize $E[\tau \mid y_0]$ (see definition (16) below) with respect to all adapted continuous functions $\hat{x}(t)$. For example, if $h(x, t) = \sin x$ in a phase estimation problem, then $L = (-\pi, \pi)$ and lock is lost when $e(t) = \pm \pi$.

We can rewrite the model equations (1), (2) in terms of the error process $e(t)$ as

$$de(t) = M_x(e(t), t) \, dt + \varepsilon \sigma \, dw(t),$$

$$dy(t) = H_x(e(t), t) \, dt + \varepsilon \rho \, d\nu(t).$$
where

\[ M_s(e(t), t) = m(\dot{x}(t) + e(t)) - \dot{x}(t) \]

\[ H_s(e(t), t) = h(\dot{x}(t) + e(t)), \]

and the filtering problem is to find \( \dot{x}(t) \), such that \( \mathbb{E}[\tau | y_0] \) is maximal.

The survival probability of a trajectory \((e(t), y(t))\) of (5) with absorption at \( \partial L \) and (6) can be expressed in terms of the pdf \( p(e, y, t | \xi, \eta, s) \) of the two-dimensional process with an absorbing boundary condition on \( \partial L \). It is the solution of the Fokker-Planck equation (FPE)

\[
\frac{\partial p(e, y, t | \xi, \eta, s)}{\partial t} = -\frac{\partial M_s(e, t)p(e, y, t | \xi, \eta, s)}{\partial e} - \frac{\partial H_s(e, t)p(e, y, t | \xi, \eta, s)}{\partial y} + \frac{\varepsilon^2\sigma^2}{2} \frac{\partial^2 p(e, y, t | \xi, \eta, s)}{\partial e^2} + \frac{\varepsilon^2\rho^2}{2} \frac{\partial^2 p(e, y, t | \xi, \eta, s)}{\partial y^2}
\]

for \( e, \xi \in L, y, \eta \in \mathbb{R} \), with the boundary and initial conditions

\[ p(e, y, t | \xi, \eta, s) = 0 \quad \text{for} \quad e \in \partial L, y \in \mathbb{R}, \xi \in L, \eta \in \mathbb{R} \quad (8) \]

\[ p(e, y, s | \xi, \eta, s) = \delta(e - \xi, y - \eta) \quad \text{for} \quad e \in L, y \in \mathbb{R}, \xi \in L, \eta \in \mathbb{R}. \quad (9) \]

The pdf is actually the joint density and probability function \( p(e, y, t | \xi, \eta, s) = \Pr\{e(t) = e, y(t) = y, \tau > t | \xi, \eta, s\} \) and thus the survival probability is

\[ \Pr\{\tau > t | \xi, \eta, s\} = S_{\tau(\cdot), \tau(\cdot)}(t) = \int_L \int_{\mathbb{R}} p(e, y, t | \xi, \eta, s) \, de \, dy, \quad (10) \]

and it decays in time.

3 Simulation with particles

To simulate the filtering problem on a finite interval \( 0 \leq t \leq T \), we discretize (11), (12) on a sequence of grids

\[ \left\{ t_i = i \Delta t, \quad i = 0, 1, \ldots, N, \quad \Delta t = \frac{T}{N} \right\}, \]

and define discrete trajectories by the Euler scheme

\[ x_N(t_{i+1}) = x_N(t_i) + \Delta t \, m(x_N(t_i), t_i) + \varepsilon \sigma \Delta w(t_i) \quad (11) \]

\[ y_N(t_{i+1}) = y_N(t_i) + \Delta t \, h(x_N(t_i), t_i) + \varepsilon \rho \Delta v(t_i), \quad (12) \]
for \( i = 0, 1, \ldots, N-1 \), where \( \Delta w(t_i) \) and \( \Delta \nu(t_i) \) are independent zero mean Gaussian random variables with variance \( \Delta t \). The discretized version of (5), (6) is

\[
e_N(t_{i+1}) = e_N(t_i) + \Delta t M_x(e_N(t_i), t_i) + \varepsilon \sigma \Delta w(t_i) \tag{13}
\]

\[
y_N(t_{i+1}) = y_N(t_i) + \Delta t H_x(e_N(t_i), t_i) + \varepsilon \rho \Delta \nu(t_i). \tag{14}
\]

Given an observed trajectory \( \{y_N(t_i)\}_{i=0}^N \), we sample \( n \) trajectories \( \{x_{j,N}(t_i)\}_{i=0}^N \) according to the scheme (11), which produce error trajectories \( \{e_{j,N}(t_i)\}_{i=0}^N \) and determine their first exit times from \( L \), denoted \( \tau_{j,N} \) (we set \( \tau_{j,N} = T \) if \( \{x_{j,N}(t_i)\}_{i=0}^N \) does not exit \( L \) by time \( T \) ) \[22, 23, 24, 25, 26\]. The conditional MTLL is defined on the ensemble by

\[
E \left[ \tau_N \wedge T \mid \left\{ y_N(t_i), i = 0, 1, \ldots, \left[ \frac{\tau_N}{\Delta t} \right] \wedge N \right\} \right] = \sum_{j=1}^n (\tau_{j,N} \wedge T) \exp \left\{ \frac{1}{\varepsilon^2 \rho^2} \sum_{k=0}^{\left[ \frac{\tau_{j,N} \wedge T}{\Delta t} \right]} H(e_{j,N}(t_{k-1}), t_{k-1}) \Delta y_{k,N} - \frac{1}{2} \int H^2(e_{j,N}(t_{k-1}), t_{k-1}) \Delta t \right\}
\]

\[
\frac{1}{\varepsilon^2 \rho^2} \sum_{k=0}^{\left[ \frac{\tau_{j,N} \wedge T}{\Delta t} \right]} H(e_{j,N}(t_{k-1}), t_{k-1}) \Delta y_{k,N} - \frac{1}{2} \int H^2(e_{j,N}(t_{k-1}), t_{k-1}) \Delta t \right\}
\]

We define

\[
E[\tau \mid y_0^T] = \lim_{T \to \infty} \lim_{n \to \infty} \lim_{N \to \infty} E \left[ \tau_N \wedge T \mid \left\{ y_N(t_i), i = 0, 1, \ldots, \left[ \frac{\tau_N}{\Delta t} \right] \wedge N \right\} \right]. \tag{16}
\]

The conditional MTLL \( E[\tau \mid y_0^T] \) is a random variable on the \( \sigma \)-algebra \( \mathcal{F}_\infty = \bigcup_{t>0} \mathcal{F}_t \), where \( \mathcal{F}_t \) is the \( \sigma \)-algebra generated by the measurements process \( y(\cdot) \) up to time \( t \). Our purpose is to find \( \hat{x}(t) \) that maximizes \( E[\tau \mid y_0^T] \) in the class of continuous adapted functions.

### 4 The joint pdf of the discrete process

The pdf of a trajectory of \( (e_N(t), y_N(t)) \) is the Gaussian

\[
p_N(e_1, e_2, \ldots, e_N; y_1, y_2, \ldots, y_N; t_1, t_2, \ldots, t_N) = \prod_{k=1}^N \left[ \exp \left\{ -\frac{B_k(x_k, x_{k-1})}{2\varepsilon^2 \Delta t} \right\} \right] \frac{1}{2\pi \varepsilon^2 \rho \sigma \Delta t}, \tag{17}
\]
where the exponent is the quadratic form
\[ B_k(x_k, x_{k-1}) = [x_k - x_{k-1} - \Delta t a_{k-1}]^T B [x_k - x_{k-1} - \Delta t a_{k-1}], \]
such that
\[ x_k = \begin{bmatrix} e_k \\ y_k \end{bmatrix}, \quad a_k = \begin{bmatrix} M_x(e_k, t_k) \\ H_x(e_k, t_k) \end{bmatrix}, \quad B = \begin{bmatrix} \sigma^{-2} & 0 \\ 0 & \rho^{-2} \end{bmatrix}. \]

The Wiener path integral \[ \text{[27], [28], [29], [30], [31]} \]
\[ p(e, y, t \mid \xi, \eta, s) = \]
\[ \lim_{N \to \infty} \int_L dc_1 \int_L dc_2 \cdots \int_L dc_{N-1} \int_\mathbb{R} dy_1 \int_\mathbb{R} dy_2 \cdots \int_\mathbb{R} dy_{N-1} \times \]
\[ \prod_{k=1}^{N} \exp \left\{ \frac{-B_k(x_k, x_{k-1})}{2\varepsilon^2 \Delta t} \right\} \]
\[ \frac{\exp \left\{ \frac{1}{\varepsilon^2 \rho^2} H_x(e_{k-1}, t_{k-1}) (y_k - y_{k-1}) - \frac{1}{2\varepsilon^2 \rho^2} H_x^2(e_{k-1}, t_{k-1}) \Delta t \right\}}{\sqrt{2\pi \Delta t \varepsilon \rho}} \]
with \( e_N = e, \ y_N = y, \ e_0 = \xi, \ y_0 = \eta \), is the solution of the FPE (7) with the boundary and initial conditions (8) and (9).

The pdf (17) can be written as
\[ p_N(e_1, e_2, \ldots, e_N; y_1, y_2, \ldots, y_N; t_1, t_2, \ldots, t_N) = \]
\[ \prod_{k=1}^{N} \frac{1}{\sqrt{2\pi \Delta t \varepsilon \sigma}} \exp \left\{ \frac{\left[ e_k - e_{k-1} - \Delta t M_x(e_{k-1}, t_{k-1}) \right]^2}{2\varepsilon^2 \sigma^2 \Delta t} \right\} \times \]
\[ \exp \left\{ \frac{1}{\varepsilon^2 \rho^2} H_x(e_{k-1}, t_{k-1}) (y_k - y_{k-1}) - \frac{1}{2\varepsilon^2 \rho^2} H_x^2(e_{k-1}, t_{k-1}) \Delta t \right\} \times \]
\[ \frac{\exp \left\{ \frac{(y_k - y_{k-1})^2}{2\varepsilon^2 \rho^2 \Delta t} \right\}}{\sqrt{2\pi \Delta t \varepsilon \rho}} \]
where, by the Feynman-Kac formula \[ \text{[27], [28], [29], [30], [31]} \], the first product gives in the
limit the function
\[ \varphi(e, t, \rho) = \lim_{N \to \infty} \int_L de_1 \int_L de_2 \cdots \int_L de_N \prod_{k=1}^{N-1} \int_L \frac{1}{\sqrt{2\pi \Delta t \varepsilon \sigma}} \times \]
\[ \exp \left\{ -\frac{[e_k - e_{k-1} - \Delta t M_\hat{x}(e_{k-1}, t_{k-1})]^2}{2 \varepsilon^2 \sigma^2 \Delta t} \right\} \times \]
\[ \exp \left\{ \frac{1}{\varepsilon^2 \rho^2} H_\hat{x}(e_{k-1}, t_{k-1})(y_k - y_{k-1}) - \frac{1}{2 \varepsilon^2 \rho^2} H^2_\hat{x}(e_{k-1}, t_{k-1}) \Delta t \right\}, \]
which is the solution of the Zakai’s equation in Stratonovich form
\[ dS \varphi(e, t, \rho) = \left\{ -[M_\hat{x}(e, t)\varphi(e, t)]_e + \frac{1}{2} \varepsilon^2 \sigma^2 \varphi(e, t)_ee - \frac{\varphi(e, t)H^2_\hat{x}(e, t)}{2 \varepsilon^2 \rho^2} \right\} dt + \frac{\varphi(e, t)H_\hat{x}(e, t)}{\varepsilon^2 \rho^2} dS y(t), \quad (20) \]
with the boundary conditions
\[ \varphi(e, t, \rho) = 0 \text{ for } e \in \partial L. \quad (21) \]
Therefore the joint density
\[ p_N(e_N, t_N; y_1, y_2, \ldots, y_N) = \]
\[ \Pr\{e_N(t_N) = e_N, \tau > t; y_N(t_1) = y_1, y_N(t_2) = y_2, \ldots, y_N(t_N) = y_N\} \]
can be written at \( t = t_N, e_N = e \) as
\[ p_N(e, t; y_1, y_2, \ldots, y_N) = [\varphi(e, t, \rho) + o(1)] \prod_{k=1}^{N-1} \frac{1}{\sqrt{2\pi \Delta t \varepsilon \rho}} \exp \left\{ -\frac{(y_k - y_{k-1})^2}{2 \varepsilon^2 \rho^2 \Delta t} \right\}, \quad (22) \]
where \( o(1) \to 0 \) as \( N \to \infty \). Equivalently,
\[ \varphi(e, t, \rho) = \prod_{k=1}^{N} \frac{p_N(e, t; y_1, y_2, \ldots, y_N)}{\sqrt{2\pi \Delta t \varepsilon \rho}} \exp \left\{ -\frac{(y_k - y_{k-1})^2}{2 \varepsilon^2 \rho^2 \Delta t} \right\} + o(1), \quad (23) \]
which can be interpreted as follows: \( \varphi(e, t, \rho) \) is the joint conditional density of \( e_N(t) \) and \( \tau > t \), given the entire trajectory \( \{y_N(t_i)\}_{i=0}^{N} \), however, the probability density of the trajectories \( \{y_N(t_i)\}_{i=0}^{N} \),
is Brownian, rather than the a priori density imposed by (5), (6).

Now,
\[
\Pr\{\tau > t_N, y_N(t_1) = y_1, y_N(t_2) = y_2, \ldots, y_N(t_N) = y_N\} =
\]
\[
\Pr\{\tau > t_N | y_N(t_1) = y_1, y_N(t_2) = y_2, \ldots, y_N(t_N)\} \times
\]
\[
\Pr\{y_N(t_1) = y_1, y_N(t_2) = y_2, \ldots, y_N(t_N) = y_N\},
\]
which we abbreviate to
\[
\Pr\{\tau > t, y_0^t\} = \Pr\{\tau > t | y_0^t\} p_N(y_0^t), \tag{24}
\]
where the density \(p_N(y_0^t) = \Pr\{y_N(t_1) = y_1, y_N(t_2) = y_2, \ldots, y_N(t_N) = y_N\}\) is defined by the system (11), (12), independently of \(\hat{x}(t)\).

We now use the abbreviated notation (24) to write
\[
\Pr\{\tau > t | y_0^t\} = \Pr\{\tau > t | y_0^t\} p_N(y_0^t)
\]
\[
= \int_L \frac{p_N(e, t; y_1, y_2, \ldots, y_N)}{p_N(y_0^t)} de
\]
\[
= \frac{p_N^B(y_0^t)}{p_N(y_0^t)} \int_L \{\varphi(e, t, \rho) + o(1)\} \, de. \tag{25}
\]
As \(N \to \infty\), both sides of eq. (25) converge to a finite limit, which we write as
\[
\Pr\{\tau > t | y_0^t\} = \alpha(t) \int_L \varphi(e, t) \, de, \tag{26}
\]
where
\[
\alpha(t) = \lim_{N \to \infty} \frac{p_N^B(y_0^t)}{p_N(y_0^t)},
\]
is a function independent of \(\hat{x}(t)\).

Next, we show that \(E[\tau | y_0^t]\), as defined in (15), (16), is given by
\[
E[\tau | y_0^t] = \int_0^\infty \Pr\{\tau > t | y_0^t\} \, dt. \tag{27}
\]
Indeed, since \(\Pr\{\tau > t | y_0^t\} \to 0\) exponentially fast as \(t \to \infty\), we can write
\[
\int_0^\infty \Pr\{\tau > t | y_0^t\} \, dt = \lim_{T \to \infty} \int_0^T \Pr\{\tau < t | y_0^t\}
\]
and
\[
\int_0^T \Pr\{\tau < t | y_0^t\} = \lim_{N \to \infty} \sum_{i=1}^N i \Delta t \Delta \Pr\{\tau < i \Delta t | y_0^{i\Delta t}\},
\]
where
\[
\Delta \Pr\{\tau < i\Delta t \mid y_0^{(i\Delta t)}\} = \Pr\{\tau < i\Delta t \mid y_0^{(i\Delta t)}\} - \Pr\{\tau < (i-1)\Delta t \mid y_0^{(i-1)\Delta t}\}.
\]

Now, we renumber the sampled trajectories \(e_{j,N}(t_i)\) in the numerator in (15) according to increasing \(\tau_{i,N}\), so that in the new enumeration \(\tau_{i,N} = i\Delta t\). Then we group together the terms in the sum that have the same \(\tau_{i,N}\) and denote their sums \(m_{i,N}\), so that (15) becomes
\[
E\left[\tau_N \wedge T \mid \{y_N(t_i), i = 0, 1, \ldots, \left\lfloor \frac{\tau_N}{\Delta t} \right\rfloor \wedge N\}\right] = \frac{\sum_{i=1}^{N} i\Delta t m_{i,N}}{N \sum_{i=1}^{N} m_{i,N}}. \tag{28}
\]

Finally, we identify
\[
\Delta \Pr\{\tau < i\Delta t \mid y_0^{(i\Delta t)}\} = \frac{m_{i,N}}{N \sum_{i=1}^{N} m_{i,N}} (1 + o(1))
\]
where \(o(1) \to 0\) as \(N \to \infty\). Hence (27) follows. Finally, we identify
\[
\Delta \Pr\{\tau < i\Delta t \mid y_0^{(i\Delta t)}\} = \frac{m_{i,N}}{\sum_{i=1}^{N} m_{i,N}} (1 + o(1))
\]
where \(o(1) \to 0\) as \(N \to \infty\). Hence (27) follows.

4.1 Asymptotic solution of Zakai’s equation and the optimal filter

For small \(\varepsilon\) the solution of (20) with the boundary conditions (21) is constructed by the method of matched asymptotics \[33, 16, 17\]. The outer solution is given by large deviations theory \[13, 28, 34, 35\] as
\[
\varphi_{\text{outer}}(e, t) = \exp \left\{ -\frac{\psi(e, t)}{\varepsilon^2} \right\},
\]
where
\[
\psi(e, t) = \inf_{e(\cdot) \in C^1_e([0, t])} \int_0^{t \wedge T} \left\{ \frac{\dot{e}(s) - M\dot{\hat{x}}(e(s), s)}{\sigma} \right\}^2 + \left\{ \frac{\dot{y}(s) - H\dot{\hat{x}}(e(s), s)}{\rho} \right\}^2 ds, \tag{29}
\]
and
\[
C^1_e([0, t]) = \{ e(\cdot) \in C^1([0, t]) : e(0) = e \}.
\]
We denote by $\tilde{e}(t)$ the minimizer of the integral on the right hand side of eq. (29). The outer solution $\varphi_{\text{outer}}(e, t)$ does not satisfy the boundary conditions (21), so a boundary layer correction $k(e, t, \varepsilon)$ is needed to obtain a uniform asymptotic approximation,

$$\varphi(e, t) \sim \varphi_{\text{uniform}}(e, t) = \varphi_{\text{outer}}(e, t, \rho)k(e, t, \varepsilon) = \exp \left\{ -\frac{\psi(e, t)}{\varepsilon^2} \right\} k(e, t, \varepsilon).$$

The boundary layer function has to satisfy the boundary and matching conditions

$$k(e, t, \varepsilon) = 0 \quad \text{for} \quad e \in \partial L, \quad \lim_{\varepsilon \to 0} k(e, t, \varepsilon) = 1 \quad \text{for} \quad e \in L,$$

uniformly on compact subsets of the interior of $L$.

Since the survival probability is

$$\Pr \{ \tau > t \mid y_0^t \} = \int_L \alpha(t) \exp \left\{ -\frac{\psi(e, t)}{\varepsilon^2} \right\} k(e, t, \varepsilon) \, de,$$

the MTLL, according to (27), is given by

$$E[\tau \mid y_0^t] = \int_0^\infty \int_L \alpha(t) \exp \left\{ -\frac{\psi(e, t)}{\varepsilon^2} \right\} k(e, t, \varepsilon) \, de \, dt.$$  

In view of (3), the minimizer $\tilde{e}(t)$ of the integral on the right hand side of (29) can be represented as $\tilde{e}(t) = \tilde{x}(t) - \hat{x}(t)$, where $\tilde{x}(t)$ is the minimizer of the integral

$$\Psi(x, t) = \inf_{x(\cdot) \in C^1_x([0, t])} \int_0^{t \wedge \tilde{\tau}} \left\{ \left[ \frac{\dot{x}(s) - m(x(s), s)}{\sigma} \right]^2 + \left[ \frac{\dot{y}(s) - h(x(s), s)}{\rho} \right]^2 \right\} ds,$$

where $\tilde{\tau} = \inf\{ t : \tilde{x}(t) - \hat{x}(t) \in \partial L \}$ and

$$C^1_x([0, t]) = \{ x(\cdot) \in C^1([0, t]) : x(0) = x \}.$$  

Writing $\psi(e, t) = \Psi(x, t)$ and $k(e, t, \varepsilon) = K(x, t, \varepsilon)$, we rewrite (32) as

$$E[\tau \mid y_0^t] = \int_0^\infty \int_{L + \hat{x}(t)} \alpha(t) \exp \left\{ -\frac{\Psi(x, t)}{\varepsilon^2} \right\} K(x, t, \varepsilon) \, dx \, dt.$$  

The integral in (34) is evaluated for small $\varepsilon$ by the Laplace method, in which the integrand is approximated by a Gaussian density with mean $\tilde{x}(t)$ and variance proportional to $\varepsilon^2$. It is obviously maximized over the functions $\hat{x}(t)$ by choosing $\hat{x}(t)$ so that the domain of integration covers as much as possible of the area under the Gaussian bell. If $L$ is an interval, then the choice $\hat{x}(t) = \tilde{x}(t)$ is optimal. We conclude that for small noise, the minimum noise energy filter $\tilde{x}(t)$ is asymptotically the maximum MTLL filter.
5 Discussion

The main result of this paper is a proof that for small noise, the minimum noise energy filter maximizes the mean time the estimation error stays within a given region, e.g., maximizes the mean time to lose lock in problems of phase tracking and synchronization. The MNE filter is not finite-dimensional, however finite discrete approximations, such as Viterbi-type algorithms [37], [38], can give arbitrary accuracy. The practical aspects of finding the true MNE filter, or otherwise adequate approximations for it, was partially dealt with in [21] and still remains an interesting issue for further studies.

Katzur et. al. [36], and subsequently Picard [11][12], have shown that for nonlinear, but monotone measurement functions, the MNE filter is to leading order identical to the extended Kalman filter. However, for measurement functions which are non-monotone, this is apparently not the case. Ezri [20] and Fischler [21] have considered the problem of phase filtering and smoothing respectively, in which the stochastic phase process $x(t)$ is measured in a low noise channel by the vector function $h(x) = [\sin(x), \cos(x)]^T$. They show that there is a huge gap between the MTLLs of the extended Kalman filter (smoother) or particle filter, and the MNE filter (smoother), respectively.

The great advantage of the MNE filter in the case of phase estimation is explained by the observation that finite-dimensional approximations to the MAP or minimal MSEE filters (the EKF or the finite dimensional filters of Katzur [36]), do not capture large deviations of the signal or of the measurements noise. They are optimal only near local maxima of the a posteriori probability density. The MNE filter, in contrast, is a global MAP estimator and can track large deviations. Thus, it is less vulnerable to loss of lock phenomena, relative to the above mentioned filters.

Acknowledgment: The authors thank B.Z. Bobrovsky, O. Zeitouni, D. Ezri, B. Nadler and A. Taffia for useful discussions.

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