Existence of HKT metrics on hypercomplex manifolds of real dimension 8

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Abstract
A hypercomplex manifold $M$ is a manifold equipped with three complex structures satisfying quaternionic relations. Such a manifold admits a canonical torsion-free connection preserving the quaternion action, called Obata connection. A quaternionic Hermitian metric is a Riemannian metric on which is invariant with respect to unitary quaternions. Such a metric is called HKT if it is locally obtained as a Hessian of a function averaged with quaternions. HKT metric is a natural analogue of a Kähler metric on a complex manifold. We push this analogy further, proving a quaternionic analogue of Buchdahl-Lamari’s theorem for complex surfaces. Buchdahl and Lamari have shown that a complex surface $M$ admits a Kahler structure iff $b_1(M)$ is even. We show that a hypercomplex manifold $M$ with Obata holonomy $SL(2,\mathbb{H})$ admits an HKT structure iff $H^{0,1}(M) = H^1(\mathcal{O}_M)$ is even.

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1 Introduction

1.1 Hypercomplex manifolds: definition and examples

Hypercomplex manifolds are the closest quaternionic counterparts of complex manifolds. They were much studied by physicists during 1980-ies and 1990-ies, but their mathematical properties still remain a puzzle. One obstacle comes from the fact that compact hypercomplex manifold are non-Kähler (unless they are hyperkähler; see [V6]). Hypercomplex manifolds appear to be one of the more-studied and better understood classes of non-Kähler manifolds, which in bigger generality remain mysterious. There are many interesting examples of hypercomplex manifolds and many general theorems, especially about manifolds admitting HKT-metrics (Definition 2.1) or with trivial canonical bundle (Subsection 2.2).

Definition 1.1: Let $M$ be a smooth manifold equipped with endomorphisms $I, J, K : TM \longrightarrow TM$, satisfying the quaternionic relation $I^2 = J^2 = K^2 = -1$. Then $M$ is called a hypercomplex manifold.

### References

[V6]: V. V. Verbitsky, Hypercomplex manifolds and hyperkähler manifolds, J. Geom. Phys. 39 (2001), 195-211.
$IJK = -\text{Id}$. Suppose that $I, J, K$ are integrable almost-complex structures. Then $(M, I, J, K)$ is called a hypercomplex manifold.

**Theorem 1.2:** (Obata, 1955, [Ob])
On any hypercomplex manifold there exists a unique torsion-free connection $\nabla$, called Obata connection, such that $\nabla I = \nabla J = \nabla K$.

**Remark 1.3:** The holonomy of Obata connection lies in $GL(n, \mathbb{H})$.

**Remark 1.4:** A torsion-free connection $\nabla$ on $M$ with $\text{Hol}(\nabla) \subset GL(n, \mathbb{H})$ defines a hypercomplex structure on $M$.

**Example 1.5:** A Hopf surface $M = \mathbb{H}\setminus 0/\mathbb{Z} \cong S^1 \times S^3$. The holonomy of Obata connection $\text{Hol}(M) = \mathbb{Z}$.

**Example 1.6:** Compact holomorphically symplectic Kähler manifolds are hyperkähler (by Calabi–Yau theorem), hence hypercomplex. Here $\text{Hol}(M) \subset Sp(n)$ (this holonomy property is equivalent to being hyperkähler).

**Proposition 1.7:** A compact hypercomplex manifold $(M, I, J, K)$ with $(M, I)$ of Kähler type also admits a hyperkähler structure.

**Proof:** [V6 Theorem 1.4].

**Remark 1.8:** In quaternionic dimension 1, compact hypercomplex manifolds are classified by C. P. Boyer ([Bo]). This is the complete list: torus, K3 surface, Hopf surface.

**Example 1.9:** The Lie groups

\[
SU(2l+1), \quad T^l \times SU(2l), \quad T^l \times SO(2l + 1), \\
T^{2l} \times SO(4l), \quad T^l \times Sp(l), \quad T^2 \times E_6, \\
T^7 \times E^7, \quad T^8 \times E^8, \quad T^4 \times F_4, \quad T^2 \times G_2,
\]

admit a left-invariant hypercomplex structure ([SSTV], [J1]). Obata holonomy of these manifolds (and other homogeneous hypercomplex manifolds constructed by Joyce) is unknown, but most likely it is maximal, that is, equal to $GL(n, \mathbb{H})$.

**Theorem 1.10:** (Soldatenkov, [Sol])
Holonomy of Obata connection on $SU(3)$ is $GL(2, \mathbb{H})$.  

An important subgroup of $GL(n, \mathbb{H})$ is its commutator $SL(n, \mathbb{H})$. In the standard representation by real matrices this is the subgroup matrices with determinant one. As noted in [Hit] it is also isomorphic to one of the real forms of $SL(2n, \mathbb{C})$ denoted by $SU^*(2n)$ in [Hel]. In the present note we focus on manifolds with holonomy in $SL(n, \mathbb{H})$.

**Example 1.11:** Many nilmanifolds (quotients of a nilpotent Lie group by a cocompact lattice) admit hypercomplex structures. In this case, $\text{Hol}(M) \subset SL(n, \mathbb{H})$ ([BDV]).

### 1.2 Main result: existence of HKT-metrics on $SL(2, \mathbb{H})$-manifolds

**Definition 1.12:** Let $(M, I, J, K)$ be a hypercomplex manifold, and $g$ a Riemannian metric. We say that $g$ is quaternionic Hermitian if $I, J, K$ are orthogonal with respect to $g$.

**Claim 1.13:** Quaternionic Hermitian metrics always exist.

**Proof:** Take any Riemannian metric $g$ and consider its average $\text{Av}_{SU(2)} g$ with respect to $SU(2) \subset \mathbb{H}^*$. ■

Given a quaternionic Hermitian metric $g$ on $(M, I, J, K)$, consider its Hermitian forms

$$\omega_I(\cdot, \cdot) = g(\cdot, I \cdot), \omega_J, \omega_K$$

(real, but not closed). Then $\Omega = \omega_J + \sqrt{-1} \omega_K$ is of Hodge type $(2,0)$ with respect to $I$.

**Remark 1.14:** If $d\Omega = 0$, the manifold $(M, I, J, K, g)$ is hyperkähler (this is one of the definitions of a hyperkähler manifold; see [Bes]).

**Definition 1.15:** (Howe, Papadopoulos, [HP])

Let $(M, I, J, K)$ be a hypercomplex manifold, $g$ a quaternionic Hermitian metric, and $\Omega = \omega_J + \sqrt{-1} \omega_K$ the corresponding $(2,0)$-form. We say that $g$ is HKT (“hyperkähler with torsion”) if $\partial \Omega = 0$.

**Remark 1.16:** HKT-metrics play in hypercomplex geometry the same role as Kähler metrics play in complex geometry.
They admit a smooth potential (locally; see [BS]). There is a notion of an “HKT-class” (similar to Kähler class) in a certain finite-dimensional cohomology group, called Bott–Chern cohomology group (Subsection 5.1). Two metrics in the same HKT-class differ by a potential, which is a function.

When \((M, I)\) has trivial canonical bundle, a version of Hodge theory is established ([V2]), giving an \(\mathfrak{sl}(2)\)-action on holomorphic cohomology \(H^*(M, \mathcal{O}(M, I))\) and analogue of Hodge decomposition and \(dd^c\)-lemma.

Originally, it was conjectured that all hypercomplex manifolds are HKT. The first counterexample to that assertion is due to Fino and Grantcharov ([FG]); for more examples of non-HKT manifolds, see [BDV] and [SV].

The main result of this paper is the following theorem.

**Theorem 1.17:** Let \((M, I, J, K)\) be a compact hypercomplex manifold with Obata holonomy in \(SL(2, \mathbb{H})\). Then \(M\) is HKT if and only if \(\dim H^1(\mathcal{O}(M, I))\) is even.

**Proof:** Corollary 7.9

**Remark 1.18:** Using the Hodge decomposition on \(H^*(\mathcal{O}(M, I))\), one can show that \(h^1(\mathcal{O}(M, I))\) is even for any \(SL(n, \mathbb{H})\)-manifold admitting an HKT-structure ([V2, Theorem 10.2]).

### 1.3 Harvey–Lawson duality argument and Lamari’s theorem

The proof of Theorem 1.17 is based on the same arguments as used by Lamari ([L]) to prove that any complex surface with even \(b_1\) is Kähler. However, in the hypercomplex case this result is (surprisingly) much easier to prove than in the complex case.

We need the following version of Hahn–Banach theorem:

**Theorem 1.19:** (Hahn–Banach separation theorem, [Sch])

Let \(V\) be a locally convex topological vector space, \(A \subset V\) an open convex subset of \(V\), and \(W\) a closed subspace of \(V\) satisfying \(W \cap A = \emptyset\). Then, there is a continuous linear functional \(\theta\) on \(V\), such that \(\theta|_A > 0\) and \(\theta|_W = 0\). □
As an illustration, we state the original Harvey–Lawson duality theorem, which is used as a template for many other similar arguments, developed since then.

**Theorem 1.20:** (Harvey, Lawson, [HL1])
Let $M$ be a compact complex non-Kähler manifold. Then there exists a positive $(n - 1, n - 1)$-current $\xi$ which is a $(n - 1, n - 1)$-part of an exact current.

**Idea of a proof:** Hahn–Banach separation theorem is applied to the set $A$ of strictly positive $(1, 1)$-forms, and the set $W$ of closed $(1, 1)$-forms, obtaining a current $\xi \in \mathcal{D}^{n-1,n-1}(M) = \Lambda^{1,1}(M)^* \text{ positive on } A \text{ (that is, positive)} \text{ and vanishing on } W$. The later condition (after some simple cohomological manipulations) becomes “$(n - 1, n - 1)$-part of an exact current”.

This approach was further developed some 15 years later by Buchdahl and Lamari, giving the following theorem.

**Theorem 1.21:** (Buchdahl–Lamari, [Bu, L])
Let $M$ be a compact complex surface. Then $M$ is Kähler if and only if $b_1(M)$ is even.

This theorem was known since mid-1980-ies, but its proof was based on Kodaira classification of complex surfaces, taking hundreds (if not thousands) of pages and a complicated result of Siu, who proved that all K3 surfaces are Kähler, and Buchdahl–Lamari (in two independent papers, [Bu] and [L]) gave a direct proof.

**Scheme of Lamari’s proof:**

**Step 1:** Evenness of $b_1(M)$ is equivalent to $dd^c$-lemma.

**Step 2:** Using regularization of positive currents ([D]), one proves that existence of Kähler current (positive, closed current $\xi$, such that $\xi - \omega$ is positive for some Hermitian form $\omega$) is equivalent to existence of a Kähler form.

**Step 3:** Existence of a Kähler current is equivalent to non-existence of a positive current $\xi$ which is a limit of $dd^c$-closed positive forms and equal to an $(1, 1)$-part of an exact current.
Step 4: Non-existence of such $\xi$ is implied by $dd^c$-lemma.

We are lucky that for HKT-manifolds the regularization of currents is not necessary and $dd^c$-lemma (or, more precisely, its quaternionic analogue) is the only non-trivial step.

2 Hypercomplex manifolds: basic notions

2.1 HKT-manifolds

The notion of an HKT-manifold was introduced by the physicists, but it proved to be immensely useful in mathematics.

A hypercomplex manifold is a manifold equipped with almost-complex structure operators $I, J, K : TM \to TM$, integrable and satisfying the standard quaternionic relations $I^2 = J^2 = K^2 = IJK = -\text{Id}_TM$.

This gives a quaternionic algebra action on $TM$; the group $Sp(1) \cong SU(2)$ of unitary quaternions acts on all tensor powers of $TM$ by multiplicativity.

A quaternionic Hermitian structure on a hypercomplex manifold is an $SU(2)$-invariant Riemannian metric. Such a metric gives a reduction of the structure group of $M$ to $Sp(n) = U(n, \mathbb{H})$.

With any quaternionic Hermitian structure on $M$ one associates a non-degenerate $(2,0)$-form $\Omega \in \Lambda^{2,0}_I(M)$, as follows.

Consider the differential forms

$$
\omega_I(\cdot, \cdot) := g(\cdot, I\cdot), \quad \omega_J(\cdot, \cdot) := g(\cdot, J\cdot), \quad \omega_K(\cdot, \cdot) := g(\cdot, K\cdot)
$$

It is easy to check that the form $\Omega := \omega_I + \sqrt{-1}\Omega_K$ is of Hodge type $(2,0)$ with respect to $I$.

If the form $\Omega$ is closed, one has $d\omega_I = d\omega_J = d\omega_K = 0$, and the manifold $(M, I, J, K, g)$ is called hyperkähler ([Bes]). The hyperkähler condition is very restrictive.

Definition 2.1: A hypercomplex, quaternionic Hermitian manifold $(M, I, J, K, g)$ is called an HKT-manifold (hyperkähler with torsion) if $\partial\Omega = 0$, where $\partial$ denotes the $(1,0)$-part of the differential with respect to $I$. In other words, a manifold is HKT if $d\Omega \in \Lambda^{2,1}_I(M)$.

The form $\Omega \in \Lambda^{2,0}_I(M)$ is called an HKT-form on $(M, I, J, K)$. 

$^1\Lambda^*(M)$ denotes the bundle of differential forms, and $\Lambda^*(M) = \bigoplus_{p,q} \Lambda^{p,q}_I(M)$ its Hodge decomposition, taken with respect to the complex structure $I$ on $M$. 

Remark 2.2: The quaternionic Hermitian form $g$ can be easily reconstructed from $\Omega$. Indeed, for any $x, y \in T^{1,0}_I(M)$, one has

$$2g(x, \overline{y}) = \Omega(x, J(y)),$$

as a trivial calculation implies.

Let $(M, I, J, K)$ be a hypercomplex manifold. We extend $J : \Lambda^1(M) \rightarrow \Lambda^1(M)$ to $\Lambda^*(M)$ by multiplicativity. Recall that

$$J(\Lambda^{p,q}_I(M)) = \Lambda^{q,p}_I(M),$$

because $I$ and $J$ anticommute on $\Lambda^1(M)$. Denote by $\partial_J : \Lambda^{p,q}_I(M) \rightarrow \Lambda^{p+1,q}_I(M)$ the operator $J^{-1} \circ \overline{\partial} \circ J$, where $\overline{\partial} : \Lambda^{p,q}_I(M) \rightarrow \Lambda^{p,q+1}_I(M)$ is the standard Dolbeault operator on $(M, I)$, that is, the $(0,1)$-part of the de Rham differential. Since $\overline{\partial}^2 = 0$, we have $\partial_J^2 = 0$. In [V2], it was shown that $\partial$ and $\partial_J$ anticommute:

$$\{\partial_J, \partial\} = 0. \tag{2.2}$$

The pair of anticommuting differentials $\partial, \partial_J$ is a hypercomplex counterpart to the pair $d, d^c := Id^c$ of differentials on a complex manifold.

2.2 An introduction to $SL(n, \mathbb{H})$-geometry

As Obata has shown ([O]), a hypercomplex manifold $(M, I, J, K)$ admits a necessarily unique torsion-free connection, preserving $I, J, K$. The converse is also true: if a manifold $M$ equipped with an action of $\mathbb{H}$ admits a torsion-free connection preserving the quaternionic action, it is hypercomplex. This implies that a hypercomplex structure on a manifold can be defined as a torsion-free connection with holonomy in $GL(n, \mathbb{H})$. This connection is called the Obata connection.

Connections with restricted holonomy are one of the central notions in Riemannian geometry, due to Berger’s classification of irreducible holonomy of Riemannian manifolds. However, a similar classification exists for general torsion-free connections ([MS]). In the Merkulov–Schwachhöfer list, only
three subgroups of $GL(n, \mathbb{H})$ occur. In addition to the compact group $Sp(n)$ (which defines hyperkähler geometry), also $GL(n, \mathbb{H})$ and its commutator $SL(n, \mathbb{H})$ appear, corresponding to hypercomplex manifolds and hypercomplex manifolds with trivial determinant bundle, respectively. Both of these geometries are interesting, rich in structure and examples, and deserve detailed study.

It is easy to see that $(M, I)$ has holomorphically trivial canonical bundle, for any $SL(n, \mathbb{H})$-manifold $(M, I, J, K)$ ([V5]). For a hypercomplex manifold with trivial canonical bundle admitting an HKT-metric, a version of Hodge theory was constructed ([V2]). Using this result, it was shown that a compact hypercomplex manifold with trivial canonical bundle has holonomy in $SL(n, \mathbb{H})$, if it admits an HKT-structure ([V5]).

In [BDV], it was shown that holonomy of all hypercomplex nilmanifolds lies in $SL(n, \mathbb{H})$. Many working examples of hypercomplex manifolds are in fact nilmanifolds, and by this result they all belong to the class of $SL(n, \mathbb{H})$-manifolds.

The $SL(n, \mathbb{H})$-manifolds were studied in [AV2] and [V7]. On such manifolds the quaternionic Dolbeault complex is identified with a part of de Rham complex (Proposition 3.7), making it possible to write a quaternionic version of the Monge-Ampère equation ([AV2]), and to use quaternionic linear algebra to study positive currents on hyperkähler manifolds ([V7]). Under this identification, $\mathbb{H}$-positive forms become positive in the usual sense, and $\partial$, $\partial_J$-closed or exact forms become $\partial, \overline{\partial}$-closed or exact (Proposition 3.7, (iv)). This linear-algebraic identification is especially useful in the study of the quaternionic Monge-Ampère equation ([AV2]).

One of the main subjects of the present paper is a quaternionic version of the $dd^c$-lemma, called "$\partial\partial_J$-lemma".

**Theorem 2.3:** Let $M$ be a compact $SL(n, \mathbb{H})$-manifold admitting an HKT metric, and $\eta$ a $\partial_J$-closed, $\partial$-exact $(p, 0)$-form. Then $\eta$ lies in the image of $\partial\partial_J$.

**Proof:** In [V2] Theorem 10.2, it was shown that for any HKT-manifold, the Laplacian $\Delta_\partial := \partial\partial^* + \partial^*\partial$ on $\Lambda^{p,0}(M) \otimes K_M^{1/2}$ can be written as $\Delta_\partial = \{\partial, \{\partial_J, \Lambda_\Omega\}\}$, where $\{\cdot, \cdot\}$ denotes the anticommutator. Then $\Delta_\partial \eta = \partial\partial_J \Lambda_\Omega \eta$. However, since $\eta$ is exact, it is orthogonal to the kernel of $\Delta_\partial$, giving $\eta = G\Delta_\partial \eta$, where $G$ is the corresponding Green operator. This gives

$$\eta = G\Delta_\partial \eta = G\partial\partial_J \Lambda_\Omega \eta = \partial\partial_J G\Lambda_\Omega \eta.$$
However, on $SL(n, \mathbb{H})$-manifold, the canonical bundle is trivial, and this result can be applied to any $\eta \in \Lambda^{p,0}(M)$. ■

3 Quaternionic Dolbeault complex on a hypercomplex manifold

3.1 Quaternionic Dolbeault complex: a definition

It is well-known that any irreducible representation of $SU(2)$ over $\mathbb{C}$ can be obtained as a symmetric power $\text{Sym}^i(V_1)$, where $V_1$ is a fundamental 2-dimensional representation. We say that a representation $W$ has weight $i$ if it is isomorphic to $\text{Sym}^i(V_1)$. A representation is said to be pure of weight $i$ if all its irreducible components have weight $i$.

Remark 3.1: The Clebsch–Gordan formula (see [Hu]) claims that the weight is multiplicative, in the following sense: if $i \leq j$, then

$$V_i \otimes V_j = \bigoplus_{k=0}^{i} V_{i+j-2k},$$

where $V_i = \text{Sym}^i(V_1)$ denotes the irreducible representation of weight $i$.

Let $M$ be a hypercomplex manifold, $\dim_{\mathbb{H}} M = n$. There is a natural multiplicative action of $SU(2) \subset \mathbb{H}^*$ on $\Lambda^*(M)$, associated with the hypercomplex structure.

Let $V^i \subset \Lambda^i(M)$ be a maximal $SU(2)$-invariant subspace of weight $< i$. The space $V^i$ is well defined, because it is a sum of all irreducible representations $W \subset \Lambda^i(M)$ of weight $< i$. Since the weight is multiplicative (Remark 3.1), $V^* = \bigoplus_i V^i$ is an ideal in $\Lambda^*(M)$.

It is easy to see that the de Rham differential $d$ increases the weight by 1 at most. Therefore, $dV^i \subset V^{i+1}$, and $V^* \subset \Lambda^*(M)$ is a differential ideal in the de Rham DG-algebra $(\Lambda^*(M), d)$.

Definition 3.2: Denote by $(\Lambda^*_+(M), d_+)$ the quotient algebra $\Lambda^*(M)/V^*$. It is called the quaternionic Dolbeault algebra of $M$, or the quaternionic Dolbeault complex (qD-algebra or qD-complex for short).
Remark 3.3: The complex $(\Lambda^*_+ (M), d_+)$ was constructed earlier by Capria and Salamon, ([CS]) in a different (and much more general) situation, and much studied since then.

3.2 The Hodge decomposition of the quaternionic Dolbeault complex

The Hodge bigrading is compatible with the weight decomposition of $\Lambda^*(M)$, and gives a Hodge decomposition of $\Lambda^*_+(M)$ ([V2]):

$$\Lambda^i_+(M) = \bigoplus_{p+q=i} \Lambda^{p,q}_{+,I}(M).$$

The spaces $\Lambda^{p,q}_{+,I}(M)$ are the weight spaces for a particular choice of a Cartan subalgebra in $\mathfrak{su}(2)$. The $\mathfrak{su}(2)$-action induces an isomorphism of the weight spaces within an irreducible representation. This gives the following result.

Proposition 3.4: Let $(M, I, J, K)$ be a hypercomplex manifold and

$$\Lambda^i_+(M) = \bigoplus_{p+q=i} \Lambda^{p,q}_{+,I}(M)$$

the Hodge decomposition of qD-complex defined above. Then there is a natural isomorphism

$$\Lambda^{p,q}_{+,I}(M) \cong \Lambda^{p+q,0}(M, I). \quad (3.1)$$

Proof: See [V2].

This isomorphism is compatible with a natural algebraic structure on

$$\bigoplus_{p+q=i} \Lambda^{p+q,0}(M, I),$$

and with the Dolbeault differentials, in the following way.

Consider the quaternionic Dolbeault complex $(\Lambda^*_+(M), d_+)$ constructed in Subsection 3.1. Using the Hodge bigrading, we can decompose this complex, obtaining a bicomplex

$$\Lambda^{*,*}_{+,I}(M) \xrightarrow{d^{1,0}_{+,I}, d^{0,1}_{+,I}} \Lambda^{*,*}_+(M)$$
where $d_{+1}^{0,0}, d_{-1}^{0,1}$ are the Hodge components of the quaternionic Dolbeault differential $d_+$, taken with respect to $I$.

**Theorem 3.5:** Under the multiplicative isomorphism

$$\Lambda_{+1}^{p,q}(M) \cong \Lambda_{+1}^{p+q,0}(M, I)$$

constructed in Proposition 3.4, $d_{+1}^{1,0}$ corresponds to $\partial$ and $d_{+1}^{0,1}$ to $\partial J$:

$$\Lambda_{+1}^{0,0}(M) \cong \Lambda_{+1}^{1,1}(M)$$

Moreover, under this isomorphism, $\omega_I \in \Lambda_{+1}^{1,1}(M)$ corresponds to $\Omega \in \Lambda_{+1}^{2,0}(M)$.

**Proof:** See [V2] or [V4].

### 3.3 Positive $(2,0)$-forms on hypercomplex manifolds

The notion of positive $(2p,0)$-forms on hypercomplex manifolds (sometimes called $q$-positive, or $\mathbb{H}$-positive) was developed in [V1] and [AV1] (see also [AV2] and [V7]). For our present purposes, only $(2,0)$-forms are interesting, but everything can be immediately generalized to a general situation

Let $\eta \in \Lambda_{+1}^{p,q}(M)$ be a differential form. Since $I$ and $J$ anticommute, $J(\eta)$ lies in $\Lambda_{+1}^{p,q}(M)$. Clearly, $J^2|_{\Lambda_{+1}^{p,q}(M)} = (-1)^{p+q}$. For $p + q$ even, $J|_{\Lambda_{+1}^{p,q}(M)}$ is an anticomplex involution, that is, a real structure on $\Lambda_{+1}^{p,q}(M)$. A form $\eta \in \Lambda_{+1}^{2p,0}(M)$ is called real if $J(\eta) = \eta$. We denote real forms in $\Lambda_{+1}^{2p,0}(M)$ by $\Lambda_{+1}^{2p,0}(M, I)$.

For a real $(2,0)$-form $\eta$,

$$\eta(x, J(\tau)) = \overline{\eta}(J(x), J^2(\tau)) = \overline{\eta}(\tau, J(x)),$$

for any $x \in T_+^{1,0}(M)$. From a definition of a real form, we obtain that the scalar $\eta(x, J(\tau))$ is always real.
Definition 3.6: A real \((2,0)\)-form \(\eta\) on a hypercomplex manifold is called positive if \(\eta(x,J(x)) \geq 0\) for any \(x \in T^1_{1,0}(M)\), and strictly positive if this inequality is strict, for all \(x \neq 0\).

An HKT-form \(\Omega \in \Lambda^2_{1,0}(M)\) of any HKT-structure is strictly positive, as follows from Remark 2.2. Moreover, HKT-structures on a hypercomplex manifold are in one-to-one correspondence with closed, strictly positive \((2,0)\)-forms.

The analogy between Kähler forms and HKT-forms can be pushed further: it turns out that any HKT-form \(\Omega \in \Lambda^2_{1,0}(M)\) has a local potential \(\varphi \in C^\infty(M)\), in such a way that \(\partial\partial J \varphi = \Omega\) ([BS], [AV1]). Here \(\partial\partial J\) is a composition of \(\partial\) and \(\partial J\) defined on the quaternionic Dolbeault complex as above (these operators anticommute).

### 3.4 The map \(V_{p,q} : \Lambda^p_{1} \Lambda^+_{q,0}(M) \rightarrow \Lambda^{n+p,n+q}_{1}(M)\) on \(SL(n,\mathbb{H})\)-manifolds

Let \((M, I, J, K)\) be an \(SL(n,\mathbb{H})\)-manifold, \(\dim \mathbb{R} M = 4n\), and

\(\mathcal{R}_{p,q} : \Lambda^p_{1} \Lambda^+_{q,0}(M) \rightarrow \Lambda^{p,q}_{1,+}(M)\)

the isomorphism induced by \(\mathfrak{su}(2)\)-action as in Theorem 3.5. Consider the projection

\[\Lambda^p_{1,0}(M) \rightarrow \Lambda_{1,+}^{p,q}(M),\tag{3.3}\]

and let \(R : \Lambda_{1}^{p,q}(M) \rightarrow \Lambda^{p+q,0}_{1}(M)\) denote the composition of (3.3) and \(\mathcal{R}_{p,q}^{-1}\).

Let \(\Phi\) be a nowhere degenerate holomorphic section of \(\Lambda^{2n,0}_{1}(M)\). Assume that \(\Phi\) is real, that is, \(J(\Phi) = \overline{\Phi}\), and positive. Existence of such a form is equivalent to \(\text{Hol}(M) \subset SL(n,\mathbb{H})\) ([V5]). It is often convenient to define \(SL(n,\mathbb{H})\)-structure by fixing the quaternionic action and the holomorphic form \(\Phi\).

Define the map

\[V_{p,q} : \Lambda^p_{1} \Lambda^+_{q,0}(M) \rightarrow \Lambda^{n+p,n+q}_{1}(M)\]

by the relation

\[V_{p,q}(\eta) \wedge \alpha = \eta \wedge R(\alpha) \wedge \overline{\Phi},\tag{3.4}\]

for any test form \(\alpha \in \Lambda^{n-p,n-q}_{1}(M)\).
The map $V_{p,p}$ is especially remarkable, because it maps closed, positive $(2p,0)$-forms to closed, positive $(n+p,n+p)$-forms, as the following proposition implies.

**Proposition 3.7:** Let $(M,I,J,K,\Phi)$ be an $SL(n,\mathbb{H})$-manifold, and

$$V_{p,q} : \Lambda^{p,q+0}(M) \rightarrow \Lambda^{4n-p,4n-q}(M)$$

the map defined above. Then

(i) $V_{p,q}(\eta) = R_{p,q}(\eta) \wedge V_{0,0}(1)$.

(ii) The map $V_{p,q}$ is injective, for all $p,q$.

(iii) $(\sqrt{-1})^{(n-p)^2}V_{p,p}(\eta)$ is real if and only $\eta \in \Lambda^{2p,0}(M)$ is real, and weakly positive if and only if $\eta$ is weakly positive.

(iv) $V_{p,q}(\partial \eta) = \partial V_{p-1,q}(\eta)$, and $V_{p,q}(\overline{\partial} \eta) = \overline{\partial} V_{p,q-1}(\eta)$.

(v) $V_{0,0}(1) = \lambda R_{n,n}(\Phi)$, where $\lambda$ is a positive rational number, depending only on the dimension $n$.

**Proof:** See [V7], Proposition 4.2, or [AV2], Theorem 3.6. ■

### 4 Quaternionic Gauduchon metrics

#### 4.1 Gauduchon metrics

**Definition 4.1:** A Hermitian metric $\omega$ on a complex $n$ manifold is called **Gauduchon** if $\partial \overline{\partial} \omega^{n-1} = 0$.

**Theorem 4.2:** Every Hermitian metric on a compact complex manifold is conformally equivalent to a Gauduchon metric, which is unique in its conformal class, up to a constant multiplier.

**Proof:** [Gä]. ■

Gauduchon metrics is one of the very few instruments available for the study of general non-Kähler manifolds, and probably the most important one.
4.2 Gauduchon metrics and hypercomplex structures

Let \( g \) be a quaternionic Hermitian metric on a hypercomplex manifold \( M \).
Consider the corresponding \((2,0)\)-form \( \Omega := \omega_J + \sqrt{-1} \omega_K \) defined as in Remark 2.2. From the definition of positive \((2,0)\)-forms it follows that this correspondence is bijective: quaternionic Hermitian metrics are in \((1,1)\)-correspondence with positive \((2,0)\)-forms.

Definition 4.3: A quaternionic Hermitian form \( g \) on a hypercomplex manifold \( M \), \( \dim_H M = n \), is called quaternionic Gauduchon if \( \partial \partial_J \Omega^{n-1} = 0 \), where \( \Omega = \omega_J + \sqrt{-1} \omega_K \) is the corresponding positive \((2,0)\)-form.

Proposition 4.4: Let \((M, I, J, K, \Phi)\) be an \( SL(n, \mathbb{H}) \)-manifold equipped with a quaternionic Hermitian form \( g \), and

\[
|\Phi|^2 := \frac{\Phi \wedge \overline{\Phi}}{(2^n 2n!)^{-1} \omega_I^{2n}}
\]

Then the following conditions are equivalent.

(i) \( g \) is quaternionic Gauduchon.

(ii) The Hermitian metric \( |\Phi|^{-1}g \) is Gauduchon on \((M, I)\).

(iii) The Hermitian metric \( |\Phi|^{-1}g \) is Gauduchon with respect to any of the induced complex structures \( L = aI + bJ + cK \).

Proof: The equivalence (i) \( \iff \) (ii) follows from

\[
\mathcal{V}_{n-1,n-1}(\Omega^{n-1}) = |\Phi|^{-1} \omega_I^{2n-1},
\]

proven in [GV] (the formula in the proof of Theorem 6.4). So, using Proposition 3.7 (iv), we have that \( \mathcal{V}_{n,n}(\partial \partial_J \Omega^{n-1}) = \partial \overline{\partial}(|\Phi|^{-1} \omega_I^{2n-1}) \).

Corollary 4.5: For any \( SL(n, \mathbb{H}) \)-manifold equipped with a quaternionic Hermitian form, there exists a unique (up to a constant multiplier) positive function \( \mu \) such that \( \mu g \) is quaternionic Gauduchon.

4.3 Surjectivity of \( f \longrightarrow \Omega^{n-1} \wedge \partial \partial_J f / \)

We are interested in quaternionic Gauduchon forms because of the following theorem.
Theorem 4.6: Let \((M, I, J, K, \Omega, \Phi)\) be a compact quaternionic Hermitian \(SL(n, \mathbb{H})\)-manifold. Assume that \(\Omega\) is quaternionic Gauduchon. Consider the map \(D : C^\infty(M) \to \Lambda^{4n}(M)\),

\[ D(f) = \partial \partial_J f \wedge \Omega^{n-1} \wedge \Phi. \]

Then \(D\) induces a bijection between \(C^\infty(M)/\text{const}\) and the space of exact \(4n\)-forms on \(M\).

**Proof:**

**Step 1:** Clearly, \(D\) is elliptic, and has index 0, because it has the same symbol as Laplacian, which is self-adjoint.

**Step 2:** E. Hopf maximum principle ([GT]) implies that \(\ker D = \text{const}\). Therefore, \(\text{coker } D\) is 1-dimensional. It remains to show that \(\text{im } D\) consists of exact \(4n\)-forms.

**Step 3:**

\[
\int_M \partial \partial_J f \wedge \Omega^{n-1} \wedge \Phi = -\int_M f \wedge \partial \partial_J (\Omega^{n-1}) \wedge \Phi = 0
\]

because \(\Omega\) is quaternionic Gauduchon. This implies that all forms in \(\text{im } D\) are exact. Converse is also true, because \(\text{codim im } D = 1\). 

5 Quaternionic Aeppli and Bott–Chern cohomology

Throughout this section, \((M, I, J, K, g)\) is a compact hypercomplex manifold equipped with a quaternionic Hermitian metric \(g\). Recall that \(\{\partial, \partial_J\} = 0\).

5.1 Quaternionic Bott–Chern cohomology

Define \(H_{BC}^{p,0}(M)\) to be the group

\[
H_{BC}^{p,0}(M) = \frac{\{\varphi \in \Lambda_J^{p,0}(M) \mid \partial \varphi = \partial_J \varphi = 0\}}{\partial \partial_J \Lambda_J^{p-2,0}(M)}.
\]

**Theorem 5.1:** The group \(H_{BC}^{p,0}(M)\) is finite dimensional.

**Proof:** We consider the following operator

\[
\Delta_{BC} = \partial^* \partial + \partial_J^* \partial_J + \partial \partial_J \partial_J^* \partial^* + \partial_J^* \partial^* \partial \partial_J + \partial^* \partial \partial_J \partial_J^* + \partial^* \partial_J \partial_J^* \partial.
\]
acting on $\Lambda^p_0(M)$. Here, $\partial^*$ (resp. $\partial_J^*$) is the adjoint of $\partial$ (resp. $\partial_J$) with respect to $g$.

We claim that $\Delta_{BC}$ is a fourth order self-adjoint elliptic operator. Using the elliptic theory, we obtain the following decomposition

$$\Lambda^p_0(M) = \mathcal{H}_{\Delta_{BC}} \oplus \text{im} \Delta_{BC},$$

$$= \mathcal{H}_{\Delta_{BC}} \oplus \text{im} \partial \partial_J \oplus (\text{im} \partial^* + \text{im} \partial_J^*),$$

where $\mathcal{H}_{\Delta_{BC}} = \{ \varphi \in \Lambda^p_0(M) | \partial \varphi = \partial_J \varphi = \partial_J^* \partial^* \varphi = 0 \}$ is the kernel of $\Delta_{BC}$.

Furthermore, for $\varphi \in \Lambda^p_0(M)$, we write $\varphi = \varphi_H + \partial \partial_J \rho + \partial^* \alpha + \partial_J^* \beta$, where $\varphi_H \in \mathcal{H}_{\Delta_{BC}}$. Then, $\partial \varphi = \partial_J \varphi = 0$ is equivalent to $\partial^* \alpha + \partial_J^* \beta = 0$. Thus, we deduce

$$\ker \partial|_{\Lambda^p_0(M)} \cap \ker \partial_J|_{\Lambda^p_0(M)} = \mathcal{H}_{\Delta_{BC}} \oplus \text{im} \partial \partial_J.$$

\section{5.2 Quaternionic Aeppli cohomology}

In a similar way, we define $H^p_0(M)$ to be the group

$$H^p_0(M) = \frac{\{ \varphi \in \Lambda^p_0(M) | \partial \partial_J \varphi = 0 \}}{\partial \Lambda^{p-1,0}_0(M) + \partial_J \Lambda^{p-1,0}_0(M)}.$$

\textbf{Theorem 5.2:} The group $H^p_0(M)$ is finite dimensional.

\textbf{Proof:} Here, we consider the operator

$$\Delta_{AE} = \partial \partial^* + \partial_J \partial_J^* + \partial \partial_J \partial^* + \partial_J \partial^* \partial \partial_J + \partial \partial_J \partial_J \partial^* + \partial_J \partial^* \partial \partial_J,$$

acting on $\Lambda^p_0(M)$. The operator $\Delta_{AE}$ is a fourth order self-adjoint elliptic operator having the same symbol as $\Delta_{BC}$. We have then

$$\Lambda^p_0(M) = \mathcal{H}_{\Delta_{AE}} \oplus \text{im} \Delta_{AE},$$

$$= \mathcal{H}_{\Delta_{AE}} \oplus \text{im} \partial_J^* \partial^* \oplus (\text{im} \partial + \text{im} \partial_J),$$

where $\mathcal{H}_{\Delta_{AE}} = \{ \varphi \in \Lambda^p_0(M) | \partial^* \varphi = \partial_J^* \varphi = \partial \partial_J \varphi = 0 \}$ is the kernel of $\Delta_{AE}$.

Moreover, if $\varphi \in \Lambda^p_0(M)$ is decomposed as $\varphi = \varphi_H + \partial_J \partial^* \rho + \partial \alpha + \partial_J \beta$, where $\varphi_H \in \mathcal{H}_{\Delta_{AE}}$, then $\partial \partial_J \varphi = 0$ is equivalent to $\partial_J \partial^* \rho = 0$. We obtain that

$$\ker \partial \partial_J|_{\Lambda^p_0(M)} = \mathcal{H}_{\Delta_{AE}} \oplus (\text{im} \partial + \text{im} \partial_J).$$
Remark 5.3: The groups $H_{BC}^{p,0}(M)$ and $H_{AE}^{2n-p,0}(M)$ are dual when $M$ is a compact $SL(n,\mathbb{H})$-manifold. Indeed, let $\Phi$ be a nowhere degenerate holomorphic section of $\Lambda_1^{2n,0}(M)$. We assume also that $\Phi$ is real and positive. We consider the pairing on $H_{BC}^{p,0}(M) \times H_{AE}^{2n-p,0}(M)$ given by

\[ ([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta \wedge \Phi. \]

One can check that this pairing is well defined (recall that $\partial \Phi = \partial J \Phi = 0$) and non-degenerate.

6 $\partial \partial_J$-lemma in $\dim \mathbb{H} = 2$.

Definition 6.1: Let $(M, I, J, K, \Omega, \Phi)$ be a compact quaternionic Gauduchon $SL(n,\mathbb{H})$-manifold, and $H_{AE}^0(M)$, $H_{BC}^0(M)$ the quaternionic Aeppli and Bott–Chern cohomology. Consider the map $\text{deg} : H_{AE}^{1,0}(M) \to \mathbb{C}$ putting $\alpha$ to $\int \partial \alpha \wedge \Omega^{n-1} \wedge \Phi$. Since $\Omega$ is quaternionic Gauduchon, $\text{deg} \alpha$ is independent from the choice of $\alpha$ in its cohomology class. We call $\text{deg}$ the degree map.

Remark 6.2: Consider the natural map $H_{AE}^{1,0}(M) \to H_{BC}^{2,0}(M)$. The kernel of this map consists of all cohomology classes $\alpha$ such that $\partial \alpha = \partial J \beta$, hence the form $\alpha - \partial J \beta$ cohomological to $\alpha$ in $H_{AE}^{1,0}(M)$ is $\partial$-closed. We obtain that the kernel of $H_{AE}^{1,0}(M) \to H_{BC}^{2,0}(M)$ is identified with the space $H_{\partial}^{1,0}(M) = \ker \partial \bigg|_{\Lambda_{1,0}(M)} / \text{im} \partial$.

Lemma 6.3: ($\partial \partial_J$-lemma for $H_{AE}^{1,0}(M)$)

Let $(M, I, J, K, \Omega, \Phi)$ be a compact quaternionic Gauduchon $SL(2,\mathbb{H})$-manifold. Let $\theta \in \Lambda_{1,0}(M)$ be a $\partial_J$-exact, $\partial$-closed form. Then $\theta = 0$.

Proof: Let $\theta = 6_J(f)$. Then $\partial \partial_J(f) = \partial \theta = 0$. However, the map $f \mapsto \frac{\partial \partial_J(f) \wedge \Omega}{\Omega^2}$ is an elliptic operator with vanishing constant term, hence any function in its kernel is constant by Hopf maximum principle ($\text{GT}$).

Corollary 6.4: On a compact $SL(2,\mathbb{H})$-manifold, the natural map

\[ H_{\partial}^{1,0}(M) \to H_{AE}^{1,0}(M) \]
is injective.

**Theorem 6.5:** Let \((M, I, J, K, \Omega, \Phi)\) be a compact quaternionic Gauduchon \(SL(2, \mathbb{H})\)-manifold. Then, the sequence

\[
0 \longrightarrow H^{1,0}_\partial(M) \longrightarrow H^{1,0}_{AE}(M) \xrightarrow{\deg} \mathbb{C}
\]

is exact. Moreover, the space \(\ker \deg\) is equal to the kernel of the natural map \(H^{1,0}_{AE}(M) \xrightarrow{\partial} H^{2,0}_{BC}(M)\).

**Proof:**

**Step 0:** By Lemma 6.3, the sequence (6.1) is exact in the first term. It remains to prove that (6.1) is exact in the second term and to show that \(\ker \deg \ker \partial\).

**Step 1:** Let \(\alpha \in \ker \deg\). By Theorem 4.6, there exists \(f \in C^\infty(M)\) such that \((\partial \alpha + \partial \partial J f) \wedge \Omega \wedge \Phi = 0\), equivalently \((\partial \alpha + \partial \partial J f) \wedge \Omega = 0\). Replacing \(\alpha\) by \(\alpha + \partial J f\) in the same cohomology class, we may assume that \(\partial \alpha \wedge \Omega = 0\).

**Step 2:** Since \(\partial \alpha\) is primitive, one has \(\int_M \partial \alpha \wedge \partial J \alpha \wedge \Phi = -\|\partial \alpha\|^2\) by a quaternionic version of Hodge–Riemann relations ([V3, Theorem 6.3]).

**Step 3:** However,

\[
\|\partial \alpha\|^2 = \int_M \partial \alpha \wedge \partial J \alpha \wedge \Phi = -\int_M \partial \partial J \alpha \wedge \alpha \wedge \Phi = 0,
\]

hence \(\partial \alpha = 0\). This implies that \(\ker \deg = \ker \partial\). \(\blacksquare\)

The \(\partial \partial J\)-lemma for even \(h^1(O_M)\) follows directly from the above theorem.

**Theorem 6.6:** Let \((M, I, J, K, \Phi)\) be a compact \(SL(2, \mathbb{H})\)-manifold. Then \(\partial \partial J\)-lemma holds on \(\Lambda^{2,0}(M)\) if and only if \(h^1(O_M)\) is even.

**Proof:**

**Step 1:** Clearly, \(\partial \partial J\)-lemma is equivalent to vanishing of \(\partial : H^{1,0}_{AE}(M) \longrightarrow H^{2,0}_{BC}(M)\), but the kernel of this map is \(H^{1,0}_\partial(M) = \ker \deg\) (Theorem 6.5), hence it suffices to show that the degree map vanishes iff \(h^1(O_M)\) is even.

**Step 2:** Since \(J\) defines the quaternionic structure on \(H^{1,0}_{AE}(M)\), this space is even-dimensional. Now, from the exact sequence

\[
0 \longrightarrow H^{1,0}_\partial(M) \longrightarrow H^{1,0}_{AE}(M) \xrightarrow{\deg} \mathbb{C},
\]

hence \(h^1(O_M)\) is even.
we obtain that deg = 0 whenever $H^{1,0}_\partial(M)$ is even-dimensional. The space $H^{1,0}_\partial(M)$ is complex conjugate to $H^1(O_M)$, hence has the same dimension. ■

7 Currents in HKT-geometry

7.1 Cohomology of currents

Definition 7.1: Let $(M, I, J, K)$ be a hypercomplex manifold. Denote by $\mathcal{D}_{p,q}(M)$ the topological dual to the Fréchet space $\Lambda^p_q(M)$. An element $T \in \mathcal{D}_{p,q}(M)$ is called a current of bidimension $(p, q)$ and it has a compact support on $M$. Denote by $\mathcal{D}_{p,q}(M) = \mathcal{D}_{2n-p,2n-q}(M)$, where $\dim_{\mathbb{R}} M = n$.

The complex structure $J$ acts naturally on $\mathcal{D}_{p,q}(M)$ as a map

$$J : \mathcal{D}_{p,q}(M) \to \mathcal{D}_{q,p}(M)$$

in the following way

$$(JT)(\varphi) = T(J\varphi),$$

for $\varphi \in \Lambda^{2n-q, 2n-p}_I(M)$ with compact support. The operators $d, \partial, \overline{\partial}$ are extended in the standard way using the Stokes theorem, for example $\partial : \mathcal{D}_{p,q}(M) \to \mathcal{D}_{p+1,q}(M)$ is expressed as $\partial T(\varphi) = (-1)^{\dim \varphi} T(\partial \varphi)$, where $\varphi \in \Lambda^{2n-p-1, 2n-q}_I(M)$. Similarly, we can define $\partial_J = J^{-1} \circ \overline{\partial} \circ J$ on $\mathcal{D}_{p,q}(M)$.

Definition 7.2: A current $T \in \mathcal{D}^{2n,0}(M)$ is called real if $JT = T$ and we denote real currents by $\mathcal{D}^{2p,0}_{\mathbb{R}}(M)$.

The following result is a currents version of local $\partial \partial_J$-lemma, due to Banos and Swann in the smooth case.

Proposition 7.3: Let $T \in \mathcal{D}^{2p,0}_{\mathbb{R}}(M)$ be a real $\partial$-closed current. Then, locally $T$ can be written in the form $T = \partial \partial_J \varphi$, for some real generalized function $\varphi$.

Proof: We use essentially the same arguments as in the proof of the main theorem in [BS]. Let $T$ as above. We write $T = T_J + \sqrt{-1} T_K$. Since $T$ is real and $\partial$-closed, a straightforward verification shows that $T_I(\cdot, \cdot) = T_J(\cdot, K \cdot)$ is $I$-invariant and that $Id T_I = J d T_J = K d T_K$.

Let $Z = M \times S^2$ be the twistor space of $M$ and we consider the current $\eta \in \mathcal{D}^{0,2}(Z)$ given by $\eta = (T_I)^{(0,2)}_z$ i.e. the $(0, 2)$-part of $T_I$ with respect to the
complex structure $\mathcal{I}|(\rho, \bar{\rho}) = aI + bJ + cK$ $(\bar{\rho} = (a, b, c) \in \mathbb{R}^3$, $a^2 + b^2 + c^2 = 1$).

A direct computation shows that $\overline{\partial}\partial I = 0$. By a 1-pseudo-convexity argument and the $\partial\bar{\partial}$-Poincaré Lemma (for currents), locally $\eta = \overline{\partial}(\alpha + \sqrt{-1}\mathcal{I}\alpha)$ where $\alpha$ is a real current defined locally in $M$. Hence, the real part of $\eta$ is given by $\frac{1}{2}(d\alpha - \mathcal{I} d\alpha)$. It follows that $d\alpha$ is a closed $I$-invariant current. Hence, by the $\partial\bar{\partial}$-Poincaré Lemma (for currents), $\mathcal{T}_I = \frac{1}{2}(dd_I \varphi + d_J d_K \varphi)$ for some real generalized function $\varphi$. By [AV1], this implies that locally $T = \partial\bar{\partial}\varphi$.

Using Definition 3.6, we give the following:

**Definition 7.4:** On a $SL(n, \mathbb{H})$-manifold $(M, I, J, K, \Phi)$, a current $T \in \mathcal{D}_{2n-2,0}^0(M)$ is said to be **positive** if (locally) $T \wedge \alpha \wedge \mathcal{I}$ is a positive measure for any choice of (local) real strictly positive $(2,0)$-form $\alpha$.

**Definition 7.5:** A generalized function is called **plurisubharmonic** if $\partial\bar{\partial}\varphi$ is a positive $(2,0)$-current.

**Theorem 7.6:** A plurisubharmonic generalized function is subharmonic with respect to any quaternionic Hermitian metric (hence, constant on any compact hypercomplex manifold).

**Proof:** [HL2, Lemma 3.6].

**Remark 7.7:** We consider the group

$$H_{\mathbb{R}}^{2,0}(M) = \left\{ u \in \mathcal{D}_{\mathbb{R}}^{2,0}(M) \mid \partial u = 0 \right\} \frac{\partial\partial}{\partial\bar{\partial}} \mathcal{D}_{\mathbb{R}}^{0,0}(M).$$

Denote by $\mathcal{H}$ the sheaf of real generalized functions satisfying $\partial\partial f = 0$. By the proof of [Lemma 6.3], elements of $\mathcal{H}$ satisfy an elliptic equation. Elliptic regularity implies that all functions in $\mathcal{H}$ are smooth.

The sheaf $\mathcal{H}$ admits two resolutions starting by

$$0 \to \mathcal{H} \to \Lambda_{\mathbb{R}}^{0,0}(M, I) \to \Lambda_{\mathbb{R}}^{2,0}(M, I) \to \Lambda^{3,0}(M, I) \to 0$$

$$\begin{array}{c}
0 \\
\mathcal{H} \\
\downarrow id \\
\Lambda_{\mathbb{R}}^{0,0}(M, I) \to \Lambda_{\mathbb{R}}^{2,0}(M, I) \\
i d \\
\Lambda^{3,0}(M, I) \\
\end{array}$$

$$0 \to \mathcal{D}_{\mathbb{R}}^{0,0}(M) \to \mathcal{D}_{\mathbb{R}}^{2,0}(M) \to \mathcal{D}_3^3(M) \to 0$$
where $i$ is the inclusion of forms in the space of currents. We deduce that

$$H^2_\mathbb{R}(M) \simeq \frac{\varphi \in \Lambda^2(M, I) \mid \partial \varphi = 0}{\partial I \Lambda^2_\mathbb{R}(M, I)}.$$ 

Let

$$H^{2n-2}_\mathbb{R}(M) = \left\{ u \in D^{2n-2}_\mathbb{R}(M) \mid \partial J u = 0 \right\}.$$ 

Then, by the same argument in [Remark 5.3], we deduce that $H^{2n-2}_\mathbb{R}(M)$ and $H^2_\mathbb{R}(M)$ are dual when $M$ is a compact $SL(n, \mathbb{H})$-manifold.

### 7.2 Harvey–Lawson’s theorem in HKT-geometry

Using the Hahn–Banach Separation Theorem [Theorem 1.19], we obtain the following.

**Theorem 7.8:** Let $(M, I, J, K, \Phi)$ be an $SL(n, \mathbb{H})$-manifold. Then $M$ admits no HKT-metric if and only if it admits a $\partial$-exact, real, positive $(2n-2,0)$-current.

**Proof:**

**Step 1:** Apply Hahn–Banach separation theorem to the space $A$ of positive, real $(2,0)$-forms and $W$ of $\partial$-closed real $(2,0)$-forms to obtain a current $\xi \in \Lambda^2_\mathbb{R}(M, I)^*$ which is positive on $A$ (hence, real and positive) and vanishes on $W$. Such a current exists iff $A \cap W = \emptyset$, or, equivalently, when $M$ is not HKT.

**Step 2:** Consider the pairing $\langle \eta, \nu \rangle = \int_M \eta \wedge \nu \wedge \Phi$ on $(p,0)$-forms. This pairing is compatible with $\partial$ and $\partial J$ and allows one to identify the currents $\Lambda^2_\mathbb{R}(M, I)^*$ with $\Lambda^{n-2}_\mathbb{R}(M, I) \otimes C^\infty(M)^*$, where $C^\infty(M)^*$ denotes generalized functions. This identification is compatible with $\partial$ and $\partial J$, and cohomology of currents are the same as cohomology of forms (Remark 7.7).

**Step 3:** Since $\langle \xi, W \rangle = 0$, for each $\eta$ one has $0 = \langle \xi, \partial \eta \rangle = \langle \partial \xi, \eta \rangle$, giving $\partial \xi = 0$. It remains to show that the cohomology class of $\xi$ in $H^2_\partial(\Lambda^* M)$ vanishes.

**Step 4:** The Serre’s duality gives a non-degenerate pairing $\langle [\xi], [\nu] \rangle \rightarrow \mathbb{R}$ on cohomology classes in $H^2_\partial(\Lambda^*_0(M))$:

$$\xi, \nu \rightarrow \int_M \xi \wedge \nu \wedge \Phi.$$
Since \( \langle [\xi], [\nu] \rangle = 0 \) for each \( \partial \)-closed \( \nu \), the cohomology class of \( \xi \) also vanishes.

**Corollary 7.9:** Let \( M \) be a compact \( SL(2, \mathbb{H}) \)-manifold. Then \( M \) admits HKT-metric if and only if \( H^1(\mathcal{O}(M, I)) \) is even-dimensional.

**Proof:** Even-dimensionality of \( H^1(\mathcal{O}(M, I)) \) for HKT-manifolds with holonomy in \( SL(n, \mathbb{H}) \) follows from [V2, Theorem 10.2]. Conversely, suppose that \( H^1(\mathcal{O}(M, I)) \) is even-dimensional, but \( M \) is not HKT. Then Theorem 7.8 implies that there exists a real, positive, exact \((2,0)\)-current \( \xi \). However, \( \xi \) is \( \partial \partial_J \)-exact by Theorem 6.6, hence \( \xi = \partial \partial_J f \), for some \( f \in C^\infty(M) \). Such \( f \) is a quaternionic plurisubharmonic function, which has to vanish by Theorem 7.6.

### 8 Examples

The known examples of manifolds with holonomy \( SL(n, \mathbb{H}) \) are either nilmanifolds ([BDV]) or obtained via the twist construction of A. Swann [S], which is based on previous examples by D. Joyce. The later construction provides also simply-connected examples. We describe briefly a simplified version of it.

Let \((X, I, J, K, g)\) be a compact hyperkähler manifold. By definition, an anti-self-dual 2-form on it is a form which is of type \((1,1)\) with respect to \( I \) and \( J \) and hence with respect to all complex structures of the hypercomplex family. Let \( \alpha_1, \ldots, \alpha_{4k} \) be closed 2-forms representing integral cohomology classes on \( X \). Consider the principal \( T^{4k} \)-bundle \( \pi : M \to X \) with characteristic classes determined by \( \alpha_1, \ldots, \alpha_{4k} \). It admits a connection \( A \) given by \( 4k \) 1-forms \( \theta_i \) such that \( d\theta_i = \pi^*(\alpha_i) \). Define an almost-hypercomplex structure on \( M \) in the following way: on the horizontal spaces of \( A \) we have the pull-backs of \( I, J, K \) and on the vertical spaces we fix a linear hypercomplex structure of the \( 4k \)-torus. The structures \( \mathcal{I}, \mathcal{J}, \mathcal{K} \) on \( M \) are extended to act on the cotangent bundle \( T^*M \) using the following relations:

\[
\begin{align*}
\mathcal{I}(\theta_{4i+1}) &= \theta_{4i+2}, & \mathcal{I}(\theta_{4i+3}) &= \theta_{4i+4}, & \mathcal{J}(\theta_{4i+1}) &= \theta_{4i+3}, & \mathcal{J}(\theta_{4i+2}) &= -\theta_{4i+4}, \\
\mathcal{I}(\pi^*\alpha) &= \pi^*(I\alpha), & \mathcal{J}(\pi^*\alpha) &= \pi^*(J\alpha),
\end{align*}
\]

for any 1-form \( \alpha \) on \( X \) and \( i = 0, \ldots, k - 1 \).

It follows from [S] or by direct and easy calculations, that \( \mathcal{I} \) is integrable iff \( \alpha_{4i+1} + i\alpha_{4i+2} \) and \( \alpha_{4i+3} + i\alpha_{4i+4} \) are of type \((2,0) + (1,1)\) with respect to \( \mathcal{I} \) and \( \mathcal{J} \).
to \( I \) for every \( i = 1, \ldots, k \). Similarly \( J \) is integrable iff \( \alpha_{4i+1} + i\alpha_{4i+3} \) and \( \alpha_{4i+2} - i\alpha_{4i+4} \) are of type \((2,0)+(1,1)\) with respect to \( J \) for every \( i = 1, \ldots, k \).

Similarly, one can define a quaternionic Hermitian metric on \( M \) from \( g \) and a fixed hyperkähler metric on \( T^4k \) using the splitting of \( T(M) \) in horizontal and vertical subspaces. As A. Swann [S] has shown the structure has a holonomy in \( SL(n, \mathbb{H}) \) and is HKT when all forms \( \alpha_l \) are self-dual (of type \((1,1)\) with respect to all structures).

As a particular case, assume \( X \) to be a \( K3 \) such that there are 3 closed integral forms which define a hyperkähler structure and a self-dual integral class, so defining a principal \( T^4 \)-bundle \( M \) over \( X = K3 \) with finite fundamental group. After passing to a finite cover, we may assume that \( M \) is simply-connected. These forms satisfy the integrability condition above. If \( \alpha_2 + i\alpha_3 \) is a \((2,0)\)-form for \( I \), then \( \pi^*(\alpha_2 + i\alpha_3) = d(\theta_2 + i\theta_3) \) is an exact \((2,0)\)-form, which defines a positive current in the definition of the previous section. Then \( M \) can not admit any HKT-metric - a fact proven by A. Swann using different arguments.

We can also calculate \( \dim(H^1(O_{(M,I)})) = h^{0,1}_I(M) \) and apply Theorem 1.17 to decide the existence of HKT-structure. One can use the Borel method of doubly graded spectral sequence from [HI], Appendix B, to determine \( h^{p,q} \), but in our case, its simpler to use a more direct approach. The vector fields \( X_1, X_2, X_3, X_4 \) on \( M \) generated by torus action which are also dual to \( \theta_i \) are hyperholomorphic, so \( \mathcal{L}_{X_i} \circ \mathcal{I} = \mathcal{I} \circ \mathcal{L}_{X_i} \). We can also choose a bundle metric, which for the vertical vectors is the flat hyperkähler 4-torus metric and on the horizontal vectors is a pull-back from the hyperkähler metric from the base \( X = K3 \). The horizontal and vertical vectors are perpendicular. Such metric is hypercomplex and \( X_i \) are Killing fields. So, since they also fix the orientation, then \( \mathcal{L}_{X_i} \) commutes with the Hodge star \* for this metric. In particular, they also commute with the \( \overline{\partial} \)-Laplace operator and \( \mathcal{L}_{X_i} \alpha \) is a harmonic form for every harmonic \( \alpha \). Since \( X_i^{(0,1)} \) is a complex vector field which preserves the structure \( I \) and transforms \((0,1)\)-form into \((0,1)\)-form, for a \( \overline{\partial} \)-harmonic form \( \alpha \), we have \( \mathcal{L}_{X_i^{(0,1)}} \alpha^{(0,1)} = \overline{\partial} f \), for the function \( f = \alpha^{(0,1)}(X_i^{(0,1)}) \). Since \( \mathcal{L}_{X_i^{(0,1)}} \alpha^{(0,1)} \) is harmonic, it vanishes. Since we can use any \((0,1)\)-vector field generated by the action and any harmonic \((0,1)\)-form, in particular \( \sqrt{-1} \alpha \), we see that the vector fields \( X_i \) preserve the harmonic \((0,1)\)-forms. Then, any such form has a representation

\[
\alpha = A_1(\theta_1 - i\theta_2) + A_2(\theta_3 - i\theta_4) + \pi^*(\varphi),
\]

where \( A_i \) are pull-backs of functions on the base and \( \varphi \) is a harmonic form on the base \( X \). Since \( X \) is K3 surface, \( \varphi = 0 \). Then, from \( d\theta_i = \alpha_i \), we
have $\overline{\partial}(\theta_1 - i\theta_2) = \alpha_1 - i\alpha_2$ if $\alpha_1 - i\alpha_2$ is $(2,0)$-form and 0 if its is $(1,1)$. On the other side, $\overline{\partial}(\theta_3 - i\theta_4) = 0$, since the other characteristic classes are $(1,1)$. As a result, we see that $h^{0,1}_T(M) = 2$, if all curvature forms are $(1,1)$ (or instantons) and $h^{0,1}_T(M) = 1$, if we have one of these forms to be of type $(2,0)$. By [Theorem 1.17] in the first case there is an HKT-metric and in the second there is none.

In the construction above we can use a flat 4-tori as a base instead of $K3$ surface. Then $M$ is a nilmanifold which corresponds to an example which appeared in [FG]. Consider the nilpotent Lie algebra $\mathbb{R} \times \mathfrak{n}_7$, where $\mathfrak{n}_7$ is the algebra of the quaternionic Heisenberg group $H_7$. Its is spanned by the left-invariant vector fields $e_1, \cdots, e_8$ and is defined by the following relation on the basis of the dual 1-forms:

$$
de^i = 0, i = 1, \ldots, 5$$
$$de^6 = e^1 \wedge e^2 + e^3 \wedge e^4,$n $$de^7 = e^1 \wedge e^3 - e^2 \wedge e^4,$n $$de^8 = e^1 \wedge e^4 + e^2 \wedge e^3$$

On a compact quotient $M = \mathbb{R} \times H_7/\Gamma$, consider the family [FG] of complex structures defined via:

$$I_t(e^1) = \frac{i-1}{t} e^2, I_t(e^3) = 0, J_t(e^5) = \frac{1}{t} e^6, J_t(e^7) = e^8,$n $$J_t(e^1) = \frac{i-1}{t} e^3, J_t(e^2) = -e^4, J_t(e^5) = \frac{1}{t} e^7, J_t(e^6) = -e^8,$n

for $t \in (0,1)$. Then, for each $t$, $I_tJ_t = -J_tI_t = K_t$ defines a hypercomplex structure on $M$. Using averaging argument in [FG], it was shown that for $t = \frac{1}{2}$ the structure is HKT and for $t \neq \frac{1}{2}$ there is no HKT-metric. Here we provide a different proof using [Theorem 7.8] and [Theorem 1.17]. The manifold $M$ has a projection on $X = T^4$ which makes it a principal bundle with fiber 4-tori and base 4-tori. Then the forms $e^1, e^2, e^3, e^4$ are pull-backs from forms on the base $X$ and the forms $e^5, e^6, e^7, e^8$ are connection forms in this bundle. So (up to a constant), the characteristic classes of the bundle are $0, e^1 \wedge e^2 + e^3 \wedge e^4, e^1 \wedge e^3 - e^2 \wedge e^4, e^1 \wedge e^4 + e^2 \wedge e^3$. Now, we note that

$$d(e^7 + ie^8) = (e^1 + ie^2) \wedge (e^3 + ie^4)$$
$$= \frac{2t-1}{2t-2} \left( e^1 + i\frac{t-1}{t} e^4 \right) \wedge (e^3 + ie^4)$$
$$- \frac{1}{2t-2} \left( e^1 - i\frac{t-1}{t} e^2 \right) \wedge (e^3 + ie^4).$$

So, when $t = \frac{1}{2}$, it is of type $(1,1)$ with respect to $I_{\frac{1}{2}}$, but for $t \neq \frac{1}{2}$, it is of type $(2,0) + (1,1)$. Moreover, the $(2,0)$ component in this case is
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\[ \partial_t (e^7 - ie^8) = \frac{2i}{2t-1} (e^1 + i\frac{t-1}{t}e^4) \wedge (e^3 + ie^4), \]
which defines a positive (2,0)-current. So, there is no HKT-structure if \( t \neq \frac{1}{2} \) by Theorem 7.8. Similarly, we can calculate the Hodge number \( h^{0,1}(M, I_t) \) to check its parity. Instead of using the fibration structure, it's easier to use the result of Console and Fino ([CF]) who proved that the Dolbeault cohomology of a nilmanifold with an invariant complex structure are isomorphic to the \( \overline{\partial} \)-cohomology of the complex of invariant forms. From the defining equations above, we see that \( e^1 + ie^2, e^3 + ie^4 \) and \( e^5 - ie^6 \) are nonzero elements of \( H^{0,1}(M, I_t) \). Also, \( \overline{\partial}_t (e^7 - ie^8) = d(e^7 - ie^8)|^{(0,2)} = \frac{2i}{2t-1} (e^1 - i\frac{t-1}{t}e^4) \wedge (e^3 - ie^4) \). So, for \( t = \frac{1}{2} \), it is non-zero in the cohomology and \( h^{0,1} = 4 \). When \( t \neq \frac{1}{2} \), it is not \( \overline{\partial} \)-closed, \( h^{0,1}(M) = 3 \) and we can apply Theorem 7.8

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