Rank and select: Another lesson learned

Szymon Grabowski and Marcin Raniszewski

Lodz University of Technology, Institute of Applied Computer Science,
Al. Politechniki 11, 90–924 Łódź, Poland
{sgrabow|mranisz}@kis.p.lodz.pl

Abstract. Rank and select queries on bitmaps are essential building bricks of many compressed data structures, including text indexes, membership and range supporting spatial data structures, compressed graphs, and more. Theoretically considered yet in 1980s, these primitives have also been a subject of vivid research concerning their practical incarnations in the last decade. We present a few novel rank/select variants, focusing mostly on speed, obtaining competitive space-time results in the compressed setting. Our findings can be summarized as follows: (i) no single rank/select solution works best on any kind of data (ours are optimized for concatenated bit arrays obtained from wavelet trees for real text datasets), (ii) it pays to efficiently handle blocks consisting of all 0 or all 1 bits, (iii) compressed select does not have to be significantly slower than compressed rank at a comparable memory use.

1 Introduction

Rank and select are essential building bricks of many compressed data structures, and text indexes in particular. In their most frequently used binary incarnation, they can be defined as follows: given a bit-vector $B[0 \ldots n-1]$, $\text{rank}_b(B, i)$ returns the number of occurrences of symbol $b$ in the prefix $B[0 \ldots i]$ and $\text{select}_b(B, i)$ returns the position of the $i$-th occurrence of symbol $b$ in $B$, where $b \in \{0, 1\}$.

Note that $\text{rank}_0(B, i) = i - \text{rank}_1(B, i)$, hence it is enough to directly support the rank only for one binary symbol (e.g., 1). There is no similar relation binding the values of $\text{select}_0(B, i)$ and $\text{select}_1(B, i)$.

It is known for at least two decades [7, 2, 9] that these operations can be performed in constant time, using the extra space of $o(n)$ bits. Raman et al. [12] showed how to compress the vector $B$ to $nH_0(B) + o(n)$ bits, where $H_0(B)$ is the order-0 entropy of $B$, and still support rank and select in constant time.

Much research has been dedicated to construct rank and select solutions, both compressed and non-compressed, to answer the queries possibly fast in practice. Especially in the compressed setting also the lower-order terms of the space matter, hence the practical questions involve two aspects: the query time and the space used by the data structure. The next section briefly recalls the history of practical solutions for rank and select, starting from the non-compressed ones.
2 Related work

The original rank solution, by Jacobson [7], which needs $O(n \log \log n / \log n)$ bits of space apart from the original bit-vector, performs three memory accesses (to one superblock counter and one block counter, plus a lookup into a table with precomputed popcount answers), which often translates into three cache misses, a significant penalty. González et al. [5] proposed a scheme with one level of blocks followed by sequential scan. In theoretical terms, this solution no longer obtains both constant time and sublinear extra space, yet it fares well practically and is very simple.

Vigna [13] interleaved data from different levels to improve access locality. Gog and Petri [4] carried this idea even further, interleaving the precomputed rank values and the bit-vector data. The sequential scans over small blocks of data are performed with an efficient 64-bit hardware popcount instruction (popcnt), available in Intel and AMD processors since 2008. In the manner of the González et al. solution, Gog and Petri store only one level of precomputed ranks, yet data from two successive cache lines can be sometimes read in their scheme. In a recent work, Grabowski et al. [6] showed a solution with one cache miss in the worst case. They achieve it with interleaving 64-bit precomputed rank fields with $512 - 64 = 448$ bits of data. As 64 bits per rank is more than needed, part of this field stores popcount values for some prefixes of the following block, thus saving on the popcnt invocations.

Kärkkäinen et al. [8] proposed a hybrid scheme for the compressed rank, where the bit-vector is divided into blocks and each block is encoded separately, choosing one of a few different methods, depending on its “local” performance. This general approach was implemented in a version with three encodings: no compression (i.e., the block kept verbatim), storing the positions of the minority bits (zeros or ones, whichever have fewer occurrences in the block), and run-length encoding for runs of zeros and ones. To make the data structure even more compact, blocks are grouped into superblocks. Thanks to it, the blocks’ header data store ranks and offsets to the beginnings of the encoded block bodies with respect to the beginning of the superblock rather than the beginning of the whole structure. Only the rank operation is supported, yet the authors mention briefly a possibility to extend their scheme in order to support selects too.

As it can be implied from the literature, an efficient select is harder to design than an efficient rank, even in the non-compressed variant. Clark [2] was the first to show a constant-time select with $3n / \lceil \log \log n \rceil + O(n^{1/2} \log n \log \log n)$ bits of extra space. The solution is relatively complicated and needs at least 60% space overhead. González et al. [5] noticed that implementing select with binary search over a rank structure is often superior (in spite of having $O(\log n)$ time complexity), both in execution times and the space overhead. Yet, for large inputs or for low densities of set bits (assuming that we focus on the $\text{select}_1$ query), Clark’s solution dominates. More recently, Gog and Petri [4] presented a

---

1 Logarithms of base 2 are used throughout the paper.
practical implementation of the Clark select idea, reducing its worst-case space overhead to less than 29%.

Okanohara and Sadakane [11] were the first to consider practical compressed implementations of rank/select structures and they introduced four novel r/s dictionaries (each of which was based on different ideas), reaching different space/time tradeoffs in theory and in practice. For example, they offer to answer rank or select queries in a few tenths of a microsecond (on a 3.4 GHz Intel Xeon) spending about 25% of extra space for densely (50%) populated bit-vectors. Vigna [13] obtained similar times, yet with about twice smaller space overhead. Navarro and Providel [10] raised the bar even higher (or, should we say, lower?), reducing the space overhead to about 10% of the original bit-vector on top of the entropy, solving in this space both rank and select. Their rank queries are handled within about 0.4 µsec and select queries within 1 µsec. They also show how to reuse sampling data between the rank and the select in a non-compressed scenario, with a benefit in space, which allows to answer these queries within around 0.2 µsec, using only 3% of extra space on top of the plain bit-vector.

A unique approach was taken by Beskers and Fischer [1], who focus on sequences with low higher-order entropies. Their solutions are likely to be competitive e.g. for representing wavelet trees for repetitive collections of strings, like individual genomes of the same species.

3 Our algorithms

In the two following subsections we present our compressed rank and select variants.

3.1 Compressed rank

The input bit-vector $B$ is conceptually divided into blocks of $k$ bytes, and each run of $h$ successive blocks is grouped in a superblock. For each superblock a fixed-size header is stored, having $h$ ranks of the prefix of $B$ up to the current block and $h$ offsets to the areas storing the successive blocks. Popcounting over a block (or its prefix), with a 64-bit built-in instruction, is used in the final phase of the operation. Several variants were implemented, depending on the header and block representations. One important optimization is used in all the variants: a block consisting entirely of zero or one bits is not stored, as, during the query, such a case is easily recognized from checking two adjacent rank values. Such a block will be called a mono-block. Our variants are presented in the successive paragraphs.

**basic** We have uncompressed rank and offsets in the superblock, each stored on 4 bytes, and uncompressed bit data. (Note that the notion of a superblock is artificial for this variant, but we keep it for consistency.)
**bch (basic with compressed headers)** The first rank and the first offset in a superblock are stored on 4 bytes, while the following \( h - 1 \) ranks are stored differentially with respect to the first rank, using 2 bytes each. Additionally, we keep \( h - 1 \) bits (rounded up to a multiple of 16 bits) to tell which blocks are mono-blocks. If the query falls into a non-mono-block, knowing the first offset and checking how many mono-blocks in the superblock precede the block in question, we immediately obtain the desired offset.

**mpe1 (mono-pair elimination v1)** The bit data are compressed in a simple manner. The input block of size \( k \) bytes is divided into 2-byte chunks and the chunks consisting of all zeros (i.e., two bytes 0) or all ones (i.e., two bytes 255) are not stored in the block. Such chunks are called mono-pairs. To make decoding possible, the block is prepended with two fields of \( k/2 \) bits each (both rounded up to the nearest multiple of 16 bits), where the bits from the first field denote if the successive 2-byte chunks are stored in the block or not, and the bits from the second field distinguish between mono-pairs with zeros and mono-pairs with ones. (Note that this encoding is somewhat redundant, as e.g. \( k/2 \) trits would be enough, but we preferred convenience over striving for maximum compression.) Some blocks are however not (well) compressible, that is, the number of mono-pairs in them is less than \( k/16 \). In such cases, compression is not applied to the block. The header contains one verbatim rank and one verbatim offset value, each stored on 4 bytes, while the following \( h - 1 \) ranks and \( h - 1 \) offsets are stored differentially, with respect to the first rank/offset, using 2 bytes each. One bit out of the 2 bytes per differentially encoded offset is spent for the flag informing if compression is applied to the block.

**mpe2** This is a little refinement of the previous variant. We separately check if it pays to remove mono-pairs of zeros, and mono-pairs of ones from the block. There are four possible cases: no compression, eliminate only mono-pairs of zeros, eliminate only mono-pairs of ones, eliminate all mono-pairs, hence the choice is marked with 2 bits (again, taken from the differentially encoded offsets).

**mpe3** The rank variants involving block compression are slower than basic or bch, due to the necessity to decode a compressed block before popcounting. To alleviate this penalty, in the current variant we apply compression only to the blocks which are reduced to at most half of their original size (we have not tried to tune this criterion). In this way, weakly compressible blocks are stored verbatim and querying them is fast.

**cf (cache-friendly)** In all the variants above, answering the rank query boils down to either noticing that the query falls into a mono-block, where the answer is immediate, or not, when the data block has to be accessed and processed appropriately. The latter case is slower, partly because it typically involves another cache miss. Therefore, if the fraction of mono-blocks in a dataset is \( 0 \leq f \leq 1 \), we may expect about \( 2 - f \) cache misses per rank query, on average. The current variant aims to reduce the average number of cache misses. Let us have \( b \) blocks
in the input, \( m \) of which are mono-blocks; the value of \( m \) is found in the first pass over the data in the construction phase. We need another scan over the blocks, and let us denote the number of mono-blocks among the first \( j \) blocks with \( m_j \) (that is, \( m = m_b \)). We stop after such \( j \)-th block that \( m_j = (b - j) - (m_b - m_j) \).

This divides \( B \) into its left (the first \( j \) blocks) and right part (the remaining \( b - j \) blocks), in such a way that the number of mono-blocks in the left part is equal to the number of non-mono-blocks in the right part. The data layout is now like that. The blocks from the left part are located sequentially, each prepended with the 4-byte rank of the prefix of the sequence. The right part stores 8 bytes per block in a contiguous area: 4 bytes are for the rank and the other 4 bytes are for the offset. These offset values point to the mono-blocks from the left part. Assuming that the mono-blocks are uniformly distributed over the whole dataset, the expected number of cache misses (or non-contiguous memory accesses) per query is 

\[
 f \times 1 + (1 - f) (1 - f) \times 1 + f (1 - f) \times 2 = 1 + f - f^2 \leq 2 - f,
\]

where the equality holds only for \( f = 1 \).

3.2 Compressed select

The proposed select variants are related to the rank ones. On a high level, our solutions make use of two integer parameters, \( \ell \) and \( \text{thr} \), and the \( \text{sel}_1(B, j) \) values for all \( j \) being a multiple of \( \ell \) are stored directly. Moreover, we distinguish between sparse blocks, that is such for which \( \text{sel}_1(B, (i + 1) \ell) - \text{sel}_1(B, i \ell) > \text{thr} \), and dense blocks for which the shown condition is not fulfilled. If a block is sparse, the select values for all \( j \) inside it, i.e., such that \( i \ell < j < (i + 1) \ell \), are stored directly. For dense blocks we store the bits \( B[\text{sel}_1(B, i \ell) + 1 \ldots \text{sel}_1(B, (i + 1) \ell) - 1] \), possibly in a compressed form. Again, block headers consist of precomputed select values and offsets to block data, and blocks are grouped into superblocks. The particular variants: basic, bch, mpe1, mpe2 and mpe3, mimic the corresponding rank variants, with some exceptions. The main difference is that the select values in superblocks are never compressed (as opposed to rank values in several rank variants). Additionally, select-bch differs to its rank counterpart in having its offsets encoded differentially (on 2 bytes) with respect to the first offset in the superblock. Such a representation for select is needed to handle varying-length blocks. Note also that the differential offset encoding enforces some restrictions on the \( \ell \) and \( \text{thr} \) parameters.

4 Experimental results

All experiments were run on a machine equipped with a 6-core Intel i7 CPU (4930K) clocked at 3.4 GHz, with 64 GB of RAM, running Ubuntu 15.10 64-bit. The RAM modules were 8 × 8 GB DDR3-1600 with the timings 11-11-11 (Kingston KVR16R11D4K4/64). The CPU cache sizes were: 6 × 32 KB (data) and 6 × 32 KB (instructions) in the L1 level, 6 × 256 KB in L2 and 12 MB in L3. One CPU core was used for the computations. All codes were written in C++ and
Fig. 1. Rank query times (in ns), averaged over 100M random queries, and used spaces for real data. The datasets on the left: concatenated balanced WTs, on the right: concatenated Huffman-shaped WTs. The hyb series parameters: 8, 16, 32 and 64.
compiled with 64-bit gcc 5.2.1, with -O3 and -mpopcnt options. Our source codes are available at [http://ranisz.iis.p.lodz.pl/indexes/ranks&selects/](http://ranisz.iis.p.lodz.pl/indexes/ranks&selects/).

The test datasets comprise concatenated bit-vectors from binary wavelet trees built over four 200 MB Pizza & Chili site texts, in two versions, with standard (balanced) and Huffman-shaped WT layout. Additionally, we used random bit-vectors with density of set bits equal to 0.05 or 0.2, respectively.

For comparisons, we took a number of rank and select variants from the well-known sdsl library [3]. In particular, for the rank experiments we used:

- *hyb*, the hybrid rank from [8], with the superblock sizes of 8, 16, 32 and 64,
- *rank-V*, a variant from [13] (called *rank9* therein),
- our variants: *basic*, *bch*, *mpe1*, *mpe2*, *mpe3* and *cf*, as described in Section 3.1.

For the select experiments we took:

- Clark’s algorithm [2] in an implementation proposed in [4], called *sel-mcl*,
- *sel-RRR* [10], called *sel-R4K* in [4], where *K* is the block size,
- our variants: *basic*, *bch*, *mpe1*, *mpe2* and *mpe3*, as described in Section 3.2.

The block size in the rank variants *basic*, *bch* and *cf* was set to *k* = 64 and to *k* = 128 in the remaining ones. The number of blocks per superblock, *h*, was set to 32 for *bch* and to 16 for the other rank variants and to all select variants.

From Fig. 1 we can see that our ranks *mpe1*, *mpe2* and *mpe3* offer similar (or slightly better) compression as *hyb* at a significantly higher speed. If even more speed is preferred, then *bch* or *cf* is the method of choice. Another Pareto-optimal variant is (in most cases) *basic*. Not surprisingly, the only non-compressed solution in the comparison, *rank-V*, is the fastest, yet the gap between it and the other top contenders is often more striking in space than in time.

The implementation of *cf* is slightly more complicated (e.g., it has the extra, non-predictable, comparison at the start, to check if the parameter points to the left or to the right part of the bit-vector), and this is why it cannot beat *basic* in speed for the concatenated balanced WTs. Table 1 shows how the fraction of mono-blocks *f* affects the average number of memory accesses per query and the total query times in the rank variants *basic* and *cf*. The gap in the average number of memory accesses gets large for most Huffman-shaped WTs and there *cf* takes the lead in speed.

The proposed select variants (Fig. 2) are about 3–4 times faster than *sel-RRR63* (and even 6–7 times faster on both XML datasets), yet offering worse compression. The numbers in the names of our variants are the used parameters, e.g., variant-128-4K denotes *ℓ* = 128 and *thr* = 4096. Compared to the non-compressed select variant, *sel-mcl*, whose space requirement is about 1.13*n* in all the cases, our solutions are both more compact and faster in six cases out of eight and obtain mixed results in the remaining two cases.

Finally, we run the rank and select algorithms on uniformly random bit-vectors of size 200 MB, with the density of bits 1 set to 5% (random-5) or 20% (random-20), as presented in Fig. 3. Our rank results (the top row) in the 20%
Fig. 2. Select query times (in ns), averaged over 100M random queries, and used spaces for real data. The datasets on the left: concatenated balanced WTs, on the right: concatenated Huffman-shaped WTs.
Table 1. The impact of the fraction of mono-blocks $f$ on the average number of memory accesses per query and the total query times in the rank variants basic and cf. The block size is 64 bytes. The numbers of memory accesses are calculated from the formulas on $f$ given in Section 3.1, in the paragraph on the cf variant. The times are expressed in nanoseconds.

| dataset         | fraction of mono-bl. [%] | basic, accesses | cf, accesses | basic, time | cf, time |
|-----------------|--------------------------|-----------------|--------------|------------|----------|
| dna200-bal      | 73.60                    | 1.2640          | 1.1943       | 24.73      | 27.97    |
| english200-bal  | 52.95                    | 1.4705          | 1.2491       | 34.47      | 36.89    |
| proteins200-bal | 45.66                    | 1.5434          | 1.2481       | 38.89      | 39.14    |
| xml200-bal      | 78.50                    | 1.2150          | 1.1688       | 23.16      | 26.34    |
| dna200-huff     | 9.03                     | 1.9097          | 1.0822       | 38.25      | 32.06    |
| english200-huff | 23.74                    | 1.7626          | 1.1811       | 42.72      | 37.31    |
| proteins200-huff| 3.84                     | 1.9616          | 1.0370       | 45.86      | 32.87    |
| xml200-huff     | 68.35                    | 1.3165          | 1.2163       | 25.51      | 29.44    |

Fig. 3. Rank (top) and select (bottom) query times (in ns), averaged over 100M random queries, and used spaces for random data with density of set bit 0.05 or 0.2. The four points in a hyb series in the top figures correspond to parameters: 8, 16, 32 and 64.
case are quite competitive (especially the variant $cf$), but they are less attractive when the set bit density is lower. The select query times (the bottom row) vary a lot for the low density case, with major differences in space as well, but our solutions are Pareto-optimal only if practically all the blocks are sparse, which is obviously not interesting. There is nothing to boast about the 20% case either.

Overall, we can conclude that no single rank or select solution dominates over a large range of data characteristics. For random data with low density, the rank $hyb$ may be the winner. For bit-vectors with higher-order redundancies, the Beskers and Fischer algorithms are the most competitive ones. Finally, for real data from FM-indexes (concatenated binary wavelet trees) the solutions proposed in this paper tend to win, sometimes by a large margin.

5 Conclusion

Rank and select are major components of many compressed data structures. In spite of a great progress in their implementations in recent years, the “ultimate” variants have not yet been found, as this paper tries to demonstrate. Not claiming big originality of our techniques, we have to point out that a number of the proposed variants represent new Pareto frontiers for query time and data structure size for real data. One of our achievements is a (moderately) compressed select variant answering queries in about 100 ns. The key idea used in our algorithms is efficient handling of aligned runs of zeros and ones of specified size: either relatively large blocks (the technique used in all our variants) or 16-bit chunks (all the $mpe$ variants). On the other hand, reducing the number of memory accesses, a leitmotif in rank/select research for many years, still has some potential, as demonstrated by the $cf$ variant.

Several aspects of our ideas require further research. The $cf$ variant for the rank operation could be modified to handle compressed blocks. The proposed select solutions work for only one binary digit (e.g., 1) and the current data structures would have to be doubled to handle both digits. Yet, it may be possible to modify them for data reuse, without a large drop in performance. Finally, the rank variants have to be embedded in a full FM-index. According to our preliminary experiments, an FM-index with Huffman-shaped binary wavelet tree and the $rank-mpe2$ variant performs the count query in time shorter by about 7–11% than the FM-hybrid with the superblock size 8. If the hybrid’s superblock is increased to 64 (when the FM-index with $rank-mpe2$ still wins in compression), the gap grows to 27–37%.

Acknowledgement

The work was supported by the Polish National Science Centre under the project DEC-2013/09/B/ST6/03117 (both authors).
References

1. K. Beskers and J. Fischer. High-order entropy compressed bit vectors with rank/select. *Algorithms*, 7(4):608–620, 2014.
2. D. Clark. *Compact Pat Trees*. PhD thesis, University of Waterloo, Ontario, Canada, 1996.
3. S. Gog, T. Beller, A. Moffat, and M. Petri. From theory to practice: Plug and play with succinct data structures. In *Proc. 13th International Symposium on Experimental Algorithms (SEA)*, LNCS 8504, pages 326–337. Springer, 2014.
4. S. Gog and M. Petri. Optimized succinct data structures for massive data. *Software–Practice and Experience*, 44(11):1287–1314, 2014.
5. R. González, S. Grabowski, V. Mäkinen, and G. Navarro. Practical implementation of rank and select queries. In *Poster Proc. 4th International Workshop on Efficient and Experimental Algorithms (WEA)*, pages 27–38. CTI Press and Ellinika Grammata, 2005.
6. S. Grabowski, M. Ranizewski, and S. Deorowicz. FM-index for dummies. *CoRR*, abs/1506.04896, 2015.
7. G. Jacobson. Space-efficient static trees and graphs. In *Proc. 30th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 549–554. IEEE Computer Society, 1989.
8. J. Kärkkäinen, D. Kempa, and S. J. Puglisi. Hybrid compression of bitvectors for the FM-index. In *Proc. Data Compression Conference (DCC)*, pages 302–311. IEEE, 2014.
9. J. I. Munro. Tables. In *Proc. 16th Conference on Foundations of Software Technology and Theoretical Computer Science*, LNCS 1180, pages 37–42. Springer, 1996.
10. G. Navarro and E. Providel. Fast, small, simple rank/select on bitmaps. In *Proc. 11th International Symposium on Experimental Algorithms (SEA)*, LNCS 7276, pages 295–306. Springer, 2012.
11. D. Okanohara and K. Sadakane. Practical entropy-compressed rank/select dictionary. In *Proc. 9th Workshop on Algorithm Engineering and Experiments (ALENEX)*. SIAM, 2007.
12. R. Raman, V. Raman, and S. S. Rao. Succinct indexable dictionaries with applications to encoding $k$-ary trees and multisets. In *Proc. 13th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 233–242. ACM/SIAM, 2002.
13. S. Vigna. Broadword implementation of rank/select queries. In *Proc. 7th International Workshop on Efficient and Experimental Algorithms (WEA)*, LNCS 5038, pages 154–168. Springer, 2008.