Graph matching: relax or not?

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Abstract

We consider the problem of exact and inexact matching of weighted undirected graphs, in which a bijective correspondence is sought to minimize a quadratic weight disagreement. This NP-hard problem is often relaxed as a convex quadratic program, in which the space of permutations is replaced by the space of doubly-stochastic matrices. However, the applicability of such a relaxation is poorly understood. We define a broad class of friendly graphs characterized by an easily verifiable spectral property. We prove that for friendly graphs, the convex relaxation is guaranteed to find the exact isomorphism or certify its inexistence. This result is further extended to approximately isomorphic graphs, for which we develop an explicit bound on the amount of weight disagreement under which the relaxation is guaranteed to find the globally optimal approximate isomorphism. We also show that in many cases, the graph matching problem can be further harmlessly relaxed to a convex quadratic program with only $n$ separable linear equality constraints, which is substantially more efficient than the standard relaxation involving $2n$ equality and $n^2$ inequality constraints. Finally, we show that both convex relaxations are generally unsuitable for matching symmetric graphs.
1 Introduction

Graphs are abstract data types that model structures by connecting with a set of pairs called edges a finite set of entities called vertices. The first to introduce graph analysis as an abstraction method for solving an actual problem was probably L. Euler in his celebrate 1736 paper on the Seven Bridges of Königsberg. Graphs are a natural abstraction in a variety of problems and are particularly useful for modeling structures, frequently arising in different domains of science and engineering. In most applications, graphs are not only used for storage of information, but have to be compared or brought into correspondence. The umbrella term graph isomorphism or, more generally, matching refers to a class of computational problems consisting of finding an optimal correspondence between the vertices of two graphs that minimizes adjacency disagreement.

The uses of graph models in general and graph matching in particular are too numerous to allow a comprehensive review within the scope of this paper. In what follows, we will just list a few prominent ones, referring the reader to a (partial) review of applications with a particular emphasis on the domain of pattern recognition [1]. In computer vision and pattern recognition, graph matching is used for stereo vision and 3D reconstruction [2], object detection and recognition [3, 4] – in particular optical character recognition [5], and image and video indexing and retrieval [6]. In biometric applications, graph-based techniques have been widely used for identification tasks implemented by means of elastic graph matching. These include, among other, face recognition and pose estimation [7], and fingerprint recognition [8]. In biomedical applications, graphs have been used to model vascular structures and, more recently, to represent connections between neurons [9]. In data mining, graphs are used to model networks, including the Web and social networks [10].

Despite the tremendous popularity of graph models, graph matching remains a computationally intensive task. In the strict sense, it is computationally intractable as no polynomial algorithms are known for its solution, except for graphs admitting certain particular structures. The increase in the available computational power of modern computers and the remarkable development of numerous efficient graph matching heuristics have made graph matching feasible for relatively large graphs, counting about a thousand of vertices. However, novel applications such as the analysis of brain graphs – the so-called connectomes, and social networks require matching of graphs with millions if not billions of vertices. These scales are far beyond the reach of existing heuristics. Furthermore, a major disadvantage of graph
matching heuristics is that, in general, they are not guaranteed to find the optimal matching, or at least to guarantee how far the found matching is from the optimal one.

Contributions. In this paper, we focus on the family of scalable graph matching heuristics based on continuous (in particular, convex) optimization [11]. We analyze the standard convex relaxation of the graph matching problem based on replacing the space of permutations by the space of doubly-stochastic matrices, and make the following contributions:

First, we establish conditions under which the relaxation is equivalent to exact graph matching, in the sense that it is guaranteed to find the exact graph isomorphism if such exists, or certify its inexistence (Theorem 1). The class of graphs on which such an equivalence holds is characterized by an easily verifiable spectral property we call friendliness, and is surprisingly large – practically, as large as the class of asymmetric graphs.

Second, we generalize this result to inexact graph matching, providing an explicit bound on the amount of weight disagreement under which the relaxation is guaranteed to find the globally optimal approximate isomorphism (Theorem 2).

Third, we show a subclass of symmetric graphs that we call hostile (again, defined through a simple spectral property), on which the convex relaxation fails in the sense that its solution space is guaranteed to contain permutations that are not isomorphisms even if the graphs are isomorphic (Theorem 3).

These three contributions establish the boundaries of applicability of the convex relaxation, which have so far been poorly understood. Finally, a byproduct of our analysis is the fact that the former results are also satisfied by a simpler convex relaxation, in which the space of permutations is replaced with an affine space of matrices we call pseudo-stochastic. This alternative relaxation leads to a simpler, and potentially more efficiently solvable, optimization problem.

Notation. The following notation will be used in the rest of the paper: vectors and matrices are denoted in bold lower and upper case, respectively, and their elements by lower and upper case italic with appropriate subscript indices. The norm $\| \cdot \|$ will denote the standard $\ell_2$ norm of a vector, and the spectral norm of a matrix (to be distinguished from the Frobenius norm, specified with the subscript $F$).
2 Graph matching

Let $A = (V, A)$ and $B = (V, B)$ be two undirected graphs build upon a common vertex set $V$, which for convenience is assumed to be $V = \{1, \ldots, n\}$. As $A$ and $B$ are fully represented by the symmetric $n \times n$ adjacency matrices $A$ and $B$, we will use the two notations interchangeably. We allow the adjacency matrices to contain non-binary edge weights, and henceforth consider this case without explicitly specifying that the graphs are weighted. Let us denote by $\mathcal{P}(n) = \{\pi : V \to V\}$ the space of vertex permutations, which can be equivalently represented by $n \times n$ permutation matrices of the form $\{\Pi \in \{0, 1\}^{n \times n} : \Pi^T \mathbf{1} = \mathbf{1}\}$, with $\mathbf{1}$ denoting a column vector of ones. With some abuse of notation, we will refer to both spaces as $\mathcal{P}(n)$, dropping the $n$ whenever possible. A permutation $\pi$ represents a bijective correspondence between the two graphs, mapping each vertex $i$ in $A$ to a vertex $\pi_i$ in $B$. Similarly, for each edge $(i, j)$, the correspondence pulls back the adjacency weight $b_{\pi_i, \pi_j}$. The latter can be stated equivalently by constructing a new adjacency matrix $\Pi^T B \Pi$, where $\Pi$ is the permutation matrix representing $\pi$. To measure the adjacency disagreement under correspondence, we define on $\mathcal{P}$ a distortion function of the form $\text{dis}_{A \to B}(\Pi) = \|A - \Pi^T B \Pi\|$, with $\|\cdot\|$ denoting some norm (for brevity, we will drop $A \to B$ whenever possible). The graphs are said to be isomorphic if there exists a zero-distortion permutation. We denote the collection of all isomorphisms relating $A$ and $B$ by $\text{Iso}(A \to B) = \{\Pi : \text{dis}(\Pi) = 0\}$.

In this notation, the graph matching (GM) problem consists of finding a zero-distortion permutation; such a permutation might not be unique if the graph possesses symmetries, as we clarify in the sequel. The closely related graph isomorphism (GI) problem consists of verifying whether a zero-distortion permutation exists. This strict setting is usually referred to as exact. Since in practical applications the matched graphs might be contaminated by noise, GM is frequently stated in the inexact flavor, consisting of finding a minimum rather than zero distortion permutation.

Computationally, GM is at least as hard as GI, since solving GM also solves GI, but not vice versa. GM is known to be NP-hard, while it is not known whether GI is in the complexity class P. In fact, GI is one of the few problems which, if P $\neq$ NP, might reside in an intermediate “GI-complete” complexity class [12]. Yet, the GI problem is known to be only “moderately exponential” [13]; furthermore, polynomial (and even linear) time algorithms exist for checking the isomorphism of various particular types of graphs, such as planar graphs [14], graphs with bounded vertex degree [15], and trees [16]. However, the constants characterizing the complexity of such
algorithms are extremely large; for example, the linear time algorithm for checking the isomorphism of graphs with vertex degree bounded by 2 is over $2 \times 10^6!$. Moreover, these results are largely inapplicable to inexact or weighted graph matching. Because of this fact, exact graph matching is not used in practical applications involving even moderately-scaled graphs, except for very particular cases. Instead, various types of heuristics are usually employed.

The common property of heuristic algorithms is that they often perform well on real problems and scale to large graphs at the expense of having no theoretical guarantee to converge to the true global minimizer of the GM problem. The wealth of literature dedicated to graph matching heuristics counts hundreds of studies published in the past four decades, and we will not attempt to review it within the scope of this paper. Instead, we refer the reader to [1] for a comprehensive review, and focus on the popular class of continuous optimization relaxations. In these heuristics, the combinatorial GM problem is replaced by an optimization problem with continuous variables, enabling the use of efficient and scalable continuous optimization algorithms [17].

3 Relaxation of graph matching

Adopting this perspective, GM can be formulated as an optimization problem

$$\Pi^* = \arg\min_{\Pi \in \mathcal{P}} \text{dis}_{A \rightarrow B}(\Pi) = \arg\min_{\Pi \in \mathcal{P}} \|A - \Pi^T B \Pi\|. \quad (1)$$

The norm in the objective is typically chosen to be the standard $\ell_1$ norm $\|X\|_1 = \sum_{i,j} |x_{ij}|$, the Frobenius ($\ell_2$) norm $\|X\|_F^2 = \sum_{i,j} x_{ij}^2$, or the min-max ($\ell_\infty$) norm $\|X\|_\infty = \max_{i,j} |x_{ij}|$. In what follows, we will adopt the Frobenius norm, henceforth defining

$$\text{dis}_{A \rightarrow B}(\Pi) = \|A - \Pi^T B \Pi\|_F = \|\Pi A - B \Pi\|_F,$$

where the second identity is possible due to unitarity of permutation matrices. For this particular choice, problem (1) can be rewritten as

$$\Pi^* = \arg\min_{\Pi \in \mathcal{P}} \|\Pi A - B \Pi\|_F^2 = \arg\max_{\Pi \in \mathcal{P}} \text{tr}(B \Pi A \Pi^T), \quad (2)$$

known as a quadratic assignment problem (QAP).

Both (1) and (2) are NP-hard due to the combinatorial complexity of the constraint $\Pi \in \mathcal{P}$. Relaxation techniques reduce this complexity by
replacing the latter constraint with a more tractable continuous set. Since the practically used norms in \( \| \cdot \|_1 \) can yield a convex minimization objective, convex relaxation techniques consist of replacing \( \mathcal{P} \) with a larger convex set, resulting in a tractable convex program. Various techniques differ mainly in the choice of the norm, the choice of the convex set (i.e., the relaxation), and the particular numerical algorithm used to solve the resulting convex program \([11]\).

A popular choice is to relax \( \mathcal{P} \) to the space of doubly-stochastic matrices \( \mathcal{D} = \{ \mathcal{P} \geq 0 : \mathcal{P} \mathcal{1} = \mathcal{P}^T \mathcal{1} = 1 \} \) constituting the convex hull of permutation matrices in \( \mathbb{R}^{n \times n} \). Combined with the \( \ell_1 \) or the \( \ell_\infty \) norms, such a relaxation leads to a linear program \([18]\), while the use of the Frobenius norm results in a linearly-constrained quadratic program (LCQP or QP for short) \([19]\). Both types of optimization problems are solvable using polynomial time algorithms, very efficient in practice \([17]\).

Along with convex relaxations of the GM problem \((1)\), there exist numerous techniques for relaxing its QAP formulation \((2)\). Note that after the relaxation the two problems are generally not equivalent! Unlike \((1)\), the objective of \((2)\) is non-convex and hence even if \( \mathcal{P} \) is replaced by a convex set, the resulting optimization problem is non-convex. One of the most celebrate relaxations of QAP is the spectral relaxation \([20]\), in which the solution matrix is constrained to constant Frobenius norm, which transforms the relaxed problem to the maximum eigenvector problem. The latter is one of the few non-convex optimization problems for which there exists algorithms with global convergence guarantees. Other non-convex relaxations of the QAP have been proposed, including restricting the matrix \( \mathcal{P} \) to the non-negative simplex \([21]\), or to the space of doubly-stochastic matrices \([19]\). All such relaxations have only local convergence guarantees.

In this paper, we consider the convex QP relaxation of GM,

\[
\mathcal{P}^* = \arg\min_{\mathcal{P} \in \mathcal{D}} \| \mathcal{P} \mathcal{A} - \mathcal{B} \mathcal{P} \|_F^2. \tag{3}
\]

In the sequel, we show that the double-stochasticity constraint can be further harmlessly relaxed. Since the solution \( \mathcal{P}^* \) of the relaxation is, generally, not a permutation matrix, it has to be projected back onto \( \mathcal{P} \) \([1]\). The orthogonal projection \( \hat{\Pi} \) onto \( \mathcal{P} \) has to maximize the standard Euclidean inner product, which can be stated as the optimization problem

\[
\hat{\Pi} = P \mathcal{P}^* = \arg\max_{\Pi \in \mathcal{P}} \langle \Pi, \mathcal{P}^* \rangle = \arg\max_{\Pi \in \mathcal{P}} \text{tr}(\Pi^T \mathcal{P}^*). \tag{4}
\]

This problem is called a linear assignment problem (LAP) and, unlike QAP, is solvable in polynomial time using a family of techniques collectively known
as the Hungarian method [22]. LAP can also be formulated and solved as a linear program, in which the linear objective is minimized over the polytope \( \mathcal{D} \) instead of \( \mathcal{P} \). The solution of such a linear program is guaranteed to be in \( \mathcal{P} \) due to a particular property of the constraints called total unimodularity.

The considered convex relaxation of graph matching can be thus summarized as the following two-step procedure, which we henceforth call the relaxed GM or RGM:

1. Solve QP (3).

2. Project the obtained solution onto the space of permutation matrices by solving the LAP (4).

We will henceforth refer to the permutation matrix \( \hat{\Pi} \) obtained from step 2 above as the solution of the RGM.

Variants of the described procedure are often used in practice; due to their relatively low computational complexity, they scale to large graphs. There is a considerable practical evidence that the RGM produces a good approximation to the exact solution of the GM problem, in the sense that \( \text{dis}(\hat{\Pi}) \approx \text{dis}(\Pi^*) \), and often \( \hat{\Pi} \approx \Pi^* \). It is therefore astonishing that no theoretical bounds exist on \( |\text{dis}(\hat{\Pi}) - \text{dis}(\Pi^*)| \), and practically nothing is known about \( \|\hat{\Pi} - \Pi^*\| \)!

One of the main goals of this paper is to establish conditions under which RGM is equivalent to the exact GM, in the sense that the projection of the solution space of (3) onto \( \mathcal{P} \) coincides with \( \text{Iso}(A \mapsto B) \).

We also investigate conditions for the converse situation, when the solution space of the relaxation contains non-zero distortion permutations, making the relaxation unusable.

4 Exact matching of asymmetric graphs

We start with the case of exact (i.e., distortion-less) matching of asymmetric graphs. An undirected graph \( A \) with the adjacency matrix \( A \) is said to possess a symmetry \( \Pi \in \mathcal{P} \) if \( \text{dis}_{A \mapsto A}(\Pi) = 0 \). This notation emphasizes that symmetries are self-isomorphisms. Symmetries form a group with the matrix multiplication operation (or with the function composition, if permutations are interpreted as bijective functions), which we refer to as the symmetry group of \( A \) and denote by \( \text{Sym}A \). The graph is called asymmetric if its symmetry group is trivial, \( \text{Sym}A = \{I\} \).

It is straightforward to show that two isomorphic graphs \( A \) and \( B \) have identical (isomorphic) symmetry groups, and if \( \Pi \in \mathcal{P} \) is an isomorphism, then \( \Pi \text{Sym}^T A = \{\Pi \text{Sym}^T : \Pi \in \text{Sym}A\} \) (or, equivalently, \( \text{Sym}B \Pi \) are...
also isomorphisms. The converse is also true: if the two graphs are related by a collection of isomorphisms \( \text{Iso}(A \mapsto B) = \{\Pi_1, \ldots, \Pi_k \in \mathcal{P}\} \), then they are symmetric with \( \text{Sym}A \) generated by \( \text{Iso}(B \mapsto A) \circ \text{Iso}(A \mapsto B) = \{\Pi_i^T\Pi_j\} \), and \( \text{Sym}B \) by \( \text{Iso}(A \mapsto B) \circ \text{Iso}(B \mapsto A) = \{\Pi_i\Pi_j^T\} \). Consequently, if \( A \) is asymmetric and \( B \) is isomorphic to it, they are related by a unique isomorphism which is the global minimizer of \( \Pi \). In what follows, we denote this unique isomorphism by \( \Pi^* \).

We emphasize that the symmetry or asymmetry of a graph has nothing to do with the fact that the adjacency matrix \( A \) is symmetric. The latter property stems from the fact that the graph is undirected, and because of it \( A \) admits unitary diagonalization of the form \( A = U\Lambda U^T \), with an orthonormal \( U = (u_1, \ldots, u_n) \) containing the eigenvectors in its columns, and a diagonal \( \Lambda = \text{diag} \{\lambda_1, \ldots, \lambda_n\} \) containing the corresponding eigenvalues.

The uniqueness of the isomorphism relating isomorphic asymmetric graphs is crucial for the results we present next. However, the existence or the absence of symmetry is not an easy property to verify. To overcome this difficulty, instead of considering the class of asymmetric graphs, we consider another class of graphs characterized by the following spectral property:

**Definition.** A graph \( A \) is called **friendly** if its adjacency matrix \( A \) has simple spectrum (i.e., all the \( \lambda_i \) are distinct), and all its eigenvectors satisfy \( u_i^T 1 \neq 0 \).

We note the following important consequence of friendliness:

**Lemma 1.** A friendly graph is asymmetric.

**Proof.** Let \( A = U\Lambda U^T \) denote the adjacency matrix of the graph, and let assume by contradiction that there exists \( \Pi \neq I \) such that \( \Pi A = A\Pi \). Then, for every eigenvector \( u_i \) of \( A \), we have \( \PiAu_i = \PiAu_i = \lambda_i\PiAu_i \), that is, \( \PiAu_i \) is also an eigenvector of \( A \). Since, due to friendliness, \( A \) has simple spectrum, the only two possibilities are \( \PiAu_i = \pm u_i \). Since we assumed \( \Pi \neq I \), there must be at least one eigenvector \( u_i \) for which \( \PiAu_i = -u_i \).

Then, \( 1^T\PiAu_i = -1^Tu_i \). On the other hand, since \( \Pi \) is a permutation matrix, \( 1^T\PiAu_i = 1^Tu_i \). Hence, \( u_i^T 1 = 0 \) in contradiction to friendliness of \( A \).

The converse is not true, as there might exist an asymmetric graph with an unfriendly adjacency matrix. For example, any regular unweighted graph has a constant eigenvector and is thus highly unfriendly; on the other hand, there exist asymmetric regular graphs such as the Frucht graph with \( n = 12 \). However, unfriendliness still seems to be a singular property, and intuition...
suggests that unfriendly graphs should have measure zero among random asymmetric weighted graphs, and the class of friendly graphs should be almost as big as that of asymmetric graphs. We do not pursue the rigorous proof of this claim, since it might delicately depend on what is meant by “random”. We only emphasize that, in contrast to asymmetry, friendliness is an easily verifiable property.

Using the notion of friendliness, we state our first result:

**Theorem 1.** Let $A$ and $B$ be friendly isomorphic graphs. Then, $GM$ and $RGM$ are equivalent.

**Proof.** We consider the relaxation (3) of $GM$, denoting by $\Pi^*$ the global minimizer of the latter. The minimizer is unique due to asymmetry. For any doubly-stochastic matrix $P$,

$$PA - BP = (P\Pi^T B - BP\Pi^T)\Pi^* = QB - BQ,$$

where $Q = P\Pi^T$. We therefore reformulate (3) in terms of $Q$ as the minimization of $f(Q) = \frac{1}{2}\|QB - BQ\|_F^2$ w.r.t $Q \in D$. Since the objective $f(Q)$ is convex in $Q$, and so is the set of double stochastic matrices $D$, the problem has a global minimum at $Q = I$. It remains to prove that the minimum is unique. Since $B$ is symmetric, simple calculus yields the gradient of $f(Q)$, $\nabla_Q f = QB^2 + B^2Q - 2BBQ$. By omitting the non-negativity and unit column sum constraints, we further relax the constraint $Q \in D$ to $Q1 = 1$, referring to such matrices as pseudo-stochastic. The Lagrangian of $f$ with the pseudo-stochasticity constraint on $Q$ is given by $L(Q, \alpha) = f(Q) + \alpha^T(Q1 - 1) = f(Q) + tr((Q1 - 1)\alpha^T)$, with $\alpha$ denoting the vector of Lagrange multipliers. Problem (3) reaches a minimum when

$$\nabla_Q L(Q, \alpha) = QB^2 + B^2Q - 2BBQ + \alpha1^T = 0.$$  

Substituting the unitary eigendecomposition $B = U\Lambda U^T$, the latter equation can be rewritten as

$$FA^2 + \Lambda^2 F - 2AFA + \gamma\nu^T = 0,$$

where $\gamma = U^T\alpha$, $\nu = U^T1$, and $F = U^TQU$. It is easy to see that (5) can be expressed coordinate-wise as

$$F_{ij}(\lambda_i - \lambda_j)^2 + v_j\gamma_i = 0.$$  

(6)
For every $i = j$, we have $v_i \gamma_i = 0$; since the friendliness assumption implies $v_i \neq 0$ for all $i$, we have $\gamma = 0$. This yields
\[ F_{ij}(\lambda_i - \lambda_j)^2 = 0 \text{ for every } i \neq j. \] (7)

Since friendliness also implies $\lambda_i \neq \lambda_j$, $F$ must be diagonal and so must $Q = UFU^T$. Because $Q$ is also pseudo-stochastic, it has to satisfy $1 = Q1 = UFU^T1$ or, equivalently, $v = Fv$. Yet, since $F$ is diagonal and, due to friendliness, $v$ has no zero elements, the above property is satisfied only if $F = I$. This implies that $Q = I$ or, equivalently, $P = \Pi^*$. Hence, $\Pi^*$ is the unique minimizer of (3). Since the solution is already in $P$, the projection $\Pi$ leaves it unchanged.

Note that in the proof we only used the pseudo-stochasticity constraint $P1 = 1$. This leads to an important consequence: instead of relaxing $P$ to the space $D$ of doubly-stochastic matrices, a coarser relaxation to pseudo-stochastic matrices is equivalent in the discussed case. Practically, this means that we can solve a simpler quadratic program
\[
P^* = \arg\min_P \|PA - BP\|_F^2 \quad \text{s.t.} \quad P1 = 1,
\] (8)
with $n^2$ variables and only $n$ equality constraints, instead of $2n$ equality constraints and $n^2$ inequality constraints in (3). In what follows, we focus on this simpler convex relaxation instead of (3) in the RGM.

While checking the friendliness condition in Theorem 1 is straightforward, checking whether the perfect isomorphism condition is satisfied is not (in fact, it is a graph isomorphism problem!) However, in practice one can simply solve relaxation (8) for the two friendly graphs, project the result onto $P$, and check whether $\text{dis}(\Pi) = 0$. If the answer is positive, $\Pi$ is guaranteed to be the unique isomorphism; otherwise, the theorem guarantees that the graphs are not isomorphic.

5 Inexact matching of asymmetric graphs

The case of perfectly isomorphic graphs, to which Theorem 1 is applicable, is often an unachievable mathematical idealization. Many practical applications of graph matching assume some amount of noise, and seek a least distortion correspondence rather than a perfect isomorphism. To formalize this notion, we say that two graphs $A$ and $B$ are $\rho$-isomorphic if there exists $\Pi^* \in \mathcal{P}$ with $\text{dis}(\Pi^*) \leq \rho$. 

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Similarly, we say that a graph \( A \) possesses a \( \rho \)-symmetry \( \Pi \in \mathcal{P} \) if \( \text{dis}_{A \to A}(\Pi) \leq \rho \). Note that unlike their exact counterparts, \( \rho \)-symmetries do not form a group, as the composition of two \( \rho \)-symmetries might have \( \text{dis} > \rho \). A graph with a trivial \( \rho \)-symmetry set is called \( \rho \)-asymmetric. The lack of symmetry of such a graph is strong enough to guarantee that a bounded perturbation of the adjacency weights does not produce new, previously inexistent symmetries.

In order to generalize our result to the case of nearly-isomorphic graphs, we define the strength of a graph’s friendliness:

**Definition.** A graph \( A \) is \((\epsilon, \delta)\)-friendly if its adjacency matrix \( A = U\Lambda U^T \) has the spectral gap \( \sigma(A) = \min_{i \neq j} |\lambda_i - \lambda_j| > \delta \), and \( \epsilon < |u_i^T 1| < \frac{1}{\epsilon} \) for \( i = 1, \ldots, n \).

Also note that our former definition of friendliness corresponds to \((\epsilon, \delta) = (0, 0)\). We refer to the case \( \epsilon, \delta > 0 \) as strong friendliness.

For the broad family of strongly friendly graphs, we first show that the result of Theorem 1 is stable in the sense that a bounded perturbation in the adjacency matrix results in a bounded perturbation of the solution:

**Lemma 2.** Let \( A \) and \( B \) be \((\epsilon, \delta)\)-friendly isomorphic graphs with spectral radius \( \sigma = \max_i |\lambda_i| \), related by the unique isomorphism \( \Pi^* \). Let \( \hat{B} \) be a perturbed version of \( B \) with \( \hat{B} = B + \rho R \), where \( R \) is symmetric with \( \|R\|_F \leq 1 \), and \( \rho \leq \min \left\{ \sqrt{2\sigma}, \frac{\delta^2 \epsilon^4}{12\sigma n^{1.5}} \right\} \). Then, the solution \( \hat{P}^* \) of the perturbed problem (8) is unique and satisfies \( \|\hat{P}^* - \Pi^*\|_F < \frac{1}{2} \).

**Proof.** The proof goes along the lines of the proof of Theorem 1. As before, we reparametrize the optimization in terms of \( Q = \Pi^* P \) instead of \( P \). Denoting by \( B = U\Lambda U^T \) the orthonormal eigendecomposition of \( B \) and \( E = U^T RU \), we substitute \( \hat{B} = U(\Lambda + \rho E)U^T \) into a perturbed version of the Lagrangian,

\[
\nabla_Q L(Q, \alpha) = Q\hat{B}^2 + B^2Q - 2B\hat{B} + \alpha 1^T = 0,
\]

and obtain a perturbed version of (5),

\[
(F\hat{A}^2 + \hat{A}^2F - 2\hat{A}F\hat{A}) + \rho(F\Lambda E + \Lambda E + 2\Lambda E - 2AFE) + \gamma v^T + \rho^2 FG = 0,
\]
where \( G = E^2 \), and \( F, v, \) and \( \gamma \) are defined as before. The system can be rewritten coordinate-wise as

\[
F_{ij}(\lambda_i - \lambda_j)^2 + v_j \gamma_i + \rho \sum_k F_{ik} (E_{kj}(\lambda_j + \lambda_k - 2\lambda_i) + \rho G_{kj}) = 0. \tag{10}
\]

Substituting \( i = j \) and re-arranging the terms yields

\[
\gamma_i = -\frac{\rho}{v_i} \sum_k F_{ik}(E_{ki}(\lambda_k - \lambda_i) + \rho G_{ki}).
\]

Substituting \( \gamma_i \) back into (10) and multiplying both sides by \( v_i \) yields

\[
F_{ij}v_i(\lambda_i - \lambda_j)^2 + \rho^2 \sum_k F_{ik}(v_i G_{kj} - v_j G_{ki}) + \rho \sum_k F_{ik} (v_i E_{kj}(\lambda_j + \lambda_k - 2\lambda_i) - v_j E_{ki}(\lambda_k - \lambda_i)) = 0.
\]

Denoting

\[
s_{ij} = \frac{1}{(\lambda_i - \lambda_j)^2} \left( E_{kj}(\lambda_j + \lambda_k - 2\lambda_i) - \frac{v_j}{v_i} E_{ki}(\lambda_k - \lambda_i) \right),
\]

\[
t_{ij} = \frac{1}{(\lambda_i - \lambda_j)^2} \left( G_{kj} - \frac{v_j}{v_i} G_{ki} \right), \tag{11}
\]

for \( i \neq j \), and \( s_{ik} = t_{ik} = 0 \), we arrive at the following perturbed linear system

\[
F_{ij} + \rho \sum_k F_{ik} (s_{jk} + \rho t_{jk}) = 0, \quad i \neq j \tag{12}
\]

\[
\sum_k F_{ik}v_k = v_i,
\]

where the second set of equations \( Fv = v \) comes, as before, from the pseudo-stochasticity constraint \( Q1 = 1 \). Also note that we absorbed the second-order perturbation into the terms \( t_{jk} \).

From this point, it remains to show that the solution \( F \) of the perturbed system (12) is unique and sufficiently close to the solution \( F_0 = I \) of the unperturbed system, for which we rely on a result in perturbation analysis of linear systems summarized as Lemma 5 in the Appendix. Denoting by \( f = (F_{11}, \ldots, F_{1n}, \ldots, F_{n1}, \ldots, F_{nn})^T \) the row stack vector representation of \( F \), equation (12) can be rewritten as

\[
(M + \rho N)f = c \tag{13}
\]
with $M = \text{diag}\{M_1, \ldots, M_n\}$ being an $n^2 \times n^2$ block-diagonal matrix, where each $M_i$ is an $n \times n$ block consisting of the identity matrix with the $i$-th row replaced by the row vector $v = (v_1, \ldots, v_n)$. Similarly, $N$ is an $n^2 \times n^2$ block-diagonal matrix with the $n \times n$ blocks $N_i = (s^i_{jk} + \rho t^i_{jk})_{jk}$, and $c$ is an $n^2 \times 1$ vector of zeros, with every $(i-1)(n+1)+1$-st element replaced by $v_i$. Due to the block-diagonal structure of $M$, we readily have that $M^{-1}$ is also block-diagonal with the same structure, where each $n \times n$ diagonal block $M^{-1}_i$ is the identity matrix with the $i$-th row replaced by the row vector $w = (v_1, \ldots, v_i, 1, -v_{i+1}, \ldots, -v_n)$. Decomposing each $M^{-1}_i$ into the sum of the identity matrix and a rank-one matrix, we have

$$\|M^{-1}_i\| \leq \|I\| + \|w\| < 1 + \frac{\sqrt{n}}{\epsilon^2},$$

where the second inequality is due to the $(\epsilon, \delta)$-friendliness assumption. Due to the block-diagonal structure of $M^{-1}$,

$$\|M^{-1}\| \leq \max_{i=1,\ldots,n} \|M^{-1}_i\| < 1 + \frac{\sqrt{n}}{\epsilon^2}. \quad (14)$$

Similarly, we obtain

$$\|N\|^2 \leq \max_{i=1,\ldots,n} \|N_i\|_F^2 = \sum_{jk} (s^i_{jk} + \rho t^i_{jk})^2$$

$$\leq 2 \left( \sum_{jk} (s^i_{jk})^2 + \rho^2 (t^i_{jk})^2 \right). \quad (15)$$

To bound the $(s^i_{jk})^2$ terms, we invoke strong friendliness again, obtaining $\frac{v_i}{v_j} \leq \frac{1}{\epsilon^2}$. Combining this result with $(\lambda_i - \lambda_j)^2 \geq \delta^2$ for $i \neq j$, $\lambda_i^2 \leq \sigma^2$, and substituting into (11) yields

$$(s^i_{jk})^2 \leq \left( \frac{2\sigma}{\delta^2} \left( 2E_{kj} + \frac{1}{\epsilon^2} E_{ki} \right) \right)^2$$

$$\leq \frac{4\sigma^2}{\delta^2} \left( 2E^2_{kj} + \frac{4}{\epsilon^2} |E_{kj} E_{ki}| + \frac{1}{\epsilon^4} E^2_{ki} \right). \quad (16)$$

For the first term, we use

$$\sum_{jk} E^2_{kj} = \|E\|_F^2 = \|U^T R U\|_F^2 = \|R\|_F^2 \leq 1,$$
from where $E_{ki}^2 \leq 1$. Using standard norm inequalities,

$$\sum_{jk} |E_{kj}E_{ki}| \leq \sum_{jk} |E_{kj}| \leq n.$$  

Substituting the latter bounds into (16) yields

$$\sum_{ijk} (s_{ijk}^i)^2 \leq \frac{4\sigma^2}{\delta^4} \left( 2 + \frac{4n}{\epsilon^2} + \frac{1}{\epsilon^4} \right). \tag{17}$$

The $(t_{ijk}^i)^2$ terms in (15) are bounded in a similar way. First, we observe that

$$(t_{ijk}^i)^2 \leq \frac{2}{\delta^4} \left( G_{kj}^2 + \frac{1}{\epsilon^4} G_{ki}^2 \right).$$

Substituting $G = E^2$ leads to

$$\sum_{jk} G_{kj}^2 = \|G\|_F^2 \leq \|E\|_F^4 \leq 1,$$

from where

$$\sum_{jk} (t_{ijk}^i)^2 \leq \frac{2}{\delta^4} \left( 1 + \frac{1}{\epsilon^4} \right). \tag{18}$$

Substituting (17) and (18) into (15) and assuming $\rho^2 \leq 2\sigma^2$ yields

$$\|N\|^2 \leq \frac{4}{\delta^4} \left( 2\sigma^2 \left( 2 + \frac{4n}{\epsilon^2} + \frac{1}{\epsilon^4} \right) + \rho^2 \left( 1 + \frac{1}{\epsilon^4} \right) \right)$$

$$\leq \frac{8\sigma^2}{\delta^4} \left( 3 + \frac{4n}{\epsilon^2} + \frac{2}{\epsilon^4} \right). \tag{19}$$

Combining bounds (14) and (19) and requiring $\epsilon \leq 1$, one has

$$\|M^{-1}\| \|N\| < \frac{\sqrt{8\sigma(\epsilon^2 + \sqrt{n})}}{\delta^2\epsilon^3} \sqrt{3\epsilon^4 + 4\epsilon^2n + 2}$$

$$\leq \frac{\sqrt{13}}{2} (1 + \sqrt{2}) \frac{\sigma n}{\delta^2\epsilon^4}. \tag{20}$$

It is easy to verify that for $n \geq 2$, demanding $\rho < \frac{\delta^2 \epsilon^4}{12\sigma n^{1.5}}$ implies

$$\rho < \frac{1}{(1 + 2\sqrt{n}) \|M^{-1}\| \|N\|}, \tag{21}$$
from where it follows that
\[
\frac{\rho \|M^{-1}\| \|N\|}{1 - \rho \|M^{-1}\| \|N\|} < \frac{1}{2\sqrt{n}}. \tag{22}
\]
Since (21) implies \(\|\rho M^{-1}N\| < \rho \|M^{-1}\| \|N\| < 1\), \(\mathbf{I} + \rho M^{-1}N\) is invertible, and so is \(\mathbf{M}(\mathbf{I} + \rho M^{-1}N) = \mathbf{M} + \rho \mathbf{N}\), from which uniqueness of the solution follows. Invertibility of the perturbed system (13) allows to invoke Lemma 5, which combined with (22) yields
\[
\|F - F_0\|_F = \|f - f_0\| \leq \frac{\rho \|M^{-1}\| \|N\| \|f_0\|}{1 - \rho \|M^{-1}\| \|N\|} < \frac{1}{2}, \tag{23}
\]
where we used \(\|f_0\| = \|F_0\|_F = \sqrt{n}\) since \(F_0 = \mathbf{I}\). Recalling that \(\mathbf{P} = \mathbf{U} \mathbf{F} \mathbf{U}^T \mathbf{\Pi}\) and that the unperturbed solution is \(\mathbf{P}_0 = \mathbf{\Pi}\), and using the orthonormality of \(\mathbf{U}\) and \(\mathbf{\Pi}\), one has \(\|\mathbf{P} - \mathbf{\Pi}\|_F = \|\mathbf{U}(\mathbf{F} - \mathbf{F}_0)\mathbf{U}^T\mathbf{\Pi}\|_F = \|\mathbf{F} - \mathbf{F}_0\|_F\), which completes the proof.

Applying the former result to matching a graph with itself (\(\mathcal{A} = \mathcal{B}\)), the following generalization of Lemma 1 can be straightforwardly obtained:

**Corollary 1.** An \((\epsilon, \delta)\)-friendly graph is \(\rho\)-asymmetric, with \(\rho\) satisfying the conditions of Lemma 2.

In fact, this property guarantees that the perturbation creates no symmetries and, thus, the perturbed version of system (7) remains full rank.

The stability of the relaxation in Lemma 2 leads directly to our second result:

**Theorem 2.** Let \(\mathcal{A}\) be an \((\epsilon, \delta)\)-friendly graph with the adjacency matrix normalized such that \(\sigma = 1\), and let \(\mathcal{B}\) be \(\rho\)-isomorphic to \(\mathcal{A}\). Then, if \(\rho < \frac{\delta^2\epsilon^4}{12n^{1.5}}\), RGM and GM are equivalent.

**Proof.** Let \(\mathbf{\Pi}\) be a \(\rho\)-isomorphism relating \(\mathcal{B}\) and \(\mathcal{A}\), and let us denote \(\mathbf{B}_0 = \mathbf{\Pi}\mathbf{A}\mathbf{\Pi}^T\) and \(\mathbf{R} = \frac{1}{\rho}(\mathbf{B} - \mathbf{B}_0)\). Then, \(\mathcal{B}_0\) is perfectly isomorphic to \(\mathcal{A}\), and \(\mathcal{B}\) is a perturbed version of \(\mathcal{B}_0\) with \(\mathbf{B} = \mathbf{B}_0 + \rho \mathbf{R}\) and \(\|\mathbf{R}\|_F = \frac{1}{\rho}\|\mathbf{B} - \mathbf{\Pi}\mathbf{A}\mathbf{\Pi}^T\|_F \leq 1\). By Corollary 1, \(\mathcal{B}\) is \(\rho\)-asymmetric and, hence, \(\mathcal{B}_0\) is asymmetric. Denoting by \(\mathbf{P}\) the solution of (8) applied to \(\mathcal{A}\) and \(\mathcal{B}\), we invoke Lemma 2 which guarantees uniqueness of \(\mathbf{P}\) and \(\|\mathbf{P} - \mathbf{\Pi}\|_F < \frac{1}{2}\). By standard norm inequalities, this implies \(\|P^*_{ij} - \Pi^*_{ij}\| < \frac{1}{2}\) element-wise for every \(i, j\). Therefore, the projection of \(\mathbf{P}\) onto \(\mathcal{P}\) coincides with \(\mathbf{\Pi}\). \(\square\)
As in the case of perfectly isomorphic graphs, checking the strong friendliness condition in Theorem 1 is straightforward, while checking the $\rho$-isomorphism of $A$ and $B$ is not. Yet, as in the previous case, one can again solve relaxation (8), project the solution onto $P$, and verify whether $\text{dis}(\hat{\Pi}) < \rho$. In case of a positive answer, $\hat{\Pi}$ is guaranteed to be the unique global minimizer of the graph matching problem; otherwise, the graphs are guaranteed not to be $\rho$-isomorphic.

6 Exact matching of symmetric graphs

The assumption of friendliness plays a crucial role in the results we have developed so far: it guarantees uniqueness of solution of the relaxation. These results cannot be directly extended to symmetric graphs, for which the solution space of the relaxation should contain several isomorphisms and their affine (in case of relaxation (8)), or convex (in case of relaxation (3)) combinations. In fact, if two isomorphic graphs $A$ and $B$ have a non-trivial symmetry group, the size of $\text{Iso}(A \mapsto B)$ has to coincide with the size of $\text{Sym}A$. This, in turn, means that the solution space of a convex relaxation can no more be of dimension zero.

In what follows, we address the case of exact matching of symmetric graphs. As before, in order to avoid verifying whether a graph is symmetric or not, we consider the easily verifiable friendliness property, referring to graphs that are not friendly as unfriendly. We define the degree of a graph’s unfriendliness as follows:

**Definition.** A graph $A$ is called $(k, m)$-unfriendly if its adjacency matrix has $d$ non-simple eigenvalues with the multiplicities $m_1 + \cdots + m_d = d + m$, and $k$ eigenvectors satisfying $u_i^T 1 = 0$.

Intuition suggests that unfriendliness is a consequence of symmetry, and we conjecture that unfriendly graphs that are not symmetric are singular cases. Although the exact relation between the spectral properties of the adjacency matrix and the geometry of the symmetry group eludes our complete understanding, the following relation between the number of non-trivial symmetries a graph possesses and the degree of its unfriendliness is easy to establish:

**Lemma 3.** Let $A$ be a graph with $l = |\text{Sym}A| - 1$ non-trivial symmetries. Then, $A$ is $(k, m)$-unfriendly with $k + m \geq l$.

**Proof.** The proof extends the proof of Lemma 1, where we showed that for every non-trivial symmetry $\Pi$, there exists at least one $i$ such that $\Pi u_i \neq u_i$.
is an eigenvector of $A$ corresponding to $\lambda_i$. Furthermore, if $u_i$ is simple, $u_i^T 1 = 0$. Therefore, each non-trivial symmetry increments by one either $k$ (in case $\lambda_i$ is simple) or $m$ (otherwise), or both.

This result suggests that graphs with richer symmetries are more unfriendly than graphs with simple ones. However, the degree of unfriendliness does not distinguish between the ways the non-simple eigenspaces of the adjacency matrix are configured; for example, a $(3, k)$-unfriendly graph might have three non-simple eigenspaces with multiplicities $m_1 = m_2 = m_3 = 2$, or two non-simple eigenspaces with multiplicities $m_1 = 3, m_2 = 2$. We call hostile unfriendly graphs with at least one $m_i > 2$. Based on numerical evidence, we conjecture that graphs exhibiting sufficiently rich symmetries are hostile.

The degree of a graph’s unfriendliness also impacts the dimension of the solution space of convex relaxations. Denoting by $S^*$ and $D^*$ the solution spaces of relaxations (8) and (3), respectively, we establish the following lower bound on the dimensions of these spaces:

**Lemma 4.** Let $A$ and $B$ be isomorphic $(k, m)$-unfriendly graphs. Then, \( \dim S^* \geq k + m \) with strict inequality if the graph has non-simple spectrum, and \( \dim D^* \geq k + m \) with strict inequality if the graph is hostile.

**Proof.** The linear system (6) yields a block-diagonal matrix $F$ with $d m_i \times m_i$ blocks corresponding to the non-simple eigenspaces, followed by diagonal entries corresponding to the simple ones. Since the eigenvectors spanning an $m_i$-dimensional eigenspace can be defined up to a rotation within it, either the entire eigenspace is orthogonal to the constant vector $1$, or such eigenvectors can be selected that none of them is orthogonal to $1$. If the eigenspace is not orthogonal to $1$, the corresponding $m_i \times m_i$ diagonal block of $F$ has $m_i(m_i - 1)$ degrees of freedom since the pseudo-stochasticity constraints $Fv = v$ add $m_i$ equations; otherwise, the constraints become trivial, adding $m_i$ degrees of freedom. This results in

\[
\dim S^* = k + \sum_{i=1}^{d} m_i(m_i - 1) \geq k + d + m,
\]

which is strictly bigger than $k + m$ if $d > 0$. A similar argument holds for $D^*$: If the eigenspace is not orthogonal to $1$, the additional set of constraints $F^Tv = v$ together with $Fv = v$ remove $2m_i - 1$ degrees of freedom (one of these constraints is dependent, as the sum of the row sums must equal the sum of the column sums), resulting in $(m_i - 1)^2$ degrees of freedom in each $m_i \times m_i$ diagonal block of $F$. Otherwise, both constraints become trivial.
adding $2m_i - 1$ degrees of freedom. Adding the non-negativity constraints $P \geq 0$, which can be thought as the projection of the affine solution space on the non-negative orthant, does not impact the dimension of $D^*$. This results in

$$\dim D^* \geq k + \sum_{i=1}^{d} (m_i - 1)^2 \geq k + m,$$

holding sharply if at least one $m_i > 2$.

Combining the results of Lemmas 3 and 4 leads to the following conditions under which the dimension of the solution space of convex relaxation is strictly bigger than the number of non-trivial symmetries:

**Corollary 2.** $\dim S^* \geq |\text{Sym}A|$ if the graph has non-simple spectrum;
$\dim D^* \geq |\text{Sym}A|$ if the graph is hostile.

Intuitively, the fact that a relaxation produces “too many” solutions implies that some of them must be “wrong”. Formally, this can be stated as follows:

**Theorem 3.** Let $A$ and $B$ be two isomorphic hostile graphs. Then, there exists a permutation matrix $\Pi \in D^*$ such that $\text{dis}(\Pi) > 0$.

**Proof.** The solution space $D^*$ is a convex polytope contained in $D$. Since every isomorphism is a global minimizer of $3$, the convex hull $\mathcal{C} = \text{Conv Iso}(A \mapsto B)$ is contained in $D^*$. By Corollary 2 $\dim D^* \geq |\text{Sym}A| - 1 \geq \dim \mathcal{C}$, implying that the orthogonal complement $\mathcal{C}^\perp$ in $D$ is non-empty. Since $\mathcal{C}^\perp$ is a subspace of $D$, it must contain at least one permutation $\Pi \notin \text{Iso}(A \mapsto B)$. \hfill \Box

Theorem 3 guarantees that for hostile graphs, RGM might return a non-zero distortion permutation; whether this happens or not in practice depends on the particular numerical optimization algorithm used to solve the relaxation and its initialization. Since $D^* \subseteq S^*$, the result applies to $S^*$ as well. The practical meaning of this result is that the considered two convex relaxations of graph matching are generally inapplicable to symmetric graphs.

### 7 Discussion and conclusion

In this paper, we considered convex relaxation of the NP-hard graph matching problem. We proposed an easy-to-verify friendliness property, and proved
that for friendly graphs, convex relaxation is equivalent to the NP-hard exact matching; the result extends to inexact matching of strongly friendly graphs. In such cases, convex relaxation is guaranteed to find the exact (or approximate) isomorphism or guarantee its inexistence. We also showed that convex relaxation is inapplicable to exact matching of graphs whose adjacency matrices have at least one non-simple eigenspace of dimension bigger than two (referred to as hostile). For such graphs, the solution space is non-trivial and is guaranteed to contain wrong solutions in addition to the correct isomorphisms; it is merely a matter of luck whether one of these solutions is returned by the optimization algorithm. The analysis tools we used are not refined enough to establish what happens to convex relaxation on the class of unfriendly graphs that are not hostile or have non-simple spectrum. We will attempt to close this gap in future studies.

Another surprising observation is that none of our results is influenced by the nonnegativity constraints $P \geq 0$. While the space of doubly-stochastic matrices is the smallest convex set containing the space $\mathcal{P}$ of $n \times n$ permutations, and is therefore the most natural convex relaxation of the latter, our findings question the utility of the non-negativity constraints in graph matching problems, and suggest relaxing $\mathcal{P}$ as the bigger affine space $\{P : P1 = P^T1 = 1\}$. Also, for the class of friendly graphs on which we were able to prove global convergence of convex relaxations, the column-wise equality constraints $P^T1 = 1$ has no utility and can be removed. The question whether these constraints are at all needed, and whether they can help extending the applicability of convex relaxation to, e.g., all non-hostile graphs requires further investigation.

From the practical perspective, the removal of the nonnegativity constraints might allow the use of simpler and better scalable convex optimization algorithms. Furthermore, the removal of the constraints $P^T1 = 1$ splits the remaining constraints $P1 = 1$ into $n$ constraints separable with respect to the rows of $P$. This allows to employ block-coordinate update schemes operating each time on $n$ variables from one row of $P$ only, thus potentially improving the algorithm scalability to large graphs.

Finally, we showed several, in our opinion, surprising, relations between spectral properties of the adjacency matrix and the geometry of the graph, such as the relation between friendliness and asymmetry, and unfriendliness degree and the size of the symmetry group. These relations between algebraic and geometric properties of graphs require further investigation. Even more surprising is the important role played by the orthogonality of the adjacency matrix eigenspaces to the constant vector. This questions the usability of convex relaxation when matching graphs with adjacency ma-
trices derived from differential operators (in particular, graph Laplacians). Having a constant eigenvector, the Laplacian has a very high degree of unfriendliness ($k = n - 1$ for a simply-connected graph). Consequently, using quantities such as diffusion distances as the adjacency weights for graph matching may require a second thought.

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8 Appendix

Lemma 5. Let $u_0$ be the solution of a full-rank linear system $Mu = c$, and let $u$ be the solution of the perturbed full-rank system $(M + \rho N)u = c$, $\rho > 0$. Then,

$$
\|u - u_0\| \leq \frac{\rho \|M^{-1}\| \|N\| \|u_0\|}{1 - \rho \|M^{-1}\| \|N\|}.
$$

(24)

Proof. Denoting $\delta = u - u_0$, we have $Mu_0 = c$ and $(M + \rho N)(u_0 + \delta) = c$, from where $(M + \rho N)\delta = -\rho Nu_0$. The latter is equivalent to $\delta = -\rho (M + \rho N)^{-1} Nu_0$ assuming an invertible $(M + \rho N)^{-1}$. From the identity

$$(M + \rho N)^{-1} = M^{-1}(I + \rho M^{-1}N)^{-1}$$

$$= M^{-1}\left(\sum_{i=0}^{\infty} (-\rho)^i (M^{-1}N)^i\right)$$

and the inequality $\|(M^{-1}N)^i v\| \leq \|M^{-1}\|^i \|N\|^i \|v\|$ holding for every $i \geq 0$ and every $v$, we have

$$
\|\delta\| = \|(M + \rho N)^{-1} Nu_0\| \leq \frac{\|M^{-1}\| \|N\| \|u_0\|}{1 - \rho \|M^{-1}\| \|N\|}.
$$

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