A critical phenomenon for sublinear elliptic equations in cone–like domains

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Abstract

We study positive supersolutions to an elliptic equation
\(-\Delta u = c|x|^{-s} u^p,\) \(p, s \in \mathbb{R},\) in cone–like domains in \(\mathbb{R}^N\) \((N \geq 2).\) We prove that in the sublinear case \(p < 1\) there exists a critical exponent \(p_* < 1\) such that equation \((*)\) has a positive supersolution if and only if \(-\infty < p < p_*\). The value of \(p_*\) is determined explicitly by \(s\) and the geometry of the cone.

1 Introduction

We study the existence and nonexistence of positive solutions and supersolutions to the equation

\[-\Delta u = \frac{c}{|x|^s} u^p \text{ in } C^\rho_\Omega.\]

Here \(p \in \mathbb{R}, s \in \mathbb{R}, c > 0\) and \(C^\rho_\Omega \subset \mathbb{R}^N\) \((N \geq 2)\) is an unbounded cone–like domain

\[C^\rho_\Omega := \{(r, \omega) \in \mathbb{R}^N : \omega \in \Omega, r > \rho\},\]

where \((r, \omega)\) are the polar coordinates in \(\mathbb{R}^N, r > 0\) and \(\Omega \subseteq S^{N-1}\) is a subdomain (a connected open subset) of the unit sphere \(S^{N-1}\) in \(\mathbb{R}^N.\) We say that \(u \in H^1_{loc}(C^\rho_\Omega)\) is a supersolution (subsolution) to equation \(\Box\) if

\[\int_{C^\rho_\Omega} \nabla u \cdot \nabla \varphi \, dx \geq (\leq) \int_{C^\rho_\Omega} \frac{c}{|x|^s} u^p \varphi \, dx \text{ for all } 0 \leq \varphi \in C^\infty_0(C^\rho_\Omega).\]
If \( u \) is a sub and supersolution to (1) then \( u \) is said to be a solution to (1). By the weak Harnack inequality any nontrivial nonnegative supersolution to (1) is positive in \( C^0_{\Omega} \).

We define critical exponents for equation (1) by

\[
\begin{align*}
p^* &= p^*(\Omega, s) = \inf\{p > 1 : \text{(1) has a positive supersolution in } C^0_{\Omega}\text{ for some } \rho > 0\}, \\
p_* &= p_*(\Omega, s) = \sup\{p < 1 : \text{(1) has a positive supersolution in } C^0_{\Omega}\text{ for some } \rho > 0\}.
\end{align*}
\]

Set \( p_* = -\infty \) if (1) has no positive supersolution in \( C^0_{\Omega} \) for any \( p < 1 \).

Remark 1. (i) One can show that if \( p < p_* \) or \( p > p^* \) then (1) has a positive solution in \( C^0_{\Omega} \) (see \([6]\) for the proof of the case \( p > 1 \) and the proofs below for the case \( p < 1 \)). The existence (or nonexistence) of positive (super) solutions at the critical values \( p_* \) and \( p^* \) is a separate issue.

(ii) Observe that in view of the scaling invariance of the Laplacian the critical exponents \( p_* \) and \( p^* \) do not depend on \( \rho > 0 \).

(iii) We do not make any assumptions on the smoothness of the domain \( \Omega \subseteq S^{N-1} \).

Let \( \lambda_1 = \lambda_1(\Omega) \geq 0 \) be the principal eigenvalue of the Dirichlet Laplace–Beltrami operator \(-\Delta_{\omega}\) on \( \Omega \). Let \( \alpha_+ \geq 0 \) and \( \alpha_- < 0 \) be the roots of the quadratic equation

\[
\alpha(\alpha + N - 2) = \lambda_1(\Omega).
\]

In the superlinear case \( p > 1 \) the value of the critical exponent is \( p^* = 1 - \frac{2 - s}{\alpha} \). Moreover, if \( s < 2 \) then (1) has no positive supersolutions in the critical case \( p = p^* \). This has been proved by Bandle and Levine \([3]\), Bandle and Essen \([2]\) and Berestycki, Capuzzo–Dolcetta and Nirenberg \([4]\) (see also \([6]\) for yet another proof of this result and for equations with measurable coefficients).

The sublinear case \( p < 1 \) has been studied in \([5,7]\). From the result of Brezis and Kamin \([5]\) it follows that for \( p \in (0, 1) \) equation (1) has a bounded positive solution in \( \mathbb{R}^N \) if and only if \( s > 2 \). It has been proved in \([7]\) (amongst other things) that for any \( p \in (-\infty, 1) \) equation (1) has a positive supersolution outside a ball in \( \mathbb{R}^N \) if and only if \( s > 2 \).

In this note, we discover a new critical phenomenon. Namely, we show that in sublinear case equation (1) exhibits a "non-trivial" critical exponent \( p_* > -\infty \) in cone-like domains. The main result of the paper reads as follows.

**Theorem 1.** For \( p \leq 1 \), the critical exponent for equation (1) is \( p_* = \min\{1 - \frac{2 - s}{\alpha}, 1\} \). If \( p_* < 1 \) then (1) has no positive supersolutions in the critical case \( p = p_* \).

Remark 2. (i) If \( \alpha_* = 0 \) then we set \( p_* = -\infty \).

(ii) If \( s > 2 \) then \( p_* = p^* = 1 \) and (1) has positive solutions for any \( p \in \mathbb{R} \) \([5,7]\). If \( s = 2 \) then \( p_* = p^* = 1 \). In this critical case (1) becomes a linear equation with the potential \( c|x|^{-2} \), which has a positive (super) solution if and only if \( c \leq \frac{(N-2)^2}{4} + \lambda_1(\Omega) \).

(iii) Let \( S_k = \{x \in S^{N-1} : x_1 > 0, \ldots x_k > 0\} \). Then \( \lambda_1(S_k) = k(k + N - 2) \) and \( \alpha_+(S_k) = k \), \( \alpha_-(S_k) = 2 - N - k \). Hence \( p_*(S_k, s) = 1 - \frac{2 - s}{k} \) and \( p^*(S_k, s) = 1 - \frac{2 - s}{2 - N - k} \). In particular, in the case of the halfspace \( S_1 \) we have \( p_*(S_1, s) = s - 1 \) and \( p^*(S_1, s) = \frac{N + 1 - s}{N - 1} \).

Applying the Kelvin transformation \( y = y(x) = \frac{x}{|x|^2} \) we see that if \( u \) is a positive solution to (1) in \( C^1_\Omega \) then \( \hat{u}(y) = |y|^{2-N}u(x(y)) \) is a positive solution to

\[
-\Delta \hat{u} = \frac{c}{|y|^s} \hat{u}^p \quad \text{in } C^1_\Omega,
\]
where \( \sigma = (N + 2) - p(N - 2) - s \) and \( \tilde{C}^1_\Omega := \{(r, \omega) \in \mathbb{R}^N : \omega \in \Omega, 0 < r < 1\} \). We define the critical exponents \( \tilde{p}^* = \tilde{p}^*(\Omega, s) \) and \( \tilde{p}_s = \tilde{p}_s(\Omega, s) \) for equation \( (2) \) similarly to \( p^*(\Omega, s) \) and \( p_s(\Omega, s) \).

In the superlinear case \( p > 1 \), Bandle and Essen [2] proved that if \( \sigma > 2 \) then \( \tilde{p}_s = 1 - \frac{2 - \sigma}{\alpha + N - 2} \) and \( (2) \) has no positive supersolutions when \( p = \tilde{p}^*(\Omega) \). In the sublinear case \( p < 1 \) by an easy computation we derive from Theorem 1 the following result.

**Theorem 2.** For \( p \leq 1 \), the critical exponent for equation \( (2) \) is \( \tilde{p}_s = \min\{1 - \frac{2 - \sigma}{\alpha + N - 2}, 1\} \). If \( \tilde{p}_s < 1 \) then \( (2) \) has no positive supersolutions in the critical case \( p = \tilde{p}_s \).

In the remaining part of the paper we prove Theorem 1.

### 2 Proof of Theorem 1

**Existence.** In the polar coordinates equation \( (1) \) reads as follows

\[
-u_{rr} - \frac{N-1}{r} u_r - \frac{1}{r^2} \Delta_\omega u = \varphi \frac{c}{r^s} u^p \quad \text{in} \quad C^1_\Omega.
\]

Let \( s \leq 2, \ p < 1 - \frac{2-s}{\alpha+1} \). Let \( 0 < \varphi \in H^1_{\text{loc}}(\Omega) \) be a positive solution to the equation

\[
-\Delta_\omega \psi - \alpha(\alpha + N - 2) \psi = \psi^p \quad \text{in} \quad \Omega,
\]

where \( \alpha := \frac{2-s}{1-p} \). Then it is readily seen that \( u := c^{-1} r^{\alpha} \psi \in H^1_{\text{loc}}(C^1_\Omega) \) is a positive solution to \( (3) \) in \( C^1_\Omega \). Thus the problem reduces to the existence of positive solutions to \( (4) \).

Note that \( 0 < \alpha(\alpha + N - 2) < \lambda_1(\Omega) \). Hence the operator \(-\Delta_\omega - \alpha(\alpha + N - 2)\) is coercive on \( H^1_0(\Omega) \) and satisfies the maximum principle. We consider separately the cases \( p \in [0,1] \) and \( p < 0 \).
Case $p \in [0, 1)$. Let $\phi_1 > 0$ be the principal Dirichlet eigenfunction of $-\Delta_\omega$ on $\Omega$. Let $\phi > 0$ be the unique solution to the problem

$$-\Delta_\omega \phi - \alpha(\alpha + N - 2)\phi = 1, \quad \phi \in H^1_0(\Omega).$$

Observe that $\phi_1, \phi \in L^\infty$.

Hence $\tau \phi_1$ is a supersolution to (4) for a large $\tau > 0$, and $\epsilon \phi_1$ is a subsolution to (4) for a small $\epsilon > 0$. Thus by the sub and supersolutions argument equation (4) has a solution $\psi \in H^1_0(\Omega)$ such that $\epsilon \phi_1 < \psi \leq \tau \phi_1$.

Case $p < 0$. Consider the problem

$$-\Delta_\omega \phi - \alpha(\alpha + N - 2)(\phi + 1) = (\phi + 1)^p, \quad \phi \in H^1_0(\Omega).$$

Let $\phi > 0$ be the unique solution to the problem

$$-\Delta \phi - \alpha(\alpha + N - 2)(\phi + 1) = 1, \quad \phi \in H^1_0(\Omega).$$

It is clear that $\phi$ is a supersolution to (5) and $\phi \equiv 0$ is a subsolution to (5). We conclude that (5) has a positive solution $\phi \in H^1_0(\Omega)$ such that $0 < \phi \leq \phi_1$. Then $\psi := \phi + 1 \in H^1_{loc}(\Omega)$ is a positive solution to (4). This completes the proof of the existence part of Theorem 1.

Nonexistence. In what follows we set $\delta := 1$ if $p < 0$ and $\delta := 0$ if $p \in [0, 1)$. Let $G \subset \mathbb{R}^N$ be a domain, $0 \notin G$. Observe that equation (1) has a positive supersolution in $G$ if and only if the equation

$$-\Delta w = \frac{c}{|x|^s}(w + \delta)^p \quad \text{in} \ G$$

has a positive supersolution. Indeed, if $u > 0$ is a supersolution to (1) in $G$ then $u$ is a supersolution to (6). If $w > 0$ is a supersolution to (6) then $u = w + \delta$ is a supersolution to (1). The main argument of the proof nonexistence rests upon the following two lemmas.

The next lemma is an adaptation a comparison principle by Ambrosetti, Brezis and Cerami [1, Lemma 3.3].

**Lemma 3.** Let $G \subset \mathbb{R}^N$ be a bounded domain, $0 \notin G$. Let $0 \leq w \in H^1_0(G)$ be a subsolution and $0 \leq \overline{w} \in H^1_{loc}(G)$ a supersolution to (6). Then $\overline{w} \leq w$ in $G$.

**Proof.** In [1, Lemma 3.3] the result was proved for a smooth bounded domain $G$ and $\overline{w}, w \in H^1_0(G)$ (and more general nonlinearities). The proof given in [1] carries over literally to the case of an arbitrary bounded domain $G$ and $\overline{w}, w \in H^1_0(G)$, or a smooth bounded domain $G, w \in H^1_0(G)$ and $0 \leq \overline{w} \in H^1(G)$. Thus we only need to extend the lemma to an arbitrary bounded domain $G$ and $\overline{w} \in H^1_{loc}(G)$.

Let $\overline{w} \in H^1_{loc}(G)$ be a supersolution to (6) in $G$. Let $(G_n)_{n \in \mathbb{N}}$ be an exhaustion of $G$, that is a sequence of bounded smooth domains such that $\overline{G}_n \subset G_{n+1} \subset G$ and $\cup_{n \in \mathbb{N}} G_n = G$. Analogously to the argument given above in the existence part of the proof, one can readily see that, for each $n \in \mathbb{N}$, there exists a solution $0 < w_n \in H^1_0(G_n)$ to (6) (e.g., by constructing appropriate sub and supersolutions). Moreover, $w_n \leq w_{n+1}$. Observe that $w_n \leq \overline{w}$ in $G_n$ by [1] Lemma 3.3].
We claim that \( \sup \| \nabla w_n \|_{L^2} < \infty \). This is clear for \( p < 0 \), since \( (w + 1)^p \leq 1 \). For \( p \in [0, 1) \), using \( w_n \) as a test function in (3), we have

\[
\int_G |\nabla w_n|^2 \, dx = \int_G \frac{c}{|x|^s} w_n^{p+1} \, dx \leq c_1 \left( \int_G |\nabla w_n|^2 \, dx \right)^{(p+1)/2},
\]

which implies the claim. It follows that \( w_n \) converges pointwise in \( G \), strongly in \( L^2(G) \) and weakly in \( H_0^1(G) \) to a positive \( w_* \in H_0^1(G) \). Clearly \( w_* > 0 \) is a solution to (6) in \( G \) and \( 0 < w_* \leq \bar{w} \) in \( G \).

Now let \( 0 \leq w \in H_0^1(G) \) be a subsolution to (6) in \( G \). By (1) Lemma 3.3] we conclude that \( \bar{w} \leq w_* \) in \( G \).

Next, consider the initial value problem

(7) \[ -v_{rr} - \frac{N - 1}{r} v_r + \frac{\lambda_1}{r^2} v = \frac{c}{r^s} \quad \text{for } r > 1; \quad v(1) = \delta, \quad v_r(1) = K; \]

where \( p < 1, s \in \mathbb{R}, c > 0, K > 1 \) and \( \delta \) as above. Let \( (1, R), R = R(\delta, K) \leq \infty, \) be the maximal right interval of existence of the solution \( v \) to (7) in the region \( \{(r, v) \in (1, +\infty) \times (\delta, +\infty)\} \).

**Lemma 4.** Let \( s < 2 \) and \( p \in [1 - \frac{2-s}{a+1}, 1) \). Then for any interval \([r_*, r^*] \subset (1, +\infty)\) there exists \( K_0 > 0 \) such that

i) for all \( K > K_0 \) one has \( r_* < R < +\infty \) and \( v(r) \to \delta \) as \( r \nrightarrow R \);

ii) for any \( M > \delta \) there exists \( K > K_0 \) such that \( \min_{[r_*, r^*]} v \geq M \).

**Proof.** Set \( \alpha := \alpha_+, v := wr^\alpha, t = r^{2-N-2\alpha} \). Then \( w \) solves the following problem

\[
w_{tt} + c_1 t^{-\sigma} w^p = 0 \quad \text{for } t \in (T, 1); \quad w(1) = \delta, \quad w_t(1) = -L,
\]

where \( \sigma = \frac{2N-2+a(p+3)-s}{N-2+2a} \geq 2, c_1 > 0, 0 \leq T = R^{2-N-2\alpha} < 1 \) and \( L = \frac{K^{1-a}\delta}{N-2+2a} \to \infty \) as \( K \to \infty \). Choose \( K_0 \) such that \( L > \delta \). Observe that \( w(t) \) is concave, hence

\[
\delta < w(t) \leq w(1) - w_t(1)(1 - t) \leq \delta + L \quad \text{for } t \in (T, 1).
\]

To see that \( T > 0 \) let \( \bar{w} := w \) for \( p < 0 \), otherwise let \( \bar{w} := w^{1-p} \). Then \( \bar{w} \) satisfies the inequality

\[
\bar{w}_{tt} + c_2 t^{-2} \bar{w}^{q} \leq 0 \quad \text{for } t \in (T, 1),
\]

with \( c_2 > 0 \) and \( q := \min\{p, 0\} \). Integrating \( \bar{w}_{tt} \) twice one can easily see that such inequality has no positive solutions in any neighborhood of zero. Thus we conclude that \( T > 0 \), hence \( w(t) \to \delta \) as \( t \searrow T \). In particular, \( w(t) \) attains its maximum on \( (T, 1) \).

Let \( T_0 \in (T, 1) \) be such that \( w_t(T_0) = -\frac{L-\delta}{2} \). Since \( \delta \leq w(t) \leq \delta + L \) for \( t \in (T_0, 1) \), it follows that

\[
\frac{L + \delta}{2} = w_t(T_0) - w_t(1) = -\int_{T_0}^{1} w_{tt} \, d\tau = c_1 \int_{T_0}^{1} \frac{w^p}{t^\sigma} \, d\tau \leq c_3 \left( \frac{1}{T_0^{p-1}} - 1 \right) \quad \text{for } t \in (T_0, 1).
\]

Hence \( T_0 \to 0 \) as \( L \to +\infty \). Therefore for any given \( t^* < 1 \) there exists \( L_0 > 1 \) such that for any \( L > L_0 \) one has \( 0 < T < T_0 < t^* \). Thus, (i) follows with \( r^* = (t^*)^{\frac{1}{N-2+2a}} \).
Observe now that for any $L > L_0$ we have
\[ -\frac{L - \delta}{2} \geq w(t) \geq -L \quad \text{for} \quad t \in (t^*, 1), \]
since $w$ is concave. Hence for any $t \in (t^*, 1)$ we obtain
\[ w(t) = w(1) - \int_t^1 w(t) dt \geq \delta + (1 - t) \frac{L - \delta}{2} \to \infty \quad \text{as} \quad L \to \infty. \]
Thus (ii) follows.

Nonexistence – completed. Let $p \in [1 - \frac{2-n}{\alpha_x}, 1)$. Fix a compact $K \subset C^1_\Omega$ and $M > 1$. There exists an interval $[r_*, r^*] \subset (1, +\infty)$ such that $K \subset C^{(r_*, r^*)}_\Omega$, where $C^{(r_1, r_2)}_\Omega$ denotes the set $\{x \in C^1_\Omega | r_1 \leq |x| \leq r_2\}$. Then by Lemma 4 there exists $v : (1, R) \to (\delta, +\infty)$ solving (7) such that $R > r^*$ and $\inf_{[r_*, r^*]} v \geq M + \delta$.

Let $\phi_1 > 0$ be the principal Dirichlet eigenvalue of $-\Delta_\omega$ on $\Omega$ with $\|\phi_1\|_\infty = 1$. Set $w_M := (v - \delta)\phi_1$. Then $0 < w_M \in H^1_0(C^{(1,R)}_\Omega)$, and direct computation shows that $w_M$ is a subsolution to (6) in $C^{(1,R)}_\Omega$. Now assume that $w > 0$ is a supersolution to (6) in $C^1_\Omega$. By Lemma 3 it follows that that $w \geq w_M$ in $C^{(1,R)}_\Omega$. By the weak Harnack inequality we have
\[ \inf_K w \geq c_H \int_K w dx \geq c_H \int_K w_M dx \geq c_2 M. \]
Since $M$ was arbitrary, we conclude that $w \equiv +\infty$ in $K$.

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References

[1] A. Ambrosetti, H. Brezis and G. Cerami, Combined effect of concave and convex nonlinearities in some elliptic problems. J. Funct. Anal. 122 (1994), 519–543.

[2] C. Bandle and M. Essén, On positive solutions of Emden equations in cone-like domains, Arch. Rational Mech. Anal. 112 (1990), 319–338.

[3] C. Bandle and H. A. Levine, On the existence and nonexistence of global solutions of reaction–diffusion equations in sectorial domains. Trans. Amer. Math. Soc. 316 (1989), 595–622.

[4] H. Berestycki, I. Capuzzo-Dolcetta and L. Nirenberg, Superlinear indefinite elliptic problems and nonlinear Liouville theorems. Topol. Methods Nonlinear Anal. 4 (1994), 59–78.

[5] H. Brezis and S. Kamin, Sublinear elliptic equations in $\mathbb{R}^N$, Manuscripta Math. 74 (1992), 87–106.

[6] V. Kondratiev, V. Liskevich and V. Moroz, Positive solutions to superlinear second–order divergence type elliptic equations in cone–like domains, Preprint, 2003.

[7] V. Kondratiev, V. Liskevich and Z. Sobol, Second–order semilinear elliptic inequalities in exterior domains, J. Differential Equations 187 (2003), 429–455.