GROUP ANALYSIS OF NONLINEAR INTERNAL WAVES IN OCEANS

I: Self-adjointness, conservation laws, invariant solutions

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Abstract. The paper is devoted to the group analysis of equations of motion of two-dimensional uniformly stratified rotating fluids used as a basic model in geophysical fluid dynamics. It is shown that the nonlinear equations in question have a remarkable property to be self-adjoint. This property is crucial for constructing conservation laws provided in the present paper. Invariant solutions are constructed using certain symmetries. The invariant solutions are used for defining internal wave beams.

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1 Introduction

We will apply Lie group analysis for investigating the system of nonlinear equations

\begin{align*}
\Delta \psi_t - g \rho_x - f v_z &= \psi_z \Delta \psi_z - \psi_z \Delta \psi_x, \\
v_t + f \psi_z &= \psi_z v_z - \psi_z v_x, \\
\rho_t + \frac{N^2}{g} \psi_x &= \psi_x \rho_z - \psi_z \rho_x
\end{align*} 

(1.1) (1.2) (1.3)

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used in geophysical fluid dynamics for investigating internal waves in uniformly stratified incompressible fluids (oceans). In particular, the system (1.1)-(1.3) with $f=0$ was used in [1] to study two non-unidirectional wave beams propagating and interacting in stratified fluid. An exact solution of the same system, again in the case when $f=0$, was employed in [2] for investigating stability of a single internal plane wave. Weakly nonlinear effects in colliding of internal wave beams were investigated in [3], [4] by using Eqs. (1.1)-(1.3) with $f=0$. The system (1.1)-(1.3) with $f \neq 0$ was used in [5] to model weakly nonlinear wave interactions governing the time behavior of the oceanic energy spectrum.

In these equations $\Delta$ is the two-dimensional Laplacian:

$$\Delta = D_x^2 + D_z^2,$$

e.g.  \[ \Delta \psi_t = \frac{\partial^2 \psi_t}{\partial x^2} + \frac{\partial^2 \psi_t}{\partial z^2} \equiv D_t(\Delta \psi), \]

and $g, f, N$ are constants. Namely, $g$ is the gravitational acceleration, $f$ is the Coriolis parameter. The quantity $N$ appears due to the density stratification of a fluid and is constant under the linear stratification hypothesis.

We will show in what follows that the system of equations (1.1)-(1.3) is self-adjoint (in the terminology of [6, 7]) and use this remarkable property of the system for calculating conservation laws associated with symmetry properties of the system (1.1)-(1.3).

In some calculations, e.g. in Sections 4.7, 4.5, 4.8 it is convenient to write Eqs. (1.1)-(1.3) by using the Jacobians $J(\psi, v) = \psi_x v_z - \psi_z v_x$, etc., in the following form:

$$\Delta \psi_t - g \rho_x - f v_z = J(\psi, \Delta \psi), \quad (1.4)$$

$$v_t + f \psi_z = J(\psi, v), \quad (1.5)$$

$$\rho_t + \frac{N^2}{g} \psi_x = J(\psi, \rho). \quad (1.6)$$

2 Self-adjointness

2.1 Preliminaries

We will use the terminology and the following definitions from [6, 7] (see also [8]).

Let $x = (x^1, \ldots, x^n)$ be $n$ independent variables, and $u = (u^1, \ldots, u^m)$ be $m$ dependent variables. The partial derivatives of $u^\alpha$ with respect to $x^i$ are denoted by $u_{(1)} = \{u^\alpha_i\}$, $u_{(2)} = \{u^\alpha_{ij}\}$, \ldots with

$$u^\alpha_i = D_i(u^\alpha), \quad u^\alpha_{ij} = D_i(u^\alpha_j) = D_i D_j(u^\alpha), \ldots,$$
where $D_i$ is the operator of total differentiation with respect to $x^i$:

$$D_i = \frac{\partial}{\partial x^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_j} + \cdots, \quad i = 1, \ldots, n. \quad (2.1)$$

Even though the operators $D_i$ are given by formal infinite sums, their action $D_i(f)$ is well defined for functions $f(x, u, u^{(1)}, \ldots)$ depending on a finite number of the variables $x, u, u^{(1)}, u^{(2)}, \ldots$. The usual summation convention on repeated indices $\alpha$ and $i$ is assumed in expressions like Eq. (2.1).

The variational derivatives (the Euler-Lagrange operator) are defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^{\alpha}_{i_1 \ldots i_s}}, \quad \alpha = 1, \ldots, m, \quad (2.2)$$

where the summation over the repeated indices $i_1 \ldots i_s$ runs from 1 to $n$.

**Definition 2.1.** The adjoint equations to nonlinear partial differential equations

$$F^*_\alpha(x, u, \ldots, u^{(s)}) = 0, \quad \alpha = 1, \ldots, m, \quad (2.3)$$

are given by (see also [9])

$$F^*_\alpha(x, u, \mu, \ldots, u^{(s)}, \mu^{(s)}) = 0, \quad \alpha = 1, \ldots, m, \quad (2.4)$$

where $\mu = (\mu^1, \ldots, \mu^m)$ are new dependent variables, and $F^*_\alpha$ are defined by

$$F^*_\alpha(x, u, \mu, \ldots, u^{(s)}, \mu^{(s)}) = \frac{\delta (\mu^\beta F^*_\beta)}{\delta u^\alpha}. \quad (2.5)$$

In the case of linear equations, Definition 2.1 is equivalent to the classical definition of the adjoint equation.

Consider the function

$$\mathcal{L} = \mu^\beta F^*_\beta(x, u, \ldots, u^{(s)}) \quad (2.6)$$

involved in (2.5). Eqs. (2.3) and their adjoint equations (2.4) can be obtained from (2.5) by taking the variational derivatives (2.2) with respect to the dependent variables $u$ and the similar variational derivatives with respect to the new dependent variables $\mu$,

$$\frac{\delta}{\delta \mu^\alpha} = \frac{\partial}{\partial \mu^\alpha} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial \mu^{\alpha}_{i_1 \ldots i_s}}, \quad \alpha = 1, \ldots, m. \quad (2.7)$$
Namely:
\[
\frac{\delta L}{\delta \mu^\alpha} = F_\alpha(x, u, \ldots, u(s)), (2.8)
\]
\[
\frac{\delta L}{\delta u^\alpha} = F_*^\alpha(x, u, \mu, \ldots, u(s), \mu(s)). (2.9)
\]

This circumstance justifies the following definition.

**Definition 2.2.** The differential function (2.6) is called a *formal Lagrangian* for the differential equations (2.3). For the sake of brevity, formal Lagrangians are also referred to as Lagrangians.

If the variables \(u\) are known, the new variables \(\mu\) are obtained by solving Eqs. (2.4) which are, according to (2.5), linear partial differential equations (2.4) with respect to \(\mu^\alpha\). Using the existing terminology (see, e.g. [10]), we will call \(\mu^\alpha\) *nonlocal variables*.

Nonlocal variables can be excluded from physical quantities such as conservation laws if Eqs. (2.3) are *self-adjoint* ([6]) or, in general, *quasi-self-adjoint* ([11]) in the following sense.

**Definition 2.3.** Eqs. (2.3) are said to be *self-adjoint* if the system obtained from the adjoint equations (2.4) by the substitution \(\mu = u\):
\[
F_*^\alpha(x, u, u, \ldots, u(s), u(s)) = 0, \quad \alpha = 1, \ldots, m, \quad (2.10)
\]
is equivalent to the original system (2.3), i.e.
\[
F_\alpha^* (x, u, u, \ldots, u(s), u(s)) = \Phi_\beta^\alpha F_\beta(x, u, \ldots, u(s)), \quad \alpha = 1, \ldots, m,
\]
with regular (in general, variable) coefficients \(\Phi_\beta^\alpha\).

**Definition 2.4.** Eqs. (2.3) are said to be quasi-self-adjoint if the system of adjoint equations (2.4) becomes equivalent to the original system (2.3) upon the substitution
\[
\mu = h(u) \quad (2.11)
\]
with a certain function \(h(u)\) such that \(h'(u) \neq 0\).

### 2.2 Adjoint system to Eqs. (1.1)-(1.3)

Let us apply the methods from Section 2.1 to Eqs. (1.1)-(1.3). In this case the formal Lagrangian (2.6) for Eqs. (1.1)-(1.3) is written
\[
\mathcal{L} = \varphi \left[ \Delta \psi_t - g \rho_x - f \psi_z - \psi_x \Delta \psi_z + \psi_z \Delta \psi_x \right]
\]
\[
+ \mu \left[ v_t + f \psi_z - \psi_x v_z + \psi_z v_x \right] + r \left[ \rho_t + \frac{N^2}{g} \psi_z - \psi_x \rho_z + \psi_z \rho_x \right], \quad (2.12)
\]
where $\varphi, \mu$ and $r$ are new dependent variables. The adjoint equations to Eqs. (1.1)-(1.3) are obtained by taking the variational derivatives of $L$, namely:

$$
\frac{\delta L}{\delta \psi} = 0, \quad \frac{\delta L}{\delta v} = 0, \quad \frac{\delta L}{\delta \rho} = 0,
$$

(2.13)

where (see (2.2); see also Eqs. (3.6))

$$
\frac{\delta}{\delta \psi} = \frac{\partial}{\partial \psi} - D_x \frac{\partial}{\partial \psi x} - D_z \frac{\partial}{\partial \psi z},
$$

$$
\frac{\delta}{\delta \rho} = \frac{\partial}{\partial \rho} - D_x \frac{\partial}{\partial \rho x} - D_z \frac{\partial}{\partial \rho z},
$$

$$
\frac{\delta}{\delta v} = \frac{\partial}{\partial v} - D_x \frac{\partial}{\partial v x} - D_z \frac{\partial}{\partial v z} + D_x D_t \frac{\partial}{\partial \psi x t} + D_z D_t \frac{\partial}{\partial \psi z t} + \cdots.
$$

Taking into account the special form (2.12) of $L$, we have:

$$
\frac{\delta L}{\delta \psi} = - D_x \frac{\partial L}{\partial \psi x} - D_z \frac{\partial L}{\partial \psi z} - (D_x^2 + D_z^2) \left[ D_t \frac{\partial L}{\partial \Delta \psi_t} + D_x \frac{\partial L}{\partial \Delta \psi_x} + D_z \frac{\partial L}{\partial \Delta \psi_z} \right]
$$

$$
= D_x \left( \varphi \Delta \psi_z + \mu v_z - \frac{N^2}{g} r + r \rho_z \right) - D_z \left( \varphi \Delta \psi_x + f \mu + \mu v_x + r \rho_x \right)
$$

$$
- (D_x^2 + D_z^2) \left[ D_t (\varphi) + D_x (\varphi \psi_z) - D_z (\varphi \psi_x) \right]
$$

$$
= \varphi_x \Delta \psi_z - \varphi_z \Delta \psi_x + \mu_x v_x - \frac{N^2}{g} r_x + r \rho_x - f \mu_x + \mu_z v_x - r_z \rho_x
$$

$$
- \Delta \varphi_t + 2 \left[ \varphi_x \varphi_{xx} + \varphi_z \varphi_{zz} - \varphi_x z \varphi_{zz} - \varphi_z x \varphi_{zz} \right],
$$

$$
\frac{\delta L}{\delta v} = - D_t \frac{\partial L}{\partial \psi t} - D_x \frac{\partial L}{\partial \psi x} - D_z \frac{\partial L}{\partial \psi z} = - \mu_t - \mu_x \psi_x + f \varphi_x + \mu_z \psi_x ,
$$

$$
\frac{\delta L}{\delta \rho} = - D_t \frac{\partial L}{\partial \rho t} - D_x \frac{\partial L}{\partial \rho x} - D_z \frac{\partial L}{\partial \rho z} = - r_t + g \varphi_x - r_x \psi_x + r_z \psi_x .
$$

Hence, the adjoint equations (2.13) can be written as follows:

$$
\Delta \varphi_t + \frac{N^2}{g} r_x + f \mu_z - \varphi_x \Delta \psi_z + \varphi_z \Delta \psi_x - \Theta = 0, \quad (2.14)
$$

$$
- \mu_t - \mu_x \psi_x + f \varphi_x + \mu_z \psi_x = 0, \quad (2.15)
$$

$$
- r_t + g \varphi_x - r_x \psi_x + r_z \psi_x = 0, \quad (2.16)
$$

5
where

$$\Theta = J(\mu, v) + J(r, \rho) + 2(\varphi_{xz}\psi_{xx} + \varphi_{zz}\psi_{xz} - \varphi_{xx}\psi_{xz} - \varphi_{xz}\psi_{zz}). \quad (2.17)$$

### 2.3 Self-adjointness of Eqs. (1.1)-(1.3)

**Theorem 2.1.** Eqs. (1.1)-(1.3) are quasi-self-adjoint.

**Proof.** Looking for (2.11) in the form of a general scaling transformation, one can readily obtain that after the transformation

$$\varphi = \psi, \quad \mu = -v, \quad r = -\frac{g^2}{N^2} \rho, \quad (2.18)$$

the quantity $\Theta$ given by Eq. (2.17) vanishes. Therefore the adjoint equations (2.14)-(2.16) become identical with Eqs. (1.1)-(1.3) after the substitution (2.18). Hence, according to Definition 2.4, Eqs. (1.1)-(1.3) are quasi-self-adjoint. Since Eqs. (2.18) are obtained just by simple scaling of the equations $\varphi = \psi, \mu = v, r = \rho$ required for the self-adjointness, we will say that Eqs. (1.1)-(1.3) are self-adjoint.

### 3 Conservation laws

#### 3.1 General discussion of conservation equations

Along with the individual notation $t, x, z$ for the independent variables, and $v, \rho, \psi$ for the dependent variables, we will also use the index notation $x^1 = t, x^2 = x, x^3 = z$ and $u^1 = v, u^2 = \rho, u^3 = \psi$, respectively. We will write the conservation laws both in the differential form

$$D_t(C^1) + D_x(C^2) + D_z(C^3) = 0 \quad (3.1)$$

and the integral form

$$\frac{d}{dt} \int \int C^1 \, dx \, dz = 0, \quad (3.2)$$

where the double integral in taken over the the $(x, z)$ plane $\mathbb{R}^2$. The equations (3.1) and (3.2) provide a conservation law for Eqs. (1.1)-(1.3) if they hold for the solutions of Eqs. (1.1)-(1.3). The vector $C = (C^1, C^2, C^3)$ satisfying the conservation equation (3.1) is termed a conserved vector. Its component $C^1$ is called the density of the conservation law due to Eq. (3.2). The two-dimensional vector $(C^2, C^3)$ defines the flux of the conservation law.
The integral form (3.2) of a conservation law follows from the differential form (3.1) provided that the solutions of Eqs. (1.1)-(1.3) vanish or rapidly decrease at the infinity on $\mathbb{R}^2$. Indeed, integrating Eq. (3.1) over an arbitrary region $\Omega \subset \mathbb{R}^2$ we have:

$$\frac{d}{dt} \iint_{\Omega} C^1 dxdz = - \iint_{\Omega} [D_x(C^2) + D_z(C^3)] dxdz.$$

According to Green’s theorem, the integral on the right-hand side reduces to the integral along the boundary $\partial \Omega$ of $\Omega$:

$$- \iint_{\Omega} [D_x(C^2) + D_z(C^3)] dxdz = \int_{\partial \Omega} C^3 dx - C^2 dz,$$

and hence vanishes as $\Omega$ expands and becomes the plane $\mathbb{R}^2$.

**Remark 3.1.** It is manifest from this discussion that one can ignore in $C^1$ “divergent type” terms because they do not change the integral in the conservation equation (3.2). Specifically if $C^1$ evaluated on the solutions of Eqs. (1.1)-(1.3) has the form

$$C^1 = \tilde{C}^1 + D_x(h^2) + D_z(h^3)$$

with some functions $h^2, h^3$, then the conservation equation (3.1) can be equivalently rewritten in the form (see [12], Paper 1, Section 20.1)

$$D_t(\tilde{C}^1) + D_x(\tilde{C}^2) + D_z(\tilde{C}^3) = 0,$$

where

$$\tilde{C}^2 = C^2 + D_t(h^2), \quad \tilde{C}^3 = C^3 + D_t(h^3).$$

Accordingly, we have

$$\iint C^1 dxdz = \iint \tilde{C}^1 dxdz,$$

and hence the integral conservation equation (3.2) provided by the conservation density $C^1$ of the form (3.3) coincides with that provided by the density $\tilde{C}^1$.

In particular, if $\tilde{C}^1 = 0$ the integral in Eq. (3.2) vanishes. This kind of conservation laws are *trivial* from physical point of view. Therefore we single out physically useless conservation laws by the following definition.
Definition 3.1. The conservation law is said to be trivial if its density $C^1$ evaluated on the solutions of Eqs. (1.1)-(1.3) is the divergence,

$$C^1 = D_x(h^2) + D_z(h^3).$$

The following statement ([13], Section 8.4.1; see also [7]) simplifies calculations while dealing with conservation equations.

Lemma 3.1. A function $F(v, \rho, \psi, v_x, v_z, \rho_x, \rho_z, \psi_x, \psi_z, \psi_{xt}, \psi_{zt}, \ldots)$ is the divergence,

$$F = D_x(C^1) + D_z(C^2), \quad (3.4)$$

if and only if satisfies the following equations:

$$\frac{\delta F}{\delta v} = 0, \quad \frac{\delta F}{\delta \rho} = 0, \quad \frac{\delta F}{\delta \psi} = 0. \quad (3.5)$$

Here the variational derivatives act on $F$ as usual (see also Section 2.2):

$$\frac{\delta F}{\delta v} = \partial F/\partial v - D_x \left( \partial F/\partial v_x \right) - D_z \left( \partial F/\partial v_z \right),$$

$$\frac{\delta F}{\delta \rho} = \partial F/\partial \rho - D_x \left( \partial F/\partial \rho_x \right) - D_z \left( \partial F/\partial \rho_z \right), \quad (3.6)$$

$$\frac{\delta F}{\delta \psi} = \partial F/\partial \psi - D_x \left( \partial F/\partial \psi_x \right) - D_z \left( \partial F/\partial \psi_z \right) + D_x D_t \left( \partial F/\partial \psi_{xt} \right) + D_z D_t \left( \partial F/\partial \psi_{zt} \right) + \cdots. \quad (3.7)$$

Corollary 3.1. A function $C^1$ is the density of a conservation law (3.1) if and only if the function

$$F = D_t(C^1)\big|_{(1.1)-(1.3)}$$

satisfies Eqs. (3.5). Here $\big|_{(1.1)-(1.3)}$ means that the quantity $D_t(C^1)$ is evaluated on the solutions of Eqs. (1.1)-(1.3).

In particular, Lemma 3.1 allows one to single out trivial conservation laws as follows.

Corollary 3.2. The conservation law (3.1) is trivial if and only if its density $C^1$ evaluated on the solutions of Eqs. (1.1)-(1.3), i.e. the quantity

$$C^1_* = C^1\big|_{(1.1)-(1.3)}$$

satisfies Eqs. (3.5),

$$\frac{\delta C^1_*}{\delta v} = 0, \quad \frac{\delta C^1_*}{\delta \rho} = 0, \quad \frac{\delta C^1_*}{\delta \psi} = 0. \quad (3.8)$$

on the solutions of Eqs. (1.1)-(1.3).
3.2 Variational derivatives of expressions with Jacobians

We will use in our calculations the following statement on the behaviour of certain expressions with Jacobians under the action of the variational derivatives (3.6).

**Proposition 3.1.** The following equations hold:

\[
\frac{\delta J(\psi, v)}{\delta v} = 0, \quad \frac{\delta J(\psi, v)}{\delta \psi} = 0, \tag{3.10}
\]

\[
\frac{\delta [vJ(\psi, v)]}{\delta v} = 0, \quad \frac{\delta [vJ(\psi, v)]}{\delta \psi} = 0, \tag{3.11}
\]

\[
\frac{\delta [\rho J(\psi, \rho)]}{\delta \rho} = 0, \quad \frac{\delta [\rho J(\psi, \rho)]}{\delta \psi} = 0, \tag{3.12}
\]

\[
\frac{\delta J(\psi, \Delta \psi)}{\delta \psi} = 0, \quad \frac{\delta [\psi J(\psi, \Delta \psi)]}{\delta \psi} = 0. \tag{3.13}
\]

**Proof.** Let us verify that the first equation (3.10) holds. We have (see (3.6)):

\[
\frac{\delta J(\psi, v)}{\delta v} = \frac{\delta (\psi_x v_z - \psi_z v_x)}{\delta v} = -D_z(\psi_x) + D_x(\psi_z) = -\psi_{xz} + \psi_{zx} = 0.
\]

Replacing \( v \) by \( \psi \) one obtains the second equation (3.10). Let us verify now that Eqs. (3.11) are satisfied. We have:

\[
\frac{\delta [vJ(\psi, v)]}{\delta v} = \frac{\delta [v(\psi_x v_z - \psi_z v_x)]}{\delta v} = \frac{\delta [v(\psi_x v_z - \psi_z v_x)]}{\delta \psi} - D_z(v \psi_x) + D_x(v \psi_z)
\]

\[= J(\psi, v) - D_z(v \psi_x) + D_x(v \psi_z) = J(\psi, v) - J(\psi, v) - v \psi_{xz} + v \psi_{zx} = 0 \]

and

\[
\frac{\delta [vJ(\psi, v)]}{\delta \psi} = -D_x(v v_x) + D_z(v v_z) = -v_x v_z - v v_z + v_x v_z + v v_z = 0.
\]

Replacing \( v \) by \( \rho \) one obtains Eqs. (3.12). Eqs. (3.13) are derived likewise even though they involve higher-order derivatives. We have:

\[
\frac{\delta J(\psi, \Delta \psi)}{\delta \psi} = \frac{\delta (\psi_x \Delta \psi_z - \psi_z \Delta \psi_x)}{\delta \psi} = \frac{\delta [\psi_x (\psi_{xxx} + \psi_{xzz}) - \psi_z (\psi_{xxx} + \psi_{xzz})]}{\delta \psi}
\]

\[= -D_x(\Delta \psi_z) + D_z(\Delta \psi_x) - D_x(D^2_x + D^2_z)(\psi_x) + D_x(D^2_x + D^2_z)(\psi_z)
\]

\[= -D_x(\Delta \psi_z) + D_z(\Delta \psi_x) - D_z(\Delta \psi_x) + D_x(\Delta \psi_z) = 0.
\]
Derivation of the second equation (3.13) requires only a simple modification of the previous calculations. Namely:

\[
\frac{\delta J(\psi, \Delta \psi)}{\delta \psi} = \frac{\delta (\psi \Delta \psi_z - \psi_z \Delta \psi_x)}{\delta \psi} = \psi_x \Delta \psi_z - \psi_z \Delta \psi_x - D_x (\psi \Delta \psi_z) + D_z (\psi \Delta \psi_x) - D_x (\psi \Delta \psi_z) + D_z (\psi \Delta \psi_x) - \frac{1}{2} \Delta D_z D_x (\psi^2) + \frac{1}{2} \Delta D_x D_z (\psi^2) = 0.
\]

3.3 Nonlocal conserved vectors

It has been demonstrated in [8, 7] that for any operator

\[ X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} \quad (3.14) \]

admitted by the system (1.1)-(1.3), the quantities

\[ C^i = \xi^i \mathcal{L} + W^\alpha \left[ \frac{\partial \mathcal{L}}{\partial u^\alpha_i} - D_j \left( \frac{\partial \mathcal{L}}{\partial u^\alpha_{ij}} \right) + D_j D_k \left( \frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}} \right) \right] \quad (3.15) \]

\[ + D_j (W^\alpha) \left[ \frac{\partial \mathcal{L}}{\partial u^\alpha_{ij}} - D_k \left( \frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}} \right) \right] + D_j D_k (W^\alpha) \left( \frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}} \right), \quad i = 1, 2, 3, \]

define the components of a conserved vector for Eqs. (1.1)-(1.3) considered together with the adjoint equations (2.14)-(2.16). Here

\[ W^\alpha = \eta^\alpha - \xi^j u^\alpha_j, \quad \alpha = 1, 2, 3. \quad (3.16) \]

The formula (3.15) is written by taking into account that the Lagrangian (2.12) involves the derivatives up to third order. Moreover, noting that the Lagrangian (2.12) vanishes on the solutions of Eqs. (1.1)-(1.3), we can drop the first term in (3.15) and use the conserved vector in the abbreviated form

\[ C^i = W^\alpha \left[ \frac{\partial \mathcal{L}}{\partial u^\alpha_i} - D_j \left( \frac{\partial \mathcal{L}}{\partial u^\alpha_{ij}} \right) + D_j D_k \left( \frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}} \right) \right] \quad (3.17) \]

\[ + D_j (W^\alpha) \left[ \frac{\partial \mathcal{L}}{\partial u^\alpha_{ij}} - D_k \left( \frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}} \right) \right] + D_j D_k (W^\alpha) \frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}}. \]
For computing the conserved vectors (3.17), the Lagrangian (2.12) containing the mixed derivatives should be written in the symmetric form

\[
\mathcal{L} = \frac{1}{3} \varphi \left[ \psi_{txx} + \psi_{txx} + \psi_{xxx} + \psi_{txz} + \psi_{zzz} - 3g\rho_x - 3fv_z 
- \psi_x (\psi_{xxx} + \psi_{xxx} + 3\psi_{zzz}) + \psi_z (3\psi_{xxx} + \psi_{xxx} + \psi_{xxx}) \right] 
+ \mu \left[ v_t + f\psi_z - \psi_x v_z + \psi_z v_x \right] + r \left[ \rho t + \frac{N^2}{g} \psi x - \psi_x \rho_z + \psi_z \rho_x \right].
\] (3.18)

Since the Lagrangian \(\mathcal{L}\), and hence the components (3.17) of a conserved vector contain the nonlocal variables \(\varphi, \mu, r\), we obtain in this way nonlocal conserved vectors.

### 3.4 Computation of nonlocal conserved vectors

The substitution of (3.18) in (3.17) yields:

\[
C^1 = W^1 \frac{\partial \mathcal{L}}{\partial v_t} + W^2 \frac{\partial \mathcal{L}}{\partial \rho_t} + W^3 \left[ D_x^2 \left( \frac{\partial \mathcal{L}}{\partial \psi_{txx}} \right) + D_z^2 \left( \frac{\partial \mathcal{L}}{\partial \psi_{zzz}} \right) \right] 
- \left[ D_x (W^3) D_x \left( \frac{\partial \mathcal{L}}{\partial \psi_{txx}} \right) + D_z (W^3) D_z \left( \frac{\partial \mathcal{L}}{\partial \psi_{zzz}} \right) \right] + D_x^2 (W^3) \frac{\partial \mathcal{L}}{\partial \psi_{txx}} + D_z^2 (W^3) \frac{\partial \mathcal{L}}{\partial \psi_{zzz}},
\]

or

\[
C^1 = W^1 \mu + W^2 r + \frac{1}{3} W^3 \left[ D_x^2 (\varphi) + D_z^2 (\varphi) \right] 
- \frac{1}{3} \left[ \varphi_x D_x (W^3) + \varphi_z D_z (W^3) \right] + \frac{1}{3} \varphi \left[ D_x^2 (W^3) + D_z^2 (W^3) \right].
\] (3.19)
Furthermore, using the same procedure, we obtain:

\[ C^2 = W^1 \frac{\partial \mathcal{L}}{\partial v_x} + W^2 \frac{\partial \mathcal{L}}{\partial p_x} + W^3 \left[ \frac{\partial \mathcal{L}}{\partial \psi_x} + D_x^2 \left( \frac{\partial \mathcal{L}}{\partial \psi_{xxx}} \right) + D_z^2 \left( \frac{\partial \mathcal{L}}{\partial \psi_{zzz}} \right) \right. \]

\[ + D_t D_x \left( \frac{\partial \mathcal{L}}{\partial \psi_{xt}} + \frac{\partial \mathcal{L}}{\partial \psi_{xx}} \right) + D_x D_z \left( \frac{\partial \mathcal{L}}{\partial \psi_{xzz}} + \frac{\partial \mathcal{L}}{\partial \psi_{zzx}} \right) \left] - D_x (W^3) D_x \left( \frac{\partial \mathcal{L}}{\partial \psi_{xxx}} \right) \right. \]

\[ - D_t (W^3) D_x \left( \frac{\partial \mathcal{L}}{\partial \psi_{xt}} \right) - D_x (W^3) D_t \left( \frac{\partial \mathcal{L}}{\partial \psi_{xt}} \right) - D_z (W^3) D_z \left( \frac{\partial \mathcal{L}}{\partial \psi_{zzz}} \right) \]

\[ - D_z (W^3) \left( \frac{\partial \mathcal{L}}{\partial \psi_{zzz}} \right) - D_x (W^3) D_z \left( \frac{\partial \mathcal{L}}{\partial \psi_{xzz}} \right) + D_z^2 (W^3) \frac{\partial \mathcal{L}}{\partial \psi_{zzz}} \]

\[ + D_x^2 (W^3) \frac{\partial \mathcal{L}}{\partial \psi_{zzz}} + D_t D_x (W^3) \left( \frac{\partial \mathcal{L}}{\partial \psi_{xt}} + \frac{\partial \mathcal{L}}{\partial \psi_{xx}} \right) + D_x D_z (W^3) \left( \frac{\partial \mathcal{L}}{\partial \psi_{xzz}} + \frac{\partial \mathcal{L}}{\partial \psi_{zzx}} \right), \]

or

\[ C^2 = W^1 \mu \psi_z + W^2 (r \psi_z - g \phi) + W^3 \left[ - \Delta \psi_z - \mu v_z + \frac{N^2}{g} r - r \rho_z \right] (3.20) \]

\[ + D_x^2 (\psi \psi_z) + \frac{1}{3} D_z^2 (\psi \psi_z) + \frac{2}{3} \psi_{xt} - \frac{2}{3} D_x D_z (\psi \psi_z) \]

\[ - D_x (W^3) \left[ D_x (\psi \psi_z) + \frac{1}{3} \psi_t - \frac{1}{3} D_z (\psi \psi_z) \right] - \frac{1}{3} D_t (W^3) \phi \]

\[ - \frac{1}{3} D_z (W^3) \left[ D_z (\psi \psi_z) - D_x (\psi \psi_z) \right] + \left[ D_x^2 (W^3) + \frac{1}{3} D_z^2 (W^3) \right] \phi \psi_z \]

\[ + \frac{2}{3} \phi D_t D_x (W^3) - \frac{2}{3} \psi \psi_z D_z D_x (W^3). \]

Likewise we get

\[ C^3 = -W^1 (\mu \psi_x + f \phi) - W^2 r \psi_x + W^3 \left[ - \Delta \psi_x + (f + v_x) \mu + r \rho_x \right] (3.21) \]

\[ - \frac{1}{3} D_x^2 (\psi \psi_x) - D_z^2 (\psi \psi_x) + \frac{1}{3} D_z^2 (\psi \psi_x) + \frac{2}{3} \psi_{xt} + \frac{2}{3} D_x D_z (\psi \psi_x) \]

\[ + \frac{1}{3} D_x (W^3) \left[ D_x (\psi \psi_x) - D_z (\psi \psi_x) \right] - \frac{1}{3} D_t (W^3) \phi \]

\[ - D_z (W^3) \left[ \frac{1}{3} \phi_x - D_z (\psi \psi_x) + \frac{1}{3} D_x (\psi \psi_x) \right] \]

\[ - \left[ \frac{1}{3} D_x^2 (W^3) + D_z^2 (W^3) \right] \psi \psi_x + \frac{2}{3} \phi D_t D_z (W^3) + \frac{2}{3} \psi \psi_z D_z D_x (W^3). \]
3.5 Local conserved vectors

The quantities (3.19)-(3.21) define a nonlocal conserved vector because they contain the nonlocal variables \( \phi, \mu, r \). In consequence, the conservation equation (3.1) requires not only the basic equations (1.1)-(1.3), but also the adjoint equations (2.14)-(2.16).

However, we can eliminate the nonlocal variables using the self-adjointness of Eqs. (1.1)-(1.3) thus transforming the nonlocal conserved vector into a local one. Namely, we substitute in Eqs. (3.19)-(3.21) the expressions (2.18) for \( \phi, \mu, r \):

\[
\varphi = \psi, \quad \mu = -v, \quad r = -\frac{g^2}{N^2} \rho.
\]

(2.18)

Then the adjoint equations (2.14)-(2.16) are satisfied for any solutions of the basic equations (1.1)-(1.3), and hence the quantities (3.19)-(3.21) satisfy the conservation equation (3.1) on all solutions of Eqs. (1.1)-(1.3).

Let us apply the procedure to \( C^1 \). We eliminate the nonlocal variables in (3.19) by substituting there the expressions (2.18) and write \( C^1 \) in the following form:

\[
C^1 = -v W^1 - \frac{g^2}{N^2} \rho W^2 + \frac{1}{3} W^3 \Delta \psi - \psi_x D_x (W^3) - \psi_z D_z (W^3) + \psi \Delta W^3,
\]

where

\[
\Delta \psi = D^2_x (\psi) + D^2_z (\psi), \quad \Delta W^3 = D^2_x (W^3) + D^2_z (W^3).
\]

We further simplify the expression for \( C^1 \) by using the identities

\[
W^3 D^2_x (\psi) = D_x [W^3 D_x (\psi)] - \psi_x D_x (W^3),
\]

\[
W^3 D^2_z (\psi) = D_z [W^3 D_z (\psi)] - \psi_z D_z (W^3)
\]

and

\[
\psi D^2_x (W^3) = D_x [\psi D_x (W^3)] - \psi_x D_x (W^3),
\]

\[
\psi D^2_z (W^3) = D_z [\psi D_z (W^3)] - \psi_z D_z (W^3).
\]

Then we have:

\[
C^1 = -v W^1 - \frac{g^2}{N^2} \rho W^2 - \psi_x D_x (W^3) - \psi_z D_z (W^3) + \frac{1}{3} \Delta (\psi W^3). \quad (3.22)
\]

Dropping in (3.22) the divergent type term

\[
\frac{1}{3} \Delta (\psi W^3) = D_x \left[ \frac{1}{3} D_x (\psi W^3) \right] + D_z \left[ \frac{1}{3} D_z (\psi W^3) \right]
\]

13
in accordance with Remark 3.1, we finally obtain:

\[ C^1 = -v W^1 - \frac{g^2}{N_2} \rho W^2 - \psi_x D_x (W^3) - \psi_z D_z (W^3). \] (3.23)

We will not dwell on a similar modification of the expressions (3.20), (3.21) for the components \( C^2 \) and \( C^3 \) of conserved vectors. We will see further in Section 4.5 that they can be found by simpler calculations when a density \( C^1 \) is known.

4 Utilization of obvious symmetries

4.1 Introduction

Eqs. (1.1)-(1.3) do not contain the dependent and independent variables explicitly and therefore they are invariant with respect to addition of arbitrary constants to all these variables. It means that Eqs. (1.1)-(1.3) admit the one-parameter groups of translations in all variables,

\[ \begin{align*}
\bar{v} &= v + a_1, \\
\bar{x} &= x + a_5, \\
\bar{\rho} &= \rho + a_2, \\
\bar{\rho} &= \rho + a_2, \\
\bar{\psi} &= \psi + a_3, \\
\bar{t} &= t + a_4, \\
\bar{x} &= x + a_5, \\
\bar{z} &= z + a_6,
\end{align*} \]

with the generators

\[ X_1 = \frac{\partial}{\partial v}, \quad X_2 = \frac{\partial}{\partial \rho}, \quad X_3 = \frac{\partial}{\partial \psi}, \quad X_4 = \frac{\partial}{\partial t}, \quad X_5 = \frac{\partial}{\partial x}, \quad X_6 = \frac{\partial}{\partial z}. \] (4.1)

One can also find by simple calculations the dilations (scaling transformations)

\[ \begin{align*}
\bar{v} &= av, \\
\bar{\rho} &= b\rho, \\
\bar{\psi} &= c\psi, \\
\bar{t} &= \alpha \bar{t}, \\
\bar{x} &= \beta \bar{x}, \\
\bar{z} &= \beta \bar{z}
\end{align*} \] (4.2)

admitted by Eqs. (1.1)-(1.3). These transformations are defined near the identity transformation if the parameters \( a, \ldots, \beta \) are positive. The dilations of \( x \) and \( z \) are taken by the same parameter \( \beta \) in order to keep invariant the Laplacian \( \Delta \). Let us find the parameters \( a, \ldots, \beta \) from the invariance condition of Eqs. (1.1)-(1.3). The transformations (4.2) change the derivatives involved in Eqs. (1.1)-(1.3) as follows:

\[ \begin{align*}
\bar{v}_t &= a\alpha v_t, \\
\bar{v}_x &= a\beta v_x, \\
\bar{v}_z &= a\beta v_z, \\
\bar{\rho}_t &= b\alpha \rho_t, \\
\bar{\rho}_x &= b\beta \rho_x, \\
\bar{\rho}_z &= b\beta \rho_z, \\
\bar{\psi}_t &= c\alpha \psi_t, \\
\bar{\psi}_x &= c\beta \psi_x, \\
\bar{\psi}_z &= c\beta \psi_z, \\
\bar{\Delta} \psi_t &= c\alpha\beta^2 \Delta \psi_t, \\
\bar{\Delta} \psi_x &= c\beta^2 \Delta \psi_x, \\
\bar{\Delta} \psi_z &= c\beta^3 \Delta \psi_z,
\end{align*} \] (4.3)
where $\bar{\Delta}$ is the Laplacian written in the variables $\bar{x}$, $\bar{z}$. The invariance of Eqs. (1.1)-(1.3) under the transformations (4.2) means that the following equations are satisfied:

\[
\bar{\Delta} \bar{\psi} \bar{t} - g \bar{\rho} \bar{x} - f \bar{v} \bar{z} - \bar{\psi} \bar{z} \bar{\Delta} \bar{\psi} + \bar{\psi} \bar{x} \bar{\Delta} \bar{\psi} = 0,
\]
\[
\bar{v} \bar{t} + f \bar{\psi} \bar{x} + \bar{x} \bar{v} \bar{z} + \bar{\psi} \bar{z} \bar{v} \bar{x} = 0,
\]
\[
\bar{\rho} \bar{t} + \frac{N^2}{g} \bar{\psi} \bar{x} - \bar{x} \bar{\psi} \bar{z} + \bar{\psi} \bar{z} \bar{\rho} \bar{x} = 0,
\]
whenever Eqs.(1.1)-(1.3) hold. Substituting here the expressions (4.3) we have:

\[
\bar{\Delta} \bar{\psi} \bar{t} - g \bar{\rho} \bar{x} - f \bar{v} \bar{z} - \bar{\psi} \bar{z} \bar{\Delta} \bar{\psi} + \bar{\psi} \bar{x} \bar{\Delta} \bar{\psi} = c \alpha \beta^2 \bar{\Delta} \bar{\psi} - b \beta g \bar{\rho} - c^2 \beta^4 \bar{\psi} \bar{x} \bar{\psi} \bar{z} \bar{\Delta} \bar{\psi} + \bar{\psi} \bar{z} \bar{\psi} \bar{x} = 0,
\]
\[
\bar{v} \bar{t} + f \bar{\psi} \bar{x} - \bar{x} \bar{\psi} \bar{z} + \bar{\psi} \bar{z} \bar{v} \bar{x} = a \alpha \bar{\psi} t + c \beta f \bar{\psi} z - ac \beta^2 \bar{\psi} \bar{x} \bar{\psi} \bar{z} = 0,
\]
\[
\bar{\rho} \bar{t} + \frac{N^2}{g} \bar{\psi} \bar{x} - \bar{x} \bar{\psi} \bar{z} + \bar{\psi} \bar{z} \bar{\rho} \bar{x} = b \alpha \bar{\rho} t + c \beta N^2 g \bar{\psi} \bar{x} - bc \beta^2 \bar{\psi} \bar{x} \bar{\rho} - \bar{\psi} \bar{z} \bar{\rho} \bar{x} = 0.
\]

These equations show that the invariance of Eqs. (1.1)-(1.3) is guaranteed by the following six equations for five undetermined parameters $a, b, c, \alpha, \beta$:

\[
c \alpha \beta^2 = b \beta = c^2 \beta^4, \quad a \alpha = c \beta = a c \beta^2, \quad b \alpha = c \beta = b c \beta^2.
\]

(4.4)

It can be verified by simple calculations that Eqs. (4.4) yield

\[
\alpha = 1, \quad b = a, \quad c = a^2, \quad \beta = \frac{1}{a},
\]

where $a$ is an arbitrary parameter. We substitute these values of the parameters in (4.2), denote the positive parameter $a$ by $e^a$, drop the tilde and conclude that Eqs. (1.1)-(1.3) admit the one-parameter non-uniform dilation group

\[
\bar{t} = t, \quad \bar{x} = xe^a, \quad \bar{z} = ze^a, \quad \bar{v} = ve^a, \quad \bar{\rho} = \rho e^a, \quad \bar{\psi} = \psi e^{2a}
\]

with the following generator:

\[
X_7 = x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} + v \frac{\partial}{\partial v} + \rho \frac{\partial}{\partial \rho} + 2 \psi \frac{\partial}{\partial \psi}.
\]

(4.5)

We will consider the operators (4.1)-(4.5) as obvious symmetries of Eqs. (1.1)-(1.3) and will compute the local conservation laws provided by these symmetries.
4.2 Translation of $v$

For the operator $X_1$ from (4.1) Eqs. (3.16) yield

$$W^1 = 1, \quad W^2 = 0, \quad W^3 = 0.$$  

Substituting these expressions in Eq. (3.23) we obtain

$$C^1 = -v.$$  

In this case the equations (3.20) and (3.21) are also simple. They are written

$$C^2 = u\psi_z, \quad C^3 = -(u\psi_x + f\varphi)$$

and upon using Eqs. (2.18) yield:

$$C^2 = -v\psi_z, \quad C^3 = v\psi_x - f\psi.$$  

Since any conserved vector is defined up to multiplication by an arbitrary constant, we change the sign of $C^1$, $C^2$, $C^3$ and obtain the following conserved vector:

$$C^1 = v, \quad C^2 = v\psi_z, \quad C^3 = f\psi - v\psi_x. \quad (4.6)$$

We have:

$$D_t(C^1) + D_x(C^2) + D_z(C^3) = v_t + v_x\psi_x + f\psi_z - v_z\psi_x.$$  

Hence, the conservation equation (3.1) coincides with Eq. (1.2).

4.3 Translation of $\rho$

For the operator $X_2$ from (4.1) Eqs. (3.16) yield

$$W^1 = 0, \quad W^2 = 1, \quad W^3 = 0.$$  

Substituting these expressions in Eq. (3.23) we obtain

$$C^1 = -\frac{g^2}{N^2} \rho.$$  

Furthermore, Eqs. (3.20), (3.21) and Eqs. (2.18) yield:

$$C^2 = -g\psi - \frac{g^2}{N^2} \rho \psi_z, \quad C^3 = \frac{g^2}{N^2} \rho \psi_x.$$  

Multiplying $C^1$, $C^2$, $C^3$ by $-N^2/g^2$ we arrive at the following conserved vector:

$$C^1 = \rho, \quad C^2 = \frac{N^2}{g} \psi + \rho \psi_z, \quad C^3 = -\rho \psi_x. \quad (4.7)$$

One can readily verify that the conservation equation (3.1) for the vector (4.7) is also satisfied. Namely, it coincides with Eq. (1.3).
4.4 Translation of $\psi$

For the operator $X_3$ from (4.1) Eqs. (3.16) yield

$$W^1 = 0, \quad W^2 = 0, \quad W^3 = 1.$$ 

Substituting these expressions in Eq. (3.23) we obtain

$$C^1 = 0.$$ 

Hence, the invariance of Eqs. (1.1)-(1.3) under the translation of $\psi$ furnishes only a trivial conservation law (see Definition 3.1).

4.5 Derivation of the flux of conserved vectors with known densities

We will show here how to find the components $C^2$ and $C^3$ of the conserved vector (4.6) without using Eqs. (3.20), (3.21), provided that we know the conserved density $C^1 = v$.

Let us first verify that $C^1 = v$ satisfies Corollary 3.1. In this case $D_t(C^1) = v_t$, and hence Eq. (3.7) yields

$$F = D_t(C^1)\bigg|_{(1.1)-(1.3)} = J(\psi, v) - f\psi_z.$$ (4.8)

Using Proposition 3.1 we see that Eqs. (3.5) are satisfied:

$$\frac{\delta F}{\delta v} = \delta F = 0, \quad \frac{\delta F}{\delta \rho} = D_z(f) = 0.$$ (4.9)

Therefore Corollary 3.1 guarantees that $F$ defined by Eq. (4.8) satisfies Eq. (3.4):

$$\psi_x\psi_z - \psi_z\psi_x - f\psi_z = D_x(H^1) + D_z(H^2)$$ (4.10)

with certain functions $H^1$, $H^2$.

In order to find $H^1$, $H^2$, we write

$$\psi_x\psi_z - f\psi_z = D_z(v\psi_x - f\psi) - v\psi_{xx}, \quad -\psi_z\psi_x = D_x(-v\psi_z) + v\psi_{zx}$$

and obtain:

$$\psi_x\psi_z - \psi_z\psi_x - f\psi_z = D_x(-v\psi_z) + D_z(v\psi_x - f\psi).$$

Thus, $H^1 = -v\psi_z$, $H^2 = v\psi_x - f\psi$. Denoting $C^2 = -H^1$, $C^3 = -H^2$, i.e.

$$C^1 = v\psi_z, \quad C^2 = f\psi - v\psi_x,$$
and invoking Eq. (4.8), we write Eq. (4.10) in the form

\[ D_t(C^1) \bigg|_{(1.1)-(1.3)} + D_x(C^1) + D_z(C^2) = 0. \]

This is precisely the conservation equation (3.1) for the vector (4.6). Thus, we have obtained the components \( C^2, C^3 \) of the conserved vector (4.6) without using Eqs. (3.20), (3.21).

The components \( C^2, C^3 \) of the conserved vector (4.7) can be derived likewise.

### 4.6 Translation of \( x \)

For the operator \( X_5 \) from (4.1) Eqs. (3.16) yield

\[ W^1 = -v_x, \quad W^2 = -\rho_x, \quad W^3 = -\psi_x. \]

Substituting these expressions in Eq. (3.23) we obtain

\[ C^1 = vu_x + \frac{g^2}{N^2} \rho \rho_x + \psi_x \psi_{xx} + \psi_z \psi_{xz} = D_x \left( \frac{1}{2} v^2 + \frac{1}{2} \frac{g^2}{N^2} \rho^2 + \frac{1}{2} \psi_x^2 + \frac{1}{2} \psi_z^2 \right). \]

Hence, the invariance of Eqs. (1.1)-(1.3) under the translation of \( x \) furnishes only a trivial conservation law (see Definition 3.1). Similar calculations show that the invariance under the translation of \( z \) provides also a trivial conservation law.

### 4.7 Time translation

For the operator \( X_4 \) from (4.1) Eqs. (3.16) yield

\[ W^1 = -v_t, \quad W^2 = -\rho_t, \quad W^3 = -\psi_t. \]

Substituting these expressions in Eq. (3.23) we obtain

\[ C^1 = vu_t + \frac{g^2}{N^2} \rho \rho_t + \psi_x \psi_{xt} + \psi_z \psi_{zt}. \]  

(4.11)

Changing the last two terms of \( C^1 \) by using the identity

\[ \psi_x \psi_{xt} + \psi_z \psi_{zt} = D_x(\psi_x \psi_{xt}) - \psi_x \psi_{xxt} + D_z(\psi_z \psi_{zt}) - \psi_z \psi_{zt} \]

\[ = D_x(\psi_x \psi_{xt}) + D_z(\psi_z \psi_{zt}) - \psi \Delta \psi_t \]  

(4.12)

and dropping the divergent type terms, we rewrite \( C^1 \) given by Eq. (4.11) in the form

\[ C^1 = vu_t + \frac{g^2}{N^2} \rho \rho_t - \psi \Delta \psi_t. \]  

(4.13)
Let us clarify if the conservation law with the density (4.13) is trivial or non-trivial. According to Definition 3.1, we have to evaluate the density (4.13) on the solutions of Eqs. (1.1)-(1.3). In this case it is convenient to use Eqs. (1.1)-(1.3) in the form (1.4)-(1.6) and replace Eq. (3.8) by

$$C^1_* = C^1|_{(1.4)-(1.6)}.$$ 

Then we have

$$C^1_* = \left\{ vJ(\psi, v) + \frac{g^2}{N^2} \rho J(\psi, \rho) - \psi J(\psi, \Delta \psi) \right\} - fD_z(v\psi) - gD_x(\rho\psi)$$

and Corollary 3.2 shows that the conservation law is trivial. Indeed, the last two terms of $C^1_*$ have the divergent form. The expression in braces for $C^1_*$ satisfies Eqs. (3.9) according to Proposition 3.1, and hence it also has the divergent form.

Thus, the invariance of Eqs. (1.1)-(1.3) under the time translation furnishes only a trivial conservation law.

### 4.8 Use of the dilation. Conservation of energy

Consider the generator (4.5) of the dilation group,

$$X_7 = x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} + v \frac{\partial}{\partial v} + \rho \frac{\partial}{\partial \rho} + 2\psi \frac{\partial}{\partial \psi}.$$ 

In this case the quantities (3.16) have the form

$$W^1 = v - xv_x - zv_z, \quad W^2 = \rho - x\rho_x - z\rho_z, \quad W^3 = 2\psi - x\psi_x - z\psi_z. \quad (4.14)$$

The substitution of (4.14) in (3.23) yields:

$$C^1 = -v^2 + xv_x + zv_z + \frac{g^2}{N^2} (\rho^2 + x\rho_x + z\rho_z) \rho - \psi^2 + x\psi_x^2 + z\psi_x^2 - \psi_x^2 + x\psi_x^2 + z\psi_x^2 + \psi_x^2. \quad (4.15)$$

We modify (4.15) by using the identities

$$xvv_x + zvv_z = \frac{1}{2} D_x (xv^2) + \frac{1}{2} D_z (zv^2) - v^2,$$

$$x\rho x + z\rho z = \frac{1}{2} D_x (x\rho^2) + \frac{1}{2} D_z (z\rho^2) - \rho^2,$$

$$x\psi_x^2 + x\psi_x^2 + z\psi_x^2 = \frac{1}{2} D_x [x (\psi_x^2 + \psi_x^2)] - \frac{1}{2} (\psi_x^2 + \psi_x^2),$$

$$z\psi_x^2 + z\psi_x^2 = \frac{1}{2} D_z [z (\psi_x^2 + \psi_x^2)] - \frac{1}{2} (\psi_x^2 + \psi_x^2).$$
Substituting these in (4.15) and dropping the divergent type terms we have:

\[ C^1 = -2 \left( v^2 + \frac{g^2}{N^2} \rho^2 + |\nabla \psi|^2 \right), \]

where

\[ |\nabla \psi|^2 = \psi_x^2 + \psi_z^2. \]

Dividing \( C^1 \) by the inessential coefficient \((-2)\) we finally obtain the following conservation law in the integral form (3.2):

\[
\frac{d}{dt} \int \int \left[ v^2 + \frac{g^2}{N^2} \rho^2 + |\nabla \psi|^2 \right] dx dz = 0. \tag{4.16}
\]

Eq. (4.16) represents the conservation of the energy with the density

\[ E = v^2 + \frac{g^2}{N^2} \rho^2 + |\nabla \psi|^2. \tag{4.17} \]

Let us find the components \( C^2 \) and \( C^3 \) of this conservation law written in the differential form (3.1). We will use the procedure suggested in Section 4.5. Let us first verify that \( E \) defined by Eq. (4.17) satisfies Corollary 3.1 for densities of conservation laws. We have

\[
D_t(E) = 2 \left( vv_t + \frac{g^2}{N^2} \rho \rho_t + \psi_x \psi_{xt} + \psi_z \psi_{zt} \right). \tag{4.18}
\]

Since the expression in the brackets in Eq. (4.18) is identical with (4.11) it can be rewritten in the form (4.13), and hence satisfies Eqs. (3.5). Corollary 3.1 guarantees that \( E \) is the density of a conservation law. It is manifest from Eq. (4.17) that this conservation law is non-trivial.

According to Corollary 3.1, \( D_t(E) \) defined by Eq. (4.18) and evaluated on the solutions of Eqs. (1.1)-(1.3) satisfies Eq. (3.4),

\[
D_t(E) \bigg|_{(1.1)-(1.3)} = D_x(H^1) + D_z(H^2), \tag{4.19}
\]

with certain functions \( H^1, H^2 \). In order to find \( H^1, H^2 \), we use Eq. (4.12),

\[
2(\psi_x \psi_{xt} + \psi_z \psi_{zt}) = D_x(2\psi \psi_{xt}) + D_z(2\psi \psi_{zt}) - 2\psi \Delta \psi_t, \tag{4.20}
\]

and write:

\[
2vv_t = 2\psi_x v \psi_x - \psi_z v \psi_x - 2f v \psi_z
= D_z(v^2 \psi_x) - D_x(v^2 \psi_z) - 2f D_z(v \psi) + 2f \psi v_z, \tag{4.21}
\]

20
\[
2 \frac{g^2}{N^2} \rho \psi_t = 2 \frac{g^2}{N^2} \left( \psi_x \rho \rho_z - \psi_z \rho \rho_x \right) - 2g \rho \psi_x \\
= \frac{g^2}{N^2} \left[ D_z(\rho^2 \psi_x) - D_x(\rho^2 \psi_z) \right] - 2g D_x(\rho \psi) + 2g \rho \psi_x,
\]

\[
-2 \psi \Delta \psi_t = -2g \rho \psi_x - 2f \psi \psi_z - 2\psi \Delta \psi_x + 2\psi \psi \Delta \psi_x \\
= -2g \rho \psi_x - 2f \psi \psi_z - D_x(\psi^2 \Delta \psi_x) + D_z(\psi^2 \Delta \psi_x).
\]

Substituting the expressions (4.21), (4.22) and (4.20), (4.23) in the right-hand side of Eq. (4.18), we arrive at Eq. (4.19) with

\[
H^1 = -v^2 \psi_z - \frac{g^2}{N^2} \rho^2 \psi_z - 2g \rho \psi + 2 \psi \psi_{xt} - \psi^2 \Delta \psi_z,
\]

\[
H^2 = v^2 \psi_x + \frac{g^2}{N^2} \rho^2 \psi_x - 2f \psi \psi + 2 \psi \psi_{zt} + \psi^2 \Delta \psi_x.
\]

Thus, denoting \( C^2 = -H^1 \), \( C^3 = -H^2 \) we arrive at the following differential form (3.1) of the conservation of energy for Eqs. (1.1)-(1.3):

\[
D_t(E) + D_x(C^2) + D_z(C^3) = 0 \quad (4.24)
\]

with the density \( E \) given by Eq. (4.17) and the flux given by the equations

\[
C^2 = 2g \rho \psi + v^2 \psi_z + \frac{g^2}{N^2} \rho^2 \psi_z - 2 \psi \psi_{xt} + \psi^2 \Delta \psi_z,
\]

\[
C^3 = 2f \psi \psi - v^2 \psi_x - \frac{g^2}{N^2} \rho^2 \psi_x - 2 \psi \psi_{zt} - \psi^2 \Delta \psi_x. \quad (4.25)
\]

## 5 Invariant solutions

### 5.1 Invariant solution based on translation and dilation

Let us find the invariant solution based on the following two operators:

\[
mX_5 - kX_6 = m \frac{\partial}{\partial x} - k \frac{\partial}{\partial z} \quad (m, k = \text{const.}),
\]

\[
X_7 = x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} + v \frac{\partial}{\partial v} + \rho \frac{\partial}{\partial \rho} + 2 \psi \frac{\partial}{\partial \psi}.
\]

We first find their invariants \( J(t, x, z, v, \rho, \psi) \) by solving the equations

\[
(mX_5 - kX_6)J = 0, \quad X_7J = 0. \quad (5.2)
\]
The characteristic equation $kdx + mdz = 0$ for the first equation (5.2) yields that the operator $mX_2 - kX_3$ has, along with $t, v, \rho, \psi$, the following invariant:

$$\lambda = kx + mz. \quad (5.3)$$

Therefore we have to find the invariants $J(t, \lambda, \rho, \psi)$ for the operator $X_7$. To this end, we write the action of $X_7$ on the variables $t, \lambda, v, \rho, \psi$ by the standard formula

$$X_7 = X_7(\lambda) \frac{\partial}{\partial \lambda} + v \frac{\partial}{\partial v} + \rho \frac{\partial}{\partial \rho} + 2\psi \frac{\partial}{\partial \psi}$$

and obtain

$$X_7 = \lambda \frac{\partial}{\partial \lambda} + v \frac{\partial}{\partial v} + \rho \frac{\partial}{\partial \rho} + 2\psi \frac{\partial}{\partial \psi}. \quad (5.4)$$

To solve the equation $X_7J(t, \lambda, v, \rho, \psi) = 0$ for the invariants, we calculate the first integrals for the characteristic system

$$\frac{d\lambda}{\lambda} = \frac{dv}{v} = \frac{d\rho}{\rho} = \frac{d\psi}{2\psi}$$

and see that a basis of invariants for the operators (5.1) is given by

$$t, \quad V = \frac{v}{\lambda}, \quad R = \frac{\rho}{\lambda}, \quad \phi = \frac{\psi}{\lambda^2}.$$ 

Accordingly, we assign the invariants $V, R, \phi$ to be functions of the invariant $t$ and arrive at the following general form of the candidates for the invariant solutions:

$$v = \lambda V(t), \quad \rho = \lambda R(t), \quad \psi = \lambda^2 \phi(t), \quad \lambda = kx + mz. \quad (5.5)$$

In order to find the functions $V(t), R(t), \phi(t)$, we have to substitute the expressions (5.5) in Eqs. (1.1)- (1.3).

We have:

$$\psi_t = \lambda^2 \phi'(t), \quad \psi_x = 2k\lambda \phi(t), \quad \psi_z = 2m\lambda \phi(t),$$

$$\nabla^2 \psi_t = 2(k^2 + m^2)\phi'(t), \quad \nabla^2 \psi_x = 0, \quad \nabla^2 \psi_z = 0,$$

$$\psi_x v_z = 2km\lambda \phi(t) V(t), \quad \psi_z v_x = 2km\lambda \phi(t) V(t),$$

$$\psi_x \rho_z = 2km\lambda \phi(t) R(t), \quad \psi_z \rho_x = 2km\lambda \phi(t) R(t).$$
Therefore Eqs. (1.1)- (1.3) yield the following system of first-order linear ordinary differential equations:

\[ 2(k^2 + m^2)\phi' - gkR - fmV = 0, \]
\[ \lambda V' + 2fm\lambda \phi = 0, \]
\[ \lambda R' + 2\frac{k\lambda}{g} N^2 \phi = 0, \]

or

\[ \phi' = \frac{1}{2(k^2 + m^2)}(gkR + fmV), \quad (5.6) \]
\[ V' = -2fm\phi, \quad (5.7) \]
\[ R' = -2\frac{k}{g} N^2 \phi. \quad (5.8) \]

Let us integrate Eqs. (5.6)-(5.8). Differentiating Eq. (5.6) and using Eqs. (5.7)-(5.8), we obtain

\[ \phi'' + \omega^2 \phi = 0, \quad (5.9) \]

where

\[ \omega^2 = \frac{k^2 N^2 + m^2 f^2}{k^2 + m^2}. \quad (5.10) \]

The general solution of Eq. (5.9) is given by

\[ \phi(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t), \quad C_1, C_2 = \text{const.} \quad (5.11) \]

Substituting (5.11) in Eqs. (5.7)-(5.8) and integrating, we obtain

\[ V = C_3 - \frac{2fm}{\omega} \left[ C_1 \sin(\omega t) - C_2 \cos(\omega t) \right], \]
\[ R = C_4 - \frac{2k}{g\omega} N^2 \left[ C_1 \sin(\omega t) - C_2 \cos(\omega t) \right]. \]

To determine the constants \( C_3 \) and \( C_4 \), we substitute in Eq. (5.6) the above expressions for \( V, R \) and the expression (5.11) for \( \phi \) and obtain

\[ fmC_3 + gkC_4 = 0. \]
Thus, the solution to Eqs. (5.6)-(5.8) has the following form:

\[ \phi(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t), \]  
\[ V(t) = \frac{2f m}{\omega} [C_2 \cos(\omega t) - C_1 \sin(\omega t)] + C_3, \]  
\[ R(t) = \frac{2k}{g \omega} N^2 [C_2 \cos(\omega t) - C_1 \sin(\omega t)] - \frac{fm}{gk} C_3. \]

Finally, substituting (5.12)-(5.14) in (5.5), we arrive at the following solution to the system (1.1)-(1.3):

\[ \rho = \frac{2k}{g \omega} N^2 [C_2 \cos(\omega t) - C_1 \sin(\omega t)] \lambda - \frac{fm}{gk} C_3 \lambda, \]  
\[ v = \frac{2f m}{\omega} [C_2 \cos(\omega t) - C_1 \sin(\omega t)] \lambda + C_3 \lambda, \]  
\[ \psi = [C_1 \cos(\omega t) + C_2 \sin(\omega t)] \lambda^2, \]  

where \( \lambda \) is given by (5.3), \( \omega \) is defined by Eq. (5.10) and \( C_1, C_2, C_3 \) are arbitrary constants.

### 5.2 Generalized invariant solution and wave beams

It is natural to generalize the candidates (5.5) for the invariant solutions and look for particular solutions of the system (1.1)-(1.3) in the following form of separated variables:

\[ v = F(\lambda)V(t), \quad \rho = \alpha(\lambda)R(t), \quad \psi = \beta(\lambda)\phi(t), \quad \lambda = kx + mz. \]

The reckoning shows that then the right-hand sides of Eqs. (1.1)-(1.3) vanish and Eqs. (1.1)-(1.3) become:

\[ (k^2 + m^2)\beta''(\lambda)\phi'(t) - gk\alpha'(\lambda)R(t) - fmF'(\lambda)V(t) = 0, \]  
\[ F(\lambda)V'(t) + fm\beta'(\lambda)\phi(t) = 0, \]  
\[ \alpha(\lambda)R'(t) + \frac{kN^2}{g} \beta'(\lambda)\phi(t) = 0. \]

Differentiating Eq. (5.19) with respect to \( t \), using Eqs. (5.20)-(5.21) and dividing by \( \beta' \), we obtain

\[ (k^2 + m^2) \frac{\beta''}{\beta'} \phi'' + \left( N^2 k^2 \alpha' \frac{\alpha'}{\alpha} + f^2 m^2 \frac{F'}{F} \right) \phi = 0. \]
Assuming that the ratios $\beta''/\beta', \alpha'/\alpha, F'/F$ are proportional with constant coefficients and have one and the same sign, we arrive at an equation of the form (5.9). For example, letting

$$\frac{\beta''}{\beta'} = \frac{\alpha'}{\alpha} = \frac{F'}{F},$$

we obtain Eq. (5.9). Then, according to (5.11), we can set in (5.18) $\phi(t) = \cos(\omega t)$ and $\phi(t) = \sin(\omega t)$, i.e.

$$\psi = A(\lambda) \cos(\omega t) \quad \text{and} \quad \psi = B(\lambda) \sin(\omega t).$$

(5.23)

For each function $\psi$ given by (5.23) we determine the functions $V(t), R(t)$ using Eqs. (5.20), (5.21), (5.21), then take the linear combinations of the resulting functions and arrive at the following form of the “generalized invariant solution” (5.18):

$$\psi = A(\lambda) \cos(\omega t) + B(\lambda) \sin(\omega t),$$

(5.24)

$$v = \frac{fm}{\omega} [B'(\lambda) \cos(\omega t) - A'(\lambda) \sin(\omega t)] + F(\lambda),$$

(5.25)

$$\rho = \frac{kN^2}{g\omega} [B'(\lambda) \cos(\omega t) - A'(\lambda) \sin(\omega t)] + H(\lambda),$$

(5.26)

where $\omega$ is given by Eq. (5.10).

The reckoning shows that the functions (5.24)-(5.26) with arbitrary $A(\lambda), B(\lambda)$ solve Eqs. (1.1)-(1.3) provided that $F(\lambda), H(\lambda)$ satisfy the following equation:

$$gkH'(\lambda) + fmF'(\lambda) = 0.$$  

(5.27)

One can readily verify that the invariant solution (5.15)-(5.17), which is a particular case of (5.24)-(5.26), obeys the condition (5.27).

### 5.3 Energy of the generalized invariant solution

If we substitute in (4.17) the generalized invariant solution (see Eqs. (5.24)-(5.26))

$$\psi = A(\lambda) \cos(\omega t) + B(\lambda) \sin(\omega t),$$

(5.28)

$$v = \frac{fm}{\omega} [B'(\lambda) \cos(\omega t) - A'(\lambda) \sin(\omega t)],$$

(5.29)

$$\rho = \frac{kN^2}{g\omega} [B'(\lambda) \cos(\omega t) - A'(\lambda) \sin(\omega t)],$$

(5.30)
where
\[ \lambda = kx + mz, \quad \omega^2 = \frac{k^2N^2 + m^2f^2}{k^2 + m^2}, \]
we obtain:
\[ E = (k^2 + m^2) \left[ A'(\lambda)^2 + B'(\lambda)^2 \right]. \]
Invoking that any conserved vector is defined up to multiplication by an arbitrary constant, we divide the above expression for \( E \) by \((k^2 + m^2)\) and obtain the following energy:
\[ E = A'(\lambda)^2 + B'(\lambda)^2. \quad (5.31) \]
Since the energy density (5.31) depends only on \( \lambda = kx + mz \), it is constant along the straight line
\[ kx + mz = \text{const}. \quad (5.32) \]
Accordingly, the “local energy” (5.31) has one and the same value at points \((x_0, z_0)\) and \((x_1, z_1)\) provided that
\[ kx_0 + mz_0 = kx_1 + mz_1. \quad (5.33) \]
The energy density (5.31) describes the local behavior of the solutions. Therefore it is significant to understand its distribution on the \((x, z)\) plane. Suppose that the functions \( A(\lambda), B(\lambda) \) and their derivatives rapidly decrease as \( \eta \to \infty \). If we take, as an example, the functions
\[ A(\lambda) = \frac{a}{1 + \lambda^2}, \quad B(\lambda) = \frac{a\lambda}{1 + \lambda^2}, \quad (5.34) \]
where \( a \) is a positive constant, then the energy density (5.31) of the wave beams has the form
\[ E = \frac{a^2}{(1 + \lambda^2)^2}. \]
Hence, the energy is localized along the straight line (5.33). Therefore we can define a wave beam through a point \((x_0, z_0)\) as the totality of the points \((x_1, z_1)\) satisfying Eq. (5.33), i.e. identify it with the straight line (5.33).
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