Noncommutative Wilson lines in higher-spin theory and correlation functions of conserved currents for free conformal fields

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Abstract

We first prove that, in Vasiliev’s theory, the zero-form charges studied in Sezgin E and Sundell P 2011 (arXiv:1103.2360 [hep-th]) and Colombo N and Sundell P 20 (arXiv:1208.3880 [hep-th]) are twisted open Wilson lines in the noncommutative Z space. This is shown by mapping Vasiliev’s higher-spin model on noncommutative Yang–Mills theory. We then prove that, prior to Bose-symmetrising, the cyclically-symmetric higher-spin invariants given by the leading order of these n-point zero-form charges are equal to corresponding cyclically-invariant building blocks of n-point correlation functions of bilinear operators in free conformal field theories (CFT) in three dimensions. On the higher spin gravity side, our computation reproduces the results of Didenko V and Skvortsov E 2013 J. High Energy Phys. JHEP04(2013)158 using an alternative method amenable to the computation of subleading corrections obtained by perturbation theory in normal order. On the free CFT side, our proof involves the explicit computation of the separate cyclic building blocks of the correlation functions of n conserved currents in arbitrary dimension d > 2 using polarization vectors, which is an original result. It is shown to agree, for d = 3, with the results obtained in Gelfond O A and Vasiliev M A 2013 Nucl. Phys. B 876 871–917 in various dimensions and where polarization spinors were used.

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1. Introduction

The conjectured holographic duality between higher spin (HS) gauge theory on $\text{AdS}_{d+1}$ background and free CFT$_d$ was spelled out, after the pioneering works [1–7] in the references [8–11], where the last work also contains a simple test of the conjecture. As stressed respectively in [6] and [9], this holographic correspondence is remarkable in that it (i) relates two weakly coupled theories and (ii) does not require any supersymmetry.

If one postulates the validity of the HS/CFT conjecture, the holographic reconstruction programme undertaken in [12], that starts from the free $O(N)$ model on the boundary, enabled the reconstruction of all the cubic vertices in the bulk [13] as well as the quartic vertex for the bulk scalar field [14]. The cubic vertices obtained in this way involve finitely many derivatives at finite values of the spins. On the other hand, without postulating the holographic conjecture, the standard tests of the HS$_4$/CFT$_3$ correspondence work well [15, 16] for the cubic vertices in Vasiliev’s theory that are fully fixed by the rigid, nonabelian higher-spin symmetry algebra of the vacuum $\text{AdS}_4$ solution; for recent results, see [17, 18].

The computations of [16] of 3-point functions show divergences in the case of the Vasiliev vertices that are not fully fixed by the pure higher-spin kinematics. These divergences have been confirmed since then from a different perspective in [19], where all the first nontrivial interactions around $\text{AdS}_4$ have been extracted from the Vasiliev equations. To date, no regularisation scheme has been shown to tame these divergences in a completely satisfactory manner; for discussions, see for example [20], and for recent progress, see [17, 18, 21]. Nonetheless, since the computation of [16] utilises techniques of AdS/CFT that rely on the existence of a standard action principle on the bulk side, i.e. an action for which the kinetic terms are given by Fronsdal Lagrangians in $\text{AdS}_4$ [22], it is suggestive of the existence of an effective action of deformed Fronsdal type, but not necessarily of any underlying path integral measure for self-interacting Fronsdal fields.

Being more circumspect, one can say that, for the Vasiliev’s equations, the issues of locality and weak-field perturbative expansion around $\text{AdS}_4$ are subtle and require more investigations, see [18, 21, 23–28] for some preliminary works in that direction. In fact, taking a closer look at the deformed Fronsdal action, it is more reminiscent of an effective than a classical action. The weakest assumption would be a duality between Vasiliev’s equations and the deformed Fronsdal theory, in the sense that the two theories would be equivalent only at the level of their respective on-shell actions subject to suitable boundary conditions. However, in view of the matching of 3-point couplings [11, 16–18], a more natural outcome would be that Vasiliev’s equations contain a deformed Fronsdal branch, obtained by a fine-tuned field redefinition [25] possibly related to a noncommutative geometric framework [29–31]. As this branch would have to correspond to a unitary (free) CFT, one should thus think of the deformed Fronsdal action as an effective action, that is, there is no need to quantise it any further — whereas further $1/N$ corrections can be obtained by altering boundary conditions. Moreover, unlike in an ordinary quantum field theory, in which the effective action has a loop expansion, the trivial nature of the $1/N$ expansion of the free conformal field theory implies that the anti-holographically dual deformed Fronsdal action has a trivial loop expansion as well. Thus, rather than to quantise the deformed Fronsdal theory as a four-dimensional field theory (only to discover that all loop corrections actually cancel), it seems more reasonable to us that the actual microscopic theory turns to be a field theory based on a classical action of the covariant Hamiltonian form proposed in [30].

In this paper, we shall examine the issue of locality of unfolded field equations from a different point of view, by studying higher spin invariants known as zero-form charges [32, 33].
It was argued in [34, 35] that rather than asking for locality in a gravitational gauge theory at the level of the field equations, the question is to determine in which way the degrees of freedom assemble themselves so as to exhibit some form of cluster decomposition, in a way reminiscent to glueball formation in the strong coupling regime of quantum chromodynamics (QCD). In the body of this paper, we shall present further evidence for that this heuristic comparison with QCD is not coincidental.

In Vasiliev’s unfolded formulation, where the spacetime dependence of the locally-defined master fields can be encoded into a gauge function, the boundary condition dictated by the spacetime physics that one wishes to address is translated into the choice of a suitable class of functions in the internal fiber space. Following this line of reasoning, the authors of [35] proposed that locality properties manifest themselves at the level of zero-form charges [36], whose leading orders in the expansion in terms of curvatures of the bulk-to-boundary propagators found in [16] hence ought to reproduce holographic correlation functions of the dual CFT as was later verified in [37] for two- and three-point functions. A related check was performed in the case of black-hole-like solutions in [29, 31, 38], whose asymptotic charges are mapped to non-polynomial functions in the fiber space, that in their turn determine zero-form charges that indeed exhibit cluster-decomposition properties, whereby the zero-form charges of two well-separated one-body solutions are perturbatively additive in the leading order of the separation parameter.

The relation between the leading orders of more general zero-form charges and general \(n\)-point functions was then established in [39] following a slightly different approach in terms of Cayley transforms, reproducing the CFT results for the 3-point correlation functions of conserved currents for free bosons and free fermions obtained in [40, 41]. In [42], the \(n\)-point functions involving the conformal weight \(\Delta = 2\) operator were also computed and a comparison was made between the leading-order zero-form charges of [37] and the results of [43], showing complete agreement. The authors of [43] used a convenient twistor basis in order to express the operator product algebra of free bosons and fermions in various dimensions, and from this they computed all the correlators. The case of free CFT was fully covered, as well as free CFT (all Lorentz spins), upon certain truncations of the \(sp(8)\) generalised spacetime coordinates. No expressions were given, however, for the \(n\)-point correlation functions in the free scalar CFT for arbitrary \(d\). The latter were conjectured in [13].

We want to stress that the approach to the computation of free CFT correlation functions initiated in [37] from the bulk side has \textit{a priori} nothing to do with the usual AdS/CFT prescription [44, 45]. From the standard approach in terms of Witten diagrams, it is rather surprising that the evaluation of some quantities in the bulk (here the zero-form charges), evaluated on the free theory, could produce \(n\)-point correlation functions on the CFT side, as no information from vertices of order \(n\) in the Vasiliev theory is being used.

It can be somehow reconciled with holography if one uses a non-standard HS action [46] that reproduces the Vasiliev equations upon variation and is non-standard in the sense that the Lagrangian density is integrated over a higher-dimensional noncommutative open manifold containing AdS as a submanifold of its boundary, and with kinetic terms that are \textit{not} of the Fronsdal type but instead of the BF type usually met in topological field theory. In this approach, the zero-form charges appear as some pieces of the on-shell action. The fact that the generating functional in the HS bulk theory reproduces, in the semi-classical limit, the free CFT generating function of correlation functions, is therefore not totally odd [37].

In the present paper, one of the tasks we undertake is to pursue the evaluation of zero-form charges along these lines, using throughout a method that enables one to evaluate subleading corrections to the free CFT correlators due to the higher orders in the weak field expansion of Vasiliev’s zero-form master field, postponing the systematic evaluation of these subleading corrections to a future work. Although the results of [39, 42] already produced the \(n\)-point
correlators for free bosons and fermions CFT3, the fact that the noncommutative Z variables at the heart of Vasiliev’s formalism [47] were discarded \textit{ab initio} does not allow the consideration of subleading corrections in Vasiliev’s equations.

The plan of the paper is as follows: After a review of Vasiliev’s bosonic model in sections 2 and 3 contains a description of observables in Vasiliev’s model and the proof (completed in appendix A) that the zero-form charges discussed in [35–37], are nothing but twisted open Wilson lines in the noncommutative twistor Z space. In section 4, we compute the twisted open Wilson lines on the bulk-to-boundary propagator computed in [16] and derive the corresponding quasi-amplitudes for arbitrary number of external legs. In the next section 5, we compute the n-point correlation functions of conserved currents of the free CFT corresponding to a set of free bosons in arbitrary dimension d. We show that, even before Bose symmetrisation, the cyclic-invariant pre-amplitudes obtained from the open Wilson lines at leading order in weak field expansion correspond to the cyclically-invariant building blocks of the correlators in the free U(N) model, obtained from the Wick contraction of the nearest-neighbours free fields inside the correlation functions. Our notation for spinor indices together with some technical results are contained in appendix B, while we relegated some other technical results in the appendix C.

2. Vasiliev’s bosonic model

Four-dimensional bosonic Vasiliev’s higher spin theories are formulated in terms of locally defined differential forms on a base manifold \( X_4 \times Z_4 \), where \( X_4 \) is a commutative spacetime manifold, with coordinates \( x^\mu \), and \( Z_4 \) is a noncommutative four-manifold co-ordinated by \( Z^\alpha \), with \( \alpha = 1,...,4 \). The fields are valued in an associative algebra generated by oscillators \( Y^\alpha \) that are coordinates of a noncommutative internal manifold \( Y_4 \). By using symbol calculus, we shall treat \( Z^\alpha \) and \( Y^\alpha \) as commuting variables, whereby the noncommutative structure is ensured by endowing the algebra of functions \( f(x, Z, Y) \) with a noncommutative associative product, denoted by \( \star \), giving rise to the following oscillator algebra:

\[
[Y^\alpha, Y^\beta]_\star = 2i \ C^{\alpha\beta}, \quad [Z^\alpha, Z^\beta]_\star = -2i \ C^{\alpha\beta}, \quad [Y^\alpha, Z^\beta]_\star = 0, \tag{2.1}
\]

where \( C^{\alpha\beta} \) is an \( Sp(4) \) invariant non-degenerate tensor used to raise and lower \( Sp(4) \) indices using the NW-SE convention

\[
V^\alpha := C^{\alpha\beta} V_\beta, \quad V^\beta := V^\beta C^\alpha_{\beta\alpha}. \tag{2.2}
\]

The \( Sp(4) \) indices are split under \( sl(2, \mathbb{C}) \) in the following way:

\[
Y^\alpha = (y^\alpha, \bar{y}^\alpha), \quad Z^\alpha = (z^\alpha, -\bar{z}^\alpha). \tag{2.3}
\]

Correspondingly, the \( Sp(4) \) invariant tensor is chosen to be

\[
C^{\alpha\beta} = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon^{\alpha\beta} \end{pmatrix}, \quad C^\alpha_{\beta\alpha} = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon^{\alpha\beta} \end{pmatrix}, \tag{2.4}
\]

where \( \epsilon^{12} = \epsilon^{12} = 1 \) and \( \epsilon^{12} = \epsilon^{12} = 1 \) for the \( sl(2, \mathbb{C}) \) invariant tensors.

The star product algebra is extended from functions to differential forms on \( X_4 \times Z_4 \) by defining

\[
dZ^\alpha \star \tilde{f} := dZ^\alpha \tilde{f}, \quad \tilde{f} \star dZ^\alpha := \tilde{f} dZ^\alpha, \tag{2.5}
\]

\(^4\) We put hats on objects that are nontrivial differential forms on \( Z_4 \).
idem \, dx^\alpha$, where \( \hat{J} \) is a differential form and the wedge product is left implicit. Thus, the differential forms are horizontal on a total space \( \mathcal{X}_4 \times \mathcal{Z}_4 \times \mathcal{Y}_4 \), sometimes referred to as the correspondence space, with fiber space \( \mathcal{Y}_4 \), base \( \mathcal{X}_4 \times \mathcal{Z}_4 \) and total horizontal differential

\[
\hat{d} := d + q, \quad d := dx^\alpha \partial^\alpha, \quad q := dZ^i \partial^i, \quad (2.6)
\]

which obeys the graded Leibniz rule

\[
\hat{d} (\hat{f} \, \star \, \hat{g}) = (\hat{d} \hat{f}) \, \star \, \hat{g} + (-)^{\text{deg} \hat{f} \, \text{deg} \hat{g}} \hat{f} \, \star \, (\hat{d} \hat{g}), \quad (2.7)
\]

with \text{deg} denoting the total form degree. The differential graded star product algebra of forms admits a set of linear (anti-)automorphisms defined by

\[
\pi(x, y, \bar{y}, z, \bar{z}) = (x, -y, \bar{y}, -z, \bar{z}), \quad (2.8)
\]

\[
\bar{\pi}(x, y, \bar{y}, z, \bar{z}) = (x, y, -\bar{y}, z, \bar{z}), \quad (2.9)
\]

\[
\tau(x, y, \bar{y}, z, \bar{z}) = (x, iy, iy, -iz, iz), \quad (2.10)
\]

and

\[
\pi(\hat{d} \hat{f}) = \hat{d} \pi(\hat{f}), \quad \pi(\hat{f} \, \star \, \hat{g}) = \pi(\hat{f}) \, \star \, \pi(\hat{g}), \quad \text{idem for} \, \bar{\pi}, \quad (2.11)
\]

\[
\tau(\hat{d} \hat{f}) = \hat{d} \tau(\hat{f}), \quad \tau(\hat{f} \, \star \, \hat{g}) = (-)^{\text{deg} \hat{f} \, \text{deg} \hat{g}} \tau(\hat{g}) \, \star \, \tau(\hat{f}), \quad (2.12)
\]

for differential forms \( \hat{f} \) and \( \hat{g} \). Let us notice that \( \tau^2 = \pi \bar{\pi} \) and demanding that \( \pi \bar{\pi}(\hat{f}) = \hat{f} \) amounts to removing all half-integer spin gauge fields on \( \mathcal{X}_4 \) from the model, leaving a bosonic model whose gauge fields on \( \mathcal{X}_4 \) have integer spin. The hermitian conjugation is the anti-linear anti-automorphism defined by

\[
(x, y, \bar{y}, z, \bar{z})^\dagger = (x, \bar{y}, y, z, \bar{z}), \quad (2.13)
\]

\[
(\hat{d} \hat{f})^\dagger = \hat{d}^\dagger \hat{f}, \quad (\hat{f} \, \star \, \hat{g})^\dagger = (-)^{\text{deg} \hat{f} \, \text{deg} \hat{g}} \hat{g}^\dagger \, \star \, \hat{f}^\dagger. \quad (2.14)
\]

In what follows, we shall use the normal ordered basis for the star product\(^5\). Among the various conventions existing in the literature, we choose to work with the explicit realisation:

\[
(\hat{f} \, \star \, \hat{g})(Z, Y) := \int \frac{d^4Ud^4V}{(2\pi)^4} \, e^{iUZ + iVY} \hat{f}(Z + U, Y + U) \hat{g}(Z - V, Y + V), \quad (2.15)
\]

for auxiliary variables \( U^\alpha := (u^\alpha, \bar{u}^\alpha) \) and \( V^\alpha := (v^\alpha, \bar{v}^\alpha) \). The space of bounded functions of \( Y \) and \( Z \) whose complex modulus is integrable (usually written \( L^1(\mathcal{Y}_4 \times \mathcal{Z}_4) \)) forms a star product algebra that admits a trace operation, given by

\[
\text{Tr} \hat{f}(Z, Y) := \int d^4Z \, d^4Y \, \hat{f}(Z, Y); \quad (2.16)
\]

indeed, it has the desired cyclicity property

\[
\text{Tr} \hat{f} \, \star \, \hat{g} = \text{Tr} \, \hat{g} \, \star \, \hat{f}. \quad (2.17)
\]

\(^5\)It provides normal ordering of \( a_2^\alpha := \frac{1}{2}(Y^\alpha + Z^\alpha) \) and \( a_2^\alpha := \frac{1}{2}(Y^\alpha - Z^\alpha) \) so that \( \hat{f} \, \star \, a_2^\alpha = \hat{f} \, a_2^\alpha \) and \( a_2^\alpha \, \star \, \hat{f} = a_2^\alpha \hat{f} \).
It has the remarkable property:
\[
\text{Tr} \left( \hat{f}(Y) \star \hat{g}(Z) \right) = \text{Tr} \left( \tilde{f}(Y) \tilde{g}(Z) \right).
\] (2.18)

In normal order, the inner Klein operators \( \hat{\kappa} \) and \( \hat{\bar{\kappa}} \), defined by
\[
\hat{\kappa} \star \hat{\kappa} = 1 = \hat{\bar{\kappa}} \star \hat{\bar{\kappa}}, \quad \pi(\hat{f}) = \hat{\kappa} \star \hat{f} \star \hat{\kappa}, \quad \tilde{\pi}(\hat{f}) = \hat{\bar{\kappa}} \star \tilde{f} \star \hat{\bar{\kappa}},
\] (2.19)
for all zero-forms \( \hat{f} \), become real-analytic functions on \( \mathcal{Y}_4 \times \mathbb{Z}_4 \), viz
\[
\hat{\kappa} := e^{i\varphi z}, \quad \hat{\bar{\kappa}} := e^{-i\varphi \bar{z}}.
\] (2.20)

As shown in [48], the inner kleinians factorise as
\[
\hat{\kappa} = \kappa_y(y) \star \kappa_z(z), \quad \hat{\bar{\kappa}} = \bar{\kappa}_y(\bar{y}) \star \bar{\kappa}_z(\bar{z}), \quad \text{with}
\]
\[
\kappa_y(y) := 2\pi \delta^2(y), \quad \kappa_z(z) := 2\pi \delta^2(z), \quad \bar{\kappa}_y(\bar{y}) := 2\pi \delta^2(\bar{y}), \quad \bar{\kappa}_z(\bar{z}) := 2\pi \delta^2(\bar{z}),
\] (2.21)
which implies that their symbols are non-real analytic in Weyl order. As we shall see, the inner kleinians define closed and central elements appearing in the equations of motion, which implies that the fields will be real-analytic on-shell in normal order but not in Weyl order. Thus, the normal order is suitable for a standard higher spin gravity interpretation of the model, which requires a manifestly Lorentz covariant symbol calculus with symbols that are real analytic at the origin of \( \mathbb{Z}_4 \times \mathcal{Y}_4 \).

The master fields describing the bosonic model consist of a connection one-form \( \hat{A} \) and a twisted-adjoint zero-form \( \hat{\Phi} \), subject to the reality conditions and bosonic projection
\[
\hat{A}^\dagger = -\hat{A}, \quad \pi(\hat{A}) = \tilde{\pi}(\hat{A}), \quad \hat{\Phi}^\dagger = \pi(\tilde{\Phi}) = \tilde{\pi}(\hat{\Phi}).
\] (2.22)
(2.23)

At the linearised level the physical spectrum consists of an infinite tower of massless particles of every integer spin, each occurring once. The bosonic model can be further projected to its minimal version, containing only even spin particles, by imposing
\[
\tau(\hat{A}) = -\hat{A}, \quad \tau(\hat{\Phi}) = \tilde{\pi}(\hat{\Phi}).
\] (2.24)

The equations of motion are given by the twisted-adjoint covariant constancy condition
\[
\hat{d}\hat{\Phi} + \hat{A} \star \hat{\Phi} - \tilde{\Phi} \star \pi(\hat{A}) = 0,
\] (2.25)
and the curvature constraint
\[
\hat{d}\hat{A} + \hat{A} \star \hat{A} + \tilde{\Phi} \star \tilde{J} = 0,
\] (2.26)
where the two-form \( \tilde{J} \) is given by
\[
\tilde{J} := -\frac{i}{4} \left( e^{i\theta} \hat{\kappa} \, dz^\alpha d\bar{z}_\alpha + e^{-i\theta} \hat{\bar{\kappa}} \, d\bar{z}^\alpha dz_\alpha \right),
\] (2.27)
with \( \theta \) a real constant caracterising the model. This element obeys
\[
\tilde{J}^\dagger = \tau(\tilde{J}) = -\tilde{J},
\] (2.28)
and
\[
\hat{d}\tilde{J} \equiv 0, \quad \hat{A} \star \tilde{J} - \tilde{J} \star \pi(\hat{A}) \equiv 0 \equiv \tilde{\Phi} \star \tilde{J} - \tilde{J} \star \pi(\tilde{\Phi}).
\] (2.29)
It follows that the field equations are compatible with the reality conditions on the master fields and the integer-spin projections, and that they are universally Cartan integrable. The latter implies invariance of (2.26) and (2.25) under the finite gauge transformation

\[ \hat{A} \rightarrow \hat{g} \ast \hat{d} \hat{g}^{-1} + \hat{g} \ast \hat{A} \ast \hat{g}^{-1}, \]  

(2.30)

\[ \hat{\Phi} \rightarrow \hat{g} \ast \hat{\Phi} \ast \pi(\hat{g}^{-1}), \quad \hat{g} = \hat{g}(x, Z, Y), \]  

(2.31)

which preserve the reality conditions on the master fields and the integer-spin projection of the model provided that

\[ \hat{g}^\dagger = \hat{g}^{-1}, \quad \pi(\hat{g}) = \bar{\pi}(\hat{g}); \]  

(2.32)

the minimal projection in addition requires that

\[ \tau(\hat{g}) = \hat{g}^{-1}. \]  

(2.33)

The parity invariant models [11] are obtained by taking \( \theta = 0 \) for the Type A model and \( \theta = \frac{\pi}{2} \) for the Type B model, in which the physical scalar field is parity even and odd, respectively.

Upon splitting the connection one-form into \( dx^\mu \) and \( dZ^\alpha \) directions:

\[ \hat{A} = \hat{U} + \hat{V}, \quad \hat{U} = dx^\mu \hat{A}_\mu, \quad \hat{V} = dx^\alpha \hat{A}_\alpha + dz^\dot{\alpha} \hat{A}_{\dot{\alpha}}, \]  

(2.34)

Vasiliev’s equations read

\[ d\hat{U} + \hat{U} \ast \hat{U} = 0, \]  

(2.35)

\[ d\hat{V} + q\hat{U} + \hat{U} \ast \hat{V} + \hat{V} \ast \hat{U} = 0, \]  

(2.36)

\[ q\hat{V} + \hat{V} \ast \hat{V} + \hat{\Phi} \ast \hat{J} = 0, \]  

(2.37)

\[ d\hat{\Phi} + \hat{U} \ast \hat{\Phi} - \hat{\Phi} \ast \pi(\hat{U}) = 0, \]  

(2.38)

\[ q\hat{\Phi} + \hat{V} \ast \hat{\Phi} - \hat{\Phi} \ast \pi(\hat{V}) = 0. \]  

(2.39)

Remarkably, unlike the case of a connection on a commutative manifold, the connection on a noncommutative (symplectic) manifold can be mapped in a one-to-one fashion to an adjoint quantity, given in Vasiliev’s theory by\(^6\)

\[ \hat{S}_\alpha := Z_\alpha - 2i\hat{A}_\alpha, \quad \hat{\Psi} := \hat{\Phi} \ast \hat{\kappa}, \quad \hat{\bar{\Psi}} := \hat{\Phi} \ast \hat{\bar{\kappa}}. \]  

(2.40)

In terms of these variables, Vasiliev’s equations read as follows:

\[ d\hat{U} + \hat{U} \ast \hat{U} = 0, \quad [\hat{S}_\alpha, \hat{S}_\alpha]_* = 0, \]  

(2.41)

\[ d\hat{S}_\alpha + [\hat{U}, \hat{S}_\alpha]_* = 0, \quad d\hat{S}_\alpha + [\hat{U}, \hat{S}_\alpha]_* = 0, \]  

(2.42)

\[ d\hat{\Psi} + [\hat{U}, \hat{\Psi}]_* = 0, \quad d\hat{\bar{\Psi}} + [\hat{U}, \hat{\bar{\Psi}}]_* = 0, \]  

(2.43)

\(^6\) The transformation of \( \hat{S}_\alpha \) can be obtained by recalling that \(-2i\hat{d}^\alpha \hat{J} = [Z_\alpha, \hat{J}]_* \).
\[
\left\{ \hat{S}_\alpha, \hat{\Psi} \right\}_* = \left[ \hat{S}_\alpha, \hat{\Psi} \right] = 0,
\left\{ \hat{S}_\alpha, \hat{\Psi} \right\}_* = \left[ \hat{S}_\alpha, \hat{\Psi} \right] = 0.
\]

\[
\left[ \hat{S}_\alpha, \hat{\beta}_\beta \right] = 2i \epsilon_{\alpha\beta} (1 - e^{i\theta} \hat{\Psi}) = 0,
\left[ \hat{S}_\alpha, \hat{\beta}_\beta \right] = 2i \epsilon_{\alpha\beta} (1 - e^{-i\theta} \hat{\Psi}) = 0.
\]

Thus, the adjoint variables \((\hat{S}_\alpha, \hat{\beta}_\beta)\) obey a generalized version of Wigner’s deformation [49, 50] of the Heisenberg algebra, as in [51], and are hence referred to as the deformed oscillators.

The vacuum solution describing the AdS$_4$ background is obtained by setting \(\hat{\Phi} = 0 = \hat{V}\) and taking \(\hat{U} = \Omega\), the Cartan connection of AdS$_4$, given by

\[
\Omega(Y|\chi) = \frac{1}{4i} (\gamma^\alpha y^\beta \omega_{\alpha\beta} + \gamma^\alpha y^\beta \bar{\omega}_{\alpha\beta} + 2 \gamma^\alpha y^\beta h_{\alpha\beta})
\]

obeying the zero-curvature condition \(d\Omega + \Omega \star \Omega = 0\). One may then perform a perturbative expansion around this background and find, at the linearised level, the central on mass-shell theorem [52] that describes, in a suitable gauge, the free propagation of an infinite tower of Fronsdal fields around AdS$_4$. For our purpose it is important to recall that the twisted adjoint zero-form is $Z$-independent at the linearised level, and will be denoted by $\Phi(x; Y)$.

### 3. Observables in Vasiliev’s theory

As the gauge transformations of Vasiliev’s theory resemble those of noncommutative Yang–Mills theory (see [53–55]), some results of the latter theory may be applied to Vasiliev’s theory. In particular, one can construct gauge invariant observables from holonomies formed from curves that are closed in $\mathcal{X}_4$ and open in $\mathcal{Z}_4$. To this end, we consider a curve

\[
\mathcal{C} : [0, 1] \rightarrow \mathcal{X}_4 \times \mathcal{Z}_4 : \sigma \rightarrow (\xi^\mu(\sigma), \xi^\alpha(\sigma))
\]

that is based at the origin, closed in the commutative directions and open along the noncommutative space, i.e.

\[
\xi^\mu(0) = \xi^\mu(1) = 0, \quad \xi^\alpha(0) = 0, \quad \xi^\alpha(1) = 2M_\alpha = 2C^{\alpha\beta}M_\beta.
\]

Here $M_\alpha = (\mu_\alpha, \bar{\mu}_\alpha)$ is seen as a momentum conjugated to $Z_\alpha$. We can associate the following Wilson line to $\mathcal{C}$:

\[
W_C(x, Z; Y) := P \exp_n \left( \int_0^1 d\sigma \left( \hat{\xi}^\mu(\sigma) \hat{A}_\mu(\sigma) + \hat{\xi}^\alpha(\sigma) \hat{A}_\alpha(\sigma) \right) \right)
\]

\[
= \sum_{n=0}^\infty \int_0^1 d\sigma_n ... \int_0^1 d\sigma_1 \hat{\star} \sum_{i=1}^{n} \left( \hat{\xi}^\mu(\sigma_i) \hat{A}_\mu(\sigma_i) + \hat{\xi}^\alpha(\sigma_i) \hat{A}_\alpha(\sigma_i) \right).
\]

where $\hat{\star}$ is defined as $\hat{f}_1 \star ... \star \hat{f}_n$ in that order. Under the gauge transformation (2.30) and (2.31) it transforms as:

\[
W_C(x, Z; Y) \rightarrow \hat{g}(x, Z; Y) \star W_C(x, Z; Y) \star \hat{g}^{-1}(x, Z + 2M; Y).
\]
Unlike open Wilson lines in commutative spaces, which cannot be made gauge invariant, their counterparts in noncommutative spaces can be made gauge invariant by star multiplying them with the generator $e^{\hat{M}Z}$ of finite translations in $\mathcal{Z}_4$, and tracing over both the fiber space and base manifold\(^7\). Thus, for any adjoint zero-form $\hat{O}(x,Z,Y)$, i.e. any operator transforming as $\hat{O} \longrightarrow \tilde{g} \ast \hat{O} \ast \tilde{g}^{-1}$, the quantity

$$\tilde{O}_C(M|x) := \text{Tr} \left[ \hat{O}(x,Z,Y) \ast W_C(x,Z,Y) \ast e^{i\hat{M}Z} \right],$$

(3.5)

which one may formally think of as the Fourier transform of the impurity, is gauge invariant, as shown in [55]. Indeed, this follows from the cyclicity of the trace (2.16), the gauge transformation law (3.4) and the following expression for the star commutator of a function $\hat{f}(x,Z,Y)$ with the exponential $e^{i\hat{M}Z}$ (see (3.29) and (3.30) for details):

$$\hat{f}(x,Z+2M,Y) \ast e^{i\hat{M}Z} = e^{i\hat{M}Z} \ast \hat{f}(x,Z,Y).$$

(3.6)

The above construction is formal, and hence the well-definiteness of the observables (3.5) depends on the properties of the considered curve and solution to Vasiliev equations. The case of the leading order pre-amplitudes will be discussed at the end of this section.

Previously [35–37], decorated Wilson loops in the commutative $\mathcal{X}_4$ space have been considered within the context of Vasiliev’s theory; in particular, for trivial loops, these reduce to invariants of the form

$$I_{m,t}(M) := \text{Tr} \left[ \tilde{\Psi}^{*\sigma} \ast (\tilde{\kappa})^\tau \ast \exp_s \left( iM\tilde{S} \right) \right],$$

(3.7)

for $t$ being zero or one, where we note the insertion of the adjoint operator $e^{iS\tilde{S}}$. As these invariants are independent of the choice of base point in $\mathcal{X}_4$, they have been referred to as zero-form charges. Let us show the equivalence between these invariants and twisted straight open Wilson lines in $\mathcal{Z}_4$, thereby providing a geometrical underpinning for the insertion of the adjoint impurity $e^{iS\tilde{S}}$ formed out of the deformed oscillator $\tilde{S}$. To this end, we consider the straight line

$$L_{2M} : [0,1] \rightarrow \mathcal{X}_4 \times \mathcal{Z}_4 : \sigma \rightarrow (0, 2\sigma M^2).$$

(3.8)

The open Wilson lines in noncommutative space form an over-complete set of observables for noncommutative Yang–Mills theory, see [53] and references therein. As stressed in the same reference, the straight open Wilson lines with a complete set of adjoint impurities inserted at one end of the Wilson line also provide such a set of observables\(^8\). In the case of Vasiliev’s theory, these observables can be written as follows:

$$\tilde{O}_{L_{2M}}(M|x) = \int d^4Z d^4Y \hat{O}(x,Z,Y) \ast W_{L_{2M}}(x,Z,Y) \ast \exp(i\hat{M}Z)$$

(3.9)

$$= \int d^4Z d^4Y \hat{O}(x,Z,Y) \ast \exp_s \left( iM\tilde{S} \right).$$

(3.10)

\(^7\)Let us point out that this $\ast$-multiplication by $e^{iM\tilde{S}}$ is not equivalent to considering the same curve, closed by a straight path linking its two ends, since the connection $A$ is not path integrated along this straight path.

\(^8\)Indeed, considering a general path (3.1), one can write

$$W_C(x,Z,Y) = W_C(x,Z,Y) \ast (W_{L_{2M}}(x,Z,Y))^{-1} \ast W_{L_{2M}}(x,Z,Y),$$

where $W_C(x,Z,Y) \ast (W_{L_{2M}}(x,Z,Y))^{-1}$ is an adjoint operator.
The last equality comes from the following result:

\[
\exp(i\mathcal{M}\hat{S}) = W_{\mathcal{M}}(x; Z; Y) \ast \exp(i\mathcal{M}Z),
\]

(3.11)
that is proved order by order in powers of the higher spin connection \( \hat{A} \) in appendix A.

From the form of the equations (2.41)–(2.45) involving \( \check{\mathcal{S}} \) and \( \check{\Psi} \), one can see that the most general adjoint operator one can build out of the master fields is equivalent on shell to:

\[
\tilde{O}_{n_0; \mathcal{M}\hat{A}}(K) := \check{\Psi}^{n_0} \ast \left( \tilde{\kappa} \right)^{\ast \mathcal{I}} \ast \left( \check{\mathcal{S}}_{\hat{A}} \right)^{\ast \mathcal{K}},
\]

(3.12)

where, as suggested by the indices, the deformed oscillators \( \check{\mathcal{S}}_{\hat{A}} \) are symmetrized. Through (3.9), this yields the following form for the most general observable of Vasiliev’s theory:

\[
\mathcal{I}_{n_0; \mathcal{M}\hat{A}}(K)(M) = \int d^4Z d^4Y \check{\Psi}^{n_0} \ast \left( \tilde{\kappa} \right)^{\ast \mathcal{I}} \ast \left( \check{\mathcal{S}}_{\hat{A}} \right)^{\ast \mathcal{K}} \ast \exp\left(i\mathcal{M}\hat{S}\right).
\]

(3.13)

However, it turns out that one can obtain all of these observables from the evaluation of the following ones, viewed as functions of \( M_{\hat{A}} \):

\[
\mathcal{I}_{n_0}(M) = \int d^4Z d^4Y \check{\Psi}^{n_0} \ast \left( \tilde{\kappa} \right)^{\ast \mathcal{I}} \ast \exp\left(i\mathcal{M}\hat{S}\right).
\]

(3.14)

Indeed, one has

\[
\left( i\mathcal{M}_\omega \right)^{\mathcal{K}} \mathcal{I}_{n_0}(M) \bigg|_{M=0} = \int d^4Z d^4Y \check{\Psi}^{n_0} \ast \left( \tilde{\kappa} \right)^{\ast \mathcal{I}} \ast (i\check{\mathcal{S}}_{\hat{A}})^{\ast \mathcal{K}}
\]

(3.15)

and the observable (3.13) can be written as an infinite sum of term of this form, upon applying (2.44, 2.45) repeatedly in order to symmetrise all the deformed oscillators. We will refer to the observables (3.14) as zero-form charges since they are nothing but the observables considered and evaluated in some special cases in [35–37]. In the weak field expansion scheme, we can write the leading order contribution to the zero-form charges as

\[
\mathcal{I}_{n_0}^{(n)}(M) = \int d^4Z d^4Y \left( \Phi \ast \tilde{\kappa} \right)^{n_0} \ast \left( \tilde{\kappa} \right)^{\ast \mathcal{I}} \ast e^{i\mathcal{K}z} \ast e^{-i\mathcal{K}z}
\]

(3.16)

\[
= \int d^4Z d^4Y \left( \bigotimes_{i=1}^{n_0} \Phi \left( \tau; (-1)^{i+1}y, n^i \right) \right) \ast e^{i\mathcal{K}z} \ast e^{-i\mathcal{K}z} \ast e^{i\mathcal{K}z} \ast e^{-i\mathcal{K}z}.
\]

(3.17)

The second line was obtained using (2.19) and defining:

\[
e := n_0 + t \mod 2, \quad t \in \{0, 1\},
\]

(3.18)

Following [37], we use the zero-form charges as building blocks for quasi-amplitudes of various orders. To define the quasi-amplitudes of order \( n \), we expand a given zero-form charge to \( n \)th order in the twisted-adjoint weak field \( \Phi \) and replace, in a way that we specify below, the \( n \) fields \( \Phi \) with \( n \) distinct external twisted-adjoint quantities \( \Phi_i, i = 1, \ldots, n \), each of which transforming as (2.31) under a diagonal higher spin group acting on all \( \Phi_i \)’s with the same parameter. The quasi-amplitude \( Q_{n_0}^{(n)}(\Phi | M) \) is now defined unambiguously as the functional of \( \Phi_1, \ldots, \Phi_n \) that is totally symmetric in its \( n \) arguments and obeys:

\[
Q_{n_0}^{(n)}(\Phi_1 | M) \bigg|_{\Phi_1 = \ldots = \Phi_n = \Phi} = \mathcal{I}_{n_0}^{(n)}(M).
\]

(3.19)
As shown in [37, 39], at the leading order, i.e. \( n = n_0 \), these reproduce the correlation functions of bilinear operators in the free conformal field theory in three dimensions for \( n = 2, 3 \) and 4.

In this paper, we are interested in more fundamental building blocks from the bulk point of view, which are referred to as pre-amplitudes and are defined as follows:

\[
\mathcal{A}_{n_0, t} (\Phi_1| M) := \int d^4 Z \, d^4 Y \left( \prod_{i=1}^{n_0} \Phi_i \left( x_i, (y^i + 1) y_i \right) \right) \star e^{iy\gamma \zeta} \star e^{-iy\bar{\gamma} \bar{\zeta}} \star e^{i\mu \zeta} \star e^{-i\mu \zeta}.
\] (3.20)

The prescription to obtain those objects is to replace \( \Phi \) in the expression (3.16) of the leading order zero-form charges with the different fields \( \Phi_i \) in a given order, conventionally with the label growing from left to right. We will not discuss their generalization to higher orders in perturbation theory. The aim of this paper is to strengthen the previous results on quasi-amplitudes to higher order zero-form charges with the different fields \( \Phi_i \) and \( \Phi_j \). Thus, the cyclic invariance follow from the integer-spin projection in (2.23).

The first step will be to show the invariance of the pre-amplitudes (3.20) under cyclic permutations of the external legs. To do so, we use the decompositions (2.21) to split the integrand into a \( Z \)-dependent part \( \mathcal{G}^t(Z| M) \) and a \( Y \)-dependent part to be specified below. From the cyclicity of the trace and the mutual star-commutativity of functions of \( Y \) and functions of \( Z \), we compute:

\[
\begin{align*}
\mathcal{A}_{n_0, t} (\Phi_1| M) &= \int d^4 Z \, d^4 Y \left( \prod_{i=1}^{n_0} \pi^{i+1} \Phi_i \right) \star (k_\gamma)^{m_0} \star (k_{\bar{\gamma}})^{t^t} \star \mathcal{G}^t(Z| M) \\
&= \int d^4 Z \, d^4 Y \left( \prod_{i=1}^{n_0} \pi^{i+1} \Phi_i \right) \star (k_\gamma)^{m_0} \star (k_{\bar{\gamma}})^{t^t} \star \mathcal{G}^t(Z| M) \Phi_1 \\
&= \int d^4 Z \, d^4 Y \left( \prod_{i=2}^{n_0} \pi^{i+1} \Phi_i \right) \star (k_\gamma)^{m_0} \star (k_{\bar{\gamma}})^{t^t} \star \Phi_1 \star \mathcal{G}^t(Z| M) \\
&= \int d^4 Z \, d^4 Y \left( \prod_{i=1}^{n_0-1} \pi^{i+1} \Phi_i \right) \star (\pi^m \pi^t \bar{\Phi}_1) \star (k_\gamma)^{m_0} \star (k_{\bar{\gamma}})^{t^t} \star \mathcal{G}^t(Z| M) \\
&= \mathcal{A}_{n_0, t} (\Phi_2, ..., \Phi_{n_0}, (\pi^t \bar{\Phi}_1| M),
\end{align*}
\] (3.21)

where the concluding line is obtained by a change of integration variables given by \( y \rightarrow -y \). Thus, the cyclic invariance follow from the integer-spin projection in (2.23).

We remark that as Weyl- and normal-ordered symbols of operators depending either only on \( Y \) or only on \( Z \) are the same, the integrations over \( Y \) and \( Z \) can be factorized. This simplifying scheme was used in [39] in deriving the leading order quasi-amplitudes for all \( n \), whereas the earlier results for \( n = 2, 3 \) in [37] were based on a different scheme adapted to the weak-field expansion of Vasiliev’s equations to higher order, for which there is no obvious factorization of the integrand. In what follows, we shall follow the latter approach, and perform an alternative derivation of the results of [39], which thus can be generalized more straightforwardly to computing subleading corrections to open Wilson lines in \( \mathcal{Z}_4 \).

To the latter end, we define \( Y \)-space momenta \( \Lambda_{\gamma} = (\lambda_{\alpha}, \lambda_{\bar{\alpha}}) \), that are complex conjugate of each other, i.e. \( \lambda, \bar{\lambda} \), and which are not affected by \( \star \)-products. Assuming that the twisted-adjoint zero-form \( \Phi \in \mathcal{S}(\mathcal{Y}_4) \), i.e. that it is a rapidly decreasing function, we have the following Fourier transformation relations:
\[ \tilde{\Phi}(\Lambda) := \int \frac{d^4Y}{(2\pi)^2} \Phi(Y) \exp(-i\Lambda Y), \]  
(3.22)

\[ \Phi(Y) = \int \frac{d^4\Lambda}{(2\pi)^2} \tilde{\Phi}(\Lambda) \exp(i\Lambda Y). \]  
(3.23)

Upon defining the following mappings:

\[ \pi_\Lambda(\lambda, \bar{\lambda}) = (-\lambda, \bar{\lambda}), \quad \bar{\pi}_\Lambda(\lambda, \bar{\lambda}) = (\lambda, -\bar{\lambda}), \quad \tau_\Lambda(\lambda, \bar{\lambda}) = (i\lambda, i\bar{\lambda}), \]  
(3.24)

the integer-spin projection and reality condition in (2.23) and the minimal bosonic projection (2.24), respectively, translate into

\[ \tilde{\Phi}^\dagger = \pi_\Lambda \tilde{\Phi} = \bar{\pi}_\Lambda \tilde{\Phi}, \]  
(3.25)

\[ \pi_\Lambda \tilde{\Phi} = \tau_\Lambda \tilde{\Phi}. \]  
(3.26)

Writing (3.20) in terms of plane waves gives

\[ A_{m,\ell}(\Phi_j|M) = \int \left( \prod_{j=1}^{n} \frac{d^4\Lambda_j}{(2\pi)^7} \right) \left( \prod_{j=1}^{n} \tilde{\Phi}(\Lambda_j) \right) F_{m,\ell}(\Lambda_j|M), \]  
(3.27)

where the quantity \( F_{m,\ell}(\Lambda_j|M) \), which one may think of as a higher spin form factor, is given by

\[ F_{m,\ell}(\Lambda_j|M) = \int d^4Z d^4Y \left( \otimes_{j=1}^{n} e^{i([-1]+i\lambda_jY+\bar{\lambda}_jZ)} \right) e^{i\lambda_jZ} \star e^{-i\bar{\lambda}_jZ}. \]  
(3.28)

In order to perform the star-products appearing above, we need some lemmas. First of all, star multiplying exponentials of linear expressions in \( Y \) and \( Z \) one has

\[ \tilde{f}(x, Z; Y) \star e^{iMZ} = e^{iMZ} \tilde{f}(x, Z - M; Y - M), \]  
(3.29)

\[ e^{iMZ} \star \tilde{f}(x, Z; Y) = e^{iMZ} \tilde{f}(x, Z + M; Y - M), \]  
(3.30)

\[ \tilde{f}(x, Z; Y) \star e^{i\Lambda Y} = e^{i\Lambda Y} \tilde{f}(x, Z + \Lambda; Y + \Lambda), \]  
(3.31)

\[ e^{i\Lambda Y} \star \tilde{f}(x, Z; Y) = e^{i\Lambda Y} \tilde{f}(x, Z + \Lambda; Y - \Lambda). \]  
(3.32)

Then one can show recursively that the following equality holds:

\[ \ast_{j=1}^{n} \exp(i\Lambda_jY) = \exp\left( i \sum_{j=1}^{n} \sum_{k=1}^{j-1} \Lambda_k \Lambda_j \right) \exp\left( i \sum_{j=1}^{n} \Lambda_j Y \right). \]  
(3.33)

The following relations, valid for any \( t \) and \( e \), will be useful as well:

\[ \tilde{f}(x, z, -\bar{z}; y, \bar{y}) \star e^{i\bar{\lambda}Z} = e^{i\bar{\lambda}Z} \tilde{f}(x, (1 - e)z - ey, -\bar{z}; (1 - e)y - ez, \bar{y}), \]  
(3.34)

\[ \tilde{f}(x, z, -\bar{z}; y, \bar{y}) \star e^{-i\lambda Z} = e^{-i\lambda Z} \tilde{f}(x, z, -(1 - t)\bar{z} - ty; y, (1 - t)\bar{y} + t\bar{z}). \]  
(3.35)
The above relations allow the factorization of $F_{n,t}(\Lambda|M)$ as follows:

$$F_{n,t}(\Lambda|M) = g_{n,t}(\Lambda) f_{n,t}(\lambda|x) f_{n,t}(\bar{\lambda}|\bar{\mu}),$$

(3.36)

where

$$g_{n,t}(\Lambda) := \exp \left( i \sum_{i<j} (-1)^{j+i} \lambda_i \lambda_j + \sum_{i<j} \bar{\lambda}_i \bar{\lambda}_j \right),$$

(3.37)

$$f_{n,t}(\lambda|x) := \int \frac{d^2 \lambda}{(2\pi)^2} \exp \left[ i(1-e) \left( \sum_i (-\lambda_i y + \mu z) \right) \right]$$

$$\exp \left[ ie \left( \lambda \sum_j (-\lambda_jy) + \mu \right) \right]$$

$$= (2\pi)^{4-2\epsilon} \left( \delta^2(\mu) \delta^2(\sum_j (-\lambda_jy)) \right)^{1-\epsilon},$$

(3.38)

$$\bar{f}_{n,t}(\bar{\lambda}|\bar{\mu}) := \int \frac{d^2 \bar{\lambda}}{(2\pi)^2} \exp \left[ i(1-t) \left( \sum_i \bar{\lambda}_i (\bar{y} - \bar{\mu}) + \bar{\mu} \bar{z} \right) + it \left( -\bar{y} + \sum_i \bar{\lambda}_i \right) \left( \bar{z} + \bar{\mu} \right) \right]$$

$$= (2\pi)^{4-2\epsilon} \left( \delta^2(\bar{\mu}) \delta^2(\sum_j \bar{\lambda}_j) \right)^{1-\epsilon}.$$

(3.39)

The above functions have the following behaviour under cyclic permutations of the $Y$-space momenta:

$$g_{n,t}(\lambda_1, \lambda_2, ..., \lambda_n|\Lambda_1, \Lambda_2, ..., \Lambda_n) = g_{n,t}(\Lambda_2, ..., \Lambda_n, -(-)^m \lambda_1|\Lambda_2, ..., \Lambda_n, -\Lambda_1),$$

(3.40)

$$f_{n,t}(\lambda_1, \lambda_2, ..., \lambda_n|\mu) = f_{n,t}(\lambda_2, ..., \lambda_n, -(-)^m \lambda_1|\mu),$$

(3.41)

$$\bar{f}_{n,t}(\bar{\lambda}_1, \bar{\lambda}_2, ..., \bar{\lambda}_n|\bar{\mu}) = \bar{f}_{n,t}(\bar{\lambda}_2, ..., \bar{\lambda}_n, \bar{\lambda}_1|\bar{\mu}).$$

(3.42)

Let us point out the fact that $f_{n,t}(\lambda|x)$ (resp. $\bar{f}_{n,t}(\bar{\lambda}|\bar{\mu})$) is given by $(2\pi)^2$ if $e = 1$ (resp. $t = 1$), or else it is given by a delta function that sets, in $g_{n,t}(\Lambda)$, the momentum $\lambda_j$ (resp. $\lambda_1$) equal to combinations of the other variables $\lambda_j$, $j = 2, ..., n_0$. As a result, one can write the cyclic property of $F_{n,t}(\Lambda|M)$ as

$$F_{n,t}(\Lambda_1, \Lambda_2, ..., \Lambda_n) = F_{n,t}(\Lambda_2, ..., \Lambda_n, -(-)^i \Lambda_1).$$

(3.43)

This behaviour under cyclic permutations can be used in (3.27) to show the cyclic invariance of $\mathcal{A}_{n,t}(\Phi|M)$ as:
The third line is a mere change of variables, while the last one uses (3.25).

The computations of this section have shown that the $M$ dependence of the pre-amplitudes $\mathcal{A}_{m,t}$ ($\Phi_i|M$) can be factorised as follows:

$$\mathcal{A}_{m,t} (\Phi_i|M) = \delta^2(\mu) \delta^2(\bar{\mu}) \mathcal{A}_{m,t} (\Phi_i).$$

(3.44)

This result presents the divergences that were first discussed in [37]. This is a consequence of the integrand of (3.20) not being in $L^1(\mathcal{Y}_4 \times Z_4)$. Let us replace the twistor plane waves $e^{iMZ}$ by a function $\mathcal{V}(Z) \in S(Z_4)$, then the following object is well defined:

$$\mathcal{A}_{m,t} \mathcal{V} (\Phi_i) := \int d^4Z \ d^4Y \left( \prod_{i=1}^{m} \Phi_i (x; (-)^{\bar{\mu}+1}y, \bar{y}) \right) \ast e^{i\mathcal{V}Z} \ast e^{-i\mathcal{V}Z} \ast \mathcal{V}(Z).$$

(3.45)

$$= \int d^4Z \ d^4Y \mathcal{V}(Z) \left( \prod_{i=1}^{m} \Phi_i (x; (-)^{\bar{\mu}+1}y, \bar{y}) \right) \ast e^{i\mathcal{V}Z} \ast e^{-i\mathcal{V}Z},$$

(3.46)

where the last line follows from the cyclicity of the trace. Indeed, it can be shown from (2.18), (3.34) and (3.35) that:

$$S(Z_4) \ast S(\mathcal{Y}_4) \ast e^{i\mathcal{V}Z} \ast e^{-i\mathcal{V}Z} \subseteq L^1(Z_4) \ast L^1(\mathcal{Y}_4) \ast e^{i\mathcal{V}Z} \ast e^{-i\mathcal{V}Z}$$

(3.47)

$$\subseteq L^1(\mathcal{Y}_4 \times Z_4) \ast e^{i\mathcal{V}Z} \ast e^{-i\mathcal{V}Z}$$

(3.48)

$$= L^1(\mathcal{Y}_4 \times Z_4).$$

(3.49)
Because of the following analogous of (3.22):

\[
\tilde{V}(M) := \int \frac{d^4Z}{(2\pi)^4} \mathcal{V}(Z) \exp(-iMZ),
\]

(3.50)

\[
\mathcal{V}(Z) = \int \frac{d^4M}{(2\pi)^4} \tilde{V}(M) \exp(iMZ).
\]

(3.51)

we have that

\[
\mathcal{A}_{\mu,t}^{\mathcal{V}}(\Phi_i) = \int d^4M \tilde{V}(M) \mathcal{A}_{\mu,t}(\Phi_i|\mathcal{M})
\]

(3.52)

\[= \tilde{V}_{t,\alpha(0),\alpha(0)} \mathcal{A}_{\mu,t}(\Phi_i).\]

(3.53)

This amounts to the regularisation scheme introduced in [37], where \(\tilde{V}(M)\) was called smearing function. The introduction of the (field-independent) smearing function does not spoil the gauge invariance. At this order in perturbation theory, its effect on the pre-amplitudes \(\mathcal{A}_{\mu,t}^{\mathcal{V}}(\Phi_i)\) is the appearance of four coupling constants, a particular case of the ones listed in table 1.

However, this does not mean that those constants are the only contributions of the regularising function that are observable. Indeed, let us show that the complete information about the function \(\tilde{V}(M)\) appear in the evaluation of the pre-amplitudes \(\mathcal{A}_{\mu,t,\alpha(k),\bar{\alpha}(\bar{k})}^{\mathcal{V}}(\Phi_i)\) obtained by applying the prescription explained below (3.20) to the most general observables (3.13), then regularising as in (3.52). Let us stress that even though those observables can be built from \(\mathcal{A}_{\mu,t}(\Phi_i|\mathcal{M})\) using (3.15), this expression (3.15) needs the \(M\) dependent (hence divergent) version of the pre-amplitudes \(\mathcal{A}_{\mu,t}(\Phi_i|\mathcal{M})\), and does not apply with the regularised one. This being said, the non-regularised version of \(\mathcal{A}_{\mu,t,\alpha(k),\bar{\alpha}(\bar{k})}^{\mathcal{V}}(\Phi_i)\) reads:

\[
\mathcal{A}_{\mu,t,\alpha(k),\bar{\alpha}(\bar{k})}(\Phi_i|\mathcal{M}) = \int d^4Z d^4Y \left( \prod_{i=1}^{m_0} \tilde{\Psi}_i \right)^{\bar{\alpha}(\bar{k})} (Z_\alpha)^K \left( \sum_{\mu,\bar{\mu}} A_{\mu} \right) (\Phi_i) |\mathcal{M}\rangle.
\]

(3.54)

From (3.44), we can extract the contributions where \((Z_\alpha)^K\) brings \(z_\alpha^K \neq \bar{z}_\alpha^K\):

\[
\mathcal{A}_{\mu,t,\alpha(k),\bar{\alpha}(\bar{k})}(\Phi_i|\mathcal{M}) = \left( (-i\partial_\alpha)^{K^2} (\bar{\mu}) \right)^{1-\varepsilon} (-\mu_\alpha)^{K^2} \left( (-i\partial_\alpha)^{K^2} (\bar{\mu}) \right)^{1-\varepsilon} (-\bar{\mu}_\alpha)^{K^2} A_{\mu,\bar{\mu}} (\Phi_i).
\]

(3.55)

Let us make precise that the absence of powers of \(\mu\) in the \(e = 1\) case is due to the symmetrization of the indices, yielding the commutation of \(\mu\) and \(\partial_\mu\), which in turn allows to put all \(\mu\) right in front of the delta function. The same argument rules out the presence of \(\bar{\mu}\) when \(t = 1\), \(\partial_\mu\), and \(\partial_{\bar{\mu}}\) when \(e = 0\) and \(\partial_{\bar{\mu}}\) when \(e = 1\). After various integrations by part, the regularised general pre-amplitudes read:
\( A^V_{n,\kappa\alpha(k),\alpha(k)}(\Phi_i) \sim \bar{V}_{\kappa\alpha(k),\alpha(k)} A_{n,t} (\Phi_i) , \) 

(3.56)

where, again, the prefactors are listed in table 1. Once again, the well-definedness of each of those observables relies on the properties of \( V \). All of them are finite since \( V \) is a rapidly decreasing function.

In the next section, we will be less general and see what happens when plugging a precise weak-field in the definition of the pre-amplitude. In that context, the question of the \( M \) dependence will be irrelevant and we will only be interested in computing \( A_{n,t} (\Phi_i) \) from the following equivalent of (3.27):

\[
A_{n,t} (\Phi_i) = \int \left( \prod_{j=1}^{n_0} \frac{d^4 \Lambda_j}{(2\pi)^2} \right) \left( \prod_{j=1}^{n_0} \bar{\Phi}(\Lambda_j) \right) F_{n,t} (\Lambda_i) ,
\]

(3.57)

where \( F_{n,t} (\Lambda_i) \) is defined by

\[
F_{n,t} (\Lambda_i | M) = \delta^2 (\mu) \delta^2 (\bar{\mu}) \delta^{1-t} F_{n,t} (\Lambda_i) .
\]

(3.58)

4. Correlators from zero-form charges

The purpose of this section is to use the bulk-to-boundary propagators of [15, 16] as weak fields inside the expression (3.57) for the pre-amplitudes, and fully evaluate the complete expression that results.

Then, in section 5 we will separately compute the \( n \)-point correlation functions of conserved currents of the free CFT\(_3\) corresponding to a set of free bosons. The latter model was conjectured in [8] to be dual to the type-A Vasiliev model (with a parity-even bulk scalar field) where the bulk scalar field obeys the Neumann boundary condition, sometimes called ‘irregular boundary condition’. We will show that both expressions, i.e. the pre-amplitude for Vasiliev’s equations on the one hand and the correlations functions of conserved currents in the free CFT\(_3\) on the other hand, exactly coincide. We stress that the pre-amplitude (3.57) only refers to the free Vasiliev equations and does not take into account the interactions incorporated into the fully nonlinear model.

After [8], Klebanov and Polyakov [9] further conjectured that the Vasiliev model where the scalar field is parity-even and obeys Dirichlet boundary condition should be dual to the critical \( U(N) \) model. As for the free \( U(N) \) model, we instead stick to the Neumann boundary condition for the bulk scalar field of the Vasiliev model, and use the boundary to bulk propagators with the conventions of [37] that we are now going to recall.

The metric of AdS is expressed in Poincaré coordinates, so that \( ds^2 = \frac{1}{r^2} \eta_{\mu\nu} dx^\mu dx^\nu \). One of the space-like coordinates in \( x^\mu \) is the radial variable \( r \) that vanishes on the boundary of AdS. The Minkowski metric components \( \eta_{\mu\nu} \) and the inverse \( g^{\mu\nu} \) are used to lower and raise world indices. We do not use the components of the complete metric \( ds^2 \), as one might expect in a geometric formulation. We characterise vectors that are tangent to the boundary by the vanishing of their \( r \)-component. In the same spirit, all spinors have a bulk notation, with dotted and undotted indices, and boundary Dirac spinors will be defined as the boundary value of bulk spinors submitted to an appropriate projection that we specify below. We use the four matrices \( \sigma^\mu \), three of which are the Pauli matrices, to link any vector \( u^\mu \) to a \( 2 \times 2 \) Hermitian matrix as \( v_{\alpha\beta} = u_{\beta\alpha} = u_\mu (\sigma^\mu)_{\alpha\beta} \). As before, we will omit most of the spinorial indices and do all contractions according to the NW-SE convention.
To every pair of points of AdS\(_3\) with respective coordinates \(x^\mu_i\) and \(x^\mu_j\), we can associate the following two sets of quantities:

\[
x^\mu_{ij} = x^\mu_i - x^\mu_j \quad \text{and} \quad \tilde{x}^\mu_{ij} = (x^\mu_i - x^\mu_j)^{-2} x^\mu_i.
\]

From now on, all the points will be taken on the boundary, except for one bulk point with coordinates \(x^\mu_0\) (in particular, its \(r\) coordinate will be denoted \(r_0\)). Of interest in this section will be the following \(2 \times 2\) matrices

\[
\Sigma_i := \sigma^r - 2 r_0 x^\mu_0,
\]

that are attached to every boundary point with coordinate \(x\). Some of the properties of the matrices \(\Sigma_i\) are collected in appendix B and will implicitly be used in the rest of this section. Among them is:

\[
\det \Sigma := \det (\Sigma_i - \Sigma_j) = \frac{4 r_0^2(x_{ij})^2}{(x_{ij})^2(x_{ij})^2).
\]

The propagator of the spin-\(s\) component of the master field \(\Phi\) from the chosen bulk point to a given boundary point of coordinates \(x^\mu_0\) was computed in [15]. The boundary conditions were chosen as follows: Neumann for the scalar field and Dirichlet for the spin-\(s > 0\) fields. For the case of the bosonic model, the bulk-to-boundary propagator of the master field \(\Phi\) was given in [37] by:

\[
K_{\epsilon}(\lambda_0, x_i, \chi_i|Y) := K_{\epsilon} e^{i \lambda_0 \Sigma_i} \sum_{\sigma_i = \pm 1} \left( e^{i \theta} e^{i \sigma_i \epsilon \mu \lambda_i} + e^{-i \theta} e^{i \sigma_i \epsilon \mu \lambda_i} \right),
\]

where \(\chi_i\) denotes the polarization spinor attached to the boundary point \(x_i\), and where

\[
K_{\epsilon} := (x_{ij})^{-2} r_0, \quad \nu_i := \sqrt{2 r_0} \Sigma_i x^\mu_0 \chi_i, \quad (\chi_i)^{\dagger} = \bar{\chi}_i = \sigma^r \chi_i, \quad (\nu_i)^{\dagger} = \bar{\nu}_i = -\Sigma_i \nu_i.
\]

The propagator \(K_{\epsilon}(x_0, x_i, \chi_i|Y)\) is an imaginary Gaussian in \(Y_i\). Hence, submitted to the usual \(i\epsilon\) prescription allowing the use of (4.20), it becomes a rapidly decreasing function, justifying the above procedure. Following the definition (3.22), the Fourier transform of \(K_{\epsilon}(x_0, x_i, \chi_i|Y)\) is given by

\[
K_{\tilde{\epsilon}}(\lambda_0, x_i, \chi_i|\Lambda_i) = K_{\tilde{\epsilon}} e^{i \lambda_0 \Sigma_i} \sum_{\sigma_i = \pm 1} \left( e^{i \theta} e^{i \sigma_i \epsilon \mu \lambda_i} + e^{-i \theta} e^{i \sigma_i \epsilon \mu \lambda_i} \right)
\]

\[
= K_{\tilde{\epsilon}} e^{i \lambda_0 \Sigma_i} \sum_{\epsilon_i \in \{0,1\}} \sum_{\sigma_i = \pm 1} e^{i \theta(1- \epsilon_i)} e^{i \sigma_i \epsilon \mu \lambda_i + (1- \epsilon_i) \sigma_i \epsilon \mu \lambda_i}.
\]

Let us emphasize that \(\tilde{K}_{\tilde{\epsilon}}\) satisfies the reality conditions (3.25) but not the minimal bosonic projection. We will take care of this projection separately at the end of this section.

Now we insert this propagator into (3.57), which yields:

\[
A_{m,t}(K_i) = \int \left( \prod_{j=1}^{m} \int \frac{d^4 \Lambda_j}{(2\pi)^4} \right) \left( \prod_{j=1}^{m} \tilde{K}_j(\Lambda_j) \right) F_{m,t} (\Lambda_i).
\]

Let us introduce the following notation:

\[
\sum_{\sigma, \epsilon} := \sum_{\epsilon \in \{0,1\}} \sum_{\sigma = \pm 1} \cdots \sum_{\epsilon_{m} \in \{0,1\}} \sum_{\sigma_{m} = \pm 1}.
\]
By making use of (3.37)–(3.39) and (3.58), we can write the expression (4.8) in the following way:

\[
A_{\alpha_{n_0}}(\mathcal{K}_i) = \sum_{\epsilon, \delta} e^{-i \bar{d} \Sigma (2 \epsilon, -1)} \int d^4 \Lambda e^{i \Lambda^T R' \Lambda + \bar{J}^T \Lambda} \left[ \delta^2 \left( \sum_{j=1}^{n_0} (-1)^j \lambda_j \right) \right]^{(1-e)} \left[ \delta^2 \left( \sum_{j=1}^{n_0} \bar{\lambda}_j \right) \right]^{(1-i)},
\]

where the pre-factor in the above expression is given by

\[
\alpha_{\alpha_{n_0}} := (2\pi)^{8 - 2n_0 - 2\epsilon - 2\delta} \left( \prod_{i=1}^{n_0} K_i \right).
\]

and the symbol \( \Lambda = (\lambda_i, \bar{\lambda}_i) \), where \( i \) and \( \bar{i} \) run from 1 to \( n_0 \), denotes a \( 4n_0 \)-dimensional column vector. Then the entries of the matrix \( R' \) and the source \( J' = (j'_i, \bar{j}'_{\bar{i}}) \) are given by

\[
R'_{ij} = (1 - \delta_{ij})(-1)^{i+j+\Theta(i,j)}, \quad R'_{ij} = (1 - \delta_{ij})(-1)^{\Theta(i,j)},
\]

\[
R'_{ij} = \delta_{ij} \Sigma_i, \quad j'_i = \epsilon_i \sigma_i \nu_{\bar{i}}, \quad \bar{j}'_{\bar{i}} = (1 - \epsilon_{\bar{i}}) \sigma_{\bar{i}} \nu_i.
\]

In this expression, \( \Theta(x, y) \) is a function whose value is 1 when \( x \) is greater than \( y \) and 0 otherwise. Since it always comes in expressions multiplied by \((1 - \delta_{ij})\), we do not need to specify the value of \( \Theta(x, y) \). In the case when \( e = 0 \) (resp. \( \bar{e} = 0 \)), in (4.10) we integrate out \( \lambda_{n_0} \) (resp. \( \bar{\lambda}_{n_0} \)) so that \( A_{\alpha_{n_0}}(\mathcal{K}_i) \) is given by

\[
A_{\alpha_{n_0}}(\mathcal{K}_i) = \sum_{\epsilon, \delta} e^{-i \bar{d} \Sigma (2 \epsilon, -1)} \int d^{2n_0 + 2\epsilon + 2\delta} \Lambda e^{i \Lambda^T R \Lambda + \bar{J}^T \Lambda},
\]

where the matrix \( R \) and the source \( J \) are given by

\[
R_{ij} = (1 - \delta_{ij})(-1)^{i+j+\Theta(i,j)}, \quad R_{ij} = (1 - \delta_{ij})(-1)^{\Theta(i,j)},
\]

\[
R_{ij} = \delta_{ij} \Sigma_i + \left( -1 - t \right) \delta_{ij} \nu_{\bar{i}} + \left( 1 - e \right) \delta_{ij} \nu_{\bar{i}} + \left( 1 - t \right) \left( 1 - e \right) \left( -1 \right) \Sigma_{n_0},
\]

\[
R_{ij} = \delta_{ij} \Sigma_i + \left( -1 - t \right) \delta_{ij} \nu_{\bar{i}} + \left( 1 - e \right) \delta_{ij} \nu_{\bar{i}} + \left( 1 - t \right) \left( 1 - e \right) \left( -1 \right) \Sigma_{n_0},
\]

\[
j_i = \epsilon_i \sigma_i \nu_i - \left( 1 - e \right) \left( -1 \right)^{i+n_0} \epsilon_{n_0} \sigma_{n_0} \nu_{n_0} \sigma_{n_0} \nu_{n_0}, \quad \bar{j}_i = \left( 1 - \epsilon_{\bar{i}} \right) \sigma_{\bar{i}} \nu_i - \left( 1 - t \right) \left( 1 - e \right) \sigma_{n_0} \nu_{n_0},
\]

where the indices \( i \) and \( \bar{i} \) labeling the entries of the matrix \( R \) and the column vector \( J \) now run over the following values:

\[
i \in \{1, \ldots, n = n_0 - (1 - e)\}, \quad \bar{i} \in \{1, \ldots, n = n_0 - (1 - t)\}.
\]

The Gaussian integration in (4.14) can be carried out via the formula

\[
\mathcal{G} := \int d^{2n_0 + 2\epsilon + 2\delta} \Lambda e^{i \Lambda^T R \Lambda + \bar{J}^T \Lambda} = \sqrt{\frac{2 \pi^2}{\text{det} R}} e^{i \Lambda^T R^{-1} \Lambda},
\]

10 We recall that each \( \lambda \) and \( \bar{\lambda} \) carries a Weyl spinor index.
11 It is possible to make a transformation \( \Lambda \rightarrow \Omega \Lambda \) such that \( \bar{R} := \Omega^T R \Omega \) is a real symmetric matrix and therefore can be diagonalised by an orthogonal matrix.
which differs from the usual gaussian integration formula by a change of sign due to the 
NW-SE convention. As we show in appendix C, the determinant and inverse of $R$ are given by

$$
\det R = 2^{4(n_0-1)} r_0^{n_0} \prod_{i=1}^{n_0} \frac{(x_{i,i+1})^2}{(x_{0,i})^2},
$$

(4.21)

$$
R_{ij}^{-1} = \sum_{\eta=\pm 1} \frac{1}{2 \det i,j+\eta} \left( -\eta \delta_{i,j} + \delta_{i+\eta,j} \xi_{i,j+\eta} \right) (\Sigma_i - \Sigma_{i+\eta}) \Sigma_{i+\eta},
$$

(4.22)

$$
R_{ij}^{-1} = \sum_{\eta=\pm 1} \frac{1}{2 \det i,j+\eta} \left( -\eta \delta_{i,j} - \delta_{i+\eta,j} \xi_{i,j+\eta} \right) (\Sigma_i - \Sigma_{i+\eta}),
$$

(4.23)

$$
R_{ij}^{-1} = \sum_{\eta=\pm 1} \frac{1}{2 \det i,j+\eta} \left( \delta_{i,j} + \eta \delta_{i+\eta,j} \xi_{i,j+\eta} \right) (\Sigma_i - \Sigma_{i+\eta}),
$$

(4.24)

$$
R_{ij}^{-1} = \sum_{\eta=\pm 1} \frac{1}{2 \det i,j+\eta} \left( \delta_{i,j} - \eta \delta_{i+\eta,j} \xi_{i,j+\eta} \right) (\Sigma_i - \Sigma_{i+\eta}),
$$

(4.25)

where it should be understood that the indices $j$ and $j + kn_0$ are identified with each other for
any integers $j$ and $k$, and where the coefficients $\xi_{i,j+\eta}$ are defined as follows:

$$
\xi_{i,j+\eta} = -\eta + t \delta_{i,j} (\eta + 1) + t \delta_{i+\eta,j} (\eta - 1).
$$

(4.26)

Using the above expressions for the inverse matrix $R^{-1}$ and the source $J$ into (4.20), we get

$$
G = \sqrt{\frac{(2\pi)^{2n+2\lambda_0} r_0^{n_0}}{2(4\pi)^{n_0} \prod_{i=1}^{n_0} \left( x_{i,i+1} \right)^4 \exp \left( -\frac{i}{4} \sum_{i=1}^{n_0} \mathcal{Q}_i \right) \mathcal{G}_p}},
$$

(4.27)

$$
\mathcal{G}_p = \exp \left( -\frac{i}{2} \sum_{i=1}^{n_0} (-)^{j_{i,i+1}} \chi_i \sigma_i \chi_{i+1} \left( 2 \xi_{i,i+1} - 1 \right) \mathcal{P}_{i,i+1} \right).
$$

(4.28)

Where the conformally-invariant variables are defined as [37]:

$$
\mathcal{P}_{i,i+1} = \chi_i \sigma^{\prime} \hat{x}_{i,i+1} \chi_{i+1}, \quad \mathcal{Q}_i = \chi_i \sigma^{\prime} \left( \hat{x}_{i,i+1} - \hat{x}_{i,i+1} \right) \chi_i.
$$

(4.29)

At this stage, as can be seen in (4.14), the next step is to sum over all values taken by $\sigma_1, \ldots, \sigma_{n_0}$. In order to do so, one can show the following two identities, holding for any $\mathcal{P}_{i,i+1}$:

$$
\sum_{\sigma_{i+1} \ldots \sigma_{n_0}} e^{\frac{i}{2} \sum_{i=1}^{n_0} \sigma_i \mathcal{P}_{i,i+1}} \equiv 2^{n_0-2} \left( \prod_{i=1}^{n_0-1} \cos \left( \frac{1}{2} \mathcal{P}_{i,i+1} \right) + r_0^{n_0-1} \sigma_1 \sigma_{n_0} \prod_{i=1}^{n_0-1} \sin \left( \frac{1}{2} \mathcal{P}_{i,i+1} \right) \right),
$$

(4.30)

$$
\sum_{\sigma_1 \ldots \sigma_{n_0}} e^{\frac{i}{2} \sum_{i=1}^{n_0} \sigma_i \mathcal{P}_{i,i+1}} \equiv 2^{n_0} \left( \prod_{i=1}^{n_0} \cos \left( \frac{1}{2} \mathcal{P}_{i,i+1} \right) + r_0^{n_0} \prod_{i=1}^{n_0} \sin \left( \frac{1}{2} \mathcal{P}_{i,i+1} \right) \right),
$$

(4.31)

where the first identity can be derived recursively on $n_0$ and the second one can be obtained from
the first relation by summing over $\sigma_1$ and $\sigma_{n_0}$. Replacing $\mathcal{P}_{i,i+1}$ by $(-1)^{\delta_{i,i+1}} \left( 2 \xi_{i,i+1} - 1 \right) \mathcal{P}_{i,i+1}$, one finds
\[
\sum_{\sigma_1, \ldots, \sigma_{n_0}} G_{\pi} = 2^{n_0} \prod_{i=1}^{n_0} \cos \left( \frac{1}{2} P_{i,j+1} \right) - (2i)^{m_0} (-1)^i \prod_{j=1}^{m_0} (1 - 2\varepsilon_j) \prod_{i=1}^{n_0} \sin \left( \frac{1}{2} P_{i,j+1} \right).
\]

At this stage, all we have to do is to sum over the 2 values (zero and one) taken by each of the variables \(\varepsilon_i, i = 1, \ldots, n_0\), or equivalently summing over the two values \(\pm 1\) taken by the \(n_0\) variables \((1 - 2\varepsilon_i), i = 1, \ldots, n_0\), so as to yield

\[
\sum_{\sigma, \varepsilon} e^{i\sigma \sum (1 - 2\varepsilon_i)} G_{\pi} = 2^{2n_0} \left( \cos \theta \right)^{n_0} \prod_{i=1}^{n_0} \cos \left( \frac{1}{2} P_{i,j+1} \right) - (-1)^i (\sin \theta)^{m_0} \prod_{i=1}^{m_0} \sin \left( \frac{1}{2} P_{i,j+1} \right).
\]

Gathering all the prefactors appearing in (4.11), (4.27) and (4.33), we finally obtain the following expression for the pre-amplitudes \(A_{\mu, \tau} (K_i)\)

\[
A_{\mu, \tau} (K_i) = \beta_{\mu, \tau} \exp \left( -\frac{i}{4} \sum_{i=1}^{m_0} Q_i \right) \left( \prod_{i=1}^{m_0} \frac{1}{|i_j|+1} \right) \times \left( \cos \theta \right)^{n_0} \prod_{i=1}^{n_0} \cos \left( \frac{1}{2} P_{i,j+1} \right) - (-1)^i (\sin \theta)^{m_0} \prod_{i=1}^{m_0} \sin \left( \frac{1}{2} P_{i,j+1} \right),
\]

(4.34)

\[
\beta_{\mu, \tau} := 4(i)^{2n_0 - 2 + \varepsilon + 1}(2\pi)^{2 + \varepsilon + 1} \prod_{j=1}^{m_0} \text{sgn} (\chi_{j+1}),
\]

(4.35)

where \(\text{sgn}(x)\) is the sign function. This expression is one of the central results of the paper. It reproduces, by restricting to the case where \(t = 0\) and up to constant coefficients, the expression obtained by combining the equations (6.19) and (6.20) of [39]. We generalise this result to the cases where the pre-amplitudes have extra insertions of \(\hat{\kappa} \kappa\), see (3.16), which corresponds to taking \(t = 1\). However, we see that at the leading order the extra insertion has no effect on the final result, except for a global sign in the B-model. The dependence on \(\theta\) was kept as a matter of convenience during the computation, but the result should be understood to hold only for the parity-invariant cases, i.e. for \(\theta = 0\) (type A model) and \(\theta = \pi/2\) (type B model). It would be interesting to understand how to modify the twisted open Wilson line in order to capture genuinely parity-breaking terms.

If one wants to restrict to the minimal bosonic model, one has to use bulk to boundary propagators \(K_{MB}^{\text{MB}}\) satisfying the minimal bosonic projection (3.26). Since it is not the case of the one defined in (4.7), we have to project it explicitly by defining the propagator for the minimal model as

\[
K_{MB}^{\text{MB}}(x_0, x_i, \chi_i | A_i) := \frac{1}{2} \sum_{\xi = 0, 1} (\pi A \tau A) \xi \hat{K}_i(x_0, x_i, \chi_i | A_i)
\]

(4.36)

\[
= \frac{1}{2} \sum_{\xi = 0, 1} \hat{K}_i(x_0, x_i, \chi_i | A_i)
\]

(4.37)

\[
= \frac{1}{2} \sum_{\xi = 0, 1} \tau_x \hat{K}_i(x_0, x_i, \chi_i | A_i).
\]

(4.38)
Then, the pre-amplitude for the minimal bosonic model is given in terms of the non-minimal one as

\[ A_{\text{MB}}^{\mu a} (K_i) = \left( \prod_{i=1}^{n_0} \frac{1}{2} \sum_{\xi_i=0,1} \phi_i^\xi \right) A_{\text{MB}} (K_i). \]  

(4.39)

We will now compute the correlation functions of conserved currents on the free CFT side, and show that, before performing Bose-symmetrisation, the result (see (5.29) below) exactly reproduces the formula (4.34).

5. Free U(N) and O(N) vector models

The purpose of this section is to compute cyclic building blocks for the amplitudes in the free U(N) vector model in a space-time of dimension \( d > 2 \), thereby proving explicitly a formula conjectured in [13], where the 3-point functions where computed. In the three-dimensional case, we find that they match the pre-amplitudes (4.34) defined in Vasiliev’s bosonic type-A model. Eventually we will show that the minimal bosonic projection of Vasiliev’s bosonic type-A case, we find that they match the pre-amplitudes (4.34) defined in Vasiliev’s bosonic type-A model.

As just stated, the computations of this section take place in a \( d \)-dimensional spacetime, whose world indices we will denote by Greek letters. This should not create any confusion with the other sections where we also use Greek letters but for base indices in \( d + 1 \) dimensions. Let \( 2 \) \( d \)-dimensional vectors \( a = a^\mu \partial_\mu \) and \( b = b^\mu \partial_\mu \). In this section we will use the notation \( a \cdot b := a^\mu b_\mu \) and \( a^2 := a^\mu a_\mu \).

The fields of the theory are complex Lorentz scalars \( \phi^i \) carrying an internal index \( i \). The theory is free and the propagators are given by

\[ \langle \phi^i (x) \phi^j (y) \rangle = 0 = \langle \phi^i_\ast (x) \phi^j_\ast (y) \rangle, \quad \langle \phi^i (x) \phi^j_\ast (y) \rangle = c_1 \frac{\delta^i_j}{|x-y|^{d-2}}, \]  

(5.1)

where \( |x| := \sqrt{x^2} \). This theory is known to be conformal. The conserved current of spin \( s \) is a traceless tensor \( J_{\mu(s)} \) containing \( s \) derivatives and a single trace in the sense of the internal algebra. Using a polarisation vector \( e^\mu \) and some weights \( \alpha_i \), one can gather all the conserved currents into a generating function, see e.g. [15, 56], [57] and references therein:

\[ \sum_{s=0}^{\infty} a_s J_{\mu(s)} (x) (e^\mu)^s = J (x, \epsilon) = \phi^\ast_\ast_\ast (x) f \left( \epsilon, \partial, \overrightarrow{\partial} \right) \phi^i (x). \]  

(5.2)

We assume that the function \( f \) is analytical. This section will involve sums over integer values, that will always be taken from zero to infinity upon identifying the inverse of diverging factorials with zero. Since the generating function \( J (x, \epsilon) \) is Lorentz invariant and the spin \( s \) current \( J_{\mu(s)} \) contains \( s \) derivatives, the function \( f(\epsilon, u, v) \) can be written as

\[ f(u, v, \epsilon) = \sum_{k,l,m,p,q} f_{k,l,m,p,q} (\epsilon \cdot u)^k (\epsilon \cdot v)^l (u \cdot v)^m (u^2 \epsilon^2)^p (v^2 \epsilon^2)^q. \]  

(5.3)

Once we are sure that all \( J_{\mu(s)} \) in (5.2) are traceless, the generating function will be left unchanged by transformations of the form \( (e^\mu)^s \rightarrow (e^\mu)^s + (\eta^\mu (2))^{\prime} (e^2)^i (e^\mu)^s 2 \epsilon \). We thus may use transformations of this type to effectively constraint the polarization vector to be null \( (\epsilon^2 = 0) \) without affecting the generating function of the currents, hence without affecting the generating functions of the correlation functions either. Thus, the only coefficients that we
need to know explicitly are \( f_{k,l,0,0} \). The tracelessness condition \( \partial^2 f = 0 \) gives several relations between the coefficients appearing in (5.3). Among those equations, we find that the \( m \) dependence of the coefficients \( f_{k,l,m,0} \) is given by

\[
(5.4)
\]

\[
f_{k,l,m+1,0,0} = -\frac{(k+1)(\ell+1)}{2(m+1)(k+\ell+m+1+\frac{d}{2})} f_{k+1,l,\ell+1,m,0,0}.
\]

Then, altogether with the conservation condition \( \partial_k \partial_s f|_{u^2=v^2=s^2=0} = 0 \), it gives the \( k \) dependence as

\[
(5.5)
\]

\[
f_{k+1,l,m,0,0} = -\frac{(\ell+1)(\ell+m+\frac{d}{2})}{(k+1)(k+m+\frac{d}{2})} f_{k,l,\ell+1,m,0,0}.
\]

Then, choosing \( b_{\ell} := f_{0,l,0,0,0} \), one can solve those two recursions and get the following expression for the on-shell part of the current:

\[
(5.6)
\]

\[
f_{k,l,m,0,0} = \frac{(-1)^k (k+\ell+2m)!}{2^m k! \ell! m!} \frac{\Gamma(k+\ell+m+\frac{d}{2}) \Gamma(\frac{d}{2})}{\Gamma(k+m+\frac{d}{2}) \Gamma(\ell+m+\frac{d}{2})} b_{k+\ell+2m}.
\]

After effectively removing \( \epsilon^2 \) from the generating function \( J(x, \epsilon) \), this amounts to rewrite (5.3) as

\[
(5.7)
\]

\[
f(u, v, \epsilon) = \sum_{s,k} b_s \left( \frac{s}{k} \right) \frac{\Gamma(s+\frac{d}{2}) \Gamma(\frac{d}{2})}{\Gamma(k+\frac{d}{2}) \Gamma(s+\frac{d}{2})} (\epsilon \cdot u)^k (\epsilon \cdot v)^{s-k}
\]

Now we choose \( b_s \) to be expressed in term of a constant \( \gamma \) (to be specified later) as

\[
(5.8)
\]

\[
\gamma = \frac{\gamma^s}{s! \Gamma(s+\frac{d}{2})}.
\]

We are interested in the connected correlation functions \( \langle J_1 \cdot J_\ell \rangle_{\text{conn}} \), which descend from the contribution of several Feynman diagrams. Since \( \partial^2 \) can only be contracted with \( \partial^\gamma \), those contributions only differ by permutations of the currents. As we will discuss later, this is to contrast with the real field theory, where each current has two possible contraction with the next one. We are interested in the cyclic building block for the correlation function, that is to say the first Wick contraction, that we define with the following normalisation:

\[
(5.9)
\]

\[
\langle J_1, \ldots, J_\ell \rangle_{\text{cyclic}} := \frac{1}{N} \prod_{i=1}^{n_0} \int f(\partial_{x_i'}, \partial_{x_i}, \epsilon_i) \prod_{j=1}^{n_0} \langle \phi^b(x_j) \phi^i_{x_{j+1}}(x'_{j+1}) \rangle |_{x'_j = x_j} \quad (5.10)
\]

\[
= \prod_{i=1}^{n_0} \int f(\partial_{x_i'}, \partial_{x_i}, \epsilon_i) \prod_{j=1}^{n_0} \left| x_j - x'_{j+1} \right|^{2-d} \quad (5.11)
\]

\[
= \prod_{i=1}^{n_0} \int f(\partial_{x_i}, \partial_{x_i+1}, \epsilon_i) \prod_{j=1}^{n_0} \left| x_{j+1} \right|^{2-d}.
\]

The \( \frac{1}{\Gamma} \) factor has disappeared in the second line because of the internal traces. We define \( x_{ij} \) as in the previous section (now it is manifestly a 3-vector). The rest of this section involves multiple sums that generally will be written as follows:
This being said, we inject (5.7) and (5.8) into (5.11) and get

\[
(J_1, \ldots, J_n)_{\text{cyclic}} = \sum_{i=1}^{n} \prod_{j=1}^{n} \gamma_{\epsilon_i} \Gamma\left(\frac{d+2}{2}\right) \frac{\gamma_{\epsilon_{i+1}} \Gamma\left(\frac{d+2}{2}\right)}{(s_i - k_i)! \Gamma\left(s_i - k_i + \frac{d+2}{2}\right)} \left(\epsilon_i \cdot \partial_{i \rightarrow i+1}\right)^{\gamma_{\epsilon_i} - k_i} \prod_{j=1}^{n} |x_{j+1}|^{2-d}.
\]

(5.12)

Additionally to some index redefinition and reorganization of the product, we made use of the following lemma for 2 null vectors \(\epsilon_i\) and \(\epsilon_{i+1}\):

\[
(\epsilon_i \cdot \partial_{\epsilon_i})^p (\epsilon_{i+1} \cdot \partial_{\epsilon_{i+1}})^q (x^2)^{-\frac{d+2}{2}} = \sum_q (-2)^{n+q} \left(\begin{array}{c} n \\ q \end{array}\right) \frac{\Gamma(n + q + \frac{d+2}{2})}{\Gamma\left(\frac{d+2}{2}\right) q!} (\epsilon_i \cdot \epsilon_{i+1})^p (\epsilon_i \cdot \epsilon_{i+1})^q (x^2)^{-\frac{d+2}{2}}.
\]

(5.13)

The \(p = 0\) version can be shown recursively, then the full one comes from a direct application of Leibniz rule. We will then need

\[
k_d(t, q, m) := \frac{\Gamma(t + q + m + \frac{d+2}{2})}{t! q! m! \Gamma(t + m + \frac{d+2}{2}) \Gamma(q + m + \frac{d+2}{2})} \frac{1}{r!}.
\]

(5.14)

The last equality is straightforward to show when \(t = 0\). In the other cases, it is proven via the recursion:

\[
k_d(t + 1, q, m) = \frac{1}{t+1} \left(k_d(t, q, m) + k_d(t, q - 1, m + 1)\right).
\]

(5.15)
This allows one to rewrite (5.14) as

\[
\langle J_1, \ldots, J_n \rangle_{\text{cyclic}} = \sum_{t,q,m,r} m \prod_{i,q,m,r} (-2\gamma)^{c_i+q_i+2m_i} (t_i-r_i)!(q_i-r_i)! m_i! r_i! \Gamma(r_i + m_i + \frac{d-2}{2}) \times \left(-\frac{1}{2} (e_i \cdot e_{i+1}) x_{i,i+1}^2\right)^{m_i} (e_i \cdot \hat{x}_{i,i+1})^2 (e_{i+1} \cdot \hat{x}_{i,i+1})^q_i |x_{i,i+1}|^{2-d}
\]

\[
= \sum_{a,b,c,r} \prod_{i=1}^n a_i! b_i! c_i! r_i! \Gamma(c_i + \frac{d-2}{2}) \times \left(-\frac{1}{2} (e_i \cdot e_{i+1}) x_{i,i+1}^2\right)^{c_i} (e_i \cdot \hat{x}_{i,i+1})^2 (e_{i+1} \cdot \hat{x}_{i,i+1})^q_i |x_{i,i+1}|^{2-d}
\]

\[
= \sum_{c} \prod_{i=1}^n \frac{1}{c_i! \Gamma(c_i + \frac{d-2}{2})} \exp (-2\gamma (e_i + e_{i+1}) \cdot \hat{x}_{i,i+1}) \times \left(-\frac{1}{2} (e_i \cdot e_{i+1}) x_{i,i+1}^2\right)^{c_i} (e_i \cdot \hat{x}_{i,i+1})^2 (e_{i+1} \cdot \hat{x}_{i,i+1})^q_i |x_{i,i+1}|^{2-d}
\]

Then, as usual one defines the conformal structures as

\[
Q_i = 2e_i \cdot (\hat{x}_{i-1,i} + \hat{x}_{i,i+1}), \quad P_{i,i+1}^2 = 4 \left( (e_i \cdot \hat{x}_{i,i+1}) (e_{i+1} \cdot \hat{x}_{i,i+1}) - \frac{1}{2} ((e_i \cdot e_{i+1}) x_{i,i+1}^2) \right).
\]

In this language, we can write the final expression for the n-point conserved-current correlation functions of the $d$-dimensional U(N) free vector model as

\[
\langle J_1, \ldots, J_n \rangle_{\text{cyclic}} = \prod_{i=1}^n \exp (-\gamma Q_i) \sum_{c_i} \frac{1}{c_i! \Gamma(c_i + \frac{d-2}{2})} (\gamma P_{i,i+1})^2 c_i |x_{i,i+1}|^{2-d}.
\]

Before specifying the dimension in order to compare with the result in the 4-dimensional bulk, let us show that this is consistent with the result conjectured in [13]. We start by rewriting (5.22) as a series expansion:

\[
\langle J_1, \ldots, J_n \rangle_{\text{cyclic}} = \sum_{c,d} \prod_{i=1}^n \frac{1}{d_i! c_i! \Gamma(c_i + \frac{d-2}{2})} (-\gamma Q_i)^{d_i} (\gamma P_{i,i+1})^{2c_i} |x_{i,i+1}|^{2-d}
\]

\[
= \sum_{c,d} \prod_{i=1}^n \frac{1}{d_i! c_i! \Gamma(c_i + \frac{d-2}{2})} (-\gamma Q_i)^{d_i-c_i} (\gamma P_{i,i+1})^{2c_i} |x_{i,i+1}|^{2-d}
\]

\[
= \sum_{c,d} \prod_{i=1}^n (-\gamma)^{d_i} (Q_i)^{d_i-c_i} (P_{i,i+1})^{2c_i} |x_{i,i+1}|^{2-d}
\]

\[
= \sum_{c,d} \prod_{i=1}^n (-\gamma)^{d_i} (Q_i)^{d_i-c_i} (P_{i,i+1})^{2c_i} |x_{i,i+1}|^{2-d}
\]
\[
= \sum_{i=1}^{n_0} \left( -\gamma \right)_i \sum_{c_i} \frac{(-)^{c_i}}{2^{|c_i|} c_i!} \left( -4P_{i,j+1} \partial Q_i \partial Q_{i+1} \right)^{c_i} \left( Q_i \right)^{n_0} |x_{i,j+1}|^{2-d} \tag{5.26}
\]

\[
= \sum_{i=1}^{n_0} \left( -\gamma \right)_i \frac{2^{d+1}}{2^{d+1}-1} \left( q_i \right)^{d} \left( - \right)^{d} J_{2d+1} (\sqrt{q_i}) \bigg|_{q_i = -4P_{i,j+1} \partial Q_i \partial Q_{i+1}} \left( Q_i \right)^{n_0} |x_{i,j+1}|^{2-d} , \tag{5.27}
\]

where \( J_\alpha (x) \) is the Bessel function of first kind. It is now clear that the Bose symmetrisation of this result is the same as in [13] up to a function of \( s_i \) appearing in front of the current \( J_\alpha \). Let us stress that this information is encoded in the weight \( N_\alpha \) of the conformal structures \( Q_\alpha \)-point function, provided one can link the two definitions up to global normalisation of the current \( J_\alpha \). Hence the freedom to fix it is not spoiled by the previous fixation (5.8) of \( b_\alpha \propto a_i, N_\alpha \).

Now let us go back to our three-dimensional holographic purpose. In this setup, one can use the following consequence of the duplication formula for Gamma functions:

\[
\Gamma (x + \frac{1}{2}) = \frac{\sqrt{\pi} (2x)!}{2^{2x} x!} , \tag{5.28}
\]

and rewrite the result (5.22) as:

\[
\langle J_\alpha, ..., J_n \rangle_{\text{cyclic}} = \prod_{i=1}^{n_0} \frac{1}{\sqrt{\pi}} \exp (-\gamma Q_\alpha) \cos (2i\gamma P_{i,j+1}) |x_{i,j+1}|^{2-d} . \tag{5.29}
\]

We already see that if we choose \( \gamma = \frac{1}{2} \), we recover the formula (4.34) for the type A-model, up to global normalisation of the \( n \)-point function, provided one can link the two definitions of the conformal structures \( Q_\alpha \) and \( P_{i,j} \). This is done by defining the polarization vector \( \epsilon_i \) in terms of the polarization spinor \( \chi_i \) of the previous section as follows:

\[
(\chi_i)_{\alpha} (\bar{\chi}_i)_{\dot{\alpha}} = \epsilon_i^\mu (\sigma_\mu)_{\alpha \dot{\alpha}} . \tag{5.30}
\]

It is formally a null 4-vector but from (4.5) we see that its \( r \)-component vanishes identically, making it a null vector tangent to the boundary, as expected.

The above translation to the language of polarisation spinors shows that the result is the same as the one given in [43]. We expect this to be the case in any dimension where the spinorial language exists (e.g. \( d = 4 \)), though the verification requires a dictionary between the generalised space-time of [43] and the standard space-time where the CFT lives.

Now we want to study the O(N)-vector model. The field is now a real scalar, with a propagator given by

\[
\langle \phi(x) \phi(y) \rangle = c_2 \frac{\delta^{ij}}{|x-y|^{d-2}} . \tag{5.31}
\]

Since in this case there are twice more contractions to consider, \( f(u,v,\epsilon) \) will be effectively projected on its part that is invariant under the exchange of \( u \) and \( v \). From (5.7) it is clear that this is equivalent to select even powers of \( \epsilon \). Thus the relevant projection is

\[
f^{O(N)} (u,v,\epsilon) = \frac{1}{2} \left( f(u,v,\epsilon) + f(u,v,-\epsilon) \right) . \tag{5.32}
\]

From (5.30), this is exactly equivalent to the minimal bosonic projection (4.37) of the propagator.
6. Conclusions

In this paper we have examined a set of higher spin invariants in Vasiliev’s theory, to the leading order in the perturbative expansion for curvatures, generated by boundary-to-bulk propagators in the polarization spinor basis. These invariants are given by straight, twisted open Wilson lines in the noncommutative twistor Z space, with the most general adjoint insertions at one end. As in noncommutative Yang–Mills theory, one can argue that these quantities form a complete set of observables in higher spin gravity on a topologically trivial X space. In the leading order, the observables are given by free (boson and fermion) CFT boundary correlation functions in the polarization spinor basis, given by bounded (trigonometric) functions times inverse powers of relative positions, integrable in three dimensions.

It would be interesting to extend our analysis to Vasiliev’s D-dimensional Type A model [58], for which one can define analogs of all the objects that we have studied in the four-dimensional case, and ask whether the decorated twisted open Wilson lines, now in the $2(D+1)$-dimensional phase space of the underlying conformal particle, reproduces the n-point correlation functions in $d = D − 1$ dimensions given in (5.22).

It would also be interesting to examine the twisted open Wilson lines at subleading orders in the curvature expansion of the zero-form charges. As the twisted, noncommutative open Wilson lines are invariant under the full higher spin gauge transformations that contain sub-leading corrections as well, one may speculate that these structures contain information about deviations of the conformal field theory dual beyond the free point. Another related issue concerns the fact that the zero-form charges can be evaluated on exact solutions to Vasiliev’s theory, which one may expect correspond to states of the conformal field theory generated by deformations using finite sources. Indeed, as the exact solutions are given using polarization spinor bases, it would be interesting to trace more carefully the observed finiteness of the zero-form charges [29, 38] to the mildly divergent nature of the correlation functions in the polarization spinor spaces.

One might go further and speculate that the free energy functional itself of Vasiliev’s theory, i.e. its on-shell action, contains similar sub-leading corrections related to non-linear sources coupling to higher-trace operators, which would thus show up as contact terms in the holographic correlation functions. Such a refinement would imply that the holographically dual theory is actually a non-trivial three-dimensional field theory already in the case of the Type A Vasiliev model. We hope to return to these issues in the near future.

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Appendix A. Twisted, straight noncommutative Wilson lines

The purpose of this appendix is to prove the central relation between observables in Vasiliev’s theory and Wilson lines in noncommutative Yang–Mills theory:

$$\exp_s\left(i\hat{M}\hat{S}\right) = W_{\ell\omega}(x, Z; Y) \ast \exp_s\left(iMZ\right).$$

(A.1)

In this section we use the convention that repeated (uncontracted) indices are totally symmetrized, e.g. $Z_{\omega}^2$ denotes $\frac{1}{2} \left( Z_{\omega}^2 \ast Z_{\omega}^2 + Z_{\omega}^2 \ast Z_{\omega}^2 \right)$.

Addressing the left hand side of (A.1) requires the use the following property of the $\ast$-product:

$$\left[Z_{\omega}, \hat{j}\right]_\ast = -2i\partial Z_{\omega} \hat{j}. \quad \text{(A.2)}$$

Its successive application can be shown recursively to yield:

$$(Z_{\omega})^m \ast \hat{A}_{\omega} = \sum_{j=0}^{m} \binom{m}{j} (-2i\partial Z_{\omega})^j \hat{A}_{\omega} \ast (Z_{\omega})^{m-j}. \quad \text{(A.3)}$$

Then we define the symbols $C(m, n, k_1, ..., k_n)$ entering the expansion of a monomial in $\hat{S}_{\omega}$ as

$$\left(Z_{\omega} - 2i\hat{A}_{\omega}\right)^m = \sum_{n=0}^{m} \sum_{k_1=0}^{m-n} \sum_{k_2=0}^{m-n-k_1} \cdots (-2i)^{n+\Sigma k} C(m, n, k_1, ..., k_n) \left(\star \left(\partial Z_{\omega}\right)^k \hat{A}_{\omega}\right) \ast (Z_{\omega})^{(m-n-\Sigma k)}, \quad \text{(A.4)}$$

where $\Sigma k$ denotes $\sum_{i=1}^{n} k_i$ and we recall that, for any set of $n$ functions $\left\{ \hat{f}_1, ..., \hat{f}_n \right\}$, the symbol $\star_{\omega} \hat{f}_1 \ast ... \ast \hat{f}_n$ is defined as $\hat{f}_1 \ast ... \ast \hat{f}_n$ in that order. The consistency of this expansion with (A.3) requires the boundary conditions:

$$C(m, 0) = C(m, m, 0, ..., 0) = 1, \quad \text{(A.5)}$$

as well as the recurrence relations:

$$C(m + 1, n, k_1, ..., k_n) = C(m, n, k_1, ..., k_n) + \binom{m + 1 - n - \sum_{i=1}^{n-1} k_i}{k_n} C(m, n - 1, k_1, ..., k_{n-1}). \quad \text{(A.6)}$$

This can be seen by plugging (A.3) into $\hat{S}_{\omega}^{(m+1)} = \hat{S}_{\omega}^m \ast \hat{S}_{\omega}$. Then defining $q := m - n - \Sigma k$ allows one to reformulate the left hand side of (A.1) as:

$$\exp_s\left(i\hat{M}\hat{S}\right) = \sum_{n=0}^{\infty} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{q=0}^{\infty} (2M_{\omega}^2)^n \Sigma_k (iM_{\omega}^2)^q \times \frac{C(q + n + \sum k, n, k_1, ..., k_n)}{(q + n + \sum k)!} \left(\star \left(\partial Z_{\omega}\right)^k \hat{A}_{\omega}\right) \ast (Z_{\omega})^q. \quad \text{(A.7)}$$
On the other hand, to address the r.h.s. of (A.1) we recall the definition of the path-ordered exponential:

\[
W_{L^2M}(x, Z; Y) = P \exp \left( (2M^2) \int_0^1 d\sigma \, \hat{A}_\omega (x, Z + 2\sigma M; Y) \right)
\]

Then we can write the Taylor expansion of the higher-spin connection and get:

\[
W_{L^2M}(x, Z; Y) = \sum_{n=0}^{\infty} \sum_{k_1=0}^{\infty} \ldots \sum_{k_n=0}^{\infty} (2M^2)^{n+\sum k} \int_0^1 d\sigma_n \, \frac{\sigma_n^{k_n}}{k_n!} \ldots \int_0^{\sigma_2} d\sigma_1 \, \frac{\sigma_1^{k_1}}{k_1!} \left( \begin{array}{c} n \\ \sum_{j=1}^{n} \end{array} \right) \left( \hat{A}_\omega (x, Z; Y) \right) \left( \hat{A}_\omega (x, Z; Y) \right) \ldots \left( \hat{A}_\omega (x, Z; Y) \right),
\]

(A.8)

Where the coefficient is given by

\[
F(\sigma \mid n, k_1, \ldots, k_n) = \int_0^\sigma d\sigma_n \, \frac{\sigma_n^{k_n}}{k_n!} \ldots \int_0^{\sigma_2} d\sigma_1 \, \frac{\sigma_1^{k_1}}{k_1!} \prod_{j=1}^n \frac{1}{k_j! (j + \sum_{j=1}^n k_j)}.
\]

(A.10)

The latter form can be shown recursively. Upon expanding:

\[
\exp \left( iMZ \right) = \sum_{q=0}^{\infty} \frac{(iMZ)^q}{q!},
\]

(A.11)

we see that (A.1) holds if and only if:

\[
C(q + n + \sum k, n, k_1, \ldots, k_n) = \frac{(q + n + \sum k)!}{q!} F(1 \mid n, k_1, \ldots, k_n)
\]

\[
= \frac{(q + n + \sum k)!}{q!} \prod_{j=1}^n \frac{1}{k_j! (j + \sum_{j=1}^n k_j)}.
\]

(A.12)

It is the case, since taking this as a definition for \( C(m, n, k_1, \ldots, k_n) \) is consistent with (A.5) and (A.6). Therefore, we have proven (A.1).

Appendix B. Spinor notation and properties of the \( \Sigma_i \) matrices

The metric of \( \text{AdS}_4 \) is expressed in Poincaré coordinates, so that \( ds^2 = \frac{1}{r^2} \eta_{\mu\nu} dx^\mu dx^\nu \). The Minkowski metric \( \eta_{\mu\nu} \) is the one which one uses in this context in order to raise and lower world indices, as well as for computing norms of Lorentz vectors.

The Pauli matrices plus the identity \( (\sigma^\mu)_{\alpha\beta} \) form a basis for Hermitian \( 2 \times 2 \) matrices. As usual, their barred counterpart are given by:
\[
\langle \sigma^\mu i \sigma^\alpha \rangle := \epsilon^{i\gamma} \epsilon^{\alpha\gamma} (\sigma^\mu)_{\gamma\gamma} \equiv (\sigma^\mu)^{\alpha\beta}.
\]

(B.1)

The main property of these matrices is

\[
(\sigma^\mu)^{\alpha\gamma} (\sigma^\nu)_{\gamma\beta} + (\sigma^\nu)^{\alpha\gamma} (\sigma^\mu)_{\gamma\beta} = 2\eta^\mu\nu \epsilon^{\alpha\beta},
\]

(B.2)

\[
(\sigma^\mu)^{i\gamma} (\sigma^\nu)_{\gamma\beta} + (\sigma^\nu)^{i\gamma} (\sigma^\mu)_{\gamma\beta} = 2\eta^\mu\nu \epsilon^{i\beta}.
\]

(B.3)

It means that they generate a Clifford algebra. Now we omit all spinorial indices, it being understood that we do all contractions following the NW-SE conventions.

A Lorentz vector \(A_\mu\) is related to two Hermitian 2 \(\times\) 2 matrices in the following way:

\[
A := A^\mu \sigma_\mu, \quad \bar{A} := A^\mu \bar{\sigma}_\mu.
\]

(B.4)

Then for 2 vector fields with \(A_\mu(x)\) and \(B_\mu(x)\) as respective component, the relations (B.2) and (B.3) become

\[
A\bar{B} + B\bar{A} = 2A^\mu B_\mu, \quad \bar{A}B + BA = 2A^\mu B_\mu.
\]

(B.5)

In particular:

\[
\det A = A\bar{A} = A^\mu A_\mu.
\]

(B.6)

Another useful algebraic property is:

\[
A(a\bar{A} + b\bar{B}) = a\det(A)\bar{B} + b\det(B)A = B(a\bar{A} + b\bar{B})A, \quad \forall a, b \in \mathbb{C}.
\]

(B.7)

Those few results are used implicitly all along the paper and are valid for any Hermitian matrices, in particular for those introduced in (B.8) and (B.11).

Now we consider a bulk point with coordinate \(x_0^i\) (among which the \(r\) component is denoted \(r_0\)) and various boundary points having coordinates \(x_i^\mu, i = 1, \ldots, n_0\). Their \(r\) component vanishes for every boundary points. We then define the matrices

\[
\Sigma_i := \sigma^\mu - 2r_0 \bar{x}_{0i} \equiv (\gamma^{\mu\nu} - 2r_0 \bar{x}_{0i}^\mu)\sigma_\mu, \quad i = 1, \ldots, n_0.
\]

(B.8)

They have unit determinant and enjoy the following useful property:

\[
\det \Sigma_i := \det (\Sigma_i - \Sigma_j) = \frac{4r_0^2 (x_{ij})^2}{(x_{0i})^2(x_{0j})^2}.
\]

(B.9)

Among the applications of (B.7), the following one is often used in our computations:

\[
(\Sigma_i - \Sigma_j) (\Sigma_k - \Sigma_i) = -\det \Sigma_i (\Sigma_k - \Sigma_j) - \det \Sigma_k (\Sigma_i - \Sigma_j) + \det \Sigma_{ij} (\Sigma_k - \Sigma_j).
\]

(B.10)

One matrix inversion given in the next appendix C uses some quantities that we now define. Let us suppose that we have \((n + 1)\) boundary points corresponding to the Hermitian matrices \(\Sigma_0, \Sigma_1, \ldots, \Sigma_n\). Note that, although they are all defined in terms of the bulk point \(x_0\), see (B.8), the matrix \(\Sigma_0\) refers to a boundary point that we also call \(x_0\). This abuse of notation should not create confusions, hopefully. In other words, the boundary point corresponding to \(\Sigma_0\) has nothing to do with the reference bulk point. Then we define:

\[
S_n := -\sum_{j=0}^{n} (-1)^j\Sigma_j, \quad S_n^{(i)} := \sum_{j=1}^{n} (1 - \delta_{ij})(-1)^{j+\Theta(i,j)}\Sigma_j.
\]

(B.11)
Among the properties of the matrices $S_n$ and $S_n^{(i)}$ used in appendix C, there are various applications of (B.7) as well as the following relations:

$$S_n^{(i)} = S_n^{(i+1)} = \pm (-1)^i (\Sigma_i - \Sigma_{i+1}), \quad S_n + S_n^{(1)} = \Sigma_1 - \Sigma_0, \quad S_n + S_n^{(n)} = \Sigma_n - \Sigma_0,$$

where the first relation is understood to hold only when both $i$ and $i \pm 1$ take value in $\{1, ..., n\}$.

**Appendix C. Inverse and determinant of the matrix $R$**

The results (4.21) to (4.25) come from various block inversions, and the procedure is not exactly the same when $e$ and $t$ take different values.

First, let us set $e = 1$. In the two relevant cases, we decompose $R$ into the following blocks:

$$R = \begin{pmatrix} A_{ij} & B_{ij} \\ C_{ij} & D_{ij} \end{pmatrix} = \begin{pmatrix} (1 - \delta_{ij})(-)^{i+j+\Theta(i,j)} & \left(\delta_{ij} - (1 - t)\delta_{j0}\right) \Sigma_i \\ (\delta_{ij} - (1 - t)\delta_{j0}) \Sigma_j & (1 - \delta_{ij})(-)^{\Theta(i,j)} \end{pmatrix}. \quad (C.1)$$

Let us remind that in those cases, the indices takes the following values:

$$i \in \{1, ..., n_0\}, \quad \bar{i} \in \{1, ..., \bar{n} = n_0 - (1 - t)\}. \quad (C.2)$$

The keypoint is that here $\bar{n}$ is always even, which provides $D$ with the following inverse matrix:

$$D^{-1}_{ij} = (1 - \delta_{ij})(-)^{i+j+\Theta(i,j)}. \quad (C.3)$$

Hence $M$ can be decomposed as:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mathbb{I} & B \\ 0 & D \end{pmatrix} \begin{pmatrix} \tilde{A} & 0 \\ D^{-1}C & \mathbb{I} \end{pmatrix}, \quad \text{with} \quad \tilde{A} := A - BD^{-1}C. \quad (C.4)$$

This allows to write its determinant and inverse matrix as:

$$\det R = \det(\tilde{A}) \det(D), \quad (C.5)$$

$$R^{-1} = \begin{pmatrix} \tilde{A}^{-1} & 0 \\ -D^{-1}C\tilde{A}^{-1} & \mathbb{I} \end{pmatrix} \begin{pmatrix} \mathbb{I} & -BD^{-1} \\ 0 & D^{-1} \end{pmatrix} = \begin{pmatrix} \tilde{A}^{-1} & -\tilde{A}^{-1}BD^{-1} \\ -D^{-1}C\tilde{A}^{-1} & D^{-1} + D^{-1}C\tilde{A}^{-1}BD^{-1} \end{pmatrix}. \quad (C.6)$$

Let us point out the fact that since barred and unbarred indices do not run the same range, the $\delta$ symbols are summed as follows:

$$\sum_{i=1}^{n_0} \delta_{ij} f(j) = (1 - (1 - t)\delta_{j0})f(i). \quad (C.7)$$

Knowing that fact, one can easily compute:

$$\tilde{A}_{ij} = (1 - \delta_{ij})(-)^{i+j+\Theta(i,j)}\Sigma_i(\Sigma_i - \Sigma_j) = \sum_{k=1}^{n_0} (\delta_{ik}\Sigma_i) \left( (1 - \delta_{kj})(-)^{k+j+\Theta(k,j)}(\Sigma_k - \Sigma_j) \right). \quad (C.8)$$

The purpose of this factorization is that the first block diagonal piece is straightforward to invert and of unit determinant. For the second piece (which we write $X^{(n_0)}$), we proceed to a
recursive block inversion, which means computing its inverse and determinant by decomposing it as:

\[ X^{(m)} = \begin{pmatrix} D_g & C_i \\ B_g & A_j \end{pmatrix} = \begin{pmatrix} (X^{(m-1)})_{g} & (X^{(m)})_{n_0} \\ (X^{(m)})_{m_0} & (X^{(m)})_{m_0n_0} \end{pmatrix} . \]  

(C.9)

As a recursion hypothesis, we postulate the determinant and inverse of \( X^{(m)} \) to be:

\[ \det(X^{(m)}) = 2^{2(n-2)} \prod_{i=1}^{m_0} \det \cdot i+1 , \]  

(C.10)

\[ \left(X^{(m)} \right)^{-1}_{g} = \sum_{\eta = \pm 1} \frac{1}{2 \det \cdot i+\eta \left( -\delta_{i+\eta} + \delta_{i+\eta} \xi_{i+\eta} \right) (\Sigma_i - \Sigma_{i+\eta})} , \]  

(C.11)

where we recall:

\[ \xi_{i+\eta} = -\eta + t \delta_{i,n_0} (\eta + 1) + t \delta_{i+\eta,n_0} (\eta - 1) , \]  

(C.12)

\[ \det \cdot i := \det(\Sigma_i - \Sigma_j) = \frac{4 \kappa_i^2 (\kappa_{ij})^2}{(\kappa_{0i})^2 (\kappa_{0j})^2} . \]  

(C.13)

When proceeding to the recursive step, it is important to note that 0 is identified with \((n_0 - 1)\) in the expression of \( \left(X^{(m-1)}\right)^{-1}_{g} \), implying \((1) - 1 = (n_0 - 1)\) and \((n_0 - 1) + 1 = 1\). Knowing that, one finds:

\[ \tilde{\lambda}' = \frac{2}{\det_{n_0 - 1,n_0}} (\Sigma_{n_0 - 1} - \Sigma_0) (\Sigma_{n_0 - 1} - \Sigma_1) (\Sigma_{n_0} - \Sigma_1) . \]  

(C.14)

Then the algebra of \( \Sigma_i \) matrices (see appendix B) gives its inverse and determinant as:

\[ \det \tilde{\lambda}' = \frac{4 \det_{n_0 - 1,n_0} \det_{1,n_0}}{\det_{n_0 - 1,n_0}} , \]  

(C.15)

\[ (\tilde{\lambda}')^{-1} = \frac{1}{2 \det_{1,n_0} \det_{n_0 - 1}} (\Sigma_{n_0} - \Sigma_1) (\Sigma_{n_0 - 1} - \Sigma_0) (\Sigma_{n_0 - 1} - \Sigma_1) . \]  

(C.16)

Then the straightforward application of (C.5) and of (C.6) allows first to confirm (C.10) and (C.11), then to find equations (4.21) to (4.25).

In the cases where \( t = 1 \), we use the following block decomposition:

\[ R = \begin{pmatrix} D_g & C_i \\ B_g & A_j \end{pmatrix} = \begin{pmatrix} (1 - \delta_{i,j})(-1)^{i+j} + \Theta(i,j) \\ (\delta_{i,j} - (1 - e)(-1)^{i+n_0}\delta_{i,n_0}) \end{pmatrix} \begin{pmatrix} (\delta_{i,j} - (1 - e)(-1)^{i+n_0}\delta_{i,n_0}) & (1 - \delta_{i,j})(-1)^{\Theta(i,j)} \\ (\delta_{i,j} - (1 - e)(-1)^{i+n_0}\delta_{i,n_0}) & (1 - \delta_{i,j})(-1)^{\Theta(i,j)} \end{pmatrix} . \]  

(C.17)

Here \( n = n_0 - 1 - e \) is the one to be always even, and the same exact method can be applied to this cases.

In the last case, the one where \( e = t = 0 \), both barred and unbarred indices run an odd number of values. Therefore, we take a different approach and start from the following block decomposition:

\[ R = \begin{pmatrix} B_g & A_j \\ D_g & C_i \end{pmatrix} = \begin{pmatrix} A_{ij} & B_{ij} \\ C_{ij} & D_{ij} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \]  

(C.18)
We decide in the rest of this appendix to write $\Sigma_n$ as $\Sigma_0$, in order to do recursions on $n := n_0 - 1$ without having to change notations between two recursive steps. As explained in appendix B, the boundary point corresponding to $\Sigma_0$ has nothing to do with the reference bulk point. The matrix to invert in this case is:

$N := (A_{ij} B_{ij}) (C_{ij} D_{ij}) = \left( \begin{array}{cc} \delta_{ij} \Sigma_i + (-1)^i \Sigma_0 & (1 - \delta_{ij}) (-1)^{i+j+\Theta(i,j)} \\ (1 - \delta_{ij}) (-1)^{\Theta(i,j)} & \delta_{ij} \Sigma_j + (-1)^j \Sigma_0 \end{array} \right)$.  

(C.19)

The first step is to recursively block invert $D$ and get:

$D_{ij}^{-1} = \delta_{ij} \Sigma_j + \frac{(-1)^i}{\det(S_n)} \Sigma_i \delta_n \Sigma_j $,  

(C.20)

$\det D = \det(S_n)$,  

(C.21)

where $S_n$ is defined by (B.11). After some algebra, the next step gives:

$\tilde{A}_{ij} = \frac{(-1)^i}{\det(S_n)} \left( S_n + (-1)^\Theta(i,j) S_n^i \right) \bar{S}_n \left( S_n - (-1)^\Theta(i,j) S_n^i \right)$,  

(C.22)

where $S_n^i$ is defined by (B.11). It is good to note that the diagonal piece does not depend on the convention we choose for $\Theta(i, j)$. Then the recursive block inversion of $X^{(q)}$ (the upper left $q \times q$ block of $\tilde{A}$) gives:

$X^{(q)}_{i+\eta} = \sum_{\eta_{i+\eta+j+1} = 1}^{\eta_{i-1}(-1)^j} \frac{2^{(q-1)}}{\det(S_n) \det(S_n + S_n^{(1)}) \det(S_n - S_n^{(q)})}$.

(C.23)

Then the straightforward application of (C.5), (C.6) and of the properties of $\Sigma$ allow once again to recover the equations (4.21) to (4.25).

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