Generating-function method for tensor products.

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Abstract: This is the first of two articles devoted to an exposition of the generating-function method for computing fusion rules in affine Lie algebras. The present paper is entirely devoted to the study of the tensor-product (infinite-level) limit of fusion rules. We start by reviewing Sharp’s character method. An alternative approach to the construction of tensor-product generating functions is then presented which overcomes most of the technical difficulties associated with the character method. It is based on the reformulation of the problem of calculating tensor products in terms of the solution of a set of linear and homogeneous Diophantine equations whose elementary solutions represent “elementary couplings”. Grobner bases provide a tool for generating the complete set of relations between elementary couplings and, most importantly, as an algorithm for specifying a complete, compatible set of “forbidden couplings”.

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1. Introduction

1.1. Orientation

Fusion rules yield the number of independent couplings between three given primary fields in conformal field theories. We are interested in fusion rules in unitary conformal field theories that have a Lie group symmetry, that is, those whose generating spectrum algebra is an affine Lie algebra at integer level. These are the Wess-Zumino-Witten models \([1,2]\). Primary fields in these cases are in 1-1 correspondence with the integrable representations of the appropriate affine Lie algebra at level \(k\). Denote this set by \(P^{(k)}_+\) and a primary field by the corresponding affine weight \(\hat{\lambda}\). Fusion coefficients \(N^{(k)}_{\hat{\lambda}\hat{\mu}\hat{\nu}}\) are defined by the product

\[
\hat{\lambda} \times \hat{\mu} = \sum_{\nu \in P^{(k)}_+} N^{(k)}_{\hat{\lambda}\hat{\mu}\hat{\nu}} \hat{\nu}
\]  

(For a review of conformal field theory and in particular fusion rules, see \([3]\); to a large extend we follow the notation of this reference.)

In the infinite-level limit and for fields with finite conformal dimensions, the purely affine condition on weight integrability is relaxed and the primary fields are solely characterised by their finite part, required to be an integrable weight of the corresponding finite Lie algebra. Recall that a finite weight \(\lambda\) is characterised by its expansion coefficients in terms of the fundamental weights \(\omega_i\)

\[
\lambda = \sum_{i=1}^{r} \lambda_i \omega_i = (\lambda_1, ..., \lambda_r)
\]  

(1.2)

where \(r\) is the rank of the algebra. The numbers \(\lambda_i\)’s are the Dynkin labels. The set of weights with non-negative Dynkin labels (the integrable weights) is denoted by \(P_+\).

In the infinite-level limit the fusion coefficients reduce to tensor-product coefficients:

\[
\lim_{k \to \infty} N^{(k)}_{\hat{\lambda}\hat{\mu}} \hat{\nu} = N_{\hat{\lambda}\hat{\mu}} \hat{\nu}.
\]  

(1.3)

where \(N_{\hat{\lambda}\hat{\mu}} \hat{\nu}\) is defined by

\[
\lambda \otimes \mu = \sum_{\nu \in P_+} N_{\hat{\lambda}\hat{\mu}} \hat{\nu} \hat{\nu}
\]  

(1.4)
By abuse of notation, we use the same symbol for the highest weight and the highest-weight representation. Notice that

\[ N_{\lambda \mu}^{\nu} = N_{\lambda \mu}^{\nu^*} \]  

(1.5)

where \( \nu^* \) denotes the highest weight of the representation conjugate to that of \( \nu \). Equivalently, \( N_{\lambda \mu}^{\nu^*} \) gives the multiplicity of the scalar representation in the triple product \( \lambda \otimes \mu \otimes \nu^* \).

A tensor-product generating function codes the information for all the tensor products of a given algebra in a single function defined by

\[ G(L, M, N) = \sum_{\lambda, \mu, \nu \in P_+} N_{\lambda \mu}^{\nu} L^\lambda M^\mu N^\nu \]  

(1.6)

where \( L^\lambda = L_1^\lambda \cdots L_r^\lambda \) and similarly for \( M^\mu \) and \( N^\nu \). \( G \) can generally be expressed as a simple closed function of its variables. For instance, for \( \text{su}(2) \), it reads

\[ G(L, M, N) = \frac{1}{(1 - LM)(1 - LN)(1 - MN)} \]  

(1.7)

An example of basic global information that can be deduced from a generating function is the integrality as well as the positivity of the tensor-product coefficients. More importantly, from our point of view, is that in the context of fusion rules, the construction of the simplest generating functions led to the discovery of the notion of threshold levels [4]. Moreover, as shown in the sequel paper, setting up a fusion generating function is a way to obtain explicit expressions for these threshold levels. Our new approach to fusion-rule generating functions, which originates from the generalisation of techniques developed in the present paper on tensor products, leads to a further new concept, that of a fusion basis.

1.2. Overview of the paper

The present article is organised as follows. We start by explaining in detail the construction of tensor-product generating functions for finite Lie algebras. The first construction which is presented is the character method developed by Sharp and his collaborators (section 2).

Although it is conceptually very simple, the character method is limited by its inherent computational difficulties: the disproportion between the simplicity of the resulting form
of the generating function and the intermediate calculations is enormous. This motivates our alternative approach to the construction of tensor-product generating function. It is based on the reformulation of the problem of calculating tensor products in terms of the solution of a set of linear and homogeneous Diophantine equations (cf. section 3). The elementary solutions of these Diophantine equations represent “elementary couplings”. For \( sp(4) \), the use of the Berenstein-Zelevinsky inequalities to obtain the elementary couplings and their relations (cf. the analysis of section 6) is new.

The key difficulty is finding the numerous relations that exist in general between the elementary solutions. From the Diophantine-equation point of view, the decomposition of a solution may not be unique because different sums of elementary solutions could yield the same result. To solve this problem we first “exponentiate” it: given a solution \( \alpha = (\alpha_1, \ldots, \alpha_k) \) to our system of linear Diophantine equations, we introduce formal variables \( X_1, \ldots, X_k \) and consider the monomial \( X_1^{\alpha_1} \cdots X_k^{\alpha_k} \). The linear span, \( R \), of all such monomials is a “model” for the generating function for the solutions to the original set of linear Diophantine equations (see section 5), since the Poincaré series of \( R \) is the required generating function. This series can be calculated using Grobner basis methods.

For \( su(N) \) there is a remarkable graphical construction for computing tensor product multiplicities, the famous Berenstein-Zelevinsky triangles. These are introduced in section 6. We also discuss the analogous construction for \( sp(4) \), whose diagrammatic representation is new. But the main interest of these re-formulations is that it yields a simple and systematic way of obtaining the elementary couplings from the construction of a vector basis. Thus we get a new way of constructing the corresponding generating functions.

2. Generating-function for tensor products: the character method

2.1. The character method for the construction of the tensor-product generating function: the \( su(2) \) case

The method developed by Sharp and collaborators for constructing generating functions for tensor products is based on manipulations of the character generating functions [5]. Although simple in principle, these manipulations become rather cumbersome as the rank of the algebra is increased. To illustrate the method, we will work in complete detail the simplest example, the \( su(2) \) case.
The first step is the derivation of the character generating function. The Weyl character formula for a general algebra of rank \( r \) and a highest-weight representation \( \lambda \) is

\[
\chi_\lambda = \frac{\xi_{\lambda+\rho}}{\xi_{\rho}} \quad (2.1)
\]

where \( \rho \) is the finite Weyl vector, \( \rho = \sum_{i=1}^{r} \omega_i \), and where the characteristic function \( \xi \) is defined as

\[
\xi_{\lambda+\rho} = \sum_{w \in W} \epsilon(w)e^{w(\lambda+\rho)} \quad (2.2)
\]

where \( \epsilon(w) \) is the signature of the Weyl reflection \( w \) and \( W \) is the Weyl group.

For \( su(2) \), \( W \) contains two elements: \( 1, s_1 \). With

\[
x = e^{\omega_1} \quad (2.3)
\]

the \( su(2) \) characteristic function \( \xi \) for the representation of highest weight \( m\omega_1 \equiv (m) \) is

\[
x^{m+1} - x^{-m-1} \quad (2.4)
\]

The character reads then

\[
\chi_m = \frac{x^{m+1} - x^{-m-1}}{x - x^{-1}} = \frac{x^m - x^{-m-2}}{1 - x^{-2}} = x^m + x^{m-2} + \ldots + x^{-m} \quad (2.5)
\]

The character generating function \( \chi_L \) is obtained by multiplying the above expression by \( L^m \) where \( L \) is a dummy variable, and summing over all positive values of \( m \):

\[
\chi_L(x) = \sum_{0}^{\infty} L^m \chi_m = \frac{1}{x - x^{-1}} \sum_{0}^{\infty} L^m (x^{m+1} - x^{-m-1})
\]

\[
= \frac{1}{1 - x^{-2}} \left( \frac{1}{1 - Lx} - \frac{x^{-2}}{1 - Lx^{-1}} \right) = \frac{1}{(1 - Lx)(1 - Lx^{-1})} \quad (2.6)
\]

We should point out here that in all generating functions in this paper, expressions of the form \( 1/(1 - a) \) should be formally expanded in positive powers of \( a \). So for example, \( 1/(1 - Lx^{-1}) = 1 + Lx^{-1} + L^2x^{-2} + \ldots \). By construction, the character of the highest weight \( (m) \) can be recovered from the power expansion of \( \chi_L \) as the coefficient of the term \( L^m \). The characteristic generating function \( \xi_L \) is defined by

\[
\chi_L(x) = \frac{\xi_L}{\xi_0} \quad (2.7)
\]
and it reads
\[ \xi_L(x) = \frac{x - x^{-1}}{(1 - Lx)(1 - Lx^{-1})} = \frac{x}{1 - Lx} - \frac{x^{-1}}{1 - Lx^{-1}} \] (2.8)
the last form being the one that results directly from (2.5).

The tensor product of two highest-weight representations can be obtained from the product of the corresponding characters:
\[ \chi_m \chi_n = \sum_\ell N_{mn} \chi_\ell \] (2.9)
This information can be extracted from the product of the corresponding generating functions. We are thus led to consider the product \( \chi_L(x)\chi_M(x)\). To simplify the analysis of the resulting expression, notice that the information concerning the representations occurring in the tensor product is coded in the leading term of the character, i.e., the term \( x^{m+1} \).

To insure that every positive power of \( x \) singles out a highest-weight representation, we can multiply both sides by \( \xi_0 \). To read off these terms, we can focus on the terms with strictly positive powers of \( x \) in the product \( \chi_L(x)\chi_M(x)\xi_0(x) \). If we require the Dynkin label of the representations (and not their shifted value), it is more convenient to divide by \( x \) before doing the projection, now restricted to the non-negative powers of \( x \). The truncation of an expression by its negative powers of \( x \) will be denoted by the MacMahon symbol \([6] \Omega\), defined by
\[ x \Omega \geq \sum_{n \geq 0} c_n x^n \] (2.10)
When there is no ambiguity concerning the variable in terms of which the projection is defined, it is omitted from the \( \Omega \) symbol.

We are thus interested in the projection of the following expression
\[ \chi_L(x)\chi_M(x)\xi_0(x)x^{-1} = \chi_L(x)\xi_M(x)x^{-1} \]
\[ = \frac{1}{(1 - Lx)(1 - Lx^{-1})} \left( \frac{1}{1 - Mx} - \frac{x^{-2}}{1 - Mx^{-1}} \right) \] (2.11)
For these manipulations, we use systematically the following simple identities:
\[ \frac{1}{(1 - A)(1 - B)} = \frac{1}{(1 - AB)} \left( \frac{1}{1 - A} + \frac{B}{1 - B} \right) \]
\[ = \frac{1}{(1 - AB)} \left( \frac{A}{1 - A} + \frac{1}{1 - B} \right) \]
\[ = \frac{1}{(1 - AB)} \left( \frac{1}{1 - A} + \frac{1}{1 - B} - 1 \right) \] (2.12)
There are two terms to analyse. The first is
\[
\frac{1}{(1 - Lx)(1 - Lx^{-1})(1 - Mx)} = \frac{1}{(1 - Lx)(1 - LM)} \left( \frac{1}{1 - Mx} + \frac{Lx^{-1}}{1 - Lx^{-1}} \right) \tag{2.13}
\]
The first part is not affected by the projection and the second can be written as
\[
\frac{Lx^{-1}}{(1 - Lx)(1 - LM)(1 - Lx^{-1})} = \frac{Lx^{-1}}{(1 - LM)(1 - L^2)} \left( \frac{Lx}{1 - Lx} + \frac{1}{1 - Lx^{-1}} \right) \tag{2.14}
\]
The second term of this expression contains only negative powers of \( x \) and can thus be ignored and the first part is unaffected by the projection. We have thus, for the first term of (2.11)
\[
\Omega \geq \frac{1}{(1 - Lx)(1 - Lx^{-1})(1 - Mx)} = \frac{1}{(1 - Lx)(1 - LM)} \left( \frac{1}{1 - Mx} + \frac{L^2}{1 - L^2} \right) \tag{2.15}
\]
The projection of the second term of (2.11) is:
\[
\Omega \geq \frac{x^{-2}}{(1 - Lx)(1 - Lx^{-1})(1 - Mx^{-1})} = \frac{x^{-2}}{(1 - Lx^{-1})(1 - LM)(1 - Lx)} \left( \frac{1}{1 - Lx} + \frac{Mx^{-1}}{1 - Mx^{-1}} \right)
\]
\[
= \frac{x^{-2}}{(1 - Lx^{-1})(1 - LM)(1 - L^2)(1 - Lx)} \left( \frac{Lx}{1 - Lx} + \frac{1}{1 - Lx^{-1}} \right) \tag{2.16}
\]
\[
= \frac{Lx^{-1}}{(1 - LM)(1 - L^2)(1 - Lx)} \left( \frac{1}{1 - Lx} - 1 \right)
\]
\[
= \frac{L^2}{(1 - LM)(1 - L^2)(1 - Lx)}
\]
Subtracting (2.16) from (2.15), we find that
\[
\Omega \geq \chi_L(x)\xi_M(x) x^{-1} = \frac{1}{(1 - LM)(1 - Lx)(1 - Mx)} \tag{2.17}
\]
Replacing \( x \) by \( N \), we thus get
\[
G^\text{su(2)}(L, M, N) = \frac{1}{(1 - LM)(1 - LN)(1 - MN)} \tag{2.18}
\]
2.2. The abstract setting: Poincaré series, elementary couplings and relations; defining a model

As we shall see it is frequently useful have a model, $R$, for a generating function $G(X_1, \ldots, X_k)$ such as (2.18). By this we mean a commutative $\mathbb{Q}$-algebra with an identity, graded by $\mathbb{N}^k$, ($\mathbb{N} = \{0, 1, 2, 3, \ldots\}$)

$$R = \bigoplus_{\alpha \in \mathbb{N}^k} R_\alpha, \quad R_\alpha R_\beta \subseteq R_{\alpha + \beta}$$

and such that its Poincaré series (also frequently called Hilbert series)

$$F(R) = \sum_{\alpha \in \mathbb{N}^k} \dim_\mathbb{Q}(R_\alpha) X^\alpha$$

satisfies

$$F(R) = G(X_1, \ldots, X_k).$$

For example, for (2.18), with $X_1 = L$, $X_2 = M$, $X_3 = N$, we can take $R = \mathbb{Q}[E_1, E_2, E_3]$, which is the polynomial ring generated by the formal variables $E_1, E_2, E_3$ (in fact all our examples $R$ is either a subring or quotient of a polynomial ring) with the grading of $E_1, E_2$ and $E_3$ being $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$. The homogeneous subspaces are spanned by $E_1^a E_2^b E_3^c$, $a, b, c \in \mathbb{N}$ with grade $(a + b, a + c, b + c)$ and so

$$F(R) = \sum_{(a, b, c) \in \mathbb{N}^3} X_1^{a+b} X_2^{a+c} X_3^{b+c} = G^{su(2)}(X_1, X_2, X_3)$$

as required.

If $R$ is generated by elements $E_1, \ldots, E_s$ and is a model for a generating function $G$ for tensor products (or fusion products) then we call $E_1, \ldots, E_s$ a set of “elementary couplings” for $G$.

It should perhaps be stressed that $a$ priori the variables $X_1, \ldots, X_k$ and $E_1, \ldots, E_s$ are unrelated. We shall refer to the $E$’s as model variables and the $X$’s as grading variables. If the grading vector of $E_i$ is $\alpha^i$, $i = 1, \ldots, s$ then there is an associated monomial in the grading variables: $X^{\alpha^i}$, for which we will use the notation $g(E_i)$. For example in the above example we have $g(E_1) = X_1^1 X_2^1 X_3^0 = LM$. However, to avoid tedious repetition when writing down generating functions we shall often write, for example, $1/(1 - E_1)$ rather than
$1/(1 - g(E_1))$. In all such cases where model variables appear in a generating function they should be replaced by the corresponding monomial in the grading variables.

In the case of tensor products we use the notation \( E : g(E) : \text{product} \) to denote a set of elementary couplings with their “exponentiated” grading and the corresponding term in the tensor product. So in the example above we would write:

\[
\begin{align*}
E_1 : LM & : (1) \otimes (1) \supset (0), \\
E_2 : LN & : (1) \otimes (0) \supset (1), \\
E_3 : MN & : (0) \otimes (1) \supset (1)
\end{align*}
\] (2.22)

Having made the distinction between grading and model variables, it should be noted that there are cases where we can identify the model as a ring generated by monomials in the grading variables. So in the above example we could define \( E_1 = LM, E_2 = LN \) and \( E_3 = MN \) and take the model for our generating function to be the subring of \( Q[L, M, N] \) generated by \( E_1, E_2 \) and \( E_3 \). However, it is not always desirable, or even possible, to make this identification.

We close this section with two examples of how models for the \( su(2) \) character generating function can be constructed.

The first method, which has been exploited by Sharp et al (see [5]) to construct character generating functions, amounts to finding an algebra \( R \) which is a module for the Lie algebra \( su(2) \) and such that, as an \( su(2) \) module, \( R \) is isomorphic to \( \oplus_{i \geq 1} V_i \) where \( V_i \) is the irreducible \( su(2) \) module of dimension \( i \).

In this case we can take \( R = \mathbb{Q}[p, q] \) with the generators of \( su(2) \) being given by differential operators:

\[
h = p \frac{\partial}{\partial p} - q \frac{\partial}{\partial q}, \quad x_- = q \frac{\partial}{\partial p}, \quad x_+ = p \frac{\partial}{\partial q}
\] (2.23)

The \( su(2) \) highest-weight vectors are \( p^i, i \geq 0 \) and a basis of the irreducible submodule of dimension \( i \) is just given by the monomials of degree \( i \) in \( p \) and \( q \). We can give \( R \) an \( \mathbb{N}^3 \) grading by taking the degree of \( p \) to be \( (1, 1, 0) \) and of \( q \) to be \( (1, 0, 1) \). Here the first grading index specifies the representation while the other two refer to a particular weight. As \( R = \mathbb{Q}[p, q] \) the Poincaré function for \( R \) is,

\[
\frac{1}{(1 - p)(1 - q)}
\]
with the understanding, as explained above, that $p$ and $q$ should be replaced by the corresponding expression in terms of the grading variables. Let us denote these grading variables here by $L$ (which exponentiates the representation index) and $x, y$ (exponentially related to the weights). The Poincaré function reads then

$$\frac{1}{(1 - Lx)(1 - Ly)}$$  \hfill (2.24)

Another way of constructing a model for the weight generating function, which makes more natural the $\mathbb{N}^3$ grading, is to observe that the complete set $SU(2)$ weight vectors of finite dimensional irreducible $su(2)$ modules are in 1-1 correspondence with one-rowed Young tableaux. If the Young tableau has $c$ boxes filled with $a$ 1’s and $b$ 2’s then there is a constraint

$$a + b - c = 0, \quad a, b, c \geq 0$$  \hfill (2.25)

and so the solutions to this linear Diophantine equation are in 1-1 correspondence with the complete set of $SU(2)$ weight vectors. Thus to find a model for the weight generating function it is sufficient to find a model for the solutions to (2.25). It is not difficult to see that every solution to this equation is a linear combination (with non-negative coefficients) of the two fundamental solutions: $(a, b, c) = (1, 0, 1)$ and $(a, b, c) = (0, 1, 1)$. Let $R$ be the subring of $\mathbb{Q}[A, B, C]$ generated by the monomials $E_1 = AC$, $E_2 = BC$. Considering the exponents of the monomials $E_1$ and $E_2$, we see that the monomials in $R$ correspond to the solutions of (2.25) and hence taking the natural grading on $R$ ensures that the Poincaré series of $R$ is the generating function for the solutions to (2.25) and hence is the required generating function. In this example there are no relations between $E_1$ and $E_2$ and so $R$ is isomorphic to the polynomial ring in two variables (as expected) and so the Poincaré function is once again (with $A \rightarrow x, B \rightarrow y, C \rightarrow L$) given by (2.24).

2.3. Multiple $su(2)$ tensor products

In order to illustrate the occurrence of linear relations between elementary couplings, consider the problem of finding the multiplicity of a given representation $\zeta$ in the triple product $\lambda \otimes \mu \otimes \nu$. In terms of character generating functions, this amounts to considering the product $\chi_L(x)\chi_M(x)\chi_N(x) \supset \chi_P(x)$, or equivalently, $\chi_L(x)\chi_M(x)\xi_N(x)x^{-1} \supset$
\( \xi_P(x)x^{-1} \). The left side is then projected onto positive powers of \( x \). We are thus led to consider

\[
\Omega \geq \frac{1}{(1-Lx)(1-Lx^{-1})(1-Mx)(1-Mx^{-1})} \left( \frac{1}{1-Nx} - \frac{x^{-2}}{1-Nx^{-1}} \right)
\]  

(2.26)

The projection of each term is worked out as previously and the resulting expression is found to be, with \( x \) replaced by \( P \):

\[
G(L, M, N, P) = \frac{1 - LMNP}{(1 - LP)(1 - MP)(1 - NP)(1 - LM)(1 - LN)(1 - MN)}
\]  

(2.27)

This is the sought for generating function. Here we would like to have a model with 6 elementary couplings corresponding to the terms in the denominator of the generating function:

- \( E_1 : LM : (1) \otimes (1) \otimes (0) \supset (0) \)
- \( E_2 : LN : (1) \otimes (0) \otimes (1) \supset (0) \)
- \( E_3 : LP : (1) \otimes (0) \otimes (0) \supset (1) \)
- \( E_4 : MN : (0) \otimes (1) \otimes (1) \supset (0) \)
- \( E_5 : MP : (0) \otimes (1) \otimes (0) \supset (1) \)
- \( E_6 : NP : (0) \otimes (0) \otimes (1) \supset (1) \)

and there must be a linear relation (in this context, such a relation is often called a syzygy in the physics literature - see in particular [5] and related works) between the following products (signalled by a term in the numerator) which has grading \( LMNP \):

\[
E_1E_6, \quad E_2E_5, \quad E_3E_4
\]  

(2.28)

(2.29)

It is not difficult to see that a model is given by \( \mathbb{Q}[e_1, e_2, e_3, e_4, e_5, e_6]/I \) where \( E_i = e_i + I, i = 1, \ldots, 6 \) and \( I = (ae_1e_6 + be_2e_5 + ce_3e_4) \) is the ideal generated by the polynomial \( ae_1e_6 + be_2e_5 + ce_3e_4 \) for any choice of \( a, b, c \in \mathbb{Q} \) not all zero.

The elements of \( R \) have the form \( m + I \) with \( m \in \mathbb{Q}[e_1, \ldots, e_6] \). However there is no canonical way of choosing the representatives \( m \). Take for example the case \( a = b = c = 1 \). (Usually we will construct a model for our generating function as explained above and this construction will fix the values of \( a, b \) and \( c \). In \( R \) we have \( E_1E_6 = -(E_2E_5 + E_3E_4) \) and so we can take as a basis for \( R \) the set of (equivalences classes of ) monomials which do not contain the product \( E_1E_6 \). In this case we say that we have chosen to make \( E_1E_6 \) a
of ‘forbidden product’. Similarly we can forbid the products $E_2E_5$ or $E_3E_4$. As we shall see later, the choice of forbidden products corresponds to a choice of term ordering.

Before leaving this example, we would like to rework it from a different point of view, as an illustration of the ‘composition’ technique of generating functions. Let $G(L, M, R)$ describe the tensor product corresponding to $\chi_L\chi_M \supset \chi_R$ and similarly let $G(Q, N, P)$ correspond to $\chi_Q\chi_N \supset \chi_P$. We are interested in the product $\chi_L(x)\chi_M(x)\chi_N(x) \supset \chi_P(x)$, but treated from the product of the two generating functions $G$. We thus want to enforce the constraint $R = Q$ in the product $G(L, M, R)G(Q, N, P)$. The idea – which is used in the references in [5] mainly in relation with the construction of generating functions for branching functions – is to multiply this product by $(1 - Q^{-1}R^{-1})^{-1}$ and, in the expansion in powers of $R$ and $Q$, keep only terms of order zero in both variables: with an obvious notation we have

$$\frac{RQ}{Q}G(L, M, R)G(Q, N, P) \frac{1}{1 - Q^{-1}R^{-1}}$$

$$= \frac{RQ}{Q} \sum_n A_n(L, M) R^n \sum_m B_m(N, P) Q^m \sum_\ell R^{-\ell}Q^{-\ell}$$

$$= \sum_p A_p(L, M) B_p(L, M)$$

(2.30)

which is manifestly equivalent to considering

$$x \frac{Q}{x} G(L, M, x)G(x^{-1}, N, P)$$

(2.31)

With the explicit expressions for the generating functions, we have thus

$$\frac{x}{Q} \frac{1}{(1 - Lx)(1 - Mx)(1 - LM)} \frac{1}{(1 - P^{-1}x)(1 - N^{-1}x)(1 - NP)}$$

(2.32)

A brief and by now standard analysis yields directly the generating function (2.27).

2.4. The $sp(4)$ case

As a final example, consider the $sp(4)$ case. With the $x_i = e^{\omega_i}$, $i = 1, 2$, the characteristic function is found to be

$$\xi(m,n) = x_1^{m+1}x_2^{n+1} - x_1^{-m-1}x_2^{m+n+2} - x_1^{n+m+5}x_2^{-n-1} + x_1^{m+2n+3}x_2^{-m-n-2}$$

$$+ x_1^{-m-2n-3}x_2^{n+m+2} - x_1^{m+1}x_2^{-m-n-2} - x_1^{-m-2n-3}x_2^{n} + x_1^{-m-1}x_2^{-n-1}$$

(2.33)
and the characteristic generating function is
\[
\xi_{L_1, L_2} = \frac{1}{(1 - L_1 x_1)(1 - L_1 x_1 x_2^{-1})(1 - L_2 x_2^{-1})(1 - L_2 x_1^{-2} x_2)}
\times \left( \frac{1 + L_2}{(1 - L_2 x_1^2 x_2^{-1})(1 - L_2 x_2^{-1})} + \frac{(1 + L_2)L_1 x_1}{(1 - L_1 x_1)(1 - L_2 x_1^{-2} x_2)} \right) (1 - L_1 x_1^{-1} x_2)
\]  
(2.34)

From this we construct the character generating function and then we can proceed to the tensor-product generating function. This is again extremely cumbersome. The result is [7]
\[
G^{sp(4)}(L_1, L_2, M_1, M_2, N_1, N_2)
= [(1 - M_1 N_1)(1 - L_1 N_1)(1 - L_1 M_1)(1 - M_2 N_2)(1 - L_2 N_2)(1 - L_2 M_2)]^{-1}
\times \left( \frac{1}{(1 - L_2 M_1 N_1)(1 - L_2 M_1^2 N_2)} + \frac{L_2 M_2 N_2^2}{(1 - L_2 M_1 N_1)(1 - L_2 M_2 N_2^2)} \right)
\]  
(2.35)

From this expression, we read off the following list of elementary couplings (recall that the first variable is a model variable and then we write the corresponding monomial in the grading variables):
\[
A_1 : M_1 N_1, \quad A_2 : L_1 N_1, \quad A_3 : L_1 M_1
\]
\[
B_1 : M_2 N_2, \quad B_2 : L_2 N_2, \quad B_3 : L_2 M_2
\]
\[
C_1 : L_2 M_1 N_1, \quad C_2 : L_1 M_2 N_1, \quad C_3 : L_1 M_1 N_2
\]
\[
D_1 : L_1^2 M_2 N_2, \quad D_2 : L_2 M_1^2 N_2, \quad D_3 : L_2 M_2 N_1^2
\]  
(2.36)

However, not all the products of the model variables can be linearly independent: there are linear relations between:
\[
C_i C_j, \quad A_k D_k, \quad A_i A_j B_k
\]
\[
D_i D_j, \quad A_k^2 B_i B_j, \quad B_k C_k^2
\]
\[
C_i D_i, \quad A_j B_k C_k, \quad A_k B_j C_j
\]  
(2.37)

for \(i, j, k\) a cyclic permutation of \(1, 2, 3\) and repeated indices are not summed. (It is plain that the three sets of products found to be linearly related must have the same Dynkin labels.) A specific form of the generating function, as expressed in terms of the elementary couplings, amounts to a specific choice of a set of forbidden couplings among those that are related by a linear relation.
3. Tensor-product descriptions

3.1. The need for a tensor-product description

It is clear that one major technical complication of the character method is that it starts at too fundamental a level, namely the character of the separate representations. One natural way to proceed is to start from a combinatorial description of the tensor-product rules. Such a description already takes into account the action of the Weyl group and encodes the various subtractions of the singular vectors.

But how do we make the connection with the generating-function approach? The key is to find a combinatorial description which can be expressed as a set of linear Diophantine inequalities. Given this set of inequalities, there is an algorithm, again due to MacMahon, for constructing a generating function. (This is an adaptation of a method developed by Elliot [8] for the analysis of linear Diophantine equalities and for this reason the algorithm is often referred to as the Elliot-MacMahon method. For a detailed discussion of the algorithm, see in particular vol. 2 section VIII of [6].) This method is conceptually similar to the character method, except that the starting point is substantially closer to the end result. See section 7.3 for a slight generalisation of this algorithm.

Although the description of tensor products via linear Diophantine equations is a more efficient route to finding the generating function than the character one, complications associated to the Ω projections remain a source of technical difficulty that severely limits the practical applicability of the method.

A more powerful approach to our problem is to use the techniques of computational algebra. We start with a description of the tensor-product multiplicities as solutions to linear Diophantine inequalities. Efficient algorithms exists for finding the fundamental solutions to these inequalities [9]. From these we find directly a model for the generating function using Grobner basis techniques. (This is roughly the inverse of MacMahon’s method which was originally conceived as a technique to generate the elementary couplings and their linear relations through the construction of the generating function. Here, the elementary couplings and their relations are first obtained and used as the input for the construction of the generating function.)
4. The LR rule \((su(N))\)

For \(su(N)\) tensor products there is a particularly convenient description based on Littlewood-Richardson tableaux supplemented by the stretched-product operation (defined below) \([10]\).

Integrable weights in \(su(N)\) can be represented by tableaux: the weight \((\lambda_1, \lambda_2, \cdots, \lambda_{N-1})\) is associated to a left justified tableau of \(N-1\) rows with \(\lambda_1 + \lambda_2 + \cdots + \lambda_{N-1}\) boxes in the first row, \(\lambda_2 + \lambda_3 + \cdots + \lambda_{N-1}\) boxes in the second row, etc. Equivalently, the tableau has \(\lambda_1\) columns of 1 box, \(\lambda_2\) columns of 2 boxes, etc. The scalar representation has no boxes, or equivalently, any number of columns of \(N\) boxes.

The Littlewood-Richardson rule is a simple combinatorial description of the tensor product of two \(su(N)\) representations \(\lambda \otimes \mu\). The second tableau \((\mu)\) is filled with numbers as follows: the first row with 1’s, the second row with 2’s, etc. All the boxes with a 1 are then added to the first tableau according to following restrictions:

1) the resulting tableau must be regular: the number of boxes in a given row must be smaller or equal to the number of boxes in the row immediately above;

2) the resulting tableau must not contain two boxes marked by 1 in the same column.

All the boxes marked by a 2 are the added to the resulting tableaux according to the above two rules (with 1 is replaced by 2) and the further restriction:

3) in counting from right to left and top to bottom, the number of 1’s must always be greater or equal to the number of 2’s.

The process is repeated with the boxes marked by a 3, 4, \cdots, \(N-1\), with the additional rule that the number of \(i\)’s must always be greater or equal to the number of \(i+1\)’s when counted from right to left and top to bottom. The resulting Littlewood-Richardson (LR) tableaux are the Young tableaux of the irreducible representations occurring in the decomposition.

These rules can be rephrased in an algebraic way as follows \([10]\). Define \(n_{ij}\) to be the number of boxes \(i\) that appear in the LR tableau in the row \(j\). The LR conditions read

\[
\lambda_{j-1} + \sum_{i=1}^{k-1} n_{i,j-1} - \sum_{i=1}^{k} n_{ij} \geq 0 \quad 1 \leq k < j \leq N
\] (4.1)
and
\[ \sum_{j=i}^{k} n_{i-1,j-1} - \sum_{j=i}^{k} n_{ij} \geq 0 \quad 2 \leq i \leq k \leq N \quad \text{and} \quad i \leq N - 1. \quad (4.2) \]
The weight \( \mu \) of the second tableau and the weight \( \nu \) of the resulting LR tableau are respectively given by
\[
\sum_{j=i}^{N} n_{ij} = \sum_{j=i}^{N-1} \mu_j \quad i = 1, 2, ..., N - 1, \\
\nu_j - \lambda_j + \sum_{i=1}^{N-1} n_{i,j+1} = \sum_{i=1}^{\min(j,N-1)} n_{ij} \quad j = 1, 2, ..., N - 1. \quad (4.3)
\]
Hence, given three weights \( \lambda, \mu \) and \( \nu \), the number of non-negative integers solutions \( \{n_{ij}\} \) satisfying the above conditions gives the multiplicity \( N_{\lambda \mu}^\nu \) of \( \nu \) in the tensor product \( \lambda \otimes \mu \).

The combined equations (4.1) and (4.2) constitute a set of linear and homogeneous inequalities. As described in [11], the Hilbert basis theorem guarantees that every solution can be expanded in terms of the elementary solutions of these inequalities.

We can construct a model for the solutions of the equations (4.1) and (4.2) by introducing new formal variables \( A_i \), \( 1 \leq i \leq t \) where \( t \) is the total number of variables in (4.1) and (4.2). Then the subring of \( \mathbb{Q}[A_i; 1 \leq i \leq t] \) generated by the monomials \( A^\alpha \) with \( \alpha \) a solution of (4.1) and (4.2) provides the required model. This ring \( R \) will be generated by a finite set of monomials \( E_j \), \( 1 \leq j \leq s \) which we call elementary couplings corresponding to the elementary solutions of (4.1) and (4.2). Thus \( R \) is isomorphic to \( \mathbb{Q}[e_1, \ldots, e_s]/I \) under the mapping \( \phi : e_i \rightarrow E_i \) where \( I \) is some ideal. Each element of \( I \) corresponds, via the map \( \phi \), to a relation between the elementary couplings.

In the case of LR tableaux, there is a nice pictorial representation of the model \( R \). Consider the set of formal linear combinations of LR tableaux with rational coefficients. It is given a ring structure by defining the stretched product of two LR tableaux (denoted by \( \cdot \)) to be the tableau obtained by fusing the two tableaux and reordering the numbers in each row in increasing order [10]. More algebraically, if we denote the empty boxes of a LR tableau by a 0, so that
\[
n_{0j} = \sum_{i=j}^{N-1} \lambda_i \quad j = 1, 2, ..., N - 1. \quad (4.4)
\]
we can characterise completely a tableau by the data \( \{n_{ij}\} \) with now \( i \geq 0 \). It is clear
the set of numbers \( \{n_{ij}\} \) with \( i \geq 0 \), or equivalently, \( \{\lambda, n_{ij}\} \) with \( i \geq 1 \), is a complete
set of variables for the description of the tensor products. Then, the tableau obtained by
the stretched product of the tableaux \( \{n_{ij}\} \) and \( \{n'_{ij}\} \) is simply described by the numbers
\( \{n_{ij} + n'_{ij}\} \). Here is a simple example:

\[
\begin{array}{ccc}
1 & 1 & 2 \\
2 & 3 & \\
4 & \\
\end{array} \cdot \begin{array}{ccc}
1 & \\
1 & 1 & 2 \\
2 & 2 & 3 \\
4 & \\
\end{array} = \begin{array}{ccc}
1 & 1 & 1 \\
1 & 2 & 2 \\
2 & 2 & 3 \\
4 & \\
\end{array}
\] (4.5)

This ring of tableaux is isomorphic to the model \( R \) constructed above and we do not
distinguish between them. Thus we specify a set of elementary couplings (i.e. a set of
generators of \( R \)) as a set of elementary LR Tableaux.

4.1. Example: the su(2) case

The complete set of inequalities for \( su(2) \) variables \( \{\lambda, n_{11}, n_{12}\} \) is simply

\[
\lambda \geq n_{12} \quad n_{11} \geq 0 \quad n_{12} \geq 0
\] (4.6)

The other weights are fixed by

\[
\mu_1 = n_{11} + n_{12} \quad \nu_1 = \lambda + n_{11} - n_{12}
\] (4.7)

By inspection, the elementary solutions of this set of inequalities are

\[
(\lambda, n_{11}, n_{12}) = (1, 0, 1), \ (1, 0, 0), \ (0, 1, 0)
\] (4.8)

which correspond respectively to \( E_1, E_2, E_3 \) in (2.22). These correspond to the following
LR tableaux:

\[
E_1 : \begin{array}{c}
1
\end{array}, \quad E_2 : \begin{array}{c}
\n\end{array}, \quad E_3 : \begin{array}{c}
1
\end{array}
\] (4.9)

It is also manifest that there are no linear relations between these couplings. The generating
function is thus simply:

\[
G^{su(2)} = \frac{1}{(1 - E_1)(1 - E_2)(1 - E_3)}
\] (4.10)
4.2. Example: multiple tensor products in the $su(2)$ case

Consider the problem of finding the multiplicity of the representation $\zeta$ in the triple product $\lambda \otimes \mu \otimes \nu \supset \zeta$. As a first step, the LR rule applies as before: with $n_{11} + n_{12} = \mu_1$, we have $\lambda_1 \geq n_{12}$. After the first product, we re-apply the LR rule with now $\lambda_1$ replaced by $\lambda_1 + n_{11} - n_{12}$ and $n_{ij}$ replaced by $m_{ij}$ with $m_{11} + m_{12} = \nu_1$. The LR gives $\lambda_1 + n_{11} - n_{12} \geq m_{12}$. The two inequalities for the $su(2)$ quadruple product are then:

$$\lambda_1 \geq n_{12} \quad \lambda_1 + n_{11} - n_{12} \geq m_{12} \quad n_{ij} \geq 0 \quad m_{ij} \geq 0 \quad (4.11)$$

The elementary solutions are then, in the order: name of the coupling, corresponding Dynkin labels and the 5-vector $(\lambda_1, n_{11}, n_{12}, m_{11}, m_{12})$:

$$E_1 : (1) \otimes (1) \otimes (0) \supset (0) \quad (1, 0, 1, 0, 0)$$
$$E_2 : (1) \otimes (0) \otimes (1) \supset (0) \quad (1, 0, 0, 0, 1)$$
$$E_3 : (1) \otimes (0) \otimes (0) \supset (1) \quad (1, 0, 0, 0, 0)$$
$$E_4 : (0) \otimes (1) \otimes (1) \supset (0) \quad (0, 1, 0, 0, 1)$$
$$E_5 : (0) \otimes (1) \otimes (0) \supset (1) \quad (0, 1, 0, 0, 0)$$
$$E_6 : (0) \otimes (0) \otimes (1) \supset (1) \quad (0, 0, 0, 1, 0)$$

The linear relation, whose existence was signalled by the character method, is

$$E_3 E_4 = E_2 E_5 : (1, 1, 0, 0, 1), \quad \neq E_1 E_6 : (1, 0, 1, 1, 0) \quad (4.13)$$

Choosing to forbid the product $E_3 E_4$, the generating function can be written in the form

$$G = \frac{1 - E_3 E_4}{(1 - E_1)(1 - E_2)(1 - E_3)(1 - E_4)(1 - E_5)(1 - E_6)}$$

$$= \left( \prod_{i=1,2,5,6} \frac{1}{1 - E_i} \right) \left( \frac{1}{1 - E_3} + \frac{E_4}{1 - E_4} \right) \quad (4.14)$$

The latter form makes manifest the absence of $E_3 E_4$.

We could represent the elementary couplings in terms of tableaux, where the boxes with 1’s refers to the $\mu$ tableau and those with 2’s originate from the $\nu$ tableau. (Warning: the resulting tableaux describing the four-products are not necessarily LR tableaux.)
Hence, $n_{1j}$ gives the number of 1's in row $j$ of the composed tableau while $m_{1k}$ gives the number of 2's in row $k$. The elementary tableaux are

$$E_1 : \begin{array}c 1 \\ 2 \\ 3 \end{array}, E_2 : \begin{array}c 2 \\ 1 \\ 3 \end{array}, E_3 : \begin{array}c \hline \hline \hline \end{array}$$

$$E_4 : \begin{array}c 1 \\ 2 \\ 2 \\ 1 \end{array}, E_5 : \begin{array}c 1 \\ 1 \\ 2 \\ 3 \end{array}, E_6 : \begin{array}c 2 \\ 1 \\ 3 \\ 2 \end{array}$$

From this representation, the relation reads

$$E_3 E_4 = E_2 E_5 : \begin{array}c 1 \\ 2 \\ \hline \hline \hline \end{array}, \quad \neq E_1 E_6 : \begin{array}c 2 \\ 1 \\ \hline \hline \hline \end{array}$$

4.3. Example: the $su(4)$ case

The $su(4)$ LR conditions are:

$$\begin{align*}
\lambda_1 & \geq n_{12} & n_{11} & \geq n_{22} \\
\lambda_2 & \geq n_{13} & n_{11} + n_{12} & \geq n_{22} + n_{23} \\
\lambda_2 + n_{12} & \geq n_{13} + n_{23} & n_{11} + n_{12} + n_{13} & \geq n_{22} + n_{23} + n_{24} \\
\lambda_3 & \geq n_{14} & n_{22} & \geq n_{33} \\
\lambda_3 + n_{13} & \geq n_{14} + n_{24} & n_{22} + n_{23} & \geq n_{33} + n_{34} \\
\lambda_3 + n_{13} + n_{23} & \geq n_{14} + n_{24} + n_{34} & & 
\end{align*}$$

The tensor-product elementary couplings are:

$$A_1 : \begin{array}c 1 \\ 2 \\ 3 \end{array}, A_2 : \begin{array}c \hline \hline \hline \end{array}, A_3 : \begin{array}c \hline \hline \hline \end{array}, B_1 : \begin{array}c 1 \\ 2 \\ \hline \hline \hline \end{array}, B_2 : \begin{array}c 1 \\ \hline \hline \hline \end{array}, B_3 : \begin{array}c \hline \hline \hline \end{array}$$

$$C_1 : \begin{array}c \hline \hline \hline \end{array}, C_2 : \begin{array}c 1 \\ 2 \\ 3 \end{array}, C_3 : \begin{array}c \hline \hline \hline \end{array}, D'_1 : \begin{array}c 1 \\ \hline \hline \hline \end{array}, D'_2 : \begin{array}c 1 \\ \hline \hline \hline \end{array}, D'_3 : \begin{array}c \hline \hline \hline \end{array}$$

together with

$$D_1 : \begin{array}c 1 \\ 2 \\ 3 \end{array}, D_2 : \begin{array}c 1 \\ 2 \\ 3 \end{array}, D_3 : \begin{array}c 1 \\ 2 \\ \hline \hline \hline \end{array}, E_1 : \begin{array}c 1 \\ 1 \\ \hline \hline \hline \end{array}, E_2 : \begin{array}c 1 \\ 1 \\ 2 \\ \hline \hline \hline \end{array}, E_3 : \begin{array}c 1 \\ 2 \\ 3 \end{array}$$
The Dynkin-label transcription of the elementary couplings reads

\[
\begin{align*}
A_1 &: (0,0,0) \otimes (0,0,1) \supset (0,0,1) \\
A_2 &: (0,0,1) \otimes (1,0,0) \supset (0,0,0) \\
A_3 &: (1,0,0) \otimes (0,0,0) \supset (1,0,0) \\
B_1 &: (0,0,0) \otimes (0,1,0) \supset (0,1,0) \\
B_2 &: (0,1,0) \otimes (0,1,0) \supset (0,0,0) \\
B_3 &: (0,1,0) \otimes (0,0,0) \supset (1,0,0) \\
C_1 &: (0,0,0) \otimes (1,0,0) \supset (1,0,0) \\
C_2 &: (1,0,0) \otimes (0,0,1) \supset (0,0,0) \\
C_3 &: (0,0,1) \otimes (0,0,0) \supset (0,0,1)
\end{align*}
\]

For \( su(4) \) there are 15 relations [12,10] :

\[
\begin{align*}
D'_i D_k &= C_i E_i & D_j D'_k &= B_i C_j C_k & E_i E_j &= B_k D_k D'_k \\
D_i E_i &= C_j B_k D_k & D'_i E_i &= B_j D'_j C_k
\end{align*}
\]

with \( i, j, k \) a cyclic permutation of 1, 2, 3.

To construct the generating function, we need to select forbidden couplings. It turns out that when there are more than one relation, complications may arise. We must ensure that the selected forbidden couplings are complete, which means that no further (usually higher-order) relations are required for a unique decomposition of a given coupling. A technique that is tailor-made for dealing with problems of that type is that of Grobner bases. This will be introduced in the next section. At this point, we simply indicate a complete choice of forbidden couplings, namely \( \{ E_i E_j, D'_i E_i, D_i E_i, D_j D'_i, D'_j D_i \} \). This yields then a model for the generating function, which then reads [12,10] :

\[
G^{su(4)} = (\prod_{i=1}^{3} \tilde{A}_i \tilde{B}_i \tilde{C}_i) (\tilde{D}'_1 \tilde{D}'_2 \tilde{D}'_3 + E_1 \tilde{E}_1 \tilde{D}'_2 \tilde{D}'_3 + D_3 \tilde{D}_3 \tilde{D}'_3 \tilde{E}_1 \\
+ D_2 \tilde{D}_2 \tilde{D}_3 \tilde{E}_1 + D_1 \tilde{D}_1 \tilde{D}_2 \tilde{D}_3 + E_3 \tilde{E}_3 \tilde{D}_1 \tilde{D}_2 + D'_1 \tilde{D}'_1 \tilde{D}_1 \tilde{E}_3 \\
+ D'_2 E_2 \tilde{E}_2 \tilde{D}'_3 + E_2 \tilde{E}_2 \tilde{D}'_2 \tilde{D}'_3 + E_2 \tilde{D}_1 \tilde{E}_2 \tilde{D}'_1 + E_2 D_3 \tilde{E}_2 \tilde{D}'_3 \\
+ D_1 D_3 E_2 \tilde{D}_1 \tilde{D}_3 \tilde{E}_2 + D_2 D'_2 \tilde{D}_2 \tilde{D}'_2 \tilde{E}_1 + D_2 D'_2 E_3 \tilde{D}_2 \tilde{D}'_2 \tilde{E}_3).
\]

where

\[
\tilde{M}_i = (1 - M_i)^{-1}.
\]
5. Diophantine inequalities: elementary couplings, relations and Grobner bases

We introduce the idea of the Grobner basis via a simple example (see also [13]). Suppose $R$ is a model for a generating function, where $R = Q[x, y, z, t]/I$ and $I = (xy - t, zy - t)$ is the ideal generated by $xy - t$ and $zy - t$, with an $\mathbb{N}^2$ grading given by $(1, 0), (0, 1), (1, 0)$ and $(1, 1)$ for $x, y, z$ and $t$. Writing $\bar{x} = x + I$ and similarly for the other variables, we have in $R$ that $\bar{x}\bar{y} = \bar{t}$ and $\bar{z}\bar{y} = \bar{t}$. These two expressions give two re-write rules: $xy \mapsto t$ and $zy \mapsto t$. These rules can be used to simplify any monomial. The aim is to find a re-write rule which, when iterated, produces unique representatives for the classes of $I$. If this is the case, then a vector space basis of $R$ would consist of terms of the form $m + I$ with $m$ a monomial which is not divisible by any of the left-hand sides of the rewrite rules.

In the example above, if we had ‘good’ rewrite rules then a basis for $R$ would be represented by monomials not containing $xy$ or $zy$, i.e. monomials of the form either $y^a t^b$ or $x^a z^b t^c$. The generating function which counts these monomials is:

$$\frac{1}{(1 - AB)} \left( \frac{B}{1 - B} + \frac{1}{(1 - A)^2} \right), \quad (5.1)$$

The exponent of $A$ carries the first grading index and $B$ the second.

However this generating function is not correct. It contains the term $2A^2B$ corresponding to the monomials $xt$ and $zt$. But the polynomial $z(xy - t) - x(zy - t) = xt - zt$ is also in $I$ and hence in $R$ we have $\bar{x}\bar{t} = \bar{z}\bar{t}$ and so the space of grade $(2, 1)$ has dimension 1 rather than 2. This problem can also be seen as a problem with the re-write rules. If we start with $xyz$ then we can use the first re-write rule: $xyz \mapsto tz$ or the second: $xyz \mapsto xt$. We cannot apply any further re-write rules and so this set of re-write rules does not produce a unique representative. The solution is to include the rule $xt \mapsto zt$. This gives a set of 3 rules: $xy \mapsto t$, $zy \mapsto t$ and $xt \mapsto zt$. It turns out that this is a ‘good’ set and so a basis for $R$ is given by (the classes of) monomials of the form $y^a t^b, x^a z^b$ and $z^a t^b$ which gives the generating function:

$$\frac{1}{(1 - AB)(1 - B)} + \frac{A}{(1 - A)^2} + \frac{A}{(1 - A)(1 - AB)}, \quad (5.2)$$

The set of ‘good’ generators, $xy - t, zy - t, xt - zt$ we have found for $I$ is known as a Grobner basis [14].
The general procedure for constructing a Grobner basis given a set of generating polynomials is as follows. First choose a term ordering, which is an ordering on monomials with the property that any chain $m_1 > m_2 > \ldots$ has finite length. For example we can order the variables by $x > y > z > t$ and then order all monomials by the corresponding lexicographic (dictionary) order, for example: $x^2y > xyz > y^3$. For each generator of our ideal $I$, select the monomial which is highest with respect to the given term ordering. This is then the term which appears on the left of the re-write rule. The lexicographic ordering gives the first two re-write rules of our example: $xy \mapsto t$ and $zy \mapsto t$. Next, for each pair of leading terms find the lowest common multiple and simplify it in the two possible ways. In this case there is only one pair of leading terms and the lowest common multiple is $xyz$ which simplifies to $xt$ and $yt$. Continue to apply the re-write rules until the terms do not simplify further. If the resulting pair of terms are the same, then proceed to the next pair of leading terms, otherwise add a new re-write rule. In this case we add $xt \mapsto yt$. Proceed until no pair of leading terms gives a new rule. This is the case for the rules we now have. For example the two rules $xy \mapsto t$ and $xt \mapsto zt$ appears to give a new rule by simplifying $xyt$ to both $t^2$ and $yzt$. However the second term can be further reduced to $t^2$ and so no new rule is required.

This algorithm for computing Grobner bases is known as Buchberger’s [14] algorithm. Improvements on this basic algorithm mean that it is now feasible to find Grobner bases for quite large sets of generating polynomials. (The web pages of the computer-algebra information network at the address [http://cand.can.nl/CAIN](http://cand.can.nl/CAIN) contain information about many of the programs currently available.)

Although it is not clear from this example, Grobner bases are a very versatile tool for performing explicit calculations. We end this section with an illustrative example relevant to our discussion of tensor-product generating functions.

Consider a set of linear Diophantine equations:

$$M\alpha = 0, \quad \alpha \geq 0$$  \hfill (5.3)

with $M$ an integer matrix and $\alpha$ a vector of non-negative integers. We would like to construct a generating function for the solutions to this set of equations:

$$\sum_\alpha x^\alpha.$$  \hfill (5.4)
A non-trivial example is given by the Diophantine equations that describe a $3 \times 3$ magic square:

\[
\begin{pmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{pmatrix}
\] (5.5)

with non-negative entries and equal row and column sums. The magic square condition (the sum of each row and each column is the same, say equal to $t$) gives the following set of equations:

\[
\begin{align*}
a + b + c &= t & a + d + g &= t \\
d + e + f &= t & b + e + h &= t \\
g + h + i &= t & c + f + i &= t
\end{align*}
\] (5.6)

With $\alpha$ standing for the column vector with entries $(a, b, c, d, e, f, g, h, i, t)$, the matrix $M$ reads

\[
M = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & -1
\end{pmatrix}
\] (5.7)

There is a straightforward algorithm for finding the basic set of solutions [9] which yields:

\[
\begin{align*}
\alpha_1 &= (0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1) & \alpha_4 &= (1, 0, 0, 0, 0, 1, 0, 1, 0, 1) \\
\alpha_2 &= (0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 1) & \alpha_5 &= (0, 1, 0, 1, 0, 0, 0, 1, 1, 0, 1) \\
\alpha_3 &= (0, 0, 1, 1, 0, 0, 1, 0, 1, 0, 1) & \alpha_6 &= (1, 0, 0, 0, 1, 0, 0, 0, 1, 1, 1)
\end{align*}
\] (5.8)

We shall use $A, B, \ldots, T$ to denote the “grading variables” of this example so that the exponent of $A$ carries the value of $a$ and so on. A model for the generating function is given by the subring $S$ of $\mathbb{Q}[A, B, C, D, E, F, G, H, I, T]$ generated by monomials corresponding to the 6 elementary solutions,

\[
\begin{align*}
E_1 &= CEGT, & E_2 &= BF GT, & E_3 &= CDHT, \\
E_4 &= AF HT, & E_5 &= BD IT, & E_6 &= AE IT
\end{align*}
\] (5.9)

The monomials in $S$ correspond to magic squares. For example $E_1^2 E_4 E_6 = A^2 C^2 E^3 F G^2 H I T^4 \in S$ corresponds to a square with row and column sums equal to 4:

\[
\begin{pmatrix}
2 & 0 & 2 \\
0 & 3 & 1 \\
2 & 1 & 1
\end{pmatrix}
\] (5.10)
Note that in this example it is convenient to construct our model as a subring of the ring of grading variables. Thus each “elementary coupling” $E_i$ is actually equal to the corresponding monomial in the grading variables.

However, there are relations between these generators and so it is not immediately clear how to construct the Poincaré series for $S$. What we require is an isomorphism of $S$ with $R = \mathbb{Q}[e_1, \ldots, e_6]/I$ such that $e_i \mapsto E_i, i = 1, \ldots, 6$ and such that we have a Grobner basis of the ideal $I$ (the ‘ideal of relations’).

Fortunately, such an isomorphism is easily constructed using Grobner-basis methods. Introduce the ring $\mathbb{Q}[A, B, C, D, E, F, G, H, I, T, e_1, \ldots, e_6]$ with the lexicographic ordering $A > B > C > D > E > F > G > H > I > T > e_1 > \ldots > e_6$ \hspace{1cm} (5.11)

Let $J$ be the ideal generated by $E_1 - e_1, \ldots, E_6 - e_6$. This is not necessarily a Grobner basis with respect to this term ordering. Let $G$ be the Grobner basis for $J$ with the given ordering. Then it can be shown [14] that $G \cap \mathbb{Q}[e_1, \ldots, e_6]$ is a Grobner basis for the ideal of relations $I$ which we require. In this case $G$ is quite large, but its intersection with $\mathbb{Q}[e_1, \ldots, e_6]$ is $e_1 e_4 e_5 - e_2 e_3 e_6$. The corresponding relation in $R$ is $E_1 E_4 E_5 - E_2 E_3 E_6$ and these two terms do indeed give the same magic square, so that indeed we have found a relation between the generators of $R$. The Poincaré series for $\mathbb{Q}[e_1, \ldots, e_6]/I$ is easily computed:

$$
\frac{1}{(1 - E_2)(1 - E_3)(1 - E_6)} \left( \frac{1}{(1 - E_1)(1 - E_4)} + \frac{E_5}{(1 - E_1)(1 - E_5)} + \frac{E_4 E_5}{(1 - E_4)(1 - E_5)} \right) \hspace{1cm} (5.12)
$$

6. Berenstein-Zelevinsky Triangles

6.1. Generalities

The previous examples make clear the usefulness of a re-expression of the tensor-product calculation in terms of Diophantine inequalities. The Littlewood-Richardson algorithm yields a set of such inequalities only for $su(N)$. Fortunately, Berenstein and Zelevinsky [15] have expressed the solution of the multiplicity of a given tensor product as a counting problem for the number of integral points in a convex polytope. For a given algebra, the polytope is formulated in terms of a characteristic set of inequalities. For $su(N)$, these reduce to the LR set of inequalities. For the other classical algebras, except $sp(4)$, the proposed set of inequalities is a conjecture.
6.2. BZ triangles for \( sp(4) \)

The combinatorial description of tensor products for \( sp(4) \) is not as simple as in the \( su(N) \) case: a standard LR product must be supplemented by a division operation and modification rules [16]. Given the BZ set of inequalities, the natural way to proceed, as just mentioned, is to interpret these as the appropriate inequalities for the description of the tensor products. These inequalities are as follows:

\[
\begin{align*}
\lambda_1 &\geq p & \mu_1 &\geq q \\
\lambda_2 &\geq r_1/2 & \mu_1 &\geq q + r_1 - r_2 \\
\lambda_2 &\geq r_1/2 + q - p & \mu_1 &\geq p + r_1 - r_2 \\
\lambda_2 &\geq r_2/2 + q - p & \mu_2 &\geq r_2/2 \\
\nu_1 &= r_2 - r_1 - 2p + \lambda_1 + \mu_1 & \nu_2 &= p - q - r_2 + \lambda_2 + \mu_2
\end{align*}
\]

(Our notation is different from that used in [15]: the relation is \( r_1 = m_1, r_2 = m_2, p = m_{12}, q = m^\dagger_{12} \).) The \( sp(4) \) tensor product coefficient \( N_{\lambda\mu\nu} \) is thus given by the number of solutions of the above system with \( r_1, r_2 \in \mathbb{N} \) et \( p, q \in \mathbb{N} \) (\( \mathbb{N} \) being the set of nonnegative integers).

A proper set of variables for a complete description of a particular tensor-product coupling is thus \( \{\lambda_1, \lambda_2, \mu_1, \mu_2, r_1, r_2, p, q\} \). We give the list of elementary couplings, adding to each coupling the corresponding four-vector \( [r_1, r_2, p, q] \):

\[
\begin{array}{lll}
A_1 &: (0,0) \otimes (1,0) \supset (1,0) & [0,0,0,0] \\
A_2 &: (1,0) \otimes (0,0) \supset (1,0) & [0,0,0,0] \\
A_3 &: (1,0) \otimes (1,0) \supset (0,0) & [0,0,1,1] \\
B_1 &: (0,0) \otimes (0,1) \supset (0,1) & [0,0,0,0] \\
B_2 &: (0,1) \otimes (0,0) \supset (0,1) & [0,0,0,0] \\
B_3 &: (0,1) \otimes (0,1) \supset (0,0) & [2,2,0,0] \\
C_1 &: (0,1) \otimes (1,0) \supset (1,0) & [0,0,0,1] \\
C_2 &: (1,0) \otimes (0,1) \supset (1,0) & [0,2,1,0] \\
C_3 &: (1,0) \otimes (1,0) \supset (0,1) & [0,0,1,0] \\
D_1 &: (2,0) \otimes (0,1) \supset (0,1) & [0,2,2,0] \\
D_2 &: (0,1) \otimes (2,0) \supset (0,1) & [2,0,0,0] \\
D_3 &: (0,1) \otimes (0,1) \supset (2,0) & [0,2,0,0]
\end{array}
\]
The unspecified linear relations mentioned in (2.37) can now be obtained. To find those products that are equal in the current situation we need only compare their corresponding sets of four-vectors \([r_1, r_2, p, q]\) (which are additive in products of couplings). We thus find for instance that

\[
C_1 C_2 = A_3 D_3 : [0, 2, 1, 1] \neq A_1 A_2 B_3 : [2, 2, 0, 0]
\]

(6.3)

Proceeding in this way for the other cases, we find the following complete list of relations:

\[
\begin{align*}
C_1 C_2 &= A_3 D_3, & C_2 C_3 &= A_1 D_1 & C_3 C_1 &= A_1 A_3 B_2 \\
D_1 D_2 &= B_3 C_3^2 & D_2 D_3 &= A_1^2 B_2 B_3 & D_1 D_3 &= B_2 C_2^2 \\
C_1 D_1 &= A_3 B_2 C_2 & C_2 D_2 &= A_1 B_3 C_3 & C_3 D_3 &= A_1 B_2 C_2
\end{align*}
\]

(6.4)

The use of the BZ inequalities to find the elementary couplings and their relations is novel. (An off-shoot of our construction is that it provides an indirect proof of the validity of the BZ inequalities since we recover from it the result of [7] derived from the character method.)

A possible choice of forbidden products is the one given in [7]:

\[\{C_i C_j, D_i D_j, C_i D_i\}\]

(6.5)

with \(i, j = 1, 2, 3\) and \(i \neq j\). It leads to the generating function:

\[
G^{sp(4)} = \left( \prod_{i=1}^{3} \tilde{A}_i \tilde{B}_i \right) \left( \tilde{C}_1 \tilde{D}_2 + D_3 \tilde{C}_1 \tilde{D}_3 + C_2 D_1 \tilde{C}_2 \tilde{D}_1 + C_2 \tilde{C}_2 D_3 + D_1 \tilde{C}_3 \tilde{D}_1 + C_3 \tilde{C}_3 \tilde{D}_2 \right)
\]

(6.6)

Of course, by modifying the ordering in the Grobner basis, we can get other choices of forbidden couplings. Here is another set of forbidden couplings that can be obtained: \(\{D_i D_j, C_i D_i, A_1 D_1, A_3 D_3, A_1 A_3 B_2\}\). The corresponding generating function reads

\[
G^{sp(4)} = \tilde{B}_1 \tilde{B}_2 \tilde{B}_3 \left[ \left( \prod_{i=1}^{3} \tilde{A}_i \right) \tilde{C}_i (1 - A_1 A_3 B_2) + D_3 \tilde{D}_3 \tilde{A}_1 \tilde{A}_2 \tilde{C}_1 \tilde{C}_2 + D_1 \tilde{D}_1 \tilde{A}_2 \tilde{A}_3 \tilde{C}_2 \tilde{C}_3 + D_2 \tilde{D}_2 \tilde{A}_1 \tilde{A}_3 \tilde{C}_1 \tilde{C}_3 (1 - A_1 A_3 B_2) \right].
\]

(6.7)

These two generating functions are equivalent when rewritten in terms of the grading variables, that is, in terms of Dynkin labels. However, they originate from two distinct models. The second one turns out to be well adapted to the fusion extension.
7. A vector basis approach to the construction of generating functions

In this section, we present a simple and systematic way of generating by hand all the elementary solutions of a set of linear homogeneous inequalities starting from the well-known construction of a vector basis. The first step amounts to reformulate the system of inequalities in terms of equalities. We then look for the elementary independent solutions by relaxing the positivity requirement. In other words, we construct the vector basis. In a final step, we find the minimal linear combinations of these vector basis elements that yield positive solutions. This will also provide an illustration of MacMahon’s projection technique. The result of this projection is the desired tensor-product generating function. Hence, this approach turns out to be a new way of constructing the tensor-product generating functions. (This generic method, referred to as being novel for tensor products, is certainly well-known in general: it is discussed in the first reference of [11].)

7.1. Graphical representations as BZ triangles for $su(N)$

Consider the direct transformation of the LR inequalities to equalities by introducing an appropriate number of new non-negative integer variables. Consider first the $su(2)$ case, for which there is a single inequality: $\lambda_1 \geq n_{12}$. We transform this into an equality by introducing the non-negative integer $a$ defined by

$$\lambda_1 = n_{12} + a \quad (7.1)$$

The expression for $\nu_1$ becomes then $\nu_1 = \lambda_1 + n_{11} - n_{12} = a + n_{11}$. Since $\mu_1 = n_{11} + n_{12}$, we are led naturally to a triangle representation of the tensor product:

$$\lambda \otimes \mu \supseteq \nu \leftrightarrow \begin{array}{c} a \\ n_{12} \\ n_{11} \end{array} \quad (7.2)$$

We read off the Dynkin label of the $\lambda$ representation from the sum of the two integers that form the left side of the triangle, that of the $\mu$ representation from the bottom of the triangle and the $\nu_1$ label is the sum of the two integers that form the right side. A more uniform notation amounts to setting $a = m_{12}$ and $n_{11} = l_{12}$, in terms of which the triangle looks quite symmetrical:

$$\begin{array}{c} m_{12} \\ n_{12} \\ l_{12} \end{array} \quad (7.3)$$
with
\[ \lambda_1 = m_{12} + n_{12} \quad \mu_1 = n_{12} + l_{12} \quad \nu_1 = m_{12} + l_{12} \] (7.4)

These numbers \( m_{12} \) and \( l_{12} \) plays the role of \( n_{12} \) in the permuted versions of the tensor product. The triangle combinatorial reformulation of the tensor product problem is as follows: the number of triangles that can be formed from non-negative integers \( n_{12}, m_{12} \) and \( l_{12} \) that add up to the Dynkin labels of the representations under study according to the above relations gives the multiplicity of the triple coupling \( \lambda \otimes \mu \supset \nu \), or equivalently, the multiplicity of the scalar representation in the product \( \lambda \otimes \mu \otimes \nu \supset (0) \) (since for \( su(2) \), \( \nu^* = \nu \)).

The situation for \( su(3) \) is somewhat more complicated. The transformation of the LR inequalities (4.1, 4.2) into equalities in this case takes the form

\[ \begin{align*}
\lambda_1 &= n_{12} + a \\
\lambda_2 &= n_{13} + b \\
\lambda_2 + n_{12} &= n_{13} + n_{23} + c
\end{align*} \]

(7.5)

The expression for the other weights becomes

\[ \begin{align*}
\mu_1 &= n_{13} + e \\
\mu_2 &= n_{22} + n_{23} \\
\nu_1 &= a + d \\
\nu_2 &= n_{22} + c
\end{align*} \]

(7.6)

Since there are two expressions for both \( n_{11} \) and \( \lambda_2 \), there follows the compatibility relations:

\[ \begin{align*}
n_{12} + d &= n_{23} + e \\
n_{23} + c &= b + n_{12}
\end{align*} \]

(7.7)

By adding these two relations, we find:

\[ c + d = b + e \]

(7.8)

Again we are led naturally to a triangle representation: with \( \zeta = \nu^* \) this reads

\[ \begin{array}{cccc}
a & n_{12} & d & \lambda_1 \\
b & c & \nu_1 & \lambda_2 \\
n_{13} & e & n_{23} & n_{22}
\end{array} \]

(7.9)

We read the Dynkin labels from the sides of the triangles, from \( \lambda_1 \) to \( \zeta_2 \) in an anti-clockwise rotation starting from the top of the triangle, exactly as for \( su(2) \), except that here there are two labels on each sides. Notice that the compatibility conditions amounts to the
equality of the sums of the extremal points of the three pairs of opposite sides of the hexagon obtained by dropping the three corners of the triangle.

Again a more symmetrical notation is:

\[ a = m_{13} \quad b = m_{23} \quad c = m_{12} \quad d = l_{23} \quad e = l_{12} \quad n_{22} = l_{13} \quad (7.10) \]

in terms of which the triangle reads

\[
\begin{array}{cccc}
m_{13} & n_{12} & l_{23} \\
m_{23} & m_{12} & n_{23} \\
n_{13} & l_{12} & n_{13} \\
\end{array}
\]

(7.11)

with labels fixed by:

\[ \lambda_1 = m_{13} + n_{12} \quad \lambda_2 = m_{23} + n_{13} \]
\[ \mu_1 = n_{13} + l_{12} \quad \mu_2 = n_{23} + l_{13} \quad (7.12) \]
\[ \zeta_1 = l_{13} + m_{12} \quad \zeta_2 = l_{23} + m_{13} \]

The hexagon conditions read:

\[ n_{12} + m_{23} = n_{23} + m_{12}, \]
\[ l_{12} + m_{23} = l_{23} + m_{12}, \quad (7.13) \]
\[ l_{12} + n_{23} = l_{23} + n_{12}. \]

In terms of triangles, the problem of finding the multiplicity of the \( su(3) \) tensor product \( \lambda \otimes \mu \otimes \zeta \supset 0 \) boils down to enumerating the number of triangles made with non-negative integers that form a bipartition of the Dynkin labels and that satisfy the above three hexagon relations. (For the \( su(N) \) generalisation, see [17]).

Here is the rationale for the labelling \( n_{ij}, m_{ij}, l_{ij} \) from the triangle point of view [18]. If \( e_i \) are orthonormal vectors in \( \mathbb{R}^N \), then the positive roots of \( su(N) \) can be represented in the form \( e_i - e_j, \ 1 \leq i < j \leq N \). The triangle encodes three sums of positive roots:

\[ \mu + \zeta - \lambda^* = \sum_{i<j} l_{ij}(e_i - e_j), \]
\[ \zeta + \lambda - \mu^* = \sum_{i<j} m_{ij}(e_i - e_j), \]
\[ \lambda + \mu - \zeta^* = \sum_{i<j} n_{ij}(e_i - e_j), \quad (7.14) \]

The hexagon relations are simply the consistency conditions for these three expansions. Clearly, the variables \( n_{ij} \) that appear in the above relations are exactly the \( n_{ij} \) that appear in the LR tableaux for the product \( \lambda \otimes \mu \supset \zeta^* = \nu \).
7.2. From a vector basis to the generating function: the \( \text{su}(3) \) case

Given the transcription of inequalities into equalities, we can easily extract the corresponding basis vectors. This is the starting point of a new method for constructing the tensor-product generating functions. To keep things concrete, we focus on the \( \text{su}(3) \) case. The goal is to first get a vector basis and then to project it to get the elementary couplings. The generating function is a direct result of this procedure.

The equality version of the LR inequalities are (7.12) and (7.13); they underlie the construction of the BZ triangle (7.11). The last hexagon condition of (7.13) is the difference of the previous two so it is not an independent relations. We thus have a total of 15 variables: \( \lambda_1, \ldots, \zeta_2, l_{12}, \ldots, n_{23} \) and 8 equations. The number of independent variables is thus 7. These will be chosen to be \( m_{13}, m_{23}, l_{13}, l_{23}, n_{12}, n_{13}, n_{23} \). The dependent variables are fixed as follows:

\[
\begin{align*}
\lambda_1 &= m_{13} + n_{12} & \lambda_2 &= m_{23} + n_{13} \\
\mu_1 &= n_{13} + n_{12} + l_{23} - n_{23} & \mu_2 &= n_{23} + l_{13} \\
\zeta_1 &= n_{12} + m_{23} + l_{13} - n_{23} & \zeta_2 &= l_{23} + m_{13} \\
l_{12} &= n_{12} + l_{23} - n_{23} & m_{12} &= n_{12} + m_{23} - n_{23}
\end{align*}
\]

We now look for the elementary solutions of this system (without invoking the constraint that all the above dependent variables should be necessarily positive). The sought basis vectors are obtained by setting one of the variable \( m_{13}, \ldots, n_{23} \) to 1 and all other set equal to zero.

This produces (in order) the triangles \( E_2, E_5, E_6, E_3, E_7, E_4 \) and \( Z_1 \) displayed below:

\[
\begin{align*}
E_2 &: (1,0)(0,0)(0,1) & E_3 &: (0,0)(1,0)(0,1) \\
E_4 &: (0,1)(1,0)(0,0) & E_5 &: (0,1)(0,0)(1,0) & E_6 &: (0,0)(0,1)(1,0)
\end{align*}
\]
These are all genuine BZ triangles except for $Z_1$ which has some negative entries. However, at this level, there are no relations between these elementary solutions (the basis vectors are independent), hence the decomposition of any solution in terms of these 7 basic ones is unique. All solutions are then freely generated from the following function:

$$G = \frac{1}{(1 - E_2)(1 - E_3)(1 - E_4)(1 - E_5)(1 - E_6)(1 - E_7)(1 - Z_1)} \quad (7.17)$$

To recover the generating function for all tensor products from the above expression, we need to project out terms that lead to triangles with negative entries. To achieve this, we introduce the grading variables associated to the above couplings (compare the above triangles with the general form given in (7.11)):

$$E_2 : M_{13} \quad E_3 : L_{12}L_{23} \quad E_4 : N_{13}$$

$$E_5 : M_{12}M_{23} \quad E_6 : L_{13} \quad E_7 : L_{12}M_{12}N_{12}$$

$$Z_1 : L^{-1}_{12}M^{-1}_{12}N_{23} \quad (7.18)$$

Our generating function follows from the projection of the above function $G$, re-expressed in terms of the grading variables, to positive powers of $L_{12}$ and $M_{12}$. Equivalently, one can re-scale $L_{12}$ by $x$ and $M_{12}$ by $y$ and project to positive powers of $x$ and $y$ and set $x = y = 1$ in the result. This is equivalent to the rescaling

$$E_3 \rightarrow xE_3 \quad E_5 \rightarrow yE_5 \quad E_7 \rightarrow xyE_7 \quad Z_1 \rightarrow x^{-1}y^{-1}Z_1 \quad (7.19)$$

We are thus led to consider

$$\frac{\frac{x}{y}}{\Omega} G(E_2, xE_3, \ldots, x^{-1}y^{-1}Z_1) \quad (7.20)$$

Keeping only those terms which depend explicitly upon $x$ or $y$, we have then

$$\frac{\frac{x}{y}}{\Omega} \frac{1}{(1 - xE_3)(1 - yE_5)(1 - xyE_7)(1 - x^{-1}y^{-1}Z_1)}$$

$$= \frac{1}{(1 - xE_3)(1 - yE_5)(1 - E_7Z_1)} \left( \frac{1}{1 - xyE_7} + \frac{x^{-1}y^{-1}Z_1}{1 - x^{-1}y^{-1}Z_1} \right) \quad (7.21)$$
No more work is needed for the first term. For the second one, we have
\[
\frac{x^y}{\Omega} \geq \frac{x^{-1} y^{-1} Z_1}{(1 - x E_3)(1 - E_7 Z_1)(1 - x^{-1} Z_1 E_5)} \left( \frac{y E_5}{1 - y E_5} + \frac{1}{1 - x^{-1} y^{-1} Z_1} \right)
\]
\[
= \frac{x^y}{\Omega} \geq \frac{x^{-1} E_5 Z_1}{(1 - E_5)(1 - E_7 Z_1)(1 - x E_3)(1 - x^{-1} Z_1 E_5)} \left( \frac{x E_3}{1 - x E_3} + \frac{1}{1 - x^{-1} Z_1 E_5} \right)
\]
\[
= \frac{x^y}{\Omega} \geq \frac{x^{-1} E_5 Z_1}{E_3 E_5 Z_1} \left( \frac{x E_3}{1 - E_5 Z_1 E_3} \right) \left( 1 - E_7 E_8 \right)
\]
\[
= \frac{x^y}{\Omega} \geq \frac{x^{-1} E_5 Z_1}{(1 - E_5)(1 - E_7 Z_1)(1 - E_3 E_5 Z_1)(1 - E_3)}
\]

We then introduce the following two new elementary couplings
\[
E_1 = E_7 Z_1, \quad E_8 = E_3 E_5 Z_1
\]

Collecting the two terms resulting from the projection, we end up with
\[
G^{su(3)} = \left( \prod_{i=1}^{8} \tilde{E}_i \right) (1 - E_7 E_8)
\]

which is indeed the \( su(3) \) tensor-product generating function.

It is worth pointing out that the Elliot-MacMahon algorithm that has been presented here as a method distinct from the vector basis, can be reinterpreted in a way that makes the two approaches equivalent. This is done in section 3 of the first reference of [11]. There, the elementary solutions are not obtained as above by setting successively one dependent variables equal to 1 and the others equal to 0, but in reading them off directly from the columns of the \( 8 \times 7 \) matrix of the matrix version of the above equation:

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & -1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & -1 & 0 & 1 \\
0 & 1 & 1 & 0 & -1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
n_{12} \\
m_{23} \\
n_{13} \\
n_{23} \\
l_{13} \\
l_{23}
\end{pmatrix}
= 
\begin{pmatrix}
\mu_1 \\
\mu_2 \\
\zeta_1 \\
\zeta_2 \\
l_{12} \\
m_{12}
\end{pmatrix}
\]

The exponentiated version of the columns gives the elementary solutions written below. This leads to the so-called ‘crude’ generating function that is then projected onto the positive solutions by the usual method.
7.3. General aspects of the vector basis construction

In general, of course, the fundamental solutions to the linear system may have non-integral values of the variables. However the corresponding terms in the generating function can be eliminated by rationalising all the denominator terms and then keeping only those terms in the numerator that have integral exponents. This suggests the following modification of MacMahon’s algorithm.

Consider the system of equations

\[ Mx = 0, \quad x \in \mathbb{N}^k \]  

where \( M \) is a matrix of rank \( s \). We thus have \( k \) variables and \( s \) relations between them. The dimension of the vector basis is thus \( k - s \). We will denote the independent (free) variables as \( x_i, i = 1, \ldots, k - s \) and the remaining ones as \( \bar{x}_j, j = 1, \ldots, s \). To find a generating function for the solutions of this system:

1. First construct a basis in \( \mathbb{Q}^k \) for the solutions of \( Mx = 0 \) by setting \( x_i = 1 \) with all other \( x_j \) zero \( (j = 1, \ldots, k - s, j \neq i) \). Denote by \( \bar{x}_j^{(1)} \) the value of the dependent variable \( \bar{x}_j \) evaluated at \( x_1 = 1 \) with all other \( x_i \) zero. The basis then reads

\[
\begin{align*}
\epsilon_1 &= (1, 0, 0, \ldots, 0; \{\bar{x}_j^{(1)}\}), \\
\epsilon_2 &= (0, 1, 0, \ldots, 0; \{\bar{x}_j^{(2)}\}), \\
&\quad \vdots \\
\epsilon_{k-s} &= (0, 0, 0, \ldots, 1; \{\bar{x}_j^{(k-s)}\})
\end{align*}
\]  

By construction, the \( \epsilon_i \)’s are linearly independent. However notice that in general the \( \bar{x}_j^{(i)} \) might be rational.

2. From the form of the \( \epsilon_i \)’s, it follows that any solution to (7.25) can be written as \( \sum_i c_i \epsilon_i \) with \( c_i \) non-negative integers. In particular this means that every solution to (7.25) corresponds to a term in the generating function:

\[ G(X) = \frac{1}{(1 - X^{\epsilon_1})(1 - X^{\epsilon_2}) \ldots (1 - X^{\epsilon_{k-s}})} \]  

where \( X_1, \ldots, X_k \) are grading variables.

3. \( G(x) \) may contain negative or fractional exponents due to the occurrence of \( \bar{x}_j^{(i)} \) in the exponents. These are eliminated by first using MacMahon’s algorithm to eliminate
any negative exponents and then rationalising denominators and keeping only terms with integral exponents in the numerators.

The result is the generating function for the solutions to (7.25). This algorithm, however, does not seem to be optimal in all case.

7.4. Multiple su(2) products from the vector basis construction

A simple and different application of the formalism just developed is furnished by the analysis of su(2) quadruple tensor products. This application is different in that it does not rely on the triangle description and as such, its formulation is less direct. This does not mean however that there are no diagrammatic representations for the quadruple product. In fact, having a set of inequalities, we can transform them into equalities, as it is done below, and from them set up a diagrammatic representation. In the present case, it would correspond to two adjacent su(2) triangles, one upside down, with their adjacent sides forced to be equal. However, our analysis will not rely on such a description. It will serve as a preparation the somewhat more complicated sp(4) example treated in the following section.

The Diophantine description of this problem has been presented in section 4.2. It is based on the two inequalities (4.11) which are readily transformed into equalities by the introduction of two non-negative integers $a_1, a_2$:

$$\lambda_1 = n_{12} + a_1 \quad \lambda_1 + n_{11} - n_{12} = m_{12} + a_2$$

(7.28)

However this system does not contain any reference to the variable $m_{11}$ and for this reason we introduce the further constraint $m_{11} \geq 0$ which calls for a new non-negative integer variable:

$$m_{11} = a_3$$

(7.29)

We have thus a total of 8 variables : $\{\lambda_1, n_{11}, n_{12}, m_{11}, m_{12}, a_1, a_2, a_3\}$ and 3 equations. There are thus 5 independent variables, chosen to be $\{a_1, a_2, a_3, n_{12}, m_{12}\}$. The basis vectors, with components ordered as follows

$$(a_1, a_2, a_3, n_{12}, m_{12}; \lambda_1, n_{11}, m_{11})$$

(7.30)
are obtained by successively setting equal to 1 one of \( \{a_1, a_2, a_3, n_{12}, m_{12}\} \) and the others equal to 0. These basis vectors together with their exponentiated version written in terms of appropriate grading variables read:

\[

d (1, 0, 0, 0; 1, -1, 0) : L_{1} N_{11}^{-1} A_1 \\
(0, 1, 0, 0; 0, 1, 0) : N_{11} A_2 \\
(0, 0, 1, 0; 0, 0, 1) : M_{11} A_3 \\
(0, 0, 0, 1; 0, 1, 0) : L_{1} N_{12} \\
(0, 0, 0, 1; 0, 1, 0) : N_{11} M_{12}
\] (7.31)

The desired generating function is obtained from the projection to positive powers of \( N_{11} \) of the function

\[
G = \frac{1}{(1 - L_{1} N_{11}^{-1} A_1)(1 - N_{11} A_2)(1 - L_{1} N_{12})(1 - N_{11} M_{12})(1 - M_{11} A_3)}
\] (7.32)

The projection operation is done by the familiar method and the result, after setting all \( A_i = 1 \) is

\[
G = \frac{1 - L_{1} N_{11} M_{12}}{(1 - L_{1} N_{12})(1 - L_{1} M_{12})(1 - L_{1})(1 - N_{11} M_{12})(1 - N_{11})(1 - M_{11})}
\] (7.33)

from which we read of the 6 elementary couplings \( E_1, \cdots, E_6 \) (in the order where they appear in the denominator) given in (4.12) and the relation \( E_3 E_4 = E_2 E_5 \). The above function is exactly the one derived in section 4.2.

7.5. \( sp(4) \) diamonds and the vector basis derivation of the generating function

The system of inequalities (6.1) pertaining to \( sp(4) \) can be transformed into a system of equations in the standard way: by setting \( r_1/2 = s_1 \) and \( r_2/2 = s_2 \) and introducing the non-negative integers \( a_i \), we get [19]:

\[
\begin{align*}
\lambda_1 &= p + a_1 & \nu_2 &= a_4 + a_8 \\
\lambda_2 &= s_1 + a_2 & a_2 + p &= a_3 + q \\
\mu_1 &= q + a_5 & a_3 + s_1 &= a_4 + s_2 \\
\mu_2 &= s_2 + a_8 & a_5 + 2s_2 &= a_6 + 2s_1 \\
\nu_1 &= a_1 + a_7 & a_6 + q &= a_7 + p
\end{align*}
\] (7.34)
This leads to a diamond-type graphical representation of the tensor product that has the advantage over the one presented in [15] of being linear in that the sum of two diamonds is also a diamond. This is illustrated in Fig. 1.

In Fig. 1, all data pertaining to the first (second) Dynkin label appear at the left (right). Dotted lines relate those two points that compose the label indicated beside it. Opposite continuous lines are constrained to be equal, with the length of a line being defined as the sum of its extremal points except for the lines delimited by the points \((a_6, s_1)\) and \((a_5, s_2)\) where the point \(s_i\) is counted twice (the little bar besides \(s_1\) and \(s_2\) being a reminder of this). Explicitly, for those lines, we have thus the constraint \(a_6 + 2s_1 = a_5 + 2s_2\). Given a triple \(sp(4)\) product, the number of such diamonds that can be drawn with non-negative entries yields the multiplicity of the product. For instance, the two diamonds that describe the triple coupling \((1, 1) \otimes (1, 1) \otimes (2, 0)\) are shown in Fig 2.

The dimension of the vector basis is 8 (18 variables and 10 equations, the last four equations above being linearly independent). As our free variables we choose the set \(\{s_1, s_2, p, q, a_1, a_3, a_6, a_8\}\). The 8 basis vectors in terms of grading variables are:

\[
\begin{align*}
E_1 : L_2M_1^2N_2A_4A_5^2S_1 & \quad E_2 : M_1^{-2}M_2N_2^{-1}A_4^{-1}A_5^{-2}S_2 \\
E_3 : L_1L_2^{-1}N_1^{-1}A_2^{-1}A_7^{-1}P & \quad E_4 : L_2M_1N_1A_2A_7Q \\
E_5 : L_1N_1A_1 & \quad E_6 : L_2N_2A_2A_3A_4 \\
E_7 : M_1N_1A_5A_6A_7 & \quad E_8 : M_2N_2A_8
\end{align*}
\]  

(7.35)

The generating function is obtained by first projecting of the function \(\prod(1 - \mathcal{E}_i)^{-1}\) to positive powers for each grading variables and then by setting all grading variables equal to 1 except for \(L_i, M_i, N_i\)’s. The \(sp(4)\) elementary couplings are simple products of the \(\mathcal{E}_i\)’s (the following \(A_{1,2,3}\) should not be confused with the above grading variables):

\[
\begin{align*}
A_1 = \mathcal{E}_7 & \quad A_2 = \mathcal{E}_5 & \quad A_3 = \mathcal{E}_3\mathcal{E}_4 \\
B_1 = \mathcal{E}_8 & \quad B_2 = \mathcal{E}_6 & \quad B_3 = \mathcal{E}_1\mathcal{E}_2 \\
C_1 = \mathcal{E}_4 & \quad C_2 = \mathcal{E}_2\mathcal{E}_3\mathcal{E}_6\mathcal{E}_7^2 & \quad C_3 = \mathcal{E}_1\mathcal{E}_3\mathcal{E}_7 \\
D_1 = \mathcal{E}_2\mathcal{E}_3^2\mathcal{E}_6^2\mathcal{E}_7^2 & \quad D_2 = \mathcal{E}_1 & \quad D_3 = \mathcal{E}_2\mathcal{E}_6\mathcal{E}_7^2
\end{align*}
\]  

(7.36)

The complete list of \(sp(4)\) elementary couplings (6.2) are thus recovered.
8. Conclusion

As was stressed in the introduction, the main purpose of this work is to prepare the ground for the analysis of fusion rules, which is the subject of a sequel paper. In this paper, we have reviewed the existing techniques for computing tensor-product generating functions and presented a comparative assessment of their virtues and limitations. We also focused on a model formulation linking generating functions to Poincaré series, an idea first introduced in [11] and extended in [10]. Our contribution has been to rephrase this program more explicitly, clarify some issues and to exemplify the procedure with many examples, some of which are new. An extended version of this article is available on the Los Alamos server [20].

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Figure 1
Figure 2