Marginal chimera state at cross-frequency locking of pulse-coupled neural networks

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We consider two coupled populations of leaky integrate-and-fire neurons. Depending on the coupling strength, mean fields generated by these populations can have incommensurate frequencies, or become frequency locked. In the observed 2:1 locking state of the mean fields, individual neurons in one population are asynchronous with the mean fields, while in another populations they have the same frequency as the mean field. These synchronous neurons form a chimera state, where part of them build a fully synchronized cluster, while other remain scattered. We explain this chimera as a marginal one, caused by a self-organized neutral dynamics of the effective circle map.

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I. INTRODUCTION

Studies of the dynamics of globally coupled populations of oscillators, pioneered more than 40 years ago by Winfree and Kuramoto [1], are in the focus of current research due to numerous applications in diverse fields from physics to neuroscience, but also due to striking effects like synchronization, collective chaos, and chimera states [2].

While typically ensembles of identical oscillators either fully synchronize or desynchronize, depending on whether the coupling is attractive or not, there are situations where oscillators produce a macroscopic mean field without full synchrony, such a regime is called partial synchronization [3, 4] (see Ref. [5] for its experimental observation). Remarkably, partial synchronization can be explained within a simplest setup of one-dimensional oscillators, described either by their phase dynamics [4], or as integrate-and-fire units [3]. Another way of breaking full synchrony is formation of a chimera state, where part of a population of identical oscillators synchronize and form a cluster, while other remain asynchronous. This regime requires self-induced bistability, so far such regimes have not been observed in one-dimensional phase models, but in oscillators described by high-dimensional equations [6].

In this paper we study mutual coupling of two populations of partially synchronous integrate-and-fire oscillators, and find a surprising chimera state in this system: at mutual 2:1 locking of two macroscopic mean fields, elements of one ensemble form a cluster and a marginally stable scattered group. This means that nontrivial chimeras can happen in populations of one-dimensional units. Below we will analyze how the mean field dynamics, yielding effective bistability in such ensembles, appears in a self-consistent way.

Before proceeding to description of the model, we mention that mutual influence of two or several populations of oscillators, generating macroscopic mean fields at significantly different frequencies, has been studied recently in the context of phase dynamics of Kuramoto model type [7]. Here we extend these studies to realistic neural models of integrate-and-fire neurons, which is applicable for explanation of cross-relations between brain waves. Indeed, in brain one observes macroscopic oscillations in different frequency ranges [8], and these regularly regular mean fields are not related to exact synchrony of individual neurons, but rather to a temporal organization of their firing events. Thus, the model of partial synchronization we consider below, appears more adequate for the neural dynamics than the Kuramoto model of phase oscillators.

II. THE MODEL

Our consideration of two neural populations is based on the model of globally coupled leaky integrate-and-fire oscillations, a prototypical example of generation of nontrivial mean fields due to partial synchronization [3]. One such population has been thoroughly studied in Refs. [3, 9, 10]. The potential $x_k$ of each neuron (here $1 \leq k \leq N$, $N$ is the size of the population) is described by the following equation

$$\dot{x}_k = a - x_k + gE_k,$$  \hspace{1cm} (1)

where the mean field $E$ is composed by contributions from all the neurons

$$\dot{E} + 2\alpha \dot{E} + \alpha^2 E = \frac{\alpha^2}{N} \sum_{j} \delta(t - t_{k,j}).$$  \hspace{1cm} (2)

The potential of neuron $k$ grows from 0 to the threshold value 1, governed by the suprathreshold input current $a > 1$ and the excitatory action from the field $E$ (Eq. (1)). When this potential reaches 1 at a time instant $t_{k,n}$, the neuron fires, contributing a delta-function pulse to the field $E$ (Eq. (2)), and is reset to 0. Linear equation (2) describes thus a sequences of so-called $\alpha$-pulses created by the spiking neurons. As the analysis of system (1,2) has shown (see Refs. [3, 9, 10] for
details), for large values of the coupling parameter $g$ (at given $\alpha$) the neurons are desynchronized: the time intervals $t_{k,n} - t_{k-1,n}$ between successive firing events of two neurons are constant (do not depend on $k$), and the field $E$ is nearly a constant (with small variations $\sim N^{-1}$). At some critical value of coupling $g$ this regime becomes unstable, and the neurons start to form a group with a smaller interval between firings, as a result the mean field $E(t)$ demonstrate macroscopic nearly periodic variations $\lbrack 3, 9 \rbrack$. This state is called partial synchronization, because neurons never synchronize fully (never fire simultaneously). In fact, in this state the dynamics is, strictly speaking, quasiperiodic, because frequency of spiking of a neuron $\omega$ (it this the same for all neurons because they are identical) is incommensurate to the frequency of the macroscopic mean field oscillations $\Omega$.

In this paper we consider two interacting populations of neurons of the described type. Contrary to Ref. $\lbrack 11 \rbrack$, where two identical populations have been considered, we study two different populations (while inside each population all the neurons are identical), therefore we use slightly changed notations: membrane potentials of the neurons in the populations will be denoted as $x_k$ and $y_k$, while the mean fields generated by them as $X$ and $Y$, respectively. Interaction is due to a mixture of the mean fields: on neurons $x$ acts the field $E_x = (1 - \varepsilon)X + \varepsilon Y$, while on neurons $y$ acts the field $E_y = (1 - \varepsilon)Y + \varepsilon X$. Parameter $\varepsilon$ describes the coupling between populations. Equivalently, one can consider this setup as one with two acting mean fields $(E_x, E_y)$, which are fed by spikes from two populations: each spike of a neuron $x$ contributes a delta-kick with amplitude $(1 - \varepsilon)$ to the field $E_x$, and a delta-kick with amplitude $\varepsilon$ to the field $E_y$, and similarly for neurons $y$. We will assume that both populations have equal number of neurons $N$. The equations of the model thus read

$$\begin{align*}
\dot{x}_k &= a_x - x_k + g_x((1 - \varepsilon)X + \varepsilon Y), \quad (3) \\
\dot{y}_k &= a_y - y_k + g_y((1 - \varepsilon)Y + \varepsilon X), \quad (4) \\
\ddot{X} + 2\alpha_x \dot{X} + \alpha_x^2 X &= \frac{\alpha_x^2}{N} \sum_{k,n_x} \delta(t - t_{k,n_x}), \quad (5) \\
\ddot{Y} + 2\alpha_y \dot{Y} + \alpha_y^2 Y &= \frac{\alpha_y^2}{N} \sum_{k,n_y} \delta(t - t_{k,n_y}). \quad (6)
\end{align*}$$

Integration of these equations can be performed semi-analytically (cf. Ref. $\lbrack 10 \rbrack$). Between the firings, equations for $X, Y$ are linear and the solution can be written explicitly, substitution of these solutions to Eqs. (3,4) allows also for an analytic representation of $x_k(t), y_k(t)$. This results in transcendent equations for determining the next firing time, which is solved numerically using the Newton method.

### III. CROSS-FREQUENCY LOCKING

In this section we focus on the effect of cross-frequency locking in two interacting populations. As has been already mentioned, one neural population demonstrates macroscopic mean-field oscillations with frequency $\Omega$; two non-interacting populations will have generally different macroscopic frequencies $\Omega_x, \Omega_y$. We demonstrate now, that the interaction can lead to a rather nontrivial regime of locking of these macroscopic oscillations. In Fig. 1 we report on the frequencies of two neural populations for $a_x = 1.5, g_x = 0.35, a_y = 1.21, g_y = 0.09, \alpha_x = \alpha_y = 10, N = 50$. The figure shows the ratio of two macroscopic frequencies $\Omega_x/\Omega_y$ as a function of the coupling constant $\varepsilon$. One clearly sees an interval of cross-frequency $2 : 1$ locking for $\varepsilon \in [0.25; 0.33]$, here $\Omega_x = 2\Omega_y$. Outside of this region a macroscopic quasiperiodic regime with an irrational ratio $\Omega_x/\Omega_y$ is observed as illustrated by the phase portraits projections on plane $(X, Y)$ in Fig. 2. One clearly distinguishes the locked state at $\varepsilon = 0.3$ from the quasiperiodic states at $\varepsilon = 0.2$ and $\varepsilon = 0.4$ on these Lissagous-type curves.

![FIG. 1. Color online. Frequency ratios $\Omega_x/\Omega_y$ (solid line) and $\omega_x/\omega_y$ (dashed line). The 2:1 locking is observed for $\varepsilon \in [0.25; 0.33]$, but in this case $\omega_x/\omega_y$ is not equal to 2.](image)

![FIG. 2. Projections of the phase portraits on plane $(X, Y)$ for $\varepsilon = 0.2$ (left panel), $\varepsilon = 0.3$ (middle panel) and $\varepsilon = 0.4$ (right panel).](image)
In population \( x \) the states of the neuron are different, what indicates that its firing frequency \( \omega_x \) is incommensurate with the frequency \( \Omega_x \) of the mean field \( X(t) \). In contradistinction to this, in population \( y \) all the states are the same (they take two values because, due to 2:1 locking, the mean field \( X \) demonstrates two oscillations within one oscillation of the mean field \( Y \)) what means that neurons in this population are locked by the mean field \( Y(t) \) and fire with the same period as the period of \( Y(t) \), i.e. \( \Omega_y = \omega_y \). This property explains the behavior of the ration of firing rates of neurons plotted in Fig. 1: in the whole region where one observes locking 2:1 of the mean fields \( X, Y \), the ratio \( \omega_x/\omega_y \) of the firing rates of the neurons remains also a constant, although not equal to 2, but slightly exceeds this value due to a small difference between \( \omega_x \) and \( \Omega_x \).

FIG. 3. (Color online) Stroboscopic observations of the states of two oscillators in two populations, taken at \( X(t_j^*) = 1.5, X > 0 \). Blue circles: values \( x(t_j^*) \), red squares: values \( y(t_j^*) \). Not all points are depicted for better visibility.

IV. MARGINAL CHIMERA STATE

Locking of the neuron states in ensemble \( y \) by the mean field acting on these neurons, at \( \varepsilon = 0.3 \), allows one to expect that these neurons form a fully synchronized cluster. Surprisingly, this is not the case. In Fig. 4 we plot the states of all neurons in two populations at a certain moment of time, in the locked regime \( \varepsilon = 0.3 \). In population \( x \) all the states are different, what corresponds to the fact that they are in a quasiperiodic mode of partial synchronization. Neurons in population \( y \) are in a chimera state: a part of them build a fully synchronized cluster \( y_1 = y_2 = \ldots = y_m \), while all the neurons \( y_k, k > m \), are distributed in some range. We have checked that this property is not an artifact, by observing it for different population sizes and for very long evolution times (up to \( 10^6 \)).

At first glance, a chimera state in a population of identical oscillators described by first-order equations (like Eqs. (3,4) of our model) is impossible. Indeed, in the case of a periodic forcing by mean fields \( (X(t), Y(t)) \), Eq. (4) for a neuron in population \( y \) reduces to a one-dimensional circle map, if a stroboscopic map is constructed from this first-order equation. This map is the same for all neurons in population \( y \). According to general theory of one-dimensional circle maps, all neurons have the same frequency (because the rotation number of one-dimensional maps does not depend on initial conditions). Moreover, for general one-dimensional maps one has a dichotomy [12]: (i) either there is an equal number of unstable and stable periodic orbits, the latter attract all the points of the circle except for those lying exactly on unstable orbits; (ii) or the dynamics is quasiperiodic and reduces according to Denjoy’s theorem to a shift on the circle, here all initially different states remain different. This dichotomy allows for quasiperiodic (partial synchronization) and fully synchronized regimes, but seemingly excludes chimera states.

In the dynamics of our two coupled populations we see, that the population \( y \) violates the dichotomy above. To clarify this point, we constructed a stroboscopic map \( y(t_0) \rightarrow y(t_0 + T) \), where \( T \) is period of the field \( Y(t) \) (Fig. 5) and \( 0 \leq y(t_0) < 1 \). One can see that this map is not of general smooth type, as it has an interval (we call it marginal domain) where \( y(t_0 + T) \approx y(t_0) \) with high accuracy. This can be seen in panel (b) of Fig. 5, where the marginal domain is enlarged. Here the deviations \( y(t + T) - y(t) \) are of order \( 10^{-5} \). Moreover, these deviations fluctuate in sign, if the stroboscopic map is built at different phases \( t_0 \) of the mean fields \( X(t_0), Y(t_0) \). These fluctuations are due to the fact, that the fields \( Y(t) \) is, strictly speaking, not exactly periodic. Indeed, due to quasiperiodicity of the population \( x \), the fields \( X(t), Y(t) \) are quasiperiodic, although deviations from the pure periodicity are extremely small and are not seen in Fig. 2. This quasiperiodicity can, e.g., be seen if at a given value of \( Y \) plots values of \( \dot{Y} \), these are spread in a small interval. Due to this small spreading, the deviations from identity map fluctuate in the marginal domain.

Existence of the marginal domain, together with a complementary interval where \( y(t_0 + T) \neq y(t_0) \), explains the observed chimera state: those neurons which have initial conditions in the non-marginal domain, are attracted to one state and form a cluster, while those in the marginal domain remain scattered and form a “cloud”. Noteworthy, the described non-general properties of the

FIG. 4. (Color online) Snapshots of states \( x_k, y_k \) in the 2:1 locked state for \( \varepsilon = 0.3 \). While all values of \( x_k \) are different, in population \( y \) the neurons with \( 1 \leq k \leq 39 \) form a cluster.
dynamics are not pre-built to the system, but appear in a self-consistent manner, because the fields \( X, Y \) are composed from the contributions of individual neurons. Quite unexpectedly, these mean fields are self-organized in such a manner, that one population is purely quasiperiodic, while another one combines properties of stable and marginal dynamics that results in a chimera.

V. CONCLUSION

In this paper we considered two populations of integrate-and-fire oscillators having definitely different frequencies of the generated mean fields. First we showed, that due to mutual coupling, a 2:1 locking of the mean fields can be observed, without synchronization between the individual neurons. Noteworthy, neurons in two populations behave differently in the locked state: while in one population they are not synchronized by the mean field and have a different frequency, in another population the period of firings is the same as the basic period of the mean field. However, this synchronous state is rather nontrivial, and this is our second main result: identical neurons in the synchronous population build a chimera: one part of them forms an identical cluster, while other oscillators do not join this cluster and remain scattered. We explain this regime as a self-sustained marginal dynamics of the driven neurons: the corresponding stroboscopic one-dimensional map has a domain where this map is practically an identity. This marginality is possibly the only way to achieve a bistability in a one-dimensional map, as the period here must be independent of initial conditions. This is another peculiarity of the marginal chimera: in other cases where chimera has been observed in identical units, the frequencies of a cluster and a scattered populations were different [6].

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FIG. 5. (Color online) (a) The stroboscopic map \( y(t) \to y(t + T) \) for neurons in population \( y \) in the 2:1 locked regime for \( \varepsilon = 0.3 \). For the chosen phase of the mean field, in the region \( 0.25 \leq y \leq 0.45 \) the map is nearly identity. This region is resolved in panel (b), where we plot values \( y(t_0 + T) - y(t_0) \) vs \( y(t_0) \) for different choices of \( t_0 \). Small vertical shifts are due to a tiny degree of quasiperiodicity in the fields \( X(t), Y(t) \).

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