HECKE CHARACTERS AND THE MEAN-PERIODICITY CORRESPONDENCE FOR CM ELLIPTIC CURVES

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Abstract. Arithmetic schemes, whose zeta functions admit meromorphic continuation and the expected functional equation, correspond to mean-periodic elements in certain functional spaces by the mean-periodicity correspondence of Fesenko–Ricotta–Suzuki. Such elements satisfy convolution equations, and for elliptic curves with additional structure (CM) this paper provides an explicit description of the convolutors using simple techniques dating back to Tate’s thesis. We thus uncover a concrete manifestation of a tentative relationship between automorphicity of $L$-functions and the mean-periodicity correspondence for zeta-functions.

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1. Introduction

The zeta function of a regular scheme of finite type over $\mathbb{Z}$ is conjectured to have analytic properties extending those of the Dedekind zeta functions. For an introduction to such zeta functions, the reader is referred to [16]. The mean-periodicity correspondence developed by Fesenko, Ricotta and Suzuki in [3] provides a passage between arithmetic schemes satisfying this expectation and mean-periodic functions in appropriate functional spaces. If the arithmetic scheme is the model of an algebraic variety over a number field, its zeta-function is closely related to the $L$-function of the variety, and automorphicity of the Hasse–Weil $L$-factors imply the expected properties of the zeta-function. Moreover, the mean-periodicity condition is known to be a consequence of automorphicity [loc. cit., Theorems 4.2, 6.5]. It is interesting to understand the interaction between automorphicity and mean-periodicity in

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In this paper we will be dealing with models of CM elliptic curves, defined over a number field \( F \). As such, the automorphic object is very simple - a Hecke character of \( F \). Through direct computation, we will show how the Hecke character is related to the solution of the convolution equation anticipated by mean-periodicity.

Briefly we acknowledge a similar and inspiring result in this direction. In [19], Suzuki proves the mean-periodicity hypothesis for an elliptic curve over \( \mathbb{Q} \) using the associated cuspidal automorphic representation. Whereas Suzuki’s work uses technical machinery of Connes and Soulé, in this paper we use simpler techniques dating back to Tate’s thesis. With this simplification we can make an unconditional statement over an arbitrary number field.

We remark that in dimension 2, the mean-periodicity correspondence has an interpretation in terms of two-dimensional adelic analysis [5, 6]. Further developments in this direction are expected lead to a proof of the mean-periodicity hypothesis in the general case, without using any automorphic properties, conjectural or established. In turn, one would deduce a direct proof of the meromorphic continuation and functional equation of the \( L \)-functions of a curve over a number field. General connections between adelic analysis and automorphicity surely exist, but are yet to be uncovered.

We devote the rest of this introduction to explaining our result and outlining the structure of this paper. Mean-periodicity is a simple generalization of periodicity. It is a property of a function viewed as an element of a particular functional space \( \mathfrak{X} \), and its definition depends in the properties of \( \mathfrak{X} \). When a locally compact topological abelian group \( G \) acts on \( \mathfrak{X} \), we can define the set of translates of \( f \in \mathfrak{X} \):

\[
T(f) := \{ g \cdot f : g \in G \}.
\]

\( f \) is mean-periodic if \( T(f) \) is not dense in \( \mathfrak{X} \). We recover the definition of periodic smooth function by considering the following action of \( G = \mathbb{R} \) on \( \mathfrak{X} = C^\infty(\mathbb{R}) \):

\[
g \cdot f(x) = f(x - g).
\]

When the Hahn-Banach theorem is verified by \( \mathfrak{X} \) (as will be the case in this paper), this is equivalent to saying that \( f \) satisfies a homogeneous convolution equation \( f * f^* = 0 \) for some nontrivial \( f^* \in \mathfrak{X}^* \).

We begin by outlining the connection between zeta- and \( L \)-functions, after which we review mean-periodicity. Section 3 provides the proof of mean-periodicity in the CM case.

2. Zeta-Functions, Boundary Functions and CM Elliptic Curves

In this section we provide all background material and definitions required for the reading of this paper.

2.1. \( L \)- and Zeta-Functions of Elliptic Curves. Let \( F \) be a number field and \( \mathfrak{p} \) denote a prime ideal in \( \mathcal{O}_F \). The inertia group at \( \mathfrak{p} \) is denoted \( I_{\mathfrak{p}} \) and \( l \) denotes a rational prime
distinct from the characteristic of the residue field at \( p \). Let \( E \) be an elliptic curve over \( F \), with \( l \)-adic Tate module \( T_l(E) \). We use the common notation
\[ V_l(E) := T_l(E) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l. \]
There is an \( l \)-adic representation \( G_F \to V_l(E) \). Let \( f_p \) denote the Frobenius at \( p \), which acts on \( V_l(E) \).

One defines the \( L \)-function \( L_E(s) \), for \( s > \frac{3}{2} \) at least, as an Euler product over prime ideals \( p \) in \( \mathcal{O}_F \):
\[ L_{E,p}(s) = \det(1 - f_p \cdot Np^{-s}|V_l(E)^f_p)^{-1}, \]
\[ L_E(s) = \prod_p L_{E,p}(s). \]
Observe that this is the \( i = 1 \) case of the \( i \)-th Hasse–Weil \( L \)-function, in which if \( X \) is a projective variety over \( F \), for \( 0 \leq i \leq 2 \dim(X) \),
\[ L_p(H^i(X), s) = \det(1 - f_p \cdot Np^{-s}|H^i_{\text{ét}}(X_{\bar{F}}, \mathbb{Q}_l)^f)^{-1}. \]
As expressed in the introduction, Hasse–Weil \( L \)-functions are expected to be automorphic. We will refer to their alternating product as the Hasse–Weil zeta-function; in our case:
\[ \zeta_E(s) = \frac{\zeta_F(s)\zeta_F(s - 1)}{L_E(s)}. \]
Automorphicity of the \( L \)-factors does not imply that of zeta-functions, and the mean-periodicity should be viewed as a candidate replacement. Some intuition behind this can be found in [6, 4.2, Remark 3]. The content in the present paper is a worked example of this philosophy.

Amongst the most fundamental problems regarding the \( L \)-function are its meromorphic continuation and functional equation. For this, one needs a completed \( L \)-function \( \Lambda_E(s) \) incorporating the \( \Gamma \)-factor, the conductor and the discriminant of \( F \). For general definitions of local factors of completed Hasse–Weil \( L \)-functions, the reader is referred to [17]. In our case, let \( D_F \) be the discriminant of \( F \), \( q_E \) denote the conductor of \( E \) and normalize the Gamma functions at an archimedean place \( v \) as follows:
\[ \Gamma_{F,v}(s) = \begin{cases} \pi^{-s/2}\Gamma(s/2) & u \text{ real}, \\ 2(2\pi)^{-s}\Gamma(s) & u \text{ complex}. \end{cases} \]
As usual, \( r_1 \) is the number of real places and \( r_2 \) is the number of complex places. The Gamma factor of \( E \) is:
\[ \Gamma_E(s) = \Gamma_C(s)^{r_1+2r_2}. \]
Observe that \( \Gamma_E(s) \) does not appear due to the fact that the Hodge decomposition of \( H^1(E(\mathbb{C}), \mathbb{C}) \) has no symmetric summands (ie. \( H^{1,s} \)). Let \( d_F \) denote the discriminant of \( F \), then the completed \( L \)-function of \( E \), \( \Lambda(E, s) \) is:
\[ \Lambda(E, s) = (N_{F/\mathbb{Q}}(q_E)|D_F|^2s/2) L_E(s)\Gamma_E(s). \]
It is conjectured to extend meromorphically to \( \mathbb{C} \) and satisfy the following functional equation:
\[ \Lambda_E(s) = \pm \Lambda_E(2 - s). \]
Automorphicity would of course imply these basic conjectures. Perhaps the most famous example of automorphicity is that an elliptic curve over \( \mathbb{Q} \) is “modular” and its \( L \)-function is the \( L \)-function of a certain modular form. Another example, more elementary in both its statement and proof, is elliptic curves with complex multiplication - see [18, Chapter 2], which will be briefly recapped below. In general, there is still a long way to go - even the \( L \)-function of an elliptic curve over a proper extension of \( \mathbb{Q} \) is not known to be modular.

On the other hand, associated to \( E \) one has an infinitude of proper regular models over \( \Spec(\mathcal{O}_F) \). Such models are regular two-dimensional schemes of finite type over \( \mathbb{Z} \), and as such have a (Hasse) zeta-function. In general, if \( S \) is a scheme of finite type over \( \mathbb{Z} \), let \( S_0 \) denote the set of closed points. For \( x \in S_0 \), the residue field \( k(x) \) is finite. The Hasse zeta-function of \( S \) is defined on the half plane \( \Re(s) > 2 \) by the following product:

\[
\zeta_S(s) = \prod_{x \in S_0} \frac{1}{1-|k(x)|^{-s}}.
\]

The completed zeta-function \( \xi_S(s) \) is also expected to have meromorphic continuation and functional equation. In the case where \( S = \mathcal{E} \) is a proper regular model of \( E \),

\[
\xi_{\mathcal{E}}(s) = \frac{2^r \pi^{\frac{r_1+r_2}{2}}}{(s-1)^{r_1+r_2}} A_{\mathcal{E}}^{s/2} \xi_{\mathcal{E}}(s),
\]

\[
\xi_{\mathcal{E}}(2-s) = \pm \xi_{\mathcal{E}}(s).
\]

where \( A_{\mathcal{E}} \) is the conductor of the arithmetic surface \( \mathcal{E} \) [2], [15], [10]. In general, the gamma factor and conductor can be deduced from those of the Hasse–Weil \( L \)-functions.

**Lemma 2.1.** Let \( E \) be an elliptic curve over \( F \) and \( \mathcal{E} \rightarrow \Spec(\mathcal{O}_F) \) be a proper regular model. The meromorphic continuation and functional equation of \( \Lambda(E, s) \) is equivalent to that of \( \xi_{\mathcal{E}}(s) \).

**Proof.** Let \( \zeta_{\mathcal{E}}(s) \) denote the Hasse–Weil zeta function of \( E \) as in 1, with completion \( \xi_{\mathcal{E}}(s) \). Introduce the following functions:

\[
n_{\mathcal{E}}(s) = \frac{\zeta_{\mathcal{E}}(s)}{\zeta_{\mathcal{E}}(1)},
\]

\[
n_{\mathcal{E}}(s) = \frac{\xi_{\mathcal{E}}(1)}{\xi_{\mathcal{E}}(s)}.
\]

By [2, Proposition 1.1, Proof], \( n_{\mathcal{E}}(2-s) = \pm n_{\mathcal{E}}(s) \). Also,

\[
\xi_{\mathcal{E}}(2-s)\xi_{\mathcal{E}}(2-(s-1)) = \xi_{\mathcal{E}}(1-(s-1))\xi_{\mathcal{E}}(1-s) = \xi_{\mathcal{E}}(s-1)\xi_{\mathcal{E}}(s),
\]

and so the result follows from

\[
\zeta_{\mathcal{E}}(s) = n_{\mathcal{E}}(s)\frac{\xi_{\mathcal{E}}(s)\xi_{\mathcal{E}}(s-1)}{L_{\mathcal{E}}(s)}.
\]

**Remark 2.2.** The gamma factor for the (Hasse–Weil and Hasse) zeta-function is much simpler than that of the \( L \)-function, in fact, for models of elliptic curves, it is a rational function as in the formula above. Moreover, the completed zeta-function does not involve the discriminant of the base field, this is explained in [2, Remark 1.3].
Unlike $L$-functions, it is expected that zeta-functions of arithmetic schemes can be studied through commutative methods, extending the techniques of Iwasawa–Tate [7], [20]. As far back as Weil [21, Foreward], it has been suspected that the only obstacle is the insufficient development of harmonic analysis on more general topological abelian groups. Recently the case of models of elliptic curves over number fields was studied by Ivan Fesenko [5, 6], using two-dimensional adelic analysis.

The two-dimensional adelic approach leads to the conjecture that certain functions are mean-periodic in appropriate function spaces - see 2.2. In this paper we are concerned with direct connections between automorphy and mean-periodicity. No knowledge of two-dimensional adelic analysis will be required.

2.2. Function Spaces, Mean-Periodicity and Boundary Functions. In 2.2.3 we demonstrate how to construct a boundary function $h_E$ associated to a proper regular model $E$ of an elliptic curve over a number field. We will fix $E$ and thus there is no harm in denoting $h := h_E$. In this paper, $h$ is a complex valued function defined on the multiplicative topological group $\mathbb{R}_+^\times$. Composing with the exponential function, one obtains a function $H$ on the additive topological group $\mathbb{R}$ and the results of this work can easily be translated to equivalent statements in that context.

Mean-periodicity is defined in definition 2.4. It is important to observe that mean-periodicity is a property of a function in a given topological space of functions. Definitions of mean-periodicity of functions in a variety of different spaces can be found in [8] and [1]. Asking which function space is most appropriate for boundary functions is an interesting question. One reason for this is that functional spaces with special properties have several equivalent definitions of mean-periodicity and some may be more convenient than others.

For example, if a space has spectral synthesis, then a function is mean-periodic if and only if it can be expressed as a limit exponential polynomials in that space; generalizing Fourier series of periodic functions. Spectral synthesis is known to hold in the space $\mathcal{C}^{\infty}_{\text{poly}}(\mathbb{R}_+^\times)$ of infinitely differentiable complex valued functions on $\mathbb{R}$, with at most polynomial growth at $0^+$ and $+\infty$ [14]. For an elliptic curve over $\mathbb{Q}$, an explicit formula for the boundary function is known [6, 4.3, Discussion 1], from which it may be possible to prove mean-periodicity using this definition. We will not use the notion of spectral synthesis in this paper, nor will we attempt such a direct approach.

We will work with the space of weak tempered distributions, which is the dual of the strong Schwartz space $\mathcal{S}(\mathbb{R}_+^\times)^*$ on $\mathbb{R}_+^\times$ (definition 2.3), without the need for spectral synthesis. This is also the decision made by Suzuki [19]. $\mathcal{S}(\mathbb{R}_+^\times)^*$ appears elsewhere in the theory of zeta-functions, for example [11]. Of course, tempered distributions have long been connected with the fundamental properties of Hecke $L$-functions [22], [12]. The strong Schwartz space is continuously injected into the dual of $\mathcal{C}^{\infty}_{\text{poly}}(\mathbb{R}_+^\times)$ - see 2.

2.2.1. The Strong Schwartz Space. For $f : \mathbb{R} \to \mathbb{C}$, and positive integers $m$ and $n$, define

$$|f|_{m,n} = \sup_{x \in \mathbb{R}} |x^m f^{(n)}(x)|.$$
The Schwartz space $S(\mathbb{R})$ on $\mathbb{R}$ is the set of all smooth complex valued functions $f$ such that $|f|_{m,n} < \infty$, for all positive integers $m$ and $n$. The set $\{ |f|_{m,n} : m, n \in \mathbb{N} \}$ is a family of seminorms on $S(\mathbb{R})$ which induce a topology for which $S(\mathbb{R})$ is a Fréchet space over $\mathbb{C}$.

The Schwartz space on $\mathbb{R}_+^\times$, and its topology can be defined via the homeomorphism

$$S(\mathbb{R}) \rightarrow S(\mathbb{R}_+^\times)$$

$$f(t) \mapsto f(-\log x).$$

**Definition 2.3.** We define the topological space of strong Schwartz functions as follows:

(1) The strong Schwartz space on $\mathbb{R}_+^\times$ is defined by

$$S(\mathbb{R}_+^\times) = \bigcap_{s \in \mathbb{R}} \{ f : \mathbb{R}_+^\times \rightarrow \mathbb{C} : [x \mapsto x^{-s} f(x)] \in S(\mathbb{R}_+^\times) \}.$$

(2) $S(\mathbb{R}_+^\times)$ is given by the topology defined by the following family of seminorms:

$$||f||_{m,n} = \sup_{x \in \mathbb{R}_+^\times} |x^m f^{(n)}(x)|.$$

$\mathbb{R}_+^\times$ naturally acts on $S(\mathbb{R}_+^\times)$ by $\tau_a f(x) = f(x/a)$.

Dual to the strong Schwartz space on $\mathbb{R}_+^\times$, we have the space $S(\mathbb{R}_+^\times)^*$. With the weak $\ast$-topology, this is the space of weak-tempered distributions. There are continuous injections:

(2)

$$\begin{align*}
C^\infty_{\text{poly}}(\mathbb{R}_+^\times) & \prec S(\mathbb{R}_+^\times)^*, \\
S(\mathbb{R}_+^\times) & \prec C^\infty_{\text{poly}}(\mathbb{R}_+^\times)^*.
\end{align*}$$

2.2.2. **Mean-Periodicity in the Strong Schwartz Space.** Let $\phi$ be a weak tempered distribution, and let $f$ be a strong Schwartz function. We have a pairing

$$< , > : S(\mathbb{R}_+^\times) \times S(\mathbb{R}_+^\times)^* \rightarrow \mathbb{C}$$

$$< f, \phi >= \phi(f).$$

The convolution $f \ast \phi$ is defined by

$$\forall x \in \mathbb{R}_+^\times, \quad (f \ast \phi)(x) = < \tau_x f, \phi >,$$

where

$$\hat{f}(x) = f(x^{-1}).$$

Explicitly, the convolution is given by the following formula:

$$(f \ast \phi)(x) = \int_0^\infty f(x/y)\phi(y)\frac{dy}{y}.$$ 

For brevity, from now on we will denote $S(\mathbb{R}_+^\times)$ by $\mathfrak{X}$. In $\mathfrak{X}$, the following two definitions are equivalent:

**Definition 2.4.** A function $f \in \mathfrak{X}$ is ($\mathfrak{X}$)-mean-periodic if it satisfies either of the following equivalent conditions:

(1) There exists a non-trivial $f^* \in \mathfrak{X}^*$ such that $f \ast f^* = 0$. This also defines the $\mathfrak{X}^*$-mean-periodicity of $f^*$. 


(2) Let \( \mathcal{T}(f) \) to be the closure of \( \text{Span}_\mathbb{C}(\{\tau_a f : a \in \mathbb{R}_+^\times\}) \). \( f \) is \( \mathfrak{X} \)-mean-periodic if \( \mathcal{T}(f) \neq \mathfrak{X} \).

In light of these definitions, we will make use of the following space, for \( h \in \mathfrak{X} \):

\[
\mathcal{T}(h)^{\perp} = \{ \tau \in S(\mathbb{R}_+^\times)^* : g * \tau = 0, \forall g \in \mathcal{T}(h) \}.
\]

2.2.3. Boundary Functions. Let \( E \) be an elliptic curve over a number field \( F \) and let \( \mathcal{E} \to \text{Spec}(\mathcal{O}_F) \) be a proper regular model. Let \( \mathcal{C}_\mathcal{E} \) denote the following constant:

\[
\mathcal{C}_\mathcal{E} = N_{F/Q}(q_E) \prod_{p \in \text{Spec}(\mathcal{O}_F)} |k(p)|^{m_p-1},
\]

where \( q_E \) is the conductor of \( E \) and \( m_p \) is the number of irreducible geometric components of the fiber over the closed point \( p \in \text{Spec}(\mathcal{O}_F) \). For \( c > 1 \), let \( (c) \) be the line \( \{ z \in \mathbb{C} : \text{Re}(z) = c \} \). For \( x \in \mathbb{R}_+^\times \), we consider the following integral:

\[
f_\mathcal{E}(x) = \frac{1}{2\pi i} \int_{(c)} \xi_F\left(\frac{s}{2} + \frac{1}{4}\right)^2 c_\mathcal{E}^{-\frac{s}{2}+\frac{1}{4}} \xi_E(s + \frac{1}{2})^2 x^{-s} ds,
\]

where \( \xi_F(s) \) denotes the completed Dedekind zeta function of \( F \). Next we will define the boundary function for \( \mathcal{E} \) - the motivation for such terminology is very well explained in [3, Section 1].

**Definition 2.5.** The “boundary function” is the following function:

\[
h_\mathcal{E} : \mathbb{R}_+^\times \to \mathbb{C}
\]

\[
h_\mathcal{E}(x) = f_\mathcal{E}(x) - x^{-1} f_\mathcal{E}(x^{-1}).
\]

This is an element of \( S(\mathbb{R}_+^\times)^* \).

Mean-periodicity of the boundary function (in appropriate spaces, such as that of weak tempered distributions) is a sufficient condition for the analytic continuation and functional equation of the zeta-function of \( \mathcal{E} \), and hence the \( L \)-function of \( E \) [5, 6]. Several more connections between mean-periodicity and zeta-functions were exposed in [3].

**Remark 2.6.** In dimension 2 Fesenko [6, Section 4] introduced a closely related function. Its key properties can be reduced to those of an integral over an adelic boundary. More precisely, let \( X \) be a Noetherian surface with function field \( K \). At a closed point \( x \) on an irreducible curve \( y \) on \( X \), one has a two-dimensional local field \( K_{x,y} \) (see [13, Sections 6, 7]). At all closed points \( x \in y \), the discrete valuation field \( K_y \) embeds into \( K_{x,y} \). This induces an adelic embedding \( \mathbb{B} \hookrightarrow \mathbb{A} \). In two-dimensional adelic analysis, one considers integrals over \( \mathbb{A}_\mathbb{K} \times \mathbb{A}_\mathbb{K} \) and \( \mathbb{B}_\mathbb{K} \times \mathbb{B}_\mathbb{K} \). \( \mathbb{A}_\mathbb{K} \times \mathbb{A}_\mathbb{K} \) can be given a (weak) topology in such a way that integrals over the boundary \( \partial(\mathbb{B}_\mathbb{K} \times \mathbb{B}_\mathbb{K}) \) encode many properties regarding analytic properties of zeta-functions. The relationship of definition 2.5 to Fesenko’s original two-dimensional boundary function is explained in [19, Remark 2.1].

The proof of our main theorem 3.1 will implement the following result:

**Proposition 2.7.** Let \( \lambda \) denote a pole of \( \xi_F(\frac{s}{2} + \frac{1}{4})^2 c_\mathcal{E}^{-\frac{s}{2}+\frac{1}{4}} \xi_E(s + \frac{1}{2})^2 \) of multiplicity \( m_\lambda \), then

\[
h_\mathcal{E}(x) = \lim_{T \to \infty} \sum_{I_{m_\lambda}(\lambda) \leq T} \sum_{m=1}^{m_\lambda} C_m(\lambda) (-1)^{m-1} (m-1)! x^{-\lambda} (\log(x))^{m-1},
\]
where \( C_m(\lambda) \) are the coefficients of \((s-\lambda)^{-m}\) in the principal part of \( \xi_F(\frac{s}{2} + \frac{1}{4})^2 c_F^{-s} \zeta(s + \frac{1}{2})^2 \) at \( s = \lambda \).

**Proof.** [3, Section 5].

We recall some notation from lemma 2.1,

\[
N_\varepsilon(s) = \frac{\zeta_F(s)}{\zeta(s)}.
\]

As a consequence:

\[
\zeta(s) = n_\varepsilon(s) \frac{\zeta_F(s) \zeta_F(s-1)}{L_E(s)}.
\]

**Lemma 2.8.** The poles \( \tau = \lambda + \frac{1}{2} \) that appear in the expansion of 2.7 are classified as follows:

1. 0 or 2, in which case, \( m_\lambda = 4 \)
2. A zero of \( \Lambda(E, s) \), different from 1, with \( n_\varepsilon(\lambda)^{-1} \neq 0 \). In this case \( m_\lambda \) is non-negative and at most the multiplicity \( M \) of the zero of \( \Lambda(E, s)^2 \) at \( s = \lambda \).
3. A zero of \( \Lambda(E, s) \), different from 1, with \( n_\varepsilon(\lambda)^{-1} = 0 \). In this case \( -2 \leq m_\lambda - 2 \leq M \).
4. A zero of \( n_\varepsilon(s)^{-1} \), different from 1, with \( \Lambda(E, \lambda) \neq 0 \). In this case, \( m_\lambda = 2 \).
5. 1 - in which case, \( -2 - 2J \leq m_\lambda - 2 - 2J \leq M \).

**Proof.** We use the formula

\[
\zeta(s) = n_\varepsilon(s) \frac{\zeta_F(s) \zeta_F(s-1)}{L_E(s)},
\]

therefore a zero of the \( L \)-function of \( E \) gives a pole of the \( \zeta \)-function of \( E \).

1. \( \zeta(s) \) and \( \xi_F(s/2) \) each have poles of order 1 at \( s = 0, 2 \), so \( \xi_F(\frac{s}{2})^2 c_F^{-s} \zeta(s)^2 \) has a pole of order 4 at these points. From now on we assume that \( s \neq 0, 2 \).
2. In the next three cases we assume \( s \neq 1 \). By the above, a zero \( \lambda \) of \( \Lambda(E, s)^2 \) gives a pole of \( \zeta(s)^2 \), of order at most the order of the zero. If \( n_\varepsilon \) does not have a pole at \( \tau \), then the order of the pole of \( \Lambda(E, \frac{s}{2})^2 c_F^{-s} \zeta(s)^2 \) is \( \geq 0 \).
3. In the situation above, if \( n_\varepsilon(s) \) does have a pole here, then the order of the pole of \( \xi_F(\frac{s}{2})^2 c_F^{-s} \zeta(s)^2 \) is \( \geq -2 \).
4. If \( \Lambda(E, s)^2 \neq 0 \) but \( n_\varepsilon(s) \) has a pole, then \( \xi_F(\frac{s}{2})^2 c_F^{-s} \zeta(s)^2 \) has a pole of order 2 at \( s \).
5. Finally, at \( s = 1 \), we have a zero of \( \Lambda(E, s) \), and hence a pole of \( \zeta(s) \) and hence a pole of \( \xi_F(\frac{s}{2})^2 c_F^{-s} \zeta(s)^2 \) with the order bounded as prescribed.

This classification of poles motivates the definition of \( w_0 \) below (equation 3).

The following example shows a relationship between mean-periodicity and automorphicity. More detail can be found in [19].

**Example 2.9.** Let \( E/\mathbb{Q} \) be an elliptic curve and \( (\pi, V_\pi) \) be the corresponding cuspidal automorphic representation, which exists by the celebrated modularity theorem. In [19, Subsection 3.1] Suzuki constructs from this information a space \( \mathcal{W}_\pi \), defined as the convolution of a fixed function \( w_0 \) with two copies of the image of an operator defined by integration of an admissible coefficient against a Schwartz-Bruhat function. We will encounter \( w_0 \) in
section 2, but instead of using the above aforementioned operator, we will use zeta integrals of Tate. The space defined by Suzuki is dual to $\mathcal{T}(h_\nu)$ in the sense that:

$$\mathcal{T}(h_\nu) \subset W_\pi^\perp$$ and $W_\pi \subset \mathcal{T}(h_\nu)^\perp$.

The proofs work by directly showing, for example, that $w \ast h_\nu = 0$ for any $w \in W_\pi$, using knowledge of zeros of automorphic L-functions. We will use a similar argument in the proof of theorem 3.1. It a very concrete sense, this first inclusion tells us that modularity is a stronger condition than mean-periodicity, correlating with the idea that the latter will be easier to prove.

In the situation when we have the equality

$$\mathcal{T}(h_\nu)^\perp = W_\pi,$$

we can say something more - see corollary 3.2.

2.3. L-Series of CM Elliptic Curves. The basic reference for CM elliptic curves required in this paper is [18, Chapter 2]. Every elliptic curve with complex multiplication is defined over an algebraic extension of $\mathbb{Q}$.

Example 2.10. Let $E$ be the elliptic curve defined by the equation

$$y^2 = x^3 + x.$$ 

$E$ then has complex multiplication by $\mathbb{Z}[i]$. Indeed, we have an isomorphism

$$\mathbb{Z}[i] \rightarrow \text{End}(E),$$

defined by

$$[i](x,y) = (-x, iy).$$

Similarly, the curve given by $y^2 = x^3 + 1$ has CM by $\mathbb{Z}[e^{2\pi i/3}]$.

In general, if an elliptic curve has CM, the endomorphism ring is isomorphic an order in the ring of integers of a quadratic imaginary field, $K$. In this case we will say the curve has CM by $K$.

Let $K$ be a quadratic imaginary field and let $\mathfrak{a}$ be a fractional ideal of $K$, an idele $x \in \mathbb{A}_K^\times$ defines a map

$$x : K/\mathfrak{a} \rightarrow K/xb$$

by commutativity of the following diagram:

$$\begin{array}{ccc}
K/\mathfrak{a} & \xrightarrow{x} & K/xb \\
\downarrow \sim & & \downarrow \sim \\
\oplus_p K_p/\mathfrak{a}_p & \xrightarrow{\sim} & \oplus_p K_p/xb_p \\
(t_p) & \mapsto & (x_p t_p).
\end{array}$$

Let $\sigma$ denote an automorphism of $\mathbb{C}$ and let $s \in \mathbb{A}_K^\times$ be mapped to $\sigma|_{K^{\text{ab}}}$ under the reciprocity map. If $E$ is an elliptic curve over $\mathbb{C}$ such that $\text{End}(E) \cong \mathcal{O}_K$, then we can fix a complex analytic isomorphism

$$f : \mathbb{C}/\mathfrak{a} \rightarrow E(\mathbb{C}).$$
where \(a\) is a fractional ideal of \(K\). The main theorem of complex multiplication says that for each \(f\) and \(\sigma\) there is a unique complex analytic isomorphism \(f' : \mathbb{C}/s^{-1}a \to E^\sigma\mathbb{C}\) such that the following diagram commutes:

\[
\begin{array}{ccc}
K/a & \xrightarrow{s^{-1}} & K/s^{-1}a \\
\downarrow f & & \downarrow f' \\
E(\mathbb{C}) & \xrightarrow{\sigma} & E^\sigma(\mathbb{C}).
\end{array}
\]

Let \(E\) be defined over \(F\) and assume that \(F\) contains \(K\). For \(x \in \mathbb{A}_F^\times\), put \(s = N_K^F x \in \mathbb{A}_K^\times\). There is a unique \(\alpha_{E/L}(x) =: \alpha \in K^\times\) such that

1. \(\alpha\mathcal{O}_K\) is the ideal of \(s \in \mathbb{A}_K^\times\)
2. For any fractional ideal \(a \subset K\) and any analytic isomorphism \(F : \mathbb{C}/a \to E(\mathbb{C})\),

the following diagram commutes:

\[
\begin{array}{ccc}
K/a & \xrightarrow{\alpha s^{-1}} & K/xa \\
\downarrow f & & \downarrow f \\
E(\mathbb{C}) & \xrightarrow{\sigma_L(x)} & E^\sigma(\mathbb{C}),
\end{array}
\]

where \(\sigma_L\) denotes the reciprocity map of \(L\).

The Hecke character associated to \(E/F\) is defined by:

\[
\psi_{E/F} : \mathbb{A}_F^\times \to \mathbb{C}^\times
\]

\[
\psi_{E/F}(x) = \alpha_{E/F}(x) N_K^F (x^{-1})_\infty.
\]

For CM elliptic curves, meromorphic continuation and functional equation are clearly a consequence of the following theorem.

**Theorem 2.11.** Let \(E\) be an elliptic curve over a number field \(F\) with CM by the quadratic imaginary number field \(K\). Let \(\psi_{E/F}\) denote the associated Hecke characters \(\mathbb{A}_F^\times \to \mathbb{C}^\times\) and denote its complex conjugate by \(\overline{\psi_{E/F}}\), then

\[
L(E/K, s) = \begin{cases} 
L(s, \psi_{E/F})L(s, \overline{\psi_{E/F}}), & \text{if } K \subseteq F, \\
L(s, \psi_{E/F}), & \text{otherwise},
\end{cases}
\]

where \(F' = FK\) in the case that \(K\) is not contained in \(F\).

We will use this identification to prove mean-periodicity of the boundary function directly from knowledge of the elliptic curve, rather than through two-dimensional adelic analysis.

**Remark 2.12.** The L-functions of CM abelian varieties are also of a simple form - see [9, Chapter 4, Section 6]. Moreover, the a mean-periodicity correspondence for all arithmetic schemes over \(\mathbb{Z}\), in particular models of abelian varieties, [3]. It is therefore a reasonable goal to prove the mean-periodicity of boundary functions associated to \(L\)-functions of CM abelian varieties - we leave this for a later work. We note that in higher dimensions we do not have the corresponding adelic analysis required to construct boundary functions (although a local theory is began in [4]).
3. Mean-Periodicity the Strong Schwartz Space

In this section we prove a relationship between the Hecke character of a CM elliptic curve and convolutors of the boundary function. In particular, we prove mean-periodicity of the boundary function in this case.

We keep the notation above: $E$ is an elliptic curve over $F$ with CM by $K$ and associated Hecke character $\psi_{E/F}$. We have a well-known surjective module map $| | : \mathbb{A}_F^\times \rightarrow \mathbb{R}_+^\times$.

For all $x \in \mathbb{R}_+^\times$, we will use the following notation:

$$(\mathbb{A}_F^\times)_x := \{ \alpha \in \mathbb{A}_F^\times : |\alpha| = x \}.$$ 

Let $\chi$ be a multiplicative character of $\mathbb{A}_F^\times$, trivial on $F^\times$. For an adelic Schwartz function $f \in S(\mathbb{A}_F)$, we can define

$$Z_{f,\chi}(x) = \int_{(\mathbb{A}_F^\times)_x} f(\alpha)\chi(\alpha)|\alpha|^s d\alpha,$$

where $s \in \mathbb{C}$ and $d$ is the usual Haar measure on the idele group of $F$. This is a map $S(\mathbb{A}_F) \rightarrow S(\mathbb{R}_+^\times)$.

We will denote the image by $\mathcal{V}_\chi$. The relationship between $Z_{f,\chi}$ and the Hecke $L$-function $L(s,\chi)$ of $\chi$ is as follows:

$$\int_0^\infty Z_{f,\chi}(x)\frac{dx}{x} = \int_{\mathbb{A}_F^\times} f(\alpha)\chi(\alpha)|\alpha|^s d\alpha = h_{f,\chi}(s)L(s,\chi),$$

for some nonzero holomorphic function $h_{f,\chi}$ depending on $f$ and $\chi$.

Define

$$(3) \quad w_0 = \frac{1}{2\pi i} \int_{(c)} (2(2\pi)^s)^{r_1+2r_2}\Gamma_E\left(\frac{s}{4}\right)^2\frac{\zeta_E(q_E)}{\eta_E(s)}s^4(s-2)^4(s-1)^2x^{-s} ds.$$ 

Finally, define

$$\mathcal{W}_{\psi_{E/F}} = \left\{ \begin{array}{ll}
  w_0 * \mathcal{V}_{\psi_{E/F}} * \mathcal{V}_{\psi_{E/F}} * \mathcal{V}_{\psi_{E/F}} * \mathcal{V}_{\psi_{E/F}} & \text{if } K \subset F, \\
  w_0 * \mathcal{V}_{\psi_{E/F'}} * \mathcal{V}_{\psi_{E/F'}} & \text{otherwise},
\end{array} \right.$$ 

where we use the following shorthand:

$$\mathcal{U} * \mathcal{V} := \text{Span}_\mathbb{C}\{ u * v : u \in \mathcal{U}, v \in \mathcal{V} \}.$$ 

This is a subspace of the strong Schwartz space, which is closed under convolution. Mean-periodicity of the boundary term in the space of weak tempered distributions is a consequence of the following result:

**Theorem 3.1.**

$$\mathcal{W}_{\psi_{E/F}} \subset \mathcal{T}(h_\mathcal{E})^\perp.$$ 

We will drop the subscript $E/F$ from $\psi$ and prove the case when $K \subset F$, the other case is completely similar. The technique of proof used below is very similar to that found in [19].
Proof. Let \( w \) be any function in \( \mathcal{W}_{\psi_{E/F}} \), we will show that its convolution with the boundary function is 0.

If \( \lambda \) is a pole of \( \xi_F(s + \frac{1}{2})^2 \), \( s = -\frac{1}{2} \zeta_F(s + \frac{1}{2})^2 \) of order \( m_\lambda \), for \( 0 \leq k \leq m_\lambda - 1 \), let \( f_{\lambda,k} \) be
\[
f_{\lambda,k}(x) = x^{-\lambda}(\log(x)^k).
\]

By proposition 2.7,
\[
h_\xi(x) = \lim_{T \to \infty} \sum_{m(\lambda) \leq T} \sum_{m=0}^{m-1} C_m(\lambda) \frac{(-1)^{m-1}}{(m-1)!} f_{\lambda,k}(x).
\]

By linearity of convolution \( w \in \mathcal{W}_\psi \), we have
\[
w * h_\xi(x) = \lim_{T \to \infty} \sum_{m(\lambda) \leq T} \sum_{m=0}^{m-1} C_m(\lambda) \frac{(-1)^{m-1}}{(m-1)!} f_{\lambda,k} * h_\xi(x),
\]
\[
w * f_{\lambda,k}(x) = \int_{0}^{\infty} w(y)f_{\lambda,k}(x/y) \frac{dy}{y}.
\]

Applying a binomial expansion to \( (\log(x/y))^k \), and using the symmetry of the convolution operation gives:
\[
w * f_{\lambda,k}(x) = \sum_{i=1}^{k} (-1)^i \binom{k}{j} x^{-\lambda}(\log(x))^{k-j} \int_{0}^{\infty} \omega(y)y^\lambda(\log(y))^j \frac{dy}{y}.
\]

Observe that:
\[
\int_{0}^{\infty} w(y)y^\lambda(\log(y))^j \frac{dy}{y} = \frac{d^j}{d\lambda^j} \int_{0}^{\infty} \omega(y)y^\lambda dy.
\]

Recall that \( \mathcal{W}_\psi = w_0 * \mathcal{V}_{\psi_{E/F}} * \mathcal{V}_{\psi_{E/F}} * \mathcal{V}_{\psi_{E/F}} \), so there are \( f_1, f_2, f_3, f_4 \in S(\mathbb{A}_F) \) such that \( w = w_0 * Z_{\psi, f_1} * Z_{\psi, f_2} * Z_{\psi, f_3} * Z_{\psi, f_4} \) and
\[
\int_{0}^{\infty} w(y)y^\lambda dy = \int_{0}^{\infty} w_0(y)y^\lambda dy \int_{0}^{\infty} Z_{\psi, f_1}(y) dy \int_{0}^{\infty} Z_{\psi, f_2}(y) dy \int_{0}^{\infty} Z_{\psi, f_3}(y) dy \int_{0}^{\infty} Z_{\psi, f_4}(y) dy.
\]

Let \( \chi \) be an Hecke character of \( F \), and \( f \) be an adelic Schwartz function, then
\[
\int_{0}^{\infty} (Z_{\chi,f})(y) \frac{dy}{y} = \int_{\mathcal{E}_F} f(\alpha)\chi(\alpha) |\alpha|^s d\alpha.
\]

We recall the following fact:
\[
\int_{\mathcal{E}_F} f(\alpha)\chi(\alpha) |\alpha|^s d\alpha = h(s, \chi)L(s, \chi),
\]
where \( h \) is a nonzero meromorphic function in \( s \).

Altogether, we obtain
\[
\int_{0}^{\infty} w(y)y^\lambda dy = \int_{0}^{\infty} w_0(y)y^\lambda h_{f_1}(s, \psi)L(s, \psi)h_{f_2}(s, \psi)L(s, \psi)h_{f_3}(s, \psi)L(s, \psi)h_{f_4}(s, \psi)L(s, \psi),
\]
for some entire functions $h_f$. Theorem 2.11 therefore gives:

$$\int_0^\infty w(y)y^\lambda dy = \int_0^\infty w_0(y)y^\lambda dy h_f(s, \psi)h_f^2(s, \psi)h_f^3(s, \bar{\psi})h_f^4(s, \bar{\psi})L(E/F)^2.$$  

The remaining integral evaluates as:

$$\int_0^\infty w_0(y)y^\lambda dy = \Gamma_E(\lambda^2)\lambda^4(\lambda - 2)^4(\lambda - 1)^2(\frac{c_E}{q_E})^\lambda n_E(\lambda)^{-2}.$$  

So, the classification of possible $\lambda$ in 1.1 implies that $w \ast h_\xi = 0$ and hence $W_\psi \subset \mathcal{T}(h_\xi)^\perp$.

Corollary 3.2. If $\mathcal{T}(h_\xi) = W_\psi^\perp$, there are no cancelations of zeros between

$$(s - 1)\hat{\zeta}(s/2)\hat{\zeta}_F(s)\hat{\zeta}_E(s - 1)$$

and

$$n_\xi(s)^{-1}\Lambda_E(s).$$

4. Further Work

Of course, the ultimate goal is to prove that boundary functions are mean-periodic in general. From this work, and that of Suzuki, one is lead to ask if the convoluter of the boundary term can always be constructed from the associated automorphic representation.

In dimension 2, independent of automorphicity, one might hope that the mean-periodicity hypothesis can be verified through the geometric methods of two-dimensional adelic analysis. For this, one should consider alternative definitions of mean-periodicity in appropriate functional spaces.

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